Three-Cluster Equation Using Two-Cluster RGM Kernel

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We propose a new type of three-cluster equation which uses two-cluster resonating-group-method (RGM) kernels. In this equation, the orthogonality of the total wave function to two-cluster Pauli-forbidden states is essential to eliminate redundant components admixed in the three-cluster systems. The explicit energy-dependence inherent in the exchange RGM kernel is self-consistently determined. For bound-state problems, this equation is straightforwardly transformed to the Faddeev equation which uses a modified singularity-free $T$-matrix constructed from the two-cluster RGM kernel. The approximation of the present three-cluster formalism can be examined with more complete calculation using the three-cluster RGM. As a simple example, we discuss three di-neutron ($3d'$) and $3\alpha$ systems in the harmonic-oscillator variational calculation. The result of the Faddeev calculation is also presented for the $3d'$ system.

§1. Introduction

All the present-day quark-model descriptions of the nucleon-nucleon ($NN$) and hyperon-nucleon ($YN$) interactions incorporate important roles of the quark-gluon degrees of freedom in the short-range region and the meson-exchange processes dominated in the medium- and long-range parts of the interaction. For example, we have introduced one-gluon exchange Fermi-Breit interaction and effective meson-exchange potentials acting between quarks, and have achieved very accurate descriptions of the $NN$ and $YN$ interactions with limited number of parameters. We hope that the derived interaction in these models can be used for a realistic calculation of few-baryon systems like the hypertriton and various types of baryonic matter. This program, however, involves a non-trivial problem of how to extract the effective two-baryon interaction from the microscopic quark-exchange kernel. The basic baryon-baryon interaction is formulated as a composite-particle interaction in the framework of the resonating-group method (RGM). If we rewrite the RGM equation in the form of the Schrödinger-type equation, the interaction term becomes non-local and energy dependent. Furthermore, the RGM equation sometimes involves redundant components due to the effect of the antisymmetrization, which is related to the existence of the Pauli-forbidden states. In such a case, the full off-shell $T$-matrix is not well defined in the standard procedure which usually assumes simple
energy-independent local potentials. Since these features are related to the characteristic description of the short-range part in the quark model, it would be desirable if one can use the quark-exchange kernel directly in the application to many-baryon systems.

In this paper, we propose a new type of three-cluster equation which uses two-cluster RGM kernel for the inter-cluster interaction. We assume, for simplicity, three identical clusters having only one Pauli-forbidden state for the inter-cluster relative motion, but the extension to general systems is rather straightforward. We first consider a two-cluster RGM equation and the structure of $T$-matrix constructed from the two-cluster RGM kernel (RGM $T$-matrix). Next, we formulate a three-cluster equation which employs the two-cluster RGM kernel and a projection operator on the pairwise Pauli-allowed space. This equation is converted to the Faddeev equation which uses a non-singular part of the RGM $T$-matrix. Finally, we show some examples of the present formulation with respect to the $0^+$ ground states of three di-neutron ($3d'$) and $3\alpha$ systems. The calculation is performed in the variational method, using the translationally-invariant harmonic oscillator basis. For the $3d'$ system, the result of the Faddeev calculation is also presented. Detailed comparison is made with respect to the more desirable three-cluster RGM calculation, and to some other approximations like “renormalized RGM” and the well-known orthogonality condition model (OCM).

§2. $T$-matrix of the two-cluster RGM kernel

We use the same notation as used in Ref. 5) and write a two-cluster RGM equation as

$$\left[ \varepsilon - H_0 - V^{\text{RGM}}(\varepsilon) \right] \chi = 0 , \quad (2.1)$$

where $\varepsilon$ is the relative energy, $\varepsilon = E - E^{\text{int}}$, between the two clusters, $H_0$ is the relative kinetic-energy operator, and

$$V^{\text{RGM}}(\varepsilon) = V + \varepsilon K , \quad (2.2)$$

is the RGM kernel composed of the direct potential $V$, the sum of the exchange kinetic-energy and interaction kernels, $G = G^K + G^V$, and the exchange normalization kernel $K$. We assume that there exists only one Pauli-forbidden state $|u\rangle$, which satisfies the eigen-value equation $K|u\rangle = \gamma |u\rangle$ with the eigen-value $\gamma = 1$. The projection operator on the Pauli-allowed space for the relative motion is denoted by $\Lambda = 1 - |u\rangle\langle u|$. Using the basic property of the Pauli-forbidden state $|u\rangle$, $(H_0 + V + G)|u\rangle = \langle u|(H_0 + V + G) = 0$, we can separate $V^{\text{RGM}}(\varepsilon)$ into two distinct parts:

$$V^{\text{RGM}}(\varepsilon) = V(\varepsilon) + v(\varepsilon) , \quad (2.3)$$

where

$$V(\varepsilon) = (\varepsilon - H_0) - \Lambda(\varepsilon - H_0) \Lambda = \varepsilon|u\rangle\langle u| + \Lambda H_0 \Lambda - H_0 ,$$

$$v(\varepsilon) = \Lambda V^{\text{RGM}}(\varepsilon) \Lambda = \Lambda (V_D + G + \varepsilon K) \Lambda . \quad (2.4)$$
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Note that \( \Lambda V(\varepsilon)A = 0 \) and \( \Lambda v(\varepsilon)A = v(\varepsilon) \); namely, \( V(\varepsilon) \) may be considered as an operator acting in the Pauli-forbidden space, while \( v(\varepsilon) \) an operator acting in the Pauli-allowed space. Using these properties, we can express Eq. (2.1) as

\[
\Lambda \left[ \varepsilon - H_0 - v(\varepsilon) \right] A \chi = 0 . \tag{2.5}
\]

The separation of \( V_{\text{RGM}}(\varepsilon) \) in Eq. (2.3) enables us to deal with the energy dependence of the exchange RGM kernel in the Pauli-forbidden space and that in the allowed space, separately. Let us generalize Eq. (2.3) to

\[
V(\omega, \varepsilon) = V(\omega) + v(\varepsilon) , \tag{2.6}
\]

which we use in the following three-cluster formulation. We will see that the energy dependence involved \( V(\omega) \) can be eliminated by the orthogonality condition to the Pauli-forbidden state.

Since the direct application of the \( T \)-matrix formalism to Eq. (2.1) leads to a singular off-shell \( T \)-matrix, we first consider the subsidiary equation

\[
\left[ \omega - H_0 - V_{\text{RGM}}(\varepsilon) \right] \chi = 0 , \tag{2.7}
\]

with \( \omega \neq \varepsilon \), and extract a singularity-free off-shell \( T \)-matrix starting from the standard \( T \)-matrix formulation for the “potential” \( V_{\text{RGM}}(\varepsilon) \). A formal solution of the \( T \)-matrix equation

\[
T(\omega, \varepsilon) = V_{\text{RGM}}(\varepsilon) + V_{\text{RGM}}(\varepsilon) G_0^{(+)}(\omega) T(\omega, \varepsilon) \tag{2.8}
\]

with \( G_0^{(+)}(\omega) = 1/(\omega - H_0 + i0) \) is given by

\[
T(\omega, \varepsilon) = \tilde{T}(\omega, \varepsilon) + (\omega - H_0)|u\rangle \frac{1}{\omega - \varepsilon} \langle u|(\omega - H_0) ,
\]

\[
\tilde{T}(\omega, \varepsilon) = T_v(\omega, \varepsilon) - \left( 1 + T_v(\omega, \varepsilon) G_0^{(+)}(\omega) \right) |u\rangle \frac{1}{\langle u|G_0^{(+)}(\varepsilon)|u\rangle} \langle u|G_0^{(+)}(\varepsilon)|u\rangle ,
\]

\[
\times \langle u| \left( 1 + G_0^{(+)}(\omega) T_v(\omega, \varepsilon) \right) , \tag{2.9}
\]

where \( T_v(\omega, \varepsilon) \) is defined by

\[
T_v(\omega, \varepsilon) = v(\varepsilon) + v(\varepsilon) G_0^{(+)}(\omega) T_v(\omega, \varepsilon) . \tag{2.10}
\]

This result is obtained through the expression for the full Green function given by

\[
G^{(+)}(\omega, \varepsilon) = \frac{1}{\omega - H_0 - V_{\text{RGM}}(\varepsilon) + i0} = G_A^{(+)}(\omega, \varepsilon) + |u\rangle \frac{1}{\omega - \varepsilon} \langle u| , \tag{2.11}
\]

where

\[
G_A^{(+)}(\omega, \varepsilon) = G_v^{(+)}(\omega, \varepsilon) - G_v^{(+)}(\omega, \varepsilon) |u\rangle \frac{1}{\langle u|G_v^{(+)}(\varepsilon)|u\rangle} \langle u|G_v^{(+)}(\varepsilon)|u\rangle \times \langle u|G_v^{(+)}(\varepsilon) , \tag{2.12}
\]
and $G_v^{(+)}(\omega, \varepsilon) = 1/(\omega - H_0 - v(\varepsilon) + i0)$ is the solution of
\[ G_v^{(+)}(\omega, \varepsilon) = G_0^{(+)}(\omega) + G_0^{(+)}(\omega) v(\varepsilon) G_v^{(+)}(\omega, \varepsilon) . \] (2.13)

In fact, the simple relationship
\[ V(\varepsilon) = V(\omega) - (\omega - \varepsilon)|u\rangle\langle u| \] (2.14)
yields
\[ \omega - H_0 - V_{\text{RGM}}(\varepsilon) = \Lambda(\omega - H_0 - v(\varepsilon)) A + (\omega - \varepsilon)|u\rangle\langle u| . \] (2.15)

Then, if one uses the property
\[ \Lambda[\omega - H_0 - v(\varepsilon)] A \cdot G^{(+)}_A(\omega, \varepsilon) = G^{(+)}_A(\omega, \varepsilon) \cdot \Lambda[\omega - H_0 - v(\varepsilon)] A = \Lambda , \] (2.16)
it is easily found that
\[ \left[ \omega - H_0 - V_{\text{RGM}}(\varepsilon) \right] G^{(+)}(\omega, \varepsilon) = \left\{ \Lambda[\omega - H_0 - v(\varepsilon)] A + (\omega - \varepsilon)|u\rangle\langle u| \right\} \left\{ G^{(+)}_A(\omega, \varepsilon) + |u\rangle \frac{1}{\omega - \varepsilon} \langle u| \right\} = \Lambda + |u\rangle\langle u| = 1 . \] (2.17)

The expression of $T(\omega, \varepsilon)$ in Eq. (2.9) is most easily obtained from
\[ G^{(+)}(\omega, \varepsilon) = G_0^{(+)}(\omega) + G_0^{(+)}(\omega) T(\omega, \varepsilon) G_0^{(+)}(\omega) \] (2.18)
or
\[ T(\omega, \varepsilon) = (\omega - H_0) G^{(+)}(\omega, \varepsilon) (\omega - H_0) - (\omega - H_0) . \] (2.19)

The basic relationship which will be used in the following is
\[ G_0^{(+)}(\omega) T(\omega, \varepsilon) = G^{(+)}(\omega, \varepsilon) V_{\text{RGM}}(\varepsilon) \]
\[ = G^{(+)}_A(\omega, \varepsilon) V_{\text{RGM}}(\varepsilon) + |u\rangle \frac{1}{\omega - \varepsilon} \langle u| V_{\text{RGM}}(\varepsilon) \]
\[ = G^{(+)}_0(\omega, \varepsilon) V(\omega, \varepsilon) - |u\rangle\langle u| + |u\rangle \frac{1}{\omega - \varepsilon} \langle u| (\omega - H_0) \]
\[ = G^{(+)}_0(\omega) \tilde{T}(\omega, \varepsilon) + |u\rangle \frac{1}{\omega - \varepsilon} \langle u| (\omega - H_0) , \] (2.20)

where $\tilde{T}(\omega, \varepsilon)$ satisfies
\[ G^{(+)}_0(\omega) \tilde{T}(\omega, \varepsilon) = G^{(+)}_A(\omega, \varepsilon) V(\omega, \varepsilon) - |u\rangle\langle u| . \] (2.21)

These can be shown by using Eqs. (2.11) and (2.14). The full $T$-matrix, $T(\omega, \varepsilon)$, in Eq. (2.9) is singular at $\varepsilon = \omega$, while $\tilde{T}(\omega, \varepsilon)$ does not have such singularity. For $\varepsilon \neq \omega$, $T(\omega, \varepsilon)$ satisfies the relationship
\[ \langle u| G^{(+)}_0(\omega) T(\omega, \varepsilon)|\phi\rangle = \langle \phi| T(\omega, \varepsilon) G^{(+)}_0(\omega)|u\rangle = 0 \] (2.22)
for the plane wave solution $|\phi\rangle$ with the energy $\varepsilon$, i.e., $(\varepsilon - H_0)|\phi\rangle = 0$. This relationship is a direct result of more general relationship

$$\langle u| [1 + G_0^{(+)}(\omega) \tilde{T}(\omega, \varepsilon)] = [1 + \tilde{T}(\omega, \varepsilon) G_0^{(+)}(\omega)] |u\rangle = 0 \, , \quad (2.23)$$

which is simply seen from Eq. (2.21).

We note that $\tilde{T}(\omega, \varepsilon)$ satisfies

$$\tilde{T}(\omega, \varepsilon) = V(\omega, \varepsilon) - |u\rangle \langle u| (\omega - H_0) + V(\omega, \varepsilon) G_0^{(+)}(\omega) \tilde{T}(\omega, \varepsilon) \, ,$$

$$\tilde{T}(\omega, \varepsilon) = V(\omega, \varepsilon) - (\omega - H_0)|u\rangle \langle u| + \tilde{T}(\omega, \varepsilon) G_0^{(+)}(\omega) V(\omega, \varepsilon) \, . \quad (2.24)$$

However, these asymmetric forms of the $T$-matrix equations do not determine the solution $\tilde{T}(\omega, \varepsilon)$ uniquely, since the resolvent kernel $\left[1 - V(\omega, \varepsilon) G_0^{(+)}(\omega)\right]^{-1}$ has a singularity related to the existence of the trivial solution $|u\rangle$:

$$\langle u| [1 - V(\omega, \varepsilon) G_0^{(+)}(\omega)] = 0 \, . \quad (2.25)$$

The driving term, $V(\omega, \varepsilon) - |u\rangle \langle u| (\omega - H_0)$, etc., guarantees the existence of the solution, except for an arbitrary admixture of $(\omega - H_0)|u\rangle$ component. In order to eliminate this ambiguity and to make $\tilde{T}(\omega, \varepsilon)$ symmetric, one has to impose some orthogonality conditions, which will be discussed in a separate paper.

§3. Three-cluster equation

Let us consider a system composed of three identical spinless particles, interacting via the two-cluster RGM kernel $V^{RGM}(\varepsilon)$. The energy dependence involved in $V^{RGM}(\varepsilon)$ should be treated properly by calculating the expectation value of the two-cluster subsystem, at least for $v(\varepsilon)$. On the other hand, the energy dependence involved in $V(\varepsilon)$ is of kinematical origin, and could be modified so as to be best suited to the three-cluster equation. The three-body equation we propose is expressed as

$$P \left[ E - H_0 - V^{RGM}_\alpha(\varepsilon_\alpha) - V^{RGM}_\beta(\varepsilon_\beta) - V^{RGM}_\gamma(\varepsilon_\gamma) \right] P \Psi = 0 \, , \quad (3.1)$$

where $H_0$ is the free three-body kinetic-energy operator and $V^{RGM}_\alpha(\varepsilon_\alpha)$ stands for the RGM kernel Eq. (2.2) for the $\alpha$-pair, etc. The two-cluster relative energy $\varepsilon_\alpha$ in the three-cluster system is self-consistently determined through

$$\varepsilon_\alpha = \langle P\Psi | h_\alpha + V^{RGM}_\alpha(\varepsilon_\alpha) | P\Psi \rangle \, , \quad (3.2)$$

using the normalized three-cluster wave function $P\Psi$ with $\langle P\Psi | P\Psi \rangle = 1$. Here $h_\alpha$ is the free kinetic-energy operator for the $\alpha$-pair. Also, $P$ is the projection operator on the $[3]$ symmetric Pauli-allowed space as defined below. We solve the eigen-value problem

$$\sum_\alpha |u_\alpha\rangle \langle u_\alpha| \Psi_\lambda \rangle = \lambda |\Psi_\lambda\rangle \quad (3.3)$$
in the [3] symmetric model space, \( |\Psi_\lambda\rangle \in [3] \), and define \( P \) as a projection on the space spanned by eigen-vectors with the eigen-value \( \lambda = 0 \):

\[
P = \sum_{\lambda=0} |\Psi_\lambda\rangle \langle\Psi_\lambda| .
\]

(3.4)

It is easy to prove that \( P \) has the following properties:

(i) \( A_\alpha P = PA_\alpha = P \) for \( \forall \alpha \),

(ii) when \( \Psi \in [3] \), \( \forall \langle u_\alpha |\Psi\rangle = 0 \Longleftrightarrow P\Psi = \Psi \),

(iii) when \( \Psi \in [3] \), \( P\Psi = 0 \Longleftrightarrow 3|f\rangle \),

such that \( \Psi = |u_\alpha\rangle |f_\alpha\rangle + |u_\beta\rangle |f_\beta\rangle + |u_\gamma\rangle |f_\gamma\rangle \).

(3.5)

Note that all these relations are considered in the [3] symmetric model space, and \( P \) and \( Q \equiv 1 - P \) are both [3] symmetric three-body operators. Using the property (i), we can simplify Eq. (3.1) as

\[
P \left[ E - H_0 - v_\alpha(\varepsilon_\alpha) - v_\beta(\varepsilon_\beta) - v_\gamma(\varepsilon_\gamma) \right] P\Psi = 0 .
\]

(3.6)

In order to derive the Faddeev equation corresponding to Eq. (3.1), it is convenient to rewrite Eq. (3.1) or (3.6) as

\[
P \left[ E - H_0 - V^{(3)}_\alpha(E, \varepsilon_\alpha) - V^{(3)}_\beta(E, \varepsilon_\beta) - V^{(3)}_\gamma(E, \varepsilon_\gamma) \right] P\Psi = 0 ,
\]

(3.7)

where \( V^{(3)}_\alpha(E, \varepsilon_\alpha) \) is the three-body operator defined by \( V(\omega, \varepsilon) \) in Eq. (2.6) through

\[
V^{(3)}_\alpha(E, \varepsilon_\alpha) = V_\alpha(E - h_\alpha, \varepsilon_\alpha) = (E - H_0) - A_\alpha(E - H_0)A_\alpha + v_\alpha(\varepsilon_\alpha) .
\]

(3.8)

Here \( h_\alpha \) is the kinetic-energy operator between the \( \alpha \)-pair and the third particle. The last equation of Eq. (3.8) is derived by using \( h_\alpha + h_\alpha = H_0 \). The validity of Eq. (3.7) is easily seen from, for example, \( PV^{(3)}_\beta(E, \varepsilon_\beta) = PA_\beta V^{(3)}_\beta(E, \varepsilon_\beta)A_\beta P = P_{V^{(3)}_\beta(E, \varepsilon_\beta)} P \), which uses the property (i) of Eq. (3.5). The expression behind the first \( P \) in the left-hand side of Eq. (3.7) is symmetric with respect to the exchange of the three particles. Thus, by applying the property (iii) of Eq. (3.5), we find

\[
\left[ E - H_0 - V^{(3)}_\alpha(E, \varepsilon_\alpha) - V^{(3)}_\beta(E, \varepsilon_\beta) - V^{(3)}_\gamma(E, \varepsilon_\gamma) \right] P\Psi = |u_\alpha\rangle |f_\alpha\rangle + |u_\beta\rangle |f_\beta\rangle + |u_\gamma\rangle |f_\gamma\rangle ,
\]

(3.9)

where \( |f\rangle \) is an unknown function, and \( |f_\beta\rangle \) and \( |f_\gamma\rangle \) are simply obtained from \( |f_\alpha\rangle \) by the cyclic permutations. Here we invoke the standard ansatz to set

\[
P\Psi = \psi_\alpha + \psi_\beta + \psi_\gamma ,
\]

(3.10)

and define \( \psi_\alpha \) as the solution of

\[
\left( E - H_0 \right) \psi_\alpha = V^{(3)}_\alpha(E, \varepsilon_\alpha) P\Psi + |u_\alpha\rangle |f_\alpha\rangle .
\]

(3.11)

* The two-cluster relative energies, \( \varepsilon_\alpha \), \( \varepsilon_\beta \) and \( \varepsilon_\gamma \), are actually all equal, since we are dealing with the three identical particles.
This equation can be written as
\[
E - H_0 - V_{\alpha}^{(3)}(E, \varepsilon_{\alpha}) \psi_{\alpha} = V_{\alpha}^{(3)}(E, \varepsilon_{\alpha})(\psi_{\beta} + \psi_{\gamma}) + |u_{\alpha}|f_{\alpha}, \quad (3.12)
\]
or by using Eq. (3.8) as
\[
A_{\alpha} \left[ E - H_0 - v_{\alpha}(\varepsilon_{\alpha}) \right] A_{\alpha} \psi_{\alpha} = V_{\alpha}^{(3)}(E, \varepsilon_{\alpha})(\psi_{\beta} + \psi_{\gamma}) + |u_{\alpha}|f_{\alpha} . \quad (3.13)
\]
The unknown function \(|f_{\alpha}|\) is determined if we multiply this equation by \(\langle u_{\alpha} \rangle\) from the left and use \(\langle u_{\alpha} \rangle V_{\alpha}^{(3)}(E, \varepsilon_{\alpha}) = \langle u_{\alpha} \rangle |E - H_0|\):
\[
|f_{\alpha}| = -\langle u_{\alpha} \rangle |E - H_0| \psi_{\beta} + \psi_{\gamma} . \quad (3.14)
\]
Thus we obtain
\[
A_{\alpha} \left[ E - H_0 - v_{\alpha}(\varepsilon_{\alpha}) \right] A_{\alpha} \psi_{\alpha} = V_{\alpha}^{(3)}(E, \varepsilon_{\alpha})(\psi_{\beta} + \psi_{\gamma}) - |u_{\alpha}\rangle \langle u_{\alpha} | E - H_0 | \psi_{\beta} + \psi_{\gamma} . \quad (3.15)
\]
By employing the two-cluster relation Eq. (2.21) in the three-cluster model space,
\[
G_{\alpha}^{(+)}(E, \varepsilon_{\alpha}) V_{\alpha}^{(3)}(E, \varepsilon_{\alpha}) = G_{\alpha}^{(+)}(E) \tilde{T}_{\alpha}^{(3)}(E, \varepsilon_{\alpha}) + |u_{\alpha}\rangle \langle u_{\alpha} | , \quad (3.16)
\]
and the relationship,
\[
G_{\alpha}^{(+)}(E, \varepsilon_{\alpha}) A_{\alpha} \left[ E - H_0 - v_{\alpha}(\varepsilon_{\alpha}) \right] A_{\alpha} G_{\alpha}^{(+)}(E, \varepsilon_{\alpha}) = A_{\alpha} , \quad (3.17)
\]
Eq. (3.15) yields
\[
A_{\alpha} \psi_{\alpha} = G_{\alpha}^{(+)}(E) \tilde{T}_{\alpha}^{(3)}(E, \varepsilon_{\alpha})(\psi_{\beta} + \psi_{\gamma}) + |u_{\alpha}\rangle \langle u_{\alpha} | \psi_{\beta} + \psi_{\gamma} . \quad (3.18)
\]
Since \(\langle u_{\alpha} | \psi_{\beta} + \psi_{\gamma} = -\langle u_{\alpha} | \psi_{\alpha} \rangle\) from Eq. (3.10), we finally obtain
\[
\psi_{\alpha} = G_{\alpha}^{(+)}(E) \tilde{T}_{\alpha}^{(3)}(E, \varepsilon_{\alpha})(\psi_{\beta} + \psi_{\gamma}) . \quad (3.19)
\]
Note that \(\tilde{T}_{\alpha}^{(3)}(E, \varepsilon_{\alpha})\) is essentially the non-singular part of the two-cluster RGM T-matrix Eq. (2.9):
\[
\tilde{T}_{\alpha}^{(3)}(E, \varepsilon_{\alpha}) = \tilde{T}_{\alpha}(E - h_{\alpha}, \varepsilon_{\alpha}) , \quad (3.20)
\]
and that the solution of Eq. (3.19) automatically satisfies \(\langle u_{\alpha} | \psi_{\alpha} + \psi_{\beta} + \psi_{\gamma} = 0\) due to Eq. (2.23). Since \(\psi_{\alpha} + \psi_{\beta} + \psi_{\gamma} \in \mathbb{R}\), the property (ii) of Eq. (3.5) yields \(\Psi = P\Psi\) if we set \(\Psi = \psi_{\alpha} + \psi_{\beta} + \psi_{\gamma}\). We can also start from Eq. (3.19) and derive Eq. (3.1) by using the properties (i) and (ii) of Eq. (3.5), thus establish the equivalence between Eq. (3.1) and Eq. (3.19).
§4. Three di-neutron system

As a simplest non-trivial example, we first consider three di-neutron \( (d') \) system, where the internal wave function of the \( d' \) is assumed to be \((0s)\) harmonic oscillator (h.o.) wave function. The normalization kernel \( K \) for the \( 2d' \) system is given by \( K = \Lambda = 1 - |u\rangle\langle u| \) and the \( \Lambda(\varepsilon K)\Lambda \) term in \( u(\varepsilon) \) disappears. Here \( |u\rangle \) is the \((0s)\) h.o. wave function given by \( u(r) = \langle r|u \rangle = (2\nu/\pi)^{3/4}e^{-\nu r^2} \). We assume a very simple two-nucleon interaction of the Serber type

\[
v_{ij} = -v_0 e^{-\kappa r^2} \frac{1}{2}(1 + P_r) ,
\]

according to Ref. 9). This paper deals with a schematic model of the almost forbidden state with \( v_0 = 90 \text{ MeV} \), but this strength is too weak to give a bound state for the \( 3d' \) system. We use the following parameter set in the present calculation: \( \nu = 0.12 \text{ fm}^{-2} \), \( \kappa = 0.46 \text{ fm}^{-2} \) and \( v_0 = 153 \text{ MeV} \). With this value of \( v_0 \), the \( 2d' \) system is slightly bound.

In order to solve the three-cluster equation (3.1), we first prepare \([3]\) symmetric translationally-invariant h.o. basis according to the Moshinsky’s method 10). The \([3]\) symmetric Pauli-allowed states, which we denote by \( \varphi_{\mu,n}^{[3]}(\rho,\lambda) \), are explicitly constructed by the diagonalization procedure in Eq. (3.3). Here, \( \rho = (X_1 - X_2)/\sqrt{2} \) and \( \lambda = (X_1 + X_2 - 2X_3)/\sqrt{6} \) are the Jacobi coordinates for the center-of-mass coordinates \( X_i \) (\( i = 1 - 3 \)) of the three \( d' \) clusters. These eigen-states are specified by the \( SU_3 \) quantum number \( (\lambda\mu) \) and a set of the other quantum numbers \( n \), which includes the total h.o. quanta \( N \). We then perform the variational calculation using these basis states. Namely, we first expand \( P\Psi \) as

\[
P\Psi = \sum_{(\lambda,\mu),n} c_{\rho}^{(\lambda,\mu)} \varphi_{\mu,n}^{[3]}(\rho,\lambda) .
\]

In the following, we use a simplified notation \( n \) to represent the set of \((\lambda\mu)\) and \( n \) (and also the possible \( K \) quantum number if the total orbital angular momentum \( L \neq 0 \)). Since \( \Psi \) is \([3]\) symmetric, the three interaction terms in Eq. (3.1) give the same contribution. This leads to the eigen-value equation

\[
\sum_{n'} (E \delta_{nn'} - H_{nn'}) c_{n'} = 0 ,
\]

\[
H_{nn'} = (H^0)_{nn'} + 3 \left[ (V_D)_{nn'} + G_{nn'} + \varepsilon K_{nn'} \right] .
\]

Here \( K_{nn'} \) term in the \( 3d' \) problem is trivially zero since the \([3]\) symmetric allowed basis does not contain the \((0s)\) component from the very beginning. This implies that our \( d'd' \) interaction is energy independent and the self-consistency for \( \varepsilon \) is not necessary. On the other hand, the \( 3\alpha \) case which will be discussed in the next section

* The \( S \)-wave phase shift for the \( 2d' \) scattering shows that the \( 2d' \) system gets bound between \( v_0 = 151 \text{ MeV} \) and \( 152 \text{ MeV} \).

** This process is most easily formulated using the theory of Double Gel’fand polynomials \([3]\).
Clebsch-Gordan (C-G) coefficients of the type \( \langle \mathcal{O}_{nn'} | \mathcal{O} | \mathcal{O}_{nn'}^\prime \rangle \) can be calculated by using the power series expansion of the complex GCM kernel and SU3 Clebsch-Gordan (C-G) coefficients of the type \( \langle (N_1 0) \ell | (N_2 0) \ell | (\lambda \mu 00) \rangle \). Fortunately, a concise expression is given by Suzuki and Hecht [12] for this particular type of SU3 C-G coefficients with \( L = 0 \).

The two-body matrix elements in the three-body space, \( \mathcal{O}_{nn'} = \langle \varphi_n^{[3]} | \mathcal{O} | \varphi_{n'}^{[3]} \rangle \), can be calculated by using the power series expansion of the complex GCM kernel and \( SU_3 \) Clebsch-Gordan (C-G) coefficients of the type \( \langle (N_1 0) \ell | (N_2 0) \ell | (\lambda \mu 00) \rangle \). Fortunately, a concise expression is given by Suzuki and Hecht [12] for this particular type of SU3 C-G coefficients with \( L = 0 \).

Table I shows the lowest eigen-values of Eq. (4.3) with an increasing number of total h.o. quanta \( N \) included in the calculation. The number of basis states rapidly increases, as the maximum \( N \) becomes larger. The listing is terminated when the number of basis states \( n_{\text{Max}} \) is over 1000, which is reached around \( N \sim 60 \).

The convergence of the 3\( d' \) system is rather slow, since the bound-state energy is especially small in this particular system. The best value obtained in the variational calculation is \( E_{3d'} = -0.4323 \) MeV, using 2,927 basis states with \( N \leq 88 \). We have also solved the Faddeev equation (3.19), and obtained \( E_{3d'} = -0.4375 \) MeV and \( -0.4378 \) MeV, when the partial waves up to \( \ell = 4 \) and 6 are included in the calculation, respectively. The final value \( E_{3d'} = -0.438 \) MeV can also be compared with \( E_{3d'}^{\text{RGM}} = -1.188 \) MeV, which is obtained by the stochastic variational method [13] for the 3\( d' \) RGM. Our result by the three-cluster equation gives 0.75 MeV less bound, compared with the full microscopic 3\( d' \) RGM calculation.

### §5. 3\( \alpha \) system

In this system, the structure of the 2\( \alpha \) normalization kernel \( K \) is more involved. In the relative \( S \)-wave we have two Pauli-forbidden states, \((0s)\) and \((1s)\), while in the \( D \)-wave only one \((0d)\) h.o. state is forbidden. The relative motion between the two \( \alpha \) clusters starts from \( N = 4 \) h.o. quanta The eigen-value \( \gamma_N \) for \( K \) is given by \( \gamma_N = 2^{n-N} - 3 \delta_{N,0} \), which is 1 \((N = 0 \text{ or } 2)\), 1/4 \((N = 4)\), 1/16 \((N = 6)\), \cdots. The rather large value \( \gamma_4 = 1/4 \) makes the self-consistent procedure through Eq. (4.4) very important. For the two-body effective interaction, we use the Volkov No.2 force with \( m = 0.59 \), following the 3\( \alpha \) RGM calculation by Fukushima and Kamimura [4]. The h.o. constant for the \( \alpha \) cluster is \( \nu = 0.275 \) fm\(^{-2} \), which gives the \( \alpha \) cluster internal energy \( E_{\alpha} = -27.3 \) MeV for the \((0s)^4 \) configuration, if the Coulomb interaction is included. (Cf. \( E_{\alpha}^{\text{exp't}} = -28.3 \) MeV.) We have carried out the 2\( \alpha \) RGM calculation by using this parameter set, and found that the present 2\( \alpha \) system is bound with the binding energy \( E_{2\alpha} = -0.245 \) MeV. (Cf. \( E_{2\alpha}^{\text{exp't}} = 92 \) keV.)

Table I lists the convergence of the lowest 3\( \alpha \) eigen-values with respect to the maximum total h.o. quanta \( N \). We find that the final values of \( E_{3\alpha} \) and \( \varepsilon \) are \( E_{3\alpha} = -5.97 \) MeV and \( \varepsilon = 9.50 \) MeV. If we compare this with \( E_{3\alpha}^{\text{RGM}} = -7.5 \) MeV (Cf. \( E_{3\alpha}^{\text{exp't}} = -7.3 \) MeV) by the 3\( \alpha \) RGM calculation [13], we find that our result is 1.5 MeV less bound. The amplitude of the lowest shell-model component with the \( SU_3 \) \((04)\) representation is \( c_{(04)} = 0.790 \). We think that the underbinding compared
to the three-cluster RGM calculation is reasonable, since our three-cluster equation misses some attractive effect due to the genuine three-cluster exchange kernel. Oryu et al. carried out $3\alpha$ Faddeev calculation using $2\alpha$ RGM kernel. They obtained very large binding energy, $E_{3\alpha} = -28.2$ MeV with the Coulomb force turned off. Since the effect of the Coulomb force is at most 6 MeV, this value is too deep. This is because they did not treat the $\varepsilon K$ term in the RGM kernel properly and the effect of $P$ in Eq. (3.1) is not fully taken into account in their Faddeev formalism. In order to see the importance of the $\varepsilon K$ term in Eq. (4.3), it is useful to see the contribution of this term in the lowest h.o. (04) configuration. The decomposition of $E_{3\alpha}^{(04)} = 12.634$ MeV with $N = 8$ in Table I is

$$H = H_0 + 3 \left( V_D + G^K + G^V + V_D^{CL} + G^{CL} + \varepsilon K \right)$$
$$= 125.4 + 3(-36.54 - 15.68 + 6.54 + 2.58 - 0.78 + 25.12/4)$$
$$= 125.4 - 3 \times 37.6 = 12.6 \text{ MeV} . \quad (5.1)$$

This example shows very clearly that the self-consistent procedure for the energy-dependence of the RGM kernel in the allowed model space is sometimes very important.

Since the present calculation employs the h.o. basis, it is very easy to examine another approximation which eliminate the explicit energy dependence involved in $v(\varepsilon)$. This approximation is related to the proper normalization of the two-cluster relative wave function $\chi$ in Eq. (2.1) through $\psi = \sqrt{1 - K}\chi$, and we call this approximation the renormalized RGM. In this formulation, the interaction generated from the RGM kernel is expressed as

$$v = \frac{1}{\sqrt{1 - K}'} \left( h_0 + V_D + G \right) \left( \frac{1}{\sqrt{1 - K}} \right)' - A h_0 A , \quad (5.2)$$

where the prime in $(1/\sqrt{1 - K})'$ implies the inversion of $\sqrt{1 - K}$ in the allowed model space. (See Ref. 7 and the discussion in Ref. 5.) The column $E_{3\alpha}^{RN}$ in Table I shows the result of this approximation. We find $E_{3\alpha}^{RN} = -4.99$ MeV, which is 0.98 MeV less bound in comparison with our result. This may result from rather inflexible choice of the $3\alpha$ Hamiltonian, caused by the lack of the self-consistency. Table I also shows the result by $3\alpha$ OCM ($E_{3\alpha}^{OCM}$), whose procedure is to use $v = A(V_D + V_D^{CL})A$. We find $E_{3\alpha}^{OCM} = -4.68$ MeV, which is further 0.31 MeV less bound. In this case, $2\alpha$ OCM gives a larger binding energy, $E_{2\alpha}^{OCM} \leq -0.4$ MeV, than the $2\alpha$ RGM. If we readjust the potential parameter to fit the $2\alpha$ binding energy, we would apparently obtain an even worse result. It was realized a long time ago that a simple choice of the direct potential $V_D$ for the effective interaction $V^{eff}$ in OCM gives a poor result.\[8]

\[\textsuperscript{*} \text{In our calculation, } E_{3\alpha} = -11.42 \text{ MeV when the Coulomb interaction is turned off, which implies that the effect of the Coulomb interaction is 5.45 MeV.}\]
§6. Summary

The main purpose of this study is to find an optimum equation for three-cluster systems interacting via pairwise two-cluster RGM kernel. This is a necessary first step to apply the quark-model baryon-baryon interactions to few-baryon systems. We have found that the orthogonality condition to the pairwise Pauli-forbidden states is a compulsory condition to assure a reasonable result. The inherent energy dependence of the two-cluster exchange RGM kernel should be treated self-consistently, if the eigen-values of the exchange normalization kernel $K$ are non-negligible in the allowed space. The proposed three-cluster equation has a nice feature that the equivalent three-cluster Faddeev equation is straightforwardly formulated using the non-singular part of the full $T$-matrix derived from the RGM kernel. We have applied this equation to simple systems composed of the three di-neutrons and three $\alpha$ clusters. The equivalent Faddeev equation is also solved for the three di-neutron system. The Faddeev calculation for the 3$\alpha$ system will be reported in a separate paper.\(^{17}\) For the ground state of the three $\alpha$ system, the obtained binding energy is 1.5 MeV less bound, in comparison with the full microscopic three-cluster RGM calculation. We think this satisfactory, since three-cluster RGM calculation of the few-baryon systems using quark-model baryon-baryon interaction is still beyond the scope of feasibility. The application to the hypertriton using our quark-model nucleon-nucleon and hyperon-nucleon interactions\(^{3}^{,}^{4}\) is now under way.

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Table I. The lowest eigen-values for $3d'$ and $3\sigma$ systems, obtained by diagonalization using [3] symmetric translationally-invariant h.o. basis. The orthogonality condition by the projection operator $P$ in Eq. (3-4) is imposed. $N$ stands for the maximum total h.o. quanta included in the calculation, and $n_{\text{max}}$ the number of the basis states with the orbital angular momentum $L = 0$. The three-cluster equation Eq. (3-1) is used for $E_{3d'}$ and $E_{3\sigma}$, while the energy-independent renormalized interaction Eq. (5-2) and $v = \Lambda (V_{D} + V_{D}^{\text{CL}}) \Lambda$ are used in Eq. (3-6) for $E_{\text{RN}}^{3\sigma}$ and $E_{\text{GCM}}^{3\sigma}$, respectively.

| $N$ | $3d'$ | $3\sigma$ |
|-----|-------|-----------|
|     | $n_{\text{max}}$ | $E_{3d'}$ | $\varepsilon$ | $E_{3\sigma}$ | $E_{\text{RN}}^{3\sigma}$ | $E_{\text{GCM}}^{3\sigma}$ |
| 4   | 1     | 3.256     |        |        |        |        |
| 6   | 3     | 2.828     |        |        |        |        |
| 8   | 6     | 0.7373    | 1      | 25.120 | 12.634 | 12.634 | 23.570 |
| 10  | 10    | 0.5585    | 3      | 18.023 | 3.615  | 4.575  | 11.422 |
| 12  | 16    | 0.1169    | 7      | 14.857 | 0.343  | 0.874  | 5.322  |
| 14  | 23    | 0.0523    | 12     | 13.046 | -2.454 | -1.194 | 1.827  |
| 16  | 32    | -0.0868   | 19     | 11.920 | -3.678 | -2.449 | -0.323 |
| 18  | 43    | -0.1351   | 28     | 11.182 | -4.439 | -3.255 | -1.703 |
| 20  | 56    | -0.1972   | 39     | 10.682 | -4.929 | -3.788 | -2.614 |
| 22  | 71    | -0.2313   | 52     | 10.339 | -5.252 | -4.148 | -3.227 |
| 24  | 89    | -0.2660   | 68     | 10.099 | -5.470 | -4.394 | -3.647 |
| 26  | 109   | -0.2899   | 86     | 9.931  | -5.619 | -4.565 | -3.939 |
| 28  | 132   | -0.3117   | 107    | 9.812  | -5.721 | -4.685 | -4.143 |
| 30  | 158   | -0.3284   | 131    | 9.727  | -5.792 | -4.769 | -4.289 |
| 32  | 187   | -0.3431   | 158    | 9.666  | -5.842 | -4.829 | -4.394 |
| 34  | 219   | -0.3550   | 188    | 9.623  | -5.878 | -4.872 | -4.469 |
| 36  | 255   | -0.3653   | 222    | 9.591  | -5.903 | -4.904 | -4.525 |
| 38  | 294   | -0.3740   | 259    | 9.568  | -5.921 | -4.926 | -4.565 |
| 40  | 337   | -0.3815   | 300    | 9.551  | -5.934 | -4.943 | -4.595 |
| 42  | 384   | -0.3879   | 345    | 9.539  | -5.943 | -4.954 | -4.618 |
| 44  | 435   | -0.3934   | 394    | 9.530  | -5.950 | -4.964 | -4.635 |
| 46  | 490   | -0.3982   | 447    | 9.524  | -5.955 | -4.971 | -4.647 |
| 48  | 550   | -0.4025   | 505    | 9.519  | -5.959 | -4.976 | -4.657 |
| 50  | 614   | -0.4061   | 567    | 9.515  | -5.962 | -4.979 | -4.664 |
| 52  | 683   | -0.4094   | 634    | 9.512  | -5.964 | -4.982 | -4.669 |
| 54  | 757   | -0.4122   | 706    | 9.510  | -5.965 | -4.984 | -4.674 |
| 56  | 836   | -0.4147   | 783    | 9.508  | -5.966 | -4.986 | -4.677 |
| 58  | 920   | -0.4169   | 865    | 9.507  | -5.967 | -4.987 | -4.679 |
| 60  | 1010  | -0.4189   | 953    | 9.506  | -5.968 | -4.988 | -4.681 |