HOFER'S LENGTH SPECTRUM OF SYMPLECTIC SURFACES

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ABSTRACT. Following a question of F. Le Roux, we consider a system of invariants $l_A : H_1(M) \to \mathbb{R}$ of a symplectic surface $M$. These invariants compute the minimal Hofer energy needed to translate a disk of area $A$ along a given homology class and can be seen as a symplectic analogue of the Riemannian length spectrum. When $M$ has genus zero we also construct Hofer- and $C^0$-continuous quasimorphisms $Ham(M) \to H_1(M)$ that compute trajectories of periodic non-displaceable disks.

1. INTRODUCTION AND RESULTS

Let $(M, \omega)$ be an open symplectic surface of finite type, possibly with boundary. Pick an open disk $D \subset M$ of Area$(D) = A$. To avoid degenerate situations we will always assume that the closure of $D$ is a closed disk in the interior of $M$. Let $\phi \in Ham(M)$ be a compactly supported Hamiltonian such that $\phi(D) = D$. We define the trajectory $\text{traj}_D(\phi) \in \pi_1(M)$ in the following way. Pick a Hamiltonian isotopy $\phi_t$ which connects the identity to $\phi$. Roughly speaking, $\text{traj}_D(\phi)$ is the free homotopy class of the trajectory of $D$ under $\phi_t$. More precisely, pick $x \in D$ and denote by $\gamma \subset D$ some curve which connects $\phi_t(x)$ with $x$. We concatenate the trajectory of $x$ under $\phi_t$ with $\gamma$,

$$\text{traj}_D(\phi) = [\{\phi_t(x)\} \ast \gamma] \in \pi_1(M).$$

It is not difficult to see that $\text{traj}_D(\phi)$ does not depend on the choices of $x$, $\gamma$ and $\phi_t$.

Denote by $\| \cdot \|$ the Hofer norm on the group $Ham(M)$,

$$\|\phi\| = \inf_{0 \leq t \leq 1} \max_{p \in M} H(p, t) - \min_{p \in M} H(p, t) dt,$$

where the infimum goes over all compactly supported Hamiltonians $H : M \times [0, 1] \to \mathbb{R}$ such that $\phi$ is the time-1 map of the corresponding flow.

Following F. Le Roux, we consider the following invariant. Given $\alpha \in \pi_1(M)$ and a disk $D$ of area $0 < A < \text{Area}(M)$, we define

$$l_A(\alpha) = \inf \{ \|\phi\| : \phi \in Ham(M) \text{ s.t. } \phi(D) = D \text{ and } \text{traj}_D(\phi) = \alpha \}.$$
Since all disks in $M$ of the same area can be mapped one to another by a symplectomorphism of $M$, $l_A$ depends only on the area $A$ and not on a choice of particular $D$.

Catenation of Hamiltonian isotopies shows that $l_A$ satisfies the triangle inequality: let $a, b$ be two based loops with the same base point, denote $c = a * b$. Then $l_A([c]) \leq l_A([a]) + l_A([b])$. Using the argument from [9], one shows that $l_A(\alpha) \geq A$ for all $\alpha \neq 0$. Namely, given $\phi \in Ham(M)$ with $\text{traj}_D(\phi) \neq 0$, the lift of $\phi$ to the universal cover displaces a lift of $D$, hence $\|\phi\| \geq A$ by the energy-capacity inequality. Therefore, $l_A$ is a nontrivial system of invariants which behaves as a symplectic analogue of the length spectrum.

Question 5 from [11] addresses a computation of $l_A$ in the case when $M$ is an annulus and $D$ is non-displaceable. Its solution appeared in [7]. This paper extends the results to general surfaces.

REMARK. One may modify the definition of $l_A$ and restrict attention to those $\phi$ that fix $D$ pointwise. All estimates and results presented in this article remain true also in this setting, the same ideas apply after some minor adjustments of the argument.

In this article we discuss a weaker version of this invariant where we consider trajectories in a class $\alpha \in H_1(M; \mathbb{Z})$. We prove the following:

**Theorem 1.** Given $0 < A < \text{Area}(M)$ and $\alpha \in H_1(M; \mathbb{Z})$, the following estimates hold:

- If $A < \text{Area}(M)/2$ or genus($M$) $> 0$, then $l_A(\alpha) \leq 2A$.
- If genus($M$) $= 0$ and $A > \text{Area}(M)/2$, then
  $$l_A : H_1(M; \mathbb{Z}) \to \mathbb{R}$$

is comparable to a homogeneous norm in $H_1(M; \mathbb{Z})$.

It is not important which norm we consider in $H_1(M; \mathbb{R})$. The first homology group is finitely generated, hence all homogeneous norms in $H_1(M; \mathbb{Z})$ are equivalent.

Partial results are available also for the homotopical version $l_A : \pi_1(M) \to \mathbb{R}$, see the discussion in Section 4.2.

We also show:

**Theorem 2.** Let $(M, \omega)$ be a disk with $k$ punctures, pick $A$ such that $\frac{1}{2} \text{Area}(M) < A < \text{Area}(M)$. There exists a family $P_A = \{\rho_A\}$ of invariants $\rho_A : \text{Ham}(M) \to H_1(M; \mathbb{R})$ that satisfy:

- $\rho_A$ is a homogeneous quasimorphism;
- $\rho_A$ is Hofer-Lipschitz and $C^0$-continuous;
- Suppose that $\phi \in \text{Ham}(M)$ has an invariant disk $D$ of $\text{Area}(D) \geq A$. Then $\rho_A(\phi) = [\text{traj}_D(\phi)] \in H_1(M; \mathbb{Z}) \subset H_1(M; \mathbb{R})$.

$P_A$ contains a continuum of such quasimorphisms that are linearly independent.

These $\rho_A$ can be seen as one of the many ways to generalize the rotation number to dimension two. The Lipschitz property $|\rho_A(\phi)| \leq c\|\phi\|$ in Theorem 2
implies a lower bound for the second case of Theorem 1. We pick a disk \( \mathcal{D} \) of area \( A \). Then
\[
I_A(\alpha) = \inf \{ \| \phi \| : \phi(\mathcal{D}) = \mathcal{D} \text{ and } |\text{traj}_\mathcal{D}(\phi)| = \alpha \}
\geq \inf \left\{ \left| \frac{\rho_A(\phi)}{c} \right| : \phi(\mathcal{D}) = \mathcal{D} \text{ and } |\text{traj}_\mathcal{D}(\phi)| = \alpha \right\} = \frac{|\alpha|}{c}.
\]

Additional properties and applications of invariants \( \rho_A \) are discussed in Section 4.4. We remark that no such quasimorphisms exist under conditions of the first case in Theorem 1.

This paper is organized as follows. In order to show the upper bounds for \( I_A \) as in Theorem 1 we describe explicit Hamiltonian isotopies that achieve desired energy bounds. Details are provided in Section 2. In Section 3 we construct \( \rho_A \) for Theorem 2 from the Calabi quasimorphism \( \rho : \text{Ham}(S^2) \to \mathbb{R} \) described in [4]. Note that the conditions on \( M \) in Theorem 2 are equivalent to the statement that \( M \) embeds to a sphere, so \( \rho \) can be pulled back to \( \text{Ham}(M) \). Section 4 discusses possible generalizations of \( I_A \) and provides a few applications of the quasimorphisms from Theorem 2.

2. Upper bounds

In this section we show the upper bounds for Theorem 1. We construct explicit Hamiltonian isotopies that translate a disk along a given homology class and remain within the prescribed energy bound.

2.1. Suppose \( A = \text{Area} \mathcal{D} < \text{Area}(M)/2 \). We show that \( I_A \leq 2A \).

Consider the following Hamiltonian in \((\mathbb{R}^2, dx \wedge dy)\). We pick two disjoint disks \( D_1 \) and \( D_2 \) of area \( A \) in the plane, connect them by two narrow non-intersecting strips (‘pipes’) as described in Figure 1. Let \( H \) be an autonomous Hamiltonian which equals one in the area bounded by the two disks and the pipes, zero outside and linearly interpolated in between. We apply to \( H \) a \( C^0 \)-small smoothing near the singular points.

\[
\text{Figure 1. Translation of a disk}
\]

The flux of the Hamiltonian flow through a curve \( \gamma \) equals the difference of values of \( H \) at the endpoints of \( \gamma \). Hence, if we choose \( \gamma \) to be a curve connecting the inner region (where \( H = 1 \)) to the outside, then the flux through \( \gamma \) is one. This implies that the resulting flow is relatively slow inside the disks and accelerates in the pipes where such a curve \( \gamma \) can be very short. For an appropriate choice of smoothing of \( H \) the time-\((A + \varepsilon)\) map of the flow of \( H \) will move \( D_1 \)
to \( D_2 \). In fact, it is possible to arrange that the two disks will be swapped. The Hofer norm is at most \( \int_0^{A+\varepsilon} (\max H - \min H) \, dt = A + \varepsilon \), where \( \varepsilon \) depends on the area of the pipes and a choice of smoothing of \( H \), and can be made arbitrarily small. Note that the energy cost of \((A + \varepsilon)\) is optimal since every Hamiltonian that displaces \( D_1 \) has energy \( \geq A \) by the energy-capacity inequality.

We augment this construction as in Figure 2: we choose the first pipe to go along an interval connecting \( D_1 \) with \( D_2 \) and let the second pipe lie in a small neighborhood of the two disks and the first pipe.

![Figure 2. Translation of a disk](image)

This way the Hamiltonian \( H \) is supported in a small neighborhood of the two disks and an interval. This construction can be embedded to any surface given two disjoint copies of a disk and a simple path connecting boundary points of the disks. This allows us to translate a disk along a simple path.

Let \( D_1, D_2 \) be two disjoint copies of \( \mathcal{D} \) in \( M \). Let \( \gamma_1 \) be a simple path connecting \( D_1 \) to \( D_2 \) and \( \gamma_2 \) be a simple path connecting \( D_2 \) to \( D_1 \). \( \gamma_1 \) and \( \gamma_2 \) may intersect one another but not the disks (see Figure 3). We apply the construction above twice: first in order to transfer \( D_1 \) to \( D_2 \) along \( \gamma_1 \) and then to move it back to \( D_1 \) along \( \gamma_2 \). The total energy cost of this process equals \( 2 \text{Area}(\mathcal{D}) + 2\varepsilon \) and the result is a translation of the disk \( D_1 \) along a trajectory in the class \([\gamma_1 \ast \gamma_2]\). (By abuse of notation we extend \( \gamma_1, \gamma_2 \) inside the disks so that they have the same endpoints and catenation makes sense.)

![Figure 3. Translation of a disk](image)

The upper bound \( l_A \leq 2A \) follows from the following topological lemma:

**Lemma 3.** Let \( M \) be a surface and \( \alpha \in \pi_1(M) \). Then there exist two simple curves \( \gamma_1, \gamma_2 \) with the same endpoints such that \( \alpha = [\gamma_1 \ast -\gamma_2] \). (\(-\gamma_2 \) denotes \( \gamma_2 \) with reversed orientation.)

Indeed, pick a class \( \alpha \in \pi_1(M) \). By the above lemma there exist two points \( p, q \in M \) and two simple curves \( \gamma_1, \gamma_2 \) connecting them so that \([\gamma_1 \ast \gamma_2] = \alpha \). We
consider small disks $D_1, D_2$ centered at $p$ and $q$ which intersect both curves only in small arcs near the endpoints. Pick a symplectic form $\omega'$ on $M$ so that both disks have area $A$ and the total area is $\int_M \omega' = \frac{1}{2} \int_M \omega$. $(M, \omega')$ is symplectomorphic to $(M, \omega)$ by Moser's argument, so it is sufficient to show the result for $(M, \omega')$. We translate $D_1$ along $\gamma_1$ to $D_2$ and then back along $\gamma_2$. The resulting deformation $\phi$ satisfies

$$\text{traj}_{D_1}(\phi) = [\gamma_1 * \gamma_2] = \alpha \in \pi_1(M)$$

at the cost of $2 \text{Area}(\partial D) + 2 \varepsilon$, where $\varepsilon$ is arbitrarily small. This finishes the proof.

**Remark.** Note that we proved the upper bound for $\alpha \in \pi_1(M)$ which, in particular, implies the same bound for the homological length spectrum.

**Proof of Lemma 3.** Pick $\alpha \in \pi_1(M)$. Let $\gamma$ be a loop representing $\alpha$ which has a finite number of transverse self-intersections. We deform $\gamma$ so that it can be cut into a pair of simple curves. Pick arbitrary point $p \in \gamma$ which will be the starting point of $\gamma_1$. We draw $\gamma_1$ by traversing $\gamma$ starting from $p$. We prevent self-intersections of $\gamma_1$ as follows. Once we arrive to a self-crossing we stretch the previously drawn path and push it in a small neighborhood in front of our current position (see Figure 4).

![Figure 4. Path deformation](image)

After passing several crossings we will have a number of strings pushed in this manner. During this process we construct a simple path which is homotopic to the corresponding arc of $\gamma$ relative the endpoints. We stop the process when we arrive to a point $q \in \gamma$ such that the arc $[q, p] \subset \gamma$ is simple. Set $\gamma_2 = [q, p] \subset \gamma$. Lemma follows since $\gamma_1$ is homotopic to $[p, q] \subset \gamma$ relative the endpoints. □

**Remark.** One may also consider the case $A = \text{Area}(\partial D) = \text{Area}(M)/2$. For a small $\varepsilon > 0$ we choose a smaller disk $D_\varepsilon \subset D$ with $\text{Area}(D_\varepsilon) = A - \varepsilon$. Using the argument above one constructs Hamiltonian isotopies $\phi_\varepsilon$ that translate $D_\varepsilon$ along a class $\alpha$ with energy $\|\phi_\varepsilon\| \leq 2A$. One shows that those $\phi_\varepsilon$ can be carefully chosen so
that the trajectories $\{\varphi_{\epsilon,t}(\mathcal{D})\}_t$ converge (in Hofer’s norm) to a trajectory which translates $\mathcal{D}$ along $\alpha$ as $\epsilon \to 0$. This shows that

$$l_A(\alpha) \leq \lim_{\epsilon \to 0} l_{A-\epsilon}(\alpha) \leq 2A.$$  

Unfortunately, the formal argument that uses this idea and is known to the author is much longer than the proof of the main case $A < \text{Area}(M)/2$, so we decided to exclude it from the present paper.

### 2.2. Arbitrary $\mathcal{D}$

We show first that one can translate a disk along a simple loop using energy $\text{Area}(\mathcal{D}) + \epsilon$. Then we consider separately the cases $\text{genus}(M) = 0$ and $\text{genus}(M) > 0$.

Consider the following Hamiltonian in $(\mathbb{R}^2, dx \wedge dy)$. Pick a disk $D \subset \mathbb{R}^2$ of area $A$ and connect two boundary points of the disk by a circular pipe. Let $H$ be an autonomous Hamiltonian which equals one in the inner region, zero outside and linearly interpolated in between. We apply to $H$ a $C^0$-small smoothing near the singular points (see Figure 5).

![Figure 5. Rotation of a disk](image)

The Hamiltonian flow of $H$ is relatively slow inside the disk $D$ and fast inside the pipe. For an appropriate choice of smoothing of $H$, there exists $\epsilon > 0$ such that $D$ is a fixed set of the time-$(A + \epsilon)$ map of the flow of $H$. The energy is at most $A + \epsilon$ and $\epsilon$ depends on the area of the pipe and a choice of smoothing thus can be made arbitrarily small.

We cut $H$ off inside the circle so that it is supported in a small neighborhood of the disk and the pipe. Note that this cutoff does not affect the flow of $H$ inside the disk $D$ and the pipe. This construction can be copied to any symplectic surface given a disk and a simple curve which connects boundary points of the disk and does not intersect it away from the endpoints. This implies the following result.

**Lemma 4.** Suppose that $\alpha \neq 0$ is represented by a simple loop. Then $l_A(\alpha) = A$.

**Proof.** Let $\gamma$ be a simple loop representing $\alpha$. As in the previous subsection, we deform $\omega$ so that there exists a disk of area $A$ whose boundary points are
connected by an arc of $\gamma$ and this arc does not intersect the disk except for the endpoints. An application of the Hamiltonian described above implies $l_A(\alpha) \leq A + \varepsilon$. The lemma follows by letting $\varepsilon \to 0$ and from the fact that $l_A(\alpha) \geq A$.

**Corollary 5.** Let $S \subseteq H_1(M; \mathbb{Z})$ be the set of classes represented by simple loops. $S$ generates $H_1(M; \mathbb{Z})$. Denote by $\| \cdot \|_S$ the norm on $H_1(M; \mathbb{Z})$ given by word length with respect to $S$. The triangle inequality and the lemma imply $l_A(\alpha) \leq A \cdot \| \alpha \|_S$. The same argument holds for the homotopical length spectrum as well.

Given any surface $M$, $H_1(M; \mathbb{Z})$ admits a basis $B \subseteq S$ represented by simple loops. The triangle inequality and the lemma imply $l_A(\alpha) \leq A \cdot \| \alpha \|_S \leq A \cdot \| \alpha \|_B$, where $\| \cdot \|_B$ is a homogeneous norm on $H_1(M; \mathbb{Z})$. This shows the upper bound for the second case in Theorem 1. A more careful argument may provide better constants. For example, in the case of an annulus $M$, the bound can be improved to

$$l_A(n \cdot \text{generator}) \leq (2A - \text{Area}(M)) \cdot |n| + \text{Area}(M)$$

(see [7]).

Suppose that $\text{genus}(M) > 0$. The upper bound for Theorem 1 follows from the lemma below and Corollary 5.

**Lemma 6.** Let $M$ be a surface of positive genus. Then every homology class in $H_1(M; \mathbb{Z})$ is represented by catenation of two simple loops.

**Proof.** Step 1. Suppose $M = S^1 \times S^1$ is a torus, $\alpha = [\{pt\} \times S^1]$ and $\beta = [S^1 \times \{pt\}]$ generate $H_1(M; \mathbb{Z}) = \mathbb{Z} < \alpha, \beta >$. Any class $n\alpha + m\beta$ can be decomposed into the sum

$$m\alpha + n\beta = ((m-1)\alpha + \beta) + (\alpha + (n-1)\beta),$$

where each summand is represented by a simple loop (see Figure 6).

![Figure 6. Torus case](image)

**Step 2.** Suppose $M = S^1 \times S^1 \setminus \{p_1, \ldots, p_k\}$ is a punctured torus. Without loss of generality we assume that all punctures lie on a horizontal meridian. $H_1(M; \mathbb{Z})$ admits a basis of $k$ vertical meridians $\alpha_1, \ldots, \alpha_k$ and a horizontal meridian $\beta$ (see Figure 7). Then every

$$\alpha = m_1\alpha_1 + \ldots + m_k\alpha_k + n\beta \in H_1(M; \mathbb{Z})$$

admits a decomposition

$$\alpha = ((m_1-1)\alpha_1 + \ldots + m_k\alpha_k + \beta) + (\alpha_1 + (n-1)\beta),$$

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where each summand is represented by a simple loop.

\[
\{\alpha_1, \ldots, \alpha_k\}
\]

\[
\beta
\]

\[
\]

**Figure 7.** Punctured torus

**Step 3.** Suppose genus(M) > 1. M can be decomposed into a connected sum of punctured tori (see Figure 8)

\[
M = T_1 \# \ldots \# T_l, \quad H_1(M;\mathbb{Z}) = H_1(T_1;\mathbb{Z}) + \ldots + H_1(T_l;\mathbb{Z}).
\]

We identify elements of \(H_1(T_i;\mathbb{Z})\) with their image in \(H_1(M;\mathbb{Z})\) by abuse of notation.

\[
\]

**Figure 8.** Positive genus surface

Pick \(\alpha \in H_1(M;\mathbb{Z})\). There exist \(\alpha_i \in H_1(T_i;\mathbb{Z})\) such that \(\alpha = \sum \alpha_i\). By the previous step we choose homology classes \(\beta_i, \beta'_i \in H_1(T_i;\mathbb{Z})\) represented by simple loops such that \(\alpha_i = \beta_i + \beta'_i\). Now

\[
\alpha = \sum \alpha_i = \sum (\beta_i + \beta'_i) = \sum \beta_i + \sum \beta'_i.
\]

Both \(\sum \beta_i\) and \(\sum \beta'_i\) are represented by simple loops: consider a connected sum of the corresponding simple loops in each \(T_i\). The result is a simple loop in \(M\) since \(T_i\) are disjoint.

\[
\]

3. CONSTRUCTION OF QUASIMORPHISMS \(\rho_A\)

Let \(G\) be a group. A function \(r : G \to \mathbb{R}\) is called a *quasimorphism* if there exists a constant \(R\) such that \(|r(fg) - r(f) - r(g)| \leq R\) for all \(f, g \in G\). \(R\) is called the *defect* of \(r\). The quasimorphism \(r\) is *homogeneous* if it satisfies \(r(g^m) = mr(g)\).
for all \( g \in G \) and \( m \in \mathbb{Z} \). Any homogeneous quasimorphism satisfies \( r(fg) = r(f) + r(g) \) for commuting elements \( f, g \) and is invariant under conjugations.

We construct quasimorphisms \( \rho_A \) described in Theorem 2. We assume that \( M = U \sim \{p_1, \ldots, p_k\} \) is a disk with \( k \) punctures. \( M \) has finite area, so without loss of generality we rescale \( \omega \) to get \( \text{Area}(M) = 1 \). \( k \) is the rank of \( H_1(M; \mathbb{Z}) \).

**Proposition 7.** Let \((\mathbb{A}, \omega)\) be an annulus (disk with one puncture). Suppose that \( \text{Area}(\mathbb{A}) = 1 \). Pick \( \frac{1}{2} < A < 1 \) and choose a generator \( \alpha \) for \( H_1(M, \mathbb{Z}) \). There exists a family \( \mathbb{P}_A = \{\rho_A\} \) of invariants \( \rho_A : \text{Ham}(\mathbb{A}) \to \mathbb{R} \) that satisfy:

- \( \rho_A \) is a homogeneous quasimorphism;
- \( \rho_A \) is Hofer-Lipschitz and \( C^0 \)-continuous;
- Suppose that \( \phi \in \text{Ham}(\mathbb{A}) \) has an invariant disk \( D \) of \( \text{Area}(D) \geq A \). Then
  \[ [\text{traj}_D(\phi)] = \rho_A(\phi) \cdot \alpha. \]

\( \mathbb{P}_A \) contains a continuum of such quasimorphisms that are linearly independent.

The proposition implies Theorem 2.

**Proof of Theorem 2.** Pick a quasimorphism \( \rho_A : \text{Ham}(\mathbb{A}) \to \mathbb{R} \) as in the proposition and denote by

\[ i_j : U \sim \{p_1, \ldots, p_k\} \hookrightarrow U \sim \{p_j\} = \mathbb{A}, \quad 1 \leq j \leq k, \]

the embeddings obtained by gluing all punctures except for \( p_j \) and identifying the resulting once punctured disk with \( \mathbb{A} \). The pullback quasimorphisms \( \rho_j = i_j^* \rho_A : \text{Ham}(U \sim \{p_1, \ldots, p_k\}) \to \mathbb{R} \) satisfy the Lipschitz and continuity properties. Given a Hamiltonian \( \phi \) with a fixed disk \( \mathcal{D} \) of area \( \geq A \), \( \rho_j(\phi) \) counts (up to a sign) the winding number of \( \text{traj}_\mathcal{D}(\phi) \) around \( p_j \). Replacing \( \rho_j \) by \( -\rho_j \), if necessary, we assume that the sign is positive. Denote by \( \gamma_j \) a small positively oriented circle in \( U \) around \( p_j \) which does not encircle other punctures. \( \{[\gamma_j]\}_{j=1}^k \) is a basis for \( H_1(U \sim \{p_1, \ldots, p_k\}; \mathbb{Z}) \). The dual basis \( \{[\gamma_j]^*\} \) allows us to write

\[ \rho_j(\phi) = [\gamma_j]^* [\text{traj}_\mathcal{D}(\phi)] \]

for all Hamiltonians \( \phi \) with a fixed disk \( \mathcal{D} \) as above. This implies

\[ [\text{traj}_\mathcal{D}(\phi)] = \sum_{j=1}^k [\gamma_j]^* [\text{traj}_\mathcal{D}(\phi)] \cdot [\gamma_j] = \sum_{j=1}^k \rho_j(\phi)[\gamma_j], \]

and

\[ \rho_A = \sum_{j=1}^k \rho_j(\cdot)[\gamma_j] : \text{Ham}(U \sim \{p_1, \ldots, p_k\}) \to H_1(U \sim \{p_1, \ldots, p_k\}; \mathbb{R}) \]

is a homogeneous quasimorphism which satisfies all properties in Theorem 2. A continuum of linearly independent choices of \( \rho_A \) implies that for a choice of \( \rho_A \).

The proof of Proposition 7 takes the rest of this section. It can be found in [7], but we give it below for the sake of completeness. Let \( \mathbb{A} = S^1 \times (0, 1) \simeq \mathbb{R}/\mathbb{Z} \times (0, 1) \) equipped with the standard symplectic form \( \omega = d\theta \wedge dh \), \( \text{Area}(\mathbb{A}) = 1 \). Without
loss of generality we assume that the generator \( \alpha \in H_1(\mathbb{A}; \mathbb{Z}) \) is represented by the positively oriented circle \( S^1 \times [0,0.5] \). Let \( \mathcal{D} \subset \mathbb{A} \) be a disk with \( \text{Area}(\mathcal{D}) = A \), and denote \( L = \partial \mathcal{D} \). Let \( S = \{ \phi \in \text{Ham}(\mathbb{A}) | \phi(\mathcal{D}) = \mathcal{D} \} \) be the stabilizer of \( \mathcal{D} \) in \( \text{Ham}(\mathbb{A}) \). We denote \( S_n = \{ \phi \in S | \|\text{traj}_\mathcal{D}(\phi)\| = na \} \).

Pick a function \( \hat{H} : \mathbb{A} \to \mathbb{R} \) with compact support such that \( \hat{H}(\theta, h) = h \) away from a small neighborhood of \( \partial \mathbb{A} \). Denote by \( \Phi \) the time-1 map of the Hamiltonian flow generated by \( \hat{H} \). It is easy to see that \( \Phi \in S \) with trajectory \( [\text{traj}_\mathcal{D}(\Phi)] = \alpha \). Note that \( S = \bigcup_{n \in \mathbb{Z}} S_n = \bigcup_{n \in \mathbb{Z}} \hat{\Phi}^n S_0 \). We construct a quasimorphism \( \rho_A \).

**Step I.** It suffices to construct a homogeneous quasimorphism \( \rho : \text{Ham}(\mathbb{A}) \to \mathbb{R} \) which satisfies:

- \( \rho \) is Hofer-Lipschitz,
- \( \rho(S_0) = 0 \), \( \rho(\Phi) = c > 0 \),
- for all disks \( U \subset \mathbb{A} \), \( \rho \) vanishes on the image of \( \text{Ham}(U) \hookrightarrow \text{Ham}(\mathbb{A}) \).

**Proof.** \( \rho \) vanishes on Hamiltonians supported in a disk. It follows from [5, Theorem 1.7] that such \( \rho \) is continuous in the \( C^0 \)-topology. Any \( \phi \in S_n \) decomposes as \( \phi = \hat{\Phi}^n \circ s \) for some \( s \in S_0 \). Hence

\[
|\rho(\phi) - cn| = |\rho(\hat{\Phi}^n \circ s) - n\rho(\Phi) - 0| = |\rho(\hat{\Phi}^n \circ s) - \rho(\hat{\Phi}^n) - \rho(s)| < R.
\]

(\( R \) denotes the defect of \( \rho \)). It follows that

\[
\rho(\phi) = c \cdot n + \delta_\phi,
\]

where \( |\delta_\phi| \leq R \). Note that \( \phi^k \in S_{nk} \), hence by homogeneity

\[
\rho(\phi) = \frac{\rho(\phi^k)}{k} = \frac{c \cdot nk + \delta_{\phi^k}}{k} = c \cdot n + \frac{\delta_{\phi^k}}{k},
\]

which implies \( \rho(\phi) = c \cdot n \) in the limit as \( k \to \infty \).

Set \( \rho_A = \frac{\rho}{c} \). Suppose that \( \phi \in \text{Ham}(\mathbb{A}) \) has an invariant disk \( D \) of \( \text{Area}(D) \geq A \).

1. We consider the case \( \text{Area}(D) = \emptyset \). Then \( \phi \in S_n \) for some \( n \in \mathbb{Z} \) and

\[
\rho_A(\phi) \cdot \alpha = \frac{\rho(\phi)}{c} \cdot \alpha = n\alpha = [\text{traj}_D(\phi)].
\]

2. The case \( \text{Area}(D) = A \). Pick \( g \in \text{Ham}(\mathbb{A}) \) such that \( g(D) = \emptyset \). Then \( g\phi g^{-1} \in S \), hence

\[
\rho_A(\phi) \cdot \alpha = \rho_A(g\phi g^{-1}) \cdot \alpha = [\text{traj}_D(g\phi g^{-1})] = [\text{traj}_D(\phi)].
\]

3. The case \( \text{Area}(D) > A \). Pick a smaller disk \( D_1 \subset D \) with \( \text{Area}(D_1) = A \). Since \( \phi(D) = D \), we can choose \( g \in \text{Ham}(D) \) such that \( g(\phi(D_1)) = D_1 \). Then \( D_1 \) is a fixed disk of area \( A \) for \( g\phi \), hence

\[
\rho_A(g\phi) \cdot \alpha = [\text{traj}_{D_1}(g\phi)] = [\text{traj}_D(\phi)].
\]

We claim that \( \rho_A(g\phi) = \rho_A(\phi) \). Indeed, the disk \( D \) is fixed for \( g\phi \), \( \phi \) and their iterates. Moreover, \( (g\phi)^n \) differs from \( \phi^n \) by a Hamiltonian \( f_n \in \mathbb{H} \).
$Ham(D)$. Hence, by the homogeneity of $\rho_A$,

$$|\rho_A((g\phi)^n) - \rho_A(\phi^n) - \rho_A(f_n)| = |n\rho_A(g\phi) - n\rho_A(\phi) - 0| = n\cdot |\rho_A(g\phi) - \rho_A(\phi)|$$

is bounded by the defect of $\rho_A$. Letting $n \to \infty$, we get $\rho_A(\phi) = \rho_A(g\phi)$. \(\square\)

**Step II.** In order to build $\rho$ we use the Calabi quasimorphism on $Ham(S^2)$ which was constructed by M. Entov and L. Polterovich in [4]. We give a brief recollection of the relevant facts.

Let $U$ be an open disk equipped with a symplectic form $\omega$. Let $F_t : U \to \mathbb{R}$, $t \in [0,1]$, be a time-dependent smooth function with compact support. We define

$$\overline{Cal}(F) = \int_0^1 \left( \int_U F_t \omega \right) dt.$$ 

As $\omega$ is exact on $U$, $\overline{Cal}$ descends to a homomorphism $Cal_U : Ham(U) \to \mathbb{R}$ which is called the Calabi homomorphism. Clearly, for $\phi \in Ham(U)$, $|Cal_U(\phi)| \leq \text{Area}(U) \cdot \|\phi\|$. 

Let $S^2$ be a sphere equipped with a symplectic form $\omega$. Suppose $\text{Area}(S^2) = 2A$. For a smooth function $F : S^2 \to \mathbb{R}$ the Reeb graph $T_F$ is defined as the set of connected components of level sets of $F$ (for a more detailed definition we refer the reader to [4]). For a generic Morse function $F$ this set, equipped with the topology induced by the projection $\pi_F : S^2 \to T_F$, is homeomorphic to a tree. We endow $T_F$ with a measure given by $\mu(X) = \int_{\pi_F^{-1}(X)} \omega$ for any $X \subseteq T_F$ with measurable $\pi_F^{-1}(X)$. $x \in T_F$ is called a median of $T_F$ if the measure of each connected component of $T_F \sim \{x\}$ does not exceed $A$. By [4] a median exists and is unique. This construction can be extended to functions $F$ such that $F|_{\text{supp}(F)}$ is Morse.

Reference [4] describes the construction of a homogeneous quasimorphism $Cal_S^2 : Ham(S^2) \to \mathbb{R}$. It has the following properties: $Cal_S^2$ is Hofer-Lipschitz ($|Cal_S^2(\phi)| \leq 2A \cdot \|\phi\|$). In the case when $\phi \in Ham(S^2)$ is supported in a disk $U$ which is displaceable in $S^2$, $Cal_S^2(\phi) = Cal_U(\phi|_U)$. Moreover, for $\phi \in Ham(S^2)$ generated by an autonomous function $F : S^2 \to \mathbb{R}$, $Cal_S^2(\phi)$ can be computed in the following way. Let $x$ be the median of $T_F$ and $X = \pi_F^{-1}(x)$ be the corresponding subset of $S^2$. Then

$$Cal_S^2(\phi) = \int_{S^2} F\omega - 2A \cdot F(X).$$

Given a symplectic embedding $j : \mathbb{A} \to S^2$ into a sphere of area $2A$, consider the pullback $Cal_j = j^*(Cal_S^2) : Ham(\mathbb{A}) \to \mathbb{R}$. Namely, given $\phi \in Ham(\mathbb{A})$, extend $j_*(\phi)$ to $\hat{\phi} \in Ham(S^2)$ by identity in the complement of $j(\mathbb{A})$. Then $Cal_j(\phi) = Cal_S^2(\phi)$. Clearly, $Cal_j$ is a homogeneous quasimorphism. It has the following properties:

- $Cal_j(\phi) = Cal_S^2(\phi|_D)$ for any $\phi$ supported in a disk $D$ of area $A$. To see that, note that the corresponding $\hat{\phi} \in Ham(S^2)$ is supported in a displaceable disk $j(D)$ in $S^2$;
which fixes a neighborhood of $L$ of $q$ of $L$ may be chosen with arbitrarily small Hofer norm. Further, we may find a $\psi$ isotopy $\psi$ topological.

Proof. Pick an open disk $\psi'$ and vanishes on $S^1 \setminus \{0\}$ and a disk of area $(2A - 1 - s)$ to $S^1 \times \{1\}$. This construction ensures that $j_s(L) = j_s(\partial S^2)$ cuts $S^2$ into two displaceable disks. We pick $0 \leq s_1 < s_2 \leq 2A - 1$ and set

$$
\rho = \rho_{s_1, s_2} = \text{Cal}_{j_2} - \text{Cal}_{j_1}.
$$

We claim that $\rho$ satisfies the conditions of Step 1. Obviously, $\rho$ is a homogeneous quasimorphism on $\text{Ham}(\mathbb{A})$ which satisfies the Lipschitz property:

$$
|\rho(\phi)| \leq |\text{Cal}_{j_2}(\phi)| + |\text{Cal}_{j_1}(\phi)| \leq 4A \cdot \|\phi\|.
$$

Consider the function $\hat{H}$ which was used to define the Hamiltonian $\hat{\Phi}$ described above. It is easy to see that the “median” level set $X$ of $\hat{H}$ which is relevant for the computation of $\text{Cal}_{j_1}(\hat{\Phi})$ is $S^1 \times \{A - s\}$. Hence

$$
\text{Cal}_{j_1}(\hat{\Phi}) = \int_{\mathbb{A}} \hat{H} \omega - 2A \cdot \hat{H}(X) = \int_{\mathbb{A}} \hat{H} \omega - 2A \cdot (A - s).
$$

This implies

$$
\rho(\hat{\Phi}) = \text{Cal}_{j_2}(\hat{\Phi}) - \text{Cal}_{j_1}(\hat{\Phi}) = 2A \cdot (- (A - s_2) + (A - s_1)) = 2A \cdot (s_2 - s_1) > 0.
$$

Suppose that $\phi \in \text{Ham}(U)$ for some disk $U \subset \mathbb{A}$. Then

$$
\rho(\phi) = \text{Cal}_{j_{2*}}(\phi) - \text{Cal}_{j_{1*}}(\phi) = \text{Cal}_{S^2}(j_{s_2*}(\phi)) - \text{Cal}_{S^2}(j_{s_1*}(\phi)) = 0
$$

since $j_{s_2*}(\phi)$ is conjugate to $j_{s_1*}(\phi)$.

It is left to show that $\rho$ vanishes on $S_0$. Denote by $S_0'$ the subgroup of $S_0$ which fixes a neighborhood of $L$ pointwise.

**Lemma 8.** Let $q$ be a homogeneous quasimorphism which is Hofer-continuous and vanishes on $S'_0$. Then $q$ vanishes on $S_0$.

**Proof.** Pick an open disk $U \subset \mathbb{A} \sim L$. $\text{Ham}(U) \subset S'_0$, therefore $q$ vanishes on $\text{Ham}(U)$. It follows from the results of [5] that $q$ is continuous in the $C^0$-topology.

Suppose $\phi \in S_0$. This implies $\phi(L) = L$ (as a set). Applying an appropriate isotopy $\psi \in S_0$ supported near $L$, we may ensure that $\psi \circ \phi = \mathbb{I}$ on $L$. Moreover, $\psi$ may be chosen with arbitrarily small Hofer norm. Further, we may find a $C^0$-small diffeomorphism $h \in \text{Ham}(\mathbb{A})$ such that $h \circ \psi \circ \phi = \mathbb{I}$ in a neighborhood of $L$. It follows that $h \circ \psi \circ \phi \in S'_0$ and $q(h \circ \psi \circ \phi) = 0$. Hofer- and $C^0$-continuity of $q$ imply that $q(\phi) = 0$. \hfill $\Box$
Let \( \phi \in S'_0 \). We show that \( \rho(\phi) = 0 \). \( \phi \) splits to a composition \( \phi = \phi_{\xi} \circ \phi_P \), where \( \phi_{\xi} \) is supported in \( \mathcal{D} \) and \( \phi_P \) is supported in the pair of pants \( P = \mathbb{A} \sim \mathcal{D} \). \( \phi_{\xi} , \phi_P \) have disjoint supports, therefore they commute. Hence \( \rho(\phi) = \rho(\phi_{\xi}) + \rho(\phi_P) \). Note that \( \phi_{\xi} \) is supported in a disk, so \( \rho(\phi_{\xi}) = 0 \).

\( \phi_P \) is Hamiltonian in \( \mathbb{A} \), but after the restriction to \( P \) we have just \( \hat{\phi}_P = \phi_P \big|_P \in \text{Sym} \text{plect}c(P) \). In the argument below we apply to \( \phi_P \) a sequence of deformations in order to get \( \phi'_P \) whose restriction \( \hat{\phi}'_P \in \text{Ham}(P) \). All deformations involved in the process preserve the value \( \rho(\phi_P) \). Finally, we show that \( \rho(\phi'_P) = 0 \) by explicit computation.

The mapping class group \( \pi_0(\text{Sym} \text{plect}c(P)) \) is isomorphic to \( \mathbb{Z}^3 \) and is generated by Dehn twists near the three boundary components. For the proof of this fact we refer the reader to \([6]\) where the authors show that \( \pi_0(\text{Diff}_c(P)) = \mathbb{Z}^3 \) and is generated by Dehn twists. Note that \( \phi, \psi \in \text{Sym} \text{plect}c(P) \) are isotopic in \( \text{Sym} \text{plect}c \) if and only if they are isotopic in \( \text{Diff}_c \). As Dehn twists belong to \( \text{Sym} \text{plect}c(P) \), the statement for \( \pi_0(\text{Sym} \text{plect}c(P)) \) follows.

Denote by \( T_1, T_0 \) Dehn twists near \( S^1 \times \{1\}, S^1 \times \{0\} \) and by \( T_L \) a Dehn twist in \( P \) near \( L = \partial \mathcal{D} \). There are Hamiltonians \( \psi_L \) in \( S'_0 \) with arbitrary small Hofer norm whose restriction to \( P \) realizes the Dehn twist \( T_L \). For example, consider a bump function supported near \( \mathcal{D} \) which has small height but is very steep in an annulus around \( L \). \( \phi_P \) is isotopic in \( \text{Sym} \text{plect}c(P) \) to some \( T_1^{k_1} T_0^{k_0} T_L^{-k_L} \) \((k_i \in \mathbb{Z})\). If \( k_L \neq 0 \) we replace the original \( \phi \in S'_0 \) by \( \psi_L^{-k_L} \circ \phi \in S'_0 \). As \( \|\psi_L\| \) can be chosen to be arbitrarily small, by continuity of \( \rho \) it is enough to show the desired statement for the deformed \( \phi \). After the replacement \( k_L \) vanishes, hence the modified \( \phi_P \sim T_1^{k_1} T_0^{k_0} \). Note that \( \phi_P \) is induced by a Hamiltonian \( \phi \in \mathcal{A} \). The definition of \( \text{traj}_{\mathcal{D}}(\phi) \) implies that \( k_1 \alpha = [\text{traj}_{\mathcal{D}}(\phi)] = -k_0 \alpha \) (recall that \( \alpha \) is the chosen generator of \( H_1(\mathcal{A}; \mathbb{Z}) \)). The minus sign appears because opposite orientations of the boundary components result in the opposite directions of the corresponding Dehn twists. Moreover, as \( \phi \in S'_0 \) \([\text{traj}_{\mathcal{D}}(\phi)] = 0 \) hence \( k_1 = k_0 = 0 \).

Therefore the restriction \( \hat{\phi}_P \) belongs to the identity component of \( \text{Sym} \text{plect}c(P) \).

Pick \( K : \mathcal{A} \to \mathcal{R} \) supported in a small neighborhood of \( \mathcal{D} \) such that \( K = 1 \) in a neighborhood of the closure \( \overline{\mathcal{D}} \). Denote by \( \chi^t \) the time-\( t \) map generated by the Hamiltonian flow of \( K \). \( \chi^t \) is supported in a disk in \( \mathcal{A} \), hence \( \rho(\chi^t) = 0 \) for all \( t \).

Consider the homomorphism \( i_* : H^1_c(P; \mathbb{R}) \to H^1_c(\mathcal{A}; \mathbb{R}) \) induced by inclusion \( i : P \to \mathcal{A} \). Both \( \chi^t, \phi_P \) are Hamiltonian in \( \mathcal{A} \), hence their fluxes are zero in \( H^1_c(\mathcal{A}; \mathbb{R}) \). After restriction to \( P \), \( \chi^t \big|_P, \phi_P \) belong to one-dimensional subspace \( \ker i_* \subset H^1_c(P; \mathbb{R}) \). \( \chi^t \big|_P, \phi_P \) are not zero flux, therefore one can find an appropriate \( t_0 \in \mathbb{R} \) such that \( \phi'_P = \chi^{t_0} \circ \phi_P \) restricts to \( \hat{\phi}'_P = \chi^{t_0} \big|_P \circ \hat{\phi}_P \) with zero flux in \( P \). The results of \([1]\) imply \( \hat{\phi}'_P \in \text{Ham}(P) \).

Pick a compactly supported function \( F_t : P \times [0,1] \to \mathcal{R} \) whose flow generates \( \hat{\phi}'_P \). Denote by \( U_s \) the complement of the closed disk \( \overline{\mathcal{D}} \) in \( S^2 \), it is a displaceable disk. \( j_{s,*}(\phi'_P) \in \text{Ham}(S^2) \) and it is supported in \( U_s \), therefore

\[
\text{Cal}_{j_s}(\phi'_P) = \text{Cal}_{U_s}(j_{s,*}(\phi'_P)) = \int_0^1 \left( \int_{U_s} j_{s,*} F_t \omega \right) dt = \int_0^1 \left( \int_P F_t \omega \right) dt
\]
is independent of $s$. Hence $\rho(\phi^s_P) = 0$. From $\phi^s_P = \chi^{t s} \circ \phi_P$ we obtain that
\[ |\rho(\phi^s_P) - \rho(\chi^{t s}) - \rho(\phi_P)| = |\rho(\phi_P)| \]
is bounded by the defect of $\rho$. It follows that $\rho$ is bounded on the subgroup $S'_0$. As $\rho$ is homogeneous, it vanishes there.

**Remark.** The choice of different parameters $s_1, s_2$ in (1) gives rise to different quasimorphisms $\rho_A = \rho_{A, s_1, s_2}$. If we fix $s_1$ and let $s_2$ vary, we get a linearly independent family. We show that by an argument from [2]: let $H(\theta, h) = H(h) : A \to \mathbb{R}$ be a function which depends only on $h$, and denote by $\phi_H \in \text{Ham}(A)$ the corresponding Hamiltonian. The median level set relevant for $\text{Cal}_j(\phi_H)$ is $S^1 \times (A - s)$, hence
\[ \rho_{A, s_1, s_2}(\phi_H) = \text{Cal}_j(s_2)(\phi_H) - \text{Cal}_j(s_1)(\phi_H) = 2A \cdot (H(A - s_1) - H(A - s_2)). \]
If we consider the set of all Hamiltonians generated by functions $H = H(h)$, the restriction of the family $\{\rho_{A, s_1, s_2}\}_{s_2}$ to this set is linearly independent.

4. Discussion

4.1. **Comparison with the Riemannian length spectrum.** In the case of a Riemannian manifold $(M, g)$, the marked length spectrum $l : \pi_1(M) \to \mathbb{R}$ contains a lot of information regarding the metric $g$. For example, in a closed negatively curved surface $(M, g)$, $g$ is completely determined by $l$ ([10]).

The symplectic analogue looks much less rewarding. The only case when $l_A : H_1(M; \mathbb{Z}) \to \mathbb{R}$ is not bounded is the case of a punctured disk. Therefore Hofer’s length spectrum is able to detect genus zero surfaces of finite area. It may happen that more precise estimates of $l_A$ provide additional information regarding the topology of $M$.

On the other side, the symplectic setting is less rich than the Riemannian one. The equivalence class of $(M, \omega)$ is determined by the genus, the number of punctures and $\text{Area}(M)$. In the case of finite area $\omega$ is just a scaling factor, hence asymptotic behavior of $l_A$ depends only on the topology of $M$. Clearly, $l_A$ is able to extract some of the topological information of $M$. This result may be interpreted as the fact that the geometry of $\text{Ham}(M)$ sees some of the topology of $M$.

4.2. **Homotopical spectrum.** It would be interesting to provide similar estimates for $l_A : \pi_1(M) \to \mathbb{R}$. When we consider the asymptotical behavior of $l_A$, it is more convenient to discuss the stabilized version
\[ \tilde{l}_A(\alpha) = \lim_{n \to \infty} \frac{l_A(n\alpha)}{n}. \]
The homological version of $\tilde{l}_A$ is a norm on $H_1$ when $\text{genus}(M) = 0$ and $2A > \text{Area}(M)$. In all other cases $\tilde{l}_A \equiv 0$.

If we replace $H_1$ by $\pi_1$, the argument from the previous sections gives the following partial information:
• Suppose that $2A \leq \text{Area}(M)$. Then $\tilde{I}_A \equiv 0$. This is true since the argument of Section 2.1 applies to $\pi_1$ and not just $H_1$.

• Suppose $2A > \text{Area}(M)$ and $\text{genus}(M) > 0$. Assume that $\alpha$ is represented by a simple nonseparating loop. Then $na$ can be represented by concatenation of two simple loops, hence $I_A(na) \leq 2A$. Therefore $I_A(\alpha) = 0$ and $I_A(ka) = 0$ for all $k \in \mathbb{Z}$.

• Suppose $2A > \text{Area}(M)$ and $\text{genus}(M) = 0$. If $\alpha$ corresponds to a nonzero homology class then $\tilde{I}_A(\alpha) > 0$.

The author does not know estimates for most of the remaining cases. For example, let $2A > \text{Area}(M)$ and $\text{genus}(M) \geq 2$. For a primitive self-intersecting class $\alpha$ the number of simple curves needed to represent $na$ is not bounded as $n \to \infty$ (see [3]), so the argument for the upper bound does not give a lot. On the other hand, we do not know any tools that may give a nontrivial lower bound. There are several constructions of quasimorphisms on positive genus surfaces, but none of them is known to respect Hofer’s metric. When $\text{genus}(M) = 0$ and $\alpha$ belongs to the commutator subgroup $[\pi_1(M, \ast), \pi_1(M, \ast)]$, the argument used in this paper fails as translation by $\alpha$ cannot be realized by an autonomous flow. It may happen that Entov-Polterovich quasimorphisms used in our argument still imply a nontrivial lower bound. Unfortunately, the author was not able to compute them on appropriate non-autonomous examples.

One may take the stabilized word length norm $\| \cdot \|_S$ in $\pi_1(M)$ with respect to the generating set of simple loops. D. Calegari observed that $A\| \cdot \|_S$ satisfies all properties of $\tilde{I}_A$ described above (we assume $2A > \text{Area}(M)$). Indeed, the discussion in Section 2 implies $\tilde{I}_A \leq A\| \cdot \|_S$ and we do not know any example where the inequality is strict.

4.3. **Higher dimensions.** Most arguments and constructions used in this paper fail when one considers balls, polydisks or tori in higher dimension manifolds. For example, upper bounds for $I_A(\alpha)$ were induced by decomposition of $\alpha$ into simple loops. In dimension $\geq 4$ any loop can be perturbed into a simple one. However, we cannot translate a ball through a pipe around such simple loop because of the ‘symplectic camel’ phenomenon.

4.4. **The quasimorphism $\rho_A$.** In Theorem 2 we consider quasimorphisms $\rho_A : \text{Ham}(M) \to H_1(M; \mathbb{R})$ for a $k$ times punctured disk $M$. $\rho_A$ can be seen as a generalization of the rotation number $\rho : \text{Homeo}_+(S^1) \to \mathbb{R}$. Indeed, $\rho$ is a homogeneous quasimorphism which satisfies the following: it is Lipschitz with respect to the $C^0$-norm on $\text{Homeo}_+(S^1)$. If $f \in \text{Homeo}_+(S^1)$ has a fixed point $x$, then $\rho(f)$ computes the degree of the trajectory of $x$ under $f$. In fact, in dimension 1 $\rho$ is determined by the second property, while there is an infinite family of linearly independent homogeneous quasimorphisms satisfying the same properties as $\rho_A$.

Let $f_t$ be a Hamiltonian isotopy and $\emptyset$ a disk of area $A$ or greater. $\rho_A(f_t)$ vanishes for those isotopies $f_t$ that fix $\partial \emptyset$ for all $t$. This implies that the value $\rho_A(f_t)$ is determined up to a bounded defect by the trajectory $\{f_t(\partial \emptyset)\}_t$. That is,
for any \( g_t \) such that \( g_t(\partial \mathcal{D}) = f_t(\partial \mathcal{D}) \), \( \rho_A(g_1) - \rho_A(f_1) \) is bounded. Therefore \( \rho_A \) can be thought of as an invariant of Hamiltonian isotopies of \( \partial \mathcal{D} \).

In particular, \( \rho_A \) nearly ignores those dynamical attributes that do not affect a large enough disk in \( M \), it sees only 'global' features that deform all large disks. In this sense it is different from most other generalizations of rotation number to dimension two.

A possible application is the ability to detect non-existence of large fixed or periodic disks. Namely, if we pick two such quasimorphisms \( \rho_A, \rho_B' \) and \( \rho_A(f) \neq \rho_B'(f) \), then \( f \) has neither fixed nor periodic disks of area \( \geq \max\{A, B\} \).

More than that, if one wishes to perturb \( f \) in order to have such a periodic disk, \( \rho_A(f) - \rho_B'(f) \) allows to estimate a lower bound on Hofer's energy of such perturbation. However, current techniques allow to compute the value of \( \rho_A \) in a very limited number of scenarios (like the case of autonomous Hamiltonians, locally supported and etc.) which limits practical use of this observation.

Another application is as follows. Pick a non-displaceable disk \( \mathcal{D} \) in \( M \) and two linearly independent quasimorphisms \( \rho_A, \rho_B' \) where \( A, B \leq \text{Area} \mathcal{D} \). Denote by \( \mathcal{L} \) the space of Lagrangians Hamiltonian isotopic to \( \partial \mathcal{D} \). Suppose that \( f_t \) is a Hamiltonian isotopy. A simple computation shows that \( \tau = \rho_A - \rho_B' \) depends (up to a bounded defect) just on \( \partial f_1(\partial \mathcal{D}) \in \mathcal{L} \). This way \( \tau \) gives rise to an invariant on \( \mathcal{L} \) which is Lipschitz with respect to the induced Hofer's norm. \( \tau \) is not bounded since it is a nonzero homogeneous quasimorphism, therefore by Lipschitz property the space \( \mathcal{L} \) is not bounded in Hofer’s norm as well (a similar argument with more details can be found in [8]).

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