Mesh generation techniques for numerical integration of arbitrary function over polygonal domain by finite element method

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Abstract. In this paper, a new approach is introduced for approximating two dimensional surface integral problems numerically over a convex, non convex regions such integrals are typically occur in boundary element method (BEM), the approach given here is domain discretized method by influence of quadrilateral mesh generation technique, then apply Gauss Legendre quadrature rule to generate Gaussian points over convex, nonconvex region. The performances of this method are illustrated with numerical examples.

1. Introduction
The problem is considered in this paper is the numerical integration of the form

\[ I = \iint_D f(x, y) \, dx \, dy \]  

Where D is a domain bounded by convex and non convex polygons with arbitrary function \( f(x, y) \) are evaluated numerically by domain discretization method or mesh generation, discretized the polygonal domain into quadrilateral mesh based on triangle mesh generator. The midpoints of arbitrary triangle are joins with its sides. The finite element method is numerical method for evaluating the differential or integral form of the equation for given domain and its requires entire geometry including the boundary region to be modeled with finite elements, many numerical methods for solving equation (1) in triangle region have been investigated in [1-4], numerical approximation of the integrals over convex, nonconvex polygonal region by line integral method, cubic spline method are discussed in [5, 6], discretized the polygonal domain into 4- node, 8-node, 12-node quadrilateral element then apply Gauss Legendre quadrature method and approximate numerically by [7-9], quadrilaterals is divided into triangles by Gauss Legendre quadrature rule and approximated numerically in [10], numerical integration of arbitrary function over 2- dimensional region with line segments by Generalized Gaussian quadrature rule are discussed in [11]. The present method of this paper introduces a domain discretization method, discretize the polygonal region into quadrilateral mesh then approximated numerically by Gauss Legendre quadrature rule and evaluate typical integral problems which cannot be integrated analytically and numerically by considering lower order quadrature methods, if increases the quadrilaterals in the domain we get more accurate results.

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2. Mathematical Formulations over Quadrilateral Elements

Numerical approximation of integrals over quadrilateral element is of the form

\[ I = \int_{-1}^{1} \int_{-1}^{1} f(x, y) \, dx \, dy \] (2)

The family of four node quadrilateral element is mapped into 2-square region. The transformation of Cartesian coordinates \((x, y)\) plane to \((\xi, \eta)\) plane is considered by

\[ x = \sum_{k=1}^{4} x_k N_k(\xi, \eta) \]
\[ y = \sum_{k=1}^{4} y_k N_k(\xi, \eta) \]

Where \(N_k(\xi, \eta)\) be the shape function for quadrilateral element

\[ N_k(\xi, \eta) = \frac{1}{4}(1 + \xi_k \xi)(1 + \eta_k \eta) \] (3)

Where \((\xi_k, \eta_k), k = 1,2,3,4) = ((-1,-1),(1,-1),(1,1),(1,1))\)

From the eqn.(2) and eqn.(3) , we have

\[ \frac{\partial x}{\partial \xi} = \sum_{k=1}^{4} x_k \frac{\partial N_k}{\partial \xi} = \frac{1}{4} \left[ (-x_1 + x_2 + x_3 - x_4) + (x_1 - x_2 + x_3 - x_4) \eta \right] \]

\[ \frac{\partial x}{\partial \eta} = \sum_{k=1}^{4} x_k \frac{\partial N_k}{\partial \eta} = \frac{1}{4} \left[ (-x_1 - x_2 + x_3 + x_4) + (x_1 - x_2 + x_3 + x_4) \xi \right] \]

Similarly

\[ \frac{\partial y}{\partial \xi} = \sum_{k=1}^{4} y_k \frac{\partial N_k}{\partial \xi} = \frac{1}{4} \left[ (-y_1 + y_2 + y_3 - y_4) + (y_1 - y_2 + y_3 - y_4) \xi \right] \]

\[ \frac{\partial y}{\partial \eta} = \sum_{k=1}^{4} y_k \frac{\partial N_k}{\partial \eta} = \frac{1}{4} \left[ (-y_1 - y_2 + y_3 + y_4) + (y_1 - y_2 + y_3 + y_4) \xi \right] \]

\[ J = \frac{\partial (x,y)}{\partial (\xi,\eta)} = \frac{\partial x}{\partial \xi} \frac{\partial y}{\partial \eta} - \frac{\partial x}{\partial \eta} \frac{\partial y}{\partial \xi} = \alpha + \beta \xi + \gamma \eta \]

Where

\[ \alpha = \frac{1}{8} \left[ (x_4 - x_2)(y_1 - y_3) + (x_3 - x_1)(y_4 - y_2) \right] \]

\[ \beta = \frac{1}{8} \left[ (x_4 - x_3)(y_2 - y_1) + (x_1 - x_2)(y_4 - y_3) \right] \]

\[ \gamma = \frac{1}{8} \left[ (x_4 - x_1)(y_2 - y_3) + (x_3 - x_2)(y_4 - y_1) \right] \]

Then equation (2) reduces to

\[ I = \int_{-1}^{1} \int_{-1}^{1} f(x, y) \, dx \, dy = \sum_{i=1}^{m} \sum_{j=1}^{n} f \left( x(\xi_i, \eta_j), y(\xi_i, \eta_j) \right) J w_i w_j \]

to test the integral of polygonal region is shown in Figure.1 with end vertex are

\[ D_1 = \{(1, 2), (1, 4), (3, 3), (2, 1)\}, \quad D_2 = \{(2, 0), (0, 1), (2, 0), (1, -1), (1, -1)\}, \]
\[ D_3 = \{(0, 0), (2, 1.7), (0.3, 1), (-2, 1.3)\}, \quad D_4 = \{(0, 2), (1, 1), (0, 0)\} \]
3. Quadrilateral Mesh Generation
Quadrilateral mesh generation is based on triangle mesh generator, every triangle is discretized into three quadrilaterals, centroid of triangle is joins with midpoint of its sides, finally generate a quadrilateral meshes in regular and irregular polygonal domain, we write a program in Maple 13 for generating quadrilateral mesh
4. Numerical Integration over polygonal domain

In this section, we derive a new quadrature formula for evaluation of integrals over convex, non-convex polygonal region. Integral form of equation (1) can be rewritten as

\[ I = \sum_{i=1}^{n} \int_{D_i} f(x, y) \, dx \, dy \]

\[ = \Delta \sum_{i=1}^{n} \sum_{j=1}^{m} w_i \, w_j \, \left( \sum_{e=1}^{3} f(x_e(x_i, \eta_j), y_e(x_i, \eta_j)) \right) J_i \, d\xi \, d\eta \]

where \( \xi_i, \eta_j \) are Gaussian points and \( w_i, w_j \) are its weight coefficients. Jacobian is same for three quadrilaterals when arbitrary triangle is discretized into three quadrilaterals, find out all Gaussian points, its weight coefficients of arbitrary triangle by Gauss legendre quadrature rule of order \( N = 10 \), are plotted in Figure 3.
5. Application Examples

In order to evaluate the integration of arbitrary function over polygonal domain numerically by Gauss Legendre quadrature rule, compared the numerical results with [5], [11] as shown in Table 1.

Table 1.

| Integral with exact value | Present method | Computed value in [5,11] |
|---------------------------|----------------|--------------------------|
| $\int_{D_1} (x+y)^{1/2} (1+x+y)^2 \, dx \, dy$ | 298.234339210033 | 298.234339210033 |
| = 298.234339210034 |                  |                          |
\[ \int_{D_1} (x + y)^{-1/2} \, dx \, dy = 3.54961302678932 \]
\[ \int_{D_2} (1 - x) \sin (10xy) \, dx \, dy = -0.013103719669957 \]
\[ \int_{D_2} \frac{(x^4 + y^3)}{(1 + x^2)} \, dx \, dy = 1.924030542632 \]
\[ \int_{D_3} \sin(y) \, dx \, dy = 1.348335321149 \]
\[ \int_{D_3} yx^4 \, e^{-y} \, dx \, dy = 0.724875983177 \]
\[ \int_{0}^{1} \int_{x}^{2-x} \sqrt{x+y} \, dx \, dy = 1.13137084989633 \]

6. Conclusions

In this paper, numerical integration of the form \( I = \int_{D} f(x, y) \, dx \, dy \) are evaluated numerically by Gauss Legendre quadrature rule by discretizing the polygonal domain into quadrilateral mesh, increase the quadrilaterals in the domain we get more accurate solution in order to integration the function

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