The Three Quasi-Static Limits of the Maxwell Equations

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It is shown that the Galilean limit \( (V \ll c, \text{ or } L/T \ll c) \) of the Maxwell equations admits three different limits: the magneto-quasi-static, electro-quasi-static, and electromagnetic-quasi-static limits, in addition to the two obvious static limits. The first two quasi-static limits have been previously identified as Galilean Electrodynamics, while the latter is also known as the Darwin approximation. Using a perturbation expansion, a generalization of Rappetti and Rousseaux [Applied Numerical Mathematics, 79, 92] orders the vacuum Maxwell equations and obtains all three limits. To order the equations, the dimensionless version of the Maxwell equations are derived using a modification of Jackson’s review of EM unit systems [Jackson, Classical Electrodynamics, Wiley, 1999, 3rd ed.] The perturbation expansion is repeated for the potential form of the Maxwell equations to emphasize the importance of gauge conditions. The integral solutions of the potentials are derived for the three limits, and the generalized Coulomb and Biot-Savart equations are derived from these solutions. It is shown that although the forms are the same as the static equations, the quasi-static forms of the Maxwell equations are recovered. The induction term is recovered when the time derivative of the vector potential is kept. The displacement current is recovered when the Lorenz gauge is used. The equivalence of this approach and Jackson’s derivation [Amer. J. of Phys., 70, 917 (2002)] of the Darwin approximation is shown. The regions of applicability of the quasi-static forms of the Maxwell equations are discussed in terms of macroscopic media.
I. INTRODUCTION

The most common interaction people have with electromagnetic fields is turning on a light switch. When the switch is closed, a current at 60 Hz (in the U.S.) flows with an EM wavelength $5 \times 10^6$ m, or more than $3/4$ of an Earth radius. The light produced by the light bulb has multiple frequencies around $600$ THz, or a wavelength of $5 \times 10^{-7}$ m. The Maxwell equations describe the physics of both the circuit and the light propagation, but in practice we use approximations for each extreme of this 13 orders of magnitude range. For the high-frequency regime, an eikonal approximation yields the ray-tracing equations. These equations are so well-understood that they serve as the basis for the multi-billion dollar video game and computer animation industries. At the other extreme is the quasi-static regime. Although the traditional realm of electrical engineering, this regime is less well understood, as evidenced by the confusion around the different quasi-static limits.

One of the first derivations of quasi-static limits was published by Einstein and Laub in a series of papers (Einstein and Laub, 1908a,b,c,d) motivated by several experiments of rotating media. Some of the more prominent examples of this are those showing the Roentgen-Eichenwald effect (Eichenwald, 1903, 1904; Roentgen, 1888, 1890), the Wilson-Wilson effect (Wilson et al., 1913), and Barnett effect (Barnett, 1915). Rousseaux (Rousseaux, 2013) provides a review that gives a modern review of these experiments in the context of quasi-static electromagnetics. Because these experiments used rotating media in a laboratory, the rotation speeds were much less than the speed of light. The Einstein-Laub papers published in 1908 served as a primary reference for “low velocity limits” for physicists. In 1920, Darwin (Darwin, 1920) derived another quasi-static limit by truncating the potentials of the far-field in studying particle motion using Lagrangian mechanics. In 1927, the relationship between the Maxwell equations and circuit equations was elucidated by Carson (Carson, 1927) using the ratio of system size to wavelength as a small parameter which, as will be shown, is closely related to quasi-static limits.

The first discussion of two distinct quasi-static limits, the electro-quasi-static (EQS) and magneto-quasi-static (MQS) regimes, was published by engineers in 1968 (Woodson and Melcher, 1968). The first physics discussion of these two regimes by physicists was published by Le Bellac and Lévy-Leblond (LBLL) (Le Bellac and Lévy-Leblond, 1973) in 1973. They were motivated by a simple question: If mechanics has a limit
where the equations become Galilean covariant, do the Lorentz-covariant Maxwell equations also have a limit where they are Galilean covariant? Their “Galilean Electromagnetics” identified two incompatible limits: the time-like limit with $E \gg cB$ (EQS), and the space-like limit with $E \ll cB$ (MQS). In more recent years, there has been a resurgence of interest in Galilean electromagnetics. A derivation using group theory was made by de Montigny (De Montigny et al., 2003). A derivation using an order of magnitude analysis was given by Rousseaux (Rousseaux, 2003; Rousseaux and Domps, 2004). More formal derivations using a perturbation analysis with macroscopic media was made by Manfredi (Manfredi, 2013) and Rapetti and Rousseaux (Rapetti and Rousseaux, 2011, 2014). A 2013 review paper by Rousseaux (Rousseaux, 2013) provides an excellent overview of the literature to that date, including both the equations and the applications to experiments of rotating media.

These formal Galilean limits do not appear in the most popular physics textbooks. In Griffith’s undergraduate textbook (Griffiths, 2005), only the MQS limit is given, and it is discussed as the quasi-static limit. This is consistent with the Einstein-Laub papers discussing only the MQS limit. Jackson (Jackson, 1975, 1999) discusses the MQS limit, although referred to as the quasi-static approximation, as well as the Darwin approximation. Rapetti and Rousseaux (Rapetti and Rousseaux, 2014) speculate that the Darwin approximation is an electromagnetic quasi-static (EMQS) limit, but do not include a formal derivation. Thus, there is no comprehensive review of all three quasi-static limits.

The reviews have also been limited in scope. The study of electrodynamics involves studying multiple mathematical forms of the Maxwell equations, as illustrated in Figure 1. The best form to use for the electrodynamical equations depends on the application at hand, and the solution technique employed, either analytic or numerical. For example, charged particle motion in an electrodynamical field may best be studied using the potential formulation if using Lagrangian mechanics, the generalized Coulomb and Biot-Savart methods if using Newtonian mechanics, the field-form of the Maxwell equations if using the FDTD numerical method (Yee, 1966), or the retarded potential solutions if using the radiation far field. In all of the previous derivations of the quasi-static limits, the complete set of the forms of the Maxwell equations are not given for each quasi-static limit. Here, a unified approach to the derivation to the EQS, MQS, and EMQS equations is presented using a quasi-static perturbation expansion for the vacuum Maxwell equations. Ignoring the effects of macroscopic media provides a simpler method and recovers the elegance found in the original LBLL.

Figure 1 The study of electromagnetics requires understanding the different, yet mathematically equivalent, forms of Maxwell equations. The best form depends on the application of the study, and this applies to the quasistatic regime as well. Table III summarizes these different forms for the three Galilean limits of Maxwell equations.
Arguably, the most common encounter of the quasi-static limits in modern physics is in the area of plasma physics. Because the electrons are free to move and shield the slower moving ions, a process known as Debye shielding, the quasi-neutral approximation is used. This approximation is equivalent to using the MQS limit for the bulk of the plasma. Near the edge of laboratory plasmas, a thin boundary layer exists, known as the plasma sheath, where the EQS limit applies. Because of this transition, and yet still quasi-static nature of the laboratory experiments, the Darwin approximation is also used in plasma physics. Despite the widespread use, the relationship of plasma equations to the quasi-static equations are often not clear, and errors such as using MQS equations with gauges other the Coulomb are frequently made. Focusing on the vacuum equations provides a useful framework for understanding the approximations before dealing the complications of plasma theory.

The framework for this review will be a perturbation analysis of the Maxwell equations. A perturbation analysis uses the expansion of the variables in a small parameter to study the properties of the equations to a given order – in this case, only the first-order equations are studied. In order to do this, we must first cast the Maxwell equations in a dimensionless form so that the relative sizes of terms can be determined appropriately. While this process is easy to understand if using a specific unit system such as MKS units, it is helpful to discuss it for arbitrary unit systems, and we start with a full discussion of the different unit systems for the Maxwell equations. After this background, we order the equations, take the appropriate limits, and show that only three first-order, quasi-static limits are possible. We then revisit the derivation in terms of potentials, and relate our derivation of the Darwin approximation to the more typical approximation given by Jackson in both his book and more recent paper (Jackson, 1975, 1999, 2002). From the potential form of the Maxwell equations, integral solutions and generalized Coulomb and Biot-Savart equations are easily derived. Finally, we relate this this derivation to the results of Rapetti and Rousseaux for macroscopic media is shown. Frequent references to the textbooks of Griffiths (Griffiths, 2005) and Jackson (Jackson, 1975, 1999) are given to make this review easier to follow.

II. UNITS AND DIMENSIONAL ANALYSIS OF THE MAXWELL EQUATIONS

The Maxwell equations have engendered considerable confusion when it comes to units. While the trend has been towards using MKSA units, the scientific literature uses many different systems. Our technique for understanding the various quasi-static limits is to perform a perturbation analysis that analyzes the terms that become small as the ordering parameter becomes small. The relative “smallness” of a term must be identified independent of units so that units do not distort which terms appear small. For the Maxwell equations, this is especially important because c, a key factor, is large, and its location changes depending on the system of units.

The key issue in dimensional analysis is the choice of fundamental, or base, units. As review, for all coherent unit systems, there are fundamental, or base, units, and the units of every other quantity are then expressed in terms of these units. There is freedom in choosing the base units. For the Maxwell equations, the natural base units are Length-Time-Charge, while most analyses are in terms of the traditional Length-Mass-Time units from mechanics (e.g., cgs and MKS). In other words, the Maxwell equations themselves do not have mass as a quantity, so including it confuses the analysis. The most systematic analysis of the EM unit system in a textbook is the Appendix of Jackson. In this paper, the analysis of Jackson is repeated using a more traditional Rayleigh-Buckingham dimensional analysis approach with Length-Time-Charge as the base units. The relationship to the mechanical base units is discussed after this analysis.

The fundamental units of common EM unit systems are given in Table I. The Maxwell equations are equations for 4 variables ($\mathbf{E, B, p, J}$) with 3 fundamental units ($L, T, Q$) where $Q$ is the unit of charge. According to the Buckingham $\Pi$ theorem, there will be (4 variables)−(3 units) = (1 dimensionless constant) to describe the system. Our goal is therefore to derive a dimensionless version of the Maxwell equations with this dimensionless number, and the expression for it for each system of units under consideration. Here, an algebraic approach is used as opposed to the (admittedly more standard) matrix-power approach of Heras (Heras and Baez, 2008) and Rapetti and Rousseaux (Rapetti and Rousseaux, 2014). The methods are equivalent, but the algebraic approach is useful in understanding the perturbation analysis in the next section.
| Base Units          | Length (L) | Time (T) | Charge (Q) | Relationship to standard base units |
|---------------------|------------|----------|------------|-------------------------------------|
| Electrostatic (esu) | centimeter | second   | statcoulomb | $Q = \left[ML^3T^{-2}\right]^{1/2}$ |
| Electromagnetic (emu)|            |          |            | $Q = \left[MLT\right]^{1/2}$          |
| Gaussian            |            |          |            | $Q = \left[ML^3T^{-2}\right]^{1/2}$ |
| Heaviside-Lorentz   |            |          |            | $Q = \left[ML^3T^{-2}\right]^{1/2}$ |
| Rationalized MKSA   | meter      | second   | coulomb    | ampere (I): $Q = [IT]$                |

Table I Summary of the base units for each of the unit systems. The analysis presented is in terms of the Maxwell base units ($LTQ$). The last column shows the fundamental base unit for each system and its relationship to the charge unit.

We start by writing the Maxwell equations with each term on the right-hand side prefixed by a (rationalized) constant:

- **Gauss’s Law**: $\nabla \cdot \mathbf{E} = 4\pi k_1 \rho$ \hspace{1cm} (1a)
- **No magnetic monopoles**: $\nabla \cdot \mathbf{B} = 0$ \hspace{1cm} (1b)
- **Faraday’s Law**: $\nabla \times \mathbf{E} = -k_3 \frac{\partial \mathbf{B}}{\partial t}$ \hspace{1cm} (1c)
- **Maxwell-Ampère Law**: $\nabla \times \mathbf{B} = K_4 \frac{\partial \mathbf{E}}{\partial t} + 4\pi K_2 \mathbf{J}$. \hspace{1cm} (1d)

It is obvious that any unit system can be described in terms of the four constants, $k_1, K_2, k_3$, and $K_4$. The notation follows that of the appendix of Jackson, with the capitalized constants being the factors that differ from Jackson to simplify the new approach. Our goal is to determine a minimal set of constants to describe the Maxwell equations. The easiest method is to determine the capitalized constants in the Maxwell-Ampère law.

Taking the divergence of the Maxwell-Ampère law, Eq. (1d), and inserting Gauss’s law, Eq. (1a), gives the continuity equation

$$\frac{\partial \rho}{\partial t} + \frac{K_2}{K_4 k_1} \nabla \cdot J = 0.$$ \hspace{1cm} (2)

The factor in front of the divergence term must be unity to match the conventional relationship between the units of $\rho$ and $\mathbf{J}$ that all unit systems obey. This yields the elimination equation

$$K_2 = K_4 k_1,$$ \hspace{1cm} (3)

which relates the constant in the Maxwell-Ampère law to the constant $k_1$ in Gauss’s Law. Taking the curl of the source-free form of the Maxwell-Ampère law, and inserting Faraday’s Law, gives the wave equation:

$$\frac{\partial^2 \mathbf{B}}{\partial t^2} - \frac{1}{K_4 k_3} \nabla^2 \mathbf{B} = 0.$$ \hspace{1cm} (4)

This equation describes the propagation of light waves, and thus the factor in front of the Laplacian must be equal to the inverse square of the speed of light. This yields the second elimination equation:

$$K_4 = \frac{1}{k_3 c^2},$$ \hspace{1cm} (5)

which relates a scale factor in the Maxwell-Ampère law to the constant $k_3$ in Faraday’s Law.

Using these relations, we rewrite the Maxwell-Ampère law as

$$\nabla \times \mathbf{B} = \frac{1}{c^2 k_3} \left[ \frac{\partial \mathbf{E}}{\partial t} + 4\pi k_1 \mathbf{J} \right].$$ \hspace{1cm} (6)
The constant $4\pi k_1$, which also prefixes the source term in Gauss’s law (Eq. (1a)), is related to the single dimensionless constant ($\hat{\alpha}$), predicted by the Buckingham $\Pi$ theorem as shown below.

The constant $k_3$ is related to unit relationships. To see this, the units of $E, B, \rho, J$ are denoted as $[E], [B], [\rho], [J]$. The units of space and time are denoted $L$ and $T$ respectively. Examination of the units of Faraday’s law, Eq. (1c), and the source-free version of the Maxwell-Ampère law Eq. (6), gives the same result:

$$\frac{[E]}{[B]} = [k_3] \frac{L}{T} \tag{7}$$

From this we see that $k_3$ serves to adjust the units of $B$ relative to $E$, with two reasonable choices being that either $k_3$ is unitless and the ratio is given by a velocity unit, or that $k_3$ is $c^{-1}$ and the units of $E$ and $B$ are the same (see Table I).

The derived units of the Maxwell equations follow from the above relations. Starting with the units for $\rho$ that are the same in all unit systems (equivalently, the units of $J$ could be used because the continuity equation is the same in all unit systems), the rest of the units are derived straightforwardly from the above equations:

$$[\rho] = QL^{-3} \quad [J] = QL^{-2}T^{-1} \quad [E] = [k_1] QL^{-2} \quad [B] = \left[ \frac{k_1}{k_3} \right] QL^{-3}T \tag{8}$$

The units for $E$ and $B$ are determined once the units for $k_1$ and $k_3$ are specified for a given unit system. Similar to Jackson, a table summarizing the different equation systems is shown in Table II. If $k_1$ contains a factor of inverse $4\pi$, then the unit system is termed a rationalized unit system. The Heaviside-Lorentz unit system is the rationalized Gaussian unit system.

Another recent dimensional analysis of the Maxwell equations is the “$\alpha\beta\gamma$-units” equations of Heras and Báez [Heras and Báez 2008]. The relationship between our system and that description is $\alpha = 4\pi k_1$, $\beta = 4\pi k_1/(k_3 c^2)$, and $\gamma = k_3$. The form of the dimensional Maxwell equations given by Eqs. (1) with either $k_3 = 1$ or $k_3 = 1/c$, uses a single dimensionless constant ($k_1$) that by the Buckingham $\Pi$ theorem is the form for the Maxwell equations with the minimal number of constants. It is important to distinguish what is necessary ($k_1$) and what is useful for scaling units ($k_3$) \footnote{One could imagine yet another constant to enable a difference in units between $\rho$ and $J$, which would allow the charge continuity equation to be written differently. Fortunately, no unit system has yet been so created; i.e., all unit systems developed have the same form of the continuity equation.}

We are now ready to de-dimensionalize the equations. To do so, we write

$$x = \bar{L} \hat{x} \quad t = \bar{T} \hat{t}$$

$$E(x,t) = \bar{E} \hat{E}(x,t) \quad B(x,t) = \bar{B} \hat{B}(x,t)$$

$$\rho(x,t) = \bar{\rho} \hat{\rho}(x,t) \quad J(x,t) = \bar{J} \hat{J}(x,t) \tag{9}$$

where the bar denotes a constant with units, and hat denotes a unitless, but spatially and temporally varying, quantity. We use $\bar{L}$ as our characteristic length scale such that the units are $[x] = [\bar{L}] = L$. The dimensionless spatial vector $\hat{x}$ should not be confused with a unit vector. Substituting these into the equations above yields:

$$\hat{\nabla} \cdot \hat{E} = \left(4\pi k_1 \frac{\bar{\rho} \bar{L}}{\bar{E}} \right) \hat{\rho}$$

$$\hat{\nabla} \cdot \hat{B} = 0$$

$$\hat{\nabla} \times \hat{E} = -\frac{\bar{L}}{cT} \frac{k_3 c \bar{B}}{\bar{E}} \frac{\partial \bar{B}}{\partial \hat{t}}$$

$$\hat{\nabla} \times \hat{B} = \frac{\bar{L}}{cT} \frac{\bar{E}}{ck_3 \bar{B}} \left[ \frac{\partial \bar{E}}{\partial \hat{t}} + 4\pi k_1 \frac{\bar{\rho} \bar{L}}{\bar{E}} \frac{\bar{J}}{c \bar{\rho} \bar{L}} \right] \tag{10}$$
Equations 7 and 8 can be used to demonstrate that these equations are dimensionless.

The scale length ratios in these equations provide both freedom and constraints. We use the freedom to the relationships to set to unity the ratios involving pure field terms or pure source terms. This is the process of de-dimensionalization. The constraints force the ratios of field magnitudes to source magnitudes, in other words, the constraints force the factors involving \( k_1 \) to have a specified value determined by \( \hat{\alpha} \).

In the relativistic limit, setting these factors to unity is clear: there is no separation between electric and magnetic fields, between charge density and current density, and between space and time. The quasi-static (non-relativistic) limit allows these quantities to be ordered separately as discussed in the next section.

The dimensionless forms of the equations can then be simply written as:

\[
\hat{\nabla} \cdot \hat{E} = \hat{\alpha} \hat{\rho} \tag{11a}
\]
\[
\hat{\nabla} \cdot \hat{B} = 0 \tag{11b}
\]
\[
\hat{\nabla} \times \hat{E} = -\frac{\partial \hat{B}}{\partial t} \tag{11c}
\]
\[
\hat{\nabla} \times \hat{B} = \frac{\partial \hat{E}}{\partial t} + \hat{\alpha} \hat{J} \tag{11d}
\]

with the single dimensionless constant given by

\[
\hat{\alpha} = 4\pi k_1 \frac{\hat{\rho}\hat{L}}{E} \tag{12}
\]

The hat serves to distinguish this alpha from Jackson’s and indicates that it is dimensionless. Manfredi (Ref. [Manfredi, 2013]) denotes this factor as \( 1/\alpha \). This parameter is to the Maxwell equations as the Reynolds number is to the Navier-Stokes equations.

This form of the Maxwell equations is useful for numerical work because the variables will be of order unity, which is useful to prevent numerical round-off issues. To solve, one would choose useful charge and current density characteristic scales (\( \hat{\rho} \) and \( \hat{J} \)) to de-dimensionalize the equations, and also set the dimensionless parameter \( \hat{\alpha} \). One can then numerically solve the equations. The conversions to dimensional units for \( E \) and \( B \) are given by \( E = 4\pi k_1 \hat{\rho} \hat{L}/\hat{\alpha} \) and \( B = (4\pi k_1 / c^2/k_3)(\hat{L}\hat{J})/\hat{\alpha} \).

To summarize the results so far, the arbitrary unit version of the Maxwell equations are given by Eqs. (1). Specifying a unit system requires choosing the constants \( k_1 \) and \( k_3 \), with \( k_3 \) being chosen as a convenient normalization between \( E \) and \( B \). The dimensionless form of the Maxwell equations, Eqs. (11), is obviously the same for any unit system, and the dimensionless number is specified by Eq. (12). A table summarizing the different equation systems is shown in Table II.

As discussed earlier, unit systems such as cgs or MKS refer to the mechanical base units (Length-Mass-Time). To relate the EM base units to the mechanical base units, the Lorentz force law in our dimensional units is written as

\[
F = q(E + k_3 v \times B) \tag{13}
\]

Using \( F = ma \) (or the relativistic version thereof), considering only the electric field, and using Eqs. (8), one can show that

\[
Q = \sqrt{ML^3 \over T^2 |k_1|} \tag{14}
\]

Substituting in the values of \([k_1]\) for the different unit systems gives the final column of Table II.

The MKSA system is unique in that the constants used in the Maxwell equations, \( \epsilon_0 \) and \( \mu_0 \), have units containing mass as seen in Table II. As discussed in the Appendix of Jackson, this is because the determination of the constant involves Newton’s law. This is an unfortunate property of the MKSA unit system; a true Rayleigh-Buckingham approach to dimensional analysis necessitates the inclusion of an additional equation. Some authors (e.g., [Heras, 2010a; Rousseaux, 2013]) have made note of the difference between a \( c \) used for units (\( c_u \)) and the fundamental constant denoting the speed of light (\( c \)). These different \( c \)‘s are a historical artifact of how the different unit systems developed, and define their fundamental constants. As seen in the
\[
\begin{align*}
\nabla \cdot \mathbf{E} &= 4\pi k_1 \rho \\
\nabla \cdot \mathbf{B} &= 0 \\
\nabla \times \mathbf{E} &= -\frac{k_3 \varphi \mathbf{E}}{\varphi t} \\
\n\nabla \times \mathbf{B} &= \frac{1}{c^2 k_3} \left[ \frac{\partial \mathbf{E}}{\partial t} + 4\pi k_1 \mathbf{J} \right]
\end{align*}
\]

\[
\begin{align*}
\nabla \cdot \mathbf{E} &= \hat{\alpha} \rho \\
\nabla \cdot \mathbf{B} &= 0 \\
\nabla \times \mathbf{E} &= -\frac{\partial \mathbf{E}}{\partial t} \\
\n\nabla \times \mathbf{B} &= \frac{1}{c^2} \left[ \frac{\partial \mathbf{E}}{\partial t} + \hat{\alpha} \mathbf{J} \right]
\end{align*}
\]

Table II Summary of the different unit systems used for Maxwell equations. Placing \(k_1\) and \(k_3\) into the equations give the specific forms. The units of \(k_1\) and \(k_3\) can be combined with Eq. (8) to determine the units of \(\mathbf{E}\) and \(\mathbf{B}\). The dimensionless Maxwell equations number for the different unit systems is unit system specific.

minimal form of the dimensional equations above, \(c\) is fundamental to the Maxwell equations, appearing in the Maxwell-Ampère law regardless of the value of \(k_1\) or \(k_3\).

The use of \(k_3\) is helpful in understanding what makes a good engineering set of units. In the third edition of Jackson (Jackson, 1999), the first 10 chapters use MKSA units, while the remaining chapters covering relativistic mechanics use cgs units. What makes cgs a good base unit system for studying relativity is that \(\mathbf{B}\) has the same units as \(\mathbf{E}\) in the same way that \(x_0 = ct\), the zeroth component of the space-time 4-vector, has the same units as the remaining components of the 4-vector. Just as Newtonian mechanics uses \(t\) and not \(x_0\), everyday experiences with electromagnetism work better with \(k_3 = 1\): \(c\mathbf{B}\) the same units as \(\mathbf{E}\). MKSA being a good set of “engineering units” has less to do with the use of mass in defining its units (which causes confusion), and more to do with convenience in the Galilean limit.

### III. LOW VELOCITY LIMIT OF THE MAXWELL EQUATIONS

#### A. Quasi-static limits and the charge continuity equation

This paper focuses on the “quasi-static limits” of the Maxwell equations where we are interested in the forms of the equations when the following parameter is small:

\[
\beta = \frac{\bar{L}}{\bar{T} c}.
\]

The particular interpretation of this parameter is problem specific. For example, the Galilean limit of the Lorentz transformation is taken by assuming that the speed of the inertial frame is small: \(V = \bar{L} / \bar{T} \ll c\), where \(c\) is the speed of light. The small parameter in this instance will be

\[
\beta_{\text{inertial}} = \frac{V}{c} = \frac{\bar{L}}{\bar{T} c} \quad \text{and} \quad \beta_{\text{inertial}} = \frac{V}{c},
\]

where the subscript inertial implies that this is the small parameter for inertial transformations, and the vector is used to denote the direction of the inertial velocity.
For the common occurrence of a source oscillating at a single frequency, \( f \), this small parameter can be written as

\[
\beta_{\text{osc}} = \frac{\bar{L}}{T f \lambda} = \frac{1}{N_{\text{osc}}} \frac{\bar{L}}{\lambda},
\]

(17)

where \( N_{\text{osc}} \) is the number of oscillations of interest and \( \bar{L}/\lambda \) is the electrical length in a vacuum. For example, consider a parallel plate capacitor that is 1 mm wide as part of an AC circuit oscillating at 60 Hz. At 60 Hz, the wavelength is \( 5 \times 10^6 \) m gives an electrical length of \( 2 \times 10^{-10} \). If this capacitor is part of an RC circuit with a charging time of 1/6 sec, then one charging time corresponds to 10 oscillations, and hence a \( \beta_{\text{osc}} = 2 \times 10^{-11} \). An expansion in this case is so good that one could reasonably believe that a static approximation would be satisfactory. This is discussed in Section VII. The number of periods of interest, \( N_{\text{osc}} \), depends on details of the specific problem, and this will be discussed throughout the paper. The electrical length being much less than one serves as a conservative estimate for whether we are in the quasi-static regime; \( N_{\text{osc}} = 1 \) is the minimum number that one would want to study when oscillating sources are present.

Other important examples of \( \beta \) as a small parameter are the laboratory electrodynamic experiments of rotating media (Barnett, 1915; de Montigny and Rousseaux, 2007; Rousseaux, 2013; Weber, 1997; Wilson, 1905; Wilson et al., 1913). In this case, the rotation speed is much slower than the speed of light, but the transformation is non-inertial. Finally, quasi-static limits can be useful in studying particle motion. Darwin (Darwin, 1920) developed a quasi-static form by using an ordering of the ratio of the particle velocity to the speed of light.

To understand how we use this small parameter to derive the quasi-static, or Galilean limits, we first review the kinematic and dynamic limits from relativistic mechanics. We then consider the charge continuity equation to show how electrodynamics gives different behavior than classical mechanics.

A powerful concept in special relativity is the invariance of 4-vector products under Lorentz transformations. For example, the “length” of the spacetime vector

\[
x^\mu x_\mu = c^2 t^2 - \mathbf{x} \cdot \mathbf{x}
\]

(18)

is invariant under any inertial transformation. If there are two events, and we place one event at the origin of the spacetime grid, then the separation between the two events corresponds to an invariant quantity. When this invariant is positive, the separation is “time-like”; when the invariant is negative, the separation is “space-like”; and when the invariant is zero, the separation is light-like, as shown in Figure 2.

De-dimensionalizing the above spacetime invariant gives

\[
\hat{t}^2 - \frac{\bar{L}^2}{T^2 c^2} \hat{x} \cdot \hat{x} = \hat{t}^2 - \beta^2 \hat{x} \cdot \hat{x} = \text{invariant}.
\]

(19)

Thus between any two inertial reference frames, this invariant is the same:

\[
\hat{t}^2 - \beta^2 \hat{x} \cdot \hat{x}' = \hat{t}'^2 - \beta^2 \hat{x}' \cdot \hat{x},
\]

(20)

where the prime denotes a different inertial reference frame, and the origin of both frames is the same to simplify the algebra. In the limit \( \beta \rightarrow 0 \), \( \hat{t}' = \hat{t} = \text{constant} \). Graphically speaking, the light cone collapses to the \( x \) axis, and the hyperbolic lines defining simultaneity become horizontal lines. This is the Galilean limit.

This can also be derived from the Lorentz transformation directly. The Lorentz transformation of the spacetime 4-vector is

\[
\hat{t}' = \gamma (\hat{t} - \beta \hat{x} \cdot \hat{x})
\]

\[
\hat{x}' = \hat{x} - \frac{\beta}{\hat{t}} \hat{t} + (\gamma - 1) \frac{1}{\beta} \beta (\hat{x} \cdot \hat{x})
\]

(21)

The electrical length is electromagnetic and not electrical, and is not a length. We use the traditional term regardless.

See, for example: Jackson (Ref. Jackson, 1975, 1999), Sec. 11.3.
where $\gamma = (1 - \beta^2)^{-1/2}$. The $\beta$ and $\beta$ here are the inertial versions presented in Eq. [16], but the subscripts “inertial” are dropped for convenience. The Galilean limits are given by the transformation equations (written here with dimensional quantities):

$$\begin{align*}
t' &= t \\
x' &= x - V t.
\end{align*}$$

Thus, the Galilean limit has only the causal time-like behavior with an event being everywhere simultaneous. In this dimensionless form, the inertial frame velocity is unity.

Conservation of momentum-energy arises because of the invariance of the 4-vector

$$p^\mu p_\mu = \frac{E^2}{c^2} - \mathbf{p} \cdot \mathbf{p} = \text{invariant.}$$

In the $\beta \to 0$ limit, the time-like and space-like solutions decouple, and we have energy and momentum conservation simultaneously. That is, Newtonian mechanics is the quasi-static limit of Special Relativity, with identical time-like and space-like limits.

Now consider the current 4-vector. It has the same invariance properties as above:

$$J^\mu J_\mu = c^2 \rho^2 - \mathbf{J} \cdot \mathbf{J} = \text{invariant.}$$

For the current 4-vector, the key physics equation is not from the invariance, but rather charge continuity:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0.$$ 

---

4 The so-called “Carroll limit” [Lévy-Leblond, 1965] is not included here because it is second-order in the expansion parameter. In contrast to Heras (Heras, 2010b), our definition of quasi-static is that it be only first order in $\beta$. 
De-dimensionalizing this equation (while keeping the scale lengths) gives

\[
\left( \frac{\bar{L}}{T_c \bar{J}} \right) \frac{\partial \bar{\rho}}{\partial \bar{t}} + \nabla \cdot \bar{J} = \left( \beta \frac{\bar{c} \bar{\rho}}{\bar{J}} \right) \frac{\partial \bar{\rho}}{\partial \bar{t}} + \nabla \cdot \bar{J} = 0. 
\]  

(26)

As \( \beta \) becomes small, the ratio of \( c \bar{\rho} / \bar{J} \) can have different relative orderings, and those different orderings have different physical consequences. Two possible regimes are of interest as \( \beta \rightarrow 0 \):

\[
\text{Time-like: } \frac{c \bar{\rho}}{\bar{J}} \sim \frac{1}{\beta} \quad \frac{\partial \bar{\rho}}{\partial \bar{t}} + \nabla \cdot \bar{J} = 0,
\]

\[
\text{Space-like: } \frac{c \bar{\rho}}{\bar{J}} \sim \beta \quad \nabla \cdot \bar{J} = 0.
\]

That is, in the quasi-static limit, the charge density and current density decouple in the same way that space and time decouple. These two limits are equivalent to the absolute value of the invariant tensor product being large; i.e., \( |J^\mu J_\mu| \gg 0 \). The middle ordering of the scale length ratio \( c \bar{\rho} / \bar{J} \sim \mathcal{O}(1) \) might appear to be a “light-like” ordering; however, “light-like” corresponds to the fully relativistic case where there is no separation between \( \rho \) and \( J \) (which is why these ratios were set to unity to get the dimensionless equations as discussed after Eq. (10)). In the low-velocity limit, \( c \bar{\rho} / \bar{J} \) being of order unity is related to the static limits as discussed in Section VII. One could expect that the ordering of the sources will lead to a large value, either positive (time-like) or negative (space-like), in the Lorentz invariant of the electromagnetic field-strength tensor (with \( k_3 = 1/c \)):

\[
F^{\mu\nu} F_{\mu\nu} = E^2 - B^2. 
\]  

(27)

This will be shown in the next section.

In summary, our small parameter is the ratio of the length to time scale that is small relative to the speed of light. For the space-time vector \( x_\mu \), only one limit exists, and this is the Galilean transformation limit. It is time-like, \( c^2 \bar{t}^2 \gg \bar{x}^2 \), and is equivalent to an event being simultaneous across all reference frames. For the momentum 4-vector, there is no difference between the time-like and space-like limits: there is only one Newtonian limit where energy and momentum conservation are decoupled. However, the charge continuity equation admits two limits. In the time-like limit, \( c \bar{\rho} \gg \bar{J} \) and full charge continuity exists. The current density is causal: a changing charge density implies a change in current density. In the space-like limit, \( c \bar{\rho} \ll \bar{J} \), the current density is only a (possibly time-dependent) divergence-free field.

**B. Perturbation expansion of the Maxwell equations**

As discussed in the previous section, the sources and fields can be expected to separate into either “space-like” or “time-like” components in the Galilean limits; thus the relative magnitudes of \( E \) to \( B \), and \( \rho \) to \( J \) will be included in the ordering of the Maxwell equations. Starting with Eqs. (10) and specifying \( k_3 = 1/c \), to make the ratio of electric and magnetic field dimensionless, yields

\[
\nabla \cdot \bar{E} = \hat{\alpha} \bar{\rho}, \\
\nabla \cdot \bar{B} = 0, \\
\n\nabla \times \bar{E} = -\left( \beta \frac{\bar{E}}{\bar{B}} \right) \frac{\partial \bar{B}}{\partial \bar{t}}, \\
\n\nabla \times \bar{B} = \left( \beta \frac{\bar{E}}{\bar{B}} \right) \frac{\partial \bar{E}}{\partial \bar{t}} + \left( \bar{E} \frac{\bar{J}}{\bar{B} \bar{c} \bar{\rho}} \right) \hat{\alpha} \bar{J}.
\]  

(28)

In the relativistic, Lorentz covariant limit, \( \beta \) is not a small parameter and it can be set to unity as well as the ratios \( \bar{E} / \bar{B} \) and \( c \bar{\rho} / \bar{J} \) to give the purely dimensionless form of the Maxwell equations.

We are interested in the limit of \( \hat{\alpha} \sim \mathcal{O}(1) \) and \( \beta \ll 1 \). Examination of the last two ordered equations shows an apparent contradiction: the time derivative terms cannot both be of order unity simultaneously. When one term is of order unity, the other term will be of order \( \beta^2 \). Similarly, keeping the current term in
the last equation implies an inverse relationship between $\vec{E}/\vec{B}$ and $\vec{J}/(c\vec{p})$. Intuitively, when the EM tensor is (time, space)-like ($\vec{E} \gg \vec{B}, \vec{E} \ll \vec{B}$ or equivalently $F_{\mu\nu}F_{\nu\mu} \gg 0, F_{\mu\nu}F_{\nu\mu} \ll 0$), the current 4-vector must also be (time, space)-like, $(c\vec{p} \gg \vec{J}, c\vec{p} \ll \vec{J})$. For the case of $\vec{E} \sim \vec{B}$, we recover the trivial cases of the electrostatic and magnetostatic limits. While these are obviously Galilean limits, they are not of interest here although we will revisit these limits in Section VII.

More formally, the quasi-static limits of the Maxwell equations can be derived by expanding the fields in terms of $\beta$ as the small parameter:

$$
E = E_0 + \beta E_1 + \cdots
$$

$$
B = B_0 + \beta B_1 + \cdots
$$

$$
\rho = \rho_0 + \beta \rho_1 + \cdots
$$

$$
J = J_0 + \beta J_1 + \cdots.
$$

These are placed into the form of Maxwell’s equations with the fields as dimensional, the operators dimensionless, and the $\beta$ parameter kept for the ordering:

$$
\hat{\nabla} \cdot E = \bar{\alpha} \rho
$$

$$
\hat{\nabla} \cdot B = 0
$$

$$
\hat{\nabla} \times E = -\beta \frac{\partial B}{\partial t}
$$

$$
\hat{\nabla} \times B = \beta \frac{\partial E}{\partial t} + \frac{\bar{\alpha}}{c} J,
$$

where $\bar{\alpha} = \hat{\alpha} \vec{E}/\vec{\rho}$ has been introduced for convenience. Following the previous discussion, there are two limits: the time-like limit when $c\vec{p} \gg \vec{J}$, and the space-like limit when $c\vec{p} \ll \vec{J}$. The first ordering is equivalent to $\rho_0 \neq 0$ and $J_0 = 0$ such that $c\vec{p}/\vec{J} \sim O(1/\beta)$. Similarly, the second limit is given by $\rho_0 = 0$ and $J_0 \neq 0$ such that $c\vec{p}/\vec{J} \sim O(\beta)$.

For the time-like limit, $J_0 = 0$ and Maxwell-Ampère law implies that $\hat{\nabla} \times B_0 = 0$. Combined with $\hat{\nabla} \cdot B_0 = 0$ implies that $B_0 = 0$, assuming appropriate boundary conditions. Faraday’s law gives $\hat{\nabla} \times E_0 = 0$, but Maxwell-Ampère law keeps all terms that are of order $\beta$:

$$
\beta \left[ \hat{\nabla} \times B_1 = \frac{\partial E_0}{\partial t} + \bar{\alpha} J_0 \right]
$$

Likewise the full Gauss’s law is kept and it is of order unity. Thus, we have that $c\vec{p}/\vec{J} \sim O(1/\beta)$ implies that $\vec{E}/\vec{B} \sim O(1/\beta)$. Because the electric field is much larger than the magnetic field, this is called the electro-quasi-static (EQS) limit.

For the space-like limit, $\rho_0 = 0$ and Gauss’s law implies that $\hat{\nabla} \cdot E_0 = 0$. Faraday’s law implies that $\hat{\nabla} \times E_0 = 0$. Having both the divergence and curl of $E_0$ be zero implies that $E_0 = 0$ (assuming appropriate boundary conditions. This will be discussed further later). Maxwell-Ampère law gives $\hat{\nabla} \times B_0 = \bar{\alpha}/cJ_0$. Faraday’s law keeps all terms that are of order $\beta$:

$$
\beta \left[ \hat{\nabla} \times E_1 = -\frac{\partial B_0}{\partial t} \right]
$$

Likewise the full Gauss’s law is kept and it is of order beta. Thus, we have that $c\vec{p}/\vec{J} \sim O(\beta)$ implies that $\vec{E}/\vec{B} \sim O(\beta)$. Because the magnetic field is much larger than the electric field, this is called the magneto-quasi-static (MQS) limit.

In summary, the resultant equations for each regime are
Time-like Space-like

Electro-Quasi-static Magneto-Quasi-static

\( \vec{E} \gg \vec{B}, c\vec{\rho} \gg \vec{J} \) \( \vec{E} \ll \vec{B}, c\vec{\rho} \ll \vec{J} \)

\( \vec{J}_0 = 0 \) \( \rho_0 = 0 \)

\( \hat{\nabla} \cdot \vec{B}_0 = 0; \hat{\nabla} \times \vec{B}_0 = 0 \) \( \hat{\nabla} \cdot \vec{E}_0 = 0; \hat{\nabla} \times \vec{E}_0 = 0 \)

\( \hat{\nabla} \cdot \vec{E}_0 = \hat{\alpha}_\rho \rho_0 \) \( \hat{\nabla} \cdot \vec{E}_1 = \hat{\alpha}_\rho \rho_1 \)

\( \hat{\nabla} \times \vec{B}_0 = 0 \) \( \hat{\nabla} \times \vec{B}_1 = 0 \)

\( \hat{\nabla} \times \vec{E}_0 = 0 \) \( \hat{\nabla} \times \vec{B}_0 = 0 \)

If the boundary conditions are such that there are no strong, static external fields, then we can ignore \((\vec{B}_0, \vec{E}_0)\) in the (EQS,MQS) limit. The use of subscripts naturally shows the LBLL regimes of validity of \( \bar{E} \gg \bar{B} \) for the EQS regimes and \( \bar{E} \ll \bar{B} \) for the MQS regime. The two regimes are merely the dropping of relevant time derivative terms. It will be shown in the next section that these equations are Galilean covariant.

Choosing \( k_3 = 1/c \), the Lorentz force law, Eq. (13), can be written as

\[ \mathbf{F} = q (\mathbf{E} + \beta \times \mathbf{B}) \]  \hspace{1cm} (33)

Because \( \beta \) is the ordering parameter, this shows that for the EQS regime, only the electric field contributes to the Lorentz force, and the force is zeroth order. For the MQS regime, the force is first order, and the zeroth-order magnetic field and the first-order electric field contribute. If the boundary conditions are such that there are strong, static external fields, then these fields can be added as a linear superposition of the fields. The implications of how the fields act on particles is discussed by LBLL [Le Bellac and Lévy-Leblond, 1973], who proposed solving the equations in both regimes and then combining.

As can be seen, the key feature of these limits is the dropping of one of the appropriate time-derivative terms. This changes the Maxwell equations from second-order in time to first-order. As an intuitive consideration of this behavior, consider driven waveguide modes where the electrical length in Eq. (17) is less than unity. In this case, the modes decay axially rather than propagate: a change from a wave solution of a second-order-in-time equation to an exponentially-decaying solution of a first-order-in-time equation. This confined-system case provides a useful method for studying the validity of the quasi-static equations [Haus and Melcher, 1989].

These two regimes of “Galilean Electrodynamics” do not include the Darwin approximation that is discussed in textbooks and widely used. This third approximation will make use of the Helmholtz decomposition, discussed in Appendix B. It decomposes the electric field into its longitudinal (irrotational) and transverse (solenoidal) parts:

\[ \vec{E} = \vec{E}_L + \vec{E}_T, \]  \hspace{1cm} (34a)

\[ \nabla \cdot \vec{E}_T = 0, \]  \hspace{1cm} (34b)

\[ \nabla \times \vec{E}_L = 0. \]  \hspace{1cm} (34c)

To motivate why the EQS and MQS regimes might not be a complete ordering, we rewrite the Maxwell equations using this decomposition:

\[ \hat{\nabla} \cdot \hat{\vec{E}}_L = \hat{\alpha}_\rho \hat{\rho} \]

\[ \hat{\nabla} \cdot \hat{\vec{B}} = 0 \]

\[ \hat{\nabla} \times \hat{\vec{E}}_T = -\frac{\partial \vec{B}}{\partial t} \]

\[ \hat{\nabla} \times \hat{\vec{B}} = \frac{\partial \hat{\vec{E}}_L}{\partial t} + \frac{\partial \hat{\vec{E}}_T}{\partial t} + \hat{\alpha}_\rho \hat{\vec{J}} \]  \hspace{1cm} (35)
The electromagnetic wave equation is formed by combining Faraday’s law and the Maxwell-Ampère law. A quasi-static approximation eliminates light waves, so it is reasonable to ask if there is a method of eliminating only the transverse component in the Maxwell-Ampère law. To investigate this question, the scale lengths in our ordering procedure are generalized to

\[ \hat{\nabla} \cdot \hat{E} = \hat{\alpha} \hat{\rho} \]
\[ \hat{\nabla} \cdot \hat{B} = 0 \]
\[ \hat{\nabla} \times \hat{E} = -\left( \beta \frac{\hat{B}}{E_T} \right) \frac{\partial \hat{B}}{\partial t} \]
\[ \hat{\nabla} \times \hat{B} = \beta \frac{\hat{E}_L}{B} \frac{\partial \hat{B}}{\partial t} + \frac{\hat{E}_T}{B} \frac{\partial \hat{E}_T}{\partial t} + \left( \frac{\hat{E} \cdot \hat{J}}{B \cdot c \hat{\rho}} \right) \hat{\alpha} \hat{\rho} \]

The constraint for the magneto-quasi-static limit that the induction term be of order unity forces only the transverse electric field to be small, and does not constrain the longitudinal component. There exists another ordering of \( \hat{E}_T / \hat{B} \sim O(\beta) \) and \( \hat{E}_L / \hat{B} \sim O(1/\beta) \). This mixed ordering eliminates light-waves, to give a quasi-static ordering, and gives

\[ \hat{\nabla} \cdot \hat{E} = \hat{\alpha} \hat{\rho} \]
\[ \hat{\nabla} \cdot \hat{B} = 0 \]
\[ \hat{\nabla} \times \hat{E} = \frac{\partial \hat{B}}{\partial t} \]
\[ \hat{\nabla} \times \hat{B} = \frac{\partial \hat{E}_L}{\partial t} + \hat{\alpha} \hat{\rho} \hat{J} \]

This is known as the Darwin approximation, and because it includes both the induction term and part of the displacement current, it will be referred to as the electromagnetic quasi-static (EMQS) regime. The relationship of this form of the Darwin approximation and to that given in Jackson is presented in Section V when the retarded potentials are discussed.

The EMQS approximation, like the EQS and MQS equations, does not support wave propagation, as was our goal. The EMQS equations also yields the full continuity equation, like the EQS limit. The MQS limit gives a continuity equation of \( \nabla \cdot \hat{J} = 0 \). A summary of these results are included at the end of this article in Table III.

### C. Galilean Covariance

The original LBLL paper derived the EQS and MQS equations by first deriving the Galilean transformation of fields, and then asking what equations were needed to satisfy those fields. Here the inverse procedure is taken, whereby we demonstrate that the EQS and MQS equations derived previously are Galilean covariant. This allows us then to discuss the transformation properties of the EMQS approximation.

The Lorentz transformation of the fields can be written as (in dimensional form with \( k_3 = 1/c \)):

\[ E' = \gamma (E + \beta \times B) - \frac{\gamma^2}{\gamma + 1} \beta (\beta \cdot E) \]
\[ B' = \gamma (B - \beta \times E) - \frac{\gamma^2}{\gamma + 1} \beta (\beta \cdot B) \]
\[ J' = J - \gamma c \rho \beta - \frac{\gamma^2}{\gamma + 1} \beta (\beta \cdot J) \]
\[ \rho' = \gamma (\rho - \beta \cdot J / c) \]

---

5 See, for example: Ref. [Griffiths, 2005], Sec. 12.3.2; Ref. [Jackson, 1975, 1999], Sec. 11.10.
These relations are for inertial transformations and thus the $\beta$ in these relationships are the same as that as defined in Eq. (16), but we will not use the inertial subscript in this section. The Galilean transformations will have two different limits because of the different limits for the magnitudes as $\beta$ goes to zero. After de-dimensionalizing these equations and taking the limits similar to the previous discussions, the Galilean transformations are

\[
\begin{align*}
\hat{E}' &= \hat{E} \\
\hat{B}' &= \hat{B} - \hat{\nabla} \times \hat{E} \\
J' &= J - \hat{\nabla} \hat{\beta} \\
\end{align*}
\]

The $\hat{V}$ in these relationships is equivalent to $\beta / \beta$. In “engineering dimensional units” with $k_3 = 1$, no $c$ will appear in the equations as expected for the Galilean limit. For “relativistic dimensional units” with $k_3 = 1/c$, the $c$ that appears is strictly to get the units correct; that is, $\hat{V}$ is de-dimensionalized by $c$ for relativistic units. As pointed out by LBLL (Le Bellac and Lévy-Leblond, 1973), the low-velocity transformations were stated incorrectly in such textbooks as Purcell (Purcell, 1965) and Landau and Lifshitz (Landau and Lifshitz, 1959), demonstrating the confusion over the two separate limits.

To show Galilean covariance, these relationships are placed into their respective limiting forms of the Maxwell equations along with the Galilean transformation of the time and space derivative:

\[
\frac{\partial}{\partial t'} = \frac{\partial}{\partial t} + \frac{\beta}{\beta} \cdot \hat{\nabla} = \frac{\partial}{\partial t} + \hat{V} \cdot \hat{\nabla}
\]

The algebra is straightforward, and the only equations of interest are those with time derivatives. To demonstrate how the algebra works, the Maxwell-Ampère law in the EQS limits is shown. Substituting in these relationships gives

\[
\begin{align*}
\hat{\nabla} \times \hat{B}' &= \frac{\partial \hat{E}'}{\partial t'} + \hat{\alpha} \hat{J}' \\
\hat{\nabla} \times \hat{B} - \hat{\nabla} \times \left( \frac{\beta}{\beta} \times \hat{E} \right) &= \frac{\partial \hat{E}}{\partial t} + \left( \frac{\beta}{\beta} \cdot \hat{\nabla} \right) \hat{E} + \hat{\alpha} \hat{J} - \hat{\alpha} \hat{\beta} \frac{\beta}{\beta} \\
\hat{\nabla} \times \hat{B} - \hat{\beta} \hat{\nabla} \cdot \hat{E} + \hat{E} \left( \hat{\nabla} \cdot \frac{\beta}{\beta} \right) - \left( \hat{E} \cdot \hat{\nabla} \right) \frac{\beta}{\beta} &= \frac{\partial \hat{E}}{\partial t} + \left( \frac{\beta}{\beta} \cdot \hat{\nabla} \right) \hat{E} + \hat{\alpha} \hat{J} - \hat{\alpha} \hat{\beta} \frac{\beta}{\beta} \\
\hat{\nabla} \times \hat{B} &= \frac{\partial \hat{E}}{\partial t} + \hat{\alpha} \hat{J},
\end{align*}
\]

where in the last step all derivatives on $\beta$ were dropped because it is an inertial transformation, and Gauss’s law was used to cancel the $\hat{\beta}$ term. There is also a cancellation of $\left( \beta \cdot \hat{\nabla} \right) \hat{E}$ term on both sides. Both cancellations are required for Galilean covariance. Similar algebra can be used to show that the entire set of EQS and MQS limits of the Maxwell equations are Galilean covariant.

The same method also shows that the EMQS equations are not Galilean covariant. The cancellations required in the Maxwell-Ampère law do not occur. The underlying reason is that the decomposition of the fields into longitudinal and transverse components is neither Lorentz nor Galilean invariant. The non-Galilean covariance of the EMQS will be revisited in Section V when the retarded potentials are discussed.

IV. POTENTIAL OF THE QUASI-STATIC MAXWELL EQUATIONS

Our goal in this section is to derive the quasi-static forms of the Maxwell equations in terms of potentials. An obvious method would be to substitute in the expression for $\mathbf{E}$ and $\mathbf{B}$ in terms of potentials into the previously derived equations. However, here we repeat the ordering to give more insight into the important
role of the gauge conditions in the quasi-static limit. We start by writing the equations for the fields from potentials in dimensional form:

\[
\begin{align*}
E &= -\nabla \phi - k_3 \frac{\partial A}{\partial t}, \\
B &= \nabla \times A.
\end{align*}
\]  

(41a)\hspace{1cm} (41b)

Putting these expressions into the generalized dimensional form of the Maxwell equations (Eqs. (41)) gives the following form for the Maxwell equations

\[
\begin{align*}
-\nabla^2 \phi(x, t) - k_3 \frac{\partial}{\partial t} \nabla \cdot A(x, t) &= 4\pi k_1 \rho(x, t) \\
-\nabla^2 A(x, t) + \nabla \nabla \cdot A(x, t) &= -\frac{1}{c^2 k_3} \left[ \frac{\partial \nabla \phi(x, t)}{\partial t} + k_3 \frac{\partial^2 A(x, t)}{\partial t^2} - 4\pi k_1 J(x, t) \right],
\end{align*}
\]

(42a)\hspace{1cm} (42b)

with Eqs. (1b) and (1c) being satisfied automatically by the forms in Eq. (41). Following the same procedure for de-dimensionalization and ordering (and using \( k_3 = 1/c \) as before) gives

\[
\begin{align*}
\hat{\nabla}^2 \hat{\phi}(\hat{x}, \hat{t}) &= -\hat{\alpha} \hat{\rho}(\hat{x}, \hat{t}) - \hat{\beta} \frac{\hat{A}}{\hat{\phi}} \frac{\partial}{\partial \hat{t}} \hat{\nabla} \cdot \hat{A}(\hat{x}, \hat{t}), \\
\hat{\nabla}^2 \hat{A}(\hat{x}, \hat{t}) &= -\hat{\alpha} \frac{\hat{J}}{c \hat{\rho} \hat{A}} + \hat{\nabla} \left[ \beta \frac{\hat{\phi}}{\hat{A}} \frac{\partial \hat{\phi}(\hat{x}, \hat{t})}{\partial \hat{t}} + \hat{\nabla} \cdot \hat{A}(\hat{x}, \hat{t}) \right] + \beta^2 \frac{\partial^2 \hat{A}(\hat{x}, \hat{t})}{\partial \hat{t}^2},
\end{align*}
\]

(43)

where, in accordance with Eq. (41), we define \( \hat{\phi} = \frac{\phi}{\bar{E} \bar{L}} \) and \( \hat{A} = \frac{A}{\bar{B} \bar{L}} \). The displacement current is responsible for the second and fourth terms in the last equation. We therefore see that in the \( \beta \to 0 \) limit, the vector potential never contributes to the displacement current.

Before taking the quasi-static ordering, we consider the fully relativistic case. The dimensionless form can be seen by taking \( \beta \to 1 \) and also setting the ratios \( \hat{\phi}/\hat{A} \) and \( c \hat{\rho}/\hat{J} \) to unity. Using the Lorenz gauge allows writing the equations in the pleasingly symmetric and uncoupled form

\[
\begin{align*}
\left( \hat{\nabla}^2 - \frac{\partial^2}{\partial \hat{t}^2} \right) \hat{\phi}(\hat{x}, \hat{t}) &= -\hat{\alpha} \hat{\rho}(\hat{x}, \hat{t}), \\
\left( \hat{\nabla}^2 - \frac{\partial^2}{\partial \hat{t}^2} \right) \hat{A}(\hat{x}, \hat{t}) &= -\hat{\alpha} \hat{J}(\hat{x}, \hat{t}),
\end{align*}
\]

(44a)\hspace{1cm} (44b)

where the Lorenz gauge is

\[
\frac{\partial \hat{\phi}(\hat{x}, \hat{t})}{\partial \hat{t}} + \hat{\nabla} \cdot \hat{A}(\hat{x}, \hat{t}) = 0.
\]

(45)

The previous form, Eqs. (42), is more general.

We now consider the quasi-static orderings. The electro-quasi-static (EQS) regime can be defined as the ordering regime of \( \hat{A}/\hat{\phi} \sim \beta \), in addition to \( c \hat{\rho}/\hat{J} \sim 1/\beta \) as discussed before, to give to lowest order

\[
\begin{align*}
\hat{\nabla}^2 \hat{\phi}(\hat{x}, \hat{t}) &= -\hat{\alpha} \hat{\rho}(\hat{x}, \hat{t}), \\
\hat{\nabla}^2 \hat{A}(\hat{x}, \hat{t}) &= -\hat{\alpha} \hat{J}(\hat{x}, \hat{t}) + \hat{\nabla} \left[ \frac{\partial \hat{\phi}}{\partial \hat{t}} + \hat{\nabla} \cdot \hat{A} \right].
\end{align*}
\]

(46a)\hspace{1cm} (46b)

The Lorenz gauge is required to place these equations into standard form involving only a Laplacian, which is the same form as the electrostatic (ES) and magnetostatic (MS) equations. Along with the Maxwell equations, the ordered equations for calculating the fields in the EQS limit are

\[
\begin{align*}
\hat{E} &= -\hat{\nabla} \hat{\phi}, \\
\hat{B} &= \hat{\nabla} \times \hat{A}.
\end{align*}
\]

(47a)\hspace{1cm} (47b)
Looking at the potential equations, the EQS equations look like the electrostatic and magnetostatic equations. The time dependence is implicit in the fields, and in the use of Lorenz gauge for the EQS approximation.

The magneto-quasi-static (MQS) regime can be defined as the ordering regime of $\bar{\phi}/\bar{A} \sim \beta$ to give to lowest order

$$\nabla^2 \phi = -\dot{\phi} - \frac{\partial \nabla \cdot A}{\partial t}, \quad (48a)$$

$$\nabla^2 \dot{A} = -\dot{\alpha} \rho + \nabla \left( \nabla \cdot \dot{A} \right). \quad (48b)$$

The Coulomb gauge is required to place these equations into the same form as the electrostatic (ES) and magnetostatic (MS) equations. The ordered equations for calculating the fields from potentials are

$$\dot{E} = -\nabla \phi - \frac{\partial \dot{A}}{\partial t}, \quad (49a)$$

$$\dot{B} = \nabla \times \dot{A}. \quad (49b)$$

In this limit, the equations relating the fields and potentials are the same as in the full EM limit.

In summary, the Maxwell equations in potential form for both the EQS and MQS limits can be written as

$$\nabla^2 \phi(\hat{x}, \hat{t}) = -\dot{\alpha} \rho(\hat{x}, \hat{t}) \quad (50a)$$

$$\nabla^2 \dot{A}(\hat{x}, \hat{t}) = -\dot{\alpha} \dot{J}(\hat{x}, \hat{t}) \quad (50b)$$

using the Lorenz and Coulomb gauges respectively. The obvious question arises: If the potential forms for the EQS and MQS limits are the same, then how do different equations sets for the Maxwell equations arise? In the EQS limit, the use of the Lorenz gauge condition instead of the Coulomb gauge condition adds the effect of the displacement current. In the MQS limit, the calculations of the $\mathbf{E}, \mathbf{B}$ fields from the potentials given by Eq. (49) include an explicit time dependence in the calculation of $\mathbf{E}$. This gives the inductive physics from Faraday’s law. This is shown in more detail in Section VI. The subtleties of the quasi-static as opposed to static equations are discussed further in Section VII.

The previous derivation of the EMQS equations, Eqs. (37), used the fact that the transverse and longitudinal components of the electric field have different quasi-static limits. The derivation of the EMQS equations in the potential formulation involves the gauge conditions. In the EQS and MQS limits (Galilean EM), the Lorenz gauge is ordered the same way as the continuity equation, Eq. (26), to obtain the nice Laplacian form of the potential form of the Maxwell equations: the Lorenz gauge is used in the time-like EQS limit (where we have the full continuity equation), and the Coulomb gauge is used in the space-like MQS limit (where we have $\nabla \cdot \mathbf{J} = 0$).

For the EMQS limit, we take advantage of the gauge freedom to order $\dot{A}/\dot{\phi} \sim \beta$, like EQS, but specify the Coulomb gauge, like MQS. This ordering and gauge in Eq. (43) gives

$$\nabla^2 \dot{\phi}(\hat{x}, \hat{t}) = -\dot{\alpha} \rho(\hat{x}, \hat{t}) \quad (51a)$$

$$\nabla^2 \dot{A}(\hat{x}, \hat{t}) = -\dot{\alpha} \dot{J}(\hat{x}, \hat{t}) + \nabla \frac{\partial \dot{\phi}(\hat{x}, \hat{t})}{\partial \hat{t}}. \quad (51b)$$

This form is not convenient because $\phi$ appears in the equation for $A$. To write this equation in a more useful form, we again look at the longitudinal and transverse components. The right side of Eq. (51b) is divergence free, as can easily be seen using the continuity equation. As a consequence, the right side can be written as proportional to the transverse current density, $J_T$. As a consequence, although not proven here, only the transverse vector potential, $\mathbf{A}_T$, is specified by this equation.

---

6 Ordering like MQS but keeping the Lorenz gauge gives an inconsistent ordering.
The formal proof that the right-side of Eq. (51b) is the transverse current density is given in Appendix A. There it is shown that the last term is equivalent to $\hat{\alpha}\hat{J}_L$, which allows writing the right-side as $\hat{\alpha}$ times the transverse current. This gives the EMQS equations in potential form as

\[ \hat{\nabla}^2 \hat{\phi}(\hat{x}, \hat{t}) = -\hat{\alpha}\hat{\rho}(\hat{x}, \hat{t}), \]  
\[ \hat{\nabla}^2 \hat{A}_T(\hat{x}, \hat{t}) = -\hat{\alpha}\hat{J}_T(\hat{x}, \hat{t}). \]

A detailed derivation of this form is also given by Jackson (Jackson, 2002), along with an illuminating discussion of the implications of using the Coulomb gauge.

By definition and because of the Coulomb gauge, the longitudinal vector potential satisfies

\[ \nabla \cdot A_L = 0, \]
\[ \nabla \times A_L = 0. \]

If the boundary conditions on $A_L$ vanish at infinity, then this leads to $A_L = 0$. In any case, $A_L$ is small and hence the longitudinal part of the electric field, $E_L$, is given by the scalar potential. Contrast this with the Lorenz gauge that when ordered gives

\[ \hat{\phi} \sim \frac{1}{\beta}. \]

Because the EQS ordering is a statement of the ratio of the magnitude of $E/B$ and $B$ is from $A_T$, the EQS ordering also implies $A_T \sim A_L$. By choosing the EQS ordering but the Coulomb gauge, one is effectively ordering $\hat{\phi} \gg A_T \gg A_L$. In Jackson’s book, the Darwin approximation is explained as an ordering of the retarded potentials. To show equivalency between our forms of the EMQS equations and the Darwin approximation, we discuss the retarded potentials next.

V. INTEGRAL SOLUTIONS OF THE POTENTIAL EQUATIONS

The full Maxwell equations in dimensionless potential form using the Lorenz gauge are given by

\[ \hat{\nabla}^2 \hat{\phi} - \frac{\partial^2 \hat{\phi}}{\partial \hat{t}^2} = -\hat{\alpha}\hat{\rho}, \]
\[ \hat{\nabla}^2 \hat{A} - \frac{\partial^2 \hat{A}}{\partial \hat{t}^2} = -\hat{\alpha}\hat{J}. \]

The solution to these equations are given by the Lorenz integral forms

\[ \hat{\phi}(\hat{x}, \hat{t}) = \frac{\hat{\alpha}}{4\pi} \int \frac{[\hat{\rho}(\hat{x}', \hat{t})]_{\text{ret}} d^3 x'}{R}, \]
\[ \hat{A}(\hat{x}, \hat{t}) = \frac{\hat{\alpha}}{4\pi} \int \frac{[\hat{J}(\hat{x}', \hat{t})]_{\text{ret}} d^3 x'}{R}, \]

where $R = |\hat{x} - \hat{x}'|$, and $[\ ]_{\text{ret}}$ indicates that the argument is evaluated at the retarded time $\hat{t}' = \hat{t} - R$. Here $R$ is unitless, but we do not use the hat notation for convenience. Dropping the time dependence gives the electrostatic and magnetostatic solutions respectively.

Because the quasi-static equations are the same as the static equations but with implicit time dependence, the solutions are the same as the static solutions with implicit time dependence:

\[ \hat{\phi}(\hat{x}, \hat{t}) = \frac{\hat{\alpha}}{4\pi} \int \frac{\hat{\rho}(\hat{x}', \hat{t}) d^3 x'}{R}, \]
\[ \hat{A}(\hat{x}, \hat{t}) = \frac{\hat{\alpha}}{4\pi} \int \frac{\hat{J}(\hat{x}', \hat{t}) d^3 x'}{R}. \]

7 See, for example, (Griffiths, 2005), Sect. 7.4.3; (Jackson, 1975), Sect. 6.4; (Jackson, 1999), Sect. 6.3.
8 See, for example, (Griffiths, 2005), Sect. 9.9.1; (Jackson, 1975), Sec. 6.5; (Jackson, 1999), Sect. 6.5.
The lack of retardation in the potentials explains why the quasi-static limits are sometimes referred to as the $c \to \infty$ limit. The solutions are indeed instantaneous, but for the EQS limit, the use of the Lorenz gauge condition and the full continuity equation indicates that some causal effects remain. The ordering of the EQS limit says that the Lorenz gauge condition and continuity equation may be used for describing slow motions.

The integral solutions to the EMQS equations, Eqs. (52), are evidently

$$
\hat{\phi}(\hat{x}, \hat{t}) = \frac{\hat{\alpha}}{4\pi} \int \frac{\hat{\phi}(\hat{x'}, t)}{\hat{R}} d^3 x',
$$

(58a)

$$
\hat{A}_T(\hat{x}, \hat{t}) = \frac{\hat{\alpha}}{4\pi} \int \frac{\hat{J}_T(\hat{x'}, t)}{\hat{R}} d^3 x'.
$$

(58b)

In this form, the only difference between the EMQS and other quasi-static limits is the use of the transverse current to give a transverse vector potential. Recall from Eq. (51b) and the discussion thereafter, the transverse current includes the time derivative of the scalar potential, and thus has an implicit time dependence. The implications of this will be discussed in the next section. After a bit of algebra (Jackson, 2002; Krause et al., 2007), the transverse current of the last expression can be rewritten and the vector potential solution of the EMQS equation is

$$
\hat{A}_T = \frac{\hat{\alpha}}{4\pi} \int \frac{1}{\hat{R}} \left[ \hat{J}(\hat{x}', t) - \hat{R} \hat{R} \cdot \hat{J}(\hat{x}', t) \right] d^3 x',
$$

(59)

where the unit vector $\hat{R}$ points from the source to the location where $\hat{A}_T$ is calculated. This is the form presented Jackson's book (Jackson, 1975, 1999) that derived the Darwin approximation as a quasi-static approximation of Eq. (56b). In this view of the EMQS equations, the non-Galilean covariance can be seen as a result of taking a specific direction; i.e., the form here is obviously not rotationally symmetric.

The integral solutions are useful for studying simple radiating systems. In these studies, it is useful to classify the different regions as:

- The near (static) zone: $d \ll \bar{L} \ll \lambda \bar{L}/\lambda \ll 1$
- The intermediate (induction) zone: $d \ll \bar{L} \sim \lambda \bar{L}/\lambda \sim 1$ where $d$ is the source dimension and
- The far (radiation) zone: $d \ll \lambda \ll \bar{L} \bar{L}/\lambda \gg 1$.

$\bar{L}$ is the region of interest. The last column expresses orderings in terms of the electrical length (that is $\beta_{osc}$ defined in Eq. (17) with $N_{osc} = 1$). Although the first region is termed the static zone, it is more properly described as the quasi-static zone as there are, necessarily, time-dependent sources. As shown in Jackson, the difference between the near-field part of the solution and the radiation part of the solution is that the near-field solution exponentially decays. Again, exponentially decaying solutions should always indicate a solution that could be described with a first-order in time partial differential equation, and hence, a quasi-static limit.

### VI. GENERALIZED COULOMB AND BIOT-SAVART LAWS IN THE QUASI-STATIC LIMIT

The generalized Coulomb and Biot-Savart Laws are derived using the solutions in the previous section. For full electromagnetics, these are known as Jefimenko’s equations (Jefimenko, 1966), and they are discussed in Griffith’s textbook (Griffiths, 2005) and the third edition of Jackson (Jackson, 1999). The derivation of the generalized Coulomb and Biot-Savart equations is straightforward by substituting the field-potential relationships in Eqs. (47) and (49) into Eqs. (57a) and (57b), and the equivalent EMQS relationships. The following standard identities are useful in the derivation:

$$
\nabla \left( \frac{1}{\hat{R}} \right) = -\frac{\hat{R}}{\hat{R}^2}; \quad \nabla \cdot \frac{\hat{R}}{\hat{R}^2} = \frac{1}{\hat{R}^2}; \quad \nabla \cdot \frac{\hat{R}}{\hat{R}^2} = 4\pi \delta; \quad \nabla \times \frac{\hat{R}}{\hat{R}^n} = 0.
$$

(60)
Here \( \dot{R} = R/R \), and as defined previously, both \( R \) and \( R \) are unitless. The \( \delta \) function is also unitless. These identities and their derivations are found in most standard classical EM textbooks.

For the EQS equations, we obtain

\[
\dot{E}(\hat{x}, \hat{t}) = \frac{\dot{\alpha}}{4\pi} \int \left[ \hat{\rho}(x', t) \frac{\dot{R}}{R^2} - \hat{\nabla} \times \frac{\dot{J}(x', t)}{R} \right] d^3x'.
\]

These have the same form as the Coulomb and Biot-Savart laws; only the implicit time dependency is added to the fields. In the MQS limit, we obtain

\[
\dot{E}(\hat{x}, \hat{t}) = \frac{\alpha}{4\pi} \int \hat{\rho}(x', t) \frac{\dot{R}}{R^2} d^3x',
\]

where the dot denotes a time derivative. The presence of the time derivative of \( J \) in the generalized Coulomb law is responsible for the induction term in Faraday’s law.

To derive the Maxwell equations in the quasi-static limits, the divergence and the curl of the above equations is taken. The algebra is straightforward using Eqs. (60); however, the equations of most interest are those with time dependence (the curl of \( E \) in the MQS limit, and the curl of \( B \) in the EQS limit), so that algebra is shown in more detail. The curl of the MQS limit of the electric field equation, Eq. (62a), is

\[
\hat{\nabla} \times \dot{E} = \frac{\dot{\alpha}}{4\pi} \int \left[ \hat{\rho}(x', t) \hat{\nabla} \times \frac{\dot{R}}{R^2} - \hat{\nabla} \times \frac{\dot{J}(x', t)}{R} \right] d^3x'
\]

\[
= -\frac{\partial}{\partial t} \left( \frac{\dot{\alpha}}{4\pi} \int \hat{J}(x', t) \times \frac{\dot{R}}{R^2} d^3x' \right)
\]

\[
= -\frac{\partial \dot{B}}{\partial t},
\]

giving Faraday’s law in the MQS limit.

We next derive the Maxwell-Ampère law in the EQS limit by taking the curl of the magnetic field in Eq. (61b). Before taking the curl of \( \dot{B} \), it is useful to recall this vector identity:

\[
\nabla \times \nabla \times A = -\nabla^2 A + \nabla \nabla \cdot A.
\]

When we take the curl of \( \dot{B} \) from this generalized Biot-Savart equation, we are effectively taking the curl-curl of the integral solutions for \( A \). However, the integral solution was derived by inverting the Laplacian for \( A \), thus there is a difference. That difference is the last term in that equation, and the divergence of \( A \) will bring in the gauge condition. With this background, the curl of Eq. (61b) is most easily done by starting with Eq. (57b):

\[
\hat{\nabla} \times \dot{B}(\hat{x}, \hat{t}) = \frac{\dot{\alpha}}{4\pi} \int \left[ \hat{\nabla} \times \hat{\nabla} \times \frac{\hat{J}(x', t)}{R} \right] d^3x'
\]

\[
= \frac{\dot{\alpha}}{4\pi} \int \left[ -\hat{\nabla}^2 \frac{\hat{J}}{R} + \hat{\nabla} \hat{\nabla} \cdot \frac{\hat{J}(x', t)}{R} \right] d^3x'
\]

\[
= \frac{\dot{\alpha}}{4\pi} \int \left[ \hat{J} 4\pi \delta(R) \right] d^3x' + \hat{\nabla} \hat{\nabla} \cdot A(\hat{x}, \hat{t})
\]

\[
= \dot{\alpha} \hat{J}(\hat{x}, \hat{t}) - \hat{\nabla} \frac{\partial \phi(\hat{x}, \hat{t})}{\partial t}
\]

\[
= \dot{\alpha} \hat{J}(\hat{x}, \hat{t}) + \frac{\partial \hat{E}(\hat{x}, \hat{t})}{\partial t}.
\]
recovering the full the Maxwell-Ampère law.

The generalized Coulomb and Biot-Savart equations for the EMQS limit are

\[
\hat{E}(\hat{x}, \hat{t}) = \frac{\hat{\alpha}}{4\pi} \int \left[ \hat{\rho}(x', t) \frac{\vec{R}}{R^2} - \frac{\hat{J}_T(x', t)}{R} \right] d^3x'.
\]

\[
\hat{B}(\hat{x}, \hat{t}) = \frac{\hat{\alpha}}{4\pi} \int \left[ \hat{J}_T(x', t) \times \frac{\vec{R}}{R^2} \right] d^3x'.
\]

The recovery of the EMQS limit of the Maxwell equations is straightforward. The induction term is recovered from the vector potential contribution to the electric field, as in the MQS limit. For the longitudinal component of the displacement current, the fact that the current is only transverse implies that the scalar potential contributes, as seen in Eq. (51b). This differs from the EQS limit, where the full displacement current appears from the use of the Lorenz gauge.

**VII. MAXWELL EQUATIONS FOR MACROSCOPIC MEDIA**

In this section, we discuss the quasi-static limits as they apply to the Maxwell equations for macroscopic media with bound electrons. This area is thoroughly studied by Rousseaux and colleagues (Montigny and Rousseaux, 2006; Rapetti and Rousseaux, 2014; Rousseaux, 2008, 2013; Rousseaux et al., 2008) so this discussion does not re-derive all of the results, but summarizes them and places into the context of the current work.

Macroscopic media are materials that have a different field within them because of the response of the material to an external field. For electrically-active media, a polarization field arises, and for magnetically-active materials a magnetization field arises. These fields are then added to the original applied fields to produce the displacement field, \( \vec{D} \), and the auxiliary magnetic field, \( \vec{H} \):

\[
\vec{D} = \frac{\vec{E}}{4\pi\kappa_1} + \lambda P
\]

\[
\vec{H} = \frac{\epsilon^2 k_3}{4\pi\kappa_1} \vec{B} - \lambda' M,
\]

where \( \lambda \) and \( \lambda' \) are either unity or \( 4\pi \) depending on whether the unit system is rationalized or not. Thankfully, all unit systems have the same form for the constitutive relations (also the same for dimensionless forms):

\[
\begin{align*}
\vec{D} &= \epsilon \vec{E} \\
\vec{H} &= \frac{1}{\mu} \vec{B}.
\end{align*}
\]

In taking the Galilean limit of the macroscopic Maxwell equations, it is important to understand their Lorentz transformation properties. As discussed extensively in Rousseaux’s review paper, Ref. (Rousseaux, 2013), the key is the Minkowski hypothesis that the constitutive constants are invariant in any inertial frame:

\[
\begin{align*}
\vec{D}' &= \epsilon \vec{E}' \\
\vec{H}' &= \frac{1}{\mu} \vec{B}'.
\end{align*}
\]

The Maxwell equations with macroscopic media are written as

\[
\begin{align*}
\hat{\nabla} \cdot \hat{\vec{D}} &= \hat{\rho} \\
\hat{\nabla} \cdot \hat{\vec{B}} &= 0 \\
\hat{\nabla} \times \hat{\vec{E}} &= -\frac{\partial \hat{\vec{B}}}{\partial \hat{t}} \\
\hat{\nabla} \times \hat{\vec{H}} &= \frac{\partial \hat{\vec{D}}}{\partial \hat{t}} + \hat{\vec{J}}.
\end{align*}
\]
These will have the same form whether dimensional or dimensionless. In generalizing the ordered Maxwell equations, the dimensions of $D$ are $\bar{D} = \bar{\epsilon} \bar{E}$, and of $H$ are $\bar{H} = \bar{B}/\bar{\mu}$.

The other constitutive relationship that will be used is $J = \sigma E = E/\eta$. Although this form is not Galilean covariant, the generalizations add an algebraic complexity that is unimportant. The dimensional units are straightforward. To match onto circuit theory, resistivity will be used instead of conductivity.

Using these relationships, the generalization of the ordered Maxwell equations in Eqs. (28) is

\[
\hat{\nabla} \cdot \hat{D} = \hat{\rho} \\
\hat{\nabla} \cdot \hat{B} = 0 \\
\hat{\nabla} \times \hat{E} = -\left(\beta \frac{\hat{B}}{E}\right) \frac{\partial \hat{B}}{\partial \hat{t}} + \left(\frac{\hat{E} c^2}{B c_m^2} \hat{D} \alpha_\eta\right) \hat{E},
\]

where $c_m^2 = 1/(\mu \epsilon)$ is the speed of light in the media. The ratio $c/c_m$ does not significantly change the results of ordering the vacuum Maxwell equations. However, the last term does cause a significant change. The factor of $\hat{\alpha}_\eta$ can be expressed in multiple ways (shown here in MKSA units):

\[
\hat{\alpha}_\eta = \frac{c \mu_0 \bar{L}}{\eta} = \frac{c \bar{L}}{\bar{D}_m} = \frac{\tau_M}{\tau_{EM}} = \frac{\bar{L}}{c \epsilon_0 \eta} \frac{\tau_{EM}}{\tau_E},
\]

where $D_m = \eta/\mu_0$ is the resistive diffusivity. As shown below, the ratio $\tau_{EM}/\tau_M$ is related to the $L/R$ time of LR circuits, while the ratio $\tau_E/\tau_{EM}$ is related to the $RC$ time of RC circuits. The constitutive relationship for $J$, however, means that the appropriate asymptotic regime depends on the value of the parameter, $\eta$. With values ranging from zero to infinity, this will put the range of parameters outside of the orderings used in the vacuum equations. The regions outside of the range of validity are the static limits, and the full EM equations. This area of validity for the regimes will be discussed in Section VII.B but first the relationship of the Maxwell equations and circuit theory is briefly reviewed to provide greater intuition.

### A. Circuit equations with simple elements

The relationship between the Maxwell equations and circuit theory is well-understood, with Carson 1927 being an early reference [Carson 1927]. In Carson, the derivation used the retarded potentials, Eqs. (56), to show that the derivation of Kirchoff’s “laws” requires assuming that the electrical length is small, and that radiation effects are negligible (see Section V). The derivation here is more heuristic and follows directly from the EQS and MQS limits. To derive these circuit equations, it is assumed that the circuit elements are small relative to the wavelength, time-dependence is important, and that the full Maxwell equations are not needed. In the discussion after Eq. (17), these assumptions were discussed more quantitatively for normal U.S. household circuits to show that these assumptions are easily satisfied.

**RC Circuits** First consider a direct current (DC) circuit with a single capacitor and resistor connected in a loop. Because of the charge on the capacitor and the relatively low magnetic field energy of the resistor, $c \hat{\rho} \gg \hat{J}$, we are in the EQS regime. The time dependence is given by the Maxwell-Ampère equation for the...
EQS equation. We integrate that equation across the cross sectional area, to calculate the effective total current (from \( I = \int J \cdot dS \)). Doing so is equivalent to Kirchoff’s “Current Law”, which is a reduced form of charge conservation:

\[
\int \nabla \times H \cdot dS = \int \frac{\partial D}{\partial t} \cdot dS + \int \frac{1}{\eta} E \cdot dS, \\
\int H \cdot dl = 0 \approx \frac{\epsilon A_c}{d} \frac{dV}{dt} + \frac{A_R}{\eta l} V \\
\approx C \frac{dV}{dt} + \frac{V}{R},
\]

(72)

where \( V \) is the voltage, \( R \) is the resistance of the resistor, and \( C \) is the capacitance of the capacitor. We approximate \( E \approx V/d \) for a capacitor, where \( d \) is the gap of the capacitor, and \( E \approx V/l \) for a resistor, where \( l \) is the length of the resistor. If there is a source, this equation becomes

\[
\tau_{RC} \frac{dV}{dt} + V = V_{source},
\]

(73)

where \( \tau_{RC} = RC \). This is a first-order differential equation with exponential solutions, as expected from the first-order EQS equations. As \( \tau_{RC} \to 0 \), the capacitor charges (or discharges) more quickly. Eventually this speed will violate the quasi-static assumption made in deriving this equation. At the other limit, the limit of \( \tau_{RC} \to \infty \) means that eventually the discharge is so slow that the electrostatic limit can be used.

The heuristic nature of the derivation is evident: the inductance of the loop is ignored; the resistance of the capacitor is ignored; and the complicated geometry of the circuit elements in the integrals is also ignored. For alternating current (AC) circuits, the relationship to \( \beta_{osc} \), Eq. (17), is set by the frequency of the current.

As discussed in the Introduction, the wavelength of 60 Hz is \( 5 \times 10^6 \text{m} \), and the size of most circuit elements is on the order of a centimeters, resulting in a large electrical length. In this case, the relevant \( T \) used in \( N_{osc} \) would be the RC time scale modified for the impedance of the capacitor, leading to a very small \( \beta_{osc} \).

Because in the EQS regime \( E = -\nabla \phi \), the electromotive force is equivalent to the potential drop across a segment:

\[
\mathcal{E} \equiv \int_{x_0}^{x_1} E \cdot dl \overset{\text{EQS}}{=} \phi(x_1) - \phi(x_0).
\]

(74)

This is not true in the MQS regime.

LR Circuits Here we consider a simple loop with an inductor, a resistor, and a voltage source. Because inductors store magnetic fields, \( \vec{B} \gg \vec{E} \) and we are in the MQS regime. The time dependence term is in Faraday’s equation. Considering an integral over the loop, we have

\[
\int \nabla \times E \cdot dS = -\int \frac{\partial B}{\partial t} \cdot dS \\
\int \eta J \cdot dl = -\frac{\partial}{\partial t} \int \vec{A} \cdot dl \\
IR = -\frac{\alpha}{r \pi} \frac{\partial}{\partial t} \int dl \cdot \int \frac{1}{R} dl' \\
IR = -L \frac{\partial I}{\partial t},
\]

(75)

where an approximate form for the integral solution for \( A \) in the MQS regime was used, Eq. (57b), and \( L \) uses the definition of inductance. With a voltage source, this equation becomes

\[
\tau_{RL} \frac{dI}{dt} + I = \frac{1}{R} V_{source},
\]

(76)

where \( \tau_{RL} = L/R \). Here, we are ignoring both the capacitive effects of the inductor geometry, and the resistance of the inductor itself. In the MQS approximation, the \( \mathcal{E} \) is not the same as the voltage drop.
The Rousseaux-Rapetti diagram drawn in terms of $\beta_{osc}$ shows the regions of validity for the various approximations heuristically. The x-axis is in terms of the small parameter. The upper vertical axis is proportional to $\tau^{-1}_{\text{RC}}$, and the lower vertical axis is proportional to $\tau^{-1}_{\text{LR}}$. Because the calculation of $E$ includes the vector potential. Nonetheless, the form in the second line of Eq. (75) shows that the derivation is equivalent to calculating the $E$ across each element and effectively using Kirchoff's voltage law. The static version of this equation corresponds to $\tau_{RL} = 0$, which requires either the inductance to go to zero or the resistance to go to infinity.

**RLC Circuits** There are two ways to derive the equations for a simple RLC circuit. As alluded to above for a simple circuit, each element of the circuit should be considered piecewise. In this case, we use the fact that the solutions of the individual EQS and MQS equations can be additive, as discussed by LBLL (Le Bellac and Lévy-Leblond, 1973). The appropriate $E$ can be calculated for each element (including the capacitor) and sum the segments. The result is

$$\frac{d^2 I}{dt^2} + \frac{1}{\tau_{RL}} \frac{dI}{dt} + \frac{1}{LC} I = \frac{1}{\tau_{RL} R} \frac{dV_{\text{source}}}{dt}. \quad (77)$$

Another derivation method uses the EMQS approximation. The key question is whether the $E$ contribution from the capacitor, which comes from the displacement current in Eq. (72), is primarily longitudinal. Because the normal derivation capacitive effects uses Gauss’s law, it is seen that it is. For more complicated geometries where the separation of capacitive, inductive, and resistive effects are not possible, the EMQS approximation provides a more useful approximation than the additive properties of EQS and MQS. This is discussed in more detail next.

**B. Static and Quasi-static regimes of validity**

There are many situations where simple circuit assumptions are violated; the most important of these occur where the size of the devices are such that each individual device has capacitive, resistive, and inductive
contributions. Figuring out the appropriate approximation to use can be complicated. As seen above, geometry is an important consideration; however, Refs. (Haus and Melcher 1989, Rapetti and Rousseaux 2011, 2014) provide useful heuristics based on the ordering parameters above:

\[
0 \leq \tilde{T} \leq \tau_{EM} \quad \text{Electromagnetic regime} \\
\tau_{EM} \ll \tilde{T} \ll \tau_{E}, \tau_{M} \quad \text{Quasi-static regime} \\
\tau_{E}, \tau_{M} \ll \tilde{T} \quad \text{Static regime.}
\] (78)

The relationship \( \tau_{EM} \ll \tilde{T} \) is equivalent to the \( \beta \ll 1 \) ordering in this paper. The relationship to the static regimes has not been previously discussed here. This can be cast into a form that is closer to this paper by relating the key parameter \( \tau_{EM} \) to \( \beta_{osc} \), Eq. (17), used in this paper:

\[
\tau_{EM} = \tilde{T} \beta_{osc}.
\] (79)

Using these relationships, a diagram heuristically defining the regimes may be drawn. This follows those in Refs. (Rapetti and Rousseaux 2011, 2014) where the key insight was to take the log of the axis such that order unity defines the origin of the axis, and factors less than unity become the negative regions. This allows the \( y = 0 \) line to correspond to the \( \eta = \infty \) for the ES/EQS regime and \( \eta = 0 \) for the MS/MQS regimes. The discontinuity of the regimes is a consequence of the separation of the time-like and space-like regions of spacetime. Graphically, the parameter space is divided into six different regimes, as seen in Fig. 3. This provides an intuitive picture of when each regime is valid.

**VIII. CONCLUSIONS**

In this paper, three quasi-static limits of the Maxwell equations – the electro-quasi-static (EQS), the magneto-quasi-static (MQS), and the electromagnetic-quasi-static (EMQS) – are derived using an ordering analysis with the small parameter given by the ratio of the characteristic length scale to \( c \) times the time scale. The formal perturbation technique is more precise and illuminating than phrasing the quasi-static as taking the \( c \to \infty \) limit. The three limits are expressed in four different forms of the Maxwell equations: the local differential equations in terms of fields and potentials, and the global integral solutions in terms of fields and potentials. Summaries of the results are presented in Table III and Appendix B. The derivation for EMQS, or Darwin’s approximation, in this paper uses the Maxwell equations, and the integral solutions use to show equivalence to the derivation in Jackson’s book that cast it as a truncation of the retarded potentials. The EMQS limit is shown to not be Galilean covariant. This approximation is obviously useful given its many applications, but care must be taken due to the lack of Galilean relativity.

This paper reviews the unit systems and issues that guide the choices of units. Whether a system is unrationlized or rationalized is a choice of whether factors of \( 4\pi \) appear in the differential equations (left side of Fig. 1) or the integral equations (right side of Fig. 1). Unit systems that have the same units for \( \vec{E} \) and \( \vec{B} \) \((k_3 = 1/c)\) are generally useful in regimes where Lorentz covariance is important, while having different units \((k_3 = 1)\) provides more useful units for the Galilean, or engineering, limit. There is only one parameter that truly controls the unit system as given by the Buckingham \( \Pi \) theorem. The MKSA unit system makes it appear as if two constants, with mass in their base unit definition, are needed. This, as well as using current as a base unit rather than charge, seems designed to create confusion for physicists. This system feels natural only in the engineering limit where everyday usage involves macroscopic media, the quasi-static limit, and force calculations involving Newton’s law.

In the original LBLP paper, the ordering parameter \( \beta \) was viewed strictly as the ratio of an inertial frame velocity to \( c \). This paper broadens that view with more general definitions of \( \beta \). With the introduction of \( \beta_{osc} \), this ordering parameter is explicitly related to the more commonly used electrical length, \( L/\lambda \) (as used in

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11 We term this diagram the Rousseaux-Rapetti diagram because the first appearance of the figure is in an unpublished Rousseaux document, and published versions are in two Rapetti-Rousseaux papers. Rapetti and Rousseaux define the \( x \)-axis as \( \log(T/\tau_{EM}) \). Our choice of \( \log(\beta_{osc}) \) as the \( x \)-axis means we are flipped relative to the published Rousseaux-Rapetti diagrams. In addition to expressing the diagram in terms of \( \beta_{osc} \), our diagram shifts things away from the \( y \)-axis to show that \( \beta \ll 1 \) is needed for the ordering.

12 The dimensionless equations presented here are rationalized because the author agrees that it is more rational.
The larger Courant-Friedrichs-Levy (CFL) limit. The use of a formal perturbation theory enables estimates smaller. If solving explicitly with time domain methods, then time steps can be made larger because of the analytic elimination of a faster time scale: the largest eigenvalue becomes better conditioned due to the analytic elimination of a faster time scale. If solving

| Regime               | Electro-QuasiStatic (EQS) | Magneto-QuasiStatic (MQS) | EM-QuasiStatic (EMQS) |
|----------------------|---------------------------|---------------------------|-----------------------|
| Ordering             | \( \mathcal{O} \left( \frac{E}{\beta} \right) \sim \frac{1}{\beta} \) | \( \mathcal{O} \left( \frac{E}{\beta} \right) \sim \beta \) | \( \mathcal{O} \left( \frac{E}{\beta} \right) \sim \beta \), \( \mathcal{O} \left( \frac{E}{\beta} \right) \sim \frac{1}{\beta} \) |
| Maxwell Equs.        | \( \nabla \cdot E = \hat{\alpha} \rho \) | \( \nabla \cdot E = \hat{\alpha} \rho \) | \( \nabla \cdot E = \hat{\alpha} \rho \) |
|                      | \( \nabla \cdot B = 0 \) | \( \nabla \cdot B = 0 \) | \( \nabla \cdot B = 0 \) |
|                      | \( \nabla \times E = 0 \) | \( \nabla \times E = -\frac{\partial B}{\partial t} \) | \( \nabla \times E = -\frac{\partial B}{\partial t} \) |
|                      | \( \nabla \times B = \frac{\partial E}{\partial t} + \hat{\alpha} J \) | \( \nabla \times B = \hat{\alpha} J \) | \( \nabla \times B = \frac{\partial E}{\partial t} + \hat{\alpha} J \) |
| Continuity Eqn.      | \( \frac{\partial \rho}{\partial t} + \nabla \cdot J = 0 \) | \( \nabla \cdot J = 0 \) | \( \frac{\partial \rho}{\partial t} + \nabla \cdot J = 0 \) |
| Galilean Transform   | \( E' = E \) | \( E' = E + V \times B \) | Not Applicable |
|                      | \( B' = B - V \times E \) | \( B' = B \) | Applicable |
|                      | \( \rho' = \rho - V \cdot J \) | \( \rho' = \rho - V \cdot J \) | |
|                      | \( J' = J - V \rho \) | \( J' = J \) | |
| Fields from Potential ordering | \( \mathcal{O} \left( \frac{\hat{\alpha}}{\beta} \right) \sim \frac{1}{\beta} \) | \( \mathcal{O} \left( \frac{\hat{\alpha}}{\beta} \right) \sim \beta \) | \( \mathcal{O} \left( \frac{\hat{\alpha}}{\beta} \right) \sim \beta \), \( \mathcal{O} \left( \frac{\hat{\alpha}}{\beta} \right) \sim \frac{1}{\beta} \) |
| Maxwell Equs. (Potential Form) | \( \nabla^{2} \phi = -\hat{\alpha} \rho \) | \( \nabla^{2} \phi = -\hat{\alpha} \rho \) | \( \nabla^{2} \phi = -\hat{\alpha} \rho \) |
|                      | \( \nabla^{2} A = -\hat{\alpha} J \) | \( \nabla^{2} A = -\hat{\alpha} J \) | \( \nabla^{2} A_{T} = -\hat{\alpha} J_{T} \) |
| Gauge Condition      | \( \frac{\partial \phi}{\partial t} + \nabla \cdot A = 0 \) | \( \nabla \cdot A = 0 \) | \( \nabla \cdot A = 0 \) |
| Potential Solutions  | \( \phi = \frac{\hat{\alpha}}{4\pi} \int \frac{\rho(x', t)}{R} d^{3}x' \) | \( \phi = \frac{\hat{\alpha}}{4\pi} \int \frac{\rho(x', t)}{R} d^{3}x' \) | \( \phi = \frac{\hat{\alpha}}{4\pi} \int \frac{\rho(x', t)}{R} d^{3}x' \) |
| (Integral form)      | \( A = \frac{\hat{\alpha}}{4\pi} \int \frac{J(x', t)}{R} d^{3}x' \) | \( A = \frac{\hat{\alpha}}{4\pi} \int \frac{J(x', t)}{R} d^{3}x' \) | \( A_{T} = \frac{\hat{\alpha}}{4\pi} \int \frac{J_{T}(x', t)}{R} d^{3}x' \) |
| Generalized          | \( E = \frac{\hat{\alpha}}{4\pi} \int \rho(x', t) \frac{R}{R^{2}} d^{3}x' \) | \( E = \frac{\hat{\alpha}}{4\pi} \int \rho(x', t) \frac{R}{R^{2}} d^{3}x' \) | \( E = \frac{\hat{\alpha}}{4\pi} \int \rho(x', t) \frac{R}{R^{2}} d^{3}x' \) |
| Coulomb and Biot-Savart Laws | \( \nabla \times J = \frac{\hat{\alpha}}{4\pi} \int J \times \frac{R}{R^{2}} d^{3}x' \) | \( B = \frac{\hat{\alpha}}{4\pi} \int J \times \frac{R}{R^{2}} d^{3}x' \) | \( B = \frac{\hat{\alpha}}{4\pi} \int J_{T} \times \frac{R}{R^{2}} d^{3}x' \) |
| Equiv. Circuit Eqn.  | \( \tau_{RC} \frac{dV}{dt} + V = 0 \) | \( \tau_{RL} \frac{dI}{dt} + I = 0 \) | \( \frac{d^{2}I}{dt^{2}} + \frac{1}{\tau_{L}} \frac{dI}{dt} + \frac{1}{\tau_{C}} I = 0 \) |

Table III: Summary of the results of this paper enables a comparison of the three quasi-static regimes of the Maxwell equations in all of their forms. In this table, the equations are written in dimensionless form, but the hats are not used, and in referring to the units in the ordering, \( k_{3} = 1/c \) is used to place \( E \) and \( B \) in the same units for simplicity.

Carson (Carson, 1927) for example). For the case of rotating media – the Wilson-Wilson experiment (Wilson et al. 1913) or the Barnett effect (Barnett, 1915) – the slow rotation speed serves as the expansion parameter. Finally, in deriving the Lagrangian of a single particle with a velocity small relative to the speed of light, this perturbation expansion applies regardless of whether the particle is moving inertially.

A major advantage of the quasi-static limits is the elimination of light waves: yielding significant analytical and computational simplicity. In the EQS and MQS limits, this is the result of changing the order of the equations from second-order to first order. Decaying EM fields suggest the applicability of the quasi-static equations as seen in such examples as non-propagating waveguide modes, near-field solutions from radiating source, and the \( RC \) and \( LR \) circuit equations. When studying a regime in which a quasi-static limit is valid, the ability to eliminate the fast light-wave time scale yields valuable numerical benefits. If solving for the EM fields with frequency-domain or implicit time-domain methods, then the resultant matrices will be better conditioned due to the analytic elimination of a faster time scale: the largest eigenvalue becomes smaller. If solving explicitly with time domain methods, then time steps can be made larger because of the larger Courant-Friedrichs-Levy (CFL) limit. The use of a formal perturbation theory enables estimates.
of the errors associated with using this approximation. These estimates can then be included within the numerical solution as an a posteriori check on validity. Although this paper provides a framework for such developments, further work is needed.

Studying quasi-static limits is also beneficial for pedagogical reasons. Here, the generalized Coulomb and Biot-Savart laws for the EQS limit are derived for the first time. This derivation illuminates the close relationship between the Lorenz gauge condition and the displacement current, as well as the induction term and the electric field contributions from the time derivative of the magnetic vector potential. As another simple example, consider the role of displacement current in a parallel plate capacitor with a time-dependent current. In the original LBLL paper, they state that the EQS equations cannot model capacitors. Their logic is that the induction is needed to cause a time-dependent voltage (i.e., current) on the opposite capacitor plate. However, this can be viewed as an EQS problem where the key is developing boundary conditions of the voltage, or electrostatic potential, consistent with the entire circuit (Verboncoeur et al., 1993). In other words, the RC circuit can be calculated from the EQS equations and the induction term is not needed, as discussed in Section VII.A.

Important experiments (Rousseaux, 2013) such as the Wilson-Wilson experiment (Wilson and Espenschied, 1930) or those describing the Barnett effect (Barnett, 1915) are in the MQS regime. Applications of the EQS regime are fewer, but one example is electrohydrodynamics (Castellanos, 2014). Another application is in the area of plasma physics where the EQS limit is used for modeling the plasma sheath region (Turner et al., 2013; Verboncoeur et al., 1993). Even though it called the electrostatic model in that field, the oscillating voltage boundary conditions mean that it is electro-quasi-static. The results here point towards a method for self-consistently including the magnetic field effects in the particle motion in plasma physics simulations. Although not discussed in this paper, but discussed by LBLL, the Lorentz force equation in the EQS limit contains only the electric field to leading order (absent a zeroth-order, static, externally applied magnetic field which is consistent with this theory, as discussed in Section III.B). LBLL discuss ways of taking advantage of this theory to combine solutions. This will be the subject of future work.

The unitless version of the inhomogeneous vacuum Maxwell equations in Lorentz-covariant form are

\[
\partial_{\mu} \hat{F}^{\mu\nu} = \hat{\alpha} \hat{J}^\mu.
\]

By providing the ordering of the vacuum Maxwell equations, we show that the three quasi-static limits of the fields depend on the properties of the source equation, as is perhaps unsurprising given this form of the equation. But every day experience with the Maxwell’s equations in the quasi-static limit, such as circuit equations or lab experiments, typically involve macroscopic media. When studying macroscopic media, the distinct nature of the electric and magnetic fields in the quasi-static limit is even more distinct — the electric field in a capacitor appears unrelated to the magnetic field in an inductor. This paper follows the work of Rapetti and Rousseaux, and Melcher and Haus, in using circuit equations to aid in providing intuition on when the quasi-static equations are useful. Using these equations allows the definition of 6 regions: EM, EMQS, EQS, MQS, ES, and MS. For more detailed understanding of how the quasi-static equations leads to quantitative comparisons with experiments, the review paper of Rousseaux (Rousseaux, 2013) serves as an excellent starting point. With the exception of the textbooks of Melcher and colleagues (Haus and Melcher, 1989; Woodson and Melcher, 1968), which are engineering textbooks, no textbook that we are aware of discusses the EQS regime, likely because of its relatively limited practical use due to the Lorentz force limitations discussed above. However, a complete understanding of electrodynamics benefits from a complete view of all three quasi-static limits.

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Appendix A: The Helmholtz Theorem

In this appendix, we review the Helmholtz decomposition of a vector and how it is used in the main text of the paper. The basic material here essentially the same as Appendix B of Griffiths’ book (Griffiths 2005), but in a proof that we find simpler. The Helmholtz Theorem says that any vector field $\mathbf{F}(x)$ can be decomposed into longitudinal (irrotational) and transverse (solenoidal) components as

$$
\mathbf{F}(x) = \mathbf{F}_L(x) + \mathbf{F}_T(x) = -\nabla U(x) + \nabla \times \mathbf{V}(x),
$$

and where the fields $U$ and $V$ are given by

$$
U(x) = \frac{1}{4\pi} \int \frac{\nabla' \cdot \mathbf{F}(x')}{R} d^3x' - \frac{1}{4\pi} \oint \frac{\mathbf{F}(x')}{R} \cdot dS',
$$

$$
V(x) = \frac{1}{4\pi} \int \frac{\nabla' \times \mathbf{F}(x')}{R} d^3x' + \frac{1}{4\pi} \oint \frac{\mathbf{F}(x')}{R} \times dS',
$$

where $R = |x - x'|$.

We begin the proof by expressing $\mathbf{F}$ as a global integral using the Green’s function kernel of $1/R$:

$$
\mathbf{F}(x) = \int \mathbf{F}(x') \delta(x - x') d^3x',
$$

$$
= -\frac{1}{4\pi} \int \frac{\mathbf{F}(x') \nabla^2 \frac{1}{R}}{R} d^3x' = -\nabla^2 \left( \frac{1}{4\pi} \int \frac{\mathbf{F}(x')}{R} d^3x' \right),
$$

where we have used relationships in Eq. (60) to convert the delta function to the Laplacian operator. Using Eq. (63), we can then write:

$$
\mathbf{F}(x) = -\nabla \left( \nabla' \cdot \frac{1}{4\pi} \int \frac{\mathbf{F}(x')}{R} d^3x' \right) + \nabla \times \left( \nabla' \times \frac{1}{4\pi} \int \frac{\mathbf{F}(x')}{R} d^3x' \right).
$$

This gives a form for $U$ and $V$, but not yet the desired form. We need to move the divergence and curl operators to inside of the integrals and have them operate over the primed variables rather than the unprimed variables. For the divergence operator, we convert as follows:

$$
\nabla \cdot \frac{\mathbf{F}(x')}{R} = \mathbf{F}(x') \cdot \nabla \frac{1}{R} = -\mathbf{F}(x') \cdot \nabla' \frac{1}{R}
$$

$$
= -\nabla' \cdot \frac{\mathbf{F}(x')}{R} + \frac{1}{R} \nabla' \cdot \mathbf{F}(x'),
$$

where we have used $\nabla \cdot \mathbf{F}(x') = 0$ in the first step. This result with Gauss’s theorem gives

$$
\nabla \cdot \left( \int \frac{\mathbf{F}(x')}{R} d^3x' \right) = \int \frac{\nabla' \cdot \mathbf{F}(x')}{R} d^3x' - \oint \frac{\mathbf{F}(x')}{R} \cdot dS'.
$$

Similarly for the term with the curl:

$$
\nabla \times \frac{\mathbf{F}(x')}{R} = -\mathbf{F}(x') \times \nabla \frac{1}{R} = \mathbf{F}(x') \times \nabla' \frac{1}{R}
$$

$$
= -\nabla' \times \frac{\mathbf{F}(x')}{R} + \frac{1}{R} \nabla' \times \mathbf{F}(x'),
$$

13 See also [Jackson 1975], Sect. 6.5; [Jackson 1999], Sect. 6.3
and then converting the volume integral to a surface integral:

\[
\nabla \times \left( \int \frac{F(x')}{R} d^3 x' \right) = \int \frac{\nabla' \times F(x')}{R} d^3 x' + \oint \frac{F(x')}{R} \times dS'.
\]

Using these results, we can write Eq. (A4) as

\[
F(x) = -\nabla \left[ \frac{1}{4\pi} \int \frac{\nabla' \cdot F(x')}{R} d^3 x' - \frac{1}{4\pi} \oint \frac{F(x')}{R} \cdot dS' \right]
+ \nabla \times \left[ \frac{1}{4\pi} \int \frac{\nabla' \times F(x')}{R} d^3 x' + \frac{1}{4\pi} \oint \frac{F(x')}{R} \times dS' \right].
\]

This effectively proves the theorem. Assuming that the surface terms vanish, this allows us to write the longitudinal and transverse components in the more commonly written form

\[
F_L(x) = -\nabla U = -\nabla \left[ \frac{1}{4\pi} \int \frac{\nabla' \cdot F(x')}{R} d^3 x' \right]
\]

(A10a)

\[
F_T(x) = \nabla \times V = \nabla \times \left[ \frac{1}{4\pi} \int \frac{\nabla' \times F(x')}{R} d^3 x' \right].
\]

(A10b)

For the electrostatic, EQS, and MQS limits, these expression to calculate \( \phi = U \) based on \( F = E \) yield

\[
\phi = \frac{1}{4\pi} \int \frac{1}{R} \nabla' \cdot F(x') d^3 x' = \frac{\hat{\alpha}}{4\pi} \int \frac{1}{R} \rho(x') d^3 x',
\]

(A11)

which is the same expressions as Eq. (56a). For the magnetostatic and MQS limit, calculating \( A = V \) based on \( F = B \) yields

\[
A = \frac{1}{4\pi} \int \frac{1}{R} \nabla' \times B(x') d^3 x' = \frac{\hat{\alpha}}{4\pi} \int \frac{1}{R} J(x') d^3 x',
\]

(A12)

which is the same expression as Eq. (56b).

For the EMQS equation, the last term in Eq. (51b), can be written as

\[
\nabla \cdot \frac{\partial \phi(\hat{x}, \hat{t})}{\partial \hat{t}} = -\frac{\hat{\alpha}}{4\pi} \nabla \cdot \left( \frac{\partial \hat{\rho}}{\partial \hat{t}} \frac{1}{R} \hat{\delta} \right) \int d^3 x'
= -\frac{\hat{\alpha}}{4\pi} \int \frac{\partial \nabla' \cdot J}{\partial \hat{t}} \frac{1}{R} \hat{\delta} d^3 x' \right.
\]

(A13)

which is the form needed to prove that the EMQS equation uses the transverse current.

The astute reader may notice that we did include the EQS limit in the set of equations that Eq. (A12) applies. That does not work because the curl of \( B \) will bring in the time derivative of \( E \). By taking the divergence of \( A \) in that case, one can show that it is zero using the continuity equation, similar to the derivation in Eq. (A13). Hence, this method of calculating \( A \) only gives the transverse current. As shown in the text, the inversion of the Laplacian as given by Eq. (56b) gives both the transverse and longitudinal components of \( A \) required to recover the full set of EQS equations as discussed in Section VI.
Appendix B: Figures summarizing results

The textbook of Griffiths (Griffiths, 2005) uses “triangle diagrams” to show the relationship between the sources, the fields, and the potentials for electrostatics and magnetostatics. As an alternative, these “electromagnetic quad diagrams” as a more useful pedagogical tool. These are shown for all six regions discussed in the paper: the two static limits, the three quasi-static limits, and the full Maxwell equations.
Integral equations

Continuity: \( \frac{\partial \phi}{\partial t} + \nabla \cdot J = 0 \)

Maxwell Equations - Potential form
\( \nabla^2 \phi = \frac{\partial \rho}{\partial t} \)

Gauge: \( \frac{\partial \phi}{\partial t} + \nabla \cdot A = 0 \)

\( E, B \)

Continuity: \( \nabla \cdot E = 0 \)

Maxwell Equations - Potential form
\( \nabla \times E = \frac{\partial B}{\partial t} \)

Gauge: \( \nabla \times B = 0 \)

\( E, B \)

Differential equations

Continuity: \( \nabla \cdot D = \rho \)

Maxwell Equations - Potential form
\( \nabla \times B = \mu_0 J \)

Gauge: \( \nabla \times E = 0 \)

\( E, B \)

Integral Solutions (aka Retarded Potentials)
\( \phi = \frac{1}{4\pi} \int \frac{\rho(r')}{|\mathbf{r} - \mathbf{r}'|} d^3 r' \)

Gauge: \( \nabla \cdot A = 0 \)

\( E, B \)

Generalized Coulomb & Biot-Savart Equations
\( \mathbf{E} = \frac{1}{4\pi} \int \frac{\mathbf{J}(r')}{|\mathbf{r} - \mathbf{r}'|} d^3 r' \)

Gauge: \( \nabla \times B = 0 \)

\( E, B \)

Magneto-Statics (MS)

Local Differential equations
\( \nabla \times \mathbf{B} = 0 \)

Continuity: \( \nabla \cdot \mathbf{D} = \rho \)

Maxwell Equations - Potential form
\( \nabla \times \mathbf{E} = \frac{\partial \mathbf{B}}{\partial t} \)

Gauge: \( \nabla \times \mathbf{H} = 0 \)

\( E, B \)

Electro-Statics (ES)

Local Differential equations
\( \nabla \cdot \mathbf{D} = \rho \)

Continuity: \( \nabla \cdot \mathbf{J} = 0 \)

Maxwell Equations - Potential form
\( \nabla \times \mathbf{E} = \mathbf{H} \)

Gauge: \( \nabla \times \mathbf{E} = 0 \)

\( E, B \)

Electro-Quasistatics (EQS)

Local Differential equations
\( \nabla \cdot \mathbf{A} = 0 \)

Continuity: \( \nabla \cdot \mathbf{E} = \rho \)

Maxwell Equations - Potential form
\( \nabla \times \mathbf{B} = \mu_0 \mathbf{J} \)

Gauge: \( \nabla \times \mathbf{B} = 0 \)

\( E, B \)

Magneto-Quasistatics (MQS)

Local Differential equations
\( \nabla \cdot \mathbf{B} = 0 \)

Continuity: \( \nabla \cdot \mathbf{D} = \rho \)

Maxwell Equations - Potential form
\( \nabla \times \mathbf{E} = \frac{\partial \mathbf{B}}{\partial t} \)

Gauge: \( \nabla \times \mathbf{E} = 0 \)

\( E, B \)