De Finetti’s theorem: rate of convergence in Kolmogorov distance

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Abstract

This paper provides a quantitative version of de Finetti law of large numbers. Given an infinite sequence \( \{X_n\}_{n \geq 1} \) of exchangeable Bernoulli variables, it is well-known that \( \frac{1}{n} \sum_{i=1}^{n} X_i \xrightarrow{a.s.} Y \), for a suitable random variable \( Y \) taking values in \([0, 1]\). Here, we consider the rate of convergence in law of \( \frac{1}{n} \sum_{i=1}^{n} X_i \) towards \( Y \), with respect to the Kolmogorov distance. After showing that any rate of the type of \( 1/n^\alpha \) can be obtained for any \( \alpha \in (0, 1] \), we find a sufficient condition on the probability distribution of \( Y \) for the achievement of the optimal rate of convergence, that is \( 1/n \). Our main result improve on existing literature: in particular, with respect to [20], we study a stronger metric while, with respect to [21], we weaken the regularity hypothesis on the probability distribution of \( Y \).

1 Introduction

In this paper, we provide a quantitative version of the law of large numbers (LLN, from now on) for exchangeable random variables (r.v.’s). For simplicity, we confine ourselves to considering an infinite sequence \( \{X_n\}_{n \geq 1} \) of Bernoulli variables defined on \((\Omega, \mathcal{F}, P)\), satisfying the exchangeability condition, namely

\[
P[X_1 = x_1, \ldots, X_n = x_n] = P[X_1 = x_{\sigma_n(1)}, \ldots, X_n = x_{\sigma_n(n)}]
\]

for all \( n \in \mathbb{N}, (x_1, \ldots, x_n) \in \{0, 1\}^n \) and permutation \( \sigma_n \) of \( \{1, \ldots, n\} \). For this kind of variables, de Finetti [5] proved that \( \frac{1}{n} \sum_{i=1}^{n} X_i \xrightarrow{a.s.} Y \) for a suitable r.v. \( Y : \Omega \to [0, 1] \)

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satisfying $P[X_1 = x_1, \ldots, X_n = x_n \mid Y] = Y^{s_n}(1 - Y)^{n-s_n}$, where $s_n := \sum_{i=1}^{n} x_i$. This identity yields the so-called de Finetti representation \cite{4}, which reads

$$P[X_1 = x_1, \ldots, X_n = x_n] = \int_{0}^{1} \theta^{s_n}(1 - \theta)^{n-s_n} \mu(d\theta)$$

(1)

for all $n \in \mathbb{N}$ and $(x_1, \ldots, x_n) \in \{0, 1\}^n$, where $\mu$—called de Finetti (or prior) measure—stands for the probability distribution (p.d.) of $Y$. See \cite{1} for a comprehensive treatment of exchangeable r.v.’s. The above LLN entails that the p.d. of $\frac{1}{n} \sum_{i=1}^{n} X_i$, say $\mu_n$, converges weakly to $\mu$ ($\mu_n \Rightarrow \mu$, in symbols), meaning that $\lim_{n \to \infty} \int_{0}^{1} \psi(\theta) \mu_n(d\theta) = \int_{0}^{1} \psi(\theta) \mu(d\theta)$ is valid for all $\psi : [0, 1] \to \mathbb{R}$ bounded and continuous. Of course, the characterization of $\mu$ as limiting distribution makes sense only in the presence of an infinite exchangeable sequence, whilst other characterizations, similar to (1), can be given also for a finite, not necessarily extendible, family of exchangeable r.v.’s. See, e.g., \cite{6, 7}.

Our aim is to provide a quantification for the convergence of $\mu_n$ to $\mu$, tacking cognizance that weak convergence is induced by some distance, so that a reasonable choice of that distance yields an explicit evaluation of the discrepancy between $\mu_n$ and $\mu$ in terms of $n$, and also a practical interpretation of the approximation from various points of view. In the present paper, we will focus on the so-called Kolmogorov distance which, for any pair $\nu_1, \nu_2$ of probability measures (p.m.’s) on $([0, 1], \mathcal{B}([0, 1]))$, is defined as

$$d_K(\nu_1; \nu_2) := \sup_{x \in [0, 1]} |\nu_1([0, x]) - \nu_2([0, x])| = \sup_{x \in [0, 1]} |F_1(x) - F_2(x)| .$$

We recall that $d_K$ metrizes the weak convergence on the space $\mathcal{P}(0, 1)$ of all p.m.’s on $([0, 1], \mathcal{B}([0, 1]))$ when the limiting p.m. has a continuous distribution function (d.f.). Moreover, to underline the attention that such a distance has received within the probabilistic literature, we mention the research dealing with the errors of the normal approximation in the central limit theorem which culminates in the well-known Berry-Esseen bounds. See, e.g., Chapters 5-6 in \cite{24} or Chapter 3 in \cite{26}. On the other hand, a very popular use of $d_K$ in statistics is related to the Kolmogorov-Smirnov test.

Other remarkable distances that metrize weak convergence on $\mathcal{P}(0, 1)$ are the Lévy distance, given by

$$d_L(\nu_1; \nu_2) := \inf\{\epsilon > 0 \mid F_1(x - \epsilon) - \epsilon \leq F_2(x) \leq F_1(x + \epsilon) + \epsilon, \forall x \in [0, 1]\} ,$$

and the Kantorovich distance (also known as Wasserstein distance of order 1), defined by

$$d_W(\nu_1; \nu_2) = \int_{0}^{1} |F_1(x) - F_2(x)|dx .$$

See, e.g., \cite{14, 26} for an overview on probability metrics, as well as for the statement of the inequalities $d_W(\nu_1; \nu_2) \leq d_K(\nu_1; \nu_2)$ and $d_L(\nu_1; \nu_2) \leq d_K(\nu_1; \nu_2)$, which clearly show that $d_K$ is stronger than the other two distances.
Despite the long history of de Finetti’s theorem, the study of the rate of convergence of \( \mu_n \) to \( \mu \) has been initiated very recently in \([20]\), where some important facts about \( d_W(\mu_n; \mu) \) has been proved in full generality. First, it is shown that the two inequalities

\[
\frac{C_1(\mu)}{n} \leq d_W(\mu_n; \mu) \leq \sqrt{\frac{C_1(\mu)}{n}} \quad (n \in \mathbb{N})
\]

are valid for any \( \mu \in \mathcal{P}(0,1) \), where \( C_1(\mu) := \int_0^1 \theta(1-\theta)\mu(d\theta) \), yielding that \( 1/n \) is the best possible rate of convergence to zero also for \( d_K(\mu_n; \mu) \). Moreover, it follows that \( d_W(\mu_n; \mu) \) goes to zero at least as fast as \( 1/\sqrt{n} \). The second main result provides sufficient conditions for the achievement of the best rate: in particular, if \( \mu \) is absolutely continuous with a density \( f \) satisfying \( \int_0^1 \theta(1-\theta)|f'(\theta)|d\theta < +\infty \), where \( f' \) stands for the (distributional) derivative of \( f \), then

\[
d_W(\mu_n; \mu) \leq \frac{C_2(\mu)}{n} \quad (n \in \mathbb{N})
\]

holds true with an explicit constant \( C_2(\mu) \). Finally, it is proved that, for every \( \delta \in [\frac{1}{2}, 1] \), there exists a suitable \( \mu \in \mathcal{P}(0,1) \) for which \( d_W(\mu_n; \mu) \sim 1/n^\delta \) as \( n \to +\infty \).

One of the merits of \([20]\) consists in having generalized bounds of the form \( d_W(\mu_n; \mu) \leq C/n \) previously obtained in \([16]\) under the restriction that \( \mu \) is a beta distribution. For the sake of notational clarity, we recall that the beta distribution with parameters \((a,b) \in (0, +\infty)^2\) is the element of \( \mathcal{P}(0,1) \) corresponding to the probability density function \((0,1) \ni \theta \mapsto \beta(\theta; a, b) := \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \theta^{a-1}(1-\theta)^{b-1}\). In point of fact, since the bounds displayed in \([16]\), with the aim to investigate the Pólya-Eggenberger urn model, are obtained by an application of the Stein method, the authors of that paper left the problem of finding a general strategy (i.e., not relying on a specific form of \( \mu \)) completely open, notably if the objective is the achievement of the optimal rate. See also \([10]\) for similar bounds obtained again via Stein’s method, and \([11]\) for the connection of these Pólya sequences with the important concept of Hoeffding decomposability given in \([23]\). Therefore, the main breakthrough in \([20]\) may be recognized in having highlighted the connection between the rate of convergence in de Finetti’s theorem and the classical Berry-Esseen bounds for the normal approximation in the central limit theorem, a connection we found very useful also in the study of \( d_K(\mu_n; \mu) \). As we shall see in Section 2, our strategy hinges on refined versions of these bounds, usually mentioned as Edgeworth expansions. See, e.g., Chapters 5-6 in \([24]\) or Chapter 3 in \([18]\).

The study of \( d_K(\mu_n; \mu) \) is indeed more delicate than that of \( d_W(\mu_n; \mu) \). To realize this fact, we start by stating a very simple result confined to beta priors—whose proof can be obtain by direct computation—which shows that any rate \( n^{-\alpha} \), with \( \alpha \in (0,1] \), is actually achieved by \( d_K(\mu_n; \mu) \).

**Proposition 1.** If \( \mu \) is equal to the beta distribution with parameters \((\alpha, 1)\) or \((1, \alpha)\) with \( \alpha > 0 \), then there exists a constant \( C_\alpha \) for which

\[
d_K(\mu_n; \mu) \leq C_\alpha \left( \frac{1}{n} \right)^{\alpha+1} \quad (n \in \mathbb{N})
\]
is fulfilled, where $\wedge$ denotes the minimum value.

We stress that for the beta distributions just considered we have $d_W(\mu_n; \mu) \sim 1/n$ in view of the statements in [16, 20], remarking once again that the asymptotic behavior of $d_K(\mu_n; \mu)$ is rather different from that of $d_W(\mu_n; \mu)$. In addition, we notice that it seems not convenient at all to resort to the inequality

$$d_K(\mu_1; \mu_2) \leq C(\nu_2) \sqrt{d_W(\nu_1; \nu_2)} \quad (3)$$

which is valid whenever $\nu_2$ has a bounded density (see, e.g., [14]): in fact, for the beta distribution with parameters $(\alpha, 1)$ or $(1, \alpha)$ with $\alpha \geq 1$, the upper bound provided by (3) would be of the form $C/\sqrt{n}$, which is worse than the bound $C/n$ given by (2).

As far as we know, there is only the paper [21] that studies the rate of convergence of $d_K(\mu_n; \mu)$ in full generality, even though it does not contain any explicit mention to de Finetti’s theorem. In the equivalent reformulation of our main problem as the finding of the rate of approximation in the Hausdorff moment problem, Theorem 2 in [21] provides the existence of a constant $C(\mu)$ for which

$$d_K(\mu_n; \mu) \leq C(\mu) \frac{1}{n} \quad (n \in \mathbb{N}) \quad (4)$$

is in force whenever $\mu$ is absolutely continuous with a density $f$ belonging to $W^{1,\infty}(0,1)$, the Sobolev space of essentially bounded functions on $(0,1)$ with an essentially bounded distributional derivative. In any case, a direct comparison of Theorem 2 in [21] with our Proposition 1 shows that the assumption $f \in W^{1,\infty}(0,1)$ is indeed too strong and far from capturing the whole class of priors for which (4) is met. To fill the gap, we state the main result of the paper which provides a more general sufficient condition for the achievement of the best rate $1/n$.

**Theorem 2.** If $\mu \in \mathcal{P}(0,1)$ has a density $f$ satisfying

$$[\theta(1-\theta)]^\gamma |f'(\theta)| \in L^\infty(0,1) \quad (5)$$

for some $\gamma \in (0,1)$, where $f'$ denotes the distributional derivative of $f$, then (4) is fulfilled with a $C(\mu)$ that depends on $\mu$ only through $\sup_{\theta \in [0,1]} |f(\theta)|$ and $\sup_{\theta \in [0,1]} [\theta(1-\theta)]^{\gamma} |f'(\theta)|$.

Apropos of the condition (5), we notice that it can be written in terms of a weighted Sobolev norm, as well as the the assumption $\int_0^1 \theta(1-\theta)|f'(\theta)|d\theta < +\infty$ made in [20], with a similar weight. Actually, for priors $\mu$ with a support strictly contained in $(0,1)$, our assumption boils down to requiring $f \in W^{1,\infty}(0,1)$, but, without this restriction, it is evident that our theorem improves on the aforesaid result in [21], by allowing $f'(\theta)$ to diverge moderately in $\theta = 0$ and $\theta = 1$. As a corollary, we state a result for beta distributions which, by agreeing with our Proposition 1, captures exactly the elements of this class for which (4) is valid. In fact, since a beta density belongs to $W^{1,\infty}(0,1)$ if and only if $a,b \geq 2$, we improve the basic assumption by the following
Corollary 3. If \( \mu \) is equal to the beta distribution with parameters \((a, b)\), then (4) is fulfilled if and only if \(a, b \geq 1\).

To conclude the introduction, we formulate a last statement dealing with a larger class of de Finetti's measures, with respect to Theorem 2. Specifically, we will show that a rate of convergence for \(d_K(\mu_n; \mu)\), although not sharp, can be obtained even in the presence of a non absolutely continuous \(\mu\), provided that its d.f. \(F\) is Hölder continuous (as it happens if \(F\) coincides, for example, with the Cantor function). The determination of the sharp rate in this less regular setting remains an interest open problem.

Proposition 4. If \(\mu \in \mathcal{P}(0, 1)\) has a \(\gamma\)-Hölder continuous d.f. for some \(\gamma \in (0, 1]\), then there exists a suitable constant \(L_\gamma(\mu)\) for which
\[
d_K(\mu_n; \mu) \leq \frac{L_\gamma(\mu)}{n^{\gamma/2}} \quad (n \in \mathbb{N}) .
\]

2 Proofs

Here, we gather the proofs of the statements formulated in Section 1.

2.1 Proof of Proposition 1

We start by dealing with the case of a beta distribution with parameters \((\alpha, 1)\). First, we note that the associated density belongs to \(W^{1,\infty}(0, 1)\) if \(\alpha \in \{1\} \cup [2, +\infty)\), so that (2) follows as a direct application of Theorem 2 in [21]. Therefore, we treat the case \(\alpha \in (0, 1) \cup (1, 2)\) by starting from the direct computation of the d.f. \(F_n\) associated to \(\mu_n\), namely
\[
F_n(x) = \sum_{k=0}^{|nx|} \int_0^1 \beta^k (1 - \theta)^{n-k} \mu(d\theta) = \int_0^1 \beta(y; \lfloor nx \rfloor + 1, n - \lfloor nx \rfloor)F(y)dy \quad (7)
\]
for all \(x \in (0, 1)\), where \(\lfloor \cdot \rfloor\) denotes the integral part. For the validity of (7), see formulae (13)-(14) in [21], or Problems 44-45 at the end of Chapter VI of [13]. Now, for \(\alpha \in (0, 1)\), we invoke Wendell's inequalities (see formula (5) in [25]) to obtain
\[
L_\alpha(\lfloor nx \rfloor, n) \leq F_n(x) \leq U_\alpha(\lfloor nx \rfloor, n)
\]
for all \(n \in \mathbb{N}\) and \(x \in (0, 1)\), where
\[
L_\alpha(\lfloor nx \rfloor, n) := \left(\frac{|nx|+1}{|nx|+1+\alpha}\right)^{1-\alpha} \times \left(\frac{|nx|+1}{n+1}\right)^\alpha \quad \text{and} \quad U_\alpha(\lfloor nx \rfloor, n)
\]
\[
:= \left(\frac{n+1+\alpha}{n+1}\right)^{1-\alpha} \times \left(\frac{|nx|+1}{|nx|+1+\alpha}\right)^\alpha .
\]
Then, we observe that
\[
d_K(\mu_n; \mu) = \max_{k \in \{1, \ldots, n\}} \sup_{x \in \left(\frac{k-1}{n}, \frac{k}{n}\right)} |F_n(x) - F(x)| = \max_{k \in \{1, \ldots, n\}} \sup_{x \in \left(\frac{k-1}{n}, \frac{k}{n}\right)} \left|F_n\left(\frac{k-1}{n}\right) - x^\alpha\right|
\]
and that, for any $k \in \{1, \ldots, n\}$,
\[
\sup_{x \in \left[\frac{k-1}{n}, \frac{k}{n}\right]} |F_n\left(\frac{k-1}{n}\right) - x^\alpha| \leq \left| U_\alpha(k-1, n) - L_\alpha(k-1, n) \right| + \left| U_\alpha(k-1, n) - \left( \frac{k}{n+1} \right)^\alpha \right|
\]
\[+ \left[ \left( \frac{k}{n} \right)^\alpha - \left( \frac{k}{n+1} \right)^\alpha \right] + \left[ \left( \frac{k}{n} \right)^\alpha - \left( \frac{k-1}{n} \right)^\alpha \right]. \tag{9}
\]

For the first two summands on the above right-hand side, we can write
\[
\left| U_\alpha(k-1, n) - L_\alpha(k-1, n) \right| + \left| U_\alpha(k-1, n) - \left( \frac{k}{n+1} \right)^\alpha \right|
\leq 2 \left[ \left( \frac{n+1+\alpha}{n+1} \right)^{1-\alpha} - 1 \right] + \left( \frac{k}{n+1} \right)^\alpha \left[ 1 - \left( \frac{k}{k+\alpha} \right)^{1-\alpha} \right]. \tag{10}
\]
At this stage, we observe that $\left( \frac{n+1+\alpha}{n+1} \right)^{1-\alpha} - 1 \leq \frac{\alpha(1-\alpha)}{n+1}$ holds for all $n \in \mathbb{N}$, while for the latter summand on the right-hand side of (10) we get
\[
\left( \frac{k}{n+1} \right)^\alpha \left[ 1 - \left( \frac{k}{k+\alpha} \right)^{1-\alpha} \right] \leq \alpha(1-\alpha)(1+\alpha)^\alpha \left( \frac{1}{n+1} \right)^\alpha.
\]
Moreover, for the third summand on the right-hand side of (9) we have $\left( \frac{k}{n} \right)^\alpha - \left( \frac{k+\alpha}{n+1} \right)^\alpha \leq \frac{\alpha}{n}$, while for the last summand on the right-hand side of the same relation we obtain $\left( \frac{k}{n} \right)^\alpha - \left( \frac{k-1}{n} \right)^\alpha \leq \left( \frac{1}{n} \right)^\alpha$. Putting these bounds together via (9)-(10), we get (2) for $\alpha \in (0, 1)$.

When $\alpha \in (1, 2)$, we start again from (8), which can be equivalently rewritten as
\[
\frac{\Gamma(n+1)}{\Gamma(n+1+\delta)} \frac{\Gamma([nx]+1+\delta)}{\Gamma([nx]+1)} \frac{|nx| + 1 + \delta}{n+1+\delta}
\]
with $\delta := \alpha - 1$. Whence,
\[
\frac{|nx| + \alpha}{n+\alpha} L_\delta([nx], n) \leq F_n(x) \leq \frac{|nx| + \alpha}{n+\alpha} U_\delta([nx], n)
\]
for all $n \in \mathbb{N}$ and $x \in (0, 1)$. Then, we can establish a bound similar to (9), which reads
\[
\sup_{x \in \left[\frac{k-1}{n}, \frac{k}{n}\right]} |F_n\left(\frac{k-1}{n}\right) - x^\alpha| \leq \frac{k+\delta}{n+\alpha} [U_\delta(k-1, n) - L_\delta(k-1, n)]
\]
\[+ \frac{k+\delta}{n+\alpha} [U_\delta(k-1, n) - \left( \frac{k}{n+1} \right)^\delta]
\]
\[+ \frac{k+\delta}{n+\alpha} \left( \left( \frac{k}{n} \right)^\delta - \left( \frac{k}{n+1} \right)^\delta \right) + \left( \frac{k}{n} \right)^\delta \left| \frac{k+\delta}{n+\alpha} - \frac{k}{n} \right|
\]
\[+ \left[ \left( \frac{k}{n} \right)^\alpha - \left( \frac{k-1}{n} \right)^\alpha \right]. \tag{11}
\]
We analyze the first two summands on the above right-hand side by resorting to (10), to obtain
\[
\frac{k + \delta}{n + \alpha} [U_\delta(k - 1, n) - L_\delta(k - 1, n)] + \frac{k + \delta}{n + \alpha} U_\delta(k - 1, n) - \left( \frac{k}{n + 1} \right)^\delta \leq 2 \left[ \left( \frac{n + 1 + \delta}{n + 1} \right)^{1-\delta} - 1 \right] + 2 \left( \frac{k}{n + 1} \right)^\alpha \left[ 1 - \left( \frac{k}{k + \delta} \right)^{1-\delta} \right].
\]

Now, the former summand on the above right-hand side has been already bounded by \(\frac{2\alpha(1-\alpha)}{n+1}\), so that we can focus the attention on the latter. Arguing as above, we get
\[
2 \left( \frac{k}{n + 1} \right)^\alpha \left[ 1 - \left( \frac{k}{k + \delta} \right)^{1-\delta} \right] \leq 2\delta(1-\delta)(1+\delta)^\alpha \frac{1}{n + 1}.
\]

Lastly, the very same arguments used to handle the case \(\alpha \in (0, 1)\) lead to conclude that also the last three terms on the right-hand side of (11) are bounded from above by a term of the type \(C_\alpha/n\) for some constant \(C_\alpha\) independent of \(k\). The proof of (2) is therefore complete in the case that the prior is a beta distribution with parameters \((\alpha, 1)\).

The case of a beta distribution with parameters \((1, \alpha)\) is easily reformulated in terms of a beta distribution with parameters \((\alpha, 1)\), in view of the following symmetry argument. First, we note that the d.f. \(F(x)\) of a beta with parameters \((1, \alpha)\) coincides with \(1-F^\star(1-x)\), where \(F^\star\) is the d.f. of a beta with parameters \((\alpha, 1)\). An analogous argument is true for the d.f. \(F_n(x)\), in the sense that it coincides, for all \(x \in [0, 1] \setminus \{0, \frac{1}{n}, \frac{2}{n}, \ldots, 1\}\), with the d.f. \(F_n^\star(x)\) of the r.v. \(\frac{1}{n} \sum_{i=1}^n X_i\), where \(X_i := 1 - X_i\) for all \(i \in \mathbb{N}\). The conclusion is reached by observing that the de Finetti measure of the (exchangeable) sequence \(\{X_i\}_{i \geq 1}\) is exactly the beta with parameters \((\alpha, 1)\), whenever the de Finetti measure of the sequence \(\{X_i\}_{i \geq 1}\) is the beta with parameters \((1, \alpha)\), and that
\[
d_K(\mu_n; \mu) = \sup_{x \in [0,1]} \left| F_n(x) - F(x) \right| = \sup_{x \in [0,1]} \left| F_n^\star(x) - F^\star(x) \right| = \sup_{x \in [0,1]} \left| F_n^\star(x) - F(x) \right|.
\]

### 2.2 Preliminaries and proof of Theorem 2

We premise a decomposition lemma for probability density functions which will be used to justify the introduction of the additional hypothesis \(f(0) = f(1) = 0\) in the first part of the proof of Theorem 2.

**Lemma 5.** Given a probability density function \(f\) on \([0, 1]\) which is expressed by polynomial, then there exist three non-negative constants \(A_\infty, A_+, A_-\) and three continuous probability density functions \(f_\infty, f_+, f_-\) on \([0, 1]\) such that:

\[
i) \quad A_\infty \leq \|f\|_\infty := \sup_{\theta \in [0,1]} |f(\theta)| \quad \text{and} \quad A_\pm \leq 1 + \|f\|_\infty;
\]
ii) \( f_\infty \in W^{1,\infty}(0, 1) \) with \( A_\infty \| f_\infty \|_\infty \leq \| f \|_\infty \) and \( A_\infty \| f_\infty \|_\infty \leq 2 \| f \|_\infty \);

iii) \( f_+, f_- \in W^{1,\infty}(0, 1), f_+(0) = f_+(1) = f_-(0) = f_-(1) = 0, A_\pm \| f_\pm \|_\infty \leq 2 \| f \|_\infty \) and, for any \( \gamma \in (0, 1) \),

\[
A_\pm \sup_{\theta \in [0, 1]} [\theta(1-\theta)]^\gamma |f_\pm'(\theta)| \leq \frac{2}{\gamma} \| f \|_\infty + \sup_{\theta \in [0, 1]} [\theta(1-\theta)]^\gamma |f'(\theta)|;
\]

iv) \( f(\theta) = A_\infty f_\infty(\theta) + A_+ f_+(\theta) - A_- f_-(\theta) \) for all \( \theta \in [0, 1] \).

**Proof of Lemma 5** If \( f(0) = f(1) = 0 \), the thesis is trivial. Otherwise, we put \( A_\infty = \frac{f(1)+f(0)}{2} \leq \| f \|_\infty \) and \( f_\infty(\theta) = \frac{f(1)-f(0)\theta+f(0)}{A_\infty} \), so that ii) holds trivially. Then, we exploit that, for any \( a, b \in \mathbb{R}, a = b + (a-b)_+ - (b-a)_- \), where \( x_+ := \max\{0, x\} \), to obtain \( g_+(\theta) := (f(\theta)-A_\infty f_\infty(\theta))_+ \) and \( g_-(\theta) := (A_\infty f_\infty(\theta) - f(\theta))_- \). Thus, we put \( A_\pm = \int_0^1 g_\pm(\theta)d\theta \) and \( f_\pm = g_\pm(\theta)/A_\pm \) with the proviso that, if \( A_+ = 0 \) (\( A_- = 0 \), respectively), the definition of \( f_+ \) (\( f_- \)) is arbitrary and can be chosen equal to \( 6x(1-x) \). By definition, point iv) is met along with \( f_+(0) = f_+(1) = f_-(0) = f_-(1) = 0 \). Moreover, we have \( A_\pm \leq 1 + A_\infty \) and point i) follows. To prove that \( f_+, f_- \in W^{1,\infty}(0, 1) \), it is enough to notice that \( f_\theta = A_\infty f_\infty(\theta) \) for finitely many \( \theta \)'s, by virtue of the fundamental theorem of algebra, and, in the complement of this set, both \( f_+ \) and \( f_- \) are again two polynomials. To check that \( A_\pm \| f_\pm \|_\infty \leq 2 \| f \|_\infty \), it is enough to observe that \( \| g_\pm \|_\infty \leq \| f \|_\infty + A_\| f_\infty \|_\infty \). Finally, we just note that, except on the finite set \( \{ \theta \in [0, 1] \mid f(\theta) = A_\infty f_\infty(\theta) \} \), we have \( A_\pm \| f_\pm'(\theta) \| \leq |f'(\theta)| + A_\| f_\infty'(\theta) \| \), so that we can deduce the validity of (12) from the previous bounds. □

Another preliminary result deals with further regularity properties of the densities that satisfy \([\mathfrak{G}]\). We observe that, since \( [x(1-x)]^{-\gamma} \in L^1(0, 1) \) if \( \gamma \in (0, 1) \), the validity of \([\mathfrak{G}]\) entails \( f \in W^{1,1}(0, 1) \) and, hence, the existence of a continuous version of the same density on the whole set \([0, 1] \). See, e.g., Theorem 8.2 in \([\mathfrak{L}]\).

**Lemma 6.** If \( f \) is a probability density function satisfying \([\mathfrak{G}]\) for some \( \gamma \in (0, 1) \), then there exists a positive constant \( R(\gamma) \) such that

\[
\sup_{x \in (0, 1), 0 < w < x(1-x)} x(1-x) \left| \frac{F(x+w) - 2F(x) + F(x-w)}{w^2} \right| \leq R(\gamma) \| f \|_{1,\gamma}
\]

is fulfilled with \( \| f \|_{1,\gamma} := \sup_{\theta \in [0, 1]} [\theta(1-\theta)]^\gamma |f'(\theta)| \). Moreover, if the additional condition \( f(0) = f(1) = 0 \) (referred to the continuous representative of \( f \)) is in force, then:

i) \( f(\theta) \leq M(f) \theta^{1-\gamma}, f(\theta) \leq M(f)(1-\theta)^{1-\gamma} \) hold for all \( \theta \in [0, 1] \), with \( M(f) := \frac{2^\gamma}{1-\gamma} \| f \|_{1,\gamma} + 2^{1-\gamma} \| f \|_\infty \);

ii) \( F(x) \leq M(f) \frac{x^{2-\gamma}}{2-\gamma}, 1 - F(x) \leq M(f) \frac{(1-x)^{2-\gamma}}{2-\gamma} \) hold for all \( x \in [0, 1] \).
**Proof of Lemma 6.** Since \( f \in W^{1,1}(0,1) \) by virtue of (5), the Taylor formula with integral remainder can be applied to obtain

\[
F(x + w) - 2F(x) + F(x - w) = \int_x^{x+w} (x + w - t)f'(t)dt + \int_x^{x-w} (x - w - t)f'(t)dt
\]

for all \( x \in (0, 1) \) and \( w \) satisfying \( 0 < w < x(1 - x) \). Whence,

\[
|F(x + w) - 2F(x) + F(x - w)| \leq \int_x^{x+w} (w + t - x)|f'(t)|dt + \int_x^{x-w} (w + x - t)|f'(t)|dt . \tag{14}
\]

At this stage, we show explicitly how to bound the former integral when \( x \in (0, 1/2] \), the other cases being analogous. Since \( 0 < x < x + w < 3/4 \), then we get \(|f'(t)| \leq 4\gamma|f|_{1,\gamma}t^{-\gamma}\) for all \( t \in [x, x + w] \), leading to

\[
\int_x^{x+w} (w + t - x)|f'(t)|dt \leq 4\gamma|f|_{1,\gamma} \left[ w \frac{(1 + \eta)^{1-\gamma} - 1}{1-\gamma} + x^2 - \frac{(1 + \eta)^2 - 1 - (2 - \gamma)\eta}{2-\gamma} - x \frac{(1 + \eta)^{1-\gamma} - 1}{1-\gamma} \right] . \tag{15}
\]

Then, we put \( \eta := w/x \) and we observe that \( \eta \in (0, 1/2) \), so that the expression inside the brackets can be written as

\[
w x^{1-\gamma} \frac{(1 + \eta)^{1-\gamma} - 1}{1-\gamma} + x^2 - \frac{(1 + \eta)^2 - 1 - (2 - \gamma)\eta}{2-\gamma} - x \frac{(1 + \eta)^{1-\gamma} - 1}{1-\gamma} = 1 + \frac{H(2 - \gamma)}{2 - \gamma} + \frac{3H(1-\gamma)}{2(1-\gamma)}
\]

The conclusion this argument by noticing that, for any \( \eta \) satisfying \(|\eta| \leq 1/2 \) and any \( \alpha > 0 \) there exists a constant \( H(\alpha) \) such that \(|(1 + \eta)^{\alpha} - 1 - \alpha\eta| \leq H(\alpha)\eta^2 \). This remark implies that the expression in (15) is bounded by

\[
\sup_{x \in (0,1/2], \ 0 < w < x(1-x)} \frac{x(1-x)}{w^2} \int_x^{x+w} (w + t - x)|f'(t)|dt \leq 2^{3\gamma-2} \left[ 1 + \frac{H(2 - \gamma)}{2 - \gamma} + \frac{3H(1-\gamma)}{2(1-\gamma)} \right] |f|_{1,\gamma} .
\]

As recalled, the treatment of the latter integral on the right-hand side of (14) for \( x \in (0, 1/2] \) is analogous. Lastly, when \( x \in [1/2, 1] \), it is enough to change the variable \( t = 1 - s \), obtaining \( \int_{x-w}^{x+w} (w + t - x)|f'(t)|dt = \int_{1-x}^{1-x-w} (w + 1 - x - s)|f'(1-s)|ds \) and \( \int_{x-w}^{x+w} (w + x - t)|f'(t)|dt = \int_{1-x}^{1-x-w} (w + s - 1 + x)|f'(1-s)|ds \), where the integrals in the new variable \( s \) are exactly the integrals studied above.
To prove i), we just write $f(\theta) = \int_0^\theta f'(y)dy$ and, confining to the case that $\theta \in [0, 1/2]$, we exploit (5) in the form $\|f'(\theta)\| \leq 2^\gamma \|f\|_1 \theta^{-\gamma}$. This proves the first bound, after noticing that $f(\theta) \leq 2^{1-\gamma} \|f\|_\infty \theta^{1-\gamma}$ is valid for any $\theta \in [1/2, 1]$. For the latter bound, we start from $-f(\theta) = \int_0^1 f'(y)dy$ and we argue in an analogous way.

To prove point ii), it is enough to integrate the bounds obtained in point i). □

The last preliminary result is a refinement of the well-known estimates of Berry-Esseen type for the characteristic function of a normalized sum of i.i.d., centered r.v.’s. In fact, the following statement can be viewed as a generalization of Lemma 4 in Chapter VI of [18] in the case that the summands possess the absolute moment for some $\alpha \geq 2$, $\alpha \geq 2$, $\gamma \geq 0$, $\alpha \geq 2$, $\gamma \geq 0$, and $\delta$ is a numerical constant independent of $\xi$ and the p.d. of $V_1$.

**Lemma 7.** Let $\{V_n\}_{n \geq 1}$ be a sequence of i.i.d. r.v.'s defined on $(\Omega, \mathcal{F}, \mathbb{P})$ such that $\beta_{3+\delta} := \mathbb{E}[|V_1|^{3+\delta}] < +\infty$ holds for some $\delta \in (0, 1)$, along with $\mathbb{E}[V_1] = 0$ and $\mathbb{E}[V_1^2] =: \sigma^2 > 0$. Upon putting $\alpha_3 := \mathbb{E}[V_1^3]$, $\psi_n(\xi) := \mathbb{E}\left[\exp \left\{i\xi \sum_{k=1}^n V_k / \sqrt{n\sigma^2}\right\}\right]$, there holds

$$\left|\psi_n(\xi) - e^{-\xi^2/2} \left(1 + \frac{\alpha_3}{6\sqrt{n\sigma^2}} (i\xi)^3\right)\right| \leq Q(\delta) \frac{\beta_{3+\delta}}{n^{(1+\delta)/2} \sigma^{3+\delta}} |\xi|^{3+\delta} (1 + |\xi|^4) e^{-\xi^2/4} \quad (16)$$

for any $\xi$ satisfying $|\xi| < \frac{1}{4} \sqrt{n} \left(\frac{\sigma^{3+\delta}}{\beta_{3+\delta}}\right)^{1/(1+\delta)}$, where $Q(\delta)$ is a numerical constant independent of $\xi$ and the p.d. of $V_1$.

The proof is based on the arguments used to prove Lemma A.2 in [9] and Lemma 3.1 in [18].

**Proof of Lemma 7** First, we put $\psi(\xi) := \mathbb{E}[e^{i\xi V_1}]$ and we observe that $\psi(\xi) = 1 - \frac{\sigma^2}{2} \xi^2 + \frac{\alpha_3}{6} (i\xi)^3 + \rho_n(\xi)$, where

$$|\rho_n(\xi)| \leq \frac{2^{1-\delta} \beta_{3+\delta}}{(1+\delta)(2+\delta)(3+\delta) |\xi|^{3+\delta}}.$$

See, e.g., Theorem 1 in Section 8.4 of [3]. Whence,

$$\psi \left(\frac{\xi}{\sqrt{n\sigma^2}}\right) = 1 - \frac{1}{2n} \xi^2 + \frac{\alpha_3}{6\sigma^3 n^{3/2}} (i\xi)^3 + \rho_{n,\delta}(\xi)$$

with

$$|\rho_{n,\delta}(\xi)| \leq \frac{2^{1-\delta} \beta_{3+\delta}}{(1+\delta)(2+\delta)(3+\delta) \sigma^{3+\delta} n^{(3+\delta)/2} |\xi|^{3+\delta}}.$$

Now, we notice that Lyapunov’s inequality entails $\sigma^{3+\delta} \leq \beta_{3+\delta}$, while Hölder’s inequality shows that

$$\frac{|\alpha_3|}{\sigma^3} \left(\frac{\sigma^{3+\delta}}{\beta_{3+\delta}}\right)^{3/(1+\delta)} \leq \frac{\beta_3}{\sigma^3} \left(\frac{\sigma^{3+\delta}}{\beta_{3+\delta}}\right)^{3/(1+\delta)} \leq \frac{\beta_3}{\sigma^3} \left(\frac{\sigma^{3+\delta}}{\beta_{3+\delta}}\right)^{1/(1+\delta)} \leq 1$$
where $\beta_3 := \mathbb{E}[V_4^2]$. Therefore, for any $\xi$ satisfying $|\xi| \leq \frac{1}{4} \sqrt{n} \left( \frac{\sigma^{3+\delta}}{\beta_{3+\delta}} \right)^{1/(1+\delta)}$, we have

$$\left| - \frac{1}{2n} \xi^2 + \frac{\alpha_3}{6\sigma^3 n^{3/2}} (i\xi)^3 + \rho_{n,\delta}(\xi) \right| \leq \frac{5}{128}.$$

Thanks to this bound, we are allowed to consider the principal logarithm $\text{Log}(1 + z) := \sum_{k=1}^{\infty} \frac{(-z)^k}{k}$, $|z| < 1$, as follows:

$$\psi_n(\xi) = \exp \left\{ n \text{Log} \left( \frac{\xi}{\sqrt{n} \sigma^2} \right) \right\} = \exp \left\{ n \text{Log} \left[ 1 - \frac{1}{2n} \xi^2 + \frac{\alpha_3}{6\sigma^3 n^{3/2}} (i\xi)^3 + \rho_{n,\delta}(\xi) \right] \right\} = e^{-\xi^2/2} \exp \left\{ \frac{\alpha_3}{6\sqrt{n} \sigma^3} (i\xi)^3 \right\} e^{\tau_{n,\delta}(\xi)}$$

where $\tau_{n,\delta}(\xi)$ is defined to be $n \rho_{n,\delta}(\xi) + n \left[ - \frac{1}{2n} \xi^2 + \frac{\alpha_3}{6\sigma^3 n^{3/2}} (i\xi)^3 + \rho_{n,\delta}(\xi) \right] \gamma \left( - \frac{1}{2n} \xi^2 + \frac{\alpha_3}{6\sigma^3 n^{3/2}} (i\xi)^3 + \rho_{n,\delta}(\xi) \right)$

with $\gamma(z) := \sum_{k=2}^{\infty} \frac{(-z)^{k-2}}{k}$ for $|z| < 1$. At this stage, we put $u_3(\xi) := \frac{\alpha_3}{6\sqrt{n} \sigma^3} (i\xi)^3$ and $\Theta(z) := e^z - 1 - z$, and we exploit the elementary inequality $|e^z - 1| \leq |z| e^{|z|}$ to obtain

$$\left| \psi_n(\xi) - e^{-\xi^2/2} \left\{ 1 + \frac{\alpha_3}{6\sqrt{n} \sigma^3} (i\xi)^3 \right\} \right| \leq e^{-\xi^2/2} e^{\left| u_3(\xi) \right|} |1 - \exp\{\tau_{n,\delta}(\xi)\}| + e^{-\xi^2/2} |\Theta(u_3)|$$

$$\leq e^{-\xi^2/2} e^{\left| u_3(\xi) \right|} |\tau_{n,\delta}(\xi)| + e^{-\xi^2/2} |\Theta(u_3)|.$$

To conclude, it is enough to notice that, for any $\xi$ satisfying $|\xi| \leq \frac{1}{4} \sqrt{n} \left( \frac{\sigma^{3+\delta}}{\beta_{3+\delta}} \right)^{1/(1+\delta)}$, we have $|u_3(\xi)| \leq \frac{1}{384}$, $|\tau_{n,\delta}(\xi)| \leq \frac{1}{4} \xi^2$, $\left| \tau_{n,\delta}(\xi) \right| \leq Q_1(\delta) \frac{\beta_{3+\delta}}{\sigma^{(1+\delta)/2 \beta_{3+\delta}}} |\xi|^{3+\delta} (1 + |\xi|^4)$

and

$$\Theta(|u_3|) \leq Q_2(\delta) \frac{\beta_{3+\delta}}{\sigma^{(1+\delta)/2 \beta_{3+\delta}}} |\xi|^{3(1+\delta)}$$

for suitable constants $Q_1(\delta)$ and $Q_2(\delta)$ independent of $\xi$ and the p.d. of $V_1$. $\square$

The way is now paved for the study of Theorem 2.

**Proof of Theorem 2** The first part of the proof, which requires the major effort, is devoted to proving [11] when the density $f$ of $\mu$, in addition to [5], satisfies also $f(0) = f(1) = 0$. We recall again that [5] entails $f \in W^{1,1}(0,1)$ and, hence, the existence of a continuous version of this density on the whole set $[0,1]$, by virtue of Theorem 8.2 in [2]. Obviously, the additional assumption $f(0) = f(1) = 0$ is referred to this version.
After these preliminaries, we get into the real proof by defining
$I(n, \gamma) := [\bar{x}_{n, \gamma}, 1 - \bar{x}_{n, \gamma}]$,
with $\bar{x}_{n, \gamma} := (1/n)^{1/\gamma}$, which is a proper interval provided that $n \geq 4$. Then, we split the
original quantity as follows:

$$d_R(\mu_n; \mu) \leq \sup_{x \in [0, \bar{x}_{n, \gamma}]} |F_n(x) - F(x)| + \sup_{x \in [1 - \bar{x}_{n, \gamma}, 1]} |F_n(x) - F(x)| + \sup_{x \in [1, \bar{x}_{n, \gamma}]} |F_n(x) - F(x)|$$

$$\leq F_n(\bar{x}_{n, \gamma}) + F(\bar{x}_{n, \gamma}) + \sup_{x \in (0, \bar{x}_{n, \gamma})} |F_n(x) - F(x)|$$

$$+ [1 - F_n(1 - \bar{x}_{n, \gamma})] + [1 - F(1 - \bar{x}_{n, \gamma})].$$

To bound $F(\bar{x}_{n, \gamma})$ and $[1 - F(1 - \bar{x}_{n, \gamma})]$, it is enough to use point ii) of Lemma 6 which
gives:

$$F(\bar{x}_{n, \gamma}) + [1 - F(1 - \bar{x}_{n, \gamma})] \leq \frac{2M(f)}{2 - \gamma} \cdot \frac{1}{n}$$

(18) for all $n \geq 4$. To bound $F_n(\bar{x}_{n, \gamma})$ and $1 - F_n(1 - \bar{x}_{n, \gamma})$, we invoke equation (17) in the proof
of Proposition 1 which, in combination with point ii) of Lemma 6 yields

$$F_n(\bar{x}_{n, \gamma}) \leq M(f) \int_0^1 \beta(y; [n\bar{x}_{n, \gamma}] + 1, n - [n\bar{x}_{n, \gamma}])y^{2-\gamma}dy$$

$$= M(f) \frac{\Gamma(n+1)}{\Gamma([n\bar{x}_{n, \gamma}] + 1)\Gamma(n - [n\bar{x}_{n, \gamma}])} \frac{\Gamma([n\bar{x}_{n, \gamma}] + 3 - \gamma)\Gamma(n - [n\bar{x}_{n, \gamma}])}{\Gamma(n + 3 - \gamma)} .$$

The last expression can be majorized by means of Wendel’s inequalities (see (5) in [25]) as

$$M(f)\frac{[n\bar{x}_{n, \gamma}] + 2 - \gamma}{n + 2 - \gamma} \left(\frac{[n\bar{x}_{n, \gamma}] + 1}{n + 1}\right)^{1-\gamma} \left(\frac{n + 2 - \gamma}{n + 1}\right)^{\gamma}$$

which is easily shown, by elementary algebra, to be less than $9M(f)/n$. To study $1 - F_n(1 - \bar{x}_{n, \gamma})$, we argue as in the proof of Lemma 6 by considering the exchangeable sequence
$\{X_n\}_{n \geq 1}$, where $X_n := 1 - X_n$ for all $n \in \mathbb{N}$. Since the de Finetti measure of this new sequence is the element of $\mathcal{P}(0, 1)$ associated to the d.f. $1 - F(1 - x)$, we resort again to (7) to obtain

$$1 - F_n(1 - \bar{x}_{n, \gamma}) \leq P\left[\frac{1}{n} \sum_{i=1}^{n} \bar{X}_i \leq \bar{x}_{n, \gamma}\right] = \frac{1}{0} \beta(y; [n\bar{x}_{n, \gamma}] + 1, n - [n\bar{x}_{n, \gamma}])[1 - F(1 - y)]dy .$$

Thus, by using the latter bound stated in point ii) of Lemma 6 and arguing exactly as above,
we conclude that $1 - F_n(1 - \bar{x}_{n, \gamma}) \leq 9M(f)/n$. Now, we study $\sup_{x \in (0, \bar{x}_{n, \gamma})} |F_n(x) - F(x)|$. First, we get $P[\sum_{i=1}^{n} X_i = k \mid Y = \theta] = \binom{n}{k} \theta^k (1 - \theta)^{n-k}$ for any $k \in \{0, \ldots, n\}$, thanks to
de Finetti’s representation. Since we can write

$$P\left[\sum_{i=1}^{n} X_i \leq nx \mid Y = \theta\right] = P\left[\frac{\sum_{i=1}^{n} (X_i - \theta)}{\sqrt{n} \theta(1 - \theta)} \leq \frac{n(x - \theta)}{\sqrt{n} \theta(1 - \theta)} \mid Y = \theta\right],$$

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we put \( u := u(x, \theta, n) := \frac{n(x-\theta)}{\sqrt{n\theta(1-\theta)}} \) and \( B_n(y; \theta) := \mathbb{P} \left[ \sum_{i=1}^{n} (X_i - \theta) \leq y \sqrt{n\theta(1-\theta)} \mid Y = \theta \right] \) for \( y \in \mathbb{R} \). To study \( B_n(; \theta) \), we make the key remark that it coincides with the d.f. of a normalized sum of i.i.d., centered r.v.’s, so that we can employ well-known results pertinent to the central limit theorem, as stated in Chapter 8 of [15], Chapters 5-6 of [24], Chapter 3 of [18], and in [22]. In particular, mimicking the main theorem in [22], we introduce the functions \( G_n(y; \theta) := \Phi(y) + H_n(y; \theta) \) and

\[
H_n(y; \theta) := \frac{1}{\sqrt{2\pi n\theta(1-\theta)}} e^{-\frac{y^2}{2n\theta(1-\theta)}} \left\{ \frac{1}{6} (1 - 2\theta)(1 - y^2) + \frac{1 - 2\theta}{6\sqrt{n\theta(1-\theta)}} \left( y^3 - 3y \right) \right\}, \tag{19}
\]

where \( \Phi(y) := \int_{-\infty}^{y} \frac{1}{\sqrt{2\pi}x} e^{-\frac{x^2}{2}} dx \) and \( S(x) := |x| - x + \frac{1}{2} \). Now, Theorem 2 in Chapter 8 of [15] (see also Theorem 2b in Chapter II of [12]) entails \( \sup_{Y \in \mathbb{R}} |B_n(y; \theta) - G_n(y; \theta)| \leq \epsilon_n(\theta) \) and provides the existence of three numerical constants \( \lambda_1, \lambda_2 > 0 \) and \( n_0 \in \mathbb{N} \) (independent of \( n \) and \( \theta \)) such that

\[
\epsilon_n(\theta) = \lambda_1 \int_{-n}^{n} \left| \frac{\hat{B}_n(\xi; \theta) - \hat{G}_n(\xi; \theta)}{\xi} \right| d\xi + \lambda_2 \sup_{y \notin \mathcal{Y}(n, \theta)} \left| \frac{\partial}{\partial y} G_n(y; \theta) \right| \\
\leq \lambda_1 \int_{-n}^{n} \left| \frac{\hat{B}_n(\xi; \theta) - \hat{G}_n(\xi; \theta)}{\xi} \right| d\xi + \frac{\lambda_2 G}{n\theta(1-\theta)} \quad (n \geq n_0) \tag{20}
\]

where \( G > 0 \) is another constant (independent of \( n \) and \( \theta \)), \( \hat{B}_n(\xi; \theta) \) and \( \hat{G}_n(\xi; \theta) \) are defined as Fourier-Stieltjes transforms of \( B_n(.; \theta) \) and \( G_n(.; \theta) \), namely \( \int_{-\infty}^{\infty} e^{\xi y} d\hat{B}_n(y; \theta) \) and \( \int_{-\infty}^{\infty} e^{i\xi y} d\hat{G}_n(y; \theta) \), respectively, and \( \mathcal{Y}(n, \theta) := \left\{ \frac{k-n\theta}{\sqrt{n\theta(1-\theta)}} \mid k \in \mathbb{Z} \right\} \) is the set of the discontinuities of both \( B_n(y; \theta) \) and \( G_n(y; \theta) \). Thus, since \( F_n(x) = \int_{0}^{1} B_n(u(x, \theta, n); \theta) f(\theta) d\theta \), we write

\[
\sup_{x \in \mathcal{I}(n, \gamma)} \left| F_n(x) - F(x) \right| \leq \sup_{x \in \mathcal{I}(n, \gamma)} \left| \int_{0}^{1} \Phi(u(x, \theta, n)) f(\theta) d\theta - F(x) \right| \\
+ \sup_{x \in \mathcal{I}(n, \gamma)} \int_{0}^{1} \left| H_n(u(x, \theta, n); \theta) f(\theta) d\theta + \int_{0}^{1} \epsilon_n(\theta) f(\theta) d\theta \right| \tag{21}
\]

and we try to bound each term on the right-hand side.

Apropos of the first term on the right-hand side of (21), we introduce a Gaussian r.v. \( Z_n : \Omega \to \mathbb{R} \) with zero mean and variance \( 1/n \), independent of \( Y \), so that

\[
\int_{0}^{1} \Phi(u(x, \theta, n)) f(\theta) d\theta = E \left[ \mathbb{P}[Y + Z_n \sqrt{Y(1-Y)} \leq x \mid Y] \right] = \mathbb{P}[Y + Z_n \sqrt{Y(1-Y)} \leq x].
\]
This d.f. (in the $x$ variable) plays an important role also in [20], where its closeness to $F$ is proved with respect to the Kantorovich distance (see Proposition 4.1 therein). In any case, the proof in [20] is strongly based on a dual representation of $d$, which does not have any analog for $d_K$. Therefore, we tackle the problem by a direct computation which, after exchanging the order of conditioning in the above identity and using some elementary algebra, leads to

$$P[Y + Z_n \sqrt{Y(1-Y)} \leq x] = \int_0^{+\infty} \sqrt{\frac{n}{2\pi}} \exp\left\{-\frac{n}{2}z^2\right\} \cdot [F(\theta_1(x,z)) + F(\theta_2(x,z))]dz$$

where

$$\theta_1(x,z) := \frac{2x + z^2 - z \sqrt{z^2 + 4x(1-x)}}{2(z^2 + 1)}, \quad \theta_2(x,z) := \frac{2x + z^2 + z \sqrt{z^2 + 4x(1-x)}}{2(z^2 + 1)}.$$ 

It is routine to check that $\theta_1(x,z), \theta_2(x,z) \in [0,1]$ whenever $x \in [0,1]$ and $z > 0$. Whence,

$$\left| \int_0^1 \Phi(u(x,\theta, n))f(\theta)d\theta - F(x) \right| \leq \int_0^{+\infty} \sqrt{\frac{n}{2\pi}} \exp\left\{-\frac{n}{2}z^2\right\}|F(\theta_1(x,z)) + F(\theta_2(x,z)) - 2F(x)|dz$$

so that, introducing $\delta_n := \sqrt{\frac{2\log(n+1)}{n}}$, we have

$$\int_\delta_n^{+\infty} \sqrt{\frac{n}{2\pi}} \exp\left\{-\frac{n}{2}z^2\right\}|F(\theta_1(x,z)) + F(\theta_2(x,z)) - 2F(x)|dz \leq 2 \int_\delta_n^{+\infty} \sqrt{\frac{n}{2\pi}} \exp\left\{-\frac{n}{2}z^2\right\}dz \leq \frac{1}{(n+1)\sqrt{\pi\log(n+1)}}. \quad (22)$$

It remains to study the integral on $[0,\delta_n]$, by noticing that, after this splitting, we can consider the variable $z^2$ much smaller than $x$ and $1-x$, whenever $x \in I(n,\gamma)$. More precisely, given $\gamma \in (0,1)$ it is possible to find an integer $N(\gamma) \geq 4$ for which $\delta_n \leq x_{n,\gamma}(1-x_{n,\gamma})$ for all $n \geq N(\gamma)$. Therefore, we have that both $\overline{\theta}_1(x,z) := x - z \sqrt{x(1-x)}$ and $\overline{\theta}_2(x,z) := x + z \sqrt{x(1-x)}$ belong to $[0,1]$ whenever $z \in [0,\delta_n], x \in I(n,\gamma)$ and $n \geq N(\gamma)$. We now have

$$|F(\theta_1) + F(\theta_2) - 2F(x)| \leq |F(\theta_1) - F(\overline{\theta}_1)| + |F(\overline{\theta}_1) + F(\overline{\theta}_2) - 2F(x)| + |F(\overline{\theta}_2) - F(\theta_2)|$$

$$\leq \|f\|_\infty [\theta_1 - \overline{\theta}_1] + [\theta_2 - \overline{\theta}_2] + z^2 \left| \frac{F(\overline{\theta}_1) - 2F(x) + F(\overline{\theta}_2)}{z^2} \right|$$

where both $|\theta_1 - \overline{\theta}_1|$ and $|\theta_2 - \overline{\theta}_2|$ are bounded from above by $\frac{3}{2}z^2$. To check this bound, it is enough to consider the quantities

$$\frac{2x + z^2 \pm z \sqrt{z^2 + 4x(1-x)} - 2x(1+z^2) \mp z(1+z^2)\sqrt{4x(1-x)}}{2(1+z^2)}$$
which are less than \(|1-2x|z^2 + \frac{1}{2}z^3 + \frac{1}{2}z[\sqrt{z^2 + 4x(1-x)} - \sqrt{4x(1-x)}]|. The desired result now follows by observing that \(|1-2x| \leq 1\), \(z\in (0,1)\) whenever \(n \geq N(\gamma)\), and 
\[\sqrt{z^2 + 4x(1-x)} - \sqrt{4x(1-x)} \leq z.\] To conclude this argument, we note that
\[
|F_{\theta_1}(x,z) - 2F(x)| \text{ is bounded by virtue of (13) with } w = z \sqrt{x(1-x)}, \text{ since } z \sqrt{x(1-x)} < x(1-x) \text{ holds under the restrictions } z \in [0, \delta_n], x \in I(n, \gamma) \text{ and } n \geq N(\gamma).\]
Whence,
\[
\int_0^{\delta_n} \sqrt{\frac{n}{2\pi}} \exp \left\{ - \frac{n}{2} z^2 \right\} |F(\theta_1(x,z)) + F(\theta_2(x,z)) - 2F(x)| \, dz
\]
\[
\leq (3\|f\|_{\infty} + R(\gamma)|f|_{1,\gamma}) \int_0^{\delta_n} z^2 \sqrt{\frac{n}{2\pi}} \exp \left\{ - \frac{n}{2} z^2 \right\} \, dz
\]
\[
= \frac{1}{2}(3\|f\|_{\infty} + R(\gamma)|f|_{1,\gamma}) \int_{\mathbb{R}} z^2 \sqrt{\frac{n}{2\pi}} \exp \left\{ - \frac{n}{2} z^2 \right\} \, dz = \frac{3\|f\|_{\infty} + R(\gamma)|f|_{1,\gamma}}{2n}. \tag{23}
\]

To bound the expression in (21) that contains \(H_n\), we can exploit the inequalities
\[y^2 \leq e^{y^2/3} \text{ and } |S(x)| \leq 1/2,\] valid for any \(x, y \in \mathbb{R}\), to get
\[
|H_n(y; \theta)| \leq \frac{5}{6\sqrt{2\pi n\theta(1-\theta)}} e^{-y^2/6} + \frac{\lambda_3}{n\theta(1-\theta)}
\]
for all \(y \in \mathbb{R}\) and \(\theta \in (0,1)\), where \(\lambda_3 := \frac{1}{2\pi} \sup_{y \in \mathbb{R}} e^{-y^2/2}|y^3 - 3y|\). Then, after writing
\[
\int_0^1 |H_n(u(x, \theta; n); \theta)| f(\theta) \, d\theta \leq \frac{5\|f\|_{\infty}}{6\sqrt{2\pi n}} \int_0^1 \frac{e^{-u(x, \theta; n)^2/6}}{\sqrt{\theta(1-\theta)}} \, d\theta + \frac{\lambda_3}{n} \int_0^1 \frac{f(\theta)}{\theta(1-\theta)} \, d\theta, \tag{24}
\]
we have only to show that the former integral on the right-hand side is \(O(1/n)\) since, for the latter integral, it is enough to notice that
\[
\int_0^1 \frac{f(\theta)}{\theta(1-\theta)} \, d\theta \leq \frac{2^{\gamma+1}}{1-\gamma} |f|_{1,\gamma} \int_0^{1/2} \theta^{-\gamma} \, d\theta + \frac{2^{\gamma+1}}{1-\gamma} |f|_{1,\gamma} \int_{1/2}^1 (1-\theta)^{-\gamma} \, d\theta \leq \frac{2^{\gamma+2}}{(1-\gamma)^2} |f|_{1,\gamma} \tag{25}
\]
by virtue of the same arguments used to prove point i) of Lemma \[6\]. Now, the above-mentioned claim about the former integral follows after checking the boundedness of the expression
\[
\mathcal{J}_n(a, b; x) := \sqrt{n} \int_a^b \exp \left\{ - \frac{n(x-\theta)^2}{6\theta(1-\theta)} \right\} \frac{d\theta}{\sqrt{\theta(1-\theta)}}
\]
by letting \(n\) and \(x\) vary in \(\mathbb{N}\) and \(I(n, \gamma)\), respectively, where \([a, b]\) coincides with either

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Therefore, taking the former case as reference, we have

\[ I_n(0, \frac{1}{2}; x) \leq \sqrt{\frac{n}{2}} \int_0^\infty \exp \left\{ -\frac{n(x - \theta)^2}{6\theta} \right\} d\theta = \sqrt{3} \int_0^\infty \exp \left\{ -\frac{\frac{1}{6}n(x)^2}{y} - y \right\} \frac{dy}{\sqrt{y}} \]

where \( K_{1/2} \) stands for the modified Bessel function of the second kind. See formula 3.471.12 in [17]. Since \( x \in I(n, \gamma) \) implies that \( nx \geq n^{\frac{1-\gamma}{2}} \geq 1 \), we notice that the expression \( \sqrt{2(nx)^{1/2}e^{nx/3}K_{1/2}(nx/3)} \) is bounded, in view of the asymptotic expansion \( K_{1/2}(z) \sim \sqrt{\frac{2}{\pi z}}e^{-z} \), which is valid as \( z \to +\infty \). Since an analogous bound holds also for \( I_n(1/2, 1; x) \), we can combine (24)-(25) with the analytical study of \( I_n(a, b; x) \) to obtain that

\[
\sup_{x \in I(n, \gamma)} \int_0^1 |H_n(u(x, \theta, n); \theta)| f(\theta) d\theta \leq \|f\|_\infty + |f|_{1, \gamma} \frac{\lambda_4}{n} \quad (n \geq 4)
\]

is valid with a numerical constant \( \lambda_4 \), independent of \( f \) and \( n \).

We conclude the first part of the proof with the analysis of the last term on the right-hand side of (21). Taking account of (20), we immediately realize that the latter summand yields \( \frac{\lambda_2 G}{n} \int_0^1 f(\theta)[\theta(1 - \theta)]^{-1} d\theta \), which is of order \( O(1/n) \) by virtue of (25). The study of the former summand in (20) is more laborious, and it will be conducted by mimicking the argument used in [18] to prove formula (3.3.10). As first step, we borrow from Section 3.3 of [18] the explicit expression of the Fourier-Stieltjes transform \( d_n(t) \) (see page 101 therein) and we combine it with the formulae displayed in Section VI.1 of [24], to obtain

\[
\hat{D}_n(\xi; \theta) := \int_{\mathbb{R}} e^{i\xi y} d_n(y; \theta) = \frac{-\xi}{\sqrt{n\theta(1-\theta)}} \sum_{r \in \mathbb{Z} \setminus \{0\}} e^{2\pi i r n \theta} \frac{2\pi r}{2\pi} \exp \left\{ -\frac{1}{2} \left[ \xi + 2\pi r \sqrt{n \theta(1-\theta)} \right]^2 \right\} \times \left[ 1 + \frac{1 - 2\theta}{6\sqrt{n\theta(1-\theta)}} (i\xi + 2\pi i r \sqrt{n \theta(1-\theta)})^3 \right]
\]

where

\[
D_n(y; \theta) := \frac{1}{\sqrt{2\pi n \theta(1-\theta)}} e^{-\frac{y^2}{2n\theta(1-\theta)}} S(n \theta + y \sqrt{n \theta(1-\theta)}) \left[ 1 + \frac{1 - 2\theta}{6\sqrt{n \theta(1-\theta)}} (y^3 - 3y) \right].
\]

Then, we split the integral in (20) into five terms, by dividing the domain \([-n, n]\) into suitable subdomains whose definitions depend on \( T_1(n, \theta) := \pi \sqrt{n \theta(1-\theta)} \) and \( T_2(n, \theta) := \sqrt{\frac{n \theta(1-\theta)}{1 - 3\theta + 3\theta^2}} \).
We observe that, since $1 - 3\theta + 3\theta^2 \geq 1/4$ for any $\theta \in [0, 1]$, the relation $T_2(n, \theta) \leq T_1(n, \theta)$ is always in force, whereas $T_1(n, \theta) \leq n$ holds whenever $n > \pi^2/4$, which we now assume. Therefore, the desired bound for the integral in (20) follows from

$$
\int_{-n}^{n} \left| \frac{\hat{B}_n(\xi; \theta) - \hat{G}_n(\xi; \theta)}{\xi} \right| \, d\xi \leq \int_{-T_2(n, \theta)}^{T_2(n, \theta)} \left| \frac{\hat{B}_n(\xi; \theta) - \hat{V}_n(\xi; \theta)}{\xi} \right| \, d\xi + \int_{-T_1(n, \theta)}^{T_1(n, \theta)} \left| \frac{\hat{D}_n(\xi; \theta)}{\xi} \right| \, d\xi
\int_{\{T_2(n, \theta) \leq |\xi| \leq n\}} \left| \frac{\hat{V}_n(\xi; \theta)}{\xi} \right| \, d\xi + \int_{\{T_2(n, \theta) \leq |\xi| \leq T_1(n, \theta)\}} \left| \frac{\hat{D}_n(\xi; \theta)}{\xi} \right| \, d\xi
\int_{\{T_1(n, \theta) \leq |\xi| \leq n\}} \left| \frac{\hat{B}_n(\xi; \theta) - \hat{D}_n(\xi; \theta)}{\xi} \right| \, d\xi
$$

(28)

where

$$
\hat{V}_n(\xi; \theta) := \int_{\mathbb{R}} e^{i\xi y} \, dy \left( \frac{1}{\sqrt{2\pi n}} e^{-\frac{1}{2} y^2} \frac{1 - 2\theta}{6\sqrt{\theta(1 - \theta)}} (1 - y^2) \right)
= e^{-\frac{1}{2} \xi^2} \left[ 1 + \frac{1 - 2\theta}{6\sqrt{n\theta(1 - \theta)}} (i\xi)^3 \right].
$$

For the derivation of $\hat{V}(\xi; \theta)$, see Section VI.1 of [24]. Moreover, with reference to that very same section, we note that the term $\frac{1 - 2\theta}{\sqrt{\theta(1 - \theta)}}$ coincides with the ratio between the third cumulant and the third power of the standard deviation of a centered Bernoulli variable with parameter $\theta$, while $y^2 - 1$ coincides with the Chebyshev-Hermite polynomial of degree 2. In addition, $T_2(n, \theta)$ coincides with the product between $\sqrt{n}$ and the square root of the ratio between the fourth power of the standard deviation and the fourth moment of the same centered Bernoulli variable.

In view of these remarks, we provide a bound for the first integral on the right-hand side of (28) by an application of Lemma 4 in Chapter VI of [24] with $s = 4$, namely

$$
\int_{-T_2(n, \theta)}^{T_2(n, \theta)} \left| \frac{\hat{B}_n(\xi; \theta) - \hat{V}_n(\xi; \theta)}{\xi} \right| \, d\xi \leq \lambda_5 \frac{1 - 3\theta + 3\theta^2}{n\theta(1 - \theta)} \int_{-T_2(n, \theta)}^{T_2(n, \theta)} (|\xi|^3 + |\xi|^9) e^{-\frac{1}{2} \xi^2} \, d\xi
$$

where $\lambda_5$ is a numerical constant specified in the proof of the quoted lemma. Whence,

$$
\int_{-T_2(n, \theta)}^{T_2(n, \theta)} \left| \frac{\hat{B}_n(\xi; \theta) - \hat{V}_n(\xi; \theta)}{\xi} \right| \, d\xi \leq \frac{\lambda_6}{n\theta(1 - \theta)}
$$

(29)

where $\lambda_6$ is another numerical constant (independent of $n$ and $\theta$).
Then, we study the second integral on the right-hand side of (28) by resorting to the explicit expression of $\hat{D}_n(\xi; \theta)$, to obtain

$$
\int_{-T_1(n, \theta)}^{T_1(n, \theta)} \frac{1}{\xi} \left| \hat{D}_n(\xi; \theta) \right| d\xi \leq \frac{1}{\sqrt{n\theta(1-\theta)}} \sum_{r \in \mathbb{Z} \setminus \{0\}} \frac{1}{2\pi|r|} [1 + 6\pi^3|r|^3n\theta(1-\theta)] \times
$$

$$
\times \int_{-T_1(n, \theta)}^{T_1(n, \theta)} \exp \left\{ -\frac{1}{2} \left[ \xi^2 + 2\pi r \sqrt{n\theta(1-\theta)} \right] \right\} d\xi . \quad (30)
$$

At this stage, we exploit that $r^2 - |r| \geq \frac{1}{2}r^2$ if $|r| \geq 2$ to write

$$
[\xi + 2\pi r \sqrt{n\theta(1-\theta)}]^2 \geq \xi^2 + (2\pi r \sqrt{n\theta(1-\theta)})^2 - 4T_1(n, \theta)\pi|r| \sqrt{n\theta(1-\theta)} = \xi^2 + 4\pi^2(r^2 - |r|)n\theta(1-\theta) \geq \xi^2 + 2\pi^2r^2n\theta(1-\theta) .
$$

After removing the two terms corresponding to $r = \pm 1$, the series on the right-hand side of (30) can be bounded by

$$
\left( \frac{2}{\pi n\theta(1-\theta)} \right)^{1/2} \sum_{r=2}^{+\infty} \frac{1 + 6\pi^3r^3n\theta(1-\theta)}{r} e^{-\pi^2n\theta(1-\theta)r^2} \quad (31)
$$

and then, taking cognizance that there is a suitable constant $K(\beta)$ such that $\sum_{r=2}^{+\infty} r^\beta e^{-\lambda r^2} \leq K(\beta)\lambda^{-(\beta+1)/2}$ holds for all $\lambda > 0$ if $\beta \geq 0$, we have that the expression in (31) can be bounded by $\lambda_7/[n\theta(1-\theta)]$, where $\lambda_7$ is a constant (independent of $n$ and $\theta$). To handle also the terms of the series corresponding to $r = \pm 1$, we take account that $(x \pm 2)^2 \geq \frac{1}{2}x^2 + \frac{1}{\beta}$ holds for all $x \in [-1, 1]$, to write $[\xi \pm 2\pi \sqrt{n\theta(1-\theta)}]^2 \geq \frac{1}{2}\xi^2 + \frac{\pi^2}{\beta}n\theta(1-\theta)$ for all $\xi \in [-T_1(n, \theta), T_1(n, \theta)]$. Lastly, the sum of the two terms in (30) corresponding to $r = \pm 1$ can be bounded by

$$
2 \left( \frac{1}{\pi n\theta(1-\theta)} \right)^{1/2} [1 + 6\pi^3n\theta(1-\theta)] e^{-\pi^2n\theta(1-\theta)} .
$$

Therefore, we can conclude that

$$
\int_{-T_1(n, \theta)}^{T_1(n, \theta)} \left| \frac{\hat{D}_n(\xi; \theta)}{\xi} \right| d\xi \leq \frac{\lambda_8}{n\theta(1-\theta)} \quad (32)
$$

holds with a suitable numerical constant $\lambda_8$ (independent of $n$ and $\theta$).

As for the third integral on the right-hand side of (28), we just use the explicit expression of $\tilde{V}_n(\xi; \theta)$ to write

$$
\int_{\{T_2(n, \theta) \leq |\xi| \leq n\}} \left| \frac{\tilde{V}_n(\xi; \theta)}{\xi} \right| d\xi \leq 2 \int_{\sqrt{n\theta(1-\theta)}}^{+\infty} e^{-\frac{1}{2}\xi^2} \xi d\xi + \frac{1}{3\sqrt{n\theta(1-\theta)}} \int_{\sqrt{n\theta(1-\theta)}}^{+\infty} \xi^2 e^{-\frac{1}{2}\xi^2} d\xi .
$$
Using that \( x^p e^{-x} \leq (p/e)^p \), which is valid whenever \( x, p > 0 \), we show that

\[
\int_{\{T_2(n, \theta) \leq |\xi| \leq n\}} \left| \hat{V}_n(\xi; \theta) / \xi \right| \, d\xi \leq \frac{\lambda_9}{n\theta(1 - \theta)} \tag{33}
\]

holds with a suitable numerical constant \( \lambda_9 \) (independent of \( n \) and \( \theta \)).

We now consider the fourth integral on the right-hand side of (28). By definition, we have

\[
\int \left| \frac{\hat{B}_n(\xi; \theta)}{\xi} \right| \, d\xi \]

and, after changing the variable by the rule \( u = \xi / \sqrt{n\theta(1 - \theta)} \) and noticing that \( (1 - 3\theta + 3\theta^2)^{-1/2} \geq 1 \) is valid for any \( \theta \in [0, 1] \), we provide the following upper bound

\[
\int_{\{1 \leq |u| \leq \pi\}} \left| \mathbb{E} \left[ \exp \left\{ \frac{i\xi}{\sqrt{n\theta(1 - \theta)}} \left( \sum_{j=1}^{n} X_j - n\theta \right) \right\} \mid Y = \theta \right] \right| \, du .
\]

Now, we just utilize the explicit form of the characteristic function of the binomial distribution with parameters \( n \) and \( \theta \) to write

\[
\left| \mathbb{E} \left[ \exp \left\{ i\xi \sum_{j=1}^{n} X_j \right\} \mid Y = \theta \right] \right| = |1 - \theta + \theta e^{i\xi}|^n = [1 - 2\theta(1 - \theta)(1 - \cos u)]^{n/2} .
\]

Whence,

\[
\int_{\{T_2(n, \theta) \leq |\xi| \leq T_1(n, \theta)\}} \left| \frac{\hat{B}_n(\xi; \theta)}{\xi} \right| \, d\xi \leq 2(\pi - 1)[1 - 2\theta(1 - \theta)(1 - \cos 1)]^{n/2} \leq \frac{2(\pi - 1)}{e(1 - \cos 1)} \frac{1}{n\theta(1 - \theta)} . \tag{34}
\]

To study of the last integral on the right-hand side of (28), we introduce the characteristic function \( \phi(\cdot; \theta) \) of the r.v. \( (X_1 - \theta) \) given \( Y = \theta \), that is \( \phi(\xi; \theta) = (1 - \theta + \theta e^{i\xi})^{n} \), so that we have \( \hat{B}_n(\xi; \theta) = [\phi(\xi / \sqrt{n\theta(1 - \theta)}; \theta)]^{n} \). After changing the variable in that integral according to \( u = \xi / \sqrt{n\theta(1 - \theta)} \) and recalling that \( \phi(-\xi; \theta) = \phi(\xi; \theta) \), we get

\[
\int_{\{T_1(n, \theta) \leq |\xi| \leq n\}} \left| \frac{\hat{B}_n(\xi; \theta) - \hat{D}_n(\xi; \theta)}{\xi} \right| \, d\xi = 2 \int_{\pi}^{\sqrt{n/[\theta(1 - \theta)]}} \left| \frac{\phi(u; \theta)^n - \hat{D}_n(u \sqrt{n\theta(1 - \theta)}; \theta)}{u} \right| \, du .
\]
At this stage, we introduce the quantity
\[
\tau(n; \theta) := \left\lfloor \frac{1}{2} \left( \frac{1}{\pi} \sqrt{\frac{n}{\theta(1-\theta)}} - 1 \right) \right\rfloor
\]
and we notice that \((2\tau(n; \theta) + 1)\pi \leq \sqrt{\frac{n}{\theta(1-\theta)}} < (2\tau(n; \theta) + 3)\pi\). In this notation, we have
\[
\sqrt{\frac{n}{\theta(1-\theta)\pi}} \int_{\pi}^{\pi} \left| \frac{\phi(u; \theta)}{u} - \hat{D}_n(u\sqrt{n\theta(1-\theta)}; \theta) \right| du \leq \sum_{k=1}^{\tau(n; \theta) + 1} J_k(n; \theta)
\]
where
\[
J_k(n; \theta) := \int_{(2k-1)\pi}^{(2k+1)\pi} \left| \frac{\phi(u; \theta)}{u} - \hat{D}_n(u\sqrt{n\theta(1-\theta)}; \theta) \right| du
\]
To bound the integrals \(J_k\)'s, we first isolate from the series \(\hat{D}_n\) defining \(\hat{D}_n(u\sqrt{n\theta(1-\theta)}; \theta)\)
the term corresponding to \(r = -k\), which reads
\[
u e^{-2\pi kn\theta} \frac{1}{2k\pi} \exp \left\{-\frac{n\theta(1-\theta)}{2} (u - 2k\pi)^2 \right\} \cdot \left[ 1 + \frac{1}{6} n\theta(1-\theta)(iu - 2\pi ki)^3 \right] := \hat{\delta}_{n,k}(u; \theta)
\]
so that we obtain
\[
J_k(n; \theta) \leq \int_{(2k-1)\pi}^{(2k+1)\pi} \left| \frac{\hat{D}_n(u\sqrt{n\theta(1-\theta)}; \theta)}{u} - \hat{\delta}_{n,k}(u; \theta) \right| du + \int_{(2k-1)\pi}^{(2k+1)\pi} \left| \frac{\phi(u; \theta)}{u} - \hat{\delta}_{n,k}(u; \theta) \right| du
\]
\[
=: J_k^{(1)}(n; \theta) + J_k^{(2)}(n; \theta)
\]
To analyze \(J_k^{(1)}(n; \theta)\), we write
\[
J_k^{(1)}(n; \theta) \leq \left( \sum_{r=-\infty}^{\infty} + \sum_{r=-\infty}^{-(k+2)} + \sum_{r=\{-(k+1)\}} + \sum_{r=\{-k+1\}} + \sum_{r=-k+2}^{-1} \right) \int_{(2k-1)\pi}^{(2k+1)\pi} \exp \left\{-\frac{n\theta(1-\theta)}{2} (u + 2\pi r)^2 \right\} \left[ 1 + \frac{1}{6} n\theta(1-\theta)|u + 2\pi r|^3 \right] du
\]
with the proviso that both the fourth and the fifth sum are void when \(k = 1\), and that the fifth sum is void when \(k = 2\).
To deal with the series in \(\hat{D}_n\) limited to \(r \in \mathbb{N}\), we observe that \((u + 2\pi r)^2 \geq u^2 + 4\pi^2 r^2\)
if \(u \in [(2k-1)\pi, (2k+1)\pi]\) and we take account that \(|z_1 + z_2|^3 \leq 4(|z_1|^3 + |z_2|^3)\), so that we deduce, for the series at issue, the upper bound
\[
\sum_{r=1}^{\infty} \frac{1}{2\pi r} e^{-2\pi n\theta(1-\theta)r^2} \int_{(2k-1)\pi}^{(2k+1)\pi} \exp \left\{-\frac{n\theta(1-\theta)}{2} u^2 \right\} \left[ 1 + \frac{2}{3} n\theta(1-\theta)(u^3 + 8\pi^3 r^3) \right] du
\]

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At this stage, recalling (35), we conclude that the sum over the index \( k \) of the last expression is majorized by

\[
\sum_{r=1}^{\infty} \frac{1}{2\pi r} e^{-2\pi^2 n\theta(1-\theta)r^2} \int_0^{\infty} \exp \left\{ -\frac{n\theta(1-\theta)}{2} u^2 \right\} \left[ 1 + \frac{2}{3} n\theta(1-\theta)(u^3 + 8\pi^3 r^3) \right] \, du
\]

\[
= \sum_{r=1}^{\infty} \frac{1}{2\pi r} e^{-2\pi^2 n\theta(1-\theta)r^2} \left[ \frac{1}{2} \sqrt{\frac{2\pi}{n\theta(1-\theta)}} + \frac{4}{3n\theta(1-\theta)} + \frac{8\sqrt{2\pi}}{3} \pi^3 r^3 \sqrt{n\theta(1-\theta)} \right].
\]

Then, we use \( x^p e^{-x} \leq (p/e)^p \), valid for any \( x, p > 0 \), with \( p = \frac{1}{2} + \varepsilon(\gamma), \varepsilon(\gamma), \frac{3}{2} + \varepsilon(\gamma) \), respectively, and \( \varepsilon(\gamma) := \frac{1-\gamma}{2} \), to produce the global bound \( S_1(\gamma)[n\theta(1-\theta)]^{-(1+\varepsilon(\gamma))} \) for the last series, where

\[
S_1(\gamma) := \frac{1}{2\pi} \left[ \sqrt{\frac{2\pi}{2\pi^2 e}} \left( \frac{\frac{3}{2} + \varepsilon(\gamma)}{2\pi^2 e} \right)^{\frac{1}{2} + \varepsilon(\gamma)} \zeta(2[1 + \varepsilon(\gamma)]) + \frac{4}{3} \left( \frac{\varepsilon(\gamma)}{2\pi^2 e} \right)^{\varepsilon(\gamma)} \zeta(1 + 2\varepsilon(\gamma)) \right]
\]

\[
+ \frac{8\pi^3 \sqrt{2\pi}}{3} \left( \frac{\frac{3}{2} + \varepsilon(\gamma)}{2\pi^2 e} \right)^{\frac{1}{2} + \varepsilon(\gamma)} \zeta(1 + 2\varepsilon(\gamma))
\]

\( \zeta(\cdot) \) denoting the Riemann zeta function. Now, we come back to (36) and we consider the remaining four sums. The change of variable \( u = s + 2k\pi \) in the integral and the inequality \( |z_1 + z_2|^3 \leq 4(|z_1|^3 + |z_2|^3) \) lead us to rewrite the expression inside the sums in (36) as

\[
\frac{1}{2\pi k} \int_{-\pi}^{\pi} \exp \left\{ -\frac{n\theta(1-\theta)}{2} [s + 2\pi(k + r)]^2 \right\} ds. \tag{37}
\]

Therefore, for the second series in (36), relative to the set \( r \leq -(k + 2) \), we have

\[
\sum_{r=\infty}^{k+2} \frac{1}{2\pi r} \left[ 1 + 6\pi^3 n\theta(1-\theta) r^3 \right] \int_{-\pi}^{\pi} \exp \left\{ -\frac{n\theta(1-\theta)}{2} [s + 2\pi(k + r)]^2 \right\} ds
\]

\[
\leq \frac{1}{2\pi k} \sum_{h=2}^{\infty} \left[ 1 + 6\pi^3 n\theta(1-\theta) h^3 \right] \int_{-\pi}^{\pi} \exp \left\{ -\frac{n\theta(1-\theta)}{2} (s - 2\pi h)^2 \right\} ds.
\]

After noticing that \( (s - 2\pi h)^2 \geq s^2 + 2\pi^2 h^2 \) for \( s \in [-\pi, \pi] \) and \( h \geq 2 \), we get the new upper bound

\[
\frac{1}{2\pi k} \int_{-\pi}^{\pi} \exp \left\{ -\frac{n\theta(1-\theta)}{2} s^2 \right\} ds \times \sum_{h=2}^{\infty} \left[ 1 + 6\pi^3 n\theta(1-\theta) h^3 \right] e^{-\pi^2 n\theta(1-\theta) h^2}
\]

which is less or equal than \( \frac{1}{k} S_2(\gamma)[n\theta(1-\theta)]^{-(1+\varepsilon(\gamma))} \) where, by another application of \( x^p e^{-x} \leq (p/e)^p \) for \( p = \frac{1}{2} + \varepsilon(\gamma) \) and \( 2 + \varepsilon(\gamma) \), respectively,

\[
S_2(\gamma) := \frac{1}{2\pi} \left[ \sqrt{2\pi} \left( \frac{\frac{3}{2} + \varepsilon(\gamma)}{\pi^2 e} \right)^{\frac{1}{2} + \varepsilon(\gamma)} \zeta(1 + 2\varepsilon(\gamma)) + 12\pi^4 \left( \frac{\frac{3}{2} + \varepsilon(\gamma)}{\pi^2 e} \right)^{2+\varepsilon(\gamma)} \zeta(2[1 + \varepsilon(\gamma)]) \right].
\]
For \( r = -(k + 1) \) the expression in (37) is majorized by
\[
\frac{1 + 6\pi^3 n\theta(1 - \theta)}{k + 1} \exp \left\{ -\frac{\pi^2 n\theta(1 - \theta)}{2} \right\}
\]
which is, in turn, less or equal than \( \frac{1}{k+1} S_3(\gamma) [n\theta(1 - \theta)]^{-(1+\varepsilon(\gamma))} \) with
\[
S_3(\gamma) := \left( \frac{2[1 + \varepsilon(\gamma)]}{\pi^2 e} \right)^{1+\varepsilon(\gamma)} + 6\pi^3 \left( \frac{2[2 + \varepsilon(\gamma)]}{\pi^2 e} \right)^{2+\varepsilon(\gamma)}.
\]
Analogously, for any \( k \geq 2 \), the expression in (37) with \( r = -k + 1 \) is majorized by
\[
\frac{1 + 6\pi^3 n\theta(1 - \theta)}{k - 1} \exp \left\{ -\frac{\pi^2 n\theta(1 - \theta)}{2} \right\}
\]
which is, in turn, less or equal than \( \frac{1}{k-1} S_3(\gamma) [n\theta(1 - \theta)]^{-(1+\varepsilon(\gamma))} \). Finally, for \( k \geq 3 \), it remains to provide an upper bound for the sum of the expression (37) as \( r \) varies from \(-k + 2\) to \(-1\). Changing the variable in the sum, according to \( h = k + r \), we obtain the equivalent expression
\[
\sum_{h=2}^{k-1} \frac{1}{2\pi(k-h)} [1 + 6\pi^3 n\theta(1 - \theta)h^3] \int_{-\pi}^{\pi} \exp \left\{ -\frac{n\theta(1 - \theta)}{2} (s + 2\pi h)^2 \right\} ds
\]
which is majorized by virtue of the inequality \((s + 2\pi h)^2 \geq s^2 + 2\pi^2 h^2\), valid for any \( s \in [-\pi, \pi] \) and \( h \geq 2 \). Then, we arrive at
\[
\int_{-\pi}^{\pi} \exp \left\{ -\frac{n\theta(1 - \theta)}{2} s^2 \right\} ds \times \sum_{h=2}^{k-1} e^{-\pi^2 n\theta(1 - \theta)h^2} \frac{1}{2\pi(k-h)} [1 + 6\pi^3 n\theta(1 - \theta)h^3]
\]
and we now realize that, in view of (35), we can exchange the order of summation according to \( \sum_{k=3}^{n(\pi;\theta)+1} \sum_{h=2}^{k-1} \sum_{r=1}^{\pi(n;\theta)+1} \). At this stage, for the inner sum, we have
\[
\sum_{k=h+1}^{n(\pi;\theta)+1} \frac{1}{k-h} \leq \sum_{r=1}^{\pi(n;\theta)} \frac{1}{r} \leq \frac{1}{2} \log \left( \frac{n}{\theta(1 - \theta)} \right).
\]
Using this upper bound, we pass to the outer sum, which is majorized by
\[
\int_{-\pi}^{\pi} \exp \left\{ -\frac{n\theta(1 - \theta)}{2} s^2 \right\} ds \times \frac{1}{4\pi} \log \left( \frac{n}{\theta(1 - \theta)} \right) \sum_{h=2}^{\infty} e^{-\pi^2 n\theta(1 - \theta)h^2} [1 + 6\pi^3 n\theta(1 - \theta)h^3].
\]
The series in the above expression has been already treated above, yielding the further upper bound
\[
\frac{S_2(\gamma)}{2} \log \left( \frac{n}{\theta(1 - \theta)} \right) [n\theta(1 - \theta)]^{-(1+\varepsilon(\gamma))}.
\]
Therefore, gathering all the bounds that follow formula (36), we get
\[
\tau(n,\theta) + 1 \sum_{k=1}^{(n+\theta)/2} \Phi_k^{(n)}(n;\theta) \leq \frac{S_1(\gamma)}{n\theta(1-\theta)^{1+\varepsilon(\gamma)}} + \frac{S_2(\gamma) + S_3(\gamma)}{n\theta(1-\theta)^{1+\varepsilon(\gamma)}} \log \left( \frac{n}{\theta(1-\theta)} \right). \quad (38)
\]

Now, we pass to analyze the integrals $\Phi_k^{(2)}$'s. We start again from the change of variable $u = s + 2k\pi$ and we exploit the fact that $\phi(s + 2k\pi; \theta) = \phi(s; \theta)e^{-2\pi ik\theta}$, to obtain
\[
\int_{(2k-1)\pi}^{(2k+1)\pi} \left| \frac{[\phi(u; \theta)]^n - \delta_n k(u; \theta)}{u} \right| \, du = \int_{-\pi}^{\pi} \left| [\phi(s; \theta)]^n - (1 + \frac{s}{2k\pi}) \exp \left\{ -\frac{n\theta(1-\theta)}{2} s^2 \right\} \left[ 1 + \frac{1-2\theta}{6n\theta(1-\theta)(is)^3} \right] \right| \, ds.
\]

The last integral is majorized by
\[
\frac{1}{\pi(2k-1)} \pi \int_{-\pi}^{\pi} \left| [\phi(s; \theta)]^n - \exp \left\{ -\frac{n\theta(1-\theta)}{2} s^2 \right\} \left[ 1 + \frac{1-2\theta}{6n\theta(1-\theta)(is)^3} \right] \right| \, ds + \frac{1}{\pi(2k-1)} \int_{0}^{\infty} s \exp \left\{ -\frac{n\theta(1-\theta)}{2} s^2 \right\} \left[ 1 + \frac{\pi}{6n\theta(1-\theta)s^2} \right] \, ds. \quad (39)
\]

For the first summand in (39), we change again the variable according to $s = \xi / \sqrt{n\theta(1-\theta)}$ to obtain the equality with
\[
\frac{1}{\pi(2k-1)\sqrt{n\theta(1-\theta)}} \int_{-T_1(n,\theta)}^{T_1(n,\theta)} \left| \hat{\Phi}_n(\xi; \theta) - \hat{\Phi}_n(\xi; \theta) \right| \, d\xi
\]
which can be bounded, as before, the splitting the above integral as
\[
\int_{-T_2(n,\theta)/4}^{T_2(n,\theta)/4} \left| \hat{\Phi}_n(\xi; \theta) - \hat{\Phi}_n(\xi; \theta) \right| \, d\xi + 2 \int_{T_2(n,\theta)/4}^{T_1(n,\theta)} \left| \hat{\Phi}_n(\xi; \theta) \right| \, d\xi + 2 \int_{T_2(n,\theta)/4}^{T_1(n,\theta)} \left| \hat{\Phi}_n(\xi; \theta) \right| \, d\xi.
\]

In fact, it is now crucial to observe that the expression
\[
T_3(n,\theta,\delta) := \frac{1}{4} \sqrt{n} \left( \frac{\theta(1-\theta)^{3+\delta}/2}{\theta(1-\theta)(1-\theta)^{2+\delta} + \theta(2+\delta)} \right)^{1/(1+\delta)},
\]
corresponding to the limitation for $|\xi|$ given in Lemma 7 when the $V_n$'s are i.i.d., centered Bernoulli variables, is not less than $T_2(n,\theta)/4$, by virtue of the Hölder inequality. In fact,
it is enough to observe that, for a centered r.v. $V$, we have $\sqrt{\sigma^2/\beta_4} \leq (\sigma^{3+\delta}/\beta_{3+\delta})^{1/(1+\delta)}$ where $\sigma^2 := \mathbb{E}[V^2]$ and $\beta_s := \mathbb{E}[|V|^s]$. Therefore, we can apply Lemma 7 with $\delta = 2(\gamma) = 1 - \gamma$, to get

$$\int_{-T_2(n,\theta)/4}^{T_2(n,\theta)/4} |\hat{B}_n(\xi; \theta) - \hat{V}_n(\xi; \theta)| d\xi \leq \frac{\lambda_{10}}{[n\theta(1-\theta)]^{1/2+\epsilon(\gamma)}}.$$

Since an analogous bound is in force also for $\int_{-T_2(n,\theta)/4}^{T_2(n,\theta)/4} |\hat{V}_n(\xi; \theta)| d\xi$ and $\int_{-T_2(n,\theta)/4}^{T_2(n,\theta)/4} |\hat{B}_n(\xi; \theta)| d\xi$, in view of the argument already used to prove (33)-(34), we conclude that

$$\sum_{k=1}^{\tau(n,\theta)+1} \frac{1}{\pi(2k-1)} \int_{-\pi}^{\pi} |[\phi(s; \theta)]^n - \exp \left\{ -\frac{n\theta(1-\theta)}{2} s^2 \right\} \cdot \left[ 1 + \frac{1 - 2\theta}{6} n\theta(1-\theta)(is)^2 \right] | \, ds \leq \frac{\lambda_{11}}{[n\theta(1-\theta)]^{1+\epsilon(\gamma)}} \log \left( \frac{n}{\theta(1-\theta)} \right).$$

As to the latter summand in (39), it is enough to notice that it equals

$$\frac{1}{\pi^2 k(2k-1)} \left( 2 + \frac{\pi}{12} \right) \frac{1}{n\theta(1-\theta)}$$

yielding in the end that

$$\sum_{k=1}^{\tau(n,\theta)+1} 3^{(1)}_k(n; \theta) \leq \frac{\lambda_{12}}{n\theta(1-\theta)} + \frac{\lambda_{13}}{[n\theta(1-\theta)]^{1+\epsilon(\gamma)}} \log \left( \frac{n}{\theta(1-\theta)} \right). \quad (40)$$

At this stage, we notice that, for any $\eta > 0$, $[\theta(1-\theta)]^n \log \left( \frac{1}{\theta(1-\theta)} \right)$ is bounded by a constant which depends only on $\eta$, and we can choose $\eta = \eta(\gamma) = (1-\gamma)/4$. Then, we collect (28)-(29)-(32)-(33)-(34)-(35)-(38)-(40) to draw the important conclusion that

$$\int_0^1 \epsilon_n(\theta) f(\theta) d\theta \leq \lambda_{14} \frac{\|f\|_\infty + |f|_{1,\gamma}}{n} \quad (41)$$

holds with a suitable constant $\lambda_{14}$ which is independent of $n$ and $f$, thanks to point i) in Lemma 6 and

$$\int_0^1 \frac{1}{[\theta(1-\theta)]^{1+\epsilon(\gamma)+\eta(\gamma)}} f(\theta) d\theta \leq \frac{2^{3+\gamma+\epsilon(\gamma)+\eta(\gamma)}}{1-\gamma} M(f).$$

Therefore, the achievement of the bound (41) concludes the first part of the proof, culminating in the validity of (4) with a suitable constant $C(\mu)$ proportional to $1 + \|f\|_\infty + |f|_{1,\gamma}$, under the additional hypothesis $f(0) = f(1) = 0$, thanks to the combination of (17)-(18), the two bound $9M(f)/n$ for both $F_n(\pi_{n,\gamma})$ and $1 - F_n(1 - \pi_{n,\gamma})$, and (21)-(22)-(23)-(25)-(26)-(41). For completeness, we note that we have proved (4) only for $n \geq N_\ast := \ldots$
max\{4, n_0, N(\gamma), |\pi^2/4| + 1\}, but now it is immediate to extend the validity of (4) to all the set of positive integer: we just add the term N_*/n to the right-hand side of (4) and we rename the new constant as C(\mu).

After proving the theorem under the additional hypothesis \( f(0) = f(1) = 0 \), we show how to get rid of this extra-condition. First, we assume that \( f \) is given by a polynomial, with generic values of \( f(0) \) and \( f(1) \), and we apply Lemma [5]. Since \( F_n(x) = P[S_n \leq nx] = \int_0^1 P[S_n \leq nx | Y = \theta] \mu(d\theta) \), we obtain \( F_n(x) = A_\infty F_{\infty,n}(x) + A_+ F_{+,n}(x) - A_- F_{-,n}(x) \) for all \( x \in [0,1] \), where \( F_{*,n}(x) := \int_0^1 P[S_n \leq nx | Y = \theta] f_*(\theta) d\theta \), for \( * = \infty, + \) and \(-\), respectively. Whence,

\[
d_K(\mu_n; \mu) \leq A_\infty \sup_{x \in [0,1]} |F_{\infty,n}(x) - F_\infty(x)| + A_+ \sup_{x \in [0,1]} |F_{+,n}(x) - F_+(x)| + A_- \sup_{x \in [0,1]} |F_{-,n}(x) - F_-(x)| \tag{42}
\]

where \( F_*(x) := \int_0^x f_*(\theta) d\theta \) for all \( x \in [0,1] \) and \(* = \infty, + \) and \(-\), respectively. At this stage, from Theorem 2 in [21], we can find a constant \( C(f_\infty) \), proportional to \( 1 + \|f_\infty\|_\infty + \|f'_\infty\|_\infty \), such that \( \sup_{x \in [0,1]} |F_{\infty,n}(x) - F_\infty(x)| \leq C(f_\infty)/n \) is in force for all \( n \in \mathbb{N} \). For the last two terms on the right-hand side of (42), since \( f_\pm(0) = f_\pm(1) = 0 \), the problem is traced back to the first part of the proof. Hence, the inequality (4) holds for all \( n \in \mathbb{N} \) with a constant \( C(\mu) \) proportional to \( 1 + \|f\|_\infty + |f|_{1,\gamma} \), thanks to the bounds provided in points i)-ii)-iii) of Lemma [7].

The final act consists in removing the regularity of \( f \) by some approximation arguments. First, we start from a probability density \( f \) belonging to \( C^1([0,1]) \) and we consider an approximating family of probability densities \( f^{(\delta)} \) expressed by a polynomial which converges to \( f \) uniformly with the first derivative, as \( \delta \to 0 \). See, e.g., [19] for classical results about this kind of approximation. Since \( \|f^{(\delta)}\|_\infty \to \|f\|_\infty \) and \( \|f^{(\delta)}\|_{1,\gamma} \to |f|_{1,\gamma} \) are obvious, we pass to analyze the behavior of \( d_K(\mu_n; \mu) \) under the approximation. After fixing \( n \), for any \( x \not\in \{0, \frac{1}{n}, \ldots, 1\} \), we have

\[
|F_n(x) - F(x)| = \lim_{\delta \to 0} |F_n^{(\delta)}(x) - F^{(\delta)}(x)| \leq \lim_{\delta \to 0} C \frac{1 + \|f^{(\delta)}\|_\infty + |f^{(\delta)}|_{1,\gamma}}{n} = C \frac{1 + \|f\|_\infty + |f|_{1,\gamma}}{n}
\]

where the inequality follows from the previous argument. This relation entails the validity of (4) for all \( n \in \mathbb{N} \) and \( f \in C^1([0,1]) \), with a constant \( C(\mu) \) proportional to \( 1 + \|f\|_\infty + |f|_{1,\gamma} \). Finally, the removal of the \( C^1([0,1]) \)-regularity follows by standard arguments based on the convolution of a regularizing kernel. \( \square \)
2.3 Proof of Proposition 4

By resorting once again to formula (7), we get

\[ d_K(\mu_n; \mu) \leq \sup_{x \in [0,1)} 1 \int_0^1 \beta(\theta; \lfloor nx \rfloor + 1, n - \lfloor nx \rfloor) |F(x) - F(\theta)| d\theta \]

\[ \leq H_\gamma(F) \sup_{x \in [0,1)} 1 \int_0^1 \beta(\theta; \lfloor nx \rfloor + 1, n - \lfloor nx \rfloor) |x - \theta|^{-\gamma} d\theta \]

where \( H_\gamma(F) \) denotes the Hölder constant of \( F \). Now, we exploit that \(|x - \theta|^{-\gamma} \leq |x - \eta_x|^{-\gamma} + |\eta_x - \theta|^{-\gamma}\), where \( \eta_x := \frac{\lfloor nx \rfloor + 1}{n+1} = \int_0^1 \beta(\theta; \lfloor nx \rfloor + 1, n - \lfloor nx \rfloor) d\theta \), to get

\[ d_K(\mu_n; \mu) \leq H_\gamma(F) \left[ \sup_{x \in [0,1)} |x - \eta_x|^{-\gamma} + \sup_{x \in [0,1)} 1 \int_0^1 |\eta_x - \theta|^{-\gamma} \beta(\theta; \lfloor nx \rfloor + 1, n - \lfloor nx \rfloor) d\theta \right]. \]

Since \( \sup_{x \in [0,1)} |x - \eta_x| \leq \frac{2}{n+1} \) follows from direct computation, we can focus on the second summand on the above right-hand side, which can be bounded by means of the Jensen inequality as follows:

\[ \int_0^1 |\eta_x - \theta|^{-\gamma} \beta(\theta; \lfloor nx \rfloor + 1, n - \lfloor nx \rfloor) d\theta \leq \left( \int_0^1 |\eta_x - \theta|^{-2\gamma} \beta(\theta; \lfloor nx \rfloor + 1, n - \lfloor nx \rfloor) d\theta \right)^{\gamma/2}. \]

The proof is completed by observing that the integral \( \int_0^1 |\eta_x - \theta|^{-2\gamma} \beta(\theta; \lfloor nx \rfloor + 1, n - \lfloor nx \rfloor) d\theta \) represents the variance of the beta distribution with parameters \((\lfloor nx \rfloor + 1, n - \lfloor nx \rfloor)\), which, being equal to \( \frac{(\lfloor nx \rfloor + 1)(n - \lfloor nx \rfloor)}{(n+1)^2(n+2)} \), is less than \( \frac{1}{n+2} \) for any \( x \in [0,1] \).

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