Probabilistic Automata over Infinite Words:
Expressiveness, Efficiency, and Decidability*

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Probabilistic $\omega$-automata are variants of nondeterministic automata for infinite words where all choices are resolved by probabilistic distributions. Acceptance of an infinite input word can be defined in different ways: by requiring that (i) the probability for the accepting runs is positive (probable semantics), or (ii) almost all runs are accepting (almost-sure semantics), or (iii) the probability measure of the accepting runs is greater than a certain threshold (threshold semantics). The underlying notion of an accepting run can be defined as for standard $\omega$-automata by means of a Büchi condition or other acceptance conditions, e.g., Rabin or Streett conditions. In this paper, we put the main focus on the probable semantics and provide a summary of the fundamental properties of probabilistic $\omega$-automata concerning expressiveness, efficiency, and decision problems.

1 Introduction

While classical finite automata can serve to recognize languages over finite discrete structures, $\omega$-automata are acceptors for languages consisting of infinite objects. They have been applied in various research areas, including the verification of reactive systems and reasoning about infinite games and decision problems for certain logics. Many variants of $\omega$-automata have been studied in the literature that can be classified according to their inputs (e.g., words or trees), their acceptance conditions (e.g., Büchi, Rabin, Streett, Muller or parity acceptance) and their branching structure (e.g., deterministic, nondeterministic, or alternating). We refer to [19, 7] for an overview of automata over infinite objects.

Probabilistic variants of $\omega$-automata for languages over infinite words have been recently introduced. Their syntax is roughly the same as for probabilistic finite automata (PFA) [15, 14], i.e., they are finite-state automaton where for each state $q$ and input letter a probability distribution specifies the probabilities for the successor states. Furthermore, they are equipped with an acceptance condition as in nondeterministic $\omega$-automata. The accepted language of a probabilistic $\omega$-automata can be defined by imposing a condition on the acceptance probability for the input words. Under the probable semantics, acceptance of an infinite word $\sigma = a_1 a_2 a_3 \ldots$ requires that the generated sample run for $\sigma$ (i.e., sequence of states that are passed in the automaton while reading $\sigma$ letter by letter) meets the acceptance condition with positive probability. The probable semantics is in the spirit of nondeterministic automata where the accepted words are those words that have at least one accepting run. The almost-sure semantics of a probabilistic $\omega$-automata can be understood as the probabilistic counterpart to universal automata as it requires that the

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accepting runs have probability measure 1, i.e., almost all runs are accepting. The \textit{threshold semantics}
follows the concept of PFA and deals with a fixed threshold $\lambda \in [0, 1]$ and classifies an input word $\sigma$ to be
accepted if the probability of the accepting runs for $\sigma$ is greater than $\lambda$.

The different semantics yield different classes of recognizable languages over infinite words. Most
powerful is the threshold semantics which covers the class of $\omega$-regular languages, but also non-$\omega$-
regular languages. Given the results for PFA which are known to be more expressive than standard
finite automata, this is not surprising. While PFA with the probable semantics agree with ordinary non-
deterministic automata, probabilistic automata with Büchi or other standard acceptance conditions and
the probable semantics are strictly more expressive than their nondeterministic counterparts. Furthermore,
there are languages $L_n$ that are recognizable by probabilistic Büchi automata of linear size, while
smallest nondeterministic $\omega$-automata for $L_n$ have exponentially many states. For nondeterministic $\omega$-
automata it is well-known that Büchi acceptance is as powerful as, e.g., Streett or Rabin acceptance,
but the transformations from nondeterministic Streett automata to nondeterministic Büchi automata can
cause an exponential blow-up \cite{17}. In contrast, there is a polynomial transformation from probabilistic
Büchi to probabilistic Streett automata, both under the probable semantics. Concerning the standard
composition operators (union, intersection and complementation), the class of languages that are rec-
ognizable by probabilistic $\omega$-automata under the probable semantics enjoys the same properties as the
class of $\omega$-regular languages. Both are closed under all three operators. Union and intersection can easily
be realized by means of sum and product constructions, respectively. Complementation, however, is
“difficult” and relies on a complex powerset construction that can cause an exponential blow-up. The
price we have to pay for the extra power of probabilistic $\omega$-automata under the probable semantics is
that all relevant decision problems (checking emptiness, universality or equivalence) are undecidable.
The undecidability results for PBA have several important consequences. First, the concept of PBA is
not adequate for solving algorithmic problems that are related to the emptiness or universality problems.
This, e.g., applies to the verification of nondeterministic systems against PBA-specifications. Second,
PBA can be viewed as a special instance of partially-observable Markov decision processes (POMDPs)
which are widely used in various areas, including robotics and stochastic planning (see, e.g., \cite{18,13,10})
and the negative results established for PBA yield the undecidability of various verification problems for
POMDPs.

For probabilistic Büchi automata with the almost-sure semantics we obtain a completely different
picture. They are less powerful and even do not cover the full class of $\omega$-regular languages, but still can
accept languages that are not $\omega$-regular. However, the emptiness and universality problem are decidable
for them. Furthermore, the class of languages that can be accepted by an almost-sure PBA is closed
under union and intersection, but not under complementation.

\textbf{Organization.} In Section\ref{section2} we briefly recall the definition of nondeterministic $\omega$-automata with Büchi,
Rabin or Streett acceptance conditions and introduces their probabilistic variants and the probable,
almost-sure and threshold semantics. The following three sections mainly deal with probabilistic automata
under the probable semantics. Results on the expressiveness and efficiency of probabilistic Büchi,
Rabin and Streett automata are summarized in Section\ref{section3}. Composition operators for PBA under the probable
semantics are addressed in Section\ref{section4}. Decision problems for PBA and the relation to POMDPs will
be discussed in Section\ref{section5}. Section\ref{section6} summarizes the main results for the almost-sure and threshold
semantics. Finally, Section\ref{section7} contains some concluding remarks.

The material of this paper is a summary of the results presented in the papers \cite{3,2}. Further details
can be found there and in the thesis by Marcus Größer \cite{8}. 
2 Nondeterministic and probabilistic \(\omega\)-automata

Throughout the paper, we assume some familiarity with classical nondeterministic automata over finite or infinite words and refer to [19, 7] for details. We first recall some basic concepts of nondeterministic \(\omega\)-automata and then adapt these concepts to the probabilistic setting.

Definition 1 Nondeterministic \(\omega\)-automata.

A nondeterministic \(\omega\)-automaton is a tuple \(\mathcal{N} = (Q, \Sigma, \delta, Q_0, \text{Acc})\), where

- \(Q\) is a finite nonempty set of states,
- \(\Sigma\) is a finite nonempty input alphabet,
- \(\delta : Q \times \Sigma \rightarrow 2^Q\) is a transition function,
- \(Q_0 \subseteq Q\) is the set of initial states,
- \(\text{Acc}\) is an acceptance condition (which will be explained below).

\(\mathcal{N}\) is called deterministic if \(|Q_0| = 1\) and \(|\delta(q, a)| = 1\) for all \(q \in Q\) and \(a \in \Sigma\).

The intuitive operational behavior of a nondeterministic \(\omega\)-automaton \(\mathcal{N}\) for an infinite input word \(\sigma = a_1 a_2 a_3 \ldots \in \Sigma^\omega\) is as follows. The computation starts in a nondeterministically chosen initial state \(q_0 \in Q_0\). Then, \(\mathcal{N}\) attempts to read the first letter \(a_1\) in state \(q_0\). If \(q_0\) does not have an outgoing \(a_1\)-transition (i.e., \(\delta(q_0, a_1) = \emptyset\)) then the automaton rejects. Otherwise, the automaton reads the first letter \(a_1\) and chooses nondeterministically some state \(q_1 \in \delta(q_0, a_1)\). It then attempts to read the remaining word \(a_2 a_3 \ldots\) from state \(q_1\). That is, the automaton rejects if \(\delta(q_1, a_2) = \emptyset\). Otherwise the automaton reads letter \(a_2\) and moves to some state \(q_2 \in \delta(q_1, a_2)\), and so on. Any maximal state-sequence \(\pi = q_0 q_1 q_2 \ldots\) that can be obtained in this way is called a run for \(\sigma\). We write \(\text{inf}(\pi)\) to denote the set of states \(p \in Q\) that appear infinitely often in \(\pi\). Each finite run \(q_0 q_1 \ldots q_l\) (where \(\mathcal{N}\) fails to read letter \(a_{l+1}\) in the last state \(q_l\) because \(\delta(q_l, a_{l+1})\) is empty) is said to be rejecting. The acceptance condition \(\text{Acc}\) imposes a condition on infinite runs and declares which of the infinite runs are accepting. Several acceptance conditions are known for nondeterministic \(\omega\)-automata. We will consider three types of acceptance conditions:

Büchi: A Büchi acceptance condition \(\text{Acc}\) is a subset \(F\) of \(Q\). The elements in \(F\) are called final or accepting states. An infinite run \(\pi = q_0 q_1 q_2 \ldots\) is called (Büchi) accepting if \(\pi\) visits \(F\) infinitely often, i.e., \(\text{inf}(\pi) \cap F \neq \emptyset\).

Streett: A Streett acceptance condition \(\text{Acc}\) is a finite set of pairs \((H_l, K_l)\) consisting of subsets \(H_l, K_l\) of \(Q\), i.e., \(\text{Acc} = \{(H_1, K_1), \ldots, (H_\ell, K_\ell)\}\). An infinite run \(\pi = q_0 q_1 q_2 \ldots\) is called (Streett) accepting if for each \(l \in \{1, \ldots, \ell\}\) we have: \(\text{inf}(\pi) \cap H_l \neq \emptyset\) or \(\text{inf}(\pi) \cap K_l = \emptyset\).

Rabin: A Rabin acceptance condition \(\text{Acc}\) is syntactically the same as a Streett acceptance condition, i.e., a finite set \(\text{Acc} = \{(H_1, K_1), \ldots, (H_\ell, K_\ell)\}\) where \(H_l, K_l \subseteq Q\) for \(1 \leq l \leq \ell\). An infinite run \(\pi = q_0 q_1 q_2 \ldots\) is called (Rabin) accepting if there is some \(l \in \{1, \ldots, \ell\}\) such that \(\text{inf}(\pi) \cap H_l = \emptyset\) and \(\text{inf}(\pi) \cap K_l \neq \emptyset\).

Using LTL-like notations, a Streett condition can be understood as a strong fairness condition and a Rabin condition as its dual.

\[
\bigwedge_{1 \leq l \leq \ell} (\Box H_l) \quad \text{(Streett)}
\]

\[
\bigvee_{1 \leq l \leq \ell} (\Box K_l \land \Box \neg H_l) \quad \text{(Rabin)}
\]
Clearly, a Büchi acceptance condition $F$ can be viewed as a special case of a Streett and Rabin condition with a single acceptance pair, namely $\{(F, Q)\}$ for the Streett condition and $\{((\emptyset, F))\}$ for the Rabin condition.

The accepted language of a nondeterministic $\omega$-automaton $N$ with the alphabet $\Sigma$, denoted $\mathcal{L}(N)$, is defined as the set of infinite words $\sigma \in \Sigma^\omega$ that have at least one accepting run in $N$.

$$\mathcal{L}(N) \overset{\text{def}}{=} \{ \sigma \in \Sigma^\omega : \text{there exists an accepting run for } \sigma \text{ in } N \}$$

In what follows, we write NBA to denote a nondeterministic Büchi automaton, NRA for nondeterministic Rabin automata and NSA for nondeterministic Streett automata. Similarly, the notations DBA, DRA and DSA are used to denote deterministic $\omega$-automata with a Büchi, Rabin or Streett acceptance condition.

It is well-known that the classes of languages that can be accepted by NBA, DRA, NRA, DSA or NSA are the same. These languages are often called $\omega$-regular and represented by $\omega$-regular expressions, i.e., finite sums of expressions of the form $\alpha \beta^\omega$ where $\alpha$ and $\beta$ are ordinary regular expressions (representing regular languages over finite words) and the language associated with $\beta$ is nonempty and does not contain the empty word. In the sequel, we will identify $\omega$-regular expressions with the induced $\omega$-regular language.

While deterministic $\omega$-automata with Rabin and Streett acceptance (DRA and DSA) cover the full class of $\omega$-regular languages, DBA are less powerful as, e.g., the language $(a+b)^*a^\omega$ cannot be recognized by a DBA. Hence, the class of DBA-recognizable languages is a proper subclass of the class of $\omega$-regular languages.

Probabilistic $\omega$-automata can be viewed as nondeterministic $\omega$-automata where the transition function $\delta$ specifies probabilities for the successor states. That is, for any state $p$ and letter $a \in \Sigma$ either $p$ does not have any $a$-successor or there is a probability distribution for the $a$-successors of $p$.

**Definition 2 Probabilistic $\omega$-automata.** A probabilistic $\omega$-automaton is a tuple $P = (Q, \Sigma, \delta, \mu_0, \text{Acc})$, where

- $Q$ is a finite nonempty set of states,
- $\Sigma$ is a finite nonempty input alphabet,
- $\delta : Q \times \Sigma \times Q \rightarrow [0, 1]$ is a transition probability function such that for all $p \in Q$ and $a \in \Sigma$ we have: $\sum_{q \in Q} \delta(p, a, q) \in \{0, 1\}$,
- $\mu_0 : Q \rightarrow [0, 1]$ is the initial distribution, i.e., $\sum_{q \in Q} \mu_0(q) = 1$,
- $\text{Acc}$ is an acceptance condition (as for nondeterministic $\omega$-automata).

We refer to the states $q_0 \in Q$ where $\mu_0(q_0) > 0$ as initial states. If $p$ is a state such that $\delta(q, a, p) > 0$ then we say that $q$ has an outgoing $a$-transition to state $p$.

Acceptance conditions can be defined as in the nondeterministic case. In this paper, we just regard Büchi, Rabin and Streett acceptance and use the abbreviations PBA, PRA and PSA for probabilistic Büchi automata, probabilistic Rabin automata, and probabilistic Streett automata, respectively.

The intuitive operational behavior of a probabilistic $\omega$-automaton $P$ for a given input word $\sigma = a_1 a_2 \ldots \in \Sigma^\omega$ is similar to the nondeterministic setting, except that all choices are resolved probabilistically: the initial state is chosen according to the initial distribution $\mu_0$, and if $q$ is the current state and $a$ the next input letter then $P$ moves with probability $\delta(q, a, p)$ to state $p$. If there is no outgoing $a$-transition from $q$, i.e., if $\sum_{p \in Q} \delta(q, a, p) = 0$, then $P$ rejects. As in the nondeterministic case, the resulting state-sequence $\pi = p_0 p_1 p_2 \ldots \in Q^* \cup Q^\omega$ is called a run for $\sigma$ in $P$. Acceptance of a run according
to a Büchi, Rabin or Streett acceptance condition is defined as in the nondeterministic setting. While acceptance of an infinite word in a nondeterministic ω-automata requires the existence of an accepting run, a probabilistic ω-automaton accepts an infinite input word σ if the probability for the generated sample run to be accepting is “sufficiently large”.

### Acceptance probability and accepted language.

Given an infinite word $σ \in Σ^ω$, the acceptance probability $Pr^P(σ)$ for $σ$ in $P$ denotes the probability measure of the accepting runs for $σ$ in $P$. The formal definition of the acceptance probability relies on the view of an input word $σ \in Σ^ω$ as a scheduler when $P$ is treated as a Markov decision process, i.e., an operational model for a probabilistic system where in definition of the acceptance probability.

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We can now apply the standard concepts for Markov chains to reason about the probabilities of infinite paths and define the acceptance probability for the infinite word $σ$ in $P$, denoted $Pr^P(σ)$ or briefly $Pr(σ)$, as the probability measure of the accepting runs for $σ$ in the Markov chain $M_σ$.

For the definition of the accepted language, we distinguish three semantics for probabilistic ω-automata. The probable semantics assigns to $P$ the set of infinite words $σ$ such that the accepting runs for $σ$ have positive measure. Under the almost-sure semantics a word $σ$ is accepted by $P$ if almost all runs for $σ$ are accepting. (The formulation “almost all runs have property $X$” means that the probability measure of the runs where property $X$ does not hold is 0.) The threshold semantics relies on a fixed threshold $λ$ that serves as strict lower bound for the acceptance probability for all accepted words:

\[
L^>0(P) \overset{\text{def}}{=} \{ σ \in Σ^ω : Pr^P(σ) > 0 \}
\]

\[
L^=1(P) \overset{\text{def}}{=} \{ σ \in Σ^ω : Pr^P(σ) = 1 \}
\]

\[
L^>λ(P) \overset{\text{def}}{=} \{ σ \in Σ^ω : Pr^P(σ) > λ \}
\]

Equivalence of ω-automata means that their accepted languages agree. The notion of the size, denoted $|P|$, of an ω-automaton $P$ is used here as follows. The size of a PBA is simply the number of states. The size of a probabilistic Rabin or Streett automaton denotes the number of states plus the number of acceptance pairs.

**Example 3 Probabilistic Büchi automata (PBA).** In the pictures for PBA we attach the probabilities $δ(q,a,p)$ to the $a$-labeled edge from $q$ to $p$, provided that $0 < δ(q,a,p) < 1$. An $a$-labeled edge from $q$ to $p$ without any probability value indicates that $δ(q,a,p) = 1$ (in which case $p$ is the unique $a$-successor of $q$). Similarly, the initial distribution is depicted by attaching the value $μ₀(q)$ to an arrow pointing to $q$, provided that $q$ is an initial state and $μ₀(q) < 1$. For PBA, we depict the accepting states (i.e., the states $q ∈ F$) by squares, non-accepting states by circles. The PBA $P$ over the alphabet $Σ = \{a,b\}$ shown in the left part of Figure has a single initial state $q₀$. Its Büchi condition is given by $F = \{q₁\}$.

Let us first observe that each word $σ$ that is accepted by $P$ must be contained in the language $(a+b)^*a^ω$ of the NBA that results from $P$ by ignoring the probabilities. Indeed all words with only
consume the next letter and almost surely contain three or fewer.

This yields e. g., for threshold 0.1

Pr and σ = PrP (σ) = 1. Suppose now that σ contains at least one b and let k be the total number of a’s that appear before the last b in σ, i.e., if σ = c1 ... cℓbaω then k = |{i ∈ {1, ..., ℓ} : ci = a}]. With probability 2−k, the current state is q0 after reading c1 ··· cℓ. But then P can read b and will almost surely move to q1 when reading the suffix aω. Thus, PrP (σ) = 2−k which yields that σ ∈ L1 (P) iff k = 0 (i.e., if σ ∈ b*ω) and that all words in (a+b)*ω belong to £0 (P).

Figure 1: Examples for PBA P (left) and P′ (right)

Regard the PBA P′ over the alphabet Σ = {a,b,c} shown in the right part of Figure 1. Let us first observe that the underlying nondeterministic Büchi automaton (NBA) that we obtain by ignoring the probabilities has an accepting run for each infinite word in (ab+ac)ω with infinitely many b’s, no matter whether there are only finitely c’s or infinitely many c’s. Thus, the accepted language of the NBA is ((ac)*ab)ω. This language is different from the accepted language of the PBA P′ under the probable and almost-sure semantics:

L0 (P′) = (ab+ac)* (ab)ω, L1 (P′) = (ab)ω.

Clearly, all accepted words σ ∈ L0 (P′) belong to ((ac)*ab)ω. Any word σ in (ab+ac)ω with infinitely many c’s is rejected by P′ as almost all runs for σ are finite and end in state p1, where the next input symbol is c and cannot be consumed in state p1. Thus, L0 (P′) ⊆ (ab+ac)* (ab)ω. Given an input word σ ∈ (ab+ac)* (ab)ω, say σ = x(ab)ω where x ∈ (ab+ac)∗, then with positive probability P′ generates the run fragment p0p1p0p2p0p1p0 when reading x. For the remaining suffix (ab)ω, P′ can always consume the next letter and almost surely P′ will visit p1 and p2 infinitely often. This yields PrP′ (σ) > 0 and σ ∈ L0 (P′).

Clearly, we have L1 (P′) ⊆ L0 (P′). Using an argument as above, it is clear that no word in L1 (P′) contains letter c. The runs for the word (ab)ω will almost surely visit state p1 infinitely often. This yields L1 (P′) = (ab)ω.

The precise acceptance probability for σ ∈ {a,b,c}ω is as follows. If σ ∉ (ab+ac)* (ab)ω then PrP′ (σ) = 0. If σ ∈ (ab+ac)* (ab)ω and letter c appears k times in σ then PrP′ (σ) = 2−k. Thus, e.g., for threshold 0.1, the accepted language L0,1 (P′) consists of all words σ ∈ (ab+ac)* (ab)ω that contain three or fewer c’s.

Figure 1: Examples for PBA P (left) and P′ (right)
3 Expressiveness and efficiency of PBA

In the following three sections, we put the focus on probabilistic $\omega$-automata with the probable semantics. Results for the almost-sure and threshold semantics are summarized in Section 6. Unless stated differently, we simply say PBA to denote a PBA with the probable semantics.

We start with a discussion on the expressiveness and efficiency of PBA compared to their nondeterministic counterparts. At the end of this section, we will show that as in the nondeterministic case, Büchi acceptance is as powerful as Streett and Rabin acceptance.

**PBA and $\omega$-regular languages.** DBA can be viewed as special instances of PBA (we just have to assign probability 1 to all edges in the DBA and deal with the initial distribution that assigns probability 1 to the unique initial state). As the language $(a+b)^*a^\omega$ is recognizable by a PBA with the probable semantics (see Example 3), PBA are strictly more expressive than DBA, i.e., the class of DBA-recognizable languages is a proper subclass of the class of languages $L^>0(P)$ for some PBA $P$. Indeed all $\omega$-regular languages can be represented by a PBA with the probable semantics:

**Lemma 4** From NBA to PBA under the probable semantics. For each NBA $N$ there exists a PBA $P$ such that $L^>0(P) = L(N)$.

**Proof:** A transformation from NBA into an equivalent PBA is obtained by using NBA that are deterministic-in-limit. These are NBA such that $\delta(q,a,p) \in \{0,1\}$ for all states $p$ and $q$ that are reachable from some accepting state and all letters $a \in \Sigma$. That is, as soon as an accepting state has been reached the behavior from then on is deterministic. Courcoubetis and Yannakakis [6] presented some kind of powerset construction which turns a given NBA $N$ into an equivalent NBA $N_{\text{det}}$ that is deterministic-in-limit. If we now resolve the nondeterministic choices in $N_{\text{det}}$ by uniform distributions then $N_{\text{det}}$ becomes a PBA that accepts the same language as $N$ (and $N_{\text{det}}$).

We now address the question whether each PBA can be transformed into an equivalent NBA. Surprisingly, this is not the case, as there are PBA where the accepted language is not $\omega$-regular. An example for a PBA $P = P_{\lambda}$ where the accepted language under the probable semantics is not $\omega$-regular is given in Figure 2.

![Figure 2: PBA $P_{\lambda}$ accepts a non-$\omega$-regular language](image)

Here, $\lambda$ is an arbitrary real number in the open interval $]0,1[$.

**Lemma 5.** The language of the PBA $P_{\lambda}$ under the probable semantics is not NBA-recognizable, i.e., $L^>0(P)$ is not $\omega$-regular.

\[1\text{If } q \text{ is a state in } N_{\text{det}} \text{ and } a \in \Sigma \text{ such that } q \text{ has } k \text{ } a\text{-successors } q_1, \ldots, q_k \text{ then we define } \delta(q,a,q_i) = \frac{1}{k} \text{ for } 1 \leq i \leq k \text{ and } \delta(q,a,p) = 0 \text{ for all states } p \notin \{q_1, \ldots, q_k\}. \text{ Similarly, if } Q_0 \text{ is the set of initial states in } N_{\text{det}} \text{ and } Q_0 \text{ is nonempty then we deal with the initial distribution } \mu_0 \text{ that assigns probability } 1/|Q_0| \text{ to each state in } Q_0.\]
Proof: The PBA $P_\lambda$ accepts the language

$$L^{>0}(P_\lambda) = \left\{ a^{k_1}ba^{k_2}ba^{k_3}b\ldots : \prod_{i=1}^\infty (1 - \lambda^{k_i}) > 0 \right\}.$$ 

The convergence condition which requires the infinite product over the values $1 - \lambda^{k_i}$ to be positive can easily be shown to be non-$\omega$-regular, i.e., $L^{>0}(P_\lambda)$ cannot be recognized by an NBA.

To see that, indeed, $L^{>0}(P_\lambda)$ agrees with the above language, let us first observe that all words in $L^{>0}(P_\lambda)$ must contain infinitely many $b$’s. Note that if a input word $\sigma$ ends with the suffix $a^\omega$ then almost all infinite runs for $\sigma$ will eventually enter state $q_1$ and stay there forever. As $P_\lambda$ cannot consume two consecutive $b$’s, all words in $L^{>0}(P_\lambda)$ have the form $a^{k_1}ba^{k_2}ba^{k_3}b\ldots$ where $k_1,k_2,\ldots$ is a sequence of positive natural numbers. We now show that

$$\Pr^{P_\lambda}(a^{k_1}ba^{k_2}ba^{k_3}b\ldots) = \prod_{i=1}^\infty (1 - \lambda^{k_i}).$$

The factors $1 - \lambda^{k_i}$ stand for the probability to move from state $q_0$ to $q_1$ when reading the subword $a^{k_i}$. With the remaining probability $\lambda^{k_i}$, the automaton $P_\lambda$ stays in state $q_0$, but then letter $b$ at position $k_1 + \cdots + k_i + i$ of the input word $a^{k_1}ba^{k_2}ba^{k_3}b\ldots$ cannot be consumed and $P_\lambda$ rejects. Hence, the probability for run fragments of the form $q_0\ldots q_0q_1\ldots q_0q_1$ that are generated while reading the subword $a^{k_i}b$ is precisely $1 - \lambda^{k_i}$. This yields that the infinite product of these values agrees with the acceptance probability for the input word $a^{k_1}ba^{k_2}ba^{k_3}b\ldots$. \hfill \square

As a consequence of Lemma 4 and Lemma 5 we get that PBA with the probable semantics are more powerful than NBA. This result should be contrasted to the case of finite automaton where the probable semantics turns PFA into ordinary NFA, and thus, PFA with the probable semantics represent exactly the class of $\omega$-regular languages.

Corollary 6. The class of languages that are accepted by a PBA strictly subsumes the class of $\omega$-regular languages.

The PBA $P_\lambda$ can also serve to illustrate that the probable semantics is sensitive to modifications of the transition probabilities. Consider two values $\lambda$ and $\nu \in [0,1]$ with $\lambda < \nu$. For any sequence $(k_i)_{i\geq 1}$ of natural numbers $k_i$ where the infinite product over the values $1 - \lambda^{k_i}$ converges to some positive value, also the infinite product over the values $1 - \nu^{k_i}$ is positive, as we have $1 - \nu^{k_i} < 1 - \lambda^{k_i}$. Thus, $L^{>0}(P_\nu) \subseteq L^{>0}(P_\lambda)$. However, whenever $\lambda < \nu$ then $L^{>0}(P_\nu)$ is a proper sublanguage $L^{>0}(P_\lambda)$ as there are sequences $(k_i)_{i\geq 1}$ such that the product of the values $1 - \lambda^{k_i}$ converges to some positive real number, while the product of the values $1 - \nu^{k_i}$ has value 0. Hence:

Lemma 7. If $\lambda < \nu$ then $L^{>0}(P_\nu) \neq L^{>0}(P_\lambda)$.

Thus, the languages of PBA are sensitive to the distributions for the successor states. That is, if we are given a PBA and modify the nonzero transition probabilities then also the accepted language might change. This property is surprising since the definition of the accepted language just relies on a qualitative criterion: the acceptance probability must be positive, but might be arbitrarily small. This should be opposed to the verification of finite-state Markov decision processes where it is known that whether or not a given linear time property holds with positive probability just depends on the underlying graph, but not on the concrete transition probabilities.
Lemma 8. Each NSA for $L_n$ has $2^n/\omega$ states under the initial distribution. All states are accepting. (Thus, any infinite run in $L_n$ is exponential. In fact, in the worst-case, the exponential blow-up cannot be avoided for the transformation from NBA to PBA as there are families $(L_n)_{n \geq 1}$ of $\omega$-regular languages that are accepted by NBA of linear size, while each PBA for $L_n$ has $\Omega(2^n)$ states. An example for such a family of languages is

$$L_n = ((a+b)^*a(a+b)^n)c^\omega.$$ 

Language $L_n$ is recognizable by an NBA with $n+1$ states which guesses nondeterministically for any word position $i$ where the input word contains an $a$ whether letter $c$ will appear at word position $i+n$. Since there is no upper bound on the distance between the word positions of the $c$’s in the words in $L_n$, any PBA for $L_n$ needs to store the positions of letter $a$ among the last $n$ letters (see [3]). Hence, the size of any PBA for $L_n$ is exponential. Vice versa, there are also examples for $\omega$-regular languages where probabilism allows for a more compact representation than nondeterminism. Let

$$L_n' = \{ xy' : x,y \in \{a,b\}^*, |y| = n \}.$$

Lemma 8. Each NSA for $L_n'$ has $2^n/n$ or more states in each NSA for $L_n'$, while there exist PBA $P_n$ consisting of $O(n)$ states with $L(n)P_n = L_n'$.

Proof: The lower bound $2^n/n$ for the number of states in any NSA for $L_n'$ is obtained by verifying that given two words $y = c_1c_2\ldots c_n$ and $z = d_1d_2\ldots d_n$ of length $n$ such that

$$d_1d_2\ldots d_n \notin \{ c_i c_{i+1} \ldots c_n c_1 \ldots c_{i-1} : 1 \leq i \leq n \}$$

then the “accepting cycles” for the words $y^\omega$, $z^\omega \in L_n'$ do not intersect.

It remains to show the existence of PBA of linear size for $L_n'$. Let $P_n$ be the following PBA. The states of $P_n$ are $1_a, \ldots, n_a, 1_b, \ldots, n_b$. Thus, $P_n$ has $2n$ states. States $1_a$ and $1_b$ are initial, both have probability 0.5 under the initial distribution. All states are accepting. (Thus, any infinite run in $P_n$ is accepting.) $P_n$ has the following transitions. From any state $k_a$ with $1 \leq k < n$ there is an $a$-transition to state $(k+1)_a$ and a $b$-transition to state $(k+1)_b$. All these transitions have probability 0.5. All states, except for state $n_b$, have an $a$-transition to state $1_a$. These transitions have probability 0.5, except for the transition from $n_a$ to $1_a$ which has probability 1. Similarly, from any state $k_b$ with $1 \leq k < n$ there is an $a$-transition and a $b$-transition to state $(k+1)_b$ with probability 0.5. All states, except for state $n_a$, have a $b$-transition to state $1_b$ with probability 0.5 except for state $n_b$ which has a $b$-transition to $1_b$ with probability 1.

The idea of this construction is as follows. While scanning an infinite input word

$$\sigma = c_1c_2c_3\ldots \in \{a,b\}^\omega,$$

$P_n$ chooses at random word positions $i$ by moving to state $1_a$ (if $c_i = a$) or state $1_b$ (if $c_i = b$) and checks whether $c_{n+i} = c_i$ via following the path

$$1_i \xrightarrow{c_{i+1}} 2_{c_i} \xrightarrow{c_{i+2}} \ldots \xrightarrow{c_{i+n-1}} n_{c_i}$$

and rejecting (if $c_{n+i} \neq c_i$) or returning to state $1_i$ (if $c_{n+i} = c_i$) and choosing the next word position $j$, and so on. If $\sigma \notin L_n'$, then there are infinitely many word positions $i$ such that $c_{n+i} \neq c_i$ and almost surely $P_n$ will pick such a word position and reject in state $n_{c_i}$. If $\sigma = c_1c_2c_3\ldots \in L_n'$, then there exists some index $\ell$ such that $c_i = c_{n+i}$ for all $i \geq \ell$. After reading the $\ell$-th letter, $P_n$ will be in state $1_{c_\ell}$ with probability $\geq 2^{-\ell}$. From then on, $P_n$ will never reject and the resulting runs are accepting. Hence, $P_n^{P_n}(\sigma) > 0$. \qed
Streett and Rabin acceptance. The three types of probabilistic $\omega$-automata (Büchi, Rabin, Streett) are equally expressive. As the Büchi acceptance condition can be rewritten as a Rabin or Streett acceptance condition, each PBA can be viewed as a PRA or as a PSA with the same accepted language. But we can establish a stronger result stating that each PBA can be transformed into a 0/1-PRA which means a PRA $P_R$ such that for each word $\sigma$, the acceptance probability for $\sigma$ is either 0 or 1. This result can be viewed as the probabilistic analogue to the well-known fact that each NBA can be transformed into an equivalent deterministic Rabin automaton. The idea for this transformation is to design a 0/1-PRA $P_R$ that generates up to $n$ sample runs of $P$ and checks whether at least one of them is accepting, where $n$ is the number of states in $P$. If so then $P_R$ accepts, otherwise it rejects. For the details of this construction we refer to [2, 8].

Theorem 9 From PBA to 0/1-PRA. For each PBA $P$ there exists a 0/1-PRA $P_R$ such that $L^{>0}(P) = L^{>0}(P_R)$.

Vice versa, there are polynomial transformations from PRA and PSA to PBA:

Theorem 10 Polynomial transformations from PBA to PRA and PSA.

(a) Given a PRA $P_R$ with $\ell$ acceptance pairs there exists a PBA $P$ of size $O(\ell |P_R|)$ such that $L^{>0}(P) = L^{>0}(P_R)$.

(b) Given a PSA $P_S$ with $\ell$ acceptance pairs there exists a PBA of size $O(\ell^2 |P_S|)$ such that $L^{>0}(P) = L^{>0}(P_S)$.

The transformation from PRA to PBA is roughly the same as in the nondeterministic case. The construction of a PBA of size $O(\ell^2 |P_S|)$ from a given PSA $P_S$, however, crucially relies on the probabilistic semantics. In fact, it is worth noting that in the nonprobabilistic case it is known (see [17]) that there are families $(L_n)_{n \geq 0}$ of languages $L_n \subseteq \Sigma^\omega$ that are recognizable by nondeterministic Streett automata of size $O(n)$, while each nondeterministic Büchi automaton for $L_n$ has $2^n$ or more states. Thus, the polynomial transformation from Streett to Büchi acceptance is specific for the probabilistic case.

4 Composition operators for PBA

The most important composition operators for any class of languages over infinite words are the standard set operations union, intersection and complementation. In fact, the class of PBA-recognizable languages is closed under all three operations.

Theorem 11. The class of languages $L^{>0}(P)$ for some PBA $P$ is closed under union, intersection and complementation.

Given two PBA $P_1$ and $P_2$ over the same alphabet with initial distributions $\mu_1$ and $\mu_2$, respectively, then a PBA $P$ for the language $L^{>0}(P_1) \cup L^{>0}(P_2)$ can be obtained by the disjoint union of $P_1$ and $P_2$ with the initial distribution $\mu(q) = \frac{1}{2}\mu_i(q)$ if $q$ is a state in $P_i$. If $F_1$ and $F_2$ are the sets of accepting states in $P_1$ and $P_2$, respectively, then $P$ requires to visit $F_1 \cup F_2$ infinitely often.

An operator for PBA with the probable semantics that realizes intersection can be designed by reusing ideas that are known for NBA. Given two PBA $P_1$ and $P_2$ over the same alphabet, we use a product construction $P_1 \times P_2$ (which runs $P_1$ and $P_2$ in parallel) and equip $P_1 \times P_2$ with a Streett acceptance condition consisting of two acceptance pairs. One of the acceptance pairs requires that an accepting
state of $P_1$ is visited infinitely often, the other one stands for the acceptance condition of $P_2$. This PSA $P_1 \times P_2$ can then be transformed into an equivalent PBA (part (b) of Theorem 10).

The most interesting operator is complementation. Given a PBA $P$ with $L = \mathcal{L}^{>0}(P) \subseteq \Sigma^\omega$, the idea for the construction of a PBA $\overline{P}$ for the language $L = \Sigma^\omega \setminus L$ is somehow similar to the complementation of NBA via Safra’s determinisation operator [16] and relies on the transformations sketched in Figure 3.

In the first step we apply the transformation mentioned in Theorem 9, while the last step relies on part (b) of Theorem 10. Recall that a 0/1-PRA denotes a PRA $P_R$ where the acceptance probabilities for all words are 0 or 1, i.e., $\Pr^{P_R}(\sigma) \in \{0, 1\}$ for each word $\sigma \in \Sigma^\omega$. Thus, $\mathcal{L}^{>0}(P_R) = \mathcal{L}^{-1}(P_R)$ and for transforming the 0/1-PRA $P_R$ into a 0/1-PSA $P_S$ for the complement of $\mathcal{L}^{>0}(P_R)$ we may simply use the duality of Rabin and Streett acceptance. That is, syntactically $P_R$ and $P_S$ agree (but $P_S$ is viewed as a Streett and $P_R$ as a Rabin automaton). The size of the resulting PBA $\overline{P}$ for $L$ can be exponentially larger than the size of $P$ due to the powerset construction used in the generation of a 0/1-PRA.

5 Decision problems for PBA

For many applications of automata-like models, it is important to have (efficient) decision algorithms for some fundamental problems, like checking emptiness or language inclusion. For instance, the automata-based approach [20] for verifying $\omega$-regular properties of a nondeterministic finite-state system relies on a reduction to the emptiness problem for NBA. Unfortunately, the emptiness problem and various other classical decision problems for automata cannot be solved algorithmically for PBA:

**Theorem 12 Undecidability of PBA.** The following problems are undecidable:

- **emptiness:** given a PBA $P$, does $\mathcal{L}^{>0}(P) = \emptyset$ hold?
- **universality:** given a PBA $P$ with the alphabet $\Sigma$, does $\mathcal{L}^{>0}(P) = \Sigma^\omega$ hold?
- **equivalence:** given two PBA $P_1$ and $P_2$, does $\mathcal{L}^{>0}(P_1) = \mathcal{L}^{>0}(P_2)$ hold?

To prove undecidability of the emptiness problem, we provided in [2] a reduction from a variant of the emptiness problem for probabilistic finite automata (PFA) which has been shown to be undecidable [11]. Undecidability of the universality problem then follows by the effectiveness of complementation for PBA. Undecidability of the PBA-equivalence problem is an immediate consequence of the undecidability of the emptiness problem (just consider $P_1 = P$ and $P_2$ a PBA for the empty language).

A consequence of Theorem 12 is that PBA are not appropriate for verification algorithms. Consider, e.g., a finite-state transition system $T$ and suppose that a linear-time property $p$ to be verified for $T$ is specified by a PBA $P$ in the sense that $\mathcal{L}^{>0}(P)$ represents all infinite behaviors where property $p$ holds. (Typically $p$ is a language over some alphabet $\Sigma = 2^{AP}$ where AP is a set of atomic propositions and the states in $T$ are labeled with subsets of AP.) Then, the question whether all traces of $T$ have property $p$ is reducible to the universality problem for PBA and therefore undecidable. Similarly, the question whether $T$ has at least one trace where $p$ holds is reducible to the emptiness problem for PBA and therefore undecidable too.
Another important consequence of Theorem 12 is that it yields the undecidability of the verification problem for partially observable Markov decision processes (POMDPs) against $\omega$-regular properties. POMDPs provide an operational model for stochastic systems with non-observable behaviors. They play a central role in many application areas such as mobile robot navigation, probabilistic planning task, elevator control, and so on. See, e.g., [18,12,13,10]. The syntax of a POMDP can be defined as for probabilistic $\omega$-automata, except that the acceptance condition has to be replaced with an equivalence relation $\sim$ on the states which formalizes which states cannot be distinguished from outside. The elements in the alphabet $\Sigma$ are viewed as action names. The goal is then to design a scheduler $S$ that chooses the actions for the current state and ensures that a certain condition $X$ holds when the choices between different enabled actions in the POMDP $M$ are resolved by $S$. For his choice the scheduler may use the sequence of equivalence classes that have been passed to reach the equivalence class of the current state. That is, the scheduler is supposed to observe the equivalence classes, but not the specific states. (Such schedulers are sometimes called “partial-information schedulers” or “observation-based schedulers”.)

The emptiness problem for PBA is a special instance for the scheduler-synthesis problem for POMDPs. Given a PBA $P = (Q, \Sigma, \delta, \mu_0, F)$, we regard the POMDP $M = (Q, \Sigma, \delta, \mu_0, \sim)$ where $\sim$ identifies all states and ask for the existence of a scheduler that ensures that $F$ will be visited infinitely often with positive probability. We first observe that the infinite words over $\Sigma$ can be viewed as schedulers for $M$, and vice versa. Hence, $L^{>0}(P)$ is nonempty if and only if there is a scheduler $S$ such that $Pr_S^M(\Box \Diamond F) > 0$, where $Pr_S^M(\Box \Diamond F)$ denotes the probability that $M$ visits $F$ infinitely often when $S$ is used to schedule the actions in $M$. Similarly, the universality problem for PBA can be viewed as a special instance of the problem where we are given a POMDP $M$ and a set $F$ of states and ask for the existence of a scheduler $S$ such that $Pr_S^M(\Diamond \Box F) = 1$ where $Pr_S^M(\Diamond \Box F)$ denotes the probability that $M$ under scheduler $S$ eventually enters $F$ and never leaves $F$ from this moment on. Thus:

**Theorem 13** Undecidability results for POMDPs.

The following problems are undecidable:

- given a POMDP $M$ and a set $F$ of states, decide whether $\exists S. Pr_S^M(\Box \Diamond F) > 0$,
- given a POMDP $M$ and a set $F$ of states, decide whether $\exists S. Pr_S^M(\Diamond \Box F) = 1$.

The result of Theorem 13 is remarkable since the corresponding questions for fully observable Markov decision processes (i.e., POMDPs where the $\sim$-equivalence classes are singletons) are decidable in polynomial time.

### 6 The almost-sure and threshold semantics

So far, we concentrated on the probable semantics of probabilistic $\omega$-automata. We will briefly summarize the main results on the almost-sure and threshold semantics.

PBA with the almost-sure semantics are less expressive than PBA with the probable semantics. They even do not cover the full class of $\omega$-regular languages. For instance, the $\omega$-regular language $(a + b)^* a^\omega$ cannot be recognized by a PBA with the almost-sure semantics. Since the complement $(a + b)^\omega$ of this language is recognizable by a deterministic Büchi automaton (and therefore also by a PBA with the almost-sure semantics), PBA with the almost-sure semantics are not closed under complementation. Furthermore, there are PBA where the almost-sure semantics yields a non-$\omega$-regular language. An example is the language

$$ L = \left\{ a^{k_1} b a^{k_2} b a^{k_3} b \ldots : \prod_{i=1}^{\infty} \left( 1 - \lambda^{k_i} \right) = 0 \right\} $$
which can be shown to be recognizable by a PBA with the almost-sure semantics. However, the class of languages $L^{=1}(P)$ for some PBA $P$ is closed under union and intersection. For PBA with the almost-sure semantics, the emptiness and universality problem are decidable. Indeed one can even show that given a POMDP $M$ and a set $F$ of states in $M$ then the questions

- does there exists a scheduler $S$ such that $Pr^M_\mathcal{S}(\square F) = 1$?
- does there exists a scheduler $S$ such that $Pr^M_\mathcal{S}(\Diamond F) > 0$?

are decidable by a certain powerset construction. Using the above mentioned fact that PBA can be viewed as special instances of POMDPs, one obtains the decidability of the emptiness and universality problem for PBA with the almost-sure semantics.

It should be noticed that the above results on the almost-sure semantics are specific for the Büchi acceptance condition. For Rabin or Streett acceptance, the almost-sure semantics is as expressive as the probable semantics. This is a consequence of Theorems 9 and 10 which show that PRA with the almost-sure semantics are as expressive as PRA (and PBA) with the probable semantics. Thus, the emptiness, universality and equivalence problems for PRA with the almost-sure semantics are undecidable.

The threshold semantics is more powerful than the probable semantics. Indeed for each PBA $P$ and threshold $\lambda$ there exists a PBA $P'$ such that $L^{>\lambda}(P) = L^{>\lambda}(P')$. Furthermore, there are transformations to stretch and relax acceptance probabilities which yields that whenever $\lambda, \nu \in [0, 1]$ and $P$ is a PBA then there exists a PBA $P'$ such that $L^{>\lambda}(P) = L^{>\nu}(P')$. That is, all thresholds define the same class of languages. Using known results on the expressiveness of probabilistic finite automata (PFA) [15, 14], one can show that there are threshold languages $L^{>\lambda}(P)$ that cannot be recognized by PBA with the probable semantics. The undecidability of all relevant algorithmic problems for PBA with the threshold semantics is clear from the undecidability of corresponding problems for PFA [11]. As far as we know, closure properties under composition operators have not yet been studied for PBA with the threshold semantics.

7 Conclusion

We gave a summary of the fundamental properties of probabilistic acceptors for infinite words formalized by probabilistic $\omega$-automata with Büchi, Rabin or Streett acceptance conditions. The results show some major differences to nondeterministic (or alternating) $\omega$-automata concerning the expressiveness, efficiency and decidability.

Beside being of theoretical interest, we believe that PBA could be useful in several application areas. We briefly sketched the connection between probabilistic $\omega$-automata and POMDPs. Since PBA arise as special instance of POMDPs all negative results for PBA (undecidability) carry over from PBA to POMDP. Vice versa, it seems that for many algorithmic problems for POMDPs, algorithmic solutions for probabilistic $\omega$-automata (e.g., PBA with the almost-sure semantics) can be combined with standard algorithms for (fully observable) Markov decision processes to obtain an algorithm that solves the analogous problem for POMDPs. Another application of probabilistic $\omega$-automata is run-time verification where special types of PBA can serve as probabilistic monitors [4]. Given the wide range of application areas of probabilistic finite automata, there might be various other applications of probabilistic $\omega$-automata. For instance, the concept of probabilistic $\omega$-automata is also related to partial-information games with $\omega$-regular winning objectives [5] or could serve as starting point for studying quantum automata over infinite inputs, in the same way as PFA yield the basis for the definition of quantum finite automata [9, 1].
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