The Graph of Annihilating Ideals

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ABSTRACT

Let $R$ be a commutative ring with identity and $AG(R)$ be the set of ideals with non-zero annihilators. The annihilating ideal graph $AG(R)$ is a graph of vertex set $AG(R)\backslash\{(0)\}$ and two distinct ideal vertices $I$ and $J$ are adjacent if and only if $IJ=(0)$. In this paper, we establish a new fundamental properties of $AG(R)$ as well as its connection with $\Gamma(R)$.

Keywords: Annihilating ideal graph, zero divisor graph, reduced rings, finite local rings, rings integer modulo $n$

1. Introduction:

Let $R$ be a commutative ring with identity, and let $Z(R)$ be its set of zero divisors. We associate a simple graph $\Gamma(R)$ to $R$ with vertices $Z^*(R)=Z(R)\backslash\{(0)\}$, the set of all non-zero zero divisors of $R$, and for distinct $x,y \in Z^*(R)$, the vertices $x$ and $y$ are adjacent if and only if $xy=0$. Thus, $\Gamma(R)$ is empty graph iff $R$ is an integral domain.

Beck introduced the concept of zero divisor graph of a commutative ring in [4]. In the recent years zero divisor graph have been extensively studied by many authors in [1,2,3,8].

An ideal $I$ of $R$ is said to be annihilating ideal if there exists a non-trivial ideal $J$ of $R$ such that $IJ=(0)$. Let $AG(R)$ be the set of annihilating ideals of $R$. The annihilating
ideal graph $AG(R)$ is a graph with vertex set $AG^*(R)=AG(R)\backslash\{(0)\}$ such that there is an edge between vertices $I$ and $J$ if and only if $I\neq J$ and $IJ=(0)$. The idea of annihilating ideal graph was introduced by Behboodi and Rakeei in [5] and [6].

In the present paper, we investigate the annihilating ideal graph $AG(R)$. We establish a new of its basic properties and its relation of $\Gamma(R)$.

Recall that:
1. $R$ is called reduced if $R$ has no non-zero nilpotent element.
2. The distance $d(u,v)$ between two vertices $u$ and $v$ of a connected graph $\Gamma$ is the minimum of the lengths of the $u-v$ paths of $\Gamma$ [7].
3. The degree of the vertex $a$ in the graph $\Gamma$ is the number of edges of $\Gamma$ incident with $a$ [7].
4. The graph $\Gamma$ is called a plane graph if it can be drawn in the plane with their edges crossing. A graph which is an isomorphic to a plane graph is called a planer graph [7].
5. A graph $\Gamma$ is bipartite graph, if it is possible to partition the vertex set of $\Gamma$ into two subsets $V_1$ and $V_2$ such that every element of edges of $\Gamma$ joins a vertex of $V_1$ to a vertex of $V_2$. A complete bipartite graph with partite sets $V_1$ and $V_2$ where, $|V_1|=m$ and $|V_2|=n$, is then denoted by $K_{m,n}$ [7].

2. Annihilating ideal graph:
In this section, we consider annihilating ideal graph, we give some of its basic properties and provide some examples.

Definition 2.1[5]: Let $R$ be a ring and let $I$ and $J$ are distinct non-trivial ideals of $R$. Then, $I$ and $J$ are adjacent ideal vertices in $AG(R)$ if $IJ=(0)$.

From now on, we shall use the symbol $I \rightarrow J$ to denote for two adjacent ideal vertices $I$ and $J$. We start this section with the following example.

Example 1: Let $Z_{24}$ be the ring of integers modulo 24. The graph $AG(Z_{24})$ can be drawn as follows:

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(3) --- (8)   (6)   (4)
(2) --- (12)
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The following result is an easy consequence of definition of 2.1.

Lemma 2.2: If $I$ and $J$ are non-trivial ideals of $R$ such that $I \cap J = (0)$, then $I \rightarrow J$ is an edge of $AG(R)$ and $I \cup J \subseteq Z(R)$.

The converse of Lemma 2.2 is not true in general, as the following example shows.

Example 2: Let $Z_{12}$ be the ring of integers modulo 12. Then, $(2) \rightarrow (6)$ is an edge of the graph $AG(Z_{12})$, but $(2) \cap (6) \neq (0)$.

We now give a sufficient condition for the converse of Lemma 2.2 to be true.

Proposition 2.3: Let $R$ be a reduced ring, and let $I \rightarrow J$ be an edge in $AG(R)$. Then, $I \cap J = (0)$.
Proof: Let $a \in I \cap J$. Then, $a \in I$ and $a \in J$, this implies that $a^2 \in IJ = (0)$, so $a^2 = 0$. Since, $R$ is a reduced ring, then $a = 0$. Therefore, $I \cap J = (0)$. ■

The next result illustrates that the distance of any two nilpotent ideal vertices of $AG(R)$ is at most 2.

Theorem 2.4: Let $I$ and $J$ be two ideal vertices of $AG(R)$. If either $I$ or $J$ is a nilpotent, then $d(I, J) \leq 2$.

Proof: Let $d(I, J) = 3$. Then, there is a path from $I$ to $J$ in $AG(R)$ say $I \overset{L}{\rightarrow} K \overset{L}{\rightarrow} J$. Let $I$ be a nilpotent ideal of $R$. Then, there exists an integer $n > 1$ such that $I^n = (0)$. Consider the sequence $L, LI, LI^2, \ldots, LI^n$. Let $m$ be the smallest integer in which $LI^m \neq (0)$. Hence, $LI^{m+1} = (0)$. Obviously, $LI^m$ adjacent to both $I$ and $J$. This contradicts the fact that $d(I, J) = 3$. Therefore, $d(I, J) \leq 2$. ■

The next result illustrates the degree of a vertex adjacent to the set of zero divisors of $R$.

Proposition 2.5: Let $R$ be a finite ring and let $Z(R)$ be an ideal of $R$. If $I \overset{Z}{\rightarrow} Z(R)$ is an edge in $AG(R)$, then $deg(I) = |AG(R)| - 1$.

Proof: Suppose that $I \overset{Z}{\rightarrow} Z(R)$ is an edge in $AG(R)$, it follows that $I \cap Z(R) = (0)$. Let $J$ be any vertex of $AG(R)$. Then, by Lemma 2.2, $J$ is a subset of $Z(R)$. This implies that $I \cap J = (0)$. Thus, $I$ is adjacent to all vertices of $AG(R)$. This means that $deg(I) = |AG(R)| - 1$. ■

Example 3: Let $Z_{16}$ be the ring of integers modulo 16. The vertices of $AG(Z_{16})$ are $I = (8)$, $J = (4)$ and $K = (2) = Z(Z_{16})$. Clearly, $deg(I) = deg(J) = |AG(Z_{16})| = 3 - 1$.

![Diagram](I J K)

The next result considers the adjacency of two minimal ideals in the graph $AG(R)$.

Proposition 2.6: Every two distinct minimal ideals of $R$ are adjacent in $AG(R)$.

Proof: Let $M$ and $N$ be two distinct minimal ideals of $R$. Since, $M$ and $N$ contain $MN$, then $MN = M = N$ or $MN = (0)$. The first case is not true because $M$ and $N$ are distinct ideals. Thus, $MN = (0)$. This means that $M$ and $N$ are adjacent vertices in $AG(R)$. ■

Example 4: Let $Z_{18}$ be the ring of integers modulo 18. Clearly, the minimal ideals of $Z_{18}$ are (6) and (9), which are adjacent vertices in $AG(Z_{18})$.

![Diagram](3 6 9 2)

The next result considers the number of minimal ideals of $R$.

Theorem 2.7: If $AG(R)$ is a planar graph, then $R$ has at most four minimal ideals.

Proof: Suppose that $R$ has five minimal ideals say $M_1, M_2, M_3, M_4$ and $M_5$. By Proposition 2.6, any two of $M_1, M_2, M_3, M_4$ and $M_5$ are adjacent. This means that
\(AG(R)\) contains the complete graph \(K_5\). This is contradiction that \(AG(R)\) is a planar graph (See the Kuratowsky Theorem in [7]). Therefore, \(R\) has at most four minimal ideals.

Example 5: Let \(Z_{16}\) be the ring of integers modulo 16. Clearly, the graph \(AG(Z_{16})\) is a planar graph and the only minimal ideal of \(Z_{16}\) is (8).

3. The graphs \(\Gamma(R)\) and \(AG(R)\)

In this section, we consider the relationship between \(\Gamma(R)\) and \(AG(R)\).

It is natural to ask whether \(\Gamma(R)\) and \(AG(R)\) are isomorphic, the answer is negative, as the following example shows.

Example 6: Let \(Z_{12}\) be the ring of integer modulo 12. Then, the number of vertices of \(\Gamma(Z_{12})\) is 7, while the number of vertices of \(AG(Z_{12})\) is 4. Obviously, \(\Gamma(Z_{12})\) and \(AG(Z_{12})\) are not isomorphic.

The next result explores the relation between the set of zero divisors of \(R\) and the vertices of \(AG(R)\).

Theorem 3.1: For any ring \(R\), \(Z(R)=\bigcup \{I:i\text{ is an ideal vertex of } AG(R)\}\).

Proof: Let \(0\neq x \in Z(R)\). Then, there exists \(y \in Z^*(R)\) such that \(xy=0\). This implies that \((x)(y)=(0)\). If \((x)=R\), then \(x\) is a unit element. This contradicts the fact that \(x \in Z^*(R)\). So, \((x) \neq R\). Since \((x)\) is adjacent to \((y)\), then \(x \in (x) \in \{I:i\text{ is an ideal vertex of } AG(R)\}\). Therefore, \(x \in \bigcup \{I:i\text{ is an ideal vertex of } AG(R)\}\).

Conversely, suppose that \(x \in \bigcup \{I:i\text{ is an ideal vertex of } AG(R)\}\). Then, \(x \in I\) for some vertex \(I\) of \(AG(R)\). By Lemma 2.2, \(x \in Z(R)\). Hence, \(Z(R)=\bigcup \{I:i\text{ is a vertex of } AG(R)\}\).

Let us give the following easy result.

Proposition 3.2: Let \(\Gamma(R)\) and \(AG(R)\) are finite graphs, then \(|\Gamma(R)| \geq |AG(R)|\).

The following result demonstrates the isomorphism between \(\Gamma(R)\) and \(AG(R)\) by considering \(R=Z_n\).

Theorem 3.3: Let \(n>1\) be a non-prime integer. Then, \(\Gamma(Z_n)\) contains a subgraph which isomorphic with \(AG(Z_n)\).

Proof: Define the graph \(G\) by \(G=\{a\rightarrow b: a-b\text{ is an edge in } \Gamma(Z_n), a|n, b|n\text{ and } a \neq b\}\). Obviously, \(G\) is a subgraph of \(\Gamma(Z_n)\). Now, define a function \(f:G \rightarrow AG(Z_n)\) by \(f(a)\equiv (a)\), with \(a \in G\). Clearly \(f\) is onto. Now, for any distinct vertices \(a, b \in G, a|n\) and \(b|n\). So, \(f(a) = (a) \neq (b) = f(b)\). Thus, \(f\) is one to one. Now, suppose that \(a-b\) is an edge in \(G\). Then, \(ab = 0\), so \((a)(b)=(0)\). This shows that \(f(a) f(b) = (0)\), and hence \(f(a) - f(b)\) is an edge in \(AG(Z_n)\). Thus \(f\) preserves the adjacency property. This proves that \(G \cong AG(Z_n)\).

The following result gives a sufficient conditions for two vertices of \(\Gamma(R)\) such that their annihilators are adjacent ideal vertices in \(AG(R)\).

Theorem 3.4: If \(a\) and \(b\) are two vertices in \(\Gamma(R)\) such that \(d(a,b)=3\), then \(\text{Ann}(a)\) and \(\text{Ann}(b)\) are adjacent ideal vertices in \(AG(R)\).
**Proof:** Since, \( a, b \in Z^*(R) \), then both \( \text{Ann}(a) \) and \( \text{Ann}(b) \) are non-zero. On the other hand, \( d(a, b) = 3 \). This means that neither \( b \in \text{Ann}(a) \) nor \( a \in \text{Ann}(b) \). Then, neither \( \text{Ann}(a) = R \) nor \( \text{Ann}(b) = R \). So, both \( \text{Ann}(a) \) and \( \text{Ann}(b) \) are nontrivial ideals. If we assume that \( \text{Ann}(a) \cap \text{Ann}(b) \neq (0) \), then there exists \( c \in \text{Ann}(a) \) and \( d \in \text{Ann}(b) \) such that \( cd \neq 0 \). Clearly \( a(cd) = b(cd) = 0 \). This means that \( a \rightarrow cd \rightarrow b \) is a path in \( \Gamma(R) \). This contradicts the fact that \( d(a, b) = 3 \). Therefore, \( \text{Ann}(a) \) and \( \text{Ann}(b) \) are adjacent ideal vertices in \( AG(R) \). □

**Example 7:** Let \( Z_{12} \) be the ring of integers modulo 12. Clearly, \( d((3), (10)) = 3 \) in \( AG(Z_{12}) \) and \( \text{Ann}(3) \cap \text{Ann}(10) = (4) \cap (6) = (0) \). This means that \( \text{Ann}(3) \) and \( \text{Ann}(10) \) are adjacent in \( AG(Z_{12}) \).

We end this paper by showing that,

**Proposition 3.5:** If \( R \) is a finite local ring, then \( AG(R) \neq K_{mn} \) for any integers \( m, n > 1 \).

**Proof:** Suppose that \( AG(R) = K_{mn} \) for some integers \( m, n > 1 \), and let \( A = \{ I_1, I_2, \ldots, I_n \} \) and \( B = \{ J_1, J_2, \ldots, J_m \} \) be the partition of \( AG(R) \). Since, \( R \) is a local ring, then by Theorem 1.2 in [9], \( Z(R) \) is an ideal of \( R \) and there exists a vertex \( a \) of \( \Gamma(R) \) such that \( a \cdot Z(R) = (0) \). It follows that \( (a) \cdot Z(R) = (0) \). Hence, \( Z(R) \) is a vertex of \( AG(R) \), yielding \( Z(R) \in A \) or \( (R) \in B \). Now, if \( (R) \in A \), then \( J_i \cdot Z(R) = (0) \) for all \( i = 1, 2, \ldots, m \). By Theorem 3.1, \( J_i \cdot J_k = (0) \) for \( i \neq k \). This contradicts the fact that \( J_i \) and \( J_k \) are not adjacent. If \( (R) \in B \), this will lead to a contradiction. Thus, \( AG(R) \neq K_{mn} \) for any integers \( m, n > 1 \). □
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