MODULI OF SYMPLECTIC INSTANTON VECTOR BUNDLES
OF HIGHER RANK ON PROJECTIVE SPACE \( \mathbb{P}^3 \)

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Abstract. Symplectic instanton vector bundles on the projective space \( \mathbb{P}^3 \) constitute a natural generalization of mathematical instantons of rank 2. We study the moduli space \( I_{n,r} \) of rank-2r symplectic instanton vector bundles on \( \mathbb{P}^3 \) with \( r \geq 2 \) and second Chern class \( n \geq r \), \( n \equiv r (\text{mod}2) \). We give an explicit construction of an irreducible component \( I^*_{n,r} \) of this space for each such value of \( n \) and show that \( I^*_{n,r} \) has the expected dimension \( 4n(r+1) - r(2r+1) \).

1. Introduction

By a symplectic instanton vector bundle of rank \( 2r \) and charge \( n \) (shortly, a symplectic \((n,r)\)-instanton) on the 3-dimensional projective space \( \mathbb{P}^3 \) we understand an algebraic vector bundle \( E = E_{2r} \) of rank \( 2r \) on \( \mathbb{P}^3 \) with Chern classes
\[
(1) \quad c_1(E) = 0, \\
(2) \quad c_2(E) = n, \quad n \geq 1,
\]
supplied with a symplectic structure and satisfying the vanishing conditions
\[
(3) \quad h^0(E) = h^1(E \otimes \mathcal{O}_{\mathbb{P}^3}(-2)) = 0.
\]
By a symplectic structure we mean an anti-self-dual isomorphism
\[
(4) \quad \phi : E \cong E^\vee, \quad \phi^\vee = -\phi,
\]
considered modulo proportionality. The vanishing of the first Chern class \((1)\) follows from the existence of a symplectic structure \((4)\), and if \( r = 1 \), then the two conditions are equivalent. We will denote the moduli space of symplectic \((n,r)\)-instantons by \( I_{n,r} \).

For \( r = 1 \) these bundles relate, via the so-called Atiyah-Ward correspondence, to rank-2 “physical” instantons over the 4-sphere \( S^4 \), these being anti-self-dual connections with structure group \( SU(2) = \text{Sp}(1) \) \([AW]\). Important results on the moduli spaces \( I_n = I_{n,1} \) of rank-2 instantons have been obtained recently: smoothness \([JV]\) for all \( n \), irreducibility \([1]\) for odd \( n \).

Much less is known about the moduli spaces \( I_{n,r} \) for \( r > 1 \). In fact the symplectic instantons with \( r > 1 \) are as natural as those with \( r = 1 \), for they are related, via the same Atiyah-Ward correspondence, to the anti-self-dual connections over \( S^4 \) with structure group \( \text{Sp}(r) \), see \([A]\). As far as we know, the present paper is the first one addressing the properties of the corresponding spaces \( I_{n,r} \). The tool we use to construct \( I_{n,r} \) is the monad method; it originates in the work of Horrocks \([H]\) and is known as the ADHM construction of instantons since \([ADHM]\). It was further sharpened in the work of Barth \([B]\), Barth and Hulek \([BH]\) and Tyurin \([Tju1]\), \([Tju2]\). This method permits to encode the instantons, usual ones or symplectic of higher rank, by hyperwebs of quadrics.

For a sample of the physical literature about symplectic instantons, see e.g. \([Mc]\).
We fix basic terminology and notation in Section 2 and introduce the hyperwebs of quadrics in Section 3. We prove that, for any \( r \geq 2 \) and for any \( n \geq r \) such that \( n \equiv r(\text{mod}\ 2) \), the moduli space \( I_{n,r} \) is nonempty and is realized as a free quotient \( MI_{n,r}/(GL(n)/\pm \text{id}) \), where \( MI_{n,r} \) is a Zariski locally closed subset of an affine space (see Theorem 3.1). Thus \( MI_{n,r} \) carries a natural structure of a reduced scheme, and \( I_{n,r} \) is an algebraic space. In Section 4 we give an explicit construction of vector bundles from \( I_{n,r} \) for the above values of \( n \) and \( r \) and introduce a component \( I^*_{n,r} \) of \( I_{n,r} \) characterized by a certain open condition (*), see Definition 4.6. In Section 5 we prove Theorem 5.3 on the irreducibility of \( I^*_{n,r} \), the main result of this paper.

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2. Notation and conventions

In many respects, we follow the exposition of [1], and we stick to the notation introduced therein. The base field \( k \) is assumed to be algebraically closed of characteristic 0. We identify vector bundles with locally free sheaves. If \( F \) is a sheaf of \( O_X \)-modules on an algebraic variety or a scheme \( X \), then \( nF \) denotes a direct sum of \( n \) copies of \( F \), \( H^i(F) \) denotes the \( i^{th} \) cohomology group of \( F \), \( h^i(F) := \dim H^i(F) \), and \( F^\vee \) denotes the dual of \( F \), that is, \( F^\vee := \text{Hom}_{O_X}(F, O_X) \). If \( X = \mathbb{P}^r \) and \( t \) is an integer, then by \( F(t) \) we denote the sheaf \( F \otimes O_{\mathbb{P}^r}(t) \). \([F]\) will denote the isomorphism class of a sheaf \( F \). For any morphism of \( O_X \)-sheaves \( f : F \to F' \) and any \( k \)-vector space \( U \) (respectively, for any homomorphism \( f : U \to U' \) of \( k \)-vector spaces) we will denote, for short, by the same letter \( f \) the induced morphism of sheaves \( \text{id} \otimes f : U \otimes F \to U \otimes F' \) (respectively, the induced morphism \( f \otimes \text{id} : U \otimes F \to U' \otimes F \)).

We fix an integer \( n \geq 1 \) and denote by \( H_n \) a fixed \( n \)-dimensional vector space over \( k \). Throughout the paper, \( V \) will be a fixed vector space of dimension 4 over \( k \), and we set \( \mathbb{P}^3 := P(V) \). We reserve the letters \( u \) and \( v \) for denoting the two morphisms in the Euler exact sequence \( 0 \to \mathcal{O}_{\mathbb{P}^3}(-1) \overset{s}{\longrightarrow} V^\vee \otimes \mathcal{O}_{\mathbb{P}^3} \overset{\delta}{\longrightarrow} T_{\mathbb{P}^3}(-1) \to 0 \). For any \( k \)-vector spaces \( U \) and \( W \) and any vector \( \phi \in \text{Hom}(U, W \otimes \Lambda^2 V^\vee) \subset \text{Hom}(U \otimes V, W \otimes V^\vee) \) understood as a linear map \( \phi : U \otimes V \to W \otimes V^\vee \) or, equivalently, as a map \( \tilde{\phi} : U \to W \otimes \Lambda^2 V^\vee \), we will denote by \( \phi \) the composition \( U \otimes \mathcal{O}_{\mathbb{P}^3} \overset{\tilde{\phi}}{\longrightarrow} W \otimes \Lambda^2 V^\vee \otimes \mathcal{O}_{\mathbb{P}^3} \overset{\epsilon}{\longrightarrow} W \otimes \Omega_{\mathbb{P}^3}(2) \), where \( \epsilon \) is the induced morphism in the exact triple \( 0 \to \Lambda^2 \Omega_{\mathbb{P}^3}(2) \overset{\Lambda^2 \nu}{\longrightarrow} \Lambda^2 V^\vee \otimes \mathcal{O}_{\mathbb{P}^3} \overset{\epsilon}{\longrightarrow} \Omega_{\mathbb{P}^3}(2) \to 0 \) obtained by taking the second wedge power of the dual Euler exact sequence.

Given an integer \( m \geq 1 \), we denote by \( S_m \) (resp. \( \Sigma_{m+1} \)) the vector space \( S^2 H^\vee_m \otimes \Lambda^2 V^\vee \) (resp. \( \text{Hom}(H^\vee_m, H^\vee_{m+1} \otimes \Lambda^2 V^\vee) \)). Abusing notation, we will denote by the same symbol a \( k \)-vector space, say \( U \), and the associated affine space \( V(U^\vee) = \text{Spec}(\text{Sym}^* U^\vee) \).

All the schemes considered in the paper are Noetherian. By a general point of an irreducible (but not necessarily reduced) scheme \( X \) we mean any closed point of some dense open subset of \( X \). An irreducible scheme is called generically reduced if it is reduced at a general point.
3. Generalities on symplectic instantons and definition of $MI_{n,r}$

In this section we enumerate some facts about symplectic instantons which are completely parallel to those for rank-2 usual instantons, see [1] Section 3.

For a given symplectic $(n, r)$-instanton $E$, the first condition (3) yields $h^0(E(-i)) = 0$, $i \geq 0$, which, together with the exact triple $0 \to E(-j - 1) \to E(-j) \to E(-j)|_{p^2} \to 0$ for $j = 0$ and (3), implies that $h^0(E(-1)|_{p^2}) = 0$, hence also $h^0(E(-i)|_{p^2}) = 0$, $i \geq 1$. The last equality for $i = 2$, together with (3) and the above triple for $j = 2$, gives $h^1(E(-3)) = 0$, hence also $h^1(E(-4)) = 0$. Then, from Serre duality and (4), we deduce:

(5) $h^i(E) = h^i(E(-1)) = h^{3-i}(E(-3)) = h^{3-i}(E(-4)) = 0$, $i \neq 1$, $h^1(E(-2)) = 0$, $i \geq 0$.

By Riemann-Roch and (3), (5), we have

(6) $h^1(E(-1)) = h^2(E(-3)) = n$, $h^1(E) = h^2(E(-4)) = 2n - 2r$.

By tensoring the dual Euler sequence by $E$ we also obtain

(7) $h^1(E \otimes \Omega^1_{\mathbb{P}^3}) = h^2(E \otimes \Omega^2_{\mathbb{P}^3}) = 2n + 2r$,

Consider a triple $(E, f, \phi)$ where $E$ is a $(n, r)$-instanton, $f : H_n \xrightarrow{\sim} H^2(E(-3))$ an isomorphism and $\phi : E \xrightarrow{\sim} E^\vee$ a symplectic structure on $E$. Two triples $(E, f, \phi)$ and $(E', f', \phi')$ are called equivalent if there is an isomorphism $g : E \xrightarrow{\sim} E'$ such that $g_* \circ f = \lambda f'$ with $\lambda \in \{1, -1\}$ and $\phi = g^\vee \circ \phi' \circ g$, where $g_* : H^2(E(-3)) \xrightarrow{\sim} H^2(E'(-3))$ is the induced isomorphism. We denote by $[E, f, \phi]$ the equivalence class of a triple $(E, f, \phi)$. It follows from this definition that the set $F[E]$ of all equivalence classes $[E, f, \phi]$ with given $[E]$ is a homogeneous space of the group $GL(H_n)/\{\pm id\}$.

Each class $[E, f, \phi]$ defines a point

(8) $A = A([E, f, \phi]) \in S^2H^\vee_n \otimes \wedge^2V^\vee$

in the following way. Consider the exact sequences

(9) $0 \to \Omega^1_{\mathbb{P}^3} \xrightarrow{i_1} V^\vee \otimes O_{\mathbb{P}^3}(-1) \to O_{\mathbb{P}^3} \to 0,$

$0 \to \Omega^2_{\mathbb{P}^3} \to \wedge^2V^\vee \otimes O_{\mathbb{P}^3}(-2) \to \Omega^1_{\mathbb{P}^3} \to 0,$

$0 \to \wedge^4V^\vee \otimes O_{\mathbb{P}^3}(-4) \to \wedge^3V^\vee \otimes O_{\mathbb{P}^3}(-3) \xrightarrow{i_2} \Omega^2_{\mathbb{P}^3} \to 0$,

induced by the Koszul complex of $V^\vee \otimes O_{\mathbb{P}^3}(-1) \xrightarrow{ev} O_{\mathbb{P}^3}$. Twisting these sequences by $E$ and taking into account (3), (5)-(7), we obtain the vanishing

(10) $h^0(E \otimes \Omega^1_{\mathbb{P}^3}) = h^3(E \otimes \Omega^2_{\mathbb{P}^3}) = h^2(E \otimes \Omega^3_{\mathbb{P}^3}) = 0$

and the diagram with exact rows

(11) $0 \xrightarrow{} H^2(E(-4)) \otimes \wedge^4V^\vee \xrightarrow{i_2} H^2(E(-3)) \otimes \wedge^3V^\vee \xrightarrow{i_2} H^2(E \otimes \Omega^2_{\mathbb{P}^3}) \xrightarrow{} 0$

$0 \xleftarrow{} H^1(E(-1)) \xrightarrow{A'} H^1(E(-1)) \otimes \wedge^4V^\vee \xrightarrow{i_1} H^1(E \otimes \Omega^3_{\mathbb{P}^3}) \xleftarrow{} 0,$

where $A' := i_1 \circ \partial^{-1} \circ i_2$. The Euler exact sequence (9) yields the canonical isomorphism $\omega_{\mathbb{P}^3} \xrightarrow{\sim} \wedge^4V^\vee \otimes O_{\mathbb{P}^3}(-4)$, and fixing an isomorphism $\tau : k \xrightarrow{\sim} \wedge^4V^\vee$ we have the isomorphisms $\tilde{\tau} : V \xrightarrow{\sim} \wedge^3V^\vee$ and $\tilde{\tau} : \omega_{\mathbb{P}^3} \xrightarrow{\sim} O_{\mathbb{P}^3}(-4)$. We define $A$ in (8) as the composition

(12) $A : H_n \otimes V \xrightarrow{\tilde{\tau}} H_n \otimes \wedge^3V^\vee \xrightarrow{\tilde{\tau}} H^2(E(-3)) \otimes \wedge^3V^\vee \xrightarrow{A'} H^1(E(-1)) \otimes \wedge^4V^\vee \xrightarrow{\phi}$
whose cohomology sheaf

\[ M(16) \]

\[ A \]

Thus applying Beilinson spectral sequence [Bei] to

\[ E_0 \]

In view of (7), dim

\[ \text{dim} \]

Theorem 3.1. The natural morphism

\[ \pi_{n,r} : MI_{n,r} \to I_{n,r}, \ A \mapsto [E_{2r}(A)] \]

is a principal \( GL(H_n) \)/\{±id\}-bundle in the étale topology. Hence \( I_{n,r} \) is a quotient stack \( MI_{n,r}/(GL(H_n))/\{±id\} \), making it an algebraic space.
Proof. See [T, Section 3].

Each fibre $F_E = \pi^{-1}_n([E])$ over an arbitrary point $[E] \in I_{n,r}$ is a principal homogeneous space of the group $GL(H_n)/\{\pm \text{id}\}$. Hence the irreducibility of $(I_{n,r})_{\text{red}}$ is equivalent to the irreducibility of the scheme $(MI_{n,r})_{\text{red}}$.

We can also state:

**Theorem 3.2.** For each $n \geq 1$, the space $MI_{n,r}$ of $(n,r)$-instanton hyperwebs of quadrics is a locally closed subscheme of the vector space $S_n$ given locally at any point $A \in MI_{n,r}$ by

$$(20) \quad \left(\frac{2n - 2r}{2}\right) = 2n^2 - n(4r + 1) + r(2r + 1)$$

equations obtained as the rank condition (i) in (18).

Note that from (20) it follows that

$$(21) \quad \dim [A] MI_{n,r} \geq \dim S_n - (2n^2 - n(4r + 1) + r(2r + 1)) = n^2 + 4n(r + 1) - r(2r + 1)$$
at any point $A \in MI_{n,r}$. Hence,

$$(22) \quad \dim [E] I_{n,r} \geq 4n(r + 1) - r(2r + 1)$$
at any point $[E] \in I_{n,r}$, since $MI_{n,r} \rightarrow I_{n,r}$ is a principal $GL(H_n)/\{\pm \text{id}\}$-bundle in the étale topology.

4. Explicit construction of symplectic instantons

4.1. Example: symplectic $(n,n)$-instantons. In this subsection we recall some known facts about symplectic $(n,n)$-instantons and their relation to usual rank-2 instantons, see [T, Sections 5-6]. We first show that each invertible hyperweb of quadrics $A \in S_n$ naturally leads to a construction of a symplectic $(n,n)$-instanton $E_{2n}(A)$ on $\mathbb{P}^3$. Given an integer $n \geq 1$, set

$$(23) \quad S^0_n := \{A \in S_n \mid A : H_n \otimes V \rightarrow H^\vee_n \otimes V^\vee \text{ is an invertible map}\}.$$ 

Then $S^0_n$ is a dense open subset of $S_n$, and it is easy to see that for any $A \in S^0_n$ the following conditions are satisfied.

(1) The morphism $\tilde{A} : H_n \otimes O_{\mathbb{P}^3}(-1) \rightarrow H^\vee_n \otimes \Omega_{\mathbb{P}^3}(1)$ induced by $A$ is a subbundle embedding, and

$$(24) \quad E_{2n}(A) := \text{coker}(\tilde{A})$$
is a symplectic $(n,n)$-instanton, that is,

$$(25) \quad [E_{2n}(A)] \in I_{n,n}.$$ 

(2) For all $i \geq 0$,

$$(26) \quad h^i(E_{2n}(A)) = h^i(E_{2n}(A)(-2)) = 0.$$
This follows from the diagram
\[
\begin{array}{ccccccccc}
0 & \rightarrow & H_\mathbb{P}^3(-1) & \rightarrow & H_\mathbb{P}^3(1) & \rightarrow & E_{2n}(A) & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & H_\mathbb{P}^3(-1) & \rightarrow & H_\mathbb{P}^3(1) & \rightarrow & E_{2n}(A) & \rightarrow & 0 \\
\end{array}
\]

Thus \( S^0_n \subset M_{I,n,n} \). In fact, the following result is true.

**Proposition 4.1.** \( S^0_n = M_{I,n,n} \). In particular, \( M_{I,n,n} \) is irreducible of dimension \( 3n^2 + 3n \), and hence \( I_{n,n} \) is irreducible of dimension \( 2n^2 + 3n \).

**Proof.** We have to show that \( M_{I,n,n} \subset S^0_n \). Let \( A \in M_{I,n,n} \). Since \( n = r \), by condition (i) from \((13)\) the rank of the hyperweb of quadrics \( A : H_n \otimes V \rightarrow H_n' \otimes V' \) is \( 2n + 2r = 4n = \dim H_n' \otimes V' \), hence \( A \) is invertible. By \((23)\), this means that \( A \in S^0_n \).

Now we proceed to spell out the relation between symplectic \((n,n)\)-instantons and usual rank-2 instantons with second Chern class \( 2n - 1 \). This relation is given at the level of spaces of hyperwebs of quadrics \( M_{I,n,n} \) and \( M_{I_{2n-1,1}} \) interpreted as spaces of monads.

We need some more notation. Let \( B \in S^0_n \). By definition, \( B \) is an invertible anti-self-dual map \( H_n \otimes V \rightarrow H_n' \otimes V' \). Then the inverse
\[
B^{-1} : H_n' \otimes V' \rightarrow H_n \otimes V
\]
is also anti-self-dual. Consider the vector space \( \Sigma_n = H_n' \otimes H_n'_{-1} \otimes \wedge^2 V' \). An element \( C \in \Sigma_n \) can be viewed as a linear map \( C : H_{n-1} \otimes V \rightarrow H_n' \otimes V' \), and its transpose \( C' \) as a map \( C' : H_n \otimes V \rightarrow H_{n-1}' \otimes V' \). As the composition \( C' \circ B^{-1} \circ C \) is anti-self-dual, we can consider it as an element of \( \wedge^2 (H_{n-1}' \otimes V') \simeq S_{n-1} \oplus \wedge^2 H_{n-1} V \otimes S^2 V' \) (cf. \((13)\)). Thus the condition
\[
C' \circ B^{-1} \circ C \in S_{n-1}
\]
makes sense.

Next, consider the upper horizontal triple in \((27)\) with \( A = B \). Twisting it by \( O_{\mathbb{P}^3}(1) \) and passing to global sections we obtain the exact triple
\[
0 \rightarrow H_n \rightarrow H_n' \otimes \wedge^2 V' \rightarrow H^0(E_{2n}(B)(1)) \rightarrow 0
\]
Besides, interpreting \( C \in \Sigma_n \) as a map \( C : H_{n-1} \rightarrow H_n' \otimes \wedge^2 V' \), we obtain the composition \( H_{n-1} \rightarrow H_n' \otimes \wedge^2 V' \rightarrow H^0(E_{2n}(B)(1)) \) which induces the morphism of sheaves
\[
\rho_{B,C} : H_{n-1} \otimes O_{\mathbb{P}^3}(-1) \rightarrow E_{2n}(B).
\]
Note also that the maps \( B : H_n \otimes V \rightarrow H_n' \otimes V' \) and \( C : H_{n-1} \otimes V \rightarrow H_n' \otimes V' \) provide a map \( (H_n \oplus H_{n-1}) \otimes V \rightarrow H_n' \otimes V' \), which induces the morphism of sheaves
\[
\tau_{B,C} : (H_n \oplus H_{n-1}) \otimes O_{\mathbb{P}^3}(-1) \rightarrow H_n' \otimes V' \otimes O_{\mathbb{P}^3}.
\]
Now set
\[(33) \quad X_n := \left\{ (B, C) \in S_n^0 \times \Sigma_n \right\} \]
with
\[
(i) \text{ the condition } \xi : \begin{array}{c}
H_{2n-1} \xrightarrow{\sim} H_n \oplus H_{n-1},
\end{array}
\]
we obtain the corresponding decomposition
\[
(34) \quad \xi : S_{2n-1} \xrightarrow{\sim} S_n \oplus \Sigma_n \oplus S_{n-1} : A \mapsto (A_1(\xi), A_2(\xi), A_3(\xi)).
\]
Thus, considering the set $MI_{2n-1}$ of $(2n-1)$-instanton hyperwebs of quadrics as a subset of $S_{2n-1}$, we obtain a natural projection
\[
(35) \quad f_n : MI_{2n-1,1} \to S_n \oplus \Sigma_n : A \mapsto (A_1(\xi), A_2(\xi)).
\]
The following result is proved in [T, Theorems 1.1, 6.1 and Remark 7.6].

**Proposition 4.2.** For a general decomposition $\xi$ in (34), there exists a dense open subset $MI_{2n-1,1}(\xi)$ of $MI_{2n-1,1}$ such that the projection $f_n$ in (35) induces an isomorphism or integral schemes
\[
(36) \quad \tilde{\xi} : S_{2n-1} \xrightarrow{\sim} X_n : A \mapsto (A_1(\xi), A_2(\xi)).
\]
The inverse isomorphism is given by the formula
\[
(37) \quad f_n^{-1} : X_n \xrightarrow{\sim} MI_{2n-1,1}(\xi) : (B, C) \mapsto \tilde{\xi}^{-1}(B, C, -C^\vee \circ B^{-1} \circ C).
\]
Besides, the projection
\[
(38) \quad pr_1 : X_n \to S_n^0 : (B, C) \mapsto B
\]
is dominant.

It is not hard to check that the morphism $\rho_{B,C} : H_{n-1} \otimes O_{\mathbb{P}^3}(-1) \to E_{2n}(B)$ defined in (31) satisfies the condition $^t \rho_{B,C} \circ \rho_{B,C} = 0$, where $^t \rho_{B,C}$ is the composition
\[
^t \rho_{B,C} : E_{2n}(B) \xrightarrow{\phi} E_{2n}(B) \xrightarrow{\rho_{B,C}^\vee} H_{n-1}^\vee \otimes O_{\mathbb{P}^3}(1)
\]
and $\phi$ is a symplectic structure on $E_{2n}(B)$ (cf. [T, formulas (71)-(72)]). In other words, we obtain an anti-self-dual monad
\[
(39) \quad 0 \to H_{n-1} \otimes O_{\mathbb{P}^3}(-1) \xrightarrow{\rho_{B,C}} E_{2n}(B) \xrightarrow{\phi} E_{2n}(B) \xrightarrow{\rho_{B,C}^\vee} H_{n-1}^\vee \otimes O_{\mathbb{P}^3}(1) \to 0
\]
with cohomology sheaf
\[
E_2(A) = E_2(B, C) := \ker ^t \rho_{B,C}/\text{im} \rho_{B,C}, \quad A = f_n^{-1}(B, C).
\]
Next, by (19) we have the natural projection
\[
(40) \quad \pi_{2n-1,1} : MI_{2n-1,1} \to I_{2n-1,1} : A \mapsto [E_2(A)].
\]
We have the following interpretation of the isomorphism (38) on the level of vector bundles:
\[
(41) \quad [E_2(B, C)] = \pi_{2n-1,1}(f_n^{-1}(B, C)).
\]
Remark 4.3. Note that, according to the definitions (16)-(18) of $MI_{2n-1,1}$ and $MI_{n,n}$, for any $A \in MI_{2n-1,1}$ and $MI_{n,n}$, one has two other anti-self-dual monads

\begin{equation}
\mathcal{M}_A: \ 0 \to H_{2n-1} \otimes O_{p^3}(-1) \xrightarrow{a_A} W_A \otimes O_{p^3} \xrightarrow{a_\gamma^\vee q_A} H_{2n-1}^\vee \otimes O_{p^3}(1) \to 0
\end{equation}

\begin{equation}
\mathcal{M}_B: \ 0 \to H_n \otimes O_{p^3}(-1) \xrightarrow{a_B} W_B \otimes O_{p^3} \xrightarrow{a_\beta^\vee q_B} H_n^\vee \otimes O_{p^3}(1) \to 0
\end{equation}

with cohomology sheaves

\begin{equation}
E_2(A) = \ker(a_\gamma^\vee \circ q_A)/\text{im} \ a_A, \ E_2n(B) = \ker(a_\beta^\vee \circ q_B)/\text{im} \ a_B
\end{equation}

respectively. Moreover, (40) and (41) provide an isomorphism $w: W_B = H^2(E_2(B) \otimes O_{p^3}) \iso H^2(E_2n(A) \otimes O_{p^3}) = W_A$. We thus obtain a commutative anti-self-dual diagram relating these monads:

\begin{equation}
0 \longrightarrow H_n \otimes O_{p^3}(-1) \xrightarrow{a_B} W_B \otimes O_{p^3} \xrightarrow{q_B} W_B^\vee \otimes O_{p^3} \xrightarrow{a_\gamma^\vee} H_n^\vee \otimes O_{p^3}(1) \longrightarrow 0
\end{equation}

where $i_\xi: H_n \hookrightarrow H_{2n-1}$ is the embedding induced by the decomposition (34). In view of (46) and the canonical isomorphism $H_{2n-1}/i_\xi(H_n) \simeq H_{n-1}$, from this diagram we obtain the monad

\begin{equation}
\mathcal{M}_{A,B}: \ 0 \to H_{n-1} \otimes O_{p^3}(-1) \xrightarrow{a_{A,B}} E_{2n}(B) \xrightarrow{\phi} E_{2n}(B)^\vee \xrightarrow{a_{A,B}^\vee} H_{2n-1}^\vee \otimes O_{p^3}(1) \to 0
\end{equation}

with cohomology sheaf

\begin{equation}
E_2(A) = \ker(a_{A,B}^\vee \circ \phi)/\text{im} \ a_A.
\end{equation}

We call (48) the quotient monad of the monads (44) and (45).

Remark 4.4. Note that, by Proposition 1.2, the set of all diagrams (17) is parametrized by the irreducible variety $I_{2n-1,1}(\xi)$.

4.2. Example: a special family of symplectic $(n, r)$-instantons. Now assume $n \geq 2$ and, for any integer $r$, $2 \leq r \leq n-1$, consider an inclusion

\begin{equation}
\tau: H_{2n-r} \hookrightarrow H_{2n-1}
\end{equation}

such that

\begin{equation}
\tau(H_{2n-r}) \supset i_\xi(H_n).
\end{equation}

We obtain a hyperweb of quadrics

\begin{equation}
A_r \in S^2 H_{2n-r}^\vee \otimes \wedge^2 V^\vee
\end{equation}

as the image of $A$ under the map $S^2 H_{2n-r}^\vee \otimes \wedge^2 V^\vee \to S^2 H_{2n-r}^\vee \otimes \wedge^2 V^\vee$ induced by $\tau$. The corresponding monad

\begin{equation}
\mathcal{M}_r: \ 0 \to H_{2n-r} \otimes O_{p^3}(-1) \xrightarrow{a_r} W_A \otimes O_{p^3} \xrightarrow{a_\gamma^\vee q_A} H_{2n-r}^\vee \otimes O_{p^3}(1) \to 0,
\end{equation}

has a rank-$2r$ cohomology bundle

\begin{equation}
E_{2r}(A_r) = \ker(a_\gamma^\vee \circ q_A)/\text{im} \ a_r.
\end{equation}
where \( a_\tau := a_A \circ \tau \). By construction, \( E_{2r}(A_\tau) \) inherits a natural symplectic structure
\[
\phi_\tau : E_{2r}(A_\tau) \xrightarrow{\sim} E_{2r}(A_\tau)^\vee.
\]
Besides, in view of (51), the monad (52) can be inserted as a middle row into the diagram (47), extending it to a three-row commutative anti-self-dual diagram. Arguing as in Remark 4.3 we obtain, in addition to the quotient monad (48), two more quotient monoids:
\[
\begin{align*}
\mathcal{M}_\tau' &: 0 \to H_{n-r} \otimes \mathcal{O}_{\mathbb{P}^1}(-1) \xrightarrow{\phi'_r} E_{2n}(B) \xrightarrow{\phi'_r} E_{2n}(B)^\vee \xrightarrow{\phi'_r} H_{n-r} \otimes \mathcal{O}_{\mathbb{P}^1}(1) \to 0, \\
\mathcal{M}_\tau'' &: 0 \to H_{r-1} \otimes \mathcal{O}_{\mathbb{P}^1}(-1) \xrightarrow{\phi''_r} E_{2r}(B) \xrightarrow{\phi''_r} E_{2r}(B)^\vee \xrightarrow{\phi''_r} H_{r-1} \otimes \mathcal{O}_{\mathbb{P}^1}(1) \to 0,
\end{align*}
\]
From (26) and (55) we easily deduce:
\[
h^{0}(E_{2r}(A_\tau)) = h^{0}(E_{2r}(A_\tau)(-2)) = 0, \quad i \geq 0, \quad c_2(E_{2r}(A_\tau)) = 2n - r.
\]
By definition, this together with (52)-(54) means that
\[
[E_{2r}(A_\tau)] \in I_{2n-r,r}.
\]
\textbf{Remark 4.5.} Observe that, in view of (51), the maps \( \tau \) belong to the set
\[
N_{n,r} := \{ \tau \in \text{Hom}(H_{2n-r}, H_{2n-1}) | \tau \text{ is injective and } \text{im } \tau \supset \text{im } i_\xi \}.
\]
When \( A \in MI_{2n-1,1}(\xi) \) is fixed, \( N_{n,r} \) parametrizes some family of hyperwebs \( A_\tau \) from \( MI_{2n-r,r} \). Since \( N_{n,r} \) is a principal \( GL(H_{2n-r}) \)-bundle over an open subset of the Grassmannian \( Gr(n - r, n - 1) \), it is irreducible. Thus, by Remark 4.4, the family of the three-row extensions of the diagrams (47) can be parametrized by the irreducible variety \( MI_{2n-1,1}(\xi) \times N_{n,r} \). Hence the family \( D_{n,r} \) of isomorphism classes of symplectic rank-2R bundles obtained from these diagrams by formula (53) is an irreducible locally closed subset of \( I_{2n-r,r} \).

Note that it is a priori not clear whether the closure of \( D_{n,r} \) in \( I_{2n-r,r} \) is an irreducible component of \( I_{2n-r,r} \).

\textbf{Definition 4.6.} Let \( 2 \leq r \leq n - 1 \). We say that \( A \in MI_{2n-r,r} \) satisfies property (*) if there exists a monomorphism \( i : H_n \hookrightarrow H_{2n-r} \) such that the image \( B \) of \( A \) under the surjection \( S_{2n-r} \twoheadrightarrow S_n \) induced by \( i \) is invertible as a homomorphism \( B : H_n \otimes V \to H_n^1 \otimes V^1 \).

The property (*) is clearly an open condition on \( A \). Moreover, since \( \pi_{2n-r,r} : MI_{2n-r,r} \to I_{2n-r,r} \) is a principal bundle (Theorem 3.1), if an element \( A \in \pi_{2n-r,r}^{-1}([E_{2r}]) \) satisfies (*), then any other point \( A' \in \pi_{2n-r,r}^{-1}([E_{2r}]) \) satisfies (*). We thus say that \([E_{2r}] \in I_{2n-r,r} \) satisfies property (*) if some (hence any) \( A \in \pi_{2n-r,r}^{-1}([E_{2r}]) \) satisfies property (*). It is obviously an open condition on \([E_{2r}] \in I_{2n-r,r} \).

\textbf{Remark 4.7.} By Proposition 4.2 and using (51), we see that any \([E_{2r}] \in D_{n,r} \), as well as any \( A \in f_n^{-1}(D_{n,r}) \) satisfies property (*). We define
\[
I_{2n-r,r}^* := I_{(1)} \cup \ldots \cup I_{(k)},
\]
where \( I_{(1)}, \ldots, I_{(k)} \) are all the irreducible components of \( I_{2n-r,r} \) whose general points satisfy property (*). By definition, \( D_{n,r} \subset I_{2n-r,r}^* \), hence \( I_{2n-r,r}^* \) is nonempty. We also set \( MI_{2n-r,r}^* = \pi_{2n-r,r}^{-1}(I_{2n-r,r}^* \cup I_{2n-r,r}^* \cup \ldots \cup I_{2n-r,r}^*) \), so that the map \( \pi_{2n-r,r} : MI_{2n-r,r}^* \to I_{2n-r,r}^* \) is a principal bundle with structure group \( GL(H_{2n-r})/\{\pm 1\} \).
5. Irreducibility of $I_{2n-r,r}^*$

5.1. A dense open subset $X_{n,r}$ of $M I_{2n-r,r}^*$. Reduction of the irreducibility of $I_{n,r}^*$ to that of $X_{n,r}$. In this section we prove the irreducibility of the component $I_{2n-r,r}^*$ of $I_{2n-r,r}$ defined in (59), see Theorem 5.3. The explicit construction of symplectic instantons in Section 4 gives us a hint to the proof. We proceed along the lines of Subsection 4.1.

Take any $B \in S_n^0$ and consider it as an invertible anti-self-dual linear map $H_n \otimes V \to H_n^\vee \otimes V^\vee$. Then $B^{-1}$ is also anti-self-dual. Let

$$
\Sigma_{n,r} := H_{n-r}^\vee \otimes H_n^\vee \otimes \wedge^2 V^\vee.
$$

An element $C \in \Sigma_n$ can be understood as a map $C : H_{n-r} \otimes V \to H_n^\vee \otimes V^\vee$, and its transpose $C^\vee$ is a map $H_n \otimes V \to H_{n-r}^\vee \otimes V^\vee$. The composition $C^\vee \circ B^{-1} \circ C$ is anti-self-dual, i.e., it is an element of $\wedge^2(H_{n-r}^\vee \otimes V^\vee) \cong S_{n-r} \oplus \wedge^2 H_{n-r}^\vee \otimes S^2 V^\vee$ (cf. (13)). We will later impose the condition

$$
C^\vee \circ B^{-1} \circ C \in S_{n-r}.
$$

Next, as in (30), we have a well defined epimorphism $\epsilon(B) : H_n^\vee \otimes \wedge^2 V^\vee \to H^0(E_{2n}(B)(1))$. Besides, interpreting the above element $C \in \Sigma_{n,r}$ as a map $^2C : H_{n-r} \to H_n^\vee \otimes \wedge^2 V^\vee$, we obtain the composition $H_{n-r} \xrightarrow{\delta_{B}} H_n^\vee \otimes \wedge^2 V^\vee \xrightarrow{\epsilon(B)} H^0(E_{2n}(B)(1))$ which induces the morphism of sheaves

$$
\rho_{B,C} : H_{n-r} \otimes O_{\mathbb{P}^3}(-1) \to E_{2n}(B).
$$

Note also that $B : H_n \otimes V \to H_n^\vee \otimes V^\vee$ and $C : H_{n-r} \otimes V \to H_n^\vee \otimes V^\vee$ define a map $(H_n \oplus H_{n-r}) \otimes V \to H_n^\vee \otimes V^\vee$ which induces the morphism of sheaves

$$
\tau_{B,C} : (H_n \oplus H_{n-r}) \otimes O_{\mathbb{P}^3}(-1) \to H_n^\vee \otimes V^\vee \otimes O_{\mathbb{P}^3}.
$$

Now set

$$
X_{n,r} := \left\{ (B,C) \in S_n^0 \times \Sigma_{n,r} \bigg| \begin{array}{l}
\text{(i) the condition (61) is satisfied,} \\
\text{(ii) $\rho_{B,C}$ in (62) is a subbundle inclusion,} \\
\text{(iii) $\tau_{B,C}$ in (63) is a subbundle inclusion.}
\end{array} \right\}
$$

By definition, $X_{n,r}$ is a locally closed subset of $S_n^0 \times \Sigma_{n,r}$. Hence it has a natural structure of reduced scheme.

Now for an arbitrary direct sum decomposition

$$
\xi : H_{2n-r} \xrightarrow{\sim} H_n \oplus H_{n-r}
$$

we obtain the corresponding decomposition

$$
\tilde{\xi} : S_{2n-r} \xrightarrow{\sim} S_n \oplus \Sigma_{n,r} \oplus S_{n-r} : A \mapsto (A_1(\xi), A_2(\xi), A_3(\xi)).
$$

Thus, considering the set $M I_{2n-r,r}$ of symplectic $(2n-r,r)$-instanton hyperwebs of quadrics as a subset of $S_{2n-r}$, we obtain a natural projection

$$
f_{n,r} : M I_{2n-r,r} \to S_n \oplus \Sigma_{n,r} : A \mapsto (A_1(\xi), A_2(\xi)).
$$

We now prove the following result parallel to Proposition 4.2.
Theorem 5.1. Let $n \geq 3$ and $2 \leq r \leq n - 1$.

(i) For a general decomposition $\xi$ in (65), there is an open dense subset $MI^{*}_{2n-r,r}(\xi)$ of $MI^{*}_{2n-r,r}$ and an isomorphism of reduced schemes

$$f_{n,r} : MI^{*}_{2n-r,r}(\xi) \xrightarrow{\sim} X_{n,r} : A \mapsto (A_1(\xi), A_2(\xi)),$$

where $A_1(\xi)$ and $A_2(\xi)$ are defined by (66).

(ii) The inverse isomorphism is given by the formula

$$f_{n,r}^{-1} : X_{n,r} \xrightarrow{\sim} MI^{*}_{2n-r,r}(\xi) : (B, C) \mapsto \tilde{\xi}^{-1}(B, C, -C^\vee \circ B^{-1} \circ C),$$

where $\tilde{\xi}$ is defined by (67).

Proof. Set $MI^{*}_{2n-r,r}(\xi) := \{A \in MI^{*}_{2n-r,r} \mid A$ satisfies property (*) for the monomorphism $i : H_n \hookrightarrow H_{2n-r}$ defined by $\xi\}$.

It follows from Definition 4.6 and Remark 4.7 that, for a general decomposition $\xi$ in (65), $MI^{*}_{2n-r,r}(\xi)$ is a dense open subset of $MI^{*}_{2n-r,r}$. Then, for this choice of $\xi$, the proof of this Theorem essentially mimics the proof of [T, Proposition 6.1] in which we make the substitution $m + 1 \mapsto n$, $m \mapsto n - r$ and change the notation accordingly.

The proof of the following theorem will be given in Subsection 5.2.

Theorem 5.2. $X_{n,r}$ is irreducible of dimension $(2n - r)^2 + 4(2n - r)(r + 1) - r(2r + 1)$.

From Theorems 5.1 and 5.2 it follows that $MI^{*}_{2n-r,r}$ is irreducible of dimension $(2n - r)^2 + 4(2n - r)(r + 1) - r(2r + 1)$ for any $n \leq 3$ and $2 \leq r \leq n - 1$. Hence $I^{*}_{2n-r,r}$ is irreducible of dimension $4(2n - r)(r + 1) - r(2r + 1)$ for these values of $n$ and $r$. Note that the irreducibility of $I^{*}_{2n-r,r}$ is also true when $r = n$, and in this case $I^{*}_{n,n}$ coincides with $I_{n,n}$. Substituting $2n - 1 \mapsto n$, we obtain the following main result of the paper.

Theorem 5.3. For any integer $r \geq 2$ and for any integer $n \geq r$ such that $n \equiv r(\text{mod}2)$, $I^{*}_{n,r}$ is an irreducible component of $I_{n,r}$ of dimension $4n(r + 1) - r(2r + 1)$.

5.2. Proof of the irreducibility of $X_{n,r}$. In this subsection we give the proof of Theorem 5.2. Define

$$\tilde{X}_{n,r} := \{(D, C) \in (S^\vee_n)^0 \times \Sigma_{n,r} \mid (C^\vee \circ D \circ C : H_{n-r} \otimes V \to H^\vee_{n-r} \otimes V^\vee) \in S_{n-r}\},$$

a closed subscheme of $(S^\vee_n)^0 \times \Sigma_{n,r}$ defined by the equations

$$C^\vee \circ D \circ C \in S_{n-r}.$$

Since the conditions (ii) and (iii) in the definition (33) of $X_{n,r}$ are open and $X_{n,r}$ is nonempty (see Theorem 5.1), the isomorphism

$$S^0_n \xrightarrow{\sim} (S^\vee_n)^0 : B \mapsto B^{-1}$$

implies that $X_{n,r}$ is a nonempty open subset of $(\tilde{X}_{n,r})_{\red}$.

$$\emptyset \neq X_{n,r} \xrightarrow{\text{open}} (\tilde{X}_{n,r})_{\red}.$$

Fix a direct sum decomposition

$$H_{n} \xrightarrow{\sim} H_{n-r} \oplus H_{r}.$$

Then any linear map

$$C \in \Sigma_{n,r} = \text{Hom}(H_{n-r}, H^\vee_n \otimes \wedge^2 V^\vee), \quad C : H_{n-r} \otimes V \to H^\vee_n \otimes V^\vee,$$
can be represented as a map
\begin{equation}
C : H_{n-r} \otimes V \to H_{n-r}^\vee \otimes V^\vee \oplus H_r^\vee \otimes V^\vee,
\end{equation}
or else as a block matrix
\begin{equation}
C = \begin{pmatrix} \phi \\ \psi \end{pmatrix},
\end{equation}
where
\begin{equation}
\phi \in \text{Hom}(H_{n-r}, H_{n-r}^\vee \otimes \wedge^2 V^\vee) = \Phi_{n-r}, \quad \psi \in \Psi_{n,r} := \text{Hom}(H_{n-r}, H_r^\vee \otimes \wedge^2 V^\vee).
\end{equation}

Similarly, any \(D \in (S_n^\vee)^0 \subset S_n^\vee = S^2 H_n \otimes \wedge^2 V \subset \text{Hom}(H_n^\vee \otimes V^\vee, H_n \otimes V)\) can be represented in the form
\begin{equation}
D = \begin{pmatrix} D_1 & \lambda \\ -\lambda^\vee & \mu \end{pmatrix},
\end{equation}
where
\begin{equation}
D_1 \in S_n^\vee \subset \text{Hom}(H_{n-r}^\vee \otimes V^\vee, H_{n-r} \otimes V), \quad \lambda \in L_{n,r} := \text{Hom}(H_r^\vee, H_{n-r} \otimes V), \quad \mu \in M_r := S^2 H_r \otimes \wedge^2 V.
\end{equation}

By (75) and (77) the composition
\begin{equation}
C^\vee \circ D \circ C : H_{n-r} \otimes V \to H_{n-r}^\vee \otimes V^\vee \quad (C^\vee \circ D \circ C \in \wedge^2 (H_{n-r}^\vee \otimes V^\vee))
\end{equation}
can be written in the form
\begin{equation}
C^\vee \circ D \circ C = \phi^\vee \circ D_1 \circ \phi + \phi^\vee \circ \lambda \circ \psi - \psi^\vee \circ \lambda \circ \phi + \psi^\vee \circ \mu \circ \psi.
\end{equation}

By (75)-(78) we have
\[ S_n^\vee \times \Sigma_{n,r} = S_{n-r}^\vee \times \Phi_{n-r} \times \Psi_{n,r} \times L_{n,r} \times M_r, \]
and there are well defined morphisms
\[ \tilde{p} : \tilde{X}_{n,r} \to L_{n,r} \times M_r : (D_1, \phi, \psi, \lambda, \mu) \mapsto (\lambda, \mu). \]

and
\[ p := \tilde{p} | \overline{X}_{n,r} : \overline{X}_{n,r} \to L_{n,r} \oplus M_r, \]
where \( \overline{X}_{n,r} \) is the closure of \( X_{n,r} \) in \((S_n^\vee)^0 \times \Sigma_{n,r} \). We now invoke the following result from [1]:

**Proposition 5.4.** Let \( n \geq 2 \). Then for any \( D \in (S_n^\vee)^0 \) and for a general choice of the decomposition \( H_n \sim H_{n-r} \oplus H_r \), the block \( D_1 \) of \( D \) in (77) is nondegenerate.

**Proof.** See [1] Proposition 7.3. By repeatedly applying this proposition \( r \) times, we can find a decomposition \( H_n \sim H_{n-r} \oplus H_r \) such that \( D_1 : H_{n-r}^\vee \otimes V^\vee \to H_{n-r} \otimes V \) in (77) is nondegenerate, i.e., \( D_1 \in (S_{n-r}^\vee)^0 \).

Let \( \mathcal{X} \) be any irreducible component of \( X_{n,r} \) and let \( \overline{\mathcal{X}} \) be its closure in \( \overline{X}_{n,r} \). Fix a point \( z = (D_1, \phi, \psi, \lambda, \mu) \in \mathcal{X} \) not lying in the components of \( X_{n,r} \) different from \( \mathcal{X} \). Consider the morphism
\begin{equation}
f : \mathbb{A}^1 \to \overline{\mathcal{X}} : t \mapsto (D_1, t^2 \phi, t\psi, t\lambda, t^2 \mu), \quad f(1) = z,
\end{equation}
which is well defined by (79). By definition, the point \( f(0) = (D_1, 0, 0, 0) \) lies in the fibre \( p^{-1}(0, 0) \). Hence, \( p^{-1}(0, 0) \cap X \neq \emptyset \). In other words,

\[
\rho^{-1}(0, 0) \neq \emptyset, \quad \text{where} \quad \rho := p|\bar{X}.
\]

Now, it follows from (79) and the definition of \( \bar{X}_{n,r} \) that

\[
\bar{p}^{-1}(0, 0) = \{(D_1, \phi, \psi) \in (S_{n-r}^\vee)^0 \times \Phi_{n-r} \times \Psi_{n,r} \mid \phi^\vee \circ D \circ \phi \in S_{n-r}\}.
\]

Consider the set

\[
Z_{n-r} = \{(D, \phi) \in (S_{n-r}^\vee)^0 \times \Phi_{n-r} \mid \phi^\vee \circ D \circ \phi \in S_{n-r}\}.
\]

It carries a natural scheme structure, where it is a closed subscheme of \((S_{n-r}^\vee)^0 \times \Phi_{n-r}\). Comparing the definition of \( Z_{n-r} \) with (82) we see that there are scheme-theoretic inclusions of schemes

\[
\rho^{-1}(0, 0) \subset p^{-1}(0, 0) \subset \bar{p}^{-1}(0, 0) = Z_{n-r} \times \Psi_{n,r}.
\]

By [T, Theorem 7.2], \( Z_{n-r} \) is an integral scheme of dimension \( 4(n-r)(n-r+2) \). This together with (83) implies that

\[
\dim \rho^{-1}(0, 0) \leq \dim p^{-1}(0, 0) \leq \dim Z_{n-r} + \dim \Psi_{n,r} = 4(n-r)(n-r+2) + 6r(n-r) = (n-r)(4n+2r+8).
\]

Hence in view of (81)

\[
\dim \bar{X} \leq \dim \rho^{-1}(0, 0) + \dim L_{n,r} + \dim M_r \leq (n-r)(4n+2r+8) + 6r(n-r) + 3r(r+1) = (2n-r)^2 + 4(2n-r)(r+1) - r(2r+1).
\]

On the other hand, formula (21), with \( 2n-r \) substituted for \( n \), and Theorem 5.1(ii) show that, for any point \( x \in X \) such that \( A := f^{-1}_{n,r}(x) \in MT_{2n-r,r}^0(\xi) \),

\[
(2n-r)^2 + 4(2n-r)(r+1) - r(2r+1) \leq \dim_A MT_{2n-r,r}^0(\xi) = \dim \bar{X}.
\]

Comparing (85) with (86), we see that all the inequalities in (84)-(86) are equalities. In particular,

\[
\dim \rho^{-1}(0, 0) = \dim(Z_{n-r} \times \Psi_{n,r}) = \dim \bar{X} - \dim(L_{n,r} \times M_r).
\]

Since by Theorem [T, Theorem 7.2] the scheme \( Z_{n-r} \) is integral and so \( Z_{n-r} \times \Psi_{n,r} \) is integral as well, (83) and (87) yield the equalities of integral schemes

\[
\rho^{-1}(0, 0) = p^{-1}(0, 0) = \bar{p}^{-1}(0, 0) = Z_{n-r} \times \Psi_{n,r}.
\]

Now we invoke one auxiliary result from [T].

**Lemma 5.5.** Let \( f : X \to Y \) be a morphism of reduced schemes, where \( Y \) is a smooth integral scheme. Assume that there exists a closed point \( y \in Y \) such that for any irreducible component \( X' \) of \( X \) the following conditions are satisfied:

(a) \( \dim f^{-1}(y) = \dim X' - \dim Y \),

(b) the scheme-theoretic inclusion of fibres \( (f|_{X'})^{-1}(y) \subset f^{-1}(y) \) is an isomorphism of integral schemes.

Then

(i) there exists an open subset \( U \) of \( Y \) containing the point \( y \) such that the morphism \( f|_{f^{-1}(U)} : f^{-1}(U) \to U \) is flat, and

(ii) \( X \) is integral.
Proof. See \cite{T, Lemma 7.4}. □

Applying assertions (i)-(ii) of this lemma to $X = X_{n,r}$, $X' = X$, $Y = L_{n,r} \times M_r$, $y = (0,0)$, $f = p$, and using \cite{57} and \cite{88}, we obtain that $X_{n,r}$ is integral of dimension $(2n-r)^2 + 4(2n-r)(r+1) - r(2r+1)$. Theorem \ref{5.2} is proved.

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