The Charmonium (Bottomonium) binding at finite $T$

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The charmonium (bottomonium) binding at finite temperature is studied with static potentials extracted from the lattice QCD data of Kaczmarek et al. The bottomonium spectrum is also studied. This is relevant for Hard Probes in Heavy Ion Collisions.

I. INTRODUCTION

This talk is motivated by the seminal hard $c\bar{c}$ probe paper of Matsui and Satz [1] by the lattice QCD data on finite temperature static potentials [2,3,4,6,7,8,9] and binding of charmonia [10,11] and by the theoretical studies [12,13,14,15] on finite temperature lattice potentials.

With a finite $T$ quark potential, and modern quark model techniques one might, approximately,

- study chiral symmetry breaking, quark mass generation, a finite $T$ notice that if one maintains a confining potential, chiral symmetry is always broken, whatever the $T$,
- compute the spectrum of any hadron at finite $T$ not only the $J/\psi$ but also light mesons, baryons, etc
- compute the interaction of any hadron-hadron at finite $T$ using cluster methods like the Resonating Group Method.

Here just study the charmonium and bottomonium as prototypes to study finite $T$ quark potentials.

The charmonium is a good starting point because

\[ m_c \gg \Lambda_{QCD} , \]
\[ m_c \gg T_c . \]

Thus it is reasonable to neglect in the bound state equation, spontaneous chiral symmetry breaking, relativistic effects, coupled channels, and temperature effects (other

\[ \Delta F_1(r,T)/T_c \]

than the potential dependence on the temperature) . We can simply solve the Schrödinger equation with static lattice QCD finite $T$ potentials

Notice that a Coulomb potential is sufficient to bind an infinite number of charmonia, and therefore the loss of the linear confinement is not sufficient to melt the charmonia states. A detailed fit of the finite temperature potentials is needed to determine the melting of charmnia.
FIG. 3: Boundstates with the Free Energy potentials. In the top left we show the energy spectrum of charmonium in the enveloping potential, and in the top right we show the wavefunctions for the ground state. In the bottom left we show the energy spectrum of bottomonium in the enveloping potential, and in the bottom right we show the wavefunctions for the ground state.

II. THE FINITE TEMPERATURE STATIC QUARK POTENTIALS

We assume that the static quark-antiquark potential $V$ is due to the length $r$ of the flux tube, related the volume $V_{\text{vol}}$ of excited QCD vacuum confined into the flux tube. Our first step consists in assessing $V$, the static quark-antiquark potential,

$$dV = -\sigma dr,$$

(2)

| $T/T_c$ | $d$ | $A$ | $\lambda$ | $\sigma$ | $\alpha$ | $\alpha'$ | $\Lambda$ | $\sigma$ |
|---|---|---|---|---|---|---|---|---|
| 1.13 | 3.94 | 6.99 | 2.64 | 0 | 0 | 0.336 |
| 1.18 | 2.35 | 4.58 | 2.27 | 0 | 0 | 0.847 |
| 1.27 | 2.64 | 5.38 | 2.87 | 0 | 0 | 0.461 |
| 1.40 | 2.23 | 5.57 | 2.91 | 0 | 0 | 0 |
| 1.64 | 1.79 | 5.60 | 3.65 | 0 | 0 | 0 |
| 1.95 | 1.51 | 5.57 | 4.15 | 0 | 0 | 0.117 |
| 2.61 | 1.21 | 6.12 | 5.44 | 0 | 0 | 0 |
| 4.50 | -40.7 | 12.6 | 5.44 | 1.00 | 0.018 | 0 |
| 7.00 | -74.0 | 15.9 | 5.44 | 0.49 | 0.021 | 0 |

TABLE I: Fitting parameters of the free energy $F_{1T}(r)$.

TABLE II: Fitting parameters of the internal energy $U_{1T}(r)$.
where here $\sigma$ is a force, generalizing the string tension. In lattice QCD, with the Polyakov Loop, one computes $F_1$, the Free Energy

$$dF_1 = -\sigma dr - SdT,$$  

which equals the potential for adiabatic transformations. Naturally close to the phase transition temperature $T_c$, the transformations are nearly isothermal and and, the potential $V$ is close to the internal energy $U_1$, while far from $T_c$ the transformations are nearly adiabatic potential is close to the free energy $F_1$. The interpolating relation for $V$ between $F_1$ and $U_1$ has been derived by Wong.

Lattice QCD provides potentials and energies, both with quenched (pure gauge) and dynamical (with

$$T/T_c \quad E_{10}(MeV) \quad B(MeV) \quad \sqrt{(r^2)}(fm)$$

| $T/T_c$       | $E_{10}$  | $B$   | $\sqrt{(r^2)}$ |
|---------------|-----------|-------|----------------|
| 0             | 660       | $-\infty$ | 0.390         |
| 1.05          | 465       | $-24.7$   | 0.921         |
| 1.17          | 465       | $-0.593$  | 4.875         |

TABLE III: Groundstate solutions of the charmonium at different temperatures, for the free energy $F_{1T}(r)$.

$T/T_c \quad E_{10}(MeV) \quad B(MeV) \quad \sqrt{(r^2)}(fm)$

| $T/T_c$       | $E_{10}$  | $B$   | $\sqrt{(r^2)}$ |
|---------------|-----------|-------|----------------|
| 0.08         | 289       | $-\infty$ | 0.242         |
| 1.05         | 245       | $-245$   | 0.278         |
| 1.20         | 217       | $-132$   | 0.314         |
| 1.50         | 189       | $-61$    | 0.397         |

TABLE IV: Groundstate solutions of the bottomonium at different temperatures for the free energy $F_{1T}(r)$.
fermions) for quarks in confined system such as mesons, diquarks, baryons, tetraquarks, pentaquarks and hybrids. Lattice also computes spectral distribution functions, to access directly, say, the bottomonium masses. However the limitations of the lattice QCD potentials are,
- static potentials only have been computed,
- a constant shift of the potential is undetermined
- few spin dependent potentials are so far computed,
- lattices have a relatively small volume.

Nevertheless we can extract the potentials from lattice data and use them to compute the spectra with the Schrödinger equation.

### III. FITTING BETWEEN THE $T=0$ ENVELOPE AND THE FINITE $T$ SATURATION

In the plots for the free energy $F_{1T}$ and the internal energy $U_{1T}$ for the different temperatures $T$ as in Figs. 1 and 2, it is clear that each set of energies is bound from above by a common enveloping function.

The enveloping functions are respectively the $T = 0$ energies, $U_{10}$ and $F_{10}$. Notice that the $T = 0$ free (internal) energy also coincides with the small distance part of the free (internal) energy functions.

In what concerns the large distance part of the free (internal) energy, for $T < T_c$ the string tension decreases, while for $T > T_c$ the string tension vanishes and the free (internal) energy saturates.

Thus we fit the free (internal) energies in three steps:
- first we fit the enveloping function
- then we fit the long distance saturation
- finally we match the long distance part to the short distance par of the energy.

We fit the enveloping function $V(r)$ of the free energies $F_{1T}(r)$ with a constant shift $c$, two Coulomb potentials, one screened for the short distance and another for the long distance part of the potential, and a linear potential,

$$V(r) = c + \frac{-\alpha + \alpha' e^{-\lambda r}}{r} + \sigma r.$$  \hfill (5)

The parameters are shown in Table I.

We fit the long distance part of the free energies $F_{1T}(r)$ with a saturation function $M_T(r)$, including a constant shift $d$ and an exponential decay,

$$M_T(r) = d_T - A e^{-\lambda_T r}.$$  \hfill (6)

The parameters are shown in Table I.

The interpolation between the short distance $F_{0}(r)$ and the long distance $M_T(r)$ is performed with the coupled channel method, where we choose the lowest eigenvalue of the matrix

$$\begin{pmatrix} V(r) & w_T \\ w_T^* & M_T(r) \end{pmatrix},$$

where the interpolating parameter $w_T$ is shown in Table I.

| $M$ [MeV] | $T/T_c$ | $l$ | $n$ | $B$ [MeV] | rms [Fm] |
|-----------|---------|----|----|-----------|---------|
| 4750      | 1.13    | 0  | 0  | -810      | 0.244   |
| 1 -224    | 0.521   |
| 2 -5      | 1.752   |
| 1 0 -400  | 0.381   |
| 1 -31     | 0.924   |
| 2 0 -118  | 0.541   |
| 1.18      | 0       | -683 | 0.252 |
| 1 -168   | 0.596   |
| 2 -10    | 1.723   |
| 1 0 -302  | 0.413   |
| 1 -30    | 1.068   |
| 2 0 -68   | 0.659   |
| 1.18      | 0       | -683 | 0.257 |
| 1 -168   | 0.596   |
| 2 -10    | 1.723   |
| 1 0 -302  | 0.413   |
| 1 -30    | 1.068   |
| 2 0 -68   | 0.659   |

| $M$ [MeV] | $T/T_c$ | $l$ | $n$ | $B$ [MeV] | rms [Fm] |
|-----------|---------|----|----|-----------|---------|
| 1300      | 1.13    | 0  | 0  | -355      | 0.458   |
| 1.18      | 0       | -273 | 0.509 |
| 1.4       | 0       | -121 | 0.618 |
| 1.64      | 0       | -78  | 0.720 |
| 1.95      | 0       | -17  | 1.199 |
| 14.5      | 0       | -1.4 | 3.3606 |
| 1752      | 1.13    | 0  | 0  | -462      | 0.389   |
| 0 1       | -316    | 7.99  |
| 1 0       | -42     | 0.820 |
| 1.18      | 0       | -365 | 0.422 |
| 0 1       | -4      | 2.981 |
| 1 0       | -33     | 1.090 |
| 1.27      | 0       | -195 | 0.477 |
| 1.4       | 0       | -136 | 0.539 |
| 1.64      | 0       | -55  | 0.687 |
| 1.95      | 0       | -22  | 0.936 |

**TABLE V:** Charmonium boundstates computed with the internal energy $U_{1T}(r)$

**TABLE VI:** Bottomonium boundstates computed with the internal energy $U_{1T}(r)$
Finally the free energy is,

\[ F_T(r) = \frac{V(r) + M_T(r) - \sqrt{[V(r) - M_T(r)]^2 + |u|^2}}{2}. \] (8)

The fit is achieved with 5 constant parameters for \( V(r) \), and 4 parameters per temperature for the long distance part \( M_T(r) \) and for the matching parameter \( w \).

In what concerns the internal energy \( U_{1T}(r) \), a fit with similar functions but with different parameters is performed, except for the higher temperatures that need an extra pair of parameters in the saturation function,

\[ M_T(r) = d_T - A e^{-\lambda_T r} - A' e^{-\lambda'_T r}. \] (9)

The parameters of the internal energy are listed in Table II.

### IV. SOLVING THE SCHRÖDINGER EQUATION

To solve the Schrödinger equation

\[ -\frac{\hbar^2}{2m} \nabla^2 \psi(r) + V(r) \psi = E \psi(r) \] (10)

we separate the eigenfunctions as

\[ \psi(r) = \frac{u(r)}{r} Y_{lm}(\theta, \varphi), \] (11)

and we get the following equation for the radial component \( u(r) \) ( \( \hbar = 1 \))

\[ -\frac{1}{2m} \frac{d^2 u_{nl}}{dr^2} + \frac{l(l+1)}{2mr^2} u_{nl} + V(r) u_{nl} = E_{nl} u_{nl}. \] (12)

To solve the radial equation we discretize it, with the finite difference substitution

\[ \frac{d^2 u}{dr^2} \rightarrow \frac{1}{a^2} (u_{i+1} - 2u_i + u_{i-1}). \] (13)

We also impose the Dirichlet boundary conditions \( u_0 = u_N = 0 \), consistent with the radial equation. So we get the linear system

\[ H_{ij} u_j = E u_i \] (14)

where \( H_{ij} \) is tridiagonal. We solve the linear system for the lowest eigenvalues, by using the inverse iteration method.

The results depend on the mass of the heavy quarks, affecting the kinetic energy and on the temperature, affecting the potential. The charm mass and the bottom mass, in different potential models, ranges respectively from 1300 MeV to 1752 MeV and from 4750 to 5100 MeV.

Since the lattice potentials have a constant energy shift undefined, we don’t show the total energy of the systems, only the binding energy, defined with the difference between the potential at infinity and the boundstate energy. For instance for \( T = 0 \), where confinement occurs, the binding energy is \(-\infty\).

### V. RESULTS

We now show the results of the boundstate equations. A first study of the melting of the boundstates can be achieved with the enveloping potentials, for the internal energy and for the free energy. Comparing the energy levels of the enveloping potentials with the saturation energies at the different temperatures, we estimate the melting temperatures. Essentially, we can estimate melting to occur when the energy level is close to the saturation energy.

Thus we first solve the Schrödinger equation with the \( V(r) \) enveloping potential for the charmonium and for the bottomonium and compare the energy levels with the potentials obtained in finite temperature. This is depicted in Figs. [9] and [10]. It occurs that only few of the temperatures \( T > T_c \) will provide binding for the charmonia, while the bottomonia survives up to higher temperatures. It also appears that the internal energy...
may provide binding up to higher temperatures than the free energy.

We then study binding for the different potentials extracted from lattice QCD simulations of Kaczmarek [2, 3, 4, 5, 6], with the free energy (quenched or dynamical) the J/ψ and ηc melt at T/T_c ≃ 1.17 and the groundstate bottomonium melts at T/T_c ≃ 1.8. With the internal energy (quenched) the groundstate charmonium melts at T/T_c ≃ 1.6, while the p-wave excitations, say the χ_c, melt at T/T_c ≃ 1.1, while the bottomonium groundstate melts at T/T_c ≃ 3.2, the p-wave χ_b melts at T/T_c ≃ 1.7, the d-wave melts at T/T_c ≃ 1.2. Examples of wavefunctions are also show in Figs. 3 and 4. The details of the binding are shown in Tables III and IV for the free energy and in Tables V, VII and VII for the internal energy.

VI. CONCLUSION

Wong [12] studied how the static potential interpolates between the free energy and the internal energy at different temperatures. Wong showed that at T ≃ T_c the free energy approximates the static potential, while at T >> T_c the static potential is closer to the internal energy.

We study the binding of charmonium and bottomonium, both for the free energy and for the internal energy, with a finite difference lattice of 200001 points. Our study reproduces the charmonium melting temperatures in the literature.

In particular the J/ψ melts above T_c (at 1.6 T_c according to lattice results) but the excited charmonium χ_c or ψ^* are probes that melt just above T_c. New in our results are the detailed studies of the bottomonium, relevant for the LHC, with a melting as high as T/T_c ≃ 3.2.

Quantitative Puzzles remain in understanding the mass shifts > 300 MeV, unseen in the experimental data, and in the importance of the the spin dependent potentials, missing in the lattice QCD data.

Possible future efforts may be to,
- calibrate the best we can the T-dependent quark-antiquark potentials,
- apply the potentials to chiral symmetry breaking/restoration,
- apply chiral restoration to the light quark mesons ρ, ω, φ and π, K.

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