FAMILIES OF EULERIAN FUNCTIONS INVOLVED IN
REGULARIZATION OF DIVERGENT POLYZETAS

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Abstract. Extending the Eulerian functions, we study their relationship with zeta function of several variables. In particular, starting with Weierstrass factorization theorem (and Newton-Girard identity) for the complex Gamma function, we are interested in the ratios of $\zeta(2k)/\pi^{2k}$ and their multiindexed generalization, we obtain an analogue situation and draw some consequences about a structure of the algebra of polyzetas values, by means of some combinatorics of words and noncommutative rational series. The same frameworks also allow to study the independence of a family of eulerian functions.

Résumé: En généralisant les fonctions euleriennes, nous étudions leurs relations avec la fonction zêta en plusieurs variables. En particulier, à partir du théorème de factorisation de Weierstrass (et l’identité de Newton-Girard) pour la fonction Gamma complexe, nous nous intéressons aux rapports $\zeta(2k)/\pi^{2k}$ et leurs généralisations. Nous obtenons une situation analogue et nous tirerons quelques conséquences sur une structure de l’algèbre des valeurs polyzêtas, au moyen de la combinatorique des mots et des séries rationnelles en variables non commutatifs. Le même cadre de travail permet également d’étudier l’indépendance d’une famille de fonctions euleriennes.

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1. Introduction

Eulerian functions are most significant for analytic number theory and they are widely applied in Probability theory and in Physical sciences. They are tightly relating to Riemann zeta functions, for instance as follows

\[ \zeta(s) = \frac{1}{\Gamma(s)} \int_0^\infty dt \frac{t^{s-1}}{e^t - 1} \quad \text{and} \quad \Gamma(s) = \int_0^\infty du \, u^{s-1} e^{-u}, \quad \text{for} \ \Re(s) > 0. \] (1.1)

The function \( \Gamma \) is meromorphic, with no zeroes and \( -\mathbb{N}^* \) as set of simple poles. Hence \( \Gamma^{-1} \) is entire and admits \( -\mathbb{N}^* \) as set of simple zeroes. Moreover, it satisfies

\[ \frac{1}{\Gamma(z)} = \frac{1}{\Gamma(z)}. \] From Weierstrass factorization [5] and Newton-Girard identity [11], we have successively

\[ 1 = \frac{1}{\Gamma(z+1)} = e^{\gamma z} \prod_{n \geq 1} \left( 1 + \frac{z}{n} \right) e^{-\frac{\pi}{2n}} = \exp\left( \gamma z - \sum_{k \geq 2} \zeta(k) (-z)^k \right). \] (1.2)

Using the following functional equation and Euler’s complement formula, i.e.

\[ \Gamma(1+z) = z \Gamma(z) \quad \text{and} \quad \Gamma(z) \Gamma(1-z) = \pi \sin\left( \frac{z \pi}{2} \right), \]

and also introducing the partial beta function defined (for any \( a, b, z \in \mathbb{C} \) such that \( \Re a > 0, \Re b > 0, |z| < 1 \)) by

\[ B(z; a, b) := \int_0^z dt \ t^{a-1}(1-t)^{b-1} \]

and then, classically, \( B(a, b) := B(1; a, b) = \Gamma(a)\Gamma(b)/\Gamma(a+b) \), one has (for any \( u, v \in \mathbb{C} \) such that \( |u| < 1, |v| < 1 \) and \( |u+v| < 1 \)) the following expression

\[ \exp\left( -\sum_{k \geq 2} \zeta(2k) \frac{(u+v)^n - (u^n + v^n)}{n} \right) = \frac{\Gamma(1-u)\Gamma(1-v)}{\Gamma(1-u-v)}, \] (1.4)

\[ = \frac{\Gamma(u+v)\pi \sin((u+v)\pi)}{\Gamma(u)\Gamma(v) \sin(u\pi) \sin(v\pi)} \] (1.5)

\[ = \frac{\pi}{B(u,v)} (\cot(u\pi) + \cot(v\pi)). \] (1.6)

In particular, for \( v = -u \ (|u| < 1) \), one gets

\[ \exp\left( -\sum_{k \geq 1} \zeta(2k) \frac{u^{2k}}{k} \right) = 1 = \frac{1}{\Gamma(1-u)\Gamma(1+u)} = \frac{\sin(u\pi)}{u\pi}. \]

Hence, taking the logarithms and considering Taylor expansions, one obtains

\[ -\sum_{k \geq 1} \zeta(2k) \frac{u^{2k}}{k} = \log\left( 1 + \sum_{n \geq 1} \frac{(ui\pi)^{2n}}{\Gamma(2n+2)} \right) \] (1.7)

\[ = \sum_{k \geq 1} \sum_{l \geq 1} (-1)^{l-1} \sum_{n_1, \ldots, n_l \geq 1, n_1 + \ldots + n_l = k} \frac{1}{\Gamma(2n_i + 2)}. \] (1.8)

\[ \text{i.e. its coefficients are real, we will see later the combinatorial content of them.} \]
One can deduce then the following expression\(^2\) for \(\zeta(2k)\):

\[
\frac{\zeta(2k)}{\pi^{2k}} = k \sum_{l=1}^{k} \frac{(-1)^{k+l-1}}{l} \sum_{n_1, \ldots, n_l \geq 1 \atop n_1 + \ldots + n_l = k} \frac{1}{\Gamma(2n_i + 2)} \in \mathbb{Q}.
\]

Now, more generally, for any \(r \in \mathbb{N}_{\geq 1}\) and \((s_1, \ldots, s_r) \in \mathbb{C}^r\), let us consider the following several variable zeta function

\[
\zeta(s_1, \ldots, s_r) := \sum_{n_1 > \ldots > n_r > 0} n_1^{-s_1} \cdots n_r^{-s_r}
\]

which converges for \((s_1, \ldots, s_r)\) in the open sub-domain of \(\mathbb{C}^r, r \geq 1, [13, 20]\)

\(\mathcal{H}_r := \{(s_1, \ldots, s_r) \in \mathbb{C}^r \ | \forall m = 1, \ldots, r, \Re(s_1) + \ldots + \Re(s_m) > m\}\).

In the convergent cases, from a theorem by Abel, for \(n \in \mathbb{N}, z \in \mathbb{C}, |z| < 1\), its values can be obtained as the following limits

\[
\zeta(s_1, \ldots, s_r) = \lim_{z \to 1^+} \text{Li}_{s_1, \ldots, s_r}(z) = \lim_{n \to +\infty} \mathcal{H}_{s_1, \ldots, s_r}(n),
\]

where the following polylogarithms are well defined

\[
\text{Li}_{s_1, \ldots, s_r}(z) := \sum_{n_1 > \ldots > n_r > 0} \frac{z^{n_1}}{n_1^{s_1} \cdots n_r^{s_r}},
\]

\[
\frac{\text{Li}_{s_1, \ldots, s_r}(z)}{1 - z} = \sum_{n \geq 0} \mathcal{H}_{s_1, \ldots, s_r}(n) z^n,
\]

and so are the Taylor coefficients\(^3\) here simply called harmonic sums

\[
\text{H}_{s_1, \ldots, s_r} : \mathbb{N} \longrightarrow \mathbb{Q} (i.e. \text{ an arithmetic function}),
\]

\[
n \longrightarrow \text{H}_{s_1, \ldots, s_r}(n) = \sum_{n \geq n_1 > \ldots > n_r > 0} n_1^{-s_1} \cdots n_r^{-s_r}.
\]

On \(\mathcal{H}_r \cap \mathbb{N}^r\), the polyzetas can be represented by the following integral representation\(^4\) over \([0, 1]\)\(^1\)\([14]\) (here, one set \(\lambda(z) := z(1 - z)^{-1}, t_0 = 1\) and \(u_{r+1} = 1\)):

\[
\zeta(s_1, \ldots, s_r) = \int_0^1 \omega_1(t_1) \log^{s_1-1}(t_0/t_1) \cdots \int_0^{t_{r-1}} \omega_1(t_r) \log^{s_r-1}(t_{r-1}/t_r) \frac{1}{\Gamma(s_1)} \cdots \frac{1}{\Gamma(s_r)} \omega_0(u_j) \lambda(u_1 \cdots u_j) \log^{s_j-1}(1/u_j)
\]

\[
= \prod_{i=1}^r \frac{1}{\Gamma(s_i)} \int_{[0,1]} \prod_{j=1}^{r} \omega_0(u_j) \lambda(u_1 \cdots u_j) \log^{s_j-1}(1/u_j),
\]

with \(\omega_0(z) = dz/z\) and \(\omega_1(z) = dz/(1 - z)\).

As for the Riemann zeta function in \((1.1)\), we observe that \((1.16)\) involves again the factors (and products) of eulerian Gamma function and also their quotients (hence, eulerian Beta function). In the sequel, in continuation with \([8, 7, 17]\), we propose to study the ratios \(\zeta(s_1, \ldots, s_r)/\pi^{s_1 + \ldots + s_r}\) (and others), an analogue of

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\(\text{Note that Euler gave another explicit formula using Bernoulli numbers.}\)

\(\text{These quantities are generalizations of the harmonic numbers } H_n = 1 + 2^{-1} \ldots + n^{-1} \text{ to which they boil down for } r = 1, s_1 = 1. \text{ They are also truncations of the zeta values } \zeta(s_1, \ldots, s_r) \text{ at order } n + 1.\)

\(\text{On } \mathcal{H}_r, \log(a/b) \text{ is replaced by } \log(a) - \log(b).\)
will be achieved as consequence of regularizations, via the values of entire functions, of divergent polyzetas and infinite sums of polyzetas (see Theorem 2.22 and Corollaries 2.24, 2.27 in Section 2.4) for which a theorem by Abel (see (1.11)) could not help any more. This achievement is justified thanks to the extensions of polylogarithms and harmonic sums (see Theorems 2.16 and 2.18 in Section 2.3) and thanks to the study of the independence of a family of eulerian functions which can be viewed as generating series of zeta values (for $r \geq 2$):

$$(1.17) \quad \frac{1}{\Gamma_{y_{1}}(z + 1)} = \sum_{k \geq 0} \zeta(r, \ldots, r)z^{kr} = \exp \left(- \sum_{k \geq 1} \zeta(kr)(-z)^{k} \right)$$

(see Propositions 2.7, 2.11 and Theorem 2.13 in Section 2.2) via the combinatorial tools introduced in Section 2.1 (see Lemma 2.2, 2.1 in Section 2.1). Finally, identities among these (convergent or divergent) generating series of zeta values are suitable to obtain relations, at arbitrary weight, among polyzetas (see Examples 2.26 and 2.28 in Section 2.4).

2. Families of eulerian functions

In all the sequel, $\mathbb{C}[[f_{i}]]$ denotes the algebra generated by $\{f_{i}\}_{i \in I}$, $\mathbb{C}[\{(g_{i})\}_{i \in I}]$ denotes the differential $\mathbb{C}$-algebra generated by the family $(g_{i})_{i \in I}$ of the $\mathbb{C}$-commutative differential ring $(A, \partial)$ ($1_{A}$ is its neutral element) and $C_{0}$ denotes a differential subring of $A$ ($\partial C_{0} \subset C_{0}$) which is an integral domain containing the field of constants. If the ring $A$ is without zero divisors then the fields of fractions $\text{Frac}(C_{0})$ and $\text{Frac}(A)$ are naturally differential fields and can be seen as the smallest ones containing $C_{0}$ and $A$, respectively, satisfying $\text{Frac}(C_{0}) \subset \text{Frac}(A)$.

2.1. Words and formal power series. Let $X$ denote either the alphabets $X := \{x_{0}, x_{1}\}$ or $Y := \{y_{k}\}_{k \geq 1}$, equipped with a total ordering, and let $X^{*}$ denote the monoid freely generated by $X$ (its unit is denoted by $1_{X^{*}}$). The set of noncommutative polynomials (resp. series) over $X$ with coefficients in a commutative ring $A$, containing $\mathbb{Q}$, is denoted by $A\langle\langle X \rangle\rangle$ (resp. $A\langle\langle Y \rangle\rangle$) by the rational operations $\{\text{conc}, +, \ast\}$ within $A\langle\langle X \rangle\rangle$ is denoted by $A^{\text{rat}}(\langle\langle X \rangle\rangle)$.

We will also consider the following Hopf algebras and, in the case of $A = k$ being a field, their Sweedler’s duality $[9, 17]$

$$$(2.1) \quad (A\langle\langle X \rangle\rangle, \text{conc}, \Delta_{\text{shuffle}}, 1_{X^{*}}, \epsilon) \quad \text{and} \quad (A\langle\langle Y \rangle\rangle, \text{conc}, \Delta_{\text{shuffle}}, 1_{Y^{*}}, \epsilon),$$$

$$$(2.2) \quad (k^{\text{rat}}(\langle\langle X \rangle\rangle), \omega_{1}, 1_{X^{*}}, \Delta_{\text{shuffle}}, \epsilon) \quad \text{and} \quad (k^{\text{rat}}(\langle\langle Y \rangle\rangle), \omega_{1}, 1_{Y^{*}}, \Delta_{\text{shuffle}}, \epsilon).$$$

In particular, using the set of Lyndon words, denoted by $\mathcal{L}_{\text{Lynd}}X^{*}$, one constructs the basis $\{P_{i}\}_{i \in \mathcal{L}_{\text{Lynd}}X^{*}}$ for $\mathcal{L}_{\text{Lynd}}A\langle\langle X \rangle\rangle$, generating the PBW-Lyndon basis $\{P_{w}\}_{w \in X^{*}}$ for $(A\langle\langle X \rangle\rangle, \text{conc}, 1_{X^{*}})$ and then the graded dual basis $\{S_{w}\}_{w \in X^{*}}$ containing the pure transcendence basis $\{S_{i}\}_{i \in \mathcal{L}_{\text{Lynd}}X}$ for the shuffle algebra $(A\langle\langle X \rangle\rangle, \omega_{1}, 1_{X^{*}})$. Similarly,
the basis \( \{ \Pi \}_{l \in \mathcal{L}ynY} \) generating the PBW-Lyndon basis \( \{ \Pi_{w} \}_{w \in Y} \) for \( (A(Y), \conc, 1_{Y} \cdot) \) and then the graded dual basis \( \{ \Sigma_{w} \}_{w \in Y} \) containing the pure transcendence basis \( \{ \Sigma_{l} \}_{l \in \mathcal{L}ynY} \) for the stuffle algebra \( (A(\langle X \rangle), \shuffle, 1_{Y} \cdot) \).

**Lemma 2.1.** (1) The algebras \( \langle \{ x^{*} \}_{x \in X} \rangle, \omega, 1_{X} \cdot \rangle \) and \( \langle X \rangle, \omega, 1_{X} \cdot \rangle \) are algebraically disjoint over \( \mathbb{C} \) and

\[
\langle \{ x^{*} \}_{x \in X} \rangle, \omega, 1_{X} \cdot \rangle \cong \langle \{ x^{*} \}_{x \in X} \rangle, \omega, 1_{X} \cdot \rangle \cong \langle \{ x^{*}, l \}_{x \in X, l \in \mathcal{L}ynX} \rangle, \omega, 1_{X} \cdot \rangle
\]

which is generated by the transcendent basis \( \{ x^{*}, l \}_{x \in X, l \in \mathcal{L}ynX} \) over \( \mathbb{C} \).

(2) Let \( K := \mathbb{C}[\{ f(x^{*}) \}_{x \in X}] \) and \( F := \mathbb{C}[\{ f(l) \}_{l \in \mathcal{L}ynX}] \).

Let \( f \) be the shuffle morphism \( \langle \mathbb{C}[\{ x^{*} \}_{x \in X}], \omega, 1_{X} \cdot \rangle \rightarrow \langle A, \times, 1_{A} \rangle \). Then the following assertions are equivalent

(a) The morphism \( f \) is injective.

(b) The algebras \( K \) and \( F \), satisfying \( K \cap F = \mathbb{C}.1_{A} \), are generated by the transcendent bases \( \{ f(x^{*}) \}_{x \in X} \) and \( \{ f(l) \}_{l \in \mathcal{L}ynX} \), respectively, over \( \mathbb{C} \).

Hence, if (2a) or (2b) holds then \( F, K \) are algebraically disjoint over \( \mathbb{C} \) and

\[
\mathbb{C}[\{ f(x^{*}) \}_{x \in X}] \mid \mathbb{C}[\{ f(l) \}_{l \in \mathcal{L}ynX}] \cong \mathbb{C}[\{ f(x^{*}), f(l) \}_{x \in X, l \in \mathcal{L}ynX}]
\]

which is generated by the transcendent basis \( \{ f(x^{*}), f(l) \}_{x \in X, l \in \mathcal{L}ynX} \) over \( \mathbb{C} \).

**Proof.** Recall that the algebras \( \langle \{ x^{*} \}_{x \in X} \rangle, \omega, 1_{X} \cdot \rangle \) and \( \langle X \rangle, \omega, 1_{X} \cdot \rangle \) are generated, respectively, by the transcendent bases \( \{ x^{*} \}_{x \in X} \) \( \mathbb{C}[9] \) and \( \mathcal{L}yn\mathcal{X} \) \( \mathbb{C}[9] \). Moreover, \( \{ x^{*} \}_{x \in X} \) is also algebraically independent over \( \mathbb{C}[\langle X \rangle] \mathbb{C}[9] \) and then \( \mathbb{C}[\{ x^{*} \}_{x \in X}] \cap \mathbb{C}[\langle X \rangle] = \mathbb{C}.1_{X} \cdot \). It follows the then expected results.

(2) Straightforward.

Now, for any \( r \geq 1 \), let us consider the following differential form

\[
(2.3) \quad \omega_{r}(z) = u_{y_{r}}(z)dz \quad \text{with} \quad u_{y_{r}} \in \mathcal{C}_{0} \subset A.
\]

Let us also consider the following noncommutative differential equation (see \( \mathbb{C}[9] \))

\[
(2.4) \quad dS = MS; \quad \langle S|1_{X} \cdot \rangle = 1_{A}, \quad \text{where} \quad M = \sum_{x \in X} u_{x}x \in \mathcal{C}_{0}X,
\]

where \( d \) is the differential operator on \( A(\langle X \rangle) \) extending \( \partial \) as follows:

\[
(2.5) \quad \forall S = \sum_{w \in X^{*}} \langle S|w \rangle w \in A(\langle X \rangle), \quad dS = \sum_{w \in X^{*}} (\partial \langle S|w \rangle) w.
\]

In order to prove Proposition \( \mathbb{C}[2.3] \), Theorems \( \mathbb{C}[2.13] \) and \( \mathbb{C}[2.16] \) below, we use the following lemma, a particular case of a general localization result to be proved in a forthcoming paper \( \mathbb{C}[9] \).

**Lemma 2.2.** Suppose that the \( \mathbb{C} \)-commutative ring \( A \) is without zero divisors and equipped with a differential operator \( \partial \) such that \( \mathbb{C} = \ker \partial \).

Let \( S \in A(\langle X \rangle) \) be a group-like solution of \( \mathbb{C}[2.4] \), in the following form

\[
S = 1_{X} \cdot + \sum_{w \in X^{*}X} \langle S|w \rangle w = 1_{X} \cdot + \sum_{w \in X^{*}X} \langle S|S_{w} \rangle P_{w} = \prod_{l \in \mathcal{L}ynX} e^{\langle S|l \rangle P_{l}}.
\]

Then
(1) If \( H \in \mathcal{A}(\langle X \rangle) \) is another group-like solution of \( (2.4) \) then there exists \( C \in \text{Lie}_e(\langle X \rangle) \) such that \( S = H e^C \) (and conversely).

(2) The following assertions are equivalent:
   
   \[(a) \{ (S|w) \}_{w \in X^*} \text{ is } \mathcal{C}_0\text{-linearly independent,} \]
   
   \[(b) \{ (S|l) \}_{l \in \mathcal{C}_0(X)} \text{ is } \mathcal{C}_0\text{-algebraically independent,} \]
   
   \[(c) \{ (S|x) \}_{x \in X} \text{ is } \mathcal{C}_0\text{-algebraically independent,} \]
   
   \[(d) \{ (S|x) \}_{x \in X \cup \{1_X\}} \text{ is } \mathcal{C}_0\text{-linearly independent,} \]
   
   \[(e) \text{ The family } \{ u_x \}_{x \in X} \text{ is such that, for } f \in \text{Frac}(C_0) \text{ and } (c_x)_{x \in X} \in \mathbb{C}(x), \]
   
   \[
   \sum_{x \in X} c_x u_x = \partial f \implies (\forall x \in X)(c_x = 0). \]
   
   \[(f) \text{ The family } (u_x)_{x \in X} \text{ is free over } \mathbb{C} \text{ and } \text{Frac}(C_0) \cap \text{span}_\mathbb{C} \{ u_x \}_{x \in X} = \{0\}. \]

**Sketch.** The first item has been treated in \( [10] \). The second is a group-like version of the abstract form of Theorem 1 of \( [4] \). It goes as follows:

- due to the fact that \( \mathcal{A} \) is without zero divisors, we have the following embeddings \( \mathcal{C}_0 \subset \text{Frac}(C_0) \subset \text{Frac}(\mathcal{A}) \), \( \text{Frac}(\mathcal{A}) \) is a differential field, and its derivation can still be denoted by \( \partial \) as it induces the previous one on \( \mathcal{A} \),

- the same holds for \( \mathcal{A}(\langle X \rangle) \subset \text{Frac}(\mathcal{A}(\langle X \rangle)) \) and \( \mathcal{A} \),

- therefore, equation \( (2.4) \) can be transported in \( \text{Frac}(\mathcal{A}(\langle X \rangle)) \) and \( M \) satisfies the same condition as previously.

- Equivalence between \( 2a \) \( 2d \) comes from the fact that \( \mathcal{C}_0 \) is without zero divisors and then, by denominator chasing, linear independences w.r.t \( \mathcal{C}_0 \) and \( \text{Frac}(C_0) \) are equivalent. In particular, supposing condition \( 2d \), the family \( \{ (S|x) \}_{x \in X \cup \{1_X\}} \) (basic triangle) is \( \text{Frac}(C_0)\)-linearly independent which imply, by the Theorem 1 of \( [4] \), condition \( 2c \).

- still by Theorem 1 of \( [4] \), \( 2c \) is equivalent to \( 2a \) implying that \( \{ (S|w) \}_{w \in X^*} \) is \( \text{Frac}(C_0)\)-linearly independent which induces \( \mathcal{C}_0\)-linear independence (i.e. \( 2a \)).

Now, let \( \mathcal{A} = \mathcal{H}(\Omega) \), the ring of holomorphic functions on a simply connected domain \( \Omega \subset \mathbb{C} \) (1(\mathcal{H}(\Omega)) is its neutral element). With the notations in \( (2.3) \) and \( (2.4) \) for any path \( z_0 \rightsquigarrow z \in \Omega \), let \( \alpha_{z_0}^z : (\mathcal{C}^0(\langle X \rangle), \omega, 1_X^* \rightarrow (\mathcal{H}(\Omega), \times, 1_\mathcal{A}) \) be the morphism defined, for any \( x_1, \ldots, x_k \in X^* \), by \( [9] \)

\[
(2.6) \alpha_{z_0}^z(x_1, \ldots, x_k) = \int_{z_0}^{z} \omega_1(z_1) \ldots \int_{z_0}^{z_k-1} \omega_k(z_k) \quad \text{and} \quad \alpha_{z_0}^z(1_X^*) = 1(\mathcal{H}(\Omega)),
\]

satisfying \( \alpha_{z_0}^z(u \omega v) = \alpha_{z_0}^z(u) \alpha_{z_0}^z(v) \), for \( u, v \in X^* \). By a Ree’s theorem \( [19] \), the Chen series of \( \{ \omega_t \}_{t \geq 1} \) and along the path \( z_0 \rightsquigarrow z \in \Omega \) is group-like:

\[
(2.7) \quad C_{z_0 \rightsquigarrow z} = \sum_{w \in X^*} \alpha_{z_0}^z(w) = \prod_{l \in \mathcal{C}_0(X)} e^{\alpha_{z_0}^z(S_l)P_l} \in \mathcal{H}(\Omega)(\langle X \rangle). \]

Since \( \partial \alpha_{z_0}^z(x_1, \ldots, x_k) = u_1(z) \alpha_{z_0}^z(x_2, x_3, \ldots, x_k) \) then \( C_{z_0 \rightsquigarrow z} \) is a solution of \( (2.4) \).

**Remark 2.3.** For any \( w \in X^* \), the value of \( \alpha_{z_0}^z(w) \) depends on \( \{ \omega_t \}_{1 \geq 1} \), or equivalently on \( \{ u_x \}_{x \in X} \) and if \( f_x(z) = \alpha_{z_0}^z(x) \) then, for any \( n \geq 0 \), one has \( [14] \)

\[
\alpha_{z_0}^z(x^n) = \alpha_{z_0}^z(x^{n+1})/n! = f_x^z(z)n! \quad \text{and then} \quad f_x(z) := \alpha_{z_0}^z(x^*) = e^{f_x(z)}. \]
2.2. Families of eulerian functions.

**Definition 2.4.** For any \( z \in \mathbb{C} \) such that \( |z| < 1 \), we put
\[
\ell_1(z) := \gamma z - \sum_{k \geq 2} \zeta(k) \frac{(-z)^k}{k} \quad \text{and for} \quad r \geq 2, \quad \ell_r(z) := -\sum_{k \geq 1} \zeta(kr) \frac{(-z)^k}{k}.
\]

For any \( k \geq 1 \), let \( \Gamma_{y_k}(1 + z) := e^{-\ell_k(z)} \) and \( B_{y_k}(a, b) := \frac{\Gamma_{y_k}(a) \Gamma_{y_k}(b)}{\Gamma_{y_k}(a + b)} \).

**Remark 2.5.**
1. \( (\ell_r)_{r \geq 1} \) is triangular\(^9\). So is \( (e^{\ell_r} - e^{\ell_r(0)})_{r \geq 1} \).
2. For any \( z \in \Omega = \mathbb{C}, |z| < 1 \) and \( k \geq 1 \), using Remark 2.3 one has

\[
\begin{array}{|c|c|c|}
\hline
\xi_{2|y_k|} & \alpha_2^z(y_k) & \alpha_2^z(y_k^* - 1) \\
\hline
1^{-e^{\ell_k}} & (1 - z)^{-1} & (1 - z)^{-1} \\
\hline
\partial \ell_k & \ell_k(z) & e^{\ell_k(z)} = \frac{\Gamma_{y_k}^{-1}(1 + z)}{\Gamma_{y_k}(1 + z)} \\
\hline
e^{\ell_k} \partial \ell_k & e^{\ell_k(z)} = \Gamma_{y_k}^{-1}(1 + z) & e^{\ell_k(z)} \\
\hline
\end{array}
\]

3. The function \( \ell_1 \) is already considered by Legendre for studying the eulerian Beta and Gamma functions \( [12] \), denoted here, respectively, by \( B_\tau \) and \( \Gamma_\tau \).
4. For any \( r \geq 1 \), one has \( \partial \ell_r = e^{-\ell_r} \partial e^{\ell_r} \).
5. For any \( n \geq 1 \), one puts classically \( \Psi_n := \partial^n \log \Gamma_\tau \).
6. Some of these functions cease (unlike \( \Gamma \)) to be hypertranscendental. For example, \( y(x) = \Gamma_{y^{-1}}(1 + x) \) is a solution of \( (1 - \pi^2 x^2)y^2 + 2x y' + x^2 y'' = 1 \).

Now, for any \( r \geq 1 \), let \( \mathcal{G}_\tau \) (resp. \( \mathcal{G}_\tau^r \)) denote the set (resp. group) of solutions, \( \{ \xi_0, \ldots, \xi_{r-1} \} \), of the equation \( z^r = (-1)^{-r-1} \) (resp. \( z^r = 1 \)). For \( r, q \geq 1 \), we will need also a system \( \mathbb{X} \) of representatives of \( \mathcal{G}_{qr^r}/\mathcal{G}_r \), i.e. \( \mathbb{X} \subset \mathcal{G}_{qr^r} \) such that
\[
\mathcal{G}_{qr^r} = \bigcup_{\tau \in \mathbb{X}} \tau \mathcal{G}_r.
\]

It can also be assumed that \( 1 \in \mathbb{X} \) as with \( \mathbb{X} = \{ e^{2i\pi/q} \}_{0 \leq k \leq q-1} \).

**Remark 2.6.** If \( r \) is odd then \( z^r = (-1)^{r-1} = 1 \) and \( G_r = G_r^* \) being an orbit, where \( \xi \) satisfies \( \xi^{-1} = -1 \) (this is equivalent to \( \xi \in \mathcal{G}_{2r} \) and \( \xi \notin \mathcal{G}_r \)).

**Proposition 2.7.**
1. For \( r \geq 1, \chi \in \mathcal{G}_r \) and \( z \in \mathbb{C}, |z| < 1 \), the functions \( \ell_r \) and \( e^{\ell_r} \) have the symmetry \( \ell_r(z) = \ell_r(\chi z) \) and \( e^{\ell_r(z)} = e^{\ell_r(\chi z)} \). In particular, for \( r \) even, as \( -1 \in \mathcal{G}_{2r} \), these functions are even.
2. For \( |z| < 1 \), we have
\[
\ell_r(z) = -\sum_{\chi \in G_r} \log(\Gamma(1 + \chi z)) \quad \text{and} \quad e^{\ell_r(z)} = \prod_{\chi \in G_r} e^{\gamma \chi z} \prod_{n \geq 1} \left( 1 + \frac{\chi^z}{n} \right) e^{-\chi z/n}.
\]

\(^9\)(\(g_i\))\(_{i \geq 1}\) is said to be triangular if the valuation of \( g_i, z(w_i) \), equals \( i \geq 1 \). It is easy to check that such a family is \( C \)-linearly free and that is also the case of families such that \( (g_i - g(0))_{i \geq 1} \) is triangular.

\(^10\)Indeed, we use the fact that \( \Gamma_{y^{-1}}(1 + x) = \sin(i\pi x)/i\pi x \) (see Example 2.26 below).
Proof. The results are known for \( r = 1 \) (i.e. for \( \Gamma^{-1} \)). For \( r \geq 2 \), we get

1. By Definition \ref{defn:1} with \( \chi \in \mathcal{G}_r \), we get
   \[
   \ell_r(z) = -\sum_{k \geq 1} \zeta(kr) \frac{(-z^r)^k}{k} = -\sum_{k \geq 1} \zeta(kr) \frac{(-z^r)^k}{k} = \ell_r(z),
   \]
   thanks to the fact that, for any \( \chi \in \mathcal{G}_r \), one has \( \chi^r = 1 \). In particular, if \( r \) is even then \( \ell_r(z) = \ell_r(-z) \), i.e. \( \ell_r \) is even.

2. If \( r \) is odd, as \( \mathcal{G}_r = \mathcal{G}_r \) and, applying the symmetrization principle\footnote{Within the same disk of convergence as \( f \), one has, \( f(z) = \sum_{n \geq 1} a_n z^n \) and \( \sum_{\chi \in \mathcal{G}_r} f(\chi z) = r \sum_{k \geq 1} a_k z^{rk} \).}, we get
   \[
   -\sum_{\chi \in \mathcal{G}_r} \ell_1(\chi z) = -\sum_{\chi \in \mathcal{G}_r} \ell_1(\chi z) = \ell_r(z).
   \]
   The last term being due to, precisely, \( r \) is odd. If \( r \) is even, we have the orbit \( \mathcal{G}_r = \xi \mathcal{G}_r \) (still with \( \xi^r = -1 \)) and then, by the same principle,
   \[
   -\sum_{\chi \in \mathcal{G}_r} \ell_1(\chi \xi z) = r \sum_{k \geq 1} \zeta(kr) \frac{(-\xi z)^k}{kr} = \sum_{k \geq 1} \zeta(kr) \frac{(-\xi z)^k}{k}.
   \]

3. Straightforward.

4. Due to the fact that the external product is finite, we get
   \[
   e^{\ell_r(z)} = \left( \prod_{\chi \in \mathcal{G}_r} e^{-\chi z} \right) \prod_{n \geq 1} \left( 1 + \frac{\chi z}{n} \right) e^{-\chi z/n} = \left( \prod_{n \geq 1} e^{-\frac{\chi z}{n}} \right) \prod_{\chi \in \mathcal{G}_r} \left( 1 + \frac{\chi z}{n} \right).
   \]
   Using the elementary symmetric functions of \( \mathcal{G}_r \), we get the expected result. \( \square \)

Proposition 2.8. Let \( L := \text{span}_\mathbb{C}\{\ell_r\}_{r \geq 1} \) and \( E := \text{span}_\mathbb{C}\{e^{\ell_r}\}_{r \geq 1} \). Let \( \mathbb{C}[L] \) and \( \mathbb{C}[E] \) be their respective algebra. One has

1. The family \( \{\ell_r\}_{r \geq 1} \) is \( \mathbb{C} \)-linearly free and free from \( 1_{\mathcal{H}(\Omega)} \).
2. The family \( \{\ell_r\}_{r \geq 1} \) and \( \{e^{\ell_r}\}_{r \geq 1} \) is \( \mathbb{C} \)-linearly free and free from \( 1_{\mathcal{H}(\Omega)} \).
3. The families \( \{\ell_r\}_{r \geq 1} \) and \( \{e^{\ell_r}\}_{r \geq 1} \) are \( \mathbb{C} \)-algebraically independent.
4. For any \( r \geq 1 \), one has

---

\( \Gamma^{-1} \)
(a) The functions \( \ell_r \) and \( e^{\ell_r} \) are \( \mathbb{C} \)-algebraically independent.
(b) The function \( \ell_r \) is holomorphic on the open unit disc, \( D_{1} \).
(c) The function \( e^{\ell_r} \) (resp. \( e^{-\ell_r} \)) is entire (resp. meromorphic), and admits a countable set of isolated zeroes (resp. poles) on the complex plane which is expressed as \( \bigcup_{x \in G} x \mathbb{Z} \leq 1 \).

(5) One has \( E \cap L = \{0\} \) and, more generally, \( \mathbb{C}[E] \cap E_c[L] = \mathbb{C}1_{\mathcal{H}(\Omega)} \).

Proof. 

(1) Suppose that there is \((a_r)_{r \geq 1} \in \mathbb{C}^{(N)}\) such that

\[
\sum_{r \geq 1} a_r \ell_r(z) = a_1 \gamma z - \sum_{k \geq 2} a_1 \zeta(k) \frac{(-1)^k}{k} z^k - \sum_{r \geq 2} \sum_{k \geq 1} a_r \zeta(kr) \frac{(-1)^k}{k} z^{rk} = 0,
\]

in which, since \( \gamma \neq 0 \) then \( a_1 = 0 \). It follows that

\[
\sum_{r \geq 2} \sum_{k \geq 1} a_r \zeta(kr) \frac{(-1)^k}{k} z^{rk} = 0
\]

in which \( \langle LS \rangle z^2 = a_2 \zeta(2)/2 \). Since \( \zeta(2) \neq 9 \) then \( a_2 = 0 \). It also follows that

\[
\sum_{r \geq 3} \sum_{k \geq 1} a_r \zeta(kr) \frac{(-1)^k}{k} z^{rk} = 0.
\]

In similar way, one proves that \( a_r = 0 \), for \( r \in \mathbb{N}^+ \). Hence, \( (\ell_r)_{r \geq 1} \) is \( \mathbb{C} \)-free.

(2) Suppose that there is \((b_i)_{i \geq 1} \in \mathbb{C}^{(N)}\) such that

\[
\sum_{i \geq 1} a_i \dot{\ell}_i = 0 \quad \text{and then} \quad \sum_{i \geq 1} a_i \dot{\ell}_i = 0
\]

(taking the logarithmic derivative). By integration, one deduces then \( (\ell_r)_{r \geq 1} \) is \( \mathbb{C} \)-linearly dependent contradicting with the item [1] It remains that \((e^{\ell_r})_{i \geq 1} \in \mathbb{C}^{(N)}\) is \( \mathbb{C} \)-free.

(3) Using Chen series of \( \{\omega_r\}_{r \geq 1} \) defined, as in Remark 2.16 by \( \{u_r, \partial \ell_r\} \), via items 2a or 2b of Lemma 2.2 \( \{e^{\ell_r}\}_{r \geq 1} \) is \( \mathbb{C} \)-algebraically independent.

(4) (a) Since \( \ell_r(0) = 0, \partial e^{\ell_r} = e^{\ell_r} \partial \ell_r \) then \( \ell_r \) and \( e^{\ell_r} \) are \( \mathbb{C} \)-algebraically independent.
(b) One has \( e^{\ell_r(z)} = \Gamma^{-1}(1 + z) \) which proves the claim for \( r = 1 \). For \( r \geq 2 \), note that \( 1 \leq \zeta(r) \leq \zeta(2) \) which implies that the radius of convergence of the exponent is 1 and means that \( \ell_r \) is holomorphic on the open unit disc. This proves the claim.
(c) \( e^{\ell_r(z)} = \Gamma^{-1}(1 + z) \) (resp. \( e^{-\ell_r(z)} = \Gamma^{-1}(1 + z) \)) is entire (resp. meromorphic) as finite product of entire (resp. meromorphic) functions and, by Proposition 2.7, Weierstrass factorization yields zeroes (resp. poles).

(5) Let \( f \in E \cap L \) and then there is \( \{c_y\}_{y \in Y} \) and \( \{d_y\}_{y \in Y} \in \mathbb{C}^{(Y)} \) such that

\[
f = \sum_{r \geq 1} c_y \ell_r = \sum_{r \geq 1} d_y e^{\ell_r}
\]

If \( f \neq 0 \) then \( \ell_{r_0}, e^{\ell_{r_0}} \) could be linearly dependent, for some \( r_0 \geq 1 \), contradicting with item [1] Hence, \( E \cap L = \{0\} \).

\( \mathbb{C}[E] \) (resp. \( \mathbb{C}[F] \)) is generated freely by \( (e^{\ell_r})_{r \geq 1} \) (resp. \( (\ell_r)_{r \geq 1} \)) which are entire (resp. holomorphic on \( D_{1} \)) functions. Moreover, any \( \mathbb{C}[E] \ni
\[ f \neq c1_{\Omega} \ (c \in \mathbb{C}) \] is entire and then \( f \notin \mathbb{C}[L] \) (and conversely). It follows the expected result.

By Lemma \ref{lem:example_2.1} Proposition \ref{prop:2.8} and Remark \ref{rem:2.5}, one deduces then

**Corollary 2.9.** The map \( \alpha_{\Omega}^0 : (\mathbb{C}(Y), \omega, 1_{Y^*}) \longrightarrow (\text{span}_\mathbb{C}\{\alpha_{\Omega}^0(w)\}_{w \in Y^*}, \times, 1_{H(\Omega)}) \) is injective, for the inputs \( \{\partial\ell_r\}_{r \geq 1} \) or \( \{e^{\ell_r}, \partial\ell_r\}_{r \geq 1} \), and then \( \{\alpha_{\Omega}^0(w)\}_{w \in Y^*} \) (resp. \( \{\alpha_{\Omega}^0(l)\}_{l \in \mathcal{Z}_{ynY}} \)) is linearly (resp. algebraically) independent over \( \mathbb{C} \).

From now on the countable set of isolated zeros (resp. poles) of the entire (resp. meromorphic) function \( e^{\ell_r} \) (resp. \( e^{-\ell_r} \)) is denoted by \( \mathcal{O}(e^{\ell_r}) \). We have

\[
\mathcal{O}(e^{\ell_r}) = \bigcup_{\chi \in G_r} \chi\mathbb{Z}_{\leq -1}.
\]

**Example 2.10.** One has

\[
\mathcal{O}(e^{\ell_1}) = \mathbb{Z}_{\leq -1},
\]
\[
\mathcal{O}(e^{\ell_2}) = -i\mathbb{Z}_{\leq -1} \cup i\mathbb{Z}_{\leq -1} = i\mathbb{Z}_{\neq 0},
\]
\[
\mathcal{O}(e^{\ell_3}) = \mathbb{Z}_{\leq -1} \cup j\mathbb{Z}_{\leq -1} \cup j^2\mathbb{Z}_{\leq -1},
\]
\[
\mathcal{O}(e^{\ell_4}) = (1 + i)\sqrt{2}\mathbb{Z}_{\neq 0} \cup (1 - i)\sqrt{2}\mathbb{Z}_{\neq 0}.
\]

**Proposition 2.11.** Let \( X \) denote any system of representatives of \( \mathcal{G}_{qr}/\mathcal{G}_r \).

1. For any \( r \geq 1 \) and odd \( q \geq 1 \), one has, for \( |z| < 1 \),

\[
e^{\ell_{qr}}(z) = \prod_{\chi \in X} e^{\ell_r}(\chi z), \quad \text{or equivalently,} \quad \Gamma_{y_{qr}}^{-1}(1 + z) = \prod_{\chi \in X} \Gamma_{y_r}^{-1}(1 + \chi z).
\]

2. \( e^{\ell_r} \) divides \( e^{\ell_{qr}} \) if and only if \( q \) is odd.

3. The full symmetry group of \( e^{\ell_r} \) for the representation \( s \ast f[z] = f(sz) \) is \( \mathcal{G}_r \).

**Proof.** (1) Let \( \xi \) be any root of \( z^r = (-1)^{r-1} \), one remarks that, in all cases \( r \) be odd or even, we have

\[
G_r = \xi\mathcal{G}_r, \quad G_{qr} = \xi\mathcal{G}_{qr}, \quad \mathcal{G}_{qr} = \bigcup_{\chi \in X} \chi\mathcal{G}_r.
\]

Then, by Proposition \ref{prop:2.7} we have

\[
\ell_{qr}(z) = \sum_{\chi \in \mathcal{G}_{qr}} \ell_1(\chi z)
\]
\[
= \sum_{\rho_1 \in \mathcal{G}_{qr}} \ell_1(\xi\rho_1 z) = \sum_{\chi \in X, \rho_2 \in \mathcal{G}_r} \ell_1(\xi\rho_2 \chi z)
\]
\[
= \sum_{\chi \in X, \rho_2 \in \mathcal{G}_r} \ell_1(\xi\rho_2 (\chi z)) = \sum_{\chi \in X} \ell_r(\chi z)
\]
\[
= \ell_r(z) + \sum_{\chi \in X \setminus \{\xi\}} \ell_r(\chi z).
\]

Last equality assumes that \( 1 \in X \). Taking exponentials, we get

\[
e^{\ell_{qr}}(z) = \prod_{\chi \in X} e^{\ell_r}(\chi z) = e^{\ell_r}(z) \prod_{\chi \in X \setminus \{1\}} e^{\ell_r}(\chi z).
\]

Again, first equality is general and the last assumes that \( 1 \in X \).
(2) The fact that $e^{\ell_r}$ divides $e^{\ell_{q,r}}$ if $q$ is odd comes from the factorization $2.9$. Now, when $q$ even, it suffices to remark, from $(2.13)$, that the opposite of any solution of $z' = -1$ is a zero of $e^{\ell_r}$ and $O(e^{\ell_{q,r}}) \cap U = -G_{q,r}$. But when $q$ is even one has $-G_{q,r} \cap -G_{q,r} = \emptyset$. Hence, in this case, $e^{\ell_r}$ cannot divide $e^{\ell_{q,r}}$.

(3) Let $G$ denote the symmetry group of $e^{\ell_r}$ and remark that the distance of $O(e^{\ell_r})$ to zero is 1. Hence, as $O(e^{\ell_r}(s,z)) = s^{-1}O(e^{\ell_r})$, we must have $G \subset U$. Then, by Remark 2.6 as $O(e^{\ell_r}) \cap U = G_r$, we must have $G \subset G_r$, the reverse inclusion is exactly the first point of Proposition 2.7.

Example 2.12. (1) For $r = 1, q = 2, X = \{1, -1\}$, one has the Euler’s complement like formula, i.e. $\Gamma_{y_1}(1 + iz) = \Gamma_{y_1}(1 + z)\Gamma_{y_1}(1 - z) = z\pi / \sin(\pi z)$. Changing $z \mapsto -iz$, one also has $\Gamma_{y_2}(1 + iz) = \Gamma_{y_1}(1 + iz)\Gamma_{y_1}(1 - iz)$.

(2) For $r = 2, q = 3, X = \{1, 1.2, \ldots\}$, one has $\Gamma_{y_3}(1 + z) = \Gamma_{y_2}(1 + z)\Gamma_{y_2}(1 + iz)\Gamma_{y_2}(1 + jz)\Gamma_{y_2}(1 + jz)\Gamma_{y_2}(1 + jz)$.

With the notations of Proposition 2.8, the algebra $C[L]$ (resp. $C[E]$) is generated freely by $(\ell_{r})_{r \geq 1}$ (resp. $(e^{\ell_r})_{r \geq 1}$) which are holomorphic on $D_{<1}$ (resp. entire) functions. Moreover,

$$E \cap L = \{0\}, \text{ and more generally, } C[E] \cap C[L] = C.1 H(\Omega).$$

We are in a position to consider the following differential subalgebras of $(H(\Omega), \partial)$:

$$\mathcal{L} := C\{((\ell_{r})_{r \geq 1})\} \quad \text{and} \quad \mathcal{E} := C\{(e^{\ell_r})_{r \geq 1}\}.$$  

Since $\partial_{\ell}^{-1} = -\ell_{r}^{-2}\partial_{\ell}$ then $\mathcal{L} = C\{((\ell_{r})_{r \geq 1}, \partial_{\ell_{r}})_{r \geq 1}\}$. Let

$$\mathcal{L}^+ := C\{(\ell_{r})_{r \geq 1}\}.$$  

This $C$-differential subalgebra $\mathcal{L}^+$ is an integral domain generated by holomorphic functions and $\text{Frac}(\mathcal{L}^+)$ is generated by meromorphic functions. Since there is $0 \neq q_{i, l, k} \in \mathcal{L}^+$ such that $(\partial_{\ell_{r}}e^{\ell_{r}})^l = q_{i, l, k}e^{\ell_{r}}$ (i, l, k $\geq 1$) then let

$$\mathcal{E}^+ := \text{span}_{C}\{(\partial_{\ell_{r}}e^{\ell_{r}})^l \ldots (\partial_{\ell_{r}}e^{\ell_{r}})^k\}_{(l, i, l, r_1) \ldots (l, i, l, r_k) \in (N^*)^k, k \geq 1}$$

$$= \text{span}_{C}\{(q_{i, l, r_1} \ldots q_{i, l, r_k}e^{\ell_{r_1}} \ldots e^{\ell_{r_k}} )_{(l, i, l, r_1) \ldots (l, i, l, r_k) \in (N^*)^k, k \geq 1}\}$$

$$\subset \text{span}_{C}\{e^{\ell_{r_1}} \ldots e^{\ell_{r_k}}\}_{(l, i, l, r_1) \ldots (l, i, l, r_k) \in (N^*)^k, k \geq 1}.$$  

Note that in $2.13$, $C$ is a differential subring of $A = H(\Omega)$ (hence, $\text{Frac}(C)$ is a differential subfield of $\text{Frac}(A)$) and

$$\mathcal{E}^+ \cap E = \{0\}.$$  

Theorem 2.13. (1) The family $(e^{\ell_r})_{r \geq 1}$ (resp. $(\ell_{r})_{r \geq 1}$) is algebraically free over $\mathcal{E}^+$ (resp. $\mathcal{L}^+$).

(2) $C[E]$ and $C[L]$ are algebraically disjoint, within $A$.

Proof. (1) Considering the Chen series of the differential forms $\omega_r$ defined, for any $r \geq 1$, by $u_{y_r} = e^{\ell_r}\partial_{\ell_r}$. Let $Q \in \text{Frac}(\mathcal{E}) \cap E$ (resp. $\text{Frac}(\mathcal{C}) \cap E$):

$$e^{\ell_r}$$

12 More precisely, denoting $U$ the unit circle, one has $O(e^{\ell_r}) \cap U = -G_r \neq \emptyset$.  

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(a) since $Q \in E$ then there is $\{c_{y_r}\}_{y \in Y} \in \mathbb{C}^{(Y)}$ such that
\begin{equation}
Q = \sum_{r \geq 1} c_{y_r} e^{\ell_r} \quad \text{and then} \quad \partial Q = \sum_{r \geq 1} c_{y_r} e^{\ell_r} \partial \ell_r,
\end{equation}
(b) since $Q \in \text{Frac}(\mathcal{L}) \supset \mathcal{L} \supset \mathcal{C}[L]$ (resp. $\text{Frac}(\mathcal{C}) \supset \mathcal{C} \supset \mathcal{E}^+$) then, by Proposition 2.8 (resp. 2.14) it remains that $Q = 0$.

Hence, by Proposition 2.8 since $\{e^{\ell_k}\}_{k \geq 1}$ is $\mathbb{C}$-free and $Q = 0$ then
(a) on the one hand, for any $r \geq 1$, one has $c_{y_r} = 0$,
(b) on the other hand, $\{\alpha_0(S_i)\}_{i \in \mathcal{L} \cap \mathcal{Y}}$ (including $\{\alpha_0(S_y)\}_{y \in Y}$) is algebraically free over $\mathcal{L}$ (resp. $\mathcal{C}$).

It follows that $\{e^{\ell_r}\}_{r \geq 1}$ is algebraically free over $\mathbb{C}[L]$ (resp. $\mathcal{E}^+$).

Now, suppose there is an algebraic relation among $\{\ell_k\}_{k \geq 1}$ over $\mathcal{L}^+$ in which, by differentiating and substituting $\partial \ell_k$ by $e^{-\ell_k} \partial e^{\ell_k}$, we get an algebraic relation among $\{e^{\ell_r}\}_{r \geq 1}$ over $\mathbb{C}[L]$ and $\mathcal{E}^+$ contradicting with previous results. It follows then $(\ell_k)_{k \geq 1}$ is $\mathcal{L}^+$-algebraically independent.

(2) $\{e^{\ell_k}\}_{k \geq 1}$ (resp. $\{\ell_k\}_{k \geq 1}$) is algebraically independent over $\mathbb{C}[L]$ (resp. $\mathbb{C}[E]$). Hence, $\{e^{\ell_k}, \ell_k\}_{k \geq 1}$ generates freely $\mathbb{C}[E + L]$ and $\mathbb{C}[E] \cap \mathbb{C}[L] = \mathbb{C}.1_{\mathcal{H}(1)}$.

It follows that $\mathbb{C}[E]$ and $\mathbb{C}[L]$ are algebraically disjoint, within $\mathcal{A}$.

\[\square\]

Corollary 2.14. (1) Using the inputs $\{\partial \ell_r\}_{r \geq 1}$ (resp. $\{e^{\ell_r} \partial \ell_r\}_{r \geq 1}$), the following morphism is injective (see also Remark 2.4)
\[
\alpha_0^* : (\mathcal{L}^+ \langle Y \rangle, \omega, 1_{Y^*}) \rightarrow (\text{span}_{\mathcal{L}^+} \{\alpha_0^*(w)\}_{w \in Y^*, x, 1_{\mathcal{H}(1)}}),
\]
(resp. $\alpha_0^* : (\mathcal{E}^+ \langle Y \rangle, \omega, 1_{Y^*}) \rightarrow (\text{span}_{\mathcal{E}^+} \{\alpha_0^*(w)\}_{w \in Y^*, x, 1_{\mathcal{H}(1)}})$).

Hence, $\{\alpha_0^*(w)\}_{w \in Y^*}$ (resp. $\{\alpha_0^*(l)\}_{l \in \mathcal{L} \cap \mathcal{Y}}$) is linearly (resp. algebraically) independent over $\mathcal{L}^+$ (resp. $\mathcal{E}^+$).

(2) Using the inputs $\{\partial \ell_r\}_{r \geq 1}$ and denoting the set of exchangeable polynomials (over $Y$ and with coefficients in $\mathbb{C}$) by $\mathcal{C}_{\text{exc}}(Y)$, (see [B] for example), the family $\{\alpha_0^*(\lambda)\}_{\lambda \in \mathcal{L} \cap \mathcal{Y}}$ is $\mathbb{C}$-algebraically independent and the restricted morphism $\alpha_0^* : (\mathcal{C}_{\text{exc}}(Y) \omega \mathbb{C}[\{y_r^*\}_{r \geq 1}], \omega, 1_{Y^*}) \rightarrow \mathbb{C}[L + E]$ is bijective.

Hence, $\{(e^{\ell_r}, \ell_r)_{r \geq 1}, (\ell_r)_{r \geq 1}\}$ is $\mathcal{C}$-algebraically independent.

(3) Let $C_k := \text{span}_{\mathcal{L}^+} \{e^{\ell_1 + \cdots + \ell_k} e^{\ell_{r_1} + \cdots + \ell_{r_k}}\}_{(l_1, r_1), \ldots, (l_k, r_k) \in \mathbb{Z}^* \times \mathbb{N}^*}$. Then
\[
C = \bigoplus_{k \geq 1} C_k
\]

Proof. (1) It is a consequence of Theorem 2.13.

(2) The free algebras $(\mathcal{C}_{\text{exc}}(Y), \omega, 1_{Y^*})$ and $(\mathbb{C}[\{y_r\}_{r \geq 1}], \omega, 1_{Y^*})$ are algebraically disjoint and their images by $\alpha_0^*$, by Proposition 2.8 are, respectively, the free algebras $\mathbb{C}[L]$ and $\mathbb{C}[E]$ which are, by Theorem 2.13 algebraically disjoint. Moreover, since $\mathcal{C}_{\text{exc}}(Y) = \mathbb{C}[\{y_r\}_{y \in Y}]$ and $Y \subset \mathcal{L} \cap \mathcal{Y}$ then we deduce the respected results.

(3) For any $k \geq 1$, let $\Phi_k := \text{span}_C \{e^{\ell_1 + \cdots + \ell_k} e^{\ell_{r_1} + \cdots + \ell_{r_k}}\}_{\text{distinct } r_1, \ldots, r_k \in \mathbb{N}^*, l_1, \ldots, l_k \in \mathbb{Z}^*}$.

Let $\mathcal{C}[\Phi]$ be the algebra of $\Phi := \text{span}_C \{e^{\ell_r} e^{\ell_{r'}}\}_{r \geq 1}$. Since $(\ell_r)_{r \geq 1}$ is $\mathbb{C}$-free then $\Phi_1 \subset \Phi_2 \subset \ldots$ and then $\mathcal{C}[\Phi] = \bigoplus_{k \geq 1} \Phi_k$. Moreover, the disjunction of $\mathbb{C}[E]$ and $\mathbb{C}[L]$ leads to $C_k \cong \mathcal{L}^+ \otimes_C \Phi_k$ and then yields the expected result.
Under suitable convergence condition this extension can be realized and \( \{2, 7, 17\} \)

\[
e_{\nu}(z) = e_{\nu}(z) \prod_{\chi \in \mathcal{X}(1)} e_{\nu}(\chi z).
\]

Since \((e_{\nu})_{\nu \geq 1}\) is algebraically free over \( E^+ \) then

\[
\prod_{\chi \in \mathcal{X}(1)} e_{\nu}(\chi z) \notin E^+[(e_{\nu(k)})_{k \geq 1}].
\]

### 2.3. Polylogarithms and harmonic sums indexed by rational series.

Using the projector \( \pi_X : (\mathbb{C}(Y), 1_Y) \rightarrow (\mathbb{C}(X), 1_X) \) defined as the concatenation morphism, mapping \( y_s \) to \( x_0^{s-1}x_1 \) and admitting \( \pi_Y \) as adjoint, one has the following one-to-one correspondences

\[
(s_1, \ldots, s_r) \in (\mathbb{N}^*)^r \leftrightarrow y_{s_1} \cdots y_{s_r} \in Y^*: \pi_X \pi_Y x_0^{s_1-1}x_1 \cdots x_0^{s_r-1}x_1 \in X^*x_1.
\]

In all the sequel, \( \Omega := \mathbb{C} \setminus \{0, 1\} \) and

\[
(\omega_0(z) := z^{-1}dz \quad \text{and} \quad \omega_1(z) := (1-z)^{-1}dz).
\]

By \((2.9)\), \( Li_{s_1 \ldots s_r}(z) = \alpha(z)(x_0^{s_1-1}x_1 \cdots x_0^{s_r-1}x_1) \), for \( s_1, \ldots, s_r \in \mathbb{N}^* \). Thus, putting \( \omega_{x_0}(z) := \log(z) \), the following morphisms are injective

\[
\begin{align*}
Li_\bullet : (\mathbb{Q}(X), \omega, 1_X) & \rightarrow (\mathbb{Q}\{Li_w\}_{w \in X^*}, 1), \\
\omega_{x_0} & \mapsto \omega_{x_0}; \\
H_\bullet : (\mathbb{Q}(Y), \omega, 1_Y) & \rightarrow (\mathbb{Q}\{H_w\}_{w \in Y^*}, 1), \\
y_{s_1} \cdots y_{s_r} & \mapsto H_{y_1 \ldots y_{s_r}} = H_{s_1 \ldots s_r}.
\end{align*}
\]

In order to extend \( Li_\bullet, H_\bullet \) (in \((2.17), (2.18)\)) over some subdomain of \( \mathbb{C}^{\text{rat}}\langle \langle X \rangle \rangle \) (resp. \( \mathbb{C}^{\text{rat}}\langle \langle Y \rangle \rangle \)), let us call \( \text{Dom}_R(Li_\bullet) \) the set of series

\[
S = \sum_{n \geq 0} S_n \quad \text{with} \quad S_n := \sum_{|w|=n} \langle S|w \rangle w
\]

such that \( \sum_{n \geq 0} Li_{s_n} \) converge uniformly in any compact of \( \Omega \).

For any \( 0 < R \leq 1 \), such that \( \sum_{n \geq 0} Li_{s_n} \) converge uniformly in the open disc \( D_{|z|<R} \), one has \( (1-z)^{-1}Li_S = \sum_{N \geq 0} a_N z^N \) converge in the same disc and then \( H_{\pi_Y(S_n)}(N) = a_N \), for \( N \geq 0 \). Hence, let us define

\[
\begin{align*}
\text{Dom}_R(Li_\bullet) & := \{ S \in \mathbb{C}1_X + \mathbb{C}\langle\langle X \rangle\rangle x_1| \sum_{n \geq 1} Li_{s_n} \text{ converge in } D_{|z|<R} \}, \\
\text{Dom}(H_\bullet) & := \pi_Y \text{Dom}^{\text{loc}}(Li_\bullet), \quad \text{where} \quad \text{Dom}^{\text{loc}}(Li_\bullet) := \bigcup_{0<R\leq1} \text{Dom}_R(Li_\bullet).
\end{align*}
\]

Under suitable convergence condition this extension can be realized and \((2,7,17)\)

1. \( \text{Dom}(Li_\bullet) \) (resp. \( \text{Dom}(H_\bullet) \)) is closed by shuffle (resp. quasi-shuffle) products.
2. \( Li_{S \uplus T} = Li_S Li_T \) and \( H_{S \uplus T} = H_S H_T \), for \( S, T \in \text{Dom}(Li_\bullet) \) (resp. \( \text{Dom}(H_\bullet) \)).
Any series $S \in \mathbb{C}(\langle \mathcal{X} \rangle)$ is syntactically exchangeable iff it is of the form

$$S = \sum_{\alpha \in \mathbb{N}(\mathcal{X}), \supp(\alpha) = \{x_1, \ldots, x_k\}} s_\alpha x_1^{\alpha(x_1)} \ldots x_k^{\alpha(x_k)}.$$  

(2.20)

The set of these series, a $\omega$-subalgebra of $A(\langle \mathcal{X} \rangle)$, will be denoted by $\mathbb{C}_{\text{ext}}(\langle \mathcal{X} \rangle)$.

**Theorem 2.16** (extension of $\text{Li}_\bullet$). Let $\mathcal{C}_\bullet := \mathbb{C}[[z^a, (1 - z)^b]_{a, b \in \mathbb{C}}]$. Then

1. The algebra $\mathcal{C}_\bullet \{\text{Li}_w\}_{w \in X^*}$ is closed under the differential operators $\theta_0 := z \partial_z$ and $\theta_1 := (1 - z) \partial_z$ and under their sections $\iota_0, \iota_1$ ($\theta_{010} = \theta_{111} = \text{Id}$).
2. The bi-integro differential algebra $(\mathcal{C}_\bullet \{\text{Li}_w\}_{w \in X^*}, \theta_0, \theta_1, \iota_0, \iota_1)$ is closed under the action of the group of transformations, $\mathcal{G}$, generated by $\{z \mapsto 1 - z, z \mapsto 1/z\}$, permuting $\{0, 1, +\infty\}$:

$$\forall h \in \mathcal{C}_\bullet \{\text{Li}_w\}_{w \in X^*}, \quad \forall g \in \mathcal{G}, \quad h(g) \in \mathcal{C}_\bullet \{\text{Li}_w\}_{w \in X^*}.$$

3. If $R \in \mathbb{C}_{\text{ext}}(\langle \mathcal{X} \rangle) \subset \mathbb{C}(\langle \mathcal{X} \rangle)$ (resp. $\mathbb{C}_{\text{ext}}(\langle \Omega \rangle)$) then $\text{Li}_R \in \mathbb{C}_\bullet \{\text{Li}_w\}_{w \in X^*}$ (resp. $\mathbb{C}_\bullet \{\log(z), \log(1 - z)\}$).
4. The family $\{\text{Li}_w\}_{w \in X^*}$ (resp. $\{\text{Li}_w\}_{w \in \mathcal{C}_{\text{yn}}X}$) is linearly (resp. algebraically) independent over $\mathcal{C}_\bullet$.

**Proof.** The three first items are immediate. Only the last one needs a proof:

Let then $B = \mathbb{C} \setminus \{0, 1\}$, $\Omega = \mathbb{C} \setminus [-\infty, 0] \cup [1, +\infty]$ and choose a basepoint $b \in \Omega$, one has the following diagram

$$
\begin{array}{ccc}
\longrightarrow & (\tilde{B}, \tilde{b}) \\
\downarrow \quad p \\
\Omega, b & \quad \iota \quad (B, b)
\end{array}
$$

Any holomorphic function $f \in \mathcal{H}(\Omega)$ such that $f' = df/dz$ admits an analytic continuation to $B$ can be lifted to $\tilde{B}$ by $\tilde{f}(z) := f(b) + \int_b^z f'(s)ds$.

Let $L$ be the noncommutative series of the polylogarithms $\{\text{Li}_w\}_{w \in X^*}$, which is group-like, and $\text{C}_{z_0 \rightarrow z}$ be the Chen series, of $\{\omega_0, \omega_1\}$ along $z_0 \rightarrow z \in \tilde{B}$, $\text{C}_{z_0 \rightarrow z} = L(z) L^{-1}(z_0)$ (see [17]).

Now, in view of Lemma 2.22 as the algebra $\mathcal{C}$ is without zero divisors and contains the field of constants $\mathbb{C}$, it suffices to prove that $\text{Li}_{z_0}, \text{Li}_{z_1}$ and $1_{\Omega}$ are $\mathcal{C}$-linearly independent. It is an easy exercise to check that $s_\alpha(f) := \tilde{f} \circ s$ coincides with the given $f$. This is the case, in particular of the functions $\log(z)$ and $\log(1 - z)^{-1}$ whose liftings will be denoted $\log_0$ and $\log_1$, respectively. So, we lift the functions $z^a$ and $(1 - z)^b$ as, respectively,

$$e_0^a(\tilde{z}) := e^{a \log_0(\tilde{z})} \quad \text{and} \quad e_1^b(\tilde{z}) := e^{b \log_1(\tilde{z})}$$

and, of course, by construction,

$$e_0^a \circ s = (z \mapsto z^a) \quad \text{and} \quad e_1^b \circ s = (z \mapsto (1 - z)^b).$$

(2.22)

We suppose a dependence relation, in $\mathcal{H}(\Omega)$

$$P_0(z^a, (1 - z)^b) \text{Li}_{z_0} + P_1(z^a, (1 - z)^b) \text{Li}_{z_1} + P_2(z^a, (1 - z)^b).1_{\Omega} = 0$$

(2.23) where $P_1 \in \mathbb{C}[X, Y]$ are two-variable polynomials. From (2.22) and the fact that $\Omega \neq \emptyset$, we get

$$P_0(e_0^a, e_1^b) \log_0 + P_1(e_0^a, e_1^b) \log_1 + P_2(e_0^a, e_1^b).1_\tilde{B} = 0.$$  

(2.24)
Theorem 2.18
indexed by series, belonging to (S (by convention path
Now we remark that (2.26)
e_{0}^{[a]} \circ D_{0}(\tilde{z}) = e_{0}^{[a]}(\tilde{z})e^{2ai\pi} \quad \text{and} \quad e_{1}^{[b]} \circ D_{0} = e_{1}^{[b]}
and, similarly
(2.27)
e_{1}^{[b]} \circ D_{1}(\tilde{z}) = e_{1}^{[b]}(\tilde{z})e^{2bi\pi} \quad \text{and} \quad e_{0}^{[a]} \circ D_{1} = e_{0}^{[a]}
so that \( P_{1}(e_{0}^{[a]}, e_{1}^{[b]}) \) remain bounded through the actions of \( D_{0}^{*} \) and \( D_{1}^{*} \), from (2.25),
we get that \( P_{1} = 0, i = 0.2 \) which proves the claim. \( \square \)

Example 2.17 ([7]). Let us use the noncommutative multivariate exponential transforms i.e., for any syntactically exchangeable series, we get
\[
\sum_{i_{0}i_{1} \geq 0} s_{i_{0}i_{1}} x_{0}^{i_{0}} x_{1}^{i_{1}} \rightarrow \sum_{i_{0}i_{1} \geq 0} s_{i_{0}i_{1}} \text{Li}_{0}^{i_{0}} \text{Li}_{1}^{i_{1}}.
\]
Hence, \( x_{n}^{a} \rightarrow \text{Li}_{x_{0}/n!}^{n} \) and \( x_{n}^{a} \rightarrow \text{Li}_{x_{1}/n!}^{n} \), for \( n \in \mathbb{N} \), yielding some polylogarithms indexed by series,
\[
\text{Li}_{x_{0}}^{a}(z) = z, \quad \text{Li}_{x_{1}}^{b}(z) = (1 - z)^{-1}, \quad \text{Li}_{(ax_{0} + bx_{1})}^{c}(z) = z^{a}(1 - z)^{-b}.
\]
Moreover, for any \( (s_{1}, \ldots, s_{r}) \in \mathbb{N}_{+}^{r} \), there exists an unique series \( R_{y_{s_{1}} \ldots y_{s_{r}}} \) belonging to \( \mathbb{Z}[x_{1}^{*}], \omega, 1x_{1}^{*} \) such that \( \text{Li}_{-s_{1}, \ldots, -s_{r}} = \text{Li}_{R_{y_{s_{1}} \ldots y_{s_{r}}}} \). More precisely
(by convention \( \rho_{0} = x_{1}^{*} - 1x_{1}^{*} \)),
\[
R_{y_{s_{1}} \ldots y_{s_{r}}} = \sum_{k_{1}=0}^{s_{1}} \cdots \sum_{k_{r}=0}^{s_{r}} (s_{1} + \cdots + s_{r} - (k_{1} + \cdots + k_{r-1})) \binom{s_{1}}{k_{1}} \cdots \binom{s_{r}}{k_{r}} \rho_{k_{1}} \omega \cdots \omega \rho_{k_{r}},
\]
and using the Stirling numbers of second kind, \( S_{2}(k_{i}, j) \), one has
\[
\rho_{k_{i}} = x_{1}^{*} \omega \sum_{j=1}^{k_{i}} S_{2}(k_{i}, j) j!(x_{1}^{*} - 1x_{1}^{*})^{\omega j}, (k_{i} \neq 0).
\]

Theorem 2.18 (extension of \( H_{*} \)). For any \( r \geq 1 \), one has, for any \( t \in \mathbb{C}, |t| < 1 \),
\[
H_{(t y_{r})^{r}} = \sum_{k \geq 0} H_{y_{r}^{k}} t^{kr} = \exp \left( \sum_{k \geq 1} H_{y_{r}} \frac{(-t)^{k-1}}{k} \right).
\]

Moreover, for \( |a_{s}| < 1, |b_{s}| < 1 \) and \( |a_{s} + b_{s}| < 1 \),
\[
H_{(\sum_{s \geq 1} (a_{s} + b_{s}) y_{s}^{r} + \sum_{r \geq 1} a_{s} b_{s} y_{s}^{r})^{r}} = H_{(\sum_{s \geq 1} a_{s} y_{s})^{r}} \cdot H_{(\sum_{s \geq 1} b_{s} y_{s})^{r}}.
\]
Hence,
\[
H_{(a_{s} y_{r} + a_{r} y_{s} + a_{s} a_{r} y_{s})^{r}} = H_{(a_{s} y_{r})^{r}} \cdot H_{(a_{r} y_{s})^{r}}, \quad H_{(-a_{s}^{2} y_{r}^{2})^{r}} = H_{(a_{s} y_{r})^{r}} \cdot H_{(-a_{s} y_{r})^{r}}.
\]
Proof. For \( t \in \mathbb{C}, |t| < 1 \), since \( \text{Li}_{(t x_{1})^{r}} \) is well defined then so is the arithmetic function, expressed via Newton-Girard formula (see [3]), for \( n \geq 0 \), by
\[
H_{(t y_{1})^{r}}(n) = \sum_{k \geq 0} H_{t^{k}}(n) t^{k} = \exp \left( -\sum_{k \geq 1} H_{y_{k}}(n) \frac{(-t)^{k}}{k} \right) = \prod_{l=1}^{n} (1 + \frac{t}{l}).
\]
Similarly, for any \( r \geq 2 \), the transcendent function \( H_{(r,y_r)} \) can be expressed via Newton-Girard formula (see [3]) once again and via Adam’s transform, by

\[
H_{(r,y_r)}(n) = \sum_{k \geq 0} H_{y_k}(n) t^{kr} = \exp\left( -\sum_{k \geq 1} H_{y_k}(n) \left( -\frac{t^r}{k} \right) \right) = \prod_{l=1}^{N} \left( 1 - \frac{t^r}{l^r} \right).
\]

Since \( \|H_{y_r}\|_{\infty} \leq \zeta(r) \) then \( -\sum_{k \geq 1} H_{y_k}(-t^r/k) \) is termwise dominated by \( \|\ell_r\|_{\infty} \) and then \( H_{(r,y_r)} \) by \( e^{\ell_r} \) (see also Theorem 2.22 below). It follows then the last results by using the following identity [9]

\[
\left( \sum_{s \geq 1} a_s y_s \right)^* \left( \sum_{s \geq 1} b_s y_s \right)^* = \left( \sum_{s \geq 1} (a_s + b_s) y_s + \sum_{r,s \geq 1} a_s b_r y_{r+s} \right)^*.
\]

From the estimations from above of the previous proof, it follows then

**Corollary 2.19.** For any \( r \geq 2 \), one has

\[
\frac{1}{y_r(1 + t)} = \sum_{k \geq 0} k \times \zeta(r, \ldots, r) t^{kr} = \exp\left( -\sum_{k \geq 1} \zeta(kr) \left( -\frac{t^r}{k} \right) \right) = \prod_{n \geq 1} \left( 1 - \frac{t^r}{n^r} \right).
\]

**Corollary 2.20.** For any \( r \geq 1 \)

\[
y_r^r = \exp \left( \sum_{k \geq 1} y_{kr} \frac{(-1)^{k-1}}{k} \right).
\]

Hence, for any \( k \geq 0 \), one has

\[
y_r^r = \frac{(-1)^n}{n!} \sum_{s_1 + \ldots + s_n = n} \frac{(-y_r)^{s_1}}{s_1!} \ldots \frac{(-y_{nr})^{s_n}}{s_n!}
\]

and, for any \( r, s \geq 1 \), one also has

\[
y_r^r \cdot y_s^s = \sum_{k=0}^{r} \binom{r}{k} \binom{s}{s-k} y_{r+s-k}.
\]

### 2.4. Extended double regularization by Newton-Girard formula.

By (2.28), the following polymorphism is, by definition, surjective (see [17])

(2.28) \( \zeta : (\mathbb{Q}1_X \oplus \mathbb{Q}Y, 1_X, 1_Y) \rightarrow (\mathbb{Z}, 1) \)

mapping both \( x_{s_1} \ldots x_{s_r} \) and \( y_{s_1} \ldots y_{s_r} \) to \( \zeta(s_1, \ldots, s_r) \), where \( \mathbb{Z} \) denotes the \( \mathbb{Q} \)-algebra (algebraically) generated by \( \{\zeta(l)\}_{l \in \mathcal{L}(\mathcal{X})} \), or equivalently, \( \{\zeta(l)\}_{l \in \mathcal{L}(\mathcal{X})} \). It can be extended as characters

(2.29) \( \zeta : (\mathbb{R}1_X, 1_X) \rightarrow (\mathbb{R}, 1) \)

(2.30) \( \zeta : (\mathbb{R}1_Y, 1_Y) \rightarrow (\mathbb{R}, 1) \)

such that, for any \( l \in \mathcal{L}(\mathcal{X}) \), one has (see [17])

(2.31) \( \zeta(l) = f.p.z \rightarrow \ln(z), \{1 - z\} \log^b(1 - z) \}_{a \in \mathbb{Z}, b \in \mathbb{N}} \)

(2.32) \( \zeta(l) = f.p.n \rightarrow \{\log^b(n) \}_{a \in \mathbb{Z}, b \in \mathbb{N}} \)

(2.33) \( \zeta(l) = f.p.n \rightarrow \{\log^b(n) \}_{a \in \mathbb{Z}, b \in \mathbb{N}} \)
It follows that, for any \( l \in \mathcal{L} \), \( \zeta_{\Psi} (l) = \zeta_{\Psi} (\pi \gamma l) = \gamma_{\pi \gamma} t = \zeta(l) \), and, for the algebraic generator \( x_0 \), \( \zeta_{\Psi} (x_0) = 0 = \log(1) \) and, for the algebraic generators \( x_1 \) and \( y_1 \) (divergent cases),

\[
\begin{align*}
(2.34) \quad & \zeta_{\Psi} (x_1) = 0 = f.p.z_{\gamma} \log (1 - z), \quad \{ (1 - z)^a \log^b (1 - z) \}_{a \in \mathbb{Z}, b \in \mathbb{N}}, \\
(2.35) \quad & \zeta_{\Psi} (y_1) = 0 = f.p.n_{\gamma} \log (1 + n), \quad \{ n^a \log^b (n) \}_{a \in \mathbb{Z}, b \in \mathbb{N}}.
\end{align*}
\]

As in [7, 17], considering a character \( \chi_\Psi \) on \( (\mathbb{C} \langle X, \omega, 1_X \rangle) \) and considering \( \text{Dom}(\chi_\Psi) \subset \mathbb{C} \langle X \rangle \) as in (2.19), we can also check easily that (2):

\[
\begin{align*}
& \mathbb{C} \langle X \rangle \subset \mathbb{C}^{\text{rat}} \langle X \rangle \subset \text{Dom}(\chi_\Psi) \text{ which is closed by shuffle product,} \\
& \text{for any } S, T \in \text{Dom}(\chi_\Psi), \text{ one has } \chi_\Psi S T = \chi_\Psi S \chi_\Psi T, \\
& \text{if } S \in \text{Dom}(\chi_\Psi) \text{ then } \exp_{\Psi}(S) \in \text{Dom}(\chi_\Psi) \text{ and } \chi \exp_{\Psi} (S) = e^{\chi S}.
\end{align*}
\]

Similarly, considering a character \( \chi_\Psi \) on \( (\mathbb{C} \langle Y, \omega, 1_y \rangle) \) and considering \( \text{Dom}(\chi_\Psi) \subset \mathbb{C} \langle Y \rangle \) as in (2.19), we can also check easily that (2):

\[
\begin{align*}
& \mathbb{C} \langle Y \rangle \subset \mathbb{C}^{\text{rat}} \langle Y \rangle \subset \text{Dom}(\chi_\Psi) \text{ which is closed by quasi-shuffle product,} \\
& \text{for any } S, T \in \text{Dom}(\chi_\Psi), \text{ one has } \chi_\Psi S T = \chi_\Psi S \chi_\Psi T, \\
& \text{if } S \in \text{Dom}(\chi_\Psi) \text{ then } \exp_{\Psi}(S) \in \text{Dom}(\chi_\Psi) \text{ and } \chi \exp_{\Psi} (S) = e^{\chi S}.
\end{align*}
\]

**Example 2.21.** For any \( z \in \mathbb{C}, |z| < 1, x \in X = \{ x_0, x_1 \}, y_r \in Y = \{ y_k \}_{k \geq 1} \), since

\[
(zx)^* = \exp_{\Psi} (z) \quad \text{and} \quad (zy_r)^* = \exp_{\Psi} \left( \sum_{k \geq 1} y_{kr} \frac{(-z)^{k-1}}{k} \right)
\]

then

\[
(\zeta_{\Psi} ((zx)^*)) = e^{z \zeta_{\Psi} (z)} \quad \text{and} \quad \gamma (zy_r)^* = \exp \left( \sum_{k \geq 1} \zeta_{\Psi} (y_{kr}) \frac{(-z)^{k-1}}{k} \right).
\]

We now in situation to state that

**Theorem 2.22** (Regularization by Newton-Girard formula). The characters \( \zeta_{\Psi} \) and \( \gamma_\Psi \) are extended algebraically as follows:

\[
\begin{align*}
& \zeta_{\Psi} : (\mathbb{C} \langle X \rangle \subset \mathbb{C}^{\text{rat}} \langle X \rangle), \omega, 1_X \rightarrow (\mathbb{C}, \cdot, 1), \\
& \forall t \in \mathbb{C}, |t| < 1, (tx_0)^*, (tx_1)^* \quad \rightarrow \quad 1_C.
\end{align*}
\]

and

\[
\begin{align*}
& \gamma_\Psi : (\mathbb{C} \langle Y \rangle \subset \mathbb{C}^{\text{rat}} \langle Y \rangle), \omega, 1_Y \rightarrow (\mathbb{C}, \cdot, 1), \\
& \forall t \in \mathbb{C}, |t| < 1, \forall r \geq 1, (tr y_r)^* \quad \rightarrow \quad \Gamma_{y_r}^{-1}(1 + t).
\end{align*}
\]

Moreover, the morphism \( (\mathbb{C}[y_r^*]_{r \geq 1}, \omega, 1_Y \rightarrow \mathbb{C}[E] \) mapping \( y_r^* \) to \( \Gamma_{y_r}^{-1} \), is injective and \( \Gamma_{y_r}(1 + \sqrt{-1} t) = \Gamma_{y_r}(1 + t) \Gamma_{y_r}(1 + \sqrt{-1} t), \) for \( r \geq 1 \).

**Proof.** By Definition 2.3, Propositions 2.8, 2.11 and Theorems 2.16, 2.18 we get the expected results (see also Proposition 2.7). \[\square\]
Example 2.23. \[7\]

\[\begin{align*}
\text{Li}_{-1,-1} &= \text{Li}_{-x_1^2+5(2x_1)^*-7(3x_1)^*+3(4x_1)^*}, \\
\text{Li}_{-2,-1} &= \text{Li}_{x_1^2-11(2x_1)^*+31(3x_1)^*-33(4x_1)^*+12(5x_1)^*}, \\
\text{Li}_{-1,-2} &= \text{Li}_{x_1^2-9(2x_1)^*+23(3x_1)^*-23(4x_1)^*+8(5x_1)^*}, \\
H_{-1,-1} &= H_{-y_1^2+5(2y_1)^*-7(3y_1)^*+3(4y_1)^*}, \\
H_{-2,-1} &= H_{y_1^2-11(2y_1)^*+31(3y_1)^*-33(4y_1)^*+12(5y_1)^*}, \\
H_{-1,-2} &= H_{y_1^2-9(2y_1)^*+23(3y_1)^*-23(4y_1)^*+8(5y_1)^*}.
\end{align*}\]

Hence, \(\zeta_{\omega}(-1, -1) = 0\), \(\zeta_{\omega}(-2, -1) = -1\), and \(\zeta_{\omega}(-1, -2) = 0\).

\[\begin{align*}
\gamma_{-1,-1} &= -\Gamma^{-1}(2) + 5\Gamma^{-1}(3) - 7\Gamma^{-1}(4) + 3\Gamma^{-1}(5) = 11/24, \\
\gamma_{-2,-1} &= -\Gamma^{-1}(2) + 11\Gamma^{-1}(3) + 31\Gamma^{-1}(4) - 33\Gamma^{-1}(5) + 12\Gamma^{-1}(6) = -73/120, \\
\gamma_{-1,-2} &= -\Gamma^{-1}(2) - 9\Gamma^{-1}(3) + 23\Gamma^{-1}(4) - 23\Gamma^{-1}(5) + 8\Gamma^{-1}(6) = -67/120.
\end{align*}\]

From Theorems \[2.18\] and \[2.22\] one deduces

Corollary 2.24. \((1)\) With the notations of \[2.20\], Definition \[2.4\] and with the differential forms \((\partial \ell_r) dz\) \(r \geq 1\), for any \(z \in \mathbb{C}, |z| < 1\), one has

\[\gamma_{\omega} \gamma_{\omega} = \prod_{r \geq 1} \gamma_{\omega} = \prod_{r \geq 1} \frac{1}{\gamma_{\omega}(1+z)} = \alpha_{\omega}^{-1}(1, \gamma_{\omega}).\]

\[(2)\] One has, for \(|a_s| < 1, |b_s| < 1\) and \(|a_s + b_s| < 1,\)

\[\gamma_{\omega}(\sum_{s \geq 1} a_s \overline{b_s}) \gamma_{\omega}(\sum_{s \geq 2} a_s \overline{b_s} + 1) = \gamma_{\omega}(\sum_{s \geq 1} a_s \overline{b_s}) \gamma_{\omega}(\sum_{s \geq 1} b_s \overline{a_s}).\]

Hence, \(\gamma_{\omega}(a_s \overline{b_s}) \gamma_{\omega}(a_s \overline{y_s}) = \gamma_{\omega}(a_s \overline{y_s} \overline{b_s}) \gamma_{\omega}(a_s \overline{b_s} \overline{y_s}).\)

Remark 2.25. The restriction \(\alpha_{\omega} : \mathbb{C}[y_r, y_r] \to \mathbb{C}[L + E]\) is injective (see Corollary \[2.14\]) while \(\ker(\gamma_{\omega}) \neq \{0\}\), over \(\mathbb{C}[Y] \mathcal{W} \mathbb{C}^{\text{rat}}(\langle Y \rangle)\).

Example 2.26. \[15\] \[16\] By Theorem \[2.22\]

\[\gamma_{(-t^2 y_2)} = \Gamma_{y_2}^{-1}(1 + it), \quad \gamma_{(t y_1)} = \Gamma_{y_1}^{-1}(1 + t), \quad \gamma_{(-t y_1)} = \Gamma_{y_1}^{-1}(1 + t).\]

Then, by Corollary \[2.21\]

\[\gamma_{(-t^2 y_2)} = \gamma_{(t y_1)} \gamma_{(-t y_1)}, \quad \text{meaning that} \quad \Gamma_{y_2}^{-1}(1 + it) = \Gamma_{y_1}^{-1}(1 + t) \Gamma_{y_1}^{-1}(1 - t).\]

Or equivalently,

\[\exp\left( - \sum_{k \geq 2} \zeta(2k) \frac{t^{2k}}{k} \right) = \sum_{k \geq 2} \zeta(2, \ldots, 2) \left( \frac{(-1)^k t^{2k}}{2 \pi i} \right) = \sum_{k \geq 1} \frac{(-1)^k t^{2k}}{(2k + 1)!}.\]

Since \(\gamma_{(-t^2 y_2)} = \zeta((-t^2 x_0 x_1)^*)\) then, identifying the coefficients of \(t^{2k}\), we get

\[\exp\left( - \sum_{k \geq 2} \zeta(2, \ldots, 2) \left( \frac{(-1)^k t^{2k}}{2 \pi i} \right) \right) = \frac{1}{(2k + 1)!} \in \mathbb{Q}.\]

Similarly, by Theorem \[2.22\]

\[\gamma_{(-t^4 y_4)} = \Gamma_{y_4}^{-1}(1 + \sqrt{-1}, \quad \gamma_{(-t^2 y_2)} = \Gamma_{y_2}^{-1}(1 + it). \quad \gamma_{(-t^4 y_4)} = \Gamma_{y_4}^{-1}(1 + it).\]

Then, by Corollary \[2.21\]

\[\gamma_{(-t^4 y_4)} = \gamma_{(-t^2 y_2)} \gamma_{(-t^2 y_2)}, \quad \text{meaning that} \quad \Gamma_{y_2}^{-1}(1 + \sqrt{-1}) = \Gamma_{y_2}^{-1}(1 + t) \Gamma_{y_2}^{-1}(1 + it).\]
Thus, by identification the coefficients of $t^{4k}$, we obtain
\[
\exp \left( - \sum_{k \geq 1} \zeta(4k) \frac{t^{4k}}{k} \right) = \sum_{k \geq 2} \zeta(4, \ldots, 4) (-1)^k \frac{(t\pi)^{4k}}{(4k + 2)!}.
\]
\[
= \frac{\sin(t\pi)\sin(t\pi)}{it\pi} = \sum_{k \geq 1} \frac{2(-4t\pi)^{4k}}{(4k + 2)!}.
\]

Since $\gamma((-t^4y_4)^*) = \zeta((-t^4y_4)^*)$, $\gamma((-t^2y_2)^*) = \zeta((-t^2y_2)^*)$, $\gamma((t^2y_2)^*) = \zeta((t^2y_2)^*)$ then, using the poly-morphism $\zeta$ and identities on rational series, we get
\[
\zeta((-t^4y_4)^*) = \zeta((-t^2y_2)^*)\zeta((t^2y_2)^*)
\]
\[
= \zeta((-t^2x_0x_1)^*)\zeta((t^2x_0x_1)^*)
\]
\[
= \zeta((-4t^4x_0^2x_1^2)^*).
\]

Thus, by identification the coefficients of $t^{4k}$, we obtain
\[
\zeta(4k) = \frac{\zeta(3, 1, \ldots, 3)}{\pi^{4k}} = \frac{2}{(4k + 2)!} \in \mathbb{Q}.
\]

Corollary 2.27 (comparison formula, [5, 17]). For any $z, a, b \in \mathbb{C}$ such that $|z| < 1$ and $\Re(a) > 0, \Re(b) > 0$, we have
\[
B(z; a, b) = \frac{\zeta((-t^4y_4)^*)}{\zeta((-t^2y_2)^*)\zeta((t^2y_2)^*)}
\]
\[
= \frac{\zeta((-t^2x_0x_1)^*)\zeta((t^2x_0x_1)^*)}{\zeta((-t^2x_0x_1)^*)\zeta((t^2x_0x_1)^*)}
\]
\[
= \frac{\zeta((-a - 1)x_0)^* \omega((-b - 1)x_1)^*}{\zeta((-a - 1)x_0)^* \omega((-b - 1)x_1)^*}
\]
\[
= \frac{\zeta((-a - 1)x_0)^* \omega((-b - 1)x_1)^*}{\zeta((-a - 1)x_0)^* \omega((-b - 1)x_1)^*}
\]
\[
= \frac{\gamma((-a - 1)y_1)^*}{\gamma((-a - 1)y_1)^*}.
\]

Proof. The results, of $B(z; a, b)$, are the computations of iterated integrals associated to different rational series, using the differential forms in (2.16). Those of $B(a, b)$, are then immediate consequences, by evaluating these iterated integrals at $z = 1$ and by using Definition 2.23 and Corollary 2.24.

Example 2.28. [5, 16] Let us consider, for $t_0, t_1 \in \mathbb{C}, |t_0| < 1, |t_1| < 1$,
\[
R := t_0x_0(t_0x_0 + t_0t_1x_1)^*(t_0t_1x_1) = t_0^2t_1x_0[t_0x_0]^*(t_0t_1x_1)]x_1.
\]

Then, with the differential forms in (2.16), we get successively
\[
\text{Li}_R(z) = t_0^2t_1 \int_0^z \frac{ds}{s} \int_0^s \left( \frac{dr}{1 - s} \right) \frac{t_0 - t_1}{1 - s} \frac{dr}{1 - r}
\]
\[
= t_0^2t_1 \int_0^z \left( 1 - s \right)^{t_0 - t_1} \cdot \left( 1 - r \right)^{t_0 - 1 - r} \cdot ds \cdot dr
\]
\[
\text{By change of variable, } r = st, \text{ we obtain then}
\]
\[
\text{Li}_R(z) = t_0^2t_1 \int_0^z \left( 1 - s \right)^{-t_0 - t_1} \int_0^1 \left( 1 - st \right)^{t_0 - 1 - r} \cdot dt \cdot ds.
\]
It follows then
\[
\zeta(R) = t_0^2 t_1 \int_0^1 \int_0^1 (1 - s)^{-t_0 t_1} (1 - st)^{t_0 t_1 - 1} t^{-t_0} dt ds.
\]
By change of variable, \( y = \frac{1 - s}{1 - st} \), we obtain also
\[
\zeta(R) = t_0^2 t_1 \int_0^1 \int_0^1 (1 - ty)^{-1} t^{-t_0} y^{-t_0 t_1} dy dt.
\]
By expending \( (1 - ty)^{-1} \) and then integrating, we get on the one hand
\[
\zeta(R) = \sum_{n \geq 1} \frac{t_0}{n - t_0} \frac{t_0 t_1}{n - t_0 t_1} = \sum_{k > l > 0} \zeta(k) t_0^k t_1^l.
\]
On the other hand, using the the expansion of \( R \), we get also
\[
\zeta(R) = \sum_{k > 0} \sum_{l > 0} \sum_{s_1 + \ldots + s_l = k} \zeta(s_1, \ldots, s_l) t_0^{s_1} t_1^{s_l}.
\]
Finally, by identification the coefficients of \( \langle \zeta(R) \rangle |_{t_0^k t_1^l} \), we deduce the sum formula
\[
\zeta(k) = \sum_{s_1 + \ldots + s_l = k} \zeta(s_1, \ldots, s_l).
\]

3. Conclusion

In this work, we illustrated a bijection, between a sub shuffle algebra of non-commutative rational series (recalled in 2.1) and a subalgebra of holomorphic functions, \( \mathcal{H}(\Omega) \), on a simply connected domain \( \Omega \subset \mathbb{C} \) containing the family of extended eulerian functions \( \{ \Gamma^{-1}_y(1 + z) \}_{y \in Y} \) and the family of their logarithms, \( \{ \log \Gamma^{-1}_y(1 + z) \}_{y \in Y} \) (introduced in 2.2), involved in summations of polylogarithms and harmonics sums (studied in 2.3) and in regularizations of divergent polyzetas (achieved, for this stage, in 2.4).

These two families are algebraically independent over a differential subring of \( \mathcal{H}(\Omega) \) and generate freely two disjoint functional algebras. For any \( y_r \in Y \), the special functions \( \Gamma^{-1}_{y_r}(1 + z) \) and \( \log \Gamma^{-1}_{y_r}(1 + z) \) are entire and holomorphic on the unit open disc, respectively. In particular, \( \Gamma^{-1}_{y_r}(1 + z) \) admits a countable set of isolated zeroes on the complex plane, i.e. \( \biguplus_{x \in G_r} x \mathbb{Z} \leq -1 \), where \( G_r \) is the set of solutions of the equation \( z^r = (-1)^{r-1} \).

These functions allow to obtain identities, at arbitrary weight, among polyzetas and an analogue situation, as the ratios \( \zeta(2k)/\pi^{2k} \), drawing out consequences about a structure of polyzetas. This work will be completed, in the forth coming works, by a study a family of functions obtained as image of rational series for which their linear representation \( (\nu, \mu, \eta) \) are such that the Lie algebra generated by the matrices \( \{ \mu(y) \}_{y \in Y} \) is solvable.

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