INFINITE-DIMENSIONAL VECTOR BUNDLES IN ALGEBRAIC GEOMETRY (AN INTRODUCTION)

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To Izrail Moiseevich Gelfand with deepest gratitude and admiration

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According to the Lamb conjecture, the key to the future development of Quantum Field Theory is probably buried in some forgotten paper published in the 30’s. Attempts to follow up this conjecture, however, will probably be unsuccessful because of the Peierls-Jensen paradox; namely, that even if one finds the right paper, the point will probably be missed until it is found independently and accidentally by experiment.

The Future of Field Theory, by Pure Imaginary Observer [PI].

1. Introduction

1.1. Subject of the article. The goal of this work is to show that there is a reasonable algebro-geometric notion of vector bundle with infinite-dimensional locally linearly compact fibers and that these objects appear “in nature”. Our approach is based on some results and ideas discovered in algebra in 1958–1972 by H. Bass, L. Gruson, I. Kaplansky, M. Karoubi, and M. Raynaud.

This article contains definitions and formulations of the main theorems, but practically no proofs. A detailed exposition will appear in [Dr].

1.2. Conventions. We use the words “S-family of vector spaces” as shorthand for “vector bundle on a scheme S” and “Tate space” as shorthand for “locally linearly compact vector space”.

1.3. Overview of the results and structure of the article.

1.3.1. General theory. In [2] we recall the Raynaud-Gruson theorem on the local nature of projectivity, which shows that there is a good notion of family of discrete infinite-dimensional vector spaces.

In [3] we introduce the notion of “Tate module” over an arbitrary ring \( R \) and show that if \( R \) is commutative one thus gets a reasonable notion of S-family of Tate spaces, \( S = \text{Spec} \, R \). One has to take in account that \( K_0 \) of the additive category of Tate \( R \)-modules may be nontrivial. In fact, it equals \( K_{-1}(R) \). We show that \( K_{-1}(R) = 0 \) if \( R \) is Henselian. We give a proof of this fact because it explains the fundamental role of the Nisnevich topology in this work. We discuss the notions of dimension torsor and determinant gerbe of a family of Tate spaces.

At least technically, the theory of Tate \( R \)-modules is based on the notion of almost projective module, which is introduced in [4]. Roughly speaking, a module is almost projective if it is projective up to finitely generated modules. Unlike Tate modules, almost projective modules are discrete. Any Tate module can be represented as the projective limit of a filtering projective system of almost projective modules with surjective transition maps.
§5 is devoted to the canonical central extension of the automorphism groups of almost projective and Tate $R$-modules. In §5.5 we discuss an interesting (though slightly vague) picture, which I learned from A. Beilinson.

1.3.2. Application to the space of formal loops. In §6 we define a class of Tate-smooth ind-schemes (morally, these are smooth infinite-dimensional algebraic manifolds modeled on Tate spaces). According to Theorem 6.3, the ind-scheme of formal loops of a smooth affine manifold $Y$ over the local field $k((t))$ is Tate-smooth over $k$. This is one of our main results. In 6.10 we use it to define a “refined” version of the motivic integral of a differential form on $Y$ with no zeros over the intersection of $Y$ with a polydisk. Unlike the usual motivic integral, the “refined” one is an object of a triangulated category rather than an element of a group.

1.3.3. Application to vector bundles on a manifold with punctures. In §7 we first show that almost projective and Tate modules appear naturally in the study of the cohomology of a family of finite-dimensional vector bundles on a punctured smooth manifold. Then we briefly explain how the canonical central extension that comes from this cohomology allows (in the case of $GL(n)$-bundles) to interpret the “Uhlenbeck compactification” constructed in [FGK, BFG] as the fine moduli space of a certain type of generalized vector bundles on $\mathbb{P}^2$ (we call them gundles).

In fact, the application to the “Uhlenbeck compactification” was one of the main motivations of this work.

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2. Families of discrete infinite-dimensional vector spaces (after Kaplansky, Raynaud and Gruson).

Is there a reasonable notion of not necessarily finite-dimensional vector bundle on a scheme? We know due to Serre [S] that a finite-dimensional vector bundle on an affine scheme $\text{Spec } R$ is the same as a finitely generated projective $R$-module. So it is natural to give the following definition.

**Definition.** A vector bundle on a scheme $X$ is a quasicoherent sheaf of $O_X$-modules $\mathcal{F}$ such that for every open affine subset $\text{Spec } R \subset X$ the $R$-module $H^0(\text{Spec } R, \mathcal{F})$ is projective.

**Key question:** is this a local notion? More precisely, the question is as follows. Let $\text{Spec } R = \bigcup U_i, U_i = \text{Spec } R_i$. Let $M$ be a (not necessarily finitely generated) $R$-module such that $M \otimes_R R_i$ is projective for all $i$. Does it follow that $M$ is projective?

The question is difficult: the arguments used in the case that $M$ is finitely generated fail for modules of infinite type. Nevertheless Grothendieck ([Gr2], Remark 9.5.8) conjectured that the answer is positive. This was proved by Raynaud and Gruson [RG] (in Ch. 1 for countably generated modules and in Ch. 2 for arbitrary ones). Moreover, they proved the following theorem, which says that projectivity is a local property for the fpqc topology (not only for Zariski).

**Theorem 2.1.** Let $M$ be a module over a commutative ring $R$ and $R'$ be a flat commutative $R$-algebra such that the morphism $\text{Spec } R' \to \text{Spec } R$ is surjective. If $R' \otimes_R M$ is projective then $M$ is.

In fact, they derived it as an easy corollary of the following remarkable and nontrivial theorem due to Kaplansky [Ka] and Raynaud-Gruson [RG], which explains what projectivity really is. Theorem 2.1 follows from the fact that for commutative rings properties (a)-(c) below are local.
**Theorem 2.2.** Let $R$ be a (not necessarily commutative) ring. An $R$-module $M$ is projective if and only if the following properties hold:

(a) $M$ is flat;
(b) $M$ is a direct sum of countably generated modules;
(c) $M$ is a Mittag-Leffler module.

The fact that a projective module can be represented as a direct sum of countably generated ones was proved by Kaplansky [Ka].

The remaining part of Theorem 2.2 is due to Raynaud and Gruson [RG]. The key notion of Mittag-Leffler module was introduced in [RG]. Here I prefer only to explain what a flat Mittag-Leffler module is. By the Govorov–Lazard lemma [Gov, L az], a flat $R$-module $M$ can be represented as the inductive limit of a directed family of finitely generated projective modules $P_i$. According to [RG], in this situation $M$ is Mittag-Leffler if and only if the projective system formed by the dual (right) $R$-modules $P_i^* := \text{Hom}_R(P_i, R)$ satisfies the Mittag-Leffler condition: for every $i$ there exists $j \geq i$ such that $\text{Im}(P_j^* \to P_i^*) = \text{Im}(P_k^* \to P_i^*)$ for all $k \geq j$.

**Remarks.** (i) One gets a slightly different definition of not necessarily finite-dimensional vector bundle on a scheme if one replaces projectivity by the property of being a flat Mittag-Leffler module. The product of infinitely many copies of $\mathbb{Z}$ is an example of a flat Mittag-Leffler $\mathbb{Z}$-module which is not projective (it is due to Baer, see p.48 and p.82 of [Ka2]). Unlike projectivity, the property of $M$ being a flat Mittag-Leffler module is a first-order property (in the sense of mathematical logic) of $R(\mathbb{N}) \otimes_R M$ viewed as a module over $\text{End}_R R(\mathbb{N})$ (here $R(\mathbb{N})$ is the right $R$-module freely generated by $\mathbb{N}$). Let me also mention that one does not need AC (the axiom of choice) to prove that a vector space over a field is a flat Mittag-Leffler module, but in set theory without AC one cannot prove that $R$ is a direct summand of a free $\mathbb{Q}$-module\(^1\) (one cannot even prove the existence of a $\mathbb{Q}$-linear embedding of $R$ into a free $\mathbb{Q}$-module $F$, for given such an embedding and using a $\mathbb{Q}$-linear retraction $F \to \mathbb{Q}$ one would get a splitting $s : R/\mathbb{Q} \to R$ of the exact sequence $0 \to \mathbb{Q} \to R \to R/\mathbb{Q} \to 0$ and therefore a non-measurable subset $s(R/\mathbb{Q}) \subset R$, but it is known [So] that the existence of such a subset cannot be proved in set theory without AC).

(ii) Instead of property (c) from Theorem 2.2 the authors of [RG] used a slightly different one, which is harder to formulate. Probably their property has some technical advantages.

(iii) Here are some more comments regarding the work [RG]. First, there is no evidence that the authors of [RG] knew that Theorem 2.1 had been conjectured by Grothendieck. Second, their notion of Mittag-Leffler module and their results on infinitely generated projective modules were probably largely forgotten (even though they deserve being mentioned in algebra textbooks). Probably they were “lost” among many other powerful and important results of [RG] (mostly in the spirit of EGA IV).

### 3. Families of Tate vector spaces and the $K_{-1}$-functor

#### 3.1. A class of topological vector spaces

We consider topological vector spaces over a discrete field $k$.

**Definition.** A topological vector space is **linearly compact** if it is the topological dual of a discrete vector space.

**Example:** $k[[t]] \simeq k \times k \times \ldots = (k \oplus k \oplus \ldots)^*$.

A topological vector space $V$ is linearly compact if and only if it has the following 3 properties:

1) $V$ is complete and Hausdorff,
2) $V$ has a base of neighborhoods of 0 consisting of vector subspaces,
3) each open subspace of $V$ has finite codimension.

\(^1\)Without AC it is not true that any free module $F$ is projective, i.e., every epimorphism $M \to F$ has a section. So without AC projectivity is not equivalent to being a direct summand of a free module.
**Definition.** A **Tate space** is a topological vector space isomorphic to $P \oplus Q^*$, where $P$ and $Q$ are discrete.

A topological vector space $T$ is a Tate space if and only if it has an open linearly compact subspace.

**Example:** $k((t))$ equipped with its usual topology (the subspaces $t^n k[[t]]$ form a base of neighborhoods of $0$). This is a Tate space because it is a direct sum of the linearly compact space $k[[t]]$ and the discrete space $t^{-1}k[t^{-1}]$, or because $k[[t]] \subset k((t))$ is an open linearly compact subspace.

Tate spaces play an important role in the algebraic geometry of curves (e.g., the ring of adeles corresponding to an algebraic curve is a Tate space) and also in the theory of infinite-dimensional vector algebras and Conformal Field Theory. In fact, they were introduced by Lefschetz (L, p.78–79) under the name of locally linearly compact spaces. The name “Tate space” was introduced by Beilinson because these spaces are implicit in Tate’s remarkable work [T]. In fact, the approach to residues on curves developed in [T] can be most naturally interpreted in terms of the canonical central extension of the endomorphism algebra of a Tate space, which is also implicit in [T].

### 3.2. What is a family of Tate spaces?

Probably this question has not been considered. We suggest the following answer. In the category of topological modules over a (not necessarily commutative) ring $R$ we define a full subcategory of Tate $R$-modules. If $R$ is commutative then we suggest to consider Tate $R$-modules as “families of Tate spaces”. This viewpoint is justified by Theorems 3.3 and 3.4 below.

#### 3.2.1. Definitions.

An **elementary Tate $R$-module** is a topological $R$-module isomorphic to $P \oplus Q^*$, where $P$, $Q$ are discrete projective $R$-modules ($P$ is a left module, $Q$ is a right one). A **Tate $R$-module** is a direct summand of an elementary Tate $R$-module. A Tate $R$-module $M$ is **quasi-elementary** if $M \oplus R^n$ is elementary for some $n \in \mathbb{N}$.

By definition, a morphism of Tate modules is a continuous homomorphism. The following lemma is very easy.

**Lemma 3.1.** Let $P, Q$ be as in the definition of Tate $R$-module. Then every morphism $Q^* \to P$ has finitely generated image. □

#### 3.2.2. Examples.

1) $R((t))^n$ is an elementary Tate $R$-module.

2) A finitely generated projective $R((t))$-module $M$ has a unique structure of topological $R((t))$-module such that every $R((t))$-linear morphism $M \to R((t))$ is continuous. This topology is called the **standard topology** of $M$. Clearly $M$ equipped with its standard topology is a Tate $R$-module. In general, it is not quasi-elementary. E.g., let $k$ be a field, $R := \{f \in k[x]|f(0) = f(1)\}$ and

$$M := \{u = u(x, t) \in k[x][[(t)]] | u(1, t) = tu(0, t)\}.$$  

Then $M$ is a finitely generated projective $R((t))$-module which is not quasi-elementary as a Tate $R$-module (see 3.5.3).

**Remark.** The precise relation between finitely generated projective $R((t))$-modules and Tate $R$-modules is explained in Theorem 3.10 below.

#### 3.2.3. Lattices and bounded submodules.

A submodule $L$ of a topological $R$-module $M$ is said to be a **lattice** if it is open and $L/U$ is finitely generated for every open submodule $U \subset L$. A subset of a Tate $R$-module $M$ is **bounded** if it is contained in some lattice. A lattice $L$ in a Tate module $M$ is **coprojective** if $M/L$ is projective.

**Remarks.** (i) One can show that a lattice $L$ in a Tate module $M$ is coprojective if and only if $M/L$ is flat.

(ii) In every Tate $R$-module lattices exist and, moreover, form a base of neighborhoods of $0$. On the other hand, a Tate $R$-module $M$ has a coprojective lattice if and only if $M$ is elementary.
Theorem 3.2. A Tate $R$-module $M$ has the following properties:

(a) $M$ is complete and Hausdorff;
(b) lattices in $M$ form a base of neighborhoods of $0$;
(c) the functor that associates to a discrete $R$-module $N$ the group $\text{Hom}(M, N)$ of continuous homomorphisms $M \to N$ is exact.

If a topological $R$-module $M$ has a countable base of neighborhoods of $0$ and satisfies (a)-(c) then it is a Tate $R$-module.

Only the last statement of the theorem is nontrivial. The countability assumption is essential in it.

Remark. If a topological $R$-module $M$ satisfies (b) then (c) is equivalent to the following property: for every lattice $L$ there is a lattice $L' \subset L$ such that the morphism $M/L' \to M/L$ admits a factorization $M/L' \to P \to M/L$ for some projective module $P$.

3.2.4. Duality. The dual of a Tate $R$-module $M$ is defined to be the right $R$-module $M^*$ of continuous homomorphisms $M \to R$ equipped with the topology whose base is formed by orthogonal complements of open bounded submodules $L \subset M$. Then $M^*$ is again a Tate module, and $M^{**} = M$ (it suffices to check this for elementary Tate modules).

3.2.5. Tate modules as families of Tate spaces.

Theorem 3.3. The notion of Tate module over a commutative ring $R$ is local for the flat topology, i.e., for every faithfully flat commutative $R$-algebra $R'$ the category of Tate $R$-modules is canonically equivalent to that of Tate $R'$-modules equipped with a descent datum.

The proof is based on the Raynaud–Gruson technique.

Theorem 3.4. Let $R$ be a commutative ring. Then every Tate $R$-module $M$ is Nisnevich-locally elementary; in other words, there exists a Nisnevich covering $\text{Spec} R' \to \text{Spec} R$ such that $R' \otimes_R M$ has a coprojective lattice $L'$. Moreover, for every lattice $L \subset M$ one can choose $R'$ and $L'$ so that $L' \supset R' \otimes_R L$.

The proof is not hard. A close statement (Theorem 3.7) will be proved in 3.4.

Let me give the definition of Nisnevich covering. A morphism $\pi : X \to \text{Spec} R$ is said to be a Nisnevich covering if it is etale and there exist closed subschemes $\text{Spec} R = F_0 \supset F_1 \supset \ldots \supset F_n = \emptyset$ such that each $F_i$ is defined by finitely many equations and $\pi$ admits a section over $F_{i-1} \setminus F_i$, $i = 1, \ldots, n$. A morphism $\pi : X \to Y$ is a Nisnevich covering if for every open affine $U \subset Y$ the morphism $\pi^{-1}(U) \to U$ is a Nisnevich covering. (If $Y$ is locally Noetherian then an etale morphism $X \to Y$ is a Nisnevich covering if and only if it admits a section over each point of $Y$; this is the usual definition.) The Nisnevich topology is weaker than etale but stronger than Zariski. The following table may be helpful:

| Topology  | Stalks of $\mathcal{O}_X$, $X = \text{Spec} R$ |
|-----------|-----------------------------------------------|
| Zariski   | Localizations of $R$                          |
| Nisnevich | Henselizations of $R$                         |
| Etale     | Strict henselizations of $R$                  |

3.2.6. Remarks on Theorem 3.4. (i) In Theorem 3.4 one cannot replace “Nisnevich” by “Zariski”. E.g., we will see in 3.5.3 that the Tate module $3.4$ is not Zariski-locally elementary.

(ii) It is easy to show that every quasi-elementary Tate module over a commutative ring $R$ is Zariski-locally elementary.

3.3. Tate $R$-modules and $K_1(R)$. How to see that a Tate $R$-module is not quasi-elementary? We will assign to each Tate $R$-module $M$ a class $[M] \in K_1(R)$ so that $[M] = 0$ if and only if $M$ is quasi-elementary. It is easy to define $[M]$ if one uses the following definition of $K_1(R)$.
3.3.1. $K_{-1}$ via Calkin category. First, introduce the following category $C^\text{all} = C^\text{all}_R$: its objects are all $R$-modules and the group $\text{Hom}_{C^\text{all}}(M, M')$ of $C^\text{all}$-morphisms $M \to M'$ is defined by

$$\text{Hom}_{C^\text{all}}(M, M') := \text{Hom}(M, M')/\text{Hom}_f(M, M'),$$

where $\text{Hom}_f(M, M')$ is the group of $R$-linear maps $A : M \to M'$ whose image is contained in a finitely generated submodule of $M'$. Let $\mathcal{C} \subset C^\text{all}$ be the full subcategory whose objects are projective modules. The idempotent completion\(^2\) of $\mathcal{C}$ (a.k.a. the Karoubi envelope of $\mathcal{C}$) will be denoted by $C^\text{Kar} \subset C^\text{Kar}_R$ and will be called the Calkin category of $R$. Let $\mathcal{C}_{R_0} \subset \mathcal{C}$ be the full subcategory of countably generated projective $R$-modules and $C^\text{Kar}_{R_0}$ its idempotent completion.

**Proposition 3.5.** Every object of $C^\text{Kar}$ is stably equivalent\(^3\) to an object of $C^\text{Kar}_{R_0}$. Two objects of $C^\text{Kar}_{R_0}$ are stably equivalent in $C^\text{Kar}$ if and only if they are stably equivalent in $C^\text{Kar}_R$. So

As a corollary, we see that $K_0(C^\text{Kar})$ is well-defined\(^4\) (even though $C^\text{Kar}$ is not equivalent to a small category), and the morphism $K_0(C^\text{Kar}_{R_0}) \to K_0(C^\text{Kar}_R)$ is an isomorphism. Now define $K_{-1}(R)$ by

$$\tag{3.2} K_{-1}(R) := K_0(C^\text{Kar}_R).$$

**Remarks.** (i) The above definition of $K_{-1}$ is slightly nonstandard but equivalent to the standard ones.

(ii) Define the algebraic Calkin ring by

$$\text{Calk}(R) := \text{End}_\mathcal{C} R^{[N]} := \text{End} R^{[N]} / \text{End}_f R^{[N]}, \quad R^{[N]} := R \oplus R \oplus \ldots$$

(Calk($R$) is an algebraic version of the analysts’ Calkin algebra, which is defined to be the quotient of the ring of continuous endomorphisms of a Banach space by the ideal of compact operators). If $P \in C^\text{Kar}$ then $\text{Hom}_{C^\text{Kar}_R}(P, R^{[N]})$ is a finitely generated projective module over Calk($R$). Thus one gets an antiequivalence between $C^\text{Kar}_{R_0}$ and the category of finitely generated projective Calk($R$)-modules, which induces an isomorphism

$$K_{-1}(R) \xrightarrow{\sim} K_0(\text{Calk}(R))$$

3.3.2. The class of a Tate $R$-module. Let $\mathcal{T}_R$ denote the additive category of Tate $R$-modules. We will define a functor

$$\Phi : \mathcal{T}_R \to C^\text{Kar}_R.$$

Let $E_R \subset \mathcal{T}_R$ be the full subcategory of elementary Tate modules. One gets a functor $\Psi : E_R \to \mathcal{C}_R$ by setting $\Psi(P \oplus Q^*) := P$ (here $P, Q$ are discrete projective modules) and defining $\Psi(f) \in \text{Hom}_\mathcal{C}(P, P_1)$, $f : P \oplus Q^* \to P_1 \oplus Q_1^*$, to be the image of the composition $P \hookrightarrow P \oplus Q^* \xrightarrow{f} P_1 \oplus Q_1^* \to P_1$ in $\text{Hom}_\mathcal{C}(P, P_1)$ (the equality $\Psi(f'f) = \Psi(f')\Psi(f)$ follows from Lemma 3.1). The functor $\Phi$ is defined to be the extension of $\Psi : E_R \to \mathcal{C}_R \subset C^\text{Kar}_R$ to $\mathcal{T}_R = E_R^\text{Kar}$. Now define the class $[M]$ of a Tate $R$-module $M$ by $[M] := [\Phi(M)] \in K_0(C^\text{Kar}_R) = K_{-1}(R)$.

\(^2\) The idempotent completion of a category $\mathcal{B}$ is the category $\mathcal{B}^\text{Kar}$ in which an object is a pair $(B, p : B \to B)$ with $B \in \mathcal{B}$ and $p^2 = p$, and a morphism $(B_1, p_1) \to (B_2, p_2)$ is a $\mathcal{B}$-morphism $\varphi : B_1 \to B_2$ such that $p_2 \circ \varphi = \varphi$. This construction was explained by P. Freyd in Exercise B2 of Ch. 2 of [Fr] a few years before Karoubi.

\(^3\) Objects $X, Y$ of an additive category $\mathcal{A}$ are said to be stably equivalent if $X \oplus Z \simeq Y \oplus Z$ for some $Z \in \mathcal{A}$.

\(^4\) $K_0$ of an additive category $\mathcal{A}$ is defined by the usual universal property. It may exist even if $\mathcal{A}$ is not equivalent to a small category, e.g., $K_0$ of the category of all vector spaces equals 0.
3.3.3. $K_0$ of the category of Tate $R$-modules.

**Theorem 3.6.** (i) A Tate $R$-module has zero class in $K_{-1}(R)$ if and only if it is quasi-elementary.

(ii) $K_0(T_R)$ is well-defined (even though $T_R$ is not equivalent to a small category).

(iii) The morphism $K_0(T_R) \to K_0(C^K_{R_0}) = K_{-1}(R)$ induced by (3.3) is an isomorphism.

(iv) Every element of $K_0(T_R) = K_{-1}(R)$ can be represented as the class of $R((t)) \otimes_{R[t,t^{-1}]} P$ for some finitely generated projective $R[t,t^{-1}]-module P$.

**Remark.** The only nontrivial point of the proof is the surjectivity of the composition

\[(3.4) \quad K_0(R[t,t^{-1}]) \to K_0(R((t))) \to K_0(T_R) \to K_{-1}(R),\]

which is used in the proof of (iii) and (iv) (in fact, to prove (iii) it suffices to use Theorem 4.1(a) below). The surjectivity of (3.4) is a standard fact\(^5\) from $K$-theory. It is proved by noticing that there is a canonical section $K_{-1}(R) \to K_0(R[t,t^{-1}])$, namely multiplication by the canonical element of $K_1(\mathbb{Z}[t,t^{-1}])$.

3.4. Nisnevich-local vanishing of $K_{-1}$. Theorem 3.4 is closely related\(^6\) to the following theorem, which I was unable to find in the literature.

**Theorem 3.7.** Let $R$ be a commutative ring. Then every element of $K_{-1}(R)$ vanishes Nisnevich-locally.

**Remarks.** (i) According to Example 8.5 of [We2] (which goes back to L. Reid’s work [Re]), it is not true that every element of $K_i(R)$, $i < -1$, vanishes Nisnevich-locally.

(ii) It is known that $K_{-1}$ commutes with filtering inductive limits. So Theorem 3.7 is equivalent to vanishing of $K_{-1}(R)$ for commutative Henselian rings $R$. I prefer the above formulation of the theorem because commutation of $K_{-1}$ with filtering inductive limits is not immediate if one defines $K_{-1}$ by (3.2), i.e., via the Calkin category.

In the proof of Theorem 3.7 given below we use the definition of $K_{-1}$ from 3.3.1, but it is also easy to prove the theorem using the definition of $K_{-1}$ given by H. Bass [Ba].

**Proof.** It suffices to show that if $P$ is an $R$-module\(^7\), $F \subset P$ is a finitely generated submodule, and $\pi \in \text{End } P$ is such that $\text{Im}(\pi^2 - \pi) \subset F$ then after Nisnevich localization there exists $\bar{\pi} \in \text{End } P$ such that $\bar{\pi}^2 = \bar{\pi}$ and $\text{Im}(\bar{\pi} - \pi) \subset F$.

The idea is to look at the spectrum of $\pi$. There exists a monic $f \in R[\lambda]$ such that $f(\pi^2 - \pi)$ annihilates $F$. Then $f(\pi^2 - \pi)(\pi^2 - \pi) = 0$. Put $g(\lambda) := (\lambda^2 - \lambda)f(\lambda^2 - \lambda)$, then there is a unique morphism $R[\lambda]/(g) \to \text{End } P$ such that $\lambda \mapsto \pi$. Put $\mathcal{S} := \text{Spec } R[\lambda]/(g) \subset \text{Spec } R \times \mathbb{A}^1$, then $\mathcal{S} \supset \emptyset \cup \mathbb{A}$, where $\emptyset = \text{Spec } R \times \{0\}$ and $\mathbb{A} = \text{Spec } R \times \{1\}$.

Suppose we have a decomposition

\[(3.5) \quad S = S_0 \amalg S_1, \quad S_0 \cap S_1 \supset \emptyset, S_0 \supset \emptyset, S_1 \supset \emptyset, \]

Then we can define $e \in R[\lambda]/(g) = H^0(S, \mathcal{O}_S)$ by $e|_{S_0} = 0$, $e|_{S_1} = 1$ and define $\bar{\pi}$ to be the image of $e$ in $\text{End } P$.

Claim: a decomposition (3.5) exists Nisnevich-locally on $\text{Spec } R$. Indeed, according to the table at the end of 3.2.5 it suffices to show that this decomposition exists if $R$ is Henselian. Let $\bar{g} \in (R/m)[\lambda]$ be the reduction of $g$ modulo the maximal ideal $m \subset R$. To get (3.5) it suffices to choose a factorization $g = \bar{g}_0 \bar{g}_1$ so that $\bar{g}_0, \bar{g}_1$ are coprime, $\bar{g}_0(0) = 0, \bar{g}_1(1) = 0$ and then lift it to a factorization $g = g_0 g_1$.

---

\(^5\)The surjectivity of (3.4) is a tautology if one uses the definition of $K_{-1}$ given by H. Bass [Ba]. But it is a theorem if one defines $K_{-1}$ by (3.2).

\(^6\)More precisely: Theorem 3.7 follows from Theorems 3.2(i) and 3.6(iii); Theorem 3.4 follows from Theorem 3.7 and (1.9(ii)).

\(^7\)We need only the case that $P$ is projective, but projectivity is not used in what follows.
3.5. The dimension torsor. Let $R$ be commutative. Then it follows from Theorem 8.5 of [We2] that there is a canonical epimorphism $K_{-1}(R) \to H^1_{et}(\text{Spec } R, \mathbb{Z})$, so a Tate $R$-module $M$ should define $\alpha_M \in H^1_{et}(\text{Spec } R, \mathbb{Z})$. We will define $\alpha_M$ explicitly as a class of a certain $\mathbb{Z}$-torsor $\dim_M$ on $\text{Spec } R$ canonically associated to $M$. $\dim_M$ is called “the torsor of dimension theories” or “dimension torsor”.

3.5.1. The case that $R$ is a field. If $M$ is a Tate vector space over a field $R$ the notion of dimension torsor is well known.\(^8\) Notice that if $L \subset M$ is open and linearly compact then usually $\dim L = \infty$ and $\dim(M/L) = \infty$. But for any open linearly compact $L$, $L' \subset M$ one has the relative dimension $d_L^{L'} := \dim(L'/L \cap L) - \dim(L/L' \cap L) \in \mathbb{Z}$.

**Definition.** A dimension theory on a Tate vector space $M$ is a function

$$d : \text{open linearly compact subspaces } L \subset M \to \mathbb{Z}$$

such that $d(L') - d(L) = d_L^{L'}$.

A dimension theory exists and is unique up to adding $n \in \mathbb{Z}$. So dimension theories on a Tate space form a $\mathbb{Z}$-torsor. This is $\dim_M$.

**Example.** Let $T$ be a $\mathbb{Z}$-torsor, let $R(T)$ be the vector space over a field $R$ freely generated by $T$. Then $\mathbb{Z}$ acts on $R(T)$, so $R(T)$ becomes a $R[\mathbb{Z}, \mathbb{Z}^{-1}]$-module (multiplication by $z$ coincides with the action of $1 \in \mathbb{Z}$). Put $M := R((z)) \otimes_{R[\mathbb{Z}, \mathbb{Z}^{-1}]} R(T)$. Then one has a canonical isomorphism

$$(3.6) \quad \dim_M \sim_{\sim} T :$$

to $t \in T$ one associates the dimension theory $d_t$ such that $d_t(L_t) = 0$, where $L_t \subset M$ is the $R[[z]]$-subspace generated by $t$.

3.5.2. The general case. If $M$ is a Tate module and $L \subset L' \subset M$ are coprojective lattices then $L'/L$ is a finitely generated projective $R$-module, so if $R$ is commutative then $d_L^{L'} := \text{rank}(L'/L) \in H^0(\text{Spec } R, \mathbb{Z})$ is well-defined.

**Definition.** Let $M$ be a Tate module over a commutative ring $R$. A dimension theory on $M$ is a rule that associates to each $R$-algebra $R'$ and each coprojective lattice $L \subset R' \otimes_R M$ a locally constant function $d_L : \text{Spec } R' \to \mathbb{Z}$ in a way compatible with base change and so that $d_{L_2} - d_{L_1} = \text{rank}(L_2/L_1)$ for any pair of coprojective lattices $L_1 \subset L_2 \subset R' \otimes_R M$. Here $R' \otimes_R M$ denotes the completed tensor product.

Theorem 3.4 implies that if the functions $d_L$ with the above properties are defined for all etale $R$-algebras then there exists a unique way to extend the definition to all $R$-algebras. It also shows that dimension theories form a $\mathbb{Z}$-torsor for the Nisnevich topology.\(^9\) It is called the dimension torsor and denoted by $\dim_M$.

One has a canonical isomorphism

$$(3.7) \quad \dim_{M_1 \oplus M_2} \sim_{\sim} \dim_{M_1} + \dim_{M_2} .$$

So one gets a morphism $K_0(T_R) = K_{-1}(R) \to H^1_{et}(\text{Spec } R, \mathbb{Z})$. It is surjective. Indeed, let $T$ be a $\mathbb{Z}$-torsor on $S := \text{Spec } R$. Then the free $O_S$-module $O_S(T)$ generated by the sheaf of sets $T$ is equipped with an action of $\mathbb{Z}$, so it is a module over $O_S[z, z^{-1}]$ (multiplication by $z$ coincides with the action of $1 \in \mathbb{Z}$). This module is locally free of rank one, so its global sections form a projective $R[z, z^{-1}]$-module $R(T)$ of rank 1. Therefore $R((z)) \otimes_{R[z, z^{-1}]} R(T)$ is a Tate $R$-module. Its dimension torsor is canonically isomorphic to $T$ (cf. (3.6)).

\(^8\)I copied the definition below from [Ka3], but the notion goes back at least to the physical concept of “Dirac sea”, which many years later became the “infinite wedge construction” in the representation theory of infinite-dimensional Lie algebras.

\(^9\)In fact, the categories of $\mathbb{Z}$-torsors for the Nisnevich, etale, fppf, and fpqc topologies are equivalent.
3.5.3. Example. Let $M$ be the Tate module \([3.1]\) over $R := \{f \in k[x]| f(0) = f(1)\}$. Then the $\mathbb{Z}$-torsor $\text{Dim}_M$ is nontrivial (its pullback to $S := \text{Spec}(R \otimes_k \bar{k})$ corresponds to the universal covering of $S$). So the class of $M$ in $K_0(\mathcal{T}_R) = K_{-1}(R)$ is nontrivial and therefore $M$ is not quasi-elementary. Moreover, it does not become quasi-elementary after Zariski localization.

3.5.4. The kernel of the morphism $K_{-1}(R) \to H^1_{\text{et}}(\text{Spec } R, \mathbb{Z})$ may be nonzero. Moreover, this can happen even if $R$ is local. Examples can be found in \([\text{We}3]\). More precisely, §6 of \([\text{We}3]\) contains examples of algebras $R$ over a field $k$ such that $H^1_{\text{et}}(\text{Spec } R, \mathbb{Z}) = 0$ but $K_{-1}(R) \neq 0$. In each of these examples $\text{Spec } R$ is a normal surface with one singular point $x$. Let $R_x$ denote the local ring of $x$. According to \([\text{We}1]\), the map $K_{-1}(R) \to K_{-1}(R_x)$ is an isomorphism, so $K_{-1}(R_x) \neq 0$.

3.6. The determinant gerbe. Given a Tate space $M$ over a field Kapranov \([\text{Ka}3]\) defines its groupoid of determinant theories. The definition is based on the notion of relative determinant of two lattices in a Tate space and goes back to J.-L. Brylinski \([\text{Br}]\) (and further back to the Japanese school and \([\text{ACK}]\)). If $M$ is a Tate module over a commutative ring $R$ then rephrasing the definition from \([\text{Ka}3]\) in the obvious way one gets a sheaf of groupoids on the Nisnevich topology of $S := \text{Spec } R$ (details will be explained in §6). This sheaf of groupoids is, in fact, an $\mathcal{O}_S^\times$-gerbe. We call it the determinant gerbe of $M$. Associating the class of this gerbe to a Tate $R$-module $M$ one gets a morphism

\[
K_0(\mathcal{T}_R) = K_{-1}(R) \to H^2_{\text{Nis}}(S, \mathcal{O}_S^\times).
\]

Probably it is well known to $K$-theorists. One can get the restriction of \([3.8]\) to $\text{Ker}(K_{-1}(R) \to H^1_{\text{et}}(\text{Spec } R, \mathbb{Z}))$ (and possibly the morphism \([3.8]\) itself) from the Brown–Gersten–Thomason spectral sequence (\([\text{TT}]\), §10.8). More details on the determinant gerbes will be given in §6

3.7. Co-Sato Grassmannian. Let $M$ be a Tate module over a commutative ring $R$. The co-Sato Grassmannian of $M$ is the following functor $\text{Gras}_M$ from the category of commutative $R$-algebras $R'$ to that of sets: $\text{Gras}_M(R')$ is the set of coprojective lattices in $R' \hat{\otimes}_R M$. Given lattices $L \subset M$ and $\hat{L} \subset M^*$ let $\text{Gras}_{M, L}^{\hat{L}}(R') \subset \text{Gras}_M(R')$ be the set of coprojective lattices in $R' \hat{\otimes}_R M$ containing $R' \hat{\otimes}_R L$ and orthogonal to $R' \hat{\otimes}_R \hat{L}$. The functor $\text{Gras}_M$ is the inductive limit of the subfunctors $\text{Gras}_{M, L}^{\hat{L}}$, and these subfunctors form a filtering family. Theorem \([3.4]\) easily implies the following proposition.

**Proposition 3.8.** (i) $\text{Gras}_{M, L}^{\hat{L}}$ is an algebraic space proper and of finite presentation over $\text{Spec } R$. Locally for the Nisnevich topology of $\text{Spec } R$ it is a projective scheme over $\text{Spec } R$.

(ii) $\text{Gras}_M$ is an ind-algebraic space ind-proper over $\text{Spec } R$.

**Remarks.** (a) A standard argument based on the Plücker embedding (see \([3.13]\)) shows that if the determinant gerbe of $M$ is trivial then $\text{Gras}_{M, L}^{\hat{L}}$ is projective over $\text{Spec } R$ and $\text{Gras}_M$ is an ind-projective ind-scheme.

(b) Using Proposition \([3.8]\) it is easy to prove ind-representability and ind-properness of the $\mathcal{F}$-twisted affine Grassmannian $G\mathcal{R}_\mathcal{F}$ of a reductive group scheme $G$ over $R$. Here $\mathcal{F}$ is a $G$-torsor on $\text{Spec } R((t))$ and $G\mathcal{R}_\mathcal{F}$ is the functor that sends a commutative $R$-algebra $R'$ to the set of extensions of $\mathcal{F} \otimes_R R'((z))$ to a $G$-torsor over $\text{Spec } R'[[z]]$ (up to isomorphisms whose restriction to $\mathcal{F} \otimes_R R'((z))$ equals the identity).

3.8. Finitely generated projective $R((t))$-modules from the Tate viewpoint. Theorem \([3.10]\) below says that a finitely generated projective $R((t))$-module is the same as a Tate $R$-module equipped with a topologically nilpotent automorphism. An endomorphism (in particular, an automorphism) of a Tate $R$-module $M$ is said to be topologically nilpotent if it satisfies the equivalent conditions of the next lemma.
Lemma 3.9. Let $M$ be a Tate $R$-module, $T \in \text{End} M$. Then the following conditions are equivalent:

(i) $T^n \rightarrow 0$ for $n \rightarrow 0$ (which means that for every lattices $L, L' \subset M$ there exists $N$ such that $T^nL' \subset L$ for all $n > N$);

(ii) there exists a (unique) structure of topological $R[[t]]$-module on $M$ such that $T$ acts as multiplication by $t$.

If $M$ is a finitely generated projective $R((t))$-module equipped with its standard topology then multiplication by $t$ is a topologically nilpotent automorphism of $M$. The next theorem says that the converse statement is also true.

Theorem 3.10. Let $M$ be a Tate $R$-module and $T : M \rightarrow M$ be a topologically nilpotent automorphism. Equip $M$ with the topological $R((t))$-module structure such that $tm = T(m)$ for $m \in M$. Then $M$ is a finitely generated projective $R((t))$-module, and the topology on $M$ is the standard one.

Theorem 3.11. Let $R$ be commutative. Then the notion of finitely generated projective $R((t))$-module is local for the fpqc topology of $\text{Spec} R$. More precisely, let $R'$ be a faithfully flat commutative $R$-algebra, $R'' := R' \otimes_R R'$, and let $f, g : R'(t)) \rightarrow R''(t)$ be defined by $f(a) := 1 \otimes a, g(a) := a \otimes 1$; then the category of finitely generated projective $R((t))$-modules is canonically equivalent to that of finitely generated projective $R'(t))$-modules $M'$ equipped with an isomorphism $R''(t)) \otimes_f M' \sim R''(t)) \otimes_g M'$ satisfying the usual cocycle condition.

This is an immediate corollary of Theorems 3.3 and 3.10.

Remark. If $R$ is of finite type over a field $k$ and the morphism $\text{Spec} R' \rightarrow \text{Spec} R$ is a Zariski covering then Theorem 3.11 is well known from the theory of non-archimedian analytic spaces $\text{BGR}$, which is applicable because $R((t))$ is an affinoid $(k((t)))$-algebra in the sense of 5.3.

3.9. The dimension torsor of a projective $R((t))$-module. Let $R$ be a commutative ring. Let $M$ be a finitely generated projective $R((t))$-module equipped with an isomorphism $\varphi : \text{det} M \sim R((t))$. If $R$ is a field then $M$ has an $R[[t]]$-stable lattice; moreover, there is a lattice $L \subset M$ such that

$$R[[t]]L \subset L, \quad \varphi(\text{det} L) = R[[t]].$$

So it is easy to see that if $R$ is a field then there is a unique dimension theory $d_\varphi$ on $M$ such that $d_\varphi(L) = 0$ for all lattices $L \subset M$ satisfying (3.9). Therefore if $R$ is any commutative ring then the $\mathbb{Z}$-torsor $\text{Dim}_M$ is trivialized over each point of $\text{Spec} R$.

Proposition 3.12. These trivializations come from a (unique) trivialization $d_\varphi$ of the $\mathbb{Z}$-torsor $\text{Dim}_M$.

By Proposition 3.12 the morphism $K_0(R((t))) \rightarrow H^1_{\text{et}}(\text{Spec} R, \mathbb{Z})$ that sends the class of a projective $R((t))$-module $M$ to the class of $\text{Dim}_M$ annihilates the kernel of the epimorphism $\text{det} : K_0(R((t))) \twoheadrightarrow \text{Pic} R((t))$, so we get a morphism

$$f : \text{Pic} R((t)) \rightarrow H^1_{\text{et}}(\text{Spec} R, \mathbb{Z})$$

such that the diagram

$$\begin{array}{c}
K_0(R((t))) \\
\downarrow \text{det} \\
H^1_{\text{et}}(\text{Spec} R, \mathbb{Z})
\end{array}$$

commutes. The composition

$$g : \text{Pic} R[t, t^{-1}] \rightarrow \text{Pic} R((t)) \xrightarrow{f} H^1_{\text{et}}(\text{Spec} R, \mathbb{Z})$$

was studied in [We2].
Remarks. (i) As explained in [We2], the kernels of (3.10) and (3.12) may be nontrivial (even if $R$ is Henselian). Example: if $k$ is a field and $R$ is either $k[x^2,x^3] \subset k[x]$ or the Henselization of $k[x^2,x^3]$ at the singular point of its spectrum then $\text{Ker } f \simeq k((t))/k[[t]]$, $\text{Ker } g \simeq k[t,t^{-1}]/k[[t]]$ (e.g., to show that $\text{Ker } f \simeq k((t))/k[[t]]$ for $R = k[x^2,x^3]$ notice that a line bundle on $\text{Spec } R(t)$ is the same as a triple consisting of a line bundle on $\text{Spec } k[x]((t))$, a line bundle on $\text{Spec } k((t))$ and an isomorphism between their pullbacks to $\text{Spec } k[x]((t))/(x^2))$. It is also explained in [We2] that $g$ has a splitting (and therefore $f$ has). Indeed, $\text{Pic } R[t,t^{-1}] = \text{H}^1_{et}(\text{Spec } R, C)$, where $C$ is the derived direct image of the etale sheaf of invertible functions on $\text{Spec } R[t,t^{-1}]$, and the morphism $Z \to C$ defined by $n \mapsto t^n$ gives a splitting.

(ii) The interested reader can easily lift the diagram (3.11) of abelian groups to a commutative diagram of appropriate Picard groupoids (in the sense of §4.2).

4. ALMOST PROJECTIVE AND 2-ALMOST PROJECTIVE MODULES.

4.1. Main definitions and results. Recall that every Tate $R$-module has a lattice but not necessarily a coprojective one. If $M$ is a Tate $R$-module and $L \subset M$ is a lattice (resp. a bounded open submodule) then $M/L$ is 2-almost projective (resp. almost projective) in the sense of the following definitions.

Definitions. An elementary almost projective $R$-module is a module isomorphic to a direct sum of a projective $R$-module and a finitely generated one. An almost projective $R$-module is a direct summand of an elementary almost projective module. An almost projective $R$-module $M$ is quasi-elementary if $M \oplus R^n$ is elementary for some $n \in \mathbb{N}$.

Definition. An $R$-module $M$ is 2-almost projective if it can be represented as a direct summand of $P \oplus F$ with $P$ a projective $R$-module and $F$ an $R$-module of finite presentation.

In fact, there is a reasonable notion of $n$-almost projectivity for any positive $n$, see Remark 3 at the end of this subsection.

Remark. It is easy to show that an almost projective module $M$ is quasi-elementary if and only if it can be represented as $P/N$ with $P$ projective and $N \subset P$ a submodule of a finitely generated submodule of $P$. It is also easy to show that for $P$ and $N$ as above $P/N$ is 2-almost projective if and only if $N$ is finitely generated.

Theorem 4.1. (a) Every almost projective $R$-module $M_0$ can be represented as $M/L$ with $M$ being a Tate $R$-module and $L \subset M$ a bounded open submodule.

(b) If $M_0$ is 2-almost projective then in such a representation $L$ is a lattice.

Theorem 4.2. (i) The notion of almost projective module over a commutative ring $R$ is local for the flat topology, i.e., for every faithfully flat commutative $R$-algebra $R'$ almost projectivity of an $R$-module $M$ is equivalent to almost projectivity of the $R'$-module $R' \otimes_R M$. The same is true for 2-almost projectivity.

(ii) For every almost projective module $M$ over a commutative ring $R$ there exists a Nisnevich covering $\text{Spec } R' \to \text{Spec } R$ such that $R' \otimes_R M$ is elementary.

(iii) For every quasi-elementary almost projective module $M$ over a commutative ring $R$ there exists a Zariski covering $\text{Spec } R = \bigcup_i \text{Spec } R_{f_i}$ such that $R_{f_i} \otimes_R M$ is elementary for all $i$.

The proof of (i) is based on the Raynaud–Gruson technique. The proofs of (ii) and (iii) are much easier. In particular, (iii) easily follows from Kaplansky’s Theorem [Ka], which says that a projective module over a local field is free (even if it is not finitely generated).

Remarks. 1) In statement (ii) of the theorem one cannot replace “Nisnevich” by “Zariski”. E.g., the quotient of the Tate $R$-module (3.1) by any open bounded submodule is an almost projective module which is not Zariski-loocally elementary (because the Tate module (3.1) is not, see 3.5.3).
2) My impression is that statement (ii) is more important than (i) even though it is much easier to prove. Statement (i) gives you a peace of mind (without it one would have two candidates for the notion of almost projectivity), but in the examples of almost projective modules that I know one can prove almost projectivity directly rather than showing that the property holds locally. The roles of Theorems 3.3 and 3.4 in the theory of Tate R-modules are similar.

3) Although we do not need it in the rest of this work, let us define the notion of n-almost projectivity for any n ∈ N: an R-module M is n-almost projective if in the derived category of R-modules M can be represented as a direct summand of $P \oplus F^r$ with P being a projective R-module and $F^r$ being a complex of projective R-modules such that $F^i = 0$ for $i > 0$ and $F^i$ is finitely generated for $i > -n$. One can show that for $n = 1, 2$ this is equivalent to the above definitions of almost projectivity and 2-almost projectivity and that if $n > 2$ then an R-module M is n-almost projective if and only if it is 2-almost projective and for some (or for any) epimorphism $f : P \to M$ with P projective Ker $f$ is $(n - 1)$-almost projective. One can also show that a module M over a commutative ring is n-almost projective if and only if it can be Nisnevich-locally represented as a direct sum of a projective module and a module $M'$ having a resolution $P_{n-1} \to P_{n-2} \to \ldots P_0 \to M' \to 0$ by finitely generated projective modules.

4.2. Class of an almost projective module in $K_{-1}$. In 3.3.1 we defined the category $C_{\text{all}}$ and its full subcategory $C$ formed by projective modules. Let $C^\text{ap} \subset C_{\text{all}}$ denote the full subcategory of almost projective modules. By definition, an almost projective module M is a direct summand of $F \oplus P$ with $F$ finitely generated and P projective, so M viewed as an object of $C^\text{ap}$ becomes a direct summand of $P \in C$. So we get a fully faithful functor $\Phi : C^\text{ap} \to C_{\text{Kar}}$ (in fact, it is not hard to prove that $\Phi$ is an equivalence). To an almost projective R-module M one associates an element $[M] \in K_{-1}(R) := K_0(C_{\text{Kar}})$, namely $[M]$ is the class of $\Phi(M) \in C_{\text{Kar}}$.

Let T be a Tate R-module and $L \subset T$ an open bounded submodule (so $T/L$ is almost projective). Then $[T/L] = [T]$.

4.3. The dimension torsor of an almost projective module. To an almost projective module one associates its dimension torsor. The definition is given below. It is parallel to the definition of the dimension torsor of a Tate R-module, but there is one new feature: the dimension torsor of an almost projective module is equipped with a canonical upper semicontinuous section.

A submodule L of an almost projective R-module M is said to be a lattice if it is finitely generated. In this case $M/L$ is also almost projective. A lattice $L \subset M$ is said to be coprojective if $M/L$ is projective. One shows that in this case $M/L$ is projective and L has finite presentation, so coprojective lattices exist if and only if M is elementary.

Now let R be commutative. We define a dimension theory (resp. upper semicontinuous dimension theory) on an almost projective R-module M to be a rule that associates to each R-algebra $R'$ and each coprojective lattice $L \subset R' \otimes_R M$ a locally constant (resp. upper semicontinuous) function $d_L : \text{Spec } R' \to \mathbb{Z}$ in a way compatible with base change and so that $d_{L_2} - d_{L_1} = \text{rank}(L_2/L_1)$ for any pair of coprojective lattices $L_1 \subset L_2 \subset R' \otimes_R M$. The notion of dimension theory (or upper semicontinuous dimension theory) does not change if one considers only etale R-algebras instead of arbitrary ones. Dimension theories on an almost projective R-module M form a $\mathbb{Z}$-torsor for the Nisnevich topology of $\text{Spec } R$, which is denoted by $\text{Dim}_M$. One defines the canonical upper semicontinuous dimension theory $d^\text{can}$ on M by $d^\text{can}_L(x) := \dim_{K_x}(K_x \otimes_R L)$, where $R'$ is an R-algebra, $L \subset R' \otimes_R M$ is a coprojective lattice, $x \in \text{Spec } R'$, and $K_x$ is the residue field of $x$. An upper semicontinuous dimension theory on M is the same as an upper semicontinuous section of $\text{Dim}_M$, by which we mean a $\mathbb{Z}$-antiequivariant morphism from the $\mathbb{Z}$-torsor $\text{Dim}_M$ to the sheaf of upper semicontinuous $\mathbb{Z}$-valued functions on $\text{Spec } R$. Clearly $d^\text{can}$ is a true (i.e., locally constant) section of $\text{Dim}_M$ if and only if the quotient

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10One can show that this definition is equivalent to the following one: an R-module M is n-almost projective if a projective resolution of M viewed as a complex in the Calkin category $C_{\text{Kar}}^R$ from 3.3.1 is homotopy equivalent to a direct sum of an object of $C_{\text{Kar}}^R$ and a complex $C'$ in $C_{\text{Kar}}^R$ such that $C' \neq 0$ only for $i \leq -n$.

11One can show that if M is 2-almost projective this is equivalent to $M/L$ being flat.
of $M$ modulo the nilradical $I \subset R$ is projective over $R/I$. In this case $d^{\text{can}}$ defines a trivialization of $\text{Dim}_M$.

If $N$ is a Tate $R$-module and $L \subset N$ is an open bounded submodule then the dimension torsor of the almost projective module $N/L$ canonically identifies with that of $N$.

5. Finer points: determinants and the canonical central extension

Subsection 5.6 (in which we discuss the canonical central extension of the automorphism group of an almost projective module) is the only part of this section used in the rest of the article, namely in §7. Therefore some readers (especially those interested primarily in spaces of formal loops and refined motivic integration) may prefer to skip this section. But it contains an interesting (though slightly vague) picture, which I learned from A. Beilinson (see §5.5).

In 5.1-5.4 we follow §2 of [BBE]. In particular, we combine the dimension torsor and the determinant gerbe into a single object, which is a Torsor over a certain Picard groupoid (these notions are defined below). The reason why it is convenient and maybe necessary to do this is explained in §5.6. Our terminology is slightly different from that of [BBE], and our determinant Torsor is inverse to that of [BBE].

5.1. Terminology. According to §1.4 of [Del], a Picard groupoid is a symmetric monoidal category $\mathcal{A}$ such that all the morphisms of $\mathcal{A}$ are invertible and the semigroup of isomorphism classes of the objects of $\mathcal{A}$ is a group. A Picard groupoid is said to be strictly commutative if for every $a \in \text{Ob}\mathcal{A}$ the commutativity isomorphism $a \otimes a \sim (a \otimes a) \sim (-1)^{p(a)p(b)} b \otimes a$.

For a scheme $S$ denote by $\mathcal{P}ic^\mathcal{Z}_S$ (resp. $\mathcal{P}ic^p_S$) the sheaf of Picard groupoids on the Nisnevich site of $S$ formed by $\mathbb{Z}$-graded invertible $\mathcal{O}_S$-modules (resp. plain invertible $\mathcal{O}_S$-modules, a.k.a. $\mathcal{O}_S^\times$-torsors).

We need more terminology. An Action of a monoidal category $\mathcal{A}$ on a category $\mathcal{C}$ is a monoidal functor from $\mathcal{A}$ to the monoidal category $\text{Funct}(\mathcal{C}, \mathcal{C})$ of functors $\mathcal{C} \to \mathcal{C}$. Suppose $\mathcal{A}$ acts on $\mathcal{C}$ and $\mathcal{C}'$, i.e., one has monoidal functors $\Phi : \mathcal{A} \to \text{Funct}(\mathcal{C}, \mathcal{C})$ and $\Phi : \mathcal{A} \to \text{Funct}(\mathcal{C}', \mathcal{C}')$. Then an $\mathcal{A}$-functor $\mathcal{C} \to \mathcal{C}'$ is a functor $F : \mathcal{C} \to \mathcal{C}'$ equipped with isomorphisms $F\Phi(a) \sim \Phi'(a)F$ satisfying the natural compatibility condition (the two ways of constructing an isomorphism $F\Phi(a_1 \otimes a_2) \sim F'(a_1 \otimes a_2)F$ must give the same result). An $\mathcal{A}$-equivalence $\mathcal{C} \to \mathcal{C}'$ is an $\mathcal{A}$-functor $\mathcal{C} \to \mathcal{C}'$ which is an equivalence.

There is also an obvious notion of Action of a sheaf of monoidal categories $\mathcal{A}$ on a sheaf of categories $\mathcal{C}$, and given an Action of $\mathcal{A}$ on $\mathcal{C}$ and $\mathcal{C}'$ there is an obvious notion of $\mathcal{A}$-functor $\mathcal{C} \to \mathcal{C}'$ and $\mathcal{A}$-equivalence $\mathcal{C} \to \mathcal{C}'$.

Definition. Let $\mathcal{A}$ be a sheaf of Picard groupoids on a site. A sheaf of categories $\mathcal{C}$ equipped with an Action of $\mathcal{A}$ is an $\mathcal{A}$-Torsor if it is locally $\mathcal{A}$-equivalent to $\mathcal{A}$.

Remark. The notion of Torsor makes sense even if $\mathcal{A}$ is non-symmetric. But $\mathcal{A}$ has to be symmetric if we want to have a notion of product of $\mathcal{A}$-Torsors.

5.2. The determinant Torsor. Let $R$ be a commutative ring, $S := \text{Spec} R$. Slightly modifying the construction of [Ka3], we will associate a Torsor over $\mathcal{P}ic^\mathcal{Z}_S$ to an almost projective $R$-module $M$. Recall that a coprojective lattice $L \subset M$ is a finitely generated submodule such that $M/L$ is projective. The set of coprojective lattices $L \subset M$ will be denoted by $G(M)$. In general, $G(M)$ may be empty, and it is not clear if every $L_1, L_2 \in G(M)$ are contained in some $L \in G(M)$. But it follows from Theorem 1.2 (ii) that these properties hold after Nisnevich localization (to show
that every $L_1, L_2 \in G(M)$ are Nisnevich-locally contained in some coprojective lattice apply statement (ii) or (iii) of Theorem 4.2 to $M/(L_1 + L_2)$. In other words, for every $x \in \text{Spec } R$ the inductive limit of $G(R' \otimes_R M)$ over the filtering category of all etale $R$-algebras $R'$ equipped with an $R$-morphism $x \to \text{Spec } R'$ is a non-empty directed set.

For each pair $L_1 \subset L_2$ in $G(M)$ one has the invertible $R$-module $\det(L_2/L_1)$. It is equipped with a $\mathbb{Z}$-grading (the determinant of an $n$-dimensional vector space has grading $n$).

**Definition.** A determinant theory on $M$ (resp. a weak determinant theory on $M$) is a rule $\Delta$ which associates to each $R$-algebra $R'$ and each $L \in G(R' \otimes_R M)$ an invertible graded $R'$-module $\Delta(L)$ (resp. an invertible $R'$-module $\Delta(L)$), to each pair $L_1 \subset L_2$ in $G(R' \otimes_R M)$ an isomorphism

$$\Delta_{L_1, L_2} : \Delta(L_1) \otimes \det(L_2/L_1) \xrightarrow{\sim} \Delta(L_2),$$

and to each morphism $f : R' \to R''$ of $R$-algebras a collection of base change morphisms $\Delta_f = \Delta_{f, L'} : \Delta(L') \to \Delta(R''L')$, $L' \in G(R' \otimes_R M)$. These data should satisfy the following conditions:

(i) each $\Delta_{f, L'}$ induces an isomorphism $R'' \otimes_R \Delta(L') \xrightarrow{\sim} \Delta(R''L')$;

(ii) $\Delta_{f_2 f_1} = \Delta_{f_2} \Delta_{f_1}$;

(iii) the isomorphisms (5.1) commute with base change;

(iv) for any triple $L_1 \subset L_2 \subset L_3$ in $G(R' \otimes_R M)$ the obvious diagram

$$\begin{array}{ccc}
\Delta(L_1) \otimes \det(L_2/L_1) \otimes \det(L_3/L_2) & \xrightarrow{\sim} & \Delta(L_1) \otimes \det(L_3/L_1) \\
\downarrow & & \downarrow \\
\Delta(L_2) \otimes \det(L_3/L_2) & \xrightarrow{\sim} & \Delta(L_3)
\end{array}$$

commutes.

**Remark.** It follows from Theorem 4.2(ii) that the notion of (weak) determinant theory does not change if one considers only etale $R$-algebras instead of arbitrary ones.

The groupoid of all determinant theories on $M$ is equipped with an obvious Action of the Picard groupoid $\mathcal{Pic}_R^\mathbb{Z}$ of invertible $\mathbb{Z}$-graded $R$-modules: $P \in \mathcal{Pic}_R^\mathbb{Z}$ sends $\Delta$ to $P \Delta$, where $(P \Delta)(L) := P \otimes_R \Delta(L)$.

Determinant theories on $R' \otimes_R M$ for all etale $R$ algebras $R'$ form a sheaf of groupoids $\text{Det}_M$ on the Nisnevich site of $S := \text{Spec } R$, which is equipped with an Action of the sheaf of Picard groupoids $\mathcal{Pic}_S^\mathbb{Z}$. It follows from Theorem 4.2(ii) that $\text{Det}_M$ is a Torsor over $\mathcal{Pic}_S^\mathbb{Z}$. We call it the determinant Torsor of $M$.

If $M$ is a Tate module (rather than an almost projective one) then the above definition of determinant theory and determinant Torsor still applies (of course, in this case the words “coprojective lattice” should be understood in the sense of 4.2 and $\otimes$ should be replaced by $\hat{\otimes}$). If $M$ is an almost projective or Tate module and $L \subset M$ is a lattice then $M/L$ is almost projective and $\text{Det}_{M/L}$ canonically identifies with $\text{Det}_M$.

**Remark.** Consider the category whose set of objects is $\mathbb{Z}$ and whose only morphisms are the identities. We will denote it simply by $\mathbb{Z}$. Addition of integers defines a functor $\mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$, so $\mathbb{Z}$ becomes a Picard groupoid. We have a canonical Picard functor from $\mathcal{Pic}_\mathbb{Z}^\mathbb{Z}$ to the constant sheaf $\mathbb{Z}$ of Picard groupoids: an invertible $\mathcal{O}_S$-module placed in degree $n$ goes to $n$. The $\mathbb{Z}$-torsor corresponding to the $\mathcal{Pic}_\mathbb{Z}^\mathbb{Z}$-Torsor $\text{Det}_M$ is the dimension torsor $\text{Dim}_M$ from 3.5, 4.3.

5.3. On the notion of determinant gerbe. We also have the forgetful functor from the category of $\mathbb{Z}$-graded invertible $R$-modules to that of plain invertible $R$-modules and the corresponding functor $F : \mathcal{Pic}_\mathbb{Z}^\mathbb{Z} \to \mathcal{Pic}_S$. Notice that $F$ is a monoidal functor, but not a Picard functor. Applying $F$ to the $\mathcal{Pic}_\mathbb{Z}^\mathbb{Z}$-Torsor $\text{Det}_M$ one gets a $\mathcal{Pic}_S$-Torsor, which is the same as an
$\mathcal{O}_\mathcal{S}^\times$-gerbe. This is the determinant gerbe considered by Kapranov\footnote{This follows from the definitions, but also from the Grothendieck-Deligne dictionary mentioned in 5.2. The complex of sheaves of abelian groups corresponding via this dictionary to the sheaf of Picard categories $\mathcal{P}ic_\mathcal{S}$ is $\mathcal{O}_\mathcal{S}[1]$, i.e., $\mathcal{O}_\mathcal{S}$ placed in degree -1.} and mentioned in 5.4. Its sections are weak determinant theories. As $F$ does not commute with the commutativity constraint, there is no canonical equivalence between the $\mathcal{O}_\mathcal{S}^\times$-gerbe corresponding to a direct sum of almost projective modules $M_i$, $i \in I$, $\text{Card} \ I < \infty$, and the product of the $\mathcal{O}_\mathcal{S}^\times$-gerbes corresponding to $M_i$, $i \in I$ (but there is an equivalence which depends on the choice of an ordering of $I$). This is the source of the numerous signs in ACK and the reason why we prefer to consider Torsors over $\mathcal{P}ic_\mathcal{S}$ rather than pairs consisting of an $\mathcal{O}_\mathcal{S}^\times$-gerbe and a $\mathbb{Z}$-torsor (as Kapranov does in \cite{Ka3}).

5.4. Fermion modules, determinant theories, and co-Sato Grassmannian. We follow §2.14 – 2.15 of BBE (in particular, see Remark (iii) at the end of §2.15 of BBE).

5.4.1. Fermion modules and weak determinant theories. Fix a Tate $R$-module $M$. Let $\text{Cl}(M \oplus M^*)$ denote the Clifford algebra of $M \oplus M^*$. Define a Clifford module to be a module $V$ over $\text{Cl}(M \oplus M^*)$ such that for any $v \in V$ the set $\{a \in M \oplus M^*|av = 0\}$ is open in $M \oplus M^*$. A Clifford module $V$ is said to be a fermion module\footnote{Motivation of the name: if $M$ is a discrete projective $R$-module then fermion modules have the form $(\bigwedge M) \otimes_R \mathcal{L}$ with $\mathcal{L}$ being an invertible $R$-module.} if $V$ is fiberwise irreducible and projective over $R$.

If $V$ is a fermion module and $L \subset M$ is a coprojective lattice let $\Delta_V(L)$ denote the annihilator of $L \oplus L^\perp$ in $V$. As explained in BBE, $\Delta_V(L)$ is a line in $V$ (i.e., a direct summand of $V$ which is an invertible $R$-module) and $\Delta_V$ is a weak determinant theory: if $L_1 \subset L_2 \subset M$ are coprojective lattices then the isomorphism \footnote{This is the source of the numerous signs in ACK and the reason why we prefer to consider Torsors over $\mathcal{P}ic_\mathcal{S}$ rather than pairs consisting of an $\mathcal{O}_\mathcal{S}^\times$-gerbe and a $\mathbb{Z}$-torsor (as Kapranov does in \cite{Ka3}).} comes from the composition $\bigwedge^r L_2 \to \bigwedge M \to \text{Cl}(M \oplus M^*)$, where $r$ is the rank of $L_2/L_1$ and $\bigwedge M$ is the exterior algebra of $M$. Thus one gets a functor $V \mapsto \Delta_V$ from the groupoid of fermion modules to that of weak determinant theories. As explained in BBE, it is an equivalence: to construct the inverse functor $\Delta \mapsto V_\Delta$ one first constructs $V_\Delta$ Nisnevich-locally, then glues the results of the local constructions, and finally uses Theorem 2.1 to prove that $V_\Delta$ is a projective $R$-module.

The equivalences $V \mapsto \Delta_V$ and $\Delta \mapsto V_\Delta$ are compatible with the Actions of the groupoid $\text{Pic}_R$ of invertible $R$-modules.

5.4.2. Graded fermion modules and determinant theories. As explained in BBE, the fermion module $V_\Delta$ corresponding to a weak determinant theory $\Delta$ is equipped with a $T$-grading, where $T$ is the dimension torsor of $M$. Given a determinant theory on $M$ rather than a weak determinant theory one gets a $\mathbb{Z}$-grading on the fermion module compatible with the $\mathbb{Z}$-grading of $\text{Cl}(M \oplus M^*)$ for which $M$ has degree 1 and $M^*$ has degree -1. Thus $\text{Det}_M$ identifies with the groupoid of $\mathbb{Z}$-graded fermion modules.

5.4.3. The Plücker embedding of the co-Sato Grassmannian. The co-Sato Grassmannian $\text{Gras}_M$ of a Tate $R$-module $M$ was defined in 3.7. Now suppose that the determinant gerbe of $M$ is trivial and fix a weak determinant theory $\Delta$ on $M$. Then we get a line bundle $\mathcal{A}_\Delta$ on $\text{Gras}_M$ whose fiber over a coprojective lattice $L$ equals $\Delta(L)$.

On the other hand, we have the fermion module $V = V_\Delta$ such that $\Delta = \Delta_V$ (see 5.4.1). Assigning to a coprojective lattice $L$ the line $\Delta_V(L)$ one gets a morphism $i : \text{Gras}_M \to \mathbb{P}$, where $\mathbb{P}$ is the ind-scheme of lines in $V$. As explained by Plücker, $i$ is a closed embedding.

Clearly $\mathcal{A}_\Delta = i^* \mathcal{O}(-1)$.

5.5. A somewhat vague picture.
5.5.1. The picture I learned from Beilinson. Let $S$ be a spectrum in the sense of algebraic topology. We put $\pi^i(S) := \pi_{-i}(S)$ and define $\tau^{\leq k}S$ to be the spectrum equipped with a morphism $\tau^{\leq k}S \to S$ such that $\pi^i(\tau^{\leq k}S) = 0$ for $i > k$ and the morphism $\pi^i(\tau^{\leq k}S) \to \pi^i(S)$ is an isomorphism for $i \leq k$. There is a notion of torsor over a spectrum $S$, which depends only on $\tau^{\leq 1}S$. Namely, an $S$-torsor is a point of the infinite loop space $L$ corresponding to $(\tau^{\leq 1}S)[1]$ (or equivalently, a morphism from the spherical spectrum to $S[1]$). A homotopy equivalence between torsors is a path connecting the corresponding points of $L$, so equivalence classes are parametrized by $\pi^1(S) := \pi_{-1}(S)$.

Beilinson’s first remark: an object of the Calkin category $\mathcal{C}_{R}^{Kar}$ (see §3.3) defines a point of the infinite loop space corresponding to the $K$-theory spectrum $K(\mathcal{C}_{R}^{Kar})$, and as $K(\mathcal{C}_{R}^{Kar}) = K(R)[1]$ it defines a $K(R)$-torsor. In particular, an almost projective $R$-module $M$ defines a $K(R)$-torsor, whose class in $\pi_{-1}(K(R)) = K_{-1}(R)$ is the class $[M]$ considered in §3.3. If $R$ is commutative then by Thomason’s localization theorem ([111], §10.8) $K(R) = RT(S, K)$, where $K$ is the sheaf of $K$-theories of $\mathcal{O}_{S}$ (this is a sheaf of spectra on the Nisnevich site of $S$). So the notion of $K(R)$-torsor should coincide with that of $\mathcal{K}$-torsor. Both of them should coincide with that of $\tau^{\leq 1}K$-torsor. By Theorem 3.7, $\mathcal{K}^1 := \mathcal{K}_{-1} = 0$, so $\tau^{\leq 1}\mathcal{K} = \tau^{\leq 0}\mathcal{K}$ and therefore we get a morphism $\tau^{\leq 1}\mathcal{K} = \tau^{\leq 0}\mathcal{K} \to \mathcal{K}_{[0,1]} := \mathcal{K}_{[-1,0]} := \tau^{\geq -1}\tau^{\leq 0}\mathcal{K}$. So to an almost projective $R$-module $M$ there should correspond a $\mathcal{K}_{[0,1]}$-torsor $\Delta_{M}$. According to Beilinson, $\mathcal{K}_{[0,1]}$ and $\Delta_{M}$ should identify with $\mathcal{P}ic_{\mathcal{K}}^{2}$ and the Torsor $Det_{M}$ from §5, via the following dictionary, which goes back to A. Grothendieck and was used in §1.4-1.5 of [Del] and in §4 of [Del87].

5.5.2. Grothendieck’s Dictionary. According to it, a Picard groupoid is essentially the same as a spectrum $X$ with $\pi_{i}(X) = 0$ for $i \neq 0, 1$. More precisely, the following two constructions become essentially inverse to each other if the first one is applied only to infinite loop spaces $X$ with $\pi_{i}(X) = 0$ for $i > 1$:

(i) to an infinite loop spaces $X$ one associates its fundamental groupoid $\Pi(X)$ viewed as a Picard groupoid\footnote{Here and in what follows I use the word “should” to indicate the parts of the picture that I do not quite understand (probably due to the fact that I have not learned the theory of sheaves of spectra).}

(ii) to a Picard groupoid one associates its classifying space viewed as an infinite loop space\footnote{If $X = \Omega Y$ the group structure on $\pi_{0}(X) = \pi_{1}(Y)$ lifts to a monoidal category structure on $\Pi(X)$. If $X = \Omega^{2}Z$ the proof of the commutativity of $\pi_{0}(X) = \pi_{2}(Z)$ “lifts” to a braiding on $\Pi(X)$, the “square of the braiding” map $t : \pi_{0}(X) \times \pi_{0}(X) \to \pi_{1}(X)$ equals the Whitehead product $\pi_{2}(Z) \times \pi_{2}(Z) \to \pi_{3}(Z)$, and therefore $t$ vanishes if $Z$ is a loop space.}

For strictly commutative Picard groupoids there is a similar dictionary and, moreover, a precise reference, namely Corollary 1.4.17 of [Del]. The statement from [Del] is formulated in a more general context of sheaves. It says that a sheaf of strictly commutative Picard groupoids is essentially the same as a complex of sheaves of abelian groups with cohomology concentrated in degrees 0 and -1.

Hopefully, there is also a sheafified version of the dictionary in the non-strictly commutative case. It should say that a sheaf $\mathcal{A}$ of Picard groupoids is essentially the same as a sheaf of spectra $S$ whose sheaves of homotopy groups $\pi_{i}$ vanish for $i \neq 0, 1$ and that the notion of $\mathcal{A}$-Torsor from §5.7 is equivalent to that of $S$-torsor.

5.5.3. Problem: make the above somewhat vague picture precise. The notion of determinant Torsor is very useful, and its rigorous interpretation in the standard homotopy-theoretic language of algebraic $K$-theory would be helpful.
5.6. The central extension for almost projective modules. Let $M$ be an almost projective module over a commutative ring $R$ and $\hat{M}$ be the corresponding quasicoherent sheaf on the Nisnevich topology of $S := \text{Spec} R$. Then the sheaf $\hat{\text{Aut}} M := \text{Aut} \hat{M}$ has a canonical central extension
\[
0 \to \mathcal{O}_S^\times \to \hat{\text{Aut}} M \to \text{Aut} M \to 0
\]
(5.2)

Its definition is similar to that of the Tate central extension of the automorphism group of a Tate vector space a.k.a. “Japanese” extension (see [Br, Ka3, PS, BBE]). Namely, if $M$ has a determinant theory $\Delta$ then $\text{Aut} M$ is the sheaf of automorphisms of $(M, \Delta)$. This sheaf does not depend (up to canonical isomorphism) on the choice of $\Delta$. This allows to define $\hat{\text{Aut}} M$ even if $\Delta$ exists only locally.

Now suppose that a group $R$-scheme $G$ acts on $M$ (i.e., one has a compatible collection of morphisms $G(R') \to \text{Aut}(R' \otimes_R M)$ for all $R$-algebras $R'$). Then (5.2) induces a canonical central extension of group schemes
\[
0 \to \mathbb{G}_m \to \hat{G} \to G \to 0
\]
(5.3)

(one first defines $\hat{G}$ as a functor $\{R$-algebras$\} \to \{\text{groups}\}$ and then notices that $\hat{G}$ is representable because it is a $\mathbb{G}_m$-torsor over $G$). If $G$ is abelian we get the commutator map $G \times G \to \mathbb{G}_m$. If $M$ is projective (in particular, if $k$ is a field) the extension (5.3) canonically splits because in this case there is a canonical determinant theory on $M$ defined by $\Delta(L) = \det L$. The following example shows that in general the extension (5.3) can be nontrivial.

Example (A. Beilinson). Let $k$ be a field, $R = k[[\varepsilon]]/((\varepsilon)^2)$. Fix $g \in k((t))$, then $L_g := R[[(t)]]/\varepsilon gR[[t]]$ is a lattice in the Tate $R$-module $R((t))$. Put $M := R((t))/L_g$. Let $G_1$ denote the multiplicative group of $R[[(t)]]$ viewed as a group scheme over $R$. On $M$ we have the natural action of $G_1$ and also the action of $\mathbb{G}_a$ such that $c \in \mathbb{G}_a$ acts as multiplication by $1 + \varepsilon g$. So $G := G_1 \times \mathbb{G}_a$ acts on $M$. The theory of the Tate extension (see, e.g., §3 of [BBE]) tells us that in the corresponding central extension (5.3) the commutator of $c \in \mathbb{G}_a$ and $u \in R [[t]]^\times$ equals $ce \cdot \text{res}(u \cdot dg)$. So the extension (5.3) is not commutative if $dg \neq 0$.

Remarks. (i) To define the central extension (5.3) it suffices to have an action of $G$ on $M$ as an object of the Calkin category $\mathcal{C}^{\text{Kar}}$ defined in 3.3.1 (see §2 of [BBE] for more details). Of course, in this setting the extension (5.3) may be nontrivial even if $R$ is a field.

(i') One can define the extension (5.3) if $G$ is any group-valued functor on the category of $R$-algebras (e.g, a group ind-scheme).

(ii) As explained in [BBE], the canonical central extension of the automorphism group of a Tate vector space should rather be considered as a “super-extension” (this is necessary to formulate the compatibility between the extensions corresponding to Tate spaces $T_1$, $T_2$, and $T_1 \oplus T_2$). The same is true for the canonical central extension of the automorphism group of an object of the Calkin category $\mathcal{C}^{\text{Kar}}$. But in the case of an almost projective module $M$ “super” is unnecessary because any automorphism of $M$ has degree 0, i.e., preserves the dimension torsor (this follows from the existence of the canonical upper semicontinuous dimension theory on $M$, see [13]).

6. Applications to spaces of formal loops. “Refined” motivic integration

In this section all rings and algebras are assumed to be commutative. We fix a ring $k$. Starting from 6.4 we suppose that $k$ is a field.

6.1. A class of schemes. We will use the following notation for affine spaces:

$\mathbb{A}^I := \text{Spec} k[x_i]_{i \in I}$, \hspace{1em} $\mathbb{A}^\infty := \mathbb{A}^\mathbb{N}$.

We say that a $k$-scheme is nice if it is isomorphic to $X \times \mathbb{A}^I$, where $X$ is of finite presentation over $k$ (the set $I$ may be infinite). An affine scheme is nice if and only if it can be defined by finitely many equations in a (not necessarily finite-dimensional) affine space.
Definition. A $k$-scheme $X$ is locally nice (resp. Zariski-locally nice, etale-locally nice) if it becomes nice after Nisnevich localization (resp. Zariski or etale localization). $X$ is differentially nice if for every open affine $\text{Spec } R \subset X$ the $R$-module $\Omega^1_R := \Omega^1_{R/k}$ is $2$-almost projective.

By Theorem 4.2(i) etale-local niceness implies differential niceness. I do not know if etale-local niceness implies local niceness. Local niceness does not imply Zariski-local niceness (see 6.3 below).

For a differentially nice $k$-scheme $X$ one defines the dimension torsor $\text{Dim}_X$: if $X$ is an affine scheme $\text{Spec } R$ then $\text{Dim}_X$ is the dimension torsor of the almost projective $R$-module $\Omega^1_R$, and for a general $X$ one defines $\text{Dim}_X$ by gluing together the torsors $\text{Dim}_{U'}$ for all open affine $U \subset X$. If $X' \subset X$ is a closed subscheme defined by finitely many equations and $X$ is differentially nice then $X'$ is; in this situation $\text{Dim}_X$, canonically identifies with the restriction of $\text{Dim}_X$ to $X'$.

In the next subsection we will see that the dimension torsor of a locally nice $k$-scheme may be nontrivial, and on the other hand, there exists a locally nice $k$-scheme with trivial dimension torsor which is not Zariski-locally nice.

6.2. Examples. (i) Define $i : \mathbb{A}^\infty \to \mathbb{A}^\infty$ by $i(x_1, x_2, \ldots) := (0, x_1, x_2, \ldots)$. Take $\mathbb{A}^1 \times \mathbb{A}^\infty$ and then glue $(0, x) \in \mathbb{A}^1 \times \mathbb{A}^\infty$ with $(1, i(x)) \in \mathbb{A}^1 \times \mathbb{A}^\infty$. Thus one gets a locally nice $k$-scheme $X$ whose dimension torsor is nontrivial and even not Zariski-locally trivial.

(ii) Let $M$ be an almost projective module over a finitely generated algebra $R$ over a Noetherian ring $k$. Let $X$ denote the spectrum of the symmetric algebra of $M$. Then $X$ is locally nice. This follows from Theorem 4.2(i) and the next theorem, which is due to H. Bass (Corollary 4.5 from [Ba2]).

Theorem 6.1. If $R$ is a commutative Noetherian ring whose spectrum is connected then every infinitely generated projective $R$-module is free.

It is easy to deduce from Theorem 6.1 that $X$ is Zariski-locally nice if and only if the class of $M$ in $K_{-1}(R)$ vanishes locally for the Zariski topology (to prove the “only if” statement consider the restriction of $\Omega^1_X$ to the zero section $\text{Spec } R \to X$). If $R$ and $M$ are as in 6.1 then we get the above Example (i).

(iii) There exists a locally nice scheme $X$ over a field $k$ which is not Zariski-locally nice but has trivial dimension torsor.\footnote{In 6.3.3 we will see that one can get such $X$ from the loop space of a smooth affine manifold.} According to (ii), to get such an example it suffices to find a finitely generated $k$-algebra $R$ and an almost projective $R$-module $M$ such that $H^1_{et}(\text{Spec } R, \mathbb{Z}) = 0$ but the class of $M$ in $K_{-1}(R)$ is not Zariski-locally trivial. §6 of [We3] contains examples of finitely generated normal $k$-algebras $R$ with $K_{-1}(R) \neq 0$. In each of them $\text{Spec } R$ has a unique singular point $x$, and according to [We1], the map $K_{-1}(R) \to K_{-1}(R_x)$ is an isomorphism. Now take any nonzero element of $K_{-1}(R)$ and represent it as a class of an almost projective $R$-module $M$.

6.3. Generalities on ind-schemes. The key notions introduced in this subsection are those of reasonable, $T$-smooth, and Tate-smooth ind-scheme (see 6.3.3–6.3.7).

6.3.1. Definition of ind-scheme and formal scheme. Functors from the category of $k$-algebras to that of sets will be called “spaces”. E.g., a $k$-scheme can be considered as a space. “Subspace” means “subfunctor”. A subspace $Y \subset X$ is said to be closed if for every (affine) scheme $Z$ and every $f : Z \to X$ the subspace $Z \times_X Y \subset Z$ is a closed subscheme.

Let us agree that an ind-scheme is a space which can be represented as $\text{lim } X_\alpha$, where $\{X_\alpha\}$ is a directed family of quasi-compact schemes such that all the maps $i_{\alpha \beta} : X_\alpha \to X_\beta$, $\alpha \leq \beta$, are closed embeddings. (Notice that if the same space can also be represented as the inductive limit of a directed family of quasi-compact schemes $X'_\alpha$ then each $X'_\alpha$ is contained in some $X_\alpha$ and each $X_\alpha$ is contained in some $X'_\beta$.) If $X$ can be represented as above so that the set of indices $\alpha$ is countable then $X$ is said to be an $\aleph_0$-ind-scheme. If $P$ is a property of schemes stable under passage to closed subschemes then we say that $X$ satisfies the ind-$P$ property if...
each $X_\alpha$ satisfies $P$. E.g., one has the notion of ind-affine ind-scheme and that of ind-scheme of ind-finite type.

Set $X_{\text{red}} := \lim X_{\alpha,\text{red}}$; an ind-scheme $X$ is said to be reduced if $X_{\text{red}} = X$.

6.3.2. $O$-modules and pro-$O$-modules on ind-schemes. A pro-module over a ring $R$ is defined to be a pro-object\(^{18}\) of the category of $R$-modules. We identify the category of Tate $R$-modules with a full subcategory of the category of pro-$R$-modules by associating to a Tate $R$-module the projective system formed by its discrete quotient modules.

An $O$-module (resp. a pro-$O$-module) $P$ on a space $X$ is a rule that assigns to a commutative algebra $A$ and a point $\phi \in X(A)$ an $A$-module (resp. an $A$-module) $P_\phi$, and to any morphism of algebras $f : A \rightarrow B$ a $B$-isomorphism $f_P : B \otimes P_\phi \sim \rightarrow P_{f\phi}$ in a way compatible with composition of $f$’s. An $O$-module on a scheme $Y$ is the same as a quasi-coherent sheaf of $O_Y$-modules, and an $O$-module $P$ on an ind-scheme $X = \lim X_\alpha$ is the same as a collection of $O$-modules $P_{X_\alpha}$ on $X_\alpha$ together with identifications $i_{\alpha,\beta}^* P_{X_\beta} = P_{X_\alpha}$ for $\alpha \leq \beta$ that satisfy the obvious transitivity property.

A pro-$O$-module is said to be a Tate sheaf if for each $\phi$ as above the pro-module $P_\phi$ is a Tate module.

The cotangent sheaf $\Omega_X^1$ of an ind-scheme $X = \lim X_\alpha$ is the pro-$O$-module whose restriction to each $X_\alpha$ is defined by the projective system $i_{\alpha,\beta}^* \Omega_{X_{\beta}}^1 \beta \geq \alpha$ (here $i_{\alpha,\beta}$ is the embedding $X_\alpha \rightarrow X_{\beta}$).

6.3.3. The notion of reasonable ind-scheme. The following definitions are due to A. Beilinson.

A closed quasi-compact subscheme $Y$ of an ind-scheme $X$ is called reasonable if for any closed subscheme $Z \subset X$ containing $Y$ the ideal of $Y$ in $O_Z$ is finitely generated. Notice that reasonable subschemes of $X$ form a directed set. An ind-scheme $X$ is reasonable if $X$ is the union of its reasonable subschemes, i.e., if it can be represented as $\lim X_\alpha$, where all $X_\alpha$’s are reasonable.

Any scheme is a reasonable ind-scheme. A closed subspace of a reasonable ind-scheme is a reasonable ind-scheme. The product of two reasonable ind-schemes is reasonable. The completion of any ind-scheme along a reasonable closed subscheme is a reasonable ind-scheme.

6.3.4. Main example: ind-scheme of formal loops. Let $Y$ be an affine scheme over $F := k((t))$. Define a functor $\mathcal{LY}$ from the category of $k$-algebras to that of sets by $\mathcal{LY}(R) := Y(R \hat{\otimes} F)$, $R \hat{\otimes} F := R((t))$. It is well known and easy to see that $\mathcal{LY}$ is an ind-affine ind-subscheme. This is the ind-scheme of formal loops of $Y$. If $Y$ is an affine scheme of finite type over $F$ then $\mathcal{LY}$ is a reasonable $\mathbb{N}_0$-ind-scheme.

6.3.5. Formal schemes. We define a formal scheme to be an ind-scheme $X$ such that $X_{\text{red}}$ is a scheme. An $\mathbb{N}_0$-formal scheme is a formal scheme which is an $\mathbb{N}_0$-ind-scheme. The completion of an ind-scheme $Z$ along a closed subscheme $Y \subset Z$ is the direct limit of closed subschemes $Y' \subset Z$ such that $Y'_{\text{red}} = Y_{\text{red}}$. In the case of formal schemes we write “affine” instead of “ind-affine”. A formal scheme $X$ is affine if and only if $X_{\text{red}}$ is affine.

Remark. As soon as you compare the above definition of formal scheme with the one from EGA I you see that they are not equivalent (even in the affine case) but the difference is not big: an $\mathbb{N}_0$-formal scheme in our sense which is reasonable in the sense of EGA I is a formal scheme in the sense of EGA I, and on the other hand, a Noetherian formal scheme in the sense of EGA I is a formal scheme in our sense.

\(^{18}\)A nice exposition of the theory of pro-objects and ind-objects of a category is given in §8 of [GV]. See also the Appendix of [AM].
6.3.6. **Formal smoothness.** Following Grothendieck ([Gr64], [Gr]), we say that $X$ is formally smooth if for every $k$-algebra $A$ and every nilpotent ideal $I \subset A$ the map $X(A) \to X(A/I)$ is surjective. A morphism $X \to Y$ is said to be formally smooth if for every $k$-algebra $k'$ and every morphism $\text{Spec } k' \to Y$ the $k'$-space $X \times_Y \text{Spec } k'$ is formally smooth. Clearly formal smoothness of any ind-scheme (resp. a reasonable ind-scheme) is equivalent to formal smoothness of its completions along all closed subschemes (resp. all reasonable closed subschemes).

**Theorem 6.2.** (i) For reasonable formal schemes formal smoothness is an étale-local property.

(ii) A reasonable closed subscheme of a formally smooth ind-scheme is differentially nice.

(iii) If a reasonable $\aleph_0$-ind-scheme $X$ is formally smooth then $\Omega^1_X$ is a Tate sheaf (the notions of cotangent sheaf of an ind-scheme, Tate sheaf, and Mittag-Leffler-Tate sheaf are defined in 6.3.2).

In the case of schemes statement (i) of the theorem was proved by Grothendieck (cf. Remark 9.5.8 from [Gr2]) modulo the conjecture on the local nature of projectivity (which was proved a few years later in [RG]). The proof of Theorem 6.2 in the general case is slightly more complicated but based on the same ideas.

6.3.7. **T-smoothness and Tate-smoothness.** We say that a reasonable ind-scheme $X$ is T-smooth if

(i) every reasonable closed subscheme of $X$ is locally nice;

(ii) $X$ is formally smooth.

A T-smooth ind-scheme $X$ is said to be Tate-smooth if its cotangent sheaf is a Tate sheaf (according to Theorem 6.2(iii), this is automatic for $\aleph_0$-ind-schemes).

**Remark.** In the above definitions we do not require every closed subscheme of $X$ to be contained in a formally smooth subscheme. It is not clear if this property holds for $\mathcal{L}(SL(n))$ or for the affine Grassmannian, even though these ind-schemes are Tate-smooth. See also Remark (ii) from 6.4.

6.3.8. **Dimension torsor.** Let $X$ be a reasonable ind-scheme such that all its reasonable closed subschemes are differentially nice (by Theorem 6.2(ii) this is true for any formally smooth reasonable ind-scheme). Then there is an obvious notion of the dimension torsor of $X$: for each reasonable closed subscheme $Y \subset X$ one has the dimension torsor $\text{Dim}_Y$, and if $Y' \subset Y$ are reasonable closed subschemes then $\text{Dim}_{Y'}$ identifies with the restriction of $\text{Dim}_Y$ to $Y'$.

6.3.9. **Relation with the Kapranov-Vasserot theory.** The notion of T-smooth ind-scheme is similar to the notion of “smooth locally compact ind-scheme” introduced by M. Kapranov and E. Vasserot (see Definition 4.4.4 from [KV]). Neither of these classes of ind-schemes contains the other one. The theory of $\mathcal{D}$-modules on smooth locally compact ind-schemes developed in [KV] renders to the class of T-smooth ind-schemes, and the same is true for the Kapranov-Vasserot theory of de Rham complexes (which goes back to the notion of chiral de Rham complex from [MSV]). According to A. Beilinson (private communication), these theories, in fact, render to the class of formally smooth reasonable ind-schemes, which contains both “smooth locally compact” ind-schemes in the sense of [KV] and T-smooth ones.

6.4. **Loops of an affine manifold.** From now on we assume that $k$ is a field (I have not checked if Theorems 6.3 and 6.4 hold for any commutative ring $k$). So $F = k((t))$ is also a field. For any affine $F$-scheme $Y$ one has the ind-scheme of formal loops $\mathcal{L}Y$ (see 6.3.4).

**Theorem 6.3.** If an affine $F$-scheme $Y$ is smooth then $\mathcal{L}Y$ is Tate-smooth.

**Remarks.** (i) The theorem is not hard. It is only property (i) from the definition of T-smoothness (see 6.3.7) that requires some efforts. See 6.7 for more details.
(ii) If $Y$ is a smooth affine $F$-scheme then by Theorem 6.3 every reasonable closed subscheme $X \subset \mathcal{L}Y$ is locally nice. But there exist $Y$ and $X \subset \mathcal{L}Y$ as above such that $X$ is not Zariski-locally nice. One can choose $Y$ and $X$ so that $\dim_X$ is not Zariski-locally trivial. But one can also choose $Y$ and $X$ so that $\dim_X$ is trivial but $X$ is not Zariski-locally nice. See 6.13 for examples of these situations. According to H. Bass [Ba], $K_0$ of a regular ring is zero, so in these examples $\mathcal{L}Y$ cannot be represented (even Zariski-locally) as the union of an increasing sequence of smooth closed subschemes.

In the next subsection we formulate an analog of Theorem 6.3 for affinoid analytic spaces (this is a natural thing to do in view of 6.6).

6.5. Loops of an affine analytic space. We will use the terminology from [BGR] (which goes back to Tate) rather than the one from [Be]. Let $F(z_1, \ldots, z_n) \subset F[[z_1, \ldots, z_n]]$ be the algebra of power series which converge in the polydisk $|z_i| \leq 1$. As $F = k((t))$ one has $F(z_1, \ldots, z_n) = k[z_1, \ldots, z_n](t))$. For every $k$-algebra $R$ the $F$-algebra $R \otimes F = R((t))$ is equipped with the norm whose unit ball is $R[[t]]$. In particular, $F(z_1, \ldots, z_n)$ is a Banach algebra. An affinoid $F$-algebra is a topological $F$-algebra isomorphic to a quotient of $F(z_1, \ldots, z_n)$ for some $n$. All morphisms between affinoid $F$-algebras are automatically continuous (see, e.g., §6.1.3 of [BGR]). The category of affinoid analytic spaces is defined to be dual to that of affinoid $F$-algebras; the affinoid space corresponding to an affinoid $F$-algebra $A$ will be denoted by $\mathcal{M}(A)$.

For an affinoid analytic space $Z = \mathcal{M}(A)$ and a $k$-algebra $R$ denote by $\mathcal{L}Z(R)$ the set of continuous $F$-homomorphisms from $A$ to the Banach $F$-algebra $R \otimes F = R((t))$. It is easy to see that the functor $\mathcal{L}Z$ is a reasonable affine $\mathbb{N}_0$-formal scheme in the sense of 6.3.3 (and therefore an affine formal scheme in the sense of EGA I). E.g., if $Z$ is the unit disk then $\mathcal{L}Z$ is the completion of the ind-scheme of formal Laurent series along the subscheme of formal Taylor series.

**Theorem 6.4.** If an affinoid space $Z$ is smooth then the formal scheme $\mathcal{L}Z$ is Tate-smooth. In particular, $(\mathcal{L}Z)_{\text{red}}$ is a locally nice scheme.

6.6. **Theorem 6.3 follows from Theorem 6.4.** Let $Y = \text{Spec } B$ be a closed subscheme of $\mathbb{A}^n = \text{Spec } F[z_1, \ldots, z_n]$. The ind-scheme $\mathcal{L}Y$ is the union of its closed subschemes $\mathcal{L}_MY$ defined by $(\mathcal{L}_MY)(R) := Y(R) \cap (t^{-n} R[[t]])^\mathbb{N} \subset R((t))^n$ for any $k$-algebra $R$. The completion of $\mathcal{L}Y$ along $\mathcal{L}_N Y$ identifies with $\mathcal{L}_N Y$, where $Y_N$ is the affinoid analytic space defined by

$$Y_N := \mathcal{M}(B_N), \quad B_N := B \otimes_{F[z_1, \ldots, z_n]} F(tN z_1, \ldots, t^N z_n)$$

(in other words, $Y_N$ is the intersection of $Y$ with the polydisk of radius $r^n$, $r := |t^{-1}| > 1$).

Therefore Theorem 6.3 follows from Theorem 6.4.

6.7. **Sketch of the proof of Theorem 6.4.** The formal smoothness of $\mathcal{L}Z$ immediately follows from the definitions. It is also easy to describe the cotangent sheaf of $\mathcal{L}Z$. Let $A$ be the affinoid $F$-algebra corresponding to $Z$. Every finite-dimensional vector bundle $E$ on $Z$ defines a Tate sheaf $\mathcal{L}E$ on $\mathcal{L}Z$: if $\text{Spec } R \subset \mathcal{L}Z$ is a closed affine subscheme and $f : A \to R((t))$ corresponds to the morphism $\text{Spec } R \to \mathcal{L}Z$ then the pullback of $\mathcal{L}E$ to $\text{Spec } R$ is the Tate $R$-module $R((t)) \otimes_A \Gamma(Z, E)$. The proof of the next lemma is straightforward.

**Lemma 6.5.** The cotangent sheaf of $\mathcal{L}Z$ identifies with the Tate sheaf $\mathcal{L}\Omega^1_Z$ corresponding to the cotangent bundle $\Omega^1_Z$ of the analytic space $Z$.

**Corollary.** Let $\text{Spec } R \subset \mathcal{L}Z$ be a reasonable closed subscheme. Let $M$ be the module of global sections of the pullback to $\text{Spec } R$ of the Tate sheaf $\mathcal{L}\Omega^1_Z$. Then $\Omega^1_R$ is the quotient of the Tate $R$-module $M$ by some lattice.

It remains to show that a reasonable closed subscheme $\text{Spec } R \subset \mathcal{L}Z$ is locally nice. It easily follows from the above Corollary and Theorem 6.3 that after Nisnevich localization $\Omega^1_R$ becomes a direct sum of a free module and a module of finite presentation. This is a linearized version of
local niceness. To deduce local niceness from its linearized version one works with the implicit function theorem.

6.8. The renormalized dualizing complex. Fix a prime $l \neq \text{char } k$. Let $D^b_c(X, \mathbb{Z}_l)$ denote the appropriately defined bounded constructible $l$-adic derived category on a scheme $X$ (see [2, J7]). For a general locally nice $k$-scheme $X$ there is no natural way to define the dualizing complex $K_X \in D^b_c(X, \mathbb{Z}_l)$. Indeed, if $X$ is the product of $\mathbb{A}^\infty$ and a $k$-scheme $Y$ of finite type and if $\pi : X \to Y$ is the projection then $K_X$ should equal $\pi^*K_Y \otimes (\mathbb{Z}_l[2](1))^{\otimes m}$, which makes no sense. But suppose that the dimension $\mathbb{Z}$-torsor $\text{dim}_X$ is trivial and that we have chosen its trivialization $\eta$. Then one can define the renormalized dualizing complex $K^n_{\eta} \in D^b_c(X, \mathbb{Z}_l)$. The definition (which is straightforward) is given below. The reader can skip it and go directly to (6.5).

First assume that $X$ is nice, i.e., there exists a morphism $\pi : X \to Y$ such that $Y$ is a $k$-scheme of finite type and $X$ is $Y$-isomorphic to $X \times \mathbb{A}^l$ for some set $I$. Let $C_X$ be the category of all such pairs $(Y, \pi)$. A morphism $f : (Y, \pi) \to (Y', \pi')$ is defined to be a morphism $f : Y \to Y'$ such that $\pi' = f \pi$. Such $f$ is unique if it exists. The category $C_X$ is equivalent to a directed set. So to define $K^n_{\pi}$ it suffices to define a functor

$$C_X \to D^b_c(X, \mathbb{Z}_l), \quad (Y, \pi) \mapsto K^n_{\eta, \pi}$$

which sends all morphisms to isomorphisms.

If $(Y, \pi) \in C_X$ then $\pi^*\Omega^1_Y \subset \Omega^1_X$ is locally of finite presentation and $\Omega^1_X/\pi^*\Omega^1_Y$ is locally free. So for every open affine $U \subset X$ one has the coprojective lattice $\Gamma(U, \pi^*\Omega^1_Y) \subset \Gamma(U, \Omega^1_X)$ and therefore a section of the torsor $\text{dim}_X$ over $U$. These sections agree with each other, so we get a global section $\eta_\pi$ of $\text{dim}_X$. Put

$$m := \eta_\pi - \eta \in H^0(X, \mathbb{Z}),$$

$$K^n_{\eta, \pi} := \pi^*K_Y \otimes (\mathbb{Z}_l[2](1))^{\otimes m},$$

Now let $f : (Y, \pi) \to (Y', \pi')$ be a morphism. One easily shows that $f : Y \to Y'$ is smooth, so one has a canonical isomorphism

$$K_Y \cong f^*K_{Y'} \otimes (\mathbb{Z}_l[2](1))^{\otimes d},$$

where $d$ is the relative dimension of $Y$ over $Y'$. It is easy to see that $\pi^*d = \eta_{\pi'} - \eta_\pi$, so (6.4) induces an isomorphism $\alpha_f : K^n_{\eta', \pi'} \to K^n_{\eta, \pi'}$. We define (6.1) on morphisms by $f \mapsto \alpha_f$.

So we have defined $K^n_{\eta}$ if $X$ is nice. The formation of $K^n_{\eta}$ commutes with etale localization of $X$. It is easy to see that $\text{Ext}^i(K^n_{\eta}, K^n_{\eta}) = 0$ for $i < 0$. So by Theorem 3.2.4 of [RBD] there is a unique way to extend the definition of $K^n_{\eta}$ to all etale-locally nice $k$-schemes $X$ so that the formation of $K^n_{\eta}$ still commutes with etale localization.

6.9. $R\Gamma_c$ of a locally nice scheme. Suppose we are in the situation of (6.8), i.e., we have a locally nice $k$-scheme $X$, a trivialization $\eta$ of its dimension torsor, and a prime $l \neq \text{char } k$. Assume that $X$ is quasicompact and quasiseparated. Then we put

$$R\Gamma^0_c(X \otimes \bar{k}, \mathbb{Z}_l) := R\Gamma(X \otimes \bar{k}, K^n_{\eta})^*,$$

where $K^n_{\eta}$ is the renormalized dualizing complex defined in (6.8) $R\Gamma^0_c(X \otimes \bar{k}, \mathbb{Z}_l)$ is an object of $D^b_c(\text{Spec } k, \mathbb{Z}_l)$, i.e., of the appropriately defined bounded constructible derived category of $l$-adic representations of $\text{Gal}(k^s/k)$, where $k^s$ is a separable closure of $k$.

Problems. 1) Define an object of the triangulated category of $k$-motives $\text{VST}$ or $\text{VAMW}$ whose $l$-adic realization equals $R\Gamma^0_c(X \otimes \bar{k}, \mathbb{Z}_l)$ for each $l \neq \text{char } k$ (Voevodsky says this can probably be done).

---

[19] Choosing a section $Y \to X$ one sees that $Y$ is $Y'$-isomorphic to a retract of $Y' \times \mathbb{A}^J$ for some $J$. So $f$ is formally smooth and therefore smooth.
2) Now suppose that the determinant gerbe of $X$ is trivial and we have fixed its trivialization $\xi$. Can one canonically lift $R^0_\xi(X \otimes k, \mathbb{Z}_l)$ to an object of the motivic stable homotopy category depending on $\eta$ and $\xi$? Or at least, can one canonically lift $R^\eta_\xi(X \otimes \bar{k}, \mathbb{Q}_l)$ to an object of the motivic stable homotopy category tensored by $\mathbb{Q}$? (The motivic stable homotopy category a.k.a. $A^1$ stable homotopy category was defined in $[V_0]$). Reason why $\xi$ is supposed to exist and to be fixed: if $k = \mathbb{R}$ this allows to define $R^\eta_\xi(X(\mathbb{R}), \mathbb{Z})$.

6.10. “Refined” motivic integration. Suppose that in the situation of Theorem 6.4 the canonical bundle $\det \Omega^1_Z$ is trivial. Choose a trivialization of $\det \Omega^1_Z$, i.e., a differential form $\omega \in H^0(Z, \det \Omega^1_Z)$ with no zeros. By 6.4, the scheme $X := (LZ)_\text{red}$ is locally nice. By 3.9 and the corollary of Lemma 6.5, our trivialization of $\det \Omega^1_Z$ induces a trivialization $\eta$ of the dimension torsor $\text{Dim } X$. We put

$$\int_Z |\omega| := R^\eta \xi(X, \mathbb{Z}_l) \subseteq D^\eta(X(\text{Spec } k, \mathbb{Z}_l),$$

where $R^\eta_\xi(X, \mathbb{Z}_l)$ is defined by (5.5). Clearly $\int_Z |\omega|$ does not depend on the choice of $X$.

6.11. Comparison with usual motivic integration. In the situation of 6.10 (i.e., integrating a holomorphic form with no zeros over an affinoid domain) the usual motivic integral $[L_\xi]$ belongs to $M_k := M_k'[L^{-1}]$, where $M_k'$ is the Grothendieck ring of $k$-varieties\footnote{\textit{M}_k'$ is generated by elements $[X]$ corresponding to isomorphism classes of $k$-schemes of finite type, and the defining relations are $[X] = [Y] + [X \setminus Y]$ for any $k$-scheme $X$ of finite type and any closed subscheme $Y \subseteq X$.} and $L \in M_k'$ is the class of the affine line. Its definition can be reformulated as follows.

Given a connected nice $k$-scheme $X$ and a trivialization $\eta$ of its dimension torsor one chooses $\pi : X \to Y$ as in 6.8 defines $m \in H^0(X, \mathbb{Z}) = \mathbb{Z}$ by (6.2) and puts $[X]^\eta := [Y] L^m \in M_k$. If $X$ is any quasicompact quasiseparated locally nice $k$-scheme choose closed subschemes $X = F_0 \supset F_1 \supset \ldots \supset F_n = \emptyset$ so that each $F_i$ is defined by finitely many equations and $F_i \setminus F_{i+1}$ is nice and connected; then put $[X]^\eta := \sum [F_i \setminus F_{i+1}]^\eta$. Finally, in the situation of 6.10 one puts

$$\int_Z |\omega|_{\text{usual}} := [X]^\eta \in M_k.$$

Clearly (6.7) is well-defined, and the images of (5.7) and (6.6) in $K_0(D^b_c(\text{Spec } k, \mathbb{Z}_l))$ are equal. So (6.7) and (6.6) can be considered as different refinements of the same object of $K_0(D^b_c(\text{Spec } k, \mathbb{Z}_l))$. Unless the map $M_k \to K_0(D^b_c(\text{Spec } k, \mathbb{Z}_l))$ is injective (which seems unlikely), the “refined” motivic integral (5.3) cannot be considered as the refinement of the usual motivic integral (6.7). This is why I am using quotation marks.

6.12. Remark. Our definition of “refined” motivic integration works only in the case of integrating a holomorphic form with no zeros over an affinoid domain (which is probably too special for serious applications).

On the other hand, in an unpublished manuscript V. Vologodsky defined a different kind of “refined motivic integration” in the case of K3 surfaces. More precisely, let $\omega \neq 0$ be a regular differential form on a K3 surface $X$ over $F = k((t))$, char $k = 0$. Let $A$ denote the Grothendieck ring of the category of Grothendieck motives over $k$, and let $I_n$ denote the motivic integral of $\omega$ over $X \otimes F k((t^{1/n}))$ viewed as an object of $A \otimes \mathbb{Q}$. Vologodsky defined objects $M_1, M_2, M_3$ of the category of Grothendieck motives so that $I_n$ is a certain linear combination of the classes of $M_1, M_2, M_3$. The objects $M_1, M_2, M_3$ depend functorially on $(X, \omega)$. His definition of $M_1, M_2, M_3$ is mysterious.

6.13. Counterexamples. Here are the examples promised in Remark (ii) of 6.4
6.13.1. Not Zariski-locally trivial dimension torsor. Put \( Y := (\mathbb{P}^1 \times \mathbb{P}^1) \setminus \Gamma_f \), where \( \mathbb{P}^1 \) is the projective line over \( F := k((t)) \) and \( \Gamma_f \) is the graph of a morphism \( f : \mathbb{P}^1 \to \mathbb{P}^1 \) of degree \( n > 0 \). Clearly \( Y \) is affine, and

\[
\text{det} \Omega^1_Y = p_1^* \mathcal{O}(-2) \otimes p_2^* \mathcal{O}(-2) = p_1^* \mathcal{O}(2n - 2),
\]

where \( p_1, p_2 : Y \to \mathbb{P}^1 \) are the projections. We claim that if \( n > 1 \) then the dimension torsor of \( \mathcal{L}Y \) is not Zariski-locally trivial. Moreover, there exists a morphism \( \phi : \text{Spec} \, R \to \mathcal{L}Y \), \( R := \{ f \in k[x] | f(0) = f(1) \} \), such that \( \phi^* \text{Dim}_{\mathcal{L}Y} \) is not Zariski-locally trivial. One constructs \( \phi \) as follows. Consider the \( R((t)) \)-module \( M \) defined by (6.1). One can represent \( M \) as a direct summand of \( R((t))^2 \). Indeed, the \( R((t)) \)-module

\[
\{ u = u(x, t) \in k[x(t)]^2 | u(1, t) = A(t)u(0, t) \}, \quad A(t) := \left( \begin{array}{cc} t & 0 \\ 0 & t^{-1} \end{array} \right)
\]

is isomorphic to \( R((t))^2 \) because there exists \( A(x, t) \in SL(2, k[x, t, t^{-1}]) \) such that \( A(0, t) \) is the unit matrix and \( A(1, t) = A(t) \) (to find \( A(x, t) \) represent \( A(t) \) as a product of elementary matrices). Representing \( M \) as a direct summand of \( R((t))^2 \) one gets a morphism

\[
g : \text{Spec} \, R((t)) \to \mathbb{P}^1.
\]

As \( p_1 : Y \to \mathbb{P}^1 \) is a locally trivial fibration with fiber \( A^1 \), one can represent \( g \) as \( p_1 \phi \) for some \( \phi : \text{Spec} \, R((t)) \to Y \). Let \( \phi : \text{Spec} \, R \to \mathcal{L}Y \) be the morphism corresponding to \( \phi \). By (6.8) and the corollary of Lemma 6.5 the \( \mathbb{Z} \)-torsor \( \phi^* \text{Dim}_{\mathcal{L}Y} \) canonically identifies with the dimension torsor of \( M^{\otimes (2n-2)} \). In particular, \( \phi \) has the desired property, i.e., \( \phi^* \text{Dim}_{\mathcal{L}Y} \) is not Zariski-locally trivial.

The class of \( \phi^* \text{Dim}_{\mathcal{L}Y} \) in \( H^1_{\text{et}}(\text{Spec} \, R, \mathbb{Z}) \) is not a generator of this group (using (6.8) and the morphism (6.10) one sees that it equals \( (2n - 2)v \), where \( v \) is a generator). Below we construct a slightly different pair \( (Y, \phi : \text{Spec} \, R \to \mathcal{L}Y) \) so that the class of \( \phi^* \text{Dim}_{\mathcal{L}Y} \) in \( H^1_{\text{et}}(\text{Spec} \, R, \mathbb{Z}) \) is a generator.

6.13.2. Modification of the above example. Let \( Y \) be the space of triples \((v, l, l')\), where \( l, l' \) are transversal 1-dimensional subspaces in \( F^2 \) and \( v \in l \). Then there exists a morphism \( \phi : \text{Spec} \, R \to \mathcal{L}Y \), \( R := \{ f \in k[x] | f(0) = f(1) \} \), such that the class of the \( \mathbb{Z} \)-torsor \( \phi^* \text{Dim}_{\mathcal{L}Y} \) is a generator of \( H^1_{\text{et}}(\text{Spec} \, R, \mathbb{Z}) \).

More precisely, define \( \pi : Y \to \mathbb{P}^1 \) by \( \pi(v, l, l') := l \), let \( \tilde{\phi} : \text{Spec} \, R((t)) \to Y \) be such that \( \pi \tilde{\phi} \) equals (6.9), and let \( \phi : \text{Spec} \, R \to \mathcal{L}Y \) be the morphism corresponding to \( \tilde{\phi} \). Then the class of \( \phi^* \text{Dim}_{\mathcal{L}Y} \) is a generator of \( H^1_{\text{et}}(\text{Spec} \, R, \mathbb{Z}) \).

6.13.3. Any “unpleasant thing” can happen. This is what the following theorem essentially says. E.g., combining statement (ii) of the theorem with Weibel’s examples mentioned in 3.5.4 one sees that for some smooth affine scheme \( Y \) over \( F = k((t)) \) with trivial canonical bundle there exists a reasonable closed subscheme of \( \mathcal{L}Y \) which is not Zariski-locally nice (even though its dimension torsor is trivial).

**Theorem 6.6.** Let \( R \) be a \( k \)-algebra and \( u \in K_{-1}(R) \).

(i) There exists a smooth affine scheme \( Y \) over \( F = k((t)) \) and a morphism \( f : \text{Spec} \, R \to \mathcal{L}Y \) such that the pullback of the cotangent sheaf of \( \mathcal{L}Y \) to \( \text{Spec} \, R \) has class \( u \).

(ii) If the image of \( u \) in \( H^1_{\text{et}}(\text{Spec} \, R, \mathbb{Z}) \) equals 0 then one can choose \( Y \) to have trivial canonical bundle (in this case the dimension torsor of \( \mathcal{L}Y \) is trivial).

**Sketch of the proof.** Consider schemes \( Y \) of the following type\(^{21}\):

\[
Y = Y_0 \otimes_k F, \quad Y_0 = (G \times V)/H, \quad G = \text{Aut}(k^m \oplus k^n), \quad H = \text{Aut} k^m \times \text{Aut} k^n, \quad m, n \in \mathbb{N},
\]

where \( G \) and \( H \) are viewed as algebraic groups over \( k \) and \( V \) is a suitable representation of \( H \). To prove statement (i) of the theorem it suffices to take \( V = \text{Lie}[H, H] \oplus W^*, \) where \( W \) is the representation of \( H \) in \( k^m \). To prove (ii) it suffices to take \( V = \text{Lie}[H, H] \oplus W^* \oplus \text{det} \, V \).

\(^{21}\)The manifold \( Y \) from 6.13.2 is a particular example of (6.10), in which \( m = n = 1 \) and \( \dim V = 1 \).
7. Application to finite-dimensional vector bundles on manifolds with punctures

7.1. The top cohomology. Let \( R \) be commutative, \(^{22}\) \( S_n := \text{Spec } R[[t_1, \ldots, t_n]] \), \( 0 \subset S_n \) the subset defined by the equations \( t_1 = \ldots = t_n = 0 \), and \( S'_n := S_n \setminus \{0\} \). Let \( \text{Vect} \) denote the category of vector bundles on \( S'_n \) (of finite rank). For \( L \in \text{Vect} \) write \( H^i(L) \) instead of \( H^i(S'_n, L) \). The cohomology functors \( H^i : \text{Vect} \to \{\text{R-modules}\} \) vanish for \( i \geq n \) and if \( n > 1 \) then \( H^{n-1} \) commutes with base change over \( \mathbb{R} \). Theorem 7.1.

If \( n > 1 \) then for every \( \mathcal{L} \in \text{Vect} \) the \( \text{R-module} \ H^{n-1}(S'_n, \mathcal{L}) \) is 2-almost projective.

7.2. Derived version. It was A. Beilinson who explained to me that such a version should exist. Consider \( D_{\text{perf}}(S'_n) = K^b(\text{Vect}) := \{\text{homotopy category of bounded complexes in } \text{Vect}\} \). We will decompose \( R\Gamma : K^b(\text{Vect}) \to D(\mathbb{R}) \) as

\[
K^b(\text{Vect}) \xrightarrow{(R\Gamma)_{\text{topol}}} K^b(\mathcal{T}_R) \xrightarrow{\text{Forget}} D(\mathbb{R}),
\]

where \( \mathcal{T}_R \) is the category of Tate \( \mathbb{R} \)-modules. First of all, we have the derived functor

\[
(7.1) \quad R\Gamma : K^b(\text{Vect}) \to D^-(\mathbb{R}[[t_1, \ldots, t_n]]) = K^-(\mathcal{P}),
\]

where \( \mathcal{P} \) is the category of projective \( \mathbb{R}[[t_1, \ldots, t_n]] \)-modules. Second, a projective module \( P \) over \( \mathbb{R}[[t_1, \ldots, t_n]] \) carries a natural topology (the strongest one such that all \( \mathbb{R}[[t_1, \ldots, t_n]] \)-linear maps from finitely generated free \( \mathbb{R}[[t_1, \ldots, t_n]] \)-modules to \( P \) are continuous), so we get a functor from \( \mathcal{P} \) to the additive category \( \mathbb{R} \)-\text{top} of topological \( \mathbb{R} \)-modules and therefore a functor

\[
(7.2) \quad K^-(\mathcal{P}) \to K^-(\mathbb{R} \text{-top}).
\]

Theorem 7.2. The composition of \((7.1)\) and \((7.2)\) belongs to the essential image of \( K^b(\mathcal{T}_R) \) in \( K^-(\mathbb{R} \text{-top}) \), so we get a triangulated functor \((R\Gamma)_{\text{topol}} : K^b(\text{Vect}) \to K^b(\mathcal{T}_R)\). If \( \mathcal{L} \in \text{Vect} \) then \((R\Gamma)_{\text{topol}}(\mathcal{L}) \subset K^{0,n-1}(\mathcal{T})\).

To formulate the basic properties of \((R\Gamma)_{\text{topol}}\) we need some notation. Let \( C^{\text{Kar}} \) denote the Calkin category of \( \mathbb{R} \) (see 3.3.1). Consider the functor \( K^b(\mathcal{T}_R) \to K^b(C^{\text{Kar}}) \) induced by \( (3.3) \). The composition

\[
(7.3) \quad K^b(\text{Vect}) \xrightarrow{(R\Gamma)_{\text{topol}}} K^b(\mathcal{T}_R) \to K^b(C^{\text{Kar}})
\]

will be denoted by \( R\Gamma_{\text{discr}} \), because the image of a Tate \( \mathbb{R} \)-module \( T \) in \( C^{\text{Kar}} \) may be viewed as the "discrete part" of \( T \). One also has the "compact part" functor from \( \mathcal{T}_R \) to the category \((C^{\text{Kar}})^{\circ} \) dual to \( C^{\text{Kar}} \); this is the composition of the dualization functor \( \mathcal{T}_R \to \mathcal{T}_R^{\circ} \) and the functor \( \mathcal{T}_R^{\circ} \to C^{\text{Kar}} \). So we get \( R\Gamma_{\text{comp}} : K^b(\text{Vect}) \to K^b((C^{\text{Kar}})^{\circ}) \).

Theorem 7.3. (i) If \( \mathcal{L} \in \text{Vect} \) then \( R\Gamma_{\text{discr}}(\mathcal{L}) \) is an object of \( C^{\text{Kar}} \) placed in degree \( n-1 \).

(ii) If \( \mathcal{L} \in \text{Vect} \) then \( R\Gamma_{\text{comp}}(\mathcal{L}) \) is an object of \( (C^{\text{Kar}})^{\circ} \) placed in degree 0.

(iii) Let \( \omega \) denote the relative (over \( \mathbb{R} \)) canonical line bundle on \( S'_n \). Then there is a canonical duality between \((R\Gamma)_{\text{topol}}(\mathcal{L}) \), \( \mathcal{L} \in K^b(\text{Vect}) \), and \((R\Gamma)_{\text{topol}}(\mathcal{L}^* \otimes \omega[n-1])\).

\(^{22}\) One can formulate and prove Theorems 7.1, 7.3 without the commutativity assumption. In this case there is no \( S'_n \), but one can define a vector bundle on \( S_n \) to be a collection of finitely generated projective modules \( P_i \) over \( \text{Spec } R[[t_1, \ldots, t_n]]/t_i^{-1} \) with a compatible system of isomorphisms \( P_i[t_i^{-1}] \to P_j[t_j^{-1}] \).
The dimension torsor corresponding to a vector bundle. Let $X = \text{Spec } R$ be an affine scheme of finite type over $\mathbb{C}$. Let $Y$ denote $X(\mathbb{C})$ equipped with the usual topology. Given a vector bundle $\mathcal{L}$ on $X \times (\mathbb{A}^n \setminus \{0\})$, $n > 1$, one has the $R$-module $M := H^{n-1}(X \times (\mathbb{A}^n \setminus \{0\}), \mathcal{L})$, which is almost projective by Theorem 7.4. So by Remark 7.4 one has the dimension torsor $\text{Dim}_M$ (which can be viewed as a torsor on $Y$) and its canonical upper semicontinuous section $d_{\text{can}}$. Here is a geometric description of $(\text{Dim}_M, d_{\text{can}})$.

(i) Notice that a complex vector bundle of any rank $m$ on the topological space $\mathbb{C}^n \setminus \{0\}$ is trivial (because $\pi_{2n-2}(GL(m, \mathbb{C})) = 0$), and the homotopy classes of its trivializations form a torsor over $\pi_{2n-1}(GL(m, \mathbb{C}))$. One has the natural morphism $\pi_{2n-1}(GL(m, \mathbb{C})) \to \pi_{2n-1}(GL(\infty, \mathbb{C})) = K^\text{top}(S^{2n-2}) = \mathbb{Z}$. So $L$ defines a $\mathbb{Z}$-torsor $T_L$ on $Y$.

(ii) More generally, a finite complex $L$ of topological vector bundles on $X \times (\mathbb{C}^n \setminus \{0\})$ defines a $\mathbb{Z}$-torsor $T_L := \sum_i (-1)^i T_{L_i}$, and a homotopy equivalence $f : L'_1 \to L'_2$ defines an isomorphism $T_{L_1} \sim T_{L_2}$ (because the dimension torsor of $\text{Cone}(f)$ is canonically trivialized). Of course, this isomorphism depends only on the homotopy class of $f$. An extension of $L$ to an object of the homotopy category of complexes of topological vector bundles on $X \times \mathbb{C}^n$ defines a trivialization of $T_L$.

(iii) Let $L$ be an algebraic vector bundle on $X \times (\mathbb{A}^n \setminus \{0\})$, $n > 1$. Let $j : \mathbb{A}^n \setminus \{0\} \to \mathbb{A}^n$ be the embedding. For each $x \in X$ the sheaf $j_x \mathcal{L}_x$ is coherent and has a finite locally free resolution (here $\mathcal{L}_x$ is the restriction of $L$ to $\{x\} \times (\mathbb{A}^n \setminus \{0\})$). So by (ii) one gets a trivialization of $T_{\mathcal{L}_x}$ for each $x$, i.e., a set-theoretical section $s$ of $T_L$.

(iv) One can show that $(T_L, s)$ is canonically isomorphic to $(\text{Dim}_M, d_{\text{can}})$ (maybe up to a sign).

7.4. Central extension.

Theorem 7.4. Let $X$ be a smooth scheme over $S := \text{Spec } R$ of pure relative dimension $n > 1$. Let $F \subset X$ be a closed subscheme which is finite over $S$ and a locally complete intersection over $S$. Let $j : X \setminus F \hookrightarrow X$ denote the open embedding. Then for any vector bundle $\mathcal{L}$ on $X \setminus F$ the $R$-module $H^0(X, R^{n-1} j_* \mathcal{L})$ is 2-almost projective.

This easily follows from Theorems 7.3 and 4.2(i).

Now let $O_F$ be the ring of regular functions on the formal completion of $X$ along $F$. In the situation of Theorem 7.4 $H^0(X, R^{n-1} j_* \mathcal{L})$ is an $O_F$-module, so it is equipped with an action of the group scheme $G := O_F^\times$. Therefore applying 7.3 one gets a central extension

$$0 \to \mathbb{G}_m \to \hat{G}_L \to O_F^\times \to 0. \tag{7.4}$$

Remarks. (i) Suppose that in the situation of Theorem 7.4 $\mathcal{L}$ extends to a vector bundle on $X$. Then the $R$-module $H^0(X, R^{n-1} j_* \mathcal{L})$ is projective, and therefore the extension 7.4 canonically splits.

(ii) Suppose that in the situation of Theorem 7.4 $F \subset \tilde{F} \subset X$ and $\tilde{F}$ satisfies the same conditions as $F$. Put $\tilde{\mathcal{L}} := \mathcal{L}|_{X \setminus \tilde{F}}$. Then we have the central extension 7.4 and a similar central extension

$$0 \to \mathbb{G}_m \to \hat{G}_{\tilde{\mathcal{L}}} \to O_{\tilde{F}}^\times \to 0. \tag{7.5}$$

Using the functor $(R\Gamma)^\text{topol}$ from 4.2 one can construct a canonical morphism from 7.5 to 7.4 which induces the restriction map $O_{\tilde{F}}^\times \to O_F^\times$ and the identity map $\mathbb{G}_m \to \mathbb{G}_m$. If $F = \emptyset$ this amounts to Remark (i) above.

7.5. Commutativity theorem. Let $c_L : O_F^\times \times O_F^\times \to \mathbb{G}_m$ be the commutator map of the central extension 7.4.

Theorem 7.5. Suppose that in the situation of Theorem 7.4 $n = 2$. Then $c_L = 1$ if and only if $\text{det } \mathcal{L}$ extends to an invertible sheaf on $X$. 

Remarks. (i) If an extension of det $\mathcal{L}$ to an invertible sheaf on $X$ exists it equals $j_* \det \mathcal{L}$. In particular, the extension is unique.

(ii) Theorems 7.4 and 7.5 are still true for vector bundles on $(\text{Spec} \ O_F) \setminus F$ instead of $X \setminus F$.

Question. What is the geometric meaning of $c_L$ and the condition $c_L = 1$ if $n > 2$?

7.6. Generalizing the notion of vector bundle on a surface. Let $G$ be a reductive group over $\mathbb{Q}$. The moduli scheme of $G$-bundles on $\mathbb{P}_\mathbb{Q}^2$ trivialized over a fixed projective line $\mathbb{P}_\mathbb{Q}^1 \subset \mathbb{P}_\mathbb{Q}^2$ has a remarkable “Uhlenbeck compactification” $\mathcal{U}_G$ constructed in [FGK, BFG], which goes back to the physical picture of “instanton gas”. It would be very important to interpret $\mathcal{U}_G$ as a moduli scheme of some kind of geometric objects. These conjectural new objects are, so to say, “$G$-bundles with singularities”. I suggest to call them $G$-bundles. The new word “gundle” can be considered as an abbreviation for “generalized $G$-bundle”. On the other hand, its first 3 letters are also the first letters of the names of D. Gaitsgory, V. Ginzburg, K. Uhlenbeck, and H. Nakajima.

It turns out that the central extension (7.4) allows to give a definition of $GL(n)$-gundle on any smooth family of surfaces over any base so that $\mathcal{U}_{GL(n)}$ identifies with the moduli scheme of $GL(n)$-gundles on $\mathbb{P}_\mathbb{Q}^2$ trivialized over $\mathbb{P}_\mathbb{Q}^1$.

Let $X$ be a scheme smooth over $S$ of pure relative dimension 2. The definition of $GL(n)$-gundle on $X$ consists of several steps. I will only explain the first one and list the other steps.

7.6.1. Pre-gundles 1. Let $F$ be as in Theorem 7.4.

Definition. A $GL(n)$-pre-gundle on $X$ nonsingular outside $F$ is a pair that consists of a rank $n$ vector bundle $\mathcal{L}$ on $X \setminus F$ and a splitting of (7.4). The groupoid of such pairs will be denoted by $\text{Pre-gun}_F(X)$.

Remark. If (7.4) admits a splitting then by Theorem 7.5 det $\mathcal{L}$ extends to a line bundle on $X$.

7.6.2. Pre-gundles 2. If $F, \tilde{F}$ are as in Theorem 7.4 and $\tilde{F} \supset F$ then one defines a fully faithful functor $\text{Pre-gun}_F(X) \to \text{Pre-gun}_{\tilde{F}}(X)$ using Remark (ii) at the end of Remark (ii).

7.6.3. Pre-gundles 3. If $X$ is projective over $S$ one defines the groupoid of pre-gundles on $X$ to be the inductive 2-limit of $\text{Pre-gun}_F(X)$ over all closed subschemes $F \subset X$ which are finite over $S$ and locally complete intersections over $S$. This groupoid is denoted by $\text{Pre-gun}(X)$, and its objects are called $GL(n)$-pre-gundles on $X$.

If $X$ is arbitrary one first defines $\text{Pre-gun}_F(X)$ for any subscheme $F \subset X$ quasi-finite over $S$ (a standard etale or Nisnevich localization technique allows to reduce this to the case of finite locally complete intersection considered above). Then one defines $\text{Pre-gun}(X)$ to be the inductive 2-limit of $\text{Pre-gun}_F(X)$ over all closed subschemes $F \subset X$ quasi-finite over $S$.

7.6.4. Remark. If $S$ is the spectrum of a field then $\text{Pre-gun}(X)$ identifies with the groupoid of pairs $(\mathcal{L}, Z)$ with $\mathcal{L}$ being a $GL(n)$-bundle on $X$ and $Z$ a 0-cycle on $X$, it being understood that an isomorphism $(\mathcal{L}_1, Z) \simto (\mathcal{L}_2, Z)$ is the same as an isomorphism $\mathcal{L}_1 \simto \mathcal{L}_2$ and that there are no isomorphisms $(\mathcal{L}_1, Z_1) \simto (\mathcal{L}_2, Z_2)$ if $Z_1 \neq Z_2$.

To see this, first notice that for any finite $F \subset X$ a vector bundle on $X \setminus F$ uniquely extends to $X$. Second, by Remark (i) from 7.3 the central extension (7.4) has a canonical splitting, so all splittings of (7.4) are parametrized by $\text{Hom}(O_X^\times, \mathbb{G}_m)$, i.e., by the group of 0-cycles on $X$ supported on $F$.

7.6.5. Pre-gundles 4. Let $S$ be again arbitrary. Associating to an $S$-scheme $S'$ the groupoid of pre-gundles on $X \times_S S'$ one gets a (non-algebraic) $S$-stack $\text{Pre-gun}_X$. This is the stack of $GL(n)$-pre-gundles on $X$.

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23E.g., such an interpretation would hopefully allow to define an analog of $\mathcal{U}_G$ for any proper smooth surface.

24Gaitsgory is an author of [FGK, BFG], and the relation of the other 3 mathematicians to these articles is explained in the introductions to them.
7.6.6. *Gundles.* One defines a closed substack $\text{Gun}_X \subset \text{Pre-gun}_X$, whose formation commutes with base change $S' \to S$. Its $S$-points are called $(GL(n))$-gundles on $X$.

By [7.6.2] if $S = \text{Spec} \, k$ with $k$ being a field then $GL(n)$-pre-gundles on $X$ identify with pairs $(\mathcal{L}, Z)$ with $\mathcal{L}$ being a $GL(n)$-bundle on $X$ and $Z$ being a 0-cycle on $X$. It turns out that *such a pair $(\mathcal{L}, Z)$ is a $GL(n)$-gundle if and only if $Z \geq 0$.*

**Remark.** I can define the closed substack $\text{Gun}_X \subset \text{Pre-gun}_X$ using the method of $[\text{F} \text{G} \text{K}, \text{B} \text{F} \text{G}]$, i.e., by working with various curves on $X$. Unfortunately, I do not know a “purely 2-dimensional” way to do it.

7.6.7. *Hope.* If $X$ is proper over $S$ then $\text{Gun}_X$ is an algebraic stack.

7.6.8. *Fact.* Now let $S = \text{Spec} \, \mathbb{Q}$ and $X = \mathbb{P}^2_\mathbb{Q}$. Fix a projective line $\mathbb{P}^1_\mathbb{Q} \subset \mathbb{P}^2_\mathbb{Q}$, and consider the open substack $U \subset \text{Gun}_X$ parametrizing those gundles which are nonsingular on a neighborhood of $\mathbb{P}^1$ and whose restriction to $\mathbb{P}^1$ is trivial. Then $U$ identifies with the quotient of the “Uhlenbeck compactification” $\mathcal{U}_{GL(n)}$ from $[\text{B} \text{F} \text{G}]$ by the action$^{25}$ of $GL(n)$. In particular, the stack $U$ is algebraic.

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\[25\] $GL(n)$ acts on $\mathcal{U}_{GL(n)}$ by changing the trivialization over $\mathbb{P}^1$. 

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