Two “correlation games” for a nonlinear network with Hebbian excitatory neurons and anti-Hebbian inhibitory neurons

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Abstract

A companion paper introduces a nonlinear network with Hebbian excitatory (E) neurons that are reciprocally coupled with anti-Hebbian inhibitory (I) neurons and also receive Hebbian feedforward excitation from sensory (S) afferents. The present paper derives the network from two normative principles that are mathematically equivalent but conceptually different. The first principle formulates unsupervised learning as a constrained optimization problem: maximization of $S - E$ correlations subject to a copositivity constraint on $E - E$ correlations. A combination of Legendre and Lagrangian duality yields a zero-sum continuous game between excitatory and inhibitory connections that is solved by the neural network. The second principle defines a zero-sum game between E and I cells. E cells want to maximize $S - E$ correlations and minimize $E - I$ correlations, while I cells want to maximize $I - E$ correlations and minimize power. The conflict between I and E objectives effectively forces the E cells to decorrelate from each other, although only incompletely. Legendre duality yields the neural network.

A companion paper [Seung, 2018] introduces a nonlinear neural network for unsupervised learning in which a population of excitatory (E) neurons is reciprocally connected with a population of inhibitory (I) neurons (Fig. 1 left). The E neurons also receive feedforward excitation from a population of sensory (S) afferents. The reciprocal $E - I$ connections allow the E neurons to inhibit each other through disynaptic $E \rightarrow I \rightarrow E$ pathways mediated by I neurons. This network motif will be called “disynaptic recurrent inhibition,” or just “disynaptic inhibition.” Excitatory connections ($S \rightarrow E$ and $E \rightarrow I$) are modified by Hebbian plasticity, and inhibitory connections ($I \rightarrow E$) by anti-Hebbian plasticity.

The companion paper investigates the computational properties of the neural network through mathematical analysis and numerical simulations. It would be helpful to attain a deeper understanding of the network as arising from some normative principle of unsupervised learning. As emphasized by Pehlevan and Chklovskii [2015], the long
history of neural network models based on Hebbian and anti-Hebbian plasticity has primarily relied on numerical simulations to demonstrate interesting self-organizing behaviors, and lacked normative principles that can be mathematically formalized as optimizations of some objective function. Normative principles are not only helpful for understanding, but also have the practical consequence of suggesting optimization algorithms other than neural networks, which could be useful in certain settings.

The companion paper takes one step towards a normative principle by “deriving” the network with disynaptic inhibition (Fig. 1, left) as an approximation to a network with all-to-all inhibition (Fig. 1, right). The latter network contains a single population of neurons that directly inhibit each other and also receive feedforward excitation from sensory afferents, as in the Seung and Zung [2017] variant of the Földiák [1990] model. The derivation involves a factorized approximation of a Lagrange multiplier matrix, which is not fully justified in the companion paper. The present paper is intended to provide the missing theoretical foundation by deriving the network from two normative principles that are mathematically equivalent but suggest different interpretations.

The first normative principle is a modification of a principle previously used by Seung and Zung [2017] to derive the network with all-to-all inhibition. They defined unsupervised learning as the maximization of input-output correlations subject to a bound constraint on output-output correlations. The bound constraint can be regarded as enforcing a decorrelated output representation. The principle led to a zero-sum “correlation game” between excitation and all-to-all inhibition in a nonlinear network that was a variant of the Földiák [1990] model. The inhibitory connections of the

Figure 1: Disynaptic inhibition versus direct all-to-all inhibition. Green arrows indicate excitatory connections, which change via Hebbian plasticity. Red arrows indicate inhibitory connections (note minus sign), which change via anti-Hebbian plasticity. (Left) $E$ neurons are reciprocally coupled with $I$ neurons, and receive input from $S$ afferents. $E \rightarrow I$ connections ($A$) and $I \rightarrow E$ connections ($-A^T$) have equal but opposite strengths. (Right) The $E$ and $I$ populations collapse to a single population of neurons directly connected by all-to-all inhibition ($-L$).
network were conjugate to output-output correlations via Lagrangian duality. The ex-
citatory connections were conjugate to input-output correlations via Legendre duality. A similar approach was taken by [Pehlevan et al., 2018], who derived linear networks performing variants of principal component analysis from the similarity matching principle.

For the disynaptic inhibition network, the first normative principle is a constrained optimization problem: maximization of input-output correlations subject to a copositivity constraint on output-output correlations. The constraint can be regarded as forcing an imperfect form of decorrelation. The principle leads to a zero-sum game solved by the disynaptic inhibition network. The $S \rightarrow E$ connections are conjugate to the input-output correlations via Legendre duality. The $I \rightarrow E$ connections are Lagrange multipliers enforcing the copositivity constraint.

The first normative principle is based on the activities of the $E$ cells only; the activities of the $I$ cells do not appear until after the duality transforms. The second normative principle is a zero-sum game between $E$ and $I$ cells. $E$ cells want to maximize $S \rightarrow E$ correlations and minimize $E \rightarrow I$ correlations. $I$ cells want to maximize $E \rightarrow I$ correlations and minimize power. There is some conflict inherent in the fact that $I$ and $E$ cells want to drive $E \rightarrow I$ correlations in opposite directions. The compromise is that $E$ cells tend to decorrelate from each other. Both excitatory and inhibitory connections arise from Legendre duality.

A theatrical metaphor describes the difference between the normative principles. The first normative principle places the $E$ cells center stage while the $I$ cells lurk back-stage. In the second normative principle, $E$ and $I$ cells are both on stage; $E$ cells are leading actors while $I$ cells are supporting actors. No connections appear in either normative principle; the network only emerges after duality transforms.

Both normative principles are mathematically equivalent to each other, but they invite different viewpoints. The first normative principle is simple because it contains $E$ cell activity only, but complex because copositivity is not a terribly intuitive idea. The second normative principle is complex because it contains both $E$ and $I$ cell activity, but simple because there is no need for copositivity.

The second normative principle is similar to two algorithms for dimensionality reduction (hard-thresholding of covariance eigenvalues and equalizing thresholded covariance eigenvalues) defined by [Pehlevan and Chklovskii, 2015]. The neural network implementation of the algorithms involved separate populations of principal neurons and interneurons, but the neurons did not obey Dale’s Law and so could not be interpreted as excitatory and inhibitory neurons.

and inhibition minimize reconstruction error.
1 Network model with disynaptic inhibition

The disynaptic inhibition network (Fig. 1, left) has the activity dynamics,

\[ x_i := \left[ (1 - dt)x_i + dt\lambda_i^{-1}\left( \sum_{a=1}^{m} W_{ia}u_a - \sum_{\alpha=1}^{r} y_{\alpha}A_{\alpha i} \right) \right]^+ \]  

(1)

\[ y_{\alpha} = \sum_{i=1}^{n} A_{\alpha i} x_i \]  

(2)

Here \( dt \) is a step size parameter, which can be set at a small constant value or adjusted adaptively. The activation function \( [z]^+ = \max\{z, 0\} \) is half-wave rectification. After the activities converge to a steady state, update the connection matrices via

\[ \Delta W_{ia} \propto x_iu_a - \gamma W_{ia} - \kappa \sum_{b} W_{ib} \]  

(3)

\[ \Delta A_{\alpha i} \propto y_{\alpha}x_j - (q^2 - p^2) A_{\alpha i} - p^2 \sum_{i} A_{\alpha i} \]  

(4)

where \( \gamma > 0, \kappa > 0, \) and \( q^2 > p^2 \). After the updates (3) and (4), any negative elements of \( W \) and \( A \) are zeroed to maintain nonnegativity. The divisive factor \( \lambda_i > 0 \) in Eq. (1) is updated via

\[ \Delta \lambda_i \propto x_i^2 - q^2 \]  

(5)

Intuitions behind the model definitions are explained in the companion paper [Seung, 2018]. The goal of the present paper is show how the network can be interpreted as a method of solving a zero-sum game.

2 Correlation game between connections

2.1 Formulation as constrained optimization

The first normative principle concerns transformation of a sequence of input vectors \( u(1), \ldots, u(T) \) into a sequence of output vectors \( x(1), \ldots, x(T) \). Both input and output are assumed nonnegative. Define the input matrix \( U = [u(1), \ldots, u(T)] \) as the matrix containing input vectors \( u(t) \) as its columns. The element \( U_{at} \) is the \( at \)th component of \( u(t) \). Similarly, define the output matrix \( X = [x(1), \ldots, x(T)] \) as containing output vectors \( x(t) \) as its columns. Define the output-input correlation matrix is

\[ \frac{XU^T}{T} = \frac{1}{T} \sum_{t=1}^{T} x(t)u(t)^T \]

Its ia element is the time average of \( x_iu_a \), or \( \langle x_iu_a \rangle \). Similarly, define the output-output correlation matrix

\[ \frac{XX^T}{T} = \frac{1}{T} \sum_{t=1}^{T} x(t)x(t)^T \]
Its \(ij\) element is the time average of \(x_i x_j\), or \(\langle x_i x_j \rangle\). Note that “correlation matrix” is used to mean second moment matrix rather than covariance matrix. In other words, the correlation matrix does not involve subtraction of mean values. This is natural for sparse nonnegative variables, but covariance matrices may be substituted in other settings.

**Problem 1** (Constrained optimization). Define the goal of unsupervised learning as the constrained optimization

\[
\max_{X \succeq 0} \Phi^* \left( \frac{XU^\top}{T} \right) \quad \text{subject to copositivity of } D - \frac{XX^\top}{T}
\]

where \(D\) is a fixed matrix and \(\Phi^*\) is a scalar-valued function that is assumed monotone nondecreasing as a function of every element of its matrix-valued argument.

Monotonicity is an important assumption because it allows us to interpret the objective of Eq. (6) as maximization of input-output correlations.

### 2.2 Copositivity vs. nonnegativity

Seung and Zung [2017] introduced the principle

\[
\max_{X \succeq 0} \Phi^* \left( \frac{XU^\top}{T} \right) \quad \text{subject to nonnegativity of } D - \frac{XX^\top}{T}
\]

which differs from Eq. (6) only by the substitution of “nonnegativity” for “copositivity.” (Here nonnegativity of a matrix is defined to mean nonnegativity of all its elements.) While the formalisms here are valid for arbitrary \(D\), a convenient choice is to set diagonal elements of \(D\) to be \(q^2\) and off-diagonal elements of \(D\) to be \(p^2\),

\[
D_{ij} = \begin{cases} q^2, & i = j, \\ p^2, & i \neq j \end{cases}
\]

If \(p\) is much smaller than \(q\), the nonnegativity constraint \(\langle x_i x_j \rangle \leq D_{ij}\) in Eq. (7) amounts to decorrelation.

A symmetric matrix \(S\) is said to be copositive when \(v^\top S v \geq 0\) for every nonnegative vector \(v \succeq 0\). This constraint is analogous to positive semidefiniteness but is more complex because it cannot be reduced to a single eigenvalue constraint. Hahnloser et al. [2003] give sufficient and necessary conditions for copositivity involving eigenvalues of submatrices.

Nonnegativity of \(S\) is a sufficient condition for copositivity of \(S\), but it is not a necessary condition. In particular, copositivity of \(S - XX^\top / T\) does not require nonnegativity, so a solution of Problem 1 may have \(\langle x_i x_j \rangle > D_{ij}\) for some \(i\) and \(j\).

A necessary condition for copositivity of \(S\) is nonnegativity of its diagonal elements, since \(e_i^\top S e_j < 0\) if \(S_{ii} < 0\) where \(e_1, \ldots, e_n\) denotes the standard basis for \(\mathbb{R}^n\). In particular copositivity of \(S = D - XX^\top / T\) requires that \(\langle x_i^2 \rangle \leq D_{ii}\) for all \(i\). These inequalities will be called “power constraints,” because they limit the power in the outputs.
If either of the diagonal elements \( S_{ii} \) and \( S_{jj} \) vanish, then a necessary condition for copositivity is nonnegativity of the off-diagonal element \( S_{ij} \). Therefore \( \langle x_i, x_j \rangle \) may exceed \( D_{ij} \) in a solution of Problem 1 only if the power constraints for \( i \) and \( j \) are not saturated.

2.3 Correlation game from Legendre-Lagrangian duality

The copositivity constraint in Eq. (6) can be enforced by introducing Lagrange multipliers \( A \) and \( \Lambda \),

\[
\max_{X \geq 0, A, \Lambda \geq 0} \{ \Phi^* \left( \frac{XU^T}{T} \right) + \frac{1}{2} \text{Tr} A \left( D - \frac{XX^T}{T} \right) A^T + \frac{1}{2} \text{Tr} \Lambda \left( D - \frac{XX^T}{T} \right) \} \tag{9}
\]

The Lagrange multiplier \( A \) is a nonnegative \( r \times n \) matrix. The outer maximum must choose \( X \) so that \( D - XX^T/T \) is copositive because otherwise the minimum with respect to \( A \) is \(-\infty\). The Lagrange multiplier \( \Lambda = \text{diag} \{ \lambda_1, \ldots, \lambda_n \} \) is a nonnegative diagonal matrix. The outer maximum must choose \( X \) so that the diagonal elements of \( D - XX^T/T \) are nonnegative because otherwise the minimum with respect to \( \Lambda \) is \(-\infty\).

As mentioned above, copositivity of \( D - XX^T/T \) by itself already implies that the diagonal elements are nonnegative. It follows that the Lagrange multiplier \( \Lambda \) is redundant for the primal problem, though it does affect the dual problem. Similarly, adding extra rows to the Lagrange multiplier \( A \) does not change the primal problem. For enforcing the copositivity constraint, it would be sufficient for \( A \) to be \( 1 \times n \). However, making \( r > 1 \) does affect the dual problem.

**Problem 2** (Game between cells and connections). Switching the order of min and max in Eq. (11) yields the dual problem,

\[
\min_{A, \Lambda \geq 0} \max_{X \geq 0} \left\{ \Phi^* \left( \frac{XU^T}{T} \right) + \frac{1}{2} \text{Tr} \left( D - \frac{XX^T}{T} \right) \left( A^T A + \Lambda \right) \right\} \tag{10}
\]

This is an upper bound for Eq. (9) by the minimax inequality.

At this point, it is convenient to define the objective function \( \Phi^* \) as the convex conjugate (Legendre-Fenchel transform) of a function \( \Phi \),

\[
\Phi^*(C) = \max_{W \geq 0} \left\{ \sum_{ia} W_{ia} C_{ia} - \Phi(W) \right\} \tag{11}
\]

The nonnegativity constraint on \( W \) in Eq. (11) guarantees that \( \Phi^*(C) \) is monotone nondecreasing as a function of every element of \( C \). The function \( \Phi \) can be interpreted as a regularizer or prior for the weight matrix \( W \).

With Legendre duality, a maximization with respect to \( W \) is implicit in Eq. (11). Switching the order of \( W \) and \( X \) maximizations yields the following equivalent problem.
**Problem 3** (Game between connections). The Lagrangian dual of the constrained optimization in Problem 1 is

\[
\min_{A,\Lambda \geq 0} \max_{W \geq 0} R(W, A^\top A + \Lambda)
\]  

(12)

with payoff function defined by

\[
R(W, L) = \max_{X \geq 0} \left\{ \frac{1}{T} \text{Tr} W^\top X U^\top - \Phi(W) + \frac{1}{2} \text{Tr} \left( D - \frac{XX^\top}{T} \right) L \right\}
\]  

(13)

The min-max problem can be interpreted as a zero-sum game between \(W\) on the one hand and \(A\) and \(\Lambda\) on the other.

Problem 3 is closely related to the correlation game previously introduced by Seung and Zung [2017],

\[
\min_{L \geq 0} \max_{W \geq 0} R(W, L)
\]  

(14)

Problem 3 constrains \(L = A^\top A + \Lambda\) for some nonnegative \(A\) and \(\Lambda\), so it is an upper bound for Eq. (14). This is the mathematical interpretation of choosing a parametrized form \(A^\top A + \Lambda\) for the Lagrange multiplier \(L\), as was done by Seung [2018].

The network model of Section 1 follows by setting

\[
\Phi(W) = \frac{T}{2} \sum_{ia} W_{ia}^2 + \frac{\kappa}{2} \sum_i \left( \sum_a W_{ia} \right)^2
\]

in Eq. (13) and applying online projected gradient ascent to perform the maximizations in Eq. (12) and online projected gradient descent to perform the minimizations. For a more general choice of \(\Phi(W)\), Eq. (3) should be replaced by

\[
\Delta W_{ia} \propto x_i u_a - \frac{\partial \Phi}{\partial W_{ia}}
\]

### 3 Correlation game between cells

The second normative principle concerns transformation of a sequence of nonnegative input vectors \(u(1), \ldots, u(T)\) into two sequences of nonnegative output vectors \(x(1), \ldots, x(T)\) and \(y(1), \ldots, y(T)\). Define the input matrix \(U = [u(1), \ldots, u(T)]\) and the two output matrices \(X = [x(1), \ldots, x(T)]\) and \(Y = [y(1), \ldots, y(T)]\).

**Problem 4** (Game between cells). Define the goal of unsupervised learning as the zero-sum game between \(X\) and \(Y\)

\[
\max_{X \geq 0} \min_{Y \geq 0} \left\{ \Phi^*(X U^\top T) - \Psi^*(Y X^\top T) + \frac{1}{2} \text{Tr} \left( YY^\top T \right) \right\}
\]  

(15)

where \(\Phi^*\) and \(\Psi^*\) are scalar-valued functions assumed monotone nondecreasing as a function of every element of their matrix-valued arguments.
Note that only nonnegativity constraints remain in Problem 4; the copositivity constraint of Problem 1 is completely hidden. This correlation game can be interpreted as follows. The $E$ cells would like to maximize $S - E$ correlations (make $\Phi^*$ large) and minimize $E - I$ correlations (make $\Psi^*$ small). The $I$ cells would like to maximize $I - E$ correlations (make $\Psi^*$ large) and minimize power (make $\text{Tr} YY^\top / T$ small). There is conflict between the $E$ and $I$ cells because $E$ cells would like to minimize $E - I$ correlations while $I$ cells would like to maximize them. The compromise is that $E$ cells incompletely decorrelate from each other.

Problem 4 is equivalent to Problem 1 given the definition of

$$
\Psi^*(C) = \max_{A \geq 0} \left\{ \sum_{ia} A_{ai} C_{ai} - \Psi(A) \right\} 
$$

as the Legendre transform of

$$
\Psi(A) = \frac{1}{2} \text{Tr} AD A^\top 
$$

**Proof.** Substituting the definitions of Eqs. (16) and (17) into Eq. (15) yields

$$
\max_{X \succeq 0, A \geq 0} \min_{Y \geq 0} \left\{ \Phi^* \left( \frac{XU^\top}{T} \right) - \frac{1}{T} \text{Tr} A^\top YY^\top + \frac{1}{2} \text{Tr} ADA^\top + \frac{1}{2} \text{Tr} YY^\top T \right\}
$$

This is minimized when $Y = AX$, attaining the value

$$
\max_{X \succeq 0, A \geq 0} \min_{Y \geq 0} \left\{ \Phi^* \left( \frac{XU^\top}{T} \right) - \frac{1}{2T} \text{Tr} A^\top AXX^\top + \frac{1}{2} \text{Tr} ADA^\top \right\}
$$

This is identical to Eq. (9), except for the omission of the Lagrange multiplier $\Lambda$ which, as mentioned previously, is redundant in the primal problem.

4 Discussion

The second normative principle is also interesting because it can be generalized to include $E - E$ and $I - I$ connections. This will be the subject of future work.

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