Minimum $L_1$-Distance Projection onto the Boundary of a Convex Set: Simple Characterization

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Abstract. We show that the minimum distance projection in the $L_1$-norm from an interior point onto the boundary of a convex set is achieved by a single, unidimensional projection. Application of this characterization when the convex set is a polyhedron leads to either an elementary minmax problem or a set of easily solved linear programs, depending upon whether the polyhedron is given as the intersection of a set of halfspaces, or as the convex hull of a set of extreme points. The outcome is an easier and more straightforward derivation of the “special case” results given in a recent paper by Briec (Ref. [1]).

Key Words. Convex sets, minimum distance projection, $L_1$-norm.

1 Introduction

In a recent paper, Briec (Ref. [1]) studied the problem of determining the minimum distance from an interior point to the boundary of a convex set. The approach taken by Briec is largely geometric and is based on the idea of separating convex sets with hyperplanes. This geometric viewpoint leads to a duality result, that is then applied to polyhedral sets. For the special case that the distance is measured in the $L_1$-norm, Briec showed that the global minimum is obtained by solving a finite set of linear programs.

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The purpose of this note is to show that, for the \( L_p \)-norm with \( p \) less than or equal to one, there is a much simpler geometric characterization of the minimum distance projection from an interior point onto the boundary of a convex set. Application of this characterization to polyhedral sets immediately renders Briec’s “special case” results.

## 2 Characterization of the Minimum \( L_1 \)-Distance Projection

Despite the vast literature on convex sets (cf. Refs.\(^2\) and\(^3\), the following simple result involving the \( L_1 \)-distance norm seems to have passed unnoticed.

### Theorem 2.1

Let \( S \) be a convex set in \( \mathbb{R}^n \) with the origin as an interior point, and let \( X \) be the set of points on its boundary \( \partial S \) with minimum \( L_1 \)-distance to the origin. Then there is an \( x \in X \) with exactly one nonzero component.

**Proof.** Let \( x \in X \) be a point on the boundary of \( S \) with minimum distance \( d = \sum_i |x_i| \) to the origin. Note that \( x \) has at least one nonzero component since the origin is an interior point.

Now consider the set \( E \), consisting of the \( 2^n \) points \( \pm de_i \), where \( e_i \) is a vector with all components equal to zero except the \( i \)th component, which is equal to one. If there is an \( \tilde{x} \in E \) that is not in the closure of \( S \), then there is a \( \lambda \in ]0,1[ \) such that \( \lambda \tilde{x} \in \partial S \). But then \( d(\lambda \tilde{x}) = \lambda d(\tilde{x}) < d \), leading to a contradiction, and so \( E \subset \text{cl} \ S \) must hold. Now assume that all the elements of \( E \) are interior points of \( S \). Any element of \( X \) can be expressed as a convex combination of elements of \( E \) as follows:

\[
x = \sum_i (|x_i|/d)\{\text{sgn}(x_i)de_i}\,
\]

and this implies that \( X \subset \text{int} \ S \). This, now leads to a contradiction, and shows that there must be an \( x \in E \) that lies on the boundary \( \partial S \), and so completes the proof. \( \square \)

This theorem implies that the minimum \( L_1 \)-distance projection from an interior point to the boundary of a convex set is achieved by a single, unidimensional projection. As the \( L_1 \)-norm is invariant under translation, the following corollary is immediate.

### Corollary 2.1

Let \( a \) be an interior point of the convex set \( S \). If \( d \) is the minimum \( L_1 \)-distance from point \( a \) to the boundary \( \partial S \), then there is an \( x \in \partial S \) with \( d = \sum_i |x_i - a_i| \), that differs in only one component from \( a \).

Theorem 2.1 and its corollary also hold when we take the \( L_p \)-norm with \( p \) less than one as the distance metric. The proof proceeds along the same lines as for the case...
when $p$ is equal to one. The argument that any $x \in X$ can be expressed as a convex combination of the elements of $E$ is a consequence of the fact that, for $p \leq 1$,

$$\max \left\{ \sum |x_i| / d \mid \left( \sum |x_i|^p \right)^{1/p} = d \right\} = 1.$$  

For $p > 1$, this maximum has a value of $n^{1-1/p}$, and shows that [the proof of] Theorem 2.1 does not generalize for the $L_p$-norm, when $p$ is larger than one. 

3 Applications to Polyhedral Sets

Briec considers two representational forms of a polyhedron: first as the intersection of some finite collection of halfspaces, and second as the convex hull of a finite set of extreme points. For the former he remarks that, in the particular case of the $L_1$-metric, it is possible to find a global solution by solving $m$ linear programs, where $m$ is the number of hyperplanes defining the polyhedron. In fact, as shown below, the global minimum can be obtained by solving an elementary minmax problem. For the latter representation as a convex hull he shows that the global minimum is obtained as the minimum of $2m$ linear programs, where $m$ is the number of extreme points defining the convex hull. In what follows, we assume for both representations that the origin is an interior point of the polyhedron.

3.1 Intersection of Halfspaces

If the polyhedron is given as the intersection of halfspaces it can be represented as

$$S = \{ x \in \mathbb{R}^n \mid Ax \leq b \}.$$  

Note that the components of the vector $b$ are strictly positive, as the origin is an interior point. Because of this, we may assume, without loss of generality, that the matrix $A$ does not contain a row that consists entirely of zeros, as this defines a redundant constraint. By Theorem 2.1 we know that there is a point on the boundary of $S$ with minimum distance to the origin that is a unidimensional projection of the form $\lambda \epsilon_j$, with $\lambda \epsilon_j \leq b$ and $|\lambda|$ maximal, for some $1 \leq j \leq n$. Taking into account the sign of $\lambda$, define

$$\lambda^-_j = \max \{ \lambda \mid -\lambda \epsilon_j \leq b \} \quad \text{and} \quad \lambda^+_j = \max \{ \lambda \mid \lambda \epsilon_j \leq b \}.$$  

This gives the minimum distance as $d = \min_j \{ \lambda^-_j, \lambda^+_j \}$. Writing out the constraints by component gives

$$\lambda^-_j = \min_{\{i \mid a_{ij} < 0\}} -b_i / a_{ij} \quad \text{and} \quad \lambda^+_j = \min_{\{i \mid a_{ij} > 0\}} b_i / a_{ij}.$$  

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We note that, in the case where an index set is empty, which can occur when the polyhedron is unbounded, one assigns to the corresponding $\lambda_j^-$ or $\lambda_j^+$ a value of infinity or some sufficiently large number. Now combine these expressions to

$$d = \min_i \frac{b_i}{\max_j |a_{ij}|},$$

and one obtains the minimum distance as the solution to a simple minmax problem.

### 3.2 Convex Hull

If the polyhedron is given as the convex hull of a finite set of points it can be represented as

$$S = \{ x \in \mathbb{R}^n \mid x = \sum_k a_k \mu_k, \sum_k \mu_k \leq 1, \mu_k \geq 0 \}.$$

Applying Theorem 2.1, while taking into account the sign of $\lambda_j$, gives the following set of linear programs:

$$\lambda_j^- = \max \left\{ \lambda \mid -\lambda e_j = A \mu, \sum_k \mu_k \leq 1, \mu_k \geq 0 \right\},$$

$$\lambda_j^+ = \max \left\{ \lambda \mid \lambda e_j = A \mu, \sum_k \mu_k \leq 1, \mu_k \geq 0 \right\},$$

and the minimum distance as $d = \min_j \{\lambda_j^-, \lambda_j^+\}$. These linear programs are the duals of the ones in Briec’s paper.

### 4 Discussion

Of course, our characterization and results do not have the generality that Briec’s duality result (Proposition 3.1) has. His result also covers the $L_p$-norm for $p > 1$, in particular the important case of the Euclidean distance ($p = 2$). However, for the special cases considered here ($p \leq 1$), our characterization does allow for an easier geometric interpretation, and this leads to the corresponding optimization problems in a straightforward manner. Note however, that the characterization does not make any assumptions on the form of the convex set, and therefore that it is applicable to convex sets other than polyhedral.

Finally, we note that a weighted version of the minimum distance projection, where one considers the sum $\sum_i w_i |x_i|^p$, with the weights $w_i$ nonnegative and $p \leq 1$, also leads to the characterization that the minimum distance projection onto the boundary of the convex set is achieved by a single, unidimensional projection.
References

1. Briec, W., *Minimum Distance to the Complement of a Convex Set: Duality Result*, Journal of Optimization Theory and Applications, Vol. 93, No. 6, pp. 301–319, 1997.

2. Gruber, P. M. and Wills, J. M., Editors, *Handbook of Convex Geometry, Vols. A and B*, North-Holland Publishing Co., Amsterdam, The Netherlands, 1993.

3. Rockafellar, R. T., *Convex Analysis*, Princeton University Press, Princeton, New Jersey, 1970.