Online Allocation Problem with Two-sided Resource Constraints

Qixin Zhang\textsuperscript{*1,2}, Wenbing Ye\textsuperscript{*1}, Zaiyi Chen\textsuperscript{*1}, Haoyuan Hu\textsuperscript{1}, Enhong Chen\textsuperscript{3}, and Yu Yang\textsuperscript{2}

1 Cainiao Network, Hang Zhou, China
2 School of Data Science City University of Hong Kong, Kowloon Hong Kong, China
3 University of Science and Technology of China

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Abstract

In this paper, we investigate the online allocation problem of maximizing the overall revenue subject to both lower and upper bound constraints. Compared to the extensively studied online problems with only resource upper bounds, the two-sided constraints affect the prospects of resource consumption more severely. As a result, only limited violations of constraints or pessimistic competitive bounds could be guaranteed. To tackle the challenge, we define a measure of feasibility $\xi^*$ to evaluate the hardness of this problem, and estimate this measurement by an optimization routine with theoretical guarantees. We propose an online algorithm adopting a constructive framework, where we initialize a threshold price vector using the estimation, then dynamically update the price vector and use it for decision-making at each step. It can be shown that the proposed algorithm is $(1 - O(\frac{\epsilon}{\xi^*}))$ or $(1 - O(\sqrt{\frac{\epsilon}{\xi^*}}))$ competitive with high probability for $\xi^*$ known or unknown respectively. To the best of our knowledge, this is the first result establishing a nearly optimal competitive algorithm for solving two-sided constrained online allocation problems with a high probability of feasibility.

1 Introduction

Online resource allocation is a prominent paradigm for sequential decision making during a finite horizon subject to the resource constraints, increasingly attracting the wide attention of researchers and practitioners in theoretical computer science (Mehta et al., 2007; Devanur and Jain, 2012; Devanur et al., 2014), operations research (Agrawal et al., 2014; Li and Ye, 2021) and machine learning communities (Balseiro et al., 2020; Li et al., 2020). In these settings, the requests arrive online and we need to serve each request via one of the available channels, which consumes a certain amount of resources and generates a corresponding service charge. The objective of the decision maker is to maximize the cumulative revenue subject to the resource capacity constraints. Such problem frequently appears in many applications including online advertising (Mehta et al., 2007; Buchbinder et al., 2007), online combinatorial auctions (Chawla et al., 2010), online linear programming (Agrawal et al., 2014; Buchbinder and Naor, 2005), online routing (Buchbinder and Naor, 2006), online multi-leg flight seats and hotel rooms allocation (Talluri et al., 2004), etc.

The aforementioned online resource allocation framework only considers the capacity (upper bound) constraints for resources. As a measure of fairness for resources expenditure, the requirements for guaranteeing a certain amount of resource allocation play important roles in real-world applications (Haitao Cui et al., 2007; Zhang et al., 2020) ranging from contractual obligations and group-level fairness to load balance. We give several examples in Appendix A for completeness. Recently, some attempts come out to alleviate the difficulties introduced by lower bound constraints. For instance, Lobos et al. (2021) propose an online mirror descent method to address this new online allocation problem with $O(\sqrt{T})$ asymptotic regret as well as $O(\sqrt{T})$ violation of lower bounds in expectation, where $T$ is the number of requests. Meanwhile, Balseiro et al. (2021) consider a more general regularized setting to satisfy fairness requirements by non-separable penalties, which also

\textsuperscript{*}Equal contribution.
leads to $O(\sqrt{T})$ asymptotic regret by wisely choosing the penalties. All these studies consider the lower bound requirements as soft threshold, i.e., there is no guarantee on the satisfaction of lower bound constraints, which remains an open problem for online resource allocation.

In this paper, we investigate the online resource allocation problem under stochastic setting, where requests are drawn independently from some unknown distribution, and arrive sequentially. The goal is to maximize the total revenue subject to the two-sided resource constraints. In the beginning, the decision maker is endowed with a limited and unreplenishable amount of resources, and agrees a minimum consumption on each resource. The decision maker can access the information of the current request and requests that have been processed before, but she is unable to get the information of future requests until their arrivals. Once observing an online request, the decision is made irrevocably to assign one available channel to serve the request, where we assume there is an oracle that determines the revenue and the amount of consumed resources depending on (request, channel) pair for convenience. A formal definition can be found in Section 3.

To address the challenges brought by two-sided resource constraints, we first define a problem dependent quantity $\gamma$ to represent i). the maximum fraction of revenue or resource consumption per request, ii). the pessimism about the satisfaction of lower bound constraints. Generally, large $\gamma$ will rule out good competitive ratio across different settings (Mehta et al., 2007; Buchbinder et al., 2007; Agrawal et al., 2014; Devanur et al., 2019). Meanwhile, we find that the margins between lower and upper bounds directly affect the hardness of the online allocation problem in stochastic setting. Inspired by the Slater’s condition and its applications (Slater, 2014; Boyd et al., 2004), we define a measure of feasibility $\xi^*$ to evaluate the hardness and present an estimator with sufficient accuracy. In the end, the proposed algorithm integrates several estimators to compensate undiscovered information. Our contribution can be summarized as follows:

1. Under the gradually improved assumptions, we propose three algorithms that return $1 - O(\xi^*/\varepsilon)$ competitive solutions satisfying the two-sided constraints w.h.p., if $\gamma$ is at most $O\left(\frac{\varepsilon^2}{\ln(K/\varepsilon^2)}\right)$, where $\xi^* \gg \varepsilon$ is the measure of feasibility and $K$ is the number of resources. To the best of our knowledge, this is the first result establishing a competitive algorithm for solving two-sided constrained online allocation problems feasibly, which is nearly optimal according to (Devanur et al., 2019).

2. To tackle the unknown parameter $\xi^*$, we propose an optimization routine in Algorithm 5. Through merging the estimate method into the previous framework, a new algorithm is proposed in Algorithm 6 with a solution achieving $1 - O(\xi^*/\sqrt{\varepsilon})$ competitive ratio when $\xi^*$ is an unknown and problem dependent constant.

3. Our analytical tools can be used to strengthen the existing models (Mehta et al., 2007; Devanur et al., 2019; Lobos et al., 2021) for online resource allocation problems.

2 Related Works

Online allocation problems have been extensively studied in theoretical computer science and operations research communities. In this section, we overview the related literature. When the incoming requests are adversarially chosen, there is a stream of literature investigating online allocation problems. Mehta et al. (2007) and Buchbinder et al. (2007) first study the AdWords problem, a special case of online allocation, and provide an algorithm that obtains a $(1 - 1/e)$ approximation to the offline optimal allocation, which is optimal under the adversarial input model. However, the adversarial assumption may be too pessimistic about the requests. To consider another application scenarios, Devanur and Hayes (2009) propose the random permutation model, where an adversary first selects a sequence of requests which are then presented to the decision maker in random order. This model is more general than the stochastic i.i.d. setting in which requests are drawn independently and at random from an unknown distribution. In this new stochastic model, Devanur and Hayes (2009) revisit the AdWords problem and present a dual training algorithm with two phases: a training phase where data is used to estimate the dual variables by solving a linear program and an exploitation phase where actions are taken using the estimated dual variables. Their algorithm is guaranteed to obtain a $1 - o(1)$ competitive ratio, which is problem dependent. Feldman et al. (2010) show that this training-based algorithm could resolve more general linear online allocation problems. Pushing these ideas one step further, Agrawal et al. (2014) consider primal and dual algorithm that dynamically updates dual variables by periodically solving a linear program using the data collected so far. Meanwhile, Kesselheim et al. (2014) take the same policy and
only consider renewing the primal variables. Recently, Devanur et al. (2019) take other innovative
techniques to geometrically update the price vector via some decreasing potential function derived
from probability inequalities. These algorithms also obtain $1 - o(1)$ approximation guarantees under
some mild assumptions. While the algorithms described above usually require solving large linear
problems periodically, there is a recent line of work seeking simple algorithms that does not need to
solve a large linear programming. Balseiro et al. (2020) study a simple dual mirror descent algorithm
for online allocation problems with concave reward functions and stochastic inputs, which attains
$O(\sqrt{KT})$ regret, where $K$ and $T$ are the number of resources and requests respectively, i.e., updating
dual variables via mirror descent algorithm and avoids solving large auxiliary linear programming.
Simultaneously, Li et al. (2020) present a similar fast algorithm that updates the dual variable via
projected gradient descent in every round for linear rewards. It is worth noting that all of these
literature only consider the online allocation with capacity constraints.

3 Preliminaries and Assumptions

In this section, we introduce some essential concepts, assumptions, and concentration inequalities
that are adopted in our framework.

3.1 Two-sided Resource Allocation Framework

We consider the following framework for offline resource allocation problems. Let $\mathcal{K}$ be the set of $K$
resources. There are $J$ different types of requests. Each request $j \in \mathcal{J}$ could be served via some channel $i \in \mathcal{I}$, which will consume $a_{ijk}$ amount of resource $k \in \mathcal{K}$ and generate $w_{ij}$ amount
of revenue. For each resource $k \in \mathcal{K}$, $L_k$ and $U_k$ denote the lower bound requirement and capacity,
respectively. The objective of the two-sided resource allocation is to maximize the revenue subject
to the two-sided resource constraints. The following is the offline integer linear programming for
resource allocation where the entire sequence of $T$ requests is given in advance:

$$
W_R = \max_x \sum_{i \in \mathcal{I}, j \in [T]} w_{ij}x_{ij}
$$

s.t. $L_k \leq \sum_{i \in \mathcal{I}, j \in [T]} a_{ijk}x_{ij} \leq U_k, \forall k \in \mathcal{K}$

$$
\sum_{i \in \mathcal{I}} x_{ij} \leq 1, \forall j \in [T]
$$

$$
x_{ij} \in \{0, 1\}, \forall i \in \mathcal{I}, j \in [T]
$$

where we denote the sample set by $[T]$. It is important to distinguish the collection of all kinds of
requests $\mathcal{J}$ and the sample set $[T]$. When $\mathcal{J}$ is used in the LP problem, the (expected) number of
requests of each type is required. As a common relaxation in online resource allocation literature
(Agrawal et al., 2014; Devanur et al., 2019; Li et al., 2020; Balseiro et al., 2020), not picking any
channel is permitted in ILP (1). We denote the no picking channel option by $\perp \in \mathcal{I}$, where $a_{i\perp k} = 0$
and $w_{i\perp j} = 0$ for all $j \in \mathcal{J}$ and $k \in \mathcal{K}$.

3.1.1 Online Two-sided Resource Allocation

To facilitate the analysis in following sections, we hereby explain the detailed assumptions for two-
sided online resource allocation problem.

For better exhibition, we denote the following expected LP problem with lower bound $L_k + \beta T\bar{a}_k$
for every resource $k \in \mathcal{K}$ by $E(\beta)$, i.e.,

$$
W_\beta = \max_x \sum_{i \in \mathcal{I}, j \in \mathcal{J}} Tp_j w_{ij}x_{ij}
$$

s.t. $L_k + \beta T\bar{a}_k \leq \sum_{i \in \mathcal{I}, j \in \mathcal{J}} Tp_j a_{ijk}x_{ij} \leq U_k, \forall k \in \mathcal{K}$

$$
\sum_{i \in \mathcal{I}} x_{ij} \leq 1, \forall j \in \mathcal{J}
$$

$$
x_{ij} \geq 0, \forall i \in \mathcal{I}, j \in \mathcal{J}
$$

3
where $\beta \in [0, \xi^*]$ is a deviation parameter and $W_\beta$ is the optimal value for problem $E(\beta)$. Meanwhile, we denote the optimal solution for $E(\beta)$ as $\{x(\beta)_{ij}, \forall i \in \mathcal{I}, j \in \mathcal{J}\}$. The motivation for adding $\beta \bar{\alpha}_k$ to the lower bound is to measure how the gap between the lower and upper bounds affects the difficulty of the problem. More insights can be found in the proof of Theorem 4.3 where we solve a factor-revealing LP problem and then build the connection between $\xi^*$ (in Assumption 3) and the competitive ratio.

**Assumption 1.** The requests arrive sequentially and are independently drawn from some unknown distribution $\mathcal{P} : \mathcal{J} \rightarrow [0, 1]$, where $\mathcal{P}(j)$ denotes the arrival probability of request $j \in \mathcal{J}$ and we denote $p_j = \mathcal{P}(j)$, $\forall j \in \mathcal{J}$.

Under Assumption 1, if we take the same policy for every request $j \in \mathcal{J}$ in ILP (1), it can be verified that $E \left[ \sum_{i \in \mathcal{I}, j \in \mathcal{J}} a_{ijk} x_{ij} \right] = \sum_{i \in \mathcal{I}, j \in \mathcal{J}} T p_j a_{ijk} x_{ij}$ and $E \left[ \sum_{i \in \mathcal{I}, j \in \mathcal{J}} w_{ij} x_{ij} \right] = \sum_{i \in \mathcal{I}, j \in \mathcal{J}} T p_j w_{ij} x_{ij}$.

Therefore, we could view the LP (2) with $\beta = 0$ as a relaxed version of the expectation of ILP (1). Moreover, $W_0$ is an upper bound of the expectation of $W_R$.

**Lemma 3.1.** $W_0 \geq E[W_R]$.

The proof of the above lemma is deferred to Appendix C. The competitive ratio of an algorithm is defined as the ratio of the expected cumulative revenue of the algorithm to $W_0$.

**Assumption 2.** In stochastic settings, we make the following reasonable assumptions.

i. When the algorithm is initialized, we know the lower and upper bound requirements $L_k$ and $U_k$ regarding every resource $k \in \mathcal{K}$ and the number of requests $T$.

ii. Without loss of generality, the revenues $w_{ij}$ and consumption of resources $a_{ijk}$ are finite, non-negative and revealed when each request arrives, $\forall i \in \mathcal{I}, j \in \mathcal{J}, k \in \mathcal{K}$. Moreover, we know $\bar{w} = \sup_{i \in \mathcal{I}, j \in \mathcal{J}} w_{ij}$ and $\bar{a}_k = \sup_{i \in \mathcal{I}, j \in \mathcal{J}} a_{ijk}$, $\forall k \in \mathcal{K}$.

The above assumptions are widely adopted in literature. We also make an assumption on the margin of every resource constraint.

**Assumption 3.** (Strong feasible condition)

There exists a $\xi > 0$ making the linear constraints of the following problem feasible,

$$
\begin{align*}
\xi^* &= \max_{\xi \geq 0, z} \xi \\
\text{s.t.} \quad &L_k + \xi T \bar{a}_k \leq \sum_{i \in \mathcal{I}, j \in \mathcal{J}} T p_j a_{ijk} x_{ij} \leq U_k, \forall k \in \mathcal{K} \\
&\sum_{i \in \mathcal{I}} x_{ij} \leq 1, \forall j \in \mathcal{J} \\
&x_{ij} \geq 0, \forall i \in \mathcal{I}, j \in \mathcal{J}.
\end{align*}
$$

We call $\xi^*$ the measure of feasibility, and assume $\xi^* \gg \varepsilon$ for simplicity, where $\varepsilon > 0$ is an error parameter.

Due to limited space, we present some frequently used concentration inequalities to predigest the theoretical analysis in Appendix B.

**4 Competitive Algorithms for Online Resource Allocation with Two-sided Constraints**

In this section, we propose a series of online algorithms for resource allocation with two-sided constraints by progressively weakening the following assumptions.

- **4.1** and **4.2** With known distribution;
- **4.3** With known optimal objective;
- **4.4** Completely unknown distribution.
Algorithm 1 Algorithm $\tilde{P}$

1: Input: $\tau = \frac{\xi}{1-\xi}$, $\mathcal{P}$
2: Output: $(x_{ij})_{i \in I, j \in J}$
3: $(x(\tau)_{i,j}^*)_{i \in I, j \in J} = \arg \max_{x(\tau)} E(\tau)$.
4: When a type $j \in J$ request comes, we assign this request to channel $i$ with probability $(1-\varepsilon)x(\tau)_{i,j}^*$.

That is, If assigning the type $j$ request to channel $i$, we set $x_{ij} = 1$, otherwise $x_{ij} = 0$.

4.1 With Known Distribution

In this section, we assume that we have the complete knowledge of the distribution $\mathcal{P}$. We first propose a high-level overview of our algorithm and outcomes as follows.

**High-level Overview:**

1. With the knowledge of $\mathcal{P}$, we could directly solve the expected problem $E(\tau)$ and obtain the optimal solution $(x(\tau)_{i,j}^*)_{i \in I, j \in J}$, where the deviation parameter $\tau = \frac{\xi}{1-\xi}$. For each fixed request $j \in J$, if $x(\tau)_{i,j} \geq x(\tau)_{i,j}^*$, we tend to assign this type of requests to channel $i_1$ rather than $i_2$. Motivated by this intuition, we design the Algorithm $\tilde{P}$ in Algorithm 1 which assigns request $j \in J$ to channel $i \in I$ with probability $(1-\varepsilon)x(\tau)_{i,j}^*$.

2. According to the Algorithm $\tilde{P}$, if we define the r.v. $X_k^\tilde{P} = \sum_{i \in I} a_{i,k} x_{i,k}$ for resource $k$ consumed by one request selected from the distribution $\mathcal{P}$ and r.v. $Y^\tilde{P}$ for the revenue, it is easy to obtain that $(1-\varepsilon)\frac{L_k}{\mathbb{E}X_k^{\tilde{P}}} + \varepsilon a_k \leq \mathbb{E}(X_k^{\tilde{P}}) \leq (1-\varepsilon)\frac{L_k}{\mathbb{E}X_k^{\tilde{P}}} + \mathbb{E}[Y^\tilde{P}] = (1-\varepsilon)\frac{L_k}{\mathbb{E}X_k^{\tilde{P}}}$. Because the expectation of resource consumption is restricted to the interval $[L_k, U_k]$, in Theorem 4.3 we could prove that the Algorithm $\tilde{P}$ generates a feasible solution with high probability for online resource problem with two-sided constraints ILP (1) via the Bernstein inequalities in Lemma 4.1. This is the reason for enlarging the lower bound $L_k$ by $\tau T a_k$ and scaling the solution with a factor $1-\varepsilon$. Meanwhile, we also verify that the cumulative revenue will be no less than $(1-2\varepsilon)W_\tau$ w.p. $1-\varepsilon$ in Lemma 4.1.

3. Since we have lifted the lower resource constraints from $L_k$ to $L_k + \tau T a_k$ in Algorithm $\tilde{P}$, it would cause the change of baseline when analyzing the accumulative revenues. We have to derive the relationship between $W_\tau$ and $W_0$ ($W_0$). By taking a sensitive analysis in Section 4.2, we obtain $W_\tau \geq (1-\varepsilon)W_0$ in Theorem 4.2 under the Assumption 3 where $\xi^*$ is the measure of feasibility. Finally, we could prove Theorem 4.3.

Similar to (Devanur et al., 2019), we first consider the competitive ratio that Algorithm 1 could achieve for a surrogate LP problem $E(\tau)$ with optimal objective $W_\tau$ according to Definition 2.

**Lemma 4.1.** Under Assumption 1 if $\forall \varepsilon > 0$ and $\gamma = O(\frac{\varepsilon^2}{\ln(1/\varepsilon)})$, Algorithm 1 achieves an objective value at least $(1-2\varepsilon)W_\tau$ and satisfies the constraints w.p. $1-\varepsilon$.

The proof is deferred to Appendix D. From Lemma 4.1, we have verified the cumulative revenue is at least $(1-2\varepsilon)W_\tau$, w.p. $1-\varepsilon$. Thus, in order to compare the revenue with $W_0$, we should derive the relationship between $W_\tau$ and $W_0$. However, due to the deviation $\tau T a_k$, it is hard to directly obtain this relationship. We will tackle this sensitive problem in the next subsection.

4.2 Sensitive Analysis

In this subsection, we demonstrate the relationship between $W_\tau$ and $W_0$. Before introducing the details, we first investigate the difference between the problem $E(\tau)$ and the problem $E(0)$. Specifically, we enlarge the lower bound for every resource $k \in K$ by an extra amount of $\tau T a_k$, and then investigate the effects of these changes.

Through a sensitive analysis, we finally found that $\xi^*$ controls the decline ratio of the $E(\tau)$ objective function value. The result can be summarized by follows.

**Theorem 4.2.** Under the strong feasible condition in Assumption 3 the optimal objective of $E(\tau)$ satisfies that

$$W_\tau \geq \left(1 - \frac{\tau}{\xi^*}\right) W_0,$$

where $\tau = \frac{\xi}{1-\xi}$.
We prove Theorem 4.2 from a geometric perspective.

**Proof.** Let \( x_{ij} = (1 - \frac{1}{\xi})x(0)_{ij} + \frac{1}{\xi}x(\xi^*)_{ij}, \forall i \in I \text{ and } j \in J \), then \( x_{ij}' \) is a convex combination between the optimal solutions of \( E(0) \) and \( E(\xi^*) \). According to the constraint set of problem \( E(0) \) and \( E(\xi) \), we can verify that \( x_{ij}' \) is non-negative and satisfies \( \sum_{i \in I,j \in J} Tp_j a_{ijk} x_{ij}' \leq U_k, \forall k \in K \), \( \sum_{i \in I} x_{ij} \leq 1, \forall j \in J \).

Meanwhile,

\[
\begin{align*}
\sum_{i \in I,j \in J} Tp_j a_{ijk} x_{ij}' & \geq \sum_{i \in I,j \in J} Tp_j a_{ijk} \left( (1 - \frac{\tau}{\xi^*})x(0)_{ij} + \frac{\tau}{\xi^*}x(\xi^*)_{ij} \right) \\
& \geq (1 - \frac{\tau}{\xi^*})L_k + \frac{\tau}{\xi^*}(L_k + \xi^* T \bar{a}k) \\
& \geq L_k + \tau T \bar{a}k, \forall k \in K.
\end{align*}
\]

Thus \( x_{ij}' \) is feasible to \( E(\tau) \), and

\[
W_\tau \geq \sum_{i \in I,j \in J} Tp_j a_{ijk} x_{ij}' \geq (1 - \frac{\tau}{\xi^*})W_0.
\]

This completes the proof of Theorem 4.2. \( \square \)

In practical problems, the influence of \( \xi^* \) on competitive ratio may be far better than the worst case bound in Theorem 4.2 since constraints usually represent different resource requirements and only affect a part of requests. It is worth mentioning that our initial proof of Theorem 4.2 was based on analyzing a factor-revealing fractional linear programming problem motivated by (Jain et al., 2003). Although the above geometric proof of Theorem 4.2 is simpler, the factor-revealing perspective tells that \( \xi^* \) represents the severity of mutual interference between the lower and upper bounds. Its effect will eventually be reflected in the competitive ratio. We put this analysis in Appendix D due to the space limit and wish to provide more insights.

Combining Lemma 4.1 with Theorem 4.2, we can show that the cumulative revenue obtained by Algorithm \( \tilde{P} \) is larger than \( (1 - 2\varepsilon)W_\tau \geq (1 - 2\varepsilon)(1 - \frac{\tau}{\xi^*})W_0 \geq (1 - (2 + \frac{2}{\varepsilon})\varepsilon)W_0 \). Therefore, we have the following theorem.

**Theorem 4.3.** Under Assumption 4.2 if \( \varepsilon > 0 \), \( \tau = \frac{\varepsilon}{1 - \varepsilon} \) and \( \gamma = \max(\frac{\gamma_0}{\varepsilon}, \frac{\gamma_0}{\varepsilon} - \frac{\gamma_0}{\varepsilon} - L_2, \frac{\gamma_0}{\varepsilon}) = O(\frac{\varepsilon^2}{\ln(K/\varepsilon)}) \), Algorithm 7 achieves an objective value of at least \( (1 - (2 + \frac{2}{\varepsilon})\varepsilon)W_0 \) and satisfies the constraints w.p. \( 1 - \varepsilon \).

Although \( \tilde{P} \) is an impractical algorithm owing to the complete knowledge of distribution \( P \), it builds a bridge to the desired competitive ratio. In the following subsections, we will release the unavailable knowledge by replacing \( \tilde{P} \) in a constructive way.

### 4.3 With Known \( W_\tau \)

In this section, we only assume the knowledge of the optimal value \( W_\tau \) and abandon the assumption of knowing the distribution \( P \). Based on this assumption, we design an algorithm \( A \) only using \( W_\tau \) in Algorithm 2 which achieves at least \( (1 - (2 + \frac{2}{\varepsilon})\varepsilon)W_0 \) objective and satisfies the constraints w.p. \( 1 - \varepsilon \).

Intuitively, we hope to construct a new algorithm \( A \) through Algorithm \( \tilde{P} \), which performs no worse than \( \tilde{P} \) with high probability. We still use the r.v. \( X_{jk}^A \) for resource \( k \) consumed by the \( j \)-th request, which is determined by algorithm \( A \), and r.v. \( Y_{jk}^A \) for the revenue. We first investigate a simple case with only one upper bound \( U_k \). Considering the intermediate step in the proof of Bernstein inequality, we can bound the probability of failing to follow this upper bound constraint.
Algorithm 2 Algorithm A

1: Input: $\varepsilon, W_T$
2: Output: $\{x_{ij}\}_{i \in \mathcal{K}, j \in [T]}$
3: Set $c_{1k} = -\frac{\ln(1-\varepsilon)}{n_{w}}, \forall k \in \mathcal{K}$ and $c_2 = -\frac{\ln(1-\varepsilon)}{w}$
4: Initialize $\phi_0 = 1, \psi_0 = 1, \forall k \in \mathcal{K},$ and $\psi_0 = 1$
5: for $j = 1, \ldots, T$ do
6: compute the optimal $i^*$ by

$$i^* = \arg\min_{i \in I^s} \sum_{k \in \mathcal{K}} \phi_k^{i-1} \exp\left(c_{1k}(a_{ijk} - U_k/T)\right)$$

$$+ \sum_{k \in \mathcal{K}} \psi_k^{i-1} \exp\left(c_{1k}(L_k/T - a_{ijk})\right)$$

$$+ \psi_j^{i-1} \exp\left(c_2\left(\frac{(1 - 2\varepsilon)W_{\tau} - w_i}{T}\right)\right)$$

7: Set $X^A_{jk} = a_{i^*jk}, Y^A_{ij} = w_i$ for
8: Update $\phi_k^i = \phi_k^{i-1} \exp\left(c_{1k}(X^A_{jk} - \frac{U_k}{T})\right), \forall k \in \mathcal{K}$
9: Update $\psi_k^i = \psi_k^{i-1} \exp\left(c_{1k}(L_k/T - X^A_{jk})\right), \forall k \in \mathcal{K}$
10: Update $\psi_j^i = \psi_j^{i-1} \exp\left(c_2\left(\frac{(1 - 2\varepsilon)W_{\tau} - Y^A_{ij}}{T}\right)\right)$
11: end for

by

$$P\left(\sum_{j=1}^{T} X^A_{jk} \geq U_k\right)$$

$$\leq \min_{t \geq 0} \mathbb{E}\left[\exp\left(t \left(\sum_{j=1}^{s} X^A_{jk} - \frac{s}{T}U_k\right) + t \left(\sum_{j=s+1}^{T} X^A_{jk}\right) - T - sU_k\right)\right].$$

Then we relax the second term $t \left(\sum_{j=s+1}^{T} X^A_{jk} - \frac{T-s}{T}U_k\right)$ by Lemma 11.1, with $t$ given in the proof of Lemma 11.1, which means we bound the failure probability for requests that still decided by Algorithm $\hat{P}$ from $s+1$ to $T$. Although $\hat{P}$ is unknown, its failure probability has been bounded explicitly. We construct Algorithm A by induction. In the beginning, Algorithm A obtains the information of the first request, and determines $X^A_{jk}$ no worse than the decision made by Algorithm $\hat{P}$. To achieve this goal, Algorithm A iterates over all possible channels to minimize the failure probability upper bound (similar to Algorithm 2), replacing $X^A_{jk}$ by $X^A_{ik}$ in the above expression. In the induction step, $s > 1$, since Algorithm A has decided the first $s-1$ requests, it will make a decision for the $s$-th request no worse than Algorithm $\hat{P}$.

In general, the key to the previous analysis is bounding the probabilities of three bad events: i) $\sum_{j=1}^{s} X^A_{jk} \geq U_k, \forall k \in \mathcal{K}$; ii) $\sum_{j=1}^{T} X^A_{jk} \leq L_k, \forall k \in \mathcal{K}$; iii) $\sum_{j=1}^{T} Y^A_{ij} \leq (1 - 2\varepsilon)W_{\tau}$. The joint probability of the events can be considered as the failure probability of the proposed algorithm. We use the Bernstein inequalities to bound the failure probability of the Algorithm $\hat{P}$. We first design a moment generating function $F(A^T\hat{P}^{T-s})$ for controlling the probability of the three bad events of the hybrid Algorithm $A^T\hat{P}^{T-s}$, which runs Algorithm A for the first $s$ requests and $\hat{P}$ for the rest $T-s$ requests. We let $F$ enjoy an inspiring monotone property for the special hybrid Algorithm $A^T\hat{P}^{T-s}$, i.e., $F(A^T\hat{P}^{T-s}) \leq F(A^T\hat{P}^{T-s-1}) \leq F(A^T\hat{P}^{T-s-2}) \leq \cdots \leq F(\hat{P}^{T-s}) \leq \varepsilon$. Hence, the $F(A^T) \leq F(A^T\hat{P}) \leq \cdots \leq F(\hat{P}^{T-s}) \leq \varepsilon$. Intuitively, Algorithm A can be viewed as minimizing the upper bound of the probability of bad events. This completes the high-level idea when $W_{\tau}$ is known. We present the details of Algorithm A in Algorithm 2.

Theorem 4.4. Under Assumption 13, if $\varepsilon > 0$, $\tau$ and $\gamma$ are defined as Theorem 4.3, the Algorithm 2 achieves an objective value at least $(1 - (2 + \frac{1}{\varepsilon})\varepsilon)W_0$ and satisfies the constraints w.p. $1 - \varepsilon$.

The proof is deferred to Appendix 13.
Algorithm 3 Objective Estimator \(\{\text{request}_i\}_r, t_r, \beta\)

1: **Input:** requests from \(r\)-th stage \(\{\text{request}_i\}_r\), request number \(t_r\), deviation parameter \(\beta\).
2: **Output:** \(W^r\)
3: Solve \(E(\beta)\) as

\[
W^r = \max_x \sum_{i \in I} \sum_{j=1}^{t_r} w_{ij} x_{ij}
\]

\[
s.t. \frac{t_r}{T} \left( L_k + \beta T a_k \right) \leq \sum_{i \in I} \sum_{j=1}^{t_r} a_{ijk} x_{ij}
\]

\[
\leq \frac{t_r}{T} U_k, \forall k \in \mathcal{K}
\]

\[
\sum_{i \in I} x_{ij} \leq 1, \forall j \in [t_r]
\]

\[
x_{ij} \geq 0, \forall i \in I, j \in [t_r]
\]

4.4 With Known \(\xi^*\) but Unknown Distribution

In this section, we consider the completely unknown distribution setting. Under this setting, we first divide the incoming \(T\) requests into multiple stages and run an inner loop similar to Algorithm 2 in each stage. The differences between the inner loop and Algorithm 2 are threefold: i) choose the deviation parameter \(\beta = \varepsilon\) instead of \(\tau\), where \(\varepsilon\) is an error parameter defined in Theorem 4.3, ii) pessimistically reduce the estimated objective value from \(W^{r-1}\) to \(Z^r\) in each stage for boosting the chance of success, where \(W^{r-1}\) is estimated by Algorithm 3, iii) consider the relative error of objective and the maximum relative error of constraints as \(\varepsilon_{y,r}\) and \(\varepsilon_{x,r}\) separately in each stage, instead of using the global error parameter \(\varepsilon\). We consider that \(\varepsilon_{y,r}\) describes the error we care about the most (objective) and \(\varepsilon_{x,r}\) describes the rest errors (constraints and objective estimates). Distinguishing these error terms allows for more refined results, but will lead to the competitive ratio of the same order in our settings. In Algorithm 4 we name the proposed algorithm by \(A_1\) to facilitate the analysis.

The high-level ideas can be summarized as follows. There are three types of errors that have to be restrained, 1). the error \(|Z_r - W_r|\) from estimating deviant objective value, 2). the error \(\varepsilon_{x,r}\) from guaranteeing the satisfaction of constraints, and 3). the error \(\varepsilon_{y,r}\) from Algorithm \(A_1\) affected by the randomness in the objective. We use an iterative learn-and-predict strategy to periodically reduce relative errors caused by insufficient samples. Let \(\tilde{W}^r\) be the objective obtained by Algorithm \(A_1\) in stage \(r\). According to the previous sections, if objective \(Z_r\) is reachable, we could prove that the loss \(\bar{\ell}_r := |\tilde{W}^r - Z^r|\) is upper bounded by \(O(T \varepsilon_{y,r}, Z_r)\) in each stage. We also need \(Z_r\) to be a good estimate for the objective. \(Z^r \geq (1 - O(\varepsilon_{x,r-1}) W_r\) ensures that the objective is not underestimated, and \(Z_r \leq W_r\) ensures that it is a reachable target and facilitates the proof for \(r = 0, \ldots, l - 1\). Besides the normal stages, a warm-up stage \(r = -1\) is added to provide an estimate \(Z_{-1}\). Then the cumulative loss \(\ell_r := \sum_{r=0}^{l-1} \ell_r\) will be no greater than \(O(T (\varepsilon_{x,r-1} + \varepsilon_{y,r} W_r))\). The algorithm is designed to balance the relative error \(\varepsilon_{x,r}, \varepsilon_{y,r}\) and their impact on the requests in each stage.

More precisely, Algorithm \(A_1\) geometrically divides \(T\) requests into \(l + 1\) stages, where the number of requests are \(t_r = \varepsilon_2^T\) for \(r = 0, \ldots, l - 1\) and \(t_{-1} = \varepsilon T\). In the initial stage \(r = -1\), we use the first \(t_r\) requests to estimate \(W_r\) and obtain \(W_r^{-1}\), assuming that none of the requests are served in worst case. In stage \(r \in \{0, 1, \ldots, l - 1\}\), the requests from stage \(r - 1\) are used to provide more and more accurate estimate \(W_r^{-1}\) of \(W_r\). By Union Bound, we set the failure probability as \(\delta = \frac{1}{T^2}\) and reduce the estimate \(W_r^{-1}\) to \(Z^r = \frac{T W_r^{-1}}{t_r (1 + O(\varepsilon_{x,r-1}))}\), which is promised between \((1 - O(\varepsilon_{x,r-1})) W_r, W_r\) w.p. \(1 - \delta\).

Then in each stage \(r\), define the error parameters for objective and constraints as \(\varepsilon_{y,r} = O\left(\sqrt{\frac{T \ln K / A}{t_r \varepsilon_2^T}}\right)\) and \(\varepsilon_{x,r} = O\left(\sqrt{\frac{2 T \ln K / A}{t_r}}\right)\) respectively. We construct a surrogate Algorithm \(\hat{A}_2\) that achieves \(\frac{t_r}{T} (1 - \varepsilon_{y,r}) Z^r\) cumulative revenue with the consumption of every resource \(k\) between \([\bar{\ell}_r (1 + \varepsilon_{y,r}) (L_k + (1 - \varepsilon_{y,r})) a_k), \bar{\ell}_r (1 + \varepsilon_{y,r}) a_k]\) with probability at least \(1 - \delta\). Similar to previous sections, the connection between the inner loop of Algorithm \(A_1\) and Algorithm \(\hat{A}_2\) is built to minimize the upper bound of failure probabilities. Finally, with probability at least
Algorithm 4 Algorithm A1

1: Input: \( \varepsilon, \gamma_1, \xi \)
2: Output: \( \{x_{i,j}\}_{i \in \mathcal{K}, j \in [T]} \)
3: Set \( t = \log_2 \left( \frac{1}{\varepsilon} \right) \), \( t_0 = \varepsilon^2 T \), \( t_{-1} = \varepsilon T \) and \( \delta = \frac{\varepsilon}{w} \)
4: for \( r = 0 \) to \( T-1 \) do
5: Set \( W^{r-1} = \text{ObjectiveEstimator}\left(\{request_j\}_{r-1}, t_{r-1}, \varepsilon\right) \)
6: Set \( Z_r = \left\{ \frac{1}{1 + \frac{1}{\varepsilon} \left( 1 + \sqrt{\frac{\varepsilon}{2W_{r-1}}} \right) t_{r-1} \gamma_1} \right\} \)
7: Set \( \varepsilon, r \), \( \varepsilon, r \) \( \varepsilon, r \) \( \varepsilon, r \) \( \varepsilon, r \)
8: Set \( c_{0,k} = \frac{\ln(1+\varepsilon, r, k)}{w_k} \) and \( c_2 = \frac{\ln(1+\varepsilon, r, k)}{w_k} \)
9: Initialize \( \phi^0_k = \exp \left( \frac{-(t_{r-1})^2 \varepsilon}{4t_{r-1}} \right) \), \( \psi^0_k = \exp \left( \frac{-(t_{r-1})^2 \varepsilon}{4t_{r-1}} \right) \)
10: for \( j = 1, \ldots, t_r \) do
11: compute the optimal \( i^* \) by
\[ i^* = \arg\min_{i, k} \left\{ \sum_{k \in \mathcal{K}} \phi^{i-1}_k \exp \left( c_{1,k,r} \left( -1 + \varepsilon, r, U_k \right) \frac{T}{k} + a_{i,k} \right) \right\} \]
12: Set \( X^k_{1,j} = a_{i,j,k}, Y^k_{j} = w_{i,j}, Z^k_{j} = \tilde{a}_k - a_{i,j,k} \)
13: Update \( \phi^0_k = \phi^0_k - \exp \left( c_{1,k,r} \left( X^k_{1,j} - \frac{(1+\varepsilon, r, U_k)}{k} \right) \frac{T}{k} \right) \)
14: \( \psi^0_k = \psi^0_k - \exp \left( - c_{1,k,r} \left( 1+\varepsilon, r, U_k \right) \frac{T}{k} \right) \)
15: \( \psi^0_k = \psi^0_k - \exp \left( \frac{1-\varepsilon, r, Z_r}{k} - Y^k_{j} \right) \\frac{T}{k} \)
16: end for
17: end for

1 - \( \delta \), the cumulative revenue is at least \( \sum_{r=0}^{t-1} t_r \varepsilon, r, (1-\varepsilon, r, U_k) \) and the cumulative consumed resource of \( k \in \mathcal{K} \) is between \( \sum_{r=0}^{t-1} t_r \varepsilon, r, (1+\varepsilon, r, U_k) \) and \( \sum_{r=0}^{t-1} t_r \varepsilon, r, U_k \). Letting \( \gamma_1 = O \left( \frac{\varepsilon^2}{\ln(\mathcal{K}, T)} \right) \), we could keep \( \sum_{r=0}^{t-1} t_r \varepsilon, r, (1+\varepsilon, r, U_k) \leq U_k \), \( \sum_{r=0}^{t-1} t_r \varepsilon, r, U_k \geq L_k \) and \( \sum_{r=0}^{t-1} t_r \varepsilon, r, (1-\varepsilon, r, U_k) \geq \left( 1 - O(\frac{\varepsilon^2}{\ln(\mathcal{K}, T)}) \right) W_0 \).

The most tricky part of Algorithm A1 is estimating \( W^r \) in Algorithm 3. Since the lower bound \( L_k \) cannot upper bound the mean of \( X_{j,k} \) as \( U_k \) does, it brings a great challenge to the theoretical analysis. To address this issue, we solve a biased LP problem, uplifting \( L_k \) by \( \varepsilon T \tilde{a}_k \), at the end of each stage. It helps the satisfaction of the lower bound in the next stage, and the influence on the objective can be determined by Theorem 4.2. This completes the high-level overview of the analysis of Algorithm A1. The theoretical result of Algorithm A1 is presented in Theorem 4.5 with explicit requirements of parameters.

Theorem 4.5. Under Assumption 2, if \( \varepsilon > 0 \) \( \tau_1 = \frac{\varepsilon}{\gamma^*} \) such that \( \tau_1 + \varepsilon \leq \gamma^* \) and \( \gamma_1 = \max \left( \frac{\delta}{U_k}, \frac{\varepsilon}{\gamma^*} \frac{1}{1+\varepsilon, T \tilde{a}_k - L_k, W_{r-1}} \right) = O \left( \frac{\varepsilon^2}{\ln(\mathcal{K}, T)} \right) \), Algorithm A1 defined in Algorithm 4 achieves an objective value of at least \( \left( 1 - O(\frac{\varepsilon^2}{\ln(\mathcal{K}, T)}) \right) W_0 \) and satisfies the constraints w. p. 1 - \( \varepsilon \).

The proof of the theorem is deferred to Appendix C.

Remark:
1. It can be shown that the problem dependent parameter \( \xi^* \) restricts the capacity of Algorithm A1 by affecting \( \varepsilon \), \( \tau_1 \), and \( \gamma_1 \). This is consistent with our intuition that the problem with two-sided constraints becomes harder if \( \xi^* \) decreases.
2. It is worth noting that the competitive ratio obtained in this paper reflects the high-probability performance of the proposed algorithms under a finite \( T \), so it is difficult to compare with the
Theorem 5.2. Under Assumption 7-8, if \( \gamma_2 = \max(\frac{a_k}{b_k}, \frac{b_k}{a_k}) = O(\sqrt{\frac{f^2}{m(K/\epsilon)}}) \), Algorithm 5 outputs \( \tilde{\xi} \) such that

\[ \tilde{\xi} \in [\xi^* - 4\epsilon_{x,r}, \xi^*] \]

w.p. \( 1 - 2\delta \), where \( \epsilon_{x,r} = \sqrt{\frac{4\gamma_2 T \ln(K/\delta)}{t_r}} \).
Algorithm 6 Algorithm $A_2$

1: Input: $\epsilon$, $L_k$, $U_k$, $\gamma_1$, $\gamma_2 = O\left(\frac{\epsilon^2}{\ln(K/\epsilon)}\right)$, $a_k$
2: Output: $\{x_{ij}\}_{i \leq K, j \in [T]}$
3: Set $l = \log_2\left(\frac{1}{\epsilon}\right)$, $t_r = \epsilon 2^r T$, $t_{-1} = \epsilon T$ and $\delta = \frac{\epsilon}{3} t_2$
4: Set $\hat{\xi}_0 = \text{FeasEstimator}\left(\{\text{request}\}_1, t_{-1}, \gamma_2, \delta\right)$
5: Execute the Algorithm $A_1(\epsilon, \gamma_1, \hat{\xi}_0)$

The proof is deferred to Appendix I. 

Now $\hat{\xi}_0$ can be viewed as a good estimate for $\xi^*$ from stage 0. Based on the estimation of $\hat{\xi}_0$, we propose an algorithm $A_2$ in Algorithm 6. Like the previous methods, we geometrically divide $T$ requests into $l + 1$ stages for $r = -1, 0, \ldots, l - 1$, where the initial stage $r = -1$ and the first stage $r = 0$ have $\epsilon T$ requests. Besides the estimate $W^{-1}$, the estimate $\hat{\xi}_0$ for $\xi^*$ is obtained by Algorithm 5 at the end of the initial stage $r = -1$. Then we consider the new expected problem $E(\hat{\xi}_0)$ and run Algorithm $A_1$ defined in Algorithm 4 for the rest requests.

Theorem 5.3. Under Assumption 2, if $\epsilon > 0$, $\tau_1 = \frac{\epsilon}{1 - \sqrt{2} \epsilon}$ such that $\tau_1 + 4 \sqrt{2} \epsilon + \epsilon \leq \xi^*$ and $\gamma_1 = \max\left(\frac{a_k}{L_2}, \frac{a_k}{U_k - L_2}, \frac{w}{W + r_{t_1}}\right) = O\left(\frac{\epsilon^2}{\ln(K/\epsilon)}\right)$, Algorithm 6 achieves an objective value of at least $\left(1 - O\left(\frac{\epsilon^2}{\ln(K/\epsilon)}\right)\right)W_0$ and satisfies the constraints, w.p. $1 - \epsilon$.

The proof can be found in Appendix I. 

Until now, we have dropped the knowledge of the entire distribution $P$, the objective value $W_r$ for problem $E(\tau)$ and the strong feasible constant $\xi^*$ step by step. In practice, the only hyperparameter $\epsilon$ can be well approximated by solving a polynomial approximation of the transcendental equation between $\gamma_1$ and $\epsilon$ in Theorem 5.3. We regard the proposed Algorithm 3 as a practical algorithm for two-sided constrained online resource allocation problems.

6 Conclusion

In this paper, we have developed a method for online allocation problems with two-sided resource constraints, which has a wide range of real-world applications. By designing a factor-revealing linear fractional programming, a measure of feasibility $\xi^*$ is defined to facilitate our theoretical analysis. We prove that Algorithm 3 holds a nearly optimal competitive ratio if the measurement is known and large enough compared with the error parameter, i.e. $\xi^* \gg \epsilon$. An estimator is also presented in the paper for the unknown $\xi^*$ scenario. We will investigate more efficient extensions of this work in the near future.

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A  The Examples of Resource Lower Bounds

In this section, we give three practical instances showing the necessity of lower bound constraints in real-world applications.

1. **Guaranteed Advertising Delivery**: In the online advertising scenario, the advertising publishers will sell the ad impressions in advance with the promise to provide each advertiser an agree-on number of target impressions over a fixed future time period, which is usually written in the contracts. Furthermore, the advertising platform considers other constraints, such as advertisers’ budgets and impressions inventories, and simultaneously maximizes multiple accumulative objectives regarding different interested parties, e.g., Gross Merchandise Volume (GMV) for ad providers and publisher’s revenue. This widely used guaranteed delivery advertising model is generally formulated as an online resource allocation problem with two-sided constraints (Zhang et al., 2020).

2.1. **Fair Channel Constraints**: We consider the online orders assignment in an e-commerce platform, where the platform allocates the orders (or called packages in applications) to different warehouses/logistics providers. It can be shown that when providers are concerned about fairness, the platform can use a simple wholesale price above its marginal cost to coordinate this channel in terms of both achieving the maximum channel profit and attaining the maximum channel utility (Haitao Cui et al., 2007). Besides, according to the governmental regulations and contracts between platforms and providers, we usually tend to set lower bounds to the daily accepted orders for special channels, such as new/small-scale providers or those in developing areas.

2.2. **Timeliness Achievement Constraints**: The delivery time for orders is highly related with the customers’ shopping experience. Thus, the online shopping platforms also take the time-effectiveness of parcel shipment into account. For every parcel, platforms usually use the timeline achievement rate $r_u \in [0, 1]$ to denote the probability of arriving at the destination in required $u$ days if we assign this order to one channel, which could be estimated by the historical data and features of order and channel/logistics providers. As we know, long delivery time will impair the consumer’s shopping experience, but reducing delivery time means increasing cost. In order to balance the customers’ shopping experience and transportation costs, platforms always set a predefined lower threshold to the timeline achievement rate, which can be modeled as lower bound requirements in order assignment.

B  Useful Concentrations

In this section, we present the classical concentration inequalities for completeness.

**Lemma B.1.** [Bernstein, 1946]

i. Suppose that $|X| \leq c$ and $E[X] = 0$. For any $t > 0$,

$$E[\exp(tX)] \leq \exp \left( \frac{\sigma^2}{c^2} (e^{tc} - 1 - ct) \right)$$

where $\sigma^2 = \text{Var}(X)$.

ii. If $X_1, X_2, \ldots, X_n$ are independent r.v., $E[X_i] = \mu$ and $P(|X_i - \mu| \leq c) = 1$, $\forall i = 1, \ldots, n$, then $\forall \varepsilon > 0$ following inequality holds

$$P \left( \left| \frac{\sum_{i=1}^{n} X_i}{n} - \mu \right| \geq \varepsilon \right) \leq 2 \exp \left( \frac{-n\varepsilon^2}{2\sigma^2 + \frac{nc^2}{2}} \right)$$

where $\sigma^2 = \frac{\sum_{i=1}^{n} \text{Var}(X_i)}{n}$.

The first result, Lemma B.1.i., is a well-known intermediate result of Bennett’s inequality. We go a few steps further to construct the algorithm.
where \( \tau \) satisfies that 

\[
| \cdot |
\]

Theorem 4.2. Under the strong feasible condition in Assumption 3, the optimal objective of version of ILP (1), we know that 

\[
W \text{ value of Sample Instance (S.I I) and (S.I II)) respectively. Because the LP (S.I II) is a relaxation of LP (S.I I), it's easy to verify that the solution } \]

\[
W \text{ is a feasible solution for LP (2), so that } \]

\[
W \text{ linear programming (1), i.e. } \]

Proof. We consider two linear programming problem, the linear relaxation of sampled integer linear programming (1), i.e.

\[
\max_x \sum_{i,j \in J} w_{ij} x_{ij} \\
\text{s.t. } L_k \leq \sum_{i,j \in I} a_{ijk} x_{ij} \leq U_k, \forall k \in K \\
\sum_{i,j \in I} x_{ij} \leq 1, \forall j \in J \\
x_{ij} \geq 0, \forall i \in I, j \in J
\]

(S.I I)

and the linear programming that considers the samples from the type of requests perspective, i.e.

\[
\max_x \sum_{i,j \in J} |\{j' | j' \in [T], j' = j\}| w_{ij} x_{ij} \\
\text{s.t. } L_k \leq \sum_{i,j \in J} |\{j' | j' \in [T], j' = j\}| a_{ijk} x_{ij} \leq U_k, \forall k \in K \\
\sum_{i,j \in I} x_{ij} \leq 1, \forall j \in J \\
x_{ij} \geq 0, \forall i \in I, j \in J
\]

(S.I II)

where \( | \cdot | \) denote the cardinality of a given set. We use the \( W_{R_1} \) and \( W_{R_2} \) to denote the optimal value of Sample Instance (S.I I) and (S.I II) respectively. Because the LP (S.I I) is a relaxation version of ILP (1), we know that \( W_{R_1} \geq W_R \). For any optimal solution \( \{x_{ij}^\ast | \forall i \in I, j \in [T]\} \) of LP (S.I I), it’s easy to verify that the solution \( \{x_{ij} | x_{ij} = \sum_{j' \in [T], j' = j} x_{ij'}^\ast, \forall i \in I, j \in J\} \) is feasible for LP (S.I II), so that \( W_{R_2} \geq W_{R_1} \). Moreover, the average of optimal solution for all possible LP (S.I II) is a feasible solution for LP (2), \( \beta = 0 \), whose optimal solution is \( W_0 \). Thus \( W_0 \geq \mathbb{E}[W_{R_2}] \geq \mathbb{E}[W_{R_1}] \geq \mathbb{E}[W_R] \). □

D Proof of Theorem 4.2

Theorem 4.2. Under the strong feasible condition in Assumption 3, the optimal objective of \( E(\tau) \) satisfies that

\[
W_\tau \geq \left( 1 - \frac{\tau}{\xi} \right) W_0,
\]

where \( \tau = \frac{\varepsilon}{1 - \varepsilon} \).

Proof. The dual problem of the expected instance, i.e., problem \( E(0) \), is

\[
\min_{\alpha, \beta, \rho} \sum_{k \in K} \alpha_k U_k - \sum_{k \in K} \beta_k L_k + \sum_{j \in J} \rho_j \\
\text{s.t. } (\alpha_k - \beta_k) T p_j a_{ijk} - T p_j w_{ij} + \rho_j \geq 0, \\
\forall i \in I, j \in J, \\
\alpha_k, \beta_k, \rho_j \geq 0, k \in K, j \in J,
\]

(7)

and the dual problem of \( E(\tau) \) is

\[
\min_{\alpha, \beta, \rho} \sum_{k \in K} \alpha_k U_k - \sum_{k \in K} \beta_k (L_k + \tau T a_k) + \sum_{j \in J} \rho_j \\
\text{s.t. } (\alpha_k - \beta_k) T p_j a_{ijk} - T p_j w_{ij} + \rho_j \geq 0, \\
\forall i \in I, j \in J, \\
\alpha_k, \beta_k, \rho_j \geq 0, k \in K, j \in J.
\]

(8)
It can be observed that LP (7) and LP (8) share the same feasible set, while LP (8) has an extra term \(-\sum_{k \in \mathcal{K}} \tau T \tilde{a}_k \beta_k\) in the objective. It is necessary to study the relationship between \(\sum_{k \in \mathcal{K}} \tau T \tilde{a}_k \beta_k\) and \(\sum_{k \in \mathcal{K}} \alpha_k U_k - \sum_{k \in \mathcal{K}} \beta_k L_k + \sum_{j \in \mathcal{J}} \rho_j\), under the dual constraints if we want to obtain the ratio of \(W_7\) to \(W_0\). Motivated by [Jain et al., 2003], we propose a Factor-Revealing Linear Programming method for this analysis.

In order to derive the competitive ratio of the cumulative revenue obtained by the Algorithm \(\tilde{P}\) to \(W_0\), we need to find a number \(c \in (0, 1)\) which makes \(W_7 \geq (1 - c)W_0\) always hold. Considering the dual LP (7) and LP (8), if we can show that \(\sum_{k \in \mathcal{K}} \tau T \tilde{a}_k \beta_k \leq c(\sum_{k \in \mathcal{K}} \alpha_k U_k - \sum_{k \in \mathcal{K}} \beta_k L_k + \sum_{j \in \mathcal{J}} \rho_j)\) for any dual feasible solution, it will give us that \(W_7 \geq (1 - c)W_0\). Hence this question can be translated to solving the following linear fractional programming

\[
\max_{\alpha, \beta, \rho} \frac{\sum_{k \in \mathcal{K}} \tau T \tilde{a}_k \beta_k}{\sum_{k \in \mathcal{K}} \alpha_k U_k - \sum_{k \in \mathcal{K}} \beta_k L_k + \sum_{j \in \mathcal{J}} \rho_j}
\]

s.t.
\[
\forall i \in \mathcal{I}, j \in \mathcal{J} \quad \alpha_k, \beta_k, \rho_j \geq 0, k \in \mathcal{K}, j \in \mathcal{J}.
\]

Since \(\sum_{k \in \mathcal{K}} \alpha_k U_k - \sum_{k \in \mathcal{K}} \beta_k L_k + \sum_{j \in \mathcal{J}} \rho_j \geq W_0 > 0\) for any dual feasible solution, we can do the following transformation

\[
\tilde{\alpha}_k = \frac{\alpha_k}{\sum_{k \in \mathcal{K}} \alpha_k U_k - \sum_{k \in \mathcal{K}} \beta_k L_k + \sum_{j \in \mathcal{J}} \rho_j}
\]

\[
\tilde{\beta}_k = \frac{\beta_k}{\sum_{k \in \mathcal{K}} \alpha_k U_k - \sum_{k \in \mathcal{K}} \beta_k L_k + \sum_{j \in \mathcal{J}} \rho_j}
\]

\[
\tilde{\rho}_j = \frac{\rho_j}{\sum_{k \in \mathcal{K}} \alpha_k U_k - \sum_{k \in \mathcal{K}} \beta_k L_k + \sum_{j \in \mathcal{J}} \rho_j}
\]

\[
z = \frac{1}{\sum_{k \in \mathcal{K}} \alpha_k U_k - \sum_{k \in \mathcal{K}} \beta_k L_k + \sum_{j \in \mathcal{J}} \rho_j}
\]

In this way, we transfer the linear fractional programming (9) into

\[
\max_{\tilde{\alpha}, \tilde{\beta}, \tilde{\rho}} \sum_{k \in \mathcal{K}} \frac{\epsilon}{1 - \epsilon} T \tilde{a}_k \tilde{\beta}_k
\]

s.t.
\[
\forall i \in \mathcal{I}, j \in \mathcal{J} \quad (\tilde{\alpha}_k - \tilde{\beta}_k)T p_j a_{ijk} - T p_j w_{ij} + \tilde{\rho}_j \geq 0,
\]

\[
\sum_{k \in \mathcal{K}} \tilde{\alpha}_k U_k - \sum_{k \in \mathcal{K}} \tilde{\beta}_k L_k + \sum_{j \in \mathcal{J}} \tilde{\rho}_j = 1
\]

\[
\tilde{\alpha}_k, \tilde{\beta}_k, \tilde{\rho}_j, z \geq 0, k \in \mathcal{K}, j \in \mathcal{J}.
\]

We investigate the dual problem of LP (10) as follows

\[
\min_t \sum_{t, \bar{d}} \bar{d}_{ij} \leq t
\]

s.t.
\[
\sum_{i \in \mathcal{I}} d_{ij} \leq t
\]

\[
\sum_{i \in \mathcal{I}, j \in \mathcal{J}} d_{ij} T p_j a_{ijk} \leq t U_k
\]

\[
\sum_{i \in \mathcal{I}, j \in \mathcal{J}} d_{ij} T p_j a_{ijk} \geq t L_k + \frac{\epsilon}{1 - \epsilon} T \tilde{a}_k
\]

\[
\forall d_{ij} \geq 0, t \in \mathbb{R}, \forall i \in \mathcal{I}, j \in \mathcal{J}.
\]

By the first constraint of LP (11), it can be observed that \(t \geq 0\) holds. With the auxiliary
variable $z_{ij}$ which makes $d_{ij} = tz_{ij}$, we reformulate LP (11) into

$$t^* = \min_{t, z} t \quad \text{s.t.} \quad \sum_{i \in I} z_{ij} - 1 \leq 0,$$

$$t \left( \sum_{i \in I, j \in J} z_{ij} Tp_j a_{ijk} - U_k \right) \leq 0,$$

$$t \left( \sum_{i \in I, j \in J} z_{ij} Tp_j a_{ijk} - L_k \right) \geq \frac{\varepsilon}{1 - \tau} T\bar{a}_k$$

under the constraint of LP (7). Therefore, $\sum_{i \in I, j \in J} a_{ijk} x_{ij} \leq U_k$ to LP (8) such that $\sum_{i \in I, j \in J} x_{ij} Tp_j a_{ijk} \leq U_k$, $\sum_{i \in I, j \in J} Tp_j a_{ijk} x_{ij} \geq L_k + \xi^* T\bar{a}_k$ and $\sum_{i \in I} x_{ij} \leq 1$, $\forall k \in K$. Therefore, $t = \frac{1}{\xi^*}$ and $z_{ij} = x_{ij} \forall i \in I$, $j \in J$ is a feasible solution to LP (12). As a result, we have $t^* \leq \frac{1}{\xi^*}$. Since LP (11) has the same optimum as LP (12), we have that

$$\sum_{k \in K} \alpha_k U_k - \sum_{k \in K} \beta_k \left( L_k + \frac{\varepsilon}{1 - \tau} T\bar{a}_k \right) + \sum_{j \in J} \rho_j L_k$$

$$= \sum_{k \in K} \alpha_k U_k - \sum_{k \in K} \beta_k L_k + \sum_{j \in J} \rho_j - \sum_{k \in K} \frac{\varepsilon}{1 - \tau} T\bar{a}_k \beta_k$$

$$\geq \left( 1 - \frac{\tau}{\xi^*} \right) \left( \sum_{k \in K} \alpha_k U_k - \sum_{k \in K} \beta_k L_k + \sum_{j \in J} \rho_j \right)$$

under the constraint of LP (7). Therefore, $W^*_\tau \geq (1 - \frac{\tau}{\xi^*})W_0$. \hfill \Box

The factor-revealing linear fractional programming analysis shows the way we develop the definition of $\xi^*$ and enlightens the design of feasibility estimator in Algorithm 5. Besides, the proof framework is reused in proving Lemma 4.1.

### E Proof of Lemma 4.1 and Theorem 4.3

We restate the Lemma 4.1 as follows.

**Lemma E.1.** Under Assumption 4.3 if $\forall \varepsilon > 0$ and $\gamma = O \left( \frac{\varepsilon^2}{m^2 K / \varepsilon^2} \right)$, Algorithm 4 achieves an objective value at least $(1 - 2\varepsilon) W_\tau$ and satisfies the constraints w.p. $1 - \varepsilon$.

**Proof.** We first prove that, for every resource $k$, the consumed resource is below the capacity $U_k$ w.h.p.

$$P \left( \sum_{j=1}^{T} X_{jk}^p \geq U_k \right)$$

$$= P \left( \sum_{j=1}^{T} \left( X_{jk}^p - E[X_{jk}^p] \right) \geq U_k - T E[X_{jk}^p] \right)$$

$$\leq \exp \left\{ \frac{\left( U_k - T E[X_{jk}^p] \right)^2}{2 T \sigma^2 + \frac{2}{3} a_k \left( U_k - T E[X_{jk}^p] \right)} \right\}$$

$$= \exp \left\{ \frac{U_k - T E[X_{jk}^p]}{2 T \sigma^2 + \frac{2}{3} a_k} \right\}$$

$$\leq \exp \left\{ \frac{U_k - T E[X_{jk}^p]}{2 T \sigma^2 + \frac{2}{3} a_k} \right\}$$

(13)
\[
\leq \exp \left\{ -\frac{\varepsilon^2}{2(1 - \frac{2}{3}\varepsilon) a_k} \right\}
\]
\[
\leq \exp \left\{ -\frac{\varepsilon^2}{2(1 - \frac{2}{3}\varepsilon) \gamma} \right\}
\]
\[
\leq \frac{\varepsilon}{2K + 1}
\]

where the first equality follows from \( \mathbb{E}[X_{1j}^\tilde{P}] = \mathbb{E}[X_{2j}^\tilde{P}] = \cdots = \mathbb{E}[X_{Tj}^\tilde{P}] \); the first inequality from Lemma [B.1] and setting \( \sigma^2 = \text{Var}(X_{jk}^\tilde{P}) \); in the second inequality, it can verified that \( \sigma^2 \leq \mathbb{E}((X_{jk}^\tilde{P})^2) \leq \tilde{a}_k \mathbb{E}[X_{jk}^\tilde{P}] \leq (1 - \varepsilon) \tilde{a}_k \mathbb{E}[X_{jk}^\tilde{P}] \), \( U_k - T\mathbb{E}[X_{jk}^\tilde{P}] \geq \varepsilon U_k \) and \( \frac{T\sigma^2}{U_k - T\mathbb{E}[X_{jk}^\tilde{P}]} \leq \tilde{a}_k 1^{-\varepsilon} \) so that \( \frac{U_k - T\mathbb{E}[X_{jk}^\tilde{P}]}{U_k - T\mathbb{E}[X_{jk}^\tilde{P}]} \leq \frac{\tilde{a}_k}{2(1 - \frac{2}{3}\varepsilon) a_k} \leq \frac{1}{2(1 - \frac{2}{3}\varepsilon) a_k} \leq \frac{\varepsilon}{2K + 1} \) - the final inequality from \( \gamma = O\left(\frac{\varepsilon^2}{\ln(1/K)}\right) \).

Next, we verify that the Algorithm \( \tilde{P} \) satisfies the lower resource bound with high probability.

\[
P\left( \sum_{j=1}^T X_{jk}^\tilde{P} \leq L_k \right)
= P\left( \sum_{j=1}^T \left( \mathbb{E}[X_{jk}^\tilde{P}] - X_{jk} \right) \geq T\mathbb{E}[X_{jk}^\tilde{P}] - L_k \right)
\leq \exp \left\{ -\frac{(T\mathbb{E}[X_{jk}^\tilde{P}] - L_k)^2}{2T\sigma^2 + \frac{2}{3}\tilde{a}_k (T\mathbb{E}[X_{jk}^\tilde{P}] - L_k)} \right\}
\leq \exp \left\{ -\frac{\varepsilon^2}{2(1 - \frac{2}{3}\varepsilon) \tilde{a}_k} \right\}
\leq \frac{\varepsilon}{2K + 1}
\]

where the first inequality from Lemma [B.1] and setting \( \sigma^2 = \text{Var}(X_{jk}^\tilde{P}) \); in the second inequality, we could verify that \( \sigma^2 = \text{Var}(X_{jk}^\tilde{P}) = \text{Var}(\tilde{a}_k - X_{jk}) \leq 2 \tilde{a}_k \mathbb{E}[\tilde{a}_k - X_{jk}] \leq (1 - \varepsilon) \tilde{a}_k (T\tilde{a}_k - L_k), \mathbb{E}[X_{jk}^\tilde{P}] - L_k \geq \varepsilon (T\tilde{a}_k - L_k), \) and \( \frac{T\sigma^2}{T\mathbb{E}[X_{jk}^\tilde{P}] - L_k} \leq \tilde{a}_k 1^{-\varepsilon} \) so that \( \frac{T\mathbb{E}[X_{jk}^\tilde{P}] - L_k}{T\mathbb{E}[X_{jk}^\tilde{P}] - L_k} \leq \tilde{a}_k \frac{1}{2(1 - \frac{2}{3}\varepsilon) a_k} \leq \frac{1}{2(1 - \frac{2}{3}\varepsilon) a_k} \leq \frac{\varepsilon}{2K + 1} \) - the final inequality from \( \gamma = O\left(\frac{\varepsilon^2}{\ln(1/K)}\right) \).

Therefore, from the previous outcomes, the consumed resource \( k \) satisfies our lower and upper bound requirements, w.h.p. Next, we investigate the revenue the Algorithm \( \tilde{P} \) brings.

\[
P\left( \sum_{j=1}^T Y_{jk}^\tilde{P} \leq (1 - 2\varepsilon)W_T \right)
\]
\[
= P\left( \sum_{j=1}^T \left( \mathbb{E}[Y_{jk}^\tilde{P}] - Y_{jk}^\tilde{P} \right) \geq T\mathbb{E}[Y_{jk}^\tilde{P}] - (1 - 2\varepsilon)W_T \right)
\leq \exp \left\{ -\frac{(T\mathbb{E}[Y_{jk}^\tilde{P}] - (1 - 2\varepsilon)W_T)^2}{2T\sigma^2 + \frac{2}{3}\tilde{a}_k (T\mathbb{E}[Y_{jk}^\tilde{P}] - (1 - 2\varepsilon)W_T)} \right\}
\]

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where the first equality follows from $\mathbb{E}[Y_i^\tilde{P}] = \mathbb{E}[Y_j^\tilde{P}] = \cdots = \mathbb{E}[Y_n^\tilde{P}]$; the first inequality from Lemma B.1 and setting $\sigma_i^2 = \text{Var}(Y_j^\tilde{P})$; in the second inequality, we could easily verify that $\sigma_i^2 \leq \mathbb{E}[Y_i^\tilde{P}])^2 \leq \bar{w}\mathbb{E}[Y_j^\tilde{P}] \leq \frac{(1-\varepsilon)\bar{w}\mathbb{E}}{\varepsilon}$ and $\mathbb{E}[Y_j^\tilde{P}] = (1-\varepsilon)\frac{\bar{w}\mathbb{E}}{\varepsilon}$ so that $- \frac{T\mathbb{E}[Y_j^\tilde{P}]- (1-2\varepsilon)\mathbb{E}}{2T\sigma_i^2} \leq -\frac{\varepsilon^2}{2\varepsilon^2 + \frac{\varepsilon}{\bar{w}\mathbb{E}}}$; the final inequality follows from $\gamma = O\left(\frac{\varepsilon^2}{\ln(\frac{\varepsilon}{\bar{w}\mathbb{E}})}\right)$. From equation (13)-(15),

$$P(\sum_{j=1}^{T} Y_j^\tilde{P} \leq (1 - 2\varepsilon)\mathbb{E}) + \sum_{k\in K} P(\sum_{j=1}^{T} X_{jk}^\tilde{P} \notin [L_k, U_k]) \leq (2K + 1) \frac{\varepsilon}{2K + 1} \leq \varepsilon$$

when $\gamma = O\left(\frac{\varepsilon^2}{\ln(\frac{\varepsilon}{\bar{w}\mathbb{E}})}\right)$, where $\gamma = \max\left(\frac{\bar{w}\mathbb{E}}{\bar{w}\mathbb{E}}, \frac{\bar{w}\mathbb{E}}{\bar{w}\mathbb{E} - L_k}, \frac{\bar{w}\mathbb{E}}{\bar{w}\mathbb{E} - U_k}\right)$. \hfill \Box

**Theorem 4.3.** Under Assumption 1, 2 if $\varepsilon > 0$, $\tau = \frac{\varepsilon}{1-\varepsilon}$ and $\gamma = \max\left(\frac{\bar{w}\mathbb{E}}{\bar{w}\mathbb{E}}, \frac{\bar{w}\mathbb{E}}{\bar{w}\mathbb{E} - L_k}, \frac{\bar{w}\mathbb{E}}{\bar{w}\mathbb{E} - U_k}\right) = O\left(\frac{\varepsilon^2}{\ln(\frac{\varepsilon}{\bar{w}\mathbb{E}})}\right)$, Algorithm 1 achieves an objective value of at least $\left(1 - (2 + \frac{1}{\tau})\varepsilon\right)\mathbb{E}$ and satisfies the constraints w.p. $1 - \varepsilon$.

With Assumption 3, Assumption 4 and Lemma 4.1, the cumulative revenue is larger than $(1 - 2\varepsilon)\mathbb{E} \geq (1 - 2\varepsilon)\left(1 - \frac{\varepsilon}{\tau}\right)\mathbb{E} \geq (1 - 2 + \frac{1}{\tau})\varepsilon)\mathbb{E}$. We finish the proof of Theorem 4.3.

**F. Proof of Theorem 4.4**

**Theorem 4.4.** Under Assumption 1, 2 if $\varepsilon > 0$, $\tau$ and $\gamma$ are defined as Theorem 4.3, the Algorithm 2 achieves an objective value of at least $\left(1 - (2 + \frac{1}{\tau})\varepsilon\right)\mathbb{E}$ and satisfies the constraints w.p. $1 - \varepsilon$.

**Proof.** We consider the good event defined by

$$G := \left\{ \sum_{j=1}^{s} X_{jk}^A + \sum_{j=s+1}^{T} X_{jk}^\tilde{P} \leq U_k, \forall k \in K \right\} \cap \left\{ \sum_{j=1}^{s} X_{jk}^A + \sum_{j=s+1}^{T} X_{jk}^\tilde{P} \geq L_k, \forall k \in K \right\} \cap \left\{ \sum_{j=1}^{s} Y_{jk}^A + \sum_{j=s+1}^{T} Y_{jk}^\tilde{P} \geq (1 - 2\varepsilon)\mathbb{E} \right\} = G_1 \cap G_2 \cap G_3,$$

which means the hybrid Algorithm $A^s P^{T-s}$ can achieve at least $(1 - 2\varepsilon)\mathbb{E}$ revenue while satisfying the two-side constraints. We will show that the probability of the complement event $G^c$ can be bounded by some moment generating functions.
For the first bad event $G_1^c$, we have

$$
P(\sum_{j=1}^s X_{jk}^A + \sum_{j=s+1}^T X_{jk}^\tilde{P} \geq U_k) \leq \min_{t > 0} \mathbb{E} \left[ \exp(t \sum_{j=1}^s X_{jk}^A + \sum_{j=s+1}^T X_{jk}^\tilde{P} - U_k) \right]
$$

$$\leq \min_{t > 0} \mathbb{E} \left[ \exp(t \sum_{j=1}^s X_{jk}^A - \frac{s}{T} U_k) + t(\sum_{j=s+1}^T X_{jk}^\tilde{P} - \frac{T-s}{T} U_k) \right]
$$

$$= \min_{t > 0} \mathbb{E} \left[ \phi_\tilde{P}^*(t) \exp(t(\sum_{j=s+1}^T (X_{jk}^\tilde{P} - \mathbb{E}[X_{jk}^\tilde{P}]) + \frac{T-s}{T} t(\mathbb{E}[X_{jk}^\tilde{P}] - U_k)) \right]
$$

where the first inequality follows from $\exp(t(\sum_{j=1}^s X_{jk}^A + \sum_{j=s+1}^T X_{jk}^\tilde{P} - U_k)) \geq 1$ when $\sum_{j=1}^s X_{jk}^A + \sum_{j=s+1}^T X_{jk}^\tilde{P} \geq U_k$; in the second equality, we set $\phi_\tilde{P}^*(t) = \exp(t(\sum_{j=s+1}^T X_{jk}^\tilde{P} - \frac{T-s}{T} U_k))$; the second inequality from Lemma B.1 and $T\mathbb{E}[X_{jk}^\tilde{P}] \leq (1-\epsilon)U_k$; the third inequality from $\sigma^2 = \text{Var}(X_{jk}^\tilde{P}) \leq (1-\epsilon)\eta U_k$; in the fourth inequality, we set $t = -\frac{\ln(1-\epsilon)}{\tilde{a}_k \eta} \leq \frac{1}{1-\epsilon}$; then the last inequality from $(1+\eta) \ln(1+\eta) - \eta \geq \frac{\eta^2}{2(1-\epsilon)}$ and the definition of $\gamma$.

Next, for the bad event $G_2^c$, we have

$$
P(\sum_{j=1}^s X_{jk}^A + \sum_{j=s+1}^T X_{jk}^\tilde{P} \leq L_k) \leq \min_{t > 0} \mathbb{E} \left[ \exp(t(L_k - \sum_{j=1}^s X_{jk}^A - \sum_{j=s+1}^T X_{jk}^\tilde{P}) \right]
$$

$$= \min_{t > 0} \mathbb{E} \left[ \exp(t \frac{s}{T} L_k - \sum_{j=1}^s X_{jk}^A + t(\frac{T-s}{T} L_k - \sum_{j=s+1}^T X_{jk}^\tilde{P})) \right]
$$

where the second inequality from the fourth inequality, we set $\phi_\tilde{P}^*(t) = \exp(t(\sum_{j=s+1}^T X_{jk}^\tilde{P} - \frac{T-s}{T} \mathbb{E}[X_{jk}^\tilde{P}]))$; the second inequality from Lemma B.1 and $T\mathbb{E}[X_{jk}^\tilde{P}] \geq (1-\epsilon)L_k + \epsilon \tilde{a}_k$; the third inequality from $\sigma^2 = \text{Var}(X_{jk}^\tilde{P}) = \text{Var}(\tilde{a}_k - \bar{a}^\tilde{a}_k - \bar{a}^\tilde{a}_k)$.
\[ X_j^a \leq \bar{a}_k E[\bar{a}_k - X_j^a] \leq \bar{a}_k \frac{(1-\epsilon)T(\bar{a}_k - L_k)}{T}; \] in the fourth inequality, we set \( t = \frac{-\ln(1-\epsilon)}{\bar{a}_k}, \eta = \frac{\epsilon}{1-\epsilon}; \)

The last inequality from \((1 + \eta)\ln(1 + \eta) - \eta \geq \frac{\eta^2}{2}\) and the definition of \(\gamma\).

Finally, we bound the probability of event \(G^a_3\) by

\[
P \left( \sum_{j=1}^{s} T_j^a + \sum_{j=s+1}^{T} \bar{T}_j^a \leq (1-\epsilon)W_T \right)
\]

\[
\leq \min_{t \geq 0} \mathbb{E} \left[ \exp(t((1-\epsilon)W_T - \sum_{j=1}^{s} T_j^a - \sum_{j=s+1}^{T} \bar{T}_j^a)) \right]
\]

\[
= \min_{t \geq 0} \mathbb{E} \left[ \exp(t \frac{s}{T} (1-\epsilon)W_T - \sum_{j=1}^{s} T_j^a) + t((1-\epsilon)W_T - \sum_{j=s+1}^{T} \bar{T}_j^a)) \right]
\]

\[
= \min_{t \geq 0} \mathbb{E} \left[ \psi^s(t) \exp(t \sum_{j=s+1}^{T} (\mathbb{E}[\bar{T}_j^a] - T_j^a) + \frac{T-s}{T} ((1-\epsilon)W_T - \sum_{j=s+1}^{T} \bar{T}_j^a)) \right]
\]

where in the second equality, we set \(\psi^s(t) = \exp(t((1-\epsilon)W_T - \sum_{j=1}^{s} T_j^a))\); the second inequality from Lemma \[\text{[F.1]}\] and \( T \mathbb{E}[\bar{T}_j^a] = (1-\epsilon)W_T \); the third inequality from \(\sigma^2 = \text{Var}(\bar{T}_j^a) \leq \bar{a}_k \mathbb{E}[\bar{T}_j^a] \leq \frac{(1-\epsilon)\bar{a}_k W_T}{\bar{a}_k} \); in the fourth inequality, we set \( t = -\frac{\ln(1-\epsilon)}{\bar{a}_k}, \eta = \frac{\epsilon}{1-\epsilon}. \) The last inequality from \((1 + \eta)\ln(1 + \eta) - \eta \geq \frac{\eta^2}{2}\) and the definition of \(\gamma\).

With the inequalities \[\text{[F.7]}, \text{[F.9]}\] and union bound in probability theory, we can show that \( P(G^c) \leq \mathcal{F}(A^a \tilde{P}_{T-s}) \) where \( \mathcal{F}(A^a \tilde{P}_{T-s}) \) is defined by

\[
\mathcal{F}(A^a \tilde{P}_{T-s}) = \mathbb{E} \left[ \sum_{k \in K} \frac{\phi^a_k}{\bar{a}_k} \exp\left( -\frac{T-s}{2(1-\frac{2+\epsilon}{3})\gamma} \right) + \sum_{k \in K} \frac{\varphi^a_k}{\bar{a}_k} \exp\left( -\frac{T-s}{2(1-\frac{2+\epsilon}{3})\gamma} \right) \right]
\]

In Lemma \[\text{[F.1]}\] we have proven that \( \mathcal{F}(\tilde{P}_T) = (2K+1) \exp\left( -\frac{\epsilon^2}{2(1-\frac{2+\epsilon}{3})\gamma} \right) \leq \epsilon, \) and we will show that \( \mathcal{F}(A^a \tilde{P}_{T-s}) \leq \mathcal{F}(A^{a-1} \tilde{P}_{T-s+1}) \) in the Lemma \[\text{[F.1]}\]. Thus, we have that \( \mathcal{F}(A^T) \leq \mathcal{F}(\tilde{P}_T) \leq \epsilon \) by induction. Substituting \( \tau = \frac{1}{1-\epsilon} \) and \( W_T \geq (1 - \frac{\epsilon}{T})W_0 \) in Theorem \[\text{[F.3]}\] we complete the proof of Theorem \[\text{[F.4]}\]. \[\square\]

**Lemma F.1.** \( \mathcal{F}(A^a \tilde{P}_{T-s}) \leq \mathcal{F}(A^{a-1} \tilde{P}_{T-s+1}) \)

**Proof.** By the definition of \( \mathcal{F}(A^a \tilde{P}_{T-s}) \), we have that

\[
\mathcal{F}(A^a \tilde{P}_{T-s}) = \mathbb{E} \left[ \sum_{k \in K} \phi^a_k \exp\left( -\frac{T-s}{2(1-\frac{2+\epsilon}{3})\gamma} \right) + \sum_{k \in K} \varphi^a_k \exp\left( -\frac{T-s}{2(1-\frac{2+\epsilon}{3})\gamma} \right) \right] \exp\left( -\frac{T-s}{2(1-\frac{2+\epsilon}{3})\gamma} \right)
\]

(20)
According to algorithm A in Algorithm 2 we allocate the s-th request to the channel \(i^*\) where

\[
i^* = \arg \min_{i \in I} \sum_{k \in K} \phi_k^{-1}(\frac{-\ln(1-\varepsilon)}{\tilde{a}_k}) \exp(-\frac{\ln(1-\varepsilon)}{\tilde{a}_k}(a_{sk} - U_k/T))
\]

\[+ \sum_{k \in K} \phi_k^{-1}(\frac{-\ln(1-\varepsilon)}{\tilde{a}_k}) \exp(-\frac{\ln(1-\varepsilon)}{\tilde{a}_k}(L_k/T - a_{sk}))
\]

\[+ \psi^s(\frac{-\ln(1-\varepsilon)}{\tilde{w}}) \exp(-\frac{\ln(1-\varepsilon)}{\tilde{w}}(1 - 2\varepsilon)W_x - w_{is}))\]

which means that

\[
\mathcal{F}(A^s\tilde{P}^{T-s}) \leq \mathbb{E} \left[ \sum_{k \in K} \phi_k^{-1}(\frac{-\ln(1-\varepsilon)}{\tilde{a}_k}) \exp(-\frac{\ln(1-\varepsilon)}{\tilde{a}_k}(X_{sk} - \tilde{P})) \right]
\]

\[+ \sum_{k \in K} \phi_k^{-1}(\frac{-\ln(1-\varepsilon)}{\tilde{a}_k}) \exp(-\frac{\ln(1-\varepsilon)}{\tilde{a}_k}(L_k/T - X_{sk}))
\]

\[+ \psi^s(\frac{-\ln(1-\varepsilon)}{\tilde{w}}) \exp(-\frac{\ln(1-\varepsilon)}{\tilde{w}}((1 - 2\varepsilon)W_x - Y_{sk}))\] exp \((-\frac{T - s}{T} \frac{\varepsilon^2}{2(1 - \frac{3}{2}\varepsilon)}\gamma). \]

(21)

For the term \(\mathbb{1}\), we have

\[
\mathbb{1} = \mathbb{E} \left[ \exp \left( -\frac{\ln(1-\varepsilon)}{\tilde{a}_k} \left( (X_{sk} - \mathbb{E}[X_{sk}]) + (\mathbb{E}[X_{sk}] - \frac{U_k}{T}) \right) \right) \right]
\]

\[\leq \mathbb{E} \left[ \exp \left( \sigma^2 \frac{T}{\tilde{a}_k} - \frac{\ln(1-\varepsilon)}{\tilde{a}_k} - \frac{\varepsilon}{T}\ln(1-\varepsilon) \right) \right]
\]

\[\leq \mathbb{E} \left[ \exp \left( \frac{(1 - \varepsilon)U_k}{\tilde{a}_k} \left( \frac{1}{1 - \varepsilon} - 1 + \ln(1-\varepsilon) \right) \right) \right]
\]

\[\leq \mathbb{E} \left[ \exp \left( \frac{(1 - \varepsilon)U_k}{\tilde{a}_k} \left( (1 + \eta)\ln(1 + \eta) - \eta \right) \right) \right]
\]

\[\leq \exp \left( \frac{-1}{T} \frac{\varepsilon^2}{2(1 - \frac{3}{2}\varepsilon)}\gamma \right). \]

(22)

where the first inequality follows from Lemma B.1 and \(\mathbb{E}[X_{sk}] \leq (1 - \varepsilon)U_k/T\), the second from \(\sigma^2 = \text{Var}(X_{sk}) \leq \frac{(1 - \varepsilon)U_k}{\tilde{a}_k}\). Next setting \(\eta = \frac{-\mathbb{E}[X_{sk}]}{\mathbb{E}[X_{sk}] - \eta} \geq \frac{\eta^2}{2 + \eta}\) and the definition of \(\gamma\). For the term \(\mathbb{2}\), we have

\[
\mathbb{2} = \mathbb{E} \left[ \exp \left( -\frac{\ln(1-\varepsilon)}{\tilde{a}_k} \left( (\mathbb{E}[X_{sk}] - \frac{L_k}{T}) - \mathbb{E}[X_{sk}] \right) \right) \right]
\]

\[\leq \mathbb{E} \left[ \exp \left( \sigma^2 \frac{T}{\tilde{a}_k} - \frac{\ln(1-\varepsilon)}{\tilde{a}_k} + \frac{\varepsilon}{T}\ln(1-\varepsilon) \right) \right]
\]

\[\leq \mathbb{E} \left[ \exp \left( \frac{(1 - \varepsilon)U_k}{\tilde{a}_k} \left( \frac{1}{1 - \varepsilon} - 1 + \ln(1-\varepsilon) \right) \right) \right]
\]

\[\leq \mathbb{E} \left[ \exp \left( \frac{(1 - \varepsilon)U_k}{\tilde{a}_k} \left( (1 + \eta)\ln(1 + \eta) - \eta \right) \right) \right]
\]

\[\leq \exp \left( \frac{-1}{T} \frac{\varepsilon^2}{2(1 - \frac{3}{2}\varepsilon)}\gamma \right). \]

(23)

where the first inequality follows from Lemma B.1 and \(\mathbb{E}[X_{sk}] \leq \frac{(1 - \varepsilon)U_k}{\tilde{a}_k + \varepsilon U_k}\), the second from \(\sigma^2 = \text{Var}(X_{sk}) \leq \frac{(1 - \varepsilon)U_k}{\tilde{a}_k}\). Next setting \(\eta = \frac{-\mathbb{E}[X_{sk}]}{\mathbb{E}[X_{sk}] - \eta} \geq \frac{\eta^2}{2 + \eta}\) and the definition of \(\gamma\).
For the term \( \gamma \), we have

\[
\gamma = \mathbb{E} \left[ \exp \left( - \frac{\ln(1 - \varepsilon)}{\varepsilon} \left( (\mathbb{E}[Y^\text{P}_s] - Y^\text{P}_s) + \frac{(1 - 2\varepsilon)W^\text{r}_s}{T} - \mathbb{E}[Y^\text{P}_s]) \right) \right] \\
\leq \mathbb{E} \left[ \exp \left( \frac{(1 - \varepsilon)W^\text{r}_s}{T} \left( \frac{1}{1 - \varepsilon} - 1 + \ln(1 - \varepsilon) + \frac{\varepsilon}{1 - \varepsilon} \ln(1 - \varepsilon) \right) \right) \right] \\
= \mathbb{E} \left[ \exp \left( - \frac{(1 - \varepsilon)W^\text{r}_s}{T} (1 + \eta) \ln(1 + \eta) - \eta \right) \right] \\
\leq \exp \left( - \frac{1}{T} \frac{\varepsilon^2}{2(1 - \frac{2}{3}\varepsilon)} \gamma \right)
\]

where the first inequality follows from Lemma \( \text{G.1} \) and \( \mathbb{E}[Y^\text{P}_s] = \frac{(1 - \varepsilon)W^\text{r}_s}{T} \), the second from \( \sigma^2 = \text{Var}(Y^\text{P}_s) \leq \frac{(1 - \varepsilon)\omega W^\text{r}_s}{} \). Next setting \( \eta = \frac{\varepsilon}{\varepsilon^2} \), the last inequality follows from \( (1 + \eta) \ln(1 + \eta) - \eta \geq \frac{\varepsilon^2}{2 + \frac{2}{3}\varepsilon} \) and the definition of \( \gamma \). According to the inequality \( [23] - [25] \), we can show that

\[
\mathcal{F}(A^s \tilde{P}^{T-s}) \leq \left( \sum_{k \in K} \phi_k^{s-1} \left( - \frac{\ln(1 - \varepsilon)}{a_k} \right) + \sum_{k \in K} \varphi_k^{s-1} \left( - \frac{\ln(1 - \varepsilon)}{a_k} \right) \right) \exp \left( - \frac{T - s + 1}{T} \frac{\varepsilon^2}{2(1 - \frac{2}{3}\varepsilon)} \gamma \right)
\]

which completes the proof.

\[ \square \]

## G Proof of Theorem 4.5

### G.1 Concentration of \( Z^r \)

In the first step, we study the relationship between \( Z^r \) and \( W^r \).

**Lemma G.1.** Under Assumption \( \text{[23]} \) if \( \tau_1 + \varepsilon \leq \xi^* \) and \( \gamma_1 = \max \left( \frac{a_k}{\gamma_1^2}, \frac{a_k}{T \bar{a}_k - T_k}, \frac{\omega}{W_{s+1}} \right) = O\left( \frac{\varepsilon^2}{\ln(K/\tau_1)} \right) \), for given measure of feasibility \( \xi^* \), we have

\[
W^r \leq \frac{l_r \omega}{T} \left( 1 + \frac{1}{\xi^* - \varepsilon} \varepsilon_{x,r} \right)
\]

with probability at least \( 1 - \delta \), where the predefined parameter \( \varepsilon > 0 \), \( \tau_1 = \frac{\omega}{1 - \sqrt{l}} \), \( \delta = \frac{\varepsilon}{\bar{w}} \), \( l = \log_2(\frac{1}{\varepsilon}) \) and \( \varepsilon_{x,r} = \sqrt{\frac{4T \gamma_1 \ln(\frac{2\varepsilon_{x,r}}{\varepsilon})}{l_r}} \).

**Proof.** We consider the definition of \( W^r \):

\[
W^r = \max_{x} \sum_{i \in I, j \in S_r} w_{ij} x_{ij} \\
\text{s.t. } \frac{l_r}{T} (L_k + \varepsilon \bar{a}_k) \leq \sum_{i \in I, j \in S_r} a_{ijk} x_{ij} \leq \frac{l_r}{T} U_k, \forall k \in K \\
\sum_{i \in I} x_{ij} \leq 1, \forall j \in S_r \\
x_{ij} \geq 0, \forall i \in I, j \in S_r
\]

where we use \( S_r \) to denote the request set in stage \( r \). The dual of LP \( [27] \) is

\[
W^r = \min_{\alpha, \beta, \rho} \sum_{k \in K} \alpha_k \frac{l_r}{T} U_k - \sum_{k \in K} \beta_k \frac{l_r}{T} (L_k + \varepsilon \bar{a}_k) + \sum_{j \in S_r} \rho_j \\
\text{s.t. } \sum_{k \in K} (\alpha_k - \beta_k) a_{ijk} - w_{ij} + \rho_j \geq 0 \forall i \in I, j \in S_r \\
\alpha_k, \beta_k, \rho_j \geq 0, k \in K, j \in S_r
\]

\[ \square \]
we have divided the equation (30) into three parts. Next, we will derive the relationship between 
\( k \) we only consider the resource 
\( \alpha_k, \beta_k, \rho_j \geq 0, k \in \mathcal{K}, j \in \mathcal{J} \)

we have 
\[ \sum_{k \in \mathcal{K}} (\alpha_k - \beta_k) a_{ijk} x^*_{ij} - w_{ij} x^*_{ij} + \rho_j^* x^*_{ij} \geq 0 \forall i \in \mathcal{T}, j \in \mathcal{S} \]

we can observe that the constraints of LP (28) is a subset of those of LP (29). We denote the primal and dual optimal solution of \( E(\varepsilon) \) as \{\( x^*_{ij} \)\} and \{\( \alpha^*_k, \beta^*_k, \rho^*_j \)\}. So \{\( \alpha^*_k, \beta^*_k, \rho^*_j \)\} is feasible for LP (28).

Hence,
\[
W^r \leq \sum_{k \in \mathcal{K}} \alpha_k^* \frac{t_r}{T} U_k - \sum_{k \in \mathcal{K}} \beta_k^* \frac{t_r}{T} (L_k + \varepsilon T \bar{a}_k) + \sum_{j \in \mathcal{S}_r} \rho_j^* \\
= \sum_{k \in \mathcal{K}} \alpha_k^* \left( \frac{t_r}{n} U_k - \sum_{i \in I, \mathcal{I} \in \mathcal{I}} a_{ijk} x^*_{ij} \right) + \sum_{k \in \mathcal{K}} \beta_k^* \left( \sum_{i \in S_r, j \in \mathcal{I}} a_{ijk} x^*_{ij} - \frac{t_r}{T} (L_k + \varepsilon T \bar{a}_k) \right) \\
+ \sum_{j \in \mathcal{S}_r} (\rho_j^* + \sum_{i \in \mathcal{I}, k \in \mathcal{K}} (\alpha_k^* - \beta_k^*) a_{ijk} x^*_{ij})
\]

We have divided the equation (30) into three parts. Next, we will derive the relationship between 
\( W^r \) and \( W^e \) by controlling these three parts. To facilitate the analysis, we first present the KKT conditions [Boyd et al. 2004] for the problem \( E(\varepsilon) \) as follows
\[
\sum_{k \in \mathcal{K}} (\alpha_k^* - \beta_k^*) a_{ijk} x^*_{ij} - w_{ij} x^*_{ij} + \rho_j^* x^*_{ij} = 0, \forall i \in \mathcal{T}, j \in \mathcal{S}_r \\
\rho_j^* (\sum_{i \in \mathcal{T}_r} x^*_{ij} - 1) = 0, \forall \mathcal{I}_r \\
\alpha_k^* \left( \sum_{ij} T p_j a_{ijk} x^*_{ij} - U_k \right) = 0, \forall k \in \mathcal{K} \\
\beta_k^* (L_k + \varepsilon T \bar{a}_k - \sum_{ij} T p_j a_{ijk} x^*_{ij}) = 0, \forall k \in \mathcal{K}.
\]

For part (1), according to the KKT conditions, if \( \sum_{i \in \mathcal{T}, j \in \mathcal{J}} T p_j a_{ijk} x^*_{ij} < U_k \), then \( \alpha_k^* = 0 \). Thus we only consider the resource \( k \) making \( \sum_{i \in \mathcal{T}, j \in \mathcal{J}} T p_j a_{ijk} x^*_{ij} = U_k \). According to the Assumption \( \mathbb{I} \) we have \( \mathbb{E}(\sum_{j \in \mathcal{S}_r, i \in \mathcal{T}} a_{ijk} x^*_{ij}) = \frac{t_r}{T} U_k \forall j \in \mathcal{S}_r \). Thus, we can show that
\[
P \left( \sum_{j \in \mathcal{S}_r, i \in \mathcal{T}} a_{ijk} x^*_{ij} \leq (1 - \varepsilon_{x,r}) \frac{t_r}{T} U_k \right) \\
= P \left( \sum_{j \in \mathcal{S}_r, i \in \mathcal{T}} a_{ijk} x^*_{ij} - \mathbb{E}[\sum_{j \in \mathcal{S}_r, i \in \mathcal{T}} a_{ijk} x^*_{ij}] \leq -\varepsilon_{x,r} \frac{t_r}{T} U_k \right) \\
\leq \exp \left( -\frac{t_r U_k^2 \varepsilon_{x,r}^2 / T^2}{2 \text{Var}(\sum_{j \in \mathcal{S}_r, i \in \mathcal{T}} a_{ijk} x^*_{ij}) / T^2} \right) \\
\leq \exp \left( -\frac{t_r U_k^2 \varepsilon_{x,r}^2 / T^2}{2(1 + \frac{\varepsilon_{x,r}^2}{T^2}) \alpha_k \varepsilon_{x,r} / T} \right) \\
\leq \exp \left( -\frac{t_r U_k^2 \varepsilon_{x,r}^2 / T^2}{2(1 + \frac{\varepsilon_{x,r}^2}{T^2}) \gamma_1} \right) \\
\leq \frac{\delta}{2K + 1}
\]
where the first inequality follows from the Bernstein inequality in Lemma [3.1] the second inequality from \( \text{Var}(\sum_{j \in \mathcal{S}_r, i \in \mathcal{T}} a_{ijk} x^*_{ij}) / t_r \leq \alpha_k \mathbb{E}[\sum_{j \in \mathcal{S}_r, i \in \mathcal{T}} a_{ijk} x^*_{ij}] / t_r = \alpha_k \frac{U_k}{T} \), the third inequality from the definition of \( \gamma_1 \) and the last from the definition of \( \varepsilon_{x,r} \).
For part 2, we only consider the $k$ making $\sum_{i \in I, j \in J} Tp_j a_{ijk} x_{ij}^* = L_k + \epsilon T\bar{a}_k$, which means 

$$E \left[ \sum_{j \in S, i \in I} (a_k - a_{ijk} x_{ij}^*) \right] = \frac{T}{T} \left( 1 - \epsilon \right) T\bar{a}_k - L_k$$

Using Bernstein inequality, we have

$$P \left( \sum_{j \in S, i \in I} (a_k - a_{ijk} x_{ij}^*) \leq (1 - \epsilon_{x,r}) \frac{T}{T} \left( 1 - \epsilon \right) T\bar{a}_k - L_k \right) \leq \exp \left( -\frac{T}{2} \right) \left( \frac{1}{E} \right)^x_{x,r}$$

$$\leq \exp \left( -\frac{T}{2} \right) \left( \frac{1}{E} \right)^x_{x,r} \leq \exp \left( -\frac{T}{2} \right) \left( \frac{1}{E} \right)^x_{x,r} \leq \frac{\delta}{2K + 1}$$

where the second inequality from $Var \left( \sum_{j \in S, i \in I} (a_k - a_{ijk} x_{ij}^*) \right) / T \leq a_k E[\sum_{j \in S, i \in I} (a_k - a_{ijk} x_{ij}^*)] / T = a_k \frac{(1 - \epsilon) T\bar{a}_k - L_k}{T}$, the third inequality from the definition of $\gamma_1$ and the last from the definition of $\epsilon_{x,r}$.

For the last part 3, from KKT conditions, it’s easy to verify that $\sum_{i \in I, j \in J} (\alpha_k^* - \beta_k^*) a_{ijk} x_{ij}^* - \sum_{i \in I} w_{ijk} x_{ij}^* + \rho_j^* \sum_{i \in I} x_{ij}^* = \sum_{i \in I, j \in J} (\alpha_k^* - \beta_k^*) a_{ijk} x_{ij}^* - \sum_{i \in I} w_{ijk} x_{ij}^* + \rho_j^* = 0$, so $\sum_{i \in I, j \in J} (\alpha_k^* - \beta_k^*) a_{ijk} x_{ij}^* = \frac{\sum_{i \in I, j \in J} \rho_j^* w_{ijk} x_{ij}^*}{W_k} \forall j \in S$. Therefore, following the similar analysis as equation (32), we have

$$P \left( \sum_{j \in S} \left( \rho_j^* + \sum_{i \in I, j \in J} (\alpha_k^* - \beta_k^*) a_{ijk} x_{ij}^* \right) \geq \frac{T}{T} W_k \epsilon_{x,r} \right) \leq \frac{T}{T} W_k \epsilon_{x,r}$$

$$\leq \frac{T}{T} W_k \epsilon_{x,r} \leq \frac{T}{T} W_k \epsilon_{x,r} \leq \frac{T}{T} W_k \epsilon_{x,r} \leq \frac{T}{T} W_k \epsilon_{x,r} \leq \frac{T}{T} W_k \epsilon_{x,r} \leq \frac{T}{T} W_k \epsilon_{x,r}$$

where the first inequality follows from the Bernstein inequality in Lemma 3.1, the second inequality from $Var(\sum_{j \in S, i \in I} w_{ijk} x_{ij}^*) / T \leq W_k \epsilon_{x,r} \frac{\sum_{j \in S, i \in I} w_{ijk} x_{ij}^*}{T} = W_k \epsilon_{x,r}$, the third inequality from the definition of $\gamma_1$ and the last from the definition of $\epsilon_{x,r}$.

Based on inequality 31-33, we have shown that the following inequalities holds with probability at least $1 - \delta$

$$\sum_{j \in S, i \in I} a_{ijk} x_{ij}^* \geq (1 - \epsilon_{x,r}) \frac{T}{T} U_k, \forall k \in K$$

$$\sum_{j \in S, i \in I} (a_k - a_{ijk} x_{ij}^*) \geq (1 - \epsilon_{x,r}) \frac{T}{T} \left( 1 - \epsilon \right) T\bar{a}_k - L_k, \forall k \in K$$

$$\sum_{j \in S} (\rho_j^* + \sum_{i \in I, j \in J} (\alpha_k^* - \beta_k^*) a_{ijk} x_{ij}^*) \leq \frac{T}{T} W_k (1 + \epsilon_{x,r}), \forall k \in K.$$
Therefore, with probability at least $1 - \delta$, we have

\[
\begin{align*}
& (1 + \underline{2} + \underline{3}) \\
& = \sum_{k \in K} \alpha_k^* \left( \frac{t_r}{T} U_k - \sum_{j \in S, i \in I} a_{ijk} x_{ij}^* \right) + \sum_{k \in K} \beta_k^* \left( \frac{t_r}{T} ((1 - \varepsilon) T \bar{a}_k - L_k) \right) - \sum_{j \in S, i \in I} (\bar{a}_k - a_{ijk} x_{ij}^*) + \underline{3} \\
& \leq \varepsilon_x r \sum_{k \in K} \left( \alpha_k^* \frac{t_r}{T} U_k + \beta_k^* \left( \frac{t_r}{T} ((1 - \varepsilon) T \bar{a}_k - L_k) \right) \right) + \underline{3} \\
& \leq \varepsilon_x r \sum_{k \in K} \left( \alpha_k^* \frac{t_r}{T} U_k + \beta_k^* \left( \frac{t_r}{T} ((1 - \varepsilon) T \bar{a}_k - L_k) \right) \right) + \frac{t_r}{T} W_x (1 + \varepsilon_x r) \\
& \leq (1 + (2 + \frac{1}{\xi^* - \varepsilon}) \varepsilon_x r) \frac{t_r}{T} W_x
\end{align*}
\]

where the first inequality from $\sum_{j \in S, i \in I} a_{ijk} x_{ij}^* \geq (1 - \varepsilon_x r) \frac{t_r}{T} U_k$ and $\sum_{j \in S, i \in I} (\bar{a}_k - a_{ijk} x_{ij}^*) \geq (1 - \varepsilon_x r) \frac{t_r}{T} ((1 - \varepsilon) T \bar{a}_k - L_k)$; the second inequality from $\sum_{j \in S, i \in I} (\rho_j^* + \sum_{k \in K} (\alpha_k^* - \beta_k^*) a_{ijk} x_{ij}^*) \leq \frac{t_r}{T} W_x (1 + \varepsilon_x r)$; the third inequality from $W_x = \sum_{k \in K} \alpha_k^* U_k - \sum_{k \in K} \beta_k^* (L_k + \varepsilon \bar{a}_k)$; and the final inequality from $\sum_{k \in K} \alpha_k^* \frac{t_r}{T} U_k - \sum_{k \in K} \beta_k^* (L_k + \varepsilon \bar{a}_k) \leq \frac{t_r}{T} W_x$, which can be shown if we follow the proof of Theorem 4.2 in Appendix D and regard the problem $E(\varepsilon)$ as an LP with $\xi^* - \varepsilon$ measure of feasibility.

\[ \text{Theorem G.2. Under Assumption Jmm, if } \tau_1 + \varepsilon \leq \xi^* \text{ and } \gamma_1 = \max \left( \frac{\bar{a}_k}{(1 - \varepsilon_x r) T \bar{a}_k - L_k}, \frac{w}{W_{r+1}} + \frac{\bar{a}_k}{w} \right) = O \left( \frac{\varepsilon^2}{(\ln K)^2} \right), \text{ with probability } 1 - 2\delta, \text{ we have} \]

\[
\frac{t_r}{T} W_x \left( 1 - (2 + \frac{1}{\xi^* - \varepsilon}) \varepsilon_x r \right) \leq W_x \leq \frac{t_r}{T} W_x \left( 1 + (2 + \frac{1}{\xi^* - \varepsilon}) \varepsilon_x r \right)
\]

where the predefined parameter $\varepsilon > 0$, $\tau_1 = \frac{\varepsilon^2}{1 - \varepsilon_x r}$, $\delta = \frac{\varepsilon^2}{\delta}$, $l = \log_2 \left( \frac{\varepsilon^2}{\delta} \right)$ and $\varepsilon_x r = \sqrt{\frac{4l T \gamma_1 \ln \left( \frac{2\varepsilon^2}{\delta} \right)}{t_r}}$.

\[ \text{Proof. RHS: } \text{we have proven the right hand side in Lemma G.1.} \]

\[ \text{LHS: For every request in stage } r, \text{ we consider to imitate Algorithm 1 to design an algorithm } \tilde{P}_1. \]

\[ \text{In Algorithm } \tilde{P}_1, \text{ we first solve the LP problem } E(\varepsilon) + \frac{\varepsilon_x r}{1 - \varepsilon_x r} \text{ to get the LP solution } x_{ij}^*. \]

\[ \text{and then assign request } j \in J \text{ to channel } i \in I \text{ with probability } (1 - \varepsilon_x r) x_{ij}^*. \]

\[ \text{Following the similar analysis in the proof of Lemma 4.2, we can prove that } x_{ij}^*, \text{ with probability } 1 - \delta, \]

\[
\begin{align*}
& P \left( \frac{t_r}{T} X_{jk} \geq \frac{t_r}{T} U_k \right) \leq \exp \left( -\frac{t_r^2}{2(1 - \frac{t_r^2}{3\varepsilon_x r}) T W_{\tau_1 + \varepsilon}} \right) \\
& P \left( \frac{t_r}{T} Y_{ij} \leq (1 - (2 + \frac{1}{\xi^* - \varepsilon}) \varepsilon_x r) \frac{t_r}{T} W_x \right) \leq \exp \left( -\frac{t_r^2}{2(1 - \frac{t_r^2}{3\varepsilon_x r}) T W_{\tau_1 + \varepsilon}} \right) \\
& P \left( \frac{t_r}{T} X_{jk} \leq \frac{t_r}{T} L_k \right) \leq \exp \left( -\frac{t_r^2}{2(1 - \frac{t_r^2}{3\varepsilon_x r}) T \frac{w}{w_{r+1}} \frac{\bar{a}_k}{w}} \right)
\end{align*}
\]

where the second inequality from the truth that the problem $E(\varepsilon)$ is satisfied with the strong feasible condition with the measure parameter $\xi^* - \varepsilon$ and $W_{\tau_1 + \varepsilon} \geq W_{\tau_1 + \varepsilon}$. Therefore, we have that

\[
\begin{align*}
& P \left( \sum_{j=1}^{t_r} Y_{ij} \leq (1 - (2 + \frac{1}{\xi^* - \varepsilon}) \varepsilon_x r) \frac{t_r}{T} W_x \right) + \sum_{k \in K} P \left( \frac{t_r}{T} X_{jk} \notin \frac{t_r}{T} L_k, \frac{t_r}{T} U_k \right) \\
& \leq (2K + 1) \exp \left( -\frac{t_r^2}{4T \gamma_1} \right) \\
& \leq \delta,
\end{align*}
\]

which means that $W_x \geq (1 - (2 + \frac{1}{\xi^* - \varepsilon}) \varepsilon_x r) \frac{t_r}{T} W_x$, w.p. $1 - \delta$. This completes the proof.
G.2 Proof with \( \xi^* \) but without the Knowledge of Distribution

**Theorem 4.5.** Under Assumption 1, if \( \gamma_1 = \max \left( \frac{\xi^2}{\ln(K/\varepsilon)}, \frac{\xi}{1-\varepsilon} \right) \) such that \( \gamma_1 + \varepsilon \leq \xi^* \) and \( \gamma_1 = \frac{\xi}{1-\varepsilon} \), Algorithm A1 defined in Algorithm 4 achieves an objective value of at least \( (1 - O(\frac{\varepsilon^2}{\ln(K/\varepsilon)}))W_0 \) and satisfies the constraints w.p. 1 - \( \varepsilon \).

Now, we prove that, at each stage \( r \), Algorithm A1 returns a solution whose cumulative revenue is at least \( \frac{t_r}{T}(L_k + (\varepsilon - \frac{\varepsilon}{1 + \varepsilon})T\bar{a}_k)(1 + \varepsilon_{x,r}) \) and \( \frac{t_r}{T}(1 + \varepsilon_{x,r}) \) with probability at least 1 - \( \delta \).

**First step:** We design a surrogate Algorithm \( \tilde{P}_2 \) that allocates request \( j \) to channel \( i \) with probability \( x(\varepsilon)^{\gamma_j}_{ij} \).

**Lemma G.3.** In the \( r \)-th stage, if \( \gamma_1 = O(\frac{\varepsilon^2}{\ln(\frac{T}{\varepsilon})}) \) and \( Z^r \leq W_\varepsilon \), the Algorithm \( \tilde{P}_2 \) returns a solution satisfying the sum over all channels and stages \( \sum_{i=1}^{r+1} X_j^{P_2} \geq (1-\varepsilon_{x,r}) \frac{t_r}{T} Z^r \) and the sum over all channels and stages \( \sum_{i=1}^{r+1} X_j^{P_2} \leq \left\{ \frac{t_r}{T} \left( (1 + \varepsilon_{x,r}) L_k + (\varepsilon(1 + \varepsilon_{x,r}) - \varepsilon_{x,r}) T\bar{a}_k \right) \right\} \), \( (1 + \varepsilon_{x,r}) \left\{ \frac{t_r}{T} U_k \right\} \) w.p. 1 - \( \delta \), where \( \delta = \frac{\gamma_1}{\ln(K/\varepsilon)} \), \( \varepsilon_{x,r} = \frac{\varepsilon}{1 + \varepsilon} \), and \( \varepsilon_{x,r} = \sqrt{\frac{\varepsilon}{2(1 + \varepsilon)} \frac{\varepsilon}{\ln(K/\varepsilon)} \gamma_1} \).

**Proof.** Following the same technique in the proof of Lemma 4.1, we use Bernstein inequality to bound the probability of upper bound violation as follows

\[
P \left( \sum_{j=t_r+1}^{t_{r+1}} X_j^{\tilde{P}_2} \geq (1 + \varepsilon_{x,r}) \frac{t_r}{T} U_k \right) \leq \exp \left( -\frac{\frac{t_r}{T} \varepsilon^2 x_{x,r}}{2(1 + \frac{\varepsilon}{1 + \varepsilon}) \frac{T}{U_k}} \right) \leq \exp \left( -\frac{\frac{t_r}{T} \varepsilon^2 x_{x,r}}{2(1 + \frac{\varepsilon}{1 + \varepsilon}) \gamma_1} \right) \leq \frac{\delta}{2K + 1}.
\]

For the lower bound,

\[
P \left( \sum_{j=t_r+1}^{t_{r+1}} X_j^{\tilde{P}_2} \leq \frac{t_r}{T} (L_k + (\varepsilon - \frac{\varepsilon x_{x,r}}{1 + \varepsilon x_{x,r}}) T\bar{a}_k)(1 + \varepsilon_{x,r}) \right) = P \left( \sum_{j=t_r+1}^{t_{r+1}} (\bar{a}_k - X_j^{\tilde{P}_2}) \geq (1 + \varepsilon_{x,r}) \frac{t_r}{T} ((1 - \varepsilon) T\bar{a}_k - L_k) \right) \leq \exp \left( -\frac{\frac{t_r}{T} \varepsilon^2 x_{x,r}}{2(1 + \frac{\varepsilon}{1 + \varepsilon}) \frac{\bar{a}_k}{(1 - \varepsilon) T\bar{a}_k - L_k}} \right) \leq \exp \left( -\frac{\frac{t_r}{T} \varepsilon^2 x_{x,r}}{2(1 + \frac{\varepsilon}{1 + \varepsilon}) \gamma_1} \right) \leq \frac{\delta}{2K + 1}.
\]
For the accumulative revenue in r-th stage,

\[
P \left( \sum_{j=t_r+1}^{t_{r+1}} Y_j^P \leq (1 - \varepsilon_{y,r}) \frac{t_r}{T} Z^r \right)
\]

\[
= P \left( \sum_{j=t_r+1}^{t_{r+1}} (E[Y_j^P] - Y_j^P) \geq \frac{t_r}{T} (TE[Y_j^P] - (1 - \varepsilon_{y,r}) Z^r) \right)
\]

\[
\leq \exp \left( - \frac{\left( TE[Y_j^P] - (1 - \varepsilon_{y,r}) Z^r \right)^2}{T \left( 2T \sigma_1^2 + \frac{2}{3} \bar{w} (TE[Y_j^P] - (1 - \varepsilon_{y,r}) Z^r) \right)} \right)
\]

\[
= \exp \left( - \frac{t_r \left( \varepsilon_{y,r} Z^r \right)^2}{T \left( 2T \sigma_1^2 + \frac{2}{3} \bar{w} \right)} \right)
\]

\[
\leq \exp \left( - \frac{t_r \varepsilon_{y,r}^2}{2(1 + \frac{\delta}{2T}) T \bar{w}} \right)
\]

\[
\leq \frac{\delta}{2K + 1}
\]

where the second inequality follows \( \sigma_1^2 = Var(Y_j^P) \leq \bar{w} \sigma \) and \( TE[Y_j^P] - (1 - \varepsilon_{y,r}) Z^r \geq \varepsilon_{y,r} \bar{w} \), the third inequality follows \( Z^r \leq W_{e} = TE[Y_j^P] \) according to the condition of the lemma, and the last inequality from \( \varepsilon_{y,r} = \sqrt{\frac{4T \ln(\frac{2K+1}{\delta})}{Z^r}} \).

Therefore,

\[
\sum_{k \in K} P \left( \sum_{j=t_r+1}^{t_{r+1}} X_{jk} \notin \left[ \frac{t_r}{T} (1 + \varepsilon_{x,r}) L_k - (\varepsilon (1 + \varepsilon_{x,r}) - \varepsilon_{x,r}) T a_k, \frac{(1 + \varepsilon_{x,r}) t_r}{T} U_r \right] \right) + P \left( \sum_{j=t_r+1}^{t_{r+1}} Y_j^P \leq (1 - \varepsilon_{y,r}) \frac{t_r}{T} Z^r \right) \leq \delta.
\]

**Second Step:** Applying the same technique in the proof of Theorem 4.4, we derive a potential function to bound the failure probability of hybrid Algorithm \( \mathcal{A}_1 \mathcal{P}^{\mathcal{F}_1 - s} \) for request in stage r.

We begin with the moment generating function for the event that the consumed resource \( k \in K \)
is larger than \((1 + \varepsilon_{x,r})\frac{L}{T}U_k\). It can be shown that

\[
P \left( \sum_{j=1+t_r}^{s+t_r} X_{jk}^A + \sum_{j=s+t_r}^{t_r+1} X_{jk}^p \geq \frac{(1 + \varepsilon_{x,r})t_r U_k}{T} \right)
\]

\[
\leq \min_{t > 0} \mathbb{E} \left[ \exp \left( t \sum_{j=1+t_r}^{s+t_r} X_{jk}^A + \sum_{j=s+t_r}^{t_r+1} X_{jk}^p - \frac{(1 + \varepsilon_{x,r})t_r U_k}{T} \right) \right]
\]

\[
\leq \min_{t > 0} \mathbb{E} \left[ \exp \left( t \sum_{j=1+t_r}^{s+t_r} X_{jk}^A - \frac{(1 + \varepsilon_{x,r})sU_k}{T} + t \sum_{j=s+t_r}^{t_r+1} X_{jk}^p - \frac{(1 + \varepsilon_{x,r})(t_r - s)U_k}{T} \right) \right]
\]

\[
\leq \min_{t > 0} \mathbb{E} \left[ \phi_k^\ast(t) \exp \left( t \sum_{j=1+t_r}^{s+t_r} (X_{jk}^A - \mathbb{E}[X_{jk}^p]) + \sum_{j=s+t_r}^{t_r+1} X_{jk}^p - \frac{(1 + \varepsilon_{x,r})(t_r - s)U_k}{T} \right) \right]
\]

\[
\leq \min_{t > 0} \mathbb{E} \left[ \phi_k^\ast(t) \exp \left( (t_r - s) \frac{\text{Var}(X_{jk}^p)}{\bar{a}_k} (1 + \eta) \ln(1 + \eta) - \eta) \right) \right]
\]

\[
\leq \mathbb{E} \left[ \phi_k^\ast \left( \frac{\ln(1 + \varepsilon_{x,r})}{\bar{a}_k} \right) \exp \left( -\frac{(t_r - s)U_k}{T} \right) \right]
\]

where \(\phi_k^\ast(t) = \exp(t \sum_{j=1+t_r}^{s+t_r} X_{jk}^A - \frac{(1 + \varepsilon_{x,r})sU_k}{T})\). It should be noted that most of the above analysis is similar as the derivation of inequality (17) except that \(\text{Var}(X_{jk}^p) \leq \bar{a}_k \mathbb{E}[X_{jk}^p] \leq \bar{a}_k \frac{L}{T}\).

Next for the lower bound, we set \(Z_{jk}^p = \bar{a}_k - X_{jk}^p\) and \(Z_{jk}^A = \bar{a}_k - X_{jk}^A\), then we have

\[
P \left( \sum_{j=1+t_r}^{s+t_r} Z_{jk}^A + \sum_{j=s+t_r}^{t_r+1} Z_{jk}^p \geq \frac{(1 + \varepsilon_{x,r})t_r ((1 - \varepsilon)\bar{a}_k - L_k)}{T} \right)
\]

\[
\leq \min_{t > 0} \mathbb{E} \left[ \exp \left( t \sum_{j=1+t_r}^{s+t_r} Z_{jk}^A + \sum_{j=s+t_r}^{t_r+1} Z_{jk}^p - \frac{(1 + \varepsilon_{x,r})t_r ((1 - \varepsilon)\bar{a}_k - L_k)}{T} \right) \right]
\]

\[
\leq \min_{t > 0} \mathbb{E} \left[ \phi_k^\ast(t) \exp \left( t \sum_{j=1+t_r}^{s+t_r} (Z_{jk}^A - \mathbb{E}[Z_{jk}^p]) + \sum_{j=s+t_r}^{t_r+1} Z_{jk}^p - \frac{(1 + \varepsilon_{x,r})(1 - \varepsilon)\bar{a}_k - L_k)}{T} \right) \right]
\]

\[
\leq \min_{t > 0} \mathbb{E} \left[ \phi_k^\ast \left( \frac{\ln(1 + \varepsilon_{x,r})}{\bar{a}_k} \right) \exp \left( -\frac{(t_r - s)\sigma^2}{4\gamma_1} \right) \right]
\]

where the third inequality follows from setting \(\phi_k^\ast(t) = \exp \left( t \sum_{j=1+t_r}^{s+t_r} Z_{jk}^A - \frac{(1 + \varepsilon_{x,r})(1 - \varepsilon)\bar{a}_k - L_k)}{T} \right)\); the fifth inequality from \(\text{TE}[Z_{jk}^p] \leq (1 - \varepsilon)\bar{a}_k - L_k\) and \(\sigma^2 = \text{Var}(Z_{jk}^p) \leq \bar{a}_k \mathbb{E}[Z_{jk}^p]\).
Then we consider the revenue and have that
\[
P(\sum_{j=1+t_r}^{s+t_r} Y^A_j + \sum_{j=s+1+t_r}^{t_r+1} Y^F_j \leq (1 - \varepsilon_{y,r}) \frac{t_r}{T} Z^r) \\
\leq \min_{t_r > 0} \mathbb{E} \left[ \exp \left( t \left( (1 - \varepsilon_{y,r}) \frac{t_r}{T} Z^r - \sum_{j=1+t_r}^{s+t_r} Y^A_j - \sum_{j=s+1+t_r}^{t_r+1} Y^F_j \right) \right) \right] \\
\leq \min_{t_r > 0} \mathbb{E} \left[ \exp \left( t \left( \frac{s}{T} (1 - \varepsilon_{y,r}) Z^r - \sum_{j=1+t_r}^{s+t_r} Y^A_j \right) + t \left( \frac{t_r - s}{T} (1 - \varepsilon_{y,r}) Z^r - \sum_{j=s+1+t_r}^{t_r+1} Y^F_j \right) \right) \right] \\
\leq \min_{t_r > 0} \mathbb{E} \left[ \psi^s(t) \exp \left( t \sum_{j=s+1+t_r}^{t_r+1} (\mathbb{E}[Y^F_j] - Y^F_j) + \frac{t_r - s}{T} t \left( (1 - \varepsilon_{y,r}) Z^r - \mathbb{E}[Y^F_j] \right) \right) \right] \tag{41} \\
\leq \min_{t_r > 0} \mathbb{E} \left[ \psi^s(t) \exp \left( (t_r - s) \frac{\sigma^2_t}{w^2} \psi(t) - (t_r - s) t \bar{w} \right) \right] \\
\leq \mathbb{E} \left[ \psi^s \left( \frac{\ln(1 + \varepsilon_{y,r})}{\bar{w}} \right) \exp \left( - \frac{(t_r - s) \mathbb{E}[Y^F_j]}{\bar{w}} (1 + \eta) \ln(1 + \eta) - \eta \right) \right] \\
\leq \mathbb{E} \left[ \psi^s \left( \frac{\ln(1 + \varepsilon_{y,r})}{\bar{w}} \right) \exp \left( - \frac{t_r - s \varepsilon_{y,r}^2 Z^r}{4 w} \right) \right]
\]

where the third inequality follows from \( \psi^s(t) = \exp(t(1 - \varepsilon_{y,r}) Z^r - \sum_{j=1+t_r}^{s+t_r} Y^A_j)) \); the fourth inequality from \( Z^r \leq T \mathbb{E}[Y^F_j] \); the fifth inequality from \( \sigma^2_t = Var(Y^F_j) \leq \bar{w}\mathbb{E}[Y^F_j] \).

With the inequalities (39)-(41), we can bound the failure probability of hybrid Algorithm \( A_t^* \bar{P}^{r-s}_2 \) in stage \( r \) by \( \mathcal{F}_r(A_t^* \bar{P}^{r-s}_2) \) which is defined as
\[
\mathcal{F}_r(A_t^* \bar{P}^{r-s}_2) = \mathbb{E} \left[ \phi^s_k \left( \frac{\ln(1 + \varepsilon_{y,r})}{\bar{w}} \right) \exp \left( - \frac{t_r - s \varepsilon_{y,r}^2}{4 \gamma_1} \right) + \phi^s \left( \frac{\ln(1 + \varepsilon_{y,r})}{\bar{w}} \right) \exp \left( - \frac{t_r - s \varepsilon_{y,r}^2 Z^r}{4 \bar{w}} \right) \right] \\
\leq \mathbb{E} \left[ \phi^s_k \left( \frac{\ln(1 + \varepsilon_{y,r})}{\bar{w}} \right) \exp \left( - \frac{t_r - s \varepsilon_{y,r}^2}{4 \gamma_1} \right) + \phi^s \left( \frac{\ln(1 + \varepsilon_{y,r})}{\bar{w}} \right) \exp \left( - \frac{t_r - s \varepsilon_{y,r}^2 Z^r}{4 \bar{w}} \right) \right] \\
= \mathbb{E} \left[ \phi^s_k \left( \frac{\ln(1 + \varepsilon_{y,r})}{\bar{w}} \right) \exp \left( - \frac{t_r - s \varepsilon_{y,r}^2}{4 \gamma_1} \right) \right] \\
\leq \mathbb{E} \left[ \psi^s \left( \frac{\ln(1 + \varepsilon_{y,r})}{\bar{w}} \right) \exp \left( - \frac{t_r - s \varepsilon_{y,r}^2 Z^r}{4 \bar{w}} \right) \right]
\]

**Lemma G.4.** \( \mathcal{F}_r(A_t^* \bar{P}^{r-s}_2) = \mathcal{F}_r(A_t^{r-1} \bar{P}^{r-s+1}_2) \)

**Proof.** By the definition of \( \mathcal{F}_r(A_t^* \bar{P}^{r-s}_2) \), we have that
\[
\mathcal{F}_r(A_t^* \bar{P}^{r-s}_2) = \mathbb{E} \left[ \phi^s_k \left( \frac{\ln(1 + \varepsilon_{y,r})}{\bar{w}} \right) \exp \left( - \frac{t_r - s \varepsilon_{y,r}^2}{4 \gamma_1} \right) \right] \\
= \mathbb{E} \left[ \psi^s \left( \frac{\ln(1 + \varepsilon_{y,r})}{\bar{w}} \right) \exp \left( - \frac{t_r - s \varepsilon_{y,r}^2 Z^r}{4 \bar{w}} \right) \right]
\]

According to algorithm A in Algorithm 2, we allocate the \( s \)-th request to the channel \( i^* \) which
minimize the $F_r(A_1^s \tilde{P}_2^{t_r-s})$. Thus we have

$$
F_r(A_1^s \tilde{P}_2^{t_r-s}) \leq \mathbb{E} \left[ \phi_k^{-1} \left( \frac{\ln(1 + \varepsilon_{x,r})}{\tilde{a}_k} \right) \exp \left( \frac{\ln(1 + \varepsilon_{x,r})}{\tilde{a}_k} (X_{sk} - (1 + \varepsilon_{x,r}) \tilde{u}_k) \right) \exp \left( -\frac{t_r - s + \varepsilon_{x,r}^2}{4 \gamma_1} \right) \right]
$$

Similarly, we can show that

$$
F_r(A_1^s \tilde{P}_2^{t_r-s}) \leq \mathbb{E} \left[ \psi_k^{-1} \left( \frac{\ln(1 + \varepsilon_{y,r})}{\tilde{w}} \right) \exp \left( -\frac{t_r - s + \varepsilon_{y,r}^2}{4 \gamma_1} \right) \right]
$$

Following the similar analysis in the inequality (39)-(41), we can show that

$$
F_r(A_1^s \tilde{P}_2^{t_r-s}) \leq \mathbb{E} \left[ \phi_k^{-1} \left( \frac{\ln(1 + \varepsilon_{x,r})}{\tilde{a}_k} \right) \exp \left( -\frac{t_r - s + \varepsilon_{x,r}^2}{4 \gamma_1} \right) \right]
$$

which completes the proof. □

In Lemma 6.3, we have proven that $F_r(\tilde{P}_2^r) \leq \delta$ and we will show that $F_r(A_1^s \tilde{P}_2^{t_r-s}) \leq F_r(A_1^{s-1} \tilde{P}_2^{t_r-s+1})$ in Lemma 6.4. Thus we have $F_r(A_1^s) \leq F_r(\tilde{P}_2^r) \leq \delta$ by induction. Meanwhile, according to Theorem 6.3, we have that

$$(1 - (4 + \frac{2}{\xi^* - \varepsilon})\varepsilon_{x,r-1})W_r \leq Z_r \leq W_r$$

with probability $1 - 2\delta$. During the stage $r$, the Algorithm $A_1$ return a solution satisfying

$$
\sum_{j=t_r+1}^{t_r+1} X_{j,k}^A_1 \leq \frac{(1 + \varepsilon_{x,r})t_r \tilde{u}_k}{T}, \forall k \in \mathcal{K}
$$

$$
\sum_{j=t_r+1}^{t_r+1} (\tilde{a}_k - X_{j,k}^A_1) \leq \frac{(1 + \varepsilon_{x,r})t_r \tilde{a}_k}{T}((1 - \varepsilon)T \tilde{a}_k - L_k), \forall k \in \mathcal{K}
$$

$$
\sum_{j=t_r+1}^{t_r+1} Y_{j}^A_1 \geq (1 - \varepsilon_{y,r}) \frac{t_r}{T} Z_r \geq (1 - \varepsilon_{y,r}) \frac{t_r}{T} \exp \left( -\frac{t_r - s + \varepsilon_{y,r}^2}{4 \gamma_1} \right)
$$

with probability at least $1 - 3\delta$, since

$$
P \left( \left\{ \sum_{j=t_r+1}^{t_r+1} X_{j,k}^A_1 \in \left[ \frac{t_r}{T} \left( (1 + \varepsilon_{x,r})L_k - (1 + \varepsilon_{x,r} - \varepsilon_{x,r})T \tilde{a}_k \right), \frac{(1 + \varepsilon_{x,r})t_r}{T} \tilde{u}_k \right], \forall k \in \mathcal{K} \right\} \right)
$$

$$
\cap \left\{ \sum_{j=t_r+1}^{t_r+1} Y_{j}^A_1 \geq (1 - \varepsilon_{y,r}) \frac{t_r}{T} Z_r \right\} \cap \left\{ Z_r \in \left[ \frac{t_r W_r}{T} \left( 1 - (4 + \frac{1}{\xi^* - \varepsilon})\varepsilon_{x,r} \right), \frac{t_r W_r}{T} (1 + \frac{1}{\xi^* - \varepsilon})\varepsilon_{x,r}) \right] \right\}
$$

$$
\geq (1 - \delta)(1 - 2\delta) \geq 1 - 3\delta.
$$

Now considering all the stages, for the upper bound, we have

$$
\sum_{r=0}^{l-1} \sum_{j=t_r+1}^{t_r+1} X_{j,k}^A_1 \leq \sum_{r=0}^{l-1} \frac{(1 + \varepsilon_{x,r})t_r}{T} U_k \leq U_k.
$$

(43)
For the lower bound, we have
\[(1 - \varepsilon)T\bar{a}_k - \sum_{r=0}^{l-1} \sum_{j=t_r+1}^{t_{r+1}} X_{jk}^{A_1} = \sum_{r=0}^{l-1} \sum_{j=t_r+1}^{t_{r+1}} (\bar{a}_k - X_{jk}^{A_1}) \leq \sum_{r=0}^{l-1} \frac{(1 + \varepsilon_{x,r})L}{T}((1 - \varepsilon)T\bar{a}_k - L_k) \leq (1 - \varepsilon)T\bar{a}_k - L_k.\] (44)
which is equivalent to
\[\sum_{r=0}^{l-1} \sum_{j=t_r+1}^{t_{r+1}} X_{jk}^{A_1} \geq L_k.\]

And for the revenue, we have
\[\sum_{r=0}^{l-1} \sum_{j=t_r+1}^{t_{r+1}} Y_j^{A_1} \geq \sum_{r=0}^{l-1} (1 - \varepsilon_{y,r}) t_r (1 - (4 + \frac{2}{\varepsilon} - \varepsilon)\varepsilon_{x,r})W_0 \geq \sum_{r=0}^{l-1} (1 - \varepsilon_{y,r}) t_r (1 - (4 + \frac{2}{\varepsilon} - \varepsilon)\varepsilon_{x,r-1}) (1 - \varepsilon)W_0 \geq (1 - O(\frac{\varepsilon}{\varepsilon^*}))W_0.\] (45)

since $\delta = \frac{\varepsilon}{\varepsilon^*}$, the inequalities (43)-(45) hold with probability at least $1 - \varepsilon$, which completes the proof of Theorem 5.2.

**H Proof of Theorem 5.2**

**Theorem 5.2.** Under Assumption 4.3 if $\gamma_2 = \max(\frac{\hat{a}_k}{T\bar{a}_k - L_k}) = O(\frac{\varepsilon^2}{\ln(K/\varepsilon^*)})$, Algorithm 5 with $t_r$ i.i.d. requests outputs $\xi$ such that
\[\hat{\xi} \in [\xi^* - 4\varepsilon_{x,r}, \xi^*] \]
w.p. $1 - 2\delta$, where $\varepsilon_{x,r} = \sqrt{\frac{4\gamma_2 T \ln(K/\varepsilon^*)}{t_r}}$.

**Proof. RHS:** This side takes the same techniques as in Lemma 5.1. First, the dual of LP (3) in Section 3.1 is
\[
\min_{\alpha, \beta, \rho} \sum_{k \in K} \alpha_k U_k - \sum_{k \in K} \beta_k L_k + \sum_{j \in J} Tp_j \rho_j
\]
s.t.
\[
\sum_{k \in K} (\alpha_k - \beta_k) a_{ijk} + \rho_j \geq 0 \quad \forall i \in I, j \in J
\]
\[
\sum_{k \in K} T\bar{a}_k \beta_k = 1
\]
\[
\alpha_k, \beta_k, \rho_j \geq 0, k \in K, j \in J.
\]
We denote the optimal solution of LP (3) in Section 3.1 and LP (46) as $(x^*_{ij}, \xi^*)$ and $(\alpha^*_k, \beta^*_k, \rho^*_k)$ respectively.

According to the KKT conditions [Boyd et al., 2004], we have that
\[
\sum_{k \in K} (\alpha^*_k - \beta^*_k) a_{ijk} x^*_{ij} + \rho^*_j x^*_{ij} = 0
\]
\[
\sum_{k \in K} T\bar{a}_k \beta^*_k = 1
\]
\[
\rho^*_j (\sum_{i \in I} x^*_{ij} - 1) = 0
\]
\[
\alpha^*_k (\sum_{ij} Tp_j a_{ijk} x^*_{ij} - U_k) = 0
\]
\[
\beta^*_k (L_k + \xi^* T\bar{a}_k - \sum_{ij} Tp_j a_{ijk} x^*_{ij}) = 0
\]

which completes the proof.
Assumption 3, we know that
\[ \sum_{i,j} (a_{ijk} - \beta_k) = 1 \]
\[ \alpha_k, \beta_k, \rho_j \geq 0, k \in K, j \in S_r \]
where $S_r$ denotes the request set in stage $r$. Since $(\alpha^*_k, \beta^*_k, \rho^*_j)$ is a feasible solution to the LP (48), the solution $(\frac{T_{a_k^*}}{tr}, \frac{T_{a_k^*}}{tr}, \frac{T_{a_k^*}}{tr})$ is feasible for the dual of sample LP (49), we have that
\[
\hat{\xi} + 2\epsilon, r = \frac{T}{tr} \left( \sum_{k \in K} \alpha_k^* T_{U_k} - \sum_{k \in K} \beta_k^* T_{L_k} + \sum_{j \in S_r} \rho_j^* \right)
\]
\[
= \frac{T}{tr} \sum_{k \in K} \alpha_k^* \left( \frac{T_{U_k}}{tr} - \sum_{i,j} a_{ijk}^* \right) + \frac{T}{tr} \sum_{k \in K} \beta_k^* \left( \sum_{j \in S_r, i \in I} a_{ijk}^* \right)
\]
\[
+ \frac{T}{tr} \sum_{j \in S_r} \rho_j^* \left( \sum_{i \in I, k \in K} (\alpha_k^* - \beta_k^*) a_{ijk}^* \right)
\]
where the final equality follows from the KKT conditions (47), i.e. $\sum_{k \in K} (\alpha_k^* - \beta_k^*) a_{ijk}^* + \rho_j^* a_{ijk}^* = 0$ and $\rho_j^* \sum_{i \in I} a_{ijk}^* = 0$, so that $\sum_{i \in I, k \in K} (\alpha_k^* - \beta_k^*) a_{ijk}^* + \sum_{i \in I} \rho_j^* a_{ijk}^* = 0$.

For those $k$ such that $L_k + \xi^* T\bar{a}_k < \sum_{i \in I, j \in J} Tp_j a_{ijk}^* < U_k$, we know that they have no effect to $\hat{\xi}$ following the complementary slackness in (47). For part 1, we only consider the resource $k$ making $\sum_{i \in I, j \in J} Tp_j a_{ijk}^* = U_k$. By Lemma B.1, it is easy to get that
\[
P\left( \sum_{j \in [r], i \in I} a_{ijk}^* \leq (1 - \epsilon, r) \frac{T_{U_k}}{tr} \right) \leq \exp \left( -\frac{T_{U_k}}{2(1 + \epsilon, r) a_{ijk}^*} \right) \]
where $E(\sum_{i \in I} a_{ijk}^*) = \frac{U_k}{tr}, \forall j \in J$.

Similarly, for part 2, we only consider the constraints $k$ making $\sum_{i \in I, j \in J} Tp_j a_{ijk}^* = L_k + \xi^* T\bar{a}_k$. Before that, we redefine the r.v. $Y_{jk} = (1 + \xi^*) a_{jk}^* - \sum_{i \in I} a_{ijk}^*$. Since $\xi^* \in [0, 1]$ from Assumption 3, we know that $|Y_{jk}| \leq (1 + \xi^*) a_{jk}$ and $E(Y_{jk}) = \frac{T\bar{a}_k - L_k}{2a_{jk}}$. Therefore, by Lemma B.1,
\[
P\left( \sum_{j \in S_r} Y_{jk} \leq (1 - \epsilon, r) \frac{T_{a_k^*} - L_k}{tr} \right) \leq \exp \left( -\frac{T_{a_k^*} - L_k}{2(1 + \epsilon, r) \frac{(1 + \xi^*) a_{jk}}{tr}} \right) \]
Since $\gamma_2 = \max(\frac{a_{jk}}{a_{jk^*}}, \frac{a_{jk}}{a_{jk^*}}) = O\left(\frac{a_{jk^*}^2}{a_{jk}}\right)$, and both lower and upper bound are achieved only if $U_k = T\bar{a}_k - L_k$, we have that
\[
\sum_{j \in S_r, i \in I} a_{ijk}^* \geq (1 - \epsilon, r) \frac{T_{U_k}}{tr}
\]
\[
\sum_{j \in S_r} ((1 + \xi^*) a_{jk} - \sum_{i \in I} a_{ijk}^*) \geq (1 - \epsilon, r) \frac{T_{T\bar{a}_k - L_k}}{tr}
\]
\[
\begin{align*}
&\text{1} + \text{2} \\
&= \frac{T}{t_r} \left( \sum_{k \in K} \alpha_k^* \left( \frac{t_r}{T} U_k - \sum_{j \in S_r, i \in \mathcal{I}} a_{ijk} x^*_i \right) + \sum_{k \in K} \beta_k^* \left( \sum_{j \in S_r, i \in \mathcal{I}} a_{ijk} x^*_i - \frac{t_r}{T} L_k \right) \right) \\
&= \frac{T}{t_r} \left( \sum_{k \in K} \alpha_k^* \left( \frac{t_r}{T} U_k - \sum_{j \in S_r, i \in \mathcal{I}} a_{ijk} x^*_i \right) + \sum_{k \in K} \beta_k^* \left( \frac{t_r}{T} (T \bar{a}_k - L_k) - \sum_{j \in S_r, i \in \mathcal{I}} ((1 + \xi^*) \hat{a}_k - \sum_{i \in \mathcal{I}} a_{ijk} x^*_i) \right) \right) \\
&\quad + \sum_{k \in K} \beta_k^* \xi^* t_r \hat{a}_k \\
&\leq \frac{T}{t_r} \left( \epsilon_{x,r} \sum_{k \in K} \alpha_k^* t_r U_k + \epsilon_{x,r} \sum_{k \in K} \beta_k^* \frac{t_r}{T} (T \bar{a}_k - L_k) + \sum_{k \in K} \beta_k^* \xi^* t_r \hat{a}_k \right) \\
&= \frac{T}{t_r} \left( \epsilon_{x,r} \sum_{k \in K} \alpha_k^* t_r U_k - \sum_{k \in K} \beta_k^* \frac{t_r}{T} L_k \right) + (\epsilon_{x,r} + \xi^*) \sum_{k \in K} \beta_k^* t_r \hat{a}_k \\
&\leq \frac{T}{t_r} \left( \epsilon_{x,r} \xi^* t_r U_k + (\epsilon_{x,r} + \xi^*) \frac{t_r}{T} \right) \\
&= \xi^* + (\xi^* + 1) \epsilon_{x,r} \\
&\leq \xi^* + 2x_r, \\
\end{align*}
\]

where the first inequality from \( \sum_{j \in S_r, i \in \mathcal{I}} a_{ijk} x^*_i \geq (1 - \epsilon_{x,r}) \frac{t_r}{T} U_k \) and \( \sum_{j \in S_r, i \in \mathcal{I}} (1 + \xi^*) \hat{a}_k - \sum_{i \in \mathcal{I}} a_{ijk} x^*_i \geq (1 - \epsilon_{x,r}) \frac{t_r}{T} (T \bar{a}_k - L_k) \); the second inequality from \( \sum_{k \in K} \beta_k^* T \bar{a}_k = 1 \) and \( \xi^* = \sum_{k \in K} \alpha_k^* U_k - \sum_{k \in K} \beta_k^* L_k \); and the last inequality follows from the fact \( \xi^* \leq 1 \).

**LHS:** We design an algorithm \( \tilde{P}_3 \) by allocating request \( j \) to channel \( i \) with probability \((1 - \epsilon_{x,r}) x^*_i \), where \((x^*_i, \xi^*)\) is the optimal solution for LP \( \square \). Following the very similar proofs in Section \( \square \), and letting \( \gamma_2 = \max \left( \frac{\gamma}{t_r}, \frac{\alpha_k^*}{T \bar{a}_k - L_k} \right), \epsilon_{x,r} = \sqrt{\frac{4\gamma^2 T \ln(K)}{t_r}}, \) we have that

\[
P \left( \sum_{j=1}^{t_r} X_{jk} \geq \frac{t_r}{T} U_k \right) \leq \exp \left( - \frac{t_r \epsilon_{x,r}^2}{2(1 - \frac{4}{3} \epsilon_{x,r}) \frac{t_\delta}{T \bar{a}_k - L_k}} \right) \leq \frac{\delta}{2K}
\]

where the second inequality follows from the definition of \( t_r \) and \( \gamma_2 = O \left( \frac{\delta^2}{T \bar{a}_k - L_k} \right) \), which result in \( \epsilon_{x,r} < 1 \). Defining \( Y_{jk} = (1 - \epsilon_{x,r})(1 + \xi^*) \hat{a}_k - X_{jk} \tilde{P}_3 \), we have that \( E(Y_{jk}) \) \( \leq (1 - \epsilon_{x,r})(1 + \xi^*) \hat{a}_k \) and \( |Y_{jk}| \leq (1 - \epsilon_{x,r})(1 + \xi^*) \hat{a}_k \), since \( |X_{jk}| \leq (1 - \epsilon_{x,r}) \hat{a}_k \). Therefore, we have

\[
P \left( \sum_{j=1}^{t_r} Y_{jk} \geq \frac{t_r}{T} (T \bar{a}_k - L_k) \right) \leq \exp \left( - \frac{t_r \epsilon_{x,r}^2}{2(1 - \frac{4}{3} \epsilon_{x,r})(1 + \xi^*) \hat{a}_k} \right) \leq \frac{\delta}{2K}
\]

Thus, with probability at least \( 1 - \delta \), we could find a solution whose consumed resource for each \( k \) is in \( \left[ \frac{t_r}{T}(L_k + (1 + \xi^*) \bar{a}_k), \frac{t_r}{T} U_k \right] \) by Algorithm \( \tilde{P}_3 \). According to the definition of \( \xi \) in Algorithm \( \square \), we have that \( \tilde{\xi} \geq \xi^* - 4 \epsilon_{x,r} \).

In conclusion, we have that \( \xi^* - 4 \epsilon_{x,r} \leq \tilde{\xi} \leq \xi^* \), w.p. \( 1 - 2\delta \).

Now \( \tilde{\xi} \) can be viewed as an good estimate for \( \xi^* \) from 0, if we have enough data.
I Proof of Theorem 5.3

Proof. We mainly consider two events, namely,

\[ G_1 = \left\{ \sum_{j=1}^{T} Y_{jA_2} \geq (1 - O(\frac{\varepsilon}{\xi^* - 4\sqrt{\varepsilon} - \varepsilon}))W_0, \sum_{j=1}^{T} X_{jk}^A \in [L_k, U_k] \forall k \in K \right\}, \]

\[ G_2 = \left\{ \xi^* - 4\sqrt{\varepsilon} \leq \hat{\xi}_0 \leq \xi^* \right\}. \]

Step 1.
When initializing Algorithm 6, we use the first \( \varepsilon T \) incoming requests to estimate the optimal measure of feasibility \( \hat{\xi}_0 \). From the Theorem 5.2, we have that \( P(\{\xi^* - 4\sqrt{\varepsilon} \leq \hat{\xi}_0 \leq \xi^*\}) \geq 1 - 2\delta \), choosing \( \delta = \frac{\varepsilon^3}{3^{\frac{1}{3}+2}} \).

Step 2.
We investigate the conditional event \( G_1|G_2 \). Under the assumption \( \sqrt{\varepsilon_1} - \sqrt{\varepsilon_2} + 4\sqrt{\varepsilon} + \varepsilon \leq \xi^* \), if \( \xi^* - 4\sqrt{\varepsilon} \leq \hat{\xi}_0 \), we have \( \hat{\xi}_0 \geq \xi^* - 4\sqrt{\varepsilon} \geq \frac{\varepsilon}{1 - \sqrt{\varepsilon}} + \varepsilon \). Besides, we know that the domain

\[ L_k + \hat{\xi}_0 T a_k \leq \sum_{i \in \mathcal{I}, j \in \mathcal{J}} T p_{a_{ijk}} x_{ij} \leq U_k, \forall k \in K \]

\[ \sum_{i \in \mathcal{I}} x_{ij} \leq 1, \forall j \in \mathcal{J} \]

\[ x_{ij} \geq 0, \forall i \in \mathcal{I}, j \in \mathcal{J}, \]

is feasible under the assumption.

When \( \gamma_3 = O\left(\frac{\varepsilon^2}{\ln(\frac{K}{\varepsilon})}\right) \), according to the Theorem 4.5, we have

\[ P\left( \left\{ \sum_{j=1}^{T} Y_{jA_2} \geq (1 - O(\frac{\varepsilon}{\xi^* - \varepsilon}))W_0 \right\} \cap \left\{ \sum_{j=1}^{T} X_{jk}^A \in [L_k, U_k] \right\} \right) \geq 1 - 3\delta, \]

where \( l = \log_2 \left(\frac{1}{l} \right) \). Due to \( \hat{\xi}_0 \geq \xi^* - 4\sqrt{\varepsilon} \), we also could derive that \( P(G_1|G_2) \geq 1 - 3\delta \).

Step 3.
Now we can verify that

\[ P(G_1^c) = P(G_1^c|G_2)P(G_2) + P(G_1^c|G_2^c)P(G_2^c) \]

\[ \leq P(G_1^c|G_2) + P(G_2^c) \]

\[ \leq 3\delta + 2\delta = \varepsilon. \]

Therefore, \( P(G_1) \geq 1 - \varepsilon \), if \( \varepsilon \geq 0 \) and \( \tau_1 = \frac{\varepsilon}{1 - \sqrt{\varepsilon}} \) such that \( \tau_1 + 4\sqrt{\varepsilon} + \varepsilon \leq \xi^* \) and \( \gamma_3 = O\left(\frac{\varepsilon^2}{\ln(\frac{1}{\varepsilon})}\right) \leq \max\left(\frac{\varepsilon}{\tau_1 (1 - \tau_1 \bar{a}_k - \tau_1 \bar{w}_k + \tau_1)}, \frac{\varepsilon}{W_0 + \bar{w}_k}\right). \]

\[ \square \]