Some Inequalities for the Generalized Parton Distribution $E(x, 0, t)$

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We discuss some constraints on the $x$ and $t$-dependence of $E(x, 0, t)$ that arise from positivity bounds in the impact parameter representation. In addition, we show that $E(x, 0, 0)$ for the nucleon vanishes for $x \to 1$ at least as rapidly as $(1-x)^4$. Finally we provide an inequality that limits the contribution from $E$ to the angular momentum sum rule.

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I. INTRODUCTION

Generalized parton distributions (GPDs) are hybrid quantities that have features in common both with form factors and with the usual parton distribution functions (PDFs). They are defined as non-forward $(p \neq p')$ matrix elements of the same operator $\hat{O} \equiv \int \frac{dx}{2\pi} e^{i\vec{x} \cdot \vec{p}} \bar{q}(\vec{x}) \gamma^+ q(\vec{x}) \gamma^2$ whose forward matrix elements (i.e. expectation value) yield the usual parton distributions

$$\langle p' | \hat{O} | p \rangle = H(x, \xi, \Delta^2) \bar{u}(p') \gamma^+ u(p)$$

$$+ E(x, \xi, \Delta^2) \bar{u}(p') \frac{i\sigma^{+\nu} \Delta^\nu}{2M} u(p).$$

Over the last few years there has been a strong interest in GPDs and meanwhile many observables have been identified that can be linked to them (for a recent review see Ref. [3]). One of the most interesting observables that can be linked to GPDs is a quantity that has identified been identified with the total (spin plus orbital) angular momentum carried by the quarks in the nucleon

$$\langle J_q \rangle = \frac{1}{2} \int_0^1 dx \int_0^1 dx \left[ H_q(x, 0, 0) + E_q(x, 0, 0) \right],$$

where the subscript $q$ indicates that Eq. [2] holds for each quark flavor separately. Of course, $H_q(x, 0, 0) = q(x)$ is well known for the relevant values of $x$, but little is known about $E_q(x, 0, 0)$.

Although GPDs can be probed in Compton scattering experiments, they usually enter experimentally measurable cross sections only in terms of some integrals and therefore there may be some difficulties in unambiguously extracting GPDs from Compton scattering data. It is therefore desirable to use as many model-independent theoretical constraints as possible to help pin down the data on GPDs.

One class of such constraints are positivity constraints, where one uses the fact that any state in a Hilbert space has a non-negative norm. By using carefully constructed states one can thus derive inequalities relating physical observables.

II. POSITIVITY CONSTRAINTS IN IMPACT PARAMETER SPACE

In the case of GPDs, the impact parameter space presentation turns out to be very useful, since GPDs (for $\xi = 0$) become diagonal in that basis. Par-ton distributions in impact parameter space are related to GPDs via a simple Fourier transform (throughout this paper, we use a notation where parton distributions in impact parameter space are denoted by script letters)

$$\mathcal{H}(x, b\perp) = \int \frac{d^2\Delta^\perp}{(2\pi)^2} H(x, 0, -\Delta^\perp_1) e^{ib\perp \cdot \Delta^\perp}$$

$$\tilde{\mathcal{H}}(x, b\perp) = \int \frac{d^2\Delta^\perp}{(2\pi)^2} \tilde{H}(x, 0, -\Delta^\perp_1) e^{ib\perp \cdot \Delta^\perp}$$

$$\mathcal{E}(x, b\perp) = \int \frac{d^2\Delta^\perp}{(2\pi)^2} E(x, 0, -\Delta^\perp_1) e^{ib\perp \cdot \Delta^\perp}.$$

In Ref. [11], it was observed that the probabilistic interpretation of parton distributions in impact parameter space implies the positivity bound

$$\frac{1}{2M} \| \nabla_{b\perp} \mathcal{E}(x, b\perp) \| \leq \mathcal{H}(x, b\perp).$$

Here $x > 0$; for $x < 0$ a similar inequality with $\mathcal{H} \to -\mathcal{H}$ holds. In Ref. [11], an even stronger bound

$$\frac{1}{(2M)^2} \| \nabla_{b\perp} \mathcal{E}(x, b\perp) \|^2 \leq \| \mathcal{H}(x, b\perp) \|^2 - \| \tilde{\mathcal{H}}(x, b\perp) \|^2$$

was derived. Although Eqs. [4] and [5] are rigorous, their practical use has been rather limited so far since they are relations between several unknown quantities. In this paper, we will manipulate these positivity bounds into a form, where they should be more directly applicable to phenomenology. For this purpose we first take Eq. [4] and simply integrate over impact parameter. Since the inequality is preserved under this operation, and since the norm is invariant under Fourier transformation (e.g. $\int d^2b\perp |\mathcal{H}(x, b\perp)|^2 = \int d^2\Delta^\perp |H(x, 0, -\Delta^\perp_1)|^2$), one immediately finds

$$\int_{-\infty}^0 dt \frac{|t|}{(2M)^2} |E(x, 0, t)|^2 \leq \int_{-\infty}^0 dt \left\{ |H(x, 0, t)|^2 - |\tilde{H}(x, 0, t)|^2 \right\}.$$
Similar expressions can be derived by repeating this procedure with additional powers of $|b_\perp|$ in the integrand.

While this result immediately deals with the GPDs rather than parton distributions in impact parameter space, its usefulness is still limited by the fact that it involves 3 unknown functions and therefore leaves too much room for model dependence. It would be much more useful if we had constraints relating $E(x, 0, t)$ to some known functions, such as the forward PDFs $q(x)$ and $\Delta q(x)$.

Deriving such relations will be the main goal in the rest of this paper.

For this purpose, we first introduce impact parameter dependent parton distributions for quarks with spins parallel (anti-parallel) to the nucleon spin. Combining Eqs. (9) and (10) yields

$$
\mathcal{H}_\perp(x, b_\perp) = \frac{1}{2} \left[ \mathcal{H}(x, b_\perp) \pm \mathcal{H}(x, -b_\perp) \right].
$$

In terms of $\mathcal{H}_\pm$, Eq. (11) can be expressed in the form

$$
\frac{1}{2M} |\nabla_{b_\perp} \mathcal{E}(x, b_\perp)| \leq 2 \sqrt{\mathcal{H}_+(x, b_\perp) \mathcal{H}_-(x, b_\perp)}.
$$

Integrating the l.h.s. of Eq. (8) over the transverse plane yields [rotational invariance implies $\mathcal{E}(x, b_\perp) = \mathcal{E}(x, b)$]

$$
\frac{1}{2M} \int d^2 b_\perp |\nabla_{b_\perp} \mathcal{E}(x, b_\perp)| = \frac{\pi}{M} \int_0^\infty d b |\partial_b \mathcal{E}(x, b)|
$$

$$
\geq \frac{\pi}{M} \int_0^\infty d b E(x, b)
$$

$$
\geq \frac{1}{4\pi M} \int d^2 \Delta_\perp \frac{E(x, 0, -\Delta_\perp^2)}{|\Delta_\perp|}
$$

$$
= \frac{1}{4M} \int_{-\infty}^0 dt \frac{E(x, 0, t)}{\sqrt{-t}}.
$$

where we used $\int d^2 b_\perp e^{-ib_\perp \cdot \Delta_\perp} = \frac{2\pi}{|\Delta_\perp|}$.

When integrating the r.h.s. of Eq. (8), we use the Schwarz inequality to obtain

$$
2 \int d^2 b_\perp \sqrt{\mathcal{H}_+(x, b_\perp) \mathcal{H}_-(x, b_\perp)}
$$

$$
\leq 2 \sqrt{\left( \int d^2 b_\perp \mathcal{H}_+(x, b_\perp) \right) \left( \int d^2 b_\perp \mathcal{H}_-(x, b_\perp) \right)}
$$

$$
= 2 \sqrt{q_+(x)q_-(x)},
$$

where $q_\pm(x) \equiv \frac{1}{2} (q(x) \pm \Delta q(x))$ are the parton distribution for quarks with spin parallel (anti-parallel) to the nucleon spin. Combining Eqs. (10) and (11) yields

$$
\frac{1}{8M} \int_{-\infty}^0 dt \frac{E(x, 0, t)}{\sqrt{-t}} < \sqrt{q_+(x)q_-(x)},
$$

which is one of the results of this paper. Like Eqs. (11) and (10), this result holds for each quark flavor.

While Eq. (11) is weaker than our starting point (9), it may still be of more use at this point because it contains only one unknown quantity $\mathcal{E}(x, 0, t)$ and relates it to $q_\pm(x)$, which are much better known from parton phenomenology.

Although the r.h.s. of Eq. (11) involves only known quantities, the l.h.s. still involves an integral. For practical applications it may be more useful to have an inequality that contains the unintegrated GPD $E$. For this purpose we now multiply Eq. (3) by $|b_\perp|$ and integrate.

For the l.h.s. we find

$$
\frac{1}{2M} \int d^2 b_\perp |b_\perp| |\nabla_{b_\perp} \mathcal{E}(x, b_\perp)| = \frac{\pi}{M} \int_0^\infty d b 2 b |\partial_b \mathcal{E}(x, b)|
$$

$$
\geq \frac{2\pi}{M} \int_0^\infty d b E(x, b)
$$

$$
= \frac{1}{M} |E(x, 0, 0)|,
$$

which involves $E(x, 0, 0)$, i.e., the quantity entering the angular momentum sum rule (2), directly.

On the r.h.s. one can invoke the Schwarz inequality in different ways, and we choose to apply it in the form

$$
2 \int d^2 b_\perp \sqrt{b_\perp^2 \mathcal{H}_+(x, b_\perp) \mathcal{H}_-(x, b_\perp)}
$$

$$
\leq 2 \sqrt{\left( \int d^2 b_\perp b_\perp^2 \mathcal{H}_+(x, b_\perp) \right) \left( \int d^2 b_\perp^2 \mathcal{H}_-(x, b_\perp) \right)}
$$

$$
= 2 \sqrt{4 \frac{d}{dt} H_+(x, 0, t)} q_-(x),
$$

where $H_\pm \equiv \frac{1}{2} (H \pm \hat{H})$. Combining Eqs. (10), (12), and (13) we thus obtain

$$
\frac{1}{4M} |E(x, 0, 0)| \leq \sqrt{q_-(x) \frac{d}{dt} H_+(x, 0, t)} \bigg|_{t=0}.
$$

While we do not know the slope of $H_+(x, 0, t)$, we know some general features. In particular, one expects that the transverse width of GPDs vanishes as $x \to 1$.

$$
\frac{d}{dt} H_+(x, 0, t) \bigg|_{t=0} \sim (1-x)^2 \quad \text{for} \quad x \to 1.
$$

The reason for the vanishing of the transverse width for $x \to 1$ is that the variable conjugate to $\Delta_\perp$ is the impact parameter $b_\perp$, which is measured w.r.t. the $\perp$ center of momentum. The latter is related to the distance from the active quark to the center of momentum of the spectators (which we denote by $B_\perp$ via the relation $b_\perp = (1 - x) B_\perp$). The distance $|B_\perp|$ between the active quark and the spectator should be roughly equal to the size of the nucleon or less. Being rescaled by a factor $(1 - x)$, the typical scale for $b_\perp$ is therefore only $(1-x)$ times that size, which leads to Eq. (15).

Making use of Eq. (15) in Eq. (14) thus yields

$$
\frac{d}{dt} H_+(x, 0, t) \bigg|_{t=0} \sim (1-x)^2 + n_+ \quad \text{for} \quad x \to 1
$$

(16)
where $n_{\pm}$ characterizes the behavior of $q_{\pm}(x)$ for $x \to 1$

$$q_{\pm}(x) \sim (1-x)^{n_{\pm}} \text{ for } x \to 1. \quad (17)$$

For example, if $n_{+} = 3$ and $n_{-} = 5$ (based on hadron helicity conservation [12]), then

$$E(x,0,0) \sim (1-x)^{1+\frac{n_{-}+n_{+}}{2}} = (1-x)^{5} \quad (18)$$

for $x \to 1$. Even if there is a small contribution to the negative helicity distribution $q_{-}(x)$ that vanishes with the same power as the positive helicity distribution $q_{+}(x)$, i.e. if $n_{+} = n_{-} = 3$, then $E(x,0,0)$ would still behave like $(1-x)^4$ and therefore vanish faster than $H(x,0,0)$ as $x \to 1$. In either case we find

$$\lim_{x \to 1} \frac{E(x,0,0)}{H(x,0,0)} = 0. \quad (19)$$

For applications to the angular momentum sum rule [2], we can also try to convert Eq. (14) into a statement about the 2nd moment of $E$. Upon multiplying Eq. (14) by $|x|$ and integrating from $-1$ to 1 (antiquarks correspond to $x < 0$), one finds

$$\frac{1}{4M} \left| \int dx E(x,0,0)x \right| \leq \frac{1}{4M} \int dx |E(x,0,0)||x|$$

$$\leq \int dx \left| \frac{d}{dt} x H_{+}(x,0,0) \right|_{t=0}$$

$$\left( \int dx x q_{-}(x) \right) \left( \int dx \frac{d}{dt} x H_{+}(x',0,0) \right|_{t=0}$$

$$\left( \int dx x q_{+}(x) \right) \left( \int dx' \frac{d}{dt} x' H_{+}(x',0,0) \right|_{t=0} \right) \right),$$

which contains only one unknown on the r.h.s., namely the slope of the second moment of $H_{+}$. To illustrate that this inequality may provide some useful bounds, let us insert some rough figures: not distinguishing between different flavors (i.e. implicitly adding all quark flavors) we approximate:

$$\int dx x q_{-}(x) \approx \int dx x q_{+}(x) \approx \frac{1}{4}$$

and

$$\int dx' \frac{d}{dt} x' H_{+}(x',0,0)|_{t=0} = \int dx x q_{+}(x) \frac{R_{+}^2}{x^2} \approx \frac{1}{4} \frac{R_{+}^2}{x^2},$$

where $R_{+}^2$ is the rms-radius corresponding to $\int dx x H_{+}$.

We do not know the value of $R_{+}$ but it should be on the order of the rms radius of the nucleon. In fact, $R_{+}$ should be smaller than that since the slope of the form factor should decrease for increasing $x$-moments (see the discussion following Eq. (16) and also Ref. [14]), i.e. we approximate $R_{+} \approx 0.5 fm$. Inserting these rough figures, we find $|\int dx E(x,0,0)| \leq \frac{R_{+}^2}{\sqrt{6}} \approx 1$. Although this is not a very strong constraint, a better estimate may be available once the slope of the second moment of $H$ is known for different quark flavors.

### III. SUMMARY

We started from positivity constraints for parton distributions in impact parameter space [5] and derived several new positivity constraints on GPDs [6, 7, 8, 9]. Although the new constraints are weaker than the standing inequality [3], the new inequalities may be more useful since they can be applied directly in momentum space, where the data is obtained. One of the new inequalities [7] relates $\int d^{4}p'E(x,0,t)$ directly to the (forward) parton distributions $q(x)$ and $\Delta q(x)$ and therefore provides a direct constraint on the shape of $E$.

The third inequality that we derived [10] is a bound on $E(x,0,0)$. Unfortunately, it involves the slope of $H(x,0,t)$ for $t = 0$, which is currently not known.

If hadron helicity conservation (HHC) holds and $q_{-}(x) \sim (1-x)^{3}$ for quarks with spin antiparallel to the nucleon spin then $E(x,0,0) \sim (1-x)^{3}$ as $x \to 1$ for valence quarks in the nucleon. Even if HHC is violated, and $q_{-}(x)$ has a small component that vanishes only like $(1-x)^{3}$ then $E$ still vanishes faster than the leading valence distributions, i.e. $E(x,0,0) \sim (1-x)^{4}$. This is a consequence of the fact that the positivity constraints in impact parameter representation [11] and [12] involve $\bar{V}_{bc}$ together with the fact that the $\perp$ width of GPDs goes to zero as $x \to 1$. Knowing that $E(x,0,0)$ vanishes faster than $q(x)$ near be useful in estimating the contribution from $E$ to the angular momentum sum rule.

Finally we derived an inequality that can be used to constrain the second moment of $E(x,0,0)$. The only unknown in this inequality is the rms-radius for the second moment of $H$.

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