Implicit regularization beyond one loop order: scalar field theories

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Implicit regularization (IR) has been shown as an useful momentum space tool for perturbative calculations in dimension specific theories, such as chiral gauge, topological and supersymmetric quantum field theoretical models at one loop level. In this paper, we aim at generalizing systematically IR to be applicable beyond one loop order. We use a scalar field theory as an example and pave the way for the extension to quantum field theories which are richer from the symmetry content viewpoint. Particularly, we show that a natural (minimal) renormalization scheme emerges, in which the infinities displayed in terms of integrals in one internal momentum are subtracted, whereas infrared and ultraviolet modes do not mix and therefore leave no room for ambiguities. A systematic cancelation of the infrared divergences at any loop order takes place between the ultraviolet finite and divergent parts of the amplitude for non-exceptional momenta leaving, as a byproduct, a renormalization group scale.

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I. INTRODUCTION

The problem of defining a consistent perturbative regularization of dimension specific quantum field theories is by itself of theoretical interest. Chiral, topological and supersymmetric gauge theories belong to this class of theories. The very concept of chirality is highly dimension dependent as there are no Weyl fermions in odd dimensions. The $\gamma_5$ matrix which enters the chiral projection operators has no straightforward generalization to arbitrary dimensions. Hence, care must be exercised in applying dimensional regularization methods. Yet the standard model as a full theory is actually anomaly free, from the practical standpoint perturbative calculations are necessary to test the standard model and extensions against the experiments. The appearance of spurious anomalies, which spoils renormalizability by violating chiral symmetry, demands the addition of symmetry restoring counterterms, order by order, in perturbation theory. This renders practical calculations very tedious. Besides dimensional regularization (DREG) explicitly breaks supersymmetry because the number of degrees of freedom of gauge bosons and gauginos does not match for $d \neq 4$. Therefore, an invariant regularization scheme becomes important in this case. An alternative approach based on algebraic renormalization, which discards an invariant regularization scheme, is based on the construction of Slavnov-Taylor identities which encodes all the essential symmetries of the model in consideration, gauge and supersymmetry included. Such constraints, in turn, deliver unambiguously the (non-invariant) counterterms to be added in order to heal the symmetry breakdowns. The main drawback of this approach is that it is much simpler to work within an invariant scheme which hitherto is missing at least to all orders in perturbation theory. Moreover, it is not a priori obvious that, in general, there are no supersymmetry anomalies \cite{1}, although the evidences quoted so far points to supersymmetry being a full symmetry of the quantum theory \cite{2}.

From the phenomenological viewpoint, it is expected that the minimal supersymmetric standard model (MSSM) can be realized at weak scale in the experiments at LHC around 1 TeV. This makes quantum effects important. That is to say, besides the direct detection of susy particles, supersymmetry can also be probed via the virtual effects of additional particles such as the electroweak precision observables within the MSSM \cite{3}. Examples are the correction for the $W$ boson mass, the effective
rely on dimensional extension of space-time dimensions, which, like Differential Renormalization, does not require space regularization independent framework, constructed \[7\]. Nonetheless, recent works have demonstrated that its validity can be extended beyond one loop order by using the quantum action principle, although it remains to be circumvented some problems related to QCD mass factorization \[\beta\]. From the pragmatical point of view, DRED is largely used to unravel supersymmetric extensions of the standard model \[\delta\]. However, it is important to note that in DRED it is unclear whether or to what extent the supersymmetry is actually preserved. It is only known that some susy relations are satisfied at the one and two loop level but even at one loop the checks do not exhaust all the Green’s functions that could be affected by a susy breaking.

An important coordinate space regularization independent framework, which operates in the physical dimension of the model, is Differential Renormalization (DR). DR basically consists of replacing coordinate-space amplitudes that are too singular to have a Fourier transform by derivatives of less singular ones as well as an integration by parts prescription. It has been proved to be a symmetry preserving scheme \[\gamma\]. This is achieved by both constraining the number of independent scales which result from each divergence and applying a set of operational rules in the introduction of basic functions (Constrained-DR), which assume the same values as the original divergent ones except for coincident points (short distance behaviour). Whilst at one loop order Constrained-DR automatically produces expressions with a single mass scale which fulfill the corresponding Ward-Slavnov-Taylor identities of the underlying model, at two loops a consistent extension of this program has not yet been constructed \[\delta\].

Implicit regularization (IR) is a momentum space regularization independent framework, which, like Differential Renormalization, does not rely on dimensional extension of space-time dimension \[\delta\]. The basic idea behind IR is to assume implicitly the presence of a regularization part of divergent amplitudes only in order to separate their regularization dependent from its finite part, by using a simple algebraic identity in the integrand of the amplitude. This operation is analogous to apply Taylor operators in the integrand of the amplitudes in the BPHZ formalism \[\gamma\], as far as mathematical rigourousness is concerned, with the advantage of not modifying the structure of the integrand. The divergences are singled out in terms of internal momentum integrals which need not be evaluated. Such procedure generalizes to higher loop order. These so-called loop integrals may be cast solely as a function of an arbitrary scale which both parametrises the freedom of separating the divergent part of an amplitude and plays the role of renormalization group scale. Accordingly, a minimal subtraction renormalization scheme in which such loop integrals are subtracted in the definition of counterterms naturally emerges. An essential by-product of isolating the divergencies in IR are surface terms expressed by differences between loop integrals of the same superficial degree of divergence. This is a crucial point of IR. Although in principle they are arbitrary numbers, such surface terms can be shown to be related to momentum routing invariance in a Feynman diagram, namely the possibility of a shift in the momentum integration variable. The surface terms play an essential role in preserving symmetries in Feynman diagram calculations: they translate into finite local counterterms, whose value is in principle arbitrary but can be determined by the symmetry requirements of the underlying theory.

For an account on the connection between gauge invariance and surface terms in IR see \[\delta\], for the study of chiral and gravitational anomalies and CPT violation see \[\gamma\] and \[\delta\], in which we also showed that IR is an ideal arena to display consistently the physics of models which have finite but undetermined parameters, as discussed by Jackiw in \[\delta\]. Applications to supersymmetry can be found in \[\delta\], where the \(\beta\)-function of the Wess-Zumino model is calculated to three loop order and in \[\delta\], for the calculation of the anomalous magnetic moment of the lepton in supergravity. A generalization to higher loop order was firstly sketched in \[\delta\]. The role of the surface terms, as undetermined parameters in effective phenomenological models, have been considered in \[\gamma\]. Along the lines of IR, a complete systematisation of one-loop four-dimensional Feynman integrals appears
in [20].

The purpose of this work is to clarify some issues for calculations beyond one loop level within IR which are basic to any field theoretical model. In doing so, we pave the way for a complete systematization of IR as an automatic invariant framework order by order in perturbation theory, i.e. that preserves gauge invariance and supersymmetry, as we have shown at one loop order.

Those issues are: 1) Can we still display the divergences as basic loop integrals in one internal momentum in any loop order, overlapping divergences included? 2) In addition, may such basic loop integrals be written as a function of an arbitrary parameter \( \lambda \) which plays the role of renormalization group scale in a general case whilst the subtractions are dictated by the BPHZ forest formula? 3) In connection to this, can the derivatives of the basic divergent integrals with respect to \( \lambda \) also be displayed by loop integrals (or constants) in the lines of Implicit Regularization? 4) How do the surface terms, which encode momentum routing invariance in the loops, look like in arbitrary loop order? 5) In dimensional methods ultraviolet and infrared divergences become mixed. How does IR deal with this problem, in general, for one loop and higher loop order amplitudes?

In order to answer these questions we use the simplest renormalizable scalar field theory namely \( \phi^3 \) theory in 6 dimensions as a working example and show how we calculate renormalization group functions in the IR framework. We also outline the IR rules to work out amplitudes to arbitrary loop order.

II. IMPLICIT REGULARIZATION AND HIDDEN PARAMETERS IN BASIC DIVERGENT INTEGRALS AT ONE LOOP LEVEL

In this section we state the basic steps of IR in one loop calculations. We also construct an arbitrary parametrization of the divergences expressed by loop integrals in order to make contact with other regularizations. Also we show how the infrared cutoff in the propagators cancels out in a subtle interplay between the divergent and finite parts of the amplitude at one loop level, the divergent part being expressed as a basic divergent integral independent of the external momenta. We deal with higher loop in the next section.

1. In order to give mathematical rigor to any algebraic manipulation performed in the amplitude, we implicitly assume that a regularization has been applied. It can be maintained implicit, the only requirement being that the integrand of the amplitude nor the dimension of the space-time is modified. After eventually performing Dirac matrix traces, group index contractions etc., we cast the momentum-space amplitude as a combination of basic integrals. Typical basic integrals are:

\[
I, I_\mu, I_{\mu\nu}, \cdots = \\
\int \frac{d^w k}{(2\pi)^2} \frac{1, k_1, k_2, \cdots}{(k + k_1)^2 - m^2} \cdots [(k + k_N)^2 - m^2],
\]

for a \( N \)-point function in a \( 2w \) integer dimensional space. The internal momenta \( k_i \) are related to the external momenta \( p_i \) by \( p_N = k_1 - k_N, p_i = k_{i+1} - k_i, i = 1, \ldots, N-1 \) if we choose \( \sum_{j=1}^N p_j = 0 \).

2. In the basic integrals, the divergent part is subtracted as basic divergent integrals which are obtained by applying recursively the identity

\[
\frac{1}{(p-k)^2 - m^2} = \frac{1}{(k^2 - m^2)} - \frac{p^2 - 2p \cdot k}{(k^2 - m^2) [(p-k)^2 - m^2]},
\]

until the divergent part is free from the external momentum \( p \) dependence in the denominator. This will assure local counter-terms. The basic divergent integrals have the general form

\[
\int \frac{d^w k}{(2\pi)^2} g_{\mu_1 \mu_2} (k^2 - m^2)^{\alpha_1} \\
\int \frac{d^w k}{(2\pi)^2} k_{\mu_1} k_{\mu_2} \cdots k_{\mu_n},
\]

where the superscript \( \Lambda \) indicates that the integral is regularized and \( w - \alpha \geq 1 \) for the first integral and \( 2w - 2\alpha + n \geq 0 \) for the second. Eventual even powers of internal momenta in the numerator are simplified by adding and subtracting a mass squared term.

3. Express the basic divergent integrals for which the internal momenta carry Lorentz indices as a function of surface terms (i.e. integrals of a total divergence). For example,
in four dimensions:

\[ \int_{\Lambda}^{\infty} \frac{d^4 k}{(2\pi)^4} \frac{k_\mu k_\nu}{(k^2 - m^2)^3} = \]

\[ = \frac{1}{4} \int_{\Lambda}^{\infty} \frac{d^4 k}{(2\pi)^4} \frac{\partial}{\partial k^\mu} \left( \frac{k_\mu}{(k^2 - m^2)^2} \right) + \]

\[ + \frac{g_{\mu\nu}}{4} \int_{\Lambda}^{\infty} \frac{d^4 k}{(2\pi)^4} \frac{1}{(k^2 - m^2)^2}. \]

Such surface terms are regularization dependent (e.g., they vanish in dimensional regularization). The possibility of making shifts in the loop momenta means that the surface terms are null which reveals momentum routing invariance (MRI). A constrained version of IR (CIR) assumes that such surface terms are canceled by local (MRI) restoring counterterms. In practice, this is automatically realized by setting them to zero from the start. CIR is an invariant regularization which preserves the Ward-Slawnov-Taylor identities of the amplitude, save when quantum symmetry breaking occurs. In this case they should be considered as finite arbitrary parameters to be fixed on physical grounds. That is because anomalies are linked to momentum routing dependence in Feynman diagram calculations.

4. The basic divergent integrals which encode the ultraviolet behavior of the amplitude need not be evaluated. We adopt the following notation:

\[ I_{\text{log}}(m^2) = \int_{\Lambda}^{\infty} \frac{d^2 w k}{(2\pi)^2} \frac{1}{(k^2 - m^2)^w}, \]

where \( b_{2w} = \frac{1}{1 - 4w} \) \( \left( \frac{1}{m^2} \right)^w \), and subtracting \( I_{\text{log}}(\lambda^2) \), \( \lambda \) playing the role of renormalization group scale in the renormalization group equation. For infrared safe massless models a systematic cancelation of \( \ln(m^2) \) stemming from \([11]\) and from ultraviolet finite part will render the amplitude well defined as \( m^2 \to 0 \).

6. The remaining ultraviolet finite integrals are evaluated as usual, using Feynman parameters or an extensive library of methods in the momentum space \([23]\).

A systematic presentation of the finite part of one loop \( N \)-point Green’s functions in four dimensions is given in \([24]\).

As a matter of illustration consider the self energy graph of the massless \( \phi^4 \) theory as an example, which is power counting quadratically divergent,

\[ -i\Sigma(p^2) = \frac{g^2}{2} \int_{k}^{\Lambda} \frac{1}{(k^2 - m^2) [(p - k)^2 - m^2]}. \]

5. These objects can be subtracted as they stand in the definition of renormalization functions through, for instance, the definition of local counterterms in the process of renormalization. A minimal, mass independent scheme is defined by substituting \( m^2 \) with \( \Lambda^2 \neq 0 \) using the regularization independent relation in \( 2w \) dimensional space-time.

\[ I_{\text{log}}(m^2) = I_{\text{log}}(\lambda^2) + b_{2w}(-1)^w \ln \left( \frac{\lambda^2}{m^2} \right), \]

where \( b_{2w} = \frac{1}{1 - 4w} \) \( \left( \frac{1}{m^2} \right)^w \), and subtracting \( I_{\text{log}}(\lambda^2) \), \( \lambda \) playing the role of renormalization group scale in the renormalization group equation. For infrared safe massless models a systematic cancelation of \( \ln(m^2) \) stemming from \([11]\) and from ultraviolet finite part will render the amplitude well defined as \( m^2 \to 0 \).

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From now on we adopt the notation \( \int_{k}^{\Lambda} \equiv \int_{k}^{\Lambda} d^2 w k / (2\pi)^{2w} \) in \( 2w \) dimensions. An infrared cut off is introduced in the propagators and the limit \( m^2 \to 0 \) will be taken at the end of the calculation. In order to separate the divergences, the identity \([11]\) is applied three times to yield

\[ -i\Sigma(p^2) = \frac{g^2}{2} \int_{k}^{\Lambda} \left\{ \frac{1}{(k^2 - m^2)^2} - \frac{p^2}{(k^2 - m^2)^3} \right\} + \frac{4(p \cdot k)^2}{(k^2 - m^2)^4} + \frac{p^4}{(k^2 - m^2)^4} \]

\[ - \frac{(p^2 - 2p \cdot k)^3}{(k^2 - m^2)^4 [(p - k)^2 - m^2]} \].

Note that the first three integrals on the r.h.s. are divergent and regularization dependent. The first one is quadratically whereas the other two logarithmically divergent integrals. The remaining integrals are ultraviolet (UV) finite and can be evaluated using Feynman parameters. However, in a massless case they are still infrared divergent. In the sense of IR, in which we do not evaluate loop integrals, we will show that a cancellation of the infrared divergences coming from the UV divergent and finite parts will always take place before taking the limit \( m^2 \to 0 \).

Let us start by discussing the regularization dependent terms. The basic one-loop regularization
dependent objects of a six dimension theory are obtained by choosing \( w = 3 \) in eq. (2) and (3).

In equation (9) there appears, besides the quadratic divergence which vanishes for a massless theory if an adequate parametrization is used, a difference between logarithmic divergent integrals. Let us write \(-i\Sigma = -i\Sigma_{\infty} - i\Sigma_F, \(-i\Sigma_{\infty} = -i\Sigma_F \) standing for the power counting (divergent) finite basic integrals. Then according to the rules of IR we have

\[
\frac{-i\Sigma_{\infty}}{g^2} = 2p^\mu p'^\nu \int_k^A \frac{k_\mu k_\nu}{(k^2 - m^2)^4} \left( \frac{p^2}{2} - I_{\log}(m^2) \right) = 2p^\mu p'^\nu \left( \Upsilon_{\mu\nu}^0 - \frac{1}{12}g_{\mu\nu}I_{\log}(m^2) \right),
\]

with

\[
\Upsilon_{\mu\nu}^0 = \int_k^A \frac{k_\mu k_\nu}{(k^2 - m^2)^4} \frac{g_{\mu\nu}}{6} I_{\log}(m^2) = -\frac{1}{6} \int_k^A \frac{\partial}{\partial k^\rho} \left( \frac{k_\nu}{(k^2 - m^2)^3} \right).
\]

whereas \( \Sigma_F \) can be easily evaluated and has a simple form in the limit \( m^2 \downarrow 0 \). If we write

\[
\Upsilon_{\mu\nu} = a g_{\mu\nu}
\]

\( a \) being an arbitrary constant, (9) reads

\[
\frac{-i\Sigma(p^2)}{g^2} = 2ap^2 - \frac{p^2}{6} I_{\log}(m^2) + \frac{b}{6} p^2 \ln \left( -\frac{p^2}{m^2} \right) + \frac{1}{9} b p^2 - \frac{4b}{3} p^2,
\]

with \( b = i/[2(4\pi)^3] \). Using relation (4) for six dimensions, \( I_{\log}(m^2) = I_{\log}(\lambda^2) - b \ln (\frac{\lambda^2}{m^2}) \), \( \lambda^2 \neq 0 \).

In the equation above, it is clear that the divergent logarithms as \( m^2 \downarrow 0 \) cancel out as we indicated before. We show in the next sections that such infrared divergence cancellation mechanism takes place for arbitrary \( N \)-point functions at one loop order as well as for higher loop order (see also [14]).

The one-loop QED vacuum polarization tensor in 4D, as it is calculated in [10] with arbitrary momentum in the internal lines, \( k_1 \) and \( k_2 \):

\[
\Pi_{\mu\nu} = \Pi(p^2)(p_\mu p_\nu - p^2 g_{\mu\nu})
\]

\[
+ 4 \left( \alpha_1 g_{\mu\nu} - \frac{1}{2}(k_1^2 + k_2^2) \alpha_2 g_{\mu\nu} \right).
\]

In the equation above, \( p = k_1 - k_2 \) is the external momentum and

\[
\Pi(p^2) = \frac{4}{3} I_{\log}(\lambda^2) + \frac{i}{(4\pi)^2} \ln \left( \frac{e^2 \lambda^2}{-p^2} \right) - \frac{i}{3(4\pi)^2}
\]

includes the basic divergent integral. We have chosen the massless limit just for the sake of simplicity. Now the momentum routing dependent terms are proportional to \( \alpha_i \)'s, namely

\[
\alpha_1 g_{\mu\nu} \equiv \int_k^A \frac{g_{\mu\nu}}{k^2 - m^2} - 2 \int_k^A \frac{k_\mu k_\nu}{(k^2 - m^2)^2},
\]

\[
\alpha_2 g_{\mu\nu} \equiv \int_k^A \frac{g_{\mu\nu}}{(k^2 - m^2)^2} - 4 \int_k^A \frac{k_\mu k_\nu}{(k^2 - m^2)^3}.
\]
and
\[ \alpha_3 g_{\mu
u g_{\alpha\beta}} \equiv \int_{k}^{\Lambda} \frac{1}{(k^2 - m^2)^2} - 24 \int_{k}^{\Lambda} \frac{k_{\mu} k_{\nu} k_{\alpha} k_{\beta}}{(k^2 - m^2)^4}. \] (15)

These parameters are surface terms. It can be easily shown that
\[ \alpha_2 g_{\mu\nu} = \int_{k}^{\Lambda} \frac{\partial}{\partial k^\mu} \left( \frac{k_{\nu}}{(k^2 - m^2)^2} \right), \] (16)
\[ \alpha_1 g_{\mu\nu} = \int_{k}^{\Lambda} \frac{\partial}{\partial k^\mu} \left( \frac{k_{\nu}}{(k^2 - m^2)} \right), \] (17)

and
\[ \int_{k}^{\Lambda} \frac{\partial}{\partial k^\beta} \left[ \frac{4 k_{\mu} k_{\nu} k_{\alpha}}{(k^2 - m^2)^3} \right] = g_{\mu\nu g_{\alpha\beta}} (\alpha_3 - \alpha_2). \] (18)

In the case of gauge symmetry, both abelian and nonabelian, a constrained version of IR in which such surface terms are set to vanish delivers gauge invariant amplitudes automatically [15], which is illustrated in the simple example above. The cancellation of these surface terms is also a requirement to preserve Supersymmetry [15]. That is to say: they represent the symmetry restoring local counterterms. In other words, momentum routing invariance seems to be the crucial property in a Feynman diagram in order to preserve symmetries. In fact such surface terms evaluate to zero should we employ DREG to explicitly evaluate them. This property somewhat reveals why DREG is manifestly gauge invariant yet it breaks supersymmetry because invariance of the action with respect to supersymmetry transformations only holds in general for specific values of the space-time dimension [29].

A particular situation, however, is the occurrence of quantum symmetry breaking (anomalies). Anomalies within perturbation theory may present some oddities such as preserving a certain symmetry at the expense of adopting a special momentum routing in a Feynman diagram e.g. in the (Adler-Bardeen-Bell-Jackiw) AV triangle anomaly [23]. In the case of chiral anomalies, IR has been shown to preserve the democracy between the vector and axial sectors of the Ward identities which is a good ‘acid test’ for regularizations. The arbitrary parameter represented by the surface term remains undetermined and floats between the axial and vector sectors of the Ward identities. That is to say, in the anomalous amplitudes, there is no possibility of restoring, at the same time, the axial and the vectorial Ward identities. The counterterm that will restore one symmetry causes the violation of the other and therefore it does not make sense to set the surface terms to zero. The answer is to be established by physical constraints on such amplitude. This feature has also been illustrated in the description of two-dimensional gravitational anomalies [16].

We end this section by showing that we can parametrize the basic divergent integrals should we wish to make contact with an explicit regularization without assigning a definite value to a regularization dependent parameter hidden in an UV divergent amplitude.

As we have argued before, in IR we need only the (loop) integral representation of the divergence and its derivatives. The latter can always be cast as a loop integral as well. Alternatively, we could construct a general parametrization for such basic divergent integrals, which we do here for the purpose of illustration and comparison with other regularization methods. It is however not necessary for calculational purposes.

For instance the basic logarithmically divergent integral at one loop order satisfies
\[ \frac{\partial}{\partial m^2} I_{\log}(m^2) = \frac{b}{m^2}, \] (19)
with \( b = i/[2(4\pi)^3] \), from which we may construct a general parametrisation by integrating the equation above. Namely
\[ I_{\log}(m^2) = -b \ln \left( \frac{\Lambda^2}{m^2} \right) + \beta, \] (20)
where \( \beta \) is an arbitrary constant and \( \Lambda \) is a mass parameter introduced for dimensional reasons, which dictates the UV behavior of the integral. It is very important to notice that a free arbitrary parameter \( \beta \) appears. The explicit parametrisation of the differences between logarithmically divergent integrals (14) and (15) which represents the surface terms will render an arbitrary constant in the end as the ultraviolet divergent logarithms are cancelled.

Now, because
\[ \frac{\partial}{\partial m^2} I_{\quad}(m^2) = 2 I_{\log}(m^2), \] (21)
a possible parametrization of \( I_{\quad}(m^2) \) is
\[ I_{\quad}(m^2) = -\frac{i}{(4\pi)^3} m^2 \left[ \ln \left( \frac{\Lambda^2}{m^2} \right) + \beta' \right]. \] (22)
Let us now evaluate the quadratic divergence in (3). Recalling that \(\text{In DREG, for dimensional reasons, we have a factor} \ \mu^{2\epsilon} \), where \(\mu\) is a mass parameter, \(\epsilon = 6 - d, \ d\) is the space-time dimension, we have

\[
\mu^{2\epsilon} I_{\text{quad}}(m^2) = - \frac{i}{(4\pi)^2} m^2 \left( \frac{1}{\epsilon} + 1 + \ln \left( \frac{-4\pi \mu^2}{m^2} \right) \right) + \gamma_E + \mathcal{O}(\epsilon),
\]

which vanishes as \(m^2 \to 0\) and resembles the parametrization (22). Now, in order to compare IR with DREG let us calculate (7). Because

\[
\mu^{2\epsilon} I_{\text{log}}(m^2) = - b \left[ \frac{1}{\epsilon} + \ln \left( \frac{-4\pi \mu^2}{m^2} \right) + \gamma_E + \mathcal{O}(\epsilon) \right],
\]

and that \(\Upsilon_{\mu}^{0, \text{DREG}} = 0\), we obtain

\[
\mu^{2\epsilon} \sum_{\mu \nu} = - \mu^{2\epsilon} \frac{1}{6} p^2 I_{\text{log}}(m^2) = - b \left[ \frac{1}{\epsilon} + \ln \left( \frac{-4\pi \mu^2}{m^2} \right) + \gamma_E + \mathcal{O}(\epsilon) \right],
\]

which should be compared with (7).

### III. ONE LOOP N-POINT FUNCTIONS

We now turn ourselves to a general one-loop amplitude for a general massless amplitude, using the procedure described above. In a theory defined in \(d\) dimensions, with \(d\) even, one will frequently deal with logarithmically ultraviolet divergent integrals of the type,

\[
I = \int \frac{d^w k}{(2\pi)^2} \frac{1}{[(k + p_n)^2 - m^2]} \cdots \frac{1}{[(k + p_n)^2 - m^2]},
\]

where \(n = w - 1 = d/2 - 1\). The following expansion can be performed,

\[
I = \int_k \frac{1}{(k^2 - m^2)^n[(k + p_n)^2 - m^2]} - \sum_{i=1}^{n-1} I_i,
\]

with

\[
I_i = \int_k \frac{p_i^2 + 2p_i \cdot k}{(k^2 - m^2)^{i+1}[(k + p_i)^2 - m^2]} \cdots \frac{1}{[(k + p_n)^2 - m^2]}).
\]

The expansion above has been obtained by using the identity (11) in all factors containing external momenta \(p_i\) dependence from \(i = 1\) to \(i = n - 1\). Each time the identity is applied, we obtain a new integral \(I_i\) that is ultraviolet finite. This is why we do not use the index \(\Lambda\) in these integrals. Next we want to show that in the sense of IR the limit \(m^2 \to 0\) is well defined. Our proof is done in two parts: first we show that the integrals \(I_i\) are well defined. A few algebraic manipulations which will lead to eq. (32) will make that clear. Then we show how to handle the \(m^2\) dependence in the regularized integral.

We begin the calculation of \(I_i\) by using Feynman’s parametrization

\[
\frac{1}{a_1 \cdots a_l} = \frac{(l + a - 2)!}{(\alpha - 1)!} \int dX (1 - x_1 - \cdots - x_{l-1})^{\alpha-1}
\]

\[
\frac{1}{[(a_1 - a_l)x_1 + (a_2 - a_l)x_2 + \cdots + a_l]^{l+\alpha-1}},
\]

where

\[
\int dX \equiv \int_0^1 dx_1 \int_0^{1-x_1} dx_2 \cdots \int_0^{1-x_{l-1}} dx_{l-1}.
\]

In the integral \(I_i\) we have \(\alpha = i+1\) and \(l = w-i+1\), so that

\[
I_i = \frac{w!}{l!} \int dX (1 - x_1 - \cdots - x_{w-i})^i \times
\]

\[
(p_i^2 - 2p_i^2 x_{w-i} - 2(p_i \cdot p_{i+1}) x_{w-i-1} \cdots - 2(p_i \cdot p_n)x_1)
\]

\[
\times \int_k \frac{1}{(k^2 + Q^2)^{w+1}},
\]

with

\[
Q^2 = p_i^2 x_1 (1 - x_1) + \cdots + p_i^2 x_{n-i+1} (1 - x_{n-i+1})
\]

\[
- 2 \sum_{i \neq j} (p_i \cdot p_j) x_{n-i+1} x_{n-j+1} - m^2.
\]

The final result is given by

\[
I_i = \frac{i}{(4\pi)^w} \frac{(-1)^w}{l!} \int dX (1 - x_1 - \cdots - x_{w-l})^l
\]

\[
(p_l^2 - 2p_l^2 x_{w-l} - 2(p_l \cdot p_{l+1}) x_{w-l-1} \cdots - 2(p_l \cdot p_n)x_1)
\]

\[
\times \frac{1}{Q^2}.
\]

It is clear from the expression above that whenever the external momenta are such that \(p_i^2 \neq 0\), the limit \(m^2 \to 0\) is well defined.

We are now left with the regularization dependent integral,

\[
I^\Lambda = \int_k \frac{1}{(k^2 - m^2)^n[(k + p_n)^2 - m^2]}.
\]
from which we can separate the appropriate basic divergence $I_{\text{log}}^d(m^2)$ (typical of this dimension) from its finite part:

$$I^\Lambda = I_{\text{log}}^d(m^2) - \int \frac{p_n^2 - 2p_n \cdot k}{(k^2 - m^2)^w[(k + p_n)^2 - m^2]},$$  \hspace{1cm} (34)

with $I_{\text{log}}^d(m^2)$ given by eq. (2), and where we must remember that $n = w - 1$. Note that the cutoff mass $m^2$ is present in both finite and divergent parts. We will now show that, due to a scale relation, the $m^2$ dependence of the divergent integral can be extracted and precisely cancels out the $m^2$ dependence of the finite part. First, we make use of the identity

$$\frac{1}{(k^2 - m^2)^w} = \frac{1}{(k^2 - \lambda^2)^w} - (\lambda^2 - m^2)$$

$$\times \sum_{i=1}^{n} \frac{1}{(k^2 - m^2)^i(k^2 - \lambda^2)^{w-i+1}},$$  \hspace{1cm} (35)

and obtain the $d$- dimensional scale relation (4),

$$I_{\text{log}}^d(m^2) = I_{\text{log}}^d(\lambda^2) + b_d(-1)^{d/2} \ln \left(\frac{\lambda^2}{m^2}\right),$$  \hspace{1cm} (36)

with $b_d = \frac{4}{(4\pi)^{d/2}} \frac{1}{\Gamma(d/2)}$. Again, like in the six dimensional case, the relation above could be obtained from a particular regularization, but, as we have seen, the relation is regularization independent. On the other hand, the calculation of the finite part yields:

$$\int \frac{p_n^2 - 2p_n \cdot k}{(k^2 - m^2)^w[(k + p_n)^2 - m^2]}$$

$$= (-1)^{d/2}b_d \left(\frac{d}{2} - 1\right) \int_0^1 dx(1 - x)^{d/2-2}$$

$$\ln \left(\frac{p_n^2 x(1 - x) - m^2}{(-m^2)}\right).$$  \hspace{1cm} (37)

We can now clearly see, by collecting the two parts together, that

$$I^\Lambda = I_{\text{log}}^d(\lambda^2) - (-1)^{d/2}b_d \left(\frac{d}{2} - 1\right) \int_0^1 dx$$

$$\left(1 - x\right)^{d/2-2} \ln \left(\frac{p_n^2 x(1 - x)}{(-\lambda^2)}\right),$$  \hspace{1cm} (38)

where the limit $m^2 \to 0$ has been taken. The whole integral,

$$I = I^\Lambda - \sum_{i=1}^{n-1} I_i,$$  \hspace{1cm} (39)

becomes then $m^2$ independent. This result, in conjunction with equation (32), will be important for the generalization of IR to higher order in order to subtract subdivergences. Also, once the subtraction scheme is appropriately chosen, the object $I_{\text{log}}^d(\lambda^2)$ can be subtracted without having to be explicitly evaluated. The calculation above establishes a procedure for dealing with massless theories at one loop order in the context of IR. This procedure can be straightforwardly adapted to higher orders when non-overlapping divergencies occur, as shown in the ref. [14], where the $\beta$-function of the massless Wess-Zumino model was calculated at three loop order. In that contribution, it was shown that a $n$-loop scale relation can be used in order to appropriately handle the dependence on the infrared cut-off.

We shall now generalize this procedure for the general case when overlapping divergencies occurs with support in the BPHZ-forest formula.

### IV. HIGHER ORDER CALCULATIONS

Renormalization is a recursive program. In a $n$-loop order calculation the main goal is to identify the typical divergence of the $n^{th}$ order and the finite part of an amplitude once the $(n-1)^{th}$ has been renormalized. In [14], we have calculated within IR the three-loop $\beta$ function for the Wess-Zumino model in which there were not overlapping divergencies to consider.

In fact IR can be applied in the sense of defining loop integrals as basic ultraviolet divergent objects also when overlapping divergencies occur. A regularization independent scale relation will also emerge in this case which serves to both cancel infrared divergences in a similar fashion as we presented in the last sections and introduce a renormalization group scale. New surface terms appear beyond one loop level and are expected to play an essential role in preserving gauge symmetry. Moreover IR rules are compatible with BPHZ forest formula, which judiciously defines the set of subtractions to remove the subdivergences. The rules to implement IR beyond one loop order can be simply stated once we adopt the version of the forest formula which is analogous to the ordinary counterterm method in which the subtraction operators are translated into a local counterterm which substitutes the subgraph. This is implemented via a recursion equation which involves disjoint renormalization parts only [21], described in textbooks,
1. Starting from one loop order, the (sub)divergences should be expressed in terms of basic divergent integrals, which can be written in terms of one internal momentum only and an arbitrary non-vanishing parameter \( \lambda \), the renormalization group scale of the method. For this purpose identity (1) is judiciously used. We assume that subdiagrams are proper (one particle irreducible) which is sufficient to discuss renormalization in general.

2. The counterterms are defined order by order through a minimal subtraction process within implicit regularization which amounts to subtracting the (sub)divergences as basic divergent integrals as defined above.

3. The subtraction of subdivergences follows the forest formula written in a way which is equivalent of the subtractions of local counterterms only. That is to say given a diagram \( G \) the subtraction operator for a certain renormalization part \( H \) has the effect of crushing \( H \) to a point and multiplying the counterterm to the resulting Feynman integral. In this recursive process only the set of disjoint renormalization parts is needed.

Mathematically speaking, a recursion relation for the subtraction of subdivergences based on the forest formula for a Feynman diagram \( G \) in which only the disjoint renormalization parts are needed can be written as

\[
\bar{R}_G F_G = \sum_\psi F_G/\psi \prod_{H \in \psi} (-t^H \bar{R}_H F_H)
\]

\[
= \sum_\psi \prod_{H \in \psi} (-t^H \bar{R}_H) F_G
\]  

(40)

where \( \bar{R}_G \) is the operation to subtract only subdivergencies, \( F_{G(H)} \) is the part of the amplitude which represents the (sub)graph \( G(H) \), \( \psi \) is a set of disjoint renormalization parts of \( G \), namely

\[
\psi = \{H/H \subset G, H = \text{proper, disjoint}, d_H \geq 0\},
\]

\( G/\psi \) representing the diagram obtained from \( G \) by crushing all \( H \) in \( \psi \) to a point. The counterterm graph so obtained can be constructed with the loop integrals characteristic of IR order by order. Of course a further operation \( 1 - t_G \) is required to the overall divergence. The natural (minimal) renormalization scheme in IR is to subtract the basic divergent loop integrals as a function of the arbitrary scale \( \lambda \).

For the massless case the basic divergent integrals are very simple. For instance the logarithmic basic divergence to \( n \)-loop order is given by

\[
I^{(n)}_{log}(m^2) = \int_\Lambda^{\Lambda} \frac{d^{2w} k}{(2\pi)^{2w}} \frac{1}{(k^2 - m^2)^n} \times \ln^{n-1} \left( \frac{k^2 - m^2}{\lambda^2} \right)
\]

(41)

and the corresponding scale relation is written as

\[
I^{(n)}_{log}(m^2) = I^{(n)}_{log}(\lambda^2) + \sum_{i=0}^{n} a_i^{(w)} \ln^i \left( \frac{m^2}{\lambda^2} \right),
\]

(42)

with the \( a_i^{(w)} \) depending on the dimension \( 2w \).

Let us consider the two-loop contribution to the self-energy in \( \phi_3^3 \) theory. The nested subdiagram in fig. (1) can be calculated by simply substituting the subdiagram by its finite part. The remaining integral, in one internal momentum, includes only the divergence of the order and the finite part.

![FIG. 1: The 2-loop diagram with a nested subdivergence for the \( \phi_3^3 \) self-energy](image)

The amplitude for the diagram depicted in fig. (1) reads

\[
-i \Sigma_1^{(2)}(p^2) = \frac{i g^4}{2} \int \frac{1}{k^4(k-p)^2} \int \frac{1}{l^2(l-k)^2}.
\]

(43)

The integral in the momentum \( l \) has been calculated before (eq. (11)):

\[
I = -\frac{k^2}{3} \left\{ I_{log}(\lambda^2) + b \ln \left( \frac{k^2}{\lambda^2} \right) - \frac{8}{3} b \right\}.
\]

(44)

We take its finite part and obtain:

\[
-i \Sigma_1^{(2)}(p^2) = -i g^4 b \int \frac{1}{k^4(k-p)^2} \times \left( \ln \left( \frac{k^2}{\lambda^2} \right) - \frac{8}{3} \right)
\]

\[
-\frac{8}{3} I(2) \end{equation}

(45)
where

\[ I^{(2)} = \int_k \frac{1}{k^2(k - p)^2} \ln \left( \frac{k^2}{\lambda^2} \right). \]  

(46)

Notice that

\[ -i\tilde{\Sigma}^{(2)}(p^2) = (1 - t)(-i\Sigma_1^{(2)}(p^2)), \]  

(47)

where \(-t\) corresponds to the subtraction operator which has the effect of crushing the nested sub-
divergence to a point and multiplying the counter-
term, which has been calculated in the previous
order, to the resulting Feynman integral.

Now we evaluate \(I^{(2)}\). We introduce an infrared
cutoff \(m^2\) as we have done before to rewrite

\[ I^{(2)} = \int_k \frac{1}{(k^2 - m^2)[(k - p)^2 - m^2]} \ln \left( \frac{k^2 - m^2}{\lambda^2} \right), \]  

(48)

which is ultraviolet quadratically divergent. The
identity expressed by eq. (41) is applied three times to yield

\[
I^{(2)} = \int_k \ln \left( \frac{k^2 - m^2}{\lambda^2} \right) \left\{ \frac{1}{(k^2 - m^2)^2} - \frac{p^2}{(k^2 - m^2)^3} + \frac{4(p \cdot k)^2}{(k^2 - m^2)^4} + \frac{p^4}{(k^2 - m^2)^4} - \frac{(p^2 - 2p \cdot k)^3}{(k^2 - m^2)^4} \right\}.

\]

(49)

We define the basic two-loop divergent integrals in
6 dimensions,

\[ I_{\log}^{(2)}(m^2) = \int_k \frac{1}{(l^2 - m^2)^2} \ln \left( -\frac{(l^2 - m^2)}{\lambda^2} \right) \]  

(50)

and

\[ I_{\text{quad}}^{(2)}(m^2) = \int_k \frac{1}{(l^2 - m^2)^2} \ln \left( -\frac{(l^2 - m^2)}{\lambda^2} \right). \]  

(51)

Clearly \(I_{\text{quad}}^{(2)}(m^2)\) vanishes for massless theories in
a regularization independent fashion. whilst the
divergent basic integrals with Lorentz indices can be
written in function of a surface term:

\[
\Theta_{\alpha\beta}(m^2) = \int_k \frac{k_\alpha k_\beta}{(k^2 - m^2)^4} \ln \left( -\frac{(k^2 - m^2)}{\lambda^2} \right)
= \frac{1}{6} \left\{ g_{\alpha\beta} \left( I_{\log}^{(2)}(m^2) + \frac{1}{3} I_{\log}(m^2) \right) - \int \frac{\partial}{\partial k_\beta} \left( \frac{k_\alpha}{(k^2 - m^2)^3} \ln \left( -\frac{(k^2 - m^2)}{\lambda^2} \right) \right) \right\}.
\]

(52)

The surface term will be set to zero for the sake of
momentum routing invariance. Thus we have

\[ I^{(2)} = -\frac{p^2}{3} I_{\log}^{(2)}(m^2) + \frac{2}{9} p^2 I_{\log}(m^2) + \tilde{I}^{(2)}, \]  

(53)

\(\tilde{I}^{(2)}\) being the finite part which is given by

\[
\tilde{I}^{(2)} = \int_k \frac{p^4}{(k^2 - m^2)^4} \ln \left( -\frac{(k^2 - m^2)}{\lambda^2} \right) \]  

\[ - \int_k \frac{(p^2 - 2p \cdot k)^3}{(k^2 - m^2)^4} \ln \left( -\frac{(k^2 - m^2)}{\lambda^2} \right). \]  

(54)

The finite integrals above can be easily evaluated
using the identity,

\[ \ln a = \lim_{\epsilon \to 0} \frac{1}{\epsilon} (a^\epsilon - 1), \]  

(55)

which after Feynman parametrization in the limit
where \(m^2 \to 0\) is given by

\[
\tilde{I}^{(2)} = \frac{b}{18} p^2 \left\{ \ln \left( \frac{m^2}{\lambda^2} \right) \left[ 6 \ln \left( \frac{p^2}{m^2} \right) - 16 \right] + 3 \ln^2 \left( \frac{p^2}{m^2} \right) - 11 \ln \left( \frac{p^2}{m^2} \right) + 11 \right\}. \]  

(56)

We see the infrared divergence, parametrized by
\(m^2\), appearing in the divergent and in the finite
part. In order to observe the cancellation of these
two contributions, we use the one-loop (51) and
the two-loop scale relation,

\[ I_{\log}^{(2)}(m^2) = I_{\log}^{(2)}(\lambda^2) + b \left\{ \frac{1}{2} \ln \left( \frac{m^2}{\lambda^2} \right) + \frac{3}{2} \ln \left( \frac{m^2}{\lambda^2} \right) \right\}, \]  

(57)

so that eq. (53) becomes

\[ I^{(2)} = \frac{p^2}{3} \left( -I_{\log}^{(2)}(\lambda^2) + \frac{2}{3} p^2 I_{\log}(\lambda^2) - \frac{b}{2} \ln^2 \left( \frac{p^2}{\lambda^2} \right) + \frac{11}{6} b \ln \left( \frac{p^2}{\lambda^2} \right) + \frac{11}{6} b \right). \]  

(58)

Now equations (44), (45) and (58) together give

\[ -i\tilde{\Sigma}_1^{(2)}(p^2) = \frac{ig^4 b p^2}{18} \left\{ I_{\log}^{(2)}(\lambda^2) - \frac{10}{3} I_{\log}(\lambda^2) + \frac{b}{2} \ln^2 \left( \frac{p^2}{\lambda^2} \right) - \frac{27}{6} b \ln \left( \frac{p^2}{\lambda^2} \right) + \frac{95}{18} b \right\}. \]  

(59)

Let us now consider the two-loop overlapping
contribution for the self-energy in \(\phi^4\) theory, rep-
resented by fig. (2). It is given by
To solve the overlapping integral $A$, it will be easier to rewrite it as

$$A = -A_1 + A_2 + \text{surface term},$$

by using the identity [23]:

$$\frac{\partial}{\partial k_a} \left( \frac{(k-l)_a}{D} \right) = \frac{1}{D} \left( 2 + \frac{l^2}{k^2} + \frac{(p-l)^2}{(p-k)^2} \right) - \frac{(k-l)^2}{k^2} - \frac{(k-l)^2}{(p-k)^2},$$

with

$$D = k^2 l^2 (k-l)^2 (p-k)^2 (p-l)^2.$$  \hfill (61)

We disregard the surface term on momentum routing invariance grounds to write

$$A_1 = \frac{1}{\Lambda} \int_{k,l} \frac{1}{k^2(p-k)^2 l^2(l-k)^2}$$

and

$$A_2 = \frac{1}{\Lambda} \int_{k,l} \frac{1}{k^4(p-k)^2 l^4(p-l)^2}.$$  \hfill (64)

In order to reduce to these two integrals, we have performed shifts (surface terms are not considered). The integral $A_2$ is actually a product of two independent one-loop integrals in the internal momenta $l$ and $k$. It gives us

$$A_2 = -\frac{p^2}{3} \left\{ I_{\log}(\lambda^2) - \frac{14}{3} b I_{\log}(\lambda^2) \right\} + 2 b \ln \left(-\frac{p^2}{\lambda^2}\right) I_{\log}(\lambda^2) + b^2 \ln^2 \left(-\frac{p^2}{\lambda^2}\right) - \frac{16}{3} b^3.$$  \hfill (65)

For the $A_1$, after the integral in $l$ is solved, we have

$$A_1 = -\frac{1}{3} \int_{k}^{\Lambda} \frac{1}{(p-k)^2} \left\{ I_{\log}(\lambda^2) + b \ln \left(-\frac{p^2}{\lambda^2}\right) - \frac{8}{3} b \right\}$$

$$= -\frac{1}{3} \int_{k}^{\Lambda} \frac{1}{(p-k)^2} \left\{ I_{\log}(\lambda^2) - \frac{8}{3} b \right\} \frac{1}{(p-k)^4}$$

$$= -\frac{b}{3} \int_{k}^{\Lambda} \frac{1}{(p-k)^4} \ln \left(-\frac{p^2}{\lambda^2}\right).$$  \hfill (66)

Using an infrared cutoff, $m^2$, the first term above is proportional to $I_{quad}(m^2)$, plus a surface term. Therefore it does not contribute in the massless limit if we call for momentum routing invariance. Hence

$$A_1 = -\frac{b}{3} \int_{k}^{\Lambda} \frac{1}{(p-k)^2 - m^2} \ln \left(-\frac{(k-m^2)}{\lambda^2}\right)$$

$$= -\frac{b}{3} \int_{k}^{\Lambda} \frac{1}{(p-k)^2 - m^2} \ln \left(-\frac{(k-m^2)}{\lambda^2}\right) \times$$

$$\left\{ \frac{1}{(k^2-m^2)^2} - \frac{p^2-2p.k}{(k^2-m^2)^2} + \frac{(p^2-2p.k)^2}{(k^2-m^2)^3} - \frac{(p^2-2p.k)^3}{(k^2-m^2)^3} \right\}.$$  \hfill (67)

It is easy to check that the first term above is $I^{(2)}$. If we use identity (11) in $I^{(2)}$, we find

$$\int_{k}^{\Lambda} \frac{p^2-2p.k}{(k^2-m^2)^2} \frac{1}{(p-k)^2 - m^2} \ln \left(-\frac{(k-m^2)}{\lambda^2}\right)$$

$$= -I^{(2)} + I_{quad}^{(2)}(m^2) = -I^{(2)}$$  \hfill (68)

and

$$\int_{k}^{\Lambda} \frac{(p^2-2p.k)^2}{(k^2-m^2)^3} \frac{1}{(p-k)^2 - m^2} \ln \left(-\frac{(k-m^2)}{\lambda^2}\right)$$

$$= I^{(2)} - I_{quad}^{(2)}(m^2) + p^2 I^{(2)}(m^2)$$

$$= I^{(2)} + p^2 I_{\log}(m^2).$$  \hfill (69)

The last term of eq. (67) is finite and can be calculated with the use of eq. (55). The final result of $A_1$ is

$$A_1 = \frac{2b}{9} p^2 \left\{ I_{\log}(\lambda^2) + b \ln \left(-\frac{p^2}{\lambda^2}\right) + \frac{7}{3} b \right\}.$$  \hfill (70)
and we have, for the overlapping diagram,

\[-i\Sigma_2^{(2)}(p^2) = i g^4 p^2 \left\{ -I_{\text{log}}^2(\lambda^2) + \frac{16}{3} b I_{\text{log}}(\lambda^2) - 2b \ln \left( -\frac{p^2}{\lambda^2} \right) I_{\text{log}}(\lambda^2) - b^2 \ln^2 \left( -\frac{p^2}{\lambda^2} \right) + \frac{16}{3} b^2 \ln \left( -\frac{p^2}{\lambda^2} \right) - \frac{34}{9} b^2 \right\}.\]

(71)

The result above refers to the complete diagram, which includes the subdivergences with non-local terms. The diagrams, according to the forest formula, for the counterterms to be added so as to cancel the subdivergences are given in fig. (3).

FIG. 3: Counterterms to cancel the subdivergences of the overlapping diagram

We have two equal contributions that furnish us

\[-i\Sigma_2^{(2)}_{\text{sub}} = -2 ig^2 I_{\text{log}}(\lambda^2)(-i\Sigma(p^2))
= \frac{ig^4 p^2}{3} \left\{ -I_{\text{log}}^2(\lambda^2) + b \ln \left( -\frac{p^2}{\lambda^2} \right) I_{\text{log}}(\lambda^2) \right\} - \frac{16}{3} b I_{\text{log}}(\lambda^2)\]

(72)

After adding the above counterterm, we finally obtain

\[-i\Sigma_2^{(2)}(p^2) = \frac{ig^4 p^2}{6} \left\{ -I_{\text{log}}^2(\lambda^2) + b \ln \left( -\frac{p^2}{\lambda^2} \right) I_{\text{log}}(\lambda^2) \right\} - \frac{16}{3} b I_{\text{log}}(\lambda^2)
- b^2 \ln^2 \left( -\frac{p^2}{\lambda^2} \right) + \frac{16}{3} b^2 \ln \left( -\frac{p^2}{\lambda^2} \right) - \frac{34}{9} b^2\]

\[\equiv (1 - t_{(a)} - t_{(b)})(-i\Sigma_2^{(2)}(p^2)),\]

(73)

where \(t_{(a)}\) and \(t_{(b)}\) are the subtraction operators corresponding to the renormalization parts in fig. (3).

As in the one loop calculation, the limit \(m^2 \to 0\) was taken in the end and we made use of a scale relation given by eq. (57). The complete self-energy at the \(g^4\) order is given by

\[-i\Sigma_2^{(2)}(p^2) = \frac{ig^4 p^2}{6} \left\{ -I_{\text{log}}^2(\lambda^2) + \frac{16}{3} b I_{\text{log}}(\lambda^2) - 2b \ln \left( -\frac{p^2}{\lambda^2} \right) I_{\text{log}}(\lambda^2) - \frac{b^2}{3} \ln \left( -\frac{p^2}{\lambda^2} \right) + \frac{10}{9} b^2 I_{\text{log}}(\lambda^2) \right\}\]

\[\equiv (1 - t_{(a)} - t_{(b)})(-i\Sigma_2^{(2)}(p^2)),\]

(74)

where the tilde refers to the finite part. We are now able to calculate the anomalous dimension \((\gamma\text{-function})\). As usual, we define the renormalization constants, \(Z_3\) and \(Z_g\), so that

\[
\phi_0 = \lambda^{1/2} \phi \quad \text{and} \quad g_0 = Z_g g,
\]

and the Lagrangian is written as

\[
\mathcal{L} = \frac{1}{2} \partial^\mu \phi \partial_\mu \phi + \frac{g}{3!} \phi^3 + \frac{1}{2} (Z_3 - 1) \partial^\mu \phi \partial_\mu \phi + \frac{1}{2} (Z_3 - 1) \frac{g}{3!} \phi^3.
\]

(76)

The expansion in terms of the coupling constant, \(g\), is given by

\[
Z_3 = 1 + \alpha_1 g^2 + \alpha_2 g^4 + \mathcal{O}(g^6)
\]

(77)

and

\[
Z_g = 1 + \rho_1 g^2 + \rho_2 g^4 + \mathcal{O}(g^6).
\]

(78)

According to our calculations, we have

\[
\alpha_1 = \frac{i}{6} I_{\text{log}}(\lambda^2),
\]

\[
\alpha_2 = \frac{1}{6} \left\{ - I_{\text{log}}^2(\lambda^2) - \frac{1}{3} b I_{\text{log}}(\lambda^2) + \frac{10}{9} b I_{\text{log}}(\lambda^2) \right\}
\]

(79)

and

\[
Z_3^{3/2} Z_g = 1 - ig^2 I_{\text{log}}(\lambda^2),\]

(81)

which gives us

\[
\rho_1 = - \frac{3}{4} i I_{\text{log}}(\lambda^2).
\]

(82)

The \(\gamma\text{-function}\) is defined as

\[
\gamma = \frac{\lambda}{2 Z_3} \frac{\partial Z_3}{\partial \lambda} = \frac{Z_3}{2 Z_3},
\]

(83)
where
\[ \dot{f} = \lambda \frac{\partial f}{\partial \lambda} = 2\lambda^2 \frac{\partial f}{\partial \lambda^2}. \] (84)

So, we have that
\[ \gamma = \frac{1}{2} \{ \dot{\alpha}_1 g^2 + (2\beta_1 \alpha_1 + \dot{\alpha}_2 - \alpha_1 \dot{\alpha}_1)g^4 \} + O(g^6), \]
with \( \beta_1 \) the coefficient of \( g^3 \) in the \( \beta \)-function, defined as
\[ \beta = \dot{g} = -\frac{\dot{g}}{Z_g}. \] (85)

Taking into account that
\[ \frac{d}{d\lambda^2}(I_{\log}^2(\lambda^2)) = \frac{2b}{\lambda^2} I_{\log}(\lambda^2), \] (86)
\[ \frac{d}{d\lambda^2}(I_{\log}^2(\lambda^2)) = \frac{1}{\lambda^2} \left( \frac{3}{2} b - I_{\log}(\lambda^2) \right) \] (87)
and
\[ \frac{d}{d\lambda^2}(I_{\log}(\lambda^2)) = \frac{b}{\lambda^2}, \] (88)
we find
\[ \beta_1 = -\frac{3}{4(4\pi)^3}. \] (89)

and
\[ \gamma = \frac{1}{12(4\pi)^3} g^2 - \frac{11}{108} \frac{4(4\pi)^6}{g^4} + O(g^6). \] (90)

Notice that \( \beta_1 \) and the \( g^2 \) contribution to the \( \gamma \)-function are universal results whereas the \( g^4 \) coefficient of the \( \gamma \)-function is renormalization scheme dependent. These results corresponds to the minimal subtraction scheme in implicit regularization.

V. CONCLUSIONS

In this paper we have addressed important issues regarding the extension of implicit regularization beyond one loop order. Namely we showed that we can display the divergencies as basic loop integrals in one internal momentum in any loop order including the case of overlapping divergencies. This is the heart of IR which enables us to work in the physical dimension of the model. Moreover we have shown the general form of surface terms which appear as finite differences of divergent integrals with the same superficial degree of divergence. They are important because they appear to relate momentum routing invariance to gauge and supersymmetry in a \( n \)-loop amplitude. We have explicitly verified this link in one loop order for both gauge and supersymmetry and to three loop order for supersymmetry. A general proof of the connection between momentum routing invariance in a general Feynman diagram and the vital symmetries to be respected by the corresponding Green’s function is interesting to be constructed
\[ \text{[25]. Besides We have also shown that basic } n \text{-loop integrals may be written as a function of an arbitrary parameter } \lambda \text{ which plays the role of renormalization group scale; they are absorbed as they stand in the definition of the renormalization constants in the light of the BPHZ forest formula in a regularization independent fashion. That is because the derivatives of the basic divergent integrals with respect to } \lambda \text{ can be displayed by loop integrals as well. Finally, contrarily to dimensional methods where ultraviolet and infrared divergencies may become mixed and lead to ambiguities, IR clearly separates ultraviolet and infrared degrees of freedom through a kind of scale relation obeyed by the basic divergent integrals.} \]

[1] D. Stockinger, Nucl. Phys. Proc. Suppl. B 160, 250 (2006) and references therein.
[2] I. Jack and D. R. T. Jones in G. L. Kane (ed.) Perspectives on Supersymmetry, 149-167, hep-ph/9707278.
[3] S. Heinemeyer, W. Hollik and G. Weiglein, Phys. Rep. 425 (2006) 265.
[4] W. Siegel, Phys. Lett. B 94, (1980) 37.
[5] J. A. Aguilar-Saavedra et al., Eur. Phys. J. C 46 (2006) 43.
[6] D. Z. Freedman, K. Johnson and J. I. Latorre, Nucl. Phys. B 371 (1992) 353, M. Pérez Victoria, Phys. Lett. B 442 (1998) 315, F. del Aguila, A. Culatti, R. Muñoz Tapia and M. Pérez Victoria, Nucl. Phys. B 504 (1997) 532, F. del Aguila, A. Culatti, R. Muñoz Tapia and M. Pérez Victoria, Phys. Lett. B 419 (1998) 263, F. del Aguila, A. Culatti, R. Muñoz Tapia and M. Pérez Victoria, Nucl. Phys. B 537 (1999) 561, Javier Mas, Manuel Perez-Victoria, Cesar Seijas, JHEP 0203 (2002) 49.
[7] C. Seijas, Two loop divergences studied with...
one loop Constrained Differential Renormalization, hep-th/0604071.

[8] O. A. Battistel, PhD thesis, Federal University of Minas Gerais (2000).

[9] O. A. Battistel, A. L. Mota, M. C. Nemes Mod. Phys. Lett. A 13 (1998) 1597.

[10] A. P. Baêta Scarpelli, M. Sampaio and M. C. Nemes, Phys. Rev. D 63 (2001) 046004.

[11] A. P. Baêta Scarpelli, M. Sampaio, B. Hiller and M. C. Nemes, Phys. Rev. D 64 (2001) 046013.

[12] M. Sampaio, A. P. Baêta Scarpelli, B. Hiller, A. Brizola, M. C. Nemes and S. Gobira, Phys. Rev. D 65 (2002) 125023.

[13] S. R. Gobira and M. C. Nemes, Int. J. Theor. Phys. 42 (2003) 2765.

[14] D. Carneiro, A. P. Baêta Scarpelli, M. Sampaio and M. C. Nemes, JHEP 12 (2003) 044.

[15] M. Sampaio, A. P. Baêta Scarpelli, J. E. Ottoni, M. C. Nemes, Int. J. Theor. Phys. 45 (2006) 436.

[16] L. A. M. Souza, Marcos Sampaio, M. C. Nemes, Phys. Lett. B 632 (2006) 717.

[17] J. E. Ottoni, A. P. B. Scarpelli, Marcos Sampaio, M. C. Nemes, Phys. Lett. B 642 (2006) 253.

[18] E. W. Dias, B. Hiller, A. L. Mota, M. C. Nemes, Marcos Sampaio, A. A. Osipov, Mod. Phys. Lett. A 21 (2006) 339.

[19] Brigitte Hiller, A. L. Mota, M. C. Nemes, A. A. Osipov, Marcos Sampaio, Nucl. Phys. A 769 (2006) 53.

[20] Orimar Battistel and G. Dallabona, Eur. Phys. J. C 45 (2006) 721.

[21] N. N. Bogoliubov, O. S. Parasiuk, Acta Math. 97 (1957) 227, N. N. Bogoliubov, D. V. Shirkov, Introduction to the theory of quantized fields (3rd ed.), John Wiley (1980), K. Hepp, Comm. Math. Phys. 2 (1966) 301, W. Zimmermann, Comm. Math. Phys. 15 (1969) 208.

[22] R. Jackiw, Int. J. Mod. Phys. B 14 (2000) 2011.

[23] V. A. Smirnov, Evaluating Feynman Integrals, Springer-Verlag (Feb. 2005) ISBN: 3540239332.

[24] V. Elias, S. B. Phillips, R. B. Mann, Phys. Lett. B 133 (1983) 83.

[25] S. Treiman, R. Jackiw, B. Zumino and E. Witten, Current Algebras and Anomalies, World Scientific, 1985.

[26] T. Muta, Foundations of QCD, World Scientific Lecture Notes in Physics, 1998.

[27] W. Siegel, Fields, (1999, 3rd edition 2005), 885 pp.

[28] E. Dias, A. P. Baêta Scarpelli, Marcos Sampaio and M. C. Nemes, work in progress

[29] The idea of associating momentum routing in the loops with symmetry properties of the Green’s functions has been exploited in a framework named Preregularization which did not call for momentum routing invariance but instead fixed the routing in order to fulfill certain Ward identities. [24]