On the derivative nonlinear Schrödinger equation with weakly dissipative structure

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Abstract. We consider the initial value problem for cubic derivative nonlinear Schrödinger equation in one space dimension. Under a suitable weakly dissipative condition on the nonlinearity, we show that the small data solution has a logarithmic time decay in $L^2$.

1. Introduction and the main result

We consider the initial value problem

$$
\begin{cases}
    i\partial_t u + \frac{1}{2}\partial_x^2 u = N(u, \partial_x u), & t > 0, \ x \in \mathbb{R}, \\
    u(0, x) = \varphi(x), & x \in \mathbb{R},
\end{cases}
$$

where $i = \sqrt{-1}$, $u = u(t, x)$ is a $\mathbb{C}$-valued unknown function on $[0, \infty) \times \mathbb{R}$. $\varphi$ is a prescribed $\mathbb{C}$-valued function on $\mathbb{R}$ which belongs to $H^3 \cap H^{2,1}$ and is suitably small in its norm. Here and later on as well, for non-negative integers $k$ and $m$, $H^k$ denotes the standard $L^2$-based Sobolev space of order $k$, and the weighted Sobolev space $H^{k,m}$ is defined by $\{ \phi \in L^2 \mid \langle \cdot \rangle^m \phi \in H^k \}$, equipped with the norm $\| \phi \|_{H^{k,m}} = \| \langle \cdot \rangle^m \phi \|_{H^k}$, where $\langle x \rangle = \sqrt{1 + x^2}$. Throughout this paper, the nonlinear term $N(u, \partial_x u)$ is always assumed to be a cubic homogeneous polynomial in $(u, \bar{u}, \partial_x u, \bar{\partial}_x u)$ with complex coefficients. We will often write $u_x$ for $\partial_x u$.

From the perturbative point of view, cubic nonlinear Schrödinger equations in one space dimension are of special interest because the best possible decay in $L^2$ of general cubic nonlinear terms is $O(t^{-1})$, so the cubic nonlinearity must be regarded as a long-range perturbation. What we can expect for general cubic nonlinear Schrödinger equations in $\mathbb{R}$ is the lower estimate for the lifespan $T_\varepsilon$ in the form $T_\varepsilon \geq \exp(c/\varepsilon^2)$

Mathematics Subject Classification: 35Q55, 35B40

Keywords: Cubic derivative nonlinear Schrödinger equation, Large time behavior, Weakly dissipative structure.

This work is partly supported by Osaka City University Advanced Mathematical Institute (MEXT Joint Usage/Research Center on Mathematics and Theoretical Physics JPMXP0619217849). The work of H. S. is supported by Grant-in-Aid for Scientific Research (C) (No. 17K05322), JSPS.
with some $c > 0$, and this is best possible in general (see [16] for an example of small data blow-up). More precise information on the lower bound is available under the restriction

$$N(e^{i\theta}, 0) = e^{i\theta} N(1, 0), \quad \theta \in \mathbb{R}.$$  \hspace{2cm} (2)

According to [29] (see also [31]), if we assume (2) and the initial condition in (1) is replaced by $u(0, x) = \varepsilon \psi(x)$ with $\psi \in H^3 \cap H^{2,1}$, then it holds that

$$\liminf_{\varepsilon \to +0} \varepsilon^2 \log T_\varepsilon \geq \frac{1}{2} \sup_{\xi \in \mathbb{R}} (|F\psi(\xi)|^2 \text{Im } \nu(\xi))$$  \hspace{2cm} (3)

with the convention $1/0 = +\infty$, where the function $\nu : \mathbb{R} \to \mathbb{C}$ is defined by

$$\nu(\xi) = \frac{1}{2\pi i} \oint_{|z|=1} N(z, i\xi z) \frac{dz}{z^2}$$  \hspace{2cm} (4)

and $F$ denotes the Fourier transform, i.e.,

$$(F\psi)(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-iy\xi} \psi(y) dy$$

for $\xi \in \mathbb{R}$. Note that (2) excludes just the worst terms $u^3$, $|u|^2 \bar{u}$, $\bar{u}$. As pointed out in [4–8,24,25], etc., these three terms make the situation much more complicated. We do not intend to pursue this case here. We always assume (2) in what follows.

In view of the right-hand side in (3), it may be natural to expect that the sign of $\text{Im } \nu(\xi)$ has something to do with global behavior of small data solutions to (1). In fact, it has been pointed out in [29] that typical results on small data global existence and large-time asymptotic behavior for (1) under (2) can be summarized in terms of $\text{Im } \nu(\xi)$ as follows.

(i) The small data global existence holds in $H^3 \cap H^{2,1}$ under the condition

$$\text{Im } \nu(\xi) \leq 0, \quad \xi \in \mathbb{R}.$$  \hspace{2cm} (A)

(ii) If the inequality in (A) is replaced by the equality, i.e.,

$$\text{Im } \nu(\xi) = 0, \quad \xi \in \mathbb{R},$$  \hspace{2cm} (A_0)

then the solution has a logarithmic phase correction in the asymptotic profile, i.e., it holds that

$$u(t, x) = \frac{1}{\sqrt{t}} \alpha^+(x/t) \exp \left( \frac{i x^2}{2t} - i |\alpha^+(x/t)|^2 \text{Re } \nu(x/t) \log t \right)$$

$$+ o(t^{-1/2})$$

as $t \to +\infty$ uniformly in $x \in \mathbb{R}$, where $\alpha^+(\xi)$ is a suitable $\mathbb{C}$-valued function of $\xi \in \mathbb{R}$. 


(iii) If the inequality in (A) is strict, i.e.,
\[ \sup_{\xi \in \mathbb{R}} \text{Im } \nu(\xi) < 0, \]  
then the solution gains an additional logarithmic time decay \[ \|u(t)\|_{L^\infty} = O((t \log t)^{-1/2}). \]

For more details on each case, see the references cited in Sect. 1 of [29]. As for the large time behavior in the sense of \( L^2_x \) under (A), it is not difficult to see that (A) implies \( \lim_{t \to +\infty} \|u(t)\|_{L^2} = 0 \), whereas (A0) implies \( \lim_{t \to +\infty} \|u(t)\|_{L^2} \neq 0 \) for generic initial data of small amplitude. However, it is not clear whether \( L^2 \)-decay occurs or not in the other cases (even for a simple example such as \( N(u, u_x) = -i |u_x|^2(u + u_x) + (u^3)_x \)), for which we have
\[ \nu(\xi) = \frac{1}{2\pi i} \oint_{|z|=1} (-i \xi^2 (1 + i \xi)) |z|^2 z + 3i \xi z^3 \frac{dz}{z^2} = -i \xi^2 + \xi^3 \]
and \( \text{Im } \nu(\xi) = -\xi^2 \). Despite the recent progress of studies on dissipative nonlinear Schrödinger equations ([1,9–12,15,17–19,22,23,30], etc.), questions on decay/non-decay in \( L^2_x \) without (A+) have not been addressed in the previous works except [20] and [21].

The aim of this paper is to fill in the missing piece between (A+ and (A0), that is, to investigate \( L^2 \)-decay property of global solutions to (1) under (2) and (A) without (A+) and (A0). Our main result is as follows.

**Theorem 1.** Suppose that \( \varepsilon = \|\varphi\|_{H^3 \cap H^{2,1}} \) is sufficiently small. Assume that (2) and (A) are satisfied but (A0) is violated. Then, for any \( \delta > 0 \), there exists a positive constant \( C \) such that the global solution \( u \) to (1) satisfies
\[ \|u(t)\|_{L^2} \leq C \varepsilon (1 + \varepsilon^2 \log(t + 2))^{1/4 - \delta} \]
for \( t \geq 0. \)

**Remark 1.** Under (2) and (A+), we can show the global solution to (1) has the stronger \( L^2 \)-decay of order \( O((\log t)^{-3/8 + \delta}) \) with arbitrarily small \( \delta > 0 \) by the same method. For the detail, see Remark 4 below.

**Remark 2.** An analogous result for the semilinear wave equation \( \Box u = F(\partial u) \) in two space dimensions has been obtained in the recent paper [28]. To be more specific, the energy decay has been shown in [28] under a suitable structural condition on the cubic part of \( F \), which is comparable to Theorem 1.

**Remark 3.** In the case of systems, the situation is much more delicate than the single case. Detailed discussions on a weakly dissipative nonlinear Schrödinger system relevant to the present work can be found in [20] and [21] (see also [27] and [26] for closely related works on a system of semilinear wave equations in two space dimensions).
2. Proof

The rest part of this paper is devoted to the proof of Theorem 1. The argument will be divided into four steps.

Step 1: We begin with the following elementary lemma, whose proof is skipped.

**Lemma 1.** Let \( p(\xi) \) be a real polynomial with \( \deg p \leq 3 \). If \( p(\xi) \geq 0 \) for all \( \xi \in \mathbb{R} \), then we have either of the following three assertions.

(a) \( p(\xi) \) vanishes identically on \( \mathbb{R} \).

(b) \( \inf_{\xi \in \mathbb{R}} p(\xi) > 0 \).

(c) There exist \( c_0 > 0 \) and \( \xi_0 \in \mathbb{R} \) such that \( p(\xi) = c_0 (\xi - \xi_0)^2 \).

For \( \nu(\xi) \) given by (4), we put \( p(\xi) = -\text{Im} \nu(\xi) \). Since we assume that \((A)\) is satisfied but \((A_0)\) is violated, we see that the case (a) in Lemma 1 is excluded. Note also that (b) is equivalent to \((A_+)\). Now, let us turn our attentions to the admissible range of the parameter \( \theta \) for convergence of the integral

\[
I_\theta = \int_{\mathbb{R}} \frac{d\xi}{p(\xi)^{\theta}(\xi)^{4-4\theta}}
\]

under (c) or (b). In the case (c), we have

\[
I_\theta = c_0^{-\theta} \int_{\mathbb{R}} \frac{d\xi}{|\xi - \xi_0|^{2\theta}} < \infty
\]

for \( \theta < 1/2 \). In the case (b), we have

\[
I_\theta \leq (\inf_{\xi \in \mathbb{R}} p(\xi))^{-\theta} \int_{\mathbb{R}} \frac{d\xi}{(\xi)^{4-4\theta}} < \infty
\]

for \( \theta < 3/4 \).

Step 2: Next we summarize the basic estimates for the global solution \( u \) to (1). In what follows, we denote various positive constants by the same letter \( C \) which may vary from one line to another.

First, we write \( J = x + it\partial_x \) and \( L = i\partial_t + \frac{1}{2} \partial_x^2 \). We note the important commutation relations \([\partial_x, J] = 1, [L, J] = 0\). Next, we set \( U(t) = \exp(i \frac{t}{2} \partial_x^2) \) and \( \alpha(t, \xi) = \mathcal{F}[U(-t)u(t, \cdot)](\xi) \) for the solution \( u \) to (1). According to the previous works ([3,9,29], etc.), we already know the following estimates.

**Lemma 2.** Let \( \epsilon = \|\varphi\|_{H^3} \geq 1 \) be suitably small. Assume that (2) and (A) are fulfilled. Then, the solution \( u \) to (1) satisfies

\[
|\alpha(t, \xi)| \leq \frac{C\epsilon}{(\xi)^2}
\]

for \( t \geq 0, \xi \in \mathbb{R} \), and

\[
\|u(t)\|_{H^3} + \|Ju(t)\|_{H^2} \leq C\epsilon(1 + t)^{\gamma}
\]

for \( t \geq 0 \), where \( 0 < \gamma < 1/12 \).
The following lemma has been obtained in [29] (see also [3]). We write \( \alpha_\omega(t, \xi) = \alpha(t, \xi/\omega) \) for \( \omega \in \mathbb{R} \setminus \{0\} \).

**Lemma 3.** Under the assumption (2), we have

\[
\mathcal{FU}(-t)(1 - \partial_x^2)N(u, u_x) = (1 + \xi^2) \frac{\nu(\xi)}{t} |\alpha|^2 \alpha + \frac{\xi e^{it\xi^2/3}}{t} \mu_1(\xi) \alpha_3^3 + \frac{\xi e^{it\xi^2/3}}{t} \mu_2(\xi)(\overline{\alpha_{-3}})^3 + \frac{\xi e^{it\xi^2}}{t} \mu_3(\xi)|\alpha_{-1}|^2 \overline{\alpha_{-1}} + R,
\]

where \( \nu(\xi) \) is given by (4), \( \mu_1(\xi), \mu_2(\xi), \mu_3(\xi) \) are polynomials in \( \xi \) of degree at most 4, and \( R(t, \xi) \) satisfies

\[
|R(t, \xi)| \leq C \frac{\varepsilon t^{3/4} (\|u(t)\|_{H^3} + \|Ju(t)\|_{H^2})^3}{t^{1+\kappa}}
\]

for \( t \geq 1, \xi \in \mathbb{R} \).

For the proof, see Lemma 4.3 in [29]. By (7) and (9), we have

\[
|\partial_t \alpha(t, \xi)| \leq C \frac{\varepsilon t^{3} (\|u(t)\|_{H^3} + \|Ju(t)\|_{H^2})^3}{t^{1+\kappa}}
\]

for \( t \geq 1, \xi \in \mathbb{R} \), where \( \kappa = 1/4 - 3\gamma > 0 \). This indicates that \( R \) can be regarded as a remainder in (8). We also observe that one \( \xi \) pops up in front of the oscillating factors in (8). This is the point where (2) plays a crucial role. As for the role of \( \nu(\xi) \), the first term of the right-hand side in (8) tells us that \( \nu(\xi) \) is responsible for the contribution from the gauge-invariant part in \( N \).

**Step 3:** We are going to make some reductions. The goal in this step is to derive the ordinary differential equation (13) (with \( \xi \in \mathbb{R} \) regarded as a parameter).

Let \( t \geq 2 \) from now on. By the relation \( \mathcal{L} = \mathcal{U}(t)i\partial_t \mathcal{U}(-t) \) and Lemma 3, we have

\[
i \partial_t \alpha(t, \xi) = \mathcal{FU}(-t)\mathcal{L}u = \langle \xi \rangle^{-2} \mathcal{FU}(-t)(1 - \partial_x^2)N(u, u_x)
\]

\[
= \frac{\nu(\xi)}{t} |\alpha(t, \xi)|^2 \alpha(t, \xi) + \eta(t, \xi) + \langle \xi \rangle^{-2} R(t, \xi),
\]

where

\[
\eta(t, \xi) = \frac{\xi e^{it\xi^2/3}}{t} \mu_1(\xi) \langle \xi \rangle^2 \alpha_3^3 + \frac{\xi e^{it\xi^2/3}}{t} \mu_2(\xi) \langle \xi \rangle^2 \overline{\alpha_{-3}}^3 + \frac{\xi e^{it\xi^2}}{t} \mu_3(\xi) \langle \xi \rangle^2 |\alpha_{-1}|^2 \overline{\alpha_{-1}}.
\]

It follows from (6), (10) and (11) that

\[
|\partial_t \alpha(t, \xi)| \leq C \frac{(\langle \xi \rangle)^3}{t} \left( \frac{C\varepsilon}{\langle \xi \rangle^2} \right)^3 + \frac{C\varepsilon^3}{t^{1+\kappa} \langle \xi \rangle^2} \leq C \frac{\varepsilon^3}{\langle \xi \rangle^2}.
\]
Also, by using the identity
\[
\frac{\xi e^{i\omega t\xi^2}}{t} f(t, \xi) = \frac{\xi \partial_t (te^{i\omega t\xi^2})}{t(1 + i\omega t\xi^2)} f(t, \xi)
\]
\[= i \partial_t \left( -\frac{-i\xi e^{i\omega t\xi^2}}{1 + i\omega t\xi^2} f(t, \xi) \right) - te^{i\omega t\xi^2} \partial_t \left( \frac{\xi f(t, \xi)}{t(1 + i\omega t\xi^2)} \right) \]
and the inequality
\[
\sup_{\xi \in \mathbb{R}} \frac{|\xi|^a}{|1 + i\omega t\xi^2|} \leq \frac{C}{(|\omega|t)^{a/2}}
\]
for \(0 \leq a \leq 2\), we see that \(\eta(t, \xi)\) can be split into
\[
\eta = i \partial_t \sigma_1 + \sigma_2; \quad |\sigma_1(t, \xi)| \leq \frac{C \varepsilon^3}{t^{1/2} \langle \xi \rangle^4}, \quad |\sigma_2(t, \xi)| \leq \frac{C \varepsilon^3}{t^{3/2} \langle \xi \rangle^4}.
\]  
(12)

With this \(\sigma_1\), we set \(\beta(t, \xi) = \alpha(t, \xi) - \sigma_1(t, \xi)\). Then, it follows from (11) that
\[
i \partial_t \beta(t, \xi) = \frac{\nu(\xi)}{t} |\beta(t, \xi)|^2 \beta(t, \xi) + \rho(t, \xi),
\]  
(13)

where
\[
\rho(t, \xi) = \frac{\nu(\xi)}{t} \left( |\alpha|^2 \alpha - |\beta|^2 \beta \right) + \sigma_2 + \langle \xi \rangle^{-2} R
\]
\[= \frac{\nu(\xi)}{t} \left( 2|\alpha|^2 \sigma_1 + |\alpha|^2 \sigma_1 - 2|\alpha|^2 \sigma_1 - \alpha \sigma_1^2 + |\sigma_1|^2 \sigma_1 \right) + \sigma_2 + \langle \xi \rangle^{-2} R.
\]

By (6), (10) and (12), we have
\[
|\rho(t, \xi)| \leq \frac{C \langle \xi \rangle^3}{t} \left( \frac{C \varepsilon}{t^{1/2} \langle \xi \rangle^4} + \frac{C \varepsilon^3}{t^{1/2} \langle \xi \rangle^4} + \frac{C \varepsilon^3}{t^{1/2} \langle \xi \rangle^4} + \frac{C \varepsilon^3}{t^{1+\kappa} \langle \xi \rangle^2} \right)
\]
\[\leq \frac{C \varepsilon^3}{t^{1+\kappa} \langle \xi \rangle^2}.
\]
Remember that \(0 < \kappa < 1/4\).

Roughly speaking, what we have seen so far is that the solution \(u\) to (1) under (2) can be expressed as
\[
u = \mathcal{U}(t) \mathcal{F}^{-1} \beta + \cdots
\]
with
\[
i \partial_t \beta = \frac{\nu(\xi)}{t} |\beta|^2 \beta + \cdots,
\]
where the terms “+ · · ·” are expected to be harmless. By this reason it would be fair to call (13) the profile equation associated with (1) under (2). The original idea of this reduction is due to Hayashi–Naumkin [2].

Final step: Before going further, let us recall the following useful lemma due to Matsumura.
Lemma 4. Let $C_0 > 0$, $C_1 \geq 0$, $q > 1$ and $s > 1$. Suppose that a function $\Phi(t)$ satisfies

$$\frac{d\Phi}{dt}(t) \leq -\frac{C_0}{t} |\Phi(t)|^q + \frac{C_1}{t^s}$$

for $t \geq 2$. Then, we have

$$\Phi(t) \leq \frac{C_2}{(\log t)^{q^*-1}}$$

for $t \geq 2$, where $q^*$ is the Hölder conjugate of $q$ (i.e., $1/q + 1/q^* = 1$), and

$$C_2 = \frac{1}{\log 2} \left( (\log 2)^q \Phi(2) + C_1 \int_2^{\infty} \frac{(\log \tau)^q}{\tau^s} d\tau \right) + \left( \frac{q^*}{qC_0} \right)^{q^*-1}.$$

For the proof, see Lemma 4.1 in [13]. At last, we are in a position to reach the conclusion. We set $\Phi(t, \xi) = p(\xi)|\beta(t, \xi)|^2$ with $p(\xi) = -\text{Im} \nu(\xi)$. Note that $\Phi(t, \xi) \geq 0$ by (A). It follows from (13) that

$$\partial_t \Phi(t, \xi) = 2p(\xi)\text{Im} \left( \overline{\beta(t, \xi)}i \partial_t \beta(t, \xi) \right)$$

$$= 2p(\xi) \left( \frac{\text{Im} \nu(\xi)}{t} |\beta(t, \xi)|^4 + \text{Im} \left( \overline{\beta(t, \xi)} \rho(t, \xi) \right) \right)$$

$$\leq -\frac{2p(\xi)^2}{t} |\beta(t, \xi)|^4 + C \langle \xi \rangle^3 \frac{C \epsilon}{\langle \xi \rangle^2 t^{1+\kappa}}$$

$$\leq -\frac{2}{t} \Phi(t, \xi)^2 + \frac{C \epsilon^4}{t^{1+\kappa} \langle \xi \rangle},$$

where $\kappa \in (0, 1/4)$. We also note that (6) yields

$$\Phi(2, \xi) \leq C \langle \xi \rangle^3 \left( \frac{C \epsilon}{\langle \xi \rangle^2} \right)^2 \leq \frac{C \epsilon^2}{\langle \xi \rangle}.$$

Therefore, we can apply Lemma 4 with $q = 2$ and $s = 1 + \kappa$ to obtain

$$0 \leq \Phi(t, \xi) \leq \frac{C}{\log t},$$

whence

$$|\alpha(t, \xi)| \leq \sqrt{\frac{\Phi(t, \xi)}{p(\xi)}} + |\sigma_1(t, \xi)|$$

$$\leq \frac{C}{\sqrt{p(\xi) \log t}} \left( 1 + \epsilon^3 \sqrt{\frac{p(\xi)}{\langle \xi \rangle^4}} \sqrt{\frac{\log t}{t}} \right)$$

$$\leq \frac{C \epsilon}{\sqrt{p(\xi) \epsilon^2 \log t}}.$$
Interpolating this with (6), we deduce that

$$|\alpha(t, \xi)| \leq \frac{C\varepsilon}{(e^2 \log t)^{\theta/2}} \frac{1}{p(\xi)^{\theta/2} \langle \xi \rangle^{2-2\theta}}$$

(14)

for $\theta \in [0, 1]$. By the $L^2$-unitarity of $\mathcal{U}(t)$ and $\mathcal{F}$, we have

$$\|u(t)\|_{L^2}^2 = \|\alpha(t)\|_{L^2}^2 \leq \frac{C\varepsilon^2}{(e^2 \log t)^\theta} I_\theta$$

(15)

for $0 \leq \theta < \frac{1}{2}$, where $I_\theta$ is given by (5). Therefore, we can take $\theta = 1/2 - 2\delta$ with $\delta > 0$ to see that

$$\|u(t)\|_{L^2} \leq \frac{C\varepsilon}{(e^2 \log t)^{1/4-\delta}}.$$

Also we obtain $\|u(t)\|_{L^2} \leq C\varepsilon$ by taking $\theta = 0$ in (15). Piecing them together, we arrive at the desired estimate.

**Remark 4.** Under (2) and the stronger condition $(\mathbf{A}_{++})$, it is possible to choose $\theta = 3/4 - 2\delta$ in (15) because $(\mathbf{A}_{++})$ implies (b) in Lemma 1 and thus the admissible range for $\theta$ in (15) becomes $0 \leq \theta < \frac{3}{4}$. That is the reason why $\|u(t)\|_{L^2}$ decays like $O((\log t)^{-3/8+\delta})$ under $(\mathbf{A}_{++})$. It is not certain whether this rate is the best or not. Indeed, it is possible to improve the exponent from $-3/8 + \delta$ to $-1/2$ if there exists a positive constant $C_*$ such that

$$\text{Im} \nu(\xi) \leq -C_*(\xi)^2, \quad \xi \in \mathbb{R} \quad (\mathbf{A}_{++})$$

(cf. Theorem 2.3 in [22]). A typical example of $N$ satisfying $(\mathbf{A}_{++})$ is $-i|u + u_x|^2u$.

It may be an interesting problem to specify the optimal $L^2$-decay rates for the solutions to (1) under (2) and (A) (with or without $(\mathbf{A}_{++})$).

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Accepted: 25 September 2020