LIMITS FOR PARTIAL MAXIMA OF GAUSSIAN
RANDOM VECTORS

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ABSTRACT. We obtain almost sure limit theorems for partial maxima
of norms of a sequence of Banach-valued Gaussian random variables.

1. INTRODUCTION

Limit theorems for various maximal functions of a sequence of random
variables or a continuous time process have a rich and extensive history.
They include distributional results as well as almost sure results in a variety
of settings, and here we obtain related results for Gaussian sequences and the
Ornstein-Uhlenbeck process with values in a separable Banach space. The
results we establish are almost sure limit theorems motivated by the work
of Berman [Ber62] and Pickands [Pic67] for real-valued random variables,
which can be viewed as laws of large numbers when the maximal functions
are appropriately centered. We also deal with some almost sure results
related to the classical Darling-Erdős Theorem [DE56].

We now state a few of the results mentioned above, so the reader can
have some points of comparison. Since the Darling-Erdős Theorem was the
earliest of those in the general one-dimensional context, we’ll start with that.

Notation: Throughout the paper we take $Lx =: \max\{\log_e x, 1\}$.

Motivated by a paper of Robbins [Rob52], Darling and Erdős [DE56]
obtained the following distributional limit theorem.

**Theorem A.1.** Let $\{ξ, ξ_j : j \geq 1\}$ be iid rv’s with $Eξ = 0, Eξ^2 = 1$ and
$E|ξ|^3 < \infty$. Further, let $S_k = \sum_{j=1}^{k} ξ_j$ and

$$
α_n = (2LLn)^{1/2} \text{ and } β_n = (2LLn + \frac{1}{2} LLLn - \frac{1}{2} L(4\pi)).
$$

Then for every $x ∈ \mathbb{R}$,

$$
\lim_{n→∞} \Pr \left( α_n(\max_{k≤n} k^{-1/2} S_k - \frac{β_n}{α_n}) ≤ x \right) = \exp\{-e^{-x}\}.
$$

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Remark 1.1. Note that this differs from a corollary of the “invariance principle”, which yields a distributional limit theorem for \( \max_{k \leq n} n^{-1/2} S_k \). Hence, as one might expect, the result obtained in (1.1) involves a number of intricate steps. The first establishes the Gaussian case, and then the Berry-Esseen theorem is used to obtain the result just stated. Along the way a form of the invariance principle, the law of the iterated logarithm, and a comparison to the Ornstein-Uhlenbeck process are employed.

The law of large numbers results by Berman [Ber62] and followed by Pickands [Pic67] are as follows.

**Theorem A2.** (Berman 1962) Let \( \{\eta_j\} \) be a stationary Gaussian process such that \( \mathbb{E} \eta_j = 0, \mathbb{E} \eta_j^2 = 1 \) and for \( \mathbb{E} \eta_k \eta_j = r_{ij}, nr_n \to 0 \). Then \( \max_{k \leq n} \eta_k - (2Ln)^{1/2} \to 0 \), in probability.

The results in Pickands extend this result to convergence a.s., and also obtains similar results for real-valued continuous time stationary Gaussian processes.

In the recent paper [DE16], Dierickx and Einmahl establish a version of the Darling-Erdős Theorem for the Euclidean norm of sums of \( \mathbb{R}^d \)-valued random vectors. Thus one might ask whether similar results might hold for Banach-valued random variables. Some first results related to almost sure limits are obtained in Corollary 4.1 below, but even for Gaussian random vectors much remains unsettled. In fact, whether the Darling-Erdős theorem is valid in the setting we are studying remains open. The potential difficulties dealing with other norms on finite dimensional spaces seem considerable - much less norms on infinite dimensional spaces. Furthermore, since the Darling-Erdős theorem deals with (partial sums of) iid rv's under a moment condition with CLT normalizations, and CLT’s in the infinite dimensional setting require restrictions on the geometry of the space, there are many additional difficulties. Hence, as was done in the study of limit theorems for Banach-valued rv's, it is natural to first consider the case of the law of large numbers in this setting. In addition, the nature of the explicit centerings and the importance of Gaussian random variables in the proofs, even for real-valued random variables and processes, suggests that tools developed for log and loglog laws for Banach-valued random vectors as in [CK76], [LT88], [LeP73] and [GK92] may be of some use. We have found this to be the case for the law of large number results established in this paper, but explicit distributional results such as those of the Darling-Erdős theorem involve additional difficulties.

Goodman’s paper [Goo88] provided additional motivation for the questions addressed here, and the first part of Theorem 2.1 in [Goo88] as it relates to the maxima of the norm of i.i.d. Gaussian random vectors in the Banach space setting emerges as a special case of (2.30) in Corollary 2.2.

In Section 2 we present some notation and strong law results for maxima of the norm of sequences of Banach-valued random vectors, and their
provides analogous results for continuous time processes, a corollary for stationary Gaussian processes in this setting, and the proofs of these results. Section 4 studies the Ornstein-Uhlenbeck process, and its implications for maximal functions of norms of normalized partial sums in the Banach space setting. These are quantities that would naturally appear in a Darling-Erdős result for normalized partial sums of centered Gaussian random vectors.

Section 4 also provides some applications to self-adjoint operator-valued Gaussian random vectors and their spectrums, and symmetrization results in Proposition 1. When combined with the centerings in our law of large number results as in Proposition 2, these symmetrizations allow us to determine the asymptotic behavior of the various medians of these partial maxima. Determining such asymptotics by direct calculation appears far less promising.

Section 5 starts with details on the sample function continuity of the Banach-valued Ornstein-Uhlenbeck process. This allows us to examine this process in enough detail that the technical assumptions for the continuous time results of Section 3 emerge as consequences of the definition of the process itself.

2. Strong Laws for Partial Maxima of Sequences

Throughout this section \( B \) is a separable Banach space over the reals with norm \( \| \cdot \| \), and its dual space is denoted by \( B^* \) with norm \( \| \cdot \|^* \). A probability measure \( \mu \) on the Borel subsets of \( B \) is a centered Gaussian measure if every linear functional \( h \in B^* \) has a Gaussian distribution with mean zero and variance \( \int_B h^2(x) d\mu(x) \). Since \( \mu \) is a centered Gaussian measure on \( B \) we recall some necessary notation that can be found in Lemma 2.1 of \cite{Kue76}.

That is, since \( \mu \) is a centered Gaussian measure, then

\[
\int_B \|x\|^2 d\mu(x) < \infty.
\]

and there is a unique Hilbert space \( H_\mu \subseteq B \) given by the completion of the linear space \( S(B^*) \) in the inner product

\[
\langle Sf, Sg \rangle_{\mu} = \int_B f(x) g(x) d\mu(x),
\]

where

\[
Sf =: \int_B x f(x) d\mu(x), f \in B^*,
\]

are Bochner integrals in \( B \). Furthermore, Lemma 2.1 in \cite{Kue76} implies

\[
\sigma(\mu) := \sup_{\|f\|^* \leq 1} \left( \int_B f^2(x) d\mu(x) \right)^{1/2} < \infty,
\]

and

\[
\|x\| \leq \sigma(\mu) \|x\|_{\mu}, x \in H_\mu,
\]
and the unit ball of $H_\mu$,

\[(2.6)\quad K =: \{ x \in H_\mu : \| x \|_\mu \leq 1 \},\]

is a compact subset of $B$ with $\Gamma =: \sup_{x \in K} \| x \| \text{ finite}$. The Hilbert space $H_\mu$ is often referred to as the Hilbert space that generates the Gaussian measure $\mu$.

The following lemma further explains the notation used in our results. It links $\Gamma$ to $\sigma(\mu)$ in (2.7) and also to the variance of a single linear functional in (2.8). This is relevant in that it links the normalizations of the terms maximized in our limit theorems to those one would use for i.i.d. real-valued centered Gaussian variables. It appears in connection with the assumptions (2.19) and (2.23) we use to obtain the liminf in our strong law results for maximums of sequences of centered Gaussian random vectors, and at least in Corollary 2.2 the lemma allows us to simplify these cumbersome assumptions because of what is known for real-valued Gaussian random variables.

**Lemma 2.1.** Let $\mu$ be a mean zero Gaussian measure on $B$ with norm $\| \cdot \|$, and assume $H_\mu$ is as above with (2.2)-(2.6) holding. If $\Gamma = \sup_{x \in K} \| x \|$, then

\[(2.7)\quad \Gamma = \sigma(\mu).\]

Moreover, if $f_0 \in B^*$ and $x_0 \in K$ with $\| x_0 \| = \Gamma, f_0(x_0) = \Gamma$, and $\| f_0 \|^* = 1$, then

\[(2.8)\quad (\int_B f_0^2(x)d\mu(x))^{1/2} = \Gamma.\]

Furthermore, if we define $\tilde{q}(x) = \| x \|/\Gamma, x \in B$, then $\tilde{q}(\cdot)$ is a norm on $B$ with dual norm $\tilde{q}^*(f) = \sup_{\{ x : \tilde{q}(x) \leq 1 \}} f(x), f \in B^*$, and

\[(2.9)\quad \Gamma_{\tilde{q}} = \sigma(\mu)_{\tilde{q}} = 1,\]

where

\[(2.10)\quad \Gamma_{\tilde{q}} =: \sup_{x \in K} \tilde{q}(x) \quad \text{and} \quad \sigma(\mu)_{\tilde{q}} =: \sup_{\tilde{q}^*(f) \leq 1} (\int_B f^2(x)d\mu(x))^{1/2} < \infty.\]

In addition, if $f_0 \in B^*$ and $x_0 \in K$ with $\tilde{q}(x_0) = 1, f_0(x_0) = 1$, and $\tilde{q}^*(f_0) = 1$, then

\[(2.11)\quad (\int_B f_0^2(x)d\mu(x))^{1/2} = 1.\]

**Remark 2.1.** Since $K$ is compact in $B$, the Hahn-Banach theorem gives us $f_0 \in B^*$ and $x_0 \in K$ with $\| x_0 \| = \Gamma, f_0(x_0) = \Gamma$, and $\| f_0 \|^* = 1$. For all such choices of $f_0$ and $x_0$ the point of the lemma is that one then always has

\[(\int_B f_0^2(x)d\mu(x))^{1/2} = \Gamma.\]
Moreover, in view of (2.9) and (2.11) there is no loss of generality in assuming that the norm $\| \cdot \|$ on $B$ is such that
\begin{equation}
\Gamma = \sup_{x \in K} \| x \| = 1. 
\end{equation}

**Proof of Lemma 2.** For each $f \in B^*$, (2.4) in \[Kue76\] implies
\[\sup_{x \in K} f(x) = \left( \int_B f^2(y) d\mu(y) \right)^{\frac{1}{2}}.\]
Taking the sup over all $f \in B^*$ with $\| f \| = 1$, and interchanging sups on the left term we immediately have (2.7). Assuming the conditions on $f_0 \in B^*$ and $x_0 \in K$ for (2.8) we also have
\[\left( \int_B f_0^2(y) d\mu(y) \right)^{\frac{1}{2}} = \sup_{x \in K} f_0(x) \leq \sup_{x \in K} \| x \| = \Gamma\]
and
\[\sup_{x \in K} f_0(x) \geq f_0(x_0) = \Gamma.\]
Thus (2.8) holds, and when combined with (2.7) the remainder of the lemma in (2.9) and (2.11) also holds. \hfill \Box

**Theorem 2.** Let $\mu, \mu_1, \mu_2, \cdots$ be centered non-degenerate Gaussian measures on $B$ with norm $\| \cdot \|$, and $X, X_1, X_2, \cdots$ be $B$-valued random vectors on some probability space with distributions $\mu, \mu_1, \mu_2, \cdots$. In addition, assume
\begin{equation}
\Gamma = \sup_{x \in K} \| x \| \quad \text{and} \quad \Gamma_n = \sup_{x \in K_n} \| x \|, \quad n \geq 1, 
\end{equation}
where $K, K_1, K_2, \cdots$ are the unit balls of the Hilbert spaces $H_\mu, H_{\mu_1}, \cdots$ that generate the Gaussian measures $\mu, \mu_1, \mu_2, \cdots$, and for $n \geq 1$ that
\begin{equation}
\bar{M}_n = \max_{1 \leq k \leq n} \frac{\| X_k \|}{\Gamma_k} \quad \text{and} \quad M_n = \max_{1 \leq k \leq n} \frac{\| X_k \|}{\Gamma}. 
\end{equation}
If $\{ \mu_n : n \geq 1 \}$ is assumed to converge weakly to $\mu$ in $B$, then
\begin{equation}
\lim_{n \to \infty} \Gamma_n = \Gamma > 0, 
\end{equation}
and with probability one
\begin{equation}
\limsup_{n \to \infty} [\bar{M}_n - \sqrt{2Ln}] \leq 0. 
\end{equation}
Moreover, if
\begin{equation}
\frac{\Gamma_k}{\Gamma} - 1 = o((\sqrt{Lk})^{-1}), 
\end{equation}
then with probability one
\begin{equation}
\limsup_{n \to \infty} [M_n - \sqrt{2Ln}] \leq 0. 
\end{equation}
In addition, if $f_n \in B^*$, $x_n \in K_n$ with $\| x_n \| = \Gamma_n$, $\| x_n \|_{\mu_n} = 1$, $f_n(x_n) = \Gamma_n$, and $\| f_n \|^* = 1$, then
\begin{equation}
\sigma_{f_n}^2 = \int_B f_n^2(x) d\mu_n(x) = \Gamma_n^2. 
\end{equation}
and if
\begin{equation}
\liminf_{n \to \infty} \max\{f_1(X_1, \Gamma_1), \cdots, f_n(X_n, \Gamma_n)\} - \sqrt{2Ln} \geq 0
\end{equation}
with probability one, we also have
\begin{equation}
\liminf_{n \to \infty} [\hat{M}_n - \sqrt{2Ln}] \geq 0
\end{equation}
with probability one and
\begin{equation}
\liminf_{n \to \infty} [M_n - \sqrt{2Ln}] \geq 0.
\end{equation}
with probability one whenever (2.17) is assumed.

An immediate corollary of Theorem 2.1 is the following.

**Corollary 2.1.** Let \( \mu \) be a centered Gaussian measure on \( B \) with norm \( \| \cdot \| \), and assume \( X_1, X_2, \cdots \) are defined on some probability space with each having distribution \( \mu \). If \( \Gamma = 1 \) as in (2.12), and
\[ M_n = \max\{\|X_1\|, \cdots, \|X_n\|\}, n \geq 1, \]
then with probability one
\begin{equation}
\limsup_{n \to \infty} [M_n - \sqrt{2Ln}] \leq 0.
\end{equation}
In addition, if \( f_0 \in B^* \), \( x_0 \in K \) with \( \|x_0\| = \|x_0\|_\mu = 1 \), \( f_0(x_0) = 1 \), and \( \|f_0\|^* = 1 \), then
\[
\sigma_{f_0}^2 = = : \int_B f_0^2(x) d\mu(x) = 1,
\]
and if
\begin{equation}
\liminf_{n \to \infty} \max\{f_0(X_1), \cdots, f_0(X_n)\} - \sqrt{2Ln} \geq 0
\end{equation}
with probability one, we also have
\begin{equation}
\liminf_{n \to \infty} [M_n - \sqrt{2Ln}] \geq 0
\end{equation}
with probability one.

**Remark 2.2.** In Theorem 2.1 and Corollary 2.1 it is not assumed the random vectors \( \{X_n : n \geq 1\} \) are jointly Gaussian, only that each is Gaussian. The limsup results of (2.16) and (2.22) are obtained using a Borel-Cantelli argument that is based on rates of convergence results for clustering and convergence of \( X_n \) to the set \( K_n \) obtained in [GK92]. Combined with (2.17) this sort of argument also yields (2.18). For real-valued \( \{X_n : n \geq 1\} \) this is fairly simple, but in the Banach space setting it is far less so. In contrast, the liminf results in the real-valued case are considerably more complex, but here they follow easily using Lemma 2.1 and the assumptions (2.19) and (2.23), which are likely to be hard (maybe even impossible) to verify in many settings. Situations where they can be simplified through a combination of Lemma 2.1 and [Pic67] appear in Corollary 2.2. Of course, using Lemma 2.1 the liminf results for for i.i.d. sequences can also be done directly.
The stationary case in Corollary 2.2 also appears in connection with results for the vector-valued Ornstein-Uhlenbeck process presented in the following sections. First we need a couple of definitions.

**Definition 2.1.** A sequence of $B$-valued random vectors $\{X_n : n \geq 1\}$ is stationary if for all integers $r \geq 1, h \geq 1$ the finite dimensional distributions of

\[(X_1, \ldots, X_r) \text{ and } (X_{1+h}, \ldots, X_{r+h}) \text{ on } B^r\]

are equal.

**Definition 2.2.** A sequence of $B$-valued random vectors $\{X_n : n \geq 1\}$ is a mean zero Gaussian sequence if for all integers $d \geq 1$ the finite dimensional distribution of $(X_1, \ldots, X_d)$ is a mean zero Gaussian measure on $B^d$.

**Corollary 2.2.** Let $\mu$ be a centered Gaussian measure on $B$ with norm $\| \cdot \|$, and assume $\{X_n : n \geq 1\}$ is a sequence of random vectors on some probability space with $\mathcal{L}(X_n) = \mu$ for $n \geq 1$. If $\Gamma = 1$ as in (2.12), and

\[M_n = \max\{\|X_1\|, \ldots, \|X_n\|\}, n \geq 1,\]

then with probability one

\[(2.27) \lim_{n \to \infty} \sup (M_n - \sqrt{2Ln}) \leq 0.\]

Furthermore, if for some $f_0 \in B^*$ such that $\|f_0\|^* = 1$ the sequence $\{f_0(X_n) : n \geq 1\}$ is a stationary mean zero Gaussian sequence with

\[\sigma_{f_0}^2 := \int_B f_0^2(x) d\mu(x) = 1,\]

and

\[(2.28) \lim_{n \to \infty} (\log n) E[f_0(X_1)f_0(X_n)] = 0,\]

then

\[(2.29) \lim_{n \to \infty} \inf (M_n - \sqrt{2Ln}) \geq 0\]

with probability one. In particular, if $\{X_n : n \geq 1\}$ are i.i.d. with $\Gamma = 1$, then with probability one

\[(2.30) \lim_{n \to \infty} (M_n - \sqrt{2Ln}) = 0.\]

**Remark 2.3.** If we assume $\{X_n : n \geq 1\}$ is a centered stationary Gaussian sequence in Corollary 2.2 then $\{f_0(X_n) : n \geq 1\}$ is a mean zero stationary real-valued Gaussian sequence for all $f_0 \in B^*$. Therefore, if we also have $\|f_0\|^* = 1$ with $f_0(x_0) = 1$ for some $x_0 \in K$, then Lemma 2.1 implies $\sigma^2(f_0) = 1$, and (2.28) then implies (2.29).
Proof of Theorem 2.1 and its Corollaries. Since the centered Gaussian measures \( \{ \mu_k : k \geq 1 \} \) are non-degenerate and converge weakly to the non-degenerate measure \( \mu \) on \( B \), (2.2) in Theorem 1 of [GK92] implies (2.15). Furthermore, for \( \epsilon > 0 \) and \( \epsilon_k = \epsilon / \sqrt{2Lk} \), under the assumptions of Theorem 2.1 (2.3) in Theorem 1 of [GK92] implies that with probability one

\begin{equation}
\frac{X_k(\omega)}{\sqrt{2Lk}} \in K_k + \epsilon_k U
\end{equation}

for all \( k \geq k_0(\omega, \epsilon) \). This implies

\begin{equation}
\frac{\|X_k(\omega)\|}{\sqrt{2Lk}} \leq \Gamma_k + \epsilon_k, \quad \text{and hence that} \quad \frac{\|X_k(\omega)\|}{\Gamma_k} \leq \sqrt{2Lk} + \epsilon
\end{equation}

for all \( k \geq k_0 =: k_0(\omega, \epsilon) \) with probability one. Since \( \tilde{M}_n(\omega) = \max_{1 \leq k \leq n} \frac{\|X_k(\omega)\|}{\Gamma_k} \), (2.16) is immediate for all \( \omega \) such that \( \sup_{n \geq 1} \tilde{M}_n(\omega) < \infty \). If \( \sup_{n \geq 1} \tilde{M}_n(\omega) = \infty \), then for \( k_0 = k_0(\omega, \epsilon) \) and all \( n \geq n_0(\omega, \epsilon) \)

\[ \tilde{M}_n(\omega) = \max_{k_0 \leq k \leq n} \frac{\|X_k(\omega)\|}{\Gamma_k} \leq \sqrt{2Ln} + \epsilon, \]

and, since \( \epsilon > 0 \) is arbitrary, for such \( \omega \) we have (2.16). Thus with probability one we have (2.16), and the limsup results in Corollaries 2.1 and 2.2, namely (2.22) and (2.27), also hold.

Now we turn to the proof of the liminf results in (2.20), and the implications for the liminf results of Corollaries 2.1 and 2.2. Given the assumptions on \( \{ f_n : n \geq 1 \} \) in Theorem 2.1 Lemma 2.1 implies \( \sigma_{f_n}^2 = \Gamma_n^2 \), and with probability one

\[ M_n(\omega) \geq \max_{1 \leq k \leq n} \frac{f_k(X_k(\omega))}{\Gamma_k}. \]

That (2.20) holds with probability one is now immediate from (2.19). The reader will note we have not used the fact that we have \( \sigma_n^2 = \Gamma_n^2 \), but we have included it since it is the normalization required to verify the analogue of assumption (2.19) in Corollary 2.2. An entirely similar argument also gives the liminf result in (2.28), so Corollary 2.1 is proven.

The liminf result in (2.29) of Corollary 2.2 follows from Theorem 3.3 in [Pic67] since \( \sigma_{f_0}^2 = 1 \), and (2.28) is assumed hold. To verify (2.30) observe that \( \Gamma = 1 \), and Lemma 2.1 implies there exists \( f_0 \in B^* \) with \( \|f_0\|^* = 1, \sigma_{f_0}^2 = 1 \), and such that \( \{ f_0(X_k) : k \geq 1 \} \) is a sequence of i.i.d Gaussian random variables with mean zero and variance one. Hence (2.28) of Corollary 2.2 is trivial, and (2.29) implies the liminf result for (2.30). Since the limsup result follows from (2.27), this proves Corollary 2.2.

What remains in the proof of Theorem 2.1 is to verify the limsup in (2.18) and the liminf in (2.21) hold with probability one when (2.17) is assumed.
For $\epsilon > 0$ and $\epsilon_k = \epsilon/\sqrt{2Lk}$, from (2.32) we have for all $\omega$ in a set of probability one and $k \geq k_0(\omega, \epsilon)$ that

$$\frac{\|X_k(\cdot, \omega)\|}{\sqrt{2Lk}} \leq \Gamma + (\Gamma - \Gamma) + \epsilon_k,$$

which implies

(2.33) $$\frac{\|X_k(\cdot, \omega)\|}{\Gamma} \leq \sqrt{2Lk} + (\frac{\Gamma_k}{\Gamma} - 1)\sqrt{2Lk} + \frac{\epsilon}{\Gamma}$$

for all $k \geq k_0(\omega, \epsilon)$ with probability one. Since we are assuming (2.17), there exists non-random $k_1$ such that $k \geq k_1(\epsilon)$ implies

$$|\frac{\Gamma_k}{\Gamma} - 1|\sqrt{2Lk} \leq \epsilon,$$

and combining with (2.33) this implies

(2.34) $$\frac{\|X_k(\cdot, \omega)\|}{\Gamma} \leq \sqrt{2Lk} + \epsilon + \frac{\epsilon}{\Gamma}$$

for all $k \geq k_2(\omega, \epsilon) =: \max\{k_0(\omega, \epsilon), k_1(\epsilon)\}$. Since

$$\max_{1 \leq k \leq n} \frac{\|X_k(\cdot, \omega)\|}{\Gamma}$$

is increasing in $n$ and (2.18) is trivial if it is bounded in $n$, we assume

(2.35) $$\sup_{n \geq 1} \max_{1 \leq k \leq n} \frac{\|X_k(\cdot, \omega)\|}{\Gamma} = \infty,$$

which implies

(2.36) $$\limsup_{n \to \infty} [\max_{1 \leq k \leq n} \frac{\|X_k(\cdot, \omega)\|}{\Gamma} - \sqrt{2Ln}] = \limsup_{n \to \infty} [\max_{k_2(\omega, \epsilon) \leq k \leq n} \frac{\|X_k(\cdot, \omega)\|}{\Gamma} - \sqrt{2Ln}] \leq \epsilon(1 + 1/\Gamma)$$

whenever (2.35) holds. Since $\epsilon > 0$ is arbitrary, when combined with the trivial case, this implies (2.18) with probability one.

Now we turn to the final step in the proof, which is to verify (2.21). Given the assumptions on the linear functionals $\{f_k\}$, we have

$$\max_{1 \leq k \leq n} \frac{\|X_k(\cdot, \omega)\|}{\Gamma} \geq \max_{1 \leq k \leq n} \frac{|f_k(X_k(\cdot, \omega))|}{\Gamma} - \sqrt{2Ln},$$

so it suffices to show that the assumption (2.19) implies that

(2.37) $$\liminf_{n \to \infty} [\max_{1 \leq k \leq n} \frac{|f_k(X_k(\cdot, \omega))|}{\Gamma} - \sqrt{2Ln}] \geq 0$$

with probability one. Since (2.19) implies

$$\sup_{n \geq 1} \max_{1 \leq k \leq n} \frac{|f_k(X_k(\cdot, \omega))|}{\Gamma_k}$$
increases to infinity with probability one, for every \( k_0 \geq 1 \) we have for \( n \geq n_0(\omega, k_0) \) that
\[
\max_{1 \leq k \leq n} \frac{|f_k(X_k(\cdot, \omega))|}{\Gamma_k} = \max_{k_0 \leq k \leq n} \frac{|f_k(X_k(\cdot, \omega))|}{\Gamma_k}
\]
with probability one, and (2.19) implies
\[
\liminf_{n \to \infty} \left[ \max_{k_0 \leq k \leq n} \frac{|f_k(X_k(\cdot, \omega))|}{\Gamma_k} - \sqrt{2Ln} \right] \geq 0
\]
with probability one. Now for each \( \epsilon > 0 \), (2.17) allows us to choose \( k_1 =: k_1(\epsilon) \geq 1 \) such that
\[
k \geq k_1 \implies |\Gamma_k - \sqrt{2Ln}| < \epsilon.
\]
In addition, note that
\[
\frac{|f_k(X_k(\cdot, \omega))|}{\sqrt{2Ln} \Gamma_k} \leq 2
\]
for all \( k \geq k_2(\omega) \) with probability one, and
\[
\frac{|f_k(X_k(\cdot, \omega))|}{\Gamma} = \frac{|f_k(X_k(\cdot, \omega))|}{\Gamma_k} \frac{\Gamma_k}{\Gamma} = \frac{|f_k(X_k(\cdot, \omega))|}{\Gamma_k} + \frac{|f_k(X_k(\cdot, \omega))|}{\Gamma_k} \left( \frac{\Gamma_k}{\Gamma} - 1 \right).
\]
Therefore, for \( k \geq k_0 = k_0(\omega, \epsilon) \geq \max\{k_1(\epsilon), k_2(\omega)\} \)
\[
\frac{|f_k(X_k(\cdot, \omega))|}{\Gamma} \geq \frac{|f_k(X_k(\cdot, \omega))|}{\Gamma_k} - 2\epsilon,
\]
so for \( n \geq n_0(\omega, k_0(\omega, \epsilon)) \) we have
\[
\max_{1 \leq k \leq n} \frac{|f_k(X_k(\cdot, \omega))|}{\Gamma} \geq \max_{k_0 \leq k \leq n} \frac{|f_k(X_k(\cdot, \omega))|}{\Gamma} \geq \max_{k_0 \leq k \leq n} \frac{|f_k(X_k(\cdot, \omega))|}{\Gamma_k} - 2\epsilon = \max_{1 \leq k \leq n} \frac{|f_k(X_k(\cdot, \omega))|}{\Gamma_k} - 2\epsilon,
\]
where the equality follows from (2.38) and our choice of \( n \). Therefore, (2.39) and (2.40) imply
\[
\liminf_{n \to \infty} \left[ \max_{1 \leq k \leq n} \frac{|f_k(X_k(\cdot, \omega))|}{\Gamma} - \sqrt{2Ln} \right] \geq \liminf_{n \to \infty} \left[ \max_{1 \leq k \leq n} \frac{|f_k(X_k(\cdot, \omega))|}{\Gamma_k} - \sqrt{2Ln} \right] - 2\epsilon \geq -2\epsilon.
\]
with probability one. Since \( \epsilon > 0 \) was arbitrary this implies (2.21), and Theorem 2.1 is proved. \( \Box \)
3. Strong Laws for Maxima of Continuous Time Processes

Applying Theorem 2.1 and its corollaries we obtain generalizations to continuous time vector-valued stochastic processes. In particular, Corollary 2.2 allows us to provide some generalizations of results for real-valued stationary Gaussian processes that appeared in [Pic67], and the references therein.

In the real-valued case the proofs of the continuous time results are more complex, so it is somewhat of a surprise that the sequence results in section two make at least part of the argument easier even for Banach-valued sample continuous Gaussian processes (see Remark 2.2 for further clarification and details). Now we need some additional notation.

Throughout this section $E$ has norm $q(\cdot)$ and $B = C_E[0,1]$ denotes the space of $E$-valued continuous functions on $[0,1]$ with norm

$$\|x\| = \sup_{t \in [0,1]} q(x(t)), x \in C_E[0,1].$$

(3.1)

Let $Y = \{Y(t) : t \geq 0\}$ denote a centered, sample continuous process with values in $(E, q(\cdot))$, and for each integer $k \geq 1$ define the processes

$$X_k(t) = Y(t + (k - 1)), t \in [0,1].$$

(3.2)

Then, $\{X_k(t) : t \in [0,1]\}$ is a sample continuous process, and its distribution is a centered measure $\mu_k$ on $B = C_E[0,1]$ with norm as in (3.1).

**Definition 3.1.** A stochastic process $Z = \{Z(t) : t \in T\}$ is said to be an $E$-valued mean zero Gaussian process if for each integer $d \geq 1$ and finite subset $\{t_1, t_2, \cdots, t_d\}$ of $T$ the finite dimensional distribution of

$$(Z(t_1), \cdots, Z(t_d))$$

(3.3)

is a mean zero Gaussian measure on $E^d$.

If $Y = \{Y(t) : t \geq 0\}$ is assumed to be a mean zero Gaussian process as in Definition 3.1 then the finite dimensional distributions of each of the processes $X_k$ are mean zero Gaussian. Since $B$ is separable in the sup-norm given in (3.1), the Borel probability measures $\mu_k = \mathcal{L}(X_k)$ all have the same mean zero Gaussian finite dimensional distributions, but are the $\mu_k$ mean zero Gaussian measures on $B$? Recall that a measure $\mu$ is a mean zero Gaussian measure on a separable Banach space $B$ if every linear functional $f \in B^*$ is a mean zero Gaussian random variable with variance

$$\int_B f^2(x) d\mu(x).$$

(3.4)

The next lemma shows this is indeed the case.

**Lemma 3.1.** If $X = \{X(t) : t \in [0,1]\}$ is an $E$-valued, mean zero, sample continuous Gaussian process per Definition 3.1 then $\mu = \mathcal{L}(X)$ is a Gaussian measure on the Borel subsets of $C_E[0,1]$. That is, for every $f \in C_E^*[0,1]$, $f(X)$ is a mean zero Gaussian random variable with variance as in (3.4) with $B = C_E[0,1]$. 
Proof. Let $X_1, X_2, \ldots, X_n$ be independent copies of $X$. Then, for each integer $d \geq 1$ and $0 \leq t_1 < \cdots < t_d \leq 1$ the finite dimensional distributions

$$\mu_X^{t_1, \cdots, t_d} = \mathcal{L}(X(t_1), \ldots, X(t_d))$$

and

$$\mu_{X_j}^{t_1, \cdots, t_d} = \mathcal{L}(X_j(t_1), \ldots, X_j(t_d)),$$

on $E^d$ are such that

$$\mu_X^{t_1, \cdots, t_d} = \mu_{X_j}^{t_1, \cdots, t_d}.$$

Moreover, since the measures are mean zero Gaussian on $E^d$, we then have

$$Z_n = (X_1 + \cdots + X_n)/\sqrt{n},$$

that

$$\mu_{Z_n}^{t_1, \cdots, t_d} = \mathcal{L}(Z_n(t_1), \ldots, Z_n(t_d)) = \mu_X^{t_1, \cdots, t_d}$$
on $E^d$.

Since equality of the finite dimensional distributions of measures on cylinder sets of $C_E[0,1]$ extends to the Borel subsets, we thus have

$$\mu = \mathcal{L}(X) = \mathcal{L}(Z_n), n \geq 1,$$

and hence for every $f \in C^*_E[0,1]$ and $n \geq 1$

$$\mathcal{L}(f(X)) = \mathcal{L}(f(Z_n)).$$

Therefore,

$$\mathcal{L}(f(X)) = \mathcal{L}(f(X_1) + \cdots + f(X_n))/\sqrt{n},$$

and by Proposition 9.1 in [Bre68], p. 186 and its extension in Problem 2 in [Bre68], p. 202, $f(X)$ is mean zero Gaussian with variance as in (3.3).

\textbf{Theorem 3.1.} Let $Y = \{Y(t) : t \geq 0\}$ denote a centered $E$-valued sample continuous Gaussian process, and for each integer $k \geq 1$ define $X_k(t)$ as in (3.2) with $\mu_k$ the centered Borel probability measure induced by $X_k$ on the Banach space $B = C_E[0,1]$ with norm $\| \cdot \|$ given by (3.1). Then, the measures $\{\mu_k : k \geq 1\}$ are Gaussian measures on $B$. In addition, assume $\mu$ is a non-degenerate Gaussian measure on $B$, and

\begin{equation}
(3.5) \quad \Gamma = \sup_{x \in K} \|x\| \text{ and } \Gamma_k = \sup_{x \in K_k} \|x\|, k \geq 1,
\end{equation}

where $K, K_1, K_2, \ldots$ are the unit balls of the Hilbert spaces $H_\mu, H_{\mu_1}, \cdots$ that generate the Gaussian measures $\mu, \mu_1, \mu_2, \cdots$, and

$$\tilde{Y}(t) = \frac{Y(t)}{\Gamma_k} I(t \in [k-1, k)), k \geq 1,$$
where \( \tilde{Y}(t) \) is understood to be zero for \( t \in [k-1,k) \) whenever \( \Gamma_k = 0 \). If \( \{\mu_k : k \geq 1\} \) converges weakly to \( \mu \) in \( B = C_E[0,1] \), then
\[
\lim_{k \to \infty} \Gamma_k = \Gamma > 0,
\]
and with probability one
\[
\limsup_{T \to \infty} \left( \sup_{0 \leq t \leq T} q(\tilde{Y}(t)) - \sqrt{2LT} \right) \leq 0.
\]
Moreover, if
\[
\frac{\Gamma_k}{\Gamma} - 1 = o((\sqrt{Lk})^{-1}),
\]
then with probability one
\[
\limsup_{T \to \infty} \left( \sup_{0 \leq t \leq T} q(\tilde{Y}(t)) - \sqrt{2LT} \right) \leq 0.
\]
In addition, if \( (2.19) \) holds with \( f_n \in B^*, x_n \in K_n \) with \( \|x_n\| = \Gamma_n, \|x_n\|_{\mu_n} = 1, f_n(x_n) = \Gamma_n, \) and \( \|f_n\|^* = 1 \), then with probability one
\[
\liminf_{T \to \infty} \left( \sup_{0 \leq t \leq T} q(\tilde{Y}(t)) - \sqrt{2LT} \right) \geq 0,
\]
and
\[
\liminf_{T \to \infty} \left( \sup_{0 \leq t \leq T} q(\tilde{Y}(t)) - \sqrt{2LT} \right) \geq 0.
\]
with probability one whenever \( (3.3) \) is assumed.

The proof of Theorem 3.1 follows easily from Theorem 2.1 and is given below. However, we first indicate a corollary for continuous time stationary Gaussian processes. We start with a definition and a lemma.

**Definition 3.2.** The process \( Y =: \{Y(t) : t \geq 0\} \) is said to be an \( E \)-valued stationary process if for each integer \( r \geq 1 \), \( 0 \leq t_1 < t_2 < \cdots < t_r < \infty \), and \( h > 0 \) the finite dimensional distributions of
\[
(Y(t_1), \cdots, Y(t_r)) \quad \text{and} \quad (Y(t_1+h), \cdots, Y(t_r+h)) \quad \text{on} \quad E^r
\]
are equal.

**Lemma 3.2.** If \( Y =: \{Y(t) : t \geq 0\} \) is a sample continuous mean zero \( E \)-valued stationary process, then the Borel probability measures \( \mu_k = L(X_k) \) on \( C_E[0,1] \) as in \( (3.2) \) are equal. Moreover, if we also assume \( Y = \{Y(t) : t \geq 0\} \) is a mean Gaussian process in the sense of Definition 3.1, then the \( X_k, k \geq 1 \), are mean zero Gaussian processes as in Definition 3.1. In addition, for every \( f \in C_E^*[0,1] \), \( f(X) \) is a mean zero Gaussian random variable with variance as in \( (3.4) \) with \( B = C_E[0,1] \).

**Proof.** For all \( d \geq 1 \) and \( 0 \leq t_1 < t_2 < \cdots < t_d \leq 1 \) a typical cylinder set of \( C_E[0,1] \) is
\[
A = \{x \in C_E[0,1] : (x(t_1), \cdots, x(t_d)) \in J \},
\]
where $J$ is a Borel subset of $E^d$. The class of all such cylinder sets form an algebra of sets, and since $C_E[0,1]$ is separable the minimal sigma algebra containing them is all the Borel subsets. To see this recall the basic fact that for $\{t_j: j \geq 1\}$ a dense subset of $[0,1]$ and $\epsilon > 0$ we have
\[
\{x \in C_E[0,1]: \|x\| = \sup_{t \in [0,1]} q(x(t)) \leq \epsilon\} = \cap_{n \geq 1} \{x \in C_E[0,1]: \sup_{1 \leq j \leq n} q(x(t_j)) \leq \epsilon\},
\]
and then argue as is usual to show that open subsets of $C_E[0,1]$ are in this minimal sigma algebra. Since $Y$ is assumed to be stationary, the finite dimensional distributions of $X_1$ agree with those for $X_k$ for all $k \geq 2$, which implies $\mu_k(A) = \mu_1(A)$ for all cylinder sets $A$, and hence $\mu_k = \mu_1$ on the Borel sets for all $k \geq 1$. Moreover, if we assume $Y$ is a mean zero $E$-valued Gaussian process, then the finite dimensional distributions of $Y$ are all mean zero Gaussian, and hence those of the $X_k$ are also mean zero Gaussian for all $k \geq 1$. Therefore, the $X_k$ are mean zero $E$-valued Gaussian processes in the sense of Definition 3.1. Since $C_E[0,1]$ is separable in the sup-norm given in (3.1), the remainder of the lemma follows from Lemma 3.1.

**Corollary 3.1.** Let $Y = \{Y(t): t \geq 0\}$ be a centered $E$-valued sample continuous non-degenerate stationary Gaussian process, and for each integer $k \geq 1$ define $X_k(t)$ as in (3.2) with $\mu_k$ the centered Borel probability measure induced by $X_k$ on $B = C_E[0,1]$ as given in (3.2). Then, the $\{\mu_k: k \geq 1\}$ are mean zero Gaussian measures on $B$ with $\mu_k = \mu$ for all $k \geq 1$. Furthermore, if $K$ is the unit ball of the Hilbert space $H_\mu$ that generates the Gaussian measure $\mu$, and
\[
\Gamma = \sup_{x \in K} \|x\|,
\]
then with probability one
\[
\limsup_{T \to \infty} \left[ \sup_{0 \leq t \leq T} q\left(\frac{Y(t)}{\Gamma}\right) - \sqrt{2LT}\right] \leq 0.
\]
Furthermore, if for some $f_0 \in B^*$ such that $\|f_0\|^* = 1$ the sequence $\{f_0(X_n): n \geq 1\}$ is a stationary mean zero Gaussian sequence with
\[
\sigma^2_{f_0} := \int_B f_0^2(x)d\mu(x) = \Gamma^2,
\]
and
\[
\lim_{n \to \infty} (\log_e n)E[f_0(X_1)f_0(X_n)] = 0,
\]
then
\[
\liminf_{T \to \infty} \left[ \sup_{0 \leq t \leq T} q\left(\frac{Y(t)}{\Gamma}\right) - \sqrt{2LT}\right] \geq 0.
\]
with probability one.
Proof of Theorem 3.1. From Lemma 3.1 we have that the measures \( \{\mu_k : k \geq 1\} \) are mean zero Gaussian measures on \( B \), and since \( \{\mu_k : k \geq 1\} \) converge weakly to the non-degenerate measure \( \mu \) on \( B = C_E[0,1] \), (2.2) in Theorem 1 of [GK92] implies (3.6). Furthermore, if \( n-1 \leq T \leq n \) then \( LT - L(n-1) \leq 1/(n-1) \) for \( n \geq 2 \), so it suffices to prove the result when \( T = n \). Then,

\[
\sup_{0 \leq t \leq n} q(\tilde{Y}(t)) = \sup_{k-1 \leq t < k, 1 \leq k \leq n} q(\tilde{Y}(t)) = \sup_{k-1 \leq t < k, 1 \leq k \leq n} q(\frac{Y(t)}{\Gamma_k}),
\]

which implies

\[
(3.16) \quad \sup_{0 \leq t \leq n} q(\tilde{Y}(t)) = \sup_{0 \leq t \leq 1, 1 \leq k \leq n} q(\frac{X_k(t)}{\Gamma_k}) = \sup_{1 \leq k \leq n} \frac{\|X_k(\cdot)\|}{\Gamma_k},
\]

where sample function continuity of \( X_k \) at \( t = 1 \) is used on the last equality. Since we assumed (2.19) here, (3.16), and (2.16) and (2.20) of Theorem 2.1 combine to establish (3.7) and (3.10) with \( T = n \).

Since we also have

\[
(3.17) \quad \sup_{0 \leq t \leq n} q(\frac{Y(t)}{\Gamma}) = \sup_{0 \leq t \leq 1, 1 \leq k \leq n} q(\frac{X_k(t)}{\Gamma}) = \sup_{1 \leq k \leq n} \frac{\|X_k(\cdot)\|}{\Gamma},
\]

assuming (3.8) also holds, (3.9) and (3.11) follow from (2.18) and (2.21). Thus Theorem 3.1 is proved.

Proof of Corollary 3.1. As before it suffices to prove the results with \( T = n \). The limsup result in (3.13) then follows using Lemma 3.2 and applying either (3.7) or (3.9) of Theorem 3.1. To verify (3.15) observe from (3.17) and that \( f_0 \in B^* \) with \( \|f_0\|^* = 1 \) implies

\[
\liminf_{n \to \infty} \left[ \sup_{0 \leq t \leq n} q(\frac{Y(t)}{\Gamma}) - \sqrt{2Ln} \right] = \liminf_{n \to \infty} \left[ \max_{1 \leq k \leq n} \frac{\|X_k(\cdot, \omega)\|}{\Gamma} - \sqrt{2Ln} \right] \geq \liminf_{n \to \infty} \left[ \max_{1 \leq k \leq n} \left| \frac{f_0(X_k(\cdot, \omega))}{\Gamma} \right| - \sqrt{2Ln} \right] \geq 0
\]

with probability one by applying Corollary 2.2 since \( \sigma_k^2 = 1 \) with \( \{f_0(X_k)/\Gamma : k \geq 1\} \) a stationary Gaussian sequence of real-valued random variables with mean zero, variance one, and (3.14) is assumed.

Remark 3.1. In the next section we provide some applications of the results in Sections 2 and 3. One example we consider in some detail is the Banach-valued Ornstein-Uhlenbeck process. In particular, we show all the assumptions in Corollary 3.1 (such as the stationarity of \( \{f_0(X_k) : k \geq 1\} \) and (3.14)) can be verified directly from the process itself. Examples showing that if assumption (3.8) fails, then (3.9) need not hold are easy to find.
4. **Banach-valued Ornstein-Uhlenbeck Processes and Applications**

The goal here is to show the Ornstein-Uhlenbeck process with values in a separable Banach space $E$ has a strong law of large numbers for its maximum as in Theorem 4.1 below, and also to derive some strong law limit theorems for maximum of normalized partial sums of Banach-valued Gaussian random vectors. In fact, we will show (3.14) (and hence (2.28)) can be verified to hold from the process itself, and is not an extra assumption for the Ornstein-Uhlenbeck process.

If $\gamma$ is a non-degenerate mean zero Gaussian measure on the Borel sets of $E$ with norm $q$, we let $W = \{W(t) : t \geq 0\}$ denote the Brownian motion in $E$ generated by $\gamma$. The stochastic process

\[ Y(t) = e^{-\frac{t}{2}} W(e^t), \quad t \geq 0, \tag{4.1} \]

is the $E$-valued Ornstein-Uhlenbeck process generated or determined by $\gamma$-Brownian motion. In particular, we assume $W$ is normalized so that the law of $W(1)$ is $\gamma$ (see Subsection 5.1 for more details). To simplify, we will sometimes say $Y = \{Y(t) : t \geq 0\}$ is a $\gamma$-generated Ornstein-Uhlenbeck process. Since we always assume $\gamma$ is a non-degenerate mean zero Gaussian measure on $E$, its support is a closed linear subspace of $E$ of dimension at least one. Hence the $\gamma$-generated Ornstein-Uhlenbeck process is also always non-trivial.

The existence of a sample continuous $E$-valued Brownian motion $W = \{W(t) : t \geq 0\}$ generated by $\gamma$ follows from [Gro67]. A precise description appears in Lemma 5.1.1 below, and more self contained proofs appear in the appendix for this paper [KZ]. This immediately implies the sample continuous Ornstein-Uhlenbeck process exists, and we assume throughout $Y = \{Y(t) : t \geq 0\}$ as in (4.1) is a sample continuous version. Lemma (5.1.1) provides the construction of the process $W$ on the probability space $(\Omega_E, F, P)$, where $\Omega_E$ consists of the $E$-valued continuous functions $x$ defined on $[0, \infty)$ with $x(0) = 0$, $F$ is the $\sigma$-field of $\Omega_E$ generated by the functions $x \to x(t), 0 \leq t < \infty$, and $P$ is the probability measure on $(\Omega_E, F)$ such that $W = \{W(t) : t \geq 0\}$ has stationary independent increments as in (5.1.1).

**Theorem 4.1.** Let $\gamma$ be a non-degenerate mean zero Gaussian measure on the Borel sets of $E$ with norm $q$, and assume $Y = \{Y(t) : t \geq 0\}$ is the $E$-valued sample continuous $\gamma$-generated Ornstein-Uhlenbeck process. Then the following hold:

(a) $Y$ is a stationary mean zero Gaussian process in the sense of Definition 3.1 and Definition 3.2.

(b) The probability measure $\mu$ induced by $\{Y(t) : 0 \leq t \leq 1\}$ on $B = C_E[0, 1]$ with norm $\| \cdot \|$ as in (3.7) is a non-degenerate mean zero Gaussian measure in the sense that every $f \in B^*$ is a mean zero Gaussian random variable with variance as in (3.4). Moreover, the sample continuous processes $\{X_k(t) :
t ∈ [0, 1] defined in (3.2) are Gaussian in the sense of Definition 3.1 and they induce mean zero Gaussian measures \( \{ \mu_k : k \geq 1 \} \) on the Borel subsets of \( B \) such that

\[
\mathcal{L}(X_k) = \mu_k = \mu, k \geq 1.
\]

(c) If \( K \) is the unit ball of the Hilbert space \( H_\mu \) that generates \( \mu \) and

\[
\Gamma = \sup_{x \in K} \|x\| = \sup_{0 \leq t \leq 1} q(x(t)),
\]

then \( \Gamma \in (0, \infty) \) and with probability one

\[
\limsup_{T \to \infty} \left( \sup_{0 \leq t \leq T} q\left( \frac{Y(t)}{\Gamma} \right) - \sqrt{2LT} \right) \leq 0,
\]

and

\[
\liminf_{T \to \infty} \left( \sup_{0 \leq t \leq T} q\left( \frac{Y(t)}{\Gamma} \right) - \sqrt{2LT} \right) \geq 0.
\]

**Corollary 4.1.** Let \( \gamma \) be a non-degenerate mean zero Gaussian measure on \( E \), and assume \( G_1, G_2, \ldots \) are i.i.d. Gaussian random vectors with distribution \( \gamma \). If \( S_k = G_1 + \cdots + G_k \) for \( k \geq 1 \) and \( \Gamma \) is as in (4.2), then with probability one

\[
\limsup_{n \to \infty} \left( \max_{1 \leq k \leq n} q\left( \frac{S_k}{\Gamma} \right) - \sqrt{2Lln} \right) \leq 0.
\]

and

\[
\liminf_{n \to \infty} \left( \max_{1 \leq k \leq n} q\left( \frac{S_k}{\Gamma} \right) - \sqrt{2Lln} \right) \geq 0.
\]

**Proof of Theorem 4.1.** If \( Y =: \{ Y(t) : t \geq 0 \} \) is sample continuous and as in (4.1), then Lemmas 5.2.1 and 5.2.2 show \( Y \) is a stationary, mean-zero Gaussian process in the sense of Definitions 3.1 and 3.2. Hence (a) in Theorem 4.1 holds. In addition, Lemmas 5.1 and 5.2 are then applicable and imply that the sample continuous processes \( \{ X_k(t) : t \in [0, 1] \} \) defined in (3.2) are Gaussian in the sense of Definition 3.1. Moreover, since \( \gamma \) is assumed non-degenerate they show that the mean zero Gaussian measures \( \{ \mu_k : k \geq 1 \} \) induced on the Borel subsets of \( B \) are also non-degenerate and such that

\[
\mathcal{L}(X_k) = \mu_k = \mu, k \geq 1.
\]

It also follows from Lemma 3.1 that for every \( f \in C_E^*[0, 1] \), \( f(X) \) is a mean zero Gaussian random variable with variance as in (3.4) with \( B = C_E[0, 1] \). Therefore, (b) also holds.

To prove (c) we first observe that \( \Gamma < \infty \) since \( K \) is a compact subset of \( C_E[0, 1] \), and it is strictly positive since \( \mu \) is non-degenerate when we assume \( \gamma \) is non-degenerate, which implies the unit ball \( K \) of the Hilbert space \( H_\mu \) is non-degenerate. Combining (b) and \( \Gamma \) as in (4.2) we now have (4.3) with probability one by (3.13) in Corollary 3.1. Finally, (4.4) holds.
with probability one from (3.13) in Corollary 3.1 since we can check (3.14). That is, for $x_0 \in K$ such that $\|x_0\| = \Gamma$ the Hahn-Banach theorem implies there is a linear functional $f_0 \in B^*$ such that $f_0(x_0) = \Gamma$, $\|f_0\|^* = 1$ and Lemma 2.1 implies
\begin{equation*}
\sigma_{f_0}^2 = \int_B f_0^2(x) d\mu(x) = \Gamma^2.
\end{equation*}
Furthermore, Lemma 5.3.1 implies the sequence of mean zero random variables \( \{f_0(X_k) : k \geq 1\} \) is stationary with variance $\Gamma^2$ and Lemma 5.4.3 shows
\begin{equation*}
\lim_{n \to \infty} (\log_{e} n) E[f_0(X_1)f_0(X_n)] = 0, \tag{4.7}
\end{equation*}
which when combined with Corollary 3.1 completes the proof of Theorem 4.1.

Proof of Corollary 4.1. Since $Y = \{Y(t) : t \geq 0\}$ is as in (4.1) with $\{W(t) : t \geq 0\}$ the $E$-valued sample continuous Brownian motion induced by $\gamma$, it follows that the sequences $\{G_k : k \geq 1\}$ and $\{W(k) - W(k - 1) : k \geq 1\}$ with $W(0) = 0$ have the same law. Therefore, the $E^\infty$-valued random vectors
\begin{equation*}
(G_1, \frac{G_1 + G_2}{\sqrt{2}}, \ldots) \text{ and } (W(1), \frac{W(2)}{\sqrt{2}}, \ldots)
\end{equation*}
have the same law, and assuming without loss of generality that $G_k = W(k) - W(k - 1)$ for all $k \geq 1$ we have $W(k)/\sqrt{k} = Y(Lk)$ with probability one for all $k \geq 1$. Hence with probability one for all $n \geq 1$ we have
\begin{equation*}
\max_{1 \leq k \leq n} q\left(\frac{G_1 + \cdots + G_k}{\sqrt{k}\Gamma}\right) = \max_{1 \leq k \leq n} q\left(\frac{Y(Lk)}{\Gamma}\right)
\end{equation*}
which implies with probability one that
\begin{equation*}
\limsup_{n \to \infty} \left[ \max_{1 \leq k \leq n} q\left(\frac{G_1 + \cdots + G_k}{\sqrt{k}\Gamma}\right) - \sqrt{2\log n}\right] = \limsup_{n \to \infty} \left[ \max_{1 \leq k \leq n} q\left(\frac{Y(Lk)}{\Gamma}\right) - \sqrt{2\log n}\right]
\end{equation*}
\begin{equation*}
\leq \limsup_{n \to \infty} \max_{1 \leq t \leq Ln} q\left(\frac{Y(t)}{\Gamma}\right) - \sqrt{2\log n} \leq 0,
\end{equation*}
where the inequality follows immediately from (4.3) with $T = Ln$. Hence (4.5) holds, and we now turn to (4.6).

Let $\Gamma$ be as in (4.2) and $\Gamma_\gamma = \sup_{z \in K_\gamma} q(z)$, where $K_\gamma = \{z : \|z\|_{H_\gamma} \leq 1\}$ is the unit ball of the Hilbert space $H_\gamma$. Then Lemma 2.1 implies $\Gamma = \sigma(\mu)$, where $\mu = \mathcal{L}(X_1)$, and hence by (5.4.29) we have
\begin{equation*}
\Gamma \leq \sigma(\gamma) = \Gamma_\gamma.
\end{equation*}
Furthermore, by the conclusion of Remark 5.4.1 we have
\begin{equation*}
\Gamma = \Gamma_\gamma, \tag{4.11}
\end{equation*}
and applying Lemma 2.1 to the mean zero Gaussian measure \( \gamma \) on \( E \), there exists \( f_0 \in E^* \), \( \|f_0\| = 1 \), \( f_0(z_0) = \Gamma_\gamma \) (\( z_0 \) as above), such that

\[
\int_E f_0^2(z) d\gamma(z) = \Gamma_\gamma^2.
\]

Since \( \Gamma = \Gamma_\gamma \) and \( S_k = G_1 + \cdots + G_k \) for \( k \geq 1 \),

\[
\max_{1 \leq k \leq n} q\left( \frac{S_k}{\sqrt{k}\Gamma_\gamma} \right) \geq \max_{1 \leq k \leq n} \frac{|f_0(S_k)|}{\sqrt{k}\Gamma_\gamma} \geq \max_{0 \leq j \leq j_n} |Y_j|,
\]

where \( j_n = \max\{j : 2^j \leq n\} \) for \( n \geq 1 \) and

\[
Y_j = \frac{f_0(G_1) + \cdots + f_0(G_{2^j})}{\sqrt{2^j\Gamma_\gamma}}, j \geq 1.
\]

Therefore, \( E(Y_j) = 0, E(Y_j^2) = 1 \), and for \( 0 \leq i \leq j < \infty \)

\[
E(Y_i Y_j) = \frac{E(f_0^2(G_1))2^i}{\Gamma_\gamma^2 \sqrt{2(i+j)}} = 2^{-(j-i)/2},
\]

which implies \( \{Y_j : j \geq 0\} \) is a mean zero-variance one stationary Gaussian sequence of real random variables with

\[
r_k = E(Y_0 Y_k) = e^{-k/2},
\]

and hence Theorem 3.3 in [Pic67] implies

\[
\liminf_{k \to \infty} \left[ \max_{0 \leq j \leq k} |Y_j - \sqrt{2Lk}| \right] \geq 0
\]

with probability one.

From (4.13) we have

\[
\liminf_{n \to \infty} \max_{1 \leq k \leq n} q\left( \frac{S_k}{\sqrt{k}\Gamma_\gamma} \right) - \sqrt{2LLn} \geq \liminf_{n \to \infty} \max_{0 \leq j \leq j_n} |Y_j - \sqrt{2Lj_n + \epsilon_n}|
\]

where \( n \to \infty \) implies

\[
|\epsilon_n| = |\sqrt{2Lj_n} - \sqrt{2LLn}| \to 0.
\]

Therefore, (4.14) combined with (4.15) implies (4.6) with probability one, and the corollary is proved.

\[
\Box
\]

4.1. An Application to Random operators. Let \( H \) be a separable Hilbert space over the complex numbers with inner product \( \langle x, y \rangle \), \( x, y \in H \), and for a bounded operator \( A \) from \( H \) to \( H \) denote the uniform operator norm by

\[
q(A) =: \sup_{h \in H, \langle h, h \rangle = 1} \langle Ah, Ah \rangle^{\frac{1}{2}}.
\]

In this sub-section we assume \( E \) is a separable Banach space over the real numbers consisting of bounded self-adjoint operators with norm \( q(\cdot) \), and
that $\gamma$ is a mean zero Gaussian measure on the Borel subsets of $(E, q)$. The Hilbert space generating $\gamma$ will be denoted by $H_\gamma$, its unit ball by $K_\gamma$, and
\begin{equation}
\Gamma_\gamma =: \sup_{z \in K_\gamma} q(z).
\end{equation}

Some standard facts about elements of $E$ are as follows. If $A$ is self-adjoint, then the spectrum of $A$, denoted by $\sigma(A)$, is a compact non-empty subset of real numbers with spectral radius
\begin{equation}
(4.17) \quad r_\sigma(A) =: \sup\{|\lambda| : \lambda \in \sigma(A)\},
\end{equation}
and
\begin{equation}
(4.18) \quad r_\sigma(A) = q(A).
\end{equation}
If $A$ is a compact self-adjoint operator from $H$ to $H$, then $\sigma(A)$ is a countable set of real numbers consisting of the eigenvalues of $A$ and
$$\sigma(A) \cap ((-\infty, \infty) - \{0\}) = \sigma(A)^+ \cup \sigma(A)^-,$$
where $\sigma(A)^+$ denotes the strictly positive eigenvalues of $A$ and $\sigma(A)^-$ is the strictly negative eigenvalues of $A$. Furthermore, zero may or may not be an eigenvalue of $A$, but if $0 \notin \sigma(A)$ and $H$ is infinite dimensional, then it is always a limit point of either $\sigma(A)^+$ or $\sigma(A)^-$.

Perhaps somewhat less well known are the following facts, and hence a detailed summary appears in the appendix [KZ].

For compact self-adjoint operators on the infinite dimensional Hilbert space $H$, let $\Sigma$ be the set-valued map on these operators defined by
$$\Sigma(A) = \sigma(A),$$
and define the Hausdorff metric distance between $\sigma(A)$ and $\sigma(B)$ by
\begin{equation}
(4.19) \quad d(\sigma(A), \sigma(B)) =: \inf\{\delta > 0 : \sigma(A) \subseteq \sigma(B) + (-\delta, \delta)
\text{ and } \sigma(B) \subseteq \sigma(A) + (-\delta, \delta)\}.
\end{equation}
Then, for $A, B$ compact self-adjoint operators on $H$ it is known that
\begin{equation}
(4.20) \quad q(A - B) < \delta \text{ implies } d(\sigma(A), \sigma(B)) < \delta.
\end{equation}

For a proof of (4.20) see Theorem 3 in [KZ]. In particular, if $E$ consists of compact self-adjoint operators on $H$ and
$$E_\sigma = \{\sigma(A) : A \in E\}$$
with distance on $E_\sigma$ the Hausdorff metric in (4.19), then the map $\Sigma : A \to \sigma(A)$ is a Lip-1 continuous map from $(E, q)$ onto $(E_\sigma, d)$ since (4.20) implies
\begin{equation}
\text{d}(\Sigma(A), \Sigma(B)) \leq 2q(A - B).
\end{equation}
Thus for $(E, q)$ a Banach space of compact self-adjoint operators and $\{A_n : n \geq 1\}$ a sequence in $E$ and $A \in E$
As usual in any metric space \((M, \rho)\), for \(x \in D, D \subseteq M\), we define \(\rho(x, D) = \inf_{a \in D} \rho(x, a)\) and the cluster set \(C(\{x_n\})\) to be the set of all limit points of the sequence \(\{x_n\} \subseteq M\) taken in \((M, \rho)\). Thus for \((E, q)\) a Banach space of compact self-adjoint operators, \(\{A_n : n \geq 1\}\) a sequence in \(E\), and \(D \subseteq E\) we have that

\[(4.22) \quad \lim_{n \to \infty} q(A_n, D) = 0 \implies \lim_{n \to \infty} d(\Sigma(A_n), \Sigma(D)) = \lim_{n \to \infty} \inf_{a \in D} d(\sigma(A_n), \sigma(a)) = 0.\]

Moreover, if \(E_\sigma = \Sigma(E) = \{\sigma(A) : A \in E\}\) with distance on \(E_\sigma\) the Hausdorff metric in \((4.19)\), then for compact subsets \(D\) of \((E, q)\)

\[(4.23) \quad \lim_{n \to \infty} q(A_n, D) = 0 \text{ and } C(\{A_n\}) = D \implies C(\{\Sigma(A_n)\}) = \Sigma(D),\]

where the cluster set \(C(\{\Sigma(A_n)\})\) is computed relative to \((E_\sigma, d)\).

From these standard facts it follows rather easily that our results have implications for the spectrums of random operators with centered Gaussian distribution \(\gamma\) on \(E\). As a sample we will present applications of \((4.20)\) in Corollary \(2.2\) and \(4.1\). The interested reader should then envision others.

**Corollary 4.2.** Let \(E\) be a separable Banach space over the real numbers consisting of bounded self-adjoint operators on the Hilbert space \(H\) in the uniform operator norm \(q(\cdot)\), and that \(\gamma\) is a mean zero Gaussian measure on the Borel subsets of \((E, q)\) with \(\Gamma_\gamma\) as in \((4.19)\). If \(A_1, A_2, \ldots\) are i.i.d. \(E\)-valued random operators with law \(\gamma\) and \(S_k = A_1 + \cdots + A_k\) for \(k \geq 1\), then with probability one

\[(4.24) \quad \lim_{n \to \infty} \max_{1 \leq k \leq n} \frac{r_\sigma(A_k)}{\Gamma_\gamma} - \sqrt{2Ln} = 0\]

and

\[(4.25) \quad \lim_{n \to \infty} \max_{1 \leq k \leq n} \frac{r_\sigma(S_k)}{\sqrt{k} \Gamma_\gamma} - \sqrt{2LLn} = 0.\]

In addition, if we assume the operators are compact and \(d(\cdot, \cdot)\) is the Hausdorff metric on \(E_\sigma\), then with probability one

\[(4.26) \quad \lim_{n \to \infty} d(\sigma(A_n) / \sqrt{2Ln}, \mathcal{K}) = 0,\]
and
\[(4.27)\]
\[C(\{\frac{\sigma(A)}{\sqrt{2Ln}}\}) = K,\]
where
\[(4.28)\]
\[K := \Sigma(K_\gamma) = \{\sigma(A) : A \in K_\gamma\}.\]
With probability one we also have
\[(4.29)\]
\[\lim_{n \to \infty} d(\frac{\sigma(S_n)}{\sqrt{2nLLn}}, K) = 0,\]
and
\[(4.30)\]
\[C(\{\frac{\sigma(S_n)}{\sqrt{2nLLn}}\}) = K.\]

Proof. The conclusion in (4.24) follows immediately from (2.30) and (4.18). Similarly, (4.25) follows from Corollary 4.1, (4.18), and that \(r_{\sigma}(\frac{S_k}{\sqrt{k}}) = \frac{r_\sigma(S_k)}{\sqrt{k}}\).

To verify (4.26) and (4.27) we observe from Theorem 1 in [GK92] or Theorem 2.1 in [GK91] for every \(\epsilon > 0\) and \(\epsilon_n = \epsilon/\sqrt{2Ln}\) that with probability one
\[(4.31)\]
\[P(\frac{A_n}{\sqrt{2Ln}} \in K_\gamma + \epsilon_nU \text{ eventually}) = 1,\]
where \(U = \{A \in E : q(A) < 1\}\). Thus with probability one we have \(\frac{A_n}{\sqrt{2Ln}}\) converging to \(K_\gamma\) in \((E, q)\), and
\[(4.32)\]
\[P(q(\frac{A_n}{\sqrt{2Ln}}, K_\gamma) < \epsilon_n \text{ eventually}) = 1.\]
Since \(\Sigma\) is Lip-1 as above and (4.28) holds, then
\[\{d(\frac{\Sigma(A_n)}{\sqrt{2Ln}}, K) < 2\epsilon_n \text{ eventually}\} \supseteq \{q(\frac{A_n}{\sqrt{2Ln}}, K_\gamma) < \epsilon_n \text{ eventually}\},\]
and hence we have (4.26) with probability one.

To verify (4.27) we first observe Théorème 4.1 of [CK76] implies
\[(4.33)\]
\[P(C(\{\frac{A_n}{\sqrt{2Ln}}\}) = K_\gamma) = 1,\]
and since \(K_\gamma\) is compact in \(E\), (4.32) and (4.33) combined with (4.21) gives (4.27) with probability one.

Finally, the proof of (4.29) and (4.30) follow as that for (4.26) and (4.27) since Theorem 4.1 in [GKZ81] implies the law of the iterated logarithm for the i.i.d. centered \(E\)-valued Gaussian random vectors \(\{A_k : k \geq 1\}\), and hence
\[(4.34)\]
\[P(\lim_{n \to \infty} q(\frac{S_n}{\sqrt{2nLLn}}, K_\gamma) = 0) = P(C(\{\frac{S_n}{\sqrt{2nLLn}}\}) = K_\gamma) = 1.\]
Thus the corollary is proved. \(\square\)
Remark 4.1. Since $\epsilon_n = \epsilon / \sqrt{2 Ln}$ the statements in (4.32-33) imply

$$P(\lim_{n \to \infty} q(A_n, \sqrt{2 LnK}) = 0) = 1$$

and

$$P(\lim_{n \to \infty} d(\Sigma(A_n), \sqrt{2 LnK}) = 0) = 1.$$  

4.2. An Application to Medians for Gaussian Maximal Functions.

Next we establish a symmetrization result, and indicate how it provides precise asymptotic behavior for the medians of the partial maxima of Gaussian vectors when combined with our results above.

Proposition 1. Assume $x_n$ is a real-valued random variable with unique median, $m_n$, and $x'_n$ is an independent copy of $x_n$. Then,

$$x_n - x'_n \to 0 \text{ in probability implies } x_n - m_n \to 0 \text{ in probability, and}$$

$$x_n - x'_n \to 0 \text{ a.s. implies } x_n - m_n \to 0 \text{ a.s.}$$

Proof. We’ll use $2 \Pr(x'_n \leq m_n) = 1$ and $2 \Pr(x'_n \geq m_n) = 1$. Fix $\epsilon > 0$.

$$\Pr(x_n > m_n + \epsilon) = 2 \Pr(x'_n \leq m_n) \Pr(x_n > m_n + \epsilon)$$

$$= 2 \Pr(x_n > m_n + \epsilon, x'_n \leq m_n) \leq 2 \Pr(x_n > x'_n + \epsilon, x'_n \leq m_n)$$

$$\leq 2 \Pr(x_n > x'_n + \epsilon) \to 0.$$

Similarly,

$$\Pr(x_n < m_n - \epsilon) = 2 \Pr(x'_n \geq m_n) \Pr(x_n < m_n - \epsilon)$$

$$= 2 \Pr(x_n < m_n - \epsilon, x'_n \geq m_n) \leq 2 \Pr(x_n < x'_n - \epsilon, x'_n \geq m_n)$$

$$\leq 2 \Pr(x_n < x'_n - \epsilon) \to 0.$$

Hence (4.35) holds.

Now assume $x_n - x'_n \to 0$, a.s. Then, by Fubini’s theorem there is a particular “$\omega'$” such that $x_n - x_n(\omega') \to 0$, a.s. and letting $b_n = x_n(\omega')$ we have, $x_n - b_n \to 0$ in probability. Since, the assumption also implies $x_n - x'_n \to 0$ in probability, we have by (4.35) that $x_n - m_n \to 0$ in probability. Therefore, $b_n - m_n \to 0$ in probability. Since these last are constants, we have $b_n - m_n \to 0$. Going back to $x_n - b_n \to 0$ a.s., we get $x_n - m_n \to 0$, a.s. \hfill \Box

Proposition 2. Let $\mu, \mu_1, \mu_2, \cdots$ be centered non-degenerate Gaussian measures on $B$ with norm $\|\cdot\|$, and $X, X_1, X_2, \cdots$ be $B$-valued random vectors on some probability space with distributions $\mu, \mu_1, \mu_2, \cdots$ such that $\{\mu_n : n \geq 1\}$ converges weakly to $\mu$ on $B$. In addition, assume $\{b_n\}$ are numbers such that

$$(4.37) \quad \lim_{n \to \infty} \|M_n - b_n\| = 0$$

where the convergence to zero is in probability, and

$$\Gamma = \sup_{x \in K} \|x\| \text{ and } M_n =: \max_{1 \leq k \leq n} \frac{\|X_k\|}{\Gamma}.$$
Then,
\[
\lim_{n \to \infty} [\text{med}(M_n) - b_n] = 0.
\]

The proof of Proposition 2 is immediate since the distribution function of \(M_n\) is continuous and strictly increasing on \((0, \infty)\) and zero elsewhere when \(B\) is separable and the Gaussian random vectors are assumed to be centered and non-degenerate. Since results in the literature as well as in Sections 2, 3, 4 (above) imply (4.37) with \(b_n\) either \(\sqrt{2L_n}\) or \(\sqrt{2LL_n}\), the asymptotic behavior of the medians of \(M_n\) in these situations are precisely determined by (4.38). Furthermore, determining precise asymptotic behavior for these medians by direct calculation does not appear to be immediate.

5. Properties of the Banach-valued OU Process

Here we present some results on sample function continuity and stationarity properties for the \(E\)-valued Ornstein-Uhlenbeck process of (4.1) that were used in the proofs in Section 4. We start with the sample function continuity of the related Brownian motion in Lemma 5.1.1 which appeared earlier in [Gro67]. In view of this, and the additional results and proofs in [KZ], we only state the necessary facts and related notation here. That the resulting \(\gamma\)-Brownian motion is a mean zero Gaussian process in the sense of Definition 3.1 can easily be seen from the proof of Lemma 5.2.1 below.

5.1. Sample Function Continuity. As before assume \(\gamma\) is a mean zero Gaussian measure on \(E\) with norm \(q\). In addition, assume \(\Omega_E\) is the space of continuous functions \(x\) from \([0, \infty)\) into \(E\) such that \(x(0) = 0\), and \(\mathcal{F}\) is the \(\sigma\)-field of \(\Omega_E\) generated by the functions \(x \to x(t), 0 \leq t < \infty\). Our next lemma provides a proof that there exists a probability measure \(P_\gamma\) on \((\Omega_E, \mathcal{F})\) such that if \(0 = t_0 < t_1 \cdots < t_n\), then the random vectors
\[
(x(t_j) - x(t_{j-1}), j = 1, \cdots, n),
\]
and \(x(t_j) - x(t_{j-1})\) has distribution \(\gamma_{t_j - t_{j-1}}\) on \(E\), where \(\gamma_s(A) = \gamma(A/\sqrt{s})\) for Borel subsets \(A\) of \(E\) when \(s > 0\) and \(\gamma_0 = \delta_0\). In particular, the stochastic process \(\{W(t) : t \geq 0\}\) defined on \((\Omega_E, \mathcal{F}, P_\gamma)\) by \(W(t, x) = x(t)\) has stationary independent mean zero Gaussian increments. We will call it \(\gamma\)-Brownian motion on \(E\).

Lemma 5.1.1. Let \(\gamma\) be a mean zero Gaussian measure on \(E\). Then, the \(E\)-valued Brownian motion \(W = \{W(t) : t \geq 0\}\) defined on \((\Omega_E, \mathcal{F}, P_\gamma)\) by \(W(t, x) = x(t)\) exists. In particular, it is sample path continuous and a mean zero Gaussian process (in the sense of Definition 3.1) with

(i) \(W(0) = 0\),

(ii) \(W\) has stationary independent mean zero Gaussian increments as indicated above, and

(iii) if the support of \(\gamma\) is a closed subspace \(F\) of \(E\), then the probability \(P_\gamma\) has support on \(\Omega_F\), the space of continuous functions on \([0, \infty)\) with values
in $F$, and $W$ is a $\gamma$ Brownian motion on $F$.

Remark 5.1.1. It is usual to assume the support of $\gamma$ to be $E$ when we define $\gamma$-Brownian motion, but in later proofs it will be convenient to keep it mind this need not be the case.

5.2. Gaussian and Stationary Properties for Ornstein-Uhlenbeck Processes. Let $Y$ be a sample continuous Ornstein-Uhlenbeck process generated by $\gamma$ as in (4.1). In this subsection we then prove $Y$ is a mean zero Gaussian process in the sense of Definition 3.1 and also a stationary process in the sense of Definition 3.2. As a result, it follows that both Lemma 3.1 and 3.2 apply to $Y$.

Lemma 5.2.1. The sample continuous Ornstein-Uhlenbeck process $\{Y(t) : t \geq 0\}$ is a mean zero Gaussian process in the sense of Definition 3.1.

Proof. For each integer $d \geq 1$ and $0 \leq t_1 < t_2, \cdots < t_d$ we need to show the finite dimensional distribution of

$$(Y(t_1), \cdots, Y(t_d))$$

is a mean zero Gaussian measure on $E^d$. Since the typical continuous linear functional on $E^d$ is of the form

$$f(x_1, \cdots, x_d) = \sum_{j=1}^{d} \theta_j(x_j),$$

where $\theta_1, \cdots, \theta_d \in E^*$, and $E^d$ is a separable Banach space, it suffices to show

$$\sum_{j=1}^{d} \theta_j(Y(t_j))$$

is a mean zero Gaussian random variable.

If $d = 1$, then

$$\theta_1(Y(t_1)) = e^{-\frac{t_1}{2}} \theta_1(W(e^{t_1})),$$

and hence $Y(t_1)$ is a mean zero Gaussian random variable since $\theta_1(W(s))$ is a mean zero Gaussian random variable for each $s \geq 0$. Moreover, by the scaling property of the $E$-valued Brownian motion

$$E[\theta_1^2(Y(t_1))] = E[\theta_1^2(W(1))].$$

To handle the situation for $d \geq 2$ it is useful to consider the following identity which is easily established by induction. That is, for $a_j, 1 \leq j \leq d$, linear functionals on $E$ and $b_j, 0 \leq j \leq d$ points in $E$ with $b_0 = 0$ we have

$$\sum_{j=1}^{d} a_j(b_j) = \sum_{j=1}^{d} \sum_{k=j}^{d} a_k(b_j - b_{j-1}) = \sum_{j=1}^{d} \left( \sum_{k=j}^{d} a_k \right)(b_j - b_{j-1}).$$
Therefore, with \( a_j = e^{-\frac{t_j}{2}}\theta_j \) and \( b_j = W(e^{t_j}) \) for \( 1 \leq j \leq d \) with \( b_0 = W(e^{t_0}) = 0 \) we have

\[
(5.2.4) \quad \sum_{j=1}^{d} \theta_j(Y(t_j)) = \sum_{j=1}^{d} e^{-\frac{t_j}{2}}\theta_j(W(e^{t_j})) = \sum_{j=1}^{d} \left( \sum_{k=j}^{d} e^{-\frac{t_k}{2}}\theta_k \right) (W(e^{t_j}) - W(e^{t_{j-1}})).
\]

Hence, by the independent increments of the \( \gamma \)-Brownian motion on \( E \) we thus have \( \sum_{j=1}^{d} \theta_j(Y(t_j)) \) is the sum of the independent mean zero Gaussian random variables

\[
(5.2.5) \quad \left( \sum_{k=j}^{d} e^{-\frac{t_k}{2}}\theta_k \right) (W(e^{t_j}) - W(e^{t_{j-1}})), 1 \leq j \leq d,
\]

which completes the proof. \( \square \)

**Lemma 5.2.2.** The sample continuous Ornstein-Uhlenbeck process \( \{Y(t) : t \geq 0\} \) is a stationary mean zero Gaussian process in the sense of Definitions [3.1 and 3.2](#).

**Proof.** Given Lemma (5.2.1) it suffices to show for each integer \( d \geq 1, h > 0, \) and \( 0 \leq t_1 < t_2, < \cdots < t_d \) that the finite dimensional distributions of

\[
(5.2.6) \quad (Y(t_1), \cdots, Y(t_d)) \text{ and } (Y(t_1 + h), \cdots, Y(t_d + h))
\]

are equal Gaussian probability measures on \( E^d \).

By (5.2.1) we know both are mean zero Gaussian distributions on \( E^d \), and hence they will be equal if the variance of every linear functional on \( E^d \) is the same for each distribution. That is, since the typical continuous linear functional on \( E^d \) is of the form

\[
(5.2.7) \quad f(x_1, \cdots, x_d) = \sum_{j=1}^{d} \theta_j(x_j),
\]

where \( \theta, \cdots, \theta_d \in E^* \), and \( E^d \) is a separable Banach space, it suffices to show

\[
(5.2.8) \quad E[\left( \sum_{j=1}^{d} \theta_j(Y(t_j)) \right)^2] = E[\left( \sum_{j=1}^{d} \theta_j(Y(t_j + h)) \right)^2].
\]

Hence, from (5.2.4), (5.2.5), and

\[
\psi_j = \sum_{k=j}^{d} e^{-\frac{t_k}{2}}\theta_k
\]

for \( 1 \leq j \leq d \), it suffices to show

\[
(5.2.9) \quad E[\{\psi(W(e^{t_j}) - W(e^{t_{j-1}}))\}^2] = E[\{e^{-\frac{h}{2}}\psi(W(e^{t_j+h}) - W(e^{t_{j-1}+h}))\}^2].
\]
Using the scaling property and independent increments of the $E$-valued Brownian motion $\{W(t) : t \geq 0\}$ we have

$$E\left[\{\psi(W(e^s)) - W(e^{s-1}))\}^2\right] = (e^s - e^{s-1})E[\psi^2(W(1))],$$

and

$$E\left[\{e^{-\frac{h}{2}}\psi(W(e^s + h)) - W(e^{s-1} + h))\}^2\right] = e^{-h}(e^s + h - e^{s-1} + h)E[\psi^2(W(1))],$$

Thus (5.2.9) holds for $1 \leq j \leq d$, which completes the proof. \□

5.3. More Stationary and Gaussian Properties for the Ornstein-Uhlenbeck Process. Let $\gamma$ be a non-degenerate mean zero Gaussian measure on the Borel sets of $E$, and assume

\[ Y(t) := e^{-\frac{h}{2}}W(e^t), t \geq 0, \]

is the $E$-valued sample path continuous Ornstein-Uhlenbeck process generated by $\gamma$ normalized so that the law of $W(1)$, and hence also $Y(1)$, is $\gamma$ (see Section 5.1 for more details). In addition, assume the $C_E[0, 1]$ valued random vectors $\{X_k : k \geq 1\}$ are defined as in (3.2).

The next lemma shows that for every $f \in C_E^*[0, 1]$ the sequence $\{f(X_k(\cdot)) : k \geq 1\}$ is a stationary sequence of real-valued mean zero Gaussian random variables. Our proof depends on a Riesz representation result for such linear functionals established by Bochner and Taylor in [BT38], and for that we need some additional notation.

Define $V(E^*, q^*)$ to be the functions $g$ mapping $[0, 1]$ into $E^*$ such that the sums

$$\sum_{j=1}^n q^*(g(t_j) - g(t_{j-1}))$$

are uniformly bounded for all partitions $\{t_j\}$, where $0 = t_0 < t_1 < \cdots < t_n = 1$. For $g \in V(E^*, q^*)$, the least upper bound of all such sums is the $E^*$ bounded variation of $g$ and is denoted by $V(g, E^*)$.

If $x \in C_E[0, 1]$ and $g \in V(E^*, q^*)$, Bochner and Taylor define the integral

\[ \int_0^1 dg(t)x(t) \]

in the usual way. That is, with $g$ fixed, for each $x \in C_E[0, 1]$ and each partition $D = \{t_j\}$ with $0 = t_0 < t_1 < \cdots < t_n = 1$ and points $\tau_j$, $t_{j-1} \leq \tau_j \leq t_j$, denote the real-valued sum

\[ S(D, x) = \sum_{j=1}^n (g(t_j) - g(t_{j-1}))x(\tau_j). \]

If we put $|D| = \max_{1 \leq j \leq n}|t_j - t_{j-1}|$, then as indicated in [BT38] in the usual way for a sequence of partitions $D_i$ with $|D_i| \to 0$ the sums $S(D_i, x)$ approach a limit (in $\mathbb{R}$) which is independent of the particular sequence
\{D_i\}. This limit is called the integral of $x(\cdot)$ with respect to $g(\cdot)$. It is written as in (5.3.2), and it follows immediately that

$$|\int_0^1 dg(t)x(t)| \leq \sup_{0 \leq t \leq 1} q(x(t))V(g, E^*).$$

(5.3.4)

The Riesz representation given by Bochner and Taylor is the following theorem. It will be used in Lemma 5.3.1 below.

**Theorem 5.3.1.** \textbf{[BT38]} If $f$ is a continuous linear functional on $C_E[0, 1]$, then there exists a function $g$ of $E^*$-bounded variation ($g \in V(E^*, q^*)$) such that for each $x \in C_E[0, 1]$

$$f(x) = \int_0^1 dg(t)x(t),$$

where the integral is of the Riemann-Stieljes type as given in (5.3.2)-(5.3.3), and

$$\sup_{\{x \in C_E[0, 1]: \|x\|=1\}} |f(x)| = V(g, E^*).$$

(5.3.5)

(5.3.6)

**Lemma 5.3.1.** Let $Y =: \{Y(t) : t \geq 0\}$ be the sample continuous Ornstein-Uhlenbeck process generated by the non-degenerate mean zero Gaussian measure $\gamma$ on $E$ as in (5.3.1), and assume the processes $\{X_k(t) : 0 \leq t \leq 1\}$ are as in (3.2). Then, for every $f \in C_E^*[0, 1]$ the sequence $\{f(X_k(\cdot)) : k \geq 1\}$ is a stationary sequence of real-valued mean zero Gaussian random variables.

**Proof.** Given Lemmas 5.2.1 and 5.2.2 Lemma 3.2 implies $\mu_k = \mathcal{L}(X_k), k \geq 1$, are identical mean zero Gaussian measures on $C_E[0, 1]$, and the random variables $f(X_k), k \geq 1$, are mean zero Gaussian random variables with the same distribution. What remains is to show they are stationary and jointly Gaussian.

Applying the Bochner-Taylor result, for each $f \in C_E^*[0, 1]$ there exists $g$ of $E^*$ bounded variation such that $f$ is as in (5.3.5), with (5.3.6) holding. Hence, for a sequence of partitions $\{D_i\}$ with $|D_i| \to 0$, for all $x \in C_E[0, 1]$

$$\lim_{i \to \infty} S(D_i, x) = f(x),$$

(5.3.7)

where

$$f_i(x) =: S(D_i, x) = \sum_{j=1}^r \phi_j(x(\tau_j)),$$

(5.3.8)

$\phi_j =: g(t_j) - g(t_{j-1}) \in E^*$, and $r + 1$ equals the number of points in $D_i$. Of course, $r$ depends on $i$, but this is omitted to simplify the notation, and for all $x \in C_E[0, 1]$ (5.3.7) becomes

$$\lim_{i \to \infty} |f(x) - f_i(x)| = 0.$$

(5.3.9)
To show \( \{ f(X_k) : k \geq 1 \} \) is stationary and Gaussian we take \( 1 \leq k_1 < \cdots < k_d < \infty \) and \( d \geq 1 \) integers, and show
\[
(5.3.10) \quad \mathcal{L}(f(X_{k_1}), \cdots, f(X_{k_d})) = \mathcal{L}(f(X_{k_1 + h}), \cdots, f(X_{k_d + h}))
\]
are equal Gaussian probabilities on \( \mathbb{R}^d \). Since (5.3.8) and (5.3.9) hold, (5.3.10) follows when we verify
\[
\mathcal{L}(\sum_{j=1}^r \phi_j(X_{k_1}(\tau_j)), \cdots, \sum_{j=1}^r \phi_j(X_{k_d}(\tau_j)))
= \mathcal{L}(\sum_{j=1}^r \phi_j(X_{k_1 + h}(\tau_j)), \cdots, \sum_{j=1}^r \phi_j(X_{k_d + h}(\tau_j)))
\]
and that they are Gaussian measures. Since \( X_k(t) = Y(t + (k - 1)) \) for all \( t \in [0, 1] \) and integers \( k \geq 1 \), it suffices to show the probability measures
\[
(5.3.11) \quad \mathcal{L}(\sum_{j=1}^r \phi_j(Y(k_1 - 1 + \tau_j)), \cdots, \sum_{j=1}^r \phi_j(Y(k_d - 1 + \tau_j)))
\]
and
\[
(5.3.12) \quad \mathcal{L}(\sum_{j=1}^r \phi_j(Y(k_1 + h - 1 + \tau_j)), \cdots, \sum_{j=1}^r \phi_j(Y(k_d + h - 1 + \tau_j)))
\]
are equal and jointly Gaussian. However, since
\[
0 \leq k_1 - 1 + \tau_1 < \cdots < k_1 - 1 + \tau_r \leq k_2 - 1 < \cdots < k_2 - 1 + \tau_r \\
\leq \cdots \leq k_d - 1 < \cdots < k_d - 1 + \tau_r
\]
and \( \{ Y(t) : t \geq 0 \} \) is stationary and Gaussian we have
\[
\mathcal{L}(Y(k_1 - 1 + \tau_1), \cdots, Y(k_1 - 1 + \tau_r), \cdots, Y(k_d - 1 + \tau_1), \\
\cdots, Y(k_d - 1 + \tau_r)),
\]
and
\[
\mathcal{L}(Y(k_1 + h - 1 + \tau_1), \cdots, Y(k_1 + h - 1 + \tau_r), \cdots, Y(k_d + h - 1 + \tau_1), \\
\cdots, Y(k_d + h - 1 + \tau_r)),
\]
are equal Gaussian probabilities on \( \mathbb{R}^{rd} \). From this, and that the \( \{ \phi_j : 1 \leq j \leq r \} \) are in \( E^* \), we immediately have the probabilities in (5.3.11) and (5.3.12) are equal and Gaussian, which completes the proof of the lemma. \( \Box \)

5.4. Covariance Decay for the Ornstein-Uhlenbeck Process. In our proof of (4.4) and (4.6) we have used Theorem 3.3 in [Pic67], which requires a rate of decay of the covariances for a sequence of stationary Gaussian random variables. Here we show that for the Ornstein-Uhlenbeck process the required rate of decay follows from the process itself, and is not an extra assumption. To estimate this covariance decay for the Ornstein-Uhlenbeck
process generated by the mean zero Gaussian measure $\gamma$ on $E$ we first show the problem can be moved to any Banach space $F$ that is linearly isometric to $E$. Then, with suitably chosen $F$ we make the estimate. First, however we indicate some additional notation.

Let $E$ and $F$ be separable Banach spaces with norms $q_E$ and $q_F$ and suppose $\Lambda: E \to F$ is a linear isometry from $E$ onto $F$ with inverse $\Lambda^{-1}$. If $x \in C_E[0,\infty)$, then we define the function $y(\cdot) = \hat{\Lambda}x(\cdot) \in C_F[0,\infty)$ by

$$\hat{\Lambda}x(t) = \Lambda(x(t)), \quad 0 \leq t < \infty,$$

and for $x \in C_E[0,b]$ and $b > 0$ we define

$$\hat{\Lambda}_b x(\cdot) = \Lambda(x(\cdot)), \quad 0 \leq t \leq b.$$}

For every $b > 0$ and $x \in C_E[0,\infty)$ we have for $y = \hat{\Lambda}(x)$ that

$$\sup_{0 \leq t \leq b} q_F(y(t)) = \sup_{0 \leq t \leq b} q_F(\hat{\Lambda}(x)(t)) = \sup_{0 \leq t \leq b} q_F(\Lambda(x(t))) = \sup_{0 \leq t \leq b} q_E(x(t)),$$

so $\hat{\Lambda}_b$ is a linear isometry from $C_E[0,b]$ onto $C_F[0,b]$. Similarly, $\hat{\Lambda}$ and $\hat{\Lambda}_b$ have linear inverses $\hat{\Lambda}^{-1}$ and $\hat{\Lambda}_b^{-1}$ from $C_F[0,\infty)$ onto $C_E[0,\infty)$ and $C_F[0,b]$ onto $C_E[0,b]$. Of course, $\Lambda_b^{-1}$ is also an isometry.

As before $\gamma$ is a non-degenerate mean zero Gaussian measure on the Borel subsets of $E$ and we define (as usual)

$$\gamma^\Lambda(A) = \gamma(\Lambda^{-1}(A)).$$

Then, the probability $\gamma^\Lambda$ is a mean zero Gaussian measure on the Borel subsets of $F$, the supports of $\gamma$ and $\gamma^\Lambda$ are closed linear subspaces of $E$ and $F$, respectively, and they are linearly isometric under $\Lambda$.

The $E$-valued $\gamma$-Brownian motion process on $(\Omega_E,\mathcal{F},P_\gamma)$ is defined as in Lemma 5.1.1 and is indicated by $W_\gamma = \{W_\gamma(t) : t \geq 0\}$ with $W_\gamma(t,x) = x(t), x \in \Omega_E$. The $E$-valued $\gamma$-Ornstein-Uhlenbeck process is given by

$$Y_\gamma(t,x) = e^{-\frac{t}{2}}W_\gamma(e^t,x) = e^{-\frac{t}{2}}x(e^t), t \geq 0, x \in \Omega_E \subseteq C_E[0,\infty),$$

and since the support of $\gamma$ is a closed subspace of $E$, the processes $W_\gamma$ and $Y_\gamma$ have support on the continuous functions on $[0,\infty)$ with values in that subspace.

Similarly, we let $\Omega_F$ be the space of continuous functions $y$ from $[0,\infty)$ into $F$ such that $y(0) = 0$, and $\mathcal{G}$ the $\sigma$-field of $\Omega_F$ generated by the functions $y \to y(t), 0 \leq t < \infty$. Then, there is a probability $P_{\gamma^\Lambda}$ on $(\Omega_F,\mathcal{G})$ such that the stochastic process $W_{\gamma^\Lambda} = \{W_{\gamma^\Lambda}(t) : t \geq 0\}$ defined on $(\Omega_F,\mathcal{G},P_{\gamma^\Lambda})$ by $W_{\gamma^\Lambda}(t,y) = y(t)$ is the $\gamma^\Lambda$-Brownian motion on $F$, and the stochastic process

$$Y_{\gamma^\Lambda}(t,y) = e^{-\frac{t}{2}}W_{\gamma^\Lambda}(e^t,y) = e^{-\frac{t}{2}}y(e^t), t \geq 0, y \in \Omega_F \subseteq C_F[0,\infty),$$
is the $F$-valued $\gamma^\Lambda$-Ornstein-Uhlenbeck process with support the space of continuous functions on $[0, \infty)$ with values in the subspace of $F$ supporting $\gamma^\Lambda$.

**Notation.** For $k \geq 1$ and $0 \leq t \leq 1$ we define the linear maps

(5.4.7)
\[
\tau_k : \Omega_E \subseteq C_E[0, \infty) \to C_E[0, 1], \quad \hat{\tau}_k : \Omega_F \subseteq C_F[0, \infty) \to C_F[0, 1],
\]
by

(5.4.8)
\[
\tau_k(x)(t) = e^{-\frac{t+(k-1)}{2}} x(e^{t+(k-1)}) \quad \text{and} \quad \hat{\tau}_k(y)(t) = e^{-\frac{t+(k-1)}{2}} y(e^{t+(k-1)}),
\]
and the processes

(5.4.9)
\[
X_k(t, x) = \tau_k(x)(t), x \in \Omega_E, \quad \text{and} \quad \hat{X}_k(t, y) = \hat{\tau}_k(y)(t), y \in \Omega_F.
\]
These maps are not only linear, but are continuous with respect to uniform convergence on compact subsets of $[0, \infty)$.

**Lemma 5.4.1.** $P_{\gamma^\Lambda} = (P_{\gamma})^\Lambda$ on $(\Omega_F, \mathcal{G})$ and $P_{\gamma} = P_{(\gamma^\Lambda)^{-1}} = (P_{\gamma^\Lambda})^{\hat{\Lambda}^{-1}}$ on $(\Omega_E, \mathcal{F})$, where $\hat{\Lambda}$ is defined in (5.4.1). Hence $P_{\gamma^\Lambda}$ is the distribution of $\hat{\Lambda}(W_{\gamma})$. Furthermore, if

(5.4.10)
\[
J(y) = [h \circ \tau_1(\hat{\Lambda}^{-1}(y))][h \circ \tau_k(\hat{\Lambda}^{-1}(y))],
\]
where $h \in C^*_F[0, 1]$ and $y \in \Omega_F$, then

(5.4.11)
\[
\int_{\Omega_F} J(y)dP_{\gamma^\Lambda}(y) = \int_{\Omega_F} J(y)dP_{\gamma}^\Lambda(y) = \int_{\Omega_E} J(\hat{\Lambda}(x))dP_{\gamma}(x)
\]
\[
= \int_{\Omega_E} [h \circ \tau_1(x)][h \circ \tau_k(x)]dP_{\gamma}(x).
\]
Moreover, for $y \in \Omega_F$ and $h \in C^*_F[0, 1]$, $J(\cdot)$ is such that

(5.4.12)
\[
J(y) = [h \circ \hat{\Lambda}^{-1}_1(\hat{\tau}_1(y))][h \circ \hat{\Lambda}^{-1}_k(\hat{\tau}_k(y))]
\]
with $h \circ \hat{\Lambda}^{-1}_1 \in C^*_F[0, 1]$. Hence, for $\hat{h} = h \circ \hat{\Lambda}^{-1}_1$ we have

(5.4.13)
\[
\int_{\Omega_F} [\hat{h} \circ \hat{\tau}_1(y)][\hat{h} \circ \hat{\tau}_k(y)]dP_{\gamma^\Lambda}(y) = \int_{\Omega_E} [h \circ \tau_1(x)][h \circ \tau_k(x)]dP_{\gamma}(x).
\]

**Proof.** By definition of the Brownian motion induced by $\gamma^\Lambda$, for $0 = t_0 < t_1 \cdots < t_n$ the random vectors

\[
y(t_j) - y(t_{j-1}), j = 1, \cdots, n, \quad \text{are independent,}
\]
and $y(t_j) - y(t_{j-1})$ has distribution $\gamma^\Lambda_{t_j-t_{j-1}}$ on $F$, where $\gamma^\Lambda_{s}(A) = \gamma^\Lambda(A/\sqrt{s})$ for Borel subsets $A$ of $F$ when $s > 0$ and $\gamma^\Lambda_0 = \delta_0$. In particular, the stochastic process $W_{\gamma^\Lambda} = \{W_{\gamma^\Lambda}(t) : t \geq 0\}$ defined on $(\Omega_F, \mathcal{G}, P_{\gamma^\Lambda})$ by $W_{\gamma^\Lambda}(t, y) = y(t)$ has stationary independent mean zero Gaussian increments and an increment of length $s > 0$ has distribution given by $\gamma^\Lambda(\cdot/\sqrt{s})$.  

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We also have by definition that the stochastic process \( \hat{\Lambda}(W_\gamma) =: \{\hat{\Lambda}(W_\gamma)(t) : t \geq 0\} \) has law on \((\Omega, \mathcal{F}, \mathbb{P})\) given by \((P_\gamma)\hat{\Lambda}\), where
\[
(5.4.14) \quad \hat{\Lambda}(W_\gamma)(t) = \Lambda(W_\gamma(t)), t \geq 0.
\]
Hence the increments of \( \hat{\Lambda}(W_\gamma) \) based on \( 0 = t_0 < t_1 \cdots < t_n \) are
\[
\hat{\Lambda}(W_\gamma)(t_j) - \hat{\Lambda}(W_\gamma)(t_{j-1}) = \Lambda(W_\gamma(t_j) - W_\gamma(t_{j-1})), \quad j = 1, \cdots, n.
\]
In addition, they are independent and an increment over an interval of length \( s > 0 \) is such that
\[
P_\gamma(x \in \Omega_E : \Lambda(x(t+s) - x(t)) \in A) = P_\gamma(x(1) \in \Lambda^{-1}(A/\sqrt{s})) = \gamma^\Lambda(A/\sqrt{s}),
\]
where \( A \) is any Borel subset of \( F \). Thus \((P_\gamma)\hat{\Lambda} = P_\gamma\Lambda\) on \((\Omega, \mathcal{F}, \mathbb{P})\). Furthermore, we therefore also have \((5.4.11)\) by the standard change of variables formula. Finally, given \((5.4.10)\), \((5.4.12)\) is immediate from the definitions of the mappings, and \((5.4.13)\) follows from \((5.4.11)\).

Since \( E \) is separable, the Banach-Mazur Theorem allows us to take the linearly isometric Banach space \( F \) to be a subspace of the sup-norm Banach space \( C[0,1] \) with norm \( q_F(y) = \sup_{0 \leq t \leq 1} |y(t)| \). Furthermore, since \( \gamma \) is a non-degenerate centered Gaussian measure on the Borel sets of \( E \), its support is a non-degenerate closed linear subspace of \( E \), and \( \gamma^\Lambda \) is a Gaussian measure on the Borel subsets of \( C[0,1] \) whose support is a closed linear subspace \( F = \Lambda(E) \). Hence applying \((5.4.12)\) and \((5.4.13)\) to the conclusion of the following lemma, we will have a decay of the covariances suitable for every linear functional on \( C_E[0,1] \). That is, the following lemma establishes the results for all linear functionals on \( C_E[0,1] \), and since \( \hat{\Lambda}_1^{-1} \) is a linear isometry from \( C_F[0,1] \) onto \( C_E[0,1] \) the norms
\[
(5.4.15) \quad ||h||^* =: ||h||_{C_F[0,1]} \quad \text{and} \quad ||h \circ \hat{\Lambda}_1^{-1}||^* =: ||h \circ \hat{\Lambda}_1^{-1}||_{C_E[0,1]}\n\]
are equal (see \((5.4.3)\)). Lemma \((5.4.3)\) below summarizes this, and hence the desired rate of decay for the covariances also holds for all linear functionals on \( C_E[0,1] \).

**Lemma 5.4.2.** Let \( \hat{\gamma} \) be a non-degenerate mean zero Gaussian measure on the sup-norm Banach space \( C[0,1] \) whose support is a closed subspace \( F \) of \( C[0,1] \), and \( Y_\gamma =: \{Y_\gamma(t) : t \geq 0\} \) be the \( F \)-valued sample continuous Ornstein-Uhlenbeck process generated by \( \hat{\gamma} \) on \( F \) as in \((5.4.6)\). Also assume the processes \( \{\hat{X}_k(t) : 0 \leq t \leq 1\} \) are as in \((5.4.9)\). Then, for each \( \hat{h} \in C_F[0,1] \) and \( k \geq 2 \)
\[
(5.4.16) \quad |E_{P_\gamma}(\hat{h}(\hat{X}_1(\cdot))\hat{h}(\hat{X}_k(\cdot)))| \leq e^{-\frac{k-2}{2} \sigma^2(\hat{\gamma})||\hat{h}||_{C_F[0,1]}^2},
\]
and
\[
(5.4.17) \quad E_{P_\gamma}(\hat{h}^2(\hat{X}_1(\cdot))) \leq ||\hat{h}||_{C_F[0,1]}^2 \sigma^2(\hat{\gamma}),
\]
where \( \sigma^2(\hat{\gamma}) \) is defined as in Lemma \((2.1)\).
Proof. Identifying \( C_{[0,1]}[0,\infty) \) with \( C(\mathbb{R} \times [0,1]) \) we then have
\[
\Omega_F = \{ y(\cdot, \cdot) : y(\cdot, s) \in C_F[0,\infty), y(t, \cdot) \in F \subseteq C[0,1], \forall t < \infty \}
\]
\[
\subseteq \{ y(\cdot, \cdot) : y(\cdot, \cdot) \in C(\mathbb{R} \times [0,1]) \},
\]
and by the Hahn-Banach theorem we let \( \hat{h} \) be a norm preserved extension of \( h \) from \( C_F[0,1] \) to all of \( C(\mathbb{R} \times I) \), where \( I = [0,1] \). Then, by the Riesz Representation theorem there is a signed measure \( \lambda \) on \( I \times I \), where \( \lambda \) has finite total variation \(|\lambda| = ||\hat{h}||_{C^*} = ||\hat{h}||_{C_F^*[0,1]} = ||\hat{h}||_{C_F^*[0,1]} \), and
\[
\hat{h}(y(\cdot, \cdot)) = \int_{I \times I} y(t, s) d\lambda(t, s), y(\cdot, \cdot) \in C(I \times I).
\]
Since we are assuming \( Y_\gamma \) is defined on \( (\Omega_F, \mathcal{G}, P_\gamma) \) as in (5.4.6), we have for \( y \in \Omega_F, t \in [0,\infty), s \in [0,1] \), with \( P_\gamma \)-probability one that
\[
Y_\gamma(t, y)(s) = e^{-\frac{1}{2}} y(\cdot, s).
\]
Using the notation defined in (5.4.8) and (5.4.9), and identifying \( y(t, s) \) with \( y(\cdot, \cdot) \), we have for \( k \geq 1, y \in \Omega_F \) that for \( 0 \leq t, s \leq 1 \), with \( P_\gamma \)-probability one
\[
\hat{\tau}_k(y)(t)(s) = \hat{\tau}_k(y)(t, s) = \hat{X}_k(t, y)(s) = Y_\gamma(k - 1 + t, y)(s).
\]
Therefore,
\[
E_{P_\gamma}[\hat{h}(\hat{X}_1(\cdot)) \hat{h}(\hat{X}_k(\cdot))] = E_{P_\gamma}[\hat{h}(\hat{X}_1(\cdot)) \hat{h}(\hat{X}_k(\cdot))] = E_{P_\gamma}[\hat{h}(\hat{\tau}_1(y)) \hat{h}(\hat{\tau}_k(y))],
\]
and hence by (5.4.19) and (5.4.20) we have
\[
E_{P_\gamma}[\hat{h}(\hat{X}_1(\cdot)) \hat{h}(\hat{X}_k(\cdot))] = E_{P_\gamma}[\int_{I \times I} \hat{\tau}_1(y)(t, s) d\lambda_\hat{h}(t, s) \int_{I \times I} \hat{\tau}_k(y)(v, u) d\lambda_\hat{h}(v, u)].
\]
Moreover, since \( \gamma \)-Brownian motion has independent mean zero Gaussian increments, for \( k \geq 2 \) we have
\[
y(e^{k-1+v}, \cdot) - y(e^t, \cdot) \text{ independent of } y(e^t, \cdot),
\]
and hence \( y(e^{k-1+v}, u) - y(e^t, u) \) and \( y(e^t, u) \) are independent mean zero random variables. Therefore, the integrability of the Gaussian random variables and vectors involved implies
\[
e^{t+k-1+v} |E_{P_\gamma}[\hat{\tau}_1(y)(t, s) \hat{\tau}_k(y)(v, u)]|
\]
\[
= |E_{P_\gamma}[y(e^t, s) y(e^{k-1+v}, u) - y(e^t, u)]| + |E_{P_\gamma}[y(e^t, s) y(e^t, u)]|
\]
\[
\leq |E_{P_\gamma}[y^2(e^t, s)] | \cdot |E_{P_\gamma}[y^2(e^t, u)] |^{\frac{1}{2}} = e^t \left[ E_{P_\gamma}[y^2(e^t, s)] \right]^{\frac{1}{2}} \left[ E_{P_\gamma}[y^2(e^t, u)] \right]^{\frac{1}{2}}
\]
\[
= e^t \left[ \int_F y^2(1, s) d\gamma(y) \right]^\frac{1}{2} \left[ \int_F y^2(1, u) d\gamma(y) \right]^\frac{1}{2} \leq e^t \sigma^2(\gamma),
\]
where the last inequality holds by the definition of \( \sigma^2(\hat{\gamma}) \) as in Lemma 2.1 and that the evaluation maps at \( s, u \) are linear functionals of norm one. Therefore, for \( t, v \in \mathcal{I} \)

\[
(5.4.23) \quad |E_{P_\gamma}[\hat{\gamma}_1(y)(t, s)\hat{\gamma}_k(y)(v, u)]| \leq e^\frac{-k-\hat{\gamma}^2}{2} \sigma^2(\hat{\gamma}) \leq e^\frac{-k-\hat{\gamma}^2}{2} \sigma^2(\hat{\gamma}).
\]

Combining (5.4.21) and (5.4.23), Fubini’s theorem implies

\[
(5.4.24) \quad |E_{P_\gamma}[\hat{\gamma}(X_1, t)\hat{\gamma}(X_k, t)]| \leq e^\frac{-k-\hat{\gamma}^2}{2} \sigma^2(\hat{\gamma})\|\gamma\|_{C_\gamma^2[0, 1]}^2,
\]

and (5.4.16) holds. The proof of (5.4.27) is similar. That is, from (5.4.21) we have

\[
E_{P_\gamma}[\hat{\gamma}(X_1, t)] = E_{P_\gamma}\left[\int_{\mathcal{I} \times \mathcal{I}} \hat{\gamma}_1(y)(t, s)d\lambda_\lambda(t, s)\right]^2
\]

\[
\leq E_{P_\gamma}\left[\int_{\mathcal{I} \times \mathcal{I}} |\hat{\gamma}_1(y)(t, s)|d\lambda_\lambda(t, s)\right] \leq E_{P_\gamma}\left[\int_{\mathcal{I} \times \mathcal{I}} e^{-t}y_2(e', s)d\lambda_\lambda(t, s)\right]|\lambda_\lambda|(\mathcal{I} \times \mathcal{I})
\]

\[
= \int_{\mathcal{I} \times \mathcal{I}} e^{-t}E_{P_\gamma}[y_2(e', s)]d\lambda_\lambda(t, s)|\lambda_\lambda|(\mathcal{I} \times \mathcal{I}) = \int_{\mathcal{I} \times \mathcal{I}} E_{P_\gamma}[y_2(1, s)]d\lambda_\lambda(t, s)|\lambda_\lambda|(\mathcal{I} \times \mathcal{I})
\]

\[
\leq |\lambda_\lambda|(\mathcal{I} \times \mathcal{I})^2 \sigma^2(\hat{\gamma}) = |\hat{\gamma}|^2_{C_\gamma^2[0, 1]} \sigma^2(\hat{\gamma}),
\]

which establishes (5.4.17). □

**Lemma 5.4.3.** Let \( Y_\gamma =: \{Y_\gamma(t) : t \geq 0\} \) be an \( E \)-valued sample continuous Ornstein-Uhlenbeck process generated by the non-degenerate mean zero Gaussian measure \( \gamma \) on \( E \) as in (5.4.9), and assume the processes \( \{X_k(t) : 0 \leq t \leq 1\} \) are as in (5.4.9). Then, for each \( h \in C_E^1[0, 1] \) and \( k \geq 2 \)

\[
(5.4.25) \quad |E_{P_\lambda}(h(X_1))h(X_k, \hat{\gamma})| \leq e^\frac{-k-\hat{\gamma}^2}{2} \sigma^2(\hat{\gamma})||h||^2_{C_\gamma^2[0, 1]},
\]

and

\[
(5.4.26) \quad E_{P_\gamma}(h^2(X_1)) \leq ||h||^2_{C_\gamma^2[0, 1]} \sigma^2(\hat{\gamma}).
\]

**Proof.** Since \( E \) is a separable Banach space, the Banach-Mazur theorem shows there is a linear isometry \( \Lambda \) mapping \( E \) onto \( F \), where \( F \) is a closed subspace of the sup-norm Banach space \( C[0, 1] \), and the \( q \)-norm on \( E \) maps to the sup-norm on \( F \). Using (5.4.11) and (5.4.12) of Lemma 5.4.1 with \( \hat{\gamma} = \gamma \Lambda \) and \( P_\gamma = P_\Lambda \), we then have for \( k \geq 2 \) that

\[
(5.4.27) \quad E_{P_\gamma}(h(X_1))h(X_k) = E_{P_\Lambda}[h \circ \tau_1(\Lambda^{-1}(y))]h \circ \tau_k(\Lambda^{-1}(y))]
\]

\[
= E_{P_\Lambda}[h \circ \Lambda_1^{-1}(\hat{\gamma}_1(y))]h \circ \Lambda_1^{-1}(\hat{\gamma}_k(y))]
\]

where \( \hat{h} = h \circ \Lambda_1^{-1} \). Combining (5.4.16) of Lemma 5.4.2 and (5.4.27) we have

\[
(5.4.28) \quad E_{P_\gamma}(h(X_1))h(X_k) \leq e^\frac{-k-\hat{\gamma}^2}{2} \sigma^2(\hat{\gamma})||\hat{h}||^2_{C_\gamma^2[0, 1]} = e^\frac{-k-\hat{\gamma}^2}{2} \sigma^2(\hat{\gamma})||h||^2_{C_\gamma^2[0, 1]}.
\]
where the equality in (5.4.28) holds since (5.4.15) implies
\[ ||h||_{C^*_E[0,1]} = ||h \circ \hat{\Lambda}^{-1}||_{C^*_E[0,1]} = ||h||_{C^*_E[0,1]}, \]
and \( \sigma^2(\gamma) = \sigma^2(\hat{\gamma}). \) Hence (5.4.25) holds for \( k \geq 2, \) and the proof of (5.4.26) is entirely similar. \( \square \)

**Remark 5.4.1.** If \( V = \{h \in C^*_E[0,1] : ||h||_{C^*_E[0,1]} \leq 1\}, \) then (5.4.26) implies
\[ J \geq \int_{C^*_E[0,1]} h_0^2(\tau_1(x))dP_\gamma(x) = \int_{C^*_E[0,1]} f_0^2(\tau_1(x))dP_\gamma(x) = \int_{E} f_0^2(w)d\gamma(w). \]

Therefore, for \( f = f_0 \) as in Lemma 2.4, with the Gaussian measure being \( \gamma, \) we have \( ||f_0||_{E^*} = 1 \) and \( J \geq \int_{E} f_0^2(x)d\gamma(x) = \sigma^2(\gamma) = \Gamma^2. \) Thus equality holds in (5.4.29) as claimed.

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