A HARNACK INEQUALITY FOR FRACTIONAL LAPLACE EQUATIONS
WITH LOWER ORDER TERMS

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ABSTRACT. We establish a Harnack inequality of fractional Laplace equations without
imposing sign condition on the coefficient of zero order term via the Moser’s iteration and
John-Nirenberg inequality.

1. INTRODUCTION

This note is devoted to a Harnack inequality of Laplace equations without imposing sign
condition on the coefficient of zero order terms.

The fractional Laplacians \((-\Delta)^{\sigma}\), \(0 < \sigma < 1\), which are the infinitesimal generators
in stable Lévy stable processes, are given by the Fourier transform \(\mathcal{F}\) as follows: for \(u \in H^{\sigma}(\mathbb{R}^n)\), \(n \geq 2\),
\[
\mathcal{F}((-\Delta)^{\sigma} u)(\xi) := |\xi|^{2\sigma} \mathcal{F}(u)(\xi) \quad \xi \in \mathbb{R}^n.
\]

Caffarelli and Silvestre \([3]\) introduced fractional extension \(v \in D^{1,2}_{\sigma}(\mathbb{R}^{n+1})\) of \(v(x,0) = u(x)\) satisfying
\[
\int_0^\infty \int_{\mathbb{R}^n} |\nabla v(x,t)|^2 \, t^{1-2\sigma} \, dt \, dx = c_\sigma \int_{\mathbb{R}^n} |\xi|^{2\sigma} |\mathcal{F}(u)(\xi)|^2 \, d\xi,
\]
where \(c_\sigma^{-1} = 2^{-1}(4\pi)^{2\sigma} \Gamma(2-2\sigma)\). Then the fractional Laplacians are realized by the
Dirichlet-Neumann map of \(v\)
\[
(-\Delta)^{\sigma} u(x) = -c_\sigma \lim_{t \to 0} t^{1-2\sigma} v_t.
\]

Let \(B_r \subset \mathbb{R}^n\) be the ball centered at origin with radius \(r\). Our main result is

**Theorem 1.1.** Let \(u \in H^{\sigma}(\mathbb{R}^n)\) be nonnegative in \(\mathbb{R}^n\) and \(C^2(B_1) \cap C^1(\overline{B_1})\). Suppose
that \(u(x)\) satisfies
\[
(-\Delta)^{\sigma} u(x) = a(x)u(x) + b(x) \quad \text{in } B_1,
\]
where \(a(x), b(x) \in L^\infty(B_1)\). Then
\[
\sup_{B_{1/2}} u \leq C \left( \inf_{B_{1/2}} u + \|b\|_{L^\infty(B_1)} \right),
\]
where \(C > 0\) depends only on \(n, \sigma, \|a\|_{L^\infty(B_1)}\).

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To prove it, we establish a Harnack inequality for the equivalent problem as follow. Let \( X = (x, t) \in \mathbb{R}^{n+1}, Q_R = B_R \times (0, R) \subset \mathbb{R}^{n+1} \) and \( \partial' Q_R = B_R \times \{0\} \). Define
\[
H(t^{1-2\sigma}, Q_R) := \left\{ U \in H^1(Q_R) : \int_{Q_R} t^{1-2\sigma}(U^2 + |\nabla U|^2) \, dX < \infty \right\}.
\]

**Theorem 1.2.** Let \( U \in H(t^{1-2\sigma}, Q_1) \) be nonnegative solution \( C^2(Q_1) \cap C^1(\overline{Q_1}) \) of
\[
\begin{align*}
\text{div}(t^{1-2\sigma} \nabla U(X)) &= 0 & \text{in } Q_1 \\
- \lim_{t\to 0^+} t^{1-2\sigma} \partial_t U(x, t) &= a(x)U(x, 0) + b(x) & \text{on } \partial' Q_1.
\end{align*}
\]

Suppose \( a, b \in L^\infty(B_1) \). Then
\[
\sup_{Q_{1/2}} U \leq C \left( \inf_{Q_{1/2}} U + \|b\|_{L^\infty(B_1)} \right),
\]
where \( C > 0 \) depends only on \( n, \sigma \) and \( \|a\|_{L^\infty(B_1)} \).

The main feature is that we do not assume the sign condition of \( a(x) \). Previously, in the case \( a(x) \equiv 0 \), Bass and Levin \([1]\) establish the Harnack inequality for nonnegative functions of a class of symmetric stable processes that are harmonic with respect to these processes, see also \([4]\) by Chen and Song. The analytic method was given by Caffarelli and Silvestre \([3]\), by employing the fractional extension of fractional harmonic functions.

We here establish the Harnack inequality as in Theorem \([1,2]\) by the Moser iteration. The proof bases on the properties of the weighted Sobolev space developed by Fabes, Kenig and Serapioni \([5]\) and the John-Nirenberg inequality in \( A_2 \) weighted \( BMO \) space obtained by Muchenhoupt and Wheeden \([10]\).

If \( \sigma = \frac{1}{2} \), the result is due to Han and Li \([7]\). After we complete our manuscript, we observe that Theorem \([1,2]\) the Harnack inequality for \( b \equiv 0 \), has been shown recently by Cabre and Sire \([2]\) through making even extension and using the result of Fabes, Kenig and Serapioni \([3]\). But our proof has independent interest.

On the other hand, since the fractional Laplacian is a nonlocal operator, the condition \( u \geq 0 \) in \( \mathbb{R}^n \) cannot be relaxed to \( u \geq 0 \) in \( B_1 \). In fact, we need all information in the complement of \( B_1 \). For example, see an counterexample of the case \( a \equiv b \equiv 0 \) in \([9]\) by Kassmann. By the Dirichlet-Neumann map, we transform \([1,3]\) to the local problem in \( \mathbb{R}^{n+1}_+ \), which grantees the identity \([1,2]\). The nonnegative assumption of \( u \) implies that its fractional extension \( v \) is nonnegative in the half space \( \mathbb{R}^{n+1}_+ \). Thus, \( v \geq 0 \) in all of cubes \( Q_R, R > 0 \). Therefore, we can obtain the desired Harnack inequality by studying the local version \([1,5]\).

The paper is organized as follows. In Section \([2]\) we demonstrate some properties in the weighted Sobolev spaces. The proofs of Theorem \([1,1,1,2]\) are given in Section \([3]\).

## 2. Preliminaries

In this section, we shall present some important weighted inequalities.

Denote \( Q_R = B_R \times (0, R) \subset \mathbb{R}^{n+1}, \partial' Q_R = B_R \times \{0\} \) and \( \partial'' Q_R = \partial Q_R \setminus \partial' Q_R \). We use capital letters like \( X = (x, t), Y = (y, s) \) to represent points in \( \mathbb{R}^{n+1} \).

Let us recall the definition of \( A_2 \) class.
Definition 2.1. Let $\omega(X)$ be a nonnegative measurable function in $\mathbb{R}^{n+1}$. We say $\omega$ being of the class $A_2$ if there exists a constant $C_\omega$ such that for any ball $B \subset \mathbb{R}^{n+1}$

$$
\left( \frac{1}{|B|} \int_B \omega(X) \, dX \right) \left( \frac{1}{|B|} \int_B \omega^{-1}(X) \, dX \right) \leq C_\omega,
$$
where $| \cdot |$ is the Lebesgue measure.

Lemma 2.1. Let $f(X) \in C^1(Q_R) \cup \partial' Q_R)$ and $\omega(X) \in A_2$. Then there exist constants $C$ and $\delta > 0$ depending only on $n$ and $C_\omega$ such that for any $1 \leq k \leq \frac{n+1}{n} + \delta$

$$
(2.1) \quad \left( \frac{1}{\omega(Q_R)} \int_{Q_R} |f|^{2k} \omega \, dX \right)^{1/2k} \leq CR \left( \frac{1}{\omega(Q_R)} \int_{Q_R} |\nabla f|^2 \omega \, dX \right)^{1/2},
$$
where $\omega(Q_R) = \int_{Q_R} \omega(X) \, dX$.

Proof. The proof of this Lemma is similar to that of Theorem 1.2 in [5]. The following inequality is the only thing we need to show.

$$
(2.2) \quad |f(X)| \leq \frac{2}{\omega_n} \int_{Q_R} \frac{|\nabla f(Y)|}{|X - Y|^n} \, dY, \quad \text{for any } X \in Q_R,
$$
where $\omega_n$ is the area of the sphere $\mathbb{S}^n$.

Extend $f$ to be zero outside $Q_R$. Let $X \in Q_R$, then (2.2) follows from

$$
(2.3) \quad f(X) = \frac{2}{\omega_n} \int_{\mathbb{R}^{n+1}} \frac{\nabla f(X - Y) \cdot Y}{|Y|^{n+1}} \, dY.
$$

Since $X - Y \in \mathbb{R}^{n+1}$, $\nabla f(X - Y)$ makes sense. Let $\xi \in S^n$, the south half sphere. For $t > 0$, note that

$$
\begin{align*}
    f(X) &= \int_0^\infty -\frac{\partial}{\partial t} f(X - \xi t) \, dt = \int_0^\infty \nabla f(X - \xi t) \cdot \xi \, dt.
\end{align*}
$$

We integrate the above over $\xi$ ranging on the south half sphere. This gives

$$
\begin{align*}
    f(X) &= \frac{2}{\omega_n} \int_{\xi \in S^n} \int_0^\infty \nabla f(X - \xi t) \cdot \xi \, dt \, d\xi.
\end{align*}
$$

Identity (2.3) follows from coordinate changing. \qed

Next we quote the following weighted Poincaré inequality which can be found in [5].

Lemma 2.2. Let $f \in C^1(Q_R)$, then any $1 \leq k \leq \frac{n+1}{n} + \delta$, we have

$$
\begin{align*}
    \left( \frac{1}{\omega(Q_R)} \int_{Q_R} |f - f_{R,\omega}|^{2k} \omega \, dX \right)^{1/2k} \leq CR \left( \frac{1}{\omega(Q_R)} \int_{Q_R} |\nabla f|^2 \omega \, dX \right)^{1/2},
\end{align*}
$$
where $f_{R,\omega} = \frac{1}{\omega(Q_R)} \int_{Q_R} f \omega$.

Finally, we prove the following trace embedding result.

Lemma 2.3. Let $f(X) \in C^1(Q_R \cup \partial' Q_R)$ and $\alpha \in (-1, 1)$. Then there exists a positive constant $\delta$ depending only on $\alpha$ such that

$$
(2.4) \quad \int_{\partial' Q_R} |f|^2 \leq \epsilon \int_{Q_R} |\nabla f|^2 t^\alpha + \frac{C(R)}{\epsilon^\delta} \int_{Q_R} |f|^2 t^\alpha,
$$
for any $\epsilon > 0$. 
Proof. For any \( 1 < p < \infty \), we have
\[
\int_{\partial' Q_R} |f|^p = -\int_{Q_R} \partial_t |f|^p = -\int_{Q_R} p|f|^{p-1}\text{sgn}f \partial_t f
\]
(2.5)
\[
\leq \varepsilon \int_{Q_R} |\nabla f|^p + C\varepsilon^{-\frac{1}{p-1}} \int_{Q_R} |f|^p.
\]
Next, we claim for \( 0 < \alpha < 1 \) and any \( \lambda > -1 \)
\[
\int_{Q_R} |f|^{2\lambda} \leq C(\lambda, \alpha) \int_{Q_R} |\nabla f|^{2\lambda}.
\]
(2.6)
In fact, by the Hölder inequality
\[
f^2(x, t) = (\int_{t}^{R} \partial_t f(x, s) \, ds)^2 \leq \int_{t}^{R} s^{-\alpha} \, ds \int_{t}^{R} |\partial_t f|^2 s^\alpha \, ds
\]
\[
\leq \frac{C}{1-\alpha} \int_{0}^{R} |\nabla f(x, s)|^2 s^\alpha \, ds.
\]
Multiplying the above by \( t^\lambda \) and integrating over \( Q_R \), we obtain
\[
\int_{Q_R} t^\lambda f^2 \leq C \int_{B_R} t^\lambda \, dt \int_{B_R} \int_{0}^{R} |\nabla f(x, s)|^2 s^\alpha \, ds \, dx
\]
\[
\leq C \int_{Q_R} |\nabla f(x, s)|^2 s^\alpha,
\]
so (2.6) follows.

We are going to prove (2.4). Let \( p \in (1, \frac{2}{1+\alpha}) \). It follows from (2.5) and the Hölder inequality that
\[
\int_{\partial' Q_R} |f|^2 = \int_{\partial' Q_R} (|f|^2)^\frac{p}{2}
\]
\[
\leq \varepsilon \int_{Q_R} |\nabla f|^\frac{p}{2} + C\varepsilon^{-\frac{1}{p-1}} \int_{Q_R} |f|^2
\]
\[
= \varepsilon \left(\begin{array}{c}
\frac{2}{p}
\end{array}\right)^p \int_{Q_R} |f|^{2-\frac{p}{2}} |\nabla f|^\frac{p}{2} + C\varepsilon^{-\frac{1}{p-1}} \int_{Q_R} |f|^2 |f|^\frac{p}{2}
\]
\[
\leq \varepsilon \left(\begin{array}{c}
\frac{2}{p}
\end{array}\right)^p \left(\int_{Q_R} |f|^2 t^{-\alpha} \right)^\frac{p-1}{p-2} \left(\int_{Q_R} |\nabla f|^2 t^\alpha\right)^\frac{2}{p}
\]
\[
+ C\varepsilon^{-\frac{1}{p-1}} \int_{Q_R} \left\{ \varepsilon^{1+\frac{1}{p-2}} |f|^2 t^{-\alpha} + \varepsilon^{-1-\frac{1}{p-2}} |f|^2 t^\alpha \right\}
\]
\[
\leq \varepsilon C \int_{Q_R} |\nabla f|^2 t^\alpha + \frac{C}{\varepsilon^{1+\frac{2}{p-2}}} \int_{Q_R} |f|^2 t^\alpha,
\]
where we used (2.6) for \( \lambda = -\frac{p\alpha}{2-p} > -1 \) and \( \lambda = -\alpha > -1 \) in the last inequality. Therefore, we complete the proof. \( \square \)
In this section, we will prove the main results by making use of the Moser’s iteration. For $p \in (0, \infty)$ denote

$$
\|U\|_{L^p(Q)} := \left( \int_{Q_R} t^{1-2\sigma} U^p \right)^{\frac{1}{p}}.
$$

**Proposition 3.1.** Let $U(X) \in H^1(Q)$ be a weak solution of

$$
\begin{align*}
\text{div}(1-2\sigma \nabla U(X)) & \geq 0 & \text{in } Q, \\
- \lim_{t \to 0^+} t^{1-2\sigma} \partial_t U(x, t) & \leq a(x)U(x, 0) + b(x) & \text{on } \partial' Q.
\end{align*}
$$

Then

$$
\sup_{Q_{1/2}} U^+ \leq C(\|U^+\|_{L^2(Q)} + \|b\|_{L^\infty(B_1)}),
$$

where $U^+ = \max\{0, U\}$, and $C > 0$ depends only on $n, \sigma, \|a\|_{L^\infty(B_1)}$.

**Proof.** Let $k, m > 0$ be some constants. Set $\overline{U} = U^+ + k$ and

$$
\overline{U}_m = \begin{cases} 
\overline{U} & \text{if } U < m, \\
k + m & \text{if } U \geq m.
\end{cases}
$$

Consider the test function

$$
\phi = \eta^2 (\overline{U}_m^\beta \overline{U} - k^{\beta+1}) \in H^1(Q),
$$

for some $\beta \geq 0$ and some nonnegative function $\eta \in C_c^1(Q_1 \cup \partial Q_1)$. Clearly, $\nabla \overline{U}_m = 0$ in $\{U < 0\}$ and $\{U \geq m\}$. A direct calculation yields

$$
\nabla \phi = \beta \eta \overline{U}_m^{\beta-1} \nabla \overline{U}_m \overline{U} + \eta \overline{U}_m^\beta \nabla \overline{U} + 2\eta \nabla \eta (\overline{U}_m^\beta \overline{U} - k^{\beta+1})
$$

$$
= \eta \overline{U}_m^{\beta} (\beta \nabla \overline{U}_m + \nabla \overline{U}) + + 2\eta \nabla \eta (\overline{U}_m^\beta \overline{U} - k^{\beta+1}).
$$

Multiplying (3.1) by $\phi$ and integrating by parts, we have

$$
0 \leq - \int_{Q_1} t^{1-2\sigma} \nabla U \nabla \phi + \int_{\partial' Q_1} a(x)U \phi + b(x) \phi
$$

$$
= - \int_{Q_1} t^{1-2\sigma} \eta^2 \nabla \overline{U}_m (\nabla \overline{U} \overline{U}_m^2 + \nabla \overline{U}) - 2 \int_{Q_1} t^{1-2\sigma} \eta \nabla \overline{U}_m \overline{U} - k^{\beta+1} \nabla \eta \nabla \overline{U}
$$

$$
+ \int_{\partial' Q_1} a(x)U \eta^2 (\overline{U}_m^\beta \overline{U} - k^{\beta+1}) + b(x) \eta^2 (\overline{U}_m^\beta \overline{U} - k^{\beta+1})
$$

$$
\leq -\frac{1}{2} \int_{Q_1} t^{1-2\sigma} \eta^2 \overline{U}_m (\nabla \overline{U}_m \overline{U}_m^2 + \nabla \overline{U}_m^2) + 4 \int_{Q_1} t^{1-2\sigma} \overline{U}_m^\beta \overline{U} \nabla \eta \nabla \overline{U}_m
$$

$$
+ \int_{\partial' Q_1} |a(x)| \eta^2 \overline{U}_m^\beta \overline{U}^2 + |b(x)| \eta^2 \overline{U}_m^\beta \overline{U},
$$

where we used the Cauchy inequality and the fact $\overline{U}_m^\beta \overline{U} - k^{\beta+1} < \overline{U}_m^\beta \overline{U}$. Choosing $k = \|b\|_{L^\infty(B_1)}$ if $b$ is not identically zero. Otherwise choose an arbitrary $k > 0$ and
eventually let \( k \to 0 \). Then we see that \( |b(x)| \eta^2 U_m^\beta \leq \eta^2 U_m^\beta \). Hence (3.3) gives
\[
\int_{Q_1} t^{1-2\sigma} \eta^2 U_m^\beta (|\nabla U_m|^2 + |\nabla U|^2)
\leq 8 \int_{Q_1} t^{1-2\sigma} \eta^2 U_m^\beta (|\nabla \eta|^2 + 2\|a\|_{L^\infty(B_1)} + 1) \int_{\partial Q_1} \eta^2 U_m^\beta.
\]
Set \( W = \frac{U}{U_m} \). Then
\[
|\nabla W|^2 \leq (1 + \beta)(\beta \eta^2 U_m^\beta |\nabla U_m|^2 + \eta^2 |\nabla U|^2).
\]
Therefore, we have
\[
\int_{Q_1} t^{1-2\sigma} \eta^2 |\nabla W|^2 \leq C(1 + \beta) \left\{ \int_{Q_1} t^{1-2\sigma} W^2 |\nabla \eta|^2 + \int_{\partial Q_1} \eta^2 W^2 \right\},
\]
or
\[
\int_{Q_1} t^{1-2\sigma} |\nabla (\eta W)|^2 \leq C(1 + \beta) \left\{ \int_{Q_1} t^{1-2\sigma} W^2 |\nabla \eta|^2 + \int_{\partial Q_1} \eta^2 W^2 \right\}.
\]
By Lemma 2.3,
\[
C(1 + \beta) \int_{\partial Q_1} \eta^2 W^2 \leq \frac{1}{\gamma} \int_{Q_1} t^{1-2\sigma} |\nabla (\eta W)|^2 + C(1 + \beta)^\delta \int_{Q_1} t^{1-2\sigma} \eta^2 W^2
\]
for some \( \delta > 1 \) depending on \( n, \sigma \). It follows that
\[
\int_{Q_1} t^{1-2\sigma} |\nabla (\eta W)|^2 \leq C(1 + \beta)^\delta \int_{Q_1} t^{1-2\sigma} (\eta^2 + |\nabla \eta|^2) W^2.
\]
By the Sobolev inequality, see Lemma 2.2 we obtain
\[
\left( \int_{Q_1} t^{1-2\sigma} |\eta W|^2 \right)^\frac{2}{\chi} \leq C(1 + \beta)^\delta \int_{Q_1} t^{1-2\sigma} (\eta^2 + |\nabla \eta|^2) W^2,
\]
where \( \chi = \frac{n+1}{\eta} > 1 \). For any \( 0 < r < R \leq 1 \), consider an \( \eta \in C_c(Q_1 \cup \partial' Q_1) \) with \( \eta = 1 \) in \( Q_r \) and \( |\nabla \eta| \leq 2/(R - r) \). Thus we have
\[
\left( \int_{Q_r} t^{1-2\sigma} W^2 \right)^\frac{2}{\chi} \leq C(1 + \beta)^\delta \frac{(R - r)^2}{r^2} \int_{Q_R} t^{1-2\sigma} W^2.
\]
or, by the definition of \( W \),
\[
\left( \int_{Q_r} t^{1-2\sigma} U_m^\beta \right)^\frac{2}{\chi} \leq C \frac{(1 + \beta)^\delta}{(R - r)^2} \int_{Q_R} t^{1-2\sigma} U_m^\beta.
\]
Noting that \( U_m \leq U \), we get
\[
\left( \int_{Q_r} t^{1-2\sigma} U_m^\chi \right)^\frac{2}{\chi} \leq C \frac{(1 + \beta)^\delta}{(R - r)^2} \int_{Q_R} t^{1-2\sigma} U_m^\beta.
\]
provided the integral in the right hand side is bounded. By letting \( m \to \infty \), we conclude that
\[
\| U \|_{L^\chi(t^{1-2\sigma}, Q_r)} \leq \left( C \frac{(1 + \beta)^\delta}{(R - r)^2} \right)^\frac{1}{\chi} \| U \|_{L^\chi(t^{1-2\sigma}, Q_R)},
\]
where $C > 0$ is a constant depending only on $n, \sigma, \|a\|_{L^\infty(B_1)}$. As in standard Moser iterating procedure, we then arrive at

$$\sup_{Q_{1/2}} U \leq C \|U\|_{L^2(t_1^{1-2\sigma}, Q_1)}$$

or

$$\sup_{Q_{1/2}} U^+ \leq C(\|U^+\|_{L^2(t_1^{1-2\sigma}, Q_1)} + k).$$

Recalling the definition of $k$, we complete the proof. $\square$

The next lemma is so called weak Harnack inequality.

**Proposition 3.2.** Let $U(X) \in H(t^{1-2\sigma}, Q_1)$ be a nonnegative weak solution of

$$\begin{cases}
\text{div}(t^{1-2\sigma}\nabla U(X)) \leq 0 & \text{in } Q_1 \\
-t^{1-2\sigma} \partial_t U(x, t) = a(x)U(x, 0) + b(x) & \text{on } \partial' Q_1.
\end{cases}
$$

Then for some $p > 0$ and any $0 < \theta < \tau < 1$ we have

$$\inf_{Q_\theta} U + \|b\|_{L^\infty(Q_1)} \geq C\|U\|_{L^p(t^{1-2\sigma}, Q_\tau)},$$

where $C > 0$ depends only on $n, \sigma, \theta, \tau, \|a\|_{L^\infty(Q_1)}$.

**Proof.** Set $\overline{U} = U + k > 0$, for some positive $k$ to be determined and $V = \overline{U}^{-1}$. Let $\Phi$ be any nonnegative function in $H(t^{1-2\sigma}, Q_1)$ with compact support in $Q_1 \cup \partial' Q_1$.

Multiplying both sides of first inequality of (3.3) by $\overline{U}^{-2} \Phi$ and integrating by parts, we obtain

$$0 \geq -\int_{Q_1} t^{1-2\sigma} \nabla U \nabla \Phi \overline{U}^{-2} + 2\int_{Q_1} t^{1-2\sigma} \nabla U \nabla \Phi \overline{U}^{-2} + \int_{\partial' Q_1} (aU + b)V \Phi \overline{U}^{-2}.$$

Note that $\nabla U = \nabla \overline{U}$ and $\nabla V = -\overline{U}^2 \nabla \overline{U}$. Therefore, we have

$$\int_{Q_1} t^{1-2\sigma} \nabla V \nabla \Phi + \int_{\partial' Q_1} \tilde{a} \nabla \Phi \leq 0,$$

where

$$\tilde{a} = \frac{aU + b}{\overline{U}}.$$

Choose $k = \|b\|_{L^\infty(Q_1)}$ if $b$ is not identical zero. Otherwise, choose an arbitrary $k > 0$ and eventually let it tend to zero. Note that $\|\tilde{a}\|_{L^\infty(Q_1)} \leq \|a\|_{L^\infty(Q_1)} + 1$. Therefore Proposition 3.1 implies that for any $\tau \in (\theta, 1)$ and any $p > 0$

$$\sup_{Q_\theta} V \leq C\|V\|_{L^p(t^{1-2\sigma}, Q_\tau)},$$

or

$$\inf_{Q_\theta} \overline{U} \geq C \left(\int_{Q_\tau} t^{1-2\sigma} \overline{U}^{-p}\right)^{-\frac{1}{p}}.$$

$$= C \left(\int_{Q_\tau} t^{1-2\sigma} \overline{U}^{-p} \int_{Q_\tau} t^{1-2\sigma} \overline{U}^{-p} \left(\int_{Q_\tau} t^{1-2\sigma} \overline{U}^{-p}\right)^{-\frac{1}{p}},
$$

where $C > 0$ depends only on $n, \sigma, p, \theta, \tau$.

The next key point is to show that there exists some $p_0 > 0$ such that

$$\int_{Q_\tau} t^{1-2\sigma} \overline{U}^{-p_0} \int_{Q_\tau} t^{1-2\sigma} \overline{U}^{-p_0} \leq C,$$
where $C > 0$ depends only on $n, \sigma, \tau$. We are going to show that for any $\tau < 1$

\[
\int_{Q_1} e^{p_0|W|} \leq C, \tag{3.4}
\]

where $W = \log \overline{U} - (\log \overline{U})_{0,\tau}$. The idea is as usual. (3.4) will follows from John-Nirenberg type lemma (see [10]) if $W \in BMO(t^{1-2\sigma} dX)$.

We first derive an equation for $W$. Multiplying both sides of first inequality of (3.3) by $\overline{U}^{-1} \Phi$ and integrating by parts, we obtain

\[
\int_{Q_1} t^{1-2\sigma} |\nabla W|^2 \Phi \leq \int_{Q_1} t^{1-2\sigma} \nabla W \nabla \Phi + \int_{\partial Q_1} \tilde{a} \Phi,
\]

where

\[
\tilde{a} = \frac{aU + b}{U}.
\]

Replace $\Phi$ by $\Phi^2$. It follows from the Cauchy inequality and the Sobolev inequality that

\[
\int_{Q_1} t^{1-2\sigma} |\nabla W|^2 \Phi^2 \leq C \int_{Q_1} t^{1-2\sigma} |\nabla \Phi|^2, \tag{3.5}
\]

where $C > 0$ depends only on $n, \sigma$. Then for any $Q_{2r}(Y) \subset Q_1$, $Y \in \partial \mathbb{R}^{n+1}_+$, choose $\Phi$ with

\[
supp(\Phi) \subset Q_{2r}(Y) \cup \partial' Q_{2r}(Y), \quad \Phi = 1 \text{ in } Q_r(Y) \cup \partial' Q_r(Y), \quad |\nabla \Phi| \leq \frac{C}{r}.
\]

We have

\[
\int_{Q_r(Y)} t^{1-2\sigma} |\nabla W|^2 \leq C \frac{r^2}{r^2} \int_{Q_r(Y)} t^{1-2\sigma}.
\]

Hence the Poincaré inequality, Lemma 2.2, implies

\[
\left( \int_{Q_r(Y)} t^{1-2\sigma} \right)^{-1} \int_{Q_r(Y)} t^{1-2\sigma} |W - W_{Y,r}|^{1/2}
\leq \left( \int_{Q_r(Y)} t^{1-2\sigma} \right)^{-1/2} \left( \int_{Q_r(Y)} t^{1-2\sigma} |W - W_{Y,r}|^{2} \right)^{1/2}
\leq r \left( \int_{Q_r(Y)} t^{1-2\sigma} \right)^{-1/2} \left( \int_{Q_r(Y)} t^{1-2\sigma} |\nabla W|^2 \right)^{1/2}
\leq C.
\]

For other $Y \in Q_1$, one can verify the above similarly. Therefore, we conclude that $W \in BMO(t^{1-2\sigma}, Q_1)$. \qed

**Proof of Theorem 1.2** The proof follows from Proposition 3.1 and 3.2. \qed

**Proof of Theorem 1.1** Since $u \geq 0$ in $\mathbb{R}^n$ be a solution of (1.3), there exists a nonnegative function $U(x, t) \in H(t^{1-2\sigma}, \mathbb{R}^{n+1}_+)$ satisfying

\[
\text{div}(t^{1-2\sigma} \nabla U(x, t)) = 0 \text{ in } \mathbb{R}^{n+1}_+
\]

and $U(x, 0) = u(x)$. It follows from (1.3) that

\[
\lim_{t \to 0^+} t^{1-2\sigma} \partial_t U(x, t) = -c_\sigma (-\Delta)^\sigma U(x, 0) = -c_\sigma (a(x)U(x, 0) + b(x)),
\]
where we used $u \in C^2(B_1)$. Hence Theorem 1.1 immediately follows from Theorem 1.2.

\[\square\]

**Theorem 3.1.** Let $0 < \sigma < 1$ and $B_R = B_R(0) \subset \mathbb{R}^n$, $n > 2\sigma$. Suppose that $a(x) \in L^\infty(\mathbb{R}^n)$, $0 \leq u \in C(\mathbb{R}^n)$ satisfies

\[(-\Delta)^\sigma u(x) = a(x)u(x), \quad x \in B_R.\]

Then for $\delta > 0$, there exists $C(n, \sigma, \delta) > 0$ such that

\[\max_{\overline{B}_{R-\delta}} u \leq C(n, \sigma, \delta) \min_{\overline{B}_{R-\delta}} u.\]

**Proof.** By rescaling, we can prove it from Theorem 1.1. See another proof in [2]. \[\square\]

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