On one nonlocal inverse boundary problem for the Benney – Luke equation with integral conditions

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Abstract. An inverse boundary value problem for the Benney-Luke equation with periodic and integral condition is investigated. The definition of a classical solution of the problem is introduced. The goal of this paper is to determine the unknown coefficient and to solve the problem of interest. The problem is considered in a rectangular domain. To investigate the solvability of the inverse problem, we perform a conversion from the original problem to some auxiliary inverse problem with trivial boundary conditions. By the contraction mapping principle we prove the existence and uniqueness of solutions of the auxiliary problem. Then we make a conversion to the stated problem again and, as a result, we obtain the solvability of the inverse problem.

1. Introduction
There are many cases where the needs of the practice bring about the problems of determining coefficients or the right hand side of differential equations from some knowledge of its solutions. Such problems are called inverse boundary value problems of mathematical physics. Inverse boundary value problems arise in various areas of human activity such as seismology, mineral exploration, biology, medicine, quality control in industry etc., which makes them an active field of contemporary mathematics. Inverse problems for various types have been studied in many papers.

The foundations of the theory and practice of studying inverse problems of mathematical physics were laid and developed in pioneering works of the outstanding scientists A. N. Tikhonov [1], M. M. Lavrent’ev [2], V. K. Ivanov [3] and V. G. Romanov [4]. At present, there are many works devoted to the study of inverse problems for partial differential equations [5–10].

Many problems of gas dynamics, theory of elasticity, theory plates and shells is reduced to the consideration of differential equations in high-order partial derivatives. Of great interest from the point of view of applications are differential equations of the fourth order. Partial differential equations of the Benneo – Luke type have applications in mathematical physics [11].

In this paper, we prove existence and uniqueness of the solution to an inverse boundary value problem for the Benney – Luke equation with integral conditions.

2. Problem statement and its reduction to an equivalent problem
Consider for the Benney – Luke equation [11]
\[ u_t(x, t) - u_{xx}(x, t) + \alpha u_{xxx}(x, t) - \beta u_{xxt}(x, t) = a(t)u(x, t) + b(t)g(x, t) + f(x, t) \quad (x, t) \in D_T \quad (1) \]

in the domain \( D_T = \{ (x, t) : 0 \leq x \leq 1, 0 \leq t \leq T \} \) an inverse boundary problem with the nonlocal initial conditions

\[ u(x, 0) = \int_0^T p(t)u(x, t)dt + \varphi(x), \quad u_t(x, 0) = \psi(x) \quad (0 \leq x \leq 1), \quad (2) \]

the periodic conditions

\[ u(0, t) = u(1, t), \quad u_x(0, t) = u_x(1, t), \quad u_{xx}(0, t) = u_{xx}(1, t) \quad (0 \leq t \leq T), \quad (3) \]

the non-local integral condition

\[ \int_0^1 u(x, t)dx = 0 \quad (0 \leq t \leq T) \quad (4) \]

and with the additional condition

\[ u(x_i, t) = h_i(t) \quad (i = 1, 2; x_1 \neq x_2; 0 \leq t \leq T) \quad (5) \]

where \( \alpha > 0, \beta > 0, x_i \in (0, 1) (i = 1, 2) \) are the given numbers, \( f(x, t), g(x, t), \varphi(x), \psi(x), p(t), h_i(t) \quad (i = 1, 2) \) are the given functions, and \( u(x, t), a(t), b(t) \) are the required functions.

We introduce the following set of functions:

\[ \tilde{C}^{4,2}(D_T) = \left\{ u(x, t) : u(x, t) \in C^2(D_T), u_{tx}(x, t), u_{txx}(x, t), u_{xxx}(x, t), u_{xxxx}(x, t) \in C(D_T) \right\}. \]

**Definition.** The triple \( u(x, t), a(t), b(t) \) is said to be a classical solution of problem (1)–(5), if the functions \( u(x, t) \in \tilde{C}^{4,2}(D_T), a(t) \in C[0, T] \) and \( b(t) \in C[0, T] \) satisfy an equation (1) in \( D_T \), the condition (2) on \([0, 1]\), and the statements (3)–(5) on the interval \([0, T]\).

For investigating problem (1)–(5), firstly we consider the following problem:

\[ g''(t) = a(t)g(t) \quad (0 \leq t \leq T), \quad (6) \]
\[ g(0) = \int_0^T p(t)g(t)dt, \quad g'(0) = 0, \quad (7) \]

where \( a(t), p(t) \in C[0, T] \) is the given functions, \( y = y(t) \) is the unknown function, and if \( y = y(t) \) is the solution of problem (6), (7) then \( y(t) \) is continuous on \([0, T]\) together with all its derivatives contained in equation (6) and satisfying conditions (6), (7) in the ordinary sense.

Analogously [8], the following lemma was proved.

**Lemma 1.** Let \( p(t) \in C[0, T], a(t) \in C[0, T], \|a(t)\|_{C[0, T]} \leq R = const \) and

\[ \left( \|p(t)\|_{C[0, T]} + RT \right) T < 1 \]

where \( R \) is a constant. Then problem (6), (7) has only a trivial solution.

In order to investigate problem (1)–(5), we first consider the following auxiliary problem: it is required to define a triplet \( u(x, t), a(t), b(t) \) of functions \( u(x, t) \in \tilde{C}^{4,2}(D_T), a(t) \in C[0, T] \) and \( b(t) \in C[0, T], \) from relations (1)–(3),
equation (1) with respect to conclude that (8) is fulfilled.

From (12), by (5) and (13), we conclude that the relation (9) is fulfilled.

Lemma 2. Let \( \varphi(x), \psi(x) \in C[0, 1], p(t) \in C[0, T], f(x, t), g(x, t) \in C(D_T), \frac{1}{0} g(x, t) dx = 0, \) \( \frac{1}{0} g(x, t) dx = 0 \) \( (0 \leq t \leq T), h_i(t) \in C^2[0, T](i = 1, 2), h(t) \equiv h_1(t)g(x_2, t) - h_2(t)g(x_1, t) \neq 0, \) and the consistency condition

\[
\int_{0}^{1} \varphi(x) dx = 0, \int_{0}^{1} \psi(x) dx = 0, \varphi(x_i) = h_i(0) - \int_{0}^{T} p(t)h_i(t) dt, \psi(x_i) = h'_i(0) (i = 1, 2)
\]

be satisfied. Then the following statements are valid:

1. Each classical solution \( u(x, t), a(t), b(t) \) of problem (1)–(3), (8), (9) as well;

2. each solution of problem (1)–(3), (8), (9) is a classical solution of the problem (1)–(5), if

\[
\left( \|p(t)\|_{C[0, T]} + 2T \|a(t)\|_{C[0, T]} \right) T < 1.
\]

Proof. Let \( u(x, t), a(t), b(t) \) be a classical solution to the problem (1)–(5). Integrating equation (1) with respect to \( x \) from 0 to 1, we have

\[
\frac{d^2}{dt^2} \int_{0}^{1} u(x, t) dx - u_x(1, t) + u_x(0, t) + \alpha(u_{xxx}(1, t) - u_{xxx}(0, t)) - \beta(u_{xtt}(1, t) - u_{xtt}(0, t)) = a(t) \int_{0}^{1} u(x, t) dx + b(t) \int_{0}^{1} g(x, t) dx + \int_{0}^{1} f(x, t) dx \quad (0 \leq t \leq T).
\]

Taking into account that \( \int_{0}^{1} f(x, t) dx = 0, \int_{0}^{1} g(x, t) dx = 0 \) \( (0 \leq t \leq T), \) and (3), (4), we conclude that (8) is fulfilled.

Setting \( x = x_i \) in (1) we obtain

\[
u_{tt}(x_i, t) - u_{xx}(x_i, t) + \alpha u_{xxxx}(x_i, t) - \beta u_{xxtt}(x_i, t) = a(t)u(x_i, t) + b(t)g(x_i, t) + f(x_i, t) \quad (i = 1, 2; 0 \leq t \leq T).
\]

Further, assuming \( h_i(t) \in C^2[0, T](i = 1, 2) \) and differentiating (5), we have

\[
u_{t}(x_i, t) = h'_i(t), \quad u_{tt}(x_i, t) = h''_i(t) \quad (i = 1, 2; 0 \leq t \leq T).
\]

From (12), by (5) and (13), we conclude that the relation (9) is fulfilled.

Now, assume that \( u(x, t), a(t), b(t) \) is the solution of (1)–(3), (8), (9). Then from (11), taking into account (3) and (8), we find

\[
\frac{d^2}{dt^2} \int_{0}^{1} u(x, t) dx = a(t) \int_{0}^{1} u(x, t) dx \quad (0 \leq t \leq T).
\]
By (2) and \( \int_0^1 \varphi(x)dx = 0, \int_0^1 \psi(x)dx = 0 \), it is obvious that

\[
\int_0^1 u(x,0)dx - \int_0^T p(t) \left( \int_0^1 u(x,t)dx \right) dt = \int_0^1 \left( u(x,0) - \int_0^T p(t)u(x,t)dt \right) dx = \int_0^1 \varphi(x)dx = 0,
\]

\[
\int_0^1 u_t(x,0)dx = \int_0^1 \psi(x)dx = 0. \tag{15}
\]

Since, by Lemma 1, problem (14), (15) has only a trivial solution, it follows that

\[
\int_0^1 u(x,t)dx = 0 (0 \leq t \leq T),
\]

i. e. the condition (4) holds.

Moreover, from (9) and (12) we find

\[
\frac{d^2}{dt^2} (u(x_i, t) - h_i(t)) = a(t)(u(x_i, t) - h_i(t)) \quad (i = 1, 2; \ 0 \leq t \leq T). \tag{16}
\]

Using (2) and \( \varphi(x_i) = h_i(0) - \int_0^T p(t)h_i(t)dt, \ \psi(x_i) = h_i'(0)(i = 1, 2) \) we have

\[
u(x_i, 0) - h_i(0) - \int_0^T p(t)(u(x_i, t) - h_i(t))dt = u(x_i, 0) - \int_0^T p(t)u(x_i, t)dt -
\]

\[
\varphi(x_i) - \left( h_i(0) - \int_0^T p(t)h_i(t)dt \right) = 0,
\]

\[
u_t(x_i, 0) - h_i'(0) + \delta (u(0, T) - h_1(T)) = 0 \quad (i = 1, 2). \tag{17}
\]

From (16) and (17), by Lemma 1, we conclude that the condition (5) is fulfilled. The lemma is proved.

3. Solvability of inverse boundary-value problem
It is known [12] that the system

\[
1, \cos \lambda_1 x, \sin \lambda_1 x, ..., \cos \lambda_k x, \sin \lambda_k x, ...
\]

is a basis in \( L_2(0,1) \), where \( \lambda_k = 2k\pi \ (k = 1, 2, ..). \)

Since the system (18) form a basis in \( L_2(0,1) \), we shall seek the first component \( u(x,t) \) of classical solution \( u(x,t), a(t), b(t) \) of the problem (1)–(3), (8),(9) in the form

\[
u(x,t) = \sum_{k=0}^{\infty} u_{1k}(t) \cos \lambda_k x + \sum_{k=1}^{\infty} u_{2k}(t) \sin \lambda_k x \quad (\lambda_k = 2\pi k), \tag{19}
\]

where

\[
u_{10}(t) = \int_0^1 u(x, t)dx, \nu_{1k}(t) = 2 \int_0^1 u(x, t) \cos \lambda_k xdx \quad (k = 1, 2, ...),
\]
\[ u_{2k}(t) = 2 \int_0^1 u(x, t) \sin \lambda_k x \, dx \quad (k = 1, 2, \ldots). \]

Then applying the formal scheme of the Fourier method, for determining of unknown coefficients \( u_{1k}(t) \quad (k = 0, 1, \ldots) \) and \( u_{2k}(t) \quad (k = 1, 2, \ldots) \) of function \( u(x, t) \) from (1) and (2) we have

\[
\begin{align*}
&u''_{10}(t) = F_{10}(t; u, a, b) \quad (0 \leq t \leq T), \\
&\beta \lambda_k^2 u''_{ik}(t) + \lambda_k^2 (1 + \alpha \lambda_k^2) u_{ik}(t) = F_{ik}(t; u, a, b) \quad (i = 1, 2; 0 \leq t \leq T; k = 1, 2, \ldots),
\end{align*}
\]

\[
u_{10}(0) = \int_0^T p(t)u_{10}(t) \, dt + \varphi_{10}, \quad u'_{10}(0) = \psi_{10},
\]

\[
u_{ik}(0) = \int_0^T p(t)u_{ik}(t) \, dt + \varphi_{ik}, \quad u'_{ik}(0) = \psi_{ik} \quad (i = 1, 2; k = 1, 2, \ldots),
\]

where

\[
F_{1k}(t) = a(t)u_{1k}(t) + b(t)g_{1k}(t) + f_{1k}(t) \quad (k = 0, 1, \ldots),
\]

\[
f_{10}(t) = \int_0^1 f(x, t) \, dx, \quad g_{10}(t) = \int_0^1 g(x, t) \, dx,
\]

\[
f_{1k}(t) = 2 \int_0^1 f(x, t) \cos \lambda_k x \, dx, \quad g_{1k}(t) = 2 \int_0^1 g(x, t) \cos \lambda_k x \, dx \quad (k = 1, 2, \ldots),
\]

\[
\varphi_{10} = \int_0^1 \varphi(x) \, dx, \quad \psi_{10} = 2 \int_0^1 \psi(x) \, dx,
\]

\[
\varphi_{1k} = 2 \int_0^1 \varphi(x) \cos \lambda_k x \, dx, \quad \psi_{1k} = 2 \int_0^1 \psi(x) \cos \lambda_k x \, dx \quad (k = 1, 2, \ldots),
\]

\[
F_{2k}(t) = a(t)u_{2k}(t) + b(t)g_{2k}(t) + f_{2k}(t) \quad (k = 1, 2, \ldots),
\]

\[
f_{2k}(t) = 2 \int_0^1 f(x, t) \sin \lambda_k x \, dx, \quad g_{2k}(t) = 2 \int_0^1 g(x, t) \sin \lambda_k x \, dx \quad (k = 1, 2, \ldots),
\]

\[
\varphi_{2k} = 2 \int_0^1 \varphi(x) \sin \lambda_k x \, dx, \quad \psi_{2k} = 2 \int_0^1 \psi(x) \sin \lambda_k x \, dx \quad (k = 1, 2, \ldots),
\]

Solving the problem (20)-(23), we fin

\[
u_{10}(t) = \int_0^T p(t)u_{10}(t) \, dt + \varphi_{10} + t\psi_{10} + \int_0^t (t - \tau)F_{10}(\tau; u, a, b) \, d\tau,
\]

\[
u_{ik}(t) = \left( \int_0^T p(t)u_{ik}(t) \, dt + \varphi_{ik} \right) \cos \beta_k t + \frac{\sin \beta_k t}{\beta_k} \psi_{ik} + \frac{\beta_k t}{\beta_k} \psi_{ik} + \ldots
\]
where
\[ \beta_k = \lambda_k \sqrt{\frac{1 + \alpha \lambda_k^2}{1 + \beta \lambda_k^2}} \quad (k = 1, 2, \ldots). \]

After substituting the expressions \( u_{1k}(t) \) \((k = 0, 1, \ldots)\) and \( u_{2k}(t) \) \((k = 1, 2, \ldots)\) into (19), for the component \( u(x, t) \) of the solution \( u(x, t), a(t), b(t) \) to the problem (1)–(3), (8), (9) we get
\[
\frac{1}{\beta_k(1 + \beta \lambda_k^2)} \int_0^t F_k(\tau; u, a, b) \sin \beta_k (t - \tau) \, d\tau \quad (i = 1, 2; k = 1, 2, \ldots),
\]
(25)

Now, using (9), (19) and (21) we have
\[
a(t) = [h(t)]^{-1} \{ g(x_2, t) \ (h''_1(t) - f(x_1, t)) - g(x_1, t) \ (h''_2(t) - f(x_2, t)) + \sum_{k=1}^{\infty} \left( \beta_k^2 u_{1k}(t) + \frac{\beta \lambda_k^2}{1 + \beta \lambda_k^2} F_{1k}(t; u, a, b) \right) (g(x_2, t) \cos \lambda_k x_1 - g(x_1, t) \cos \lambda_k x_2) + \sum_{k=1}^{\infty} \left( \beta_k^2 u_{2k}(t) + \frac{\beta \lambda_k^2}{1 + \beta \lambda_k^2} F_{2k}(t; u, a, b) \right) (g(x_2, t) \sin \lambda_k x_1 - g(x_1, t) \sin \lambda_k x_2),
\]
(27)
\[
b(t) = [h(t)]^{-1} \{ h_1(t) \ (h''_1(t) - f(x_2, t)) - h_2(t) \ (h''_2(t) - f(x_1, t)) + \sum_{k=1}^{\infty} \left( \beta_k^2 u_{1k}(t) + \frac{\beta \lambda_k^2}{1 + \beta \lambda_k^2} F_{1k}(t; u, a, b) \right) (h_1(t) \cos \lambda_k x_2 - h_2(t) \cos \lambda_k x_1) + \sum_{k=1}^{\infty} \left( \beta_k^2 u_{2k}(t) + \frac{\beta \lambda_k^2}{1 + \beta \lambda_k^2} F_{2k}(t; u, a, b) \right) (h_1(t) \sin \lambda_k x_2 - h_2(t) \sin \lambda_k x_1) \}.
\]
(28)

We substitute expression (25) into (27), (28) and have
\[
a(t) = [h(t)]^{-1} \{ g(x_2, t) \ (h''_1(t) - f(x_1, t)) - g(x_1, t) \ (h''_2(t) - f(x_2, t)) + \]
\[
+ \frac{1}{\beta_k(1 + \beta \lambda_k^2)} \int_0^t F_k(\tau; u, a, b) \sin \beta_k (t - \tau) \, d\tau \quad (i = 1, 2; k = 1, 2, \ldots),
\]
(25)
\[ + \sum_{k=1}^{\infty} (g(x_2, t) \cos \lambda_k x_1 - g(x_1, t) \cos \lambda_k x_2) \left[ \beta_k^2 \left[ \left( \varphi_{1k} + \int_0^T p(t) u_{1k}(t) \, dt \right) \cos \beta_k t + \right. \right. \]

\[ + \frac{\sin \beta_k t}{\beta_k} \psi_{1k} + \frac{1}{\beta_k(1 + \beta \lambda_k^2)} \int_0^t F_{1k}(\tau; u, a, b) \sin \beta_k (t - \tau) \, d\tau \right] + \frac{\beta \lambda_k^2}{1 + \beta \lambda_k^2} F_{1k}(t; u, a, b) \right] + \]

\[ + \sum_{k=1}^{\infty} (g(x_2, t) \sin \lambda_k x_1 - g(x_1, t) \sin \lambda_k x_2) \left[ \beta_k^2 \left[ \left( \varphi_{2k} + \int_0^T p(t) u_{2k}(t) \, dt \right) \cos \beta_k t + \right. \right. \]

\[ + \frac{\sin \beta_k t}{\beta_k} \psi_{2k} + \frac{1}{\beta_k(1 + \beta \lambda_k^2)} \int_0^t F_{2k}(\tau; u, a, b) \sin \beta_k (t - \tau) \, d\tau \right] + \frac{\beta \lambda_k^2}{1 + \beta \lambda_k^2} F_{2k}(t; u, a, b) \right] \right), \tag{29} \]

\[ b(t) = [h(t)]^{-1} \{ h_1(t) (h_2''(t) - f(x_2, t)) - h_2(t) (h_1''(t) - f(x_1, t)) + \]

\[ + \sum_{k=1}^{\infty} (h_1(t) \cos \lambda_k x_2 - h_2(t) \cos \lambda_k x_1) \left[ \beta_k^2 \left[ \left( \varphi_{1k} + \int_0^T p(t) u_{1k}(t) \, dt \right) \cos \beta_k t + \right. \right. \]

\[ + \frac{\sin \beta_k t}{\beta_k} \psi_{1k} + \frac{1}{\beta_k(1 + \beta \lambda_k^2)} \int_0^t F_{1k}(\tau; u, a, b) \sin \beta_k (t - \tau) \, d\tau \right] + \frac{\beta \lambda_k^2}{1 + \beta \lambda_k^2} F_{1k}(t; u, a, b) \right] + \]

\[ + \sum_{k=1}^{\infty} (h_1(t) \sin \lambda_k x_2 - h_2(t) \sin \lambda_k x_1) \left[ \beta_k^2 \left[ \left( \varphi_{2k} + \int_0^T p(t) u_{2k}(t) \, dt \right) \cos \beta_k t + \right. \right. \]

\[ + \frac{\sin \beta_k t}{\beta_k} \psi_{2k} + \frac{1}{\beta_k(1 + \beta \lambda_k^2)} \int_0^t F_{2k}(\tau; u, a, b) \sin \beta_k (t - \tau) \, d\tau \right] + \frac{\beta \lambda_k^2}{1 + \beta \lambda_k^2} F_{2k}(t; u, a, b) \right] \right). \tag{30} \]

Thus, the problem (1)–(3), (8), (9) is reduced to solving the system (26), (29), (30) with respect to the unknown functions \( u(x, t), a(t) \) and \( b(t) \).

Using the definition of the solution of the problem (1)–(3), (8), (9), we prove the following lemma.

**Lemma 3.** If \( u(x, t), a(t), b(t) \) is any solution to the problem (1)–(3), (8), (9), then the functions

\[ u_{10}(t) = \int_0^1 u(x, t) \, dx, u_{1k}(t) = 2 \int_0^1 u(x, t) \cos \lambda_k x \, dx \quad (k = 1, 2, ...), \]

\[ u_{2k}(t) = 2 \int_0^1 u(x, t) \sin \lambda_k x \, dx \quad (k = 1, 2, ...). \]

satisfy the system (24), (25) in \([0, T]\).
Remark. It follows from lemma 3 that to prove the uniqueness of the solution to the problem (1)–(3), (8), (9), it suffices to prove the uniqueness of the solution to the system (26), (29), (30).

In order to investigate the problem (1)–(3), (8), (9), consider the following spaces. Denote by $B^{5}_{2,T}$ [9] the set of all functions of the form

$$u(x,t) = \sum_{k=0}^{\infty} u_{1k}(t) \cos \lambda_k x + \sum_{k=1}^{\infty} u_{2k}(t) \sin \lambda_k x \quad (\lambda_k = 2\pi k)$$

defined on $D_T$ such that the functions $u_{1k}(t)$ ($k = 0, 1, 2, \ldots$), $u_{2k}(t)$ ($k = 1, 2, \ldots$) are continuous on $[0, T]$ and

$$\parallel u_{10}(t) \parallel_{C[0,T]} + \left( \sum_{k=1}^{\infty} (\lambda_k^{5} \parallel u_{1k}(t) \parallel_{C[0,T]}{2}) \right)^{\frac{1}{2}} + \left( \sum_{k=1}^{\infty} (\lambda_k^{5} \parallel u_{2k}(t) \parallel_{C[0,T]}{2}) \right)^{\frac{1}{2}} < +\infty.$$

The norm on this set is given by

$$\parallel u(x,t) \parallel_{B^{5}_{2,T}} = \parallel u_{10}(t) \parallel_{C[0,T]} + \left( \sum_{k=1}^{\infty} (\lambda_k^{5} \parallel u_{1k}(t) \parallel_{C[0,T]}{2}) \right)^{\frac{1}{2}} + \left( \sum_{k=1}^{\infty} (\lambda_k^{5} \parallel u_{2k}(t) \parallel_{C[0,T]}{2}) \right)^{\frac{1}{2}}.$$

Denote by $E_{5}^{T}$ the space $B^{5}_{2,T} \times C[0,T]$ of the vector-functions $z(x,t) = \{u(x,t), a(t), b(t)\}$ with the norm

$$\parallel z(x,t) \parallel_{E_{5}^{T}} = \parallel u(x,t) \parallel_{B^{5}_{2,T}} + \parallel a(t) \parallel_{C[0,T]} + \parallel b(t) \parallel_{C[0,T]}.$$

It is known that $B^{5}_{2,T}$ and $E_{5}^{T}$ are Banach spaces.

Now, in the space $E_{5}^{T}$ consider the operator

$$\Phi(u,a,b) = \{\Phi_{1}(u,a,b), \Phi_{2}(u,a,b), \Phi_{3}(u,a,b)\},$$

where $\Phi_{1}(u,a,b) = \tilde{a}(x,t) = \sum_{k=0}^{\infty} \tilde{u}_{1k}(t) \cos \lambda_k x + \sum_{k=1}^{\infty} \tilde{u}_{2k}(t) \sin \lambda_k x$, $\Phi_{2}(u,a,b) = \tilde{a}(t)$, $\Phi_{3}(u,a,b) = \tilde{b}(t)$ and $\tilde{u}_{10}(t), \tilde{u}_{ik}(t)$ ($i = 1, 2; k = 1, 2, \ldots$), $\tilde{a}(t), \tilde{b}(t)$, equal to the right hand sides of (24), (25), (29), (30), respectively.

It is easy to see that

$$\parallel \tilde{u}_{10}(t) \parallel_{C[0,T]} \leq |\varphi_{10}| + T \parallel p(t) \parallel_{C[0,T]} \parallel u_{10}(t) \parallel_{C[0,T]} + T |\psi_{10}| +$$

$$+ T \left[ \sqrt{T} \left( \int_{0}^{T} |f_{10}(\tau)|^2 d\tau \right)^{\frac{1}{2}} + T \parallel a(t) \parallel_{C[0,T]} \parallel u_{10}(t) \parallel_{C[0,T]} +$$

$$+ \parallel b(t) \parallel_{C[0,T]} \sqrt{T} \left( \int_{0}^{T} |g_{10}(\tau)|^2 d\tau \right)^{\frac{1}{2}} \right],$$

$$\left( \sum_{k=1}^{\infty} (\lambda_k^{5} \parallel \tilde{u}_{ik}(t) \parallel_{C[0,T]}{2}) \right)^{\frac{1}{2}} \leq \sqrt{6} \left[ \sum_{k=1}^{\infty} (\lambda_k^{5} |\varphi_{ik}|^2) \right]^{\frac{1}{2}} +$$

$$+ T \parallel p(t) \parallel_{C[0,T]} \left( \sum_{k=1}^{\infty} (\lambda_k^{5} \parallel u_{ik}(t) \parallel_{C[0,T]}{2}) \right)^{\frac{1}{2}} + \sqrt{6} \left( \sum_{k=1}^{\infty} (\lambda_k^{5} |\psi_{ik}|^2) \right)^{\frac{1}{2}} +$$
$$\frac{1}{\beta} \sqrt{6(1+\beta)/\alpha} \left[ \sqrt{T} \left( \int_0^T \sum_{k=1}^\infty |\lambda_k^f f_{ik}(\tau)|^2 \,d\tau \right)^{\frac{1}{2}} + T \|a(t)\|_{C[0,T]} \times \right. $$

$$\left. \times \left( \sum_{k=1}^\infty \left( \frac{\lambda_k^f}{C[T]} \right)^2 \right)^{\frac{1}{2}} + \sqrt{T} \|b(t)\|_{C[0,T]} \left( \int_0^T \sum_{k=1}^\infty |\lambda_k^g g_{ik}(\tau)|^2 \,d\tau \right)^{\frac{1}{2}} \right], \quad (32)$$

$$\|\tilde{a}(t)\|_{C[0,T]} \leq \|h(t)^{-1}\|_{C[0,T]} \left\{ \left\| (h_1^f(t) - f(x_1,t)) \,g(x_2,t) - (h_2^f(t) - f(x_2,t)) \,g(x_1,t) \right\|_{C[0,T]} + \right.$$  

$$+ \|g(x_1,t)| + \|g(x_2,t)||_{C[0,T]} \left( \sum_{k=1}^\infty \lambda_k^{-2} \right)^{\frac{1}{2}} \sum_{i=1}^2 \left\{ \frac{1+\alpha}{\beta} \left( \sum_{k=1}^\infty |\lambda_k^f |\varphi_{ik}|^2 \right)^{\frac{1}{2}} + \right.$$  

$$+ T \|p(t)\|_{C[0,T]} \left( \sum_{k=1}^\infty \lambda_k^f \|u_{ik}(t)\|_{C[0,T]} \right)^2 \right\} \left[ \sqrt{T} \left( \int_0^T \sum_{k=1}^\infty |\lambda_k^f f_{ik}(\tau)|^2 \,d\tau \right)^{\frac{1}{2}} + T \|a(t)\|_{C[0,T]} \times \right.$$  

$$\left. \times \left( \sum_{k=1}^\infty \left( \frac{\lambda_k^f}{C[T]} \right)^2 \right)^{\frac{1}{2}} + \sqrt{T} \|b(t)\|_{C[0,T]} \left( \int_0^T \sum_{k=1}^\infty |\lambda_k^g g_{ik}(\tau)|^2 \,d\tau \right)^{\frac{1}{2}} \right] \right\}, \quad (33)$$

$$\|\tilde{b}(t)\|_{C[0,T]} \leq \|h(t)^{-1}\|_{C[0,T]} \left\{ \left\| h_1(t) \,(h_2^f(t) - f(x_2,t)) - h_2(t) \,(h_1^f(t) - f(x_1,t)) \right\|_{C[0,T]} + \right.$$  

$$+ \|h_1(t)| + \|h_2(t)||_{C[0,T]} \left( \sum_{k=1}^\infty \lambda_k^{-2} \right)^{\frac{1}{2}} \sum_{i=1}^2 \left\{ \frac{1+\alpha}{\beta} \left( \sum_{k=1}^\infty |\lambda_k^f |\varphi_{ik}|^2 \right)^{\frac{1}{2}} + \right.$$  

$$+ T \|p(t)\|_{C[0,T]} \left( \sum_{k=1}^\infty \lambda_k^f \|u_{ik}(t)\|_{C[0,T]} \right)^2 \right\} \left[ \sqrt{T} \left( \int_0^T \sum_{k=1}^\infty |\lambda_k^f f_{ik}(\tau)|^2 \,d\tau \right)^{\frac{1}{2}} + T \|a(t)\|_{C[0,T]} \times \right.$$  

$$\left. \times \left( \sum_{k=1}^\infty \left( \frac{\lambda_k^f}{C[T]} \right)^2 \right)^{\frac{1}{2}} + \sqrt{T} \|b(t)\|_{C[0,T]} \left( \int_0^T \sum_{k=1}^\infty |\lambda_k^g g_{ik}(\tau)|^2 \,d\tau \right)^{\frac{1}{2}} \right] \right\}.$$
\[
+ \left( \sum_{k=1}^{\infty} (\lambda_k^2 \| f_{ik}(t) \|_{C[0,T]} )^2 \right)^{1/2} + \| a(t) \|_{C[0,T]} \left( \sum_{k=1}^{\infty} (\lambda_k^2 \| u_{ik}(t) \|_{C[0,T]} )^2 \right)^{1/2} + \\
+ \| b(t) \|_{C[0,T]} \left( \sum_{k=1}^{\infty} (\lambda_k^2 \| g_{ik}(t) \|_{C[0,T]} )^2 \right)^{1/2} \right) \right) \} .
\]

(34)

Suppose that the data of the problem (1)–(3), (8), (9) satisfy the following condition:

Q1. \( \alpha > 0, \beta > 0, p(t) \in C[0,T] \).

Q2. \( \varphi(x) \in C^4[0, 1], \varphi(5)(x) \in L_2(0, 1), \varphi(0) = \varphi(1), \varphi'(0) = \varphi'(1), \varphi''(0) = \varphi''(1), \varphi'''(0) = \varphi'''(1) \).

Q3. \( \psi(x) \in C^5[0, 1], \psi(4)(x) \in L_2(0, 1), \psi(0) = \psi(1), \psi'(0) = \psi'(1), \psi''(0) = \psi''(1), \psi'''(0) = \psi'''(1) \).

Q4. \( f(x, t), f_x(x, t) \in C(D_T), f_{xx}(x, t) \in L_2(D_T), f(0, t) = f(1, t), f_x(0, t) = f_x(1, t)(0 \leq t \leq T) \).

Q5. \( g(x, t), g_x(x, t) \in C(D_T), g_{xx}(x, t) \in L_2(D_T), g(0, t) = g(1, t), g_x(0, t) = g_x(1, t)(0 \leq t \leq T) \).

Q6. \( h_1(t) \in C^2[0, T](t = 1, 2), h(t) \equiv h_1(t)g(x_2, t) - h_2(t)g(x_1, t) \neq 0(0 \leq t \leq T) \).

Then, from (31)–(34), we get

\[
\| \tilde{u}(x, t) \|_{B_{2,T}^2} \leq A_1(T) + B_1(T) \| a(t) \|_{C[0,T]} \| u(x, t) \|_{B_{2,T}^1} + C_1(T) \| u(x, t) \|_{B_{2,T}^1} + D_1(T) \| b(t) \|_{C[0,T]} ,
\]

(35)

\[
\| \tilde{u}(t) \|_{C[0,T]} \leq A_2(T) + B_2(T) \| a(t) \|_{C[0,T]} \| u(x, t) \|_{B_{2,T}^1} + C_2(T) \| u(x, t) \|_{B_{2,T}^1} + D_2(T) \| b(t) \|_{C[0,T]} ,
\]

(36)

\[
\| \tilde{b}(t) \|_{C[0,T]} \leq A_3(T) + B_3(T) \| a(t) \|_{C[0,T]} \| u(x, t) \|_{B_{2,T}^1} + C_3(T) \| u(x, t) \|_{B_{2,T}^1} + D_3(T) \| b(t) \|_{C[0,T]} ,
\]

(37)

where

\[
A_1(T) = \| \varphi(x) \|_{L_2(0,1)} + T \| \psi(x) \|_{L_2(0,1)} + T \sqrt{\alpha} \| f(x, t) \|_{L_2(D_T)} + \\
+ 2\sqrt{6} \| \varphi(5)(x) \|_{L_2(0,1)} + 2 \sqrt{6(1 + \beta) \alpha} \| \psi(4)(x) \|_{L_2(0,1)} + 2 \sqrt{6T(1 + \beta) \alpha} \| f_{xx}(x, t) \|_{L_2(D_T)} ,
\]

\[
B_1(T) = \left( T + \frac{ \alpha }{ \beta } \frac{ 6(1 + \beta) }{ \alpha } \right) T, \quad C_1(T) = T \| p(t) \|_{C[0,T]} (1 + 2 \sqrt{6}) ,
\]

\[
D_1(T) = T \| g(x, t) \|_{L_2(D_T)} + 2 \sqrt{6T} \| g_{xx}(x, t) \|_{L_2(D_T)} ,
\]

\[
A_2(T) = \| [\tilde{h}(t)]^{-1} \|_{C(0,T)} \left( \| h_1(t) - f(x_1, t) \|_{C[0,T]} + \| h_2(t) - f(x_2, t) \|_{C[0,T]} \right) + \\
+ \| g(x_1, t) \|_{C(0,T)} + \| g(x_2, t) \|_{C(0,T)} \left( \sum_{k=1}^{\infty} \lambda_k^{-2} \left( \frac{ 2(1 + \alpha) }{ \beta } \left( \| \varphi(5)(x) \|_{L_2(0,1)} + \sqrt{\frac{1 + \beta}{ \alpha } \| \psi(4)(x) \|_{L_2(0,1)} } + \\
+ \sqrt{\frac{1 + \beta}{ \alpha } \| f_{xx}(x, t) \|_{L_2(D_T)} } \right) \right) \right) ,
\]

\[
+ \| f_{xx}(x, t) \|_{C[0,T]} \| g(x_1, t) \|_{C[0,T]} + \| g(x_2, t) \|_{C[0,T]} \right) ,
\]

(38)
where

\[ \sup_{t \in [0, T]} \sup_{x \in R^2} \left( \sum_{k=1}^{\infty} \lambda_k^{-2} \right)^{\frac{1}{2}} \left( \frac{1 + \alpha}{\beta^2} \sqrt{\frac{1 + \beta}{\alpha} T + 1} \right), \]

\[ C_2(T) = \left\| h(t) \right\|_{C[0,T]} \left( \sum_{k=1}^{\infty} \lambda_k^{-2} \right)^{\frac{1}{2}} \left\| g(x, t) \right\|_{C[0,T]} \left( \frac{1 + \alpha}{\beta} \right) \left\| p(t) \right\|_{C[0,T]} T, \]

\[ D_2(T) = 2 \left\| h(t) \right\|_{C[0,T]} \left\| g(x, t) \right\|_{C[0,T]} \left( \sum_{k=1}^{\infty} \lambda_k^{-2} \right)^{\frac{1}{2}} \times \]

\[ \times \left( \frac{1 + \alpha}{\beta^2} \sqrt{\frac{T(1 + \beta)}{\alpha} g_{xx}(x, t)} \right) \left\| L_2(D_T) + \left\| g_{xx}(x, t) \right\|_{C[0,T]} \right \|_{L_2(0,1)} \right), \]

\[ A_3(T) = \left\| h(t) \right\|_{C[0,T]} \left\{ \left\| h_1(t) \left( h_0^2(t) - f(x, t) \right) - h_2(t) \left( h_1^2(t) - f(x, t) \right) \right\|_{C[0,T]} + \right. \]

\[ + \left\| h_1(t) + h_2(t) \right\|_{C[0,T]} \left\{ \left\| (1 + \alpha) \right\|_{L_2(0,1)} + \left\| \frac{1 + \alpha}{\beta} \right\|_{L_2(0,1)} + \right. \]

\[ + \left\| f_{xx}(x, t) \right\|_{L_2(D_T)} \} + \left\| f_{xx}(x, t) \right\|_{C[0,T]} \left\| L_2(0,1) \right\} \}, \]

\[ C_3(T) = \left\| h(t) \right\|_{C[0,T]} \left( \sum_{k=1}^{\infty} \lambda_k^{-2} \right)^{\frac{1}{2}} \left\| h_1(t) \right\|_{C[0,T]} \left\| h_2(t) \right\|_{C[0,T]} \frac{1 + \alpha}{\beta} \left\| p(t) \right\|_{C[0,T]} T, \]

\[ D_3(T) = 2 \left\| h(t) \right\|_{C[0,T]} \left\| h_1(t) \right\|_{C[0,T]} \left\| h_2(t) \right\|_{C[0,T]} \left( \sum_{k=1}^{\infty} \lambda_k^{-2} \right)^{\frac{1}{2}} \times \]

\[ \times \left( \frac{1 + \alpha}{\beta^2} \sqrt{\frac{T(1 + \beta)}{\alpha} g_{xx}(x, t)} \right) \left\| L_2(D_T) + \left\| g_{xx}(x, t) \right\|_{C[0,T]} \right \|_{L_2(0,1)} \right). \]

It follows from the inequalities (35)–(37) that

\[ \left\| \bar{u}(x, t) \right\|_{B^2_{L, T}} \leq \left\| A(T) + B(T) \right\|_{C[0,T]} \left\| u(x, t) \right\|_{B^2_{L, T}} + \left\| C(T) \right\|_{C[0,T]} \left\| u(x, t) \right\|_{B^2_{L, T}} + \left\| D(T) \right\|_{C[0,T]} \right. \]

where

\[ A(T) = A_1(T) + A_2(T) + A_3(T), B(T) = B_1(T) + B_2(T) + B_3(T), \]

\[ C(T) = C_1(T) + C_2(T) + C_3(T), D(T) = D_1(T) + D_2(T) + D_3(T). \]

So, we can prove the following theorem:
Theorem 1. Let conditions $Q_1 - Q_6$ be satisfied, and
\[
(B(T)(A(T) + 2) + C(T) + D(T))(A(T) + 2) < 1. \tag{39}
\]

Then problem (1)-(3), (8), (9) has a unique solution in the sphere $K = K_R(||z||_{E^5_T} ≤ R = A(T) + 2)$ of the space $E^5_T$.

Proof. In the space $E^5_T$ consider the equation
\[
z = Φz, \tag{40}
\]
where $z = \{u, a, b\}$, the components $Φ_i(u, a, b)(i = 1, 2, 3)$ of the operator $Φ(u, a, b)$ are determined by the right hand sides of equations (26), (29) and (30). Consider the operator $Φ(u, a, b)$ in the sphere $K = K_R$ from $E^5_T$. Similar to (38), we get that for any $z_1, z_2, z_3 ∈ K_R$ the following estimations are valid:
\[
||Φz||_{E^5_T} ≤ A(T) + B(T)||a(t)||_{C[0,T]}||u(x, t)||_{B^5_{2,T}} + C(T)||u(x, t)||_{B^5_{2,T}} + D(T)||b(t)||_{C[0,T]} ≤ \\
≤ A(T) + B(T)(A(T) + 2)^2 + C(T)(A(T) + 2) + D(T)(A(T) + 2), \tag{41}
\]
\[
||Φz_1 - Φz_2||_{E^5_T} ≤ B(T)R(||a_1(t) - a_2(t)||_{C[0,T]} + ||u_1(x, t) - u_2(x, t)||_{P^5_{2,T}}) + \\
+C(T)||u_1(x, t) - u_2(x, t)||_{P^5_{2,T}} + D(T)||b_1(t) - b_2(t)||_{C[0,T]} \tag{42}
\]

Then, it follows from (39) together with the estimates (41) and (42) that the operator $Φ$ acts in the ball $K = K_R$ and is contractive. Therefore, in the ball $K = K_R$ the operator $Φ$ has a unique fixed point $z = \{u, a, b\}$, that is a unique solution to the equation (40), i. e. a unique solution to the system (26), (29), (30).

The function $u(x, t)$, as an element of the space $B^5_{2,T}$, is continuous and has continuous derivatives $u_x(x, t), u_{xx}(x, t), u_{xxx}(x, t), u_{xxxx}(x, t)$ in $D_T$.

It can be shown that has continuous derivatives $u_t(x, t), u_{tt}(x, t)$ in $D_T$.

It is easy to verify that the equation (1) and conditions (2), (3), (6), (7) are satisfied in the ordinary sense. Consequently, $\{u(x, t), a(t), b(t)\}$ is a solution to the problem (1)–(3), (6), (7), and by Lemma 3 it is unique in the ball.

By Lemma 1 the unique solvability of the initial problem (1)–(5) follows from the theorem. The theorem is proved.

Theorem 2. Let all the conditions of Theorem 1 be fulfilled and
\[
\int_0^1 f(x, t)dx = 0, \int_0^1 g(x, t)dx = 0(0 ≤ t ≤ T), \int_0^1 \varphi(x)dx = 0, \int_0^1 ψ(x)dx = 0,
\]
\[
\varphi(x_i) = h_i(0) - \delta h_i'(0), ψ(x_i) = h_i'(T) - \int_0^T p(t)h_i(t)dt(i = 1, 2),
\]
\[
T \left(2T + \delta)(A(T) + 2) + (T + \delta)||p(t)||_{C[0,T]} \right) < 1.
\]

Then the problem (1)–(5) has a unique classical solution in the ball $K = K_R(||z||_{E^5_T} ≤ R = A(T) + 2)$ of the space $E^5_T$. 

References

[1] Tikhonov A N 1943 *Doklady AN SSSR* **39** 195–198 (in Russian)
[2] Lavrent’ev M M 1964 *Doklady AN SSSR* **157** 520–521 (in Russian)
[3] Ivanov V K, Vasin V V and Tanana V P 1978 *Theory of linear ill-posed problems and its applications* (in Russian)
[4] Romanov V G 1984 *Inverse Problems of Mathematical Physics* (in Russian)
[5] Kozhanov A I 2004 *Computational Mathematics and Mathematical Physics* **44** pp 657–675
[6] Hazanee A, Lesnic D, Ismailov M I and Kerimov N B 2015 *Applied Mathematical Modelling* **39** 6258–6272
[7] Aliev Z S and Mehraliev Y T 2014 *Doklady Mathematics* **90** pp 513–517
[8] Mehraliyev Y T and Kanca F 2014 *Mathematical Modelling and Analysis* **19** 241–256
[9] Mehraliyev Ya T 2011 *Vest Tverskogo Gos Univ Ser prikladnaya matematika* **23** 25–38 (in Russian)
[10] Eskin G 2017 *Bulletin of Mathematical Sciences* **7** 247–307
[11] Benney D J and Luke J C 1964 *Journal of Mathematical Physics* **43** 309–313
[12] Budak B M, Samarskii A A and Tikhonov A N 1972 *A Collection of Problems in Mathematical Physics* (in Russian)