A GLOBAL TORELLI THEOREM FOR RIGID HYPERHOLOMORPHIC SHEAVES

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Abstract. Let $X$ be an irreducible holomorphic symplectic manifold of $K3^{[n]}$ deformation type, $n \geq 2$. There exists over $X \times X$ a rank $2n-2$ rigid and stable reflexive sheaf $A$ of Azumaya algebras, constructed in [Ma5], such that the pair $(X \times X, A)$ is deformation equivalent to the following pair $(M \times M, A_M)$. The manifold $M$ is a smooth and compact moduli space of stable sheaves over a $K3$ surface $S$, and $A_M$ is the reflexive sheaf, whose fiber, over a pair $(F_1, F_2) \in M \times M$ of non-isomorphic stable sheaves $F_1$ and $F_2$, is $\text{End}(\text{Ext}_S^1(F_1, F_2))$. We prove in this paper the following uniqueness result.

Let $A_1$ and $A_2$ be two rigid Azumaya algebras over $X \times X$, which are $(\omega, \omega)$-slope-stable with respect to the same Kähler class $\omega$ on $X$, and such that $(X \times X, A_i)$ is deformation equivalent to $(M \times M, A_M)$, for $i = 1, 2$. Assume that the singularities of $A_1$ and $A_2$ along the diagonal are of the same infinitesimal type prescribed by $A_M$. Then $A_1$ is isomorphic to $A_2$ or $A_2^\ast$. Furthermore, if $\text{Pic}(X)$ is trivial, or cyclic generated by a class of non-negative Beauville-Bogomolov-Fujiki degree, then there exists a unique pair $\{A, A^\ast\}$, such that $(X \times X, A)$ is deformation equivalent to $(M \times M, A_M)$, and $A$ is $(\omega, \omega)$-slope-stable with respect to every Kähler class $\omega$ on $X$.

The above is the main example of a more general global Torelli theorem proven. The result is used in the authors forthcoming work on generalized deformations of $K3$ surfaces.

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1. Introduction

An irreducible holomorphic symplectic manifold is a simply connected compact Kähler manifold \( X \), such that \( H^0(X, \wedge^2 T^* X) \) is spanned by an everywhere non-degenerate holomorphic 2-form. The second cohomology \( H^2(X, \mathbb{Z}) \) admits a symmetric non-degenerate integral bi-linear pairing of signature \((3, b_2(X) - 3)\), called the Beauville-Bogomolov-Fujiki pairing \([Be]\). The pairing is positive on the Kähler cone, and normalized, so that \( \langle \alpha, \beta \rangle = 1 \) for \( \alpha, \beta \in H^2(X, \mathbb{Z}) \).

The pairing is invariant under deformations of the complex structure of \( X \), and is thus monodromy invariant. Given a Kähler \( K3 \) surface \( S \), the Hilbert scheme (or Douady space) \( S^{[n]} \) of length \( n \) subschemes of \( S \) is an example of an irreducible holomorphic symplectic manifold. An irreducible holomorphic symplectic manifold is said to be of \( K3^{[n]} \)-type, if it is deformation equivalent to \( S^{[n]} \), for some (hence any) Kähler \( K3 \) surface \( S \).

Let \( X_0 \) be an irreducible holomorphic symplectic manifold. Fix a lattice \( \Lambda \) isometric to \( H^2(X_0, \mathbb{Z}) \), endowed with its Beauville-Bogomolov-Fujiki pairing. Let \( \mathfrak{M}_\Lambda \) be the moduli space of marked irreducible holomorphic symplectic manifolds with integral second cohomology isometric to \( \Lambda \). A point of \( \mathfrak{M}_\Lambda \) parametrizes an isomorphism class of a pair \( (X, \eta) \), where \( \eta : H^2(X, \mathbb{Z}) \to \Lambda \) is an isometry \([Hu1]\).

Fix a positive integer \( d \). In section 4 we construct a coarse moduli space \( \tilde{\mathfrak{M}}_\Lambda \), consisting of isomorphism classes of triples \((X, \eta, E)\), where \( (X, \eta) \) is a marked pair as above and \( E \) is a reflexive sheaf of Azumaya algebras over the \( d \)-th Cartesian product \( X^d \), which is stable with respect to some Kähler class, \( c_2(E) \) is invariant under the diagonal action of a finite index subgroup of the monodromy group of \( X \), and \( E \) satisfies the technical open Condition 1.6. We introduce next the necessary background needed for the statement of Condition 1.6. The main results are then stated using that condition.

**Definition 1.1.** A reflexive sheaf of Azumaya\(^1\) \( O_X \)-algebras of rank \( r \) over a Kähler manifold \( X \) is a sheaf \( E \) of reflexive coherent \( O_X \)-modules, with a global section \( 1_E \), and an associative multiplication \( m : E \otimes E \to E \) with identity \( 1_E \), admitting an open covering \( \{U_\alpha\} \) of \( X \), and an isomorphism \( \eta_\alpha : E|_{U_\alpha} \to \text{End}(F_\alpha) \) of unital associative algebras, for some reflexive sheaf \( F_\alpha \) of rank \( r \) over each \( U_\alpha \).

From now on the term a sheaf of Azumaya algebras will mean a reflexive sheaf of Azumaya \( O_X \)-algebras. A subsheaf \( P \) of a sheaf \( E \) of Azumaya algebras is a sheaf of maximal parabolic subalgebras if, in the notation of Definition 1.1, \( \eta_\alpha(P|_{U_\alpha}) \) is the sheaf of subalgebras of \( \text{End}(F_\alpha) \)

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\(^1\)Caution: The standard definition of a sheaf of Azumaya \( O_X \)-algebras assumes that \( E \) is a locally free \( O_X \)-module, while we assume only that it is reflexive.
leaving invariant a non-zero subsheaf $F'_\alpha \subset F_\alpha$ of lower rank, for all $\alpha$. Let $\pi_i : X^k \to X$ be the projection onto the $i$-th factor.

**Definition 1.2.** Given a sheaf $E$ of rank $r$ Azumaya algebras over $X^d$ and a Kähler class $\omega$ on $X$, we say that $E$ is $\omega$-slope-stable, if every non-trivial subsheaf of maximal parabolic subalgebras of $E$ has negative $(\sum_{i=1}^d \pi_i^* \omega)$-slope, and the rank $r^2$ sheaf $E$ is $(\sum_{i=1}^d \pi_i^* \omega)$-slope-polystable.

Let $M$ be a complex manifold and $E$ a rank $r$ sheaf of reflexive Azumaya algebras over $M$. Let $U \subset M$ be the open subset where $E$ is locally free. $E$ corresponds to a class in $H^1(U, PGL(r))$. The connecting homomorphism of the short exact sequence $1 \to \mu_r \to SL(r) \to PGL(r) \to 1$ associates to $E$ a characteristic class in $H^2(U, \mu_r)$, which is isomorphic to $H^2(M, \mu_r)$, since the singular locus has co-dimension $\geq 3$. We will refer to this class as the characteristic class of $E$ in $H^2(M, \mu_r)$. The image of this class in $H^*(M, \mathcal{O}_M^*)$ is called the Brauer class of $E$.

**Proposition 1.3.** [Ma5, Prop. 7.8] Let $E$ be a sheaf of Azumaya algebras over a Kähler manifold $M$. If the order of the Brauer class of $E$ in $H^2(M, \mathcal{O}_M^*)$ is equal to the rank of $E$, then $E$ is $\omega$-slope-stable with respect to every Kähler class $\omega$ on $M$.

Let $r$ be a positive integer and $\tilde{\theta}$ an element of order $r$ in $\Lambda^{\otimes d}/r\Lambda^{\otimes d}$. Denote by $\mu_r \subset \mathbb{C}^*$ the group of $r$-th roots of unity. Let $\iota : H^2(X^d, \mu_r) \to H^2(X^d, \mathcal{O}_X^*)$ be the natural homomorphism. Given a marked pair $(X, \eta)$, let $\tilde{\eta} : H^2(X^d, \mathbb{Z}/r\mathbb{Z}) \to \Lambda^{\otimes d}/r\Lambda^{\otimes d}$ be the isomorphism induced by the marking $\eta$. Denote by

$$\theta_{(X, \eta)}$$

the class $\iota \left( \exp \left( \frac{-2\pi i}{r} \tilde{\eta}^{-1}(\tilde{\theta}) \right) \right)$ in $H^2(X^d, \mathcal{O}_X^*)$.

**Remark 1.4.** Note that $\theta_{(X, \eta)}$ has order $r$, if $\text{Pic}(X)$ is trivial. In that case any rank $r$ Azumaya algebra with Brauer class $\theta(X, \eta)$ is slope-stable with respect to every Kähler class on $X$, by Proposition 1.3.

**Definition 1.5.** The monodromy group $\text{Mon}(X)$ of $X$ is the subgroup, of the automorphism group of the cohomology ring $H^*(X, \mathbb{Z})$, generated by monodromy operators $g$ of families $X \to B$ (which may depend on $g$) of irreducible holomorphic symplectic manifolds deforming $X$.

Let $E$ be a sheaf of Azumaya algebras over a complex manifold $M$ and $F$ a (possibly twisted) reflexive sheaf, such that $E$ is isomorphic to $\mathcal{E}nd(F)$. Let $\text{Ext}^1_0(F, F)$ be the kernel of the trace homomorphism $\text{Ext}^1(F, F) \to H^1(M, \mathcal{O}_M)$. Then $\text{Ext}^1_0(F, F)$ parametrizes infinitesimal deformations of $E$ as a sheaf of Azumaya algebras over $M$. Set $T^1_0(E) := \text{Ext}^1_0(F, F)$.

**Condition 1.6.** Let $X$ be an irreducible holomorphic symplectic manifold and $E$ a sheaf of rank $r$ Azumaya algebras over $X^d$.

1. The Chern class $c_2(E) \in H^{2,2}(X^d, \mathbb{Z})$ is invariant under the diagonal action of a finite index subgroup of $\text{Mon}(X)$.

2. The characteristic class of $E$ in $H^2(X^d, \mu_r)$ is $\exp \left( \frac{-2\pi i}{r} \tilde{\eta}^{-1}(\tilde{\theta}) \right)$, for some marking $\eta$.

In particular, the Brauer class of $E$ is $\theta_{(X, \eta)}$, given in Equation (1.1).

3. $T^1_0(E) = 0$, so that the Azumaya algebra $E$ is infinitesimally rigid over the fixed $X^d$.

4. $E$ is $\omega$-slope-stable with respect to some Kähler class $\omega$ on $X$.  

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(5) If \( d = 1 \), then \( E \) is locally free. If \( d > 1 \) then \( E \) is locally free away from the diagonal image of \( X \) in \( X^d \). If \( E \) is not locally free along the diagonal, then its singularity along the diagonal satisfies the infinitesimal constraints listed in Section 1.6.

Let \( E \) be a \( \omega \)-slope-stable sheaf of Azumaya algebras over \( X^d \) satisfying Condition 1.6. Parts (1) and (4) of Condition 1.6 imply that the sheaf \( E \) is \( \omega \)-stable hyperholomorphic in the sense of Verbitsky \[Ve6\]. Associated to the Kähler class \( \omega \) in part (1) above is a twistor family \( \pi : X \to \mathbb{P}^1 \) over \( \mathbb{P}^1 \) and a reflexive sheaf \( E \) of Azumaya algebras over the \( d \)-th fiber product \( X \times_{\mathbb{P}^1} X \times_{\mathbb{P}^1} \ldots \times_{\mathbb{P}^1} X \) of \( X \). Note that the above theorem is unconditional when \( d = 1 \), as Conjecture 1.7 is known when \( E \) is locally free away from the diagonal image of \( X \) in \( X^d \).

Conjecture 1.7. \( H^1(\mathcal{X}^d_\vartheta, E_t) = (0) \), for all \( t \in \mathbb{P}^1 \).

Verbitsky proved the above conjecture in case \( E \) is locally free \[Ve4\, Cor. 8.1\]. In the reflexive sheaf case it is known that the moduli space of \( E \) of Verbitsky \[Ve6\, Ma5\, Lemma 6.14\], which is flat over \( \mathbb{P}^1 \) by Proposition 3.2. Part (3) of Condition 1.6 implies that \( H^1(X^d, E) = 0 \). Denote by \( E_t, t \in \mathbb{P}^1 \), the restriction of \( E \) to the fiber \( X_t \) of \( X \) over \( t \).

\[ \text{Theorem 1.8. (Theorem 4.13)} \text{ Assume Conjecture 1.7. There exists a coarse moduli space } \mathcal{M}_{X, \eta, E} \text{ which is a complex non-Hausdorff manifold, parametrizing isomorphism classes of triples } (X, \eta, E) \text{ satisfying Condition 1.6.} \]

We denote by \( \mathcal{M}_X \) the union of \( \mathcal{M}_{X, \eta, E} \), as \( \theta \) varies over all elements of order \( r \) in \( \Lambda^{\oplus d} / r \Lambda^{\oplus d} \). Let \( \Omega \) be the period domain

\[ \Omega := \{ x \in \mathbb{P}[\Lambda \otimes \mathbb{Z} \mathbb{C}] : (x, x) = 0, \text{ and } (x, \bar{x}) > 0 \} \]

associated to the lattice \( \Lambda \). We get the period map

\[ P : \mathcal{M}_X \to \Omega, \]

\[ P(X, \eta) = \eta(H^2,0(X)) \], which is a local homeomorphism \[Be\]. Given a period \( x \in \Omega \), denote by \( \Lambda_{x} \) the sub-lattice of \( \Lambda \) consisting of classes orthogonal to the line \( x \). Let \( \phi : \mathcal{M}_X \to \mathcal{M}_X \) be the morphism sending the isomorphism class of a triple \((X, \eta, E)\) to that of the marked pair \((X, \eta)\). Fix a connected component \( \mathcal{M}_X^0 \) of \( \mathcal{M}_X \) and let \( \mathcal{M}_X^0 \) be the connected component of \( \mathcal{M}_X \) containing \( \phi(\mathcal{M}_X^0) \).

\[ \text{Theorem 1.9. (Theorem 6.1)} \text{ Assume that Conjecture 1.7 holds.} \]

(1) The morphism \( \tilde{P} := P \circ \phi : \mathcal{M}_X^0 \to \Omega \) is surjective and a local homeomorphism. Any two points in the same fiber of \( P \) are inseparable in \( \mathcal{M}_X^0 \).

(2) If \( \Lambda_{x} \) is cyclic generated by a class of non-negative self intersection, then the fiber \( \tilde{P}^{-1}(x) \) consists of a single separable point of \( \mathcal{M}_X^0 \).

Note that the above theorem is unconditional when \( d = 1 \), as Conjecture 1.7 is known when \( E \) is locally free. When \( X \) is a \( K3 \) surface and \( d = 1 \), Theorem 1.9 generalizes the Torelli theorem for \( K3 \) surfaces \[PS, BR\] and it may be viewed as a global analogue of Mukai’s result that the moduli space of \( H \)-stable vector bundles with Mukai vector \( v \) of self intersection \(-2\) on a fixed polarized \( K3 \) surface \((X, H)\) is non-empty, smooth, connected, and zero dimensional (a
Conjecture 1.7 or the following conjecture involving only isolated singularities.

We consider in our global analogue only rank $r$ Azumaya algebras (or equivalently $\mathbb{P}^{r-1}$ bundles) with characteristic class of order $r$ in $H^2(X, \mu_r)$, as assumed in Condition 1.6 (2).

The locally free case of Theorem 1.9 follows easily from Verbitsky’s proof of his global Torelli theorem [Ve7], and Verbitsky’s work on hyperholomorphic vector bundles [Ve4]. Most of the effort in this paper goes toward the reflexive but non-locally-free cases. When the dimension of $X$ is larger than 2, the invariance of $c_2(E)$ assumed in Condition 1.6 (1) is highly restrictive, and the only interesting example known to the authors is the non-locally free example, which we now describe.

Definition 1.10. Let $(X_1, E_1)$ and $(X_2, E_2)$ be two pairs each consisting of an irreducible holomorphic symplectic manifold $X_i$ and a reflexive sheaf $E_i$ of Azumaya algebras over $X^d_i$. We say that the two pairs are deformation equivalent, if there exist markings $\eta_i$, such that the two triples $(X_i, \eta_i, E_i)$, $i = 1, 2$, belong to the same connected component of the moduli space $\mathfrak{M}_\Lambda$.

Following is our main application. In this case we are able to state a version of Theorem 1.9 for isomorphism classes of pairs $(X, E)$, forgetting the marking. Let $S$ be a Kähler $K3$ surface and $S^{[n]}$ the Hilbert scheme of length $n$ subschemes of $S$. Denote by $\mathcal{U}$ the ideal sheaf of the universal subscheme in $S \times S^{[n]}$. Let $\pi_{ij}$ be the projection from $S^{[n]} \times S \times S^{[n]}$ onto the product of the $i$-th and $j$-th factors. Let

$$F := \mathcal{E}xt^1_{\pi_{12}}(\pi_{12}^* \mathcal{U}, \pi_{23}^* \mathcal{U})$$

be the relative extension sheaf over $S^{[n]} \times S^{[n]}$. $F$ is a reflexive sheaf of rank $2n - 2$ [Ma5, Prop. 4.5]. Let $\tau$ be the involution of $S^{[n]} \times S^{[n]}$ interchanging the two factors. Then $\tau^* F \cong F^*$, by Grothendieck-Verdier duality and the triviality of the canonical line bundle $\omega_{\pi_{12}}$. Let $E$ be the sheaf $\mathcal{E}nd(F)$ and set $E^* := \mathcal{E}nd(F^*)$. The two sheaves $E$ and $E^*$ are isomorphic as sheaves, but if $n > 2$ they are not isomorphic as sheaves of Azumaya algebras.

Theorem 1.11. (1) $E$ is a sheaf of Azumaya algebras over $S^{[n]} \times S^{[n]}$, which satisfies Condition 1.6 and is slope-stable with respect to every Kähler class on $S^{[n]}$, if $\text{Pic}(S)$ is trivial.

(2) Let $X$ be of $K3^{[n]}$-type. There exists a sheaf $E'$ of Azumaya algebras over $X \times X$, which satisfies Condition 1.6 and such that $(X, E')$ is deformation equivalent to $(S^{[n]}, E)$.

(3) Let $X$ be of $K3^{[n]}$-type and $E_1$ and $E_2$ be two sheaves of Azumaya algebras over $X \times X$, such that $(X, E_i)$ is deformation equivalent to $(S^{[n]}, E)$ and satisfies Condition 1.6, $i = 1, 2$. Assume that one of the following conditions hold.

(a) $\text{Pic}(X)$ is trivial, or cyclic generated by a line-bundle of non-negative Beauville-Bogomolov-Fujiki degree.

(b) $E_1$ and $E_2$ are $\omega$-slope-stable with respect to the same Kähler class $\omega$ on $X$.

Then $E_1$ is isomorphic to $E_2$ or $E_2^*$, as a sheaf of Azumaya algebras.

The theorem is proven in section 7. Parts 2 and 3 of the theorem are conditional on either Conjecture 1.7 or the following conjecture involving only isolated singularities.

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2In the set-up of the theorem, where the marking is forgotten, the characteristic class $\tilde{\theta}$ in Condition 1.6 (2) is replaced by a monodromy orbit of characteristic classes in $\Lambda^{[n]}/(2n - 2)\Lambda^{[n]}$, where $\Lambda = H^2(S^{[n]}, \mathbb{Z})$. The orbit is determined in Equation 1.6.
Conjecture 1.12. Let $X$ be an irreducible holomorphic symplectic manifold, $\omega$ a Kähler class on $X$, and $E$ an $\omega$-stable hyperholomorphic reflexive sheaf on $X$, which is locally free away from a single point $x_0 \in X$. Assume that $H^1(X, E) = 0$. Then $H^1(X_1, E_1) = 0$, for every marking $t$ in the base $\mathbb{P}^1_\omega$ of the hyperholomorphic deformation $(X_t, E_t)$ of $(X, E)$.

We will see that Condition 1.6 is open (Lemma 2.15), and so the condition holds more generally for the sheaf $E$ in Theorem 1.11 over $S^n \times S^n$ for K3 surfaces $S$ in a dense open subset of moduli and, in particular, for a generic projective K3 surface. Let $M$ be a more general smooth and projective connected component of the moduli space of sheaves on a projective K3 surface $S$ and $\mathcal{U}$ a (possibly twisted) universal sheaf over $S \times M$. Set $n := \dim(M)/2$ and assume that $n \geq 2$. We get the rank $2n - 2$ reflexive sheaf $F$ over $M \times M$, as above, and the proof of the above Theorem shows that $E_M := \mathcal{E}nd(F)$ satisfies Condition 1.6 except possibly for the stability Condition 1.6 (4). When the Brauer class of $\mathcal{U}$ has order $2n - 2$, then $E_M$ satisfies Condition 1.6 by Proposition 1.3. Explicit examples are provided in [Ma5, Theorem 1.5].

We expect $E_M$ to always satisfy Condition 1.6 regardless of the order of the Brauer class of $\mathcal{U}$. Given any smooth and projective component $M$ of the moduli space of sheaves over a K3 surface $S_1$, the pairs $(M, E_M)$ and $(S^{[n]}, E)$ are known to be deformation equivalent, where $S$ and $E$ are as in Theorem 1.11 above [Y]. Consequently, if $E_M$ satisfies Condition 1.6 (4), then for every marking $\eta_1$ of $M$, there exists a marking $\eta_2$ of $S^{[n]}$, such that the triples $(M, \eta_1, E_M)$ and $(S^{[n]}, \eta_2, E)$ belong to the same connected component of the moduli space $\mathfrak{M}_\Lambda$.

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1.1. Infinitesimal constraints on the singularities of a reflexive sheaf. Let $X$ be a complex manifold, $\Delta \subset X$ a smooth subvariety, and $\beta : Y \to X$ the blow-up of $X$ centered along $\Delta$. Let $G$ be a locally free sheaf over $Y$ and set $F := \beta_* G$. Recall that the Atiyah class $At_F$ is the extension class in $\text{Ext}^1(F, F \otimes T^* X)$ of the extension

$$0 \to F \otimes T^* X \to J^1(F) \to F \to 0,$$

where $J^1(F)$ is the sheaf of first order jets of $F$. The local extension sheaf $\mathcal{E}xt^1(F, F \otimes T^* X)$ is isomorphic to $\mathcal{H}om(TX, \mathcal{E}xt^1(F, F))$. The image of $At_F$ in $H^0(X, \mathcal{E}xt^1(F, F \otimes T^* X)$ may thus be interpreted as a sheaf homomorphism

$$a_F : TX \to \mathcal{E}xt^1(F, F).$$

Given an open Stein subset $U$ of $X$ and a section $\xi$ of $TU$ we get the infinitesimal action morphism $\xi : U \times \text{Spec}(\mathbb{C}[e]/(e^2)) \to U$, as well as the projection $\pi_1 : U \times \text{Spec}(\mathbb{C}[e]/(e^2)) \to U$. The sheaf $\pi_1^* \xi^* F$ is an extension of $F$ by $F$, whose extension class is $a_F(\xi)$.

Let $\mathcal{D}^1(G) := \mathcal{H}om(J^1(G), G)$ be the sheaf of differential operators of order $\leq 1$. We have the symbol map $\sigma_G : \mathcal{D}^1(G) \to TY \otimes \mathcal{E}nd(G)$. Let $D^1(G) \subset \mathcal{D}^1(G)$ be the subsheaf with scalar symbol. Define $\mathcal{D}^1(F)$ and $D^1(F)$ similarly. There is a natural homomorphism

$$\beta_* : \beta_\ast D^1(G) \to D^1(\beta_* G).$$
Definition 1.13. We say the sheaf $F$ is $\beta$-tight, if the homomorphism $a_F$, given in (1.4), is surjective and the natural homomorphism $\beta_*: \beta_*\mathcal{D}^1(G) \rightarrow \mathcal{D}^1(F)$ is an isomorphism.

Definition 1.14. (1) Let $W$ be a coherent sheaf over $\mathbb{P}^n$. The pair $(\mathbb{P}^n, W)$ is said to be infinitesimally rigid, if the differential $sl_{n+1} \rightarrow \text{Ext}^1(W, W)$, of the pullback action of $\text{Aut}(\mathbb{P}^n)$, is surjective.

(2) Let $W$ be a coherent sheaf over $\mathbb{P}(V \otimes \mathbb{C}^m)$, where $V$ is a symplectic vector space. The symplectic group $Sp(V)$ acts on the first factor $V$ and the group $SL(m)$ acts on the second factor $\mathbb{C}^m$ yielding an action of $Sp(V) \times SL(m)$ on $V \otimes \mathbb{C}^m$. We say that $W$ is $sl(m)$-invariant, if the kernel of the differential $sl(V \otimes \mathbb{C}^m) \rightarrow \text{Ext}^1(W, W)$ contains $sl(m)$. We say that $W$ is $sp(V)$-equivariant, if the above kernel is an $sp(V)$-subrepresentation.

Let $\pi: X \rightarrow B$ be a smooth morphism of complex analytic spaces with connected fibers. Let $T_\pi$ be the vertical tangent bundle. Set $D := \mathbb{P}(T_\pi^{[d-1]})$ and let $p: D \rightarrow X$ be the natural morphism.

Definition 1.15. Given a vector bundle $W$ over $D$, we consider it as a family over $B$ of pairs $(W_{(x^p=1,y^p)}, \mathcal{X}_0)$, where the first term is the restriction of $W$ to the fiber $\mathbb{P}(\mathcal{X}_0)^{[d-1]}$ of $\pi \circ p$ over $b \in B$. We say that the family $W$ is locally trivial in the topology of $X$, if for each point $x \in X$ there exist open neighborhoods $U$ of $x$ in $X$ and $\mathcal{U}$ of $\pi(x)$ in $B$, as well as an isomorphism $f: U \rightarrow (U \cap \mathcal{X}_{\pi(x)}) \times \mathcal{U}$, such that $\pi \circ f^{-1}$ is the projection to $\mathcal{U}$, and an isomorphism

$$
\tilde{f}: \pi_1^*\mathcal{X}_pW \rightarrow \iota^*W,
$$

where $\iota: p^{-1}(U) \rightarrow D$ and $\iota: p^{-1}(U \cap \mathcal{X}_{\pi(x)}) \rightarrow D$ are the inclusions, and the projection $\pi_1: p^{-1}(U) \rightarrow p^{-1}(U \cap \mathcal{X}_{\pi(x)})$, from the projectivized relative tangent bundle onto the projectivized tangent bundle of $U \cap \mathcal{X}_{\pi(x)}$, is defined using the isomorphism $p^{-1}(U) \cong p^{-1}(U \cap \mathcal{X}_{\pi(x)}) \times \mathcal{U}$ induced by $f$.

Keep the notation of Condition 1.6. Following are the constraints on the singularities along the diagonal of the sheaf $E$ of Azumaya algebras over $X^d$ in Condition 1.6 (5). Let $\beta: Y \rightarrow X^d$ be the blow-up of the diagonal and let $D \subset Y$ be the exceptional divisor. Let $F$ be a reflexive twisted sheaf, such that $E \cong \mathcal{E}nd(F)$.

Condition 1.6 (5): The quotient $G := (\beta^*F)/\text{torsion}$ is a locally free sheaf and $F$ is $\beta$-tight. $G$ restricts to each fiber $D_x$ of $D \rightarrow X$, which is the projective space $\mathbb{P}(T_xX \otimes \mathbb{C}^{d-1})$, as the same stable unobstructed vector bundle $W$, modulo the action of the automorphism group of the projective space. The $\text{Aut}(D_x)$-orbit $[W]$ of the isomorphism class of the vector bundle $W$ has the following properties:

(a) There exist nonnegative integers $m$ and $m'$, such that

$$
H^0(W \otimes \mathcal{O}_{D_x}(j)) = 0, \quad \text{for } j < m, \tag{1.7}
$$

$$
H^i(W \otimes \mathcal{O}_{D_x}(j)) = 0, \quad \text{for } i > 0, \text{ and for } j \geq m, \tag{1.8}
$$

the analogues of (1.7) and (1.8) holds for $W'$ and $m'$, and $W \otimes \mathcal{O}_{D_x}(m)$ is generated by its global sections.

3The condition is needed for flatness of the deformation $\mathcal{E}_t$, $t \in \mathbb{P}^1$, of the sheaf $E$ over the twistor line $\mathbb{P}^1$ (Proposition 3.2) as well as for the condition to be open.
(b) The pair \((D_x, W)\) is infinitesimally rigid and \(W\) is \(\mathfrak{sl}(d-1)\)-invariant as well as \(\mathfrak{sp}(T_x X)\)-equivariant. Finally, there exists a positive integer \(k\), such that the traceless endomorphism bundle \(\mathcal{E}nd_0(W)\) satisfies

\[
\begin{align*}
H^0(\mathcal{E}nd_0(W) \otimes \mathcal{O}_{D_x}(j)) &= 0, \quad \text{for } j < k, \\
H^i(\mathcal{E}nd_0(W) \otimes \mathcal{O}_{D_x}(j)) &= 0, \quad \text{for } i > 0, \text{ and for } j \geq 1,
\end{align*}
\]

and \(\mathcal{E}nd_0(W) \otimes \mathcal{O}_{D_x}(k)\) is generated by its global sections.\(^4\)

(c) Let \(\pi: X \to B\) be a smooth and proper family of irreducible holomorphic symplectic manifolds over an analytic space \(B\). If a vector bundle \(W\) over \(D := \mathbb{P}(T_x^{d-1})\) restricts as a vector bundle in the orbit \([W]\) to each fiber of \(p: D \to X\), then \(W\) is locally trivializable in the topology of \(X\) in the sense of Definition \((1.15)\).

A simple example of a vector bundle \(W\) over a projective space, having the above properties, is provided in Equation \((7.1)\). In Condition \((1.6)\) we view the pair \((D_x, W)\) as an infinitesimal structure on the tangent space \(T_x X\) of \(X\). In the case of the vector bundle in Equation \((7.1)\) this structure is equivalent to a symplectic structure, up to a scalar, as shown in section \((7)\).

The \(\beta\)-tightness condition is needed in Proposition \((3.5)\). There we show that the vanishing of \(H^1(X^d, E_t)\) in Conjecture \((1.7)\) implies that \(E_t\) is infinitesimally rigid.

2. Families of reflexive sheaves of Azumaya algebras

We define and study in this section a class of families of reflexive sheaves with good base change properties. Throughout this section \(S\) will be an analytic space (not necessarily reduced) and \(\pi: X \to S\) will be a smooth morphism with connected fibers. In the applications \(\pi\) will be proper, but we do not assume properness in this section.

2.1. Families of reflexive sheaves. Given a coherent sheaf \(\mathcal{F}\) on \(X\) and a morphism \(T \to S\), set \(X_T := T \times_S X\) and let \(\mathcal{F}_T\) be the pullback of \(\mathcal{F}\) by the natural morphism from \(X_T\) to \(X\).

**Definition 2.1.** A family \(\mathcal{F}\) of reflexive sheaves over \(\pi\) is a coherent sheaf \(\mathcal{F}\) over \(X\) satisfying the following conditions.

1. Both \(\mathcal{F}\) and its dual \(\mathcal{F}^* := \mathcal{H}om(\mathcal{F}, \mathcal{O}_X)\) are flat over \(S\).
2. The natural homomorphisms \((\mathcal{F}^*)_T \to (\mathcal{F}_T)^*\) and \(\mathcal{F}_T \to (\mathcal{F}_T)^*\) are isomorphisms, for every morphism \(T \to S\).

If \(\mathcal{F}\) is a twisted coherent sheaf over \(X\) satisfying the conditions above, we say that \(\mathcal{F}\) is a family of reflexive twisted sheaves over \(\pi\).

In the rest of this subsection we provide a construction of families of reflexive sheaves. Let \(Z \subset X\) be a subscheme, smooth and proper over \(S\), of relative co-dimension \(c \geq 2\). Let \(\beta : Y \to X\) be the blow-up of \(X\) with center \(Z\). Denote by \(D \subset Y\) the exceptional divisor, \(e : D \to Y\) the closed immersion, and let \(p : D \to Z\) be the natural morphism. Note that \(p\) is a \(\mathbb{P}^{c-1}\) bundle. Let \(V\) be a locally free sheaf over \(Y\). \(V\) may be \((\beta^*\theta)\)-twisted by the pullback \(\beta^*\theta\) of a \(\check{\text{C}}\v{e}ch\) 2-co-cycle \(\theta\) for the sheaf \(\mathcal{O}_Y^2\). In the latter case the higher direct images \(R^i \beta_* V\) are \(\varpi\)-twisted coherent sheaves, for all \(i \geq 0\).

\(^4\) Caution: There is a subtlety involved in the pullback of sheaves of Azumaya algebras. The condition implies that \((\beta^* E)/\text{torsion}\) is isomorphic to \([\mathcal{E}nd_0(G) \otimes \mathcal{O}_Y(-kD)] \oplus \mathcal{O}_Y\), where \(G := (\beta^* F)/\text{torsion}\) (see Lemma \((2.3)(2)\)). In particular, \((\beta^* E)/\text{torsion}\) is not a sheaf of Azumaya algebras, if \(E\) is not locally free, as \(k > 0\).
Lemma 2.2. Assume that $R^i\beta_*e^*(V(-jD)) = 0$, for $i > 0$ and for $j \geq 0$. Then $R^i\beta_*V = 0$, for $i > 0$, and $\beta_*V$ is flat over $S$.

Proof. We prove first that $R^i\beta_*V = 0$, for $i > 0$. Let $I := \mathcal{O}_Y(-D)$ be the ideal sheaf of $D$. Let $V_n := V \otimes (\mathcal{O}_Y/I^{n+1})$ be the restriction of $V$ to the $n$-th order infinitesimal neighborhood of $D$. By Grothendieck’s comparison theorem, it suffices to prove that $R^i\beta_*(V_n)$ vanishes, for all $n \geq 0$. Tensor by $V$ the short exact sequence

$$0 \rightarrow I^n/I^{n+1} \rightarrow \mathcal{O}_Y/I^{n+1} \rightarrow \mathcal{O}_Y/I^n \rightarrow 0$$

to get the short exact sequence

$$0 \rightarrow e_*e^*(V(-nD)) \rightarrow V_n \rightarrow V_{n-1} \rightarrow 0.$$ 

Now $R^i\beta_*(e_*e^*(V(-nD))) = R^i\beta_*(V(-nD))$ vanishes, by assumption, and we get the isomorphism $R^i\beta_*(V_n) \cong R^i\beta_*(V_{n-1})$, for all $i > 0$ and for all $n \geq 1$. Consequently, $R^i\beta_*(V_n) \cong R^i\beta_*(V_0) = R^i\beta_*(e_*e^*V) = R^i\beta_*(e^*V) = 0$, for all $i > 0$.

It remains to prove that $\beta_*V$ is flat over $S$. Given a morphism $g : T \rightarrow S$ we get the cartesian diagram

$$\begin{array}{ccc}
Y_T & \overset{h}{\longrightarrow} & Y \\
\beta_T & \downarrow & \beta \\
X_T & \overset{f}{\longrightarrow} & X \\
\downarrow & & \downarrow \\
T & \overset{g}{\longrightarrow} & S.
\end{array}$$

The vanishing of $R^i\beta_*V$, for $i > 0$, implies that $\beta_T_*(h^*V) \cong Lf^*\beta_*V$, by [III] Prop. 6.3]. Therefore, $Lf^*(\beta_*V)$ is concentrated in degree 0. Considering a closed point $T$ of $S$ we get that $\mathcal{T}or^i_S(\mathbb{C}_T, \beta_*V) = 0$, for $i > 0$. Flatness of $\beta_*V$ over $S$ now follows from the local criterion for flatness [Mat] Ch. 8 Theorem 49].

Lemma 2.3. (1) Assume that there exist integers $k \geq 0$ and $\ell \geq 1 - c$, such that the following conditions hold.

(a) $p_*e^*(V(-jD)) = 0$, for $j < k$.
(b) $R^i p_*e^*(V(-jD)) = 0$, for $i > 0$ and for $j \geq k$.
(c) $p_*e^*(V(-jD)) = 0$, for $j < \ell$.
(d) $R^i p_*e^*(V(-jD)) = 0$, for $i > 0$ and for $j \geq \ell$.

Then $F := \beta_*V$ is a family of reflexive sheaves over $\pi$.

(2) Assume [1a] and [1b] above and that the counit $p^*p_* \rightarrow id$ for the adjunction $p^* \dashv p_*$ induces a surjective homomorphism $p^*p_*e^*V(-kD) \rightarrow e^*V(-kD)$. Then $\beta_*V = \beta_*V(-kD)$ and the analogous natural homomorphism

$$\beta^*F = \beta^*\beta_*V(-kD) \rightarrow V(-kD)$$

is surjective and its kernel is supported on $D$.

Proof. We prove first the flatness of $F$ and $F^*$. Assumption [1a] implies that $F$ is isomorphic to $\beta_*(V(-kD))$. The sheaf $\beta_*V(-kD)$ is flat over $S$ and $R\beta_*V(-kD)$ is isomorphic to
\( \beta_* V(-kD) \), by assumption \((\text{Id})\) and Lemma 2.2.

We have the isomorphisms

\[
F^* \cong \mathcal{H}^0[\mathcal{R}Hom(\beta_* V(-kD), \mathcal{O}_X)] \cong \mathcal{H}^0[\mathcal{R}Hom(\beta_* V(-kD), \mathcal{O}_X)] \\
\cong \mathcal{H}^0[\beta_*(V^*(kD) \otimes \omega_\beta)] \cong \beta_*(V^*((k + c - 1)D)) \cong \beta_*(V^*(-\ell D)),
\]

where the first is clear, the second follows from the isomorphism \( \beta_* V(-kD) \cong R\beta_* V(-kD) \) established above, the third is Grothendieck-Verdier Duality \( RRV \), the fourth follows from the isomorphism \( \omega_\beta \cong \mathcal{O}_X((c-1)D) \), and the last follows from assumption \((\text{Id})\). Now \( \beta_*(V^*(-\ell D)) \) is flat over \( S \), by assumption \((\text{Id})\) and Lemma 2.2.

We prove next the isomorphism \( f^*(\mathcal{F}) \cong (f^* \mathcal{F})^* \). We may assume that \( k = 0 \), possibly after replacing \( V \) by \( V(-kD) \) and replacing \( \ell \) by \( \ell + k \). Given a morphism \( g : T \rightarrow S \) we get the cartesian diagram (2.1). We have the isomorphisms

\[
(2.3) \quad f^* \beta_* V \cong Lf^* \beta_* V \cong Lf^* R\beta_* V \cong R\beta_T_* (h^* V) \cong \beta_T_*(h^* V),
\]

where the first (leftmost) follows from flatness of \( \mathcal{F} \), the second by assumption \((\text{Id})\), the third by cohomology and base change, and the fourth by assumption \((\text{Id})\) again. Hence, \( f^* \mathcal{F} \cong \beta_T_*(h^* V) \). We also have the isomorphisms

\[
f^*(\mathcal{F}) \cong f^*(\beta_* (V^*(-\ell D))) \cong R\beta_T_* (h^* [V^*(-\ell D)]) \cong \beta_T_* ((h^* V^*(-\ell D_T))),
\]

where the left was established above, the middle one follows from a sequence analogous to the one in equation (2.3), and the right by assumption \((\text{Id})\). Now, \( \beta_T_* ((h^* V^*(-\ell D_T))) \) is isomorphic to \( \beta_T_*(h^* V^* \otimes \omega_\beta) \), by assumption \((\text{Id})\), the sheaf \( \beta_T_*(h^* V^* \otimes \omega_\beta) \) is equal to the sheaf \( \mathcal{H}^0[\beta_T_*(h^* V \otimes \omega_\beta)] \), which is isomorphic to \( \mathcal{H}^0[\mathcal{R}Hom(\beta_T_*(h^* V), \mathcal{O}_{X_T})] \), by Grothendieck-Verdier duality, and the latter is isomorphic to \( \mathcal{H}^0(\beta_T_*(h^* V), \mathcal{O}_{X_T}) \), which in turn is isomorphic to \( \beta_T_* h^* V \). We conclude the short exact sequence

\[
\mathcal{O}_T \rightarrow V(-kD) \rightarrow e_* e^* V(-kD) \rightarrow 0.
\]

We get the short exact sequence

\[
0 \rightarrow \beta_* V(-(k + 1)D) \rightarrow \beta_* V(-kD) \rightarrow \beta_* e_* e^* V(-kD) \rightarrow 0.
\]
Pulling back via $\beta^*$ we get the commutative diagram with right exact top row:

$$
\begin{array}{c}
\beta^*\beta_*V(-(k+1)D) \\
\downarrow\text{ev} \\
V(-kD) \\
\gamma \\
\end{array}
\begin{array}{c}
\beta^*\beta_*V(-kD) \\
\beta^*\beta_*e_*e^*V(-kD) \\
0 \\
\end{array}
\begin{array}{c}
\beta^*\beta_*e_*e^*V(-kD) \\
\end{array}
\begin{array}{c}
0 \\
\end{array}
$$

The vertical homomorphism are associated to the co-unit natural transformation $\beta^*\beta_* \to \text{id}$. Let $\zeta : Z \hookrightarrow X$ be the closed immersion. Then $\beta \circ e = \zeta \circ p$. Hence, $\beta^*\beta_*e_*e^*V(-kD) = \beta^*\zeta_*p_*e^*V(-kD)$. Now $p_*e^*V(-kD)$ is a locally free sheaf over $Z$, by assumption (13), and $\beta^*\zeta_*\mathcal{O}_Z \cong e_*p^*\mathcal{O}_Z$. We get the right isomorphism below:

$$
\beta^*\beta_*e_*e^*V(-kD) = \beta^*\zeta_*p_*e^*V(-kD) \cong e_*p^*p_*e^*V(-kD).
$$

The homomorphism $\gamma$ is the composition of the above isomorphism with the homomorphism $e_*p^*p_*e^*V(-kD) \to e_*e^*V(-kD)$. The latter is surjective, by assumption. Hence, the homomorphism $\gamma$ is surjective. We conclude that the composition $\rho \circ \text{ev} : \beta^*\beta_*V(-kD) \to e_*e^*V(-kD)$ is surjective, by the commutativity of the diagram above. The surjectivity of the homomorphism $\text{ev}$ on stalks of points of $D$ follows, by Nakayama’s Lemma. The homomorphism $\text{ev}$ clearly induces an isomorphism on stalks of points outside the exceptional divisor $D$. \qed

**Remark 2.4.** Let $M \to S$ be a smooth morphism with connected fibers of dimension $\geq 3$. Set $X := M \times_S M$, $\pi : X \to S$ the natural morphism, $Z \subset X$ the diagonal, $\beta : Y \to X$ the blow-up along $Z$, and $V$ a vector bundle over $Y$ satisfying the assumptions of Lemma 2.3. Then $F := \beta_*V$ is a family of reflexive sheaves over $\pi$, by Lemma 2.3. Let $\pi_i : X \to M$ be the projection, $i = 1, 2$. We claim that $F$ is a family of reflexive sheaves over $\pi_i$ as well. Indeed, $Z$ is smooth over $M$ as well being a section of $\pi_i$. Hence, Lemma 2.3 applies with the morphisms $\pi_i$, $i = 1, 2$, as well.

2. Some basic properties of families of reflexive sheaves.

**Lemma 2.5.** Let $F$ be a family of reflexive sheaves over $\pi$. Let $U \subset X$ be the open subset over which $F$ is locally free and set $Z := X \setminus U$. Given a closed point $s \in S$, let $X_s$ and $Z_s$ be the fibers of $X$ and $Z$ over $s$. Then the co-dimension of $Z_s$ in $X_s$ is at least 3.

**Proof.** The restriction $F_s$ of $F$ to $X_s$ is reflexive, by definition. Hence, $F_s$ is locally free away from a closed analytic subset of co-dimension $\geq 3$. If $F_s$ is locally free at a point $x \in X_s$, then $F$ is locally free at that point. Indeed, let $\{f_1, \ldots, f_r\}$ be a basis of the stalk $F_{s,x}$ of $F_s$ at $x$ and $\phi : \oplus_{i=1}^r \mathcal{O}_{X_s} \to F_{s,x}$ the corresponding isomorphism of stalks. Choose a subset $\{\tilde{f}_1, \ldots, \tilde{f}_r\}$ of the stalk $F_x$ of $F$ at $x$, which maps to the above basis, and let $\tilde{\phi} : \oplus_{i=1}^r \mathcal{O}_{X_s} \to F_x$ be the corresponding homomorphism of stalks. Then $\tilde{\phi}$ is surjective, by Nakayama’s lemma. Let $N$ be the kernel of $\tilde{\phi}$. Tensoring the exact sequence

$$
0 \to N \to \oplus_{i=1}^r \mathcal{O}_{X_s} \xrightarrow{\tilde{\phi}} F_x \to 0
$$

by the stalk $\mathcal{O}_{S_s}$ we get the long exact sequence

$$
\mathcal{T}or_1^{\mathcal{O}_{X_s}}(F_x, \mathcal{O}_{S_s}) \to N \otimes_{\mathcal{O}_{X_s}} \mathcal{O}_{S_s} \to \oplus_{i=1}^r \mathcal{O}_{X_{s,x}} \xrightarrow{\phi} F_{s,x} \to 0.
$$
Now $\text{Tor}_i^{O_X}(F_x, O_{S_x})$ vanishes, by flatness of $F$ over $S$, and $\phi$ is an isomorphism. Thus, $N$ restricts to zero along $X_{s,x}$. Hence, $N = 0$, by Nakayama’s lemma. Consequently, $\tilde{\phi}$ is injective as well, and $F$ is locally free at $x$. We conclude that the co-dimension of $Z \cap X_s$ in $X_s$ is at least 3.

Let $Z$ be a closed analytic subset of $X$ and $I$ its ideal sheaf. Given a point $x \in X$, denote by $I_x$ the stalk of $I$ at $x$ and by $O_{X,x}$ the stalk of $O_X$ at $x$. Note that $O_{X,x}$ is a noetherian ring [Serre Prop. 1]. Let $E$ be a coherent sheaf on $X$ and $E_x$ its stalk at $x$. Denote by $\text{depth}_{I_x}(E_x)$ the maximal length of a regular sequence in $I_x$ for the $O_{X,x}$-module $E_x$. Given a closed analytic subset $Z \subset X$, set $\text{depth}_Z(E) := \inf_{x \in Z} \text{depth}_{I_x}(E_x)$ [1 Ha3 Cor. 3.6].

**Lemma 2.6.** Assume that $Z$ intersects each fiber of $\pi$ in a subset of codimension $\geq c$. Then $\text{depth}_Z(O_X) \geq c$. Equivalently, given a coherent sheaf $Q$ supported set theoretically on $Z$, the extension sheaves $\text{Ext}^i(Q, O_X)$ vanish, for $i < c$.

**Proof.** Let $x$ be a point of $X$ and let $Q_x$ and $\text{Ext}^i(Q, O_X)_x$ be the stalks at $x$. We have the isomorphism $\text{Ext}^i(Q, O_X)_x \cong \text{Ext}^i(Q_x, O_{X,x})$, by [Ha0 Prop. III.6.8]. There exists a regular sequence $a_1, \ldots, a_c$ of length $c$ in $I_x$, by [Mat Cor. 20.F], since $Z$ has relative co-dimension $c$ in $X$ over $S$ and $X$ is smooth over $S$. Thus, $\text{Ext}^i(Q_x, O_{X,x}) = 0$, for $i < c$, by [Mat Ch. 6 Theorem 28]. The equivalence with the inequality $\text{depth}_Z(O_X) \geq c$ follows from [Ha3 Prop. 3.3]

**Lemma 2.7.** Let $F$ be a family of reflexive sheaves over $\pi$. Then, locally over $X$, it can be included in an exact sequence $0 \to F \to E \to G \to 0$, where $E$ is locally free and $G$ is a subsheaf of a locally free sheaf.

**Proof.** We follow the first part of the proof of [Ha2 Prop. 1.1]. The statement is local, so we may assume that there exists a right exact sequence $V_1 \xrightarrow{d} V_0 \to F^* \to 0$, where $V_0$ and $V_1$ are locally free. Taking duals we get the left exact sequence $0 \to F^{**} \to V_0^* \xrightarrow{d^*} V_1^*$. Set $E := V_0^*$ and let $G$ be the image of $d^*$. The isomorphism $F \cong F^{**}$ yields the desired short exact sequence.

**Lemma 2.8.** Let $F$ be a family of reflexive sheaves over $\pi$. Let $Z \subset X$ be a closed analytic subset such that $\text{depth}_Z(O_X) \geq 2$. Then $\text{depth}_Z(F) \geq 2$.

**Proof.** We follow the proof of [Ha2 Prop. 1.3]. Recall that $\text{depth}_Z(F) \geq k$, if and only if the sheaf $\text{Ext}^i(N, F)$ vanishes, for every coherent sheaf $N$ supported on $Z$ and for all $i < k$ [Ha3 Prop. 3.3]. The statement is local and we may assume that there exists a short exact sequence $0 \to F \to E \to G \to 0$ with $E$ locally free and $G$ a subsheaf of a locally free sheaf, by Lemma 2.7. Let $N$ be a coherent sheaf supported on $Z$. We get the exact sequence

$$\text{Hom}(N, G) \to \text{Ext}^1(N, F) \to \text{Ext}^1(N, E).$$

Now $\text{Ext}^i(N, W)$ vanishes, for $i \leq 1$, for any locally free sheaf $W$, since $\text{depth}_Z(O_X) \geq 2$. Hence, $\text{Ext}^1(N, E)$ vanishes and so do $\text{Hom}(N, F)$ and $\text{Hom}(N, G)$, since both $F$ and $G$ are subsheaves of locally free sheaves. We conclude that $\text{Ext}^i(N, F)$ vanishes for $i = 0$ and $i = 1$.

**Lemma 2.9.** Let $F$ be a family of reflexive sheaves over $\pi$. Let $U \subset X$ be the open subset where $F$ is locally free, and let $i : U \to X$ be the inclusion. Then the natural homomorphism $F \to i_* i^* F$ is an isomorphism.
Proof. Set $Z := X \setminus U$. Recall that depth$_Z(F) \geq k$, if and only if the cohomology sheaves $H^i_Z(F)$ with support along $Z$ vanish for $i < k$ [H1, Theorem 3.8]. We know that depth$_Z(O_X) \geq 2$, by Lemmas 2.5 and 2.6. Hence, depth$_Z(F) \geq 2$, by Lemma 2.8, and the sheaves $H^i_Z(F)$ vanish for $i = 0$ and $i = 1$. The statement thus follows from the long exact sequence

$$0 \to H^0_Z(F) \to F \to \iota_*i^*F \to H^1_Z(F) \to 0$$

[Serre, Section 3].

\[\square\]

2.3. Families of Azumaya algebras. Keep the notation of section 2.1. In particular, $Z$ is a subscheme of $X$, smooth of relative co-dimension $c$ over $S$, and $\beta : \mathcal{Y} \to \mathcal{X}$ is the blow-up centered at $Z$. Assume that $c \geq 3$. Let $V_1$ and $V_2$ be coherent sheaves over $\mathcal{Y}$. Assume that $V_1$ is locally free. Given an open subset $U$ of $X$ we get the homomorphism

$$\Gamma(\beta_*(V_1^* \otimes V_2), U) := \Gamma(V_1^* \otimes V_2, \beta^{-1}(U)) = \Gamma(\text{Hom}(V_1, V_2), \beta^{-1}(U)) \xrightarrow{\beta_*} \Gamma(\text{Hom}(\beta_*V_1, \beta_*V_2), U).$$

We denote the corresponding sheaf homomorphism by

$$\beta_* : \beta_*V_1^* \otimes V_2 \to \text{Hom}(\beta_*V_1, \beta_*V_2).$$

The above homomorphism is induced on the cohomology in degree zero by the following natural transformation of exact functors from $D^b(\mathcal{Y})$ to $D^b(\mathcal{X})$. Let $\eta : id \to \beta^!R\beta_*$ be the unit of the adjunction $R\beta_* \dashv \beta^!$. We get

$$R\beta_*R\text{Hom}(V_1, V_2) \to R\beta_*[R\text{Hom}(V_1, \beta^!R\beta_*V_2)] \cong R\text{Hom}(R\beta_*V_1, R\beta_*V_2),$$

where the left arrow is $R\beta_*R\text{Hom}(V_1, \eta_{V_2})$ and the right isomorphism is provided by Grothendieck-Verdier duality.

Taking $V_1 = V_2 = V$, we get the homomorphism

$$(2.4) \quad \beta_* : \beta_*V^* \otimes V \to \text{Hom}(\mathcal{F}, \mathcal{F}).$$

Lemma 2.10. Assume that the sheaves $\mathcal{F} := \beta_*V$ and $\beta_*V^* \otimes V$ are reflexive over $\pi$. Then the homomorphism $\beta_*$ given in (2.4) is an isomorphism.

Proof. The homomorphism $\beta_*$ restricts as an isomorphism over $U := X \setminus Z$. Hence, the kernel of $\beta_*$ is supported over $Z$. But the reflexive sheaf $\beta_*V^* \otimes V$ does not have a non-zero subsheaf supported over $Z$, by Lemma 2.8. Hence, $\beta_*$ is injective.

Let $Q$ be the co-kernel of $\beta_*$ and consider the short exact sequence:

$$0 \to \beta_*V^* \otimes V \to \text{Hom}(\mathcal{F}, \mathcal{F}) \to Q \to 0.$$

$Q$ is supported, set theoretically, on a subset of $Z$. Hence, the sheaf $\mathcal{E}xt^1(Q, \beta_*V^* \otimes V)$ vanishes, by Lemma 2.8, and the above short exact sequence splits, locally over $\mathcal{X}$. But any local section of $\text{Hom}(\mathcal{F}, \mathcal{F})$, supported over $Z$, has an image subsheaf of $\mathcal{F}$, supported over $Z$. Now $\mathcal{F}$ is assumed reflexive, hence it does not have non-trivial subsheaves supported over $Z$. Hence, $Q$ vanishes and $\beta_*$ is an isomorphism. $\square$

Definition 2.11. A family $\mathcal{A}$ of reflexive sheaves of Azumaya algebras over $\pi$ is a family of reflexive sheaves over $\pi$, with an associative multiplication $m : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ and a unit section $1$, such that locally over $\mathcal{X}$, $(\mathcal{A}, m, 1)$ is isomorphic to $(\text{End}(\mathcal{F}), m', id)$, for some reflexive sheaf $\mathcal{F}$ over $\pi$, where the multiplication $m' : \text{End}(\mathcal{F}) \otimes \text{End}(\mathcal{F}) \to \text{End}(\mathcal{F})$ is given by composition.
We conclude that the homomorphism \( \iota \nu \) is coherent. Let
\[ \text{Corollary 2.12.} \]
Assume further that there exists an integer \( t \geq 0 \), such that \( V^t \otimes V \) satisfies the conditions of Lemma 2.10 with both \( k \) and \( \ell \) replaced by \( t \). Then \( \beta_+ (\End(V)) \) is a family of reflexive sheaves of Azumaya algebras over \( \pi \), which is isomorphic to \( \End(\beta_+ V) \).

Proof. Both \( \mathcal{F} := \beta_+ V \) and \( \beta_+ (V^t \otimes V) \) are reflexive over \( \pi \), by Lemma 2.3. The sheaf \( \beta_+ (V^t \otimes V) \) is globally isomorphic to \( \End(\mathcal{F}) \), by Lemma 2.10. \( \square \)

2.4. Every family of reflexive Azumaya algebras comes from a family of reflexive sheaves. Keep the notation of the previous subsection. Assume that the co-dimension \( c \) of \( Z \) in \( \mathcal{X} \) is at least 3. Let \( \nu_X \subset \mathcal{O}_X \) be the nilpotent radical ideal subsheaf. Assume that \( \nu_X^m = 0 \), for some positive integer \( m \).

Lemma 2.13. Let \( W \) be a Stein open subspace of \( \mathcal{X} \), set \( U := \mathcal{X} \setminus Z \), and let \( \iota : [W \cap U] \hookrightarrow W \) be the inclusion. Denote by \( \Pic(W \cap U) \) the subset of \( \Pic(W \cap U) \) consisting of line bundles \( L \), such that \( \iota_* (L) \) is a coherent \( \mathcal{O}_W \)-module. Then \( \iota^* : \Pic(W) \to \Pic(W \cap U) \) is an isomorphism.

Proof. The pullback homomorphism \( \iota^* : \Pic(W) \to \Pic(W \cap U) \) is injective, and its image is contained in \( \Pic(W \cap U)_\mathbb{Z} \), since \( \iota_* \iota^*(L) \) is isomorphic to \( L \), by Lemma 2.9. Assume first that \( S \) is reduced. Then so is \( \mathcal{X} \). The inequality \( \text{depth}_Z (\mathcal{O}_X) \geq 3 \) holds, by Lemma 2.6. Hence, \( H^i_{W \cap Z}(\mathcal{O}_W) \) vanishes, for \( i \leq 2 \), and the homomorphism \( \iota^* : H^i(W, \mathcal{O}_W) \to H^i(W \cap U, \mathcal{O}_{W \cap U}) \) is an isomorphism, for \( i \leq 1 \), by [H3] Theorem 3.8. The space \( H^i_{W \cap Z}(W, \mathbb{Z}) \) vanishes, for \( i \leq 5 \), by the last example in chapter I of [H3], where we use the assumption that both \( Z \) and \( \mathcal{X} \) are smooth over the base \( S \). Hence, the homomorphism \( \iota^* : H^i(W, \mathbb{Z}) \to H^i(W \cap U, \mathbb{Z}) \) is an isomorphism, for \( i \leq 4 \). Considering the long exact sequences associated to the exponential sequence we get the commutative diagram:

\[
\begin{array}{cccccc}
H^1(W, \mathbb{Z}) & \longrightarrow & H^1(W, \mathcal{O}_W) & \longrightarrow & H^1(W, \mathcal{O}_W^*) & \longrightarrow & H^2(W, \mathbb{Z}) \\
\downarrow \cong & & \downarrow \cong & & \downarrow \cong & & \\
H^1(W \cap U, \mathbb{Z}) & \longrightarrow & H^1(W \cap U, \mathcal{O}_W) & \longrightarrow & H^1(W \cap U, \mathcal{O}_W^*) & \longrightarrow & H^2(W \cap U, \mathbb{Z}).
\end{array}
\]

We conclude that the homomorphism \( \iota^* : \Pic(W) \to \Pic(W \cap U) \) is an isomorphism.

We prove the general case next. Let \( L \) be a line bundle on \( W \cap U \) and assume that \( \iota_* (L) \) is coherent. Let \( \nu_L \) be the subsheaf of \( L \) analogous to \( \nu_W \). It is well defined, since \( L \) is determined by some 1-co-cycle for \( \mathcal{O}_W \) and a gluing transition function maps (locally) the subsheaf \( \nu_W \) of \( \mathcal{O}_W \) to itself. We get the short exact sequence of coherent \( \mathcal{O}_W \)-modules

\[
0 \to \iota_* (\nu_L) \to \iota_* (L) \to \iota_* (L/\nu_L) \to 0.
\]

Set \( \mathcal{T} := L/\nu_L \). Note that \( \mathcal{T} \) is isomorphic to the restriction of \( L \) to the reduced subscheme \([W \cap U]_\text{red} \) with structure sheaf \( \mathcal{O}_{W \cap U}/\nu_{W \cap U} \).

The restriction homomorphism \( \Pic(W) \to \Pic(W_\text{red}) \) is an isomorphism, by [Ba] Prop. 1. There exists a line bundles \( \bar{L} \) over \( W \), and an isomorphism \( \bar{g} \) from the restriction of \( \bar{L} \) to \([W \cap U]_\text{red} \) onto \( \mathcal{T} \), by the reduced case proven above. We get the short exact sequence

\[
0 \to \bar{L}^{-1} \otimes \iota_* (\nu_L) \to \bar{L}^{-1} \otimes \iota_* (L) \to \bar{L}^{-1} \otimes \iota_* (L/\nu_L) \to 0
\]

and the long exact sequence

\[
H^0(W, \bar{L}^{-1} \otimes \iota_* (L)) \to H^0(W, \bar{L}^{-1} \otimes \iota_* (\mathcal{T})) \to H^1(W, \bar{L}^{-1} \otimes \iota_* (\nu_L)).
\]
The right hand space vanishes, since $W$ is Stein. Hence, there exists a lift of $\tilde{g}$ to a homomorphism $g : \tilde{L} \to \iota_*(L)$ of $O_W$-modules. The restriction of $g$ to $W \cap U$ is an isomorphism of line-bundles, since it further restricts to an isomorphism $\tilde{g}$ over the reduced subscheme. It follows that $\iota_*(L)$ is isomorphic to $\iota_*\iota^*(\tilde{L})$, which in turn is isomorphic to $\tilde{L}$, by Lemma 2.9.

We conclude that $\iota^*$ maps Pic($W$) onto Pic($W \cap U$).

\begin{proposition}
Let $\mathcal{A}$ be a family of reflexive sheaves of Azumaya algebras over $\pi$. There exists a family $\mathcal{F}$ of twisted reflexive sheaves over $\pi$, such that $\mathcal{A}$ is isomorphic to $\text{End}(\mathcal{F})$ as $O_X$-algebras. Furthermore, $\mathcal{F}$ is unique up to tensorization by a line bundle.
\end{proposition}

The above statement is well known when $\mathcal{A}$ is locally free [Cal, Theorem 1.3.5] or when $S$ is a point.

\textbf{Proof.} Step 1: Let $W$ be an open subset of $X$ and $\mathcal{F}_1, \mathcal{F}_2$ two families of reflexive sheaves over the restriction of $\pi$ to $W$. Let $\iota : [W \cap U] \to W$ be the inclusion, where $U := X \setminus Z$ is the open subset where $\mathcal{F}$ is locally free. Assume that there exist isomorphisms $\varphi_1 : \text{End}(\mathcal{F}_1) \to (\mathcal{A})|_W$ and $\varphi_2 : \text{End}(\mathcal{F}_2) \to (\mathcal{A})|_W$ of unital algebras. Then there exists a line bundle $L$ over $W \cap U$ and an isomorphism $g : \iota^*\mathcal{F}_1 \to \iota^*\mathcal{F}_2 \otimes L$, such that $Ad_g : \iota^*\text{End}(\mathcal{F}_1) \to \iota^*\text{End}(\mathcal{F}_2)$ is the restriction of $\varphi_2^{-1}\varphi_1$, by the locally free case of the proposition [Cal, Theorem 1.3.5].

The push-forward of $L$ to $W$ is coherent, since $L$ is the restriction of the kernel of the homomorphism

$$\text{Hom}(\mathcal{F}_2, \mathcal{F}_1) \to \text{Hom}(\mathcal{A}|_W, \text{Hom}(\mathcal{F}_2, \mathcal{F}_1))$$

sending a homomorphism $f$ to the homomorphism $\tilde{f}$ given by $\tilde{f}(a) := f\varphi_2^{-1}(a) - \varphi_1^{-1}(a)f$.

Denote by the same letter $L$ also a the line bundle on $W$ restricting to $L$ on $W \cap U$, which exists by Lemma 2.13. We get the isomorphisms

$$\mathcal{F}_1 \cong \iota_*\iota^*\mathcal{F}_1 \cong \iota_*\iota^*[\mathcal{F}_2 \otimes L] \cong \mathcal{F}_2 \otimes L,$$

where the first and last isomorphisms follow from Lemma 2.9 and the middle one is $\iota_*(g)$.

Step 2: There exists an open covering $\{W_\alpha\}$ of $X$ and a family $\mathcal{F}_\alpha$, of reflexive sheaves over the restriction of $\pi$ to $W_\alpha$, each admitting an isomorphism $\varphi_\alpha : \text{End}(\mathcal{F}_\alpha) \to \mathcal{A}_\alpha$, where $\mathcal{A}_\alpha$ is the restriction of $\mathcal{A}$ to $W_\alpha$, by Definition 2.11. Set $W_{\alpha\beta} := W_\alpha \cap W_\beta$ and $U_{\alpha\beta} := W_{\alpha\beta} \cap U$. Let $\iota_{\alpha\beta} : U_{\alpha\beta} \to W_{\alpha\beta}$ be the inclusion. There exist line bundles $L_{\alpha\beta}$ over $W_{\alpha\beta}$ and isomorphisms

$$g_{\alpha\beta} : (\mathcal{F}_\alpha)|_{W_{\alpha\beta}} \to (\mathcal{F}_\beta)|_{W_{\alpha\beta}} \otimes L_{\alpha\beta},$$

such that $Ad_{g_{\alpha\beta}} : \text{End}(\mathcal{F}_\alpha)|_{W_{\alpha\beta}} \to \text{End}(\mathcal{F}_\beta)|_{W_{\alpha\beta}}$ and $\varphi_\alpha^{-1}\varphi_\beta$ agree over $U_{\alpha\beta}$, by step 1. The equality

$$Ad_{g_{\alpha\beta}} = \varphi_\alpha^{-1}\varphi_\beta$$

over $W_{\alpha\beta}$ follows, by Lemma 2.9.

We may choose the line bundle $L_{\beta\alpha}$ to be $L_{\alpha\beta}^{-1}$. We claim that

1. the collection $\{L_{\alpha\beta}\}$ defines a gerbe over $X$ [H1, Sec. 1.2], and
2. the collection $\{(\mathcal{F}_\alpha, g_{\alpha\beta})\}$ defines a sheaf on this gerbe (i.e., a twisted sheaf [Cal]).

The first statement means that the line bundles $L_{\alpha\beta\gamma} := L_{\alpha\beta}L_{\beta\gamma}L_{\gamma\alpha}$ over $W_{\alpha\beta\gamma}$ come with nowhere vanishing global sections $\theta_{\alpha\beta\gamma}$. Furthermore, $\delta\theta$ is trivial, i.e.,

$$\theta_{\alpha\beta\gamma}^{-1}\theta_{\alpha\beta\delta}\theta_{\alpha\gamma\delta}\theta_{\beta\gamma\delta} = 1,$$
as a section of the naturally trivial line bundle \( L_{αβγ}^{-1}L_{αβδ}L_{αγδ}^{-1}L_{βγδ} \) over \( W_{αβγδ} \).

The sections \( θ_{αβγ} \) arise as follows. Set

\[ (δg)_{αβγ} := g_{αβγ}g_{γδ}g_{δα} : (F_{α})|_{W_{αβγ}} \to (F_{α})|_{W_{αβγ}} \otimes L_{αγδ}. \]

Equation (2.5) implies that \( Ad(δg)_{αβγ} \) restricts to \( U_{αβγ} \) as the identity endomorphism of \( (F_{α})|_{W_{αβγ}} \). Hence, the isomorphism \( (δg)_{αβγ} \) restricts to \( U_{αβγ} \) as tensorization by a nowhere vanishing section \( θ_{αβγ} \) of \( L_{αβγ} \). The section extends to a nowhere vanishing section of \( L_{αβγ} \) over \( W_{αβγ} \), denoted again by \( θ_{αβγ} \), since \( \text{depth}(O_{X}) \geq 2 \). It follows that \( (δg)_{αβγ} \) is equal to tensorization by \( θ_{αβγ} \) over \( W_{αβγ} \), by Lemma 2.9. The proof of Equation (2.6) is similar.

The fact that \( (F_{α},g_{αβ}) \) is a sheaf over the gerbe means that \( (δg)_{αβγ} \) is tensorization by \( θ_{αβγ} \). This holds, by construction of \( θ_{αβγ} \).

\[ \square \]

2.5. Condition 1.6 (5) is open. We keep the notation of section 2.1. Let \( A \) be a locally free Azumaya algebra over \( Y \). Set \( E := β_{s}A \). Given a point \( s ∈ S \), denote the fibers of \( X, Y, Z, D \) and \( β : Y → X \) over \( s \) by \( X_{s}, Y_{s}, Z_{s}, D_{s} \) and \( β_{s} : Y_{s} → X_{s} \). Let \( A_{s} \) and \( E_{s} \) be the restrictions of \( A \) and \( E \) to the fibers over \( s \).

Lemma 2.15. Assume that \( E_{0} \) satisfies Condition 1.6 (5) for some point \( 0 ∈ S \) and the morphism \( π \) is proper. Then there exists an open neighborhood \( U \) of 0 in \( S \), such that \( E \) is a family of reflexive sheaves of Azumaya algebras over \( π : X|_{U} → U \) and \( E_{s} \) satisfies Condition 1.6 (5) for all \( s ∈ U \).

Proof. Let \( V \) be a (possibly twisted) locally free sheaf over \( Y \), such that \( A \cong \text{End}(V) \). Denote its restrictions to the fibers over \( s ∈ S \) by \( V_{s} \). Let \( F_{0} \) be a reflexive sheaf such that \( E_{0} \cong \text{End}(F_{0}) \). The assumption that \( E_{0} \) satisfies Condition 1.6 (5) implies that \( (β_{s}^{*}F_{0})/\text{torsion} \) is locally free over \( Y_{0} \) and \( \text{End}(V_{0}) \cong \text{End}(β_{s}^{*}F_{0})/\text{torsion} \). We may thus assume that the restrictions of \( V \) to fibers \( D_{x_{0}} \) of \( p : D → X \) over \( x_{0} ∈ X_{0} \) are isomorphic to the restrictions of \( (β_{s}^{*}F_{0})/\text{torsion} \) to \( D_{x_{0}} \), possibly after replacing \( Y \) by \( Y(nD) \), for some integer \( n \). We conclude that \( E_{0} \) is isomorphic to \( \text{End}(β_{s_{0}} V_{0}) \), by Corollary 2.12 (applied in the case where the base \( S \) is a point). We may thus replace \( F_{0} \) by \( β_{s_{0}}^{*}V_{0} \) and assume their equality.

Set \( F := β_{s}V \). Given a point \( x ∈ X \), denote by \( W_{x} \) the restriction of \( V \) to \( D_{x} \). Fix a point \( x_{0} ∈ X_{0} \). We know that \( W_{x_{0}} \) has the cohomological properties listed in Condition 1.6 (5). Hence, there exists an open subset \( U \) of \( X \), containing \( x_{0} \), such that the pairs \( (D_{x}, W_{x}) \) and \( (D_{x_{0}}, W_{x_{0}}) \) are isomorphic, for every \( x ∈ U \), since \( W_{x_{0}} \) is assumed to be stable and unobstructed and the pair \( (D_{x_{0}}, W_{x_{0}}) \) is assumed to be infinitesimally rigid in Condition 1.6 (5). The fiber \( X_{0} \) is contained in \( U \), since \( E_{0} \) satisfies Condition 1.6 (5). Hence, there exists an open neighborhood \( U \) of 0 in \( S \), such that the pairs \( (D_{x}, W_{x}) \) and \( (D_{x_{0}}, W_{x_{0}}) \) are isomorphic, for every \( x ∈ X_{s} \), for all \( s ∈ U \), since the morphism \( π \) is proper.

\( F := β_{s}V \) is a family of reflexive sheaves over \( X|_{U} → U \), by Lemma 2.3 (1), and \( β_{s}A = β_{s}\text{End}(V) \) is isomorphic to \( \text{End}(E) := \text{End}(β_{s}V) \), by Corollary 2.12. Furthermore, the homomorphism \( β_{s}^{*}F_{s}/(\text{torsion}) \) is isomorphic to \( V_{s}(-mD_{s}) \) is surjective, where \( m \) is the integer in Equation (1.7), by Lemma 2.3 (2). It follows that \( β_{s}^{*}F_{s}/(\text{torsion}) \) is isomorphic to \( V_{s}(-mD_{s}) \) and is hence locally free.

\[ \square \]
3. Condition \[1.6\] is preserved under twistor deformations

3.1. Hyperholomorphic sheaves. Let \(M\) be a hyper-Kähler manifold. Then \(M\) is endowed with a metric and an action of the algebra \(\mathbb{H}\) of quaternions on \(T^R M\), such that the group of unit quaternions acts via integrable complex structures, and the metric is Kähler with respect to each of these complex structures. We will refer to these as induced complex structures. The unit quaternions form a 2-sphere \(S^2\) and the induced complex structures on \(M\) come from a complex structure on \(M := M \times S^2\). The projection on the second factor is holomorphic, yielding the twistor family \(M \to \mathbb{P}^1\) [HKLR]. A point of \(M\) determines a section of the projection \(M \times S^2 \to S^2\), which is a holomorphic section of the twistor family \(M \to \mathbb{P}^1\), called a horizontal section [HKLR Sec. 3(F)]. If \(M\) is an irreducible holomorphic symplectic manifold and \(\omega\) a Kähler class on \(M\), then the Kähler metric associated to \(\omega\) is a hyper-Kähler metric [Be]. In this case we will denote the twistor family by \(M \to \mathbb{P}^1\) to emphasize the dependence on the Kähler class \(\omega\).

Let \(M\) be a hyper-Kähler manifold, \(I_0\) one of the induced complex structures, and \(V\) a holomorphic vector bundle over \((M, I_0)\). A hyperholomorphic connection on \(V\) is a Hermitian connection \(\nabla\), whose curvature form is of Hodge type \((1,1)\) with respect to all the induced complex structures [Ve5]. This means that the \((0,1)\)-part of the connection, with respect to every induced complex structure on \(M\), is an integrable complex structure on \(V\), and \(V\) extends to a holomorphic bundle \(\mathcal{V}\) over the twistor family \(M\).

Given a reflexive coherent sheaf \(F_{I_0}\) on \((M, I_0)\), Verbitsky defines the notion of a hyperholomorphic connection on \(F_{I_0}\) as a connection over the open subset \(U \subset M\) where \(F_{I_0}\) is locally free, with a condition on the curvature and the associated metric along the singular locus \(Z := M \setminus U\) [Ve5 Definition 3.15]. If the sheaf \(F_{I_0}\) is stable, with respect to the Kähler form associated to \(I_0\), and the first two Chern classes of \(F_{I_0}\) remain of Hodge type with respect to all the induced complex structures, then \(F_{I_0}\) admits a hyperholomorphic connection [Ve5 Theorem 3.19]. The singular locus \(Z\) of \(F_{I_0}\) remains analytic with respect to all induced complex structures [Ve5 Claim 3.16], and the reflexive coherent sheaf \(F_I\) over \((M, I)\), for each induced complex structure \(I\), is determined by the \((0,1)\)-part with respect to \(I\) of the hyperholomorphic connection over \(U\).

Let \(F_{I_0}\) be a reflexive coherent sheaf admitting a hyperholomorphic connection. Assume that \(F_{I_0}\) has an isolated singularity at a point \(x\) of \(M\). We get a reflexive coherent sheaf \(F_I\) on \((M, I)\), with an isolated singularity at \(x\), for each complex structure \(I\) in the twistor family. Let \(\beta_I : \tilde{M}_I \to (M, I)\) be the blow-up at the point \(x\) and \(D_I := \mathbb{P}(T^E_x(M, I))\) the exceptional divisor in \(\tilde{M}_I\).

Denote by \(\mathbb{H}\) the subalgebra of global endomorphisms of the real tangent bundle \(T^R M\), which is isomorphic to the algebra of quaternions, associated to the hyper-Kähler structure. Note that \(I\) is an element of \(\mathbb{H}\). Let \(\mathbb{H}^*\) be the multiplicative group of invertible elements of \(\mathbb{H}\). Given \(g \in \mathbb{H}^*\), set \(g(I) := gIg^{-1}\). Then \(g : (T^E_x M, I) \to (T^E_x M, g(I))\) is an isomorphism of complex vector spaces, which descends to an isomorphism

\[\tilde{g} : D_I \to D_{g(I)}\]

of the exceptional divisors of \(\tilde{M}_I\) and \(\tilde{M}_{g(I)}\).
Theorem 3.1. [Ve5 Theorems 6.1 and 8.15] The sheaf \( \tilde{F}_I := (\beta_I^* F_I)/\text{torsion} \) is locally free, for all induced complex structures \( I \). Furthermore, \( \tilde{g}^*(\tilde{F}_{g(I)})|_{D_{g(I)}} \) is isomorphic to \( (\tilde{F}_I)|_{D_I} \), for all \( g \in H^* \).

Proof. The sheaf \( \tilde{F}_I \) is locally free, by [Ve5 Theorem 6.1]. Set \( Q := (\mathbb{T}_x^\mathbb{R} M \setminus 0)/\mathbb{H}^* \). \( Q \) is isomorphic to the quaternionic projective space. Let \( \mathbb{C}^*_I \) be the subgroup of \( \mathbb{H}^* \) associated to the complex structure \( I \), so that \( D_I = (\mathbb{T}_x^\mathbb{R} M \setminus 0)/\mathbb{C}^*_I \). The quotient map \( (\mathbb{T}_x^\mathbb{R} M \setminus 0) \to Q \) factors through the quotient map

\[
q_I : D_I \to Q.
\]

Verbitsky constructs a finite set of complex vector bundles \( \{B_j\} \) over \( Q \), with special connections \( \nabla_j \) and non-negative integers \( k_j \) with the following properties. Endow the pullback \( q_I^* B_j \) with the holomorphic structure associated to \( q_I^* \nabla_j \). Then the direct sum \( \oplus_j (q_I^* B_j) \otimes \mathcal{O}_{D_I}(k_j) \) is isomorphic to \( (\tilde{F}_I)|_{D_I} \), for every complex structure \( I \) in the twistor family [Ve5 Theorem 8.15]. The isomorphism between \( (\beta_I^* F_I)|_{D_I} \) and \( \tilde{g}^*(\beta_{g(I)}^* F_{g(I)})|_{D_{g(I)}} \) would thus follow from the equality of \( q_I^*(B_j, \nabla_j) \) and \( \tilde{g}^* q_{g(I)}^*(B_j, \nabla_j) \). The latter equality follows from the equality \( q_I = q_{g(I)}\tilde{g} \). \( \square \)

3.2. Preservation of Condition 1.6 (5).

Proposition 3.2. Let \( E \) be a reflexive sheaf of Azumaya algebras over \( X^d \) satisfying Condition 1.6 (5). Assume that \( E \) is \( \omega \)-hyperholomorphic. Let \( \pi : \mathcal{X} \to \mathbb{P}^1_\omega \) be the twistor deformation of \( X \). Then the hyperholomorphic deformation \( \mathcal{E} \) of \( E \) is a (flat) family of reflexive sheaves of Azumaya algebras over \( \mathcal{X}_\pi^d \to \mathbb{P}^1_\omega \) (Definition 2.11). Furthermore, \( \mathcal{E}_t \) satisfies Condition 1.6 (5), for all \( t \in \mathbb{P}^1_\omega \).

Proof. The sheaf \( \mathcal{E} \) is reflexive, by Verbitsky’s construction. \( \mathcal{E} \) admits the structure of a reflexive sheaf of Azumaya algebras, extending that of \( E \), by [Ma5 Lemma 6.5 (3)].

Step 1: We prove first the flatness of \( \mathcal{E} \). Let \( \mathcal{E}_0 \subset \mathcal{E} \) and \( E_0 \subset E \) be the kernels of the trace homomorphisms. Flatness of \( \mathcal{E} \) would follow once we prove that \( \mathcal{E}_0 \) is flat over \( \mathbb{P}^1_\omega \).

Let \( F \) and \( W \) be as in Condition 1.6 (5). Set

\[
A_0 := \text{End}_0(\beta^* F/\text{torsion}).
\]

Then \( \beta_\ast A_0 \) is reflexive, by Lemma 2.3 (1). Now \( E_0 \) and \( \beta_\ast A_0 \) are isomorphic over the complement of \( \Delta \). Hence, \( \beta_\ast A_0 \) is isomorphic to \( E_0 \), as both are reflexive.

The vanishing in equation (1.9) implies that the homomorphism \( \beta_\ast (A_0 \otimes \mathcal{O}_Y(-kD)) \to \beta_\ast A_0 \) is an isomorphism. Furthermore, the co-unit of the adjunction \( \beta^* \dashv \beta_* \) induces an isomorphism

\[
(\beta^* E_0)/\text{torsion} = (\beta^* \beta_\ast [A_0 \otimes \mathcal{O}_Y(-kD)])/\text{torsion} \to A_0 \otimes \mathcal{O}_Y(-kD),
\]

by Lemma 2.3 (2), and the assumption that \( \text{End}_0(W) \otimes \mathcal{O}_D(k) \) is generated by global sections.

Let \( \tilde{\Delta} \subset \mathcal{X}_\pi^d \) be the relative diagonal and \( \tilde{\beta} : \mathcal{Y} \to \mathcal{X}_\pi^d \) the blow-up of \( \mathcal{X}_\pi^d \) centered at \( \tilde{\Delta} \). Given a complex structure \( I \in \mathbb{P}^1_\omega \), denote by \( \beta_I : \mathcal{Y}_I \to \mathcal{X}_\pi^d \) the blow-up of the diagonal \( \Delta_I \) in \( \mathcal{X}_\pi^d \). Denote by \( D \subset \mathcal{Y} \) the exceptional divisor and let \( D_I \) be its fiber over \( I \). Denote by \( \mathcal{E}_{0,I} \) the restriction of \( \mathcal{E}_0 \) to \( \mathcal{X}_\pi^d \).

The fiber \( f_x \) of the differentiable projection \( \mathcal{X} \cong X \times S^2 \to X \) over a point \( x \in X \) is a holomorphic section of the twistor family \( \pi : \mathcal{X} \to \mathbb{P}^1_\omega \), called a horizontal section [IKLR Sec. 3(F)]. The preimage of \( f_x \), via the projection \( \mathcal{X}_\pi^d \to \mathcal{X} \) onto the \( d \)-th factor, is the
twistor deformation \( \mathcal{X}^{d-1}_n \rightarrow \mathbb{P}^1_\omega \). The restriction of \( E_0 \) to \( X^{d-1} \times \{ x \} \) is a hyperholomorphic reflexive sheaf and the restriction of \( E_0 \) to \( \pi^{-1}(f_x) \) is its hyperholomorphic deformation along \( \mathcal{X}^{d-1}_n \rightarrow \mathbb{P}^1_\omega \). The projection \( \tilde{\pi} : \mathcal{X}^d_n \rightarrow \mathcal{X} \) onto the \( d \)-th factor is a smooth and proper morphism with hyperkähler fibers, each intersecting transversally the singular locus \( \tilde{D} \) of \( E \) at a single point. Theorem 3.1 states that \( \left[ (\tilde{\beta}^*E_0)_{|\tilde{\beta}^{-1}(\pi^{-1}(f_x))} \right] /_{\text{torsion}} \) is locally free. It follows that \( (\tilde{\beta}^*E_0)/_{\text{torsion}} \) is locally free over an open subset of \( \mathcal{X}^d_n \), by Nakayama’s lemma. Theorem 3.1 thus applies to conclude that the pullback \( \tilde{\beta}^*(E_0)/_{\text{torsion}} \) is locally free, and the isomorphism class of its restriction to each fiber \( D_z \), of \( D \) over \( z \in \tilde{\Delta} \), belongs to the same \( \text{Aut}(D_z) \)-orbit. This restriction is isomorphic to \( \mathcal{E}_{0|Y}(W) \otimes \mathcal{O}_{D_z}(k) \), by the computation for the special fiber in Equation (3.2). It follows that the restriction of each \( \mathcal{E}_{0,I} \), \( I \in \mathbb{P}^1_\omega \), to the fibers of \( p : D \rightarrow \tilde{\Delta} \) has the properties in Condition 1.10 (5) with the same \( W \). Set

\[
(3.3) \quad A_0 := (\tilde{\beta}^*E_0/_{\text{torsion}}) \otimes \mathcal{O}_Y(kD).
\]

We get that both \( \tilde{\beta}_*A_0 \) and \( \tilde{\beta}_*[A_0 \otimes \mathcal{O}_Y(-kD)] \) are isomorphic to \( E_0 \), by the same argument used for the special fiber. The higher sheaf cohomologies of \( \mathcal{E}_{0|Y}(W) \otimes \mathcal{O}_{D_z}(k) \) vanish, by Equation (1.10). Hence, the higher direct image sheaves \( R^i\tilde{\beta}_*[A_0 \otimes \mathcal{O}_Y(-kD)] \) vanish, for \( i > 0 \), and \( E_0 \) is flat over \( \mathbb{P}^1_\omega \), by Lemma 2.2.

Step 2: We lift next the structure of an Azumaya algebra from \( E \) to the locally free sheaf \( A := A_0 \oplus \mathcal{O}_Y \). Such a structure exists over \( U := \mathcal{Y}\setminus \mathcal{D} \), as \( A \) coincides with the pullback of \( E \) over that open set. We need to show that the multiplication homomorphism \( m : (A \otimes A)|_U \rightarrow A|_U \) extends to a regular homomorphism \( \tilde{m} \) over \( \mathcal{Y} \). The extension is clear for the restriction of \( m \) to the summands \( A_0 \otimes \mathcal{O}_Y, \mathcal{O}_Y \otimes A_0, \) and \( \mathcal{O}_Y \otimes \mathcal{O}_Y \). Hence, it suffices to prove that the restriction of \( m \) to \( A_0 \otimes A_0 \) extends. The latter decomposes as the sum of two meromorphic sections:

\[
\tilde{m}_1 : A_0 \otimes A_0 \rightarrow \mathcal{O}_Y,
\]

\[
\tilde{m}_2 : A_0 \otimes A_0 \rightarrow A_0.
\]

Denote by \( m_i \) the corresponding summand of \( m \) over \( U \). Then \( m_1 \) corresponds to the isomorphism \( m_1 : E \rightarrow E^* \). The regularity of \( \tilde{m}_1 \) is equivalent to an extension of \( m_1 \) to an isomorphism \( \tilde{m}_1 : A_0 \rightarrow A_0 \). Note that the determinant line bundle \( \text{det}(A_0) \) is trivial, as it restricts to a trivial line bundle over \( U \) and over the special fiber \( X^{d} \), by the isomorphism in Equation (3.2). Furthermore, the polar divisor of \( \tilde{m}_1 \) is disjoint from \( U \) and from the special fiber. The polar divisor must be empty, since \( \mathcal{D} \) is irreducible and meets the special fiber. Hence, \( \tilde{m}_1 \) extends to an isomorphism over the whole of \( \mathcal{Y} \).

The summand \( m_2 \) comes from a homomorphism \( m_2 \) in \( \text{Hom}(\mathcal{E}_0, \mathcal{H}om(\mathcal{E}_0, \mathcal{E}_0)) \). Dualizing, we get the homomorphism \( m_2^* \) in \( \text{Hom}(\mathcal{H}om(\mathcal{E}_0, \mathcal{E}_0), \mathcal{E}_0) \). The equality (3.3) induces a homomorphism \( \tilde{\beta}^*(m_2^*) \) in

\[
\text{Hom}(\tilde{\beta}^*\mathcal{H}om(\mathcal{E}_0, \mathcal{E}_0), A_0(-kD)).
\]

\(^{5}\text{Aut}(D_z) \) acts via pullback, and an \( \text{Aut}(D_{z_i}) \)-orbit is equal to an \( \text{Aut}(D_{z_2}) \)-orbit, if there exists an isomorphism \( D_{z_2} \rightarrow D_{z_1} \), which pulls back the restriction of \( \tilde{\beta}^*(E_0)/_{\text{torsion}} \) to \( D_{z_1} \) to the restriction of \( \tilde{\beta}^*(E_0)/_{\text{torsion}} \) to \( D_{z_2} \). The equality of the two orbits follows directly from Theorem 3.1 when \( z_1 \) and \( z_2 \) both belong to \( f_x \), for the same point \( x \) in the special fiber \( X \) of the twistor family. The equality of the two orbits holds, by assumption, if \( z_1 \) and \( z_2 \) both belong to the diagonal \( \Delta \) of \( X^{d} \), since \( E \) is assumed to satisfy Condition 1.10 (5). Hence it holds for all \( z_1 \) and \( z_2 \) in \( \tilde{\Delta} \).
Hence, the regularity of the extension \( \tilde{m}_2 \) would follow, once we construct a regular homomorphism
\[
\text{Hom}(A_0, A_0(-kD)) \to \tilde{\beta}^* \text{Hom}(E_0, E_0)
\]
extending the identity homomorphism over \( U \). We have the isomorphisms
\[
\text{Hom} \left( \text{Hom}(A_0, A_0(-kD)), \tilde{\beta}^* \text{Hom}(E_0, E_0) \right) \cong \text{Hom} \left( \tilde{\beta}_* \left[ \omega_{\beta} \otimes \text{Hom}(A_0, A_0(-kD)) \right], \text{Hom}(E_0, E_0) \right);
\]
by Grothendieck-Verdier duality. Now, \( \omega_{\beta} \) is isomorphic to \( O_Y((c-1)D) \), where \( c \) is the codimension \((d-1)\dim(X)\) of \( \Delta \) in \( X^d \). We have the isomorphisms
\[
\tilde{\beta}_* \left[ \text{Hom}(A_0, A_0((c-1-k)D)) \right] \cong \tilde{\beta}_* \left[ \text{Hom}(\tilde{\beta}^*E_0, A_0((c-1-2k)D)) \right] \cong \\
\text{Hom}(E_0, \tilde{\beta}_* [A_0((c-1-2k)D)]),
\]
where the first follows from Equation (3.3) and the second from the adjunction \( \tilde{\beta}^* \dashv \tilde{\beta}_* \). We have a regular homomorphism
\[
\text{Hom}(E_0, \tilde{\beta}_* [A_0((c-1-2k)D)]) \to \text{Hom}(E_0, E_0),
\]
since \( \tilde{\beta}_*[A_0((j)D)] \) is a subsheaf of \( E_0 \) for every integer \( j \), by Equation (1.9). This completes the construction of the homomorphism (3.4), and hence the proof of the regularity of \( \tilde{m}_2 \), as well as the construction of the multiplication \( \tilde{m}_n \). The axioms of an Azumaya algebra are satisfied by \( \tilde{m}_n \), since they hold over the dense open subset \( U \). Hence, \( A \) is an Azumaya algebra over \( Y \).

Step 3: We show next that \( E \) arises via the construction of Corollary 2.12. This will complete the proof that \( E \) is a family of reflexive sheaves of Azumaya algebras over \( X^d \to \mathbb{P}^1_\omega \). Let \( V \) be a (possibly twisted) locally free sheaf over \( Y \), such that \( A \) is isomorphic to \( \mathcal{E}nd(V) \). Let \( W \) be the restriction of \( V \) to \( D \). Let \( z \) be a point in the diagonal of the special fiber \( X^d \) and \( D_z \) the fiber of \( D \) over \( z \). The assumption that \( E \) satisfies Condition 1.6 (5) states that the sheaf \( \beta^*(F)/\text{torsion} \) is locally free over \( Y \). The Azumaya algebra \( A := \mathcal{E}nd(\beta^*(F)/\text{torsion}) \) is, by construction, the restriction of the Azumaya algebra \( A \) to \( Y \). Hence, the restriction \( W_2 \) of \( V \) to \( D_z \) is isomorphic to \( W \otimes O_{D_z}(j) \), for some integer \( j \), where \( W \) is the restriction of \( \beta^*(F)/\text{torsion} \) to \( D_z \). We may assume that \( j = 0 \), possibly after replacing \( V \) by \( V(-jD) \).

We have already seen that \( A \) restricts as the same Azumaya algebra \( \mathcal{E}nd(W) \) to the fibers of \( p : D \to \Delta \), modulo pullback by automorphisms of the fibers. Hence, \( V \) restricts as the same vector bundle \( W \) to fibers of \( p \), again modulo pullback by automorphisms of the fibers. The vector bundle \( W \) is assumed to have the properties of Condition 1.6 (5). We conclude that \( V \) satisfies the assumptions of Corollary 2.12. Set \( F := \tilde{\beta}_* V \). We conclude that \( F \) is a family of reflexive sheaves over \( X^d \to \mathbb{P}^1_\omega \) and \( \beta_* A \cong \mathcal{E}nd(F) \), by Corollary 2.12. We already know that \( E \) is isomorphic to \( \tilde{\beta}_* A \). Consequently, \( E \) is a family of reflexive sheaves of Azumaya algebras over \( X^d \to \mathbb{P}^1_\omega \) (Definition 2.11).

The fibers \( E_t \) satisfy Condition 1.6 (5), except possibly the tightness condition, for all \( t \in \mathbb{P}^1_\omega \), since \( \tilde{\beta}^* F/\text{torsion} \) is isomorphic to \( V(-mD) \), by Lemma 2.8 (2). The tightness condition will be proven in section 3.3. ∎
3.3. **Locally trivial families of reflexive Azumaya algebras.** Let $X$ be an irreducible holomorphic symplectic manifold, $F$ a (possibly twisted) reflexive sheaf over $X^d$, and let $\beta : Y \to X^d$ be the blow-up of the diagonal image $\Delta$ of $X$. Assume that $F$ satisfies Condition 1.6, with the possible exception of the $\beta$-tightness condition. Then $G := (\beta^*F)/\text{torsion}$ is locally free. Let $\mathcal{D}^1(G)$ be the sheaf of differential operators of order $\leq 1$ with scalar symbol.

**Lemma 3.3.**

(1) The sheaf $R^1\beta_*(\mathcal{D}^1(G))$ vanishes.

(2) The vector space $H^0(X^d; R^1\beta_*\mathcal{E}nd(G))$ vanishes.

**Proof.** (1) Consider the short exact sequence of the symbol map

$$0 \to \mathcal{E}nd(G) \to \mathcal{D}^1(G) \to TY \to 0.$$ 

Let $e : D \to Y$ be the inclusion of the exceptional divisor $D := \mathbb{P}(N\Delta/X^d)$ and let $p : D \to \Delta$ be the natural morphism. Let $\iota : \Delta \to X^d$ be the inclusion. We have the short exact sequence

$$0 \to TY \to \beta^*T[X^d] \to e_*Q \to 0,$$

where $Q := (p^*N\Delta/X^d)/\mathcal{O}_D(-1)$ is the tautological quotient bundle (this is well known and it will be proven in detail in Diagram (4.2)). Hence, $\beta_*(e_*Q)$ is isomorphic to $\iota_*N\Delta/X^d$ and the homomorphism $\beta_*(\beta^*T[X^d]) \to \beta_*(e_*Q)$ is surjective. The sheaf $R^1\beta_*(\beta^*T[X^d])$ vanishes. The sheaf $R^1\beta_*TY$ thus vanishes. We get the exact sequences

$$0 \to \beta_*\mathcal{E}nd(G) \to \beta_*\mathcal{D}^1(G) \to \beta_*TY \xrightarrow{\gamma} R^1\beta_*(\mathcal{E}nd(G)) \to R^1\beta_*(\mathcal{D}^1(G)) \to 0,$$

$$0 \to \beta_*TY \to T[X^d] \to \iota_*N\Delta/X^d \to 0.$$

The latter is obtained via push-forward of the short exact sequence (3.5) above.

It remains to prove that the homomorphism $\gamma$ in Equation (3.6) above is surjective. Consider the commutative diagram with short exact rows:

$$\begin{array}{ccc}
0 & (\beta_*TY) \cdot I_{\Delta} & T[X^d] \otimes I_{\Delta} \\
\downarrow & \downarrow & \downarrow \\
0 & \beta_*TY & T[X^d] \\
\downarrow & \downarrow & \downarrow \\
0 & \iota_*N\Delta/X^d & 0.
\end{array}$$

The left and middle vertical homomorphisms are injective, and the right one vanishes. The snake lemma yields the long exact sequence

$$0 \to \iota_* \left[ N\Delta/X^d \otimes N\Delta/X^d \right] \to \iota_* e^* \beta_* TY \to \iota_* e^* T[X^d] \to \iota_* N\Delta/X^d \to 0.$$ 

We get the short exact sequence over $\Delta$

$$0 \to N\Delta/X^d \otimes N\Delta/X^d \to e^* \beta_* TY \to T\Delta \to 0. $$

The restriction homomorphism $TY \to e_* e^* TY$ induces the homomorphism

$$\beta_* (TY) \to \beta_* e_* e^* TY = \iota_* p_* e^* TY.$$ 

We claim that the latter homomorphism is surjective. Indeed $e^* TY$ fits in the short exact sequence

$$0 \to TD \to e^* TY \to \mathcal{O}_D(D) \to 0.$$
Now \( p_*(\mathcal{O}_D(D)) \) vanishes. Hence, \( p_*e^*TY \) is isomorphic to \( p_*TD \). The tangent bundle of the exceptional divisor fits in

\[
0 \to T_p \to TD \to p^*T\Delta \to 0.
\]

The sheaf \( R^1p_*T_p \) vanishes. We get the short exact sequence

\[
0 \to p_*T_p \to p_*e^*TY \to T\Delta \to 0.
\]

Note that \( p_*T_p \) is naturally isomorphic to \( \mathcal{E}nd_0(N_{\Delta/X^d}) \). We conclude the surjectivity of \( (3.5) \) from comparison of the short exact sequence above to the short exact sequence \( (3.7) \).

The sheaf \( R^i\beta_*(\mathcal{E}nd(G)(-D)) \) vanishes for \( i > 0 \), by Equation \( (1.10) \) in Condition \( 1.6 \). We get the isomorphism

\[
(3.9)
R^1\beta_*(\mathcal{E}nd(G)) \cong \iota_*R^1p_*(\mathcal{E}nd(e^*G))
\]

from the short exact sequence

\[
0 \to \mathcal{E}nd(G)(-D) \to \mathcal{E}nd(G) \to e_*e^*\mathcal{E}nd(G) \to 0.
\]

We conclude that the homomorphism \( \gamma \) factors through the pushforward \( \iota_* (\tilde{\gamma}) \) of a homomorphism \( \tilde{\gamma} : p_*e^*TY \to R^1p_*(\mathcal{E}nd(e^*G)) \), via the surjective homomorphism \( (3.8) \). It suffices to prove that \( \tilde{\gamma} \) is surjective. We have seen that \( p_*e^*TY \) is naturally isomorphic to \( p_*TD \). Denote by

\[
\tilde{a} : p_*TD \to R^1p_*(\mathcal{E}nd(e^*G))
\]

the pullback of \( \tilde{\gamma} \). Clearly, \( \tilde{a} \) corresponds to the infinitesimal pullback action on \( e^*G \) of local automorphisms of the exceptional divisor \( D \).

Let \( W_z \) be the restriction of \( G \) to the fiber \( D_z \) of \( p \) over \( z \in \Delta \). The assumed infinitesimal rigidity of the pairs \( (D_z, W_z) \) in Condition \( 1.6 \) implies that \( R^1p_*(\mathcal{E}nd(e^*G)) \) is locally free and the homomorphism

\[
(3.10)
a : \mathcal{E}nd_0(N_{\Delta/X^d}) \to R^1p_*(\mathcal{E}nd(e^*G))
\]

is surjective, where \( a \) is the infinitesimal action of the automorphism groups of the fibers of \( p \). Now \( a \) is the restriction of \( \tilde{a} \) and so \( \tilde{a} \) is surjective as well. It follows that \( \tilde{\gamma} \) is surjective, which implies that so is \( \gamma \), by the reduction established above.

[2] The vector bundle \( N_{\Delta/X^d} \) is isomorphic to \( T\Delta \otimes_{\mathbb{C}} \mathbb{C}^{d-1} \) and thus \( SL(d-1) \) naturally embeds in the automorphism group of the normal bundle. The bundle \( \ker(a) \) contains the trivial \( \mathfrak{sl}(d-1) \) subbundle of \( \mathcal{E}nd_0(N_{\Delta/X^d}) \), by the assumed invariance of the bundles \( W_z \) in Condition \( 1.6 \) \( [5] \). It follows that the homomorphism \( H^0(\Delta, \ker(a)) \to H^0(\Delta, \mathcal{E}nd_0(N_{\Delta/X^d})) \) is surjective. The vector bundle \( TX \) of an irreducible holomorphic symplectic manifold is stable (with respect to every Kähler class). Hence, \( \mathcal{E}nd_0(N_{\Delta/X^d}) \) is polystable. The kernel \( \ker(a) \) is a subbundle of degree 0, by the \( sp(T\gamma X) \)-equivariance assumption in Condition \( 1.6 \) \( [5] \). Thus, \( \mathcal{E}nd_0(N_{\Delta/X^d}) \) decomposes as the direct sum of the kernel and image of \( a \). We conclude the vanishing of \( H^0(R^1p_*(\mathcal{E}nd(e^*G))) \). The vanishing of \( H^0(X^d, R^1\beta_*(\mathcal{E}nd(G))) \) follows from Equation \( (3.9) \).

We keep the notation of Proposition \( 3.2 \). In particular, \( \mathcal{E} \) is the hyperholomorphic deformation of \( E \) over the \( d \)-th fiber product of the twistor family \( \pi : \mathcal{X} \to \mathbb{P}^1_{\omega} \). Let \( \mathcal{F} \) be a family of reflexive sheaves over \( \varphi : \mathcal{X}^d_{\pi} \to \mathbb{P}^1_{\omega} \), such that \( \mathcal{E} \cong \mathcal{E}nd(\mathcal{F}) \). Let \( 0 \in \mathbb{P}^1_{\omega} \) be the point corresponding to the complex structure of \( X \), so that \( \mathcal{E}_0 \) is \( E \). We have the blow-up morphism \( \tilde{\beta} : \mathcal{Y} \to \mathcal{X}^d_{\pi} \).
and $\mathcal{F}$ is isomorphic to $\tilde{\beta}\mathcal{V}$, where $\mathcal{V}$ is a locally free sheaf over $\mathcal{Y}$, such that $\tilde{\beta}^*\mathcal{F}/\text{torsion}$ is isomorphic to $\mathcal{V}(-mD)$, where $D$ is the exceptional divisor and $m$ is the integer in Equation (1.7). Let $W$ be the restriction of $\mathcal{V}$ to $\mathcal{D}$.

**Proposition 3.4.** Assume that $E$ satisfies Condition (1.6) (5), with the possible exception of the $\beta$-tightness condition. Then the family $\mathcal{E}$ is locally, in the topology of $\mathcal{X}^d_{\pi}$, trivial over $\mathbb{P}^1$. In other words, for every point $x \in \mathcal{X}^d_{\pi}$ there are open neighborhoods $U$ of $x$ and $\overline{U}$ of $\varphi(x)$, an isomorphism $f : U \to \overline{U} \times (U \cap \mathcal{X}^d_{\varphi(x)})$, such that $\varphi \circ f^{-1}$ is the projection to $\overline{U}$, and an isomorphism

$$\tilde{f} : \iota_U^*\mathcal{E} \to \pi_2^*\iota_{\overline{U}}^*\mathcal{E},$$

where $\iota_U : U \to \mathcal{X}^d_{\pi}$ and $\iota_{\overline{U}} : U \cap \mathcal{X}^d_{\varphi(x)} \to \mathcal{X}^d_{\pi}$ are the inclusions and $\pi_2$ is the projection from $\overline{U} \times (U \cap \mathcal{X}^d_{\varphi(x)})$ onto the second factor.

**Proof.** $\mathcal{E}$ is locally free away from the diagonal. Hence, it suffices to prove the statement locally around diagonal points of $\mathcal{X}^d_{\pi}$. For every point $x \in \mathcal{X}$ there exists an open neighborhood $U_1 \subset \mathcal{X}$, an open contractible neighborhood $\overline{U} \subset \mathbb{P}^1$ of $\pi(x)$, an isomorphism $f_1 : \overline{U} \times (U_1 \cap \mathcal{X}^d_{\pi(x)}) \to U_1$, such that $\pi \circ f_1$ is the projection onto $\overline{U}$, and an isomorphism

$$(3.11) \quad \tilde{f}_1 : \pi_1^*(\mathcal{W}|_{\mathcal{Y}_1 \cap \mathcal{X}^d_{\pi}}) \to \mathcal{W}|_{\mathcal{Y}_1 \cap \mathcal{X}^d_{\pi}}$$

over the open subset $p^{-1}(U_1)$ of $\mathcal{D}$, as in Equation (1.6), by Condition (1.6) (5). We get the open neighborhood $U := (U_1)^{\pi}_{\pi}$ of the diagonal image of $x$ in $\mathcal{X}^d_{\pi}$. We may assume that $U_1$ and $\overline{U}$ are Stein. Then so is $U$.

A theorem of Flenner and Kosarew states that for each $t \in \overline{U}$ there exists a maximal analytic subspace $S \subset \overline{U}$ containing $t$, such that the restriction of $\mathcal{E}$ to the subspace $\varphi^{-1}(S) \cap U$ is isomorphic to a trivial family, i.e., to the pullback of a coherent sheaf over $\mathcal{X}^d_{\pi} \cap U$ via the projection $U \to \mathcal{X}^d_{\pi} \cap U$ induced by $f_1$ [FK, Remark 5.4(2)]. Now $\overline{U}$ is smooth and one-dimensional. Hence, either $S$ is a fat subscheme supported on the point $t$, or $S$ contains an open neighborhood of $t$ in $\overline{U}$. Let $z$ be a local parameter at $t$ in $\overline{U}$ and let $S_k \subset \overline{U}$ be $\text{Spec}(\mathbb{C}[z]/(z^{k+1}))$. It suffices to prove that the restriction of $\mathcal{E}$ to the subspace $\varphi^{-1}(S_k) \cap U$ is isomorphic to a trivial family, for all $k > 0$.

Set $\tilde{\mathcal{U}} := \tilde{\beta}^{-1}(U)$. The trivialization $f_1$ of $U_1 \to \overline{U}$ induces a trivialization of the blow-up $f : \tilde{\mathcal{U}} \times (\mathcal{Y}_1 \cap \tilde{\mathcal{U}}) \to \tilde{\mathcal{U}}$, such that $\varphi \circ \tilde{\beta} \circ f$ is the projection onto $\overline{U}$. Let $\psi : \tilde{\mathcal{U}} \to \mathcal{Y}_1 \cap \tilde{\mathcal{U}}$ be the projection induced by $f$. We have the left exact sequence of set valued sheaves

$$0 \to H^1(\mathcal{Y}_1 \cap \tilde{\mathcal{U}}, \psi^*_sGL_r(\mathcal{O}_{\tilde{\mathcal{U}}})) \to H^1(\tilde{\mathcal{U}}, GL_r(\mathcal{O}_{\tilde{\mathcal{U}}})) \to H^0(\mathcal{Y}_1 \cap \tilde{\mathcal{U}}, R^1\psi^*_sGL_r(\mathcal{O}_{\tilde{\mathcal{U}}}))$$

The sheaf $R^1\psi^*_sGL_r(\mathcal{O}_{\tilde{\mathcal{U}}})$ is trivial, since the inverse image via $\psi$ of a contractible Stein open subset is contractible and Stein, and every vector bundle over a contractible Stein manifold is trivial, by Grauert’s Theorem [Ca, Theorem A1]. Let $\mathcal{V}$ be a vector bundle over $\mathcal{Y}_1$, such that $\mathcal{F}$ is isomorphic to $\tilde{\beta}_s\mathcal{V}$ and $W$ is the restriction of $\mathcal{V}$ to $\mathcal{D}$. We conclude that the restriction of $\mathcal{V}$ to $\tilde{\mathcal{U}}$ is represented by a cohomology class in $H^1(\mathcal{Y}_1 \cap \tilde{\mathcal{U}}, \psi^*_sGL_r(\mathcal{O}_{\tilde{\mathcal{U}}}))$.

Set $\tilde{U}_k := \tilde{\mathcal{U}} \cap (\varphi \circ \tilde{\beta})^{-1}(S_k)$. Denote the restriction of $\psi$ by $\psi_k : \tilde{U}_k \to (\mathcal{Y}_1 \cap \tilde{\mathcal{U}})$. Then the restriction of $\mathcal{V}$ to $\tilde{U}_k$ is represented by a class $c_k$ in $H^1(\mathcal{Y}_1 \cap \tilde{\mathcal{U}}, \psi^*_kGL_r(\mathcal{O}_{\tilde{U}_k}))$. It suffices to prove...
that \( c_k \) belongs to the image of \( H^1(\mathcal{Y}_t \cap \tilde{U}, GL_r(O_{\mathcal{Y}_t \cap \tilde{U}})) \) via the pullback sheaf homomorphism

\[
\psi^*_k : GL_r(O_{\mathcal{Y}_t \cap \tilde{U}}) \to \psi_{k,*}GL_r(O_{\tilde{U}_k}).
\]

The proof is by induction on \( k \). The case \( k = 0 \) is clear. Assume that \( k > 0 \) and the statement holds for \( k - 1 \). Denote by \( \mathcal{V}_0 \) the restriction of \( \mathcal{V} \) to \( \mathcal{Y}_t \cap \tilde{U} \). Note that \( \mathcal{V}_0 \) is represented by the class \( c_0 \). We have the short exact sequence

\[
0 \to z^k : GL_r(O_{\tilde{U}_k}) \to \psi_{k,*}GL_r(O_{\tilde{U}_k}) \to \psi_{k-1,*}GL_r(O_{\tilde{U}_{k-1}}) \to 0.
\]

Two 1-cocycles \( \theta_k \) and \( \tilde{\theta}_k \) of the sheaf \( \psi_{k,*}GL_r(O_{\tilde{U}_k}) \) map to the same cocycle \( \theta_{k-1} \) of the sheaf \( \psi_{k-1,*}GL_r(O_{\tilde{U}_{k-1}}) \), which restricts to \( \mathcal{Y}_t \cap \tilde{U} \) as a 1-cocycle \( \theta_0 \) representing \( \mathcal{V}_0 \), if and only if \( \theta_k \) and \( \tilde{\theta}_k \) differ by a \( z^k \) multiple of a 1-cocycle of \( \operatorname{End}(\mathcal{V}_0) \). On the level of cohomology one checks that fibers of the restriction map

\[
H^1(\mathcal{Y}_t \cap \tilde{U}, \psi_{k,*}GL_r(O_{\tilde{U}_k})) \to H^1(\mathcal{Y}_t \cap \tilde{U}, \psi_{k-1,*}GL_r(O_{\tilde{U}_{k-1}}))
\]

are \( H^1(\mathcal{Y}_t \cap \tilde{U}, \operatorname{End}(\mathcal{V}_0)) \)-torsors. The classes \( c_k \) and \( \psi^*_k c_0 \) belong to the same fiber, by the induction hypothesis.

Let \( \psi_k : D \cap \tilde{U}_k \to D_t \cap \tilde{U} \) be the restriction of \( \psi_k \). Let \( \tilde{c}_k \) be the restriction of the class \( c_k \) to \( H^1(D_t \cap \tilde{U}, \psi_{k,*}GL_r(O_{D_t \cap \tilde{U}})) \). The classes \( \tilde{c}_k \) and \( \psi^*_k \tilde{c}_0 \) are equal, since we have the trivialization \( f_1 \) given in (3.11). Hence, the difference between \( c_k \) and \( \psi^*_k c_0 \) is a class in the kernel of the restriction homomorphism

\[
H^1(\mathcal{Y}_t \cap \tilde{U}, \operatorname{End}(\mathcal{V}_0)) \to H^1(D_t \cap \tilde{U}, \operatorname{End}(\mathcal{W}_0)).
\]

The latter fits as the middle vertical homomorphism in the following commutative diagram with left exact rows:

\[
\begin{array}{ccc}
H^1(X^d_t \cap U, \beta_* \operatorname{End}(\mathcal{V}_0)) & \longrightarrow & H^1(\mathcal{Y}_t \cap \tilde{U}, \operatorname{End}(\mathcal{V}_0)) \\
\downarrow & & \downarrow \\
H^1(\Delta_t \cap U, \beta_* \operatorname{End}(\mathcal{W}_0)) & \longrightarrow & H^1(D_t \cap \tilde{U}, \operatorname{End}(\mathcal{W}_0)).
\end{array}
\]

The spaces in the left column vanish, since \( \Delta_t \cap U \) and \( X^d_t \cap U \) are Stein. The right vertical homomorphism is an isomorphism, as shown in Equation (3.9). It follows that the middle vertical homomorphism is injective, and the classes \( c_k \) and \( \psi^*_k c_0 \) are equal.

**Completion of the proof of Proposition 3.2:** It remain to prove that \( F_t \) is \( \beta_t \)-tight (Definition 1.13). Now the set of \( t \in \mathbb{P}^1_{\omega} \), where \( F_t \) is \( \beta_t \)-tight is open and non-empty, since \( E \) is assumed to satisfy Condition Condition [1.6] [5]. The set of \( t \in \mathbb{P}^1_{\omega} \), where \( F_t \) is not \( \beta_t \)-tight, is open as well, by the local triviality Proposition 3.4. The latter set must be empty, since \( \mathbb{P}^1_{\omega} \) is connected.

**3.4. Completion of the proof of the preservation of Condition 1.6.** Keep the notation of Proposition 3.2.

**Proposition 3.5.** Assume that \( E \) satisfies Condition [1.6] [5]. Then \( H^0(X^d_t, \operatorname{Ext}^1(F_t, F_t)) \) vanishes, for all \( t \in \mathbb{P}^1_{\omega} \).
Proof. $F_t$ satisfies Condition 1.6 [3], for all $t \in \mathbb{P}^1$, by Proposition 3.2. Set $F := F_t$. Keep the notation of Lemma 3.3. The functor $R\beta_* : D^b(Y) \to D^b(X^d)$ maps the object $G \otimes \mathcal{O}_Y(-mD)$ to $F$, by Condition 1.6 [3] and Lemma 2.2. We get the induced homomorphism $\text{Hom}(G \otimes \mathcal{O}_Y(-mD), G \otimes \mathcal{O}_Y(-mD)[1]) \to \text{Hom}(F, F[1])$, and hence the homomorphism $	ext{Ext}^1(G, G) \to \text{Ext}^1(F, F)$. The analogous homomorphisms, over open subsets of $X^d$, induce the sheaf homomorphism $\beta_* : R^1\beta_*(\mathcal{E}nd(G)) \to \mathcal{E}xt^1(F, F)$. The latter fits in the following commutative diagram with exact rows:

\[
\begin{array}{c}
0 \longrightarrow \beta_*\mathcal{E}nd(G) \longrightarrow \beta_*\mathcal{D}^1(G) \longrightarrow \beta_\gamma \gamma \longrightarrow R^1\beta_*(\mathcal{E}nd(G)) \longrightarrow 0 \\
0 \longrightarrow \mathcal{E}nd(F) \longrightarrow \mathcal{D}^1(F) \longrightarrow \mathcal{T}X^d \longrightarrow 0.
\end{array}
\]

The left vertical homomorphism is an isomorphism, by Lemma 2.10. The homomorphism $\gamma$ is surjective, by Lemma 3.3 [0]. The equality $\text{ker}(a_F) = \text{Im}(\sigma_F)$ is seen as follows. We have the long exact sequence

\[
0 \to \mathcal{E}nd(F) \longrightarrow \mathcal{D}^1(F) \xrightarrow{\beta_*} \mathcal{E}nd(F) \otimes \mathcal{T}X^d \xrightarrow{a_F} \mathcal{E}xt^1(F, F),
\]

by definition of $\mathcal{D}^1(F)$ as $\mathcal{H}om(\mathcal{J}(1)^1(F), F)$. Now,

\[
\text{Im}(\sigma_F) = \text{Im}(\bar{\sigma}_F) \cap \text{id}_F \cdot \mathcal{T}X^d = \text{ker}(\bar{a}_F) \cap \text{id}_F \cdot \mathcal{T}X^d = \text{ker}(a_F).
\]

The homomorphism $a_F$ is surjective and the middle vertical homomorphism $\beta_\gamma$ in the above diagram is an isomorphism, by the $\beta$-tightness assumption in Condition 1.6 [3]. Note that the three left vertical arrows are injective and the left two are surjective. The Snake Lemma (applied to the quotient of Diagram (3.12) by its left column) yields that $\beta_*^1$ is injective and its co-kernel is $\iota_*\mathcal{N}_{\Delta/X^d}$.

\[
0 \to R^1(\beta_*\mathcal{E}nd(G)) \xrightarrow{\beta_*^1} \mathcal{E}xt^1(F, F) \longrightarrow \iota_*\mathcal{N}_{\Delta/X^d} \to 0.
\]

Now $H^0(\mathcal{N}_{\Delta/X^d})$ vanishes, and $H^0(X^d, R^1(\beta_*\mathcal{E}nd(G)))$ vanishes, by Lemma 3.3 [2]. We conclude the desired vanishing of $H^0(X^d, \mathcal{E}xt^1(F, F))$. □

The following tightness criterion will be needed in Section 7.

**Lemma 3.6.** Assume that $F$ satisfies Condition 1.6 [2], with the possible exception of $\beta$-tightness. Then $F$ is $\beta$-tight, if and only if the sheaf $\mathcal{E}xt^1(F, F)$ fits in the short exact sequence (3.13), and the resulting isomorphism $\psi : \text{coker}(\beta_*^1) \to \iota_*\mathcal{N}_{\Delta/X^d}$ is the inverse of the homomorphism $\bar{a}_F : \iota_*\mathcal{N}_{\Delta/X^d} \to \text{coker}(\beta_*^1)$ induced by the homomorphism $a_F$ in Diagram (3.12).

**Proof.** If $F$ is $\beta$-tight, then the sheaf $\mathcal{E}xt^1(F, F)$ fits in the short exact sequence (3.13), by the proof of Proposition 3.5. Assume that $\mathcal{E}xt^1(F, F)$ fits in the short exact sequence (3.13) and $\psi^{-1} = \bar{a}_F$. Then $a_F$ is surjective and $\beta_*^1$ is injective. It follows the $\sigma_F$ induces an isomorphism from the cokernel of $\beta_*$ onto the kernel of $\bar{a}_F$, by the Snake Lemma applied to the quotient of Diagram (3.12) by its left column. The kernel of $\bar{a}_F$ vanishes, since $\bar{a}_F$ is an isomorphism. Hence, $\beta_* : \beta_*\mathcal{D}^1(G) \to \mathcal{D}^1(F)$ is surjective. □
Proposition 3.7. (Conditional on Conjecture 1.7). Let $E$ be a reflexive sheaf of Azumaya algebras satisfying Condition 1.6 and $\omega$ a Kähler class, such that $E$ is $\omega$-slope-stable. Then every member $E_t$, $t \in \mathbb{P}_1^d$, of the twistor deformation of $E$ satisfies Condition 1.6.

Proof. (1) This part was established in Proposition 3.2.

(2) See [Ma5, Construction 6.7].

(3) Let $F_t$ be a reflexive sheaf over $X_t$ such that $E_t \cong \text{End}(F_t)$. We have the left exact sequence
\[
0 \to H^1(X_t, E_t) \to \text{Ext}^1(F_t, F_t) \to H^0(X_t, \text{Ext}^1(F_t, F_t)).
\]
The vanishing of $\text{Ext}^1(F_t, F_t)$ follows from Conjecture 1.7 and Proposition 3.5.

(4) This fact is due to Verbitsky [Ve3, Ma5, Cor. 6.10].

We provide next a criterion for the reduction of Conjecture 1.7 to Conjecture 1.12 in case $d = 2$. Let $E$ be a reflexive sheaf of Azumaya algebras over $X \times X$, satisfying Condition 1.6.

Let $F$ be a twisted reflexive sheaf, such that $E \cong \text{End}(F)$. Given a point $x \in X$, denote by $F_x$ the restriction of $F$ to $\{x\} \times X$. Let
\[
(3.14) \quad \kappa_x : T_x X \to \text{Ext}^1(F_x, F_x)
\]
be the Kodaira-Spencer homomorphism.

Lemma 3.8. (1) Assume that $E$ is not locally free along the diagonal and that the Kodaira-Spencer homomorphism $\kappa_x$ is surjective, for all $x \in X$. Then $H^1(E_x)$ vanishes for all $x \in X$.

(2) Assume that $F_x$ is $\omega$-slope stable and $H^1(E_x)$ vanishes for all $x \in X$. Then Conjecture 1.7 for $E$ follows from Conjecture 1.12.

Proof. (1) Let $\beta : Y \to X \times X$ be the blow-up of the diagonal $\Delta$. $F$ is $\beta$-tight, by Condition 1.6. Let $e_1 : T \Delta \to (TX^2)_{\Delta}$ be the inclusion as the first direct summand (not as the subbundle tangent to the diagonal). Set $G := \beta^*F/torsion$. Let $\beta^t_\Delta$ be the homomorphism in Diagram 3.12. The composition
\[
T \Delta \xrightarrow{\psi} (TX^2)_{\Delta} \to N_{\Delta/X^2} \xrightarrow{\beta^t_\Delta} \text{coker}(\beta^t_\Delta)
\]
is an isomorphism, since the right homomorphism is an isomorphism, by the short exact sequence 3.13. Restriction of the above isomorphism to $\{x\} \times X$ yields the isomorphism
\[
T_x X \cong (N_{\Delta/X^2})_x \cong H^0(X, \text{Ext}^1(F_x, F_x)/\beta^t_\Delta[R^1\beta_*\text{End}(G_x)]),
\]
where $G_x$ is the restriction of $G$ to the strict transform in $Y$ of $\{x\} \times X$ and $\beta$ now denotes the blow-up of $G$ at $x$. The above isomorphism coincides with the composition
\[
T_x X \xrightarrow{\kappa} \text{Ext}^1(F_x, F_x) \xrightarrow{\psi} H^0(X, \text{Ext}^1(F_x, F_x)) \to H^0(X, \text{Ext}^1(F_x, F_x)/\beta^t_\Delta[R^1\beta_*\text{End}(G_x)]).
\]
Hence, $\kappa_x$ is injective, and so an isomorphism. It follows that the natural homomorphism $\psi$ above is injective. The cohomology $H^1(X, \text{End}(F_x))$ thus vanishes, by the left exactness of the sequence
\[
0 \to H^1(X, \text{End}(F_x)) \to \text{Ext}^1(F_x, F_x) \xrightarrow{\psi} H^0(X, \text{Ext}^1(F_x, F_x)).
\]
Definition 4.1. Let \( f : V \to W \) be a morphism of smooth and compact analytic spaces. Define \( \mathcal{D}ef(V, f, W) : (\text{GAn}) \to (\text{Sets}) \) to be the deformation functor from the category of complex analytic germs to sets, which associates to a germ \( S \) the set of isomorphism classes of deformations \( \tilde{f} : V \to \tilde{W} \) over \( S \) of \( f \).

We shall consistently use the following notation below. For a complex analytic manifold \( V \), \( \mathcal{D}ef(V) \) will stand for the set-valued functor of flat deformations of \( V \) over complex analytic germs. The Kuranishi deformation space of \( V \) will be denoted \( \mathcal{D}V \).

Lemma 4.2. Let \( f : V \to W \) be a morphism as above satisfying the conditions

1. \( R^i f_* \mathcal{O}_V \cong \mathcal{O}_W \) for \( i > 0 \);
(2) \( H^0(W, T_W) = 0 \).

Assume that the Kuranishi deformation family of \( V \) is universal. Then, the forgetful morphism \( \alpha : \text{Def}(V, f, W) \to \text{Def}(V) \) of the two deformation functors is an isomorphism. In particular, \( \text{Def}(V, f, W) \) is representable by a universal deformation family.

**Proof.** The morphism \( \alpha \) is formally smooth, by hypothesis (1) and \([\text{Ran}]\) Theorem 3.3. Furthermore, the relative dimension of \( \alpha \) is \( \dim(H^0(V, f^*T_W)) \) ([\text{Ran}] (2.2)), which is 0 by hypothesis (2).

Denote by \( D_V \) and \( D_W \) the Kuranishi deformation spaces and let \( v : V \to D_V \) and \( w : W \to D_W \) be the Kuranishi families of \( V \) and \( W \). According to \([\text{KM}]\) Prop. 11.4, there are natural morphisms \( F \) and \( \hat{f} \) making the following diagram commutative

\[
\begin{array}{ccc}
V & \xrightarrow{\hat{f}} & W \\
v \downarrow & & \downarrow w \\
D_V & \xrightarrow{F} & D_W,
\end{array}
\]

and such that \( \hat{f}|_V = f \). Consequently, we get a family of morphisms

\[
\begin{array}{ccc}
V & \xrightarrow{\tilde{f}} & F^*W \\
\downarrow & & \downarrow \\
D_V & \xrightarrow{} & \end{array}
\]

We claim that the above family represents \( \text{Def}(V, f, W) \), which, of course, also implies the lemma. To see this, let us recall the construction of the family \((4.1)\). Denote by \( \Gamma \subset V \times W \) the graph of the morphism \( f \), and let \( D \) be the component of the relative Douady space of \( V \times W / D_V \times D_W \) parametrizing graphs of morphisms which contains \( \Gamma \). Kollar-Mori argue that the projection morphism

\[
pr : [\Gamma] \in D \to 0 \in D_V
\]

admits an analytic section \( \sigma : D_V \to D \), which yields \((4.1)\). Now, let \( \tilde{V} \xrightarrow{\tilde{g}} \tilde{W} \to T \) be a deformation of the morphism \( \tilde{f} \) over a base \( T \). This corresponds to a section \( \tilde{\sigma} \) of \( pr \) over \( T \), in the sense that it corresponds to a commutative diagram

\[
\begin{array}{ccc}
\tilde{V} & \xrightarrow{\tilde{\sigma}} & D \\
\tilde{g} \downarrow & & \downarrow pr \\
T & \xrightarrow{\kappa} & D_V
\end{array}
\]

But, as the morphism \( \alpha \) is formally smooth of relative dimension zero, the section \( \tilde{\sigma} \) must coincide with \( \sigma \circ \kappa \) in the formal neighborhood of \( 0 \in T \). Thus, the family \( \tilde{V} \xrightarrow{\tilde{g}} \tilde{W} \to T \) is isomorphic to the pullback of the family \((4.1)\) via the classifying morphism. This completes the proof of the claim. \( \Box \)

As in section \([1]\), let \( E \) be a reflexive sheaf of Azumaya algebras on \( X^d \) satisfying Condition \([1.6]\), \( F \) a reflexive twisted sheaf representing \( E \), and \( A = \text{End}(\beta^*F/tor) \) the locally-free sheaf of Azumaya algebras on \( Y \), the blow-up of \( X^d \) along the small diagonal.
Lemma 4.3. $A$ is rigid as a sheaf of Azumaya algebras.

Proof. $E$ is rigid by assumption, and $\beta_* A = E$ from Corollary 2.12. The space $H^0(X^d, R^1 \beta_* A)$ vanishes, by Lemma 3.3 (2). The Leray spectral sequence yields the left exact sequence

$$0 \to H^1(X^d, \beta_* A) \to H^1(Y, A) \to H^0(X^d, R^1 \beta_* A).$$

We conclude the vanishing of $H^1(Y, A)$. \hfill \qed

Definition 4.4. Let $A = End(\beta^* F/tor)$ be as above. Define $Def(Y, A) : (GAN) \to (Sets)$ to be the deformation functor from the category of complex analytic germs to sets, which associates to a germ $S$ the isomorphism classes of deformations $(\tilde{Y}, \tilde{A})$ over $S$ of the pair $(Y, A)$.

Lemma 4.5. The natural morphism $Def(X) \to Def(Y)$ is an isomorphism, as is the forgetful morphism $Def(Y, A) \to Def(Y)$. Consequently, we get the isomorphism $Def(Y, A) \to Def(X)$ of the two deformation functors.

Proof. Denote by $f : \mathbb{P}_A \to Y$ the projective bundle on $Y$ associated to $A$. We shall construct isomorphisms between the following pairs of deformation functors:

1. $Def(Y)$ and $Def(X)$,
2. $Def(Y, A)$ and $Def(\mathbb{P}_A, f, Y)$,
3. $Def(\mathbb{P}_A, f, Y)$ and $Def(\mathbb{P}_A)$,
4. $Def(\mathbb{P}_A)$ and $Def(Y)$,

such that the composition $\alpha_1 \circ \alpha_2$ : $Def(Y, A) \to Def(X)$ is the forgetful morphism, where $\alpha_i$ denotes the morphism from the first deformation functor to the second in the $i$-th pair.

The obvious morphism from $Def(X)$ to $Def(Y)$ will be shown to be an isomorphism. Most of the proof involves the cohomological identification of the differential $d\alpha_1$. Let $K$ be the sheaf of germs of tangent vectors to $X^d$ that are tangent to the small diagonal $\Delta$. The sheaf $K$ controls the deformations of the embedding $\Delta \subset X^d$: $H^1(X^d, K)$ is the tangent space of $Def(\Delta, i, X^d)$, and $H^2(X^d, K)$ its obstruction space [Sern 3.4.4]. The Cartesian square

$$
\begin{array}{ccc}
D & \longrightarrow & Y \\
\downarrow p & & \downarrow \beta \\
\Delta & \longrightarrow & X^d
\end{array}
$$

with $\beta$ the natural morphism.
gives rise to the diagram

\[
\begin{array}{ccccccccc}
0 & \rightarrow & \beta^* K/tor & \rightarrow & \beta^* T_{X^d} & \rightarrow & p^* N_{\Delta/X^d} & \rightarrow & 0 \\
& & 0 & \rightarrow & T_Y & \rightarrow & \beta^* T_{X^d} & \rightarrow & E \text{xt}^1(\Omega_\beta, \mathcal{O}_Y) & \rightarrow & 0 \\
& & \mathcal{O}(D)|_D & \rightarrow & 0 & \rightarrow & 0 \\
\end{array}
\]

Note that \(\Omega_\beta\) fits in the short exact sequence \(0 \rightarrow \beta^* \Omega_{X^d} \rightarrow \Omega_Y \rightarrow \Omega_\beta \rightarrow 0\), which explains the lower horizontal sequence above. Taking cohomology, (4.2) yields the diagram

\[
\begin{array}{ccccccccc}
0 & \rightarrow & H^1(K) & \rightarrow & H^1(T_{X^d}) & \rightarrow & H^1(N_{\Delta/X^d}) & \rightarrow & 0 \\
& & \cong & \rightarrow & \cong & \rightarrow & H^1(T_Y) & \rightarrow & H^1(T_{X^d}) & \rightarrow & H^1(\text{ext}^1(\Omega_\beta, \mathcal{O}_Y)) & \rightarrow & 0 \\
& & \mathcal{O}(D)|_D & \rightarrow & 0 & \rightarrow & 0 \\
\end{array}
\]

with exact rows, as we now explain. The top row is exact, since \(H^0(D, p^* N_{\Delta/X^d})\) is isomorphic to \(H^0(\Delta, N_{\Delta/X^d})\), which vanishes, and the homomorphism \(H^1(X^d, T_{X^d}) \rightarrow H^1(\Delta, N_{\Delta/X^d}) \cong H^1(X, T_X)^{d-1}\) is surjective, by the natural decomposition of \(T_{X^d}\). The left and right vertical arrows in diagram (4.3) are isomorphisms, since \(\mathcal{O}(D)|_D\) is cohomologically trivial. We conclude that the bottom row is exact as well. Unfurling the bottom row further thus gives the inclusion \(0 \rightarrow H^2(T_Y) \rightarrow H^2(T_{X^d}) = 0\), whence \(\text{Def}(Y)\) is unobstructed. One also easily reads off from the diagram (4.3) that each of the projections \(H^1(T_{X^d}) = H^1(T_X)^d \rightarrow H^1(T_X)\) composes with the inclusion \(H^1(T_Y) \rightarrow H^1(T_X)\) to an isomorphism \(d_\alpha: H^1(T_Y) \rightarrow H^1(T_X)\), yielding the desired identification of the differential of \(\alpha_1\). We conclude that \(\alpha_1\) is an isomorphism, as \(\text{Def}(X)\) and \(\text{Def}(Y)\) are unobstructed.

(2) There are obvious morphisms of functors in both directions which are inverse to each other.

(3) We use Lemma 4.2 to prove that the forgetful morphism \(\alpha_3: \text{Def}(\mathbb{P}_A, f, Y) \rightarrow \text{Def}(\mathbb{P}_A)\) is an isomorphism. Clearly hypotheses (1) and (2) of Lemma 4.2 hold. Let us verify that

\[\text{Def}(Y)\] is unobstructed.

\[\text{Def}(X)\] is unobstructed.
$H^0(T_{\mathbb{P}A}) = 0$, so that the Kuranishi family of $\mathbb{P}A$ is universal [BHPV Thm 10.5]. The following sequence is the push-forward of the relative Euler sequence of $f$ on $Y$:

(4.4) \[ 0 \to \mathcal{O}_Y \to A \to f^*T_Y \to 0. \]

This implies that $H^0(T_f)$ consists of traceless sections of $A$. But $A$ is simple, so $H^0(T_f) = 0$. The vanishing of $H^0(T_{\mathbb{P}A})$ now follows from the sequence

(4.5) \[ 0 \to T_f \to T_{\mathbb{P}A} \to f^*T_Y \to 0, \]

using the fact that $T_Y$ has no nontrivial global sections.

\[ \text{Composing the isomorphism } \alpha^{-1} : \text{Def}(\mathbb{P}A) \to \text{Def}(\mathbb{P}A, f, Y) \text{ with the forgetful morphism, we get a natural morphism } \alpha : \text{Def}(\mathbb{P}A) \to \text{Def}(Y), \text{ whose differential is the following map in the long exact cohomology sequence of (4.5):} \]

\[ H^1(T_f) \to H^1(T_{\mathbb{P}A}) \to H^1(T_Y) \to \]

Thus, $d\alpha$ is injective and $h^1(T_{\mathbb{P}A}) \leq h^1(T_Y) = h^1(T_X)$, where the equality uses the isomorphism of the pair \([1]\). We claim that the composition \([1][4]: \text{Def}(\mathbb{P}A) \to \text{Def}(X) \]

induces a surjective morphism $\xi : D_{\mathbb{P}A} \to D_X$ between the Kuranishi deformation spaces. Indeed, any point $x$ in $D_X$ can be joined to the origin by an open connected subset $C^0$ of a path $C$ of generalized twistor lines that is wholly contained within $D_X$ by [Hu2 Prop. 3.10]. Then, given our assumptions on $E = \beta_A$, the construction of Verbitsky [Ye5, Ma5 Cor. 6.10] (see Section 3) produces a projective bundle satisfying Condition \([1.6]\) over this path. Thus, the point $x$ lies in the image of $\xi$. The conclusion is that $h^1(T_X) \leq h^1(T_{\mathbb{P}A})$, so that these two dimensions are equal and $d\alpha$ is an isomorphism. The result follows, as we have already established that both functors $\text{Def}(\mathbb{P}A)$ and $\text{Def}(Y)$ are unobstructed.

4.2. Local moduli space of reflexive Azumaya algebras. Keep the notation of the previous subsection. Let $A = \text{End}(\beta^*F/tor)$ be the locally free Azumaya algebra over $Y$ as above. For each point $x \in X \hookrightarrow X^d$ in the diagonal, let $D_x := \beta^{-1}(x) \cong \mathbb{P}^{2n-1}$, and write $W_x$ for the restriction of $\beta^*F/tor$ to this fiber. Let us reiterate the vanishing in Condition \([1.6]\). There exists a non-negative integer $k$, such that the traceless endomorphism bundle $\mathcal{E}nd_0(W) = (A_0)_x$ satisfies

(4.6) \[ H^0((A_0)_x \otimes \mathcal{O}_{D_x}(j)) = 0, \quad \text{for } j < k, \]

(4.7) \[ H^i((A_0)_x \otimes \mathcal{O}_{D_x}(j)) = 0, \quad \text{for } i > 0, j \geq 1 \]

and $(A_0)_x \otimes \mathcal{O}_{D_x}(k)$ is generated by its global sections. We note that these conditions are open in families $(Y, A)$.

Lemma 4.6. Let $\tilde{A}$ be a family of locally free sheaves of Azumaya algebras as above, satisfying the conditions of the previous paragraph. Then $\tilde{\beta}_xA$ is a flat family of sheaves of reflexive Azumaya algebras on $X \times_S \cdots \times_S X$.

Proof. This is an immediate consequence of Lemma \([2.3]\). Indeed, $\tilde{A} \cong \tilde{A}^*$, so \((4.6)\) and \((4.7)\) furnish the hypotheses of that result (with $k = l$). \qed
Remark 4.7. Suppose \( \overline{\beta} : \overline{Y} \to \overline{X}/S \) is a deformation of \( \beta : Y \to X^d \) over \( S \). Denote by the subscript \( T \) base-changes under any morphism \( i : T \to S \). At several points below, we shall use without comment the isomorphism \( i^*\overline{\beta}^*\overline{A} \sim (\overline{\beta}_T)_*(\overline{A}_T) \) established in equation (2.3) of the proof of Lemma 2.3.

Lemma 4.8. Let \( E \) be a reflexive sheaf of Azumaya algebras on \( X^d \) satisfying Condition 1.6, and \( \overline{A} \) a sideways deformation of \( A = \text{End}(\beta^*F/\text{tor}) \), over \( \overline{Y} \to S \). Then \( \overline{E} := \overline{\beta}_*\overline{A} \) is a family of reflexive sheaves of Azumaya algebras over \( S \) which satisfies Condition 1.6 fiberwise over an open neighborhood of 0 in \( S \).

Proof. As noted above, Conditions (4.6) and (4.7) are open, so may be assumed to hold fiberwise over \( S \). The previous lemma shows that \( \overline{\beta}_*\overline{A} \) is a flat family of reflexive sheaves. Items (1) and (2) are open in families, being topological, while (3) is open by semi-continuity. Item (4) is well-known to be open. Thus, these conditions hold over \( S \). The family \( \overline{E} \) satisfies item (5) of Condition 1.6 over an open neighborhood of 0 in \( S \), by Lemma 2.15. □

Let \( \Lambda_0 \) be an Artin local ring, \( m_0 \subset \Lambda_0 \) its maximal ideal, and \( \mathbb{C} \) its residue field. Let \((\Lambda, m) \to (\Lambda_0, m_0)\) be an extension with kernel \( I \) satisfying \( I\mathbb{C} = 0 \), so that \( I \) is a \( \mathbb{C} \)-vector space. Suppose \( p : Z \to \text{Spec} \Lambda \) is a flat and proper morphism of analytic spaces or schemes, and \( G_0 \) a coherent twisted sheaf on \( Z_0 := Z \otimes_\Lambda \Lambda_0 \) that is \( \Lambda_0 \)-flat. Denote by \( Z_{00} \) the reduction \( Z \otimes_\Lambda \mathbb{C} \); in general, let the subscript “00” denote restriction of an object to the closed fiber of \( p \).

Proposition 4.9. There is an obstruction class \( o(G_0) \) in \( I \otimes_\mathbb{C} \text{Ext}^2_{Z_{00}}(G_{00}, G_{00}) \) such that \( o(G_0) = 0 \) if and only if \( G_0 \) has a deformation \( F \) as a twisted sheaf over \( Z_0 \). If the obstruction class vanishes, the set of isomorphism classes of deformations is an affine space over \( I \otimes_\mathbb{C} \text{Hom}_{Z_{00}}(G_{00}, G_{00}) \). If a deformation \( G \) exists, then the automorphism group \( \text{Aut}(G/G_0) \) is naturally isomorphic to \( I \otimes_\mathbb{C} \text{Hom}_{Z_{00}}(G_{00}, G_{00}) \).

Proof. This may be found in Section 2.2.5 of Lieblich’s thesis [Lieb]. The proof is very general and works in our analytic setting too. □

The following theorem of Artin will be used in the next result.

Theorem 4.10. [Art, Theorem 1.5] Let

\[
\begin{array}{ccc}
U & \to & V \\
\downarrow & & \downarrow \\
W & & \\
\end{array}
\]

be a diagram of analytic spaces, where the solid arrows are given maps. Consider the problem of finding a map \( \phi : U \to V \) locally at a point \( x \in U \) which makes the diagram commute. If a formal lift \( \overline{\phi} \) exists, then for any integer \( c > 0 \), there is a map \( \phi \) locally, which agrees with \( \overline{\phi} \) (modulo \( \hat{m}_{U,x}^c \)).

Returning to our situation, suppose \( E = \text{End}(F) \) is a reflexive sheaf of Azumaya algebras on \( X^d \) satisfying Condition 1.6. Recall that part (3) of that condition assumes that \( \text{Ext}^1(F, F) = 0. \)
Lemma 4.11. Let $\pi : \tilde{X} \to S$ be a deformation of $X$ and let $\tilde{E}$, $\tilde{E}'$ be families of reflexive sheaves of Azumaya algebras over $\tilde{X}_\pi^d \to S$, both restricting to the special fiber as $E$. Then there is an isomorphism $\tilde{E} \cong \tilde{E}'$ over an open neighborhood of 0 in $S$.

Proof. There exist families of twisted sheaves $\tilde{F}$, $\tilde{F}'$ such that $\tilde{E} \cong \mathcal{E}nd(\tilde{F})$, $\tilde{E}' \cong \mathcal{E}nd(\tilde{F}')$ by Proposition 2.14. We shall prove the equivalent statement that $\tilde{F}$ and $\tilde{F}'$ agree over an open neighborhood of 0 in $S$, up to a line bundle.

Denote the restrictions of $\tilde{F}$, $\tilde{F}'$ to the central fiber by $F_0$, $F_0'$. There exists a unique line bundle $L_0$ over the central fiber such that $F_0 \cong F_0' \otimes L_0$, by the first step of the proof of Proposition 2.14. Let $\tilde{M}$ be the line bundle $\det(\tilde{F}) \otimes \det(\tilde{F}')^{-1}$ (see [KnM]). Then, $L_0$ is an $n$-th root of the restriction of $\tilde{M}$ to the central fiber $M_0$, where $n = \text{rk}(F)$, since determinants base-change correctly. The obstructions to deforming $M_0$ sideways to successive infinitesimal neighborhoods are $n$ times the obstructions to infinitesimally deforming $L_0$ sideways [BuF Section 7.1]. The former clearly vanish, so the latter do too. Therefore, $L_0$ deforms formally to $\tilde{L}$ and this deformation is unique, by Proposition 4.9 as $H^1(\mathcal{O}_{X_d}) = 0$. The relative Picard functor of the family $\tilde{X}_\pi^d \to S$ is representable by an analytic space $p : \mathcal{P}ic \to S$ by [Bi]. There is a formal section of $p$ given by $\tilde{L}$; this can be extended to a section over an open neighborhood of 0 in $S$, by Theorem 4.10 or what is the same thing, a line bundle $\tilde{L}$ over this open neighborhood. We may assume that this neighborhood is the whole of $S$. Note that we have used here the uniqueness of $\tilde{L}$. We claim that $\tilde{L}$ is unique too. Indeed, if $\tilde{L}'$ is another line bundle over $S$ extending $\tilde{L}$, then the completion $\mathcal{H}om_\pi(\tilde{L}, \tilde{L}')^\vee$ of the sheaf of fiberwise homomorphisms $\mathcal{H}om_\pi(\tilde{L}, \tilde{L}')$ at the origin is non-zero, by the Formal Functions Theorem. Thus its stalk is non-zero. This is enough to say that $\tilde{L}$ and $\tilde{L}'$ are isomorphic over an open neighborhood of 0, since these sheaves are simple, and are known to be isomorphic over the central fiber.

Consider now the twisted sheaves $\tilde{F}$, and $\tilde{F}'' := \tilde{F} \otimes \tilde{L}$. These are deformations of $F_0$, and as $\text{Ext}^1(F_0, F_0) = 0$, they agree formally by Proposition 4.9. This formal agreement, together with the fact that $\tilde{F}$ and $\tilde{F}'$ are simple, implies that these sheaves agree over a neighborhood of 0, by the same argument used above to prove the uniqueness of the line bundle $\tilde{L}$. □

Definition 4.12. Let $E$ be a reflexive sheaf of Azumaya algebras on $X$ which satisfies Condition [1.6] Define $\mathcal{D}ef(X, E) : (\text{GAn}) \to (\text{Sets})$ to be the deformation functor which associates to a germ $S$ the set of isomorphism classes of deformations $(\tilde{X}, \tilde{E})$ over $S$ of the pair $(X, E)$. Here $\pi : \tilde{X} \to S$ is a flat deformation of $X$, and $\tilde{E}$ is a family of reflexive Azumaya algebras over $\tilde{X}_\pi^d \to S$.

Let $\varrho : \mathcal{D}ef(Y, A) \to \mathcal{D}ef(X, E)$ be the assignment which takes a pair $(\tilde{Y}, \tilde{A})$ over $S$ to $(\tilde{X}, \mathcal{E}nd(\tilde{F}))$, where $\tilde{F}$ is a reflexive twisted sheaf such that $\beta_*\tilde{A} = \mathcal{E}nd(\tilde{F})$. Note that $\tilde{F}$ exists by Proposition 2.14. This is a natural transformation of functors, since $\tilde{A}$ base changes correctly as mentioned in Remark 4.7. Write $\phi : \mathcal{D}ef(X, E) \to \mathcal{D}ef(X)$ for the forgetful functor.

Corollary 4.13. The transformation $\varrho : \mathcal{D}ef(Y, A) \to \mathcal{D}ef(X, E)$ is an isomorphism of deformation functors.

Proof. The composite $\mathcal{D}ef(Y, A) \to \mathcal{D}ef(X, E) \to \mathcal{D}ef(X)$ is an isomorphism by Lemma 4.5, so $\varrho$ is injective. Lemma 4.11 says that the forgetful functor $\phi$ is also injective. Thus $\varrho$ is surjective. □
4.3. A global moduli space. The proof of the next Theorem will require the following terminology. Let \( \omega \) be a Kähler class on \( X \). The twistor line \( \mathbb{P}^1_\omega \) associated to \( \omega \) (in Section 3) is generic, if the Picard group \( \text{Pic}(\mathcal{X}_t) \) is trivial for some fiber \( \mathcal{X}_t \) of the twistor family \( \mathcal{X} \to \mathbb{P}^1_\omega \). A path of twistor lines in \( \mathcal{M}_\Lambda \) is generic, if it consists of generic twistor lines and each pair of consecutive twistor lines meets at a marked pair with a trivial Picard group.

**Theorem 4.14.** There exists a coarse moduli space \( \widetilde{\mathcal{M}}_\Lambda \) parametrizing triples \((X_0, \eta_0, E)\) where \( X_0 \) is a holomorphic symplectic manifold of \( K3^{[n]} \)-type, \( \eta_0 : H^2(X_0, \mathbb{Z}) \to \Lambda \) is a marking, and \( E \) is an Azumaya algebra on \( X_0^d \) satisfying Condition 1.6. Let \( \mathcal{M}_\Lambda^0 \) be a connected component of \( \mathcal{M}_\Lambda \) and \( \mathcal{M}_\Lambda^0 \) the connected component of \( \mathcal{M}_\Lambda \) containing the image of \( \mathcal{M}_\Lambda^0 \) via the forgetful morphism

\[
\phi : \mathcal{M}_\Lambda^0 \to \mathcal{M}_\Lambda.
\]

Then the morphism \( \phi : \mathcal{M}_\Lambda^0 \to \mathcal{M}_\Lambda^0 \) is surjective and a local homeomorphism.

**Proof.** Putting together Corollary 4.13 and Lemma 4.15 we have that \( \text{Def}(X, E) \) and \( \text{Def}(X) \) are naturally isomorphic. Thus the deformation problem \( \text{Def}(X, E) \) admits a universal local moduli space. Adding the marking and glueing the local charts \( \mathcal{D}_{(X, \eta, E)} \) by the same procedure as used in the construction of \( \mathcal{M}_\Lambda \) (see [Hu1 1.18]), we obtain \( \mathcal{M}_\Lambda \). As a topological space, \( \mathcal{M}_\Lambda \) is the quotient \( Z/\sim \) of the disjoint union \( Z \) of the local Kuranishi spaces \( \mathcal{D}_{(X, \eta, E)} \) by the equivalence relation \( \sim \) corresponding to isomorphism of triples. The map \( \mathcal{D}_{(X, \eta, E)} \to \mathcal{M}_\Lambda \) is injective, since each composition

\[
\mathcal{D}_{(X, \eta, E)} \to \mathcal{D}_{(X, \eta)} \xrightarrow{P} \Omega
\]

is injective, and these compositions glue to a well defined map \( \widetilde{P} : \mathcal{M}_\Lambda \to \Omega \). The equivalence relation \( \sim \) is open \( ^7 \) so the quotient map \( Z \to \mathcal{M}_\Lambda \) is an open map. We conclude that each Kuranishi deformation space \( \mathcal{D}_{(X, \eta, E)} \) is embedded as an open subset of the quotient space \( \mathcal{M}_\Lambda \). The gluing transformations are holomorphic, as each of the compositions \( \mathcal{D}_{(X, \eta, E)} \to \mathcal{D}_{(X, \eta)} \to \Omega \) is an open holomorphic embedding.

The statement that \( \phi \) is a local homeomorphism is clear from the construction.

The morphism \( \phi : \mathcal{M}_\Lambda^0 \to \mathcal{M}_\Lambda^0 \) is surjective. The proof is identical to that [Ma5 Theorem 7.11]. There it is shown that given any marked pair \((X, \eta)\) in \( \mathcal{M}_\Lambda^0 \), there exists a generic twistor path \( C \) from \((X_0, \eta_0)\) to \((X, \eta)\), and a flat deformation \( \mathcal{E} \) of \( E_0 \) along this path, via slope-stable hyperholomorphic reflexive sheaves \( \mathcal{E}_t, t \in C \), of Azumaya algebras. The properties in Condition 1.6 hold for all \( \mathcal{E}_t, t \in C \), by Proposition 3.7. The family \( \mathcal{E} \) provides a lift of the path \( C \) to a path in \( \mathcal{M}_\Lambda^0 \). Hence, \( \phi \) is surjective. The proof of Theorem 7.11 uses Theorem 7.10 in [Ma5], which establishes the existence of a triple \((X_0, \eta_0, E_0)\) determining the component \( \mathcal{M}_\Lambda^0 \), such that the order of the Brauer class \( \theta_{(X_0, \eta_0)} \) is equal to the rank of \( E_0 \). Here the existence of such a triple, in any non-empty component \( \mathcal{M}_\Lambda^0 \), is established as follows (eliminating the need

\[ ^7 \text{The statement that the relation is open translates, by definition, to the following statement. If } t'_1 := (X'_1, \eta'_1, \mathcal{E}'_1) \in \mathcal{D}_{(X, \eta, \mathcal{E})} \text{ is isomorphic to } t'_2 := (X'_2, \eta'_2, \mathcal{E}'_2) \in \mathcal{D}_{(X_0, \eta_0, E)}, \text{ then for every open neighborhood } V_1 \text{ of } t'_1 \text{ in } Z, \text{ there exists an open neighborhood } V_2 \text{ of } t'_2 \text{ in } Z, \text{ such that for any } t'_3 := (X''_3, \eta''_3, \mathcal{E}''_3) \in V_2 \text{ there exists a point } t'_3 \in \mathcal{V} \text{ in } V_1, \text{ such that } (X''_3, \eta''_1, \mathcal{E}''_3) \text{ is isomorphic to } (X''_3, \eta''_1, \mathcal{E}''_3). \text{ This is known for marked pairs, and extends to marked triples by Lemma 4.11.} \]
for an analogue of [Ma5, Theorem 7.10]). The order of \( \theta(X, \eta, E) \) is \( r \), whenever \( \text{Pic}(X) \) is trivial, by our assumption that the element \( \bar{\theta} \) has order \( r \) in Condition 1.6 (2). Points in \( \mathcal{M}_\Lambda^0 \) with a trivial Picard group form a dense subset. Hence, the existence of such a triple \((X_0, \eta_0, E_0)\) follows from the fact that the morphism \( \phi : \mathcal{M}_\Lambda^0 \to \mathcal{M}_\Lambda^0 \) is a local homeomorphism. \( \square \)

5. Stability and separability

Recall that a sheaf of Azumaya algebras may be represented as the endomorphism sheaf of some twisted sheaf, which is unique, up to tensorization by a line bundle, by Proposition 2.14. The following is an analogue of [KO] Prop. 6.6.5.

**Proposition 5.1.** Let \( E_1, E_2 \) be two reflexive sheaves of Azumaya algebras on \( X^d \) satisfying Condition 1.6 (and so with the same Brauer class 1.7). Assume that the points associated to \((X, \eta, E_1)\) and \((X, \eta, E_2)\) are non-separated in \( \mathcal{M}_\Lambda \). Then there exist twisted reflexive sheaves \( F_i \), such that \( E_i \cong \text{End}(F_i) \), \( i = 1, 2 \), and \( \det(F_i^* \otimes F_2) \) is the trivial line bundle, as well as non-trivial homomorphisms \( \varphi : F_1 \to F_2 \) and \( \psi : F_2 \to F_1 \), satisfying \( \varphi \psi = 0 \) and \( \psi \varphi = 0 \).

**Proof.** Our assumption implies that there exists a family \( \pi : \mathcal{X} \to S \), over a one dimensional disk \( S \), with special fiber \( X_0 \) isomorphic to \( X \), families \( \mathcal{E}_1 \) and \( \mathcal{E}_2 \) of reflexive sheaves of Azumaya algebras over \( X_0^d \), and isomorphisms of \( E_i \) with the restriction of \( \mathcal{E}_i \) to \( X_0^d \), as well as an isomorphism \( f : (\mathcal{E}_1)_U \to (\mathcal{E}_2)_U \) between the restrictions of the families to the complement \( U \subset X_0^d \) of the fiber \( X_0 \).

Let \( \mathcal{F}_i \) be a family of \( \theta \)-twisted reflexive sheaves over \( X_0^d \to S \), such that \( \text{End}(\mathcal{F}_i) \) is isomorphic to \( \mathcal{E}_i \) as sheaves of Azumaya algebras. Such sheaves \( \mathcal{F}_i \) exist, by Proposition 2.14. We may assume that \( (\mathcal{F}_1)_U \) is isomorphic to \( (\mathcal{F}_2)_U \), possibly after tensoring \( \mathcal{F}_2 \) by a line bundle \( L \) over \( X_0^d \). Indeed, the existence of the isomorphism \( f \) implies that there exists a line bundle \( L \) over \( U \), such that \( (\mathcal{F}_1)_U \) is isomorphic to \( (\mathcal{F}_2)_U \otimes L \), by the uniqueness statement in Proposition 2.14. The existence of such a line bundle \( L \) over \( X_0^d \) follows from the surjectivity of the restriction homomorphism from \( \text{Pic}(X_0^d) \) to \( \text{Pic}(U) \). Let \( F_i \) be the restriction of \( \mathcal{F}_i \) to the special fiber \( X_0^d \). The existence of the non-trivial homomorphisms \( \varphi \) and \( \psi \) follows by the semi-continuity theorem. The sheaves \( F_i \) are simple, by the stability Condition 1.6 (1). Hence, \( \varphi \psi \) and \( \psi \varphi \) either vanish or they are isomorphisms. The vanishing follows, since we assumed that the Azumaya algebras \( E_1 \) and \( E_2 \) are not isomorphic. The triviality of \( \det(F_i^* \otimes F_2) \) follows, by continuity. \( \square \)

**Corollary 5.2.** Let \( E \) be a sheaf of Azumaya algebras over \( X^d \), satisfying Condition 1.6 with a Brauer class \( \theta \in H^2(X^d, \mathcal{O}_{X^d}^*) \). If the rank of \( E \) is equal to the order of \( \theta \), then \((X, \eta, E)\) is a separated point of \( \mathcal{M}_\Lambda \), provided \((X, \eta)\) is a separated point of \( \mathcal{M}_\Lambda \).

**Proof.** Assume that \((X, \eta, E)\) and \((X', \eta', E')\) are non-separated points of \( \mathcal{M}_\Lambda \). Then \((X, \eta)\) and \((X', \eta')\) are non-separated points of \( \mathcal{M}_\Lambda \) and are hence equal. Let \( F \) be a \( \theta \)-twisted sheaf representing \( E \). Then \( F \) does not admit any non-trivial subsheaf of lower rank, since the rank of any subsheaf divides the order of \( \theta \). The Corollary now follows from Proposition 5.1. \( \square \)

**Lemma 5.3.** Let \((X, \eta)\) be a point of \( \mathcal{M}_\Lambda \), not necessarily separated, and \( E_1, E_2 \) two reflexive sheaves of Azumaya algebras over \( X^d \) satisfying Condition 1.6. Set \( t_i = (X, \eta, E_i), i = 1, 2 \).
$E_1$ and $E_2$ are both $\omega$-slope-stable, with respect to the same Kähler class $\omega$ on $X$, then either $t_1 = t_2$, or the points $t_1$ and $t_2$ are separated in $\mathcal{M}_\Lambda$.

**Proof.** The proof is by contradiction. Assume that $t_1 \neq t_2$ and that the points $t_1$ and $t_2$ are non-separated. Let $F_i$ be twisted sheaves satisfying $\mathcal{E}nd(F_i) \cong E_i$, $i = 1, 2$, such that the line bundle $\det(F_i^* \otimes F_2)$ is trivial, admitting non-zero homomorphisms $\varphi : F_1 \to F_2$ and $\psi : F_2 \to F_1$ satisfying $\varphi \psi = 0$ and $\psi \varphi = 0$, as in Proposition 5.1.

The slope function is additive with respect to tensor products and $\mu(F_1^* \otimes F_2) = 0$. Hence,

$$\mu[\text{Hom}(\text{Im}(\varphi), F_1)] = \mu[\text{Hom}(\text{Im}(\varphi), F_1) \otimes F_1^* \otimes F_2] = \mu[\text{Hom}(\text{Im}(\varphi), F_2)].$$

We conclude the equality of degrees $\deg [\text{Hom}(\text{Im}(\varphi), F_1)] = \deg [\text{Hom}(\text{Im}(\varphi), F_2)]$. The degree function is additive with respect to short exact sequences, and so $\deg [\text{Hom}(\text{Im}(\varphi), F_1)] = - \deg [\text{Hom}(\text{ker}(\varphi), F_1)]$. We get the equality

$$\deg [\text{Hom}(\text{Im}(\varphi), F_2)] = - \deg [\text{Hom}(\text{ker}(\varphi), F_1)].$$

The left hand side of the above equality is strictly positive, since $F_2$ is stable, while the right hand side is strictly negative, since $F_1$ is $\omega$-slope-stable. A contradiction. $\Box$

Once Theorem 1.9 (1) is established, we conclude the equality $t_1 = t_2$ in the statement of Lemma 5.3.

6. A global Torelli theorem

Let $\tilde{\mathcal{M}}^0_{\Lambda}$ be a connected component of $\mathcal{M}_{\Lambda}$. Then $\mathcal{M}_{\Lambda}$ contains a point associated to an isomorphism class of a triple $(X_0, \eta_0, E_0)$, where the order of the Brauer class $\theta_{X_0, \eta_0} \in H^2(X_0^d, \mathcal{O}_{X_0^d}^*)$ is equal to the rank $r$ of $E_0$, as shown in the proof of Theorem 4.14. Let $\mathcal{M}_{\Lambda}$ be the connected component, of the moduli space $\mathcal{M}_{\Lambda}$, containing the image of $\tilde{\mathcal{M}}^0_{\Lambda}$ via the forgetful morphism. Denote the forgetful morphism by

$$\phi : \tilde{\mathcal{M}}^0_{\Lambda} \to \mathcal{M}_{\Lambda}.$$

Let $\Omega \subset \mathbb{P}(\Lambda \otimes \mathbb{Z} \mathbb{C})$ be the period domain (1.2) and $P : \mathcal{M}_{\Lambda} \to \Omega$ the period map. Let $\tilde{P} : \tilde{\mathcal{M}}^0_{\Lambda} \to \Omega$ be the composition $P \circ \phi$.

Consider the relation $(X_1, \eta_1, E_1) \sim (X_2, \eta_2, E_2)$ between any pair of inseparable points in $\tilde{\mathcal{M}}^0_{\Lambda}$. The relation $\sim$ is clearly symmetric and reflexive. Following is the main result of this paper.

**Theorem 6.1.** Assume that Conjecture 1.8 holds.

1. The relation $\sim$ is an equivalence relation. The quotient space $\tilde{\mathcal{M}}^0_{\Lambda} : = \tilde{\mathcal{M}}^0_{\Lambda} / \sim$ admits a natural structure of a Hausdorff complex manifold, and the quotient map $\tilde{\mathcal{M}}^0_{\Lambda} \to \tilde{\mathcal{M}}^0_{\Lambda}$ is open. The period map $\tilde{P}$ factors through a morphism $\overline{P} : \tilde{\mathcal{M}}^0_{\Lambda} \to \Omega$.

2. The morphism $\overline{P} : \tilde{\mathcal{M}}^0_{\Lambda} \to \Omega$ is an isomorphism of complex manifolds.

3. Let $x$ be a point of $\Omega$, such that $\Lambda_{1,1}(x) = 0$, or $\Lambda_{1,1}(x)$ is cyclic generated by a class of non-negative self intersection. Then the fiber $\overline{P}^{-1}(x)$ consists of a single separable point of $\tilde{\mathcal{M}}^0_{\Lambda}$. 


Proof of part 1 of Theorem 6.1. The morphism $\phi$ is a local homeomorphism, by Theorem 4.14. Hence, so is $\tilde{P} : \mathcal{M}_\Lambda^0 \to \Omega$. We show next that a point $(X, \eta, E)$ of $\mathcal{M}_\Lambda^0$ is a separated point, if $\text{Pic}(X)$ is trivial. The point $(X, \eta)$ of $\mathcal{M}_\Lambda^0$ is known to be separated [Ve7, Hu2, Prop. 4.7]. Condition 1.6 (2) assures that the Brauer class of the Azumaya algebra $E$ has order $r$ (Remark 1.4). The triple $(X, \eta, E)$ thus corresponds to a separated point, by Corollary 5.2 above.

We follow the procedure of Hausdorff reduction used by Verbitsky in his proof of the Global Torelli Theorem [Ve7, Hu2, Prop. 4.9 and Cor. 4.10]. There it is proven that the analogous relation $\sim$ for $\mathcal{M}_\Lambda^0$ is an equivalence relation, $\mathcal{M}_\Lambda^0 / \sim$ is a complex Hausdorff manifold, the quotient morphism $\mathcal{M}_\Lambda^0 \to \mathcal{M}_\Lambda^0 / \sim$ is open, and the period map $P$ factors through $\mathcal{M}_\Lambda^0 / \sim$.

The only two facts used, are that $P : \mathcal{M}_\Lambda^0 \to \Omega$ is a local homeomorphism, and that points of $\mathcal{M}_\Lambda^0$, such that $\text{Pic}(X)$ is trivial, are separated. These two facts hold for $\tilde{\mathcal{M}}_\Lambda^0$ as well, as shown above. Hence, the proofs of [Hu2, Prop. 4.9 and Cor. 4.10] apply verbatim to prove part 1. □

Proof of part 2 of Theorem 6.1

Proposition 6.2. Consider an equivalence class $[X, \eta, E] \in \mathcal{M}_\Lambda^0$ of a triple $(X, \eta, E)$ and assume that its period $\overline{P}([X, \eta, E])$ is contained in a generic twistor line $Tw \subset \Omega$. Then there exists a lift $\iota : Tw \to \mathcal{M}_\Lambda^0$, such that $\overline{P} \circ \iota : Tw \to \Omega$ is the inclusion, and $\iota(Tw)$ contains $[X, \eta, E]$.

Proof. The analogous statement for the Hausdorff reduction of $\mathcal{M}_\Lambda^0$ is [Hu2, Prop. 5.4]. There a simple point set topology argument reduces the proof to the case where $\text{Pic}(X)$ is trivial. In that case the existence of a unique lift $\iota : Tw \to \mathcal{M}_\Lambda^0$ through $(X, \eta)$ follows from the fact that the Kähler cone of $X$ is equal to its positive cone, and from the construction of a twistor family through $(X, \eta)$ for any Kähler class $\omega$ of $X$. Thus, it remains to prove that when $\text{Pic}(X) = (0)$, any twistor line through $(X, \eta)$ lifts further to a twistor line through $(X, \eta, E)$ in $\mathcal{M}_\Lambda^0$. The triviality of $\text{Pic}(X)$ implies that the order of $\theta(X, E)$ is $r$, hence $E$ is $\omega$-slope-stable with respect to every Kähler class on $X$ [Ma3, Prop. 7.8]. The lifting of each twistor line to a deformation of the triple $(X, \eta, E)$ over the twistor family is proven, using results of Verbitsky, in [Ma3, Cor. 6.10 and Prop. 7.8]. The lifted deformation lies in $\mathcal{M}_\Lambda^0$, by Proposition 5.7. □

The proof of part 2 of Theorem 6.1 is now identical to the argument in [Hu2, Sec.5.4] establishing Corollary [Hu2, Prop. 5.4] by Proposition 6.2 above.

In the language of Verbitsky, Proposition 6.2 establishes the fact that the map $\overline{P} : \mathcal{M}_\Lambda^0 \to \Omega$ is compatible with generic hyperkähler lines [Ve7, Definition 6.2]. Verbitsky proves that any map $\psi : M \to \Omega$ from a Hausdorff manifold $M$, which is compatible with generic hyperkähler lines, is a covering [Ve7, Theorem 6.14]. Theorem 6.1 (2) follows, since $\Omega$ is simply connected. □

Proof of part 3 of Theorem 6.1. The case $\Lambda^{1,1}(x) = 0$ follows from Corollary 5.2 and Remark 1.4. Assume that $\Lambda^{1,1}(x)$ is spanned by the non-zero class $\tilde{c}$, and $(\tilde{c}, \bar{c}) \geq 0$. The fiber $P^{-1}(x)$ intersects the connected component $\mathcal{M}_\Lambda^0$ in a single separable point $(X, \eta)$, by Verbitsky’s Global Torelli Theorem [Ve7, Hu2, Ma7, Theorem 2.2]. Let $t_1 := (X, \eta, E_1)$ and $t_2 := (X, \eta, E_2)$ be points in the intersection of the fiber $\tilde{P}^{-1}(x)$ with the connected component $\mathcal{M}_\Lambda^0$. Then either $t_1 = t_2$, or $t_1$ and $t_2$ are inseparable, by part 2 of the Theorem. Let $\omega_i$
be a Kähler class, such that $E_i$ is $\omega_i$-slope-stable, $i = 1, 2$. It suffices to show that $E_2$ is also $\omega_1$-slope-stable, by Lemma 7.3.

Let $c := \eta^{-1}(\hat{c}) \in H^{1,1}(X, \mathbb{Z})$ be a generator. We may assume that $c$ belongs to the closure of the positive cone, possibly after replacing $\hat{c}$ by $-\hat{c}$. Let $B \subset E_2$ be a maximal parabolic sheaf of Lie subalgebras. Then $\int_X \omega_2^{2n-1} c_1(B) < 0$, since $E_2$ is $\omega_2$-slope-stable. For every Kähler class $\omega$ and for every class $\alpha$ in the closure of the positive cone of $X$ in $H^{1,1}(X, \mathbb{R})$, the following inequality holds:

$$\int_X \omega^{2n-1} \alpha > 0,$$

by [Hull (1.10)]. Hence, $c_1(B) = kc$, for some negative integer $k$. We conclude that the $\omega_1$-degree $\int_X \omega_1^{2n-1} c_1(B) = k \int \omega_1^{2n-1} c$ of $B$ is negative as well. Hence, $E_2$ is $\omega_1$-slope-stable. □

7. PROOF OF THEOREM 1.1 establishing the main example

We will need the following Lemmas. Let $V$ be a complex vector space of even dimension $2n$, $n \geq 2$, and $\sigma$ a non-degenerate anti-symmetric bilinear pairing on $V$. Denote by $\mathcal{V}$ the trivial bundle over $\mathbb{P}V$ with fiber $V$ and let $\ell$ be the tautological line sub-bundle of $\mathcal{V}$. Denote by $\ell^\perp$ the sub-bundle of $\mathcal{V}$, which is $\sigma$-orthogonal to $\ell$. Note that $\sigma$ induces a non-degenerate anti-symmetric bilinear pairing on the quotient bundle

$$(7.1) \quad W := \ell^\perp/\ell.$$

**Lemma 7.1.**

1. $H^i(\ell^{j+1} \otimes \ell^\perp) = 0$, if $i > 0$, $j \leq 2$, and $(i, j) \neq (1, 0)$.

2. $H^1(\ell \otimes \ell^\perp)$ is one-dimensional.

3. $H^0(\ell^{j+1} \otimes \ell^\perp) \cong \ker [\text{Sym}^{-j-1} V^* \otimes V^* \to \text{Sym}^{-j} V^*]$, for $j \leq -2$, and it vanishes for $j > -2$.

**Proof.** Consider the short exact sequence $0 \to \ell^{j+1} \otimes \ell^\perp \to \ell^{j+1} \otimes V^* \xrightarrow{\alpha} \ell^j \to 0$, where we embed $\ell^\perp$ as a sub-bundle of $V^*$ via $\sigma$. The middle and right terms do not have higher cohomology for $j \not\in \{2n - 1, 2n\}$, and in particular for $j \leq 2$. The homomorphism $\alpha$ induces a surjective homomorphism on global sections, except when $j = 0$. □

**Lemma 7.2.**

1. $H^i(\ell^\perp \otimes \ell^\perp \otimes \ell^j) = 0$, for $j \leq 0$, for all $i > 0$, except when $(i, j) = (1, 0)$.

2. $H^0(\ell^\perp \otimes \ell^\perp \otimes \ell^j) = 0$, for $j \geq -1$.

**Proof.**

1. Consider the short exact sequence

$$0 \to \ell^\perp \otimes \ell^\perp \otimes \ell^j \to V^* \otimes \ell^\perp \otimes \ell^j \xrightarrow{\alpha} \ell^\perp \otimes \ell^{j-1} \to 0.$$

The higher cohomologies of the middle and right terms vanish for $j \leq 0$, by Lemma 7.1 (1).

When $j = 0$, $H^0(V \otimes \ell^\perp \otimes \ell^j) = 0$ and $H^0(\ell^\perp \otimes \ell^{j-1}) \cong \text{Sym}^{2} V^*$, by Lemma 7.1 (3). When $j \leq 0$, $H^0(\ell^\perp \otimes \ell^{-1})$ is equal to the image of $\text{Sym}^{-j}(V^* \otimes 2 \wedge V^*)$ in $\text{Sym}^{j+2} V^*$, by Lemma 7.1 (3).

Similarly, $H^0(V^* \otimes \ell^\perp \otimes \ell^j)$ is the image of $V^* \otimes \text{Sym}^{-j-1}(V^*) \otimes 2 \wedge V^*$ in $\text{Sym}^{j+1} V^*$. Thus, the homomorphism $\alpha$ induces a surjective homomorphism on global sections, for $j \leq -1$.

2. The vanishing is clear for $j \geq 0$. When $j = -1$, the homomorphism $\alpha$ induces an isomorphism on global sections, by Lemma 7.1 (4). □
Lemma 7.3.  

(1) \( H^0(\mathbb{P}V, \mathcal{E}nd_0(W) \otimes \mathcal{O}_{\mathbb{P}V}(j)) = 0 \), for \( j < 2 \).

(2) \( H^0(\mathbb{P}V, \mathcal{E}nd(W)) \) is one dimensional, \( H^1(\mathbb{P}V, \mathcal{E}nd(W)) \cong \Lambda^2 V^*/\sigma \), \( H^2(\mathbb{P}V, \mathcal{E}nd(W)) \) vanishes, and the pair \((\mathbb{P}V, W)\) is infinitesimally rigid.

(3) The sheaf \( \mathcal{E}nd_0(W) \otimes \mathcal{O}_{\mathbb{P}V}(2) \) is generated by its global sections.

(4) \( H^i(\mathbb{P}V, \mathcal{E}nd_0(W) \otimes \mathcal{O}_{\mathbb{P}V}(j)) = 0 \), for \( i > 0 \) and for \( j \geq 1 \).

(5) \( H^i(W \otimes \mathcal{O}_{\mathbb{P}V}(j)) \) vanishes, for \( i > 0 \) and \( j \geq 0 \). \( W \otimes \mathcal{O}_{\mathbb{P}V}(1) \) is generated by global sections.

Proof. Consider the short exact sequence

\[(7.2) 0 \to \ell \to \ell^1 \to W \to 0.\]

Tensoring with \( W \otimes \ell^j \) we get the short exact sequence

\[(7.3) 0 \to W \otimes \ell^{j+1} \to \ell^1 \otimes W \otimes \ell^j \to W \otimes W \otimes \ell^j \to 0.\]

Note that \( \mathcal{E}nd(W) \) is isomorphic to \( W \otimes W \). We need to compute the zero-th sheaf cohomologies of \( W \otimes W \otimes \ell^j \) for \( j \geq -2 \) and the higher sheaf cohomology for \( j = 0 \) and \( j \leq -2 \). We have the short exact sequences

\[(7.4) 0 \to \ell^{j+2} \to \ell^{j+1} \otimes \ell^1 \to W \otimes \ell^{j+1} \to 0,\]

\[(7.5) 0 \to \ell^1 \otimes \ell^{j+1} \to \ell^1 \otimes \ell^1 \otimes \ell^j \to \ell^1 \otimes W \otimes \ell^j \to 0.\]

Part (1): When \( j \leq -1 \), the higher cohomologies of the left and middle terms in the two sequences above vanish, by Lemmas 7.1 and 7.2. Hence, so do the higher cohomologies of the right terms. The latter are the left and middle terms of the sequence (7.3). Hence, the higher cohomologies of \( W \otimes W \otimes \ell^j \) vanish, for \( j \leq -1 \).

Part (2): The quotient homomorphism \( \ell^1 \otimes \ell \to W \otimes \ell \) induces an isomorphism \( H^i(\ell^1 \otimes \ell) \cong H^i(W \otimes \ell) \), for all \( i \), as seen from the exact sequence (7.4) and the vanishing of \( H^1(\ell^2) \), for all \( i \). Hence, \( H^i(W \otimes \ell) \) vanishes, for \( i \neq 1 \), and \( H^1(W \otimes \ell) \) is one dimensional, by Lemma 7.1. \( H^0(\ell^1 \otimes W) \) clearly vanishes. Consider now the long exact sequence of cohomologies associated to the sequence (7.3) when \( j = 0 \). In that case we have seen that \( H^1(W \otimes \ell^{j+1}) \) is one-dimensional, and \( H^0(\ell^1 \otimes W \otimes \ell^j) \) vanishes, and so \( H^0(W \otimes W) \) is one-dimensional, as it injects into the one-dimensional space \( H^1(W \otimes \ell^j) \).

The vector space \( H^1(\ell^1 \otimes W) \) is isomorphic to \( \Lambda^2 V^*/\sigma \), by the computation of the first sheaf cohomologies of the left and middle terms in the short exact sequence (7.5) in Lemmas 7.1 and 7.2. \( H^1(\ell^1 \otimes W) \to H^1(W \otimes W) \) is an isomorphism, using the long exact cohomology sequence associated to the short exact sequence (7.3), the vanishing of \( H^2(\ell \otimes W) \), and the established fact that \( H^0(W \otimes W) \to H^1(\ell \otimes W) \) is an isomorphism. We conclude that \( H^1(W \otimes W) \) is isomorphic to \( \Lambda^2 V^*/\sigma \).

The differential \( \text{sl}(V) \to H^1(\mathcal{E}nd(W)) \) of the pullback action of \( \text{Aut}(\mathbb{P}V) \) is identified as the composition

\[
\text{sl}(V) \to V^* \otimes V \xrightarrow{\sigma \otimes 1} V^* \otimes V^* \to \Lambda^2 V^* \xrightarrow{2} \Lambda^2 V^*/\mathbb{C}\sigma \cong H^1(\mathcal{E}nd(W)),
\]

where the right isomorphism is the one constructed above. The differential is thus surjective. Hence, the pair \((\mathbb{P}V, W)\) is infinitesimally rigid.
It remains to prove the vanishing of $H^2(W \otimes W)$. The homomorphism $H^2(\ell^\perp \otimes W) \to H^2(W \otimes W)$ is bijective, by the vanishing of $H^i(\ell \otimes W)$ established above for $i \geq 2$. In turn, $H^2(\ell^\perp \otimes W)$ is isomorphic to $H^2(\ell^\perp \otimes \ell^\perp)$, since $H^i(\ell^\perp \otimes \ell)$ vanishes for $i \geq 2$, by Lemma 7.4. Finally, $H^2(\ell^\perp \otimes \ell^\perp)$ vanishes, by Lemma 7.2.

Part (1): The vanishing of $H^0(\text{End}_0(W))$ has been established in part (2). Hence, the space $H^0(\text{End}_0(W) \otimes \ell^j)$ vanishes, for $j \geq 0$. It remains to prove the vanishing of $H^0(W \otimes W \otimes \ell^{-1})$. $H^i(\ell)$ and $H^j(\ell^\perp)$ vanish for all $i$. Hence, $H^i(W)$ vanish for all $i$, by the exactness of the sequence (7.2). We get the isomorphism $H^0(W \otimes W \otimes \ell^{-1}) \cong H^0(\ell^\perp \otimes W \otimes \ell^{-1})$, by the exactness of the sequence (7.3). The vector space $H^0(\ell^\perp \otimes W \otimes \ell^{-1})$ is isomorphic to $H^0(\ell^\perp \otimes \ell^\perp \otimes \ell^{-1})$, by the exactness of sequence (7.5). Now $H^0(\ell^\perp \otimes \ell^\perp \otimes \ell^{-1})$ vanishes, by Lemma 7.2 (2).

Part (3): It suffices to prove that $W \otimes \ell^{-1}$ is generated by its global sections. This was proven in [Ma5, Lemma 4.6].

Part (5): The vanishing was proven in the proof of part (4). The generation is proven in [Ma5, Lemma 4.6].

Denote by $Sp(V,\sigma)$ the subgroup of $GL(V)$ leaving $\sigma$ invariant.

Lemma 7.4. The vector bundle $W$ is slope stable.

Proof. Given a subsheaf $F$ of $W$, denote by $\tilde{F}$ its inverse image subsheaf in $\ell^\perp$. Clearly, the degree of $W$ is zero, and $\deg(\ell^\perp) = -1$. Let $F$ be a saturated non-zero subsheaf of $W$ of degree $d$ and rank $< 2n - 2$. Then $\deg(\tilde{F}) = d - 1$. The sheaf $\tilde{F}$ is a saturated subsheaf of the trivial bundle $V$. Hence, $d - 1 \leq 0$, and $d - 1 = 0$, if and only if $\tilde{F}$ is a trivial bundle. Now $H^0(\tilde{F})$ vanishes, since so does $H^0(\ell^\perp)$. Hence, $\tilde{F}$ is non-trivial and $d - 1 < 0$. We conclude that $d \leq 0$ and $W$ is semi-stable.

The sub-bundle $\ell$ and the symplectic form $\sigma$ are invariant with respect to the $Sp(V,\sigma)$ action on $V$, which corresponds to the diagonal action on $\mathbb{P}V \times V$. Hence, so is $\ell^\perp$, and the bundle $W$ is homogeneous with respect to an induced $Sp(V,\sigma)$-action. Let $F \subset W$ be the maximal polystable subsheaf [H14, Lemma 1.5.5]. Then $F$ is an $Sp(V,\sigma)$-invariant subsheaf. Given a point $x \in \mathbb{P}V$, the stabilizer of $x$ in $Sp(V,\sigma)$ acts transitively on the set of non-zero vectors in the fiber of $W$ over $x$. Hence, $F = W$ and $W$ is polystable. Finally, $\text{End}(W)$ is one-dimensional, by Lemma 7.3 and so $W$ is stable.

Let $M$ be the coarse moduli space of slope stable vector bundles over $\mathbb{P}(V)$ of rank $\dim(V) - 2$ and Chern character equal to $\dim(V) - 2 \sum_{i=0}^{\dim(V)/2} h^{2i}/(2i)!$, where $h$ is the first Chern class of $\mathcal{O}_{\mathbb{P}(V)}(1)$. Let $\Sigma \subset \mathbb{P} \left( \varprojlim V^* \right)$ be the Zariski open subset of lines in $\varprojlim V^*$ spanned by non-degenerate 2-forms. A point $\bar{\sigma}$ of $\Sigma$ determines the vector bundle $W_{\bar{\sigma}}$ given in Equation (7.2). A relative analogue of the above construction combines with Lemma 7.3 to yield a morphism

$$\kappa : \Sigma \to M,$$

sending the point $\bar{\sigma}$ to the isomorphism class of the vector bundle $W_{\bar{\sigma}}$.

Lemma 7.5. The morphism $\kappa$ is an open embedding.

Proof. The obstruction space $H^2(\mathbb{P}(V),\mathcal{E}nd(W_{\bar{\sigma}}))$ vanishes, by Lemma 7.3 (2). Hence, the image of $\kappa$ is contained in the smooth locus of $M$. The differential $d\kappa_{\bar{\sigma}} : T_{\bar{\sigma}} \Sigma \to H^1(\mathbb{P}(V),\mathcal{E}nd(W_{\bar{\sigma}}))$ is an isomorphism, by Lemma 7.3 (2) again. Hence the image of $\kappa$ is an open subset of $M$. 
The Euler characteristic $\chi(W_\sigma \otimes \ell)$ is $-1$, as seen in the proof of Lemma 7.3 (2). Consequently, there exists a universal family $W$ over $\mathbb{P}(V) \times M$, by the appendix in [Mu].

Let $p_2 : \mathbb{P}(V) \times M \to M$ be the projection. Denote by $\mathfrak{sl}(V)_M$ the trivial vector bundle over $M$ with fiber $\mathfrak{sl}(V)$. The action of $PGL(V)$ on $\mathbb{P}(V)$ induces an action of $PGL(V)$ on $M$, and we denote by

$$a : \mathfrak{sl}(V)_M \to R^1p_{2*}\text{End}(W)$$

the differential of this action. We have seen that the homomorphism $a$ is surjective over the open subset $\kappa(\Sigma)$. The fiber of $\ker(a)$ over $\kappa(\Sigma)$ is the Lie sub-algebra $\mathfrak{sp}(V, \bar{\sigma}) \subset \mathfrak{sl}(V)$ leaving the line $\bar{\sigma} \subset \Lambda^2 V^*$ invariant. The composition $\ker(a) \to \mathfrak{sl}(V)_M \to \text{End}
\left(\frac{\Lambda^2 V^*}{M}\right),$ corresponds to a homomorphism $\varphi : \left(\frac{\Lambda^2 V^*}{M}\right) \to \text{Hom}
\left(\ker(a), \left(\frac{\Lambda^2 V^*}{M}\right)\right),$ whose kernel is a line subbundle $L$ of the trivial vector bundle $\left(\frac{\Lambda^2 V^*}{M}\right)$. The line subbundle $L$ corresponds to a morphism $M \to \Sigma$, which is the inverse of $\kappa$. \qed

Let $\pi : \mathcal{X} \to B$ be a smooth and proper family of irreducible holomorphic symplectic manifolds of relative dimension $2n$ over a complex analytic space $B$. Let $T_\pi$ be the vertical tangent bundle. Set $D := \mathbb{P}(T_\pi)$ and let $p : D \to \mathcal{X}$ be the natural morphism. Let $\ell$ be the tautological line subbundle of $p^*T_\pi$. Let $L \subset \Lambda^2 T_\pi$ be the image of the canonical homomorphism $\pi^*\pi_* \Lambda^2 T_\pi \to \Lambda^2 T_\pi$. Then $L$ is a line subbundle of $\Lambda^2 T_\pi$ and each fiber of $L$ is spanned by the holomorphic symplectic form of the fiber of $\pi$. Let $\ell^\perp$ be the symplectic orthogonal to $\ell$ with respect to $L$. Set $\mathcal{W} := \ell^\perp$.

Let $\ell_x$ be the restriction of $\ell$ to the fiber $\mathbb{P}[T_x(\mathcal{X}_\pi(x))]$ of $p$ over $x \in \mathcal{X}$. Given a non-degenerate element $\sigma_x \in \Lambda^2 T_x^* \mathcal{X}_\pi(x)$, let $\ell_x^\perp_{\sigma_x}$ be the subbundle, of the trivial bundle with fiber $T_x^* \mathcal{X}_\pi(x)$ over $\mathbb{P}[T_x(\mathcal{X}_\pi(x))]$, which is $\sigma_x$-orthogonal to $\ell_x$.

**Lemma 7.6.** Let $\mathcal{W}'$ be a rank $2n - 2$ vector bundle over $D$. Given a point $x \in \mathcal{X}$, denote by $\mathcal{W}'_x$ the restriction of $\mathcal{W}'$ to the fiber $\mathbb{P}[T_x(\mathcal{X}_\pi(x))]$ of $p$ over $x$. Assume that for every $x \in \mathcal{X}$, the bundle $\mathcal{W}'_x$ is isomorphic to $\ell_x^{\perp_{\sigma_x}} / \ell_x$, for some non-degenerate element $\sigma_x \in \Lambda^2 T_x^* \mathcal{X}_\pi(x)$. Then $\mathcal{W}'$ is isomorphic to $\mathcal{W} \otimes p^*Q$, for some line bundle $Q$ over $\mathcal{X}$.

**Proof.** It suffices to show $\mathcal{W}'_x$ is isomorphic to $\mathcal{W}_x$, for all $x \in \mathcal{X}$. The vector bundle $\mathcal{W}'$ determines a line subbundle $L'$ of $\Lambda^2 T_\pi$, each fiber of which is spanned by a non-degenerate form, by Lemma 7.5. We thus have a natural isomorphism $T_\pi \otimes L' \to T^*_\pi$. It follows that the tensor power $(L')^{2n}$ is isomorphic to the square $\omega^2_\pi$ of the relative canonical bundle over $\mathcal{X}$. The Picard group of $\mathcal{X}_b$ is torsion free, for all $b \in B$, and the canonical line bundle of $\mathcal{X}_b$ is trivial. Hence, $L'$ restricts to $\mathcal{X}_b$ as the trivial line bundle, for all $b \in B$. The vector bundle $\Lambda^2 T^* \mathcal{X}_b$ admits a unique trivial line subbundle. Hence $L_b = L'_b$, for all $b \in B$. We conclude that the line subbundles $L$ and $L'$ are equal, and hence $\mathcal{W}'_x$ and $\mathcal{W}_x$ are isomorphic for all $x \in \mathcal{X}$. \qed

**Corollary 7.7.** Let $\mathcal{W}$ and $\mathcal{W}'$ be the vector bundles over $D$ given in Lemma 7.6. Then $\mathcal{W}'$ is locally trivial in the topology of $\mathcal{X}$ in the sense of Definition 7.15.
Proof. It suffices to prove the statement for $\mathcal{W}$, by Lemma 7.6. A smooth family of holomorphic symplectic manifolds always admits local symplectic trivializations. Hence, each point $x \in \mathcal{X}$ admits an open neighborhood $U \subset \mathcal{X}$ and an open neighborhood $\overline{U} \subset \mathbb{P}_\omega$ of $\pi(x)$, with an isomorphism
\[
f : U \to (U \cap \mathcal{X}_{\pi(x)}) \times \overline{U},\]
such that $\pi \circ f^{-1}$ is the projection on $\overline{U}$, and $f$ restricts to a symplectomorphism
\[
f_b : (U \cap \mathcal{X}_{\pi(x)}) \to (U \cap \mathcal{X}_b),\]
for all $b \in \overline{U}$. The vector bundle $\mathcal{W}$ depends only on the symplectic structure and hence admits a trivializing isomorphism $\tilde{f} : \pi^*(\mathcal{W}|_{p^{-1}(U \cap \mathcal{X}_{\pi(x)})}) \to \mathcal{W}|_{p^{-1}(U)}$, of the form required in Equation (1.6) over the inverse image $p^{-1}(U)$ of $U$ in $D$. \hfill $\square$

Proof of Theorem 1.11. Part 1 Condition 1.6 (1) is verified in [Ma5, Prop. 4.2]. The sheaf $F$, given in Equation (1.3), is a simple and infinitesimally rigid reflexive sheaf of rank $2n - 2$ [Ma5 Prop. 4.5] and [MM1, Lemma 4.2]. Condition 1.6 (3) thus holds. The sheaf $F$ does not have any non-zero subsheaf of lower rank, when $\text{Pic}(S)$ is trivial [Ma8], and so Condition 1.6 (4) holds.

Condition 1.6 (5): We prove first that $F$ is $\beta$-tight (Definition 1.13). The extension sheaves $\mathcal{E}xt^i(F, F)$ where calculated in [MM1 Prop. 3.15], for all $i$, and the short exact sequence (3.13) for $\mathcal{E}xt^1(F, F)$ was established there. The sheaf $F$ is thus $\beta$-tight, by Lemma 3.6.

Let $\ell$ be the tautological line sub-bundle of the trivial vector bundle over $\mathbb{P}[T_2S^{[n]}]$ with fiber $T_2S^{[n]}$. Let $\ell^\perp$ be the symplectic orthogonal of $\ell$ and set $W := \ell^\perp / \ell$. The sheaf $\beta^*F$/torsion restricts to $\mathbb{P}[T_2S^{[n]}]$ as $W \otimes \ell^{-1}$, by [Ma5, Prop. 4.5]. Condition 1.6 (5) now follows, with $m = 0, m' = 2$, and $k = 2$, from Lemmas 7.3 and 7.4.

It remains to verify Condition 1.6 (2). Let $\tilde{H}(S, Z)$ be the Mukai vector of the ideal sheaf of a length $n$ subscheme, $v^\perp$ the sublattice of $\tilde{H}(S, Z)$ orthogonal to $v$, and
\[
\mu : v^\perp \to H^2(S^{[n]}, \mathbb{Z})\]
the Mukai isometry [1] Eq. (1.6) and Theorem 8.1. Set $\Lambda := v^\perp$ and let $\eta_0 : H^2(S^{[n]}, \mathbb{Z}) \to \Lambda$ be the inverse $\mu^{-1}$. Let $w \in v^\perp$ be the Mukai vector $(1, 0, n - 1)$ and set
\[
\tilde{\theta} := (-w, w) + (2n - 2)[v^\perp \oplus v^\perp].\]
Then $\tilde{\theta}$ is a class of order $2n - 2$ in $\Lambda^{\otimes 2}/(2n - 2)\Lambda^{\otimes 2}$. Given a marked pair $(X, \eta)$, deformation equivalent to $(S^{[n]}, \eta_0)$, we get the class $\theta(X, \eta)$ in $H^2(X \times X, \mathcal{O}_{X \times X}^*)$, given in (1.1). The sheaf $E := \mathcal{E}nd(F)$, the marking $\eta$, and the class $\tilde{\theta}$ satisfy Condition 1.6 (2), by [Ma5, Lemma 7.3].

We claim that the sheaf $E := \mathcal{E}nd(F)$ satisfies Condition 3.9. Indeed, the Kodaira-Spencer class $\kappa_x$, given in Equation (3.14), is surjective for all $x \in S^{[n]}$, by [MM1, Lemma 4.3]. Lemma 3.8 implies the vanishing of $H^1(E_x)$, for all $x \in S^{[n]}$. Slope-stability of $E_x$, for all $x \in S^{[n]}$, for some Kähler class on $S^{[n]}$, is shown as follows. Fix a Kähler class $\omega$, such that the twistor line $\mathbb{P}_\omega^1$ is generic. Let $\mathcal{E}$ be a hyperholomorphic deformation of $E$ along $\mathbb{P}_\omega^1$. There exists a point $t \in \mathbb{P}_\omega^1$ such that $\text{Pic}(\mathcal{X}_t)$ is trivial, by our choice of $\omega$. Let $\mathcal{E}_{t,x'}$ be the restriction of $\mathcal{E}_t$ to $\{x'\} \times \mathcal{X}_t$, $x' \in \mathcal{X}_t$. The Brauer class of the sheaves $\mathcal{E}_{t,x'}$ has order $2n - 2$, for all $x' \in \mathcal{X}_t$. Here we used the condition on the class $\tilde{\theta}$ in Condition 3.9, which requires that it projects to a class of order.
The pairs $(X, E_1)$ and $(X, E_2)$ are both assumed to be deformation equivalent to $(S[^n], E)$. This assumption implies that for every marking $\eta_1$ of $X$, there exists an element $g \in \text{Mon}^2(X)$, such that $(X, \eta_1, E_1)$ and $(X, \eta_1 g, E_2)$ belong to the same connected component $\mathfrak{M}_\Lambda^1$. Thus, $(X, \eta_1 g, E_1^{(s \text{cov}(g))})$ and $(X, \eta_1 g, E_2)$ belong to the intersection of the same fiber of $\tilde{P}$ with $\mathfrak{M}_\Lambda^1$. Now $E_1^{(s \text{cov}(g))}$ and $E_2$ are both $\omega$-slope-stable, by assumption. Hence, $E_1^{(s \text{cov}(g))}$ and $E_2$ are isomorphic, by Lemma 5.3 and Theorem 6.1 part 2.

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