Profinite completion and double-dual: isomorphisms and counter-examples
Colas Bardavid

To cite this version:
Colas Bardavid. Profinite completion and double-dual: isomorphisms and counter-examples. 2008.
<hal-00208000>

HAL Id: hal-00208000
https://hal.archives-ouvertes.fr/hal-00208000
Submitted on 18 Jan 2008

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L’archive ouverte pluridisciplinaire HAL, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d’enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.
Profinite completion and double-dual: isomorphisms and counter-examples

Colas Bardavid
IRMAR — UMR 6625 du CNRS
Université de Rennes 1 Campus de Beaulieu
35042 Rennes CEDEX FRANCE

Abstract – We define, for any group $G$, finite approximations; with this tool, we give a new presentation of the profinite completion $\hat{\pi} : G \to \hat{G}$ of an abstract group $G$. We then prove the following theorem: if $k$ is a finite prime field and if $V$ is a $k$-vector space, then, there is a natural isomorphism between $\hat{V}$ (for the underlying additive group structure) and the additive group of the double-dual $V^{**}$. This theorem gives counter-examples concerning the iterated profinite completions of a group. These phenomena don’t occur in the topological case.

(1) Introduction.

In this paper, we study the profinite completion of a certain class of groups, namely, the additive groups of vector spaces over $\mathbb{F}_p$. The principal result is that, in this case, the profinite completion equals the double-dual. This study is based on a “dual” definition of the profinite completion of a group.

(2) Brief survey of the classical point of view for profinite completion.

As explained in [Ser02] or [RZ00], one usually defines the profinite completion of a group $^1 G$ as follows. The profinite completion $\hat{G}$ of $G$ is the projective limit (ie the inverse limit) of the finite quotients of $G$:

$$\hat{G} = \lim_{\overset{\longrightarrow}{N \in G}} G/N.$$ 

There is a more explicit form for this definition. Indeed, if $N, M$ are two normal subgroups of $G$ with $N \subset M$, we have a natural factorisation $\varphi_{N \subset M}$ of the canonical projection $\pi_M$:

$$\begin{array}{ccc}
G & \xrightarrow{\pi_N} & G/N \\
\pi_M & \downarrow & \varphi_{N \subset M} \\
G/M & & \\
\end{array}$$

$^1$If we set in the category of topological groups, we should precise: "... of a discrete group $G".$
One can then write:

\[ \hat{G} = \left\{ (x_N) \in \prod_{\substack{N \in G \setminus \{N:G\}\leq \infty \}} G/N \mid \forall N \subset M, \varphi_{N \subset M} (x_N) = x_M \right\}, \]

(3) Finite approximations and profinite completion.

In this paper, we will use a “dual” (but equivalent) point of view for the profinite completion of a group. To begin with, we introduce the notion of “finite approximation”, which will lead naturally to the concept of profinite completion.

(3.1) Definition. If \( G \) is a group, we call finite approximation of \( G \) every couple \( \nu = (F, \varphi) \) where \( F \) is a finite group and \( \varphi : G \to F \) a morphism. We denote \( F = F_{\nu} \) and \( \varphi = \varphi_{\nu} \). We say that \( f : \nu \to \nu' \) is a morphism between \( \nu \) and \( \nu' \) if it is an arrow that makes the following diagram commute:

\[ \begin{array}{ccc}
F_{\nu} & \xleftarrow{\varphi} & F_{\nu'} \\
\downarrow{f} & & \downarrow{f} \\
G & \xleftarrow{\varphi_{\nu}} & G
\end{array} \]

We denote \( \text{App}_{f}(G) \) the category of finite approximations of \( G \).

Intuitively, a finite approximation of \( G \) allows the mathematician to get some information about \( G \) by only dealing with finite objects. Here are some examples, from various areas of mathematics, of finite approximations:

a) \( \mathbb{R}^* \to \mathbb{Z}/2\mathbb{Z} \) the sign of a real number.

b) The reduction modulo \( n \), \( \mathbb{Z} \to \mathbb{Z}/n\mathbb{Z} \) and all the derived morphisms and generalizations, such that \( \mathbb{Z}_{(p)} \to \mathbb{Z}/p\mathbb{Z} \), such that \( GL_m(\mathbb{Z}) \to GL_m(\mathbb{Z}/n\mathbb{Z}) \) or such that \( \mathcal{O}_K \to \mathcal{O}_K/\mathfrak{P} \) if \( K \) is a number field.

c) If \( X \) a topological space with a finite number of connected components, we can consider the “trace” on \( \pi_0(X) \) of an automorphism : \( \text{Aut}(X) \to \pi_0(X) \) \( \phi \mapsto \pi_0(\phi) \).

d) If we denote \( \mathfrak{S}(\mathbb{N}) = \lim_{\rightarrow n} \mathfrak{S}_n \) the group of permutation of \( \mathbb{N} \) with finite support, we can still define a signature \( \mathfrak{S}_n(\mathbb{N}) \to \mathbb{Z}/2\mathbb{Z} \).

e) Finally, if \( K/\mathbb{Q} \) is a Galois extension, then \( \text{Gal}([\mathbb{Q}]/\mathbb{Q}) \to \text{Gal}(K/\mathbb{Q}) \), \( \sigma \mapsto \sigma|_K \) is a finite approximation.
(3.2) **Profinite completion.** Then, one can define very naturally the profinite completion of $G$ as the projective limit of all the finite approximations of $G$. More precisely, (and without dealing with any problem of set theory)

$$\hat{G} = \left\{ (g_v)_{v \in \text{App}_f(G)} \in \prod_v F_v \mid \forall \psi : v \rightarrow w, \psi(g_v) = g_w \right\}$$

which comes with the profinite projection

$$\hat{\pi} : G \rightarrow \hat{G} ; g \mapsto (\varphi_v(g))_{v \in \text{App}_f(G)}.$$

Intuitively, this object is what remains from $G$ when one can only deal with information of finite type; some elements will be identified but, at the same time, some new elements will appear. Formally, in general, $\hat{\pi}$ is not surjective or injective.

(3.3) **Surjective finite approximations.** Among the finite approximations, some are surjective; they form a full subcategory $\text{App}_s^f(G)$ of $\text{App}_f(G)$. In the same way that we have defined the profinite completion, we can then define the “surjective” profinite completion

$$\hat{G}^s = \lim_{\leftarrow v \in \text{App}_s^f(G)} F_v.$$

The important fact about this object is that we have the following fact, whose proof is not difficult.

(3.4) **Proposition.** The natural morphism $\hat{G} \rightarrow \hat{G}^s$ is an isomorphism.

---

(4) **Profinite completion of the additive group of a vector space over $F_p$.**

(4.1) **Profinite completion and double-dual.** Before looking at what happens in the situation where the base field is $F_p$, let us remark that, in the general case, there is a morphism of comparison between the profinite completion of an “additive group” and its double-dual. Let $k$ be a finite field and $V$ a vector space over $k$. We still denote by $V$ the underlying additive group.

Let $f$ and $g$ be two linear forms of $V$ and let $\lambda \in k$. The forms $f$, $g$ and $f + \lambda \cdot g$ are, in particular, finite approximations of $V$ (in the additive group of $k$) and we denote by $v_f$, $v_g$ and $v_{f+\lambda \cdot g}$ the corresponding approximations. Now, let $x = (x_v)_v \in \hat{V}$ be a “profinite” element.

(4.2) **Fact.** $x_{v_{f+\lambda \cdot g}} = x_{v_f} + \lambda \cdot x_{v_g}$.

**Proof:** Indeed, we have the following diagram of morphisms of finite approximations
Then, if we denote by \( w \) the approximation \( V \xrightarrow{(f,g)} k^2 \), the definition of the profinite completion imposes that \( x_v = p_1(x_w) \) and \( x_{v^q} = p_2(x_w) \) and \( x_{v+f+\lambda} = (p_1+\lambda \cdot p_2)(x_w) \), that is
\[
x_{v+f+\lambda} = x_v + \lambda \cdot x_{v^q}.
\]

Using this fact, one can define the morphism of comparison:
\[
\Psi : \hat{V} \longrightarrow V^{**}
\]
\[
(x_v) \longmapsto \left( \begin{array}{c} V^* \longrightarrow k \\ f \longmapsto x_v \end{array} \right)
\]

(4.3) The case where \( k = F_p \). From now on, \( p \) is a prime number and \( k = F_p \). The interesting case is when \( V \) is of infinite dimension. A good way to understand what happens is to consider \( V = (\mathbb{Z}/2\mathbb{Z})^N \).

The first thing to do is to see that if \( \varphi : V \rightarrow F \) is a finite surjective approximation, then \( F \) is isomorphic to (the additive group of) \((F_p)^n\) for some \( n \). Indeed, first of all, since \( F \) is the homomorphic image of \( V \), \( F \) is abelian. Moreover, all the elements of \( F \) satisfy \( x^p = e \). Thus, the classification of the abelian finite groups gives the conclusion.

We can now prove:

(4.4) Theorem Let \( V \) be a vector space over \( F_p \). Then, \( \Psi : \hat{V} \rightarrow V^{**} \) is an isomorphism.

Proof: We first prove that \( \Psi \) is injective: let \( x = (x_v) \in \hat{V} \) such that for all linear form \( f : V \rightarrow k \), \( x_v(f) = 0 \). Let \( v \) be a finite surjective approximation of \( V \); we can suppose that \( v = (k^n, \varphi) \), where \( \varphi : V \rightarrow k^n \) is any morphism. By composing \( \varphi \) with the \( n \) projections \( p_i \) to the factors \( k \), one obtain \( n \) morphisms. If we prove that the \( n \) corresponding elements are equal to 0, then, it will follow that \( x_v \) is equal to 0 and, thus, that \( \Psi \) is injective. But, and it is the (easy) key point, a morphism \( V \rightarrow k \) of groups is actually a linear form, since we can rewrite the condition \( \varphi(\lambda \cdot \bar{v}) = \lambda \cdot \varphi(\bar{v}) \) as \( \varphi(\bar{v} + \cdots + \bar{v}) = \varphi(\bar{v}) + \cdots + \varphi(\bar{v}), \) for our base field is \( F_p \). And, by assumption, all the \( x_{v_i} = 0 \).

For the surjectivity, let \( \Theta \in V^{**} \) be a double-dual element. We would like to find a profinite element \( x = (x_v) \in \hat{V} \) such that, for all linear form \( f \) of \( V \), one have \( x(v)(f) = \Theta(f) \). So, let \( v = (k^n, \varphi) \) (as we can suppose it) be a finite approximation of \( V \). Let denote \( p_1, \ldots, p_n \) the \( n \) projections of \( k^n \) to the factors \( k \). Naturally, we define \( x_v \) by reconstructing it from the linear forms \( p_i \circ \varphi \):
\[
x_v := (\Theta(p_1 \circ \varphi), \Theta(p_2 \circ \varphi), \ldots, \Theta(p_n \circ \varphi)) \in k^n.
\]

Now, we just have to check that the family \( (x_v) \) is "compatible". So let \( v = (k^n, \varphi) \) and \( w = (k^n, \psi) \) be two finite approximations and \( g \) a morphism between them:

\[
\begin{array}{ccc}
V & \xrightarrow{g} & k^m \\
\phi & \downarrow & \downarrow \psi \\
k^n & \xleftarrow{\theta} & k^n
\end{array}
\]

We want to prove that \( g(x_v) = x_w \). By composing with the \( m \) projections \( q_i \) of \( k^m \), it suffices to prove it in the case where \( m = 1 \):

\[
\begin{array}{ccc}
V & \xrightarrow{g} & k \\
\phi & \downarrow & \downarrow q_1 \\
k^n & \xleftarrow{\theta} & k
\end{array}
\]
So, we are brought to this situation

\[
\begin{array}{ccc}
V & \xrightarrow{\varphi} & k^n \\
\downarrow & & \downarrow \\
\psi & \xleftarrow{\psi} & k
\end{array}
\]

where we know that \( g \) can be written as \( g = \lambda_1 \cdot p_1 + \cdots + \lambda_n \cdot p_n \), with \( \lambda_i \in k \). The fact that the previous diagram commutes tells us that \( \psi = \sum_i \lambda_i \cdot (p_i \circ \varphi) \); and, now:

\[
\begin{align*}
g(x_v) &= g((\Theta (p_1 \circ \varphi), \Theta (p_2 \circ \varphi), \ldots, \Theta (p_n \circ \varphi))) \\
&= \sum_i \lambda_i \cdot \Theta (p_i \circ \varphi) \\
&= \Theta \left( \sum_i \lambda_i \cdot (p_i \circ \varphi) \right) = \Theta (\psi) \\
&= x_w,
\end{align*}
\]

which concludes the proof. \( \blacksquare \)

**(4.5) \( \hat{\pi} \) and the canonical injection** \( i : V \to V^{**} \). We denote \( i : V \to V^{**} \) the canonical injection defined by \( i(\vec{v})(f) = f(\vec{v}) \). One can improve a bit the theorem \( \hat{\pi} \): the isomorphism \( \Psi \) between \( \hat{V} \) and \( V^{**} \) through \( \Psi \) identifies \( \hat{\pi} \) with \( i \). The proof is easy.

**Theorem.** Let \( V \) be a vector space over \( \mathbb{F}_p \). Then, \( \Psi : \hat{V} \to V^{**} \) is an isomorphism and the diagram

\[
\begin{array}{ccc}
\hat{V} & \xrightarrow{\Psi} & V^{**} \\
\downarrow & & \downarrow \\
V & \xleftarrow{i} & V^{**}
\end{array}
\]

commutes.

**Remark.** One can prove the theorem \( \hat{\pi} \) with more abstracted arguments. To begin with, we know (cf. for example [Par70, §2.7, theorem 2]) that, in a general category \( C \), if the limits exist, we always have the natural isomorphism

\[
\text{Hom} \left( \lim_{\longrightarrow} X_i, X \right) \cong \lim_{\longleftarrow} \text{Hom} (X_i, X).
\]

Moreover, in the case of \( k \)-vector spaces, this isomorphism is linear; thus, for a system of \( k \)-vector space \( V_i \), we have:

\[
\left( \lim_{\longrightarrow} V_i \right)^* \cong \lim_{\longleftarrow} (V_i^*).
\]

Let \( k \), from now on, be a field and \( V \) a \( k \)-vector space. If we denote by \( (Y_i) \) the system of finite-dimensional subvector spaces of \( V^* \), we have \( V^* = \lim_{\longrightarrow} Y_i \) and thus, thanks the previous isomorphism:

\[
V^{**} \cong \lim_{\longrightarrow} (Y_i^*).
\]

Moreover, there is a natural bijection between the finite-dimensional subspaces of \( V^* \) and the finite-codimensional subspaces of \( V \), via the application

\[
Y \mapsto Y^\perp := \{ v \in V \mid \forall \varphi \in Y, \varphi(v) = 0 \}.
\]
Besides, if \( Y \) is a finite-dimensional subspace of \( V^* \) then the dual \( Y^* \) is naturally isomorphic to \( V/Y^\bot \). Consequently, if we denote by \((Z_j)\) the system of finite-codimensional subspaces of \( V \), we have:

\[
V^{**} \cong \lim_{\leftarrow j} (V/Z_j)
\]

But, if \( k = \mathbb{F}_p \) for a prime number \( p \), one can identify the \( k \)-vector space \( V \) with its underlying additive group\(^2\) \( \omega(V) \), its dual \( V^* \) with \( \text{Hom}_{\text{Gr}}(\omega(V), \omega(\mathbb{F}_p)) \), and its finite dimensional quotients with the finite quotient of \( \omega(V) \). We thus finally get the expected alternative proof of the theorem 4.4.

---

(5) A family of counter-examples.

One would like to know if, given a group \( G \), one have \( \hat{\hat{G}} \cong \hat{G} \). This fact is known to be false (cf. example 4.2.13 of [RZ00]), but as we will see, it is still false, in general, after taking \( i \) times the profinite completion.

(5.1) The sequence of \( i \)-th profinite completions. We introduce the following notation. If \( G \) is a group, we denote \( \hat{\hat{G}} = \hat{\hat{G}}^1 \) and \( \hat{\hat{G}}^{i+1} = \hat{\hat{G}}^i \). These groups come with projections, as follows:

\[
\begin{array}{cccccccc}
G & \overset{\hat{\hat{\pi}}^1}{\longrightarrow} & \hat{\hat{G}}^1 & \overset{\hat{\hat{\pi}}^2}{\longrightarrow} & \hat{\hat{G}}^2 & \cdots & \overset{\hat{\hat{\pi}}^{i+1}}{\longrightarrow} & \hat{\hat{G}}^{i+1} & \cdots \\
\end{array}
\]

We will prove that, in general, none of the \( \hat{\hat{\pi}}^i \) is an isomorphism.

(5.2) Proposition. Let \( p \) be a prime number and \( k = \mathbb{F}_p \). Let \( V \) be (the additive group of) a \( k \)-vector space of infinite dimension. Then, in the following sequence

\[
\begin{array}{cccccccc}
V & \overset{\hat{\pi}^1}{\longrightarrow} & \hat{V}^1 & \overset{\hat{\pi}^2}{\longrightarrow} & \hat{V}^2 & \cdots & \overset{\hat{\pi}^{i+1}}{\longrightarrow} & \hat{V}^{i+1} & \cdots \\
\end{array}
\]

all the \( \hat{\pi}^i \) are injective but non-surjective morphisms.

Proof : This follows from the identification of the arrows \( \hat{\pi}^i \) with the canonical injections of a vector space in its double-dual, and from the fact that these injections are injective but non-surjective when the vector spaces are of infinite dimension, cf. Théorème 6, §7, n°5 of [Bou62].

\[\square\]

\(^2\)We denote \( \omega : k \rightarrow \text{Vs} \rightarrow \text{Gr} \) the forgetful functor from the category of \( k \)-vector spaces to the category of groups.
(6) Conclusion : abstract setting vs. topological setting.

This study has been given for groups but a similar point of view can be applied to topological groups. In this case, we start with a topological group \( G \) and we consider the category \( \text{App}_{\text{discr}}(G) \) of finite and discrete approximations: they are couples \( v = (F, \varphi) \), where \( F \) is a discrete and finite topological group and \( \varphi : G \to F \) a continuous morphism of groups.

One obtain the (topological) profinite completion of \( G \), wich is, as well-known, a topological group, compact and totally disconnected (cf. [Ser02]), and one obtain a profinite projection, which is a continuous morphism:

\[
\hat{\pi}^{\text{top}} : G \to \hat{G}^{\text{top}}.
\]

More generally, as previously done, one can define the sequence of iterated (topological) profinite completions:

\[
G \overset{\hat{\pi}^{[1], \text{top}}}{\longrightarrow} \hat{G}^{[1], \text{top}} \longrightarrow \cdots \longrightarrow \hat{G}^{[i], \text{top}} \overset{\hat{\pi}^{[i+1], \text{top}}}{\longrightarrow} \hat{G}^{[i+1], \text{top}} \longrightarrow \cdots.
\]

The situation is then totally different than before. Indeed, we have:

(6.1) Proposition. Let \( G \) be a topological group. Then, for all \( i \geq 2 \), the arrows \( \hat{\pi}^{[i], \text{top}} \) are isomorphisms of topological groups.

(6.2) Profinite groups: abstract setting and topological setting. There is a synthetical way to see the fundamental difference between the propositions 4.2 and 6.1. For this sake, we introduce two notions of profinite groups. We will say that a group \( G \) is profinite if it is the projective limit of a system of finite groups; we will say that a topological group \( G \) is topologically profinite if it is the projective limit of a system of finite and discrete groups. We then have:

(6.3) Theorem. Let \( G \) be a topological group. Then:

\[
G \text{ is topologically profinite } \iff \hat{\pi}^{\text{top}} : G \to \hat{G}^{\text{top}} \text{ is an isomorphism.}
\]

(6.4) Proposition. Let \( G \) be a group. Then:

\[
G \text{ is profinite } \iff \hat{\pi} : G \to \hat{G} \text{ is an isomorphism}
\]

\[
G \text{ is profinite } \Rightarrow \hat{\pi} : G \to \hat{G} \text{ is an isomorphism.}
\]

(6.5) A positive answer. One could legitimately be disapointed by the non-equivalence of \( G \) being profinite and \( \hat{\pi} \) being an isomorphism. Indeed, on the one hand, there is the very classical definition of a profinite group and, on the other hand, there is the deep property for a group to have its profinite projection \( \hat{\pi} \) to be an isomorphism (such a group, in a way, is separated — for \( \hat{\pi} \) is injective — and complete — for \( \hat{\pi} \) is surjective). One would have expected these two to coincide...

Fortunately, there is a positive result in this direction. It is a difficult result, which has been published in 2007 by Nikolay Nikolov and Dan Segal, cf. [NS07a] and [NS07b], and whose proof uses the classification of finite simple groups. In order to state their result, let us remark that if \( G \) is an (abstract) profinite group, if we write \( G = \lim_{\longrightarrow} F_i \), where the \( F_i \)'s are finite, and if we endow each of the \( F_i \)'s with the discrete topology, then we can view \( G \) as a topological group.

(6.6) Theorem. Let \( G \) be an (abstract) profinite group, which is topologically of finite type for the associated topology. Then, \( \hat{\pi} : G \to \hat{G} \) is an isomorphism.
(6.7) Acknowledgments. My first acknowledgments go to Xavier Caruso for many helpful discussions. I would like also to thank the referee for many valuable comments and for making me known the alternative proof.

References

[Bou62] Nicolas Bourbaki. *Éléments de mathématique. Première partie. Fascicule VI. Livre II: Algèbre. Chapitre 2: Algèbre linéaire*. Troisième édition, entièrement refondue. Actualités Sci. Indust., No. 1236. Hermann, Paris, 1962.

[NS07a] Nikolay Nikolov and Dan Segal. On finitely generated profinite groups. I. Strong completeness and uniform bounds. *Ann. of Math. (2)*, 165(1):171–238, 2007.

[NS07b] Nikolay Nikolov and Dan Segal. On finitely generated profinite groups. II. Products in quasisimple groups. *Ann. of Math. (2)*, 165(1):239–273, 2007.

[Par70] Bodo Pareigis. *Categories and functors*. Translated from the German. Pure and Applied Mathematics, Vol. 39. Academic Press, New York, 1970.

[RZ00] Luis Ribes and Pavel Zalesskii. *Profinite groups*, volume 40 of *Ergeb. Math. Grenzgeb. (3)*. Springer-Verlag, Berlin, 2000.

[Ser02] Jean-Pierre Serre. *Galois cohomology*. Springer Monogr. Math. Springer-Verlag, Berlin, english edition, 2002. Translated from the French by Patrick Ion and revised by the author.