Testing spontaneous wave-function collapse models on classical mechanical oscillators

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We show that the heating effect of spontaneous wave-function collapse models implies an experimentally significant increment $\Delta T_{sp}$ of equilibrium temperature in a mechanical oscillator. The obtained form $\Delta T_{sp}$ is linear in the oscillator’s relaxation time $\tau$ and independent of the quantum state which can be a classical superpositions of quantum states of massive degrees of freedom, also called Schrödinger Cat states, decay at (model dependent) universal rates. These models, the particular gravity-related (or DP) model \[2\] and the continuous spontaneous localization (CSL) model \[4, 7\] predict the progressive violation of the quantum mechanical superposition principle for massive degrees of freedom. For atomic degrees of freedom this violation is irrelevant while for massive degrees of freedom it becomes significant though usually masked by the environmental noise. The preparation of Schrödinger Cat states is extremely demanding hence the direct experimental test of spontaneous collapse has not yet been achieved despite relentless efforts, see, e.g., \[8–14\], and \[15, 16\] for the state-of-the-art. Quite recently, Bahrami et al. \[17\] suggested a different approach, not requesting laboratory Schrödinger Cat states. Nimmrichter et al. \[18\] discuss the optomechanical sensing of spontaneous momentum diffusion caused by collapse models. We further elucidate and simplify these considerations and come to new results. We emphasize that momentum diffusion is classical and frequency $\Omega$, with Hamiltonian

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\Omega^2\hat{x}^2. \quad (1)$$

If the mass is subject to spontaneous collapse, the density matrix $\hat{\rho}$ satisfies the following master equation:

$$\frac{d\hat{\rho}}{dt} = -\frac{i}{\hbar} [\hat{H}, \hat{\rho}] - \frac{D_{sp}}{\hbar^2} \left[\hat{x}, [\hat{x}, \hat{\rho}] \right] \quad (2)$$

where $D_{sp}$ governs the strength (rate) of spontaneous decoherence. This $\dot{x}$-decoherence is observable: it is simply equivalent with $\dot{p}$-diffusion of diffusion constant $D_{sp}$.

From now on and through our work, we assume that the oscillator is in the classical domain. Therefore we can describe it by the classical Liouville density $\rho(x, p)$ and the quantum master equation \[2\] can be replaced by the Liouville equation

$$\frac{d\rho}{dt} = \{H, \rho\} + D_{sp} \frac{\partial^2}{\partial p^2} \rho. \quad (3)$$

$H(x, p)$ is the classical Hamilton function of the oscillator, the Poisson bracket $\{H, \rho\}$ stands for $(p/m) \frac{\partial}{\partial x} \rho - m\Omega^2 x \frac{\partial^2}{\partial p^2} \rho$. In a realistic situation, the mechanical oscillator is in a thermal environment of temperature $T$, which will modify the Liouville equation:

$$\frac{d\rho}{dt} = \{H, \rho\} + D_{sp} \frac{\partial^2}{\partial p^2} \rho + \eta \frac{\partial}{\partial p} \rho + D_{th} \frac{\partial^2}{\partial p^2} \rho, \quad (4)$$

where $\eta$ is the damping rate of oscillations and $D_{th} = \eta mk_BT$ is the constant of thermal momentum-diffusion. With $D_{sp} = 0$ we would get the classical Fokker-Planck equation whose stationary solution is the Gibbs canonical distribution $\mathcal{N}\ exp(-H/k_BT)$. It is trivial to see that
with $D_{sp} > 0$ the stationary solution is the Gibbs canonical distribution
\[
\rho_\infty(x, p) = N \exp \left( -\frac{H(x, p)}{k_B T'} \right)
\]
(5)

at the higher temperature
\[
T' = \left( 1 + \frac{D_{sp}}{D_{th}} \right) T \equiv T + \Delta T_{sp}.
\]
(6)

This result can be interpreted as the extension of the Einstein-Smoluchowski relationship $D_{th} = \eta m k_B T$ for $D_{th} + D_{sp} = \eta m k_B T'$, supported by the underlying Fokker-Planck equation.

The increment $\Delta T_{sp} > 0$ over the environmental temperature $T$ is the contribution of spontaneous heating, this is the very observable quantity that we wish to test. From Eq. (6), we can express it as
\[
\Delta T_{sp} = \frac{D_{sp}}{mk_B} \tau,
\]
(7)

where $\tau = 1/\eta$ will stand for the (energy) relaxation time of the oscillator. Our classical description is valid as long as the spontaneous heating concerns many quanta of the oscillator:
\[
k_B \Delta T_{sp} \gg \hbar \Omega.
\]
(8)

Measurement. Since we restrict ourselves for the classical domain of spontaneous heating $\Delta T_{sp}$, a single-shot classical (or quantum) measurement of precision $\delta T_m$ would detect $\Delta T_{sp}$ provided $\delta T_m \lesssim \Delta T_{sp}$. If this condition does not hold, we repeat the same measurement many times, like in quantum state tomography. Observe that quantum state monitoring is not necessary, tomography is the more suitable means to detect spontaneous temperature increase of the previously prepared equilibrium oscillator state. Cumulative precision of tomography is not limited quantum theoretically.

For completeness, nonetheless, let us recapitulate the features of monitoring which is usually accompanied by some classical and/or quantum noise (back-action). We characterize this back-action by a further diffusion constant $D_m$. The complete Liouville equation reads:
\[
\frac{d\rho}{dt} = \{H, \rho\} + \frac{\partial}{\partial p} \rho \frac{\partial}{\partial p} (D_{sp} + D_{th} + D_m) \frac{\partial^2}{\partial p^2} \rho.
\]
(9)

Suppose we start to measure the temperature of the oscillator at $t = 0$. The initial state of the oscillator is the Gibbs state of temperature $T + \Delta T_{sp}$. When the ‘thermometer’ is switched on, the measurement noise starts to heat the oscillator towards the new stationary Gibbs state of temperature increased by
\[
\Delta T_m = \frac{D_m}{mk_B} \tau.
\]
(10)

Trivial dynamics of heating follows from Eq. (9) in the limit $\eta \ll \Omega$:
\[
T'(t) = T + \Delta T_{sp} + (1 - e^{-t/\tau}) \Delta T_m.
\]
(11)

Observe that the temperature effect of back-action is gradually reaching its steady state value. Back-action can be ignored for times much shorter than $\Delta T_m/\Delta T_{sp}$ times $\tau$.

There is no fundamental limitation on the measurement precision (fluctuations) $\delta T_m$ in the classical domain. There is a quantum tradeoff between the spectral components of $\delta T_m$ and $\Delta T_m$ at a chosen frequency $\omega$:
\[
\delta T_m \Delta T_m \geq \frac{\hbar^2}{4 k_B^2} \left( \Omega^2 - \omega^2 + i \eta \omega / 2 \right)^2 / \eta^2.
\]
(12)

as it follows from Refs. [20], cf. also Ref. [18]. The minimum of $\delta T_m + \Delta T_m$ is achieved when
\[
\delta T_m = \Delta T_m = \frac{\hbar}{2 k_B} \left( \frac{\Omega^2 - \omega^2 + i \eta \omega / 2}{\eta} \right) \equiv \Delta T_{SQL}
\]
(13)

which is called the standard quantum limit. This limitation concerns the steady state spectral component of the precision and back-action, respectively. For monitoring duration much shorter than $\tau$ (yet sufficient to gather significant data on $\Delta T_{sp}$) the back-action won’t influence the system, we can choose finer precisions $\delta T_m$ than $\Delta T_{SQL}$.

| $10^2$ | $10^3$ | $10^4$ | $10^5$ | $10^6$ |
|--------|--------|--------|--------|--------|
| $10^5$Hz | $10^{-8}$K | $10^{-7}$K | $10^{-6}$K | $10^{-5}$K | $10^{-4}$K |
| $10^4$Hz | $10^{-7}$K | $10^{-6}$K | $10^{-5}$K | $10^{-4}$K | $10^{-3}$K |
| $10^3$Hz | $10^{-6}$K | $10^{-5}$K | $10^{-4}$K | $10^{-3}$K | $10^{-2}$K |
| $10^2$Hz | $10^{-5}$K | $10^{-4}$K | $10^{-3}$K | $10^{-2}$K | $10^{-1}$K |
| $1$Hz | $10^{-4}$K | $10^{-3}$K | $10^{-2}$K | $10^{-1}$K | $1$K |

TABLE I: Magnitudes of spontaneous heating effect $\Delta T_{DP}$ of the DP-model on classical oscillators are shown at currently available or nearly available combinations of frequencies $\Omega$ and quality factors $Q$. The spatial resolution $\sigma_{DP} = 10^{-12}$ cm assumes the strongest effect, lattice constant is set to 500 pm. Data around the upper-left corner (it brackets) are not in the classical domain $k_B \Delta T_{DP} \gg \hbar \Omega$. Data above the millikelvin range are enhanced (typed in boldface) because their detection may not request millikelvin cooling or cooling at all.

Spontaneous heating: DP-model. In the gravity-related spontaneous collapse model (DP-model), the spontaneous diffusion is proportional to the Newton constant $G$. For the simple example of oscillator mass considered in [18]:
\[
D_{DP} = \frac{\hbar}{2 m \omega_G^2} = \frac{\hbar}{2 m} \frac{4 \pi G \rho G}{3} \left( \frac{a}{2 \sqrt{\pi \sigma_{DP}}} \right)^3
\]
(14)
where $g$ is the mass density, and $a$ is the lattice constant, while $\omega_G$ is the effective parameter used by $\sigma_{DP}$ in the DP-model, conjectured to be in the following range $[3,4]$:

$$10^{-12} \text{cm} \lesssim \sigma_{DP} \lesssim 10^{-5} \text{cm}. \quad (15)$$

The expression (14) is valid for $\sigma_{DP} \ll a$. In this range, $D_{DP}$ is independent of the shape of the mass while it depends on its microscopic structure. Using (14) for $D_{sp}$, we can write (7) as

$$\Delta T_{DP} = \frac{\hbar \omega_G^2}{2 k_B} \tau, \quad (16)$$

where $\omega_G^2$ is read out from (14). It is remarkable that $\Delta T_{DP}$ does not depend on the mass $m$.

Now we assume the strongest possible DP-decoherence, i.e., we take the finest conjectured spatial resolution $\sigma_{DP} = 10^{-12}$ cm, also favored by particular arguments $[3,4]$. If the lattice constant is set to $a = 5 \times 10^{-8}$ cm, for concreteness, we obtain $\omega_G \approx 1.3$ kHz for the effective parameter. The spontaneous heating effect (16) can be written as

$$\Delta T_{DP} \approx \tau [s] \times 4.0 \times 10^{-5} K. \quad (17)$$

This is a convenient expression of the effect $\Delta T_{DP}$ to discuss possible choices of the frequency $\Omega$ and the quality factor $Q = \Omega \tau$ of the oscillator. The mass $m$ has, as we noticed before, canceled from $\Delta T_{DP}$.

**Experimental implications.** Applying Eq. (17) to a broad range of frequencies $\Omega$ and quality factors $Q$, we calculated the spontaneous heating $\Delta T_{DP}$ in Table II.

| System                          | $m$  | $\Omega/2\pi$ (Hz) | $Q$  | $T$ (K) | $\Delta T_{DP}$ (K) |
|---------------------------------|------|---------------------|------|---------|---------------------|
| gravitational wave detector $[22]$ | 40 kg | 1                   | 25000| 300     | 0.16                |
| suspended disc $[19]$           | 5 mg | 0.5                 | $5 \times 10^6$ | 300 | 6.4              |
| SiN membrane $[23]$            | 34 ng | $1.6 \times 10^6$  | 1100 | 4.9     | $[4.4 \times 10^{-9}]$ |
| aluminium membrane $[24]$      | 48 pg | $1.1 \times 10^7$  | $3.3 \times 10^5$ | 0.015 | $[1.9 \times 10^{-7}]$ |

TABLE II: Spontaneous heating $\Delta T_{DP}$ for the selection of opto-mechanical setups quoted in $[15]$. Values $\Delta T_{DP}$ are calculated from Eq. (16), assuming the largest spontaneous decoherence rates considered for the time being, corresponding to $\omega_G = 1.3$ kHz. Two of the data (in brackets) are not in the classical domain $k_B \Delta T_{DP} \gg \hbar \Omega$.

The lesson is transparent. If $\Delta T_{DP} \gg \hbar \Omega/k_B$, and this is the case except for a few highest $\Omega$ and lowest $Q$ examples (in brackets), the DP-effect would prevent us from ground state cooling. This should be a significant detectable effect. But we do not need to try ground state cooling, the heating effect $\Delta T_{DP}$ equally shows up far from the ground state. Low frequency oscillators with high quality factors are the favorable testbed. If the ring-down time $\tau = Q/\Omega$ of the oscillator is chosen between $10^2$ s and $10^6$ s, the spontaneous heating $\Delta T_{DP}$ scales between $1$ mK and $10$ K, respectively. This is a striking result. It is clear that classical (non-quantum) ‘thermometers’ of precision $\delta T_m \sim 1$ mK should exist. Technically, nonetheless, we might need to operate the measurement device in the quantum domain especially when the oscillator itself cooled and/or controlled via high precision quantum devices. Even in this case the oscillator is assumed to stay away from its ground state since the effect $\Delta T_{DP}$ is robust classical.

Following Ref. $[18]$, and for a selection of experiments considered therein, we calculated the effect $\Delta T_{DP}$, see Table II. The experiments $[22]$ and $[19]$, both performed at room temperature $T = 300$ K, might be the promising ones. On the one hand, cooling is a reserve of higher sensitivity of detecting $\Delta T_{DP}$. On the other hand, the experiment $[19]$ even at room temperature must be sensitive to the $6.4$ K spontaneous warming up.

As we mentioned before, monitoring may be neither convenient nor sufficient for detection. Let us consider the constraint (12) at the detection band around $\omega = 2\pi \times 500$, yielding $\delta T_m \Delta T_m = \Delta T_{SQL} = (37 K)^2$. Such a standard quantum limit $37$ K gives insufficient precision on the steady state, i.e.: in monitoring of duration much longer than $\tau = 1.6 \times 10^5$ s. If we choose $\delta T_m = 1$ K the duration of monitoring must be limited to the order of hundred seconds before the back-action reaches the range of $1$ K. This is obviously not the way to go in general. In this particular experiment measurement precisions below $1$ K are not available by standard quantum monitoring. A single-pulse measurement must be considered instead, where state preparation is followed by a single one-shot measurement and the preparation-detection cycle is repeated many times.

**CSL-model.** In the CSL model the diffusion constant is proportional to the rate parameter $\lambda_{CSL}$. For the perpendicular momentum diffusion of a disk of thickness $d$
Recall that the strongest DP-effect turned out to be discussed in Ref. [18] in the context of the CSL model. The so far hypothetic spontaneous wave-function collapse on massive degrees of freedom possesses the classical effect must be testable classically, without facing the standard quantum limitations of sensing. Therefore we must get spontaneous diffusion in the cross hairs instead of spontaneous collapse. We have derived the spontaneous heating $\Delta T_{sp}$ for mechanical oscillators in classical thermal state, only using the classical Einstein-Smoluchowski relation, and found that $\Delta T_{sp}$ is proportional to the relaxation (ring-down) time, is independent of the mass. Experimental implications become transparent for both leading models DP and CSL of spontaneous collapse. We conclude that currently available extreme low-loss mechanical oscillators can already confirm the presence of spontaneous diffusion if its rate is close to the conjectured maximum. Alternatively, they enforce the update of the current constraints, cf. in Refs. [1, 21], on collapse model’s parameters. The requested measurement precisions $1 \text{mK} - 1 \text{K}$ may not be reached in standard steady state quantum monitoring. We suggested that state tomography will fit the demands. 

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\[ D_{\text{CSL}} = \lambda_{\text{CSL}} \frac{\hbar^2}{m_0} 4\pi \sigma_{\text{CSL}}^2 \frac{\theta m}{d}. \]  

(18)

where $m_0$ is the standard atomic unit. The value of the CSL collapse rate parameter has been constrained by a lower [6] and an upper estimate [5], cf. also [1]:

\[ 2.2 \times 10^{-17} \text{Hz} \lesssim \lambda_{\text{CSL}} \lesssim 2.2 \times 10^{-8 \pm 2} \text{Hz}. \]  

(19)

Using $D_{\text{CSL}}$ for $D_{sp}$ in (17) yields

\[ \Delta T_{\text{CSL}} = \lambda_{\text{CSL}} \frac{\hbar^2}{m_0 k_B} 4\pi \sigma_{\text{CSL}}^2 \frac{\theta}{d}. \]  

(20)

Note that the shape (thickness) of the oscillator matters, the mass $m$ does not.

Suppose the strongest CSL decoherence rate from the range [19], let’s take the estimate $\lambda_{\text{CSL}} = 2.2 \times 10^{-8 \pm 2}$ Hz [7]. Using this value in (20) we obtain

\[ \Delta T_{\text{CSL}} \approx \tau[s] \frac{\sigma[\text{g/cm}^3]}{d[\text{cm}]} \times 3.2 \times 10^{-6 \pm 2} K. \]  

(21)

Recall that $d \gg \sigma_{\text{CSL}} = 10^{-5} \text{cm}$, hence the strongest heating effect is achieved when $d \approx \sigma_{\text{CSL}}$, leading to

\[ \Delta T_{\text{CSL}} \approx \tau[s] \times 6.2 \times 10^{-1 \pm 2} K, \]  

(22)

where we kept $\sigma = 2 \text{ g/cm}^3$ as before. Comparing this result with (17) we conclude that, in classical oscillators, the strongest conjectured CSL effect $\Delta T_{\text{CSL}}$ would exceed the strongest conjectured DP effect $\Delta T_{\text{DP}}$ by at least two orders of magnitude.

Let us consider the $\Omega = 3.14$ Hz oscillator [19], also discussed in Ref. [18] in the context of the CSL model. Recall that the strongest DP-effect turned out to be $\Delta T_{\text{DP}} = 6.4 \text{ K}$, cf. Table 1. This oscillator has the high quality factor $Q = 5 \times 10^8$, the ring-down time is extreme long: $\tau = 1.6 \times 10^9 \text{s}$. The resonator is a $5 \text{ mg}$ disk of thickness $d = 0.2 \text{ mm}$, Eq. (21) yields the spontaneous heating $\Delta T_{\text{CSL}} = 5.1 \times 10^{1 \pm 2} \text{ K}$, corresponding to the rates $\lambda_{\text{CSL}} = 2.2 \times 10^{-8 \pm 2}$, respectively. Obviously the values $\lambda_{\text{CSL}} \gtrsim 10^{-8}$ are not compatible with the experiment and the values $\lambda_{\text{CSL}} \sim (10^{-9} - 10^{-10})$ remain to be challenged.

Summary. The so far hypothetic spontaneous wave-function collapse on massive degrees of freedom possesses a complementary classical effect: classical momentum diffusion. This produces a certain spontaneous increase $\Delta T_{\text{sp}}$ of the equilibrium temperature. This typical classical effect must be testable classically, without facing the standard quantum limitations of sensing. Therefore we must get spontaneous diffusion in the cross hairs instead of spontaneous collapse. We have derived the spontaneous heating $\Delta T_{\text{sp}}$ for mechanical oscillators in
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