ROOT PARAMETRIZED DIFFERENTIAL EQUATIONS FOR THE CLASSICAL GROUPS

MATTHIAS SEISS

Abstract. Let $C(t_1, \ldots, t_l)$ be the differential field generated by $l$ differential indeterminates $t = (t_1, \ldots, t_l)$ over an algebraically closed field $C$ of characteristic zero. We develop a lower bound criterion for the differential Galois group $G(C)$ of a matrix parameter differential equation $\partial(y) = A(t)y$ over $C(t_1, \ldots, t_l)$ and we prove that every connected linear algebraic group is the Galois group of a linear parameter differential equation over $C(t_1)$. As a second application we compute explicit and nice linear parameter differential equations over $C(t_1, \ldots, t_l)$ for the groups $SL_{l+1}(C)$, $SP_{2l}(C)$, $SO_{2l+1}(C)$, $SO_{2l}(C)$, i.e. for the classical groups of type $A_l$, $B_l$, $C_l$, $D_l$, and for $G_2$ (here $l = 2$).

Introduction

Let $C$ be an algebraically closed field of characteristic zero and consider a linear algebraic group $G$ over $C$. In differential Galois theory one of the classical questions asks whether we can realize $G(C)$ as a differential Galois group over some differential field $F$ with constants $C$ and if the answer is positive can we construct an explicit linear differential equation with differential Galois group $G(C)$? Answers are only partially known for some groups and fields.

In some settings bound criteria for the differential Galois group play an essential role. A well-established upper bound criterion is given in Proposition 1.1. It states that if the defining matrix $A$ of a linear differential equation $\partial(y) = Ay$ is contained in the Lie algebra $g(F)$ of $G(F)$, then the differential Galois group is a subgroup of $G(C)$. In the literature there is also a well-known lower bound criterion. Roughly speaking Proposition 1.2 says that a suitable differential conjugate of the defining matrix $A$ lies in the Lie algebra of the differential Galois group. But for a successful application of Proposition 1.2 we need to guarantee that a geometric condition is satisfied. This condition is automatically fulfilled if the differential base field is the field of rational functions $C(z)$ with standard derivation $\frac{d}{dz}$. Using these bounds C. Mitschi and M. Singer developed in [18] a constructive method for the realization of connected linear algebraic groups over $C(z)$ and the general inverse problem over the same differential base field could be solved by J. Hartmann in [7].

Let $C(t)$ be the differential field generated by $l$ differential indeterminates $t = (t_1, \ldots, t_l)$ over $C$ and for a differential indeterminate $y$ denote by $C(t\{y\})$ the differential ring generated by $y$ over $C(t)$. In the following we mean by a matrix parameter differential equation a matrix differential equation $\partial(y) = A(t)y$ with $A(t) \in C(t)^{n \times n}$ and by a linear parameter differential equation an equation of the form

$$L(y, t) = y^{(n)} - \sum_{i=0}^{n-1} a_i(t)y^{(i)} \in C(t_1, \ldots, t_l\{y\}).$$

It is well-known that two such types of equations can be converted into each other (see for instance [14]). Considering the structure of the Lie algebras of the classical groups we found matrix parameter differential equations which define naturally very nice and at the
same time very general linear parameter differential equations (see Theorem 0.2 below). Motivated by this observation we want to determine the differential Galois group of a matrix parameter differential equation. As in [15] we intend to apply bound criteria for the differential Galois group to the defining matrix $A(t)$. As an upper bound we can use again Proposition 1.1. But unfortunately, we cannot use Proposition 1.2 as a lower bound criterion, since we have no information whether the assumption of Proposition 1.2 is satisfied or not. In the first part of this article we develop a lower bound criterion for the differential Galois group of a matrix parameter differential equation. For a defining matrix $A(t) \in C(t)^{n \times n}$, let $R_1$ be a localization of $C\{t\}$ by a finitely generated multiplicative subset of $C\{t\}$ such that $A(t) \in R_1^{n \times n}$, where $C\{t\}$ denotes the differential ring generated by the indeterminates $t$ over $C$. We denote by $G(C)$ the differential Galois group of $\partial(y) = A(t)y$ over $C(t)$. Further let $\sigma : C\{t\} \rightarrow C[z]$ be a specialization of the differential indeterminates to the differential subring $C[z]$ of $C(z)$ such that it extends to a specialization $\sigma : R_1 \rightarrow R_2$ where $R_2$ is a suitable finitely generated localization of $C[z]$. One result of this article is the following theorem (see also Theorems 1.4 and 4.3).

**Theorem 0.1.** (Specialization Bound)
The differential Galois group $H(C)$ of the specialized equation $\partial(y) = A(\sigma(t))y$ over $C(z)$ is a subgroup of $G(C)$.

The proof uses differential maps from the coordinate rings for GL$_n$ over $R_1$ and $R_2$ to suitable rings of power series, where the differential structure on these coordinate rings is defined by $A(t)$ and $A(\sigma(t))$ accordingly. These mappings are obtained from Taylor maps, which we will construct in Chapter 3. The images of these Taylor maps will then define Picard-Vessiot rings. In Chapter 4 we will show that we can specialize the coefficients of the corresponding power series appropriately such that $\sigma$ extends to a specialization of Picard-Vessiot rings. We prove then our specialization bound. We want to note that L. Goldman presented in [8] a similar approach using so-called analytic specializations.

As a first application we show in Chapter 5 that for every connected linear algebraic group there exists a linear parameter differential equation in one parameter. In the special case of a semisimple linear algebraic group of Lie rank $l$, we prove that there exists a parameter equation in $l$ parameters which specializes to all differential equations of a specific type over $C(z)$ and that we cannot remove a parameter without loosing some of these equations.

In the second part of this article we present a more interesting application of our specialization bound. We will prove Theorem 0.2 below. It gives explicit linear parameter differential equations for the groups $\text{SL}_{l+1}(C)$, $\text{SP}_{2l}(C)$, $\text{SO}_{2l+1}(C)$, $\text{SO}_{2l}(C)$, i.e. for the classical groups of type $A_l$, $B_l$, $C_l$, $D_l$, and for $G_2$ (here $l = 2$).

**Theorem 0.2.** The linear parameter differential equation

1. $L(y, t) = y^{(l+1)} - \sum_{i=1}^{l} t_i y^{(i-1)} = 0$ has $\text{SL}_{l+1}(C)$ as differential Galois group over $C(t_1, \ldots, t_l)$.

2. $L(y, t) = y^{(2l)} - \sum_{i=1}^{l} (-1)^{i-1} (t_i y^{(l-i)})^{(l-i)} = 0$ has $\text{SP}_{2l}(C)$ as differential Galois group over $C(t_1, \ldots, t_l)$.

3. $L(y, t) = y^{(2l+1)} - \sum_{i=1}^{l} (-1)^{i-1} ((t_i y^{(l+i-l)})^{(l-i)} + (t_i y^{(l-i)})^{(l+1-i)}) = 0$ has $\text{SO}_{2l+1}(C)$ as differential Galois group over $C(t_1, \ldots, t_l)$. 
The substitutions parameters. For further differential equations with group $G$ can be obtained from our equation for $C$.

Differential equation over $E$. Frenkel and B. Gross introduced a uniform construction of a rigid irregular differential Galois group. Our construction of the defining matrices is a generalization of the Specialization Bound theorem. It will be used in combination with the $C$ over $E$ of Theorem 0.2 for each group separately.

The proof of Theorem 0.2 is organized in the following way: In Chapter 6 we prove our so-called Transformation Lemma which is an essential tool for the proof of the theorem. It will be used in combination with the $C$ over $E$ for the differential Galois group is given by the following proposition (see $[19]$, Proposition 1.31).

Let $F$ denote a differential field with field of constants $C$. An upper bound criterion for the differential Galois group is given by the following proposition (see $[19]$, Proposition 1.31).

**Proposition 1.1.** Let $H$ be a connected linear algebraic group over $C$ and let $A \in \mathfrak{h}(F)$ in the Lie algebra of $H(F)$. Then the differential Galois group $G(C)$ of the differential equation $\partial(y) = Ay$ is contained (up to conjugation) in $H(C)$.

Let $\partial(y) = Ay$ be a differential equation with differential Galois group $G$ and denote by $L$ a Picard-Vessiot ring for $\partial(y) = Ay$. By Kolchin’s Structure Theorem (see $[19]$, Theorem 1.28) we know that the affine group scheme $Z = \text{Spec}(L)$ over $F$.

---

**PART I**

**The Specialization Bound**

**1. Bounds for the differential Galois group**

In this section we present bound criteria for the differential Galois group. More precisely, besides upper and lower bound criteria known from the literature, we introduce our specialization bound and sketch the main ideas. A detailed proof then follows in the subsequent sections.

Let $F$ denote a differential field with field of constants $C$. An upper bound criterion for the differential Galois group is given by the following proposition (see $[19]$, Proposition 1.31).

**Proposition 1.1.** Let $H$ be a connected linear algebraic group over $C$ and let $A \in \mathfrak{h}(F)$ in the Lie algebra of $H(F)$. Then the differential Galois group $G(C)$ of the differential equation $\partial(y) = Ay$ is contained (up to conjugation) in $H(C)$.

Let $\partial(y) = Ay$ be a differential equation with differential Galois group $G$ and denote by $L$ a Picard-Vessiot ring for $\partial(y) = Ay$. By Kolchin’s Structure Theorem (see $[19]$, Theorem 1.28) we know that the affine group scheme $Z = \text{Spec}(L)$ over $F$. 
is a $G$-torsor. In the situation when $Z$ has a $F$-rational point or equivalently when $Z$ is the trivial torsor, the following proposition (for a proof see [19], Corollary 1.32) is a lower bound criteria for the differential Galois group.

**Proposition 1.2.** Let $L$ be a Picard-Vessiot ring for $\partial(y) = Ay$ over $F$ with connected differential Galois group $G(C)$ and let $Z$ be the associated torsor. Let $A \in \mathfrak{h}(F)$ for a connected linear algebraic group $H(C) \supset G(C)$. If $Z$ is the trivial torsor, then there exists $B \in H(F)$ such that $\partial(B)B^{-1} + BAB^{-1} \in \mathfrak{g}(F)$.

For a differential ground field of cohomological dimension at most one, the associated torsor is automatically the trivial torsor (see [22], Chapter 2.4). Thus over such differential fields, e.g. over the rational function field $C(z)$ with standard derivation $\frac{d}{dz}$, Proposition [12] can be considered as a lower bound criterion. In [18] C. Mitschi and M.F. Singer applied Proposition 1.1 and 1.2 successfully to connected semisimple linear algebraic groups for the differential ground field $C(z)$. Unfortunately in our situation, i.e. for the differential base field $C(t_1, \ldots, t_i)$, we have no information if the associated torsor is trivial or not. Therefore we cannot use Proposition [12] directly as a lower bound criterion. But we can consider a specialization of a differential equation $\partial(y) = A(t)y$ over $C(t)$ to a differential equation $\partial(y) = A(r)y$ over $C(z)$ which we obtain from evaluating the differential indeterminates $t = (t_1, \ldots, t_i)$ at elements $r = (r_1, \ldots, r_l) \in C[z]^l$ with the property that $A(r)$ is well-defined.

**Notation and Definition 1.3.** We denote in the following by $F_1$ the differential field $C(t)$ and by $F_2$ the standard differential field $C(z)$. Further, we denote by $R_1$ a localization of $C(t)$ by a finitely generated multiplicative subset of $C[t]$. If we evaluate the elements of this multiplicative subset at some $r \in C[z]^l$, which we choose such that the generators do not vanish, we obtain a multiplicative subset of $C[z]$ and we denote by $R_2$ the localization of $C[z]$ by this subset. This construction then yields a differential $C$-algebra homomorphism $\sigma : R_1 \to R_2$, $t \mapsto r$ which we call a $R_1$-specialization.

For a ring $R$ and a matrix of indeterminates $X = (X_{ij})$ we denote by $R[GL_n]$ the ring $R[X_{ij}, \det(X_{ij})^{-1}]$. For an ideal $I$ of a ring $R$ and a ring extension $R \subset S$ we mean by $(I)$ the ideal $I \cdot S$ of $S$.

We sketch the main ideas of our specialization bound: Let $A(t) \in R_1^{n \times n}$ and so we obtain $A(\sigma(t)) \in R_2^{n \times n}$ for a $R_1$-specialization $\sigma$. We equip the rings $R_1[GL_n]$ and $R_2[GL_n]$ with a well-defined differential structure which is given by the matrix equations $\partial(X) = A(t)X$ and $\partial(X) = A(\sigma(t))X$. It is then possible to extend $\sigma$ to a differential ring homomorphism $\sigma : R_1[GL_n] \to R_2[GL_n]$. We choose now a differential ideal $I_1$ of $R_1[GL_n]$ maximal with the property that $I_1 \cap R_1 = (0)$. Then, since $\sigma$ is a differential homomorphism, $\sigma(I_1)$ is a differential ideal of $R_2[GL_n]$. Assuming that $\sigma(I_1)$ is a differential ideal with $\sigma(I_1) \cap R_2 = (0)$, we can choose a differential ideal $I_2$ of $R_2[GL_n]$ maximal with the property $I_2 \cap R_2 = (0)$ and which satisfies $\sigma(I_1) \subset I_2$. By construction the quotient ring $S_i$ of $R_i[GL_n]$ by $I_i$ does not have any non-trivial differential ideals $Q_i$ with $Q_i \cap R_i = (0)$. Further, the ideal $(I_i)$ is a maximal differential ideal of $F_i[GL_n]$ by the maximality of $I_i$ and so the differential ring $S_i$ injects into the Picard-Vessiot ring

$$L_i = F_i[GL_n]/(I_i).$$

Now, since the differential Galois group $G_i(C)$ of $L_i$ has to stabilize the ideal $I_i$ and $\sigma(I_1) \subset I_2$, we expect intuitively that the differential Galois group $G_2(C)$ of $L_2$ has to satisfy the same and even more conditions than the differential Galois group $G_1(C)$ of $L_1$. Therefore $G_2(C)$ should be a subgroup of $G_1(C)$.
Theorem 1.4. Let \( \partial(y) = A(t)y \) be a matrix parameter differential equation over \( F_1 \) with differential Galois group \( G(C) \) and suppose \( A(t) \in R_1^{n \times n} \). Let \( \sigma \) be a surjective \( R_1 \)-specialization. Then the differential Galois group of \( \partial(y) = A(\sigma(t))y \) over \( F_2 \) is a subgroup of \( G(C) \).

For a formalization and completion of the proof of Theorem 1.4 we will consider in Section 3 Taylor maps from the differential rings \( R_n[GL_n] \) to suitable rings of power series. The purpose of these Taylor maps is to show that there exists a differential ideal \( I_1 \) maximal with the property \( I_1 \cap R_1 = (0) \) such that its specialization satisfies \( \sigma(I_1) \cap R_2 = (0) \).

2. Differential rings

Let \( R \) be a differential ring with field of constants \( C \). Suppose that \( R \) is an integral domain and denote \( \text{Quot}(R) \) by \( F \). For \( \partial(y) = Ay \) with \( A \in R^{n \times n} \) let a differential structure on \( R[GL_n] \) be defined by \( \partial(X) = AX \). We consider in the following differential ideals \( I \) in \( R[GL_n] \) which satisfy the following condition:

Condition 2.1:
(a) We have \( I \cap R = (0) \).
(b) If for \( q \in I \) there exists \( r \in R \) such that \( r^{-1}q \in R[GL_n] \), then \( r^{-1}q \in I \).

Definition 2.1.
(a) A differential ideal \( I \) of \( R[GL_n] \) which satisfies Condition 2.1 (a) and is maximal with this property is called a relatively maximal differential ideal.
(b) Let \( S \) be a differential ring and \( R \) a differential subring of \( S \). Then \( S \) is called a \( R \)-simple differential ring if \( S \) contains no non-trivial differential ideals \( I \) with the property \( I \cap R = (0) \).

Remark 2.2. If \( I \subset R[GL_n] \) is a relatively maximal differential ideal, then \( I \) satisfies automatically Condition 2.1 (b), since for any differential ideal \( I \subset R[GL_n] \) which satisfies Condition 2.1 (a) the expanded ideal \( I_{\text{exp}} := (I) \cap R[GL_n] \), where \( (I) \subset \text{Quot}(R)[GL_n] \), is a differential ideal and satisfies Condition 2.1 (a) and (b).

A Picard-Vessiot ring is usually defined over a differential field. We give a definition of a Picard-Vessiot ring over the differential ring \( R \).

Definition 2.3. A differential ring \( S \) over \( R \) is called a Picard-Vessiot ring for the differential equation \( \partial(y) = Ay \) with \( A \in R^{n \times n} \) if

1. \( S \) is a \( R \)-simple differential ring.
2. \( S \) contains a fundamental solution matrix \( Z \in GL_n(S) \) and \( S \) is generated as a ring by the entries of \( Z \) and the inverse of the determinant.
3. The field of constants of \( S \) is \( C \).

If \( S \) is a Picard-Vessiot ring over \( R \), then the differential ring \( L := S \otimes_R F \) is a simple differential ring, since if \( L \) would have a non-trivial differential ideal, then its non-trivial intersection with \( S \otimes_R R \) would yield a differential ideal of \( S \). It has therefore \( C \) as its field of constants and since it contains a fundamental solution matrix \( Z \) and it is generated by the entries of \( Z \) over \( F \), it follows that \( L \) is a Picard-Vessiot ring for \( \partial(y) = Ay \) over \( F \). Conversely, if \( L \) is a Picard-Vessiot ring for \( \partial(y) = Ay \) with \( A \in R^{n \times n} \) over \( F \), then the \( R \)-algebra generated by the entries of a fundamental solution matrix and the inverse of the determinant is a Picard-Vessiot ring \( S \) over \( R \) for \( \partial(y) = Ay \). In fact, \( S \) is \( R \)-simple, since if \( S \) would have a non-trivial differential ideal \( I \) with \( I \cap R = (0) \), then the ideal \( (I) \) would be a non-trivial differential ideal of \( L \) and so \( (I) = L \). But then \( 1 \in (I) \) and so there exists nonzero \( r \in R \) with \( r \in I \) which contradicts to \( I \cap R = (0) \).
Definition 2.4. The differential Galois group \( \text{Gal}_d(S/R) \) of a differential equation \( \partial(y) = Ay \) with \( A \in \mathbb{R}^{n \times n} \) is defined as the group of differential \( R \)-algebra automorphisms of a Picard-Vessiot ring \( S \) over \( R \) for the equation.

Let \( S \) be a Picard-Vessiot ring over \( R \) and let \( L = S \otimes_R F \) be the Picard-Vessiot ring over \( F \) obtained from \( S \) for \( \partial(y) = Ay \). Then there is an obvious bijection

\[
\text{Gal}(S/R) \leftrightarrow \text{Gal}(L/F).
\]

Lemma 2.5 and Lemma 2.7 below are well known in the case of Picard-Vessiot rings over differential fields.

Lemma 2.5. Let \( R \) be a differential ring with the properties that \( R \) is an integral domain, the constants of \( R \) and \( \text{Quot}(R) \) are \( C \). We extend the derivation \( \partial \) of \( R \) to a derivation on \( R[Y, \det(Y)^{-1}] \) by \( \partial(Y) = 0 \) where \( Y := (Y_i) \) are indeterminates. Then the map \( \delta : I \mapsto (I) \) from the set of ideals in the subring \( C[Y, \det(Y)^{-1}] \) to the set of differential ideals in \( R[Y, \det(Y)^{-1}] \) which satisfy Condition 2.1 is a bijection with inverse map

\[
\delta^{-1} : J \mapsto J \cap C[Y, \det(Y)^{-1}].
\]

Proof. The map \( \delta^{-1} \) is obviously well-defined. To prove that \( \delta \) is also well-defined, we need to show that the image of an ideal \( I \) under \( \delta \) satisfies Condition 2.1. We fix a \( C \)-basis \( \{e_i \mid i \in N\} \) of \( R \) and we extend it to a \( C \)-basis \( \{e_i \mid i \in M\} \) of \( \text{Quot}(R) \). Then the first basis is also a basis of the \( C[Y, \det(Y)^{-1}] \)-module \( R[Y, \det(Y)^{-1}] \) and the second one of the \( C[Y, \det(Y)^{-1}] \)-module \( \text{Quot}(R)[Y, \det(Y)^{-1}] \). Let \( q \in (I) \) and assume that \( r^{-1}q \in R[Y, \det(Y)^{-1}] \) for \( r \in R \). It is clear that \( q \) has a unique expression \( q = \sum_{i \in N} q_i e_i \) with \( q_i \in I \). We have

\[
r^{-1}q = r^{-1}(\sum_{i \in N} q_i e_i) = \sum_{i \in N} q_i(r^{-1}e_i) = \sum_{i \in N} q_i(\sum_{j \in M} c_{ij}e_j) = \sum_{i \in M} \tilde{q}_i e_i
\]

with \( r^{-1}e_i = \sum_{j \in M} c_{ij}e_j \) for \( c_{ij} \in C \) and \( \tilde{q}_i \in I \). Since \( r^{-1}q \in R[Y, \det(Y)^{-1}] \), we conclude that \( \tilde{q}_i \) can only be nonzero for indices of \( N \) and so \( r^{-1}q \in (I) \).

We need to show that the two maps are inverse to each other.

(a) We have to show that for an ideal \( I \) of \( C[Y, \det(Y)^{-1}] \) it follows that \( I = (I) \cap C[Y, \det(Y)^{-1}] \). It is clear that \( I \) is contained in the right side. For the other inclusion let \( \{e_i \mid i \in N\} \) be a \( C \)-basis of \( I \). This basis is also a free basis of the \( R \)-module \( (I) \). It is easy to see that an element \( q = \sum_{i \in N} r_i e_i \) of \( (I) \) is a constant if and only if all \( r_i \) are elements of \( C \).

(b) We have to show that an ideal \( J \) of \( R[Y, \det(Y)^{-1}] \) which satisfies Condition 2.1 is generated by the ideal \( I := J \cap C[Y, \det(Y)^{-1}] \). To this purpose we fix a \( C \)-basis \( \{e_i \mid i \in N\} \) of \( C[Y, \det(Y)^{-1}] \) which is also a basis of the free \( R \)-module \( R[Y, \det(Y)^{-1}] \). Then \( q \in J \) writes for \( r_i \in R \) uniquely as \( q = \sum_{i \in N} r_i e_i \). We show by induction on the length \( l(q) \), that is the number of nonzero \( r_i \) in the expression for \( q \), that \( q \in (I) \). Let \( l(q) = 1 \), that is, \( q = r_i e_i \). Then Condition 2.1 (b) forces \( q \in (I) \). Let \( l(q) > 1 \). If all \( r_i \) in the expression of \( q \) are elements of \( C \) or all \( r_i \) are of shape \( r_i = r_c \) with \( r \in R \) and \( c_i \in C \), then Condition 2.1 (b) yields \( q \in (I) \). Without loss of generality let \( r_{i_1} \) be not a constant and let \( r_{i_1} \) and \( r_{i_2} \) be \( C \)-linearly independent, that is, there exists no \( c \in C \) such that \( r_{i_1} = cr_{i_2} \). Note that \( r_{i_1} \partial(r_{i_2}) - r_{i_2} \partial(r_{i_1}) = 0 \) if and only if \( r_{i_1}, r_{i_2} \) are \( C \)-linearly dependent. Then

\[
\tilde{q} := r_{i_1} \partial(q) - \partial(r_{i_1})q = \sum_{i \in M \setminus \{i_1\}} \tilde{r}_i e_i \neq 0.
\]

The induction assumption implies that \( \tilde{q} \in (I) \). The same holds for \( \tilde{r}_{i_2} q - r_{i_2} \tilde{q} \). We conclude that \( \tilde{r}_{i_2} q \in (I) \). Since \( (I) \) satisfies Condition 2.1 (b), it follows that \( q \in (I) \).
Before we prove Lemma 2.6 below, we show the following lemma.

**Lemma 2.6.** Let $I_1$ and $I_2$ be two differential ideals of $R[\text{GL}_n]$ which satisfy Condition 2.1 (a) and (b).

1. Then the $R$-module $R[\text{GL}_n]/I_1 \otimes_R R[\text{GL}_n]/I_2$ is torsion-free.
2. If $I_1$ is a relatively maximal differential ideal and $S = R[\text{GL}_n]/I_1$, then the $S$-module $S \otimes_R R[\text{GL}_n]/I_2 \cong S[\text{GL}_n]/(I_2)$ is torsion-free.

**Proof.** (1) Since the ideals $I_1$ and $I_2$ satisfy Condition 2.1 (b), the ideal $J := (I_1 \otimes_R R[\text{GL}_n]) + (R[\text{GL}_n] \otimes_R I_2)$ of $R[\text{GL}_n] \otimes_R R[\text{GL}_n]$ also satisfies Condition 2.1 (b). Now, since $R[\text{GL}_n] \otimes_R R[\text{GL}_n]$ has no $R$-torsion, the $R$-module $R[\text{GL}_n] \otimes_R R[\text{GL}_n]/J$ is torsion-free if and only if $J$ satisfies Condition 2.1 (b). Then the statement in (1) follows from isomorphism of $R$-algebras

$$R[\text{GL}_n]/I_1 \otimes_R R[\text{GL}_n]/I_2 \cong R[\text{GL}_n] \otimes_R R[\text{GL}_n]/J.$$ (2) It follows from (1) and

$$S[\text{GL}_n]/(I_2) \cong S \otimes_R R[\text{GL}_n]/I_2 \cong R[\text{GL}_n]/I_1 \otimes_R R[\text{GL}_n]/I_2$$

that $S[\text{GL}_n]/(I_2)$ is a torsion-free $R$-module. We consider the $S$-module homomorphism $\phi : S[\text{GL}_n]/(I_2) \to S[\text{GL}_n]/(I_2) \otimes_R F$. Since $S[\text{GL}_n]/(I_2)$ has no $R$-torsion, $\phi$ is an injective $R$-module homomorphism and so it is also injective as a $S$-module homomorphism. The differential ring $(S \otimes R F)$ is $\partial$-simple and so [17], Theorem 4.3 yields that the differential $(S \otimes_R F)$-module $(S \otimes_R F) \otimes_R R[\text{GL}_n]/I_2$ is projective. Thus, it is a torsion-free $(S \otimes R F)$-module and so it is also a torsion-free $S$-module. We conclude with

$$S[\text{GL}_n]/(I_2) \otimes_R F \cong (S \otimes_R R[\text{GL}_n]/I_2) \otimes_R F \cong (S \otimes R F) \otimes_R R[\text{GL}_n]/I_2$$

that $S[\text{GL}_n]/(I_2) \otimes_R F$ is a torsion-free $S$-module and so, since $\phi$ is an injective $S$-module homomorphism, $S[\text{GL}_n]/(I_2)$ is a torsion-free $S$-module. $\square$

**Lemma 2.7.** Let $S$ be a Picard-Vessiot ring over $R$. Then the map

$$\iota : I \mapsto (I)$$

from the set of ideals in $R[\text{GL}_n]$ which satisfy Condition 2.1 to the set of $\text{Gal}(S/R)$ invariant ideals in $S[\text{GL}_n]$ which fulfill Condition 2.1 is a bijection where $(I)$ means the ideal $I \cdot S[\text{GL}_n]$ of $S[\text{GL}_n]$. Its inverse map is

$$\iota^{-1} : J \mapsto J \cap R[\text{GL}_n].$$

**Proof.** It is easy to verify that the map $\iota^{-1}$ is well-defined. Since $S[\text{GL}_n]/I \cdot S[\text{GL}_n]$ has no $S$-torsion by Lemma 2.6 (2), the ideal $I \cdot S[\text{GL}_n]$ satisfies Condition 2.1 (b). Hence, the map $\iota$ is also well-defined.

(a) For an ideal $I$ of $R[\text{GL}_n]$ which satisfies Condition 2.1 we show that $I = I \cdot S[\text{GL}_n] \cap R[\text{GL}_n]$. It is clear that $I \subset I \cdot S[\text{GL}_n] \cap R[\text{GL}_n]$. For the other inclusion we consider the extension $S_F := F \otimes_R S$ and the ideal $I_F := I \cdot F[\text{GL}_n]$ where $F$ denotes $\text{Quot}(R)$. One chooses a $F$-basis of the module $I_F$. This basis is also a basis of the $S_F$-module $I_F \cdot S_F[\text{GL}_n]$. We conclude similar as in Lemma 2.5 that $I_F = I_F \cdot S_F[\text{GL}_n] \cap F[\text{GL}_n]$. Now let $f \in I \cdot S[\text{GL}_n] \cap R[\text{GL}_n]$. Trivially it holds $f \in I_F \cdot S_F[\text{GL}_n] \cap F[\text{GL}_n]$ and so $f \in I_F$. Since $f \in R[\text{GL}_n]$ it follows that $f \in I_F \cap R[\text{GL}_n]$ which is equal to $I$ since $I$ satisfies Condition 2.1.

(b) For a $\text{Gal}(S/R)$-invariant ideal $I$ of $S[\text{GL}_n]$ which satisfies Condition 2.1 it is clear that $I := J \cap R[\text{GL}_n]$ satisfies also Condition 2.1. Recall that then the ideal $I \cdot S[\text{GL}_n]$ of $S[\text{GL}_n]$ also satisfies Condition 2.1 (b) by Lemma 2.6 (2). We have to show that $J$ is equal to $I \cdot S[\text{GL}_n]$. Let $\{e_i \mid i \in N\}$ be a $R$-basis of $R[\text{GL}_n]$ which
is also a $S$-module basis of $S[\text{GL}_n]$. Then every $f \in J$ writes uniquely as $\sum_{i \in N} s_i e_i$ with $s_i \in S$. We proceed by induction on the length $l(f)$, that is, on the number of nonzero $s_i$ in the expression of $f$. If $f = s_i e_i$, then $e_i \in J \cap R[\text{GL}_n]$ and so in $I \cdot S[\text{GL}_n]$. Now let $l(f) > 1$. If all $s_i$ in the expression of $f$ lie in $R$ or if $f = s \sum r_i e_i$ with $r_i \in R$, then clearly $f$ is in $I \cdot S[\text{GL}_n]$. Without loss of generality let $s_{i_1}$ and $s_{i_2}$ be $R$-linearly independent and let $s_{i_1}$ be not in $R$. Then there is $\gamma \in \text{Gal}(S/R)$ such that $\gamma(s_{i_1}) s_{i_2} \neq s_{i_1} \gamma(s_{i_2})$. The length of $g := s_{i_1} \gamma(f) - \gamma(s_{i_1}) f \neq 0$ is smaller than $l(f)$ and so $g$ lies by induction assumption in $I \cdot S[\text{GL}_n]$. Further, the length of $(s_{i_1} \gamma(s_{i_2}) - \gamma(s_{i_1}) s_{i_2}) f - s_{i_2} g$ is smaller than $l(f)$ and so it is also in $I \cdot S[\text{GL}_n]$. We conclude that $(s_{i_1} \gamma(s_{i_2}) - \gamma(s_{i_1}) s_{i_2}) f$ lies in $I \cdot S[\text{GL}_n]$. Since $I \cdot S[\text{GL}_n]$ satisfies Condition 2.1, we have that $f$ is an element of $I \cdot S[\text{GL}_n]$.

\[ \square \]

3. Formal Taylor maps

As we have seen in Chapter 1 the proof of the specialization bound is based on the idea to specialize a relatively maximal differential ideal of $R_1[\text{GL}_n]$ to a differential ideal of $R_2[\text{GL}_n]$ and then to choose a relatively maximal differential ideal above it. But to make this idea work, we need to guarantee that there is a relatively maximal differential ideal of $R_1[\text{GL}_n]$ such that its specialization is a differential ideal of $R_2[\text{GL}_n]$ satisfying Condition 2.1 (a). To ensure that such an ideal exists, we will construct in this section Taylor maps from $R_1[\text{GL}_n]$ and $R_2[\text{GL}_n]$ in appropriate rings of power series. It will turn out that the images of the two maps are Picard-Vessiot rings and so their kernels are relatively maximal differential ideals. The construction will be done in such a way that we can specialize the coefficients of the power series solutions representing $R_1[\text{GL}_n]$ to the coefficients of the ones representing $R_2[\text{GL}_n]$ and that it commutes with the specialization of $R_1[\text{GL}_n]$ to $R_2[\text{GL}_n]$. From this it will follow that the specialization of the kernel of the Taylor map for $R_1[\text{GL}_n]$ is contained in the one for $R_2[\text{GL}_n]$, that is, it is a relatively maximal differential ideal of $R_1[\text{GL}_n]$ with the desired property.

(a) In a first part we construct a Taylor map for $R_1[\text{GL}_n]$. Since $R_1$ is generated by differential indeterminates and since later on we are going to specialize them, a Taylor map for $R_1$ has to map the indeterminates to power series which are also differentially algebraically independent over $C$ and whose coefficients can be specialized. In order to construct a differential embedding of $R_1$ in a ring of power series and to have enough freedom for specializations, we need to introduce new indeterminates, that is, we consider a field extension $C(\beta)/C$ generated over $C$ by infinitely many algebraically independent elements $\beta = \{\beta_{ik} | 1 \leq i \leq l, k \in N_0\}$. It will turn out that the Taylor map

$$R_1 \to C(\beta)[[T]], \ t_i \mapsto p_i := \sum_{k \in N_0} 1/k! \ \beta_{ik} T^k$$

is a differential monomorphism (see Corollary 3.4) where $C(\beta)[[T]]$ denotes the ring of power series in $T$ over $C(\beta)$ with derivation $\partial = \frac{d}{dT}$. Since the $\beta_{ik}$ are indeterminates, the power series $p_i$ can obviously be specialized to every power series in $C[[T]]$.

The Taylor map on $R_1[\text{GL}_n]$ mentioned above will be the unique extension of the Taylor map for $R_1$ to a differential homomorphism such that the image of $X$ is congruent to $1_n$ modulo $T$. This Taylor map has the nice property that after arbitrary specializations of the coefficients the image of $X$ still is invertible, hence a fundamental solution matrix for a specialized equation. The hard task, however, is to show that the image of $R_1[\text{GL}_n]$ under this Taylor map is indeed a Picard-Vessiot
ring with constants $C$.

In order to prove this statement, we need to extend in a first step for technical reasons the constants $C$ of $R_1$ to $C(\beta)$, that is, we will construct a Taylor map for the differential ring

$$R_{\beta} := R_1 \otimes_C C(\beta).$$

Once we have seen that the kernel of this Taylor map is a relatively maximal differential ideal of $R_{\beta}[\text{GL}_n]$, we show in a second step that the kernel of the restriction to $R_1[\text{GL}_n]$ is a relatively maximal differential ideal (Theorem 3.10).

We start with the construction of the Taylor map for $R_{\beta}$ for $\partial_{ij}$ restricted to $R_{\beta}$.

Let $P_1 := (\partial^k(t_i) - \beta_{ik} | 1 \leq i \leq l, k \in \mathbb{N}_0)$ be a maximal ideal and so the image of an element $r$ of $R_{\beta}$ under the quotient map $R_{\beta} \to R_{\beta}/P_1$ is the evaluation of $r$ at $\beta$. We denote this image by $\tau(P_1)$.

**Lemma 3.1.** For $A \in R_1^{n \times n}$ let a derivation on $R_{\beta}[\text{GL}_n]$ be defined by the matrix differential equation $\partial(X) = AX$. Let $P_1$ be the maximal ideal of $R_{\beta}$ as above and let $X(P_1) \in \text{GL}_n(C(\beta))$ be initial values. Then the kernel of the differential ring homomorphism

$$\tau : R_{\beta}[\text{GL}_n] \to C(\beta)[[T]],$$

$t_j \mapsto \sum_{k \in \mathbb{N}_0} 1/k! \partial^k(t_j)(P_1), \ X_{ij} \mapsto \sum_{k \in \mathbb{N}_0} 1/k! \partial^k(X_{ij})(P_1)$

is a relatively maximal differential ideal of $R_{\beta}[\text{GL}_n]$.

**Proof.** The map

$$\tau : R_{\beta} \to C(\beta)[[T]], \ r \mapsto \sum_{k \in \mathbb{N}_0} 1/k! \partial^k(r)(P_1)T^k$$

is clearly a differential ring homomorphism and by Corollary 3.3 its image is contained in a purely differential transcendental extension of $C(\beta)$. Hence the map $\tau$ is a differential monomorphism.

We extend $\tau$ to $R_{\beta}[\text{GL}_n]$. The recursion

$$A_1 := A \text{ and } A_k := \partial(A_{k-1}) + A_{k-1}A \text{ for all } k \geq 2$$

provides us with all higher derivatives $\partial^k(X) = A_kX$ and if we choose initial values $X(P_1) \in \text{GL}_n(C(\beta))$, we obtain values $\partial^k(X)(P_1) \in C(\beta)^{n \times n}$. So the Taylor map $\tau$ extends to a differential ring homomorphism

$$\tau : R_{\beta}[\text{GL}_n] \to C(\beta)[[T]], \ X_{ij} \mapsto \sum_{k \in \mathbb{N}_0} 1/k! \partial^k(X_{ij})(P_1)T^k.$$ 

It is left to show that the kernel of $\tau$ is a relatively maximal differential ideal. Since $\tau(R_{\beta}) \subset C(\beta)[[T]]$ is an integral domain, we can take its quotient field $\text{Quot}(\tau(R_{\beta})) \subset C(\beta)((T))$ which is a differential field with field of constants $C(\beta)$.

We prove that $S := \text{Quot}(\tau(R_{\beta}))|_{\tau(X)}[\text{det}(\tau(X))^{-1}]$ is a $\partial_{ij}$-simple differential ring, that is, $S$ has no proper, non-trivial differential ideals. By construction the matrix $\tau(X)$ is a fundamental solution matrix for the differential equation $\partial(y) = \tau(A)y$ and $S$ is generated by the entries of $\tau(X)$ and the inverse of its determinant. Further, since $S$ is a subring of $C(\beta)((T))$, it is an integral domain and the constants of its field of fractions $\text{Quot}(S) \subset C(\beta)((T))$ coincide with the field of constants of $\text{Quot}(\tau(R_{\beta}))$. We apply now [3, Corollary 2.7], and obtain that $S$ is a $\partial_{ij}$-simple differential ring. Hence, the differential ring $\tau(R_{\beta}[\text{GL}_n])$ is $\tau(R_{\beta})$-simple and so $\text{kern}(\tau)$ is a relatively maximal differential ideal of $R_{\beta}[\text{GL}_n]$. $\square$
To complete the proof of Lemma 3.1, we need to show that the power series $p_i$ generated a purely differential transcendental extension of $C(\beta)$.

Lemma 3.2. For every $1 \leq i \leq l$ the power series $p_i$ is differentially algebraically independent over the differential ring $C(\beta)[T]$ with derivation $\partial = \frac{d}{dT}$.

Proof. Assume $p_i$ is differentially algebraic. Then the transcendence degree of the field $C(\beta)(T, p_i, \partial(p_i), \ldots)$ is finite over $C(\beta)$ and so the same holds for the field $C(\beta)(p_i, \partial(p_i), \ldots)$. Thus $p_i$ satisfies a differential equation coming from $C(\beta)[y]$. Since the coefficients of this equation are contained in a finitely generated subfield of $C(\beta)$, we obtain by the same argument that $p_i$ satisfies a relation from $C[y]$.

By [14], Theorem 4.1, there exist power series in $\mathbb{Q}[T]$, for example $\sum_k T^k$, which are not differentially algebraic over $\mathbb{Q}$. Thus if we specialize the coefficients of $p_i$ to the coefficients of such a power series, we get a contradiction. \hfill \Box

Corollary 3.3. For every $1 \leq i \leq l$ the power series $p_i$ is differentially algebraically independent over the differential ring $C(\beta)\{p_1(T), \ldots, p_{l-1}(T)\}$, i.e. there exists a non-trivial differential polynomial

$$h(y) \in C(\beta)\{p_1(T), \ldots, p_{l-1}(T)\}\{y\}$$

such that $h(p_i(T)) = 0$. Then the coefficients of $h(y)$ are power series with infinitely many coefficients in $C[\beta_{ij}, \ldots, \beta_{l-1,j} \mid j \in \mathbb{N}_0]$ and finitely many coefficients in $C(\beta)$. Denote by $C(\beta_i)$ the subfield of $C(\beta)$ which is generated by $\beta_i = \{\beta_{ij} \mid j \in \mathbb{N}_0\}$ over $C$. We define a specialization $\sigma : C(\beta) \rightarrow C(\beta_i)$ by $\beta_{ij} \mapsto \beta_{ij}$ for $j \in \mathbb{N}_0$ and $\beta_{ij} \mapsto c_{ij}$ for $i \in \{1, \ldots, l-1\}$ and $j \in \mathbb{N}_0$, where we choose finitely many $c_{ij} \in C^\times$ and infinitely many $c_{ij} = 0$ such that $\sigma(h)(y) \in C(\beta_i)[T]$ has no pole and does not vanish. Thus we obtain a non-trivial differential algebraic relation with

$$\sigma(h(p_i(T))) = \sigma(h)(p(T)) = 0.$$ 

But $p_i(T)$ is differentially transcendental over the differential field $C(\beta_i)[T]$ by Lemma 3.2. \hfill \Box

Corollary 3.4. The differential field extension $C(\beta)(p_1, \ldots, p_l)/C(\beta)$ is a purely differentially transcendental extension.

In Lemma 3.1 we constructed a Taylor map $\tau$ for $R_\beta[\text{GL}_n]$ such that its kernel is a relatively maximal differential ideal of $R_\beta[\text{GL}_n]$. In order to prove that the kernel of the restriction of $\tau$ to $R_1[\text{GL}_n]$ is also a relatively maximal differential ideal, we construct an auxiliary Taylor map. The difference to the previous Taylor map is that we choose finitely many coefficients in the Taylor series representing the differential indeterminates of $R_\beta$ to be elements of $C$. More precisely, for $r \in \mathbb{N}_0$ and $c_{ij} \in C$ with $0 \leq j < r$ we consider the ideal

$$\mathcal{P}_1 := \langle \partial^i(t_i) - c_{ij}, \partial^k(t_k) - \beta_{ik} \mid 1 \leq i \leq l, 0 \leq j < r, k \in \mathbb{N}, k \geq r \rangle$$

of $R_\beta$. It is clearly a maximal ideal of $R_\beta$ and so the image under the quotient map $R_\beta \rightarrow R_\beta/\mathcal{P}_1$ of an element $r \in R_\beta$ is the evaluation of $r$ at $c_{ij}$ and $\beta_{ik}$. We denote its image by $r(\mathcal{P}_1)$.

Lemma 3.5. For $A \in R_1^{n \times n}$ let a derivation on $R_\beta[\text{GL}_n]$ be defined by the differential equation $\partial(X) = AX$ and let $\mathcal{P}_1$ be the maximal ideal of $R_\beta$ as above. Let $X(\mathcal{P}_1) \in \text{GL}_n(C(\beta))$ be arbitrary initial values. Then the kernel of the differential
The proof just works as the proof of Lemma 3.1 where we use this time the ideal \( \tilde{\mathcal{C}} \) differential transcendental extension of ring homomorphism \( \mathcal{C} \) Let \( \text{Lemma 3.6} \).

Let \( m \) be the transcendence degree of \( \mathcal{C} \), that the transcendence degree of \( \mathcal{C} \) is a differential isomorphism.

Proof. We show now that if we choose the same initial values the image \( s \) of \( \tilde{\mathcal{C}} \) are, for two elements \( \tilde{\mathcal{C}} \) of \( \mathcal{C} \) and \( \tilde{\mathcal{C}} \) be the Taylor maps of Lemma 3.1 and 3.5 with initial values \( \tau \) \( \text{Lemma 3.7} \).

For a matrix differential equation \( \partial(y) = Ay \in R^{n \times n} \) let \( \tau \) and \( \tilde{\tau} \) be the Taylor maps of Lemma 3.1 and 3.5 with initial values \( X(P_1) = X(\tilde{P}_1) = 1_n \). Then the map

\[ \Psi : \text{im}(\tau) \to \text{im}(\tilde{\tau}), \quad \tau(t) \mapsto \tilde{\tau}(t) \text{ and } \tau(X) \mapsto \tilde{\tau}(X) \]

is a differential isomorphism.

Proof. Note that the constants of the Picard-Vessiot rings \( \text{im}(\tau) \) and \( \text{im}(\tilde{\tau}) \) are in both cases \( \mathcal{C}(\beta) \). The map

\[ \psi : \tau(R_\beta) \to \tilde{\tau}(R_\beta), \quad \tau(t) \mapsto \tilde{\tau}(t) \]

is obviously a differential isomorphism, since \( \tau(R_\beta) \) and \( \tilde{\tau}(R_\beta) \) are subrings of purely differential transcendental extensions of \( \mathcal{C}(\beta) \) by Corollary 3.3 and Lemma 3.6.

Let \( \text{im}(\tau) \otimes_{R_\beta} \text{im}(\tilde{\tau}) \) be the tensor product of \( \text{im}(\tau) \) and \( \text{im}(\tilde{\tau}) \) defined via \( \psi \), that is, for two elements \( p \in \tau(R_\beta) \) and \( \tilde{p} \in \tilde{\tau}(R_\beta) \) we have the rule

\[ p \otimes \tilde{p} = pv^{-1}(\tilde{p}) \otimes 1 = 1 \otimes \psi(p)\tilde{p}. \]

The ring \( \text{im}(\tau) \otimes_{R_\beta} \text{im}(\tilde{\tau}) \) becomes a differential ring in the obvious way, that is, for \( s \otimes \tilde{s} \in \text{im}(\tau) \otimes_{R_\beta} \text{im}(\tilde{\tau}) \) the derivation is defined by \( \partial(s \otimes \tilde{s}) = \partial(s) \otimes \tilde{s} + s \otimes \partial(\tilde{s}) \) and it has no \( R_\beta \)-torsion by Lemma 2.6 (1). Let \( Z_{ij} \) be the entries of the matrix

\[ Z := \tau(X)^{-1} \otimes \tilde{\tau}(X). \]

One can show now as in 4, Lemma 2.4, that \( U := \mathcal{C}(\beta)[Z_{ij}, \det(Z_{ij})^{-1}] \) is the algebra of constants of the differential ring \( \text{im}(\tau) \otimes_{R_\beta} \text{im}(\tilde{\tau}) \) and that the map

\[ \Phi : \text{im}(\tau) \otimes_{\mathcal{C}(\beta)} U \to \text{im}(\tau) \otimes_{R_\beta} \text{im}(\tilde{\tau}), \quad (s \otimes u) \mapsto (s \otimes 1)u \]

is a differential isomorphism. In the last step of the proof one uses that \( \text{im}(\tau) \otimes_{R_\beta} \text{im}(\tilde{\tau}) \) has no \( \text{im}(\tau) \)-torsion by Lemma 2.6 (2). The map \( \Phi \) defines a bijection.
between the maximal relatively differential ideals of $\text{im}(\tau) \otimes_{C(\beta)} U$ and $\text{im}(\tau) \otimes_{R_\beta} \text{im}(\tilde{\tau})$. By [16], Lemma 10.7, the relatively maximal differential ideals of $\text{im}(\tau) \otimes_{C(\beta)} U$ correspond to the maximal ideals of $U$, since $\text{im}(\tau)$ is relatively $\partial$-simple. We show that

$$Q = \langle Z_{ij} - \delta_{ij} \otimes \delta_{ij} \rangle$$

is a maximal ideal of $U$. For this purpose let us consider the ring homomorphism

$$\phi : \text{im}(\tau) \otimes_{C(\beta)} \text{im}(\tilde{\tau}) \to \text{im}(\tau) \cdot \text{im}(\tilde{\tau}) \subset C(\beta)[[T]], \ s \otimes \tilde{s} \mapsto \tilde{s}.$$ 

The subset of all power series with constant term zero of $\text{im}(\tau) \cdot \text{im}(\tilde{\tau})$ is a Picard-Vessiot ring with constants $\mathcal{F}$. For a relatively maximal differential ideal $J \supset \text{im}(\tau) \otimes_{R_\beta} \text{im}(\tilde{\tau})$ and since $J := \Phi(\text{im}(\tau) \otimes_{C(\beta)} \text{im}(\tilde{\tau}))$ is a relatively maximal differential ideal of $\text{im}(\tau) \otimes_{R_\beta} \text{im}(\tilde{\tau})$. It is easy to see that the image of $Z$ in $\text{im}(\tau) \otimes_{R_\beta} \text{im}(\tilde{\tau})/J$ under the quotient map is $1_n \otimes 1_n$. So the differential isomorphisms $\varphi : \text{im}(\tau) \to \text{im}(\tau) \otimes_{R_\beta} \text{im}(\tilde{\tau})/J$ and $\tilde{\varphi} : \text{im}(\tilde{\tau}) \to \text{im}(\tau) \otimes_{R_\beta} \text{im}(\tilde{\tau})/J$ satisfy $\varphi(\tau(X)) = \tilde{\varphi}(\tau(X))$ and consequently $\varphi(\text{im}(\tau)) = \tilde{\varphi}(\text{im}(\tilde{\tau}))$. We conclude that

$$\Psi : \text{im}(\tau) \to \text{im}(\tilde{\tau}), \ \tau(X) \mapsto \tilde{\tau}(X)$$

is a differential isomorphism. □

**Corollary 3.8.** For $A \in R_1^{n \times n}$ let a derivation on $R_\beta[GL_n]$ be defined by $\partial(X) = AX$. Then for the Taylor maps $\tau$ and $\tilde{\tau}$ of Lemma 3.7 and 3.5 with initial values $X(P_1) = X(\tilde{P}_1) = 1_n$, we have

$$\text{kern}(\tau) = \text{kern}(\tilde{\tau}).$$

**Proof.** From Lemma 3.7 we obtain a differential isomorphism $\Psi : \text{im}(\tau) \to \text{im}(\tilde{\tau})$ which satisfies $\Psi \circ \tau = \tilde{\tau}$, since $\Psi(\tau(X)) = \tilde{\tau}(X)$. From this it follows that $\text{kern}(\tau) \subset \text{kern}(\tilde{\tau})$ and $\text{kern}(\tilde{\tau}) \subset \text{kern}(\tau)$. □

**Lemma 3.9.** Let $\tau$ be the Taylor map of Lemma 3.7 for differential equation $\partial(y) = Ay$ with initial values $X(P_1) = 1_n$ and let $C(\beta)$ be an algebraic closure of $C(\beta)$. Then there exists a matrix $M \in GL_n(C(\beta))$ such that the kernel of the Taylor map

$$\tau' : R_1[GL_n] \to C(\beta)[[T]], \ X \mapsto \tau(X)M$$

is a relatively maximal differential ideal of $R_1[GL_n]$, that is, $\text{im}(\tau')$ is a Picard-Vessiot ring with constants $C$.

**Proof.** For a relatively maximal differential ideal $I$ of $R_1[GL_n]$ the ring $R_1[GL_n]/I$ is a Picard-Vessiot ring with constants $C$. The tensor product of $R_1[GL_n]/I$ with an arbitrary $C$-algebra $A$ produces a new differential ring $R_1[GL_n]/I \otimes_C A$ where the derivation on the second factor is trivial. By [16], Lemma 10.7, this differential ring is a $R_1 \otimes_C A$-simple differential ring, that is, it is a Picard-Vessiot ring over $R_1 \otimes_C A$ for the differential equation $\partial(y) = Ay$ with constants $C \otimes_C A \cong A$. We conclude that

$$S_1 := R_1[GL_n]/I \otimes_C C(\beta)$$

is a Picard-Vessiot ring with an algebraically closed field of constants $C(\beta)$. By Lemma 3.1 there is a Taylor map

$$\tau : R_\beta[GL_n] \to C(\beta)[[T]]$$
such that im(τ) is a Picard-Vessiot ring with field of constants \( C(β) \). The same argumentation as above yields then a Picard-Vessiot ring
\[
S_2 := \text{im}(τ) \otimes_{C(β)} C(β)
\]
with constants \( \overline{C(β)} \). Now by \([19]\), Proposition 1.20.2, the two differential rings \( S_1 \) and \( S_2 \) for the differential equation \( \partial(y) = A(y) \) are differentially isomorphic over the constants \( \overline{C(β)} \). More precisely, if \( Y \otimes 1_n \) is the fundamental matrix of \( S_1 \), then the isomorphism is given by
\[
φ : S_1 → S_2, \quad Y \otimes 1_n → τ(X) \otimes M
\]
where \( M ∈ GL_n(\overline{C(β)}) \). We conclude that for the initial values \( M ∈ GL_n(\overline{C(β)}) \) we obtain a new Taylor map
\[
τ' : R_1[GL_n] \otimes_C \overline{C(β)} → \overline{C(β)}[[T]], \quad X \otimes 1_n → τ(X)M
\]
with the property that \( \ker(τ') = I \otimes_C \overline{C(β)} \). Since \( I \) is a relatively maximal differential ideal of \( R_1[GL_n] \), the image of \( R_1[GL_n] \otimes 1 \) under \( τ' \) is a Picard-Vessiot ring with constants \( C \).

**Theorem 3.10.** Let \( τ \) be the Taylor map of Lemma 3.9 with initial values \( X(R_1) = 1_n \) for a matrix differential equation \( \partial(y) = A(t)y \) with \( A ∈ R_1^{n×n} \) and denote by \( τ_1 \) the restriction of \( τ \) to \( R_1[GL_n] \). Then the kernel of
\[
τ_1 : R_1[GL_n] → C(β)[[T]]
\]
is a relatively maximal differential ideal.

**Proof.** By Lemma 3.9 there is a matrix \( M ∈ GL_n(\overline{C(β)}) \) such that the kernel of the Taylor map
\[
τ' : R_1[GL_n] → \overline{C(β)}[[T]], \quad X → τ(X)M
\]
is a relatively maximal differential ideal of \( R_1[GL_n] \). The matrix \( M ∈ GL_n(\overline{C(β)}) \) depends only on finitely many \( β_{ij} \)'s. This means that for some \( r ∈ \mathbb{N} \) there is a subring
\[
C[β_{ij}] | 1 ≤ i ≤ l, \ 0 ≤ j ≤ r
\]
of \( C[β] \) and a finite algebraic extension \( A \) of a finitely generated localization of this ring such that \( M ∈ GL_n(A) \). For \( 1 ≤ i ≤ l \) and \( 0 ≤ j ≤ r \) it is possible to choose elements \( c_{ij} \) of \( C \) such that the specialization
\[
φ : C[β] → C[β_{ij}] | j > r
\]
defined by \( β_{ij} → c_{ij} \) for \( 0 ≤ j ≤ r \) and \( β_{ij} → β_{ij} \) for \( j > r \) induces a well defined specialization
\[
φ : A → C[β_{ij}] | j > r
\]
with the property that \( φ(M) ∈ GL_n(C) \). Let \( \vec{τ} \) be the Taylor map from \( R_β[GL_n] \) to \( C(β)[[T]] \) of Lemma 3.9 where we choose in this situation the ideal \( P_1 \) of \( R_β \) as
\[
P_1 = \langle \partial^j(t_i) - c_{ij}, \ \partial^k(t_i) - β_{ik} | 0 ≤ j ≤ r, \ k ∈ \mathbb{N}, \ k > r \rangle
\]
and the initial values \( X(P_1) = φ(M) \). Obviously we have by construction
\[
\vec{τ}(X) = φ(τ(X))φ(M).
\]
Every multivariate polynomial \( f(X, t) ∈ \ker(τ') \) satisfies
\[
f(\vec{τ}(X), \vec{τ}(t)) = f(φ(τ(X))φ(M), φ(τ(t))) = φ(f(τ(X)M, τ(t))) = 0
\]
and so \( \ker(τ') ∈ \ker(\vec{τ}) \). Since \( \ker(τ') ⊕ C(β) \) is a relatively maximal differential ideal, we conclude that \( \ker(\vec{τ}) = \ker(τ') ⊕ C(β) \). Lemma 3.9 provides us with a Taylor map
\[
\vec{τ} : R_β[GL_n] → C(β)[[T]]
\]
where we choose the ideal $\tilde{P}_1$ of $R_2[\text{GL}_n]$ as above, but this time the initial values $X(\tilde{P}_1) = 1_n$. The fundamental matrices of $\text{im}(\tilde{\tau}')$ and $\text{im}(\tilde{\tau})$ satisfy obviously the relation

$$\tilde{\tau}'(X) = \tilde{\tau}(X) \varphi(M)$$

and so, since $\text{ker}(\tau')$ is an ideal of $R_1[\text{GL}_n]$ and $\varphi(M) \in \text{GL}_n(C)$, it follows from $\text{ker}(\tilde{\tau}') = \text{ker}(\tau') \otimes_C C(\beta)$ that $\text{ker}(\tilde{\tau}) = I \otimes_C C(\beta)$ where $I$ is a relatively maximal differential ideal of $R_1[\text{GL}_n]$. Finally, we apply Corollary 3.8 to $\tilde{\tau}$ and to the Taylor map $\tau$ of Lemma 3.11 with initial values $X(P_1) = 1_n$. The restriction of $\tau$ to $R_1[\text{GL}_n]$ is then the desired Taylor map. \hfill $\square$

(b) We construct now in a second part a Taylor map for $R_2[\text{GL}_n]$. Since $R_2$ is a localization of the polynomial ring $C[z]$ by a finitely generated multiplicative subset, there are infinitely many $c \in C$ such that

$$P_2 := \langle z - c \rangle$$

is a maximal ideal in $R_2$, that is, $c$ is not a zero of any generator of the multiplicative subset. We choose now such a $P_2$. Similar as in the previous constructions of Taylor maps, for an element $r \in R_2$, we denote by $r(P_2)$ the image of $r$ under the quotient map from $R_2$ to $R_2/P_2$.

**Lemma 3.11.** Let a derivation on $R_2[\text{GL}_n]$ be defined by $\partial(y) = Ay$ with $A \in R_2^{n \times n}$ and let $P_2$ be a maximal ideal of $R_2$ as above. Then for arbitrary initial values $X(P_2) \in \text{GL}_n(C)$ the kernel of the differential ring homomorphism

$$\tau_2 : R_2[\text{GL}_n] \to C[[T]], \ r \mapsto \sum_{k \geq 0} \frac{1}{k!} \partial^k(r)(P_2)T^k, \ X_{ij} \mapsto \sum_{k \in \mathbb{N}_0} \frac{1}{k!} \partial^k(X_{ij})(P_2)T^k$$

is a relatively maximal differential ideal of $R_2[\text{GL}_n]$.

**Proof.** The construction of the Taylor map works just as in Lemma 3.1 but now for the matrix $A \in R_2^{n \times n}$ and the ideal $P_2$. One shows with same arguments as in Lemma 3.1 that $\text{ker}(\tau_2)$ is a relatively maximal differential ideal of $R_2[\text{GL}_n]$. \hfill $\square$

### 4. The specialization bound

In this section we prove the specialization bound. We use the results of the previous section to prove that there exists a relatively maximal differential ideal of $R_1[\text{GL}_n]$ such that its image under a $R_1$-specialization is contained in a relatively maximal differential ideal of $R_2[\text{GL}_n]$.

**Proposition 4.1.** Let $\partial(y) = Ay$ be a matrix differential equation with $A \in R_1^{n \times n}$ and let $\sigma : R_1 \to R_2$ be a surjective $R_1$-specialization.

1. Then there exist Taylor maps

$$\tau_1 : R_1[\text{GL}_n] \to \text{im}(\tau_1) \subset C(\beta)[[T]] \quad \text{and} \quad \tau_2 : R_2[\text{GL}_n] \to \text{im}(\tau_2) \subset C[[T]]$$

such that $\text{im}(\tau_1)$ is a Picard-Vessiot ring for $\partial(y) = \tau_1(A)y$ with constants $C$ and $\text{im}(\tau_2)$ is a Picard-Vessiot ring for $\partial(y) = \tau_2(\sigma(A))y$.

2. In the situation of (1). There exists a surjective differential homomorphism $\hat{\sigma}$ such that the following diagram commutes:

$$\begin{array}{ccc}
R_1[\text{GL}_n] & \xrightarrow{\tau_1} & R_2[\text{GL}_n] \\
\downarrow \tau_1 & & \downarrow \tau_2 \\
\text{im}(\tau_1) & \xrightarrow{\hat{\sigma}} & \text{im}(\tau_2)
\end{array}$$
Proof. For the ideal $P_1 \subset R_{\mathfrak{g}}$ of Section 3 and initial values $X(P_1) = 1_n$ we obtain from Theorem 3.10 a Taylor map

$$\tau_1 : R_1[\text{GL}_n] \rightarrow C(\mathfrak{g})[[T]]$$

such that $\text{im}(\tau_1)$ is a Picard-Vessiot ring for $\partial(y) = \tau_1(A)y$ with constants $C$. Further if we choose a maximal ideal $P_2 = \{z - c\}$ of $R_2$ as in Section 3, that is, $c$ is not a zero of any element in the multiplicative subset generating the localization $R_2$, and apply Lemma 3.11 to $\partial(y) = \sigma(A)y$ with initial values $X(P_2) = 1_n$, we get a second Taylor map

$$\tau_2 : R_2[\text{GL}_n] \rightarrow C[[T]]$$

such that $\text{im}(\tau_2)$ is a Picard-Vessiot ring for $\partial(y) = \tau_2(\sigma(A))y$. This proves the first part.

Let the $R_1$-specialization be given by

$$\sigma(t) = r = (r_1(z), \ldots, r_l(z)) \in C[z]^l$$

and let $c_{ij} := (\partial^i(r_i))(c)$ be the evaluation of the polynomial $(\partial^j(r_i))(z)$ at $c$. Since $R_1$ is a localization of $C\{t\}$ by a finitely generated multiplicative subset, the coefficients of the power series in $\text{im}(\tau_1)$ are elements of a localization $\mathcal{A}$ of $C[\mathfrak{g}]$ by a finitely generated multiplicative subset of $C[\mathfrak{g}]$. We extend $\sigma$ to a specialization

$$\sigma : R_1 \otimes_C \mathcal{A} \rightarrow R_2$$

by $\sigma : 1 \otimes \beta_{ij} \mapsto c_{ij}$. Note that the elements in the multiplicative subset for the localization $\mathcal{A}$ do not vanish under $\sigma$, because $c$ is not a zero of the elements in the multiplicative subset generating the localization $R_2$. We denote in the following the smaller ideal $P_1 \cap (R_1 \otimes_C \mathcal{A})$ also by $P_1$. Since for $r(z) \in R_2$ the polynomial $r(z) - r(c)$ has obviously a zero at $c$ and since $r(z) - \hat{c}$ is the trivial polynomial, if $r(z) = \hat{c} \in C$ is a constant, we obtain that

$$\sigma(t_{ij} \otimes 1 - 1 \otimes \beta_{ij}) = \partial^i(r_i) - c_{ij} \in P_2 = \{z - c\}$$

and so $\sigma(P_1) \subset P_2$. From the surjectivity of $\sigma$ which means that there is at least one $r_i(z)$ of degree grater than one, it follows that $\sigma(P_1) = P_2$. For the initial values it trivially holds $\sigma(X)(P_1) = X(P_2)$.

We define now the map $\hat{\sigma}$ by

$$\hat{\sigma} : \text{im}(\tau_1) \rightarrow \text{im}(\tau_2), \quad \sum_{k \in \mathbb{N}_0} 1/k! \partial^k(s)(P_1)T^k \mapsto \sum_{k \in \mathbb{N}_0} 1/k! \partial^k(\sigma(s))(\sigma(P_1))T^k$$

where $s \in R_1[\text{GL}_n]$. We have to show that $\hat{\sigma}$ is well defined, i.e., for $s_1, s_2 \in R_1[\text{GL}_n]$, with $\partial^k(s_1)(P_1) = \partial^k(s_2)(P_1)$ for all $k \in \mathbb{N}_0$ it has to follow that $\partial^k(\sigma(s_1))(P_2) = \partial^k(\sigma(s_2))(P_2)$ for all $k \in \mathbb{N}_0$. Since $\sigma(P_1) = P_2$, the fundamental theorem of homomorphisms yields that there exists a homomorphism $\varphi$ such that the following diagram commutes

$$R_1[\text{GL}_n] \otimes \mathcal{A} \xrightarrow{\sigma} R_2[\text{GL}_n]$$

$$\xrightarrow{\pi_1} \xrightarrow{\pi_2} (R_1[\text{GL}_n] \otimes \mathcal{A})/P_1 \xrightarrow{\varphi} R_2[\text{GL}_n]/P_2$$

We conclude that $\hat{\sigma}$ is well defined. Since $\hat{\sigma}$ is induced by $\tau_2 \circ \sigma$, it is a differential homomorphism. From the definition of $\hat{\sigma}$ it follows that the diagram commutes. □

**Corollary 4.2.** Let $\partial(y) = Ay$ be a matrix differential equation over $F_1$ with $A \in R_1^{n \times n}$ and let $\sigma : R_1 \rightarrow R_2$ be a surjective $R_1$-specialization. One extends $\sigma$ to a surjective differential homomorphism from $R_1[\text{GL}_n]$ to $R_2[\text{GL}_n]$ where the
derivation on \( R_2[\text{GL}_n] \) is defined by \( \partial(X) = \sigma(A)(X) \). Then there exist relatively maximal differential ideals \( I_1 \subset R_1[\text{GL}_n] \) and \( I_2 \subset R_2[\text{GL}_n] \) with the property that 
\[ \sigma(I_1) \subseteq I_2 \]
and \( \sigma \) can be extended to a surjective differential homomorphism
\[ \sigma : R_1[\text{GL}_n]/I_1 \to R_2[\text{GL}_n]/I_2. \]

Proof. Using the notation of Proposition 4.1 take \( I_1 := \ker(\tau_1) \) and \( I_2 := \ker(\tau_2) \).
Then \( I_1 \) is a relatively maximal differential ideal of \( R_1[\text{GL}_n] \) by Theorem 3.10 and \( I_2 \) is a relatively maximal differential ideal of \( R_2[\text{GL}_n] \) by Lemma 3.11. Since the diagram in Proposition 4.1 commutes, it follows for \( f \in \ker(\tau_1) \) that \( \sigma(f) \in \ker(\tau_2) \) and so \( \sigma(I_1) \subseteq I_2 \). The second statement follows from the fundamental theorem of homomorphisms.

Theorem 4.3. Let \( \partial(y) = Ay \) be a differential equation over \( F_1 \) with \( A \in R_1^{n \times n} \) and let \( \sigma : R_1 \to R_2 \) be a surjective \( R_1 \)-specialization. Then the differential Galois group \( G_2(C) \) of the specialized equation \( \partial(y) = \sigma(A)y \) over \( F_2 \) is a subgroup of the differential Galois group \( G_1(C) \) of the original equation \( \partial(y) = Ay \) over \( F_1 \).

Proof. We define a differential structure on the rings \( R_1[\text{GL}_n] \) and \( R_2[\text{GL}_n] \) by the matrix differential equations \( \partial(y) = Ay \) and \( \partial(y) = \sigma(A)y \). Corollary 4.2 then yields that there exist a relatively maximal differential ideal \( I_1 \) in \( R_1[\text{GL}_n] \) and a relatively maximal differential ideal \( I_2 \) in \( R_2[\text{GL}_n] \) such that \( \sigma(I_1) \subseteq I_2 \). The rings \( S_k = R_k[\text{GL}_n]/I_k \) are then Picard-Vessiot rings and the specialization \( \sigma \) extends to a surjective specialization
\[ \sigma : S_1 \to S_2. \]

Let \( Z := (Z_{ij}) \in \text{GL}_n(S_1) \) be the matrix whose entries \( Z_{ij} \) are the images of the variables \( X_{ij} \) in \( S_1 \) under the quotient map \( R_1[\text{GL}_n] \to R_1[\text{GL}_n]/I_1 \). The specialization \( \sigma \) then maps the fundamental matrix \( Z \) to the image of \( X \) under the quotient map \( R_2[\text{GL}_n] \to R_2[\text{GL}_n]/I_2 \) which is a fundamental solution matrix of \( S_2 \). We write \( \sigma(Z) \) for it. For \( k = 1, 2 \) we consider the rings
\[ S_k[Y, \det(Y)^{-1}] = S_k[X, \det(X)^{-1}] \supset R_k[X, \det(X)^{-1}] \]
and
\[ C[Y, \det(Y)^{-1}] \subset S_k[Y, \det(Y)^{-1}]. \]

The relations between the new variables \( X := (X_{ij}) \) and \( Y := (Y_{ij}) \) in these rings are defined by
\[ X = Z \cdot Y \text{ and } X = (\sigma(Z) \cdot Y) \]
respectively. We specify the derivations and Galois actions on these rings for \( k = 1 \). The derivations and Galois actions on the rings with \( k = 2 \) are defined analogously where one uses the matrix \( \sigma(A) \) and the fundamental matrix \( \sigma(Z) \) instead. The derivation on \( R_1[X, \det(X)^{-1}] \) and \( S_1[X, \det(X)^{-1}] \) is defined by \( \partial(X) = AX \) and by the derivation on \( S_1 \). Computing \( \partial(ZY) \) shows that the derivation on \( Y \) is trivial. Therefore, \( \partial \) acts also trivially on \( C[Y, \det(Y)^{-1}] \) and the derivation on \( S_1[Y, \det(Y)^{-1}] \) is defined by the derivation on \( S_1 \). The action of the Galois group \( \text{Gal}(S_1/R_1) \) on the above rings is induced by the action on \( S_1 \). More precisely, \( \text{Gal}(S_1/R_1) \) acts trivially on \( R_1[X, \det(X)^{-1}] \) and, since for \( \gamma_1 \in \text{Gal}(S_1/R_1) \) the Galois action is given by \( \gamma_1(Z) = ZM_1 \) with \( M_1 \in \text{GL}_n(C) \), it acts via \( \gamma_1(Y) = M_1^{-1} \cdot Y \) on the rings \( S_1[Y, \det(Y)^{-1}] \) and \( C[Y, \det(Y)^{-1}] \). Lemma 2.7 yields a bijection between the set of differential ideals of \( R_k[X, \det(X)^{-1}] \) which satisfy Condition 2.1 and the set of \( \text{Gal}(S_k/R_k) \)-invariant differential ideals of
\[ S_k[X, \det(X)^{-1}] = S_k[Y, \det(Y)^{-1}] \]
which also satisfy Condition 2.1. Thus, since \( I_1 \) is a relatively maximal differential ideal, the ideal

\[
\tilde{I}_1 = \{ f(Z \cdot Y) \mid f \in (I_1) \} \subseteq S_1[Y, \det(Y)^{-1}]
\]

is a maximal Gal\((S_1/R_1)\)-invariant differential ideal which satisfies Condition 2.1. Analogously we obtain that

\[
\tilde{I}_2 = \{ f(\sigma(Z) \cdot Y) \mid f \in (I_2) \} \subseteq S_2[Y, \det(Y)^{-1}].
\]

is a maximal Gal\((S_2/R_2)\)-invariant differential ideal satisfying Condition 2.1. By Lemma 2.5 there is a bijection between the differential ideals of \( S_k[Y, \det(Y)^{-1}] \) which satisfy Condition 2.1 and the ideals of \( C[Y, \det(Y)^{-1}] \). Hence, the ideal

\[
Q_k = \tilde{I}_k \cap C[Y, \det(Y)^{-1}]
\]

is a maximal Gal\((S_k/R_k)\)-invariant ideal of \( C[Y, \det(Y)^{-1}] \). By its maximality \( Q_k \) defines the differential Galois group \( G_k \) of \( S_k \). For a detailed explanation we refer to the proof of Theorem 1.28 in [19]. The specialization \( \sigma \) extends trivially to a surjective specialization

\[
\sigma : S_1[Y, \det(Y)^{-1}] \rightarrow S_2[Y, \det(Y)^{-1}], \quad Y_{ij} \mapsto \tilde{Y}_{ij}.
\]

With \( \sigma(I_1) \subseteq I_2 \) we conclude that \( \sigma(\tilde{I}_1) \subseteq \tilde{I}_2 \) and since \( \sigma \) acts trivially on the ring \( C[Y, \det(Y)^{-1}] \), we obtain that \( Q_1 \subseteq Q_2 \) and so \( G_1 \supseteq G_2 \).

5. Parameter equations for connected linear algebraic groups

Let \( G \) be a connected linear algebraic group defined over \( C \). In a first part of this section we use the specialization bound to prove that there exists a linear parameter differential equation over \( C(t_1) \) such that its differential Galois group is \( G \).

For a successful application of the specialization bound we need a differential equation over \( F_2 \) with group \( G \) to which we can specialize a suitable equation over \( C(t_1) \).

In [18] C. Mitschi and M. Singer proved that every connected linear algebraic group can be realized as a differential Galois group over a differential field which has \( C \) as its field of constants and is of finite, non-zero transcendence degree over \( C \). For the differential field \( F_2 \) the following proposition is a special case of [18], Theorem 1.1.

**Proposition 5.1.** Let \( G \) be a connected linear algebraic group defined over \( C \). Then there exists a Picard-Vessiot extension of \( F_2 \) with differential Galois group \( G \).

We can prove now that every connected linear algebraic group occurs as a differential Galois group over \( C(t_1) \).

**Theorem 5.2.** Let \( G \) be a connected linear algebraic group defined over \( C \). Then there exists a linear parameter differential equation

\[
L(y, t_1) = 0
\]

over \( C(t_1) \) with differential Galois group \( G \).

**Proof.** Proposition 5.1 yields that there exists a matrix \( A(z) \in F_2^{n \times n} \) such that the differential Galois group of

\[
\partial(y) = A(z)y
\]

is \( G(C) \). Since the cohomological dimension of \( F_2 \) is at most one, we may assume without loss of generality that \( A(z) \) is an element of the Lie algebra \( g(F_2) \) of \( G \).

If we now substitute the indeterminate \( z \) in the entries of \( A(z) \) by the differential indeterminate \( t_1 \), we get a matrix \( A(t_1) \) whose entries lie in \( C(t_1) \) and a new matrix differential equation

\[
\partial(y) = A(t_1)y
\]
over $C(t_1)$. We determine the differential Galois group $H(C)$ of $\partial(y) = A(t_1)y$. By construction we have that $A(t_1) \in g(C(t_1))$ and so $H(C) \subseteq G(C)$ by Proposition 1.1. We choose now a finitely generated multiplicative subset of $C\{t_1\}$ which contains all denominators appearing in the entries of $A(t_1)$. Then the localization $R_1$ of $C\{t_1\}$ by this set satisfies $A(t_1) \in g(R_1)$ and by construction the map $\sigma : R_1 \to R_2, \ t_1 \mapsto z$
is surjective $R_1$-specialization with $\sigma(A(t_1)) = A(z) \in R_2^{n \times n}$. We apply now Theorem 4.3 to $\partial(y) = A(t_1)y$, $R_1$ and $\sigma$. By construction the differential Galois group of $\partial(y) = A(\sigma(t_1))y$ over $F_2$ is $G(C)$. Hence, we get from Theorem 4.3 that $G(C) \subseteq H(C)$ and so we deduce with $H(C) \subseteq G(C)$ that $H(C) = G(C)$. Finally we apply the Cyclic Vector Theorem (see for instance [14]) and obtain a linear parameter equation $L(y, t_1) = 0$ with differential Galois group $G$. □

In a second part of this section we show that for every connected semisimple linear algebraic group $G$ of Lie rank $l$ there exists a parameter differential equation in $l$ parameters with the property that we can specialize the parameters such that we get all differential equations with group $G$ of a specific type over $F_2$.

We begin to recall some structure theory of a semisimple linear algebraic group $G$. Let $\Phi$ be the root system of $G$ and denote by $T$ a maximal torus of $G$. Then from the adjoint action of $T$ on the Lie algebra $g$, we obtain a root space decomposition

$$g(C) = h(C) \oplus \bigoplus_{\alpha \in \Phi} g_{\alpha}(C)$$

where $h(C) = \text{Lie}(T)$ is a Cartan subalgebra and for $\alpha \in \Phi$ we denote by $g_{\alpha}(C)$ the one dimensional root space of $g(C)$. Let $\Delta$ be a basis of the root system $\Phi$ with simple roots $\alpha_i \in \Delta$. According to the above root space decomposition we can choose a Chevalley basis

$$\{H_{\alpha_i} \ | \ \alpha_i \in \Delta\} \cup \{X_{\alpha_i} \ | \ \alpha \in \Phi\}$$

of $g$ where $h(C) = \langle H_{\alpha_1}, \ldots, H_{\alpha_l} \rangle$ and $g_{\alpha_i}(C) = \langle X_{\alpha_i} \rangle$.

Using the structure of a connected semisimple linear algebraic group $G$, C. Mitschi and M. Singer constructed in [13] specific matrix differential equations over $F_2$ with group $G$. The following proposition is a modification of [13], Proposition 3.5.

**Proposition 5.3.** Let $G$ be a connected semisimple linear algebraic group and define $A_0 = \sum_{\alpha_i \in \Delta} \langle X_{\alpha_i} + X_{-\alpha_i} \rangle$. Then there exists $A_1 \in h(C)$ such that the differential equation $\partial(y) = (A_0 + A_1z)y$ over $F_2$ has $G(C)$ as its differential Galois group.

A proof can be found in [21], Proposition 3.1.

**Theorem 5.4.** Let $G$ be a connected semisimple linear algebraic group of Lie rank $l$ over $C$. Then there exists a parameter differential equation

$$L(y, t_1) = 0$$

over $F_1$ with differential Galois group $G(C)$ such that we obtain from a suitable specialization of the parameters every Picard-Vessiot extension which is defined by an equation of Proposition 5.3.

**Proof.** We define a parameter differential equation $\partial(y) = A(t)y$ by

$$A(t) = A_0 + \sum_{i=1}^{l} t_i H_{\alpha_i} \in g(C[t]).$$

Proposition 1.1 yields that the Galois group $H(C)$ of $A(t)$ is a subgroup of $G(C)$. On the other hand there exists $A_1 \in h(C)$ by Proposition 5.3 such that the differential
The Galois group of $\partial(y) = (A_0 + zA_1)y$ is $G(C)$. Let $a_1, \ldots, a_l \in C$ such that $A_1 = \sum_{i=1}^l a_i H_{\alpha_i}$. Then we obtain for the specialization 

$$\sigma : C\{t_1, \ldots, t_l\} \rightarrow C[z], \ (t_1, \ldots, t_l) \mapsto (a_1 z, \ldots, a_l z)$$

that the differential Galois group of $\sigma(A(t))$ is $G(C)$. By Theorem 4.3 we have that $G(C) \subseteq H(C) \subseteq G(C)$, that is $H(C) = G(C)$. We apply now the Cyclic Vector Theorem (see for instance [14]) and obtain a linear parameter differential equation $L(y, t)$. By construction we can specialize the parameters of $A(t)$ such that we obtain all equations of Proposition 5.3 and therefore a suitable specialization of $L(y, t)$ yields every Picard-Vessiot extension defined by such an equation. 

\[ \square \]

\section*{PART II}

\textbf{PARAMETER DIFFERENTIAL EQUATIONS FOR THE CLASSICAL GROUPS}

6. \textbf{THE TRANSFORMATION LEMMA}

In this and in the following chapters we will prove Theorem 0.2 from the introduction. To this purpose let $G$ be one of the groups occurring in Theorem 0.2 and keep the notations of the preceding section. With respect to a Cartan decomposition of $\Phi$, $\Phi^+$ is defined by all positive (respectively negative) roots of $\Phi$. Further, we denote by $b^+ = h + u^+$ (respectively $b^-$) the maximal solvable subalgebra of $g$ which contains the maximal nilpotent subalgebra $u^+$ (respectively $u^-$) and the Cartan subalgebra $h$. Let $X \in g$ and denote by $s$ a subspace of $g$. Then we call the subset $X + s$ a plane of $g$. We denote by $A_0^+ = \sum_{i=1}^l X_{\alpha_i}$ (respectively $A_0^- = \sum_{i=1}^l X_{-\alpha_i}$) the sum of all basis elements belonging to the positive (respectively negative) simple roots. For a root $\alpha = \sum_{\alpha_i \in A} n_{\alpha_i}(\alpha) \alpha_i \in \Phi$, where $n_{\alpha_i}(\alpha) \in \mathbb{Z}$ are all negative or positive, we denote by $ht(\alpha) = \sum_{\alpha_i \in A} n_{\alpha_i}(\alpha) \in \mathbb{Z}$ the height of $\alpha$.

The proof of Theorem 0.2 is organized in the following way: In this chapter we show that for every group $G$ in Theorem 0.2 there are $l$ negative roots $\gamma_i \in \Phi^-$ of specific heights such that every matrix of the plane $A_0^+ + b^-$ is gauge equivalent to a matrix of the plane $A_0^+ + \sum_{i=1}^l g_{\gamma_i}$.

Afterwards, i.e. in the Chapters 7-11, we prove Theorem 0.2 for each group separately. We will determine the roots $\gamma_i$ and the explicit shape of the matrix $A_G(t)$, where $A_G(t)$ is the parameterization of the above plane. Finally, we will show that $\partial(y) = A_G(t)y$ is equivalent to the corresponding linear parameter differential equation in Theorem 0.2 and has $G(C)$ as differential Galois group over $C\{t_1, \ldots, t_l\}$. Let $X, Y \in g$. Then we write $[X, Y]$ for the usual bracket product and ad$(X)$ for the endomorphism $\text{ad}(X) : Y \mapsto [X, Y]$. The adjoint action for an element $B \in G \subseteq \text{GL}_n$ on $g$ is denoted by $\text{Ad}(B) : g \rightarrow g, X \mapsto BXB^{-1}$. For $X \in g$ nilpotent, let 

$$\exp(\text{ad}(X)) = \sum_{j \geq 0} \frac{1}{j!} \text{ad}^j(X)$$

be the exponential of $\text{ad}(X)$, which is an automorphism of $g$, and for a root $\beta \in \Phi$ the exponential map from $g_{\beta}$ to the root group $U_{\beta}$ is defined by 

$$\exp : g_{\beta} \rightarrow U_{\beta}, \ X_{\beta} \mapsto \sum_{j \geq 0} \frac{1}{j!} X_{\beta}^j.$$
For a parameter $\rho$ we denote by $u_\rho(\rho)$ the root group element $\exp(\rho X_\beta)$. We have the well known identity $\exp(\text{ad}(X_\beta))(Y) = \text{Ad}(\exp(X_\beta))(Y)$. Finally, we write $g^X$ for the centralizer of $X$ in $g$.

Now, by [12, §5.2], there exists a unique element $H_0$ in $\mathfrak{h}$ such that $\alpha_i(H_0) = 1$ for all $\alpha_i \in \Delta$. Therefore, for $\alpha \in \Phi$ we have $\alpha(H_0) = \text{ht}(\alpha)$ and $\text{ad}(H_0)(X_\alpha) = \text{ht}(\alpha)X_\alpha$. This yields a decomposition of $g$ into eigenspaces $g^{(j)}$ of $\text{ad}(H_0)(X_\alpha)$ with eigenvalues $j \in \mathbb{Z}$. More precisely, Lemma 6.1 below gives a decomposition of $g$ into a direct sum of subspaces where each subspace is the sum of root spaces of the same height $j$, i.e. we have

$$g = \sum_{\alpha \in \Phi, \, \text{ht}(\alpha) = j} g_\alpha.$$ 

A proof can be found in [11], page 369.

**Lemma 6.1.** The maximal nilpotent subalgebras $u^+$, $u^-$ and the Cartan algebra $\mathfrak{h}$ are the following direct sums of eigenspaces of $\text{ad}(H_0)(X_\alpha)$:

$$u^+ = \sum_{j > 0} g^{(j)}, \quad u^- = \sum_{j < 0} g^{(j)}, \quad \mathfrak{h} = g^{(0)}.$$ 

Additionally, for two eigenvalues $i, j \in \mathbb{Z}$ we have the following relation:

$$[g^{(i)}, g^{(j)}] \subset g^{(i+j)}.$$ 

Let us consider the ring of polynomials on $g$ as a $G$-module in the obvious way. Then by a theorem of Chevalley its ring of invariants is generated by $l$ homogeneous polynomials $u_i$ of degree $\deg(u_i) = m_i + 1$. The integers $m_i$ are such that

$$pc_G(x) = \prod_{i=1}^{l} (1 + x^{2m_i + 1})$$

is the Poincaré polynomial of $G$ and are called the exponents of $g$ (see [12]). The exponents can also be recovered as the eigenvalues of a particular element in the Weyl group which is known as a Coxeter–Killing transformation (see [2]).

Recall that $A_0^+ = \sum_{i=1}^{l} X_{-\alpha_i}$ is the sum of basis elements for all negative simple roots and $A_0^- = \sum_{i=1}^{l} X_{\alpha_i}$ for all positive roots respectively. Then Theorem 6.2 below characterizes a basis of the centralizer $g^{A_0^+}$ in terms of exponents and the above eigenspace decomposition. For a proof see [11], Theorem 5.

**Theorem 6.2.** There exists a basis $\{Z_i \mid i = 1, \ldots, l\}$ of $g^{A_0^+}$ such that $Z_i \in g^{(m_i)}$, where the integers $m_i$ are the exponents of $g$. In particular, $g^{A_0^+} \subset b^+$. 

Later we need the exponents $m_i$ of the Lie algebras $\mathfrak{sl}_{l+1}, \mathfrak{so}_{2l+1}, \mathfrak{sp}_{2l}, \mathfrak{so}_{2l}$ and $\mathfrak{g}_2$, i.e. the Lie algebras of type $A_l, B_l, C_l, D_l$ and $G_2$, for the explicit computation of the equations in Theorem 6.2. We want to note that it is also possible to read off the exponents from the root system. The following procedure was discovered by A. Shapiro and R. Steinberg (see [20]): For $k = 1, \ldots, \text{ht}(\beta)$, where $\beta \in \Phi^+$ is the maximal root, let

$$c_k = \lvert \{\alpha \in \Phi^+ \mid \text{ht}(\alpha) = k\} \rvert,$$

i.e. $c_k$ is the number of roots $\alpha \in \Phi^+$ such that $\text{ht}(\alpha) = k$. Then $k$ is $c_k - c_{k+1}$ times an exponent of $g$. A proof of correctness of this empirical method follows from [12], Corollary 8.7.

**Lemma 6.3.** The subalgebra $b^+$ is the direct sum of the centralizer $g^{A_0^+}$ and the image of $u^+$ under $\text{ad}(A_0^-)$, i.e.

$$b^+ = g^{A_0^+} + \text{ad}(A_0^-)(u^+).$$
For a proof of Lemma 6.4 above we refer to [11], Lemma 12.
We will now determine a basis of the subspace \( \text{ad}(A_0^-)(u^+) \) (see also [11], the beginning of the proof of Proposition 19). To this purpose, we rename the basis elements \( \{X_{\alpha} \mid \alpha \in \Phi^+\} \) of \( u^+ \) into \( X_i \) so that \( X_i \in g^{(j)} \) for some \( j > 0 \). We denote by \( r(i) \) the positive number such that \( X_i \in g^{(r(i))} \) and by \( r = |\Phi^+| \) the number of positive roots. We can now order the basis such that \( r(i) \leq r(i + 1) \) for all \( i = 1, \ldots, r - 1 \). We define \( W_i := \{X_i, A_0^\pm\} \). Then, when we interchange the role of the positive and negative roots, Theorem 6.2 yields \( g^{A_0^\pm} \subset b^- \) and therefore \( g^{A_0^\pm} \cap u^+ = (0) \). We conclude that \( W_i \in g^{r(i)-1} \) and that the set \( \{W_i \mid i = 1, \ldots, r\} \) is a basis of \( \text{ad}(A_0^-)(u^+) \).

**Lemma 6.4.** There are \( l \) roots \( \gamma_i \in \Phi^+ \) of \( \text{ht}(\gamma_i) = m_i \), where the integers \( m_i \) are the exponents of \( g \), such that \( b^+ \) is the direct sum of \( \tau := \sum_{i=1}^{l}(X_{\gamma_i}) \) and \( \text{ad}(A_0^-)(u^+) \), i.e.

\[
b^+ = \tau + \text{ad}(A_0^-)(u^+)
\]

**Proof.** Theorem 6.2 yields basis elements \( Z_i \) of \( g^{A_0^\pm} \) such that \( Z_i \in g^{(m_i)} \) for \( i = 1, \ldots, l \). We choose now \( i \in \{1, \ldots, l\} \) and fix it. Let \( m := m_i \) and denote by \( i_1, \ldots, i_s \) all elements of \( \{1, \ldots, l\} \) such that \( m_{i_k} = m \). Then the set

\[
\{Z_{i_1}, \ldots, Z_{i_s}\} \cup \{W_h \mid W_h \in g^{(m)}\}
\]

is a basis of \( g^{(m)} \) by Lemma 6.3 where the first set is a basis of \( g^{A_0^\pm} \cap g^{m} \) and the second one of \( \text{ad}(A_0^-)(g^{(m+1)}) \). On the other hand \( g^{(m)} \) is the direct sum of one-dimensional root spaces \( g_\alpha \) for \( \alpha \in \Phi^+ \) with \( \text{ht}(\alpha) = m \). We denote the set of this roots by \( \Lambda^{(m)} = \{\gamma \in \Phi^+ \mid \text{ht}(\gamma) = m\} \). Then by basis extension there exist \( s \) roots \( \gamma_1, \ldots, \gamma_s \in \Lambda^{(m)} \) such that the set

\[
\{W_h \mid W_h \in g^{(m)}\} \cup \{X_{\gamma_k} \mid k = 1, \ldots, s\}
\]

is a basis of \( g^{(m)} \).

If we repeat this procedure for all distinct \( m_i \), we obtain a basis for all eigenspaces \( g^{(m_i)} \). A basis for all remaining eigenspaces \( g^{(j)} \) in the direct sum \( b^+ = \sum_{j \geq 0} g^{(j)} \), i.e. for all \( g^{(j)} \) such that \( j \geq 0 \) and \( j \neq m_i \) for all \( i = 1, \ldots, l \), is simply given by \( \{W_h \mid W_h \in g^{(j)}\} \). \( \square \)

**Definition 6.5.** We call \( l \) roots \( \gamma_1, \ldots, \gamma_l \) of \( \Phi^+ \), which satisfy the conditions in Lemma 6.4, complementary roots of \( \Phi^+ \) and the subspace \( \tau = \sum_{i=1}^{l} \Phi_{\gamma_i} \) a root space complement to \( \text{ad}(A_0^-)(u^+) \).

Now let \( F \) be a differential field with constants \( C \) and \( \partial(y) = Ay \) be a matrix differential equation with \( A \in F^{n \times n} \).

**Definition 6.6.** The map

\[
l\delta : \text{GL}_n(F) \to F^{n \times n}, B \mapsto \partial(B)B^{-1}
\]

is called the logarithmic derivative.

**Proposition 6.7.** Let \( G \subset \text{GL}_n \) be a linear algebraic group. Then the restriction of \( l\delta \) to \( G \) maps \( G \) to its Lie algebra \( g \), that is

\[
l\delta \mid_G : G \to g
\]

A proof for Proposition 6.7 can be found in [13], page 585. We prove now the Transformation Lemma.
Lemma 6.8. Let $A \in A_0^\nu + b^+(F)$. Then $A$ is gauge equivalent to a matrix of the plane $A_0^\nu + \tau(F)$, i.e. there exists $u \in U^+$ such that

$$\text{Ad}(u)(A) + b\delta(u) \in A_0^\nu + \tau(F),$$

where $U^+$ is the maximal unipotent subgroup of $G$ with $\text{Lie}(U^+) = u^+$ and $\tau(F)$ is a root space complement to $\text{ad}(A_0^\nu)(u^+)$. 

Proof. Note that by Lemma 6.4 we can express every matrix $A_0^\nu \in b^+(F)$ in terms of basis elements $\{W_h, X_{\gamma_i} \mid 1 \leq h \leq r, 1 \leq i \leq l\}$. This allows us to make the following inductive assumption on an integer $k = 1, \ldots, r$: For $A = A_0^\nu + \sum_{i=1}^r w_i W_i + \sum_{j=1}^l a_j X_{\gamma_j}$ there exists $u \in U^+$ such that

$$\text{Ad}(u)(A) + b\delta(u) = A_0^\nu + \sum_{i=1}^r \bar{w}_i W_i + \sum_{j=1}^l \bar{a}_j X_{\gamma_j} \text{ with } \bar{w}_i = 0 \text{ for all } 1 \leq i \leq k.$$

Let $k > 1$ and assume the assumption holds for $k - 1$. We only have to consider the case $w_k \neq 0$, otherwise there is nothing to show. The element $X_k$ forms a basis of a one-dimensional root space. We denote the root which belongs to this root space by $\beta$. Now let $u_\beta(\rho)$ be a parameterized root group element

$$u_\beta(\rho) = \exp(\rho X_k) = \sum_{j=0}^{\infty} \frac{1}{j!} \rho^j (X_k)^j.$$

We show that we can choose $\rho \in F$ such that the coefficient of $W_k$ in the expression $\text{Ad}(u_\beta(\rho))(A) + b\delta(u_\beta(\rho))$ vanishes. Note that

$$\text{Ad}(u_\beta(\rho))(A) = \exp(\rho \text{ad}(X_k))(A) = \sum_{j=0}^{\infty} \frac{1}{j!} \rho^j \text{ad}^j(X_k)(A)$$

and that this expression is linear in $A$. We compute the image of $A$ under $\text{Ad}(u_\beta(\rho))$ separately for the three main summands $A_0^\nu$, $\sum_{i=1}^r w_i W_i$ and $\sum_{j=1}^l a_j X_{\gamma_j}$.

We determine first the image of $A_0^\nu$ under $\text{Ad}(u_\beta(\rho))$. For $j = 1$ we have $\rho W_k = \rho \text{ad}(X_k)(A_0^\nu)$ by construction of the basis elements $W_h$ and for $j \geq 2$ the image $\text{ad}^j(X_k)(A_0^\nu)$ lies in root spaces which belong to roots of height greater equal than $\text{ht}(\beta)$. The last assertion is implied by the following argument: If $j \geq 2$ the expression $j\beta - \alpha_i$ is a root of $\Phi$ for some $\alpha_i \in \Delta$, then $\text{ht}(j\beta - \alpha_i) \geq r(k)$. Note that $W_k \in g^{(r(k)-1)}$. We conclude that

$$\text{Ad}(u_\beta(\rho))(A_0^\nu) = \sum_{j=0}^{\infty} \frac{1}{j!} \rho^j \text{ad}^j(X_k)(A_0^\nu) \in A_0^\nu + \rho W_k + \sum_{j=0}^{\infty} g^{(r(k)+j)}.$$

Next, we consider $\text{Ad}(u(\rho))(\sum_{i=k}^r w_i W_i)$. Here, we omitted all indices $1 \leq i \leq k - 1$, since by the induction assumption $w_i = 0$ for all these indices. Recall that by construction $r(k) \geq 1$. For $X_k \in g^{(r(k))}$ and $W_i \in g^{(r(i)-1)}$ ($k \leq i \leq m$) the relation in Lemma 6.4 implies that $\text{ad}(X_k)(W_i) \in g^{(r(k)+r(i)-1)}$. We define $p = \max\{i \mid r(i) = r(k)\}$. Clearly $p \geq k$ and for $i \geq p + 1$ we have $W_i \in g^{(s)}$ with $s \geq r(k)$. These arguments show

$$\text{Ad}(u_\beta(\rho))(\sum_{i=k}^r w_i W_i) \in \sum_{i=k}^p w_i W_i + \sum_{j=0}^{\infty} g^{(r(k)+j)}.$$
Now we determine the image of $\sum_{j=1}^{l} a_j X_{\gamma_j}$ under $Ad(u_\beta(\rho))$. The relation in Lemma 6.4 yields $Ad(X_k)(X_{\gamma_i}) \in g^{(r(k) + 1)}$ for $X_k \in g^{(r(k))}$ and $X_{\gamma_i} \in g^{(l_{\gamma_i})}$. Thus, with $q = \max\{m_i \mid m_i \leq r(k) - 1\}$ we obtain
\[ Ad(u_\beta(\rho)) \left( \sum_{j=1}^{l} a_j X_{\gamma_j} \right) \in \sum_{j=1}^{q} a_j X_{\gamma_j} + \sum_{j \geq 0} g^{(r(k) + j)}. \]

Finally, we compute the image of $u_\beta(\rho)$ under the logarithmic derivative. By construction the Lie algebra of the root subgroup $U_\beta$ is $g_\beta$. Thus, $\delta(u_\beta(\rho)) \in g_\beta$ by Proposition 6.7 and therefore $\delta(u_\beta(\rho)) \in g^{(r(k))}$. Summing up our results, we obtain that
\[ Ad(u_\beta(\rho))(A) + \delta(u_\beta(\rho)) \in A_0 + \rho W_k + \sum_{i=k}^{p} w_i W_i + \sum_{j=1}^{q} a_j X_{\gamma_j} + \sum_{j \geq 0} g^{(r(k) + j)}. \]

Hence, for $\rho = -w_k$ the induction assumption follows for $k > 1$. The same argumentation shows that the induction assumption also holds for $k = 1$. Thus, the assertion of the lemma follows for $k = r$.

\[ \square \]

7. THE EQUATION FOR $\text{SL}_{d+1}(C)$

Let $\epsilon_1, \ldots, \epsilon_{l+1}$ be the standard orthonormal basis of $\mathbb{R}^{l+1}$ with respect to the usual inner product $(\cdot, \cdot)$ of $\mathbb{R}^{l+1}$. From [1], VI, Planch. I, we obtain that the root system $\Phi$ of type $A_l$ consists of the vectors $\epsilon_i - \epsilon_j$ $(1 \leq i, j \leq l + 1$ and $i \neq j)$. Further, the elements $\alpha_i := \epsilon_i - \epsilon_{i+1}$ $(1 \leq i \leq l$) form a basis $\Delta$ of $\Phi$ and with respect to $\Delta$ the root system of type $A_l$ is
\[ \Phi = \{ \pm \alpha_i \mid i = 1, \ldots, l \} \cup \{ \pm \alpha_i \pm \alpha_{i+1} \pm \cdots \pm \alpha_j \mid 1 \leq i < j \leq l \}. \]

A basis of a Cartan decomposition of $\mathfrak{sl}_{l+1}$ can also be taken from [1]. More precisely, the discussion in [1], VII, §13.1, yields that the set of matrices
\[ \{ E_{ij} \mid 1 \leq i, j \leq l + 1, i \neq j \} \cup \{ E_{ii} - E_{i+1,i+1} \mid 1 \leq i \leq l \} \]
forms a basis of $\mathfrak{sl}_{l+1}$ and that the matrices $H_i := E_{ii} - E_{i+1,i+1}$( $1 \leq i \leq l$) form a basis of a Cartan algebra $\mathfrak{h}$. Then, with respect to $\mathfrak{h}$ the elements $E_{ij}$ generate the one-dimensional root spaces $(\mathfrak{sl}_{l+1})_{\alpha}$ of the corresponding Cartan decomposition where $\alpha = \epsilon_i - \epsilon_j$ for $1 \leq i, j \leq l + 1$ and $i \neq j$.

Lemma 7.1. The $l$ roots
\[ \gamma_1 := \alpha_1, \gamma_2 := \alpha_{l+1} + \alpha_1, \ldots, \gamma_l := \alpha_1 + \cdots + \alpha_l. \]
are complementary roots of $\Phi^+$. 

Proof. In [1], VI, Planch. I, it follows that the maximal root in $\Phi^+$ has height $l$ and that for $k = 1, \ldots, l$ the number of roots of height $k$ is $l+1-k$. We conclude with Shapiro’s method that the exponents of $\mathfrak{sl}_{l+1}$ are $m_i = i$ for $1 \leq i \leq l$ (alternatively see [1], IV, Planches I). Then, for $1 \leq i \leq l$ the roots $\gamma_i$ satisfy $ht(\gamma_i) = m_i$. Following the proof of Lemma 6.3, we need to show that we can extend the set \{ $W_k \mid W_k \in \mathfrak{sl}_{l+1}^{(m_i)}$ \} to a basis of $\mathfrak{sl}_{l+1}^{(m_i)}$ by adjoining $X_{\gamma_i}$. We proceed by induction on $l$.

In case $l = 2$, the root system $\Phi$ consists of the roots $\pm \alpha_1, \pm \alpha_2$ and $\pm (\alpha_1 + \alpha_2)$. Then the single element $W_2 = [A_0, X_{\alpha_1 + \alpha_2}]$ forms a basis of $\text{ad}(A_0)(u^+)$ $\cap \mathfrak{sl}_{l+1}^{(1)}$ and has a non-zero component in $\mathfrak{sl}_{l+1}^{(\alpha_1)}$, where we recall that here $A_0 = X_{-\alpha_1} + X_{-\alpha_2}$.

We conclude that \{ $W_3, X_{\alpha_2}$ \} is a basis of $\mathfrak{sl}_{l+1}^{(1)}$. Finally, since $\alpha_1 + \alpha_2$ is the maximal root, the set \{ $X_{\alpha_1 + \alpha_2}$ \} is obviously a basis for $\mathfrak{sl}_{l+1}^{(2)}$.

Now assume the assertion is true for $l - 1$. The Dynkin diagram of type $A_l$ shows that the subset $\Phi' = \{ \alpha \in \Phi \mid n_{\alpha_1}(\alpha) = 0 \}$ $\subset \Phi$ is a root system of type $A_{l-1}$ which
is generated by the simple roots \( \alpha_2, \ldots, \alpha_l \). The induction assumption yields for 
\( 1 \leq i \leq l - 1 \) that the vector \( X_{\gamma_i} \) and the vectors \( W_h \) with \( W_h \in sl^{(m_i)}_l \) form a basis of \( sl^{(m_i)}_l \). The new elements in \( \Phi^+ \) are the roots \( \beta_k := \sum_{j=1}^k \alpha_j \) \( (1 \leq k \leq l) \). Thus, for \( 1 \leq i \leq l - 1 \) the root \( \beta_{m_{l+1}} \) is the only root \( \beta \in \Phi^+ \) such that \( \beta - \alpha_i = \beta_{m_i} \) for some \( \alpha_i \in \Delta \). Hence, the vector \( W = [A^0_0, X_{\beta_{m_{l+1}}} \} \) and the vectors of the set 
\( \{ W_h \mid W_h \in sl^{(m_i)}_l \} \) form a basis of \( ad(A^0_0)(u^+) \cap sl^{(m_i)}_l \) and \( W \) is the only element in \( \{ W_h \mid W_h \in sl^{(m_i)}_l \} \) with a non-zero component in \( (sl^{(m_i)}_{l+1})_{\beta_{m_i}} \). Combining this with the induction assumption shows that the set 
\( \{ W_h \mid W_h \in sl^{(m_i)}_l \} \cup \{ X_{\gamma_i} \} \)
is a basis of \( sl^{(m_i)}_l \) for every \( 1 \leq i \leq l - 1 \). Since \( \gamma_l \) is the maximal root, it is clear 
that \( \{ X_{\gamma_i} \} \) is a basis of \( sl^{(m_i)}_l \). Summing up we have shown that the roots \( \gamma_i \) are 
complementary roots and therefore Lemma 6.8 holds for the root space complement \( r = \sum_{i=1}^l (sl^{(m_i)}_l)_{\gamma_i} \).

**Theorem 7.2.** The linear parameter differential equation

\[
L(y, t) = g^{(l+1)} + \sum_{j=1}^l t_j g^{(l-j)} = 0
\]

has differential Galois group \( SL_{l+1}(C) \) over \( F_1 \).

**Proof.** We consider the differential equation \( \partial(y) = A_{SL_{l+1}}(t)y \) over \( F_1 \), where \( A_{SL_{l+1}}(t) \) is defined by

\[
A_{SL_{l+1}}(t) := A^+_0 + \sum_{i=1}^l t_i X_{-\gamma_{l+1} - i}
\]

and we denote its differential Galois group by \( H(C) \). We prove that \( H(C) = SL_{l+1}(C) \).

By construction \( A_{SL_{l+1}}(t) \in sl_{l+1}(F_1) \) and so Proposition 1.10 yields that \( H(C) \) is a 
subgroup of \( SL_{l+1}(C) \), that is, \( H(C) \subseteq SL_{l+1}(C) \).

Conversely, Proposition 6.8 implies that there exists a matrix differential equation 
\( \partial(y) = Ay \) over \( F_2 \) such that \( A \in A^+_0 + b^- \) and such that its differential Galois 
group is \( SL_{l+1}(C) \). We are going to apply Lemma 6.8. If we interchange the role of the 
positive and negative roots, Lemma 6.8 yields that \( A \) is a gauge equivalent to a matrix in the 
plane \( A^+_0 + r \), where by Lemma 7.1 the space \( r := \sum_{i=1}^l \mathfrak{g}^-_{\gamma_i} \)
is a root space complement of \( ad(A^+_0)(u^-) \). Hence, there exists a specialization \( \sigma : R_1 \rightarrow R_2 \) 
such that the differential Galois group of \( \partial(y) = A_{SL_{l+1}}(\sigma(t))y \) over \( F_2 \) is \( SL_{l+1}(C) \). 

Theorem 6.9 then asserts that \( SL_{l+1}(C) \subseteq H(C) \).

Combining this with the relation from above we obtain that \( H(C) = SL_{l+1}(C) \).

Finally, we need to show that the matrix differential equation \( \partial(y) = A_{SL_{l+1}}(t)y \) 
is equivalent to the linear parameter differential equation in the statement of the 
theorem. But this is clearly satisfied (see for instance [19], Chapter 1.2), since the 
defining matrix

\[
A_{SL_{l+1}}(t) = \sum_{i=1}^l E_{i, l+1} + \sum_{i=1}^l t_i E_{l+1, i}
\]

has the shape of a companion matrix with trace zero. \( \square \)

8. The Equation for \( SP_{2l}(C) \)

Let \( e_1, \ldots, e_{2l} \) be the standard orthonormal basis of \( \mathbb{R}^{2l} \) with respect to the usual 
inner product \( \langle \cdot, \cdot \rangle \) of \( \mathbb{R}^{2l} \). Then, from [1], IV, Planche III, we obtain that the
root system $\Phi$ of type $C_l$ consists of the vectors $\pm 2\epsilon_i$ ($1 \leq i \leq l$) and $\pm \epsilon_i \pm \epsilon_j$ ($1 \leq i < j \leq l$). A basis $\Delta$ for $\Phi$ is given by the vectors

$$\alpha_1 = \epsilon_1 - \epsilon_2, \alpha_2 = \epsilon_2 - \epsilon_3, \ldots, \alpha_{l-1} = \epsilon_{l-1} - \epsilon_l, \alpha_l = 2\epsilon_l$$

and with respect to $\Delta$ the positive roots are

$$\epsilon_i - \epsilon_j = \sum_{i \leq k < j} \alpha_k \quad (1 \leq i < j \leq l),$$

$$\epsilon_i + \epsilon_j = \sum_{i \leq k < j} \alpha_k + 2 \sum_{j \leq k < l} \alpha_k + \alpha_l \quad (1 \leq i < j \leq l)$$

$$2\epsilon_l = 2 \sum_{i \leq k < l} \alpha_k + \alpha_l \quad (1 \leq i \leq l).$$

The negative roots of $\Phi$ are obtained by interchanging all signs in the expressions of the positive roots.

We take a Cartan decomposition of the Lie algebra $sp_{2l}$ from [1], VII, §13.3. To this purpose we renumber the rows and columns of the matrices $E_{ij} \in C^{2l \times 2l}$ into $(1, 2, \ldots, l, -l, \ldots, -2, -1)$. Then, by [1], VII, §13.3, a Cartan subalgebra $\mathfrak{h}$ is generated by the basis elements $H_i = E_{i,i} - E_{-i,-i}$ ($1 \leq i \leq l$) and for $\alpha \in \Phi$ the root system $\Phi$ consists of the roots $\lambda_i$ generated by the basis elements $X_{\lambda_i} = A_{\gamma_i}^\alpha$, where $\gamma_i$ is a root of maximal height in $\Phi$. In order to show that the set

$$\{X_{\gamma_i} \cup \{W_h \mid W_h \in sp_{2l}(m_1)\}$$

is a basis of $sp_{2l}(m_1)$ for $1 \leq i \leq l$. The proof is done by induction on $l$.

For $l = 2$ the root system $\Phi$ consists of the roots $\pm \alpha_1$, $\pm \alpha_2$, $\pm (\alpha_1 + \alpha_2)$ and $\pm (2\alpha_1 + \alpha_2)$. Then the vector

$$W_3 = [A_{\gamma_1}^\alpha, X_{\alpha_1 + \alpha_2}] = c_1 X_{\alpha_1} + c_2 X_{\alpha_2}$$

has a non-zero component in $(sp_4)\alpha_1$ and forms a basis of $ad(A_{\gamma_1}^\alpha)(u^+) \cap sp_4^{(1)}$. Therefore, the set $\{W_3\} \cup \{X_{\alpha_1}\}$ is a basis of $sp_4^{(1)}$. Finally, since $2\alpha_1 + \alpha_2$ is the root of maximal height in $\Phi$, the set $\{X_{2\alpha_1 + \alpha_2}\}$ is obviously a basis of $sp_4^{(1)}$.

Let $l > 2$. We define the subset $\Phi'$ of the root system $\Phi$ by $\Phi' := \{\alpha \mid n_{\alpha_1}(\alpha) = 0\}$. Then the Dynkin diagram shows that $\Phi'$ is a root system of type $C_{l-1}$ and is generated by the simple roots $\alpha_2, \ldots, \alpha_l$. From the shapes of the roots in $\Phi'$ we conclude that the set $\Phi' \setminus \Phi'^{+}$ consists of the following roots:

$$\alpha_1, \alpha_1 + \alpha_2, \ldots, \alpha_1 + \cdots + \alpha_l,$$

$$\alpha_1 + \cdots + \alpha_{l-2} + 2\alpha_{l-1} + \alpha_l, \ldots, 2\alpha_1 + \cdots + 2\alpha_{l-1} + \alpha_l.$$
Hence, we obtain for 
\[ W = \{ W_h | W_h \in \mathfrak{sp}_{2l-2}^{(ht(\alpha))} \} \cup \{ W \} \]
is a basis of \( \text{ad}(A_0) \) and \( W \) is the only basis element that has a non-zero component in the root space \( (\mathfrak{sp}_{2l})_\alpha \). Further, the induction assumption yields for \( 1 \leq k \leq l-1 \) that 
\[ \{ W_h | W_h \in \mathfrak{sp}_{2l-2}^{(m_h)} \} \cup \{ X_{\gamma_l} \} \]
is a basis of \( \mathfrak{sp}_{2l-2}^{(m_h)} \). Hence, the set \( \{ W_h | W_h \in \mathfrak{sp}_{2l}^{(m_h)} \} \cup \{ X_{\gamma_l} \} \) is a basis of \( \mathfrak{sp}_{2l}^{(m_h)} \). Since \( \gamma_l \) is the root of maximal height in \( \Phi^+ \), it is trivial that \( \{ X_{\gamma_l} \} \) is a basis of \( \mathfrak{sp}_{2l}^{(m_h)} \). This completes the induction. 
\[ \square \]

**Lemma 8.2.** The matrix parameter differential equation \( \partial(y) = A_{SP_{2l}}(t)y \) over \( F_1 \), where 
\[ A_{SP_{2l}}(t) := A_0^+ + \sum_{i=1}^{l} (-t_i) X_{-\gamma_i} \]
is equivalent to the linear parameter differential equation 
\[ L(y, t) = y^{(2)} - \sum_{i=1}^{l} (-1)^{i-1} (t_iy^{(l-i)})(l-i) = 0. \]

**Proof.** Let \( y = (y_1, \ldots, y_l, y_{l+1}, \ldots, y_{2l})^T \). Then the matrix differential equation \( \partial(y) = A_{SP_{2l}}(t)y \), where 
\[ A_{SP_{2l}}(t) = (\sum_{i=1}^{l} E_{i,i+1} - E_{-1-1,-1}) + E_{l-1} - \sum_{i=1}^{l} t_i E_{-l-1+i,i+1-i}, \]
expands into the following system of differential equations:
\[ y_1 = y_2 \]
\[ y_{l+1} = t_1y_l - y_{l+2} \]
\[ \vdots \]
\[ y_{l-1} = y_l \]
\[ y_{2l-1} = t_{l-1}y_{2l-2} - y_{2l} \]
\[ y_{2l} = y_{l+1} \]
For \( 1 \leq k \leq l-1 \) we consider the subsystem
\[ y_{-k+1} = y_{-k+2}, \ldots, y_k = y_k, \]
\[ y_{l+1} = t_1y_l - y_{l+2}, \ldots, y_{l+k} = t_ky_{l+1-k} - y_{l+1+k} \]
and prove by induction on \( k \) that this system yields the following linear differential equation:
\[ y_{l+1-k}^{(2k)} = \sum_{i=1}^{k} (-1)^{i-1} (t_iy_{l-k+i}^{(k-i)})(k-i) + (-1)^{k} y_{l+1+k}. \]

Letting \( k = 1 \), the subsystem consists of the two equations \( y_0' = y_1 + 1 \) and \( y_{l+1} = t_1y_l - y_{l+2} \). We differentiate the first equation and we substitute in this expression the right hand side of the second equation for \( y_{l+1} \), i.e. we obtain \( y_1' = t_1y_l - y_{l+2} \). Now let \( k > 1 \). Then for \( k-1 \) the induction assumption, applied to the subsystem
\[ y_{-k+2} = y_{-k+3}, \ldots, y_k = y_k, \]
\[ y_{l+1} = t_1y_l - y_{l+2}, \ldots, y_{l+k-1} = t_ky_{l+2-k} - y_{l+k} \],
yields the linear differential equation

\begin{equation}
(1) \quad y_{l-k+2}^{(2k-2)} = \sum_{i=1}^{k-1} (-1)^{i-1} (t_i y_{l-k+2}^{(k-1)}(k-1-i)) + (-1)^{k-1} y_{l+k}.
\end{equation}

The equation \(y_{l-k+1} = y_{l-k+2}\) implies that we can replace \(y_{l-k+2}\) by \(y_{l-k+1}\) in Equation (1) and if we differentiate this expression, we obtain

\begin{equation}
(2) \quad y_{l-k+1}^{(2k)} = \sum_{i=1}^{k-1} (-1)^{i-1} (t_i y_{l-k+1}^{(k-i)}(k-i)) + (-1)^{k-1} y_{l+k}'.
\end{equation}

Differentiating Equation (2) and making the substitution \(y_{l+k} = t_k y_{l+1-k} - y_{l+1+k}\) completes the induction.

Now we consider the full system of equations. Ignoring the first and last equation, we get from the induction for \(k = l - 1\) the following equation:

\begin{equation}
\sum_{i=1}^{l-1} (-1)^{i-1} (t_i y_{l}^{(l-i)}) = 0.
\end{equation}

As above, the first equation of the full system implies that we can replace \(y_2\) by \(y_1\) which yields an expression in \(y_2\) and derivatives of \(y_1\). Thus, if we differentiate this expression and make the substitution \(y_{2l} = t_l y_1\), which is given by the last equation of the full system, we obtain the linear differential equation in the assertion of the lemma. \(\square\)

**Theorem 8.3.** The linear parameter differential equation

\[ L(y, t) = y^{(2l)} - \sum_{i=1}^{l} (-1)^{i-1} (t_i y_{l}^{(l-i)}) = 0 \]

has differential Galois group \(\text{SP}_{2l}(C)\) over \(F_1\).

**Proof.** The proof just works as the proof of Theorem 8.2. For \(A_{2l+1}(t)\) as in Lemma 8.2 one shows with Proposition 5.3, Lemma 8.1 and Lemma 6.8 (interchanging the role of the positive and negative roots) that \(\partial(y) = A_{2l+1}(t) y\) specializes to a differential equation over \(F_2\) with group \(\text{SP}_{2l}(C)\). If one applies then Theorem 4.3 and Proposition 1.1 one obtains that the differential Galois group of \(\partial(y) = A_{2l+1}(t) y\) is \(\text{SP}_{2l}(C)\). Finally use Lemma 8.2 to complete the proof. \(\square\)

**9. The equation for \(SO_{2l+1}(C)\)**

Let \(\epsilon_1, \ldots, \epsilon_l\) be the standard orthonormal basis of \(\mathbb{R}^l\) with respect to the standard inner product on \(\mathbb{R}^l\) and let

\[ \Phi = \{ \pm \epsilon_i \mid 1 \leq i \leq l \} \cup \{ \pm \epsilon_i \pm \epsilon_j \mid 1 \leq i < j \leq l \}. \]

Then by [1], IV, Planche II, the set \(\Phi\) is a root system of type \(B_l\) and a basis \(\Delta\) is given by the vectors \(\alpha_1 := \epsilon_1 - \epsilon_2\), \(\alpha_2 := \epsilon_2 - \epsilon_3\), \ldots, \(\alpha_{l-1} := \epsilon_{l-1} - \epsilon_l\) and \(\alpha_l := \epsilon_l\).

With respect to \(\Delta\) the positive roots of \(\Phi\) are

\[ \sum_{h \leq k \leq l} \alpha_k = \epsilon_h, \quad \sum_{i \leq k < j} \alpha_k = \epsilon_i - \epsilon_j \quad \text{and} \quad \sum_{i \leq k < j} \alpha_k + 2 \sum_{j \leq k \leq l} \alpha_k = \epsilon_i + \epsilon_j, \]

where \(1 \leq h \leq l\) and \(1 \leq i < j \leq l\). Note that we obtain all negative roots of \(\Phi\) in terms of the \(\alpha_i\) by interchanging all signs in the expressions for the positive roots.

Let us now renumber the rows and columns of the matrices \(E_{ij} \in C^{2l+1 \times 2l+1}\) into \((1, \ldots, l, 0, -l, \ldots, -1)\). Then a basis of a Cartan decomposition for \(so_{2l+1}\) is given by the following matrices (see [1], VII, §13.2.): The diagonal matrices \(H_i := E_{ii} - E_{-i,-i} (1 \leq i \leq l)\) generate a Cartan subalgebra, which we denote
by \( h \), and for a root \( \alpha \in \Phi \) the root space \((\mathfrak{so}_{2l+1})_\alpha\) with respect to \( h \) is generated correspondingly by one of the following matrices:

\[
\begin{align*}
X_{\gamma_0} &= 2E_{h,0} + E_{0,-h}, \\
X_{\gamma_i} &= E_{i,j} - E_{j,-i}, \\
X_{\gamma_{l-i}} &= E_{i,j} - E_{j,-i}, \\
X_{\gamma_{l+i}} &= E_{i,j} - E_{j,-i},
\end{align*}
\]

where \( 1 \leq h \leq l \) and \( 1 \leq i < j \leq l \).

**Lemma 9.1.** Let \( \gamma_1 = \alpha_l \) and \( \gamma_{l+1} = \alpha_{l-i} + 2\alpha_{l+1-i} + \cdots + 2\alpha_l \) for \( 1 \leq i \leq l - 1 \). The \( l \) roots \( \gamma_1, \ldots, \gamma_l \) are complementary roots of \( \Phi \).

**Proof.** From [1], IV, Planche II, we obtain that the exponents of \( \mathfrak{so}_{2l+1} \) are

\[ m_1 = 1, m_2 = 3, m_3 = 5, \ldots, m_l = 2l - 1 \]

and it is easy to check that the roots \( \gamma_i \) satisfy \( \text{ht}(\gamma_i) = m_i \). We make now the following inductive assumption on \( l \geq 2 \): For all \( 1 \leq i \leq l \) the set

\[ \{ W_h \mid W_h \in \mathfrak{so}_{2l+1}^{(m_i)} \} \cup \{ X_{\gamma_i} \} \]

is a basis of \( \mathfrak{so}_{2l+1} \).

In case \( l = 2 \), that is, the root system \( \Phi \) is of type \( B_2 \), the positive roots of \( \Phi \) are \( \alpha_1, \alpha_2, \alpha_1 + \alpha_2 \) and \( \alpha_1 + 2\alpha_2 \). It follows that the element \( W_3 = [A_0^5, X_{\alpha_1+\alpha_2}] \) forms a basis of \( \text{ad}(X_{\alpha_1+\alpha_2})(u^+) \cap \mathfrak{so}_5 \) and that \( W_3 \) has a non-zero component in the root space \( \mathfrak{so}_5 \). Hence, the set \( \{ W_3 \} \cup \{ X_{\alpha_2} \} \) is a basis of \( \mathfrak{so}_5 \) and the matrix \( X_{\alpha_1+2\alpha_2} \) forms obviously a basis of \( \mathfrak{so}_5 \), since \( \alpha_1 + 2\alpha_2 \) is the root of maximal height.

Let \( l > 2 \). We consider the subset \( \Phi' = \{ \alpha \in \Phi \mid n_{\alpha_1}(\alpha) = 0 \} \) of the root system \( \Phi \) of type \( B_l \). The Dynkin diagram shows that \( \Phi' \) is a subsystem of type \( B_{l-1} \) which is generated by the simple roots \( \alpha_2, \ldots, \alpha_l \). It is easy to check that \( \Phi^+ \setminus \Phi'^+ \) consists of the roots \( \alpha_1 + \cdots + \alpha_j \) for \( 1 \leq j \leq l \) and of

\[ \alpha_1 + \cdots + \alpha_{j-1} + 2\alpha_j + \cdots + 2\alpha_l \] for \( 2 \leq j \leq l \).

Note that the roots occurring in \( \Phi'^+ \) are of heights \( 1, \ldots, 2l - 1 \). For \( 1 \leq k \leq 2l - 1 \) there is a unique root \( \beta \) in \( \Phi^+ \setminus \Phi'^+ \) such that \( \text{ht}(\beta) = k \) which we denote by \( \beta_k \).

Now, for \( \beta_k \in \Phi^+ \setminus \Phi'^+ \) with \( 1 \leq k \leq 2l - 2 \), let \( \alpha \in \Phi^+ \) such that \( \alpha - \alpha_j = \beta_k \) for some \( \alpha_j \in \Delta \). Since \( n_{\alpha_1}(\alpha) \) has to be non-zero, we conclude that \( \alpha \in \Phi^+ \setminus \Phi'^+ \) and therefore we have \( \alpha = \beta_{k+1} \). This implies that \( W := [A_0^5, X_{\beta_{k+1}}] \) is the only basis element in \( \{ W_h \mid W_h \in \mathfrak{so}_{2l+1}^{(k)} \} \) of \( \text{ad}(A_0^5)(u^+) \cap \mathfrak{so}_{2l+1}^{(k)} \) such that its component in \( \mathfrak{so}_{2l+1} \) is non-zero. The induction assumption yields that for \( 1 \leq i \leq l - 1 \) the set

\[ \{ W_h \mid W_h \in \mathfrak{so}_{2l+1}^{(m_i)} \} \cup \{ X_{\gamma_i} \} \]

is a basis of \( \mathfrak{so}_{2l-1} \). Combining this with the above argument implies that \( X_{\gamma_i} \) and the vectors of \( \{ W_h \mid W_h \in \mathfrak{so}_{2l+1}^{(m_i)} \} \) form a basis of \( \mathfrak{so}_{2l+1}^{(m_i)} \). For \( m_l \) the set \( \{ X_{\gamma_l} \} \) is obviously a basis of \( \mathfrak{so}_{2l+1} \). This shows that the induction assumption holds for every \( l \in \mathbb{N} \). \( \square \)

**Lemma 9.2.** The matrix parameter differential equation \( \partial(y) = A_{\mathfrak{so}_{2l+1}}(t)y \) over \( F_1 \) is equivalent to the linear parameter differential equation

\[
L(y,t) = y^{(2l+1)} - \frac{1}{2} \sum_{i=1}^{l} t_i X_{\gamma_i} = 0,
\]

where the matrix \( A_{\mathfrak{so}_{2l+1}}(t) \) is defined by

\[
A_{\mathfrak{so}_{2l+1}}(t) := A_0^5 + \frac{1}{2} \sum_{i=1}^{l} t_i X_{\gamma_i}.
\]
Proof. With respect to the explicit basis of $\mathfrak{so}_{2l+1}$ the matrix differential equation $\partial(y) = A_{\mathfrak{so}_{2l+1}}(t) y$ is equivalent to the following system of equations:

\begin{align*}
y_k' &= y_{k+1} \quad (1 \leq k \leq l-1) \\
y_0' &= 2y_0, \quad y_0' = \frac{1}{2}t_1y_1 + y_{-l}, \quad y_{-l}' = -\frac{1}{2}t_2y_{-1} + t_1y_0 - y_{-l+1} \\
y_{-l+k}' &= -\frac{1}{2}t_{k+2}y_{-l-1-k} + \frac{1}{2}t_{k+1}y_{1-k} - y_{-l+1+k} \quad (1 \leq k \leq l-2) \\
y_{-1}' &= \frac{1}{2}t_1y_2.
\end{align*}

We consider first the case $l \geq 3$ and prove by induction on $1 \leq k \leq l-2$ that the subsystem defined by the integer $k$ is equivalent to the single differential equation

$$y_{l-k}' = \sum_{i=1}^{k+1} (-1)^{i-1} \left( (t_i y_{l-k}) (k+2-i) + (t_i y_{k-l}) (k+1-i) \right) + (-1)^{k+1} (t_{k+2}y_{l-k-1} + 2y_{-l+k+1}).$$

Letting $k = 1$, the equations $y_{l-1}' = y_1$ and $y_0' = 2y_0$ imply $y_{l-2}' = 2y_0$. We differentiate again and plug in $\frac{1}{2}t_1y_{l-1} + y_{-l}$ for $y_0'$, where we made the substitution $y_0 = y_{l-1}'$. We obtain $y_{l-2}' = t_1 y_{l-2}' + 2y_{-l}$. Differentiating again and replacing $y_{l-1}'$ by the right hand side of the last equation in Line (3) yields

$$y_{l-1}'^{(3)} = (t_1 y_{l-1}')' - t_2 y_{l-1} + t_1 y_{l-2}' - 2y_{-l+1},$$

where we substituted $y_0$ by $\frac{1}{2} y_{l-2}'$. Differentiating Equation (5), replacing $y_{l-1}'$ by the right hand side of Equation (5) and plugging in $y_{l-1}'$ for $y_{l-1}'$ yields

$$y_{l-1}'^{(5)} = (t_1 y_{l-1}')'' + (t_1 y_{l-1}')' - (t_2 y_{l-1}') - (t_2 y_{l-1}')' + t_3 y_{l-2} - 2y_{-l+2}.$$

We note that in case $l = 2$, we have to omit Equation (5) and consider instead $y_{l-1}' = \frac{1}{2} t_2 y_2$. More precisely, differentiating Equation (5) and making the substitution implied by $y_{l-1}' = \frac{1}{2} t_2 y_2$ proves the lemma for $l = 2$.

We come back to the case $l \geq 3$ and assume now $k > 1$. Then the induction assumption yields for the subsystem defined by the integer $k - 1$ the differential equation

$$y_{l-k+1}'^{(2k+1)} = \sum_{i=1}^{k} (-1)^{i-1} \left( (t_i y_{l-k+1}) (k+1-i) + (t_i y_{l-k}) (k-i) \right) + (-1)^{k} (t_{k+1} y_{l-k} + 2y_{-l+k}).$$

The subsystem defined by the integer $k$ contains the additional differential equation $y_{l-k}' = y_{l-k+1}$. We replace now $y_{l-k+1}$ by $y_{l-k}'$ and obtain

$$y_{l-k}'^{(2k+2)} = \sum_{i=1}^{k} (-1)^{i-1} \left( (t_i y_{l-k}') (k+1-i) + (t_i y_{l-k}) (k-i) \right) + (-1)^{k} (t_{k+1} y_{l-k} + 2y_{-l+k}).$$
Differentiating and replacing \( y'_{-l+k} \) by the right hand side of Equation (5) with \( j = k \) proves the induction assumption.

Finally, if we consider the subsystem of the full system of equations defined by the integer \( k = l - 2 \), we obtain the differential equation

\[
y_2^{(2l-1)} = \sum_{i=1}^{l-1} (-1)^{i-1} \left( t_i y_2^{(l-1-i)}(t-i) + (t_i y_2^{l-i})(t-1-i) \right) + (-1)^{l-1} (t_i y_1 + 2 y_{-1}) .
\]

The first equation of the full system implies the substitution \( y_2 = y'_1 \) in Equation (7).

We use this substitution also for the last equation of the full system and if we differentiate and replace \( y'_{-1} \) by the right hand side of this equation, it follows that the full system of equations is equivalent to the linear parameter differential equation of the lemma.

Theorem 9.3. The linear parameter differential equation

\[
L(y, t) = y^{(2l+1)} - \sum_{i=1}^{l} (-1)^{i-1} \left( (t_i y^{(l+1-i)}(t-i) + (t_i y^{l-i})(t-1-i) \right) = 0
\]

has differential Galois group \( \text{SO}_{2l+1}(C) \) over \( F_1 \).

Proof. The proof works as the proof of Theorem 7.2. Let \( \text{ASO}_{2l+1}(t) \) be as in Lemma 9.2. To prove that the equation \( \partial(y) = \text{ASO}_{2l+1}(t)y \) specializes to a differential equation over \( F_2 \) with group \( \text{SO}_{2l+1}(C) \) one uses Proposition 5.3, Lemma 9.1 and Lemma 6.8 (interchange the role of the positive and negative roots). Then Theorem 4.3 and Proposition 1.1 imply that the differential Galois group of \( \partial(y) = \text{ASO}_{2l+1}(t)y \) is \( \text{SO}_{2l+1}(C) \). Finally one applies Lemma 9.2.

10. The equation for \( \text{SO}_{2l}(C) \)

We denote the standard orthonormal basis of \( \mathbb{R}^l \) with respect to the standard inner product \( \langle \cdot, \cdot \rangle \) by \( e_1, \ldots, e_l \). Let us assume for the rest of this section that \( l \geq 3 \). Then by [P], Planche IV, the vectors \( \pm e_i \pm e_j \) \( (1 \leq i < j \leq l) \) form a root system \( \Phi \) of type \( D_l \) and a basis \( \Delta \) of \( \Phi \) is given by the \( l \) elements \( \alpha_1 = e_1 - e_2 \), \( \alpha_2 = e_2 - e_3 \), \( \ldots \), \( \alpha_{l-1} = e_{l-1} - e_l \), \( \alpha_l = e_{l-1} + e_l \). Then the positive roots can be expressed in terms of these basis elements in the following way:

\[
\begin{align*}
\epsilon_i - \epsilon_j &= \sum_{i < k < j} \alpha_k & (1 \leq i < j \leq l), \\
\epsilon_i + \epsilon_j &= \sum_{i < k < j} \alpha_k + 2 \sum_{j < k \leq l-1} \alpha_k + \alpha_{l-1} + \alpha_l & (1 \leq i < j < l), \\
\epsilon_i + \epsilon_l &= \sum_{i \leq k \leq l-2} \alpha_k + \alpha_l & (1 \leq i < l).
\end{align*}
\]

The negative roots are obtained by simply interchanging all signs in the above expressions. We take a Cartan decomposition of \( \mathfrak{so}_{2l} \) from [P], VII, §13.4. Renumbering the rows and columns of the matrices \( E_{ij} \in C^{2l \times 2l} \) into \( 1, \ldots, l, -l, \ldots, -1 \), [P], VII, §13.4 yields that the matrices \( H_i = E_{i,i} - E_{-i,-i} \) \( (1 \leq i \leq l) \) generate a Cartan subalgebra \( \mathfrak{h} \) of \( \mathfrak{so}_{2l} \) and, for the roots \( \alpha = \pm \epsilon_i \pm \epsilon_j \), where \( 1 \leq i < j \leq l \), the root spaces \( \langle \mathfrak{so}_{2l} \rangle_\alpha \) with respect to \( \mathfrak{h} \) are generated by the following matrices:

\[
\begin{align*}
X_{\epsilon_i - \epsilon_j} &= E_{i,j} - E_{-j,-i}, & X_{-\epsilon_i - \epsilon_j} &= -E_{i,j} + E_{-i,-j}, \\
X_{\epsilon_i + \epsilon_j} &= E_{i,j} - E_{-j,-i}, & X_{-\epsilon_i + \epsilon_j} &= -E_{i,j} + E_{-i,-j}.
\end{align*}
\]
Let now $\gamma_1 := \alpha_1$, $\gamma_2 := \alpha_{l-2} + \alpha_{l-1} + \alpha_1$ and for $3 \leq i \leq l-1$ let
\[
\gamma_i := \alpha_{l-1} + 2 \sum_{j=l+1-i}^{l-2} \alpha_j + \alpha_{l-1} + \alpha_1.
\]
Finally, we define $\bar{\gamma} := \alpha_1 + \cdots + \alpha_{l-2} + \alpha_l$. The following lemma shows that for these $l$ roots Lemma 6.3 and Lemma 6.8 hold.

**Lemma 10.1.** The $l$ roots $\gamma_1, \ldots, \gamma_{l-1}$ and $\bar{\gamma}$ are complementary roots of $\Phi$.

**Proof.** From [1], IV, Planche IV, we obtain that the exponents of the root system of type $D_l$ are $m_1 = 1$, $m_2 = 3$, $m_3 = 5, \ldots, m_{l-2} = 2l - 5$, $m_{l-1} = 2l - 3$ and $\bar{m} = l - 1$. For $1 \leq i \leq l - 1$ we have the relation $m_i = 2i - 1$. Note that in case $l$ is even, the value $l - 1$ occurs twice as an exponent of $D_l$, i.e. we have $\bar{m} = m_{\frac{l}{2}} = l - 1$. Now, for $1 \leq i \leq l - 1$ the roots $\gamma_i$ satisfy $ht(\gamma_i) = m_i$ and the root $\bar{\gamma}$ fulfills $ht(\bar{\gamma}) = \bar{m}$.

To shorten notation we define for $1 \leq i \leq l - 1$ the sets of matrices
\[
B_{l,m_i} := \{W_h \mid W_h \in \mathfrak{so}_{2l}^{(m_i)}\} \text{ and } B_{l,\bar{m}} := \{W_h \mid W_h \in \mathfrak{so}_{2l}^{(\bar{m})}\}.
\]
Recall that $B_{l,m_i}$ (respectively $B_{l,\bar{m}}$) is the basis of $\text{ad}(A_0^{-})^{(m_i)}$ (respectively of $\text{ad}(A_0^{-})^{(\bar{m})}$) which we constructed at the beginning of Chapter 6. In order to prove the lemma we need to show that for each exponent $m$ we can extend the basis $B_{l,m}$ to a basis of $\mathfrak{so}_{2l}$ by adjoining suitable root vectors $X_{\gamma_i}, X_{\bar{\gamma}}$. The proof is done by induction on $l$ where we differentiate between an odd and even $l$. We make the following inductive assumption on $l \in \mathbb{N}$, $l \geq 4$:

(a) If $l$ is odd, then $B_{l,m_i} \cup \{X_{\gamma_i}\}$ is a basis of $\mathfrak{so}_{2l}^{(m_i)}$ for $1 \leq i \leq l - 1$ and $B_{l,\bar{m}} \cup \{X_{\bar{\gamma}}\}$ is a basis of $\mathfrak{so}_{2l}^{(\bar{m})}$.

(b) If $l$ is even, then $B_{l,m_i} \cup \{X_{\gamma_i}\}$ is a basis of $\mathfrak{so}_{2l}^{(m_i)}$ for $i \in \{1, \ldots, l-1\} \setminus \{\frac{1}{2}\}$ and $B_{l,\bar{m}} \cup \{X_{\bar{\gamma}}, X_{\gamma_2}^{-}\}$ is a basis of $\mathfrak{so}_{2l}^{(\bar{m})}$.

Let $l = 4$. We start with the exponent $m_1 = 1$. The roots of height 2 in $\Phi$ are $\beta_1 = \alpha_1 + \alpha_2$, $\beta_2 = \alpha_2 + \alpha_3$ and $\beta_3 = \alpha_2 + \alpha_4$ and a basis of $\mathfrak{so}_{8}^{(1)}$ is given by the four root vectors corresponding to the simple roots, i.e. by the set $B := \{X_{\alpha_j} \mid 1 \leq j \leq 4\}$. We express the basis elements $W_h$ of $B_{4,1}$ in terms of the basis elements in $B$. For $1 \leq h \leq 3$ let
\[
W_h = [A_0^{-}, X_{\beta_h}] = \sum_{j=1}^{4} c_{\beta_h, \alpha_j} X_{\alpha_j},
\]
where the coefficient $c_{\beta_h, \alpha_j}$ is zero if and only if $\beta_h - \alpha_j \notin \Phi$. It is easy to check that the coordinates of the vectors $W_1$, $W_2$, $W_3$ and $X_{\alpha_4}$ with respect to the basis $B$ are
\[
M_1 := \begin{pmatrix}
c_{\beta_1,1} & 0 & 0 & 0 \\
c_{\beta_1,2} & c_{\beta_2,2} & 0 & 0 \\
0 & c_{\beta_2,3} & 0 & 0 \\
0 & 0 & c_{\beta_3,4} & 1
\end{pmatrix}.
\]
Since the determinant of $M_1$ is non-zero, it follows that $B_{4,1} \cup \{X_{\alpha_4}\}$ is a basis of $\mathfrak{so}_{8}^{(1)}$.

We consider now the exponents $m_2 = \bar{m} = 3$. The only root of height 4 in $\Phi$ is $\beta_1 = \alpha_1 + \cdots + \alpha_4$ and the roots of height 3 are $\bar{\beta}_1 = \alpha_1 + \alpha_2 + \alpha_3$, $\bar{\beta}_2 = \alpha_1 + \alpha_2 + \alpha_4$ and $\bar{\beta}_3 = \alpha_2 + \alpha_3 + \alpha_4$. Thus, the set $\{X_{\bar{\beta}_j} \mid 1 \leq j \leq 3\}$ is a basis of $\mathfrak{so}_{8}^{(3)}$ and the
coordinates \(c_{\beta_1,j}\) of
\[
W_{\beta_1} = [A_0^-, X_{\beta_1}] = \sum_{j=1}^{3} c_{\beta_1,j} X_{\beta_j}
\]
with respect to \(\{X_{\beta_j} \mid 1 \leq j \leq 3\}\) are all non-zero. We obtain the matrix
\[
M_2 := \begin{pmatrix}
c_{\beta_1,1} & 0 & 0 \\
c_{\beta_1,2} & 1 & 0 \\
c_{\beta_1,3} & 0 & 1
\end{pmatrix},
\]
where the first column represents \(W_{\beta_1}\) and the last two columns represent the elements \(X_{\beta_2}\) and \(X_{\beta_3}\). We conclude that the set \(B_k \cup \{X_{\beta_2}, X_{\beta_3}\}\) is a basis of \(\mathfrak{so}_8^{(3)}\), since the determinant of \(M_2\) is non-zero.

Finally, the root \(\gamma_3\) is the maximal root of \(\Phi\) and therefore, for the exponent \(m_3 = 5\), the set \(\{X_{\gamma_3}\}\) is trivially a basis of \(\mathfrak{so}_8^{(3)}\). This proves the lemma for \(l = 4\).

Now let \(l > 4\). We consider the set \(\Phi' = \{\alpha \in \Phi \mid n_1(\alpha) = 0\}\). The Dynkin diagram shows that \(\Phi' \subset \Phi\) is a root system of type \(D_{l-1}\) and that it is generated by the simple roots \(\alpha_2, \ldots, \alpha_l\). From the shapes of the roots in \(\Phi\) we deduce that the set \(\Phi^+ \setminus \Phi'^+\) consists of the following roots:
\[
\alpha_1, \alpha_1 + \alpha_2, \ldots, \alpha_1 + \cdots + \alpha_{l-1}, \alpha_1 + \cdots + \alpha_{l-2} + \alpha_l \text{ and }
\sum_{1 \leq k < l} \alpha_k + 2 \sum_{j \leq k < l-1} \alpha_k + \alpha_{l-1} + \alpha_l \quad (2 \leq j < l-2).
\]
We deduce that for \(k \in \{1, \ldots, 2l-3\} \setminus \{l-1\}\) there exists a unique root \(\beta \in \Phi^+ \setminus \Phi'^+\) such that \(h(\beta) = k\) (Recall that \(2l - 3\) is the maximal height of roots in \(\Phi^+\)). We denote this root by \(\tilde{\beta}_k\). For \(k = l-1\) the two roots \(\delta_1 := \alpha_1 + \cdots + \alpha_{l-1}\) and \(\delta_2 := \alpha_1 + \cdots + \alpha_{l-2} + \alpha_l\) are the only roots in \(\Phi^+ \setminus \Phi'^+\) such that \(h(\delta_i) = l-1\). We are going to apply the induction assumption to the root system \(\Phi'\) of rank \(l-1\). This to purpose we denote the exponents of the root system \(\Phi'\) by \(m_i'\) and \(\tilde{m}_i\) and the roots of the corresponding assertion of the lemma by \(\gamma_i'\) and \(\tilde{\gamma}_i\). Comparing the shapes of the roots \(\gamma_i'\) with \(\gamma_i\) and the values of the exponents \(m_i'\) with \(m_i\) we see that \(\gamma_1 = \gamma_1'\) and \(m_i = m_i'\) for \(1 \leq i \leq l-2\).

Now let \(p \in \mathbb{N}\) be the biggest index of all exponents \(m_i\) with the property that \(m_i \leq l-3\) and let \(q \in \mathbb{N}\) be the smallest index of all exponents \(m_i\) such that \(m_i \geq l\). A small computation shows that \(1 \leq p \leq \frac{1}{2}(l-2)\) and \(l-2 \geq q \geq \frac{1}{2}(l+1)\). Hence, for \(i \in \{1, \ldots, p\} \cup \{q, \ldots, l-2\}\) the induction assumption yields that a basis of \(\mathfrak{so}_{2l-2}^{(m_i)}\) is given by \(B_{l-1,m_i'} \cup \{X_{\gamma_i'}\}\). Since \(m_i + 1 \leq l - 2\) or \(m_i + 1 \geq l - 1\), the roots \(\beta_{m_i}\) and \(\beta_{m_i+1}\) are unique in \(\Phi^+ \setminus \Phi'^+\) with \(h(\beta_{m_i}) = m_i\) and \(h(\beta_{m_i+1}) = m_i + 1\). Thus the vector \(W = [A_0^-, X_{\beta_{m_i+1}}]\) is the only basis element in \(B_{l,m_i} = B_{l-1,m_i'} \cup \{W\}\) that has a non-zero component in the root space \(\mathfrak{so}_{2l}^{(\beta_{m_i})}\). We conclude that \(B_{l,m_i} \cup \{X_{\gamma_1}\}\) is a basis of \(\mathfrak{so}_{2l}^{(m_i)}\).

To complete the proof we still need to consider the exponents with possible values \(l-2\), \(l-1\) and \(2l-3\). We start with the roots of height \(l-2\). If \(l\) is even, then \(l-2\) is not an exponent of \(\mathfrak{so}_{2l}\) and there is nothing to show. So let \(l\) be odd. Then we have that \(m_{p+1} = l - 2\). The induction assumption yields for the root system \(\Phi'\) that
\[
B_{l-1,l-2} \cup \{X_{\gamma_1'}, X_{\gamma_{l-1}'}\}
\]
is a basis of \(\mathfrak{so}_{2l-2}^{(l-2)}\). In the following we denote by \(\Phi^+_{m_i}\) the set of all positive roots of height \(n\) for a root system \(\Phi\). Since
\[
\dim(\mathfrak{so}_{2l-2}^{(l-2)}) = |\Phi^+_{l-2}| \text{ and } \dim(\mathfrak{so}_{2l-2}^{(l-1)}) = |\Phi^+_{l-1}| = |B_{l-1,l-2}|
\]
we conclude that $|\Phi^+_{l-1}| + 2 = |\Phi^+_{l-1}|$. The root $\alpha_1 + \cdots + \alpha_{l-2}$ is the only root of height $l-2$ in $\Phi^+ \setminus \Phi^+$ and the roots of height $l-1$ in $\Phi^+ \setminus \Phi^+$ are $\delta_1$ and $\delta_2$. We obtain that $|\Phi^+_{l-1}| + 1 = |\Phi^+_{l-2}|$, i.e., we have to adjoin one root vector to $B_{l,l-2}$ to get a basis of $\mathfrak{so}_2$. Now, both vectors $W_{\delta_1} = [A_0^-, X_{\delta_1}]$ and $W_{\delta_2} = [A_0^-, X_{\delta_2}]$ have a non-zero component in $(\mathfrak{so}_2)_{\beta_{l-2}}$. On the other hand, only $W_{\delta_2}$ has a non-zero component in the root space of $\bar{\alpha}$.

We conclude that:

$$B_{l-1,l-2} \cup \{W_{\delta_1}, W_{\delta_2}, X_{\gamma_{l-1}}\} = B_{l,l-2} \cup \{X_{\gamma_{l-1}}\}$$

is a basis of $\mathfrak{so}_2^{(l-2)}$.

In the next step we consider the roots of height $l-1$. If $l$ is even, then $\tilde{m} = m_{l-1} = l - 1$, i.e., we have that the exponent $l - 1$ occurs twice when we consider the full system $\Phi$. Note that $l - 1$ is odd and therefore $\frac{1}{2}(l - 1)$ is not an integer and for the subsystem $\Phi'$ of type $D_{l-1}$ we have that $\tilde{m}' = l - 2$ and $m_{l-2}' = l - 1$. The induction assumption part (a) yields for $\Phi'$ and $i = l/2$ that $B_{l-1,i-1} \cup \{X_{\gamma_{l/2}}\}$ is a basis of $\mathfrak{so}_2^{(l-1)}$. Since $\delta_1$ and $\delta_2$ are the roots of height $l - 1$ in $\Phi^+ \setminus \Phi^+$, we have that $\dim(\mathfrak{so}_2^{(l-1)}) + 2 = \dim(\mathfrak{so}_2^{(l-1)})$. On the other hand, $\beta_l = \alpha_1 + \cdots + \alpha_l$ is the only root of height $l$ in $\Phi^+ \setminus \Phi^+$. Therefore $W = [A_0^- X_{\beta_l}]$ is the only new basis element in $B_{l-1}$, i.e., we have $B_{l-1} \setminus B_{l-1,i-1} = \{W\}$. Further, $W$ is the only basis element of $B_{l-1}$ that has non-zero components in $(\mathfrak{so}_2)_{\delta_1}$ and $(\mathfrak{so}_2)_{\delta_2}$. Combining these arguments we obtain that:

$$B_{l-1,i-1} \cup \{X_{\gamma_{l/2}}\} \cup \{W, X_{\delta_2}\} = B_{l-1} \cup \{X_{\gamma_{l/2}}, X_{\delta_2}\}$$

is a basis of $\mathfrak{so}_2^{(l-1)}$, where we have that $\delta_2 = \bar{\gamma}$ and $\gamma_{l/2} = \gamma_{l/2}$. Let $l$ now be odd. Then $l - 1$ is an exponent of $D_l$. By the induction assumption we obtain for the root system $\Phi'$ of type $D_{l-1}$, where $l - 1$ is even and not an exponent of $D_{l-1}$, that $B_{l-1,i-1}$ is a basis of $\mathfrak{so}_2^{(l-1)}$. The roots $\delta_1$ and $\delta_2$ are the only new roots of height $l - 1$ in $\Phi^+ \setminus \Phi^+$ and therefore $\dim(\mathfrak{so}_2^{(l-1)}) = |B_{l-1,i-1}| + 2$. Since $\beta_l$ is the only root of height $l$ in $\Phi^+ \setminus \Phi^+$, the vector $W = [A_0^- X_{\beta_l}]$ is the only basis element in $B_{l-1}$ such that its components in $(\mathfrak{so}_2)_{\delta_1}$ and $(\mathfrak{so}_2)_{\delta_2}$ are non-zero. We conclude that:

$$B_{l-1,i-1} \cup \{W, X_{\delta_2}\} = B_{l,i-1} \cup \{X_{\gamma}\}$$

is a basis of $\mathfrak{so}_2^{(l-1)}$. Finally, we come to the exponent $m_{l-1} = 2l - 3$. Since $\gamma_{l-1}$ is the root of maximal height in $\Phi$, the vector $X_{\gamma_{l-1}}$ forms trivially a basis of $\mathfrak{so}_2^{m_{l-1}}$.

Note that we proved the lemma for $l \geq 4$. In case $l = 3$, the root system is

$$\Phi = \{\pm \alpha_1, \pm \alpha_2, \pm \alpha_3, \pm (\alpha_1 + \alpha_2), \pm (\alpha_1 + \alpha_3), \pm (\alpha_1 + \alpha_2 + \alpha_3)\}$$

from which we can conclude easily that $\gamma_1 = \alpha_3, \gamma_2 = \alpha_1 + \alpha_2 + \alpha_3$ and $\bar{\gamma} = \alpha_1 + \alpha_3$ are complementary roots of $\Phi$. 

\[\square\]

**Lemma 10.2.** The matrix parameter differential equation $\partial(y) = A_{\mathfrak{so}_2}(t)y$ over $F_1$, where

$$A_{\mathfrak{so}_2}(t) := A_0^+ t_1 X_{-\gamma} + \sum_{i=1}^{l-1} -t_{i+1} X_{-\gamma_i}$$
and the roots \( \gamma_i \) and \( \bar{\gamma}_i \) are as in Lemma 10.1, is equivalent to the linear parameter differential equation

\[
L(y, t_1, ..., t_l) = y^{(2l)} - 2 \sum_{i=3}^{l} (-1)^i ((t_i y^{(l-i)})^{(l+2-i)} + (t_i y^{(l+1-i)})^{(l+1-i)})
\]

\[
- (t_2 y^{(l-2)} + t_1 y^{(l)}) - ((-1)^{l} t_1 z_1 + z_2) - \sum_{i=0}^{l-2} (t_2^{(l-2-i)} z_1^{(i)}).
\]

The coefficients \( z_1 \) and \( z_2 \) are given by

\[
z_1 = y^{(l-1)} - t_2 y^{(l-2)} - t_1 y
\]

\[
z_2 = \frac{(t_2^{(l-2)} + (-1)^{l-2} t_1^{(1)})}{t_2^{(l-2)} + (-1)^{l-2} t_1} \left( y^{(2l-1)} - 2 \sum_{i=3}^{l} (-1)^i ((t_i y^{(l-i)})^{(l+1-i)})
\]

\[
+ (t_i y^{(l+1-i)})^{(l-i)} - (t_2 y^{(l-2)} + t_1 y^{(l)})^{(l-1)} - \sum_{i=0}^{l-3} (t_2^{(l-3-i)} z_1^{(i)})\right).
\]

Proof. Using the representation of \( so_{2l} \) given at the beginning of this section, we obtain for \( y = (y_1, ..., y_{2l})^{(1)} \) that the matrix differential equation \( \partial(y) = A_{so_{2l}}(t) y \) is equivalent to the following system of linear differential equations:

\[
(k) \quad y_{k} = y_{k+1} + 1 \quad (1 \leq k \leq l - 2)
\]

\[
(\ell - 1) \quad y'_{\ell - 1} = y_{\ell} + y_{\ell + 1}
\]

\[
(\ell) \quad y'_{\ell} = -y_{\ell + 2}
\]

\[
(\ell + 1) \quad y'_{\ell + 1} = t_1 y_1 + t_2 y_{\ell - 1} - y_{\ell + 2}
\]

\[
(\ell + 2) \quad y'_{\ell + 2} = t_3 y_{\ell - 2} - t_2 y_{\ell - 3}
\]

\[
(\ell + k) \quad y'_{\ell + k} = t_{k+1} y_{\ell - k} - t_k y_{\ell - k + 2} - y_{\ell + k + 1} \quad (3 \leq k \leq l - 1)
\]

\[
(2\ell) \quad y'_{2\ell} = -t_1 y_2 - t_1 y_
\]

We show that \( y_1 \) is a cyclic vector. From the Equations (1) - (\ell - 2) we deduce that

\[
y_{(i-1)}^{(1)} = y_{i} \quad \text{for} \ 1 \leq i \leq l - 1.
\]

In particular, we have \( y_1^{(l-2)} = y_{l-1}. \) Differentiating \( y_1^{(l-2)} = y_{l-1} \) and substituting \( y_{l-1} \) by the right hand side of Equation (\ell - 1) yields \( y_{(i-1)}^{(1)} = y_{l} + y_{l+1}. \) We differentiate the last expression. Using Equation (\ell) and (\ell + 1) we obtain that

\[
y_1^{(l)} = t_1 y_1 + t_2 y_{l-1} - 2y_{l+2}.
\]

Thus, we have \( y_1^{(l)} = t_1 y_1 + t_2 y_{l-2} - 2y_{l+2} \) and if we solve for \(-2y_{l+2}, \) we get that

\[
-2y_{l+2} = y_1^{(l)} - t_1 y_1 - t_2 y_1^{(l-2)} = z_1
\]

We differentiate \( y_1^{(l)} = t_1 y_1 + t_2 y_{l-2} - 2y_{l+2} \) and substitute \( y_{l+2} \) by the right hand side of Equation (\ell + 2). This yields

\[
y_1^{(l+1)} = (t_1 y_1 + t_2 y_1^{(l-2)})' - 2t_3 y_1^{(l-3)} + 2t_2 y_1 + 2y_{l+3}.
\]

In case \( l \geq 4, \) we prove the following claim: For \( 1 \leq k \leq l - 3 \) the system

\[
y_{l+3}^{(1)} = t_4 y_{l-3} - t_3 y_{l-1} - y_{l+4}
\]

\[
\vdots
\]

\[
y_{l+2+k}^{(1)} = t_{k+3} y_{l-k-2} - t_{k+2} y_{l-k} - y_{l+k+3}
\]
together with the equations
(a) \[ y_1^{(i+1)} = z' - 2t_3y_1^{(l-3)} + 2t_2y + 2y_{l+3} \]
(b) \[ y_1^{(i-1)} = y_i \quad (l - k - 2 \leq i \leq l - 1) \]
(c) \[ y_1' = -y_{l+2} \]
(d) \[ -2y_{l+2} = y_1^{(l)} - t_1y_1 - t_2y_1^{(l-2)} =: z_1 \]
yields the differential equation
\[ y_1^{(l+k+1)} = z^{(k+1)} + 2 \sum_{i=3}^{k+2} (-1)^i ((t_iy_1^{(l-1)})^{(k+2-i)} + (t_iy_1^{(l+1)})^{(k+1-i)}) + \sum_{i=0}^{k-1} (t_i^{(k-i-1)}z_1)^{(i)} \]
where we denote in the following by \( z^{(m)} \) the term \( (t_1y_1 + t_2y_1^{(l-2)})^{(m)} \) for \( m \in \mathbb{N} \).
The proof is done by induction on \( 1 \leq k \leq l - 3 \).
Let \( k = 1 \). We differentiate \( y_1^{(l+1)} = z' - 2t_3y_1^{(l-3)} + 2t_2y + 2y_{l+3} \) and substitute \( y_1' = -y_{l+2} \) by the right hand side of Equation (1). We obtain the equation
\[ y_1^{(l+2)} = z^{(2)} - 2(t_3y_1^{(l-3)})^{(1)} - 2t_3y_{l-1} + 2t_4y_{l-3} + 2t_2y + 2t_2y_1 - 2y_{l+4} \]
Using the Equations (1) and (2) for the substitution of \( y_{l-1}, y_{l-3} \) and \( y_1' \) and afterwards Equation (3) for the substitution of \( y_{l+2}, y_{l+4} \), we get that
\[ y_1^{(l+2)} = z^{(2)} - 2(t_3y_1^{(l-3)})^{(1)} - 2t_3y_1^{(l-2)} + 2t_4y_1^{(l-4)} + 2t_2y_1 + t_2z_1 - 2y_{l+4} \]
Now let \( 1 < k < l - 3 \). For \( k - 1 \) we obtain a subsystem of the above system formed by the corresponding equations (1)-(k-1) and (a), (b'), (c) and (d). Then the induction assumption yields for this subsystem the differential equation
\[ y_1^{(l+k)} = z^{(k)} + 2 \sum_{i=3}^{k+1} (-1)^i ((t_iy_1^{(l-1)})^{(k+2-i)} + (t_iy_1^{(l+1)})^{(k+1-i)}) + \sum_{i=0}^{k-2} (t_i^{(k-i-1)}z_1)^{(i)} \]
We differentiate this equation and substitute \( y_1' = y_{l+k+2} \) by the right hand side of Equation (k-1). It follows that
\[ y_1^{(l+k+1)} = z^{(k+1)} + 2 \sum_{i=3}^{k+1} (-1)^i ((t_iy_1^{(l-1)})^{(k+3-i)} + (t_iy_1^{(l+1)})^{(k+2-i)}) + \sum_{i=0}^{k-2} (t_i^{(k-i-2)}z_1)^{(i+1)} \]
The equations in (3) show that we can replace \( y_{l-k-2} \) by \( y_1^{(l-k-3)} \) and \( y_{l-k} \) by \( y_1^{(l-k-1)} \) in the above expression. The Equations (3) and (4) imply that \( 2y_1' = -2y_{l+2} = z_1 \).
Using these substitutions, the above equation simplifies to
\[ y_1^{(l+k+1)} = z^{(k+1)} + 2 \sum_{i=3}^{k+2} (-1)^i ((t_iy_1^{(l-1)})^{(k+3-i)} + (t_iy_1^{(l+1)})^{(k+2-i)}) + \sum_{i=0}^{k-1} (t_i^{(k-i-1)}z_1)^{(i)} \]
and the induction is complete.

Now, in case \( l \geq 4 \) the claim yields for \( k = l - 3 \) the differential equation

\[
y_1^{(2l-2)} = z^{(l-2)} + 2 \sum_{i=3}^{l-1} (-1)^i ((t_1 + t_2) y_1^{(l-i-1)} + t_4 y_1^{(l-i+1)} (t_1 z_1))
+ 2((-1)^l t_1 y_1 + (-1)^l t_2 y_2 + t_2^2 y_3) + \sum_{i=0}^{l-4} (t_2^{l-4-i} z_1)^{(i)}.
\]

Note that for \( l = 3 \) this equation is equal to Equation 4 and the proof in case \( l = 3 \) continues here. We differentiate the last equation and in the new expression we plug in \(-t_1 y_1 - t_1 y_1 \) for \( y_2 y_1 \) (see Equation 22). Afterwards, we simplify the so obtained equation. Summing up, we get that

\[
y_1^{(2l-1)} = z^{(l-1)} + 2 \sum_{i=3}^{l-1} (-1)^i ((t_1 + t_2) y_1^{(l-i+1)} + t_4 y_1^{(l-i+1)} (t_1 z_1))
+ 2((-1)^l t_1 + t_2^{l-2}) y_1 + \sum_{i=0}^{l-3} (t_2^{l-3-i} z_1)^{(i)}.
\]

In order to obtain a suitable expression of \( y_l \) in terms of derivatives of \( y_1 \) for the last step, we solve the above equation for \( 2y_l \) and multiply with \((-1)^l t_1 + t_2^{l-2}\)', i.e. we obtain the following relation:

\[
2((-1)^l t_1 + t_2^{l-2}) y_1 = \frac{((-1)^l t_1 + t_2^{l-2})}{(-1)^l t_1 + t_2^{l-2}} \cdot (y_1^{(2l-1)} - z^{(l-1)})
- 2 \sum_{i=3}^{l} (-1)^i ((t_1 + t_2) y_1^{(l-i+1)} + t_4 y_1^{(l-i+1)} (t_1 z_1)) - \sum_{i=0}^{l-3} (t_2^{l-3-i} z_1)^{(i)} =: z_2.
\]

Finally, we differentiate the penultimate equation and replace \( 2((-1)^l t_1 + t_2^{l-2}) y_l \) by \( z_2 \) and \( 2y_l \) by \( z_1 \) (see above). This yields the equation in the assertion of the lemma. \( \square \)

**Theorem 10.3.** The linear parameter differential equation

\[
L(y, t) = y^{(2l)} - 2 \sum_{i=3}^{l} (-1)^i ((t_1 + t_2) y^{(l-i+1)} + t_4 y^{(l-i+1)} (t_1 z_1))
- (t_2 y^{(l-2)} + t_3 y^{(l)}) - ((-1)^l t_1 z_1 + z_2) - \sum_{i=0}^{l-2} (t_2^{l-2-i} z_1)^{(i)} = 0
\]

has SO\(_{2l}(C)\) as differential Galois group over \( F_1 \) where the functions \( z_1 \) and \( z_2 \) are as in Lemma 10.2.

**Proof.** The proof is very similar to the proof of Theorem 7.2. Let \( A_{SO_{2l}}(t) \) be the matrix of Lemma 10.2. One uses Proposition 6.3, Lemma 10.1 and Lemma 6.8 (interchange the role of the positive and negative roots) to show that \( \partial(y) = A_{SO_{2l}}(t)y \) specializes to an equation over \( F_2 \) with group SO\(_{2l}(C)\). It follows then from Proposition 1.1 and Theorem 4.3 that the differential Galois group of \( \partial(y) = A_{SO_{2l}}(t)y \) is SO\(_{2l}(C)\). To complete the proof apply Lemma 10.2. \( \square \)

11. The equation for \( G_2 \)

Let \( \epsilon_1, \epsilon_2, \epsilon_3 \) be the standard orthonormal basis of \( \mathbb{R}^3 \) with respect to the standard inner product \((\cdot, \cdot)\). Then from [1], IV, Planches IX, we obtain that the root system \( \Phi \) of type \( G_2 \) consists of the twelve vectors \( \pm(\epsilon_1 - \epsilon_2), \pm(\epsilon_1 - \epsilon_3), \pm(\epsilon_2 - \epsilon_3), \pm(2\epsilon_1 - \epsilon_2 - \epsilon_3), \pm(2\epsilon_2 - \epsilon_1 - \epsilon_3) \) and \( \pm(2\epsilon_3 - \epsilon_1 - \epsilon_2) \). Further, as a
basis $\Delta$ of $\Phi$ we can take the two vectors $\alpha_1 = \epsilon_1 - \epsilon_2$ and $-2\epsilon_1 + \epsilon_2 + \epsilon_3$. Then, with respect to $\Delta$ the roots in $\Phi$ have the following shapes:

$$
\pm \alpha_1 = \pm (\epsilon_1 - \epsilon_2), \quad \pm \alpha_2 = \pm (-2\epsilon_1 + \epsilon_2 + \epsilon_3), \\
\pm (\alpha_1 + \alpha_2) = \pm (-\epsilon_1 + \epsilon_3), \quad \pm (2\alpha_1 + \alpha_2) = \pm (-2\epsilon_2 + \epsilon_3), \\
\pm (3\alpha_1 + \alpha_2) = \pm (-2\epsilon_2 + \epsilon_1 + \epsilon_3), \quad \pm (3\alpha_1 + 2\alpha_2) = \pm (2\epsilon_3 - \epsilon_1 - \epsilon_2).
$$

In [8], Chapter 19.3, J. Humphreys presents an irreducible representation of the Lie algebra $\mathfrak{g}_2$ of type $G_2$ as a subalgebra of $\mathfrak{gl}_7$. Following his argumentation, we obtain that a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}_2$ is generated by the two diagonal matrices $H_1 = -E_{2,2} + E_{5,5} + 2E_{3,3} - 2E_{6,6} - E_{4,4} + E_{7,7}$ and $H_2 = E_{3,3} - E_{6,6} - E_{4,4} + E_{7,7}$ and that with respect to $\mathfrak{h}$ we can choose the root vectors corresponding to the positive roots as

$$
X_{\alpha_1} = \sqrt{2}(E_{1,3} - E_{6,1}) + (E_{2,7} - E_{4,5}), \quad X_{\alpha_2} = E_{3,4} - E_{7,6}, \quad X_{2\alpha_1 + \alpha_2} = \sqrt{2}(E_{1,4} - E_{7,1}) - (E_{2,6} - E_{3,5}), \quad X_{3\alpha_1 + \alpha_2} = E_{2,3} - E_{6,5}, \quad X_{2\alpha_1 + 2\alpha_2} = \sqrt{2}(E_{2,1} - E_{1,5}) - (E_{7,3} - E_{6,4}), \quad X_{3\alpha_1 + 2\alpha_2} = E_{2,4} - E_{7,5}.
$$

The root vectors corresponding to the negative roots are

$$
X_{-\alpha_1} = X_{\alpha_1}^t, \quad X_{-\alpha_2} = X_{\alpha_2}^t, \quad X_{-\alpha_1 - \alpha_2} = X_{\alpha_1 + \alpha_2}^t, \quad X_{-2\alpha_1 - \alpha_2} = X_{2\alpha_1 + \alpha_2}^t, \quad X_{-3\alpha_1 - \alpha_2} = X_{3\alpha_1 + \alpha_2}^t, \quad X_{-3\alpha_1 - 2\alpha_2} = X_{3\alpha_1 + 2\alpha_2}^t.
$$

In [8], Chapter 19.3, the corresponding matrices $g_k, g_{ij}$ are not assigned to the roots $\alpha \in \Phi$. To obtain such an assignment as above one can use the relations (1)-(5) given there. Summing up, we have a Cartan decomposition $\mathfrak{g}_2 = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi} \langle \mathfrak{g}_2 \rangle_\alpha = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi} \langle X_\alpha \rangle$

given by the explicit matrices $X_\alpha$ from above for $\alpha \in \Phi$.

**Lemma 11.1.** The two roots $\gamma_1 = \alpha_2$ and $\gamma_2 = 3\alpha_1 + 2\alpha_2$ of the root system $\Phi$ of type $G_2$ are complementary roots.

**Proof.** The exponents of the root system of type $G_2$ are $m_1 = 1$ and $m_2 = 5$ (see [1], IV, Planches IX). Thus, the roots $\gamma_i$ satisfy $\text{ht}(\gamma_i) = m_i$ ($i = 1, 2$). We have that $\alpha_1 + \alpha_2$ is the only root in $\Phi^+$ of height 2 and therefore $W_3 = [A_0^+, X_{\alpha_1 + \alpha_2}]$ forms a basis of $\text{ad}(A_0^+) \cap \langle \mathfrak{g}_2 \rangle_{(m_1)}$. Since $(\alpha_1 + \alpha_2) - \alpha_2 = \alpha_1$, $W_3$ has a non-zero component in the root space $\langle \mathfrak{g}_2 \rangle_{\alpha_1}$. We conclude that the set $\{W_3, X_{\alpha_2}\}$ is a basis of $\langle \mathfrak{g}_2 \rangle_{(m_2)}$. Finally, since the root $\gamma_2$ is the maximal root, it follows that $X_{\alpha_2}$ forms a basis of $\langle \mathfrak{g}_2 \rangle_{(m_2)}$. \hfill \Box

**Lemma 11.2.** The matrix parameter differential equation $\partial(y) = A_{G_2}(t_1, t_2) y$ over $C(t_1, t_2)$, where $A_{G_2}(t_1, t_2) := A_0^+ + t_1 X_{-\gamma_1} + t_2 X_{-\gamma_2}$, is equivalent to the linear parameter differential equation

$$
y^{(7)} = 2t_2 y' + 2(t_2 y')' + (t_1 y^{(4)})' + (t_1 y')^{(4)} - (t_1 (t_1 y'))' \cdot
$$

**Proof.** With respect to the above representation of $\mathfrak{g}_2$ the matrix differential equation $\partial(y) = A_{G_2}(t_1, t_2) y$ is equivalent to the following system of differential equations:

$$
y_1^{(1)} = \sqrt{2} y_3, \quad y_2^{(1)} = y_7, \quad y_3^{(1)} = y_4, \quad y_4^{(1)} = t_2 y_2 + t_1 y_3 - y_5, \quad y_5^{(1)} = -t_2 y_7, \quad y_6^{(1)} = -\sqrt{2} y_1 - t_1 y_7, \quad y_7^{(1)} = -y_6.
$$
We take $y_3$ as a cyclic vector. More precisely, we differentiate successively the equation $y_2^{(1)} = y_7$ and using the above equations we substitute accordingly until we get an expression only in derivatives of $y_2$. After the first step we obtain $y_2^{(2)} = -y_6$, where we substituted $y_7^{(1)}$ by $-y_6$. Differentiating this equation and using the relation

$$y_6^{(1)} = -\sqrt{2}y_1 - t_1 y_7 = -\sqrt{2}y_1 - t_1 y_2^{(1)}$$

we get $y_2^{(3)} = \sqrt{2}y_1 + t_1 y_2^{(1)}$. Differentiating this expression and plugging in $y_1^{(1)} = \sqrt{2}y_1$ shows that $y_2^{(4)} = (t_1 y_2^{(1)})^{(1)} + 2y_1$. If we differentiate again and substitute $y_3^{(1)}$ by $y_4$, we obtain $y_2^{(5)} = (t_1 y_2^{(1)})^{(2)} + 2y_4$. Differentiating the last equation yields

$$y_2^{(6)} = (t_1 y_2^{(1)})^{(3)} + 2t_2 y_2^{(1)} + t_1 y_2^{(4)} - t_1 (t_1 y_2^{(1)})^{(1)} - 2y_5,$$

where we used $y_4^{(1)} = t_2 y_2^{(1)} + t_1 y_3 - y_5$ and $y_3 = y_2^{(4)} - (t_1 y_2^{(1)})^{(1)}$. With the relation $y_5^{(1)} = -t_3 y_7 = -t_2 y_2^{(1)}$ we get finally that

$$y_2^{(7)} = (t_1 y_2^{(1)})^{(4)} + 2(t_2 y_2^{(1)})^{(1)} + (t_1 y_2^{(4)})^{(1)} - (t_1 (t_1 y_2^{(1)})^{(1)})^{(1)} + 2t_2 y_2^{(1)}.$$  

\[\square\]

**Theorem 11.3.** The differential Galois group of the linear parameter differential equation

$$L(y, t_1, t_2) = y^{(7)} - 2t_2 y' - 2(t_2 y)^{'} - (t_1 y^{(4)})^{'} - (t_1 y y')^{'} + (t_1 t_1 y')^{'} = 0$$

is $G_2(C)$ over $C(t_1, t_2)$.

**Proof.** The proof works as the proof of Theorem 7.2. Let $A_{G_2}(t_1, t_2)$ be the matrix of Lemma 11.2. Using Proposition 5.3, Lemma 11.1 and Lemma 6.8 (interchange the role of the positive and negative roots) one shows that there exists a specialization of the differential equation $\partial(y) = A_{G_2}(t_1, t_2) y$ to an equation over $F_2$ with differential Galois group $G_2(C)$. It follows then from Theorem 11.3 and Proposition 11.1 that the differential Galois group of $\partial(y) = A_{G_2}(t_1, t_2) y$ is $G_2(C)$. Finally, we apply Lemma 11.2.  

\[\square\]

**References**

[1] N. Bourbaki, *Groupes et Algèbres de Lie*, Springer-Verlag, Berlin Heidelberg, 1975.

[2] H.S.M. Coxeter, *The product of the generators of a finite group generated by reflections*, Duke Mathematical Journal, vol. 18 (1951), pp. 349-356.

[3] M. Dettweiler and S. Reiter, *The classification of orthogonally rigid $G_2$-local systems and related differential operators*, Trans. Amer. Math. Soc. 366, no. 11 (2014), 5821-5851.

[4] T. Dyckerhoff, *The inverse problem of differential Galois theory over the field $\mathbb{R}(z)$*, arXiv:0802.2597v1 [math.CA], 2008.

[5] E. Frenkel and B. Gross, *A rigid irregular connection on the projective line*, Annals of Math. 170 (2009), 1469-1512.

[6] L. Goldman, *Specialization and Picard-Vessiot theory*, Trans. Am. Math. Soc., 85:327-356, 1957.

[7] J. Hartmann, *On the inverse problem in differential Galois theory*, J. Reine Angew. Math. (Crelle) 586, 21–44 (2005).

[8] J. E. Humphreys, *Introduction to Lie Algebras and Representation Theory*, Springer-Verlag, New York, 1972.

[9] L. Juan and A. Ledet, *On generic differential $SO_n$-extensions*, Proc. Amer. Math. Soc., 136:1145–1153, 2008.

[10] N. Katz, *Differential Equations and Exponential Sums*, Volume 124 of Annals of Mathematics Studies, Princeton University Press, Princeton 1990.

[11] B. Kostant, *Lie Group Representations on Polynomial Rings*, Amer. J. Math., 85, (1963), 327-404.
[12] B. Kostant, *The Principal Three-Dimensional Subgroups and the Betti Numbers of a Complex Simple Lie Group*, Amer. Jour. of Math., 81, (1959), 973-1032.

[13] J. Kovacic, *The inverse problem in the Galois theory of differential fields*, Ann. Math., 89:583-608, 1969.

[14] J. Kovacic, *Cyclic vectors and Picard-Vessiot extensions*, technical report, Prolifics Inc., 1996.

[15] L. Lipshitz and L. A. Rubel, *A Gap Theorem for Power Series Solutions of Algebraic Differential Equations*, American Journal of Mathematics, Vol. 108, No. 5 (1986), 1193-1213.

[16] A. Maurischat, *Galois theory for iterative connections and nonreduced Galois groups*, Transactions of the AMS, 362 (2010), no. 10, pp. 5411-5453.

[17] A. Maurischat, *Picard-Vessiot theory of differentially simple rings*, Journal of Algebra 409 (2014), pp. 162-181.

[18] C. Mitschi and M.F. Singer, *Connected groups as differential Galois groups*, J. Algebra 184 (1996), 333-361.

[19] M. van der Put and M. F. Singer, *Galois theory of linear differential equations*, Springer-Verlag, Berlin-Heidelberg-New York, 2003.

[20] R. Steinberg, *Finite Reflection groups*, Transactions of the American Mathematical Society, vol. 91 (1959), pp. 493-504.

[21] M. Seiß, *A root parametrized differential equation for the special linear group*, arXiv:1405.0925v1.

[22] J.-P. Serre, *Galois cohomology*, Springer-Verlag, Berlin Heidelberg, 1997.

E-mail address: matthias.seiss@mathematik.uni-kassel.de

Universität Kassel, Fachbereich 10, Heinrich Plett Str. 40, 34132 Kassel, Germany.