States of non-zero ghost number in $c < 1$ matter
coupled to 2-D Gravity

Suresh Govindarajan, T. Jayaraman, Varghese John and Parthasarathi Majumdar
The Institute of Mathematical Sciences
C.I.T. Campus, Taramani
Madras 600113
INDIA

We study $c < 1$ matter coupled to gravity in the Coulomb gas formalism using the
double cohomology of the string BRST and Felder BRST charges. We find that states
outside the primary conformal grid are related to the states of non-zero ghost number by
means of descent equations given by the double cohomology. Some aspects of the Virasoro
structure of the Liouville Fock space are studied. As a consequence, states of non-zero
ghost number are easily constructed by “solving” these descent equations. This enables
us to map ghost number conserving correlation functions involving non-zero ghost number
states into those involving states outside the primary conformal grid.

December 91

1 email: suresh, jayaram, john, partha@imsc.ernet.in
1. Introduction

While the matrix model approach to $c \leq 1$ matter coupled to gravity has made considerable progress the same level of clarity and computability has not yet been achieved in the continuum Liouville approach to these theories. The true Liouville theory is certainly not a free field theory. However important progress has been made by recognising that at least in some domain the Liouville theory behaves like a modified free field theory \[1\],\[2\]. A significant success of this approach has been the calculation of the torus partition function of minimal models coupled to 2-d gravity \[3\]. Further Polyakov has pointed out that at least some part of the target space scattering amplitudes may be obtained by purely free field techniques\[4\].

But even in this free field approach major questions still remain to be answered satisfactorily. In particular the issue of the identification of physical operators and the evaluation of their correlation functions is still not completely resolved. Two important steps have however been taken in addressing these questions. Lian and Zuckerman \[5\] have shown that there exist an infinite number of BRST invariant states in $c < 1$ theories coupled to gravity. The Liouville momenta of these states are such as to provide the gravitational dressing of the matter null states of the minimal model. Further there is one state at ghost number $\pm n$ for every matter Virasoro representation whose Liouville momenta are $\beta > \beta_0$ for ghost number $+n$ states and $\beta < \beta_0$ for ghost number $-n$ states, where $\beta_0$ is the background charge of the Liouville theory. Using the theorem of Lian and Zuckerman, an explicit construction of the ghost number $\pm 1$ states has been given by Imbimbo, Mahapatra and Mukhi \[6\]. The three-point correlation functions of the states of ghost number zero have been computed using the Coulomb gas representation of the matter part of the theory by a number of authors \[7\],\[8\],\[9\],\[10\]. Surprisingly, in these calculations, free field techniques supplemented by the technique of negative number of screening contours in the Liouville sector are sufficient to produce results in agreement with those computed from the matrix models. Dotsenko and Kitazawa \[8\],\[9\] noticed that for the $c < 1$ minimal models coupled to gravity the matter vertex operator momenta could be continued to values outside the primary grid, with non-zero results for the correlation function. These operators have the same Liouville momenta as the Lian-Zuckerman states, at least in part, for the appropriate choice of gravitational dressing. Thus there appears to be two types of states, which we refer to as the LZ (Lian-Zuckerman) and DK (Dotsenko-Kitazawa) type states. The corresponding operators are referred to as LZ and DK operators. The ghost number zero operators are of course the same in both LZ and DK form.
In this letter we study the $c < 1$ models coupled to Liouville using the Coulomb Gas formalism for the matter part of the theory. We show using the rederivation of the Lian-Zuckerman theorem by Bouwknegt, McCarthy and Pilch \cite{11} that the DK type states are related to LZ states by a set of descent equations in a doubly graded complex associated with two BRST operators. One of these is the usual string BRST operator for the theory and the other is the BRST operator associated with the screening operators of the Coulomb gas formalism as in the work of Felder\cite{12}. For this construction, we also need a more detailed knowledge of the Virasoro structure of the Liouville Fock space. This is studied using the techniques of Kac determinants for Fock spaces ala Kato and Matsuda \cite{13}. This in turn provides us with an algorithm to explicitly construct the LZ type states, beginning with DK type states. The DK type states are, of course, trivial to construct. We illustrate our construction of LZ states from DK states. We then demonstrate the use of these descent equations to show that the correlation function of LZ operators may be reduced to correlation of DK operators.

2. Matter and Liouville Fock Spaces

Following \cite{11}, we consider two scalars $\phi^M$ and $\phi^L$ with charges $\alpha_0$ and $\beta_0$ respectively at infinity. The corresponding energy-momentum tensors are given by

\begin{align}
T^M &= -\frac{1}{4} \partial \phi^M \partial \phi^M + i\alpha_0 \partial^2 \phi^M , \\
T^L &= -\frac{1}{4} \partial \phi^L \partial \phi^L + i\beta_0 \partial^2 \phi^L ,
\end{align}

with central charges $c_M = 1 - 24\alpha_0^2$ and $c_L = 1 - 24\beta_0^2$. For the $(p, p+1)$ unitary minimal models

\[ \alpha_0^2 = \frac{1}{4p(p+1)} \text{ and } \beta_0^2 = -\frac{(2p+1)^2}{4p(p+1)} . \]

The vertex operators $e^{i\alpha\phi^M}$ and $e^{i\beta\phi^L}$ have conformal weights $\alpha(\alpha - 2\alpha_0)$ and $\beta(\beta - 2\beta_0)$ respectively. The usual screening charges for matter are

\[ \alpha_+ = \sqrt{\frac{p+1}{p}} \text{ and } \alpha_- = -\sqrt{\frac{p}{p+1}} . \]

The screening charges for the Liouville sector are given by

\[ \beta_+ = i\alpha_+ \text{ and } \beta_- = -i\alpha_- . \]
Following Felder [12], we consider the complex of Fock spaces (hereafter referred to as the Felder complex) in the matter sector given by $\bigoplus F_{m', \pm m + 2np}$, where $F_{m', m}$ is the Fock space built over the primary associated with the vertex operator $e^{i\alpha_{m', m} \phi}$. Here

$$\alpha_{m', m} = \frac{1 - m'}{2} \alpha_- + \frac{1 - m}{2} \alpha_+ .$$

We will also need the dual Fock space obtained by $F_{-m', -m}$. There exists an identity under the change of label given by $(m', m) \rightarrow (m' + p + 1, m + p)$ for the matter sector and $(m', m) \rightarrow (m' + p + 1, m - p)$ in the Liouville sector. These two seemingly distinct labels refer to the same vertex operator. This identity will prove to be useful later. There is one such Felder complex for every $m', m$ restricted to the range $1 \leq m' \leq p$ and $1 \leq m \leq (p - 1)$.

The screening operators $Q_+^{m} = \int \prod_{i=1}^{m} dz_i e^{i \alpha_+ \phi(z_i)}$ and similarly $Q_+^{p-m}$ act on these Fock spaces. The irreducible Virasoro module $L(m', m)$ (for a given $c_M$ labelled by $p$) is given by $Ker Q_+^{m}/Im Q_+^{p-m}$ on this complex. We shall refer to the screening operators loosely as $Q_F$ except when necessary. We also have the Fock spaces of the Liouville and ghost sectors denoted by $F(\beta)$ and $F(gh)$ respectively. The string BRST operator $Q_B$ given by

$$Q_B = \oint : c(z)(T^M(z) + T^L(z) + \frac{1}{2} T_{gh}(z)) :$$

acts on the tensor product $F(\alpha) \otimes F(\beta) \otimes F(gh)$.

We now proceed to consider the Virasoro structure of these Fock spaces. In the matter case, this is encoded in the diagram given in Figure 1. In each Fock tower, $v_{m', m}$ is the highest weight vector associated with the operator $e^{i\alpha_{m', m}}$, with an irreducible Virasoro module under it. $u_{m', m}$ is a level $(p + 1 - m')(p - m)$ singular vector over $v_{m', m}$ with an irreducible Virasoro module over it. It is generically given by an expression of the form

$$L_{(p+1-m')(p-m)} v_{m', m} M = (L_{-(p+1-m')(p-m)} + \cdots) v_{m', m} M ,$$

where the ellipsis refer to a polynomial of $L_n$’s at level $(p + 1 - m')(p - m)$. The second Virasoro null vector at level $m'm$ is identically vanishing in the Fock space. At that level, we have a Fock space state, $w_{m', m}$ of level $m'm$ which cannot be obtained by the action of $L_n$’s on $v_{m', m}$. $w_{m', m}$ can be obtained as the limit (see for instance [14])

$$|w_{m', m}\rangle = \lim_{(\alpha \rightarrow \alpha_{m', m})} \frac{1}{(\alpha - \alpha_{m', m})} L_{-m'm} |v^{\alpha}\rangle ,$$

3
where \( v^{\alpha} = e^{i \alpha \phi^{M}} \). Thus, \( w_{m',m} \) is the analog of the discrete primary in the \( c_M = 1 \) case. There these states became BRST closed on being given an appropriate Liouville dressing. However, we will find that they play a slightly different role here. The pattern of \( v_{m',m} \) with a null vector of type \( u_{m',m} \) and a Fock state \( w_{m',m} \) is repeated in every Fock tower.\(^2\) For details, see [12].

We would like to understand the null state structure of the Liouville primaries. For that, we note that Kato and Matsuda\(^3\) have provided the connection between Virasoro secondaries in the Fock space and the Fock states (those created by the action of creation and annihilation operators). We may write

\[
L_I |\alpha_{m',m}\rangle = \sum C_{IJ} a_J |\alpha_{m',m}\rangle ,
\]

where \( L_I \) are products of Virasoro raising operators of level \( N \), \( I = 1, \ldots, p(N) \). \( a_J \) are the products of Fock creation operators of level \( N \) with \( J = 1, \ldots, p(N) \). \( C_{IJ} \) is a matrix representation of the Virasoro secondaries in term of the Fock states. The zeros of the det \( C_{IJ} \) corresponds to the null vectors of the Virasoro structure which do not map into the Fock space. The det is given by the expression

\[
\text{det} C_{IJ} = \text{const.} \prod_{1 \leq m' m \leq N} (\alpha - \alpha_{m',m})^{p(N-m'm)} \quad \text{for} \quad m', m > 0 ,
\]

where the zeros of the determinant are explicitly shown. The non-vanishing null is obtained by examining the det for the dual Fock space.

\[
\text{det} \hat{C}_{IJ} = \text{const.} \prod_{1 \leq m' m \leq N} (\alpha - \alpha_{m',m})^{p(N-m'm)} \quad \text{for} \quad m', m < 0 .
\]

For example, consider the matter primary given by \( \alpha_{m',m} \). Note that (2.6) and (2.7) have a zero at levels \( m'm \) and \( (p + 1 - m')(p - m) \) respectively. Hence, it has a vanishing null at level \( m'm \) and a non-vanishing null at level \( (p + 1 - m')(p - m) \). They interchange their roles for the dual Fock primary which is given by \( \alpha_{p+1-m',p-m} \). Now, we shall apply this to the Liouville Fock primary labelled by \( \beta_{m',m} \) We consider three cases.

Case 1: \( \beta_{m',m} \) with \( 0 < m' < p, \ 0 < m < (p - 1) \)

Since (2.6) and (2.7) do not have any zeros for these values or the corresponding dual, there is no null over these primaries. This is true in all the cases when the Liouville dresses

---

\(^2\) The \( v, u, w \) in our notation correspond to the \( v_0, u_1, w_0 \) in Felder’s notation.

\(^3\) \( p(N) \) refers to the number of partitions of \( N \).
a matter primary inside the conformal grid. This includes the case of the cosmological constant[15].

Case 2: $\beta_{m',m}$ with $0 < m' < p$, $0 < m < (p - 1)$

Now (2.6) has a zero at level $m'm$ indicating a vanishing null. The dual has a non-vanishing null at the same level. Note that the $m'$, $m$ labels of the dual are both negative and cannot be both made simultaneously positive by means of the symmetry discussed earlier. For the case of the vanishing null, $\beta < \beta_0$ and $\beta > \beta_0$ for the other case. This is the case studied in [6].

Case 3: $\beta_{m',m}$ with $0 < m' < p$, $m > (p - 1)$

Now (2.6) has zeros at levels $m'm$ and $(n(p+1)+m')(m-np)$ where $n$ is fixed so that $(m-np)$ lies inside the conformal grid. Thus, in this case we have two vanishing nulls. The dual has two non-vanishing nulls given by the zeros of (2.7). They are at the same levels as in the vanishing case. Note again that the $m'$, $m$ of the dual are both negative and can never be made simultaneously positive. Once again, for the case of the vanishing null, $\beta < \beta_0$ and $\beta > \beta_0$ for the other case.

It appears possible to extend this argument to completely determine the Virasoro structure of the Liouville sector. It would appear that Liouville should belong to the $\text{III}_+(-)$ and $\text{III}_+(+)$ case of Feigin and Fuchs[16]. We shall however leave detailed considerations for the future[17]. For the work at hand, the first two nulls over the Liouville Fock primary are sufficient.

3. Double Cohomology of $Q_B$ and $Q_F$

The cohomology of the string BRST for $c < 1$ matter coupled to Liouville is the object of interest here. The tensor product of the Fock spaces $F(\alpha) \otimes F(\beta) \otimes F(gh)$ is labelled by two gradings. One is the usual ghost number $G(b$ has ghost number $-1$, $c$ has ghost number $+1$ and the state $c_1|0\rangle_{gh}$ is chosen to have ghost number $0$). The other grading related to the Felder cohomology is given by the distance of the given Fock tower from the central tower in the Felder complex. We shall call it tower number, with positive to the right of the central tower and negative to the left (refer Figure 1). One may study this cohomology directly on the relevant Fock spaces before truncating the matter Fock space to the Felder cohomology. There is one obvious class of states that are in the $Q_B$ cohomology. They are given by the states

$$|\alpha_{m',m \pm 2np}\rangle_M \otimes |\beta\rangle_L \otimes c_1|0\rangle_{gh},$$

(3.1)
where $\beta$ is chosen to obey the gravitational dressing condition
\[
\alpha(\alpha - 2\alpha_0) + \beta(\beta - 2\beta_0) = 1 .
\]

So all the $v_{m',m}$'s from every Fock tower in the Felder complex suitably dressed are in the cohomology. Under the action of $Q_F$, the states given in (3.1) are either exact or non-closed except in the central Fock tower. However, they map to or map from states that are not in the $Q_B$ cohomology. Thus the set of states in (3.1) (which are in fact the DK states) form representatives of the cohomology classes denoted by
\[
H^{(n)}(H^{(0)}_{rel}(F(\alpha)_M \otimes F(\beta)_L \otimes F_{gh}, Q_B), Q_F) .
\] (3.2)

Bouwknegt, McCarthy and Pilch\cite{ref11} have shown that there is an isomorphism between cohomology classes given by
\[
H^{(n)}_{rel}(H^{(0)}(F(\alpha)_M \otimes F(\beta)_L \otimes F_{gh}, Q_F), Q_B)
\simeq H^{(n)}(H^{(0)}_{rel}(F(\alpha)_M \otimes F(\beta)_L \otimes F_{gh}, Q_B), Q_F) .
\] (3.3)

Rewriting the left hand side as
\[
H^{(n)}_{rel}(L(m',m) \otimes F(\beta)_L \otimes F_{gh}, Q_B)
\]
we find that the states belonging to (3.4) are the LZ states, the objects of interest. Here, we take the isomorphism further by explicitly relating the DK states to the LZ states by means of descent equations in the double complex of $Q_B$ and $Q_F$. This isomorphism in particular, would relate states at non-zero ghost number(LZ) to those of zero ghost number(DK). We illustrate the descent for ghost number $-1$. Take a DK state of tower number $-1$ given by $|v_{m',2p-m}\rangle_M \otimes |v_{-m',2p-m}\rangle_L \otimes c_1|0\rangle_{gh}$. The choice of Liouville dressing here belongs to case 2 of the previous section with a vanishing null. Then
\[
Q_F|DK\rangle = Q_F|v_{m',2p-m}\rangle_M \otimes |v_{-m',2p-m}\rangle_L \otimes c_1|0\rangle_{gh}
\]
\[
= |u_{m',m}\rangle_M \otimes |v_{-m',2p-m}\rangle_L \otimes c_1|0\rangle_{gh}
\] (3.5)

Then one can construct a LZ state such that
\[
Q_B|LZ\rangle = |u_{m',m}\rangle_M \otimes |v_{-m',2p-m}\rangle_L \otimes c_1|0\rangle_{gh} .
\] (3.6)

(3.3) and (3.6) imply that
\[
Q_B|LZ\rangle = Q_F|DK\rangle .
\] (3.7)
Note further that

\[ Q_F |LZ\rangle = 0 , \quad Q_B |DK\rangle = 0 \quad . \] (3.8)

(3.7) and (3.8) give the descent equations for ghost number \(-1\). This procedure works for all the examples constructed in [3]. The general descent equation for ghost number \(-n\) is given by

\[ Q_B |LZ\rangle = Q_F |I_1\rangle \quad , \]
\[ Q_B |I_1\rangle = -Q_F |I_2\rangle \quad , \]
\[ \vdots \]
\[ Q_B |I_{n-1}\rangle = (-)^{n+1} Q_F |DK\rangle \quad . \] (3.9)

Such a descent equation follows from the statement that \((|LZ\rangle + |I_1\rangle + \ldots + |DK\rangle)\) is closed under \((Q_B - (-)^G Q_F)\). Note that the sum of the tower number and ghost number is invariant for \(|LZ\rangle, |DK\rangle, |I_m\rangle\). As we shall see later, for ghost number \(+n\), the situation is different. There, the descent equations are valid but vacuous.

Since the DK states are easy to construct, one can now “solve” (3.9) to obtain the LZ state. This now provides a systematic procedure for generating LZ states of negative ghost number. Establishing every equation in (3.9) will require knowing the appropriate null state equations in the matter and Liouville Fock towers. We shall demonstrate this construction for ghost numbers \(-1, -2\) in the next section. Interestingly, the condition \(\beta < \beta_0\) as the dressing of the DK state is required in “solving” (3.9), which is also the condition given by Lian and Zuckerman. This always corresponds to choosing the vanishing null over the Liouville primary.

The states of positive ghost number require a slightly different construction. All the positive ghost number states are obtained from the oscillator states \(|w\rangle\) which appear in place of the vanishing Virasoro null vector (in the matter sector). As pointed out earlier, these states can be defined by means of a limit procedure. Note that this is equivalent to defining them by means of Fock creation operators.

We shall first describe the situation for ghost number \(+1\). We have

\[ |DK\rangle = Q_F |w_{m',m}\rangle_M \otimes |\beta\rangle_L \otimes c_1 |0\rangle_{gh} \quad , \]
\[ |LZ\rangle = Q_B |w_{m',m}\rangle_M \otimes |\beta\rangle_L \otimes c_1 |0\rangle_{gh} \quad , \] (3.10)

where the Liouville dressing is fixed to be \(\beta > \beta_0\) in order to agree with Lian and Zuckerman. Unlike in the negative ghost number case, we find that this choice is not essential.
for our construction to work. The state $|LZ\rangle$ in (3.10) is not truly $Q_B$ exact since $w_{m',m}$ is not in the Felder cohomology. A similar argument holds for $|DK\rangle$ which is not truly $Q_F$ exact. It is easy to see that state with $w_{m',m}$ in the matter sector would become a discrete primary in the $c = 1$ limit and would be $Q_B$ closed.

This procedure is easy to generalise for ghost number $+n$.  

$$
|DK\rangle = Q_F|W^{(1)}\rangle , \\
|I_1\rangle = Q_B|W^{(1)}\rangle , \\
|I_1\rangle = Q_F|W^{(2)}\rangle , \\
|I_2\rangle = Q_B|W^{(2)}\rangle , \\
\vdots , \\
|I_{n-1}\rangle = Q_F|W^{(n-1)}\rangle , \\
|LZ\rangle = Q_B|W^{(n-1)}\rangle ,
$$

(3.11)

where $|W^{(m)}\rangle = L^c|w^{(m)}\rangle \otimes |\beta\rangle_L \otimes c_1|0\rangle_{gh}$ and $w^{(m)}$ is the first oscillator state associated with matter Fock tower $m$ towers to the left of $|DK\rangle$ (i.e., tower number $(n - m)$).  Further, $L^c$ is a polynomial of ghost number $(m - 1)$ in $L_{-n}$’s and appropriate $c_n$’s.

Finally, note that we have also used the isomorphism between the irreducible Verma modules $L(m', m)$ and $L(p' - m', p - m)$. In the construction of the LZ states we have used the first Virasoro irrep for one half of the states and the other irrep for the rest.

4. Examples

In this section, we present explicit construction of LZ states from DK states. We obtain examples for states of ghost number $\pm 1$ and $\pm 2$ to illustrate the construction. We compare the ghost number $\pm 1$ states obtained by our construction to those in [3].

First, we construct the ghost number $-1$ state for $(m', m) = (-2, -1)$ for arbitrary $c_M$. The descent equation is given by

$$
Q_B|LZ\rangle = Q_F|DK\rangle
$$

(4.1)

4 This is very much like the zero momentum dilaton in the bosonic string where the dilaton can be written as $Q_B(c_0 - \bar{c}_0)|0\rangle$. Yet it is not a truly BRST exact state since $(c_0 - \bar{c}_0)|0\rangle$ is not annihilated by $(b_0 - \bar{b}_0)$ and hence is not an allowed state in the Hilbert space.
The DK state is given by
\[ |DK\rangle = |v_{-2,1}\rangle_M \otimes |v_{2,1}\rangle_L \otimes c_1|0\rangle_{gh} \quad . \] (4.2)

We then have
\[ Q_F |DK\rangle = |u_{-2,-1}\rangle_M \otimes |v_{2,1}\rangle_L \otimes c_1|0\rangle_{gh} \quad , \]
\[ = \left( L_M^{\perp -2} - tL_M^{L \perp -2} \right) |v_{-2,-1}\rangle_M \otimes |v_{2,1}\rangle_L \otimes c_1|0\rangle_{gh} \quad , \] (4.3)

where \( t = \frac{p}{p+1} \). The corresponding LZ-state is given by solving the descent equation (4.1)
\[ |LZ\rangle = \left( t(L_M^{L \perp -1} - L_M^{L \perp -1}) b_{-1} + b_{-2} \right) |v_{-2,-1}\rangle_M \otimes |v_{2,1}\rangle_L \otimes c_1|0\rangle_{gh} \quad . \] (4.4)

In checking equation (4.1), one encounters the Liouville null state given by
\[ (L_M^{L \perp -2} + tL_M^{L \perp -2}) |v_{-2,3}\rangle_L \quad , \] (4.5)

which is zero since the choice of Liouville dressing is made so as to have a vanishing null. This also corresponds to Lian and Zuckerman’s choice of \( \beta < \beta_0 \).

We shall now construct a ghost number \(-2\) LZ state. For convenience, we choose the case of \( c_M = 0 \). The descent equation is
\[ Q_B |LZ\rangle = Q_F |I\rangle \quad , \]
\[ Q_B |I\rangle = -Q_F |DK\rangle \quad . \] (4.6)

Figure 2 shows the location of the matter part of these states in the Felder complex. Consider the DK state obtained from dressing \( |v_{2,5}\rangle_M \). “Solving” (4.6) and after tedious but straightforward algebra, we obtain
\[ |DK\rangle = |v_{2,5}\rangle_M \otimes |v_{-2,5}\rangle_L \otimes c_1|0\rangle_{gh} \quad , \]
\[ |I\rangle = \mathcal{L}_4^b |v_{2,3}\rangle_M \otimes |v_{-2,5}\rangle_L \otimes c_1|0\rangle_{gh} \quad , \]
\[ |LZ\rangle = \mathcal{L}_5^{2b} |v_{2,1}\rangle_M \otimes |v_{-2,5}\rangle_L \otimes c_1|0\rangle_{gh} \quad , \] (4.7)

where
\[ \mathcal{L}_4^b = \frac{94}{3} b_{-4} + b_{-3} \left( \frac{61}{3} L_M^{L \perp -1} + 3L_M^{M \perp -1} \right) \\
+ b_{-2} \left( 4L_M^{L \perp -2} - 4L_M^{L \perp -2} + \frac{20}{3} L_M^{L \perp -2} \right) \\
+ b_{-1} \left( -3L_M^{L \perp -3} - \frac{41}{3} L_M^{M \perp -3} - \frac{20}{3} L_M^{L \perp -1} L_M^{L \perp -2} \right) \\
+ \frac{20}{3} L_M^{L \perp -2} L_M^{L \perp -1} + L_M^{L \perp -3} - L_M^{L \perp 2} L_M^{M \perp -1} + L_M^{L \perp 1} L_M^{M \perp 2} - L_M^{L \perp 3} \right) \quad . \] (4.8)
and
\[
\mathcal{L}^{2b} = -\frac{4}{3} b_{-1} b_{-1} + 4 b_{-3} b_{-2} + b_{-3} b_{-1} \left( \frac{2}{3} L_{-1}^L - 15 L_{-1}^M \right) \\
+ b_{-2} b_{-1} \left( -4 L_{-2}^L - \frac{2}{3} L_{-1}^L - 6 L_{-1}^L L_{-1}^M + L_{-1}^M \right). \tag{4.9}
\]

These expressions are unique up to $Q_B$ exact pieces that are also $Q_F$ closed. In the process of checking that \((4.7)\) satisfies \((4.6)\), vanishing nulls encountered in both the Liouville and matter sectors. The relevant null state equations are
\[
|u_{2,3}\rangle_M = \mathcal{L}^M_4 |v_{2,3}\rangle_M, \\
(L_{-2}^M - \frac{3}{2} L_{-1}^M) |v_{2,1}\rangle_M = 0, \\
\mathcal{L}^L_4 |v_{-2,5}\rangle_L = 0, \mathcal{L}^L_3 |v_{-2,5}\rangle_L = 0, \tag{4.10}
\]
where
\[
\mathcal{L}^M_4 = 4 L_{-4}^L - 4 L_{-3}^L L_{-1}^L - 4 L_{-2}^M L_{-1}^2 + \frac{20}{3} L_{-2}^L L_{-1}^M - L_{-1}^M, \\
\mathcal{L}^L_4 = \frac{76}{3} L_{-4}^L + \frac{52}{3} L_{-3}^L L_{-1}^L + 4 L_{-2}^L L_{-1}^2 + \frac{20}{3} L_{-2}^L L_{-1}^M + L_{-1}^M, \\
\mathcal{L}^L_3 = L_{-3}^L + \frac{1}{2} L_{-2}^L L_{-1}^L + \frac{1}{12} L_{-1}^3.
\]

This procedure can be used to generate LZ-states at arbitrary negative ghost number. Of course, in practice this entails the use of higher null state equations that rapidly increase in complexity which makes computations tedious.

As mentioned earlier, a slightly different procedure is required to construct states at positive ghost number. We shall now illustrate the construction for ghost numbers $+1$, $+2$. Consider the ghost number $+1$ case. We begin by considering the $w_{1,1}$ in the matter $(1,1)$ Fock tower.
\[
|w_{1,1}\rangle_M = \lim_{\alpha \to \alpha_{1,1}} \frac{1}{\alpha - \alpha_{1,1}} L_{-1}^M |v_{1,1}\rangle_M. \tag{4.11}
\]

The LZ and DK states are obtained from the above matter state (appropriately dressed by a Liouville primary). They are
\[
|LZ\rangle_{+1} = Q_B |w_{1,1}\rangle_M \otimes |v_{-1,-1}\rangle_L \otimes c_1 |0\rangle_{gh} \\
= -4 \alpha_0 c_{-1} |v_{1,1}\rangle_M \otimes |v_{-1,-1}\rangle_L \otimes c_1 |0\rangle_{gh}, \tag{4.12}
\]
\[
|D\rangle = |v_{1,-1}\rangle_M \otimes |v_{-1,-1}\rangle_L \otimes c_1 |0\rangle_{gh}.
\]

This agrees with the corresponding LZ-state given in \([4]\). We now construct another LZ-state at ghost number $+1$. Consider the $w_{2,1}$ in the Fock tower $(m', m) = (2, 1)$.
\[
|w_{2,1}\rangle_M = \lim_{\alpha \to \alpha_{2,1}} \frac{1}{\alpha - \alpha_{2,1}} \left( L_{-2}^M - \frac{3}{2(2\Delta+1)} L_{-1}^M \right) |v_{2,1}\rangle_M. \tag{4.13}
\]

\footnote{The Liouville state $|v_{-2,5}\rangle_L \equiv |v_{1,3}\rangle_L$ has two vanishing nulls over it as given by Case 3 in the section 2.}
where $\Delta$ is the weight of $|v_{1,2}\rangle_M$. The corresponding LZ and DK states are given by

$$|LZ\rangle_{+1} \propto c_{-2}|v_{2,1}\rangle_M \otimes |v_{-2,-1}\rangle_L \otimes c_1|0\rangle_{gh},$$

$$|DK\rangle = |v_{2,-1}\rangle_M \otimes |v_{-2,-1}\rangle_L \otimes c_1|0\rangle_{gh}.$$  \hfill (4.14)

This differs from the corresponding state in [6] by a $Q$-exact piece. The exact part is obtained by acting with $Q_B$ on the level two Virasoro secondary (orthogonal to the non-vanishing Virasoro null) over the primary $|v_{-2,-1}\rangle_L$. Note that since this term is given by $Q_B$ acting on a state in the Felder cohomology, it is truly exact.

We now will construct a ghost number +2 state. We shall do this for the case of $c_M = 0$ and $(m', m) = (1, 1)$. Figure 2 shows the location of the matter part of these states in the Felder complex. Consider the DK state

$$|DK\rangle = |v_{1,-3}\rangle_M \otimes |v_{-1,-3}\rangle_L \otimes c_1|0\rangle_{gh}.$$  \hfill (4.15)

Following our construction, $Q_F|w_{1,-1}\rangle_M = |v_{1,-3}\rangle_M$ with

$$|w_{1,-1}\rangle_M = \lim_{\alpha \to \alpha_{1,-1}} \frac{1}{\alpha - \alpha_{1,-1}} L^M_4|v_{1,-1}\rangle_M,$$  \hfill (4.16)

where $L^M_4$ is as given after (4.10). The state $|I\rangle$ is obtained as follows

$$|I\rangle_{+1} = Q_B|w_{1,-1}\rangle_M \otimes |v_{-1,-3}\rangle_L \otimes c_1|0\rangle_{gh}$$

$$= L^c_{-4}|v_{1,-1}\rangle_M \otimes |v_{-1,-3}\rangle_L \otimes c_1|0\rangle_{gh}.$$  \hfill (4.17)

where

$$L^c_4 = -\frac{5}{\sqrt{6}}\{c_{-1}(2L^M_{-3} - \frac{20}{3}L^M_{-2}L^M_{-1} + 2L^M_{-1}3) + c_{-2}(-2L^M_{-2} + \frac{7}{3}L^M_{-1}2) - \frac{10}{3}c_{-3}L^M_{-1}\}$$

The intermediate oscillator state (again replacing a vanishing null) encountered is given by using

$$|I\rangle_{+1} = Q_F|\tilde{w}_{1,1}\rangle \otimes |v_{-1,-3}\rangle_L \otimes c_1|0\rangle_{gh}.$$  \hfill (4.18)

We obtain

$$|\tilde{w}_{1,1}\rangle = L^c_{-4}|w_{1,1}\rangle_M$$

$$= \lim_{\alpha \to \alpha_{1,-1}} \frac{1}{\alpha - \alpha_{1,-1}} L^c_{-4}L^M_{-1}|v_{1,1}\rangle_M.$$  \hfill (4.19)

The LZ state is given by

$$|LZ\rangle_{+2} = Q_B|\tilde{w}_{1,-1}\rangle$$

$$= \frac{5}{6}\{28c_{-4}c_{-1} + 56c_{-3}c_{-2} - 36c_{-2}c_{-1}L^M_{-2}\}|v_{1,1}\rangle_M \otimes |v_{-1,-1}\rangle_L \otimes c_1|0\rangle_{gh}.$$  \hfill (4.20)

It is straightforward to check that the norm $-2\langle LZ | c_0 | LZ\rangle_{+2}$ is non-zero. This is a simple check to show that the states are in the cohomology.
5. Correlation Functions

The isomorphism of LZ states and DK states can now be used in interpreting at least some of the correlation functions calculated by Dotsenko and Kitazawa. Let us begin with a simple example of the three-point function of operators with ghost number +1 and −1 and a ghost number zero operator. We show that this can be converted into a three-point function with all the operators in the DK form.

Denote by $\Phi^{(n)}_{m',m}$ the operator of ghost number $n$ whose matter part is a state with charge $\alpha_{m',m}$ that creates an LZ state from the vacuum. We write the three-point function

$$X \equiv \langle \Phi^{(-1)}_{m',m} \Phi^{(0)}_{n',n} \Phi^{(+1)}_{k',k} \rangle$$

(5.1)

We can now write the operators explicitly in the Coulomb gas language. We may write $\Phi^{(0)}_{n',n}$ such that in the matter part we have a suitably screened vertex operator ala Felder defined as

$$V^{r',r}_{n',n}(z) = \int V_{n',n}(z)V_{\alpha_-(u_1)}\ldots V_{\alpha_-(u_{r'})}V_{\alpha_+(v_1)}\ldots V_{\alpha_+(v_r)} \prod_{i=1}^{r'} du_i \prod_{j=1}^{r} dv_j ,$$

(5.2)

where $V_{n',n} = e^{i\alpha m',m} \phi^M$, $V_{\alpha_{\pm}} = e^{i\alpha \pm}$, $2r' = n' + m' - k' - 1$ and $2r = n + m - k - 1$. Writing $Q_B = \oint J_B(z)$ and using the construction of the ghost number +1 state given earlier we obtain

$$X = \langle \Phi^{(-1)}_{m',m} \Phi^{(0)}_{n',n} \oint J_B(z)W_{k',k}(0) \rangle ,$$

(5.3)

where $W_{k',k}$ is the appropriate state associated with the Fock secondary. We can now deform the contour so that it picks up the contribution from the other vertex operators. There is no contribution from $\Phi^{(0)}_{n',n}$. The only non-zero contribution arises from the contour enclosing $\Phi^{(-1)}_{m',m}$. Thus, we obtain

$$X = \langle \oint J_B(z)\Phi^{(-1)}_{m',m} \Phi^{(0)}_{n',n} W_{k',k} \rangle .$$

(5.4)

On using the descent equation (3.7), we get

$$X = \langle Q_{p-m} \Phi^{(D)}_{m',-m+2p} \Phi^{(0)}_{n',n} W_{k',k} \rangle ,$$

(5.5)

where $\Phi^{(D)}_{m',-m+2p}$ is the corresponding DK operator. We can now deform the contour of $Q_{p-m}$ through to the right (picking up phase factors) to give

$$X = \langle \Phi^{(D)}_{m',-m+2p} \Phi^{(0)}_{n',n} Q_{p-k}W_{k',k} \rangle = \langle \Phi^{(D)}_{m',-m+2p} \Phi^{(0)}_{n',n} \Phi^{(D)}_{k',-k} \rangle .$$

(5.6)
The screened vertex operator associated with the matter part of $\Phi_{n',n}^{(0)}$ is actually modified to $V_{n',n}$ where now $2r = n - m + k - 1$ and $r'$ remains unchanged. Thus we have converted a three point function with LZ operators to a computation involving DK operators. This has been computed in [8][9]. Note that we have assumed that negative number of screenings operators in the Liouville sector do not affect any of the contour deformation arguments. This can be justified by first doing these operations and then analytically continuing to negative screening. In the Liouville sector, the number of screenings remains the same as we have not moved out of the original Fock tower.

It is clear that the process would work out for the general case also, where we have a set of three operators such that the total ghost number is zero. Note that ghost number conservation for LZ operators translates to tower number conservation for DK type operators.

The case of two-point functions is particularly simple. We can compute the norm as in [6] and as we have seen it is manifestly non-zero. In terms of operators it is particularly easy to compute as the vertex operator charge of the matter and Liouville sectors precisely differ by a $2\alpha_0$ and a $2\beta_0$ when we write them as DK type operators, thus requiring no screening in either sector.

6. Discussion

We have made clear the relation between the LZ states and the DK states and how correlators with one kind of operators is mapped into correlators of the other kind. It is clear that the calculation of 3-point functions in the language of DK operators really implements the non-decoupling of null states as pointed out by Polyakov [4]. This in a calculation with only LZ operators would be difficult to see. Further, the LZ operators involve Virasoro secondaries and are therefore non-covariant in their construction. Here by the explicit mapping from LZ states to DK states, we have provided a covariantisation prescription implemented through the use of $Q_B$. However several questions remain to which we now turn. In a calculation with DK states we can set up a class of 3-point functions with only positive number of screening operators. This happens, for instance, in the $c_M = 0$ case for all operators with $\beta < \beta_0$. This correlation function can be calculated using the Coulomb gas formalism. However this cannot be mapped to a three-point function with only LZ operators. In fact, the corresponding LZ states are all of negative ghost number and the three-point function of such objects would be expected to be zero. It is not clear...
how ghost number conservation could be effectively violated in a theory with LZ operators alone. On the other hand, a correlation function of LZ operators would if transformed to a correlation function of DK operators, necessarily require negative screening in the Liouville sector. Thus ghost number conservation may be intimately related to negative screening in the Liouville sector.

A similar set of descent equations can clearly arise in the case of $W_N$ gravity. It has been pointed out by Kalyana Rama\cite{15} that the Ising fermion appears in $W_3$ gravity as a LZ type state of ghost number $+1$. We expect that the same descent procedure, using the screening operators of $W_3$ and $Q_B$, would relate the LZ state to a DK state at ghost number zero. It would be interesting to apply our descent procedure in the case of $W_N$ gravity coupled to matter.

The other interesting point is the fact that the mapping from LZ to DK states relates massive states to massless states at specified momenta. A similar feature has been noticed in the analysis of states in the black hole string theory by Distler and Nelson \cite{19}. Indeed, one may look for such features in all cases where there is a non-trivial tachyon and dilaton background in critical strings. If the OPE of such massless states generates a non-trivial symmetry algebra, this symmetry algebra may in fact be understood as a symmetry of at least a subset of the tower of massive states of the theory.

There is a curious analogy here with the work of Distler on $c=-2$ coupled to Liouville\cite{20}. There the operators in the $-1$ picture are simple vertex operators of DK type. However, the operators may be picture-changed to the $n-1$ picture for the $O_n$ operator, in Distler’s notation. The correlations can be computed in the $-1$ picture with appropriate screening operators being added. In our setting, the DK type states are all in a ‘fixed picture,’ measured by the difference of the charges of matter and Liouville vertex operators, which is, in fact, zero. The LZ states are in different pictures, the difference in Liouville and matter charge varying with ghost number and the level of the null state in the matter sector that is dressed. However the operators have non-zero ghost number making the analogy somewhat imprecise.

We would like to thank Swapna Mahapatra, Parameswaran Sankaran and Ashoke Sen for useful discussions.
References

[1] J. Polchinski, “Remarks on the Liouville field theory,” Texas preprint UTTG-19-90, in Proceedings of Strings ’90.
[2] N. Seiberg, “Notes on Quantum Liouville Theory and Quantum Gravity,” Prog. of Theo. Phys., 102(1990), 319.
[3] M. Bershadsky and I. Klebanov, Phys. Rev. Lett. 65 (1990), 3088; Nuclear Phys. B360 (1991), 559.
[4] A. M. Polyakov, Mod. Phys. Lett. A6 (1991), 635-644.
[5] B. Lian and G. Zuckerman, Phys. Lett. B254 (1991), 417.
[6] C. Imbimbo, S. Mahapatra and S. Mukhi, “Construction of Physical States of Non-trivial ghost number in c < 1 String Theory,” Tata preprint TIFR/TH/91-41 (1991).
[7] M. Goulian and B. Li, Phys. Rev. Lett. 66 (1991), 2051.
[8] V. Dotsenko, “Three Point Correlation Functions of the Minimal Conformal Theories coupled to 2D gravity,” Paris preprint PAR-LPTHE 91-18 (1991).
[9] Y. Kitazawa, Phys. Lett. B265 (1991), 262.
[10] K. Aoki and E. D’Hoker, “On the Liouville approach to correlation functions for 2-D quantum gravity,” UCLA preprint UCLA/91/TEP/32 (1991).
[11] P. Bouwknegt, J. McCarthy and K. Pilch, “BRST analysis of physical states for 2d gravity coupled to c < 1 matter,” CERN preprint CERN-TH.6162/91 (1991).
[12] G. Felder, Nuclear Phys. B317 (1989), 215.
[13] M. Kato and S. Matsuda in Advanced Studies in Pure Mathematics, Vol. 16, ed. H. Morikawa (1988), 205.
[14] U. H. Danielsson and D. Gross, Nuclear Phys. B366 (1991), 3.
[15] E. S. Gardner, unpublished.
[16] B. Feigin and D. Fuchs, “Representations of the Virasoro algebra,” in Seminar on Supermanifolds No.5, ed. D. Leites (1988), Univ. of Stockholm Report No. 25.
[17] S. Govindarajan, T. Jayaraman, V. John, and P. Majumdar, work in progress.
[18] S. Kalyana Rama, “New special operators in W-gravity theories,” Tata preprint TIFR/TH/91-41 (1991).
[19] J. Distler and P. Nelson, “New discrete states of strings near a black hole,” Penn preprint UPR-0462T (1991).
[20] J. Distler, Nuclear Phys. B342 (1990), 523.