Fluctuations and stability in front propagation

E. Khain\(^1\), Y. T. Lin\(^2\) and L. M. Sande\(^r\)\(^3\)(a)

\(^1\) Physics Department, Oakland University - Rochester, MI 48309, USA
\(^2\) Department of Physics, University of Michigan - Ann Arbor, MI 48109-1040, USA
\(^3\) Center for the Study of Complex Systems, University of Michigan - Ann Arbor, MI 48109-1040, USA

received 27 September 2010; accepted in final form 3 January 2011
published online 3 February 2011

PACS 82.40.Ck - Pattern formation in reaction with diffusion, flow and heat transfer
PACS 87.23.Cc - Population dynamics and ecological pattern formation
PACS 02.50.Ey - Stochastic processes

Abstract – Propagating fronts arising from bistable reaction-diffusion equations are a purely deterministic effect. Stochastic reaction-diffusion processes also show front propagation which coincides with the deterministic effect in the limit of small fluctuations (usually, large populations). However, for larger fluctuations propagation can be affected. We give an example, based on the classic spruce budworm model, where the direction of wave propagation, i.e., the relative stability of two phases, can be reversed by fluctuations.

Copyright © EPLA, 2011

Introduction. – The study of front propagation arising from reaction-diffusion equations is a fundamental problem in nonequilibrium physics. In this work we consider cases with two equilibria, a bistable system. If such a deterministic system forms a propagating front, we can say that the “stable” state invades the “metastable” state, and the theory [1] gives a simple criterion for which is which.

However, we can also consider a stochastic system where fluctuations play a role, such that in the limit of small noise, we approach the deterministic system. Then the situation is not so simple, and fluctuations can play a role in determining the velocity and even the direction of motion of the front, because fluctuations can give rise to spontaneous transitions between equilibria. That is, fluctuations can reverse the stability of the two equilibria. In this paper we will discuss the mechanism for such reversals, and give an explicit example based on the well-known spruce budworm problem [1-3].

Noise-driven transition between equilibria can be treated using the formalism of rare events. In systems with stochastic birth-death processes this approach has recently attracted considerable attention. In these systems, fluctuations may lead to extinction, the phenomenon that is not described by the continuum rate equations [4]. The interplay between stochastic and deterministic effects becomes much more intriguing when spatial degrees of freedom are introduced and agents can diffuse on a lattice [5]. Here we focus on the role of fluctuations in the phenomenon of front propagation.

Deterministic definition of stability. – The standard deterministic treatment of moving fronts in reaction-diffusion systems [1] begins with an equation of the form

\[ u_t = D u_{xx} + f(u). \]  

(1)

Bistability means that

\[ V(u) = - \int_0^u dw f(w) \]  

(2)

has two minima, \( u_1, u_3 \) separated by a maximum, \( u_2 \). We will refer to \( V \) as a potential.

A moving front occurs when the system is divided into regions where \( u \) is at different equilibria. For example, \( u = u_3 \) for \( x < 0 \), \( u = u_1 \) for \( x > 0 \). To analyze the dynamics, we seek a traveling front solution of eq. (1): \( u(x,t) = u(\xi) = x - vt \), where \( v \) is the front velocity. Substituting this into eq. (1) we get

\[ Du'' + v u' + f(u) = 0. \]  

(3)

The prime indicates the derivative with respect to \( \xi \). Multiplying eq. (3) by \( u' \) and integrating from \( \xi = -\infty \) to \( \infty \), we have:

\[ v = -\int_{u_3}^{u_1} du f(u) \left[ \int_{-\infty}^{\infty} d\xi \frac{D[u'']}{2} \right] \times V(u_1) - V(u_3). \]  

(4)

The velocity will be positive, i.e. the state at \( u_3 \) will invade \( u_1 \) if \( V_1 - V_3 \) is positive, so that \( u_3 \) corresponds to a lower potential; thus \( u_3 \) is stable, and \( u_1 \) metastable if \( V_3 < V_1 \). If \( V_1 - V_2 = 0 = \int_{u_3}^{u_1} du f(u) \) the front will not move (area rule). We refer to this as the “stall point”;
the stall point does not depend on $D$, but only on $V$. If $V_1 - V_3$ is negative, the front will move towards negative $x$, i.e. $u_1$ will invade $u_3$. In some models, e.g., domain wall movement in the Ising model, the stall point is determined by symmetry – for Ising it occurs at zero magnetic field. This is not the case in the models we treat here.

We will need to actually compute $v$ below, i.e. we need the denominator in the eq. (4). Note that eq. (3) can be thought of as a dynamical system; there are two stable fixed points: $u = u_1, u' = 0$ and $u = u_3, u' = 0$. To calculate $v$ we need to find the heteroclinic orbit connecting the two fixed points. We do this by using shooting [6].

**Stochastic definition of stability.** – Equations like eq. (1) often arise as a mean-field description of a stochastic process such as population dynamics in spatially extended systems. To be specific, consider a birth-death process such as population dynamics in spatially extended systems. The horizontal box size is taken to be the width of the interface, which is proportional to $\sqrt{D}$.

The smaller time will correspond to the smaller barrier. Based on this, we can give a different definition of stability: if $t_{3\to1} > t_{1\to3}$ we say that $u_3$ is stable, and vice versa. For example, $u_3$ will be more stable than $u_1$ if the barrier is larger, i.e., $\Phi_3 < \Phi_1$.

This discussion is for a single site. However, spontaneous transitions can move the front if they occur at the interface between the two states. If there are transitions far from the front, and they are not too frequent, i.e. the jump rate due to diffusion is large compared to the spontaneous transition rate, the “hole” will quickly be filled. We have seen examples of this in the simulations to be described below.

**Competition.** – The two potentials, $V$ and $\Phi$ are similar, but not identical. In particular, there is no reason for them to give the same result for stability. Thus a deterministic wave could move from $u_3$ to $u_1$, but noise-induced barrier climbing could cause it to move the other way. In this regime, the two effects compete. In the next section we give a numerical example where this occurs. Also, fluctuations will shift the stall point.

We note that there is a natural length scale for the system given by the width of the wavefront, $w$. On dimensional grounds, $w \propto \sqrt{D}$. This leads us to imagine dividing the system into boxes of width $w$; see fig. 1. Focus on the central box at the front interface. Suppose the continuum front is stalled. If the probability for spontaneous jumps $u_3 \to u_1$ in the left box is larger than the probability for spontaneous jumps $u_1 \to u_3$ in the right box, the discrete front would move to the left, and the system will not be stalled. Thus the observed stall point will be different from that for a purely deterministic front. By the same token, if the two transition times are equal, the velocity will be exactly deterministic, but for other parameters the velocity will be increased or decreased by spontaneous transitions.

Fig. 1: (Colour on-line) Front profile from simulations of the lattice model. The horizontal box size is taken to be the width of the interface, which is proportional to $\sqrt{D}$.

\[ \Phi(u) = -\int u^w \ln \left( \frac{\lambda(w)}{\mu(w)} \right) \approx -2\int u^w \left( f(w) \lambda(w) + \mu(w) \right). \]

The last line is the limiting form when $f/(\lambda + \mu)$ is small (the Fokker-Planck limit). The transition time from state 3 to state 1 is of the form $t_{3\to1} = t_0 \exp(A[\Phi(u_2) - \Phi(u_3)])$, where $t_0$ is a slowly varying prefactor. The transition time depends on the barrier height $\Phi_2 - \Phi_3$. A similar expression holds for transitions from $u_1$ to $u_3$, $t_{1\to3}$. (Note that in the computations below we did not use eq. (5) but the exact solution from [8].)
The relative importance of fluctuations usually depends on the size of the population. For large populations on each site, so that $n_1, n_2 \gg 1$, fluctuations are negligible. As populations decrease stochastic effects will start to affect the stall point and $v$. Finally, for finite systems (perhaps of length $< w$) and very large fluctuations the picture of wave motion will break down altogether, and transitions of the system as a whole will be the dominant path for transitions from metastable to unstable.

We will be interested here in the first two regimes, and the development of waves. This corresponds to a range of $D$. We need $D$ to be large enough that there is front propagation at all, as noted above. For very large $D$ we expect stochastic effects to become negligible. We will discuss this limit below. All of our qualitative notions about the effects of fluctuations on waves will be illustrated by an example in the next section.

In this work we use periodic boundary conditions. With absorbing boundary conditions, $u = 0$ at ends of the system, a small system may not be bistable at all [3].

**Waves in the spruce budworm problem.** – There is a classic model in the literature, the spruce budworm model [1–3], which we will use to illustrate the effects we have discussed. The model is based on real experience in forestry where it is found that a pest which damages balsam fir trees can exist in two states, the “refuge” state (small numbers), and the “outbreak” state, larger numbers. Bistability is thought to be due to the non-linear effects of birds on controlling the insect population [2].

The spatially extended model [3] is written as follows:

$$\frac{\partial n}{\partial t} = D \frac{\partial^2 n}{\partial x^2} + r_b n \left( 1 - \frac{n}{K_b} \right) - \frac{B n^2}{A^2 + n^2}. \quad (6)$$

The first term on the right-hand side of eq. (1) schematically describes migration, assuming random motion. The second term describes the usual logistic growth: insects proliferate with rate $r_b$, but there is some carrying capacity $K_b$, related to available food, which restricts exponential proliferation. The third term represents predation by birds, which saturates at large $n$: birds are not able to consume more than some maximal number of insects per unit time. When $n$ is small, the rate of predation is very small, since birds prefer other regions with a larger population of insects. Introducing dimensionless time $t = B t / A$, and distance $\bar{x} = x / a$, we have

$$\frac{\partial n}{\partial t} = \bar{D} \frac{\partial^2 n}{\partial \bar{x}^2} + r_n \left( 1 - \frac{n}{\bar{K}_n} \right) - \frac{\bar{B} n^2}{\bar{A}^2 + n^2}, \quad (7)$$

where $r = r_b A / B$, $\bar{D} = DA / (Ba^2)$, $q = K_B / A$. Further, if we scale the coordinate $y = \bar{x} / \sqrt{\bar{D}}$, and the number of insects $u = n / A$, we arrive at

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial y^2} + r u \left( 1 - \frac{u}{q} \right) - \frac{u^2}{1 + u^2}. \quad (8)$$

In case there is no spatial dependence (a single-site problem), there exists a region in the parameter space $(r, q)$, such that there are two stable states: $u_1$ describes the normal population size and $u_3 \gg u_1$ corresponds to the outbreak [2,3]. We consider now this region in parameter space and apply the formalism outlined above to eq. (8). Figure 2 shows the front velocity as a function of the birth parameter $r$ (solid line). For a specific birth parameter, $r^*$, the two states coexist and the front stalls, $v = 0$.

Next we analyze the same phenomena, based on eq. (7), as a continuous time Markov process for agents (insects) on a lattice with lattice constant $a$. Every site of the one-dimensional lattice can be occupied by any number of insects. At each time step, an insect on some site, $j$, is picked at random. It can either jump to a neighboring site (to the right or to the left), proliferate, or die with probabilities related to the diffusion, birth and death rates on the site:

$$p_{\text{birth}} = \frac{r}{(r + \mu + 2D)},$$
$$p_{\text{death}} = \frac{\mu}{(r + \mu + 2D)},$$
$$p_{\text{right}} = p_{\text{left}} = \frac{D}{(r + \mu + 2D)},$$
$$\mu = \frac{rn(j)}{qA} + \frac{An(j)}{\bar{A}^2 + n(j)^2}. \quad (9)$$

The birth parameter, $r$, is what we called $\lambda$ above. The death rate per insect, $\mu$, represents the negative terms in eq. (7). After every event, the time is advanced by $1/|n_{\text{total}}(r + \mu + 2D)|$, where $n_{\text{total}}$ is the total number of insects. Clearly, eq. (7) is the continuum analog of this discrete lattice model.

We performed many simulations of the discrete model. To compute the front velocity, we did a time-average for a single run by plotting the space-time diagram, see fig. 3, and computing the slope. Then we averaged the results over many (50–100) simulations. Figure 2 presents
the front velocity as a function of the birth parameter $r$ (dotted line). For the parameters that we consider the front width is many lattice spacings even for small $D$. For example, for $D = 2.5$ we find front widths of order 20–30 sites. Thus it is reasonable to compare our results to eq. (6). We do not expect, and do not find, that discrete effects such as lattice trapping play a role in this regime.

Strikingly, in fig. 2, the discrete stall point $r^*_d$ is not equal to the continuum stall point: $r^*_d > r^*$. This is precisely the result presented above: since the jump probability from $n_1$ to $n_3$ is not equal to the jump probability from $n_3$ to $n_1$, spontaneous jumps contribute to the velocity. Since the probability decreases exponentially with the population scale $A$, the stall point shift $r^*_d - r^*$ should tend to zero when the number of particles increases. Simulations of the discrete model show that this is indeed the case: $r^*_d$ tends to its continuum value $r^*$ as $n$ increases; see below and fig. 6.

We suggested above that the “continuum” and “discrete” velocities in fig. 2 should be equal when the mean transition time from $n_1$ to $n_3$, $t_{1 \rightarrow 3}$, equals the time for the reverse transition, $t_{3 \rightarrow 1}$. Figure 4 shows $k = (t_{1 \rightarrow 3}/t_{3 \rightarrow 1} - 1)$ vs. the birth parameter $r$. Note that the times are equal at approximately $\bar{r} = 0.5075$. This suggests that at this value of birth parameter $r$, there is no stochastic correction to the continuum velocity. Our simulations support this prediction: fig. 2 shows that the continuum and the discrete velocities are equal at $r = \bar{r}$.

As was mentioned above, the continuum stall point, $r^*$, does not depend on $D$. This is not the case in discrete lattice system: fig. 5 shows the $v(r)$ dependence for various $D$; $r^*_d$ depends on $D$. We might expect that the shift $\Delta r = r^*_d - r^*$ would tend to zero as the diffusion coefficient increases, and effectively coarse-grains the system. We might argue as follows: stochastic transitions contribute to the front velocity when they occur in all the sites inside the left or the right box, see fig. 1. The typical front width scales as $\sqrt{D}$. The larger the box, the smaller the probability for such a collective jump. Therefore, as $D$ increases, the stall point shift should tend to zero. However, fig. 5 shows a small discrepancy for large $D$ and no sign of convergence to $r^* = 0.4605$. In fact, our simulations (not shown) reveal that the equilibria, $n_1, n_3$ for the discrete model do not approach those for the continuum model for large $D$, but are uniformly shifted.
by a small amount. We do not understand the large $D$ limit for this system.

However, the large-$A$ limit works exactly as we expect, fig. 6: $\Delta r \to 0$. The approach is consistent with $\Delta r \propto \exp(-CA)$ with $C$ a constant.

There is another effect that we observed in our simulations. We are dealing with quite small numbers, $n$, so that spontaneous transitions are reasonably common. In this case “islands” of the stable state can appear ahead of the front and either disappear, presumably because they are smaller than some critical nucleation size, or be enveloped by the advancing front. This has been observed previously in other stochastic wavefront problems [10].

Discussion. – It is well known that fluctuations can strongly affect front propagation for situations when a stable state invades an unstable state [11–13], the “pushed case”. Here we are dealing with “pushed” fronts, and the qualitative effects are more subtle, and only occur in a limited region of parameter space. Nevertheless, we have shown that they can occur. They should be considered whenever small populations of discrete agents are involved in a spatially spreading process.

***

EK thanks B. MEERSON and Y. LOUZOUN and LMS thanks C. DOERING and D. LUBENSKY for useful discussions.

REFERENCES

[1] Murray J. D., Mathematical Biology I: An Introduction, 3rd edition (Springer, New York, Berlin, Heidelberg) 2003.
[2] Ludwig D., Jones D. and Holling C., J. Anim. Ecol., 47 (1978) 315.
[3] Ludwig D., Aronson D. and Weinberger H., J. Math. Biol., 8 (1979) 217.
[4] Assaf M. and Meerson B., Phys. Rev. Lett., 97 (2006) 200602; Kessler D. A. and Shnerb N. M., J. Stat. Phys., 127 (2007) 861; Kamenev A., Meerson B. and Shkolovskii B., Phys. Rev. Lett., 101 (2008) 268103; Assaf M., Kamenev A. and Meerson B., Phys. Rev. E, 78 (2008) 041123; Dykman M. I., Schwartz I. B. and Landsman A. S., Phys. Rev. Lett., 101 (2008) 078101; Kamenev A. and Meerson B., Phys. Rev. E, 77 (2008) 061107; Khasin M. and Dykman M. I., Phys. Rev. Lett., 103 (2009) 068101; Khasin M., Meerson B. and Sasorov P. V., Phys. Rev. E, 81 (2010) 031126.
[5] Castelvecchi F. and Wio H. S., Europhys. Lett., 38 (1997) 91; Cardy J. L. and Tauber U. C., J. Stat. Phys., 90 (1998) 1; Shnerb N. M., Louzoun Y., Bettelheim E. and Solomon S., Proc. Natl. Acad. Sci. U.S.A., 97 (2000) 10322; Shnerb N. M., Bettelheim E., Louzoun Y., Agam O. and Solomon S., Phys. Rev. E, 63 (2001) 021103; Escudero C., Buceta J., de la Rubia F. J. and Lindenberg K., Phys. Rev. E, 69 (2004) 021908; Elgart V. and Kamenev A., Phys. Rev. E, 74 (2006) 041101; Callaghan T., Khrain E., Sander L. M. and Zipp R. M., J. Stat. Phys., 122 (2006) 909; Mobilia M., Georgiev I. T. and Tauber U. C., J. Stat. Phys., 128 (2007) 447; Kessler D. A. and Sander L. M., Phys. Rev. E, 80 (2009) 041907; Mendez V., Llopis I., Campos D. and Horsthemke W., Theor. Popul. Biol., 77 (2010) 250; Waddell J. N., Sander L. M. and Doering C. R., Theor. Popul. Biol., 77 (2010) 279; Evron G., Kessler D. A. and Shnerb N. M., Physica A, 389 (2010) 428.
[6] Sauer T., Numerical Analysis (Addison-Wesley, Boston) 2005.
[7] Karzazi M. A., Lemarchand A. and Mareschal M., Phys. Rev. E, 54 (1996) 4888.
[8] Doering C. R., Sargsyan K. V. and Sander L. M., Multiscale Model. Simul., 3 (2005) 283.
[9] Doering C. R., Sargsyan K. V., Sander L. M. and Vanden-Eijnden E., J. Phys.: Condens. Matter, 19 (2007) 065145.
[10] Khain E., Sander L. M. and Schneider-Mizell C. M., J. Stat. Phys., 128 (2007) 299; Khain E. and Sander L. M., Phys. Rev. E, 77 (2008) 051129.
[11] Brunet E. and Derrida B., Phys. Rev. E, 56 (1997) 2597.
[12] Kessler D. A., Ner Z. and Sander L. M., Phys. Rev. E, 58 (1998) 167.
[13] Panja D., Phys. Rep., 393 (2004) 87.