Abstract

Independent Component Analysis (ICA), a well-known approach in statistics, assumes that data is generated by applying an affine transformation of a fully independent set of random variables, and aims to recover the orthogonal basis corresponding to the independent random variables. We consider a generalization of ICA, wherein the data is generated as an affine transformation applied to a product of distributions on two orthogonal subspaces, and the goal is to recover the two component subspaces. Our main result, extending the work of Frieze, Jerrum and Kannan, is an algorithm for generalized ICA that uses local optima of high moments and recovers the component subspaces. When one component is on a $k$-dimensional “relevant” subspace and satisfies some mild assumptions while the other is “noise” modeled as an $(n-k)$-dimensional Gaussian, the complexity of the algorithm is $T(k, \epsilon) + \text{poly}(n)$ where $T$ depends only on the $k$-dimensional distribution. We apply this result to learning a $k$-subspace junta, i.e., an unknown 0-1 function in $\mathbb{R}^n$ determined by an unknown $k$-dimensional subspace. This is a common generalization of learning a $k$-junta in $\mathbb{R}^n$ and of learning an intersection of $k$ halfspaces in $\mathbb{R}^n$, two important problems in learning theory.

Our main tools are the use of local optima to recover global structure, a gradient-based algorithm for optimization over tensors, and an approximate polynomial identity test. Together, they significantly extend ICA and the class of $k$-dimensional labeling functions that can be learned efficiently.
1 Introduction

Independent Component Analysis (ICA) [25] is a statistical approach that models data in $\mathbb{R}^n$ as generated by a distribution consisting of $n$ linear combinations of $n$ independent univariate component distributions, $y = Ax$ with $x, y \in \mathbb{R}^n$, $x_i$ are independent random variables and $A$ is an invertible $n \times n$ matrix; in other words, an affine transformation of a product distribution. The goal is to recover the underlying component distributions of the $x_i$ given only a set of observations $y$. Special cases of ICA are of interest in many application areas with large or high-dimensional data sets [24]. An important feature of ICA, as we will presently see, is that it can provide an insightful representation even when Principal Component Analysis (PCA) does not.

In this paper, we consider generalized ICA, where instead of $n$ independent one-dimensional distributions, we only assume two independent distributions on complementary subspaces. This natural extension of ICA provides a common generalization of two fundamental problems in high-dimensional learning, where one sees labeled points (examples) from an unknown distribution labeled by an unknown 0-1 function and the goal is to find a labeling function that agrees on most of the distribution [40]. The first, introduced by A. Blum [8], is learning a function of $k$ coordinates in $\mathbb{R}^n$, known as a $k$-junta. The second is the problem of learning an intersection of $k$ halfspaces in $\mathbb{R}^n$ [6, 7] ($k = 1$ is the classic problem of learning a halfspace). Although the complexity of both problems is far from settled, there has been much progress in recent years for special cases, as we discuss in Section 1.1. Indeed, generalized ICA can be applied to the problem of learning an unknown function of an unknown $k$-dimensional subspace of $\mathbb{R}^n$, provided the distribution on points can be factored into independent distributions on the $k$-dimensional “relevant” subspace and the $(n-k)$-dimensional “noise” subspace.

We give an algorithm for generalized ICA that can be viewed as a tensor version of PCA applied to higher moments, specifically local optima of moment functions to infer the component distributions. The algorithm uses a second-order gradient descent method and an approximate version of the Schwartz-Zippel polynomial identity test, while its analysis needs tools from convex geometry and probability. Before we describe our results and techniques in detail, we summarize the known algorithmic approaches to ICA.

For the problem of identifying the source components given only their linear combinations as data, PCA suggests the approach of using principal components of the data as candidates for the component directions. This would indeed recover the components if the covariance matrix of the data has distinct nonzero eigenvalues. However, if variances along two or more directions are equal, then the principal components are not uniquely defined and PCA does not work. In more detail, assume that the data is centered, i.e., its mean is zero. Then PCA can be viewed as finding vectors on the unit sphere that are local optima of the second moment of the projection, $\max_{x \in S^{n-1}} \|Ax\|^2$ where $A$ is $m \times n$ with each row being a data point. These maxima are eigenvalues of $A^TA$, the covariance matrix of $A$ and hence attain at most $n$ distinct values. The values and the corresponding vectors can be approximated to arbitrary accuracy efficiently.

What to do when eigenvalues are repeated? To address this, the idea in ICA is to consider a broader class of functions to optimize. A natural choice is higher moments. The use of local optima of fourth moments was suggested as early as 1991 [30, 15]. When the component distributions are sufficiently far from being Gaussian, the local optima of a family of functions on the unit sphere are the component directions [37, 17] (if component distributions are Gaussians, then their linear combinations are also Gaussian and the linear transformation $A$ might not uniquely defined). This approach can be turned into an a polynomial-time algorithm for unraveling a product distribution of a wide class of one-dimensional distributions.
We now describe generalized ICA, which significantly weakens the ICA assumption of a full product distribution. Namely, we assume that the distribution $F$ in $\mathbb{R}^n$ can be factored into a product of two independent marginal distributions $F_V$ and $F_W$ on unknown orthogonal subspaces $V$ and $W = V^\perp$, i.e., $F = F_V F_W$. We call such an $F$ factorizable. Thus, a random point in $F$ is generated by first picking its coordinates in $V$ according to $F_V$ and then independently picking coordinates in $W$ according to $F_W$. The corresponding problem is the following.

**Problem 1** (Factoring distributions). Given (unlabeled) samples from a factorizable distribution $F = F_V F_W$ over $\mathbb{R}^n$ (with $V$ and $W$ unknown), recover a factorization of $F$.

If $F$ in fact factorizes further into product of more distributions, or even a full product distribution of one-dimensional component distributions as in ICA, an algorithm for the above problem can be applied recursively to find the full factorization. We will give an algorithm for this problem under further mild assumptions (roughly speaking, at least one of $F_V, F_W$ is sufficiently different from being a Gaussian). Our approach is based on viewing PCA as a second moment optimization problem, then extending this to higher moments (alternatively, optimization over tensors). Although such tensor optimization is intractable in general, for our setting, it will turn out that local optima provide valuable information, and can be approximated efficiently.

The factoring problem above has direct applications to learning in high dimension. Let $\pi_V$ denote projection to a subspace $V$. We consider labeling functions $\ell : \mathbb{R}^n \rightarrow \{0, 1\}$ of the form $\ell(x) = \ell(\pi_V(x))$. We are given points according to some distribution $F$ over $\mathbb{R}^n$ along with their labels $\ell(x) = \ell(\pi_V(x))$ for some unknown subspace $V$ of dimension $k$ (the ‘relevant’ subspace), and wish to learn the unknown concept $\ell$, i.e., find a function that agrees with $\ell$ on most of $F$. We call this the problem of learning a $k$-subspace junta. We further assume that $F$ is factorizable as $F = F_V F_W$, with $W = V^\perp$ (the ‘irrelevant’ subspace). The justification for this factorizability assumption is that coordinates in the $W$ subspace are not relevant to the labeling function and can be considered to be noisy attributes. The full statement of our learning problem is as follows:

**Problem 2** (Learning a $k$-subspace junta). For $\epsilon, \delta > 0$, given samples drawn from a factorizable distribution $F = F_V F_W$, and labeled by $\ell = \ell \circ \pi_V$, find a 0-1 function $f$ such that with probability at least $1 - \delta$,

$$\Pr_{x \sim F}(\ell(x) \neq f(x)) \leq \epsilon.$$ 

Our algorithm for generalized ICA leads to an efficient algorithm for learning $k$-subspace juntas for a large class of ambient distributions $F$.

1.1 Related work

Jutten and Herault formalized the ICA problem [25] and mention in their paper that variants of this problem had appeared in a variety of different fields prior to this (the earliest such mention is in [3]). The notion that random variables should be far from being Gaussian pervades ICA research. By the central limit theorem, sums of independent random variables converge to a Gaussian, whereas individually the latent random variables are not Gaussian. Thus finding directions that maximize some notion of non-Gaussianity might reveal the latent variables. This intuition is formalized by introducing functions which serve as a proxy for non-Gaussianity, called “contrast functions” in the ICA literature. The definition of a contrast function is that maximizing a contrast function will give an independent component. Some examples of contrast functions include the kurtosis (4th order analogue of variance)[30, 15], various cumulants, and functions based on the so-called negentropy.
Additionally, there are a variety of tensor methods and maximum likelihood methods used for ICA, some of which appear to perform well in practice (e.g., FastICA [23]). While there are many algorithms proposed for ICA, some of which appear to perform well in practice (e.g., FastICA [23]), there are almost no explicit time complexity bounds. Frieze, Jerrum and Kannan [19] were the first to give a polynomial complexity bound for this special case of ICA, namely a product of uniform distributions on intervals, which can also be viewed as the problem of learning an unknown paralleloped from samples. They used fourth moments, an idea presented earlier in several papers in the ICA literature; the key structural lemma is already present in [17], which was inspired by [37] (Lemma of our paper is a generalization). Subsequently, Nguyen and Regev [33] simplified Frieze et al’s gradient descent algorithm and provided some cryptographic applications.

A different motivation for our work comes from computational learning theory, where learning a $k$-junta is a fundamental problem [8]. In this problem, one is given points from some distribution over $\{0, 1\}^n$, labeled by a Boolean function that depends only on $k$ of the $n$ coordinates. The goal is to learn the relevant $k$ coordinates and the labeling function. Naive enumeration of $k$ subsets of the coordinates leads to an algorithm of complexity roughly $n^k$. Mossel et al [32] gave an algorithm of complexity roughly $O(n^{0.7k})$ assuming the uniform distribution over $\{0, 1\}^n$. For other special cases of Problem 2, previous authors have applied standard low-dimensional representation techniques, low-degree polynomials, random projection and Principal Component Analysis (PCA) to identify $V$ under strong distributional assumptions [1, 27, 6, 11, 43]. The strongest result in this line achieves a fixed polynomial dependence on $n$ by applying PCA to learn convex concepts over Gaussian input distributions [42]. Unfortunately, standard PCA does not work for other distributions or more general concept classes, in part because PCA does not provide useful information when the covariance matrices of the positive and negative samples are equal. In fact, the problem appears to be quite hard with no assumptions on the input distribution, even for small values of $k$, e.g., a single halfspace can be PAC-learned via linear programming, but learning an intersection of two halfspaces (a 2-subspace junta) in polynomial time is an open problem.

There have been a number of extensions of PCA to tensors [29] analogous to SVD, although no method is known to have polynomial complexity. One approach is to view PCA as an optimization problem. The top eigenvector is the solution to a matrix optimization problem:

$$\max_{\|v\|=1} v^T Av = \sum_{i_1, i_2} (A)_{i_1, i_2} v_{i_1} v_{i_2}$$

where $A$ is the covariance matrix. A higher moment method optimizes the multilinear form defined by the tensors of higher moments:

$$\max_{\|v\|=1} A(v, \ldots, v) = \sum_{i_1, \ldots, i_r} A_{i_1, \ldots, i_r} v_{i_1} \ldots v_{i_r}.$$ 

Unlike the bilinear case, finding the global maximum of a multilinear form is hard. For $\alpha > 16/17$, it is NP-hard to approximate the optimum to better than factor $\alpha^{\lfloor r/4 \rfloor}$ [10], and the best known approximation factor is roughly $n^{r/2}$. Several local search methods have been proposed for this problem as well [28].

## 1.2 Results

To state our results formally, we need to define the distance of a distribution from a Gaussian via moments. For a random vector $x \in \mathbb{R}^n$ with distribution $F$, the $m^{th}$ moment tensor $M^m$ is a tensor
of order \( m \) with \( n^m \) entries given by:

\[
M_{i_1,\ldots,i_m}^m = \mathbb{E}(x_{i_1} \ldots x_{i_m}).
\]

Let \( \Gamma^n \) be the standard Gaussian distribution over \( \mathbb{R}^n \) and \( \gamma_m \) denote the \( m \)th moment of a standard Gaussian random variable: \( \gamma_m = (m-1)!! \) when \( m \) is even and 0 when \( m \) is odd.

The \( m \)th-moment distance of two distributions \( F, G \) over \( \mathbb{R}^n \) is defined as

\[
d_m(F, G) = \max_{||u||=1} |\mathbb{E}_F((x^T u)^m) - \mathbb{E}_G((x^T u)^m)| = \|M_F^m - M_G^m\|_2.
\]

We say that a distribution \( F \) over \( \mathbb{R}^k \) is \((m, \eta)\)-moment-distinguishable along unit vector \( u \in \mathbb{R}^k \), if either there exists \( j \leq m \):

\[
|\mathbb{E}_F((x^T u)^j) - \gamma_j| \geq \eta
\]

or there exist unit vectors \( \{v_1, \ldots, v_t\} \subset u^\perp \) where \( t \leq m \) such that

\[
|\mathbb{E}_F((x^T u)^m - t \Pi_{i=1}^t (x^T v_i)) - \mathbb{E}_F((x^T u)^m - t) \mathbb{E}(\Pi_{i=1}^t (x^T v_i))| \geq \eta.
\]

In words, \( F \) differs from a Gaussian either along some direction \( u \), or by exhibiting a correlation between its marginal along \( u \) and vectors orthogonal to \( u \) (for a Gaussian such subsets have zero correlation). The rationale for this definition is that if two continuous distributions are identical (or close) in many moments, then one would expect them to be close in \( L^1 \) distance. For example, the following holds for one-dimensional logconcave distributions via an explicit bound on the number of moments required.

**Lemma 1** (\( L^1 \) distance from Gaussian). Fix \( m \) and \( \epsilon > 0 \). Let \( f : \mathbb{R} \to \mathbb{R} \) be an isotropic logconcave density, whose first \( m \) moments satisfy \( |\mathbb{E}_f(x^m) - \gamma^m| < \epsilon \), then:

\[
\|f - g\|_1 \leq \left( \frac{c}{m^{1/8}} + c'm^m \epsilon^2 \right)^{1/2} \log m \leq c \left( \frac{\log m}{m^{1/16}} + c'm^m \right)
\]

We are now ready to state our first main result: we can efficiently factorize distributions assuming the distribution on the relevant subspace is moment-distinguishable and the distribution on the irrelevant noisy attributes is some Gaussian. In what follows, it might be illustrative to regard \( k \) as a constant independent of \( n \). Let \( C_F(n, m, \epsilon) \) be the number of samples needed to estimate each entry of the \( m \)th moment tensor of \( F \) to within additive error \( \epsilon \) and \( M \) be an upper bound on the \( m \)th moment along any direction.

**Theorem 1** (Factoring, Gaussian noise). Let \( F = F_VF_W \) be a distribution over \( \mathbb{R}^n \) where \( V \) is a subspace of dimension \( k \), and \( F_W = \Gamma^{n-k} \). Suppose that \( F_V \) is \((m, \eta)\)-moment-distinguishable for each unit vector \( u \in V \). Then for any \( \epsilon, \delta \geq 0 \), in time \( C_F(n, m, \epsilon) \) poly\((n, \eta, 1/\epsilon, \log(1/\delta), M)\), Algorithm FactorUnderGaussian finds a subspace \( U \) of dimension at most \( k \) such that for \( j \leq m \),

\[
d_j(F, F_U) \leq j(M + \gamma_j)\epsilon \text{ with probability at least } 1 - \delta.
\]

In addition, for any vector in \( u \in U \), \( \|\pi_V(u)\| \geq 1 - \epsilon \).

Next we turn to learning. For a distribution \( F \) and a \( k \)-dimensional concept class \( \mathcal{H} \), we say that the triple \((k, F, \mathcal{H})\) is \((m, \eta)\)-moment-learnable if:

1. \( F = F_VF_W \) is a factorizable distribution with \( \text{dim}(V) = k \).
2. \( \mathcal{H} \) is a set of \( k \)-subspace juntas whose relevant subspaces are contained in \( V \).

3. For \( \ell \in \mathcal{H} \) with minimal (with respect to dimension) relevant subspace \( P \subseteq V \), for each unit vector \( u \in P \) either \( F_V \) or \( F^+_V \) (the distribution over the positive samples) is \((m, \eta)\)-moment distinguishable along \( u \).

In words, the third condition says that if \( F_V \) resembles a Gaussian in its first \( m \) moments along every direction, then \( F^+_V \) does not. We will see examples of concept classes and distributions for which \( m \) is bounded under this definition. Indeed, we conjecture that a concept class \( \mathcal{H} \) with bounded VC dimension \( d \) is \((m, \eta)\) moment-learnable where \( m \) depends only on \( d \) and \( \eta \).

To state our learning guarantee, we need one more definition: A triple \((k, F, \mathcal{H})\) is called robust if for any subspace \( U \) of dimension at most \( k \) with orthonormal basis \( \{u_i\} \) where \( |u_i^T \pi_V(u_i)| \geq 1 - \epsilon \), then \( \ell(\pi_U(x)) \) labels correctly \( 1 - g(\epsilon) \) fraction of \( \mathbb{R}^n \) under \( F \) where \( g(\epsilon) < \epsilon^c \) for constant \( c > 0 \) and sufficiently small \( \epsilon \). The definition requires the distribution \( F \) and labeling function \( \ell \) to be robust under small perturbations of the relevant subspace. Once we identify the relevant subspace approximately, we can project samples to it and use an algorithm that can learn \( \ell \) in spite of a \( g(\epsilon) \) fraction of noisy labels.

**Theorem 2** (Learning, Gaussian noise). Let \( \epsilon, \delta > 0 \), let \( \ell \in \mathcal{H} \) where \((k, F, \mathcal{H})\) is \((m, \eta)\)-moment-learnable and robust, and let \( F_W = \Gamma^{n-k} \) be Gaussian. Suppose that we are given labeled examples from \( F \), then Algorithm **LearnUnderGaussian** identifies a subspace \( U \) and a hypothesis \( h \) such that \( h \) correctly classifies \( 1 - \epsilon \) of \( F \) according to \( \ell \) with probability at least \( 1 - \delta \). The time and sample complexity of the algorithm are bounded by \( T(k, \epsilon) + C_F(n, m, \epsilon)\text{poly}(n, \eta, k, 1/\epsilon, \log(1/\delta), M) \) where \( T \) is the complexity of learning the \( k \)-dimensional concept class \( \mathcal{H} \).

We note here that for a concept class of VC-dimension \( d \), a standard reduction implies that the complexity of learning with \( \epsilon \) arbitrary noise is at most \((2/\epsilon)^{O(d \log(1/\epsilon))} \) times the complexity of learning with no noise (Proposition 3). Our algorithms run in polynomial-time in \( n \) provided \((k, F, \mathcal{H})\) satisfy the moment-learnable condition. Some special cases of this result were previously known, e.g., when \( F \) is a Gaussian and \( \mathcal{H} \) is a convex concept class [27, 42]. The application of PCA to learning convex bodies in [12] can be viewed as the assertion that convex concepts in \( \mathbb{R}^k \) are moment-learnable: under a Gaussian distribution, the positive distribution \( F^+ \) has variance less than 1 along any direction. The following two examples further illustrate Theorem 2.

- When the full distribution in the relevant subspace is uniform in an ellipsoid, then robust concept classes can be learned in time \( T(k, \epsilon) + C_{k, \epsilon} \cdot n^2 \). Here \( T \) depends on the \( k \) and concept class, and \( C \) is a constant fixed by \( k \) and \( \epsilon \) and independent of the concept class. Thus we can learn general concept classes beyond convex bodies and low-degree polynomials for uniform distributions over a ball in the relevant subspace.

- When the distribution on the positive examples \( F^+ \) has bounded support, i.e., the positive labels lie in a ball of radius \( r(k) \), such robust concepts can be learned in time \( T(k, \epsilon) + C_{k, \epsilon} \cdot n^{O(r(k)^2)} \) for an arbitrary distribution in the relevant subspace. Previously, for logconcave \( F \), learning an intersection of \( k \) half-spaces was known to have complexity growing as \( n^{O(k)} \) [13, 26].

### 1.3 Techniques

Our strategy is to identify the relevant subspace \( V \) to examine higher moments of the distribution.
As mentioned earlier, our approach is inspired by viewing PCA as finding the global maxima and minima of the bilinear form defined by the covariance matrix. Instead of trying to compute \textit{global} optima of the multilinear form, we use \textit{local} optima. These local optima turn out to be highly structured. The use of local optima can be viewed as an effective realization of higher-order PCA that leads to efficient algorithms. Previous algorithmic problems have all required the use of global optima — for example, the planted clique algorithms of \cite{20, 11}. We prove that local optima of the $m^{th}$ moment $f_m(u) = \mathbb{E}((x^T u)^m)$ must lie entirely in $V$ or its complement $W$ (Lemma 4) unless its first $m$ moments are identical to those of a Gaussian.

To make these ideas algorithmic, we use a local search method that increases the function value by performing first-order moves along the gradient and then second-order moves in the direction of the top eigenvector of the Hessian matrix. These second-order moves allow us to avoid saddle points and other critical points which arise in higher dimensions. Saddle points have a gradient of zero and look like maxima in some directions and minima in others. While searching for a local maximum, one could end up in a saddle point. The top eigenvector of the Hessian shows directions of greatest quadratic increase, and hence will move us from the saddle point to a true local maximum.

Another component in our algorithms is an approximate version of the well-known Schwartz-Zippel polynomial identity test. Observing that $f_m(u)$ is a polynomial of degree $m$ in the variables $u_1, \ldots, u_m$, in principle we can test whether $f_m$ is a constant function by evaluating $f_m$ at random points. We use a robust version of this test (Lemma 12) derived via a result of Carbery and Wright \cite{12}.

### 2 Structure of local optima

We derive a representation of $f_m(u) = \mathbb{E}((x^T u)^m)$ in Lemma \ref{lem:structure}. Using this representation, we show in Lemma \ref{lem:local_optima} that each local optimum lies in $V$ or $W$ exclusively. Finding a sequence of orthogonal local optima will give us basis vectors for the relevant subspace.

For convenience we often use $u_V = \pi_V(u)$ for the projection of $u$ onto $V$, $u_W$ for the projection onto the orthogonal subspace $W$, and $u^0$ for the unit vector in the direction of $u$.

We may assume that $\mathbb{E}(x) = 0$: if otherwise, then we can apply a translation $x - \mathbb{E}(x)$.

**Lemma 2** (Translation of product distributions). Let $x \in \mathbb{R}^n$ be a random vector drawn from $F = F_V F_W$, a product distribution. Then $x - \mathbb{E}(x)$ has a product distribution over $V$ and $W$.

**Proof of Lemma 2** Take our translation $y = T_a(x) = x + a$, for Borel sets $B_1$ and $B_2$:

$$
\Pr(y_V \in B_1 \land y_W \in B_2) = \Pr(x_V + a_V \in B_1 \land x_W + a_W \in B_2) \\
= \Pr(x_V \in B_1 - a_V \land x_W \in B_2 - a_W) \\
= \Pr(x_V \in B_1 - a_V) \Pr(x_W \in B_2 - a_W) \\
= \Pr(y_V \in B_1) \Pr(y_W \in B_2)
$$

We can combine this with a linear transformation to obtain an isotropic distribution, given by $y = \Sigma^{-1/2}(x - \mu)$ where $\mu$ is the expectation vector. This simplifies subsequent calculations because the covariance matrix for $y$ is $I_n$. The following lemma, inspired by \cite{19, 17, 37}, provides the main insight for the structural theorem.
Lemma 3 (Representation of $f_m$). Let $F = F_V F_W$. Suppose that $x$ has the same $j$th moments as a Gaussian for all integers $j < m$, then for $u \in S^{n-1}$:

$$f_m(u) = \|u_V\|^m (\mathbb{E} (x_V^T u_V^0)^m) - \gamma_m) + \|u_W\|^m (\mathbb{E} (x_W^T u_W^0)^m) - \gamma_m + \gamma_m \quad (1)$$

Proof of Lemma 3 Consider the case when $m$ is odd:

$$f_m(u) = \|u_V\|^m \mathbb{E} (x_V^T u_V^0)^m + \|u_W\|^m \mathbb{E} (x_W^T u_W^0)^m.$$

When $m$ is even, we need the following formula:

$$\sum_{i=0}^{m} \binom{m}{i} \|u_V\|^i \|u_W\|^{m-i} \gamma_i \gamma_{m-i} = \gamma_m$$

This follows from $\mathbb{E} ((aX + bY)^m) = \gamma_m$ where $a^2 + b^2 = 1$ and $X$ and $Y$ are independent standard normal variables:

$$f_m(u) = \sum_{i=0}^{m} \binom{m}{i} \|u_V\|^i \|u_W\|^{m-i} \mathbb{E} (x_V^T u_V^0)^i \mathbb{E} (x_W^T u_W^0)^{m-i}$$

$$= \|u_V\|^m \mathbb{E} (x_V^T u_V^0)^m + \|u_W\|^m \mathbb{E} (x_W^T u_W^0)^m + \sum_{i=1}^{m-1} \binom{m}{i} \|u_V\|^i \|u_W\|^{m-i} \gamma_i \gamma_{m-i}$$

$$= \|u_V\|^m (\mathbb{E} (x_V^T u_V^0)^m) - \gamma_m) + \|u_W\|^m (\mathbb{E} (x_W^T u_W^0)^m) - \gamma_m) + \sum_{i=0}^{m} \binom{m}{i} \|u_V\|^i \|u_W\|^{m-i} \gamma_i \gamma_{m-i}$$

Using this representation, we can characterize all local optima of $f_m$.

Lemma 4 (Support). Let distribution $F = F_V F_W$ have the same first $m-1$ moments as a Gaussian but a different $m$th moment. Then for a local maximum (local minimum) $u^*$ of $f_m$ restricted to the unit sphere, where $f_m(u^*) > \gamma_m$ ($f_m(u^*) < \gamma_m$), either $\|u_V^*\| = 1$ or $\|u_W^*\| = 1$. 

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Proof of Lemma \[4\] Consider the curve \( C = \{ s(u_V^0)^0 + t(u_W^0)^0 : s^2 + t^2 = 1, s \geq 0, t \geq 0 \} \). The point \( u^* \) lies on \( C \); thus if \( u^* \) is a local maximum in full space, it had better be a local maximum on \( C \). On the other hand, we will show that there are no local maxima interior to \( C \), whence we must have \( \|u_V^*\| = 1 \) or \( \|u_W^*\| = 1 \).

Let us denote \( a_v = \mathbb{E} \left( (x^Tu_V^0)^m \right) - \gamma_m \) and \( a_w = \mathbb{E} \left( (x^Tu_w^0)^m \right) - \gamma_m \). By the assumption that \( f_m(u^*) > \gamma_m \), we know that at least one of \( a_v \) or \( a_w \) is positive. Suppose that \( s \neq 0 \) and \( s \neq 1 \): we form the associated Lagrangian with positive real multiplier \( \lambda \):

\[
\mathcal{L} = a_vs^m + a_wt^m + \gamma_m - \lambda(s^2 + t^2 - 1)
\]

At every critical point in the interior of \( C \), we must have \( D\mathcal{L} = 0 \):

\[
\begin{pmatrix}
ma_vs^{m-1} - 2\lambda s \\
ma_wt^{m-1} - 2\lambda t
\end{pmatrix} = 0
\]

If we consider only the interior critical points where \( s,t > 0 \), then both \( a_v > 0 \) and \( a_w > 0 \) (otherwise we would have \( \lambda > 0 \) and \( \lambda \leq 0 \)). There is only one solution:

\[
s = \frac{a_v^{1/(m-2)}}{\sqrt{\frac{2}{(m-2)}} + a_w^{2/(m-2)}} \\
t = \frac{a_v^{1/(m-2)}}{\sqrt{\frac{2}{(m-2)}} + a_w^{2/(m-2)}} \\
\lambda = (m/2)(a_vs^m + a_wt^m)
\]

If we now consider the Hessian on the tangent plane orthogonal to the gradient of the constraint (equivalent to considering the bordered Hessian), we see that it is positive definite for \( m > 3 \) (when \( m = 3 \), differentiating \( a_v s^3 + a_w (1 - s^2)^{1.5} \) twice at the critical point gives a positive value):

\[
D^2\mathcal{L} = \begin{pmatrix}
m(m-1)a_vs^{m-2} - 2\lambda & 0 \\
0 & m(m-1)a_wt^{m-2} - 2\lambda
\end{pmatrix} = \frac{m(m-3)a_va_w}{\left[\frac{a_v^2}{m-2} + \frac{a_w^2}{m-2}\right]^{(m-2)/2}} I > 0
\]

In particular, there are no local maxima interior to \( C \), that is, \( \|u_V^*\| = 1 \) or \( \|u_W^*\| = 0 \). \( \square \)

3 Finding a basis

Our two basic algorithms exploit the property that local optimum to \( f_m(u) = \mathbb{E} \left( (x^Tu)^m \right) \) on the unit sphere must lie in either \( V \) or \( W \) (Lemma \[3\]). In this section, we assume that the algorithms have access to exact moment tensors and can compute exact local optima. We provide efficient algorithms (with error analysis) in Section \[4\].

The basic idea of the algorithm is to start with a random direction and evaluate the \( j \)'th moment in that direction. If it is different from a Gaussian we go to a local max or local min (whichever keeps it different from a Gaussian) and thus find a vector of interest; if many random unit vectors have Gaussian moments, then all directions have Gaussian moments due to the Schwartz-Zippel Lemma and we go to the next higher moment. At the end of the algorithm we have a subset of an orthogonal basis consistent with \( V \) and \( W \), and the property that all orthogonal directions have Gaussian moments.

**Lemma 5** (Schwartz-Zippel\[36\]). Let \( P \in F[x_1, \ldots, x_n] \) be a nonzero polynomial of degree \( dn \geq 0 \) over field \( F \). Let \( S \) be a finite subset of \( F \) and let \( r_1, \ldots, r_n \) be selected randomly from \( S \). Then:

\[
\Pr(P(r_1, \ldots, r_n) = 0) \leq \frac{d}{|S|}.
\]
Algorithm 1 FindBasis

Input: Moment bound \( m \), Distribution \( F \)
1: Orthonormal vectors \( B \leftarrow \phi \), moment \( j \leftarrow 2 \).
2: while \( |B| < n \) and \( j \leq m \) do
3: Compute the \( j \)th moment tensor \( M^B_j \) orthogonal to \( B \), so that for any \( v \in B^\perp \), \( f_j(v) = \mathbb{E} \left( (x^Tv)^j \right) = M^B_j(v, \ldots, v) \).
4: if \( f_j(v/\|v\|) \leq \gamma_j \) then
5: \( j \leftarrow j + 1 \)
6: else
7: if \( f_j(v) > \gamma_j \) for some \( v \) then
8: Compute a local maximum \( u^* \) to \( f_j \) starting from \( v \).
9: else
10: Compute a local minimum \( u^* \) to \( f_j \) starting from \( v \).
11: \( B \leftarrow B \cup \{u^*\} \).
12: return \( B \)

For Line 3, let \( A : \mathbb{R}^n \to \mathbb{R}^n \) denote the linear map that projects orthogonal to \( B \). Then

\[
M(Au, \ldots, Au) = \sum_{i_1, \ldots, i_m} M_{i_1, \ldots, i_m}(Au)_{i_1} \cdots (Au)_{i_m} = \sum_{j_1, \ldots, j_m} \left( \sum_{i_1, \ldots, i_m} M_{i_1, \ldots, i_m} A_{i_1, j_1} \cdots A_{i_m, j_m} \right) u_{j_1} \cdots u_{j_m}
\]

The identity check in Line 4 is performed by selecting a random vector \( x \) with i.i.d. uniform coordinates from \( \{1, \ldots, 2m\} \) and evaluating the polynomial \( f_j(x/\|x\|) - \gamma_j \). Repeating \( O(\log(n/\delta)) \) times gives a \( 1 - \delta \) probability of success.

Algorithm FindBasis does not suffice on its own. Although every direction orthogonal to \( B \) looks Gaussian up to the \( m \)th moment, it is possible that some directions are correlated with vectors in \( B \). The next procedure identifies such directions.

Theorem 3 (Find Basis). Let \( F = F_VF_W \) be a factorizable distribution over \( \mathbb{R}^n \). Then, with probability at least \( 1 - \delta \), each vector in the output of FindBasis lies in either \( V \) or \( W \).

Proof of Theorem. From the above comment, at each step, with probability at least \( 1 - \delta/n \) (hence total failure probability \( \delta \)), we are able to find a point \( u \) where \( f_r(u) \neq \gamma_m \). In particular, if \( f_r(u) > \gamma_m \), then we find a local maximum \( u^* \). By Lemma \( u^* \) is contained entirely within \( V \) or \( W \). The analysis is identical when our initial point \( u \) satisfies \( f_r(u) \leq \gamma_m \).

Observe that \( F_{V\setminus\text{span}(B)} F_{W\setminus\text{span}(B)} \) is a factorizable distribution over \( \pi_{B^\perp}(x) \), and hence a local optimum in \( B^\perp \) also will lie in either \( V \) or \( W \). \( \square \)

The next theorem states that ExtendBasis finds all vectors which are correlated with \( S \subseteq V \), and that all remaining vectors at the end of the algorithm are uncorrelated up to the \( m \)th moment.

Theorem 4 (Basis Extension). The output \( S' \) of ExtendBasis on input \( S \subseteq V, T \subseteq W \) satisfies:

1. \( S \subseteq S' \subseteq V \).
2. For \( \{v_i\} \subset S' \) and \( \{u_i\} \subset (S')^\perp \):

\[
\mathbb{E} \left( \prod_{i=1}^{j-l} (x^Tu_i) \prod_{t=1}^l (x^Tv_i) \right) = \mathbb{E} \left( \prod_{i=1}^{j-l} (x^Tu_i) \right) \mathbb{E} \left( \prod_{t=1}^l (x^Tv_t) \right)
\]
Algorithm 2 \textbf{ExtendBasis}
\textbf{Input:} Moment bound $m$, distribution $F$, candidate vectors $S$ and non-Gaussian directions $T$. 
\begin{algorithmic}[1]
\State $S' \leftarrow S$, $j \leftarrow 2$. 
\While{$|S'| < k$ and $j \leq m$} 
\For{each choice (with repetition) $\{v_1, \ldots, v_l\} \subseteq S'$ where $1 \leq l < j$.}
\State Compute the $(j-l)$ tensor $M_{ij}^{S',T}$ so that for any $u \in (S' \cap T)^\perp$,
\State $g(u) = \mathbb{E}\left((x^T u)^{j-l} \prod_{t=1}^l (x^T v_t)\right) - \mathbb{E}\left((x^T u)^{j-l}\right) \mathbb{E}\left(\prod_{t=1}^l (x^T v_t)\right) = M_{ij}^{S',T}(u, \ldots, u, v_1, \ldots, v_l)$.
\EndFor
\If{$g(u) \equiv 0$} 
\State Continue with next choice of $\{v_i\}$.
\Else 
\If{$g(u) > 0$ for any $u$} 
\State Compute a local maximum $u^*$ to $g$ starting with $u/\|u\|$.
\Else 
\State Compute a local minimum $u^*$ to $g$ starting with $u/\|u\|$.
\EndIf
\State $S' \leftarrow S' \cup \{u^*\}$ and restart the while loop with $j = 3$.
\EndIf
\EndWhile
\State $j \leftarrow j + 1$.
\State \textbf{return} $S'$.
\end{algorithmic}

\textit{Proof of Theorem 4.} The Schwartz-Zippel lemma returns a correct decision at every iteration (there are at most $n^k$ of these, so if we pick our domain to be of size $2n^k$ and run $O(\log n^k/\delta)$ iterations each time, then we have a correct decision for all iterations with probability at least $1 - \delta$.

Let $u^*$ be a local maximum found by \textbf{ExtendBasis} using the $j^{th}$ moment. Consider the $\{v_1, \ldots, v_l\}$ where $u^*$ was found. 
\[
g(u) = \mathbb{E}\left((x^T u)^{(j-l)} \prod_{t=1}^l (x^T v_t)\right) - \mathbb{E}\left((x^T u)^{(j-l)}\right) \mathbb{E}\left(\prod_{t=1}^l (x^T v_t)\right)
= \sum_{i=0}^{j-l} \binom{j-l}{i} \mathbb{E}\left((x^T u v_i)^{(j-l)}\right) \mathbb{E}\left((x^T u v_i)^{(j-l-i)} \prod_{t=1}^l (x^T v_t)\right) - \mathbb{E}\left(\prod_{t=1}^l (x^T v_t)\right) \mathbb{E}\left((x^T u v_i)^{(j-l-i)}\right)
\]
Since $u^*$ was found at moment $j$, then for all $0 < i < j - l$:
\[
\mathbb{E}\left((x^T u v_i)^{(j-l-i)} \prod_{t=1}^l (x^T v_t)\right) = \mathbb{E}\left(\prod_{t=1}^l (x^T v_t)\right) \mathbb{E}\left((x^T u v_i)^{(j-l-i)}\right)
\]
Only the first and last terms survive:
\[
g(u) = \mathbb{E}\left((x^T u v_0)^{(j-l)} \prod_{t=1}^l (x^T v_t)\right) - \mathbb{E}\left(\prod_{t=1}^l (x^T v_t)\right) \mathbb{E}\left((x^T u v_0)^{(j-l)}\right)
= \|u_v\|^{j-l} \left[ \mathbb{E}\left((x^T u v_0)^{(j-l)} \prod_{t=1}^l (x^T v_t)\right) - \mathbb{E}\left(\prod_{t=1}^l (x^T v_t)\right) \mathbb{E}\left((x^T u v_0)^{(j-l)}\right) \right]
\]
Having a positive local maximum implies that $\|u_v\| = 1$. 

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For the second part of this lemma: we already know that all the remaining vectors must have Gaussian moments. Fix $j \leq m$ and a choice of $\{v_1, \ldots, v_l\}$ from $S'$ and consider the symmetric tensor $\hat{M}$ represented by $f(u) - \mathbb{E}((x^T u)^{j-l}) \mathbb{E}\left(\prod_{t=1}^{l} (x^T v_t)\right)$. We require the following claim for symmetric tensors where for any permutation $\sigma : [m] \to [m]$:

$$\mathbb{E}\left(\prod_{k=1}^{m} x_{i_k}\right) = \mathbb{E}\left(\prod_{k=1}^{m} x_{\sigma(k)}\right).$$

Claim 6. If $A$ is a symmetric order $r$ tensor, then:

$$\max_{\|v\|=1} A(v, \ldots, v) = \max_{\|v_1\|=1, \ldots, \|v_r\|=1} A(v_1, \ldots, v_r)$$

Using Claim 6:

$$\max_{\|u\|=1} \hat{M}(u, \ldots, u) = \max_{\|u_1\|=1, \ldots, \|u_{j-l}\|=1} \hat{M}(u_1, \ldots, u_{j-l})$$

In particular, there exists $\{u_i\}$ such that

$$\mathbb{E}\left(\prod_{i=1}^{j-l} (x^T u_i) \prod_{t=1}^{l} (x^T v_t)\right) > \mathbb{E}\left(\prod_{i=1}^{j-l} (x^T u_i)\right) \mathbb{E}\left(\prod_{t=1}^{l} (x^T v_t)\right)$$

if and only if there exists $u$ such that

$$\mathbb{E}\left((x^T u)^{j-l} \prod_{t=1}^{l} (x^T v_t)\right) > \mathbb{E}\left((x^T u)^{j-l}\right) \mathbb{E}\left(\prod_{t=1}^{l} (x^T v_t)\right)$$

But at the end of the algorithm, we know that there are no such $u$, hence there can be no such $u_i$ either. Thus, we can factor any $u \not\in S'$ through the expectations which contain only $v_i$ from $S'$.

3.1 Generalized ICA

We can now give an algorithm for generalized ICA, assuming access to exact moment tensors and local optima. If $F$ is factorizable, Algorithm Factor will provide a factoring into subspaces such that the marginal distributions look independent up to $m$ moments. The output of FindBasis is a set of vectors that each lie in $V$ or in $W$. We try all possibilities for the subset from $V$, then extend this using ExtendBasis, consider the resulting decomposition and take the option that gives a product factorization. The factorization found will be $U, U^\perp$ for some $U \subseteq V$.

Theorem 5 (Factoring, general noise). For any $\epsilon, \delta > 0$, given the moment tensors of distribution $F$ over $\mathbb{R}^n$ and the ability to compute exact local optima, if there exists a subspace $V$ with $\dim(V) = k$ such that for $j = 1, \ldots, m$ $d_j(F, F_V F_W) = 0$, Algorithm Factor finds a subspace $U$ such that $d(F, F_U F_{U^\perp}) = 0$ with probability at least $1 - \delta$. The time and sample complexity of the algorithm are bounded by $n^{O(k+m)}$.

Proof of Theorem 5. In the enumeration of all subsets of size at most $k$ subsets of $B$ at line 2, we encounter $T = B \cap V$. By Theorem 4 the output $S'$ of ExtendBasis contains only vectors in $V$.
Algorithm 3 Factor

Input: Highest moment $m$, distribution $F$.
1: $B \leftarrow \text{FindBasis}(m, F)$.
2: for every subset $T \subseteq B$ of at most $k$ vectors do
3: $S' \leftarrow \text{ExtendBasis}(m, F, T, B - T)$.
4: $T' \leftarrow \text{ExtendBasis}(m, F, B - T, S')$.
5: if $|S'| > k$ or $|T'| > n - k$ then
6: Continue with the next choice of $T$.
7: Augment $S'$ with $k - |S'|$ orthonormal vectors from $\mathbb{R}^n - \text{span}(T')$, forming basis $U$.
8: Compute $m$ moments of $F$, $F_U$ and $F_{U\perp}$:
9: for $l \leq m$ do
10: Compute:
$$\Delta^l_U = \sum_{(i_1, \ldots, i_l)} \left( \mathbb{E}_F(x_{i_1} \cdots x_{i_l}) - \mathbb{E}_{F_U}(x_{p_1} \cdots x_{p_j}) \mathbb{E}_{F_{U\perp}}(x_{p_{j+1}} \cdots x_{p_l}) \right)^2$$
where $\{x_{p_1}, \ldots, x_{p_k}\}$ correspond to coordinates in $U$, and $\{x_{p_{k+1}}, \ldots, x_{p_l}\}$ correspond to coordinates in the $U\perp$ subspace.
11: return $U$ with lowest $\Delta^3_U$, breaking ties by considering $\Delta^4_U, \Delta^5_U, \ldots$

and $T'$ contains only vectors from $W$. By Part 2 of Theorem 4, the following holds for any choice of vector $\{u_i\} \subset S^{n-1} - \text{span}(S', T')$, we have $\mathbb{E}((x^T u_i)^2) = \gamma_m$ for $j \leq m$ and that:
$$\mathbb{E} \left( \prod_{i=1}^{j-l} (x^T u_i) \prod_{l=1}^{l}(x^T v_l) \right) = \mathbb{E} \left( \prod_{i=1}^{j-l} (x^T u_i) \right) \mathbb{E} \left( \prod_{l=1}^{l}(x^T v_l) \right)$$
for $v_l \in S' \cup T'$. In particular, every such $u$ is independent from $S'$ and $T'$ up to the $m$th moment. In the augmented basis, the expectations separate into the products over the two subspaces:
$$\mathbb{E}(x_{i_1} \cdots x_{i_l}) = \mathbb{E}(x_{p_1} \cdots x_{p_j}) \mathbb{E}(x_{p_{j+1}} \cdots x_{p_l})$$
where $\{x_{p_1}, \ldots, x_{p_k}\}$ correspond to coordinates in the $U$ subspace, and $\{x_{p_{k+1}}, \ldots, x_{p_l}\}$ correspond to coordinates in the $U\perp$ subspace. In particular, the entries of the moment tensor of $F_U F_{U\perp}$, and hence will return $\Delta^j = 0$. \qed

4 Gaussian noise model

We now give a complete algorithm assuming $F_W$ is a Gaussian, assuming we only have access to $F$ through samples (not exact moment tensors). The main difficulty is handling the error accumulation over multiple iterations, as in each round we can only hope to approximately compute moments and find approximate local optima. The idea is that FindBasis and ExtendBasis find vectors where $\mathbb{E}(x^T u)^m \neq \gamma_m$. If $F_W$ is Gaussian, our algorithms only find directions in $V$. Thus, the error accumulates over only $k$ steps, and the total error depends on $k$ rather than $n$.  

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4.1 Local search

To compute approximate local optima, we perform gradient ascent, moving in the direction of the gradient. If moving along the gradient does not increase the function value by a certain value, we switch to second-order moves based on the Hessian. We will use the notation that $Dg_u$ is the gradient of $g$ at $u$ and $D^2g_u$ for the Hessian matrix. The top eigenvalue of a matrix on a subspace orthogonal to a vector can be computed via a coordinate transformation. We denote $M = \|M^m\|_2$.

**LocalOpt** terminates in polynomial time when the parameters $\epsilon_1, r_1, \epsilon_2$ and $r_2$, the thresholds and step sizes for the first-order moves and second-order moves are chosen appropriately. Note that $\epsilon_2$ varies with the function value, but the remaining parameters are fixed.

**Lemma 7** (Local search termination). Let $g(u)$ satisfy $g(tu) = t^m g(u)$ for some integer $m$. Suppose that for our starting point $u$ that $g(u) \geq \eta > 0$. Choose the parameters as follows:

$$
\epsilon_1 \leq \left( \frac{81m(m-1)^2\eta^2}{1048M} \right)^2
$$

$$
\epsilon_2 = \frac{3m(m-1)g(u)}{4}
$$

$$
r_1 \leq \frac{\sqrt{\epsilon_1}}{4m^2M}
$$

$$
r_2 \leq \frac{9\eta}{256(m-2)M}
$$

where $M$ is an upper bound for $g$ on the unit sphere. Then **LocalOpt** will terminate in at most $\text{poly}(M, m, 1/\epsilon_1, 1/r_1, 1/r_2, 1/\eta)$ iterations.

**Proof of Lemma 7** Consider an iteration of the algorithm where the first derivative condition is unsatisfied, and we make a step of size $r_1$ in the direction of $v/\|v\|$ (call the step $h$). The new function value at this point $u + h$ is given by the Taylor series expansion with error (where $\zeta$ lies between $u$ and $u + h$):

$$
g(u + h) = g(u) + Dg_u \cdot h + \frac{1}{2} h^T(D^2g_{\zeta})h
$$

Algorithm 4 LocalOpt

**Input:** Function $g$, error parameter $\epsilon_1$.

1: $u \leftarrow$ uniformly at random over unit sphere.
2: while $|\langle u, Dg_u \rangle| \leq (1 - \epsilon_1) \|Dg_u\|$ or $\lambda_{\text{max}}(D^2g_u) \geq \epsilon_2$ on $u^\perp$ do
3: if $|\langle u, Dg_u \rangle| \leq (1 - \epsilon_1) \|Dg_u\|$ then
4: Direction $v \leftarrow \pi_{u^\perp}(Dg_u)$.
5: $u \leftarrow u + r_1 v/\|v\|$.
6: Renormalize $u \leftarrow u/\|u\|$.
7: else if $\lambda_{\text{max}}(D^2g_u) \geq \epsilon_2$ on $u^\perp$ then
8: Direction $v \leftarrow$ top eigenvector of $D^2g_u$ on $u^\perp$.
9: $u \leftarrow u - r_2 v/\|v\|$.
10: Renormalize $u \leftarrow u/\|u\|$.
11: return $u$. 

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The increase in function value is lower bounded as follows:

\[
Dg_u \cdot h + \frac{1}{2} h^T D^2 g \zeta h \geq r_1 \left( Dg_u, \frac{v}{\|v\|} \right) - \frac{1}{2} r_1^2 (v/\|v\|)^T D^2 g \zeta (v/\|v\|)
\]

\[
\geq r_1 \sqrt{\epsilon_1} - \frac{1}{2} r_1^2 m^2 M
\]

\[
\geq r_1 \sqrt{\epsilon_1} - \frac{1}{8} r_1 \sqrt{\epsilon_1}
\]

\[
\geq \frac{7}{8} r_1 \sqrt{\epsilon_1}
\]

Thus, we have lower bounded the increase in the function value. When we rescale \( u + h \) back to norm 1, we can apply the \( m \)-homogeneity of \( f \) to deduce that:

\[
g\left( \frac{u + h}{\|u + h\|} \right) = \frac{1}{\|u + h\|^{m/2}} g(u + h)
\]

We can compute \( \|u + h\| = 1 + r_1^2 \) because \( r_1 \) is perpendicular to \( u \). Hence:

\[
g\left( \frac{u + h}{\|u + h\|} \right) = \frac{1}{(1 + r_1^2)^{m/2}} g(u + h)
\]

\[
\geq \left( 1 - \frac{m + 2}{2} r_1^2 \right) g(u + h)
\]

\[
\geq g(u) \left( 1 + \frac{7}{8 g(u)} r_1 \sqrt{\epsilon_1} \right) \left( 1 - \frac{m + 2}{2} r_1^2 \right)
\]

where we used the estimate:

\[
(1 + x)^k \geq 1 + (k + 1/2)x
\]

for \( x \leq 2/k^2 \). To finish this calculation, we simply substitute our value for \( r_1 \) in terms of \( \epsilon_1 \):

\[
g\left( \frac{u + h}{\|u + h\|} \right) \geq g(u) \left( 1 + \frac{7}{8 g(u)} r_1 \sqrt{\epsilon_1} \right) \left( 1 - \frac{1}{8M} r_1 \sqrt{\epsilon_1} \right)
\]

\[
\geq g(u) \left( 1 + \frac{5}{8M} r_1 \sqrt{\epsilon_1} \right)
\]

Hence, there are at most at most a polynomial number of iterations of this form. Consider now an iteration where the second derivative condition is unsatisfied (and the first derivative condition must be satisfied). We take the same Taylor series expansion with error term (to one further term), where \( h = r_2 v/\|v\| \):

\[
g(u + h) = g(u) + Dg_u \cdot h + \frac{1}{2} h^T D^2 g_u h + \frac{1}{6} D^3 g \zeta(h, h, h)
\]

We will show that the contributions from the first and third derivative terms are small, and that the second derivative term dominates. In the first derivative term, note that \( h \) is orthogonal to \( u \),
and hence the component of $Dg_u$ parallel to $h$ has norm at most $\sqrt{2\epsilon_1 - \epsilon_1^2} \|Dg_u\|$. We estimate the other terms as before:

$$Dg_u \cdot h + \frac{1}{2} h^T D^2 g_u h + \frac{1}{6} D^3 g_u(h, h, h) \geq -\sqrt{2\epsilon_1 - \epsilon_1^2} m M + \frac{1}{2} r_2^2 \epsilon_2 - \frac{m(m-1)(m-2)}{6} r_2^3 M$$

$$\geq -\frac{1}{128} r_2^2 \epsilon_2 + \frac{1}{2} r_2^2 \epsilon_2 - \frac{1}{128} r_2^2 \epsilon_2$$

$$\geq \frac{31}{64} r_2^2 \epsilon_2$$

Once again, we have to rescale back to norm 1. In this case:

$$g\left(\frac{u + h}{\|u + h\|}\right) \geq g(u) \left(1 + \frac{31}{64g(u)} r_2^2 \epsilon_2 - \frac{m + 1}{2} r_2^2 \right)$$

$$\geq g(u) \left(1 + \frac{93}{256} m(m-1) r_2^2 - \frac{m + 1}{2} r_2^2 \right)$$

$$\geq g(u) \left(1 + \frac{93}{256} r_2^2 \right)$$

The last bound follows because the worst possible lower bound occurs at $m = 3$. Hence, there are only a polynomial number of iterations of this form as well. \qed

### 4.2 Exact moments, approximate local optima

We are now ready to extend the analysis of Theorem 3 to the case when we have access to the exact moment tensor, but instead of using exact moments, we will use LocalOpt with appropriately chosen $\epsilon_1$. On the other hand, using a weaker local optimum algorithm will also give us weaker guarantees on the quality of the output, giving a weaker form of Lemma 4. Over $\mathbb{R}^n$ (instead of $S^{n-1}$), Lemma 3 gives us a formula of the form:

$$f_m(u) = \|u_V\|^m (\mathbb{E}(x_V^T u_V^0)^m) - \gamma_m + \|u_W\|^m (\mathbb{E}(x_W^T u_W^0)^m) - \gamma_m + \gamma_m \|u\|^m$$

We will optimise the function $g(u) = f_m(u) - \gamma_m \norm{u}^m$ using LocalOpt over the unit sphere. This is equivalent to optimising $f_m$ and simplifies our derivative calculations.

**Lemma 8** (Exact moments, inexact optima). Let $F = F_V F_W$ have the same first $m - 1$ moments as a Gaussian but a different $m$th moment. Suppose we run LocalOpt on $g(u)$, starting from a point $u$ where $g(u) \geq \eta$, setting $\epsilon_1 \leq m \epsilon_2^2 / (m-2) / M^{2/(m-2)}$. After $\text{poly}(n, 1/\epsilon_1, \eta)$ iterations, we will have a point $u^*$ where either $\|\pi_V(u^*)\| \geq 1 - 16\epsilon_1$ or $\|\pi_W(u^*)\| \geq 1 - 16\epsilon_1$.

**Proof.** We proceed as in Lemma 4. $u^*$ lies on a curve $C = \{s(u_V^0) + t(u_W^0) : s^2 + t^2 = 1, s \geq 0, t \geq 0\}$. We will show that neither $s$ nor $t$ is bounded away from 0 and 1.

Restricted to the curve $g(u) = g(s, t) = a_v s^m + a_w t^m$. Suppose that $a_w \leq 0$, then we must have that $s \geq (\eta/M)^{1/m}$. In this case, a direct calculation comparing $\langle Dg_u, u \rangle = m a_v s^{m-1} + m a_w t^{m-1}$ with $\|Dg_u\| = m \sqrt{a_v^2 s^{2(m-1)} + a_w^2 t^{2(m-1)}}$ will yield $s \geq 1 - 2\epsilon_1$. Thus, we may assume that both $a_v$ and $a_w$ are positive, and that $a_v \geq a_w$.

Suppose that for a unit vector $u$, we have $s, t \geq 16\sqrt{\epsilon_1}$, and the first-order gradient condition:

$$\langle Dg_u, u \rangle / \|Dg_u\| \geq 1 - \epsilon_1,$$
then,
\[ \lambda_{\text{max}}(D^2 g_u) \geq \frac{3m(m-1)g(u)}{4} \]

(where the eigenvalue is taken only in the subspace orthogonal to \( u \)). Thus, the algorithm continues making progress at such a vector \( u \). To do this, we lower bound the top eigenvalue by the quadratic form in the direction \(-tu^0_v + su^0_w\), which is orthogonal to \( u \).

\[
\lambda_{\text{max}}(D^2 g_u) \geq ( -tu^0_v + su^0_w )^T D^2 g_u ( -tu^0_v + su^0_w )
= m(m-1)(a_v s^m-2 t^2 + a_w s^2 t^{m-2})
= m(m-1)(a_v s^m-1, a_w t^{m-1}) ( \frac{t^2}{s^2} )
\]

By construction, \( u \) has two nonzero coordinates, taking values \( s \) and \( t \) and all other coordinates zero. \( Dg_u \) has partial derivatives \( ma_v s^{m-1} \) and \( ma_w t^{m-1} \) in these directions. Thus,

\[
\frac{\langle Dg_u, u \rangle}{\| Dg_u \|} \leq \frac{ (a_v s^{m-1} \quad T ) (s \quad t) }{ \| (a_v s^{m-1} \quad a_w t^{m-1}) \| } 
\]

Thus we obtain the condition that:

\[
\begin{pmatrix} a_v s^{m-1} \\ a_w t^{m-1} \end{pmatrix} = (1 - \epsilon) \begin{pmatrix} a_v s^{m-1} \\ a_w t^{m-1} \end{pmatrix} + \sqrt{2\epsilon - \epsilon^2} \begin{pmatrix} a_v s^{m-1} \\ a_w t^{m-1} \end{pmatrix} r
\]

where \( 0 \leq \epsilon \leq \epsilon_1 \) and \( r \) is a unit vector orthogonal to \((s, t)\). Substituting this into the previous equation:

\[
\lambda_{\text{max}}(D^2 g_u) \geq m(m-1) \left[ (1 - \epsilon) \left\| \begin{pmatrix} a_v s^{m-1} \\ a_w t^{m-1} \end{pmatrix} \right\| + \sqrt{2\epsilon - \epsilon^2} \left\| \begin{pmatrix} a_v s^{m-1} \\ a_w t^{m-1} \end{pmatrix} \right\| r^T \left( \frac{t^2}{s^2} \right) \right]
\]

\[
\geq m(m-1) \left( \begin{pmatrix} a_v s^{m-1} \\ a_w t^{m-1} \end{pmatrix} \right) \left( (1 - \epsilon) - \sqrt{2\epsilon - \epsilon^2} \left( \frac{1}{s} + \frac{1}{t} \right) \right)
\]

\[
\geq m(m-1) \left( \begin{pmatrix} a_v s^{m-1} \\ a_w t^{m-1} \end{pmatrix} \right) \left( (1 - \epsilon) - 2\sqrt{2\epsilon - \epsilon^2} \frac{1}{16\epsilon} \right)
\]

\[
\geq \frac{3m(m-1)g(u)}{4}
\]

where the last estimate follows from the Cauchy-Schwartz inequality.

\[\square\]

### 4.3 Approximate moments and approximate local optima

By using the robust Schwartz Zippel Lemma (Lemma 12) instead of the usual form, and **LocalOpt** at Lines 10 and 11 of **FindBasis** and Lines 13 and 15 of **ExtendBasis**, we can obtain an efficient randomized algorithm. The major difficulty remaining is that we must bound the error incurred every time we call **LocalOpt**. The error analysis is technical: the idea is to obtain approximate versions of Lemmas 9 and 10 and to show that **LocalOpt** behaves well on these approximate versions. Consider the first iteration:
Lemma 9 (Two steps). Let \( x \) have the same first \( m-1 \) moments as a Gaussian but a different \( m^{th} \) moment. Let \( u_1 = \sqrt{1 - \delta v_1} - \sqrt{\delta w_1} \) be the vector found in the first iteration of FindBasis, where \( v_1 \) and \( w_1 \) are unit vectors in \( V \) and \( W \) respectively. Suppose we run LocalOpt on \( g(u) = f_m(u) - \gamma_m \|u\|^m \) on the orthogonal subspace \( u_1^\perp \), starting from a point \( u \) where \( g(u) \geq \eta = M\delta^{1/16} \), setting \( \epsilon_1 \leq \frac{m\epsilon_0^{2/(m-2)}}{M^2(M^{1/2} - \delta^{1/8})} - 60m^2M^2\delta^{5/16} \) as the error parameter in LocalOpt. After \( \text{poly}(n, 1/\epsilon_1, \eta) \) iterations, we will have a point \( u^* \) where either \( \|\pi_V(u^*)\| \geq 1 - \delta^{1/8} \) or \( \|\pi_W(u^*)\| \geq 1 - \delta^{1/8} \).

The sequence of ideas in this proof is not unlike the proofs in Section 3 first we derive a nice representation of \( f_m \) (cf Lemma 3) then we analyse the support of a local optimum under this representation (cf Lemma 4) – we are not able to claim that the local optimum found is contained wholly in \( V \) or \( W \); but since we are satisfied with approximate local optima, we can bound the components around 0 and 1. All through this, we must bound the error, and try to push through the calculations of Lemma 3.

Proof of Lemma 9 First, we will construct an orthonormal basis which includes \( u_1 \): extend \( \{v_1\} \) and \( \{w_1\} \) to orthonormal bases \( \{v_i\} \) and \( \{w_i\} \) of \( V \) and \( W \) respectively. Replace \( v_1 \) and \( w_1 \) with the following two vectors:

\[
\begin{align*}
\hat{u}_1 &= \sqrt{1 - \delta}v_1 - \sqrt{\delta}w_1 \\
\hat{u}_1 &= \sqrt{\delta}v_1 + \sqrt{1 - \delta}w_1
\end{align*}
\]

Thus our basis will be \( \{u_1, \hat{u}_1, v_2, \ldots, v_k, w_2, \ldots, w_l\} \). For a vector \( x = (x_1, \ldots, x_n) \) in the basis of \( \{v_i\} \) and \( \{w_i\} \), we now have:

\[
x = (\sqrt{1 - \delta}x_1 - \sqrt{\delta}x_2, \sqrt{\delta}x_1 + \sqrt{1 - \delta}x_2, x_3, \ldots, x_n)
\]

which is simply a rotation (unitary transformation) in the first two coordinates.

We evaluate the \( m^{th} \) moment on the subspace orthogonal to \( u_1 \). Let \( \xi \) be a point on this orthogonal subspace: note that \( \xi \) has 0 component in the first coordinate:

\[
f_m(\xi) = E((x^T \xi)^m) = E\left((\sqrt{\delta}x_1 \xi_2 + \sqrt{1 - \delta}x_2 \xi_2 + \sum_{i=2}^{k} x_{v_i} \xi_{v_i} + \sum_{i=2}^{l} x_{w_i} \xi_{w_i})^m\right)
\]

We can break up the argument into two dot products, which are independent of each other. Moreover, observe that the norm of the two constituent parts of the \( \xi \) vector taken together is still 1.

\[
x^T \xi = (x_1, x_{v_2}, \ldots, x_{v_k})^T(\sqrt{\delta}x_2, \xi_2, \ldots, \xi_v) + (x_2, x_{w_2}, \ldots, x_{w_l})^T(\sqrt{1 - \delta}x_2, \xi_2, \ldots, \xi_w)
\]

Hence, we can apply Lemma 3 this gives a perturbed version of Lemma 3:

\[
f_m(\xi) = \left(\delta_2^2 + \sum_{i=2}^{k} \xi_{v_i}^2\right)^{m/2} E\left(((x_1, x_{v_2}, \ldots, x_{v_k})^T(\sqrt{\delta}x_2, \xi_2, \ldots, \xi_v)^0)^m - \gamma_m\right) + \\
\left((1 - \delta) \xi_2^2 + \sum_{i=2}^{l} \xi_{w_i}^2\right)^{m/2} E\left(((x_2, x_{w_2}, \ldots, x_{w_l})^T(\sqrt{1 - \delta}x_2, \xi_2, \ldots, \xi_w)^0)^m - \gamma_m\right) + \gamma_m
\]
Fixing a point $\xi^* \in u_1^+ \cap S^{n-1}$, we will give a curve $C$ which passes through this point and remains on the unit sphere. We will analyse the value of the $g(\xi) = f_m(\xi) - \gamma_m \|\xi\|^m$ on this curve – as before, every point which is a local optimum on $S^{n-1}$ has to be a local optimum on $C$ as well. Thus by studying the local optima over $C$, we will be able to describe the structure of the local optima in full space.

We may assume that all the $\xi^*_i$ are nonnegative – otherwise we can pick simply negate the associated basis vector. We take the following as the components for $C$:

$$\xi^*_v = \frac{1}{\sqrt{\sum_{i=2}^k (\xi^*_i)^2}} (0, 0, \xi^*_v, \ldots, \xi^*_v, 0, \ldots, 0)$$
$$\xi^*_w = \frac{1}{\sqrt{(1 - \delta)(\xi^*_2)^2 + \sum_{i=2}^l (\xi^*_w)^2}} (0, \sqrt{1 - \delta} \xi^*_2, 0, \ldots, 0, \xi^*_w, \ldots, \xi^*_v)$$
$$\xi^*_l = (1, 0, \ldots, 0)$$

Since these are the only three directions that change along $C$, we will use these three vectors as an orthonormal basis. Now, defining the following quantity:

$$\alpha = (\xi^*_2)^2 / \left( (1 - \delta)(\xi^*_2)^2 + \sum_{i=2}^l (\xi^*_w)^2 \right)$$

we can write our curve $C$ as:

$$C = \{ y\xi^*_v + z\xi^*_w + \sqrt{\alpha \delta} z\xi^*_l^* : y^2 + (1 + \delta \alpha)z^2 = 1, y, z \geq 0 \}$$

Specifically, we will use the basis $\xi^*_w$ and $(1 + \alpha \delta)^{-1}(\xi^*_w + \xi^*_l)$. Note that in this basis, $y$ is precisely the coordinate along the first basis vector and $(1 + \delta \alpha)^{1/2}z$ is the coordinate along the second basis vector. Denote this latter quantity by $z'$, then by the chain rule, we have that $\partial / \partial z' = (1 + \delta \alpha)^{-1/2} \partial / \partial z$.

Restricted to $C$, the expectation terms simplify: note that $(\sqrt{1 - \delta} \xi_2, \xi_{w_2}, \ldots, \xi_{w_1})^0$ remains constant on $C$, so the second expectation term reduces to a constant, which we will denote with $a_w$. The first expectation term does not remain constant, because there is an additional component in the direction of $v_1$, but this component always has a small magnitude. With a change of basis, we can simplify this expression to involving only $y$ and $z$:

$$\mathbb{E} \left( (x_1, x_{v_2}, \ldots, x_{v_k})^T (\sqrt{\delta} \xi_2, \xi_{v_2}, \ldots, \xi_{v_k})^0 \right)^m = \mathbb{E} \left( (x_1, x_{\xi}^*)^T (\sqrt{\alpha \delta} z, y)^0 \right)^m$$

We will denote the first expectation term by $a_v$. In full, our objective function on $C$ is given by:

$$g(\xi) = [\delta \alpha z^2 + y^2]^{(m/2)} \mathbb{E} \left( (x_1, x_{\xi})^T (\sqrt{\alpha \delta} z, y)^0 \right)^m - \gamma_m + a_w z^m$$

Next we will examine the local optima on $C$: let $\xi$ be the output of of LocalOpt: we will show that $\xi$ has large projection with either the $V$ or $W$ subspace (cf Lemma [4]). We will analyse the following cases:

1. $y^2 \leq \delta^{1/4}$ or $z^2 \leq \delta^{1/4}$. 


2. \(y^2 \geq \delta^{1/4}\) and \(z^2 \geq 1/3\).

3. \(z^2 \geq \delta^{1/4}\) and \(y^2 \geq 1/3\).

Case 1: Suppose that \(y^2 \leq \delta^{1/4}\), then we must have \(z \geq \sqrt{1 - \delta^{1/4} - \alpha \delta}\). The approximate local optimum \(u\) that we compute has projection at least \(\sqrt{1 - \delta}\) on this local optimum, and hence, the projection of \(u\) onto \(w\) is at least:

\[
\|\pi_W(u)\| \geq \sqrt{(1 - \delta)(1 - \delta^{1/4} - \alpha \delta)} - \sqrt{\delta} \\
\geq 1 - \delta/2 - \delta^{1/4}/2 - \alpha \delta - \sqrt{\delta} \\
\geq 1 - \delta^{1/4}
\]

In this case, for sufficiently small \(\delta\), we have:

\[
\|\pi_W(u)\|^2 \geq 1 - \delta^{1/8}
\]

The argument for when \(z^2 \leq \delta^{1/4}\) is identical.

Case 2: We will prove that \textbf{LocalOpt} cannot terminate in this region by carrying out the calculations of Lemma \[\text{[lemma number]}\] whilst keeping track of errors. Thus, let \(\xi\) be a point in this range, we will show that if the first derivative condition in \textbf{LocalOpt} is satisfied, then the second derivative condition is unsatisfied, thus \textbf{LocalOpt} cannot terminate at \(\xi\). First, let us examine how \(f\) changes over \(C\):

**Claim 10** (First partials under perturbations). In the range where \(y^2 \geq \delta^{1/4}\) and \(z^2 \geq 1/3\),

\[
\left| \frac{\partial g}{\partial y} - ma_v y^{m-1} \right| \leq 3mM \sqrt{\delta} \\
\left| \frac{\partial g}{\partial z} - ma_w z^{m-1} \right| \leq 4mM \sqrt{\delta}
\]

As a corollary, via the triangle inequality, we have that:

\[
\|(g_y, g_z)\| \geq m \|(a_v y^{m-1}, a_w z^{m-1})\| - 5mM \sqrt{\delta}
\]

**Claim 11** (Second partials under perturbations). In the range where \(y^2 \geq \delta^{1/4}\) and \(z^2 \geq 1/3\):

\[
\left| \frac{\partial^2 g}{\partial y^2} - m(m-1)a_v y^{m-2} \right| \leq c_{vv} m^2 M \sqrt{\delta} \\
\left| \frac{\partial^2 g}{\partial z^2} - m(m-1)a_w z^{m-2} \right| \leq c_{ww} m^2 M \sqrt{\delta} \\
\left| \frac{\partial^2 g}{\partial y \partial z} \right| \leq c_{vw} m^2 M \sqrt{\delta}
\]

where \(c_{vv}, c_{vw}\) and \(c_{ww}\) are absolute constants bounded by 20.

Throughout the rest of this calculation, we will use the basis of \(n - 1\) vectors consisting of \(\{\xi^*_v, (1 + \alpha \delta)^{-1}(\xi^*_w + \xi^*_v)\}\), and any orthonormal extension to \(u^*_v\). In particular, in this basis, \(\xi = (y, z', 0, \ldots, 0)\).
As before, we will lower bound the contribution of the second derivative term. Our direction of movement will be \((-z', y, 0, \ldots, 0)\). This vector is clearly a unit vector orthogonal to \(\xi\), where top eigenvalue is taken orthogonal to \(\xi\).

\[
\lambda_{\text{max}}(D^2 g_\xi) \geq \langle -z', y \rangle D^2 g_\xi \begin{pmatrix} -z' \\ y \end{pmatrix} \\
\geq (-\sqrt{1 + \alpha \delta z}, y) \begin{pmatrix} g_{yy} & g_{z' y} \\ g_{z' y} & g_{z' z'} \end{pmatrix} \begin{pmatrix} -\sqrt{1 + \alpha \delta z} \\ y \end{pmatrix}
\]

We can further use Claim 11 to simplify the other components of the quadratic form:

\[
\lambda_{\text{max}}(D^2 g_\xi) \geq (1 + \delta \alpha) z^2 g_{yy} + y^2 g_{z' z'} - 2c_{vw}m^2M\sqrt{\delta} \\
\geq (1 + \delta \alpha) m(m - 1)(a_v y^{m-1}, a_w z^{m-1}) \left( \frac{z^2}{y}, \frac{y^2}{z} \right) - (1 + \delta \alpha)(c_{zz} + c_{yy} + 2c_{zw})m^2M\sqrt{\delta}
\]

Our first derivative condition is given by:

\[
\frac{\langle Dg_\xi, \xi \rangle}{\|Dg_\xi\|} \geq 1 - \epsilon_1
\]

Since \(\xi = (y, z', 0, \ldots, 0)\) has only two nonzero components, we need only evaluate two components of the derivative: furthermore, we can lower bound the norm \(\|Dg_\xi\| \geq \|(g_y, g_{z'})\|\), which gives the following lower bound:

\[
\frac{(g_y, g_{z'}) \left( \frac{y}{z'} \right)}{\|(g_y, g_{z'})\|} \geq 1 - \epsilon_1
\]

Rearranging, and applying Claim 11 yields:

\[
m(a_v y^{m-1}, a_w z^{m-1}) \begin{pmatrix} y \\ z \end{pmatrix} \geq (1 - \epsilon_1) \|(g_y, g_{z'})\| - 7mM\sqrt{\delta} \\
\geq m(1 - \epsilon_1) \|(a_v y^{m-1}, a_w z^{m-1})\| - 12mM\sqrt{\delta} \\
\geq m(1 - \epsilon_1 - \frac{12M\sqrt{\delta}}{\eta}) \|(a_v y^{m-1}, a_w z^{m-1})\|
\]

Thus, we can rewrite this relationship for unit vector \(r\) orthogonal to \((a_v y^{m-1}, a_w z^{m-1})\) and \(0 \leq \epsilon \leq \epsilon_1 + \frac{12M\sqrt{\delta}}{\eta}:

\[
\begin{pmatrix} a_v y^{m-1} \\ a_w z^{m-1} \end{pmatrix} = (1 - \epsilon) \|(a_v y^{m-1}, a_w z^{m-1})\| \begin{pmatrix} y \\ z \end{pmatrix} + \sqrt{2\epsilon - \epsilon^2} \|(a_v y^{m-1}, a_w z^{m-1})\| r
\]

Substituting this back into our lower bound for \(\lambda_{\text{max}}\) yields:

\[
\lambda_{\text{max}} \geq (1 + \delta \alpha)(1 - \epsilon) \|(a_v y^{m-1}, a_w z^{m-1})\| \left( z^2 + y^2 \right) - \sqrt{2\epsilon - \epsilon^2} \|(a_v y^{m-1}, a_w z^{m-1})\| \left( \frac{1}{y} + \frac{1}{z} \right) \\
- 80m^2M\sqrt{\delta} \\
\geq (1 + \delta \alpha)(1 - \delta^{1/6})m(m - 1)f(\xi) - \sqrt{2\delta^{1/4}} - 80m^2M\sqrt{\delta} \\
\geq \frac{3}{4}m(m - 1)f(\xi)
\]

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where we used the Cauchy-Schwartz inequality for:
\[
\| (a_v y^{m-1}, a_w z_m - 1) \| \geq a_v y^m + a_w z^m \geq g(\xi) - mM\sqrt{\alpha \delta}
\]

Case 3: This case follows from the exactly the same analysis as above. It is in fact substantially easier, as the denominator terms \(\alpha \delta z + y\) are in fact all bounded by constants now, and hence the numerator is small enough in almost all cases above to bound the terms.

We now provide the proofs for the claims regarding the coefficients \(a_v\) and \(a_w\). In explicitly taking derivatives, it is important to note the following:

\[
a_v = E \left( \left( (x_1, x_v)^T (\sqrt{\alpha \delta z}, y)^0 \right)^m \right) - \gamma_m
\]

\[
= \frac{1}{(\alpha \delta z^2 + y^2)^{(m/2)}} E \left( \left( (x_1, x_v)^T (\sqrt{\alpha \delta z}, y)^0 \right)^m \right) - \gamma_m
\]

For ease of notation, denote \(\phi = (\sqrt{\alpha \delta z}, y)\), we will suppress all but one \(\phi\) argument in our moment tensors, thus we will write \(A(\phi)\) instead of \(A(\phi, \ldots, \phi)\), and \(A(\phi, e_1)\) instead of \(A(\phi, \ldots, \phi, e_1)\). If \(A\) is a \(m\)th order tensor, its derivative has components given by \((DA)_i = mA(\phi, e_i)\) where \(A\) takes \((m - 1)\) copies of \(\phi\). We also have the Hessian \(D^2\): \((D^2A)_{ij} = m(m - 1)A(\phi, e_i, e_j)\). We can bound the spectral norm of \(D^2A\) using Claim \([7]\) which yields \(\lambda_{max}(D^2A) \leq m(m - 1)M\).

**Proof of Claim \([7]\)** Firstly, we have:

\[
\frac{\partial y}{\partial y} = my(\alpha \delta z^2 + y^2)^{(m/2) - 1} a_v + (\alpha \delta z^2 + y^2)^{(m/2)} \frac{\partial a_v}{\partial y}
\]

\[
\frac{\partial y}{\partial z} = mz^{m-1} a_w + m\alpha \delta z (\alpha \delta z^2 + y^2)^{(m/2) - 1} a_v + (\alpha \delta z^2 + y^2)^{(m/2)} \frac{\partial a_v}{\partial z}
\]

The \(m\alpha \delta z(\alpha \delta z^2 + y^2)^{(m/2) - 1} a_v\) is upper bounded in absolute value in \(mM\delta\). Similarly, it is also clear that:

\[
\left| my(\alpha \delta z^2 + y^2)^{(m/2) - 1} a_v - ma_v y^{m-1} \right| \leq mM\sqrt{\delta}
\]

Thus it remains to show that the partial derivative terms are small:

\[
(\alpha \delta z^2 + y^2)^{(m/2)} \frac{\partial a_v}{\partial y} = (\alpha \delta z^2 + y^2)^{(m/2)} \left[ \frac{-my}{(\alpha \delta z^2 + y^2)^{(m/2) + 1}} A(\phi, \phi) + \frac{m}{(\alpha \delta z^2 + y^2)^{(m/2)}} A(\phi, e_1) \right]
\]

\[
= m \left( -\sqrt{\alpha \delta z} A(\phi, e_2) - y^2 A(\phi, e_1) + \sqrt{\alpha \delta z} A(\phi, e_1) \right)
\]

\[
= m \left( -\sqrt{\alpha \delta z} A(\phi, e_2) + \alpha \delta z^2 A(\phi, e_1) \right)
\]

\[
\frac{\alpha \delta z^2 + y^2}{\alpha \delta z^2 + y^2}
\]

When we have a term like \(A(\phi, \ldots, \phi, e_1)\), the arguments are not normalised. In particular:

\[
A(\phi, \ldots, \phi, e_1) = (\alpha \delta z^2 + y^2)^{(m-1)/2} A(\phi^0, \ldots, \phi^0, e_1)
\]

Thus, normalising gives:

\[
\left| (\alpha \delta z^2 + y^2)^{(m/2)} \frac{\partial a_v}{\partial y} \right| \leq mM\sqrt{\delta} + m\delta M
\]

\[
\leq 2mM\sqrt{\delta}
\]

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For the other partial derivative, we want to compute:

\[(\delta z^2 + y^2)^{\frac{m}{2}} \frac{\partial a_v}{\partial z} = (\delta z^2 + y^2)^{\frac{m}{2}} \left[ \frac{-m\alpha \delta z}{(\alpha \delta z^2 + y^2)^{\frac{m}{2}+1}} A(\phi, \phi) + \frac{m\sqrt{\delta}}{(\alpha \delta z^2 + y^2)^{\frac{m}{2}}} A(\phi, e_2) \right] \]

\[= m\sqrt{\delta} \left( \frac{\sqrt{\alpha \delta z}A(\phi, \phi) + \alpha \delta z^2 A(\phi, e_2) + y^2 A(\phi, e_2)}{\alpha \delta z^2 + y^2} \right) \]

Applying the same method:

\[\left| (\delta z^2 + y^2)^{\frac{m}{2}} \frac{\partial a_v}{\partial z} \right| \leq 3mM \sqrt{\delta} \]

Hence combining this with our earlier bound, we have the desired inequality. \(\square\)

**Proof of Claim 17.** By direct calculation, we obtain:

\[\frac{\partial^2 y}{\partial y^2} = (\delta z^2 + y^2)^{\frac{m}{2}} \frac{\partial^2 a_v}{\partial y^2} + 2my(\delta z^2 + y^2)^{\frac{m}{2}-1} \frac{\partial a_v}{\partial y} + ma_v(\delta z^2 + y^2)^{\frac{m}{2}-2}(\delta z^2 + (m-1)y^2)\]

We now estimate the three terms in this sum – the first two terms will be of order \(\sqrt{\delta}\), and the last term will give us approximately \(m(m-1)a_v y^{m-2}\).

\[(\delta z^2 + y^2)^{\frac{m}{2}} \frac{\partial^2 a_v}{\partial y^2} = (\delta z^2 + y^2)^{\frac{m}{2}} \left\{ -\frac{m^2 y}{(\alpha \delta z^2 + y^2)^{\frac{m}{2}+1}} \left[ \frac{-y\sqrt{\alpha \delta z}A(\phi, e_2)}{\alpha \delta z^2 + y^2} + \frac{\alpha \delta z^2 A(\phi, e_1)}{\alpha \delta z^2 + y^2} \right] + \frac{m}{(\alpha \delta z^2 + y^2)^{\frac{m}{2}}} \left[ -\frac{\sqrt{\alpha \delta z}A(\phi, e_2)}{\alpha \delta z^2 + y^2} - \frac{2y\alpha \delta z^2 A(\phi, e_1)}{(\alpha \delta z^2 + y^2)^2} \right] \right\} \]

\[= (-m^2 y) \left[ \frac{-y\sqrt{\alpha \delta z}A(\phi, e_2)}{(\alpha \delta z^2 + y^2)^{\frac{m}{2}}} + \frac{\alpha \delta z^2 A(\phi, e_1)}{(\alpha \delta z^2 + y^2)^{\frac{m}{2}}} \right] + m \left[ -\frac{\sqrt{\alpha \delta z}A(\phi, e_2)}{\alpha \delta z^2 + y^2} - \frac{2y\alpha \delta z^2 A(\phi, e_1)}{(\alpha \delta z^2 + y^2)^2} \right] \]

We will bound the magnitude of every term in this sum. Consider the first term of the form:

\[\left| (-m^2 y) \frac{-y\sqrt{\alpha \delta z}A(\phi, e_2)}{(\alpha \delta z^2 + y^2)^{\frac{m}{2}}} \right| \leq m^2 \left| \frac{y^2 \sqrt{\delta}}{\alpha \delta z^2 + y^2} \right| \]

Thus, since \(m \geq 3:\)

\[\left| (-m^2 y) \frac{-y\sqrt{\alpha \delta z}A(\phi, e_2)}{(\alpha \delta z^2 + y^2)^{\frac{m}{2}}} \right| \leq 3m^2 M \left| \frac{y^2 \sqrt{\delta}}{\alpha \delta z^2 + y^2} \right| \]

Now, \(y^2/(\alpha \delta z^2 + y^2) \leq 1\), hence:

\[\left| (-m^2 y) \frac{-y\sqrt{\alpha \delta z}A(\phi, e_2)}{(\alpha \delta z^2 + y^2)^{\frac{m}{2}}} \right| \leq 3m^2 M \sqrt{\delta} \]
Of the seven terms in the sum, the first, third and fifth terms can be analysed exactly as above, and their sum can be upper bounded by $15m^2M\sqrt{\delta}$. For the remaining terms we have to use our lower bound on $y$, for example:

$$\left|\frac{(\alpha\delta z^2 A(\phi, e_1))}{(\alpha\delta z^2 + y^2)^2}\right| \leq m \left|\frac{\delta y}{\alpha\delta z^2 + y^2}\right|$$

$$\leq m \left|\frac{\delta}{y}\right|$$

$$\leq mM\delta^{7/8}$$

By this reasoning, we can bound all seven terms by $m^2M\sqrt{\delta}$, hence this term in $\partial^2 g/\partial y^2$ contributes is bounded in absolute value by $7m^2M\sqrt{\delta}$. For the second term in that expression, the analysis is almost identical to the previous claim and gives

$$\left|2my(\alpha\delta z^2 + y^2)^{(m/2)-1}\frac{\partial y}{\partial y}\right| = 2m^2 \left|y(\alpha\delta z^2 + y^2)^{(m/2)-1}\frac{(-y\sqrt{\alpha\delta z A(\phi, e_2)} + \alpha\delta z^2 A(\phi, e_1))}{(\alpha\delta z^2 + y^2)^{3/2}}\right|$$

$$\leq 2m^2 \left|y\frac{(-y\sqrt{\alpha\delta z A(\phi, e_2)} + \alpha\delta z^2 A(\phi, e_1))}{(\alpha\delta z^2 + y^2)}\right|$$

$$\leq 2m^2M\sqrt{\delta} + 2m^2 \left|\frac{\delta}{y}\right|$$

$$\leq 4m^2M\sqrt{\delta}$$

Thus, we have:

$$\left|\frac{\partial^2 g}{\partial y^2} - ma_v(\alpha\delta z^2 + y^2)^{(m/2)-2}(\delta\alpha z^2 + (m-1)y^2)\right| \leq 19m^2M\sqrt{\delta}$$

By applying the triangle inequality:

$$\left|ma_v(\alpha\delta z^2 + y^2)^{(m/2)-2}(\delta\alpha z^2 + (m-1)y^2) - m(m-1)a_v y^{m-2}\right| \leq m^2M\sqrt{\delta}$$

Thus we have the desired result for the second partial with respect to $y$. The other second derivatives are computed in a similar way.

Using the above, we are now examine what happens after $t$ iterations of FindBasis. The following theorem shows that after $k$ iterations of FindBasis, our error blows up at most doubly exponentially in $k$. The proof holds for ExtendBasis as well.

**Theorem 6** (Multiple iterations). Suppose FindBasis finds $j \leq k$ orthogonal vectors $\{u_1, \ldots, u_j\}$ of $g(u)$ taking $\epsilon_1$ such that $\eta \leq M\epsilon_1^{1/16^j}$ for each call of LocalOpt, then $\|v(\epsilon_1^n)\|^2 \geq 1 - \epsilon_1^{(1/16)^j}$.

**Proof of Theorem 6**. After $t$ iterations, we have a basis of orthonormal vectors $\{u_1, \ldots, u_t\}$ where each $u_i$ is close to some vector in $V$:

$$u_1 = a_{11}v_1 + b_{11}w_1$$

$$u_2 = a_{21}v_1 + a_{22}v_2 + b_{21}w_1 + b_{22}w_2$$

$$\vdots$$

$$u_t = a_{t1}v_1 + \cdots + a_{tt}v_t + b_{t1}w_1 + \cdots b_{tt}w_t$$

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We use the orthonormal basis \( \{u_i\} \), \( \{v_{t+1}, \ldots, v_k\} \), the remaining vectors in \( W \{w_{t+1}, \ldots, w_{n-k}\} \), and approximate copies of \( \{w_1, \ldots, w_t\} \). This last set is given by:

\[
w'_1 = c_1 w_1 + \sum_{i=1}^{t} d_{1i} v_i + \sum_{i=1}^{t} e_{1i} w_i
\]

\[
\vdots \quad \vdots
\]

\[
w'_t = c_t w_t + \sum_{i=1}^{t} d_{ti} v_i + \sum_{i=1}^{t} e_{ti} w_i
\]

In these sums we have \( d_{ii} = e_{ii} = 0 \), and we have orthonormality between these vectors. Consider the inner product \( x^T \xi \), where \( \xi \) is of unit length and lies in the space orthogonal to \( \{u_i\} \):

\[
x^T \xi = \sum_{i=t+1}^{k} x_{vi} \xi_{vi} + \sum_{i=t+1}^{n-k} x_{wi} \xi_{wi} + \sum_{i=1}^{t} \xi_{wi'} (c_i x_{wi} + \sum_{j=1}^{t} d_{ij} x_{wj} + \sum_{j=1}^{t} e_{ij} x_{wj})
\]

\[
= (x_{v_{t+1}}, \ldots, x_{v_k}, x_{v_{t+1}}, \ldots, x_{v_k})^T (\sum_{i=1}^{t} \xi_{wi'} d_{i1}, \ldots, \sum_{i=1}^{t} \xi_{wi'} d_{it}, \xi_{v_{t+1}}, \ldots, \xi_{v_k}) +
\]

\[
+ (x_{w_1}, \ldots, x_{w_{t+1}}, \ldots, x_{w_{n-k}})^T (\xi_{w_1} c_1 + \sum_{i=1}^{t} \xi_{w_i} e_{i1}, \xi_{w_1} c_t + \sum_{i=1}^{t} \xi_{w_i} e_{ij})
\]

The two vectors formed from \( \xi \) have total norm 1. Now, we can apply Lemma 3 to obtain:

\[
f_m(\xi') = \left( \sum_{j=t+1}^{k} \xi_{vj}^2 + \sum_{j=t+1}^{n-k} \left( \sum_{i=1}^{t} \xi_{wi'} d_{ij} \right)^2 \right)^{m/2} a_v + \left( \sum_{j=t+1}^{n-k} \xi_{wj}^2 + \sum_{j=1}^{t} \left( \xi_{wi'} c_j + \sum_{i=1}^{t} \xi_{wi'} e_{ij} \right)^2 \right)^{m/2} a_w + \gamma_m
\]

where the expectation term \( a_v \) is given by:

\[
a_v = \mathbb{E} \left( \left( (x_{v_{t+1}}, \ldots, x_{v_k}, x_{v_{t+1}}, \ldots, x_{v_k})^T (\sum_{i=1}^{t} \xi_{wi'} d_{i1}, \ldots, \sum_{i=1}^{t} \xi_{wi'} d_{it}, \xi_{v_{t+1}}, \ldots, \xi_{v_k}) \right)^m - \gamma_m \right)
\]

(and similarly for \( a_w \)). As in the single iteration case, we restrict to a curve. Fix an output \( \xi' \) of FindBasis: we will fix the ratio of the components \( \{\xi_{wi}\} \) in the ratio of \( \xi^* \), and similarly, we will fix the ratios of \( \{\xi_{wi}', \ldots, \xi_{wi'}, \xi_{w_{t+1}}, \ldots, \xi_{w_{n-k}}\} \) according to \( \xi^* \) as well. This gives the following restriction on our curve after subtracting \( \gamma_m \|\xi\|^m \).

\[
g(\xi') = a_v \left[ (y')^2 + (z')^2 \left( \sum_{j=1}^{t} \left( \sum_{i=1}^{t} d_{ij} \xi_{wi}^*/l \right)^2 \right)^{m/2} + a_w (z')^m \right]
\]

where \( l \) is a constant given by:

\[
l = \frac{1}{\left( \sum_{j=t+1}^{n-k} (\xi^*_{wj})^2 + \sum_{j=1}^{t} \left( \xi^*_{wi'} c_j + \sum_{i=1}^{t} \xi^*_{wi'} e_{ij} \right)^2 \right)}
\]
The coefficient of \( z^2 \) is bounded by at most \( 2t(\epsilon_1^{1/16})^t \), hence using the previous lemma for a single iteration, the output produced here is a \((t + 1)\)th vector \( u_{t+1} \) such that:

\[
\langle u_{t+1}, u^* \rangle \geq 1 - \left( 2t(\epsilon_1^{1/16})^t \right)^{1/8} \geq 1 - (\epsilon_1^{1/16})^{t+1}
\]

for sufficiently small \( \epsilon_1 \) (relative to \( k \)).

\[\square\]

4.4 Algorithms

Using \textbf{LocalOpt}, we have an algorithm for factoring (Problem 1). To deal with the errors introduced by sampling and approximate local optima, we replace the Schwartz-Zippel step in \textbf{FindBasis} with the robust version in Lemma 12, where we set the error parameter of the robust Schwartz-Zippel lemma to be \((\eta - \|M^m\|_2 \epsilon)/n^m\). We will use the following robust version of the Schwartz-Zippel identity test.

\textbf{Lemma 12} (Robust Schwartz-Zippel). Let \( p \) be a degree \( m \) polynomial over \( n \) variables and \( K \) a convex body in \( \mathbb{R}^n \). If there exists \( x \in K \) such that \( |p(x)| > \epsilon(2cn)^m \), then for \( l \) random points \( s_i \),

\[
\Pr(\forall s_i : |p(s_i)| \leq \epsilon) \leq 2^{-l}.
\]

4.5 Robust Schwartz-Zippel lemma

\textit{Proof of Lemma 12} Let \( \mu \) denote the uniform measure over \( K \), by Corollary 2 of Carbery and Wright [12]:

\[
\max_{x \in K} |p(x)|^{1/m} \epsilon^{-1/m} \mu(\{x \in K : |p(x)| \leq \epsilon\}) \leq cn
\]

Consider our \( l \) samples – there are two possibilities:

1. \( \mu(\{x \in K : |p(x)| \leq \epsilon\}) \geq 1/2 \). In this case, we have \( |p(x)| \leq \epsilon(2cn) \) from the bound above.
2. \( \mu(\{x \in K : |p(x)| \leq \epsilon\}) \leq 1/2 \). Then, \( \Pr(\forall i |p(x_i)| \leq \epsilon) \leq 1/2^l \).

\[\square\]

We can of course amplify this probability by repeating the test (or simply taking \( l \) larger).

\textbf{Algorithm 5} FactorUnderGaussian

\textbf{Input:} Highest moment \( m \), distribution \( F \).

1: \( B \leftarrow \text{FindBasis}(m, F) \).
2: \( U \leftarrow \text{ExtendBasis}(m, F, B, \phi) \).
3: \text{return} \( U \)

\textit{Proof of Theorem 4} We choose \( \epsilon_1 \) (the first step local iteration) to be:

\[
(\epsilon_1)^{1/16} \leq \min\{\epsilon, \eta - \|M^m\|_2 \epsilon\}
\]

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where $\|M^m\|_2$ is the 2-norm (spectral norm) of the $m^{th}$ moment tensor. We take enough samples so that each estimated moment in $W$ is within $\min(e_1, \eta - \|M^m\|_2)/n^m$ of the Gaussian moment, and every moment in $V$ is off by at most $\min(e_1/2, \eta/2)$. In particular, note that all sampled gradients and Hessian matrices take a value which differ by no more than $e_1/2$ from their true values. Thus, we can simply absorb this as part of the error arising from local search. Also, this gives us an upper bound on sample complexity – the number of samples it takes to estimate the $m^{th}$ moments of a Gaussian distribution to accuracy $\epsilon$ in $\mathbb{R}^n$ is given as $C_m \epsilon^{-2} n^{m/2} \log n$ [22], which when evaluated becomes $n^{O(m)}$.

At each iteration of the algorithm, we run the Robust Schwartz-Zippel test $\log(k/\delta)$ times with Schwartz-Zippel parameter $\eta - \|M^m\|_2 \epsilon$. With probability at least $1 - \delta$, either each iteration produces a $u$, where $|E(F((x^T u)^m)) - \gamma_m| \geq \eta - \|M^m\|_2 \epsilon$ or we correctly deduce that there are no more directions whose moments differ from a Gaussian by more than $(\eta - \|M^m\|_2 \epsilon)/n^m$. In the latter case, by moment distinguishability, every vector in $P$, the minimally relevant subspace, has large projection on the basis $\{u_i\}$.

In the former case, we know that every unit vector in $\{u_i\}^\perp$ with projection at least $1 - \epsilon$ takes value which is bounded away from $\gamma_m$ by at least $\eta - \|M^m\|_2 \epsilon$, thus every such vector is still moment distinguishable. Applying Theorem 6 then, we sequentially generate a sequence of at most $k$ orthogonal $u_i$ such that:

$$|\langle u_i, \pi_V(u_i) \rangle| \geq 1 - (e_1)^{(1/16)}^i$$

We need to show that in addition $d_m(F, \hat{F}_U, \hat{F}_U^\perp) \leq \epsilon$. Let $F' = \hat{F}_U, \hat{F}_U^\perp$: the moment-distance between the true and sampled distributions differ by at most $e_1$, it suffices for us to prove that $d_m(F, F') \leq \epsilon$. To this end, we will apply the representation formula to $F'$ for some fixed unit vector $u$. As before, we have:

$$E_{F'}((x^T u)^m) = E_{F'}((x^T u_U)^m)) + E_{F'}((x^T u_U^\perp)^m) - \gamma_m \|u_U\|^m - \gamma_m \|u_U^\perp\|^m + \gamma_m$$

Hence, comparing with a similar expression for $E_F((x^T u)^m)$:

$$|E_{F'}((x^T u)^m) - E_F((x^T u)^m)| \leq |E_{F'}((x^T u_U)^m)) - E_F((x^T u_U)^m))| +$$

$$+ |E_{F'}((x^T u_U^\perp)^m) - E_F((x^T u_U^\perp)^m)|$$

$$+ |\gamma_m \|u_U\|^m - \|u_U\|^m| + \gamma_m \|u_U^\perp\|^m - \|u_U^\perp\||$$

Now, viewing $E_F((x^T u)^m)$ as the tensor applied to $u$, we see that we can bound these terms by the tensor spectral norm:

$$|E_{F'}((x^T u)^m) - E_F((x^T u)^m)| \leq (\|M^m\|_2 + m\gamma_m) \|u_U - u_V\| + (\|M^m\|_2 + m\gamma_m) \|u_U^\perp - u_V^\perp\|$$

By choice of $U$, we have $\|u_U - u_V\| \leq \epsilon$, and similarly for the orthogonal component, thus we have our bound.

We now apply these methods to learning the concept class $\mathcal{H}$ (Problem 2). After applying an isotropic transformation, $F$ will have Gaussian moments in every direction orthogonal to $V$, and hence the output basis of $\text{FindBasis}$ and $\text{ExtendBasis}$ returns only vectors in the $V$ subspace.

The proof of this algorithm is straightforward given the proof of the factoring algorithm under Gaussian noise and our robustness assumptions. We will use the following proposition on robust learnability (see e.g., [2]).
Algorithm 6 LearnUnderGaussian

Input: Highest moment \( m \), distribution \( F \).

1: \( B_1 \leftarrow \text{FindBasis}(m, F) \).
2: \( B_2 \leftarrow \text{FindBasis}(m, F^+) \) on the space orthogonal to \( B_1 \).
3: Alternately run \text{ExtendBasis} on \( F \) and \( F^+ \) to find a basis \( U \) of size at most \( k \). Extend this to a basis of dimension \( k \).
4: Draw sufficient samples \( S \) to learn \( H \) on this \( k \) dimensional subspace. Project \( S \) to span \( (U) \).
5: Learn \( H \) over \( U \).

Proposition 7 (VC dimension). Let \( H \) be a hypothesis class with VC dimension \( d \). Let \( \ell \in H \) be a subspace junta with relevant subspace \( V \), where \( \dim(V) = k \). Let \( U \) be a \( k \) dimensional subspace where \( \ell(\pi_U) \) labels a \( 1 - \epsilon \) fraction of points correctly. Then we can learn \( \ell \) with sample complexity 

\[
(1/\epsilon)^{c_2 d \log(1/\epsilon) + c_2 \log(2/\delta)}
\]

with probability at least \( 1 - \delta \).

Proof of Theorem 2. \( H \) is robust; by assumption there exists \( \epsilon' \) which is polynomial in \( \epsilon \) such that \( \epsilon' + g(\epsilon') \leq \epsilon/2 \). We will take this \( \epsilon' \) and will use the following \( \epsilon_1 \) for all our calls to \text{LocalOpt}:

\[
(\epsilon_1)(1/16)^k \leq \min\{\epsilon', \eta - \| M^m \|_2 \epsilon \}
\]

Under these parameters, the proof for the factoring steps of Lines 1-3 are as in \text{FactorUnderGaussian}. Thus with probability at least \( 1 - \delta \) we will obtain an orthonormal basis \( \{u_i\} \) where 

\[
|\langle u_i, \pi_V(u_i) \rangle| \geq 1 - (\epsilon_1)^{(1/16)^k}.
\]

By moment learnability, the set of \( \{u_i\} \) discovered is approximately a basis for \( P \), the minimal dimension relevant subspace. By our choice of \( \epsilon_1 \) above, we satisfy the robustness condition, i.e., \( \epsilon_1^{16k} \leq \epsilon' \), in which case only \( \epsilon/2 \) fraction of the points are mislabeled over span \( \{u_i\} \). Finally, we allow the remaining \( \epsilon/2 \) error to the learning algorithm, to obtain an output hypothesis which correctly labels \( 1 - \epsilon \) fraction of \( F \).

5 Moment distance

In our algorithms, we terminate if all remaining directions are Gaussian in the \( m^{th} \) moment (for some fixed \( m \)). We would like a guarantee that when we do this, that the random variable is in fact very close to being Gaussian. What follows is a set of results which quantify this idea. We first restrict ourselves to one random variable to introduce the analytic tools we need. In what follows, we use the following normalisation for our fourier transforms in \( \mathbb{R}^n \):

\[
\hat{f}(\xi) = \int e^{i\xi \cdot x} f(x) dx
\]

This implies that the Parseval/Plancherel theorem takes the following form:

\[
\int |f(x)|^2 dx = \frac{1}{(2\pi)^n} \int |\hat{f}(\xi)|^2 d\xi
\]

for \( f \in L^2(\mathbb{R}^n) \).

The core of the proof is the following statement, whose proof employs Fourier analytic techniques. We need the following standard theorem on characteristic functions (see for example [35]):
Theorem 8 (Characteristic functions). Let $\xi$ be a random variable with distribution function $F = F(x)$ and $\phi(t) = \mathbb{E}(e^{it\xi})$ its characteristic function. Let $\mathbb{E}(|\xi|^n) < \infty$ for some $n \geq 1$, then $\phi^{(r)}$ exists for all $r \leq n$ and

$$\phi^{(r)}(t) = \int (ix)^re^{itx}dF(x)$$

Moreover, we have an expression for the derivatives at 0:

$$\mathbb{E}(\xi^r) = \frac{\phi^{(r)}(0)}{i^r}$$

And finally we have the following Taylor series estimate with error:

$$\phi(t) = \sum_{r=0}^{n} \frac{(it)^r}{r!} \mathbb{E}(\xi^r) + \frac{(it)^n}{n!} \epsilon_n(t)$$

where the error term $\epsilon_n(t) \to 0$ as $n \to \infty$ and is bounded:

$$|\epsilon_n(t)| \leq 3\mathbb{E}(|\xi|^n)$$

Now:

Lemma 13 ($L^2$ distance from a Gaussian). Let $f \in L^2(\mathbb{R})$ be a probability density over $\mathbb{R}$ whose first $m$ moments match those of a standard Gaussian (whose probability density we will denote $g$). Suppose that the Fourier transform $\hat{f}$ satisfies a tail bound that $|\hat{f}(\xi)| < c/|\xi|$ for some $c > 0$, then:

$$\int_{\mathbb{R}} |f(x) - g(x)|^2 dx \leq \frac{c'}{m^{1/8}}$$

Proof. We will assume for the sake of simplicity that $m$ is even. By Parseval’s formula, we have:

$$\int |f(x) - g(x)|^2 dx = \frac{1}{\sqrt{2\pi}} \int |(f - g)(\xi)|^2 d\xi$$

Both $f$ and $g$ have tail bounds: $f$ by hypothesis, and $g$ because the Fourier transform of a Gaussian is still a Gaussian. Thus if we truncate the tails in an interval $[-L, L]$ where $L = m^{1/8}$:

$$\int_{\mathbb{R}/[-L,L]} |\hat{f}(\xi)|^2 d\xi \leq 2 \int_{L}^{\infty} \frac{1}{\xi^2} d\xi \leq \frac{4}{L}$$

The Fourier transform of a Gaussian is a Gaussian, and by applying a standard Gaussian tail bound [18]:

$$\frac{1}{\sqrt{2\pi}} \int_{x}^{\infty} e^{-t^2/2} dt \leq \frac{1}{x} \frac{e^{-x^2/2}}{\sqrt{2\pi}}$$
We can then combine these estimates using the triangle inequality:

\[
\int_{\mathbb{R}/[-L,L]} |f - g(\xi)|^2 d\xi \leq \int_{\mathbb{R}/[-L,L]} |\hat{f}(\xi)|^2 + |\hat{g}(\xi)|^2 d\xi
\]

\[
\leq \frac{6}{L}
\]

In the interval \([-L,L]\), we now apply Theorem 8:

\[
(\hat{f} - \hat{g})(\xi) = \sum_{k=0}^{m} \frac{E_f(x^k) - E_g(x^k)}{k!}(i\xi)^k + (\epsilon_f(t) - \epsilon_g(t)) \frac{(i\xi)^m}{m!}
\]

\[
= (\epsilon_f(t) - \epsilon_g(t)) \frac{(i\xi)^m}{m!}
\]

Now we can bound the integral:

\[
\int_{-L}^{L} |(f - g)(\xi)|^2 d\xi \leq \int_{-L}^{L} \left| (\epsilon_f(\xi) - \epsilon_g(\xi)) \frac{(i\xi)^m}{m!} \right|^2 d\xi
\]

\[
\leq 6 \left( \frac{E(x^m)}{m!} \right)^2 \int_{-L}^{L} t^{2m} dt
\]

\[
\leq \frac{6}{(2m/2)!} \frac{2L^{2m+1}}{2m+1}
\]

\[
\leq \frac{12}{2m+1} \exp \left( (2m + 1) \log(L) - m \log(2) - m \log \left( \frac{m}{2} \right) + m \right)
\]

\[
\leq \frac{c}{m} e^{-m}
\]

We can also give an approximate version of this theorem:

**Lemma 14 (Approximate moments).** Fix \(\epsilon > 0\), let \(f \in L^2(\mathbb{R})\) be a probability density over \(\mathbb{R}\) whose first \(m\) moments satisfy:

\[
\left| E_f(x^k) - \gamma_k \right| \leq \epsilon
\]

Suppose that the Fourier transform \(\hat{f}\) satisfies a tail bound that \(|\hat{f}(\xi)| < c/|\xi|\) for some \(c > 0\), then (for a standard Gaussian \(g\)):

\[
\int_{\mathbb{R}} |f(x) - g(x)|^2 dx \leq \frac{c'}{m^{1/8}} + c'' m^2 e^2 e^m
\]

**Proof.** We proceed as in the previous lemma. It suffices for us to bound the integral over the interval \([-L,L]\). We apply Cauchy-Schwarz for a termwise estimate.

\[
\int_{-L}^{L} \sum_{k=0}^{m} \frac{E_f(x^k) - E_g(x^k)}{k!}(i\xi)^k + (\epsilon_f(t) - \epsilon_g(t)) \frac{(i\xi)^m}{m!} \right|^2 d\xi
\]

\[
\leq m \int_{-L}^{L} \sum_{k=0}^{m} \left( \frac{E_f(x^k) - E_g(x^k)}{k!}, \xi^k \right)^2 + \left( (\epsilon_f(t) - \epsilon_g(t)) \frac{\xi^m}{m!} \right)^2 d\xi
\]
We can now partition the moments into powers of $2$, so consider the moments where $k \in [m/2^{i+2}, m/2^i]$; the integral of each contributing term is now:

$$\int_{-L}^{L} \left( \frac{\mathbb{E}_f (x^k) - \mathbb{E}_g (x^k)}{k!} e^{k} \right)^2 d\xi = \frac{2(\mathbb{E}_f (x^k) - \mathbb{E}_g (x^k))^2 L^{2k+1}}{(2k+1)k!}$$

$$\leq 2 \left( \frac{\mathbb{E}_f (x^k) - \mathbb{E}_g (x^k)}{k!} \right)^2 \exp \left( \frac{2k + 1}{8} \log(m) - k \log k + k \right)$$

$$\leq 2 \left( \frac{\mathbb{E}_f (x^k) - \mathbb{E}_g (x^k)}{k!} \right)^2 \exp \left( \frac{m/2^i}{4} + \frac{2}{m} \log(m) - \frac{m}{2^{i+2}} \log \left( \frac{m}{2^{i+2}} + \frac{m}{2^i} \right) \right)$$

$$\leq 2 \left( \frac{\mathbb{E}_f (x^k) - \mathbb{E}_g (x^k)}{k!} \right)^2 \exp \left( \frac{m}{2^{i-1}} \right)$$

Both of our lemmas so far in this section use a tail bound for the Fourier transform. One way to obtain such a tail-bound is to examine logconcave probability densities:

**Lemma 15** (Log-concave densities). Let $f : \mathbb{R} \to \mathbb{R}$ be a logconcave density which is isotropic and differentiable, then $|\hat{f}(\xi)| \leq 2/|\xi|$.

**Proof.** We start by bounding the magnitude of the Fourier transform by the integral of the derivative.

$$\hat{f}(\xi) = \int_{\mathbb{R}} e^{i \xi x} f(x) dx$$

$$= \int_{\mathbb{R}} \frac{1}{i \xi} e^{i \xi x} f(x) dx$$

$$= \int_{\mathbb{R}} \frac{1}{i \xi} e^{i \xi x} \frac{df(x)}{dx} dx$$

where the third line follows by integration by parts and noting that in the limit $f(x) \to 0$ as $x \to \pm \infty$. This allows us to bound $\hat{f}(\xi)$:

$$\left| \hat{f}(\xi) \right| \leq \frac{1}{|\xi|} \int_{\mathbb{R}} \left| f'(x) \right| dx$$

Let us now turn to logconcave densities. Since $f$ is logconcave, we can write it as $f(x) = e^{h(x)}$ where $h$ is concave. Because $f$ is a probability density, we must have $h(x) \to -\infty$ as $x \to \pm \infty$, in which case since $h$ is concave there exists a unique interval $[a, b]$ where $h(x)$ takes a maximum. This fully determines the sign of the derivative: $h'(x) = 0$ in this interval $h'(x) < 0$ for $x < a$ and $h'(x) > 0$ for $x > b$. The same signs pattern holds for $f'$, as multiplication by $e^{-h(x)}$ does not change the sign. We can now compute the integral by applying the fundamental theorem of calculus:

$$\int_{\mathbb{R}} |f'(x)| dx = \int_{-\infty}^{a} f'(x) dx + \int_{a}^{b} f'(x) dx + \int_{b}^{\infty} -f'(x) dx$$

$$= \lim_{t \to \infty} (f(a) - f(-t)) + (f(b) - f(a)) + (-f(t) + f(b))$$

$$= f(a) + f(b)$$

$$= 2f(a)$$
We now apply the following lemma \[31\], which yields the desired result.

**Lemma 16** (Upper bound on logconcave functions). *Let \( f \) be an isotropic logconcave density in one dimension, then \(|f(x)| \leq 1\).*

\[\square\]

Then as a corollary to Lemma 13:

**Corollary 17** (**\(L^2\) distance for logconcave densities**). *Let \( f : \mathbb{R} \rightarrow \mathbb{R} \) be an isotropic logconcave density whose first \( m \) moments match a Gaussian \( g \), then:*

\[
\|f - g\| \leq \frac{c}{m^{1/8}}
\]

**Proof.** First, consider the case when \( f(x) \) is differentiable. We already know that \( f \in L^1(\mathbb{R}) \); since \( f(x) \) is bounded by 1 (Lemma 16), then we have that \( f(x) \in L^2(\mathbb{R}) \) because \( f(x)^2 \leq |f(x)| \). We can now apply Theorem 13 with the tail bound guaranteed by Lemma 15.

For the case when \( f(x) \) is not differentiable, we can perturb by a small Gaussian random variable: let \( X \sim f \), and let \( Z \sim N(0,1) \) be an independent normal variable. Fix a parameter \( \tau \in [0,1] \):

\[
Y_{\tau} = (1 - \tau)X + \sqrt{2\tau + \tau^2}Z
\]

is isotropic. Moreover, since this the sum of two independent logconcave random variables, its density is also logconcave. Let \( h_1 \) denote the density of \((1 - \tau)X\) and \( h_2 \) the density of \( \sqrt{2\tau + \tau^2}Z \), then the density of our new random variable is given by:

\[
h_1 * h_2(x) = \int_{-\infty}^{\infty} h_1(x-t)h_2(t) dt
\]

The convolution of these two distributions is (infinitely) differentiable because \( h_2 \) is (infinitely) differentiable:

\[
\frac{d}{dx}(h_1 * h_2) = \left( \frac{d}{dx}h_1 \right) * h_2 = h_1 * \left( \frac{d}{dx}h_2 \right)
\]

Thus \( Y_\tau \) satisfies the hypotheses of Lemma 15 and we have a tail bound for \( Y_\tau \) as long as \( \tau > 0 \).

The first \( m \) moments of \( Y \) are also close to those of \( X \): if we compute the \( j^{th} \) moment for example:

\[
\mathbb{E}(Y_j^j) = \mathbb{E}\left( (1 - \tau)X + \sqrt{2\tau + \tau^2}Z \right)^j
\]

\[
= (1 - \tau)^j\mathbb{E}(X^j) + \sum_{i=1}^{j} \binom{i}{j}(1 - \tau)^j(\sqrt{2\tau + \tau^2})^{i-j}\mathbb{E}(X^i)\mathbb{E}(Z^{i-j})
\]

Thus we can pick \( \tau \) small enough so that:

\[
|\mathbb{E}(Y_j^j) - \mathbb{E}(X^j)| \leq \epsilon
\]
for any $\epsilon > 0$. In the proof of Lemma $13$ then, instead of the moment differences from the first $m$ terms of the characteristic function being $0$, we can make them arbitrarily small by choosing smaller $\tau$. Thus we have the conclusion of Lemma $13$ for $Z$. To conclude, we note that:

$$\lim_{\tau \to 0} \|h_1 * h_2 - f\|_2 = 0$$

in which case, taking $\tau$ small enough allows us to apply the triangle inequality to:

$$\|f - g\| \leq \|f - h\| + \|h - g\|$$

We also need a lemma to convert our $L^2$ estimates to $L^1$ estimates. This is not general in possible, but since logconcave functions have exponential tailbounds:

**Lemma 18** ($L^2$ to $L^1$). Let $f, g : \mathbb{R} \to \mathbb{R}$ isotropic logconcave densities such that for some $m > 0$ that:

$$\int |f(x) - g(x)|^2 dx \leq \frac{1}{m}$$

then:

$$\int |f(x) - g(x)| dx \leq \frac{c \log(m)}{\sqrt{m}}$$

for some absolute constant $c > 0$.

**Proof.** Fix $L = (\frac{1}{m}) \log(m)$, then as before:

$$\int |f(x) - g(x)| dx = \int_{|x| \leq L} |f(x) - g(x)| dx + \int_{|x| > L} |f(x) - g(x)| dx$$

We can now use tail bound for logconcave functions over the tail [21], in particular, for isotropic logconcave random variables $X$ in $\mathbb{R}^n$, we have (for some fixed absolute constants $c, C > 0$):

$$\Pr \left( \|x\| - \sqrt{n} \geq t \sqrt{n} \right) \leq C \exp \left(-cn^{\frac{1}{2}} \min(t, t^3) \right)$$

In one dimension, this shows that the integral of our tail is bounded by $C/m$ (after application of triangle inequality). Now inside the interval $[-L, L]$, we will apply the Cauchy-Schwartz inequality:

$$\int_{[-L, L]} |f(x) - g(x)| dx \leq \left( \int_{[-L, L]} |f(x) - g(x)|^2 dx \right)^{1/2} \left( \int_{[-L, L]} 1 dx \right)^{1/2} \leq \frac{\sqrt{2}}{c\sqrt{m}} \log(m)$$

**Proof of Theorem 7.** The proof follows from Lemma 14, Corollary 17 and Lemma 18, noting that the technique of Corollary 17 can be applied to Lemma 14 in the same way as Lemma 32.
6 Applications

In this section, we give some applications of our general theorems and we some explicit bounds for moment-learnable triples and the running time of our algorithms on these triples. We make explicit in our analysis the three key contributions to runtime – how many moments are required, how efficiently these moments can be sampled, and how efficiently the hypothesis can be learned in the $k$-dimensional relevant subspace.

6.1 Moment estimation

In this section, we highlight some further consequences and subtleties of using moments in algorithms. The use of moments is a very natural way of studying random variables. For example, the inequalities of Markov, Chebyshev and Chernoff are statements about the relationship between a finite sequence of moments and the tail of a distribution. If we consider an infinite sequence of moments, often these will determine the distribution uniquely (the moments problem).

One of the critical terms in the runtime given in our main theorems is $C_{F}(m, \epsilon)$: the sample complexity of approximating the $m^{th}$ moment tensor of distribution $F$ to within accuracy (in the moment metric above). The competitiveness of our algorithm with other learning algorithms depends on the number of moments we need (ie the previous section), and the number of samples we need to attain the required accuracy. This latter problem is well-studied, and there is an impressive body of literature surrounding it. In particular, when $m = 2$, the problem is of interest to random matrices community, who have provided strong bounds in a number of important cases. We will provide a brief overview of these results, but this by no means is intended to be a comprehensive survey of the literature! When the distribution $F$ is isotropic and almost surely supported in a ball of radius $O(\sqrt{n})$, Rudelson [35] gave a very strong bound on $C_{F}(n, \epsilon)$ to achieve the following guarantee:

$$\mathbb{E}\left(\left\|\frac{1}{N} \sum_{i=1}^{N} x_{i}x_{i}^{T} - I\right\|\right) \leq \epsilon.$$  

Rudelson required only $O(n \log(n))$ samples when $F$ is almost surely supported on a ball of radius $O(\sqrt{n})$, and where the constant is dependent on $\epsilon$. Adamczak et al. [1] were able to improve this bound of $O(n)$ samples. Their assumptions were support on a ball of radius $O(\sqrt{n})$ as before, and a subexponential moment condition:

$$\sup_{\|v\|=1} \mathbb{E}\left(\left\| (x^{T}v)^p \right\| \right)^{1/p} = O(p)$$

As an application, they showed that logconcave distributions satisfy these assumptions, and thus their covariance matrices can be sampled very efficiently. Subsequent work by Vershynin and collaborators [39, 44] has broadened the class of efficiently samplable covariance matrices to distributions where $2 + \epsilon$ moments exist and also to distributions where the $m^{th}$ moment is bounded by $K^m$ for some constant $K$.

Finally, in the setting of higher moments, there is the result of Guedon and Rudelson [22], which gives the sample complexity of sampling for higher moments of logconcave distributions. Their result is that $O(n^{m/2} \log(n))$ samples are necessary to approximate moments in all directions up to an $1 + \epsilon$ factor. In particular, this leads to the observation that explicitly computing a sample moment tensor from $n^{m/2}$ samples is actually less efficient than simply storing the points,
computing the inner products to the appropriate powers and summing. This last result is used in our applications in Section 6 as it allows us to handle many distributions efficiently, including Gaussians and uniform distributions over convex bodies.

6.2 Robust learning

For learning over a $k$-dimensional subspace, we have the following proposition:

**Proposition 9** (VC dimension). Let $\mathcal{H}$ be a hypothesis class with VC dimension $d$. Let $\ell \in \mathcal{H}$ be a subspace junta with relevant subspace $V$, where $\dim(V) = k$. Let $U$ be a $k$ dimensional subspace where $\ell(\pi_U(x))$ labels a $1 - \epsilon$ fraction of points correctly. Then we can learn $\ell$ with sample complexity $(1/\epsilon)^{c_2d\log(1/\epsilon)+c_2\log(2/\delta)}$ with probability at $1 - \delta$.

**Proof.** To come up with a hypothesis over $U$, we take a new set of samples $S$ of size $m$ and project them onto $U$. By robustness of $\mathcal{H}$ under $F$, we know that $\Pr(\ell(\pi_U(x)) = \ell(x)) \geq 1 - \epsilon$. Then we guess the correct labels by trying all relabelings of subsets of size $\epsilon m$. One of these relabelings will give us a labeling consistent with $\ell$ viewed as a function of the $k$-coordinates in $U$. For each relabeling we attempt to learn the labeling function. On the correct relabeling, we can learn $\ell$ to with at most $\epsilon$ fraction of errors. By the theorem above, our total error over $\mathbb{R}^n$ is $2\epsilon$.

To bound $m$, we apply an idea from [9] via a slight extension (Theorem 5 of [2]). The required bound is $m \geq (32/\epsilon) \log(C[m]) + (32/\epsilon) \log(2/\delta)$ where $C[m]$ is the maximum number of distinct labelings obtainable using concepts in $\mathcal{H}$ over $\mathbb{R}^k$. In particular, we have $C[m] \leq \sum_{i=0}^{d} \binom{m}{i}$, whence $C[m] \leq m^d$. A computation reveals that $m \geq c(d/\epsilon) \log(1/\epsilon) + (d/\epsilon) \log(2/\delta)$ suffices. The number of relabelings is $\binom{m}{\epsilon m}$, which is upper bounded by $(m/\epsilon)^{\epsilon m} \leq (1/\epsilon)^{c_2d\log(1/\epsilon)+c_2\log(2/\delta)}$. \qed

As mentioned previously, we can view the work of [42] as a specialization of our algorithms to the $j = m = 2$ case in FindBasis. We give examples here where the second moment does not suffice, and we must use higher moments to resolve the relevant subspace $V$. Our examples are: (1) hyperrectangles (cuboids) in balls, (2) subsets of balls, and (3) concepts which have compact support. In all our examples, the algorithm used is LearnUnderGaussian. We will prove that we can find the relevant subspaces by running FindBasis on either the full distribution or distribution conditioned on positive labels (the “positive” distribution).

We use the uniform distribution over a ball in $\mathbb{R}^k$ in the relevant subspace. We need the following elementary fact.

**Claim 19** (Isotropic balls). Let $F$ be the uniform distribution (with density $\rho$) over $B_R(0) \subset \mathbb{R}^n$ where $R = \sqrt{n+2}$, then $E((x^T u)^2) = 1$ for any unit vector $u$.

By a hyperrectangle, we refer to a region of space which is the Cartesian product of closed intervals i.e. $S = [a_i, b_i] \times \cdots \times [a_k, b_k] \subset \mathbb{R}^k$.

**Application 1** (Hyperrectangles in balls). Let $F = F_VF_W$ where $F_V$ is a uniform distribution over a ball $B$ and $k = \dim(V)$, $F_W$ is any Gaussian over $n - k$ dimensions. Let $S \subset B$ denote a (hyper)rectangle in $V$. Take the hypothesis class $\mathcal{H} = \{\chi_S(\pi_V)(x) : S \subset B\}$ to be the set of functions which assigns positive labels to points whose projection to $V$ lies in the interior of rectangle $S$.

**Proposition 10.** The triple $(k, F, \mathcal{H})$ as defined in Application 7 is $(4, 6/(5k))$ moment-learnable with time and sample complexity $\text{poly}(k, 1/\epsilon) + C_{k, \epsilon} n^2$. 34
Proof. Without loss of generality, we may assume that \( B = B_{\sqrt{n/2}}(0) \) after isotropic transformation, and that the Gaussian over \( F_W \) is a standard \( n \)-dimensional Gaussian. Furthermore, we may assume that \( S \) is centered on the origin as well (i.e. we apply Lemma 2 to the positively labeled points).

Suppose we now run \texttt{LearnUnderGaussian} on the positively labeled samples. We start with the second moment \((r = 2)\) in our algorithm \texttt{FindBasis}: the second moments of a uniform distribution over a rectangle are fully determined by the second moments along the axes of the rectangle. In particular, \texttt{FindBasis} using the second moments will simply give us every axis of the rectangle where the second moment is not 1. A simple calculation of the moments of a uniform distribution over a rectangle along axis \( x_i \) where the rectangle has length \( 2S_i \) gives:

\[
E(x_i^2) = \int_{-S_i}^{S_i} x_i^2 \frac{1}{2S_i} dx_i = \frac{S_i^2}{3}.
\]

Thus, using the second moment will give us all the axes of our hyperrectangle except where the rectangle has length \( 2S_i = 2\sqrt{3} \). Projecting orthogonally to these axes, we now consider the third moments \((r = 3)\): the third moment of our uniform rectangle is clearly 0 in every direction by symmetry of the rectangle. Thus, we turn to the fourth moment – note that fixing \( S_i = \sqrt{3} \) fixes the fourth moment along each axis of the rectangle, in particular:

\[
E(x_i^4) = \int_{-S_i}^{S_i} x_i^4 \frac{1}{2S_i} dx_i = \frac{9}{5}.
\]

Unfortunately, the equality of the fourth moment along the axes of a rectangle does not necessarily imply the same fourth moment in every direction. However, iterating Lemma 3 allows us to bound the fourth moments away from the fourth moment of a Gaussian \( \gamma_4 = 3 \):

\[
E((x^T u)^4) = \left( \frac{9}{5} - \gamma_4 \right) \sum_{i \in R'} u_i^4 + \gamma_4
\]

where the sum is taken over directions corresponding to axes where \( S_i = \sqrt{3} \). Now by applying the Lagrangian style techniques of Lemma 4 we can bound this by:

\[
E((x^T u)^4) \leq \gamma_4 - \frac{6}{5k}
\]

Thus, we have our moment learnability using only the fourth moment! Now that we have the relevant subspace \( V \), we can simply learn our rectangle in a dimension \( k \) space, which takes \( \text{poly}(k) \) time. Moreover, note that since all the distributions are logconcave, we can apply the moment sampling results of Guedeon and Rudelson mentioned in Section 6.1 – in particular, we can take the number of samples required to be \( C_F(m, \epsilon) = C_c n^2 \). Thus this gives a final runtime of \( \text{poly}(k) + C_{k,c} n^2 \) where \( C_{k,c} \).

The key point here is that we have very low polynomial dependence in \( n \). This conforms well with our model where we think of \( k \) as being small compared to \( n \). We can, in fact, prove a stronger result — we can always find the relevant subspace if \( F_V \) is a uniform distribution over a ball:

**Application 2** (Uniform distributions over balls). Let \( F = F_V F_W \) where \( F_V \) is a uniform distribution over a ball \( \mathbb{B} \) and \( k = \dim(V) \), \( F_W \) is a Gaussian. Let \( \mathcal{H} \) be a robust hypothesis class which we can learn with complexity bounded by \( T(k, \epsilon) \).
**Proposition 11.** The triple \((k, F, H)\) as defined in Application \(2\) is \((4, \Omega(1))\) moment-learnable, with the time and sample complexity bounded by \(T(k, \varepsilon) + C_{k, \varepsilon} n^2\).

**Proof.** We will examine what happens when we run **FindBasis** on the full distribution (as opposed to the positive distribution in the previous example). We compute the fourth moment of a ball of radius \(R = \sqrt{n+2}\). For simplicity, we will assume that \(k = 2l + 1\) for some positive integer \(l\) if \(k\) is odd:

\[
\mathbb{E}(x_1^4) = \int_{B_R(0)} x_1^4 \rho dx = \int_{-R}^R \int_{B_{R^2 - x_1}^{k-1}(0)} x_1^4 \rho dx_2 \cdots dx_k dx_1 \]

\[
= \frac{1}{\operatorname{vol}(B_R^k(0))} \int_{-R}^R x_1^4 \operatorname{vol}(B_{R^2 - x_1}^{k-1}(0)) dx_1
\]

\[
= \frac{\operatorname{vol}(B_R^{k-1}(0))}{\operatorname{vol}(B_R^k(0))} \int_{-R}^R x_1^4 \left(1 - \frac{x_1^2}{R^2}\right)^r dx_1
\]

We first examine the volume ratio: using the recurrence:

\[
\operatorname{vol}(B_R^{k}(0)) = 2\pi R^2 \frac{k}{k-1} \operatorname{vol}(B_R^{k-2}(0))
\]

and unrolling the recurrence, we have:

\[
\frac{\operatorname{vol}(2l)}{\operatorname{vol}(2l+1)} = \frac{(2l+1)!!}{2R(2l)!!} = \frac{1}{2R} \frac{(2l+2)!!}{(l+1)!2^{l+1}l!} = \frac{1}{2R} \frac{(2l+2)!!}{(l+1)!!2^{2l+1}}
\]

Applying Stirling’s approximation, we have:

\[
\frac{\operatorname{vol}(2l)}{\operatorname{vol}(2l+1)} = \frac{1}{2R} \sqrt{\frac{2\pi(2l+2)}{2^{2l+1}l!}} \left(\frac{2l+2}{e}\right)^{2l+2} \left(\frac{e}{l+1}\right) \left(\frac{e}{l+1}\right)^r
\]

\[
= \frac{1}{2R} \frac{1}{\sqrt{\pi l}} e \left(\frac{l+1}{l}\right)^r (l+1)
\]

\[
= \frac{1}{R\sqrt{\pi}} \frac{l+1}{\sqrt{l}}
\]

\[
= \frac{1}{\sqrt{\pi}} \left(1 + \frac{1}{\sqrt{l(l+2)}}\right)
\]

Returning to the integrand, we can simplify it somewhat:

\[
\int_{-R}^R x_1^4 \left(1 - \frac{x_1^2}{R^2}\right)^r dx_1 = 2 \int_0^R x_1^4 \left(1 - \frac{x_1^2}{R^2}\right)^r dx_1
\]
By explicitly taking the integral (using a computer algebra system), we have:

\[
\int_0^R x^4 \left( 1 - \frac{x^2}{R^2} \right) dx = \frac{3\sqrt{\pi}(2l + 3)^{5/2}\Gamma(l + 1)}{8\Gamma(l + 7/2)}
\]

where \(\Gamma\) here is the usual gamma function. The behavior of this function is as follows:

\[
\lim_{l \to \infty} \frac{3\sqrt{\pi}(2l + 3)^{5/2}\Gamma(l + 1)}{8\Gamma(l + 7/2)} = 3\sqrt{\frac{\pi}{2}}
\]

Moreover, the function is monotonic increasing for \(l > 0\), and takes on the value \(56\sqrt{7}/45\) at \(l = 2\). Thus, combining these facts with the estimate of the volume ratios, we can see that the fourth moment of a ball is bounded away from the fourth moment of a standard Gaussian by a constant, hence we can take \(\eta = \Omega(1)\). Once we have the relevant subspace \(V\), we can project the samples to \(V\) and learn in time \(T(k, \epsilon)\). The runtime in this case is \(T(k, \epsilon) + C_{k, \epsilon}n^2\).

As a specialization, when the positive examples are determined by a convex subset of the unit ball, \(T(k, \epsilon) \leq (k/\epsilon)^O(k)\). In a \(k\)-dimensional subspace, we can learn a convex subset of the ball by simply taking the convex hull of \((k/\epsilon)^O(k)\) random positive points. From the classical approximation theory of convex bodies \([34]\), we obtain an approximation to the true convex body to within relative error \(\epsilon\), giving total runtime \((k/\epsilon)^O(k) + C_{k, \epsilon}n^2\). This complements \([12]\) which provides a PCA-based algorithm for learning convex bodies when the distribution in the relevant subspace is also Gaussian. In that paper, it is mentioned that standard PCA fails if the full distributions is not a Gaussian.

We now present an example that relies on boundedness – either of the full distribution in the relevant subspace, or the positive distribution. This rather general result uses relatively many moments.

**Application 3** (Compact distribution in relevant subspace). Let \(F = F_V F_W\) where \(F_W\) is any Gaussian over \(n - k\) dimensions. Take \(\mathcal{H}\) to be a robust hypothesis class learnable with complexity \(T(k, \epsilon)\). Assume that either \(F_V\) or \(\mathcal{H}\) has its support contained in \(B_{g(k)}(0)\).

**Proposition 12.** The triple \((k, F, \mathcal{H})\) described in Application [3] is \((g(k), \Omega(1))\) moment-learnable with complexity \(T(k, \epsilon) + C_{k, \epsilon}n^{O(g(k)^2)}\).

**Proof.** Suppose we run **FindBasis** on the full distribution or the positive distribution, whichever is contained in a ball of radius \(g(k)\). Consider the relevant subspace. If we fix some even moment \(m\) then we can give explicit bounds on the moments:

\[
\mathbb{E}((x_t)^m) \leq g(k)^m.
\]

On the other hand, the even moments of a Gaussian are given by \((m - 1)!! = m!/(m/2)!!2^{m/2}\) which grows much more rapidly. If we take logarithms on both sides, then we can find \(m = m(k)\) such that:

\[
m \log(g(k)) \leq \log \left( \frac{m!}{(m/2)!!2^{m/2}} \right)
\]
Applying Stirling’s approximation yields:

\[
m \frac{m}{2} \log(g(k)^2) \leq m \log(m) - m - m \frac{m}{2} \log \left( \frac{m}{2} \right) + m - m \frac{m}{2} \log(2) \\
\leq m \frac{m}{2} \log(m) - m \frac{m}{2}
\]

So if we pick \( m = 2g(k)^2 \), then the difference in the moments should be \( \Omega(1) \). Thus, simply running \texttt{FindBasis} on the full distribution will allow us to recover the relevant subspace, at which point we can learn \( \mathcal{H} \) in \( \mathbb{R}^k \) (doable in time \( T(k) \)). It remains to prove that we can sample the first \( 2g(k)^2 \) moments of a bounded distribution efficiently: since it is bounded, all moments exist. In particular, if we require \( 2g(k)^2 \) moments, then the \( 4g(k)^2 \) moment is bounded by \( g(k)^{4g(k)^2} \). Then by applying Chebyshev’s inequality, we see that we need at most \( g(k)^{O(g(k)^2)} \) samples in the relevant subspace. The overall runtime for this algorithm is then \( T(k, \epsilon) + C_{k, \epsilon} n^{O(g(k)^2)} \).

\[ \Box \]

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