Modeling ringdown II: non-precessing binary black holes

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The aftermath of binary black hole coalescence is a perturbed remnant whose gravitational radiation rings down, encoding information about the new black hole’s recent history and current state. It is expected that this ringdown radiation will be composed primarily of Kerr quasi-normal modes, and thereby enable tests of general relativity. Here, the author presents the first ringdown model that captures the amplitude and relative phase of the dominant and subdominant quasi-normal modes for non-precessing binary black holes systems. For the first time it is noticed that the dominant mode’s excitation is a remarkably simple function of system parameters, suggesting that an analytic treatment may be within reach. Application of the model to parameter estimation indicates that some mode amplitudes and relative phases are in general very difficult to constrain, while others are well constrained, even for low SNR signals. GW150914 is discussed as an example case.

Introduction – Recent direct detections of gravitational waves by LIGO and Virgo bring the possibility of testing General Relativity’s (GR’s) detailed predictions [1–5]. With prospective detectors such as LIGO-India [6], KAGRA [7], Einstein Telescope (ET) [8] and LISA [9], it is likely that there will be many high signal-to-noise ratio (SNR) detections, allowing for increasingly stringent tests of GR [10–14]. To this end, the final moments of binary black hole coalescence are of particular interest. Shortly after two black holes (BHs) merge, the remnant is expected to be a perturbed BH whose gravitational radiation rings down with frequencies predicted by Teukolsky’s equations [15, 16]. In particular, classical linear perturbations of the Kerr spacetime induce transient radiative modes, the most dominant of which are the exponentially damped and oscillatory Quasi-Normal Modes (QNMs) [17–19]. The damped ringing of these modes is colloquially named ringdown [17].

It is expected that the spatiotemporal dependence of each QNM is determined by the remnant’s mass and spin, which in turn determine the matter-free background metric (e.g. [20]). Consequently, direct observation of two or more QNMs has been linked to testing the No-Hair Theorem, thereby quantifying the underlying spacetime geometry relevant for the source [21–23]. Yet just as the ringing of a bell depends not only on the bell’s shape, but how hard it was struck, the excitation amplitude of each QNM is influenced by the pre-merger binary system. Thus direct observation of more than one QNM may confer information about the remnant spacetime, as well detailed information about the progenitor system.

With the advent of Numerical Relativity (NR) simulations, it has been demonstrated that initial binary parameters can be mapped to QNM excitation amplitudes, resulting in a model for ringdown waveforms [24–26]. While initial work focused on remnant BHs with spin, it was pointed out in Ref. [27] that results were represented in an angular basis appropriate for non-spinning BHs. In particular, it was shown that a complete model for QNM amplitudes and relative phases could instead be constructed in the basis of spheroidal harmonics, the angular eigenfunctions of Teukolsky’s equations. However, this work was limited to initially non-spinning BH binaries, while generic astrophysical binaries are expected to spin and precess.

In this paper, a model of QNM excitations for spinning but non precessing BH binaries is presented. Because this model outputs the expected ringdown of non-precessing binary black hole (BBH) systems, we will refer to it as RDNP. Its primary use is expected to be in testing GR before and during LIGO’s third observing run (O3) [13, 14, 21, 22]. However, as there is presently no complete gravitational wave signal model which explicitly captures spheroidal harmonic information in ringdown, the results of this work may also be used to construct inspiral-merger-ringdown models with more accurate post-mergers [28–31]. While there is a focus here on ground based detectors, the primary results of this work apply to proposed space based detectors such as LISA [23].

Methods – Starting from the Weyl scalar ψ4(t) in geometric units (G = c = 1), strain QNM amplitudes are calculated and then modeled via the following workflow. 101 nonprecessing simulations from the Georgia Tech catalog are used [32]. Among them, 42 are nonspinning, 31 have different dimensionless spins on each BH, and 28 have equal spin on each BH. Mass ratios vary between 1:1 and 1:15, and component spins vary between -0.8 and 0.8. For each simulation, t = 0 is defined to locate the peak of the dominant spin -2

![Figure 1. Time (Top Panel) and frequency domain (Bottom Panel) inspiral-merger-ringdown and ringdown waveform (highlighted blue) for a m1/m2 = 1.2 nonspinning binary black hole. In both cases, all multipoles with ℓ ≤ 5 are used to represent the gravitational radiation at an orientation of (i, φ) = (14π/5, 0). The bottom panel’s ringdown waveform was obtained by directly Fourier transforming the top panel’s ringdown portion.](image-url)
spherical harmonic multipole moment, $|h_{22}|$, where

$$h_{22}(t) = d_{22} \int_{\Omega} -Y_{22}^* (\theta, \phi) h(t, \theta, \phi) d\Omega.$$  \hspace{1cm} (1)

In Eq. (1), $d_{22}$ is the source’s luminosity distance, $M_i = M_1 + M_2$ is the sum of the component BH masses, $M_1 = 1$ by convention, $-Y_{22}^*$ denotes the complex conjugate of $-Y_{22}$, and $h = h_{1} - i h_{2}$ is the complex strain [33]. The choice of $t = 0$ used here differs from the Ref. [34], where $t = 0$ is defined to be the location of $|h_{22}|$’s peak. Ringdown is conservatively defined to begin at $T_0 = 20M$. This is consistent with recent bounds within the calibration set [35]. Fig. (1) shows this convention for a 1:2:1 mass-ratio non-spinning NR simulation. Here the waveform amplitude prior to $20M$ is seen to significantly differ from damped exponential behavior, and the frequency domain representation of ringdown is generally not a good approximation of the inspiral-merger-ringdown waveform’s high frequency region, meaning that ringdown may only be well defined in the time domain. The termination of ringdown is defined at $T_1$, located just before the waveform becomes dominated by numerical noise.

For each $\psi_{l\ell m} = -\frac{d^2}{dt^2} h_{l\ell m}$, the region defined by $[T_0, T_1]$ is modeled as a sum of QNMs. For this purpose, expert knowledge and stepwise regression are used [27,36]. Specifically, given the final mass and dimensionless spin of each remnant BH [37–39], each $\psi_{l\ell m}$ is modeled as a linear superposition of QNMs with $(\ell, m, n)$ such that $\ell - \ell' \in [0,1]$, $m \in \bar{m}$ and $n \in [0,1]$: \hspace{1cm} (2)

In Eq. (2), $m = \bar{m}$ results from the spherical and spheroidal harmonic’s orthogonality in $m$. The constraints on $\ell$ and $n$ reflect approximate orthogonality in $\ell$ of the spheroidal harmonics [40]. The complex QNM frequency, $\bar{\omega}_{l\ell m}$, is given by $\bar{\omega}_{l\ell m} = (\omega_{l\ell m} + i \tau_{l\ell m})/M_f$, where the numerator is the dimensionless QNM frequency via Ref. [15], and $M_f$ is the remnant BH’s mass. The complex valued $A_{l\ell m}$ is found via matrix least-squares fitting in the frequency domain [27]. This is found to be less sensitive to noise than naive time domain least-squares fitting, where the associated Vandermonde matrix (i.e. with elements $e^{i\bar{\omega}_{l\ell m} t}$) is pseudo-inverted, and then applied to $\psi_{l\ell m}(t)$. Following the initial fit, a backward greedy process is applied to counter over-modeling [41]. Given the resulting QNM content, the fit is re-applied over fitting regions with $T_0 \rightarrow T_0'$ on $[T_0, T_0 + 10 M]$. This enables the identification of incidental QNMs which do not satisfy time translational symmetry (i.e. $A_{l\ell m}$ varies significantly with $T_0'$, when it should be constant). While not physical, these incidental QNMs capture information that can be attributed either to the pre-QNM regime, or to time dependent numerical noise not of interest for modeling. This cross-validation step enables QNMs of true interest to be identified. In particular, it is found that the overtones, while inconsistent with noise, do not display time translational symmetry. The median of $A_{l\ell m}$ is stored over the cross validation region. By convention, the phase of each $A_{l\ell m}$, $\arg(A_{l\ell m})$, is written such that the binary’s orbital phase is measured relative to the recoil direction at $t = T_0$ [27]. For each $b^{th}$ NR simulation with initial parameter list $\lambda_b = [M_1, M_2, \chi_{1 b}^{(3)}, \chi_{2 b}^{(3)}]$, the above results in a chart $(\lambda \rightarrow A_{l\ell m})_b$. Here, the dimensionless spin, $\chi_{f}^{(3)}$, is the $f^{th}$ BH spin’s z-component divided by $M_f^2$. With guidance from Post-Newtonian (PN) theory, the symmetric mass ratio, mass difference, symmetric and anti-symmetric spins are considered, i.e. $\lambda_b = [\eta, \delta, \chi_{1 b}, \chi_{2 b}]$, where, $\eta = M_1 M_2 / M_f^2$; $\delta = \sqrt{1 - 4\bar{\eta}} \chi_{1 b} = (M_1 \chi_{1 b}^{(3)} + M_2 \chi_{2 b}^{(3)}) / M_f$ and $\chi_{a b} = (M_1 \chi_{a b}^{(3)} + M_2 \chi_{b a}^{(3)}) / M_f$.

The desired QNM amplitude model, $A_{l\ell m}(\lambda)$, interpolates over $(\lambda \rightarrow A_{l\ell m})_b$. Each $A_{l\ell m}(\lambda)$ is found to be well represented by a PN-like multinomial series: $A_l(\lambda) = \sum a_{\ell m} C_{\ell m}(\lambda)$, where each $C_{\ell m}(\lambda)$ represents a unique product of $\lambda$’s elements to some power (e.g. $C_{\ell m} \in \{1, \eta, \chi_{1 m}, \chi_{2 m}, \eta, \ldots, \chi_{k m}\}$, and $k$ encodes $(\ell, m, l, n)$. From this perspective, determining each $A_l(\lambda)$ is equivalent to finding each $a_{\ell m}$. As this problem is linear in $C_l(\lambda)$, $a_{\ell m}$ are determined using matrix least-squares regression in the same modulated step-wise manner applied to Eq. (2).

$$A_{2220} = \eta \left(-0.6537 \chi_{1} + (-4.0071)\right)$$

$$A_{21210} = \eta \left(3.4884 e^{2.6631} \delta + (0.8011 e^{5.7070}) \chi_{a}ight)$$

$$A_{33330} = \eta \left(5.284 e^{5.5224} \eta \delta + (1.1774 e^{0.2454}) \chi_{a} \delta + (0.6260 e^{5.3457}) \chi_{a} \eta \right)$$

$$A_{33330} = \eta \left(2.4612 e^{2.9980} \delta + (1.6030 e^{0.6655}) \delta^2 \right)$$

$$A_{22320} = \eta \left(2.5707 e^{4.1427} \eta + (9.4216 e^{0.8076}) \eta^2 \right)$$

$$A_{32320} = \eta \left(5.973 e^{3.1816} \chi_{a} \right) + (2.014 e^{0.9043}) \chi_{a}^2$$

$$A_{32220} = \eta \left(1.3407 e^{2.9466} \eta + (0.0717 e^{5.5304}) \right)$$

$$A_{32220} = \eta \left(0.1061 e^{0.6432} \chi_{a}^2 + (0.9894 e^{2.9294}) \chi_{a} \delta \right)$$

$$A_{44440} = \eta \left(1.3284 e^{2.6831} \delta^2 + (1.1619 e^{0.4142}) \delta^3 \right)$$

$$A_{33330} = \eta \left(1.2790 e^{4.7226} \chi_{a} \chi_{a} \delta + (1.2387 e^{0.5616}) \chi_{a} \chi_{a}^3 \delta \right)$$

$$A_{44330} = \eta \left(0.0411 e^{2.6441} \chi_{a} + (0.0486 e^{3.2085}) \chi_{a}^3 \right)$$

$$A_{44330} = \eta \left(0.5665 e^{3.3992} \delta + (0.1457 e^{4.7476}) \delta \right)$$

$$A_{44330} = \eta \left(0.8239 e^{1.8174} \chi_{a} \right) + (0.0507 e^{1.7495}) \chi_{a} \delta$$

$$A_{44330} = \eta \left(0.9806 e^{0.6029} \delta^3 + (10.1676 e^{6.6185}) \delta \right)$$

Results – Results for each $A_{l\ell m}(\eta, \delta, \chi_{1 b}, \chi_{2 b})$ are shown in Eqs. (3-10). Residuals for each fit are found to be approximately gaussian, zero centered, with an average standard deviation of 4.66% in both real and imaginary parts. As in Ref. [27] finite radius effects were found to be much smaller, on the order of 0.01% in both real and imaginary parts. These results enable the evaluation of RDIP according to

$$h_{l\ell m}(t, \theta, \phi) = \frac{GM}{d_l c^3} \sum_{l\ell m} A_{l\ell m} e^{i \omega_{l\ell m} t} -2Y_{l\ell m}(t, \theta, \phi).$$  \hspace{1cm} (11)

Eq. (11) is limited to the indices present in Eqs. (3-10) with the exception that non-precessing symmetry allows $\bar{m} \leq 0$ terms to be derived from $h_{l\ell m} = (-1)^l h_{l\ell m}$. [42].

Verification that calibration points are consistent with linear BH perturbation theory is taken as a prerequisite result.
As noted in Ref. [43], this can be quantified by comparing numerical results with BH perturbation theory’s analytic predictions. Fig. (2)’s bottom left panel shows such a comparison derived from the presence of the \((\ell, m, n) = (2, 2, 0)\) QNM within the \((\ell, \tilde{m}) = (3, 2)\) spherical moment, \(\psi_{32}(t)\). Namely, it can be shown that each \(A_{\ell\tilde{m}n} = \mu_{\ell\tilde{m}n}\mathcal{A}_{\ell\tilde{m}n}\), where \(\mu_{\ell\tilde{m}n} = \int_{0}^{2\pi} \frac{2}{\ell+1} \sin^{\ell+1} \tilde{m} \sin^{2\ell+1} \Omega \mathcal{A}_{\ell\tilde{m}n}\) is the true QNM amplitude, and \(-2\tilde{m} \sin \Omega\) is the spin weighted spheroidal harmonic [44]. Fig. (2)’s bottom left panel shows that there is good agreement between the numerical ratio, \(\rho_{\ell\tilde{m}n} = A_{\ell\tilde{m}n}/A_{\ell\tilde{m}n}\), and the analytic \(\rho_{\ell\tilde{m}n} = \mu_{\ell\tilde{m}n}/\mu_{\ell\tilde{m}n}\). This quantity may be computed for RDNP in the same way, yielding agreement with analytic predictions for \(\rho_{\ell\tilde{m}n}\) within 5% in amplitude, and 15% in phase. One has the option of enforcing the theoretical prediction for the phase by shifting the phase of \(A_{\ell\tilde{m}n}\) by the demonstrated phase difference.

Fig. (2)’s top left and center display select \(|A_{\ell\tilde{m}n}|\). It is discovered that \(A_{3220}\) is remarkably simple: for non-spinning systems, \(A_{3220}(\eta, 0) \approx -4\eta\). This appears to be the first time such a simple relationship has been found for the dominant QNM amplitude, and it is a direct result of this work’s convention of \(t = 0\). The result for \(A_{3220}\) (Eq. 4) shows that this QNM excitation is strongly dependent on the progenitor spins, corroborating Ref. [34].

As in Ref. [27], \(A_{3220}\) and \(A_{44440}\) are found to have non-monotonic amplitudes which correspond to sigmoidal relative phases for changing mass-ratio. In Fig. (2)’s center panels, we see for the first time that this is a robust feature of the non-precessing parameter space.

Fig. (2)’s top right panel shows validation of RDNP against 5 select non-calibration NR waveforms from the BAMBAM code [45, 46]. Here, NR ringdown plays the role of a hypothetical signal at inclination \(i\), and RDNP plays the role of a template at the same inclination, with independent polarization and orbital phase. The normalized inner-product, or match, \((\sqrt{\text{RDNP}}/\text{NR})\), is weighted by the anticipated Advanced LIGO (Adv. LIGO) zero-detuned noise power spectrum at design sensitivity [47] and calculated following Eq. (46) of Ref. [48], with a starting frequency \(f_{\text{min}} = 30\) Hz for the integral. RDNP is evaluated at the same intrinsic parameters as the NR waveform such that there are no spin components within the orbital plane. RDNP matches extremely well with NR cases in and out of the calibration region, often having matches above 0.998. This is the case even for a precessing waveform, “q1.2”, similar to GW150914 [49]. The high spin aligned validation case \((m_1/m_2 = 8, \chi_s = 0.85)\) breaks from this trend. The nonlinear regime for this system extends to approx. 40 \(M_\odot\). When taking this into account, RDNP matches as low as 0.97 for \(i \approx \pi/2\), but 0.99 and well above for \(|\pi/2 - i| > \pi/6\).

Fig. (2)’s bottom right panel compares signal power in \(h_+\) to power in noise for three single interferometer (IFO) cases. The amplitude of each mock signal reflects the presence of multiple QNMs. These cases demonstrate that single IFO ringdown SNRs will approximately double between Adv.
LIGO’s O1 and O3, before increasing another 30 fold under the ET [50, 51].

Discussion — RDNP has been presented to model the ringdown of non-precessing BBH systems. While RDNP matches well with NR simulations, there are multiple avenues for improvement. RDNP does not model precession. RDNP also does not model the apparent nonlinear QNMs reported in Ref. [27]. This may be most important for systems with high aligned spins, where the nonlinear regime is extended. Future ringdown models should be calibrated to a larger set of more accurate NR simulations. Like Ref. [11], the current work is limited by quality concerns between simulations of different numerical codes. RDNP and related techniques may be of use in constructing NR-tuned full signal models with accurate post-mergers. Primarily, it is expected that RDNP will be of imminent use aiding tests of GR before and during LIGO’s third observing run. In this setting, many practical questions regarding ringdown will be pertinent.

The following are briefly considered: How much SNR is in the ringdown? How much SNR can be attributed to subdominant QNMs? Can the QNM amplitudes be constrained? Can their relative phases? To proceed, RDNP is applied to inferred posteriors of GW150914’s parameters via a higher-multipole inspiral-merger-ringdown model, PhenomHM [32]. PhenomHM is not fully tuned to NR, and has accurate but imprecise ringdowns. Here, PhenomHM is applied to the Bayesian inference of GW150914 [53], according to Ref. [49], and then posterior samples are input to RDNP. This approach allows GR predictions for GW150914 to be quantified independently of resolvability criteria for each QNM.

Fig. (3)’s top left panel shows the posterior distribution for $\rho_{\text{opt}}$, the estimated ringdown network SNR (red) for GW150914 [54, 55]. Additional posteriors are shown for $\rho_{\text{Im,n,m}} = \rho_{\text{opt}} - \rho_{\text{opt,n,m}}$, where the single IFO $\rho_{\text{opt,n,m}}$ is the inner product between a RDNP template with all QNMs, and without the $(\ell, m, n)$ QNM. (Top Right) GR predictions for absolute QNM amplitudes via Eqs. (3-10). Here medians and credible intervals are scaled by $10^{31}$. (Bottom Right) GR predictions for QNM relative phases.

Fig. (3)’s top right panel shows GR predictions for QNM amplitudes. For the first time it can be seen that QNMs with odd $m$ have amplitudes and relative phases which are different from those with even $m$. This order 1 contributed SNR is explained by $\rho_{\text{opt}}$ having cross-terms that are proportional to $\rho_{\text{opt},2,-2} \rho_{\text{opt},2,-2} \rho_{\text{opt},2,-2}$, meaning that the larger $\rho_{\text{opt},2,-2}$, the larger the effect of $\rho_{\text{opt},2,-2}$ on $\rho_{\text{opt}}$. This suggests that GW150914 and similar events may be of cumulative use for QNM science with current and future detectors.

Fig. (3)’s top right panel shows GR predictions for QNM amplitudes. For the first time it can be seen that QNMs with odd $m$ have amplitudes and relative phases which are difficult to constrain. Eqs. (3-10), along with well know difficulty measuring component spins (e.g. [56]), yield a straightforward explanation: uncertainty in $\chi_{\text{nr}}$ (odd $m$) is dominated by uncertainty in $\chi_{\text{nr}}$. In the case of Fig. (3)’s bottom left panel, $(2, -1, 2, -1, 0)$’s positive and negative phase values directly correlate with positive and negative $\chi_{\text{nr}}$. Detector networks with greater sensitivity and more IFOs may yet overcome this limitation. Intriguingly, this example highlights that the relative phases, not just the amplitudes, encode information about the progenitor. Consequently, it is proposed that future tests of GR with ringdown use these relative phases in addition to the QNM frequencies, and amplitudes.

The amount of SNR attributed to higher QNMs and the possibility of using QNM relative phase to test GR illuminate a need to further development of analysis pipelines. While much work has been done in this regard (e.g. [10, 21]), there has yet to be demonstrated a pipeline for testing GR with ringdown under the current LIGO algorithm library [57].

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Figure 3. IMR posterior sample post-processing for GW150914: All panels show normalized posterior distributions, with related medians and 90% credible intervals in legends. Median values are shown with dotted lines. (Top Left) SNR attributed to each QNM with indices $(\ell, m, n)$ within a spherical multipole with indices $(\bar{\ell}, \bar{m})$. The attributed SNR for the 2 most significant QNMs are shown. The optimal SNR is shown for reference. (Bottom Left) Difference between the optimal SNR and the SNR contributed by the $(\ell, \bar{m}, \ell, m, n)$ QNM. (Top Right) GR predictions for absolute QNM amplitudes via Eqs. (3-10). Here medians and credible intervals are scaled by $10^{31}$. (Bottom Right) GR predictions for QNM relative phases.
rently, the simplicity of $A_{2220}$, and the robust presence of non-monotonic amplitudes, point to a potential and urgent need for theory to catch up with numerical results. Each avenue is under active investigation.

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