Composition series of affine manifolds and $n$–gerbes.

by

Tsemo Aristide.

The Abdus Salam center for theoretical Physics
Strada Costiera 11
34014 Trieste, Italy.
tsemo@ictp.trieste.it

Abstract.
In this paper, we study $n$–composition series of affine manifolds these are sequences $(M_n, \nabla_{M_n}) \rightarrow (M_{n-1}, \nabla_{M_{n-1}}) \rightarrow ... \rightarrow (M_1, \nabla_{M_1})$, where each affine map $f_i : (M_{i+1}, \nabla_{M_{i+1}}) \rightarrow (M_i, \nabla_{M_i})$ is surjective. One composition series are classified by gerbe theory. It is natural to think that $n$–composition series must be classified by $n$–gerbe theory. In the last section of this paper we propose a notion of abelian $n$–gerbe theory.

Introduction.
An affine bundle is a surjective affine map between affine manifolds. A composition serie of affine manifolds is a sequence $(M_n, \nabla_{M_n}) \rightarrow (M_{n-1}, \nabla_{M_{n-1}}) \rightarrow ... \rightarrow (M_2, \nabla_{M_2}) \rightarrow (M_1, \nabla_{M_1})$, where each map $f_i : (M_{i+1}, \nabla_{M_{i+1}}) \rightarrow (M_i, \nabla_{M_i})$ is an affine bundle.

When the source space $M_i$ of an affine bundle is compact, it becomes a locally trivial differentiable bundle by a well-known Ehresmann theorem (see [God] theorem 2.11 p.16). Let $B$ and $F$ be respectively the base and the fiber spaces of an affine bundle with compact total space. If moreover the second homotopy group of $B$ is trivial, then by the Serre bundle theorem, one deduces that the first homotopy group $\pi_1(M)$ of $M$ is an extension of $\pi_1(B)$ by $\pi_1(F)$. In particular this happens when $(B, \nabla_B)$ is geodesically complete.

Auslander has conjectured that the fundamental group of a compact geodetically complete affine manifold is polycyclic. The existence of a non trivial affine map, on a finite cyclic galoisian cover of a $n$–compact and complete affine manifold ($n > 2$) endowed with a complete structure eventually different from the pull-back, implies the Auslander conjecture [T4]. The classification of affine bundles whose total spaces are compact and complete and more generally of composition series of affine manifolds will conjecturally allow us to know all compact and complete affine manifolds, up to a finite cover, as we know the 2–closed and complete affine manifolds.

The main goal of this paper is to study composition series of affine manifolds.

First we study affine bundles.

Let $\pi_1(F)$ and $\pi_1(B)$ be two groups. Write $\mathbb{R}^{m+l} = \mathbb{R}^m \oplus \mathbb{R}^l$. We denote by $Aff(\mathbb{R}^m, \mathbb{R}^l)$ the group of affine maps of $\mathbb{R}^{m+l}$ which preserve $\mathbb{R}^l$, and by
Aff_f (R^n, R^l) the subgroup of Aff (R^n, R^l) whose restriction on R^n is the
identity.

An algebra problem related to this classification problem of affine bundles is
the following:

Given two representations \( \pi_1 (F) \to Aff (R^l) \), and of \( \pi_1 (B) \to Aff (R^n) \),
classify all commutative diagrams:

\[
\begin{array}{cccc}
1 & \to & \pi_1 (F) & \to & \pi_1 (M) & \to & \pi_1 (B) & \to & 1 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \to & Aff_f (R^n, R^l) & \to & Aff (R^n, R^l) & \to & Aff (R^m) & \to & 1
\end{array}
\]

where the first line is an exact sequence.

There are many ways to solve the classification problem of affine bundles.
First, we give a classification, of affinely locally trivial affine bundles, (see defi-
nition 2.2) after we solve the general case.

Let’s present the classification of affinely locally trivial affine bundles. Let
us consider an affinely locally trivial affine bundle with base \((B, \nabla_B)\) and typ-
ical fiber \((F, \nabla_F)\). We denote by \( T_F \) the group of translations of \((F, \nabla_F)\) (see
section 3). The affine bundle \( f \) gives rise to representations \( \pi_f : \pi_1 (B) \to
Aff (F, \nabla_F) / T_F, \pi'_f : \pi_1 (B) \to Gl(T_F), \) and to a flat bundle \( \hat{T}_F \), with typ-
ical fiber \( T_F \) over \( B \) associated to \( \pi'_f \). We denote by \( T'_F \) the sheaf of affine sections
of \( \hat{T}_F \).

In [T5], we gave a classification of affinely locally trivial affine bundles using
Hochschild cohomology classes for a representation of \( \pi_1 (B) \).

In this paper, we will give another using Cech cohomological classes via gerbe
theory.

In fact it seems to us that the second classification fits best to our prob-
lem. Inspired by the philosophy of groupoid sheaves, we canonically associate
to any representation \( \pi : \pi_1 (B) \to Aff (F, \nabla_F) / T_F, \) a gerbe with lien \( T_F \) which
describes the gluing problem related to the existence of affinely locally trivial
affine bundles associated to \( \pi \). The 2–cocycle described in Giraud’s classifica-
tion theorem of the associated gerbe, is given by an element of \( H^2 (B, T'_F) \). This
class is the obstruction to the existence of affinely locally trivial affine bundles
associated to \( \pi \). When it vanishes, each element of \( H^1 (B, T'_F) \) defines an affinely
locally trivial affine bundle. In this case the classification of the affinely locally
trivial affine bundles is given by the orbits of elements of \( H^1 (B, T'_F) \) under a
gauge group.

After the classification of affine bundles, we classify composition series of
affine manifolds in which each map \( f_i : (M_{i+1}, \nabla_{M_{i+1}}) \to (M_i, \nabla_{M_i}) \) is an
affinely locally trivial affine bundle. Since the classification of affinely locally
trivial affine bundle has been done using gerbe theory, it is natural to think that
the theory involved in the classification of composition series of affine manifolds
is \( n \)–gerbe theory. In the last section of our work, we build a commutative
\( n \)–gerbe theory.

This is the plan of our paper:
0. Introduction.

I. AFFINE BUNDLES.

1. Background.
2. Generality.
3. The classification of affinely locally trivial affine bundles.
4. The general case.

II. COMPOSITION SERIES OF AFFINE MANIFOLDS.

1. 3 composition series.
2. The general case ($n$–composition series of affine manifolds).
3. The conceptualization (commutative $n$–gerbe theory).

I. AFFINE BUNDLES.

1. Background.

An $n$–connected affine manifold $(M, \nabla_M)$, is an $n$–connected differentiable manifold endowed with a connection $\nabla_M$, whose curvature and torsion forms vanish identically. The connection $\nabla_M$ defines on $M$ an atlas (affine) whose transition functions are locally affine transformations of $\mathbb{R}^n$.

Let $(M, \nabla_M)$ and $(N, \nabla_N)$ be two affine manifolds respectively associated to the affine atlas, $(U_i, \phi_i)$ and $(U'_j, \phi'_j)$. An affine map between $(M, \nabla_M)$ and $(N, \nabla_N)$ is a differentiable map $f : M \to N$ such that $\phi'_j \circ f \circ \phi_i^{-1}$ is an affine map. We denote by $\text{App}((M, \nabla_M), (N, \nabla_N))$ the set of affine maps between $(M, \nabla_M)$ and $(N, \nabla_N)$, and by $\text{Aff}(M, \nabla_M)$ the space of affine automorphisms of $(M, \nabla_M)$.

The affine structure of $M$ pulls back to its universal cover $\tilde{M}$, and defines on it an affine structure $(\tilde{M}, \tilde{\nabla}_M)$, for which the universal cover map $p_M : \tilde{M} \to M$ is an affine map. The affine structure of $(\tilde{M}, \tilde{\nabla}_M)$ is defined by a local diffeomorphism $D_M : \tilde{M} \to \mathbb{R}^n$ called the developing map.

The developing map gives rise to a representation $A_M : \text{Aff}(\tilde{M}, \tilde{\nabla}_M) \to \text{Aff}(\mathbb{R}^n)$ which makes the following diagram commute

\[
\begin{array}{ccc}
(\tilde{M}, \tilde{\nabla}_M) & \xrightarrow{g} & (\tilde{M}, \tilde{\nabla}_M) \\
\downarrow D_M & & \downarrow D_M \\
\mathbb{R}^n & \xrightarrow{A_M(g)} & \mathbb{R}^n
\end{array}
\]

where $g$ is an element of $\text{Aff}(\tilde{M}, \tilde{\nabla}_M)$. The restriction of $A_M$ to the fundamental group $\pi_1(M)$ of $M$, is the holonomy representation $h_M$. The linear part $L(h_M)$ of $h_M$, is the linear holonomy of $(M, \nabla_M)$. It is in fact the holonomy of the connection $\nabla_M$ in the classical sense.

Definitions 1.1.

- The affine manifold $(M, \nabla_M)$ is complete, if and only if the developing map is a diffeomorphism. This is equivalent to saying that the connection $\nabla_M$ is geodesically complete.
- The affine manifold \((M, \nabla_M)\) is unimodular, if its linear holonomy lies in \(\text{Sl}(n, \mathbb{R})\). Markus has conjectured that a compact affine manifold is complete if and only if it is unimodular.

- Let \(f\) and \(g\) be two affine bundles with the same base space \((B, \nabla_B)\), and respectively total spaces \((M, \nabla_M)\), and \((N, \nabla_N)\). An affine isomorphism between \(f\) and \(g\), is an affine isomorphism between \((M, \nabla_M)\) and \((N, \nabla_N)\) which sends a fiber of \(f\) onto a fiber of \(g\), and gives rise to an automorphism of \((B, \nabla_B)\).

2. Generalities.

This paragraph is devoted to some basic properties of affine bundles. In the sequel, we will suppose that all the fibers of a given affine bundle are diffeomorphic to each other.

Let \(f : (M, \nabla_M) \to (M', \nabla_{M'})\) be an affine map, the map \(f\) pulls back to a map \(\hat{f} : (\hat{M}, \nabla_{\hat{M}}) \to (\hat{M}', \nabla_{\hat{M}'})\) which makes the following diagram commute:

\[
\begin{array}{ccc}
(M, \nabla_M) & \xrightarrow{f} & (M', \nabla_{M'}) \\
\downarrow \text{PM} & \downarrow \text{PM'} & \downarrow \\
(M, \nabla_M) & \xrightarrow{f} & (M', \nabla_{M'})
\end{array}
\]

**Proposition 2.1.** [T4]. Let \((M, \nabla_M)\) be the domain of an affine map. Suppose that \(M\) is compact. We denote by \(\text{df}_x\), the differential \(\text{df}\) of \(f\), at \(x\). Then the distribution \(Df\) of \(M\) defined by

\[
Df_x = \{v \in T_x M/\text{df}_x(v) = 0\}
\]

defines on \(M\) an affine bundle whose fibers are the leaves of the foliation defined by \(Df\).

**Sketch of proof.**

As \(M\) is compact, the space of fibers is a differentiable manifold, say \(B\). The transverse affine structure of the foliation \(Df\), pushes forward to \(B\) and defines on it an affine connection \(\nabla_B\), which makes the projection \((M, \nabla_M) \to (B, \nabla_B)\) an affine map.

This proposition implies that an affine bundle with compact total space gives rise to a locally trivial differentiable bundle by a well-known Ehresmann result [God]. Denote by \(F\) the typical fiber. Applying the Serre bundle sequence to this bundle, we obtain the following short exact sequence:

\[
\pi_2(B) \to \pi_1(F) \to \pi_1(M) \to \pi_1(B) \to 1.
\]

If we suppose that \(\pi_2(B) = 1\), then we obtain that \(\pi_1(M)\) is an extension of \(\pi_1(B)\) by \(\pi_1(F)\). In particular this happens when \((M, \nabla_M)\) is complete. Remark that if \((M, \nabla_M)\) is complete, then the fibers and the base of the induced bundle are also complete.

Auslander has conjectured that the fundamental group of a compact and complete affine manifold is polycyclic. In [T4], we have conjectured that we
can change the complete affine structure of a galoisian cyclic finite cover of a 
$n$—compact, \( n > 2 \) and complete affine manifold to another complete one, 
so that it becomes the domain of a non trivial affine map. Non trivial means 
that the distribution \( Df \) is neither 0, nor the whole space. This conjecture 
implies the Auslander conjecture (see [T4]). As mentioned in the introduction, 
the classification of affine compact bundles will conjecturally allow us to know 
all compact and complete affine manifolds up to a finite cover.

In fact, there are examples of affine manifolds, which are total spaces of 
more than one non isomorphic affine bundle. The following is an example of 
this situation in dimension 3.

Let \( C = (e_1, e_2, e_3) \) be a basis of \( \mathbb{R}^3 \). Consider the subgroup \( \Gamma \) of \( \text{Aff} \left( \mathbb{R}^3 \right) \), 
generated by \( f_1, f_2 \) and \( f_3 \), whose expressions in \( C \) are:

\[
\begin{align*}
  f_1(x, y, z) &= (x + 1, y, z) \\
  f_2(x, y, z) &= (x, y + 1, z) \\
  f_3(x, y, z) &= (x + y, y, z + 1).
\end{align*}
\]

The quotient of \( \mathbb{R}^3 \) by \( \Gamma \) is a compact affine manifold, \( M^3 \). The projections 
\( p_2(x, y, z) = y \), and \( p_3(x, y, z) = z \), define projections of \( M^3 \) over the circle 
endowed with its canonical complete structure. The bundles defined by those 
projections are not isomorphic. If they were isomorphic, there would exist an 
element of \( \text{Aff} \left( \mathbb{R}^3 \right) \) of the form \( (x, y, z) \mapsto (ax + b, f(y, z) + d(x)) \) where \( f \) is an 
element of \( \text{Aff} \left( \mathbb{R}^2 \right) \), and \( d \) a linear map \( \mathbb{R} \to \mathbb{R}^2 \) which conjugates the map 
\( (x, y, z) \mapsto (x + 1, y, z) \) to the map \( (x, y, z) \mapsto (x + 1, y + z, z) \). This is evidently 
impossible. (For each bundle, we have adapted the expression of \( \Gamma \) in a basis 
\( (e_1', e_2', e_3') \) such that the vector subspace \( \mathbb{R} e_1' \) pulls forward on the base of each 
fibration).

**Definition 2.2.**

Let \( f : (M, \nabla_M) \to (B, \nabla_B) \) be an affine bundle, We will say that the bundle 
\( f \) is an affinely locally trivial affine bundle, if and only if there exists an affine 
manifold \( (F, \nabla_F) \) such that each element \( x \) of \( B \), is contained in an open set \( U_x \), 
such that there exists an affine isomorphism 
\[
  f^{-1}(U_x) \to U_x \times (F, \nabla_F)
\]
and the restriction of the projection on \( f^{-1}(U_x) \) is the first projection \( U_x \times 
(F, \nabla_F) \to U_x \) via this isomorphism.

When the total space is compact, the last definition is equivalent to saying 
that one can build the Cech cocycle which defines the locally trivial differentiable 
structure of the affine bundle by affine maps.

In the previous examples the bundle defined by \( p_3 \) is affinely locally trivial, 
but not the one defined by \( p_2 \).

Let \( f : (M, \nabla_M) \to (M', \nabla_M') \) be an affine map, where \( M \) and \( M' \) are 
respectively an \( n \) and an \( n' \)–manifold. The map \( f \) pulls back to a map \( \hat{f} : 
(M, \nabla_M) \to (M', \nabla_{M'}) \). There exists an affine map \( f' : \mathbb{R}^n \to \mathbb{R}^{n'} \) which 
makes the following diagram commute:
Let \((M, \nabla_M)\) be the total space of an affine bundle \(f\). The foliation \(\mathcal{F}_f\) defined by the leaves of \(f\), pulls back to a foliation \(\tilde{\mathcal{F}}_f\) on \(\tilde{M}\), which is the pull-back of a foliation \(D_M(\tilde{\mathcal{F}}_f)\) of \(\mathbb{R}^n\), by parallel \(l\)-affine subspaces, here \(n\) and \(l\) are the dimensions of \(M\) and of the fibers of the bundle.

Write \(\mathbb{R}^n = \mathbb{R}^m \oplus \mathbb{R}^l\), where \(m\) is the dimension of the base of the bundle.

For every element \(\gamma\) of \(\pi_1(M)\), we have:

\[
h_M(\gamma)(x,y) = (A_\gamma(x) + a_\gamma, B_\gamma(y) + C_\gamma(x) + d_\gamma)
\]
as \(D_M(\tilde{\mathcal{F}}_f)\) is stable under the holonomy.

An element \(\gamma\) of \(\pi_1(M)\) which preserves a fiber of \(\tilde{f}\), preserves all the other fibers as the foliation \(\mathcal{F}_f\) does not have holonomy. We obtain that

\[
h_M(\gamma)(x,y) = (x, B_\gamma(y) + C_\gamma(x) + d_\gamma).
\]

In fact we obtain a representation \(\pi_1(F) \to \text{Aff}(\mathbb{R}^l)\) for each \(x \in \mathbb{R}^m\).

We deduce that, as they have been supposed to be diffeomorphic, all the fibers have the same linear holonomy.

The map

\[
\pi_1(F) \to \mathbb{R}^l
\]

\[
\gamma \to C_\gamma(x) + d_\gamma
\]
is a 1-cocycle with respect to the linear holonomy. It defines the map

\[
r : \mathbb{R}^m \to H^1(\pi_1(F), \mathbb{R}^l)
\]

\[
x \to [C_\gamma(x) + d_\gamma]
\]

where \(H^*(\pi_1(F), \mathbb{R}^l)\) is the * cohomology group, with respect to the linear holonomy of the fibers. The cohomology class \(r(x)\) is often called the radiance obstruction of the affine holonomy of the fiber over \(p_B(x)\).

If the fibers are compact and complete, the image of \(r\) is contained in the algebraic subvariety \(L\), of \(H^1(\pi_1(F), \mathbb{R}^l)\) defined by

\[
L = \{c \in H^1(\pi_1(F), \mathbb{R}^l)/\Lambda^l c \neq 0\}.
\]

See [F-G-H] theorem 2.2.

The fact that \(f\) is an affinely locally trivial affine bundle, is equivalent to the fact that the map \(r\) is a constant map when the total space is complete.

**Question.** Are the fibers of an affine bundle isomorphic if its total space is compact?

The following theorem was inspired by the last question:
Theorem 2.3. [T5]. Suppose that the total space of an affine bundle is an $n$—compact and complete affine manifold, and moreover the fundamental group of the fibers are nilpotent. Then all the fibers are isomorphic to each other.

Given an element $\gamma$ of $\pi_1(B)$, and an element $x \in \mathbb{R}^m$, the restriction of the affine holonomy representation of $\pi_1(F)$ to $x \times \mathbb{R}^l$ and $(h_B(\gamma))(x) \times \mathbb{R}^l$ are conjugated by an element of $Aff(\mathbb{R}^l)$, since they define the same affine structure (we choose $x \in D_B(\hat{B})$). This leads to define a gauge group for the linear representation $L(h_F)$.

Consider the subgroup $G$ of automorphisms of $\pi_1(F)$ such that for every element $g$ of $G$, there is a linear map $B_g$ such that $L(h_F)(g(\gamma)) = B_g \circ L(h_F)(\gamma) \circ B_g^{-1}$.

The group which elements are $B_g$ will be called the gauge group of $L(h_F)$.

We associate to every $B_g \in G$ the following linear map of $H^p(\pi_1(F), \mathbb{R}^l)$: for each $c \in H^p(\pi_1(F), \mathbb{R}^l)$ we define $B_g^*c$ by

$$B_g^*(\gamma_1, \ldots, \gamma_p) = B_g(c(g^{-1}(\gamma_1), \ldots, g^{-1}(\gamma_p))).$$

Two complete affine structures $\nabla_1$ and $\nabla_2$ on $F$, with same linear holonomy $L(h_F)$, and holonomy $h_1$ and $h_2$, are isomorphic if and only if there is an element $B_g$ of the gauge group such that $B_g^*c_1 = c_2$, where $c_1$ and $c_2$ are respectively the radiance obstruction of $h_1$ and $h_2$.

3. Affinely locally trivial affine bundles.

Recall that, an affine bundle with complete total space is said to be an affinely locally trivial affine bundle, if and only if its pull-back of the bundle to the universal cover of the base is a trivial affine bundle.

In the sequel, $(M, \nabla_M)$ will be the compact total space of an affinely locally trivial affine bundle, with base space $(B, \nabla_B)$ and typical fiber $(F, \nabla_F)$.

The following proposition emphasizes the importance of the category of affinely locally trivial affine bundles.

Proposition 3.1. [T5] Let $f$ be an affine bundle whose total space is a complete affine $n$—manifold (not necessarily compact). If we suppose that the fibers are 2—tori and moreover their linear holonomy is the linear holonomy of a complete structure of the 2—torus distinct from the flat Riemannian one, then $f$ is an affinely locally trivial affine bundle.

Let’s go back to the classification problem.

Recall that $\pi_1(F)$ is a normal subgroup of $\pi_1(M)$. Let $\gamma$ and $\gamma_1$ be respectively two elements of $\pi_1(F)$ and $\pi_1(M)$. We can write

$$h_M(\gamma) = (x, B_{\gamma}(y) + d_{\gamma})$$

and

$$h_M(\gamma_1)(x, y) = (A_{\gamma_1}(x) + a_{\gamma_1}, B_{\gamma_1}(y) + C_{\gamma_1}(x) + d_{\gamma_1}),$$
where $A_{\gamma_1}$ is an automorphism of $\mathbb{R}^m$, $B_{\gamma}$ and $B_{\gamma_1}$ are automorphisms of $\mathbb{R}^d$.

One has

$$h_M(\gamma_1)^{-1}(x,y) = (A_{\gamma_1}^{-1}(x) - A_{\gamma_1}^{-1}(a_{\gamma_1}), B_{\gamma_1}^{-1}(y) - B_{\gamma_1}^{-1}(d_{\gamma_1}) - B_{\gamma_1}^{-1}C_{\gamma_1}(A_{\gamma_1}^{-1}(x) - A_{\gamma_1}^{-1}(a_{\gamma_1}))).$$

Using the fact that $h_M(\pi_1(F))$ is a normal subgroup of $h_M(\pi_1(M))$, one obtains

$$h_M(\gamma_1^{-1}) \circ h_M(\gamma) \circ h_M(\gamma_1) = (x, B_{\gamma_1}^{-1}B_{\gamma_1}B_{\gamma_1}(y) + B_{\gamma_1}^{-1}B_{\gamma_1}(B_{\gamma_1}(x) + B_{\gamma_1}^{-1}B_{\gamma_1}(d_{\gamma_1}) + B_{\gamma_1}^{-1}(d_{\gamma_1}) - B_{\gamma_1}^{-1}C_{\gamma_1}(x) - B_{\gamma_1}^{-1}(d_{\gamma_1})).$$

This implies that

$$B_{\gamma_1}^{-1}B_{\gamma_1}C_{\gamma_1}(x) - B_{\gamma_1}^{-1}C_{\gamma_1}(x) = 0;$$

we deduce that $C_{\gamma_1}(x) \in H^0(\pi_1(F), \mathbb{R}^F)$. The linear space $\text{Applin}(\mathbb{R}^m, H^0(\pi_1(F), \mathbb{R}^F))$ of linear maps $\mathbb{R}^m \to H^0(\pi_1(F), \mathbb{R}^F)$ has a left $\pi_1(B)$-module structure defined by

$$\gamma_1(D) = B_{\gamma_1} \circ D$$

and a right $\pi_1(B)$-module structure defined by

$$\gamma_1(D) = D \circ A_{\gamma_1},$$

where $\gamma_1$ is an element of $\pi_1(M)$ over an element $\gamma_1'$ of $\pi_1(B)$.

We denote by $T_F$ the connected component of the group of affine maps of $(F, \nabla_F)$, which pull-back on translations of $\mathbb{R}^d$. The linear map of $H^0(\pi_1(F), \mathbb{R}^F)$ defined by

$$t \rightarrow B_{\gamma_1}t$$

induces a linear map of $T_F$. This induces a $\pi_1(B)$ left structure on $\text{Applin}(\mathbb{R}^m, T_F)$. The right structure of $\pi_1(B)$ on $\text{Applin}(\mathbb{R}^m, H^0(\pi_1(F), \mathbb{R}^F))$ also induces a right $\pi_1(B)$ structure on $\text{Applin}(\mathbb{R}^m, T_F)$. Let $\mathcal{L}\pi_1(B)$ be the group algebra of the group $\pi_1(B)$. The vector space $\text{Applin}(\mathbb{R}^m, T_F)$ is endowed with a $\mathcal{L}\pi_1(B)$ Hochschild module structure.

The bundle is supposed to be affinely locally trivial; this implies that his lifts on $\hat{B}$, is $\hat{B} \times (F, \nabla_F)$. The action of $\pi_1(B)$ on $\hat{B} \times (F, \nabla_F)$ is made by affine maps. We deduce that $\pi_1(F)$ is normal in $\pi_1(M)$, and a representation $\pi_F : \pi_1(B) \to Aff(F, \nabla_F)/T_F$ induced by its action on $\hat{B} \times (F, \nabla_F)$.

Recall the following problem stated in [Bry]. Let

$$1 \rightarrow A \rightarrow G \rightarrow H \rightarrow 1$$

be an exact sequence of groups, where $A$ is commutative. Given an $H$–bundle over the manifold $X$, we want to classify all the bundles over $X$, with structural group $G$, which are lifts of the previous bundle.

8
Our problem is quite similar: We have seen that an affinely locally trivial affine bundle $f$ gives rise to a representation $\pi_f : \pi_1(B) \to \text{Aff}(F, \nabla_F)/T_F$. We denote by $\pi'_f$ the flat bundle induced by $\pi_f$. Given such a representation $\pi : \pi_1(B) \to \text{Aff}(F, \nabla_F)/T_F$ with associated $\text{Aff}(F, \nabla_F)/T_F$ bundle $\pi'$, we want to classify all affinely locally trivial affine bundles associated.

To make our theory fit in gerbe theory with the band $T'_F$, the sheaf of affine sections of $T'_F$, we will consider first, the classification of affine bundles up to $T'_F$ isomorphisms.

Now let’s recall some general facts of sheaf of groupoids and descent theory. Our exposition follows the treatment of [Bry]. The general philosophy is to express gluing conditions in terms of covering maps. We remark that if a manifold is an affine manifold, so are its covers, the category of affine manifolds is stable under affine fiber product.

For every locally isomorphic affine map $g : (Y, \nabla_Y) \to (B, \nabla_B)$, we can pull-back the affinely locally trivial affine bundle $\pi'$ over $(B, \nabla_B)$ in a bundle $\pi'_Y$ over $(Y, \nabla_Y)$ (associated to $\pi_Y$). The total space of the pull-back is $Q = \pi' \times_B (Y, \nabla_Y)$ and the fiber map the canonical projection.

Consider $Y \times_B Y$ with the two canonical projections $p_1, p_2 : Y \times_B Y \to Y$. We denote by $p_i^*Q i = \{1,2\}$ the pull-back of $Q$ by $p_i$. There is a natural isomorphism $\phi : p_1^*Q \to p_2^*Q$. This isomorphism satisfies the following cocycle condition

$$p_{13}^*(\phi) = p_{23}^*(\phi) \circ p_{12}^*(\phi),$$

an equality of morphisms $p_1^*Q \to p_2^*Q$ of affine bundles over $Y \times_B Y \times_B Y$, where $p_1$, $p_2$ and $p_3$ are the canonical projections on the three factors, and $p_{12}$, $p_{13}$ and $p_{23}$ are the canonical projections of $Y \times_B Y \times_B Y$ over $Y \times_B Y$.

Conversely, given an affine bundle $Q \to Y$ which satisfies the condition (2), we recover an affine bundle over $(B, \nabla_B)$.

In fact one obtains:

**Proposition 3.2.** Let $g : (Y, \nabla_Y) \to (B, \nabla_B)$ be a local isomorphism of affine manifolds. The pull-back functor $g^*$ induces an equivalence of categories between the category of affine bundles over $(B, \nabla_B)$ and the category of affine bundle bundles over $(Y, \nabla_Y)$ equipped with a descent isomorphism $\phi : p_1^*Q \to p_2^*Q$ satisfying the cocycle condition (2).

The general definition of torsor adapted to this case is:

**Définition 3.3.** A $T'_F$ torsor, will be a sheaf $H$ on $(B, \nabla_B)$, together with a $T'_F$ action such that every point of $B$ has a neighborhood $U$ with the property that for every $V \subset U$ open, the space $H(V)$ is an affine principal bundle with structural group $T'_{F|V}$.

The isomorphism classes of $T'_F$ torsors are given by $H^1(B, T'_F)$, where $H^*(B, T'_F)$ is the $*$ cohomology group of $T'_F$ related to the usual Cech cohomology.
As the sheaf $T'_F$ is a locally constant sheaf, the notion of $T'_F$ torsor in our case is similar to a notion of affine $T'_F$ bundle over $(B, \nabla_B)$.

We can associate to a representation $\pi : \pi_1(B) \to \text{Aff}(F, \nabla_F)/T'_F$, the following sheaf of groupoids, $B_\pi$. To every local affine isomorphism $(Y, \nabla_Y) \to (B, \nabla_B)$, we associate the category $Y_\pi$, whose objects are affinely locally trivial affine bundles over $(Y, \nabla_Y)$ with typical fiber $(F, \nabla_F)$, associated to $\pi_Y$. The (auto)morphisms are $T'_F$-automorphisms.

It is easy to show that the following properties are satisfied:

(i) For every diagram $(Z, \nabla_Z) \xrightarrow{g} (Y, \nabla_Y) \xrightarrow{h} (B, \nabla_B)$ of local affine isomorphisms, there is a functor $g^{-1}_\pi : Y_\pi \to Z_\pi$;

(ii) For every diagram $(W, \nabla_W) \xrightarrow{k} (Z, \nabla_Z) \xrightarrow{g} (Y, \nabla_Y) \xrightarrow{h} (B, \nabla_B)$ there is an invertible natural transformation $\theta_{g,k} : k^{-1}g^{-1} \to (gk)^{-1}$.

This makes our category a presheaf category. Moreover properties a la (2) are satisfied to ensure that some kind of Haefliger 1-cocycles is satisfied, in order to make our presheaf of category a sheaf of category.

One can more usually define a sheaf of category. It is a map $C$ on the family of open subsets of $B$

$$U \mapsto C(U)$$

which assigns to any open subset $U$ of $B$ a category $C(U)$. For every open subset $V \subset U$, there is a composition of morphisms from $C(U)$ to $C(V)$. When $U = V$, this composition is just the composition of morphisms. This defines the presheaf of category. Moreover a descent condition is needed to make the presheaf a sheaf.

In fact, our sheaf of category is a gerbe with band $T'_F$. It means that the following properties are satisfied

(G1) Given any object of $Y_\pi$, the sheaf of local automorphisms of this object is a sheaf of groups which is locally isomorphic to $T'_F$.

(G2) Given two objects $Q_1$ and $Q_2$ of $Y_\pi$, there exists a local isomorphism surjective map $g : Z \to Y$ such that $g^{-1}Q_1$ and $g^{-1}Q_2$ are locally isomorphic.

(G3) There is a local isomorphism surjective affine map $Y \to X$ such that the category $Y_\pi$ is not empty.

One say that our sheaf of category is a gerbe with band or lien $T'_F$.

Remark.

To ensure the axiom (G3) to be satisfied, one may show an affinely local trivial affine bundle with typical fiber $(F, \nabla_F)$, over $\hat{B}$. This bundle is just the trivial one.

Let’s now state the first classification theorem which is an adaptation of the Giraud classification theorem.

**Theorem 3.4.** The set of equivalence classes of the gerbes is in one to one correspondence with $H^2(B, T'_F)$.

**Proof.**
Let's consider a cover \((U_i)_{i \in I}\) of \(B\) by open \(-\)connected affine charts. The \(T'_F\)\(-\)automorphisms of an object \(P_i\) of \(C(U_i)\), is isomorphic to the restriction of \(T'_F\) to \(U_i\).

There is a \(T'_F\)\(-\)isomorphism 
\[
u_{ij} : (P_i)|_{C(U_{ij})} \to (P_j)|_{C(U_{ij})}
\]
in the category \(C(U_{ij})\). We define a section \(h_{ijk}\) of \(T'_F\) by,
\[
h_{ijk} = u_{ik}^{-1} u_{ij} u_{jk} \in \text{Aut}(P_k) \simeq T'_F.
\]

In fact \(h = (h_{ijk})\) is a \(T'_F\)\(-\)valued Cech \(-\)cocycle. The corresponding class in \(H^2(B,T'_F)\) is independent of all the choices. We will show that this correspondence defines an isomorphism between the group of equivalence classes, and the set of isomorphic gerbes with band \(T'_F\).

To show the injectivity of this map, one remarks that if the cohomology class defined by \(h\) is trivial, then one can modify the isomorphisms \(u_{ij}\) such that \(u_{ik} = u_{ij} u_{jk}\). We then obtain an affine \(T'_F\) torsor over \((B,\nabla_B)\) which represents a trivial gerbe.

To prove the surjectivity, we construct a gerbe associated to a \(2\)\(-\)Cech cocycle \(h = (h_{ijk})\) with values in \(T'_F\). It is sufficient to find a family of elements \(u_{ij}\) of \(T'_F\)\(-\)automorphisms of \(U_{ij}\) such that the condition \(u_{ik}^{-1} u_{ij} u_{jk} = h_{ijk}\) is satisfied.

This is our classification theorem for affinely locally trivial affine bundles, up to \(T'_F\)\(-\)isomorphisms.

**Theorem 3.5.** Let \(\pi : \pi_1(B) \to Aff(F,\nabla_F)/T_F\) be a representation. Then there are affine bundles over \(\pi'\), if and only if its associated gerbe is trivial. In this case the \(T'_F\)\(-\)isomorphism classes of affine bundles are given by \(H^1(B,T'_F)\).

**Proof.**

If there exists an affine bundle associated to the representation \(\pi\), the boundary of the cocycle which defines the fibration represents the associated gerbe, so this gerbe is trivial.

On the other hand, the \(2\)\(-\)cocycle associated can be described as follows:

Consider a trivialisation of the flat bundle \(\pi'\), associated to \(\pi\).

For every \(i, j\) such that \(U_i \cap U_j \neq \emptyset\), we have
\[
U_i \cap U_j \times Aff(F,\nabla_F)/T_F \longrightarrow U_i \cap U_j \times Aff(F,\nabla_F)/T_F
\]
\[
(x,y) \longrightarrow (x,g'_{ij}(y)).
\]

We denote by \(g_{ij}(x)\), an element of \(Aff(F,\nabla_F)\) over \(g'_{ij}\) which depends affinely on \(x\). We set
\[
h_{ijk} = g_{ik}^{-1} g_{ij} g_{jk}.
\]

We have seen that if the cocycle is trivial, one can find a family of maps \(w_{ij} : U_i \cap U_j \to T'_F\) such that
\[
(g_{ik} + w_{ik}) = (g_{ij} + w_{ij})(g_{jk} + w_{jk}).
\]
Consider the family of maps

$$\phi_{ij} : U_i \cap U_j \times (F, \nabla_F) \longrightarrow U_i \cap U_j \times (F, \nabla_F)$$

$$(x, y) \longmapsto (x, (g_{ij} + w_{ij}(x))(y))$$

We have

$$\phi_{ij} \circ \phi_{jk}(x, y) = (x, (g_{ij}g_{jk} + w_{ij}(x) + g_{ij}w_{jk}(x))(y)).$$

One sees that the Čech cocycle condition is verified.

For every other 1-cocycle $\{(v_{ij})\}$, one gets an affine bundle by setting

$$\phi'_{ij} : U_i \cap U_j \times (F, \nabla_F) \longrightarrow U_i \cap U_j \times (F, \nabla_F)$$

$$(x, y) \longmapsto (x, (g_i + (w_{ij} + v_{ij}))(x))(y)).$$

Two different cocycles used to define affinely locally trivial affine bundles can define isomorphic affine bundles.

Let $f_1$ and $f_2$ be two isomorphic affinely locally trivial affine bundles whose total spaces are $n$-compact, and which induce the same sheaf $T_F$. There is an affine transformation $g$ of $f_1$ which preserves the foliation $\mathcal{F}_{f_1}$ (where $\mathcal{F}_{f_1}$ is the pull-back of the the foliation induced by $f_1$) and conjugates the Deck transformations which defines the total space of $f_1$, in those which define the one of $f_2$. As both bundles induce $T'_F$, their induced representations $\pi_1(B) \to Aff(F, \nabla_F)/T_F$ coincide. The cohomology class defined by $f_1$ is changed in the class defined by $f_2$ by $g$, one has

**Proposition 3.6.** The isomorphism classes of affine bundles are given by the quotient of $H^1(B, T'_F)$ by the action of a gauge group. This group is the group of affine automorphisms of $f_1$ which preserve its fibers, are pull back of automorphisms of $(B, \nabla_B)$ and give rise to the same bundle $\pi'$.

4. The general case.

In the previous section of our paper, we have classified affinely locally trivial affine bundles. In the following section, we consider the more general situation, when the total space is supposed to be only compact and complete.

Given an affine bundle with compact and complete total space say $(M, \nabla_M)$ and base space $(B, \nabla_B)$, we have seen that the fibers inherit affine structures from the total space which are not necessarily isomorphic, but which have the same linear holonomy. Let $\gamma$ be an element of $\pi_1(M)$, set $\mathbb{R}^n = \mathbb{R}^m \oplus \mathbb{R}^l$ where $n, m,$ and $l$ are respectively the dimension of $M, B$ and the typical fiber $F$. We have:

$$h_M(\gamma)(x, y) = (A_\gamma(x) + a_\gamma, B_\gamma(y) + C_\gamma(x) + d_\gamma),$$

where $A_\gamma$ and $B_\gamma$ are respectively affine automorphisms of $\mathbb{R}^m$ and $\mathbb{R}^l$, $a_\gamma$ and $d_\gamma$ are respectively elements of $\mathbb{R}^m$ and $\mathbb{R}^l$, and $C : \mathbb{R}^m \to \mathbb{R}^l$ is a linear map.
If \( \gamma \) lies in \( \pi_1(F) \), then \( A_\gamma = I_{\mathbb{R}^m} \) and \( a_\gamma = 0 \).

Now consider an \( m \)-affine manifold \((B, \nabla_B)\) compact and complete, a compact \( l \)-manifold \( F \) and a representation \( L(h_F) : \pi_1(F) \rightarrow \text{Gl}(l, \mathbb{R}) \) which is the linear holonomy of a complete affine structure of \( F \). We want to classify all affine bundles with complete total space with base \((B, \nabla_B)\) and whose fibers are diffeomorphic to \( F \) and inherit affine structures from the total space whose linear holonomy is \( L(h_F) \).

The natural question which arises is: Does every map \( r : \mathbb{R}^m \rightarrow H^1(\pi_1(F), \mathbb{R}^l) \) give rise to an affine bundle?

For every element \( \gamma_1 \) of \( \pi_1(B) \), the affine representations defined by the cocycles \( r(x) \) and \( r(\gamma_1(x)) \) must be isomorphic.

Recall that we have defined a gauge group \( G \) of the representation \( L(h_F) \) as follows: it is a group of total maps such that for every element \( B_g \in G \) there is an automorphism \( g \) of \( \pi_1(F) \) which satisfies

\[
L(h_F)(g(\gamma)) = B_g L(h_F)(\gamma) B_g^{-1}.
\]

The group \( G \) acts on \( L(h_F) \) one cocycles by setting

\[
(B_g^* c)(\gamma) = B_g c(g^{-1}(\gamma)).
\]

Let \( \gamma_1 \) and \( \gamma \) be respectively elements of \( \pi_1(M) \) and \( \pi_1(F) \). We have

\[
\gamma_1 \circ \gamma \circ \gamma_1^{-1}(x, y) = \\
(x, B_{\gamma_1} B_{\gamma} B_{\gamma}^{-1}(y) - B_{\gamma_1} B_{\gamma} B_{\gamma}^{-1} C_{\gamma_1}(A_{\gamma_1}(x) - A_{\gamma_1}(a_{\gamma_1})) - B_{\gamma_1} B_{\gamma} B_{\gamma}^{-1}(d_{\gamma_1}) + \\
B_{\gamma_1} C_{\gamma}(A_{\gamma_1}(x) - A_{\gamma_1}(a_{\gamma_1})) + B_{\gamma_1}(d_2) + C_{\gamma}(A_{\gamma_1}(x) - A_{\gamma_1}(a_{\gamma_1})) + d_{\gamma_1})
\]

The map

\[
i'(\gamma_1) : \pi_1(F) \rightarrow \pi_1(F)
\]

\[
\gamma \rightarrow \gamma_1 \gamma \gamma_1^{-1}
\]

is an automorphism associated to the element of the gauge group \( B_{\gamma_1} \). If \( \gamma_1 \) lies in \( \pi_1(F) \) the induced map on \( H^1(\pi_1(F), \mathbb{R}^l) \) is trivial. We deduce a map

\[
i : \pi_1(B) \rightarrow \text{Gl}(H^1(\pi_1(F), \mathbb{R}^l))
\]

\[
\gamma_1 \rightarrow a i'(\gamma_1),
\]

where \( \gamma_1 \) pulls back to \( \gamma_1' \) and \( a i'(\gamma_1) \) is the action of \( \gamma_1' \) on \( H^1(\pi_1(F), \mathbb{R}^l) \) induced by \( B_{\gamma_1} \).

We have \( r(A_{\gamma_1'}(x) + a_{\gamma_1'}) = [B_{\gamma_1}, r(x)] \).

It follows that the following square is commutative

\[
\begin{array}{ccc}
\mathbb{R}^m & \xrightarrow{\gamma_1} & \mathbb{R}^m \\
\downarrow r & & \downarrow r \\
H^1(\pi_1(F), \mathbb{R}^l) & \xrightarrow{i(\gamma_1)} & H^1(\pi_1(F), \mathbb{R}^l)
\end{array}
\]
The representation $i$ allows also to construct a bundle over $(B, \nabla_B)$ with typical fiber $H^1(\pi_1(F), B^l)$. We also denote by $i$ this bundle. The map $r$ can also be viewed as a section of this bundle.

We will classify all affine bundles for a given representation $i$, and a map $r$ such that each $r(x)$ defines an affine structure.

The map $r$ defines a representation $\pi_1(F) \to Aff(B^m)$ such that the quotient $B^m/\pi_1(F)$ is an affine bundle over $B^m$. We assume that for each $\gamma \in \pi_1(B)$, there is an element of $Aff(B^m/\pi_1(F))$ which induces $i(\gamma)$.

Let $U$ be an open set of $(B, \nabla_B)$. We define the following sheaf of categories

$$U \to C(U).$$

Where $C(U)$ is the set of affine bundles such that the canonical bundle with typical fiber $H^1(\pi_1(F), B^l)$ associated, is the restriction of $i$ to $U$, and whose lifts on the universal cover of $U$ is the restriction of $B^m/\pi_1(F)$ to it.

The sheaf of categories $C$ is a gerbe with band $Aff(B^m/\pi_1(F))_0$, which denotes the sheaf induced by the connected component of the affine automorphisms of $B^m/\pi_1(F)$ which pushes forward on the identity of $B^m$.

Let us explain why the band is $Aff(B^m/\pi_1(F))_0$.

Consider a trivialization of $i$, $\bar{h}_{ij} : U_i \cap U_j \times H^1(\pi_1(F), B^l) \to U_i \cap U_j \times H^1(\pi_1(F), B^l) (x, y) \mapsto (x, \bar{h}_{ij}(y))$.

Here the $U_i$ are connected open sets of affine charts. So we can restrict the bundle $B^m/\pi_1(F)$ to each $U_i$. We denote by $U^F_i$ this restriction.

To $\bar{h}_{ij}$ we associate an element $h_{ij}$ of $Aff(U^F_i)$ which pushes forward on the identity of $U_i$, and gives rise to $\bar{h}_{ij}$. The maps

$$h^{-1}_{ik} h_{ij} h_{jk}$$

are the obstructions of the existence of an affine bundle associated to $i$ and $r$.

**Proposition 4.1.** The map $h_{ijk} = h^{-1}_{ik} h_{ij} h_{jk}$ is an element of the restriction of $Aff(B^m/\pi_1(F))_0$ to $U_i \cap U_j \cap U_k$.

**Proof.**

We deduce this, from the fact that, the map $h_{ijk}$ gives rise to the identity of $H^1(\pi_1(F), B^l)$, since it is shown in [T2] that the set of affine automorphisms which commutes with the holonomy of a compact and complete affine manifold is a cover of the connected component of its affine automorphism group.

This is our classification theorem in the general case.

**Theorem 4.2.** For each representation $i$ and map $r$, there is a 2–Cech cocycle which is the obstruction of the existence of an affine bundle associated to $i$ and $r$. When it vanishes the set of isomorphisms classes of affine bundles
are given by the orbit of element of $H^1(B, Aff(\mathbb{R}^n/\pi_1(F)))_0$ under a gauge group.

II. COMPOSITION SERIES OF AFFINE MANIFOLDS.

Recall that a $n$–affine manifold is said to be complete if and only if the connection $\nabla_M$ is complete. This is equivalent to saying that $M$ is the quotient of $\mathbb{R}^n$ by a group $\Gamma_M$ of affine automorphisms which act properly and freely on $\mathbb{R}^n$. In this part, we will only consider complete affine manifolds.

The representation $h_M : \Gamma_M = \pi_1(M) \rightarrow Aff(\mathbb{R}^n)$ is called the holonomy of the affine manifold $(M, \nabla_M)$. Its linear part $L(h_M)$ is called the linear holonomy.

It has been conjectured by Auslander that the fundamental group of a compact and complete affine manifold is polycyclic.

In [T4], we have conjectured that each compact and complete $n > 2$ affine manifold $(M, \nabla_M)$ has a finite cyclic and galoisian cover $M'$ endowed with a complete affine structure $(M', \nabla'_{M'})$ eventually different from the pull back such that $(M', \nabla'_{M'})$ is the source space of a non trivial affine map. Non trivial means that the fibers of $f$ are neither $M'$ nor points of $M'$. This conjecture implies the Auslander conjecture.

If we suppose that the source space $(M, \nabla_M)$, of a non trivial affine map is compact, then $(M, \nabla_M)$ is the source space of a non trivial affine surjection $f$ over a manifold $(B, \nabla_B)$. We deduce from a well-known Ehresmann theorem, that $f$ is also a locally trivial differentiable fibration. All the fibers of an affine fibration inherit affine structures from $(M, \nabla_M)$ with same linear holonomy.

The last conjecture leads to the following problem: Given two affine manifolds $(B, \nabla_B)$ and $(F, \nabla_F)$ classify every affine surjection $f : (M, \nabla_M) \rightarrow (B, \nabla_B)$ such that the differentiable structure of the fibers is $F$ and their linear holonomy is the one of $(F, \nabla_F)$. Or more generally, Given $n$ affine manifolds $(F_i, \nabla_{F_i})$, classify all composition series $(M_{n+1}, \nabla_{M_{n+1}}) \rightarrow (M_n, \nabla_{M_n}) \rightarrow \ldots \rightarrow (M_1, \nabla_{M_1})$ such that every map $f_i : (M_{i+1}, \nabla_{M_{i+1}}) \rightarrow (M_i, \nabla_{M_i})$ in the last sequence is an affine surjection with fiber diffeomorphic to $F_i$, and which inherits from $(M_{i+1}, \nabla_{M_{i+1}})$ an affine structure which linear holonomy is the linear holonomy of $(F_i, \nabla_{F_i})$.

We have classify in the first part affine surjections with compact total spaces using gerbe theory. The purpose of this part is to classify affine composition series of affine manifolds. We restrict to composition series $(M_{n}, \nabla_{M_{n}}) \rightarrow (M_{n-1}, \nabla_{M_{n-1}}) \rightarrow \ldots \rightarrow (M_1, \nabla_{M_1})$ such that the projection $f_i : (M_{i+1}, \nabla_{M_{i+1}}) \rightarrow (M_i, \nabla_{M_i})$ is an affinely locally trivial affine (a.l.t) fibration. This means that the holonomy of the fibers is fixed.

Two composition series

$$(M_n^j, \nabla_{M_n^j}) \rightarrow (M_{n-1}^j, \nabla_{M_{n-1}^j}) \rightarrow \ldots (M_2^j, \nabla_{M_2^j}) \rightarrow (M_1, \nabla_{M_1}), \; j = 1, 2$$

are equivalent if and only if the bundle $f_i^1$ and $f_i^2$ are isomorphic in respect to $T_{F_i}$ isomorphisms. This means that we consider isomorphisms of affine fibrations which act by translations on the fibers and project on the identity of the base space.
The classification of affinely locally trivial affine bundles where made using commutative gerbe theory. It is natural to think that the classification of affinely locally $n-$ composition series must be done using $n-$gerbe theory. In this part, we will give a classification of $n-$ composition series of affine manifolds, and after conceptualize the ideas involved to give a theory of commutative $n-$gerbes.

1. 3 composition series of affine manifolds.

Let’s recall first the classification of affinely locally trivial affine bundles up to translational isomorphisms with given fiber and base space.

We have two affine manifolds $(B, \nabla_B)$ and $(F, \nabla_F)$ which represent respectively the base space and the fiber of the affine bundles we intend to classify.

The structure of a locally trivial affine bundle gives rise to a map $\pi_{BF}: \pi_1(B) \to \text{Aff}(F, \nabla_F)/T_F$ which defines a locally flat bundle $PB_F$ over $B$ with typical fiber $\text{Aff}(F, \nabla_F)/T_F$. This principal bundle induces a flat bundle $BT_F$ over $B$ with typical $T_F$.

Let $T'_F$ be the sheaf of affine sections of $BT_F$. We define a commutative gerbe $\mathcal{G}$ with lien $T'_F$ as follows:

To each open set $U$ of $B$, we associate the category $\mathcal{C}(U)$ of affinely locally trivial affine bundles with typical fiber $(F, \nabla_F)$ such that the canonical $UT_F$ bundle associated to it, is the restriction of $BT_F$ to $U$.

To represent the classifying two cocycle associated to $\mathcal{C}$, we consider an open covering $U_k$ of $B$ by connected affine charts, in each category $\mathcal{C}(U_k)$ we choose an objet which is an affine bundle isomorphic to $U_k \times (F, \nabla_F)$.

The trivialization of the bundle $PB_F$ gives rise to:

$$U_k \cap U_l \times \text{Aff}(F, \nabla_F)/T_F \to U_k \cap U_l \times \text{Aff}(F, \nabla_F)/T_F$$

$$(x, y) \mapsto (x, \tilde{t}_{kl}(y)),$$

where $\tilde{t}_{kl}$ is an element of $\text{Aff}(F, \nabla_F)/T_F$. Taking for each $k, l$ a map $t_{kl}$ over $\tilde{t}_{kl}$ which depends affinely of $x$, we obtain:

$$U_k \cap U_l \times (F, \nabla_F) \to U_k \cap U_l \times (F, \nabla_F)$$

$$(x, y) \mapsto (x, t_{kl}(x)y)$$

Then we have the family of maps

$$t_{klm}: U_k \cap U_l \cap U_m \to T_F$$

$$x \mapsto t_{kl}t_{lm}t_{km}^{-1}$$

This family of maps $t_{klm}$ is a 2–cocycle which classifies the gerbe $\mathcal{G}$. It is the obstruction to the existence of a locally trivial affine bundle over $(B, \nabla_B)$ associated to $BT_F$. In this case, the set of isomorphic classes of translational affine bundles (or the classes of $T'_F$ isomorphic bundles) with typical fiber $(F, \nabla_F)$ and base space $(B, \nabla_B)$ are given by the Cech cohomology group $H^1(B, T'_F)$ of the sheaf $T'_F$.
Remark.
Consider the composition series \((M_3, \nabla_{M_3}) \to (M_2, \nabla_{M_2}) \to (M_1, \nabla_{M_1})\). The fact that the affine bundles \((M_3, \nabla_{M_3}) \to (M_2, \nabla_{M_2})\) and the affine bundle \((M_2, \nabla_{M_2}) \to (M_1, \nabla_{M_1})\) are affinely locally trivial affine bundles does not imply that the bundle \((M_3, \nabla_{M_3}) \to (M_1, \nabla_{M_1})\) is a locally trivial affine bundle.

This can be illustrated by the following example. Consider the subgroup \(\Gamma\) of \(\mathbb{R}^3\) generated by the three maps \(f_1, f_2\) and \(f_3\) defined by

\[
\begin{align*}
f_1(x, y, z) &= (x + 1, y, z) \\
f_2(x, y, z) &= (x, y + 1, z) \\
f_3(x, y, z) &= (x + y, y, z + 1).
\end{align*}
\]

In the canonical basis \((e_1, e_2, e_3)\) of \(\mathbb{R}^3\). The quotient of \(\mathbb{R}^3\) by \(\Gamma\) is a three compact affine manifold \(M^3\). The projection \(p_1\) of \(\mathbb{R}^3\) on its subvector space \(V_2\) generated by \(e_2\) and \(e_3\) parallel to the one generated by \(e_1\), defines an affinely locally trivial affine bundle over the torus \(T^2\) endowed with its canonical riemannian flat structure.

The projection \(p_2\) of \(V_2\) on the line generated by \(e_2\) parallel to the one generated by \(e_3\) defines an affinely locally trivial affine bundle of the torus over the circle.

It is easy to see that the projection \(p_2 \circ p_1\) defines an affine bundle which is not an affinely locally trivial affine bundle over the circle.

Before to go to the general case, we will treat \(3\)– series of composition. So we have a sequence \((M_3, \nabla_{M_3}) \to (M_2, \nabla_{M_2}) \to (M_1, \nabla_{M_1})\) of affine maps such that \(f_{i-1} : (M_i, \nabla_{M_i}) \to (M_{i-1}, \nabla_{M_{i-1}})\) defines an affinely locally trivial affine bundle.

We have supposed that the total spaces of our bundles are compact and complete affine manifolds. This implies that \(\pi_1(F_2)\) is normal in \(\pi_1(M_3)\) and \(\pi_1(F_1)\) is normal in \(\pi_1(M_2)\), thus we have the following exact sequences

\[
1 \to \pi_1(F_2) \to \pi_1(M_2) \to \pi_1(M_1) \to 1,
\]

and

\[
1 \to \pi_1(F_1) \to \pi_1(M_3) \to \pi_1(M_2) \to 1.
\]

We denote by \(n_i, i = 1, 2, 3\) the dimensions of \(M_i\), and by \(l_i, i = 1, 2\) the dimensions of \(F_i\). We put \(\mathbb{R}^{n_3} = \mathbb{R}^{n_1} \oplus \mathbb{R}^{l_1} \oplus \mathbb{R}^{l_2}\).

Let \(\gamma\) be an element of \(\pi_1(M_3)\); the \((M_3, \nabla_{M_3})\) holonomy’s action of \(\gamma\) on \(\mathbb{R}^{n_3}\) is given by

\[
\gamma(x, y, z) = (A_1^i(x_1) + a_1^i, A_2^i(x_2) + B_2^i(x_1) + a_2^i, A_3^i(x_3) + B_3^i(x_1, x_2) + a_3^i),
\]

where \(A_1^i\) is an automorphism of \(\mathbb{R}^{n_1}\), \(A_2^i, A_3^i, i = 2, 3\) is an automorphism of \(\mathbb{R}^{l_2}\), \(i = 2, 3\), \(B_2^i : \mathbb{R}^{n_1} \to \mathbb{R}^{l_1}\) is a linear map, and \(B_3^i : \mathbb{R}^{n_1} \oplus \mathbb{R}^{l_1} \to \mathbb{R}^{l_2}\) is a linear map.
If $\gamma$ belongs to $\pi_1(F_2)$, then the restriction of $\gamma$ to $\mathbb{R}^{n_1} \oplus \mathbb{R}^{l_i}$ is the identity and $B_2^2 = 0$, since we have supposed the fibration $(M_3, \nabla_{M_3}) \rightarrow (M_2, \nabla_{M_2})$ to be an affinely locally trivial affine fibration.

The holonomy of $(M_2, \nabla_{M_2})$ is given by the action of $\pi_1(M_2)$ on $\mathbb{R}^{n_1} \oplus \mathbb{R}^{l_1}$ induced by the holonomy representation of $(M_3, \nabla_{M_3})$. In fact the restriction of $h_{M_3}$ to $\mathbb{R}^{n_1} \oplus \mathbb{R}^{l_1}$ factor through $\pi_1(M_2)$.

We deduce that the $(M_2, \nabla_{M_2})$ holonomy’s action of an element $\gamma$ of $\pi_1(M_2)$, is

$$(A_1^1(x) + a_1^1, A_2^2(y) + B_2^2(x) + d_2^2).$$

If $\gamma$ is an element of $\pi_1(F_1)$, then $A_1^1 = id$, $a_1^1 = 0$ and $B_2^2 = 0$ since we have supposed that the fibration $(M_2, \nabla_{M_2}) \rightarrow (M_1, \nabla_{M_1})$ is an affinely locally trivial affine fibration.

Now, considering the holonomy of $(M_2, \nabla_{M_2})$, we write the fact that $\pi_1(F_1)$ is normal in $\pi_1(M_2)$, then we obtain that the image of $B_2^2$ is contained in $H^0(\pi_1(F_1), \mathbb{R}^{l_1})$. Here the group $H^*(\pi_1(F), \mathbb{R}^{l_1})$, is the cohomology group of $\pi_1(F_1)$ related to its linear holonomy.

Considering the holonomy of $(M_3, \nabla_{M_3})$, and writing that $\pi_1(M_2)$ is a normal subgroup of $\pi_1(M_3)$, we obtain that $B_2^2$ is contained in $H^0(\pi_1(F_2), \mathbb{R}^{l_2})$.

We have the representation

$$\pi_1(M_1) \rightarrow Aff(F_1, \nabla_{F_1})/T_{F_1}$$

given by the first fibration. It leads to a flat bundle $\pi_{11}(M_1)$ over $M_1$ with typical fiber $Aff(F_1, \nabla_{F_1})/T_{F_1}$.

The bundle $(M_3, \nabla_{M_3}) \rightarrow (M_2, \nabla_{M_2})$, gives rise to a representation

$$\pi_1(M_2) \rightarrow Aff(F_2, \nabla_{F_2})/T_{F_2}$$

this leads to a representation

$$\pi_{F_1}: \pi_1(F_1) \rightarrow Aff(F_2, \nabla_{F_2})/T_{F_2}$$

and to a flat principal bundle $\pi_{12}(F_1)$ over $F_1$ with typical fiber $Aff(F_2, \nabla_{F_2})/T_{F_2}$ associated to $\pi_{F_1}$. Since $\pi_1(F_1)$ is a normal subgroup of $\pi_1(M_2)$, the action of $\pi_1(M_1)$ on this last bundle leads to a representation

$$\pi_1(M_1) \rightarrow Aut(\pi_{12}(F_1))$$

where $Aut(\pi_{12}(F_1))$ is the group of automorphisms of the bundle $\pi_{12}(F_1)$.

We will classify 3–composition series of affine manifolds given:

The base space $(M_1, \nabla_{M_1})$ and the fibers spaces $(F_1, \nabla_{F_1}), (F_2, \nabla_{F_2})$.

The representations $\pi_1(M_1) \rightarrow Aff(F_1, \nabla_{F_1})/T_{F_1}, \pi_1(F_1) \rightarrow Aff(F_2, \nabla_{F_2})/T_{F_2}$, and the representation $\pi_1(M_1) \rightarrow Aut(\pi_{12}(F_1))$. 

18
The main tool we will use to make the classification of 3 composition series is 2 gerbe, on this purpose, let recall some definitions.

**Definitions 1.1.**

A category is a family of objects and for each pair of objects $X, Y$, a set of arrows $\text{Hom}(X, Y)$, which satisfy usual rules.

An 2 category is of a family of objects, and for each pair of objects $X, Y$ the arrows is a category $C(X, Y)$ which satisfies usual rules see [Bry-Mc].

An 2 gerbe on a manifold $M$ is a sheaf of 2 categories $C$ on $M$ which satisfies the following see [Bry-Mc], [Bre]:

1. For every element $z$ of $M$, there exists an open set $U_z$ which contains $z$ such that $C(U_z)$ is not empty.
2. Let $U$ be an open set, for any pair of object $x$ and $y$, contained in $C(U)$, there is an open covering $(U_i) \in I$ of $U$ such that, for any $i$, the set of arrows between the restriction of $x$ and $y$ to $U_i$ is not empty.
3. For any 1 arrow $f : x \to y$ in $C(U)$, there is an inverse $g : y \to x$ up to a 2 arrow.
4. The two arrows are invertible.

We will associate to our problem a sheaf of 2 categories.

First we define the following sheaf of categories $C_1$ on $M_1$:

To every 1–connected open set $U$ of $M_1$ of affine chart, we associate the category of affinely locally trivial affine bundles over $U$ with typical fiber $(F_1, \nabla_{F_1})$ such that the induced $\pi_{11}(U)$ bundle, is the restriction of $\pi_{11}(M_1)$ to $U$.

We now define on $M_1$ the sheaf of 2 categories $C_2$.

For every open set $U$, consider an element $e$ of $C_1(U)$. It is an affinely locally trivial affine bundle over $U$ with typical fiber $(F_1, \nabla_{F_1})$. We associate to $e$ the sheaf of category $C_2(e)$ which objects are a.l.t bundles over $e$ with typical fiber $(F_2, \nabla_{F_2})$ such that on $e$, the flat bundle with typical fiber $\text{Aff}(F_2, \nabla_{F_2})/T_{F_2}$ induced, is the one induced by the representation $\pi_1(F_1) \to \text{Aff}(F_2, \nabla_{F_2})/T_{F_2}$.

Gluing conditions for the 2 sheaf $C_2$ are done using the representation $\pi_1(M_1) \to \text{Aut}(\pi_{12}(F_1))$.

This enables to associate to the open set $U$, $C_2(U) := \cup C_2(e), e \in C_1(U)$.

We will see that $C_2$ is in fact a 2 gerbe. An important fact is that this 2 gerbe is defined recursively, that is, we have first define the gerbe $C_1$. This will considerably simplify the expression of the classifying 3 cocycle which in this case will be an usual Cech 3 cocycle.

Let’s precise now what 1 and 2 arrows are in the 2 category $C_2$.

Let $e_1$ and $e_2$ be objects of $C_1(U)$ where $U$ is a simply connected open set of affine chart. A map between $e_1$ and $e_2$ can be represented by an affine map $t : U \to T_{F_1}$ acting on $U \times (F_1, \nabla_{F_1})$ as follows:

\[
U \times (F_1, \nabla_{F_1}) \longrightarrow U \times (F_1, \nabla_{F_1})
\]

\[
(x, y) \mapsto (x, t(x)y).
\]
The map $t$ can be viewed as an automorphism of the site of open sets of $U \times F_1$, so the map $t$ induces a functor between the sheaf of categories $C_2(e_1)$ and $C_2(e_2)$, such maps $t$ are 1 arrows of our 2 category.

A 2 arrow of our 2 category $C_2(U)$ over the one arrow $t$ can be represented as a family of affine arrows

$$t_{ij} : U \times U_{ij} \times (F_2, \nabla F_2) \to U \times t(U_{ij}) \times (F_2, \nabla F_2)$$

$$(x, y, z) \mapsto (x, t(x)y, t_1(x,y)z),$$

where $(U_{ij})$ is an open covering of $(F_1, \nabla F_1)$ by one connected open set of affine charts, and $t_1 : U \times U_{ij} \to T_{F_2}$ is an affine map.

It is easy to see that our category satisfy the axioms which defines 2 gerbes, so we have:

**Proposition 1.2.** The 2-sheaf $C_2$ is a 2 gerbe.

The classifying three cocycle.

The representation $\pi_1(M_1) \to Aff(F_1, \nabla F_1)/T_{F_1}$ induces an affine flat bundle $V_1$ over $(M_1, \nabla M_1)$ with typical fiber $T_{F_1}$, we will call $S_1$ the sheaf of affine sections of this bundle. The representation $\pi_1(F_1) \to Aff(F_2, \nabla F_2)/T_{F_2}$ defines a flat bundle $V_2$ over $(F_1, \nabla F_1)$ with typical fiber $T_{F_2}$; we will call $S_12$ the sheaf of affine sections of this bundle.

The representation $\pi_1(M_1) \to Aut(\pi_12)(F_1)$ induces the sheaf $S_{123}$ of affine maps $U \to S_{12}$, where $U$ is an open set of $M_1$. This sheaf is a locally constant sheaf over $M_1$.

Now we consider an open covering $(U_i)_{i \in I}$ of $(M_1, \nabla M_1)$ by 1 connected affine charts. For each $i$ the 2 category $C_2(U_i)$ is not empty. We will choose in each $U_i$ an element $e_i$ of $C_2(U_i)$. Let $i, j$ such that $U_i \cap U_j$ is not empty. The restriction of $e_i$ and $e_j$ to $U_i \cap U_j$ gives rise to an arrow $\phi_{ij} : e_i|_{U_i \cap U_j} \to e_j|_{U_i \cap U_j}$.

This arrow can be expressed as a map

$$U_i \cap U_j \times (F_1, \nabla F_1) \to U_i \cap U_j \times (F_1, \nabla F_1)$$

$$(x, y) \mapsto (x, t_{ij}(x)y).$$

Recall that the map $\phi_{ij}$ can also be viewed as a functor $C_2(e_i)_{U_i \cap U_j} \to C_2(e_j)_{U_i \cap U_j}$.

Now we consider the restriction of the functor $\phi_{ij} \circ \phi_{jk} \circ \phi_{ik}^{-1} = \psi_{ijk}$ to the family of $U_i \cap U_j \cap U_k$.

It can be represented by a family of maps

$$U_i \cap U_j \cap U_k \times U_{ij} \times (F_2, \nabla F_2) \to U_i \cap U_j \cap U_k \times t_{ij}t_{jk}t_{ik}^{-1}(U_{ij}) \times (F_2, \nabla F_2)$$

$$(x, y, z) \mapsto (x, t_{ij}(x)t_{jk}(x)t_{ik}(x)(y), u_{ijk}(x, y)(z)),$$

where $U_{ij}$ is a one connected open set of affine chart of $(F_1, \nabla F_1)$, $u_{ijk} : U_i \cap U_j \cap U_k \times U_{ij} \to T_{F_2}$ is an affine map.
Now, we can restrict $\psi_{ijk}$ to $U_i \cap U_j \cap U_k \cap U_l = U_{ijkl}$, writing the boundary of the chain $\psi_{ijk}$, we obtain $\rho_{ijkl} = \psi_{ijkl} \psi_{ikl}^{-1} \psi_{ijl}^{-1} \psi_{ijl}^{-1}$, $\rho_{ijkl}$ can be viewed as a map

$$U_{ijkl} \times U_i \times (F_2, \nabla_{F_2}) \longrightarrow U_{ijkl} \times U_i \times (F_2, \nabla_{F_2})$$

$$(x, y, z) \longmapsto (x, y, \psi_{ijkl}(x, y)z),$$

as the family of map $t_{ij}t_{jk}t_{kl}$ defines a 2 Cech cocycle of $S_1$. The family $\rho_{ijkl}$ can be viewed as sections of the bundle $S_{123}$.

**Theorem 1.3.** The family of maps $\rho_{ijkl}$ that we have just define is a 3 Cech cocycle.

**Proof.**

We must calculate the boundary of the family of $\rho_{ijkl}$.

Let $U_{ijklm}$ be the intersection $U_i \cap U_j \cap U_k \cap U_l \cap U_m$, we have:

$$d(\rho_{ijklm}) =$$

$$\rho_{ijklm} - \rho_{iklm} - \rho_{ijkm} - \rho_{ijkl} + \rho_{ijkl}$$

$$= \psi_{ijklm}^{-1} \psi_{iklm}^{-1} \psi_{ijkm}^{-1} \psi_{ijkl}^{-1}$$

$$- \psi_{iklm}^{-1} \psi_{ikm}^{-1} \psi_{ijm}^{-1} \psi_{ijl}^{-1}$$

$$+ \psi_{ijkm}^{-1} \psi_{ijm}^{-1} \psi_{ijl}^{-1}$$

$$- \psi_{ijklm}^{-1} \psi_{ijkl}^{-1} = 0.$$

The associated 3 cocycle $\rho_{ijkl}$ is not the obstruction to the existence of a composition serie, suppose that it vanishes.

This means that there is a family of maps

$$h_{ijk} : U_{ijk} \times U_i \times (F_2, \nabla_{F_2}) \longrightarrow U_{ijk} \times (U_i \times (F_2, \nabla_{F_2})$$

$$(x, y, z) \longmapsto (x, \psi_{ijk}(x)y, h'_{ijk}(x, y)z),$$

(where the map $h'_{ijk} : U_{ijk} \times U_i \to T_{F_2}$ is an affine map which boundary is $\rho_{ijkl}$) which is a 2 cocycle of $S_1 \oplus S_{123}$.

We have:

**Theorem 1.4.** If the cocycle $\rho_{ijkl}$ is trivial, then two cocycle $h_{ijk}$ that we have just define is the obstruction to the existence of a composition serie associated to the bundles $S_1$, $S_{12}$ and $S_{123}$.

**Proof.**

If the cocycle $h_{ijk}$ is trivial, then there exists a family of maps $b_{ij} : U_i \cap U_j \longrightarrow S_1 \oplus S_{123}$ such that the family of map $(t_{ij} + b_{ij})$ define a 1 Cech cocycle. This implies that the family of maps $t_{ij}$ is a 1 cocycle up to a 1 boundary, then it defines an affine bundle over $(M_1, \nabla_{M_1})$ with typical fiber $(F_1, \nabla_{F_1})$ associated to $\pi_{11}$. Let $(M_2, \nabla_{M_2})$ be its total space. The obstruction of the existence of an affine bundle over $(M_2, \nabla_{M_2})$ with typical fiber $(F_2, \nabla_{F_2})$ associated to $S_1$, $S_{12}$ and $S_{123}$ is given by $h_{ijk}$, since the open set $U \times U_i$ used to build the obstruction $\rho$ can be viewed as open subsets of $M_2$.  

21
Remark.
When the obstruction $\rho$ vanishes, the cocycle $h_{ijk}$ define on $M_1$ a gerbe which can be viewed as trivial 2 gerbe.

2. The general case.
In this part, we will classify composition series $(M_n, \nabla_{M_n}) \to \ldots \to (M_2, \nabla_{M_2}) \to (M_1, \nabla_{M_1})$.

We will denote by $(F_i, \nabla_{F_i})$ the fiber of the a.l.t affine bundle $f_i : (M_{i+1}, \nabla_{M_{i+1}}) \to (M_i, \nabla_{M_i})$.

Let $i < j \leq n$, the map $f_{ij} = f_{j-1} \circ f_{j-2} \circ \ldots \circ f_i$, is an affine map $f_{ij} : (M_j, \nabla_{M_j}) \to (M_i, \nabla_{M_i})$. Since this map is a submersion and $M_j$ is compact, we deduce that $(M_j, \nabla_{M_j})$ is the total space of a locally trivial differentiable fibration over $(M_i, \nabla_{M_i})$. The Serre bundle theorem implies the following exact sequence

$$1 \to \pi_1(F_i) \to \pi_1(M_{i+1}) \to \pi_1(M_i) \to 1$$

when $j = i + 1$.

The affine map $f_{ij}$ define an affine bundle which is not necessarily an affinely locally trivial affine bundle.

Recall that the map $f_i$ gives rise to a representation $\pi_i : \pi_1(M_i) \to \text{Aff}(F_i, \nabla_{F_i})/T_{F_i}$, and to a flat bundle $V_i$ over $M_i$ with typical $T_{F_i}$.

Thus we have a flat bundle $V_{n-1}$ over $M_{n-1}$ with typical fiber $T_{F_{n-1}}$. The group $\pi_1(F_{n-2})$ is a subgroup of $\pi_1(M_{n-1})$, thus we have a flat bundle $V_{n-1,n-2}$ over $F_{n-2}$ with typical fiber $T_{F_{n-2}}$ induced by $V_{n-1}$.

The fundamental group $\pi_1(M_{n-2})$ of $M_{n-2}$, acts on $V_{n-1,n-2}$ via the representation $\pi_{n-1}$. This action defines a flat $V_{n-1,n-2}$ over $M_{n-2}$ with typical fiber $V_{n-1,n-2}$. This bundle also gives rise to a bundle $V_{n-1,n-2,n-3}$ over $F_{n-3}$ as $\pi_1(F_{n-3})$ is a subgroup of $\pi_1(M_{n-2})$. Recursively, we can define bundle $V_{n-1 \ldots n-i}$ over $n-i$ with typical fiber $V_{n-1 \ldots n-i+1}$ over $F_i$.

We can also define the representation $\pi_{n-1,n-2}$ which is the restriction of $\pi_{n-1}$ to $\pi_1(F_{n-2})$. This representation induces on $F_{n-2}$ a flat bundle $\pi'_{n-1,n-2}$ with typical fiber $\text{Aff}(F_{n-1}, \nabla_{F_{n-1}})/T_{F_{n-1}}$, $\pi_1(M_{n-2})$ acts on this bundle via $\pi_{n-1}$, one deduces a flat bundle over $M_{n-2}$ with typical fiber $\pi'_{n-1,n-2}$ which induce a flat bundle $\pi'_{n-1,n-2,n-3}$ over $F_{n-3}$. Recursively, we can define bundle $\pi'_{n-1,\ldots,1}$.

Remark also that considering the composition serie $(M_j, \nabla_{M_j}) \to \ldots \to (M_1, \nabla_{M_1})$ for $j < n$, one can also define the bundle $V_{j-1 \ldots i}$ with typical fiber $V_{j-1 \ldots i+1}$ and base space $F_i$.

Let let $S_{n-1,n-2}$ be the sheaf of affine sections of $V_{n-1,n-2}$ one may define the sheaf $S'_{n-1,n-2,n-3}$ of affine sections of $S_{n-1,n-2}$ over $F_{n-3}$. Recursively, we can also define the sheaf $S'_{n-1, \ldots, i}$ of affine sections of $S_{n-1, \ldots, i}$ over $M_i$. The gluing conditions for those sheaves are given by the bundles $\pi'_{n-1, \ldots, i}$. One can also define in the same way the bundles $S_{i, \ldots, 1}$, $i \leq 1$.

Remark.
The bundles $V_{j-1}$ and $S_{j-1}$ depend only of the affine structures of $(F_1, \nabla_{F_1}), \ldots, (F_j, \nabla_{F_j})$ and $(M_1, \nabla_{M_1})$. They can be defined without suppose the existence of a composition serie.

The classifying $n$–cocycle.

Given bundles $V_{i-1}, \ldots, V_1$, $1 \leq i \leq n$ as above,

We want to classify all composition series $(M_n, \nabla_{M_n}) \to \ldots \to (M_1, \nabla_{M_1})$ associated to the family of bundles $V_i$. To make this classification, we are first going to define an $n$–cocycle.

First we consider the trivialization of the flat bundle over $M_1$ with typical fiber $Aff(F_1, \nabla_{F_1})/T_{F_1}$ over $M_1$ induced by $\pi_1$.

It is defined by

$$U_i \cap U_j \times Aff(F_1, \nabla_{F_1})/T_{F_1} \to U_i \cap U_j \times Aff(F_1, \nabla_{F_1})/T_{F_1}$$

$$(x, y) \mapsto (x, t_{ij}(y))$$

Consider for each $x$ in $U_i \cap U_j$ an element $t_{ij}(x)$ of $Aff(F_1, \nabla_{F_1})$ over $t_{ij}$ which depends affinely of $x$, one may define the 2 cocycle

$$t_{ijk} : U_i \cap U_j \cap U_k \to T_{F_1}$$

$$x \mapsto t_{ij}(x)t_{jk}(x)t_{ki}(x).$$

The family of $t_{ijk}$ may be considered as local sections of the bundle $S_1$. It is the cocycle associated to the gerbe which at $U_i$ associated the category of a.l.t affine bundles over $U_i$ with typical fiber $(F_1, \nabla_{F_1})$, such that the bundle over $U_i$ with typical fiber $T_{F_1}$ associated is the restriction of $V_1$.

The map $t_{ijk}$ induces a functor on the category of open sets of $U_i \cap U_j \cap U_k \times (F_1, \nabla_{F_1})$.

Consider a covering $U_{i_1}$ of $(F_1, \nabla_{F_1})$ by affine 1 connected affine charts.

Let $U_{ijk} = U_i \cap U_j \cap U_k$. We associate to $U_{ijk} \times U_i$ the category of a.l.t affine bundles with typical fiber $(F_2, \nabla_{F_2})$ such that the bundle with typical fiber $T_{F_2}$ associated is induced $V_2$.

On $U_{ijkl} = U_i \cap U_j \cap U_k \cap U_l$, the boundary of $t_{ij}t_{jk}t_{kl}$ gives rise to the map

$$v_{ijkl} : U_{ijkl} \times U_{i_1} \times (F_2, \nabla_{F_2}) \to U_{ijkl} \times U_{i_1} \times (F_2, \nabla_{F_2})$$

$$(x, x_1, x_2) \mapsto (x, x_1, u_{ijkl}(x, x_1)(x_2)),$$

where the map $u_{ijkl}$ are affine sections of the bundle $S_{i_21}$. The family of maps $u_{ijkl}$ is a 3 cocycle.

Let $U_{i_1, \ldots, i} = U_1 \cap U_2, \ldots, U_i$. For each $j$, we will consider an open covering $U_{i_j}$ of $(F_j, \nabla_{F_j})$ by 1– connected open sets of affine charts.

Suppose that we have defined a family of maps

$$v_{i_1, \ldots, i} : U_{i_1, \ldots, i} \times U_{i_1, \ldots, i} \times (F_{j-2}, \nabla_{F_{j-2}}) \to U_{i_1, \ldots, i} \times U_{i_1, \ldots, i} \times U_{i_1, \ldots, i} \times (F_{j-2}, \nabla_{F_{j-2}})$$

23
Suppose that the chain\( S \)Whitney sum of the bundles \( U \)successively is a \( j \)sequence associated to the family of bundles \( \{ U_i \} \)
the boundary \( d \)considered as an element of \( T \)are affinely locally trivial affine bundles with typical fiber \(( \mathbb{A}^1, \mathbb{R}^2 \)\).

It is the boundary of a chain which represent a \( j \)of an

\[ d(T) \]

Proposition 2.1. The Cech chain \( v_{1...j+1} \) that we have just define recursively is a \( j \)Cech cocycle.

Proof.
The proof will be made recursively. We have already verify the result if \( j = 2 \).

Suppose that the chain \( v_{1...j} \) is a Cech cocycle for \( k \leq j \), then the writing the boundary \( d(v_{1...j+1}) \) of \( v_{1...j+1} \), we obtain

\[
(x, x_1, ..., x_{j+2}) \mapsto (x, x_1, ..., x_{j+3}, u_{1...j+3}(x, x_1, ..., x_{j+2})(x_{j+2}))
\]

(where the family of maps \( u_{1...j-3} \) are local affine sections of the bundle \( S_{j-1} \) which represent a \( j-1 \) Cech cocycle. Then on \( U_{1...j} \times U_i \times \cdots \times (F_{j-2}, \nabla_{F_{j-2}}) \), one can define the sheaf of category \( C_j \) such that the objects \( C_j(U_{1...j} \times \cdots \times U_{i-1}) \) are affinely locally trivial affine bundles with typical fiber \(( F_{j-1}, \nabla_{F_{j-1}} ) \) over \( U_{1...j} \times U_i \times \cdots \times (F_{j-2}, \nabla_{F_{j-2}}) \), and the canonical flat vector bundle with typical fiber \( T_{F_{j-1}} \) associated is the restriction of \( V_{j-1} \).

The map \( v_{1...j} \) induces a functor \( u_{1...j} \) in the category \( C_j(U_{i-1} \times \cdots \times U_{i-2}) \).

The restriction of the composition \( u_{2...j+1} \circ \cdots \circ u_{1...j} \) on \( U_{1...j+1} \times U_i \times \cdots \times U_j \) induces a map

\[
v_{1...j+1} : U_{1...j+1} \times U_i \times \cdots \times U_{j+2} \times (F_{j-1}, \nabla_{F_{j-1}}) \rightarrow U_{1...j+1} \times U_i \times \cdots \times U_{j+2} \times (F_{j-1}, \nabla_{F_{j-1}})
\]

\[
(x, x_1, ..., x_{j+2}) \mapsto (x, x_1, ..., x_{j-2}, u_{1...j-2}(x, ..., x_{j-2})(x_{j-1}))
\]

The cocycle \( v_{1...n+1} \) is not the obstruction of the existence of a composition sequence associated to the family of bundles \( S_{n-1} \). If its cohomology class is zero, its means that there exists a chain \( a_{1...n} \) which boundary is \( v_{1...n+1} \). Suppose that the chain \( z_{n-1} = a_{1...n} + v_{1...n} \) considered as an element of the Whitney sum of the bundles \( S_{n-1} \oplus S_{n-2} = T_{n-1} \) is an \( n-1 \) cocycle.

If the cohomology class of the cocycle \( z_{n-1} \) is zero, then it is the boundary of an \( n-2 \) cocycle \( a_{1...n-1} \). We can define the chain \( z_{n-2} = a_{1...n-1} + v_{1...n-1} \) considered as an element of \( T_{n-1} \oplus S_{n-2} = T_{n-1} \oplus S_{n-2} \).

Suppose that we have define the cocycle \( z_{n-2} \) and its cohomology class is zero. It is the boundary of a chain \( a_{1...n-i} \). This means that the chain \( z_{n-(i+1)} = a_{1...n-i} + v_{1...n-i} \) is an \( n-i \) cocycle viewed as an element of \( T_{n-1} \oplus S_{n-(i+2)} = T_{n-1} \oplus S_{n-(i+2)} \).

24
Theorem 2.2. Suppose that we can construct a chain $z_2$ by the processus that we have just describe; then it is the obstruction to the existence of a composition serie associated to the family $S_{n-1}$. When it vanishes the set of equivalence classes of composition series is given by $H^1(M_1, T_{n-1})$.

Proof.
Suppose that we can define the cocycle $z_2$ and its cohomology class is zero, then it is the boundary of an element $z_1$ of $T_{n-1}$.

This implies that up to a boundary the chain $t_{ij}t_{jk}t_{ki}$ is zero, we deduce that we can solve the first extension problem, there exists a bundle $(M_2, \nabla_{M_2}) \rightarrow (M_1, \nabla_{M_1})$ associated to $S_1$.

The obstruction of the existence of a composition serie $(M_3, \nabla_{M_3}) \rightarrow (M_2, \nabla_{M_2}) \rightarrow (M_1, \nabla_{M_1})$ is also given by the class of $z_2$ which is zero, so we can solve the second extension problem.

Suppose that we can solve the extension problem $(M_i, \nabla_{M_i}) \rightarrow \ldots \rightarrow (M_1, \nabla_{M_1})$ then obstruction to solve the extension problem $(M_{i+1}, \nabla_{M_{i+1}}) \rightarrow \ldots \rightarrow (M_1, \nabla_{M_1})$ is also given by the cohomology class of $z_2$ which is zero.

Recursively, we deduce that we can solve the extension problem $(M_n, \nabla_{M_n}) \rightarrow \ldots \rightarrow (M_1, \nabla_{M_1})$.

3. The Conceptualization.

The resolution of the extension problem, given the base space $(M_1, \nabla_{M_1})$ and the bundle $S_1$ has been done using the gerbe theory. It is natural to think that the existence of a composition serie $(M_n, \nabla_{M_n}) \rightarrow \ldots \rightarrow (M_1, \nabla_{M_1})$ must be solve using $n-1$-gerbe theory. In this part, we are going to build a commutative $n$-gerbe theory.

On $n$–Categories.

Supposed that the notion of $i$ category is defined. An $i+1$ category $C_{i+1}$, is given by

The class of objects $O(C_{i+1})$,

the morphisms $Hom_{C_{i+1}}(x, y)$ between two objects $x$ and $y$ of $C_{i+1}$ which is an $i$ category,

For objects $x, y, z$ in $C_{i+1}$, the composition $i$ functor

$\alpha_i : Hom_{C_{i+1}}(x, y) \times Hom_{C_{i+1}}(y, z) \rightarrow Hom_{C_{i+1}}(x, z)$

For each objects $x_1, \ldots, x_{i+4}$ in $C_{i+1}$, we will assume that the following strict condition: the composition

$Hom_{C_{i+1}}(x_{i+3}, x_{i+4}) \times \ldots \times Hom_{C_{i+1}}(x_2, x_3) \times Hom_{C_{i+1}}(x_1, x_2) \rightarrow Hom_{C_{i+1}}(x_1, x_{i+4})$

does not depend of the order in which it is made. More others conditions need to be specified, but we don’t need them.
The notion of sheaf of $i$ categories.

Now we define recursively the notion of sheaf of $i$ categories on a topological space $M$.

We assume known the notion of sheaves of sets.

Supposed that we have already defined the notion of sheaves of $i$ categories.

An sheaf of $i+1$ categories $C_{i+1}$ on the topological space $M$, will be a map which assign to each open set $U$ a $i+1$ category $C_{i+1}(U)$, such that

For each inclusion $U \hookrightarrow V$, there exists a $i+1$ functor

$$c_{U,V} : C_{i+1}(V) \to C_{i+1}(U)$$

which satisfies $c_{U,V} \circ c_{V,W} = c_{U,W}$ for any open sets $U, V$ and $W$ such that $U \hookrightarrow V \hookrightarrow W$.

Gluing condition for objects. Consider an open covering $(U_j)$ of $M$, and in each $U_i$ an object $A_j$, we restrict each object to $U_{j_1}...j_{i+3}$, we assume that if the composition

$$\text{Hom}_{C_{U_{j_1}...j_{i+3}}}(A_{j_1}+2, A_{j_1}+3) \times \text{Hom}_{C_{U_{j_1}...j_{i+3}}}(A_{j_1}+1, A_{j_1}) \to \text{Hom}_{C_{U_{j_1}...j_{i+3}}}(A_{j_1}, A_{j_1}+3)$$

does not depend on the order in which it is made then there exists a global object $A$ which restriction on each $U_i$ is $A_i$.

Gluing condition for arrows. For each pair of global objects $x$ and $y$, $\text{Hom}_{C_{i+1}(M)}(x, y)$ is a sheaf of $i$ categories.

The notion of $n$ gerbe.

Now, we will define the notion of $n$ gerbe where $n \geq 2$.

Consider a topological space $M$, endowed with a sheaf of $i$ categories $C_i$ for each $i = 1, ..., n$ such that

$g_1$

For each open set $U$, the set of objects of the category $C_i(U)$ is the same for each $i = 1, ..., n$.

$g_2$

For each $x \in M$, we suppose that there is a neighborhood $U_x$ of $x$ such that $C_i(U_x)$ is not empty.

$g_3$

The sheaf of categories category $C_1$ is a gerbe with lien the abelian sheaf $T_1$ over $M$.

$g_4$

The arrows between two objects $x, y$ considered as elements of the 2 category $C_2(U)$ is a category $\text{Hom}_{C_2(U)}(x, y)$ which objects are elements of $\text{Hom}_{C_1(U)}(x, y)$ the arrows between the objects $x$ and $y$ considered as elements of $C_1(U)$. The composition $\circ_2 : \text{Hom}_{C_2(U)}(x, y) \times \text{Hom}_{C_2(U)}(y, z) \to \text{Hom}_{C_2(U)}(x, z)$ transforms two 1 arrows $f$ and $g$ to $gf$ where the product $gf$ is considered in respect to the one of $\text{Hom}_{C_1(U)}(x, y) \times \text{Hom}_{C_1(U)}(y, z) \to \text{Hom}_{C_1(U)}(x, z)$.

We recall that for each $x$ and $y$ in $C_2(U)$ the 1 arrows between $x$ and $y$ are the objects of the category $\text{Hom}_{C_2(U)}(x, y)$, we have just precise how the functor
\( o_2 \) acts on objects, not how it acts on maps. We deduce that the product is known up to 2 arrows.

The 1 arrows between \( x \) and \( y \) considered as elements of \( C_3(U) \) is \( \text{Hom}_{C_1(U)}(x,y) \), the 2 arrows are the 2 arrows of \( \text{Hom}_{C_2(U)}(x,y) \).

The product \( \alpha_3 : \text{Hom}_{C_3(U)}(x,y) \times \text{Hom}_{C_3(U)}(y,z) \to \text{Hom}_{C_3(U)}(x,z) \) transform two 1 arrows \( f \) and \( g \) onto \( gf \), the product is considered in respect to the one of \( C_1(U) \) and two 2 arrows \( h \) and \( k \) in \( kh \) the product, considered is the one of \( C_2(U) \). We can also say that the product is known up to 3 arrows.

Recurively, suppose that we have defined the 1, ..., \( i \) arrows of the category \( C_i(U) \), then the 1 arrows of \( C_{i+1}(U) \) are the arrows of \( C_1(U) \), ..., the \( i \) arrows of the category \( C_{i+1}(U) \) is the \( i \) arrows of the category \( C_i(U) \).

Suppose also that we have define recursively the product of \( l \leq i \) arrows of \( \text{Hom}_{C_i(U)}(x,y) \). Then the product \( \alpha_{i+1} : \text{Hom}_{C_{i+1}(U)}(x,y) \times \text{Hom}_{C_{i+1}(U)}(y,z) \to \text{Hom}_{C_{i+1}(U)}(x,z) \) is an \( i \) functor which send two \( l \) arrows \( l \leq i \) in \( C_{i+1}(U) \) onto the one with respect to the composition in \( C_1(U) \). We can also remark that the product \( \alpha_{i+1} \) is defined up to \( i + 1 \) arrows.

We suppose that in \( C_1(U) \) the arrows are invertible, in \( C_2(U) \), a 1 arrow is invertible up to a 2 arrow, a 2 arrow is invertible, in \( C_1(U) \), a 1 arrow is invertible up to a 2 arrow, a 2 arrow is invertible up to a 3 arrow, ... a \( i - 1 \) arrow is invertible up to a \( i \) arrow, and an arrow are invertible.

Given an object \( f \) of \( \text{Hom}_{C_2}(x,x) \), where \( x \) is an object of \( C_2(U) \), the set of morphisms of \( f \) is isomorphic to \( T_2(U) \) where \( T_2 \) is a sheaf over \( M \). More generally, let \( g \) be an \( i - 1 \) map in the category \( C_i(U) \), the set of morphisms of \( g \) is isomorphic to \( T_i(U) \) where \( T_i \) is a sheaf over \( M \).

Any two objects of \( C_i(U) \) \( i \leq n \) are locally isomorphic.

**Definition 3.1.**

A family of sheaves of \( i \) categories \( C_i \) for each \( i \leq n \) which satisfy the conditions \( g_1, ..., g_8 \), will be called an \( n \)–gerbe.

**The classifying \( n + 1 \)–cocycle.**

In this part, we are going to consider \( n \) gerbes such that for each \( i \), the sheaf \( T_i \) is a commutative sheaf.

We are going to associate to each \( n \)–gerbe a classifying \( n + 1 \)–Cech cocycle which takes values in \( T_n \).

We will assume that \( M \) is a manifold, \( U_i \) an open covering of \( M \), such that \( C_i(U) \) is not empty and for each family \( \{i_1, ..., i_k\} \) we set \( U_{i_1, ..., i_k} = U_{i_1} \cap U_{i_2} \cap ... \cap U_{i_k} \).

We first choose in each open subset \( U_i \) an object \( x_i \), we may consider \( x_i \) as an object of \( C_1(U_i) \). If \( U_{ij} \) is not empty, then there is an arrow of \( t^i_{ij} \) between
considered as a

j to the sheaf of

n processus above. The fact that the

n recursively the existence of an

U such that for every open set

by 

H

n j by an

C arrows of

x a

if its cohomology class is trivial, it implies that it is the boundary of a chain

It results from

the

d
n it is equivalent to saying that the

n c classifying cocycle, and

a is the boundary of a chain

\[ \phi \text{ isomorphism} \]

Definitions 3.2.

An n-gerbe, will be said n-trivial if the cohomology class of the associated

n + 1-cocycle \( c_{n+1} \) that we have just define is trivial.

Let \( C \) and \( C' \) two n-gerbes associated to the family of sheaves \( T_1, ..., T_n \),

which are locally isomorphic i.e, each \( x \) in \( M \), has an open neighborhood \( U_x \)

such that \( C_i(U_x) \) and \( C_i'(U_x) \) are not empty, and objects of \( C_i(U_x) \) and \( C_i'(U_x) \)

are locally isomorphic. We will say that the locally isomorphic n-gerbes \( C \) and \( C' \) are equivalent if and only if for every open set \( U \) of \( M \), there is a n

isomorphism \( \phi_n(U) : C_n(U) \rightarrow C'_n(U) \) such that the restrictions of \( \phi_n(U) \) and

\( \phi_n(V) \) on \( U \cap V \) coincide with \( \phi_n(U \cap V) \).

Suppose that the class \([c_{n+1}]\) of the cocycle \( c_{n+1} \) is trivial. It implies that it

is the boundary of a chain \( a_n \). Let consider the chain \( z_n = a_n + c_n \) of \( T_{n-1} \oplus T_n \),

if its cohomology class is trivial, it implies that it is the boundary of a chain

\( a_{n-1} \), we set \( z_{n-1} = a_{n-1} + c_{n-1} \). Suppose that recursively we have defined the chain

\( z_{n-i} \).

The n gerbe is said n-i trivial, if we can define the cocycle \( z_{n-i} \) by the

processus above. The fact that the n gerbe is n-i trivial means that it can be

considered as a n-i-1 gerbe. as follows: we consider the n-i-1 gerbe \( C' \)

such that for every open set \( U \), \( C'_j(U) = C_j(U) \) if \( j < n-i-1 \). We define the

n-i-1 maps of \( C_{n-i-1}(U) \) to be \( T_{n-i-1}(U) \oplus ... \oplus T_n(U) \).

Proposition 3.3.

The set of equivalence classes of locally isomorphic n trivial gerbes is given

by \( H^{n+1}(M, T_n) \).

Proof.

We have assigns to every n-gerbe a n + 1 cocycle \( c_{n+1} \) if it class is trivial,

it is equivalent to saying that the n-gerbe is trivial. This implies that the map

\( C \rightarrow c_{n+1} \) is injective.

On the other hand, given a cocycle \( c_{n+1} \), we can find a family of cocycles
\( c_2, ..., c_n \), such that \( c_{i+1} \) is induced by \( c_i \) by the processus described to build the

classifying cocycle, and \( c_n \) induces \( c_{n+1} \).
We can extend our definition and defines $\infty$-gerbe.

**Definition 3.4.**

The family $C_n$ $n \in \mathbb{N}$ of $n$-sheaves of categories over the manifold $M$ is an $\infty$-gerbe $C$ if and only if for each $n \in \mathbb{N}$, the family $C_1, \ldots, C_n$ is an $n$-gerbe and the family of sheaves $T_n$, $n \in \mathbb{N}$ is an inductive system such that the map $i_n : T_n \to T_{n+1}$ sends $c_{n+1}$ onto $c_{n+2}$, where $c_{n+1}$ is the classifying cocycle associated to the gerbe $C_1, \ldots, C_n$. We will call the inductive limit of $c_n$ the classifying cocyle.

**Acknowledgements.**

The author would acknowledge the Abdus Salam ICTP, Trieste, Italy for support. This work has been done at ICTP.

**Bibliography.**

[Bo] Borel, A. Linear algebraic groups. W.A. Benjamin, Inc, New York-Amsterdam 1969.

[Bre] Breen, L. On the classification of $2$-gerbes and $2$-stacks. Asterisque, 225 1994.

[Br] Bredon, G. E. Sheaf theory. McGraw-Hill Book Co., 1967.

[Bry] Brylinski, J.L Loops spaces, Characteristic Classes and Geometric Quantization, Progr. Math. 107, Birkhauser, 1993.

[Br-Mc] Brylinski, J.L, Mc Laughlin D.A, The geometry of degree four characteristic classes and of line bundles on loop spaces I. Duke Math. Journal. 75 (1994) 603-637.

[Ca] Carriere, Y. Autour de la conjecture de L. Markus sur les variétés affines. Invent. Math. 95 (1989) 615-628.

[De] Deligne, P. Theorie de Hodge III, Inst. Hautes Etudes Sci. Publ. Math. 44 (1974), 5 – 77.

[Du] Duskin, J. An outline of a theory of higher dimensional descent, Bull. Soc. Math. Bel. Série A 41 (1989) 249-277.

[F] Fried, D. Closed similarity affine manifolds. Comment. Math. Helv. 55 (1980) 576-582.

[F-G] Fried, D. Goldman, W. Three-dimensional affine crystallographic groups. Advances in Math. 47 (1983), 1-49.

[F-G-H] Fried, D. Goldman, W. Hirsch, M. Affine manifolds with nilpotent holonomy. Comment. Math. Helv. 56 (1981) 487-523.

[G1] Goldman, W. Two examples of affine manifolds. Pacific J. Math. 94 (1981) 327-330.

[G2] Goldman, W. The symplectic nature of fundamental groups of surfaces. Advances in in Math. 54 (1984) 200-225.

[G3] Goldman, W. Geometric structure on manifolds and varieties of representations. 169-198, Contemp. Math., 74

29
[G-H1] Goldman, W. Hirsch, M. The radiance obstruction and parallel forms on affine manifolds. Trans. Amer. Math. Soc. 286 (1984), 629-949.

[G-H2] Goldman, W. Hirsch, M. Affine manifolds and orbits of algebraic groups. Trans. Amer. Math. Soc. 295 (1986), 175-198.

[Gr] Grothendieck, A. Pursuing stacks, preprint available at University of Bangor.

[God] Godbillon, C. Feuilletages. Etudes géométriques. Progress in Mathematics, 98.

[K] Koszul, J-L. Variétés localement plates et convexité. Osaka J. Math. (1965), 285-290.

[K] Koszul, J-L. Déformation des connexions localement plates. Ann. Inst. Fourier 18 (1968), 103-114.

[Mc] Maclane, S. Homology. Springer-Verlag, 1963.

[Ma] Margulis, G. Complete affine locally flat manifolds with a free fundamental group. J. Soviet. Math. 134 (1987), 129-134.

[Mi] Milnor, J. W. On fundamental groups of complete affinely flat manifolds, Advances in Math. 25 (1977) 178-187.

[S-T] Sullivan, D. Thurston, W. Manifolds with canonical coordinate charts: some examples. Enseign. Math 29 (1983), 15-25.

[T1] Tsemo, A. Thèse, Université de Montpellier II. 1999.

[T2] Tsemo, A. Automorphismes polynomiaux des variétés affines. C.R. Acad. Sci. Paris Série I Math 329 (1999) 997-1002.

[T3] Tsemo, A. Décomposition des variétés affines. Bull. Sci. Math. 125 (2001) 71-83.

[T4] Tsemo, A. Dynamique des variétés affines. J. London Math. Soc. 63 (2001) 469-487.

[T5] Tsemo, A. Fibrés affines to be published in Michigan J. Math. vol 49.