THE GHIRLANDA-GUERRA IDENTITIES WITHOUT AVERAGING

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Abstract. The Ghirlanda-Guerra identities are one of the most mysterious features of spin glasses. We prove the GG identities in a large class of models that includes the Edwards-Anderson model, the random field Ising model, and the Sherrington-Kirkpatrick model in the presence of a random external field. Previously, the GG identities were rigorously proved only ‘on average’ over a range of temperatures or under small perturbations.

1. Introduction

Consider the Sherrington-Kirkpatrick model of spin glasses [11] in the presence of an external field, defined as follows. Let $N$ be a positive integer. Let $(g_{ij})_{1 \leq i < j \leq N}$ and $(g_i)_{1 \leq i \leq N}$ be collections of i.i.d. standard Gaussian random variables. Fix three real numbers $\beta, \gamma, h$. Define a Hamiltonian $H_N$ on $\{-1, 1\}^N$ as

$$-H_N(\sigma) := \frac{\beta}{\sqrt{N}} \sum_{1 \leq i < j \leq N} g_{ij} \sigma_i \sigma_j + h \sum_{1 \leq i \leq N} \sigma_i + \gamma \sum_{1 \leq i \leq N} g_i \sigma_i.$$

Let $G_N$ be the (random) probability measure on $\{-1, 1\}^N$ that puts mass

$$G_N(\{\sigma\}) \propto e^{-H_N(\sigma)}$$

at each configuration $\sigma$. This is the Gibbs measure of the SK model. As usual, for a function $f : (\{-1, 1\}^N)^n \to \mathbb{R}$ we write

$$\langle f \rangle := \sum_{\sigma_1, \ldots, \sigma^n} f(\sigma^1, \ldots, \sigma^n) G_N(\{\sigma^1\}) \cdots G_N(\{\sigma^n\}).$$

Following the notation in Talagrand [12], we write

$$\nu(f) := \mathbb{E}(f).$$
Let $\sigma^1, \sigma^2, \ldots$ be i.i.d. configurations from the Gibbs measure. In the spin glass parlance, these are known as replicas. The overlap between two replicas $\sigma^\ell$ and $\sigma^{\ell'}$ is defined as

$$R_{\ell, \ell'} := \frac{1}{N} \sum_{i=1}^{N} \sigma_i^\ell \sigma_i^{\ell'}.$$ 

Now fix a number $n$, and a bounded measurable function $f : \mathbb{R}^{n(n-1)/2} \to \mathbb{R}$. Let $\mathcal{R}_n$ denote the collection $(R_{\ell, \ell'})_{1 \leq \ell < \ell' \leq n}$. Writing $f = f(\mathcal{R}_n)$ for simplicity, let

$$\delta_N(\beta, \gamma, h) := \nu(R_{1,n+1}f) - \frac{1}{n} \nu(R_{1,2}) \nu(f) - \frac{1}{n} \sum_{2 \leq \ell \leq n} \nu(R_{1, \ell} f).$$

The Ghirlanda-Guerra identities [7] claim that for any such $n$ and $f$, for almost all $\beta, \gamma, h$,

$$\lim_{N \to \infty} \delta_N(\beta, \gamma, h) = 0.$$ 

There is an extended form of the GG identities that claims more: in the infinite volume limit, the conditional distribution of $R_{1,n+1}$ given $\mathcal{R}_n$ is

$$\frac{1}{n} L_{R_{1,2}} + \frac{1}{n} \sum_{2 \leq \ell \leq n} \delta_{R_{1, \ell}},$$

where $L_{R_{1,2}}$ is the (unconditional) law of $R_{1,2}$, and $\delta_{R_{1, \ell}}$ is the point mass at $R_{1, \ell}$. Clearly, (2) is a special case of this extended version.

There is an early version of the GG identities that are sometimes called the Aizenman-Contucci identities; these appeared in [1]. The paper [1] also introduced the important notion of stochastic stability in spin glass models. An interesting connection between stochastic stability and the GG identities was discovered by Arguin [2].

The importance of the GG identities stems from their universal nature. The identities are supposed to hold in a wide array of spin glass models, both mean field and short range. A striking recent discovery of Panchenko [9], following on the insightful work of Arguin and Aizenman [3], indicates that the GG identities may actually solve the long-standing mystery of ultrametricity in spin glasses.

It is not difficult to show (see e.g. Talagrand [12], Section 2.12) that for any $a > 0$, and any $\beta, h$,

$$\lim_{N \to \infty} \int_{-a}^{a} |\delta_N(\beta, x, h)| dx = 0.$$ 

Similar results were proved for a general class of spin glass models, including the short range Edwards-Anderson model, by Contucci and Giardinà [5]. However, this does not prove (2). The best advance till date is due to Talagrand [13] who showed that for any $\beta, \gamma, h$, there exists a sequence
\( \gamma_N \to \gamma \) such that
\[
\lim_{N \to \infty} \delta_N(\beta, \gamma_N, h) = 0.
\]

Let \( \psi_N \) be the free energy per particle, defined as
\[
\psi_N := \frac{1}{N} \log \sum_{\sigma} e^{-H_N(\sigma)}.
\]

Let \( p_N = p_N(\beta, \gamma, h) := \mathbb{E}(\psi_N) \). As easy extension of the Guerra-Toninelli argument \([8]\) shows that \( \lim_{N \to \infty} p_N(\beta, \gamma, h) \) exists for all \( \beta, \gamma, h \). Let us call this limit \( p(\beta, \gamma, h) \). Then \( p \) is a convex function of each argument.

**Theorem 1.1.** Suppose \( p \) is differentiable in \( \gamma \) at a certain value of \( (\beta, \gamma, h) \), where \( \gamma \neq 0 \). Then at this value the Ghirlanda-Guerra identities hold, that is, \( \lim_{N \to \infty} \delta_N(\beta, \gamma, h) = 0 \).

Clearly, the major drawback of the above result is that it only works for \( \gamma \neq 0 \). This is problematic since \( \gamma = 0 \) is the most interesting case. A minor drawback is that the theorem holds for almost every value of \( \gamma \) instead of all values. But this is not a serious problem since it is not clear whether the GG identities hold at points of phase transition, and the original paper of Ghirlanda and Guerra \([7]\) does not make such a claim either. And lastly, it will be an important breakthrough if one can prove a similar statement for the extended GG identities, but that seems to be out of reach at present.

The proof of Theorem 1.1 admits a vast generalization, which we state below.

For each \( N \), suppose \( \mu_N \) is a probability measure on \( \{-1, 1\}^N \). Suppose \( \mathcal{A}_N \) is a sequence of finite sets such that \( |\mathcal{A}_N| = O(N) \). For each \( N \), suppose \( (f_\alpha)_{\alpha \in \mathcal{A}_N} \) is a collection of functions from \( \{-1, 1\}^N \) into \([-1, 1]\) and \( (g_\alpha)_{\alpha \in \mathcal{A}_N} \) is a collection of i.i.d. standard Gaussian random variables. Fix a parameter \( \gamma \). Let \( G_N \) be the probability measure on \( \{-1, 1\}^N \) satisfying
\[
G_N(\{\sigma\}) \propto \mu_N(\{\sigma\}) \exp\left( \gamma \sum_{\alpha \in \mathcal{A}_N} g_\alpha f_\alpha(\sigma) \right).
\]

Let \( \psi_N \) be the free energy per particle in this model, that is,
\[
\psi_N(\gamma) := \frac{1}{N} \log \sum_{\sigma} \mu_N(\sigma) \exp\left( \gamma \sum_{\alpha \in \mathcal{A}_N} g_\alpha f_\alpha(\sigma) \right).
\]

Let \( p_N = \mathbb{E}(\psi_N) \). Assume that \( p(\gamma) := \lim_{N \to \infty} p_N(\gamma) \) exists.

Next, for two replicas \( \sigma^1 \) and \( \sigma^2 \), define the generalized overlap as
\[
R_{1,2} := \frac{1}{N} \sum_{\alpha \in \mathcal{A}_N} f_\alpha(\sigma^1) f_\alpha(\sigma^2).
\]

Assume that \( R_{1,1} \) is a deterministic constant for all \( \sigma^1 \). Given \( n \) and a bounded measurable function \( f : \mathbb{R}^{n(n-1)/2} \to \mathbb{R} \), let \( \delta_N(\gamma) \) be defined as in \([1]\), in terms of the generalized overlaps defined above.
Theorem 1.2. Suppose $p$ is differentiable at a point $\gamma$. Then the Ghirlanda-Guerra identities hold at this $\gamma$, that is, $\lim_{N \to \infty} \delta_N(\gamma) = 0$.

Note that this result includes the Sherrington-Kirkpatrick model and Derrida’s $p$-spin models under a random external field. It also covers lattice models like the random field Ising model and the Edwards-Anderson model [6]. For example, in the Edwards-Anderson model, the spins are located on a subset of the $d$-dimensional integer lattice; $\mathcal{A}_N$ is the set of edges in this subset, and for an edge $\alpha = (i, j) \in \mathcal{A}_N$, $f_\alpha(\sigma) = \sigma_i \sigma_j$. The measure $\mu_N$ is just the uniform distribution on $\{-1, 1\}^N$ in this case. In this model, our generalized overlap is simply what’s known as the bond overlap, and $\gamma$ is the inverse temperature parameter. Here we do not need to apply a random external field. The existence of the limit of the free energy per particle in the lattice models follow from simple arguments, e.g. those in [10], Chapter 2.

2. Proof

We will only prove Theorem 1.2 since Theorem 1.1 is a special case of Theorem 1.2. Let

$$H = H(N) := \sum \frac{g_\alpha f_\alpha(\sigma)}{N}.$$ 

It is easy to prove via integration by parts (see e.g. [12], Section 2.12) that the Ghirlanda-Guerra identities hold if

$$\lim_{N \to \infty} \nu(|H - \nu(H)|) = 0. \tag{5}$$

We prove (5) in two steps. First, we show that

$$\lim_{N \to \infty} \mathbb{E}|\langle H \rangle - \nu(H)| = 0. \tag{6}$$

This part requires no new ideas. The proof goes as follows. Note that

$$\langle H \rangle = \psi'_N(\gamma) \quad \text{and} \quad \nu(H) = p'_N(\gamma). \tag{7}$$

Fix some $\gamma' > \gamma$. By the convexity of $\psi_N$ and (7), we have

$$\langle H \rangle \leq \frac{\psi_N(\gamma') - \psi_N(\gamma)}{\gamma' - \gamma}. \tag{8}$$

Again, by standard concentration of measure (see e.g. [12], Theorem 2.2.4) we know that for any $\gamma$,

$$\mathbb{E}|\psi_N(\gamma) - p_N(\gamma)| \leq \sqrt{\text{Var}(\psi_N(\gamma))} \leq C|\gamma| \sqrt{\frac{|\mathcal{A}_N|}{N^2}}, \tag{9}$$
where $C$ is a universal constant. Thus,
\[
E \left| \frac{\psi_N(\gamma') - \psi_N(\gamma)}{\gamma' - \gamma} - p'(\gamma) \right| 
\leq C \frac{(|\gamma| + |\gamma'|)}{\gamma' - \gamma} \sqrt{\frac{|A_N|}{N^2}} + \frac{|p_N(\gamma') - p(\gamma')| + |p_N(\gamma) - p(\gamma)|}{\gamma' - \gamma} 
\quad + \left| \frac{p(\gamma') - p(\gamma)}{\gamma' - \gamma} - p'(\gamma) \right|.
\]

Since $|A_N| = O(N)$ and $p_N \to p$ pointwise, we get
\[
\limsup_{N \to \infty} E \left| \frac{\psi_N(\gamma') - \psi_N(\gamma)}{\gamma' - \gamma} - p'(\gamma) \right| \leq \left| \frac{p(\gamma') - p(\gamma)}{\gamma' - \gamma} - p'(\gamma) \right|.
\]

Combining with (8), we see that
\[
\limsup_{N \to \infty} E(\langle H \rangle - p'(\gamma))_+ \leq \left| \frac{p(\gamma') - p(\gamma)}{\gamma' - \gamma} - p'(\gamma) \right|.
\]

Here $x_+$ denotes the positive part of a real number $x$. Since this bound holds for any $\gamma' > \gamma$ and $p$ is differentiable at $\gamma$, we get
\[
\lim_{N \to \infty} E(\langle H \rangle - p'(\gamma))_+ = 0.
\]

Similarly, considering $\gamma' < \gamma$ and repeating the steps, we can show that the limit of the negative part is zero as well. Thus,
\[
\lim_{N \to \infty} E|\langle H \rangle - p'(\gamma)| = 0.
\]

By Jensen’s inequality, this gives
\[
\lim_{N \to \infty} |\nu(H) - p'(\gamma)| = 0.
\]

Combining the last two identities, we get (9). This completes the first step. In the next part of the proof, we show that
\[
\lim_{N \to \infty} \nu(|H - \langle H \rangle|) = 0.
\]

Combined with (6), this will complete the proof of (5) and hence the theorem. To show (10), it suffice to prove that
\[
\lim_{N \to \infty} E(\langle H - \langle H \rangle \rangle^2) = 0.
\]

The key new idea at this stage is the use of the following result. Essentially, this is the Parseval identity for the $L^2$ norm of a Gaussian functional expressed as a sum of squares of its Fourier coefficients in the orthogonal basis of multidimensional Hermite polynomials.
Theorem 2.1 (I). Let $g = (g_1, \ldots, g_n)$ be a vector of i.i.d. standard Gaussian random variables, and let $f$ be a $C^\infty$ function of $g$ with bounded derivatives of all orders. Then

$$\text{Var}(f) = \sum_{k=1}^{\infty} \frac{1}{k!} \sum_{1 \leq i_1, \ldots, i_k \leq n} \left( \mathbb{E} \left( \frac{\partial^{k} f}{\partial g_{i_1} \cdots \partial g_{i_k}} \right) \right)^2.$$  

The convergence of the infinite series is part of the conclusion.

The formula in its above form is stated and proved in [4]. Different versions of the identity were observed by a number of authors; let us refer to Section 3.8 of [4] for a discussion of these observations and alternate proofs.

We will now prove (11) using the above theorem. In the following, we will simply write $f_\alpha$ instead of $f_\alpha(\sigma)$. Integration by parts gives

$$\mathbb{E} \langle (H - \langle H \rangle)^2 \rangle = \mathbb{E} \langle H^2 \rangle - \mathbb{E} \langle (H)^2 \rangle$$

$$= \frac{1}{N^2} \sum_{\alpha, \alpha'} \mathbb{E} \langle g_\alpha g_{\alpha'} (\langle f_\alpha f_{\alpha'} \rangle - \langle f_\alpha \rangle \langle f_{\alpha'} \rangle) \rangle$$

$$= \frac{1}{N^2} \sum_{\alpha, \alpha'} \mathbb{E} \left( \frac{\partial^2}{\partial g_\alpha \partial g_{\alpha'}} (\langle f_\alpha f_{\alpha'} \rangle - \langle f_\alpha \rangle \langle f_{\alpha'} \rangle) \right)$$

$$+ \frac{1}{N^2} \sum_{\alpha} \mathbb{E} \langle f_\alpha^2 \rangle - \langle f_\alpha \rangle^2 \rangle.$$

Now note that

$$\frac{1}{N} \langle (f_\alpha f_{\alpha'} - \langle f_\alpha \rangle \langle f_{\alpha'} \rangle) \rangle = \frac{1}{\gamma^2} \frac{\partial^2 \psi_N}{\partial g_\alpha \partial g_{\alpha'}},$$

where $\psi_N$ is the free energy per particle defined in [4]. Thus, by the Cauchy-Schwarz inequality and Theorem 2.1 we get

$$\mathbb{E} \langle (H - \langle H \rangle)^2 \rangle \leq \frac{1}{N^2} \sum_{\alpha, \alpha'} \mathbb{E} \left( \frac{\partial^4 \psi_N}{\partial g_\alpha^2 \partial g_{\alpha'}^2} \right) + \frac{2|A_N|}{N^2}$$

$$\leq \frac{|A_N|}{N^2} \left( \sum_{\alpha, \alpha'} \mathbb{E} \left( \frac{\partial^4 \psi_N}{\partial g_\alpha^2 \partial g_{\alpha'}^2} \right)^2 \right)^{1/2} + O(1/N)$$

$$\leq \frac{|A_N|}{N^2} \sqrt{4! \text{Var}(\psi_N)} + O(1/N).$$

The proof is now completed by applying the variance inequality (9).

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References

[1] Aizenman, M. and Contucci, P. (1998). On the stability of the quenched state in mean-field spin-glass models. J. Stat. Phys., 92 nos. 5/6, 765–783.
[2] Arguin, L.-P. (2008). Competing particle systems and the Ghirlanda-Guerra identities. *Electron. J. Probab.* 13 no. 69, 2101–2117.

[3] Arguin, L.-P. and Aizenman, M. (2009). On the structure of quasi-stationary competing particle systems. *Ann. Probab.* 37 no. 3, 1080–1113.

[4] Chatterjee, S. (2009). Disorder chaos and multiple valleys in spin glasses. *Preprint*. Available at [http://arxiv.org/abs/0907.3381](http://arxiv.org/abs/0907.3381)

[5] Contucci, P. and Giardina, C. (2007). The Ghirlanda-Guerra identities. *J. Stat. Phys.* 126 no. 4-5, 917–931.

[6] Edwards, S. F. and Anderson, P. W. (1975). Theory of spin glasses. *J. Phys. F*, 5 965–974.

[7] Ghirlanda, S. and Guerra, F. (1998). General properties of overlap probability distributions in disordered spin systems. Towards Parisi ultrametricity. *J. Phys. A* 31 no. 46, 9149–9155.

[8] Guerra, F. and Toninelli, F. L. (2002). The thermodynamic limit in mean field spin glass models. *Comm. Math. Phys.* 230 no. 1, 71–79.

[9] Panchenko, D. (2009). A connection between Ghirlanda-Guerra identities and ultrametricity. *Preprint*. Available at [http://arxiv.org/abs/0810.0743](http://arxiv.org/abs/0810.0743)

[10] Ruelle, D. (1999). *Statistical mechanics. Rigorous results*. Reprint of the 1989 edition. World Scientific Publishing Co., Inc., NJ.

[11] Sherrington, D. and Kirkpatrick, S. (1975). Solvable model of a spin glass. *Phys. Rev. Lett.* 35 1792–1796.

[12] Talagrand, M. (2003). *Spin glasses: a challenge for mathematicians. Cavity and mean field models*. Springer-Verlag, Berlin.

[13] Talagrand, M. (2009). Construction of pure states in mean field models for spin glasses. *Preprint*.

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