On Grothendieck–Serre’s conjecture concerning principal $G$-bundles over reductive group schemes: II

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Abstract. A proof of the Grothendieck–Serre conjecture on principal bundles over a semi-local regular ring containing an infinite field is given in [1]. That proof is heavily based on Theorem 1.0.3 stated below in the introduction and proved in the present paper.

Theorem 1.0.3 itself is a consequence of two purity theorems 1.0.1 and 1.0.2 which are of completely independent interest and which are proved below. The purity theorem 1.0.1 covers all the known results of this shape and looks like a final one.

Keywords: reductive group schemes, principal bundles, Grothendieck–Serre conjecture.

To the genial mathematician J.-P. Serre

§ 1. Introduction

Recall ([2], Exp. XIX, Definition 2.7) that an $R$-group scheme $G$ is called reductive (respectively, semi-simple; respectively, simple), if it is affine and smooth as an $R$-scheme and if, moreover, for each ring homomorphism $s: R \to \Omega(s)$ to an algebraically closed field $\Omega(s)$, its scalar extension $G_{\Omega(s)}$ is a connected and reductive (respectively, semi-simple; respectively, simple) algebraic group over $\Omega(s)$. We stress that all the groups $G_{\Omega(s)}$ are connected. The class of reductive group schemes contains the class of semi-simple group schemes, which in turn contains the class of simple group schemes. This notion of a simple $R$-group scheme coincides with the notion of a simple semi-simple $R$-group scheme from Demazure–Grothendieck ([2], Exp. XIX, Definition 2.7 and Exp. XXIV, 5.3). Throughout the paper $R$ denotes an integral Noetherian domain and $G$ denotes a reductive $R$-group scheme, unless explicitly stated otherwise.

A well-known conjecture due to J.-P. Serre and A. Grothendieck ([3], Remarque, p. 31, [4], Remarque 3, pp. 26, 27, and [5], Remarque 1.11.a) asserts that given a regular local ring $R$ and its field of fractions $K$ and given a reductive group scheme $G$ over $R$ the map

$$H^1_{\text{ét}}(R, G) \to H^1_{\text{ét}}(K, G),$$

induced by the inclusion of $R$ in $K$, has trivial kernel.
A proof of the Grothendieck–Serre conjecture on principal bundles over a semi-local regular ring containing an infinite field is given in [1]. That proof is heavily based on Theorem 1.0.3 stated below in the introduction and proved in the present paper.

Theorem 1.0.3 itself is a consequence of two purity theorems 1.0.1 and 1.0.2 which are of completely independent interest and which are proved below.

**Theorem 1.0.1** (Theorem A). Let $k$ be an infinite field. Let $\mathcal{O}$ be the semi-local ring of finitely many closed points on a $k$-smooth irreducible affine $k$-variety $X$. Let $K = k(X)$. Let

$$
\mu: G \to C
$$

be a smooth $\mathcal{O}$-morphism of reductive $\mathcal{O}$-group schemes, with a torus $C$. Suppose additionally that the kernel of $\mu$ is a reductive $\mathcal{O}$-group scheme. Then the sequence

$$
\{0\} \to C(\mathcal{O})/\mu(G(\mathcal{O})) \to C(K)/\mu(G(K)) \to \sum_{\text{resp}} \bigoplus_p C(K)/[C(\mathcal{O}_p) \cdot \mu(G(K))] \to \{0\}
$$

(1)

is exact, where $p$ runs over the height 1 primes of $\mathcal{O}$ and $\text{resp}_p$ is the natural map (the projection to the factor group).

Let $\mathcal{O}$ and $K$ be as in Theorem 1.0.1. Let $G$ be a semi-simple $\mathcal{O}$-group scheme. Let $i: Z \hookrightarrow G$ be a closed subgroup scheme of the centre Cent($G$). It is known that $Z$ is of multiplicative type. Let $G' = G/Z$ be the factor group, $\pi: G \to G'$ the projection. It is known that $\pi$ is finite surjective and strictly flat. Thus the sequence of $\mathcal{O}$-group schemes

$$
\{1\} \to Z \xrightarrow{i} G \xrightarrow{\pi} G' \to \{1\}
$$

(2)

induces an exact sequence of group sheaves in the fppt-topology. Thus for every $\mathcal{O}$-algebra $R$ the sequence (21) gives rise to a boundary operator

$$
\delta_{\pi,R}: G'(R) \to H^1_{\text{ftp}}(R, Z).
$$

(3)

One can check that it is a group homomorphism (compare [3], Ch. II, § 5.6, Corollary 2). Set

$$
\mathcal{F}(R) = H^1_{\text{ftp}}(R, Z)/\text{Im}(\delta_{\pi,R}).
$$

(4)

**Theorem 1.0.2.** Let $k$, $\mathcal{O}$ and $K$ be as in Theorem 1.0.1. The sequence

$$
\{0\} \to \frac{H^1_{\text{ftp}}(\mathcal{O}, Z)}{\text{Im}(\delta_{\pi,\mathcal{O}})} \to \frac{H^1_{\text{ftp}}(K, Z)}{\text{Im}(\delta_{\pi,K})} \to \sum_{\text{resp}} \bigoplus_p \frac{H^1_{\text{ftp}}(K, Z)}{[H^1_{\text{ftp}}(\mathcal{O}, Z) + \text{Im}(\delta_{\pi,K})]} \to \{0\}
$$

(5)

is exact, where $p$ runs over the height 1 primes of $\mathcal{O}$.

The exactness of sequences (1) and (5) at their middle terms yields the following result.
Theorem 1.0.3. Let \( k \) be an infinite field. Let \( \mathcal{O} \) be the semi-local ring of finitely many closed points on a \( k \)-smooth irreducible affine \( k \)-variety \( X \). Let \( K = k(X) \). Assume that for all semi-simple simply connected reductive \( \mathcal{O} \)-group schemes \( H \) the pointed set map

\[
H^1_{\text{ét}}(\mathcal{O}, H) \to H^1_{\text{ét}}(k(X), H),
\]

induced by the inclusion of \( \mathcal{O} \) into its field of fractions \( k(X) \), has trivial kernel. Then for any reductive \( \mathcal{O} \)-group scheme \( G \) the pointed set map

\[
H^1_{\text{ét}}(\mathcal{O}, G) \to H^1_{\text{ét}}(K, G),
\]

induced by the inclusion of \( \mathcal{O} \) into its field of fractions \( K \), has trivial kernel.

Remark 1.0.4. The proof of the latter theorem is subdivided into two steps. Firstly, given a semi-simple \( \mathcal{O} \)-group scheme \( G \) we prove that the Grothendieck–Serre conjecture holds for \( G \) provided it holds for its simply-connected cover \( G^{\text{sc}} \) and all inner forms of that simply-connected \( \mathcal{O} \)-group scheme \( G^{\text{sc}} \).

Secondly, given a reductive \( \mathcal{O} \)-group scheme \( G \) we prove that the Grothendieck–Serre conjecture holds for \( \mathcal{O} \)-group scheme \( G \) provided it holds for the derived \( \mathcal{O} \)-group scheme \( G^{\text{der}} \) of \( G \) and for all inner forms of that derived \( \mathcal{O} \)-group scheme \( G^{\text{der}} \) of \( G \).

After the pioneering articles \([6] \) and \([7] \) on purity theorems for algebraic groups, various versions of purity theorems were proved in \([8]–[11] \). The most general result in the so-called constant case was given in \([10] \), Example 3.3. This result now follows from our Theorem 1.0.1 by taking \( G \) to be a \( k \)-rational reductive group, \( C = \mathbb{G}_{m,k} \) and \( \mu: G \to \mathbb{G}_{m,k} \) a dominant \( k \)-group morphism. The papers \([9]–[11] \) contain results for the non-constant case. However, they only consider specific examples of algebraic group scheme morphisms \( \mu: G \to C \).

Let \( F \) be a covariant functor from commutative rings to abelian groups. Let \( R \) be a regular local ring and \( K \) its field of fractions. We say that \( F \) satisfies purity for \( R \), if the sequence \( F(R) \to F(K) \to \bigoplus_p F(K)/\text{Im}(F(R_p)) \) is exact, where \( p \) runs over the height 1 primes of \( R \). Let \( \mu: G \to C \) be a smooth morphism of reductive \( R \)-group schemes with an \( R \)-torus \( C \). Consider the functor \( S \mapsto \mathcal{F}(S) = C(S)/\mu(G(S)). \) We expect that \( \mathcal{F} \) satisfies purity for \( R \).

Let us point out that we use transfers for the functor \( R \mapsto C(R) \), but no use at all is made of the norm principle for the homomorphism \( \mu: G \to C \). A big open question is whether the morphism \( \mu \) in Theorem 1.0.1 satisfies the norm principle even for finite separable field extensions. So, we are not able to say that the functor \( \mathcal{F} \) is a presheaf with transfers in the sense of Voevodsky or in any other weaker sense. That is the major trouble for any attempt to prove Theorem 1.0.1.

Here is our approach. Given an element \( \xi \in C(K) \) such that its image \( \overline{\xi} \in \mathcal{F}(K) \) is an \( \mathcal{O} \)-unramified element one can find a rather refined finite correspondence of the form

\[
\mathbb{A}^1_{\mathcal{O}} \xleftarrow{\sigma} X' \xrightarrow{q} X
\]

(see the diagram \((11)\)) and use it in the ‘constant group’ case to write down a good candidate \( \overline{\xi}_\mathcal{O} \in \mathcal{F}(\mathcal{O}) \) (see \((19)\)) for a lift of the element \( \overline{\xi} \) to \( \mathcal{F}(\mathcal{O}) \). In general, since the group scheme \( G \) does not come from the ground field we need to equate its two
pullbacks \((\text{pr}_O \circ \sigma)^*(G)\) and \((q'_X)^*(G)\) over \(X'\). We need to do the same with the torus \(C\). Due to these requirements our construction of the finite correspondence and of a good candidate \(\xi_O \in \mathcal{F}(O)\) (see (20)) is quite involved. The finite surjective morphism \(\sigma\) of the \(O\)-schemes has the following property: for the corresponding fraction field extension \(K(u) \subset K\) the element
\[
\zeta_u := N_{K/K(u)}((q'_X)^*(\xi)) \in C(K(u))
\]
is such that its image \(\bar{\zeta}_u \in \mathcal{F}(K(u))\) is \(K[u]\)-unramified. The latter yields that the element \(\bar{\zeta}_u\) is constant, that is, it belongs to \(\mathcal{F}(K)\). Thus the evaluations of \(\bar{\zeta}_u\) at \(u = 0\) and at \(u = 1\) coincide. Simple computations now show that the element \(\xi_O \in \mathcal{F}(O)\) is indeed a lift of the element \(\xi \in \mathcal{F}(K)\). Details are given in §8.

The paper is organized as follows. In §2 we construct norm maps following a method from [12], Section 6. In §3 we recall a geometric lemma from [13] (see Lemma 3.0.6 below). In §4 we discuss unramified elements. A key point here is Lemma 4.0.11. In §5 we discuss specialization maps. A key point here is Corollary 5.0.17. In §6 an equating statement is formulated. In §7 a convenient technical tool is introduced. In §8 Theorem 1.0.1 is proved. In §9 Theorem 9.0.26 is proved. It extends Theorem 1.0.1 to the case of local regular domains (not to semi-local domains) containing an infinite perfect field. In §10 we consider the functor (23) and prove a purity Theorem 10.0.30 for that functor. In §11 Theorem 1.0.3 is proved. Finally in §12 we collect several examples illustrating the two Purity Theorems.

The logic in the paper is this. Firstly the exactness of sequences (1) and (3) at their middle terms is proved. This yields Theorem 1.0.3. The latter theorem is used in the proof of [1], Theorem 1. In turn, [1], Theorem 1 yields the injectivity at the left-hand term of sequence (1). The exactness of (1) now follows from [14]. This yields the exactness of (3) as explained at the end of the proof of Theorem 10.0.30.

§2. Norms

Let \(k \subset K \subset L\) be field extensions and assume that \(L\) is finite separable over \(K\). Let \(K^{\text{sep}}\) be a separable closure of \(K\) and
\[
\sigma_i : K \rightarrow K^{\text{sep}}, \quad 1 \leq i \leq n,
\]
the different embeddings of \(K\) into \(L\). Let \(C\) be a commutative \(k\)-smooth algebraic group scheme defined over \(k\). We can define a norm map
\[
\mathcal{N}_{L/K} : C(L) \rightarrow C(K)
\]
by
\[
\mathcal{N}_{L/K}(\alpha) = \prod_{i=1}^{n} C(\sigma_i)(\alpha) \in C(K^{\text{sep}})^{G(K)} = C(K).
\]

Following Suslin and Voevodsky ([12], Section 6) we generalize this construction to finite flat ring extensions.

Let \(p: X \rightarrow Y\) be a finite flat morphism of affine schemes. Suppose that its rank is constant, equal to \(d\). Denote by \(S^d(X/Y)\) the \(d\)-th symmetric power of \(X\) over \(Y\).
Lemma 2.0.5. There is a canonical section

\[ N_{X/Y} : Y \to S^d(X/Y) \]

with the following three properties.

(i) Base change: for any map \( f : Y' \to Y \) of affine schemes, putting \( X' = X \times_Y Y' \), we have a commutative diagram

\[
\begin{array}{ccc}
Y' & \xrightarrow{N_{X'/Y'}} & S^d(X'/Y') \\
\downarrow f & & \downarrow S^d(Id_{X' \times f}) \\
Y & \xrightarrow{N_{X/Y}} & S^d(X/Y).
\end{array}
\]

(ii) Additivity: if \( f_1 : X_1 \to Y \) and \( f_2 : X_2 \to Y \) are finite flat morphisms of degree \( d_1 \) and \( d_2 \) respectively, then, putting \( X = X_1 \coprod X_2 \), \( f = f_1 \coprod f_2 \) and \( d = d_1 + d_2 \), we have a commutative diagram

\[
\begin{array}{ccc}
S^{d_1}(X_1/Y) \times S^{d_2}(X_2/Y) & \xrightarrow{\sigma} & S^d(X/Y) \\
\downarrow & & \downarrow \\
N_{X_1/Y} \times N_{X_2/Y} & \xrightarrow{N_{X/Y}} & Y
\end{array}
\]

where \( \sigma \) is the canonical imbedding.

(iii) Normalization: if \( X = Y \) and \( p \) is the identity, then \( N_{X/Y} \) is the identity.

Proof. We construct a map \( N_{X/Y} \) and check that it has the desired properties. Let \( B = k[X] \) and \( A = k[Y] \), so that \( B \) is a locally free \( A \)-module of finite rank \( d \). Let \( B^{\otimes d} = B \otimes_A B \otimes_A \cdots \otimes_A B \) be the \( d \)-fold tensor product of \( B \) over \( A \). The permutation group \( \mathfrak{S}_d \) acts on \( B^{\otimes d} \) by permuting the factors. Let \( S^d_A(B) \subseteq B^{\otimes d} \) be the \( A \)-algebra of all the \( \mathfrak{S}_d \)-invariant elements of \( B^{\otimes d} \). We consider \( B^{\otimes d} \) as an \( S^d_A(B) \)-module via the inclusion \( S^d_A(B) \subseteq B^{\otimes d} \) of \( A \)-algebras. Let \( I \) be the kernel of the canonical homomorphism \( B^{\otimes d} \to \bigwedge^d_A(B) \) mapping \( b_1 \otimes \cdots \otimes b_d \) to \( b_1 \wedge \cdots \wedge b_d \). It is well-known (and easily checked locally on \( A \)) that \( I \) is generated by all the elements \( x \in B^{\otimes d} \) such that \( \tau(x) = x \) for some transposition \( \tau \). If \( s \) is in \( S^d_A(B) \), then \( \tau(sx) = \tau(s)\tau(x) = sx \), hence \( sx \) is in \( S^d(B) \) too. In other words, \( I \) is an \( S^d_A(B) \)-submodule of \( B^{\otimes d} \). The induced \( S^d_A(B) \)-module structure on \( \bigwedge^d_A(B) \) defines an \( A \)-algebra homomorphism

\[ \varphi : S^d_A(B) \to \text{End}_A \left( \bigwedge^d_A(B) \right). \]

Since \( B \) is locally free of rank \( d \) over \( A \), \( \bigwedge^d_A(B) \) is an invertible \( A \)-module and we can canonically identify \( \text{End}_A(\bigwedge^d_A(B)) \) with \( A \). Thus we have a map

\[ \varphi : S^d_A(B) \to A \]
and we define

\[ N_{X/Y} : Y \to S^d(X/Y) \]

as the morphism of \( Y \)-schemes induced by \( \varphi \). The verification of properties (1), (2) and (3) is straightforward.  □

Let \( k \) be an infinite field. Let \( \mathcal{O} \) be the semi-local ring of finitely many points on a smooth affine irreducible \( k \)-variety \( X \). Let \( C \) be an affine smooth commutative \( \mathcal{O} \)-group scheme, let \( p : X \to Y \) be a finite flat surjective morphism of a constant degree \( d \) of affine \( \mathcal{O} \)-schemes and let \( f : X \to C \) be any \( \mathcal{O} \)-morphism. We define the norm \( N_{X/Y}(f) \) of \( f \) as the composite map

\[
Y \xrightarrow{N_{X/Y}} S^d(X/Y) \to S^d_\mathcal{O}(X) \xrightarrow{S^d_\mathcal{O}(f)} S^d_\mathcal{O}(C) \xrightarrow{\times} C.
\]

(6) Here we write ‘\( \times \)’ for the group law on \( C \). The norm maps \( N_{X/Y} \) satisfy the following conditions:

(i′) base change: for any map \( f : Y' \to Y \) of affine schemes, putting \( X' = X \times_Y Y' \) we have a commutative diagram

\[
\begin{array}{ccc}
C(X) & \xrightarrow{(\text{id} \times f)^*} & C(X') \\
N_{X/Y} \downarrow & & \downarrow N_{X'/Y'} \\
C(Y) & \xrightarrow{f^*} & C(Y');
\end{array}
\]

(ii′) multiplicativity: if \( X = X_1 \amalg X_2 \) then the following diagram commutes:

\[
\begin{array}{ccc}
C(X) & \xrightarrow{(\text{id} \times f)^*} & C(X_1) \times C(X_2) \\
N_{X/Y} \downarrow & & \downarrow N_{X_1/Y}N_{X_2/Y} \\
C(Y) & \xrightarrow{\text{id}} & C(Y);
\end{array}
\]

(iii′) normalization: if \( X = Y \) and the map \( X \to Y \) is the identity then \( N_{X/Y} = \text{id}_{C(X)} \).

§ 3. One Lemma

Lemma 3.0.6 below is a refinement of [15], Lemma 2. It is proved in [13], Lemma 4.3.

Lemma 3.0.6. Let \( k \) be an infinite field, and let \( \mathcal{O} \) be a domain which is a semi-local essentially smooth \( k \)-algebra with maximal ideals \( \mathfrak{m}_i \), \( 1 \leq i \leq n \). Let \( A \supseteq \mathcal{O}[t] \) be another domain, smooth as an \( \mathcal{O} \)-algebra and finite over \( \mathcal{O}[t] \). Assume that for each \( i \) the \( \mathcal{O}/\mathfrak{m}_i \)-algebra \( A_i = A/\mathfrak{m}_iA \) is equidimensional of dimension 1.

Let \( \varepsilon : A \to \mathcal{O} \) be an \( \mathcal{O} \)-augmentation and \( I = \text{Ker}(\varepsilon) \). Given an \( f \in A \) with

\[ 0 \neq \varepsilon(f) \in \bigcap_{i=1}^n \mathfrak{m}_i \subset \mathcal{O} \]
and such that the $\mathcal{O}$-module $A/fA$ is finite, one can find an element $u \in A$ satisfying the following conditions:

1. $A$ is a finite projective module over $\mathcal{O}[u]$;
2. $A/uA = A/I \times A/J$ for some ideal $J$;
3. $J + fA = A$;
4. $(u - 1)A + fA = A$;
5. setting $N(f) = N_{A/\mathcal{O}[u]}(f)$, we have $N(f) = fg \in A$ for some $g \in A$;
6. $fA + gA = A$;
7. the composite map $\varphi: \mathcal{O}[u]/(N_{A/\mathcal{O}[u]}(f)) \to A/(N_{A/\mathcal{O}[u]}(f)) \to A/(f)$ is an isomorphism.

**Proof.** See [13], Lemma 4.3. □

**Corollary 3.0.7.** Under the hypotheses of Lemma 3.0.6 let $K$ be the field of fractions of $\mathcal{O}$, $A_K = K \otimes_{\mathcal{O}} A$ and $\varepsilon_K = id_K \otimes \varepsilon: A_K \to K$. Consider the inclusion $K[u] \subset A_K$. Then the norm $N(f) = N_{A_K/K[u]}(f) \in K[u]$ does not vanish at the points 1 and 0 of the affine line $A_K$.

**Proof.** The condition (4) of Lemma 3.0.6 implies that $N(f)$ does not vanish at the point 1. Since $\varepsilon_K(f) \neq 0 \in K$ the conditions (2) and (3) imply that $N(f)$ does not vanish at 0 either. □

### § 4. Unramified elements

Let $k$ be an infinite field, $\mathcal{O}$ the $k$-algebra from Theorem 1.0.1 and $K$ the field of fractions of $\mathcal{O}$. Let $\mu: G \to C$ be the morphism of reductive $\mathcal{O}$-group schemes from Theorem 1.0.1. We work in this section with the category of commutative Noetherian $\mathcal{O}$-algebras. For a commutative $\mathcal{O}$-algebra $S$ set

$$\mathcal{F}(S) = C(S)/\mu(G(S)). \quad (7)$$

Let $S$ be an $\mathcal{O}$-algebra which is a domain and let $L$ be its field of fractions. Define the subgroup of $S$-unramified elements of $\mathcal{F}(L)$ as

$$\mathcal{F}_{\text{nr},S}(L) = \bigcap_{p \in \text{Spec}(S)^{(1)}} \text{Im}[\mathcal{F}(S_p) \to \mathcal{F}(L)], \quad (8)$$

where $\text{Spec}(S)^{(1)}$ is the set of height 1 prime ideals in $S$. Obviously the image of $\mathcal{F}(S)$ in $\mathcal{F}(L)$ is contained in $\mathcal{F}_{\text{nr},S}(L)$. In some cases $\mathcal{F}(S_p)$ injects into $\mathcal{F}(L)$ and in those cases $\mathcal{F}_{\text{nr},S}(L)$ is simply the intersection of all $\mathcal{F}(S_p)$.

For an element $\alpha \in C(S)$ we will write $\overline{\alpha}$ for its image in $\mathcal{F}(S)$. In this section we will write $\mathcal{F}$ for the functor (7), the only exception being Lemma 4.0.12. We will repeatedly use the following result due to Nisnevich.

**Theorem 4.0.8** [16]. Let $S$ be an $\mathcal{O}$-algebra which is a discrete valuation ring with field of fractions $L$. Then the map $\mathcal{F}(S) \to \mathcal{F}(L)$ is injective.

**Proof.** Let $H$ be the kernel of $\mu$. Since $\mu$ is smooth and $C$ is a torus, the group scheme sequence

$$1 \to H \to G \to C \to 1$$
gives rise to a short exact sequence of group sheaves in the étale topology. In turn that sequence of sheaves induces a long exact sequence of pointed sets. So, the boundary map $\partial: C(S) \to H^1_{\text{ét}}(S, H)$ fits into a commutative diagram

$$
\begin{array}{ccc}
C(S)/\mu(G(S)) & \longrightarrow & C(L)/\mu(G(L)) \\
\downarrow & & \downarrow \\
H^1_{\text{ét}}(S, H) & \longrightarrow & H^1_{\text{ét}}(L, H)
\end{array}
$$

in which the vertical arrows have trivial kernels. The bottom arrow has trivial kernel by a theorem from [16], since $H$ is a reductive $O$-group scheme. Thus the top arrow has trivial kernel too. □

Lemma 4.0.9. Let $\mu: G \to C$ be the above morphism of our reductive group schemes. Let $H = \text{Ker}(\mu)$. Then for an $O$-algebra $L$, where $L$ is a field, the boundary map $\partial: C(L)/\mu(G(L)) \to H^1_{\text{ét}}(L, H)$ is injective.

Proof. For an $L$-rational point $t \in C$ set $H_t = \mu^{-1}(t)$. The action by left multiplication of $H$ on $H_t$ makes $H_t$ into a left principal homogeneous $H$-space and moreover $\partial(t) \in H^1_{\text{ét}}(L, H)$ coincides with the isomorphism class of $H_t$. Now suppose that $s, t \in C(L)$ are such that $\partial(s) = \partial(t)$. This means that $H_t$ and $H_s$ are isomorphic as principal homogeneous $H$-spaces. We must check that for certain $g \in G(L)$ one has $t = s\mu(g)$.

Let $L^{\text{sep}}$ be a separable closure of $L$. Let $\psi: H_s \to H_t$ be an isomorphism of left $H$-spaces. For any $r \in H_s(L^{\text{sep}})$ and $h \in H_s(L^{\text{sep}})$ one has

$$(hr)^{-1}\psi(hr) = r^{-1}h^{-1}h\psi(r) = r^{-1}\psi(r).$$

Thus for any $\sigma \in \text{Gal}(L^{\text{sep}}/L)$ and any $r \in H_s(L^{\text{sep}})$ one has

$$r^{-1}\psi(r) = (r\sigma)^{-1}\psi(r\sigma) = (r^{-1}\psi(r))\sigma,$$

which means that the point $u = r^{-1}\psi(r)$ is a $\text{Gal}(L^{\text{sep}}/L)$-invariant point of $G(L^{\text{sep}})$. So $u \in G(L)$. The following relation shows that $\psi$ coincides with right multiplication by $u$. In fact, for any $r \in H_s(L^{\text{sep}})$ one has $\psi(r) = r\psi(r) = ru$. Since $\psi$ is right multiplication by $u$, one has $t = s\mu(u)$, which proves the lemma. □

Let $k, O$ and $K$ be as above in this section. Let $\mathcal{K}$ be a field containing $K$ and let $x: \mathcal{K}^* \to \mathbb{Z}$ be a discrete valuation vanishing on $K$. Let $A_x$ be the valuation ring of $x$. Clearly, $O \subset A_x$. Let $\hat{A}_x$ and $\mathcal{K}_x$ be the completions of $A_x$ and $\mathcal{K}$ with respect to $x$. Let $i: \mathcal{K} \hookrightarrow \mathcal{K}_x$ be the inclusion. By Theorem 4.0.8 the map $\mathcal{F}(\hat{A}_x) \to \mathcal{F}(\mathcal{K}_x)$ is injective. We will identify $\mathcal{F}(\hat{A}_x)$ with its image under this map. Set

$$\mathcal{F}_x(\mathcal{K}) = i_*^{-1}(\mathcal{F}(\hat{A}_x)).$$

The inclusion $A_x \hookrightarrow \mathcal{K}$ induces a map $\mathcal{F}(A_x) \to \mathcal{F}(\mathcal{K})$, which is injective by Lemma 4.0.8. So both groups $\mathcal{F}(A_x)$ and $\mathcal{F}_x(\mathcal{K})$ are subgroups of $\mathcal{F}(\mathcal{K})$. The following lemma shows that $\mathcal{F}_x(\mathcal{K})$ coincides with the subgroup of $\mathcal{F}(\mathcal{K})$ consisting of all elements unramified at $x$. 

Lemma 4.0.10. \( \mathcal{F}(A_x) = \mathcal{F}_x(\mathcal{K}) \).

Proof. We only have to check the inclusion \( \mathcal{F}_x(\mathcal{K}) \subseteq \mathcal{F}(A_x) \). Let \( a_x \in \mathcal{F}_x(\mathcal{K}) \) be an element. It determines the elements \( a \in \mathcal{F}(\mathcal{K}) \) and \( \hat{a} \in \mathcal{F}(\hat{A}_x) \), which coincide when regarded as elements of \( \mathcal{F}(\hat{K}_x) \). We denote this common element in \( \mathcal{F}(\hat{K}_x) \) by \( \hat{a}_x \). Let \( H = \ker(\mu) \) and let \( \partial: C(-) \to H^1_{\text{ét}}(-, H) \) be the boundary map.

Let \( \xi = \partial(a) \in H^1_{\text{ét}}(\mathcal{K}, H) \), \( \hat{\xi} = \partial(\hat{a}) \in H^1_{\text{ét}}(\hat{A}_x, H) \) and \( \hat{\xi}_x = \partial(\hat{a}_x) \in H^1_{\text{ét}}(\hat{K}_x, H) \). Clearly, \( \hat{\xi} \) and \( \xi \) both coincide with \( \hat{\xi}_x \) when regarded as elements of \( H^1_{\text{ét}}(\hat{K}_x, H) \). Thus one can glue \( \xi \) and \( \hat{\xi} \) to get a \( \xi_x \in H^1_{\text{ét}}(A_x, H) \) which maps to \( \xi \) under the map induced by the inclusion \( A_x \hookrightarrow \mathcal{K} \) and maps to \( \hat{\xi} \) under the map induced by the inclusion \( A_x \hookrightarrow \hat{A}_x \).

We now show that \( \xi_x \) has the form \( \partial(a'_x) \) for a certain \( a'_x \in \mathcal{F}(A_x) \). In fact, observe that the image \( \zeta \) of \( \xi \) in \( H^1_{\text{ét}}(\mathcal{K}, G) \) is trivial. By a theorem of Nisnevich [16] the map

\[
H^1_{\text{ét}}(A_x, G) \to H^1_{\text{ét}}(\mathcal{K}, G)
\]

has trivial kernel. Therefore the image \( \zeta_x \) of \( \xi_x \) in \( H^1_{\text{ét}}(A_x, G) \) is trivial. Thus there exists an element \( a'_x \in \mathcal{F}(A_x) \) with \( \partial(a'_x) = \xi_x \in H^1_{\text{ét}}(A_x, H) \).

We now prove that \( a'_x \) coincides with \( a_x \) in \( \mathcal{F}_x(\mathcal{K}) \). Since \( \mathcal{F}(A_x) \) and \( \mathcal{F}_x(\mathcal{K}) \) are both subgroups of \( \mathcal{F}(\mathcal{K}) \), it suffices to show that \( a'_x \) coincides with the element \( a \) in \( \mathcal{F}(\mathcal{K}) \). By Lemma 4.0.9 the map

\[
\mathcal{F}(\mathcal{K}) \xrightarrow{\partial} H^1_{\text{ét}}(\mathcal{K}, H)
\]

is injective. Thus it suffices to check that \( \partial(a'_x) = \partial(a) \) in \( H^1_{\text{ét}}(\mathcal{K}, H) \). This is indeed the case because \( \partial(a'_x) = \xi_x \) and \( \partial(a) = \xi \), and \( \xi_x \) coincides with \( \xi \) when regarded as being over \( \mathcal{K} \). We have proved that \( a'_x \) coincides with \( a_x \) in \( \mathcal{F}_x(\mathcal{K}) \). Thus the inclusion \( \mathcal{F}_x(\mathcal{K}) \subseteq \mathcal{F}(A_x) \) is proved, whence the lemma. □

Let \( k, O \) and \( K \) be as above in this section.

Lemma 4.0.11. Let \( A \subset B \) be a finite extension of Dedekind \( K \)-algebras. Let \( 0 \neq f \in B \) be such that \( B/fB \) is finite over \( K \) and reduced.

Suppose that \( N_{B/A}(f) = fg \in B \) for a certain \( g \in B \) coprime to \( f \). Suppose that the composite map \( A/N(f)A \to B/N(f)B \) is an isomorphism. Let \( F \) and \( E \) be the fields of fractions of \( A \) and \( B \) respectively. Let \( \beta \in C(B_f) \) be such that \( \overline{\beta} \in \mathcal{F}(E) \) is \( B \)-unramified. Then, for \( \alpha = N_{E/F}(\beta) \), the class \( \overline{\alpha} \in \mathcal{F}(F) \) is \( A \)-unramified.

Proof. The only primes at which \( \overline{\alpha} \) could be ramified are those which divide \( N(f) \). Let \( p \) be one of them. We check that \( \overline{\alpha} \) is unramified at \( p \).

To do this we consider all primes \( q_1, q_2, \ldots, q_n \) in \( B \) lying over \( p \). Let \( q_1 \) be the unique prime dividing \( f \) and lying over \( p \). Then

\[
\hat{B}_p = \hat{B}_{q_1} \times \prod_{i \neq 1} \hat{B}_{q_i}
\]

with \( \hat{B}_{q_1} = \hat{A}_p \). If \( F, E \) are the fields of fractions of \( A \) and \( B \) then

\[
E \otimes F_p = \hat{E}_{q_1} \times \cdots \times \hat{E}_{q_n}
\]
and $\hat{E}_q = \hat{F}_p$. We will write $\hat{E}_i$ for $\hat{E}_{q_i}$ and $\hat{B}_i$ for $\hat{B}_{q_i}$. Let $\beta \otimes 1 = (\beta_1, \ldots, \beta_n) \in C(\hat{E}_i) \times \cdots \times C(\hat{E}_n)$. Clearly for $i \geq 2$, $\beta_i \in C(\hat{B}_i)$ and $\beta_1 = \mu(\gamma_1)\beta'_1$ with $\beta'_1 \in C(\hat{B}_1) = C(\hat{A}_p)$ and $\gamma_1 \in G(\hat{E}_1) = G(\hat{F}_p)$. Now $\alpha \otimes 1 \in C(\hat{F}_p)$ coincides with the product

$$\beta_1 N_{L_2/K_p} (\beta_2) \cdots N_{L_n/K_p} (\beta_n) = \mu(\gamma_1) [\beta'_1 N_{L_2/K_p} (\beta_2) \cdots N_{L_n/K_p} (\beta_n)].$$

Thus $\alpha \otimes 1 = \beta'_1 N_{L_2/K_p} (\beta_2) \cdots N_{L_n/K_p} (\beta_n) \in \mathcal{F}(\hat{A}_p)$. Let $i: F \hookrightarrow \hat{F}_p$ be the inclusion and $i_*: \mathcal{F}(F) \to \mathcal{F}(\hat{F}_p)$ be the induced map. Clearly $i_*(\alpha) = \alpha \otimes 1$ in $\mathcal{F}(\hat{F}_p)$. Now Lemma 4.0.10 shows that the element $\alpha \in \mathcal{F}(F)$ belongs to $\mathcal{F}(A_p)$. Hence $\alpha$ is $A$-unramified. □

**Lemma 4.0.12** (Unramifiedness Lemma). Let $R$ be a commutative ring. Let $\mathcal{F}$ be a covariant functor from the category of commutative $R$-algebras to the category of abelian groups. Let $S'$ and $R'$ be two $R$-algebras which are Noetherian domains with fields of fractions $K'$ and $L'$ respectively. Let $S' \hookrightarrow R'$ be an injective flat $R$-algebra homomorphism of finite type and let $j: K' \to L'$ be the induced inclusion of the field of fractions. Then for each localization $R'' \supset R'$ of $R'$ the map

$$j_*: \mathcal{F}(K') \to \mathcal{F}(L')$$

takes $S'$-unramified elements to $R''$-unramified elements.

**Proof.** Let $v \in \mathcal{F}(K')$ be an $S'$-unramified element and let $r$ be a height 1 prime of $R'$. Then $q = R' \cap r$ is a height 1 prime of $R'$. Let $p = S' \cap q$. Since the $S'$-algebra $R'$ is flat of finite type, one has $\text{ht}(q) \geq \text{ht}(p)$. Thus $\text{ht}(p)$ is 1 or 0. The commutative diagram

$$\begin{array}{ccc}
\mathcal{F}(K') & \xrightarrow{j_*} & \mathcal{F}(L') \\
\uparrow & & \uparrow \\
\mathcal{F}(S'_p) & \longrightarrow & \mathcal{F}(R''_r)
\end{array}$$

shows that the class $j_*(v)$ is in the image of $\mathcal{F}(R''_r)$ and hence the class $j_*(v) \in \mathcal{F}(L')$ is $R''$-unramified. □

§ 5. Specialization maps

Let $k$ be an infinite field, $\mathcal{O}$ the $k$-algebra from Theorem 1.0.1 and $K$ the field of fractions of $\mathcal{O}$. Let $\mu: G \to C$ be the morphism of reductive $\mathcal{O}$-group schemes from Theorem 1.0.1. We work in this section with the category of commutative $K$-algebras and with the functor

$$\mathcal{F}: S \mapsto C(S)/\mu(G(S))$$

defined on the category of $K$-algebras. So, we assume in this section that each ring in this section is equipped with a distinguished $K$-algebra structure and each ring homomorphism from this section respects that structure. Let $S$ be
a $K$-algebra which is a domain and let $L$ be its field of fractions. Define the *subgroup of $S$-unramified elements* $\mathcal{F}_{\text{nr},S}(L)$ of $\mathcal{F}(L)$ by formulae (8).

For a regular domain $S$ with field of fractions $K$ and each height 1 prime $p$ in $S$ we construct *specialization maps* $s_p : \mathcal{F}_{\text{nr},S}(K) \to \mathcal{F}(K(p))$, where $K$ is the field of fractions of $S$ and $K(p)$ is the residue field of $R$ at the prime $p$.

**Definition 5.0.13.** Let $Ev_p : C(S_p) \to C(K(p))$ and $ev_p : \mathcal{F}(S_p) \to \mathcal{F}(K(p))$ be the maps induced by the canonical $K$-algebra homomorphism $S_p \to K(p)$. Define a homomorphism $\alpha : \mathcal{F}_{\text{nr},S}(K) \to \mathcal{F}(K(p))$ by $s_p(\alpha) = ev_p(\alpha)$, where $\alpha$ is a lift of $\alpha$ to $\mathcal{F}(S_p)$. Theorem 4.0.8 shows that the map $s_p$ is well defined. It is called the specialization map. The map $ev_p$ is called the evaluation map at the prime $p$.

Obviously for $\alpha \in C(R_p)$ one has $s_p(\alpha) = Ev_p(\alpha) \in \mathcal{F}(K(p))$.

**Lemma 5.0.14** [17]. Let $H'$ be a smooth linear algebraic group over the field $K$. Let $S$ be a $K$-algebra which is a Dedekind domain with field of fractions $K$. If $\xi \in H^1_\text{ét}(K, H')$ is an $S$-unramified element for the functor $H^1_\text{ét}(-, H')$ (see (8) for the definition), then $\xi$ can be lifted to an element of $H^1_\text{ét}(S, H')$.

**Proof.** Patching. □

**Theorem 5.0.15** ([8], Proposition 2.2). Let $G' = G_K$, where $G$ is the reductive $O$-group scheme from this section (it is connected and even geometrically connected, since we are following [2], Exp. XIX, Definition 2.7). Then

$$\text{Ker}[H^1_\text{ét}(K[t], G') \to H^1_\text{ét}(K(t), G')] = \ast.$$

We need the following theorem.

**Theorem 5.0.16** (Homotopy invariance). Let $S \mapsto \mathcal{F}(S)$ be the functor defined by the formulae (10) and let $\mathcal{F}_{\text{nr},K[t]}(K(t))$ be defined by the formulae (8). Let $K(t)$ be the rational function field in one variable. Then one has

$$\mathcal{F}(K) = \mathcal{F}_{\text{nr},K[t]}(K(t)).$$

**Proof.** The injectivity is clear, since the composite

$$\mathcal{F}(K) \to \mathcal{F}_{\text{nr},K[t]}(K(t)) \overset{s_0}{\to} \mathcal{F}(K)$$

coincides with the identity (here $s_0$ is the specialization map at the point zero defined in Definition 5.0.13).

It remains to check the surjectivity. Let

$$\mu_K = \mu \otimes_O K : G_K = G \otimes_O K \to C \otimes_O K = C_K.$$

Let $a \in \mathcal{F}_{\text{nr},K[t]}(K(t))$ and let $H_K = \text{Ker}(\mu_K)$. Since $\mu$ is smooth the $K$-group $H_K$ is smooth. Since $G_K$ is reductive it is $K$-affine. Whence $H_K$ is $K$-affine. Clearly, the element $\partial(a) \in H^1_\text{ét}(K(t), H_K)$ is a class which for every closed point $x \in A^1_K$ belongs to the image of $H^1_\text{ét}(O_x, H_K)$. Thus by Lemma 5.0.14, $\xi := \partial(a)$ can be
represented by an element $\tilde{\xi} \in H^1_{\text{et}}(K[t], H_K)$, where $K[t]$ is the polynomial ring. Consider the diagram

\[
\begin{array}{c}
1 \longrightarrow \mathcal{F}(K[t]) \xrightarrow{\partial} H^1_{\text{et}}(K[t], H_K) \longrightarrow H^1_{\text{et}}(K[t], G_K) \\
1 \longrightarrow \mathcal{F}(K(t)) \xrightarrow{\partial} H^1_{\text{et}}(K(t), H_K) \longrightarrow H^1_{\text{et}}(K(t), G_K) \\
\end{array}
\]

in which all the maps are canonical, the horizontal lines are exact sequences of pointed sets and $\text{Ker}(\eta) = *$ by Theorem 5.0.15. Since $\xi$ goes to the trivial element in $H^1_{\text{et}}(K(t), G_K)$, one concludes that $\eta(\tilde{\xi}) = *$. Whence $\tilde{\xi} = *$ by Theorem 5.0.15. Thus there exists an element $\tilde{a} \in \mathcal{F}(K[t])$ such that $\partial(\tilde{a}) = \tilde{\xi}$. The map $\mathcal{F}(K(t)) \to H^1_{\text{et}}(K(t), H_K)$ is injective by Lemma 4.0.9. Thus $\varepsilon(\tilde{a}) = a$. The map $\mathcal{F}(K) \to \mathcal{F}(K[t])$ induced by the inclusion $K \hookrightarrow K[t]$ is surjective, since the corresponding map $C(K) \to C(K[t])$ is an isomorphism. Whence there exists an $a_0 \in \mathcal{F}(K)$ such that its image in $\mathcal{F}(K(t))$ coincides with the element $a$. □

**Corollary 5.0.17.** Let $S \mapsto \mathcal{F}(S)$ be the functor defined in (7). Let

\[s_0, s_1: \text{Fr}_{\text{nr}, K[t]}(K(t)) \to \mathcal{F}(K)\]

be the specialization maps at zero and at one (at the primes $(t)$ and $(t - 1)$). Then $s_0 = s_1$.

**Proof.** It is an obvious consequence of Theorem 5.0.16. □

§ 6. Equating group schemes

The following proposition is a straightforward analogue of Proposition 7.1 in [15].

**Proposition 6.0.18.** Let $S$ be a regular semi-local irreducible scheme. Let $\mu_1: G_1 \to C_1$ and $\mu_2: G_2 \to C_2$ be two smooth $S$-group scheme morphisms with tori $C_1$ and $C_2$. Assume also that $G_1$ and $G_2$ are reductive $S$-group schemes which are forms of each other. Assume that $C_1$ and $C_2$ are forms of each other. Let $T \subset S$ be a connected non-empty closed subscheme of $S$, and let $\varphi: G_1|_{T} \to G_2|_{T}$, $\psi: C_1|_{T} \to C_2|_{T}$ be $S$-group scheme isomorphisms such that $(\mu_2|_{T}) \circ \varphi = \psi \circ (\mu_1|_{T})$. Then there exists a finite étale morphism $\tilde{S} \xrightarrow{\pi} S$ together with a section $\delta: T \to \tilde{S}$ over $T$ and $\tilde{S}$-group scheme isomorphisms $\Phi: \pi^*G_1 \to \pi^*G_2$ and $\Psi: \pi^*C_1 \to \pi^*C_2$ such that

(i) $\delta^*(\Phi) = \varphi$. 

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(ii) $\delta^*(\Psi) = \psi$,
(iii) $\pi^*(\mu_2) \circ \Phi = \Psi \circ \pi^*(\mu_1) : \pi^*(G_1) \to \pi^*(C_2)$.

We refer to [13], Proposition 5.1 for the proof of a slightly weaker statement. The proof of Proposition 6.0.18 can be carried out in the same style with some additional technicalities.

§ 7. Nice triples

In the present section we study certain packages of geometric data and morphisms of those packages. The concept of ‘nice triples’ is very close to that of ‘standard triples’ ([18], Definition 4.1) and is inspired by the latter. Let $k$ be an infinite field, $X/k$ a smooth geometrically irreducible variety, and $x_1, x_2, \ldots, x_n \in X$ closed points. Let $O_{X,\{x_1, x_2, \ldots, x_n\}}$ be the corresponding semi-local ring.

Definition 7.0.19. Let

$U := \text{Spec}(O_{X,\{x_1, x_2, \ldots, x_n\}})$.

A nice triple over $U$ consists of the following family of data:

(i) a smooth morphism $q_U : X \to U$, where $X$ is an irreducible scheme;
(ii) an element $f \in \Gamma(X, O_X)$;
(iii) a section $\Delta$ of the morphism $q_U$.

These data must satisfy the following conditions:

(a) each component of each fibre of the morphism $q_U$ has dimension 1;
(b) the $\Gamma(X, O_X)/f \cdot \Gamma(X, O_X)$ is a finite $O_{X,\{x_1, x_2, \ldots, x_n\}}$-module;
(c) there exists a finite surjective $U$-morphism $\Pi : X \to \mathbb{A}^1 \times U$;
(d) $\Delta^*(f) \neq 0 \in \Gamma(U, O_U)$.

A morphism between two nice triples $(q'_U : \mathcal{X}' \to U, f', \Delta') \to (q_U : X \to U, f, \Delta)$ is an étale morphism of $U$-schemes $\theta : \mathcal{X}' \to \mathcal{X}$ such that

(1) $q'_U = q_U \circ \theta$;
(2) $f' = \theta^*(f) \cdot g'$ for an element $g' \in \Gamma(\mathcal{X}', O_{\mathcal{X}'})$ (in particular, $\Gamma(\mathcal{X}', O_{\mathcal{X}'})/\theta^*(f)$ is a finite $O_{X,\{x_1, x_2, \ldots, x_n\}}$-module);
(3) $\Delta = \theta \circ \Delta'$.

(We stress that there are no conditions concerning an interaction of $\Pi'$ and $\Pi$.)

Theorem 7.0.20. Let $k$ be an infinite field. Let $X$ be an affine $k$-smooth irreducible $k$-variety, and let $x_1, x_2, \ldots, x_n$ be closed points in $X$. Let $U = \text{Spec}(O_{X,\{x_1, x_2, \ldots, x_n\}})$. Given a non-zero function $f \in k[X]$, vanishing at each point $x_i$ there is a diagram of the form

$$
\begin{array}{ccc}
\mathbb{A}^1 \times U & \xleftarrow{\sigma} & X \\
\downarrow{pr_U} & & \downarrow{q_U} \\
U & \xleftarrow{\Delta} & X
\end{array}
$$

with an irreducible scheme $\mathcal{X}$, a smooth morphism $q_U$, a finite surjective morphism $\sigma$, a function $f \in q_\mathcal{X}(f)k[X]$, and an essentially smooth morphism $q_X$, which enjoy the following properties:
We are given, in particular, the smooth geometrically irreducible affine scheme $\Delta \subseteq \mathbb{A}^n$. The diagram is constructed in Section 6, (16), where $\Delta$ is the diagonal morphism. Let $(\mathcal{X}, f, \Delta)$ be the nice triple $(q_U : \mathcal{X} \to U, f, \Delta)$ is a semi-local essentially smooth $k$-algebra with maximal ideals $m_i$ where $i$ runs from 1 to $n$. Let $(\mathcal{X}, f, \Delta)$ be the nice triple (17) over $U$. We show that it gives rise to certain data satisfying the hypotheses of Lemma 3.0.6.

Let $A = \Gamma(\mathcal{X}, \mathcal{O}_\mathcal{X})$. It is an $\mathcal{O}$-algebra via the ring homomorphism $(q_U)^* : \mathcal{O} \to \Gamma(\mathcal{X}, \mathcal{O}_\mathcal{X})$. Moreover this $\mathcal{O}$-algebra is smooth and $A$ is a domain. The triple $(\mathcal{X}, f, \Delta)$ is a nice triple. Thus there exists a finite surjective $U$-morphism $\Pi : \mathcal{X} \to \mathbb{A}^1_U$. It induces the corresponding inclusion of $\mathcal{O}$-algebras $\mathcal{O} [t] \subseteq \Gamma(\mathcal{X}, \mathcal{O}_\mathcal{X}) = A$ so that $A$ is finitely generated as an $\mathcal{O}[t]$-module. For each index $i$ the $\mathcal{O}/m_i$-algebra $A/m_iA$ is equidimensional of dimension 1 since $(\mathcal{X}, f, \Delta)$ is a nice triple. Let $\varepsilon = (\Delta)^* : A \to \mathcal{O}$ be the $\mathcal{O}$-algebra homomorphism induced by the section $\Delta$ of the morphism $q_U$. Clearly, that $\varepsilon$ is an augmentation. Further, $\varepsilon(f) \neq 0 \in \mathcal{O}$ since $(\mathcal{X}, f, \Delta)$ is a nice triple. Set $I = \text{Ker}(\varepsilon)$. By Claim 6.1 in §6 in [13], $f$ vanishes at all closed points of $\Delta(U)$. The $\mathcal{O}$-module $A/fA$ is finite since $(\mathcal{X}, f, \Delta)$ is a nice triple. So, we are under the hypotheses of Lemma 3.0.6. Thus we may use the conclusion of that lemma.

So, there exists an element $u \in A$ satisfying properties (1) to (7) in Lemma 3.0.6. The function $u$ defines a finite surjective morphism $\sigma : \mathcal{X} \to U \times \mathbb{A}^1$. Using the morphisms $q_X : \mathcal{X} \to X$, $\Delta$, can: $U \to X$, $q_U : \mathcal{X} \to U$ and $\text{pr}_U : \mathbb{A}^1_U \to U$ we get a diagram of the form (11). By Lemma 3.0.6 that diagram is the desired one. $\square$
Theorem 7.0.21. Let $k$ be an infinite field. Let $U$ be as in Definition 7.0.19. Let $(\mathcal{X}, f, \Delta)$ be a nice triple over $U$. Let $G_{\mathcal{X}}$ be a reductive $\mathcal{X}$-group scheme, let $G_U := \Delta^*(G_{\mathcal{X}})$ and let $G_{\text{const}}$ be the pullback of $G_U$ to $\mathcal{X}$. Let $C_{\mathcal{X}}$ be an $\mathcal{X}$-torus, let $C_U := \Delta^*(C_{\mathcal{X}})$ and let $C_{\text{const}}$ be the pullback of $C_U$ to $\mathcal{X}$. Let $\mu_{\mathcal{X}}: G_{\mathcal{X}} \to C_{\mathcal{X}}$ be an $\mathcal{X}$-group scheme morphism smooth as a scheme morphism. Let $\mu_U = \Delta^*(\mu_{\mathcal{X}})$ and let $\mu_{\text{const}}: G_{\text{const}} \to C_{\text{const}}$ be the pullback of $\mu_U$ to $\mathcal{X}$.

Then there exist a morphism $\theta: (\mathcal{X}', f', \Delta') \to (\mathcal{X}, f, \Delta)$ of nice triples over $U$ and isomorphisms

$$\Phi: \theta^*(G_{\text{const}}) \to \theta^*(G_{\mathcal{X}}), \quad \Psi: \theta^*(C_{\text{const}}) \to \theta^*(C_{\mathcal{X}})$$

of $\mathcal{X}'$-group schemes such that $(\Delta')^*(\Phi) = \text{id}_{G_U}$, $(\Delta')^*(\Phi) = \text{id}_{G_U}$ and

$$\theta^*(\mu_{\mathcal{X}}) \circ \Phi = \Psi \circ \theta^*(\mu_{\text{const}}).$$  \hspace{1cm} (13)

Basically, this theorem is a consequence of Proposition 6.0.18.

Proof of Theorem 7.0.21. We can start by almost literally repeating the arguments from the proof of Lemma 8.1 in [19], which involve the following purely geometric lemma ([19], Lemma 8.2).

For the reader’s convenience we state that lemma below, adapting the notation to those in §7. Namely, let $U$ be as in Definition 7.0.19 and let $(\mathcal{X}, f, \Delta)$ be a nice triple over $U$. Further, let $G_{\mathcal{X}}$ be a simple simply connected $\mathcal{X}$-group scheme, let $G_U := \Delta^*(G_{\mathcal{X}})$, and let $G_{\text{const}}$ be the pullback of $G_U$ to $\mathcal{X}$. Finally, by the definition of a nice triple there exists a finite surjective morphism $\Pi: \mathcal{X} \to A^1 \times U$ of $U$-schemes.

Lemma 7.0.22. Let $\mathcal{Y}$ be a closed non-empty subscheme of $\mathcal{X}$, finite over $U$. Let $\mathcal{V}$ be an open subset of $\mathcal{X}$ containing $\Pi^{-1}(\Pi(\mathcal{Y}))$. Then there exists an open set $\mathcal{W} \subseteq \mathcal{V}$ still containing $\Pi^{-1}(\Pi(\mathcal{Y}))$ and endowed with a finite surjective morphism $\mathcal{W} \to A^1 \times U$ (in general $\neq \Pi$).

Let $\Pi: \mathcal{X} \to A^1 \times U$ be the above finite surjective $U$-morphism. The following diagram summarizes the situation:

\[
\begin{array}{ccc}
Z & \xrightarrow{q_U} & \mathcal{Y} \\
\downarrow & & \downarrow \\
\mathcal{X} & \xrightarrow{\Pi} & A^1 \times U.
\end{array}
\]

Here $Z$ is the closed subscheme defined by the equation $f = 0$. By assumption, $Z$ is finite over $U$. Let $\mathcal{Y} = \Pi^{-1}(\Pi(Z \cup \Delta(U)))$. Since $Z$ and $\Delta(U)$ are both finite over $U$ and since $\Pi$ is a finite morphism of $U$-schemes, $\mathcal{Y}$ is also finite over $U$. 
Denote by \( y_1, \ldots, y_m \) its closed points and let \( S = \text{Spec}(O_{X,y_1,\ldots,y_m}) \). Set \( T = \Delta(U) \subseteq S \). Further, let \( G_X, G_U = \Delta^*(G_X) \) and \( G_{\text{const}} \) be as in the hypotheses of Theorem 7.0.21. Let \( C_X, C_U = \Delta^*(G_X) \) and \( C_{\text{const}} \) be as in the hypotheses of Theorem 7.0.21. Finally, let

\[
\varphi: G_{\text{const}}|_T \rightarrow G_X|_T, \quad \psi: C_{\text{const}}|_T \rightarrow C_X|_T
\]

be the canonical isomorphisms. Recall that by assumption \( X \) is \( U \)-smooth, and thus \( S \) is regular.

By Proposition 6.0.18 there exist a finite étale covering \( \theta_0: \tilde{S} \rightarrow S \), a section \( \delta: T \rightarrow \tilde{S} \) of \( \theta_0 \) over \( T \) and isomorphisms

\[
\Phi_0: \theta_0^*(G_{\text{const}}|_S) \rightarrow \theta_0^*(G_X|_S), \quad \Psi_0: \theta_0^*(C_{\text{const}}|_S) \rightarrow \theta_0^*(C_X|_S)
\]

such that \( \delta^*\Phi_0 = \varphi, \delta^*\Psi_0 = \psi \) and

\[
\theta_0^*(\mu_X|_S) \circ \Phi_0 = \Psi_0 \circ \theta_0^*(\mu_{\text{const}}|_S): \theta_0^*(G_{\text{const}}|_S) \rightarrow \theta_0^*(C_X|_S). \tag{14}
\]

Replacing \( \tilde{S} \) by a connected component of \( \tilde{S} \) which contains \( \delta(T) = \delta(\Delta(U)) \) we may and will assume that \( \tilde{S} \) is irreducible. We can extend these data to a neighbourhood \( \mathcal{V} \) of \( \{y_1, \ldots, y_n\} \) and get the diagram

\[
\begin{array}{ccc}
\tilde{S} & \xrightarrow{\delta} & \mathcal{V} \\
\scriptstyle{\theta_0} \downarrow & & \downarrow \scriptstyle{\theta} \\
T & \rightarrow & \mathcal{V} \rightarrow \mathcal{X}
\end{array}
\tag{15}
\]

where \( \theta: \mathcal{V} \rightarrow \mathcal{X} \) is finite étale, and isomorphisms

\[
\Phi: \theta^*(G_{\text{const}}) \rightarrow \theta^*(G_X), \quad \Psi: \theta^*(C_{\text{const}}) \rightarrow \theta^*(C_X)
\]

such that

\[
\theta^*(\mu_X|\mathcal{V}) \circ \Phi = \Psi \circ \theta^*(\mu_{\text{const}}|\mathcal{V}) \tag{16}
\]

(the latter equality holds since the equality (14) holds and \( \mathcal{V} \) is irreducible).

Since \( T \) projects isomorphically onto \( U \), it is still closed when viewed as a sub-scheme of \( \mathcal{V} \). Note that since \( \mathcal{V} \) is semi-local and \( \mathcal{V} \) contains all its closed points, \( \mathcal{V} \) contains \( \Pi^{-1}(\Pi(\mathcal{V})) = \mathcal{V} \). By Lemma 7.0.22 there exists an open subset \( \mathcal{W} \subseteq \mathcal{V} \) containing \( \mathcal{V} \) and endowed with a finite surjective \( U \)-morphism \( \Pi^*: \mathcal{W} \rightarrow \mathbb{A}^1 \times U \).

Let \( X' = \theta^{-1}(\mathcal{W}), f' = \theta^*(f), q'_U = q_U \circ \theta \), and let \( \Delta': U \rightarrow X' \) be the section of \( q'_U \) obtained as the composite of \( \delta \) with \( \Delta \). We claim that the triple \( (X', f', \Delta') \) is a nice triple. Let us verify this. Firstly, the structure morphism \( q'_U: X' \rightarrow U \) coincides with the composite

\[
X' \xrightarrow{\theta} \mathcal{W} \hookrightarrow X \xrightarrow{q_U} U.
\]

Thus, it is smooth. The element \( f' \) belongs to the ring \( \Gamma(X', \mathcal{O}_{X'}) \), the morphism \( \Delta' \) is a section of \( q'_U \). Each component of each fibre of the morphism \( q_U \) has
dimension 1, the morphism $\mathcal{X}' \xrightarrow{\theta} \mathcal{W} \hookrightarrow \mathcal{X}$ is étale. Thus, each component of each fibre of the morphism $q'_U$ is also of dimension 1. Since $\{f = 0\} \subset \mathcal{W}$ and $\theta: \mathcal{X}' \to \mathcal{W}$ is finite, $\{f' = 0\}$ is finite over $\{f = 0\}$ and hence also over $U$. In other words, the $O$-module $\Gamma(\mathcal{X}', O_{\mathcal{X}'})/f' \cdot \Gamma(\mathcal{X}', O_{\mathcal{X}'})$ is finite. The morphism $\theta: \mathcal{X}' \to \mathcal{W}$ is finite and surjective. We have constructed above the finite surjective morphism $\Pi^*: \mathcal{W} \to A^1 \times U$. It follows that $\Pi^* \circ \theta: \mathcal{X}' \to A^1 \times U$ is finite and surjective.

Clearly, the étale morphism $\theta: \mathcal{X}' \to \mathcal{X}$ is a morphism of nice triples, with $g = 1$.

Denote the restriction of $\Phi$ to $\mathcal{X}'$ simply by $\Phi$. The equality $(\Delta')^*\Phi = \id_{G_U}$ holds by the very construction of the isomorphism $\Phi$. Denote the restriction of $\Psi$ to $\mathcal{X}'$ simply by $\Psi$. The equality $(\Delta')^*\Psi = \id_{G_U}$ holds by the very construction of the isomorphism $\Psi$. Finally, the equality $\theta^*(\mu_\mathcal{X}) \circ \Phi = \Psi \circ \theta^*(\mu_{\const})$ follows directly from the equality (16) above. The theorem follows. □

**Theorem 7.0.23.** Let $k$ be an infinite field. Let $X, \{x_1, x_2, \ldots, x_n\} \subset X, f \in k[X]$ and $U = \text{Spec}(O_{X, \{x_1, x_2, \ldots, x_n\}})$ be as in Theorem 7.0.20. Let $G$ be a reductive $X$-group scheme and let $G_U := \text{can}^*(G)$. Let $C$ be an $X$-torus and let $C_U := \text{can}^*(C)$. Let $\mu: G \to G$ be an $X$-group scheme morphism smooth as a scheme morphism. Let $\mu_{G_U} = \text{can}^*(\mu): G_U \to C_U$.

Then there exists a diagram of the form (11) with an irreducible scheme $X$, a smooth morphism $q_U$, a finite surjective morphism $\sigma$, a function $f' \in q_X^* f \cdot k[X]$ and an essentially smooth morphism $q_X$, which enjoy the conditions (a) to (f) in Theorem 7.0.20, and additionally there are $\mathcal{X}$-group scheme isomorphisms

$$
\Phi: q_U^*(G_U) \to q_X^*(G), \quad \Psi: q_U^*(C_U) \to q_X^*(C)
$$

such that $(\Delta^*)^*\Phi = \id_{G_U}, (\Delta^*)^*\Psi = \id_{C_U}$ and $q_X^*(\mu) \circ \Phi = \Psi \circ q_U^*(\mu_U)$.

**Proof.** We are given, in particular, the smooth geometrically irreducible affine $k$-scheme $X$, the finite family of points $x_1, x_2, \ldots, x_n$ on $X$, and the non-zero function $f \in k[X]$ vanishing at each point $x_i$. Recall, that beginning with these data a nice triple

$$
(q_U: \mathcal{X} \to U, f, \Delta)
$$

over $U$ is constructed in [13], Section 6, (16), where $\mathcal{X} = U \times_S X$ for a $k$-smooth affine scheme $S$ and a smooth morphism $X \to S$. This is done by shrinking $X$ and securing properties (i) to (iv) at the same time. Recall that $q_U: \mathcal{X} = U \times_S X \to U$ is the projection to $U$, $\Delta: U \to \mathcal{X} = U \times_S X$ is the diagonal morphism. Let $q_X: \mathcal{X} = U \times_S X \to X$ be the projection to $X$. Clearly, $q_X \circ \Delta: U \to X$ is the canonical inclusion.

Set $G_X := q^*(G), C_X := q^*(C), \mu_X := q^*(\mu)$. By Theorem 7.0.21 there exist a morphism $\theta: (\mathcal{X}', f', \Delta') \to (\mathcal{X}, f, \Delta)$ of nice triples over $U$ and isomorphisms

$$
\Phi: \theta^*(G_{\const}) \to \theta^*(G_X), \quad \Psi: \theta^*(C_{\const}) \to \theta^*(C_X)
$$

of $\mathcal{X}'$-group schemes such that $(\Delta')^*\Phi = \id_{G_U}, (\Delta')^*\Psi = \id_{G_U}$ and

$$
\theta^*(\mu_X) \circ \Phi = \Psi \circ \theta^*(\mu_{\const}).
$$

(18)
Set \( q'_{U} = q_{U} \circ \theta \), \( q'_{X} = q_{X} \circ \theta \), \( A' = \Gamma(\mathcal{X}',\mathcal{O}_{\mathcal{X}'}) \). Repeating literally the arguments from the end of Theorem 7.0.20 one can find an element \( u' \in A' \) satisfying properties (1) to (7) from Lemma 3.0.6. The function \( u' \) defines a finite surjective morphism \( \sigma': \mathcal{X}' \to U \times \mathbb{A}^{1} \). Using the morphisms \( q'_{X}: \mathcal{X} \to X \), \( \Delta': U \to \mathcal{X}' \), can: \( U \to X \), \( q'_{U}: \mathcal{X}' \to U \) and \( \pi_{U}: \mathbb{A}^{1}_{U} \to U \) we get a diagram of the form (11). Set \( f' := \theta^*(q'_{X}(f)) \). By Lemma 3.0.6 that diagram is the desired one.

Also there are \( \mathcal{X}' \)-group scheme isomorphisms

\[
\Phi: (q')^*_U(G_U) \to (q')^*_X(G), \quad \Psi: (q')^*_U(C_U) \to (q')^*_X(C)
\]

such that \( (\Delta')^*(\Phi) = \text{id}_{G_U}, (\Delta')^*(\Psi) = \text{id}_{C_U} \) and \( (q')^*_X(\mu) \circ \Phi = \Psi \circ (q')^*_U(\mu_U) \). \( \square \)

§ 8. Proof of Theorem 1.0.1

Proof. We begin with the following data. Fix a smooth irreducible affine \( k \)-scheme \( X \), a finite family of closed points \( x_1, x_2, \ldots, x_n \) on \( X \), and set \( \mathcal{O} := \mathcal{O}_X, \{x_1, x_2, \ldots, x_n\} \) and \( U := \text{Spec}(\mathcal{O}) \). Replacing \( k \) by its algebraic closure in \( \Gamma(X, \mathcal{O}_X) \) we may and will assume that \( X \) is \( k \)-smooth and geometrically irreducible.

Further, consider the reductive \( U \)-group scheme \( G \), the \( U \)-torus \( C \) and the smooth \( U \)-group scheme morphism \( \mu: G \to C \). Firstly we prove Theorem 1.0.1 under an additional assumption that \( \mu \) is ‘constant’. By this assumption we mean the following: there are a reductive group \( G_0 \), a torus \( C_0 \) over the field \( k \) and an algebraic \( k \)-group morphism \( \mu_0 \) and \( U \)-group scheme isomorphisms

\[
\Phi: G_{0,U} = G_0 \times_{\text{Spec}(k)} U \to G, \quad \Psi: C_{0,U} = C_0 \times_{\text{Spec}(k)} U \to C
\]

such that \( \Psi \circ \mu_{0,U} = \mu \circ \Phi \).

Let \( K \) be the field of fractions of \( \mathcal{O} \). Let \( a_K \in C(K) \) be such that the element \( \sigma_K \in C(K) \) is \( \mathcal{O} \)-unramified (see (8)). Shrinking \( X \) if necessary, we may secure the following properties:

(i) the points \( x_1, x_2, \ldots, x_r \) are still in \( X \);

(ii) the element \( a_K \) is defined over \( X_{t} \), that is, there is given an element \( a \in C_0(k[X]_t) \) for a non-zero function \( f \in k[X] \) such that the image of \( a \) in \( C_0(K) = C(K) \) coincides with the element \( a_K \); we may assume further that \( f \) vanishes at each \( x_i \) and the \( k \)-algebra \( k[X]/(f) \) is reduced;

(iii) we may also assume that the element \( \tilde{a} \in \overline{C_0}(k[X]_t) \) is \( k[X] \)-unramified.

Let \( \eta: \text{Spec}(K) \to X_{t} \) be the morphism induced by the inclusion \( k[X_t] \hookrightarrow k(X) = K \) and let \( \eta_U: \text{Spec}(K) \to U \) be the morphism induced by the inclusion \( \mathcal{O} \hookrightarrow K \). Clearly, \( a_K = \eta^*(a) \in C_0(K) = C(K) \).

Our aim is to find an element \( a_U \in C_0(U) \) such that \( \eta_U^*(\tilde{a}_U) = \tilde{a}_K \in \overline{C_0}(K) \).

We will construct such an element \( a_U \) rather explicitly in (19). At the moment we are given, in particular, the smooth geometrically irreducible affine \( k \)-scheme \( X \), the finite family of points \( x_1, x_2, \ldots, x_n \) on \( X \), and the non-zero function \( f \in k[X] \) vanishing at each point \( x_i \). By Theorem 7.0.20 we have the diagram (11) and the function \( f \in q'_{X}(f )k[X] \) subject to the conditions (a) to (f).

Since the schemes \( \mathcal{X} \) and \( \mathcal{Y} := U \times \mathbb{A}^{1} \) are regular schemes and \( \sigma \) is finite surjective, it is finite and flat by a theorem of Grothendieck [20], Corollary 18.17.
So, for any $U$-map $t: \mathcal{Y} \to \mathcal{Y}$ of affine $U$-schemes, setting $\mathcal{X}' = \mathcal{X} \times \mathcal{Y} \mathcal{Y}'$ we have the norm map given by the construction (6) from §2:

$$N_{\mathcal{X}'/\mathcal{Y}'}: C_0(\mathcal{X}') \to C_0(\mathcal{Y}').$$

Set $\mathcal{D} = \text{Spec}(A/J)$ and $\mathcal{D}_1 = \text{Spec}(A/(u-1)A)$. Clearly $\mathcal{D}_1$ is the scheme-theoretic pre-image of $U \times \{1\}$ and the disjoint union $\mathcal{D}_0 := \mathcal{D} \coprod \Delta(U)$ is the scheme-theoretic pre-image of $U \times \{0\}$ under the morphism $\sigma$. In particular, $\mathcal{D}$ is finite-flat over $U$. We will write $\Delta$ for $\Delta(U)$ and $\mathcal{Z}$ for the vanishing locus of $f$ in $\mathcal{X}$.

Set $\alpha := q_f^*(a) \in C_0(\mathcal{X}_f)$ where $q_f: \mathcal{X}_f \to X_f$ is the restriction of $q_X$ to $\mathcal{X}_f$ and set

$$a_U := N_{\mathcal{D}_1/U}(\alpha|_{\mathcal{D}_1}) \cdot N_{\mathcal{D}_0/U}(\alpha|_{\mathcal{D}_0})^{-1} \in C_0(U).$$

Claim 8.0.24. Let $\eta_U: \text{Spec}(k(X)) \to \text{Spec}(\mathcal{O}) = U$ and $\eta: \text{Spec}(k(X)) \to X_f$ be generic points of $U$ and $X_f$ respectively. Then $\eta^*_U(\overline{\alpha}_U) = \eta^*(\overline{\alpha}) \in \overline{C}_0(k(X))$.

Since $\eta^*(a) = a_K$, this claim completes the proof of Theorem 1.0.1 in the constant case. To prove the claim consider the scheme $\mathcal{X}$ and its closed and open subschemes as $U$-schemes via the morphism $q_U$. Set $K = k(X)$. Taking the base change of $\mathcal{X}$, $A^1_U$ and $\sigma$ via the morphism $\eta_U: \text{Spec}(K) \to U$ we get a morphism of $K$-schemes $A^1_K \xrightarrow{\sigma_K} \mathcal{X}_K$. Recall that the class $\overline{\alpha} \in \overline{C}_0(X_f)$ is $X$-unramified. By Lemma 4.0.12 the class $\overline{\alpha} \in \overline{C}_0(\mathcal{X}_f)$ is $\mathcal{X}$-unramified. Hence its image $\overline{\sigma}_K$ in $\overline{C}_0(K(\mathcal{X}_K))$ is $X_K$-unramified too. The items (5), (6) and (7) of Lemma 3.0.6 and Lemma 4.0.11 show that for the element $\beta_u := N_K(\mathcal{X}_K)/K(\mathcal{X}_K)) \in C_0(\mathcal{K}(\mathcal{X}_K))$ the class $\overline{\beta}_u \in \overline{C}_0(\mathcal{K}(u))$ is $\mathcal{A}^1_K$-unramified. By Theorem 5.0.16 the class $\overline{\beta}_u$ is constant, that is, it comes from the field $K$. By Corollary 5.0.17 its specializations at the $K$-points 0 and 1 of the affine line $A^1_K$ coincide: $s_0(\overline{\beta}_u) = s_1(\overline{\beta}_u) \in \overline{C}_0(K)$. The items (2), (3) and (4) from Lemma 3.0.6 and the equality $q_X \circ \Delta = \text{can}: U \to X$ show that $\mathcal{D}_1, \mathcal{D}_0, \Delta(\text{Spec}(K)) \subset (\mathcal{X}_f)_K$. Thus there is a Zariski-open neighbourhood $V$ of the $K$-points 0 and 1 in $A^1_K$ such that $W := (\sigma_K)^{-1}(V) \subset (\mathcal{X}_f)_K$. Hence for $\beta_V := N_{W/V}(\alpha|_W)$, one has the equality $\beta_V = \beta_u$ in $C(K(u))$. Thus one has equalities

$$\beta(1) = s_1(\overline{\beta}_u) = s_0(\overline{\beta}_u) = \beta(0)$$

(see the remark at the end of Definition 5.0.13). By the properties (i'), (ii') and (iii') of the norm maps (see §2) one has equalities

$$N_{\mathcal{D}_1/K}(\alpha|_{\mathcal{D}_1}) = \beta(1),$$

$$N_{\mathcal{D}_0,K}(\alpha|_{\mathcal{D}_0}) = N_{\mathcal{D}_0,K}(\alpha|_{\mathcal{D}_0}) \cdot \Delta_K^*(\alpha_K).$$

By the base-change property of the norm maps one has the equality

$$\eta^*_U(a_U) = N_{\mathcal{D}_1,K}(\alpha|_{\mathcal{D}_1}) \cdot [N_{\mathcal{D}_0,K}(\alpha|_{\mathcal{D}_0})]^{-1}.$$

Hence $\Delta_K^*(\overline{\alpha}_K) = \eta^*_U(\overline{\alpha}_U) \in \overline{C}_0(k(X))$. Finally, the composite map $\text{Spec}(K) \xrightarrow{\Delta_K} (\mathcal{X}_f)_K \to \mathcal{X}_f \xrightarrow{q_f} X_f$ coincides with the canonical map $\eta: \text{Spec}(K) \to X_f$. Hence
\[ \Delta^*_K(\overline{\alpha}_K) = \eta^*(\overline{a}), \] which proves the claim. Whence Theorem 1.0.1 in the ‘constant’ case.

In the rest of the section we will prove the general case of Theorem 1.0.1. Let \( K \) be the field of fractions of \( \mathcal{O} \). Let \( a_K \in C(K) \) be such that the element \( \overline{\alpha}_K \in \overline{C}(K) \) is \( \mathcal{O} \)-unramified (see (8)). Shrinking the scheme \( X \) if necessary, we may secure the following properties: (i), (ii'), (iii), where (i), (ii), (iii) are as in the ‘constant’ case and (ii') is this: there are given a reductive \( X \)-group scheme \( G_X \), an \( X \)-torus \( C_X \), a smooth \( X \)-group scheme morphism \( \mu_X : G_X \to C_X \) such that \( H_X := \text{Ker}(\mu_X) \) is a reductive \( X \)-group scheme and \( G = U \times_X G_X, C = U \times_X C_X, \mu = U \times_X \mu_X \). In particular, for any \( U \)-scheme \( S \) one has \( G(S) = G_X(S), C(S) = C_X(S), \overline{C}(S) = \overline{C}_X(S) \), where \( \overline{C}_X(S) := C_X(S)/\mu_X(G_X(S)) \).

There exists a diagram of the form (11) with an irreducible scheme \( \mathcal{X} \), a smooth morphism \( q_\mathcal{U} \), a finite surjective morphism \( \sigma \), a function \( f' \in q_\mathcal{X}(f)k[\mathcal{X}] \) and an essentially smooth morphism \( q_X \), which enjoy the conditions (a) to (f) in Theorem 7.0.20, and additionally there are \( \mathcal{X} \)-group scheme isomorphisms

\[ \Phi : q_\mathcal{U}^*(G_\mathcal{U}) \to q_X^*(G_X), \quad \Psi : q_\mathcal{U}^*(C_\mathcal{U}) \to q_X^*(C_X) \]

such that \( \Delta^*(\Phi) = \text{id}_{G_\mathcal{U}}, \Delta^*(\Psi) = \text{id}_{C_\mathcal{U}} \) and \( q_X^*(\mu) \circ \Phi = \Psi \circ q_\mathcal{U}^*(\mu_\mathcal{U}) \).

Now there are two functors on the category of \( \mathcal{X} \)-schemes, namely, \( \overline{C} \) and \( \overline{C}_U \). If \( r : \mathcal{Y} \to \mathcal{X} \) is a scheme morphism, then

\[ \overline{C}(\mathcal{Y}) := C_X(\mathcal{Y})/(\mu(G_X(\mathcal{Y}))), \quad \overline{C}_U(\mathcal{Y}) := U\overline{C}(\mathcal{Y})/(\mu(UG(\mathcal{Y}))). \]

Here \( \mathcal{Y} \) is regarded as an \( X \)-scheme via the morphism \( q_X \circ r \) and is regarded as an \( U \)-scheme via the morphism \( q_\mathcal{U} \circ r \). The \( \mathcal{X} \)-group scheme isomorphisms \( \Phi \) and \( \Psi \) in Theorem 7.0.23 induce a group isomorphism

\[ \overline{\eta}_U : \overline{C}(\mathcal{Y}) \to \overline{C}(\mathcal{Y}), \]

which respects \( \mathcal{X} \)-scheme morphisms. Moreover, if the scheme \( U \) is regarded as an \( \mathcal{X} \)-scheme via the morphism \( \Delta \), then the isomorphism \( \overline{\eta}_U \) is the identity. And similarly for any \( U \)-scheme \( g : W \to U \) regarded as an \( \mathcal{X} \)-scheme via the morphism \( \Delta \circ g \), the isomorphism \( \overline{\eta}_W \) is the identity.

Set \( \alpha := q_f^*(a) \in C(\mathcal{X}_f) \) where \( q_f : \mathcal{X}_f \to X_f \) is as above in this proof. Let \( \alpha = U C(\mathcal{X}) \) be the unique element such that \( \overline{C}_U(\alpha) = \alpha \). Set

\[ a := N_{D_1/U}((\nu_\alpha)|_{D_1}) \cdot N_{D_0/U}((\nu_\alpha)|_{D_0})^{-1} \in \nu C(U), \quad a_U := \Psi_U(\nu a) \in C_X(U). \]

We leave to the reader to proof the following claim.

**Claim 8.0.25.** Let \( \eta_U : \text{Spec}(k(X)) \to \text{Spec}(\mathcal{O}) = U \) and \( \eta : \text{Spec}(k(X)) \to X_f \) be as above in this proof. Then

\[ \eta_U^*(\overline{\alpha}_U) = \eta^*(\overline{a}) \in \overline{C}(k(X)). \]

Since \( \eta^*(a) = a_K \), this claim completes the proof of Theorem 1.0.1. □
§ 9. An extension of Theorem 1.0.1

The main aim of the present section is to prove the following result.

Theorem 9.0.26 (Theorem B). Let $R$ be a regular local domain containing an infinite perfect field $k$. Let

$$
\mu : G \rightarrow C
$$

be a smooth $R$-group scheme morphism of reductive $R$-group schemes, with a torus $C$. Set $H = \text{Ker}(\mu)$ and suppose additionally that $H$ is a reductive $R$-group scheme. The functor

$$
\mathcal{F} : S \mapsto C(S)/\mu(G(S))
$$

defined on the category of $R$-algebras satisfies purity for $R$, that is, the sequence (1) is exact at its middle term.

To prove Theorem 9.0.26 we now recall a celebrated result of Dorin Popescu (see [21] or, for a self-contained proof, [22]).

Let $k$ be a field and $R$ a local $k$-algebra. We say that $R$ is geometrically regular if $k' \otimes_k R$ is regular for any finite extension $k'$ of $k$. A ring homomorphism $A \rightarrow R$ is called geometrically regular if it is flat and for each prime ideal $q$ of $R$ lying over $p$, $R_q/pR_q = k(p) \otimes_A R_q$ is geometrically regular over $k(p) = A_p/p^p$.

Observe that any regular local ring containing a perfect field $k$ is geometrically regular over $k$.

Theorem 9.0.27 (Popescu’s theorem). A homomorphism $A \rightarrow R$ of Noetherian rings is geometrically regular if and only if $R$ is a filtered direct limit of smooth $A$-algebras.

Proof of Theorem 9.0.26. Let $R$ be a regular local ring containing an infinite perfect field $k$. Since $k$ is perfect one can apply Popescu’s theorem. So, $R$ can be represented as a filtered direct limit of smooth $k$-algebras $A_\alpha$ over the infinite field $k$. We first observe that we may replace the direct system of the $A_\alpha$ by a system of essentially smooth local $k$-algebras. In fact, if $m$ are all maximal ideals of $R$ and $S = R - m$ we can replace each $A_\alpha$ by $(A_\alpha)_m$, where $S_\alpha = S \cap A_\alpha$. Note that in this case the canonical morphisms $\varphi_\alpha : A_\alpha \rightarrow R$ are local (serving the maximal ideal to the maximal one) and every $A_\alpha$ is a regular local ring, in particular a factorial ring.

Now let $L$ be the field of fractions of $R$ and, for each $\alpha$, let $K_\alpha$ be the field of fractions of $A_\alpha$. For each index $\alpha$ let $a_\alpha$ be the kernel of the map $\varphi_\alpha : A_\alpha \rightarrow R$ and $B_\alpha = (A_\alpha)_{a_\alpha}$. Clearly, for each $\alpha$, $K_\alpha$ is the field of fractions of $B_\alpha$. The composite map $A_\alpha \rightarrow R \rightarrow L$ factors through $B_\alpha$ and hence it also factors through the residue field $k_\alpha$ of $B_\alpha$. Since $R$ is a filtering direct limit of the $A_\alpha$, we see that $L$ is a filtering direct limit of the $B_\alpha$. We will write $\psi_\alpha$ for the canonical morphism $B_\alpha \rightarrow L$.

Let $\xi \in C(L)$ be such that the class $\overline{\xi} \in \mathcal{F}(L)$ is $R$-unramified. We need the following two lemmas.

Lemma 9.0.28. Let $B$ be a regular local ring and let $K$ be its field of fractions. Let $m$ be a maximal ideal of $B$ and $\overline{B} = B/m$. For an element $\theta \in \mathcal{F}(B)$ write
\[ \bar{\varnothing} \text{ for its image in } \mathcal{F}(\overline{B}) \text{ and } \theta_K \text{ for its image in } \mathcal{F}(K). \] Let \( \eta, \rho \in \mathcal{F}(B) \) be such that \( \eta_K = \rho_K \in \mathcal{F}(K) \). Then \( \overline{\eta} = \overline{\rho} \in \mathcal{F}(\overline{B}). \)

**Lemma 9.0.29.** There exists an index \( \alpha \) and an element \( \xi_\alpha \in C(B_\alpha) \) such that \( \psi_\alpha(\xi_\alpha) = \xi \) and the class \( \overline{\xi}_\alpha \in \mathcal{F}(K_\alpha) \) is \( A_\alpha \)-unramified.

Assuming these two lemmas we complete the proof as follows. Consider a commutative diagram

\[
\begin{array}{ccc}
A_\alpha & \xrightarrow{\varphi_\alpha} & R \\
\downarrow & & \downarrow \\
B_\alpha & \xrightarrow{k_\alpha} & L \\
\downarrow & & \downarrow \\
K_\alpha. & & \\
\end{array}
\]

By Lemma 9.0.29 the class \( \overline{\xi}_\alpha \in \mathcal{F}(K_\alpha) \) is \( A_\alpha \)-unramified. Hence by Theorem 1.0.1 there exists an element \( \eta \in \mathcal{C}(A_\alpha) \) such that \( \overline{\xi}_\alpha = \overline{\eta} \in \mathcal{F}(K_\alpha) \). By Lemma 9.0.28 the elements \( \overline{\xi}_\alpha \) and \( \overline{\eta} \) have the same image in \( \mathcal{F}(k_\alpha) \). Hence \( \overline{\xi} \in \mathcal{F}(L) \) coincides with the image of the element \( \varphi_\alpha(\overline{\eta}) \) in \( \mathcal{F}(L) \). It remains to prove the two lemmas.

**Proof of Lemma 9.0.28.** Induction on \( \text{dim}(B) \). The case of dimension 1 follows from Theorem 4.0.8 applied to the local ring \( B \). To prove the general case choose an \( f \in B \) such that \( \eta = \rho \in \mathcal{F}(B_f) \). Let \( \pi \in B \) be such that \( \pi \) is a regular parameter in \( B \), having no common factors with \( f \). Let \( B' = B/\pi B \). Then for the image \( (\eta - \rho)' \) of \( \eta - \rho \) in \( \mathcal{F}(B') \) we have \( (\eta - \rho)' = 0 \in \mathcal{F}(B'_f) \). By the inductive hypotheses one has \( (\eta - \rho)' = 0 \in \mathcal{F}(\overline{B}) \). Since \( (\eta - \rho) = (\eta - \rho)' \in \mathcal{F}(\overline{B}) \), one has \( \overline{\eta} = \overline{\rho} \in \mathcal{F}(\overline{B}) \). \( \square \)

**Proof of Lemma 9.0.29.** Choose an \( f \in R \) such that \( \xi \) is defined over \( R_f \). Then \( \xi \) is ramified at most at those height 1 primes \( p_1, \ldots, p_r \) which contain \( f \). Since the class \( \overline{\xi} \in \mathcal{F}(L) \) is \( R \)-unramified, there exists, for any \( p_i \), an element \( \sigma_i \in G(L) \) and an element \( \xi_i \in \mathcal{C}(R_{p_i}) \) such that \( \xi = \mu(\sigma_i)\xi_i \in \mathcal{C}(L) \). We may assume that \( \xi_i \) is defined over \( R_{h_i} \) for some \( h_i \in R - p_i \) and that \( \sigma_i \) is defined over \( R_{g_i} \) for some \( g_i \in R \).

We can find an index \( \alpha \) such that \( A_\alpha \) contains lifts \( f_\alpha, h_{1,\alpha}, \ldots, h_{r,\alpha}, g_{1,\alpha}, \ldots, g_{r,\alpha} \) and moreover

1. \( \mathcal{C}(A_{\alpha,f_\alpha}) \) contains a lift \( \xi_\alpha \) of \( \xi \),
2. \( \mathcal{C}(A_{\alpha,h_{i,\alpha}}) \) contains a lift of \( \xi_{i,\alpha} \) of \( \xi_i \),
3. \( \mathcal{C}(A_{\alpha,g_{i,\alpha}}) \) contains a lift of \( \sigma_{i,\alpha} \) of \( \sigma_i \).

Since none of the \( f_\alpha, h_{1,\alpha}, \ldots, h_{r,\alpha}, g_{1,\alpha}, \ldots, g_{r,\alpha} \) vanish in \( R \), the elements \( \xi_\alpha, \xi_{1,\alpha}, \ldots, \xi_{r,\alpha} \) and \( \sigma_{1,\alpha}, \ldots, \sigma_{r,\alpha} \) may be regarded as elements of \( \mathcal{C}(B_\alpha) \) and \( G(B_\alpha) \) respectively.

We know that \( \xi_{i,\alpha} \mu(\sigma_{i,\alpha}) \) and \( \xi_\alpha \) map to the same element in \( \mathcal{C}(L) \). Hence, replacing \( \alpha \) by a larger index, we may assume that \( \xi_\alpha = \xi_{i,\alpha} \mu(\sigma_{i,\alpha}) \in \mathcal{C}(B_\alpha) \). We claim that the class \( \overline{\xi}_\alpha \in \mathcal{F}(K_\alpha) \) is \( A_\alpha \)-unramified. To prove this note that the only
primes at which $\xi_\alpha$ could be ramified are those which divide $f_\alpha$. Let $q_\alpha$ be one of them. We check that $\xi_\alpha$ is unramified at $q_\alpha$. Let $q_\alpha \in A_\alpha$ be a prime element such that $q_\alpha A_\alpha = A_\alpha$. Then $q_\alpha r_\alpha = f_\alpha$ for an element $r_\alpha$. Thus $qr = f \in R$ for the images of $q_\alpha$ and $r_\alpha$ in $R$. Since the homomorphism $\varphi_\alpha: A_\alpha \to R$ is local, $q \in m_R$. The relation $qr = f$ shows that $q \in p_i$ for some index $i$. Thus $q_\alpha \in \varphi_\alpha^{-1}(p_i)$ and $q_\alpha \subset \varphi_\alpha^{-1}(p_i)$. On the other hand $h_{i,\alpha} \in A_\alpha - \varphi_\alpha^{-1}(p_i)$, because $h_i \in R - p_i$. Thus $h_{i,\alpha} \in A_\alpha - q_\alpha$. The relation $\xi_\alpha = \xi_{i,\alpha} \mu(\sigma_{i,\alpha}) \in C(B_\alpha)$ with $\xi_{i,\alpha} \in C(A_\alpha,h_{i,\alpha})$ now shows that $\xi_\alpha$ is unramified at $q_\alpha$. Thus $\xi_\alpha$ is unramified at each height 1 prime in $A_\alpha$ containing $f_\alpha$. Since $\xi_\alpha \in C(A_\alpha,f_\alpha)$ we conclude that $\xi_\alpha$ is $A_\alpha$-unramified. The lemma follows. The theorem is proved. \[\boxend\]

§ 10. One more purity result

In this section we prove another purity theorem for reductive group schemes. Let $k$ be an infinite field. Let $O$ be the semi-local ring of finitely many closed points on a $k$-smooth irreducible $k$-variety $X$ and let $K$ be the field of fractions of $O$. Let $G$ be a semi-simple $O$-group scheme. Let $i: Z \hookrightarrow G$ be a closed subgroup scheme of the centre $\text{Cent}(G)$. It is known that $Z$ is of multiplicative type. Let $G' = G/Z$ be the factor group, $\pi: G \to G'$ the projection. It is known that $\pi$ is finite surjective and strictly flat. Thus the sequence of $O$-group schemes

$$\{1\} \to Z \xrightarrow{i} G \xrightarrow{\pi} G' \to \{1\} \quad (21)$$

induces an exact sequence of group sheaves in the fppt-topology. Thus for every $O$-algebra $R$ the sequence (2) gives rise to a boundary operator

$$\delta_{\pi,R}: G'(R) \to H^1_{\text{fppt}}(R,Z). \quad (22)$$

One can check that it is a group homomorphism (compare [3], Ch. II, § 5.6, Corollary 2). Set

$$\mathcal{F}(R) = H^1_{\text{ftp}}(R,Z)/\text{Im}(\delta_{\pi,R}). \quad (23)$$

Clearly we get a functor on the category of $O$-algebras.

**Theorem 10.0.30.** The functor $\mathcal{F}$ satisfies purity for the ring $O$ above. If $K$ is the field of fractions of $O$, this statement can be restated in an explicit way as follows: given an element $\xi \in H^1_{\text{fppt}}(K,Z)$ suppose that for each height 1 prime ideal $p$ in $O$ there exist $\xi_p \in H^1_{\text{ftp}}(O_p,Z)$, $g_p \in G'(K)$ with $\xi = \xi_p + \delta_{\pi}(g_p) \in H^1_{\text{fppt}}(K,Z)$. Then there exist $\xi_m \in H^1_{\text{ftp}}(O,Z)$, $g_m \in G'(K)$, such that

$$\xi = \xi_m + \delta_{\pi}(g_m) \in H^1_{\text{fppt}}(K,Z).$$

**Proof.** The group $Z$ is of multiplicative type. So we can find a finite étale $O$-algebra $A$ and a closed embedding $Z \hookrightarrow R_{A/O}(G_m, A)$ into the permutation torus $T^+ = R_{A/O}(G_m, A)$. Let $G^+ = (G \times T^+)/Z$ and $T = T^+/Z$, where $Z$ is embedded
in \( G \times T^+ \) diagonally. Clearly \( G^+/G = T \). Consider the commutative diagram

\[
\begin{array}{ccccccccc}
\{1\} & \rightarrow & G' & \overset{id}{\rightarrow} & G' \\
\uparrow & & \downarrow{\pi} & & \downarrow{\pi^+} \\
\{1\} & \rightarrow & G & \overset{j^+}{\rightarrow} & G^+ & \overset{\mu^+}{\rightarrow} & T & \rightarrow & \{1\} \\
\uparrow{\ i} & & \downarrow{\ i^+} & & \downarrow{\ id} \\
\{1\} & \rightarrow & Z & \overset{j}{\rightarrow} & T^+ & \overset{\mu}{\rightarrow} & T & \rightarrow & \{1\} \\
\uparrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\{1\} & \rightarrow & \{1\} & \rightarrow & \{1\} & \rightarrow & \{1\}
\end{array}
\]

with exact rows and columns. By Lemma 11.0.31 and Hilbert’s Theorem 90 for the semi-local \( O \)-algebra \( A \) one has \( H^1_{\text{fppt}}(O, T^+) = H^1_{\text{et}}(O, T^+) = H^1_{\text{et}}(A, G_{m,A}) = \{\ast\} \). So, the latter diagram gives rise to a commutative diagram of pointed sets:

\[
\begin{array}{ccccccccc}
H^1_{\text{fppt}}(O, G') & \overset{id}{\rightarrow} & H^1_{\text{fppt}}(O, G') \\
\downarrow{\pi_*} & & \downarrow{\pi^*} & & \downarrow{\pi^*} \\
G^+(O) & \overset{\mu^+_O}{\rightarrow} & T(O) & \overset{\delta^+_O}{\rightarrow} & H^1_{\text{fppt}}(O, G) & \overset{j^+_O}{\rightarrow} & H^1_{\text{fppt}}(O, G^+) \\
\downarrow{\ i^+_*} & & \downarrow{\ id} & & \downarrow{\ i_*} & & \downarrow{\ i^+_*} \\
T^+(O) & \overset{\mu_O}{\rightarrow} & T(O) & \overset{\delta_O}{\rightarrow} & H^1_{\text{fppt}}(O, Z) & \overset{\mu}{\rightarrow} & \{\ast\} \\
\downarrow{\ delta^+_*} & & \downarrow{\ i_*} & & \downarrow{\ i_*} & & \downarrow{\ i_*} \\
G'(O) & & & & & & & & & &
\end{array}
\]

with exact rows and columns. It follows that \( \pi^+_* \) has trivial kernel and one has a chain of group isomorphisms

\[
H^1_{\text{fppt}}(O, Z)/\text{Im}(\delta_{\pi,O}) = \text{Ker}(\pi) = \text{Ker}(j^+_O) = T(O)/\mu^+(G^+(O)).
\]

Clearly these isomorphisms respect \( O \)-homomorphisms of semi-local \( O \)-algebras.

The morphism \( \mu^+: G^+ \rightarrow T \) is a smooth \( O \)-morphism of reductive \( O \)-group schemes, with the torus \( T \). The kernel \( \text{Ker}(\mu^+) \) is equal to \( G \) and \( G \) is a reductive \( O \)-group scheme. The functor \( O' \mapsto T(O')/\mu^+(G^+(O')) \) satisfies purity for the regular semi-local \( O \)-algebra \( O \) by Theorem 1.0.1. Hence the functor \( O' \mapsto H^1_{\text{fppt}}(O', Z)/\text{Im}(\delta_{\pi,O'}) \) satisfies purity for \( O \). \( \square \)

\section{Proof of Theorem 1.0.3}

\textbf{Proof of Theorem 1.0.3} in the case of semi-simple reductive group schemes. Let \( O \) and \( G \) be the same as in Theorem 1.0.3 and assume additionally that \( G \) is...
semi-simple. We need to prove that

$$\text{Ker}[H_{\text{ét}}^1(\mathcal{O}, G) \to H_{\text{ét}}^1(K, G)] = \ast. \quad (24)$$

Let $G^{\text{sc}}$ be the corresponding simply-connected semi-simple $\mathcal{O}$-group scheme and let $\pi: G^{\text{sc}} \to G$ be the corresponding $\mathcal{O}$-group scheme morphism. Let $Z = \text{Ker}(\pi)$. It is known that $Z$ is contained in the centre $\text{Cent}(G^{\text{sc}})$ of $G^{\text{sc}}$ and $Z$ is a finite group scheme of multiplicative type. It is known that $G = G^{\text{sc}}/Z$ and $\pi$ is finite surjective and strictly flat. Thus the sequence of $\mathcal{O}$-group schemes

$$\{1\} \to Z \xrightarrow{i} G^{\text{sc}} \xrightarrow{\pi} G \to \{1\}, \quad (25)$$
gives rise to an exact sequence of pointed sets

$$H_{\text{fppt}}^1(\mathcal{O}, Z)/\partial(G(\mathcal{O})) \to H_{\text{fppt}}^1(\mathcal{O}, G^{\text{sc}}) \to H_{\text{fppt}}^1(\mathcal{O}, G) \to H_{\text{fppt}}^2(\mathcal{O}, Z). \quad (26)$$

By Theorem 10.0.30 the functor

$$F(R) = H_{\text{ét}}^1(R, G)/\partial(G(R))$$

satisfies purity for the ring $\mathcal{O}$. The following result is known (see [23], Theorem 11.7).

**Lemma 11.0.31.** Let $R$ be a Noetherian ring. Then for a reductive $R$-group scheme $H$ and for $n = 0, 1$ the canonical map $H_{\text{ét}}^n(R, H) \to H_{\text{fppt}}^n(R, H)$ is a bijection of pointed sets. For an $R$-torus $T$ and for each integer $n \geq 0$ the canonical map $H_{\text{ét}}^n(R, T) \to H_{\text{fppt}}^n(R, T)$ is an isomorphism.

**Lemma 11.0.32.** For the ring $\mathcal{O}$ above one has

$$\text{Ker}[H_{\text{fppt}}^2(\mathcal{O}, Z) \to H_{\text{fppt}}^2(K, Z)] = \ast. \quad \text{Proof.} \quad \text{See [14], Theorem 4.3.} \quad \square$$

We continue with the proof of the equality (24). We have the exact sequence of pointed sets

$$H_{\text{fppt}}^1(\mathcal{O}, Z)/\partial(G(\mathcal{O})) \to H_{\text{fppt}}^1(\mathcal{O}, G^{\text{sc}}) \to H_{\text{fppt}}^1(\mathcal{O}, G) \to H_{\text{fppt}}^2(\mathcal{O}, Z) \quad (27)$$

and furthermore a commutative diagram with exact arrows:

$$
\begin{array}{cccccccccc}
H_{\text{fppt}}^1(\mathcal{O}, Z)/\partial(G(\mathcal{O})) & \xrightarrow{i} & H_{\text{fppt}}^1(\mathcal{O}, G^{\text{sc}}) & \xrightarrow{\pi_*} & H_{\text{fppt}}^1(\mathcal{O}, G) & \xrightarrow{\partial} & H_{\text{fppt}}^2(\mathcal{O}, Z) \\
\alpha & & \beta & & \gamma & & \delta \\
H_{\text{fppt}}^1(\mathcal{O}_p, Z)/\partial(G(\mathcal{O}_p)) & \xrightarrow{i_p} & H_{\text{fppt}}^1(\mathcal{O}_p, G^{\text{sc}}) & \xrightarrow{\pi_*} & H_{\text{fppt}}^1(\mathcal{O}_p, G) & \xrightarrow{\partial} & H_{\text{fppt}}^2(\mathcal{O}_p, Z) \\
\alpha_p & & \beta_p & & \gamma_p & & \delta_p \\
H_{\text{fppt}}^1(K, Z)/\partial(G(K)) & \xrightarrow{i''} & H_{\text{fppt}}^1(K, G^{\text{sc}}) & \xrightarrow{\pi_*} & H_{\text{fppt}}^1(K, G) & \xrightarrow{\partial} & H_{\text{fppt}}^1(K, Z). \\
\end{array}
$$

(28)
Here $p \subset \mathcal{O}$ is a height 1 prime ideal in $\mathcal{O}$. The maps $i_*, i'_*$ and $i''_*$ are injective (compare [3], Ch. I, Section 5, Proposition 39 and Corollary 1 of Proposition 40). Set $\alpha_K = \alpha_p \circ \alpha$, $\beta_K = \beta_p \circ \beta$, $\gamma_K = \gamma_p \circ \gamma$, $\delta_K = \delta_p \circ \delta$. By a theorem of Nisnevich [16] and Lemma 11.0.31 one has

$$\text{Ker}(\beta_p) = \text{Ker}(\gamma_p) = \ast. \quad (29)$$

Thus $\text{Ker}(\alpha_p) = \ast$. By the assumptions of Theorem 1.0.3 and by Lemma 11.0.31 one has

$$\text{Ker}[H^1_{\text{fppt}}(\mathcal{O}, G^{\text{sc}}) \to H^1_{\text{fppt}}(K, G^{\text{sc}})] = \ast. \quad (30)$$

By Lemma 11.0.32 the map $\delta_K$ is injective. As mentioned immediately before Lemma 11.0.31 the functor $\mathcal{F}(R) = H^1_{\text{fppt}}(R, Z)/\partial(G(R))$ satisfies purity for the ring $\mathcal{O}$. We are now ready to perform a diagram chase.

Let $\xi \in \text{Ker}(\gamma_K)$. Then $\partial(\xi) \in \text{Ker}(\delta_K)$. By [14], Theorem 4.3, one has $\text{Ker}(\delta_K) = \ast$, whence $\partial(\xi) = \ast$ and $\xi = \pi_*(\zeta)$ for a $\zeta \in H^1_{\text{ét}}(\mathcal{O}, G^{\text{sc}})$. Since $\gamma_K(\xi) = \ast$ and $\text{Ker}(\gamma_p) = \ast$ we see that $\gamma(\xi) = \ast$. Thus $\pi_*(\beta(\zeta)) = \ast$ and $\beta(\zeta) = i'_*(\varepsilon_p)$ for an $\varepsilon_p \in H^1_{\text{fppt}}(\mathcal{O}_p, Z)/\partial(G(\mathcal{O}_p))$. A diagram chase shows that there exists a unique element $\varepsilon_K \in H^1_{\text{fppt}}(K, Z)/\partial(G(K))$ such that for each height 1 prime ideal $p$ of $\mathcal{O}$ one has

$$\alpha_p(\varepsilon_p) = \varepsilon_K \in H^1_{\text{fppt}}(K, Z)/\partial(G(K)).$$

By Purity Theorem 10.0.30 there exists an element $\varepsilon \in H^1_{\text{fppt}}(\mathcal{O}, Z)/\partial(G(\mathcal{O}))$ such that $\alpha_K(\varepsilon) = \varepsilon_K$. The element $\varepsilon_K$ has the property that $i''_*(\varepsilon_K) = \beta_K(\zeta)$. Whence $\beta_K(i_*(\varepsilon)) = \beta_K(\zeta)$. The map $\beta_K : H^1_{\text{ét}}(\mathcal{O}, G^{\text{sc}}) \to H^1_{\text{ét}}(K, G^{\text{sc}})$ is injective since by the hypotheses of Theorem 1.0.3 such a map has trivial kernel for all semi-simple simply-connected reductive $O$-group schemes. Whence $i_*(\varepsilon) = \zeta$ and $\xi = i_*(\pi_*(\varepsilon)) = \ast$. The semi-simple case of Theorem 1.0.3 is proved. □

**Claim 11.0.33.** Under the hypotheses of Theorem 1.0.3, for all semi-simple reductive $O$-group schemes $G$ the map $H^1_{\text{ét}}(\mathcal{O}, G) \to H^1_{\text{ét}}(K, G)$ is injective.

Indeed, let $\xi, \zeta \in H^1_{\text{ét}}(\mathcal{O}, G)$ be two elements whose images $\xi_K, \zeta_K$ in $H^1_{\text{ét}}(K, G)$ are equal. Let $\xi G, \zeta G$ be the corresponding principal $G$-bundles over $\mathcal{O}$ and $G(\zeta)$ the inner form of the $O$-group scheme $G$ corresponding to $\zeta$. The $O$-scheme $\text{Iso}(\xi G, \zeta G)$ is a principal $G(\zeta)$-bundle over $\mathcal{O}$ which is trivial over $K$. Since $G(\zeta)$ is semi-simple reductive over $\mathcal{O}$, the $O$-scheme $\text{Iso}(\xi G, \zeta G)$ has an $O$-point. Whence the claim.

**Proof of Theorem 1.0.3.** Let $\mathcal{O}$ and $G$ be as in Theorem 1.0.3. Consider the short sequence of reductive $O$-group schemes

$$\{1\} \to G_{\text{der}} \xrightarrow{i} G \xrightarrow{\mu} C \to \{1\}, \quad (31)$$

where $G_{\text{der}}$ is the derived $O$-group scheme of $G$ and $C = \text{corad}(G)$ is a torus over $\mathcal{O}$ and $\mu = f_0$ (see [2], Exp. XXII, Theorem 6.2.1). By that theorem the morphism $\mu$ is smooth and its kernel is the reductive $O$-group scheme $G_{\text{der}}$. Moreover, $G_{\text{der}}$ is a semi-simple $O$-group scheme. By Claim 11.0.33 the map

$$H^1_{\text{ét}}(\mathcal{O}, G_{\text{der}}) \to H^1_{\text{ét}}(K, G_{\text{der}}) \quad (32)$$
is injective. We need to prove that

$$\text{Ker}[H^1_{\text{et}}(\mathcal{O}, G) \to H^1_{\text{et}}(K, G)] = \ast.$$  \hspace{1cm} (33)

The sequence (31) of $\mathcal{O}$-group schemes gives a short exact sequence of the corresponding sheaves in the étale topology on the big étale site. That sequence of sheaves gives rise to a commutative diagram with exact arrows of pointed sets:

$$
\begin{array}{cccc}
1 & \longrightarrow & C(O)/\mu(G(O)) & \longrightarrow & H^1_{\text{et}}(O, G_{\text{der}}) & \longrightarrow & H^1_{\text{et}}(O, G) & \longrightarrow & H^1_{\text{et}}(O, C) \\
\alpha & & \beta & & \gamma & & \delta & & \\
1 & \longrightarrow & C(O_p)/\mu(G(O_p)) & \longrightarrow & H^1_{\text{et}}(O_p, G_{\text{der}}) & \longrightarrow & H^1_{\text{et}}(O_p, G) & \longrightarrow & H^1_{\text{et}}(O_p, C) \\
\alpha_p & & \beta_p & & \gamma_p & & \delta_p & & \\
1 & \longrightarrow & C(K)/\mu(G(K)) & \longrightarrow & H^1_{\text{et}}(K, G_{\text{der}}) & \longrightarrow & H^1_{\text{et}}(K, G) & \longrightarrow & H^1_{\text{et}}(K, C).
\end{array}
$$  \hspace{1cm} (34)

Here $p \subset \mathcal{O}$ is a height 1 prime ideal in $\mathcal{O}$. Set $\alpha_K = \alpha_p \circ \alpha$, $\beta_K = \beta_p \circ \beta$, $\gamma_K = \gamma_p \circ \gamma$, $\delta_K = \delta_p \circ \delta$. By a theorem of Nisnevich [16] one has

$$\text{Ker}(\alpha_p) = \text{Ker}(\beta_p) = \text{Ker}(\gamma_p) = \ast.$$  \hspace{1cm} (35)

Let $\xi \in \text{Ker}(\gamma_K)$. Then $\mu(\xi) \in \text{Ker}(\delta_K)$. By [14] one has $\text{Ker}(\delta_K) = \ast$, whence $\mu(\xi) = \ast$ and $\xi = i_*(\zeta)$ for a $\zeta \in H^1_{\text{et}}(O, G_{\text{der}})$. Since $\gamma_K(\xi) = \ast$ and $\text{Ker}(\gamma_p) = \ast$, we see that $\gamma(\xi) = \ast$. Whence $i_*(\beta(\xi)) = \ast$ and $\beta(\xi) = \partial(\varepsilon_p)$ for an $\varepsilon_p \in C(O_p)/\mu(G(O_p))$. A diagram chase and Lemma 4.0.9 show that there exists a unique element $\varepsilon_K \in C(K)/\mu(G(K))$ such that for each height 1 prime ideal $p$ of $\mathcal{O}$ one has $\alpha_p(\varepsilon_p) = \varepsilon_K \in C(K)/\mu(G(K))$. By Theorem 1.0.1 there exists an element $\varepsilon \in C(\mathcal{O})/\mu(G(\mathcal{O}))$ such that $\alpha_K(\varepsilon) = \varepsilon_K$. The element $\varepsilon_K$ has the property that $\partial(\varepsilon_K) = \beta_K(\zeta)$. Whence $\beta_K(\partial(\varepsilon)) = \beta_K(\xi)$. The map $\beta_K$ is injective as indicated in the beginning of the proof. Whence $\partial(\varepsilon) = \zeta$ and $\xi = i_*(\partial(\varepsilon)) = \ast$.

The proof of Theorem 1.0.3 is complete. $\square$

§ 12. Examples

We follow here the notation of [24]. The field $k$ is a field of characteristic zero. The functors (36) to (50) satisfy purity for regular local rings containing $k$, as follows either from Theorem 10.0.30 or from Theorem 1.0.1.

(1) Let $G$ be a simple algebraic group over the field $k$, $Z$ a central subgroup, $G' = G/Z$, $\pi: G \to G'$ the canonical morphism. For any $k$-algebra $A$ let $\delta_{\pi, R}: G'(R) \to H^1_{\text{et}}(R, Z)$ be the boundary operator. One has a functor

$$R \mapsto H^1_{\text{et}}(R, Z)/\text{Im}(\delta_{\pi, R}).$$  \hspace{1cm} (36)

(2) Let $(A, \sigma)$ be a finite separable $k$-algebra with an orthogonal involution. Let $\pi: \text{Spin}(A, \sigma) \to \text{PGO}^+(A, \sigma)$ be the canonical morphism of the spinor $k$-group scheme to the projective orthogonal $k$-group scheme. Let $Z = \text{Ker}(\pi)$. For a $k$-algebra $R$ let $\delta_R: \text{PGO}^+(A, \sigma)(R) \to H^1_{\text{et}}(R, Z)$ be the boundary operator. One has a functor

$$R \mapsto H^1_{\text{et}}(R, Z)/\text{Im}(\delta_R).$$  \hspace{1cm} (37)
In (2a) and (2b) below we describe this functor somewhat more explicitly following [24].

(2a) Let $C(A, \sigma)$ be the Clifford algebra. Its centre $l$ is an étale quadratic $k$-algebra. Assume that $\deg(A)$ is divisible by 4. Let $\Omega(A, \sigma)$ be the extended Clifford group (see [24], the definition given just below (13.19)). Let $\sigma$ be the canonical involution of $C(A, \sigma)$ as described in [24] just above (8.11). Then $\sigma$ is either orthogonal or symplectic by [24], Proposition 8.12. Let $\mu: \Omega(A, \sigma) \to R_{l/k}(G_{m,l})$ be the multiplier map defined in [24], just above (13.25), by $\mu(\omega) = \sigma(\omega) \cdot \omega$. Set $R_l = R \otimes_k l$. For a field or a local ring $R$ one has $H^1_{\text{ét}}(R, Z)/\text{Im}(\delta_R) = R^X_l / \mu(\Omega(A, \sigma)(R))$ by [24], the diagram in (13.32). Consider the functor

$$R \mapsto R^X_l / \mu(\Omega(A, \sigma)(R)).$$

(38)

It coincides with the functor $R \mapsto H^1_{\text{ét}}(R, Z)/\text{Im}(\delta_R)$ on local rings containing $k$.

(2b) Now let $\deg(A) = 2m$ with $m$ odd. Let $\tau: l \to l$ be the involution of $l/k$. The kernel of the morphism $R_{l/k}(G_{m,l}) \xrightarrow{\text{id} - \tau} R_{l/k}(G_{m,l})$ coincides with $\mathbb{G}_{m,k}$. Thus $\text{id} - \tau$ induces a $k$-group scheme morphism which we denote $\text{id} - \tau: R_{l/k}(G_{m,l})/\mathbb{G}_{m,k} \hookrightarrow R_{l/k}(G_{m,l})$. Let $\mu: \Omega(A, \sigma) \to R_{l/k}(G_{m,l})$ be the multiplier map defined in [24], just above (13.25), by $\mu(\omega) = \sigma(\omega) \cdot \omega$. Let $\kappa: \Omega(A, \sigma) \to R_{l/k}(G_{m,l})/\mathbb{G}_{m,k}$ be the $k$-group scheme morphism described in [24], Proposition 13.21. The composite $(\text{id} - \tau) \circ \kappa$ lands in $R_{l/k}(G_{m,l})$. Let $U \subset \mathbb{G}_{m,k} \times R_{l/k}(G_{m,l})$ be the closed $k$-subgroup consisting of all $(\alpha, z)$ such that $\alpha^4 = N_{l/k}(z)$.

Set

$$\mu_* = (\mu, [(\text{id} - \tau) \circ \kappa] \cdot \mu(\omega)) : \Omega(A, \sigma) \to \mathbb{G}_{m,k} \times R_{l/k}(G_{m,l}).$$

This $k$-group scheme morphism lands in $U$. So we get a $k$-group scheme morphism $\mu_*: \Omega(A, \sigma) \to U$. At the level of $k$-rational points it coincides with the one described in [24], just above (13.35). For a field or a local ring one has

$$H^1_{\text{ét}}(R, Z)/\text{Im}(\delta_R) = U(R)/\{(N_{l/k}(\alpha), \alpha^4) \mid \alpha \in R^X_l \} \cdot \mu_*(\Omega(A, \sigma)(R)).$$

Consider the functor

$$R \mapsto U(R)/\{(N_{l/k}(\alpha), \alpha^4) \mid \alpha \in R^X_l \} \cdot \mu_*(\Omega(A, \sigma)(R)).$$

(39)

It coincides with the functor $R \mapsto H^1_{\text{ét}}(R, Z)/\text{Im}(\delta_R)$ on local rings containing $k$.

(3) Let $\Gamma(A, \sigma)$ be the Clifford group $k$-scheme of $(A, \sigma)$. Let $\text{Sn}: \Gamma(A, \sigma) \to \mathbb{G}_{m,k}$ be the spinor norm map. It is dominant. Consider the functor

$$R \mapsto R^X / \text{Sn}(\Gamma(A, \sigma)(R)).$$

(40)

Purity for this functor was originally proved in [10], Theorem 3.1. In fact, $\Gamma(A, \sigma)$ is $k$-rational.

(4) We follow [24], §23. Let $A$ be a separable finite-dimensional $k$-algebra with centre $l$ and $k$-involution $\sigma$ such that $k$ coincides with all $\sigma$-invariant elements of $l$, that is, $k = l^{\sigma}$. Consider the $k$-group schemes of similitudes of $(A, \sigma)$:

$$\text{Sim}(A, \sigma)(R) = \{a \in A^X_R \mid a \cdot \sigma_R(a) \in l^X_K\}.$$
We have a $k$-group scheme morphism $\mu: \operatorname{Sim}(A, \sigma) \rightarrow \mathbb{G}_{m,k}$, $a \mapsto a \cdot \sigma(a)$. It gives an exact sequence of algebraic $k$-groups

$$\{1\} \rightarrow \operatorname{Iso}(A, \sigma) \rightarrow \operatorname{Sim}(A, \sigma) \rightarrow \mathbb{G}_{m,k} \rightarrow \{1\}.$$ 

One has a functor

$$R \mapsto R^\times / \mu(\operatorname{Sim}(A, \sigma)(R)). \quad (41)$$

Purity for this functor was originally proved in [11], Theorem 1.2. Various particular cases are obtained considering unitary, symplectic and orthogonal involutions.

(4a) In the case of an orthogonal involution $\sigma$, the connected component $\operatorname{GO}^+(A, \sigma)$ ([24], (12.24)) of the similitude $k$-group scheme $\operatorname{GO}(A, \sigma) := \operatorname{Sim}(A, \sigma)$ has index two in $\operatorname{GO}(A, \sigma)$. The restriction of $\mu$ to $\operatorname{GO}^+(A, \sigma)$ is still a dominant morphism to $\mathbb{G}_{m,k}$. One has a functor

$$R \mapsto R^\times / \mu(\operatorname{GO}^+(A, \sigma)(R)). \quad (42)$$

It seems that its purity does not follow from [11], Theorem 1.2. In fact we do not know whether or not the norm principle holds for $\mu: \operatorname{GO}^+(A, \sigma) \rightarrow \mathbb{G}_{m,k}$.

(5) Let $A$ be a central simple algebra (csa) over $k$ and $\operatorname{Nrd}: \operatorname{GL}_{1,A} \rightarrow \mathbb{G}_{m,k}$ the reduced norm morphism. One has a functor

$$R \mapsto R^\times / \operatorname{Nrd}(\operatorname{GL}_{1,A}(R)). \quad (43)$$

Purity for this functor was originally proved in [8], Theorem 5.2.

(6) Let $(A, \sigma)$ be a finite separable $k$-algebra with a unitary involution such that its centre $l$ is a quadratic extension of $k$. Let $U(A, \sigma)$ be the unitary $k$-group scheme. Let $U_l(1)$ be an algebraic torus given by $N_{l/k} = 1$. One has a functor

$$R \mapsto U_l(1)(R)/\operatorname{Nrd}(U_{A,\sigma}(R)) = \{\alpha \in R_l^\times \mid N_{l/k}(\alpha) = 1\}/\operatorname{Nrd}(U_{A,\sigma}(R)), \quad (44)$$

where $\operatorname{Nrd}$ is the reduced norm map. Purity for this functor was originally proved in [10], Theorem 3.3.

(7) With the notation of example (5) choose an integer $d$ and consider the $k$-group scheme morphism $\mu: \operatorname{GL}_{1,A} \times \mathbb{G}_{m,k} \rightarrow \mathbb{G}_{m,k}$ given by $(\alpha, a) \mapsto \operatorname{Nrd}(\alpha) \cdot a^d$. One has a functor

$$R \mapsto R^\times / \mu([\operatorname{GL}_{1,A} \times \mathbb{G}_{m,k}](R)) = R^\times / \operatorname{Nrd}(A_R^\times) \cdot R^\times d. \quad (45)$$

Purity for this functor was originally proved in [10], Theorem 3.2.

(8) With the notation of example (5) choose an integer $d$ and consider the functor

$$R \mapsto U_l(1)(R)/[\operatorname{Nrd}(U_{A,\sigma}(R)) \cdot \{\alpha \in R_l^\times \mid N_{l/k}(\alpha) = 1\}]^d. \quad (46)$$

Purity for this functor was originally proved in [10], Theorem 3.2.

(9) Let $G_1, G_2, C$ be affine $k$-group schemes. Assume that $C$ is commutative and let $\mu_1: G_1 \rightarrow C$, $\mu_2: G_2 \rightarrow C$ be $k$-group scheme morphisms and $\mu: G_1 \times G_2 \rightarrow C$ be given by $\mu(g_1, g_2) = \mu_1(g_1) \cdot \mu_2(g_2)$. One has a functor

$$R \mapsto C(R)/[\mu([G_1 \times G_2](R)) = C(R)/[\mu_1(G_1(R)) \cdot \mu_2(G_2(R))]. \quad (47)$$
In this style one could get a lot of curious examples of functors, one of which is given here.

(10) Let \((A_1, \sigma_1)\) be a finite separable \(k\)-algebra with an orthogonal involution. Let \(A_2\) be a csa over \(k\). Let \(\mu_1\) be the multiplier map for \((A_1, \sigma_1)\) and \(\text{Nrd}_2\) the reduced norm for \(A_2\). One has a functor

\[ R \mapsto R^\times / [\mu_1(\text{GO}^+(A_1, \sigma_1)(R)) \cdot \text{Nrd}_2(A_{2,R}^\times) \cdot R^\times d]. \]  

Note that the connected component of \(\text{Iso}(N)\) is a simply connected algebraic \(k\)-group of type \(E_6\).

(11) Let \(A\) be a csa of degree 3 over \(k\), \(\text{Nrd}\) the reduced norm and \(\text{Trd}\) the reduced trace. Consider the cubic form on the 27-dimensional \(k\)-vector space \(A \times A \times A\) given by \(N := \text{Nrd}(x) + \text{Nrd}(y) + \text{Nrd}(z) - \text{Trd}(xyz)\). Let \(\text{Iso}(A, N)\) be the \(k\)-group scheme of isometries of \(N\) and \(\text{Sim}(N)\) the \(k\)-group scheme of similitudes of \(N\). It is known that \(\text{Iso}(N)\) is a normal algebraic subgroup in \(\text{Sim}(A, N)\) and the factor group coincides with \(\mathbb{G}_{m,k}\). So we have a canonical \(k\)-group morphism (the multiplier) \(\mu: \text{Sim}(N) \to \mathbb{G}_{m,k}\). One now has a functor

\[ R \mapsto R^\times / \mu(\text{Sim}(N)(R)). \]

(12) Let \((A, \sigma)\) be a csa of degree 8 over \(k\) with a symplectic involution. Let \(V \subset A\) be the subspace of all skew-symmetric elements. It is of dimension 28. Let \(\text{Pfd}\) be the reduced Pfaffian on \(V\) and \(\text{Trd}\) the reduced trace on \(A\). Consider the degree 4 form on the space \(V \times V\) given by \(F := \text{Pfr}(x) + \text{Pfr}(y) - 1/4\text{Trd}((xy)2) - 1/16\text{Trd}(xy)^2\). Consider the symplectic form on \(V \times V\) given by \(\phi((x_1, y_1), (x_2, y_2)) = \text{Trd}(x_1 y_2 - x_2 y_1)\). Let \(\text{Iso}(F)\) (resp. \(\text{Iso}(\phi)\)) be the \(k\)-group scheme of isometries of the pair \(F\) (resp. of \(\phi\)). Let \(\text{Sim}(F)\) (resp. \(\text{Sim}(\phi)\)) be the \(k\)-group scheme of similitudes of \(F\) (resp. of \(\phi\)). Set \(G = \text{Iso}(F) \cap \text{Iso}(\phi)\) and \(G^+ = \text{Sim}(F) \cap \text{Sim}(\phi)\). It is known that \(G\) is a normal algebraic subgroup in \(G^+\) and the factor group coincides with \(\mathbb{G}_{m,k}\). So we have a canonical \(k\)-group morphism \(\mu: G^+ \to \mathbb{G}_{m,k}\). One now has a functor

\[ R \mapsto R^\times / \mu(G^+(R)). \] 

Note that \(G\) is a simply connected group of type \(E_7\).

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