Relations Among Universal Equations For Gromov-Witten Invariants

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It is well known that relations in the tautological ring of moduli spaces of pointed stable curves give partial differential equations for Gromov-Witten invariants of compact symplectic manifolds. These equations do not depend on the target symplectic manifolds and therefore are called universal equations for Gromov-Witten invariants. In the case that the quantum cohomology of the symplectic manifolds are semisimple, it is expected that higher genus Gromov-Witten invariants are completely determined by such universal equations and genus-0 Gromov-Witten invariants. This has been proved for genus-1 (cf. [DZ]) and genus-2 (cf. [L2]) cases. Universal equations also play very important role in the understanding of the Virasoro conjecture (cf. [EHX]). The genus-0 Virasoro conjecture for all compact symplectic manifolds follows from a universal equation called the genus-0 topological recursion relation (cf. [LT]). For projective varieties, we expect that such universal equations reduce higher genus Virasoro conjecture to an $SL(2)$ symmetry for the generating function of the Gromov-Witten invariants. Again this has been proved for genus-1 ([L1]) and genus-2 ([L2]) cases.

In this paper, we will discuss the relation among known universal equations for Gromov-Witten invariants. We hope that the understanding of such relations would be helpful to the study of both Gromov-Witten invariants and the topology of the moduli spaces of pointed curves. Relations among genus-2 universal equations were studied in [L2]. It was proved that the three universal equations in [G2] and [BP] implies certain complicated genus-1 relations (see equations (4) and (6) below). Modulo these genus-1 relations, the three known genus-2 equations can be reduced to only two equations. The main result of this paper is that the genus-1 relations derived in [L2] follow from a known genus-1 relation found in [G1] and the genus-0 and genus-1 topological recursion relations. This completes the discussion of relations among genus-2 equations in [L2].

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1 Generating functions for Gromov-Witten invariants

Let $M$ be a compact symplectic manifolds. For simplicity, we assume $H^{\text{odd}}(M; \mathbb{C}) = 0$. The big phase is by definition the product of infinite copies of $H^*(M; \mathbb{C})$, i.e.

$$P := \prod_{n=0}^{\infty} H^*(M; \mathbb{C}).$$

Fix a basis $\{\gamma_1, \ldots, \gamma_N\}$ of $H^*(M; \mathbb{C})$ with $\gamma_1 = 1$ be the identity of the ordinary cohomology ring of $M$. Then we denote the corresponding basis for the $n$-th copy of $H^*(M; \mathbb{C})$ in $P$ by $\{\tau_n(\gamma_1), \ldots, \tau_n(\gamma_N)\}$. We call $\tau_n(\gamma_\alpha)$ a descendant of $\gamma_\alpha$ with descendant level $n$. We can think of $P$ as an infinite dimensional vector space with basis $\{\tau_n(\gamma_\alpha) | 1 \leq \alpha \leq N, n \in \mathbb{Z}_{\geq 0}\}$ where $\mathbb{Z}_{\geq 0} = \{n \in \mathbb{Z} | n \geq 0\}$. Let $\left( t^\alpha_n \right)_{1 \leq \alpha \leq N, n \in \mathbb{Z}_{\geq 0}}$ be the corresponding coordinate system on $P$. For convenience, we identify $\tau_n(\gamma_\alpha)$ with the coordinate vector field $\frac{\partial}{\partial t}$. If $n < 0$, $\tau_n(\gamma_\alpha)$ is understood as the 0 vector field. We also abbreviate $\tau_0(\gamma_\alpha)$ as $\gamma_\alpha$. Any vector field of the form $\sum f_\alpha \gamma_\alpha$, where $f_\alpha$ are functions on the big phase space, is called a primary vector field. We use $\tau_+$ and $\tau_-$ to denote the operator which shift the level of descendants, i.e.

$$\tau_\pm \left( \sum_{n,\alpha} f_{n,\alpha} \tau_n(\gamma_\alpha) \right) = \sum_{n,\alpha} f_{n,\alpha} \tau_{n \pm 1}(\gamma_\alpha)$$

where $f_{n,\alpha}$ are functions on the big phase space.

We will use the following notational conventions: Lower case Greek letters, e.g. $\alpha, \beta, \mu, \nu, \sigma, \ldots$, etc., will be used to index the cohomology classes. The range of these indices is from 1 to $N$. Lower case English letters, e.g. $i, j, k, m, n, \ldots$, etc., will be used to index the level of descendants. Their range is the set of all non-negative integers, i.e. $\mathbb{Z}_{\geq 0}$. All summations are over the entire ranges of the indices unless otherwise indicated. Let

$$\eta_{\alpha\beta} = \int_V \gamma_\alpha \cup \gamma_\beta$$

be the intersection form on $H^*(V; \mathbb{C})$. We will use $\eta = (\eta_{\alpha\beta})$ and $\eta^{-1} = (\eta^{\alpha\beta})$ to lower and raise indices. For example,

$$\gamma^\alpha := \eta^{\alpha\beta} \gamma_\beta.$$

Here we are using the summation convention that repeated indices (in this formula, $\beta$) should be summed over their entire ranges.

Let

$$\langle \tau_{n_1}(\gamma_{\alpha_1}) \tau_{n_2}(\gamma_{\alpha_2}) \cdots \tau_{n_k}(\gamma_{\alpha_k}) \rangle_g$$

be the genus-$g$ descendant Gromov-Witten invariant associated to $\gamma_{\alpha_1}, \ldots, \gamma_{\alpha_k}$ and non-negative integers $n_1, \ldots, n_k$ (cf. [W], [RT], [LiT], and [M]). Let $\overline{M}_{g,k}(M; d)$ be the moduli space of stable maps from genus-$g$ $k$-pointed curves to $M$ with degree $d \in H_2(M; \mathbb{Z})$, and
\[ F_g = \sum_{k \geq 0} \frac{1}{k!} \sum_{\alpha_1, \ldots, \alpha_k, n_1, \ldots, n_k} t^{\alpha_1}_{n_1} \cdots t^{\alpha_k}_{n_k} \langle \tau_{n_1}(\gamma_{\alpha_1}) \tau_{n_2}(\gamma_{\alpha_2}) \cdots \tau_{n_k}(\gamma_{\alpha_k}) \rangle_g. \]

This function is understood as a formal power series of \( t^{\alpha}. \)

Introduce a \( k \)-tensor \( \langle \langle \cdots \rangle \rangle \) defined by

\[
\langle \langle W_1 W_2 \cdots W_k \rangle \rangle_g := \sum_{m_1, \alpha_1, \ldots, m_k, \alpha_k} f^{i}_{m_1, \alpha_1} \cdots f^{k}_{m_k, \alpha_k} \frac{\partial^k}{\partial t^{\alpha_1}_{m_1} \partial t^{\alpha_2}_{m_2} \cdots \partial t^{\alpha_k}_{m_k}} F_g,
\]

for (formal) vector fields \( W_i = \sum_{m, \alpha} f^i_{m, \alpha} \frac{\partial}{\partial t^{\alpha}} \) where \( f^i_{m, \alpha} \) are (formal) functions on the big phase space. We can also view this tensor as the \( k \)-th covariant derivative of \( F_g \). This tensor is called the \( k \)-point (correlation) function.

For any vector fields \( W_1 \) and \( W_2 \) on the big phase space, the quantum product of \( W_1 \) and \( W_2 \) is defined by (cf. [L2])

\[
W_1 \cdot W_2 := \langle \langle W_1 W_2 \gamma^\alpha \rangle \rangle_0 \gamma^\alpha.
\]

Define

\[
T(W) := \tau_+(W) - \langle \langle W \gamma^\alpha \rangle \rangle_0 \gamma^\alpha
\]

for any vector field \( W \). The operator \( T \) was introduced in [L2] as a convenient tool in the study of topological recursion relations. Let \( \overline{M}_{g,k} \) be the moduli space of genus-\( g \) stable curves with \( k \)-marked points, and \( \psi_i \) the first Chern class of the tautological line bundle over \( \overline{M}_{g,k} \) whose geometric fiber over a stable curve is the cotangent space of the curve at the \( i \)-th marked point. When we translate a relation in the tautological ring of \( \overline{M}_{g,k} \) to differential equations for generating functions of Gromov-Witten invariants, the \( \psi \) classes correspond to the operator \( T \). The reason for such correspondence lies in the relation between the classes \( \Psi_i \in H^2(\overline{M}_{g,k}(M,d); \mathbb{Q}) \) and \( \psi_i \in H^2(\overline{M}_{g,k}; \mathbb{Q}) \). There is a canonical map \( St : \overline{M}_{g,k}(M,d) \rightarrow \overline{M}_{g,k} \) which forgets the map to the target manifold \( M \) and stabilizes the domain curve for each element of \( \overline{M}_{g,k}(M,d) \). The difference \( \Psi_i - St^*(\psi_i) \) is represented by a cycle containing elements whose domain curves consist of one genus-\( g \) and one genus-0 components, with the \( i \)-th marked point lying on the genus-0 component (cf. [G2] and [KM]). When we interpret this relation for Gromov-Witten invariants, \( \Psi_i \) corresponds to \( \tau_+ \), the \( i \)-th marked point is associated with a cohomology class, which will be extended to a vector field \( W \) by linearity, and the node joining the two irreducible components are associated with the diagonal class \( \gamma^\alpha \otimes \gamma_\alpha \) in \( H^*(M \times M; \mathbb{C}) \). Consequently, \( St^*(\psi_i) \) must correspond to \( T \).
The operator $T$ also has an algebraic interpretation. Let
\[ S := - \sum_{m, \alpha} \tilde{t}_m^\alpha \tau_{m-1}(\gamma_\alpha) \]
be the string vector field, where
\[ \tilde{t}_m^\alpha := t_m^\alpha - \delta_{m,1} \delta_{\alpha,1}. \]
By the second derivative of the string equation (see Section 2.4), the operator $T$ can be rewritten as
\[ T(W) = \tau_+(W) - S \cdot \tau_+(W). \]
Therefore $T$ measures the difference between $S$ and the identity of the quantum product on the big phase space, which actually does not exist by our definition of the quantum product.

Let $\nabla$ be the trivial flat connection on the big phase space with respect to which $\tau_n(\gamma_\alpha)$ are parallel vector fields for all $\alpha$ and $n$. Then the covariant derivative of the quantum product is given by
\[ \nabla_3(W_1) \cdot W_2 = (\nabla_3 W_1) \cdot W_2 + W_1 \cdot (\nabla_3 W_2) + \langle \langle W_1 W_2 W_3 \gamma_\alpha \rangle \rangle_0 \gamma_\alpha \tag{1} \]
and the covariant derivative of the operator $T$ is given by
\[ \nabla_2 T(W_1) = T(\nabla_2 W_1) - W_2 \cdot W_1 \tag{2} \]
for any vector fields $W_1, W_2$ and $W_3$ (cf. [L2, Equation (8) and Lemma 1.5]).

## 2 Universal equations

We will write universal equations of Gromov-Witten invariants as equations among tensors on the big phase space defined by generating functions of Gromov-Witten invariants. The set of tensors whose vanishing are equivalent to some universal equations form an ideal of contra-variant tensor algebra on the big phase space. We call this ideal the ideal of universal relations and denote it by $U$. This ideal is closed under the covariant differentiation defined by $\nabla$, and therefore is a differential ideal.

If $T_1,\ldots,T_k$ are $k$ contra-variant tensors, we use $I_a[T_1,\ldots,T_k]$ to denote the ideal algebraically generated by $T_1,\ldots,T_k$, i.e. $I_a[T_1,\ldots,T_k]$ is the set of tensors of the form $\sum_{i=1}^k S_i \otimes T_i$ where $S_i$ and $S''_i$ are arbitrary contra-variant tensors on the big phase space. We also use $I_d[T_1,\ldots,T_k]$ to denote the ideal differentially generated by $T_1,\ldots,T_k$, i.e. the smallest ideal which contains $T_1,\ldots,T_k$ and is closed under the covariant differentiation.

In the theory of Gromov-Witten invariants, we often need to prove two complicated expressions are equal by using certain universal equations. Since universal equations of Gromov-Witten invariants are very complicated, it is very cumbersome to write out explicitly derivations in each step. For convenience of presentation, we introduce the
following phrases: If one expression can be derived from another one by using certain universal equations \( T_1 = 0, \ldots, T_k = 0 \), but not using their covariant derivatives, then we say that these two expressions are equal modulo \( I_a[T_1, \ldots, T_k] \). If in the derivation, we also need covariant derivatives of the corresponding universal equations, we say that these two expressions are equal modulo \( I_d[T_1, \ldots, T_k] \).

2.1 genus-0 universal relations

Define

\[
\rho_0(W_1, W_2, W_3) := \langle \langle T(W_1) W_2 W_3 \rangle \rangle_0
\]

where \( W_i \) are vector fields on the big phase space. The genus-0 topological recursion relation is equivalent to \( \rho_0 = 0 \). Hence \( \rho_0 \in U \), and so does the tensor \( C_0 \) defined by

\[
C_0(W_1, W_2, W_3, W_4) := (\nabla_{W_3} \rho_0)(W_1, W_2, W_4) - (\nabla_{W_2} \rho_0)(W_1, W_3, W_4).
\]

Using equation (2), it is straightforward to show that

\[
C_0(W_1, W_2, W_3, W_4) = \langle \langle W_1 \bullet W_2 \rangle \rangle_0 \gamma_0 - \langle \langle W_1 (W_2 \bullet \gamma_0) \rangle \rangle_0 - \langle \langle W_2 \bullet \gamma_0 \rangle \rangle_0 \gamma_0 + 1/4 \langle \langle W_1 (W_2 \bullet \gamma_0) \rangle \rangle_0 \gamma_0 - \langle \langle W_2 \bullet \gamma_0 \rangle \rangle_0 \gamma_0 + 1/6 \langle \langle W_1 (W_2 \bullet \gamma_0) \rangle \rangle_0 \gamma_0 - \langle \langle W_2 \bullet \gamma_0 \rangle \rangle_0 \gamma_0.
\]

for any vector fields \( W_1, \ldots, W_4 \). Therefore \( C_0 = 0 \) implies that the quantum product "\( \bullet \)" on the big phase space is associative.

2.2 genus-1 universal relations

The genus-1 topological recursion relation is equivalent to \( \rho_1 = 0 \) where \( \rho_1 \) is defined by

\[
\rho_1(W) = \langle \langle T(W) \rangle \rangle_1 - \frac{1}{24} \langle \langle W \gamma_0 \gamma_0 \rangle \rangle_0.
\]

Define

\[
G(W_1, W_2, W_3, W_4) := \sum_{g \in S_4} \left\{ 3 \langle \langle W_{g(1)} \bullet W_{g(2)} \rangle \langle W_{g(3)} \bullet W_{g(4)} \rangle \rangle_1 
- 4 \langle \langle W_{g(1)} \bullet W_{g(2)} \bullet W_{g(3)} \rangle \rangle_1 \langle \langle W_{g(4)} \rangle \rangle_0 
- \langle \langle W_{g(1)} \bullet W_{g(2)} \rangle \langle W_{g(3)} \bullet W_{g(4)} \rangle \gamma_0 \rangle_0 \langle \langle \gamma_0 \rangle \rangle_1 
+ 2 \langle \langle W_{g(1)} W_{g(3)} \gamma_0 \rangle \rangle_0 \langle \langle \gamma_0 \bullet W_{g(4)} \rangle \rangle_1 
+ \frac{1}{6} \langle \langle W_{g(1)} W_{g(3)} \gamma_0 \rangle \rangle_0 \langle \langle \gamma_0 \gamma_0 \rangle \rangle_0 \langle \langle \gamma_0 \rangle \rangle_0 
+ \frac{1}{24} \langle \langle W_{g(1)} W_{g(2)} \gamma_0 \rangle \rangle_0 \langle \langle \gamma_0 \gamma_0 \rangle \rangle_0 \langle \langle \gamma_0 \rangle \rangle_0 
- \frac{1}{4} \langle \langle W_{g(1)} \gamma_0 \gamma_0 \rangle \rangle_0 \langle \langle \gamma_0 \rangle \rangle_0 \rangle \right\}
\]

for any vector fields \( W_1, \ldots, W_4 \). Then the genus-1 equation discovered by Getzler \([G1]\) is equivalent to \( G = 0 \). So \( \rho_1 \in U \) and \( G \in U \).
2.3 genus-2 universal relations

Define

\[
\rho_{2,1}(W) := \langle \langle T^2(W) \rangle \rangle_2 - A_1(W), \\
\rho_{2,3}(W, V) := \langle \langle T(W) T(V) \rangle \rangle_2 - 3 \langle \langle T(W \cdot V) \rangle \rangle_2 - A_2(W, V), \\
\rho_{2,3}(W_1, W_2, W_3) := 2 \langle \langle W_1 \cdot W_2 \cdot W_3 \rangle \rangle_2 - 2 \langle \langle W_1 W_2 W_3 \gamma^\alpha \rangle \rangle_0 \langle \langle T(\gamma_\alpha) \rangle \rangle_2 \\
+ \sum_{g \in S_3} \langle \langle W_{g(1)} T(W_{g(2)} \cdot W_{g(3)}) \rangle \rangle_2 - \langle \langle T(W_{g(1)} \{W_{g(2)} \cdot W_{g(3)}\} \rangle \rangle_2 \\
- B(W_1, W_2, W_3),
\]

where

\[
A_1(W) = \frac{7}{10} \langle \langle \gamma_\alpha \rangle \rangle_1 \langle \langle \{\gamma^a \cdot W\} \rangle \rangle_1 + \frac{1}{10} \langle \langle \gamma_\alpha \{\gamma^a \cdot W\} \rangle \rangle_1 \\
- \frac{1}{240} \langle \langle W \{\gamma_\alpha \cdot \gamma^a\} \rangle \rangle_1 + \frac{13}{240} \langle \langle W \gamma_\alpha \gamma^a \gamma^\beta \rangle \rangle_0 \langle \langle \gamma_\beta \rangle \rangle_1 \\
+ \frac{1}{960} \langle \langle W \gamma^a \gamma_\alpha \gamma^\beta \rangle \rangle_0,
\]

\[
A_2(W, V) = \frac{13}{10} \langle \langle W V \gamma^a \gamma^\beta \rangle \rangle_0 \langle \langle \gamma_\alpha \rangle \rangle_1 \langle \langle \gamma_\beta \rangle \rangle_1 + \frac{4}{5} \langle \langle W \gamma^a \rangle \rangle_1 \langle \langle \{\gamma_\alpha \cdot \gamma^a\} \rangle \rangle_1 \\
+ \frac{1}{5} \langle \langle \gamma_\alpha \rangle \rangle_1 \langle \langle \{\gamma_\alpha \cdot W\} \rangle \rangle_1 - \frac{4}{5} \langle \langle \gamma_\alpha \{\gamma^a \cdot \gamma^a\} \rangle \rangle_1 \\
+ \frac{23}{240} \langle \langle W V \gamma^a \gamma_\alpha \gamma^\beta \rangle \rangle_0 \langle \langle \gamma_\beta \rangle \rangle_1 + \frac{1}{48} \langle \langle W \gamma^a \gamma_\alpha \gamma^\beta \rangle \rangle_0 \langle \langle \gamma_\beta \gamma \rangle \rangle_1 \\
+ \frac{1}{48} \langle \langle V \gamma^a \gamma_\alpha \gamma^\beta \rangle \rangle_0 \langle \langle \gamma_\beta \rangle \rangle_1 - \frac{1}{80} \langle \langle W \gamma^a \gamma_\alpha \gamma^\beta \rangle \rangle_0 \langle \langle \gamma_\beta \rangle \rangle_1 \\
+ \frac{1}{30} \langle \langle \{\gamma_\alpha \{\gamma^a \cdot \gamma^a\} \} \rangle \rangle_1 - \frac{1}{30} \langle \langle \gamma_\alpha \{\gamma^a \cdot \gamma^a\} \rangle \rangle_1 \\
+ \frac{1}{576} \langle \langle W V \gamma^a \gamma_\alpha \gamma^\beta \rangle \rangle_0,
\]

and

\[
B(W_1, W_2, W_3) \\
= \frac{1}{5} \langle \langle W_1 W_2 W_3 \gamma^a \gamma^\beta \rangle \rangle_0 \langle \langle \gamma_\alpha \rangle \rangle_1 \langle \langle \gamma_\beta \rangle \rangle_1 - \frac{6}{5} \langle \langle W_1 W_2 W_3 \gamma^a \rangle \rangle_0 \langle \langle \gamma_\alpha \gamma^\beta \rangle \rangle_1 \langle \langle \gamma_\beta \rangle \rangle_1 \\
+ \frac{1}{120} \langle \langle W_1 W_2 W_3 \gamma^a \gamma_\alpha \gamma^\beta \rangle \rangle_0 \langle \langle \gamma_\beta \rangle \rangle_1 - \frac{1}{120} \langle \langle W_1 W_2 W_3 \gamma^a \gamma_\alpha \gamma^\beta \rangle \rangle_0 \langle \langle \gamma_\beta \rangle \rangle_1 \\
+ \frac{1}{10} \langle \langle W_1 W_2 W_3 \gamma_\alpha \gamma^a \gamma^\beta \rangle \rangle_0 \langle \langle \gamma_\beta \rangle \rangle_1 - \frac{1}{20} \langle \langle W_1 W_2 W_3 \gamma^a \gamma^\beta \rangle \rangle_0 \langle \langle \gamma_\alpha \gamma^\beta \gamma_\beta \rangle \rangle_1 \\
- \frac{1}{5} \sum_{\sigma \in S_3} \langle \langle W_{\sigma(1)} W_{\sigma(2)} \gamma_\alpha \gamma^a \gamma^\beta \rangle \rangle_0 \langle \langle \gamma_\alpha \rangle \rangle_1 \langle \langle \gamma_\beta \rangle \rangle_1 \langle \langle W_{\sigma(3)} \rangle \rangle_1.
\]
\[\sum_{\sigma \in S_3} \left\langle \left\langle \{W_{\sigma(1)} \cdot \gamma_\alpha\} \right\rangle_1 \left\langle \gamma^\alpha \cdot W_{\sigma(2)} \cdot W_{\sigma(3)} \right\rangle_1 \right\rangle_1 \]

\[-\frac{3}{5} \sum_{\sigma \in S_3} \left\langle \left\langle W_{\sigma(1)} \cdot \{W_{\sigma(2)} \cdot \gamma^\alpha\} \right\rangle_1 \left\langle \gamma_\alpha \cdot W_{\sigma(3)} \right\rangle_1 \right\rangle_1 \]

\[+ \frac{3}{10} \sum_{\sigma \in S_3} \left\langle \left\langle W_{\sigma(1)} \cdot \gamma_\alpha \right\rangle_1 \left\langle \gamma^\alpha \cdot \{W_{\sigma(2)} \cdot W_{\sigma(3)}\} \right\rangle_1 \right\rangle_1 \]

\[+ \frac{1}{5} \sum_{\sigma \in S_3} \left\langle \left\langle \gamma_\alpha \right\rangle_1 \left\langle \gamma^\alpha \cdot W_{\sigma(1)} \cdot \{W_{\sigma(2)} \cdot W_{\sigma(3)}\} \right\rangle_1 \right\rangle_1 \]

\[-\frac{3}{5} \sum_{\sigma \in S_3} \left\langle \left\langle W_{\sigma(1)} \cdot \gamma_\alpha \gamma^\alpha \gamma^\beta \right\rangle_1 \left\langle \gamma_\beta \cdot W_{\sigma(3)} \right\rangle_1 \right\rangle_1 \]

\[+ \frac{1}{80} \sum_{\sigma \in S_3} \left\langle \left\langle W_{\sigma(1)} \cdot \gamma_\alpha \gamma^\alpha \gamma^\beta \right\rangle_1 \left\langle \gamma_\beta \cdot W_{\sigma(2)} \cdot W_{\sigma(3)} \right\rangle_1 \right\rangle_1 \]

\[-\frac{1}{20} \sum_{\sigma \in S_3} \left\langle \left\langle W_{\sigma(1)} \cdot \gamma_\alpha \gamma_\beta \right\rangle_1 \left\langle \gamma^\alpha \gamma^\beta \cdot W_{\sigma(2)} \cdot W_{\sigma(3)} \right\rangle_1 \right\rangle_1 \]

\[+ \frac{1}{60} \sum_{\sigma \in S_3} \left\langle \left\langle W_{\sigma(1)} \cdot W_{\sigma(2)} \cdot \gamma_\alpha \{\gamma^\alpha \cdot W_{\sigma(3)}\} \right\rangle_1 \right\rangle_1 \]

\[-\frac{1}{120} \sum_{\sigma \in S_3} \left\langle \left\langle W_{\sigma(1)} \cdot \gamma_\alpha \gamma_\alpha \{W_{\sigma(2)} \cdot W_{\sigma(3)}\} \right\rangle_1 \right\rangle_1 \cdot \]

Note that the 3 complicated tensors \(A_1, A_2, B\) are symmetric tensors which depend on genus-0 and genus-1 data.

The universal equation corresponding to a formula due to Mumford is equivalent to \(\rho_{2,1} = 0\) (cf. [G2]). Another equation due to Getzler [G2] is equivalent to \(\rho_{2,2} = 0\). The equation \(\rho_{2,3} = 0\) is due to Belorousski and Pandharipande [BP]. Therefore \(\rho_{2,i} \in U\) for \(i = 1, 2, 3\).

### 2.4 String and dilaton equations

There are also universal equations which do not come from relations in the tautological ring of the moduli space of stable curves. For example, the string equation, dilaton equation and divisor equation. In this paper, we will only need the string and dilaton equations.

The string equation has the form

\[\left\langle \left\langle S \right\rangle \right\rangle_g = \frac{1}{2} \delta_{g,0} \eta_{\alpha\beta} t^\alpha_0 t^\beta_0 \]

for \(g \geq 0\). Since \(\nabla W S = -\tau_-(W)\), taking covariant derivative on both sides of the string equation, we obtain

\[\left\langle S W_1 \cdots W_k \right\rangle_g = \sum_{i=1}^k \left\langle W_1 \cdots \{\tau_-(W_i)\} \cdots W_k \right\rangle_g + \delta_{g,0} \nabla^k_{W_1, \ldots, W_k} \left(\frac{1}{2} \eta_{\alpha\beta} t^\alpha_0 t^\beta_0 \right) .\]
for any vector fields $W_1, \ldots, W_k$. The *dilaton vector field* is defined by
\[ \mathcal{D} = T(S). \]

The *dilaton equation* has the form
\[ \langle \langle \mathcal{D} \rangle \rangle_g = (2g - 2) F_g + \frac{1}{24} \chi(V) \delta_{g,1} \]
for $g \geq 0$. Since $\nabla_W \mathcal{D} = -W$, taking covariant derivative on both sides of the dilaton equation, we obtain
\[ \langle \langle \mathcal{D} W_1 \cdots W_k \rangle \rangle_g = (k + 2g - 2) \langle \langle W_1 \cdots W_k \rangle \rangle_g. \]

3 Relations among lower genus universal equations

3.1 Relation among genus-2 universal equations

Relations among genus-2 universal equations were studied in [L2]. Using the string and dilaton equations, it was proved that
\[ \rho_{2;2}(S, W) = \rho_{2;1}(\tau-(W)) \]
for any vector field $W$ (cf. [L2, Theorem 2.6]). Since $\tau-$ is surjective, this shows that the universal equation $\rho_{2;1} = 0$ follows from $\rho_{2;2} = 0$. On the other hand, it is straightforward to show that
\[ (\nabla_{T(W_1)} \rho_{2;1})(W_2) - \rho_{2;2}(W_1, T(W_2)) \]
\[ = \{3 \rho_0(W_2, W_1, \gamma^a) - \rho_0(W_1, W_2, \gamma^a)\} \langle \langle T(\gamma_a) \rangle \rangle_2 - \rho_0(W_2, T(W_1), \gamma^a) \langle \langle \gamma_a \rangle \rangle_2 \]
\[ + A_2(W_1, T(W_2)) - (\nabla_{T(W_1)} A_1)(W_2) \] (3)

for any vector fields $W_1$ and $W_2$. Two conclusions can be drawn from this equation: First, we obtain a genus-1 universal relation
\[ A_2(W_1, T(W_2)) - (\nabla_{T(W_1)} A_1)(W_2) = 0 \] (4)

for any vector fields $W_1$ and $W_2$. Secondly, the universal equation $\rho_{2;2}(W_1, T(W_2)) = 0$ follows from $\rho_{2;1} = 0$, the genus-0 topological recursion relation and the genus-1 equation (4).

Note that a special case of the genus-0 topological recursion relation is the following
\[ T(W_1) \cdot W_2 = \rho_0(W_1, W_2, \gamma^a) \gamma_a = 0 \]
for all vector fields $W_1$ and $W_2$. Taking covariant derivative of this equation with respect to $W_3$, we have
\[ \langle \langle T(W_1) W_2 W_3 \gamma_a \rangle \rangle_0 \gamma_a = W_1 \cdot W_2 \cdot W_3 \]
for all vector fields $W_1$, $W_2$, and $W_3$. Using these two equations, we obtain

$$\rho_{2,3}(W_1, W_2, T(W_3)) + (\nabla_{W_1 \cdot W_2} \rho_{2,1})(W_3) - \rho_{2,2}(W_1 \cdot W_2, W_3) = -B(W_1, W_2, T(W_3)) - (\nabla_{W_1 \cdot W_2} A_1)(W_3) + A_2(W_1 \cdot W_2, W_3).$$  \hspace{1cm} (5)

Three conclusions can be drawn from this equation: First, we obtain a genus-1 universal equation

$$B(W_1, W_2, T(W_3)) - A_2(W_1 \cdot W_2, W_3) + (\nabla_{W_1 \cdot W_2} A_1)(W_3) = 0$$ \hspace{1cm} (6)

for all vector fields $W_1$, $W_2$, and $W_3$. Secondly, the universal equation

$$\rho_{2,3}(W_1, W_2, T(W_3)) = 0$$

follows from $\rho_{2,1} = 0$, $\rho_{2,2} = 0$, the genus-0 topological recursion relation, and the genus-1 equation (6). Thirdly, since any vector field $W$ can be written as

$$W = S \cdot W + T(\tau - (W)),$$

combining equation (5) and equation (3), we obtain the result that the universal equation $\rho_{2,2} = 0$ follows from $\rho_{2,3} = 0$ and $\rho_{2,1} = 0$ together with genus-0 and genus-1 universal equations. This is precisely [L2, Theorem 2.9].

### 3.2 Relation among genus-1 universal equations

In this section, we will study the relations between genus-1 universal equations in section 2.2 and equations (4) and (6). The main result is the following

**Theorem 3.1** Equations (4) and (6) can be derived from known genus-0 and genus-1 relations described in section 2.1 and 2.2.

To prove this theorem, we need first use genus-0 and genus-1 topological recursion relations and their covariant derivatives to get rid of the $T$ operator in equations (4) and (6). The first three covariant derivatives of the genus-0 topological recursion $\rho_0 = 0$ gives

$$\langle \langle T(W_1) W_2 W_3 W_4 \rangle \rangle_0 = \langle \langle W_1 \cdot W_2 \rangle \rangle_0$$ \hspace{1cm} (7)

and

$$\langle \langle T^i W_2 \ldots W_5 \rangle \rangle_0 = \langle \langle W_1 \cdot W_2 \rangle \rangle_0 W_3 W_4 W_5 + \langle \langle W_1 \cdot W_3 \rangle \rangle_0 W_2 W_4 W_5 + \langle \langle W_1 \cdot W_4 \rangle \rangle_0 W_2 W_3 W_5 + \langle \langle W_1 \cdot W_5 \rangle \rangle_0 W_2 W_3 W_4,$$ \hspace{1cm} (8)

and

$$\langle \langle T(W_1) W_2 \ldots W_6 \rangle \rangle_0 = \sum_{i=2}^{4} \langle \langle W_1 \cdot W_i \rangle \rangle_0 W_2 \ldots W_6 + \langle \langle W_1 \cdot W_2 \cdot W_3 \cdot W_4 \rangle \rangle_0 W_5 W_6 + \langle \langle W_1 \cdot W_2 \cdot W_3 \cdot W_4 \cdot W_5 \rangle \rangle_0 W_6 + \langle \langle W_1 \cdot W_2 \cdot W_3 \cdot W_4 \cdot W_5 \cdot W_6 \rangle \rangle_0. \hspace{1cm} (9)$$
It’s obvious that the left hand sides of these equations are symmetric with respect to all arguments except $\mathcal{W}_1$. This symmetry property is not apparent for the right hand sides of these equations. Therefore when apply these equations to get rid of operator $T$, we may get two seemingly different but actually equal expressions. For the convenience of presentation, in this section we will follow the rule that when applying these equations, we always take the form which gives the smallest number of terms. A typical example is the following: When applying equation (9) to $\langle \langle T(\mathcal{W}) \gamma_\alpha \gamma_\beta \gamma_\beta \rangle \rangle_0$, we choose $\mathcal{W}_1 = \mathcal{W}$, $\mathcal{W}_2 = \gamma_\alpha$, $\mathcal{W}_3 = \gamma^\alpha$, $\mathcal{W}_4 = \gamma_\beta$, $\mathcal{W}_5 = \gamma^\beta$, $\mathcal{W}_6 = \mathcal{V}$. Keeping in mind that we are summing over repeated indices, using apparent symmetries, we obtain
\[
\langle \langle T(\mathcal{W}) \mathcal{V} \gamma_\alpha \gamma^\alpha \gamma_\beta \gamma^\beta \rangle \rangle_0
\]
\[= 3 \langle \langle (\mathcal{W} \cdot \gamma_\alpha) \gamma^\alpha \gamma_\beta \gamma^\beta \mathcal{V} \rangle \rangle_0 + \langle \langle \mathcal{W} \gamma_\alpha \gamma^\alpha \gamma_\beta (\gamma^\beta \cdot \mathcal{V}) \rangle \rangle_0
\]
\[+ \langle \langle \mathcal{W} \gamma_\alpha \gamma^\alpha \gamma^\beta \mathcal{V} \rangle \rangle_0 + 2 \langle \langle \mathcal{W} \gamma_\alpha \gamma_\beta \gamma^\beta \mathcal{V} \rangle \rangle_0 \langle \langle \gamma_\mu \gamma^\alpha \gamma^\beta \mathcal{V} \rangle \rangle_0. \tag{10}
\]
If we instead choose $\mathcal{W}_1 = \mathcal{W}$, $\mathcal{W}_2 = \mathcal{V}$, $\mathcal{W}_3 = \gamma_\alpha$, $\mathcal{W}_4 = \gamma^\alpha$, $\mathcal{W}_5 = \gamma_\beta$, $\mathcal{W}_6 = \gamma^\beta$, we obtain
\[
\langle \langle T(\mathcal{W}) \mathcal{V} \gamma_\alpha \gamma^\alpha \gamma_\beta \gamma^\beta \rangle \rangle_0
\]
\[= \langle \langle (\mathcal{W} \cdot \mathcal{V}) \gamma_\alpha \gamma^\alpha \gamma_\beta \gamma^\beta \mathcal{V} \rangle \rangle_0 + 2 \langle \langle (\mathcal{W} \cdot \gamma_\alpha) \gamma^\alpha \gamma_\beta \gamma^\beta \mathcal{V} \rangle \rangle_0 + \langle \langle \mathcal{W} \mathcal{V} \gamma_\alpha \gamma_\alpha (\gamma_\beta \cdot \gamma^\beta) \rangle \rangle_0
\]
\[+ 2 \langle \langle \mathcal{W} \mathcal{V} \gamma^\alpha \gamma^\beta \mathcal{V} \rangle \rangle_0 \langle \langle \gamma_\mu \gamma_\beta \gamma^\beta \mathcal{V} \rangle \rangle_0 + \langle \langle \mathcal{W} \gamma_\alpha \gamma_\alpha \gamma^\beta \mathcal{V} \rangle \rangle_0 \langle \langle \gamma_\mu \gamma_\beta \gamma_\beta \mathcal{V} \rangle \rangle_0. \tag{11}
\]
Although the right hand sides of equation (11) and equation (10) look very different from each other, they are actually equal since the left hand sides of these equations are the same. The right hand side of equation (11) has 5 terms. It is slightly more complicated than the right hand side of equation (10) which has only 4 terms. So following the rule stated earlier, we should use equation (10) rather than equation (11).

To eliminate operator $T$ in genus-1 correlation functions, we need the first three covariant derivatives of the genus-1 topological recursion relation $\rho_1 = 0$, which have the following forms:
\[
\langle \langle T(\mathcal{W}) \mathcal{V} \rangle \rangle_1 = \langle \langle \{\mathcal{W} \cdot \mathcal{V}\} \rangle \rangle_1 + \frac{1}{24} \langle \langle \mathcal{W} \mathcal{V} \gamma^\mu \gamma_\mu \rangle \rangle_0, \tag{12}
\]
\[
\langle \langle T(\mathcal{W}) \mathcal{V}_1 \mathcal{V}_2 \rangle \rangle_1 = \langle \langle \{\mathcal{W} \cdot \mathcal{V}_1\} \mathcal{V}_2 \rangle \rangle_1 + \langle \langle \{\mathcal{W} \cdot \mathcal{V}_2\} \mathcal{V}_1 \rangle \rangle_1 + \langle \langle \mathcal{W} \mathcal{V}_1 \mathcal{V}_2 \gamma^\alpha \rangle \rangle_0 \langle \langle \gamma_\alpha \rangle \rangle_1
\]
\[+ \frac{1}{24} \langle \langle \mathcal{W} \mathcal{V}_1 \mathcal{V}_2 \gamma^\mu \gamma_\mu \rangle \rangle_0, \tag{13}
\]
and
\[
\langle \langle T(\mathcal{W}) \mathcal{V}_1 \mathcal{V}_2 \mathcal{V}_3 \rangle \rangle_1
\]
\[= \sum_{i=1}^{3} \left\{ \langle \langle (\mathcal{W} \cdot \mathcal{V}_i) \mathcal{V}_1 \cdots \mathcal{V}_i \cdots \mathcal{V}_3 \rangle \rangle_1 + \langle \langle \mathcal{W} \mathcal{V}_1 \cdots \mathcal{V}_i \cdots \mathcal{V}_3 \gamma^\alpha \rangle \rangle_0 \langle \langle \gamma_\alpha \mathcal{V}_i \rangle \rangle_1 \right\}
\]
\[+ \langle \langle \mathcal{W} \mathcal{V}_1 \mathcal{V}_2 \mathcal{V}_3 \gamma^\alpha \rangle \rangle_0 \langle \langle \gamma_\alpha \rangle \rangle_1 + \frac{1}{24} \langle \langle \mathcal{W} \mathcal{V}_1 \mathcal{V}_2 \mathcal{V}_3 \gamma^\alpha \gamma_\alpha \rangle \rangle_0. \tag{14}
\]
for all vector fields. Note that both sides of these equations are symmetric with respect to vector fields $V_i$'s. There is no ambiguity on how to apply these formulas.

Besides the genus-0 and genus-1 topological recursion relations and their covariant derivatives, we also need universal equations $C_0=0$ and $G=0$ (see Section 2.1 and 2.2 for the definitions of tensors $C_0$ and $G$). The application of $C_0=0$ is straightforward. Most times we only need a slightly weaker version of this equation, i.e. the associativity of the quantum product. However it is less obvious how to apply the covariant derivatives of $C_0=0$. This turns out to be one of the most subtle points in the proof of Theorem 3.1. In the proof, we will include tensors $\nabla^iC_0$ in the relations among genus-1 universal tensors. It is then apparent where and how covariant derivatives of $C_0=0$ are used.

The tensors for the covariant derivatives of $C_0$ are defined by

$$C_i(W_1, \ldots, W_{i+4}) := (\nabla_{W_{i+4}}C_{i-1})(W_1, \ldots, W_{i+3})$$

for $i \geq 1$. These tensors lie in $U$, the ideal of tensors given universal equations. We need the first three covariant derivatives of $C_0$ in the proof of Theorem 3.1. Using equation (1), we can compute these tensors more explicitly. For example,

$$C_1(W_1, \ldots, W_5) = \langle (W_1 \cdot W_2) W_3 W_4 W_5 \rangle_0 + \langle (W_1 W_2 (W_3 \cdot W_4) W_5) \rangle_0$$

and

$$C_2(W_1, \ldots, W_6) = \langle (W_1 \cdot W_2) W_3 \cdots W_6 \rangle_0 + \langle W_1 W_2 (W_3 \cdot W_4) W_5 W_6 \rangle_0$$

for any vector fields $W_1, \ldots, W_6$. The third derivative $C_3$ also has a similar expression, but more complicated. So we omit the explicit expression for $C_3$.

We are now ready to prove Theorem 3.1.

**Proof of Theorem 3.1:** First using genus-0 and genus-1 topological recursion relations and their derivatives, we can get rid of operator $T$ in the expression

$$A_2(W_1, T(W_2)) - \left( \nabla_{T(W_1)} A_1 \right) (W_2).$$

We then check that the resulting expression is equal to certain combinations of tensors $G$ and $C_1$. More precisely, modulo $I_a[\rho_0, \nabla \rho_0, \nabla^2 \rho_0, \nabla^3 \rho_0, \rho_1, \nabla \rho_1, \nabla^2 \rho_1, C_0]$, we have

$$120 \left\{ A_2(W_1, T(W_2)) - \left( \nabla_{T(W_1)} A_1 \right) (W_2) \right\}$$

$$= -\frac{1}{24} G(W_1, W_2, \gamma^\alpha, \gamma_\alpha) + \frac{1}{6} C_1(W_2, \gamma_\alpha, W_1, \gamma^\alpha, \gamma_\beta)$$

$$+ \frac{19}{2} C_1(W_1, \gamma_\beta, \gamma_\alpha, \gamma^\alpha, W_2) - \frac{21}{2} C_1(W_2, \gamma_\beta, \gamma^\alpha, \gamma_\alpha, W_1) \langle \langle \gamma^\beta \rangle \rangle_1$$

(15)
for any vector fields \( W_1 \) and \( W_2 \). This shows that equation (4) follows from genus-0 and genus-1 universal equations in Section 2. The proof of equation (15) is straightforward, but somehow tedious.

Equation (6) can be treated in a similar fashion. We first use genus-0 and genus-1 topological recursion relations and their derivatives to get rid of operator \( T \) in the expression

\[
B(W_1, W_2, T(V)) - A_2(W_1 \bullet W_2, V) + (\nabla_{W_1} \circ W_2 A_1)(V),
\]

then check that the resulting expression is equal to certain combinations of tensors \( G, \nabla G, C_1, C_2, \) and \( C_3 \). More precisely, modulo

\[
I_a[\rho_0, \nabla \rho_0, \nabla^2 \rho_0, \rho_1, \nabla^2 \rho_1, \nabla^3 \rho_1, C_0],
\]

we have

\[
720 \{ B(W_1, W_2, T(V)) - A_2(W_1 \bullet W_2, V) + (\nabla_{W_1} \circ W_2 A_1)(V) \} = \frac{1}{4} \bigl\{ \langle \nabla V G \rangle(W_1, W_2, 2\gamma, \gamma) - \langle \nabla V G \rangle(V, W_2, 2\gamma, \gamma) - \langle \nabla W_1 G \rangle(V, W_1, 2\gamma, \gamma) \bigr\}
\]

\[
-3 \langle \nabla V \gamma \rangle \bigl\{ C_1(W_1, W_2, \gamma, \gamma, \gamma) - C_1(W_1, \gamma, \gamma, \gamma) \bigr\}
\]

\[
+ \langle \nabla W_1 \gamma \rangle \bigl\{ 27 C_1(W_1, \gamma, \gamma, \gamma) - 34 C_1(V, \gamma, \gamma, \gamma) \bigr\}
\]

\[
+ \langle \nabla W_2 \gamma \rangle \bigl\{ 27 C_1(W_1, \gamma, \gamma, \gamma) - 34 C_1(V, \gamma, \gamma, \gamma) \bigr\}
\]

\[
- 7 C_1(W_1, \gamma, \gamma, \gamma)
\]

\[
- 8 \langle \gamma \rangle \bigl\{ 2 C_1(W_1, W_2, \gamma, \gamma, \gamma) - C_1(W_1, V, W_2, \gamma, \gamma) \bigr\}
\]

\[
- 120 \langle \gamma \rangle \langle \gamma \rangle \bigl\{ 3 C_1(W_1, W_2, \gamma, \gamma, \gamma) - C_1(W_1, V, W_2, \gamma, \gamma) \bigr\}
\]

\[
- \langle \gamma \rangle \bigl\{ 19 C_2(W_2, \gamma, W_1, \gamma, \gamma) + 19 C_2(W_1, \gamma, \gamma, \gamma) \bigr\}
\]

\[
+ 19 C_2(W_1, W_2, \gamma, \gamma, \gamma) - 12 C_2(V, \gamma, \gamma, \gamma, W_1, W_2)
\]

\[
- 3 C_2(W_1, V, W_2, \gamma, \gamma, \gamma) - 3 C_2(W_1, \gamma, \gamma, \gamma, \gamma)
\]

\[
- 28 C_2(W_1, \gamma, \gamma, \gamma, \gamma) - 28 C_2(W_2, \gamma, \gamma, \gamma, \gamma)
\]

\[
- C_3(W_1, W_2, \gamma, \gamma, \gamma)
\]

(16)

for any vector fields \( W_1, W_2, \) and \( V \). This implies that equation (6) follows from genus-0 and genus-1 universal equations in Section 2. The proof of equation (16) is very tedious but nevertheless straightforward. All what is needed is to plug the expressions for the relevant tensors into both sides of equation (16) and check that they are equal after obvious cancellations. □
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