Symmetries of relativistic world-lines

Benjamin Koch,† Enrique Muñoz,‡ Ignacio A. Reyes†,‡

1Pontificia Universidad Católica de Chile
Instituto de Física, Pontificia Universidad Católica de Chile,
Casilla 306, Santiago, Chile

2Insitut für Theoretische Physik und Astrophysik,
Julius-Maximilians-Universität Würzburg, Am Hubland, 97074, Germany

Symmetries are essential for a consistent formulation of many quantum systems. In this paper we discuss a previously unnoticed symmetry, which is present for any Lagrangian term that involves $x^2$. As a basic model that incorporates the fundamental symmetries of quantum gravity and string theory, we consider the Lagrangian action of the relativistic point particle. A path integral quantization for this seemingly simple system has for long presented notorious problems. Here we show that those problems are overcome by taking into account the newly discovered additional symmetry, leading directly to the exact Klein-Gordon propagator.

I. INTRODUCTION

Gauge symmetries and the global symmetry of special relativity are the essential ingredients of modern quantum physics, providing the most fundamental description of nature [1]. However, up to now, any straightforward attempt to truly unify those concepts has failed. The most prominent example for this failure is that, a consistent quantum description of General Relativity, whose gauge symmetry is a local generalization of the global symmetry of special relativity, is still missing. Similar technical problems arise in the context of String Theory [2], mainly due to the square-root kinetic energy term in the Lagrangian action. The problem does not seem to be only with General Relativity or String Theory itself, but rather with the unification of the fundamental concepts of gauge symmetries and relativity within quantum mechanics. Clearly, a crucial step towards achieving this unification is the correct analysis of symmetries in the most simple theory with general covariance, the Lagrangian description of relativistic point particle world-lines whose action is given by

$$I[x(\lambda)] = -m \int d\lambda \sqrt{-\dot{x}^{\mu} \dot{x}^{\nu} \eta_{\mu\nu}}. \quad (1.1)$$

Due to the non-quadratic form of this action, the path integral formulation of this apparently basic problem has posed significant difficulties. By discovering and applying an additional symmetry present in (1.1), we were able to derive an exact explicit expression for the corresponding relativistic propagator, thus filling one of the missing pieces for several of the most pressing problems in modern theoretical physics.

Even though the Path Integral (PI) [3] of (1.1) is the natural generalization of its non-relativistic counterpart, it has presented many difficulties. With the advent of Quantum Field Theory (QFT) part of the problems where put aside since one knows what the result should be, namely the propagator for the free massive scalar field from QFT [4]. We shall work in Minkowski spacetime with signature $\eta = (-, +, \ldots, +)$. Despite of the simplicity of the action (1.1), there remain some essential problems with the path integral formulation of this theory. As is well known also in statistical mechanics of relativistic particles, the propagator fails to satisfy a naive application of the Chapman-Kolmogorov relation $(x_f | 0) = \int d^d x' (x_f | x') (x' | 0)$, seemingly implying that the theory is not unitary [5]. A “technical” problem is also relevant: A direct evaluation of a path integral with a square root action is highly non-trivial [6, 7]. Thus, for the case of the relativistic point particle one is limited to evaluate the path integral of an alternative quadratic action which is equivalent to (1.1) at the classical level [2, 3, 8, 9], or to use some other kind of approximation [10]. This is similar to the situation in string theory, where the presence of the square root has prevented a direct evaluation of the Nambu-Goto path integral, and the Polyakov action was introduced precisely to surmount these difficulties [2].

In summary, a direct PI calculation of (1.1) is still lacking. The purpose of this letter is to fill in this gap by a very simple observation: “Any Lagrangian term of the form $\dot{x}^2$ has a symmetry that must be accounted for.” The paper is organized as follows. In the next section the crucial symmetry aspects of the problem are discussed. This is followed by an explicit Faddeev-Popov procedure for the PI of (1.1), taking into account those symmetries. In the final section, we summarize the results and comment on technical and conceptual implications.

II. SYMMETRIES OF FREE PARTICLES

As is well known, the action (1.1) possesses invariance under reparametrizations: if $\lambda \rightarrow \lambda'(\lambda)$ for an arbitrary function $\lambda'(\lambda)$ (we assume monotonic, differentiable, and integrable), the action remains unchanged. For infinitesimal transformations where $\lambda' = \lambda - h(\lambda)$
with $h(\lambda)$ small, the induced transformation of the fields are
$$\delta_x \mu = h(\lambda)v^\mu \text{ where } v^\mu = \dot{x}^\mu / \sqrt{-g},$$
and then reparametrizations are generated by the first class constraint
$$\phi = p^\mu p_\mu + m^2,$$
via Poisson brackets,
$$\delta_x x^\mu = \{x^\mu, h\phi\} = 2hp^\mu,$$
$$\delta_x p_\mu = \{p_\mu, \phi\} = 0.$$

Let us now examine a different symmetry of this action, which will play a fundamental role in the argument below. Any kinetic term $v^\mu v_\mu$ possesses an additional symmetry: one can locally rotate the velocity to $v^\mu$ with the constraint that $v^2 = v^2$, i.e. local $SO(1, d-1)$ rotations of the velocity. This transformation involves $d-1$ arbitrary functions of the parameter, one for each of the angles of the $S^{d-1}$ sphere. We will refer to these as ‘local velocity rotations’. Infinitesimally, this condition is $v_\gamma \delta v = 0$, and the most general variation $\delta_x v^\mu$ orthogonal to the velocity is
$$\delta_x v^\mu (\lambda) = f^\mu (\lambda) - (f \cdot v) \frac{v^\mu}{|v|^2}.$$
The function $f^\mu (\lambda)$ is assumed to be well behaved (integrable, differentiable, and monotonic). Integrating these equations gives the transformation of the fundamental fields $\delta_x x^\mu$. Thus, the symmetry is local in the velocities, but non-local in the position variables, unlike usual gauge symmetries. However, the point is that if one factorizes this symmetry out from the path integral, the inconsistency with a naive Chapman-Kolmogorov relation can be solved and the calculation of the exact propagator results in a straightforward way.

It is instructive to count how many extra degrees of freedom are subtracted from the action, due to this new symmetry. The local transformations of $SO(1, d-1)$ contain $d(d-1)/2$ degrees of freedom. However, those contain the subgroup $SO(1, d-2)$ with $(d-1)(d-2)/2$ parameters which leave a given velocity vector $v^\mu$ constant, and which are thus acting trivially, so they must not be fixed in the path integral. The remaining non-trivial degrees of freedom (which actually change $v^\mu$) correspond precisely to the $d-1$ transformations that are orthogonal to the velocity.

As it is shown below, by factorizing this additional symmetry out from the path integral, a standard Faddeev-Popov calculation leads to the correct propagator for the relativistic point particle.

### III. PATH INTEGRAL: FADDEEV-POPOV METHOD

The object to be computed is
$$\langle x_f | 0 \rangle = \int_0^{x_f} D x \; e^{iI[x]},$$
where we have used the global translation invariance in space-time to set $x^\mu(\lambda_i) = 0$ and $x^\mu(\lambda_f) = x^\mu_f$. This can be rewritten by introducing a Dirac delta identity,
$$\langle x_f | 0 \rangle = \int_0^{x_f} D x \; \int_{-\infty}^{\infty} dS \; e^{iS} \delta(S - I[x])$$
$$= \int_{-\infty}^{\infty} dS \; e^{iS} \Omega(S),$$
where the sum over histories is now expressed as an ordinary integral over the values of the action $S$. Here, we have defined the volume or multiplicity $\Omega(S)$ of trajectories connecting the points 0 and $x_f$, that share the same value of the action $S$ as
$$\Omega(S) = \int_0^{x_f} D x (\lambda) \delta \left(S + m \int_{\lambda_i}^{\lambda_f} \sqrt{-g} \dot{x}^\mu \dot{x}_\mu d\lambda\right).$$

By explicitly computing $\Omega$ we will solve the path-integral defined in (3.2).

We now turn to the Faddeev-Popov procedure for factoring out the redundancy of the PI. In the calculation of $\Omega$, we can start by exploiting reparametrization symmetry by choosing as a convenient parametric scale the interval (or proper time) of the particle ($c = 1$), $d\tau = \sqrt{-dx^\mu dx_\mu}$. We fix this choice by inserting the Faddeev-Popov functional identity (see Appendix A)

$$1 = \int Dv(\lambda) \delta \left[v(\lambda)^2 + \dot{x}^\mu \dot{x}_\mu\right] \det \left[\frac{\delta \left(v(\lambda)^2 + \dot{x}^\mu \dot{x}_\mu\right)}{\delta v(\lambda)}\right]$$
$$= \int Dv(\lambda) \delta \left[v(\lambda)^2 + \dot{x}^\mu \dot{x}_\mu\right] \det [2\delta(\lambda - \lambda')v(\lambda)].$$

Notice that, in a given discretization $\lambda \in [\lambda_i, \lambda_f] \rightarrow \{\lambda_j\}$, the values of the function $v(\lambda) \rightarrow v(\lambda_j) = v_j$, and hence the determinant possesses the simple structure $\det [2\delta(\lambda - \lambda')v(\lambda)] = \prod_j 2v_j$. Let us define the differential proper time as
$$d\tau = \sqrt{-dx^\mu dx_\mu}.$$ sport. The invariance of the total interval upon reparameterization, in differential form $v(\lambda)d\lambda = d\tau$ implies
$$\int_{\lambda_i}^{\lambda_f} v(\lambda)d\lambda = \int_{\tau_i}^{\tau_f} d\tau(\lambda) = \int_0^{x_f} \sqrt{-dx^\mu dx_\mu}. (3.6)$$

Moreover, the reparameterization introduces, via chain-rule, a multiplicative scale factor in the velocities:
$$\dot{x}_\mu \dot{x}^\mu = \left(\frac{d\tau}{d\lambda}\right)^2 \frac{dx_\mu}{d\tau} \frac{dx^\mu}{d\tau} = v^2(\lambda(\tau)) \frac{dx_\mu}{d\tau} \frac{dx^\mu}{d\tau}. (3.7)$$

Therefore, upon inserting Eq.(3.4) the volume becomes
\[ \Omega(S) = \int_0^{\tau_f} Dx(\lambda) \int Dv(\lambda) \det [2\delta(\lambda - \lambda')v(\lambda)] \delta \left[ v^2(\lambda) + \dot{x}^2 \right] \delta \left( S + m \int_{\lambda}^{\lambda_f} \sqrt{-\dot{x}^2} d\lambda \right) \]

\[ = \int Dv(\lambda(\tau)) \int_0^{\tau_f} Dx(\tau) \delta \left[ v^2(\lambda(\tau)) \left( 1 + \frac{dx^\mu}{d\tau} \frac{dx_\mu}{d\tau} \right) \right] \delta \left[ 2\delta(\lambda - \lambda')v(\lambda) \right] \delta \left( S + m \int_{\tau_i}^{\tau_f} d\tau \sqrt{-\frac{dx^\mu}{d\tau} \frac{dx_\mu}{d\tau}} \right) \]

\[ = \left[ \int Dv(\lambda(\tau)) \int_0^{\tau_f} Dx(\tau) \delta \left[ 1 + \frac{dx^\mu}{d\tau} \frac{dx_\mu}{d\tau} \right] \delta \left( S + m \int_{\tau_i}^{\tau_f} d\tau \sqrt{-\frac{dx^\mu}{d\tau} \frac{dx_\mu}{d\tau}} \right) \right] \]

(3.8)

Thus, clearly the (infinite) ‘volume’ \( V_r = \int Dv(\tau) \) associated to reparametrization symmetry has factored out. Let us now proceed by renaming the integration variables that define the trajectories in the path-integral, upon defining the “momenta”

\[ p^\mu(\tau) \equiv m \frac{dx^\mu}{d\tau}. \]  

(3.9)

This definition implies the global identity (see Appendix B)

\[ \int_{\tau_i}^{\tau_f} p^\mu(\tau) d\tau = m \int_0^{\tau_f} dx^\mu = m x^\mu_f. \]  

(3.10)

Notice that, in terms of the momenta, and using the definition of the proper time, the action functional acquires the simpler form:

\[ S[x] = - \int_{\tau_i}^{\tau_f} \sqrt{\frac{dx^\mu}{d\tau} \frac{dx_\mu}{d\tau}} d\tau = - \int_{\tau_i}^{\tau_f} \sqrt{-\frac{dx^\mu}{d\tau} \frac{dx_\mu}{d\tau}} d\tau \]

\[ = \int_{\tau_i}^{\tau_f} \frac{dx^\mu}{d\tau} \frac{dx_\mu}{d\tau} d\tau = \frac{1}{m^2} \int_{\tau_i}^{\tau_f} p^\mu(\tau)p_\mu(\tau) d\tau, \]  

(3.11)

where we have used the definition of the differential proper time (\texttt{E}). Thus, changing the integration measure \( Dx(\tau) \rightarrow Dp(\tau) \) (see Appendix B), we have (up to an action-independent Jacobian) the expression

\[ \frac{\Omega(S)}{V_r} = \int Dp(\tau) \delta \left[ m^2 + p^2(\tau) \right] \delta \left( S - \frac{1}{m} \int_{\tau_i}^{\tau_f} p_\mu(\tau)p^\mu(\tau) d\tau \right). \]  

(3.12)

Finally, we still need to factor out the local velocity \( SO(1, d-2) \) rotations as explained in the introduction.

Given an arbitrary constant d-vector \( k^\mu \) satisfying \( k^2 = -m^2 \), there is a unique rotation in \( G = SO(1, d-2) \) that connects it to each d-momenta along the trajectory, i.e. there exists a matrix \( \Lambda^\mu_\nu(\tau) \) such that \( p^\mu(\tau) = \Lambda^\mu_\nu(\tau) k^\nu \).

We can parametrize each rotation, in the vicinity of the identity, by a set of infinitesimal antisymmetric parameters as \( \Lambda^\mu_\nu(\tau) = \delta^\mu_\nu + \omega^\mu_\nu(\tau) \). Therefore, we have the functional identity (see Appendix [A])

\[ 1 = \int_G D\Lambda \delta[p^\mu(\tau) - \Lambda^\mu_\nu(\tau)k^\nu] \]  

\[ \det \left[ \frac{\delta}{\delta \omega^\mu_\nu(\tau)} (p^\mu(\tau) - \Lambda^\mu_\nu(\tau)k^\nu) \right] \]

\[ = \int D\omega(\tau) \delta[p^\mu(\tau) - \Lambda^\mu_\nu(\tau)k^\nu] \Delta[k], \]  

(3.14)

where we have defined the Fadeev-Popov determinant

\[ \Delta[k] = \det \left[ \delta(\tau - \tau') \delta^\mu_\nu \delta^\nu_\mu k^\nu \right]. \]  

(3.15)

After Eq. 3.14 and Eq. 3.15, we notice that \( \Delta[k] \) is independent of \( p^\mu(\tau) \). Therefore, let us now define the constant

\[ C = \int d^d k (\Delta[k])^{-1} \]

\[ = \int d^d k \int_G D\Lambda \delta[p^\mu(\tau) - \Lambda^\mu_\nu(\tau)k^\nu]. \]  

(3.16)

Inserting Eq. 3.16 into 3.12, we obtain

\[ \Omega(S) = \frac{V_r}{C} \int d^d k \int_G D\Lambda \int Dp(\tau) \delta[p^\mu(\tau) - \Lambda^\mu_\nu(\tau)k^\nu] \delta \left[ m^2 + p^2(\tau) \right] \delta \left( S - \frac{1}{m} \int_{\tau_i}^{\tau_f} p_\mu(\tau)p^\mu(\tau) d\tau \right) \]

\[ = \frac{V_r}{C} \int d^d k \delta \left( m^2 + k^2 \right) \int_G D\Lambda \int Dp(\tau) \delta[p^\mu(\tau) - \Lambda^\mu_\nu(\tau)k^\nu] \delta \left( S - \frac{k_\mu}{m} \int_{\tau_i}^{\tau_f} \left[ \Lambda^{-1}(\tau) \right]_{\nu}^\mu p^\nu(\tau) d\tau \right). \]  

(3.17)

Now, let us change the momenta within the path integral by the rotation \( p^\mu(\tau) \rightarrow \Lambda^\mu_\nu(\tau)p^\nu(\tau) \equiv p^\mu_A(\tau) \). Thus,
we obtain
\[
\Omega(S) = \frac{V_c}{C} \int d^d k \delta \left( m^2 + k^2 \right) \int_D \Delta \int \mathcal{D} p \Lambda(\tau) \\
\delta \left[ \mathcal{P}_R(\tau) - \Lambda_0(\tau) \mathcal{P} \right] \delta \left( S - \frac{k}{m} \int_t^{t_f} \Lambda^{-1}_0 \mathcal{P} \right) \\
= \frac{V_c}{C} \int d^d k \delta \left( m^2 + k^2 \right) \int_D \Delta \int \mathcal{D} p \mathcal{P} \\
\delta \left[ \Lambda_0^\mu (\tau) \left( p^\mu (\tau) - k^\mu \right) \right] \delta \left( S - \frac{k}{m} \int_t^{t_f} \Lambda_0^\mu (\tau) \right) \\
= \frac{V_c}{C} \int d^d k \delta \left( m^2 + k^2 \right) \int_D \Delta \int \mathcal{D} \Lambda \mathcal{P} \\
\delta \left[ \Lambda_0^\mu (\tau) \left( p^\mu (\tau) - k^\mu \right) \right] \delta \left( S - \frac{k}{m} \int_t^{t_f} \Lambda_0^\mu (\tau) \right).
\]

Here, we have used the invariance of the path-integral measure \( \mathcal{D} \Lambda(\tau) = \mathcal{D} \mathcal{P}(\tau) \) (see Appendix B), since for an element of \( SO(1, d - 2) \) we have \( \det \Lambda(\tau) \) = 1. We also made use of the global identity Eq. (2.10). Thus, we can separate the remaining integrals in the form

\[
\Omega(S) = \frac{V_c}{C} \int \mathcal{D} \Lambda \int d^d k \delta \left( m^2 + k^2 \right) \delta \left( S - k \cdot x_f \right) \\
\left[ \int \mathcal{D} \mathcal{P}(\tau) \delta \left[ \mathcal{P}_R(\tau) - k^\mu \right] \right].
\]

The path integral in square brackets is evaluated by making use of the functional delta, to yield

\[
\int \mathcal{D} \mathcal{P}(\tau) \delta \left[ \mathcal{P}_R(\tau) - k^\mu \right] = 1.
\]

Therefore, in Eq. (3.18) we have, modulo an action independent normalization factor \( \frac{V_c}{C} \int \mathcal{D} \Lambda \) representing pure redundancy, that the desired phase-space volume of trajectories with equal action is given by

\[
\Omega(S) = \int d^d k \delta \left( k^2 + m^2 \right) \delta \left( S - k \cdot x_f \right). \tag{3.20}
\]

Eq. (3.18) is a remarkable result: after factoring out the redundancies associated to both reparametrizations and local velocity Lorentz transformations, the quantum volume of paths of equal action that are physically inequivalent is equal to the classical density of states \( \Omega(S) \) of the Hamilton-Jacobi theory for the same action.

Now that \( \Omega \) has been determined, it only remains to plug it back into (3.22),

\[
\langle x_f | 0 \rangle = N \int_{-\infty}^{\infty} dS \, e^{iS} \Omega(S) \tag{3.21}
\]

\[
= N \int d^{d-1} k \int_{-\infty}^{\infty} dk_0 \delta \left( -k^2_0 + k^2 + m^2 \right) \\
\times \int_{-\infty}^{\infty} dS e^{iS} \delta \left( S - k \cdot x_f \right) \\
= N \int d^{d-1} k \left( \frac{e^{ik \cdot x_f}}{2\sqrt{k^2 + m^2}} \bigg|_{k_0 = \sqrt{k^2 + m^2}} + \frac{e^{ik \cdot x_f}}{2\sqrt{k^2 + m^2}} \bigg|_{k_0 = -\sqrt{k^2 + m^2}} \right). \tag{3.22}
\]

This is precisely the parity-even solution for the Klein-Gordon propagator \( \langle x_f | 0 \rangle = \Delta_1 | x_f \rangle \), as given for example in [4]. It is exactly the parity even propagator since it was generated by paths which are all connected by continuous transformations. If one would like to obtain the parity-odd Klein-Gordon propagator, one would have to modify the measure of the path integral, including virtual paths, that are connected to the two different sectors of the Lorentz group. This technically complicated procedure can be circumvented by simply changing one of the poles in the propagator (3.22).

IV. DISCUSSION

The apparent incompatibility between the relativistic propagator and unitarity. In non-relativistic quantum mechanics, the propagator fulfills the Chapman-Kolmogorov relation

\[
\langle x_f | x_1 \rangle = \int d^d x \langle x_f | x_1 \rangle \langle x_1 | x_f \rangle, \tag{4.1}
\]

where the integral is realized in spatial dimensions only. In contrast, the relativistic propagator does not fulfill such a naive Chapman-Kolmogorov relation, which is of course disturbing since it seems to indicate the collapse of probability conservation. This inconsistency was noted by [3, 4], who circumvented the problem by turning to a phase space formulation or by introducing a spherical constraint. It has also been argued that it is simply impossible to formulate a probability conserving relativistic quantum mechanics and one has to go to quantum field theory right away. Taking the problem more seriously it has also been argued that the usual notion of probability has to be changed [13–17].

However, those problems are solved when one realizes that most paths that appear in a naive realization of the Chapman-Kolmogorov relation on the right hand side of (4.1) are actually equivalent through local velocity Lorentz transformations of the kind (2.3). They should not be integrated over and over again. It is this type of overcounting which produces the seemingly non-conservation of probability in the path integral of the relativistic point particle. Once, one takes into account this issue of equivalent intermediate steps, the quantum propagation becomes unitary. An explicit proof of this argument can be performed in a stepwise realization of (3.22), as shown in [18].

V. CONCLUSION

This work is based on making notice that any term of the form \( \hat{x}^2 \) has a non-trivial symmetry of its own, as explained above. Accounting for this symmetry allows one solve the practical problem of computing the path integral of the relativistic point particle (containing the square root) in a direct manner.
We leave for a forthcoming paper the consequences that taking this larger symmetries into account might have in other systems such as Yang Mills, gravity, or string theory.

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Appendix A: Fadeev-Popov determinants

Along the main body of the text, we have made use of a general functional identity that is commonly used in the Fadeev-Popov [12] technique in Quantum Field Theory. Let \( \varphi(x) \) be a scalar field, and \( F[\varphi] \) a functional of these fields. Then, the following integral identity holds [19]

\[
\int \mathcal{D}\varphi(x) \delta[F[\varphi]] \det \left( \frac{\delta F[\varphi(x)]}{\delta \varphi(x')} \right) = 1. \tag{A1}
\]

The determinant of the functional derivative that appears in the left-hand-side of Eq.(A1) is commonly referred as the Fadeev-Popov determinant in the context of Abelian and Yang-Mills Field Theories [12, 19].

Appendix B: Change of integration variables

In this appendix, we provide the details of the change of variables from coordinates \( x^\mu(\tau) \) to momenta \( p^\mu(\tau) \). In the continuum representation, we have defined the momenta as derivatives with respect to the proper time,

\[
p^\mu(\tau) = m \frac{dx^\mu}{d\tau}. \tag{B1}
\]

In a given discretization of the proper time, we have \( d\tau = \epsilon (\tau_f - \tau_i)/M \), with \( M \to \infty \). Thus, each time step is defined as \( \tau_k = \tau_i + k\epsilon \), with \( 0 \leq k \leq M \), and the instantaneous coordinates become a set of discrete variables \( x^\mu(\tau_k) \equiv x^\mu_k \). In the propagator, the initial and final conditions are fixed as

\[
x^\mu_M = x^\mu_f, \quad x^\mu_0 = 0, \tag{B2}
\]

and hence only \( M - 1 \) coordinates \( x^\mu_k \) are integrated along the trajectories. The discrete, finite differences version of Eq.(B1) is

\[
p^\mu_k = m (x^\mu_k - x^\mu_{k-1})/\epsilon, \quad 1 \leq k \leq M \tag{B3}
\]

Despite Eq.(B3) suggests that we have \( M \) momenta, there exists a global constraint that reduces the total number of independent momenta to \( M - 1 \),

\[
\sum_{k=1}^{M} \epsilon p^\mu_k = m \sum_{k=1}^{M} (x^\mu_k - x^\mu_{k-1}) = m x^\mu_f. \tag{B4}
\]

Here, in the second step we have applied the telescopic property of the sum. In the continuum limit, Eq.(B4) becomes

\[
\int_{\tau_i}^{\tau_f} d\tau p^\mu(\tau) = m x^\mu_f. \tag{B5}
\]

The functional measure for the path-integrals over space trajectories (notice that the positions at \( k = 0 \) and \( k = M \) are fixed, by Eq.(B2)) is defined as

\[
\mathcal{D}x(\tau) = \prod_{k=1}^{M-1} \prod_{\mu=0}^{d-1} dx^\mu_k = \frac{\partial(x_1, \ldots, x_{M-1})}{\partial(p_1, \ldots, p_{M-1})} \prod_{k=1}^{M-1} \prod_{\mu=0}^{d-1} dp^\mu_k
\]

\[
= \left( \frac{\epsilon}{m} \right)^{d(M-1)} \prod_{k=1}^{M-1} \prod_{\mu=0}^{d-1} dp^\mu_k \equiv \mathcal{D}p(\tau) \tag{B6}
\]

The momenta functional measure is invariant under local transformations of \( SO(1, d-2) \), of the form \( A(p) \) with \( \det(A(\tau)) = 1 \), such that \( p^\mu(\tau) \to A^\mu_\nu(\tau)p^\nu(\tau) \equiv p^\mu_A(\tau) \). Clearly, from the discrete definition of the measure Eq.(B6),

\[
\mathcal{D}p_A(\tau) = \left( \frac{\epsilon}{m} \right)^{d(M-1)} \prod_{k=1}^{M-1} \prod_{\mu=0}^{d-1} dp^\mu_A_k
\]

\[
= \left( \frac{\epsilon}{m} \right)^{d(M-1)} \prod_{k=1}^{M-1} \prod_{\mu=0}^{d-1} \frac{\partial(p^0_{A_k}, \ldots, p^{d-1}_{A_k})}{\partial(p^0_k, \ldots, p^{d-1}_k)} \prod_{\mu=0}^{d-1} dp^\mu_k
\]

\[
= \mathcal{D}p(\tau) \tag{B7}
\]

where in the last two lines we have used the property that the Jacobian of the transformation \( \frac{\partial(p^0_{A_k}, \ldots, p^{d-1}_{A_k})}{\partial(p^0_k, \ldots, p^{d-1}_k)} = \det[A_k] = 1 \).

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