Bias and Extrapolation in Markovian Linear Stochastic Approximation with Constant Stepsizes

Dongyan (Lucy) Huo,§ Yudong Chen,† Qiaomin Xie,‡

§School of Operations Research and Information Engineering, Cornell University
†Department of Computer Sciences, University of Wisconsin-Madison
‡Department of Industrial and Systems Engineering, University of Wisconsin-Madison

Abstract

We consider Linear Stochastic Approximation (LSA) with constant stepsize and Markovian data. Viewing the joint process of the data and LSA iterate as a time-homogeneous Markov chain, we prove its convergence to a unique limiting and stationary distribution in Wasserstein distance and establish non-asymptotic, geometric convergence rates. Furthermore, we show that the bias vector of this limit admits an infinite series expansion with respect to the stepsize. Consequently, the bias is proportional to the stepsize up to higher order terms. This result stands in contrast with LSA under i.i.d. data, for which the bias vanishes. In the reversible chain setting, we provide a general characterization of the relationship between the bias and the mixing time of the Markovian data, establishing that they are roughly proportional to each other.

While Polyak-Ruppert tail-averaging reduces the variance of LSA iterates, it does not affect the bias. The above characterization allows us to show that the bias can be reduced using Richardson-Romberg extrapolation with \( m \geq 2 \) stepsizes, which eliminates the \( m - 1 \) leading terms in the bias expansion. This extrapolation scheme leads to an exponentially smaller bias and an improved mean squared error, both theoretically and empirically. Our results immediately apply to the Temporal Difference learning algorithm with linear function approximation, and stochastic gradient descent applied to quadratic functions.

1 Introduction

We consider the following Linear Stochastic Approximation (LSA) iteration driven by Markovian data:

\[
\theta_{k+1} = \theta_k + \alpha (A(x_k)\theta_k + b(x_k)), \quad k = 0, 1, 2, \ldots,
\]

where \((x_k)_{k \geq 0}\) is a Markov chain representing the underlying data stream, \(A\) and \(b\) are deterministic functions, and \(\alpha > 0\) is a constant stepsize. LSA is an iterative data-driven procedure for approximating the solution \(\theta^*\) to the linear fixed point equation \(\tilde{A}\theta^* + \tilde{b} = 0\), where \(\tilde{A} := \sum_i \pi_i A(i), \tilde{b} := \sum_i \pi_i b(i)\), and \(\pi\) is the unique stationary distribution of the chain \((x_k)_{k \geq 0}\).

Stochastic Approximation (SA), which uses stochastic updates to solve fixed-point equations, is a fundamental algorithmic paradigm in many areas including stochastic control and filtering [KY03, Bor08], approximate dynamic programming and reinforcement learning (RL) [Ber19, SB18]. For example, the celebrated Temporal Difference (TD) learning algorithm [Sut88] in RL, potentially equipped with linear function approximation, is a special case of LSA and an important algorithm primitive in RL. Variants of the TD algorithm such as TD(\(\lambda\)) and Gradient TD can also be cast as LSA [LS18]. Another important special case of SA is stochastic gradient descent (SGD). When applied to a quadratic objective function, SGD can be viewed as an LSA procedure.

Classical work on SA focuses on settings with diminishing stepsizes, which allow for asymptotic convergence of \(\theta_k\) to \(\theta^*\) [RM51, Blh54, BM00]. More recently, SA with constant stepsizes has attracted

*Emails: dh622@cornell.edu, yudong.chen@wisc.edu, qiaomin.xie@wisc.edu
attention due to its simplicity, fast convergence, and good empirical performance. A growing line of work has been devoted to this setting and established non-asymptotic results valid for finite values of $k$ [LS18, SY19, CMSS21b, BRS21]. These results provide upper bounds on the mean-squared error (MSE) $\mathbb{E}\|\theta_k - \theta^*\|^2$, and such bounds typically consist of the sum of two terms: a finite-time term\(^1\) that decays with $k$, and a steady-state MSE bound that is independent of $k$.

In this work, we study LSA with constant stepsizes through the lens of Markov chain theory. This perspective allows us to delineate the convergence and distributional properties of LSA viewed as a stochastic process. Consequently, we provide a more precise characterization of the MSE in terms of the decomposition
\[
\mathbb{E}\|\theta_k - \theta^*\|^2 \approx \mathbb{E}\theta_k - \mathbb{E}\theta^*_\infty\|_2^2 + \text{variance} + \text{asymptotic bias}^2,
\]
where the random variable $\theta^*_\infty$ denotes the limit (as $k \to \infty$) of the LSA iterate $\theta_k$ with stepsize $\alpha$. Our main results, summarized below, characterize the behavior of the above three terms.

**Convergence and optimization error.** With a constant stepsize $\alpha$, the process $(x_k, \theta_k)_{k \geq 0}$ is a time-homogeneous Markov chain. Under appropriate conditions, we show that the sequence of $(x_k, \theta_k)$ converges to a unique limiting random variable $(x_\infty, \theta^*_\infty)$ in distribution and in $W_2$, the Wasserstein distance of order 2. Moreover, the distribution of $(x_\infty, \theta^*_\infty)$ is the unique stationary distribution of the chain $(x_k, \theta_k)_{k \geq 0}$. We provide non-asymptotic bounds on the distributional distance between $\theta_k$ and $\theta^*_\infty$ in $W_2$, which in turn upper bounds the optimization error $\mathbb{E}\|\theta_k - \theta^*_\infty\|_2$. Both bounds decay exponentially in $k$ thanks to the use of a constant stepsize. We emphasize that the existence of the limit $\theta^*_\infty$ and the convergence rate cannot be deduced from existing upper bounds on the MSE $\mathbb{E}\|\theta_k - \theta^*\|^2$, which does not vanish as $k \to \infty$.

**Variance and asymptotic bias.** The variance $\text{Var}(\theta_k)$ is of order $O(1)$ as $k$ grows. The variance can be made vanishing by averaging the LSA iterates. For example, the Polyak-Ruppert tail-averaged iterate $\bar{\theta}_k := \frac{1}{k/2} \sum_{t=k/2}^{k-1} \theta_t$ has variance of order $O(1/k)$. Consequently, for large $k$, the MSE of $\bar{\theta}_k$ is dominated by the asymptotic bias, i.e., $\mathbb{E}\|\bar{\theta}_k - \theta^*_\|_2^2 \approx \|\mathbb{E}\theta^*_\infty - \theta^*_\|_2^2 = \|\mathbb{E}\theta^*_\| - \theta^*_\|_2^2$. Our second main result establishes that the asymptotic bias is proportional to the stepsize $\alpha$ (up to a second order term):
\[
\mathbb{E}\theta^*_\| - \theta^*_\| = \alpha B^{(1)} + O(\alpha^2),
\]
where $B^{(1)}$ is a vector independent of $\alpha$ and admits an explicit expression in terms of $A, b$ and the transition kernel $P$ of the underlying Markov chain $(x_k)_{k \geq 0}$. Crucially, the first order term in equation (1.1) is an equality rather than an upper bound. This equality implies that the asymptotic bias is not affected by averaging the LSA iterates.

**Bias expansion and extrapolation.** While the asymptotic bias persists under Polyak-Ruppert averaging, the equality (1.1) implies that bias can be reduced using a simple technique called Richardson-Romberg (RR) extrapolation: run LSA with two stepsizes $\alpha$ and $2\alpha$, compute the respective averaged iterates $\bar{\theta}_k^{(\alpha)}$ and $\bar{\theta}_k^{(2\alpha)}$, and output their linear combination $\tilde{\theta}_k^{(\alpha)} := 2\bar{\theta}_k^{(\alpha)} - \bar{\theta}_k^{(2\alpha)}$. Doing so cancels out the leading term in the bias characterization (1.1), resulting in an order-wise smaller bias $\mathbb{E}\theta^*_\| - \theta^*_\| = O(\alpha^2)$.

In fact, the bias characterization (1.1) can be generalized to higher orders. We establish that the bias admits the following infinite series expansion:
\[
\mathbb{E}\theta^*_\| - \theta^*_\| = \alpha B^{(1)} + \alpha^2 B^{(2)} + \alpha^3 B^{(3)} + \cdots,
\]
where the vectors $\{B^{(i)}\}$ are independent of $\alpha$. Consequently, RR extrapolation can be executed with $m \geq 2$ stepsizes to eliminate the $m - 1$ leading terms in equation (1.2), reducing the bias to the order $O(\alpha^m)$.

\(^1\)This term is sometimes called “(finite-time) bias” in prior work. It should not be confused with the asymptotic bias discussed below.
Put together, the above results show that the combination of constant stepsize, averaging, and extrapolation allows one to approach the *best of three worlds*: (a) using a constant stepsize leads to fast, exponential-in-$k$ convergence of the optimization error, (b) tail-averaging eliminates the variance at an (optimal) $1/k$ rate, and (c) RR extrapolation order-wise reduces the asymptotic bias. We highlight that the $m$ iterate sequences used in RR extrapolation can be computed in parallel, using the same data stream $(x_k)_{k \geq 0}$. Compared with standard LSA, the above combined procedure is data efficient (in terms of the sample complexity $k$ for achieving a given MSE), does not require sophisticated tuning of the stepsize, and incurs a minimal increase in the computational cost.

The results above should be contrasted with the setting of LSA with i.i.d. data, where the $x_k$’s are sampled independently from some distribution $\pi$. In this setting, it has been shown (sometimes implicitly) in existing work that the asymptotic bias is zero [LS18, MLW$^+$20]. Similar results are known for SGD applied to a quadratic function with gradients contaminated by independent noise [BM13, DDB20]. It is perhaps surprising that using Markovian data leads to a non-zero bias, even when the LSA iteration is linear in $\theta_k$. To provide intuition, we plot the dependency graphs for LSA with i.i.d. data and Markovian data in Figure 1. It can be seen that in the Markovian setting, the correlation between $x_k$’s leads to additional correlation among $\theta_k$’s; in particular, the iterate sequence $(\theta_k)_{k \geq 0}$ is no longer a Markov chain by itself. As such, $\theta_{k+1}$ has an implicit, nonlinear dependence on $\theta_k$ through $(x_k, x_{k+1})$. This non-linearity is the source of the asymptotic bias.

Bias and mixing time. We quantify the observations above by relating the magnitude of the asymptotic bias to the mixing time of the underlying Markov chain $(x_k)_{k \geq 0}$ and the absolute spectral gap of the transition kernel $P$, denoted by $\gamma^*$($P$). We show that the leading coefficient $B^{(1)}$ in the expansion (1.2) has a norm upper bounded by $O(\frac{1}{\gamma^*(P)})$. It is well known that the mixing time of $(x_k)_{k \geq 0}$ can be tightly upper and lower bounded by functions of $\gamma^*(P)$ [Pau15, LP17]. Consequently, the faster the underlying chain $(x_k)_{k \geq 0}$ mixes, the smaller the asymptotic bias is. As a special case, LSA with i.i.d. data has zero mixing time and $1 - \gamma^*(P) = 0$, hence zero bias.

Our results hold for LSA driven by a Markov chain $(x_k)_{k \geq 0}$ on a general (possibly continuous) state space. These results immediately apply to Markovian settings of the TD algorithm with linear function approximation and SGD for quadratic functions. Furthermore, we provide numerical results for LSA, TD and SGD, which corroborate our theory and demonstrate the benefit of using constant stepsizes, tail averaging, and RR extrapolation.

**Paper Organization:** In Section 2, we review existing results related to our work. After setting up the problem and assumptions in Section 3, we present our main results in Section 4. In Section 5, we provide numerical results for LSA, TD and SGD. We outline the proofs of the main results in Section 6. The paper is concluded in Section 7 with a discussion of future directions. The proofs of our theoretical results are provided in the Appendix.

## 2 Related Work

In this section, we review existing results that are most related to our work. The literature on SA and SGD is vast. Here we mainly discuss prior works in the non-asymptotic and constant stepsize regime, with a focus on the Markovian noise setting.
2.1 Classical Results on SA and SGD

The study of stochastic approximation and stochastic gradient descent dates back to the seminal work of Robbins and Monro [RM51]. Convergence results in classical works typically assume that the stepsize sequence \((\alpha_k)_{k \geq 1}\) satisfies \(\sum_{k=1}^{\infty} \alpha_k = \infty\) and \(\sum_{k=1}^{\infty} \alpha_k^2 < \infty\). This assumption implies that the stepsize sequence is diminishing. Under suitable conditions, Robbins and Monro prove that SA and SGD algorithms asymptotically converge in the \(L^2\) sense [RM51], and Blum shows that the convergence holds almost surely [Blu54]. Subsequent works [Rup88, Pol90] propose the technique now known as the Polyak-Ruppert (PR) averaging. A Central Limit Theorem (CLT) for the asymptotic normality of the averaged iterates is established in [PJ92]. Borkar and Meyn [BM00] introduce the Ordinary Differential Equation (ODE) technique for analyzing SA algorithms. Utilizing the ODE technique, the recent work [BCD+21] establishes a functional CLT for SA driven by Markovian noise.

The asymptotic theory of SA and SGD is well-developed and covered in several excellent textbooks [KY03, Bor08, BMP12, WR22]. Our work, in comparison, focuses on the setting of constant stepsizes and provides non-asymptotic bounds.

2.2 SA and SGD with Constant Stepsizes

A growing body of recent works considers the constant stepsize setting of SA and the closely related SGD algorithm. A majority of works in this line assume i.i.d. noise, with a number of finite-time results. The work in [LS18] analyzes LSA and establishes finite-time upper and lower bounds on the MSE. The work [MLW+20] provides refined results with the optimal dependence on problem-specific constants, as well as a CLT for the averaged iterates with an exact characterization of the asymptotic covariance matrix. Using new results on random matrix products, the work [DMN+21] establishes tight concentration bounds of LSA, which are extended to LSA with iterate averaging in [DMNS22].

Closely related to our work is [DDB20], which studies constant stepsize SGD for strongly convex and smooth functions. By connecting SGD to time-homogeneous Markov chains, they establish that the iterates converge to a unique stationary distribution. This result is generalized to non-convex and non-smooth functions with quadratic growth in the work [YBVE21], which further establishes asymptotic normality of the averaged SGD iterates. Subsequent work [CMM22] studies the limit of the stationary distribution as stepsize goes to zero. Note that these aforementioned results are established under the i.i.d. noise setting.

More recent works study constant-stepsize SA and SGD under Markovian noise. The work [SY19] provides finite-time bounds on the MSE of LSA. The work [MPWB21] considers LSA with averaging and establishes instance-dependent MSE upper bounds with tight dimension dependence. The papers [SY19, DMNS22] establish bounds on higher moments of LSA iterates. Going beyond linear SA, the work [CMSS20] considers general SA with contractive mapping and provides finite-time convergence results. The work [BJN+20] studies SGD for linear regression problems with Markovian data and constant stepsizes. Most of these results focus on the upper bounds of the MSE and do not decouple the effect of the asymptotic bias.

A portion of our results are similar in spirit to [DDB20, Proposition 2] and [DMN+21, Theorem 3], in that we both study LSA and SGD with constant stepsizes in the lens of Markov chain analysis. A crucial difference is that we consider the Markovian data setting whereas they consider i.i.d. data. Arising naturally in stochastic control and RL problems, the Markovian setting leads to non-zero asymptotic bias and new analytical challenges, which are not present in the i.i.d. setting. In particular, our analysis involves more delicate coupling arguments and builds on the Lyapunov function techniques from [SY19]. Along the way, we obtain a refinement of the MSE bounds from the work [SY19]. We discuss these analytical challenges and improvements in greater detail after stating our theorems; see Sections 4 and 6.

2.3 Applications in Reinforcement Learning and TD Learning

Many iterative algorithms in RL solve for the fixed point of Bellman equations and can be viewed as special cases of SA [SB18, Ber19]. The TD algorithms [Sut88] with linear function approximation, including TD(0) and more generally TD(\(\lambda\)), are LSA procedures. Our results can be specialized to TD learning and hence are related to existing works in this line.

Classical results on TD Learning, similarly to those on SA, focus on asymptotic convergence under diminishing stepsizes [Sut88, Day92, DS94, TVR97]. More recent works provide finite-time results.
work [DSTM18] is among the first to provide MSE and concentration bounds for linear TD learning in its original form, and their analysis assumes diminishing stepsize and i.i.d. noise. The work [BRS21] presents finite-time analysis of TD(0) under both i.i.d. and Markovian noise, with both diminishing and constant stepsizes. Their results require a projection step in TD(0) to ensure boundedness. The Lyapunov analysis in [SY19] on LSA, when specialized to TD(0), removes this projection step and proves upper bounds on the MSE. The recent work in [CMSS21a, CMSS21b] uses Lyapunov theory to study the tabular TD and obtains finite sample convergence guarantees. The paper [KPR+21] provides sharp, instance-dependent $\ell_\infty$ error bounds for the tabular TD algorithm with i.i.d. data.

We mention in passing that Q-learning [WD92], another standard algorithm in RL, can be viewed as a nonlinear SA procedure with contractive mappings. Q-learning has been studied in both classical and recent work, e.g., [Tsi94, Sze97, EDB03] and [CMSS21b, CBD22]. Generalizing our results to nonlinear SA and Q-learning is an interesting future direction.

## 3 Set-up and Assumptions

We formally set up the problem and introduce the assumptions and notations used in the sequel.

### 3.1 Problem Set-up

Let $(x_k)_{k \geq 0}$ be a Markov chain on a general state space $\mathcal{X}$. Consider the following linear stochastic approximation iteration:

$$
\theta_{k+1} = \theta_k + \alpha \left( A(x_k)\theta_k + b(x_k) \right), \quad k = 0, 1, \ldots, 
$$

(3.1)

where $A : \mathcal{X} \to \mathbb{R}^{d \times d}$ and $b : \mathcal{X} \to \mathbb{R}^d$ are deterministic functions, and $\alpha > 0$ is a constant stepsize. In what follows, we omit the superscript in $\bar{\theta}_k^{(a)}$ when the dependence on $\alpha$ is clear from the context. The initial distribution of $\theta_0$ may depend on $x_0$ but is independent of $(x_k)_{k \geq 1}$ given $x_0$.

Let $\pi$ be the stationary distribution of the Markov chain $(x_k)$ and define the shorthands

$$
\bar{A} := \mathbb{E}_\pi[A(x)] \in \mathbb{R}^{d \times d} \quad \text{and} \quad \bar{b} := \mathbb{E}_\pi[b(x)] \in \mathbb{R}^d,
$$

(3.2)

where $\mathbb{E}_\pi[\cdot]$ denotes the expectation with respect to $x \sim \pi$. The iterative procedure (3.1) computes an approximation of the target vector $\theta^*$, defined as the solution to the steady-state equation

$$
\bar{A}\theta + \bar{b} = 0.
$$

(3.3)

Our general goal is to characterize the relationship between the iterate $\theta_k$ and the target solution $\theta^*$.

The stochastic process of the LSA iterates, $(\theta_k)_{k \geq 0}$, is not a Markov chain itself: given $\theta_k$, the random variables $\theta_{k+1}$ and $\theta_{k-1}$ are correlated through the underlying Markov process $(x_0, x_1, \ldots, x_k)$. However, direct calculation verifies that the joint process $(x_k, \theta_k)_{k \geq 0}$ is a Markov chain on the state space $\mathcal{X} \times \mathbb{R}^d$. This chain is time-homogeneous as the stepsize $\alpha$ is independent of $k$.

**Remark 1.** The LSA iteration (3.1) covers as a special case the SGD algorithm applied to minimizing a quadratic function $f(\theta) = -\frac{1}{2}\theta^T \bar{A}\theta - \bar{b}\theta = \mathbb{E}_\pi \left[ -\frac{1}{2}\theta^T A(x)\theta - b(x) \right]$, where $-\bar{A}$ is the symmetric expected Hessian matrix. Note that LSA is more general than SGD for quadratic minimization, as $A(x)$ need not be symmetric, in which case $\theta \mapsto -(\bar{A}\theta + \bar{b})$ is not a gradient field.

**Remark 2.** One primary advantage of LSA (and SGD) is the low computational cost of the iteration (3.1), particularly in forming the matrix-vector product $A(x_k)\theta_k$. For example, in TD(0) the matrix $A(x_k)$ is rank-one (and in addition 2-sparse in the tabular case); see Section 4.4. In comparison, the expected matrix $\bar{A}$, as well as its running empirical estimate $\bar{A}_k := \frac{1}{k} \sum_{i=0}^{k-1} A(x_i)$, are typically dense and full-rank.

### 3.1.1 General State Space Markov Chains

As the underlying Markov chain $(x_k)_{k \geq 0}$ is on a general state space $\mathcal{X}$, we review some relevant concepts and notations. We assume throughout the paper that the $\mathcal{X}$ is Borel, i.e., the $\sigma$-algebra $\mathcal{B}(\mathcal{X})$ is Borel.
Let $P$ denote the transition kernel of the chain. A distribution $\pi$ is the stationary/invariant distribution if $\int_X \pi(dx)P(x,B) = \pi(B), \forall B \in \mathcal{B}(X)$. For a function $f : \mathcal{X} \rightarrow \mathbb{R}^d$, we write $\pi(f) = \int_X \pi(dx)f(x)$. Define the $\pi$-weighted inner product $(f,g)_{L^2(\pi)} = \int_X \pi(dx)f^\top(x)g(x)$ and the induced norm $\|f\|_{L^2(\pi)} = ((f,f)_{L^2(\pi)})^{1/2}$.

Let $L^2(\pi) = \{ f : \|f\|_{L^2(\pi)} < \infty \}$ denote the corresponding Hilbert space of $\mathbb{R}^d$-valued, square-integrable, and measurable functions on $\mathcal{X}$.\footnote{As customary, two functions $f$ and $g$ are identified as the same element in $L^2(\pi)$ if they are equal $\pi$-almost everywhere, i.e., $\|f-g\|_{L^2(\pi)} = 0$.}

For an operator $T : L^2(\pi) \rightarrow L^2(\pi)$, its operator norm is defined as $\|T\|_{L^2(\pi)} = \sup_{\|f\|_{L^2(\pi)} = 1} \|Tf\|_{L^2(\pi)}$. The transition kernel $P$ is a bounded linear operator on $L^2(\pi)$ with norm $\|P\|_{L^2(\pi)} = 1$. Also, define the kernel/operator $\Pi = 1_{\mathcal{X}} \times \pi - \pi \times 1_{\mathcal{X}}$ and the kernel/operator $\Pi = 1 \otimes \pi$ by $\Pi(x,\cdot) = \pi$; equivalently, $(\Pi f)(x) = \pi f, \forall x \in \mathcal{X}$.

When the state space $\mathcal{X}$ is Borel, there exists a kernel $P^*$ as a regular conditional probability that satisfies $\int_A \pi(dx)P(x,B) = \int_B \pi(dy)P^*(y,A), \forall A,B \in \mathcal{B}(X)$, and $P^*$ defines the probability law for the time-reversed chain of $(x_k)_{k \geq 0}$ \cite[Chapter 21.4, Theorem 19]{FG97}. Moreover, $P^*$ is the adjoint operator to $P$ in $L^2(\pi)$, i.e., $(f,Pg)_{L^2(\pi)} = (P^*f,g)_{L^2(\pi)}$. The Markov chain $(x_k)_{k \geq 0}$ is called reversible with respect to $\pi$ if $P$ is self-adjoint, i.e., $P^* = P$.

Define the spectrum of $P$ as $\text{Spec}(P) = \{ \lambda \in \mathbb{C} \setminus 0 : (\lambda I - P)^{-1} \text{ does not exist as a bounded linear operator on } L^2(\pi) \}$. The set $\text{Spec}(P)$ contains the eigenvalues of $P$. The absolute spectral gap of $P$ is defined as

$$
\gamma^*(P) = \begin{cases} 
1 - \sup\{|\lambda| : \lambda \in \text{Spec}(P), \lambda \neq 1\} & \text{if eigenvalue 1 has multiplicity 1,} \\
0 & \text{otherwise.}
\end{cases}
$$

(3.4)

When $P$ has a unique invariant distribution $\pi$, the eigenvalue 1 has multiplicity 1 \cite[Proposition 22.1.2]{DMPS18} and hence $\gamma^*(P) > 0$. When $P$ is reversible with respect to $\pi$, Spec($P$) lies on the real line and we have the expression $\gamma^*(P) = 1 - \|P - \Pi\|_{L^2(\pi)}$.

We remark that all definitions above coincide with the familiar ones when the state space $\mathcal{X}$ is finite. For example, we have $\gamma^*(P) = 1 - |\lambda_2(P)|$, where $|\lambda_2(P)|$ is the second largest eigenvalue modulus (SLEM) of the transition probability matrix $P$.

### 3.2 Assumptions

We now state the assumptions needed for our main theorems.

**Assumption 1.** $(x_k)_{k \geq 0}$ is a uniformly ergodic Markov chain on a Borel state space $(\mathcal{X}, \mathcal{B}(X))$ with transition kernel $P$ and a unique stationary distribution $\pi$. The initial state $x_0$ is drawn from $\pi$.

Recall that a Markov chain is called uniformly ergodic if $\sup_{x \in \mathcal{X}} \|P^n(x,\cdot) - \pi\|_{TV} \rightarrow 0$ as $n \rightarrow \infty$, where $\| \cdot \|_{TV}$ is the total variation norm. A uniformly ergodic chain satisfies the following seemingly stronger condition \cite[Theorem 16.0.2]{MT09}: there exist constants $r \in [0,1)$ and $R > 0$ such that

$$
\sup_{x \in \mathcal{X}} \|P^k(x,\cdot) - \pi\|_{TV} \leq Rr^k,
$$

(3.5)

that is, the chain converges to $\pi$ from any initial $x_0$ at a uniform geometric rate. All irreducible, aperiodic, and finite state space Markov chains are uniformly ergodic. Uniform ergodicity also allows for the chain to have transient states in addition to a single recurrent class. The uniform ergodicity assumption is used in the prior work \cite{BRS21, DT22, DMNS22}. It is possible to further relax this assumption (e.g., as in \cite{SY19, MPWB21}); we do not pursue this direction in this paper.

The additional stationarity assumption $x_0 \sim \pi$, which has been used in a number of previous papers \cite{BRS21, MPWB21}, is imposed merely to simplify several mathematical expressions. This assumption is not essential; it can be removed by applying our analysis to the joint Markov chain $(x_k, \theta_k)_{k \geq 0}$ after the marginal $(x_k)_{k \geq 0}$ has approximately mixed, which happens quickly thanks to the geometric mixing property (3.5).

Our following two assumptions are similar to those in the work \cite{SY19, DMNS22}. Let $\| \cdot \|$ denote the Euclidean norm for vectors and the spectral norm (i.e., the largest singular value) for matrices.
Assumption 2. It holds that
\[ A_{\max} := \sup_{x \in \mathcal{X}} \|A(x)\| \leq 1 \quad \text{and} \quad b_{\max} := \sup_{x \in \mathcal{X}} \|b(x)\| < \infty. \]

Assumption 2 implies the bounds \( \|A\| \leq A_{\max} \leq 1 \) and \( \|b\| \leq b_{\max} \). The constant 1 in the assumption is chosen for convenience and can be relaxed to other positive constants by rescaling the LSA update (3.1).

Playing an important role in our analysis is the mixing time of the Markov chain \((x_k)_{k \geq 0}\) with respect to the functions \(A(\cdot)\) and \(b(\cdot)\), defined as follows.

**Definition 3.1.** For \( \epsilon \in (0, 1) \), the \( \epsilon \)-mixing time of \((x_k)_{k \geq 0}\) with respect to \((A, b)\) is defined to be the smallest number \( \tau_{\epsilon} \geq 1 \) satisfying
\[
\begin{align*}
\|\mathbb{E}[A(x_k) \mid x_0 = x] - A\| &\leq \epsilon \cdot A_{\max}, \quad \forall x \in \mathcal{X}, \forall k \geq \tau_{\epsilon}, \\
\|\mathbb{E}[b(x_k) \mid x_0 = x] - b\| &\leq \epsilon \cdot b_{\max}, \quad \forall x \in \mathcal{X}, \forall k \geq \tau_{\epsilon}.
\end{align*}
\]

Under Assumptions 1 and 2, the \( \epsilon \)-mixing time satisfies \( \tau_{\epsilon} \leq K \log \frac{1}{\epsilon} \) for all \( \epsilon \in (0, 1) \), where the number \( K \geq 1 \) is independent of \( \epsilon \). This fact can be seen in the inequality
\[
\|\mathbb{E}[A(x_k) \mid x_0 = x] - A\| \leq A_{\max} \left( 2 \sup_{x \in \mathcal{X}} \|P^k(x, \cdot) - \pi\|_{TV} \right) \leq 2A_{\max}R\tau_{\epsilon}^k,
\]
where the last step follows from equation (3.5); a similar argument applies to \( b(x_k) \).

In the sequel, unless specified otherwise, we always choose \( \epsilon = \alpha \) and write \( \tau \equiv \tau_{\alpha} \).

**Assumption 3.** The matrix \( \bar{A} \) is Hurwitz, i.e., all eigenvalues have strictly negative real parts.

Assumption 3 is standard in the study of the stability of dynamical systems. Under this assumption, it is well known that there exists a symmetric positive definite matrix \( \Gamma \) satisfying the Lyapunov equation \( \bar{A}^\top \Gamma + \Gamma \bar{A} = -I \), where \( I \) is the \( d \)-by-\( d \) identity matrix. Denote by \( \gamma_{\min} \) and \( \gamma_{\max} \) the minimum and maximum eigenvalues of the matrix \( \Gamma \) respectively. We have \( \gamma_{\min} > 0 \) and
\[
\gamma_{\min} \|v\|^2 \leq v^\top \Gamma v \leq \gamma_{\max} \|v\|^2, \quad \forall v \in \mathbb{R}^d.
\]
Moreover, the matrix \( \bar{A} \) is invertible and with smallest singular value \( s_{\min}(\bar{A}) > 0 \), and the target solution \( \theta^* \) to steady-state equation (3.3) is unique.

### 3.3 Notations

In general, we adopt the notational convention that upper-case letters (e.g., \( A \)) denote matrices and lowercase letters (e.g., \( b \)) denote vectors or scalars. The lowercase letter \( c \) and its derivatives \( c', c_0, \text{etc.} \) denote universal numerical constants, whose values may change from line to line. Recall that \( \| \cdot \| \) denotes the Euclidean norm for vectors and the spectral norm for matrices, and \( \| \cdot \|_{L^2(\pi)} \) denotes the norm on \( L^2(\pi) \) and the induced operator norm.

We generally use \( B \equiv B(A, b, P) \) and its derivatives to denote quantities (vectors or matrices) that depend only on \( A, b, \) and \( P \), but independent of the stepsize \( \alpha \) and the iteration index \( k \). As we are primarily interested in how various quantities scale with \( \alpha \) and \( k \), we make use of the following big-O notation: for a quantity \( h \) that may depend on all problem parameters, we write \( h = O(f(\alpha, k)) \) if it holds that \( \|h\| \leq B(A, b, P) \cdot f(\alpha, k) \) for some \( B(A, b, P) \) independent of \( \alpha \) and \( k \), where \( f \) is a function of \( \alpha \) and \( k \). For example, \( h = O(\alpha/k) \) means \( \|h\| \leq B(A, b, P) \cdot \alpha/k \). In addition, when presenting results on the relationship to mixing time, we use \( C \equiv C(A, b, \pi) \) to denote a quantity that may depend on \( \pi \) but not other properties of \( P \) (e.g., its mixing time and spectral gap).

We use \( \mathcal{L}(z) \) to denote the law of a random variable \( z \), and \( \text{Var}(z) \) its covariance matrix. We write \( z_1 \perp \perp z_2 \mid z_3 \) if the random variables \( z_1 \) and \( z_2 \) are conditionally independent given \( z_3 \). Let \( \mathcal{P}_2(\mathbb{R}^d) \) be the space of square-integrable distributions on \( \mathbb{R}^d \). Let \( \mathcal{P}_2(\mathcal{X} \times \mathbb{R}^d) \) be the set of distributions \( \nu \) on \( \mathcal{X} \times \mathbb{R}^d \) with the property that the marginal of \( \nu \) on \( \mathbb{R}^d \) is square-integrable.
4 Main Results

In this section, we present our main results. In Section 4.1, we study the convergence of the LSA iterates \((x_k, \theta_k)_{k \geq 0}\) to a unique limiting distribution. In Section 4.2, we characterize the above limit and its relationship with the stepsize and mixing time. We explore the implications of these results for PR tail averaging and RR extrapolation in Section 4.3. We apply our results to TD(0) Learning in Section 4.4 and to SGD in Section 4.5.

4.1 Convergence to Limit Distribution

Our convergence results are based on the Wasserstein distance [Vil09]. The Wasserstein distance of order 2 between two probability measures \(\mu\) and \(\nu\) in \(\mathcal{P}_2(\mathbb{R}^d)\) is defined as

\[
W_2(\mu, \nu) = \inf_{\xi \in \Pi(\mu, \nu)} \left( \int_{\mathbb{R}^d} \|u - v\|^2 \, d\xi(u, v) \right)^{1/2}
\]

where \(\Pi(\mu, \nu)\) denotes the set of all couplings between \(\mu\) and \(\nu\), i.e., the collection of joint distributions in \(\mathcal{P}_2(\mathbb{R}^d \times \mathbb{R}^d)\) with marginal distributions \(\mu\) and \(\nu\). To study the joint process \((x_k, \theta_k)_{k \geq 0}\), we extend the above distance to the space \(\mathcal{P}_2(\mathcal{X} \times \mathbb{R}^d)\). Under the uniform ergodicity Assumption 1, it is natural to metricize the space \(\mathcal{X} \times \mathbb{R}^d\):

\[
d((x, \theta), (x', \theta')) := \sqrt{d_0(x, x') + \|\theta - \theta'\|^2}.
\]

For a pair of distributions \(\bar{\mu}\) and \(\bar{\nu}\) in \(\mathcal{P}_2(\mathcal{X} \times \mathbb{R}^d)\), we consider the Wasserstein-2 distance w.r.t. \(d\):

\[
W_2(\bar{\mu}, \bar{\nu}) = \inf \left\{ \left( \mathbb{E}[d_0(x, x') + \|\theta - \theta'\|^2] \right)^{1/2} : \mathcal{L}(x, \theta) = \bar{\mu}, \mathcal{L}(x', \theta') = \bar{\nu} \right\}.
\]

It follows immediately from definition that \(W_2(\mathcal{L}(\theta), \mathcal{L}(\theta')) \leq W_2(\mathcal{L}(x, \theta), \mathcal{L}(x', \theta'))\). Note that convergence in \(W_2\) or \(W_2\) implies the usual convergence in distribution plus the convergence of the first two moments [Vil09, Definition 6.8 and Theorem 6.9].

Our first theorem establishes the convergence of the Markov chain \((x_k, \theta_k)_{k \geq 0}\) in \(W_2\) to a unique stationary distribution and characterizes the geometric convergence rate.

**Theorem 4.1.** Suppose that Assumptions 1, 2 and 3 hold, and the stepsize \(\alpha\) satisfies

\[
\alpha \tau_\alpha < \frac{0.05}{95 \gamma_{\max}}.
\]

1. Under all initial distributions of \(\theta_0\), the sequence of random variables \((x_k, \theta_k)_{k \geq 0}\) converges in \(W_2\) to a unique limit \((x_\infty, \theta_\infty) \sim \bar{\mu}\). Moreover, it holds that

\[
\text{Tr}(\text{Var}(\theta_\infty)) \leq \alpha \tau_\alpha \kappa, \quad \text{where } \kappa := \frac{\gamma^2_{\max} \cdot s_{\min}^2(\bar{A}) \gamma_{\max}^2}{\gamma_{\min}},
\]

2. \(\bar{\mu}\) is the unique stationary distribution of the Markov chain \((x_k, \theta_k)_{k \geq 0}\).

3. Let \(\mu := \mathcal{L}(\theta_\infty)\) be the second marginal of \(\bar{\mu}\). For all \(k \geq \tau_\alpha\), it holds that

\[
W_2^2(\mathcal{L}(\theta_k), \mu) \leq W_2^2(\mathcal{L}(x_k, \theta_k), \bar{\mu}) \leq 20 \frac{\gamma_{\max}^2}{\gamma_{\min}} \left( \mathbb{E}[\|\theta_0 - \mathbb{E}[\theta_\infty]\|^2] + \text{Tr}(\text{Var}(\theta_\infty)) \right) \left(1 - \frac{0.9 \alpha}{\gamma_{\max}}\right)^k.
\]

We outline the proof of Theorem 4.1 in Section 6.2, deferring the complete proof to Appendix A.2. This convergence result is valid under the stepsize condition (4.2), stated as an upper bound on the product \(\alpha \tau_\alpha\). Since \(\tau_\alpha \leq K \log \frac{1}{\alpha}\) for some constant \(K \geq 1\) independent of \(\alpha\) (see Section 3.2), the condition (4.2) is
satisfied for sufficiently small $\alpha$. Note that the limiting distribution $\mu$ is in general not a product distribution of its marginals $\pi$ and $\mu$.

We remark on the techniques for proving Theorem 4.1. To establish the convergence of a Markov chain to a unique stationary distribution, a standard approach is to show that the chain is positive recurrent by verifying irreducibility and Lyapunov drift conditions. This approach has been developed for Markov chains on general state spaces [MT09] and is adopted in the prior work [YBVE21, BCD+21, LM22]. However, it is unclear how to implement this approach for the general LSA iteration (3.1). For example, suppose that the stepsize $\alpha$ and the functions $A$ and $b$ take on rational values. If the initial $\theta_0$ is rational, then $\theta_k$ remains irrational. As such, it seems challenging to certify $\psi$-irreducibility and recurrence for the chain $(x_k, \theta_k)_{k \geq 0}$. Instead, we prove weak convergence through the convergence in the Wasserstein distance, which can be bounded via coupling arguments. The Wasserstein distance has also been used in the prior work [DDB20, BJN21] to study SGD and LSA in the i.i.d. data setting. Their analysis relies heavily upon the i.i.d. assumption and the one-step contraction property $W^2_2(\mathcal{L}(\theta_{k+1}), \mu) < W^2_2(\mathcal{L}(\theta_k), \mu)$. Establishing this property in our Markovian setting is difficult if not impossible—we elaborate in Section 6.2. Our proof makes use of a different and more delicate coupling argument.

As a corollary of Theorem 4.1, we obtain geometric convergence for the first two moments of $\theta_k$. 

**Corollary 4.2.** Under the setting of Theorem 4.1, for all $k \geq \tau_\alpha$ we have

$$
\|E[\theta_k] - E[\theta_\infty]\| \leq C \left(1 - \frac{0.9\alpha}{\gamma_{\text{max}}}\right)^{k/2} \quad \text{and} \quad \|E[\theta_k \theta_k^\top] - E[\theta_\infty \theta_\infty^\top]\| \leq C' \left(1 - \frac{0.9\alpha}{\gamma_{\text{max}}}\right)^{k/2},
$$

for some $C \equiv C(A, b, \pi)$ and $C' \equiv C'(A, b, \pi)$ that are independent of $\alpha$ and $k$.

The proof of Corollary 4.2 is given in Appendix A.3.

### 4.2 Expansion and Characterization of Bias

Having shown that $\theta_k^{(\alpha)}$ converges in distribution to a limit $\theta_\infty^{(\alpha)}$, our next theorem characterizes the asymptotic bias $E[\theta_\infty^{(\alpha)}] - \theta^*$. Some additional notations are needed to write down an explicit expression for the bias. Recall that $L^2(\pi)$ is the set of $\mathbb{R}^d$-valued square-integrable functions on $\mathcal{X}$, $P^*$ is the adjoint of $P$ as an operator on $L^2(\pi)$, and $\Pi = 1 \otimes \pi$. Let $D : L^2(\pi) \to L^2(\pi)$ denote the operator given by $(Df)(x) = A(x)f(x)$ for each $x \in \mathcal{X}$, $f \in L^2(\pi)$, with its normalized version $\hat{D}$ is given by $(\hat{D}f)(x) = A^{-1}A(x)f(x)$. We define the operator $\Xi : L^2(\pi) \to L^2(\pi)$, the function $\nu : \mathcal{X} \to \mathbb{R}^d$ and the vectors $B^{(i)} \in \mathbb{R}^d$, $i = 1, 2, \ldots$ as

$$
\Xi := (I - P^* + \Pi)^{-1}(P^* - \Pi)D(I - \Pi \hat{D}), \tag{4.6a}
$$

$$
\nu := (I - P^* + \Pi)^{-1}(P^* - \Pi)(A^\omega + b), \tag{4.6b}
$$

$$
B^{(i)} := -\pi \hat{D} \Xi^{-1} \nu. \tag{4.6c}
$$

Note that $\Xi$, $\nu$ and $B^{(i)}$ are independent of $\alpha$.

**Theorem 4.3.** Suppose that Assumptions 1, 2 and 3 hold, and $\alpha$ satisfies equation (4.2). Then $\Xi$, $\nu$ and $B^{(i)}$, $i = 1, 2, \ldots$ are bounded operators on $L^2(\pi)$. Moreover:

1. For each $m = 1, 2, \ldots$, we have the expansion

$$
E[\theta_\infty^{(\alpha)}] - \theta^* = \sum_{i=1}^{m} \alpha^i B^{(i)} + \mathcal{O}(\alpha^{m+1}). \tag{4.7}
$$

2. Suppose in addition that $\alpha < 1/\|\Xi\|_{L^2(\pi)}$. We have the infinite series expansion

$$
E[\theta_\infty^{(\alpha)}] - \theta^* = \sum_{i=1}^{\infty} \alpha^i B^{(i)} = -\alpha \pi \hat{D}(I - \alpha \Xi)^{-1} \nu. \tag{4.8}
$$

\footnotetext{See equation (A.46) for an explicit sufficient condition for $\alpha < 1/\|\Xi\|_{L^2(\pi)}$.}
Theorem 4.3 is akin to a Taylor series expansion of \( \mathbb{E}[\theta_\infty^{(\alpha)}] \) with respect to the stepsize \( \alpha \). The existence of such an expansion is non-trivial: \( \theta_\infty^{(\alpha)} \) is undefined at \( \alpha = 0 \), and it is not clear a priori whether \( \mathbb{E}[\theta_\infty^{(\alpha)}] \) is a differentiable and analytic function of \( \alpha \). We emphasize that equations (4.7) and (4.8) are equalities, hence the bias is non-zero whenever \( B^{(i)} \neq 0 \) for some \( i \geq 1 \). In particular, averaging the LSA iterates \( \theta_k \) does not affect this bias and only reduces the variance.

The proof of Theorem 4.3, outlined in Section 6.3 and completed in Appendix A.4, is based on the following idea. As discussed in Section 1, the asymptotic bias arises due to the implicit nonlinear dependence between \( \theta_{k+1} \) and \( \theta_k \) as both of them depend on the state \( x_k \) of the underlying Markov chain. If \( \theta_k \) were independent of \( x_k \), the bias would be zero. This observation suggests that the bias is determined by the strength of dependence between \( \theta_k \) and \( x_k \), which can be quantified by the variation of the conditional expectation \( \mathbb{E}[\theta_k | x_k = x] \) as a function of \( x \in \mathcal{X} \). Therefore, our analysis is based on understanding this conditional expectation in steady state, namely \( \mathbb{E}[\theta_\infty | x_\infty = x] \). We characterize this quantity using the Basic Adjoint Relationship (BAR) [Har85, HWS7, DD11] for the steady state with a specific choice of test functions.

Theorem 4.3 provides an explicit expression (4.6) for the coefficients \( B^{(i)} \) in the bias expansion. In Sections 4.2.1 and 4.2.2 below, we use this expression to further characterize the magnitude of the bias and its relationship to the mixing time of \((x_k)_{k \geq 0}\). On the other hand, even without knowing the functional form of \( B^{(i)} \), we can still use Richardson-Romberg extrapolation to cancel out the lower order terms of \( \alpha \) in the expansions (4.7) and (4.8), reducing the bias to a higher order term of \( \alpha \). These results are presented in Section 4.3.

4.2.1 Bias and Mixing Time

As mentioned, the bias \( \mathbb{E}[\theta_\infty] - \theta^* \) arises due to the Markovian correlation in \((x_k)_{k \geq 0}\). If the chain \((x_k)_{k \geq 0}\) mixes quickly, the correlation is weak and intuitively one should expect a small bias. We now rigorously quantify the relationship between the bias and the mixing time of \((x_k)_{k \geq 0}\).

We focus on the setting where the chain \((x_k)_{k \geq 0}\) is reversible, i.e., \( P^* = P \), and relate the bias to the absolute spectral gap \( \gamma^*(P) \). The gap \( \gamma^*(P) \) is in turn related to the mixing time \( \tau_\epsilon \) via

\[
\frac{1 - \gamma^*(P)}{\gamma^*(P)} \cdot K' \log(1/\epsilon) \leq \tau_\epsilon \leq \frac{1}{\gamma^*(P)} \cdot K'' \log(1/\epsilon),
\]

where \( K' \) and \( K'' \) are independent of \( \epsilon \), inequality (i) holds for general state space \( \mathcal{X} \), and inequality (ii) is valid for finite \( \mathcal{X} \) [Pau15, Proposition 3.3]. Theorem below provides upper bounds on the coefficients \( B^{(i)} \) in the bias expansions (4.7)–(4.8) in terms of \( \gamma^*(P) \).

**Theorem 4.4.** Suppose that Assumptions 1, 2 and 3 hold, \( \alpha \) satisfies equation (4.2), and the Markov chain \((x_k)_{k \geq 0}\) is reversible. For each \( i = 1, 2, \ldots \), we have

\[
\|B^{(i)}\| \leq \left( C \cdot \frac{1 - \gamma^*(P)}{\gamma^*(P)} \right)^i
\]

for some number \( C \equiv C(A, b, \pi) > 0 \) that depends only on \( A, b, \pi \).

Theorem 4.4 together with the bias expansion (4.8) imply the bound

\[
\|\mathbb{E}[\theta_\infty] - \theta^*\| \leq 2C \cdot \alpha \frac{1 - \gamma^*(P)}{\gamma^*(P)}
\]

for a small stepsize \( \alpha \). In light of the relationship (4.9), we see that the bias is roughly proportional to the product of \( \alpha \) and \( \tau_\epsilon \). In the extreme case where the \( x_k \)'s are independent with distribution \( \pi \), i.e., \( P = 1 \otimes \pi \), we have \( 1 - \gamma^*(P) = 0 \), hence \( B^{(i)} = 0 \) for all \( i \) and the bias is zero. This zero-bias property is implicit in the results in [LS18, Theorem 1] and [MLW+20, Theorem 1], which study LSA in the i.i.d. setting.

Theorem 4.4 is proved in Appendix A.5. The proof uses the spectral property \( 1 - \|P - \Pi \|_2 \leq \gamma^*(P) \) of a reversible kernel \( P \). The theorem can be readily extended to the non-reversible setting by considering the multiplicatively reversibilization \( P^*P \) and the pseudo-spectral gap, \( \gamma_{ps}(P) := \max_{k \geq 1} \{ \gamma^*(P^*P^k)/k \} \) [Pau15]. We omit the details.

---

*The proof of Theorem 4.4 provides an explicit formula for \( C \).*
Remark 3. As the bias is due to the correlation in \( x_k \)'s, one may consider running LSA with the subsampled (and thus less correlated) data \( (x'_k)_{k \geq 0} \), where \( x'_k := x_{ck} \) and \( c \geq 2 \) is an integer. Doing so reduces the mixing time of \( x'_k \) and in turn the bias by a factor of \( c \), but \( c \) times more data is used. The overall effect is essentially equivalent to using the smaller stepsize \( \alpha/c \), as we have shown that the bias and the exponent of geometric convergence in (4.4) are both proportional to the stepsize. We emphasize that all our results including Richardson-Romberg extrapolation can be applied on top of such subsampling or stepsize adjustment.

4.2.2 Zero Bias with Markovian Data

While Markovian LSA has asymptotic bias in general, there are important special cases where the bias vanishes even when \( (x_k)_{k \geq 0} \) is Markovian. By Theorem 4.3, the bias is zero for all \( \alpha \) if and only if \( B_i = 0 \) for all \( i \). Thanks to the explicit expression (4.6) for \( B_i \), we see that a sufficient condition for zero bias is \( (P^* - \Pi)(A\theta^* + b) = 0 \). This condition can be expressed in the more familiar notation of conditional expectation.

Corollary 4.5. Under the assumptions of Theorem 4.3, if

\[
\mathbb{E}\left[A(x_k)\theta^* + b(x_k) \mid x_{k+1} = x\right] = 0, \quad \forall x \in X,
\]

then \( \mathbb{E}[\theta_\infty] - \theta^* = 0 \).

It is clear that if \( P \) is reversible, then one may replace \( x_{k+1} \) by \( x_{k-1} \) in the condition (4.10) and the corollary continues to hold.

The condition (4.10) above is trivially satisfied when \( x_k \overset{i.i.d.}{\sim} \pi \), in which case \( P = \Pi \) and \( \mathbb{E}_{x_k \sim \pi}[A(x_k)\theta^* + b(x_k)] = 0 \) by definition of \( \theta^* \) in (3.3). Importantly, it is possible for the condition (4.10) to hold even when \( (x_k)_{k \geq 0} \) is correlated. We discuss two such settings in Sections 4.4 and 4.5 on the TD(0) and SGD algorithms.

4.3 Averaging and Extrapolation

We exploit the results above to study the performance of LSA in conjunction with Polyak-Ruppert tail averaging and Richardson-Romberg extrapolation. We focus on corollaries of the convergence bounds in Theorem 4.1 and the bias expansion with order \( m = 1 \) in Theorem 4.3, namely \( \mathbb{E}[\theta_k^{(\infty)}] = \theta^* + \alpha B^{(1)} + \mathcal{O}(\alpha^2) \) (analogous results can be obtained from higher order expansions). In particular, we decompose the MSE into the optimization error, squared bias and variance, and study how these quantities interplay with constant stepsizes, averaging, and extrapolation.

As our focus is dependence on the stepsize \( \alpha \) and iteration count \( k \), we follow the notation convention in Section 3.3. In particular, \( B, B' \) and \( B'' \) denote multiplicative factors independent of \( \alpha \) and \( k \), whose values may change from line to line; the big-\( \mathcal{O} \) notation hides such factors.

4.3.1 Results for Polyak-Ruppert Tail Averaging

Polyak-Ruppert averaging [Rup88, PJ92] is a classical approach for reducing the variance and accelerating the convergence of stochastic approximation. Here we consider the tail-averaging variant of PR averaging [JKK+18]. Given a user-specified burn-in period \( k_0 \geq 0 \), define the tail-averaged iterates

\[
\bar{\theta}_{k_0,k} := \frac{1}{k - k_0} \sum_{t=k_0}^{k-1} \theta_t, \quad \text{for } k = k_0 + 1, k_0 + 2, \ldots
\]

The following corollary provides non-asymptotic characterization for the first two moments of \( \bar{\theta}_{k_0,k} \). The proof can be found in Appendix A.6.
Corollary 4.6. Under the setting of Theorem 4.1, the following bounds hold for all \( k_0 \geq \frac{4\gamma_{\max}}{\alpha} \) and \( k \geq k_0 + \tau_{\alpha} \):

\[
\mathbb{E}[\tilde{\theta}_{k_0,k}] - \theta^* = \alpha B + \mathcal{O}\left(\alpha^2 + \frac{1}{\alpha(k-k_0)} \exp\left(-\frac{\alpha k_0}{4\gamma_{\max}}\right)\right),
\]

\[
\mathbb{E}\left[ (\tilde{\theta}_{k_0,k} - \theta^*) (\tilde{\theta}_{k_0,k} - \theta^*)^\top \right] = \alpha^2 B' + \mathcal{O}\left(\alpha^3 + \frac{\tau_{\alpha}}{k-k_0} + \frac{1}{\alpha(k-k_0)^2} \exp\left(-\frac{\alpha k_0}{4\gamma_{\max}}\right)\right). \tag{4.12}
\]

To parse the above results, we fix \( k_0 = k/2 \) and take the trace of both sides of equation (4.12), which gives the following characterization for the raw LSA iterates, \( \bar{\theta}_{k,k} \):

\[
\mathbb{E} \left[ \|\bar{\theta}_{k/2,k} - \theta^*\|^2 \right] = \frac{\alpha^2 B'' + \mathcal{O}(\alpha^3)}{\tau_{\alpha} k} + \mathcal{O}\left(\frac{\tau_{\alpha}}{k} \exp\left(-\frac{\alpha k}{8\gamma_{\max}}\right)\right). \tag{4.13}
\]

Here the term \( T_1 \) corresponds to the asymptotic squared bias \( \|\mathbb{E}\tilde{\theta}_{\infty/2,\infty} - \theta^*\|^2 = \|\mathbb{E}\tilde{\theta}_{\infty} - \theta^*\|^2 \), which is not affected by averaging. The term \( T_2 \) roughly corresponds to the variance \( \text{Var}(\tilde{\theta}_{k/2,k}) \), which enjoys a \( 1/k \) decay rate due to averaging. The term \( T_3 \) is associated with the optimization error \( \|\mathbb{E}\tilde{\theta}_{k/2,k} - \tilde{\theta}_{\infty/2,\infty}\|^2 \), which decays geometrically in \( k \) thanks to using a constant stepsize \( \alpha \) and only averaging the last \( k/2 \) iterates. Note that for large values of \( k \), the squared bias \( T_1 \) is the dominating term in the MSE bound (4.13).

We make several quick remarks. Firstly, since \( \tau_{\alpha} \approx \log(1/\alpha) \), the burn-in period \( k_0 \) required in Corollary 4.6 is roughly \( \mathcal{O}(\tau_{\alpha}/\alpha) \). This is the time by which both the LSA iterates \( (\bar{\theta}_k)_{k \geq 0} \) and the underlying chain \( (x_k)_{k \geq 0} \) become well mixed, at which point the effect of tail-averaging kicks in.

Secondly, our MSE bound (4.13) for the averaged iterates, which follows readily from Theorems 4.1 and 4.3, is comparable to the sharp results in [MPWB21] in terms of scaling with \( \alpha, k \), and \( \tau_{\alpha} \), though the more complicated analysis in [MPWB21] gives tighter dependence on other parameters.

Lastly, by setting \( k_0 = k - 1 \) in Corollary 4.6 (and relaxing the requirement \( k \geq k_0 + \tau \) via a more refined argument), we obtain the following characterization for the raw LSA iterates, \( \tilde{\theta}_{k,k+1} = \theta_k \):

\[
\mathbb{E} \left[ \|\theta_k - \theta^*\|^2 \right] = \alpha^2 B'' + \mathcal{O}(\alpha \tau_{\alpha}) + \mathcal{O}\left(e^{-\alpha k/(4\gamma_{\max})}\right). \tag{4.14}
\]

This result is consistent with existing MSE upper bounds in [SY19, BRS21, CMSS20]. The power of our results in (4.11)–(4.14) lies in that the first right hand side term therein features an equality rather than merely an upper bound. As such, our results decouple the contribution of the squared bias \( \alpha^2 B'' \) from that of the variance \( \mathcal{O}(\alpha \tau_{\alpha}) \). This decoupling is crucial in understanding the effect of tail-averaging (in Corollary 4.6) and RR extrapolation (in Corollary 4.7 to follow).

4.3.2 Results for Richardson-Romberg Extrapolation

We next show that RR extrapolation [SB02] can be used to reduce the bias to a higher order term of \( \alpha \). Let \( \tilde{\theta}_{k_0,k}^{(\alpha)} \) and \( \tilde{\theta}_{k_0,k}^{(2\alpha)} \) denote the tail-averaged iterates computed using two stepsizes \( \alpha \) and \( 2\alpha \) with the same data stream \( (x_k)_{k \geq 0} \). The RR extrapolated iterates are defined as

\[
\tilde{\theta}_{k_0,k}^{(\alpha)} = 2\tilde{\theta}_{k_0,k}^{(\alpha)} - \tilde{\theta}_{k_0,k}^{(2\alpha)}.
\]

With \( k_0, k \to \infty \), Theorems 4.1 and 4.3 ensure that \( \tilde{\theta}_{k_0,k}^{(\alpha)} \) converges to \( 2\bar{\theta}_{\infty}^{(\alpha)} - \bar{\theta}_{\infty}^{(2\alpha)} \), which has bias

\[
2\left(\mathbb{E}\tilde{\theta}_{\infty}^{(\alpha)} - \theta^*\right) - \left(\mathbb{E}\tilde{\theta}_{\infty}^{(2\alpha)} - \theta^*\right) = 2\left(\alpha B^{(1)} + \mathcal{O}(\alpha^2)\right) - \left(2\alpha B^{(1)} + \mathcal{O}(4\alpha^2)\right) = \mathcal{O}(\alpha^2).
\]

Note that the extrapolation cancels out the first-order term of \( \alpha \), reducing the bias by a factor of \( \alpha \).

The following corollary formalizes the above argument and provides non-asymptotic characterization for the first two moments of \( \tilde{\theta}_{k_0,k}^{(\alpha)} \). The proof can be found in Appendix A.7.
Corollary 4.7. Under the setting of Theorem 4.1, the RR extrapolated iterates with stepsizes \( \alpha \) and \( 2\alpha \) satisfy the following bounds for all \( k_0 \geq \frac{2\max \{ \alpha \} \log \left( \frac{1}{\alpha \gamma} \right)}{\alpha} \) and \( k \geq k_0 + \tau_\alpha \):

\[
E \left[ \tilde{\theta}_{k_0,k}^{(\alpha)} - \theta^* \right] = \mathcal{O}(\alpha^2) + \mathcal{O} \left( \frac{1}{\alpha(k-k_0)} \exp \left( -\frac{\alpha k_0}{4\gamma_{\max}} \right) \right),
\]

\[
E \left[ (\tilde{\theta}_{k_0,k}^{(\alpha)} - \theta^*)^\top (\tilde{\theta}_{k_0,k}^{(\alpha)} - \theta^*) \right] = \mathcal{O} \left( \frac{\tau_\alpha}{k-k_0} \right) + \mathcal{O} \left( \frac{1}{\alpha(k-k_0)^2} \exp \left( -\frac{\alpha k_0}{4\gamma_{\max}} \right) \right). 
\] (4.15)

Comparing the bound (4.15) with (4.12), we see that RR extrapolation reduces the squared bias by a factor of \( \alpha^2 \) while retaining the \( 1/k \) and \( \exp(-k) \) convergence rates for the variance and optimization error, respectively.

Thanks to higher order expansion in Theorem 4.3, RR extrapolation can in fact be applied to more than two stepsizes, which further reduces the bias. Let \( A = \{ \alpha_1, \alpha_2, \ldots, \alpha_m \} \) be a set of \( m \geq 2 \) distinct stepsizes and \( \alpha = \max_{1 \leq i \leq m} \alpha_i \). Let \( (h_1, h_2, \ldots, h_m) \in \mathbb{R}^m \) be the solution to the following linear equation system:

\[
\sum_{i=1}^m h_i = 1; \quad \sum_{i=1}^m h_i \alpha_i^t = 0, \quad t = 1, 2, \ldots, m - 1. 
\] (4.16)

The solution is unique since the coefficient matrix of the system is a Vandermonde matrix. The RR extrapolated iterates with stepsizes in \( A \) and the burn-in period \( k_0 \) are given by

\[
\tilde{\theta}_{k_0,k}^A = \sum_{i=1}^m h_i \cdot \tilde{\theta}_{k_0,k}^{(\alpha_i)}. 
\] (4.17)

This procedure eliminates the first \( m - 1 \) terms in the bias expansion (4.8), reducing the bias to

\[
E \left[ \tilde{\theta}_{k_0,\infty}^A - \theta^* \right] = \sum_{i=1}^m h_i \cdot \left( E \left[ \theta_{\infty}^{(\alpha_i)} \right] - \theta^* \right) = \mathcal{O}(\alpha^m).
\]

One can derive non-asymptotic bounds similar to Corollary 4.7—we omit the details. In Section 5, we numerically verify the efficacy of this high-order RR extrapolation approach.

4.4 Implications for TD Learning

TD(0) is an iterative algorithm in RL for evaluating a given policy for a Markov Decision Process (MDP), or equivalently for computing the value function of a Markov Reward Process (MRP) [Ber19, SB18]. Potentially equipped with function approximation, TD(0) is a special case of LSA. Consequently, all the results in the previous sections can be specialized to TD(0), as we show below.

Consider an MRP \((S, P^S, r, \gamma)\), with Borel state space \( S \), transition kernel \( P^S \), bounded deterministic reward function \( r : S \to [-r_{max}, r_{max}] \), and discount factor \( \gamma \in [0,1) \). We assume that the kernel \( P^S \) is uniformly ergodic with unique stationary distribution \( \pi^S \). Note that we allow for a general uncountable state space \( S \), which generalizes many existing works that focus on finite or countable state spaces [TVR97, BRS21]. The value function \( V : S \to \mathbb{R} \) is defined as \( V(s) = \mathbb{E} \left[ \sum_{i=0}^{\infty} \gamma^i r(s_i) | s_0 = s \right] \), where \((s_k)_{k \geq 0}\) is the Markov chain with kernel \( P^S \). It is common to assume that \( V(s) \) can be approximated by a linear function as \( V(s) \approx \phi(s)\top \theta \), where \( \phi = (\phi_1, \ldots, \phi_d)\top : S \to \mathbb{R}^d \) is a known feature map and \( \theta \) is an unknown weight vector. We assume that \( \phi \) is in \( L^2(\pi^S) \) and has a finite rank \( d \). The latter means that the functions \( \{\phi_i\} \) are linearly independent, i.e., \( \sum_{i=1}^d c_i \phi_i = 0 \) implies \( c_i = 0 \), \( \forall i \in \{1, \ldots, d\} \). When \( S \) is finite or countable, our assumption reduces to the matrix \( \Phi = [\phi(1) \ \phi(2) \ \cdots \ \phi(|S|)]\top \in \mathbb{R}^{(|S| \times d)} \) having full column rank, which is standard in literature [TVR97, BRS21, SY19]. Lastly, we assume for simplicity that \( s_0 \sim \pi^S \) and the feature map is normalized such that \( \phi_{\max} := \sup_{s \in S} \| \phi(s) \| \leq \frac{1}{\sqrt{1+\gamma}} \).
Given a single Markovian data stream \((s_k)_{k \geq 0}\), the linear TD(0) algorithm computes the update
\[
\theta_{k+1} = \theta_k + \alpha \left[ r(s_k) + \gamma \phi(s_{k+1})^T \theta_k - \phi(s_k)^T \theta_k \right] \phi(s_k).
\] (4.18)

TD(0) computes an approximation of the solution \(\theta^*\) of the projected Bellman equation \(\Phi \theta = \Pi_\phi (r + \gamma P^S \phi \theta)\), where \(\Pi_\phi\) is the projection operator w.r.t. \(\|\cdot\|_{L_2(\pi^S)}\) onto the subspace spanned by \(\{\phi_i\}\). It is easy to see that the TD(0) update (4.18) is a special case of the LSA update (3.1) with
\[
x_k = (s_k, s_{k+1}), \quad A(x_k) = \phi(s_k) \left( (\gamma \phi(s_{k+1}) - \phi(s_k))^T \right), \quad b(x_k) = r(s_k) \phi(s_k),
\]
and \(\mathcal{X} = \mathcal{S} \times \mathcal{S}\). Below we verify that TD(0) satisfies the required assumptions.

- **Assumption 1**: The state space \(\mathcal{X}\) is Borel since the product of Borel spaces remains Borel [FG97, Chapter 21.4, Proposition 20]. The uniform ergodicity of the chain \((s_k)_{k \geq 0}\) implies that of the augmented chain \((x_k)_{k \geq 0}\). The assumption \(s_0 \sim \pi^S\) implies \(x_0 \sim \pi\), where \(\pi\) denotes the stationary distribution of \((x_k)_{k \geq 0}\).

- **Assumption 2** follows from the normalization \(\phi_{\text{max}} \leq \frac{1}{\sqrt{1 + \gamma}}\) and direction calculation:
\[
A_{\text{max}} = \sup_{s, s' \in \mathcal{S}} \|\phi(s) (\gamma \phi(s') - \phi(s))^T\| \leq (1 + \gamma) \phi_{\text{max}}^2 \quad \text{and} \quad b_{\text{max}} = \sup_{s \in \mathcal{S}} \|r(s) \phi(s)\| \leq r_{\text{max}} \phi_{\text{max}}.
\]

- **Assumption 3**: By direct calculation we have \(\bar{A}_{ij} = \langle \phi_i, (I - \gamma P^S) \phi_j \rangle_{L_2(\pi^S)}, i, j = 1, \ldots, d\), so \(\bar{A}\) is negative definite and hence Hurwitz. The proof is given in Appendix A.8.

Consequently, all the results in Sections 4.1–4.3 apply to TD(0) with linear function approximation, constant stepsizes, and Markovian data.

Our work generalizes many existing non-asymptotic results on TD(0) in the i.i.d. data setting, e.g., [DSTM18, BRS21, KPR+21, DMN+21]. Under this setting, TD(0) corresponds to the update
\[
\theta_{k+1} = \theta_k + \alpha \left[ r(s_k) + \gamma \phi(s_{k+1})^T \theta_k - \phi(s_k)^T \theta_k \right] \phi(s_k),
\] (4.19)

where the data \(x_k = (s_k, s_{k+1}^\text{next})\) is independent across \(k\) and follows the distribution \(s_k \sim \pi^S, s_{k+1}^\text{next} \sim P^S(s_k, \cdot)\). In this setting, Theorem 4.4 implies that TD(0) with a constant stepsize has no asymptotic bias, i.e., \(\mathbb{E} \left[ \theta_{\infty} \right] = \theta^*\).

### 4.4.1 The Semi-Simulator Setting

Using the explicit bias characterization in Section 4.2, we show that TD(0) may admit zero bias beyond the i.i.d. data setting.

Specifically, consider the update rule (4.19). This time we assume that \((s_k)_{k \geq 0}\) is a Markov chain, conditioned on \(s_k\). \(s_{k+1}^\text{next}\) is sampled independently from \(P^S(s_k, \cdot)\). We call this the **semi-simulator setting**. For simplicity, we present our results for the tabular setting of TD(0) on a finite state space, i.e., \(d = |\mathcal{S}| < \infty\) and \(\phi(s) = e_s \in \mathbb{R}^{|\mathcal{S}|}\) is the \(s\)-th standard basis vector. In this case, the target vector \(\theta^* = V\) is the true value function and satisfies the vanilla Bellman equation \(\theta^* = r + \gamma P^S \theta^*\), which can be explicitly written as
\[
\theta^*(s) = r(s) + \gamma \mathbb{E}_{s_{\text{next}} \sim P^S(s)} \left[ \theta^*(s_{\text{next}}) \right], \quad \forall s \in \mathcal{S}.
\] (4.20)

**Corollary 4.8.** Under the semi-simulator setting, the TD(0) update (4.19) satisfies \(\mathbb{E} [\theta_{\infty}] - \theta^* = 0\).

**Proof.** We verify the condition (4.10) in Corollary 4.5. Recall that \(x_k = (s_k, s_{k+1}^\text{next})\). By definitions of \(A, b\) and \(\phi\) and the law of total expectation, we have
\[
\mathbb{E} \left[ A(x_k) \theta^* + b(x_k) \mid x_{k+1} \right] = \mathbb{E} \left[ \phi(s_k) (\gamma \phi(s_{k+1}^\text{next}) - \phi(s_k))^T \theta^* + r(s_k) \phi(s_k) \mid x_{k+1} \right]
\]
\[
= \mathbb{E} \left[ \phi(s_k) (\gamma \theta^*(s_{k+1}^\text{next}) - \theta^*(s_k) + r(s_k)) \mid x_{k+1} \right]
\]
\[
= \mathbb{E} \left[ \phi(s_k) \mathbb{E} [\theta^*(s_{k+1}^\text{next}) - \theta^*(s_k) + r(s_k) \mid x_{k+1}, s_k] \mid x_{k+1} \right]
\]

With \(s_{k+1}^\text{next} \perp x_{k+1} \mid s_k\) and the Bellman equation (4.20), the inner expectation above satisfies
\[
\mathbb{E} [\gamma \theta^*(s_{k+1}^\text{next}) - \theta^*(s_k) + r(s_k) \mid x_{k+1}, s_k] = \gamma \mathbb{E} [\theta^*(s_{k+1}^\text{next}) \mid s_k] - \theta^*(s_k) + r(s_k) = 0.
\]
Combining pieces verifies the desired condition \(\mathbb{E} [A(x_k) \theta^* + b(x_k) \mid x_{k+1}] = 0\).

\(\square\)
4.5 Implications for Markovian SGD

SGD is widely used in optimization and machine learning problems. When minimizing a quadratic objective function, the update step of SGD is linear in the iterate and can be cast as LSA. Markovian data arise in many sequential decision-making settings, such as those with experience replay and auto-regressive dynamics [BJN+20].

Specifically, we consider SGD applied to the minimization of a quadratic function

$$\ell(\theta) := -\frac{1}{2} \mathbb{E}_{x \sim \pi} \left[ \theta^\top A(x) \theta + b(x)^\top \theta + c(x) \right],$$

where $A : \mathcal{X} \to \mathbb{S}^{d \times d}$, $b : \mathcal{X} \to \mathbb{R}^d$ and $c : \mathcal{X} \to \mathbb{R}$ are deterministic functions, and $\mathbb{S}^{d \times d}$ denotes the set of $d \times d$ symmetric matrices. Given a Markovian data stream $(x_k)_{k \geq 0}$ with stationary distribution $\pi$, the constant stepsize SGD algorithm $\theta_{k+1} = \theta_k + \alpha (A(x_k) \theta_k + b(x_k))$ is a special case of the LSA iteration (3.1). We impose the same Assumptions 1–3 on $A$, $b$ and $(x_k)_{k \geq 0}$. Since the matrix $\bar{A}$ is symmetric, Assumption 3 implies that $-\bar{A}$ is positive definite. In this case, the objective function $\ell$ is strongly convex, with a unique minimizer $\theta^*$ satisfying the stationarity condition $\nabla \ell(\theta) = 0$, which can be seen to coincide with the steady-state equation (3.3). Note that we do not need the matrix $-A(x)$ to be positive definite for each individual $x$.

All the results in Sections 4.1–4.3 apply to the above setting. In particular, we establish the distributional convergence of SGD with constant stepsizes and characterize the bias, variance, and optimization error under averaging and extrapolation. These results generalize several quadratic minimization results in [BM13, DDB20], which focus on the i.i.d. setting. In the Markovian data setting, our results imply that in general there exists an asymptotic bias proportional to the stepsize $\alpha$, i.e., $\mathbb{E}[\theta_\infty] - \theta^* = \alpha B(1) + O(\alpha^2)$. This result is a generalization and a more precise version of [BJN+20, Theorem 4], which considers SGD for least squares regression—a special case of quadratic minimization—and establishes the lower bound $\|\mathbb{E}[\theta_\infty] - \theta^*\| \geq c\alpha$ under Markovian data. Our result explains the root of this error lower bound and suggests RR extrapolation as a way to reduce the error.

4.5.1 Least Squares Regression with Independent Additive Noise

We specialize our results to the least squares regression problem. The explicit bias characterization in Section 4.2 allows us to identify an interesting setting with Markovian data but nevertheless zero bias.

Suppose that we observe data pairs $(g_k, y_k)_{k \geq 0}$, where $g_k \in \mathbb{R}^d$ is the covariate vector sequentially sampled from a uniformly ergodic Markov chain, $y_k = \langle g_k, \theta^*\rangle + n_k$ is the scalar response variable, $\theta^* \in \mathbb{R}^d$ is the true regression vector, and $n_k$ is the additive noise. We assume that the noise $n_t$ is zero mean and independent from $\{g_t\}$ and $\{n_t : t \neq k\}$. The goal is to estimate the minimizer $\theta^* = \theta^* \in \mathbb{R}^d$ of the least squares objective $\ell(\theta) = \mathbb{E}_{(g,y) \sim \pi_0} \left[ (\langle g, \theta \rangle - y)^2 \right]$, where $\pi_0$ is the stationary distribution of the process $(g_k, y_k)_{k \geq 0}$. The corresponding SGD step is

$$\theta_{t+1} = \theta_t - \alpha g_t (\theta_t - y_t).$$

(4.21)

This step can be cast into the Markovian LSA iteration (3.1) with

$$x_t = (g_t, n_t), \quad A(x_t) = -g_t g_t^\top, \quad \text{and} \quad b(x_t) = g_t y_t = g_t \theta_t + g_t n_t.$$

Note that the $x_t$’s are correlated since the covariates $g_t$’s are. Nevertheless, SGD turns out to have zero asymptotic bias, $\mathbb{E}[\theta_\infty] - \theta^* = 0$, due to the structure of the noise component $n_t$. We prove this claim by verifying the condition (4.10) in Corollary 4.5:

$$\mathbb{E}[A(x_t) \theta^* + b(x_t) \mid x_{t+1}] = -\mathbb{E}[g_t g_t^\top \mid g_{t+1}, n_{t+1}] \theta^* + \mathbb{E}[g_t g_t^\top \theta^* + g_t n_t \mid g_{t+1}, n_{t+1}]
= \mathbb{E}[g_t \mid g_{t+1}] \cdot \mathbb{E}[n_t] = 0,$$

(4.22)

where the last two steps follow from the independence and zero mean assumptions on $n_t$. Note that a similar observation is made in [BJN+20, Theorem 3] using a more specialized argument.

In fact, the bias is zero under the more general setting of conditionally independent noise:
Corollary 4.9. In the regression setting above, if the additive noise satisfies
\[ n_t \perp g_t \mid (g_{t+1}, n_{t+1}) \quad \text{and} \quad \mathbb{E}[n_t \mid g_{t+1}, n_{t+1}] = 0, \]
then the asymptotic bias of the SGD update (4.21) satisfies \( \mathbb{E}[\theta_\infty] - \theta^* = 0. \)

Proof. The last two steps in equation (4.22) hold under the assumption of the corollary.

Finally, we reiterate that when the condition in Corollary 4.9 is not satisfied and the bias is nonzero, one may consider employing Richardson-Romberg extrapolation to reduce the bias.

5 Numerical Experiments

In this section, we provide numerical experiment results for LSA, TD(0) with linear function approximation, and SGD applied to least squares regression.

5.1 Experiments for LSA

We consider the LSA update (3.1) in dimension \( d = 4 \) for a finite state, irreducible, and aperiodic Markov chain with \( n = 8 \) states. We construct the transition probability matrix \( P \) and the functions \( A \) and \( b \) randomly; see Appendix C.1 for the details. Given \( P \), we generate a single trajectory of the Markov chain \( (x_k)_{k=1}^K \) of length \( K = 10^5 \), and run the LSA iteration with initialization \( \theta_0^{(\alpha)} = 0 \) and stepsizes \( \alpha \in \{0.2, 0.4, 0.8\} \).

In Figure 2(a), we plot the error \( \|\theta_k^{(\alpha)} - \theta^*\| \) for the raw LSA iterates \( \theta_k^{(\alpha)} \), the error for the tail-averaged (TA) iterates \( \bar{\theta}_{k/2}^{(\alpha)} \), and the error for the RR extrapolated iterates \( \tilde{\theta}_k^{(\alpha)} \) with stepsizes \( \alpha \) and \( 2\alpha \). For comparison, we also include the errors for LSA with a diminishing stepsize \( \alpha_k = 0.2/k^{0.75} \). We see that the raw LSA iterates with constant stepsizes oscillate, whereas the tail averaged iterates converge to a limit, with a smaller error for a smaller stepsize. Moreover, the final TA error, which corresponds to the asymptotic bias, is roughly proportional to the stepsize (note the equal spacing in the log scale between the three TA lines). Finally, RR extrapolation with two stepsizes further reduces the bias, as can be seen by comparing, e.g., the dashed red line (TA with \( \alpha = 0.4 \)) and the solid red line (RR with \( \alpha = 0.4 \) and \( 0.8 \)). These observations are consistent with our theory. Finally, the tail-averaged iterates with constant stepsizes have significantly faster initial convergence than the iterates with a diminishing stepsize.

We next investigate how the error depends on the spectral gap and mixing time of \( P \). As the state space is finite, we work with the SLEM \( |\lambda_2| \) of \( P \), which satisfies \( \gamma^*(P) = 1 - |\lambda_2| \). Given \( P \) generated above and its stationary distribution \( \pi \), we construct another transition probability matrix parameterized by \( \beta \in [0, 1] \) as follows:
\[ P^{(\beta)} = \beta \cdot P + (1 - \beta) \cdot 1\pi^\top. \]

Note that \( P^{(1)} = P \), and \( P^{(\beta)} \) has \( \pi \) as the stationary distribution for any \( \beta \). As \( \beta \) decreases from 1 to 0, the SLEM \( |\lambda_2| \) of \( P^{(\beta)} \) decreases towards 0. For different values of \( \beta \), we run the LSA with \( P^{(\beta)} \) as the transition probability matrix of the chain \( (x_k)_{k \geq 0} \). In Figure 2(b), we plot the corresponding errors of the tail-averaged iterates. We see that a smaller \( |\lambda_2| \) leads to a smaller final error. Moreover, when \( \lambda_2 = 0 \), which corresponds to the i.i.d. data setting, the error is converging to zero, which indicates a vanishing asymptotic bias. These observations are consistent with Theorem 4.4 on the relationship between the asymptotic bias and mixing time.

5.2 Experiments for TD(0) with Linear Function Approximation

We perform a similar set of experiments as in the previous sub-section on the TD(0) algorithm. In particular, we consider the classical “Problematic MDP” from [KP00, LP03], and use TD(0) with linear function approximation to estimate the value function of a given policy. See Appendix C.2 for the details of the MDP, the policy, and the choice of the feature vectors.

In Figure 3, we plot the errors of the raw TD(0) iterates, tail-averaged iterates, and RR extrapolated iterates with different stepsizes \( \alpha \). The results are qualitatively similar to those in Figure 2(a). In addition,
(a) The errors of the raw LSA iterates, tail-averaged (TA) iterates, and RR extrapolated iterates with different step-sizes $\alpha$.

(b) The errors of the raw LSA iterates and tail-averaged (TA) iterates under different SLEM $[\lambda_2]$. The stepsize $\alpha$ is fixed at 0.8.

Figure 2: Experiment results for LSA

Figure 3: The errors of the raw TD(0), tail-averaged (TA), and RR extrapolated iterates with different stepsizes $\alpha$. 
the TA iterates with a larger stepsize have faster initial convergence, which is consistent with theoretical prediction in Corollary 4.6.

We further investigate the benefit of higher-order RR extrapolation with more than 2 stepsizes, using the procedure described in equations (4.16) and (4.17). Specifically, we compare the errors of the tail-averaged iterates and the RR extrapolated iterates with 2 to 6 stepsizes. The results are shown in Figure 4. Here we use a set of large stepsizes (of similar magnitudes), which give fast initial convergence. We see that using more stepsizes in RR extrapolation reduces the final errors by a significant margin. In particular, the error of RR extrapolation with 6 stepsizes is smaller by 3 orders of magnitude than TA with the same stepsizes. We emphasize that this error reduction is obtained almost for free, as we can run the six TD(0) iterations in parallel using the same data.

![Figure 4: Comparison between tail-averaging (TA) and RR extrapolation with m stepsizes, for m = 2, . . . , 6.](image)

The setting for each line in the plot is given by its line style (representing the number of stepsizes used in RR) and line color (representing the smallest stepsize involved). For example, the dash-dotted green line corresponds to TA with stepsize \(\alpha = 2.1\), and the dashed red line corresponds to RR with four stepsizes \(\alpha \in \{1.9, 2.1, 2.3, 2.5\}\).

### 5.3 Experiments for Least Squares Linear Regression

Finally, we consider SGD applied to the regression model \(y_k = \langle g_k, \theta^{reg}\rangle + n_k\) as described in Section 4.5.1, where \(\theta^{reg} = 0\), and \(g_k \in \mathbb{R}^2\) is obtained from a Metropolis-Hastings (MH) sampler on \([-1, 1]^2\); see Appendix C.3 for the details. We generate a single trajectory of the Markov chain \((g_k, n_k)_{k=1}^K\) with \(K = 10^8\) and apply SGD with constant stepsizes \(\alpha \in \{0.01, 0.02, 0.04\}\). The target vector is the minimizer \(\theta^*\) of the least squares objective \(\ell(\theta) = \mathbb{E}_{(g, n) \sim \pi}[\|\langle g, \theta \rangle - \langle g, \theta^{reg}\rangle - n\|^2]\).

We consider two different settings for the noise \(n_k\) and study its impact on the error \(\|\theta_k - \theta^*\|\). In the first setting, we assume \(n_k \sim \text{Unif}[-1, 1]\). In this case, Corollary 4.9 predicts that the asymptotic bias is zero despite the Markovian correlation in \(\{g_k\}\). In Figure 5, we plot the errors for the TA iterates \(\bar{\theta}^{(\alpha)}_{k/2, k}\) with a constant stepsize and for the raw LSA iterates with a diminishing stepsize \(\alpha_k = 0.01/k^{0.75}\). We see that the constant stepsize TA iterates converge to zero, as predicted by our theory, and the convergence speed is faster than using a diminishing stepsize.

In the second setting, the noise \(n_k\) is correlated with \(g_k\) as follows: we set \(n_k = \text{sign}(g_k(1) + g_k(2))\), where \(g_k(i)\) denotes the \(i\)-th coordinate of \(g_k\). In Figure 6, we plot the errors for TA iterates \(\bar{\theta}^{(\alpha)}_{k/2, k}\) and RR extrapolated iterates \(\bar{\theta}^{(\alpha)}_k\). The results here indicate a nonzero bias proportional to the stepsize, in contrast to the first noise setting. We also see that RR extrapolation reduces the bias.
6 Proof Outline

In this section, we outline the proofs for Theorem 4.1 (convergence of LSA) and Theorem 4.3 (bias expansion). The proofs make use of a pilot result Proposition 6.1, stated in Section 6.1, which serves as the basis for subsequent analysis. The complete proofs of these results and other main theorems/corollaries are given in the appendix.

6.1 A Pilot Result

We have the following non-asymptotic upper bound on the MSE $\mathbb{E}[\|\theta_k - \theta^*\|^2]$.

Proposition 6.1. Under Assumptions 1, 2 and 3, if $\alpha$ satisfies equation (4.2), then the following bound holds for all $k \geq \tau$,

$$
\mathbb{E}[\|\theta_k - \theta^*\|^2] \leq 10 \frac{\gamma_{\text{max}}}{\gamma_{\text{min}}} \left(1 - \frac{0.9\alpha}{\gamma_{\text{max}}} \right)^k \left(\mathbb{E}[\|\theta_0 - \theta^*\|^2] + s^{-2}_{\min}(A)b_{\text{max}}^2\right) + \alpha \tau \cdot \kappa,
$$

with $\kappa$ defined in equation (4.3).

Proposition 6.1 is a moderate improvement of [SY19, Theorem 7]. When $b_{\text{max}} = 0$ (which means $b(x) = 0$, $\forall x$), Proposition 6.1 guarantees that $\kappa = 0$, in which case $\theta_k$ converges in mean squared to $\theta^*$ as $k \to \infty$. This fact plays an important role in proving the distributional convergence result in Theorem 4.1 in the setting with a general $b$ and nonzero $b_{\text{max}}$. In particular, the proof of Theorem 4.1 employs a coupling argument that constructs another process with $b_{\text{max}} = 0$. In comparison, the bound in [SY19, Theorem 7] gives a non-zero value of $\kappa$ even when $b_{\text{max}} = 0$ and hence is insufficient for running the coupling argument. Moreover, the stepsize condition (4.2) required by Proposition 6.1 (and by all our other results) does not involve $b_{\text{max}}$, which correctly reflects the translation invariance of the LSA update (3.1). The stepsize condition in [SY19, Theorem 7], on the other hand, has a superfluous dependence on $b_{\text{max}}$.

The proof of Proposition 6.1 is similar to that of [SY19, Theorem 7] with more refined arguments. For completeness, we provide the proof in Appendix A.1. One key refinement in our proof is to avoid invoking inequalities of the form $2u \leq u^2 + 1$, and to use instead $2u \leq \beta^2u^2 + 1/\beta^2$ with a judicious choice of $\beta$ that respects the translation invariance of the LSA update (3.1).
6.2 Proof Outline of Theorem 4.1

We sketch the main ideas in proving Theorem 4.1. The complete proof is given in Appendix A.2.

The proof consists of bounding Wasserstein distances of the form \( \bar{W}_2(\mathcal{L}(x_k, \theta_k), \mathcal{L}(x_{k+1}, \theta_{k+1})) \) and \( \bar{W}_2(\mathcal{L}(x_k, \theta_k), \mathcal{L}(x_{\infty}, \theta_{\infty})) \). Since the Wasserstein distance is defined by the optimal coupling, it can be upper bounded by constructing a particular coupling. With this strategy in mind, we consider coupling two Markov chains \((x_k^{[1]}, \theta_k^{[1]})_{k \geq 0}\) and \((x_k^{[2]}, \theta_k^{[2]})_{k \geq 0}\), which are two copies of the LSA iteration (3.1). We make use of two types of coupling in the proof.

The first type of coupling is constructed by letting the two Markov chains above share the same underlying data stream \((x_k)_{k \geq 0}\), i.e., letting \(x_k^{[1]} = x_k^{[2]} = x_k\) for all \(k \geq 0\). Explicitly, the iterates \(\theta_k^{[1]}\) and \(\theta_k^{[2]}\) are given by the updates

\[
\begin{align*}
\theta_k^{[1]} &= \theta_k^{[1]} + \alpha(A(x_k) \theta_k^{[1]} + b(x_k)), \\
\theta_k^{[2]} &= \theta_k^{[2]} + \alpha(A(x_k) \theta_k^{[2]} + b(x_k)), \\
\end{align*}
\]

Taking the difference of the two equations above, we see that the difference \(\omega_k := \theta_k^{[1]} - \theta_k^{[2]}\) satisfies the following recursion

\[
\omega_{k+1} = (I + \alpha A(x_k)) \cdot \omega_k, \quad k = 0, 1, \ldots
\]

Our key observation is that the above recursion is a special case of the LSA iteration (3.1) with \(\omega_k\) as the variable and \(b_{\text{max}} = \sup_{x \in X} \|b(x)\| = 0\). Consequently, the pilot result in Proposition 6.1 can be invoked to obtain the following geometric convergence bound for \(\omega_k\):

\[
E[\|\omega_k\|^2] \leq C(A, b, \pi) \left(1 - \frac{0.9 \alpha}{b_{\text{max}}}\right)^k E[\|\omega_0\|^2].
\]

We next judiciously choose the conditional distribution of \(\theta_0^{[2]}\) given \((x_0, \theta_0^{[1]})\) such that \((x_k, \theta_k^{[2]}) \overset{d}{=} (x_{k+1}, \theta_k^{[1]})\) for all \(k \geq 0\), where \(d\) denotes equality in distribution. Then, it follows from the above geometric convergence bound that

\[
\bar{W}_2^2(\mathcal{L}(x_k, \theta_k), \mathcal{L}(x_{k+1}, \theta_k^{[1]})) \leq E[\|\theta_k^{[1]} - \theta_k^{[2]}\|^2] \to 0 \quad \text{as} \ k \to \infty.
\]

As such, \((x_k, \theta_k^{[1]})_{k \geq 0}\) is a Cauchy sequence and hence converges to a unique limit \((x_{\infty}, \theta_{\infty})\) with the limiting distribution \(\bar{\mu} := \mathcal{L}(x_{\infty}, \theta_{\infty}))\). This proves Part 1 of Theorem 4.1.

We next show that \(\bar{\mu}\) is the invariant distribution of the Markov chain \((x_k, \theta_k)_{k \geq 0}\). This invariance property would follow easily if one could establish the one-step contraction property

\[
\bar{W}_2(\mathcal{L}(x_1, \theta_1), \mathcal{L}(x_1', \theta_1')) \leq \rho \cdot \bar{W}_2(\mathcal{L}(x_0, \theta_0), \mathcal{L}(x_0', \theta_0'))
\]

for any two copies of the LSA trajectory (3.1) and some \(\rho \in (0, 1)\). In fact, this is the approach taken in [DDB20] for analyzing SGD under i.i.d. noise. For our Markovian data setting, however, establishing one-step contraction is challenging if not impossible. Thankfully, to prove invariance of \(\bar{\mu}\), it suffices to have the following weaker property

\[
\bar{W}_2^2(\mathcal{L}(x_1, \theta_1), \mathcal{L}(x_1', \theta_1')) \leq \rho_1 \cdot \bar{W}_2^2(\mathcal{L}(x_0, \theta_0), \mathcal{L}(x_0', \theta_0')) + \sqrt{\rho_2} \cdot \bar{W}_2^2(\mathcal{L}(x_0, \theta_0), \mathcal{L}(x_0', \theta_0'))
\]

where \(\mathcal{L}(x_0, \theta_0) = \bar{\mu}\) and the quantities \(\rho_1\) and \(\rho_2\) are finite and independent of \(\mathcal{L}(x_0', \theta_0')\). We establish the property (6.1) by using a second type of coupling between \((x_k, \theta_k)_{k \geq 0}\) and \((x_k', \theta_k')_{k \geq 0}\), such that

\[
\bar{W}_2^2(\mathcal{L}(x_0, \theta_0), \mathcal{L}(x_0', \theta_0')) = E[\|d_0(x_0, x_0') + \|\theta_0 - \theta_0'\|^2]\]

and

\[
\begin{cases}
    x_{k+1} = x_{k+1}' & \text{if } x_k = x_k', \quad \forall k \geq 0,
\end{cases}
\]

That is, the two underlying Markov chains \((x_k)_{k \geq 0}\) and \((x_k')_{k \geq 0}\) evolve separately until they reach the same state, after which they coalesce and follow the same trajectory. Given the property (6.1), for any \(k \geq 0\), if we set \(\mathcal{L}(x_0, \theta_0) = \bar{\mu}\) and \(\mathcal{L}(x_0', \theta_0') = \mathcal{L}(x_k, \theta_k)\), then

\[
\bar{W}_2^2(\mathcal{L}(x_1, \theta_1), \mathcal{L}(x_{k+1}, \theta_{k+1})) \leq \rho_1 \cdot \bar{W}_2^2(\bar{\mu}, \mathcal{L}(x_k, \theta_k)) + \sqrt{\rho_2} \cdot \bar{W}_2^2(\bar{\mu}, \mathcal{L}(x_k, \theta_k)).
\]
It follows from the triangle inequality of Wasserstein distance that
\[
\bar{W}_2(\mathcal{L}(x_1, \theta_1), \bar{\mu}) \leq \bar{W}_2(\mathcal{L}(x_1, \theta_1), \mathcal{L}(x_{k+1}, \theta_{k+1})) + \bar{W}_2(\mathcal{L}(x_{k+1}, \theta_{k+1}), \bar{\mu})
\]
\[
\leq \sqrt{\rho_1 \cdot \bar{W}_2^2(\bar{\mu}, \mathcal{L}(x_{k}, \theta_k)) + \sqrt{\rho_2 \cdot \bar{W}_2^2(\bar{\mu}, \mathcal{L}(x_{k}, \theta_k)) + \bar{W}_2(\mathcal{L}(x_{k+1}, \theta_{k+1}), \bar{\mu})} 
\]
\[
\rightarrow 0 \quad \text{as} \quad k \rightarrow \infty,
\]
which establishes the invariance of \(\bar{\mu}\) and proves Part 2 of Theorem 4.1.

Finally, the non-asymptotic bound in Part 3 of Theorem 4.1 follows from the non-asymptotic bound on \(\omega_k\) and invariance property of \(\bar{\mu}\) established above.

### 6.3 Proof Outline of Theorem 4.3

We outline the proof of Theorem 4.3. The complete proof is given in Appendix A.4.

As discussed in Section 4.2, our proof centers around the condition expectations \(E[\theta_\infty | x_\infty = x], x \in \mathcal{X}\). To characterize these quantities, we make use of the Basic Adjoint Relationship (BAR), which states that under the stationary distribution \(\bar{\mu}\) and for any test function \(f: \mathbb{R}^d \times \mathcal{X} \rightarrow \mathbb{R}\) with at most quadratic growth, it holds that
\[
E[\bar{\mu} \ f(x_\infty, \theta_\infty)] = E[\bar{\mu} \ f(x_{\infty+1}, \theta_{\infty+1})].
\]

For the purpose of characterizing the first moment of \(\theta_\infty\), it suffices to consider the test functions \(f(E)(x, \theta) := \theta \cdot 1\{x \in E\}\) for each measurable subset \(E \in \mathcal{B}(\mathcal{X})\). This choice allows us to establish
\[
E[\theta_\infty | x_\infty = x] = E[\theta_{\infty+1} | x_{\infty+1} = x],
\]
which leads to the following recursive relationship:
\[
E[\theta_\infty | x_\infty = x] = \int_{s \in \mathcal{X}} P^s(x, ds) \left( E[\theta_\infty | x_\infty = s] + \alpha \left( A(s) E[\theta_\infty | x_\infty = s] + b(s) \right) \right).
\]

Define the function \(z: \mathcal{X} \rightarrow \mathbb{R}^d\) such that \(z(x) = E[\theta_\infty | x_\infty = x]\). The above recursion can be written succinctly as \(z = P^*(z + \alpha(Dz + b))\). Inserting the identity \(\Pi(Dz + b) = (1 \otimes \pi)(Dz + b) = 0\), we obtain
\[
z = P^*z + \alpha(P^* - \Pi)(Dz + b). \quad (6.2)
\]

Next, we choose \(\pi z = E[\theta_\infty]\) as the reference point and define the function \(\delta: \mathcal{X} \rightarrow \mathbb{R}^d\) by \(\delta(x) = z(x) - \pi z\). We subtract \(\pi z\) from both sides of (6.2) to obtain
\[
\delta = (P^* - \Pi)\delta + \alpha(P^* - \Pi)(Dz + b).
\]

Consolidating the terms involving \(\delta\) and using the invertibility of the operator \((I - P^* + \Pi)\) under the uniform ergodicity assumption, we obtain
\[
\delta = \alpha(I - P^* + \Pi)^{-1}(P^* - \Pi)(Dz + b). \quad (6.3)
\]

On the other hand, due to the structure of \(\delta\), we can obtain the following identity:
\[
\pi z = \theta^* - \pi D\delta. \quad (6.4)
\]

Substituting (6.4) into the definition \(\delta := z - \pi z\), we obtain
\[
z = \theta^* + (I - \Pi D)\delta. \quad (6.5)
\]

We now combine (6.5) with (6.3) to eliminate the variable \(z\), thereby establishing the following self-expressing equation for \(\delta\):
\[
\delta = \alpha \nu + \alpha \Xi \delta,
\]
where \( \Xi \) and \( \upsilon \) are defined in (4.6a) and (4.6b), respectively. Continuing this bootstrapping argument for \( m \) steps, we obtain the expansion

\[
\delta = \sum_{i=1}^{m} \alpha^i \Xi^{i-1} \upsilon + \alpha^m \Xi \delta. \tag{6.6}
\]

We next note that equation (6.3), together with the bound \( \mathbb{E}[\theta_\infty] = \mathcal{O}(1) \) (which follows from Theorem 4.1), imply the coarse bound \( \|\delta\|_{L^2(\pi)} = \mathcal{O}(\alpha) \). Applying this bound to the second RHS term in (6.6) and substituting the resulting expression for \( \delta \) into (6.4), we obtain the desired expansion for \( \mathbb{E}[\theta_\infty] \) given in Theorem 4.3:

\[
\mathbb{E}[\theta_\infty] = \theta^* + \sum_{i=1}^{m} \alpha^i B^{(i)} + \mathcal{O}(\alpha^{m+1}). \tag{6.7}
\]

Moreover, when the stepsize \( \alpha \) satisfies \( \|\alpha \Xi\|_{L^2(\pi)} < 1 \), one can take \( m \rightarrow \infty \) in the expansions (6.6) and (6.7) and obtain the infinite series expansion.

7 Conclusion

In this paper, we study linear stochastic approximation with constant stepsizes and Markovian data. We analyze the convergence rates to a limiting distribution and identify the existence of an asymptotic bias. We characterize the bias as a function of the stepsize and mixing time, and rigorously establish the benefit of Richardson-Romberg extrapolation. Our results provide a refined characterization of linear stochastic approximation, elucidating the effect of stepsize, averaging, and extrapolation on the optimization error, variance, and bias.

Based on our work, immediate next steps include tightening the dimension dependence in our bounds and relaxing the uniform ergodicity assumption. Further future directions include: (a) study higher moments of the errors and provide high probability bounds; (b) investigate extension of our results to nonlinear stochastic approximation; (c) exploit our results to guide the choice and scheduling of the stepsize.

Acknowledgements

Y. Chen is supported in part by NSF CAREER Award CCF-2233152 and grant CCF-1704828. Q. Xie is supported in part by NSF grant CNS-1955997 and a J.P. Morgan Faculty Research Award. We would like to thank Jim Dai for inspiring discussion.

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A Proofs

In this section, we prove our pilot result in Section 6 and our main results in Section 4.

Recall that $\tau \equiv \tau_{\alpha}$ is the $\alpha$-mixing time defined in Section 3.2. In the sequel, we frequently use the fact that $\alpha \tau \leq \frac{1}{4}$ if $\alpha$ satisfies the condition (4.2). This fact follows from combining the condition (4.2) with the lower bound

$$\gamma_{\text{max}} \geq \gamma_{\text{min}} \geq \frac{1}{2} \left( \frac{1}{2} \right)^{2}\geq \frac{1}{2},$$

where the inequality (i) is given in [Sha74], and the inequality (ii) holds under Assumption 2.

We also repeatedly use the following independence property:

$$(\theta_0, x_0, \theta_1, x_1, \ldots, \theta_k) \perp \perp (x_{k+1}, x_{k+2}, \ldots) \mid x_k, \quad \forall k \geq 1.$$  \hfill (A.2)

Consequently, we have $\theta_k \perp \perp x_{k+1} \mid x_k$ for all $k \geq 1$. These facts can be proved by direct calculation. Alternatively, one may verify that the joint distribution of $(x_k, \theta_k)_{k \geq 0}$ obeys the Markov property with respect to the directed acyclic graph in the right pane of Figure 1, hence the aforementioned independence properties follow from standard results on directed probabilistic graphical models [CDLS99, Corollary 5.11 and Theorem 5.14].

A.1 Proof of Proposition 6.1

We prove our pilot result in Proposition 6.1, which upper-bounds the MSE $E[\|\theta_k - \theta^*\|^2]$.

We argue that it suffices to prove Proposition 6.1 in the special case where the expected value $\bar{b}$ defined in (3.2) is assumed to be 0. When the LSA update rule in equation (3.1) has a general $\bar{b}$, we can center the update by subtracting $\theta^*$ from both sides of (3.1), which gives

$$\theta_{k+1} - \theta^* = \theta_k - \theta^* + \alpha [A(x_k)(\theta_k - \theta^*) + b(x_k) + A(x_k)\theta^*].$$

Setting $\theta'_k := \theta_k - \theta^*$ and $b'(x_k) := b(x_k) + A(x_k)\theta^*$, we rewrite equation (A.3) as

$$\theta'_{k+1} = \theta'_k + \alpha [A(x_k)\theta'_k + b'(x_k)].$$

Equation (A.4) is an LSA update in the variable ($\theta'_k$) and satisfies

$$\bar{b}' := \lim_{k \to \infty} E[b'(x_k)]$$

$$= \lim_{k \to \infty} E[b(x_k)] + E[A(x_k)]\theta^*$$

$$= \bar{b} + \bar{A}\theta^* = 0,$$

where the last equality holds since $\theta^*$ is defined as the solution to $E_{\pi}[A(x)]\theta + E_{\pi}[b(x)] = 0$. As such, we have obtained a LSA of $\theta'_k$ with $\bar{b}' = 0$.

Let $b'_{\text{max}} := \sup_{x \in \mathcal{X}} \|b'(x)\|$. The convergence rate of the new LSA update (A.4) is given in the following proposition, which is a centered version of Proposition 6.1.

**Proposition A.1.** Under Assumptions 1, 2 and 3, if $\alpha$ satisfies equation (4.2), then the update (A.4) with $\bar{b}' = 0$ satisfies for all $k \geq \tau$,

$$E[\|\theta'_k\|^2] \leq \frac{\gamma_{\text{max}}}{\gamma_{\text{min}}} \left( 1 - \frac{0.9\alpha}{\gamma_{\text{max}}} \right)^{k-\tau} \left( 5E[\|\theta'_0\|^2] + (b'_{\text{max}})^2 \right) + \frac{\gamma_{\text{max}}}{0.9\gamma_{\text{min}}} \cdot \alpha \tau \left( 160\gamma_{\text{max}}(b'_{\text{max}})^2 \right).$$

We prove the above proposition in Appendix A.1.1. Taking Proposition A.1 as given, we now complete the proof of the general Proposition 6.1.

**Proof of Proposition 6.1.** By definition of $b'$, we have $\|b'(x)\| \leq \|b(x)\| + \|A(x)\|\|\theta^*\|, \forall x \in \mathcal{X}$, whence

$$b'_{\text{max}} \leq b_{\text{max}} + A_{\text{max}}\|\theta^*\|$$

$$\leq (1 + A_{\text{max}}/s_{\text{min}}(A))b_{\text{max}} \leq 2s_{\text{min}}(A)b_{\text{max}}.$$
Substituting $\theta'_k = \theta_k - \theta^*$ and the above bound into Proposition A.1, we obtain that for all $k \geq \tau$,
\[
\mathbb{E}[\|\theta_k - \theta^*\|^2] \leq 5 \gamma_{\max} \left( 1 - \frac{0.9\alpha}{\gamma_{\max}} \right)^k \left( \mathbb{E}[\|\theta_0 - \theta^*\|^2] + s_{\min}(\bar{A})b_{\max}^2 \right) + \frac{\gamma_{\max}}{0.9\gamma_{\min}} \cdot \alpha \tau \cdot (640\gamma_{\max}s_{\min}^{-2}(\bar{A})b_{\max}^2).
\]

We can simplify the above expression using the following simple bound, whose proof is postponed to the end of this sub-sub-section.

Claim 1. We have $\left( 1 - \frac{0.9\alpha}{\gamma_{\max}} \right)^{-\tau} \leq 2$.

Using Claim 1 and the definition of $\kappa$ in equation (4.3), we obtain that for all $k \geq \tau$,
\[
\mathbb{E}[\|\theta_k - \theta^*\|^2] \leq 10 \gamma_{\max} \left( 1 - \frac{0.9\alpha}{\gamma_{\max}} \right)^k \left( \mathbb{E}[\|\theta_0 - \theta^*\|^2] + s_{\min}(\bar{A})b_{\max}^2 \right) + \alpha \tau \cdot \kappa.
\]

As such, we have completed the proof of Proposition 6.1.

Proof of Claim 1. Observe that
\[
\frac{0.9\alpha}{\gamma_{\max}} \leq \frac{0.9\alpha \tau}{\gamma_{\max}} \leq 2\alpha \tau \leq \frac{\tau}{\tau},
\]
where step (i) holds since $\tau \geq 1$, step (ii) follows from the bound (A.1), and step (iii) holds since $\alpha \tau \leq \frac{1}{4}$ under the stepsize condition (4.2). To proceed, we use the Bernoulli inequality $\left(1 + x\right)^t \geq 1 + xt, \forall x \geq -1, t \geq 1$, which is equivalent to $(1 - x)^{-t} \leq (1 - xt)^{-1}, \forall x \in (0, 1), t \in [1, 1/x)$. In light of equation (A.5), the Bernoulli inequality holds with $x = \frac{0.9\alpha}{\gamma_{\max}}$ and $t = \tau$, hence
\[
\left( 1 - \frac{0.9\alpha}{\gamma_{\max}} \right)^{-\tau} \leq \frac{1}{\frac{0.9\alpha}{\gamma_{\max}}} \leq 2,
\]
where the last step follows from (A.5). We have completed the proof of Claim 1.

A.1.1 Proof of Proposition A.1

To prove Proposition A.1, we need the following technical lemmas.

Lemma A.2. Given any $t \geq 1$, if $\alpha \cdot t \leq \frac{1}{4}$, then the following inequalities hold for all $k \geq t$,
\[
\begin{align*}
\|\theta_k - \theta_{k-t}\| & \leq 2\alpha t\|\theta_k - \theta_0\| + 2\alpha tb_{\max}, \\
\|\theta_k - \theta_{k-t}\| & \leq 4\alpha t\|\theta_k\| + 4\alpha tb_{\max}, \\
\|\theta_k - \theta_{k-t}\|^2 & \leq 32\alpha^2 t^2\|\theta_k\|^2 + 32\alpha^2 t^2b_{\max}^2.
\end{align*}
\]

Lemma A.3. The following inequality holds for any $k \geq 0$,
\[
\|\theta_{k+1} - \theta_k\| \leq 2\alpha^2 \gamma_{\max}\|\theta_k\|^2 + 2\alpha^2 \gamma_{\max}b_{\max}^2.
\]

Lemma A.4. The following inequality holds for all $k \geq \tau$, with $\alpha$ chosen sufficiently small such that $\alpha \tau \leq \frac{1}{4}$,
\[
\mathbb{E} \left[ \theta_k^\top \Gamma(A(x_k) - \bar{A})\theta_k \right] \leq \kappa_1 \mathbb{E}[\|\theta_k\|^2] + \kappa_2,
\]
where
\[
\kappa_1 = 88\alpha^2 \gamma_{\max} \quad \text{and} \quad \kappa_2 = 64\alpha^2 \gamma_{\max}b_{\max}^2.
\]

Lemma A.5. The following inequality holds for all $k \geq \tau$, with $\alpha$ chosen sufficiently small such that $\alpha \tau \leq \frac{1}{4}$,
\[
\mathbb{E} \left[ \theta_k^\top \Gamma(b(x_k) - \bar{b}) \right] \leq \bar{\kappa}_1 \mathbb{E}[\|\theta_k\|^2] + \bar{\kappa}_2,
\]
where
\[
\bar{\kappa}_1 = 5\alpha^2 \gamma_{\max} \quad \text{and} \quad \bar{\kappa}_2 = 15\alpha^2 \gamma_{\max}b_{\max}^2.
\]
The proofs of the technical lemmas above are delayed to Appendix A.1.2. Note that all lemmas above hold for the LSA update (3.1) with general $\bar{b}$. Below we shall apply these lemmas to the centered LSA update (A.4) for $\theta'_k$ with $\bar{b}' = 0$ to prove Proposition A.1.

**Proof of Proposition A.1.** Consider the following drift:

\[
\mathbb{E}[\theta'_{k+1}^\top \Gamma \theta'_{k+1} - \theta'_{k}^\top \Gamma \theta'_k] = 2\mathbb{E}[\theta'_k^\top \Gamma (\theta'_{k+1} - \theta'_k)] + 2\mathbb{E}[\theta'_k^\top \Gamma (\theta'_k(x_k) - \bar{A})\theta'_k] + 2\mathbb{E}[\theta'_k^\top \Gamma (\bar{A} \theta'_k) + \mathbb{E}[(\theta'_{k+1} - \theta'_k)^\top \Gamma (\theta'_{k+1} - \theta'_k)]].
\]

We can bound $T_1$ using Lemma A.4, $T_2$ using Lemma A.5, and $T_4$ using Lemma A.3. For $T_3$, we note that by the property of the Lyapunov equation in Assumption 3,

\[
2\alpha \mathbb{E}[\theta'_k^\top \Gamma \theta'_k] = \alpha \mathbb{E}[\theta'_k^\top (\bar{A}^\top \Gamma + \Gamma \bar{A}) \theta'_k] - \alpha \mathbb{E}[||\theta'_k||^2].
\]

Combining the above bounds, we derive that

\[
\mathbb{E}[\theta'_{k+1}^\top \Gamma \theta'_{k+1} - \theta'_{k}^\top \Gamma \theta'_k] = T_1 + T_2 + T_3 + T_4 \\
\leq 2\alpha (\kappa_1 \mathbb{E}[||\theta'_k||^2] + \kappa_2) + 2\alpha (\tilde{\kappa}_1 \mathbb{E}[||\theta'_k||^2] + \tilde{\kappa}_2) - \alpha \mathbb{E}[||\theta'_k||^2] + 2\alpha^2 \gamma_{\text{max}} \mathbb{E}[||\theta'_k||^2] + 2\alpha^2 \gamma_{\text{max}} \mathbb{E}[|\theta'_k||^2] + 2\alpha (\kappa_2 + \tilde{\kappa}_2 + \alpha \gamma_{\text{max}}) \mathbb{E}[||\theta'_k||^2] + 2\alpha (\kappa_2 + \tilde{\kappa}_2 + \alpha \gamma_{\text{max}}) \mathbb{E}[||\theta'_k||^2]
\]

We simplify the above bound by noting that

\[
\kappa_1 + \tilde{\kappa}_1 + \alpha \gamma_{\text{max}} = 88\alpha \tau \gamma_{\text{max}} + 5\alpha \tau \gamma_{\text{max}} + \alpha \gamma_{\text{max}} \leq 95\alpha \tau \gamma_{\text{max}},
\]

and

\[
\kappa_2 + \tilde{\kappa}_2 + \alpha \gamma_{\text{max}} |b'_\text{max}|^2 = 64\alpha \tau \gamma_{\text{max}} |b'_\text{max}|^2 + 15\alpha \tau \gamma_{\text{max}} |b'_\text{max}|^2 + \alpha \gamma_{\text{max}} |b'_\text{max}|^2 \leq 80\alpha \tau \gamma_{\text{max}} |b'_\text{max}|^2.
\]

Combining with the fact that $\alpha$ satisfies (4.2), we obtain that for all $k \geq \tau$,

\[
\mathbb{E}[\theta'_{k+1}^\top \Gamma \theta'_{k+1} - \theta'_{k}^\top \Gamma \theta'_k] \leq \left(1 - \frac{0.9\alpha}{\gamma_{\text{max}}} \right) \mathbb{E}[\theta'_{k}^\top \Gamma \theta'_k] + 160\alpha^2 \tau \gamma_{\text{max}} |b'_\text{max}|^2,
\]

or equivalently

\[
\mathbb{E}[\theta'_{k+1}^\top \Gamma \theta'_{k+1}] \leq \left(1 - \frac{0.9\alpha}{\gamma_{\text{max}}} \right) \mathbb{E}[\theta'_{k}^\top \Gamma \theta'_k] + 160\alpha^2 \tau \gamma_{\text{max}} |b'_\text{max}|^2.
\]

Next, we recursively apply the above inequality to obtain

\[
\mathbb{E}[\theta'_{k}^\top \Gamma \theta'_k] \leq \left(1 - \frac{0.9\alpha}{\gamma_{\text{max}}} \right) \mathbb{E}[\theta'_{k}^\top \Gamma \theta'_k] + \sum_{t=0}^{(k-\tau)-1} \left(1 - \frac{0.9\alpha}{\gamma_{\text{max}}} \right)^t \cdot \left(160\alpha^2 \tau \gamma_{\text{max}} |b'_\text{max}|^2 \right).
\]

We then apply the properties in (3.8) to the above inequality and obtain the following bounds in terms of $||\theta'_k||^2$, for $k \geq \tau$,

\[
\mathbb{E}[||\theta'_k||^2] \leq \gamma_{\text{max}} \mathbb{E}[||\theta'_k||^2] + \gamma_{\text{min}} \mathbb{E}[(\theta'_{k+1} - \theta'_{k})^\top \Gamma (\theta'_{k+1} - \theta'_{k})].
\]
Lastly, we have
\[
\|\theta_t'\|^2 \leq (\|\theta_t' - \theta_0\| + \|\theta_0\|)^2 \\
\overset{(i)}{\leq} ((1 + 2\alpha \tau)\|\theta_0\| + 2\alpha \tau b_{\max}^{'})^2 \\
\overset{(ii)}{\leq} (1.5\|\theta_0\| + 0.5b_{\max}^{'})^2 \leq 5\|\theta_0\|^2 + (b_{\max}')^2,
\]
where in step (i) we make use of Lemma A.2 to bound \(\|\theta_t' - \theta_0\|\) with \(\|\theta_0\|\), and step (ii) holds for \(\alpha\) is chosen according to (4.2) such that \(\alpha \tau < \frac{1}{4}\). Therefore, we have
\[
E[\|\theta_t'\|^2] \leq \frac{\gamma_{\max}}{\gamma_{\min}} \left(1 - \frac{0.9\alpha}{\gamma_{\max}}\right)^{k-\tau} (2E[\|\theta_0\|^2] + (b_{\max}')^2) + \frac{\gamma_{\max}}{0.9\gamma_{\min}} \cdot \alpha \tau (160\gamma_{\max}b_{\max}')^2.
\]
This concludes the proof for Proposition A.1.

### A.1.2 Proof of Technical Lemmas

We prove the technical lemmas stated at the beginning of the previous sub-sub-section.

**Proof of Lemma A.2.** Since \(\theta_{k+1} = \theta_k + \alpha(A(x_k)\theta_k + b(x_k))\), we have
\[
\|\theta_{k+1}\| \leq \|I + \alpha A(x_k)\|\|\theta_k\| + \alpha\|b(x_k)\| \leq (1 + \alpha)\|\theta_k\| + \alpha b_{\max}.
\]

Therefore, for \(k - t < i \leq k\), we have
\[
\|\theta_i\| \leq (1 + \alpha)^{i-(k-t)}\|\theta_{k-t}\| + \alpha b_{\max} \sum_{j=0}^{(i-1)-(k-t)} (1 + \alpha)^j \\
\overset{(i)}{\leq} (1 + \alpha)^t\|\theta_{k-t}\| + \alpha b_{\max} \sum_{j=0}^{l-1} (1 + \alpha)^j \leq (1 + 2\alpha t)\|\theta_{k-t}\| + 2\alpha t b_{\max},
\]
where step (i) holds since \(\alpha t \leq \frac{1}{4}\) \(\leq \log 2\). It follows that
\[
\|\theta_k - \theta_{k-t}\| = \left\| \sum_{i=k-t}^{k-1} \theta_{i+1} - \theta_i \right\| \leq \sum_{i=k-t}^{k-1} \|\theta_{i+1} - \theta_i\| = \alpha \sum_{i=k-t}^{k-1} \|A(x_k)\theta_i + b(x_k)\| \\
\leq \alpha A_{\max} \left( \sum_{i=k-t}^{k-1} \|\theta_i\| \right) + \alpha t b_{\max} \overset{\text{Assumption 2}}{=}
\leq \alpha A_{\max} \left( \sum_{i=k-t}^{k-1} (1 + 2\alpha t)\|\theta_{i-t}\| + 2\alpha t b_{\max} \right) + \alpha t b_{\max} \overset{\text{by (A.9)}}{=}
\overset{(i)}{\leq} 2\alpha t (A_{\max}\|\theta_{k-t}\| + b_{\max}) < 2\alpha t\|\theta_{k-t}\| + 2\alpha t b_{\max},
\]
where step (ii) holds since \(2\alpha t < 1\). As such, we have established (A.6).

With (A.6), it is easy to see that
\[
\|\theta_k - \theta_{k-t}\| \leq 2\alpha t\|\theta_{k-t}\| + 2\alpha t b_{\max} \leq 2\alpha t (\|\theta_k - \theta_{k-t}\| + \|\theta_k\|) + 2\alpha t b_{\max}.
\]
Reorganizing the above inequality gives \((1 - 2\alpha t)\|\theta_k - \theta_{k-t}\| \leq 2\alpha t\|\theta_k\| + 2\alpha t b_{\max}\). Together with \(\alpha t \leq \frac{1}{4}\), we obtain (A.7). Lastly, we use the inequality \((a + b)^2 \leq 2(a^2 + b^2)\) to obtain
\[
\|\theta_k - \theta_{k-t}\|^2 \leq (4\alpha t\|\theta_k\| + 4\alpha t b_{\max})^2 \leq 32\alpha^2 t^2\|\theta_k\|^2 + 32\alpha^2 t^2 b_{\max}^2,
\]
thereby proving the desired bound in (A.8).
Proof of Lemma A.3. We have
\[
\left|(\theta_{k+1} - \theta_k)\Gamma(\theta_{k+1} - \theta_k)\right| \leq \gamma_{\text{max}} \|\theta_{k+1} - \theta_k\|^2 \\
= \alpha^2 \gamma_{\text{max}} \|A(x_k)\theta_k + b(x_k)\|^2 \\
\leq \alpha^2 \gamma_{\text{max}} (A_{\text{max}} \|\theta_k\| + b_{\text{max}})^2 \\
\leq 2\alpha^2 \gamma_{\text{max}} \|\theta_k\|^2 + 2\alpha^2 \gamma_{\text{max}} b_{\text{max}}^2.
\]
This completes the proof of Lemma A.3.

Proof of Lemma A.4. Let us decompose the quantity of interest as
\[
\mathbb{E}[\theta_k^\top \Gamma(A(x_k) - \bar{A})\theta_k] = \mathbb{E}[((\theta_k - \theta_{k-\tau}) + \theta_{k-\tau})\Gamma(A(x_k) - \bar{A})(\theta_k - \theta_{k-\tau}) + \theta_{k-\tau}\Gamma(A(x_k) - \bar{A})\theta_{k-\tau}]
\]
\[
\quad = \mathbb{E}[(\theta_k - \theta_{k-\tau})\Gamma(A(x_k) - \bar{A})(\theta_k - \theta_{k-\tau})] + \mathbb{E}[\theta_{k-\tau}\Gamma(A(x_k) - \bar{A})\theta_{k-\tau}]
\]
\[
\quad \quad \quad \quad + \mathbb{E}[(\theta_k - \theta_{k-\tau})\Gamma(A(x_k) - \bar{A})\theta_{k-\tau}] + \mathbb{E}[\theta_{k-\tau}\Gamma(A(x_k) - \bar{A})(\theta_k - \theta_{k-\tau})].
\]
We bound each of the RHS terms respectively.
For \( T_1 \), we have
\[
T_1 = \mathbb{E}[(\theta_k - \theta_{k-\tau})\Gamma(A(x_k) - \bar{A})(\theta_k - \theta_{k-\tau})]
\]
\[
\quad \quad \quad \quad \leq 2\gamma_{\text{max}} \mathbb{E}[\|\theta_k - \theta_{k-\tau}\|^2]
\]
\[
\quad \quad \quad \quad \leq 2\gamma_{\text{max}} \left(32\alpha^2 \gamma_{\text{max}} \mathbb{E}[(\|\theta_k\|^2) + 32\alpha^2 \gamma_{\text{max}} b_{\text{max}}^2\right)
\]
\[
\quad \quad \quad \quad \leq 64\gamma_{\text{max}} \alpha^2 \gamma_{\text{max}} \mathbb{E}[\|\theta_k\|^2] + 64\gamma_{\text{max}} \alpha^2 \gamma_{\text{max}} b_{\text{max}}^2,
\]
where (i) holds since both \( \|A(x_k)\| \leq 1 \) and \( \|\bar{A}\| \leq 1 \) by Assumption 2 and \( \Gamma \) is symmetric with top eigenvalue \( \gamma_{\text{max}} \) by Assumption 3, and (ii) follows from equation (A.8) in Lemma A.2.
For \( T_2 \), we have
\[
T_2 = \mathbb{E}[(\theta_k^\top \Gamma(A(x_k) - \bar{A})\theta_{k-\tau} | \theta_{k-\tau}, x_{k-\tau})]
\]
\[
\quad = \mathbb{E}[\theta_{k-\tau}^\top \Gamma \mathbb{E}[A(x_k) - \bar{A} | \theta_{k-\tau}, x_{k-\tau}] \theta_{k-\tau}]
\]
\[
\quad \quad \quad \quad \quad \equiv \mathbb{E}[\theta_{k-\tau}^\top \Gamma \mathbb{E}[A(x_k) - \bar{A} | x_{k-\tau}] \theta_{k-\tau}]
\]
where step (iii) follows from conditional independence property \( x_k \perp \perp \theta_{k-\tau} | x_{k-\tau} \) shown in equation (A.2). Since \( \Gamma \) has largest eigenvalue \( \gamma_{\text{max}} \) by Assumption 3 and \( \tau \equiv \tau_{\alpha} \) is the \( \alpha \)-mixing time, which by definition ensures that \( A(x_k) \) is sufficiently close to \( \bar{A} \) in expectation, it follows that
\[
T_2 \leq \alpha \gamma_{\text{max}} \mathbb{E}[\|\theta_{k-\tau}\|^2]
\]
\[
\quad \leq \alpha \gamma_{\text{max}} \mathbb{E}[(\|\theta_k - \theta_{k-\tau}\| + \|\theta_k\|)^2]
\]
\[
\quad \leq \alpha \gamma_{\text{max}} \mathbb{E}[(4\alpha \tau \|\theta_k\| + 4\alpha \tau b_{\text{max}} + \|\theta_k\|)^2] \quad \text{by (A.7)}
\]
\[
\quad \leq \alpha \gamma_{\text{max}} \cdot 2 \left((1 + 4\alpha \tau)^2 \mathbb{E}[\|\theta_k\|^2] + 16\alpha^2 \tau^2 b_{\text{max}}^2\right)
\]
\[
\quad \leq 8\alpha \gamma_{\text{max}} \mathbb{E}[\|\theta_k\|^2] + 32\alpha^3 \tau^2 \gamma_{\text{max}} b_{\text{max}}^2,
\]
where (iv) follows from the inequality \((a + b)^2 \leq 2(a^2 + b^2)\), and the last step holds since \( \alpha \tau \leq \frac{1}{4} \) and \( \tau \geq 1 \).
We first make use of the law of total expectation and obtain that

\[
T_3 = \mathbb{E} \left[ (\theta_k - \theta_{k-\tau})^\top \Gamma(A(x_k) - \bar{A}) \theta_{k-\tau} \right]
\]

\[
\leq 2\gamma_{\text{max}} \mathbb{E} \left[ \|\theta_k - \theta_{k-\tau}\| \cdot (\|\theta_k - \theta_{k-\tau}\| + \|\theta_k\|) \right]
\]

\[
\leq 2\gamma_{\text{max}} \mathbb{E} \left[ (4\alpha \tau \|\theta_k\| + 4\alpha \tau b_{\text{max}})(4\alpha \tau \|\theta_k\| + 4\alpha \tau b_{\text{max}} + \|\theta_k\|) \right] \quad \text{by (A.7)}
\]

\[
= 8\alpha \tau + 4\alpha \tau \gamma_{\text{max}} \mathbb{E} \left[ \|\theta_k\|^2 \right] + 8\alpha \tau + 8\alpha \tau \gamma_{\text{max}} b_{\text{max}} \mathbb{E} [\|\theta_k\|] + 32\alpha^2 \tau^2 \gamma_{\text{max}} b_{\text{max}}^2
\]

\[(v)\]

\[
\leq 8\alpha \tau (1 + 4\alpha \tau) \gamma_{\text{max}} \mathbb{E} \left[ \|\theta_k\|^2 \right] + 4\alpha \tau (1 + 8\alpha \tau) \gamma_{\text{max}} \left( b_{\text{max}}^2 + \mathbb{E}[\|\theta_k\|^2] \right) + 32\alpha^2 \tau^2 \gamma_{\text{max}} b_{\text{max}}^2
\]

\[
= 4\alpha \tau \gamma_{\text{max}} \left( (1 + 4\alpha \tau) + (1 + 8\alpha \tau) \right) \mathbb{E}[\|\theta_k\|^2] + 4\alpha \tau \gamma_{\text{max}} (1 + 8\alpha \tau) + 8\alpha \tau) b_{\text{max}}^2
\]

\[(vi)\]

\[
\leq 32\alpha \tau \gamma_{\text{max}} \mathbb{E}[\|\theta_k\|^2] + 20\alpha \tau \gamma_{\text{max}} b_{\text{max}}^2
\]

where (v) utilizes the inequality \(2b_{\text{max}} \mathbb{E}[\|\theta_k\|^2] \leq b_{\text{max}}^2 + \mathbb{E}[\|\theta_k\|^2]\), and (vi) holds with \(\alpha \tau \leq \frac{1}{4}\).

Similarly, for \(T_4\), we have for \(\alpha \tau \leq \frac{1}{4}\),

\[
T_4 = \mathbb{E} \left[ \theta_k^\top \Gamma(A(x_k) - \bar{A}) (\theta_k - \theta_{k-\tau}) \right]
\]

\[
\leq 32\alpha \tau \gamma_{\text{max}} \mathbb{E}[\|\theta_k\|^2] + 20\alpha \tau \gamma_{\text{max}} b_{\text{max}}^2.
\]

Combining the bounds for \(T_1 - T_4\), we obtain that

\[
\mathbb{E} \left[ \theta_k^\top \Gamma(A(x_k) - \bar{A}) \theta_k \right] = T_1 + T_2 + T_3 + T_4
\]

\[
\leq (64\gamma_{\text{max}} \alpha^2 \tau^2 \mathbb{E}[\|\theta_k\|^2] + 64\gamma_{\text{max}} \alpha^2 \tau^2 b_{\text{max}}^2) + (8\alpha \tau \gamma_{\text{max}} \mathbb{E}[\|\theta_k\|^2] + 32\alpha^2 \tau^2 \gamma_{\text{max}} b_{\text{max}}^2)
\]

\[
+ 2 (32\alpha \tau \gamma_{\text{max}} \mathbb{E}[\|\theta_k\|^2] + 20\alpha \tau \gamma_{\text{max}} b_{\text{max}}^2)
\]

\[
\leq 88\alpha \tau \gamma_{\text{max}} \mathbb{E}[\|\theta_k\|^2] + 64\alpha \tau \gamma_{\text{max}} b_{\text{max}}^2,
\]

where the last step holds with \(\alpha \leq 1\) and \(\alpha \tau \leq \frac{1}{4}\). This completes the proof of Lemma A.4.

**Proof of Lemma A.5.** We first make use of the law of total expectation and obtain that

\[
\mathbb{E} \left[ \theta_k^\top \Gamma(b(x_k) - \bar{b}) \mid \theta_k, x_k \right] = \mathbb{E} \left[ \mathbb{E} \left[ \theta_k^\top \Gamma(b(x_k) - \bar{b}) \mid \theta_k, x_k \right] \mid \theta_k \right].
\]

We decompose the inner expectation as

\[
\mathbb{E} \left[ \theta_k^\top \Gamma(b(x_k) - \bar{b}) \mid \theta_k, x_k \right] = \mathbb{E} \left[ \theta_k^\top \Gamma(b(x_k) - \bar{b}) \mid \theta_k, x_k \right]
\]

\[
= \mathbb{E} \left[ \theta_k^\top \Gamma(b(x_k) - \bar{b}) \mid \theta_k, x_k \right] + \mathbb{E} \left[ \theta_k^\top \Gamma(b(x_k) - \bar{b}) \mid \theta_k, x_k \right].
\]

We separately bound \(T_1\) and \(T_2\). For \(T_1\), we have

\[
\mathbb{E} \left[ (\theta_k - \theta_k)^\top \Gamma(b(x_k) - \bar{b}) \mid \theta_k, x_k \right] \leq 2b_{\text{max}} \gamma_{\text{max}} \mathbb{E} [\|\theta_k - \theta_k\| \mid \theta_k, x_k] \leq 2b_{\text{max}} \gamma_{\text{max}} (2\alpha \tau \|\theta_k\| + 2\alpha \tau b_{\text{max}}),
\]

where we use (A.6) to obtain the last inequality. For \(T_2\), we have

\[
\mathbb{E} \left[ \theta_k^\top \Gamma(b(x_k) - \bar{b}) \mid \theta_k, x_k \right] = \theta_k^\top \Gamma \mathbb{E} \left[ (b(x_k) - \bar{b}) \mid \theta_k, x_k \right] \leq \alpha \gamma_{\text{max}} b_{\text{max}} \|\theta_k\|.
\]

Combining the two terms, we have

\[
\mathbb{E} \left[ \theta_k^\top \Gamma(b(x_k) - \bar{b}) \mid \theta_k, x_k \right] \leq \alpha \gamma_{\text{max}} b_{\text{max}} \|\theta_k\| + 2b_{\text{max}} \gamma_{\text{max}} (2\alpha \tau \|\theta_k\| + 2\alpha \tau b_{\text{max}})
\]

\[
= \alpha \gamma_{\text{max}} b_{\text{max}} (1 + 4\tau) \|\theta_k\| + 4\alpha \tau \gamma_{\text{max}} b_{\text{max}}^2
\]

\[
\leq \alpha \gamma_{\text{max}} b_{\text{max}} (1 + 4\tau) (\mathbb{E} [\|\theta_k - \theta_k\| \mid \theta_k, x_k] + \mathbb{E} [\|\theta_k\| \mid \theta_k, x_k]) + 4\alpha \tau \gamma_{\text{max}} b_{\text{max}}^2
\]

\[(i)\]

\[
\leq \alpha \gamma_{\text{max}} b_{\text{max}} (1 + 4\tau) ((1 + 4\alpha \tau) \mathbb{E} [\|\theta_k\| \mid \theta_k, x_k] + 4\alpha \tau b_{\text{max}}) + 4\alpha \tau \gamma_{\text{max}} b_{\text{max}}^2
\]

\[
\leq 10\alpha \gamma_{\text{max}} b_{\text{max}} \mathbb{E} [\|\theta_k\| \mid \theta_k, x_k] + 9\alpha \gamma_{\text{max}} b_{\text{max}}^2,
\]

\[32\]
where we use (A.7) to obtain (i), and $\alpha \tau \leq \frac{1}{4}$, $\alpha \leq 1$ and $\tau \geq 1$ to obtain the last inequality. Using the inequality $2b_{\max} \|\theta_k\| \leq b_{\max}^2 + \|\theta_k\|^2$, we simplify the above display equation to

$$
\mathbb{E}\left[\theta_k^\top \Gamma(b(x_k) - \tilde{b}) \mid \theta_{k-\tau}, x_{k-\tau}\right] \leq 5\alpha \tau \gamma_{\max}(b_{\max}^2 + \mathbb{E}\left[\|\theta_k\|^2 \mid \theta_{k-\tau}, x_{k-\tau}\right]) + 9\alpha \tau \gamma_{\max} b_{\max}^2 \\
\leq 5\alpha \tau \gamma_{\max} \mathbb{E}\left[\|\theta_k\|^2 \mid \theta_{k-\tau}, x_{k-\tau}\right] + 15\alpha \tau \gamma_{\max} b_{\max}^2.
$$

Lastly, we take expectations on both sides of the last display equation to obtain

$$
\mathbb{E}\left[\theta_k^\top \Gamma(b(x_k) - \tilde{b})\right] \leq 5\alpha \tau \gamma_{\max} \mathbb{E}[\|\theta_k\|^2] + 15\alpha \tau \gamma_{\max} b_{\max}^2.
$$

This completes the proof of Lemma A.5.

\[\square\]

### A.2 Proof of Theorem 4.1

In this sub-section, we prove Theorem 4.1 on the convergence of LSA to a limit.

#### A.2.1 Coupling and Geometric Convergence

Recall that $(x_k)_{k \geq 0}$ is the underlying Markov chain that drives the LSA iteration (3.1). We consider a pair of coupled Markov chains, $(x_k, \theta^{[1]}_k)_{k \geq 0}$ and $(x_k, \theta^{[2]}_k)_{k \geq 0}$, defined as

$$
\theta^{[1]}_{k+1} = \theta^{[1]}_k + \alpha (A(x_k) \theta^{[1]}_k + b(x_k)), \\
\theta^{[2]}_{k+1} = \theta^{[2]}_k + \alpha (A(x_k) \theta^{[2]}_k + b(x_k)),
$$

$k = 0, 1, \ldots$ \hspace{1cm} (A.10)

Note that $(\theta^{[1]}_k)_{k \geq 0}$ and $(\theta^{[2]}_k)_{k \geq 0}$ are two sample paths of the LSA iteration (3.1), coupled by sharing the underlying process $(x_k)_{k \geq 0}$. We assume that the initial iterates $\theta^{[1]}_0$ and $\theta^{[2]}_0$ may depend on each other and on $x_0$, but are independent of subsequent $(x_k)_{k \geq 1}$ given $x_0$.

It follows from the definition that

$$
\theta^{[1]}_{k+1} - \theta^{[2]}_{k+1} = (I + \alpha A(x_k)) \cdot (\theta^{[1]}_k - \theta^{[2]}_k),
$$

$k = 0, 1, \ldots$

If we define the shorthand $\omega_k := \theta^{[1]}_k - \theta^{[2]}_k$, then the above equation becomes

$$
\omega_{k+1} = (I + \alpha A(x_k)) \cdot \omega_k, \hspace{1cm} k = 0, 1, \ldots
$$

(A.11)

Our key observation is that equation (A.11) is a special case of the LSA iteration (3.1) with $\omega_k$ as the variable and $b_{\max} = \sup_{x \in X} \|b(x)\| = 0$. Applying Proposition 6.1 to this LSA iteration, we obtain the following finite-time geometric bound.

**Corollary A.6.** Suppose that $\alpha$ satisfies (4.2). Then, for all $k \geq \tau$, we have

$$
W_2^2\left(\mathcal{L}(\theta^{[1]}_k), \mathcal{L}(\theta^{[2]}_k)\right) \leq W_2^2\left(\mathcal{L}(x_k, \theta^{[1]}_k), \mathcal{L}(x_k, \theta^{[2]}_k)\right)
$$

$$
\leq \mathbb{E}\left[\|\theta^{[1]}_k - \theta^{[2]}_k\|^2\right]$$

$$
\leq 10 \frac{\gamma_{\max}}{\gamma_{\min}} \left(1 - \frac{9\alpha}{\gamma_{\max}}\right)^k \mathbb{E}\left[\|\theta^{[1]}_0 - \theta^{[2]}_0\|^2\right].
$$

**Proof of Corollary A.6.** The inequality (i) follows from the definition of $W_2$ and $\tilde{W}_2$. The inequality (ii) holds since the Wasserstein distance is defined by an infimum as in equation (4.1). Inequality (iii) follows from applying Proposition 6.1 with $b_{\max} = 0$ to the LSA iteration (A.11).

\[\square\]

With Corollary A.6, we are ready to prove Theorem 4.1 on the convergence of the Markov chain $(x_k, \theta_k)_{k \geq 0}$. Theorem 4.1 has three parts, whose proofs are given in the next three sub-sub-sections.
A.2.2 Part 1: Existence of Limiting Distribution

Note that Corollary A.6 is valid under any joint distribution of initial iterates \((x_0, \theta_0^{[1]}, \theta_0^{[2]})\). Arbitrarily fix the distribution of \((x_0, \theta_0^{[1]})\). Given \((x_0, \theta_0^{[1]})\), we shall judiciously choose the conditional distribution of \(\theta_0^{[2]}\) in a way that ensures \((x_k, \theta_k^{[2]}) \overset{d}{=} (x_{k+1}, \theta_{k+1}^{[1]})\) for all \(k \geq 0\), where \(\overset{d}{=}\) denotes equality in distribution. Specifically, recall that the adjoint operator \(P^*\) is the transition probability matrix for the time-reversed Markov chain of \((x_k)_{k \geq 0}\) and that the initial distribution of \(x_0\) is assumed to be the stationary distribution \(\pi\); see Sections 3.1 and 3.2. Given \(x_0\), let \(x_{-1}\) be sampled from \(P^*(x_0, \cdot)\). Let \(\theta_0^{[2]}\) be a random variable which satisfies \(\theta_0^{[2]} \overset{d}{=} \theta_0^{[1]}\) and is independent of \((x_k)_{k \geq -1}\). Finally, set \(\theta_0^{[2]}\) as

\[
\theta_0^{[2]} = \theta_0^{[1]} + \alpha \left( A(x_{-1}) \theta_{-1}^{[2]} + b(x_{-1}) \right). 
\]  

(A.12)

We argue that this initialization has the desired property.

Claim 2. Under the assumptions in Theorem 4.1 and the initialization (A.12), we have \((x_k, \theta_k^{[2]} \overset{d}{=} (x_{k+1}, \theta_{k+1}^{[1]})\) for all \(k \geq 0\).

Proof of Claim 2. From standard results on time-reversed Markov chains, we have \((x_k)_{k \geq -1} \overset{d}{=} (x_k)_{k \geq 0}\). Since by construction \(\theta_0^{[2]} \overset{d}{=} \theta_0^{[1]}\) and \(\theta_0^{[2]}\) is independent of \((x_k)_{k \geq -1}\), the claim follows from comparing the update rules for \((\theta_k^{[1]})_{k \geq 0}\) and \((\theta_k^{[2]})_{k \geq -1}\) given in equations (A.10) and (A.12).

Using the above claim, we have for all \(k \geq \tau\),

\[
W_2^2 \left( \mathcal{L}_k(x_k, \theta_k^{[1]}), \mathcal{L}_k(x_{k+1}, \theta_{k+1}^{[1]}) \right) = W_2^2 \left( \mathcal{L}_k(x_k, \theta_k^{[1]}), \mathcal{L}_k(x_k, \theta_k^{[2]}) \right) 
\leq 10 \frac{\gamma_{\max}}{\gamma_{\min}} \left( 1 - \frac{0.9\alpha}{\gamma_{\max}} \right)^k \mathbb{E}[\|\theta_0^{[1]} - \theta_0^{[2]}\|^2],
\]

where in the second step above we use Corollary A.6. It follows that

\[
\sum_{k=0}^{\infty} W_2^2 \left( \mathcal{L}_k(x_k, \theta_k^{[1]}), \mathcal{L}_k(x_{k+1}, \theta_{k+1}^{[1]}) \right) \leq \sum_{k=0}^{\tau-1} W_2^2 \left( \mathcal{L}_k(x_k, \theta_k^{[1]}), \mathcal{L}_k(x_{k+1}, \theta_{k+1}^{[1]}) \right) + 10 \frac{\gamma_{\max}}{\gamma_{\min}} \sum_{k=\tau}^{\infty} \left( 1 - \frac{0.9\alpha}{\gamma_{\max}} \right)^k \mathbb{E}[\|\theta_0^{[1]} - \theta_0^{[2]}\|^2] < \infty,
\]

where the last step holds since \(\frac{0.9\alpha}{\gamma_{\max}} \in (0, 1)\) under the assumption (4.2). The inequality above means that \((\mathcal{L}_k(x_k, \theta_k^{[1]}))_{k \geq 0}\) is a Cauchy sequence in the metric \(\tilde{W}_2\). Since the space \(P_2(\mathcal{X} \times \mathbb{R}^d)\) endowed with \(\tilde{W}_2\) is a Polish space [Vil09, Theorem 6.18], every Cauchy sequence converges. Furthermore, convergence in Wasserstein distance implies weak convergence [Vil09, Theorem 6.9]. We conclude that the sequence \((\mathcal{L}_k(x_k, \theta_k^{[1]}))_{k \geq 0}\) converges weakly to a limit \(\tilde{\mu} \in P_2(\mathcal{X} \times \mathbb{R}^d)\).

We next show that the limit \(\tilde{\mu}\) is independent of the initial distribution of \(\theta_0^{[1]}\). Recall that \(x_0\) is initialized from its unique stationary distribution \(\pi\) by Assumption 1. Suppose that another sequence \((x_k, \tilde{\theta}_k^{[1]})_{k \geq 0}\) with a different initial distribution converges to a limit \(\tilde{\mu}\), then following from the triangle inequality property for Wasserstein distance, we obtain

\[
W_2(\tilde{\mu}, \tilde{\mu}) \leq W_2 \left( \tilde{\mu}, \mathcal{L}(x_k, \theta_k^{[1]}) \right) + W_2 \left( \mathcal{L}(x_k, \theta_k^{[1]}), \mathcal{L}(x_k, \tilde{\theta}_k^{[1]}) \right) + W_2 \left( \mathcal{L}(x_k, \tilde{\theta}_k^{[1]}), \tilde{\mu} \right) \overset{k \to \infty}{\longrightarrow} 0, 
\]  

(A.13)

where the last step holds since \(W_2(\mathcal{L}(x_k, \theta_k^{[1]}), \mathcal{L}(x_k, \tilde{\theta}_k^{[1]})) \overset{k \to \infty}{\longrightarrow} 0\) by Corollary A.6. Therefore, we have \(W_2(\tilde{\mu}, \tilde{\mu}) = 0\) and hence the limit \(\tilde{\mu}\) is unique.

Finally, the bound on \(\text{Var}(\theta_\infty)\) follows from the lemma below. Recall that \(\kappa\) is defined in (4.3).
Lemma A.7. Under Assumptions 1, 2 and 3, and when $\alpha$ is chosen according to (4.2), we have

$$\text{Tr}(\text{Var}(\theta_\infty)) \leq E[\|\theta_\infty - \theta^*\|^2] \leq \alpha \tau \cdot \kappa \quad \text{(A.14)}$$

and

$$\left( E[\|\theta_\infty\|^2] \right)^2 \leq E[\|\theta_\infty\|^2] \leq C(a, b, \pi) \quad \text{(A.15)}$$

for some $C(a, b, \pi)$ that is independent of $\alpha$.

Proof of Lemma A.7. We have shown that the sequence $(\theta_k)_{k \geq 0}$ converges weakly to $\theta_\infty$ in $P_2(\mathbb{R}^d)$. It is known that weak convergence in $P_2(\mathbb{R}^d)$ is equivalent to convergence in distribution and the convergence of the first two moments [Vil99, Definition 6.8]. Consequently, we have

$$E[\|\theta_\infty - \theta^*\|^2] = \lim_{k \to \infty} E[\|\theta_k - \theta^*\|^2]. \quad \text{(A.16)}$$

Proposition 6.1 presents the following upper bound on $E[\|\theta_\infty - \theta^*\|^2]$ that

$$E[\|\theta_k - \theta^*\|^2] \leq 10 \frac{\gamma_{\max}}{\gamma_{\min}} \left(1 - \frac{0.9a}{\gamma_{\max}}\right)^k \left[ E[\|\theta_0 - \theta^*\|^2] + s_{\min}(A)\sigma^2 \right] + \alpha \tau \cdot \kappa.$$ 

Taking $k \to \infty$ and combining with equation (A.16) give $E[\|\theta_\infty - \theta^*\|^2] \leq \alpha \tau \cdot \kappa \leq \frac{1}{4} \kappa$, where the last step holds since $\alpha \tau \leq \frac{1}{4}$. Equation (A.14) follows since $\theta^*$ is a deterministic quantity.

Furthermore, we have

$$\left( E[\|\theta_\infty\|^2] \right)^2 \leq E[\|\theta_\infty - \theta^*\|^2] \leq 2E[\|\theta_\infty - \theta^*\|^2] + 2\|\theta^*\|^2 \leq \frac{1}{2} \kappa + 2\|\theta^*\|^2. \quad \text{(A.17)}$$

Equation (A.15) then follows from noting that $\gamma_{\max}, \gamma_{\min}, \kappa$ and $\theta^*$ only depend on $A, b$, and $\pi$. $\square$

We have proved part 1 of Theorem 4.1.

A.2.3 Part 2: Invariance

We next show that $\mu$ is the unique invariant distribution. Suppose that the initial distribution of $(x_0, \theta_0)$ is $\tilde{\mu}$. By the triangle inequality of Wasserstein distance, we have

$$\bar{W}_2(\mathcal{L}(x_1, \theta_1), \tilde{\mu}) \leq \bar{W}_2(\mathcal{L}(x_1, \theta_1), \mathcal{L}(x_{k+1}, \theta_{k+1})) + \bar{W}_2(\mathcal{L}(x_{k+1}, \theta_{k+1}), \tilde{\mu}). \quad \text{(A.18)}$$

We proceed by noting the following lemma, whose proof is given at the end of this sub-sub-section.

Lemma A.8. Let $(x_k, \theta_k)_{k \geq 0}$ and $(x'_k, \theta'_k)_{k \geq 0}$ be two copies of the LSA trajectory (3.1), where $\mathcal{L}(x_0, \theta_0) = \tilde{\mu}$ and $\mathcal{L}(x'_0, \theta'_0) \in P_2(\mathcal{X} \times \mathbb{R}^d)$ is arbitrary. Under Assumptions 1, 2 and 3, and when $\alpha$ is chosen according to equation (4.2), we have

$$\bar{W}_2^2(\mathcal{L}(x_1, \theta_1), \mathcal{L}(x'_1, \theta'_1)) \leq \rho_1 \cdot \bar{W}_2^2(\mathcal{L}(x_0, \theta_0), \mathcal{L}(x'_0, \theta'_0)) + \sqrt{\rho_2 \cdot \bar{W}_2^2(\mathcal{L}(x_0, \theta_0), \mathcal{L}(x'_0, \theta'_0))}, \quad \text{(A.19)}$$

where the quantities $\rho_1 := 1 + 2(1 + \alpha)^2 + 16\alpha^2 \sigma_{\max}^2 < \infty$ and $\rho_2 := 16\alpha^2 \cdot E_{\theta \sim \mu} [\|\theta_0\|^2] < \infty$ are independent of $\mathcal{L}(x'_0, \theta'_0)$. In particular, for any $k \geq 0$, if we set $\mathcal{L}(x'_0, \theta'_0) = \mathcal{L}(x_k, \theta_k)$, then

$$\bar{W}_2^2(\mathcal{L}(x_1, \theta_1), \mathcal{L}(x_{k+1}, \theta_{k+1})) \leq \rho_1 \cdot \bar{W}_2^2(\tilde{\mu}, \mathcal{L}(x_k, \theta_k)) + \sqrt{\rho_2 \cdot \bar{W}_2^2(\tilde{\mu}, \mathcal{L}(x_k, \theta_k))}. \quad \text{(A.20)}$$

Applying Lemma A.8 to bound the first term on the RHS of equation (A.18), we obtain that

$$\bar{W}_2(\mathcal{L}(x_1, \theta_1), \tilde{\mu}) \leq \sqrt{\rho_1 \cdot \bar{W}_2(\tilde{\mu}, \mathcal{L}(x_k, \theta_k)) + \sqrt{\rho_2 \cdot \bar{W}_2(\tilde{\mu}, \mathcal{L}(x_k, \theta_k)) + \bar{W}_2(\mathcal{L}(x_{k+1}, \theta_{k+1}), \tilde{\mu})}} \xrightarrow{k \to \infty} 0,$$

where the last step follows from the weak convergence result established in the last sub-sub-section. We therefore conclude that $\bar{W}_2(\mathcal{L}(x_1, \theta_1), \tilde{\mu}) = 0$ and hence $\tilde{\mu}$ is an invariant distribution of the Markov chain $(x_k, \theta_k)_{k \geq 0}$. The uniqueness of the invariant distribution follows from a similar argument as in equation (A.13). We have proved part 2 of Theorem 4.1.
Proof of Lemma A.8. We choose a coupling between the two processes \((x_k, \theta_k)_{k \geq 0}\) and \((x'_k, \theta'_k)_{k \geq 0}\) such that

\[
\hat{W}_2^2(\mathcal{L}(x_0, \theta_0), \mathcal{L}(x'_0, \theta'_0)) = \mathbb{E} \left[ d_0(x_0, x'_0) + \|\theta_0 - \theta'_0\|^2 \right] \quad \text{and} \quad x_{k+1} = x'_{k+1} \quad \text{if} \quad x_k = x'_k, \quad \forall k \geq 0. \tag{A.21}
\]

The existence of a coupling satisfying equation (A.21) at step \(k = 0\) is a standard result in optimal transport [Vil09, Theorem 4.1]. We can ensure equation (A.22) by further coupling the two processes for the subsequent steps \(k \geq 1\), such that the two underlying Markov chains \((x_k)_{k \geq 0}\) and \((x'_k)_{k \geq 0}\) evolve separately (subject to the above coupling at step \(k = 0\)) until they reach the same state, after which they coalesce and follow the same trajectory.

To prove Lemma A.8, we begin by observing that

\[
\hat{W}_2^2(\mathcal{L}(x_1, \theta_1), \mathcal{L}(x'_1, \theta'_1)) \leq \mathbb{E} \left[ d_0(x_1, x'_1) + \|\theta_1 - \theta'_1\|^2 \right]. \tag{A.23}
\]

thanks to the definition (4.1) of \(\hat{W}_2\) using an infimum. Recalling the definition of the discrete metric \(d_0(x'_0, x_0) := \mathbb{1} \{x'_0 \neq x_0\}\), we have the identities

\[
A(x_0) = A(x'_0) + d_0(x'_0, x_0) \cdot \left( A(x_0) - A(x'_0) \right) \quad \text{and} \quad b(x_0) = b(x'_0) + d_0(x'_0, x_0) \cdot (b(x_0) - b(x'_0)).
\]

The update rule (3.1) together with the above identities implies that

\[
\theta_1 - \theta'_1 - \theta_0 + \alpha \left( A(x_0) - A(x'_0) \right) - \theta_0 - \alpha \left( A(x'_0) - b(x'_0) \right) = (I + \alpha A(x'_0)) \cdot (\theta_0 - \theta'_0) + \alpha d_0(x'_0, x_0) \cdot \left( (A(x_0) - A(x'_0)) \theta_0 + b(x_0) - b(x'_0) \right),
\]

whence

\[
\|\theta_1 - \theta'_1\| \leq \|I + \alpha A(x'_0)\| \cdot \|\theta_0 - \theta'_0\| + \alpha d_0(x'_0, x_0) \cdot \left\| (A(x_0) - A(x'_0)) \theta_0 + b(x_0) - b(x'_0) \right\|
\]

\[
\leq (1 + \alpha) \|\theta_0 - \theta'_0\| + \alpha d_0(x'_0, x_0) \cdot 2 \left( \|\theta_0\| + b_{\max} \right),
\]

where the last step follows from the boundedness Assumption 2. Also note that \(d_0(x_1, x'_1) \leq d_0(x_0, x'_0)\) thanks to the coupling in equation (A.22). Combining the above inequalities gives

\[
\mathbb{E} \left[ d_0(x_1, x'_1) + \|\theta_1 - \theta'_1\|^2 \right] 
\]

\[
\leq \mathbb{E} [d_0(x_0, x'_0)] + 2(1 + \alpha)^2 \cdot \mathbb{E} \left[ \|\theta_0 - \theta'_0\|^2 \right] + 2\alpha^2 \cdot \mathbb{E} \left[ d_0(x'_0, x_0) \cdot 8\|\theta_0\|^2 + b_{\max}^2 \right]. \tag{A.24}
\]

By Cauchy-Schwarz’s inequality, we have

\[
\mathbb{E} \left[ d_0(x'_0, x_0) \cdot \|\theta_0\|^2 \right] \leq \sqrt{\mathbb{E} [d_0(x'_0, x_0)]} \sqrt{\mathbb{E}_{\theta_0 \sim \mu} \|\theta_0\|^4}. \tag{A.25}
\]

Moreover, we claim that

\[
\mathbb{E}_{\theta_0 \sim \mu} \|\theta_0\|^4 = \mathbb{E} \|\theta_\infty\|^4 < \infty. \tag{A.26}
\]

This claim follows from a moderate tightening of the result in [SY19, Theorem 9], which provides sufficient conditions for the existence of higher moments of \(\theta_\infty\). In Appendix B, we explain how to tighten their result to show that the 4th moment exists under our stepsizes condition (4.2).

Combining equations (A.24) and (A.25) and recalling the values of \(\rho_1\) and \(\rho_2\) given in the statement of the lemma, we obtain that

\[
\mathbb{E} \left[ d_0(x_1, x'_1) + \|\theta_1 - \theta'_1\|^2 \right] 
\]

\[
\leq \rho_1 \cdot \mathbb{E} \left[ d_0(x_0, x'_0) + \|\theta_0 - \theta'_0\|^2 \right] + \sqrt{\rho_2} \cdot \mathbb{E} \left[ d_0(x_0, x'_0) + \|\theta_0 - \theta'_0\|^2 \right]
\]

\[
= \rho_1 \cdot \hat{W}_2^2 (\mathcal{L}(x_0, \theta_0), \mathcal{L}(x'_0, \theta'_0)) + \sqrt{\rho_2} \cdot \hat{W}_2^2 (\mathcal{L}(x_0, \theta_0), \mathcal{L}(x'_0, \theta'_0)), \tag{A.27}
\]

where the last step from our choice of coupling in equation (A.21). Combining equations (A.23) and (A.27) proves the first equation (A.19) in Lemma A.8. The second equation (A.20) is then immediate. \qed
A.2.4 Part 3: Convergence Rate

We have established that the joint sequence \((L(x_k, \theta_k^{(1)}))_{k \geq 0}\) converges weakly to the invariant distribution \(\bar{\mu} \in P_2(\mathcal{X} \times \mathbb{R}^d)\). Consequently, \((L(\theta_k^{(1)}))_{k \geq 0}\) converges weakly to \(\mu \in P_2(\mathbb{R}^d)\), where \(\mu\) is the marginal distribution of \(\bar{\mu}\) over \(\mathbb{R}^d\). We now characterize the convergence rate.

Again consider the coupled processes defined in equation (A.10). Suppose that the initial distribution of \((x_0, \theta_0^{(2)})\) is the invariant distribution \(\bar{\mu}\), hence \(L(x_k, \theta_k^{(2)}) = \bar{\mu}\) and \(L(\theta_k) = \mu\) for all \(k \geq 0\). Applying Corollary A.6, we have for all \(k \geq \tau\),

\[
W_2^2(L(\theta_k^{(1)}), \mu) = W_2^2(L(\theta_k^{(1)}), L(\theta_k^{(2)})) \\
\leq W_2^2(L(x_k, \theta_k^{(1)}), L(x_k, \theta_k^{(2)})) \\
\leq 10 \frac{\gamma_{\text{max}}}{\gamma_{\text{min}}} \left(1 - \frac{0.9\alpha}{\gamma_{\text{max}}}\right)^k \mathbb{E}[\|\theta_0^{(1)} - \theta_0^{(2)}\|^2] \\
\leq 20 \frac{\gamma_{\text{max}}}{\gamma_{\text{min}}} \left(1 - \frac{0.9\alpha}{\gamma_{\text{max}}}\right)^k \left(\mathbb{E}[\|\theta_0^{(1)} - c\|^2] + \mathbb{E}[\|\theta_\infty - c\|^2]\right),
\]

where \(c\) is an arbitrary constant, and the last step above holds since the chain \((x_k, \theta_k^{(2)})_{k \geq 0}\) is at stationarity and hence \(\mathbb{E}[\|\theta_0^{(2)}\|^2] = \mathbb{E}[\|\theta_\infty\|^2]\).

Hence, taking \(c = \mathbb{E}[\theta_\infty]\), we have now proven equation (4.4) in part 3 of the theorem,

\[
W_2^2(L(\theta_k^{(1)}), L(\theta_\infty)) \leq 20 \frac{\gamma_{\text{max}}}{\gamma_{\text{min}}} \left(1 - \frac{0.9\alpha}{\gamma_{\text{max}}}\right)^k \left(\mathbb{E}[\|\theta_0^{(1)} - \mathbb{E}[\theta_\infty]\|^2] + \mathbb{E}[\|\theta_\infty - \mathbb{E}[\theta_\infty]\|^2]\right) \\
\leq 20 \frac{\gamma_{\text{max}}}{\gamma_{\text{min}}} \left(1 - \frac{0.9\alpha}{\gamma_{\text{max}}}\right)^k \left(\mathbb{E}[\|\theta_0^{(1)} - \mathbb{E}[\theta_\infty]\|^2] + \text{Tr}(\text{Var}(\theta_\infty))\right).
\]

A.3 Proof of Corollary 4.2

Proof of Corollary 4.2. By Lemma A.7, we have \(\mathbb{E}[\|\theta_\infty\|^2] = \mathcal{O}(1)\). Combining this bound with equation (4.4) in Theorem 4.1, we obtain that for \(k \geq \tau\),

\[
W_2^2(L(\theta_k), \mu) \leq C(A, b, \pi) \cdot \left(1 - \frac{0.9\alpha}{\gamma_{\text{max}}}\right)^k.
\]

By [Vil09, Theorem 4.1], there exists a coupling between \(\theta_k\) and \(\theta_\infty\) such that \(W_2^2(L(\theta_k), \mu) = \mathbb{E}[\|\theta_k - \theta_\infty\|^2]\). Utilizing the above bounds and applying Jensen’s inequality twice, we obtain that

\[
\|\mathbb{E}[\theta_k - \theta_\infty]\|^2 \leq (\mathbb{E}[\|\theta_k - \theta_\infty\|^2])^2 \leq \mathbb{E}[\|\theta_k - \theta_\infty\|^2] \leq C(A, b, \pi) \cdot \left(1 - \frac{0.9\alpha}{\gamma_{\text{max}}}\right)^k.
\]

The above bound then implies the first moment bound in equation (4.5).

Turning to the second moment, we observe that

\[
\|\mathbb{E}[\theta_k] - \mathbb{E}[\theta_\infty]\|^2
= \|\mathbb{E}[\theta_k - \theta_\infty] + \theta_\infty - \mathbb{E}[\theta_\infty]\|^2
\leq \|\mathbb{E}[\theta_k - \theta_\infty]\|^2 + \|\theta_\infty - \mathbb{E}[\theta_\infty]\|^2
\leq \mathbb{E}[\|\theta_k - \theta_\infty\|^2] + \text{Var}(\theta_\infty)
\leq \mathbb{E}[\|\theta_k - \theta_\infty\|^2] + 2\mathbb{E}[\|\theta_k - \theta_\infty\|^2] + \mathbb{E}[\|\theta_\infty\|^2]\]

where the last inequality (A.28) holds true by Cauchy-Schwarz inequality. On the other hand, we have already established that for \(k \geq \tau\),

\[
\mathbb{E}[\|\theta_k - \theta_\infty\|^2] \leq C(A, b, \pi) \left(1 - \frac{0.9\alpha}{\gamma_{\text{max}}}\right)^k \quad \text{and} \quad \mathbb{E}[\|\theta_\infty\|^2] \leq C'(A, b, \pi).
\]
Substituting the above bounds into the right-hand side of inequality (A.28), we obtain equation (4.5) in Corollary 4.2.

### A.4 Proof of Theorem 4.3

In this sub-section, we prove Theorem 4.3 on characterizing the asymptotic bias of LSA. The proof is divided into four steps, which are given in Appendices A.4.1–A.4.4 to follow.

#### A.4.1 Step 1: Basic Adjoint Relationship

Following the strategy discussed after Theorem 4.3, we begin by deriving a recursive relationship for the function \( z : \mathcal{X} \to \mathbb{R}^d \) given by

\[
z(x) := \mathbb{E}[\theta_\infty | x_\infty = x],
\]

which is well-defined by the Doob-Dynkin Lemma. To put the derivation in context and to avoid measurability issues on general state space, we present using the language of Basic Adjoint Relationship (BAR).

Recall that under Assumption 1, \((x_k)_{k \geq 0}\) is a time-homogeneous Markov chain with transition kernel \(P\) and unique stationary distribution \(\pi\). Theorem 4.1 has demonstrated that the Markov chain \((x_k, \theta_k)_{k \geq 0}\) also has a unique stationary distribution \(\bar{\mu}\), and \((x_k, \theta_k)\) converges in distribution to a limit \((x_\infty, \theta_\infty) \sim \bar{\mu}\), where \(\theta_\infty \sim \mu\) and \(x_\infty \sim \pi\). Given \((x_\infty, \theta_\infty)\), let \(x_{\infty+1}\) be the random variable with conditional distribution \(P(x_\infty, \cdot)\), and \(\theta_{\infty+1} = \theta_\infty + \alpha (A(x_\infty)\theta_\infty + b(x_\infty))\); that is, \((x_{\infty+1}, \theta_{\infty+1})\) is the state following \((x_\infty, \theta_\infty)\).

Denote by \(Q\) the transition kernel of \((x_k, \theta_k)_{k \geq 0}\). Since \(\bar{\mu}\) is invariant for \(Q\), they satisfy BAR

\[
\bar{\mu}(I - Q)f = 0
\]

for any test function \(f : \mathcal{X} \times \mathbb{R}^d \to \mathbb{R}^d\) satisfying \(|f(x, \theta)| \leq C(1 + ||\theta||^2), \forall (x, \theta)\) for some \(C \in \mathbb{R}\) [Vil09, Definition 6.8 and Theorem 6.9]. The above BAR can be written equivalently as

\[
\mathbb{E}[f(x_\infty, \theta_\infty)] = \bar{\mu}f = \bar{\mu}Qf = \mathbb{E}[f(x_{\infty+1}, \theta_{\infty+1})].
\]

(A.29)

It is known that equation (A.29) with a sufficiently large class of test functions \(f\) completely characterizes the invariant distribution \(\bar{\mu}\) [Har85, HW87, DD11].

For characterization of the first moment \(\mathbb{E}[\theta_\infty]\), it suffices to consider the test functions of the form

\[
f^{(E)}(x, \theta) = \theta \cdot 1\{x \in E\} \quad \text{and} \quad f^{(E,S)}(x, \theta) = 1\{\theta \in S\} \cdot 1\{x \in E\}
\]

for \(E \in \mathcal{B}(\mathcal{X})\) and \(S \in \mathcal{B}(\mathbb{R}^d)\). Substituting \(f^{(E)}\) into the BAR (A.29) gives

\[
\mathbb{E}[\theta_\infty \cdot 1\{x_\infty \in E\}] = \mathbb{E}[\theta_{\infty+1} \cdot 1\{x_{\infty+1} \in E\}].
\]

(A.30)

We now compute the left and right-hand sides of equation (A.30) above. For the left-hand side, we have

\[
\mathbb{E}[\theta_\infty \cdot 1\{x_\infty \in E\}] = \mathbb{E}\left[\mathbb{E}[\theta_\infty \cdot 1\{x_\infty \in E\} | x_\infty]\right] = \int_\mathcal{X} \mathbb{E}[\theta_\infty | x_\infty](x) \mathbb{E}[1\{x_\infty \in E\}(x) \pi(dx)] dx.
\]

For the right-hand side, we similarly obtain

\[
\mathbb{E}[\theta_{\infty+1} \cdot 1\{x_{\infty+1} \in E\}] = \int_\mathcal{X} \mathbb{E}[\theta_{\infty+1} | x_{\infty+1}](x) \pi(dx).
\]

Plugging back to equation (A.30), we obtain that \(\int_E \mathbb{E}[\theta_\infty | x_\infty](x) \pi(dx) = \int_E \mathbb{E}[\theta_{\infty+1} | x_{\infty+1}](x) \pi(dx)\) for any \(E \in \mathcal{B}(\mathcal{X})\). By [Fol99, Proposition 2.23(b)], we conclude that

\[
\mathbb{E}[\theta_\infty | x_\infty](x) = \mathbb{E}[\theta_{\infty+1} | x_{\infty+1}](x) \quad \pi\text{-a.e.}
\]

(A.31)

Repeating the above argument for the test function \(f^{(E,S)}\), we obtain that for all \(S \in \mathcal{B}(\mathbb{R}^d)\):

\[
\mathbb{E}[1\{\theta_\infty \in S\} | x_\infty](x) = \mathbb{E}[1\{\theta_{\infty+1} \in S\} | x_{\infty+1}](x) \quad \pi\text{-a.e.}
\]

(A.32)
A.4.2 Step 2: Set up System of $z$

We derive another relationship between $E[\theta_\infty \mid x_\infty]$ and $E[\theta_{\infty+1} \mid x_{\infty+1}]$ using the update rule $\theta_{\infty+1} = \theta_\infty + \alpha(A(x_\infty)\theta_\infty + b(x_\infty))$.

As the state space $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ is Borel, the conditional expectation $E[\mathbb{1}\{\theta_\infty \in \cdot\} \mid x_\infty = x]$ can be taken as a regular conditional probability measure, which we denote as $\nu(\cdot, x_\infty = x)$. Similarly define the regular conditional probability measure $\nu'(\cdot, x_\infty = x, x_{\infty+1} = x')$. By (A.32), we have that for all $S \in \mathcal{B}(\mathbb{R}^d)$:

$$
\nu(\theta_{\infty+1} \in S, x_\infty = x) = \nu(\theta_{\infty+1} \in S, x_{\infty+1} = x).
$$

Using the time-reversed kernel $P^*$, we can write

$$
\nu(\theta_{\infty+1} \in S, x_{\infty+1} = x) = \int_{s \in \mathcal{X}} \nu'(\theta_{\infty} \in S', x_\infty = s, x_{\infty+1} = x) P^*(x, ds)
$$

where the set $S' \in \mathcal{B}(\mathbb{R}^d)$ is determined by $S$ and $x$ via the aforementioned update rule, and the equality (i) follows from $\theta_{\infty} \perp x_{\infty+1} \mid x_\infty$. Using the above property of $\nu$, we obtain

$$
E[\theta_{\infty+1} \mid x_{\infty+1} = x] = \int_{\theta \in \mathbb{R}^d} \theta \nu(d\theta, x)
$$

Applying (A.31) to the left-hand side above, we obtain the following equation for $E[\theta_\infty \mid x_\infty]$:

$$
E[\theta_\infty \mid x_\infty = x] = \int_{s \in \mathcal{X}} P^*(x, ds) \left( E[\theta_\infty \mid x_\infty = s] + \alpha \left( A(s)E[\theta_\infty \mid x_\infty = s] + b(s) \right) \right).
$$

Using the function $z$ and the operator $D$ defined in Section 4.2, we can write equation (A.33) compactly as

$$
z = P^*(z + \alpha(Dz + b)).
$$

A.4.3 Step 3: Setting up System of $\delta$

Using $\pi z = \int_{\mathcal{X}} \pi(dx)z(x) = E[\theta_\infty]$ as a reference point, we define the function $\delta : \mathcal{X} \rightarrow \mathbb{R}^d$ by

$$
\delta(x) = z(x) - \pi z,
$$

which is a centered version of $z$. Applying the operator $(P^* - \Pi)$ to both sides of (A.35) gives

$$
(P^* - \Pi)z = (P^* - \Pi)\delta.
$$

Subtracting $\Pi z$ from both sides of (A.34) and invoking (A.36), we obtain

$$
\delta = (P^* - \Pi)z + \alpha P^*(Dz + b) = (P^* - \Pi)\delta + \alpha P^*(Dz + b).
$$

On the other hand, applying $\pi$ to both sides of equation (A.34) gives

$$
\pi z = \pi \left( P^*(z + \alpha(Dz + b)) \right) = \pi z + \alpha \pi(Dz + b),
$$

which implies that

$$
\pi(Dz + b) = 0.
$$
Therefore, we can subtract the vanishing quantity $\pi(Dz + b)$ from the right-hand side of (A.37) and obtain

$$\delta = (P^* - \Pi)\delta + \alpha(P^* - \Pi)(Dz + b).$$

Consolidating the $\delta$ terms, we have

$$(I - P^* + \Pi)\delta = \alpha(P^* - \Pi)(Dz + b). \quad (A.39)$$

To proceed, we observe that:

**Claim 3.** $(I - P^* + \Pi)^{-1}$ exists as a bounded operator in $L^2(\pi)$.

When the state space $\mathcal{X}$ is finite or countable, Claim 3 is a well-known fact. For completeness, below we provide the proof of Claim 3 for uniformly ergodic chains on general state spaces. Taking Claim 3 as given, we can rewrite (A.39) as

$$\delta = \alpha(I - P^* + \Pi)^{-1}(P^* - \Pi)(Dz + b). \quad (A.40)$$

We then obtain the following bound on $\delta$:

$$\|\delta\|_{L^2(\pi)} = \alpha\|(I - P^* + \Pi)^{-1}P^*(Dz + b)\|_{L^2(\pi)}$$

$$\leq \alpha\|(I - P^* + \Pi)^{-1}\|_{L^2(\pi)}\|P^*\|_{L^2(\pi)}\left(\|D\|_{L^2(\pi)}\|z\|_{L^2(\pi)} + \|b\|_{L^2(\pi)}\right)$$

$$\leq \alpha\|(I - P^* + \Pi)^{-1}\|_{L^2(\pi)}\left(A_{\text{max}}\|z\|_{L^2(\pi)} + b_{\text{max}}\right)$$

$$\leq \alpha C(A, b, \pi), \quad (A.41)$$

where in the step we use the bound

$$\|z\|_{L^2(\pi)}^2 = \int_{\mathcal{X}} \pi(dx)|E[\theta_{\infty} | x_\infty = x]|^2 \leq \int_{\mathcal{X}} \pi(dx)E[\|\theta_{\infty}\|^2 | x_\infty = x] = E[\|\theta_{\infty}\|^2] \leq C'(A, b, \pi),$$

with the last inequality following from Theorem 4.1.

**Proof of Claim 3.** Since $P$ is uniformly ergodic by Assumption 1, $P$ is $L^2(\pi)$-exponentially convergent [DMPS18, Proposition 22.3.5]. That is, there exist $r \in [0, 1)$ and $R < \infty$ such that $\|P^n - \Pi\|_{L^2(\pi)} \leq R r^n$ for all $n \in \mathbb{N}$. Recall that $P^*$ is the adjoint operator of $P$, and note that $\Pi$ is self-adjoint. We therefore have

$$\langle f, (P^n - \Pi)g \rangle_{L^2(\pi)} = \langle (P^n - \Pi)^*f, g \rangle_{L^2(\pi)} = \langle ((P^*)^n - \Pi)f, g \rangle_{L^2(\pi)},$$

which implies that $\|(P^*)^n - \Pi\|_{L^2(\pi)} = \|P^n - \Pi\|_{L^2(\pi)}$. Combining pieces, we have the following bound on the Neumann series:

$$\left\|\sum_{n=0}^{\infty} (P^* - \Pi)^{n} \right\|_{L^2(\pi)} \leq \sum_{n=0}^{\infty} \|(P^* - \Pi)^{n}\|_{L^2(\pi)} = \sum_{n=0}^{\infty} \|(P^*)^n - \Pi\|_{L^2(\pi)} = \sum_{n=0}^{\infty} \|P^n - \Pi\|_{L^2(\pi)} \leq \frac{R}{1 - r} < \infty.$$ 

Therefore, $I - P^* + \Pi$ is invertible and its inverse is given by the Neumann series. □

**A.4.4 Step 4: Bootstrapping**

We derive another relationship between $z$ and $\delta$, which can be combined with (A.40) to give an equation for $\delta$.

Recall that $\theta^*$ is the unique solution to $\bar{\Delta}\theta^* + \bar{b} = 0$. Together with (A.38), using which we can write $\bar{b} = \pi b$ in terms of $A$ and $z$, we obtain the following relationship between $\theta^*$ and $z$:

$$\theta^* = \pi \bar{D} z, \quad (A.42)$$

where the operator $\bar{D}$ is defined in Section 4.2. Substituting $z(\cdot) = \delta(\cdot) + \pi z$ into (A.42), we have

$$\theta^* = \int_{x \in \mathcal{X}} \pi(dx)\bar{A}^{-1} A(x)(\delta(x) + \pi z) = \pi \bar{D} \delta + \pi z.$$
Reorganizing the equation above, we obtain
\[ \pi z = \theta^* - \pi \bar{D} \delta. \quad (A.43) \]

It follows that
\[ z(x) = \delta(x) + \pi z = \delta(x) + \left( \theta^* - \pi \bar{D} \delta \right) = \theta^* + \left( \delta(x) - \pi \bar{D} \delta \right), \quad \forall x \in X, \]
which can be written compactly as
\[ z = \theta^* + (I - \Pi \bar{D}) \delta. \quad (A.44) \]

Substituting \((A.44)\) into the RHS of \((A.40)\), and recalling the definitions of \(\nu\) and \(\Xi\) in \((4.6)\), we obtain
\[ \delta = \alpha (I - P^* + \Pi)^{-1} (P^* - \Pi) (Dz + b) \]
\[ = \alpha (I - P^* + \Pi)^{-1} (P^* - \Pi) (A\theta^* + D(I - \Pi \bar{D}) \delta + b) \]
\[ = \alpha (I - P^* + \Pi)^{-1} (P^* - \Pi) (A\theta^* + b) + \alpha (I - P^* + \Pi)^{-1} (P^* - \Pi) D(I - \Pi \bar{D}) \delta, \]
\[ = \alpha \nu + \alpha \Xi \delta. \quad (A.45) \]

Equation \((A.45)\), which expresses \(\delta\) in terms of itself, plays a crucial role in the sequel.

Substituting \((A.45)\) into the RHS of equation \((A.43)\), we obtain
\[ E[\theta_{\infty}] = \theta^* - \pi \bar{D} \delta = \theta^* - \alpha \pi \bar{D} \nu - \alpha \pi \bar{D} \Xi \delta. \]

Using the bound \((A.41)\), we obtain that
\[ \|\pi \bar{D} \Xi \delta\| \leq \|\bar{D} \Xi\| \|\pi \delta\| \leq \|\bar{D} \Xi\| \|\delta\|_{L^2(\pi)} = O(\alpha). \]

Combining the last two equations, and recalling the definition of \(B^{(1)}\) in \((4.6c)\), we prove the base case in Theorem 4.3, i.e., \(E[\theta_{\infty}] = \theta^* + \alpha B^{(1)} + O(\alpha^2)\).

We now bootstrap the above argument. Plugging \((A.45)\) back to the RHS of \((A.43)\), we obtain the following equation for \(\delta\):
\[ \delta = \sum_{i=1}^{m} \alpha^i \Xi^{i-1} \nu + \alpha^m \Xi^m \delta. \]

Plugging the above equation into the RHS of equation \((A.43)\), and using again the bound \(\|\pi \delta\| \leq \|\delta\|_{L^2(\pi)} = O(\alpha)\), we obtain the \(m\)-th order bias expansion:
\[ E[\theta_{\infty}] = \theta^* - \pi \bar{D} \delta = \theta^* - \pi \bar{D} \left( \sum_{i=1}^{m} \alpha^i \Xi^{i-1} \nu + \alpha^m \Xi^m \delta \right) \]
\[ = \theta^* + \pi \bar{D} \sum_{i=1}^{m} \alpha^i \Xi^{i-1} \nu + O(\alpha^{m+1}). \]

We have proven the first part of Theorem 4.3.

To prove the infinite series expansion in the second part, we use the assumption \(\alpha < 1/\|\Xi\|_{L^2(\pi)}\) to establish that the following Neumann series converges:
\[ \lim_{m \to \infty} \alpha \cdot \left\| \sum_{i=0}^{m} (\alpha \Xi)^i \nu \right\| \leq \lim_{m \to \infty} \alpha \cdot \left( \sum_{i=0}^{m} \|\alpha \Xi^i\| \|\nu\| \right) < \infty. \]

Therefore, the inverse operator \((I - \alpha \Xi)^{-1}\) is well defined and given by the above Neumann series. We can then solve equation \((A.45)\) for \(\delta\) to obtain
\[ \delta = \alpha (I - \alpha \Xi)^{-1} \nu = \alpha \left( \sum_{i=0}^{\infty} (\alpha \Xi)^i \right) \nu. \]
Finally, we substitute the above expansion for $\delta$ into the RHS of equation (A.43), which gives the desired infinite expansion for $\mathbb{E}[\theta_\infty]$:

$$
\mathbb{E}[\theta_\infty] = \theta^* - \pi \bar{D} \delta = \theta^* - \alpha \pi \bar{D} \left( I - \alpha \overline{\Xi} \right)^{-1} v \\
= \theta^* - \alpha \left( \pi \bar{D} \sum_{i=0}^\infty (\alpha \overline{\Xi})^i v \right) \\
= \theta^* + \sum_{i=1}^\infty \alpha^i B^{(i)},
$$

where the last step follows from the definition of $B^{(i)}$ in (4.6). This completes the proof of Theorem 4.3.

Before concluding the section, we derive an explicit upper bound on $\|\Xi\|_{L^2(\pi)}$. Recall that the uniform ergodicity of $P$ implies the bound $\|P^n - \Pi\|_{L^2(\pi)} \leq R r^n, \forall n \in \mathbb{N}$ for some $r < 1$. It follows that

$$
\|\Xi\|_{L^2(\pi)} = \|(I - P^* + \Pi)^{-1} (P^* - \Pi) D (I - \Pi \bar{D})\|_{L^2(\pi)} \\
\leq \|(I - P^* + \Pi)^{-1}\|_{L^2(\pi)} \|P^* - \Pi\|_{L^2(\pi)} \|D\|_{L^2(\pi)} \|I - \Pi \bar{D}\|_{L^2(\pi)} \\
\leq \frac{R r}{1 - r} A_{\max}(1 + s_{\min}(A)^{-1} A_{\max}).
$$

Therefore, a sufficient condition for $\alpha < 1/\|\Xi\|_{L^2(\pi)}$ is

$$
\alpha < \left( \frac{R r}{1 - r} A_{\max}(1 + s_{\min}(A)^{-1} A_{\max}) \right)^{-1}. \quad (A.46)
$$

### A.5 Proof of Theorem 4.4

In this section, we prove Theorem 4.4 on the relationship between the bias and the absolute spectral gap of the chain $(x_k)_{k \geq 0}$.

Under the assumption that $P$ is uniformly ergodic and reversible, i.e., $P^* = P$, we have the following relationship [DMPS18, Proposition 22.2.5]:

$$
1 - \|P - \Pi\|_{L^2(\pi)} = \gamma^*(P) > 0. \quad (A.47)
$$

Therefore, the operator norm of $(I - P + \Pi)^{-1}$ satisfies,

$$
\|(I - P + \Pi)^{-1}\|_{L^2(\pi)} = \left\| \sum_{i=0}^{\infty} (P - \Pi)^i \right\|_{L^2(\pi)} \leq \sum_{i=0}^{\infty} \|P - \Pi\|_{L^2(\pi)} = \sum_{i=0}^{\infty} \left( 1 - \gamma^*(P) \right)^i = \frac{1}{\gamma^*(P)}. \quad (A.48)
$$

Recall the definitions of $\Xi, v$ and $B^{(i)}$ in (4.6). Using the submultiplicativity of the norm $\|\cdot\|_{L^2(\pi)}$, we obtain

$$
\|\Xi\|_{L^2(\pi)} = \|\Xi\|_{L^2(\pi)} \\
\leq \frac{1}{\gamma^*(P)} \cdot (1 - \gamma^*(P)) \cdot \|D(I - \Pi \bar{D})\|_{L^2(\pi)}
$$

and

$$
\|v\|_{L^2(\pi)} = \|v\|_{L^2(\pi)} \\
\leq \frac{1}{\gamma^*(P)} \cdot (1 - \gamma^*(P)) \cdot \|D(I - \Pi \bar{D})\|_{L^2(\pi)}.
$$
It follows that for each $i = 1, 2, \ldots$,
\[
\left\| B^{(i)} \right\| = \left\| \pi \bar{D} \bar{\Xi}^{i-1} \nu \right\| \leq \left\| D \bar{\Xi}^{i-1} \nu \right\|_{L^2(\pi)}^{(i)} \\
\leq \left\| D \right\|_{L^2(\pi)} \left\| \bar{\Xi} \right\|_{L^2(\pi)}^{i-1} \left\| \nu \right\|_{L^2(\pi)} \\
\leq \left\| D \right\|_{L^2(\pi)} \left( \frac{1 - \gamma^*(P)}{\gamma^*(P)} \right)^i \left\| D(I - \Pi D) \right\|_{L^2(\pi)} \left\| A\theta^* + b \right\|_{L^2(\pi)},
\]
where step (i) follows from Jensen’s inequality. Setting
\[
C(A, b, \pi) = \max \left\{ \left\| D \right\|_{L^2(\pi)} \left\| A\theta^* + b \right\|_{L^2(\pi)}, \left\| D(I - \Pi D) \right\|_{L^2(\pi)} \right\},
\]
which only depends on $A, b$ and $\pi$, we obtain
\[
\left\| B^{(i)} \right\| \leq C(A, b, \pi)^i \left( \frac{1 - \gamma^*(P)}{\gamma^*(P)} \right)^i
\]
as claimed.

### A.6 Proof of Corollary 4.6

We prove the first and second moment bounds in Corollary 4.6.

#### A.6.1 First Moment

We first have
\[
\mathbb{E} [\bar{\theta}_{k_0, k}] - \theta^* = (\mathbb{E}[\theta_{\infty}] - \theta^*) + \frac{1}{k - k_0} \sum_{t=k_0}^{k-1} \mathbb{E}[\theta_t - \theta_{\infty}].
\]

To bound $T_1$, we recall the inequality proven in (4.5), that for $k \geq \tau$,
\[
\left\| \mathbb{E}[\theta_k] - \mathbb{E}[\theta_{\infty}] \right\| \leq C(A, b, \pi) \cdot \left( 1 - \frac{0.9\alpha}{\gamma_{\text{max}}} \right)^{k/2}.
\]

As the burn-in period satisfies $k_0 \geq \tau$, we have the following bound,
\[
\left\| T_1 \right\| = \left\| \sum_{t=k_0}^{k-1} \mathbb{E}[\theta_t - \theta_{\infty}] \right\| \leq \sum_{t=k_0}^{k-1} \left\| \mathbb{E}[\theta_t] - \mathbb{E}[\theta_{\infty}] \right\|
\]
\[
\quad \leq C'(A, b, \pi) \cdot \frac{1}{\alpha} \cdot \exp \left( -\frac{\alpha k_0}{4\gamma_{\text{max}}} \right), \quad \text{(A.49)}
\]
Together with (4.7), we obtain the desired equation (4.11), that is,
\[
\mathbb{E} [\bar{\theta}_{k_0, k}] - \theta^* = \alpha B(A, b, P) + O(\alpha^2) + O \left( \frac{1}{\alpha(k - k_0)} \exp \left( -\frac{\alpha k_0}{4\gamma_{\text{max}}} \right) \right).
\]
A.6.2 Second Moment

To bound the second moment of the tail-averaged iterate, we make use of the following decomposition:

\[ E \left[ (\bar{\theta}_{k_0,k} - \theta^*) (\bar{\theta}_{k_0,k} - \theta^*)^\top \right] = \frac{1}{k - k_0} \left( \sum_{t=k_0}^{k-1} E [\theta_t - \theta_\infty] (E[\theta_\infty] - \theta^*)^\top \right) \]

\[ \overset{(iv)}{=} \mathcal{O} \left( \frac{1}{\alpha(k - k_0)} \exp \left( -\frac{\alpha k_0}{4\gamma_{\max}} \right) \right) \left( \alpha B(A, b, P) + \mathcal{O}(\alpha^2) \right) \]

\[ = \mathcal{O} \left( \frac{1}{k - k_0} \exp \left( -\frac{\alpha k_0}{4\gamma_{\max}} \right) \right), \]

where step (iv) is due to equations (A.49) and (4.7). The term \( T_3 \) can be analyzed in the same fashion.

For \( T_4 \), we observe that

\[ E \left[ (E[\theta_\infty] - \theta^*) (E[\theta_\infty] - \theta^*)^\top \right] = (E[\theta_\infty] - \theta^*) (E[\theta_\infty] - \theta^*)^\top \]

\[ \overset{(v)}{=} \alpha B(A, b, P) + \mathcal{O}(\alpha^2) \]

\[ = \alpha^2 B'(A, b, P) + \mathcal{O}(\alpha^3), \]

where step (v) holds by equation (4.7).

It remains to bound \( T_1 \). We have

\[ T_1 = \frac{1}{(k - k_0)^2} \left( \sum_{t=k_0}^{k-1} (\theta_t - E[\theta_\infty]) (\theta_t - E[\theta_\infty])^\top \right) \]

\[ \overset{T_1}{=} \frac{1}{(k - k_0)^2} \left( \sum_{t=k_0}^{k-1} E [\theta_t - E[\theta_\infty]] (\theta_t - E[\theta_\infty])^\top \right) \]

\[ + \frac{1}{(k - k_0)^2} \left( \sum_{t=k_0}^{k-1} \sum_{l=t+1}^{k-1} \left( E [(\theta_t - E[\theta_\infty])(\theta_l - E[\theta_\infty])^\top] + E [(\theta_l - E[\theta_\infty])(\theta_t - E[\theta_\infty])^\top] \right) \right). \]

Below we control \( T_1' \) and \( T_2' \) respectively. For \( T_1' \), we start with the following decomposition,

\[ E \left[ (\theta_t - E[\theta_\infty]) (\theta_t - E[\theta_\infty])^\top \right] \]

\[ = (E[\theta_t \theta_\infty^\top] - E[\theta_\infty \theta_\infty^\top]) + (E[\theta_\infty \theta_\infty^\top] - E[\theta_\infty]E[\theta_\infty^\top] - E[\theta_\infty]E[\theta_\infty^\top]-2E[\theta_\infty]E[\theta_\infty^\top]) \]

\[ = (E[\theta_t \theta_\infty^\top] - E[\theta_\infty \theta_\infty^\top]) + \text{Var}(\theta_\infty) - E[\theta_t - \theta_\infty]E[\theta_\infty^\top] - E[\theta_\infty]E[(\theta_t - \theta_\infty)^\top]. \]  

(A.52)
By Corollary 4.2 and Lemma A.7, the following bounds hold when $t \geq \tau$:

\[
\mathbb{E}[\|\theta_t - \theta_\infty\|] \leq C(A, b, \pi) \cdot \left(1 - \frac{0.9\alpha}{\gamma_{\max}}\right)^{t/2}, \tag{A.53}
\]

\[
\|\mathbb{E}[(\theta_t)'] - \mathbb{E}[\theta_\infty']\| \leq C'(A, b, \pi) \cdot \left(1 - \frac{0.9\alpha}{\gamma_{\max}}\right)^{t/2},
\]

\[
\mathbb{E}[\|\theta_\infty\|] \leq C''(A, b, \pi),
\]

\[
\text{Var}(\theta_\infty) \leq C'''(A, b, \pi) \cdot \alpha \tau. \tag{A.54}
\]

Plugging these bounds into equation (A.52), we obtain that

\[
\mathbb{E}\left[(\theta_t - \mathbb{E}[\theta_\infty]) (\theta_t - \mathbb{E}[\theta_\infty])'\right] = \mathcal{O}\left(\left(1 - \frac{0.9\alpha}{\gamma_{\max}}\right)^{t/2} + \alpha \tau\right). \tag{A.55}
\]

Now with (A.55) on hand, we obtain the following bound for $T_1'$:

\[
T_1' = \frac{1}{(k - k_0)^2} \sum_{t = k_0}^{k-1} \mathcal{O}\left(\left(1 - \frac{0.9\alpha}{\gamma_{\max}}\right)^{t/2} + \alpha \tau\right)
\]

\[
= \mathcal{O}\left(\frac{1}{(k - k_0)^2} \sum_{t = k_0}^{k_0/2} \left(1 - \frac{0.9\alpha}{\gamma_{\max}}\right)^{t/2}\right) + \mathcal{O}\left(\frac{\alpha \tau}{k - k_0}\right)
\]

\[
= \mathcal{O}\left(\frac{1}{(k - k_0)^2} \cdot 2\gamma_{\max} \cdot \frac{1}{0.9\alpha} \cdot \left(1 - \frac{0.9\alpha}{\gamma_{\max}}\right)^{k_0/2}\right) + \mathcal{O}\left(\frac{\alpha \tau}{k - k_0}\right)
\]

\[
= \mathcal{O}\left(\frac{1}{\alpha (k - k_0)^2} \exp\left(-\frac{\alpha k_0}{4\gamma_{\max}}\right) + \frac{\alpha \tau}{k - k_0}\right).
\]

To analyze $T_2'$, we first study each term in the summation. Observe that for $l > t$, we have

\[
\mathbb{E}\left[(\theta_t - \mathbb{E}[\theta_\infty]) (\theta_l - \mathbb{E}[\theta_\infty])'\right] = \mathbb{E}\left[\mathbb{E}\left[(\theta_t - \mathbb{E}[\theta_\infty]) (\theta_l - \mathbb{E}[\theta_\infty])' | \theta_t\right]\right]
\]

\[
= \mathbb{E}\left[(\theta_t - \mathbb{E}[\theta_\infty]) (\mathbb{E}[\theta_l | \theta_t] - \mathbb{E}[\theta_\infty])'\right].
\]

We now make the following claim, whose proof we delay to the end of this sub-sub-section.

**Claim 4.** For $t \geq \frac{4\gamma_{\max}}{\alpha}$ and $l \geq t + \tau$, we have

\[
\left\|\mathbb{E}\left[(\theta_t - \mathbb{E}[\theta_\infty]) (\theta_l - \mathbb{E}[\theta_\infty])'\right]\right\| = \mathcal{O}\left(\alpha \tau \cdot \left(1 - \frac{0.9\alpha}{\gamma_{\max}}\right)^{(l-t)/2}\right). \tag{A.56}
\]

Taking this claim as given, we now complete the analysis of $T_2'$, which has the form $T_2' = T_2'' + (T_2')'$. We observe that

\[
T_2'' = \frac{1}{(k - k_0)^2} \left[\sum_{t = k_0}^{k-1} \left(\sum_{l = t + \tau}^{t + \tau} \mathbb{E}\left[(\theta_t - \mathbb{E}[\theta_\infty]) (\theta_l - \mathbb{E}[\theta_\infty])'\right] + \sum_{l = t + \tau + 1}^{k - 1} \mathbb{E}\left[(\theta_t - \mathbb{E}[\theta_\infty]) (\theta_l - \mathbb{E}[\theta_\infty])'\right]\right)\right]
\]

\[
+ \left[\sum_{t = k - \tau}^{k - 1} \sum_{l = t + \tau}^{k - \tau} \mathbb{E}\left[(\theta_t - \mathbb{E}[\theta_\infty]) (\theta_l - \mathbb{E}[\theta_\infty])'\right] + \sum_{l = t + \tau + 1}^{k - 1} \sum_{t = k - \tau}^{k - 1} \mathbb{E}\left[(\theta_t - \mathbb{E}[\theta_\infty]) (\theta_l - \mathbb{E}[\theta_\infty])'\right]\right]
\]

\[
= \frac{1}{(k - k_0)^2} \left[\sum_{t = k_0}^{k-1} \left(\sum_{l = t + \tau}^{t + \tau} \mathbb{E}\left[(\theta_t - \mathbb{E}[\theta_\infty]) (\theta_l - \mathbb{E}[\theta_\infty])'\right] + \sum_{l = t + \tau + 1}^{k - 1} \mathbb{E}\left[(\theta_t - \mathbb{E}[\theta_\infty]) (\theta_l - \mathbb{E}[\theta_\infty])'\right]\right)\right]
\]

\[
+ \frac{1}{(k - k_0)^2} \left[\sum_{t = k_0}^{k-1} \sum_{l = t + \tau}^{k - 1} \mathbb{E}\left[(\theta_t - \mathbb{E}[\theta_\infty]) (\theta_l - \mathbb{E}[\theta_\infty])'\right]\right]
\]

\[=: T_a + T_b.\]
Above, we group the summand based on the $\tau$ burn-in requirements. For the $T_b$ terms, which satisfies the burn-in requirements, we use Claim 4 to obtain

$$T_b = \frac{1}{(k-k_0)^2} \sum_{t=k_0}^{k-\tau-1} \sum_{i=t+\tau+1}^{k-1} O\left( \alpha\tau \left( 1 - \frac{0.9\alpha}{\gamma_{\max}} \right)^{(i-t)/2} \right)$$

$$= \frac{\alpha\tau}{(k-k_0)^2} \sum_{t=k_0}^{k-\tau-1} \sum_{i=t+\tau+1}^{k-1} O\left( \left( 1 - \frac{0.9\alpha}{\gamma_{\max}} \right)^{1/2} \right)$$

$$\leq \frac{\alpha\tau}{k-k_0} \sum_{t=0}^{\infty} O\left( \left( 1 - \frac{0.9\alpha}{\gamma_{\max}} \right)^{1/2} \right) = O\left( \frac{\tau}{k-k_0} \right).$$

The $T_a$ term involves a finite number of cross terms. We bound them using the Cauchy-Schwarz inequality,

$$\mathbb{E} \left[ \left\| (\hat{\theta}_t - \mathbb{E}[\theta_\infty]) (\hat{\theta}_t - \mathbb{E}[\theta_\infty])^\top \right\| \right] \leq \sqrt{\mathbb{E}[\|\theta_t - \mathbb{E}[\theta_\infty]\|^2]} \sqrt{\mathbb{E}[\|\theta_t - \mathbb{E}[\theta_\infty]\|^2]}.$$

As $t \geq k_0$ and $k_0 \geq \frac{4\gamma_{\max}}{\alpha} \log\left( \frac{1}{\alpha\tau} \right)$, we can bound the above RHS using (A.55) and obtain

$$\mathbb{E} \left[ \left\| (\hat{\theta}_t - \mathbb{E}[\theta_\infty]) (\hat{\theta}_t - \mathbb{E}[\theta_\infty])^\top \right\| \right] = O(\alpha\tau).$$

Hence, we have

$$T_a \leq \frac{1}{k-k_0} O(\tau \cdot \alpha\tau) + \frac{\tau^2}{(k-k_0)^2} O(\alpha\tau) = O\left( \frac{\tau}{k-k_0} \right),$$

where in the last step we make use of $\alpha\tau \leq O(1)$ and $\tau \leq k-k_0$.

Therefore, by combining the above pieces of information, we obtain the following bound for $T_2'$:

$$T_2' = T_2'' + (T_2'')^\top = (T_a + T_b) + (T_a + T_b)^\top = O\left( \frac{\tau}{k-k_0} \right).$$

With the bounds for $T_1'$ and $T_2'$, we can bound the term $T_1$ in (A.50) for $k_0 \geq \frac{4\gamma_{\max}}{\alpha} \log\left( \frac{1}{\alpha\tau} \right)$ and $k \geq k_0 + \tau$:

$$T_1 = T_1' + T_2' = O\left( \frac{1}{\alpha(k-k_0)^2} \exp\left( -\frac{\alpha k_0}{4\gamma_{\max}} \right) \right) + O\left( \frac{\tau\alpha}{k-k_0} \right)$$

$$= O\left( \frac{\tau\alpha}{k-k_0} + \frac{1}{\alpha(k-k_0)^2} \exp\left( -\frac{\alpha k_0}{4\gamma_{\max}} \right) \right),$$

where in the second step we use $\alpha \leq \alpha\tau_\alpha < 1$ from equations (4.2) and (A.1).

Combining all the pieces, we obtain

$$\mathbb{E} \left[ (\hat{\theta}_{k_0,k} - \theta^*) (\hat{\theta}_{k_0,k} - \theta^*)^\top \right] = T_1 + T_2 + T_3 + T_4$$

$$= \alpha^2 B'(A, b, P) + O(\alpha^3) + O\left( \frac{1}{k-k_0} \exp\left( -\frac{\alpha k_0}{4\gamma_{\max}} \right) \right)$$

$$+ O\left( \frac{1}{\alpha(k-k_0)^2} \exp\left( -\frac{\alpha k_0}{4\gamma_{\max}} + \frac{\tau}{k-k_0} \right) \right),$$

which is the desired equation (4.12). We have completed the proof of Corollary 4.6.

**Proof of Claim 4.** First note that when $t \geq \frac{4\gamma_{\max}}{\alpha} \log\left( \frac{1}{\alpha\tau} \right)$, which is the assumption in Corollary 4.6, we have $\left( 1 - \frac{0.9\alpha}{\gamma_{\max}} \right)^{t/2} \lesssim \alpha\tau$. Therefore, (A.55) simplifies to

$$\mathbb{E} \left[ (\hat{\theta}_t - \mathbb{E}[\theta_\infty]) (\hat{\theta}_t - \mathbb{E}[\theta_\infty])^\top \right] = O(\alpha\tau).$$
With the above bound, we can use the convergence rate (4.4) in Theorem 4.1 to derive the following: for 

\[ t \geq \frac{4\gamma_{\text{max}}}{\alpha} \log \left( \frac{1}{\alpha} \right) \] 

and \( k \geq \tau \),

\[ W_2^2 \left( \mathcal{L}(\theta_{k+t}), \mu \right) \leq 20 \frac{\gamma_{\text{max}}}{\gamma_{\text{min}}} \left( E[\|\theta_t - E[\theta_\infty]\|^2] + \text{Tr}(\text{Var}(\theta_\infty)) \right) \cdot \left( 1 - \frac{0.9\alpha}{\gamma_{\text{max}}} \right)^k \leq \mathcal{O} \left( \alpha \tau \cdot \left( 1 - \frac{0.9\alpha}{\gamma_{\text{max}}} \right)^k \right). \]

Taking the optimal coupling that achieves the Wasserstein distance between \((\theta_{k+t}, x_{k+t})\) and \((\theta_\infty, x_\infty)\), we obtain

\[ \left\| E[\theta_{k+t}] - E[\theta_\infty] \right\|^2 \leq E \left[ \left\| \theta_{k+t} - \theta_\infty \right\|^2 \right] \leq \mathcal{O} \left( \alpha \tau \cdot \left( 1 - \frac{0.9\alpha}{\gamma_{\text{max}}} \right)^k \right). \]

Consequently, for \( l \geq t + \tau \) and \( t \geq \frac{4\gamma_{\text{max}}}{\alpha} \log \left( \frac{1}{\alpha} \right) \), we obtain

\[
\left\| E \left[ (\theta_t - E[\theta_\infty]) (\theta_t - E[\theta_\infty])^T \right] \right\|
\leq E \left[ \|\theta_t - E[\theta_\infty]\| \cdot \|E[\theta_t] - E[\theta_\infty]\| \right]
\leq E \left\| \theta_t - E[\theta_\infty] \right\| \cdot \sqrt{20 \frac{\gamma_{\text{max}}}{\gamma_{\text{min}}} \left( \|\theta_t - E[\theta_\infty]\|^2 + \text{Tr}(\text{Var}(\theta_\infty)) \right) \cdot \left( 1 - \frac{0.9\alpha}{\gamma_{\text{max}}} \right)^{l-t}}
\leq C(A, b, \pi) \cdot E \left[ \|\theta_t - E[\theta_\infty]\| \cdot \left( \|\theta_t - E[\theta_\infty]\| + \sqrt{\text{Tr}(\text{Var}(\theta_\infty))} \right) \cdot \left( 1 - \frac{0.9\alpha}{\gamma_{\text{max}}} \right)^{(l-t)/2} \right]
\leq C(A, b, \pi) \cdot \left( E \left[ \|\theta_t - E[\theta_\infty]\|^2 \right] \cdot \left( 1 - \frac{0.9\alpha}{\gamma_{\text{max}}} \right)^{(l-t)/2} + \sqrt{\alpha \tau} \cdot E \left[ \|\theta_t - E[\theta_\infty]\| \right] \cdot \left( 1 - \frac{0.9\alpha}{\gamma_{\text{max}}} \right)^{(l-t)/2} \right)
= \mathcal{O} \left( \alpha \tau \cdot \left( 1 - \frac{0.9\alpha}{\gamma_{\text{max}}} \right)^{(l-t)/2} \right).
\]

As such, we have proven the desired claim.

\[ \square \]

**A.7 Proof of Corollary 4.7**

We prove the first and second moment bounds in Corollary 4.7.

**A.7.1 First Moment**

We have

\[
E[\bar{\theta}_{k_0, k}^{(\alpha)}] - \theta^* = \left( 2\bar{\theta}_{k_0, k}^{(\alpha)} - \bar{\theta}_{k_0, k}^{(2\alpha)} \right) - \theta^* = 2 \left( \bar{\theta}_{k_0, k}^{(\alpha)} - \theta^* \right) - \left( \bar{\theta}_{k_0, k}^{(2\alpha)} - \theta^* \right)
\]

\[
\overset{(i)}{=} 2 \left( \alpha B(A, b, P) + \mathcal{O}(\alpha^2) + \mathcal{O} \left( \frac{1}{\alpha(k - k_0)} \exp \left( - \frac{\alpha k_0}{4\gamma_{\text{max}}} \right) \right) \right) - \left( 2\alpha B(A, b, P) + \mathcal{O}(\alpha^2) + \mathcal{O} \left( \frac{1}{\alpha(k - k_0)} \exp \left( - \frac{\alpha k_0}{2\gamma_{\text{max}}} \right) \right) \right)
\]

\[ = \mathcal{O}(\alpha^2) + \mathcal{O} \left( \frac{1}{\alpha(k - k_0)} \exp \left( - \frac{\alpha k_0}{4\gamma_{\text{max}}} \right) \right), \]

where (i) holds following from equation (4.11).

**A.7.2 Second Moment**

We first introduce the following shorthands:

\[ u_1 := \bar{\theta}_{k_0, k}^{(\alpha)} - E \left[ \theta_{\infty}^{(\alpha)} \right], \quad u_2 := \bar{\theta}_{k_0, k}^{(2\alpha)} - E \left[ \theta_{\infty}^{(2\alpha)} \right] \quad \text{and} \quad v := 2E \left[ \theta_{\infty}^{(\alpha)} \right] - E \left[ \theta_{\infty}^{(2\alpha)} \right] + \theta^*. \]
With these notations, we write $\tilde{\theta}_{k_0,k} - \theta^* = 2u_1 - u_2 + v$ and observe the bound
\[
\left\| E \left[ \left( \tilde{\theta}_{k_0,k} - \theta^* \right) \left( \tilde{\theta}_{k_0,k} - \theta^* \right)^\top \right] \right\| = \left\| E \left[ (2u_1 - u_2 + v) (2u_1 - u_2 + v)^\top \right] \right\| 
\leq E \left\| 2u_1 \right\|^2 + 3E \left\| u_2 \right\|^2 + 3 \left\| v \right\|^2.
\]

By equation (A.57) we have
\[
E \left\| u_1 \right\|^2 = \text{Tr} \left( E \left[ u_1 u_1^\top \right] \right) = \mathcal{O} \left( \frac{\tau_\alpha}{k - k_0} + \frac{1}{\alpha(k - k_0)^2} \exp \left( -\frac{\alpha k_0}{4\gamma_{\max}} \right) \right),
\]
and similarly,
\[
E \left\| u_2 \right\|^2 = \mathcal{O} \left( \frac{\tau_\alpha}{k - k_0} + \frac{1}{\alpha(k - k_0)^2} \exp \left( -\frac{\alpha k_0}{4\gamma_{\max}} \right) \right).
\]

Furthermore, by equation (4.7) we have $\left\| v \right\|^2 = \mathcal{O}(\alpha^4)$.

Combining these bounds and noting that $\tau_{2\alpha} \leq \tau_\alpha$, we obtain
\[
E \left[ \left( \tilde{\theta}_{k_0,k} - \theta^* \right) \left( \tilde{\theta}_{k_0,k} - \theta^* \right)^\top \right] = \mathcal{O} \left( \frac{\tau_\alpha}{k - k_0} \right) + \mathcal{O} \left( \frac{1}{\alpha(k - k_0)^2} \exp \left( -\frac{\alpha k_0}{4\gamma_{\max}} \right) \right) + \mathcal{O}(\alpha^4).
\]

We have completed the proof of Corollary 4.7.

### A.8 Proof of Negative Definiteness in TD Learning

In this subsection, we show that
\[
-\tilde{A} = -E_{(s_k,s_{k+1}) \sim \pi}[\phi(s_k) (\gamma \phi(s_{k+1}) - \phi(s_k)) \top] \in \mathbb{R}^{d \times d}
\]
is a positive definite matrix. We start by noting the following property of $I - \gamma P^S$ on $L^2(\pi^S)$.

**Claim 5.** For any $v \in L^2(\pi^S)$ and $v \neq 0$, we have $\langle v, (I - \gamma P^S)v \rangle_{L^2(\pi^S)} \geq (1 - \gamma) \left\| v \right\|^2_{L^2(\pi^S)}$. Therefore, $I - \gamma P^S$ is a positive operator on $L^2(\pi^S)$.

**Proof of Claim 5.** When $v \neq 0$, we have
\[
\langle v, (\gamma P - I)v \rangle_{L^2(\pi^S)} = \gamma \langle v, P^S v \rangle - \left\| v \right\|^2_{L^2(\pi^S)} \leq (\gamma - 1) \left\| v \right\|^2_{L^2(\pi^S)} < 0,
\]
where the last step holds since $\gamma < 1$.

We next observe the following consequence of the linear independence of $\{\phi_i\}$.

**Claim 6.** There exists some $\rho > 0$ such that
\[
\left\| \sum_{i=1}^d u_i \phi_i \right\|_{L^2(\pi^S)} \geq \rho \| u \|, \forall u \in \mathbb{R}^d.
\]

**Proof of Claim 6.** Define the matrix $\Phi \in \mathbb{R}^{d \times d}$ by $\Phi_{ij} = \langle \phi_i, \phi_j \rangle_{L^2(\pi^S)}$. We can write
\[
\left\| \sum_{i=1}^d u_i \phi_i \right\|^2_{L^2(\pi^S)} = \left( \sum_{i=1}^d u_i \phi_i \right)^\top \left( \sum_{i=1}^d u_i \phi_i \right)_{L^2(\pi^S)} = \sum_{i=1}^d \sum_{j=1}^d u_i u_j \Phi_{ij} = u^\top \Phi u.
\]

It follows that
\[
\left\| \sum_{i=1}^d u_i \phi_i \right\|^2_{L^2(\pi^S)} \geq \rho \| u \|^2, \forall u \in \mathbb{R}^d,
\]
where $\rho$ is the smallest eigenvalue of $\tilde{\Phi}$. Note that $\rho \geq 0$ since the left hand side of the last display equation is non-negative. For the sake of deriving a contradiction, assume that $\rho = 0$ and let $a \in \mathbb{R}^d$ be the corresponding unit-norm eigenvector. Then, we have
\[
\left\| \sum_{i=1}^d a_i \phi_i \right\|^2_{L^2(\pi^S)} = a^\top \Phi a = 0,
\]
which implies that $\sum_{i=1}^d a_i \phi_i = 0$ (see footnote 2), contradicting the linear independence of $\{\phi_i\}$. \(\square\)
We are ready to prove $-\bar{A}$ is positive definite. Using the definitions of $A$ and $\bar{A}$, we have

$$
-u\bar{A} = \sum_{i=1}^{d} \sum_{j=1}^{d} u_i u_j \langle \phi_i, (I - \gamma P^S) \phi_j \rangle_{L^2(\pi^S)}
$$

$$
= \left( \sum_{i=1}^{d} u_i \phi_i, (I - \gamma P^S) \left( \sum_{i=1}^{d} u_i \phi_i \right) \right)_{L^2(\pi^S)}
$$

$$
\geq (1 - \gamma) \left\| \sum_{i=1}^{d} u_i \phi_i \right\|^2_{L^2(\pi^S)}
$$

$$
\geq (1 - \gamma) \rho \|u\|_{L^2(\pi^S)}^2,
$$

where the last two steps follow from Claims 5 and 6, respectively. Since $\rho > 0$, the claim follows.

### B Existence of Higher Moments

The result in [SY19, Theorem 9] provides a sufficient condition for the existence of the $m$-th moment of the LSA iterates $\theta_k$. Their condition turns out to be more restrictive than necessary. By tightening several intermediate steps in their proof, we can establish the following Proposition B.1, which gives a more relaxed condition. In Appendix B.1 to follow, we explain how to modify the proof of [SY19, Theorem 9] to prove Proposition B.1.

**Proposition B.1.** Assume the stepsize $\alpha$ satisfies the condition (4.2). Then, for each positive integer $m$ obeying

$$
m \cdot \alpha \tau < \frac{1}{4\sqrt{\gamma_{\max}}} \left( \frac{1}{\sqrt{\gamma_{\min}}} + 1 \right)^{-1}, \quad \text{(B.1)}
$$

it holds for all $k \geq k_m$ that

$$
\mathbb{E}[\|\theta_k\|^{2m}] \leq (2m - 1)!!(\alpha \tau)^m,
$$

where

$$
k_m = m \tau + \frac{\bar{c}}{\alpha} \left( \log \frac{1}{\alpha} \right) \sum_{t=1}^{m} \frac{1}{t},
$$

and both $c$ and $\bar{c}$ are constants independent of $\alpha$ and $m$.

In the proof of Theorem 4.1, we make use of the existence of the 4th moment. Taking $m = 2$ in Proposition B.1, we see that the condition (B.1) becomes

$$
\alpha \tau < \frac{1}{8\sqrt{\gamma_{\max}}} \left( \frac{1}{\sqrt{\gamma_{\min}}} + 1 \right)^{-1}.
$$

Recall our stepsize condition (4.2): $\alpha \tau \leq \frac{0.05}{95\sqrt{\gamma_{\max}}}$. Using the inequality $\gamma_{\max} \geq \gamma_{\min} \geq \frac{1}{2}$ established in equation (A.1), we have

$$
\frac{0.05}{95\sqrt{\gamma_{\max}}} = \frac{0.05}{95\sqrt{\gamma_{\max}}} \cdot \frac{1}{\sqrt{\gamma_{\min}}} \leq \frac{0.1}{95\sqrt{\gamma_{\max}}} \leq \frac{1}{32\sqrt{\gamma_{\max}}} < \frac{1}{8\sqrt{\gamma_{\max}}} \left( \frac{1}{\sqrt{\gamma_{\min}}} + 1 \right)^{-1}.
$$

Therefore, the condition (4.2) implies that the condition (B.1) holds with $m = 2$, which in turn ensures the existence of a finite 4th moment and proves the claim in equation (A.26).  

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\section{Proof of Proposition B.1}

The proof is similar to that of [SY19, Theorem 9]. We only point out the differences. In the proof of [SY19, Theorem 9], the key constraint on $\sigma$ and $m$ to ensure a finite $m$-th moment arises when bounding $\mathbb{E}[\|\Psi_0\|^{2m}]$, where $\Psi_k = \Gamma^{1/2} \theta_{k+r}$; see [SY19, Appendix D.4]. Below we provide a refinement of the arguments therein.

We start with the following decomposition

\begin{equation}
\|\Psi_0\|^{2m} - \|\Psi_k\|^{2m} = \sum_{t=0}^{2m-1} \left( \|\Psi_0\|^{2m-t} \|\Psi_k\|^t - \|\Psi_0\|^{2m-(t+1)} \|\Psi_k\|^{t+1} \right)
\end{equation}

where we use Lemma A.2 in step (i). Hence, for the $t$-th summand on the RHS of equation (B.2), we have

\begin{align*}
\|\Psi_0\|^{2m-(t+1)} \|\Psi_k\|^t & (\|\Psi_0\| - \|\Psi_k\|) \\
\leq & 2\alpha k \sqrt{\gamma_{\max}} (\|\Psi_0\| + b_{\max}) \\
\leq & 2\alpha k \sqrt{\gamma_{\max}} \left( \frac{1}{\sqrt{\min}} \|\Psi_0\| + b_{\max} \right),
\end{align*}

where we use Lemma A.2 in step (i). Hence, for the $t$-th summand on the RHS of equation (B.2), we have

\begin{align*}
\|\Psi_0\|^{2m-(t+1)} \|\Psi_k\|^t (\|\Psi_0\| - \|\Psi_k\|) \\
\leq & 2\alpha k \sqrt{\gamma_{\max}} \|\Psi_0\|^{2m-t} \|\Psi_k\|^t + b_{\max} \|\Psi_0\|^{2m-(t+1)} \|\Psi_k\|^t) \\
\leq & 2\alpha k \sqrt{\gamma_{\max}} \left( \frac{1}{\sqrt{\min}} (\|\Psi_0\|^{2m} + \|\Psi_k\|^{2m}) + b_{\max} (\|\Psi_0\|^{2m-1} + \|\Psi_k\|^{2m-1}) \right).
\end{align*}

We further note the following bound:

\begin{align*}
\frac{1}{\sqrt{\min}} \|\Psi_0\|^{2m} + b_{\max} \|\Psi_0\|^{2m-1} = & \|\Psi_0\|^{2(m-1)} \left( \frac{1}{\sqrt{\min}} \|\Psi_0\|^2 + b_{\max} \|\Psi_0\| \right) \\
\leq & \|\Psi_0\|^{2(m-1)} \left( \frac{1}{\sqrt{\min}} \|\Psi_0\|^2 + (b_{\max}^2 + \|\Psi_0\|^2) \right) \\
= & \left( \frac{1}{\sqrt{\gamma_{\min}}} + 1 \right) \|\Psi_0\|^{2m} + b_{\max}^2 \|\Psi_0\|^{2(m-1)}. \tag{B.3}
\end{align*}

Similarly, we have

\begin{align*}
\frac{1}{\sqrt{\min}} \|\Psi_k\|^{2m} + b_{\max} \|\Psi_k\|^{2m-1} \leq & \left( \frac{1}{\sqrt{\gamma_{\min}}} + 1 \right) \|\Psi_k\|^{2m} + b_{\max}^2 \|\Psi_k\|^{2(m-1)}. \tag{B.4}
\end{align*}

Combining (B.3) and (B.4), the $t$-th summand on the RHS of (B.2) admits the following upper bound:

\begin{align*}
\|\Psi_0\|^{2m-(t+1)} \|\Psi_k\|^t (\|\Psi_0\| - \|\Psi_k\|) \\
\leq & 2\alpha k \sqrt{\gamma_{\max}} \left( \frac{1}{\sqrt{\gamma_{\min}}} + 1 \right) (\|\Psi_0\|^{2m} + \|\Psi_k\|^{2m}) + b_{\max}^2 \frac{1}{\sqrt{\min}} \|\Psi_0\|^{2(m-1)} + \|\Psi_k\|^{2(m-1)} \right) \\
\leq & 2\alpha k \sqrt{\gamma_{\max}} \left( \frac{1}{\sqrt{\gamma_{\min}}} + 1 \right) (\|\Psi_0\|^{2m} + \|\Psi_k\|^{2m}) + b_{\max}^2 \left( \frac{1}{\sqrt{\min}} \|\Psi_0\|^{2(m-1)} + \|\Psi_k\|^{2(m-1)} \right).
\end{align*}
Boundedness Assumption 2, we normalize with \( e_i E \lambda \) obtain a Hurwitz matrix, \( \bar{M} \) start with a random matrix \( M \) on an 8-state finite state space as follows.

We first illustrate the steps we take to generate the transition matrix \( P \), then \( \bar{M} \) is Hurwitz and we set it as \( \bar{M} \). Substituting the above bound back into equation (B.2), we have

\[
\|\Psi_0\|^{2m} - \|\Psi_k\|^{2m} \leq 4m a \kappa \sqrt{\gamma_{\text{max}} \left( \frac{1}{\sqrt{\gamma_{\text{min}}} + 1} \right) \left( \|\Psi_0\|^{2m} + \|\Psi_k\|^{2m} + b^2_{\text{max}} (\|\Psi_0\|^{2(m-1)} + \|\Psi_k\|^{2(m-1)}) \right)}.
\]

Set \( C \equiv C(A, b, \pi) = 4\sqrt{\gamma_{\text{max}}} \left( \frac{1}{\sqrt{\gamma_{\text{min}}} + 1} \right) \) and \( C' \equiv C'(A, b, \pi) = \sqrt{\gamma_{\text{max}} b^2_{\text{max}}} \). We have the inequalities

\[
\|\Psi_0\|^{2m} - \|\Psi_k\|^{2m} \leq makC(\|\Psi_0\|^{2m} + \|\Psi_k\|^{2m}) + makC'(\|\Psi_0\|^{2(m-1)} + \|\Psi_k\|^{2(m-1)}),
\]

\[
(1 - makC)\|\Psi_0\|^{2m} \leq (1 + makC)\|\Psi_k\|^{2m} + makC'(\|\Psi_0\|^{2(m-1)} + \|\Psi_k\|^{2(m-1)}),
\]

\[
\|\Psi_0\|^{2m} \leq 1 + makC \|\Psi_k\|^{2m} + \frac{makC'}{1 - makC}(\|\Psi_0\|^{2(m-1)} + \|\Psi_k\|^{2(m-1)}).
\]

Therefore, the constraint on \( m \) arises as we set \( \tau = k \) and require \( mak\tau < 1 \). Hence, to ensure a finite \( m \)-th moment, we require \( mak\tau < \frac{1}{C} \), which corresponds to the condition (B.1) in the statement of Proposition B.1.

C Details for Numerical Experiments

In this section, we provide the details for the setup of the numerical experiments in Section 5.

C.1 Setup for LSA Experiments

In this section, we provide the details for the setup of the numerical experiments in Section 5.

For the experiments on LSA, we generate the transition probability matrix \( P \) and functions \( A \) and \( b \) randomly on an 8-state finite state space as follows.

We first illustrate the steps we take to generate the transition matrix \( P \). For a given \( n (= |\mathcal{X}|) \), we start with a random matrix \( M^{(P)} \in [0,1]^{n \times n} \) with entries \( m^{(P)}_{ij} \text{i.i.d.} U[0,1] \), and normalize it to obtain a stochastic matrix \( \hat{M}^{(P)} = (\hat{m}^{(P)}_{ij}) \) with \( \bar{m}^{(P)}_{ij} = \frac{m^{(P)}_{ij}}{\sum_{k=1}^{n} m^{(P)}_{ik}} \). We then examine the period and reducibility of the stochastic matrix \( \hat{M}^{(P)} \) to ensure that it is aperiodic and irreducible, which gives a uniformly ergodic Markov chain as required in Assumption 1. If \( \hat{M}^{(P)} \) is not aperiodic or irreducible, we then repeat the above procedure until we obtain one, and set \( P := \hat{M}^{(P)} \). Now with \( P \) generated, we compute the stationary distribution \( \pi \).

Next, we proceed to generate \( A(x) \) for \( x \in \mathcal{X} \). As we also need \( \bar{A} = \mathbb{E}_x[A(x)] \) Hurwitz as required in Assumption 3, we start with generating the Hurwitz matrix \( \bar{A} \) and then add noise to obtain the respective \( A(x) \). We first generate a random matrix \( M^{(A)} \in \mathbb{R}^{d \times d} \) with \( m^{(A)}_{ij} \text{i.i.d.} N(0,1) \), and examine the eigenvalues \( \lambda_i(M^{(A)}) \), as Hurwitz matrix has eigenvalues all with strictly negative real parts. If \( \text{Re}(\lambda_i(M^{(A)})) < 0 \) for all \( i = 1, \ldots, d \), then \( M^{(A)} \) is Hurwitz and we set it as \( \bar{A} := M^{(A)} \). Otherwise, we adjust \( M^{(A)} \) to obtain a Hurwitz matrix, \( \bar{A} := M^{(A)} - \max(\text{Re}(\lambda_i(M^{(A)}))) \cdot I_d \). With \( \bar{A} \) generated, we add a noise matrix \( E(x) \in [-1,1]^{d \times d} \) to \( \bar{A} \) to obtain \( A(x) \), i.e., \( A(x) = \bar{A} + E(x) \). As \( \mathbb{E}_x[E(x)] = 0 \), we only generate \( E(x) \) with \( e_{ij} \text{i.i.d.} U[-1,1] \) for \( x = 1, \ldots, n-1 \), and set \( A(n) = \bar{A} - \sum_{x=1}^{n-1} \pi_x E(x) \). Lastly, to align with our boundedness Assumption 2, we normalize \( A(x) \) by the following procedure,

\[
A(x) \leftarrow A(x)/\max_x \|A(x)\|, \quad \bar{A} \leftarrow \bar{A}/\max_x \|A(x)\|,
\]

which ensures that \( A_{\text{max}} := 1 \).

Lastly, we generate \( b(x) \in \mathbb{R}^d \) with \( b(x) \text{i.i.d.} [-1,1] \) and obtain \( \bar{b} = \sum_x \pi_x b(x) \) and \( b_{\text{max}} = \max_x \|b(x)\| \).

C.2 Setup for TD(0) Experiments

We consider the TD(0) algorithm applied to the so-called “problematic MDP” considered in the work [KP00, LP03]. This MDP involves \( n^S = 4 \) states, \( S = \{1, 2, 3, 4\} \), arranged from left to right. At each state, there are two available actions, “Left” (L) and “Right” (R). When the action L is chosen, with probability 0.9 the
state transitions to the left (or stays at the same position if the current state is the leftmost state 1), and
with probability 0.1 the state transitions in the opposite direction (or stay at the same position if the current
state is the rightmost state 4). The dynamics under the action R is defined symmetrically. The reward
function is given by \( r(1) = 0, r(2) = 1, r(3) = 3, r(4) = 0 \), with a discount factor \( \gamma = 0.9 \). We consider
evaluating the policy that takes the actions R, R, L, and L at states 1, 2, 3, 4, respectively (this policy is the
optimal policy for this MDP). The induced MRP is illustrated in Figure 7.

![Figure 7: The Problematic MDP under “RRLL” Policy.](image)

We apply TD(0) with linear function approximation to the above MRP. For each state \( s \in \{1, 2, 3, 4\} \),
the corresponding \( d = 3 \) dimensional feature vector is given by
\[
\phi(s) = (1, s, s^2)^	op,
\]
which is used in the work [KP00]. We then normalize each row of the feature matrix \( \Phi \in \mathbb{R}^{n \times d} \) to have
unit norm; explicitly, we set
\[
\phi(s)_i \leftarrow \frac{\phi(s)_i}{\sum_{s=1}^{4} \phi(s)_i}, \quad i = 1, 2, 3, 4.
\]
Note that one may ensure the condition \( \max_{s \in S} \|\phi(s)\| \leq \frac{1}{\sqrt{1+\gamma}} \) required by our theory by further rescaling
the entire matrix \( \Phi \). In our experiments, we ignore this rescaling step, as it is equivalent to simply rescaling
the stepsize and iterates.

C.3 Setup for SGD Experiments

For the experiments on SGD applied to least squares regression, the data \( g_t \in \mathbb{R}^2 \) is sequentially generated from an independent Metropolis-Hastings (MH) sampler. The target stationary distribution of the MH sampler is Uniform\([-1, 1]\) \( \times \) Uniform\([-1, 1]\). The states of the MH sampler are generated by employing a
sampling distribution \( q \) for \( h \in \mathbb{R}^2 \), where each coordinate \( h(i) \) has the density
\[
q(h(i)) = \begin{cases} 
1/4 & \text{if } -1 \leq h(i) < 0 \\
3/4 & \text{if } 0 \leq h(i) < 1,
\end{cases} \quad \text{for } i = 1, 2.
\]
Given state \( g_t \), the next state \( g_{t+1} \) is generated as follows. We first generate a new sample state \( h \) coordinate
by coordinate from the sampling distribution \( q \). Then, we accept \( h \) and set \( g_{t+1} = h \) with probability
\[
\min \left\{ \frac{q(g_t(1)) q(g_t(2))}{q(h(1)) q(h(2))}, 1 \right\};
\]
otherwise, we set \( g_{t+1} = g_t \).