VIRTUALLY SMALL SPECTRAL PACKAGE OF A RIEMANNIAN MANIFOLD

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Abstract. For a Morse function $f$ on a closed orientable Riemannian manifold $(M, g)$ one introduces the virtually small spectral package, an analytic object consisting of a finite number of analytic quantities derived from $(g, f)$ which, in principle, can be calculated. One shows that they determine the Torsion of the underlying space $M$, a parallel to the result that the dimensions of the spaces of harmonic forms calculate the Euler-Poincaré characteristic of $M$ and extend the Poincaré duality between harmonic forms and between Betti numbers for a closed oriented Riemannian manifold.

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1. Introduction

For a compact ANR $X$, the integral homology $H_r(X; Z)$ is a finitely generated abelian group of a finite rank $\beta_r(X)$ whose set of finite order elements has a finite cardinality $\text{Tor}_r(X)$. The following two numbers are remarkable topological invariants

$$\chi(X) := \sum (-1)^i \beta_i(X) \quad \text{and} \quad \text{Tor}(X) := \prod \langle \text{Tor}_i(X) \rangle^{(-1)^i}.$$ 

If $M^n$ is an $n$-dimensional closed orientable (topological) manifold, the Poincaré duality implies $\beta_r(M) = \beta_{n-r}(M)$ and $\text{Tor}_r(M) = \text{Tor}_{n-1-r}(M)$ and therefore for $n$ odd $\chi(M^n) = 0$ and for $n$ even $\text{Tor}(M^n) = 1$.

If $(M, g)$ is an orientable closed Riemannian manifold and $\mathcal{H}^r(M, g) \subset \Omega^r(M)$ denotes the space of harmonic forms of degree $r$ and $\mathcal{H}^r_0 \subset \mathcal{H}^r$ the subspace of integral forms, then

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$T_\gamma(M, g) := \mathcal{H}^r(M)/\mathcal{H}_\gamma^r(M)$ is the compact torus of dimension $\beta_\gamma(M)^2$ as a Riemannian manifold with a metric induced from the scalar product on $\Omega^r(M)$ provided by $g$. Let $V_\gamma(M, g) := \text{Vol}(T_\gamma(M, g))$ and $\mathcal{V}(M, g) := \prod(V_\gamma(M, g))^{-1} \in \mathbb{R}_{>0}$. This is a Riemannian invariant. In view of the Hodge-de Rham theorem $\chi(M)$ is also a Riemannian invariant.

If $f : M \to \mathbb{R}$ is a smooth real-valued function, $f$ defines a deformation (parametrization) $(\Omega^r(M), d^*t)$ of the de Rham complex $(\Omega^*(M), d^*)$ with $d^*t(t) := e^{-t}f e^{t} = d^* + tdh \wedge \cdot$, and therefore the one parameter family of second order elliptic differential operators $\Delta^q_{g,f}(t)$:

$$\Delta^q_{g,f}(t) := d^{q-1}(t) \cdot \delta^q_g(t) + \delta^{q+1}_g(t) \cdot d^q(t) = \Delta^q_g + t(L^q_X + L^q_X) + t^2||X||^2,$$

where

1. $\delta^q_g(t) := (-1)^{n(q-1)+1} \star^q_g - e^t f \cdot e^{-t} \star^q_g$,
2. $X = -\text{grad}_g f$, $L_X$ the Lie derivative in direction $X$, $L^q_X := (-1)^{(n+1)q+1} \star^q_g \cdot L^q_g \cdot \star^q_g$,
3. $||X|| : M \to \mathbb{R}_{\geq 0}$ the length of the vector field $X = -\text{grad}_g f(x)$.

These operators remain self-adjoint, nonnegative elliptic differential operators on $L^2(\Omega^q(M))$, the $L^2$-completion of $\Omega^q(M)$ with $\Delta^q_{g,f}(t)$ a zero order perturbation of the standard Laplace-Beltrami operator $\Delta^q_g$.

When both $g$ and $f$ are implicit from the context, one abbreviates $\Delta^q_{g,f}(t), \Delta^q_g, \delta^q_g, \star^q_g$ to $\Delta^q(t), \Delta^q, \delta^q, \star^q$ for simplicity in writing.

The operators $\Delta^q(t)$, referred below as Witten Laplacians, provide a holomorphic family of type A of self-adjoint operators in the sense of Kato cf. [3] and therefore, in view of a theorem of Reillich-Kato, cf. Theorem 3.9 chapter 7 in [3], one has:

**Theorem 1.1.** (Rellich - Kato) There exist a collection of non-negative real-valued functions $\lambda^q_\gamma(t)$, unique up to permutation, and a collection of norm one $q$-differential form-valued maps $\omega^q_\gamma(t) \in \Omega^q(M)$, analytic in $t \in \mathbb{R}$, indexed by $\alpha \in \mathcal{A}^q$, $\mathcal{A}^q$ a countable set, each with holomorphic extension to a neighborhood of the real line $\mathbb{R} \subset \mathbb{C}$

such that:

1. $\Delta^q(t)\omega^q_\gamma(t) = \lambda^q_\gamma(t)\omega^q_\gamma(t)$,
2. for any $t$ the collections $\lambda^q_\gamma(t)$ represent all repeated eigenvalues of $\Delta^q(t)$ and the collection $\omega^q_\gamma(t)$ form a complete orthonormal family of associated eigenvectors for the operator $\Delta^q(t)$,
3. exactly $\beta_\gamma = \dim \mathcal{H}^q(M; \mathbb{R})$ eigenvalue functions $\lambda^q_\gamma(t)$ are identically zero and all others are strictly positive.

These analytic maps $\lambda^q_\gamma(t)$ and $\omega^q_\gamma(t)$ are called branches, eigenvalue branch and eigenform branch respectively, with extensions to holomorphic maps in the neighborhood of $\mathbb{R} \subset \mathbb{C}$. The maps $\omega^q_\gamma(t)$ have $\pm$ ambiguity and if the branch $\lambda^q_\gamma(t)$ has multiplicity $\geq 2$, i.e. $\lambda^q_\gamma(t) = \lambda^q(t)$

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1. In view of the Hodge de-Rham theorem
2. $L_X$ and $L_X$ are order one differential operators but $L_X + L_X$ is of order zero as well as the multiplication by the smooth function $||X||^2$
3. Holomorphic extension means extensions $\lambda^q(z) \in \mathbb{C}$, $\omega^q(z) \in \Omega(M) \otimes C$ for $z$ in a neighborhood of $\mathbb{R}$ in $\mathbb{C}$ which for $t \in \mathbb{R}$ is a real number and $\omega^q(t) \in \Omega(M) \otimes 1$
for a finite set of indices \( \alpha \), then the family of finite dimensional vector spaces spanned by the corresponding \( \omega^q_\alpha(t) \) is unique and of course analytic in \( t \).

If \( f \) is a Morse function with \( c_q \) critical points of Morse index \( q \), then in view of a result of Witten [9], cf [3] Proposition 5.2 or [4] Theorem 2.8 for details, for any \( q \) exactly \( c_q \) eigenvalue branches of \( \Delta^q(t) \) go exponentially fast to zero and all others go at least linearly fast to \( \infty \). Moreover each eigenvalue branch which converges to zero corresponds to a critical point and its corresponding eigenform branch concentrates to this critical point. We index these analytic functions as \( \lambda^\ind x(t) \) and \( \omega^\ind x(t) \) and refer to the finite collection of branches

\[
\{ \lambda^\ind x(t), \omega^\ind x(t), x \in Cr(f) \}
\]

as the virtually small spectral package of \((M, g, f)\) and to the finite set

\[
\{ \lambda^\ind x(0), \omega^\ind x(0), x \in Cr(f) \}
\]

the subset of the infinite set \( \{ \lambda^q_\alpha, \omega^q_\alpha \} \), the spectral package of \((M, g)\), as the virtually small spectral package of \((M, g)\) determined by \( f \).

Note that for \( t \) large enough, in view of the spectral gap theorem, cf Theorem ?? stated in Section ??, the eigenvalues \( \lambda^q(t)x \in Cr_q(f) \), exhaust the first \( c_q = \sharp Cr_q(f) \), possibly repeated, smallest eigenvalues of \( \Delta^q(t) \), however this is not true for \( t = 0 \) as the Example in section ?? shows. This explains the name virtually small for the collection \( \{ \lambda^\ind x(0), \omega^\ind x(0), x \in Cr(f) \} \).

We denote by \( \Omega^\ind \times(M)(t) \) the span of the eigenforms \( \omega^\ind x(t)x \in Cr(f) \) inside \( \Omega^\ind \times(M) \). They generate a finite dimensional sub-complex \( (\Omega^\ind \times(M)(t), d^*(t)) \) of \( (\Omega^\ind(M), d^*(t)) \) with the \( q \)-component of dimension \( c_q \), which is an analytic family of cochain complexes with cohomology of constant dimension \( \beta_q(M) \). Since \( f \) is a Morse function, for any \( x \in Cr(f) \) the stable/unstable set \( W^\pm_x \) of the vector field \(-grad_x f\) are submanifolds diffeomorphic to \( \mathbb{R}^{\ind x} \). For any \( x \) choose an orientation \( O_x \) for \( W^-_x \) and for any \( x, y \in Cr_q(f) \) consider the integral

\[
A^q(x, y)(t) := \int_{W^-_y} e^{tf} \omega^q_x(t),
\]

which a priory might not be convergent but when convergent for any \( x, y \in Cr_q(f) \) provide the non-negative number

\[
a^q(M, g, f)(t) := | det ||| A^q(x, y)(t)(||| | \geq 0,
\]

where ||| \( A^q(x, y)(t) \)||| is a \( c_q \times c_q \) matrix with \( c_q = \sharp Cr_q(f) \). Changing of the orientation \( O_y \) changes the sign of the integrals \( A^q(x, y) \) for all \( x \) but leave \( a^q(M, g, f) \) unchanged when defined.

**Proposition 1.2.**

1. If the vector field \(-grad_x f\) is Morse-Smale, then the integral \( (1.1) \) is uniformly convergent and both \( A^q(x, y)(t) \) as well as \( a^q(M, g, f)(t) \) are analytic in \( t \), with the last being independent on the choice of \( \omega^q_x(t) \) and the orientations \( O_x \).
2. The analytic function \( a^q(M, g, f)(t) \) is non-negative with at most finitely many zeros in any interval \([T, \infty)\).
In particular
\[ a(M, g, f)(t) := \prod (a^q(M, g, f)(t))^{(-1)^q} \]
is a priori a nonnegative meromorphic function in \( t \) with at most finitely many zeros and poles in any interval \([T, \infty)\).

The main result of this Note is the following theorem:

**Theorem 1.3.** Suppose that the vector field \(-\nabla g f\) is Morse-Smale. Then the following holds true.

1. The meromorphic map \( a(M, g, f)(t) \) is strictly positive and has no zeros and no poles.
2. The virtually small spectral package determined by \( f \) together with the numbers \( a(M, g, f) \) and \( V(M, g) \) are all analytic invariants and determine the topological invariant \( \text{Tor}(M) \) by the formula:

\[
\log \text{Tor}(M) = 1/2 \sum_q (-1)^{q+1} q \left( \sum_{\alpha \in A_{vs,+}^q} \log \lambda_{\alpha}^q(0) \right) + \log a(M, g, f) - \log V(M, g),
\]

where \( a(M, g, f) := a(M, g, f)(0) \) and \( A_{vs,+}^q \subset A^q \) is the set of indices \( \alpha \)'s such that \( \lambda_{\alpha}^q(t) \) belongs to the virtually spectral package with \( \lambda_{\alpha}^q(t) > 0 \).

**Conjecture 1:** The statement remains true without the hypothesis that \(-\nabla g f\) is Morse-Smale.

**Conjecture 2:** Under the hypothesis that \(-\nabla g f\) is Morse-Smale one has \( a^q(M, g, f)(t) \neq 0 \).

Note that if Conjecture 2 holds true for \( t = 0 \), then it can be shown that the Morse complex defined by \((g, f)\) can be canonically realized as a sub-complex of the de Rham complex equipped with the scalar product defined by the metric \( g \). Recall that Hodge-de Rham theorem implies that the complex \((H^r(M), 0)\) can be realized in this way as the sub-complex of harmonic forms.

As shown in Section 4, for an oriented closed Riemannian manifold the Hodge star operator
\[ \star : \Omega^q(M) \to \Omega^{n-q}(M) \]
identifies the virtually small \( q \)-spectral package of \((M, g, f)\) to the the virtually small \((n - q)\)-spectral package of \((M, g, -f)\). This can be viewed as an extension of Poincaré duality.

## 2. Proof of Proposition 1.2 and Theorem 1.3

**Proof of Proposition 1.2**

One says that the vector field \( X = -\nabla g f \) is Morse-Smale if for any \( x, y \in Cr(f) \) the unstable set \( W_x^- \) and the stable set \( W_y^+ \) are transversal, which implies that \( T(x, y) = (W_x^- \cap W_y^+)/\mathbb{R}^4 \) the space of trajectories from \( x \) to \( y \), is a manifold of dimension \( \text{ind}(x) - \text{ind}(y) - 1 \). Under the hypothesis that \( X \) is Morse-Smale, it is shown in [2] or [6] that the embedding

\[ \mathbb{R}^4 \]
This implies the uniform convergence of the integral (1.1) and the analyticity of manifold with corners whose interior is $W_x^-$. Hence, it follows that

$$\int_{W_y^-} e^{tf} \omega^q_x(t) = \int_{W_y^-} (\hat{i}_x)^* (e^{tf} \omega^q_x(t)).$$

This implies the uniform convergence of the integral (1.1) and the analyticity of $A^q(x, y)(t)$ and of $a^q(M, g, f)(t)$. To conclude the independence on the choices of $\omega^q_x(t)$ it suffices to note that the matrices $\|A^q(\cdot, \cdot)(t)\|$ for two choices differ one from the other by composition by an orthogonal matrix, hence have the same determinant up to sign. The results in [2] shows also that the partition $M = \bigcup_x W_x^-$ provides a CW structure of $M$ with open cells $W_x^-$ and if one equips each cell $W_x^-$ with the orientation $O_x$ and one denotes by $C^q$ the $\mathbb{R}$-vector space $\text{Maps}(C_q(f), \mathbb{R})$, then $\text{Int}^q(t) : \Omega^q(M) \to C^q$ defined by

$$\text{Int}^q(t)(\omega)(x) = \int_{W_x^-} \omega$$

provides a linear map, and the collection of the linear maps $\text{Int}^q(t) : \Omega^q(M) \to C^q$ define a map of cochain complexes $\text{Int}^q(t) : (\Omega^*(M), d^*(t)) \to (C^*, \partial^*)$, which by de Rham theorem is a quasi-isomorphism. Of course this involves the explicit description of the corner structures of $\hat{W}_x^-$. The cochain complex morphism $\text{Int}^q(t)$ is clearly analytic in $t$ and restricts to $(\Omega^*(M), d^*(t))$ an analytic family of quasi-isomorphisms $(\Omega^*_{\text{an}}(M), d^*(t)) \to (C^*, \partial^*)$.

If one considers $\omega^q_x(t)$’s as a basis for $\Omega^q_{\text{an}}(M)(t)$ and the characteristic functions of the set $C_{\text{an}}(f)$ as a basis for $C^q$, then one realizes that the matrix representation of $\text{Int}^q(t)$ is exactly the matrix $\|A^q(x, y)(t)\|$. For $t$ large enough Witten-Helffer-Sjöstrand results imply that $\text{Int}^q(t)$ restricted to $\Omega^q_{\text{an}}(M)(t)$ is an isomorphism, (for details [5] theorem 5.5. item 5, or [1] theorem 3.1), which shows that, for $t$ large enough, $a^q(M, g, f)(t) \neq 0$. In view of analyticity item 2 follows as stated.

Proof of Theorem (1.3):

First observe the following facts.

1. For an isomorphism $\varphi : (V, \langle \cdot, \cdot \rangle_V) \to (W, \langle \cdot, \cdot \rangle_W)$ between two finite dimensional vector spaces equipped with scalar product, let $\text{Vol}(\varphi) := \sqrt{\det(\varphi^\sharp \cdot \varphi)^{1/2}} = \sqrt{\det(\varphi \cdot \varphi^\sharp)^{1/2}}$ with $\varphi^\sharp$ the adjoint of $\varphi$.

2. If $\varphi(t) : (V(t), \langle \cdot, \cdot \rangle_{V(t)}) \to (W(t), \langle \cdot, \cdot \rangle_{W(t)})$ is a continuous/analytic family of isomorphisms between finite dimensional vector spaces equipped with scalar products, then the function $\text{Vol}(\varphi(t))$ is continuous/analytic in $t$.

3. For a cochain complex $\mathcal{C} = (C^*, d^*)$ of finite dimensional vector spaces equipped with scalar products

$$\mathcal{C} : 0 \longrightarrow (C^0, \langle \cdot, \cdot \rangle_0) \xrightarrow{d^0} (C^1, \langle \cdot, \cdot \rangle_1) \xrightarrow{d^1} \cdots \longrightarrow (C^n, \langle \cdot, \cdot \rangle_n) \longrightarrow 0,$$

\footnote{for example $V(t)$ resp.$W(t)$ appear as images in $V$ resp.$W$, of an analytic/continuous family of bounded projectors $P(t) : V \to V$ resp. $Q(t) : W \to W$ for $V$ resp $W$ topological vector spaces; this give meaning to "analytic family"}
one denotes by \( \Delta_q' := \delta^{q+1} \cdot d^q + d^{q-1} \cdot \delta^q \), \( \delta \) the adjoint of \( d \), and by \( \det' \Delta_q' \neq 0 \) the product of nonzero eigenvalues of \( \Delta_q' \). The product
\[
T(C) := \prod (\det' \Delta_q')^{\frac{1}{2} q(-1)^{q+1}}
\]
is referred to as the torsion of \( C \). Here \( \det \Delta' \) denotes the product of nonzero eigenvalues of \( \Delta \). For a continuous/analytic family of cochain complexes \( C(t) = (C^*, d^*(t)) \) such that \( \dim C^q(t) \) and \( \dim H^q(C(t)) \) are constant in \( t \) for any \( q \), the function \( T(C(t)) \) is continuous/analytic in \( t \).

The verifications of items (2) and (3) above are straightforward from definitions.

(4) Suppose that \( \varphi : C_1 \to C_2 \) is a morphism of cochain complexes of finite dimensional vector spaces with scalar products, where \( C_i = (C_i^*, d_i^*), i = 1, 2, \) and \( \varphi = \{ \varphi^q : C_1^q \to C_2^q \} \). Suppose that \( \varphi^q \) is an isomorphism for any \( q \). Then \( \varphi \) induces the isomorphism \( H^q(\varphi) : H^q(C_1) \to H^q(C_2) \) between vector spaces equipped with induced scalar product. Let
\[
\Vol(\varphi) := \prod (\Vol(\varphi^q))^{(-1)^q}
\]
and
\[
\Vol(H(\varphi)) := \prod (\Vol(H^q(\varphi)))^{(-1)^q}.
\]

As verified in \[Proposition 2.5\] one has
\[
T(C_2)/T(C_1) = \Vol(H(\varphi))/\Vol(\varphi).
\] (2.1)

(5) For a continuous/analytic family of isomorphisms \( \varphi(t) : C_1(t) \to C_2(t), t \in \mathbb{R}, \) with \( \dim C_1^q(t) = \dim C_2^q(t) \) and \( \dim H^q(C_1(t)) = \dim H^q(C_2(t)) \) constant in \( t \), the real-valued functions \( T(C_1(t)), T(C_2(t)), \Vol(\varphi(t)), \Vol(H(\varphi(t))) \) are nonzero and continuous/analytic.

We consider \( \varphi(t) = Int^*(t) : (\Omega^*_m(M)(t), d^*_t) \to (C^*, \partial^*) \) with \( * = 0, 1, \ldots, \dim M \). The first cochain complex is equipped with the scalar products defined by the metric \( g \), and the second with the unique scalar product which makes the characteristic functions of the critical points orthonormal. In view of item (4) the function
\[
\frac{T(\Omega^*_m(M)(t), d^*_t) \cdot \Vol(H(\varphi(t)))}{T(C^*, \partial^*)}
\]
is a strictly positive analytic function and in view of (4) agrees with \( a(t) \) for all \( t \) but the finite collection which might be a zero or a pole for \( a(t) \). Hence the meromorphic function \( a(t) \) has no zeros and no poles. This establishes \[Theorem 1.3\] part 1. Together with \( (2.1) \) it also implies
\[
\frac{T(\Omega^*_m(M)(t), d^*_t)}{a(t)} \cdot \Vol(H(\varphi(t))) = T(C^*, \partial^*).
\]
Evaluation at \( t = 0 \) combined with the observation that \( \text{Tor}(M) = T(C^*, \partial^*) \) implies
\[
\frac{T(\Omega^*_m(M), d^*)}{a(0)} \cdot \Vol(H(\varphi(0))) = \text{Tor}(M).
\]

Taking "log", one derives \[Theorem 1.3\] part 2.
3. Poincaré duality for the virtually small spectral package

For a closed oriented Riemannian manifold \((M, g)\) and smooth function \(f : M \to \mathbb{R}\) one has the Hodge star operator \(\ast : \Omega^q(M) \to \Omega^{n-q}(M)\) which satisfies the following properties.

1. \(\ast^{n-q} \cdot \ast^q = (-1)^q(n-q) \text{Id}\),
2. \((-1)^q(n-q) \ast^q \Delta^q \ast^{n-q} = \Delta^{n-q}\),
3. \((-1)^q(n-q) \ast^q \Delta^q_{g,f}(t) \ast^{n-q} = \Delta^{n-q}_{g,f}(t),\)
4. \(\Delta^q_{g,f}(t) = \Delta^q_{g,-f}(t)\).

As a consequence, the Hodge operator \(\ast^q\) identifies the \(q\)-spectral package of \((M, g, f)\) with the \((n-q)\)-spectral package of \((M, g, -f)\) and the \(q\)-virtually small spectral package

\[
\left\{ \lambda_{g,f,x}^q(t), \omega_{g,f,x}^q(t), \ x \in Cr_q(f) \right\}
\]

of \((M, g, f)\) with the \((n-q)\)-virtually small spectral package

\[
\left\{ \lambda_{g,-f,x}^{n-q}(t), \omega_{g,-f,x}^{n-q}(t), \ x \in Cr_{n-q}(-f) \right\}
\]

of \((M, g, -f)\). More precisely, it holds that for \(x \in Cr_q(f) = Cr_{n-q}(-f)\),

\[
\lambda_{g,f,x}^{n-q}(t) = \lambda_{g,f,x}^q(t), \quad \omega_{g,f,x}^{n-q}(t) = \ast^q \omega_{g,f,x}^q(t).
\]

**Poincaré duality**

The above identification can be regarded as a refinement of the Poincaré duality which states that \(\beta^q(M)\) viewed as the multiplicity of the eigenvalue 0 of \(\Delta^q_{g,f}(0)\) is equal to \(\beta^{n-q}(M)\) viewed as the multiplicity of the eigenvalue 0 of \(\Delta^{n-q}_{g,f}(0)\).

4. Virtually small eigenvalues versus the smallest eigenvalues

We are going to show that the virtually small eigenvalues may not be equal to the smallest eigenvalues by giving an example. For \(S^1 := \mathbb{R}/(2\pi \mathbb{Z})\), we consider a torus \(M = S^1 \times S^1\) equipped with the flat metric \(g_0\) induced from the canonical metric on \(\mathbb{R} \times \mathbb{R}\). The function \(f(\theta_1, \theta_2) = \sin(2\theta_1) + \sin(2\theta_2), (\theta_1, \theta_2) \in S^1 \times S^1\), is a Morse function on \(M\) having four critical points of index 0, four critical points of index 2 and eight critical points of index 1. The sequence of eigenvalues of \(\Delta^q_{g,f}\) in increasing order is \(0 \leq 1 \leq 1 \leq 1 \leq 4 \cdots\). The virtually small 0-eigenvalues consist of four real numbers. The first one is \(\lambda_1(0) = 0\), the next two are \(\lambda_2(0) = \lambda_3(0) \geq 1\) and the final one is \(\lambda_4(0) = 2\mu_2(0)\), where \(\mu_2(0) \geq 1\). This shows that the virtually small eigenvalues are not the same as the smallest eigenvalues.

**Proof.** We first observe that \(h : S^1 \to \mathbb{R}\) given by \(h(\theta) = \sin 2\theta\) is a Morse function on \(S^1\), whose Witten Laplacian \(\Delta^0(t)\) is

\[
\Delta^0(t) = -\partial^2/\partial \theta^2 + 4t \sin 2\theta + 4t^2(\cos 2\theta)^2.
\]
Then, $\Delta^0(t)$ has two virtually small eigenvalue branches. One of them is $\mu_1(t) \equiv 0$ and the other is $\mu_2(t) > 0$ because $\beta^0(S^1) = 1$ and $h$ has two critical points of index 0. Observe that the eigenvalues of $\Delta^0(0)$ are

$$0, 1, 2^2, 3^2, \ldots n^2, \ldots,$$

where 0 has the multiplicity 1 and all others have multiplicity 2. Since $\mu_1(0)$ and $\mu_2(0)$ are among the above eigenvalues, one has $\mu_2(0) \geq 1$. In view of the definition of $f(\theta_1, \theta_2)$ and that of $\Delta^0(t)$ for $(S^1 \times S^1, g_0, f)$, the four virtually small eigenvalues branches $\lambda_1(t), \lambda_2(t), \lambda_3(t), \lambda_4(t)$ are of the form $\mu_i(t) + \mu_j(t)$ with $i, j \in \{1, 2\}$, and hence the virtually small 0-eigenvalues are $\lambda_1(0) = 0, \lambda_2(0) = \lambda_3(0) = \mu_1(0)$ and $\lambda_4(0) = 2\mu_1(0)$. □

When $M^2$ is a 2-dimensional oriented closed Riemannian manifold, the above example shows that the virtually small spectral package of $\Delta^0$ is not the same as the collection of the smallest eigenvalues. By the Poincaré duality, the nonzero eigenvalues of $\Delta^1$ on $M^2$ are two times of the nonzero eigenvalues of $\Delta^0$ or $\Delta^2$, and hence the result remains the same for $\Delta^1$.

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