\textbf{$L^p$-resolvent estimate for finite element approximation of the Stokes operator}

Tomoya Kemmochi

March 20, 2023

In this paper, we will show the $L^p$-resolvent estimate for the finite element approximation of the Stokes operator for $p \in \left(\frac{2N}{N+2}, \frac{2N}{N-2}\right)$, where $N = 2, 3$ is the dimension of the domain. It is expected that this estimate can be applied to error estimates for finite element approximation of the non-stationary Navier–Stokes equations, since studies in this direction are successful in numerical analysis of nonlinear parabolic equations. To derive the resolvent estimate, we obtain local energy error estimate according to a novel localization technique and establish global $L^p$-type error estimates. The restriction for $p$ is caused by the treatment of lower-order terms appearing in the local energy error estimate. Our result may be a breakthrough in the $L^p$-theory of finite element methods for the non-stationary Navier–Stokes equations.

1. Introduction

Let $Ω ⊂ \mathbb{R}^N$ ($N = 2, 3$) be a bounded domain with smooth boundary and consider the Stokes resolvent problem

\[
\begin{cases}
\lambda u - \Delta u + \nabla \varphi = f & \text{in } \Omega, \\
\text{div } u = 0 & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega,
\end{cases}
\]  
(1.1)

where $f \in L^p(Ω)$, $p \in (1, \infty)$, $λ \in Σ_δ := \{z \in \mathbb{C} \setminus \{0\} \mid |\arg z| < π - δ\}$, and $δ \in (0, π)$. Then, as is well-known, the $L^p$-resolvent estimate

\[(|λ| + 1) \|u\|_{L^p(Ω)} + |λ|^{1/2} \|\nabla u\|_{L^p(Ω)} + \|\nabla^2 u\|_{L^p(Ω)} + \|\nabla \varphi\|_{L^p(Ω)} \leq C \|f\|_{L^p(Ω)} \]  
(1.2)

holds for $p \in (1, \infty)$ and $δ \in (0, π)$ (see e.g. [34, 12, 17]). The resolvent estimate (1.2) implies the analyticity of the Stokes semigroup and well-definedness of the fractional powers of the Stokes operator, which are widely applied to analysis of the Navier–Stokes equations; see e.g. [7, 1, 37].

In this paper, we consider the finite element method of the Stokes resolvent problem (1.1). Throughout this paper, we assume that $Ω$ is \textit{convex} to simplify the argument. Then, let $T_h$ be the triangulation of $Ω$ and $Ω_h \subset Ω$ be polygonal approximation of $Ω$ that consists of elements of $T_h$. We introduce a pair of finite element spaces $(V_h, Q_h) \subset
$H^1_0(\Omega_h)^N \times L^2(\Omega_h)$ and set $Q^0_h = Q_h \cap L^2_0(\Omega_h)$. With these notations, finite element approximation of (1.1) is the variational problem to find $(u_h, \varphi_h) \in V_h \times Q^0_h$ satisfying

\[ \begin{cases} 
\lambda(u_h, v_h)_h + (\nabla u_h, \nabla v_h)_h - (\varphi_h, \text{div } v_h)_h = (f, v_h)_h, \\
(\text{div } u_h, \psi_h)_h = 0,
\end{cases} \quad \forall v_h \in V_h, \forall \psi_h \in Q_h, \tag{1.3} \]

where $\lambda \in \Sigma_\delta$, and $(\cdot, \cdot)_h$ is the $L^2$-inner product over $\Omega_h$. Precise definitions will be given later.

The main result of the present paper is to establish the resolvent estimate

\[ (|\lambda| + 1)\|u_h\|_{L^p(\Omega_h)} + |\lambda|^{1/2}\|\nabla u_h\|_{L^p(\Omega_h)} + \|\nabla \varphi_h\|_{L^p(\Omega_h)} \leq C\|f\|_{L^p(\Omega)} \tag{1.4} \]

for $p \in \left(\frac{2N}{N+2}, \frac{2N}{N-2}\right)$, where $C$ is independent of $h$. Here, the restriction for $p$ is a technical assumption, which will be mentioned later.

Our result may be a breakthrough in the $L^p$-theory of finite element methods for the non-stationary Navier–Stokes equations. Indeed, for the parabolic case, the $L^p$-resolvent estimates [2], analyticity of the semigroup [32, 35], and moreover maximal regularity [9, 19, 23, 25] are established for the finite element approximation of the elliptic operators, and such results are applied to analysis of finite element methods for nonlinear problems [6, 30, 10, 22, 26]. Moreover, our result may make it possible to show that numerical solution is bounded in $L^\infty$ in time and $L^4$ in space, which implies that there exists a smooth solution to the three-dimensional Navier–Stokes equations, as proved in [24].

In the $L^p$-theory of the finite element method, an approach by the regularized Green’s function is widely used. See [31, 4, 36] and references therein for elliptic and parabolic problems. This strategy is also successful for the steady Stokes equations [13, 5]. In these studies, the stability and error estimates are reduced to the weighted or localized energy error estimates for the regularized Green’s functions, which imply $L^1$-estimates for the regularized Green’s functions.

In this paper, however, we will develop a novel approach. Instead of introducing regularized Green’s functions, we will address the $L^p$-norms of the finite element solution directly. Let $(u, \varphi)$ be the solution of (1.1) and then we will show the error estimate

\[ |\lambda|^{1/2}\|u - u_h\|_{L^p(\Omega_h)} + \|\nabla (u - u_h)\|_{L^p(\Omega_h)} \leq C|\lambda|^{-1/2}\|f\|_{L^p(\Omega)} \tag{1.5} \]

for $p \in \left[2, \frac{2N}{N-2}\right)$, which is the most difficult part of the present paper. It is clear that this implies the resolvent estimate for the velocity. To obtain the pressure estimate, we will also prove other kinds of error estimates (see Theorem 2.2), which themselves may be useful for error estimates of non-stationary Stokes equations (cf. [29, 21]).

To prove (1.5), we consider a covering of $\Omega_h$ by balls of radius $O(h)$, which differs from the existing approach that introduces a dyadic decomposition of the domain. Then, according to the approach developed in [35], we derive local energy error estimates for the velocity (Propositions 5.1 and 5.2), which generates local norms of lower-order terms. It is the best if we can address all of the lower-order terms locally. However, we cannot do that at present and instead we address some of them globally, which requires the restriction for $p$ mentioned above.

The rest of the present paper is organized as follows. In Section 2, we will present precise definition of the finite element spaces and then state the main results including error estimates. In Section 3, we collect some preliminaries related to the Stokes resolvent problem. In Section 4, we will show that the error estimate (1.5) is reduced to local
energy error estimate, which is proved in Section 5. Section 6 is devoted to local and
global estimates for the pressure term appearing in the local energy error estimate. We
present global $L^p$- and $W^{-1,p}$-estimates for the velocity in Section 7 by global duality
arguments. Summarizing these estimates, we show the error estimates in Section 8, and
finally we conclude this paper by showing the resolvent estimate (1.4) in Section 9.

Throughout this paper, all of functions are complex-valued. We denote the $L^2$-inner
product and norm by $(u, v)_D = \int_\Omega u \overline{v} \, dx$ and $\|v\|_D = \sqrt{(v, v)_D}$ respectively, where $\overline{v}$ is
the complex conjugate of $v$. For a function $v$, $\nabla^2 v$ denotes the tensor whose components
are $l$-th order derivatives of $v$. Generic constants are denoted by $C$, and if we want to
express the dependency on a certain parameter $a$, we denote it by $C_a$.

2. Main results

In this section, we state the resolvent estimate and related error estimates after introdu-
cing the finite element method on a smooth domain.

Let $\Omega \subset \mathbb{R}^N$ ($N \geq 2$) be a bounded convex domain with smooth boundary $\partial \Omega$. Let
\{${\mathcal{T}}_h$\}_h be a family of conforming triangulations of $\Omega$, where
$h = \max_{T \in \mathcal{T}_h} \text{diam} \, T$, and let
$\Omega_h$ be a polygonal approximation of $\Omega$ consisting of the elements in $\mathcal{T}_h$. Namely,
$\mathcal{T}_h$ is a family of open disjoint face-to-face $N$-simplices in $\Omega$ such that all the boundary vertices
of $\Omega_h$ belong to $\partial \Omega$. Hereafter, we write $(\cdot, \cdot)_h := (\cdot, \cdot)_{\Omega_h}$ for short.

It is clear that $\Omega_h \subseteq \Omega$ from the convexity of $\Omega$. This requires the boundary-skin
estimate of the form
$$
\|v\|_{L^p(\Omega \setminus \Omega_h)} \leq C \left( h^\frac{2}{p} \|v\|_{L^p(\partial \Omega)} + h^2 \|\nabla v\|_{L^p(\Omega \setminus \Omega_h)} \right)
$$
for all $v \in W^{1,p}(\Omega \setminus \Omega_h)$ and $p \in [1, \infty]$, which can be proved by the same argument as in [20, Theorem 8.3]. Together with the trace inequality, we obtain
$$
\|v\|_{L^p(\Omega \setminus \Omega_h)} \leq C h^\frac{2}{p} \|v\|_{W^{1,p}(\Omega)},
$$
for all $v \in W^{1,p}(\Omega)$ and $p \in [1, \infty]$. Moreover, if $v \in W^{1,p}_0(\Omega)$ and $p \in [1, \infty]$, we have
$$
\|v\|_{L^p(\partial \Omega_h)} \leq C h^{\frac{2-p}{2}} \|\nabla v\|_{L^p(\Omega \setminus \Omega_h)}
$$
by [20, Theorem 8.2], and together with (2.2), we obtain
$$
\|v\|_{L^p(\partial \Omega_h)} \leq C h^2 \|\nabla^2 v\|_{L^p(\Omega)}
$$
for $v \in W^{1,p}_0(\Omega) \cap W^{2,p}(\Omega)$ and $p \in [1, \infty]$. We also remark that $|\Omega \setminus \Omega_h| \leq C h^2$ holds.

We assume that the family $\{\mathcal{T}_h\}_h$ is shape-regular and quasi-uniform:

(H1) (Inverse inequality) There exists $C > 0$ such that $Ch \leq \rho_T$ for all $T \in \mathcal{T}_h$ and
$h > 0$, where $\rho_T$ is the radius of the maximum inscribed ball.

We then introduce finite element spaces. Let $V_h$ and $Q_h$ be spaces of piecewise
polynomials associated to $\mathcal{T}_h$ that satisfies $V_h \subset H^1_0(\Omega_h)^N$ and $Q_h \subset H^1(\Omega_h)$, which are approximation spaces for the velocity and the pressure, respectively, and we set
$Q_h^0 := Q_h \cap L^2_0(\Omega_h)$. Notice that we assume that not only $V_h$ but also $Q_h$ is a subset of
$H^1(\Omega_h)$. Moreover, we extend $v_h \in V_h$ by zero outside of $\Omega_h$, and thus $V_h \subset H^1_0(\Omega)^N$
holds. By assumption (H1), the inverse inequalities
$$
\|\nabla v_h\|_{L^p(T)} \leq C h^{-1} \|v_h\|_{L^p(T)},
$$
(2.5)
We finally assume that \( \tilde{H}_2 \) (Quasi-preservation of divergence). For all \( v \in W^{1,1}(\Omega)^N \),

\[
\|\nabla^l v_h\|_{L^p(T)} \leq C h^{-\frac{l}{N} + \frac{N}{p}} \|\nabla^l v_h\|_T, \quad l = 0, 1
\]  

(2.6)

hold for all \( p \in [1, \infty] \), \( v_h \in V_h \), and \( T \in T_h \).

Moreover, we assume there exist quasi-interpolation operators \( I_h : W^{1,1}(\Omega)^N \to V_h \) and \( J_h : W^{1,1}(\Omega) \to Q_h \) that satisfy the following conditions:

\((H2)\) (Quasi-preservation of divergence). For all \( v \in W^{1,1}(\Omega)^N \),

\[
(\text{div}(v - I_h v), \psi_h)_h = (v \cdot n, \hat{\psi}_h)_{\partial \Omega_h}, \quad \forall \psi_h \in Q_h,
\]

(2.7)

where \( \hat{\psi}_h \in L^\infty(\Omega_h) \) is a function determined by \( \psi_h \) that satisfies \( \text{supp} \hat{\psi}_h = \text{supp} \psi_h \) and

\[
\|\hat{\psi}_h\|_{L^p(\partial \Omega_h)} \leq C h^{-\frac{1}{p}} \|\psi_h\|_{L^p(\Omega_h)},
\]

(2.8)

\((H3)\) (Local \( L^p \) approximation and stability). For each \( T \in T_h \), there exists a macro-element \( \Delta_T \) including \( T \) that satisfies the following estimates for all \( p \in [1, \infty] \):

(i) For any \( v \in W^{1,p}_0(\Omega)^N \),

\[
\|v - I_h v\|_{L^p(T)} \leq Ch \|\nabla v\|_{L^p(\Delta_T)},
\]

(2.9)

\[
\|\nabla I_h v\|_{L^p(T)} \leq C \|\nabla v\|_{L^p(\Delta_T)},
\]

(2.10)

\[
\|\nabla(v - I_h v)\|_{L^p(T)} \leq C h \|\nabla v\|_{W^{1,p}(\Delta_T)}, \quad \text{if} \ v \in W^{2,p}(\Delta_T)^N,
\]

(2.11)

where

\[
\Delta_T = \Delta_T \cup \left\{ x \in \Omega \setminus \Omega_h \mid \arg\min_{S \in \mathcal{T}_h} \text{dist}(x, S) \subset \Delta_T \right\}.
\]

(ii) For any \( \psi \in W^{1,p}(\Delta_T) \)

\[
\|\psi - J_h \psi\|_{L^p(T)} \leq C h \|\nabla \psi\|_{L^p(\Delta_T)},
\]

(2.12)

\[
\|\nabla J_h \psi\|_{L^p(T)} \leq C \|\nabla \psi\|_{L^p(\Delta_T)},
\]

(2.13)

\[
\|\nabla(\psi - J_h \psi)\|_{L^p(T)} \leq C h \|\nabla^2 \psi\|_{L^p(\Delta_T)}, \quad \text{if} \ \psi \in W^{2,p}(\Delta_T).
\]

(2.14)

Here, \( C > 0 \) is independent of \( h, T \) and each \( \Delta_T \) includes at most \( L \) elements of \( \mathcal{T}_h \) with \( L \) independent of \( h, T \).

We further assume there exists another quasi-interpolation operator \( \tilde{I}_h : C^0(\overline{\Omega}_h)^N \to V_h \) that satisfies the similar properties as above:

\((H2)'\) For all \( v \in W^{1,1}_0(\Omega)^N \),

\[
(\text{div}(v - \tilde{I}_h v), \psi_h)_h = 0, \quad \forall \psi_h \in Q_h.
\]

(2.15)

\((H3)'\) For each \( T \in T_h \), there exists a macro-element \( \Delta_T \) including \( T \) that satisfies

\[
\|\nabla \tilde{I}_h v\|_{L^p(T)} \leq C \|\nabla v\|_{L^p(\Delta_T)}, \quad \forall v \in W^{1,p}(\Delta_T)^N,
\]

for all \( p \in [1, \infty] \).

We finally assume that \( \tilde{I}_h \) and \( J_h \) satisfy the super-approximation estimates:
Remark 2.1. When $\Omega$ is a polygonal or polyhedral domain (i.e., $\Omega_h = \Omega$), it is shown in [15] that the operator $I_h$ satisfying

$$(\text{div}(v - I_h v), \psi_h)_h = 0, \quad \forall \psi_h \in Q_h,$$  

and (H3) can be constructed for the Taylor–Hood $P^k$-$P^{k-1}$ elements with $k \geq N$, $N = 2, 3$. For $P^2$-$P^1$ element in $\mathbb{R}^3$, it may be possible to construct $I_h$ according to [13, §6.1], if the triangulation consists of hexahedra. For the lower-order MINI element, see [14] and [13, §6.2]. It is also known that the operator $I_h$ satisfying (H3) and (H4) can be constructed by a small modification (see [13]). For these elements, the quasi-interpolation operator of Scott–Zhang [33] can be used as $J_h$ that satisfies (H3) and (H4).

However, in our case, $\Omega$ is not polygonal and thus $\Omega_h \neq \Omega$, which makes impossible to construct an operator satisfying (2.19). Indeed, if (2.19) holds, then, letting $\psi_h \equiv 1$, one has

$$0 = (\text{div}(v - I_h v), 1)_h = \int_{\partial \Omega_h} (v - I_h v) \cdot ndS = \int_{\partial \Omega_h} v \cdot ndS, \quad \forall v \in H^1_0(\Omega)^N,$$

where $n$ is the outward unit normal vector of $\partial \Omega_h$. It is clear that this is not true for general $v \in H^1_0(\Omega)^N$.

We thus need a modified version of (2.19), which is nothing but (2.7). We can construct the interpolation $I_h$ satisfying (H2) and (H3) by a small modification of the literature, at least for the Taylor–Hood and MINI elements in the two dimensional case. For the Taylor–Hood elements, we can define $\hat{\psi}_h$ by

$$\hat{\psi}_h|_T = \frac{1}{|T|} \int_T \psi_h dx, \quad \text{for all triangles } T \in \mathcal{T}_h,$$

and for the MINI elements, we set $\hat{\psi}_h = \psi_h$. We will give the construction in Appendix.

On the other hand, we can use the same operators $\hat{I}_h$ and $J_h$ as in the literature. Indeed, $\hat{I}_h$ is defined for functions in $\Omega_h$ independently of $\Omega$, and $J_h$ is independent of the boundary condition.

We then consider the finite element approximation of the Stokes resolvent problem. Let $f \in L^p(\Omega)^N$, $\delta \in (0, \pi)$, and $\lambda \in \Sigma_d$. Then, approximate problem is:

Find $(u_h, \varphi_h) \in V_h \times Q_h^d$ that satisfies the variational equation (1.3).

This problem is uniquely solvable owing to the inf-sup condition derived from (2.7) and (2.10) (see [14, Chapter II]). We now state the main result of the present paper.
**Theorem 2.1.** Let \( \Omega \subset \mathbb{R}^N (N = 2, 3) \) be a bounded convex domain with smooth boundary, \( \mathcal{T}_h \) be a triangulation of \( \Omega \) satisfying (H1), and \((V_h, Q_h)\) be a pair of finite element spaces associated to \( \mathcal{T}_h \) that satisfies assumptions (H2)–(H4). Moreover, let \( p \in \left( \frac{2N}{N+2}, \frac{2N}{N-2} \right) \), \( \delta \in (0, \pi) \), \( \lambda \in \Sigma_\delta \), and \( f \in L^p(\Omega)^N \). Then, there exists \( h_0 > 0 \) such that for all \( h \leq h_0 \), the solution \((u_h, \varphi_h) \in V_h \times Q_h^0 \) of (1.3) satisfies the resolvent estimate (1.4), where \( C \) and \( h_0 \) depend only on \( N \), \( p \), \( \delta \), and the constants appearing in (H1)–(H4), and is independent of \( h \), \( \lambda \), and \( f \).

**Remark 2.2.** We have some remarks concerning the main theorem.

(i) The restriction for \( p \) is just a technical assumption.

(ii) If \( N = 2 \), then the result is valid for all \( p \in (1, \infty) \). If \( N = 3 \), then the range is \( p \in (6/5, 6) \), which includes \( p = 3 \).

(iii) The required regularity of \( \partial \Omega \) is at most \( C^4 \) and this assumption may be impossible to remove, because we will use \( W^{3,p} \times W^{2,p} \)-regularity of the Stokes system. See Lemma 3.2 below.

(iv) The convexity of \( \Omega \) may be unessential. If \( \Omega \) is a non-convex domain, it is necessary to address errors of domain perturbation as discussed in [18].

To show Theorem 2.1, let \((u, \varphi)\) be the solution of (1.1) for the same external force \( f \in L^p(\Omega)^N \). Then, the resolvent estimate (1.2) holds for \( p \in (1, \infty) \). Moreover, we set \( e := u - u_h \), \( \rho := \varphi - \varphi_h \). It is easy to check that the Galerkin orthogonality (or compatibility)

\[
\begin{align*}
\lambda(e, v_h)_h + (\nabla e, \nabla v_h)_h - (\rho, \text{div } v_h)_h &= 0, \quad \forall v_h \in V_h, \\
(\text{div } e, \psi_h)_h &= 0, \quad \forall \psi_h \in Q_h
\end{align*}
\]  

(2.20)

holds. Then, the proof of Theorem 2.1 is reduced to showing the following error estimates, which themselves are useful for error estimates of the time-dependent Stokes equations (see [29, 21]).

**Theorem 2.2.** Under the same hypotheses of Theorem 2.1, there exists \( h_0 > 0 \) such that

\[
\begin{align*}
|\lambda|^{1/2} e_{L^p(\Omega_h)} + \|\nabla e\|_{L^p(\Omega_h)} &\leq C \min\{h, |\lambda|^{-1/2}\} \|f\|_{L^p(\Omega)}, \\
\|e\|_{L^p(\Omega_h)} &\leq C \min\{h^2, h|\lambda|^{-1/2}, |\lambda|^{-1}\} \|f\|_{L^p(\Omega)}, \\
\|e\|_{W^{-1,p}(\Omega)} &\leq Ch|\lambda|^{-1} \|f\|_{L^p(\Omega)}, \\
\|\rho\|_{L^p(\Omega_h)} &\leq Ch \|f\|_{L^p(\Omega)}
\end{align*}
\]  

(2.21)–(2.24)

holds for all \( h \geq h_0 \), where \( C \) and \( h_0 \) depend only on \( N \), \( p \), \( \delta \), and the constants appearing in (H1)–(H4), and is independent of \( h \), \( \lambda \), and \( f \).

3. Preliminaries

In this section, we collect some preliminary results related to the Stokes resolvent problem (1.1).
3.1. Bogovskii operator

Let $D \subset \mathbb{R}^N$ a bounded domain. For a given $f \in L^p_0(D)$, a solution of the divergence equation

\[
\begin{aligned}
\text{div } v &= f \quad \text{in } D \\
v &= 0 \quad \text{on } \partial D
\end{aligned}
\]

is explicitly given by Bogovskiǐ [3], provided that the domain $D$ satisfies some conditions. We denote the solution operator by $B_D: L^p_0(D) \to W^{1,p}_0(D)^N$, i.e., $\text{div } B_D f = f$, $B_D f \in W^{1,p}_0(D)^N$, which we call the Bogovskiǐ operator. If the domain $D$ is star-like with respect to a ball, then the upper bound of the operator norm of $B_D$ is described by the circle of the ball (cf. [8, Lemma III.3.1]).

**Lemma 3.1.** Let $p \in (1, \infty)$ and let $D \subset \mathbb{R}^N$ be a bounded domain. Assume that $D$ is star-like with respect to a ball of radius $R$. Then, the Bogovskiǐ operator $B_D: L^p_0(D) \to W^{1,p}_0(D)^N$ is bounded with a bound

\[
\|\nabla B_D f\|_{L^p(D)} \leq C \left( \frac{\text{diam } D}{R} \right)^N \left( 1 + \frac{\text{diam } D}{R} \right) \|f\|_{L^p(D)}, \quad \forall f \in L^p_0(D), \tag{3.1}
\]

where $C$ depends only on $N$ and $p$.

3.2. Higher regularity of the Stokes resolvent problem

In the present paper, we will use $W^{3,p} \times W^{2,p}$-regularity of the resolvent problem (1.1). To the best of our knowledge, we could not find literature that states the following estimate for bounded domains, and thus we here give a proof briefly.

**Lemma 3.2.** Let $\Omega$ be a bounded domain of $C^4$-regularity and let $\delta \in (0, \pi)$ and $p \in (1, \infty)$. Then, for any $\lambda \in \Sigma_\delta$ and $f \in W^{1,p}(\Omega)^N$, the solution $(u, \varphi)$ of (1.1) satisfy $u \in W^{3,p}(\Omega)^N$ and $\varphi \in W^{2,p}(\Omega)$ with the estimate

\[
|\lambda||u||_{W^{1,p}(\Omega)} + |\lambda|^{1/2}||u||_{W^{2,p}(\Omega)} + ||u||_{W^{3,p}(\Omega)} + ||\varphi||_{W^{2,p}(\Omega)} \leq C ||f||_{W^{1,p}(\Omega)}. \tag{3.2}
\]

**Proof.** The resolvent problem (1.1) can be viewed as the steady Stokes equations

\[
\begin{aligned}
-\Delta u + \nabla \varphi &= f - \lambda u \quad \text{in } \Omega, \\
\text{div } u &= 0 \quad \text{in } \Omega, \\
u &= 0 \quad \text{on } \partial \Omega.
\end{aligned}
\]

Then, from the $W^{3,p} \times W^{2,p}$-regularity for the steady Stokes equations (cf. [8, Theorem IV.6.1]), we obtain $u \in W^{3,p}(\Omega)^N$, $\varphi \in W^{2,p}(\Omega)$, and the estimate

\[
||u||_{W^{3,p}(\Omega)} + ||\varphi||_{W^{2,p}(\Omega)} \leq C (||f||_{W^{1,p}(\Omega)} + |\lambda||u||_{W^{1,p}(\Omega)}).
\]

Hence, it suffices to show

\[
|\lambda||u||_{W^{1,p}(\Omega)} \leq C ||f||_{W^{1,p}(\Omega)}. \tag{3.3}
\]

We prove (3.3) by localization argument according to the proof given in [11, Section 4]. Throughout this proof, we may assume $f \in W^{1,p}(\Omega)^N \cap L^p_0(\Omega)$ since the Helmholtz projection is bounded in $W^{1,p}$ [28]. Let us first consider the half-space case. Assume
that $\Omega = \mathbb{R}^N_+ = \{x \in \mathbb{R}^N \mid x_N > 0\}$ and let $(u, \varphi)$ be the solution to the Stokes resolvent problem (1.1) with the source term $f \in W^{1,p}(\mathbb{R}^N_+) \cap L^p_\sigma(\mathbb{R}^N_+)$. Then,

$$|\lambda||u||_{W^{1,p}(\mathbb{R}^N_+)} \leq C||f||_{W^{1,p}(\mathbb{R}^N_+)},$$

holds by [27, Theorem 8.1] for $N = 3$. One can prove the same estimate for $N = 2$ by modifying the argument in [27].

Let us then consider the estimate for bounded domains. By the same argument as in [11, Section 4], one can show that there exists $R_\delta > 0$ such that (3.3) holds for $|\lambda| \geq R_\delta$. Here, the $C^4$-regularity of the boundary is required. Moreover, owing to the usual resolvent estimate (1.2), we have, for $|\lambda| \leq R_\delta$,

$$\|u\|_{W^{1,p}(\Omega)} \leq C|\lambda|^{-1/2}\|f\|_{L^p(\Omega)} \leq CR_\delta^{1/2}|\lambda|^{-1}\|f\|_{L^p(\Omega)},$$

which implies (3.3) for $|\lambda| \leq R_\delta$. Hence we obtain (3.3) for all $\lambda \in \Sigma_\delta$ and thus we complete the proof.

\[ \square \]

4. Starting point of the proof

The most difficult part of the proof of Theorem 2.1 is showing the $W^{1,p}$-error estimate (2.21). In this section, we show that the proof is reduced to a local energy estimate. Hereafter, we write $B(x_0, r) = \{x \in \mathbb{R}^N \mid |x - x_0| < r\}$.

To localize the estimates, we cover the domain $\Omega_h$ by a finite number of open balls $\{B_j\}_{j=1}^M$ that satisfy the following conditions (C1)–(C3):

(C1) The radius of each ball is $d = Kh$ with a constant $K \geq 1$ independent of $h$.

(C2) Either $B_j \subset \Omega_h$ or $x_j \in \partial \Omega_h$ holds for each $j$, where $x_j$ is the center of $B_j$.

(C3) Let $B_j' = B(x_j, 2d)$. Then, there exists $M' \in \mathbb{N}$ such that, for any $h$ and $j$,

$$\# \{i \mid B_i' \cap B_j' \neq \emptyset\} \leq M'. \quad (4.1)$$

The constant $K \geq 1$ in (C1) will be fixed large enough (but independently of $h$) later. By the Besicovitch covering theorem, one can see that such a covering can be constructed for any $d$. We then set

$$D_j := B_j \cap \Omega_h, \quad D_j' := B_j' \cap \Omega_h.$$ 

We remark that the condition (4.1) implies

$$\left[ \sum_j \|v\|^p_{D_j'} \right]^{1/p} \leq M' \left[ \sum_j \|v\|^p_{D_j} \right]^{1/p}, \quad \forall v \in L^p(\Omega) \quad (4.2)$$

for $p \in [1, \infty)$.

Assume $p > 2$ and let

$$\|v\| := \left[ \sum_j \left( h^{-\frac{N}{p}} + \frac{N}{p} \|v\|_{D_j} \right)^p \right]^{1/p}$$

for $p \in [1, \infty)$. Assume $p > 2$ and let

$$\|v\| := \left[ \sum_j \left( h^{-\frac{N}{p}} + \frac{N}{p} \|v\|_{D_j} \right)^p \right]^{1/p}$$
2.6) and (4.2), we have
\[ \|\nabla^l v_h\|_{L^p(\Omega)} \leq C \|\nabla^l v\| \] (4.3)
for \( l = 0, 1 \) and for all \( v_h \in V_h \), where \( M' \) is absorbed into \( C \). Moreover, by the Hölder inequality, we have
\[ \|v\| \leq CK^{N-\frac{N}{p}} \|v\|_{L^p(\Omega)}, \quad \forall v \in L^p(\Omega) \] (4.4)
since \( |D_j| \leq Cd^N \) and \( d/h = K \).

We now address \( \|e\|_{L^p(\Omega_h)} \). Let \( z = u - I_h u \) and \( z_h = e - z = I_h u - u_h \in V_h \). Then, we have
\[ \|e\|_{L^p(\Omega_h)} \leq C \|z_h\| + \|z\|_{L^p(\Omega_h)} \leq C \|e\| + C_K \|z\|_{L^p(\Omega_h)} \]
by (4.3) and (4.4), and similarly,
\[ \|\nabla e\|_{L^p(\Omega_h)} \leq C \|\nabla e\| + C_K \|\nabla z\|_{L^p(\Omega_h)} \]
Hence, we obtain
\[ |\lambda|^{1/2} \|e\|_{L^p(\Omega_h)} + \|\nabla e\|_{L^p(\Omega_h)} \leq C \left( |\lambda|^{1/2} \|e\| + \|\nabla e\| \right) + C_K \left( |\lambda|^{1/2} \|z\|_{L^p(\Omega_h)} + \|\nabla z\|_{L^p(\Omega_h)} \right) \] (4.5)
and replacing \( I_h u \) by zero, we have
\[ |\lambda|^{1/2} \|e\|_{L^p(\Omega_h)} + \|\nabla e\|_{L^p(\Omega_h)} \leq C \left( |\lambda|^{1/2} \|e\| + \|\nabla e\| \right) + C_K \left( |\lambda|^{1/2} \|u\|_{L^p(\Omega)} + \|\nabla u\|_{L^p(\Omega)} \right) \] (4.6)
Therefore, it suffices to address the local energy norm \( |\lambda|^{1/2} \|e\|_{D_j} + \|\nabla e\|_{D_j} \).

5. Local energy estimates

In this section, we fix \( j \) and address local energy norm on \( D_j \). For \( s > 0 \), we set \( D_{j,s} := B(x_{j,s}, (1+s)d) \cap \Omega_h \). It is clear that \( D_j' = D_{j,1} \). We introduce a cut-off function \( \omega = \omega_j \in C^\infty(\Omega) \) such that
\[ \text{supp} \omega \subset D_{j,1/4}, \quad \omega|_{D_j} \equiv 1, \quad 0 \leq \omega \leq 1, \quad |\nabla \omega| \leq Cd^{-1}. \]
Then, by (2.10) and (2.13), we may assume \( K \) is large so that
\[ \text{supp} I_h(\omega^2 v_h) \subset D_{j,2/4}, \quad \text{supp} J_h(\omega^2 \psi_h) \subset D_{j,2/4} \] (5.1)
hold for all \( v_h \in V_h \) and \( \psi_h \in Q_h \).

This section is devoted to the proof of the following local energy error estimates. The proof is based on that of [16, Lemma 3.1]. Hereafter, we set \( \zeta := \varphi - J_h \varphi \) and \( \zeta_h := J_h \varphi - \varphi_h \) so that \( \rho = \zeta + \zeta_h \).

We consider two cases: \( |\lambda| \leq O(h^{-2}) \) and \( |\lambda| \geq O(h^{-2}) \). The local energy estimate in the first case is stated as follows.
Proposition 5.1. Let $\lambda \in \Sigma_\delta$ for $\delta \in (0, \pi)$ and assume $|\lambda| \leq d^{-2}$. Let $z = u - I_h u$ and $z_h = e - z = I_h u - u_h \in V_h$. Moreover, let $\zeta_j \in C$ arbitrarily and $\tilde{\zeta}_{h,j} = \zeta_h + \zeta_j$. Then, for any $\varepsilon > 0$, we have

$$
|\lambda||e||D_j| + \|\nabla e||D_j| \leq C K^{-1}|\lambda||e||D_j| + \varepsilon^2 \|\nabla e||D_j|
+ C \varepsilon^{-2} d^{-2} \|\xi||D_j| + C \left( \varepsilon^{-2} d^{-2} \|\xi||D_j| + \|\nabla \xi||D_j| + \|\zeta||D_j| \right)
+ \varepsilon^2 \|\tilde{\zeta}_{h,j}\|^2_{D_j} + C \varepsilon^{-2} h^{-1} \|u\|^2_{\partial \Omega_h \cap D_j},
$$

(5.2)

where $C$ is independent of $j$, $h$, $d$, $K$, $\varepsilon$, and $\zeta_j$.

Proof. We start by the estimate

$$
|\lambda||\omega e||\Omega_h| + \|\omega \nabla e||\Omega_h| \leq C \delta |\lambda||\omega e||\Omega_h| + \|\omega \nabla e||\Omega_h|,
$$

(5.3)

which is derived by an elementary inequality $|ae^{i\theta} + b| \geq (a+b) \sin \frac{\theta}{2}$ for $\theta \in (-\pi+\delta, \pi-\delta)$ and $a, b > 0$. Now we will show

$$
\lambda||\omega e||\Omega_h| + \|\omega \nabla e||\Omega_h| = \sum_{i=1}^{6} R_i,
$$

(5.4)

where

$$
R_1 = \lambda(e, \omega^2 z)_h + (\nabla e, \omega^2 \nabla z)_h - (\nabla e, (\omega^2) z_h)_h,
$$

$$
R_2 = \lambda(e, \omega^2 z_h - \tilde{I}_h(\omega^2 z_h))_h + (\nabla e, \nabla (\omega^2 z_h - \tilde{I}_h(\omega^2 z_h)))_h,
$$

$$
R_3 = (\zeta, \text{div} \tilde{I}_h(\omega^2 z_h))_h,
$$

$$
R_4 = (\tilde{\zeta}_{h,j}, \nabla (\omega^2) z_h)_h,
$$

$$
R_5 = (\omega^2 \tilde{\zeta}_{h,j} - J_h(\omega^2 \tilde{\zeta}_{h,j}), \text{div} z_h)_h,
$$

$$
R_6 = - (\psi_h, u \cdot n)|_{\partial \Omega_h},
$$

where $\psi_h = J_h(\omega^2 \tilde{\zeta}_{h,j})$ and $\hat{\psi}_h$ is defined as in (H2).

Noting that $e = z + z_h$, we have

$$
\lambda||\omega e||\Omega_h| + \|\omega \nabla e||\Omega_h| = \lambda(e, \omega^2 e)_h + (\nabla e, \omega^2 \nabla e)_h
= R_1 + \lambda(e, \omega^2 z_h)_h + (\nabla e, \nabla (\omega^2 z_h))_h.
$$

Moreover, Galerkin orthogonality (2.20) yields

$$
\lambda(e, \omega^2 z_h)_h + (\nabla e, \nabla (\omega^2 z_h))_h = R_2 + (\rho, \text{div} \tilde{I}_h(\omega^2 z_h))_h.
$$

Because of (2.15) and $(\omega^2 z_h)|_{\partial \Omega_h} \equiv 0$, we have

$$
(\rho, \text{div} \tilde{I}_h(\omega^2 z_h))_h = R_3 + (\zeta_h, \text{div} (\omega^2 z_h))_h
= R_3 + (\zeta_h + \zeta_j, \text{div} (\omega^2 z_h))_h \quad (\forall \zeta_j \in C),
$$

and

$$
(\tilde{\zeta}_{h,j}, \text{div} (\omega^2 z_h))_h = (\tilde{\zeta}_{h,j}, \nabla (\omega^2) z_h)_h + (\omega^2 \tilde{\zeta}_{h,j}, \text{div} z_h)_h
= R_4 + R_5 + (J_h(\omega^2 \tilde{\zeta}_{h,j}), \text{div} z_h)_h.
$$
Finally, since $u_h$ satisfies the second equation of (1.3), we have

$$(J_h(\omega^2 \tilde{c}_{h,j}), \text{div } z_h)_h = (J_h(\omega^2 \tilde{c}_{h,j}), \text{div } I_h u)_h = R_6$$

together with (2.7). Hence (5.4) is proved.

We address each $R_i$. By the Hölder and Young inequalities, we have

$$|R_1| \leq \frac{1}{2C_\delta^2} (|\lambda|\|\omega e\|^2_{\Omega_h} + \|\omega \nabla e\|^2_{\Omega_h}) + C \left( |\lambda| \|z\|^2_{D_j'} + \|\nabla z\|^2_{D_j'} \right) + C \delta^{-2} \|z_h\|^2_{D_j'} \quad (5.5)$$

where $C_\delta$ is as in (5.3). Here we used the assumption $|\lambda| \leq d^{-2}$. By the superapproximation estimates (2.16) and (2.17), we have

$$|R_2| \leq \frac{1}{2C_\delta} (|\lambda|\|e\|^2_{D_{j,1/4}} \cdot C h d^{-1} \|z_h\|_{D_{j,3/4}} + \|\nabla e\|_{D_{j,3/4}} \cdot C d^{-1} \|z_h\|_{D_{j,3/4}}) \quad (5.6)$$

$$\leq C K^{-1} |\lambda|\|e\|^2_{D_j'} + \varepsilon \|\nabla e\|^2_{D_j'} + C \varepsilon^{-2} \delta^{-2} \|e\|^2_{D_j'} + C \delta^{-2} \|z\|^2_{D_j'},$$

where we used (5.1) and $hd^{-1} = K^{-1} \leq 1$. By the standard arguments, we have

$$|R_3| \leq \varepsilon^2 \|\nabla e\|^2_{D_j'} + C \delta^{-2} \|z\|^2_{D_j'} + C \varepsilon^{-2} \|\nabla e\|^2_{D_j'} + C \delta^{-2} \|\nabla e\|^2_{D_j'} \quad (5.7)$$

and

$$|R_4| \leq \varepsilon^2 \|\tilde{c}_{h,j}\|^2_{D_{j,3/4}} + C \varepsilon^{-2} d^{-2} \|z\|^2_{D_j'} + C \varepsilon^{-2} d^{-2} \|e\|^2_{D_j'} \quad (5.8)$$

Furthermore, by (2.18), (2.5), and (5.1), we have

$$|R_5| \leq C h^{-1/2} \|\tilde{c}_{h,j}\|_{D_{j,3/4}} \|\nabla z_h\|_{D_{j,3/4}} \leq C h^{-1} \|\tilde{c}_{h,j}\|_{D_{j,3/4}} \|z_h\|_{D_j'} \quad (5.9)$$

Finally, noting that

$$\|J_h(\omega^2 \tilde{c}_{h,j})\|_{D_{j,1/4}} \leq \|J_h(\omega^2 \tilde{c}_{h,j}) - \omega^2 \tilde{c}_{h,j}\|_{D_{j,3/4}} + \|\omega^2 \tilde{c}_{h,j}\|_{D_{j,1/4}} \leq C \|\tilde{c}_{h,j}\|_{D_{j,3/4}} \quad (5.10)$$

by (2.18) and $hd^{-1} \leq 1$, we have

$$|R_6| \leq C h^{-1/2} \|J_h(\omega^2 \tilde{c}_{h,j})\|_{\Omega_h} \|u\|_{\partial \Omega_h \cap D_{j,1/4}} \quad (5.11)$$

$$\leq \varepsilon^2 \|\tilde{c}_{h,j}\|^2_{D_{j,3/4}} + C \varepsilon^{-2} h^{-1} \|u\|^2_{\partial \Omega_h \cap D_j'}$$

together with (2.8). Substituting (5.5), (5.6), (5.7), (5.8), (5.9), and (5.11) into (5.3) and (5.4), we obtain (5.2), and thus we complete the proof. \qed

We next consider the case $|\lambda| \geq O(h^{-2})$. We here choose the following constant $\zeta_0$ independently of $j$.

**Proposition 5.2.** Let $\lambda \in \Sigma_\delta$ for $\delta \in (0, \pi)$. Fix $\alpha > 0$ arbitrarily and assume $|\lambda| \geq ah^{-2}$. Moreover, let $\zeta_0 \in C$ so that $\tilde{c}_h = \zeta_0 + \zeta_0 \in Q_h^0$. Then, for any $\varepsilon > 0$, we have

$$|\lambda|\|e\|^2_{D_j'} + \|\nabla e\|^2_{D_j'} \leq (\varepsilon^2 + C K^{-1}) |\lambda|\|e\|^2_{D_j'} + C K^{-1} \|\nabla e\|^2_{D_j'}$$

$$+ C \left( |\lambda|\|u\|^2_{D_j'} + \|\nabla u\|^2_{D_j'} + \varepsilon^{-2} |\lambda|^{-1} \|\nabla \zeta\|^2_{D_j'} \right) + C \varepsilon^{-2} |\lambda|^{-1} d^{-2} \|\tilde{c}_h\|^2_{D_j'}, \quad (5.12)$$

where $C$ is independent of $j$, $h$, $d$, $K$, and $\varepsilon$. 11
Proof. By the same argument as in Proposition 5.1, we obtain

\[ \lambda \| \omega e \|_{\Omega_h}^2 + \| \omega \nabla e \|_{\Omega_h}^2 = \sum_{l=7}^{11} R_l, \]  

where

\[ R_7 := \lambda(e, \omega^2 u)_h + (\nabla e, \omega^2 \nabla u)_h, \]
\[ R_8 := -\lambda(e, \omega^2 u - \tilde{I}_h(\omega^2 u_h))_h - (\nabla e, \nabla(\omega^2 u_h - \tilde{I}_h(\omega^2 u_h)))_h, \]
\[ R_9 := (\nabla \zeta, \tilde{I}_h(\omega^2 u_h))_h, \]
\[ R_{10} := -(\tilde{\zeta}_h, \nabla(\omega^2 u_h))_h, \]
\[ R_{11} := -(\omega^2 \tilde{\zeta}_h - J_h(\omega^2 \tilde{\zeta}_h), \text{div } u_h)_h. \]

Indeed, one obtains (5.13) by replacing \( z \) by \( u \) and \( z_h \) by \( -u_h \) in the proof of (5.4).

To address each \( R_l \), we first notice that

\[ d^{-1} \leq \alpha^{-1/2} K^{-1} |\lambda|^{1/2} \]  

holds owing to \( d = Kh \) and the assumption \( \lambda \geq \alpha h^{-2} \). Then, by standard calculations, we obtain

\[ |R_7| \leq |\lambda| \| \omega e \|_{\Omega_h} \| \omega u \|_{\Omega_h} + \| \omega \nabla e \|_{\Omega_h} \| \omega \nabla u \|_{\Omega_h} + C d^{-1} \| \omega \nabla e \|_{\Omega_h} \| u_h \|_{D_j} \]
\[ \leq \frac{1}{2C_\delta} (|\lambda| \| \omega e \|_{\Omega_h}^2 + \| \omega \nabla e \|_{\Omega_h}^2) + C K^{-2} |\lambda| \| e \|_{D_j}^2 + C |\lambda| \| \omega u \|_{D_j}^2 + C \| \nabla e \|_{D_j}^2, \]  

where \( C_\delta \) is as in (5.3). By the super-approximation estimates (2.16) and (2.17), we have

\[ |R_8| \leq C d^{-1} \| \lambda \| \| e \|_{D_j} \| u_h \|_{D_j} + C d^{-1} \| \nabla e \|_{D_j} \| u_h \|_{D_j} \]
\[ \leq C K^{-1} \left( |\lambda| \| e \|_{D_j}^2 + \| \nabla e \|_{D_j}^2 \right) + C |\lambda| \| u \|_{D_j}^2. \]  

Note that

\[ \| \tilde{I}_h(\omega^2 u_h) \|_{D_j, 2/4} \leq C \| u_h \|_{D_j} \]

holds by the same calculation as (5.10). Therefore, we obtain

\[ |R_9| \leq C \| \nabla \zeta \|_{D_j} \| u_h \|_{D_j} \]
\[ \leq \varepsilon^2 |\lambda| \| e \|_{D_j}^2 + C |\lambda| \| u \|_{D_j}^2 + C \varepsilon^{-2} |\lambda|^{-1} \| \nabla \zeta \|_{D_j}^2. \]  

Finally, we obtain

\[ |R_{10}| + |R_{11}| \leq C d^{-1} \| \tilde{\zeta}_h \|_{D_j} \| u_h \|_{D_j} \]
\[ \leq \varepsilon^2 |\lambda| \| e \|_{D_j}^2 + |\lambda| \| u \|_{D_j}^2 + C \varepsilon^{-2} |\lambda|^{-1} \| \tilde{\zeta}_h \|_{D_j}^2 \]  

by the same arguments as the estimates for \( R_4 \) and \( R_5 \). Hence, summarizing (5.3), (5.13), (5.15), (5.16), (5.17), and (5.18), we establish the desired estimate (5.12), and we complete the proof.
6. Local and global estimates for the pressure term

The first goal of this section is to show the local $L^2$-estimate of $\tilde{\zeta}_{h,j} = \zeta_j + \zeta_j$ appearing in (5.2). The proof is similar to that of [16, Lemma 3.2]

**Lemma 6.1.** Assume $|\lambda| \leq d^{-2}$. For each $j$, we choose $\zeta_j \in \mathbb{C}$ so that $\tilde{\zeta}_{h,j} \in L_0^2(D_{j,3/4})$. Then, we have

$$\|\tilde{\zeta}_{h,j}\|_{D_{j,3/4}} \leq C \left( |\lambda|^{1/2} \|e\|_{D_{j}} + \|\nabla e\|_{D_{j}} + \|\zeta\|_{D_{j}} \right),$$

(6.1)

where $C$ is independent of $d$, $j$, and $h$.

**Proof.** Let $\tilde{D}_j := D_{j,3/4}$. By the convexity of $\Omega_h$ and the condition (C2), it is clear that $\tilde{D}_j$ includes a ball of radius $Cd$ with $C$ depending only on $\Omega$ and is star-like with respect to the ball. Therefore, we can apply Lemma 3.1. Letting $w = B_{\tilde{D}_j} \tilde{\zeta}_{h,j} \in H^1_0(\tilde{D}_j)$, we have

$$\begin{cases}
\text{div } w = \tilde{\zeta}_{h,j} & \text{in } \tilde{D}_j, \\
\text{w} = 0 & \text{on } \partial \tilde{D}_j,
\end{cases}$$

(6.2)

by the choice of $\zeta_j$, where $C$ is independent of $d$ and $h$. We extend $w$ by zero outside $\tilde{D}_j$ so that $w|_{\partial \Omega_h} = 0$. This implies

$$(\psi_h, \text{div}(w - I_h w))_h = 0, \quad \forall \psi_h \in Q_h$$

owing to (2.7). Then, we have

$$\|\tilde{\zeta}_{h,j}\|_{D_{\tilde{D}_j}}^2 = (\tilde{\zeta}_{h,j}, \text{div } w)_{\tilde{D}_j} = (\zeta_j, \text{div } w)_{\tilde{D}_j} = (\zeta_j, \text{div } I_h w)_h = (\lambda, \text{div } I_h w)_h$$

(6.3)

$= (\rho, \text{div } I_h w)_h - (\zeta_j, \text{div } I_h w)_h$.

Since $\text{supp } w = \tilde{D}_j$, we can choose $K$ so that $\text{supp } I_h w \subset D_j$. Therefore, by (2.9), we have

$$|\lambda(e, I_h w)_h| \leq |\lambda(e, I_h w - w)_h| + |\lambda(e, w)_h|$$

$$\leq |\lambda|\|e\|_{D_j} \cdot C\|\nabla w\|_{\tilde{D}_j} + |\lambda|\|e\|_{D_j} \|w\|_{\tilde{D}_j}.$$  

By the Poincaré inequality on $\tilde{D}_j$, we have

$$\|w\|_{\tilde{D}_j} \leq C d \|\nabla w\|_{\tilde{D}_j},$$

which implies, together with $h \leq d \leq |\lambda|^{-1/2}$,

$$|\lambda(e, I_h w)_h| \leq C d |\lambda|\|e\|_{D_j} \|\nabla w\|_{\tilde{D}_j} \leq C |\lambda|^{1/2} \|e\|_{D_j} \|\nabla w\|_{\tilde{D}_j}.$$  

The estimates of the remainder terms in (6.3) are simple and we have

$$|\nabla e, \nabla I_h w| \leq C \|\nabla e\|_{D_j} \|\nabla I_h w\|_{D_j},$$

$$|\zeta, \text{div } I_h w| \leq C \|\zeta\|_{D_j} \|\nabla I_h w\|_{D_j},$$

by (2.10). Hence we obtain (6.1) by the above estimates and (6.2).

Next we show a global $L^p$-estimate for $\tilde{\zeta}_h$. \hfill \Box
Lemma 6.2. Let $\tilde{\zeta}_h \in Q_h^0$ be as in Proposition 5.2. Then, for every $p \in (1, \infty)$, we have
\[
\|\tilde{\zeta}_h\|_{L^p(\Omega_h)} \leq C \left(\|\lambda\|_{L^p(\Omega)} + \|
abla e\|_{L^p(\Omega_h)} + \|\zeta\|_{L^p(\Omega_h)}\right). \tag{6.4}
\]

Proof. We note that Lemma 3.1 is valid on $\Omega_h$ by the convexity of $\Omega_h$. Fix $\phi \in L^p_0(\Omega_h)$ arbitrarily and let $w = B_{\Omega_h} \phi \in W^{1,p'}_0(\Omega_h)$, where $p'$ is the Hölder conjugate of $p$. Then, from (3.1), $w$ satisfies
\[
\begin{cases}
\text{div } w = \phi & \text{in } \Omega_h, \\
 w = 0 & \text{on } \partial \Omega_h, \\
\|\nabla w\|_{L^{p'}(\Omega_h)} \leq C\|\phi\|_{L^{p'}(\Omega_h)},
\end{cases} \tag{6.5}
\]
where $C$ is independent of $h$. Then, noting that $w|_{\partial \Omega_h} \equiv 0$, we have
\[
(\tilde{\zeta}_h, \phi)_h = \lambda(e, I_h w)_h + (\nabla e, \nabla I_h w)_h - (\zeta, \text{div } I_h w)_h
\]
by the same calculation as in (6.3), which yields
\[
|\langle \tilde{\zeta}_h, \phi \rangle_h| \leq C \left(\|\lambda\|_{W^{-1,p}(\Omega)} + \|
abla e\|_{L^p(\Omega_h)} + \|\zeta\|_{L^p(\Omega_h)}\right)\|\phi\|_{L^{p'}(\Omega_h)}
\]
from (2.10) and (6.5). Hence we complete the proof of (6.4). \hfill \Box

7. Global $L^p$ duality arguments

We present global $L^p$- and $W^{-1,p}$-error estimates for the velocity by global duality arguments. We first show the $L^p$-estimate.

Lemma 7.1. Let $p \in (1, \infty)$. Then, we have
\[
\|e\|_{L^p(\Omega_h)} \leq C h \left(\|\lambda\|_{L^{p'}(\Omega_h)} + \|
abla e\|_{L^p(\Omega_h)} + h\|f\|_{L^p(\Omega)} + h^{1-\frac{1}{p}} \|\tilde{\zeta}_h\|_{L^p(\Omega_h)}\right), \tag{7.1}
\]
where $\tilde{\zeta}_h \in Q_h^0$ is as in Proposition 5.2.

Proof. Fix $\phi \in C_0^\infty(\Omega_h)^N$ arbitrarily and consider the dual problem
\[
\begin{cases}
\lambda w - \Delta w + \nabla q = \phi & \text{in } \Omega, \\
\text{div } w = 0 & \text{in } \Omega, \\
w = 0 & \text{on } \partial \Omega, \tag{7.2}
\end{cases}
\]
Then, by the resolvent estimate (1.2), we have
\[
|\lambda\|_{L^{p'}(\Omega)} + |\lambda|^{1/2}\|\nabla w\|_{L^{p'}(\Omega)} + \|
abla^2 w\|_{L^{p'}(\Omega)} + \|\nabla q\|_{L^{p'}(\Omega)} \leq C\|\phi\|_{L^{p'}(\Omega)}.
\tag{7.3}
\]
We will show
\[
(e, \phi)_h = \sum_{l=1}^6 R'_l, \tag{7.4}
\]
where
\[
\begin{align*}
R'_1 &:= \lambda(e, w - I_h w)_h, & R'_2 &:= (\nabla e, \nabla (w - I_h w))_h, \\
R'_3 &:= -(\text{div } e, q - J_h q)_h, & R'_4 &:= (\zeta, \text{div } (I_h w - w))_h, \\
R'_5 &:= -(\tilde{\zeta}_h, w \cdot n)_{\partial \Omega_h}, & R'_6 &:= \lambda(u, w)_{\Omega_h} + (\nabla u, \nabla w)_{\Omega_h,\Omega_h},
\end{align*}
\]
where \( \tilde{\zeta}_h \in L^\infty(\Omega_h) \) is the function \( \hat{\psi}_h \) in (2.8) for \( \hat{\psi}_h = \tilde{\zeta}_h \). By the definition of \((w, q)\), we have
\[
(e, \phi)_h = (e, \phi)_\Omega = \lambda(e, w)_\Omega + (\nabla e, \nabla w)_\Omega - \langle \text{div} e, q \rangle_\Omega.
\]
Since \( u_h|_{\Omega \setminus \Omega_h} \equiv 0 \) and \( \text{div} u \equiv 0 \), we have
\[
\lambda(e, w)_{\Omega \setminus \Omega_h} + (\nabla e, \nabla w)_{\Omega \setminus \Omega_h} - \langle \text{div} e, q \rangle_{\Omega \setminus \Omega_h} = R'_6,
\]
which implies
\[
(e, \phi)_h = \lambda(e, w)_h + (\nabla e, \nabla w)_h - \langle \text{div} e, q \rangle_h + R'_6.
\]
Moreover, the Galerkin orthogonality (2.20) yields
\[
\lambda(e, w)_h + (\nabla e, \nabla w)_h - \langle \text{div} e, q \rangle_h = R'_1 + R'_2 + R'_3 + \langle \rho, \text{div} I_h w \rangle_h
\]
\[
= \sum_{l=1}^{3} R'_l + \langle \rho, \text{div}(I_h w - w) \rangle_h = \sum_{l=1}^{5} R'_l.
\]
Here, we used \( \text{div} w = 0 \) and (2.7). Therefore, we obtain (7.4).

Then, by the error estimates (2.9), (2.11) and (2.12) and the resolvent estimate (7.3), we have
\[
|R'_1| \leq Ch|\lambda|\|e\|_{L^p(\Omega_h)}\|\nabla w\|_{L^p'(\Omega)} \leq Ch|\lambda|^{1/2}\|e\|_{L^p(\Omega_h)}\|\phi\|_{L^p'(\Omega)},
\]
\[
|R'_2| \leq Ch|\nabla e|_{L^p(\Omega_h)}\|\nabla^2 w\|_{L^p'(\Omega)} \leq Ch|\nabla e|_{L^p(\Omega_h)}\|\phi\|_{L^p'(\Omega)},
\]
\[
|R'_3| \leq Ch|\nabla e|_{L^p(\Omega_h)}\|\nabla q\|_{L^p'(\Omega)} \leq Ch|\nabla e|_{L^p(\Omega_h)}\|\phi\|_{L^p'(\Omega)},
\]
\[
|R'_4| \leq Ch|\zeta|\|e\|_{L^p(\Omega_h)}\|\nabla^2 w\|_{L^p'(\Omega)} \leq Ch^2\|f\|_{L^p(\Omega)}\|\phi\|_{L^p'(\Omega)}.
\]
Here, we used the estimate
\[
\|\zeta\|_{L^p(\Omega_h)} \leq Ch\|\nabla \phi\|_{L^p(\Omega_h)} \leq Ch\|f\|_{L^p(\Omega)},
\]
which is implied by (2.12) and (1.2). Moreover, the boundary-skin estimate (2.4) and and resolvent estimate (7.3) yield
\[
\|w\|_{L^p'(\partial \Omega_h)} \leq Ch^2\|\nabla^2 w\|_{L^p'(\Omega)} \leq Ch^2\|\phi\|_{L^p'(\Omega)},
\]
which implies
\[
|R'_6| \leq Ch^{2-1/2}||\tilde{\zeta}_h||_{L^p(\Omega_h)}\|\phi\|_{L^p'(\Omega)},
\]
together with (2.8). Finally, the boundary-skin estimate (2.2) and resolvent estimate (1.2) yield
\[
|\lambda|^{1/2}\|u\|_{L^p(\Omega_h)} + \|\nabla u\|_{L^p(\Omega_h)} \leq Ch^2\left|\lambda|^{1/2}\|\nabla u\|_{L^p(\Omega)} + \|\nabla^2 u\|_{L^p(\Omega)}\right|
\]
\[
\leq Ch^2\|f\|_{L^p(\Omega)},
\]
and similarly,
\[
|\lambda|^{1/2}\|w\|_{L^p'(\Omega_h)} + \|\nabla w\|_{L^p'(\Omega_h)} \leq Ch^2\|\phi\|_{L^p'(\Omega)}.
\]
Therefore, we have
\[
|R'_6| \leq Ch^2\|f\|_{L^p(\Omega)}\|\phi\|_{L^p'(\Omega)}.
\]
Summarizing above, we obtain
\[
|<e, \phi>_h| \leq Ch \left( |\lambda|^{1/2} ||e||_{L^p(\Omega_h)} + ||\nabla e||_{L^p(\Omega_h)} + h ||f||_{L^p(\Omega)} + h^{1 - \frac{2}{p}} ||\tilde{\zeta}_h||_{L^p(\Omega_h)} \right) ||\phi||_{L^{p'}(\Omega)}.
\]
Since \( \phi \in C_0^\infty(\Omega_h)^N \) is arbitrary, we complete the proof of (7.1).

Next we address the \( W^{-1,p} \)-norm of \( e \) appearing in (6.4).

**Lemma 7.2.** Let \( p \in (1, \infty) \). Then, we have
\[
||e||_{W^{-1,p}(\Omega)} \leq Ch |\lambda|^{-1/2} \left( |\lambda|^{1/2} ||e||_{L^p(\Omega_h)} + ||\nabla e||_{L^p(\Omega_h)} + |\lambda|^{-1/2} ||f||_{L^p(\Omega)} \right) + Ch^{\frac{1}{2}} |\lambda|^{-1} ||\tilde{\zeta}_h||_{L^p(\Omega_h)},
\]
where \( \tilde{\zeta}_h \in Q_h^0 \) is as in Proposition 5.2.

**Proof.** Fix \( \phi \in C_0^\infty(\Omega)^N \) arbitrarily and consider the dual problem (7.2). Then, the \( W^{3,p} \times W^{2,p'} \)-regularity estimate (3.2) yields
\[
|\lambda| ||w||_{W^{1,p'}(\Omega)} + |\lambda|^{1/2} ||w||_{W^{2,p'}(\Omega)} + ||w||_{W^{3,p'}(\Omega)} + ||q||_{W^{2,p'}(\Omega)} \leq C ||\nabla \phi||_{L^{p'}(\Omega)}. \tag{7.7}
\]
By the same calculation as in the proof of Lemma 7.1, we have
\[
<e, \phi>_\Omega = \sum_{l=1}^{5} R'_l,
\]
where
\[
\begin{align*}
R''_1 &:= \lambda(e, w - I_h w)_h, & R''_2 &:= (\nabla e, \nabla (w - I_h w))_h, \\
R''_3 &:= -(\div e, q - J_h q)_h, & R''_4 &:= (\zeta, \div I_h w)_h, \\
R''_5 &:= -(\hat{\zeta}_h, w \cdot n)_{\partial \Omega_h}, & R''_6 &:= \lambda(u, w)_{\Omega_h} + (\nabla u, \nabla w)_{\Omega_h},
\end{align*}
\]
where \( \hat{\zeta}_h \) is as in the proof of Lemma 7.1. The interpolation error estimates (2.9) and (2.11) and the resolvent estimate (7.7) yield
\[
\begin{align*}
|R''_1| &\leq Ch |\lambda| ||e||_{L^p(\Omega_h)} ||\nabla w||_{L^{p'}(\Omega)} \leq Ch ||e||_{L^p(\Omega_h)} ||\nabla \phi||_{L^{p'}(\Omega)}, \\
|R''_2| &\leq Ch ||\nabla e||_{L^p(\Omega_h)} ||\nabla^2 w||_{L^{p'}(\Omega)} \leq Ch |\lambda|^{-1/2} ||\nabla e||_{L^p(\Omega_h)} ||\nabla \phi||_{L^{p'}(\Omega)}.
\end{align*}
\]
Noting that \( \div u \equiv 0 \) and \( u_h|_{\partial \Omega_h} = 0 \), we have
\[
R''_3 = (\div u_h, q - J_h q)_h = -(u_h, \nabla (q - J_h q))_h,
\]
which implies
\[
|R''_3| \leq Ch ||u_h||_{L^p(\Omega_h)} ||\nabla^2 q||_{L^{p'}(\Omega)} \leq Ch \left( ||e||_{L^p(\Omega_h)} + |\lambda|^{-1} ||f||_{L^p(\Omega)} \right) ||\nabla \phi||_{L^{p'}(\Omega)},
\]
by (2.14), (1.2), and (7.7). From (7.5) and (2.10), we have
\[
|R''_4| \leq Ch ||f||_{L^p(\Omega)} ||\nabla w||_{L^{p'}(\Omega)} \leq Ch |\lambda|^{-1} ||f||_{L^p(\Omega)} ||\nabla \phi||_{L^{p'}(\Omega)}.
\]
Moreover, from (2.8), (2.3), and (7.7), we have

$$|R''_6| \leq Ch\frac{1}{2}\|\tilde{\zeta}_h\|_{L^p(\Omega_h)}\|\nabla w\|_{L^p'(\Omega)} \leq Ch\frac{1}{2}|\lambda|^{-1}\|\tilde{\zeta}_h\|_{L^p(\Omega_h)}\|\nabla \phi\|_{L^p'(\Omega)}.$$  

We finally address two terms of $R''_6$. First, the boundary-skin estimate (2.1) and regularity estimate (7.7) yield

$$|\lambda(u, w)_{\Omega\setminus\Omega_h}| \leq |\lambda||u||_{L^p(\Omega)} \cdot Ch^{2}\|\nabla w\|_{L^p'(\Omega)} \leq Ch^{2}|\lambda|^{-1}\|f||_{L^p(\Omega)}\|\nabla \phi\|_{L^p'(\Omega)}$$

Moreover, when $p \geq 2$, by using (2.2) for $\nabla w$, we obtain

$$\|\nabla w\|_{W^{2,p'}(\Omega)} \leq Ch\|\nabla \phi\|_{L^p'(\Omega)}.$$  

When $p \leq 2$, we obtain the same estimate by changing the role of $u$ and $w$. Therefore, we have

$$|R''_6| \leq Ch|\lambda|^{-1}\|f||_{L^p(\Omega)}\|\nabla \phi\|_{L^p'(\Omega)}.$$  

Summarizing the above estimates, we obtain

$$|(e, v)_{\Omega}| \leq C \left(h\|e||_{L^p(\Omega_h)} + Ch|\lambda|^{-1/2}\|\nabla e\|_{L^p(\Omega_h)} + h|\lambda|^{-1}\|f||_{L^p(\Omega)} + h^{1/2}|\lambda|^{-1}\|\tilde{\zeta}_h\|_{L^p(\Omega_h)}\right)\|\nabla \phi\|_{L^p'(\Omega)},$$

which proves the desired estimate (7.6). \qed

Lemmas 6.2 and 7.2 lead to the global $L^p$-estimate for $\tilde{\zeta}_h$.

Corollary 7.1. Let $\tilde{\zeta}_h \in Q^0_b$ be as in Proposition 5.2 and $p \in (1, \infty)$. Then, there exists $h_0 > 0$ such that for all $h \leq h_0$, we have

$$\|\tilde{\zeta}_h\|_{L^p(\Omega_h)} \leq Ch|\lambda|^{1/2}\left(|\lambda|^{1/2}\|e||_{L^p(\Omega_h)} + \|\nabla e\|_{L^p(\Omega_h)}\right) + C\|\nabla e||_{L^p(\Omega_h)} + Ch\|f||_{L^p(\Omega)}.$$  

(7.8)

Proof. Substituting (7.6) into (6.4) and using (7.5), we have

$$\|\tilde{\zeta}_h\|_{L^p(\Omega_h)} \leq Ch|\lambda|^{1/2}\left(|\lambda|^{1/2}\|e||_{L^p(\Omega_h)} + \|\nabla e\|_{L^p(\Omega_h)}\right) + C\|\nabla e||_{L^p(\Omega_h)} + Ch\|f||_{L^p(\Omega)} + h^{1/2}|\lambda|^{-1}\|\tilde{\zeta}_h\|_{L^p(\Omega_h)}.$$  

Therefore, we obtain (7.8) if $h$ is sufficiently small. \qed

8. Proof of the error estimates

This section is devoted to the proof of Theorem 2.2. We will give the proof in the following order: (2.21) for $p \geq 2$, (2.22), (2.21) for $p < 2$, (2.23), and finally (2.24).

8.1. $W^{1,p}$-error estimate for the velocity for $p \geq 2$

We show (2.21) with the basis of the local energy error estimate (5.2). In the proof, we will consider the two cases: (1) $|\lambda| \leq \alpha h^{-2}$ for sufficiently small $\alpha$ and (2) $|\lambda| \geq \alpha h^{-2}$ for arbitrary $\alpha$. In the following, coefficients of $\varepsilon$ is omitted since we can make $\varepsilon$ smaller if necessary.

Proof of (2.21) for $p \geq 2$. Let $p \geq 2$. 

17
Case 1: $|\lambda| \leq O(h^{-2})$ We first assume $|\lambda| \leq d^{-2}$. In this case, it suffices to show the error estimate

$$|\lambda|^{1/2} \|e\|_{L^p(\Omega_h)} + \|\nabla e\|_{L^p(\Omega_h)} \leq C h \|f\|_{L^p(\Omega)}. \quad (8.1)$$

We first observe that the local energy error estimate

$$|\lambda| \|e\|_{D_j}^2 + \|\nabla e\|_{D_j}^2 \leq (CK^{-1} + \varepsilon^2)|\lambda| \|e\|_{D_j}^2 + \varepsilon^2 \|\nabla e\|_{D_j}^2$$
$$+ \varepsilon \varepsilon^2 |\lambda| \|e\|_{D_j}^2 + \varepsilon \|\nabla z\|_{D_j}^2 + \|\zeta\|_{D_j}^2 + C_\varepsilon h^{-1} \|u\|_{\partial\Omega_h \cap D_j}^2. \quad (8.2)$$

holds owing to (5.2) and (6.1).

Step 1. Global energy error estimate by a kick-back argument. Summing up (8.2) with respect to $j$, we have

$$|\lambda|^{1/2} \|e\| + \|\nabla e\| \leq \left(CK^{-1/2} + \varepsilon\right) |\lambda|^{1/2} \|e\| + \varepsilon \|\nabla e\| + C_\varepsilon d^{-1} \|e\|$$
$$+ C_\varepsilon (d^{-1} \|z\| + \|\nabla z\| + \|\zeta\|) + C_\varepsilon h^{-1/2} \|u\|'$$
by (4.2), where

$$\|u\|' := \left[ \sum_j \left( h^{-\frac{N}{p} + \frac{N}{2}} \|u\|_{\partial\Omega_h \cap D_j} \right)^p \right]^{1/p}.$$ 

Therefore, letting $K$ be sufficiently large (but independently of $h$) and letting $\varepsilon$ sufficiently small, we obtain

$$|\lambda|^{1/2} \|e\| + \|\nabla e\| \leq Cd^{-1} \|e\| + C(d^{-1} \|z\| + \|\nabla z\| + \|\zeta\|) + Ch^{-1/2} \|u\|'.$$

Step 2. Global $L^p$-estimate. First, (4.4) and (7.5) lead to

$$|\lambda|^{1/2} \|e\|_{L^p(\Omega_h)} + \|\nabla e\|_{L^p(\Omega_h)} \leq Cd^{-1} K^{\frac{N}{p} - \frac{N}{p}} \|e\|_{L^p(\Omega_h)}$$
$$+ C(\varepsilon) \|\nabla z\|_{L^p(\Omega_h)} + \|\nabla z\|_{L^p(\Omega_h)} + h \|f\|_{L^p(\Omega)} + C_\varepsilon h^{-1 + \frac{\varepsilon}{2}} \|u\|_{L^p(\partial\Omega_h)},$$
since $|\partial\Omega_h \cap D_j| \leq Cd^{N-1}$. Therefore, together with (4.5) and (2.4), we obtain

$$|\lambda|^{1/2} \|e\|_{L^p(\Omega_h)} + \|\nabla e\|_{L^p(\Omega_h)} \leq Cd^{-1} K^{\frac{N}{p} - \frac{N}{p}} \|e\|_{L^p(\Omega_h)}$$
$$+ C_\varepsilon h \|f\|_{L^p(\Omega)} + C(\varepsilon) \|\nabla z\|_{L^p(\Omega_h)} + \|\nabla z\|_{L^p(\Omega_h)},$$

where we used the assumption $|\lambda| \leq d^{-2}$ again. Furthermore, since $z = u - I_h u$ is the interpolation error, (2.9), (2.11), and (1.2) yield

$$h^{-1} \|z\|_{L^p(\Omega_h)} + \|\nabla z\|_{L^p(\Omega_h)} \leq Ch \|f\|_{L^p(\Omega)}.$$

Thus, we obtain

$$|\lambda|^{1/2} \|e\|_{L^p(\Omega_h)} + \|\nabla e\|_{L^p(\Omega_h)} \leq Cd^{-1} K^{\frac{N}{p} - \frac{N}{p}} \|e\|_{L^p(\Omega_h)} + C_\varepsilon h \|f\|_{L^p(\Omega)}.$$ 

(8.3)

Step 3. Treatment of the $L^p$-error. We then address $\|e\|_{L^p(\Omega_h)}$ in the right hand side of (8.3). Since $|\lambda| \leq d^{-2}$, the $L^p$-pressure estimate (7.8) leads to

$$\|\tilde{z}_h\|_{L^p(\Omega_h)} \leq C \left( |\lambda|^{1/2} \|e\|_{L^p(\Omega_h)} + \|\nabla e\|_{L^p(\Omega_h)} + Ch \|f\|_{L^p(\Omega)} \right).$$
Substituting this into (7.1), we obtain
\[ \|e\|_{L^p(\Omega_h)} \leq C h \left( |\lambda|^{1/2} \|e\|_{L^p(\Omega_h)} + \|\nabla e\|_{L^p(\Omega_h)} + C h \|f\|_{L^p(\Omega)} \right). \] (8.4)

Step 4. Final estimate. Together with (8.3) and (8.4), we have
\[ |\lambda|^{1/2} \|e\|_{L^p(\Omega_h)} + \|\nabla e\|_{L^p(\Omega_h)} \leq C K^{\frac{N}{2} - \frac{N}{p} - 1} \left( |\lambda|^{1/2} \|e\|_{L^p(\Omega_h)} + \|\nabla e\|_{L^p(\Omega_h)} \right) + C K h \|f\|_{L^p(\Omega)}. \]

Hence, setting \( K \) sufficiently large, we establish the desired estimate (8.1), provided that \( p \geq 2 \) satisfies \( \frac{N}{2} - \frac{N}{p} - 1 < 0 \), namely \( p < \frac{2N}{N-2} \).

**Case 2:** \( |\lambda| \geq O(h^{-2}) \) We then assume \( |\lambda| \geq \alpha h^{-2} \) for arbitrarily fixed \( \alpha \). In this case, the desired estimate is
\[ |\lambda|^{1/2} \|e\|_{L^p(\Omega_h)} + \|\nabla e\|_{L^p(\Omega_h)} \leq C |\lambda|^{-1/2} \|f\|_{L^p(\Omega)}. \] (8.5)

We note that we can choose \( K \) independently of the previous case.

Step 1. Global energy error estimate by a kick-back argument. Summing up (5.12) with respect to \( j \) and using (4.2) and (4.4), we obtain
\[ |\lambda|^{1/2} \|e\| + \|\nabla e\| \leq \left( \varepsilon + C K^{-1/2} \right) |\lambda|^{1/2} \|e\| + C K^{-1/2} \|\nabla e\|
+ C K \left( |\lambda|^{1/2} \|u\|_{L^p(\Omega)} + \|\nabla u\|_{L^p(\Omega)} + |\lambda|^{-1/2} \|\nabla \xi\|_{L^p(\Omega)} \right)
+ C \|C_{h} K^{\frac{N}{2} - \frac{N}{p} - 1} |\lambda|^{-1/2} \|\tilde{\zeta}_h\|_{L^p(\Omega_h)} \right). \]

Thus, letting \( \varepsilon \) be small enough and \( K \) be sufficiently large, we have
\[ |\lambda|^{1/2} \|e\| + \|\nabla e\| \leq C K \left( |\lambda|^{1/2} \|u\|_{L^p(\Omega)} + \|\nabla u\|_{L^p(\Omega)} + |\lambda|^{-1/2} \|\nabla \xi\|_{L^p(\Omega)} \right)
+ C K^{\frac{N}{2} - \frac{N}{p} - 1} |\lambda|^{-1/2} \|\tilde{\zeta}_h\|_{L^p(\Omega_h)}. \] (8.6)

Step 2. Global \( L^p \)-estimate. Substituting (8.6) into (4.6), we have
\[ |\lambda|^{1/2} \|e\|_{L^p(\Omega)} + \|\nabla e\|_{L^p(\Omega)} \leq C K |\lambda|^{-1/2} \|f\|_{L^p(\Omega)} + C K^{\frac{N}{2} - \frac{N}{p} - 1} |\lambda|^{-1/2} \|\tilde{\zeta}_h\|_{L^p(\Omega_h)}, \] (8.7)
together with the resolvent estimate (1.2) and the stability estimate (2.13).

Step 3. Treatment of the pressure term. From (7.8), we have
\[ d^{-1} |\lambda|^{-1/2} \|\tilde{\zeta}_h\|_{L^p(\Omega_h)} \leq Ch^{-1} \left( |\lambda|^{1/2} \|e\|_{L^p(\Omega_h)} + \|\nabla e\|_{L^p(\Omega_h)} \right)
+ C d^{-1} |\lambda|^{-1/2} \|\nabla e\|_{L^p(\Omega_h)} + Ch^{-1} |\lambda|^{-1/2} \|f\|_{L^p(\Omega)}. \]

Therefore, recalling \( hd^{-1} = K^{-1} \leq 1 \) and (5.14), we have
\[ d^{-1} |\lambda|^{-1/2} \|\tilde{\zeta}_h\|_{L^p(\Omega_h)} \leq C K^{-1} \left( |\lambda|^{1/2} \|e\|_{L^p(\Omega_h)} + \|\nabla e\|_{L^p(\Omega_h)} \right)
+ C |\lambda|^{-1/2} \|f\|_{L^p(\Omega)}. \] (8.8)

Step 4. Final estimate. Together with (8.7) and (8.8), we obtain
\[ |\lambda|^{1/2} \|e\|_{L^p(\Omega_h)} + \|\nabla e\|_{L^p(\Omega_h)} \leq C K |\lambda|^{-1/2} \|f\|_{L^p(\Omega)}
+ C K^{\frac{N}{2} - \frac{N}{p} - 1} \left( |\lambda|^{1/2} \|e\|_{L^p(\Omega_h)} + \|\nabla e\|_{L^p(\Omega_h)} \right). \]

Therefore, letting \( K \) sufficiently large again, we establish the desired estimate (8.5), provided that \( p \geq 2 \) satisfies \( p < \frac{2N}{N-2} \). Hence we complete the proof of (2.21) for \( p \geq 2 \).
8.2. \textit{L}^p\text{-error estimate for the velocity}

We next show (2.22). For \( p \geq 2 \), this is a consequence of (7.1) and (2.21). For \( p < 2 \), we will perform duality argument.

\textbf{Proof of (2.22).} Let us first assume \( p \geq 2 \). Then, we have already shown that

\[
\|e\|_{L^p(\Omega_h)} \leq C \min\{h\lambda^{-1/2}, |\lambda|^{-1}\} \|f\|_{L^p(\Omega)}
\]

holds. This implies

\[
\|\tilde{\zeta}_h\|_{L^p(\Omega_h)} \leq Ch\|f\|_{L^p(\Omega)}
\]

(8.9) owing to (7.8). Therefore, combining these estimates with (7.1), we obtain

\[
\|e\|_{L^p(\Omega_h)} \leq Ch^2\|f\|_{L^p(\Omega)}.
\]

(8.10) Hence we obtain (2.22) for \( p \geq 2 \).

Next we will consider the case \( p < 2 \). For arbitrarily fixed \( \phi \in C_0^\infty(\Omega)^N \), we consider the dual problem

\[
\begin{aligned}
&\bar{\lambda}U - \Delta U + \nabla \Phi = \phi & &\text{in } \Omega, \\
&\text{div} U = 0 & &\text{in } \Omega, \\
&U = 0 & &\text{on } \partial\Omega,
\end{aligned}
\]

(8.11)

and

\[
\begin{aligned}
&\lambda(v_h, U_h)_h + (\nabla v_h, \nabla U_h)_h - (\text{div} v_h, \Phi)_h = (v_h, \phi)_h, & &\forall v_h \in V_h, \\
&(\psi_h, \text{div} U_h)_h = 0, & &\forall \psi_h \in Q_h.
\end{aligned}
\]

(8.12)

Then, testing \( u \) to (8.11) and \( u_h \) to (8.12), respectively, we have

\[
(e, \phi) = (f, U - U_h),
\]

which implies

\[
|\langle e, \phi \rangle| \leq \|f\|_{L^p(\Omega)} \left( \|U - U_h\|_{L^{p'}(\Omega_h)} + \|U\|_{L^{p'}(\Omega \setminus \Omega_h)} \right).
\]

Since \( p < 2 \), we have

\[
\|U - U_h\|_{L^{p'}(\Omega_h)} \leq Ca(h, \lambda)\|\phi\|_{L^{p'}(\Omega)},
\]

where \( a(h, \lambda) = \min\{h^2, h|\lambda|^{-1/2}, |\lambda|^{-1}\} \). Moreover, using the boundary-skin estimates (2.1) and (2.2), we obtain

\[
\|U\|_{L^{p'}(\Omega \setminus \Omega_h)} \leq \min\left\{ \|U\|_{L^{p'}(\Omega)}, h^2\|\nabla U\|_{L^{p'}(\Omega \setminus \Omega_h)}, h^{2+\frac{2}{p}}\|\nabla U\|_{W^{1,p'}(\Omega)} \right\}
\]

\[
\leq Ca(h, \lambda)\|\phi\|_{L^{p'}(\Omega)},
\]

together with the resolvent estimate for \( U \). Therefore, we have

\[
|\langle e, \phi \rangle| \leq a(h, \lambda)\|f\|_{L^p(\Omega)}\|\phi\|_{L^{p'}(\Omega)},
\]

which implies the desired estimate (2.22) for \( p < 2 \) since \( \phi \in C_0^\infty(\Omega)^N \) is arbitrary. Hence we complete the proof. \hfill \( \square \)
8.3. $W^{1,p}$-error estimate for the velocity for $p < 2$

We will prove (2.21) for $p < 2$ by using the inverse inequality and the $L^p$-estimate (2.22).

**Proof of (2.21) for $p < 2$.** Let $p < 2$. We again consider the two cases: (1) $|\lambda| \leq h^{-2}$ and (2) $|\lambda| \geq h^{-2}$. Let $z = u - I_h u$ and $z_h = e - z = I_h u - u_h$. Then, it is clear that

$$
\| \nabla e \|_{L^p(\Omega_h)} \leq \| \nabla z \|_{L^p(\Omega_h)} + C h^{-1} \| z \|_{L^p(\Omega_h)} + C h^{-1} \| e \|_{L^p(\Omega_h)}
$$

(8.13)

by the inverse inequality (2.5).

We first assume $|\lambda| \leq h^{-2}$ and show (8.1). Then, by (2.9) and (2.11), we have

$$
\| \nabla z \|_{L^p(\Omega_h)} + h^{-1} \| z \|_{L^p(\Omega_h)} \leq C h \| \nabla^2 u \|_{L^p(\Omega)} \leq C h \| f \|_{L^p(\Omega)}.
$$

Therefore, together with (2.22) and (8.13), we obtain (8.1).

We then assume $|\lambda| \geq h^{-2}$ and show (8.5). In this case, we have

$$
\| \nabla z \|_{L^p(\Omega_h)} + h^{-1} \| z \|_{L^p(\Omega_h)} \leq C \| \nabla u \|_{L^p(\Omega_h)} \leq C |\lambda|^{-1/2} \| f \|_{L^p(\Omega)}
$$

by (2.9) and (2.10). Together with (2.21) and (8.13), we obtain (8.5). Hence we complete the proof. \qed

8.4. $W^{-1,p}$-error estimate for the velocity

The proof of the $W^{-1,p}$-error estimate (2.23) is now a consequence of the previous results.

**Proof of (2.23).** Since the $W^{1,p}$-estimate (2.21) leads to

$$
|\lambda|^{1/2} \| e \|_{L^p(\Omega_h)} + \| \nabla e \|_{L^p(\Omega_h)} \leq C |\lambda|^{-1/2} \| f \|_{L^p(\Omega)},
$$

(8.14)

we then obtain the desired estimate together with (7.6) and (8.9). \qed

8.5. $L^p$-error estimate for the pressure

Finally we will show the $L^p$-estimate for the pressure.

**Proof of (2.24).** Let $\zeta_0$ be as in Proposition 5.2. Then, noting that $\int_{\Omega_h} \varphi_h dx = \int_{\Omega} \varphi dx = 0$, we have

$$
\zeta_0 = -\frac{1}{|\Omega_h|} \int_{\Omega_h} \zeta_h dx = \frac{1}{|\Omega_h|} \int_{\Omega_h} \zeta dx - \frac{1}{|\Omega_h|} \int_{\Omega \setminus \Omega_h} \varphi dx =: \zeta_{0,1} + \zeta_{0,2}.
$$

By the Hölder inequality and (7.5), we have

$$
|\zeta_{0,1}| \leq C \| \zeta \|_{L^p(\Omega_h)} \leq C h \| f \|_{L^p(\Omega)}.
$$

Moreover, the boundary-skin estimate (2.2) and $|\Omega \setminus \Omega_h| \leq Ch^2$ lead to

$$
|\zeta_{0,2}| \leq C h^2 \| \varphi \|_{W^{1,p}(\Omega)} \leq C h^2 \| f \|_{L^p(\Omega)}.
$$

Thus we obtain

$$
\| \zeta_h \|_{L^p(\Omega_h)} \leq C h \| f \|_{L^p(\Omega)}
$$

(8.15)

together with (8.9). Hence we obtain the desired estimate (2.24) by (7.5) and (8.15). \qed
9. Proof of the resolvent estimate

We are now ready to show (1.4).

Proof of Theorem 2.1. The resolvent estimate for the velocity is clear. Indeed, we obtain

\[ |\lambda| \|u_h\|_{L^p(\Omega_h)} + |\lambda|^{1/2} \|\nabla u_h\|_{L^p(\Omega_h)} \leq C \|f\|_{L^p(\Omega)} \]

from the resolvent estimate (1.2) and the error estimate (8.14). Moreover, the \( L^p \)-error estimate (8.10) implies

\[ \|u_h\|_{L^p(\Omega_h)} \leq C \|f\|_{L^p(\Omega)} . \]

Finally, by the consequence of the stability estimate (2.13), the inverse inequality (2.5), and the error estimate (8.15), we obtain the pressure estimate

\[ \|\nabla \varphi_h\|_{L^p(\Omega_h)} \leq C \|\nabla \varphi\|_{L^p(\Omega)} + Ch^{-1} \|\zeta_h\|_{L^p(\Omega_h)} \leq C \|f\|_{L^p(\Omega)} . \]

Hence we complete the proof of the main theorem.

A. Construction of the quasi-interpolation operator \( I_h \)

In this section, we construct the operator \( I_h \) that fulfills the assumptions (H2) and (H3) for the Taylor–Hood and MINI elements on a planar convex domain \( \Omega \) with smooth boundary. The result may hold even for non-convex domains, which will be given elsewhere. Extension to the three-dimensional case may be straightforward, so we omit the detail.

We first collect the notation. Let \( T_h \) be the triangulation of \( \Omega \subset \mathbb{R}^2 \) that satisfies (H1) and \( \Omega_h \subset \Omega \) be the polygonal approximation of \( \Omega \) consisting of triangles in \( T_h \). For a triangle \( T \subset \Omega_h \), we denote the smallest macro-element including \( T \) by \( \Delta_T \).

The sets of vertices and edges of \( T_h \) are denoted by \( V_h \) and \( E_h \), respectively, and let

\[ V_h^0 := \{ P \in V_h | P \in \partial \Omega_h \}, \quad V_h := V_h \setminus V_h^0, \]
\[ E_h^0 := \{ e \in E_h | e \subset \partial \Omega_h \}, \quad E_h := E_h \setminus E_h^0. \]

Since \( \Omega \) is convex, there exists a homeomorphism \( \pi: \partial \Omega_h \to \partial \Omega \) based on the normal vector of \( \partial \Omega_h \). For boundary edge \( e \in E_h^0 \), we set

\[ \tilde{e} = \pi(e), \quad S_e := \{(1-t)y + t\pi(y) | y \in e, t \in (0,1)\}. \]

Note that \( S_e \subset \Omega \setminus \Omega_h \) and \( \partial S_e = e \cup \tilde{e} \) holds. Moreover, define

\[ T_h^0 := \{ T \in T_h | \exists e \in E_h^0 \text{ such that } e = \partial T \cap \partial \Omega_h \}, \quad T_h^0 := T_h \setminus T_h^0. \]

We write \( e(T) = \partial T \cap \partial \Omega_h \) if \( T \in T_h^0 \).

Finally, we denote by \( \mathcal{P}^k(D) \) the space of polynomials of degree at most \( k \) for a subset \( D \subset \Omega_h \). We also denote by \( \mathcal{P}^k(T_h) \subset C^0(\Omega_h) \) the space of conforming piecewise polynomials of degree at most \( k \) associated to \( T_h \).
A.1. Preliminary: boundary-skin estimates

Lemma A.1. Let \( p \in [1, \infty] \) and \( v \in W^{1,p}_0(\Omega) \). Then, there exists \( C > 0 \) such that, for every \( e \in \mathcal{E}_h \),

\[
\|v\|_{L^p(e)} \leq C h^{2-\frac{2}{p}} \|\nabla v\|_{L^p(S_e)}
\]  

(A.1)

holds. In addition, if \( v \in W^{2,p}_0(\Omega) \), then

\[
\|v\|_{L^p(e)} \leq C h^{2-\frac{1}{p}} \|
abla v\|_{L^p(T_e)} + C h^{\frac{3}{p}} \|\nabla^2 v\|_{L^p(S_e \cup T_e)},
\]  

(A.2)

where \( T_e \in \mathcal{T}_h \) is a triangle such that \( e = \partial T_e \cap \partial \Omega_h \).

Proof. We may assume that \( e = [0, h] \times \{0\} \) and there exists \( f \in C^4([0, h]) \) such that

\[
e = \{(x, f(x)) \in \mathbb{R}^2 \mid 0 \leq x \leq h\}, \quad f(0) = f(h) = 0, \quad 0 \leq f(x) \leq C h^2,
\]

since \( e \) is the Lagrange interpolation of \( \tilde{e} \) (cf. [20, Proposition 8.2]). In this setting, we have \( v(x, f(x)) = 0 \) for \( x \in [0, h] \) and thus

\[
|v(x, 0)| \leq \int_0^{f(x)} |\partial_y v(x, y)| dy \leq C h^{2-\frac{2}{p}} \left( \int_0^{f(x)} |\nabla v(x, y)|^p dy \right)^{1/p}
\]

by the Hölder inequality. Integrating the both sides with respect to \( x \), we obtain the first estimate (A.1).

We then show (A.2). Let \( w \in W^{1,p}(S_e) \) and \( (x, y) \in S_e \). Then, we have

\[
|w(x, y)| \leq |w(x, 0)| + \int_0^y |\nabla w(x, \eta)| d\eta 
\]

\[
\leq |w(x, 0)| + C h^{2-\frac{2}{p}} \left( \int_0^{f(x)} |\nabla w(x, \eta)|^p d\eta \right)^{1/p}.
\]

Integrating with respect to \( y \), we have

\[
\left( \int_0^{f(x)} |w(x, y)|^p dy \right)^{1/p} \leq |f(x)|^{1/p} \left[ |w(x, 0)| + C h^{2-\frac{2}{p}} \left( \int_0^{f(x)} |\nabla w(x, \eta)|^p d\eta \right)^{1/p} \right]
\]

\[
\leq C h^{\frac{2}{p}} |w(x, 0)| + C h^2 \left( \int_0^{f(x)} |\nabla w(x, \eta)|^p d\eta \right)^{1/p},
\]

and then integrating with respect to \( x \), we obtain

\[
\|w\|_{L^p(S_e)} \leq C h^{\frac{2}{p}} \|w\|_{L^p(e)} + C h^2 \|\nabla w\|_{L^p(S_e)}.
\]

Therefore, we have

\[
\|\nabla v\|_{L^p(S_e)} \leq C h^{\frac{2}{p}} \|\nabla v\|_{L^p(e)} + C h^2 \|\nabla^2 v\|_{L^p(S_e)},
\]

which implies

\[
\|v\|_{L^p(e)} \leq C h^2 \|\nabla v\|_{L^p(e)} + h^{\frac{4}{p}} \|\nabla^2 v\|_{L^p(S_e)}
\]  

(A.3)

together with (A.1).

Recall that, in general, the trace inequality with scaling

\[
\|w\|_{L^p(e')} \leq C h^{\frac{1}{p}} \|w\|_{L^p(T')} + h^{1-\frac{1}{p}} \|\nabla w\|_{L^p(T')}
\]  

(A.4)

holds for \( p \in [1, \infty] \), \( T' \in \mathcal{T}_h \), \( e' \in \mathcal{E}_h \) with \( e' \subset \partial T' \), and \( w \in W^{1,p}(T') \). Therefore, (A.3) and (A.4) yield the desired estimate (A.2). \( \square \)
A.2. Taylor–Hood element

In this section, we let \( V_h = P^2(T_h) \cap H^1_0(\Omega)^N \) and \( Q_h = P^1(T_h) \). The same interpolation operator can be used for the higher-order elements, since the lower-order error estimates (H3) are required.

We first introduce an auxiliary operator \( \Pi_h \) according to [15, §3.1]. For \( P \in \mathcal{V}_h \) and \( e \in \mathcal{E}_h \), we define \( \varphi_P, \varphi_e \in P^2(T_h) \) as basis functions that satisfies

\[
\varphi_P(Q) = \delta_{P,Q}, \quad \varphi_e(Q) = 0, \quad \forall Q \in \mathcal{V}_h,
\]

\[
\int_f \varphi_P ds = 0, \quad \int_f \varphi_e ds = \delta_{e,f}, \quad \forall f \in \mathcal{E}_h,
\]

where \( \delta_{a,b} \) is the Kronecker delta. Fix \( P \in \mathcal{V}_h \) and define \( \kappa = \kappa(P) \in \mathcal{E}_h \) by

\[
\kappa = \begin{cases} 
\text{one of } e \in \mathcal{E}_h \text{ with } P \in \partial e, & \text{if } P \in \mathcal{V}_h^0, \\
\text{one of } e \in \mathcal{E}_h^0 \text{ with } P \in \partial e, & \text{if } P \in \mathcal{V}_h^\partial,
\end{cases}
\]

(A.5)

and let \( Q \in \mathcal{V}_h \) be the other endpoint of \( \kappa \). Let \( \{ \psi_P, \psi_Q, \psi_\kappa \} \subset P^2(\kappa) \) be the dual basis of \( \{ \varphi_P, \varphi_Q, \varphi_\kappa \} \). That is, these functions satisfy

\[
\int_\kappa \psi_a \varphi_b ds = \delta_{a,b}, \quad a, b \in \{ P, Q, \kappa \}.
\]

We then define auxiliary operators \( \tilde{\Pi}_h, \Pi_h : W^{1,1}(\Omega) \to P^2(T_h) \) by

\[
\tilde{\Pi}_h v = \sum_{P \in \mathcal{V}_h} \left( \int_{\kappa(P)} \psi_P vds \right) \varphi_P + \sum_{e \in \mathcal{E}_h} \left( \int_e vds \right) \varphi_e,
\]

\[
\Pi_h v = \sum_{P \in \mathcal{V}_h^0} \left( \int_{\kappa(P)} \psi_P vds \right) \varphi_P + \sum_{e \in \mathcal{E}_h^0} \left( \int_e vds \right) \varphi_e,
\]

and extend \( \tilde{\Pi}_h v \) and \( \Pi_h v \) for a vector-valued function \( v \) by componentwise operation. It is clear that \( \Pi_h v \in \mathcal{V}_h \) for \( v \in W^{1,1}(\Omega)^N \) and

\[
\tilde{\Pi}_h v|_T = \Pi_h v|_T, \quad \forall T \in T^0_h
\]

(A.6)

holds. The first goal of this section is to show the following properties.

**Proposition A.1.** Let \( p \in [1, \infty] \) and \( v \in W^{1,p}_0(\Omega)^N \). Then, the following properties hold for each \( T \in T_h \).

(i) If \( T \in T^0_h \),

\[
\int_T \text{div}(\Pi_h v - v) dx = -\int_{\partial(T)} v \cdot nds,
\]

where \( n \) is the outward unit normal vector of \( \partial \Omega_h \). Otherwise,

\[
\int_T \text{div}(\Pi_h v - v) dx = 0.
\]

(A.7)

(A.8)

(ii) The following stability and error estimates hold:

\[
\| \nabla \Pi_h v \|_{L^p(T)} \leq C \| \nabla v \|_{L^p(\Delta_T)},
\]

(A.9)
Therefore, we obtain
\[
\|\Pi_h v - v\|_{L^p(T)} \leq Ch\|\nabla v\|_{L^p(\hat{\Delta}_T)},
\]
(A.10)
where
\[
\hat{\Delta}_T = \Delta_T \cup \left( \bigcup_{T' \subset \Delta_T, T' \in \mathcal{T}_h^0} S_{e(T')} \right).
\]
(A.11)
(iii) If in addition \( v \in W^{2,p}(\Omega)^N \), then
\[
\|\Pi_h v - v\|_{L^p(T)} + h\|\nabla(\Pi_h v - v)\|_{L^p(T)} \leq Ch^2\left( \|\nabla v\|_{L^p(\Delta_T)} + \|\nabla^2 v\|_{L^p(\hat{\Delta}_T)} \right).
\]
(A.12)

Proof. To show (i), we first recall that \( \tilde{\Pi}_h \) satisfies
\[
\int_T \text{div}(\Pi_h v - v)dx = 0
\]
for all \( T \in \mathcal{T}_h \) and \( v \in W^{1,1}(\Omega)^N \) (see [15]). Therefore, (A.8) holds owing to (A.6). Thus let us see (A.7). Let \( T \in \mathcal{T}_h^0 \) and \( e_0 = e(T) \). We denote the edges of \( T \) other than \( e_0 \) by \( e_1, e_2 \). Then, by the definition, we have
\[
\int_{e_0} \Pi_h v ds = 0, \quad \int_{e_j} \Pi_h v ds = \int_{e_j} v ds, \quad j = 1, 2.
\]
Therefore, we obtain (A.7) together with the divergence theorem.

We next show (ii) and (iii). We first observe that
\[
\|\nabla^k \varphi_P\|_{L^p(T)} \leq Ch^{-k+\frac{2}{p}}, \quad \|\nabla^k e_P\|_{L^p(T)} \leq Ch^{-k+\frac{2}{p}}, \quad k = 0, 1, 2
\]
hold for every vertex \( P \in \partial T \) and edge \( e \subset \partial T \), which can be shown by mapping \( T \) to the reference triangle. Moreover, \( \psi_P \) satisfies
\[
\|\psi_P\|_{L^\infty(\kappa(P))} \leq Ch^{-1},
\]
(A.14)
owing to [33, Lemma 3.1].

Now, we may assume \( T \in \mathcal{T}_h^0 \) since \( \Pi_h \) satisfies (A.9), (A.10), and (A.12) with \( \hat{\Delta}_T \) replaced by \( \Delta_T \) [15]. Let \( T \in \mathcal{T}_h^0 \) and \( v \in W^{1,p}_0(\Omega)^N \). Then, it is clear that
\[
\|\nabla^k (\Pi_h v - v)\|_{L^p(T)} \leq \|\nabla^k \Pi_h (v - q)\|_{L^p(T)} + \|\nabla^k (\Pi_h q - q)\|_{L^p(T)} + \|\nabla^k (v - q)\|_{L^p(T)}
\]
(A.15)
holds for any \( q \in \mathcal{P}^2(\Omega_h)^N \) and for \( k = 0, 1 \).

We address the first term of (A.15). For every \( w \in W^{1,p}(\Omega_h) \), we have
\[
\|\nabla^k \Pi_h w\|_{L^p(T)} \leq Ch^{-k+\frac{2}{p}} \left( \sum_{P \in \mathcal{P}_h, P \subset \partial T} \|\psi_P w\|_{L^1(\kappa(P))} + \sum_{e \in \mathcal{E}_h^0, e \subset \partial T} \|w\|_{L^1(e)} \right)
\leq Ch^{-k+\frac{2}{p}} \sum_{e \in \mathcal{E}_h^0, e \subset \partial T} \|w\|_{L^1(e)}
\]
for every \( w \in W^{1,p}(\Omega_h) \).
for $k = 0, 1$ by (A.13) and (A.14). Therefore, this implies
\[
\|\nabla^k \Pi_h (v - q)\|_{L^p(T)} \leq C h^{-k} (\|v - q\|_{L^p(\Delta_T)} + h \|\nabla (v - q)\|_{L^p(\Delta_T)})
\]  
(A.16)
together with the Hölder inequality and the trace inequality with scaling (A.4).

We then consider the second term of (A.15). For $q \in P^2(\Omega_h)$, one can see that
\[
\Pi_h q|_T = q|_T - \sum_{P \in \mathcal{V}_h^e, P \in \partial T} \left( \int_{\kappa(p)} \psi_P q ds \right) \varphi_P - \sum_{e \in \mathcal{E}_h^p, e \in \partial T} \left( \int_{e} q ds \right) \varphi_e
\]
holds, which implies, by (A.13), (A.14), and (A.4) again,
\[
\|\nabla^k (\Pi_h q - q)\|_{L^p(T)} \leq C h^{-k-1+\frac{2}{p}} \sum_{e \in \mathcal{E}_h^p, e \in \partial \Delta_T} \|q\|_{L^1(e)}
\]
\[
\leq C h^{-k} (\|v - q\|_{L^p(\Delta_T)} + h \|\nabla (v - q)\|_{L^p(\Delta_T)}) + C h^{-k-1+\frac{2}{p}} \sum_{e \in \mathcal{E}_h^p, e \in \partial \Delta_T} \|v\|_{L^1(e)}
\]  
(A.17)

for $k = 0, 1$. Note that $\kappa(P) \in \mathcal{E}_h^p$ but in general $\kappa(P) \not\subset \partial T$, and thus the summation of the last term is over $e \in \partial \Delta_T$, not $e \subset \partial T$.

Combining (A.15), (A.16), and (A.17), we obtain
\[
\|\nabla^k (\Pi_h v - v)\|_{L^p(T)} \leq C h^{-k} (\|v - q\|_{L^p(\Delta_T)} + h \|\nabla (v - q)\|_{L^p(\Delta_T)})
\]
\[
+ C h^{-k+1+\frac{1}{p}} \sum_{e \in \mathcal{E}_h^p, e \subset \partial \Delta_T} \|v\|_{L^p(e)}.
\]  
(A.18)

Here we applied the Hölder inequality to the last term. Then, together with the boundary-skin estimate (A.1), we obtain
\[
\|\nabla^k (\Pi_h v - v)\|_{L^p(T)} \leq C h^{-k} (\|v - q\|_{L^p(\Delta_T)} + h \|\nabla (v - q)\|_{L^p(\Delta_T)})
\]
\[
+ C h^{-k+2+\frac{1}{p}} \sum_{e \in \mathcal{E}_h^p, e \subset \partial \Delta_T} \|\nabla v\|_{L^p(T_e)}.
\]

Therefore, we obtain (A.9) and (A.10) by the Bramble–Hilbert lemma (cf. [4, Lemma 4.3.8]).

Assume in addition $v \in W^{2,p}(\Omega)^N$. Then, by (A.18) and (A.2), we obtain
\[
\|\nabla^k (\Pi_h v - v)\|_{L^p(T)} \leq C h^{-k} (\|v - q\|_{L^p(\Delta_T)} + h \|\nabla (v - q)\|_{L^p(\Delta_T)})
\]
\[
+ C h^{-k+2} \sum_{e \in \mathcal{E}_h^p, e \subset \partial \Delta_T} \|\nabla v\|_{L^p(T_e)} + C h^{-k+3} \sum_{e \in \mathcal{E}_h^p, e \subset \partial \Delta_T} \|\nabla^2 v\|_{L^p(T_e \cup T_e')},
\]

which implies the desired estimate (A.12) by the Bramble–Hilbert lemma again.

We can now construct the operator $I_h$ satisfying (H2) and (H3) according to the proof of [15, Theorem 2.1].

**Theorem A.1.** Assume that $\mathcal{T}_h$ is shape-regular and quasi-uniform, and each triangle $T \in \mathcal{T}_h$ has at least one interior vertex. Then, there exists an operator $I_h : W^{1,1}_0(\Omega)^N \to V_h$ that satisfies (H2) and (H3) with $\hat{\psi}_h$ defined by
\[
\hat{\psi}_h|_T = \frac{1}{|T|} \int_T \psi_h dx, \quad \text{for all triangles } T \in \mathcal{T}_h.
\]
The proof is almost the same as that of [15, Theorem 2.1]; however, we give the proof for the reader’s convenience.

Proof. Define \( \tau : Q_h \to L^1(\Omega_h) \) by

\[
\tau \psi_h |_T = \psi_h - \frac{1}{|T|} \int_T \psi_h dx, \quad \forall T \in T_h
\]

and set \( \tilde{Q}_h = \tau Q_h \). Moreover, for each \( P_i \in \mathcal{P}_h^d \ (i = 1, 2, \ldots) \), let \( \mathcal{O}_i \) be the union of triangles including \( P_i \). Finally, we set

\[
\tilde{Q}_h(\mathcal{O}_i) := \{(\tau \psi_h)|_{\mathcal{O}_i} \mid \psi_h \in Q_h\},
\]

\[
\tilde{V}_h := \left\{ v_h \in V_h \mid \int_T \text{div} v_h dx = 0, \forall T \in T_h \right\},
\]

\[
\tilde{V}_h(\mathcal{O}_i) := \{ v_h \in \tilde{V}_h \mid \text{supp} v_h \subset \mathcal{O}_i \},
\]

\[
\tilde{V}_{h,\sigma}(\mathcal{O}_i) := \{ v_h \in \tilde{V}_h(\mathcal{O}_i) \mid (\text{div} v_h, \tilde{\psi}_h)_{\mathcal{O}_i} = 0, \forall \tilde{\psi}_h \in \tilde{Q}_h(\mathcal{O}_i) \}.
\]

Then, by [15, Lemma 3.1], the local inf-sup condition

\[
\inf_{\tilde{\psi}_h \in \tilde{Q}_h(\mathcal{O}_i)} \sup_{v_h \in \tilde{V}_h(\mathcal{O}_i)} \frac{(\text{div} v_h, \tilde{\psi}_h)_{\mathcal{O}_i}}{\lVert \nabla v_h \rVert_{L^p(\mathcal{O}_i)} \lVert \tilde{\psi}_h \rVert_{L^p(\mathcal{O}_i)}} \geq C
\]

holds. Note that this statement is independent of the shape of \( \Omega \). Moreover, by the Hölder and the inverse inequalities, one has

\[
\lVert \nabla v_h \rVert_{L^p(\mathcal{O}_i)} \lVert \tilde{\psi}_h \rVert_{L^p(\mathcal{O}_i)} \geq C \lVert \nabla v_h \rVert_{L^p(\mathcal{O}_i)} \lVert \tilde{\psi}_h \rVert_{L^p(\mathcal{O}_i)}
\]

for any \( p \in [1, \infty] \), which implies

\[
\inf_{\tilde{\psi}_h \in \tilde{Q}_h(\mathcal{O}_i)} \sup_{v_h \in \tilde{V}_h(\mathcal{O}_i)} \frac{(\text{div} v_h, \tilde{\psi}_h)_{\mathcal{O}_i}}{\lVert \nabla v_h \rVert_{L^p(\mathcal{O}_i)} \lVert \tilde{\psi}_h \rVert_{L^p(\mathcal{O}_i)}} \geq C.
\] (A.19)

We then define the macro element \( \Delta_i \) by

\[
\Delta_1 = \mathcal{O}_1,
\]

\[
\Delta_i = \bigcup \{ T \in T_h \mid T \subset \mathcal{O}_i, \ T \not\subset \Delta_j, \ j = 1, \ldots, i-1 \} \quad (i \geq 2).
\]

Clearly, the family \( \{ \Delta_i \} \) is disjoint and satisfies

\[
\Omega_h = \bigcup_i \Delta_i, \quad \Delta_i \subset \mathcal{O}_i.
\]

Now, fix \( v \in H^1(\Omega) \) arbitrarily. Then, the operator

\[
\tilde{Q}_h(\mathcal{O}_i) \ni \tilde{\psi}_h \mapsto (\text{div} (v - \Pi_h v), \tilde{\psi}_h)_{\Delta_i}
\]

is a bounded linear functional on \( \tilde{Q}_h(\mathcal{O}_i) \). Therefore, by the inf-sup condition (A.19), there exists \( c_{h,i} v \in \tilde{V}_h(\mathcal{O}_i) \subset W_0^{1,p}(\mathcal{O}_i) \) that satisfies

\[
(\text{div} c_{h,i} v, \tilde{\psi}_h)_{\mathcal{O}_i} = (\text{div} (v - \Pi_h v), \tilde{\psi}_h)_{\Delta_i}
\]

with the estimate

\[
\lVert \nabla c_{h,i} v \rVert_{L^p(\mathcal{O}_i)} \leq C \lVert \text{div} (v - \Pi_h v) \rVert_{L^p(\Delta_i)}.
\] (A.20)
We extend $c_{h,i}$ by zero outside of $O_i$ and thus $c_{h,i} \in V_h$. Then, we define a operator $c_h$ by

$$c_h v = \sum_i c_{h,i} v.$$  

By the definition, $c_h v \in \tilde{V}_h$ for any $v \in H^1(\Omega_h)^N$. We emphasize that $c_h v|_{\partial \Omega_h} = 0$ even when $v|_{\partial \Omega_h} \neq 0$.

Then, for any $\tilde{\psi}_h \in \tilde{Q}_h$, we have

$$(\text{div } c_h v, \tilde{\psi}_h)_h = \sum_i (\text{div } c_{h,i} v, \tilde{\psi}_h)_{O_i}$$

$$= \sum_i (\text{div } (v - \Pi_h v), \tilde{\psi}_h)_{\Delta_i}$$

$$= (\text{div } (v - \Pi_h v), \tilde{\psi}_h)_h.$$  

Moreover, for any $\psi_h \in Q_h$,

$$(\text{div } c_h v, \tau \psi_h)_h = \sum_{T \in \mathcal{T}_h} (\text{div } c_h v, \tau \psi_h)_T$$

$$= \sum_{T \in \mathcal{T}_h} \left[ (\text{div } c_h v, \psi_h)_T - \frac{1}{|T|} \int_T \text{div } c_h v dx \int_T \psi_h dx \right]$$

$$= \sum_{T \in \mathcal{T}_h} (\text{div } c_h v, \psi_h)_T$$

$$= (\text{div } c_h v, \psi_h)_h$$  

since $c_h v \in \tilde{V}_h$, and similarly,

$$(\text{div } (v - \Pi_h v), \tau \psi_h)_h = \sum_T (\text{div } (v - \Pi_h v), \tau \psi_h)_T$$

$$= \sum_T \left[ (\text{div } (v - \Pi_h v), \psi_h)_T - \frac{1}{|T|} \int_T \text{div } (v - \Pi_h v) dx \int_T \psi_h dx \right]$$

$$= \sum_T \left[ (\text{div } (v - \Pi_h v), \psi_h)_T - \frac{1}{|T|} \int_{\partial T \cap \partial \Omega_h} v \cdot n dx \int_T \psi_h dx \right]$$

$$= (\text{div } (v - \Pi_h v), \psi_h)_h - (v \cdot n, \psi_h)_{\partial \Omega_h}$$

due to (A.7). Therefore,

$$(\text{div } c_h v, \psi_h)_h = (\text{div } (v - \Pi_h v), \psi_h)_h - (v \cdot n, \psi_h)_{\partial \Omega_h}, \quad \forall \psi_h \in Q_h$$

holds for all $v \in W^{1,1}(\Omega)$. Thus, if we define $I_h = \Pi_h + c_h$, then this satisfies (H2). Moreover, by the Poincaré inequality on $O_i$ and (A.20), we have

$$\|c_h v\|_{L^p(O_i)} + h\|\nabla c_h v\|_{L^p(O_i)} \leq C h \|\text{div } (v - \Pi_h v)\|_{L^p(D_i)},$$

where

$$D_i = \bigcup_{O_j \cap O_i \neq \emptyset} \Delta_j.$$  

Therefore, we obtain (H3) by (A.10) and (A.12). Hence we succeeded in constructing the desired operator $I_h$. \qed
A.3. MINI element

We next address the MINI element in two dimensional domains. For each triangle $T \in \mathcal{T}_h$, let $\varphi_j \in \mathcal{P}^1(T)$ ($j = 0, 1, 2$) be the nodal basis function, and then define the bubble function $b_T$ by $b_T = \varphi_0 \varphi_1 \varphi_2$ and $B(T_h) = \text{span}\{b_T \mid T \in \mathcal{T}_h\}$. Then, throughout this section, we set $V_h = (\mathcal{P}^1(T_h) \oplus B(T_h))^2 \cap H^1_0(\Omega_h)^2$ and $Q_h = \mathcal{P}^1(T_h)$.

We first construct the quasi-interpolation operator of Scott–Zhang [33]. For each node $P \in V_h$, let $\varphi_P \in \mathcal{P}^1(T_h)$ be the nodal basis function, $\kappa = \kappa(P) \in \mathcal{E}_h$ be defined by (A.5), and let $Q \in V_h$ be the other endpoint of $\kappa$. Then, let $\{\psi_P, \psi_Q\} \in \mathcal{P}^1(\kappa)$ be the dual basis of $\{\varphi_P, \varphi_Q\}$. Namely, define $\psi_P, \psi_Q$ as affine functions satisfying

$$\int_\kappa \psi_a \varphi_b ds = \delta_{a,b}, \quad a, b \in \{P, Q\}.$$ 

We then define interpolation operators of Scott–Zhang $\tilde{I}^\text{SZ}_h, I^\text{SZ}_h : W^{1,1}(\Omega) \to \mathcal{P}^2(T_h)$ by

$$\tilde{I}^\text{SZ}_h v = \sum_{P \in V_h} \left( \int_{\kappa(P)} \psi_P v ds \right) \varphi_P, \quad I^\text{SZ}_h v = \sum_{P \in V_h} \left( \int_{\kappa(P)} \psi_P v ds \right) \varphi_P,$$

and extend $\tilde{I}^\text{SZ}_h v$ and $I^\text{SZ}_h v$ for a vector-valued function $v$ by componentwise operation. It is clear that $I^\text{SZ}_h v \in V_h$ for $v \in W^{1,1}(\Omega)^N$ and

$$I^\text{SZ}_h v|_T = \tilde{I}^\text{SZ}_h v|_T, \quad \forall T \in \mathcal{T}_h^0$$

holds.

The auxiliary operator $I^\text{SZ}$ satisfies the following estimates. The proof is almost the same as in Proposition A.1, and thus we omit the detail.

**Proposition A.2.** Let $p \in [1, \infty]$ and $v \in W^{1,p}_0(\Omega)^N$. Then, the following properties hold for each $T \in \mathcal{T}_h$.

(i) The following stability and error estimates hold:

$$\|\nabla \tilde{I}^\text{SZ}_h v\|_{L^p(T)} \leq C \|\nabla v\|_{L^p(\tilde{\Delta}_T)}, \quad \|I^\text{SZ}_h v - v\|_{L^p(T)} \leq C h \|\nabla v\|_{L^p(\tilde{\Delta}_T)},$$

where $\tilde{\Delta}_T$ is defined by (A.11).

(ii) If in addition $v \in W^{2,p}(\Omega)^N$, then

$$\|I^\text{SZ}_h v - v\|_{L^p(T)} + h \|\nabla (I^\text{SZ}_h v - v)\|_{L^p(T)} \leq C h^2 \left( \|\nabla v\|_{L^p(\tilde{\Delta}_T)} + \|\nabla^2 v\|_{L^p(\tilde{\Delta}_T)} \right).$$

We now construct the desired operator. Let $v \in W^{1,1}_0(\Omega)$. We define $I_h v \in V_h$ by

$$(I_h v)|_T := (I^\text{SZ}_h v)|_T + \alpha_T b_T, \quad \alpha_T := \left( \int_T b_T dx \right)^{-1} \int_T (v - I^\text{SZ}_h v) dx,$$

for each $T \in \mathcal{T}_h$ (cf. [14, Lemma II.4.1]).

Then, $I_h$ satisfies the desired properties.
Theorem A.2. Assume that $T_h$ is shape-regular and quasi-uniform. Then, the operator $I_h: W^{1,p}_0(\Omega) \to V_h$ satisfies (H2) and (H3) with $\hat{\psi}_h = \psi_h$.

Proof. We first show (H2). Let $v \in W^{1,p}_0(\Omega)$. Observe that

$$\int_T I_h v \, dx = \int_T v \, dx$$

for each $T \in T_h$, by the definition of $I_h$. This implies, for each $T \in T_h$ and $\psi_h \in Q_h$,

$$(\text{div } I_h v, \psi_h)_h = -(I_h v, \nabla \psi_h)_h = (v, \nabla \psi_h)_h - (v \cdot n, \psi_h)_{\partial \Omega_h},$$

since $(\nabla \psi_h)|_T$ is constant for each $T \in T_h$. Hence we obtain (H2).

We then show (H3). By mapping a triangle to the reference element, one can see that

$$\int_T b_T dx \geq C h^{-N}, \quad \|\nabla^k b_T\|_{L^p(T)} \leq C h^{-k+\frac{k}{p}}$$

for $k = 0, 1, T \in T_h$, and $p \in [1, \infty]$, which imply

$$\|\nabla^k (ab_T)\|_{L^p(T)} \leq C h^{-k - (1 - \frac{1}{p})N} \|I^{SZ}_h v - v\|_{L^1(T)} \leq C h^{-k} \|I^{SZ}_h v - v\|_{L^p(T)}.$$ 

Therefore, we obtain

$$\|\nabla (I_h v)\|_{L^p(T)} \leq \|\nabla I^{SZ}_h v\|_{L^p(T)} + C h^{-1} \|I^{SZ}_h v - v\|_{L^p(T)},$$

$$\|\nabla^k (I_h v - v)\|_{L^p(T)} \leq \|\nabla^k (I^{SZ}_h v - v)\|_{L^p(T)} + C h^{-k} \|I^{SZ}_h v - v\|_{L^p(T)}, \quad (k = 0, 1),$$

which imply (H3) together with Proposition A.2. Hence we complete the proof. \qed

References

[1] H. Amann. On the strong solvability of the Navier-Stokes equations. J. Math. Fluid Mech., 2(1):16–98, 2000.
[2] N. Y. Bakaev, V. Thomée, and L. B. Wahlbin. Maximum-norm estimates for resolvents of elliptic finite element operators. Math. Comp., 72(244):1597–1610, 2003.
[3] M. E. Bogovskiǐ. Solution of the first boundary value problem for an equation of continuity of an incompressible medium. Dokl. Akad. Nauk SSSR, 248(5):1037–1040, 1979.
[4] S. C. Brenner and L. R. Scott. The mathematical theory of finite element methods. Springer, New York, third edition, 2008.
[5] H. Chen. Pointwise error estimates for finite element solutions of the Stokes problem. SIAM J. Numer. Anal., 44(1):1–28, 2006.
[6] K. Eriksson, C. Johnson, and S. Larsson. Adaptive finite element methods for parabolic problems. VI. Analytic semigroups. SIAM J. Numer. Anal., 35(4):1315–1325, 1998.
[7] H. Fujita and T. Kato. On the Navier-Stokes initial value problem. I. Arch. Rational Mech. Anal., 16:269–315, 1964.
[8] G. P. Galdi. An introduction to the mathematical theory of the Navier-Stokes equations. Steady-state problems. Springer, New York, second edition, 2011.

[9] M. Geissert. Discrete maximal $L_p$ regularity for finite element operators. SIAM J. Numer. Anal., 44(2):677–698, 2006.

[10] M. Geissert. Applications of discrete maximal $L_p$ regularity for finite element operators. Numer. Math., 108(1):121–149, 2007.

[11] M. Geissert, M. Hess, M. Hieber, C. Schwarz, and K. Stavrakidis. Maximal $L_p$-$L_q$-estimates for the Stokes equation: a short proof of Solonnikov’s theorem. J. Math. Fluid Mech., 12(1):47–60, 2010.

[12] Y. Giga. Analyticity of the semigroup generated by the Stokes operator in $L_p$ spaces. Math. Z., 178(3):297–329, 1981.

[13] V. Girault, R. H. Nochetto, and R. Scott. Maximum-norm stability of the finite element Stokes projection. J. Math. Pures Appl. (9), 84(3):279–330, 2005.

[14] V. Girault and P.-A. Raviart. Finite element methods for Navier-Stokes equations. Theory and algorithms, volume 5. Springer-Verlag, Berlin, 1986.

[15] V. Girault and L. R. Scott. A quasi-local interpolation operator preserving the discrete divergence. Calcolo, 40(1):1–19, 2003.

[16] J. Guzmán and D. Leykekhman. Pointwise error estimates of finite element approximations to the Stokes problem on convex polyhedra. Math. Comp., 81(280):1879–1902, 2012.

[17] M. Hieber and J. Saal. The Stokes equation in the $L_p$-setting: well-posedness and regularity properties. In Handbook of mathematical analysis in mechanics of viscous fluids, pages 117–206. Springer, Cham, 2018.

[18] T. Kashiwabara and T. Kemmochi. Pointwise error estimates of linear finite element method for Neumann boundary value problems in a smooth domain. Numer. Math., 144(3):553–584, 2020.

[19] T. Kashiwabara and T. Kemmochi. Stability, analyticity, and maximal regularity for parabolic finite element problems on smooth domains. Math. Comp., 89(324):1647–1679, 2020.

[20] T. Kashiwabara, I. Oikawa, and G. Zhou. Penalty method with P1/P1 finite element approximation for the Stokes equations under the slip boundary condition. Numer. Math., 134(4):705–740, 2016.

[21] T. Kemmochi. On the finite element approximation for non-stationary saddle-point problems. Jpn. J. Ind. Appl. Math., 35(2):423–439, 2018.

[22] T. Kemmochi and N. Saito. Discrete maximal regularity and the finite element method for parabolic equations. Numer. Math., 138(4):905–937, 2018.

[23] B. Li. Analyticity, maximal regularity and maximum-norm stability of semi-discrete finite element solutions of parabolic equations in nonconvex polyhedra. Math. Comp., 88(315):1–44, 2019.
[24] B. Li. A bounded numerical solution with a small mesh size implies existence of a smooth solution to the Navier-Stokes equations. *Numer. Math.*, 147(2):283–304, 2021.

[25] B. Li. Maximal regularity of multistep fully discrete finite element methods for parabolic equations. *IMA J. Numer. Anal.*, 42(2):1700–1734, 2022.

[26] B. Li and W. Sun. Maximal $L^p$ error analysis of FEMS for nonlinear parabolic equations with nonsmooth coefficients. *Int. J. Numer. Anal. Model.*, 14(4-5):670–687, 2017.

[27] M. McCracken. The resolvent problem for the Stokes equations on halfspace in $L^p$. *SIAM J. Math. Anal.*, 12(2):201–228, 1981.

[28] H. Morimoto. Quelques propriétés de la projection de $L_r(\Omega)$ associée à la décomposition de Helmholtz. *J. Fac. Sci. Univ. Tokyo Sect. IA Math.*, 28(1):65–69, 1981.

[29] H. Okamoto. On the semidiscrete finite element approximation for the nonstationary Stokes equation. *J. Fac. Sci. Univ. Tokyo Sect. IA Math.*, 29(1):241–260, 1982.

[30] N. Saito. Error analysis of a conservative finite-element approximation for the Keller-Segel system of chemotaxis. *Commun. Pure Appl. Anal.*, 11(1):339–364, 2012.

[31] A. H. Schatz. Pointwise error estimates and asymptotic error expansion inequalities for the finite element method on irregular grids. I. Global estimates. *Math. Comp.*, 67(223):877–899, 1998.

[32] A. H. Schatz, V. Thomée, and L. B. Wahlbin. Stability, analyticity, and almost best approximation in maximum norm for parabolic finite element equations. *Comm. Pure Appl. Math.*, 51(11-12):1349–1385, 1998.

[33] L. R. Scott and S. Zhang. Finite element interpolation of nonsmooth functions satisfying boundary conditions. *Math. Comp.*, 54(190):483–493, 1990.

[34] V. A. Solonnikov. Estimates for solutions of nonstationary Navier-Stokes equations. *J. Sov. Math.*, 8:467–529, 1977.

[35] V. Thomée and L. B. Wahlbin. Stability and analyticity in maximum-norm for simplicial Lagrange finite element semidiscretizations of parabolic equations with Dirichlet boundary conditions. *Numer. Math.*, 87(2):373–389, 2000.

[36] Vidar Thomée. *Galerkin finite element methods for parabolic problems*. Springer-Verlag, Berlin, second edition, 2006.

[37] T.-P. Tsai. *Lectures on Navier-Stokes equations*. American Mathematical Society, Providence, RI, 2018.