A Parity-Conserving Canonical Quantization for the Baker’s Map

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Abstract

We present here a complete description of the quantization of the baker’s map. The method we use is quite different from that used in Balazs and Voros [BV] and Saraceno [S]. We use as the quantum algebra of observables the operators generated by \{exp (2\pi i\hat{x}), exp (2\pi i\hat{p})\} and construct a unitary propagator such that as \(\hbar \to 0\), the classical dynamics is returned. For Planck’s constant \(\hbar = 1/N\), we show that the dynamics can be reduced to the dynamics on an \(N\)-dimensional Hilbert space, and the unitary \(N \times N\) matrix propagator is the same as given in [BV] except for a small correction of order \(\hbar\). This correction is shown to preserve the symmetry \(x \to 1 - x\) and \(p \to 1 - p\) of the classical map for periodic boundary conditions.
A. The Classical Baker’s Map and its Covering Map

The classical baker’s map is a mapping of the unit torus onto itself defined as follows. Let $x$ and $p$ be the coordinates on the torus and take

$$
(x, p) \rightarrow (x', p') = \begin{cases} 
(2x, p/2), & 0 \leq x < 1/2; \\
(2x - 1, p/2 + 1/2), & 1/2 \leq x < 1.
\end{cases}
$$

(1)

This map describes a stretching in $x$, shrinking in $p$, and chopping and stacking, similar to the way bakers make certain pastries. The motion on the torus is completely chaotic with a positive Liapunov exponent $\log 2$, and in fact is a paradigm for the study of classical chaos. For more details on the classical baker’s map, including a description of the map as a dynamics on binary digits, we refer the reader to [BV].

A quantum version of the map was introduced by Balazs and Voros [BV] and then by Saraceno [S]. In this description, the dynamics is quantized for values of Planck’s constant satisfying $h = 1/N$, by constructing a quantum propagator $u_{mn}$ as a unitary $N \times N$ matrix. The classical limit is demonstrated numerically for $N \rightarrow \infty$. Below we present a quantum propagator which “reduces” to the finite dimensional matrix propagator given in [BV] at the point $\theta = (0, 0)$, corresponding to periodic boundary conditions, except for a correction of order $\hbar$. This correction is what preserves the classical symmetry broken in the quantization scheme presented in [BV]. At other points on the $\theta$-torus, the question of whether the propagator has a finite-dimensional “fixed point” remains open. It is the author’s guess that this is the case, and it remains an interesting and fairly straightforward extension of this work.

Our quantization procedure uses the fact that the baker’s map has a natural lift to the universal covering $\mathbb{R}^2$ of $\mathbb{T}^2$ given by

$$
\beta : (x, p) \rightarrow (x', p') = \begin{cases} 
(2x, p/2), & (x, p) \in l \cap e_p; \\
(2x - 1, p/2 + 1/2), & (x, p) \in r \cap e_p; \\
(2x + 1, p/2 + 1/2), & (x, p) \in l \cap o_p; \\
(2x, p/2), & (x, p) \in r \cap o_p,
\end{cases}
$$

(2)

so that for $a, b \in \mathbb{Z}$,

$$
\beta^* e^{2\pi i(ax + bp)} = e^{4\pi iax} e^{i\pi p} \left( \chi_l(x) + (-1)^b \chi_r(x) \right) \left( \chi_e(p) + (-1)^b \chi_o(p) \right),
$$

(3)
where

\[ l : = \langle [0, 1/2) + \mathbb{Z} \rangle \times \mathbb{R}, \]
\[ r : = \langle [1/2, 1) + \mathbb{Z} \rangle \times \mathbb{R}, \]
\[ e_p : = \mathbb{R} \times \langle [0, 1) + 2\mathbb{Z} \rangle, \]
\[ o_p : = \mathbb{R} \times \langle [1, 2) + 2\mathbb{Z} \rangle. \]

(4)

and

\[
\chi_l(x) = \begin{cases} 
1, & x \in [0, 1/2) + \mathbb{Z} \\
0, & \text{otherwise}
\end{cases}
\]
\[
\chi_r(x) = \begin{cases} 
1, & r \in [1/2, 1) + \mathbb{Z} \\
0, & \text{otherwise}
\end{cases}
\]
\[
\chi_{e_p}(p) = \begin{cases} 
1, & p \in [0, 1) + 2\mathbb{Z} \\
0, & \text{otherwise}
\end{cases}
\]
\[
\chi_{o_p}(p) = \begin{cases} 
1, & p \in [1, 2) + 2\mathbb{Z} \\
0, & \text{otherwise}
\end{cases}
\]

The inverse of this map is as follows:

\[
\beta^{-1} : (x, p) \rightarrow (x', p') = \begin{cases} 
(x/2, 2p), & (x, p) \in e_x \cap b; \\
(x/2 - 1/2, 2p - 1), & (x, p) \in o_x \cap b; \\
(x/2 + 1/2, 2p - 1), & (x, p) \in e_x \cap t; \\
(x/2, 2p), & (x, p) \in o_x \cap t,
\end{cases}
\]

(5)

where we have used the “conjugate” regions

\[
b : = \mathbb{R} \times \langle [0, 1/2) + \mathbb{Z} \rangle, \]
\[
t : = \mathbb{R} \times \langle [1/2, 1) + \mathbb{Z} \rangle, \]
\[
e_x : = \langle [0, 1) + 2\mathbb{Z} \rangle \times \mathbb{R}, \]
\[
o_x : = \langle [1, 2) + 2\mathbb{Z} \rangle \times \mathbb{R}. \]

(6)

Observe that these subsets of \( \mathbb{R}^2 \) satisfy the following relations:

\[
l \cup r = b \cup t = e_p \cup o_p = e_x \cup o_x = \mathbb{R}^2, \]
\[
l \cap r = b \cap t = e_p \cap o_p = e_x \cap o_x = \emptyset.
\]
B. The Quantum Propagator

The outline of our quantization can now be described as follows. The quantum algebra of observables is restricted to the set of operators generated by \( \{ U = \exp (2\pi i\hat{x}), V = \exp (2\pi i\hat{p}) \} \) (the quantization of the classical algebra of periodic functions). We construct a quantum propagator by quantizing the dynamics of the covering map \( \mathbf{2} \). The quantum dynamics induced on the algebra of observables for the quantum torus is the quantum baker’s map. In this sense, this quantization is similar to one given in [DEG].

We now construct the quantum propagator \( F \). The kinematics is already given: the Hilbert space is the usual \( L^2(\mathbb{R}) \). For the dynamics, we work in the Heisenberg picture, and first give the quantum analogs of equations (4) and (6). We define the following projection operators:

\[
L : = \int_{[0,1/2)+\mathbb{Z}} |x\rangle \langle x| \, dx,
\]

\[
R : = \int_{[1/2,1)+\mathbb{Z}} |x\rangle \langle x| \, dx,
\]

\[
B : = \int_{[0,1/2)+\mathbb{Z}} |p\rangle \langle p| \, dp,
\]

\[
T : = \int_{[1/2,1)+\mathbb{Z}} |p\rangle \langle p| \, dp,
\]

and

\[
E_x : = \int_{[0,1)+2\mathbb{Z}} |x\rangle \langle x| \, dx,
\]

\[
O_x : = \int_{[1,2)+2\mathbb{Z}} |x\rangle \langle x| \, dx,
\]

\[
E_p : = \int_{[0,1)+2\mathbb{Z}} |p\rangle \langle p| \, dp,
\]

\[
O_p : = \int_{[1,2)+2\mathbb{Z}} |p\rangle \langle p| \, dp.
\]

Observe that

\[
L + R = B + T = E_x + O_x = E_p + O_p = I,
\]
and
\[ LR = BT = E_x O_x = E_p O_p = 0. \]

We next define appropriate “shift” operators. A shift in \( p \), or a shift in \( x \), by unity is achieved by the following unitary operators, respectively:
\[ X = e^{i\hat{x}/\hbar}, \quad Y = e^{i\hat{p}/\hbar}. \]

Note that \( X \) and \( Y \) commute with the algebra \( \mathfrak{A}_h \) generated by \( U \) and \( V \). We shall also need the following commutation relations:
\[ X L = LX, \quad Y L = LY, \quad X R = RX, \quad Y R = RY, \]  
\[ Y^{1/2}L = RY^{1/2}, \quad X^{1/2}B = TX^{1/2}, \]  
\[ Y E_x = O_x Y, \quad X E_p = O_p X. \]  

We demonstrate one of these commutation relations explicitly. The others involve similar calculations:
\[ Y L = e^{i\hat{p}/\hbar} \int_{[0,1/2)+Z} |x\rangle \langle x| \ dx e^{-i\hat{p}/\hbar} e^{i\hat{p}/\hbar} \]
\[ = \int_{[0,1/2)+Z} |x - 1\rangle \langle x - 1| \ dx e^{i\hat{p}/\hbar} \]
\[ = \int_{[0,1/2)+Z} |x\rangle \langle x| \ dx e^{i\hat{p}/\hbar} = LY. \]

We next find the unitary operator \( S \) which takes \( \hat{x} \) to \( 2\hat{x} \) and \( \hat{p} \) to \( \hat{p}/2 \). We construct this operator by intuition and appealing to the corresponding classical action which takes \( x \to 2x \) and \( p \to p/2 \). Consider the commutator of \( \hat{x} \) with \( (\hat{x}\hat{p} + \hat{p}\hat{x})^n \). We find for any integer \( n \)
\[ \hat{x} (\hat{x}\hat{p} + \hat{p}\hat{x})^n = (\hat{x}\hat{p} + \hat{p}\hat{x} + 2i\hbar)^n \hat{x} \]
thus formally, for any operator expandable as a Taylor series in \( \hat{x}\hat{p} + \hat{p}\hat{x} \), we find
\[ \hat{x} f (\hat{x}\hat{p} + \hat{p}\hat{x}) = f (\hat{x}\hat{p} + \hat{p}\hat{x} + 2i\hbar) \hat{x}. \]

Thus, if we define the operator
\[ S := \exp \left( -\frac{i \log 2}{2\hbar} (\hat{x}\hat{p} + \hat{p}\hat{x}) \right) \]  
(10)
we see formally

\[ \hat{x}S = 2S\hat{x}, \]
\[ \hat{p}S = S\hat{p}/2, \]

or

\[ S^\dagger\hat{x}S = 2\hat{x}, \]
\[ S^\dagger\hat{p}S = \hat{p}/2. \]

We can make this argument rigorous with straightforward continuity argument, which we omit here. Observe also that the operator \( S \) is unitary since \( \hat{x}\hat{p} + \hat{p}\hat{x} \) is Hermitian.

We are now in a position to write down a propagator for the baker’s map. Based on equation 3, we have the following definition.

**Definition 1** *(Baker’s Map Propagator)* Let the operator \( F \) be defined as follows:

\[
F = S(L + X^{-1}R)(E_p + Y^{-1/2}O_p) \quad (11)
\]
\[
= (E_x + X^{-1/2}O_x)(B + Y^{-1}T)S \quad (12)
\]

**Lemma 2** \( F \) is unitary.

**Proof.** Observe that \( E_x = E_x^\dagger, O_x = O_x^\dagger, B = B^\dagger, T = T^\dagger \). It follows that

\[
(L + X^{-1}R)(L + X^{-1}R)^\dagger = L + R = I,
\]
\[
(E_p + Y^{-1/2}O_p)(E_p + Y^{-1/2}O_p)^\dagger = E_p + O_p = I.
\]

Thus \( F \) is the product of three unitary operators, hence is unitary. \( \blacksquare \)

**I. The Classical Limit**

Because of the piecewise continuity of the baker’s map (and its covering map), the classical limit requires more thought. The basic problem comes from the fact that the projection operators \( L \) and \( E_p \) (for example) do not commute as \( \hbar \to 0 \) (even weakly). This fact comes basically from scaling - each term contributes less, but the number of terms increases.
All is not lost, however, as we can take a more constrained view of what constitutes a quantum state with a classical limit (see, for example, [Hep]). We take as our quantum state the coherent state $|\phi; x_0, p_0\rangle$ centered around the point $(x_0, p_0)$ with a width of $\sqrt{\hbar}$. Recall that a coherent state can be written

$$\phi_{x_0, p_0}(x) = \langle x | \phi; x_0, p_0 \rangle = \frac{1}{(\pi \hbar)^{1/4}} e^{-(x-x_0)^2/2\hbar} e^{ip_0 x/\hbar}$$

with a Fourier transform

$$\tilde{\phi}_{x_0, p_0}(p) = \langle p | \phi; x_0, p_0 \rangle = \frac{1}{(\pi \hbar)^{1/4}} e^{-(p-p_0)^2/2\hbar} e^{-ip_0 x_0/\hbar}.$$ 

We are now in a position to define the classical limit.

**Definition 3** We define a quantum propagator $F$ to have a weak classical limit if for any $A \in \mathfrak{X}_0$, and for almost every $x_0, p_0$,

$$\lim_{\hbar \to 0} (\langle \phi_h; x_0, p_0 | F^\dagger T_h(A) F | \phi_h; x_0, p_0 \rangle - \langle \phi_h; \beta(x_0, p_0) | T_h(A) | \phi_h; \beta(x_0, p_0) \rangle) = 0.$$

where $\beta$ is the classical evolution.

In other words, if, as $\hbar \to 0$, all observables have the same values under classical and quantum evolution for almost all wave packets, we say the quantum mechanics yields the classical mechanics. Note also that this differs from the definition given in [Hep] by the use of the “almost all” caveat.

With this definition, we have the following theorem.

**Theorem 4** The propagator $F$ defined in (11) has a weak classical limit in the sense of definition (3).

Proof. We give the proof for a harmonic $A = U^a V^b$. The general case will follow by linearity and continuity. We divide the proof into steps.

Step 1. We first calculate the expectation value of the operator $U^a V^b$ in the coherent states. We see

$$\langle \phi_h; x_0, p_0 | U^a V^b | \phi_h; x_0, p_0 \rangle$$

$$= \frac{1}{(\pi \hbar)^{1/2}} \int dx dx' e^{-(x-x_0)^2/2\hbar} e^{-ip_0 x/\hbar} e^{2\pi i ax} \delta(x-x') - 2\pi bh) e^{-(x'-x_0)^2/2\hbar} e^{ip_0 x'/\hbar}$$

$$= e^{2\pi ibp_0} e^{2\pi i ax_0} e^{-2\pi \hbar} e^{-4\pi^2 \hbar (b+ia)^2} \to e^{2\pi ibp_0} e^{2\pi i ax_0} \text{ as } \hbar \to 0.$$
Step 2. Observe that acting on these states, we see
\[ \| L |\phi_\hbar; x_0, p_0 \rangle \|^2 = \langle \phi_\hbar; x_0, p_0 | L |\phi_\hbar; x_0, p_0 \rangle \]
\[ = \frac{1}{(\pi\hbar)^{1/2}} \sum_{k \in \mathbb{Z}} \int_0^{1/2} e^{-(x+k-x_0)^2/\hbar} dx. \]

Suppose \( x_0 \neq l/2 \) for any \( l \in \mathbb{Z} \). Now choose \( \epsilon > 0 \) and \( l \in \mathbb{Z} \) such that \( x_0 \in [l/2 + \epsilon, (l + 1)/2 - \epsilon) \) for \( l \in \mathbb{Z} \) and \( \epsilon > 0 \).

For the case of \( l \) odd, we see that the value of the integral is bounded by
\[ \left| \frac{1}{(\pi\hbar)^{1/2}} \sum_{k \in \mathbb{Z}} \int_0^{1/2} e^{-(x+k-x_0)^2/\hbar} dx \right| \leq \frac{2}{\sqrt{\pi}} \int_{\epsilon/\sqrt{\hbar}}^{\infty} e^{-x^2} dx. \]

A bound on this integral can easily be given for \( \epsilon/\sqrt{\hbar} > 1 \). We see that
\[ \int_{\epsilon/\sqrt{\hbar}}^{\infty} e^{-x^2} dx \leq \int_{\epsilon/\sqrt{\hbar}}^{\infty} xe^{-x^2} dx = \frac{1}{2} \int_{\epsilon/\sqrt{\hbar}}^{\infty} e^{-u} du = e^{-\epsilon^2/\hbar}/2. \]

Thus we see that for \( l \) odd, the limit of the integral is zero as \( \hbar \to 0 \).

Now consider \( x_0 \in [l/2 + \epsilon, (l + 1)/2 - \epsilon) \) with \( l \) even. Then we see that
\[ \| L |\phi_\hbar; x_0, p_0 \rangle \|^2 = \| (I - R) |\phi_\hbar; x_0, p_0 \rangle \|^2 \]
\[ = 1 - \| R |\phi_\hbar; x_0, p_0 \rangle \|^2 \]
\[ \to 1 \quad \text{as} \quad \hbar \to 0. \]

Similar results hold for all the projection operators \( L, R, B, T, E_x, O_x, E_p, O_p \) defined in equations 7 and 8. We let \( \tilde{l}, \tilde{r}, \tilde{b}, \tilde{t}, \tilde{e}_x, \tilde{o}_x, \tilde{e}_p, \tilde{o}_p \) denote the interior of the regions given in 4 and 6, that is the regions with the boundaries removed. (For instance, \( \tilde{l} \) does not contain \( x = 0 \) or \( x = 1/2 \).) Note that \( (\tilde{l} \cap \tilde{o}_p) \cup (\tilde{r} \cap \tilde{o}_p) \cup (\tilde{l} \cap \tilde{o}_p) \cup (\tilde{r} \cap \tilde{o}_p) \) is dense in \( \mathbb{R}^2 \).
We can summarize these results in the following table.

| Operator (\( \mathcal{O} \)) | \((x_0, p_0) \in \tilde{\mathcal{r}} \cap \tilde{\mathcal{o}}_p \) | \( \lim_{\hbar \to 0} \| \mathcal{O} | \phi_h; x_0, p_0 \rangle \| \) |
|-------------------------------|---------------------------------|---------------------------------|
| \( E_x \)                   | \( \vec{e}_x \)              | 1                              |
| \( \vec{e}_x \)             | \( \vec{e}_x \)              | 0                              |
| \( O_x \)                   | \( \vec{o}_x \)              | 0                              |
| \( \vec{o}_x \)             | \( \vec{o}_x \)              | 1                              |
| \( E_p \)                   | \( \vec{e}_p \)              | 1                              |
| \( \vec{e}_p \)             | \( \vec{e}_p \)              | 0                              |
| \( O_p \)                   | \( \vec{o}_p \)              | 0                              |
| \( \vec{o}_p \)             | \( \vec{o}_p \)              | 1                              |
| \( L \)                     | \( \tilde{l} \)              | 1                              |
| \( \tilde{l} \)             | \( \tilde{l} \)              | 0                              |
| \( R \)                     | \( \tilde{r} \)              | 1                              |
| \( \tilde{r} \)             | \( \tilde{r} \)              | 0                              |
| \( B \)                     | \( \tilde{b} \)              | 1                              |
| \( \tilde{b} \)             | \( \tilde{b} \)              | 0                              |
| \( T \)                     | \( \tilde{t} \)              | 1                              |
| \( \tilde{t} \)             | \( \tilde{t} \)              | 0                              |

Step 3. Now suppose \((x_0, p_0) \in \tilde{\mathcal{r}} \cap \tilde{\mathcal{o}}_p \). Then consider the quantum evolution. We see

\[
\langle \phi_h; x_0, p_0 | \left( E_p + Y^{1/2}O_p \right) (L + XR) S^1 U^a V^b S (L + X^{-1}R) \left( E_p + Y^{-1/2}O_p \right) | \phi_h; x_0, p_0 \rangle
\]

\[
= \langle \phi_h; x_0, p_0 | \left( E_p + Y^{1/2}O_p \right) (L + XR) U^{2a} V^{b/2} (L + X^{-1}R) \left( E_p + Y^{-1/2}O_p \right) | \phi_h; x_0, p_0 \rangle
\]

Multiplying out, we see 16 terms in the expansion. Consider one of these terms. We see from the chart that

\[
\| \| \langle \phi_h; x_0, p_0 | E_p X R U^{2a+N} V^{b-N/2} L^{-1/2} O_p | \phi_h; x_0, p_0 \rangle \|
\]

\[
= \| \| \langle \phi_h; x_0, p_0 | E_p R U^{2a+N} V^{b-N/2} L^{-1/2} O_p | \phi_h; x_0, p_0 \rangle \|
\]

\[
\leq \| \| \langle \phi_h; x_0, p_0 | E_p | \phi_h; x_0, p_0 \rangle \| \|
\]

\[
\times \| \| \langle \phi_h; x_0, p_0 | \left( R U^{2a+N} V^{b-N/2} L^{-1/2} O_p \right) \dagger \left( R U^{2a+N} V^{b-N/2} L^{-1/2} O_p \right) | \phi_h; x_0, p_0 \rangle \|
\]

\[
\to 0 \quad \text{as} \quad \hbar \to 0.
\]

Similarly, 15 of the terms vanish as \( \hbar \to 0 \). The only surviving term for \((x_0, p_0) \in \tilde{\mathcal{r}} \cap \tilde{\mathcal{o}}_p \) is

\[
\langle \phi_h; x_0, p_0 | O_p Y^{1/2} L U^{2a} V^{b/2} L Y^{-1/2} O_p | \phi_h; x_0, p_0 \rangle
\]
Step 4. Now consider the classical evolution. We have, for \((x_0, p_0) \in \tilde{r} \cap \tilde{o}_p\)
\[
\langle \phi; \beta \left(x_0, p_0\right) | U^a V^b | \phi; \beta \left(x_0, p_0\right) \rangle = \langle \phi; 2x_0, p_0/2 | U^a V^b | \phi; 2x_0, p_0/2 \rangle \rightarrow e^{2\pi i(2ax_0+(b/2)p_0)}.
\]

The calculations for the other three regions are similar, and we omit the details here. This concludes the proof. ■

II. Planck’s Constant = \(1/N\)

A remarkable set of properties can be associated with quantum dynamics on a torus if we let Planck’s constant satisfy the integrality condition
\[
h = 1/N.
\]

This fact is evidenced by the quantization schemes presented in [BV] and [S] for the baker’s map, and [BV] for the cat maps. In [KLMR] and [LRS], an explicit construction similar to what we presented in the previous section was given for the cat map, kick maps, and Harper maps. The Hilbert space is taken to be the standard \(L^2(\mathbb{R})\). The quantum torus is defined as the algebra of observables, or operators on \(L^2(\mathbb{R})\), generated by \(\{\exp(2\pi i\hat{x}), \exp(2\pi i\hat{p})\}\).

In [KLMR] a propagator is found which yields the classical dynamics as \(\hbar \rightarrow 0\) for the cat map. This quantization scheme was called “the quantum cat map.” It is valid for all \(\hbar\), but was shown to reduce to the quantizations given in [BV] for \(h = 1/N\). Here we present a similar result for the baker’s map. We shall find for the finite-dimensional matrix propagator for \(N\) even
\[
\left( \Phi_n^{(0,0)} , F\Phi_m^{(0,0)} \right)_P = \begin{cases} 
\sum_{a=0}^{N/2-1} \left( F^N \right)_{na}^{-1} \left( \begin{array}{cc} F^{N/2}^{-1} & 0 \\ 0 & F^{N/2}^{-1} \end{array} \right) & n \text{ even} \\
\sum_{a=0}^{N/2-1} \left( F^N \right)_{na}^{-1} \left( \begin{array}{cc} e^{i\pi(n-2m)/N} F^{N/2}^{-1} & 0 \\ 0 & e^{i\pi(n-(2m-N))/N} F^{N/2}^{-1} \end{array} \right) & n \text{ odd}. 
\end{cases}
\]

where \(F^N\) is the matrix for the \(N\)-dimensional discrete Fourier transform, and \(\Phi_n^{(0,0)}\) is a basis vector of the Hilbert space \(\mathcal{H}_\hbar(0) \cong \mathbb{C}^N\) defined as the periodic \(\delta\)-comb \([18]\) and \((\cdot, \cdot)\) is the inner product defined in equation \([20]\).
This propagator is shown here to preserve the symmetry $x \rightarrow 1 - x$ and $p \rightarrow 1 - p$, which is not preserved in the original quantization given by Balazs and Voros [BV]. (In Saraceno [S], an anti-periodic quantization is formulated which does preserve this symmetry.) Here we see that with a small ($O(\hbar)$) correction to the Balazs-Voros matrices, the periodic quantization can also be made to preserve the classical symmetries. Note that along the classical trajectories $n = 2m$ or $n = 2m - N$, the extra phase we obtain using this quantization vanishes.

III. The $\theta$-torus

We let $U = \exp(2\pi i\hat{x})$ and $V = \exp(2\pi i\hat{p})$ be operators on an infinite dimensional Hilbert space (Bargmann space $\mathcal{H}^2(\mathbb{C}, d\mu_\hbar)$), with $[\hat{x}, \hat{p}] = i\hbar$. Observe that for $h = 1/N$ the algebra has a natural center generated by

$$
X = U^N, \\
Y = V^N.
$$

That is

$$
[X, Y] = [X, U] = [X, V] = [Y, U] = [Y, V] = 0.
$$

In [KLMR], this insight was used to show that Bargmann space (the Hilbert space of entire functions on the plane) can be decomposed via the following eigenvalue problem:

$$
X\phi(z) = e^{2\pi i\theta_1} \phi(z), \\
Y\phi(z) = e^{2\pi i\theta_2} \phi(z),
$$

where $\theta = (\theta_1, \theta_2) \in \mathbb{T}^2$. As in [KLMR], let $\mathcal{H}_h(\theta)$ denote the space of (non-normalizable) independent eigenvectors with fixed $\theta$. The space $\mathcal{H}_h(\theta)$ was shown to have a natural inner product defined as an integral over the fundamental domain $D = [0, 1] \times [0, 1] \subset \mathbb{C}$ of Bargmann space given by

$$
(\phi_1, \phi_2)_P = \int_D \overline{\phi_1(z)}\phi_2(z) d\mu_h(z),
$$

where $d\mu_h(z) = (\pi\hbar)^{-1} \exp(-|z|^2/\hbar) d^2z$. An explicit isomorphism $\kappa : \mathcal{H}^2(\mathbb{C}, d\mu_\hbar) \rightarrow \int_{T^2} \mathcal{H}_h(\theta) d\theta$ was also derived between $\mathcal{H}^2(\mathbb{C}, d\mu_\hbar)$ and the
direct integral of the spaces $\mathcal{H}_h(\theta)$. The isomorphism was shown to be inner product preserving,

$$ (\psi_1, \psi_2)_{\mathcal{H}_2} = \int_{T^2} (\kappa \psi_1(\theta), \kappa \psi_2(\theta))_P d^2\theta , $$

where $\psi_i = \int_{T^2} \kappa \psi_i(\theta) d\theta$. The following lemma was proved:

**Lemma 5** (i) The following functions are elements of $\mathcal{H}_h(\theta)$ of unit norm:

$$ \phi_m^{(\theta)}(z) = C_m(\theta) e^{-N\pi z^2 + 2\sqrt{2\pi}(\theta_1 + m)z} \sum_{k \in \mathbb{Z}} e^{-N\pi k^2 - 2\pi(\theta_1 + i\theta_2 + m)k + 2\sqrt{2\pi}kz} , \quad (16) $$

where

$$ C_m(\theta) := (2/N)^{1/4} e^{-\pi(\theta_1 + m)^2/N - 2\pi i\theta_2 m/N} . $$

They are periodic in $m$,

$$ \phi_{m+N} = \phi_m^{(\theta)} , \quad (17) $$

and furthermore,

$$ \phi_0^{(\theta)}, \ldots, \phi_{N-1}^{(\theta)} $$

are orthogonal vectors in $\mathcal{H}_h(\theta)$.

(ii) The space $\mathcal{H}_h(\theta)$ has dimension $N$. Consequently the functions $\Phi_m^{(\theta)}$ form an orthonormal basis for $\mathcal{H}_h(\theta)$.

Composing the Bargmann transformation $\mathcal{H}^2(\mathbb{C}, d\mu_h) \to L^2(\mathbb{R}, dx)$ with this isomorphism, we can construct the transformation between $L^2(\mathbb{R}, dx)$ and $\int_{T^2} \mathcal{H}_h(\theta) d\theta$. Applying the Bargmann transformation to the basis functions $\phi_m^{(\theta)} \in \mathcal{H}_h(\theta)$, we find

$$ \Phi_m^{(\theta)}(x) = B^{-1} \phi_m^{(\theta)}(x) = \frac{e^{2\pi i\theta_2 m/N}}{N^{1/2}} \sum_{k \in \mathbb{Z}} e^{2\pi i\theta_2 k} \delta \left( x - \frac{m + \theta_1 + NK}{N} \right) . \quad (18) $$

This is the $\delta$-comb wavefunctions described informally in the physics literature (see, for example [HB]). We use the following notation for these vectors:

$$ \Phi_m^{(\theta)} = \frac{e^{2\pi i\theta_2 m/N}}{N^{1/2}} \sum_{k \in \mathbb{Z}} e^{2\pi i\theta_2 k} \left\langle \frac{\theta_1 + m}{N} + k \right\rangle_x . $$

We can, of course, just as easily work in momentum representation. In fact, for $h = 1/N$, a rather interesting calculational identity can be found.
Lemma 6 For $h = 1/N$,
\[
\Phi^{(\theta)}_m = e^{-2\pi i \theta_1/N} \sum_{n=0}^{N-1} \mathcal{F}^N_{mn} \tilde{\Phi}^{(\theta)}_n,
\]
where \( \{ \tilde{\Phi}^{(\theta)}_n \}_{0 \leq n \leq N-1} \) are the momentum-state wave functions on the torus,
\[
\tilde{\Phi}^{(\theta)}_n = e^{-2\pi i n \theta_1/N} \sqrt{N} \sum_k e^{-2\pi i \theta_1 k} \left| \frac{\theta_2 + n}{N} + k \right>_p,
\]
and \( \mathcal{F}^N_{mn} \) is the matrix for the discrete Fourier transform,
\[
\mathcal{F}^N_{mn} = \frac{e^{-2\pi i mn/N}}{\sqrt{N}}.
\]

Remark 1 We see in particular that for the subsets $\theta_1 = 0$ or $\theta_2 = 0$, changing coordinates from momentum representation to position representation is simply a discrete Fourier transform.

Proof. The proof is a direct calculation. We have
\[
\int_{\mathbb{R}} |p\rangle \langle p| \Phi^{(\theta)}_m dp = e^{2\pi i \theta_2 m/N} \sum_{k \in \mathbb{Z}} e^{2\pi i \theta_2 k} \int_{\mathbb{R}} |p\rangle \langle p| \frac{m + \theta_1}{N} + k\langle p| dp
\]
\[
= e^{2\pi i \theta_2 m/N} \sum_{k \in \mathbb{Z}} \left| \frac{\theta_2 + k}{N} \right>_p \exp \left\{ -2\pi i \left( \frac{\theta_2 + k}{N} \right) (m + \theta_1) \right\}.
\]
We now let $k \to n + kN$, with $n \in \{0, \ldots, N-1\}$, and $k \in \mathbb{Z}$, to find
\[
\int_{\mathbb{R}} |p\rangle \langle p| \Phi^{(\theta)}_m dp = e^{-2\pi i \theta_1 \theta_2/N} \sum_n e^{-2\pi i mn/N} e^{-2\pi i \theta_1/N} \sum_k e^{-2\pi i \theta_1 k} \left| \frac{\theta_2 + n}{N} + k \right>_p
\]
\[
= e^{-2\pi i \theta_1 \theta_2/N} \sum_n \mathcal{F}^N_{mn} \tilde{\Phi}^{(\theta)}_n,
\]
as claimed. ■

Analogous to (15), we can also find an explicit expression for the inner product over the $N$-dimensional Hilbert space at each point on the $\theta$-torus as an integral over the fundamental domain $[0, 1]$ of the real line.
The inner product defined in 15 can be written as

\[(\Psi_1(\theta), \Psi_2(\theta))_p = \int_0^1 \overline{\Psi_1(x, \theta)}(K\Psi_2)(x, \theta)dx,\]  

(20)

where

\[K\Psi_2(x, \theta) = \int_{-\infty}^{\infty} K(x, y)\Psi_2(y, \theta)dy,\]

\[K(x, y) = \frac{1}{2\pi\hbar}g\left(\frac{x - y}{2\hbar}\right)\]

and

\[g(r) = \frac{\sin r}{r} e^{-br^2 + i\theta}.\]

We see this via a direct calculation:

\[\int_D \overline{\psi_1(z, \theta)}\psi_2(z, \theta)d\mu_{\hbar}(z) = \int_D \overline{B\Psi_1(z, \theta)}\Psi_2(z, \theta)d\mu_{\hbar}(z) = \]

\[= \frac{1}{2(\pi\hbar)^{3/2}} \sum_{k \in \mathbb{Z}} \int_0^1 dx \int_{-\infty}^{\infty} dy \overline{\Psi_1(x + k, \theta)}\Psi_2(y, \theta)e^{-(x+k)^2+y^2)/2\hbar} \]

\[\times \int_D e^{-(u^2-(x+k)(u-i\theta)+y(u+i\theta))/\hbar}dudv.\]

We next use the fact that both \(\Psi_1\) and \(\Psi_2\) satisfy \(X\Psi_i = e^{2\pi i\theta_1}\Psi_i\) and \(Y\Psi_i = e^{2\pi i\theta_2}\Psi_i\). Substituting in, we find

\[\frac{1}{2(\pi\hbar)^{3/2}} \sum_{k \in \mathbb{Z}} \int_0^1 dx \int_{-\infty}^{\infty} dy e^{-2\pi i k \theta_2}\overline{\Psi_1(x, \theta)}\Psi_2(y, \theta)e^{-(x+k)^2+y^2)/2\hbar} \]

\[\times \int_D e^{-(u^2-(x+k)(u-i\theta)+y(u+i\theta))/\hbar}dudv \]

\[= \frac{1}{2(\pi\hbar)^{3/2}} \sum_{k \in \mathbb{Z}} \int_0^1 dx \int_{-\infty}^{\infty} dy \Psi_1(x, \theta) \sum_{k \in \mathbb{Z}} \int_{-\infty}^{\infty} dy \Psi_2(y - k, \theta)e^{-((x+k)^2+y^2)/2\hbar} \]

\[\times \int_D e^{-(u^2-(x+k)(u-i\theta)+y(u+i\theta))/\hbar}dudv \]

\[= \frac{1}{2(\pi\hbar)^{3/2}} \int_0^1 dx \overline{\Psi_1(x, \theta)} \int_{-\infty}^{\infty} dy \Psi_2(y, \theta) \sin \left(\frac{x-y}{2\hbar}\right) \frac{e^{-(x+y^2+i(x-y))/2\hbar}}{\left(\frac{x-y}{2\hbar}\right)^3} \]
\[
\times \int_{-\infty}^{\infty} e^{-((u-k)^2-(x+y)(u-k))/\hbar} du
= \int_{0}^{1} \Psi_1(x, \theta)(K\Psi_2)(x, \theta) dx.
\]

We see that the kernel \( K(x, y) \) is a type of quantum diffraction in keeping with the uncertainty principle. In fact it can be shown that as \( \hbar \to 0 \), \( K(x, y) \to \delta(x - y) \). Observe also that with respect to this inner product, the basis elements \( \{ \Phi^{(\theta)}_m \} \) are orthonormal: \( (\Phi^{(\theta)}_m, \Phi^{(\theta)}_n)_P = \delta_{mn} \).

IV. Dynamics at \( \theta = (0, 0) \)

The point \( \theta = (0, 0) \) of the \( \theta \)-torus corresponds to the \( N \)-dimensional vector space \( \mathcal{H}_\hbar(0) \) of periodic \( \delta \)-combs. In fact, for the quantum baker’s map, we now show that \( \theta = (0, 0) \) is a fixed point of the dynamics on the \( \theta \)-torus for \( N \) even. That is, the set of periodic \( \delta \)-combs is mapped onto itself by our propagator \( F \). We have

\[
 XF\Phi^{(0,0)}_m = FX^2\Phi^{(0,0)}_m = F\Phi^{(0,0)}_m
\]

and also

\[
 YF\phi^{(0,0)}_m = (O_x + X^{-1/2}E_x) (B + Y^{-1}T) SY^{1/2}\Phi^{(0,0)}_m
= (E_x + X^{1/2}O_x) (B + Y T) SXY\Phi^{(0,0)}_m
= (E_x + X^{-1/2}O_x) (B + Y^{-1}T) SXY\Phi^{(0,0)}_m
= F\Phi^{(0,0)}_m.
\]

Furthermore, at \( \theta = (0, 0) \) the observables corresponding to the algebra \( \mathfrak{A}_\hbar \) generated by \( \exp(2\pi i \hat{x}) \) and \( \exp(2\pi i \hat{p}) \) characterized by the equations \([X, A] = [Y, A] = 0 \) for \( A \in \mathfrak{A}_\hbar \) is also preserved by the quantum dynamics. That is, we have the following lemma.

**Lemma 7** For \( N \) even, and \( A \in \mathfrak{A}_\hbar \).

\[
[X, F^\dagger AF] \Phi^{(0,0)}_m = [Y, F^\dagger AF] \Phi^{(0,0)}_m = 0.
\]
Remark 2 Observe that we can write any $A$ as $\sum_{j,k} \gamma_{jk} U^j V^k$. Letting $j = a + Nc$ and $k = b + Nd$ with $0 \leq a, b \leq N - 1$ and $c, d \in \mathbb{Z}$, we see from equation [14] that

$$U^{a+Nc} V^{b+Nd} \Phi_m^{(0,0)} = U^a V^b X^c Y^d \Phi_m^{(0,0)} = U^a V^b \Phi_m^{(0,0)}.$$

Thus, acting on the subspace $\mathcal{H}(0)$, the algebra $\mathfrak{A}_\hbar$ is reduced to a set of $N^2$ operators. This is isomorphic to the algebra of $N \times N$ matrices.

Proof. The proof is a straightforward calculation using the commutation relations [3]. We provide here the case of a pure harmonic $U^m V^n$. The general case follows immediately by linearity and continuity.

$$XF^\dagger A F \Phi_m^{(0,0)}$$

$$= X S^\dagger (B + Y T) (E_x + X^{1/2} O_x) U^m V^n (E_x + X^{-1/2} O_x) (B + Y^{-1} T) S \Phi_m^{(0,0)}$$

$$= S^\dagger (-1)^n (T + Y B) (E_x + X^{1/2} O_x) U^m V^n (E_x + X^{-1/2} O_x) (T + Y^{-1} B) S \Phi_m^{(0,0)}$$

$$= S^\dagger (B + Y T) (E_x + X^{1/2} O_x) U^m V^n (E_x + X^{-1/2} O_x) (B + Y^{-1} T) S \Phi_m^{(0,0)}$$

$$= F^\dagger A F \Phi_m^{(0,0)}.$$ 

Likewise,

$$Y F^\dagger A F \Phi_m^{(0,0)} = F^\dagger A F \Phi_m^{(0,0)}.$$ 

We can now determine the matrix elements for the dynamics at this fixed point. This result should be compared to the baker's map quantum propagator given in [BV].

Theorem 8 The matrix elements for the propagator $F$ on the subspace $\mathcal{H}(0)$ are given by

$$\left( \Phi_n^{(0,0)}, F \Phi_m^{(0,0)} \right)_P$$

$$= \begin{cases} 
\sum_{a=0}^{N-1} \left( \mathcal{F}^N \right)^{-1}_{na} \begin{pmatrix} \mathcal{F}^{N/2} & 0 \\
0 & \mathcal{F}^{N/2} \end{pmatrix} & n \text{ even} \\
\sum_{a=0}^{N-1} \left( \mathcal{F}^N \right)^{-1}_{na} \begin{pmatrix} 0 \\
\mathcal{F}^{N/2} & \mathcal{F}^{N/2} \end{pmatrix} & n \text{ odd} 
\end{cases} \quad (21)$$

$$= \begin{cases} 
\sum_{a=0}^{N-1} \left( \mathcal{F}^N \right)^{-1}_{na} \begin{pmatrix} e^{i\pi (n-2m)/N} \mathcal{F}^{N/2} & 0 \\
0 & e^{i\pi (n-(2m-N))/N} \mathcal{F}^{N/2} \end{pmatrix} & n \text{ odd} 
\end{cases} \quad (22)$$

16
Proof. We divide this calculation into different cases. For $0 \leq m < N/2$, we have

$$F \Phi_m^{(0,0)} = (B + Y^{-1}T) \left( E_x + X^{-1/2}O_x \right) S \Phi_m^{(0,0)}$$

$$= (B + Y^{-1}T) \left( E_x + X^{-1/2}O_x \right) \frac{1}{\sqrt{N}} \sum_{k \in \mathbb{Z}} \left| \frac{m}{N} + k \right>$$

$$= (B + Y^{-1}T) \left( E_x + X^{-1/2}O_x \right) \sqrt{\frac{2}{N}} \sum_{k \in \mathbb{Z}} \left| \frac{2m}{N} + 2k \right>$$

$$= \frac{1}{\sqrt{2}} \left( B + Y^{-1}T \right) \left( E_x + X^{-1/2}O_x \right) \sqrt{2} \left( e^{-2\pi im/N} \Phi_m^{(0,1/2)} \right)$$

Now, observe that

$$B \Phi_m^{(0,1/2)} = \frac{e^{i\pi m/N}}{\sqrt{N}} \sum_{k \in \mathbb{Z}} (-1)^k \int_{[0,1/2]+\mathbb{Z}} |p\rangle \langle p| \frac{m}{N} + k \rangle_x \ dp$$

$$= e^{i\pi m/N} \int_{[0,1/2]+\mathbb{Z}} |p\rangle \left( \sum_{k \in \mathbb{Z}} e^{2\pi ik(1/2-Np)} \right) e^{-2\pi ipm} \ dp$$

$$= \frac{e^{i\pi m/N}}{\sqrt{N}} \sum_{a=0}^{N/2-1} e^{-2\pi i(a+1/2)m/N} \frac{1}{\sqrt{N}} \sum_{k \in \mathbb{Z}} \frac{a + 1/2}{N} + k \rangle_p$$

$$= \frac{1}{\sqrt{N}} \sum_{a=0}^{N/2-1} e^{-2\pi i am/N} \tilde{\Phi}_a^{(0,1/2)},$$
where we have used the “p-state” δ-comb given in (19). Thus, we see that

\[(B - T) \Phi^{(0,1/2)}_m = \frac{1}{\sqrt{N}} \left( \sum_{a=0}^{N/2-1} e^{-2\pi i a m/N} - \sum_{a=N/2}^{N-1} e^{-2\pi i a m/N} \right) \sum_{b=0}^{N-1} (\mathcal{F}^{-1})_{ab} \Phi^{(0,1/2)}_b,\]

and

\[
F \Phi^{(0,0)}_m = \frac{1}{\sqrt{2}} \Phi^{(0,0)}_{2m} + \frac{e^{-2\pi i m/N}}{\sqrt{2N}} \left( \sum_{a=0}^{N/2-1} e^{-2\pi i a(2m)/N} - \sum_{a=N/2}^{N-1} e^{-2\pi i a(2m)/N} \right) \times \sum_{b=0}^{N-1} (\mathcal{F}^{-1})_{ab} \left( E_x + (-1)^b O_x \right) \Phi^{(0,1/2)}_b.
\]

Now, observe

\[
E_x \Phi^{(0,0)}_m = \frac{1}{\sqrt{N}} \sum_{k \in \mathbb{Z}} \chi_{E_x} \left( \frac{m}{N} + k \right) \left| \frac{m}{N} + k \right>_x
\]

So for \(m \in [0, N-1] + 2N\mathbb{Z}\), this yields

\[
E_x \Phi^{(0,0)}_m = \frac{1}{\sqrt{N}} \sum_{k \in \text{even}} \left| \frac{m}{N} + k \right>_x = \frac{1}{\sqrt{N}} \sum_{k \in \mathbb{Z}} \left| \frac{m}{N} + 2k \right>_x
\]

\[
= \frac{\Phi^{(0,0)}_m + e^{-i\pi m/N} \Phi^{(0,1/2)}_m}{2}.
\]

For \(m \in [N, 2N-1] + 2N\mathbb{Z}\), however, we find

\[
E_x \Phi^{(0,0)}_m = \frac{\Phi^{(0,0)}_m - e^{-i\pi m/N} \Phi^{(0,1/2)}_m}{2}.
\]

We let \([m/N]\) be the integer part of \(m/N\), and observe that

\[
E_x \Phi^{(0,0)}_m = \frac{1}{2} \left( \Phi^{(0,0)}_m + e^{-i\pi(m/N-[m/N])} \Phi^{(0,1/2)}_m \right),
\]

\[
O_x \Phi^{(0,0)}_m = \frac{1}{2} \left( \Phi^{(0,0)}_m - e^{-i\pi(m/N-[m/N])} \Phi^{(0,1/2)}_m \right).
\]
and

\[(E_x - O_x) \Phi^{(0,0)}_m = e^{-i\pi(m/N - [m/N])} \Phi^{(0,1/2)}_m.\]

Having checked that this is consistent with \(\Phi^{(\theta)}_{m+N} = \Phi^{(\theta)}_m\), we now restrict ourselves to the original basis vectors, \(m \in [0, N - 1]\).

From the identity \((E_x - O_x)^2 = I\), it follows immediately that

\[(E_x - O_x) \Phi^{(0,1/2)}_m = e^{i\pi m/N} \Phi^{(0,0)}_m.\]

Thus,

\[
F \Phi^{(0,0)}_m = \frac{1}{\sqrt{2}} \Phi^{(0,0)}_{2m} + e^{-2\pi im/N} \sqrt{2/N} \left( \sum_{a=0}^{N/2-1} e^{-2\pi i a(2m)/N} - \sum_{a=N/2}^{N-1} e^{-2\pi i a(2m)/N} \right) \sum_{b \text{ even}}^{N-1} (F^{-1})_{ab} \Phi^{(0,1/2)}_b + e^{-2\pi im/N} \sqrt{2/N} \left( \sum_{a=0}^{N/2-1} e^{-2\pi i a(2m)/N} - \sum_{a=N/2}^{N-1} e^{-2\pi i a(2m)/N} \right) \sum_{b \text{ odd}}^{N-1} e^{i\pi b/N} (F^{-1})_{ab} \Phi^{(0,0)}_b.
\]

Consider just the middle term. Since we have already shown that \(\theta = (0, 0)\) is a fixed point of the dynamics, we should see this term exactly vanishing.

In fact a direct calculation readily shows this. For \(b\) even

\[
e^{-2\pi im/N} \sqrt{2/N} \left( \sum_{a=0}^{N/2-1} e^{-2\pi i a(2m)/N} e^{-i\pi ab/N} - \sum_{a=0}^{N/2-1} e^{-2\pi i(a+N/2)(2m)/N} e^{i\pi (a+N/2)b/N} \right) = 0.
\]

Thus,

\[
F \Phi^{(0,0)}_m = \frac{1}{\sqrt{2}} \Phi^{(0,0)}_{2m} + e^{-2\pi im/N} \sqrt{2/N} \left( \sum_{a=0}^{N/2-1} e^{-2\pi i a(2m)/N} - \sum_{a=N/2}^{N-1} e^{-2\pi i a(2m)/N} \right) \sum_{b \text{ odd}}^{N-1} e^{i\pi b/N} (F^{-1})_{ab} \Phi^{(0,0)}_b.
\]
We next calculate the matrix elements. We see

\[
(\Phi_n^{(0,0)}, F\Phi_m^{(0,0)})_P = \begin{cases} 
\sum_{a=0}^{N/2-1} (\mathcal{F}^N)^{-1}_{na} \left( \begin{array}{cc} 0 & 0 \\ 0 & 0 \end{array} \right) & n \text{ even} \\
e^{i\pi(n-2m)/N} \sum_{a=0}^{N/2-1} (\mathcal{F}^N)^{-1}_{na} \left( \begin{array}{cc} 0 & 0 \\ 0 & 0 \end{array} \right) & n \text{ odd}
\end{cases}
\]

where

\[
\mathcal{F}_{mn} = \frac{e^{-2\pi i mn/N}}{\sqrt{N}}.
\]

For the case \(N/2 \leq m < N\), we see

\[
F\Phi_m^{(0,0)} = (E_x + X^{-1/2}O_x)(B + Y^{-1}T) \sqrt{2 \frac{N}{k} \sum_{k \in \mathbb{Z}} \left| \frac{2m-N}{N} + 2k + 1 \right|}
\]

\[
= \frac{1}{\sqrt{2}} \Phi_m^{(0,0)} - \frac{e^{-2\pi i m/N}}{\sqrt{2N}} \left( \sum_{a=N/2}^{N-1} e^{-2\pi i a(2m)/N} - \sum_{a=N/2}^{N-1} e^{-2\pi i a(2m)/N} \right)
\]

\[
\times \sum_{b \text{ odd}}^{N-1} e^{i\pi b/N} (\mathcal{F}^{-1})_{ab} \Phi_b^{(0,1/2)}
\]

Thus,

\[
(\Phi_n^{(0,0)}, F\Phi_m^{(0,0)})_P = \begin{cases} 
\sum_{a=0}^{N/2-1} (\mathcal{F}^N)^{-1}_{na} \left( \begin{array}{cc} 0 & 0 \\ 0 & 0 \end{array} \right) & n \text{ even} \\
e^{i\pi(n-(2m-N))/N} \sum_{a=0}^{N/2-1} (\mathcal{F}^N)^{-1}_{na} \left( \begin{array}{cc} 0 & 0 \\ 0 & 0 \end{array} \right) & n \text{ odd}
\end{cases}
\]

Combining all our previous results, we see that equation 21 holds for any \(m\).

This completes the proof of the theorem. \(\blacksquare\)

Lemma 9 The matrix \(B_{nm} = (\Phi_n^{(0,0)}, F\Phi_m^{(0,0)})\) is unitary.

Proof. Consider the case \(0 \leq n, m < N/2\)

\[
\sum_{j=0}^{N-1} \left( F^\dagger \right)_{nj} F_{jm} = \sum_{j=0}^{N-1} F^+_{jn} F_{jm}
\]

20
\[
\sum_{j \text{ even}} \delta_{j,2n} \delta_{j,2m} + \sum_{j \text{ odd}} \left( \frac{1}{N/\sqrt{2}} e^{-i\pi(j-2n)/N} \sum_{a=0}^{N/2-1} e^{-2\pi i a/N} e^{2\pi i a n/(N/2)} \right) \times \left( \frac{1}{N/\sqrt{2}} e^{-i\pi(j-2n)/N} \sum_{a=0}^{N/2-1} e^{-2\pi i a/N} e^{2\pi i a n/(N/2)} \right)
\]
\[
= \frac{1}{2} \delta_{n,m} + \frac{2e^{2\pi i(n-m)/N}}{N^2} \sum_{j \text{ odd}} \sum_{a,b=0}^{N/2-1} e^{2\pi i j(b-a)/N} e^{2\pi i a(n-bm)/N/2}
\]
\[
= \frac{1}{2} \delta_{n,m} + \frac{e^{2\pi i(n-m)/N}}{N} \sum_{a=0}^{N/2-1} e^{2\pi i a(n-m)/N/2}
\]
\[
= \delta_{n,m}.
\]

The remaining cases use an analogous calculation. We omit the details here.

\section*{V. Parity Conservation in the Quantum Dynamics}

We can see explicitly that at the fixed point \( \theta = (0,0) \) the dynamics is invariant under the symmetry \( x \to 1-x \) and \( p \to 1-p \). To see this, we define the parity operator \( P \) such that
\[
P \ket{x} = \ket{1-x},
\]
\[
P \ket{p} = \ket{1-p}.
\]

Then observe that
\[
P F \phi_m^{(0,0)} = P(E_x + X^{-1/2}O_x)(B + Y^{-1} T)SP^{-1} P \phi_m^{(0,0)}
\]
\[
= (O_x + X^{1/2}E_x)(T + YB)SP \phi_m^{(0,0)}
\]
\[
= (E_x + X^{-1/2}O_x)(B + YT)SP \phi_m^{(0,0)}.
\]

Since
\[
YTSP \phi_m^{(0,0)} = Y^{-1} TSPY \phi_m^{(0,0)} = Y^{-1} TSPY \phi_m^{(0,0)}
\]
we see that
\[
P F \phi_m^{(0,0)} = FP \phi_m^{(0,0)}
\]

21
Thus we see that on the subspace $\mathcal{H}_h(0)$, the dynamics commutes with the parity operator, hence is conserved.

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