SEGAL CONDITIONS FOR GENERALIZED OPERADS

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Abstract. This note is an introduction to several generalizations of the dendroidal sets of Moerdijk–Weiss. Dendroidal sets are presheaves on a category of rooted trees, and here we consider indexing categories whose objects are other kinds of graphs with loose ends. We examine the Segal condition for presheaves on these graph categories, which is one way to identify those presheaves that are a certain kind of generalized operad (for instance wheeled properad or modular operad). Several free / forgetful adjunctions between different kinds of generalized operads can be realized at the presheaf level using only the left Kan extension / restriction adjunction along a functor of graph categories. These considerations also have bearing on homotopy-coherent versions of generalized operads, and we include some questions along these lines.

1. Introduction

A simplicial set $X$ is said to satisfy the Segal condition if the maps

\[
X_2 \xrightarrow{d_2,d_0} X_1 \times_{X_0} X_1
\]

\[
X_3 \xrightarrow{d_2d_3,d_0d_3,d_0d_0} X_1 \times_{X_0} X_1 \times_{X_0} X_1
\]

\[
\vdots
\]

\[
X_n \rightarrow X_1 \times_{X_0} X_1 \times_{X_0} \cdots \times_{X_0} X_1
\]

are bijections for every $n \geq 2$. Such simplicial sets are the same thing as categories: $X_0$ is the set of objects, $X_1$ is the set of morphisms, and $d_1 : X_2 \rightarrow X_1$ provides a composition operation $X_1 \times_{X_0} X_1 \rightarrow X_1$ (see [Gro61, Proposition 4.1]). This situation can be fruitfully generalized to simplicial spaces by replacing bijections with weak equivalences, leading toward a model for ($\infty, 1$)-categories due to Rezk [Rez01]. This approach was extended by Cisinski and Moerdijk to provide a model for ($\infty, 1$)-operads in [CM13a].

In this note, we discuss further extensions the Segal condition (in both the weak and strict sense) from categories and (colored) operads to other sorts of operadic structures. Though generalizations of operads have appeared in many guises [YJ15, KW17, BM15, BB17, Get09], we take here an example-driven approach. Our focus will be on colored operads, dioperads/symmetric polycategories [Gan03, Gar08], cyclic operads [GK95], properads [Val07, Dun06], wheeled properads [MMS09], modular operads [GK98], and props [Mac65]. Note that the generalizations of operads in the above sources are usually focused on the monochrome case (or at least the case where morphisms fix the color set), while we consider the ‘category-like’ version where operations are only composable if they have compatible colors/types.

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This setting is useful for applications, see for instance [FBSD21, Spi13, VSL15], and is formally clarifying.

Given a kind of generalized operad, the Segal condition is then formulated using some category of graphs; for categories, we can regard the simplicial category $\Delta$ as a category of linear trees, and, for operads, the Moerdijk–Weiss dendroidal category $\Omega$ is a category of rooted trees. We will present appropriate graph categories for each of the kinds of operadic structures we are focusing on (with the exception of props; see Question 5.9). Each such graph category has a natural notion of Segal presheaf. Additionally, these graph categories are generalized Reedy categories in the sense of [BM11], which has model-categorical implications for the homotopy-coherent situation.

This note is, in part, an expansion of the material from my talk “Interactions between different kinds of generalized operads” in the AMS Special Session on Higher Structures in Topology, Geometry and Physics in March 2022, and in part a distillation of the core ideas from [Hac]. That paper introduced a new description of morphisms for several graph categories (most of which were introduced earlier by the author in joint work with Robertson and Yau), and was structured to allow efficient comparison with pre-existing definitions. In contrast, the present paper is designed as an overview for someone coming to the subject for the first time. Our goals are to give basic definitions of graph categories in elementary terms, explain the Segal condition for presheaves, and discuss the connections between the directed and undirected contexts. Pointers are provided to the relevant literature, and we highlight several interesting questions near the end. For particular graph categories, there are other introductory accounts, each having a different focus: [Wei11], [Moe10], and the first part of [HM22] are good starting points for the dendroidal theory (the book is a comprehensive account of the theory), [HR18] is focused on $\infty$-properads, while the forthcoming [BR] concerns modular $\infty$-operads.

1.1. What is a generalized operad? This is a question we don’t wish to answer too directly or precisely. As a first approximation, ‘generalized operad’ should include all of the mathematical objects in MSC2020 code 18M85 as well as some closely related objects. At its core, a generalized operad should consist of a collection of operations, each of which comes equipped with some boundary information (for instance coloring by some set, an ‘input’ or ‘output’ tag, or so on), as well as specified ways of connecting (compatible) boundaries of operations to form new operations. See Figure 1.

In a moment we’ll give a brief list of several of the structures that will be discussed, but first let’s note some general features. Operad-like structures will always be colored or typed (by a set or involutive set), and, as with functors between categories, maps between them are not expected to be the identity on color sets. Each color will have an associated identity operation. We will always be discussing the symmetric version of concepts, as for many structures (e.g., properads, modular operads) the non-symmetric version is quite different. Some structures will be directed in the sense that there is a strong distinction between inputs and outputs of operations, while others will be undirected and may be considered to just have ‘puts’ (or ‘ports’).

In the following list of examples, we’ll begin with directed operadic structures, roughly in order of increasing complexity, before turning to undirected operadic

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18M85 Polycategories/dioperads, properads, PROPs, cyclic operads, modular operads
structures. Several references to precise definitions in the literature will be given just below the list.

**Category:** Morphisms in a category are of the form \( f : c \to d \). Given two morphisms \( f \) and \( g \), there are at most two ways to compose the morphisms: \( g \circ f \) and \( f \circ g \), provided they exist.

**Operad:** An operad is a generalization of a category, where operations now can have multiple (or no) inputs, they are of the form \( f : c_1, \ldots, c_n \to d \). There are compositions

\[
P(c_1, \ldots, c_n; d) \times P(d_1, \ldots, d_m; c_i) \xrightarrow{\circ} P(c_1, \ldots, c_i-1, d_1, \ldots, d_i, c_{i+1}, \ldots, c_n; d)
\]

which may give many ways to compose two given operations.

**Dioperad:** A dioperad, also called a polycategory, expands operads to allow multiple outputs as well. To compose two operations, one chooses a single input of the first which agrees with a single input of the second, and connects them. The relevant structure map when \( c_i = b_j \) is \( i \circ j \) below.

\[
P(c_1, \ldots, c_n; d_1, \ldots, d_m) \times P(a_1, \ldots, a_k; b_1, \ldots, b_l) \\
\downarrow_{i \circ j}
\]

\[
P(c_1, \ldots, c_{i-1}, a_1, \ldots, a_k, c_{i+1}, \ldots, c_n; b_1, \ldots, b_{j-1}, d_1, \ldots, d_m, b_{j+1}, \ldots, b_l)
\]

**Properads:** Arguably this is the most complicated structure on this list. The underlying operation sets are the same as those of dioperads, but now one can connect one or more inputs of the first operation to the same number of outputs of the second operation.

**Props:** These are like properads, but now one can connect zero or more inputs of the first operation to the same number of inputs of the second operation. More simply, props are just those symmetric monoidal categories whose set of objects is a free monoid under tensor. The set of colors is the free generating set for the monoid of objects.

**Wheeled properads:** A wheeled properad has additional contractions that act on a single operation. They take the form

\[
P(c_1, \ldots, c_n; d_1, \ldots, d_m) \to P(c_1, \ldots, \hat{c}_i, \ldots, c_n; d_1, \ldots, \hat{d}_j, \ldots, d_m)
\]

where \( c_i = d_j \). The properadic compositions in a wheeled properad all decompose into simpler ones. Namely, a composition connecting \( k > 0 \) inputs of one operation to \( k \) outputs of another composes as a dioperadic composition, followed by \( k - 1 \) contractions. Wheeled properads could instead be called ‘wheeled dioperads.’
Involutive category: A dagger category is a category $C$ equipped with an identity-on-objects functor $C \to C^{\text{op}}$ so that $C \to C^{\text{op}} \to (C^{\text{op}})^{\text{op}} = C$ is the identity functor $\text{id}_C$. That is, every $f: c \to d$ has an associated $f^\dagger: d \to c$ (with $(f^\dagger)^\dagger = f$), and this allows you to exchange the role of input and output of morphisms. More generally, an involutive category relaxes the requirement that $C \to C^{\text{op}}$ is the identity-on-objects, so every $f: c \to d$ has an associated $f^\dagger: d^\dagger \to c^\dagger$.

Cyclic operad: A cyclic operad $P$ is an operad equipped with an involution $\dagger$ on its set of colors, along with additional structure maps

$$P(c_1, \ldots, c_n; c_0) \xrightarrow{\sim} P(c_2, \ldots, c_n, c_0^\dagger; c_1^\dagger)$$

that combine into a $\mathbb{Z}/(n+1)\mathbb{Z}$-action on operations of arity $n$. This allows one to exchange the roles of inputs and outputs. If the underlying operad $P$ is just a category, this notion coincides with that of being an involutive category.

Augmented cyclic operad: If one can exchange the roles of inputs and outputs, is there any real distinction between the two? Operations in an augmented cyclic operad dispense with this, and regard inputs and outputs on entirely the same footing. Operation sets take the form $P(c_1, \ldots, c_n)$ (for $n \geq 0$), and compositions take the form

$$P(c_1, \ldots, c_n) \times P(d_1, \ldots, d_m) \xrightarrow{\circ_i} P(c_1, \ldots, c_{i-1}, d_j+1, \ldots, d_m, c_0, \ldots, d_{j-1}, c_{i+1}, \ldots, c_n)$$

whenever $c_i = d_j^\dagger$. The real difference is that in a cyclic operad you cannot compose two operations which only have outputs but no inputs, while in an augmented cyclic operad this is possible, and the result will land in the set $P( )$ which has no analogue for cyclic operads.

Modular operads: These are augmented cyclic operads, together with contraction operations

$$P(c_1, \ldots, c_n) \to P(c_1, \ldots, \hat{c}_i, \ldots, \hat{c}_j, \ldots, c_n)$$

for $i < j$ with $c_i = c_j^\dagger$.

Definitions in these terms for the directed structures may be found, for instance, in Chapter 11 of [YJ15]; see also [GH18, §2.2], [Dun06, §6.1], and [HR15]. For cyclic operads see [Shu20, Definition 7.4] and [CGR14, Definition 3.3], for augmented cyclic operads see [DCH21, Definition 2.3], and for modular operads see [Ray21, Definition 1.24]. But please don’t go and look all of this up: according to the contents of Section 5, many of these operadic structures could instead be defined as Segal presheaves over the graph categories we introduce here, just as in the case of categories. There are of course a number of other interesting and useful operadic structures that do not appear on the above list, some of which can even fit into the frameworks we lay out below.

2. Graphs

In operadic contexts, the correct notion of graphs to use is one with ‘loose ends.’ There are a number of different combinatorial models for this, and we now recall one that has a simple associated notion of morphism, due to Joyal and Kock [JK11, §3].
The arc set $A$ will frequently exaggerate the size of the vertices. We freely use the terms ‘connected’ where the two end maps are monomorphisms. We call elements of $E$ with $\uparrow$ evident involution swapping $n$.

A corresponding encoding of directed graph with loose ends was introduced by Kock as Definition 1.1.1 of [Koc16]. (Graphs of this type have recently been used to give new perspectives on Petri nets [BGMS21, Koc22].)

**Definition 2.2** (Directed graphs). A directed graph $G$ is a diagram of finite sets of the form

$$E_G \leftarrow I_G \rightarrow V_G \leftarrow O_G \rightarrow E_G,$$

where the two end maps are monomorphisms. We call elements of $E_G$ the edges of the graph.

Every directed graph determines a canonical associated undirected graph, with $A_G := E_G \amalg E_G$ together with the swap involution, and $D_G := I_G \amalg O_G$. We will give important examples in a moment, but for now let us introduce some subsidiary notions.

**Definition 2.3.** Suppose $G$ is an undirected graph.

- If $v \in V_G$ is a vertex, then $\text{nb}(v) := t^{-1}(v) \subseteq D_G$ is the neighborhood of $v$.
- The boundary of $G$ is the set $\partial(G) = A_G \setminus D_G$.
- The set of edges is the set of $\uparrow$-orbits.

$$E_G = \{[a, a \uparrow] \mid a \in A_G\} \cong A_G/\sim$$

- An edge $[a, a \uparrow]$ is an internal edge if neither of $a, a \uparrow$ are elements of $\partial(G)$.

Suppose $G$ is a directed graph.

- If $v \in V_G$ is a vertex, then $\text{in}(v) \subseteq I_G$ is the preimage of $v$ under $I_G \rightarrow V_G$, called the set of inputs of $v$. Likewise, $v$ has a set of outputs out$(v) \subseteq O_G$.
- The set of input edges of $G$ is defined to be $\text{in}(G) := E_G \setminus O_G$ and the set of output edges of $G$ is $\text{out}(G) := E_G \setminus I_G$. All other edges are internal edges.

Each undirected graph $G$ determines a topological pair $\mathcal{X}_G = (X_G, V_G)$, called the geometric realization. Each arc in $A_G$ gets an open interval, each element of $D_G$ gets a half-open interval, each vertex gets a point, and these are glued together in the manner described by the diagram for $G$ (e.g., if $t \in (0, 1)$ is in the interval associated to $a$, then it will be identified with $1-t \in (0, 1)$ in the interval associated to $a \uparrow$). We mention this mostly in that we will draw pictures of our graphs, and we will frequently exaggerate the size of the vertices. We freely use the terms ‘connected’ and ‘simply-connected’ to refer to properties of the topological space $X_G$ (though these can also be described combinatorially).

**Example 2.4.** Figure 2 gives a picture of an undirected graph $G$ with loose ends. The arc set $A_G = \{n, n \uparrow \mid 1 \leq n \leq 9\}$ has eighteen elements, together with the evident involution swapping $n$ and $n \uparrow$, while the vertex set is $V_G = \{u, v, w, x\}$. The set $D_G$ is given below

$$D_G = \{1^1, 3^1, 4.6^4, 8.7.4^4, 5.5^1, 6.8^1, 9\}$$

- nb$(u)$
- nb$(v)$
- nb$(w)$
- nb$(x)$
and the indicated partition specifies the function \( t: D_G \to V_G \). The boundary set is \( \partial(G) = \{1, 2, 2^1, 3, 7, 9^1\} \), which has six elements.

**Remark 2.5.** A different combinatorial definition of (undirected) graph with loose ends is common in the literature (see, e.g., [KM94, 6.6.1] or [BM96, Definition 1.1]). This is simply a diagram of finite sets

\[
\begin{array}{c}
c \subset F \to V
\end{array}
\]

where the self-map of \( F \) is an involution. The fixed points of the involution represent the loose ends of the graph, and the free orbits represent edges between vertices. Given such a graph, we can form a graph as in Definition 2.1 by setting \( D := F \), and to give \( A \) we double up the fixed points of the involution on \( F \) to turn it into a free involution. However, not all undirected graphs arise from this process: we never see graphs which have edges that float free. See [BB17, §15.3] for more details.

**Definition 2.6 (Subgraph).** Suppose \( G \) is a graph, either directed or undirected. A **subgraph** of \( G \) is determined by a pair of subsets \( E_H \subseteq E_G \) and \( V_H \subseteq V_G \) subject to a single condition.

- Suppose \( G \) is undirected, and let \( A_H \subseteq A_G \) denote the set of all arcs appearing in the edges in \( E_H \); this is naturally an involutive subset of \( A_G \). Further, let \( D_H := t^{-1}(V_H) \subseteq D_G \) be the set of those elements mapping into \( V_H \). Then \( (E_H, V_H) \) determines a subgraph just when \( D_H \to D_G \to A_G \) lands in the subset \( A_H \).

\[
\begin{array}{c}
\uparrow \subset A_H \xleftarrow{t} D_H \longrightarrow V_H \\
\downarrow \subset A_G \xleftarrow{t} D_G \longrightarrow V_G
\end{array}
\]
• Suppose $G$ is directed, and let $I_H \subseteq I_G$ and $O_H \subseteq O_G$ be the preimages of the subset $V_H$ along the functions $I_G \to V_G$ and $V_G \leftarrow O_G$. Then $(E_H, V_H)$ determines just when the dashed arrows in the following diagram exist.

$$
\begin{array}{cccc}
E_H & \leftarrow \cdots & I_H & \longrightarrow & V_H & \leftarrow & O_H & \longrightarrow & E_H \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
E_G & \leftarrow & I_G & \longrightarrow & V_G & \leftarrow & O_G & \longrightarrow & E_G
\end{array}
$$

The top row in (1) or (2) exhibits $H$ as an undirected or directed graph.

Of course we could write these requirements in other ways, such as declaring that a subgraph is a triple or quadruple of subsets satisfying some properties. But we prefer to emphasize that subgraphs are always determined by a set of edges and a set of vertices. Notice also that a subgraph of a directed graph is nothing but a subgraph of its underlying undirected graph.

We now describe two kinds of important graphs, which we term elementary. For a non-negative integer $n$, we will write $n$ for the set \{1, 2, ..., $n$\}.

**Example 2.7 (Edge).** The canonical undirected edge, here denoted $\updownarrow$, is

$$
\updownarrow \subset 2 \leftarrow 0 \longrightarrow 0.
$$

That is, this graph has no vertices, and all arcs are in the boundary. The canonical directed edge, here denoted $\downarrow$, is

$$
1 \leftarrow 0 \rightarrow 0 \leftarrow 0 \leftarrow 1.
$$

This graph has no vertices, and a single edge which is both an input and an output of the graph. We also refer to any connected graph without vertices as an edge.

**Example 2.8 (Stars or corollas).** Let $n, m \geq 0$ be integers. The $n$-star, $\star_n$, is the following undirected graph

$$
\updownarrow \subset n + n \leftarrow n \longrightarrow 1,
$$

which has a single vertex with an $n$-element neighborhood, and $\partial(\star_n) = n$. The $n,m$-star, $\star_{n,m}$, is the following directed graph

$$
n + m \leftarrow n \rightarrow 1 \leftarrow m \rightarrow n + m.
$$

The star $\star_{n,m}$ has a single vertex with $n$ inputs and $m$ outputs, and $\text{in}(\star_{n,m}) = n$, $\text{out}(\star_{n,m}) = m$. We will also refer to any connected graph with one vertex and no internal edges as a star.

Every edge and star (including graphs isomorphic to these canonical forms) will be called elementary. The elementary graphs are precisely those graphs with no internal edges. Two examples of stars are given in Figure 4; we will always follow the convention that in a directed graph, the flow goes from top to bottom (in graphs with loops, this rule will apply locally at each vertex).

If $e \in E_G$ is any edge in a graph $G$, we may consider $e$ as a subgraph as in Definition 2.6. But vertices in a graph need not span a star-shaped subgraph in the same way. For example, in Figure 2, the smallest subgraph containing the vertex $w$ has one vertex and arc set \{4, 4, 6, 6, 5, 5\}, and is not a star (the other three vertices all span star-shaped subgraphs). If $v$ is a vertex in a graph $G$, we
Figure 4. The stars $\star_5$ and $\star_{4,2}$

Figure 5. The first five linear graphs $L_0, L_1, L_2, L_3, L_4$

will write $\star_v$ for a star with $D_{\star_v} := \text{nb}(v)$ (when $G$ is undirected) or $I_{\star_v} := \text{in}(v)$ and $O_{\star_v} := \text{out}(v)$ (when $G$ is directed). Despite the fact that $\star_v$ might not be a subgraph of $G$, there is always a map (see Definition 2.12 below) $\iota_v : \star_v \rightarrow G$.

Example 2.9 (Linear graphs). The linear directed graph $L_n$ with $n$ vertices (see Figure 5) is presented as follows.

$$\{0, 1, \ldots, n\} \xleftarrow{\text{id}^{-1}} n \xrightarrow{\text{id}} n \xleftarrow{\text{id}^0} \{0, 1, \ldots, n\}$$

We have $\text{in}(L_n) = \{0\}$, $\text{out}(L_n) = \{n\}$, while the vertex $k$ has $\text{in}(k) = \{k - 1\}$ and $\text{out}(k) = \{k\}$. We also call the associated undirected graph a linear graph.

Definition 2.10 (paths, cycles, trees). Let $G$ be an undirected graph.

- A **weak path** in $G$ is a finite alternating sequence of edges and vertices of $G$ so that if $e$ and $v$ are adjacent then some arc of $e$ appears in $\text{nb}(v)$, and so that the pattern $vev$ can only appear in the path if *both* arcs of $e$ are in $\text{nb}(v)$.
- A **path** in $G$ is a finite alternating sequence of arcs and vertices of $G$ so that if $av$ appears in the sequence then $t(a) = v$, and if $va$ appears in the sequence then $t(a^\dagger) = v$.
- A **cycle** is a path of length strictly greater than one that begins and ends at the same arc or same vertex.
- The graph $G$ is a **tree** if and only if it is connected and does not have any paths which are cycles.

Each path gives a weak path by replacing the arcs with the edges they span, and each weak path has at least one associated path. The only ambiguity is how to handle the patterns of the form $vev$. In a tree, there is no real difference between paths and weak paths. The property of being connected is equivalent to having a weak path between every pair of distinct elements of $E_G \cup V_G$.

Definition 2.11. Let $G$ be a directed graph.

- An **undirected path** in $G$ is a weak path in the associated undirected graph.
- A **directed path** in $G$ is a weak path in the associated undirected graph, so that the following hold:
If $ev$ appears in the path, then $e \in \text{in}(v)$.

If $ve$ appears in the path, then $e \in \text{out}(v)$.

- A directed cycle is a directed path of length strictly greater than one that begins and ends at the same edge or vertex.

- A directed graph $G$ is acyclic if it is connected and does not contain any directed cycles.

2.1. Étale maps and embeddings. Étale maps are certain functions between graphs that preserve structure and preserve arity of vertices (that is, the cardinality of $\text{nb}(v)$ and $\text{nb}(f(v))$ coincide, and similarly for the directed case). The following definitions appear in [JK11] and [Koc16, 1.1.7].

**Definition 2.12.** An étale map $G \rightarrow G'$ between undirected graphs is a triple of functions $A_G \rightarrow A_{G'}$, $D_G \rightarrow D_{G'}$, and $V_G \rightarrow V_{G'}$ so that the diagram below left commutes with right hand square a pullback.

$$
\begin{array}{c}
A_G \downarrow \quad A_G \downarrow \quad D_G \rightarrow V_G \\
\downarrow \quad \downarrow \quad \downarrow \\
A_{G'} \quad A_{G'} \quad D_{G'} \rightarrow V_{G'}
\end{array}
\begin{array}{c}
E_G \quad I_G \quad V_G \quad O_G \quad E_G \\
\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\
E_{G'} \quad I_{G'} \quad V_{G'} \quad O_{G'} \quad E_{G'}
\end{array}
$$

Similarly, an étale map $G \rightarrow G'$ between directed graphs is a quadruple of functions $E_G \rightarrow E_{G'}$, $I_G \rightarrow I_{G'}$, $O_G \rightarrow O_{G'}$, and $V_G \rightarrow V_{G'}$ so that the diagram above right commutes and both middle squares are pullbacks.

Étale maps induce local homeomorphisms between geometric realizations. If $G$ is an undirected graph, then the set of étale maps $\downarrow \rightarrow G$ is in bijection with $A_G$; likewise if $G$ is directed then $\{\downarrow \rightarrow G\} \cong E_G$. Similarly, étale maps from stars (modulo isomorphism in the domain) classify vertices of a graph. Every étale map between directed graphs induces an étale map between the associated undirected graphs. It is immediate that every subgraph in the sense of Definition 2.6 determines an étale map. But the latter are strictly more general.

**Example 2.13.** In this example, we will refer to two étale maps $H \rightarrow G \leftarrow K$ between directed graphs appearing in Figure 6. Following our convention, the graphs have vertices with two inputs and two outputs, and also vertices with two inputs and three outputs. The edges 2 and 4 in $H$ do not touch, and this graph is only drawn this way to make the étale map easier to see. Since $G$ has only one $(2,2)$ vertex and one $(2,3)$ vertex, the actions of the maps on vertices are forced. The actions on the labelled middle edges are given by preservation of parity, and the action on the inputs and outputs of the graphs are the visually expected ones. We could likewise consider these as étale maps of undirected graphs. These examples show that étale maps do not need to be injective on vertices, and even if they are, they do not need to be injective on edges or arcs.
Remark 2.14. Étale maps between undirected graphs take paths to paths (one does not even need the pullback condition for this). It follows that \( G \to G' \) is an étale map between undirected graphs with \( G' \) a tree, then each connected component of \( G \) is a tree as well. Likewise, since étale maps between directed graphs preserve directed paths, if \( G \to G' \) is such a map with \( G' \) acyclic, then each connected component of \( G \) is acyclic as well. Of course in this case if \( G' \) is simply-connected, then so is each component of \( G \), by passing to the associated undirected graphs.

We are mainly interested in connected graphs in what follows, and we make the following definition.

Definition 2.15. An embedding between connected graphs is an étale map which is injective on vertices. We denote an embedding by \( G \hookrightarrow G' \).

Since monomorphisms are stable under pullback, if \( G \hookrightarrow G' \) is an embedding between undirected graphs, then \( D_G \to D_G' \) is a monomorphism (and likewise for \( I_G \to I_G' \) and \( O_G \to O_G' \) in the directed case). Embeddings between connected graphs yield injective local homeomorphisms between geometric realizations. Every subgraph inclusion in the sense of Definition 2.6 is an embedding, but embeddings are strictly more general. For instance, Figure 7 indicates two consecutive embeddings (from left to right), where the second indicates the kind of clutching behavior possible in embeddings, which can identify some arcs. The first is simply a subgraph inclusion, while the second is a bijection on \( D \) and \( V \), but on arcs is not a monomorphism.

Any étale map between trees is a monomorphism (see, for instance, [Ray21, Corollary 4.23]), hence an embedding. For acyclic directed graphs the situation is more complicated: there can be étale maps between such which are not embeddings, and embeddings which are not injective, as we saw in Example 2.13 / Figure 6.

Construction 2.16 (Lifting embeddings). Suppose \( G \) is an undirected graph and \( H \) is a directed graph, and temporarily let \( U(H) \) be the undirected graph associated to \( H \). If \( f: G \to U(H) \) is an embedding, then there is a directed graph \( G' \) and an embedding \( f': G' \to H \), so that \( G \cong U(G') \to U(H) \) is the map \( f \). Moreover, the graph \( G' \) and embedding \( f' \) are essentially unique with respect to this property. We leave most details to the reader. Our starting data is the map \( f \), as below.

\[
\begin{array}{c}
A_G & \hookrightarrow & A_G & \to & D_G & \to & V_G \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
E_H \amalg E_H & \leftrightarrow & E_H \amalg E_H & \leftrightarrow & I_H \amalg O_H & \to & V_H
\end{array}
\]

Figure 7. Two embeddings
We set \( E_{G'} := E_G \) and \( V_{G'} := V_G \), and define \( I_{G'} \) and \( O_{G'} \) as subsets of \( D_G \) fitting into the following pullbacks.

\[
\begin{array}{cccc}
I_{G'} & \rightarrow & D_G & \rightarrow & V_G \\
\downarrow & & \downarrow & & \downarrow \\
I_H & \hookrightarrow & I_H \amalg O_H & \rightarrow & V_H \\
\end{array}
\]

Using that \( f \) is an embedding, one can show that \( G' \) is a directed graph (that is, \( I_{G'} \rightarrow E_{G'} \) and \( O_{G'} \rightarrow E_{G'} \) are injective). Further, \( f' \): \( G' \rightarrow H \) is an embedding.

We now introduce a set associated to graphs which will be used to describe all manner of graphical maps below.

**Definition 2.17.** If \( G \) is an directed or undirected connected graph, we define \( \text{Emb}(G) \) to be the set of embeddings with codomain \( G \), modulo isomorphisms in the domain.

There is a partial order on \( \text{Emb}(G) \) determined by factorization of embeddings. As a partially-ordered set, \( \text{Emb}(G) \) has a unique maximal element given by the identity of \( G \), and each edge determines a minimal element (if \( G \) has at least one edge, that is, if it is not \( *_0 \), then the set of minimal elements is precisely \( E_G \)).

In light of Construction 2.16 if \( G \) is a directed graph and \( U(G) \) is its associated undirected graph, then \( \text{Emb}(G) \rightarrow \text{Emb}(U(G)) \) is an isomorphism. For this reason we do not distinguish notationally between the two: \( \text{Emb}(G) \) is fundamentally a set associated to an undirected graph.

Every connected subgraph (in the sense of Definition 2.6) of \( G \) determines an element of \( \text{Emb}(G) \), and if \( G \) is a tree, then \( \text{Emb}(G) \) coincides with the set of connected subgraphs. Thus if \( T \) is a subtree of a tree \( G \), we will usually write \( T \in \text{Emb}(G) \) for the class of the associated inclusion \( T \hookrightarrow G \).

For an arbitrary graph \( G \), a vertex \( v \) determines a subgraph \( *_v \) just when there are no loops at the vertex \( v \), but a vertex always determines an embedding \( \iota_v : *_v \hookrightarrow G \) picking out the vertex (\( *_v \) is a star with \( D_{*_v} = \text{nb}(v) \subseteq D_G \)). There are canonical inclusions

\[
E_G \hookrightarrow \text{Emb}(G) \quad V_G \hookrightarrow \text{Emb}(G)
\]

of edges and vertices into the embedding set, which we regard as subset inclusions.

**Definition 2.18.** Suppose that \( h : H \rightarrow G \) and \( k : K \rightarrow G \) are two embeddings. An embedding \( \ell : L \rightarrow G \) is called a union of \( h \) and \( k \) if

1. \([\ell]\) is an upper bound for both \([h]\) and \([k]\) in the poset \( \text{Emb}(G) \) (that is, there is a factorization \( H \rightarrow L \rightarrow G \) of \( h \) and likewise for \( k \)), and
2. \( \ell(V_L) = h(V_H) \cup k(V_K) \).

The two embeddings are vertex disjoint if

1. \( h(V_H) \cap k(V_K) = \emptyset \).

Unions need not be unique. In Figure 8 the two subgraphs to the left and right of the graph \( G \) in the center have three possible unions in \( \text{Emb}(G) \). These are pictured in the center column, and include the maximum \( [\text{id}_G] \in \text{Emb}(G) \). Note that these subgraphs do not have a least upper bound in \( \text{Emb}(G) \); also note that least upper bounds that do exist may fail to be unions. As another example, the embedding pictured in Figure 9 has four possible self-unions, determined by which of the two snipped edges one decides to glue together.
If $H$ and $K$ are subgraphs of a graph $G$ in the sense of Definition 2.6, then there is a natural notion of union subgraph obtained by setting $E_{H\cup K} = E_H \cup E_K \subseteq E_G$ and $V_{H\cup K} = V_H \cup V_K \subseteq V_G$. If $H$ and $K$ are both connected subgraphs of a connected graph $G$, then $H \cup K$ will be a union in the above sense just when $H$ and $K$ overlap, that is, just when at least one of $E_H \cap E_K$ or $V_H \cap V_K$ is inhabited. In fact, $H \cup K$ is the only union which is also a subgraph, by essentially the same proof as [Hac, Lemma B.7]. In particular, if $G$ is a tree then we know all embeddings come from subtrees, hence any pair $S, T \in \text{Emb}(G)$ has at most one union.

2.2. Boundary-compatibility. We end this section with one further construction that we will use in our definition of graph maps, that of boundary-compatibility in Definition 2.19. If $f: H \hookrightarrow G$ is an embedding between undirected graphs, then the composite

$$\partial(H) \hookrightarrow A_H \rightarrow A_G$$

is a monomorphism (see Lemma 1.20 of [HRY20a]), and we write $\partial(f) \subseteq A_G$ for its image, which we call the boundary of $f$. For example, if $G$ is the graph from Example 2.4 (see Figure 3) and $f$ is the star-embedding picking out the vertex $w$, then $\partial(f) = \{4, 6^1, 5, 5^1\}$. Likewise, if $f: H \hookrightarrow G$ is an embedding between directed graphs, then $\text{in}(H) \rightarrow E_G$ and $\text{out}(H) \rightarrow E_G$ are also monomorphisms, and we write $\text{in}(f), \text{out}(f) \subseteq E_G$ for the images of these sets.
The assignment $\bar{\varphi}$ on embeddings descends to a function $\text{Emb}(G) \to \varphi(A_G)$ landing in the power set of $A_G$. It is more convenient to think about a multi-set structure, so we instead consider the function

$$\bar{\varphi}: \text{Emb}(G) \to \varphi(A_G) \to \mathbb{P}(A_G)$$

landing in the free commutative monoid on $A_G$ (whose elements are finite unordered lists of elements of $A_G$). Likewise, we have functions

$$\text{in}, \text{out}: \text{Emb}(G) \to \varphi(E_G) \to \mathbb{N}E_G$$

when $G$ is directed.

**Definition 2.19.** Suppose $G$ and $G'$ are two undirected connected graphs, $\varphi_0: A_G \to A_G'$ is an involutive function, and $\hat{\varphi}: \text{Emb}(G) \to \text{Emb}(G')$ is a partially-defined function. We say that the pair $\varphi := (\varphi_0, \hat{\varphi})$ is **boundary-compatible** if the diagram

$$\begin{array}{ccc}
\text{Emb}(G) & \xrightarrow{\bar{\varphi}} & \mathbb{P}(A_G) \\
\downarrow{\hat{\varphi}} & & \downarrow{\varphi_0} \\
\text{Emb}(G') & \xrightarrow{\bar{\varphi}} & \mathbb{P}(A_G')
\end{array}$$

commutes on the domain of definition of $\hat{\varphi}$. Likewise, if $G$ and $G'$ are directed connected graphs, $\varphi_0: E_G \to E_G'$ is a function, and $\hat{\varphi}: \text{Emb}(G) \to \text{Emb}(G')$ is a partially-defined function, then we say that the pair $\varphi := (\varphi_0, \hat{\varphi})$ is **boundary-compatible** if the diagram

$$\begin{array}{ccc}
\mathbb{N}E_G & \xleftarrow{\text{in}} & \text{Emb}(G) \xrightarrow{\text{out}} \mathbb{N}E_G \\
\downarrow{\varphi_0} & & \downarrow{\hat{\varphi}} \\
\mathbb{N}E_G' & \xleftarrow{\text{in}} & \text{Emb}(G') \xrightarrow{\text{out}} \mathbb{N}E_G'
\end{array}$$

commutes on the domain of definition of $\hat{\varphi}$.

We are writing the diagrams above using partial maps for conciseness, since we will often want to discuss functions between subsets of $\text{Emb}(G)$ and $\text{Emb}(G')$ (Definition 3.1, Definition 4.8, Proposition A.6), and work with the restrictions of $\bar{\varphi}$ (or in and out) to those subsets.

The standard example of a boundary compatible pair is one associated to an embedding $f: G \to G'$. The function $\varphi_0$ is defined to be $A_G \to A_G'$ (or $E_G \to E_G'$ if $G, G'$ are directed), and the (total) function $\hat{\varphi}: \text{Emb}(G) \to \text{Emb}(G')$ is defined by post-composition with $f$.

### 3. Categories of Simply-connected Graphs

**Definition 3.1.** Suppose $G$ and $G'$ are undirected trees.

1. A **tree map** is a boundary-compatible pair $(\varphi_0, \varphi_1)$ consisting of an involutive function $\varphi_0: A_G \to A_G'$ and a function $\varphi_1: V_G \to \text{Emb}(G')$.

2. Suppose $\varphi := (\varphi_0, \hat{\varphi})$ is a pair consisting of an involutive function $\varphi_0: A_G \to A_G'$ and a function $\hat{\varphi}: \text{Emb}(G) \to \text{Emb}(G')$. We say that $\varphi$ is a **full tree map** if it is boundary-compatible, and, if two subtrees $S$ and $T$ of $G$ overlap, then so do $\hat{\varphi}(S)$ and $\hat{\varphi}(T)$ and we have $\hat{\varphi}(S \cup T) = \hat{\varphi}(S) \cup \hat{\varphi}(T)$.

The full tree maps form a category via composition of pairs, which we denote by $U_0$. Trees with inhabited boundary span the full subcategory $U_{\text{cyc}} \subset U_0$. 

Every full tree map determines a tree map by restriction of \( \hat{\phi} \) along \( V_G \to \text{Emb}(G) \), and we show in Proposition A.6 of Appendix A.2 that this gives a bijection between tree maps and full tree maps. In light of this, we will use the shorter term ‘tree map’ also for full tree maps. Since subtrees are uniquely determined by their boundaries, boundary-compatibility implies that tree maps send edges to edges.

It turns out that tree maps also preserve intersections between overlapping subtrees. This follows from Theorem A.7 and Proposition B.21 of [Hac], but we provide an elementary, direct proof in Appendix A.1.

We now give an alternative description of \( U_{\text{cyc}} \) as a full subcategory of the category of cyclic operads, \( \text{Cyc} \), and likewise of \( U_0 \) as a full subcategory of augmented cyclic operads, \( \text{Cyc}^+ \). The distinction between the two is that augmented cyclic operads \( P \) are allowed to have a meaningful set \( P(\ ) \) of operations, which cannot compose with any other operations. The term ‘augmented cyclic operad’ follows [HV02], and would be called ‘entries-only \( \ast \)-polycategories’ in [Shu20] or simply ‘cyclic operads’ in [DCH21, Definition 2.3]; see [Shu20, Example 7.7] for some discussion on terminology.

If \( C \) is an involutive set of colors, we write \( \text{Cyc}_C \subset \text{Cyc}^+_C \) for categories of \( C \)-colored objects and identity-on-colors morphisms.

Each tree \( G \) determines a free object \( C(G) \) in \( \text{Cyc}^+_A \). The generating set of \( C(G) \) is precisely the set \( V_G \), where \( v \in V_G \) is considered as an element of \( C(G)(\partial(G_v)) = C(G)(\text{nb}(v)^\dagger) \). A variation of [HRY20, §2.2] allows one to show that this becomes a functor \( C : U_0 \to \text{Cyc}^+ \).

**Proposition 3.2.** The category \( U_0 \) is equivalent to the full subcategory of augmented cyclic operads \( \text{Cyc}^+ \) on the objects \( C(G) \) (where \( G \) ranges over all trees). This equivalence restricts to an equivalence between \( U_{\text{cyc}} \) and the full subcategory of cyclic operads \( \text{Cyc} \subset \text{Cyc}^+ \) on the objects \( C(G) \), where \( G \) ranges over all trees with inhabited boundary.

The second part of this is simply the observation that \( C(G) \) is a cyclic operad if and only if \( G \) has inhabited boundary. The first part of the statement will be proved in Appendix A.3.

Turning now to directed trees, we introduce the ‘dioperadic graphical category.’

**Definition 3.3.** Suppose \( G \) and \( G' \) are directed trees. A **tree map** from \( G \) to \( G' \) is a boundary-compatible pair \( \varphi = (\varphi_0, \hat{\varphi}) \) consisting of a function \( \varphi_0 : E_G \to E_{G'} \) and a function \( \hat{\varphi} : \text{Emb}(G) \to \text{Emb}(G') \) which preserves unions: if \( S \) and \( T \) overlap, then \( \hat{\varphi}(S \cup T) = \hat{\varphi}(S) \cup \hat{\varphi}(T) \). These form a category of all directed trees, which we denote by \( O_0 \).

Of course one can equivalently define tree maps by replacing \( \hat{\varphi} \) with a function \( \varphi_1 : V_G \to \text{Emb}(G') \), as in Definition 3.1. A version of Proposition 3.2 then holds in this setting, with essentially the same proof: the category \( O_0 \) is equivalent to a full subcategory (spanned by the directed trees) on the category of dioperads.

By restricting the objects further, we conclude that the dendroidal category from [MW07], originally defined as a category of the operads associated to rooted trees, is equivalent to a full subcategory of \( O_0 \).

**Definition 3.4.** The Moerdijk–Weiss **dendroidal category**, denoted by \( \Omega \), is the full subcategory of \( O_0 \) consisting of those trees \( G \) so that \( \text{out}(v) \) is a singleton

\[\text{This was called } G_{\text{sc}} \text{ in [CH22], and is equivalent to the category } \Theta \text{ in [HRY15] §6.3.5].}\]
set for every \( v \in V_G \). The simplicial category \( \Delta \) can be identified with the full subcategory of \( O_0 \) on the linear graphs from Example 2.9.

Of course if one is only interested in rooted trees, simpler descriptions are possible; see, for example, [Koc11].

4. ON SEVERAL CATEGORIES OF GRAPHS

We now extend Definition 3.1 and Definition 3.3 to maps between graphs that are not simply-connected. This formulation first appeared in [Hac].

**Definition 4.1.** Let \( G \) and \( G' \) be connected undirected (resp. directed) graphs. A **graph map** \( \varphi: G \to G' \) is a pair \( (\varphi_0, \hat{\varphi}) \) consisting of a function \( \hat{\varphi}: \text{Emb}(G) \to \text{Emb}(G') \) as well as

1. an involutive function \( \varphi_0: A_G \to A_{G'} \) if \( G \) and \( G' \) are undirected, or
2. a function \( \varphi_0: E_G \to E_{G'} \) if \( G \) and \( G' \) are directed.

The pair \( (\varphi, \hat{\varphi}) \) must satisfy the following four conditions:

1. The function \( \hat{\varphi} \) sends edges to edges.
2. The function \( \hat{\varphi} \) preserves unions.
3. The function \( \hat{\varphi} \) takes vertex disjoint pairs to vertex disjoint pairs.
4. The pair \( (\varphi, \hat{\varphi}) \) is boundary-compatible (in the sense of Definition 2.19).

Composition of graph maps is given by composition of the two constituent functions. This yields a category \( U \) of undirected connected graphs and their graph maps, as well as a category \( O \) of directed connected graphs and their graph maps.

**Proposition 4.2.** The category \( U \) of undirected connected graphs and their graph maps is equivalent to the graphical category from [HRY20a, Definition 1.31]. The category \( O \) of directed connected graphs and their graph maps is equivalent to the wheeled properadic graphical category from [HRY18, §2].

**Proof.** This result appears in [Hac]: the first part is Theorem 4.15, and the second part combines Proposition 5.10 and Theorem 5.13. ‘Wheeled properadic graphical category’ refers to the category \( B \) from [HRY18]. \( \square \)

The next proposition uses that tree maps send edges to edges and preserve vertex-disjoint pairs (Lemma A.2).

**Proposition 4.3.** The category \( U_0 \) is the full subcategory of \( U \) on the undirected trees. The category \( O_0 \) is the full subcategory of \( O \) on the directed trees. \( \square \)

**Remark 4.4.** If \( C \) is a category, let \( \hat{C} := \text{Fun}(C^{op}, \text{Set}) \) be the category of set-valued presheaves. If \( X \in \hat{C} \) is a presheaf and \( G \in C \) is an object, we will write \( X_G \) for the value of \( X \) at \( G \). The Yoneda embedding \( C \to \hat{C} \) takes an object \( G \) to the representable presheaf \( C(\cdot, G) \). An important example, when \( C = U \) or some full subcategory, is the **orientation presheaf** \( o \). This is defined by

\[
o_G = \{ \text{involutive functions } A_G \to \{+1, -1\} \}
\]

where the set \( \{+1, -1\} \) is equipped with the free involution. This additional data determines a directed structure on \( G \), see [Hac, Construction 5.4]. Meanwhile, every
directed graph determines a canonical element of the orientation presheaf. In fact, we have equivalences of categories

$$O \xrightarrow{\sim} U \times \hat{U} / \sigma$$

$$O_0 \xrightarrow{\sim} U_0 \times \hat{U}_0 / \sigma$$

whose first projections forget the directed structure. In other words, the category of elements of $\sigma \in \hat{U}$ (resp. $\sigma \in \hat{U}_0$) is $O$ (resp. $O_0$). See [Hac, Proposition 5.10].

A similar construction appears in the context of Feynman categories and Borisov–Manin graph morphisms in [KL17, 6.4.1], and for étale maps in [Ray21, §4.5].

**Definition 4.5 (Active and inert maps).** A graph map $\varphi = (\varphi_0, \hat{\varphi}) : G \to G'$ is an **active map** if $\hat{\varphi} : \text{Emb}(G) \to \text{Emb}(G')$ preserves the maximum ($\hat{\varphi}([\text{id}_G]) = [\text{id}_{G'}]$).

We write $U_{\text{act}} \subset U$ and $O_{\text{act}} \subset O$ for the wide subcategories of active maps, and decorate the arrows as $G \xrightarrow{\varphi} G'$. A graph map $\varphi$ is **inert** if it comes from an embedding; in other words, if the the dashed map in the diagram

$$\begin{array}{ccc}
V_G & \xrightarrow{\varphi} & V_{G'} \\
\downarrow & & \downarrow \\
\text{Emb}(G) & \xrightarrow{\hat{\varphi}} & \text{Emb}(G')
\end{array}$$

exists. We write $U_{\text{int}} \subset U$ and $O_{\text{int}} \subset O$ for the subcategories of inert maps, and decorate the arrows by $G \xrightarrow{\varphi} G'$.

Every graph map $\varphi : G \to G'$ factors as an active map followed by an inert map. Indeed, if $\hat{\varphi}([\text{id}_G]) = [H \hookrightarrow G']$, then the factorization takes on the form $G \to H \to G'$. Constructing the active map $G \to H$ is delicate in general since $A_H \to A_{G'}$ (resp. $E_H \to E_{G'}$) need not be a monomorphism. Nevertheless, we have the following (see [HRY20a, Theorem 2.15]).

**Theorem 4.6.** The pair $(U_{\text{act}}, U_{\text{int}})$ is an orthogonal factorization system\footnote{An orthogonal factorization system on a category $C$ is a pair $(L, R)$ of classes of maps, each closed under composition and containing all of the isomorphisms. These have the property that any morphism of $C$ factors as $r \circ \ell$, with $r \in R$ and $\ell \in L$, and this factorization is unique, up to unique isomorphism.} on the category $U$, as is $(O_{\text{act}}, O_{\text{int}})$ on $O$. These factorization systems restrict to factorization systems on the subcategories $U_{\text{cyc}} \subset U_0 \subset U$ and $\Delta \subset \Omega \subset O_0 \subset O$.

**4.1. Properadic graphical category.** We briefly describe the properadic graphical category $G$, whose objects are connected acyclic directed graphs, in the sense of Definition 2.11. It requires a somewhat subtle notion of structured subgraph, which we give in a reformulation first appearing as [Koc16, 1.6.5]; see also [CH22, Remark 2.2.4].

**Definition 4.7 (Structured subgraph).** A naïve morphism of directed graphs has the same data as an étale map Definition 2.12, but we do not require the middle squares to be pullbacks. Suppose $G$ is an acyclic directed graph. A connected subgraph $H \subseteq G$ is a **structured subgraph** if the inclusion $H \hookrightarrow G$ is right orthogonal in the category of naïve morphisms of directed graphs to the maps $\downarrow \Pi \downarrow \to L_n$ (for all $n \geq 0$) picking out the input and output edge in the linear graph

\[ \]
In other words, given a commutative square as below (whose horizontal arrows may just be naïve morphisms), there is a unique dashed lift.

\[
\begin{array}{ccc}
& & H \\
\downarrow & & \downarrow \\
L_n & \rightarrow & G \\
\end{array}
\]

We write \( \text{sSb}(G) \subseteq \text{Emb}(G) \) for the set of structured subgraphs of \( G \).

This can alternately be phrased in terms of ‘graph substitution’ (see [CH22, Remark 2.2.4]). Every edge and star subgraph is a structured subgraph, so we have \( E_G, V_G \subseteq \text{sSb}(G) \). Note that if \( T \) is a simply-connected directed graph (i.e., an object in \( \mathbf{O}_0 \)), then every connected subgraph is a structured subgraph, so we may identify \( \text{sSb}(T) \) with \( \text{Emb}(T) \).

The following formulation is due to Hongyi Chu and the author [CH22, Definition 2.2.11], who also showed it is equivalent to the graphical category from [HRY15, Chapter 6]. This description is akin to the simpler Definition 3.3, but predates it.

**Definition 4.8.** The proaradic graphical category, denoted \( \mathbf{G} \), has objects the acyclic directed graphs. A morphism \( \varphi: G \to G' \) consists of a pair of functions \( \varphi_0: E_G \to E_{G'} \) and \( \hat{\varphi}: \text{sSb}(G) \to \text{sSb}(G') \) so that \( (\varphi_0, \hat{\varphi}) \) is boundary-compatible and \( \hat{\varphi} \) preserves unions. By ‘preserving unions,’ we mean that if \( H_1, H_2 \in \text{sSb}(G) \) are such that \( H_1 \cup H_2 \) is a structured subgraph, then \( \hat{\varphi}(H_1 \cup H_2) = \hat{\varphi}(H_1) \cup \hat{\varphi}(H_2) \).

Composition in \( \mathbf{G} \) is given by composition of pairs.

It turns out that given a map in \( \mathbf{G} \), one can extend \( \hat{\varphi}: \text{sSb}(G) \to \text{sSb}(G') \) to a function \( \text{Emb}(G) \to \text{Emb}(G') \) in a canonical way. This is one ingredient in the following (see [Hac] for a proof).

**Theorem 4.9.** The proaradic graphical category \( \mathbf{G} \) may be identified with the subcategory of \( \mathbf{O} \) with

1. objects the acyclic directed graphs, and
2. morphisms those \( \varphi: G \to G' \) so that \( \hat{\varphi}(G) \in \text{sSb}(G') \).

Since \( \text{sSb}(G) \) and \( \text{Emb}(G) \) coincide when \( G \) is simply-connected, Proposition 4.3 implies the following (or directly compare Definition 3.3 and Definition 4.8).

**Corollary 4.10.** The full subcategory of \( \mathbf{G} \) on the directed trees is \( \mathbf{O}_0 \). □

The notions of active and inert maps from Definition 4.5 can be modified in this context, where now maps in \( \mathbf{G}_{\text{int}} \) are those maps which are isomorphic to a structured subgraph inclusion. This again is a factorization system \( (\mathbf{G}_{\text{act}}, \mathbf{G}_{\text{int}}) \), restricted from the one on \( \mathbf{O} \) from Theorem 4.6 (see [Koc16, 2.4.14]).

5. THE SEGAL CONDITION AND GENERALIZED OPERADS

In this section, we turn to the main topic of the paper, which is the interpretation of generalized operads as Segal presheaves for appropriate graph categories.

First, let us give an idea about how to understand certain elements in a presheaf. Let \( X \in \hat{\mathbf{U}} \) be an \( \mathbf{U} \)-presheaf. We interpret the set \( X_\downarrow \) as a set of colors. Since \( \downarrow \) has a non-trivial automorphism in \( \mathbf{U} \), the set \( X_\downarrow \) attains a potentially non-trivial involution. (If instead \( X \) was an \( \mathbf{O} \)-presheaf, then we would not expect to have an involution on \( X_\downarrow \), as \( \downarrow \in \mathbf{O} \) only possesses the identity automorphism.) We interpret
$X_{\star_n}$ as operations with arity $n$. Using the maps $\downarrow : \star_n \to \star_n$ classifying each boundary arc in the $n$-element set $\partial(\star_n) = n$, we obtain a function

$$X_{\star_n} \to \prod_\mathbb{n} X_{\uparrow}$$

giving the boundary profile of an operation. (Likewise, if $X \in \hat{O}$ then we have functions $X_{\star_{n,m}} \to \prod_\mathbb{n} X_\downarrow$ and $X_{\star_{n,m}} \to \prod_\mathbb{m} X_\downarrow$ giving the inputs and outputs.)

Suppose now that $G \in U$ has two vertices $u, v$ and exactly one internal edge, as in Figure 10. We then have a span of sets as follows:

$$X_G$$

$$X_{\star_G} \leftarrow X_{\star_u} \times X_{\star_v}$$

The leftward leg is associated to an active map $\star_G \to G$ from a star graph (we may take $\star_G$ to be a star with $\partial(\star_G) = \partial(G)$), and the rightward leg is associated to the two inert maps $\iota_u : \star_u \to G$ and $\iota_v : \star_v \to G$. This span represents a kind of generalized way of ‘composing’ operations $X$. If the leg on the right happens to be a bijection, then this is a legitimate function

$$X_{\star_G} \leftarrow X_{\star_u} \times X_{\star_v}$$

giving composition of two operations in $X$. This is the kind of fundamental operation one has in a(n augmented) cyclic operad. In the directed setting, the evident variation gives a dioperadic composition.

Instead, suppose that $G \in U$ has one vertex $v$ and exactly one internal edge, which is necessarily a loop at $v$, as in Figure 11. We can again form a span

$$X_G$$

$$X_{\star_G} \leftarrow X_{\star_v} \subset X_{\star_v}$$

whose right leg lands in the subset $X_{\star_v}$ of $X_{\star_v}$ consisting of those elements which agree along the two embeddings $j_a : \downarrow \to \star_v$ with $\iota_v j_a = \iota_v j_a' : \downarrow \to \star_v \to G$. If this rightward map is a bijection, this gives a contraction of suitable operations

$$X_{\star_G} \leftarrow X_{\star_v}.$$

---

4When considering a $\textbf{G}$-presheaf $X$, it is more appropriate to work with acyclic graphs $G$ having two vertices, but an arbitrary number of internal edges, resulting in a span of the form $X_{\star_G} \leftarrow X_G \to X_{\star_u} \times \prod X_\downarrow X_{\star_v}$, where $\prod X_\downarrow$ is the product indexed by the inner edges of $G$. 
Figure 11. Graph with one vertex and one internal edge

These are precisely the sort of contractions that one would expect to see in a modular operad. In the directed case, the loop must go from an output of the vertex to an input, and we arrive at the contractions for a wheeled properad.

There are of course axioms that these compositions and contractions should satisfy (unitality, associativity, etc.), which can likewise be governed by graphs. Let us give the full abstract definition.

**Definition 5.1 (Segal presheaves).** Suppose $\mathcal{C}$ is one of the graph categories we have discussed above ($U, U_0, U_{\text{cyc}}, O, G, O_0, \Omega, \Delta$), and let $\mathcal{C}_{\text{el}} \subseteq \mathcal{C}_{\text{int}}$ denote the category whose objects are elementary graphs, that is, edges and stars, and whose morphisms are inert maps.

- If $G \in \mathcal{C}$ is a graph, write $\mathcal{C}_{\text{el}}^G := \mathcal{C}_{\text{el}} \times \mathcal{C}_{\text{int}}^G$ for the category whose objects are inert maps $K \rightarrow G$ from an elementary object $K$ to $G$, and whose morphisms are inert maps.

- If $X \in \hat{\mathcal{C}}$ is a presheaf, then the **Segal map** at $G$ is the function
  \[ X_G \rightarrow \lim_{K \rightarrow G} X_K. \]

- A presheaf $X \in \hat{\mathcal{C}}$ is said to satisfy the **Segal condition** if the Segal map is a bijection for all $G \in \mathcal{C}$.

We also say that $X$ is a **Segal presheaf** in this case. We write $\text{Seg}(\mathcal{C}) \subset \hat{\mathcal{C}}$ for the full subcategory on the Segal presheaves.

One can unravel that if $\mathcal{C} = U$, then the rightward maps in the spans (3) and (4) are equivalent to the Segal maps for these graphs. This uses a final subcategory argument (we can omit the objects of $\mathcal{C}_{\text{el}}^G$ corresponding to boundary edges without changing the limit). Likewise, if $\mathcal{C} = \Delta$, the Segal condition reduces to the one at the beginning of the introduction (the Segal maps on $X_1$ and $X_0$ are always identities). As an exercise, the reader should determine for which graph categories $\mathcal{C}$ all of the representable presheaves $\mathcal{C}(-, G)$ are Segal.

5.1. **The general situation.** There are now abstract settings for Segal conditions, which cover many situations of interest, including all of those above. Chu and Haugseng gave an extensive study of the abstract Segal condition for $\infty$-categories in [CH21], and the following appears as Definition 2.1 therein.

**Definition 5.2.** An algebraic pattern is an $\infty$-category $\mathcal{P}$ equipped with an inert-active factorization system $(\mathcal{P}_{\text{int}}, \mathcal{P}_{\text{act}})$, along with some specified full subcategory $\mathcal{P}_{\text{el}} \subseteq \mathcal{P}_{\text{int}}$ whose objects are called elementary.

Given a functor $X: \mathcal{P} \rightarrow \mathcal{S}$, one says that $X$ is Segal if $X(p) \rightarrow \lim_{e \in (\mathcal{P}_{\text{el}})_p} X(e)$ (where here the limit is in the $\infty$-categorical sense) is an equivalence for all $p \in \mathcal{P}$. If $\mathcal{C}$ is any one of the graph categories under discussion, then $\mathcal{P} = \mathcal{C}^{\text{op}}$ is an algebraic pattern in this sense, and this condition agrees with that from Definition 5.1.
Less general are the hypermoment categories of Berger \cite{Ber}, a framework that still includes the graph categories. To state the definition, recall that the skeletal category of finite pointed sets $\mathcal{F}_*$ has an inert-active factorization system \cite[Remark 2.1.2.2]{Lur}. A map $\{\ast, 1, 2, \ldots, n\} = \langle n \rangle \to \langle m \rangle$ is active when only the basepoint maps to the basepoint, and inert if exactly one element of $\langle n \rangle$ maps to each non-basepoint element of $\langle m \rangle$. We are interested in the corresponding active-inert factorization system on $\mathcal{F}_*^{op}$.

**Definition 5.3** (Definition 3.1 of \cite{Ber}). A (unital) hypermoment category consists of a category $\mathcal{C}$, an active-inert factorization system on $\mathcal{C}$, and a functor $\gamma: \mathcal{C} \to \mathcal{F}_*^{op}$ respecting the factorization systems. These data must satisfy the following:

1. For each $c \in \mathcal{C}$ and each inert map $\langle 1 \rangle \to \gamma(c)$ in $\mathcal{F}_*^{op}$, there is an essentially unique inert lift $u \to c$ where $u$ is a unit. Here, unit means that $\gamma(u) = \langle 1 \rangle$ and any active map with codomain $u$ has precisely one inert section.

2. For each $c \in \mathcal{C}$, there is an essentially unique active map $u \to c$ whose domain is a unit.

Given a hypermoment category $\mathcal{C}$, one can define $\mathcal{C}_{\text{int}} \subset \mathcal{C}_{\text{inj}}$ to be the full subcategory spanned by the units and the nilobjects, those objects living over $\langle 0 \rangle \in \mathcal{F}_*^{op}$. Imitating Definition 5.1 gives a notion of Segal presheaf on $\mathcal{C}$.

The graph categories we have discussed so far become hypermoment categories by means of the functor $V: \mathcal{C} \to \text{FinSet}^{op}$ that sends a graph $G$ to $(V_G)^+$. If $\varphi: G \to G'$ is a morphism and $w \in V_G$, then $V(\varphi)(w) = v$ just when $w$ is a vertex in $\varphi(\ast_v)$. Otherwise $w$ is sent to the basepoint of $(V_G)^+$. By vertex-disjointness, this is a well-defined function $(V_G)^+ \to (V_G)^+$. The units are precisely the stars, and the maps $\ast_G \to G$ are those from \cite{Ber,Hac}. See \cite{Ber,Hac} for details.

### 5.2. Nerves

In Section 3, we saw a construction taking a tree $G$ to the (augmented) cyclic operad $\mathcal{C}(G)$ which is freely generated by the vertices and arcs of $G$. This induced a (fully faithful) functor $U_0 \to \mathcal{Cyc}^\dagger$. This construction be imitated for each of the graph categories, yielding the following collection of functors into categories of generalized operads.

| Functor | Structure | Functor | Structure |
|---------|-----------|---------|-----------|
| $O \to \text{WPpd}$ | Wheeled properad | $U \to \text{ModOp}$ | Modular operad |
| $G \to \text{Ppd}$ | Properad | $U_0 \to \mathcal{Cyc}^\dagger$ | Aug. cyclic operad |
| $O_0 \hookrightarrow \text{DiOp}$ | Dioperad | $U_0 \hookrightarrow \mathcal{Cyc}$ | Cyclic operad |
| $\Omega \hookrightarrow \text{Opd}$ | Operad |
| $\Delta \hookrightarrow \text{Cat}$ | Category |

**Remark 5.4.** Most of the functors in the preceding table are fully faithful, with the exceptions of $O \to \text{WPpd}$, $U \to \text{ModOp}$, and $G \to \text{Ppd}$. We can uniquely factor each of them into an identity-on-objects functor, followed by a fully faithful functor. Two of these,

\[
\begin{align*}
U \xrightarrow{\text{idem-on-obj}} & \quad R \xleftarrow{\text{f.f.}} \text{ModOp} \\
G \xrightarrow{\text{idem-on-obj}} & \quad K \xleftarrow{\text{f.f.}} \text{Ppd}
\end{align*}
\]
have appeared in the work of Raynor (see [Ray21, §8.4], where \( R \) is called \( \Xi \)) and Kock (see [Koc16, 2.4.14], where \( K \) is called \( \tilde{Gr} \)). These sources include detailed descriptions of working in these categories, which are somewhat more complicated in structure than those they extend. For example, these categories possess weak factorization systems, in contrast with the orthogonal factorization systems present on \( U \) and \( G \). Further, there are not any clear functors into \( F^{op} \), since vertex-disjointness fails (see [CH22, Remark 7.1.10]).

Each of the functors in the table induces a nerve functor. For example, suppose \( M: U \rightarrow \text{ModOp} \) is the functor that takes an undirected graph \( G \) to the \( A_G \)-modular operad \( M(G) \) freely generated by its vertices. Then we can define a functor

\[
N: \text{ModOp} \rightarrow \hat{U}
\]

by \( NP = \text{ModOp}(M(\_), P): U^{op} \rightarrow \text{Set} \). Practically by definition, we have that

\[
\lim_{\mathcal{K} \in \mathcal{U}/G} M(K) \rightarrow M(G)
\]

is an isomorphism in \( \text{ModOp} \), which formally implies that \( NP \) is a Segal presheaf.

This construction and argument applies to all of the graph categories and functors in the table above, giving nerve functors as follows.

| ModOp | \( \hat{U} \) | WPpd | \( \hat{O} \) | Opd | \( \hat{\Omega} \) |
|-------|-------------|------|----------|-----|------------|
| Cyc* | \( \hat{U}_0 \) | Ppd | \( \hat{G} \) | Cat | \( \hat{\Delta} \) |
| Cyc | \( \hat{U}_{cyc} \) | DiOp | \( \hat{O}_0 \) |

Each of these factors through a full subcategory \( \text{Seg}(C) \hookrightarrow \hat{C} \) of Segal presheaves. Generalizing the classical equivalence \( \text{Cat} \simeq \text{Seg}(\Delta) \) from the beginning of the introduction, we have the following.

**Theorem 5.5** (Nerve theorem). The nerve functors

\[
\text{WPpd} \rightarrow \text{Seg}(O), \text{Ppd} \rightarrow \text{Seg}(G), \text{Opd} \rightarrow \text{Seg}(\Omega), \text{and} \text{ModOp} \rightarrow \text{Seg}(U)
\]

are equivalences of categories.

**Proof.** These appear in [HRY18], [HRY15], [Web07], and [HRY20a]. \( \square \)

**Slogan 5.6.** In the presence of the nerve theorem, one can work entirely at the level of Segal presheaves, rather than with the original operadic structure.

**Remark 5.7.** It is expected that the three missing cases \( \text{DiOp} \rightarrow \text{Seg}(O_0) \), \( \text{Cyc}^* \rightarrow \text{Seg}(U_0) \), and \( \text{Cyc} \rightarrow \text{Seg}(U_{cyc}) \) are also equivalences. I have been told that the latter two equivalences will appear in the PhD thesis of Patrick Elliott, along with a comparison with the ‘strict inner-Kan condition.’ Meanwhile, the nerve theorem for dioperads follows from the nerve theorem for augmented cyclic operads (see Proposition 6.4 below). Alternatively, in light of Proposition 3.2 (and its dioperadic analogue), it is expected that one can bring to bear the tools of abstract nerve theory to recover these three cases.

The aforementioned abstract nerve theory originated in work of Weber [Web07] and unpublished work of Leinster. See also [BMW12, BG19], or the overview of the key points appearing in [Ray21, §2.1]. In our restricted setting, one can consider \( \hat{U}_{el} \) as a category of graphical species ([JK11, §4] and [Ray21, §1.1]) and \( \hat{O}_{el} \) as
digraphical species \[ \text{[Koc16, §2.1]}, \text{which are a kind of collection with morphisms built in for the coloring maps. Then } \text{ModOp can be considered as the Eilenberg–Moore category of algebras } \text{Alg}(T) \text{ for the free modular operad monad } T \text{ on } \hat{U}_{el}. \text{For particularly nice monads (strongly cartesian), one can automatically produce from this set-up a full subcategory } \Theta_T \subset \text{Alg}(T) \text{ which satisfies a nerve theorem } \text{Alg}(T) \xrightarrow{\sim} \text{Seg}(\Theta_T) \subset \Theta_T. \text{This applies to produce, for instance, } \Delta, \Omega \text{ (see Example 2.14 and Example 4.19 of [Web07]), and also other important categories such as Joyal’s cell category } \Theta_n.\]

**Remark 5.8.** There are also nerve theorems for the categories \( K \) and \( R \) from Remark 5.4. The nerve functors (with a similar definition as above)
\[
N: \text{ModOp} \to \hat{R} \quad N: \text{Ppd} \to \hat{K}
\]
are fully faithful, and the essential image of each is the subcategory of Segal presheaves. The first of these is \[ \text{[Ray21] Theorem 8.2} \text{ (see also [HRY20b, §4]), while the second is } \text{[Koc16] Theorem 2.3.9}. \text{Both of these results utilize the framework for abstract nerve theorems, though there are intricate subtleties in both instances, and neither is a straightforward application of existing theory.}\]

**Question 5.9.** So far in the main text of this paper, we have not mentioned \( \text{props at all [Mac65, HR15]}, \text{yet these are one of the most widespread and expressive kinds of operadic structures. Nor have we mentioned wheeled \( \text{props [MMS09] or nc (non-connected) modular operads [KW17, 2.3.2]}. \text{All of these structures are based on disconnected graphs. A major motivation for the development of Definition 4.1 was to facilitate extensions of the categories } \text{G, O, and U to categories of disconnected graphs } \text{G_{dis}, O_{dis}, and U_{dis}. Specifically, we hope for an appropriate notion of embedding between disconnected graphs, so that we can simply imitate Definition 4.1 directly to obtain } \text{O_{dis} and U_{dis} (and similarly a disconnected version of structured subgraph, to obtain } \text{G_{dis}). We give a wish-list for such a construction:}\]

1. Each of the three categories would have a naturally occurring active-inert orthogonal factorization system, with inert maps the disconnected embeddings (or the appropriate maps for \( \text{G_{dis}} \)).
2. For each of these categories, there is a hypermoment category structure as in Definition 5.3 given by a vertex functor \( V \) into \( \text{FinSet^{op}} \). In particular, there is a natural notion of Segal presheaf.
3. Each of these categories would be a dualizable generalized Reedy category in the sense of [BMT11], as is the case for their connected counterparts ([HRY15 §6.4], [HRY18 Theorem 1.2], [HRY20a §2.2])
4. The nerve theorem holds, that is, we may regard the categories of props, wheeled props, and nc modular operads as the categories of Segal presheaves \( \text{Seg(G_{dis}), Seg(O_{dis}), and Seg(U_{dis}).}\)

Such categories would provide a sound foundation for an approach to homotopy-coherent versions of each operadic structure mentioned, either by appropriately localizing the Reedy model structure on simplicial presheaves, or by localizing \( \infty \)-category of Segal presheaves \( \text{Seg}^\infty(C) \subset \text{Fun(C^{op},S)} \) of [CH21] to impose a Rezk-completeness condition (as in [CH20, CH22]). We note that an entirely different approach to \( \infty \)-props has recently been proposed in [HK].

These proposed categories of disconnected graphs should also be related to the double category of graphs of [BK 2.5], which has as a key component the
total dissection. There is also the category of forests from [HHM16] (where each component is a rooted tree), but we are unsure of the precise relationship to expect.

6. Interactions

In this final section, we turn to some interactions between the operadic structures that have appeared so far.

**Theorem 6.1.** Suppose \( f : C \to D \) is one of the following three functors:

\[
\Omega \to U_{cyc} \quad O_0 \to U_0 \quad O \to U,
\]

and consider the associated adjoint string

\[
\hat{D} \xleftarrow{f} \hat{C} \xrightarrow{f!} \hat{D}
\]

where \( f^* \) is restriction along \( f \), and \( f! \) and \( f_* \) are left and right Kan extension. The following hold:

1. If \( X \in \hat{D} \) is a presheaf, then \( X \) is Segal if and only if \( f^*X \in \hat{C} \) is Segal.
2. If \( Y \in \hat{C} \) is Segal, then so is \( f!Y \in \hat{D} \).

**Proof.** Proposition 6.17, Theorem 6.23, and Proposition 6.24 of [Hac].

In light of the equivalence in Remark 4.4, a directed graph is essentially just the data of an undirected graph along with an element of the orientation presheaf \( o \). If \( x \in o_G \) where \( G \) is an undirected graph, for the moment we will write \( G_x \) for the associated directed graph. If \( G \) is an undirected tree and \( r \in \partial(G) \), then there is a unique directed structure on \( G \) which is a rooted tree with output \( r \); we write \( G_r \) for this rooted tree. The main component of Theorem 6.1 (2) is the following description of the left Kan extension (see [Hac, §6.1]).

**Proposition 6.2.** Let \( f : C \to D \) be one of the forgetful functors \( O_0 \to U_0 \) or \( O \to U \). If \( Z \in \hat{C} \) is \( C \)-presheaf, then the left Kan extension \( f!Z \in \hat{D} \) is given by the formula

\[
(f!Z)_G = \coprod_{x \in o_G} Z_{G_x}.
\]

For the functor \( f : \Omega \to U_{cyc} \), we have

\[
(f!Z)_G = \coprod_{r \in \partial(G)} Z_{G_r}.
\]

**Question 6.3.** Is the analogue of Theorem 6.1 also true in the \( \infty \)-categorical setting? (See Definition 5.2.) That is, for these \( f \), does \( f! : \text{Fun}(C^{op}, S) \to \text{Fun}(D^{op}, S) \) restrict to a functor \( \text{Seg}^{\infty}(C) \to \text{Seg}^{\infty}(D) \) between the \( \infty \)-categories of Segal presheaves from [CH21]?

As is explained in [DCH21 §4.3], the category of dioperads may be viewed as a certain (non-full) subcategory of augmented cyclic operads \( \text{DiOp} \to \text{Cyc}^+ \). It is better to describe this as a certain slice category, that is, \( \text{DiOp} \cong \text{Cyc}^+_{/\text{IO}} \). Here, \( \text{IO} \) is the terminal object of \( \text{Cyc}^+_{\{i,o\}} \) where the color set \( \{i,o\} \) possesses the non-trivial
involution. Another way to say this is that $\mathcal{IO}$ is isomorphic to the image of the terminal dioperad under $\text{DiOp} \to \text{Cyc}^+$, and induces an equivalence

$$\text{DiOp} \simeq \text{DiOp}_{/\ast} \overset{\simeq}{\to} \text{Cyc}_{/\mathcal{IO}}^+.$$  

The map to $\mathcal{IO}$ of course forces all cyclic operads to have free involution on color sets, and also yields a splitting of the profile of an operation into ‘inputs’ and ‘outputs.’ This follows a similar approach for wheeled properads $\text{WPpd} \simeq \text{ModOp}_{/\mathcal{IO}}$ from Example 1.29 of [Ray21].

We use this to establish the following provisional statement (see Remark 5.7).

**Proposition 6.4.** The nerve theorem for augmented cyclic operads implies that for dioperads.

**Proof.** Let $f : \mathcal{O}_0 \to \mathcal{U}_0$ be the functor forgetting the directed structure. Notice that $N(\mathcal{IO}) \cong \mathcal{O}$ is nothing but the orientation $\mathcal{U}_0$-presheaf from Remark 4.4. We then have the following commutative diagram of categories, where on presheaves we use that $f_!(\ast) = \mathcal{O}$ (by inspection, or application of Proposition 6.2).

$$
\begin{array}{cccc}
\text{DiOp} & \overset{\simeq}{\to} & \text{Cyc}_{/\mathcal{IO}}^+ & \overset{\simeq}{\to} & \text{Cyc}^+ \\
& \downarrow N & & \downarrow N \\
\text{Seg}(\mathcal{O}_0) & \overset{\text{t.f.}}{\to} & \text{Seg}(\mathcal{U}_0)_{/N(\mathcal{IO})} & \overset{\text{t.f.}}{\to} & \text{Seg}(\mathcal{U}_0) \\
\hat{\mathcal{O}}_0 & \overset{\simeq}{\to} & \hat{\mathcal{U}}_{0/\mathcal{IO}} & \overset{\text{t.f.}}{\to} & \hat{\mathcal{U}}_0
\end{array}
$$

The equivalence on the top left was mentioned just above, while that on the bottom left is a standard fact on slices of presheaf categories (see, for instance, Exercise 8 in Chapter III of [MM92]) using the second equivalence of categories appearing in Remark 4.4. If $N : \text{Cyc}^+ \to \text{Seg}(\mathcal{U}_0)$ assumed to be an equivalence (that is, if the nerve theorem holds for augmented cyclic operads), then the induced map on slices $\text{Cyc}_{/\mathcal{IO}}^+ \to \text{Seg}(\mathcal{U}_0)_{/N(\mathcal{IO})}$ is also an equivalence. By the left cancellation property for fully faithful functors, $N : \text{DiOp} \to \text{Seg}(\mathcal{O}_0)$ is fully faithful. But this functor is also essentially surjective: if $X \in \text{Seg}(\mathcal{O}_0)$, then we know $f_! X \to N(\mathcal{IO})$ in $\text{Seg}(\mathcal{U}_0)_{/N(\mathcal{IO})}$ is isomorphic to the image of some dioperad $D \in \text{DiOp}$. We can conclude that $X \cong N(D)$ since $\text{Seg}(\mathcal{O}_0) \to \text{Seg}(\mathcal{U}_0)_{/N(\mathcal{IO})}$ is injective on isomorphism classes of objects. Hence $N : \text{DiOp} \to \text{Seg}(\mathcal{U}_0)$ is an equivalence, as desired.  

Essentially the same argument shows that the nerve theorem for modular operads from [HRY20b] implies that for wheeled properads from [HRY18].

If $\mathcal{C}$ is a category, write $s\hat{\mathcal{C}} := \text{Fun}(\mathcal{C}^{op}, \Delta)$ for the category of simplicial presheaves. The following question was posed by Drummond-Cole and the author [DCH21] Remark 6.8], and has implications in the theory of cyclic 2-Segal objects [Wal21, Remark 5.0.20].

**Question 6.5.** Let $f : \Omega \to \mathcal{U}_{\text{cyc}}$ be the functor which forgets the root. Consider the operadic model structure on dendroidal sets $\hat{\Omega}$ from [CM11] and the dendroidal Rezk model structure on $s\hat{\Omega}$ from [CM13a] §6]. We have two adjoint strings associated to

\[ s\hat{\mathcal{C}} \]
the functor $f$:

\[
\begin{array}{c}
\widehat{U}_{cyc} \xleftarrow{f} \widehat{\Omega} \\
\mathcal{U}_{cyc} \xrightarrow{f^*} \widehat{\Omega}
\end{array}
\]

Is the endo-adjunction $f^* f_* \dashv f_* f^*$ on $\widehat{\Omega}$ (resp. on $s\widehat{\Omega}$) a Quillen adjunction? The formula from Proposition 6.2 may well help answer this question. If $f^* f_*$ is a left Quillen functor, then the adjoint string lifting results from [DCH19] will imply the existence of a lifted model structure on $\widehat{U}_{cyc}$ (resp. on $s\widehat{U}_{cyc}$). Further, this should yield a Quillen equivalence with the category of simplicially-enriched cyclic operads, lifting those for simplicially-enriched operads from [CM13b].

For the potential relationship with cyclic 2-Segal objects mentioned in [Wal21], one should instead use planar and non-symmetric versions of trees and (cyclic) operads everywhere (including the model structure on planar dendroidal sets from [Moe10, §8.2]). It is likely that the planar/non-symmetric versions follow from the symmetric versions via an appropriate slicing, as in [Gag15].

**Question 6.6.** In [Ray21, Corollary 8.14], Raynor gives a model structure on the category of simplicial presheaves $s\widehat{\mathcal{R}}$ where the fibrant objects are the projectively fibrant presheaves which satisfy a weak version of the Segal condition. Similarly, the category $s\widehat{\mathcal{U}}$ admits Segal-type model structure, obtained by taking a left Bousfield localization of the projective model structure (a variation of this appears in [HRY20a, Proposition 3.10]). It would be interesting to know if the restriction map $s\widehat{\mathcal{R}} \to s\widehat{\mathcal{U}}$ is a right Quillen equivalence with respect to these model structures. This would reflect the equivalence of categories $\text{Seg}(\mathcal{R}) \to \text{Seg}(\mathcal{U})$ between ordinary simplicial presheaves (as follows from Remark 5.8 and Theorem 5.5).

### Appendix A. Proofs

**A.1. Full tree maps preserve intersections.** In this section, we show that it is automatic that full tree maps preserve intersections. This was shown in a round-about way in [Hac], which went through the properadic graphical category. Here we give an elementary direct proof, that deals only with trees. It is important that our notion of boundary-compatibility refers to involutive maps $A \to A'$ rather than maps $E_G \to E_{G'}$ (compare [HRY19, Definition 1.12]).

**Lemma A.1.** Suppose $G$ and $G'$ are undirected trees, and $(\varphi_0, \varphi_1)$ is a tree map from $G$ to $G'$. If $u \neq v$ are distinct vertices of $G$, then the subtrees $\varphi_1(u)$ and $\varphi_1(v)$ of $G'$ do not share any vertices in common.

**Proof.** By way of contradiction, suppose that $w$ is a vertex in both subtrees $S = \varphi_1(u)$ and $T = \varphi_1(v)$. For simplicity, we first treat the case when $u$ and $v$ are adjacent vertices. That is, there is a (unique) arc $\tilde{a} \in D_G$ with $t(\tilde{a}) = v$ and $t(\tilde{a}^\dagger) = u$. Let $a = \varphi_0(\tilde{a})$. By boundary-compatibility we have $a \in \partial(S) = \partial(\varphi_1(u))$ and $a^\dagger \in \partial(T) = \partial(\varphi_1(v))$. Since $S$ is a tree, there is a unique path in $S$

\[
P_S := v_0 a_0 v_1 a_1 \cdots v_n a_n
\]
with \( v_0 = w \) and \( a_n = a \). By path in \( S \) we mean that \( v_i \in V_S \), \( a_i \in A_S \), and these elements satisfy the equations \( t(a_i) = v_{i+1} \) and \( t(a'_i) = v_i \) whenever both sides are defined in \( S \) (see Definition 2.10). Likewise, there is a unique path in \( T \)
\[
\phi_T := v'_0 a'_0 v'_1 \cdots v'_m a'_m
\]
with \( v'_0 = w \) and \( a'_m = a' \).

Then we have a composite path in \( G \)
\[
P_S P_T = v_0 a_0 v_1 a_1 \cdots v_n a_n v'_m (a'_{m-1})\cdots v'_{m-1} \cdots (a')\cdots v'_1 (a'_0) v'_0
\]
\[
= v_0 a_0 v_1 a_1 \cdots v_n a_n v_{n+1} a_{n+1} v_{n+2} \cdots a_{n+m-1} v_{n+m} a_{n+m} v_{n+m+1}
\]
where \( v_{n+k} = v'_{m-k+1} \) for \( 1 \leq k \leq m + 1 \) and \( a_{n+j} = (a'_{m-j})' \) for \( 0 \leq j \leq m \). Since \( v_0 = w = v_{n+m+1} \), this composite path \( P_S P_T \) is a cycle in \( G' \). This is impossible, since \( G' \) is a tree.

The general case, when \( u \) and \( v \) are not adjacent in \( G \), is proved using similar ideas. Namely, there is a unique path in \( G \)
\[
u_0 \tilde{a}_0 u_1 \tilde{a}_1 \cdots \tilde{a}_\ell u_{\ell+1}
\]
with \( u_0 = u \) and \( u_{\ell+1} = v \). We define \( P_S \) to be the path in \( S \) starting at \( w \) and ending at \( \varphi_0(\tilde{a}_0) \), and \( P_T \) to be the path in \( T \) starting at \( w \) and ending at \( \varphi_0(\tilde{a}_\ell) \). For \( 1 \leq j \leq \ell \) there is a unique path \( P_j \) in \( \varphi_1(u_j) \) which starts with \( \varphi_0(\tilde{a}_j) \) and ends with \( \varphi_0(\tilde{a}_j) \). The composite path \( P_S P_1 \cdots P_\ell P_T \) will be a cycle at \( w \), which again is impossible since \( G' \) is a tree.

**Lemma A.2.** Suppose \( G \) and \( G' \) are undirected trees, and \((\varphi_0, \varphi)\) is a full tree map from \( G \) to \( G' \). If \( S, T \in \text{Emb}(G) \) are subtrees of \( G \) so that \( \varphi(S) \) and \( \varphi(T) \) have a vertex \( w \in V_{G'} \) in common, then there is a unique vertex \( v \in V_S \cap V_T \subset V_G \) so that \( w \in \varphi(\star)v \).

**Proof.** Suppose \( w \) is a vertex of \( \varphi(S) \cap \varphi(T) \). Writing \( S \) as an iterated union of distinct stars
\[
S = (\cdots ((\star \cdots \star^{k-1} \cup \star^3) \cup \cdots) \cup \star^{n-1}) \cup \star^n
\]
we must have \( w \in \varphi(\star^k) \) for some \( k \) since \( \varphi(S) \) is the union of the \( \varphi(\star^k) \). Likewise, \( w \in \varphi(\star^k) \) for some \( \star \rightarrow T \). By Lemma A.1 applied to \((\varphi_0, \varphi|_{V_G})\), we conclude that \( \star = \star^k \). Then \( v \) is the vertex of the star \( \star = \star^k \) which is in both \( S \) and \( T \). Uniqueness is guaranteed by Lemma A.1.

For subtrees of a graph \( G \), the graph \( S \cap T \) is either empty or again a subtree. In particular, if \( S \cap T \) does not contain a vertex, then it is either empty or a single edge.

**Proposition A.3.** Full tree maps preserve intersections between overlapping subtrees. That is, if \((\varphi_0, \varphi): G \rightarrow G' \) is a full tree map and \( S, T \in \text{Emb}(G) \) are subtrees with \( S \cap T \) non-empty, then \( \varphi(S \cap T) = \varphi(S) \cap \varphi(T) \).

**Proof.** Since \( \varphi: \text{Emb}(G) \rightarrow \text{Emb}(G') \) preserves unions, it also preserves the partial order. As \( S \cap T \) is inhabited, it is a subtree of both \( S \) and \( T \), so
\[
\varphi(S \cap T) \subseteq \varphi(S) \cap \varphi(T).
\]
Notice that both sides are subtrees of \( G' \). If \( w \) is a vertex in \( \varphi(S) \cap \varphi(T) \), then \( w \) is also in \( \varphi(S \cap T) \) by Lemma A.2. Since subtrees containing a vertex are completely
determined by their vertices, we conclude that [4] is an equality. If \( \varphi(S) \cap \varphi(T) \) does not contain any vertices, then it is a single edge, so again [5] is an equality. □

A.2. Tree maps and full tree maps coincide. Our goal is to prove that there is no real difference between tree maps and full tree maps from Definition 3.1.

**Lemma A.4.** Suppose \( S \) and \( T \) are subtrees of a tree \( G \), and that \( S \) and \( T \) intersect at a single edge \( e = [a,a'] \), where \( a \in \partial(S) \) and \( a' \in \partial(T) \). If neither \( S \) nor \( T \) is an edge, then
\[
\partial(S \cup T) = (\partial(S) \setminus \{a\}) \cup (\partial(T) \setminus \{a'\}).
\]
If \( S \) is an edge then \( \partial(S \cup T) = \partial(T) \) and if \( T \) is an edge then \( \partial(S \cup T) = \partial(S) \).

**Proof.** If \( S \) is an edge then \( S \cup T = T \), and similarly \( S \cup T = S \) when \( T \) is an edge, so the final statement is immediate. We suppose that neither \( S \) nor \( T \) is an edge and compute the boundary of the union.

Since \( S \) is not an edge and \( a \in \partial(S) \), there is a vertex \( v \in V_S \) with \( t(a^1) = v \). Since \( v \) is also in \( V_{S \cup T} \), it follows that \( a^1 \in \partial(S \cup T) \). A symmetric argument shows that \( a \notin \partial(S \cup T) \).

Suppose \( a' \in \partial(S) \setminus \{a\} \), so that \( t((a')^1) \in V_S \). We either have \( a' \in \partial(G) \), or \( t(a') = v \) is a vertex of \( G \) which is not in \( S \). If \( a' \in \partial(S) \), then \( a' \in \partial(S \cup T) \). If \( t(a') = v \in V_S \setminus V_T \), then either \( v \in V_T \) or \( v \notin V_T \). If \( v \in V_T \), then \( e' = [a',(a')^1] \) is an edge in \( S \cap T \) distinct from \( e = [a,a'] \), contrary to our assumption. If \( v \notin V_T \), then \( e \notin V_{S \cup T} \), hence \( a' \in \partial(S \cup T) \). Since \( a' \) was arbitrary, we have shown \( \partial(S) \setminus \{a\} \subset \partial(S \cup T) \); reversing roles gives \( \partial(T) \setminus \{a^1\} \subset \partial(S \cup T) \).

It remains to observe that \( \partial(S \cup T) \subset \partial(S \cup T) \); since \( S \cup T \) is not an edge, we have \( t((a')^1) \in V_{S \cup T} = V_S \cup V_T \). Without loss of generality suppose \( t((a')^1) \in V_S \). Since \( a' \) is \( \partial(S \cup T) \), either \( a' \in \partial(G) \) or \( a' \in D_G \) and \( t(a') = w \in V_G \setminus V_{S \cup T} \subseteq V_G \setminus V_S \). In both cases, we conclude that \( a' \in \partial(S) \), as desired. □

**Lemma A.5.** Suppose \( G \) is a tree and \( T \) is a proper subtree. Then there exists a star subtree \( \star \subset G \) and a proper subtree \( R \subset G \) so that \( R \) overlaps with \( \star \), the union \( R \cup \star \) is equal to \( G \), and \( T \subset R \).

**Proof.** Note that an edge does not have proper subtrees, so \( G \) has at least one vertex. If \( G \) is a star, then \( T \) must be an edge, and one takes \( \star = G \) and \( R = T \). If \( G \) has more than one vertex and \( T \) is an edge, then one can take \( \star \) to be any extremal vertex of \( G \) which does not contain \( T \) as a boundary, and \( R \) to be the complement of \( \star \).

We now suppose \( T \) contains a vertex. Let \( v \in V_G \setminus V_T \) be a vertex of minimum positive distance from \( T \), that is, there is an edge \( e \) between \( v \) and some vertex \( w \) of \( T \). Consider the collection of all paths of \( G \) which contain \( e \) but do not contain \( w \), and let \( S \subset G \) be the subtree consisting of all vertices appearing on such paths, and all arcs appearing on edges on such paths. Notice that \( v \in V_S \). If \( S \) is a star then \( v \) is extremal in \( G \), and we set \( \star = S \) and let \( R \) be its complement. Otherwise, let \( u \in V_S \) be an extremal vertex of \( S \) other than \( v \) (which may or may not be extremal in \( S \)). The vertex \( u \) is also extremal in \( G \), and we let \( \star \) be its star and \( R \) the complement. In either case, the three conditions hold. □

**Proposition A.6.** Suppose \( G \) and \( G' \) are undirected trees. The assignment \((\varphi_0, \varphi) \mapsto (\varphi_0, \varphi|_{V_G})\) is a bijection between tree maps and full tree maps from Definition 3.1.
Proof. Let $\text{Emb}^n(G) \subseteq \text{Emb}(G)$ denote set of subtrees having $n$ or fewer vertices. This gives a (finite) filtration of the set $\text{Emb}(G)$, starting with

$$\text{Emb}^0(G) = E_G \subseteq E_G \cup V_G = \text{Emb}^1(G).$$

Given a tree map $(\varphi_0, \varphi_1)$, we inductively define $\check{\varphi}_n : \text{Emb}^n(G) \to \text{Emb}(G)'$ and show that it satisfies boundary-compatibility and preserves unions that exist in $\text{Emb}^n(G)$. The first map $\check{\varphi}_0 : E_G \to E_{G}' \subseteq \text{Emb}(G)'$ is induced from $\varphi_0 : A_G \to A_{G'}$, while $\check{\varphi}_1$ is defined to be $\check{\varphi}_0 \circ \varphi_1$. These both satisfy boundary-compatibility. Any unions $S \cup T \in \text{Emb}^1(G)$ for overlapping $S, T$ which exist must either have $S = T$ or one of the two terms is an edge. If $S$ is an edge or if $S = T$, then $S \cup T = T$, and also $\check{\varphi}_1(S) \cup \check{\varphi}_1(T) = \check{\varphi}_1(T)$ since either $\check{\varphi}_1(S)$ is an edge or $\check{\varphi}_1(S) = \check{\varphi}_1(T)$. This $\check{\varphi}_1$ preserves unions.

We continue inductively. Suppose $\check{\varphi}_n : \text{Emb}^n(G) \to \text{Emb}(G)$ is defined, preserves unions that exist in the $n$th filtration level, and $(\varphi_0, \check{\varphi}_n)$ is boundary-compatible. Let $n \geq 1$. Any subtree $R$ having $n + 1$ vertices can be written as $S \cup \star$, where $\star$ is a star and $S$ has $n$ vertices. It is automatic that $S$ and $\star$ intersect only at an edge $e = [a, a']$. We make a provisional definition $\check{\varphi}_{n+1}(R) := \check{\varphi}_n(S) \cup \check{\varphi}_1(\star)$, which of course depends on a choice of $S \subseteq R$ having $n$ vertices. If $n + 1 = 2$, then $S$ is itself a star and by symmetry there is no actual choice involved in this definition. Given a different subtree $T \subseteq R$ having $n \geq 2$ vertices, the tree $S \cap T$ will have $n - 1$ vertices. We have $R = T \cup \star'$ for a star $\star' \subset S \subset R$, with $T$ and $\star'$ intersecting at an edge $e'$. Note that $e, e'$ are both in $S \cap T$. We then have

$$\check{\varphi}_n(T) \cup \check{\varphi}_1(\star') = \check{\varphi}_n((S \cap T) \cup \star) \cup \check{\varphi}_1(\star')$$

$$= \check{\varphi}_{n-1}(S \cap T) \cup \check{\varphi}_1(\star) \cup \check{\varphi}_1(\star')$$

$$= \check{\varphi}_n((S \cap T) \cup \star') \cup \check{\varphi}_1(\star) = \check{\varphi}_n(S) \cup \check{\varphi}_1(\star)$$

which shows that $\check{\varphi}_{n+1}$ is well-defined.

Further, observe that $\check{\varphi}_n(S)$ and $\check{\varphi}_1(\star)$ (as above) intersect only at the single edge $\tilde{e} = [\varphi_0(a), \varphi_0(a')]$. Indeed, the proof of Lemma A.2 works equally well for $\check{\varphi}_n$ as it does for $\varphi_n$, so $\check{\varphi}_n(S)$ and $\check{\varphi}_1(\star)$ cannot share any vertices.

We can use this to show that $\check{\varphi}_{n+1}$ is boundary-compatible. As above, suppose that $R = S \cup \star$ has $n + 1$ vertices, $\star$ is a star, $S$ has $n$ vertices, and $S$ and $\star$ intersect only at the edge $e = [a, a']$. Here we take $a \in \partial(S)$ and $a' \in \partial(\star)$; write $\tilde{a} = \varphi_0(a)$. Boundary-compatibility implies that

$$\partial(S) \xrightarrow{\check{\varphi}_0} \partial(\check{\varphi}_n(S)) \quad \partial(\star) \xrightarrow{\check{\varphi}_1} \partial(\check{\varphi}_1(\star))$$

are bijections. If neither $\check{\varphi}_n(S)$ nor $\check{\varphi}_1(\star)$ is an edge, then we have the desired bijection below by Lemma A.4

$$\partial(S \cup \star) \longleftrightarrow (\partial(S) \setminus \{a\}) \amalg (\partial(\star) \setminus \{a'\})$$

$$\partial(\check{\varphi}_{n+1}(S \cup \star)) \longleftrightarrow (\partial(\check{\varphi}_n(S) \cup \check{\varphi}_1(\star)) \longrightarrow (\partial(\check{\varphi}_n(S)) \setminus \{\tilde{a}\}) \amalg (\partial(\check{\varphi}_1(\star)) \setminus \{\tilde{a}'\})$$

We must still address the case when $\check{\varphi}_n(S)$ is an edge or when $\check{\varphi}_1(\star)$ is an edge. These cases are entirely analogous, so we suppose that $\check{\varphi}_1(\star)$ is an edge. By boundary-compatibility, $\partial(\star) = \{a', a\}$ with $\varphi_0([a,a']) = \varphi_0(a) = \tilde{a}$, so $\varphi_0(a') = \tilde{a}'$. Thus

$$\partial(S \cup \star) = (\partial(S) \setminus \{a\}) \amalg (\partial(\star) \setminus \{a'\}) = (\partial(S) \setminus \{a\}) \amalg \{a'\} \to \partial(\check{\varphi}_n(S))$$

$\square$
is a bijection, as required. We have thus shown that \((\varphi_0, \varphi_{n+1})\) is boundary-compatible.

Finally, we must show that \(\hat{\varphi}_{n+1}\) preserves unions that exist in \(\text{Emb}^{n+1}(G)\). Suppose that \(S \cup T \in \text{Emb}^{n+1}(G)\) is a union of overlapping subtrees \(S\) and \(T\). We may assume that \(S \cup T\) has \(n+1\) vertices, for if it has \(n\) or fewer vertices then \(\hat{\varphi}_{n+1}(S \cup T) = \hat{\varphi}_n(S \cup T) = \hat{\varphi}_n(S) \cup \hat{\varphi}_n(T)\) by the induction hypothesis. We further assume that \(S \neq T\), since if \(S = T\) then \(\hat{\varphi}_{n+1}(S) \cup \hat{\varphi}_{n+1}(T) = \hat{\varphi}_{n+1}(S) = \hat{\varphi}_{n+1}(S \cup T)\). Without loss of generality, suppose there is a vertex of \(S\) which is not in \(T\). By Lemma A.5, there is a decomposition \(S \cup T = R \cup \star\) where \(T\) is a subtree of \(R\), and \(R\) is a proper subtree of \(S \cup T\). Notice also that \(\star \subset S\). Of course \(S \cap R\) is a subtree since it is non-empty (it contains \(S \cap T\)). We then have

\[
S \cup T = (\star \cup S \cap R) \cup T = \star \cup (S \cap R) \cup T.
\]

The union \((S \cap R) \cup T\) contains exactly \(n\) vertices, and we find

\[
\hat{\varphi}_{n+1}(S \cup T) = \hat{\varphi}_1(\star) \cup \hat{\varphi}_n((S \cap R) \cup T)
= \hat{\varphi}_1(\star) \cup (\hat{\varphi}_n(S \cap R) \cup \hat{\varphi}_n(T))
= (\hat{\varphi}_1(\star) \cup \hat{\varphi}_n(S \cap R)) \cup \hat{\varphi}_n(T)
= \hat{\varphi}_{n+1}(\star) \cup (S \cap R) \cup \hat{\varphi}_n(T) = \hat{\varphi}_{n+1}(S) \cup \hat{\varphi}_{n+1}(T).
\]

This uses that \(\hat{\varphi}_n\) preserves unions and the definition of \(\hat{\varphi}_{n+1}\).

If \(G\) has \(k\) vertices, then \(\text{Emb}^k(G) = \text{Emb}(G)\) and we have thus defined a full tree map from \(G\) to \(G'\). This construction is a section to the assignment from the proposition statement. But notice that it is the unique such section: our \(\hat{\varphi}\) must preserve unions, and we defined the section so that it would preserve particular unions. Thus the map in question is a bijection. \(\square\)

A.3. The (augmented) cyclic operad associated to a tree. Our goal in this section is to prove Proposition 3.2. The reader should recall the category of augmented cyclic operads from Definition 2.3 of [DCH21], which we have denoted by \(\text{Cyc}^+\). If \(G\) is an undirected tree, then the (augmented) cyclic operad \(\mathcal{C}(G) \in \text{Cyc}^+_G\) is obtained by choosing for each \(v \in V_G\) an ordering \(a_0, a_1, \ldots, a_n\) of \(\partial(\star_v) = \text{nb}(v)^\dagger\), and considering \(v\) as a free generator in \(\mathcal{C}(G)(a_0, a_1, \ldots, a_n)\). This choice of order does not matter for the isomorphism class of \(\mathcal{C}(G)\).

Lemma A.7. The operations in \(\mathcal{C}(G)\) are in bijection with subtrees \(T\) of \(G\) equipped with a total ordering on the boundary \(\partial(T)\).

Proof. Let \(P = \mathcal{C}(G)\). We have identity elements \(\text{id}_a \in P(a^1, a)\) and \(\text{id}_{a^1} \in P(a, a^1)\), and these correspond to the two possible orderings for the boundary of the edge subtree \([a, a^1]\).

Working inductively by composing with one generator at a time, each non-edge subtree \(T\) determines an element of \(P(\partial(T))\) for some ordering of \(\partial(T)\) by Lemma A.4. The axioms of a cyclic operad guarantee that this element does not depend on the choice of order of attaching vertices to construct the tree \(T\). This is an element \(T \in P(\partial(T)) = P(a_0, \ldots, a_n)\), where the ordering of \(\partial(T)\) is determined by the chosen orderings on \(\partial(\star_v)\). The cyclic operad axioms imply that applying a permutation at any intermediate step simply has the effect of applying some
permutation at the last step

\[ T \in P(a_0, a_1, \ldots, a_n) \rightarrow P(a_{\sigma(0)}, a_{\sigma(1)}, \ldots, a_{\sigma(n)}). \]

Thus all subtrees of equipped with an ordering of the boundary may regarded as (necessarily distinct, by \cite[Lemma A.3]{Hac}) operations of \( P \); we refer to these as simply subtrees in the next paragraph.

Now suppose that \( S, T \) are two non-edge subtrees of \( G \), considered as elements

\[ S \in P(a'_0, \ldots, a'_m) = P(\partial(S)) \quad T \in P(a_0, \ldots, a_n) = P(\partial(T)) \]

with \( a'_i = a^\dagger_j \), so that we can form the composition \( S \circ T \). The paths argument in the first two paragraphs of the proof of Lemma A.1 shows that \( S \) and \( T \) do not share any vertices, hence the composite of \( S \) and \( T \) uses each generator at of \( P \) at most once. It follows that the set of generators used is \( V_S \cup V_T \), and the composite is just the union \( S \cup T \), along with the relevant ordering of \( \partial(S \cup T) \). Since composites of subtrees are again subtrees, all iterated composites of the generators of \( P \) are subtrees. \( \Box \)

Proof of Proposition 3.2. If \( H \) and \( G \) are undirected trees, then a map of cyclic operads \( F: C(H) \rightarrow C(G) \) consists of an involutive function \( f_0: A_H \rightarrow A_G \), and a function \( f \) from \( V_H \) to the operations of \( C(G) \). As we are regarding \( v \in V_H \) as living in \( C(H)(\partial(\ast)) = C(H)(a_0, \ldots, a_n) \), the pair of functions needs only satisfy color-compatibility, that is, \( f(v) \) should live in \( C(G)(f_0(a_0), \ldots, f_0(a_n)) \). By Lemma A.7, \( f(v) \) is a subtree \( T \) of \( G \) together with an ordering of its boundary \( \partial(T) = \{ f_0(a_0), \ldots, f_0(a_n) \} \). But this ordering is not actual data of the morphism, rather is just serves to establish boundary-compatibility. Thus, this assignment is exactly the same thing as a tree map of Definition 3.1. Since \( C \) is a functor, this establishes the desired equivalence. Lemma A.7 also implies that \( C(G) \) is a cyclic operad (rather than an augmented cyclic operad) if and only if \( G \) has inhabited boundary, giving the statement about \( ucyc \). \( \Box \)

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