GLUING THEOREMS FOR ASD METRICS

A.G. KOVALEV AND M.A. SINGER

In 1991, Andreas Floer showed how methods of non-linear elliptic PDEs might be used to obtain anti-self-dual (ASD) metrics on the connected sum of \( l \geq 3 \) copies of \( \mathbb{C}P^2 \). Much subsequent work on ASD metrics has depended upon gluing theorems but has relied more on the twistor-based methods pioneered by Donaldson and Friedman \([DF]\). (A very important exception is Taubes’s ‘stable existence theorem’ for ASD metrics \([T]\).)

The purpose of this note is to report on work that goes in the direction of realizing the full potential of the techniques introduced by Floer. Full proofs will not be given here; the details, along with generalizations and examples, will appear elsewhere \([KS]\). The present exposition is intended to make accessible to a wider audience the important insights that Floer brought to this problem by stating some relatively non-technical, but representative results.

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1. ASD CONFORMAL STRUCTURES

Recall that a Riemannian metric \( g \) on an oriented 4-manifold \( X \) is said to be ASD if the self-dual part \( W^+(g) \) of its Weyl tensor vanishes. The condition is conformally invariant because \( W^+(\hat{g}) = W^+(g) \) for any two metrics \( g \) and \( \hat{g} = e^f g \) in the same conformal class \( c \) on \( M \); for this we regard \( W^+ \) as a section of the bundle \( E^2 \subset \Lambda^2 T^* \otimes \text{End}(T) \) which corresponds, by raising an index, to \( S^2_0 \Lambda^+ \). (For more details and background, see for example \([B, AHS, T1, T2] \) or \([G] \) for the state of the art about 1992.)

The condition \( W^+(g) = 0 \) is a non-linear PDE of second order in \( g \) that is elliptic modulo the action of the diffeomorphism group of \( X \). More precisely, given an ASD metric \( g \), there is an elliptic complex (to be called the deformation complex \([KK]\))

\[
C^\infty(X, E^0) \xrightarrow{L_g} C^\infty(X, E^1) \xrightarrow{D_g} C^\infty(X, E^2)
\]

where \( E^0 = TX \), \( E^1 \) is the bundle of symmetric trace-free endomorphisms of \( TX \), and \( E^2 \) was defined in the previous paragraph. A section of \( E^0 \) yields in the standard way an infinitesimal diffeomorphism of \( X \), while a section of \( E^1 \) yields a tangent to the space of conformal structures on \( X \) at \( g \) via

\[
g(1 + th)(\xi, \eta) := g(\xi, \eta) + t g(\xi, h\eta)
\]
For small $t$. The operator $L_g$ gives the infinitesimal action of the diffeomorphism group on the space of conformal structures—that is, the Lie derivative, while $D_g$ gives the infinitesimal change in $W^+$:

$$L_g \xi = (L \xi) g_0, \quad D_g h = \frac{d}{dt} W^+ [g (1 + th)] |_{t=0}.$$  

$((\cdot)_0 =$ trace-free part$.)$ With this choice of bundles, $L$ and $D$ are exactly conformally invariant:

$$L_g = L_g, \quad D_g = D_g$$

where $g$ and $\hat{g}$ are related as above.

The cohomology groups of $(1.1)$ will be denoted by $H^*_c(X)$ or just $H^*_c$ ($c$ standing for the conformal class of $g$), and play a crucial role in the theory. In particular the vanishing of the obstruction space $H^2_c$ is the basic hypothesis that enters the statement of our gluing theorems. Note that when $X$ is compact, the $H^*_c$ are finite-dimensional vector spaces.

The entire discussion goes through in a standard way to include oriented 4-dimensional orbifolds, with smooth ASD orbifold metrics. The significance of this is that such ASD orbifolds are plentiful, cf. Remark (ii) below. From now on we shall allow ourselves to consider 4-orbifolds, but only those with isolated (i.e. codimension 4) singular points.

Our first extension of Floer’s work is the following:

**Theorem I.** For $i = 1, 2$, let $\overline{X}_i$ be a compact 4-orbifold, with ASD conformal structure $c_i$ and $H^2_{c_i} = 0$. Suppose that $x_i$ is a point of $\overline{X}_i$ such that $x_1$ and $x_2$ are complementary. Then the generalized connected sum $(\overline{X}_1, x_1)\sharp(\overline{X}_2, x_2)$ admits ASD (orbifold) metrics.

As a point of notation, from now on we shall denote compact spaces with overlines to distinguish them from the non-compact manifolds which we shall be working with in §§2–3. Moreover the qualification ‘generalized’ of the term connected sum will usually be omitted.

To explain the word ‘complementary’ in Theorem I recall that by definition of orbifold, there exists a neighbourhood $U_i$ of $x_i$ in $\overline{X}_i$ and homeomorphisms $\varphi_i : \mathbb{R}^4 / \Gamma_i \to U_i$ (with $\varphi_i(0) = x_i$) where $\Gamma_i \subset SO(4)$ is a finite subgroup, such that the induced action of $\Gamma_i$ on $S^3 \subset (\mathbb{R}^4 - 0)$ has no fixed points. Then $x_1$ and $x_2$ are called complementary if there exists an orientation-reversing element of $O(4)$ that conjugates $\Gamma_1$ into $\Gamma_2$. On this definition, any pair of smooth points will be complementary as will any pair of singularities modelled on $\mathbb{R}^4 / \{\pm 1\}$.

The connected sum $(\overline{X}_1, x_1)\sharp(\overline{X}_2, x_2)$ is obtained by deleting small balls centred on $x_1$ and joining the two boundaries by a cylinder (or neck) $Y_1 \times [-l, l]$. Here $Y_1$ is the link of $x_i$ in $\overline{X}_i$; it will be a spherical space-form $S^3 / \Gamma_i$. It is clear that for this to make sense, that is, for $(\overline{X}_1, x_1)\sharp(\overline{X}_2, x_2)$ to have an orientation compatible with the given orientations of the $\overline{X}_i$, one end of the cylinder must be joined by an orientation-reversing map. The condition that the $x_i$ be complementary precisely guarantees the existence of an orientation-reversing isometry $Y_1 \to Y_2$.

**Remarks.** (i) For ASD 4-manifolds, this result was proved in [DF], by twistor methods. These methods were generalized to deal with orbifolds with $\Gamma_i = \{\pm 1\}$ in [LS2], and to more general cyclic singularities by Jian Zhou [Z]. A drawback of the twistor approach is its increasing complication with the complexity of the singularities of $X_i$. By contrast, the analytic approach is insensitive to the complication of the singularities—indeed the same methods prove Theorem I below, in which, roughly speaking, the cross-section $Y$ of the neck is allowed to be a rather general compact, oriented 3-manifold.
(ii) The hypothesis $H^2_c = 0$ is satisfied by many important classes of ASD orbifolds: conformal compactifications of ALE gravitons, 4-dimensional quaternion-Kähler spaces, and ALE ASD–Kähler spaces (such as the family constructed in [L]). The vanishing proof for this last family of examples is a variant of the analysis in [LS] and is based on unpublished work of LeBrun and the second author; the details will appear elsewhere.

(iii) It follows from the proof that the metrics constructed by Theorem I themselves have vanishing obstruction spaces. That is, Theorem I remains true for connected sums of any (finite) number of ASD orbifolds performed simultaneously along pairs of complementary singularities.

(iv) If $H^2_c ≠ 0$, then the same methods go through to solve ‘the infinite-dimensional part’ of the ASD equations for metrics on $X_1 # X_2$, and it may yet be possible to obtain ASD metrics after some additional work [DF, §6.4]. In fact, these same methods yield satisfactory answers to the natural questions about the moduli spaces of ASD metrics on $(X_1, x_1) # (X_2, x_2)$ that are raised by the construction (cf. for example the discussion in [DK, Ch. 7] of the same issue in the case of ASD Yang–Mills connections). These points will be treated carefully in [KS].

The proof of Theorem I involves the construction of an approximately ASD metric on the connected sum and then an application of the implicit function theorem (IFT) to find a nearby exactly ASD metric. The nub of the matter is appropriate choices of the geometric construction of the connected sum and function spaces in which to apply the IFT. The crucial thing is to have the linear problem (coming from (1.1)) under uniform control as the diameter of the neck of the connected sum shrinks to 0. The issues involved are well discussed in references such as [DK, §1, 11, 12]. We shall therefore concentrate on obtaining control of the linear problem, following Floer’s elegant approach. This involves a cylindrical model of the connected sum and the theory of elliptic operators on manifolds with cylindrical ends developed by Lockhart and McOwen. This emphasis on carrying out all the analysis on non-compact manifolds is in contrast to the approach employed by Donaldson [DK], or Taubes [T1, T2], who work as far as possible with compact spaces.

2. Manifolds with cylindrical ends

By a manifold with a cylindrical end (or CE-manifold) $X = X^- \cup_Y X^+$, we mean a non-compact manifold $X$ decomposed as a compact manifold $X^-$ with boundary $Y$, and a (half-)cylinder $X^+ = Y \times [0, \infty)$, with $X_\pm$ attached along the common boundary $\bar{Y} = \partial X^- \cong \bar{Y} \times \{0\} \subset X^+$. A CE-metric $g$ on $X$ is one which approaches a Riemannian product metric $g_0$ on $X^+$ at an exponential rate:

$$\sup_{Y \times \{t\}} |g - g_0|_{g_0} \leq Ce^{-\eta t}, \quad \sup_{Y \times \{t\}} |\nabla^k g|_{g_0} \leq Cke^{-\eta t}, \quad \text{for } t > 0 \text{ and } k = 1, 2, \ldots . \quad (2.2)$$

Here the point-wise norms and the covariant derivative are those of $g_0$ and $\eta > 0$ is some constant. The discussion extends in an obvious way to deal with CE-orbifolds and CE-manifolds with more than one end. It will always be assumed, however, that $Y$ is smooth.

Given a compact orbifold $\overline{X}$ with a marked point $x$ and smooth orbifold metric $\overline{g}$, one obtains a CE-manifold $X$ with CE-metric $g$ by a conformal rescaling that is singular at $x$. Since the transformation is conformal, $g$ is ASD whenever $\overline{g}$ is so. Indeed, choose a small geodesic ball $B(r_0)$ about $x$, let $Y = \partial B = S^3/T$ be the link of $x$ in $\overline{X}$ and let $r$ be the geodesic distance from $x$. Then in $B$ the metric takes the form

$$\overline{g} = dr^2 + r^2(h_0 + r^2h_2)$$
where $h_0$ is the induced metric of constant positive curvature on $Y$ and $h_2$ is some smooth $r$-dependent family of metrics on $Y$. On $B - x$ set

$$g = r^{-2}\bar{g} = dt^2 + h_0 + e^{-2t}h_2,$$

where $r/r_0 = e^{-t}$.

In particular $g$ is a CE-metric with $\eta = 2$ and $g_0$ the standard product metric on $(S^3/\Gamma) \times [0, \infty)$. To complete the formal description, $X^- = \bar{X} - B$. In what follows, we shall refer to $X$ as the conformal cylindrification of $(\overline{X}, \sigma)$. (It is understood here that $r$ is to be continued smoothly to a positive function on $X$.)

For another class of examples, for which we are indebted to Claude LeBrun, let $Y$ be an oriented 3-manifold of constant sectional curvature and admit an orientation-reversing isometric involution $\iota$ with only isolated fixed points. Let $\hat{X} = Y \times \mathbb{R}$ with the product metric; because $Y$ has constant curvature, $\hat{X}$ is conformally flat. Now let $X = \hat{X}/\langle (\iota, -1) \rangle$ with the induced (orbifold) metric. Then $X$ is a conformally flat CE-orbifold with just one end $X^+ = Y \times [0, \infty)$ and singular points corresponding to the fixed points of $\iota$ on $Y \times \{0\}$.

Lockhart and McOwen \cite{LM} gave a package of Fredholm theorems for a class of elliptic operators on CE-manifolds; these are the operators that are naturally adapted to the product geometry of the cylindrical end, and will be called CE-operators.

By a CE (differential) operator $A : C^\infty(E) \to C^\infty(F)$ (where $E$ and $F$ are smooth vector bundles over a CE-manifold $X$) we mean a (differential) operator which is asymptotic, at an exponential rate, to a ‘time-independent’ operator $B$ (say) on $Y \times \mathbb{R}$. This statement is understood relative to a choice of isomorphism of $E$ with the pull-back by the projection $Y \times [0, \infty) \to Y$ of $E|_Y \times 0$, similarly for $F$. Any differential operator canonically defined by a CE-metric will automatically be a CE differential operator; in particular, $L_g$ and $D_g$ in \cite{LM} have this property.

For the analysis of elliptic CE-operators, a crucial invariant is the asymptotic spectrum $\Sigma(A) = \Sigma(B)$. This is the set of complex numbers $\lambda$ such that

$$B(e^{i\lambda t}u(y)) = 0$$

has a solution for some $0 \neq u \in C^\infty(Y, E)$.

It is a standard fact that $\Sigma(A)$ is a discrete subset of $\mathbb{C}$ which meets every horizontal strip $\delta_1 < \text{Im}\, \lambda < \delta_2$ in a finite set of points. The other essential ingredient is the introduction of the weighted Sobolev spaces $L^p_{k, \delta}$ defined by completing the space of functions (or sections) with compact support in the norm

$$\|u\|_{L^p_{k, \delta}} = \sum_{\tau=0}^k \|e^{\delta t}\nabla^\tau u\|_p.$$  

Here the conventions are that $t$ is equal to the standard coordinate on $X^+$ (as above) and is smoothly cut off to zero on the compact piece $X^-$. Second, $\nabla$ denotes a CE-covariant-derivative operator that preserves a CE-bundle metric. Third the $L^p$-norm on the RHS is calculated with this CE-metric on the bundle and a CE-metric on the base $X$.

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1. About the same time Melrose and Mendoza \cite{Me, Theorem (51)} obtained similar results through the development of a calculus of ‘totally characteristic pseudo-differential operators’. This appears to be the more fruitful viewpoint in that there are important generalizations to geometric situations in which the CE-model is inappropriate \cite{Ma, Me}. For the purposes of this paper, we shall stay with the CE framework. Note however that Proposition \ref{prop:ce-operator} uses CE pseudo-differential operators and hence relies on the Melrose–Mendoza theory.
The basic result of [LM] is that an elliptic CE-operator $A$ of order $m$ (say) extends to a bounded Fredholm map

$$A_\delta : L^p_{k,\delta}(X, E) \to L^p_{k-m,\delta}(X, F)$$

if and only if $\delta$ is not the imaginary part of any $\lambda \in \Sigma(A)$; accordingly we shall call $\delta$ an exceptional weight (for $A$) if and only if $A_\delta$ is not Fredholm. It will be convenient to denote by $\delta_0 = \delta_0(A)$ the first positive exceptional weight for $A$. As in the compact case, the index is independent of $p$ and $k$ and $\text{Ker}(A_\delta)$ consists of smooth sections. There is a weighted version of the Fredholm alternative, identifying $\text{Coker}(A_\delta)$ with $\text{Ker}(A^*_{-\delta})$ where $A^*$ is the formal $L^2$-adjoint of $A$. The index depends strongly upon $\delta$, however, and jumps according to the formula

$$\text{index}(A_\delta) - \text{index}(A_{\delta'}) = \sum_{\delta < \text{Im} \lambda < \delta'} d(\lambda),$$

(2.3)

for any non-exceptional $\delta < \delta'$ where $d(\lambda)$ is the dimension of the space of all elements of the kernel of $B$ of the form

$$\exp(i \lambda t) \sum_0^N u_n(y) t^n.$$

(2.4)

Notation: it will be convenient later also to write $\text{Ker}_\delta(A) = \text{Ker}(A_\delta)$ and similarly for $\text{Coker}$ and index.

We are now ready to formulate a generalization of Theorem I. Let $X$ be an ASD CE-manifold (or orbifold) with CE-metric $g$ satisfying (2.2). Replace the complex (1.1) by the operator

$$D_g = (D_g, L^*_g) : C^\infty(X, E^1) \to C^\infty(X, E^2) \oplus C^\infty(X, E^0)$$

($L^*_g$ being the formal adjoint of $L_g$). On a compact manifold the kernel and cokernel of $D_g$ would of course be respectively isomorphic to $H^1_c$ and $H^2_c \oplus H^0_c$; in the CE case, pick any weight $\delta > 0$ less than $\min(\eta, \delta_0(D_g))$ and set

$$H^1_g = \text{Ker}_\delta(D_g), \quad H^2_g \oplus H^0_g = \text{Coker}_\delta(D_g).$$

(2.5)

We may now state

**Theorem II.** For $i = 1, 2$, let $X_i = X_i^- \cup Y_1^- X_i^+$ be ASD CE-orbifolds with CE-metrics $g_i$ and suppose there exists an orientation-reversing isometry $\iota : Y_1 \to Y_2$. Then if $H^2_{g_i} = 0$, there exist ASD metrics on the glued orbifold

$$X(l) = X_1^- \cup Y_1 \cup [-l, l] \cup Y_2 X_2^-$$

for all $l$ sufficiently large.

(2.6)

Here the role of $\iota$ is to glue the right-hand end of the cylinder $Y \times [-l, l]$ onto the boundary of $X_2^-$ so that $X(l)$ has an orientation compatible with the given orientations of the $X_i$ (cf. the discussion after Theorem I). We have not placed any geometric conditions on the $Y_i$ in this statement. However, the decay condition (2.2) clearly entails the existence of a metric $h$ on $Y$ such that the product metric on $Y \times \mathbb{R}$ is ASD, and one may show that any such ASD product metric is conformally flat. This in turn forces $h$ to be a metric of constant sectional curvature.

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2This operator is elliptic of mixed order, but such operators are treated explicitly in [LM].
Philosophical remark. One might have expected an additional hypothesis concerning the vanishing of ‘obstructions from the neck’ to enter in the statement of this theorem. Although this does not occur explicitly, note that in general $H^2_{\tilde{g}} = \ker - \delta (D^*_g)$ may contain elements that do not decay along the cylinder $Y \times [0, \infty)$. In this sense, it seems appropriate to think of the hypothesis $H^2_{\tilde{g}} = 0$ as including the hypothesis that there are no obstructions from the neck. We make no attempt here to make this notion more precise, however.

Remark. Remarks (iii) and (iv) following Theorem I apply, mutatis mutandis, to this Theorem also. (In particular, $X_i$ and $Y_i$ are allowed to have multiple components.)

Theorem I is contained in Theorem II by taking $X_i$ to be the ‘conformal cylindrification’ of $(\overline{X}_i, x_i)$. Then the existence of $i$ is equivalent to the complementarity of $x_1$ and $x_2$. The remaining point is to compare the cohomology groups of (1.1) with (2.5). This is achieved by the following

Theorem III. Let $(\overline{X}, x)$ be a compact ASD 4-orbifold with metric $\overline{g}$ and let $X$ be the conformal cylindrification, with metric $g$. Then we have

$$H^*_g(X) = H^*_c(\overline{X}, x)$$

where on the RHS we have the cohomology of the complex

$$C^\infty(\overline{X}, E^0) \xrightarrow{L^0} C^\infty(\overline{X}, E^1) \xrightarrow{L^1} C^\infty(\overline{X}, E^2)$$

and the subscript denotes sections which vanish at $x$.

Since $H^2_c(\overline{X}, x) = H^2_c(\overline{X})$, it follows at once that the hypotheses regarding the vanishing of $H^2$ in Theorems I and II agree.

Remark. Theorem I may be viewed as the natural basic gluing theorem for ASD metrics in ‘the CE category’. We have seen that any compact ASD orbifold $X$ with a marked point $x$ can be regarded as an ASD CE orbifold. With this in mind, our main results can be summarized by:

Theorem I + conformal cylindrification + Theorem III $\implies$ Theorem I.

This organization of the material seems to be the logical conclusion of Floer’s methods and exposes most clearly the structure of the argument. In any case, Theorem I and its proof are of independent interest.

3. Structure of the proofs of Theorems I and II

Theorem I and, accordingly, Theorem II produce solutions of a non-linear elliptic PDE on a manifold $X$ from those on its component pieces $X_i$, $i = 1, 2$. The arguments therefore involve applications of analysis—carefully tailored to the geometry of $X$. More specifically, the theme is the comparison of the linearized problem on $X(l)$ and on its components, corresponding to decomposition (2.6). By way of preparation, before explaining the main ingredients in the proof we need to introduce some notation.

Suppose, for simplicity, that in the statement of Theorem I the $X_i^-$ are smooth compact manifolds with boundary, and the $Y_i$ have just one cylindrical end. First we construct an approximately ASD metric $g(l)$ on $X(l)$. For this, fix a standard cut-off function $\alpha(t) : \mathbb{R} \to [0, 1]$, equal to 1 for $t \leq 0$ and 0 for $t \geq 1$. Denote by $t_i$ the standard coordinate along the cylinder $X_i^+$. By cutting off the exponentially decaying part of $g_i$ with $\alpha(t_i - l + 1)$ we obtain
a smooth metric \( g_i(l) \) on \( X_i \) which is equal to the cylinder metric for \( t_i \geq l \). Therefore the map

\[
(y, t_1) \mapsto (\iota(y), 2l - t_2)
\]  

(3.8)
is an orientation-preserving isometry from a small neighbourhood of \( \{ t_1 = l \} \) to a corresponding neighbourhood of \( \{ t_2 = l \} \). Let \( X(l) \) be the compact manifold obtained by attaching \( X_1 \) to \( X_2 \) by (3.8), and denote by \( g(l) \) the obvious metric induced from the \( g_i(l) \) on \( X(l) \). In

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{manifold.png}
\caption{The manifold \( X(l) \) with the cut-off functions used to construct \( g(l) \).}
\end{figure}

addition to the \( t_i \) it is useful to introduce the coordinate \( \tau \) on the neck \( Y \times [-l, l] \) of \( X(l) \), so that \( \tau = t_1 - l = l - t_2 \). Then, for each large enough \( l \), we have \( \tau = 0 \) in the middle of the neck, and the neck itself is given as the region \( |\tau| \leq l \) and thus has length \( 2l \) (Fig. 1).

It is easy to see that for each \( p \) and \( k \), we have

\[
\| W^+[g(l)] \|_{p,k} \leq C_{p,k} \exp(-\eta l)
\]

(3.9)

where the \( L^k_p \)-norm on \( X(l) \) is defined by \( g(l) \). In other words, the \( g(l) \) form an improving sequence of approximately ASD metrics on \( X(l) \), as \( l \to \infty \). To fit in with the CE-Fredholm theory outlined in §2, however, we need a variant of (3.9), involving the introduction of a weight-function \( w(l) \) on \( X(l) \). Put \( w(l) \) equal to unity on \( X_1 \) and \( X_2 \), and extend it to a smoothed version of the function \( \exp(\delta(l - |\tau|)) \) on the neck. Define \( L^k_p(w(l))(X(l)) \) to be the \( w(l) \)-weighted Sobolev space using the metric \( g(l) \). Of course, since \( X(l) \) is compact, for any fixed \( l \) this weighted norm is equivalent to the unweighted \( L^k_p \)-norm; but since \( w(l) \approx e^{\delta l} \) near \( \tau = 0 \), this equivalence is not uniform as \( l \to \infty \). Let

\[
U(l) = L^p_{2,w(l)}(X(l), E^1), \quad V(l) = L^p_{0,w(l)}(X(l), E^2), \quad W(l) = L^p_{1,w(l)}(X(l), E^0),
\]

(3.10)

and denote by

\[
\mathcal{D}(l) : U(l) \rightarrow V(l) \oplus W(l)
\]

the extension of \( (D_{g(l)}, L^*_g(l)) \) to a bounded operator between these spaces. Then (3.3) implies

\[
\| W^+[g(l)] \|_{V(l)} \leq C \exp(- (\eta - \delta) l),
\]

(3.11)

so that if \( \delta < \eta \) the \( g(l) \) still form an improving sequence of approximately ASD metrics, as measured by the norm of \( V(l) \).
Returning to (3.10), consider three operators $D_0, D_1$ and $D_2$,

$$D_a : U_a \to V_a \oplus W_a, \quad a = 0, 1, 2,$$

where

$$U_i = L^{p}_{2,\delta}(X_i, E^1), \quad V_i = L^{p}_{0,\delta}(X_i, E^2), \quad W_i = L^{p}_{1,\delta}(X_i, E^0),$$

and

$$U_0 = L^{p}_{2,w}(X_0, E^1), \quad V_0 = L^{p}_{0,w}(X_0, E^2), \quad W_0 = L^{p}_{1,w}(X_0, E^0),$$

where $X_0 = Y \times \mathbb{R}$ (with coordinate $\tau$ on $\mathbb{R}$) and $w$ is (a smoothed version of) the function $e^{-\delta|\tau|}$ on $X_0$. The aim is now to understand $D(l)$ in terms of the $D_a$.

Write

$$H_a = \text{Ker}(D_a), \quad J_a \oplus K_a = \text{Coker}(D_a)$$

(relative to the direct-sum decomposition $V_a \oplus W_a$). Now $J_0 \oplus K_0 = 0$ since this cokernel can be identified with $\text{Ker}_{w^{-1}}(D^0_0)$; and it is impossible for a non-zero element of the kernel of a $\tau$-independent operator to decay exponentially at both ends of $Y \times \mathbb{R}$ (this is an easy consequence of the results of [K, MP]). In terms of previous notation,

$$H_i = H^1_{g_i}, \quad J_i = H^2_{g_i}, \quad K_i = H^0_{g_i}. $$

Thus the hypothesis of Theorem II is $J_i = 0$; for simplicity, we shall assume also that $K_i = 0$. (One can always reduce to the case $K_i = 0$ by the device of framing the problem at a finite number of points, precisely as one can deal with the reducible connections in gauge theory.)

The comparison of $D(l)$ with the $D_a$ involves a topological idea and an analytical idea. The topology enters through index theory; in §3.1, an appropriate version of the excision property of the index is used to obtain the simple formula

$$\text{index}(D(l)) = \text{index}(D_0) + \text{index}(D_1) + \text{index}(D_2). \quad (3.12)$$

The analytical step involves the construction of a subspace $U^\perp(l) \subset U(l)$ with the following properties:

(i) The main estimate holds: there exists $C > 0$ such that for all $l \geq l_0$, and $h \in U^\perp(l)$,

$$\|D(l)h\|_l \geq C\|h\|_l; \quad (3.13)$$

(ii) $U^\perp(l)$ is of the ‘correct’ codimension:

$$U(l)/U^\perp(l) \cong H_0 \oplus H_1 \oplus H_2. \quad (3.14)$$

(In the statement of (3.13) we have used an obvious notational simplification for the norms involved.)

It now follows from (3.12), (3.14) and (3.13) that the restriction of $D(l)$ to $U^\perp(l)$ is surjective, with uniformly bounded inverse $G(l)$, say, satisfying $\|G(l)\| \leq C^{-1}$ for $l \geq l_0$. This, together with (3.11), is precisely what is needed for a successful application of a standard modification of the IFT (cf. [DK, Lemma (7.2.23)] or [F, Lemma 4.2]).
3.1. The excision property. The main idea behind the proof of (3.12) is the following claim.

Proposition 3.15 (excision property). Let $X$ be a $CE$-manifold, $E$ and $F$ vector bundles over $X$ and $A : C^\infty(E) \to C^\infty(F)$ an elliptic differential $CE$-operator. Suppose that there exists an open subset $U$ of $X$ and trivializations

$$
\alpha : E|U \to U \times \mathbb{C}^n, \quad \beta : F|U \to U \times \mathbb{C}^n.
$$

Then for every $V \subseteq U$ and $D > 0$ there exists an elliptic $CE$ pseudo-differential operator $P : C^\infty_0(E) \to C^\infty_0(F)$ (of order zero) such that

(a) for $|\delta| < D$, $A_\delta$ is Fredholm if and only if $P_\delta$ is Fredholm, and $\text{index}(P_\delta) = \text{index}(A_\delta)$;

(b) $P$ is equal to the identity over $V$, i.e. for every $u \in C^\infty(E)$ with support contained in $V$,

$$
P u = \beta^{-1}\alpha u.
$$

This is an extension of the usual excision principle for elliptic operators \cite[§8]{AS} to the non-compact setting of the $CE$-category. In order to motivate its use in deriving (3.12), let us consider how the standard excision principle yields a relative index formula for the change in the index under the operation of connected sum.

To be specific, let $I(\overline{X})$ denote the index of (1.1) over a compact 4-manifold $\overline{X}$,

$$
I(\overline{X}) := \dim H^1 - \dim H^2 - \dim H^3.
$$

Given compact smooth 4-manifolds, $\overline{X}_1, \overline{X}_2, \overline{X}_1', \overline{X}_2'$ say, the standard excision principle yields a formula

$$
I(\overline{X}_1 \sharp \overline{X}_2) - I(\overline{X}_1') \sharp I(\overline{X}_2') = I(\overline{X}_1') \sharp I(\overline{X}_2') + I(\overline{X}_1) - I(\overline{X}_2),
$$

(3.16)

since the bundles involved are trivial over the neck $V$ of each connected sum. (Cf. \cite[§7.1]{DK} for an analogous application in gauge theory.) Now take $\overline{X}_1' = \overline{X}_2 = S^4$, to obtain

$$
I(\overline{X}_1 \sharp \overline{X}_2) = I(\overline{X}_1) + I(\overline{X}_2) - I(S^4).
$$

(3.17)

The formula (3.17) extends to the (generalized) connected sums of orbifolds $(\overline{X}_1, x_1) \sharp (\overline{X}_2, x_2)$, with $S^4$ replaced by $S^4/\Gamma$, where $\Gamma$ is the local isotropy of $x_1$ (or $x_2$).

Now consider the situation of interest in Theorem 1 and (3.12) where one has a ‘generalized connected sum’, the neck having cross-section $Y$. Since the tangent bundle of any compact, oriented 3-manifold is trivial and the $E^i$ are all associated to the tangent bundle of the ambient 4-manifold, the restrictions of the $E^i$ to subsets of the form $Y \times I$ of $X(l)$ and of the $X_i$ are all trivial. One deduces from this observation and Proposition 3.13 a generalization of (3.16) in which some or all of the compact manifolds $\overline{X}$ may be replaced by $CE$-manifolds, $I(\overline{X})$ being replaced by $\text{index}_\delta(\mathcal{D}_g)$ for some appropriate choice of $\delta$. In particular, replacing $S^4$ in (3.17) by the half-cylinder $Y \times [0, \infty)$, we obtain

$$
\text{index}(\mathcal{D}(l)) = \text{index}(\mathcal{D}_1) + \text{index}(\mathcal{D}_2) - \text{index}_{w^{-1}}(\mathcal{D}_0)
$$

where $w^{-1}$ is the weight $e^{\delta|t|}$ on $Y \times \mathbb{R}$. To prove (3.12), it remains, therefore, to show the equality

$$
\text{index}_{w^{-1}}(\mathcal{D}_0) = -\text{index}_w(\mathcal{D}_0).
$$

But, for any non-exceptional $\delta$, the operator $\mathcal{D}_0$ is invertible on the $\exp(\delta t)$-weighted Sobolev spaces over the full cylinder $\mathbb{R} \times \mathbb{R}$; see also \cite{LM}. In particular $\text{index}_\delta(\mathcal{D}_0) = 0$. Then, (2.3) gives

$$
\text{index}_{w^{-1}}(\mathcal{D}_0) - \text{index}_\delta(\mathcal{D}_0) = -n(-\delta, \delta) = -[\text{index}_w(\mathcal{D}_0) - \text{index}_\delta(\mathcal{D}_0)].
$$
and this completes the proof of (3.12).

3.2. The main estimate. First we explain how to construct $U^\perp(l)$. It will be defined by 3 types of orthogonality conditions, corresponding to $H_0$, $H_1$, $H_2$. The obvious transversal to $H_i$ in $U_0$ is the $L^2$ orthogonal complement (with respect to $g_i$) of $H_i$. For technical reasons we modify this: since $H_i$ consists of exponentially decaying sections, there exists $L > 0$ such that the orthogonal complement of $H_i$ is also transverse to $H_i$, where $H_i$ consists of elements of the form $\alpha(t_i - L)e_i$, where $e_i \in H_i$. We know also that $H_0$ is spanned by sections over $X_0$ of the form (2.4), so we can choose a space $\tilde{H}_0$ of the same dimension as $H_0$, but consisting of sections supported in $\{|\tau| \leq \varepsilon\}$ and such that $\tilde{H}_0^\perp$ is transverse to $H_0$. Transferring these conditions in the obvious way to $X(l)$ (for $l \geq L + 1$) we define $U^\perp(l)$ to consist of $h \in U(l)$ such that

$$h \perp e_i, \quad \text{for any } e_i \in \tilde{H}_i, \quad i = 1, 2,$$

$$h \perp \tilde{H}_0, \quad \text{for any } \tilde{h}_0 \in \tilde{H}_0, \quad (3.18a)$$

Since the supports of the $\alpha_i(l)$ are disjoint and do not meet $\{|\tau| < \varepsilon\}$, it is clear that $U^\perp(l)$ satisfies (3.14).

Remark that in the basic example of $Y = S^4/\Gamma$, one can show that all elements of $H_0$ are independent of $\tau$. Then condition (3.18b) can be defined more naturally by orthogonality conditions on the restriction of $h$ to $\tilde{Y} \times \{\tau = 0\}$ [Eqn (4.5) (1)].

The main estimate is proved as follows. If (3.13) fails, then there is a sequence $h_n \in U^\perp(l_n)$ with $l_n \to \infty$ and such that

$$\|h_n\|_n = 1, \quad \|D(l_n)h_n\|_n \to 0 \quad \text{as } n \to \infty. \quad (3.19)$$

(Here again we are using obvious notational simplifications.) The first step is to obtain control of the $h_n$ near the middle of the neck, $\tau = 0$. More precisely, we have:

**Lemma 3.20.** Given a sequence $h_n$ satisfying (3.19), there exists a subsequence $h_{n_j}$ with the following property. Given $T > \varepsilon$ (as in the definition of $H_0$), let $K = \tilde{Y} \times \{\tau \leq T\} \subset X(l)$ for $l > T$; then

$$\lim_{j \to \infty} \|\exp(\delta l_n)h_{n_j}\|_{L_p^p(K,E^i)} = 0. \quad (3.21)$$

**Proof.** Observe first that on $K$ as in the lemma, every coefficient of $D(l_n) - D_0$ decays like $\exp(-\eta n)$ as $n \to \infty$. Note similarly that the $L_p^p$-norm of $h_n|K$ is uniformly comparable to the $L_p^p(K)$-norm of $\exp(\delta l_n)h_n$ for all $n$. Combining these two with the basic elliptic estimate for $D_0$, we obtain

$$\|\exp(\delta l_n)h_n\|_{L_p^p(K)} \leq C(K)(\|D_n h_n\|_n + \|h_n\|_{L_p^p(K)}),$$

with $C(K)$ independent of $n$. So given (3.19), the lemma will be proved if we find a subsequence $h_{n_j}$ such that

$$\lim_{j \to \infty} \|\exp(\delta l_n)h_{n_j}\|_{L_p(K)} = 0 \quad (3.22)$$

For this, define $h_n^{(0)} = \psi_n \exp(\delta l_n)h_n$, where $\psi_n = \alpha(|\tau| - l_n)$. Regard $h_n^{(0)}$ as a sequence of elements of $U_0$ by identifying the regions $|\tau| < l_n$ of $X(l)$ and $X_0$. From the definitions of the weights $w(l)$ and $w$ and Eqn. (3.19) this is a bounded sequence in $U_0$; hence there is a subsequence such that $h_n^{(0)}$ converges weakly to $h_\infty^{(0)} \in U_0$. With $T$ and $K$ as before, put
\( f_j = h_{n_j}^{(0)}|K \) and \( f_\infty = h_\infty^{(0)}|K \). Using again the two observations made at the beginning of the proof this weak convergence is enough to give \( \mathcal{D}_0 f_\infty = 0 \). Since \( T \) was arbitrary, \( \mathcal{D}_0 h_\infty^{(0)} = 0 \). But the conditions (3.18b) are preserved in the weak limit, so \( h_\infty^{(0)} = 0 \).

Reformulating this slightly, we conclude that \( \exp(\delta h_{n_j}) h_{n_j} \) is weakly convergent to 0 in \( L^p_2(K) \). Since \( K \) is compact this gives strong convergence to 0 in \( L^1_1(K) \), proving (3.22). \( \square \)

An analogous, but simpler argument controls the behaviour of \( h_n \) over the \( X_i \). With \( h_n \) now denoting the subsequence given by the lemma, define \( h_n^{(i)} = \alpha_{i,n} h_n, (\alpha_{i,n} = \alpha(t_i - l_n + 2)) \) and regard \( h_n^{(i)} \) as a sequence of sections over \( X_i \). By (3.19), the sequence is bounded in \( U_i \), and clearly satisfies (3.18b) once \( l_n > L + 1 \). We have

\[
\mathcal{D}_i h_n^{(i)} = \alpha_{i,n} \mathcal{D}_i h_n + [\mathcal{D}_i, \alpha_{i,n}] h_n = \alpha_{i,n} \mathcal{D}(l_n) h_n + [\mathcal{D}_i, \alpha_{i,n}] h_n
\]

since \( \mathcal{D}(l_n) = \mathcal{D}_i \) on the support of \( \alpha_{i,n} \). As \( n \to \infty \), both terms on the RHS tend to zero (in the norm of \( V_i \oplus W_i \); the first because of (3.19), the second because of (3.21) for the operator \([\mathcal{D}_i, \alpha_{i,n}] \) is supported in the neighbourhood \( Y \times [-2, 2] \) of the middle of the neck. Thus

\[
\lim_{n \to \infty} \|h_n^{(i)}\|_{U_i} = 0.
\]  

Combining (3.21) and (3.23) contradicts the first part of (3.19), as required.

3.3. The proof of Theorem [III]. Unlike the excision formula and the main estimate, which are very general features of the behaviour of linear elliptic operators over manifolds with long necks, Theorem [III] exploits specific features of the ASD equations, above all their conformal invariance. The argument is facilitated by making the special choice \( \delta = 2 - 4/p, 0 < \delta < 1 \) (so \( 2 < p < 4 \)). Then, the natural inclusion \( j : X \hookrightarrow \overline{X} \) induces the following Banach-space isomorphisms:

\[
L^p(X, E^2) = L^p_\delta(X, E^2), \quad \{ h \in L^p_\delta(X, E^1) : h(x) = 0 \} = L^p_{2,\delta}(X, E^1).
\]  

Since \( j \) is merely a conformal map, it is essential to keep track of the ‘conformal weights’ of the bundles \( E^i \) and the way in which the point-wise and global norms change when pulled back by \( j \). The first isomorphism of (3.24) may be thought of as a generalization of the familiar fact that the (global) \( L^2 \)-norm of the \( W(g) \) (or of \( W^\pm(g) \)) is a conformal invariant; that is the case \( \delta = 0, p = 2 \). By contrast \( E^1 \) has conformal weight zero and its point-wise norm is conformally invariant. It follows that the two spaces in the second isomorphism of (3.24) may be treated as Sobolev spaces of ordinary functions, and then the equality is due to Biquard [B].

The isomorphisms in (3.24) make it straightforward to compare the second cohomology groups. The comparison of zeroth cohomology is quite elementary and comparison of first cohomology follows [K, Prop. 3.2], together with an application of Proposition 3.15. Full details will appear in [KS].

3.4. Completion of proof of Theorem [II]. As mentioned at the end of §I, a version of the implicit function theorem is used to go from the approximately ASD metric \( g(l) \) to a genuine ASD metric. To be more precise, one uses the IFT to solve the equations

\[
W^+[g(l)(1 + h(l))] = 0, L^*_g h(l) = 0
\]  

such that the \( U(l) \)-norm of \( h(l) \) is small. In 4 dimensions \( L^p_2 \subset C^0 \) if \( p > 2 \), so a small \( U(l) \)-norm ensures that \( \tilde{g}(l) := g(l)(1 + h(l)) \) is a non-singular \( L^p_2 \)-metric on \( X(l) \). It then
follows from an elliptic regularity argument with (1.23) that \( \tilde{g}(l) \) is smooth. These remarks show why it is necessary to take \( p > 2 \) in our choice of function spaces.

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[Z] Jian Zhou. PhD thesis, SUNY at Stony Brook.

Department of Mathematics and Statistics, University of Edinburgh

E-mail address: agk@maths.ed.ac.uk, michael@maths.ed.ac.uk