Linking dissipation, anisotropy and intermittency in rotating stratified turbulence

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Analyzing a large data base of high-resolution three-dimensional direct numerical simulations of decaying rotating stratified flows, we show that anomalous mixing and dissipation, marked anisotropy, and strong intermittency are all observed simultaneously in an intermediate regime of parameters in which both waves and eddies interact efficiently nonlinearly. A critical behavior governed by the stratification occurs at Richardson numbers of order unity, close to the linear shear instability threshold, and with an accumulation of data points in its vicinity. This confirms the central dynamical role, in such turbulent flows, of strong large-scale intermittency in the vertical velocity and temperature fluctuations, as well as for their gradients, as an adjustment mechanism of the energy transfer in the presence of strong waves.
I. INTRODUCTION, EQUATIONS AND DIAGNOSTICS

The atmosphere and the ocean are both known for their large-scale intermittency, with strong non-Gaussian wings of the Probability Distribution Functions (PDFs) of the velocity and temperature fields, as observed in the nocturnal Planetary Boundary Layer, and with strong spatial and temporal variations of the rate of kinetic energy dissipation, as for example in oceanic ridges. Such large-scale intermittency is also found in high-resolution Direct Numerical Simulations (DNS) of stratified flows, in the presence or not of rotation, with a direct correlation to high levels of dissipation, as observed for example in the vicinity of the Hawaiian ridge. However, isotropy is classically assumed when estimating energy dissipation of turbulent flows, from laboratory experiments to oceanic measurements, and yet it has been known for a long time that small-scale isotropy recovers slowly in terms of the controlling parameter, such as in wakes, boundary layers, and pipe or shear flows.

A lack of isotropy can be associated with intermittency, and with the long-range interactions between large-scale coherent structures and small-scale dissipative eddies. In the purely rotating case, vertical Taylor columns form and, using particle image velocimetry, space-time dependent anisotropy has been shown to be important. In the case of pure stratification, its role on small-scale anisotropy was studied experimentally in detail. At low Reynolds number, the ratio of stream-wise strain rates of the horizontal and vertical velocity increases with Froude number. Spectral data and dissipation data are mostly stream-wise anisotropic because of the shear, on top of the anisotropy induced by the vertical direction of stratification. The vertical integral length scale does not grow, contrary to its horizontal counterpart, and vertical scales are strongly intermittent.

Different components of the energy dissipation tensor have been evaluated, for purely stably stratified flows, as a function of governing parameters (e.g. and references therein), and a slow return to isotropy is found only for rather high buoyancy Reynolds number, of the order of \( R_B \approx 10^3 \) (see next section for definitions of parameters). With strong imposed shear and using anisotropic boxes, anisotropy is found to be strongest when turbulence is weakest, as expected, and anisotropic eddies in the small scales depend on the effective scale-separation, assimilated to the buoyancy Reynolds number. Part of the difficulty in assessing the return to isotropy in either the large or the small scales, however, is that there is a strong coupling between scales, through the interactions of gravity waves and
fine-structure shear layers\textsuperscript{[15]}, as well as in fronts.

In this context, we evaluate quantitatively the link between mixing and dissipation, anisotropy and intermittency in the presence of both rotation and stratification, and as a function of the intensity of the turbulence. This is accomplished in the framework of a large series of unforced DNS runs for the Boussinesq equations, with data analyzed within a 2.5\% change in the total dissipation around its temporal peak when the turbulence is fully developed. This ensures a lack of correlation between data points within the parametric study. With $\mathcal{P}$ the total pressure, $\mathbf{u} = \mathbf{u}_\perp + w\hat{e}_z$ the velocity, $\theta$ the temperature fluctuations (normalized to have dimensions of a velocity), and $\nabla \cdot \mathbf{u} = 0$ because of incompressibility, we have in the unforced case:

\begin{align}
\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \times \mathbf{u} + 2\Omega \times \mathbf{u} &= -N\theta\hat{e}_z - \nabla \mathcal{P} + \nu \nabla^2 \mathbf{u}, \quad (1) \\
\frac{\partial \theta}{\partial t} + \mathbf{u} \cdot \nabla \theta &= Nw + \kappa \nabla^2 \theta, \quad (2)
\end{align}

with $\nu$ the viscosity, $\kappa$ the diffusivity, $\omega = \nabla \times \mathbf{u}$ the vorticity and $N$ the Brunt-Väisälä frequency. Rotation, of intensity $\Omega = f/2$, and stratification are in the vertical ($z$) direction.

We use the pseudo-spectral Geophysical High Order Suite for Turbulence (GHOST) code with hybrid MPI/OpenMP/CUDA parallelization and linear scaling up to at least 130,000 cores\textsuperscript{[16]}. The GHOST-generated database considered here consists of fifty-six simulations on grids of $1024^3$, as well as three at $512^3$, twelve at $256^3$, and two at $128^3$ resolution, all in a triply periodic box (see Tables 1 and 2 in\textsuperscript{[17]}). Initial conditions for most runs are isotropic in the velocity (thus at $t=0$, $w/u_\perp \lesssim 1$, and with zero temperature fluctuations, so that $\theta$ develops in a dynamically consistent way. Initial conditions in quasi-geostrophic (QG) equilibrium have also been considered, in that case with $N/f \approx 5$, $w(t=0) = 0$ and $\theta(t=0) \neq 0$ (see\textsuperscript{[18]} for details on how quasi-geostrophy is achieved at $t = 0$). The analysis of the QG set of runs, indicated in the figures by star symbols, has not introduced any major change in the conclusions\textsuperscript{[17]}, although it displays more intermittency and anisotropy (see Fig. 4 and Fig. 5 below). Finally, with $\perp$ referring to the horizontal direction, $k = \sqrt{|k_\perp|^2 + k_z^2}$ is the isotropic wavenumber.

The dimensionless parameters of the problem are the Reynolds, Froude, Rossby and Prandtl numbers:

$$
Re = \frac{U_0 L_{int}}{\nu}, \quad Fr = \frac{U_0}{L_{int}N}, \quad Ro = \frac{U_0}{L_{int}f}, \quad Pr = \frac{\nu}{\kappa}, \quad (3)
$$
FIG. 1.  Left: Variation with Richardson number of the kinetic energy dissipation efficiency $\beta$. Right: Variation with buoyancy interaction parameter of the mixing efficiency $\Gamma_f$. Colored symbols indicate Rossby number ranges (see inset). At left, the Roman numerals at the bottom delineate the three regimes of rotating stratified turbulence identified in $^\text{17}$.

where $U_0$ is the rms velocity and $L_{\text{int}}$ the integral scale, both evaluated at the peak of dissipation, and we set $Pr = 1$. The kinetic, potential and total energies $E_V, E_P$ and $E_T = E_V + E_P$, of respective isotropic Fourier spectra $E_{V,P,T}(k)$, and their dissipation rates $\epsilon_{V,P,T}$ are:

\[ E_V = \langle |u|^2/2 \rangle, \ E_P = \langle \theta^2/2 \rangle, \ \epsilon_V = DE_V/Dt = \nu \langle |\omega|^2 \rangle, \ \epsilon_P = DE_P/Dt = \kappa \langle |\nabla \theta|^2 \rangle, \ \epsilon_T = \epsilon_V + \epsilon_P. \]

Spectra can also be expressed in terms of $k_\perp$ or $k_z$ (as in equation (6) below). The Richardson number $R_i$, buoyancy Reynolds number $R_B$, buoyancy interaction parameter $R_{IB}$ and gradient Richardson number $R_{ig}$ are written as:

\[ R_i = [N/S]^2, \ R_B = Re Fr^2, \ R_{IB} = \epsilon_V/[\nu N^2], \ R_{ig} = N(N - \partial_z \theta)/S^2, \]  \(4\)
with $S = \langle \partial_z u_\perp \rangle$ representing the internal shear that develops in a dynamically consistent way. $Ri_g$ is a point-wise measure of instability; it can be negative when the vertical temperature gradient is locally larger than the imposed Brunt-Väisälä frequency, indicative of strong local overturning. We also define $\beta$ as a global measure of the efficiency of kinetic energy dissipation, with respect to its dimensional evaluation $\epsilon_D = U_0^3/L_{int}$:

$$\beta = \epsilon_V/\epsilon_D = \tau_{NL}/T_V, \quad R_{IB} = \beta R_B.$$  (5)

$\tau_{NL} = L_{int}/U_0$ and $T_V = E_V/\epsilon_V$ are the two characteristic times defining nonlinear transfer and energy dissipation; one can also define the waves periods $\tau_{BV} = 2\pi/N$ and $\tau_f = 2\pi/f$. Note that, in fully developed turbulence (FDT), one has $T_V = \tau_{NL}$. We showed in $^{19}$ that the characteristic times, associated with the velocity and temperature and based on their respective dissipation rates, $T_V$ and $T_P = E_P/\epsilon_P$, vary substantially with governing parameters, being comparable in a narrow range of Froude numbers when large-scale shear layers destabilize.

Simulations cover a wide range of parameters: $10^{-3} \leq Fr \leq 5.5$, $2.4 \leq N/f \leq 312$ and $1600 \leq Re \leq 18590$. $R_B$ and $R_{IB}$ vary roughly from $10^{-2}$ to $10^5$, values which, at the upper end, are relevant to the ocean and atmosphere. A few purely stratified runs are considered as well.

Anisotropy has been studied extensively for a variety of flows (see e.g., $^{21}$ and references therein), and many diagnostics have been devised. Here, we concentrate on the following set, with $\mu$ representing $z, \perp$:

$$L_{int,\mu} = \frac{\Sigma k_\mu^{-1} E_v(k_\mu)}{2\pi}, \quad b_{ij} = \langle u_i u_j \rangle - \delta_{ij}, \quad d_{ij} = \langle \partial_k u_i \partial_k u_j \rangle - \frac{\delta_{ij}}{3}, \quad g_{ij} = \langle \partial_\theta \partial_j \theta \rangle - \frac{\delta_{ij}}{3}, \quad v_{ij} = \langle \omega_i \omega_j \rangle - \frac{\delta_{ij}}{3}.$$  (6)

$L_{int,\mu}$ represent the integral scale for the isotropic case, as well as for vertical and horizontal velocity components, and we are concerned primarily with the ratio $L_{int,z}/L_{int,\perp}$. The integral scale is known to increase with time in FDT, and it has been shown to do the same in rotating and/or stratified turbulence. This is a manifestation of the interactions between widely separated scale that feed the large-scale flow through what is known as eddy noise together with, in the rotating case in the presence of forcing, the occurrence of an inverse cascade of energy. For reference, we also write the point-wise dissipation, $\epsilon_V(x) = 2\nu s_{ij}s^{ij}$, where $s_{ij}(x) = \frac{1}{2}(\partial_i u_j + \partial_j u_i)$ is the strain rate tensor.
FIG. 2. Joint PDFs of point-wise kinetic energy dissipation $\epsilon_V(x)$ and gradient Richardson number $Ri_g(x)$ in regimes II (left) and III (right), with in inset a run in regime I. Except in I, the PDFs are centered on $Ri_g \lesssim 1/4$. Runs identifications are 32, 58 and 5 (see Table I in15). The vertical lines indicate the linear instability threshold of $Ri_g = 0.25$.

Finally, we define as usual the second and third-order invariants of a tensor $T_{ij}$ as $T_\text{II} = T_{ij}T_{ji}$ and $T_\text{III} = T_{ij}T_{jk}T_{ki}$. For the tensors above, they are denoted respectively $b_{\text{II,III}}$, $d_{\text{II,III}}$, $g_{\text{II,III}}$, and $v_{\text{II,III}}$ (see for example6,11,21,22 for details and interpretation). They refer in particular to the geometry of the fields (one-dimensional or 1D vs. 2D, 3D, and axisymmetric, oblate or prolate).

In what follows, all anisotropy tensors and their invariants are computed from a snapshot of the data cube at the peak of enstrophy for each run, as are all PDFs and quantities associated with buoyancy flux, e.g., $\Gamma_f$. All other quantities that are plotted are computed based on spectra that are averaged in time over the peak in enstrophy.
FIG. 3. As a function of buoyancy interaction parameter $R_{IB} = \epsilon_V/\nu N^2$, we plot: (a) Ratio of vertical to horizontal integral scales; (b) $d_{II}^{1/2}$; and (c) $v_{II}^{1/2}$. See equation (6) for definitions of second tensor invariants for the velocity and vorticity.

II. AT THE THRESHOLD OF SHEAR INSTABILITIES

Rotating stratified turbulence (RST) consists of an ensemble of interacting inertia-gravity waves and nonlinear eddies. It can be classified into three regimes, I, II, and III, with dominance of waves in I for small Froude number, and dominance of eddies in III for high $R_{IB}$: then, the waves play a secondary role and dissipation recovers its fully developed turbulence isotropic limit $\epsilon_D$, within a factor of order unity\textsuperscript{[23]}. In the intermediate regime II, one finds (i) $\beta \sim Fr$, as required by weak turbulence arguments; this is the first central result in\textsuperscript{[24]}, together with the following two other laws: (ii) kinetic and potential energies are proportional (but not equal), with no dependence on governing parameters in regime II where waves and nonlinear eddies strongly interact; and (iii) similarly for the ratio of
vertical to total kinetic energy, $E_z/E_V$.

With these three constitutive laws, one can recover and establish a large number of scaling relationships, such as the ratio of characteristic scales, or for the mixing efficiency defined as $\Gamma_f \equiv B_f/\epsilon_V$, with $B_f = N \langle w\theta \rangle$ the buoyancy flux. One finds $\Gamma_f \sim R_{IB}^{-1} \sim Fr^{-2}$ in regimes I and II, and $\sim R_{IB}^{-1/2} \sim Fr^{-1}$ in regime III. Such scalings, predicted from simple physical arguments in\textsuperscript{17} have been observed at high $R_{IB}$, for example in oceanic data\textsuperscript{23}. Defining $\Gamma_* \equiv \epsilon_f/\epsilon_V$ provides another simple measure of irreversible mixing by looking at how much dissipation occurs in the potential and kinetic energy respectively. It is easily shown using the laws given above that, for the saturated regime III, $\Gamma_* \sim Fr^{-2}$ since the Ellison scale $L_{Ell} = 2\pi \theta_{rms}/N$ becomes comparable to $L_{int}$ in that case (see Fig. 6 in\textsuperscript{17}). These scaling laws extend smoothly to the purely stratified flows we have analyzed, where, for regime II, the reduced mixing efficiency was found in\textsuperscript{4} to scale linearly with $Fr$. These results are also compatible with other results obtained for that case (e.g.\textsuperscript{13,14,25–27} and references therein).

We thus begin our investigation by examining mixing and dissipation. We show in Fig. 1 the dissipation efficiency $\beta$ as a function of Richardson number. Unless specified otherwise, data is binned in Rossby number (refer to the legend in Fig. 1(left)), as in most subsequent scatter plots, with roughly the same number of runs in each bin. For runs initialized with random isotropic conditions, the color and symbol of a given data point both indicate together which Rossby number bin it resides in. Star symbols indicate quasi-geostrophic initial conditions, with a balance between pressure gradient, Coriolis force and gravity, and the color alone indicates the bin range it belongs to. For all scatter plots, the size of a symbol is proportional to the viscosity of the run, with the smallest symbols denoting runs on grids of $1024^3$ points and higher Reynolds numbers, and the largest denoting runs on grids of $128^3$ points and lower $Re$.

Note in the plot of $\beta(Ri)$ the presence of an inflection point for $Ri \lesssim 1/4$, and the two plateaux starting at $Ri \approx 10^{-2}$ and $\approx 10$ with an approximate scaling $\beta \sim Ri^{-1/2}$ in the intermediate regime, consistent with $\beta \sim Fr$, as found in\textsuperscript{17}. As stated earlier, this defines the three regimes of rotating stratified turbulence, I, II and III, in a similar fashion as for the case of purely stratified turbulence\textsuperscript{20}.

The mixing efficiency $\Gamma_f$ is plotted in Fig. 1(right) as a function of buoyancy interaction parameter. It also follows approximately two scaling laws. It can become singular in the quasi-absence of kinetic energy dissipation (when measured in terms of buoyancy flux), and
indeed $\Gamma_f$ takes high values for the runs at low $Fr$. Its slower decay with $R_{IB}$ for strongly turbulent flows starts at a pivotal value of $R_{IB} \approx 1$, a threshold which will be present in most of the data analyzed herein. The decay of $\Gamma_f$ to low values is inexorable in the absence of forcing and with zero initial conditions in the temperature field which, at high $R_{IB}$, becomes decoupled from the velocity and evolves in time in a way close to that of a passive scalar.

Joint PDFs of the point-wise gradient Richardson number and kinetic energy dissipation, for a run in each of the three regimes, I–III, are shown in Fig. 2. For regime II (left), most points in the flow are close to the threshold of shear instability, $Ri_g \lesssim 1/4$, indicated in all three plots by a thin vertical line. Kinetic energy dissipation is $\approx 10^{-2}$ but covers locally a range of values more than two orders of magnitude wide for $Ri_g \approx 1$. For runs in regime I, depicted in the inset of the plot at right, no data point reaches $Ri_g = 1$, and rather a smaller range of dissipation values is found in a narrow band extending to high $Ri_g$. On the other hand, in the opposite case of strongly turbulent flows, the bulge of points around $Ri_g \approx 1$ is much narrower with a flow almost everywhere at the brink of linear instability. Furthermore, the average dissipation is a bit higher (right plot), and with again a large extension in its local values, indicative of intermittent behavior, as we shall see below in Fig. 4. This accumulation of data points, for a given run, around the value $Ri \approx 1/4$ has been noted before by several authors. It has recently been interpreted as a manifestation of self-organized criticality, with flow destabilization occurring in a wide range of intensity displaying power-law behavior, as analyzed on oceanic data.28

Large-scale anisotropy can be measured by the ratio $L_{int,z}/L_{int,\perp}$. As shown in Fig. 3(a), it increases with $R_{IB}$ at a slow rate, starting at $R_{IB} \approx 1$ before settling sharply to a value close to unity for high $R_{IB} \approx 10^3$. The larger vertical integral scale (with respect to its horizontal counterpart), indicative of a lesser anisotropy for strong rotation and stratification (blue triangles), can be attributed to initial conditions that are isotropic together with, in that range, weak nonlinear coupling. Note that, at a given $R_{IB}$, vertical scales are almost a factor of 2 larger for stronger rotation, with a clear clustering of points with $Ro \leq 0.3$ (blue triangles) at intermediate values of $R_{IB}$. This can be associated with a stronger inverse energy transfer due to rotation, although an inverse energy cascade is not directly observed in the absence of forcing, but can appear, for long times, as an envelope to the temporal decay behavior of a turbulent flow.29

In Fig. 3(b)-(c) are shown the second invariants, $d_{II}^{1/2}$ and $v_{II}^{1/2}$ of the velocity gradient
FIG. 4. Left: PDFs of $\partial_z \theta$ with binning in $N/f$ (see legend). Right: Kurtosis of vertical velocity as a function of $R_{IB}$, with binning in Froude number as indicated in the legend.

and vorticity tensors (see equation (6) for definitions), again as functions of $R_{IB}$. While anisotropy expressed in terms of $d_{II}^{1/2}$ seems to show an approximate power law decrease towards isotropy (with power law index $-1/3$), in $u_{II}^{1/2}$ the three regimes of mixing are again visible. In the latter, a sharp transition is observed at $R_{IB} \gtrsim 100$. In terms of Froude number, the intermediate regime is bounded by $Fr \in [0.03, 0.2]$, and in terms of $R_B$ it is bounded by $R_B \in [10, 300]$. Note that the $Fr$ bounds encompass that for which the intermittency is strongest in the case of purely stratified forced flows, as measured by the kurtosis of the vertical Lagrangian velocity (see also Fig. 4). Note also that, for the highest values of the interaction parameter, $d_{II}^{1/2} \approx 10^{-3}$, whereas in terms of the vorticity anisotropy tensor, the tendency toward isotropy is much slower, with a lowest value of order $10^{-1}$, indicative of vorticity structures that retain a signature of the imposed anisotropy.

In terms of control parameter, this variable anisotropy associated with strong mixing properties is also accompanied by marked intermittency, which we now analyze for the temperature field. The PDFs of vertical temperature gradients, binned in $N/f$, are given in Fig. 4 (left). As stratification becomes stronger, the PDFs have a lower peak with much wider non-Gaussian wings (see legend giving interval values for $N/f$). Gradients favor small scales, but large scales are intermittent as well, as found already for purely stratified flows, at least for an interval of parameters 4. As an example, we show the kurtosis of the vertical component of the velocity at the peak of dissipation, defined as $K_w = \langle w^4 \rangle / \langle w^2 \rangle^2$. It is close to its Gaussian value of 3 or a bit higher for most runs. When considering only the
runs with isotropic initial conditions, the increase in $K_w$ is rather smooth and with a peak at $R_{IB} \approx \mathcal{O}(10)$. What is particularly striking, however, is the “bursty” behavior seen in the runs with QG initial conditions (indicated by stars) with a peak of $K_w \approx 7.5$ at $R_{IB} \approx 1$, or at $Fr \approx 0.07$, in good agreement with what is found in [3] for forced flows. The high values we see in $K_w$ are comparable to those observed in the atmosphere, and we note that the peaks in $K_w, K_\theta$ found in [3] are intermittent in time, whereas our analysis is done at a fixed time close to the maximum dissipation of the flows, in order to maximize the effective Reynolds number of each run. The behavior of the QG runs with significantly higher kurtosis is probably due to the fact that their initial conditions are two-dimensional, and with $w = 0$; in such a case, for small Froude number and at least for small times, the advection term leads to smooth fields, and the flow has to develop strong vertical excitation characteristic of stratified turbulence, through local instabilities, in order to catch up with energy dissipation and with emerging tendencies towards isotropy in the small scales. The temperature (not shown) displays for most runs a relatively flat kurtosis at close to its Gaussian value, $K_\theta^{(G)} \approx 3$, but still exhibits a rather sharp increase to $K_\theta \gtrsim 4.2$ in the QG-initialized runs at $R_{IB} \approx 1$, as well as for smaller values of Froude number and buoyancy interaction parameter.

We provide in Fig. 5 the parametric variations for some of the velocity- and temperature-related anisotropy tensor invariants defined in equ. (6). Fig. 5(a)-(b) show $b_{II}$ as a function of $R_{IB}$, and $g_{II}$ as a function of $Ri$, respectively. Both have a peak at $R_{IB} \approx 1$, $Ri \approx 1$ (corresponding also to $Fr \approx 0.075$, $R_B \approx 10$, not shown); however, we note that $g_{II,III}$ have a maxima for slightly smaller values of $Fr$. The final transition to a plateau approaching isotropic values, seen in Fig. 5(a), occurs for high $R_{IB} \approx 10^3$, as advocated on the basis of oceanic and estuary measurements in [31], or from DNS in [11,13].

Having scaled nonlinearly both the second and third invariants of tensors in order for them to have the same physical dimensions, we find that third invariants have similar scaling with control parameters, except that they can and do become negative, in ways comparable to what is found in [11]. We illustrate this in Fig. 5(c) in a scatter plot of the second and third invariants of $b_{ij}$ that, to a large degree, fills in Fig. 6 of [11] for $b_{II}^{1/2} < 0.2$, and highlights the fact that at the peak of enstrophy, the majority of our runs are dominated by oblate axisymmetric structures, in the form of sheets. This is complementary to what is performed in [11] where, by using many temporal snapshots, one can probe more of the permissible
We do note that there are two straggler points at high $b_{II}, g_{II}$, and low negative $b_{III}$, and in $v_{II}$ as seen in Fig. 3(c). These runs, indicated by blue stars, have quasi-geostrophic initial conditions and are at low Froude, Rossby and buoyancy Reynolds number ($RB \lesssim 1$); specifically, they are runs Q9 and Q10 of Table 2 in [17]. Again, the quasi two-dimensional nature of such flows at the peak of enstrophy is confirmed in Fig. 5(c), which places these QG-initialized runs on the upper left branch. This indicates that these flows are dominated by quasi two-dimensional sheets (see, e.g., Fig. 6 of [11]). Indeed, the high anisotropy observed in the vicinity of $RB \approx 1, R_{IB} \approx 1, Fr \approx 0.07$ in Fig. 3(c) corresponds to two-dimensional structures in the form of shear layers with strong quasi-vertical gradients at low $Fr$, and which eventually roll-up as they become unstable.

Fig. 5(d) shows the dependence of the kinetic energy dissipation efficiency, $\beta$, on the second invariant of the velocity anisotropy tensor, this time with binning in $Fr$. The figure serves to compliment both Fig. 1(c) in [17] and Fig. 1(right), illustrating behavior in the three RST regimes, where $\beta$ is low at reasonably high measures of (large-scale) anisotropy (as measured by $b_{ij}$) in regime I, approaches its highest value at largely constant $b_{II}^{1/2}$ in regime II, and as anisotropy begins to diminish at the end of regime II, remains essentially constant in regime III, as the anisotropy continues to decrease as stratification decreases.

Finally, in order to render more explicit the correlation between mixing and anisotropy, we show in Fig. 5(e) the mixing efficiency, $\Gamma_f$, displayed against $b_{II}^{1/2}$, again with binning in Froude number. One observes an approximate power law increase in mixing efficiency as anisotropy grows with stratification, from large to moderate $Fr$, with a best fit slope of $\approx 1$. Using the definitions for $\beta$, and $\Gamma_f$ in terms of the buoyancy flux, we can write $\Gamma_f = \frac{1}{\beta Fr} \frac{\langle w\theta \rangle}{u^2}$. Noting again that in regime III, $\beta$ is independent of anisotropy (Fig. 5(d)), and that $b_{zz}$ is remarkably linear in (indeed, nearly equal to) $b_{II}^{1/2}$ for all runs (not shown), the power law dependence of $\Gamma_f$ on $b_{II}^{1/2}$ mainly results from the increasingly passive nature of the scalar in transitioning from regime II to III and continuing to larger $Fr$. There is also an abrupt increase in $\Gamma_f$ in the smallest $Fr$ range, corresponding to regime I with negligible kinetic energy dissipation. The transitory regime (green diamonds) in the vicinity of the peak of vertical velocity kurtosis also corresponds to maximum $b_{II}$, i.e. maximum anisotropy, together with mixing efficiency of order unity. The accumulation of points for Froude numbers in the intermediate range of values has large $b_{II}$ and a mixing efficiency
around unity, with quasi-balanced vertical buoyancy flux and kinetic energy dissipation.

III. CONCLUSION AND DISCUSSION

We have shown in this paper that, in rotating stratified turbulence, a sharp increase in dissipation and mixing efficiency is associated, in an intermediate regime of parameters, with large-scale anisotropy and large-scale intermittency and with a slow return to isotropy which takes place mostly for buoyancy parameters larger than $\approx 10^3$, as already conjectured in\cite{11}. Rotation plays a role in the large scales, with a larger vertical integral scale at a given Froude number for small Rossby numbers (see Fig. 3(a)). The return to large-scale isotropy, as measured by $L_z/L_\perp$, is very sharp. These results evoke threshold behavior and avalanche dynamics, as analyzed for numerous physical systems (see, e.g.,\cite{32} for review, and\cite{33,34,35} in the context of the solar wind), and as found as well recently in observational oceanic data\cite{28}. In order to determine whether a given system is undergoing self-organized criticality (SOC) in the form of so-called avalanches, and if so what SOC class the system belongs to, one needs to resort to spatio-temporal analysis, although proxies are possible. Furthermore, different conclusions may be drawn whether one examines structures in the inertial range of turbulent flows, or whether one is in the dissipative range (see, e.g.,\cite{33}). Perhaps localized Kelvin-Helmoltz overturning vortices merge into larger regions, as a reflection of nonlocality of interactions in these flows, together with sweeping of small eddies by large-scale ones, close to the linear instability for $Ri_g = 1/4$, and leading to rare large-amplitude dissipative (avalanche) events. In that context, long-time dynamics, in the presence of forcing, should be investigated to see whether correlations emerge. A threshold analysis could be performed in these flows in terms of the number of excited sites, say above a local dissipation rate $\epsilon_C$, as a function of a control parameter, likely the local gradient Richardson number. Temporal dynamics should also be analyzed in terms of life-time of over-turning structures, as performed classically for example for pipe flows (see, e.g.,\cite{36,37} and references therein).

The burstiness of these rotating stratified flows is accompanied by a turbulence collapse once the energy has been dissipated at a rate close to that of homogeneous isotropic turbulence but dependent on the ratio of the wave period controlling the waves, to the turn-over time in an intermediate regime of parameters. This type of behavior has been studied e.g. for shear flows, emphasizing both the inter-scale interactions between large and small eddies.
FIG. 5. Velocity and temperature invariants defined in eq. (6): (a) $b_{II}^{1/2}$ versus $R_{IB}$; (b) $g_{II}^{1/2}$ vs. $R_i$; (c) $b_{II}^{1/2}$ vs. $b_{III}^{1/3}$; (d) $\beta$ vs. $b_{II}^{1/2}$, showing the three regions as in Fig. 1 and (e) mixing efficiency $\Gamma_f$ vs. $b_{II}^{1/2}$, with a best-fit reference line provided for $Fr > 0.05$. In (a,b,c), color binning is done in terms of Rossby number (see inset in Fig. 1), whereas in (d,e) it is in terms of $Fr$. 

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with rather similar statistics as well as the importance of sharp edges in frontal dynamics as suggested in \cite{39}. This has been analyzed in the laboratory at the onset of instabilities including for Taylor-Couette flows or for pipe flows, and it may be related to frontal dynamics observed in the atmosphere and ocean, given the tendencies of such flows to be, at least in the idealized dynamical setting studied herein, at the margin of such instabilities.

Recent observations and numerous DNS (see e.g. \cite{18}) indicate that indeed the gradient Richardson number resides mainly around its classical threshold for linear instability (\(\approx \frac{1}{4}\)), as also observed in our results, exhibiting a strong correlation with dissipation. In that light, it may be noted that the range of parameters for the mixing efficiency to be comparable to its canonical value observed in oceanic data is close to the instability threshold: \(\Gamma_f \approx 0.2\) for \(0.02 \lesssim Fr \lesssim 0.1\). Similarly, the kurtosis of the temperature and vertical velocity \(K_{\theta,w}\) are high in a narrow window around \(0.07 \leq Fr \leq 0.1\), also found in \cite{4}. As a specific example of marginal instability behavior in the framework of a classical model of turbulence extended to the stratified case, it is shown in \cite{19} that the flow remains close to the stable manifold of a reduced system of equations governing the temporal evolution of specific field gradients, involving in particular the vertically sheared horizontal flows through the second and third invariants of the velocity gradient matrix, and a cross-correlation velocity-temperature gradient tensor.

The link between local intermittency, anisotropy and dissipation is also found in fully developed turbulence, in the form of strong vortex filaments, non-Gaussianity of velocity gradients and localized dissipative events. The new element in rotating stratified flows is what the wave dynamics brings about, namely a fluid in a state of marginal instability, almost everywhere close to the threshold of linear instability in terms of \(Ri_g \approx 1/4\). It is already known that in magnetohydrodynamics (MHD), when coupling the velocity to a magnetic field leading to the propagation of Alfvén waves, there is stronger intermittency than for FDT, as found in models of MHD \cite{46,47}, in DNS \cite{48,49} as well as in observations of the solar wind \cite{50}. In RST, the added feature is having intermittency in the vertical component of the velocity and temperature fluctuations themselves, thus at large scale, as found in many observations in the atmosphere and in climatology as well \cite{51,52}, and delimited to a narrow range of parameters centered on the marginal instability threshold. Thus, not only does this interplay between waves and nonlinear eddies not destroy these characteristic features of turbulent flows, but in fact it acts in concert with them and can rather enhance them as
well.

The large data base we use is at a relatively constant Reynolds number, \( Re \approx 10^4 \), and thus an analysis of the variation of anisotropy with \( Re \), for fixed rotation and stratification remains to be done, in the spirit of earlier pioneering studies\cite{55,54} for fluids. Also, scale by scale anisotropy might be best studied with Fourier spectra. This will be accomplished in the future, together with a study of the role of forcing.

This paper is centered on a large parametric study of rotating stratified turbulence. Each flow taken individually is strongly intermittent in space, and thus presents zones that are active as well as zones that are quiescent. It was proposed recently to partition a given flow in such zones, with strong layers delimiting such patches, depending on the buoyancy interaction parameter \( R_{IB} \), and with threshold values of roughly 1, 10 and 100\cite{55}. The intermediate range corresponds, in our DNS runs, to the peak of anisotropy and intermittency together with mixing efficiency being close to its canonical value, \( \Gamma_f \approx 0.2 \). In that light, it will be of interest to perform such a local study for a few given runs of our data base in the three regimes.

Many other extensions of this work can be envisaged. For example, one could perform a wavelet decomposition to examine the scale-by-scale anisotropy and intermittency in such flows, as done in\cite{56}. Moreover, kinetic helicity, the correlation between velocity and vorticity, is created by turbulence in rotating stratified flows\cite{57,58}. It is the first breaker of anisotropy, since flow statistics depends only on the modulus of wavenumbers, but two defining functions (energy and helicity density) are necessary to fully describe the dynamics. In FDT, helicity is slaved to the energy in the sense that \( H_V(k)/E_V(k) \sim 1/k \), i.e. isotropy is recovered in the small scales at the rate \( 1/k \). In the stratified case, its scale distribution changes with Brunt-Väisälä frequency\cite{59}, as measured for example in the PBL\cite{60}, and it undergoes a direct cascade to small scales while energy goes to large scales in the presence of strong rotation and forcing\cite{61}. What role helicity and the nonlinear part of potential vorticity, namely \( \omega \cdot \nabla \theta \), will play in the fast destabilization of shear layers, their intermittency, anisotropy and criticality are topics for future work.

We conclude by noting that a deeper understanding of the structure of small-scale rotating stratified turbulence, and of the nonlocal interactions between small scales and large scales, will allow for better modeling in weather and climate codes. Many models of anisotropic flows have been proposed, extending isotropic formulations for kinetic energy dissipation by
adding several off-diagonal terms, and assuming (or not) isotropy in the orthogonal plane (see e.g.\textsuperscript{8,9} and see\textsuperscript{21} for two-point closures). It has already been found useful in models of turbulent mixing in oceanic simulations\textsuperscript{62,63}.

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