General Solution of Functional Equations Defined by Generic Linear-fractional Mappings $F_1 : \mathbb{C}^N \to \mathbb{C}^N$ and by Generic Maps birationally equivalent to $F_1$.

Konstantin V. Rerikh*

Bogoliubov Laboratory of Theoretical Physics, JINR, 141980, Dubna, The Moscow Region, Russian Federation

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Abstract

We consider a system of birational functional equations (BFEs) (or finite-difference equations at $w = m \in \mathbb{Z}$) for functions $y(w)$ of the form

$$y(w + 1) = F_n(y(w)), \quad y(w) : \mathbb{C} \to \mathbb{C}^N, \quad n \overset{\text{def}}{=} \deg F_n(y), \quad F_n \in \text{Bir}(\mathbb{C}^N),$$

where the map $F_n$ is a given birational one of the group of all automorphisms of $\mathbb{C}^N \to \mathbb{C}^N$. The relation of the BFEs with ordinary differential equations is discussed. We present a general solution of the above BFEs for $n = 1$, $\forall N$ and of the ones with the map $F_n$ birationally equivalent to $F_1 : F_n \equiv V \circ F_1 \circ V^{-1}$, $\forall V \in \text{Bir}(\mathbb{C}^N)$.

1 Introduction. Set of the Problem.

By this paper we start a discussion of a general problem of integrability of birational functional equations for functions $y(w) : \mathbb{C} \to \mathbb{C}^N$ in one complex variable $w$ of the form

$$y(w + 1) = F_n(y(w)), \quad y(w) : \mathbb{C} \to \mathbb{C}^N, \quad w \in \mathbb{C}, \quad F_n \in \text{Bir}(\mathbb{C}^N). \quad (1)$$

For $w = m \in \mathbb{Z}$ the above BFEs are a dynamical system with a discrete time or cascade. Here the map $F_n : y \mapsto y' = F_n(y) = \frac{f_i(y)}{f_{i+1}(y)}, i = 1, 2, \ldots, N$, $f_i(y)$ for $\forall i$ are polynomials in $y$, $\deg F_n(y) = \max_{i=1}^{N+1} \{\deg(f_i(y))\} = n$, is a given birational one of the group of all automorphisms of $\mathbb{C}^N \to \mathbb{C}^N$ with coefficients from $\mathbb{C}$. By the way, the all said above is valid and for the coefficients from any algebraically closed field $K$. This fact considerably extends the frames of possible applications and using of BFEs (1).

The BFEs (1) are equivalent to BFEs of more general form with a change $(w + 1) \to \psi(w)$ where the map $\psi$ is a given one from the linear-fractional group $\text{Aut}(\mathbb{C})$. Actually, the change $w \mapsto \tau(w) = \ln((w-w_1)/(w-w_2))/\ln(\lambda_1/\lambda_2)$, where the terms $w_i, \lambda_i, i = 1, 2$ are the fixed points and the eigenvalues of the map $\psi(w)$ at these points transform the map $\psi$ into: $\psi(\tau) : \tau \mapsto \tau' = \tau + 1$.

Note also that discretization of a standard autonomous differential equation corresponding to a vector field (\overset{\text{def}}{=} dx/d\tau, \tau \in \mathbb{C})

$$\dot{x} = v(x), \quad x \in U \subset V,$$

according to $\dot{x} \mapsto \frac{x(\tau+h)-x(\tau)}{h}$ and $\tau \mapsto hw$, $x(hw) \mapsto y_h(w)$ gives us a functional equation for $y(w)$:

$$y_h(w + 1) = F_n(y_h(w)), \quad \text{where} \quad F_n(y_h(w)) \overset{\text{def}}{=} y_h(w) + hv(y_h(w)).$$

\*rerikh@thsun1.jinr.ru
Thus, if we limit ourselves to the vector field $v(y) \overset{\text{def}}{=} \frac{F_n(y) - y}{n}$ where $F_n(y) \in \text{Bir}(C^N)$, then we derive the BFEs (1). (Of course, there are and other views of discretization.)

The dynamical systems with a discrete time (at $w = m \in Z$) and BFEs of the type (1) are an object of many investigations of problem of their integrability both algebraic (see (2)-(12) and others) and non-algebraic (see (13)-(17)).

The algebraic integrability of BFEs (1) and dynamical systems of this type at $N = 2$ and for $\forall n \geq 2$ will be a subject of another paper.

The problem of integrability of the BFEs (1) for $n = 1$ and for any $N$ is fully solved by the following theorem.

2 Main Result

Let us perform a transition from the mapping $F_n$ in $C^N$ to the mapping $\Phi_n$ in $\mathbb{C}P^N$.

**Definition 1** Birational maps $\Phi_n, \Phi_n^{-1}$ are the images of the maps $F_n, F_n^{-1}$ in $\mathbb{C}P^N$ $y \mapsto z : y_i = z_i/z_{N+1}, i \in (1, 2, \cdots, N)$ and are defined below:

\[
\Phi_n : \quad z \mapsto z', \quad z'_1 : \cdots : z'_{N+1} = \phi_1(z) : \cdots : \phi_{N+1}(z), \quad z, z' \in \mathbb{C}P^N,
\]

\[
\phi_i(z) = z_i^{n_i}f_i(z_i/z_{N+1}), \quad i \in (1, 2, \cdots, N+1), \quad l \in (1, 2, \cdots, N),
\]

and $\phi_i(z)$ are homogeneous polynomials in $z$ without any common factors. The map $\Phi_n^{-1} : z' \mapsto z \sim \phi^{(-1)}(z')$ is defined analogously.

**Theorem 1** The system of BFEs (1) at $n = \deg F(y) = 1, \quad \phi(z) = Az$, where $A$ is a complex matrix $(N+1) \times (N+1)$, has a general solution rationally depending on $w$ and linear-fractionally on $N$ periodic arbitrary functions $I_j(w), \quad j \in (1, \cdots, N)$ of $w$. We assume that the matrix $A, \det(A) \neq 0$, is preliminary reduced to the normal Jordan form (see (18)):

\[
D = UAU^{-1}, \quad D = \text{diag}(D^{(1)}, \cdots, D^{(r)}), \quad r \geq 1, \quad \text{dim}(D^{(i)}) = k_i,
\]

\[
D^{(i)}_{s,t} = \lambda_i\delta_{s,t} + \delta_{s,t-1}, \quad s, t \in (1, \cdots, k_i), \quad \sum_{i=1}^{r} k_i = (N+1).
\]

Then this solution has the form:

\[
y_i(w) = \frac{\sum_{l=1}^{N+1} U_{i,l}^{-1}Y_i(w)}{\sum_{l=1}^{N+1} U_{N+1,l}^{-1}Y_i(w)}, \quad i \in (1, \cdots, N),
\]

\[
Y(w) = \{Y_1^{(1)}(w), \cdots, Y_{k_1}^{(1)}(w), \cdots, Y_1^{(r)}(w), \cdots, Y_{k_r}^{(r)}(w)\},
\]

\[
Y_i^{(i)}(w) = \left(\frac{\lambda_i}{\lambda_r}\right)^w \sum_{m=1}^{k_i} C_{l,m}^{(i)}(w)Y_m^{(i)}(w), \quad \text{where}
\]

\[
I_l^{(i)}(w+1) = I_l^{(i)}(w), \quad l \in (1, \cdots, k_i),
\]

\[
C_{l,m}^{(i)} = \frac{\Gamma(w+1)\lambda_l^{-m-l}}{\Gamma(w+1-m-l+1)}, \quad \text{where}
\]

\[
C_{m,m}^{(i)} = 1, \quad C_{l,m}^{(i)} = 0 \text{ for } l > m.
\]

In (7)-(9) the functions $Y_k^{(i)}(w), I_k^{(i)}(w)$ are identically equal to 1.

Proof: Let us consider the equation for $z(w), \quad z(w) : C \rightarrow C\mathbb{P}^N$,

\[
z(w+1) \sim U^{-1} \circ D \circ U z(w).
\]
Then supposing \( Y(w) = Y^1(w), Y^2(w), \ldots, Y^r(w) = Uz(w) \) we have equations for the function \( Y(w) : C \hookrightarrow \mathbb{C}P^N \) and the function \( Y^i(w) : C \hookrightarrow \mathbb{C}P^{k_i} \):

\[
Y(w + 1) \sim DY(w), \quad Y^i(w + 1) \sim D^iY^i(w).
\] (11)

Remark that the symbol \( \sim \) means a projective similarity of vectors \( z(w + 1), Y(w + 1), Y^i(w + 1) \) to vectors in the right-hand side of equations (10), (11). Then the substitution \( Y^{(i)}(w) \) from (8) transforms equation (11) into identity. The functions \( Y_{k_i}^{(r)}(w) \) and \( I_{k_i}^{(r)}(w) \) are normalized to 1 due to a homogeneous dependence of the numerator and the denominator of expression for \( y_i(w) \) (7) from the functions \( Y(w) \).

We can easily generalize this result. Let us introduce the following definition.

**Definition 2** Let call birational mapping \( F_n \) birationally equivalent to another birational map \( F_{n'} \) if there exists such a birational mapping \( V \) such that the following equality holds:

\[
F_n = V \circ F_{n'} \circ V^{-1}.
\] (12)

The following theorem is valid.

**Theorem 2** Let BFE (1) be given by the mapping \( F_n \) birationally equivalent to the mapping \( F_1 \) from Theorem 1: \( F_n = V \circ F_1 \circ V^{-1} \). Then the general solution of (1) for the function \( \tilde{y}(w) \) is equal to \( \tilde{y}(w) = V(y(w)) \), where \( y(w) \) is given by formulae (7)-(9).

The proof is obvious.

**Remark 1** Let \( F_n^k = F_n \circ \cdots \circ F_n \) be a \( k \)-iteration of the map \( F_n \) and \( m = \deg V \) be a degree of the map \( V \) from Theorem 2. Then it is obvious that a boundedness of \( \deg F_n^k \) is a necessary condition for a birational equivalence of the map \( F_n \) to the map \( F_1 \) since \( F_n^k = V \circ F_1^k \circ V^{-1} \), \( \deg F_1^k = 1 \), i.e. \( \deg F_n^k \leq m^2 \).

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