Abstract

The forward estimation problem for stationary and ergodic time series \( \{X_n\}_{n=0}^{\infty} \) taking values from a finite alphabet \( \mathcal{X} \) is to estimate the probability that \( X_{n+1} = x \) based on the observations \( X_i, 0 \leq i \leq n \) without prior knowledge of the distribution of the process \( \{X_n\} \).

We present a simple procedure \( g_n \) which is evaluated on the data segment \( (X_0, \ldots, X_n) \) and for which, error\( (n) = |g_n(x) - P(X_{n+1} = x|X_0, \ldots , X_n)| \rightarrow 0 \) almost surely for a subclass of all stationary and ergodic time series, while for the full class the Cesaro average of the error tends to zero almost surely and moreover, the error tends to zero in probability.

Le problème d’estimation future d’une série de temps ergodique et stationnaire \( \{X_n\}_{n=0}^{\infty} \), qui prend ses valeurs dans un alphabet fini \( \mathcal{X} \), est d’estimer la probabilité que \( X_{n+1} = x \), connaissant les \( X_i \) pour \( 0 \leq i \leq n \) mais sans connaissance préalable de la distribution du processus \( \{X_i\} \). Nous présentons un procédé simple \( g_n \), évalué dur les données \( (X_0, \ldots, X_n) \), pour lequel erreur\( (n) = |g_n(x) - P(X_{n+1} = x|X_0, \ldots , X_n)| \rightarrow 0 \) presque sûrement pour une sous-classe de toutes les séries de temps ergodiques et stationnaires, tandis que pour la classe entière la moyenne de Cesaro de l’erreur tend vers zéro presque sûrement. De plus, l’erreur tend vers zéro en probabilité.

**Key words:** Nonparametric estimation, stationary processes

**Mathematics Subject Classifications (2000):** 62G05, 60G25, 60G10
1 Introduction

T. Cover [6] posed two fundamental problems concerning estimation for stationary and ergodic binary time series \( \{X_n\}_{n=\infty}^{\infty} \). (Note that a stationary time series \( \{X_n\}_{n=0}^{\infty} \) can be extended to be a two sided stationary time series \( \{X_n\}_{n=-\infty}^{\infty} \).) Cover’s first problem was on backward estimation.

**Problem 1** Is there an estimation scheme \( f_n \) for the value \( P(X_1 = 1|X_{-n}, \ldots, X_0) \) such that \( f_n \) depends solely on the observed data segment \( (X_{-n}, \ldots, X_0) \) and

\[
\lim_{n \to \infty} |f_n(X_{-n}, \ldots, X_0) - P(X_1 = 1|X_{-n}, \ldots, X_0)| = 0
\]

almost surely for all stationary and ergodic binary time series \( \{X_n\}_{n=-\infty}^{\infty} \)?

This problem was solved by Ornstein [20] by constructing such a scheme. (See also Bailey [5].) Ornstein’s scheme is not a simple one and the proof of consistency is rather sophisticated. For an even more general case, a much simpler scheme and proof of consistency were provided by Morvai, Yakowitz, Győrfi [19]. (See also Algoet [1] and Weiss [24].) Note that none of these schemes are reasonable from the data consumption point of view.

Cover’s second problem was on forward estimation.

**Problem 2** Is there an estimation scheme \( f_n \) for the value \( P(X_{n+1} = 1|X_0, \ldots, X_n) \) such that \( f_n \) depends solely on the data segment \( (X_0, \ldots, X_n) \) and

\[
\lim_{n \to \infty} |f_n(X_0, \ldots, X_n) - P(X_{n+1} = 1|X_0, \ldots, X_n)| = 0
\]

almost surely for all stationary and ergodic binary time series \( \{X_n\}_{n=-\infty}^{\infty} \)?

This problem was answered by Bailey [5] in a negative way, that is, he showed that there is no such scheme. (Also see Ryabko [22], Győrfi, Morvai, Yakowitz [11] and Weiss [24].) Bailey used the technique of cutting and stacking developed by Ornstein [21] and Shields [23]. Ryabko’s construction was based on a function of an infinite state Markov-chain.

Morvai [16] addressed a modified version of Problem 2. There one is not required to predict for all time instances rather he may refuse to predict for certain values of \( n \). However, he is expected to predict infinitely often. Morvai [16] proposed a sequence of stopping times \( \lambda_n \) and he managed to estimate the conditional probability \( P(X_{\lambda_n+1} = 1|X_0, \ldots, X_{\lambda_n}) \) in the pointwise sense, that is, for his estimator along the proposed stopping time sequence, the error tends to zero as \( n \) increases, almost surely. Another estimator was
proposed for this modified Problem 2 by Morvai and Weiss \[17\] for which the \( \lambda_n \) grow more slowly, but the consistency only holds for a certain subclass of all stationary binary time series.

In this paper we consider the original Problem 2 but we shall impose an additional restriction on the possible time series. The conditional probability \( P(X_1 = 1 \mid \ldots, X_{-1}, X_0) \) is said to be continuous if a version of it is continuous with respect to metric \( \sum_{i=0}^{\infty} 2^{-i-1}|x_i - y_i| \), where \( x_i, y_i \in \{0, 1\} \).

**Problem 3** Is there an estimation scheme \( f_n \) for the value \( P(X_{n+1} = 1 \mid X_0, \ldots, X_n) \) such that \( f_n \) depends solely on the data segment \( (X_0, \ldots, X_n) \) and

\[
\lim_{n \to \infty} |f_n(X_0, \ldots, X_n) - P(X_{n+1} = 1 \mid X_0, \ldots, X_n)| = 0
\]

almost surely for all stationary and ergodic binary time series \( \{X_n\}_{n=-\infty}^{\infty} \) with continuous conditional probability \( P(X_1 = 1 \mid \ldots, X_{-1}, X_0) \)?

We will answer this question in the affirmative. This class includes all \( k \)-step Markov chains. It is not known if the schemes proposed by Bailey \[5\], Ornstein \[20\], Morvai, Yakowitz, Györfi \[19\] solve Problem 3 or not.

**Problem 4** Is there an estimation scheme \( f_n \) for the value \( P(X_{n+1} = 1 \mid X_0, \ldots, X_n) \) such that \( f_n \) depends solely on the data segment \( (X_0, \ldots, X_n) \) and

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} |f_i(X_0, \ldots, X_i) - P(X_{i+1} = 1 \mid X_0, \ldots, X_i)| = 0
\]

almost surely for all stationary and ergodic binary time series \( \{X_n\}_{n=-\infty}^{\infty} \)?

Bailey \[5\] (cf. Algoet \[2\] also) showed that any scheme that solves Problem 1 can be easily modified to solve Problem 4 (indeed, just exchange the data segment \( (X_{-n}, \ldots, X_0) \) for \( (X_0, \ldots, X_n) \)), but apparently not all solutions of Problem 4 arise in this fashion. For further reading cf. Algoet \[1\], \[3\], Morvai, Yakowitz, Györfi \[19\], Györfi et. al. \[8\], Györfi, Lugosi and Morvai \[10\], Györfi and Lugosi \[9\] and Weiss \[24\].

**Problem 5** Is there an estimation scheme \( f_n \) for the value \( P(X_{n+1} = 1 \mid X_0, \ldots, X_n) \) such that \( f_n \) depends solely on the data segment \( (X_0, \ldots, X_n) \) and for arbitrary \( \epsilon > 0 \),

\[
\lim_{n \to \infty} P(\{|f_i(X_0, \ldots, X_i) - P(X_{i+1} = 1 \mid X_0, \ldots, X_i)| > \epsilon\}) = 0.
\]

for all stationary and ergodic binary time series \( \{X_n\}_{n=-\infty}^{\infty} \)?
By stationarity, for any scheme that solves Problem 1, the shifted version of it solves Problem 5. (Just replace the data segment \((X_{-n}, \ldots, X_0)\) by \((X_0, \ldots, X_n)\).)

There are existing schemes that solve Problem 4 (e.g. Bailey [5], Ornstein [20], and even for a more general case Morvai, Yakowitz, Györfi [19], Algoet [1], Györfi and Lugosi [9]) and there are schemes that solve Problem 5 (e.g. Bailey [5], Ornstein [20] and for even more general case Morvai, Yakowitz, Györfi [19], Algoet [1], Morvai, Yakowitz and Algoet [18]). In this paper we propose a reasonable, very simple algorithm that simultaneously solves Problem 3, 4 and 5. Note that the schemes given by Bailey [5], Ornstein [20], Morvai, Yakowitz, Györfi [19], Algoet [1] and Weiss [24] are not reasonable at all, they consume data extremely rapidly, cf. Morvai [15] and it is not known if their schemes solve Problem 3 or not.

2 Preliminaries and Main Results

Let \(\{X_n\}_{n=-\infty}^{\infty}\) be a stationary time series taking values from a finite alphabet \(X\). (Note that all stationary time series \(\{X_n\}_{n=0}^{\infty}\) can be thought to be a two sided time series, that is, \(\{X_n\}_{n=-\infty}^{\infty}\).) For notational convenience, let \(X_m = (X_m, \ldots, X_n)\), where \(m \leq n\). Note that if \(m > n\) then \(X_m\) is the empty string.

Let \(g : X \rightarrow (-\infty, \infty)\) be arbitrary.

Our goal is to estimate the conditional expectation \(E(g(X_{n+1})|X_n^n)\) from samples \(X^n_0\).

For \(k \geq 1\) define the stopping times \(\tau^k_t(n)\) which indicate where the \(k\)-block \(X_{n-k+1}^n\) occurs previously in the time series \(\{X_n\}\). Formally we set \(\tau^k_0(n) = 0\) and for \(i \geq 1\) let

\[
\tau^k_i(n) = \min\{t > \tau^k_{i-1}(n) : X_{n-k+1-t}^n = X_{n-k+1}^n\}. \tag{1}
\]

Let \(K_n \geq 1\) and \(J_n \geq 1\) be sequences of nondecreasing positive integers tending to \(\infty\) which will be fixed later.

Define \(\kappa_n\) as the largest \(1 \leq k \leq K_n\) such that there are at least \(J_n\) occurrences of the block \(X_{n-k+1}^n\) in the data segment \(X_0^n\), that is,

\[
\kappa_n = \max\{1 \leq k \leq K_n : \tau^k_{J_n}(n) \leq n - k + 1\} \tag{2}
\]

if there is such \(k\) and 0 otherwise.
Define $\lambda_n$ as the number of occurrences of the block $X^n_{n-\kappa_n+1}$ in the data segment $X^n_0$, that is,

$$\lambda_n = \max\{1 \leq j : \tau^n_j \leq n - \kappa_n + 1\}$$

if $\kappa_n > 0$ and zero otherwise. Observe that if $\kappa_n > 0$ then $\lambda_n \geq J_n$.

Our estimate $g_n$ for $E(g(X_{n+1})|X^n_0)$ is defined as $g_0 = 0$ and for $n \geq 1$,

$$g_n = \frac{1}{\lambda_n} \sum_{i=1}^{\lambda_n} g(X_{n-\tau^n_i(n)+1})$$

if $\kappa_n > 0$ and zero otherwise.

Let $\mathcal{X}^*$ be the set of all one-sided sequences, that is,

$$\mathcal{X}^* = \{\ldots, x_{-1}, x_0 : x_i \in \mathcal{X} \text{ for all } -\infty < i \leq 0\}.$$ 

Define the function $G : \mathcal{X}^* \rightarrow (-\infty, \infty)$ as

$$G(x_0^{-\infty}) = E(g(X_1)|X_0^{-\infty} = x_0^{-\infty}).$$

Note that as a conditional expectation this is only defined almost surely. E.g. if $g(x) = 1_{\{x = z\}}$ for a fixed $z \in \mathcal{X}$ then $G(y_0^{-\infty}) = P(X_1 = z|X_0^{-\infty} = y_0^{-\infty})$.

Define a distance on $\mathcal{X}^*$ as

$$d^*(x_0^{-\infty}, y_0^{-\infty}) = \sum_{i=0}^{\infty} 2^{-i-1}1_{\{x_i \neq y_i \}}.$$ 

**Definition** The conditional expectation $G(X_0^{-\infty})$ is said to be continuous if a version of it is continuous on the set $\mathcal{X}^*$ with respect to metric $d^*(\cdot, \cdot)$. Since this space is compact, in fact, continuity is equivalent to uniform continuity.

The processes with continuous conditional expectation are essentially the Random Markov Processes of Kalikow [12], or the continuous g-measures studied by Mike Keane [13].

**Theorem** Let $\{X_n\}$ be a stationary and ergodic time series taking values from a finite alphabet $\mathcal{X}$. Assume $K_n = \max(1, [0.1 \log|\mathcal{X}| n])$ and $J_n = \max(1, [n^{0.5}])$. Then
(A) if the conditional expectation \( G(X^0_{-\infty}) \) is continuous with respect to metric \( d^*(\cdot, \cdot) \) then
\[
\lim_{n \to \infty} |g_n - E(g(X_{n+1})|X^0_n)| = 0 \quad \text{almost surely},
\]

(B) without any continuity assumption,
\[
\lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} |g_i - E(g(X_{i+1})|X^i_0)| = 0 \quad \text{almost surely},
\]

(C) without any continuity assumption, for arbitrary \( \epsilon > 0 \),
\[
\lim_{n \to \infty} P(|g_n - E(g(X_{n+1})|X^0_n)| > \epsilon) = 0.
\]

Remarks:
Note that these results are valid without the ergodic assumption. One may use the ergodic decomposition throughout the proofs, cf. Gray [7] p. 268.

We note that from the proof of Ryabko [22] and Györfi, Morvai, Yakowitz [11] it is clear that the continuity condition in the first part of the Theorem can not be relaxed. Even for the class of all stationary and ergodic binary time-series with merely almost surely continuous conditional probability \( P(X_1 = 1|\ldots, X_{-1}, X_0) \) one can not solve Problem 2 in the Introduction. (An almost surely continuous conditional probability is such that as a function restricted to a set \( C \) with full measure, it is continuous on \( C \).)

We do not know if the shifted version of our proposed scheme \( g_n \) solves Problem 1 or not. (That is, in the case when \( g_n \) is evaluated on \( (X_{-n}, \ldots, X_0) \) rather than on \( (X_0, \ldots, X_n) \).

If \( \mathcal{X} \) is a countably infinite alphabet then there is no scheme that could achieve similar result to part (A) in the Theorem for all bounded \( g(\cdot) \), even if you assume that the resulting \( G(\cdot) \) is continuous, and the time series is in fact a first order Markov chain. Indeed, whenever a new state appears which has not occurred before, you are unable to predict, cf. Györfi, Morvai, Yakowitz [11].
3 Auxiliary Results

For \( k \geq 1, n \geq 0 \) and \( j \geq 0 \) it will be useful to define auxiliary processes \( \tilde{X}^{(k,n,j)}_i \) as follows. Let

\[
\tilde{X}^{(k,n,j)}_i = X_{n-t_j^{k}(n)+i} \quad \text{for} \quad -\infty < i < \infty. \tag{5}
\]

For an arbitrary stationary time series \( \{Y_n\} \) for \( k \geq 1 \) let \( \tilde{\tau}^k_0(Y_{-\infty}^\infty) = 0 \) and for \( i \geq 1 \) define

\[
\tilde{\tau}^k_i(Y_{-\infty}^\infty) = \min\{t > \tilde{\tau}^k_{i-1}(Y_{-\infty}^\infty) : Y_{-k+1+t}^t = Y_{-k+1}^0\}. \tag{6}
\]

If it is obvious on which time series \( \tilde{\tau}^k_i(Y_{-\infty}^\infty) \) is evaluated, we will write \( \tilde{\tau}^k_i \).

Let \( T \) denote the left shift, that is, \( (Tx_{-\infty}^\infty)_i = x_{i+1} \).

We will need the next lemmas for later use.

**Lemma 1** Let \( \{X_i\} \) be a stationary time series taking values from a finite alphabet \( \mathcal{X} \). For \( k \geq 1, n \geq 0 \) and \( j \geq 0 \), the time series \( \{\tilde{X}^{(k,n,j)}_i\}_{i=-\infty}^\infty \) has the same distribution as \( \{X_i\}_{i=-\infty}^\infty \).

**Proof** Note that by (1), and (6),

\[
T^{m-s}\{X_{n-t_j^{k}(n)+l}^m = x^m_l, \tilde{\tau}^k_j(n) = s\} = \{X^m_l = x^m_l, \tilde{\tau}^k_j = s\}
\]

where \( \tilde{\tau}^k_j \) is evaluated on time series \( \{X_i\}_{i=-\infty}^\infty \). Now by (5) and stationarity,

\[
P(\tilde{X}^{(k,n,j)}_l = x_l, \ldots, \tilde{X}^{(k,n,j)}_m = x_m) = \sum_{s=0}^\infty P(X_{n-t_j^{k}(n)+l}^m = x^m_l, \tilde{\tau}^k_j(n) = s) = \sum_{s=0}^\infty P(X^m_l = x^m_l, \tilde{\tau}^k_j = s) = P(X^m_l = x^m_l)
\]

and the proof of Lemma 1 is complete.

**Lemma 2** Let \( \{X_i\} \) be a stationary and ergodic time series taking values from a finite alphabet \( \mathcal{X} \). Assume \( K_n \to \infty, J_n \to \infty \) and \( \lim_{n \to \infty} \frac{J_n}{m} = 0 \). Then

\[
\lim_{n \to \infty} \kappa_n = \infty \quad \text{almost surely.}
\]
Proof We argue by contradiction. Suppose, that $\kappa_{n_i} = K$, $X_{n_i-K} = x_0^{-K}$ for a subsequence $n_i$. Then a simple frequency count (in the data segment $X_0^{n_i}$ there are less than $J_{n_i}$ occurrences of block $x_0^{-K}$) yields that

$$P(X_0^{-K} = x_0^{-K}) \leq \lim_{n \to \infty} \frac{J_n}{n} = 0.$$ 

The set of sequences that contain a block with zero probability has zero probability and thus Lemma 2 is proved.

4 Pointwise Consistency

Proof of Theorem (A). By Lemma 2, for large $n$,

$$|g_n(x) - E(g(X_{n+1})|X_0^n)|$$. 

$$= \left| \frac{1}{\lambda_n} \sum_{j=1}^{\lambda_n} g(X_{n-\tau_j^k(n)+1}) - E(g(X_{n+1})|X_0^n) \right|$$ 

$$\leq \max_{j=J_1, \ldots, J_{n_t+1}} \max_{k=1, \ldots, K_n} \left| \frac{1}{J} \sum_{j=1}^{J} [g(X_{n-\tau_j^k(n)+1}) - G(X_{-\infty}^{n-\tau_j^k(n)})] \right|$$ 

$$+ \left| \frac{1}{\lambda_n} \sum_{j=1}^{\lambda_n} G(X_{-\infty}^{n-\tau_j^k(n)}) - E(g(X_{n+1})|X_0^n) \right|.$$ 

Concerning the first term, by (1), (6) and (5),

$$\frac{1}{J} \sum_{j=1}^{J} [g(X_{n-\tau_j^k(n)+1}) - G(X_{-\infty}^{n-\tau_j^k(n)})]$$ 

$$= \frac{1}{J} \sum_{j=0}^{J-1} [g(\tilde{X}_{\tau_j^k+1}^{(k,n,J)}) - G(\ldots, \tilde{X}_{\tau_j^k+1}^{(k,n,J)}, \tilde{X}_{\tau_j^k}^{(k,n,J)})]$$

(7)

where $\tilde{\tau}_j^k$ is evaluated on $\{X_i^{(k,n,J)}\}_{i=-\infty}^\infty$.  Since by Lemma 1

$$G(\ldots, \tilde{X}_{\tau_j^k-1}^{(k,n,J)}, \tilde{X}_{\tau_j^k}^{(k,n,J)}) = E(g(\tilde{X}_{\tau_j^k+1}^{(k,n,J)})|\ldots, \tilde{X}_{\tau_j^k-1}^{(k,n,J)}, \tilde{X}_{\tau_j^k}^{(k,n,J)}),$$

the pair $\left( \Gamma_j = g(\tilde{X}_{\tau_j^k+1}^{(k,n,J)}) - G(\ldots, \tilde{X}_{\tau_j^k-1}^{(k,n,J)}, \tilde{X}_{\tau_j^k}^{(k,n,J)}), \mathcal{F}_j = \sigma(\tilde{X}_{-\infty}^{\tau_j^k}) \right)$ forms a martingale difference sequence ($E(\Gamma_j|\mathcal{F}_j) = 0$ and $\Gamma_j$ is measurable with
respect to \( \mathcal{F}_{j+1} \) for which Azuma’s exponential bound (cf. Azuma [4]) yields

\[
P \left( \left| \frac{1}{j} \sum_{j=0}^{J-1} [g(\tilde{X}_{j+1}^{(k,n,J)}) - G(\ldots, \tilde{X}_{j+1}^{(k,n,J)})] > \epsilon \right| \right) \leq 2e^{-\frac{\epsilon^2 j}{B}}
\]

for any \( B \) such that \( \max_{x \in X} |g(x)| < B \). Now by \( \text{(7)} \)

\[
P \left( \max_{j=J,\ldots,n} \max_{k=1,\ldots,K_n} \left| \frac{1}{j} \sum_{j=1}^{J} [g(X_{n-\tau_j^{k,n}(n)+1}) - G(\ldots, X_{-\infty}^{n-\tau_j^{k,n}(n)})] > \epsilon \right| \right) \leq nK_n2e^{-\frac{\epsilon^2 J_n}{B}}
\]

and by assumption \( nK_n2e^{-\epsilon^2 J_n/B} \) sums up and the Borel-Cantelli Lemma yields almost sure convergence to zero. Concerning the second term,

\[
\left| \frac{1}{\lambda_n} \sum_{j=1}^{\lambda_n} [G(X_{-\infty}^{n-\tau_j^{k,n}(n)}) - E(G(X_{-\infty}^{n})|X_0^n)] \right| \to 0 \quad \text{almost surely}
\]

since \( \kappa_n \) tends to infinity by Lemma 2, \( X_{n-\tau_j^{k,n}(n)-\kappa_n+1}^{n-\tau_j^{k,n}(n)} = X_{n-\kappa_n+1}^{n-\kappa_n+1} \) for \( 0 \leq j \leq \lambda_n \), and the conditional expectation \( G(\cdot) \) is in fact uniformly continuous on \( \mathcal{X}^{**} \) with respect to \( d^*(\cdot, \cdot) \). The proof of Theorem (A) is complete.

\section{5 Time Average Performance}

If the process does not have continuous conditional expectations then the last step in the proof of Theorem (A) is not valid. It can be carried out for most time instances \( n \) by using the typical behaviour of almost every realization \( x_{-\infty}^{\infty} \). More specifically, for every \( \delta > 0 \), the probability of the set of those \( x_{-\infty}^{0} \) for which

\[
|E(g(X_1|X_{-k+1}^0 = X_{-k+1}^0) - G(X_{-\infty}^0)| < \delta \quad \text{for all} \quad k \geq K
\]

tends to one as \( K \) tends to infinity. The typical behaviour we are after is the statement that most of the times \( t = n - \tau_j^{k,n}(n) \) the sequence \( T^tx_{-\infty}^t \) belongs to the above mentioned set. While this need not be the case for all \( n \), it is true for most \( n \)'s and the next lemma makes this precise. For the analysis we will fix a value of \( \kappa_n \) at \( k \).
Define the set of good indexes $M_n(\delta, K) \subseteq \{K - 1, \ldots, n - 1\}$ as

$$M_n(\delta, K) = \{K - 1 \leq i \leq n - 1 : |E(g(X_{i+1})|X^i_{i-k+1}) - G(X_i^\infty)| < \delta \text{ for all } k \geq K\}.$$  

We will analyze the behaviour of our algorithm for $\kappa_n = k$ for each $i \leq n$ by first dividing up the indices $\{1, 2, \ldots, n\}$ according to the value of $X^i_{i-k+1} = y^0_{i-k+1}$, and considering what happens for each of these. Let $y^0_{i-k+1} \in \mathcal{X}^k$. Define the set of indexes $I^k_n(y^0_{i-k+1}) \subseteq \{k - 1, \ldots, n - 1\}$, where you can find the pattern $y^0_{i-k+1}$, that is,

$$I^k_n(y^0_{i-k+1}) = \{k - 1 \leq i \leq n - 1 : X^i_{i-k+1} = y^0_{i-k+1}\}.$$  

Define $D_k(i)$ as

$$D_k(i) = \begin{cases} \{\tau^k_j(i) : \tau^k_j(i) \leq i - k + 1 \text{ and } 1 \leq j \leq i + 1\} & \text{if } \tau^k_j(i) \leq i - k + 1 \\ \emptyset & \text{otherwise.} \end{cases}$$

Let $E^k_n(\delta, K)$ be defined as

$$E^k_n(\delta, K) = \{0 \leq i \leq n - 1 : |D_k(i) \cap M_n(\delta, K)| \geq (1 - \delta^{0.5})|D_k(i)|\}.$$  

If the number of occurrences of $y^0_{i-k+1}$ prior to $i$ was not enough for our algorithm then $D_k(i)$ will be empty. This is rare, and can be expressed as follows: Let

$$F^k_n = \{0 \leq i \leq n - 1 : D_k(i) = \emptyset\}.$$  

It is immediate that

$$|F^k_n| \leq |\mathcal{X}|^k J_n. \quad (8)$$

**Lemma 3** Assume $|M_n(\delta, K)| \geq (1 - \delta)n$. Then

$$|\{0 \leq l \leq n - 1 : l \notin E^k_n(\delta, K) \text{ and } l \notin F^k_n\}| \leq \delta^{0.5} n.$$  

**Proof** Fix $\delta$, $K$, $k$ and $x \in \mathcal{X}$. Temporarily fix also $y^0_{i-k+1} \in \mathcal{X}^k$. Let $z = |I^k_n(y^0_{i-k+1})|$ and let $k \leq i_1 \leq i_2 \leq \cdots \leq i_z$ denote the elements of $I^k_n(y^0_{i-k+1})$. Let $i_j(y^0_{i-k+1})$ be the largest element $i_{j'}$ of $I^k_n(y^0_{i-k+1})$ such that $D_k(i_{j'}) \neq \emptyset$ and

$$|\{0 \leq l \leq n - 1 : l \in D_k(i_{j'}) \text{ and } l \notin M_n(\delta, K)\}| \geq \delta^{0.5}|D_k(i_{j'})|.$$  

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Define $S$ to be the set of these indexes as $y_{-k+1}^0$ varies over all element $X^k$. It is clear that if $i, j \in S$, $i \neq j$ then $D_k(i) \cap D_k(j) = \emptyset$ since different blocks $y_{-k+1}^0$ are involved. It follows from the construction that $\{D_k(i)\}_{i \in S}$ is a disjoint cover of $\{0 \leq l \leq n - 1 : l \notin E^k_n(\delta, K) \text{ and } l \notin F^k_n\}$. It follows that

$$n\delta \geq |\{0 \leq l \leq n - 1 : l \notin M_n(\delta, K)\}|$$

$$\geq \sum_{i \in S} |\{0 \leq l \leq n - 1 : l \notin M_n(\delta, K)\}|$$

$$\geq \delta^{0.5} \sum_{i \in S} |D_k(i)| = \delta^{0.5} \left| \bigcup_{i \in S} D_k(i) \right| .$$

Now

$$\left| \{0 \leq l \leq n - 1 : l \notin E^k_n(\delta, K) \text{ and } l \notin F^k_n\} \right| \leq \left| \bigcup_{i \in S} D_k(i) \right| \leq \delta^{0.5} n$$

and the proof of Lemma 3 is complete.

**Proof of Theorem (B).** Consider

$$\frac{1}{n} \sum_{i=0}^{n-1} \left| g_i(x) - E(g(X_{i+1})|X_0^i) \right|$$

$$\leq \frac{1 - E(g(X_1)|X_0)}{n}$$

$$+ \frac{1}{n} \sum_{i=1}^{n-1} 1_{\{\kappa_i < \kappa_i\}}$$

$$+ \frac{1}{n} \sum_{i=1}^{n-1} \max_{j=1,...,\lambda_i} \left| \frac{1}{j} \sum_{j=1}^{j} [g(x_i, x_{i-j+1} - \kappa_i) - G(x_i, x_{i-j+1} - \kappa_i)] \right|$$

$$+ \frac{1}{n} \sum_{i=1}^{n-1} \frac{1}{\lambda_i} \sum_{j=1}^{\lambda_i} G(x_i, x_{i-j+1} - \kappa_i) - E(g(x_{i+1})|X_i^i) 1_{\{\kappa_i = \kappa_i\}}$$

$$+ \frac{1}{n} \sum_{i=1}^{n-1} \left| E(g(x_{i+1})|X_i^i) - E(g(x_{i+1})|X_0^i) \right|. $$

The first term tends to zero. The second term tends to zero since by 8

$$|F^K_n|/n \leq |X|^K_n J_n/n \to 0.$$
Concerning the third term, by (7) and by Azuma’s exponential bound (cf. Azuma [4])

\[
\sum_{j=J_1}^i P \left( \left| \frac{1}{j} \sum_{j=1}^j \left[ g(\tilde{X}_{j+1}^{2K_i+j}) - G(\ldots, \tilde{X}_{j+1}^{2K_i+j}, \tilde{X}_{j+1}^{2K_i+j}) \right) \right| > \epsilon \right) \leq 2ie^{-\epsilon^2j/2n}
\]

(where \( B \) is any real such that \( 2\max_{x\in\mathcal{X}} |g(x)| < B \) and the right hand side is summable, hence the Borel-Cantelli Lemma yields almost sure convergence to zero. By Toeplitz lemma the average also converges to zero.

Now we deal with the fourth term. Let \( 0 < \epsilon < 1 \) be arbitrary. Choose the integer \( d \) large enough such that \( |\mathcal{X}|^{-10d-1} < \epsilon \). Let \( \delta = \frac{\epsilon}{2} \). Let \( K \) and \( N_0 \) be so large that \( \frac{|M_n(\delta, K)|}{n} > (1-\delta) \) for all \( n \geq N_0 \). (There exist such \( K \) and \( N_0 \) since by the ergodic theorem and the martingale convergence theorem \( \lim_{k \to \infty} \lim_{n \to \infty} \frac{|M_n(\delta, K)|}{n} = 1 \) almost surely.) Now let \( N_1 \geq N_0 \) be so large that \( K_n - d + 2 \geq K \) and \( |\mathcal{X}|^{10(K_n-d-1)} \geq N_0 \) for all \( n \geq N_1 \). Assume \( n \geq N_1 \). The sum

\[
\frac{1}{n} \sum_{i=1}^{n-1} \left| \frac{1}{\lambda_i} \sum_{j=1}^{\lambda_i} G(X^{i-j}i^{K_i(i)}) - E(g(X_{i+1})|X_{i-K_{i+1}}) \right| 1_{\{\kappa_i = K_i\}}
\]

that we are trying to estimate will be divided into blocks according to the value of \( K_i \). In fact only values in the range \( [K_n-d+2, K_n] \) need be considered since the sum up to \( |\mathcal{X}|^{10K_n-d+1} \) can be estimated by \( |\mathcal{X}|^{10K_n-d+1} 2\max_{y\in\mathcal{X}} |g(y)| \) and so by our assumption on \( d \), after dividing by \( n \) this will be at most \( \epsilon 2\max_{y\in\mathcal{X}} |g(y)| \). For \( i \) in the range \([|\mathcal{X}|^{10k-1}, |\mathcal{X}|^{10k})\) for \( K_n - \delta + 2 \leq k \leq K_n \), and \( \kappa_i = K_i \), if \( i \in E_{[|\mathcal{X}|]}^{k-1} \) then we get for more than \( (1-\sqrt{\delta})\delta \) \( D_k(i) \) terms an upper bound of \( \delta \) while for the rest we may use \( 2\max_{y\in\mathcal{X}} |g(y)| \).

This gives an upper bound of

\[
\frac{\delta |D_k(i)| + \sqrt{\delta} |D_k(i)| 2\max_{y\in\mathcal{X}} |g(y)|}{|D_k(i)|}
\]

Using Lemma 3 we can estimate the sum over all \( i \) in the interval \([|\mathcal{X}|^{10k-1}, |\mathcal{X}|^{10k})\) by

\[
n(\delta + \sqrt{\delta} \max_{y\in\mathcal{X}} |g(y)|) + \sqrt{\delta} \max_{y\in\mathcal{X}} |g(y)|.
\]

Dividing by \( n \), we have an upper bound:

\[
\delta + \sqrt{\delta} \max_{y\in\mathcal{X}} |g(y)| + \sqrt{\delta} \max_{y\in\mathcal{X}} |g(y)|.
\]
The same argument yields the same upper bound for the $i$'s in the range $[|X|^10K_n, n)$.

Summing over $k$ in the range $[K_n - d + 2, K_n + 1]$ yields an upper bound:

$$d\delta + d\sqrt{\delta} \max_{y \in X} |g(y)| + d\sqrt{\delta} \max_{y \in X} |g(y)|.$$

Recall that $\sqrt{\delta}d = \sqrt{\epsilon}$ and this yields an upper bound:

$$\epsilon + \sqrt{\epsilon} \max_{y \in X} |g(y)| + \sqrt{\epsilon} \max_{y \in X} |g(y)|.$$

Since $\epsilon$ was arbitrary, the fourth term tends to zero.

Now we deal with the last term. Since by the martingale convergence theorem, $E(g(X_1)|X^0_i) \to G(X^0_\infty)$ almost surely, thus

$$\lim_{n \to \infty} |E(g(X_1)|X^0_{-K_i+1}) - E(g(X_1)|X^0_i)| = 0$$

and applying Breiman’s generalized ergodic theorem, cf. Maker [14] (or Algoet [2]),

$$\lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \left| E(g(X_{i+1})|X^i_{i-K_i+1}) - E(g(X_{i+1})|X^i_0) \right| = 0$$

almost surely and the proof of Theorem (B) is complete.

6 Weak Consistency

Proof of Theorem (C).

In order to show that for all ergodic stationary processes our estimate $g_n$ converges in probability we follow the steps in the proof of Theorem (A). The probability that

$$(|g_n(x) - E(g(X_{n+1})|X^n_0)| > 3\epsilon)$$

can be estimated as the sum of the probability of several sets,

$$P \left( \max_{j=J_n,...,n} \max_{k=1,...,K_n} \left| \frac{1}{J} \sum_{j=1}^{J} g(X_{n-\tau^{k}_{j}(n)+1} - G(X^{n-\tau^{k}_{j}(n)}) \right| > \epsilon \right),$$
\[ P(\kappa_n < K_n), \]

\[ P(\left| E(g(X_{n+1})|X_0^n) - E(g(X_{n+1})|X_{n-K_n+1}^n) \right| > \epsilon) \]

and

\[ P \left( \frac{1}{\lambda_n} \sum_{j=1}^{\lambda_n} G(X_{-\infty}^{n-x_j(n)}) - E(g(X_{n+1})|X_{n-K_n+1}^n) > \epsilon, \kappa_n = K_n \right). \]

For the first, the argument given there suffices. Concerning the second, it tends to zero by Lemma 4 in the Appendix. (Apply it with \( A = \{X_{n-K_n+1}^n = x_{n-K_n+1}^n\} \), \( D = J_n \). Then sum over all possible \( x_{n-K_n+1}^n \) to get that this second probability in question is not greater than \( |\lambda_j| n J_n/n \) which tends to zero.) For the third, it is easy to see that it tends to zero by stationarity and by the martingale convergence theorem which implies that

\[ \lim_{n \to \infty} P(\left| E(g(X_1)|X_0^n) - E(g(X_1)|X_{n-K_n+1}^n) \right| > \epsilon) = 0. \]

We concentrate on the last probability. Recall the notations from the proof of Theorem (B). The main thing is to show that with probability at least \( 1 - \epsilon \), for \( n \) sufficiently large, most of the elements \( l \in I_{K_n}^n(X_{n-K_n+1}^n) \) are such that \( T^l x_{-\infty} \) does not belong to the set

\[ \tilde{M}_n(\epsilon) = \{x_{-\infty}^n : |E(g(X_1)|X_{n-K_n+1}^n = x_{n-K_n+1}^n) - G(x_{-\infty}^n)| > \epsilon \text{ for some } k \geq K_n \} \]

as neither does \( T^n x_{-\infty} \) itself. By the martingale convergence theorem, the probability of the set \( \tilde{M}_n(\epsilon) \) tends to zero as \( n \) tends to infinity. Let \( n \) be so large that this probability in question is less than \( \epsilon^2/2 \). Let

\[ B_n = \{x_{-\infty}^n : |\{l \in I_{K_n}^n(x_{n-K_n+1}^n) : x_{-\infty}^n \in T^{-l} \tilde{M}_n(\epsilon)\}| > \epsilon|I_{K_n}^n(x_{n-K_n+1}^n)|\}. \]

The probability of \( B_n \) will be evaluated using the ergodic theorem along the orbit of a typical point. Let \( x_{-\infty}^n \) be such a typical orbit and \( N \) be a very large number. Fix \( y_{0-K_n+1}^n \), and note those elements in \( I_{K_n}^n(y_{0-K_n+1}^n) \) that belong to \( B_n \). We will cover them with disjoint blocks of length \( K_n \), beginning on the right end \( N - 1 \) in the obvious way. These sets (subsets of \( I_{K_n}^n(y_{0-K_n+1}^n) \)) we call \( C_r(y_{0-K_n+1}^n) \) where \( r = 1, 2, \ldots \). Formally, let \( \cdots <
\[ l_2 < l_1 \] denote the elements of \( I_{K_n}^{y_0-K_{n+1}} \). Let \( C_0(y_{-K_{n+1}}^0) = \emptyset \). For \( r \geq 1 \) we define \( C_r(y_{-K_{n+1}}^0) \) recursively. Let \( l \) be the largest index such that \( l \geq n, l \notin \bigcup_{r \leq r'} C_{r'}(y_{-K_{n+1}}^0) \) and \( x_{-\infty}^\infty \in T^{-n+1}B_n \). If there is such \( l \) then set \( C_r(y_{-K_{n+1}}^0) = \{ l - n + K_n - 1 \leq i \leq l \text{ for } i = 1, 2, \ldots \} \). Let \( R(y_{-K_{n+1}}^0) \) be the largest \( r \) for which \( C_r(y_{-K_{n+1}}^0) \) is defined. Let

\[ I_N(\tilde{M}_n(\epsilon)) = \{ 0 \leq l \leq N - 1 : T^lx_{-\infty}^\infty \in \tilde{M}_n(\epsilon) \} \]

Then by the construction of \( C_r(y_{-K_{n+1}}^0) \), for each \( 1 \leq r \leq R(y_{-K_{n+1}}^0) \),

\[ |\{ l \in C_r(y_{-K_{n+1}}^0) : T^lx_{-\infty}^\infty \in \tilde{M}_n(\epsilon) \}| > \epsilon |C_r(y_{-K_{n+1}}^0)|. \]

Since \( x_{-\infty}^\infty \) is typical, for large \( N \), \( |I_N(\tilde{M}_n(\epsilon))| \leq \epsilon^2 N \) and

\[ \epsilon^2 N \geq \sum_{y_{-K_{n+1}}^0 \in \mathcal{X}^K_n} \sum_{r=1}^{R(y_{-K_{n+1}}^0)} |\{ l \in C_r(y_{-K_{n+1}}^0) : T^lx_{-\infty}^\infty \in \tilde{M}_n(\epsilon) \}| \]

\[ \geq \epsilon \sum_{y_{-K_{n+1}}^0 \in \mathcal{X}^K_n} \sum_{r=1}^{R(y_{-K_{n+1}}^0)} |C_r(y_{-K_{n+1}}^0)|. \]

Let

\[ I_N(B_n) = \{ n \leq l \leq N - 1 : T^lx_{-\infty}^\infty \in B_n \}. \]

But those \( n \leq l \leq N - 1 \), such that \( T^lx_{-\infty}^\infty \in B_n \) are covered by this union - thus \( \epsilon |I_N(B_n)| \leq \epsilon^2 N \) and thus

\[ P(B_n) = \lim_{N \to \infty} \frac{|I_N(B_n)|}{N} \leq \epsilon \]

since \( x_{-\infty}^\infty \) was typical. The proof of the Theorem is complete.

7 Appendix

Lemma 4 Let \( \{ X_n \} \) be stationary and ergodic. For an arbitrary set \( A \) measurable with respect to \( \sigma(X_0^n) \), the probability of the event

\[ \tilde{A}(n, D) = \{ x_{-\infty}^\infty \in A : \sum_{i=0}^{n-1} 1_A(T^ix_{-\infty}^\infty) < D \} \]

is not greater than \( D/n \).
Proof Fix a typical orbit $x^\infty_{-\infty}$. Let

$$I_N(\tilde{A}(n, D)) = \{ n \leq l \leq N - 1 : T^lx^\infty_{-\infty} \in \tilde{A}(n, D) \}.$$ 

We make a disjoint cover. Let $\ldots, l_2 < l_1$ denote the elements of $I_N(\tilde{A}(n, D))$. Set $E_r = \emptyset$ and for $r = 1, 2, \ldots$, define $E_r$ recursively. Let $l$ denote the largest element of $I_N(\tilde{A}(n, D))$ such that $l \notin \bigcup_{r < s} E_r$ if there is such and let

$$E_r = \{ l - n \leq l_i < l : \text{for } i = 1, 2, \ldots \}.$$ 

Now let $R$ denote the largest $r$ for which $E_r$ has been defined. Since the cover is disjoint, $R(n + 1) \leq N$. Then clearly,

$$\frac{I_N(\tilde{A}(n, D))}{N} \leq \frac{RD}{R(n + 1)} \leq \frac{D}{(n + 1)}$$ 

and the left hand side tends to $P(\tilde{A}(n, D))$. The proof of Lemma 4 is complete.

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