A zeta function approach to the relation between the numbers of symmetry planes and axes of a polytope

J.S. Dowker

Department of Theoretical Physics,
The University of Manchester, Manchester, England.

Abstract

A derivation of the Cesàro-Fedorov relation from the Selberg trace formula on an orbifolded 2-sphere is elaborated and extended to higher dimensions using the known heat-kernel coefficients for manifolds with piecewise-linear boundaries. Several results are obtained that relate the coefficients, $b_i$, in the Shephard-Todd polynomial to the geometry of the fundamental domain. For the 3-sphere we show that $b_4$ is given by the ratio of the volume of the fundamental tetrahedron to its Schläfli reciprocal.

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1. Introduction.

In an earlier work [1] we have indicated how the relation between the numbers of symmetry planes and symmetry axes of a solid discovered by Cesàro [2], and Fedorov [3,4] can be derived analytically. In the present work we wish to further analyse the method and extend it to higher dimensions.

This work can be looked upon as a discussion of the heat-kernel and \( \zeta \)-function on certain manifolds and so should be relevant to quantum theory.

Cesàro’s proof of the formula relies on an earlier result of his [5] relating the number of symmetry planes to the number and order of symmetry axes lying in a single symmetry plane. For completeness we repeat his findings here. Let \( b_1 \) be the total number of symmetry planes and let there be, on the \( i \)-th such plane, \( n^{(i)}_m \) symmetry axes of order \( m \), then Cesàro [5] shows that (see later)

\[
b_1 = 1 + \sum_m n^{(i)}_m (m - 1), \quad \forall i.
\]  

(1)

Summing this formula over all symmetry planes and using the fact that

\[
\sum_i n^{(i)}_m = mn_m,
\]

where \( n_m \) is the total number of symmetry axes of order \( m \), we find one form of the C-F relation,

\[
b_1(b_1 - 1) = \sum_m n_m m(m - 1).
\]  

(2)

Fedorov [3,4] refers, somewhat confusingly, to \( x(x - 1) \) as the ‘function’ of \( x \). (2) is his ‘zonohedral formula’. (See [6].)

Another form of the relation follows by noting that, for the regular solids at least, we have the orbit-stabiliser result

\[
g = 2mn_m, \quad \forall m,
\]  

(3)

where \( 2g \) is the order of the complete symmetry group, \( \Gamma \). Therefore

\[
2b_1(b_1 - 1) = g \sum_m (m - 1) = g(p + q + r - 3)
\]  

(4)

for the three types of axis of orders \( p, q \) and \( r \).

Relations (2) and (4), and their generalisations, are our interest in the present paper. Because the theory of polytopes might not be too familiar, a certain amount of historical and technical commentary might be of benefit, if only to the writer. Everything goes back to Schlӓfli.
2. A survey of the geometrical method
We recall the standard geometric situation \([7], [6] \S 3-5, \S 4-5\). The regular polyhedron \(\{p, q\}\) is projected onto its circumsphere giving a spherical tessellation. The planes of symmetry of \(\{p, q\}\) divide this tessellation further into \(|\Gamma|\) congruent spherical (Möbius) triangles, \((pqr)\) (with interior angles \(\pi/p, \pi/q\) and \(\pi/r\)), which are transitively permuted under the action of \(\Gamma = [p, q]\). This produces a simplicial decomposition (triangulation) of the sphere. The ‘\(q\)-vertex’ (0) of \((pqr)\), and its images under \(\Gamma\), correspond to the vertices of \(\{p, q\}\) and the ‘\(p\)-vertex’ (2) likewise to the vertices of the reciprocal solid, \(\{q, p\}\); \(i.e.\) to the mid-points of the faces of \(\{p, q\}\). The ‘\(r\)-vertex’ (1) is associated with the mid-points of the edges and, for the usual solids, \(r = 2\) so that \(g = 4n_r\). The edge mid-points form the vertices of the quasi-regular polyhedron \(\{pq\}\) which projects to a spherical tessellation of edge length \(2\pi/h\), where \(h\) is the Coxeter number ([6] \S 2-3, \S 2-6, 5-91).

Equation (3) says that there are \(g/m\) images of the \(m\)-fold vertex of \((pqr)\) under \([p, q]\). This follows because the stability group of this vertex has order \(2m\).

Steinberg [8] gives a simple geometrical derivation of (2) which appears in the second edition of Coxeter’s book [9]. In the first edition, [6], Coxeter just indicates the general nature of such a proof. Curiously, in [9], he drops the references to Cesàro and Fedorov, presumably because their relations are superseded by Steinberg’s and some further work of his own, [10].

The geometrical argument is worth repeating. The \(b_1\) symmetry circles intersect in \(b_1(b_1 - 1)\) (possibly coincident) points (the vertices of the triangulation). In order that a vertex correspond to a symmetry axis of order \(m\), it is necessary that \(m\) symmetry circles cross there. That is to say, there must be a coincidence of \(m(m - 1)/2\) intersections at the vertex. Equation (2) expresses the distribution of the total number of intersections among such vertices, and there are two vertices per axis. We might term \(m(m - 1)/2\) the degeneracy of the intersection.

It is clear that this argument is essentially the same as Cesàro’s. Indeed (1) follows from a counting argument applied to a single reflecting circle. The total number of such circles is just one more than the number that cross a chosen circle. Further, trivially, the number that cross this circle at a vertex of order \(m\) is \(m - 1\) (discounting the original one).

We might note the relations between the \(n_m\) and the numbers of vertices, edges, and faces, \(N_0, N_1\) and \(N_2\) respectively, of the polyhedron \(\{p, q\}\), ([6] 5-81),

\[
2n_p = N_0, \quad 2n_q = N_2, \quad 2n_r = N_1.
\]

The statement that \(qN_0 = 2N_1 = pN_2\) is a graph-theoretic one (see \(e.g.\) [11] Chap.8).

It is worth outlining the further development for completeness [6], [8]. The associated polyhedron \(\{\frac{p}{q}\}\) has \(N_1\) vertices, \(2N_1\) edges (from graph theory) and \(N_0 + N_2\) faces (of two sorts, \{q\} and \{p\}). Every edge belongs to just one equatorial
$h$-gon of which, therefore, there are $2N_1/h$ in total. Since there are four faces at each vertex, the intersections of the $h$-gons are simple (degeneracy of one) and the total number of intersections is just the total number of vertices, i.e.

$$\frac{2N_1}{h} \left( \frac{2N_1}{h} - 1 \right) = N_1$$

Equivalently, the number of equators is one plus the number of intersections with one equator i.e. $2N_1/h = 1 + h/2$. Thus

$$2g = h(h + 2). \quad (6)$$

It is easily seen that each equator on $\{p, q\}$ contains $h$ points 1 and crosses $h$ segments 02. The total number of reflecting circles on $\{p, q\}$ may now be obtained by adding up the crossings on a complete circumnavigation of an equator. There are two for each 1 point and one for each 02 segment crossed. Taking into account the antipodal points we get $b_1 = (2h + h)/2$,

$$b_1 = \frac{3h}{2}. \quad (7)$$

From (7), (6) and (4) we find Steinberg’s formula [8], [9],

$$h + 2 = \frac{24}{12 - p - q - r} \quad (8)$$

for the Coxeter number in terms of the angles of the Möbius triangle.

3. The analytical method

Let $\zeta(s)$ be the $\zeta$–function for the Laplacian on the orbifolded 2-sphere, i.e. on the Möbius triangle $(pqr)$. The analytical method consists of identifying $\zeta(0)$ calculated in two ways – one from the expression in terms of the degrees $d$ of the symmetry group, $\Gamma$, (this involves the Barnes $\zeta$–function) and another that comes either by using the expression for the constant term in the heat-kernel, derived by differential geometric methods for any manifold and then applied to the orbifolded sphere, or by using the particular form of the $\zeta$–function on the orbifolded sphere in terms of the symmetry axes, i.e. fixed points, as given in [12]. We could say that the Cesàro-Fedorov (C-F) formula is an application of the Selberg trace formula to these spherical domains.

The formula referred to in [12] gives a constructional form to the $\zeta$–function in terms of ‘cyclic’ $\zeta$–functions. As derived, the formula is for the rotational subgroup $\Gamma^+$ and a special conformal coupling. It comes originally from the conjugacy class decomposition of the heat-kernel and is actually independent of the coupling.

Although convenient, using this particular form of $\zeta$ on $S^2/\Gamma$ is, in some ways, not as satisfying as an appeal to a more universal formula such as that for the
constant term in the heat-kernel expansion used in [1]. We need no knowledge of
group theory nor of conjugacy class decomposition to use or derive this formula. Of
course, the significance of the corner terms (see later) is, at root, a fixed point one.

The number of reflecting planes is introduced analytically through the results
of Shephard and Todd [13] and Solomon [14] who give this number as the sum of
the exponents, \((d_i - 1)\), of the finite reflection group. Our approach should thus be
valid for solids other than the regular ones.

Clearly the C-F relation would emerge whatever the coupling (we shall verify
this shortly) and also there is no need to use the full, reflecting \(\Gamma\). We could have
obtained the C-F formula from the results of [12].

Another point we shall investigate concerns the value of the \(\zeta\)-function at other
arguments, in particular at negative integers. One would expect to get different
relations between the orders \(m\) and the number of symmetry planes.

4. Conformal–rotational derivation

In this section we rederive the C-F relation using the special conformal coupling
\(\xi = 1/8\) and the corresponding rotational \(\zeta\)-function on \(S^2/\Gamma\) derived in [12]. We
refer to equation (26) of [12] which we give again with a slightly different notation

\[
\zeta(s) = \frac{1}{g} \left[ \sum_m m n_m \zeta_m(s) - 2 \left( \sum_m n_m - 1 \right) \zeta_R(2s - 1, \frac{1}{2}) \right].
\] (9)

\(\zeta_m(s)\) is the \(\zeta\)-function for the rotational cyclic group, \(Z_m\), and its values at
the negative integers are shown in equation (32) of [12]. (See the appendix to the
present paper). In particular we calculate \(\zeta_m(0)\)

\[
\zeta_m(0) = \frac{2m^2 - 1}{12m}
\] (10)

and so we get from (9),

\[
\zeta(0) = \frac{1}{12g} \left[ 2 \sum_m n_m (m^2 - 1) + 1 \right].
\] (11)

On the other hand we know that the rotational \(\zeta(s)\) is given by the sum of the
Neumann and Dirichlet \(\zeta\)-functions,

\[
\zeta(s) = \zeta_N(s) + \zeta_D(s)
\] (12)

where

\[
\zeta_N(s) = \zeta_d(2s, \frac{d - 1}{2} | d) \] (13)

\[
\zeta_D(s) = \zeta_d(2s, \sum d_i - \frac{d - 1}{2} | d).
\] (14)
For $s$ a negative integer, or zero, the two terms in (12) are equal.

Barnes \cite{15} gives the values, (we use both Barnes’ notation and the more standard one of Erdelyi, \cite{16}, for the generalised Bernoulli polynomials)

$$\zeta_d(-k, a \mid d) = \frac{(-1)^d}{k+1} dS_{1+k}^{(1)}(a) = \frac{(-1)^d}{\prod d_i (d+k)!} B_{d+k}^{(d)}(a \mid d)$$  \hspace{1cm} (15)

where

$$dS_{1}^{(d+1)}(a) = \frac{1}{\prod d_i}, \hspace{1cm} dS_{1}^{(d)}(a) = -dS_{1}^{(d)}(\sum d_i - a) = \frac{2a - \sum d_i}{2\prod d_i},$$

$$dS_{1}^{(d-1)}(a) = dS_{1}^{(d-1)}(\sum d_i - a) = \frac{1}{12\prod d_i} (6a^2 - 6a \sum d_i + \sum d_i^2 + 3 \sum i<j d_id_j)$$

and so we get from (12), (15) and (16), with $g = d_1 d_2$ and $b_1 = d_1 + d_2 - 1$,

$$\zeta(0) = \frac{1}{12g} [3 - 6(d_1 + d_2) + 2(d_1^2 + d_2^2 + 3d_1 d_2)] = \frac{1}{12g} [2b_1(b_1 - 1) - 1 + 2g].$$  \hspace{1cm} (17)

The equation that must hold, therefore, is

$$b_1(b_1 - 1) = \sum m n_m (m^2 - 1) + 1 - g.$$  \hspace{1cm} (18)

Recalling the total rotational order equation

$$\sum m (m - 1)n_m = g - 1$$  \hspace{1cm} (19)

we get

$$b_1(b_1 - 1) = \sum m n_mm(m - 1)$$

as required.

Compared with Steinberg’s geometrical argument, this is a rather complicated derivation. Actually, it is not so very different in content. We are simply re-extracting the information that has already gone into the construction of the $\zeta$–function (9). This remark leads onto the next point. We note that the C-F relation has been derived in its first form, (2). The derivation in \cite{1} yields the second form (4). Let us see why this is so.
5. Heat kernel method

For the differential operator $-\Delta^2 + \xi R$, the general formula for $\zeta(0)$ on a two-dimensional domain, $\mathcal{M}$, with boundary $\partial \mathcal{M} = \cup \partial \mathcal{M}_i$ is

$$\zeta(0) = \frac{1 - 6\xi}{24\pi} \int_{\mathcal{M}} R \, dA + \frac{1}{12\pi} \sum_i \int_{\partial \mathcal{M}_i} \kappa(l) \, dl + \frac{1}{24\pi} \sum_\theta \frac{\pi^2 - \theta^2}{\theta}$$

(20)

where the $\theta$ sum runs over all inward facing angles at the corners of $\partial \mathcal{M}$. For minimal coupling $\xi = 0$, while for the special conformal coupling in $[12]$, $\xi = 1/8$.

The corner term was derived by Brownell [17] and Fedosov [18] and again by Kac [19] as the constant term in the short-time expansion of the heat-kernel. The angles $\theta$ are quite general.

Restricting attention to the spherical triangle, $\mathcal{M} = (pqr)$, $\kappa$ must vanish since the sides are geodesic. Further, if the triangle tiles the two-sphere once,

$$\frac{1 - 6\xi}{24\pi} \int_{\mathcal{M}} R \, dA = \frac{1 - 6\xi}{6g},$$

$2g$ being the number of triangles.

Choosing $\xi = 1/8$ (not essential), so that $\zeta(s) = \zeta_N(s)$, the identity becomes

$$g \sum_\theta \left( \frac{\pi}{\theta} - \frac{\theta}{\pi} \right) = 2(b_1(b_1 - 1) + g - 1)$$

(21)

or, writing the angle $\theta$ typically as $\pi/m$,

$$g \sum_m \left( m - \frac{1}{m} \right) = 2(b_1(b_1 - 1) + g - 1).$$

(22)

Reorganising the left-hand side slightly just to see how things work out in detail

$$g \sum_m (m - 1) + 3g - g \sum_m \frac{1}{m} = 2(b_1(b_1 - 1) + g - 1).$$

(23)

The Gauss-Bonnet, or area, equation reads

$$\sum_m \frac{1}{m} = \frac{2}{g} + 1$$

(24)

and we see that the $2(g - 1)$ term on the right of (23) will cancel against a term on the left to regain (4), bypassing explicit mention of the numbers $n_m$. We do have to know the area formula for a spherical triangle, however.
The \( n_m \) can be reinstated using (3), when we see that (19) is equivalent to (24).

For future reference, we note that the quantity \( g - 1 \) is the number of elements of the complete symmetry group that fix an axis but not a plane in \( \mathbb{R}^3 \). It is just the number of nontrivial rotational elements \( \text{i.e.} \) all the elements in \( \Gamma^+ \) except the identity. We denote it by \( b_2 \) and write (19) as

\[
b_2 = \sum_m n_m (m - 1)
\]  

which can be looked upon as an identity similar to, but more elementary than, (2).

Further numerical expressions, which are easily obtained, give the numbers of vertices, \( V \), edges, \( E \), and faces, \( F \), of the triangulation in terms of the group order, (cf [6] p 67)

\[
V = 2 + g, \quad E = 3g, \quad F = 2g.
\]  

Equation (24) can be interpreted in the following way. Think of the combination \( 2g \sum (1/2m) \) as a sum over all \( 2g \) fundamental domains of a quantity, \( \sum (1/2m) \), associated with one domain. If \( 2g \) Möbius triangles are assembled to form the simplicial decomposition of \( S^2 \), each vertex will appear with a multiplicity of \( 2m \). The \( 1/2m \) factor reduces this multiplicity to unity and so \( 2g \sum (1/2m) \) equals the number of vertices, \( V \).

6. Other relations. Higher dimensions

Other relations can be derived using (12) and our general knowledge of the heat-kernel coefficients, or the specific form of the \( \zeta \)-function on \( S^d/\Gamma \), such as (9).

The standard facts we need are that, if the heat-kernel expansion is

\[
K(\tau) \cong \frac{1}{(4\pi \tau)^{d/2}} \sum_{k=0,1/2,\ldots}^{\infty} C_k \tau^k,
\]

then

\[
\zeta(-k) = \frac{(-1)^k k!}{(4\pi)^{d/2}} C_{k+d/2} - n_0 \delta_{k0},
\]

where \( n_0 \) is the number of zero modes, and, further, that \( \zeta_d(s) \) has poles at \( s = (d - m)/2 \), for \( m = 0, 1, \ldots, d - 1 \) and \( m = d + 1, d + 3, \ldots \), with residues

\[
\frac{(4\pi)^{-d/2}}{\Gamma((d - m)/2)} C_{m/2}.
\]
using standard analytical methods. Our discussion is neither comprehensive nor
systematic.

Two basic dimension- and coupling-independent results are

\[ C_0 = |\mathcal{M}|, \quad C_{1/2} = \pm \frac{\sqrt{\pi}}{2} \sum_i |\partial \mathcal{M}_i|. \]  

(30)
The ± signs are for Neumann and Dirichlet conditions.

From (29) we see that \( C_0 \) and \( C_{1/2} \) are given by the residues in \( \zeta(s) \) at \( s = d/2 \)
and \( s = (d - 1)/2 \) respectively. For simplicity we might as well use the conformal
\( \zeta \)-functions given in (12). Therefore we need the pole structure of the Barnes \( \zeta \)-
function [15] which is, as \( s \to k/2 \),

\[ \zeta_d(2s, a \mid d) \to \frac{(-1)^{d+k}}{2(k-1)!} \frac{dS_1^{(k+1)}(a)}{s - k/2} \]  

(31)
where \( k \) is one of 1, 2, 3, \ldots, \( d \).

Comparing (31) and (29), cf [12] eqn. (76),

\[ C_{m/2} = (-1)^m \frac{2^m \pi^{(d+1)/2}}{\Gamma((d - m + 1)/2)} dS_1^{(d-m+1)}(a) \]

\[ = (-1)^m 2^{m-1} \pi^{m/2} |S^{d-m}| dS_1^{(d-m+1)}(a), \]  

(32)
where \( a \) is \( (d-1)/2 \) for Neumann conditions and \( \sum d_i - (d-1)/2 \) for Dirichlet. For
simplicity, we usually work with Neumann conditions. In view of (16) and (30) we
get in two dimensions, for example,

\[ |\mathcal{M}| = \frac{2\pi}{d_1 d_2}, \quad |\partial \mathcal{M}| = 2\pi \frac{d_1 + d_2 - 1}{d_1 d_2} = \frac{2\pi b_1}{g}. \]  

(33)
These equalities are easily confirmed from the trivial geometrical fact that the
two-sphere is covered by \( 2g \) spherical triangles, the vertices of which are formed by
the intersections of the \( b_1 \) symmetry great circles. The sum of the circumferences
of these circles is half the total perimeter of \( 2g \) fundamental domains, as is quickly
appreciated by drawing in the sides of each triangle separately – equivalent to
splitting each great circle into two (cf [6] p232). More formally, one can use the
measure-theoretic principle

\[ |A \cup B| + |A \cap B| = |A| + |B| \]  

(34)
applied to the edges.
Dividing in (33) we find
\[ b_1 = \frac{\partial M}{|M|}, \]
which is Laporte’s rule, [20], for the number of symmetry circles, \textit{i.e.} the nodal lines of the lowest odd mode (the Jacobian of degree \( b_1 \)).

The extension of these elementary results to higher dimensions is immediate. The relevant analytical fact is that, for all \( d \),
\[ aS_{1}^{(d+1)}(a) = \frac{1}{g}, \quad aS_{1}^{(d)}(\sum d_i - (d - 1)/2) = \frac{b_1}{2g} \tag{35} \]
so that
\[ |M| = \frac{1}{2g} |S^d| \tag{36} \]
and
\[ |\partial M| = \frac{b_1}{g} |S^{d-1}|. \tag{37} \]

The generalised Laporte rule is then
\[ b_1 = \frac{|S^d|}{2|S^{d-1}|} \frac{|\partial M|}{|M|}. \tag{38} \]

The geometric situation in \( \mathbb{R}^{d+1} \) is the generalisation of that in \( \mathbb{R}^3 \), recalled in section 1. We outline it here for any \( d \) (see [21], [6] pp 130, 137-140, [22] §4·1, [23] chap.12). We are most interested in \( d = 3 \).

The projection of the \((d+1)\)-dimensional regular polytope, \( \{k_1, k_2, \ldots, k_d\} \) (in \( \mathbb{R}^{d+1} \)) onto its circumscribing hypersphere yields a spherical tessellation, or honeycomb, the cells of which are the projections of the \( d \)-faces of the polytope. The symmetry \( d \)-flats (‘reflecting hyperplanes’) of \( \{k_1, k_2, \ldots, k_d\} \) intersect the circumsphere, \( S^d \), in a set of reflecting great \((d - 1)\)-spheres constituting the boundaries of \( |\Gamma| \) spherical \( d \)-simplexes that are transitively permuted by the complete symmetry group \( \Gamma = [k_1, k_2, \ldots, k_d] \) and which make up a simplicial subdivision of \( S^d \), [24]. (This decomposition could also be termed an irregular tessellation.) The \((d - 1)\)-dimensional boundary of the fundamental domain is the union of \( d \) pieces of the reflecting great \((d - 1)\)-spheres. Topologically, the honeycomb is the boundary, or frontier, of the ‘solid’ polytope. For \( d = 3 \) the fundamental domain is a spherical tetrahedron. Goursat [25] gives a detailed account of the different \( S^3 \) honeycombs.

Note that the symbol \( \{k_1, k_2, \ldots, k_d\} \) can refer to a \((d+1)\)-dimensional polytope in \( \mathbb{R}^{d+1} \) or to a spherical \((d+1)\)-dimensional polytope in \( S^{d+1} \) or to a \( d \)-dimensional honeycomb on \( S^d \), depending on context ([6] p 138).

Equation (37) expresses the fact that the measure of \textit{all} the reflecting great spheres is one half the measure of the total boundary of \( 2g \) fundamental domains.
The next identity arises on setting $m = 2$ in (32),

$$C_1 = 2\pi |S^{d-2}|_d S_1^{(d-1)}((d - 1)/2). \tag{39}$$

The $C_1$ coefficient in two dimensions has already been discussed in section 2 but we wish to extend the argument to $d$-dimensions and so write out its general form, valid for any coupling and region $\mathcal{M}$,

$$C_1 = \frac{1 - 6\xi}{6} \int_{\mathcal{M}} R dV + \frac{1}{3} \sum_i \int_{\partial \mathcal{M}_i} \kappa dS + \frac{1}{6} \sum_{i<j} \frac{\pi^2 - \theta_{ij}^2}{\theta_{ij}} |\partial \mathcal{M}_i \cap \partial \mathcal{M}_j|. \tag{40}$$

Here $\theta_{ij}$ is the (arbitrary) dihedral angle between the boundary components $\partial \mathcal{M}_i$ and $\partial \mathcal{M}_j$. For simplicity we have assumed that this angle is constant along the intersection. This is so for a fundamental domain. Fedosov’s calculation, [18], of the final term is for a flat polyhedral region so that the $\theta_{ij}$ are constant for him. For $d = 3$ the term could be referred to as an edge term.

We again employ the conformal coupling of [12] so that $\xi = (d - 1)/4d$. For $\mathcal{M}$ a fundamental domain, the first term simplifies to

$$\frac{1 - 6\xi}{6} \int_{\mathcal{M}} R dV = \frac{(3-d)(d-1)}{24g} |S^d| \tag{41}$$

and, because the boundary is geodesic, the second term in (40) vanishes.

We now look at the right-hand side of (39) which can be written more geometrically in terms of the number of reflecting $d$-planes and the number of those group elements that fix a $(d - 1)$-plane but not a $d$-plane. We denote these numbers by $b_1$ and $b_2$ respectively. A reflecting $(d - 1)$-plane intersects the circumsphere in a reflecting great $(d - 2)$-sphere.

The complete symmetry group $\Gamma$ is generated by $(d + 1)$ reflections, $R_1, \ldots, R_{d+1}$, in $(d + 1)$ hyperplanes in $\mathbb{R}^{d+1}$. Generally, $b_r$ is the number of elements of $\Gamma$ that fix a $(d + 1 - r)$-flat, but no flat of higher dimension, in $\mathbb{R}^{d+1}$. Equivalently, $b_r$ is the number of elements that are expressible as products of $r$ (but no fewer) reflections, i.e. those that have length $r$.

The Coxeter element, $R_1 R_2 \ldots R_{d+1}$, has period $h$, the Coxeter number, and characteristic roots $\exp(2\pi i m_j/h)$ where the $m_j$ ($j = 1, \ldots, d+1$) are the exponents, related to the degrees by $m_j = d_j - 1$.

Solomon’s theorem, [14], states that

$$\prod_{i=1}^{d+1} (1 + m_i t) = \sum_{r=0}^{d+1} b_r t^r. \tag{42}$$

We have chosen the last exponent, $m_{d+1}$, to be the one that equals unity. Therefore
\[ b_1 = \sum_{i} m_i = \sum_{i} m_i + 1, \]
\[ b_2 = \sum_{i<j} m_i m_j + \sum_{i} m_i \]

and
\[ b_3 = \sum_{i<j<k} m_i m_j m_k, \quad b_4 = \sum_{i<j<k<l} m_i m_j m_k m_l. \]

Because the characteristic roots occur in complex conjugate pairs, we have
\[ b_1 = \sum_{i} m_i = (d + 1)h/2, \quad (43) \]
in terms of the Coxeter number (cf (7)). This result also follows from a generalisation of the equator argument ([9] p 229-231, [26]). We also note that the largest exponent equals \( h - 1 \).

The breakdown
\[ 2g = \sum_{r} b_r \]
is obvious. Setting \( t = -1 \) in (42) we have
\[ \sum_{0}^{d+1} (-1)^{r} b_r = 0 \quad (44) \]
and so
\[ g = \sum_{\text{even}} b_r = \sum_{\text{odd}} b_r. \quad (45) \]
The first sum gives the order of the rotation subgroup, and the second the (equal) number of remaining elements.

Returning to (39), simple algebra yields
\[ d S_1^{(d-1)}((d - 1)/2) = \frac{1}{12g} (b_1(b_1 - 1) + b_2 + (3 - d)/2). \quad (46) \]
(The systematic evaluation of the Bernoulli polynomials is discussed in the appendix.)

When (46) is substituted into (39), the last term cancels against (41) finally giving the identity
\[ g \sum_{i<j} \left( \frac{\pi}{\theta_{ij}} - \frac{\theta_{ij}}{\pi} \right) |\partial M_i \cap \partial M_j| = |S^{d-2}|(b_1(b_1 - 1) + b_2). \quad (47) \]
As a simple check we can set \(d = 2\), when \(b_2 = g - 1\). Using \(|S^0| = 2\), (47) reduces to (22). (As a minor point, we note that for \(d = 2\), the identity does not come from residues.)

For \(d = 3\), the easiest case to visualise, the \(\partial M_i\), \((i = 0, 1, 2, 3)\), are the spherical triangular faces of \(M\), the spherical tetrahedron fundamental domain of the simplicial decomposition of the 3-honeycomb \(\Pi_4 = \{k_1, k_2, k_3\}\). The edges, \(\partial M_i \cap \partial M_j\), are the six, circular sides of these faces. The situation is depicted in [6] Fig. 7.9A, [22] Fig. 4.3B, [25]. Verbal descriptions can be found in [9] p 229, Todd [27] p 216, and Sommerville [21] p 188, [28].

The vertices \(P_i\) \((i = 0, 1, 2, 3)\) of the fundamental domain are obtained as follows. \(P_3\) is the centre of a cell of the honeycomb, i.e. of the projection onto the circumsphere of a 3-face of the 4-polytope, \(\Pi_4\). This cell is itself a regular (spherical) polytope, \(\Pi_3\), and \(P_2\) is the centre of a cell of its boundary. This cell is a 2-face (‘face’) of \(\Pi_3\) (and of \(\Pi_4\)) and is also a regular polytope, \(\Pi_2\). The cells, \(\Pi_1\)’s, of its boundary are 1-faces (‘edges’) and \(P_1\) is the centre of one of them. Finally, \(P_0\) is the end of such an edge and is the one point that corresponds to a vertex of the original polytope, \(\Pi_4\). An edge of a cell of the honeycomb has length \(2|P_0P_1|\). A nested set of \(\Pi_i\) \((i = 0, 1, 2, 3)\) is called a flag. Cells are sometimes referred to as facets.

Goursat [25] gives perhaps the most comprehensive description of the geometry. He refers to elements of the rotational subgroup, \(\Gamma^+\), as ‘transformations droites’ \((i.e. ~even)\) and to the remaining elements as ‘transformations gauches’ \((i.e. ~odd)\) and expresses them in terms of, up to four, inversions.

If a face of \(M\) be labelled dually by the opposing vertex, then the dihedral angles, \(\theta_{ij}\), between faces \(i\) and \(j\) are given by \(\theta_{01} = \pi/k_1\), \(\theta_{12} = \pi/k_2\), \(\theta_{23} = \pi/k_3\) with the rest being \(\pi/2\)’s ([27], [6], [28]). \(\theta_{ij}\) can also be characterised as the dihedral angle opposite to the edge \(P_iP_j\).

We call the edge lengths \(L_{ij} = |P_iP_j|\), and expand the identity (47) for \(d = 3\) as

\[
\begin{align*}
g \left( (k_1 - \frac{1}{k_1})L_{23} + (k_2 - \frac{1}{k_2})L_{03} + (k_3 - \frac{1}{k_3})L_{01} + \frac{3}{2}(L_{12} + L_{02} + L_{13}) \right) \\
= 2\pi(b_1(b_1 - 1) + b_2).
\end{align*}
\]

(48)

To check (47), or (48), we need the edge lengths. Some of these are given in [6], p 139 and Table I (ii).

It is always the case that \(L_{01} + L_{23} + L_{03}\) and \(L_{12} + L_{02} + L_{13}\) are separately commensurable with \(\pi\). For example, in \(\{3^3\}\) these quantities are both equal to \(\pi\) (so that the total perimeter is \(2\pi\)) and (48) is then easily checked numerically using \(g = 60, b_1 = 10\) and \(b_2 = 35\).

General formulae for \(L_{ij}\) exist due to Sommerville [28]. They are handy numerically and, for convenience, are repeated here in the appendix. We have used them to confirm (48) in all cases.
7. Geometrical meaning

A blind, numerical verification is all very well but equation (47) has a geometrical interpretation, or, if one prefers, a geometrical derivation. The left-hand side is clearly a sum over all the fundamental domains of a quantity associated with one domain and the right-hand side must be a global measure of the same total quantity. We know, for example, that \( b_1(b_1 - 1)/2 \) is the number of \( S^{d-2} \) intersections of \( b_1 \) \( S^{d-1} \)'s, including coincidences. The corresponding identity will be derived from (47) to see how the various terms fit together. Our treatment is not the most direct.

For simplicity, attention is restricted to \( d = 3 \). It is convenient to depart from the traditional notation and define

\[
    k_{kl} = \frac{\pi}{\theta_{ij}}, \quad (ijkl) \text{ cyclic.} \tag{49}
\]

The two-dimensional case suggests that we seek to evaluate the term

\[
    2g \sum_{k<l} \frac{1}{2k_{kl}} L_{kl}
\]

in (47). Assembling \( 2g \) fundamental domains, the \( 1/2k_{kl} \) factor cancels the multiplicity of appearance of the arc \( P_kP_l \) and the result is the total circumference of all the circular intersections of the reflecting \( S^{2} \)'s. Hence

\[
    2\pi b'_2 = g \sum_{k<l} \frac{1}{k_{kl}} L_{kl} \tag{50}
\]

where \( b'_2 \) is the total number of complete \( S^1 \) boundaries in the simplicial decomposition of the honeycomb. [For \( \{3^3\} \), \( b'_2=25 \).

The same reasoning applied to the perimeter gives the formula

\[
    2g \sum_{k<l} L_{kl} = 2 \sum_{k<l} N^{kl} k_{kl} L_{kl} \tag{51}
\]

where \( N^{kl} \) is the number of \( P_kP_l \) arcs in the simplicial decomposition. This agrees with

\[
    N^{kl} = \frac{g}{k_{kl}} \tag{52}
\]

which follows directly from the orbit-stabiliser theorem. Using (50), (47) becomes (for \( d = 3 \))

\[
    g \sum_{k<l} k_{kl} L_{kl} = 2\pi \left( b_1(b_1 - 1) + b_2 + b'_2 \right). \tag{53}
\]
Define, for convenience,

\[ D = g \sum_{k<l} L_{kl} - 2\pi (b_2 + b'_2) \]  

(54)

to get the intermediate result

\[ 2\pi b_1 (b_1 - 1) = g \sum_{k<l} (k_{kl} - 1) L_{kl} + D. \]  

(55)

We will show that \( D = 0 \), proving, incidentally, that the perimeter of the fundamental domain is commensurate with \( \pi \). We already have this because the perimeter is half the sum of the four face-perimeters each of which is known to be commensurate with \( \pi \). We then see that \( b_2 + b'_2 \) is the number of S\(^1\)'s on all the bounding S\(^2\)'s, counting as distinct, circles that belong to two, or more, S\(^2\)'s.

Proceeding in the geometrical vein, and noting that the intersection degeneracy of the reflecting S\(^2\)'s at the \((kl)\)-edge is \( k_{kl}(k_{kl} - 1)/2 \), the total perimeter of the S\(^1\) intersections, allowing for coincidences, is

\[ 2\pi b_1 (b_1 - 1) = \sum_{k<l} N_{kl} k_{kl}(k_{kl} - 1) L_{kl}. \]  

(56)

Hence, using (52), a comparison with (55) shows that \( D = 0 \), which equality can be put into the form

\[ 2\pi b_2 = \sum_{k<l} N_{kl} (k_{kl} - 1) L_{kl}. \]  

(57)

We have thus obtained identities, (56) and (57), precisely analogous to (2) and (25).

In \( \mathbb{R}^4 \) the analogue of a rotation axis in \( \mathbb{R}^3 \) is a rotation plane, \( i.e. \) an S\(^1\) boundary on the circumsphere, S\(^3\). Such a circle is divided into a number of arcs each with an associated order of rotation, one of the \( k_{kl} \). In words, (57) says that \( b_2 \) is obtained by counting the number of nontrivial rotations at every arc, weighted with that arc’s relative size.

Although we can look upon our development as a proof of (57) it ought to be derivable directly. However, we have been unable, so far, to provide a geometrical reason for (57). It might be related to the various extensions of the Euler formula such as the Dehn-Sommerville equations and other angle-sum relations, of which there seem to be quite a variety, \( e.g. \) [29].

To set the geometrical picture more closely, the general situation is again recalled.

Each reflecting \((d-1)\)-sphere is divided by its intersections with all the others into a certain number of \((d-1)\)-simplexes taken from those forming the boundary of
the fundamental domain. The topological structure is the same for all the reflecting spheres although the composition may differ metrically.

From (37) we find that the number of \((d-1)\)-simplexes in one reflecting \((d-1)\)-sphere is \((d+1)g/b_1\). If we assume (43), this number equals \(2g/h\), ((6), [9] pp 231-232, [26]).

A simple example is the \(\{3,4\}\) 2-honeycomb for which there are two types of reflecting circles. One has the vertex content \(02120212\) and the other \(01010101\). (See [6] p 65.) Both have \(2g/h = h + 2 = 8\) segments. Incidentally this is a good example in which to check (1).

We need to prescribe the polytope in more detail. This is done in terms of its configurational numbers \(N_{ij}\) ([6] pp 12, 130, [21], [28], [23] §12·6). A regular polytope, \(\Pi_n\), can be thought of as a nested sequence of regular polytopes, of type \(\Pi_i\), \((i = 0, 1, \ldots, n)\). (See [6] pp 127-129.) An element, \(\Pi_j\), belongs to \(N_{jk}\) \(\Pi_k\)’s for each \(k > j\) and contains \(N_{jk}\) \(\Pi_k\)’s for each \(k < j\). The number \(N_i\), which for symmetry’s sake can be written \(N_{ii}\), is just the number of \(\Pi_i\)’s in the polytope. \(N_0\) is the number of vertices, for example, and \(N_n\) is one.

For example, the regular simplex, \(\alpha_{d+1} = \{3^d\}\), has a complete symmetry group of order \((d + 2)!\) and numbers, \(N_i\),

\[
N_i = \binom{d + 2}{i + 1}.
\]

(See [21] p 96, [6].) Notationally, Coxeter in [30] has \(\binom{i}{d+1}\) for \(N_i\).

If we set \(n = d + 1\) and take \(\Pi_{d+1}\) to be the Euclidean polytope that projects to the \(d\)-dimensional honeycomb, then the vertices, \(O_i\), \((i = 0, 1, \ldots, d)\) of the \((d+1)\)-simplex in the decomposition of \(\Pi_{d+1}\) project to the vertices \(P_i\) mentioned earlier (for \(d = 3\)). \(O_{d+1}\) is the centre of \(\Pi_{d+1}\).

For the \((d+1)\)-polytope \(\Pi_{d+1} = \{k_1, k_2, \ldots k_d\}\) we can put

\[
\Pi_i = \{k_1, k_2, \ldots, k_{i-1}\}
\]

or, for the ascending construction,

\[
\Pi'_i = \{k_{d-i+2}, k_{d-i+3}, \ldots, k_d\},
\]

for the constituent polytope elements. The relation between the orders of the complete symmetry groups of these \(\Pi_i\)’s and the numbers \(N_i\) is

\[
N_i = \frac{|[k_1, \ldots, k_d]|}{|[k_1, \ldots, k_{i-1}]| \cdot |[k_{i+2}, \ldots, k_d]|}
\]

(59)

which follows from the orbit-stabiliser theorem. The first factor in the denominator is the order of the ‘internal’ symmetry group of \(\Pi_i\) while the second is the order of
the stabiliser of $\Pi_i$ as a whole. The ratio is then the index of the subgroup leaving the centre of $\Pi_i$, that is $P_i$, or $O_i$, invariant i.e. it is the number of $\Pi_i$’s. We must remember that the internal symmetry group of an edge, $\{\}$, has order 2. Equation (58) is an example of (59). We note that $N_i = N_{d-i}$.

Two special elements are $\Pi_d$, which is the ‘cell’ of $\Pi_{d+1}$ (sometimes called the ‘bounding figure’), and $\Pi'_d$, which is the ‘vertex figure’. A property of the regular polytope is that the vertex figure of the bounding figure is the bounding figure of the vertex figure. Continuing with the terminology, we may refer to $\Pi'_i$ as the $(d + 1 - i)$-th vertex figure (cf [6] pp 133, 134.) and the $\Pi_i$ as the $(d + 1 - i)$-th bounding figure. The $\Pi_i$ are the perischemons of Schlafli.

8. Volume and degrees
A fundamental result of Schlafli’s, [31], concerns the variation of the volume (or content) of a spherical polytope, $\Pi_d$, as the defining parameters, such as the positions of the vertices, are altered slightly. For arbitrary small displacements, the change in volume is

$$d|\Pi_d| = \frac{1}{d-1} \sum |\Pi_{d-2}|d\theta$$

(60)

where the sum is over the facets, $\Pi_{d-2}$, of the various bounding $\Pi_{d-1}$’s. (These facets are Schlafli’s secondary perischemons.) The angle $\theta$ is that between the $(d-1)$-planes intersecting in the facet. (In our previous notation $\theta_{ij} = \theta$ and $\partial M_i \cap \partial M_j$ is a $\Pi_{d-2}$. ) Equation (60) can be iterated on $d$.

We move quickly to $d = 3$ and choose a 3-simplex for $\Pi_3$, i.e. a general spherical tetrahedron, for which the result was also given by Richmond [32], [33]. Denoting the edges and corresponding dihedral angles by $L$ and $\theta$, the volume variation is

$$dV = \frac{1}{2} \sum_{\text{edges}} Ld\theta.$$  

(61)

The tetrahedron is uniquely fixed by specifying the six $\theta$’s or, equivalently, the six $L$’s.

Schlafli points out the following ‘reciprocity’. Consider another tetrahedron whose angles are $\pi - L$ and whose edges are $\pi - \theta$. The change in its volume is

$$dV' = -\frac{1}{2} \sum_{\text{edges}} (\pi - \theta)dL.$$  

(62)

and so $d(V + V')$ is a perfect differential which integrates to

$$V + V' = \pi^2 - \frac{1}{2} \sum (\pi - \theta)L.$$  

(63)

The constant of integration can be determined by setting all the $\theta$’s, and $L$’s, equal to $\pi/2$ giving the self-reciprocal orthant of volume $\pi^2/8$. Schlafli, [34] p 281, fixes
the constant by choosing the sides of \( V \) vanishingly small, in which case \( V' \) becomes half a full sphere.

We wish to apply (63) to a doubly rectangular tetrahedron (an orthoscheme), such as a fundamental domain, for which the angles are \( \alpha, \beta, \gamma \) and three \( \pi/2 \)'s. Schl"afli denotes the volume by \( \pi^2 f(\alpha, \beta, \gamma)/8 \), so normalising \( f \) to unity for an orthant.

The reciprocal of a doubly rectangular tetrahedron is a tetrahedron (not doubly rectangular in general) in which three consecutive ‘nonplanar’ edges equal \( \pi/2 \), the rest being \( \pi - \alpha, \pi - \beta \) and \( \pi - \gamma \). We write its volume as \( \pi^2 F/8 \).

Now we choose a fundamental domain, \( \alpha = \pi/k_1, \beta = \pi/k_2, \gamma = \pi/k_3 \), and get from (63)

\[
f + F = 8 - \frac{4}{\pi} \sum_{k<l} (1 - \frac{1}{k kl}) L_{kl},
\]

where on the right we have reverted to our previous notation. Then, from (57), using the fact that the volume of \( 2g \) fundamental domains is \( 2\pi^2 \), so that \( gf = 8 \), we find

\[
b_2 = g - 1 - \frac{F}{f}.
\]

We recall that, in the two-dimensional (i.e. \( \mathbb{R}^3 \)) case, \( b_2 = g - 1 \) on the simple geometric grounds that rotations fix an axis but not a plane. In \( \mathbb{R}^4 \), the elements that fix a plane but not a hyperplane will certainly belong to the rotation subgroup but now we have the extra possibility of a rotation (a product of four reflections) fixing a point but not a plane or hyperplane. These must also be removed, as well as the identity. Hence we have

\[
b_2 = g - 1 - b_4,
\]

which we already know, (45), and whence the curious relation

\[
F = b_4 f.
\]

The volume of the reciprocal tetrahedron is \( b_4 \) times that of the fundamental orthoscheme. Numerically, \( b_4 \) is the product of the exponents, \( m_j \). For \( \{3^3\} \), \( f = 2/15 \) and \( b_4 = 24 \). At the moment, we have no direct geometrical interpretation of (66). The reciprocal orthoscheme does not seem to be a particularly useful construct.

Schl"afli [34] interestingly uses (63), applied to a polytope, to deduce several numerical results for \( f \).

We remark that the Schl"afli function, \( f \), has been used occasionally in physics, [35], [36], [37].
9. A further identity
A further identity follows on setting \( m = 3 \) in (32),

\[
C_{3/2} = \frac{8\pi^{(d+1)/2}}{\Gamma(d/2 - 1)} dS_1^{(d-2)}
\]

\[
= \frac{\pi^{3/2}}{3g} |S^{d-3}| B_3^{(d)} ((d - 1)/2 | \textbf{d}) = \frac{\pi^{3/2}}{24g} |S^{d-3}| (2b_1(b_1 - b_2 + d - 3)),
\]

(see the appendix). This will come into play for \( d \geq 3 \).

We can only state the result for the (Neumann) coefficient \( C_{3/2} \) in the incomplete form,

\[
C_{3/2} = \frac{\sqrt{\pi}}{192} \sum_i \int_{\partial M_i} (6\text{tr}({\kappa}^2) - 3\kappa^2 - 4\widehat{R} + 12(8\xi - 1)R)dS + \text{corner terms}
\]

(68)

where \( \widehat{R} \) is the intrinsic curvature of the boundary parts.

In \( d \)-dimensions, the curvature part of \( C_{3/2} \) reduces to

\[
\frac{\sqrt{\pi}}{24} (d - 1)(d - 2)|\partial M|.
\]

(69)

This combines with the last term in (67), if we use (38) for \( b_1 \), to give for the (unknown) corner terms, the rather simple value

\[
\text{corner terms} = \frac{\pi^{3/2}}{12g} |S^{d-3}| b_1(b_1 - b_2 - 2) < 0,
\]

(70)
in this particular geometry. We check that the right-hand side vanishes for \( d < 3 \).

To the author’s knowledge, no general form for the corner term is known. An expression in the case that \( M \) is a polygonal cylinder is given by Pathria [38] and Baltes, [39], but this is not enough. There are some general results of Cheeger [40]. For the time being, we can just take the attitude that we are calculating these corner terms for a special geometry.

The derivation of (40) can be traced back to work of Sommerfeld on propagation in a wedge of arbitrary angle. To get the missing terms in (68) it would be necessary to discuss a general trihedral corner.

It would be interesting to use the result (70) to get a handle on the form of the corner terms. If \( d = 3 \), the right-hand side of (70) must have a combinatorial-geometric significance in terms of the vertices of the simplicial decomposition of \( S^3 \). In terms of the order \( g \), we find the value of \( b_1(b_2 - b_1 + 2) \) to be \( 9g/2 \) for \( \{3^3\} \), \( 6g \)
for \( \{3^2\,4\}, 7g \) for \( \{343\} \) and \( 9g \) for \( \{3^2\,5\} \). For the record, the corresponding values of \( b_2 - b_1 + 2 \) are 27, 72, 168 and 1080.

Let us recapitulate the identities. So far we have (36), (37), (47) and (67) (with (68)). Assume that the volume of \( \mathcal{M} \), and in fact all the geometry of \( \mathcal{M} \), is known in terms the dihedral angles. Then, with Schlafli, the order, \( 2g \), of the symmetry group can be found from (36). Next, \( b_1 \) follows from (37), and \( b_2 \) from (47) but \( b_3 \) is curiously absent from (67).

### 11. Higher relations

We return to two dimensions and continue with the discussion in section 4.

More generally than looking at \( \zeta(0) \), we have from the equality of \( g \zeta(-k) \),

\[
\sum_m m n_m \zeta_m(-k) = -2 \left( \sum_m m - 1 \right) \zeta_m(-2k - 1, 1/2) = \frac{2}{(2 + k)(1 + k)} B_{2+k}^{(2)} \left( \frac{1}{2} \mid d_1 d_2 \right).
\]

(71)

The Bernoulli functions are ‘non-topological’.

We repeat the expression for \( \zeta_m(-k) \) given in [12]

\[
\zeta_m(-k) = -\frac{1}{m(k + 1)} B_{2k+2}(1/2) + m^{2k-1} \left( \frac{2}{2k+1} \right) \sum_{p=0}^{m-1} (2p + 1) B_{2k+1}((2p + 1)/2m). \quad (72)
\]

The following identity, derived in the appendix to [12], is needed in order to rewrite (72) in a form more suitable for substituting into (71).

\[
\sum_{l=0}^{k} \binom{2k}{2l} (m^{2l} - 1) B_{2k-2l}(\frac{1}{2}) B_{2l} = k m^{2k-2} \sum_{p=0}^{m-1} (2p + 1) B_{2k-1}((2p + 1)/2m). \quad (73)
\]

For (71) we then find

\[
\frac{2}{(2 + k)} B_{2+k}^{(2)} \left( \frac{1}{2} \mid d_1 d_2 \right) =
\]

\[
\frac{1}{2k + 1} \sum_m m^{k+1} \sum_{l=1}^{m} \binom{2k + 2}{2l} (m^{2l} - 1) B_{2l} B_{2k+2-2l} \left( \frac{1}{2} \right) - B_{2k+2} \left( \frac{1}{2} \right). \quad (74)
\]

For \( k = 0 \) this is our previous result of course. We write out the explicit form when \( k = 1 \),

\[
240 B_{3}^{(2)} \left( \frac{1}{2} \mid d_1 d_2 \right) = -4 \sum_m m (2(m^4 - 1) + 5(m^2 - 1)) - 21 < 0. \quad (75)
\]
Generally, looking at the form of (74), we see that by a recursion argument we can derive an expression for the sum $\sum l n(m^{2^l} - 1)$ in terms of the degrees. Then, making use of the relations (3), and of (24), this expression will yield an equation for the sums of all odd powers of the three orders, $\sum l m^{2^l-1}$, in terms of the two degrees.

There is no doubt that these relations are numerically satisfied. After all they are identities. They have nevertheless been checked for the regular solids. (The dihedral case $[m]$ may also be used as an example. For this, $(p, q, r) = (m, m, 1)$ and $d_1 = m, d_2 = 1$. Heuristically we can set $n_p = n_m = 1/2$ rather than make this a special case.)

$B^2_3(\frac{1}{2} | d_1 d_2)$ can be calculated or looked up. It comes out directly as

$$240 B^2_3(\frac{1}{2} | d_1 d_2) = -8(d_1^4 + d_2^4) + 40d_1^2 d_2^2 - 120d_1 d_2(d_1 + d_2) +$$

$$60(d_1^2 + d_2^2 + 3d_1 d_2) - 60(d_1 + d_2) + 15 \quad (76)$$

but it should be rewritten in terms of $g$ and $b_1$. We have the easily verified relation

$$d_1^4 + d_2^4 = b_1^4 + 4b_1^3 - 4gb_1^2 + 6b_1^2 + 2g^2 - 8gb_1 - 4g + 4b_1 + 1$$

and we find for the left-hand side of (75),

$$-8b_1^4 - 32b_1^3 + 12b_1^2 + 28b_1 + 32gb_1^2 - 56gb_1 + 24g^2 - 28g + 7. \quad (77)$$

The resulting identity cannot be obtained from the C-F relation. So far as I can see it does not have a geometrical interpretation. Since there are only a well defined set of orders $(p, q, r)$ it is perhaps not surprising that there are many particular relations.

12. Conclusion
Since the approach depends on the expansion of the heat-kernel, it is restricted by the availability of explicit forms for the coefficients. It is certainly of interest to obtain the unknown corner terms.

Because of our inability to derive (57) directly (it may of course be obvious), we would like to claim a modest amount of novelty for our method.

At one level, it might be said that all we are doing is checking known coefficients in a specific situation. There is, however, a little more to it than that. To check the numbers, it is not necessary to rewrite the right-hand side of (40) in terms of the $b_r$. It could have been left in the degrees. However the dramatic simplification of the Bernoulli functions and the various cancellations suggest some interesting general relations that have yet to be uncovered.
If it is wished to check numerically the identity for \( d = 4 \), say, then it will be necessary to work out the areas of the 2-faces between neighbouring 3-face cells of the boundary of the 4-simplex fundamental domain.

It does not seem possible to extend easily the methods of this paper to the stellated polytopes, such as the Kepler-Poinsot polyhedra. The difficulty is the evaluation of the heat-kernels on the branched coverings. In such highly symmetrical situations it might be possible to obtain closed forms for the heat-kernels but we have not been able to make any progress in this direction. Laporte [20] has some useful comments on the mode problem.

In the two-dimensional case, Rubinowicz [41] claims to have a method of integrating the wave-equation on an arbitrary Riemann surface, but, in fact, only discusses the case of one branch point in detail and says that the general case follows upon a process of ‘Zusammenstücklung’, which, unfortunately, he does not exhibit.

**Appendix**

We give here Sommerville’s expressions [28] for the edge lengths of the fundamental simplex of the \( d \)-dimensional spherical honeycombs. For \( \{3, 3, \ldots, w\} = \{3^{d-1}, w\} \),

\[
\sin^2 L_{ij} = \frac{j - i}{j + 1} \cdot \frac{1 - d \cos \left( \frac{2\pi}{w} \right)}{1 - (d - i - 1) \cos \left( \frac{2\pi}{w} \right)}
\]  
(78)

while, for the reciprocal, \( \{w, \ldots, 3, 3\} = \{w, 3^{d-1}\} \),

\[
\sin^2 L_{ij} = \frac{j - i}{d - i + 1} \cdot \frac{1 - d \cos \left( \frac{2\pi}{w} \right)}{1 - (j - 1) \cos \left( \frac{2\pi}{w} \right)}.
\]  
(79)

For the more general case, \( \{v, 3, 3, \ldots, 3, 3, w\} = \{v, 3^{d-2}, w\} \),

\[
\cos^2 L_{ij} = \frac{1 + (1 + j - d) \cos \left( \frac{2\pi}{w} \right)}{1 + (1 + i - d) \cos \left( \frac{2\pi}{w} \right)} \cdot \frac{1 + (1 - i) \cos \left( \frac{2\pi}{v} \right)}{1 + (1 - j) \cos \left( \frac{2\pi}{v} \right)}
\]  
(80)

Finally for the case \( \{3, 4, 3, 3, \ldots, 3\} = \{3, 4, 3^{d-2}\} \) we have

\[
\cos^2 L_{ij} = \frac{d - j + 1}{d - i + 1} \cdot \frac{5 - i}{5 - j} \quad (1 \leq i < j \leq d)
\]

\[
\cos^2 L_{0j} = \frac{d - j + 1}{5 - j}
\]  
(81)

and for its reciprocal, \( \{3, 3, \ldots, 3, 4, 3\} = \{3^{d-2}, 4, 3\} \),

\[
\cos^2 L_{ij} = \frac{5 + j - d}{5 + i - d} \cdot \frac{i + 1}{j + 1} \quad (i < j \leq d - 1)
\]
\[ \cos^2 L_{id} = \frac{i + 1}{5 + i - d}. \quad (82) \]

We now turn to the connection of the generalised Bernoulli numbers with the Todd Polynomials.

Hirzebruch \[42\] gives the relation

\[ T_k(c_1, \ldots, c_k) = (-1)^k \frac{k!}{k} B_k^{(d)}(d_1, \ldots, d_d) \quad k \leq d \quad (83) \]

where, the \( c_s \) are the elementary symmetric functions of the degrees \( d_i \) \((i = 1, \ldots, d)\).

Using the known formulae for the \( T_k \) we can obtain the Bernoulli quantities if some transformations are made. We wish to write the quantities in terms of the elementary symmetric functions, \( b_r \), of the exponents \( m_i \) \((i = 1, \ldots, d + 1)\) so we firstly need the connection between the \( c_s \) and the \( b_r \). This is easily found as follows.

First divide the polynomial,

\[ \prod_{i=1}^{d} (1 + m_i t) = \frac{1}{1 + t} \sum_{r=0}^{d+1} b_r t^r = \sum_{0}^{d} b_r' t^r \]

where

\[ b_r' = (-1)^{r+1} \sum_{i=r+1}^{d+1} (-1)^i b_i = (-1)^r \sum_{0}^{r} (-1)^i b_i \]

using (44). Then set \( t \to t/(1 + t) \) to get

\[ \sum_{0}^{d} b_r' t^r (1 + t)^{d-r} = \prod_{1}^{d} (1 + d_i t) = \sum_{0}^{d} c_s t^s. \]

Expanding the \((1 + t)\), the \( c_s \) can be read off and a rearrangement of the summations gives

\[ c_s = \sum_{r=0}^{s} (-1)^r b_r \sum_{i=r}^{s} (-1)^i \binom{d-i}{d-s}. \quad (84) \]

As a point of more than technical interest, Lemmas 1 and 2 of Todd \[43\] provide an alternative means of obtaining the elementary symmetric functions of quantities translated by a constant.

More generally than is needed here we have the following result

\[ c_s(\{d_i + \lambda\}) = \sum_{i=0}^{s} \lambda^i \binom{d+i-s}{i} c_{s-i}(\{d_i\}) \]
which follows directly upon the replacement \( t \rightarrow t/(1 + \lambda t) \) in

\[
\prod_1^d (1 + d_i t) = \sum_o c_s(\{d_i\}) t^s
\]

or from a Taylor expansion and then use of the standard formula

\[
\frac{\partial c_s(\{d_i\})}{\partial d_i} = s c_s(\{\hat{d}_i\})
\]

where the hat signifies omission of the term, (cf Lemma 2 in [43]). Note that Todd’s \( h_s \) is a sum of all homogeneous products i.e.

\[
\sum_0^d h_s t^s = \frac{1}{\sum_0^d (-1)^s c_s t^s} = \frac{1}{\prod(1 - d_i t)}.
\]

The quantity we actually require is the Bernoulli function \( B_r^{(d)}(x|\mathbf{d}) \) (at \( x = (d - 1)/2 \)) which is a polynomial in \( x \) whose coefficients are the Bernoulli numbers in (83),

\[
B_r^{(d)}(x|\mathbf{d}) = (-1)^r r! \sum_{l=0}^r (-1)^l \frac{x^l}{l!} T_{r-l}(c_1, \ldots, c_{r-l}). \tag{85}
\]

From the listed expressions for the Todd polynomials in the \( c_s \), using (84) and (85), we can generate the Bernoulli polynomials at \( x = (d - 1)/2 \) in terms of the \( b_r \). These have been used in the main text. We collect a few results here.

\[
B_2^{(d)}((d - 1)/2 | \mathbf{d}) = \frac{1}{6} (b_1(b_1 - 1) + b_2 + (3 - d)/2)
\]

\[
B_3^{(d)}((d - 1)/2 | \mathbf{d}) = \frac{1}{4} b_1(b_1 - b_2 + (d - 3)/2) \tag{86}
\]

\[
B_4^{(d)}((d - 1)/2 | \mathbf{d}) = -\frac{1}{240} (8b_4 - 24b_3 - 8b_3b_1 - 24b_2^2 - 4b_2(8b_2^2 - 16b_1 - 5d + 9) + 4b_1(b_1 - 1)(2b_1^2 + 10b_1 + 5d - 13) - (5d - 3)(d - 5)).
\]

It is probably more instructive to evaluate the Bernoulli polynomials directly rather than go through the Todd polynomials. We can use the general theory of multiplicative series as developed by Hirzebruch [42] and this will be detailed elsewhere.
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