A NEW CHARACTERIZATION OF CALABI COMPOSITION
OF HYPERBOLIC AFFINE HYPERSPHERES

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Abstract. In this paper, we mainly prove a theorem with a corollary establishing two characterizations of the Calabi composition of hyperbolic hyperspheres, where the second characterization (i.e., the corollary) has been given via a dual correspondence theorem earlier but now we would like to use a very direct method. Note that Z.J. Hu, H.Z. Li and L. Vrancken also gave a characterization of the 2-factor Calabi composition in a different manner.

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1. Introduction

As we know, affine hyperspheres are the most important objects studied in affine differential geometry of hypersurfaces, drawing great attention of many geometers. In fact, affine hyperspheres seems simple in definition but they do form a very large class of hypersurfaces, the study of which is fruitful in recent twenty years. For example, the proof of the Calabi’s conjecture ([13], [14]), the classification of hypersurfaces of constant affine curvatures ([21], [22], [11]), and in [10] the complete classification of locally strongly convex hypersurfaces with parallel Fubini-Pick forms as a special class of hyperbolic affine hyperspheres (for some earlier partial results, see [1], [4], [9]). As for the general nondegenerate case, there also have been some interesting partial classification results, see for example the series of published papers by Z.J. Hu etc: [6], [7] and [8].

In 1972, E. Calabi ([2]) found a composition formula by which one can construct new hyperbolic affine hyperspheres from any two given ones. The present author has generalized Calabi construction to the case of multiple factors (See [17], published in Chinese). Later in 1994 F. Dillen and L. Vrancken [3] generalized Calabi original composition to any two proper affine hyperspheres and gave a detailed study of these composed affine hyperspheres. They also mentioned that their construction applies to the case of multiple factors but with no detail of this. In 2008, in order to establish their later classification in [10] mentioned above, Z.J. Hu, H.Z. Li and L. Vrancken proved a characterization of the Calabi composition of hyperbolic hyperspheres ([5]) by special decompositions of the tangent bundle. We would like to remark
that, by using the similar idea of [3], H.Z. Li and X.F. Wang has in a way characterized the so called Calabi product of parallel Lagrangian submanifolds in the complex projective space \( \mathbb{C}P^n \) [16].

In a previous paper, we explicitly defined the Calabi composition of multiple factors of hyperbolic hyperspheres, possibly including some point factors viewing as “0-dimensional hyperbolic hyperspheres”, and made it in detail for the computation of the basic affine invariants of this composition. In this article, by using those basic affine invariants, we prove a theorem (see Theorem 3.1) which provides a new and more natural characterization of the Calabi composition of multiple hyperbolic hyperspheres. In the case of affine symmetric factors this characterization turns out to be much more simple (see Corollary 3.2).

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2. Preliminaries

2.1. The equiaffine geometry of hypersurfaces. In this subsection, we brief some basic facts in the equiaffine geometry of hypersurfaces. For details the readers are referred to some text books, say, [16] and [20].

Let \( x : M^n \to \mathbb{R}^{n+1} \) be a nondegenerate hypersurface. Then there are several basic equiaffine invariants of \( x \) among which are: the affine metric (Berwald-Blaschke metric) \( g \), the affine normal \( \xi := \frac{1}{n} \Delta g x \), the Fubini-Pick 3-form (the so called cubic form) \( A \in \bigodot^3 T^*M^n \) and the affine second fundamental 2-form \( B \in \bigodot^2 T^*M^n \). By using the index lifting by the metric \( g \), we can identify \( A \) and \( B \) with the linear maps \( A : TM^n \to \text{End}(TM^n) \) or \( A : TM^n \odot TM^n \to TM^n \) and \( B : TM^n \to TM^n \), respectively, by

\[
g(A(X)Y, Z) = A(X, Y, Z) \quad \text{or} \quad g(A(X, Y), Z) = A(X, Y, Z), \quad g(B(X), Y) = B(X, Y),
\]

for all \( X, Y, Z \in TM^n \). Sometimes we call the corresponding \( B \in \text{End}(TM^n) \) the affine shape operator of \( x \). In this sense, the affine Gauss equation can be written as follows:

\[
R(X, Y)Z = \frac{1}{2} (g(Y, Z)B(X) + B(Y, Z)X - g(X, Z)B(Y) - B(X, Z)Y) - [A(X), A(Y)](Z),
\]

where, for any linear transformations \( T, S \in \text{End}(TM^n) \),

\[
[T, S] = T \circ S - S \circ T.
\]

Each of the eigenvalues \( B_1, \ldots, B_n \) of the linear map \( B : TM^n \to TM^n \) is called the affine principal curvature of \( x \). Define

\[
L_1 := -\frac{1}{n} \text{tr} B = -\frac{1}{n} \sum_i B_i.
\]

Then \( L_1 \) is referred to as the affine mean curvature of \( x \). A hypersurface \( x \) is called an (elliptic, parabolic, or hyperbolic) affine hypersphere, if all of its affine principal curvatures are equal to one (positive, 0, or negative) constant. In this case we have

\[
B(X) = L_1 X, \quad \text{for all } X \in TM^n.
\]

It follows that the affine Gauss equation (2.2) of an affine hypersphere assumes the following form:

\[
R(X, Y)Z = L_1 (g(Y, Z)X - g(X, Z)Y) - [A(X), A(Y)](Z),
\]

Furthermore, all the affine lines of an elliptic affine hypersphere or a hyperbolic affine hypersphere \( x : M^n \to \mathbb{R}^{n+1} \) pass through a fix point \( o \) which is refer to as the affine center of \( x \); Both the elliptic affine hyperspheres and the hyperbolic affine hyperspheres are called proper affine hyperspheres, while the parabolic affine hyperspheres are called improper affine hyperspheres.

For each vector field \( \eta \) transversal to the tangent space of \( x \), we have the following direct decomposition

\[
x^*T\mathbb{R}^{n+1} = x_*(TM) \oplus \mathbb{R} \cdot \eta.
\]
This decomposition and the canonical differentiation $\bar{D}^0$ on $\mathbb{R}^{n+1}$ define a bilinear form $h \in \bigodot^2 T^* M^n$ and a connection $D^n$ on $TM^n$ as follows:

$$\bar{D}^0 X Y = x_*(D^n_Y) + h(X,Y)\eta, \quad \forall X, Y \in TM^n.$$  \hfill (2.7)

(2.7) can be referred to as the affine Gauss formula of the hypersurface $x$. In particular, in case that $\eta$ is parallel to the affine normal $\xi$, the induced connection $\nabla := D^n$ is independent of the choice of $\eta$ and is referred to as the affine connection of $x$.

In what follows we make the following convention for the range of indices:

$$1 \leq i, j, k, l \leq n.$$

Let $\{e_i, e_{n+1}\}$ be a local unimodular frame field along $x$ with $\eta := e_{n+1}$ parallel to the affine normal $\xi$, and $\{\omega^i, \omega^{n+1}\}$ be its dual coframe. Then we have connection forms $\omega^A_B$, $1 \leq A, B \leq n + 1$, defined by

$$d\omega^A_B = \omega^R_B \wedge \omega^A_R, \quad d\omega^A_B = \sum_{C=1}^{n+1} \omega^R_A \wedge \omega^C_R, \quad \omega^{n+1} = 0.$$  \hfill (2.8)

Furthermore, the above invariants can be respectively expressed locally as

$$g = \sum g_{ij} \omega^i \omega^j, \quad A = \sum A_{ijk} \omega^i \omega^j \omega^k, \quad B = \sum B_{ij} \omega^i \omega^j,$$  \hfill (2.9)

subject to the following basic formulas:

$$\sum_{i,j} g^{ij} A_{ijk} = 0 \quad \text{(the apolarity)},$$

$$A_{ijk,l} - A_{ijl,k} = \frac{1}{2} (g_{ik} B_{jl} + g_{jl} B_{ik} - g_{il} B_{jk} - g_{jk} B_{il}),$$

$$\sum_l A_{ij,l}^l = n \left( L_1 g_{ij} - B_{ij} \right),$$  \hfill (2.10)

where $A_{ijk,l}$ are the covariant derivatives of $A_{ijk}$ with respect to the Levi-Civita connection of $g$.

Write $h = \sum h_{ij} \omega^i \omega^j$ and $H = |\det(h_{ij})|$. Then

$$g_{ij} = H^{-\frac{n-1}{n+2}} h_{ij}, \quad \xi = H^{-\frac{1}{n+2}} e_{n+1}.$$  \hfill (2.11)

Define

$$\sum_k h_{ijk} \omega^k = dh_{ij} + h_{ij} \omega_{n+1} - \sum_k h_{kj} \omega_i^k - \sum_k h_{ik} \omega_j^k.$$  \hfill (2.12)

Then the Fubini-Pick form $A$ can be determined by the following formula:

$$A_{ijk} = -\frac{1}{2} H^{-\frac{1}{n+2}} h_{ijk}.$$  \hfill (2.13)

Define the normalized scalar curvature $\chi$ and the Pick invariant $J$ by

$$\chi = \frac{1}{n(n-1)} \sum g^{ij} g^{jk} R_{ijkl}, \quad J = \frac{1}{n(n-1)} \sum A_{ijk} A_{pqrs} g^{ip} g^{jr} g^{ks}.$$  \hfill (2.14)

Then the affine Gauss equation can be written in terms of the metric and the Fubini-Pick form as follows

$$R_{ijkl} = (A_{ijk,l} - A_{ijl,k}) + (\chi - J) (g_{il} g_{jk} - g_{ik} g_{jl})$$

$$+ \frac{2}{n} \sum (g_{ik} A_{jm,m} - g_{il} A_{jm,k,m}) + \sum_m (A_{ik}^m A_{jm,l} - A_{il}^m A_{jm,k}).$$  \hfill (2.15)

The following existence and uniqueness theorems are well known:

**Theorem 2.1.** ([15]) (The existence) Let $(M^n, g)$ be a simply connected Riemannian manifold of dimension $n$, and $A$ be a symmetric 3-form on $M^n$ satisfying the affine Gauss equation (2.15) and the apolarity condition (2.9). Then there exists a locally strongly convex immersion $x : M^n \to \mathbb{R}^{n+1}$ such that $g$ and $A$ are the affine metric and the Fubini-Pick form for $x$, respectively.
Theorem 2.2. (15) (The uniqueness) Let \( x : M^n \to \mathbb{R}^{n+1}, \bar{x} : \bar{M}^n \to \mathbb{R}^{n+1} \) be two locally strongly convex hypersurfaces of dimension \( n \) with respectively the affine metrics \( g, \bar{g} \) and the Fubini-Pick forms \( A, \bar{A} \), and \( \varphi : (M^n, g) \to (\bar{M}^n, \bar{g}) \) be an isometry between Riemannian manifolds. Then \( \varphi^* \bar{A} = A \) if and only if there exists a unimodular affine transformation \( \Phi : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1} \) such that \( \bar{x} \circ \varphi = \Phi \circ x \), or equivalently, \( \bar{x} = \Phi \circ x \circ \varphi^{-1} \).

Remark 2.1. The necessity part of Theorem 2.2 is proved in [15]. Here we give a proof for the sufficient part as follows:

Choose an orthonormal frame field \( \{e_i; 1 \leq i \leq n\} \) on \( M^n \) with its dual coframe \( \{\omega^i; 1 \leq i \leq n\} \). Let \( \xi, \bar{\xi} \) be respectively the affine normal of \( x \) and \( \bar{x} \). Then \( \{e_1, \ldots, e_n, \xi\} \) is unimodular. Define \( \bar{e}_i = \varphi_*(e_i) \), \( \bar{\omega}^i = (\varphi^{-1})^* \omega^i \), \( 1 \leq i \leq n \). Then \( \{\bar{\omega}^i; 1 \leq i \leq n\} \) is the dual coframe of \( \{\bar{e}_i; 1 \leq i \leq n\} \). Since \( \varphi \) is an isometry, \( \{\bar{e}_1, \ldots, \bar{e}_n, \bar{\xi}\} \) is also unimodular.

Under the condition that \( \bar{x} = \Phi \circ x \circ \varphi^{-1} \), we claim that \( \bar{\xi} = (\Phi_*(\xi)) \circ \varphi^{-1} \). In fact

\[
\bar{e}_j(\bar{e}_i \bar{x}) = \varphi_*(e_j)((\varphi_*(e_i))(\Phi \circ x \circ \varphi^{-1})) = \varphi_*((e_j(e_i)(\Phi \circ x))) \circ \varphi^{-1} = (e_j(e_i)(\Phi \circ x)) \circ \varphi^{-1}.
\]

Denote respectively by \( \nabla, \bar{\nabla}, \Delta \) and \( \bar{\nabla}, \bar{\bar{\nabla}}, \bar{\Delta} \) the affine connections of \( x, \bar{x} \), the Riemannian connections and the Laplacians of \( g, \bar{g} \). Then we find

\[
\bar{\xi} = \frac{1}{n} \bar{\Delta} \bar{x} = \frac{1}{n} \left( \sum_i (\bar{e}_i(\bar{e}_i \bar{x}) - (\bar{\nabla}_e e_i)(\bar{x})) \right)
\]

\[
= \frac{1}{n} \left( \sum_i (e_i(e_i(\Phi \circ x))) \circ \varphi^{-1} - \varphi_*(\bar{\nabla}_e e_i)(\bar{x})) \right)
\]

\[
= \frac{1}{n} \left( \sum_i (e_i(\Phi_*(e_i)) \circ \varphi^{-1} - \varphi_*(\bar{\nabla}_e e_i)(\Phi \circ x)) \right)
\]

\[
= \frac{1}{n} \Phi_*(\sum_i (e_i(e_i(x))) \circ \varphi^{-1} - (\Phi_*(\bar{\nabla}_e e_i)(x)) \circ \varphi^{-1})
\]

\[
= \frac{1}{n} \Phi_* \left( \sum_i (e_i(e_i(x))) \circ \varphi^{-1} \right) \circ \varphi^{-1} = \Phi_*(\Delta x) \circ \varphi^{-1} = \Phi_*(\xi) \circ \varphi^{-1}.
\]

On the other hand, by (2.10) and the affine Gauss formula (2.7) of \( x \)

\[
\bar{e}_j \bar{e}_i \bar{x} = (\varphi_*(e_j(\Phi \circ x)) \circ \varphi^{-1} = (e_j \Phi_*(e_i(x))) \circ \varphi^{-1} = (\Phi_*(e_j e_i(x))) \circ \varphi^{-1}
\]

\[
= (\Phi_*(x_*(\nabla e_i e_j) + \delta_j \xi)) \circ \varphi^{-1} = (\Phi_*(x_*(\nabla e_i e_j))) \circ \varphi^{-1} + \delta_j (\Phi_*(\xi)) \circ \varphi^{-1}.
\]

But, by the affine Gauss formula of \( \bar{x} \),

\[
\bar{e}_j \bar{e}_i \bar{x} = \bar{x}_*(\bar{\nabla e_i e_j}) + \delta_j \bar{\xi} = \Phi_*(x_*(\varphi_*(\nabla e_i e_j))) \circ \varphi^{-1} + \delta_j (\Phi_*(\xi)) \circ \varphi^{-1}.
\]

It follows that

\[
\Phi_*(x_*(\nabla e_i e_j)) = \Phi_*(x_*(\varphi_*(\bar{nabla e_j e_i}))).
\]

Therefore

\[
\varphi^{-1}(\bar{\nabla e_j e_i}) = \nabla e_j e_i, \text{ or equivalently, } \varphi_*(\nabla e_j e_i) = \bar{\nabla e_j e_i},
\]

from which we find that

\[
\bar{A}(\bar{e}_i, \bar{e}_j, \bar{e}_k) = g(\bar{\nabla}_e \bar{e}_i, \bar{e}_j, \bar{e}_k) = g(\bar{\nabla}_e \bar{e}_i - \bar{\hat{\nabla}}_e \bar{e}_i, \bar{e}_k)
\]

\[
= g(\varphi^{-1}(\nabla e_j e_i) - \varphi^{-1}(\bar{\nabla} e_j \bar{e}_i), e_k) = g(\nabla e_j e_i - \bar{\nabla} e_j \bar{e}_i, e_k) = A(e_i, e_j, e_k).
\]
Consequently
\[ A = \sum A(e_i, e_j, e_k)\bar{\omega}^i \bar{\omega}^j \bar{\omega}^k = \sum A(e_i, e_j, e_k)(\varphi^{-1})^\ast \bar{\omega}^i (\varphi^{-1})^\ast \bar{\omega}^j (\varphi^{-1})^\ast \bar{\omega}^k \]
\[ = (\varphi^{-1})^\ast \left( \sum A(e_i, e_j, e_k)\bar{\omega}^i \bar{\omega}^j \bar{\omega}^k \right) = (\varphi^{-1})^\ast A, \quad (2.19) \]
or equivalently, \( \varphi^\ast A = A \). We are done.

Given \( c \in \mathbb{R} \) and a Riemannian manifold \( (M^d, g) \), denote by \( \mathfrak{S}_{(M^d, g)}(c) \) the set of all \( TM^d \)-valued symmetric bilinear forms \( A \in \Gamma(\bigotimes^2(T^*M^d) \otimes(TM^d)) \), satisfying the following conditions:

(1) Under the metric \( g \), the corresponding 3-form \( A \in \Gamma(\bigotimes^2(T^*M^d) \otimes(T^*M^d)) \) is totally symmetric, that is, \( A \in \Gamma(\bigotimes^3(T^*M^d)) \);

(2) Affine Gauss equation, that is, for any \( X, Y, Z \in \mathfrak{X}(M^d) \)
\[ R(X, Y)Z = c(g(Y, Z)X - g(X, Z)Y) - [A(X), A(Y)](Z). \quad (2.20) \]

Moreover, by adding to \( \mathfrak{S}_{(M^d, g)}(c) \) the following so called apolarity condition

(3) \( \text{tr}_g(A) = 0 \),

we define
\[ \mathfrak{S}_{(M^d, g)}(c) = \{ A \in \mathfrak{S}(M^d, g)(c), \text{tr}_g(A) \equiv 0 \}. \]

From Theorem 2.1 and Theorem 2.2, we have

**Corollary 2.1.** For each \( A \in \mathfrak{S}_{(M^d, g)}(c) \), there uniquely exists one affine hypersphere \( x : M^d \to \mathbb{R}^{d+1} \) with affine metric \( g \), Fubini-Pick form \( A \) and affine mean curvature \( c \).

Motivated by Theorem 2.2, we introduce the following modified equiaffine equivalence relation between nondegenerate hypersurfaces:

**Definition 2.1.** Let \( x : M^n \to \mathbb{R}^{n+1} \) be a nondegenerate hypersurface with the affine metric \( g \). A hypersurface \( \tilde{x} : M^n \to \mathbb{R}^{n+1} \) is called affine equivalent to \( x \) if there exists a unimodular transformation \( \Phi : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1} \) and an isometry \( \varphi \) of \( (M^n, g) \) such that \( \tilde{x} = \Phi \circ x \circ \varphi^{-1} \).

To end this section, we would like to recall the following concept:

**Definition 2.2.** (19) A nondegenerate hypersurface \( x : M^n \to \mathbb{R}^{n+1} \) is called affine symmetric (resp. locally affine symmetric) if

(1) the pseudo-Riemannian manifold \( (M^n, g) \) is symmetric (resp. locally symmetric) and therefore \( (M^n, g) \) can be written (resp. locally written) as \( G/K \) for some connected Lie group \( G \) of isometries with \( K \) one of its closed subgroups;

(2) the Fubini-Pick form \( A \) is invariant under the action of \( G \).

2.2. The multiple Calabi product of hyperbolic affine hyperspheres. For later use we make a brief review of the Calabi composition of multiple factors of hyperbolic affine hypersurfaces. Detailed proofs of the formulas in this subsection has been given in the preprint [15].

Now let \( r, s \) be two nonnegative integers with \( K := r + s \geq 2 \) and \( x_\alpha : M_\alpha^n \to \mathbb{R}^{n_\alpha+1}, 1 \leq \alpha \leq s, \) be hyperbolic affine hyperspheres of dimension \( n_\alpha > 0 \) with affine mean curvatures \( L_1 \) and with the origin their common affine center. For convenience we make the following convention:

\[ 1 \leq a, b, c \cdots \leq K, \quad 1 \leq \lambda, \mu, \nu \leq K - 1, \quad 1 \leq \alpha, \beta, \gamma \leq s, \quad \tilde{\alpha} = \alpha + r, \quad \tilde{\beta} = \beta + r, \quad \tilde{\gamma} = \gamma + r. \]

Furthermore, for each \( \alpha = 1, \cdots, s, \) set \( \tilde{i}_\alpha = i_\alpha + K - 1 + \sum_{\beta < \alpha} n_{\beta} \) with \( 1 \leq i_\alpha \leq n_\alpha. \)

Define
\[ f_{\alpha} := \begin{cases} a, & 1 \leq a \leq r; \\ \sum_{\beta \leq \alpha} n_{\beta} + \tilde{\alpha}, & r + 1 \leq a = \tilde{\alpha} \leq r + s, \end{cases} \]
and
\[ e_a := \exp \left( -\frac{t_{a-1}}{n_a+1} + \frac{t_a}{f_a} + \frac{t_{a+1}}{f_{a+1}} + \cdots + \frac{t_{K-1}}{f_{K-1}} \right), \quad 1 \leq a \leq K = r + s \]

In particular,
\[ e_1 = \exp \left( \frac{t_1}{f_1} + \frac{t_2}{f_2} + \cdots + \frac{t_{K-1}}{f_{K-1}} \right), \quad e_K = \exp \left( -\frac{t_{K-1}}{n_K+1} \right). \]

Put \( n = \sum_a n_a + K - 1 \) and \( M^n = R^{K-1} \times M^n_1 \times \cdots \times M^n_K \). For any \( K \) positive numbers \( c_1, \ldots, c_K \), define a smooth map \( x : M^n \to \mathbb{R}^{n+1} \) by
\[ x(t^1, \ldots, t^{K-1}, p_1, \ldots, p_s) := (c_1 e_1, \ldots, c_r e_r, c_{r+1} e_{r+1} x_1(p_1), \ldots, c_K e_K x_s(p_s)), \]
\[ \forall (t^1, \ldots, t^{K-1}, p_1, \ldots, p_s) \in M^n. \quad (2.21) \]

**Proposition 2.1.** ([LS]) The map \( x : M^n \to \mathbb{R}^{n+1} \) defined above is a new hyperbolic affine hypersphere with the affine mean curvature
\[ L_1 = -\frac{1}{(n+1)C}, \quad C := \left( \frac{1}{n+1} \prod_{a=1}^r c_a^2 \prod_{\alpha=1}^s \frac{c_{r+\alpha}^{2(n_{\alpha}+1)}}{(n_{\alpha}+1)(n_{\alpha}+1)(-L_1)n_{\alpha}+2} \right)^{\frac{1}{n+7}}. \quad (2.22) \]

Moreover, for given positive numbers \( c_1, \ldots, c_K \), there exists some \( c > 0 \) and \( c' > 0 \) such that the following three hyperbolic affine hyperspheres
\[ x := (c_1 e_1, \ldots, c_r e_r, c_{r+1} e_{r+1} x_1, \ldots, c_K e_s x_s), \]
\[ \tilde{x} := c(e_1, \ldots, e_r, e_{r+1} x_1, \ldots, e_s x_s), \]
\[ \tilde{x} := (e_1, \ldots, e_r, e_{r+1} x_1, \ldots, c' e_s x_s) \]
are equiaffine equivalent to each other.

**Definition 2.3.** ([LS]) The hyperbolic affine hypersphere \( x \) is called the Calabi composition of \( r \) points and \( s \) hyperbolic affine hyperspheres.

Denote by \( \{ v_i^\alpha : i = 1, \ldots, n_{\alpha} \} \) the local coordinate system of \( M_\alpha, \alpha = 1, \ldots, s \). Then we have

**Proposition 2.2.** ([LS]) The affine metric \( g \), the affine mean curvature \( L_1 \) and the possibly nonzero components of the Fubini-Pick form \( A \) of the Calabi composition \( x : M^n \to \mathbb{R}^{n+1} \) of \( r \) points and \( s \) hyperbolic affine hyperspheres \( x_\alpha : M_\alpha \to \mathbb{R}^{n_{\alpha}+1}, \alpha = 1, \ldots, s \), are given as follows:

\[
g_{\lambda\mu} = \begin{cases} 
\frac{\lambda + 1}{n_1 + 1} C \delta_{\lambda\mu}, & 1 \leq \lambda \leq r - 1; \\
\frac{\lambda}{r(n_1 + 1)} C \delta_{r\mu}, & \lambda = r;
\end{cases}
\]
\[
g_{\lambda\mu} = \begin{cases} 
\sum_{\beta \leq a+1} n_{\beta} + \tilde{\alpha} + 1 C \delta_{\lambda\mu}, & r + 1 \leq \lambda = \tilde{\alpha} \leq r + s - 1.
\end{cases}
\]
\[
g_{i_{\alpha} j_{\beta}} = (n_{\alpha} + 1)(-L_1)^{\alpha} C g_{i_{\alpha} j_{\beta}} \delta_{\alpha\beta}, \quad g_{i_{\alpha} i_{\alpha}} = 0. \quad (2.24)
\]
\[
A_{\lambda\lambda\lambda} = \begin{cases} 
1 - \frac{\lambda^2}{n_1 + 1} C, & 1 \leq \lambda \leq r - 1, \\
\frac{1}{r^2} C, & \lambda = r;
\end{cases}
\]
\[
A_{\lambda\lambda\lambda} = \left( \frac{\sum_{\beta \leq a+1} n_{\beta} + \tilde{\alpha} + 1}{n_{\alpha} + 1}(\sum_{\beta \leq a} n_{\beta} + \tilde{\alpha}) \right) \left( \frac{1}{n_{\alpha} + 1} C, & r + 1 \leq \lambda = \tilde{\alpha} \leq r + s - 1. \quad (2.25)
\right.
\]
Then it is not hard to see that
\[ x \]
example
Then by Corollary 2.2 the affine metric
\[ \text{affine hypersphere of dimension } n \]
spheres
Thus the Pick invariant of
\[ \alpha = 1, \ldots, s. \]
From Proposition 2.2 the following corollary is easily derived (cf. [15]):

**Corollary 2.2.** The Calabi composition \( x : M^n \to \mathbb{R}^{n+1} \) of \( r \) points and \( s \) hyperbolic affine hyperspheres \( x_\alpha : M^{n_\alpha} \to \mathbb{R}^{n_\alpha+1}, 1 \leq \alpha \leq s, \) is affine symmetric if and only if each positive dimensional factor \( x_\alpha \) is symmetric.

The next example will be used later:

**Example 2.1.** Given a positive number \( C_0 \), let \( x_0 : \mathbb{R}^{n_0} \to \mathbb{R}^{n_0+1} \) be the well known flat hyperbolic affine hypersphere of dimension \( n_0 \) which is defined by
\[ x^1 \cdots x^{n_0} x^{n_0+1} = C_0, \quad x^1 > 0, \ldots, x^{n_0+1} > 0. \]
Then it is not hard to see that \( x_0 \) is the Calabi composition of \( n_0 + 1 \) points. In fact, we can write for example
\[ x_0 = (e_1, \ldots, e_{n_0}, C_0 e_{n_0+1}). \]
Then by Corollary 2.2 the affine metric \( g_0 \), the affine mean curvature \( L_1 \) and the Fubini-Pick form \( A \) of \( x_0 \) are respectively given by (cf. [15])
\[
\begin{align*}
\lambda + 1 \alpha
\end{align*}
\[ \frac{1}{\lambda} \]
\[ \frac{1}{L_1} = -\frac{1}{(n_0 + 1)C_0} = -(n_0 + 1)^{-1} \frac{n_0+1}{n_0+2} C_0^{-1} \frac{2}{n_0+2}, \]
\[ A_{\lambda \mu \nu} = \begin{cases} 
-\frac{\lambda^2 - 1}{\lambda \nu} \left( \frac{C_0^2}{n_0+1} \right)^{\frac{1}{n_0+2}}, & \text{if } \lambda = \mu = \nu; \\
\frac{\lambda+1}{\lambda \nu} \left( \frac{C_0^2}{n_0+1} \right)^{\frac{1}{n_0+2}}, & \text{if } \lambda = \mu < \nu; \\
0, & \text{otherwise.}
\end{cases} \]
Thus the Pick invariant of \( x_0 \) is
\[
(0) \quad J = \frac{1}{n_0(n_0-1)} g_{\lambda_1 \lambda_2} (0) g_{\mu_1 \mu_2} (0) g_{\nu_1 \nu_2} (0) A_{\lambda_1 \mu_1 \nu_1} (0) A_{\lambda_2 \mu_2 \nu_2} = (n_0 + 1)^{-1} \frac{n_0+1}{n_0+2} C_0^{-1} \frac{2}{n_0+2} = -L_1.
\]
By restrictions, \( g \) defines a flat metric \( g_0 \) on \( \mathbb{R}^{K-1} \) with matrix \( (g_{\lambda \mu}) \) and, for each \( \alpha \), a metric \( g_0 \) on \( M_\alpha \) with matrix \( (g_{\alpha \alpha}) = (g_{\alpha \alpha}) \) and inverse matrix \( (g_{\alpha \alpha}^{-1}) \), which is conformal to the original metric
important classifications. It turns out that this special characterization theorem is needed in some related hypersphere locally to be the Calabi composition of several hyperbolic affine hyperspheres, possibly equivalent to that the holonomy algebra $\mathfrak{h}$ be identified with a $TM^\alpha$-valued symmetric 2-form $A : TM^\alpha \times TM^\alpha \rightarrow TM^\alpha$. For each ordered triple $\alpha, \beta, \gamma \in \{0, 1, \cdots, s\}$. A defines one $TM^\gamma$-valued bilinear map $A^\alpha_{\alpha\beta} : TM^\alpha \times TM^\beta \rightarrow TM^\gamma$, which is the $TM^\gamma$-component of $A_{\alpha\beta}$, the restriction of $A$ to $TM^\alpha \times TM^\beta$. For $\alpha = 1, \cdots, s$, define

$$H_\alpha = \frac{1}{n_{\alpha}} \text{tr}_{g_\alpha} A^0_{\alpha\alpha} = \frac{1}{n_{\alpha}} g^{\alpha\beta\gamma} A^0_{\alpha\alpha} \left( \frac{\partial}{\partial v^{\alpha}_{\beta}}, \frac{\partial}{\partial v^{\alpha}_{\gamma}} \right),$$

where the metric $g_\alpha$ is given by (2.34). Then we have (18).

**Proposition 2.3.** Let $x : M^n \rightarrow \mathbb{R}^{n+1}$ be the Calabi composition of $r$ points and $s$ hyperbolic affine hyperspheres and $g$ the affine metric of $x$. Then

1. The Riemannian manifold $M^n \equiv (M^n, g)$ is reducible, that is

$$(M^n, g) = \mathbb{R}^q \times (M_1, g_1) \times \cdots \times (M_s, g_s), \quad q + s \geq 2;$$

2. There must be a positive dimensional Euclidean factor $\mathbb{R}^q$ in the de Rham decomposition (2.36) of $M^n$, that is, $q > 0$;

3. $q \geq s - 1$ with the equality holding if and only if $r = 0$;

4. $A^\alpha_{\alpha\beta} \equiv 0$ if $(\alpha, \beta, \gamma)$ is not one of the following triples: $(0, 0, 0)$, $(\alpha, 0, 0)$, $(0, 0, \alpha)$, $(\alpha, \alpha, 0)$ or $(\alpha, \alpha, \alpha)$.

5. For any $p = (p_0, p_1, \cdots, p_s) \in M^n$ and each $\alpha = 1, \cdots, s$, it holds that

$$A^0_{\alpha\alpha}(R^M_\alpha(X_\alpha, Y_\alpha)Z_\alpha, W_\alpha) + A^0_{\alpha\alpha}(Z_\alpha, R^M_\alpha(X_\alpha, Y_\alpha)W_\alpha) = 0,$$

$$\forall X_\alpha, Y_\alpha, Z_\alpha, W_\alpha \in T_{p_\alpha}M_\alpha$$

(2.37) equivalent to that the holonomy algebra $\mathfrak{h}_\alpha$ of $(M_\alpha, g_\alpha)$ acts on $A^0_{\alpha\alpha}$ trivially, that is $\mathfrak{h}_\alpha \cdot A^0_{\alpha\alpha} = 0$.

6. The vector-valued functions $H_\alpha, \alpha = 1, \cdots, s$, defined by (2.35) satisfy the following equalities:

$$H_\alpha = -\frac{f_\alpha-1}{f_\alpha C} \frac{\partial}{\partial v^{\alpha}-1} + \frac{1}{C} \sum_{s-1 \leq \beta \geq \alpha} \frac{n_{\beta+1} + 1}{f_{\beta+1}} \frac{\partial}{\partial v^{\beta}},$$

$$g(H_\alpha, H_\alpha) = C^{-1} \left( \frac{1}{n_{\alpha} + 1} - \frac{1}{f_K} \right) = \frac{n - n_{\alpha}}{n_{\alpha} + 1} (-L_1),$$

(2.39)

$$g(H_\alpha, H_\beta) = L_1 \quad \text{for} \ \alpha \neq \beta;$$

(2.40)

7. $A^0_{\alpha\alpha}$ is identical to the $TM^\alpha$-valued symmetric bilinear form defined by the Fubini-Pick form $A$ of $x_\alpha$.

In the next section we shall prove as the main result that a locally strongly convex affine hypersurface $x : M^n \rightarrow \mathbb{R}^{n+1}$ is locally the Calabi composition of some points and hyperbolic affine hyperspheres if and only if the above conditions (1), (4) and (5) hold.

3. A characterization of the Calabi composition

We are mainly to establish a necessary and sufficient condition for a locally strongly convex hyperbolic hypersphere locally to be the Calabi composition of several hyperbolic affine hyperspheres, possibly including point factors. It turns out that this special characterization theorem is needed in some related important classifications.
Theorem 3.1 (The Main Theorem). A locally strongly convex hyperbolic hypersphere \( x : M^n \to \mathbb{R}^{n+1} \), with the affine metric \( g \) and the Fubini-Pick form \( A \in \Gamma(T \ast M^n \otimes T^* M^n \otimes T^* M^n) \), is locally affine equivalent to the Calabi composition of some hyperbolic affine hyperspheres, possibly including point factors, if and only if the following three conditions hold:

1. The Riemannian manifold \((M^n, g)\) is reducible, that is
\[
(M^n, g) = \mathbb{R}^q \times (M_1, g_1) \times \cdots \times (M_s, g_s), \quad q + s \geq 2,
\]
where \((M_\alpha, g_\alpha)\), \(\alpha = 1, \ldots, s\), are irreducible Riemannian manifolds;

2. Denote \(M_0 = \mathbb{R}^q\) and by \(A^\alpha_{\beta\gamma}\), \(0 \leq \alpha, \beta, \gamma \leq s\), the \(T^* M_\gamma\)-component of the restriction of \(A\) to \(T M_\alpha \times T M_\beta\). Then \(A^\alpha_{\beta\gamma} = 0\) if the oriented triple \((\alpha, \beta, \gamma)\) is not one of the following:
\[
(0, 0, 0), \quad (\alpha, 0, \alpha), \quad (0, \alpha, \alpha), \quad (\alpha, \alpha, \alpha).
\]

3. For each \(\alpha = 1, \ldots, s\), the holonomy algebra \(\mathfrak{h}_\alpha\) of \((M_\alpha, g_\alpha)\) acts on \(A^0_{\alpha\alpha}\) trivially: \(\mathfrak{h}_\alpha \cdot A^0_{\alpha\alpha} = 0\), or equivalently, for any \(p = (p_0, p_1, \ldots, p_s) \in M^n\) and for all \(X_\alpha, Y_\alpha, Z_\alpha, W_\alpha \in T_{p_\alpha} M_\alpha\), it holds that
\[
A^0_{\alpha\alpha}(R^{M_\alpha}(X_\alpha, Y_\alpha)Z_\alpha, W_\alpha) + A^0_{\alpha\alpha}(Z_\alpha, R^{M_\alpha}(X_\alpha, Y_\alpha)W_\alpha) = 0.
\]

Proof. Since our consideration is local here, we suppose that \(M^n\) is connected and simply connected. The necessary part of the theorem is clearly from Proposition 2.3. To prove the sufficient part, we first note that, the affine mean curvature \(L_1\) of a hyperbolic affine hypersphere is a negative constant.

Claim: \(M^n\) must have an Euclidean factor \(\mathbb{R}^q\), \(q > 0\), in its de Rham decomposition (3.1).

In fact it suffices to show that \(M^n\) would be irreducible if \(q = 0\). To do this we suppose that \(s > 1\). Then for non-vanishing \(X \in TM_1\) and \(Y \in TM_2\), we have by the affine Gauss equation (2.6) that
\[
0 = R(X, Y)Y = L_1(g(Y, Y)X - g(X, Y)Y) - [A(X), A(Y)](Y) = L_1g(Y, Y)X,
\]
where we have used the fact that \(A(Y)(Y) = A(Y, Y) \in TM_2\) and \(A(X)(Y) = A(X, Y) = 0\) due to the condition (2) in the theorem. Clearly (3.3) is not possible since \(L_1 < 0\) and \(Y \neq 0\). This contradiction proves the claim.

By the assumption, \(A\) can be decomposed as
\[
A = \sum_{\alpha = 0} A^\alpha_{\alpha\alpha} + \sum_{\alpha = 1} A^0_{\alpha\alpha} + \sum_{\alpha = 1} A^\alpha_{0\alpha} + \sum_{\alpha = 1} A^\alpha_{\alpha 0}.
\]

Same as in the last section, for \(\alpha = 1, \ldots, s\), we define \(H_\alpha = \frac{1}{12} \text{tr}_{g_\alpha}(A^0_{\alpha\alpha})\) and denote \(\bar{c}_\alpha = |H_\alpha|\).

Then we have

Lemma 3.1. \(A^0_{00} \in \mathfrak{S}_{\mathbb{R}^q}(L_1)\) and the following identities hold:
\[
A^0_{\alpha\alpha}(X_\alpha, Y_\alpha) = g(X_\alpha, Y_\alpha)H_\alpha, \quad g_0(H_\alpha, H_\beta) = L_1,
\]
if \(1 \leq \alpha \neq \beta \leq s\),
\[
A^0_{\alpha0}(X_\alpha, Z_0) = A^\alpha_{0\alpha}(Z_0, X_\alpha) = g_0(Z_0, H_\alpha)X_\alpha.
\]
\[
A^0_{00}(Z_0, H_\alpha) = g_0(Z_0, H_\alpha)H_\alpha + L_1Z_0,
\]
where \(Z_0 \in T_{p_0} M_0, X_\alpha, Y_\alpha \in T_{p_\alpha} M_\alpha,\) and \(g_0\) is the flat metric on \(\mathbb{R}^q\).

Proof of Lemma 3.1. First note that the first conclusion is direct from the fact that \(A \in \mathfrak{S}_{(M^n, g)}(L_1)\) together with the decomposition (3.1).

Let \(p = (p_0, p_1, \ldots, p_s) \in M^n\) be an arbitrary point with \(p_\alpha \in M_\alpha, \alpha = 0, 1, \ldots, s\). Let \(\mathfrak{h}\) be the holonomy algebra of \((M^n, g)\), and \(\mathfrak{h}_1, \ldots, \mathfrak{h}_s\) the holonomy algebras of \((M_1, g_1), \ldots, (M_s, g_s)\), respectively. Then by (3.1) we have
\[
\mathfrak{h} = \mathfrak{h}_1 + \cdots + \mathfrak{h}_s,
\]
Since Riemannian manifold \((M_\alpha, g_\alpha)\) \((1 \leq \alpha \leq s)\) is irreducible, \(h_\alpha\) acts irreducible on \(T_{p_\alpha} M_\alpha\) for every point \(p_\alpha \in M_\alpha\) and, at the same time, acts trivially on any other \(T_{p_\beta} M_\beta\), \(\beta \neq \alpha\).

For each \(\alpha = 1, \ldots, s\) and for all \(T \in h_\alpha\), \(X_\alpha, Y_\alpha \in T_{p_\alpha} M_\alpha\), (3.12) gives that
\[
A^0_{\alpha \alpha}(TX_\alpha, Y_\alpha) + A^0_{\alpha \alpha}(X_\alpha, TY_\alpha) = 0.
\]
(3.9)

By using the irreducibility of \(h_\alpha\) on \(T_{p_\alpha} M_\alpha\), we get from (3.9)
\[
\langle A^0_{\alpha \alpha}(X_\alpha, Y_\alpha), e_\alpha \rangle = c^\alpha_\alpha g_\alpha(X_\alpha, Y_\alpha)
\]
for some constants \(c^\alpha_\alpha \in \mathbb{R}\), where \(\{e_\alpha\}\) is an orthonormal basis for \(T_{p_\alpha} \mathbb{R}^q \equiv \mathbb{R}^q\). Therefore
\[
A^0_{\alpha \alpha}(X_\alpha, Y_\alpha) = g_\alpha(X_\alpha, Y_\alpha) \sum c^\alpha_\alpha e_\alpha.
\]

On the other hand, it is seen that \(H_\alpha = \sum c^\alpha_\alpha e_\alpha\), proving the first equality in (3.5). This with the symmetry of \(A\) gives (3.6).

Now put \(Z_0 \in T_{p_\alpha} M_\alpha\), \(X_\alpha, Y_\alpha \in T_{p_\alpha} M_\alpha\). Then the affine Gauss equation (2.6) of \(x\) tells that
\[
0 = L_1 g_\alpha(X_\alpha, Y_\alpha)Z_0 - \langle A(Z_0), A(X_\alpha) \rangle(Y_\alpha).
\]
(3.11)

But by the first equality of (3.5) and (3.6) we find
\[
\langle A(Z_0), A(X_\alpha) \rangle(Y_\alpha) = A(Z_0, A(X_\alpha, Y_\alpha)) - A(X_\alpha, A(Z_0, Y_\alpha))
\]
\[
= A(Z_0, A^0_{\alpha \alpha}(X_\alpha, Y_\alpha) + A^\alpha_{\alpha \alpha}(X_\alpha, Y_\alpha)) - A(X_\alpha, A^0_{\alpha \alpha}(Z_0, Y_\alpha))
\]
\[
= A(Z_0, g_\alpha(X_\alpha, Y_\alpha)H_\alpha) - A(X_\alpha, g_\alpha(Z_0, H_\alpha)Y_\alpha))
\]
\[
= A_\alpha(X_\alpha, Y_\alpha)A^0_{\alpha 00}(Z_0, H_\alpha) + A^\alpha_{\alpha 00}(Z_0, A^\alpha_{\alpha \alpha}(X_\alpha, Y_\alpha))
\]
\[
- g(Z_0, H_\alpha)(A^0_{\alpha \alpha}(X_\alpha, Y_\alpha) + A^\alpha_{\alpha \alpha}(X_\alpha, Y_\alpha))
\]
\[
= A_\alpha(X_\alpha, Y_\alpha)A^0_{\alpha 00}(Z_0, H_\alpha) + g(Z_0, H_\alpha)A^\alpha_{\alpha \alpha}(X_\alpha, Y_\alpha)
\]
\[
- g(Z_0, H_\alpha)g_\alpha(X_\alpha, Y_\alpha)H_\alpha - g(Z_0, H_\alpha)A^\alpha_{\alpha \alpha}(X_\alpha, Y_\alpha)
\]
\[
= g_\alpha(X_\alpha, Y_\alpha)(A^0_{\alpha 00}(Z_0, H_\alpha) - g(Z_0, H_\alpha)H_\alpha).
\]
Putting this equality into (3.11) we obtain (3.7).

Similarly, for \(1 \leq \alpha \neq \beta \leq s\), let \(X_\alpha \in T_{p_\alpha} M_\alpha\) and \(Y_\beta, Z_\beta \in T_{p_\beta} M_\beta\). Then
\[
0 = L_1 g_\beta(Y_\beta, Z_\beta)X_\alpha - \langle A(X_\alpha), A(Y_\beta) \rangle(Z_\beta).
\]
(3.12)

But
\[
\langle A(X_\alpha), A(Y_\beta) \rangle(Z_\beta) = A(X_\alpha, A(Y_\beta, Z_\beta)) - A(Y_\beta, A(X_\alpha, Z_\beta))
\]
\[
= A(X_\alpha, A^0_{\beta \beta}(Y_\beta, Z_\beta) + A^\beta_{\beta \beta}(Y_\beta, Z_\beta))
\]
\[
= A^\alpha_{\alpha 00}(X_\alpha, g_\beta(Y_\beta, Z_\beta)H_\beta) - g_\beta(Y_\beta, Z_\beta)g(H_\alpha, H_\beta)X_\alpha.
\]
Comparing this with (3.12) gives the second formula in (3.5).

For the given point \(p = (p_0, p_1, \ldots, p_s) \in M^n\), denote by \(\mathcal{H}_p\) the subspace of \(T_{p_\alpha} M_0 \equiv \mathbb{R}^q\) generated by \(H_1(p_1), \ldots, H_s(p_s)\) and by \(\mathcal{H}^\perp_p\) the orthogonal complement of \(\mathcal{H}_p\) in \(T_{p_\alpha} M_0\). Thus \(T_{p_\alpha} M_0 = \mathcal{H}_p \oplus \mathcal{H}^\perp_p\).

Then \(A^0_{\alpha 00}(p)\) can be decomposed into the sum of its \(\mathcal{H}_p\)-component \(A^\mathcal{H}_p\) and its \(\mathcal{H}^\perp_p\)-component \(A^\mathcal{H}^\perp_p\), that is, \(A^0_{\alpha 00}(p) = A^\mathcal{H}_p + A^\mathcal{H}^\perp_p\).

**Lemma 3.2.** Let \(q, s\) be as above. Then \(q \geq \dim \mathcal{H}_p \geq s - 1\). Furthermore, \(q \geq s\) if and only if \(\dim \mathcal{H}_p = s\).
Proof of Lemma 3.2. For simplicity we omit the point \( p \) in the symbols. To prove the first part of the lemma, it suffices to show that the set of the \( s \) nonzero vectors \( H_1, \ldots, H_s \) in \( TM_0 \equiv \mathbb{R}^q \) has a rank not less than \( s - 1 \). This is equivalent to show that the \( s \)-th order matrix

\[
\begin{pmatrix}
g(H_1, H_1) & g(H_2, H_1) & \cdots & g(H_s, H_1) \\
g(H_1, H_2) & g(H_2, H_2) & \cdots & g(H_s, H_2) \\
& \ddots & \ddots & \ddots \\
g(H_1, H_s) & g(H_2, H_s) & \cdots & g(H_s, H_s)
\end{pmatrix}
= \begin{pmatrix}
e_1^2 & L_1 & \cdots & L_1 \\
L_1 & e_2^2 & \cdots & L_1 \\
\vdots & \vdots & \ddots & \vdots \\
L_1 & L_1 & \cdots & e_s^2
\end{pmatrix}
\]

has a rank equal to or larger than \( s - 1 \).

Indeed, by deleting the last line and the second last column, we find a \((s - 1)\)-minor of the above matrix:

\[
\begin{vmatrix}
e_1^2 & L_1 & \cdots & L_1 \\
L_1 & e_2^2 & \cdots & L_1 \\
\vdots & \vdots & \ddots & \vdots \\
L_1 & L_1 & \cdots & e_s^2
\end{vmatrix} = \begin{vmatrix}
e_1^2 & L_1 & \cdots & L_1 \\
L_1 & e_2^2 & \cdots & L_1 \\
\vdots & \vdots & \ddots & \vdots \\
L_1 & L_1 & \cdots & e_s^2
\end{vmatrix} = \begin{vmatrix}
1 & 0 & \cdots & 0 \\
0 & e_2^2 - L_1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
1 & 1 & \cdots & 1
\end{vmatrix}
\]

\[= L_1((e_1^2 - L_1)(e_2^2 - L_1)) < 0. \tag{3.13}\]

Furthermore, if \( q \geq s \), and \( \dim \mathcal{H} = s - 1 \), then \( r - 1 := \dim \mathcal{H} = 1 \). Consider the restriction \( \tilde{A}_0^\mathcal{H} \) of \( A_0^\mathcal{H} \) to the subspace \( \mathcal{H} \times \mathcal{H} \). Define \( H_0 = \text{tr} \tilde{A}_0^\mathcal{H} \) and \( e_0 = |H_0| \). Then, for any unit vector \( e_0 \in \mathcal{H} \) and each \( \alpha = 1, \ldots, s \), we have

\[
g(\tilde{A}_0^\mathcal{H}(e_0, e_0), H_\alpha) = g(A(e_0, e_0), H_\alpha) = g(A_0^\mathcal{H}(e_0, H_\alpha), e_0)
\]

implying that

\[
g(H_0, H_\alpha) = L_1, \quad \alpha = 1, \ldots, s.
\]

Then in the same way as in proving that the rank of the matrix \((g(H_\alpha, H_\beta))_{1 \leq \alpha, \beta \leq s}\) is no less than \( s - 1 \), we can obtain that the rank of the \((s + 1)\)-th order matrix \((g(H_\alpha, H_\beta))_{0 \leq \alpha, \beta \leq s}\) is no less than \( s \). Since \( \{H_\alpha; 0 \leq \alpha \leq s\} \subset \mathcal{H} \), it follows that \( \dim \mathcal{H} \geq s \) which contradicts the assumption. \( \square \)

Since \( q \) is fixed, we have

Corollary 3.1. \( \dim \mathcal{H}_p \) and \( \dim \mathcal{H}_p \) are independent of the point \( p \). Thus they form two subbundles \( \mathcal{H} \) and \( \mathcal{H} \) of the tangent bundle \( TM_0 \).

Define \( \tilde{A}_0^{\mathcal{H} \times \mathcal{H}} = A_0^{\mathcal{H} \times \mathcal{H}} |_{\mathcal{H} \times \mathcal{H}} \). By (3.7), for any \( X_\mathcal{H}, Y_\mathcal{H} \in \mathcal{H} \), \( A_0^\mathcal{H}(X_\mathcal{H}, Y_\mathcal{H}) \in \mathcal{H} \), and for any \( Y_\mathcal{H} \in \mathcal{H} \), \( A_0^\mathcal{H}(X_\mathcal{H}, Y_\mathcal{H}^\perp) \in \mathcal{H} \). Therefore, \( A_0^\mathcal{H} \) can be decomposed into the following components:

\[
A_0^\mathcal{H} = A_0^{\mathcal{H} \times \mathcal{H}} + A_0^{\mathcal{H} \times \mathcal{H} \perp} + A_0^{\mathcal{H} \perp \times \mathcal{H}} + A_0^{\mathcal{H} \perp \times \mathcal{H} \perp} + A_0^\mathcal{H} \big|_{\mathcal{H} \times \mathcal{H}}. \tag{3.14}
\]

Lemma 3.3. Define \( r = \text{rank} \mathcal{H} = 1 \). Then \( \tilde{A}_0^{\mathcal{H} \perp} \in S_{r-1} \left( \frac{(n+1)L_1}{r} \right) \) if \( \text{rank} \mathcal{H} \geq 1 \).

Proof of Lemma 3.3. Since \( \text{rank} \mathcal{H} \geq 1 \), it holds by Lemma 3.2 that

\[
q = \text{rank} \mathcal{H} + \text{rank} \mathcal{H} \geq s - 1 + 1 = s.
\]

Making use of Lemma 3.2 once again we have that rank \( \mathcal{H} = s \) and therefore \( \{H_1, \ldots, H_s\} \) is a frame for the vector bundle \( \mathcal{H} \). Set \( h_{\alpha \beta} = g(H_\alpha, H_\beta), 1 \leq \alpha, \beta \leq s \), and \( (h^{\alpha \beta}) = (h_{\alpha \beta})^{-1} \).
We first compute $\bar{A}_0^H$. Write

$$\bar{A}_0^H(X, Y) = \sum C_{XY}^0 H_\alpha, \quad \forall X, Y \in \mathcal{H}^\perp.$$  

Then we have

$$g(\bar{A}_0^H(X, Y), H_\alpha) = \sum C_{XY}^0 g(H_\beta, H_\alpha) = \sum C_{XY}^0 h_{\beta \alpha};$$

$$g(\bar{A}_0^H(X, Y), H_\alpha) = g(A_0^0(X, Y), H_\alpha) = g(A_0^0(X, H_\beta), Y)$$

$$= g(g(X, H_\alpha)H_\alpha + L_1 X, Y) = L_1 g(X, Y),$$

implying that $\sum C_{XY}^0 h_{\beta \alpha} = L_1 g(X, Y)$ or equivalently

$$C_{XY}^0 = L_1 g(X, Y) \sum h_{\alpha \beta}.$$  

It follows that

$$\bar{A}_0^H(X, Y) = L_1 g(X, Y) \sum_{\alpha, \beta} h_{\alpha \beta} H_\alpha, \quad \forall X, Y \in \mathcal{H}^\perp.$$  

(3.15)

Thus we have

$$H_0 = \frac{1}{r - 1} \text{tr} (\bar{A}_0^H) = L_1 \sum_{\alpha, \beta} h_{\alpha \beta} H_\alpha.$$  

(3.16)

On the other hand, by using (3.4) and the fact that $\text{tr} (A) = 0$, it is seen that

$$\text{tr} A_0^0 + \sum_{\alpha} \text{tr} A_\alpha^0 = 0,$$

which with the decomposition (3.14) gives

$$\text{tr} (\bar{A}_0^H)^\perp = 0;$$

(3.17)

$$(r - 1)H_0 + \sum_{\alpha} h_{\alpha \beta} A_0^0 (H_\alpha, H_\beta) + \sum_{\alpha} n_\alpha H_\alpha = 0.$$  

(3.18)

But by (3.7),

$$A_0^H(H_\alpha, H_\beta) = g(H_\alpha, H_\beta)H_\beta + L_1 H_\alpha = h_{\alpha \beta} H_\beta + L_1 H_\alpha.$$  

Thus (3.18) can be rewritten as

$$(r - 1)H_0 + \sum_{\alpha} (1 + L_1 \sum_{\beta} h_{\alpha \beta}) H_\alpha + \sum_{\alpha} n_\alpha H_\alpha = 0.$$  

(3.19)

Comparing (3.16) and (3.19) gives that

$$\sum_{\alpha} \left( rL_1 \sum_{\beta} h_{\alpha \beta} + (n_\alpha + 1) \right) H_\alpha = 0,$$

or equivalently

$$rL_1 \sum_{\beta} h_{\alpha \beta} + (n_\alpha + 1) = 0, \quad \alpha = 1, \cdots, s.$$  

(3.20)

It follows that

$$\sum_{\beta} h_{\alpha \beta} = -\frac{n_\alpha + 1}{rL_1}, \quad \forall \alpha$$

(3.21)

which with (3.15) gives

$$\bar{A}_0^H(X, Y) = -g(X, Y) \sum_{\alpha} \frac{n_\alpha + 1}{r} H_\alpha, \quad \forall X, Y \in \mathcal{H}^\perp.$$  

(3.22)
and thus
\[ H_0 = \frac{1}{r-1} \text{tr}(\bar{A}_0^H) = - \sum_{\alpha} \frac{n_\alpha + 1}{r} H_\alpha. \] (3.23)

Since we have shown that \( \text{tr}(\bar{A}_0^H) = 0 \) (Equation (3.17)), to complete the proof of Lemma 3.3 it now suffices to show that
\[ \frac{(n+1)L_1}{r}(g(Y,Z)X - g(X,Z)Y) - [\bar{A}_0^H(X), \bar{A}_0^H(Y)](Z) = 0 \] (3.24)
for all \( X, Y, Z \in \mathcal{H}^\perp \).

In deed, the fact that \( A_{00}^0 = 0 \) (Equation (3.17)) implies
\[ L_1(g(Y,Z)X - g(X,Z)Y) - [A_{00}^0(X), A_{00}^0(Y)](Z) = 0. \] (3.25)

But by the decomposition (3.11)
\[ A_{00}^0(X)(A_{00}^0(Y)(Z)) = A_{00}^0(X, A_{00}^0(Y, Z)) \]
\[ = A_{00}^0(X, \bar{A}_0^H(Y, Z)) + A_{00}^0(X, \bar{A}_0^H(Y, Z)) \]
\[ = \bar{A}_0^H(X, \bar{A}_0^H(Y, Z)) + A_{00}^0(X, \bar{A}_0^H(Y, Z)) + A_{00}^H(X, \bar{A}_0^H(Y, Z)) \]
\[ = \bar{A}_0^H(X)(\bar{A}_0^H(Y)(Z)) - \frac{1}{r} g(X, \bar{A}_0^H(Y, Z)) \sum_\alpha (n_\alpha + 1) H_\alpha \]
\[ - \frac{1}{r} g(Y, Z) \sum_\alpha (n_\alpha + 1) A_0^H(X, H_\alpha) \]
\[ = \bar{A}_0^H(X)(\bar{A}_0^H(Y)(Z)) - \frac{1}{r} g(X, \bar{A}_0^H(Y, Z)) \sum_\alpha (n_\alpha + 1) H_\alpha \]
\[ - \frac{1}{r} g(Y, Z) \sum_\alpha (n_\alpha + 1) L_1 g(Y, Z) X \]
\[ = \bar{A}_0^H(X)(\bar{A}_0^H(Y)(Z)) - \frac{1}{r} g(X, \bar{A}_0^H(Y, Z)) \sum_\alpha (n_\alpha + 1) H_\alpha \]
\[ - \frac{n-r+1}{r} L_1 g(Y, Z) X \]
where we have used (3.17), (3.22) and the definition of \( r \).

Since \( g(X, \bar{A}_0^H(Y, Z)) = g(Y, \bar{A}_0^H(X, Z)) \), we find that
\[ [A_{00}^0(X), A_{00}^H(Y)](Z) = A_{00}^0(X)(A_{00}^H(Y)(Z)) - A_{00}^0(Y)(A_{00}^H(X)(Z)) \]
\[ = \bar{A}_0^H(X)(\bar{A}_0^H(Y)(Z)) - \bar{A}_0^H(Y)(\bar{A}_0^H(X)(Z)) \]
\[ - \frac{n-r+1}{r} L_1 g(Y, Z) X - g(X, Z) Y \]
\[ = [\bar{A}_0^H(X), \bar{A}_0^H(Y)](Z) - \frac{n-r+1}{r} L_1 g(Y, Z) X - g(X, Z) Y. \] (3.26)

Inserting the above equality into (3.25) we obtain the equation (3.24), which completes the proof of Lemma 3.3. \( \square \)

**Lemma 3.4.** All the vectors \( H_0, H_1, \ldots, H_s \) are constant vectors in \( \mathcal{H} \subset \mathbb{R}^3 \); In particular, \( \bar{c}_0, \bar{c}_1, \ldots, \bar{c}_s \) are all constants. Furthermore,
\[ A_{\alpha\alpha}^\alpha \in S_{M_\alpha}(L_1 - c_\alpha^2). \] (3.27)
Proof of Lemma 3.4. The first conclusion of the lemma is easily obtained by taking the derivatives of (3.23) on $M_{\alpha}$ for each $\alpha = 0, 1, \cdots, s$.

We first note that the conditions (1) and (3) for (3.27) come correspondingly from those for $A \in S_{(M^\alpha,g)}(L_1)$.

To verify the condition (2) for (3.27), we use the affine Gauss equation (2.6) to see that

$$R^{M_{\alpha}}(X, Y)Z = L_1(g_a(Y, Z)X_a - g_a(X, Z)aY_a) - [A(X), A(Y)](Z_a).$$  

(3.28)

But by (3.4), (3.5) and (3.6) we find

$$[A(X), A(Y)](Z_a) = A(X, A(Y, Z_a)) - A(Y, A(X, Z_a))$$

$$= A(X, A_{0a}(Y, Z_a)) + A_{0a}(Y, Z_a))$$

$$= A_{0a}(X, g_a(Y, Z_a)H_a) + A_{0a}(X, A_{0a}(Y, Z_a))$$

$$+ A_{0a}(X, A_{0a}(Y, Z_a)) - A_{0a}(Y_a, g_a(X, Z_a)H_a)$$

$$= g_a(Y, Z_a)g_a(H_a, Z_a)X_a + g_a(X, A_{0a}(Y, Z_a)H_a)$$

$$+ A_{0a}(X, A_{0a}(Y, Z_a)) - A_{0a}(Y_a, g_a(X, Z_a)H_a)$$

Inserting the above into (3.28) we obtain

$$R^{M_{\alpha}}(X, Y)Z = (L_1 - c_a^2)(g_a(Y, Z_a)X_a - g_a(X, Z_a)Y_a)$$

$$- [A_{0a}(X), A_{0a}(Y)](Z_a),$$

implying the condition (2) for $A_{0a} \in S_{M_{\alpha}}(L_1 - c_a^2)$. \hfill \Box

Lemma 3.5. The vector-valued symmetric bilinear form $A \in S_{(M^\alpha,g)}(L_1)$ is uniquely, up to equivalence, determined by the metrics $g_a$, the flat metric $g_0$ on $\mathbb{R}^s$, the bilinear forms $A_{0a}$ ($\alpha = 1, \cdots, s$) and the affine mean curvature $L_1$.

Proof of Lemma 3.5. Since $A_{0a} \in S_{(M^\alpha,g)}(L_1 - c_a^2)$, we see that the constant $c_a$ is completely determined by $g_a$, $L_1$ and $A_{0a}$ via

$$R_{ga}(X, Y)Z = (-L_1 + c_a^2)(g_a(Y, Z_a)X_a - g_a(X, Z_a)Y_a)$$

$$+ [A_{0a}(X), A_{0a}(Y)](Z_a),$$

where $X_a, Y_a, Z_a \in TM_{\alpha}$ and $R_{ga}$ is the curvature tensor of $g_a$.

On the other hand, up to an orthogonal transformation on $H \subset TM_0 = \mathbb{R}^s$, the constant vectors $H_1, \cdots, H_s$ are uniquely given by the matrix equality

$$
\begin{pmatrix}
  g(H_1, H_1) & g(H_1, H_2) & \cdots & g(H_1, H_s) \\
  g(H_2, H_1) & g(H_2, H_2) & \cdots & g(H_2, H_s) \\
  \vdots & \vdots & \ddots & \vdots \\
  g(H_s, H_1) & g(H_s, H_2) & \cdots & g(H_s, H_s)
\end{pmatrix}
\begin{pmatrix}
  c_1^2 & L_1 & \cdots & L_1 \\
  L_1 & c_2^2 & \cdots & L_1 \\
  \vdots & \vdots & \ddots & \vdots \\
  L_1 & L_1 & \cdots & c_s^2
\end{pmatrix}.
$$

Furthermore, it is easily seen from (3.5), (3.9), (3.7) and (3.15) that $A_{0a}, A_{0a}, A_{0a}, A_{0a}, A_{0a}, A_{0a}$ are completely determined by the flat metric $g_0$, the vectors $H_1, \cdots, H_s$ and the affine mean curvature $L_1$. 

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Finally, since $A_{0}^{\alpha}\in\mathcal{S}_{\mathbb{R}^{r-1}}\left(\frac{(n+1)L_{1}}{L_{1}}\right)$, it is realized as the Fubini-Pick form of a flat hyperbolic affine hypersphere in $\mathbb{R}^{r}$. Then a theorem of L. Vranken, A-M. Li and U. Simon in [21] (also see [12]) assures that any of such flat affine hypersphere is equiaffine equivalent to the hyperbolic hypersphere in Example 2.1. Thus $\overline{\gamma}$ is also unique up to isometries on $\mathbb{R}^{r-1}$. It then follows that $A_{0}^{\alpha}$ is completely determined by the flat metric $g$, the sections $H_{1},\cdots,H_{s}$ and the affine mean curvature $L_{1}$ up to isometries on $\mathbb{R}^{r}$.

Summing up, we have proved the conclusion of Lemma 3.5.

Now we are in a position to complete the proof of Theorem 3.1.

Let $C$ be given by (2.22). Suitably choosing the constants $c_{a}(1\leq a\leq r)$, $c_{r+\alpha}$, $(\alpha)\ L_{1}(1\leq\alpha\leq s)$, we can also assume the first equality. For each $\alpha = 1,\cdots,s$, fix one Riemannian metric

\[
\gamma^{(\alpha)} = \left(\frac{n+1}{L_{1}}\right) g_{\alpha}
\]

on $M_{\alpha}$. Then by (3.27), have find that

\[
A_{\gamma\alpha}^{\alpha} = \sum_{(\alpha)}(n_{\alpha}+1)(L_{1}-c_{\alpha}^{2})\left(\frac{n_{\alpha}+1}{n+1}L_{1}\right).
\]

We claim that

\[
\frac{(n_{\alpha}+1)(L_{1}-c_{\alpha}^{2})}{(n+1)L_{1}} = 1,
\]

that is, $c_{\alpha}^{2} = \frac{n-n_{\alpha}}{1-n_{\alpha}}L_{1}$. (3.29)

In fact, multiplying $h_{\gamma\alpha}$ to the both sides of (3.21) and then taking sum over $\alpha$ we have

\[
1 = \sum_{\alpha,\beta}h_{\gamma\alpha}^{\beta}h_{\alpha\gamma} = -\sum_{\alpha}\frac{n_{\alpha}+1}{rL_{1}}h_{\alpha\gamma}.
\]

Since, by (3.5), $h_{\gamma\alpha} = c_{\gamma}^{2}$ and $h_{\gamma\alpha} = L_{1}$ for $\alpha \neq \gamma$, the right hand side of (3.30) is

\[
-\sum_{\alpha}\frac{n_{\alpha}+1}{rL_{1}}h_{\gamma\alpha} = \frac{n_{\alpha}+1}{rL_{1}}c_{\gamma}^{2} - \sum_{\alpha\neq\gamma}\frac{n_{\alpha}+1}{rL_{1}}L_{1}
\]

\[
= \frac{n_{\alpha}+1}{rL_{1}}c_{\gamma}^{2} - \frac{1}{r}\sum_{\alpha\neq\gamma}(n_{\alpha}+1)
\]

\[
= \frac{n_{\alpha}+1}{rL_{1}}c_{\gamma}^{2} + \frac{1}{r}(n_{\alpha}+1) - \frac{1}{r}\sum_{\alpha}(n_{\alpha}+1)
\]

\[
= \frac{n_{\alpha}+1}{rL_{1}}c_{\gamma}^{2} + \frac{1}{r}(n_{\alpha}+1) - \frac{1}{r}(n-r+1)
\]

\[
= \frac{n_{\alpha}+1}{rL_{1}}c_{\gamma}^{2} - \frac{1}{r}(n-n_{\gamma}) + 1
\]

(3.31)

From (3.30) and (3.31) we easily prove the claim (3.29).

The equality (3.29) shows that $A^{\alpha}_{\gamma\alpha} = \sum_{(\alpha)}(n_{\alpha}+1)(L_{1}-c_{\alpha}^{2})\left(\frac{n_{\alpha}+1}{n+1}L_{1}\right)$. It follows from Corollary 2.1 that, for each $\alpha = 1,\cdots,s$, there exists a hyperbolic affine hypersphere $x_{\alpha} : M_{\alpha}^{n_{\alpha}} \rightarrow \mathbb{R}^{n_{\alpha}+1}$ having $g_{\alpha}, L_{1}$ and $A^{\alpha}$ as its affine metric, affine mean curvature and Fubini-Pick form respectively.

Suitably choosing the parameters $t^{\gamma},\cdots,t^{K-1}$, $K = r+s$, the original flat metric $g_{0}$ on $\mathbb{R}^{K-1}$ can be written as $g_{0} = \sum_{\alpha,\beta}g_{\lambda\mu}dt^{\lambda}dt^{\mu}$ where $g_{\lambda\mu}$ is defined by (2.22).

Now we consider the Calabi composition $\tilde{x}$ of $r$ points and the $s$ hyperbolic affine hyperspheres $x_{\alpha}$, with the previously chosen constants $c_{a}, c_{r+\alpha}$ mentioned. Then it follows that the original hyperbolic
affine hypersphere $x$ is equiaffine equivalent to the Calabi composition $\bar{x}$ since they have the same affine metric and Fubini-Pick form by Proposition 2.2.

As an application of Theorem 3.1, we can easily recover the following result in a direct manner:

**Corollary 3.2.** (19) A locally strongly convex and affine symmetric hypersurface $x : M^n \to \mathbb{R}^{n+1}$ is locally affine equivalent to the Calabi composition of some hyperbolic affine hyperspheres possibly including point factors if and only if $M^n$ is reducible as a Riemannian manifold with respect to the affine metric.

In fact, if $x$ is affine symmetric, then the holonomy algebra $\mathfrak{h}$ for $(M^n, g)$ acts trivially on the Fubini-Pick form $A$ which, together with the fact that, for each $\alpha = 1, \cdots, s$, $h_\alpha$ acts irreducibly on $TM_\alpha$ and trivially on other $TM_\beta (\beta \neq \alpha)$, directly implies (2) and (3) in Theorem 3.1.

**References**

[1] N. Bokan, K. Nomizu and U. Simon, Affine hypersurfaces with parallel cubic forms, Tôhoku Math. J. 42 (1990), 101-108, MR 1036477, Zbl0696.53006.

[2] E. Calabi, Complete affine hypersurfaces I, Symposia Math., 10(1972), 19-38.

[3] F. Dillen and L. Vrancken, Calabi-type composition of affine spheres, Diff. Geom. appl., 4(1994), 303-328.

[4] F. Dillen, L. Vrancken and S. Yaprak, Affine hypersurfaces with parallel cubic form, Nagoya Math. J. 135 (1994), 153-164. MR 1295822, Zbl0806.53008.306

[5] Z. J. Hu, H. Z. Li and L. Vrancken, Characterizations of the Calabi product of hyperbolic affine hyperspheres, Result. Math. 52 (2008), 299C314.

[6] Z. J. Hu, C.C. Li, The classification of 3-dimensional Lorentian affine hypersurfaces with parallel cubic form, Result. Math. 52(2008), 299C314.

[7] Z. J. Hu, C.C. Li, H. Z. Li and L. Vrancken, The classification of 4-dimensional nondegenerate affine hypersurfaces with parallel cubic form, Journal of Geometry and Physics, 61(2011), 2035C2057.

[8] Z. J. Hu, C.C. Li, H. Z. Li and L. Vrancken, Lorentzian affine hypersurfaces with parallel cubic form. Res. Math., 59(2011), 577C620.

[9] Z. Hu, H. Li, U. Simon, and L. Vrancken, On locally strongly convex affine hypersurfaces with parallel cubic form, I, Diff. Geom. Appl. 27(2009), no2, 188-205.

[10] Z. J. Hu, H. Li, L. Vrancken, Locally strongly convex affine hypersurfaces with parallel cubic form, J. Diff. Geom., 87(2011), 239-307.

[11] C. P. Wang, Lorentzian affine hyperspheres with constant affine sectional curvature, Trans. Amer. Math. Soc., 352(1999), no.1, 1581-1599.

[12] A-M. Li, Some theorems in affine differential geometry, Acta Math. Sinica, N.S., 5(1989), 345-354.

[13] A-M. Li, Calabi conjecture on hyperbolic affine hyperspheres, Math. Z. 203(1990), 483-491.

[14] A-M. Li, Calabi conjecture on hyperbolic affine hyperspheres (2), Math. Ann. 293(1992), 485-493.

[15] A-M. Li, U. Simon and G. S. Zhao, Global affine differential geometry of hypersurfaces, de Gruyter Expositions in Mathematics, vol. 11, Walter de Gruyter and Co., Berlin, 1993.

[16] H.Z., Li and X.F. Wang, Calabi product Lagrangian immersions in complex projective space and complex hyperbolic space. Results Math., 59(2011), 453-470.

[17] X. X. Li, The composition and the section of hyperbolic affine spheres, J. Henan Normal University (Natural Science Edition, in Chinese), 21(1993), no.2, 8-12.

[18] X. X. Li, On the Calabi composition of multiple affine hyperspheres, preprint, 2011.

[19] X. X. Li, On the correspondence between symmetric equiaffine hyperspheres and the minimal symmetric Lagrangian submanifolds, preprint in Chinese, 2013. To appear.

[20] K. Nomizu and T. Sasaki, Affine Differential Geometry. Cambridge University Press, Cambridge (1994).

[21] L. Vrancken, A-M. Li and U. Simon, Affine spheres with constant affine sectional curvature, Math. Z. 206(1991), 651-658.

[22] C. P. Wang, Canonical equiaffine hypersurfaces in $\mathbb{R}^{n+1}$, Math. Z., 214(1993), 579-592.