MINIMUM VARIANCE ESTIMATION OF GALAXY POWER SPECTRUM IN REDSHIFT SPACE

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ABSTRACT

We study an efficient way to enhance the measurability of the galaxy density and/or velocity power spectrum in redshift space. It is based on the angular decomposition with the tripolar spherical harmonic (TripoSH) basis and applicable even to galaxy distributions in wide-angle galaxy surveys. While nontrivial multipole-mode mixings are inevitable in the covariance of the Legendre decomposition coefficient commonly used in the small-angle power spectrum analysis, our analytic computation of the covariance of the TripoSH decomposition coefficient shows that such mixings are absent by virtue of high separability of the TripoSH basis, yielding the minimum variance. Via the simple signal-to-noise ratio assessment, we confirm that the detectability improvement by the TripoSH decomposition approach becomes more significant at higher multipole modes, and, e.g., the hexadecapole of the density power spectrum has two orders of magnitude improvement. The TripoSH decomposition approach is expected to be applied not only currently available survey data but also forthcoming wide-angle one, and to bring about something new or much more accurate cosmological information.

Keywords: cosmology; observations — cosmology: theory — large-scale structure of universe — methods: statistical

1. INTRODUCTION

The power spectrum, i.e., the Fourier transform of the two-point correlation function, of redshift-space galaxy density and velocity fields is an indispensable statistic for precise understanding of the Universe (Peebles 1980; Strauss & Willick 1995; Hamilton 1997). Thanks to recent precise measurements, various cosmological parameters have been tightly constrained, and the nature of gravity or the particle content of the universe has been severely tested (see e.g., Weinberg et al. 2013, for review).

In most previous data analysis mentioned above, a traditional Legendre-decomposition-based estimator (Yamamoto et al. 2006) has been employed. The observed power spectrum is, originally, characterized by two independent angles: \(\hat{k} \cdot \hat{s}_1\) and \(\hat{k} \cdot \hat{s}_2\), where \(\hat{s}_1\) and \(\hat{s}_2\) denote the line-of-sight (LOS) directions toward two fields, due to the redshift-space distortion (RSD) (Hamilton 1997). In this estimator, however, these angles are identified with each other as \(\hat{k} \cdot \hat{s}_1 = \hat{k} \cdot \hat{s}_2 \equiv \mu\); thus, the angular dependence of the power spectrum is decomposed using the Legendre polynomials \(L_\ell(\mu)\). This approximation, dubbed the plane-parallel (PP) approximation, makes the data analysis pipeline much simpler. At the same time, however, this gives rise to nontrivial couplings between different \(\ell\) modes in the covariance (see e.g., Taruya et al. 2010), obstructing the minimization of the estimator variance. There are a few studies that performed the cosmological analysis of observed galaxy clustering without relying on the PP approximation in configuration space (e.g., Matsubara et al. 2000; Pope et al. 2004; Okumura et al. 2008).

In this Letter, we propose a way to achieve the minimum variance. It is done by the angular decompositions using the tripolar spherical harmonic (TripoSH) basis \(\{Y_\ell(k) \otimes \{Y_{\ell_1}(\hat{s}_1) \otimes Y_{\ell_2}(\hat{s}_2)\}_{\ell_1 \ell_2}\}_{\ell 00}\) (Varshalovich et al. 1988). Its crucial difference from the Legendre polynomial basis is that there are two additional multipoles \(\ell_1\) and \(\ell_2\) that can decompose the dependences on \(\hat{k} \cdot \hat{s}_1\) and \(\hat{k} \cdot \hat{s}_2\) separably. The remaining multipole \(\ell\) represents the total angular momentum of \(\ell_1\) and \(\ell_2\), and is identified with the Legendre multipole under the PP approximation. We then find that, owing to high separability of the TripoSH basis, each \(\ell\) mode does not tangle even at the covariance level and the minimum variance is consequently realized. We also compare the SNR of the TripoSH coefficient with that of the Legendre one for each \(\ell\) mode, and test how much the detectability is improved by use of the TripoSH coefficient. We show that the gain of the SNR is found to be more significant for higher \(\ell\) modes, and for example, the two orders of magnitude improvement is found for the \(\ell = 4\) (hexadecapole) mode.

2. PRELIMINARIES

\(^1\) See e.g., Castorina & White (2018a); Beutler et al. (2019); Castorina & White (2019) for studies on other decomposition approaches to the wide-angle effect.
All the analyses in this Letter are done on the basis of the linear theory because we are primarily interested in large-scale galaxy clustering; therefore, the galaxy number density fluctuation, \( \delta(s) \equiv n(s)/\bar{n}(s) - 1 \), and the LOS peculiar velocity field, \( u(s) \equiv \mathbf{v}(s) \cdot \hat{s} \), in redshift space are expressed as (Hamilton 1997; Yoo & Seljak 2015)

\[
X(s) = \int \frac{d^3k}{(2\pi)^3} e^{i\mathbf{k} \cdot \hat{s}} F^X(k, \hat{s}),
\]

where \( X = \{ \delta, u \} \) and

\[
F^\delta(k, \hat{s}) = \left[ b - i \frac{\alpha(s)}{k} \hat{k} \cdot \hat{s} \right] f + (\hat{k} \cdot \hat{s})^2 f \delta_m(k), \quad F^u(k, \hat{s}) = i \frac{aH}{k} (\hat{k} \cdot \hat{s}) f \delta_m(k),
\]

with \( b \) the linear bias parameter, \( f \) the linear growth rate, \( \delta_m \) the matter density perturbation in real space, \( a \) the scale factor and \( H \) the Hubble parameter. The selection function \( \alpha \) is defined in terms of the mean number density \( \bar{n}(s) \) of the galaxy sample as \( \alpha(s) \equiv d \ln \bar{n}(s)/d \ln s + 2 \). Here and hereinafter, we omit time, redshift, or the radial distance in the arguments of variables unless the parameter dependence is nontrivial. For convenience, let us rewrite these by use of the Legendre polynomials as

\[
F^X(k, \hat{s}) = \sum_j c^X_j(k) L_j(\hat{k} \cdot \hat{s}) \delta_m(k),
\]

with

\[
c^\delta_0 = b + \frac{1}{3} f, \quad c^\delta_1 = -i \frac{\alpha}{k} f, \quad c^\delta_2 = \frac{2}{3} f, \quad c^\delta_{\ell \geq 3} = 0,
\]

\[
c^u_0 = i \frac{aH}{k} f, \quad c^u_0 = c^u_{\ell \geq 2} = 0.
\]

Note that \( F^X(k, \hat{s}) \) is not equal to the Fourier counterpart of \( X(s) \) because there still remains the position dependence. Throughout this Letter, we assume that the real-space matter power spectrum takes a statistically homogeneous and isotropic form:

\[
\langle \delta_m(k_1) \delta_m(k_2) \rangle = (2\pi)^3 \delta^{(3)}(k_1 + k_2) P_m(k_1).
\]

The redshift-space two-point correlation function then takes the form:

\[
\xi_{X_1X_2}(s_{12}, \hat{s}_1, \hat{s}_2) \equiv \langle X_1(s_1) X_2(s_2) \rangle
\]

\[
= \int \frac{d^3k}{(2\pi)^3} e^{i\mathbf{k} \cdot s_{12}} P^{X_1X_2}(k, \hat{s}_1, \hat{s}_2),
\]

where \( s_{12} \equiv s_1 - s_2 \) and

\[
P^{X_1X_2}(k, \hat{s}_1, \hat{s}_2) = \sum_{j_1j_2} c^{X_1}_{j_1}(k)(-1)^{j_2} c^{X_2}_{j_2}(k) L_{j_1}(\hat{k} \cdot \hat{s}_1) L_{j_2}(\hat{k} \cdot \hat{s}_2) P_m(k).
\]

Again, this \( P^{X_1X_2} \) differs from the Fourier counterpart of \( \xi_{X_1X_2} \) and they should not be confused with each other. In this Letter, we also do not take unequal time correlators into account although these are also nonzero and should enhance the SNR; therefore, the radial distances appearing in \( c^X_2 \) are fixed to be \( s_1 \simeq s_2 \).

3. TRIPSH DECOMPOSITION

Let us introduce the TripSH basis with zero total angular momentum using the Wigner 3j symbol (Vershalovich et al. 1988):

\[
\chi_{\ell_1\ell_2}(s_{12}, \hat{s}_1, \hat{s}_2) \equiv \langle Y_{\ell_1}^* (\hat{s}_1) \otimes (Y_{\ell_2} (\hat{s}_2) \otimes Y_{\ell_2}^* (\hat{s}_2)) \rangle_{\ell_1 \ell_2}
\]

\[
= \sum_{m_1m_2} (-1)^{m_1 + m_2} \left( \begin{array}{ccc} \ell_1 & \ell_2 & \ell \\ m_1 & m_2 & m \end{array} \right) Y_{\ell m_1}(\hat{s}_1) Y_{\ell m_2}(\hat{s}_2).
\]

The TripSH decomposition is done according to

\[
\xi_{X_1X_2}(s_{12}, \hat{s}_1, \hat{s}_2) = \sum_{\ell_1\ell_2} \Xi_{\ell_1\ell_2} X_{\ell_1\ell_2}(s_{12}, \hat{s}_1, \hat{s}_2).
\]

In this process, the dependences on \( \hat{s}_1 \) and \( \hat{s}_2 \) are characterized by \( \ell_1 \) and \( \ell_2 \), respectively, and \( \ell \) denotes the total angular momentum of \( \ell_1 \) and \( \ell_2 \). The orthonormality of \( \chi_{\ell_1\ell_2}(s_{12}, \hat{s}_1, \hat{s}_2) \) yields an inverse formula:

\[
\Xi_{\ell_1\ell_2} X_{\ell_1\ell_2}(s_{12}, \hat{s}_1, \hat{s}_2) = \int d^2\hat{s}_2 \int d^2\hat{s}_1 \int d^2\hat{s}_2 \Xi_{\ell_1\ell_2}(s_{12}, \hat{s}_1, \hat{s}_2)
\]

\[
\times \chi_{\ell_1\ell_2}(\hat{s}_2, \hat{s}_1, \hat{s}_2).
\]

This is connected to the decomposition coefficients, defined by

\[
\Pi_{\ell_1\ell_2} X_{\ell_1\ell_2}(k, \hat{s}_1, \hat{s}_2) \equiv \int d^2\hat{s}_1 \int d^2\hat{s}_2 P^{X_1X_2}(k, \hat{s}_1, \hat{s}_2)
\]

\[
\times \chi_{\ell_1\ell_2}(\hat{k}, \hat{s}_1, \hat{s}_2),
\]

via the Hankel transformation:

\[
\Xi_{\ell_1\ell_2} X_{\ell_1\ell_2}(s_{12}, \hat{s}_1, \hat{s}_2) = \ell \int_0^\infty \frac{k^2 dk}{2\pi^2} j_\ell(k s_{12}) \Pi_{\ell_1\ell_2} X_{\ell_1\ell_2}(k),
\]

with \( j_\ell(x) \) the spherical Bessel functions.

In the following, for convenience, the TripSH coefficients, obtained from Eq. (11), are utilized via an additional transformation:

\[
P^X_{\ell_1\ell_2}(k) \equiv \frac{h_{\ell_1\ell_2}}{4\pi} \Pi_{\ell_1\ell_2} X_{\ell_1\ell_2}(k),
\]

where

\[
h_{\ell_1\ell_2} = \sqrt{(2\ell_1 + 1)(2\ell_2 + 1)(2\ell_1 + 1)} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} / 4\pi
\]

\( \ell_1 \ell_2 \ell_3 \)

Taking the PP limit \( \hat{s}_1 = \hat{s}_2 = \hat{s} \), these reduced coefficients are related to the conventional Legendre decomposition ones through a simple summation:

\[
P^X_\ell X_\ell = \sum_{\ell_1\ell_2} P^X_{\ell_1\ell_2};
\]

therefore, are useful especially for the comparison with the PP-limit results. Note that this is derived employing an identity \( \chi_{\ell_1\ell_2}(k, \hat{s}, \hat{s}) = h_{\ell_1\ell_2} L_\ell(\hat{k} \cdot \hat{s})/(4\pi) \).

The above decomposition scheme follows the same format as the previous studies on the density field (e.g., Szapudi 2004; Shiraishi et al. 2017), and is newly extended to the analysis of the velocity field here. Performing the angular integrals in Eq. (11), the power spectrum signal is decomposed into the TripSH coefficients. In our case,
as the power spectrum takes the simple form (7), these are done analytically as
\[
\int d^2\delta_1 \int d^2\delta_2 X'_{\ell_1 \ell_2} (\hat{k}, \hat{s}_1, \hat{s}_2) L_{j_1} (\hat{k} \cdot \hat{s}_1) L_{j_2} (\hat{k} \cdot \hat{s}_2) = \left( \frac{2\pi}{4\pi h_{\ell_1 \ell_2}} \right) \delta_{\ell_1,j_1} \delta_{\ell_2,j_2}.
\] (16)
We then finally obtain
\[
P_{X_1 X_2} (k) = \frac{4\pi (-1)^{\ell_2} h_{\ell_1 \ell_2}^2}{(2\pi)^4 \delta_{\ell_1,0} \delta_{\ell_2,0}} c_{X_1} (k) c_{X_2} (k) P_m (k).
\] (17)
Due to the selection rule of \(h_{\ell_1 \ell_2}\), i.e., \(|\ell_1 - \ell_2| \leq \ell_1 + \ell_2\) and \(\ell_1 + \ell_2 + \ell = \) even, and a nonvanishing condition of \(c_{X}^2\) [see Eq. (4)], only 14 \(P_{\ell_1 \ell_2}\), 5 \(P_{\ell_1 \ell_2}^{\text{noise}}\), and 2 \(P_{\ell_1 \ell_2}^{\text{uu}}\) listed below can take nonzero values:
\[
\begin{align*}
P_{00}^{00}, & \quad P_{01}^{00}, P_{02}^{00}, P_{10}^{00}, P_{11}^{00}, P_{12}^{00}, P_{20}^{00}, P_{21}^{00}, P_{22}^{00}, P_{10}^{01}, P_{11}^{01}, P_{12}^{01}, P_{20}^{01}, P_{21}^{01}, P_{22}^{01}, P_{01}^{02}, P_{02}^{02}, P_{11}^{02}, P_{12}^{02}, P_{21}^{02}, P_{22}^{02}.
\end{align*}
\] (18)
Note that, since the transformation (15) is done without loss of generality, \(\Pi_{\ell_1 \ell_2}^{X_1 X_2}\) has the equivalent information and thus takes nonzero values only at the same multipole configurations.

4. TRIPOSH COVARIANCE

It is expected from Eq. (6) that \(\xi_{X_1 X_2}\) extracted from given data takes the form:
\[
\hat{\xi}_{X_1 X_2} (s_1, s_2) = \int \frac{d^3k}{(2\pi)^3} e^{ik \cdot s_2} P_{X_1 X_2} (k, s_1, s_2),
\] (19)
where \(P_{X_1 X_2} (k, s_1, s_2) = P_{X_1} (k, s_1) P_{X_2} (-k, s_2) / V - P_{X_1 X_2}^{\text{noise}}\) with \(V\) the survey volume and \(P_{X_1 X_2}^{\text{noise}}\) the contribution of the shot noise. Here and hereinafter, \(\hat{\xi}_{X_1 X_2}\), \(P_{X_1 X_2}\), and their decomposition coefficients with hat denote quantities calculated from a single realization, and they should not be confused with the unit vector. Supposing Gaussianity of \(F^X\), the covariance of \(\hat{P}_{X_1 X_2}\) is computed as
\[
\left\langle \hat{P}_{X_1 X_2} (k, s_1, s_2), \hat{P}_{X_1 X_2} (k, s_1, s_2) \right\rangle_c = 4\pi \delta_{\ell_1,j_1} \delta_{\ell_2,j_2} \left[ (\delta^2)_{\ell_1,0} k^2 \right] P_{X_1 X_2}^{\text{tot}} (k, s_1, s_2) \left[ (\delta^2)_{\ell_2,0} k^2 \right] P_{X_1 X_2}^{\text{tot}} (-k, s_2, s_2)
\] (20)
where \(N_k = V k^2 dk / (2\pi)^2\) and \(P_{X_1}^{\text{tot}} = P_{X_1} + P_{X_1}^{\text{noise}}\).
It is convenient to rewrite \(P_{X_1 X_2}^{\text{tot}}\) into
\[
P_{X_1 X_2}^{\text{tot}} (k, s_1, s_2) = \sum_{j_1 j_2} (-1)^{j_2} P_{j_1 j_2}^{X_1 X_2} (k)
\times L_{j_1} (k \cdot s_1) L_{j_2} (k \cdot s_2),
\] (21)
where
\[
P_{j_1 j_2}^{X_1 X_2} (k) = c_{X_1}^{j_1} (k) c_{X_2}^{j_2} (k) P_m (k) + P_{X_1 X_2}^{\text{noise}} \delta_{j_1,0} \delta_{j_2,0}.
\] (22)
The covariance of \(\hat{P}_{X_1 X_2}^\ell\) is obtained via the double TripoSH decomposition of Eq. (20). Again, the resulting angular integrals can be simplified so much owing to the analytic formula (16). The bottom-line form reads
\[
\left\langle \hat{P}_{X_1 X_2}^\ell (k) \right\rangle_c = \frac{\delta_{\ell_1,j_1} \delta_{\ell_2,j_2}}{N_k} \left\langle \xi_{X_1 X_2} (s_1, s_2) \right\rangle_c.
\] (23)
where
\[
\left\langle \xi_{X_1 X_2} (s_1, s_2) \right\rangle_c = \frac{2 \mu_0^2 (12)_{\ell_1,0} (12)_{\ell_2,0} k^2}{(2\pi)^4 \delta_{\ell_1,0} \delta_{\ell_2,0}}
\times \left[ \delta_{\ell_1 j_1} \delta_{\ell_2 j_2} \right] P_{X_1 X_2} (k, s_1, s_2).
\] (24)
The covariance of \(\hat{\xi}_{X_1 X_2}\) is obtained by following the transformations (12) and (13). One very interesting finding from this expression is that, at some specific multipole configurations, e.g., where none or only one of \(\ell_1\) or \(\ell_2\) takes 0, the covariance is free from the shot noise term. In such a situation, the covariance is minimized as
\[
\left\langle \hat{P}_{X_1 X_2}^\ell (k) \right\rangle_c = 2 \mu_0^2 (12)_{\ell_1,0} (12)_{\ell_2,0} k^2.
\] (25)
We stress that this is not the case if one uses the Legendre decomposition, which is adopted by most of preceding studies,
\[
\left\langle \hat{P}_{X_1 X_2}^\ell (k) \right\rangle_c = \frac{2 \mu_0^2 (12)_{\ell_1,0} (12)_{\ell_2,0} k^2}{(2\pi)^4 \delta_{\ell_1,0} \delta_{\ell_2,0}}.
\] (26)
The identification between \(k \cdot s_1\) and \(k \cdot s_2\) causes non-trivial mode mixings between different \(\ell\) modes at the covariance level. Let us focus on \(P_{X_1}^{uu}\) for example. The Legendre decomposition of \(P_{X_1}^{uu}\) under the PP approximation yields two different nonvanishing multipoles: \(P_{X_1}^{uu}\) and \(P_{X_2}^{uu}\), while the covariance of \(P_{X_1}^{uu}\) contains not only \(P_{X_1}^{uu}\) but also \(P_{X_1}^{\text{noise}} (k) + P_{X_2}^{\text{noise}} (k)\) as
\[
\left\langle \hat{P}_{X_1}^{uu} (k) \right\rangle_c = \left[ 10 (P_{X_1}^{uu} (k) + P_{X_1}^{\text{noise}} (k))^2 + \frac{40}{7} (P_{X_1}^{uu} (k) + P_{X_2}^{\text{noise}} (k))^2 + \frac{30}{7} (P_{X_2}^{uu} (k))^2 \right].
\] (27)
On the other hand, as is evident from Eq. (25), the covariance of \(P_{X_1}^{uu}\) is given by \(P_{X_1}^{uu}\) alone. The similar different-mode contamination also occurs in the covariance of \(\hat{P}_{X_1}^\ell\) and \(\hat{P}_{X_2}^\ell\) (see e.g., Taruya et al. 2010, for practical expressions). This causes the loss of detectability.

5. EFFICIENCY

How much is the TripoSH decomposition [Eq. (10)] efficient? To see it, we estimate the SNR for each TripoSH coefficient according to
\[
\left( \frac{S}{N} \right)_X^2 = V \int_{k_{\min}}^{k_{\max}} \frac{k^2 dk}{2\pi^2} \left( \frac{P_{X_1 X_2} (k)}{P_{\ell_1 \ell_2}^{\text{tot}} (k)} \right)^2.
\] (28)
Some representative values at $b = 2$ are summarized in Table 1. Here, a realistic survey at $z = 0.1$ (first line) and an ideal noiseless ones at several redshift slices (second and subsequent lines) are assumed. For the first case, we model the noise spectrum as $P_{\delta \delta}^{\text{noise}} = 1/\bar{n}$, $P_{\delta u}^{\text{noise}} = \sigma_\delta^2/\bar{n}$ and $P_{\delta u}^{\text{noise}} = P_{\delta u}^{\text{noise}} = 0$, and assume $\sigma_u = 300 \text{ km/s}$ and $\bar{n} = 5 \times 10^{-4} \text{ h}^3 \text{Mpc}^{-3}$ in anticipation of a LSST-level survey (Graziani et al. 2020). In contrast, for the second and subsequent cases, $P_{\delta \delta}^{\text{noise}} = P_{\delta u}^{\text{noise}} = P_{\delta u}^{\text{noise}} = 0$ are adopted. One can see from this that $E^2_\ell X^1_1 X^2_2$ is drastically enhanced at higher $\ell$. This is because, as explained above, the big contamination due to lower $\ell$ modes in the higher $\ell$ elements of the covariance matrix, which appears in the Legendre decomposition case, is completely absent in the TripoSH one.

Figure 1 draws the dependence of $E^2_\ell X^1_1 X^2_2$ on $z$ (left panel) and $b$ (right panel) assuming a noiseless survey. It is visually apparent that $E^2_\ell$, $E^3_\ell$ and $E^4_\ell$ increase as $z$ decreases or $b$ increases. Decreasing $z$, namely decreasing the growth rate $f$, or increasing $b$ relatively enhances $P_{\delta \delta}$ and $P_{\delta u}$ as compared with $P_{\delta \delta}$ and $P_{\delta u}$ and the contamination of the Legendre coefficient covariance gets worse, degrading the Legendre SNRs, i.e., the denominators of the efficiency indices. In contrast, the numerators, i.e., Eq. (30), remain unchanged, and consequently $E^2_\ell$, $E^3_\ell$ and $E^4_\ell$ are enhanced.

The similar comparison can also be done in configuration space by computing the SNR of $\Xi^{X^1_1 X^2_2}_{\ell_1 \ell_2}$. Since $\Xi^{X^1_1 X^2_2}_{\ell_1 \ell_2}$ is linked with $P^{X^1_1 X^2_2}_{\ell_1 \ell_2}$ through the simple Hankel transformation formula (12), the similar results are expected.

### 6. SUMMARY AND DISCUSSIONS

In this Letter, we have computed the covariance of the TripoSH-decomposed density/velocity power spectrum for the first time. We have shown that, by virtue of the complete angular decomposition using the TripoSH basis, nontrivial mode mixings at the covariance level, as seen in the usual Legendre decomposition, can be fully disentangled, and as a result, the covariance at each multipole mode is minimized. Via the simple SNR estimation, we have found that the detectability improvements by our decomposition approach are more significant for higher multipole modes, and there are some order of magnitude improvements at the hexadecapole of the density auto power spectrum and the octopole of the density-velocity cross one. In addition, odd (even) multipoles of the density auto (density-velocity cross) power spectrum, which vanish in the Legendre decomposition, are distinctive modes of the TripoSH one, producing additional gains of the total SNR. The obtained results encourage reanalyzing the currently available data based on the TripoSH decomposition approach instead of the Legendre one. Moreover, its application to the upcoming wide-angle galaxy surveys such as SPHEREx (Doré et al. 2014), Euclid (Laureijs et al. 2011) and WFIRST (Spergel et al. 2013) is expected to further improve the detectability.

In the real data analysis, however, extra artificial signals due to specific survey geometry would remain as a residue to some extent even after the subtraction process.

### Table 1

Numerical values of the efficiency index $E^2_\ell X^1_1 X^2_2$ defined as the ratio of SNR for the TripoSH coefficient to that for the corresponding Legendre coefficient.

| $\ell$ | $b$ | $E^2_{00}$ | $E^2_{01}$ | $E^2_{10}$ | $E^2_{11}$ | $E^2_{20}$ | $E^2_{21}$ | $E^2_{22}$ | $E^2_{30}$ | $E^2_{31}$ | $E^2_{32}$ |
|-------|-----|------------|------------|------------|------------|------------|------------|------------|------------|------------|------------|
| 0     | ✓   | 1.1        | 1.7        | 2.5        | 1.4        | 2.2        | 1.5        | 3.2        | 1.8        | 3.1        | 3.4        |
| 0.1   |     | 1.0        | 7.5        | 2.4        | 1.4        | 2.1        | 1.3        | 2.2        | 1.8        | 2.1        | 2.3        |
| 0.5   |     | 1.0        | 5.9        | 1.4        | 1.7        | 1.3        | 2.2        | 1.8        | 2.1        | 2.3        | 2.5        |
| 1.0   |     | 1.0        | 5.2        | 1.4        | 1.5        | 1.3        | 2.2        | 1.8        | 2.1        | 2.3        | 2.5        |
| 2.0   |     | 1.0        | 4.9        | 1.4        | 1.4        | 1.3        | 2.2        | 1.8        | 2.1        | 2.3        | 2.5        |
| 3.0   |     | 1.0        | 4.8        | 1.4        | 1.3        | 2.2        | 1.8        | 2.1        | 2.3        | 2.5        | 2.7        |

Note. — We show seven possible $E^2_\ell X^1_1 X^2_2$ at $b = 2$ in a LSST-level survey at $z = 0.1$ (first line) and a noiseless one at several redshift slices (second and subsequent lines).
and might make a nontrivial impact. Even in the theoretical analysis, the applicability of our covariance formula to the nonlinear regime (Castorina & White 2018b; Taruya et al. 2020), the general relativistic effect (e.g., Bertacca et al. 2012), or the statistically-anisotropic Universe (Shiraishi et al. 2017) is not trivial. Together with the development of an efficient and feasible estimator of the TriposH coefficient, these interesting issues will also be addressed in our future publications.

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