Noncommutative Geometry from D0-branes in a Background B-field

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We study D0-branes in type IIA on $T^2$ with a background B-field turned on. We calculate explicitly how the background B-field modifies the D0-brane action. The effect of the B-field is to replace ordinary multiplication with a noncommutative product. This enables us to find the matrix model for M-theory on $T^2$ with a background 3-form potential along the torus and the lightlike circle. This matrix model is exactly the non-local 2+1 dim SYM theory on a dual $T^2$ proposed by Connes, Douglas and Schwarz. We calculate the radii and the gauge coupling for the SYM on the dual $T^2$ for all choices of longitudinal momentum and membrane wrapping number on the $T^2$.

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1. Introduction

Last fall Connes, Douglas and Schwarz \[1\] made a very interesting proposal relating the matrix theory of M-theory on $T^2$ with a background three form potential, $C_{-12}$, to a gauge theory on a noncommutative torus. Shortly after Douglas and Hull justified this claim by relating these theories to a theory on a D-string [2]. They also mentioned that this could be seen in the original 0-brane picture.

The purpose of the present paper is to precisely incorporate the background B-field in the dynamics of 0-branes. In this way we will obtain the matrix model for M-theory on $T^2$ with $C_{-12} \neq 0$. It confirms the claims made by Connes, Douglas and Schwarz. The theory is a SYM theory on a dual torus with a modified interaction depending on the B-field. The theory contains higher derivative terms of arbitrarily high power and is thus non-local. We calculate the radii of the dual torus and the gauge coupling constant. We get a noncommutative gauge theory for any choice of longitudinal momentum and number of membranes wrapped around the $T^2$. The radii and gauge coupling depend on these numbers.

Aspects of the connection between compactifications of M-theory and noncommutative geometry has, among others, been worked out in [3,4,5].

2. Zero-branes on $T^2$ with background B-field

Let us consider M-theory on $T^2$ with radii, $R_1, R_2$. The torus is taken to be rectangular for simplicity. Making the torus oblique does not introduce anything interesting. We are interested in the matrix description of this theory with a background $C$-field, $C_{-12} \neq 0$. Here - denotes the lightlike circle and 1, 2 the directions along the torus. Let the Plank mass be $M$ and the radius of the lightlike circle be $R$. Following [3,4], we take this to mean a limit of spatial compactifications. We also perform a rescaling to keep the interesting energies finite. The upshot is that we consider M theory on $T^2 \times S^1$ with Plank mass $\tilde{M}$ and radii $\tilde{R}_1, \tilde{R}_2, \tilde{R}$ in the limit $\tilde{R} \to 0$ with

\begin{align}
\tilde{M}^2 \tilde{R} &= M^2 R \\
\tilde{M} \tilde{R}_i &= MR_i & i = 1, 2. \tag{2.1}
\end{align}

This turns into type IIA on $T^2$ with string mass $m_s$, coupling $\lambda$ and radii $r_1, r_2$ given by

\begin{align}
m_s^2 &= \tilde{M}^3 \tilde{R} \\
\lambda &= (\tilde{M} \tilde{R})^{\frac{3}{2}} \\
r_i &= \tilde{R}_i & i = 1, 2. \tag{2.2}
\end{align}
Furthermore there is a flux of the $B_{ij}$ field through the torus. We call the flux $B$:

$$B = RR_1 R_2 C_{-12}. \quad (2.3)$$

We are interested in the sector of theory labelled by two integers, namely the number of D0-branes, $N_0$, and the number of D2-branes, $N_2$, wrapped on $T^2$. In this section we will solely be interested in the case $N_2 = 0$. In the next section we will treat the general case. Let us for the moment set $N_0=1$. This makes us avoid some essentially irrelevant indices. The generalization to general $N_0$ is straightforward.

The method for dealing with this situation has been developed in [8,9]. We work with the covering space of $T^2$, namely $\mathbb{R}^2$ and place 0-branes in a lattice (Figure 1).

![Fig. 1: 0-branes on the covering space $\mathbb{R}^2$](image)

Let us label the 0-branes $(a, b)$ where $a, b$ are integers. The open strings obey Dirichlet boundary condition on the 0-branes. This is the point where we need $N_2 = 0$. If there had been D2-branes, the 0-branes would have been dispersed as fluxes inside the 2-branes and the open strings would have Dirichlet boundary conditions on the 2-branes and the above picture does not apply.

The fields in the theory come from quantizing the open strings and calculating their interactions. In the limit we are taking, $m_s \to \infty$, only the lowest modes survive and when $B = 0$ the theory becomes a SYM quantum mechanics [10,11,12,13,14]. The gauge group is ”$U(\infty)$” since there are infinitely many 0-branes. To be more precise let us define a Hilbert space on which the fields will be operators. There is a basis vector for each 0-brane, i.e. the basis is $|a, b >$ where $a, b \in \mathbb{Z}$. Let $\phi$ be any field in theory, then the matrix
Fig. 2: String starting at 0-brane \((a_1, b_1)\) and ending at 0-brane \((a_2, b_2)\) element \(\phi_{a_1b_1, a_2b_2}\) has the interpretation as a field which annihilates a state of an open string starting at \(a_1b_1\) and ending at \(a_2b_2\), see figure 2.

The fields of the theory are
1. bosons: \(X^i\) \(i = 1, \ldots, 8\)
2. fermions: \(\Psi_\alpha\) \(\alpha = 1, \ldots, 16\)
3. The gauge field: \(A_0\).

The fields are constrained to obey the following conditions:

\[
U^{-1}_i X^a U_i = X^a + 2\pi r_a \delta^{ai} \quad i = 1, 2; \\
U^{-1}_i \Psi^\alpha U_i = \Psi^\alpha, \\
U^{-1}_i A_0 U_i = A_0;
\]

where \(U_i\) are translation operators on the states in the Hilbert space:

\[
U_1|a, b> = |a + 1, b> \\
U_2|a, b> = |a, b + 1>.
\]

The gauge field \(A^0\) can be gauged away, and we will work in the gauge \(A^0 = 0\). When \(B = 0\) the action is

\[
L = \frac{m_s}{2\lambda} \text{Tr}[\dot{X}^a \dot{X}^a + \frac{m_s^4}{(2\pi)^2} \sum_{a<b} [X^a, X^b]^2] \\
+ \frac{m_s^2}{2\pi} \Psi T \dot{\Psi} - \frac{m_s^4}{(2\pi)^2} \Psi^T \Gamma_a [X^a, \Psi].
\]

What about \(B \neq 0\)? We will now show how to incorporate \(B\)-dependence in the action. The two-form \(B_{ij}\) couples to the worldsheet through the interaction \(\int_{W.S.} B_{ij}\), i.e. the \(B\)-field is pulled back to the worldsheet and integrated. Let us look at the example shown in Figure 3 below.

This interaction is represented, in the case \(B = 0\), by a term:

\[
\kappa \phi^{(3)}_{ik} \phi^{(2)}_{kj} \phi^{(1)}_{ji}.
\]
Fig. 3: The interaction between these three strings give rise to a cubic vertex.

Fig. 4: The worldsheet for a cubic vertex

where $\kappa$ is a constant. This term could, for instance, annihilate string 1 and 2 and create 3 with opposite orientation. The worldsheet would look as shown in figure 4.

To calculate $\int_{W.S.} B_{ij}$ we just need the projection into the plane of the torus, since this is the only direction in which $B_{ij} \neq 0$. This projection is exactly given by the area between the three strings in Figure 3. The important point is that $B_{ij}$ is closed so $\int B_{ij}$ only depends on the homotopy type of the worldsheet imbedding and is insensitive to the finer
details of how the interaction takes place. For the example in Figure 3, $\int_{W.S.} B_{ij} = \frac{1}{2} B$, where we remark that $B$ was defined to be the flux through the torus. This means that the interaction eq.(2.7) now is replaced with

$$e^{i\frac{1}{2} B \phi_{ik}^{(3)} \phi_{kj}^{(2)} \phi_{ji}^{(1)}}.$$  \hspace{1cm} (2.8)

It is now a straightforward exercise to figure out what happens to a general interaction between the fields:

$$\phi_{a_1 b_1, a_k b_k} \cdots \phi_{a_3 b_3, a_2 b_2} \phi_{a_2 b_2, a_1 b_1}.$$  \hspace{1cm} (2.9)

We have to find the integral of the $B$-field through the polygon shown in Figure 5.

![Fig. 5: Generic form of a vertex with k strings](image)

We should remember to count with sign. Orienting a polygon oppositely would change the sign of $\int_{W.S.} B_{ij}$. It is easily deduced that the result is

$$\int_{W.S.} B_{ij} = \frac{1}{2} B \sum_{i=2}^{k} \begin{vmatrix} a_{i+1} - a_i & a_i - a_1 \\ b_{i+1} - b_i & b_i - b_1 \end{vmatrix}.$$  \hspace{1cm} (2.10)

where $\begin{vmatrix} a \ b \\ c \ d \end{vmatrix} = ad - bc$. This means that the interaction eq.(2.9) now becomes

$$\phi_{a_1 b_1, a_k b_k} \cdots \phi_{a_3 b_3, a_2 b_2} \phi_{a_2 b_2, a_1 b_1}.$$  \hspace{1cm} (2.11)
The reason for distributing the exponentials in this way will become clear in a moment. One could put an exponential between \( \phi^{(k)} \) and \( \phi^{(k-1)} \), but it would be identically 1, so we omitted it. We can introduce a notation which will make this look simpler. First note that the interactions always appear with a sum over indices.

\[
\sum_{a_1b_1,a_2b_2} \phi^{(k)}_{a_1b_1} \phi^{(k-1)}_{a_2b_2} \phi^{(k-2)}_{a_3b_2} \cdots \phi^{(3)}_{a_4b_4} \phi^{(2)}_{a_5b_5} \phi^{(1)}_{a_6b_6}
\]

(2.12)

If we think of the fields as matrices this is just

\[
\text{Tr}(\phi^{(k)} \cdot \phi^{(k-1)} \cdots \phi^{(2)} \cdot \phi^{(1)})
\]

(2.13)

Now we define a product, called \(*\), by

\[
(\phi^{(2)} \ast \phi^{(1)})_{a_3b_3,a_1b_1} = \sum_{a_2b_2} \frac{\frac{1}{2}}{B} \begin{vmatrix}
    a_3 - a_2 & a_2 - a_1 \\
    b_3 - b_2 & b_2 - b_1
\end{vmatrix} \phi^{(2)}_{a_3b_3} \phi^{(1)}_{a_2b_2,a_1b_1}.
\]

(2.14)

Now the interaction with a \( B \)-field, eq.(2.11), can be written

\[
\text{Tr}(\phi^{(k)} \ast \phi^{(k-1)} \ast \cdots \ast \phi^{(2)} \ast \phi^{(1)})
\]

(2.15)

This is really nice. It shows that to generalize the action eq(2.14) to include a \( B \)-field we just need to replace ordinary matrix product with \(*\)-product. For \( B = 0 \) the \(*\)-product coincides with the ordinary product. This point of view, that the fields take value in another algebra, was of course the main point of [1].

In the case \( B = 0 \) the fields with the constraints eq.(2.4) and action can be rewritten to a \( SYM \) theory on a dual \( T^2 \) [3]. Let us briefly repeat that construction for the case, \( B \neq 0 \). Let us first express the basis of the Hilbert space in another form. We want to think of the Hilbert space as \( L_2 \) functions on a dual torus with radii, \( \frac{1}{m_s^2}, \frac{1}{m_s^2} \). Let the basis vector \( |ab> \) correspond to \( e^{iax m_s^2 r_1} e^{iby m_s^2 r_2} \), then the operators \( U_1, U_2 \) become multiplication operators

\[
U_1 = e^{ix m_s^2 r_1}, \quad U_2 = e^{iy m_s^2 r_2}.
\]

(2.16)

It is now easy to solve the constraints for the fields eq.(2.4)

\[
X^1 = -i2\pi \frac{1}{m_s^2} \frac{\partial}{\partial x} + \sum_{a,b} X^1_{ab,00} e^{iam_m^2 r_1 x} e^{ibm_m^2 r_2 y}
\]

\[
X^2 = -i2\pi \frac{1}{m_s^2} \frac{\partial}{\partial y} + \sum_{a,b} X^2_{ab,00} e^{iam_m^2 r_1 x} e^{ibm_m^2 r_2 y}
\]

\[
X^j = \sum_{a,b} X^j_{ab,00} e^{iam_m^2 r_1 x} e^{ibm_m^2 r_2 y}
\]

\[
\Psi = \sum_{a,b} \Psi_{ab,00} e^{iam_m^2 r_1 x} e^{ibm_m^2 r_2 y}.
\]

(2.17)
We see that $X^1$, $X^2$ become covariant derivatives and all other fields are multiplication operators. These are exactly the fields of 2+1 dim SYM on a torus of radii $\frac{r_1}{m^2r_1}$, $\frac{r_2}{m^2r_2}$.

All this is independent of $B$. We saw that the only $B$-dependence was to change products of fields to the $*$-product. Let us see how the $*$-product looks in this basis. We only need to consider the types of operators which appear in the action. We see from eq.(2.17) that these are the differential operators, $\frac{\partial}{\partial x}$, $\frac{\partial}{\partial y}$, and multiplication operators. $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ are diagonal and it is easily seen from eq.(2.14) that for diagonal operators the $*$-product is equal to the usual product. Let us now look at two multiplication operators $\phi^{(1)}(x, y)$ and $\phi^{(2)}(x, y)$. We have

$$\phi^{(i)}(x, y) = \sum_{a,b} \phi^{i}_{ab,00} e^{iam^2r_1x + ibm^2r_2y}$$  \hspace{1cm} (2.18)

with $\phi^{i}_{a_2b_2,a_2b_2} = \phi^{i}_{(a_2-a_1)(b_2-b_1),00}$. Plugging into eq.(2.14) it is seen that

$$(\phi^{(2)} \ast \phi^{(1)})(x, y) = e^{-\frac{i}{2} \frac{B}{m^2r_1r_2} (\frac{\partial}{\partial x_2} \frac{\partial}{\partial y_1} - \frac{\partial}{\partial x_1} \frac{\partial}{\partial y_2})} \phi^{(2)}(x_2, y_2) \phi^{(1)}(x_1, y_1) \bigg|_{x_2=x_1=x, y_2=y_1=y}.$$  \hspace{1cm} (2.19)

Let us recapitulate what we have obtained so far. We consider one 0-brane on a $T^2$, $N_0 = 1$, and no membranes, $N_2 = 0$. The flux of the $B_{ij}$-field through the torus is $B$. We consider the limit coming from matrix theory. If $B = 0$ the resulting theory is a 2+1 dim. SYM on a dual $T^2$. In terms of the M-theory variables the radii of the $T^2$ are

$$r_1' = \frac{1}{m^2r_1} = \frac{1}{M^3 R R_1}$$
$$r_2' = \frac{1}{m^2r_2} = \frac{1}{M^3 R R_2}$$

and the gauge coupling is

$$\frac{1}{g^2} = \frac{m_s r_1 r_2}{\lambda} = \frac{R_1 R_2}{R}.$$  \hspace{1cm} (2.20)

The gauge bundle on $T^2$ is trivial. This is a consequence of the fact that all the fields in eq.(2.17) are functions on $T^2$ instead of sections of a non-trivial bundle. Equivalently $c_1 = \frac{1}{2\pi} \int tr F = 0$. For any $B$ the only difference is that every time two fields are being multiplied in the action one should instead use the $*$-product.

When $B = 0$ the $*$-product, of course, coincides with the usual product. Looking at eq.(2.14) we see that the product has a periodicity in $B$ of $4\pi$. For $B = 2\pi$ it is different from $B = 0$. At first sight this is problematic, since the theory is known to have a periodicity in $B$ of $2\pi$. The puzzle is resolved by noting that there is a field redefinition which takes the theory at $B = 2\pi$ into $B = 0$. The field redefinition is

$$\phi_{a_2b_2,a_1b_1} \rightarrow (-1)^{(a_2-a_1)(b_2-b_1)} \phi_{a_2b_2,a_1b_1}.$$  \hspace{1cm} (2.22)
Thus the gauge theories actually have the correct $2\pi$ periodicity in $B$.

So far we have only discussed the case with $N_0 = 1$ and $N_2 = 0$, i.e. one 0-brane and no 2-branes. The case with any $N_0$ goes in exactly the same way. It is now a $U(N_0)$ theory instead of a $U(1)$ theory. Nothing else is changed. Especially the same form of the $\ast$-product should be used, except that now the fields are $N_0 \times N_0$ matrices.

3. Non-trivial gauge bundles

In the previous section we only considered cases with no membranes, $N_2 = 0$. What about $N_2 \neq 0$? In the case $B = 0$ we know the answer. Here the final theory is obtained by T-duality on both circles. After T-duality we get the decoupled theory of $N_0$ D2-branes with $N_2$ D0-branes dispersed in the 2-branes. 0-branes in 2-branes is just magnetic flux. In other words now it is a $U(N_0)$ theory with a non-trivial bundle on $T^2$. The first Chern class is $c_1 = \frac{1}{2\pi} \int tr F = N_2$. In the previous section we saw that for $B \neq 0$ and $N_2 = 0$ the theory became a $U(N_0)$ theory with $c_1 = 0$ and deformed by the $\ast$-product. All these theories have radii and coupling given by eq.(2.20). The obvious guess is now that the case with $B \neq 0$ and $N_2 \neq 0$ was described by a $U(N_0)$ theory with $c_1 = N_2$ and an action deformed by the $\ast$-product. However this cannot be true, at least not in this naive sense. The reason is that if the bundle is non-trivial we really need to define the fields in coordinate patches. The $\ast$-product does not transform correctly under change of patch. To make it do so we would have to replace $\frac{\partial}{\partial x}$, $\frac{\partial}{\partial y}$ by covariant derivatives. Even if this is possible there are other reasons to doubt that this is correct. Firstly $B \rightarrow B + 2\pi$ is not a symmetry in the presence of 2-branes. It is only a symmetry if one changes the number of 0-branes: $N_0 \rightarrow N_0 - N_2$. For $N_2 = 1$ one could always change to $N_0 = 0$. If the above guess was correct this would lead to strange connections between “$U(0)$” and $U(N)$ theories.

Another reason to doubt this naive guess is the following. When one uses Sen’s and Seiberg’s prescription [34] to derive a matrix model the energy has the form

$$E = \sqrt{(\frac{N}{R})^2 + P_\perp^2 + m^2} = \frac{N}{R} + \frac{1}{2} \frac{R}{N}(P_\perp^2 + m^2) + \ldots . \quad (3.1)$$

Here it is written for uncompactified M-theory, but a similar expression is valid for all compactifications. The point is that in the limit $R \rightarrow 0$, the first term goes to infinity, the second term stays finite (after rescaling of all energies) and the terms indicated by dots vanish. The first term goes to infinity but is fixed and independent of any dynamics. Therefore we can ignore it and just keep the second term. A matrix theory Hamiltonian
always gives the second term. When we change N the theory changes drastically. For instance the gauge group changes. In other words when the infinite term is changed we expect the finite term to change drastically. Let us now look at our situation. With $N_0$ 0-branes, $N_2$ 2-branes and a $B_{ij}$-field flux $B$. Here the energy takes the form

$$E = \frac{N_0 + BN_2}{R} + \text{finite}. \quad (3.2)$$

For $B \neq 0$ the infinite term changes when $N_2$ is changed. So following the remarks above we expect the theory to change drastically. Specifically it is probably not enough to change the bundle, but also radii and gauge coupling changes.

Whether or not the case of $N_2 \neq 0$ is solved by just changing the first Chern class, there is another way of solving it. This is the subject of next section.

4. Incorporating 2-branes

In this section we will obtain the matrix model for the general case, with $N_0$ 0-branes, $N_2$ 2-branes and a flux $B$. We will do that by performing a T-duality to transform to the case $N_2 = 0$.

For a review of T-duality, see [15]. The T-duality group for IIA on $T^2$ contains an $SL(2, \mathbb{Z})$ subgroup which acts as follows. It leaves the complex structure of $T^2$ invariant. Define a complex number in the upper halfplane, $\rho = B + iV$. Here $V$ is the volume of the torus measured in string units and $B$ is the flux of $B_{ij}$ through the torus. In our case $V = m^2 s r_1 r_2$. An element

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z})$$

acts as follows

$$\rho' = \frac{a \rho + b}{c \rho + d},$$

$$\begin{pmatrix} N_0' \\ -N_2' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} N_0 \\ -N_2 \end{pmatrix}. \quad (4.1)$$

Let us use this in our case. Let $Q$ be the greatest common divisor of $N_0$ and $N_2$. Write

$$N_2 = Q \tilde{N}_2,$$

$$N_0 = Q \tilde{N}_0. \quad (4.2)$$

Since $\tilde{N}_0, \tilde{N}_2$ are relatively prime we can choose $a, b$ such that $a \tilde{N}_0 - b \tilde{N}_2 = 1$. Let us now perform a T-duality transformation with the matrix

$$\begin{pmatrix} a & b \\ \tilde{N}_2 & \tilde{N}_0 \end{pmatrix}. \quad (4.3)$$
An easy calculation gives the new radii, $B_{ij}$ flux, 0-brane and 2-brane numbers

\[
\begin{align*}
    r_1' &= \frac{r_1}{\tilde{N}_0 + \tilde{N}_2 B} \\
    r_2' &= \frac{r_2}{\tilde{N}_0 + \tilde{N}_2 B} \\
    B' &= \frac{aB + b}{\tilde{N}_0 + \tilde{N}_2 B} \\
    N_0' &= Q \\
    N_2' &= 0.
\end{align*}
\]

(4.4)

The string mass is invariant

\[
m_s' = m_s
\]

(4.5)

and the new coupling is

\[
\lambda' = \frac{\lambda}{\tilde{N}_0 + \tilde{N}_2 B'}.
\]

(4.6)

In these formulas we have taken the matrix theory limit in the quantities which have a non-zero limit. We remark that the denominator $\tilde{N}_0 + \tilde{N}_2 B$ is positive since

\[
P_- = \frac{\tilde{N}_0 + \tilde{N}_2 B}{R}
\]

(4.7)

and $P_-$ is positive as always in matrix theory. We now see that the parameters of the theory go to zero and infinity in exactly the same way as in last section. This means that we are in exactly the same situation as in last section.

In other words the matrix theory is a 2+1 dim. SYM on a $T^2$ with gauge group $U(Q)$ where $Q$ is the greatest common divisor of $N_0$ and $N_2$. The action is deformed with the $*$-product with a value of $B$ equal to

\[
B' = \frac{aB + b}{\tilde{N}_0 + \tilde{N}_2 B'}.
\]

(4.8)

The $T^2$ has radii

\[
\begin{align*}
    r_1'' &= \frac{1}{m_s'^2 r_1} = \frac{\tilde{N}_0 + \tilde{N}_2 B}{M^3 RR_1} \\
    r_2'' &= \frac{1}{m_s'^2 r_2} = \frac{\tilde{N}_0 + \tilde{N}_2 B}{M^3 RR_2}
\end{align*}
\]

(4.9)

and the gauge coupling is

\[
\frac{1}{g^2} = \frac{m_s' r_1'' r_2''}{\lambda'} = \frac{R_1 R_2}{R(\tilde{N}_0 + \tilde{N}_2 B)}.
\]

(4.10)

The SL(2,Z) duality employed has a very easy geometric interpretation if one performs a T-duality on a single circle as in [2]. Here $N_0, N_2$ parametrize which homology cycle the D-strings wrap. The factor $\tilde{N}_0 + \tilde{N}_2 B$ is just the length of the D-string. The T-duality transformation is just a geometric change of $\tau$-parameter of the torus.
5. Conclusion

It was explained in [8,9] how to describe 0-branes on $T^2$ by working on the covering space $\mathbb{R}^2$ and modding out by translations. We did this in the presence of a background B-field. This enabled us to get a matrix theory of M-theory on $T^2$ with a background $C_{-12}$. The result agrees with [1,2] and is a gauge theory on a noncommutative torus.

There are some interesting aspects of this. In the case $B = 0$ this procedure leads to a 2+1 dim SYM which is exactly the same as the theory on the D2-brane in the T-dual picture. In other words the procedure of compactifying the 0-branes agrees with T-duality. For $B \neq 0$ this is not the case. T-duality does not give a theory on a finite torus when $B \neq 0$. This is the whole reason for all this interest in $B \neq 0$. This means that working with 0-branes on the covering space is not the same as T-duality. We believe, of course, that T-duality still is true. The point is just that the T-dual description is not simpler. The T-dual description is the theory on D2-branes wrapped on a dual $T^2$ which is again shrinking. To extract a well defined action out of that one has to expand the full Born-Infeld action as advocated in [13]. It would indeed be interesting to use the noncommutative theory to put constraints on the full Born-Infeld action. All the higher derivative terms should come out of this.

Originally it was thought that compactifications of M-theory could be gotten by compactifying the 0-brane quantum mechanics. That was indeed the case for toroidal compactifications up to $T^3$. For other compactifications certain degrees of freedom were missing. It was later realized that the correct way of obtaining the matrix model was to use string dualities in order to realise the theory as a theory living on a brane decoupled from gravity. In the case of $C_{-12} \neq 0$ we are in some sense back to the first philosophy. We can obtain the final theory starting with the 0-brane theory but we do not know how to realise it as a sensible limit of a theory on a brane.

It is an interesting question whether these new theories make sense as renormalizable quantum theories. In the case of $B = 0$ we know that the procedure of putting 0-branes on the covering space gives a renormalizable theory up to $T^3$ and not for higher tori. So certainly arguing that this procedure should give a well defined theory is wrong. However, one might hope that the question of renormalisability is related to the “number of degrees of freedom”. In that sense the theory on $T^d$ with $B \neq 0$ behaves like the theory on $T^d$ with $B = 0$ and we might expect that the noncommutative theories are well defined up to $T^3$. Realizing these theories as theories on branes would resolve this issue, but as discussed above this might require knowing the full Born-Infeld action.

It will be very interesting to see what the methods of noncommutative geometry can teach us about string theory and the other way round.
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