A quantum stochastic equivalence leads to a retrocausal model for measurement consistent with macroscopic realism

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While classical retrocausal dynamics can account for violation of Bell inequalities, such models are not generally thought to be equivalent to quantum mechanics. If retrocausal mechanisms exist, it remains unclear why retrocausality does not manifest at a macroscopic level. It is also unclear whether macroscopic realism, a requirement for macroscopic causality, holds. In this paper, we show how retrocausality arises naturally from within quantum mechanics, and explains quantum measurement consistently with macroscopic realism. We analyze a measurement \( \hat{x} \) on a system prepared in a superposition of eigenstates \( |x_j\rangle \) where the measurement is modeled by amplification. By deriving a path-integral theorem, we prove an equivalence between a quantum probability distribution \( Q(x, p, t) \) and simultaneous back-in-time and forward-in-time stochastic equations for amplitudes \( x(t) \) and \( p(t) \), respectively. The backward and forward trajectories are linked at the initial-time boundary. A Deutch-like ‘causal consistency’ and Born’s rule emerge naturally. A feature is the vacuum noise associated with the eigenstate. Unlike the eigenvalue, this noise is not amplified and is not measurable, the precise fluctuations originating from past and future boundary conditions. We find consistency with macroscopic realism: For macroscopic superpositions, the macroscopic outcome of the measurement \( \hat{x} \) is considered determined prior to the onset of the measurement. This leads to hybrid macro-causal and micro-retrocausal relations, and other models of realism. Our results support that the ‘collapse’ of the wave function occurs with amplification: The distribution \( Q_j(x, p, 0) \) for the initial ‘state’ postselected on the outcome \( x \), is not a quantum state but approaches the eigenstate \( |x_j\rangle \) for a macroscopic superposition. The full irreversible collapse is simulated by coupling to a meter. We discuss Einstein-Podolsky-Rosen and Bell correlations.

I. INTRODUCTION

Following from Bohr [1] who considered delayed-choice experiments [2] in quantum mechanics, Wheeler speculated that retrocausality due to future boundary conditions may explain quantum paradoxes [3, 4]. Retrocausal dynamics was studied in classical physics even earlier, especially by Dirac, Feynman and Wheeler in their work on electrodynamics [5, 6]. Bell later proved that all local causal theories could be falsified, if quantum predictions allowing a violation of Bell inequalities were correct [8, 10]. Violations of Bell inequalities can arise from classical fields, however, using retrocausal, i.e. advanced, solutions from absorber theory [11, 14]. Bell’s work motivated the question of whether superluminal disturbances, or communications, were possible, leading to no-signaling theorems [15]. These studies have inspired many analyses of retrocausality in quantum physics [2, 16–40].

Such questions are closely related to the measurement problem: For a system in a superposition of eigenstates of the measurement operator, the state after the measurement ‘collapses’ to just one of these states. Is there a theory to describe how this occurs? Different resolutions have been proposed, including theories which allow non-locality [47]; many worlds [48]; and decoherence collapse mechanisms additional to quantum mechanics [49, 50]. One can also suggest a purely statistical interpretation, but this leads to Born’s question of “what is the reality which our theory has been invented to describe?” [51], and related objections by Bell [52].

More recently, this work has motivated research into the causal structure of quantum mechanics [53, 74]. It is possible to account for Bell violations using classical causal models, but this entails solutions based on, for instance, superluminal causal influences [47, 75–77], retrocausality [11, 24, 42], or superdeterminism [78]. A criticism is that these mechanisms require a fine-tuning of causal parameters to explain an observed statistical independence of variables [55–57] – an exception to this criticism being retrocausal effects that induce cyclic causal loops. Causal structure has been used to better understand quantum networks [79], paradoxes [34, 45, 59, 81] and quantum information and computation [82, 83], while hidden causal loops may explain quantum speed-up [86, 88]. Regardless, the mystery remains that retrocausality is not observed at any practical, or macroscopic, level. Quantum causal models have been developed to circumvent these problems [58, 64]. It is not clear however whether one can present a unified framework combining causality and realism, and at what level (from microscopic to macroscopic) causal concepts hold.

In this paper, we analyze quantum measurement using an objective field model motivated by phase space dynamics based on the \( Q \) function [50, 91]. We derive a path-integral theorem linking the evolution of the quantum phase-space distribution function \( Q \) to the stochastic dynamics of real amplitudes involving both forward and backward trajectories in time. The model provides an explanation of quantum measurement based on retrocausality due to future boundary conditions – there is no sudden “collapse” of the wave-function. The model allows for cyclic causation, and is therefore immune to
the fine-tuning arguments in which directed acyclic graph theory excludes retrocausality as a solution \cite{55,57}. In the treatment, the forward-backward stochastic behavior is derived mathematically from within quantum mechanics, emerging as a result of microscopic noise sources generated by interactions. This distinguishes the theory from previous analyses involving toy models \cite{24,92}, and allows an analysis of causal structure and models of realism, given in terms of the dynamics of the amplitudes. The analysis reveals consistency between the forward and backward propagation, and the validity of macroscopic realism and causality.

We consider a single-mode field, for which $Q$ is the Husimi $Q$ function $Q(x,p)$, and solve for the trajectories of $x$ and $p$, for a measurement process. Following earlier work \cite{89}, the measurement of a field quadrature $\hat{x}$ is modeled by a parametric amplification interaction $H_A$, which amplifies the $\hat{x}$ observable. The amplified variable $x$ corresponds to a backward propagating trajectory, and is sampled according to a future boundary condition determined by the measurement setting. The variable $p$ is attenuated and corresponds to a forward-propagating trajectory (Figure 1). A distinctive feature of the objective field model is the existence of vacuum noise $\xi(t)$, of order $\hbar$, associated with eigenstates $\{x_j\}$ of $\hat{x}$.

In order to analyze the measurement problem, we consider a system initially in a superposition $|\psi\rangle$ of eigenstates $|x_j\rangle$. We solve for the dynamics under $H_A$, which gives an explicit physical model of the meter. In the simulations, the amplified trajectories for $x$ converge to the final measurement outcome $x_j$, with a probability density given by Born’s rule. The vacuum noise $\xi(t)$ however is not amplified. The precise values $\xi_x(t)$ for the trajectory of $x$ are determined retrocausally, from future boundary conditions (Figure 1). We also treat momentum measurements, showing the expected complementary behavior.

The stochastic approach allows new results. We prove that the $x$ and $p$ trajectories are connected according to an initial-time boundary value, in a way not previously calculable from quantum mechanics. A conditional distribution $P_Q(p|x)$ defined at the initial time $t=0$ (just prior to the measurement) dictates the nature of the coupling. For a system prepared in a superposition of eigenstates $|x_j\rangle$ at time $t=0$, the $x$ and $p$ trajectories are linked: a backward propagating trajectory for $x$ couples to a given set of $p$ trajectories. If prepared in a mixture of eigenstates $|x_j\rangle$, the $x$ and $p$ trajectories are uncorrelated. The joint probability density for the linked $x$ and $p$ trajectories can be computed, for a given time $t$ during the measurement dynamics, and confirmed to correspond to the correct quantum $Q$ function, $Q(x,p,t)$. Hence, the probability density $Q$ obtained from the combination of the forward (causal) and backward (retrocausal) trajectories depends on the time $t$ from the initial preparation time $t=0$, and is hence causal, being independent of the time $t_f$ of the future boundary condition. Thus, ‘causal consistency’ is not imposed, as in analyses of closed time-like curves \cite{93}, but arises naturally in the theory. This is reminiscent of similar results in absorber theory \cite{6,7,14}, and is linked to the observation that the vacuum noise $\xi(t)$ is not amplifiable, being at a constant level throughout the measurement dynamics.

The results motivate consideration of different models of realism that in their interpretation extend beyond quantum mechanics. On postselecting on a given final measurement outcome $x_j$, the correlated trajectories for $x$ and $p$ define a unique distribution $Q_j(x,p,0)$ at the time $t=0$, which we demonstrate cannot correspond to a quantum state - the amplitudes $x$ and $p$ are defined more precisely than permitted by the uncertainty principle. After correcting for growth due to dynamics, the inferred density of the measured amplitudes for $x_j$ is shown to correspond to the probability $P_B(x_j) = |\langle x_j | \psi \rangle|^2$ of
detecting the result $x_j$, as predicted by Born’s rule for quantum mechanics. This motivates a statistical interpretation – objective field realism (OFR) – that there is an underlying reality in which amplitudes $x$ and $p$ correspond to a ‘state’ at time $t$ for the system, with a probability $Q(x, p, t)$. The model is considered objective, in the same way that Wheeler-Feynman’s absorber theory, based on classical electrodynamics, is an objective theory, with fields assumed to exist independent of any observer. The amplitudes $x$ and $p$ however cannot be directly measured. In the OFR interpretation, this is not due to any lack of reality, but is caused by the fact that the system and measuring instruments have vacuum noise, and interact with the system being measured. While the part of the objective field that represents the eigenvalue $x_j$ for the eigenstate $|x_j\rangle$ is amplified, the stochastic noise $\xi_x(t)$ that contributes to the value of $x$ by an amount of order $\hbar$ is not, being determined by a future boundary condition.

We are motivated to consider other models, with the concrete definition of realism closer to the macroscopic realism understood by an observer. In particular, we propose a macroscopic realism (MR) model which attributes to the system prepared in a macroscopic superposition

$$|\psi_{\text{sup}}\rangle = (|x_1\rangle + |x_2\rangle)/\sqrt{2}$$

the property of macroscopic realism. Here $|x_1 - x_2| \to \infty$. Macroscopic realism asserts that at the time $t = 0$, just prior to the measurement $\hat{x}$, the system has a definite predetermined value for the amplified outcome of $\hat{x}$ (i.e. allowing the inference of either $x_1$ or $x_2$). The definition is different to that introduced in some earlier works, where the system is assumed to be in one of two macroscopically distinct but well-defined states (e.g. $|x_1\rangle$ or $|x_2\rangle$) [91]. The MR model is supported by the solutions of this paper. The causal structure is determined from the tracking of the trajectories of $x$ and $p$ (Figure 2). The model has a hybrid form: given as causal macroscopically, but retrocausal microscopically. We explain how this model does not satisfy Bell’s local hidden variable assumptions, and is not in conflict with the violation of Bell [8] or Leggett-Garg inequalities [92, 93], including those reported at a macroscopic level [95, 97, 102]. Essentially, this is because here macroscopic realism is specified for the state at the single time $t = 0$, just prior to amplification but after the unitary dynamics associated with the choice of measurement setting.

The Schrödinger cat paradox [103] is analyzed by simulating the trajectories for $|\psi_{\text{sup}}\rangle$ and for the superposition $|\text{cat}\rangle$ of two coherent states. The paradox is explained in the hybrid macroscopic-realism model not as a failure of macroscopic realism: Rather, the paradox arises because the function $Q_j(x, p, 0)$ inferred for $x$ and $p$ at the initial time $t = 0$, postselected on the final outcome $x_j$, cannot describe a quantum state. We show however that $Q_j(x, p, 0)$ approaches the $Q$ function of the eigenstate $|x_j\rangle$, as $|x_1 - x_2| \to \infty$. Hence, the hybrid MR model explains the “collapse” of the wave function as arising from amplification. For $|\psi_{\text{sup}}\rangle$ where $|x_1 - x_2|$ is small, the system evolves into a macroscopic superposition under the measurement $H_A$ (Figure 1).

A third model for realism, which we refer to as deterministic contextual realism (DCR), follows closely from the macroscopic-realism model. This is that the system prepared in a superposition of eigenstates $|x_j\rangle$ has a predetermined value for the (amplified) eigenvalue $x_j$ at the time $t_0$, once the measurement setting has been specified. Such models have been examined elsewhere (e.g. [100, 101], and shown not to conflict with Bell violations, since the choice of measurement setting involves unitary dynamics. This implies that the microscopic system of Figure 1 is at time $t > 0$ in a state with a definite final outcome $x_j e^{\alpha t}$ regardless of the value of $|x_1 - x_2|$.
The measurement by amplification $H_A$ is a special case of this example, since for $t > 0$, the measurement setting is specified. A deterministic causal relation $x_j \rightarrow Gx_j$ follows, along with a microscopic retrocausal relation. The simulations of this paper are consistent with this model.

Finally, we note that amplification due to $H_A$ alone cannot explain a full “collapse”, since any amount of amplification by $H_A$ can be reversed. The irreversibility of the “collapse” can be explained by a coupling to a second mode. We simulate the trajectories for the measurement made on a correlated entangled pair of modes in the state $|\psi_{ent}\rangle = (|\beta\rangle|x_1\rangle + |\beta\rangle|x_2\rangle)$ $\sqrt{2}$. The second mode acts as a meter which measures the amplitude $\hat{x}$ of the first mode; $|\pm \beta\rangle$ are coherent states for the meter, where $\beta \rightarrow \infty$. We evaluate the inferred state of the signal mode, postselected on the final outcome $x_j^\beta$ of the meter. The evaluation involves calculation of the marginal for the signal mode from the two-mode $Q$ function, which implies loss of information. We find that the inferred state is precisely the eigenstate $|x_j\rangle$.

The layout of the paper is as follows. In Section II, we outline our approach to the measurement problem and summarize the different models for realism relevant to this paper. In Section III, we give details of the stochastic method and of the path-integral Theorem. In Section IV, we solve the equations modeling a measurement $\hat{x}$ on the system prepared in a superposition of eigenstates of $\hat{x}$, using the amplification model $H_A$. The results are generalized to superpositions of coherent states, and to the measurement of the complementary observable $\hat{p}$. The realization of Born’s rule is explained. The coupling of the $x$ and $p$ trajectories is calculated in Section V. We give details of the realism models, the causal relations and the cat paradox in Section VI. The simulation of the measurement on the system entangled with a meter is presented in Section VII. In Section VIII, we discuss the possibility of EPR entanglement and the violation of Bell inequalities. A conclusion is given in Section IX.

II. MEASUREMENT PROBLEM AND MODELS OF REALISM

A. Measurement problem and summary of approach

In this paper, the system dynamics is described by a unified stochastic model motivated by the $Q$ function $[105]$. We will investigate the interpretation of the measurement postulate for this stochastic model, which we refer to as the objective field model $[59]$. The examples of this paper use bosonic fields. The approach requires the measurement process to be included in the dynamics. For this purpose, we use a model of a common measuring device that produces macroscopic outputs, namely the parametric amplifier $[106]$. These are not fundamental physical restrictions. Fermionic $Q$-functions exist $[107] [108]$, and have similar dynamical properties and equations. We expect other models of meters to have analogous behavior.

The measurement problem addresses a quantum state expressed as a linear combination of eigenstates $|x_j\rangle$ of $\hat{x}$

$$|\psi_{sup}\rangle = \sum_j c_j|x_j\rangle$$

where $c_j$ are probability amplitudes. The Copenhagen measurement postulate asserts that for such a state, the set of possible outcomes of the measurement $\hat{x}$ is the set $\{x_j\}$ of eigenvalues of $\hat{x}$. The probability of a given outcome $x_j$ is $P_j = |c_j|^2$. After the measurement, the system “collapses” into the eigenstate $|x_j\rangle$ associated with that outcome $x_j$. The measurement problem is to understand the transition from the state $|\psi\rangle$ to the final state $|x_j\rangle$. To treat physical states, we replace the eigenstates $|x_j\rangle$ with squeezed states $|x_j, r\rangle$ that have a finite precision, so that eigenstates are obtained for $r \rightarrow \infty$. The squeezed states are defined precisely, in Section IV. In the approach of this paper, the quantum state is epistemological, and no physical collapse occurs.

Firstly, we outline different models for realism associated with the objective field model, and explain the links with other approaches. To do this, we summarize the

Figure 3. The analytic $Q$ function for the superposition $|\psi_{sup}\rangle$ of two eigenstates of $\hat{x}$, with $r = 2$. In fact we show $[4.15]$ for $r = 2$ and using $x_2 = -x_1$. The upper plot is for $|x_1 - x_2| = 4 \gg 1$. The lower plot is for $|x_1 - x_2| = 1$, so the $x$ values are almost the same, but there are strong interference fringes in the complementary variable $p$. 

essential results of our approach. Restricting our analysis to a single mode for clarity, we define the $\hat{x}$ and $\hat{p}$ observables
\begin{equation}
\hat{x} = \hat{a} + \hat{a}^\dagger \\
\hat{p} = (\hat{a} - \hat{a}^\dagger)/i
\end{equation}
where $\hat{a}$ is the boson destruction operator \[114\]. The coherent state $|\alpha\rangle$ satisfies $\hat{a}|\alpha\rangle = \alpha |\alpha\rangle$ where $\alpha = (x + \imath p)/2$. The single-mode $Q$ function $Q(x, p) = \frac{1}{\pi} |\langle \alpha |\psi\rangle|^2$ defines the quantum state $|\psi\rangle$ uniquely as a positive probability distribution \[105\].

Without losing essential features, we consider $|\psi_{sup}\rangle$ with two states $|\pm x_1, r\rangle$, where $r$ indicates the level of squeezing, and let $c_1$ be real and $c_2 = i|c_2|$. This gives
\begin{align}
Q_{sup}(x, p) &= \frac{e^{-p^2/2\sigma_p^2}}{2\pi\sigma_x\sigma_p} \left\{ |c_1|^2 e^{-(x-x_1)^2/2\sigma_x^2} \\
&+ |c_2|^2 e^{-(x+x_1)^2/2\sigma_x^2} \\
&- 2c_1c_2|e^{-(x^2+z_1^2)/2}\sin(px_1/\sigma_x^2) \right\},
\end{align}
where $\sigma_x^2 = 1 + e^{-2r}$, and $\sigma_p^2 = 1 + e^{-2r}$. In the limit of $r \to \infty$ where the system becomes the superposition $|\psi_{sup}\rangle$ of two eigenstates, the $Q$ function has two Gaussian peaks with fixed variance $\sigma_x^2 = 1$, centered at the eigenvalues $x_1$ and $x_2$, along with a central peak centered at $\hat{x} = 0$ whose amplitude has a fringe pattern which is damped by a term $e^{-x^2/2}$. Plots are shown in Figure 3.

We will take a simple model of measurement using a parametric amplifier in a rotating frame, as in current low noise photonic or superconducting experimental measurements. To measure $\hat{x}$, the system prepared in $|\psi_{sup}\rangle$ is amplified according to a unitary interaction \[106\]
\begin{equation}
H_A = \frac{i\hbar g}{2} [\hat{a}^2 - \hat{a}^{\dagger 2}]
\end{equation}
where $g > 0$ is real. The system evolves according to $H_A$ for a time $t_f$, to give the final measurement output. The measurement is completed simply by the fields being amplified to become macroscopic, and hence detectable to macroscopic devices or observers. No special “collapse” is required, only normal physical processes.

It is shown in this paper that the following results hold:

1. The dynamics is modeled by a set of stochastic trajectories $x(t)$ that have boundaries in the future, and $p(t)$ with boundaries in the past.

2. The $Q$ function and hence the quantum moments are equivalently evaluated by averaging over the trajectories.

3. The density of the trajectories $x(t_f)$ at the time $t_f \to \infty$ is proportional to the probability for an outcome $x$ of $\hat{x}$.

We will examine the trajectories for both $x$ and $p$ as the measurement of $\hat{x}$ evolves (Figure 1). The $Q$ function evolves according to $H_A$ to give a final distribution: The probability for observing $x$ at the final time $t_f$ is
\begin{equation}
P(x, t_f) \to |c_1|^2 e^{-(x-Gx_1)^2/2} + |c_2|^2 e^{-(x-Gx_2)^2/2}
\end{equation}
where $G = e^{-|g|t}$. This gives two sharp peaks at $Gx_1$ and $Gx_2$ respectively, with relative probabilities $|c_1|^2$ and $|c_2|^2$. The values $x_j$ which correspond to the means of the Gaussians in \[2.3\] are amplified according to
\begin{equation}
x_j \to X_j = e^{\imath t_f} x_j
\end{equation}
and are therefore ultimately observed; see the top plot of Figure 1 and Figure 2. The means of the Gaussians correspond to the eigenvalues $x_j$, which are the results of the measurement. In the amplification process $H_A$, the interference term of \[2.3\] is attenuated because of the proportionality to terms $e^{-x^2}$, which decay on amplification of $x_j$ where $x_j \to e^{\imath t_f} x_j$. The measurement of $\hat{x}$ amplifies the $x_j$ variables, but attenuates the $p$ variables to the minimum possible level of the vacuum (the lower plot of Figure 1). The complementary observable $p$ therefore does not appear in the final measured probability. The inferred $x_j$ at the time $t_f$ therefore have a final probability distribution
\begin{equation}
P(x) = P(x, t_f) = |\langle x_j |\psi\rangle|^2,\end{equation}
in agreement with the quantum prediction.

While the eigenvalues $x_j$ are amplified, the noise about these values is not amplified. Hence, the vacuum fluctuation $x_\delta = x - x_j$ is not measurable, and does not lead to a macroscopic value. The values for these trajectories are determined by a future boundary condition \[143, 89\], implying retrocausality as shown in the top plot of Figure 1 and Figure 2.

It is also necessary to consider the measurement of the complementary observable $\hat{p}$ (Figure 2). The $\hat{p}$ measurement takes place by amplifying $p_j$, using $H_A$ with $g = 0$. We will show that for $|\psi_{sup}\rangle$ the probability for observing $p$ at the time $t_f$ after evolution is
\begin{equation}
P(p, t_f) \to 1 - \sin(|p|e^{-|g|t}(x_1 - x_2)),
\end{equation}
which gives a fringe pattern in the amplified variable $Gp$ (top plot of Figure 1). This is in agreement with the quantum prediction for the measurement $\hat{p}$,
\begin{equation}
P(p_j) = P(p_j, t_f) = |\langle p_j |\psi_{sup}\rangle|^2,\end{equation}
as it is inferred after taking into account the amplification by the measurement process. Here, $|p_j\rangle$ is the eigenstate for $\hat{p}$. The details of these calculations are given in the later sections.

\begin{figure}[h]
\caption{The two models for realism}
\end{figure}

As explained in Section VIII, the model presented in this paper cannot be interpreted as a Bell local hidden
variable theory. This is because of the boundary conditions imposed in the future in our implementation of the meter trajectory interpretation. The model therefore is a contextual model in terms of the outcomes for measurements of $\hat{x}$ and $\hat{p}$. This motivates us to consider the compatibility with different precisely-defined models for realism, which we examine below.

1. Objective field model

The first model specifies hidden variables, allowing for a broader conception of realism [89, 90]. In the present case, the joint density of values for $x$ and $p$ at the time $t$ corresponds to the $Q(x, p, t)$ function. This gives the correct result for the quantum prediction $P(x)$ for a measurement of $\hat{x}$, after including the dynamics of the measurement process which amplifies $\hat{x}$. The interpretation of this model for realism is that the physical system at any time $t_0$ is in an objective field state specified by the values $x$ and $p$ with probability $Q(x, p, t_0)$. The $Q$ function is a joint probability distribution for the objective fields $x$ and $p$, but $x$ and $p$ do not directly correspond to the measurable outcomes until one adds a model for the meter itself. This is because the microscopic values are different to the macroscopic meter outputs. The macroscopic outputs are the values recorded by the observer. The single-mode $Q$ function gives a probability for amplitudes $x$ and $p$ at a given time. These amplitudes determine the future measurement outcomes, subject to boundary conditions in the past and future, and on the measurement setting or Hamiltonian, that is, whether $\hat{x}$ or $\hat{p}$ is measured.

In this model, the amplitudes $x$ and $p$ provide an objective realistic picture. Measurement has no special treatment, but it must involve a physical amplification by a meter that has a macroscopic output in order to be regarded as a measurement. This leads to contextuality, as the meter interacts with the quantum system.

We term our model retrocausal, since the equations of motion depend in part on future boundary values, not just on past boundary values as in Bell’s model [8]. Our definition is similar to Dirac’s theory of radiation reaction [110]. This also has future boundary values, which Dirac described as follows: “We now have a striking departure from the usual ideas of mechanics. We must obtain solutions of our equations of motion for which the initial position and velocity of the electron are prescribed, together with its final acceleration, instead of solutions with all the initial conditions prescribed.”

Similarly, we call the model “objective”, in the same way that Dirac’s theory, or Wheeler-Feynman’s absorber theory, based on classical electrodynamics, is also considered an objective theory. We introduce quantum features through noise sources not present in such earlier classical theories. The simplest interpretation is that these are objective trajectories which are time-symmetric. Such an objective model, like the fundamental physical laws, has no preference for past or future. It makes no distinction between measurement or any other physical process, and, it is argued, does not need human agency to define reality.

Another way to define realism is via experimental preparation and measurement. We regard such conventional definitions as an epistemological “ladder” [110] to microscopic pictures.

2. Deterministic contextual realism

The second model (DCR) adopts the stronger, more conventional definition of realism: that the outcome for a future measurement $\hat{x}$ on the state given by $|\psi_{sup}\rangle$ or its $Q$ function at time $t_0$ is determined at the time $t_0$. The outcome can be described by a variable $\lambda$. Here, realism
is linked with causality, that the future is affected by the present and past (not vice versa). In such a model, a given system at time $t_0$ is in a state with a definite value $\lambda$ for the measurement of $\hat{x}$ (Figure 5). The definite value is one of the values $\{x_j\}$ that are the eigenvalues of the measurement. Hence, the set of $\lambda$ is the set $\{x_j\}$. The model does not imply that the state is the eigenstate $|x_j\rangle$, but allows a different formulation for the state. In such a model, the probability that the system is in a state giving outcome $x_j$ is indeed $|c_j|^2$. We will see that for consistency with the simulations we present, this model is to be combined with microscopic retrocausal-causal relations (see II.B.4 below).

The DCR model needs qualification. In its simplest form, the model would apparently contradict the known contextuality of quantum mechanics [111, 112]. The wave function $|\psi_{sup}\rangle$ can be written as a linear combination of eigenstates of either $\hat{x}$ or $\hat{p}$, or other observables $\hat{O}$. The simplest form of the model would then imply simultaneous specification (at time $t_0$) of the outcomes for $\hat{x}$ and $\hat{p}$ and $\hat{O}$, which can be negated.

The $Q$ function is a positive distribution function of variables $x$ and $p$, but this does not necessarily correspond to a positive joint probability for outcomes $x = x_j$ and $p = p_j$ at time $t_0$. Hence, the DCR model is specified to account for contextuality. The deterministic contextual realism model specifies that the interpretation of realism can be given for the final amplified value $Gx_j$ (or $Gp_j$), but only in a context where the measurement Hamiltonian and setting is specified.

Similar such models have been discussed elsewhere (e.g. 104, 113, 114). A unitary interaction $U(\theta)$ is required as the first stage of the measurement, in order to specify the measurement setting, for instance, as in the selection of a polarizer angle $\theta$ in a Bell-inequality experiment which measures a spin $\hat{S}_\theta$. It can then be argued that the system satisfies deterministic realism for that measurement $\hat{S}_\theta$ after this interaction $U(\theta)$ has occurred. Recent work examines such models in the context of macroscopic Bell tests with cat states, where macroscopic realism can be imposed to strengthen the argument for the validity of realism after the measurement setting is specified [45, 99, 100]. The simulations and solutions of this paper are not inconsistent with the DCR model of realism. The model has also been shown consistent with Bell violations [100, 104, 113, 114].

3. Macroscopic realism model for the pointer measurement

We give a third model for realism linked with macroscopic realism, which builds from the model of deterministic contextual realism in a way that naturally takes into account the importance of the measurement setting. We consider a macroscopic superposition

$$|\psi_m\rangle = (c_1|x_1\rangle + c_2|x_2\rangle)/\sqrt{2} \tag{2.10}$$

Figure 5. Causal relations for the measurement of $\hat{x}$ for the superposition $|\psi_{sup}\rangle$ of eigenstates $|x_j\rangle$ of $\hat{x}$, as given by the trajectories in Figures 1 and 2. The diagram is for measurement by amplification. The arrows imply the direction of cause and effect and the direction from left to right is the direction of time. Here, $X$ is the outcome for the measurement $\hat{x}$. The rectangle represents the system, which is described by the $Q$ function $Q_{sup}(x,p)$ at time $t_0$. The solid blue arrow represents a deterministic causal relation where the eigenvalue $\lambda = x_j$ for $\hat{x}$ becomes the result $G\lambda$ for $X$ on the measurement $\hat{x}$, which gives amplification by a factor $G = e^{\theta i t}$. The red dashed lines represent retrocausal (right to left) and causal (left to right) influences for $x$ and $p$. The $p$, $\delta x$ and $\delta p$ are not measured and are not amplified. In fact, a conditional distribution $P_Q(p|x)$ at the boundary $t_0$ determines whether there is a causal connection between $x$ and $p$ at $t_0$, which connects the trajectories. Depicted is the relationship for the superposition $|\psi_{sup}\rangle$. For the mixture $\rho_{mix}$ of eigenstates $|x_j\rangle$, the $x$ and $p$ are independent at the boundary and the trajectories for $x$ and $p$ completely decouple.
of two eigenstates, the term macroscopic meaning that $|x_1 - x_2| \gg 1$. Here units are such that $\hbar = 1$. The outcome of the measurement $\hat{x}$ is either $x_1$ or $x_2$. The macroscopic realism model postulates that the system in the macroscopic state $|\psi_m\rangle$ can be ascribed a definite value for the final outcome of the amplified measurement, which we call the pointer measurement $|\psi_{qm}\rangle$ [65] [100]. The value of $\lambda$ is then either $x_1$ or $x_2$, indicating the two final possible amplified outcomes. We call this a macroscopic realism (MR) model, after the definition given by Leggett and Garg [95]. The present definition of macroscopic realism is however refined over that presented in [95], to apply specifically to the state at time $t_0$ prior to amplification (after the determination of the measurement setting), and makes no assumptions about the nature of the states corresponding to $\lambda$. This model is macroscopically causal in that the value of $\lambda$ at time $t_0$ determines the measurement outcome at the later time $t_f$.

The measurement process is modeled as a dynamical interaction $H_A$. Once it begins, at some time $t_g$ there is an amplification of the system to give a state such as $|\psi_{sup}\rangle = (c_1|x_1\rangle + c_2|x_2\rangle)/\sqrt{2}$. This is the approach given in the Schrödinger cat paradox [103]. Hence the adoption of the macroscopic realism model is consistent with the adoption of the deterministic contextual realism model, because the amplification of $x$ (as opposed to $p$) amounts to the specification of measurement setting. The solutions given in this paper support the MR model.

4. Hybrid models: micro-retrocausal and macro-causal

The macroscopic realism model implies the outcome of the measurement $\hat{x}$ is specified at time $t_0$ i.e. the causal relation is that the outcome at time $t_f$ depends on the state at $t_0$. A similar causal relation holds for the DCR model. This would seem to counter the claim of this paper of a retrocausal model of measurement. Here, the simulations of the trajectories provide insight. Examining the actual procedure by which the simulations are carried out reveals both macroscopic (deterministic) causal relations consistent with the DCR and MR models, as well as microscopic retrocausal-causal relations. This is depicted in Figure 5.

The trajectories for the amplified variable $x$ propagate backward in time, independently of $p$. This means that the starting point for each trajectory is at the time $t_f$. The initial values $x$ at time $t_f$ for each trajectory are sampled based on the final marginal $P(x,t_f) = \int Q(x,p,t_f)dp$ for $x$, where $Q(x,p,t_f)$ is the evolved $Q$ function under $H_A$. This simple approach is possible because the measurement Hamiltonian $H_A$ allows a decoupling of the amplified and attenuated variables, $x$ and $p$, through the dynamics.

As we will see, the trajectories for $x$ that begin from the amplified value $x_f \sim Gx_j$ at time $t_f$ correspond to the initial Gaussian peak for $x$ with a mean of $x_j$, where $j = 1, 2$ (Figure 2). In fact, this is true regardless of the separation $|x_1 - x_2|$. Hence, we argue that the causal interpretation of the macroscopic realism model is possible: For the superposition state $|\psi_{sup}\rangle = (c_1|x_1\rangle + c_2|x_2\rangle)/\sqrt{2}$, the value for the outcome at $t_f$, whether $\sim Gx_1$ or $\sim Gx_2$, is determined for a given prepared state at the time $t_0$. This is the DCM and MR models. The sampling for the trajectories is consistent with such models. We will see that (based on $P(x,t_f)$ which depends on $Q(x,p,t_0)$) the initial values $Gx_1$ and $Gx_2$ are selected with probabilities $|c_1|^2$ and $|c_2|^2$, identically as for a system that at time $t_0$ is in a statistical mixture of the two states $|x_1\rangle$ and $|x_2\rangle$.

On the other hand, we cannot argue that the selection of the precise value $x_f - Gx_j$ is causal because this value originates at the time $t_f$ in the simulation. The constraint on the selection of this noise term that comes from the earlier time $t_0$ is only that of the variance $\sigma^2_x = 1$ (in units where $\hbar = 1$). We therefore argue that there is retrocausality but at the microscopic level of $h$. The trajectories for the attenuated variable $p$ however are forward propagating and causal (lower plot, Figure 1).

C. Links with other approaches

It is often thought that retrocausal models are unlikely to be correct because retrocausality is not observed in everyday life, at a macroscopic level. Yet irreversibility is known to occur even in classical statistical mechanics at a macroscopic level, despite the reversibility of Newton’s laws. The solutions we give show how retrocausation may exist microscopically, while macroscopic realism and causality hold macroscopically. This is simply due to imperfect experimental control on the future boundary conditions [11].

A second criticism is that classical causal models allowing retrocausation require fine-tuning of the parameters of the causal model in order for the model to be consistent with the observed independences of variables e.g. no-signaling [55]. Such proofs are restricted to acyclic causal models. These do not take into account the micro-macro hybrid nature of the trajectories, and our use of conditional boundaries. In Section VI, we give more details of the causal relations associated with the DCR and MR realism models.

III. OBJECTIVE Q MODEL FOR DYNAMICS

A. Stochastic model for dynamics

We now analyze the measurement using a $Q$ function model. The $Q$ function probability density $Q(\lambda, t)$ for a phase-space coordinate $\lambda$ is defined with respect to a non-orthogonal operator basis $\hat{\Lambda}(\lambda)$ as

$$ Q(\lambda, t) \equiv Tr \left( \hat{\Lambda}(\lambda) \hat{\rho}(t) \right) , \quad (3.1) $$

where $\rho$ is the density operator of the system. As $Q(\lambda, t)$ is normalized to unity, it is necessary to normalize the
basis so it integrates to unity, and the normalization condition is that \( \int \hat{A}(\lambda) \mathcal{d}\lambda = 1 \). The basis satisfies \( \hat{A}^2(\lambda) = \mathcal{N}(\lambda) \hat{A}(\lambda) \), which is different to the condition for projectors that \( \hat{P}^2 = \hat{P} \), because it is a continuous non-orthogonal basis. From the Schrödinger equation, the dynamics of the probability distribution is obtained from the usual equation \( i\hbar \partial_t \hat{\rho} = \big[ \hat{H}, \hat{\rho} \big] \). As a result, one obtains an equation for the \( Q \)-function time-evolution:

\[
\frac{dQ(\lambda, t)}{dt} = \frac{i}{\hbar} \mathcal{T}_\tau \big\{ \big[ \hat{H}, \hat{\Lambda}(\lambda) \big] \hat{\rho}(t) \big\} .
\]

This is equivalent to a zero-trace diffusion equation for the variables \( \lambda \), of form \( \dot{Q}(\lambda, t) = \mathcal{L}(\lambda)Q(\lambda, t) \), where \( \mathcal{L}(\lambda) \) is the differential operator for \( Q \)-function dynamics. While the examples given here are bosonic, such distributions can be extended to include fermions as well \[107, 108, 115\].

We can calculate directly how the \( Q \)-function evolves in time for a Hamiltonian \( H \). To solve for single-mode evolution, we use the bosonic function \[104\]

\[
Q(\alpha) = \frac{1}{4\pi} \langle \alpha | \rho(t) | \alpha \rangle ,
\]

developed with respect to the nonorthogonal basis of coherent states \( |\alpha\rangle \). Here the phase-space coordinates \( \lambda \) are the real coordinates \( x \) and \( p \), given by \( \alpha = (x + ip)/\sqrt{2} \). The \( Q \) function is written as \( Q(x, p) \), and is normalized with respect to integration over the \( x, p \) variables. The moments evaluated from the \( Q \) function distribution are antinormally ordered variables.

We regard the \( Q \) function amplitudes \( \lambda(x \text{ and } p \text{ in this paper}) \) as representing a realization of the superposition state at the time \( t_0 \), immediately prior to the measurement, as well as during and after the measurement. The evolution of the \( Q \) function is given by equation \[32\]. There is an equivalent time-symmetric stochastic action principle for \( Q(\lambda, t) \), leading to probabilistic path integrals \[43\]. These have sample trajectories \( \lambda \) that define the realistic path for all times. The important step is to solve the dynamical \( Q \)-function equations for trajectories of the amplitudes \( \lambda(x \text{ and } p) \) with time as the measurement process evolves. For this purpose, we derive Theorems I and II below, allowing us to determine the stochastic dynamics.

### B. Conditional path integral equivalence theorem

#### 1. Definitions and notation

For \( M \) bosonic modes, the phase-space vector \( \lambda \) is a \( 2M \)-dimensional real vector, and \( p, x \) are \( M \)-dimensional real vectors with \( \Lambda = (x, p) \). The generalized Fokker-Planck equation (GFPE) satisfied by a \( Q \)-function is given by:

\[
\dot{Q}(\lambda, t) = \mathcal{L}(\lambda)Q(\lambda, t) .
\]

The differential operator \( \mathcal{L}(\lambda) \) has both forward and backward components,

\[
\mathcal{L}(\lambda) = \mathcal{L}_p(p) - \mathcal{L}_x(x) ,
\]

where \( \mathcal{L}_p, \mathcal{L}_x \) are defined as:

\[
\mathcal{L}_p = \sum_j \left[ -\frac{\partial_j a_p'(p)}{2} + \frac{1}{2} \frac{\partial_j^2 a_p'(p)}{2} \right] .
\]

Here \( \mathcal{L}_x \) is identical to \( \mathcal{L}_p \) except for the substitution of \( x \) for \( p \), and we define \( \partial_j^2 \equiv \partial_j / \partial p^i \partial_j = \partial / \partial \partial x^i \). For the meter Hamiltonians in this paper, we have \( a_p = a_p(p) \) and \( a_x = a_x(x) \), with diagonal diffusion \( d \).

We define the total action as:

\[
\mathcal{S}_{kN} = \mathcal{S}_{kN}^x + \mathcal{S}_{kN}^y = \sum_{m=0}^{k+1} \mathcal{S}_m .
\]

The “one-step” action for the \( n \)-th step is given by \( \mathcal{S}_n = \mathcal{S}_n^x + \mathcal{S}_n^x \), with \( \mathcal{S}_n^x \) having the same form as \( \mathcal{S}_n^x \) except for the changed superscripts:

\[
\mathcal{S}_n^x = \mathcal{S}_n^x(\lambda_{n-1}, \lambda_n) = \sum_j \left[ \frac{\epsilon}{2d_j} |v_n^j|^2 + \ln \left( \sqrt{N^{ij}} \right) \right] .
\]

The normalization factor is \( N^{ij} = (2\pi \epsilon d_i^2) \), and the relative velocity fields \( v \equiv (v^x, v^y) \) in the action are functions of neighboring coordinates:

\[
v_n^x = \frac{1}{\epsilon} (p_n - p_{n-1}) - a^y(p_{n-1})
\]

\[
v_n^y = \frac{1}{\epsilon} (x_n - x_{n-1}) - a^x(x_{n-1}) .
\]

The two-time propagator \( G \) is then defined as:

\[
G \left( \lambda, t_k \left| \hat{\lambda}, \hat{t} \right. \right) = \lim_{\epsilon \to 0} \int \delta(\lambda - \lambda_k) \delta(\hat{\lambda} - \lambda_{0N}) \mathcal{P}[\hat{\lambda}] d\lambda_k .
\]

Here, \( d\lambda = \prod_{n=0}^{N} d\lambda_n = \prod_{n=0}^{N} \prod_{k=1}^{M} dx_n^k dp_n^k \), \( \lambda_{MN} = (x_N, p_0) \), while \( \hat{\lambda} = (x_f, p_f) \) gives the initial momentum \( p \) and final position \( x_f \) at times \( \hat{t} = (t_i, t_f) = (t_0, t_N) \). The N-step total path probability \( \mathcal{P} \) for a path \( \lambda = (\lambda_0, \ldots \lambda_N) \) at times \( t_k = t_0 + k\epsilon \), with \( k = 0, \ldots, N \), is

\[
\mathcal{P} = e^{-S_{0N}} .
\]

Marginals have the notation \( P(x, t) = \int Q(\lambda, t) dp \), with the variables that are integrated removed from the arguments. Boundary conditions are imposed on the final marginal \( P(x, t_f) \) and initial conditional, \( P(p|x, t_i) = Q(\lambda, t_i) P^{-1}(x, t_i) \).

#### 2. Theorem I

Any path-integral \( Q \)-function \( Q_{pi} \) obtained from multiplication by the joint probabilities at the boundaries satisfies the \( Q \)-function dynamical equations, where:

\[
Q_{pi}(\lambda, t) = \int G \left( \lambda, t \left| \hat{\lambda}, \hat{t} \right. \right) P(\hat{\lambda}, \hat{t}) d\hat{\lambda} .
\]
The quantum Q-function solution, \( Q = Q_{pi} \) is found on solving simultaneous equations for the joint distribution \( P(\lambda, t) \), and hence for \( Q_{pi} \), where:

\[
\int Q_{pi}(\lambda, t_f) d\mathbf{p} = P(\mathbf{p}, t_f), \tag{3.12}
\]

and:

\[
\frac{Q_{pi}(\lambda, t_i)}{\int Q_{pi}(\lambda, t_i) d\mathbf{p}} = P(\mathbf{p}|x, t_i). \tag{3.13}
\]

**Proof:**

We wish to show that \( Q_{pi} = Q \), by proving that \( Q_{pi} \) satisfies the GFPE \( \{3.4\} \), and has an initial condition that corresponds to the Q-function for the initial quantum state. From linearity, proving that \( Q_{pi}(\lambda) \) satisfies the GFPE \( \{3.4\} \) is achieved by verifying this for the propagator, \( G(\lambda, t | \lambda, \tilde{t}) \). To show this, define advanced and retarded propagators for \( \lambda \) at \( t = t_j \) with \( \tilde{p} = p_0 \) at \( t_i = t_0 \) and \( \tilde{x} = x_N \), as

\[
G^r(\mathbf{p}, t | \tilde{p}, t_i) = \lim_{\epsilon \to 0} \int e^{-Sp_{1:n}} \prod_{n=1}^{N-1} d\mathbf{p}_n, \tag{3.14}
\]

Due to its path-integral construction, \( G^r \) satisfies a forward Kolmogorov equation in \( \mathbf{p} \) \( \{10\} \). By reversing the sign of \( t \), we find that \( G^a \) also satisfies a forward Kolmogorov equation in \( x \), but in the negative time direction:

\[
\begin{align*}
\hat{G}^r(\mathbf{p}, t | \tilde{p}, t_i) &= \mathcal{L}^p G^r(\mathbf{p}, t | \tilde{p}, t_i) \\
\hat{G}^a(\mathbf{x}, t | \tilde{x}, t_f) &= -\mathcal{L}^x G^a(\mathbf{x}, t | \tilde{x}, t_f). \tag{3.15}
\end{align*}
\]

The normalization of the advanced Gaussian propagator terms in the path integral for \( x_n \) with \( n < j \) means that all these past time factors integrate to unity. Similarly, the retarded propagator is independent of \( p_n \) for future time points \( n > j \), as these also integrate to give unity. As a result, we can write that:

\[
\begin{align*}
G^r(\mathbf{p}, t_k | \tilde{p}, t_0) &= \lim_{\epsilon \to 0} \int e^{-Sp_{1:n}} \delta(\mathbf{p} - \mathbf{p}_k) \delta(\tilde{\mathbf{p}} - \mathbf{p}_0) d\mathbf{p}_., \\
G^a(\mathbf{x}, t_k | \tilde{x}, t_f) &= \lim_{\epsilon \to 0} \int e^{-Sp_{1:n}} \delta(\mathbf{x} - \mathbf{x}_k) \delta(\tilde{\mathbf{x}} - x_N) d\mathbf{x}. \tag{3.16}
\end{align*}
\]

From these results and the definitions above, it follows that the total propagator factorizes as \( G(\lambda, t | \lambda, \tilde{t}) = G^r(\mathbf{p}, t | \tilde{p}, t_0) G^a(\mathbf{x}, t | \tilde{x}, t_f) \). Hence, using the chain rule for differentiation and Eq. \( \{3.15\} \), the required time-evolution can be written as:

\[
\begin{align*}
\hat{G}(\lambda, t | \lambda) &= \hat{G}^r(\mathbf{p}, t | \tilde{p}, t_0) G^a(\mathbf{x}, t | \tilde{x}, t_f) \\
&\quad + G^r(\mathbf{p}, t | \tilde{p}, t_0) \hat{G}^a(\mathbf{x}, t | \tilde{x}, t_f) \\
&\quad + (\mathcal{L}^p - \mathcal{L}^x) G(\lambda, t | \lambda, \tilde{t}). \tag{3.17}
\end{align*}
\]

Due to linearity, any integral of \( G \) over its boundary values also satisfies the GFPE. Therefore, the path integral construction must obey the required GFPE,

\[
\hat{Q}_{pi}(\lambda, t) = (\mathcal{L}^p - \mathcal{L}_x(x)) Q_{pi}(\lambda, t). \tag{3.18}
\]

Provided the joint probability \( P(\lambda, t) \) satisfies the boundary equations, one can verify that at the initial time, \( Q_{pi}(\lambda, t_0) = Q(\lambda, t_0) \). Due to uniqueness of the solutions to a first order differential equations, \( Q_{pi} \) is equal to the quantum-mechanical Q-function solution for all times.

As a further check, in the short-time limit of \( t_0 = t_f \) we find that

\[
\lim_{t_0, t_f \to t} G(\lambda, t | \lambda) = \delta(\lambda - \lambda), \tag{3.19}
\]

hence, as \( t_0, t_f \to t \), one has that:

\[
\lim_{t_0, t_f \to t} Q_{pi}(\lambda, t) = \int \delta(\lambda - \lambda) P(\lambda, t_f) d\lambda, \tag{3.20}
\]

From the definition of the conditional probability, this is the initial Q-function.

### 3. Theorem II:

The path-integral solution corresponds to a time-symmetric stochastic differential equation (TSSDE) with a specified conditional initial distribution in \( \mathbf{p} \) and final marginal distribution in \( x \):

\[
\begin{align*}
p^j(t) &= p^j(t_0) + \int_{t_0}^{t} a^j_p(t') dt' + \int_{t_0}^{t} dw^j_p, \\
x^j(t) &= x^j(t_f) + \int_{t_f}^{t} a^j_x(t') dt' + \int_{t}^{t_f} dw^j_x. \tag{3.21}
\end{align*}
\]

Here \( x(t_f) \) is distributed as \( P(x, t_f) \), while \( p(t_0) \) is distributed conditionally as \( C(p | x(t_0), t_0) \), and \( [du^m] = [dw^a, dw^p] \) are independent real Gaussian noises correlated as

\[
\langle du^m du'^n \rangle = \delta^{mn} d^l \delta, \tag{3.22}
\]

where \( \mu = 1, 2M \) and \( d^{j+M} = d^j \).
Proof:

The TSSDE is obtained by discretizing the equation for times \( t_k = t_0 + k\epsilon \), with \( k = 0, \ldots, N \), and then taking the limit of \( \epsilon \to 0 \). We define \( [\mathbf{\lambda}] = [\mathbf{\lambda}_0, \mathbf{\lambda}_1, \ldots, \mathbf{\lambda}_N] \) as the stochastic path. The discretized solutions are then given as the simultaneous solutions of the equations, for \( k = 1, \ldots N \)

\[
\begin{align*}
\mathbf{p}_k &= \mathbf{p}_{k-1} + a^p (\mathbf{p}_{k-1}) \epsilon + \Delta_k^p \\
\mathbf{x}_{k-1} &= \mathbf{x}_k + a^x (\mathbf{x}_k) \epsilon + \Delta_k^x,
\end{align*}
\]

(3.23)

To obtain an equivalent path integral to the TSSDE, we first obtain the Dirac delta functions, conditioned on random noises \( [\Delta] \). This is a product of Dirac delta functions,

\[
\mathcal{G} ([\mathbf{\lambda}] | [\Delta]) = \prod_{j=1}^n \delta^{2M} (v_j - \Delta_j),
\]

(3.24)

which gives a normalized probability conditioned on a specific noise vector \( [\Delta] \). Solving for the resulting set of trajectory values \( [\mathbf{\lambda}] \) that satisfy the delta-function constraints is straightforward in the parametric amplifier case, due to the decoupling of forward and backward equations.

In a Fourier transform representation with \( k_j \equiv (k_j^1, k_j^2, \ldots, k_j^{2M}) \), one can expand the delta-functions as

\[
\mathcal{G}_n ([\mathbf{\lambda}] | [\Delta]) = \prod_{j=1}^n \int \frac{dk_j}{(2\pi)^{2M}} e^{-ik_j(v_j - \Delta_j)}.
\]

(3.25)

The \( 2M \) real Gaussian noises \( \Delta_k \) at each step in time, for \( k > 0 \), are distributed as:

\[
P (\Delta_k) = \frac{1}{(2\pi\epsilon)^M} e^{-|\Delta_k|^2/(2\epsilon)}.\]

(3.26)

On integration over \( k_j \), one obtains the path probability result as defined previously:

\[
\mathcal{G}_n ([\mathbf{\lambda}]) = \int \mathcal{G}_n ([\mathbf{\lambda}] | [\Delta]) P ([\Delta]) d[\Delta].
\]

(3.27)

By construction, the initial values of \( \mathbf{x}_N \) and \( \mathbf{P}_0 \) are sampled according to the required joint and conditional probabilities. Hence the probability of a TSSDE solution is \( Q_{pi} \), which is equal to the quantum average \( Q \) from Theorem 1.

C. Numerical \( \chi^2 \) tests

The analytic theorems obtained above were verified numerically using \( \chi^2 \) tests in several cases described here. The forward-backward stochastic equations were integrated by first propagating \( x \) backwards in time, generating a conditional sample, then propagating \( p \) forwards in time.

All trajectories plotted use 40 sample trajectories, to provide an intuitive demonstration of how they behave. In order to verify the quantitative accuracy of the trajectory probability distributions, large numbers of samples were generated and plotted using binning methods.

These samples were statistically tested with \( \chi^2 \) methods \([118, 119]\) to compare them with analytic solutions. The test cases used \( N_s = 2 \times 10^6 \) sample trajectories to obtain good statistics for the numerically sampled distributions,

\[
p_{ijk}^{samp} = \frac{N_{ijk}}{N_s},
\]

(3.28)

Here \( N_{ijk} \) is the number of trajectories in the bin at \( x_i, p_j \), and sampled time \( t_k \). Such verification requires binning on a three-dimensional \((x, p, t)\) grid, to obtain numerical estimates of the integrated analytic probabilities \( p_{ijk} \), where:

\[
p_{ijk} = \int_A d\delta x d\delta p Q (x_i + \delta x, p_j + \delta p, t_k) .
\]

(3.29)

This was evaluated by numerical integration of the analytic \( Q \) distribution, using a two-dimensional Simpson’s rule integrator in each bin.

In order to treat the dynamics of the \( Q \)-function, time-averages were evaluated to give a definitive overall result. Due to the correlations inside each trajectory, the range of fluctuations of time-averaged statistics are reduced compared to one-time tests, which exactly follow the \( \chi^2 \) distribution. Individual tests at each time-point are in agreement with \( \chi^2 \) statistics, and will be reported elsewhere.

We therefore define:

\[
\chi^2 = \frac{1}{N_t} \sum_{i,j,k} \left\langle \frac{|p_{ijk} - p_{ijk}^{samp}|^2}{\sigma_{ijk}^2} \right\rangle.
\]

(3.30)

Here, \( \sigma_{ijk}^2 = p_{ijk}/N_s \) is the expected probability variance for \( N_s \) total samples and \( N_t \) time points, with a phase-space bin area of \( A \).

As an example, the simulations described in Figure 1 used 30 time-steps of \( gdt = 0.1 \), combined with a midpoint stochastic integration method for improved accuracy \([120]\). No significant discretization error improvements were found with smaller time-steps. Comparisons were made between the analytic and numerically sampled \( Q (x, p, t) \) distributions with \( dx = 0.02 \) and \( dp = 0.05 \). This gave an average of \( \sim 55,100 \) comparison grid-points at each time step, after discarding bins with non-significant sample populations of \( N < 10 \) \([121]\).

In a typical test with \( 2 \times 10^6 \) sample trajectories and \( A = 10^{-3} \), the time-averaged statistical error was \( \bar{\chi}^2 = 55.2 \times 10^3 \), with \( 1.7 \times 10^9 \) valid comparisons. There were an average of \( k = 55.1 \times 10^3 \) significant points per time-step. This shows that \( \bar{\chi}^2 \) is within the expected range of \( \left\langle \chi^2 \right\rangle = k \pm \sqrt{2k} \).
Hence, as expected, there is good agreement between the analytic $Q$-function probability from quantum theory and the ensemble averaged stochastic trajectories. We recall that each combined forward-backward trajectory is regarded in our model as a possible objective element of reality.

IV. AMPLIFICATION MODEL FOR MEASUREMENT OF $\hat{x}$

A. Model Hamiltonian: parametric amplification

We consider a single mode field. Complementary quadrature phase amplitude operators are defined as $\hat{x} = \hat{a} + \hat{a}^\dagger$ and $\hat{p} = (\hat{a} - \hat{a}^\dagger)/i$ where $\hat{a}$, $\hat{a}^\dagger$ are annihilation and creation operators for the boson field. This implies $\Delta \hat{x} \Delta \hat{p} \geq 1$. We will show that our measurement model corresponds to the usual quantum properties.

We first analyze the simplest measurement procedure that can take place – that of direct amplification to a macroscopic signal. We model this by the parametric Hamiltonian $H_A = \frac{Ig}{2} [\hat{a}^2 - \hat{a}^2]$ given by (2.4) where $g > 0$ is real, which gives a amplification of $\hat{x}$ [106]. For $g > 0$, it is known that the dynamics of $H_A$ gives solutions that amplify the “position” $\hat{x}$ but attenuate the orthogonal “momentum” quadrature $\hat{p} = (\hat{a} - \hat{a}^\dagger)/2i$. This is clear from the standard operator Heisenberg equations which give the solutions

$$\hat{x} (t) = \hat{x} (0) e^{gt}$$
$$\hat{p} (t) = \hat{p} (0) e^{-gt}. \quad (4.1)$$

The Hamiltonian $H_A$ is equivalent to the Hamiltonian required to induce squeezing in $\hat{p}$. The solutions give for the means $\overline{x}$ and variances

$$\langle [\hat{x}^2] \rangle = \langle \hat{x}^2 (t) - \overline{x}^2 \rangle = e^{2gt}$$
$$\langle [\hat{p}^2] \rangle = \langle \hat{p}^2 (t) - \overline{p}^2 \rangle = e^{-2gt}$$
$$\overline{x} = \langle \hat{x} \rangle = \langle \hat{x} (0) \rangle e^{gt}$$
$$\overline{p} = \langle \hat{p} \rangle = \langle \hat{p} (0) \rangle e^{-gt}. \quad (4.2)$$

It will be useful to write the variances in terms of antinormally ordered products [122]. The antinormal ordering of $\hat{x}$ is given by

$$\langle [\hat{x}^2] \rangle = \langle \hat{x}^2 (0) + 1 \rangle = e^{2gt} \langle \hat{x}^2 (0) \rangle + 1. \quad (4.3)$$

Hence, if $\sigma^2_x (t) = \langle [\hat{x}^2 (t)] - \overline{x}^2 \rangle$ is the antinormally ordered variance, then

$$\sigma^2_x (t) = 1 + e^{2gt} \langle \hat{x}^2 (0) - \overline{x}^2 \rangle = 1 + e^{2gt} (\sigma^2_x (0) - 1), \quad (4.4)$$

and similarly,

$$\sigma^2_p (t) = \langle [\hat{p}^2 (t)] - \overline{p}^2 \rangle = 1 + e^{-2gt} (\sigma^2_p (0) - 1). \quad (4.5)$$

The antinormally ordered variances $\sigma^2_x (t)$ and $\sigma^2_p (t)$ are precisely the variances of $x$ and $p$ as defined by the $Q$ function $Q(x, p, t)$.

B. The superposition state

We will consider a measurement on the system prepared at time $t_0$ in the superposition $|\psi_{\text{sup}} \rangle = \sum c_i |x_i \rangle$, where $|x_i \rangle$ is an eigenstate of $\hat{x}$ of the “position” quadrature with eigenvalue $x_j$. Applying the definition (3.3) and using the overlap function [92]

$$\langle x_j | \alpha \rangle = \frac{1}{\pi^{1/4}} \exp\left( - \frac{(x - x_j)^2}{4} + i \frac{x}{2} (x_j - 2x) \right), \quad (4.6)$$

the $Q$ function for this state is given by

$$Q(x, p) = \frac{1}{\pi \sqrt{2}} \sum c_j e^{-\frac{1}{4}(x-x_j)^2} e^{i \frac{x}{2}(x_j - 2x)} \cdot (4.7)$$

This is a sum of Gaussians centered at each eigenvalue $x_j$ along with additional interference cross-terms. For the sake of simplicity without losing the essential features, we examine the simple superposition

$$|\psi_{\text{sup}} \rangle = |c_1 x_1 \rangle + |c_2 x_2 \rangle - x_1). \quad (4.8)$$

We have also taken, without loss of the essential features, $x_2 = -x_1$. Here the $c_j$ are complex amplitudes satisfying $|c_1|^2 + |c_2|^2 = 1$. The $Q$ function simplifies to

$$Q_{\text{sup}}(x, p) \sim |c_1|^2 e^{-|x_1 - x|^2 / 2} + |c_2|^2 e^{-|x_1 + x|^2 / 2}$$
$$-2c_1c_2 e^{-x_1^2 / 2} e^{-x^2 / 2} \sin(x_1 p). \quad (4.9)$$

where we take $c_1$ as real and $c_2 = i|c_2|$. A different choice of the phase of $c_2$ introduces an unimportant phase shift in the sinusoidal term. The $Q$ function for the superposition differs from the $Q$ function, $Q_{\text{mix}}(x, p)$, for the mixture of eigenstates,

$$\rho_{\text{mix}} = \frac{1}{2} \{ |x_1 \rangle \langle x_1 | + |x_2 \rangle \langle x_2 | \}, \quad (4.10)$$

only by the addition of the third term corresponding to fringes. Without a momentum cutoff of $|p| \leq p_m \gg 1$, the distribution $Q_{\text{sup}}(x, p)$ is not normalizable, as usual with pure position eigenstates in quantum mechanics. Since the limiting two-dimensional $Q$ function cannot be normalized due to the infinite variance in $\hat{p}$, in the last line of (4.9) we have written the normalized projection along the $x$ axis for a given $p$.

To obtain a more physical solution for (4.8), we model the position eigenstates as highly squeezed states in $\hat{x}$. The squeezed state is defined by [106]

$$|\psi(\beta_j, z) \rangle_{\text{sq}} = D(\beta_j) S(z) |0 \rangle. \quad (4.11)$$

Here, $|0 \rangle$ is the vacuum state satisfying $\hat{a} |0 \rangle = 0$, and $D(\beta_j) = e^{\beta_j \hat{a}^\dagger - \beta^*_j \hat{a}}$ and $S(z) = e^{z^2 (\hat{a}^\dagger - \hat{a})^2}$ are the displacement and squeezing operators respectively, where $z$ and $\beta_j$ are complex numbers. For the state with squeezed fluctuations in $\hat{x}$, we note that $z = r$ is a real and positive number that determines the amount of squeezing.
in $\dot{x}$. Defining $\bar{x} = \langle \dot{x} \rangle$, $\bar{p} = \langle \dot{p} \rangle$, we find $\langle (\Delta \bar{x})^2 \rangle = (\bar{x}^2 (t) - \bar{x}^2) = e^{-2r}$, $\langle (\Delta \bar{p})^2 \rangle = (\bar{p}^2 (t) - \bar{p}^2) = e^{2r}$ and $\langle \dot{a} \rangle = (\bar{x} + ip)/2 = \beta j$. We write $\beta j = (x_j + ip_j)/2$ where $x_j$ and $p_j$ are real. A position eigenstate $|x_j\rangle$ is thus a squeezed state with $r \to \infty$. We take $x_1$ and $x_2$ real, so that with $p_j = 0$. The superposition (4.8) then becomes the superposition of two squeezed states

$$|\psi_{\text{sup}}(r)\rangle = c_1|\psi(x_1, r)\rangle_{sq} + i|c_2|\psi(-x_1, r)\rangle_{sq}. \quad (4.12)$$

We select $c_2 = i|c_2|$ for convenience so that the normalization procedure gives the above form for all values of $r$ and $x_1$. Otherwise, the normalization involves an extra term which vanishes in the limit where the two states forming the superposition are orthogonal. Here, this requires large $r$, i.e. $r \to \infty$.

The $Q$ function for the squeezed state (4.11) is

$$Q(x, p) = \frac{1}{2\pi\sigma_x\sigma_p} e^{-\frac{(x-x_0)^2}{2\sigma_x^2}} e^{-\frac{(p-p_0)^2}{2\sigma_p^2}}, \quad (4.13)$$

where $\sigma_x^2 = 2(1 + \tanh r)^{-1}$ and $\sigma_p^2 = 2(1 - \tanh r)^{-1}$. Here, $\sigma_x^2$ and $\sigma_p^2$ are the variances of $x$ and $p$ for the $Q$ function distribution, which are given as

$$\sigma_p^2 = 1 + e^{2r}, \quad \sigma_x^2 = 1 + e^{-2r}. \quad (4.14)$$

This is in agreement with the different variances $\langle (\Delta \bar{x})^2 \rangle = e^{-2r}$ and $\langle (\Delta \bar{p})^2 \rangle = e^{2r}$ for the squeezed state (4.11), once antinormal ordering is accounted for. The full $Q$ function for (4.11) is

$$Q_{\text{sup}}(x, p) = \frac{e^{-p^2/2\sigma_p^2}}{4\pi\sigma_x\sigma_p} \left\{ |c_1|^2 e^{-\frac{(x-x_1)^2}{2\sigma_x^2}} + |c_2|^2 e^{-\frac{(x+x_1)^2}{2\sigma_x^2}} - 2|c_1|c_2| e^{-\frac{(x^2+x_1^2)}{2\sigma_x^2}} \sin(px_1/\sigma_x^2) \right\}, \quad (4.15)$$

which agrees with (4.9) for the idealized superposition (4.8) in the limit of large squeezing $r$. This function is plotted in Figure [3].

C. $Q$ function stochastic equations

For the system evolving according to the Hamiltonian $H_A = \hbar \frac{2g}{\sigma} [\dot{a}^2 - \dot{a}^2]$ given by (2.4), a dynamical equation for the $Q$ function can be derived [89, 90]. Applying the correspondence rules to transform operators into differential operators, one obtains a generalized Fokker-Planck type equation in terms of complex coherent state variables $a$:

$$\frac{dQ}{dt} = - \left[ g \frac{\partial}{\partial \alpha^*} \alpha^* + g \frac{\partial^2}{\partial \alpha^2} + h.c. \right] Q. \quad (4.16)$$

Using the quadrature definitions where the vacuum has unit noise in $x$ and $p$, one has $\alpha = (x + ip)/2$, or $\dot{x} = \dot{a} + \dot{\alpha}^\dagger$. We obtain

$$\frac{dQ}{dt} = \left[ \partial_p (gp) - \partial_x (gx) + g (\partial_p^2 - \partial_x^2) \right] Q. \quad (4.17)$$

This demonstrates a diffusion matrix which is traceless and equally divided into positive and negative definite parts, and a drift matrix:

$$A = \begin{bmatrix} \dot{x} \\ \dot{p} \end{bmatrix} = \begin{bmatrix} gx \\ -gp \end{bmatrix}. \quad (4.18)$$

D. Forward-backward stochastic equations

Using the above equivalence Theorem, we solve the measurement dynamics for the dynamics of $H_A$. Here, the $x$ and $p$ dynamics decouple. However, to obtain a mathematically tractable equation for the traceless noise matrix, we follow [89] and the sign of $t$ is reversed in the amplified dynamics of $x$.

The corresponding integrated stochastic equations for $g > 0$ are:

$$p(t) = p(t_0) - \int_{t_0}^{t} gp(t') + \int_{t_0}^{t} dw_p,$$

$$x(t) = x(t_f) - \int_{t_f}^{t} gx(t') + \int_{t_f}^{t} dw_x \quad (4.19)$$

where $\langle dw^a dw^a \rangle = 2g\delta_{\mu\nu}dt$.

Alternatively, in differential form, we obtain [89, 90]

$$\frac{dp}{dt} = -gp + \sqrt{2g}\xi_p \quad (4.20)$$

with a boundary condition in the past, and

$$\frac{dx}{dt_-} = -gx + \sqrt{2g}\xi_x \quad (4.21)$$

with a boundary condition in the future, where $t_- = t$. Defining $\xi = (\xi_p, \xi_x) = [\xi^\mu]$, the Gaussian random noises $\xi^\mu(t)$ satisfy: $\langle \xi^\mu(t) \xi^\nu(t') \rangle = \delta_{\mu\nu}\delta(t - t')$. Thus, there is a forward-backwards stochastic differential equation (FBSDE), for individual trajectories. This describes two individual stochastic trajectories, such that the average of the dynamical trajectories equals the $Q$-function averages. The trajectories are decoupled dynamically, with decay and stochastic noise occurring in each of the time directions. One propagates forward, and one backwards in time.

This leads us to consider a case where the trajectory in $x$ has a future marginal, $P(x, t_f)$. The trajectory in $p$ has a past conditional distribution, $P(p|x, t_0)$, which depends on $x$ in the future (Figure [5]), giving acyclic causal behavior.
E. Solutions for the forward and backward trajectories

To solve the trajectories for the field amplitudes, we stochastically sample according to the noise terms $\xi^x(t)$. From Eq. (4.21), we see that the $x$ solution at $t$ depends on the boundary condition for $x$ imposed in the future final time $t_f$, after the measurement has been completed. We solve this equation first, by propagating the trajectories for $x$ backwards in time. We refer to these trajectories as backward trajectories.

For sufficiently amplified fields $x$, we justify modeling the final stage of measurement as a direct detection of the amplified amplitude $x$ at $t_f$. This model is justified because the amplitudes for $x$ at the final time have a distribution given by the final $Q$ function, $P(x,t_f)$, which one can show is precisely that corresponding to the quantum prediction $P_{\text{sup}}(x,t_f) \rightarrow P(x) = |\langle x|\psi_{\text{sup}}\rangle|^2$, as $t_f \rightarrow \infty$. We proved this for a superposition of $\hat{x}$ eigenstates in Section II, and illustrate further by examples below, and in Section IV.F.

To evaluate the sampling distribution for the future boundary at time $t_f$, we evaluate the evolving $Q$ function $Q_{\text{sup}}(x,p,t)$ at the time $t = t_f$, from the Hamiltonian $H_A$ of (2.4) that amplifies $\hat{x}$. The original $Q$ function in the present time $t_0$ is given by (4.15). We evaluate the $Q$ function for the amplified system with respect to $x$, after the measurement $H_A$ has acted for a time $t$. This calculation can be done in two ways. The state formed after the unitary evolution $H_A$ is $e^{-iH_A t/\hbar}|\psi_{\text{sup}}(r)\rangle$, for which the $Q$ function can be evaluated directly as

$$Q_{\text{sup}}(x,p,t) = \frac{e^{-p^2/2\sigma_x^2(t)}}{4\pi \sigma_x(t) \sigma_p(t)} \left\{ |c_1|^2 e^{-(x-G(t)x_1)^2/2\sigma_x^2(t)} + |c_2|^2 e^{-(x-G(t)x_2)^2/2\sigma_x^2(t)} - 2|c_1||c_2| e^{-(x^2+G^2(t)x_2^2)/2\sigma_x^2(t)} \times \sin\left(\frac{\sigma_p G(t)x_1}{\sigma_x^2(t)}\right) \right\}. \quad (4.22)$$

Here $G(t) = e^{gt}$ is the amplification factor, and the variances becomes

$$\sigma_x^2(t) = 2(1 \pm \tanh(r-gt))^{-1} = 1 + e^{\pm 2(gt-r)}. \quad (4.23)$$

The solutions can also be found from the dynamical equation (4.2) of the $Q$ function. We denote the $Q$ function at the future time $t = t_f$ as $Q_{\text{sup}}(x,p,t_f)$.

Due to the separation of variables $x$ and $p$ in Eqs. 4.20-4.21, the marginal distribution $P_{\text{sup}}(x,t_f) = \int Q_{\text{sup}}(x,p,t_f)dp$ for $x$ at the future time $t = t_f$ determines the initial sampling distribution for the backward $x$ trajectories. This implies a forward causal relation, since the initial $Q$ function at time $t_0$ involves a set of Gaussians with means corresponding to the eigenvalues $x_j$. The final $Q$ function at time $t_f$ directly follows from the initial $Q$ function, with the means being amplified to $Gx_j$. This is depicted by the forward relation given by the solid blue line in Figures 3. The marginal for $x$ at the time $t \geq t_0$ is

$$P_{\text{sup}}(x,t) = \frac{1}{\sqrt{2\pi \sigma_x(t)}} \left\{ |c_1|^2 e^{-(x-G(t)x_1)^2/2\sigma_x^2(t)} + |c_2|^2 e^{-(x+G(t)x_2)^2/2\sigma_x^2(t)} \right\}. \quad (4.24)$$

We note the variance can be written as $\sigma_x^2(t) = 1 + G^2(t) \left[ \sigma_x^2(0) - 1 \right]$ where $G(t) = e^{gt}$. It is worth noting that the marginal and hence the resulting trajectories for the measured variable $x$ are identical to those given if the system were initially in the mixture $\rho_{\text{mix}}$ (Eq. (4.10)) of the two position eigenstates.

The backward trajectories for $r = 2$ are shown in Figure 6 and are depicted by the backward relation shown by the dashed red arrow in Figure 5. The key result is that regardless of the separation $\sim |x_1|$ between the eigenvalues $x_j$ and $-x_j$, the...
the eigenstates are always distinguishable upon measurement for large $gt$. In our example, this feature occurs in the asymptotic limit of very large squeezing, $r \to \infty$, where the squeezed states become eigenstates. The result arises because the fundamental vacuum noise (given in Eq. (4.14) by $\sigma_x = 1$) associated the $Q$ function eigenstate (4.13) is not amplified by $H_A$. This feature ensures the natural realization of the measurement postulate and Born’s rule, as we explain further in Section IV.H.

The forward trajectories for the attenuated variable $p$ are shown in Figure 8. These are depicted in Figure 5 by the forward-going red dashed arrows. The trajectories are calculated by solving (4.20) using the boundary condition at time $t_0 = 0$, which we refer to as the past, or present, time. The forward and backward equations decouple, and hence it is the marginal $P_{\text{sup}}(p, 0) = \int Q_{\text{sup}}(x, p) dx$ for $p$ at time $t_0 = 0$ that is relevant to the sampling. We find

$$P_{\text{sup}}(p, 0) = \frac{e^{-p^2/2\sigma_p^2}}{\sqrt{2\pi\sigma_p}} \left\{ 1 - e^{-x_1^2/2\sigma_p^2} \sin(px_1/\sigma_p^2) \right\}.$$ 

(4.25)

From this solution, we see that with increasing separation of the two eigenstates $|x_1\rangle$ and $|-x_1\rangle$ (so that $|x_1|$ increases), the fringes become less prominent (Figure 8). The evolution of $P_{\text{sup}}(p, t)$ is shown in Figure 9. The trajectories decay to a level given by the vacuum noise, as shown in Figure 8.

F. Cat states

It is useful to consider the superposition of two coherent states

$$|\text{cat}\rangle = \frac{e^{-i\pi/4}}{\sqrt{2}} \left\{ |\alpha_0\rangle + e^{i\pi/2} |\alpha_0\rangle \right\},$$ 

(4.26)

which for large $\alpha_0$ is the "cat state" [123]. We take $\alpha_0$ real, which corresponds to $r = 0$ in the expression (4.12), with $x_1 = 2\alpha_0$. The $Q$ function is given by (4.12), with $r = 0$ and $x_1 = 2\alpha_0$. 

Figure 7. Measurement of $\hat{x}$. Plot of forward propagating trajectories for the system prepared in a superposition $|\psi_{\text{sup}}\rangle$ (Eq. (4.12)) of two position eigenstates, with $r = 2$, and $x_1 = 1$. The top plot shows trajectories with $t_f = 3/g$. The second plot shows the reduction in the variance $\sigma_p^2$ to the value of 1.

Figure 8. Measurement of $\hat{x}$. As in Fig[7], except showing the corresponding marginal $P_{\text{sup}}(p, 0)$ for $p$ at the initial time $t = t_0 = 0$, where fringes are evident due to quantum interference.

Figure 9. Measurement of $\hat{x}$. As for Figure 1 with $r = 2$ and $x_1 = 1$. The plot shows the evolution of the distribution $P_{\text{sup}}(p, t)$ for $p$ as evaluated from $10^6$ multiple trajectories, a sample of which is given in Figure 8. The fringes are evident at $t = t_0 = 0$.
The amplification process acts similarly to that of the superposition of two position eigenstates, except that the quantum noise associated with the \( Q \) function has two contributions: The first noise contribution is that which exists for the eigenstate itself, and is not amplified. The second noise contribution is that for the coherent state, corresponding to the measured noise level of \( \langle (\Delta \hat{x})^2 \rangle = 1 \) given for a coherent state. Necessarily, since it is measurable, this noise level is amplified by the measurement interaction \( H_A \) given by (2.4), as evident from Figure 10.

We demonstrate the effectiveness of the model \( H_A \) for the measurement of \( \hat{x} \) by evaluating the final distribution \( P_{sup}(\hat{x}, t_f) \). The final marginal distribution \( P_{sup}(x, t_f) \) in the large amplification limit is given by (4.26), which corresponds to \( P_B(x) = \langle |x| \text{cat} \rangle^2 \) as predicted by quantum mechanics (Born’s rule). Here, \( |x\rangle \) is the eigenstate for \( \hat{x} \). We see this analytically by noting that the marginal for \( x \) at large amplification where \( gt \to \infty \) can be written in terms of the scaled variable \( \bar{x} = x/e^{gt} \) as

\[
P_{sup}(\bar{x}, t_f) = \frac{1}{2\sqrt{2\pi}} \left\{ e^{-(\bar{x} - x_1)^2/2} + e^{-(\bar{x} + x_1)^2/2} \right\},
\]

where here we use the result that \( \sigma_x^2(t) \to e^{2gt} \). This is in agreement with the \( P_B(x) = |\langle x | \text{cat} \rangle|^2 \) predicted by quantum mechanics (Born’s rule), as evaluated in [94] using \( x_1 = 2\alpha_0 \) (after correcting to deduce the inferred result \( \hat{x} \) for the outcome \( x \), given the amplification \( e^{gt} \)). The equivalence with \( P_B(x) \) is shown in Figure 11 for \( \alpha_0 = 2 \).

From Figure 12, we see that the trajectories for \( p \) for the cat state when \( \hat{x} \) are measured are attenuated. The effect is less pronounced when compared to that of the superposition of position eigenstates (Figure 8), because there is a reduced noise in \( p \) at the initial time. The measurement of \( \hat{x} \) as given by \( H_A \) amplifies \( \hat{x} \) and squeezes \( \hat{p} \). In fact, the noise levels for the initial cat state are approximately at the vacuum level \( \langle (\Delta \hat{p})^2 \rangle \sim 1 \), and the measurement Hamiltonian has the effect of squeezing the fluctuations in \( \hat{p} \), as shown by the plot of the variance in \( p \) in Figure 12.

G. Measurement of \( \hat{p} \)

We have analyzed measurement of \( \hat{x} \). We now consider the complementary measurement, \( \hat{p} \). This is incompatible with an \( \hat{x} \) measurement simply because it requires a different meter setting, which leads to a different measurement Hamiltonian. The resulting outputs have the complementary feature of interference fringes.

The \( \hat{p} \) measurement requires amplification of \( p \), with a negative \( g \) in the Hamiltonian \( H_A \). We use

\[
H_A = i\hbar g \left[ \hat{a}^2 - \hat{a}^2 \right],
\]

where \( g \) is real and \( g < 0 \). The dynamics from the standard operator Heisenberg equations gives the solutions

\[
\hat{x}(t) = \hat{x}(0) e^{-|g|t},
\]

\[
\hat{p}(t) = \hat{p}(0) e^{i|g|t},
\]

and we see that \( \hat{p} \) is amplified. The solutions for the dynamics of the \( x \) and \( p \) variables of the \( Q \) function are as above, except that \( x \) and \( p \) exchange roles. The trajectories for \( p \) are amplified and propagate back in time. Those for \( x \) are attenuated and propagate forward in time.

If we measure \( \hat{p} \) by amplifying the \( \hat{p} \) quadrature so that \( g < 0 \), then the full state at the later time is evaluated directly as before. From (4.22), we find

\[
Q_{sup}(x, p, t) = \frac{e^{-p^2/2\sigma_p^2(t)}}{4\pi\sigma_p(t)\sigma_x(t)} \left\{ e^{-(x - G(t)x_1)^2/2\sigma_x^2(t)} + e^{-(x + G(t)x_1)^2/2\sigma_x^2(t)} \right\} + 2e^{-2G(t)x_1^2/2\sigma_x^2(t)} \sin \left( \frac{pG(t)x_1}{\sigma_x^2(t)} \right),
\]

(4.30)
We may write the solution as

\[ P_B(x) = |\langle x | \psi_{sup} \rangle|^2 \]

predicted by quantum mechanics for the cat state. Here, \( t_f = 4/g \).

where \( G(t) = e^{gt} \) and \( \sigma^2 \langle x/g \rangle (t) = 2(1 \pm \tanh(r - gt))^{-1} = 1 + e^{\pm 2(gt - r)} \), except that now \( g < 0 \). Therefore \( G(t) = e^{-|g|t} \to 0 \) and \( \sigma^2(t) = 1 + e^{-2|g|t} \to 1 \), in the limit of \( |g|t \to \infty \). Hence, the future marginal in \( p \) at time \( t_f \) is

\[ P_{sup}(p, t) = e^{-\beta^2/2\sigma^2_p(t)} \{ 1 - e^{-\alpha^2 \sin^2(pG(t)x_1)} \} \]

\[ \to e^{-\beta^2/2\sigma^2_p(t)} \{ 1 - \sin(pG(t)x_1) \} \] \hspace{1cm} (4.31)

We may write the solution as

\[ P_{sup}(\tilde{p}, t) \to e^{-\beta^2/(2e^{2r})} \{ 1 - \sin(\tilde{p}x_1) \} \] \hspace{1cm} (4.32)

using the scaled variable \( \tilde{p} = p/e^{\beta t} = p/\sigma_p(t) \) and noting that \( \sigma_p(t) = e^{\beta t}e^{2\sigma^2} \) for large \( |gt| \), with \( g < 0 \).

We compare with the quantum prediction for the distribution \( P_B(p) \) for the outcome \( p \) upon measurement of \( \tilde{P} \), given as \( P_B(p) = |\langle p | \psi_{sup} \rangle|^2 \), where \( |p\rangle \) is the eigenstate of \( \tilde{P} \). We first compare for the cat state (4.26)

\[ P_B = \frac{e^{-\beta^2/\pi}}{2\sqrt{2\pi}} \{ 1 - \sin(2\alpha p \tilde{p}) \} \] \hspace{1cm} (4.33)

is in agreement with (4.32) (after accounting for the amplification \( e^{\beta t} \)), upon noting that with the choice of quadrature scaling, \( x_1 = 2\alpha_0 \). Figure [13] shows the forward trajectories and the trajectories for \( p \), for large \( |gt| \). As expected, the fringes are prominent.

The comparison is done for the superposition \( |\psi_{sup} \rangle \) of two eigenstates in Figure [4], giving exact agreement with the standard quantum prediction. The Figure also shows the forward trajectories in the complementary variable \( x \).

**H. Born’s rule for measurements \( \hat{x} \) and \( \hat{p} \)**

As the amplification \( G \) increases to a macroscopic level, the probability distribution evaluated by sampling over the trajectories for \( x \) becomes that of \( P_B(x) \), given by quantum mechanics. Here, \( P_B(x) = |\langle x | \psi \rangle|^2 \) gives the
Figure 13. Plotted are the backward trajectories for $p$ for the measurement of $\hat{p}$, using the Hamiltonian $H_A$ where $g < 0$. We consider the cat state ($\text{cat}$) given by Eq. (4.26) where $r = 0$, with well separated coherent states given by $x = 4$. This corresponds to $\alpha_0 = 2$. Here, $t_f = 4/g$. The final distribution $P_{\text{sup}}(p,t_f)$ is given in the lower plot. The fringes are sharply defined in agreement with Born’s rule for the measurement postulate. The amplified, scaled marginal distribution is $P_B(x)$ because in this limit of large amplification the vacuum noise term given by 1 in the expression (4.14) for $\sigma_z^2$ is not amplified, whereas $x_1$ and any extra noise above 1 is amplified. The analysis can be extended for an arbitrary expansion $|\psi\rangle = \sum c_i |x_i\rangle$ where $|x_i\rangle$ are eigenstates of $\hat{x}$, to validate Born’s rule for $\hat{x}$. The example of validation of $P_B(x) = \langle x | \psi_{\text{sup}} \rangle^2$ where there is a continuum of states $|x_i\rangle$ is given by the example of the cat state, in Figure 11.

We similarly demonstrate Born’s rule for measurement $\hat{p}$. We expand in eigenstates $|p\rangle$ of $\hat{p}$ as $|\psi_{\text{sup}}\rangle_p = \sum_j d_j |p_j\rangle$, where $d_j$ are probability amplitudes. Using the overlap function

$$\langle p_j | \alpha \rangle = \frac{1}{\pi^{1/4}} \exp\left(-\frac{(p-p_j)^2}{4} - i \frac{x}{2}(p_j-2p)\right),$$

the $Q$ function for this state is then given by

$$Q(x,p) = \frac{1}{\pi^{3/2}} \sum_j d_j e^{-\frac{1}{4}(p-p_j)^2} e^{-i\frac{x}{2}(p_j-2p)^2}.$$  \hfill (4.38)

This is a sum of Gaussians combined with interference cross terms. By analogy with the analysis of Section II for $\hat{x}$, we see that on amplification of $\hat{p}$ the final distribution $Q(x,p,t_f)$ will become a set of narrow Gaussians in $p$ centered at $p_j$ with weighting $|d_j|^2$. The interference terms vanish in the limit of large amplification. The realization of Born’s rule for $\hat{p}$ is evident in the examples of Section IV.G, shown in Figure 13.

Born’s rule arises naturally if one can interpret that at the time $t_0$ the system is in a state with a definite outcome $x_j$ for $\hat{x}$ with probability $|c_j|^2$. This requires that the amplified outcome $g x_j$ be determined by the state, at the time $t_0$. From Section II, we see that this is the case for the contextual deterministic realism and macroscopic-realism models.

V. MODELING THE COLLAPSE

How then do the results for the trajectories differ from that of the mixture $p_{\text{mix}}$ of the eigenstates $|x_i\rangle$ and how do the results elucidate the meaning of ‘collapse’? We
examine the coupling between the \( x \) and \( p \) trajectories, where one measures \( \hat{x} \).

At time \( t = t_0 = 0 \), the \( Q \) functions \( (4.9) \) and \( (4.15) \) for the superpositions \( |\psi_{\text{sup}}\rangle \) and \( |\psi_{\text{sup}}(r)\rangle \) give correlations between \( x \) and \( p \). We note this is not the case for the mixture \( \rho_{\text{mix}} \) \( (4.10) \), because the \( Q \) function for the single eigenstate \( |x_1\rangle \) is separable between \( x \) and \( p \). The \( x \) and \( p \) trajectories for the superposition (but not for a mixture \( \rho_{\text{mix}} \)) are therefore correlated i.e. connected. The causal relations are depicted in Figure 5.

A. Conditional distribution at the boundary

We see that a given \( x_f \) from the future time \( t_f \) propagates for each trajectory to a single \( x_p \) in the present time, at \( t_0 \). For each \( x_p \), there is a set of \( p_p \) at the present time. This set is given by the conditional distribution \( P_Q(p_p|x_p) = Q(x_p,p_p,0)/P(x_p) \) evaluated from the \( Q \) function in the present, where \( t = t_0 = 0 \). Here, \( P(x_p) \) is the marginal \( (4.24) \) at time \( t_0 \): \( P(x_p) \equiv P(x_p,0) \). Evaluating for the superposition \( |\psi_{\text{sup}}(r)\rangle \), we find

\[
P_Q(p_p|x_p) = P(x, p, 0)/P(x, 0) = e^{-p_p^2/2\sigma_p^2} \left\{ 1 - \sin(p_p x_1/\sigma_p^2) \right\} \cosh(x_p x_1/\sigma_p^2), \tag{5.1}
\]

which becomes \( \sim 1 - \sin(p_p x_1) \text{sech}(x_p x_1) \) for \( r \) large. Fringes are evident, these becoming finer for large \( x_1 \), and also increasingly damped, provided \( x_p \neq 0 \). For smaller \( x_1 \), the fringes will be more prominent, regardless of \( G \). The conditional distribution implies that the trajectories for \( x \) and \( p \) are coupled i.e. correlated. For a set of values of \( x_f \) at the time \( t_f \), we can match the set with a set of \( p \) trajectories, by propagating each given \( p_p \) from the sample generated by \( P(p_p|x_p) \), back to the future \( t_f \). We then have sets of variables \( \{x_f, x_p, p_p, p_f\} \) and all intermediate values on the trajectories. Such sets of trajectories are plotted in Figure 14.

The coupling of the \( p \) trajectories with those for \( x \) is determined by \( P_Q(p_p|x_p) \). We ask how does this depend on which state \( |x_1\rangle \) or \( |-x_1\rangle \) the system is measured to be in? The function \( P_Q(p_p|x_p) \) is independent of the sign of \( x_p \) and hence is not sensitive to which state is measured. This is evident in Figure 14 which plots the distribution for the trajectories of \( p \) conditioned on a positive final outcome i.e. \( x_f > 0 \).

In the causal relations for the deterministic-contextual-realism model (or for large \( x_1 \), the macroscopic realism models), this implies that the conditional relation giving \( p \) from \( \delta x \) is not dependent on \( \lambda = x_j \) (Figure 5). It is important to note that the conditional distribution \( P_Q(p_p|x_p) \) for the mixture \( \rho_{\text{mix}} \) of the two eigenstates is given by

\[
P_Q(p_p|x_p) = e^{-p_p^2/2\sigma_p^2}/\sigma_p \sqrt{2\pi}, \tag{5.2}
\]

which implies complete independence of the trajectories for \( x \) and \( p \) (Figure 5). The coupling of these trajectories at the boundary \( t = t_0 \) distinguishes a superposition \( |\psi_{\text{sup}}\rangle \) from a mixture \( \rho_{\text{mix}} \). The trajectories for \( x \) follow directly from the marginals \( P(x,t_f) \) in each case, and are hence identical (since the \( P(x,t_f) \) are identical).

Next, we ask what happens if we post-select on the condition that the final outcome of the measurement \( \hat{x} \) is \( x_1 \): that is, \( x_f > 0 \)? We evaluate the sets \( \{x_f, x_p, p_p, p_f\} \) conditioned on \( x_f > 0 \), for large amplification \( G \), which corresponds to a measurement. In Figure 14 we depict such sets for \( x_f > 0 \), and the forward trajectories in \( p \) for the set \( x_f > 0 \), and also the related distribution \( P_q(p) \) for the forward trajectories at the given times \( t \). We next ask what can we say about the state at the initial time \( t_0 \) i.e. in the present, given this post-selection?

---

![Figure 14](image_url)  
Figure 14. Plots show trajectories for \( x \) and \( p \) for the system prepared in a superposition \( |\psi_{\text{sup}}\rangle \), conditioned on a positive outcome for measurement of \( \hat{x} \). Here, we take \( |\psi_{\text{sup}}(r)\rangle \) of \( (4.12) \) with \( r = 2, x_1 = 8 \) and \( t_f = 2/g \).

B. Inferred initial state for \( x \) and \( p \) on postselection of measurement outcomes

We can bin the final outcomes into positive and negative \( x_f \), which would allow the final observer to infer either \( x_1 \) or \( -x_1 \) as the “result of the measurement”, in the limit of large amplification \( g t \). In our reality model, one can calculate the “state” of the original distribution...
(at $t = t_0$) corresponding to connected trajectories in $x$ and $p$, conditioned on the positive outcome $x_1$.

In this section, we evaluate the joint distribution $Q_+(x, p, t_0)$ at $t = t_0 = 0$ corresponding to the sets of connected positive trajectories for $x$ and $p$, based on the postselected positive outcome $+x_1$ for the measurement $\hat{x}$. As explained in the last section, these are evaluated by way of the conditional $P_Q(p|x)$ at time $t = t_0 = 0$. The distribution $Q_+(x, p, 0)$ has the interpretation in the deterministic-contextual and macroscopic-realism models of being the “state” inferred at time $t_0 = 0$, given the positive outcome $x_1$ for $\hat{x}$. Here, we evaluate the variances $\sigma_{x,+}^2$ and $\sigma_{p,+}^2$ of the inferred distribution $Q_+(x, p, 0)$, and show that they correspond to variances below the level allowed by the Heisenberg uncertainty principle.

We proceed as follows. For sufficiently large $gt$, each $x_f$ is either positive or negative, associated with the outcome $x_1$ or $-x_1$ which we denote by $+$ or $-$. We trace the trajectories in $x$ back to the time $t_0 = 0$ given the post-

selection of $x_f > 0$, and construct the distribution of $x$ at time $t_0 = 0$ for all such trajectories, as explained in the last sections. At the boundary in the present time $t_0 = 0$, each value of $x$ is connected to a set of trajectories in $p$, according to the conditional distribution (5.1). We thus construct the joint distribution $Q_+(x, p, 0)$ describing the correlated values in $x$ and $p$ at the time $t_0 = 0$ (Figure 15). We then determine the variances $\sigma_{x,+}^2 \equiv \sigma_{x,+}^2(0)$ and $\sigma_{p,+}^2 \equiv \sigma_{p,+}^2(0)$ for $x$ and $p$ for this distribution, and define the associated observed variance for the present $x_p$ and $p_p$ once antinormal ordering is accounted for:

\[
[\Delta(x_p|+)]^2 = \sigma_{x,+}^2(0) - 1
\]

\[
[\Delta(p_p|+)]^2 = \sigma_{p,+}^2(0) - 1. \quad (5.3)
\]

Similar variances $\sigma_{x,-}^2(0)$, $\sigma_{p,-}^2(0)$, $[\Delta(x_p|-)]^2$ and $[\Delta(x_p|-)]^2$ could be determined for the trajectories post-selected on the $x_f < 0$ corresponding to the outcome $-x_1$. This tells us what we infer about the original state (in the reality model) at time $t = 0$ based on the measurement outcome, whether $+$ or $-$. Here, by subtracting the vacuum term 1 associated with the antinormal ordering operators, we evaluate the variances that would be associated with a measurement of $\hat{x}$.

The variances are given in Figure 16 versus $x_1$ (which gives the separation between the states of the superposition) for a large value of $gt$. We also define the uncertainty product for the inferred initial state:

\[
\epsilon = \Delta(x_p|+)\Delta(p_p|+). \quad (5.4)
\]

From the Figures 16 see that $\epsilon < 1$ for all $\alpha$, although $\epsilon \rightarrow 1$ as $x_1 \rightarrow \infty$.

The figures show what happens if we postselect on the positive outcome, $x_1$, for a measurement of $\hat{x}$. This result is highly sensitive to the initial separation (given by $x_1$) of the eigenstates, or of the coherent states $|\alpha_0\rangle$ and $|-\alpha_0\rangle$. For the cat state where $r = 0$ and $x_1 = 2\alpha_0$ is large, the $x$-variance $[\Delta(x_p|+)]^2$ is reduced almost to the vacuum level of 1, as would be expected. This is explained as follows. The overall variance in $x$ at the time $t_0 = 0$ is large, due to there being two states comprising the superposition, but the final amplified outcome of either $x_1 = 2\alpha_0$ or $-x_1 = -2\alpha_0$ (Figure 11) links the trajectory back to only one of these states, $|\alpha_0\rangle$ or $|-\alpha_0\rangle$, which have a variance in $x$ of 1.

In fact, a further reduction for the variance in $x_p$ is observed because of the truncation that occurs with the Gaussian function at $t_f$. The conditioning is done for $x_f > 0$, which does not take account of the negative values in $x_f$ associated with the Gaussian centered at $x_f = gx_1$. This effect becomes negligible for a superposition of true eigenstates of $\hat{x}$ ($r \rightarrow \infty$) in the limit of a true measurement corresponding to $gt \rightarrow \infty$, as observed when comparing the results of Figure 16. However, the variance $[\Delta(p_p|+)]^2$ in $p$ occurs from the distribution for $p$ that has interference fringes. This is more pronounced with smaller separation $x_1$. The fringe pattern leads to a reduction in the variance, as compared to the simple

\[\text{Figure 15. Measurement of } \hat{x}. \text{ The upper plot shows the distribution for } p \text{ versus } gt \text{ conditioned on the positive outcome of measurement, for } x_1 = 1 \text{ and } t_f = 3/g. \text{ Despite the conditioning on a positive outcome for } \hat{x}, \text{ the lower plot is indistinguishable from Figure 9. The lower plot is the distribution } Q_+(x, p, 0) \text{ for the inferred state at time } t_0 = 0, \text{ postselected on the positive outcome } x_1 \text{ for } \hat{x}, \text{ that is, } x_f > 0 \text{ at } t_f, \text{ for the system prepared in the superposition } |\psi_{sup}\rangle \text{ of two } \hat{x} \text{ eigenstates. In practice we take } |\psi_{sup}(r)\rangle \text{ (Eq. (1.12)) with } r = 2.\]
the product, approach 1 in the limit of large $x_1$, where one has a true macroscopic superposition — a “cat state”.

VI. MACROSCOPIC REALISM, CAUSAL MODELS AND FRINGES

We now examine the results so far in order to make conclusions about the models for realism presented in Section II.

A. Deterministic-contextual and Macroscopic-realism models

We ask what is the interpretation for the state $|\psi_{\text{sup}}\rangle$ of eigenstates $|x_f\rangle$ of $\hat{x}$ at the time $t_0 = 0$, prior to the measurement of $\hat{x}$? Can we show consistency with the models for realism outlined in Section II? We considered the system prepared in the superposition

$$\psi_{\text{sup}}(r) = \frac{1}{\sqrt{2}} \{ |\psi(x_1, r)\rangle_{\text{sq}} + i |\psi(-x_1, r)\rangle_{\text{sq}} \} \quad (6.1)$$

deﬁned by Eq. (4.12), the squeezed states being eigenstates $|x_1\rangle$ and $|-x_1\rangle$ of $\hat{x}$ with eigenvalues $x_1$ and $-x_1$, respectively, in the limit of large $r$. The $Q$ function $Q_{\text{sup}}(x, p)$ for this state prepared at time $t = t_0 = 0$ is given by Eq. (4.15) and is depicted by way of the density of $x$ in Figure 3. The full $Q$ function is depicted in Figure 3.

We ﬁrst consider the macroscopic realism (MR) model. For macroscopic superpositions, the separation $\sim |x_1\rangle$ between the peaks associated with the states $|\psi(x_1, r)\rangle_{\text{sq}}$ and $|\psi(-x_1, r)\rangle_{\text{sq}}$ is much greater than the peak variance $\sigma_x^2(0) = 1$. We see that, essentially, the trajectories stemming from a positive (negative) $x$ in the future time $t_f$ link directly back to a positive (negative) $x$ in the present time $t_0$. There is a one-to-one correspondence between the initial and final positive (and negative) regions. Therefore for a measurement of $\hat{x}$, the inferred “state” for the present time $t = 0$, conditioned on a measurement outcome $Gx_1$, is exactly the state with positive $x$ values, centered close to $x_1$. We thus argue that there is consistency with the macroscopic-realism models of Section II: the system at time $t = 0$ is in a state that will give a deﬁnite result, either $x_1$ or $-x_1$, for a future measurement of $\hat{x}_1$. One can assign a hidden variable $\lambda_M$ to that system, where the value $\lambda_M = 1$ implies the outcome $x_1$ for $\hat{x}$, and the value $\lambda_M = -1$ implies the outcome $-x_1$ for $\hat{x}$.

We next consider consistency with the deterministic-contextual realism model (DCR). We have seen that re-

Gaussian in $p$. This effect is stable for $gt$ and $r$, although at greater $r$ we see that the optimal dip in the variance occurs at smaller separations $x_1$ of eigenstates.

The overall result is that the Heisenberg uncertainty principle is not satisﬁed for the coupled $x$ and $p$ trajectories: i.e., the post-selected $Q$ function distribution $Q_\perp(x, p, 0)$ does not reﬂect a “true” quantum state $\psi$ in the standard sense. Due to the fringes becoming ﬁner as $x_1 \to \infty$, the product approaches 1 with increasing separation of the eigenstates. This reveals that the variables $x_p$ and $p_p$ that we may identify as “elements of reality” are inconsistent with a standard quantum state, but become so in the limit of large $x_1$, where one has a true macroscopic superposition — a “cat state”.

Figure 16. Inferred state at time $t_0$ given the positive outcome for $\hat{x}$, i.e., $x_f > 0$. Here we plot the variances $\Delta(x_p|x_f > 0)$, $\Delta(p_p|x_f > 0)$ and the uncertainty product $\epsilon = \Delta(x_p|x_f > 0)\Delta(p_p|x_f > 0)$ conditioned on a positive outcome $x_1$ for $\hat{x}$, using Eq. (4.12). The upper dashed-dotted line is for a superposition $|\psi_{\text{sup}}\rangle$ of two eigenstates of $\hat{x}$ with $r = 2$, and $x_1 = 1$. The dashed line is for $|\psi_{\text{sup}}(r)\rangle$ with $r = 1$. The solid line is for $r = 0$. In each case, $gt = 4$, which corresponds to an effective measurement of $\hat{x}$. The two parallel lines indicate the upper and lower error bounds from sampling errors, with $1.2 \times 10^7$ trajectories.
gardless of separation $\sim |x_1|$ of the eigenstates, the amplified values for $x$ at $t_f$ are dichotomic i.e. can be binned into two categories $+$ and $-$, which the experimentalist designates as measuring either $x_1$ or $-x_1$ (Figure 1). This is due to the property that the noise $\sigma_x^2(0) = 1$ associated with each Gaussian peak in the $Q$ function at $t = 0$ is not amplified, while the eigenvalue amplitude $x_j$ itself is, thus enhancing the signal to noise ratio. Where overlap of the peaks in the initial distribution for the $Q$ function occurs ($|x_1| \ll 1$), however, there is no longer a direct correspondence between final and initial positive regions for $x$. Important however is that the trajectories for the amplified variable $x$ are identical to those produced from the $Q$ function for the mixture $\rho_{\text{mix}}$ of eigenstates $|x_j\rangle$. Hence, each realization of the system at time $t_0$ can be considered to be in a "state" with a definite value (one of the $x_j$) for the eigenvalue i.e. the amplified value $Gx_j$ is determined at the time $t_0$. The system at time $t_0$ is hence probabilistically in a state with a given eigenvalue $x_j$, with probability $|c_j|^2$. There is thus consistency with the DCR model, which postulates that at the time $t_0 = 0$, the final amplified outcome $x_1$ or $-x_1$ is determined, once the measurement setting is specified. This implies that the system can be assigned a hidden variable $\lambda_M$, as above, regardless of the size of the superposition i.e. regardless of the separation $|x_1|$. The $p$ trajectories for $\rho_{\text{mix}}$ and $|\psi_{\sup}\rangle$ are different, however, which indicates that the "state" of the system at time $t_0$ in each case is different.

The consistency of the trajectories with the MR and DCR models is established only for this particular example, $H_A$, of a measurement $\hat{x}$, which is a direct amplification. This assumes that the measurement setting has already been fixed at time $t = 0$. We refer to this as a pointer measurement. The measurement, $H_A$, leads to a separation of the dynamics of the $x$ and $p$ trajectories. A measurement process involving a further change of measurement setting may directly couple the dynamics for $x$ and $p$. However, such changes in the measurement setting involve unitary interactions, which occur over a time interval, and therefore produce a new final state, prior to an amplification process $H_A$. This explains why the above consistency with macroscopic realism is not in conflict with results that report violations of macrorealism [95] [99] [100] [102], or violations of macroscopic Bell inequalities [97] [99] [100]. More details of this distinction are given in [45] [100].

B. Causal relations

A question might be "by how much can noise from the future boundary affect the present reality?". For measurements where the separation of outcomes $x_1$ and $-x_1$ is beyond 1 (in units where $\hbar = 1$), the answer is zero. However, microscopic differences for the trajectories are apparent.

We consider the superposition $|\psi_{\sup}\rangle$ (Eq. 6.1). Where a hidden variable $\lambda_M$ can be assigned to the state at time $t = 0$ and this determines the outcome $x_1$ or $-x_1$ for $\hat{x}$, we argue that there is a causal relation. This is postulated by both the deterministic contextual realism (DCR) (for all $x_1$) and macroscopic-realism (MR) models (for $x_1$ large), and is consistent with the results of this paper. However, it is a different story for the microscopic fluctuations. The sampling from the future boundary at $t = t_f$ ensures a retrocausality. This is at the level of the quantum vacuum noise.

The causal relations are depicted in Figure 17. The model is classical in its form. The model parameters are the measurement setting $\theta$ (in this case, the choice of $\hat{x}$ or $\hat{p}$) and the variables that describe the state of the system, at the times $t = t_0 = 0$, and $t = t_f$. The $Q$ function specifies the system variables at time $t_0$ (or just prior) to be $x$ and $p$, with joint probability $Q(x, p, t_0)$. We present the causal diagram associated with the deterministic contextual realism model. An individual system is then also specified by $\lambda$, whose value is one of the set of eigenvalues $\{x_j\}$ (or $\{p_j\}$) of the measurement $\hat{x}$ (or $\hat{p}$). We define the conditional

$$P(\lambda | \theta), \quad (6.2)$$

where $P(\lambda = x_j) = |c_j|^2$ if $\theta \equiv \hat{x}$ and $P(\lambda = p_j) = |d_j|^2$ if $\theta \equiv \hat{p}$. Here, $|c_j|$ and $|d_j|$ are fully determined by the $Q$ function $Q(x, p)$, from the marginals $P(x)$ (or $P(p)$) found by integrating over the complementary variable ($p$
or $x$, but are given quantum mechanically by $c_j = \langle x|\psi\rangle$ and $d_j = \langle p|\psi\rangle$. We note we do not define an underlying joint distribution $P(\lambda_x, \lambda_p)$ for variables that correspond to $\lambda = \lambda_x$ if $\theta \equiv \dot{x}$, and $\lambda = \lambda_p$ if $\dot{p}$.

The final measured outcome value will be $X$, which is the amplified value of $\lambda$. In the model, $X$ is also a system variable at the time $t_f$. The relation between $X$ and $\lambda$ (and $\theta$) is deterministic, ie.

$$X = e^{i|\theta|t}\lambda,$$

where $\lambda = x_j$ for some $j$, if $\theta \equiv \dot{x}$; and $\lambda = p_j$ for some $j$, if $\theta \equiv \dot{p}$. This causal deterministic relation is represented by the blue forward-going solid arrow in Figure[17] With the choice of $\theta$, there is a value $\lambda$ selected, with a certain probability. This may be regarded as a macroscopic causal relation (especially for macroscopic superpositions, where $\lambda$ is macroscopic) since it relates to a measured quantity.

The system at time $t_f$ includes the fluctuations $\delta q$ that are not measured by amplification, where $q$ is either $x$ or $p$. The value of $\delta q$ is then sampled using a random Gaussian function $P_G$ with mean 0 and variance $\sigma_x^2 = 1$ at time $t_f$. Hence

$$P(\delta q(t_f)) = P_G(0, \sigma_x).$$

The value of the system variable $q$ is $q = q_j + \delta q(t_f)$, and $q_j$ is either $x_j e^{i|\theta|t}$ or $p_j e^{i|\theta|t}$, depending on the value of $\theta$. The backward trajectory is given by

$$\frac{dq}{dt} = -|g|q + \sqrt{2g}\xi_1(t),$$

as in (6.6), with the initial condition being the value of $q$ at time $t_f$. The Gaussian random noise $\xi_1(t)$ satisfies $\langle \xi_1(t)\xi_1(t')\rangle = \delta(t - t')$ and decouples from the complementary variable $q_c$ (which is either $p$ or $x$). We note however we can solve (6.6) by substituting $q = q_0 + \delta q$, where $q_0$ satisfies

$$\frac{dq_0}{dt} = -|g|q_0$$

with initial condition $q_0(t_f) = q_j$, and

$$\frac{d(\delta q)}{dt} = -|g|\delta q + \sqrt{2g}\xi_1(t),$$

with initial condition $\delta q = \delta q(t_f)$. Clearly, the solution for $q_0$ is the deterministic function $q_0 = q_je^{i|\theta|t}$, which is either $q_0 = x_j e^{i|\theta|t_f}$ or $q_0 = p_j e^{i|\theta|t_f}$ depending on the value of $\theta$, the measurement setting. This decaying solution with respect to $t_f$ is evident in the trajectories for $x$ plotted in Figures[2] and [3]. The trajectory for $\delta q$ has a stochastic solution involving the noise $\xi_1$. This noise does not depend on the value of $q_0$ or $q$, but rather has constant size. The consequence is that the $\delta q$ has the same initial condition and trajectory equations, regardless of whether the measured quantity $q$ is $x$ or $p$.

The trajectory for $\delta q$ propagates back to $t_0$ with a value $\delta q(t_0) = q_p$ (in the present time). Hence, we write

$$P(\delta q(t_0)) = P(\delta q(t_0)|\delta q(t_f)),$$

noting that the noise value $\delta q$ is independent of the value of $X$ (that is, $x_j$ or $p_j$) and, perhaps surprisingly, $\theta$. This defines the backward retrocausal relation marked by the red dashed arrow in the Figure[17]

However, $q$ is either $x$ or $p$ - which is determined at the present time $t_0$, from $\theta$. The value of the complementary variable that is not measured is denoted $q_c$. Here, $q_c$ is either $p$ or $x$. The value $\delta q_c(t_0)$ can be specified according to the conditional $P_Q(q_c|q)$ at time $t_0$. The $\delta q(t_0)$ defines the value of $q$ at time $t_0$: $q = \lambda + \delta q(t_0)$. This determines $q_c$ at time $t_0$. Denoting the values of $x$ and $p$ at time $t_0$ as $x(t_0) = x_p$ and $p(t_0) = p_p$ (meaning $x$ and $p$ in the present time), we write if $q = x$ and $q_c = p$ that

$$P_Q(p(t_0)|x(t_0)) = P_Q(p(t_0)|x(t_0)) = P_Q(p_p|x_p),$$

and if $q = p$ and $q_c = x$ that

$$P_Q(x(t_0)|p(t_0)) = P_Q(x(t_0)|p(t_0)) = P_Q(x_p|p_p).$$

This causal relation is given by the short red dashed downward arrow at the boundary $t = t_0$ in Figure[17]. The value of $q_c(t_0)$ determines the starting point of the trajectory that results in a value $q_c(t_f)$ for the complementary variable $q_c$ at time $t_f$. The probability

$$P(q_c(t_f)|q_c(t_0))$$

is well defined stochastically by the forward dynamics given by equation (4.20), which involves the noise function $\xi_2$. This relation is causal and is marked on Figure[17] by the forward arrow. The relations given by the red dashed arrows can be regarded as microscopic, or hidden, since they govern quantities that are not measured.

C. Causality and the meaning of retrocausality

The system as it has evolved at time $t$ under the effect of the measurement interaction $H_A$ is described by the state $|\psi(t)\rangle = e^{-iH_A t/\hbar}|\psi_{\text{sup}}\rangle$ which is equivalently given by the $Q$ function $Q(x,p,t)$. The description $|\psi(t)\rangle$ ($Q(x,p,t)$) is causal: the function depends on $t$. We have demonstrated that this state $Q(x,p,t)$ is also equivalent to the density obtained by averaging over the suitably connected trajectories for $x$ and $p$, at the time $t$.

It seems at first counterintuitive that a system which is retrocausal, with trajectories depending on boundary conditions at a future time $t_f$, can be consistent with the causal description, which is independent of $t_f$. The causal description, combined with the deterministic-contextual realism model, would imply that results of the measurement $\dot{x}$ (say) at the time $t$ cannot be affected by the choice of future time $t_f$. Hence, the solutions..
show there is no actual macroscopic retrocausality, in this sense.

The behavior is understood on noting that the noise \( \xi_1(t) \) that enters the retrocausal equations (6.5) at the future boundary \( t_f \) has a fixed magnitude \( \sigma_{x/p} \) at the level of the vacuum, which does not depend on the measurement setting \( \theta \), and is independent of the gain \( g t_f \). The forward equations in the complementary variable are more complex, but these are causal equations. Similarly, the nature of the coupling of the trajectories for \( x \) and \( p \) is determined by the boundary condition in the present, at time \( t_0 \).

### D. Schrödinger’s cat paradox: macroscopic realism and the incompleteness of quantum mechanics

The cat paradox seeks to interpret the reality of the system in a macroscopic superposition [103, 123]. Such a state is prepared as part of the measurement process. Figure 18 shows this realization, in the model for measurement given by amplification \( H_A \). The state formed at time \( t_g \) in the Figure is (taking \( r \) large)

\[
e^{-iH_A t/\hbar} |\psi_{\text{sup}}\rangle = \frac{1}{\sqrt{2}} \left( e^{-iH_A t/\hbar}|x_1\rangle + ie^{-iH_A t/\hbar}|-x_1\rangle \right).
\]

(6.12)

For sufficient amplification \( G = e^{gt} \) after a time \( t_g \), this state becomes a macroscopic superposition state as represented by the trajectories of Figures 2 and 6. The macroscopic realism model interprets the state at that time \( t_g \) as satisfying macroscopic realism.

This model allows the interpretation of macroscopic realism for the cat in the Schrödinger cat paradox [103], without needing to consider decoherence. The realism of the “cat”, that it be “dead” or “alive”, is achieved with the amplification \( H_A \). For the amplified state \( |\psi(t)\rangle \) at time \( t_g \), the value for the measurement \( \hat{x} \) is determined to be either \( x_1 \) or \( -x_1 \). Macroscopic realism is satisfied, in that the system is in one or other state that will yield an outcome of either \( x_1 \) or \( -x_1 \).

However, a simple analysis shows that the “states” comprising the amplified system at time \( t_g \) cannot be quantum states. We follow an argument given in [13, 100, 126]. Let us examine the macroscopic superposition \( |\psi_{\text{sup}}\rangle \) given by (6.1) with \( x_1 \) large. In the model, we assume this system satisfies macroscopic realism, and is in one of two “states” giving either \( x_1 \) or \( -x_1 \). If these states are indeed quantum states, then this would imply the state \( |\psi_{\text{sup}}\rangle \) can be expressed as a probabilistic mixture of type

\[
\rho_{\text{mix}}^Q = P_+ \rho_+ + P_- \rho_-,
\]

where \( \rho_\pm \) are quantum states giving an outcome \( \pm x_1 \) for the measurement \( \hat{x} \), and \( P_\pm \) are probabilities. However, the expression can be falsified for the predictions of the superposition state \( |\psi_{\text{sup}}\rangle \). The uncertainty relation \( \Delta \hat{x} \Delta \hat{p} \geq 1 \) holds for each \( \rho_\pm \), and therefore for the mixture. Therefore, if the \( P_\pm(x) \) distributions associated with the positive and negative \( x_1 \) are measured as Gaussians with variance \( \Delta x \leq 1 \) (as in Figure 11), it would be necessary that the distributions for \( P(p) \) satisfy \( P(p) \geq 1 \). However, the fringe distribution of Figure 13 gives \( \Delta \hat{p} < 1 \). The argument can be made that if macroscopic realism (MR) holds, then quantum mechanics is incomplete.

The apparent contradiction between the completeness of quantum mechanics and macroscopic realism is supported by the evaluation of the connected trajectories for \( x \) and \( p \), in Section V. The macroscopic system described by these coupled sets of trajectories, at the time \( t_g \), has a macroscopically determinate value of \( x \), which is either positive or negative (as in Figure 18). Macroscopically, however, the trajectories do not correspond to quantum states, because the uncertainty product \( \Delta x \Delta \hat{p} \) of the post-selected state at time \( t = 0 \) reduces below 1.

The amplification creating the macroscopic superposition state as in Figure 18 at the time \( t_g \) is reversible. One may return to the original state \( |\psi_{\text{sup}}\rangle \) at a time \( t_{2g} \) by attenuating \( x \), applying the Hamiltonian \( H_A \) of (2.4) with \( g < 0 \). This creates the forward trajectories for \( x \). Although the initial state is returned, we note that because the process is stochastic, the individual trajectories do not return to the original value precisely. It is understood that the complete process of the “collapse” to the eigenstate occurs when the process is not reversible. We expect that this occurs when there is loss of information from the system due to coupling to a second system, which motivates the next section.
E. Interpretation of fringes

Fringes are observed in the conditional distribution \( P(p|x) \) for \( p \) evaluated at the boundary corresponding to time \( t = t_0 = 0 \). The conditional distribution is given by (5.1). We see from Figure 14 that these fringes do not vanish if one considers the inferred distribution for \( p \) post-selected on the outcome of \( \hat{x} \) e.g. for \( x_f > 0 \). It is this feature that gives the reduced variance in \( \hat{p} \), leading to the conclusion of incompleteness, that the post-selected state cannot be a quantum state.

Intuitively, it may be thought that the fringes vanish once it is known “which state the system is in”, in analogy to the two-slit experiment. This motivates the next section, which examines precisely a set-up where one may indeed infer “which state the system is in” and simultaneously perform a measurement \( \hat{p} \) on the system \( A \).

VII. MEASUREMENT OF AN ENTANGLED SYSTEM AND METER

We now address a standard model of measurement, where one couples the system to a second macroscopic system \( B \), a meter. The measurement on the meter is then used to infer a result for a measurement on the first system, which we refer to as the signal.

A. Correlated state for the system and meter

First, we consider a system \( A \) prepared in a superposition state \( |\psi_{\text{sup}}(r)\rangle = \frac{1}{\sqrt{2}}(|\psi(x_1, r)\rangle_{\text{sq}} + i|\psi(-x_1, r)\rangle_{\text{sq}}) \) given by (4.12). In the limit of large squeezing \( r \), this becomes \( |\psi_{\text{sup}}\rangle = \frac{1}{\sqrt{2}}(|x_1\rangle + i| - x_1\rangle) \) where \( |x_1\rangle \) and \( | - x_1\rangle \) are eigenstates of \( \hat{x}_A \). Suppose a measurement is made on system \( A \) to infer “which of the two states the system \( A \) is in”, \( |x_1\rangle \) or \( | - x_1\rangle \). Such a measurement is made by coupling the system \( A \) to a meter \( B \). A prototype for the state after such a coupling is the entangled two-mode state

\[
|\psi_{\text{ent}}\rangle_{\text{sq}} = \frac{1}{\sqrt{2}} \{ |\psi(x_1, r)\rangle_{\text{sq}} |\beta_0\rangle + i|\psi(-x_1, r)\rangle_{\text{sq}} | - \beta_0\rangle \},
\]

which becomes in the limit of large \( r \)

\[
|\psi_{\text{ent}}\rangle = \frac{1}{\sqrt{2}} \{ |x_1\rangle |\beta_0\rangle + i| - x_1\rangle | - \beta_0\rangle \}.
\]

We take \( x_1 \) and \( \beta_0 \) to be real, and \( |\beta\rangle \) and \( | - \beta_0\rangle \) to be coherent states for mode \( B \). Normally, it is understood that for an effective measurement, \( \beta_0 \) would become large. The measurement of the quadrature phase amplitude \( \hat{x}_B = \hat{b} + \hat{b}^\dagger \) of mode \( B \) would indicate “whether the system is in the state \( |\beta_0\rangle \) or \( | - \beta_0\rangle \)”, and hence also a measurement to indicate the state of the system \( A \).

“whether \( |x_1\rangle \) or \( | - x_1\rangle \).” Here, \( \hat{b} \) is the destruction operator for field mode \( B \). When \( r = 0 \), the system is a two-mode entangled cat state

\[
|\text{cat}_2\rangle = \frac{1}{\sqrt{2}} \{ |\alpha_0\rangle |\beta_0\rangle + i| - \alpha_0\rangle |- \beta_0\rangle \},
\]

where \( \alpha_0 = x_1/2 \), and \( |\alpha_0\rangle \) and \( | - \alpha_0\rangle \) are coherent states for mode \( A \). Motivated by this model, we seek to examine the nature of the ‘collapse’ of the system \( A \) based on the direct measurement \( \hat{x}_B \) on system \( B \). The Q function of the entangled cat system (7.3) is

\[
Q_{\text{ent}}(\lambda, t_0) = \frac{e^{-((x_A + p_B)/2)^2}}{2\pi} \{ e^{-((x_A - 2\alpha_0)/2)^2} - e^{-((x_A + 2\alpha_0)/2)^2} \}
\]

where \( \lambda = (x_A, x_B, p_A, p_B) \). This can be written succinctly for the more general state (7.1) as

\[
Q(x, p) = N \left[ \cosh \left( \frac{x \cdot x_0}{\sigma_x^2} \right) - \sinh \left( \frac{p \cdot x_0}{\sigma_x^2} \right) \right]
\]

with the prefactor

\[
N = \frac{1}{4\pi^2} \prod_{i} |\sigma_{p,i}\sigma_{x,i}| e^{-\left( p^2/2\sigma_p^2 + x^2/2\sigma_x^2 \right)}.
\]

Here we use the vector notation \( x_0 = (x_1, 2\beta_0) \equiv (x_{01}, x_{02}), x = (x_A, x_B) \equiv (x_1, x_2) \) and \( p = (p_A, p_B) \equiv (p_1, p_2) \) and define \( x \cdot x_0/2\sigma_x^2 = \sum_{i} x_{i0}x_i/2\sigma_x^2, p \cdot x_0/2\sigma_p^2 = \sum_{i} p_{i0}x_i/2\sigma_p^2 \), \( x^2/2\sigma_x^2 = \sum_{i} x_{i0}^2/2\sigma_x^2 \) and \( x^2/2\sigma_x^2 = \sum_{i} x_{i0}^2/2\sigma_x^2 \). Also, \( \sigma_{x,1}^2 \) and \( \sigma_{x,2}^2 \) (\( \sigma_{p,1}^2 \) and \( \sigma_{p,2}^2 \)) are the variances for \( x_A \) and \( x_B \) (\( p_A \) and \( p_B \)) respectively.

For the general state (7.1), the variances are

\[
\sigma_{x,1}^2 = 1 + e^{-2r_1}, \quad \sigma_{x,2}^2 = 1 + e^{-2r_2}, \quad x_{01} = x_1, \ x_{02} = 2\beta_0,
\]

where \( r_i \) is the squeezing parameter \( r \) defined for the squeezed state for each mode. We have taken in Eq. (7.1) that \( r_1 = r \) and \( r_2 = 0 \).

B. Measurement on the meter and system

Following Section IV, the measurement of \( \hat{x}_B \) of the meter field \( B \) is modeled by the local interaction

\[
H_B = \frac{\hbar g_2}{2} \left[ \hat{b}^2 - \hat{b}^4 \right],
\]
The correlation between the sign of the final outcomes \( x_f^{(A)} \) and \( x_f^{(B)} \) is maximum for large \( \beta_0 \sim 2 \), so that there are negligible scatter points with \( x_f^{(A)} < 0 \). For \( \hat{x}_A \) in the lower plot, we take \( x_1 = 1 \) and \( r = 1.5 \), so it is a squeezed \( x \)-state, or near eigenstate. For the meter \( \hat{x}_B \), we take \( \beta_0 = x_2/2 = 2 \).

where \( g_2 \) is real and \( g_2 > 0 \). It is also possible to measure \( \hat{x}_A \) of system \( A \), via the local Hamiltonian

\[
 H_A = \frac{\hbar g_1}{2} (\delta x^2 - \delta^2)
\]

where \( g_1 > 0 \), as described in Section IV. With \( g_1 < 0 \), the local interaction \( H_A \) describes the measurement of \( \hat{p}_A \). The solution for the amplified \( Q \) function of the two-mode system after local interactions \( H_A \) and \( H_B \) for a time \( t \) at site \( A \) and \( B \) respectively is solved using the approach outlined in Section IV.

The solution is given by the \( Q \) function (7.5) with the means and variances becoming

\[
\begin{align*}
\sigma_{x,i}^2 &= 1 + e^{2(gt-r_i)} \\
\sigma_{p,i}^2 &= 1 + e^{-2(gt-r_i)} \\
x_{01} &= x_1 e^{eta_0} t, \\x_{02} &= 2\beta_0 e^{eta_0} t,
\end{align*}
\]

where we take \( r_1 = r \) and \( r_2 = 0 \).

We first consider joint measurements of \( \hat{x}_A \) and \( \hat{x}_B \). As before, we begin with trajectories starting with an \( x_f^{(B)} \) and an \( x_f^{(A)} \) sampled according to the marginal distribution for \( x_f^{(B)} \) and \( x_f^{(A)} \) of the \( Q \) function for the amplified system at time \( t_f \). Following the procedure to derive equations (4.21) and (4.20), the evolution for \( x_A \) decouples from that of \( p_A, x_B \) and \( p_B \). The equations for the measurement \( \hat{x}_B \) on the meter at \( B \) are

\[
\frac{dx_B}{dt} = -g_2 x_B + \sqrt{2g_2} \xi_{B1},
\]

where \( t_- = -t \) with a boundary condition at time \( t_- = -t_f \), and

\[
\frac{dp_B}{dt} = -g_2 p_B + \sqrt{2g_2} \xi_{B2},
\]

with a boundary condition at time \( t = t_0 \). The Gaussian random noises \( \xi_{\mu}(t) \) satisfy \( \langle \xi_{\mu}(t) \xi_{\nu}(t') \rangle = \delta_{\mu\nu} \delta(t - t') \). Similarly, the trajectories for \( x_A \) decouple from those of \( p_A, x_B \) and \( p_B \). The equations for the measurement \( \hat{x}_A \) at \( A \) are

\[
\frac{dx_A}{dt} = -g_1 x_A + \sqrt{2g_1} \xi_{A1},
\]

where \( t_- = -t \) with a boundary condition at time \( t_f \), and

\[
\frac{dp_A}{dt} = -g_1 p_A + \sqrt{2g_1} \xi_{A2},
\]

with a boundary condition at time \( t_0 \). The Gaussian random noises \( \xi_{\mu}(t) \) satisfy \( \langle \xi_{\mu}(t) \xi_{\mu}(t') \rangle = \delta_{\mu\nu} \delta(t - t') \).
While the stochastic evolution of the trajectories at the sites $A$ and $B$ are independent, the initial state at the future boundary is correlated, the correlations given by the $Q$ function (7.5) using (7.9). Since $p$ decouples from $x$, the relevant boundary condition for the trajectories $x_A$ and $x_B$ is determined by the future marginals for $x_A$ and $x_B$. The marginal for $x$ at time $t$ can be written on integrating over $p$, eliminating the second term, and therefore giving two two-mode Gaussians:

$$P_x(x, t) = \frac{e^{-\left(x^2 + x_0^2\right)/2\sigma_x^2}}{2\pi \prod_i \sigma_{x_i}} \cosh \left[\frac{x \cdot x_0}{\sigma_x^2}\right]. \quad (7.14)$$

The backward trajectories follow the same path as for a mixture $\rho_{mix}$ of the two states $|\psi(x_1, r)\rangle_{sq}|\beta_0\rangle$ and $|\psi(x_1, r)\rangle_{sq}|\beta_0\rangle$. There is clearly a correlation between the outcomes for the measurement $\hat{x}_A$ and $\hat{x}_B$ for large $\beta_0$ and $x_1$. This can be seen from the final amplified state and also from the initial state (7.1). In fact, provided $\beta_0$ is sufficiently large, the correlation between the final outcomes $x_A$ and $x_B$ is maximum even for small $x_1$ (where there is overlap of the two peaks of the $Q$ function) and for large $r$, where $|\psi(x_1, r)\rangle_{sq}$ becomes an eigenstate of $x_1$. This is seen from Figures 19 and 20, which plots the trajectories for $x_A$ and $x_B$ conditioned on $x_f > 0$, for different levels of squeezing in the signal field.

C. Inferred state for the system at time $t_0$ given an outcome for the meter

What can be inferred from the measurement at $B$ about the state at $A$ as it exists at the initial time $t_0 = 0$? We ask what is inferred for the state of system $A$ in the realism models presented in Section II, if the outcome at $B$ for $\hat{x}_B$ is positive $\beta_0$? We make the distinction that we are inferring the state of system $A$ prior to its direct measurement, with respect to the realism models, where it is assumed that the amplitudes $x$ and $p$ describe the ‘state’ of the system at the given time $t_0$. This has a different meaning to evaluating the $Q$ function for system $A$ conditioned on the outcome $\beta_0$ for $\hat{x}_B$, which can be calculated from standard quantum mechanics by projection. For the system prepared in the two-mode cat state (7.3) for example, where $\beta_0 \to \infty$, the state at $A$ conditioned on the outcome $\beta_0$ for $\hat{x}_B$ is $|\alpha_0\rangle$.

The trajectories for $x_B$ alone can be evaluated by integrating the $Q$ function (7.14) over $x_A$ to evaluate the marginal for $x_B$ at time $t_f$. The marginal for $x_B$ is found to be

$$P(x_B, t_f) = \frac{1}{2\sqrt{2\pi \sigma_x(t)}} \left\{ e^{-\left(x-2G(t_f)\beta_0\right)^2/2\sigma_x^2} + e^{-\left(x+2G(t_f)\beta_0\right)^2/2\sigma_x^2} \right\}, \quad (7.15)$$

where $G(t) = e^{\sigma x t}$ and $\sigma_x^2$ is given by (7.9) evaluated at $t = t_f$. This provides the distribution from which the sampling for the initial value for the backward trajectory proceeds. As expected, the trajectory for large $\beta_0$ connects back to the positive initial values centered around $2\beta_0$, at time $t_0 = 0$ (Figures 19 and 20). The $Q$ function of the state in the present time $t_0$ allows evaluation of the distribution $P_{cond}$ for $x_A, p_A$ and $p_B$ conditioned on $x_B$ in the present, similar to the method given in Section V. However, these are attenuated and are not of direct interest to us.

Our interest is the evaluation of the state at $A$ at the initial time $t_0 = 0$, postselected on the measurement outcome for the meter $B$. We take the measurement outcome for the meter $B$ to be positive i.e. $x_f^B > 0$. If the system is prepared in the state (7.1), we then expect the system $A$ would be found to be in the state $|\psi(x_1, r)\rangle_{sq}$. For large $r$, this state is the eigenstate $|x_1\rangle$; for $r = 0$, this state is the coherent $|\alpha_0\rangle$. Integrating over the variable $p_B$, we can arrive at the inferred $Q$ function $Q_{+in}^{(A)}$ for the system $A$, based on a future value $x_f^B$. This is evaluated as follows. There is a set of backward trajectories emanating from the set $x_f^B > 0$. For each such trajec-
We gain analytical insight by examining the limit of a meter, where \( \beta_0 \gg \alpha_0 \). We also consider that \( x_B \) is justified to be positive, based on the scatter plots of the trajectories for \( x_B \) that emanate from \( x_f^{(B)} > 0 \) for large \( x_f^{(B)} \). Then we see that the fringe term is damped and the conditional \( Q_{inf} \) becomes

\[
Q_{inf}(x_A, p_A|x_B) = \frac{e^{-p_A^2/4}}{8\pi \cosh(\beta_0 x_B)} \left\{ e^{-(x_A-2\alpha_0)^2/4}e^{\beta_0 x_B} + e^{-(x_A+2\alpha_0)^2/4}e^{-\beta_0 x_B} - 2e^{-(x_A^2+4\alpha_0^2+4\beta_0^2)/4}\sin(\alpha_0 p_A) \right\}.
\]

(7.16)

The inferred state of system \( A \) is \( |\alpha_0\rangle \), which is in agreement with the state projected from \( (7.3)^{(B)} \), using standard quantum mechanics. The limiting inferred state does not violate the uncertainty principle for large \( \beta_0 \), which is similar to the result for the limiting inferred state of Section V where \( x_1 \) and \( \alpha_0 \) are large.

The inferred distribution function \( Q_{+inf}^{(A)} \) for the state of system \( A \) given the measurement at \( B \) can be fully calculated from the trajectories in \( x_B \). This is plotted in Figure 21 for the system prepared in the state \( |\alpha_0\rangle \) with \( \beta_0 \) and \( \alpha_0 \) large, in the case of a quasi-eigenstate with \( r = 1.5 \). Indeed, as \( \beta_0 \) becomes large, the function becomes that of the \( Q \) function for the squeezed state \( |x_1, r\rangle \). This is true even for small \( x_1 \), where the \( Q \)-function peaks associated with each eigenstate overlap.

We conclude that in the realism models discussed in Section II (the objective field and DCR models), the state of system \( A \) at time \( t_0 \) conditioned on the outcome \( \beta_0 \) for \( \hat{x}_B \) corresponds to the “collapsed” or “projected” state \( |x_1, r\rangle \), as predicted by the measurement postulate. The conditional state for system \( A \) agrees with the quantum prediction.

In the realism models, however, each amplitude \( x \) and \( p \) is a possible realization of a “state” for the system, at the time \( t_0 \). Hence, the realism models give the extra interpretation about when the measured system \( A \) “collapsed” to the state \( |x_1, r\rangle \). This does not happen with the final measurement at \( B \), when \( x_f^{(B)} \) is detected. Rather, the system \( A \) is in one or other states with amplitudes \( x_A \) giving the final outcome \( x_1 \) or \(-x_1\) for \( \hat{x}_A \), but correlated with the outcome \( 2\beta_0 \) or \(-2\beta_0 \) for \( \hat{x}_B \) of system \( B \), at the time \( t_0 \). This we see for the special case of Eq. (7.3) where \( r = 0 \), from the two-mode \( Q \) function (7.4), which for \( \beta_0 \to \infty \) becomes two two-mode Gaussians with correlated means \( (2\alpha_0, 2\beta_0) \) and \((-2\alpha_0, -2\beta_0) \). The system has collapsed to the final state \( |\alpha_0\rangle \) or \(-|\alpha_0\rangle \) in a limiting sense for \( \beta_0 \) large, by the time \( t_0 \). The essential aspect of the “collapse” was created by a prior interaction \( H_P \), which coupled the system \( A \) to the macroscopic meter system \( B \). Figure 22 depicts the causal relations and the correlation for the coupled system.

The “collapse” that has been brought about by the interaction \( H_P \) is not complete, however, in the sense that the interaction \( H_P \) is unitary and can in principle be reversed. This is possible because the fringe terms although small do not completely vanish, even for large \( \beta_0 \). However, if the system \( A \) is decoupled from \( B \), then reversibility is not possible. The decoupling amounts to a loss of information for system \( A \). The \( Q \) function for \( A \) in this case is found by integrating over both \( x_B \) and \( p_B \). The resulting \( Q \) function becomes that of a statistical mixture...
\( \rho_{\text{mix}} \) of the two states \(|x_1, r\rangle |\beta_0\rangle \) and \(|-x_1, r\rangle |-\beta_0\rangle \), as we see for \( r = 0 \) from \((7.15)\) with \( G(t_f) = 1 \). The fringe terms vanish completely. At this stage, the system \( A \) is precisely in one or other states \(|x_1, r\rangle \) or \(|-x_1, r\rangle \) and the “collapse” is completed, being irreversible in the decoupled limit.

D. Fringes and which-way information

We now examine the question raised in Section VLE, about how or when the fringes disappear. In the above example, it is possible to measure both \( \hat{x}_A \) and \( \hat{p}_A \) simultaneously, because the outcome for \( \hat{x}_A \) is inferred by the measurement \( \hat{x}_B \) on the meter. Which-way information is gained when \( \beta_0 \) is large, and this corresponds directly to the decay of the fringes \([18][29][40]\).

We see from Eq. \((7.16)\) that fringes are present in the state inferred for \( A \) given the coupling to the meter at time \( t_0 \), but are damped by the factor \( e^{-\beta_0^2} \). In effect, the fringes vanish (but not irreversibly) once the system \( A \) has been entangled with the macroscopic meter. Here, macroscopic means \( \beta_0 \) is large, which is necessary to ensure effective measurement of \( \hat{x}_A \) by the meter. The coupling to the meter implies that the measurement of \( \hat{x}_A \) can be made at any future time, by measuring \( \hat{x}_B \) of system \( B \). The conclusion is that (within the interpretation given by realism models) the measurement of system \( A \) has taken place once the system is entangled with the macroscopic meter, at the time \( t_0 \).

The decay of the fringes can be observed experimentally by making joint measurements of \( p_A \) on system \( A \) and \( x_B \) on system \( B \). For \( \beta_0 \) large, the value of \( \hat{x}_B \) is inferred to correspond to that of \( \hat{x}_A \), based on the correlation. Where \( \beta \) is large, the postselection conditions on the outcome for \( \hat{x}_B \) being \( \beta_0 \) or \(-\beta_0 \) which implies “which way” information. The distribution shows no observable fringes. For smaller \( \beta_0 \) fringes appear, but this does not correspond to an effective measurement outcome for \( \hat{x}_A \), so that which-way information is lost.

VIII. EPR AND BELL CORRELATIONS

A challenge for any model of measurement that postulates objectivity is to explain the known violation of Bell inequalities. Here, we give a brief explanation of how the model presented in this paper accounts for continuous variable Einstein-Podolsky-Rosen (EPR) entanglement. We then identify how the model differs from the local hidden variable (or local causal) theories considered by Bell, thus explaining why the predictions of the objective model will not be constrained by Bell inequalities.

A. Einstein-Podolsky-Rosen entanglement

Consider the measurement of EPR correlations between two modes \( A \) and \( B \). The EPR correlations can be created from the two-mode squeezed state \([128][129]\)

\[
|\psi\rangle = (1 - T^2)^{1/2} \sum_{n=0}^{\infty} \tanh^n r \ |n\rangle_A |n\rangle_B, \tag{8.1}
\]

where \( T = \tanh r \) and for which the \( Q \) function is

\[
Q_{\text{epr}}(\lambda, 0) = \frac{(1 - T^2)}{16\pi^2} e^{-\frac{1}{2}(x_A - x_B)^2(1+T)} e^{-\frac{1}{2}(p_A + p_B)^2(1+T)}
\times e^{-\frac{1}{2}(x_A + x_B)^2(1-T)} e^{-\frac{1}{2}(p_A - p_B)^2(1-T)}
\to \frac{(1 - T^2)}{16\pi^2} e^{-(x_A - x_B)^2/4} e^{-(p_A + p_B)^2/4}. \tag{8.2}
\]

Here \( \lambda = (x_A, x_B, p_A, p_B) \). The last step shows the limit of \( r \to \infty \), where the state becomes a simultaneous eigenstate of \( \hat{x}_0 = \hat{x}_A - \hat{x}_B \) and \( \hat{p}_0 = \hat{p}_A + \hat{p}_B \). The variances of \( x_\pm = x_A \pm x_B \) and \( p_\pm = p_A \pm p_B \) are

\[
\langle [\Delta(x_A \pm x_B)]^2 \rangle = \sigma_\pm^2 (0) = 2(1 + e^{\pm 2r})
\]

\[
\langle [\Delta(p_A \pm p_B)]^2 \rangle = \sigma_\pm^2 (0) = 2(1 + e^{\mp 2r}). \tag{8.3}
\]

The corresponding measured variances of the operators \( \hat{x}_\pm = \hat{x}_A \pm \hat{x}_B \) and \( \hat{p}_\pm = \hat{p}_A \pm \hat{p}_B \) are

\[
[\Delta(\hat{x}_A \pm \hat{x}_B)]^2 = 2e^{\pm 2r}
\]

\[
[\Delta(\hat{p}_A \pm \hat{p}_B)]^2 = 2e^{\mp 2r}. \tag{8.4}
\]

The argument of EPR is based on local realism (LR) \([130]\). If one can predict with certainty the outcome of a measurement of \( \hat{x} \) (or \( \hat{p} \)) at a site \( A \) without disturbing that system, then the premise of EPR’s realism implies there exists an “element of reality”, or hidden variable \( \lambda^A_0 \) (\( \lambda^A_0 \)), that determines the result for that measurement should it be performed \([19]\). For spatially separated sites \( A \) and \( B \), locality implies that the outcome for \( \hat{x} \) (or \( \hat{p} \)) is predicted with certainty, by performing a measurement on system \( B \). This follows from the correlations given by the \((8.4)\). EPR therefore argued that simultaneous hidden variables \( \{\lambda^A_0, \lambda^A_0\} \) exist to describe the system \( A \). EPR then argued that quantum mechanics is incomplete, since such variables ascribed to system \( A \) at the time \( t_0 \) violate the Heisenberg uncertainty relation.

We ask what does the objective field model presented in this paper predict for EPR’s argument? For the EPR case, we first consider two-sided measurements in \( \hat{x} \), where the fields \( A \) and \( B \) are spatially separated and measurements of \( \hat{x}_A \) and \( \hat{x}_B \) are made at the respective sites. The measurements at the sites are local and given by \( H_A = i\hbar \left[ \hat{a}^2 - \hat{a}^2 \right] / 2 \) and \( H_B = i\hbar \left[ \hat{b}^2 - \hat{b}^2 \right] / 2 \) where \( g \) is real and \( g > 0 \). This results in an independent amplification of \( \hat{x}_A \) and \( \hat{x}_B \) at each site, as in Figure \([22]\). We can understand what is expected for the trajectories from the previous analyses of measurement of \( \hat{x} \) on a single system. The trajectories for \( x \) are governed
by backward propagating equations. The measurement of \( \hat{x}_- \) at \( A \) and \( B \) is such that the amplified outcomes \( gx_A \), \( gx_B \) have zero relative noise for large enough \( G \).

The systems are correlated in \( \hat{x} \) so that the measurement of \( \hat{x} \) at \( B \) implies the result at \( A \) for any \( x_1 \) and \( x_2 \), no matter how small the difference \( |x_1 - x_2| \), despite the existence of the quantum noise in the \( Q \) function at \( t_0 \). Thus, in the continuous variable description given by (8.1), the outcomes for the \( \hat{x}_A \) and \( \hat{x}_B \) are precisely correlated, as are those of \( \hat{p}_A \) and \( \hat{p}_B \).

Similar to the interpretation of the single-mode eigenstate \( |x_j\rangle \) given in the Sections II and IV, one can postulate models of realism for the two-mode EPR eigenstates \( |x_1 - x_2\rangle \), \( |p_1 + p_2\rangle \), and examine whether they would be justified by the trajectories using the amplitudes \( A \). Do we have a local hidden variable model of the type implied by EPR’s local realism? First, we see that the amplitudes \( x_A \) and \( p_A \) of system \( A \) as defined for the \( Q \) function \( Q(x_A, x_B, p_A, p_B) \) of the EPR state are not themselves perfectly correlated with those of system \( B \). This is due to the fundamental noise \( (\sigma_x = 1 \text{ and } \sigma_p = 1) \) for the \( x_- \) and \( p_+ \) that appears in the initial \( Q \) function. However, it is clear that the final amplified values \( gx_A \) and \( gx_B \) are perfectly correlated, for sufficient \( g \). Similarly, the amplified values \( |g| p_A \) and \( |g| p_B \) are perfectly anticorrelated after the amplification of \( \hat{p} \) at each site \( (g < 0) \).

The deterministic-contextual and hybrid macroscopic realism models imply that the amplified values of \( x_A \) at a time \( t_g \) are locally determined (refer Figure 18). The trajectories for the EPR case would support these models, that the system is in a state with a definite future value for the outcomes of \( \hat{x}_A \) or \( \hat{p}_A \), once the measurement setting is selected. Hence, the models do not specify the local hidden variables \( \{\lambda_x^A, \lambda_p^A\} \) that are implied by EPR’s argument, because the \( x \) and \( p \) are not amplified by the same Hamiltonian. In summary, in the objective field model, trajectories support that the system is in a state with a definite future value \( \lambda_2^A \) for the outcomes of measurement \( (\hat{x}_A \text{, say}) \), once the measurement setting is selected.

B. Quantum nonlocality: Bell’s premises

This leaves us to understand how (or whether) Bell nonlocality may arise from the objective field model studied in this paper. At first glance, the model appears to be a local realistic theory, and hence could not explain violation of a Bell inequality. In this section, while we do not give an explicit illustration of a Bell violation, we show how Bell’s assumption of local realistic theories breaks down for the model.

In the first proof [5], Bell adopted EPR’s version of local realism. This is that the outcome of a measurement \( \hat{x}_0 \) on system \( A \) can be specified by a hidden variable \( \lambda_0^A \), if it is possible to predict that value with certainty, by making an appropriate measurement on a space-like separated system, \( B \). Here, \( \theta \) is the measurement setting at \( A \) e.g. whether \( \hat{x} \) or \( \hat{p} \) is measured. By considering two spin 1/2 systems in a correlated singlet state, EPR’s local realism allowed Bell to assert, for the bipartite system, the existence of predetermined outcomes (i.e. hidden variables) for three different measurement settings of system \( A \) – an assertion which is then falsified by Bell’s theorem. Bell’s theorem therefore falsifies EPR’s version of local realism. However, this does not rule out the objective field model, in the form of the deterministic-contextual-realism or macroscopic-realism models, since these models are based on a weaker form of local realism: The hidden variable \( \lambda_\phi \) is established only when the measurement setting \( \theta \) has been determined e.g. by amplification. For bipartite systems, one cannot determine simultaneously three measurement settings, and Bell’s assertion would therefore not follow. Violations of Bell inequalities have been obtained for observables \( \hat{x}_\theta \) that are linear combinations of \( \hat{x} \) and \( \hat{p} \) [33][34], but here also the violations required three or more different measurement settings \( \theta \). This suggests that the dynamics associated with the choice of measurement setting is important in giving rise to Bell nonlocality.

Bell also derived an inequality based on a more general definition of local realistic theories. To observe nonlocality as a violation of a Bell inequality, one measures the joint probability \( P(X_\theta, X_\phi) \) for outcomes \( X_\theta \) and \( X_\phi \) at two sites \( A \) and \( B \) respectively. The \( \theta \) and \( \phi \) denote the measurement settings at each site. Bell’s theorem is based on the premise that hidden variables \( \{\lambda\} \) describe the correlated systems, according to a distribution \( \rho(\lambda) \) which is independent of measurement settings. The assumption of locality is also made: the probability \( p_A(X_\theta|\lambda) \) for an outcome \( X \) at \( A \), given the system is specified by the set of variables \( \{\lambda\} \), is independent of the choice of measurement setting \( \phi \) at the location \( B \). A distribution \( p_B(X_\phi|\lambda, \phi) \) is defined similarly. Mathematically, Bell’s assumptions imply that [3][10][33][34]

\[
P(X_\theta, X_\phi) = \int \rho(\lambda) d\lambda \ p_A(X_\theta|\lambda, \theta) p_B(X_\phi|\lambda, \phi) \quad (8.5)
\]

This assumption implies that Bell inequalities must hold.

In the models presented in this paper, the hidden variables are the phase space amplitudes denoted \( \lambda \equiv (x_A, x_B, p_A, p_B) \). How would this phase-space model violate the condition (8.5)? The trajectories given by the model are based on the independent amplification of either \( x \) or \( p \) at each site. In the sense that the choice of measurement setting is independent at each site, there is no nonlocal mechanism. We may also define \( \rho(\lambda) \) as being the \( Q \) function \( Q(\lambda) \), which is independent of the future settings. For each set \( \lambda \), there is a probability \( P(x_0|\lambda) \) for an outcome \( X_0 \) at \( A \), given the setting \( \theta \) at \( A \). However, the \( Q \) is not directly a function of the variables \( x_0 \) associated with the measurement \( X_\theta \), and it is not clear that this \( P(x_0|\lambda) \) is independent of \( \phi \). For a nonentangled state, the \( Q \) function \( Q(\lambda, 0) \) at the time \( t_0 \) can be expressed as a mixture of factorizable functions (refer Eq. (7.4) without the third term), and the Bell as-
of functions factorizable with respect to systems state, then the function cannot be expressed as a mixture sumption (8.5) holds. If $Q(\lambda, 0)$ is that of an entangled state, then the function cannot be expressed as a mixture of functions factorizable with respect to systems $A$ and $B$ (refer Eq. (7.4)).

Suppose one considers measurement of $\hat{x}$ (or $\hat{p}$) by amplification. At any time $t$ such that $t_0 \leq t \leq t_f$, we can then identify a trajectory value $x$ (or $p$). It might seem that there is a local realistic representation of the dynamical development of the correlations from an initial positive distribution $Q(\lambda, 0)$ for the amplitudes. However, the trajectories for the measured observables originate from a boundary condition in the future, at time $t_f$. The correlation between the trajectories at $A$ and $B$ will be determined by the relevant marginal of the $Q$ function as evaluated at time $t_f$. This function however depends on the measurement settings, and cannot therefore take the role of a $\rho(\lambda)$ in (8.5).

IX. CONCLUSION

We have presented a model (the objective field model, or $Q$ model) for a quantum measurement which provides a description involving trajectories of hidden variables, $x$ and $p$. In this paper, we treat a measurement of $\hat{x}$ or $\hat{p}$ on a single field mode, and consider the dynamics of a measurement interaction $H_A$ which amplifies the $\hat{x}$ or $\hat{p}$. The model is based on the $Q$ function $Q(x, p)$ which uniquely represents the quantum state, and on the equivalence of the dynamical equation for the $Q$ function with forward-backward stochastic trajectories for the amplitudes $x$ and $p$. The approach has the promising feature that, being derivable from within the framework of quantum field theory, it is Lorenz invariant and can therefore be compatible with the standard model of particle physics. The equivalence is proved in two theorems. A technique is introduced whereby the forward and backward trajectories are linked according to a conditional distribution $P_Q(p|x)$ at the boundary corresponding to the initial time $t_0 = 0$ (the time of commencement of the interaction $H_A$). The measured variable ($x$ or $p$) is amplified, and requires a future boundary condition at time $t_f$ (corresponding to the final time of interaction $H_A$).

We solve the dynamical trajectories for several examples, illustrating the equivalence to the evolved $Q$ function $Q(x, p, t)$ throughout the dynamics. This provides a check on the theorems, verified through $\chi^2$ statistical tests. Here, the description of the model as “retrocausal” refers only to the use of future boundary conditions. The theorems prove that there is a causal consistency: The forward and backward trajectories at the time $t$ are equivalent to the quantum state given by $Q(x, p, t)$, which depends on $t$, not the future time $t_f$. Measurement of both $\hat{x}$ and $\hat{p}$ on a superposition of eigenstates of $\hat{x}$ is examined, with Born’s rule validated. The analysis focuses on the superposition of two eigenstates. Regardless of the separation between the eigenstates, the final outcomes for the measurement clearly distinguish between the eigenstates. The analysis is extended to cat states.

The trajectories provide the possibility for a model of reality that is more complete than quantum mechanics. This is shown for the superposition of two eigenstates of $\hat{x}$, by analyzing the connected sets of trajectories for $x$ and $p$, and evaluating the variances associated with those distributions, postselected on the final outcome for the measurement of $\hat{x}$. The variance product reduces below that allowed by the Heisenberg uncertainty relation, implying that the linked trajectories associated with the definite outcome cannot be modeled by a quantum state $|\psi\rangle$. Different models of realism are presented, and analyzed for compatibility with the objective field $Q$ model, based on the trajectories. Most convincingly, the analysis supports the validity of macroscopic realism and macroscopic causality, despite the fact that retrocausality is embodied in the model. The interpretation is that macroscopic realism / causality holds: A system prepared in a superposition of two macroscopically distinct eigenstates of $\hat{x}$ at the time $t_0$ has a well defined outcome for the result of the pointer measurement $\hat{x}$. However, there is microscopic noise (at the level of $\hbar$) present in the final outcomes at the time $t_f$. This noise is also present in the microscopic system and the values are determined retrocausally.

The hybrid-macroscopic-realism model provides an interpretation of the Schrödinger cat paradox. In this model, macroscopic realism holds for the macroscopic superposition state, but the state of the system at the microscopic level is such that it cannot be described completely by quantum mechanics. The hybrid model (based on the objective field $Q$ model) provides a more complete description of the state of the “cat” than is given by standard quantum mechanics. This is because the microscopic fluctuations rely on retrocausality. The incompleteness of quantum mechanics to describe the state of the cat system is shown by the reduced variances in $x$ and $p$ for the trajectories conditioned on the final measurement outcome. The model therefore provides a resolution of Einstein-Podolsky-Rosen-type paradoxes that argue for the completion of quantum mechanics, based on the validity of macroscopic realism [100]. A resolution of similar related paradoxes may also be possible. The hybrid model does not conflict with the known violations of macrorealism and of macroscopic local realism (as shown through violations of Leggett-Garg [95] and macroscopic Bell inequalities [97, 99, 100]). In those cases, extra assumptions (apart from the macroscopic realism assumed in the hybrid model) are involved in the premises, and it is likely that these may be shown to be falsifiable. Possible mechanisms for this are discussed in [99, 100].

A similar model for realism that is supported by the trajectories of the $Q$ model is the deterministic contextual realism model (DCR), which postulates that the system in a superposition of two eigenstates for $\hat{x}$ is in a state with well-defined outcome for measurement $\hat{x}$, once the measurement setting is determined. This means that a suitable unitary interaction has taken place to deter-
mine the measurement setting. The hybrid macroscopic-realism model is an illustration of this case, since the choice of amplification Hamiltonian $H_A$ determines the measurement setting. Other unitary interactions are possible and the meaning of this form of realism is hence more general than that of the hybrid macroscopic-realism model.

The trajectories of the $Q$ model show consistency with contextual explanations of Bell violations [104]. These trajectories support the deterministic-contextual-realism and macroscopic-realism models. The theorems prove consistency with causality, that the forward and backward trajectories at the time $t$ ($t > t_0$) are equivalent to the quantum state given by $Q(x,p,t)$. The realism models assert that the value $x_j$ for the measurement of $\hat{x}$ is determined for the system in the state $Q(x, p, t)$ at the time $t > 0$ (after the measurement setting is determined), which counters any notion of a retrocausal influence changing the actual result for a measurement, for $t > 0$.

Einstein-Podolsky-Rosen (EPR) correlations between two field modes $A$ and $B$ are possible for quantum states which are simultaneously eigenstates of both $\hat{x}_A - \hat{x}_B$ and $\hat{p}_A + \hat{p}_B$. EPR argued that quantum mechanics was incomplete, based on the validity of the premise of local realism. We have examined how the objective $Q$ model explains these correlations, although is not fully consistent with either the EPR or Bell versions of local realism.

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