The missing piece: a new relativistic wave equation, the Pearcey equation

To cite this article: Ambra Lattanzi et al 2019 J. Phys.: Conf. Ser. 1194 012065

View the article online for updates and enhancements.
The missing piece: a new relativistic wave equation, the Pearcey equation.

Ambra Lattanzi¹, Decio Levi² and †Amalia Torre³
¹H. Niewodniczański Institute of Nuclear Physics, Polish Academy of Sciences, ul. Eljasza-Radzikowskiego 152, PL 31342 Kraków, Poland
²University Rome Tre, Dip. Mathematics and Physics and INFN, Sezione Roma Tre, via della Vasca Navale 84, 000146 Rome, Italy
³ENEA FSN-FUSPHYS-TSM via E. Fermi 45, 00044 Frascati (Rome), Italy
E-mail: ambra.lattanzi@ifj.edu.pl

Abstract. The relativistic wave equations currently used in physics are the Dirac equation and the Klein-Gordon equation. Only recently, the Salpeter equation has stimulated the interest of the scientific community. In this paper a new relativistic wave equation for particles with non-zero rest mass, the Pearcey equation, is presented. There is great formal similarity between the Pearcey equation, the Schrödinger equation and the Salpeter equation. In particular, the Pearcey equation allows us to understand the onset of the relativistic behaviour in a similar way as the Salpeter equation. It is possible also to keep the parallelism between optics and quantum mechanics at which we are used in non relativistic quantum mechanics.

1. Introduction
In 1905, starting from the observation of Michelson and Morley in 1887, Einstein proposed the theory of special relativity, revolutionizing the physics of the time. Then, the problem of unifying Einstein’s special relativity and quantum mechanics arose with the development of quantum mechanics. Different attempts were made to generalize the Schrödinger equation [1] in a form consistent with Einstein’s theory of relativity. The Klein-Gordon [2], the Dirac [3] and the Salpeter equation [4–6] were proposed as relativistic wave equations, each having its own advantages and disadvantages.

Although the Salpeter equation is a relativistic version of the Schrödinger equation, only recently it has stimulated the interest of the scientific community [7–12]. This is due to the mathematical complexity we face for its non local nature. However, the non locality does not disturb the light cone structure [10, 11] but makes it difficult to obtain rigorous analytical statements about the time-dependent and stationary solutions of the equation. Another difficulty arises from the fact that the equation is not covariant. However, it has been proven that the Hilbert space of its solutions is invariant under the Lorentz-group of transformations [13]. Summing up, the Salpeter equation makes it possible to have a quantum relativistic description which does not conflict with the principles of the special relativity, even if it is a non-local and non-covariant equation.

From the study of the Salpeter equation, we can present a new wave-like equation, the Pearcey equation [10, 11]. This equation shows the emergence of the relativistic features. Moreover, this equation preserves the analogy between quantum mechanics and optics.
The paper is organized as follows. Section 2 is dedicated to the presentation of the Pearcey equation and its fundamental solution for a free spinless particle. In Section 3 we focus our attention on the evolution of an initial Gaussian wave-packet. In this way we can compare the different features appearing in the evolution of the Pearcey equation with respect to those of the Salpeter equation. In Section 4 we derive the general expression of the solutions of the Pearcey equation with a linear time-dependent potential. Some conclusions are presented in Section 5.

2. The Pearcey equation

The relativistic expression for the energy of a free-particle is

\[ E = \sqrt{m^2c^4 + p^2c^2}, \]

where \( m \) and \( p \) denote respectively the rest mass and the momentum of the particle and \( c \) is the speed of light.

By with the standard quantization rules

\[ E \rightarrow i\hbar \frac{\partial}{\partial t}, \quad x \rightarrow x, \quad p \rightarrow -i\hbar \frac{\partial}{\partial x} \]

consistent with the Newton-Wigner localization scheme [14, 15], (1) yields the spinless Salpeter equation [4–6]

\[ i\hbar \frac{\partial \Psi(x,t)}{\partial t} = \sqrt{m^2c^4 - c^2\hbar^2 \frac{\partial^2}{\partial x^2}} \Psi(x,t). \]

Let us consider the series expansion of the relativistic energy-momentum relation (1) for a free-particle with respect to \( \frac{p}{mc} \) up to fourth order, namely

\[ E = mc^2 \left[ 1 + \frac{1}{2} \left( \frac{p}{mc} \right)^2 - \frac{1}{8} \left( \frac{p}{mc} \right)^4 \right]. \]

In accord with the above standard quantization rules (2) we obtain the (1+1)D Pearcey equation:

\[ i\hbar \frac{\partial \psi(x,t)}{\partial t} = mc^2 \left[ 1 - \frac{\hbar^2}{2m^2c^2} \frac{\partial^2}{\partial x^2} - \frac{\hbar^4}{8m^4c^4} \frac{\partial^4}{\partial x^4} \right] \psi(x,t), \]  

which in the momentum-space representation reads

\[ i\hbar \frac{\partial \tilde{\psi}(p,t)}{\partial t} = mc^2 \left[ 1 + \frac{1}{2} \left( \frac{p}{mc} \right)^2 - \frac{1}{8} \left( \frac{p}{mc} \right)^4 \right] \tilde{\psi}(p,t). \]

Equation (4) is an evolution equation in time and thus it requires the knowledge of the initial condition for fixing the dynamics of the system. We denote with \( \psi(x,0) = \psi_0(x) \) the initial condition.

In order to define the solution of (4), we resort to one of the most common techniques for linear partial differential equations (PDEs): the Fourier transform method. Exploiting the Fourier transform-based procedure, we get the general solution of (4)

\[ \psi(x,t) = e^{-\frac{i}{\hbar} mc^2}{\int_{-\infty}^{+\infty} \hat{\psi}_0(p) = \psi_0(p) \sqrt{2\pi} \hbar} \left( \frac{p}{mc} \right) d\xi, \]

where \( \tilde{\psi}_0(p) \) is the Fourier transform of the initial wave function. Equation (6) holds under the requirement that \( \tilde{\psi}_0(\xi) \) - the initial wave function in space coordinate - tends to zero sufficiently fast as \( \xi \rightarrow \pm \infty. \)
To simplify the analysis of the equation, we introduce the dimensionless variables $\xi$, $\tau$ and $\kappa$, expressed in terms of the reduced Compton wavelength $\lambda_C = \frac{\hbar}{mc}$:

$$
\xi = \frac{x}{\lambda_C}, \quad \tau = \frac{ct}{\lambda_C}, \quad \kappa = \frac{p}{mc}
$$

(7)

so that (6) becomes:

$$
\psi(\xi,\tau) = e^{-i\tau} \sqrt{2\pi} \int_{-\infty}^{+\infty} e^{i\tau \left(4\kappa^4 - 2\kappa^2 + \kappa \xi\right)} \tilde{\psi}_0(\kappa) d\kappa.
$$

(8)

Choosing $\psi_0(\xi) = \delta(\xi)$ ($\tilde{\psi}_0(\kappa) = \frac{1}{\sqrt{2\pi}}$), the above equation yields the response of the system to an impulse, i.e.

$$
S(\xi,\tau) = e^{-i\tau} \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{i\tau \left(4\kappa^4 - 2\kappa^2 + \kappa \xi\right)} d\kappa.
$$

(9)

$S$ represents the fundamental solution of the Pearcey equation (8), and hence also the kernel of the transformation. This integral is already well-known in literature: it is the Pearcey integral defined [16–21] by

$$
P(x,t) = \int_{-\infty}^{+\infty} e^{i(s^4 + x^2 s^2 + y s)} ds,
$$

(10)

where $x$, $y$ and $s$ may be in general complex numbers. By the change of the variable $\kappa = s \left(\frac{8}{\tau}\right)^{\frac{1}{4}}$ in the integral (9), as $\tau > 0$, we have:

$$
S(\xi,\tau) = \frac{e^{-i\tau}}{\pi (2\tau)^{\frac{1}{4}}} P\left(-\sqrt{2\tau}, \left(\frac{8}{\tau}\right)^{\frac{1}{4}} \xi\right).
$$

(11)

This is the reason why we use the name *Pearcey equation* for the wave-like equation (4). The correspondence between the Pearcey integral and the fundamental solution of the Pearcey equation allows us to establish a direct link between optics and quantum mechanics. Equation (11) can be considered as the relativistic counterpart of the analogy between the Schrödinger equation and the paraxial wave equation. Figure 1 presents the uniformly distributed *granularity*,

Figure 1. $(\xi,\tau)$-contourplots of $|S(\xi,\tau)|^2$. 
which is also present in the contourplot of the fundamental solution of the Salpeter equation [10, 11]. In Fig. 1 we can appreciate their symmetric distribution due to the homogeneity of the spacetime. Moreover, these spots are positioned at the intersections of the parallels to the edges of the light cone resembling the discrete-like diffraction patterns arising from the impulse response (i.e. single state exitation) in periodic photonic lattices, when the first and the second order coupling interactions are involved [11].

3. Evolution of the Gaussian wave packet

We study here the evolution ruled by the Pearcey equation (8) of a Gaussian wave function

$$\psi^G_0(\xi) = \frac{1}{\sqrt{2\pi w^2}} e^{-\frac{\xi^2}{2w^2}} \quad (w > 0),$$

such that \( \lim_{\xi \to 0} \psi^G_0(\xi) = \delta(\xi) \).

The parameter \( w \) enters in the definition of the widths of the Gaussian initial function and of its Fourier transform. The widths of these two functions are inversely proportional, as prescribed by the uncertainty product and therefore we find:

$$\sigma_\xi = \frac{w}{\sqrt{2}}, \quad \sigma_\kappa = \frac{1}{\sqrt{2w}},$$

where \( \sigma_\xi \) ans \( \sigma_\kappa \) are respectively the square root of the variance in the space and Fourier domain.

The evolution of the initial Gaussian input (12) governed by the Pearcey equation is

$$\psi^G(\xi, \tau) = \frac{e^{-i\tau}}{2\pi} \int_{-\infty}^{+\infty} e^{i\left(\frac{1}{8}\kappa^4 - \frac{1}{2}(\tau - i\omega)\kappa^2 + \kappa\xi\right)} d\kappa.$$  

Figure 2. \((\xi, \tau)\)-contourplots of \(|\psi^G(\xi, \tau)|^2\) for \((a) w = 0.2\) and \((b) w = 2.0\).

Figure 2 shows the \((\xi, \tau)\)-contourplots of \(|\psi^G(\xi, \tau)|^2\) for specific value of \(w\). We see that for small \(w\) the wave function behaves much like the fundamental solution \(S(\xi, \tau)\). With increasing \(w\), the isolates spots tend to merge into a more compact structure as observed in the solutions of the Salpeter equation [10–12].
4. Formal expression of the solution in case of linear potential

In this Section we study the Pearcey equation with a linear potential of the form:

$$ V(x, t) = f(t)x, $$

where the linear energy density $f(t)$ takes into account any eventual time dependence.

In the configuration space we have the equation

$$ i\hbar \frac{\partial \tilde{\psi}(x, t)}{\partial t} = mc^2 \left[ 1 - \frac{1}{2} \left( \frac{p}{mc} \right)^2 - \frac{1}{8} \left( \frac{p}{mc} \right)^4 + f(t)x \right] \tilde{\psi}(x, t), \quad \tilde{\psi}(x, 0) = \tilde{\psi}_0(x). $$

Despite its apparent simplicity, the linear potential in relativistic quantum mechanics is very significant. It plays a key role in explaining the quark confinement in quantum chromodynamics (QCD) [22].

Nambu [23] suggested a model for the quark confinement using a linear potential interaction for large enough distance between the quarks. This interaction can be interpreted as a string tension able to "tie together" quarks inside the hadrons, the so called flux tubes. These strings are narrow tubes of color flux lines in which all field energy is concentrated.

Numerical simulations of QCD have demonstrated [24] the correctness of Nambu’s conjecture, but until now there is no completely accepted theory [25]. Following this idea, for enough distant quarks, the coefficient of the linear potential $f(t)$ can be defined as a string tension, which has the dimension of an energy density.

In [7, 8, 10, 26], the analysis was performed in the case of a spinless Salpeter equation. Here we want to observe how the Pearcey equation behaves for a time-dependent linear potential.

Working in Fourier space, the initial value problem (16) can be recast as

$$ i\hbar \frac{\partial \tilde{\psi}(p, t)}{\partial t} = mc^2 \left[ 1 + \frac{1}{2} \left( \frac{p}{mc} \right)^2 - \frac{1}{8} \left( \frac{p}{mc} \right)^4 + i\hbar f(t) \frac{\partial}{\partial p} \right] \tilde{\psi}(p, t), \quad \tilde{\psi}(p, 0) = \tilde{\psi}_0(p). $$

To solve (17) we introduce the unitary operator $\tilde{U}$

$$ \tilde{U}(p, t) = e^{F(t) \frac{\partial}{\partial p}}, $$

where the time-dependent function $F(t) = \int^t_0 f(\tau) d\tau$ has the dimension of a momentum, and the function $\tilde{\phi}$ is defined by

$$ \tilde{\psi}(p, t) = \tilde{U}(p, t) \tilde{\phi}(p, t). $$

Evidently, $\tilde{U}$ is a translational operator in $p$ and $\tilde{\phi}_0(p) = \tilde{\psi}_0(p)$. Replacing (19) into (17), we get an equation for $\tilde{\phi}$

$$ i\hbar \frac{\partial \tilde{\phi}(p, t)}{\partial t} = \left\{ \tilde{U}^{-1} mc^2 \left[ 1 + \frac{1}{2} \left( \frac{p}{mc} \right)^2 - \frac{1}{8} \left( \frac{p}{mc} \right)^4 \right] \tilde{U} \right\} \tilde{\phi}(p, t), $$

with the $p$-derivative absorbed in $\tilde{U}$. In order to evaluate the action on the operator inside the square bracket, we can exploit the following formula which is derived from the Hadamard relation

$$ e^{\alpha A} f(B)e^{-\alpha A} = f(B + \alpha c), $$

valid in the particular case of two operators $A$ and $B$ such that their commutator is a $c$-number [27]. In our case we have $[\frac{\partial}{\partial p}, p] = 1$ and (20) can be recast as

$$ i\hbar \frac{\partial \tilde{\phi}(p, t)}{\partial t} = mc^2 \left[ 1 + \frac{1}{2} \left( \frac{p - F(t)}{mc} \right)^2 - \frac{1}{8} \left( \frac{p - F(t)}{mc} \right)^4 \right] \tilde{\phi}(p, t). $$
Now we can solve the latter equation by exponentiation and we get:

$$\tilde{\phi}(p, t) = \exp\left[-\frac{i}{\hbar} \int_0^t mc^2 \left[1 + \frac{1}{2} \left(\frac{p - F(t')}{mc}\right)^2 - \frac{1}{8} \left(\frac{p - F(t')}{mc}\right)^4\right] dt'\right] \phi_0(p).$$

(22)

Considering the definition of $\tilde{\phi}$ in (19), we finally obtain the solution of (17) in the form

$$\tilde{\psi}(p, t) = \exp\left[-\frac{i}{\hbar} \int_0^t mc^2 \left[1 + \frac{1}{2} \left(\frac{p - F(t') + F(t)}{mc}\right)^2 - \frac{1}{8} \left(\frac{p - F(t') + F(t)}{mc}\right)^4\right] dt'\right] \tilde{\psi}_0(p + F(t)).$$

(23)

The corresponding solution in the configuration space is obtained by the inverse Fourier transform

$$\psi(x, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left[-\frac{i}{\hbar} \int_0^t mc^2 \left[1 + \frac{1}{2} \left(\frac{p - F(t') + F(t)}{mc}\right)^2 - \frac{1}{8} \left(\frac{p - F(t') + F(t)}{mc}\right)^4\right] dt'\right] \tilde{\psi}_0(p + F(t)) e^{ipx} dp.$$

(24)

Figure 3. $(\xi, \tau)$-contourplots of $|\psi^G(\xi, \tau)|^2$ in (a) the potential strenght $(f(t)$ in adimensional unit) is $1$ and in (b) is $2$.

In Fig. 3 we show how the Gaussian initial input evolves in presence of a linear potential. As shown in Fig. 3, the behavior is similar to what is observed in [7,10] for the Salpeter equation, confirming the correctness of the result obtained. As the field strenght increases, the lines have a smoother profile and the initial spot, Fig. 3a, is absorbed into a more tubular structure. Spots tends to merge as in the free-Gaussian evolution.

5. Conclusions

In this work we presented a new relativistic wave equation, the Pearcey equation, obtained from the fourth-order approximation with respect to $\frac{p}{mc}$ of the relativistic energy of a free-particle under the standard quantization rules. Although it differs from the Schrödinger equation only by a term, the Pearcey equation encloses and amplifies the features of the relativistic evolution observed in the Salpeter equation [10,11].

The evolutions, both in the free case and in the linear potential case, look like the evolution ruled by the Salpeter equation thus allowing to circumvent the problem of the non locality given...
by the square root without losing the essence of the relativistic behavior. One of the most intriguing features of the relativistic evolution is the "granularity". In the evolution described by the Salpeter equation this granularity is restricted only on the edges of the light cone structure, whereas in the Pearcey function it is free to spread inside the time-like region. With considerations similar to those which gave the Pearcey equation (4) it will be possible to create a relativistic counterpart of the heat equation and the paraxial wave equation. So the Pearcey equation can have interesting developments and it can also play a key role as an equation for the relativistic Brownian motion of a particle. Work is in progress in this direction.

6. Acknowledgements.
AL acknowledges the support of the NCN research project OPUS 12 no. UMO-2016/23/B/ST3/01.
DL has been supported by INFN IS-CSN4 Mathematical Methods of Nonlinear Physics.
AL wishes to thank Dr. K. Górska, Dr. A. Horzela and Dr. G. Dattoli for valuable comments and discussions.

References
[1] Schrödinger E Phys. Rev. 1926 28 pp 1049-1070
[2] Klein O 1926 Zeitschrift für Physik 37 pp 895-906
[3] Dirac P A M and R H Fowler 1928 Proc. R. Soc. Lond. A 117 pp 610-624
[4] Salpeter E E and Bethe H A 1951 Phys. Rev. 84 pp 1232-1242
[5] Salpeter E E 1952 Phys. Rev. 87 pp 328-343
[6] Greiner W and Reinhardt J 1994 Quantum Electrodynamics (Berlin Heidelberg: Springer-Verlag)
[7] Kowalski K and Rembieliński J 2011 Phys. Rev. A 84 p 012108
[8] Dattoli G, Sabia E, Górska K, Horzela A and Fenson K A 2015 J. Phys. A. Math. Theor. 48 p 125203
[9] Kowalski K and Rembieliński J 2010 Phys. Rev. A 81 p 012118
[10] Lattanzi A 2016 Thesis, University of Roma Tre (I)
[11] Torre A, Lattanzi A and Levi D 2017 Annalen der Physik 529 p 1600231
[12] Torre A, Lattanzi A and Levi D 2017 Time-Dependent Free-Particle Salpeter Equation: Features of the Solutions Quantum Theory and Symmetries with Lie Theory and Its Applications in Physics Volume 2 (Springer Proceeding in Mathematics & Statistics vol 255) ed V Dobrev (Singapore: Springer) pp 297-307
[13] Foldy L L 1956 Phys. Rev. 102 pp 568-581
[14] Newton T D and Wigner E P 1949 Rev. Mod. Phys. 21 pp 400-406
[15] Hermann L O Localization in relativistic quantum theories, at http://philsci-archive.pitt.edu/id/eprint/5727
[16] Pearcey T 1946 Phil. Mag. S. 737 pp 311–317
[17] Poston T and Stewart I 1997 Catastrophe Theory and its Applications (New York: Dover Publications Inc.)
[18] Connor J N L and Curtis P R 1982 J. Phys. A: Math. Gen. 15 pp 1179-1190
[19] Connor J N L and Hobbs C A 2004 Chem. Phys. 23 pp 13-19
[20] Kirk R 2008 J. Opt. Soc. Am. 25 pp 1682-1690
[21] Greensite J 2011 An introduction to the confinement problem (Lecture Notes in Physics 821) (Berlin Heidelberg: Springer-Verlag)
[22] Bali G S et al. 2000 Phys. Rev. D 62 p 054503
[23] Hothi N and Bisht S I 2013 Electronic Journal of Theoretical Physics 10 pp 81–100
[24] Dattoli G, Ottaviani P L, Torre A and Vázquez L 1997 La Rivista del Nuovo Cimento 20 p 3.