Boundedness from below conditions for a general scalar potential of two real scalars fields and the Higgs boson

Yisheng Song¹, Liqun Qi²,³

¹ School of Mathematical Sciences, Chongqing Normal University, Chongqing, P.R. China, 401331; Email: yisheng.song@cqnu.edu.cn
² Department of Mathematics, School of Science, Hangzhou Dianzi University, Hangzhou 310018, P. R. China;
³ Department of Applied Mathematics, The Hong Kong Polytechnic University, Hung Hom, Kowloon, Hong Kong;
Email: maqilq@polyu.edu.hk

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Abstract

The most general scalar potential of two real scalar fields and a Higgs boson is a quartic homogeneous polynomial about 3 variables, which defines a 4th order 3 dimensional symmetric tensor. Hence, the boundedness from below of such a scalar potential involves the positive (semi-)definiteness of the corresponding tensor. So, we mainly discuss analytical expressions of positive (semi-)definiteness for such a special 4th order 3-dimension symmetric tensor in this paper. Firstly, an analytically necessary and sufficient condition is given to test the positive (semi-)definiteness of a 4th order 2 dimensional symmetric tensor. Furthermore, by means of such a result, the necessary and sufficient conditions of the boundedness from below are obtained for a general scalar potential of two real scalar fields and the Higgs boson.

Keyword: scalar potentials; boundedness from below; 4th order Tensors; Positive definiteness; Homogeneous polynomial; Analytical expression.

1 Introduction

The boundedness from below of a scalar potential makes physical sense, which simply implies that such a scalar potential is positive (or non-negative). The

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polynomial degree of the potential is 4 when one keeps the scalar interactions renormalizable \[1\]. Then the condition for the potential of \(n\) real scalar fields \(\phi_i\) \((i = 1, 2, \cdots, n)\) to be bounded from below in the strong sense is equivalent to the requirement that

\[V(\phi) = \sum_{i,j,k,l=1}^{n} v_{ijkl} \phi_i \phi_j \phi_k \phi_l > 0.\]  

(1)

Let \(V = (v_{ijkl})\). Then \(V\) is a 4th order symmetrical tensor, and hence, the above requirement \((1)\) is the positive definiteness of the tensor \(V\). Qi \[2, 3\] first used and introduced the positive definiteness and copositivity of tensors. An \(n\)th order \(n\) dimensional real tensor \(V = (v_{i_1i_2\cdots i_m})\) is said to be

(i) positive semi-definite if \(Vx^m = \sum_{i_1,i_2,\cdots,i_m=1}^{n} v_{i_1i_2\cdots i_m} x_{i_1}x_{i_2}\cdots x_{i_m} \geq 0\) for all \(x \in \mathbb{R}^n\) and an even number \(m\);

(ii) positive definite if \(Vx^m > 0\) for all \(x \in \mathbb{R}^n \setminus \{0\}\) and an even number \(m\);

(iii) copositive if \(Vx^m \geq 0\) for all \(x \geq 0\);

(iv) strictly copositive if \(Vx^m > 0\) for all \(x \geq 0\) and \(x \neq 0\).

Kannike \[4–6\] presented the vacuum stability conditions of general scalar potentials of two real scalar fields \(\phi_1\) and \(\phi_2\) and the Higgs boson \(H\), and studied the sufficient condition of boundedness from below for scalar potential of the **Standard Model** (for short, **SM**) Higgs \(H_1\), an inert doublet \(H_2\) and a complex singlet \(S\). In fact, such two problems were solved by Kannike \[4\], where the first problem involves the positive definiteness of the corresponding symmetric tensor and the second problem requires the copositivity of the corresponding symmetric tensor. Chauhan \[7\] gave an analytical vacuum stability condition of the left-right symmetric model for successful symmetry breaking. Ivanov \[8\] presented the stability conditions in multi-Higgs potentials. Bahl et.al. \[9\] provided the analytically sufficient conditions of the vacuum stability for the two-Higgs-doublet potential with CP conservation, and showed a vacuum stability condition for the two-Higgs-doublet potential with CP violation depends on the Lagrange multiplier \(\zeta\). Recently, Song \[10\] established the boundedness from below conditions of scalar potential for the two-Higgs-doublet potential with explicit CP violation. Song \[11\] obtained the boundedness from below conditions of scalar potential for a general two-Higgs-doublet, which includes necessary conditions and sufficient conditions. Also see Faro-Ivanov \[12\], Belanger-Kannike-Pukhov-Raidal \[13, 14\], Ivanov- Köpke-Mühlleitner \[1\] for more details. In Refs. \[8, 15–18\], one can construct only one quadratic term and five quartic terms for the Higgs potential with the help of three Higgs doublets with equal electroweak quantum numbers, which is a quartic polynomial with real coefficients defined on complex field. Toorop-Bazzocchi-Merlo-Paris \[19, 20\] and Degee-Ivanov-Keus \[21\] turned such a polynomial from complex field to real field. In fact, they were trying to look for the analytical condition of such a polynomial to be positive.
Recently, Song-Qi [22] and Liu-Song [23] gave a different sufficient condition of copositivity for 4th order 3 dimensional symmetric tensors to find the boundedness from below conditions of scalar potential of the SM Higgs $H_1$, an inert doublet $H_2$ and a complex singlet $S$. Very recently, Qi-Song-Zhang [24] presented a necessary and sufficient condition of copositivity for such a tensor given by the above particle physical model. Song-Li [25] provided an analytically necessary and sufficient condition of the boundedness from below for such a scalar potential model.

In the past decades, many numerical algorithms were established to find some $H$-($Z$-)eigenvalues of a tensor [2, 26–38], and were applied to test the positive definiteness of such an even order tensor by means of the sign of the smallest $H$-($Z$-)eigenvalue. On the other hand, some classes of tensors with special structure may be determined directly their positive definiteness such as Hilbert tensor [39], diagonal dominant tensor [2], B-tensor [40–42] and others. However, the practical matters such as the vacuum stability of general scalar potentials of a few fields require analytical expressions. The most general scalar potential of two real scalar fields $\phi_1$ and $\phi_2$ and the Higgs doublet $H$ (Kamkik [4–6]) is

$$V(\phi_1, \phi_2, |H|) = \lambda_H|H|^4 + \lambda_{H20}|H|^2\phi_1^2 + \lambda_{H11}|H|^2\phi_1\phi_2 + \lambda_{H02}|H|^2\phi_2^2$$
$$+ \lambda_{40}\phi_1^4 + \lambda_{31}\phi_1^3\phi_2 + \lambda_{22}\phi_1^2\phi_2^2 + \lambda_{13}\phi_1\phi_2^3 + \lambda_{04}\phi_2^4. \tag{2}$$

Clearly, such a quartic homogeneous polynomial defines a 4th order 3 dimensional symmetric tensor $V = (v_{ijkl})$.

$$v_{1111} = \lambda_{40}, \ v_{2222} = \lambda_{04}, \ v_{3333} = \lambda_H, \ v_{1112} = \frac{1}{4}\lambda_{31}, \ v_{1222} = \frac{1}{4}\lambda_{13},$$
$$v_{1133} = \frac{1}{6}\lambda_{H20}, \ v_{1122} = \frac{1}{6}\lambda_{22}, \ v_{2233} = \frac{1}{6}\lambda_{H02},$$
$$v_{1233} = \frac{1}{12}\lambda_{H11}, \ v_{ijkl} = 0 \text{ for the others.} \tag{3}$$

and hence, the boundedness from below of such a scalar potential involves the positive (semi-)definiteness of such a tensor $V$. So this requires an analytic condition of positive (semi-)definiteness. For a 4th order 2 dimensional symmetric tensor, the analytical condition of the positive definiteness traced back to ones of Refs. Gadem-Li [43], Ku [44] and Jury-Mansour [45]. Wang-Qi [46] improved their proof and conclusions. However, the above result depends on the discriminant of such a polynomial. Recently, Guo [47] showed a new necessary and sufficient condition without the discriminant. Very recently, Qi-Song-Zhang [48] gave a new necessary and sufficient condition other than the above results. Hasan-Hasan [49] claimed that a necessary and sufficient condition of positive definiteness was proved without the discriminant. However, there is a problem in their argumentations. In 1998, Fu [50] pointed out that Hasan-Hasan’s results are sufficient only. Song [51] gave several analytically sufficient conditions of the positive definiteness of 4th order 3 dimensional symmetric tensor. Until now, peoples have not found an analytically necessary and sufficient condition of positive definiteness for a 4th order 3 dimensional symmetric tensor.
In this paper, we mainly concentrate on the analytical expressions of positive definiteness for a special 4th order tensor given by (13). More precisely, by means of Qi-Song-Zhang’s result, we first show an analytically necessary and sufficient condition of positive (semi-)definiteness of 4th order 2 dimensional symmetric tensors. Secondly, with the help of this conclusion, we discuss positive (semi-)definiteness of a 4th order 3-dimension symmetric tensor defined by (13). Then these analytic conditions are the necessary and sufficient conditions of the boundedness from below for a scalar potential (12) of two real scalar fields \(\phi_1\) and \(\phi_1\) and the Higgs doublet \(H\).

2 4th order symmetric real tensor

A 4th order 3 dimensional real tensor \(V\) consists of 81 entries in the real field \(\mathbb{R}\), i.e.,

\[
V = (v_{ijkl}), \quad v_{ijkl} \in \mathbb{R}, \quad i, j, k, l = 1, 2, 3.
\]

A tensor \(V\) is said to be symmetric if its entries \(v_{ijkl}\) are invariant for any permutation of its indices. It is well-known that a 4th order 3 dimensional symmetric tensor \(V\) is composed of 15 independent entries only,

\[
v_{1111}, v_{2222}, v_{3333}, v_{1222}, v_{1122}, v_{1113}, v_{2333},
\]

\[
v_{2223}, v_{1122}, v_{1133}, v_{2233}, v_{1223}, v_{1123}, v_{1233}.
\]

A 4th order 2 dimensional symmetric tensor \(V\) is composed of 5 independent entries only,

\[
v_{1111}, v_{2222}, v_{1122}, v_{1112}, v_{1122}.
\]

It is obvious that there is a consistent one-to-one match between a 4th order 3 dimensional symmetric tensor and a quartic homogeneous polynomial with 3 variables. Such a homogeneous polynomial, denoted as \(Vx^4\), i.e.,

\[
Vx^4 = \sum_{i,j,k,l=1}^{3} v_{ijkl}x_i x_j x_k x_l.
\]

Let \(\| \cdot \|\) denote any norm on \(\mathbb{R}^n\). Then the following conclusions on unit sphere are known [2,40,57].

Let \(V\) be a 4th order symmetric tensor and let \(S\) be the unit sphere on \(\mathbb{R}^n\), \(S = \{x \in \mathbb{R}^n : \|x\| = 1\}\). Then

(i) \(V\) is positive semi-definite if and only if \(Vx^4 \geq 0\) for all \(x \in S\);

(ii) \(V\) is positive definite if and only if \(Vx^4 > 0\) for all \(x \in S\).
3 Non-negativeness of quadratic and quartic polynomial

Let $P_2(t)$ be a quadratic polynomial,

$$P_2(t) = at^2 + bt + c,$$  \hspace{1cm} (6)

with $a > 0$. Then the following results should be well-known, which is showed hundreds of years ago. Also see Qi-Song-Zhang [48].

(1) $P_2(t) > 0$ for all $t \geq 0$ if and only if

$$b \geq 0 \text{ and } c > 0; \quad b < 0 \text{ and } 4ac - b^2 > 0.$$  \hspace{1cm} (7)

(2) $P_2(t) \geq 0$ for all $t \geq 0$ if and only if

$$b \geq 0 \text{ and } c \geq 0; \quad b < 0 \text{ and } 4ac - b^2 \geq 0.$$  \hspace{1cm} (8)

Let $P_4(t)$ be a quartic polynomial,

$$P_4(t) = at^4 + bt^3 + ct^2 + dt + e,$$  \hspace{1cm} (9)

where $a > 0$ and $e > 0$. Then the positivity (non-negativeness) of $P_4(t)$ were proved by Qi-Song-Zhang [48], recently.

(3) $P_4(t) \geq 0$ for all $t \in \mathbb{R}$ if and only if

$$\Delta \geq 0, \quad |b\sqrt{e} - d\sqrt{a}| \leq 4\sqrt{ace + 2ae\sqrt{ae}} \text{ and}$$

$$-2\sqrt{ae} \leq c \leq 6\sqrt{ae};$$ \hspace{1cm} (10)

$$c > 6\sqrt{ae} \text{ and } |b\sqrt{e} + d\sqrt{a}| \leq 4\sqrt{ace - 2ae\sqrt{ae}},$$  \hspace{1cm} (11)

where

$$\Delta = 4(12ae - 3bd + c^2)^3 - (72ace + 9bcd - 2c^3 - 27ad^2 - 27b^2e)^2.$$  \hspace{1cm} (12)

(4) $P_4(t) > 0$ for all $t \in \mathbb{R}$ if and only if

$$\Delta = 0, b\sqrt{e} = d\sqrt{a}, b^2 + 8a\sqrt{ae} = 4ac < 24a\sqrt{ae};$$  \hspace{1cm} (13)

$$\Delta > 0, |b\sqrt{e} - d\sqrt{a}| \leq 4\sqrt{ace + 2ae\sqrt{ae}} \text{ and}$$

$$-2\sqrt{ae} \leq c \leq 6\sqrt{ae},$$  \hspace{1cm} (14)

$$c > 6\sqrt{ae} \text{ and } |b\sqrt{e} + d\sqrt{a}| \leq 4\sqrt{ace - 2ae\sqrt{ae}}.$$  \hspace{1cm} (15)

$$\Delta = 0, b\sqrt{e} = d\sqrt{a}, b^2 + 8a\sqrt{ae} = 4ac < 24a\sqrt{ae};$$  \hspace{1cm} (16)

$$\Delta > 0, |b\sqrt{e} - d\sqrt{a}| \leq 4\sqrt{ace + 2ae\sqrt{ae}} \text{ and}$$

$$-2\sqrt{ae} \leq c \leq 6\sqrt{ae},$$  \hspace{1cm} (17)

$$c > 6\sqrt{ae} \text{ and } |b\sqrt{e} + d\sqrt{a}| \leq 4\sqrt{ace - 2ae\sqrt{ae}}.$$  \hspace{1cm} (18)
4 Boundedness from below of scalar potential of two real scalar and a Higgs boson

The most general scalar potential of two real scalar fields $\phi_1$ and $\phi_2$ and the Higgs doublet $H$ (Kannike [4, 5]) is

$$V(\phi_1, \phi_2, |H|) = \lambda_{40} \phi_1^4 + \lambda_{31} \phi_1^3 \phi_2 + \lambda_{22} \phi_1^2 \phi_2^2 + \lambda_{13} \phi_1 \phi_2^3 + \lambda_{04} \phi_2^4 + \lambda_H |H|^4 + \lambda_{H20} |H|^2 \phi_1^2 + \lambda_{H02} |H|^2 \phi_2^2.$$  (18)

This scalar potential defines a 4th order 3 dimensional symmetric tensor $V = (v_{ijkl})$ with its entries,

$$v_{1111} = \lambda_{40}, \ v_{2222} = \lambda_{04}, \ v_{3333} = \lambda_H, \ v_{1112} = \frac{1}{4} \lambda_{31}, \ v_{1222} = \frac{1}{4} \lambda_{13},$$

$$v_{1133} = \frac{1}{6} \lambda_{H20}, \ v_{1112} = \frac{1}{6} \lambda_{22}, \ v_{2233} = \frac{1}{6} \lambda_{H02},$$

$$v_{1233} = \frac{1}{12} \lambda_{H11}, \ v_{ijkl} = 0 \text{ for the others.} \quad (19)$$

In this section, we mainly discuss analytical expressions of positive definiteness of 4th order tensor $V = (v_{ijkl})$ given by (19). Furthermore, we present a necessary and sufficient condition of the boundedness from below of scalar potential of two real scalar fields $\phi_1$ and $\phi_2$ and the Higgs doublet $H$.

4.1 Positive definiteness of 4th order 2 dimensional symmetric tensors

Let $V = (v_{ijkl})$ be a 4th order 2 dimensional symmetric tensor with $v_{1111} > 0$ and $v_{2222} > 0$. For a vector $x = (x_1, x_2)^T$ such that

$$\|x\| = \sqrt{x_1^2 + x_2^2} = 1,$$

we may assume $x_2 \neq 0$ without loss of generality. We have

$$Vx^4 = \sum_{i,j,k,l=1}^{2} v_{ijkl} x_i x_j x_k x_l$$

$$= v_{1111} x_1^4 + 4v_{1112} x_1^3 x_2 + 6v_{1122} x_1^2 x_2^2 + 4v_{1222} x_1 x_2^3 + v_{2222} x_2^4,$$

and hence,

$$\frac{Vx^4}{x_2^4} = v_{1111} \left(\frac{x_1}{x_2}\right)^4 + 4v_{1112} \left(\frac{x_1}{x_2}\right)^3 + 6v_{1122} \left(\frac{x_1}{x_2}\right)^2 + 4v_{1222} \left(\frac{x_1}{x_2}\right) + v_{2222}.$$  

Clearly, $Vx^4 > 0$ if and only if

$$P(t) = at^4 + bt^3 + ct^2 + dt + e > 0, \text{ for all } t \in \mathbb{R}$$
\[ a = v_{1111}, \ b = 4v_{1112}, \ c = 6v_{1122}, \ d = 4v_{1222}, \ e = v_{2222}. \]

Then
\[ \Delta = 4(12ae - 3bd + c^2)^3 - (72ace + 9bcd - 2e^3 - 27ad^2 - 27b^2c)^2 \]
\[ = 4(12v_{1111}v_{2222} - 48v_{1112}v_{1222} + 36v_{1222}^2)^3 - (72 \times 6v_{1111}v_{1122}v_{2222} - 72 \times 6v_{1122}v_{2222}^2 - 72 \times 6v_{1122}v_{1122}^2 - 72 \times 6v_{1122}v_{1122}^2)^2 \]
\[ = 4 \times 12^3(I^3 - 27J^2), \]

where
\[ I = v_{1111}v_{2222} - 4v_{1112}v_{1222} + 3v_{1122}^2, \]
\[ J = v_{1111}v_{1122}v_{2222} + v_{1112}v_{1222}^2 - v_{1222}v_{1222}^2 + 3v_{1122}^2 - v_{1111}v_{1122}^2 - v_{1112}v_{2222}^2. \]

and hence, the sign of \( \Delta \) is the same as one of \( (I^3 - 27J^2) \). So, it follows from Eqs. (14) - (17) that by simply calculating, \( \mathcal{V} \) is positive definite, i.e., \( \mathcal{V}x^4 > 0 \) for all \( x \in \mathbb{R}^2 \) if and only if

(I) \[
\begin{align*}
I^3 - 27J^2 &= 0, \\
v_{1112} \sqrt{v_{2222}} &= v_{1222} \sqrt{v_{1111}}, \\
v_{1112}^2 + 2v_{1111}v_{2222} &= 6v_{1111}v_{1122} < 6v_{1111}v_{2222}; \\
I^3 - 27J^2 &> 0, \\
|v_{1112} \sqrt{v_{2222}} - v_{1222} \sqrt{v_{1111}}| &\leq \sqrt{6v_{1111}v_{1222}v_{2222} + 2(v_{1111}v_{2222})^3}, \\
(i) - \sqrt{v_{1111}v_{2222}} &\leq \sqrt{v_{1111}v_{2222}} \\
(ii) v_{1222} &> \sqrt{v_{1111}v_{2222}} \\
|v_{1112} \sqrt{v_{2222}} + v_{1222} \sqrt{v_{1111}}| &\leq \sqrt{6v_{1111}v_{1222}v_{2222} - 2(v_{1111}v_{2222})^3}. \\
\end{align*}
\]

Similarly, it follows from Eqs. (10) - (12) that \( \mathcal{V} = (v_{ijkl}) \) is positive semi-definite, i.e., \( \mathcal{V}x^4 \geq 0 \) for all \( x \in \mathbb{R}^2 \) if and only if

(II) \[
\begin{align*}
I^3 - 27J^2 &\geq 0, \\
|v_{1112} \sqrt{v_{2222}} - v_{1222} \sqrt{v_{1111}}| &\leq \sqrt{6v_{1111}v_{1222}v_{2222} + 2(v_{1111}v_{2222})^3}, \\
(i) - \sqrt{v_{1111}v_{2222}} &\leq \sqrt{v_{1111}v_{2222}} \\
(ii) v_{1222} &> \sqrt{v_{1111}v_{2222}} \\
|v_{1112} \sqrt{v_{2222}} + v_{1222} \sqrt{v_{1111}}| &\leq \sqrt{6v_{1111}v_{1222}v_{2222} - 2(v_{1111}v_{2222})^3}. \\
\end{align*}
\]

Next we give an analytically necessary and sufficient condition of the boundedness from below of scalar potential of two real scalar fields \( \phi_1 \) and \( \phi_2 \). The most general scalar potential of two real scalar fields \( \phi_1 \) and \( \phi_2 \) may be written as (Kannike [46])
\[
V(\phi_1, \phi_2) = \lambda_{40} \phi_1^4 + \lambda_{31} \phi_1^3 \phi_1 + \lambda_{22} \phi_1^2 \phi_2^2 + \lambda_{13} \phi_1 \phi_2^3 + \lambda_{04} \phi_2^4. \quad (20)
\]
Let $V = (v_{ijkl})$ is the coupling tensor with its entries

$$v_{1111} = \lambda_{40}, \quad v_{2222} = \lambda_{04}, \quad v_{1112} = \frac{1}{4} \lambda_{31}, \quad v_{1122} = \frac{1}{6} \lambda_{22}, \quad v_{1222} = \frac{1}{4} \lambda_{13}. \quad (21)$$

In fact, the boundedness from below of two real scalar fields $\phi_1$ and $\phi_2$ is equivalent to the positive definiteness of the coupling tensor $V = (v_{ijkl})$. Then we have

$$\Delta' = 4(12\lambda_{40}\lambda_{04} - 3\lambda_{31}\lambda_{13} + \lambda_{22}^2)^3 - (72\lambda_{40}\lambda_{22}\lambda_{04} + 9\lambda_{31}\lambda_{22}\lambda_{31} - 2\lambda_{22}^3 - 27\lambda_{40}\lambda_{13}^2 - 27\lambda_{31}\lambda_{04}^2). \quad (22)$$

Then from Conditions (I) and (II), the following results are easy to obtain.

Let $\lambda_{40} > 0$, $\lambda_{04} > 0$. Then $\bar{V}(\phi_1, \phi_2) > 0$ if and only if

$$\begin{aligned}
& \Delta' = 0, \quad \lambda_{31}\sqrt[3]{\lambda_{04}} = \lambda_{13}\sqrt[3]{\lambda_{40}}, \\
& \lambda_{31}^2 + 8\lambda_{40}\sqrt[3]{\lambda_{40}\lambda_{04}} = 4\lambda_{40}\lambda_{22} < 24\lambda_{40}\sqrt[3]{\lambda_{40}\lambda_{04}}, \\
& \Delta' > 0, \\
& (i) \quad -2\sqrt[3]{\lambda_{40}\lambda_{04}} \leq \lambda_{22} \leq 6\sqrt[3]{\lambda_{40}\lambda_{04}}, \\
& (ii) \quad \lambda_{22} > 6\sqrt[3]{\lambda_{40}\lambda_{04}} \text{ and} \\
& |\lambda_{31}\sqrt[3]{\lambda_{04}} + \lambda_{13}\sqrt[3]{\lambda_{40}}| \leq 4\sqrt[3]{\lambda_{40}\lambda_{22}\lambda_{04} - 2\lambda_{40}\lambda_{04}\sqrt[3]{\lambda_{40}\lambda_{04}}}. \\
\end{aligned} \quad (III)$$

$\bar{V}(\phi_1, \phi_2) \geq 0$ if and only if

$$\begin{aligned}
& \Delta' \geq 0, \\
& |\lambda_{31}\sqrt[3]{\lambda_{04}} - \lambda_{13}\sqrt[3]{\lambda_{40}}| \leq 4\sqrt[3]{\lambda_{40}\lambda_{22}\lambda_{04} + 2\lambda_{40}\lambda_{04}\sqrt[3]{\lambda_{40}\lambda_{04}}}, \\
& (i) \quad -2\sqrt[3]{\lambda_{40}\lambda_{04}} \leq \lambda_{22} \leq 6\sqrt[3]{\lambda_{40}\lambda_{04}}, \\
& (ii) \quad \lambda_{22} > 6\sqrt[3]{\lambda_{40}\lambda_{04}} \text{ and} \\
& |\lambda_{31}\sqrt[3]{\lambda_{04}} + \lambda_{13}\sqrt[3]{\lambda_{40}}| \leq 4\sqrt[3]{\lambda_{40}\lambda_{22}\lambda_{04} - 2\lambda_{40}\lambda_{04}\sqrt[3]{\lambda_{40}\lambda_{04}}}. \\
\end{aligned} \quad (IV)$$

In fact, the analytical condition (III) are the boundedness from below in the stronger sense for the scalar potential \[21\] of two real scalar fields $\phi_1$ and $\phi_1$. The analytical condition (IV) are the analytical boundedness from below condition.

4.2 Boundedness from below of two real scalar and a Higgs boson

The most general scalar potential of two real scalar fields $\phi_1$ and $\phi_2$ and the Higgs doublet $H$ (Kamniye [11-5]) is

$$V(\phi_1, \phi_2, |H|) = \lambda_{H}|H|^4 + \lambda_{H_{20}}|H|^2\phi_1^2 + \lambda_{H_{11}}|H|^2\phi_1\phi_2 + \lambda_{H_{02}}|H|^2\phi_2^2 + \lambda_{40}\phi_1^4 + \lambda_{31}\phi_1^4\phi_2 + \lambda_{22}\phi_1^2\phi_2^2 + \lambda_{13}\phi_1\phi_2^3 + \lambda_{04}\phi_2^4, \quad (23)$$

$$= \lambda_{H}|H|^4 + M(\phi_1, \phi_2)|H|^2 + \bar{V}(\phi_1, \phi_2), \quad (24)$$
where
\[ M(\phi_1, \phi_2) = \lambda_{H20}\phi_1^2 + \lambda_{H11}\phi_1\phi_2 + \lambda_{H02}\phi_2^2 \] (25)
and
\[ \bar{V}(\phi_1, \phi_2) = V(\phi_1, \phi_2, 0) = \lambda_{40}\phi_1^4 + \lambda_{31}\phi_1^3\phi_2 + \lambda_{22}\phi_1^2\phi_2^2 + \lambda_{13}\phi_1\phi_2^3 + \lambda_{04}\phi_2^4. \] (26)

Recently, Kannike [4, 5] studied the boundedness from below of \( V \) and \( \bar{V} \) and gave a analytical condition of \( V \) with respect to two variables \( \phi \)
moreover, the analytic necessary and sufficient conditions are showed for the
definiteness for this special 4th order 3 dimensional symmetric tensor (19), and
Which may be regarded as a quadratic polynomial with respect to \( t \)
H and the Higgs doublet \( \bar{H} \).

Let \( x = (\phi_1, \phi_2, |H|)^\top \). Then \( V(\phi_1, \phi_2, |H|) = \mathcal{V} x^4 \), where \( \mathcal{V} = (v_{ijkl}) \) is a 4th order 3 dimensional symmetric tensor given by [19]. Clearly, the tensor
given by \( \bar{V}(\phi_1, \phi_2) \) is a 4th order 2 dimensional principal sub-tensor of \( \mathcal{V} \).

Let \( \lambda_H > 0 \). It follows from the equation (24) that
\[ \mathcal{V} x^4 = \lambda_H |H|^4 + M(\phi_1, \phi_2)|H|^2 + \bar{V}(\phi_1, \phi_2). \]
Which may be regarded as a quadratic polynomial with respect to \( t = |H|^2 \),
\[ P_2(t) = at^2 + bt + c, \]
where \( a = \lambda_H, \ b = M(\phi_1, \phi_2), \ c = \bar{V}(\phi_1, \phi_2). \)

So from Eqs. (7) and (8), it yields to the following conclusion.
\[ V(\phi_1, \phi_2, |H|) = \mathcal{V} x^4 > 0 \] for all \( \phi_1, \phi_2, H \) if and only if for all \( \phi_1, \phi_2, \)
\[ (\text{V}) \]
\[ \begin{cases} M(\phi_1, \phi_2) \geq 0 \text{ and } \bar{V}(\phi_1, \phi_2) > 0; \\ M(\phi_1, \phi_2) < 0 \text{ and } 4\lambda_H \bar{V}(\phi_1, \phi_2) - (M(\phi_1, \phi_2))^2 > 0. \end{cases} \]
\[ V(\phi_1, \phi_2, |H|) = \mathcal{V} x^4 \geq 0 \] for all \( \phi_1, \phi_2, H \) if and only if for all \( \phi_1, \phi_2, \)
\[ (\text{VI}) \]
\[ \begin{cases} M(\phi_1, \phi_2) \geq 0 \text{ and } \bar{V}(\phi_1, \phi_2) \geq 0; \\ M(\phi_1, \phi_2) < 0 \text{ and } 4\lambda_H \bar{V}(\phi_1, \phi_2) - (M(\phi_1, \phi_2))^2 \geq 0, \end{cases} \]

It is obvious that \( M(\phi_1, \phi_2) = \lambda_{H20}\phi_1^2 + \lambda_{H11}\phi_1\phi_2 + \lambda_{H02}\phi_2^2 \) is a quadric form
with respect to two variables \( \phi_1, \phi_2 \), and hence, the inequality \( M(\phi_1, \phi_2) \geq 0 \) is
equivalent to positive semi-definiteness of its coefficient matrix,
\[ \begin{pmatrix} \lambda_{H20} & \frac{1}{2}\lambda_{H11} \\ \frac{1}{2}\lambda_{H11} & \lambda_{H02} \end{pmatrix}. \]
which is equivalent to
\[ \lambda_{H20} \geq 0, \; \lambda_{H02} \geq 0, \; \lambda_{H20} \lambda_{H02} - \frac{1}{4} \lambda_{H11}^2 \geq 0. \] (27)

Similarly, the inequality \( M(\phi_1, \phi_2) < 0 \) is equivalent to negative definiteness of its coefficient matrix, i.e., the matrix
\[
\begin{pmatrix}
-\lambda_{H20} & -\frac{1}{2} \lambda_{H11} \\
-\frac{1}{2} \lambda_{H11} & -\lambda_{H02}
\end{pmatrix}
\]
is positive definite if and only if
\[
\lambda_{H20} < 0, \; \lambda_{H02} < 0, \; \lambda_{H20} \lambda_{H02} - \frac{1}{4} \lambda_{H11}^2 > 0.
\] (28)

At the same time, the inequality \( V(\phi_1, \phi_2) > 0 \) can be obtained by the condition \((\text{III})\), and \( V(\phi_1, \phi_2) \geq 0 \) can be obtained by the condition \((\text{IV})\). Next we only need show
\[ 4\lambda_H V(\phi_1, \phi_2) - (M(\phi_1, \phi_2))^2 \geq 0 \; (> 0) \] for all \( \phi_1, \phi_2 \).

In order to proving this inequality holds, we take
\[
\begin{cases}
\lambda_{40} = 4\lambda_{40} \lambda_H - \lambda_{H20}^2, & \lambda_{04} = 4\lambda_{04} \lambda_H - \lambda_{H02}^2, \\
\lambda_{31} = 4\lambda_H \lambda_{31} - 2\lambda_{H20} \lambda_{H11}, & \lambda_{13} = 4\lambda_H \lambda_{13} - 2\lambda_{H02} \lambda_{H11}, \\
\lambda_{22} = 4\lambda_H \lambda_{22} - 2\lambda_{H20} \lambda_{H02} - \lambda_{H11}^2, \\
\Delta'' = 4(12\lambda_{40} \lambda_{04} - 3\lambda_{31} \lambda_{13} + \lambda_{22})^3 \\
& - (72\lambda_{40} \lambda_{22} \lambda_{04} + 9\lambda_{31} \lambda_{13}^2 - 2\lambda_{40}^2 - 27\lambda_{40} \lambda_{13}^2 - 27\lambda_{31} \lambda_{04})^2.
\end{cases}
\] (29)

Let \( V'(\phi_1, \phi_2) = 4\lambda_H V(\phi_1, \phi_2) - (M(\phi_1, \phi_2))^2 \). We may expand the polynomial \( V'(\phi_1, \phi_2) \) as follow,
\[
V'(\phi_1, \phi_2) = 4\lambda_H V(\phi_1, \phi_2) - (M(\phi_1, \phi_2))^2 \\
= (4\lambda_H \lambda_{40} - \lambda_{H20}^2) \phi_1^4 + (4\lambda_H \lambda_{31} - 2\lambda_{H20} \lambda_{H11}) \phi_1^2 \phi_2^2 \\
+ (4\lambda_H \lambda_{22} - 2\lambda_{H20} \lambda_{H02} - \lambda_{H11}^2) \phi_1^2 \phi_2^2 \\
+ (4\lambda_H \lambda_{13} - 2\lambda_{H02} \lambda_{H11}) \phi_1 \phi_2^4 + (4\lambda_{04} \lambda_H - \lambda_{H02}^2) \phi_2^4 \\
= \lambda_{40} \phi_1^4 + \lambda_{31} \phi_1^2 \phi_2^2 + \lambda_{22} \phi_1^2 \phi_2^2 + \lambda_{13} \phi_1 \phi_2^4 + \lambda_{04} \phi_2^4.
\]

So this definite a 4th order 2 dimensional symmetric tensor \( \mathcal{V} = (v_{ijkl}) \) with its entries
\[
v_{1111} = \lambda_{40}, \; v_{2222} = \lambda_{04}, \; v_{1112} = \frac{1}{4} \lambda_{31}, \; v_{1122} = \frac{1}{6} \lambda_{22}, \; v_{1222} = \frac{1}{4} \lambda_{13}.
\]

Let \( \lambda_{40} > 0, \; \lambda_{04} > 0 \). From the condition \((\text{I})\) or \((\text{III})\), we easily obtain the following conclusions.

10
\( V'(\phi_1, \phi_2) = 4\lambda_H V(\phi_1, \phi_2) - (M(\phi_1, \phi_2))^2 \geq 0 \) if and only if

\[
\begin{align*}
\Delta'' &= 0, \lambda_3' \sqrt{\lambda_{40}} = \lambda_3' \sqrt{\lambda_{40}}, \\
\lambda_3' + 8\lambda_{40} \sqrt{\lambda_{40} \lambda_{40}} &= 4\lambda_{40} \lambda_{22} < 24\lambda_{40} \sqrt{\lambda_{40} \lambda_{40}}; \\
\Delta'' &> 0, \\
|\lambda_3' \sqrt{\lambda_{40}} - \lambda_3' \sqrt{\lambda_{40}}| &\leq 4\sqrt{\lambda_2' \lambda_{22} \lambda_{40} + 2\lambda_2' \lambda_{40} \lambda_{40}} \sqrt{\lambda_{40} \lambda_{40}}. \\
\end{align*}
\]

(II)

\[
\begin{align*}
|\lambda_3' \sqrt{\lambda_{40}} - \lambda_3' \sqrt{\lambda_{40}}| &\leq 4\sqrt{\lambda_{40} \lambda_{22} \lambda_{40} + 2\lambda_{40} \lambda_{40} \lambda_{40}}. \\
(1) \ & 2\sqrt{\lambda_{40} \lambda_{40}} \leq \lambda_{22} \leq 6\sqrt{\lambda_{40} \lambda_{40}}; \\
(2) \ & \lambda_{22} > 6\sqrt{\lambda_{40} \lambda_{40}} \\
\end{align*}
\]

(III)

\[
\begin{align*}
\Delta'' &= 0, \lambda_3' \sqrt{\lambda_{40}} = \lambda_3' \sqrt{\lambda_{40}}, \\
\lambda_3' + 8\lambda_{40} \sqrt{\lambda_{40} \lambda_{40}} &= 4\lambda_{40} \lambda_{22} < 24\lambda_{40} \sqrt{\lambda_{40} \lambda_{40}}; \\
\Delta'' &> 0, \\
|\lambda_3' \sqrt{\lambda_{40}} - \lambda_3' \sqrt{\lambda_{40}}| &\leq 4\sqrt{\lambda_2' \lambda_{22} \lambda_{40} + 2\lambda_2' \lambda_{40} \lambda_{40}} \sqrt{\lambda_{40} \lambda_{40}}. \\
\end{align*}
\]

(V)

\[
\begin{align*}
|\lambda_3' \sqrt{\lambda_{40}} - \lambda_3' \sqrt{\lambda_{40}}| &\leq 4\sqrt{\lambda_{40} \lambda_{22} \lambda_{40} + 2\lambda_{40} \lambda_{40} \lambda_{40}}. \\
(1) \ & 2\sqrt{\lambda_{40} \lambda_{40}} \leq \lambda_{22} \leq 6\sqrt{\lambda_{40} \lambda_{40}}; \\
(2) \ & \lambda_{22} > 6\sqrt{\lambda_{40} \lambda_{40}} \\
\end{align*}
\]

(VI)

\[
\begin{align*}
\Delta'' &= 0, \lambda_3' \sqrt{\lambda_{40}} = \lambda_3' \sqrt{\lambda_{40}}, \\
\lambda_3' + 8\lambda_{40} \sqrt{\lambda_{40} \lambda_{40}} &= 4\lambda_{40} \lambda_{22} < 24\lambda_{40} \sqrt{\lambda_{40} \lambda_{40}}; \\
\Delta'' &> 0, \\
|\lambda_3' \sqrt{\lambda_{40}} - \lambda_3' \sqrt{\lambda_{40}}| &\leq 4\sqrt{\lambda_2' \lambda_{22} \lambda_{40} + 2\lambda_2' \lambda_{40} \lambda_{40}} \sqrt{\lambda_{40} \lambda_{40}}. \\
\end{align*}
\]

(VII)

\[
\begin{align*}
|\lambda_3' \sqrt{\lambda_{40}} - \lambda_3' \sqrt{\lambda_{40}}| &\leq 4\sqrt{\lambda_{40} \lambda_{22} \lambda_{40} + 2\lambda_{40} \lambda_{40} \lambda_{40}}. \\
(1) \ & 2\sqrt{\lambda_{40} \lambda_{40}} \leq \lambda_{22} \leq 6\sqrt{\lambda_{40} \lambda_{40}}; \\
(2) \ & \lambda_{22} > 6\sqrt{\lambda_{40} \lambda_{40}} \\
\end{align*}
\]

(VIII)

Altogether, combing the conditions (III), (V), (VII) and Eqs. (27)–(28), the analytical necessary and sufficient condition is established for the boundedness from below in the stronger sense of scalar potential of two real scalar fields \( \phi_1 \) and \( \phi_2 \) and the Higgs doublet \( \mathbf{H} \). Let \( \lambda_H > 0, \lambda_{40} > 0 \) and \( \lambda_{40} > 0 \). Then \( V(\phi_1, \phi_2, \mathbf{H}) > 0 \) for all \( \phi_1, \phi_2, \mathbf{H} \) if and only if
In this paper, for a scalar potential of two real scalar fields $\phi_1$ and $\phi_2$ and the Higgs doublet $H$, the analytically necessary and sufficient condition is built for the boundedness from below of such a scalar potential (18) also. Let $\lambda_H > 0$, $\lambda_{40} > 0$ and $\lambda_{04} > 0$. Then $V(\phi_1, \phi_2, |H|) \geq 0$ for all $\phi_1, \phi_2, H$ if and only if

\[
\begin{align*}
\text{(1) } & \lambda_{H20} \geq 0, \lambda_{H02} \geq 0, 4\lambda_{H20}\lambda_{H02} - \lambda_{H11}^2 \geq 0, \\
& \Delta' \geq 0,
\end{align*}
\]

\[
\begin{align*}
|\lambda_{31}\sqrt{\lambda_{40}} - \lambda_{13}\sqrt{\lambda_{40}}| & \leq 4\sqrt{\lambda_{40}\lambda_{22}\lambda_{04} + 2\lambda_{40}\lambda_{04}\sqrt{\lambda_{40}\lambda_{04}}}, \\
-2\sqrt{\lambda_{40}\lambda_{04}} & \leq \lambda_{22} \leq 6\sqrt{\lambda_{40}\lambda_{04}},
\end{align*}
\]

\[
\lambda_{22} > 6\sqrt{\lambda_{40}\lambda_{04}} \text{ and }
\]

\[
\begin{align*}
|\lambda_{31}\sqrt{\lambda_{40}} + \lambda_{13}\sqrt{\lambda_{40}}| & \leq 4\sqrt{\lambda_{40}\lambda_{22}\lambda_{04} + 2\lambda_{40}\lambda_{04}\sqrt{\lambda_{40}\lambda_{04}}}, \\
-2\sqrt{\lambda_{40}\lambda_{04}} & \leq \lambda_{22} \leq 6\sqrt{\lambda_{40}\lambda_{04}}.
\end{align*}
\]

\[
\lambda_{22} > 6\sqrt{\lambda_{40}\lambda_{04}} \text{ and }
\]

\[
\begin{align*}
|\lambda_{31}\sqrt{\lambda_{40}} + \lambda_{13}\sqrt{\lambda_{40}}| & \leq 4\sqrt{\lambda_{40}\lambda_{22}\lambda_{04} + 2\lambda_{40}\lambda_{04}\sqrt{\lambda_{40}\lambda_{04}}}, \\
-2\sqrt{\lambda_{40}\lambda_{04}} & \leq \lambda_{22} \leq 6\sqrt{\lambda_{40}\lambda_{04}}.
\end{align*}
\]

(2) $\lambda_{H20} < 0$, $\lambda_{H02} < 0$, $4\lambda_{H20}\lambda_{H02} - \lambda_{H11}^2 > 0$, $\lambda'_{40} > 0$, $\lambda'_{04} > 0$, $\Delta'' \geq 0,$

\[
\begin{align*}
|\lambda'_{31}\sqrt{\lambda'_{40}} - \lambda'_{13}\sqrt{\lambda'_{40}}| & \leq 4\sqrt{\lambda'_{40}\lambda'_{22}\lambda'_{04} + 2\lambda'_{40}\lambda'_{04}\sqrt{\lambda'_{40}\lambda'_{04}}}, \\
-2\sqrt{\lambda'_{40}\lambda'_{04}} & \leq \lambda'_{22} \leq 6\sqrt{\lambda'_{40}\lambda'_{04}}.
\end{align*}
\]

\[
\lambda'_{22} > 6\sqrt{\lambda'_{40}\lambda'_{04}} \text{ and }
\]

\[
\begin{align*}
|\lambda'_{31}\sqrt{\lambda'_{40}} + \lambda'_{13}\sqrt{\lambda'_{40}}| & \leq 4\sqrt{\lambda'_{40}\lambda'_{22}\lambda'_{04} + 2\lambda'_{40}\lambda'_{04}\sqrt{\lambda'_{40}\lambda'_{04}}}, \\
-2\sqrt{\lambda'_{40}\lambda'_{04}} & \leq \lambda'_{22} \leq 6\sqrt{\lambda'_{40}\lambda'_{04}}.
\end{align*}
\]

5 Conclusions

In this paper, for a scalar potential of two real scalar fields $\phi_1$ and $\phi_2$ and the Higgs doublet $H$, the analytically necessary and sufficient conditions of the boundedness from below are achieved with the help of the analytical expressions of positive definiteness for 4th order 2-dimension symmetric tensors. More precisely, for a 4th order 2-dimension symmetric tensor,

- The condition (I) is an analytically necessary and sufficient condition of positive definiteness;

- The condition (II) is an analytically necessary and sufficient condition of positive semi-definiteness.

For a scalar potential of two real scalar fields $\phi_1$ and $\phi_2$,

- The condition (III) is an analytically necessary and sufficient condition of the boundedness from below in the stronger sense;

- The condition (IV) is an analytically necessary and sufficient condition of the boundedness from below.

For a scalar potential of two real scalar fields $\phi_1$ and $\phi_2$ and the Higgs doublet $H$,

- The condition (V) is an analytically necessary and sufficient condition of the boundedness from below;

- the condition (IV) is an analytically necessary and sufficient condition of the boundedness from below in the stronger sense.
**Competing interest**

The authors declared that they have no conflict of interest.

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