Groups With At Most Twelve Subgroups

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In [2] I showed that the finite groups with a specified number of subgroups can always be described as a finite list of similarity classes. Neil Sloane suggested that I submit the corresponding sequence, the number of similarity classes with \( n \) subgroups, to his Online Encyclopedia of Integer Sequences [4]. I thought this would be a quick calculation until I discovered that my “example” case in the paper (\( n = 6 \)) was wrong (as noted in [3]). I realized that producing a reliable count required a full proof. This note contains the proof behind my computation of the first 12 terms of the sequence.

For completeness, we include a section of [2]:

Now, we saw [above] that cyclic Sylow subgroups of \( G \) which are direct factors allow many non-isomorphic groups to have the same number of subgroups. [This lemma] tells us that these are precisely the cyclic Sylow subgroups which lie in the center of \( G \). Consequently, if \( p_1, \ldots, p_c \) are the primes which divide \( |G| \) such that a Sylow \( p_i \)-subgroup is cyclic and central, then we can write

\[
G = P_1 \times P_2 \times \cdots \times P_c \times \tilde{O}(G)
\]

where each \( P_i \) is a Sylow \( p_i \)-subgroup, the set \( \pi = \{p_1, \ldots, p_c\} \) and \( \tilde{O}(G) \) is the largest normal subgroup of \( G \) with order not divisible by any prime in \( \pi \). We will write \( \tilde{G} = O_{\pi'}(G) \). In other words, \( \tilde{G} \) is the unique subgroup of \( G \) left after factoring out the cyclic, central Sylow subgroups. It is the part of \( G \) that we can hope to restrict in terms of the number of subgroups. On the other hand, if we substituted a different prime (relatively prime to \( |G| \)) for any one of the \( p_i \), it would not affect the number of subgroups of \( G \). Thus, we are led to define the following equivalence relation (which we will call “similar” for this note).

**Definition.** Let \( G \) and \( H \) be two finite groups. Write \( G \) as a product of cyclic central Sylow subgroups and \( \tilde{G} \) as above. Hence, \( G = P_1 \times P_2 \times \cdots \times P_c \times \tilde{G} \), and similarly, \( H = Q_1 \times Q_2 \times \cdots \times Q_d \times \tilde{H} \). We say that \( G \) is *similar* to \( H \) if, and only if, the following three conditions hold:

1. \( \tilde{G} \) is isomorphic to \( \tilde{H} \);
2. \( c = d \);
3. $n_i = m_i$ for some reordering, where $|P_i| = p_i^{n_i}$ and $|Q_i| = q_i^{m_i}$.

From the comments before the definition, we see that if $G$ is similar to $H$, then they will have the same number of subgroups. Also, note that the equivalence class of $G$ is determined by $\tilde{G}$ and the multiset $[n_1, \ldots, n_c]$. If we have a bound on the number of subgroups of $G$, then this will bound $c$ and each $n_i$. Therefore, the only remaining hole in our theorem will be filled with the following lemma.

**Lemma.** Given $k > 1$ there are a finite number of isomorphism classes of groups $G = \tilde{G}$ (i.e. $G$ has no cyclic, central Sylow subgroups) having at most $k$ subgroups.

We wish to identify all similarity classes of groups with at most 12 subgroups. To begin, we will find all groups with $G = \tilde{G}$ and at most 12 subgroups.

### 1 Abelian Groups

If $G$ is abelian, then no Sylow $p$-subgroup can be cyclic. This means that if $G$ is not a $p$-group, then it has at least $p + 1$ subgroups of order $p$ and $q + 1$ subgroups of order $q$ for some distinct primes $p$ and $q$ and so at least $(p + 1)(q + 1) \geq 12$ non-trivial subgroups. So, $G$ must be a non-cyclic $p$-group.

If $G$ contains $Z_p \times Z_p$, then it has at least $2p^2 + 2p + 4 \geq 16$ subgroups. Consequently, $G = Z_{p^r} \times Z_{p^s}$ with $r \geq s \geq 1$.

Now, $Z_{p^2} \times Z_{p^2}$ has $p^2 + 3p + 5 \geq 15$ subgroups, so we must have $s = 1$. Furthermore, $Z_{p^3} \times Z_p$ has $4p + 6 \geq 14$ subgroups, so $r \leq 3$.

Hence, the abelian groups with $G = \tilde{G}$ and at most 12 subgroups are:

- $r = 1$: $(p + 3$ subgroups) $p = 2, 3, 5, 7$
- $r = 2$: $(2p + 4$ subgroups) $p = 2, 3$
- $r = 3$: $(3p + 5$ subgroups) $p = 2$

### 2 Non-Abelian Groups

Now we consider non-abelian $G$. Let $q$ be any prime dividing $|G|$. If Sylow $q$ is not normal, there are at least $q + 1$ Sylow $q$-subgroups. If Sylow $q$ is normal, there is a $q$-complement. If the complement normal, then the Sylow $q$ cannot be cyclic (else $G \neq \tilde{G}$) and so has at least $q + 1$ maximal subgroups. Finally, if the complement is not normal, then we have at least $q$ complements and 1 Sylow $q$. So, in every case, with the trivial subgroup and $G$, the number of subgroups is at least $q + 3$. This implies $q \leq 9$.

??? Can $|G|$ be div by all of 2, 3, 5, and 7 ??? As below, at least 2 must be normal and their product has a complement with 2 primes that can have at most 3 subgroups $\Rightarrow \Leftarrow$ impossible.

Suppose $|G|$ is divisible by 3 distinct primes $p, q, r$.

If none of the Sylows are normal, we have at least $(p + 1) + (q + 1) + (r + 1) \geq 13$ subgroups, so at least one is normal. Call it an $r$-subgroup $N$. Complementing $N$, we have a two prime group $H$ in $G$ and every subgroup of $H$ gives rise to at least two subgroups of $G$ (itself and itself times $N$). So, $H$ has at most 6 subgroups.
Lemma. A group $H$ divisible by 2 primes with at most 6 subgroups is either $\mathbb{Z}_p \times \mathbb{Z}_q$, $\mathbb{Z}_p \times \mathbb{Z}_q^2$, or $S_3$.

Proof. If $H$ is abelian, one sylow has 2 subgroups, the other 2 or 3 subgroups, so $H = \mathbb{Z}_p \times \mathbb{Z}_q$ or $\mathbb{Z}_p \times \mathbb{Z}_q^2$. If $H$ has a non-abelian sylow, the sylow has at least 3 maximal subgroups, 1 prime order, 1 triv, 1 whole subgrp, plus the other sylow $\Rightarrow \geq 7$ subgrps, which is too many. So both sylows of $H$ are abelian and so at least one is not normal. Assume WLOG $p < q$. Then there are at least $p+1$ non-normal sylows, one other sylow, the whole group, and trivial $\Rightarrow \geq p + 4$ subgroups. So we must have equality on our estimate which means, $p = 2$, sylow 2 is not normal, $p + 1$ divides $|G|$, so $q = 3$, sylow 3 is normal, sylows have no non-trivial subgroups $\Rightarrow H = S_3$.

If $H$ is abelian with cyclic sylows, then neither Sylow can act trivially on $N$ or it would be central cyclic in $G$. Therefore $N$ cannot normalize the sylows or $H$ and so the number of conjugates of each is at least $r$. This gives at least $3r$ subgroups of $G$. Furthermore we have $G, N, 1,$ and $NH_p$ and $NH_q$, so at least $3r + 5$ subgroups. Consequently, we must have $r = 2$.

Our list of subgroups includes only $N$ and 1 from subgroups of $N$. So, if $N$ has at least 4 subgroups, then $G$ will have at least 13 subgroups. Therefore, $N$ is cyclic of order 2 or 4. But odd primes cannot act non-trivially on a cyclic group of order 2 or 4, so this situation is impossible.

Last, we consider the case of $H = S_3$. In this case, $r \geq 5$. If $H$ acts trivially on $N$, then $N$ cannot be cyclic and so has at least 8 subgroups. Since $S_3$ has 6 subgroups, $G$ will have at least 48 subgroups. So, $H$ acts non-trivially on $N$ and a sylow 2, $H_2$, must also act non-trivially on $N$. In particular, $N$ does not normalize $H$ or $H_2$ and, as above, this gives us at least $2r$ subgroups of $G$. We also have $G, N,$ and 1. Since $r$ is at least 5, this implies $G$ has more than 12 subgroups.

Consequently, $|G|$ must be divisible by at most two primes.

3 Groups with two primes in the order

Suppose $|G| = p^aq^b$, $p < q$ primes and $G$ is non-abelian.

The argument for two primes is more complicated than I would like. I’ll use indentation to organize assumptions (like in computer code).

If neither sylow is cyclic, then we have $(p + 1) + (q + 1)$ maximals in the sylows, the two sylows and the whole group and trivial. That is, the number of subgroups is at least $p + q + 6$. This can only be less than or equal to 12 when $p = 2, q = 3$. Furthermore, the close estimate means each sylow must be normal and prime squared order. That forces $G$ to be abelian $\Rightarrow \Leftarrow$.

If sylow $p$ is cyclic,
If sylow $p$ is normal, then the sylow $q$ cannot act non-trivially and so the sylow $p$ is cyclic and central $\Rightarrow \Leftarrow$.

else sylow $p$ is not normal, then the number of sylow $p$ is $\geq q$.

If sylow $q$ not cyclic, we have at least $q + 1$ maximals and so at least $2q + 4$ subgroups. Again the estimate forces sylow $q$ to be $Z_3 \times Z_3$ and $p = 2$, at least 10 subgroups. Furthermore, we cannot have 9 sylow 2’s, so there is a central subgroup of order 3. Counting again, we have whole, trivial, one 9, four 3’s, three sylow 2’s, three sylow 2 times central 3 $\Rightarrow \geq 13$ subgroups. $\Rightarrow \Leftarrow$

else sylow $q$ is cyclic, then $G$ must be supersolvable since both sylows are cyclic and it follows the sylow $q$ must be normal. Sylow $p$ must act non-trivially on $q$, so $(p, q) = (2, 3), (2, 5), (2, 7), \text{ or } (3, 7)$.

The only case where the action is not that of an element of order $p$ is when $p = 2, q = 5$ when $Z_4$ acts faithfully on $Z_5$. In this case, $G$ will have a quotient group of order 20 isomorphic to $Z_4$ acting faithfully on $Z_5$. Since this group has 14 subgroups, we do not need to consider such $G$. Therefore, we can assume the sylow $p$ acts as an element of order $p$. The sylow $p$ fixes only the identity in sylow $q$ (only possible non-triv action). This means the sylow $p$ is self-normalizing and so the number of sylow $p$ subgroups is $q^b$. Now the subgroup of order $p^{a-1}$ centralizes the sylow $q$ and so is the intersection of the sylow $p$’s. Hence the non-triv $p$-subgroups are order $p, \ldots, p^{a-1}$, and $q^b$ subgroups order $p^a$. There are $b$ non-triv $q$-subgroups and $(a - 1) \cdot b$ proper abelian subgroups divisible by $pq$. Furthermore, for each non-triv, proper $q$-subgroup, order $q^c$, we can multiply that by the sylow $p$’s to get subgroups of order $p^aq^c$. There will be $q^c$ different sylow $p$’s giving the same subgroup of order $p^aq^c$ and so a total of $q^{b-c}$ such subgroups. Consequently, the number of (non-abelian) proper subgroups divisible by $pq$ will be $q + q^2 + \cdots + q^{b-1} = (q^b - q)/(q - 1)$. Including trivial and whole group, the number of subgroups is:

$$(a - 1 + q^b) + b + (a - 1)b + \frac{q^b - q}{q - 1} + 2 = q^b + \frac{q^b - q}{q - 1} + a(b + 1) + 1.$$ 

The corresponding values are shown in the following tables.

| $q = 3$ | $q = 5$ | $q = 7$ |
|---------|---------|---------|
| $a$     | $b$     | $a$     |
| 1       | 1       | 1       |
| 1       | 6       | 1       |
| 2       | 8       | 2       |
| 3       | 10      | 2       |
| 4       | 12      | 3       |
| 5       | 14      | 4       |
| 16      | 19      | 10      |
| 22      | 37      | 37      |
| 25      | 40      | 40      |
| 28      | 43      | 43      |
|         | 34      | 60      |
|         | 10      | 12      |
|         | 63      | 63      |
|         | 66      | 66      |

Since the $q = 7$ table applies to both $p = 2$ and $p = 3$ we see there are 11 groups with at most 12 subgroups in this case.
else sylow $p$ is not cyclic, and so sylow $q$ must be cyclic and not central.

If neither sylow is normal, then number of sylow $p$ is at least $q$ and the number of sylow $q$ is at least $q + 1$. Including trivial and whole, we have at least $2q + 3$ subgroups, which means $q < 5$ and so we have $q = 3, p = 2$. We have at least three sylow 2’s and at least four sylow 3’s, triv, whole, and one sylow 2 will contain three subgroups of order 2. However, that’s already 12 subgroups and we have more order 2 subgroups in the other sylow 2’s. \(\Rightarrow\) \(\Leftarrow\)

If sylow $p$ is normal, then it cannot be cyclic, else it would be central. Furthermore, $Z_p \times Z_p \times Z_p$ has too many subgroups, so the sylow $p$ must be a two-generator group. So, the sylow $p$ has $p + 1$ maximal subgroups, there are at least $q + 1$ sylow $q$'s and with sylow $p$ itself, triv, and whole, we have at least $p + q + 5$ subgroups. If $p \geq 3$, this is too many, so we must have $p = 2$. If $q \geq 5$, then our count gives at least 12 subgroups. The only way to avoid going over 12 would be to have the sylow $p$ be $Z_2 \times Z_2$, sylow $q$ be $Z_5$. But $Z_5$ has no non-trivial action on $Z_2 \times Z_2$, and so we are left only with the case $p = 2, q = 3$, and at least 10 subgroups. Now, the alternating group $A_4$ satisfies these conditions and has 10 subgroups. If sylow 3 has order larger than 3, then there will be a central 3-subgroup whose product with the various 2-subgroups will give more than 12 subgroups. So, the sylow 3 must be order 3. If sylow 2 has order larger than 4, then the frattini subgroup of the sylow 2 and the frattini subgroup times the various sylow 3-subgroups give more than 12 subgroups. So, $A_4$ is the only group in this case.

If sylow $q$ is normal, we are already assuming the sylow $q$ is cyclic. Furthermore, the sylow $p$ is not normal and not cyclic. So we have at least $q$ sylow $p$'s and the sylow $p$'s have at least $p + 1$ maximal subgroups.

If two sylow $p$'s have a common maximal subgroup, $M$, then the normalizer of $M$ will contain at least two sylow $p$-subgroups of $G$ and so must be divisible by $q$. Thus the intersection of the normalizer with the sylow $q$-subgroup is a non-trivial central $q$ subgroup. Therefore, our subgroups include $q$ sylow $p$'s, $q$ sylow $p$'s times the central $q$, at least $p + 1$ maximal subgroups of sylow $p$, each of those $p + 1$ times the central $q$, the sylow $q$, the central $q$, triv, and whole for at least $2q + 2p + 6 \geq 16$ subgroups. \(\Rightarrow\) \(\Leftarrow\)

else no shared maximal subgroups, which means we have at least $q(p + 1)$ maximal subgroups of sylow $p$'s altogether. Therefore we have at least these $q(p + 1)$ subgroups, $q$ sylow $p$-subgroups, one sylow $q$, triv, and whole. That is, $qp + 2q + 3 \geq 15$ subgroups. \(\Rightarrow\) \(\Leftarrow\)

Hence, the non-abelian groups with $G = \bar{G}$, $|G|$ divisible by at least two primes, and at most 12 subgroups are:

- $Z_p \ltimes Z_q$: $(q + 3$ subgroups) $(p, q) = (2, 3), (2, 5), (2, 7), (3, 7)$
- $Z_p^2 \ltimes Z_q$ with action of order $p$: $(q + 5$ subgroups) $(p, q) = (2, 3), (2, 5), (2, 7), (3, 7)$
- $Z_8 \ltimes Z_q$ with action of order 2: $(q + 7$ subgroups) $q = 3$ or $5$
- $Z_{16} \ltimes Z_3$ with action of order 2: $(12$ subgroups)$
- $A_4$: $(10$ subgroups)$
4 $p$-Groups

Finally, we have the case of $G$ a non-abelian $p$-group, $|G| = p^n$.

The group $Z_p \times Z_p \times Z_p$ has at least 16 subgroups, so $G$ must be a two-generator group. Consequently, we have $G$, $p+1$ maximal, the Frattini subgroup, and triv $\Rightarrow \geq p+4$ subgroups.

When $p$ is odd we will have at least $p + 1$ order $p$, one of which might be the Frattini, so we’d have at least $2p + 4 \Rightarrow p = 2$ or 3.

Assume $|G'| = p$ and $Z(G)$ cyclic. Then $G' \subset Z(G)$ and for $x, y \in G$, $[x^p, y] = [x, y]^p = 1$ so $x^p$ is central and $\Phi(G) \subset Z(G)$. Now $|G : \Phi(G)| = p^2$ and so we must have $\Phi(G) = Z(G)$ is cyclic order $p^n$.

If $G$ has a single subgroup of order $p$, then $p = 2$ and $G$ is generalized quaternion. Now, each generalized quaternion 2-group contains the next smaller as a subgroup. So, we can just check: $Q_8$ has 6 subgroups, $Q_{16}$ has 11 subgroups, and $Q_{32}$ has 20. That is, $G$ must be $Q_8$ or $Q_{16}$.

Otherwise, $G$ has more than one subgroup of order $p$ and so we can choose $S \subset G$ with $S$ order $p$ and not in the center. Then $M = SZ(G)$ is an abelian subgroup isomorphic to $Z_p \times Z_{p^{n-2}}$. From the abelian case above, we see that the number of subgroups of $M$ is given in the following table.

| $p$ | 2  | 3  | 4  | 5  |
|-----|----|----|----|----|
| $n$ | 3  | 5  | 8  | 11 |

In addition to the subgroups in $M$, $G$ also has $p$ other maximal subgroups, and $G$ itself.

So, increasing each table entry by $p + 1$ we see the only possibilities for at most 12 subgroups are $|G| = 8$, 16, or 27.

By brute force check we find for order 8 the dihedral group $D_8$ with 10 subgroups, for order 16 a group with presentation $\langle a, b | a^2, b^8, b^a = b^5 \rangle$ having 11 subgroups, and for order 27 the extraspecial group of exponent 9 with 10 subgroups.

Thus we have found five groups with $|G'| = p$ and cyclic center. Now any non-abelian finite $p$-group will have such a group as a homomorphic image.

Note that no generalized quaternion group can have a smaller generalized quaternion group as a homomorphic image. One easy way to see this is to note that $Q_n/Z(Q_n) = D_{n/2}$ and any non-abelian image of a dihedral group will have more than one involution. We will use this fact several times below.

Suppose $|G| = 3^5$ and choose $K$ maximal such that $\overline{G} = G/K$ is non-abelian. It follows that $\overline{G}$ will have $|\overline{G}| = 3$ and $Z(\overline{G})$ cyclic. If $|K| \leq 3$, it follows from above that $\overline{G}$, and so $G$, will have more than 12 subgroups. If $|K| = 3^2$, then $|G| = 3^3$ and so must be the extraspecial group of exponent $3^2$. Thus, $\overline{G}$ has 10 subgroups. The only way we could have at most 12 subgroups in $G$ is if $K$ is cyclic and every subgroup either contains $K$ or is order 3 or 1 in $K$. However, this implies that $G$ has only one subgroup of order 3, which is impossible. Consequently, no non-abelian 3-group of order at least $3^5$ can have at most 12 subgroups. We know from above that there is only one such group of order $3^3$ and a computer check shows that there are no examples of order $3^4$ (these groups have at least 14 subgroups).

Now suppose $|G| = 2^6$ and choose $K$ maximal such that $\overline{G} = G/K$ is non-abelian. As above, $\overline{G}$ will have $|\overline{G}| = 2$ and $Z(\overline{G})$ cyclic. If $|K| \leq 2$, it follows from above that $\overline{G}$, and so
$G$, will have more than 12 subgroups. If $|K| = 2^2$, then $|G| = 2^4$ and so we see from above that $G$ is one of two groups each of which have 11 subgroups. Since $K$ has at least 2 proper subgroups, $G$ must have 13 or more subgroups. Finally, consider $|K| = 2^3$. Then $G$ must be $D_8$ or $Q_8$ with 10 or 6 subgroups respectively. Since $K$ has at least 3 proper subgroups, $G$ cannot be $D_8$ and so must be $Q_8$ with 6 subgroups. Even if all of the subgroups of $G$ either contained $K$ or were contained in $K$, then $K$ would have to have at most 7 subgroups (6 proper). However, the only groups of order 2^3 with 7 or fewer subgroups are $Z_8$ and $Q_8$. Since each of these have a single subgroup of order 2, our assumption would force $G$ to be generalized quaternion, which it clearly is not. Thus we see that no non-abelian 2-group of order at least 2^6 can have number of subgroups less than or equal to 12. A computer check shows there are no examples of order 2^5 (these groups have at least 14 subgroups) and only the two groups mentioned above for order 2^4.

Hence, the non-abelian $p$-groups with $G = \tilde{G}$ and at most 12 subgroups are:

- $D_8$: (10 subgroups)
- $Q_8$: (6 subgroups)
- $Q_{16}$: (11 subgroups)
- $Z_2 \ltimes Z_8$: (11 subgroups)

$E_{27}$ extraspecial order 27, exponent 9: (10 subgroups)

5 Conclusion

Collecting all of our results, we have:

| $n$ | Groups with $G = \tilde{G}$ and $n$ subgroups |
|-----|---------------------------------------------|
| 1   | Trivial group                               |
| 2   |                                             |
| 3   |                                             |
| 4   |                                             |
| 5   | $Z_2 \times Z_2$, $Z_3 \times Z_3$, $S_3$, $Q_8$ |
| 6   |                                             |
| 7   | $Z_2 \times Z_4$, $Z_5 \times Z_5$, $D_{10}$, $Z_4 \ltimes Z_3$ |
| 8   |                                             |
| 9   | $Z_3 \times Z_9$, $Z_7 \times Z_7$, $D_{14}$, $A_4$, $Z_3 \ltimes Z_7$, $Z_4 \ltimes Z_5$, $Z_8 \ltimes Z_3$, $E_{27}$ |
| 10  | $Z_2 \times Z_8$, $Q_{16}$, $Z_2 \ltimes Z_8$ |
| 11  | $Z_4 \ltimes Z_7$, $Z_9 \times Z_7$, $Z_8 \times Z_5$, $Z_{16} \times Z_3$ |

In particular the sequence of the number of groups with $G = \tilde{G}$ and $n$ subgroups would be:

$$1, 0, 0, 0, 1, 3, 0, 4, 0, 8, 3, 4, \ldots$$

Forming the direct product with a coprime, cyclic group of order $p^k$ will multiply the number of subgroups by $k + 1$. Thus we find groups with $n$ subgroups corresponding to various factorizations of $n$. 


Similarity class representatives with $n$ subgroups

| $n$ | Similarity classes with $n$ subgroups |
|-----|--------------------------------------|
| 1   | Trivial group                         |
| 2   | $Z_2$                                 |
| 3   | $Z_2^2$                               |
| 4   | $Z_2^3$, $Z_2 \times Z_3$             |
| 5   | $Z_2^4$, $Z_2 \times Z_2$             |
| 6   | $Z_2^5$, $Z_2 \times Z_3^2$, $Z_3 \times Z_3$, $S_3$, $Q_8$ |
| 7   | $Z_2^6$                               |
| 8   | $Z_2^7$, $Z_2 \times Z_3^3$, $Z_2 \times Z_3 \times Z_5$, $Z_2 \times Z_4$, $Z_5 \times Z_5$, $D_{10}$, $Z_4 \ltimes Z_3$ |
| 9   | $Z_2^8$, $Z_2^2 \times Z_3^2$         |
| 10  | $Z_2^9$, $Z_2 \times Z_3^4$, $Z_2 \times Z_2 \times Z_3$, $Z_3 \times Z_0$, $Z_7 \times Z_7$, $D_{14}$, $A_4$, $Z_3 \ltimes Z_7$, $Z_4 \ltimes Z_5$, $Z_8 \ltimes Z_3$, $E_{27}$ |
| 11  | $Z_2^{10}$, $Z_2 \times Z_8$, $Q_16$, $Z_2 \ltimes Z_8$ |
| 12  | $Z_2^{11}$, $Z_2 \times Z_3^5$, $Z_3 \times Z_3 \times Z_2$, $S_3 \times Z_5$, $Q_8 \times Z_3$, $Z_2 \times Z_3^3$, $Z_2 \times Z_3 \times Z_5^2$, $Z_4 \ltimes Z_7$, $Z_9 \ltimes Z_7$, $Z_8 \ltimes Z_5$, $Z_16 \ltimes Z_3$ |

In the following version of the previous table we represent classes using $p, q, r$ to represent primes which do not occur anywhere else in the group order. This makes it a bit easier to recognize the infinite classes.

| $n$ | Similarity classes with $n$ subgroups |
|-----|--------------------------------------|
| 1   | Trivial group                         |
| 2   | $Z_p$                                 |
| 3   | $Z_p^2$                               |
| 4   | $Z_p^3$, $Z_p \times Z_q$             |
| 5   | $Z_p^4$, $Z_2 \times Z_2$             |
| 6   | $Z_p^5$, $Z_p \times Z_q^2$, $Z_3 \times Z_3$, $S_3$, $Q_8$ |
| 7   | $Z_p^6$                               |
| 8   | $Z_p^7$, $Z_p \times Z_q^3$, $Z_p \times Z_q \times Z_r$, $Z_2 \times Z_4$, $Z_5 \times Z_5$, $D_{10}$, $Z_4 \ltimes Z_3$ |
| 9   | $Z_p^8$, $Z_p^2 \times Z_q^2$         |
| 10  | $Z_p^9$, $Z_p \times Z_q^4$, $Z_2 \times Z_2 \times Z_p$, $Z_3 \times Z_9$, $Z_7 \times Z_7$, $D_{14}$, $A_4$, $Z_3 \ltimes Z_7$, $Z_4 \ltimes Z_5$, $Z_8 \ltimes Z_3$, $E_{27}$ |
| 11  | $Z_p^{10}$, $Z_2 \times Z_8$, $Q_16$, $Z_2 \ltimes Z_8$ |
| 12  | $Z_p^{11}$, $Z_p \times Z_q^5$, $Z_3 \times Z_3 \times Z_p$, $S_3 \times Z_p$, $Q_8 \times Z_p$, $Z_p^2 \times Z_q^3$, $Z_p \times Z_q \times Z_r^2$, $Z_4 \ltimes Z_7$, $Z_9 \ltimes Z_7$, $Z_8 \ltimes Z_5$, $Z_16 \ltimes Z_3$ |

In conclusion, the sequence of number of similarity classes with a given number of subgroups begins:

$$1, 1, 1, 2, 2, 5, 1, 7, 2, 11, 4, 11, \ldots$$

Note: In [1], Miller lists the groups with specified number of subgroups where the number of subgroups runs from 1 through 9. His lists agree with ours.

### References

[1] G.A. Miller, Groups having a small number of subgroups, *Proc. Natl. Acad. Sci. U S A*, vol. 25 (1939) 367–371.

[2] M.C. Slattery, On a property motivated by groups with a specified number of subgroups, *Amer. Math. Monthly*, vol. 123 (2016) 78–81.
[3] M.C. Slattery, Editor’s End Notes, *Amer. Math. Monthly*, vol. 123 (2016) 515.

[4] N.J.A. Sloane, The on-line encyclopedia of integer sequences, [http://oeis.org](http://oeis.org).