CONTRACTIONS OF 3-FOLDS: DEFORMATIONS AND INVARIANTS

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Abstract. This note discusses recent new approaches to studying flopping curves on 3-folds. In a joint paper [DW1], the author and Michael Wemyss introduced a 3-fold invariant, the contraction algebra, which may be associated to such curves. It characterises their geometric and homological properties in a unified manner, using the theory of noncommutative deformations. Toda has now clarified the enumerative significance of the contraction algebra for flopping curves, calculating its dimension in terms of Gopakumar–Vafa invariants [Tod]. Before reviewing these results, and others, I give a brief introduction to the rich geometry of flopping curves on 3-folds, starting from the resolutions of Kleinian surface singularities. This is based on a talk given at VBAC 2014 in Berlin.

1. Introduction

In joint work [DW1], the author and Michael Wemyss associated a certain algebra to a contraction of a smooth rational curve in a complex 3-fold, and began an investigation of the importance of this contraction algebra in studying the geometry of the 3-fold. In particular, the algebra was shown to represent a certain noncommutative deformation functor associated to the curve. Further recent work extends this to general 3-fold flopping contractions [DW3], and to certain higher-dimensional contractions [DW2], but for simplicity I focus in this note on flopping contractions of 3-folds with exceptional locus a smooth rational curve.

Given a flopping contraction, the 3-fold X which is contracted has a birational partner X′ called a flop. For smooth X, work of Bridgeland then yields an associated canonical symmetry of the derived category of X, known as the flop–flop functor. We use the contraction algebra to give a description of this symmetry, intrinsic to the geometry of X.

This same contraction algebra generalises and unifies known invariants of flopping curves, including the width invariant of Reid [R83] and the Dynkin type [KM, Kaw]. Recent work of Toda shows that the dimension of the contraction algebra is determined by the genus zero Gopakumar–Vafa invariants associated to
the contracted curve [Tod]. In the other direction, the contraction algebra determines these Gopakumar–Vafa invariants for curves of Dynkin types $A$ and $D$.

For flopping curves with Dynkin type $A$, the contraction algebra turns out to be of the simple form $\mathbb{C}[\epsilon]/\epsilon^d$, whereas for Dynkin types $D$ and $E$, it is always noncommutative. This note gives a compact introduction to the geometry of flopping curves, along with simple examples of contraction algebras.

I now give preliminary statements, deferring details and definitions until Section 3. Consider a contraction $X \to X_{\text{con}}$ of 3-folds which maps a curve $C \cong \mathbb{P}^1$ to a point $p$, and is an isomorphism elsewhere. We give an explicit construction of a $\mathbb{C}$-algebra $A_{\text{con}}$ (Definition 4.1), which we refer to as the contraction algebra, and prove the following theorem.

**Theorem A (4.3).** [DW1] The $\mathbb{C}$-algebra $A_{\text{con}}$ represents a functor of noncommutative deformations associated to the curve $C$.

For flopping contractions, under the assumption that $X$ is projective with at worst Gorenstein terminal singularities, the following is a result of Toda.

**Theorem B (5.3).** [Tod] For a flopping curve $C$, the dimension $\dim(A_{\text{con}})$ may be calculated from genus zero Gopakumar–Vafa invariants $n_j$, and the length invariant $l$, by the formula

$$\dim(A_{\text{con}}) = \sum_{j=1}^{l} j^2 n_j.$$  

Furthermore, the dimension of the abelianisation $\dim(A_{\text{con}}^{\text{ab}})$ is $n_1$.

Theorem B gives a new enumerative criterion for the noncommutativity of the algebra $A_{\text{con}}$, namely that it is noncommutative precisely when one of the higher Gopakumar–Vafa invariants $n_j$ for $j > 1$ is non-zero.

Toda’s proof of Theorem B makes use of the following result, which gives a description of the flop–flop functor. Stating and proving this was a key motivation for the definition of $A_{\text{con}}$.

**Theorem C (6.1).** [DW1] Assuming $X$ is projective with at worst Gorenstein terminal singularities, the Bridgeland–Chen flop–flop functor

$$F' \circ F : \text{D}(X) \xrightarrow{\sim} \text{D}(X') \xrightarrow{\sim} \text{D}(X)$$

is inverse to a symmetry $T_C$ of the derived category $\text{D}(X)$ constructed using the algebra $A_{\text{con}}$. The action of $T_C$ on $E \in \text{D}(X)$ may be calculated by the following formula, where $E \in \text{coh}(\mathcal{O}_X \otimes_{\mathbb{C}} A_{\text{con}})$ denotes the universal family for the noncommutative deformations of $\mathcal{O}_C(-1)$.

$$T_C(E) \cong \text{Cone} \left( \mathbf{R}\text{Hom}_X(\mathcal{E}, E) \otimes^L_{A_{\text{con}}} E \xrightarrow{\text{ev}} E \right)$$
Outline. In §2, I give some orientation on surface singularities and their resolutions and deformations, which have an elegant representation-theoretic description. I explain in §3 how this may be used to understand contractions of 3-folds, by viewing them as families of contractions of surfaces. In §4, two approaches to the contraction algebra $A_{\text{con}}$ are given, via a tilting algebra associated to the contraction, and via noncommutative deformations. In §5, I discuss enumerative invariants associated to $A_{\text{con}}$. Finally in §6, the contraction algebra is related to the flop–flop functor, and I indicate how Toda uses this relation to prove Theorem B above. No knowledge of the derived category is assumed until this final section.

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2. Resolutions of surfaces

2.1. Du Val resolutions. Take a surface $S$ with a Kleinian singularity. Such singularities are given complete locally by quotients $\mathbb{A}^2/G$, with $G$ a finite subgroup of $\text{SL}(2, \mathbb{C})$. These subgroups are classified by simply-laced Dynkin diagrams $\Gamma$ [McK, R99]. The types $A$ and $D$ correspond to cyclic and binary dihedral groups respectively, and the types $E_6$, $E_7$, and $E_8$ correspond to binary Platonic groups. There exists a resolution of singularities $\tilde{S} \to S$ obtained by replacing the singular point with a tree of projective lines, whose intersections are determined by the diagram $\Gamma$ [DV]. This is shown in Figure 1 for the $A_2$ and $D_4$ cases. Here the relevant groups are cyclic of order 3 and binary dihedral of order 8 respectively, with the latter being isomorphic to the quaternion group.

2.2. Versal deformations. Our goal is to understand 3-fold geometries via 1-parameter deformations of these resolutions. Such deformations are controlled by versal deformations, which have a representation-theoretic description as follows. Associated to the diagram $\Gamma$ is a complex semisimple Lie algebra, and a choice of Cartan subalgebra, denoted by $U$, carries an action of a Weyl group $W$. The singular surface $S$ has a versal deformation $Y$ parametrised by the space of $W$-orbits
in $U$. After base change from this parameter space $U/W$ to $U$ itself, the fibres of the family $Y$ have a simultaneous resolution [Slo] with central fibre recovering the resolution $\hat{S} \rightarrow S$. Denoting this by $\hat{Y}$, there is a Cartesian square as follows, and the resolution $\hat{S} \rightarrow S$ is recovered by taking the fibre of the top line over $0 \in U/W$.

\[
\begin{array}{ccc}
\hat{Y} & \rightarrow & Y \\
\downarrow & & \downarrow \\
U & \rightarrow & U/W
\end{array}
\]

2.3. Case $A_1$. I explain this case in detail as it immediately gives the simplest 3-fold geometry which will be of interest to us. We may take $S = \mathbb{A}^2/G$, where $G = \mathbb{Z}/2\mathbb{Z}$, acting by $\pm \text{Id}$. Taking suitable quadratic invariants of this $G$-action, namely $x = u^2 - v^2$, $y = 2uv$, and $z = u^2 + v^2$, the singularity $S$ may be written as a hypersurface

$$ S = \{x^2 + y^2 = z^2\} \subset \mathbb{C}^3, $$

and has a versal deformation

$$ Y = \{x^2 + y^2 = z^2 - q\} \subset \mathbb{C}^4, $$

with parameter $q$. The total space of this deformation $Y$ is smooth.

Now in this case we have $U = \mathbb{C}$ with Weyl group $W = S_2$ acting by $\pm \text{Id}$. Writing $q = t^2$ for a new parameter $\pm t \in \mathbb{C}/S_2$, we may view $Y$ as a family over $U/W$. Base changing to the parameter $t \in \mathbb{C}$, a new deformation

$$ Y' = \{x^2 + y^2 = z^2 - t^2\} \subset \mathbb{C}^4 \quad (2.A) $$

is obtained, with a singularity at 0. This has a simultaneous resolution $\hat{Y}$ which can be constructed by a blowup in one of the ideals $(x + iy, z \pm t)$, and is sketched in Figure 2. This $\hat{Y}$ is our first example of a smooth 3-fold with a contraction. It contains a single complete curve $C$ in the fibre over $t = 0$, with normal bundle $\mathcal{O}_C(-1) \oplus \mathcal{O}_C(-1)$. This is an example of the simplest sort of 3-fold contractible curve, and is referred to as a $(-1, -1)$-curve.
2.4. **Other cases.** In the $A_1$ case above, the resolution introduced a single complete curve over $0 \in U$, and was an isomorphism over $U\setminus\{0\}$. In general, the locus in $U$ parametrising those fibres in which complete curves are introduced is more complicated. It is known as the *discriminant locus*.

**Example 2.1.** The discriminant locus for the $A_2$ case is illustrated in Figure 3, where $U = \{t \in \mathbb{C}^3 : t_1 + t_2 + t_3 = 0\}$, with the natural action of $W = S_3$. A selection of fibres of the simultaneous resolution $\hat{Y}$ over $U$ are shown. The resolution $\hat{S}$ is the fibre over the origin in $U$. The two intersecting projective lines in the exceptional locus of $\hat{S}$ deform over two components of the discriminant locus $L$, and the two lines deform to a single smooth curve over a third component.

![Fig 2](image1.png)  
**Figure 2.** Simultaneous resolution $\hat{Y}$ of deformation of $A_1$ singularity

![Fig 3](image2.png)  
**Figure 3.** Simultaneous resolution $\hat{Y}$ for type $A_2$
2.5. Partial resolutions. The resolution $\hat{S}$ of Section 2.1 is minimal amongst the resolutions of the singular surface $S$. There also exist partial resolutions $S'$ of $S$, which may be constructed by contracting choices of curves in $\hat{S}$ corresponding to vertices $\Gamma' \subseteq \Gamma$ in the Dynkin diagram. The deformations of these partial resolutions have a description as follows $[\text{KM}]$. Take $W$ the Weyl group acting on $U$, and now let $W'$ be the subgroup of $W$ associated to the vertices $\Gamma'$. Taking $Y$ once again to be the versal deformation of the singularity $S$ over the base $U/W$ and now base changing to $U/W'$, there exists a partial resolution of the singularities of $Y$, yielding a versal deformation of $S'$. Denoting this by $Y'$, there is a commutative diagram of Cartesian squares as follows, and the partial resolution $S' \to S$ is recovered by taking the fibre of the top line over $0 \in U/W$.

$$
\begin{array}{ccc}
\hat{Y} & \longrightarrow & Y' \\
\downarrow & & \downarrow \\
U & \longrightarrow & U/W'
\end{array}
$$

Example 2.2. In the $D_4$ case illustrated in Figure 4, we blow down the black curves $\Gamma'$ in the Dynkin diagram $\Gamma$ to leave a single remaining curve. This blow-down then factors the resolution $\hat{S} \to S$, as shown.

![Figure 4. Partial resolution for type $D_4$](image)

Example 2.3. For the $A_2$ case in Example 2.1, choosing $\Gamma'$ to be one of the two curves, $Y'$ may be constructed by contracting the codimension 2 locus in $\hat{Y}$ consisting of a deformation of that curve lying over the corresponding component of $L$, and then quotienting by $W' = S_2$.

As before, a generic fibre of $Y'$ contains no complete curves, but $U/W'$ has a locus parametrising those deformations which do contain such curves, or singularities. This locus is given by the quotient of the discriminant locus $L$ in $U$, and denoted also by $L$.

3. Contractions of 3-folds

We now return to considering a 3-fold contraction of a curve $C \cong \mathbb{P}^1$, as in the introduction. I review the description of the geometry of such curves, and deformations of their contractions, which may be used to replace a non-generic curve $C$ with a collection of generic curves, following $[\text{R83}]$ and $[\text{BKL}]$. 

3.1. Setup. A contraction is here taken to be a projective birational morphism \( f : X \to X_{\text{con}} \) of 3-folds, with the singularities of \( X_{\text{con}} \) being rational. A contraction of a curve \( C \) is a contraction which maps a curve \( C \) to a point \( p \), and is an isomorphism from \( X \setminus C \) onto \( X_{\text{con}} \setminus p \). We say that \( f \) is a flopping contraction if the canonical bundle \( K_X \) is trivial in a neighbourhood of \( C \). In the following, \( C \cong \mathbb{P}^1 \).

For simplicity, I take \( X \) to be smooth and \( X_{\text{con}} = \text{Spec} \, R \), a complete local ring, though the theorems presented later also work in a global setting, and under weaker assumptions on singularities, which I identify in Remarks 4.2 and 4.4.

Remark 3.1. In the situation above, the scheme-theoretic fibre \( f^{-1}(p) \) may be a finite-order thickening of \( C \). This is key to much of the geometric subtlety which follows.

3.2. Slicing the singularity. Given a flopping contraction \( f \) of a curve \( C \cong \mathbb{P}^1 \) to a point \( p \), we take a generic hypersurface \( H \) through \( p \in X_{\text{con}} \). A result of Reid [R83, Theorems 1.1, 1.14] gives that \( H \) has a Kleinian singularity at \( p \), so that we may apply the discussion of Section 2. Write \( X_H \) for the preimage of \( H \) under \( f \), so that there is a map

\[
f : X_H \to H \ni p.
\]

Now the reduced fibre of \( f \) above \( p \) is exactly \( C \cong \mathbb{P}^1 \), and so this map cannot be the resolution \( \hat{H} \to H \), unless the singularity of \( H \) is of the simplest Dynkin type \( A_1 \). Indeed in general it is a partial resolution, factoring \( \hat{H} \to H \) as follows:

\[
\hat{H} \xrightarrow{q} X_H \xrightarrow{f} H.
\]

Here \( q \) is a blowdown of certain projective lines in \( \hat{H} \) as in Section 2.5, and identifies one of these lines with our curve \( C \).

Recalling that the projective lines in \( \hat{H} \) correspond to the vertices of a Dynkin diagram, we thereby associate to our contraction \( f \) the data of a Dynkin diagram \( \Gamma \) with a marked point \( i \). This data is referred to as the Dynkin type of the floppable curve \( C \).

Example 3.2. [DW1, 3.14] I present a family of examples of type \( D_4 \) flopping curves, which we will return to throughout. For each \( k \in \mathbb{N} \), put

\[
X_{k,\text{con}} = \{ u^2 + v^2 y = x(x^2 + y^{2k+1}) \} \subset \mathbb{C}^4.
\]

This singularity can be blown up in an ideal to give a threefold \( X_k \) with a single complete curve \( C \cong \mathbb{P}^1 \) [R83, Example 5.15]. Figure 5 depicts the generic hypersurface section \( H \) through the singularity, along with its partial resolution \( X_H \), and full resolution \( \hat{H} \).

It turns out that only flopping curves of types \( A_1, D_4, E_6, E_7, \) and \( E_8 \) may arise. Furthermore, in the cases \( D_4, E_6, \) and \( E_7 \), the marked point must be the central vertex of the Dynkin diagram, whereas in the case \( E_8 \), the marked point
Figure 5. Slicing a singularity by a generic hyperplane, type $D_4$

may move one place along the long arm of the diagram. See Katz–Morrison [KM] or Kawamata [Kaw] for an account of the possibilities.

3.3. Characterising the 3-fold. We now obtain the desired description of the contraction $f$ in terms of deformations of the surfaces in Section 2, as follows.

Writing $\phi = 0$ for the equation of our hypersurface $H \subset X_{con}$, the 3-fold $X$ may be viewed as a family over a formal neighbourhood $\Delta \subset \mathbb{A}^1$ via $\phi \circ f$. Now indeed this is a deformation of a partial resolution $S'$ associated to the marked Dynkin diagram $(\Gamma, i)$, and so by the description of the versal deformation $Y'$ in Section 2.5 we obtain a classifying map $g: \Delta \to U/W'$ for an appropriate subgroup $W'$ of the Weyl group. This may be viewed as a germ of a $\mathbb{C}$-parametrised path in the orbit space $U/W'$.

To summarise, we have associated to a flopping contraction of a curve $C \cong \mathbb{P}^1$, the data $(\Gamma, i, g)$ of a Dynkin diagram with a marked vertex, and a path of $W'$-orbits in the associated $U$, as follows.

$$(X, f) \leftrightarrow (\Gamma, i, g)$$

In the other direction, the data $(\Gamma, i, g)$ may be used to construct a contraction, by pulling back the versal deformation $Y'$ via $g$.

Remark 3.3. For the $D_4$ case of Example 3.2, the classifying map to the versal deformation base is given explicitly by Aspinwall and Morrison [AM, Section 4.2].

3.4. Case $A_1$. I spell this case out as it gives the easiest class of examples. Here $\Gamma$ consists of a single vertex, and we must therefore take this for the marked vertex $i$. Then $W' = \{\text{Id}\}$ with $U = \mathbb{C}$. Our path is therefore given by a function $g: \mathbb{C} \to U = \mathbb{C}$. Taking $g(w) = w^d$, base change of (2.A) in our previous type $A_1$ example gives the equation of a singularity

$$\{x^2 + y^2 = z^2 - w^{2d}\} \subset \mathbb{C}^4. \quad (3.A)$$
For $d = 1$, the 3-fold $X$ is just the simultaneous resolution of $(2.A)$ discussed in Section 2.3, containing a single $(-1, -1)$-curve. For $d \geq 2$, base change gives a simultaneous resolution of the family of singularities $(3.A)$ with a single complete curve $C$ in the fibre $w = 0$. This curve has normal bundle $\mathcal{O}_C(-2) \oplus \mathcal{O}_C$, and is referred to as a $(-2, 0)$-curve. The number $d$ is called the \textit{width} by Reid [R83, Definition 5.3].

3.5. \textbf{Deforming the 3-fold.} We may now take $\tilde{g}$, a new path in $U/W'$ transverse to the discriminant locus $L$, whose germ is a small deformation of $g$, as in Figure 6. Base changing the versal deformation $Y'$ by $\tilde{g}$, a new deformed 3-fold $\tilde{X}$ is obtained. Similarly, a singular base $\tilde{X}_{\text{con}}$ is obtained by base changing the versal deformation $Y$.

In the $A_1$ case of the previous Section 3.4 the discriminant locus $L = \{0\} \subset \mathbb{C}$. A non-transverse classifying map $g$ with $d > 1$ may be deformed to a transverse map $\tilde{g}$. The roots of $\tilde{g}$ then correspond to $d$ disjoint $(-1, -1)$-curves in the deformed 3-fold $\tilde{X}$. This is sketched in Figure 7.

Generally, non-transverse intersections with the discriminant locus in $U/W'$ correspond to non-generic curves, whereas transverse intersections correspond to $(-1, -1)$-curves. I now indicate how such intersections may be counted.
3.6. **Counting curves in the deformation.** The curves in the deformed 3-fold $\tilde{X}$ may be characterised in terms of certain enumerative invariants on $X$. As preparation, we produce a sequence of infinitesimal neighbourhoods of $C \subset X$ as follows. Taking $I$ to be the ideal of definition of $C \subset X$, we define a thickened curve $C^{(j)}$ for $j \in \mathbb{N}$ by the $j^{th}$ symbolic power $I^{(j)}$ of $I$, viewed as a subscheme of $X$.

Each $C^{(j)}$ corresponds to an isolated point in $\text{Hilb}(X)$, the Hilbert scheme of closed subschemes of $X$. Write $n_j$ for the multiplicity of this point. These $n_j$ coincide with the genus zero Gopakumar–Vafa invariants associated to $C$, as defined by Katz [Kat].

Using the multiplicities $n_j$, Bryan–Katz–Leung [BKL] describe intersections of a transverse deformation $\tilde{g}$ with the components of the discriminant locus $L$ in $U/W'$, and also show that the curves in the associated 3-fold $\tilde{X}$ are all $(-1,-1)$-curves, with different curve classes $j[C]$ corresponding to intersections of $\tilde{g}$ with different components of $L$. This is achieved by a careful analysis of the geometry of the discriminant locus in different Dynkin types, using explicit presentations of the simultaneous resolutions due to Katz–Morrison [KM]. Their result is the following.

**Proposition 3.4.** [BKL, §2.1] Writing $n_j$ for the multiplicities associated to the curve $C \subset X$ as above, the exceptional locus of a generic deformed contraction $\tilde{X} \to \tilde{X}_{\text{con}}$ consists of $(-1,-1)$-curves, with $n_1$ curves of class $[C]$, $n_2$ of class $2[C]$, and so on, up to $l[C]$.

Here the length $l$ of $C$ is the length of the scheme-theoretic fibre $f^{-1}(p)$ at the generic point of $C$. In type $A$, the length $l = 1$, and the multiplicity $n_1$ coincides with the width $d$ of $C$. In types $D_4$, $E_6$, and $E_7$, the length $l = 2$, 3, and 4, respectively. In type $E_8$ the length takes values 5 or 6, if the marked point of the Dynkin diagram is adjacent to the central vertex (on the long arm), or the central vertex itself, respectively.

### 4. The contraction algebra $A_{\text{con}}$

Having explored how isolated contractible curves arise, we introduce a new invariant associated to them. As an example application we will calculate the Gopakumar–Vafa invariants $n_j$ for our $D_4$ example in the following Section 5, using a result of Toda [Tod].

As in the previous Section 3, we consider a 3-fold contraction $f : X \to X_{\text{con}} = \text{Spec } R$, a complete local ring, contracting a curve $C \cong \mathbb{P}^1$. Unlike the previous section, the following definitions and theorems do not require the contraction $f$ to be flopping: indeed they extend to the case of singular flops and flips (see Remarks 4.2 and 4.4).
4.1. **Via a tilting algebra.** In this setting there exists a tilting bundle $\mathcal{T}$ on $X$ by a construction of Van den Bergh [VdB, Theorem A] given explicitly in [DW1, §2.3]. The usefulness of this bundle stems from the fact that the algebra $A := \text{End}_X(\mathcal{T})$ is derived equivalent to $X$. The bundle $\mathcal{T}$ decomposes as $\mathcal{T} = \mathcal{O}_X \oplus N$. Writing $N$ for the $R$-module of global sections of $N$, we find that $A \cong \text{End}_R(R \oplus N)$ using [VdB, 4.2.1]. The contraction algebra $A_{\text{con}}$ may then be defined as follows.

**Definition 4.1.** [DW1, §2.3–§2.4] For a contraction of a curve $C \cong \mathbb{P}^1$ as in Section 3.1, put $A_{\text{con}} = A/I_{\text{con}}$, where $I_{\text{con}}$ is the two-sided ideal of $A$ generated by endomorphisms $\phi \in A \cong \text{End}_R(R \oplus N)$ which factor through a direct summand of a free $R$-module.

**Remark 4.2.** We do not need smoothness of $X$ to make this definition, or for the Theorem 4.3 which follows. These are treated in [DW1] under the weaker assumption that $X$ is normal, with only Cohen–Macaulay canonical singularities.

Examples of contraction algebras $A_{\text{con}}$ are given in Section 4.3.

4.2. **Via noncommutative deformations.** The contraction algebra is also characterised, up to isomorphism, by the noncommutative deformations of the curve $C$. Roughly speaking, $A_{\text{con}}$ plays the role of an algebra of functions on a noncommutative space parametrising the deformations of the sheaf $\mathcal{O}_C$ in coherent sheaves on $X$. To be precise, we find that $A_{\text{con}}$ represents a certain deformation functor. I give a partial statement of the required formalism here, referring the reader to [DW1, §2] for full details.

First we take a category $\text{Art}_1$ of finite-dimensional $\mathbb{C}$-algebras $B$, not necessarily commutative. These are endowed with extra structure, in particular an ideal $n$ which should be considered as the unique closed point of the noncommutative space corresponding to $B$. For a given coherent sheaf $E$ on $X$ we then construct a noncommutative deformation functor

$$\text{Def}_E : \text{Art}_1 \to \text{Sets},$$

taking $(B, n)$ to the set of $B$-families $\mathcal{F}$ which recover $E$ after reduction modulo the ideal $n$. As part of their definition, such $\mathcal{F}$ are objects of $\text{coh}(\mathcal{O}_X \otimes_{\mathbb{C}} B)$.

Note that restricting $\text{Def}_E$ to a category of commutative $\mathbb{C}$-algebras, the usual deformation functor for the sheaf $E$ is obtained. This is referred to here as the classical deformation functor, and denoted by $c\text{Def}_E$. We then have the following.

**Theorem 4.3.** [DW1, 1.1, 2.13(1), §3.1] Given a contraction of a curve $C \cong \mathbb{P}^1$ as in Section 3, the noncommutative deformation functor $\text{Def}_{\mathcal{O}_C(-1)}$ is represented by the contraction algebra $A_{\text{con}}$ of Definition 4.1, and this algebra is finite-dimensional. Furthermore, the classical deformation functor $c\text{Def}_{\mathcal{O}_C(-1)}$ is represented by the abelianisation $A_{\text{con}}^{\text{ab}}$, given by quotienting $A_{\text{con}}$ by its two-sided ideal of commutators.
Remark 4.4. The statement above is a special case of the theorem in [DW1]. There we only ask that $X$ is normal and Cohen–Macaulay canonical, and we allow $X_{\text{con}}$ to be quasi-projective, rather than complete local. In particular, the theorem covers both flops and flips in singular settings.

4.3. Results and examples. The following theorem may be obtained using knowledge of the number of generators and relations of the algebra $A_{\text{con}}$, and applying a powerful result of Smoktunowicz [S13], or alternatively by a careful analysis of hyperplane sections. Both approaches are discussed in [DW1, Section 3.4].

Theorem 4.5. *The contraction algebra* $A_{\text{con}}$ *is commutative if and only if the contracted curve* $C$ *is of Dynkin type A.*

Example 4.6. *If a flopping curve* $C$ *is of type A (as in Section 3.4 for example) then* $A_{\text{con}} \cong \mathbb{C}[e]/e^d$, *where* $d$ *is the width of* $C$.

Example 4.7. *For the* $D_4$ *case of Example 3.2,*

$$A_{\text{con}} \cong \frac{\mathbb{C}(x, y)}{xy = -yx, x^2 = y^{2k+1}}.$$ 

A vector space basis for $A_{\text{con}}$ is given by the monomials $x^ay^b$ with $0 \leq a \leq 2$ and $0 \leq b \leq 2k$. We have

$$A_{\text{con}}^{ab} \cong \frac{\mathbb{C}[x, y]}{xy = 0, x^2 = y^{2k+1}}.$$ 

Both these algebras, and the associated deformations, are illustrated in Figure 8.

![Figure 8. Deformations of type $D_4$ curve for $k = 2$](image_url)

5. Numerical invariants

5.1. Widths. We refer to the dimensions of the contraction algebra $A_{\text{con}}$ and its abelianisation $A_{\text{con}}^{ab}$ as the *noncommutative width* and *commutative width* respectively, and denote these numbers by $\text{wid}(C)$ and $\text{cwid}(C)$. For the type $A$ flopping curves of Section 3.4, that is the case in which the width of Reid is defined, we
CONTRACTIONS OF 3-FOLDS: DEFORMATIONS AND INVARIANTS

find that all three notions of width coincide. More generally, the noncommutative and commutative widths may differ, as shown by the following example.

**Example 5.1.** In our type $D_4$ example, $\text{wid}(C) = 3(2k+1)$, and $\text{cwid}(C) = 2k+3$, from the results in Example 4.7.

An analysis of hyperplane sections gives lower bounds for $\text{wid}(C)$ according to Dynkin type, as explained in [DW1, Remark 3.17]. For type $D_4$ the bound is 4, and for types $E_6$, $E_7$, and $E_8$, the bounds are 12, 24, and 40, respectively.

**Remark 5.2.** The noncommutative width may also be obtained by a calculation of Ext groups on $X_{\text{con}}$. See [DW1, Remark 5.2].

5.2. Gopakumar–Vafa invariants. I now state the result of Toda from the introduction and explain how it sheds light on the width invariants defined above. As before, let $l$ be the length of the scheme-theoretic fibre $f^{-1}(p)$ at the generic point of $C$.

**Theorem 5.3.** [Tod] For a flopping contraction of a curve $C \cong \mathbb{P}^1$ as in Section 3, on a 3-fold $X$ with at worst Gorenstein terminal singularities, we have that

$$\text{wid}(C) = \sum_{j=1}^{l} j^2 n_j,$$

and $\text{cwid}(C) = n_1$, where the $n_j$ are the multiplicities defined in Section 3.6, coinciding with genus zero Gopakumar–Vafa invariants.

Recalling from Proposition 3.4 that the $n_j$ also coincide with the numbers of $(-1,-1)$-curves of class $j[C]$ in a generic deformation, we see that the noncommutative width $\text{wid}(C)$ counts the curves in such a deformation, with weights given by the squares of their curve classes.

**Example 5.4.** For instance, in our running type $D_4$ case from Example 4.7, we deduce that $n_1 = 2k + 3$, and $n_2 = k$. In particular, the multiplicity of the thickened curve $C^{(2)}$ in its Hilbert scheme is $k$. This is illustrated in Figure 9. We also deduce that under a generic deformation, the flopping curve $C$ deforms to $2k + 3$ curves of class $[C]$, and $k$ curves of class $2[C]$.

6. THE DERIVED CATEGORY

In this section, I briefly indicate the application of $A_{\text{con}}$ to describing symmetries of the bounded derived category of coherent sheaves $D(X)$, and how this is used in Toda’s proof of Theorem 5.3.

6.1. Autoequivalences. Unpacking the statement of Theorem 4.3, we obtain a universal family $E \in \text{coh}(\mathcal{O}_X \otimes_C A_{\text{con}})$ for the noncommutative deformations of $\mathcal{O}_C(-1)$. Then we have the following theorem.
Theorem 6.1. [DW1, 1.5, 7.14] Given a flopping contraction of $C \cong \mathbb{P}^1$ for a projective 3-fold $X$ with at worst Gorenstein terminal singularities, there exists an autoequivalence $T_C$ of $D(X)$ which acts on objects as follows.

$$T_C(E) \cong \text{Cone} \left( \text{RHom}_X(E, E) \otimes_{A_{\text{con}}} \mathcal{E} \xrightarrow{\text{ev}} E \right)$$

Remark 6.2. Replacing $A_{\text{con}}$ with the commutative algebra $A_{\text{ab}}^{\text{con}}$ in the above theorem does not give an autoequivalence of $D(X)$, except in Type $A$ cases, where $A_{\text{con}}$ and $A_{\text{ab}}^{\text{con}}$ coincide. This is demonstrated in [DW1, Theorem 3.15].

For a flop $X \dashrightarrow X'$ there are equivalences of categories

$$F: D(X) \leftrightarrow D(X') : F'$$

given by the Fourier–Mukai transforms naturally associated to the graph of the birational map. This is a result of Bridgeland [Bri], generalised by Chen to the Gorenstein terminal setting [Che]. We prove the following.

Theorem 6.3. [DW1, 7.18] The autoequivalence $T_C$ of Theorem 6.1 is an inverse to the flop–flop functor $F' \circ F$.

6.2. Application. I finish with a brief description of Toda’s proof [Tod] of Theorem 5.3, relating the noncommutative width $\text{wid}(C)$ and the invariants $n_j$. Interpolating the classifying maps $g, \tilde{g}: C \rightarrow U/W'$ by a continuous family $g^t$, we may obtain a 1-parameter family of 3-fold contractions $X^t \rightarrow X^t_{\text{con}}$. (For this we should replace the versal deformations with suitable localisations, so as to recover the complete local fibre of the contraction that we started with, at $t = 0$.) In this family $X^t$ there is a family flop of the exceptional locus, and hence a family of flop–flop functors. On the central fibre, the flop–flop is inverse to $T_C$ by Theorem 6.3. On the generic fibre, the flop–flop is inverse to a composition of spherical twists [ST], one for each curve. Toda obtains the identities of Theorem 5.3 by analysing the Fourier–Mukai kernels of these autoequivalences, and equating Hilbert polynomials.
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