SELECTION PRINCIPLES AND THE MINIMAL TOWER PROBLEM

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ABSTRACT. We study diagonalizations of covers using various selection principles, where the covers are related to linear quasiorderings (τ-covers). This includes: equivalences and nonequivalences, combinatorial characterizations, critical cardinalities and constructions of special sets of reals. This study leads to a solution of a topological problem which was suggested to the author by Scheepers (and stated in [15]) and is related to the Minimal Tower problem.

We also introduce a variant of the notion of τ-cover, called τ∗-cover, and settle some problems for this variant which are still open in the case of τ-covers. This new variant introduces new (and tighter) topological and combinatorial lower bounds on the Minimal Tower problem.

1. Introduction

1.1. Combinatorial spaces. We consider zero-dimensional sets of real numbers. For convenience, we may consider other spaces with more evident combinatorial structure, such as the Baire space $\mathbb{N}\mathbb{N}$ of infinite sequences of natural numbers, and the Cantor space $\mathbb{N}\{0, 1\}$ of infinite sequences of “bits” (both equipped with the product topology). The Cantor space can be identified with $P(\mathbb{N})$ using characteristic functions. We will often work in the subspace $P_\infty(\mathbb{N})$ of $P(\mathbb{N})$, consisting of the infinite sets of natural numbers. These spaces, as well as any separable, zero-dimensional metric space, are homeomorphic to sets of reals, thus our results about sets of reals can be thought of as talking about this more general case.

1.2. Selection principles. Let $\mathcal{U}$ and $\mathcal{V}$ be collections of covers of a space $X$. The following selection hypotheses have a long history for the case when the collections $\mathcal{U}$ and $\mathcal{V}$ are topologically significant.

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1
S₁(\mathcal{U}, \mathcal{V}): For each sequence \( \{U_n\}_{n \in \mathbb{N}} \) of members of \( \mathcal{U} \), there is a sequence \( \{V_n\}_{n \in \mathbb{N}} \) such that for each \( n \in \mathbb{N} \), \( V_n \in U_n \), and \( \{V_n\}_{n \in \mathbb{N}} \in \mathcal{V} \).

S_{fin}(\mathcal{U}, \mathcal{V}): For each sequence \( \{U_n\}_{n \in \mathbb{N}} \) of members of \( \mathcal{U} \), there is a sequence \( \{F_n\}_{n \in \mathbb{N}} \) such that each \( F_n \) is a finite (possibly empty) subset of \( U_n \), and \( \bigcup_{n \in \mathbb{N}} F_n \in \mathcal{V} \).

U_{fin}(\mathcal{U}, \mathcal{V}): For each sequence \( \{U_n\}_{n \in \mathbb{N}} \) of members of \( \mathcal{U} \) which do not contain a finite subcover, there exists a sequence \( \{F_n\}_{n \in \mathbb{N}} \) such that for each \( n \), \( F_n \) is a finite (possibly empty) subset of \( U_n \), and \( \bigcup F_n \in \mathcal{V} \).

We make the convention that

The space \( X \) is infinite and all covers we consider are assumed not to have \( X \) as an element.

An \( \omega \)-cover of \( X \) is a cover such that each finite subset of \( X \) is contained in some member of the cover. It is a \( \gamma \)-cover if it is infinite, and each element of \( X \) belongs to all but finitely many members of the cover.

Following [7] and [13], we consider the following types of covers:

- \( O \) (respectively, \( B \)): The collection of countable open (respectively, Borel) covers of \( X \).
- \( \Omega \) (respectively, \( B_\Omega \)): The collection of countable open (respectively, Borel) \( \omega \)-covers of \( X \).
- \( \Gamma \) (respectively, \( B_\Gamma \)): The collection of countable open (respectively, Borel) \( \gamma \)-covers of \( X \).

The inclusions among these classes can be summarized as follows:

\[
B_\Gamma \rightarrow B_\Omega \rightarrow B \\
\uparrow \quad \uparrow \quad \uparrow \\
\Gamma \rightarrow \Omega \rightarrow O
\]

These inclusions and the properties of the selection hypotheses lead to a complicated diagram depicting how the classes defined this way interrelate. However, only a few of these classes are really distinct. Figure 1 contains the distinct ones among these classes, together with their critical cardinalities, which were derived in [7] and in [13]; see definition in Section 3. The only unsettled implications in this diagram are marked with dotted arrows.

1.3. \( \tau \)-covers. A cover of a space \( X \) is large if each element of \( X \) is covered by infinitely many members of the cover. Following [15], we consider the following type of cover. A large cover \( \mathcal{U} \) of \( X \) is a \( \tau \)-cover of \( X \) if for each \( x, y \in X \) we have either \( x \in U \) implies \( y \in U \) for all but finitely many members \( U \) of the cover \( \mathcal{U} \), or \( y \in U \) implies \( x \in U \) for all but finitely many \( U \in \mathcal{U} \).
A quasiordering $\preceq$ on a set $X$ is a reflexive and transitive relation on $X$. It is linear if for all $x, y \in X$ we have $x \preceq y$ or $y \preceq x$. A $\tau$-cover $\mathcal{U}$ of a space $X$ induces a linear quasiordering $\preceq$ on $X$ by:

$$x \preceq y \iff x \in U \to y \in U$$

for all but finitely many $U \in \mathcal{U}$.

If a countable $\tau$-cover is Borel, then the induced $\preceq = \{\langle x, y \rangle : x \preceq y\}$ is a Borel subset of $X \times X$. We let $T$ and $B_T$ denote the collections of countable open and Borel $\tau$-covers of $X$, respectively. We have the following implications.

$$B_T \to B_T \to B_\Omega \to B$$

$$\uparrow \quad \uparrow \quad \uparrow \quad \uparrow$$

$$\Gamma \to T \to \Omega \to \mathcal{O}$$

There is a simple hierarchy between the selection principles: For each $\mathcal{U}, \mathcal{W}$ in $\{\mathcal{O}, \Omega, T, \Gamma\}$ or in $\{B, B_\Omega, B_T, B_\Gamma\}$, we have that $S_1(\mathcal{U}, \mathcal{W}) \to S_{fin}(\mathcal{U}, \mathcal{W}) \to U_{fin}(\mathcal{U}, \mathcal{W})$. The implication $S_{fin}(\mathcal{U}, \mathcal{W}) \to U_{fin}(\mathcal{U}, \mathcal{W})$ needs a little care when $\mathcal{W}$ is $T$ or $B_T$: It holds due to the following lemma.
Lemma 1.1. Assume that $\mathcal{U} = \bigcup_{n \in \mathbb{N}} \mathcal{F}_n$, where each $\mathcal{F}_n$ is finite, is a $\tau$-cover of a space $X$. Then either $\bigcup \mathcal{F}_n = X$ for some $n$, or else $\mathcal{V} = \{\bigcup \mathcal{F}_n\}_{n \in \mathbb{N}}$ is also a $\tau$-cover of $X$.

Proof. Assume that $\bigcup \mathcal{F}_n \neq X$ for all $n$. Then, as $\mathcal{U}$ is an $\omega$-cover of $X$, so is $\mathcal{V}$. In particular, $\mathcal{V}$ is a large cover of $X$. Now fix any $x, y \in X$ such that $x \in U \to y \in U$ for all but finitely many $U \in \mathcal{U}$, and let $F = \{n : (\exists U \in \mathcal{F}_n) x \in U$ and $y \not\in U\}$. Then $F$ is finite and contains the set of $n$'s such that $x \in \bigcup \mathcal{F}_n$ and $y \not\in \bigcup \mathcal{F}_n$. \hfill \Box

1.4. Equivalences. The notion of $\tau$-covers introduces seven new pairs—namely, $(T, \mathcal{O})$, $(T, \Omega)$, $(T, T)$, $(T, \Gamma)$, $(\mathcal{O}, T)$, $(\Omega, T)$, and $(\Gamma, T)$—to which any of the selection operators $S_1$, $S_{\text{fin}}$, and $U_{\text{fin}}$ can be applied. This makes a total of 21 new selection hypotheses. Fortunately, some of them are easily eliminated, using the arguments of [11] and [7]. We will repeat the reasoning briefly for our case. The details can be found in the cited references.

First, the properties $S_1(\mathcal{O}, T)$ and $S_{\text{fin}}(\mathcal{O}, T)$ imply $S_{\text{fin}}(\mathcal{O}, \Omega)$, and thus hold only in trivial cases (see Section 6 of [18]). Next, $S_{\text{fin}}(T, \mathcal{O})$ is equivalent to $U_{\text{fin}}(T, \mathcal{O})$, since if the finite unions cover, then the original sets cover as well. Now, since finite unions can be used to turn any countable cover which does not contain a finite subcover into a $\gamma$-cover [7], we have the following equivalences$^1$:

- $U_{\text{fin}}(T, \Gamma) = U_{\text{fin}}(\Gamma, \Gamma)$,
- $U_{\text{fin}}(\mathcal{O}, T) = U_{\text{fin}}(\Omega, T) = U_{\text{fin}}(T, \mathcal{T}) = U_{\text{fin}}(\Gamma, T)$,
- $U_{\text{fin}}(T, \Omega) = U_{\text{fin}}(\Gamma, \Omega)$; and
- $U_{\text{fin}}(T, \mathcal{O}) = U_{\text{fin}}(\Gamma, \mathcal{O})$.

In Corollary 2.5 we get that $S_1(T, \Gamma) = S_{\text{fin}}(T, \Gamma)$. We are thus left with eleven new properties, whose positions with respect to the other properties are described in Figure 2. In this Figure, as well as in the one to come, there still exist quite many unsettled possible implications.

1.5. Equivalences for Borel covers. For the Borel case we have the diagram corresponding to Figure 2, but in this case, more equivalences are known [13]: $S_1(\mathcal{B}_\Gamma, \mathcal{B}_\Gamma) = U_{\text{fin}}(\mathcal{B}_\Gamma, \mathcal{B}_\Gamma)$, $S_1(\mathcal{B}_\Gamma, \mathcal{B}_\Omega) = S_{\text{fin}}(\mathcal{B}_\Gamma, \mathcal{B}_\Omega) = U_{\text{fin}}(\mathcal{B}_\Gamma, \mathcal{B}_\Omega)$, and $S_1(\mathcal{B}_\Gamma, \mathcal{B}) = U_{\text{fin}}(\mathcal{B}_\Gamma, \mathcal{B})$. In addition, each selection principle for Borel covers implies the corresponding selection principle for open covers.

$^1$We identify each property with the collection of sets satisfying this property. Thus, for properties $P$ and $Q$, we may write $X \in P$, $P \subseteq Q$, etc.
This paper is divided into two parts. Part 1 consists of Sections 2–4, and Part 2 consists of the remaining sections. In Section 2 we study subcover-type properties and their applications to the study of the new selection principles. In Section 3 we characterize some of the properties in terms of combinatorial properties of Borel images. In Section 4 we find the critical cardinalities of most of the new properties, and apply the results to solve a topological version of the minimal tower problem, which was suggested to us by Scheepers and stated in [15].

It seems that some new mathematical tools are required to solve some of the remaining open problems, as the special properties of $\tau$-covers usually do not allow application of standard methods developed during the study of classical selection principles. For this very reason, we believe that these are the important problems which must be addressed in the future. However, we suggest in the second part of this paper two relaxations of the notion of $\tau$-cover, which are easier to work with and may turn out useful in the study of the original problems. We
demonstrate this by proving results which are still open for the case of usual $\tau$-covers.

Part 1. $\tau$-covers

2. Subcovers with stronger properties

Definition 2.1. Let $X$ be a set of reals, and $\mathcal{U}$, $\mathcal{V}$ collections of covers of $X$. We say that $X$ satisfies $(\mathcal{U} : \mathcal{V})$ (read: $\mathcal{U}$ choose $\mathcal{V}$) if for each cover $U \in \mathcal{U}$ there exists a subcover $V \subseteq U$ such that $V \in \mathcal{V}$.

Observe that for any pair $\mathcal{U}$, $\mathcal{V}$ of collections of countable covers we have that the property $S_{\text{fin}}(\mathcal{U}, \mathcal{V})$ implies $(\mathcal{U} : \mathcal{V})$. Gerlits and Nagy [6] proved that for $\mathcal{U} = \Omega$ and $\mathcal{V} = \Gamma$, the converse also holds, in fact, $S_{1}(\Omega, \Gamma) = (\Omega : \Gamma)$. But in general the property $(\mathcal{U} : \mathcal{V})$ can be strictly weaker than $S_{\text{fin}}(\mathcal{U}, \mathcal{V})$.

A useful property of this notion is the following.

Lemma 2.2 (Cancellation Laws). For collections of covers $\mathcal{U}$, $\mathcal{V}$, $\mathcal{W}$,

1. $(\mathcal{U} : \mathcal{V}) \cap (\mathcal{V} : \mathcal{W}) \subseteq (\mathcal{U} : \mathcal{W})$,
2. $(\mathcal{U} : \mathcal{V}) \cap S_{\text{fin}}(\mathcal{V}, \mathcal{W}) \subseteq S_{\text{fin}}(\mathcal{U}, \mathcal{W})$,
3. $S_{\text{fin}}(\mathcal{U}, \mathcal{V}) \cap (\mathcal{V} : \mathcal{W}) \subseteq S_{\text{fin}}(\mathcal{U}, \mathcal{W})$; and
4. $(\mathcal{U} : \mathcal{V}) \cap S_{1}(\mathcal{V}, \mathcal{W}) \subseteq S_{1}(\mathcal{U}, \mathcal{W})$,
5. If $\mathcal{W}$ is closed under taking supersets, then $S_{1}(\mathcal{U}, \mathcal{V}) \cap (\mathcal{V} : \mathcal{W}) \subseteq S_{1}(\mathcal{U}, \mathcal{W})$.

Moreover, if $\mathcal{U} \supseteq \mathcal{V} \supseteq \mathcal{W}$, then equality holds in (1)–(5).

Proof. (1) is immediate. To prove (2), we can apply $S_{\text{fin}}(\mathcal{V}, \mathcal{W})$ to $\mathcal{V}$-subcovers of the given covers. (4) is similar to (2).

(3) Assume that $U_{n} \in \mathcal{U}$, $n \in \mathbb{N}$, are given. Apply $S_{\text{fin}}(\mathcal{U}, \mathcal{V})$ to choose finite subsets $F_{n} \subseteq U_{n}$, $n \in \mathbb{N}$, such that $V = \bigcup_{n \in \mathbb{N}} F_{n} \in \mathcal{V}$. By $(\mathcal{V} : \mathcal{W})$, there exists a subset $W$ of $V$ such that $W \in \mathcal{W}$. Then for each $n$ $W \cap F_{n}$ is a finite (possibly empty) subset of $U_{n}$, and $\bigcup_{n \in \mathbb{N}} (W \cap F_{n}) = W \in \mathcal{W}$. To prove (5), observe that the resulting cover $V$ contains an element of $\mathcal{W}$, and as $\mathcal{W}$ is closed under taking supersets, $V \in \mathcal{W}$ as well.

It is clear that reverse inclusion (and therefore equality) hold in (1)–(5) when $\mathcal{U} \supseteq \mathcal{V} \supseteq \mathcal{W}$.

Corollary 2.3. Assume that $\mathcal{U} \supseteq \mathcal{V}$. Then the following equivalences hold:

1. $S_{\text{fin}}(\mathcal{U}, \mathcal{V}) = (\mathcal{U} : \mathcal{V}) \cap S_{\text{fin}}(\mathcal{V}, \mathcal{U})$. 

Proof. (1) is immediate. To prove (2), we can apply $S_{\text{fin}}(\mathcal{V}, \mathcal{W})$ to $\mathcal{V}$-subcovers of the given covers. (4) is similar to (2).

(3) Assume that $U_{n} \in \mathcal{U}$, $n \in \mathbb{N}$, are given. Apply $S_{\text{fin}}(\mathcal{U}, \mathcal{V})$ to choose finite subsets $F_{n} \subseteq U_{n}$, $n \in \mathbb{N}$, such that $V = \bigcup_{n \in \mathbb{N}} F_{n} \in \mathcal{V}$. By $(\mathcal{V} : \mathcal{W})$, there exists a subset $W$ of $V$ such that $W \in \mathcal{W}$. Then for each $n$ $W \cap F_{n}$ is a finite (possibly empty) subset of $U_{n}$, and $\bigcup_{n \in \mathbb{N}} (W \cap F_{n}) = W \in \mathcal{W}$. To prove (5), observe that the resulting cover $V$ contains an element of $\mathcal{W}$, and as $\mathcal{W}$ is closed under taking supersets, $V \in \mathcal{W}$ as well.

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(3) Assume that $U_{n} \in \mathcal{U}$, $n \in \mathbb{N}$, are given. Apply $S_{\text{fin}}(\mathcal{U}, \mathcal{V})$ to choose finite subsets $F_{n} \subseteq U_{n}$, $n \in \mathbb{N}$, such that $V = \bigcup_{n \in \mathbb{N}} F_{n} \in \mathcal{V}$. By $(\mathcal{V} : \mathcal{W})$, there exists a subset $W$ of $V$ such that $W \in \mathcal{W}$. Then for each $n$ $W \cap F_{n}$ is a finite (possibly empty) subset of $U_{n}$, and $\bigcup_{n \in \mathbb{N}} (W \cap F_{n}) = W \in \mathcal{W}$. To prove (5), observe that the resulting cover $V$ contains an element of $\mathcal{W}$, and as $\mathcal{W}$ is closed under taking supersets, $V \in \mathcal{W}$ as well.

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It is clear that reverse inclusion (and therefore equality) hold in (1)–(5) when $\mathcal{U} \supseteq \mathcal{V} \supseteq \mathcal{W}$. 

Proof. (1) is immediate. To prove (2), we can apply $S_{\text{fin}}(\mathcal{V}, \mathcal{W})$ to $\mathcal{V}$-subcovers of the given covers. (4) is similar to (2).

(3) Assume that $U_{n} \in \mathcal{U}$, $n \in \mathbb{N}$, are given. Apply $S_{\text{fin}}(\mathcal{U}, \mathcal{V})$ to choose finite subsets $F_{n} \subseteq U_{n}$, $n \in \mathbb{N}$, such that $V = \bigcup_{n \in \mathbb{N}} F_{n} \in \mathcal{V}$. By $(\mathcal{V} : \mathcal{W})$, there exists a subset $W$ of $V$ such that $W \in \mathcal{W}$. Then for each $n$ $W \cap F_{n}$ is a finite (possibly empty) subset of $U_{n}$, and $\bigcup_{n \in \mathbb{N}} (W \cap F_{n}) = W \in \mathcal{W}$. To prove (5), observe that the resulting cover $V$ contains an element of $\mathcal{W}$, and as $\mathcal{W}$ is closed under taking supersets, $V \in \mathcal{W}$ as well.
(2) If $\mathcal{U}$ is closed under taking supersets, then $S_1(\mathcal{U}, \mathcal{U}) = \left(\begin{array}{c} \mathcal{U} \\ \mathcal{U} \end{array}\right) \cap S_1(\mathcal{V}, \mathcal{U})$.

Proof. We prove (1). Clearly $S_{fin}(\mathcal{U}, \mathcal{U})$ implies $\left(\begin{array}{c} \mathcal{U} \\ \mathcal{U} \end{array}\right)$ and $S_{fin}(\mathcal{V}, \mathcal{U})$. On the other hand, by applying the Cancellation Laws (2) and then (3) we have that

$$\left(\begin{array}{c} \mathcal{U} \\ \mathcal{U} \end{array}\right) \cap S_{fin}(\mathcal{V}, \mathcal{U}) \subseteq S_{fin}(\mathcal{U}, \mathcal{U}) \cap \left(\begin{array}{c} \mathcal{U} \\ \mathcal{U} \end{array}\right) \subseteq S_{fin}(\mathcal{U}, \mathcal{U}).$$

$\square$

2.1. When every $\tau$-cover contains a $\gamma$-cover.

Theorem 2.4. The following equivalences hold:

1. $S_1(\Gamma, T) = \left(\begin{array}{c} T \\ \Gamma \end{array}\right) \cap S_{fin}(\Gamma, T)$,
2. $S_1(\mathcal{B}_T, \mathcal{B}_T) = \left(\begin{array}{c} \mathcal{B}_T \\ \mathcal{B}_T \end{array}\right) \cap S_{fin}(\mathcal{B}_T, \mathcal{B}_T)$.

Proof. (1) By the Cancellation Laws 2.2, $\left(\begin{array}{c} T \\ \Gamma \end{array}\right) \cap S_{fin}(\Gamma, T) \subseteq S_{fin}(\Gamma, \Gamma)$. In [7] it was proved that $S_{fin}(\Gamma, \Gamma) = S_1(\Gamma, \Gamma)$. Thus, $\left(\begin{array}{c} T \\ \Gamma \end{array}\right) \cap S_{fin}(\Gamma, T) \subseteq \left(\begin{array}{c} T \\ \Gamma \end{array}\right) \cap S_1(\Gamma, \Gamma)$, which by the Cancellation Laws is a subset of $S_1(\Gamma, \Gamma)$. The other direction is immediate.

(2) is similar. $\square$

Corollary 2.5. The following equivalences hold:

1. $S_1(\Gamma, T) = S_{fin}(\Gamma, T)$;
2. $S_1(\mathcal{B}_T, \mathcal{B}_T) = S_{fin}(\mathcal{B}_T, \mathcal{B}_T)$.

Using similar arguments, we have the following.

Theorem 2.6. The following equivalences hold:

1. $S_1(\mathcal{U}, \mathcal{U}) = \left(\begin{array}{c} \mathcal{U} \\ \mathcal{U} \end{array}\right) \cap S_{fin}(\mathcal{U}, \mathcal{U})$;
2. $S_1(\mathcal{B}_T, \mathcal{B}_T) = \left(\begin{array}{c} \mathcal{B}_T \\ \mathcal{B}_T \end{array}\right) \cap S_{fin}(\mathcal{B}_T, \mathcal{B}_T)$.

2.2. When every $\omega$-cover contains a $\tau$-cover.

Theorem 2.7. The following inclusions hold:

1. $\left(\begin{array}{c} \mathcal{U} \\ \mathcal{U} \end{array}\right) \subseteq S_{fin}(\Gamma, T)$.
2. $\left(\begin{array}{c} \mathcal{B}_T \\ \mathcal{B}_T \end{array}\right) \subseteq S_{fin}(\mathcal{B}_T, \mathcal{B}_T)$.

Proof. We will prove (1) (the proof of (2) is identical). Assume that $X$ satisfies $\left(\begin{array}{c} \mathcal{U} \\ \mathcal{U} \end{array}\right)$. If $X$ is countable then it satisfies all of the properties mentioned in this paper. Otherwise let $x_n, n \in \mathbb{N}$, be distinct elements in $X$. Assume that $U_n = \{U_n\}_{m \in \mathbb{N}}, n \in \mathbb{N}$, are open $\gamma$-covers of $X$. Define $\hat{U}_n = \{U_n \setminus \{x_n\}\}_{m \in \mathbb{N}}$. Then $\mathcal{U} = \bigcup_{n \in \mathbb{N}} \hat{U}_n$ is an open $\omega$-cover.
of $X$, and thus contains a $\tau$-cover $\mathcal{V}$ of $X$. Let $\preceq$ be the induced quasiordering.

**Lemma 2.8.** If $\langle X, \preceq \rangle$ has a least element, then $\mathcal{V}$ contains a $\gamma$-cover of $X$.

**Proof.** Write $\mathcal{V} = \{V_n\}_{n \in \mathbb{N}}$. Let $x_0$ be a least element in $\langle X, \preceq \rangle$. Consider the subsequence $\{V_{n_k}\}_{k \in \mathbb{N}}$ consisting of the elements $V_n$ such that $x_0 \in V_n$. Since $\tau$-covers are large, this sequence is infinite. For all $x \in X$ we have $x_0 \preceq x$, thus $x_0 \in V_n \rightarrow x \in V_n$ for all but finitely many $n$. Since $x_0 \in V_{n_k}$ for all $k$, we have that for all but finitely many $k$, $x \in V_{n_k}$.

There are two cases to consider.

**Case 1.** For some $n x_n$ is a least element in $\langle X, \preceq \rangle$. Then $\mathcal{V}$ contains a $\gamma$-cover $\hat{\mathcal{V}}$ of $X$. In this case, for all $n x_n$ belongs to all but finitely many members of $\hat{\mathcal{V}}$, thus $\hat{\mathcal{V}} \cap \hat{\mathcal{U}}_n$ is finite for each $n$, and $\mathcal{W} = \{U : (\exists n)U \setminus \{x_n\} \in \hat{\mathcal{V}}\}$ is a $\gamma$-cover of $X$.

**Case 2.** For each $n$ there exists $x \neq x_n$ with $x \preceq x_n$. For each $n$, $\mathcal{U}_n$ is a $\gamma$-cover of $X$, thus $x$ belongs to all but finitely many members of $\mathcal{V} \cap \mathcal{U}_n$. Since $x_n$ does not belong to any of the members in $\mathcal{V} \cap \mathcal{U}_n$, $\mathcal{V} \cap \mathcal{U}_n$ must be finite. Thus, $\mathcal{W} = \{U : (\exists n)U \setminus \{x_n\} \in \hat{\mathcal{V}}\}$ is a $\tau$-cover of $X$.

If $\binom{n}{T} \subseteq S_{\text{fin}}(T, \Omega)$, then by Corollary 2.3 $\binom{n}{T} = S_{\text{fin}}(\Omega, T)$.

**Problem 2.9.** Is $\binom{n}{T} = S_{\text{fin}}(\Omega, T)$?

### 3. Combinatorics of Borel images

In this section we characterize several properties in terms of Borel images in the spaces $^\mathbb{N}\mathbb{N}$ and $P_{\infty}(\mathbb{N})$, using the combinatorial structure of these spaces.

#### 3.1. The combinatorial structures

A quasiorder $\preceq^*$ is defined on the Baire space $^\mathbb{N}\mathbb{N}$ by eventual dominance:

$$f \preceq^* g \quad \text{if } f(n) \leq g(n) \text{ for all but finitely many } n.$$ 

A subset $Y$ of $^\mathbb{N}\mathbb{N}$ is called *unbounded* if it is unbounded with respect to $\preceq^*$. $Y$ is *dominating* if it is cofinal in $^\mathbb{N}\mathbb{N}$ with respect to $\preceq^*$, that is, for each $f \in ^\mathbb{N}\mathbb{N}$ there exists $g \in Y$ such that $f \preceq^* g$. $b$ is the minimal size of an unbounded subset of $^\mathbb{N}\mathbb{N}$, and $\overline{d}$ is the minimal size of a dominating subset of $^\mathbb{N}\mathbb{N}$.

Define a quasiorder $\subseteq^*$ on $P_{\infty}(\mathbb{N})$ by $a \subseteq^* b$ if $a \setminus b$ is finite. An infinite set $a \subseteq \mathbb{N}$ is a *pseudo-intersection* of a family $Y \subseteq P_{\infty}(\mathbb{N})$ if for each $b \in Y$, $a \subseteq^* b$. A family $Y \subseteq P_{\infty}(\mathbb{N})$ is a *tower* if it is linearly
quasiordered by $\subseteq^*$, and it has no pseudo-intersection. $t$ is the minimal size of a tower.

A family $Y \subseteq P_\infty(\mathbb{N})$ is centered if the intersection of each (nonempty) finite subfamily of $Y$ is infinite. Note that every tower in $P_\infty(\mathbb{N})$ is centered. A centered family $Y \subseteq P_\infty(\mathbb{N})$ is a power if it does not have a pseudo-intersection. $p$ is the minimal size of a power.

3.2. The property $(S_0^{\mathbb{N}})$. For a set of reals $X$ and a topological space $Z$, we say that $Y$ is a Borel image of $X$ in $Z$ if there exists a Borel function $f : X \to Z$ such that $f[X] = Y$. The following classes of sets were introduced in [8]:

- $P$: The set of $X \subseteq \mathbb{R}$ such that no Borel image of $X$ in $P_\infty(\mathbb{N})$ is a power,
- $B$: The set of $X \subseteq \mathbb{R}$ such that every Borel image of $X$ in $\mathbb{N}\mathbb{N}$ is bounded (with respect to eventual domination);
- $D$: The set of $X \subseteq \mathbb{R}$ such that no Borel image of $X$ in $\mathbb{N}\mathbb{N}$ is dominating.

For a collection $\mathcal{J}$ of separable metrizable spaces, let $\text{non}(\mathcal{J})$ denote the minimal cardinality of a separable metrizable space which is not a member of $\mathcal{J}$. We also call $\text{non}(\mathcal{J})$ the critical cardinality of the class $\mathcal{J}$. The critical cardinalities of the above classes are $p$, $b$, and $d$, respectively. These classes have the interesting property that they transfer the cardinal inequalities $p \leq b \leq d$ to the inclusions $P \subseteq B \subseteq D$.

Definition 3.1. For each countable cover of $X$ enumerated bijectively as $\mathcal{U} = \{U_n\}_{n \in \mathbb{N}}$ we associate a function $h_\mathcal{U} : X \to P(\mathbb{N})$, defined by $h_\mathcal{U}(x) = \{n : x \in U_n\}$.

$\mathcal{U}$ is a large cover of $X$ if, and only if, $h_\mathcal{U}[X] \subseteq P_\infty(\mathbb{N})$. As we assume that $X$ is infinite and is not a member of any of our covers, we have that each $\omega$-cover of $X$ is a large cover of $X$. The following lemma is a key observation for the rest of this section. Note that $h_\mathcal{U}$ is a Borel function whenever $\mathcal{U}$ is a Borel cover of $X$, and $h_\mathcal{U}$ is continuous whenever all elements of $\mathcal{U}$ are clopen.

Lemma 3.2 ([15]). Assume that $\mathcal{U}$ is a countable large cover of $X$.

1. $\mathcal{U}$ is an $\omega$-cover of $X$ if, and only if, $h_\mathcal{U}[X]$ is centered.
2. $\mathcal{U}$ contains a $\gamma$-cover of $X$ if, and only if, $h_\mathcal{U}[X]$ has a pseudo-intersection.
3. $\mathcal{U}$ is a $\tau$-cover of $X$ if, and only if, $h_\mathcal{U}[X]$ is linearly quasiordered by $\subseteq^*$.
Moreover, if $f : X \to P(\mathbb{N})$ is any function, and $A = \{O_n\}_{n \in \mathbb{N}}$ is the clopen cover of $P(\mathbb{N})$ such that $x \in O_n \iff n \in x$, then for $U = \{f^{-1}[O_n]\}_{n \in \mathbb{N}}$ we have that $f = h_U$.

This Lemma implies that $P = (B_0)^T$ [13].

**Corollary 3.3.** $P = (B_0) \cap (B_T)^T$.

It is natural to define the following notion.

$\mathcal{T}$: The set of $X \subseteq \mathbb{R}$ such that no Borel image of $X$ in $P_\infty(\mathbb{N})$ is a tower.

**Theorem 3.4.** $T = (B_T)^T$.

*Proof.* See [15] for the clopen version of this theorem (a straightforward usage of Lemma 3.2). The proof for the Borel case is similar. \(\square\)

**Corollary 3.5.** $\text{non}(B_T) = \text{non}(T) = t$.

*Proof.* By Theorem 3.4, $t \leq \text{non}(B_T)$. In [15] we defined $\mathcal{T}$ to be the collection of sets for which every countable clopen $\tau$-cover contains a $\gamma$-cover, and showed that $\text{non}(\mathcal{T}) = t$. But $(B_T)^T \subseteq (T)^T \subseteq T$. \(\square\)

Clearly, $P \subseteq T$. The cardinal inequality $p \leq t \leq b$ suggests pushing this further by showing that $T \subseteq B_\gamma$; unfortunately this is false. Sets which are continuous images of Borel sets are called *analytic*.

**Theorem 3.6.** Every analytic set satisfies $T$. In particular, $\mathcal{N}$ is in $T$.

*Proof.* According to [15], no continuous image of an analytic set is a tower. In particular, towers are not analytic subsets of $P_\infty(\mathbb{N})$. Since Borel images of analytic sets are again analytic sets, we have that every analytic set satisfies $T$. \(\square\)

The following equivalences hold [13]:

- $S_1(B_\Gamma, B_\Gamma) = B$,
- $S_1(B_\Gamma, B) = D$,
- $S_1(B_\Omega, B_\Gamma) = P$.

Theorem 3.6 rules out an identification of $T$ with any of the selection principles. However, we get the following characterization of $S_1(B_T, B_\Gamma)$ in terms of Borel images.

**Theorem 3.7.** $S_1(B_T, B_\Gamma) = T \cap B$.

*Proof.* By the Cancellation Laws and Theorem 3.4, $S_1(B_T, B_\Gamma) = (B_T)^T \cap S_1(B_\Gamma, B_\Gamma) = T \cap B$. \(\square\)
3.3. **The property** ($B_{\Omega}$). For a subset $Y$ of $P_\infty(\mathbb{N})$ and $a \in P_\infty(\mathbb{N})$, define

$$Y \upharpoonright a = \{ y \cap a : y \in Y \}.$$ 

If all sets in $Y \upharpoonright a$ are infinite, we say that $Y \upharpoonright a$ is a **large restriction** of $Y$.

**Theorem 3.8.** For a set $X$ of real numbers, the following are equivalent:

1. $X$ satisfies ($B_{\Omega}$)
2. For each Borel image $Y$ of $X$ in $P_\infty(\mathbb{N})$, if $Y$ is centered, then there exists a large restriction of $Y$ which is linearly quasiordered by $\subseteq^*$.

**Proof.** $1 \Rightarrow 2$: Assume that $\Psi : X \to P_\infty(\mathbb{N})$ is a Borel function, and let $Y = \Psi[X]$. Assume that $Y$ is centered, and consider the collection $\mathcal{A} = \{ O_n \}_{n \in \mathbb{N}}$ where $O_n = \{ a : n \in a \} \cap Y$ for each $n \in \mathbb{N}$. If the set $a = \{ n : Y = O_n \}$ is infinite, then $a$ is a pseudo-intersection of $Y$ and we are done. Otherwise, by removing finitely many elements from $\mathcal{A}$ we get that $\mathcal{A}$ is an $\omega$-cover of $Y$.

Setting $U_n = \Psi^{-1}[O_n]$ for each $n$, we have that $\mathcal{U} = \{ U_n \}_{n \in \mathbb{N}}$ is a Borel $\omega$-cover of $X$, which thus contains a $\tau$-cover $\{ U_{a_n} \}_{n \in \mathbb{N}}$ of $X$. Let $a = \{ a_n \}_{n \in \mathbb{N}}$, and define a cover $V = \{ V_n \}_{n \in \mathbb{N}}$ of $X$ by

$$V_n = \begin{cases} U_n & n \in a \\ \emptyset & \text{otherwise} \end{cases}$$

Then $V$ is a $\tau$-cover of $X$, and by Lemma 3.2, $\Psi[X] \upharpoonright a = h_\mathcal{U}[X] \upharpoonright a = h_V[X]$ is linearly quasiordered by $\subseteq^*$.

$2 \Rightarrow 1$: Assume that $\mathcal{U} = \{ U_n \}_{n \in \mathbb{N}}$ is an $\omega$-cover of $X$. By Lemma 3.2, $h_\mathcal{U}[X]$ is centered. Let $a = \{ a_n \}_{n \in \mathbb{N}}$ be a large restriction of $h_\mathcal{U}[X]$ which is linearly quasiordered by $\subseteq^*$, and define $V$ as in $1 \Rightarrow 2$. Then $h_\mathcal{U}[X] \upharpoonright a = h_V[X]$. Thus all elements in $h_V[X]$ are infinite (i.e., $V$ is a large cover of $X$), and $h_V[X]$ is linearly quasiordered by $\subseteq^*$ (i.e., $V$ is a $\tau$-cover of $X$). Then $V \setminus \{ \emptyset \} \subseteq \mathcal{U}$ is a $\tau$-cover of $X$. \qed

**Remark 3.9.** Replacing the “Borel sets” by “clopen sets” and “Borel functions” by “continuous functions” in the last proof we get that the following properties are equivalent for a set $X$ of reals:

1. Every countable clopen $\omega$-cover of $X$ contains a $\tau$-cover of $X$.
2. For each continuous image $Y$ of $X$ in $P_\infty(\mathbb{N})$, if $Y$ is centered, then there exists a large restriction of $Y$ which is linearly quasiordered by $\subseteq^*$.

We do not know whether the open version of this result is true.
3.4. The property $U_{\text{fin}}(B_T, B_T)$.

**Definition 3.10.** A family $Y \subseteq \mathbb{N}^\mathbb{N}$ satisfies the excluded middle property if there exists $g \in \mathbb{N}^\mathbb{N}$ such that:

1. For all $f \in Y$, $g \nless f$;
2. For each $f, h \in Y$, one of the situations $f(n) < g(n) \leq h(n)$ or $h(n) < g(n) \leq f(n)$ is possible only for finitely many $n$.

**Theorem 3.11.** For a set $X$ of real numbers, the following are equivalent:

1. $X$ satisfies $U_{\text{fin}}(B_T, B_T)$;
2. Every Borel image of $X$ in $\mathbb{N}^\mathbb{N}$ satisfies the excluded middle property.

**Proof.** $1 \Rightarrow 2$: For each $n$, the collection $U_n = \{U^m_n : m \in \mathbb{N}\}$, where $U^m_n = \{f \in \mathbb{N}^\mathbb{N} : f(n) \leq m\}$, $m \in \mathbb{N}$, is an open $\gamma$-cover of $\mathbb{N}^\mathbb{N}$. Assume that $\Psi$ is a Borel function from $X$ to $\mathbb{N}^\mathbb{N}$. By standard arguments we may assume that $\Psi^{-1}[U^m_n] \neq X$ for all $n$ and $m$. Then the collections $U_n = \{\Psi^{-1}[U^m_n] : m \in \mathbb{N}\}$, $n \in \mathbb{N}$, are Borel $\gamma$-covers of $X$. By $U_{\text{fin}}(B_T, B_T)$, there exist finite sets $F_n \subseteq \mathbb{N}$, $n \in \mathbb{N}$, such that $U = \bigcup_{m \in F_n} \Psi^{-1}[U^m_n] : n \in \mathbb{N}$ is a $\tau$-cover of $X$. Let $A = \{n : F_n \neq \emptyset\}$.

Note that for each $n \in A$, $\bigcup_{m \in F_n} \Psi^{-1}[U^m_n] = \Psi^{-1}[\bigcup_{m \in F_n} U^m_n]$. Define $g \in \mathbb{N}^\mathbb{N}$ by

$$g(n) = \begin{cases} \max F_n + 1 & n \in A \\ 0 & \text{otherwise} \end{cases}$$

For all $x \in X$, as $U$ is a large cover of $X$, there exist infinitely many $n \in A$ such that $\Psi(x) \in \bigcup_{m \in F_n} U^m_n$ (that is, $\Psi(x)(n) < g(n)$). Let $\preceq$ be the linear quasiordering of $X$ induced by the $\tau$-cover $U$. Then for all $x, y \in X$, either $x \preceq y$ or $y \preceq x$. In the first case we get that for all but finitely many $n$ $\Psi(x)(n) < g(n) \Rightarrow \Psi(y)(n) < g(n)$, and in the second case we get the same assertion with $x$ and $y$ swapped. This shows that $\Psi[X]$ satisfies the excluded middle property.

$2 \Rightarrow 1$: Assume that $U_n = \{U^m_n : m \in \mathbb{N}\}$, $n \in \mathbb{N}$, are Borel covers of $X$ which do not contain a finite subcover. Replacing each $U^m_n$ with the Borel set $\bigcup_{k \leq m} U^k_n$ we may assume that the sets $U^m_n$ are monotonically increasing with $m$. Define a function $\Psi$ from $X$ to $\mathbb{N}^\mathbb{N}$ so that for each $x$ and $n$:

$$\Psi(x)(n) = \min\{m : x \in U^m_n\}.$$ 

Then $\Psi$ is a Borel map, and so $\Psi[X]$ satisfies the excluded middle property. Let $g \in \mathbb{N}^\mathbb{N}$ be a witness for that. Then $U = \{U^n_{g(n)-1} : n \in \mathbb{N}, g(n) > 0\}$ is a $\tau$-cover of $X$: For each $x \in X$ we have that
$g \not\leq^* \Psi(x)$, thus $U$ is a large cover of $X$. Moreover, for all $x, y \in X$, we have by the excluded middle property that at least one of the assertions $\Psi(x)(n) < g(n) \leq \Psi(y)(n)$ or $\Psi(y)(n) < g(n) \leq \Psi(y)(n)$ is possible only for finitely many $n$. Then the first assertion implies that $x \preceq y$, and the second implies $y \preceq x$ with respect to $U$. \hfill \Box

Remark 3.12. The analogue clopen version of Theorem 3.11 also holds. We do not know whether there exist an analogue characterization of $U_{\text{fin}}(\Gamma, T)$ (the open version) in terms of continuous images.

4. Critical cardinalities

Theorem 4.1. $\text{non}(S_{\text{fin}}(B_T, B_\Omega)) = \text{non}(S_{\text{fin}}(T, \Omega)) = \delta$.

Proof. $S_{\text{fin}}(B_\Omega, B_\Omega) \subseteq S_{\text{fin}}(B_T, B_\Omega) \subseteq S_{\text{fin}}(T, \Omega) \subseteq S_{\text{fin}}(\Gamma, \Omega)$, and according to [7] and [13], $\text{non}(S_{\text{fin}}(B_\Omega, B_\Omega)) = \text{non}(S_{\text{fin}}(\Gamma, \Omega)) = \delta$. \hfill \Box

Theorem 4.2. $\text{non}(S_1(B_T, B_T)) = \text{non}(S_1(T, \Gamma)) = t$.

Proof. By Theorem 2.4, $S_1(T, \Gamma) = (T) \cap S_1(\Gamma, \Gamma)$, thus by Corollary 3.5, $\text{non}(S_1(T, \Gamma)) = \min\{\text{non}(T), \text{non}(S_1(\Gamma, \Gamma))\} = \min\{t, b\} = t$. The proof for the Borel case is similar. \hfill \Box

Definition 4.3. $\tau$ is the minimal cardinality of a family $Y \subseteq ^\omega \omega$ which does not satisfy the excluded middle property.

Therefore $b \leq \tau \leq \delta$. A family $Y \subseteq P_\infty(\mathbb{N})$ is splitting if for each infinite $a \subseteq \mathbb{N}$ there exists $s \in Y$ which splits $a$, that is, such that the sets $a \cap s$ and $a \setminus s$ are infinite. $s$ is the minimal size of a splitting family. In [14] it is proved that $\tau = \max\{s, b\}$.

Theorem 4.4. $\text{non}(U_{\text{fin}}(B_T, B_T)) = \text{non}(U_{\text{fin}}(\Gamma, T)) = \tau$.

Proof. By Theorem 3.11, $\text{non}(U_{\text{fin}}(B_T, B_T)) = \tau$. Thus, our theorem will follow from the inclusion $U_{\text{fin}}(B_T, B_T) \subseteq U_{\text{fin}}(\Gamma, T)$ once we prove that $\text{non}(U_{\text{fin}}(\Gamma, T)) \leq \tau$. To this end, consider a family $Y \subseteq ^\omega \omega$ of size $\tau$ which does not satisfy the excluded middle property, and consider the monotone $\gamma$-covers $U_n, n \in \mathbb{N}$, of $^\omega \omega$ defined in the proof of Theorem 3.11. Then, as in that proof, we cannot extract from these covers a $\tau$-cover of $Y$. Thus, $Y$ does not satisfy $U_{\text{fin}}(\Gamma, T)$. \hfill \Box

Definition 4.5. Let $\kappa_{\omega_1}$ be the minimal cardinality of a centered set $Y \subseteq P_\infty(\mathbb{N})$ such that for no $a \in P_\infty(\mathbb{N})$, the restriction $Y \restriction a$ is large and linearly quasiordered by $\subseteq^*$. It is easy to see (either from the definitions or by consulting the involved selection properties) that $\kappa_{\omega_1} \leq \delta$ and $p = \min\{\kappa_{\omega_1}, t\}$. In [14] it is proved that in fact $\kappa_{\omega_1} = p$. 


Lemma 4.6. \(\text{non}\left(\left(\mathcal{B}_\Omega, \mathcal{B}_T\right)\right) = \text{non}\left(\left(\Omega, \mathcal{T}\right)\right) = p.\)

Proof. Let \(\mathcal{P}_{\omega r}\) denote the property that every clopen \(\omega\)-cover contains a \(\gamma\)-cover. Then \(\left(\mathcal{B}_\Omega\right) \subseteq \left(\Omega, \mathcal{T}\right) \subseteq \mathcal{P}_{\omega r}.\) By Theorem 3.8 and Remark 3.9, 
\(\text{non}\left(\left(\mathcal{B}_\Omega\right)\right) = \text{non}\left(\mathcal{P}_{\omega r}\right) = \kappa_{\omega r} = p.\) □

Theorem 4.7. \(\text{non}\left(\text{S}_{\text{fin}}(\mathcal{B}_\Omega, \mathcal{B}_T)\right) = \text{non}\left(\text{S}_{\text{fin}}(\Omega, \mathcal{T})\right) = p.\)

Proof. By Corollary 2.3 and Theorem 4.1,
\[
\text{non}\left(\text{S}_{\text{fin}}(\Omega, \mathcal{T})\right) = \min\{\text{non}\left(\left(\Omega, \mathcal{T}\right)\right), \text{non}\left(\text{S}_{\text{fin}}(\mathcal{T}, \Omega)\right)\} = \min\{\kappa_{\omega r}, \mathcal{d}\} = \kappa_{\omega r} = p.
\]
The proof for the Borel case is the same. □

Corollary 4.8. \(\text{non}\left(\text{S}_1(\mathcal{B}_\Omega, \mathcal{B}_T)\right) = \text{non}\left(\text{S}_1(\Omega, \mathcal{T})\right) = p.\)

Proof. \(\text{S}_1(\mathcal{B}_\Omega, \mathcal{B}_T) \subseteq \text{S}_1(\mathcal{B}_\Omega, \mathcal{B}_T) \subseteq \text{S}_1(\Omega, \mathcal{T}) \subseteq \left(\Omega, \mathcal{T}\right).\) □

5. Topological variants of the Minimal Tower problem

Let \(c\) denote the size of the continuum. The following inequalities are well known [4]:
\[
p \leq t \leq b \leq d \leq c.
\]
For each pair except \(p\) and \(t\), it is well known that a strict inequality is consistent.

Problem 5.1 (Minimal Tower). Is it provable that \(p = t\)?

This is one of the major and oldest problems of infinitary combinatorics. Allusions to this problem can be found in Rothberger’s works (see, e.g., [10]).

We know that \(\text{S}_1(\Omega, \Gamma) \subseteq \text{S}_1(\mathcal{T}, \Gamma),\) and that \(\text{non}\left(\text{S}_1(\Omega, \Gamma)\right) = p,\) and \(\text{non}\left(\text{S}_1(\mathcal{T}, \Gamma)\right) = t.\) Thus, if \(p < t\) is consistent, then it is consistent that \(\text{S}_1(\Omega, \Gamma) \neq \text{S}_1(\mathcal{T}, \Gamma).\) Thus the following problem, which was suggested to us by Scheepers, is a logical lower bound on the difficulty of the Minimal Tower problem.

Problem 5.2 ([15]). Is it consistent that \(\text{S}_1(\Omega, \Gamma) \neq \text{S}_1(\mathcal{T}, \Gamma)\)?

We also have a Borel variant of this problem.

Problem 5.3. Is it consistent that \(\text{S}_1(\mathcal{B}_\Omega, \mathcal{B}_\Gamma) \neq \text{S}_1(\mathcal{B}_\Gamma, \mathcal{B}_\Gamma)\)?

We will solve both of these problems.

For a class \(\mathcal{J}\) of sets of real numbers with \(\cup \mathcal{J} \notin \mathcal{J},\) the additivity number of \(\mathcal{J}\) is the minimal cardinality of a collection \(\mathcal{F} \subseteq \mathcal{J}\) such that \(\cup \mathcal{F} \notin \mathcal{J}.\) The additivity number of \(\mathcal{J}\) is denoted \(\text{add}(\mathcal{J}).\)
Lemma 5.4 ([15]). Assume that $Y \subseteq P_\infty(\mathbb{N})$ is linearly ordered by $\subseteq^*$, and for some $\kappa < t$, $Y = \bigcup_{\alpha < \kappa} Y_\alpha$ where each $Y_\alpha$ has a pseudo-intersection. Then $Y$ has a pseudo-intersection.

Theorem 5.5. $\text{add}((^T_\text{B} )) = \text{add}((^T_\text{B} )) = t$.

Proof. By Theorem 3.4 and Lemma 5.4, we have that $\text{add}((^T_\text{B} )) = \text{add}(T) = t$. The proof that $\text{add}((^T_\text{B} )) = t$ is not as elegant and requires a back-and-forth usage of Lemma 3.2. Assume that $\kappa < t$, and let $X_\alpha$, $\alpha < \kappa$, be sets satisfying $(^T_\Gamma )$. Let $\mathcal{U}$ be a countable open $\tau$-cover of $X = \bigcup_{\alpha < \kappa} X_\alpha$. Then $h_\mathcal{U}[X] = \bigcup_{\alpha < \kappa} h_\mathcal{U}[X_\alpha]$ is linearly quasiordered by $\subseteq^*$. Since each $X_\alpha$ satisfies $(^T_\Gamma )$, for each $\mathcal{U}$ contains a $\gamma$-cover of $X_\alpha$, that is, $h_\mathcal{U}[X_\alpha]$ has a pseudo-intersection. By Lemma 5.4, $h_\mathcal{U}[X]$ has a pseudo-intersection, that is, $\mathcal{U}$ contains a $\gamma$-cover of $X$. □

Theorem 5.6. $\text{add}(S_1(\text{B}_T, \text{B}_T)) = t$.

Proof. By Theorem 2.4, $S_1(\text{B}_T, \text{B}_T) = (^T_\text{B}_\text{T} ) \cap S_1(\text{B}_T, \text{B}_T)$, and according to [19], $\text{add}(S_1(\text{B}_T, \text{B}_T)) = b$. By Theorem 5.5, we get that

$$\text{add}(S_1(\text{B}_T, \text{B}_T)) \geq \min\{t, b\} = t.$$

On the other hand, by Theorem 4.2 we have

$$\text{add}(S_1(\text{B}_T, \text{B}_T)) \leq \text{non}(S_1(\text{B}_T, \text{B}_T)) = t.$$

□

In [12] Scheepers proves that $S_1(\Gamma, \Gamma)$ is closed under taking unions of size less than the distributivity number $\mathfrak{d}$. Consequently, we get that $\text{add}(S_1(\text{T}, \text{G})) = t$ [19]. As it is consistent that $S_1(\text{O}, \text{G})$ is not closed under taking finite unions [5], we get a positive solution to Problem 5.2. We will now prove something stronger: Consistently, no class between $S_1(\text{B}_\text{O}, \text{B}_T )$ and $(^T_\text{T} )$ (inclusive) is closed under taking finite unions. This solves Problem 5.2 as well as Problem 5.3.

Theorem 5.7 (CH). There exist sets of reals $A$ and $B$ satisfying $S_1(\text{B}_\text{O}, \text{B}_T )$, such that $A \cup B$ does not satisfy $(^T_\text{T} )$. In particular, $S_1(\text{T}, \text{G}) \neq S_1(\text{O}, \text{G})$, and $S_1(\text{B}_T, \text{B}_T ) \neq S_1(\text{B}_\text{O}, \text{B}_T )$.

Proof. By a theorem of Brendle [3], assuming CH there exists a set of reals $X$ of size continuum such that all subsets of $X$ satisfy $\mathcal{P}$. (Recall that $\mathcal{P} = S_1(\text{B}_\text{O}, \text{B}_T ).$

As $\mathcal{P}$ is closed under taking Borel (continuous is enough) images, we may assume that $X \subseteq [0, 1]$. For $Y \subseteq [0, 1]$, write $Y + 1 = \{y + 1 : y \in Y\}$ for the translation of $Y$ by 1. As $|X| = \mathfrak{c}$ and only $\mathfrak{c}$ many out of the $2^\mathfrak{c}$ many subsets of $X$ are Borel, there exists a subset $Y$ of $X$
which is not $F_\sigma$ neither $G_\delta$. By a theorem of Galvin and Miller [5], for such a subset $Y$ the set $(X \setminus Y) \cup (Y + 1)$ does not satisfy $S_1(\Omega, \Gamma)$. Set $A = X \setminus Y$ and $B = Y + 1$. Then $A$ and $B$ satisfy $S_1(\mathcal{B}_\Omega, \mathcal{B}_\Gamma)$, and $A \cup B$ does not satisfy $S_1(\Omega, \Gamma) = \left(\frac{\Omega}{\Gamma}\right) \cap \left(\frac{T}{\Gamma}\right)$. By Theorem 5.5, $A \cup B$ satisfies $\left(\frac{T}{\Gamma}\right)$ and therefore it does not satisfies $\left(\frac{\Omega}{T}\right)$. But by Theorem 5.6, the set $A \cup B$ satisfies $S_1(\mathcal{B}_T, \mathcal{B}_\Gamma)$.

\[\square\]

6. Special elements

6.1. The Cantor set $C$. Let $C \subseteq \mathbb{R}$ be the canonic middle-third Cantor set.

**Proposition 6.1.** Cantor’s set $C$ does not satisfy $S_{fin}(\Gamma, T)$.

**Proof.** Had it satisfied this property, we would have by Theorem 3.6 that $C \in \left(\frac{T}{\Gamma}\right) \cap S_{fin}(\Gamma, T) = S_1(\Gamma, \Gamma)$, contradicting [7].

Thus $C$ satisfies $S_{fin}(T, \Omega)$ and $U_{fin}(\Gamma, T)$, and none of the other new properties.

6.2. A special Lusin set. In [1, 19] we construct, using cov$(\mathcal{M}) = c$, special Lusin sets of size $c$ which satisfy $S_1(\mathcal{B}_\Omega, \mathcal{B}_\Omega)$. The meta-structure of the proof is as follows. At each stage of this construction we define a set $Y^*_\alpha$ which is a union of less that cov$(\mathcal{M})$ many meager sets, and choose an element $x_\alpha \in G_\alpha \setminus Y^*_\alpha$ where $G_\alpha$ is a basic open subset of $\mathbb{N}^\mathbb{Z}$.

**Theorem 6.2.** If cov$(\mathcal{M}) = c$, then there exists a Lusin set satisfying $S_1(\mathcal{B}_\Omega, \mathcal{B}_\Omega)$ but not $U_{fin}(\Gamma, T)$.

**Proof.** We modify the aforementioned construction so to make sure that the resulting Lusin set $L$ does not satisfy the excluded middle property. As we do not need to use any group structure, we will work in $\mathbb{N}^{\mathbb{N}}$ rather than $\mathbb{N}^\mathbb{Z}$.

**Lemma 6.3.** Assume that $A$ is an infinite set of natural numbers, and $f \in \mathbb{N}^{\mathbb{N}}$. Then the sets

\[
M_{f,A} = \{ g \in \mathbb{N}^{\mathbb{N}} : [g \leq f] \cap A \text{ is finite} \}
\]

\[
\tilde{M}_{f,A} = \{ g \in \mathbb{N}^{\mathbb{N}} : [f < g] \cap A \text{ is finite} \}
\]

are meager subsets of $\mathbb{N}^{\mathbb{N}}$.

**Proof.** For each $k$, the sets

\[
N_k = \{ g \in \mathbb{N}^{\mathbb{N}} : (\forall n > k) \ n \in A \rightarrow f(n) < g(n) \}
\]

\[
\tilde{N}_k = \{ g \in \mathbb{N}^{\mathbb{N}} : (\forall n > k) \ n \in A \rightarrow g(n) \leq f(n) \}
\]

are nowhere dense in $\mathbb{N}^{\mathbb{N}}$. Now, $M_{f,A} = \bigcup_{k \in \mathbb{N}} N_k$, and $\tilde{M}_{f,A} = \bigcup_{k \in \mathbb{N}} \tilde{N}_k$.

\[\square\]
Consider an enumeration $\langle f_\alpha : \alpha < c \rangle$ of $\mathcal{N}$ which uses only even ordinals. At stage $\alpha$ for $\alpha$ even, let $Y_\alpha^*$ be the set defined in [19], and let $\tilde{Y}_\alpha^*$ be the union of $Y_\alpha^*$ and the two meager sets $M_{f_\alpha,\mathcal{N}} = \{ g \in \mathcal{N} \cap [f_\alpha] : [g \leq f_\alpha] \text{ is finite} \}$ and $\tilde{M}_{f_\alpha,\mathcal{N}} = \{ g \in \mathcal{N} : [f_\alpha < g] \text{ is finite} \}$.

Then $\tilde{Y}_\alpha^*$ is a union of less than $\text{cov}(\mathcal{M})$ many meager sets. Choose $x_\alpha \in G_\alpha \setminus \tilde{Y}_\alpha^*$. In step $\alpha + 1$ of the construction let $Y_{\alpha+1}^*$ be defined as in [19], and let $\tilde{Y}_{\alpha+1}^*$ be the union of $Y_{\alpha+1}^*$ with the meager sets $M_{f_\alpha,\mathcal{N}}[f_\alpha < x_\alpha] = \{ g \in \mathcal{N} : [g \leq f_\alpha < x_\alpha] \text{ is finite} \}$ and $\tilde{M}_{f_\alpha,\mathcal{N}}[x_\alpha \leq f_\alpha] = \{ g \in \mathcal{N} : [x_\alpha \leq f_\alpha < g] \text{ is finite} \}$.

Now choose $x_{\alpha+1} \in G_{\alpha+1} \setminus \tilde{Y}_{\alpha+1}^*$. Then $x_\alpha$ and $x_{\alpha+1}$ witness that $f_\alpha$ does not avoid middles in the resulting set $L = \{ x_\alpha : \alpha < c \}$. Consequently, $L$ does not satisfy $U_{\text{fin}}(\Gamma, T)$.

The proof that $L$ satisfies $S_1(\mathcal{B}_\Omega, \mathcal{B}_\Omega)$ is as in [19].

6.3. Sierpinski sets. If a Sierpinski set satisfies $T_1(\Gamma)$, then it satisfies $S_1(T, \Gamma)$ but not $S_1(O, O)$. Such a result would give another solution to Problem 5.2. However, as towers are null in the usual measure on $\mathcal{N}\{0,1\}$, it is not straightforward to construct a Sierpinski set which is a member of $T_1(\Gamma)$.

**Problem 6.4.** Does there exist a Sierpinski set satisfying $T_1(\Gamma)$?

6.4. Unsettled implications. The paper [13] ruled out the possibility that any selection property for the open case implies any selection property for the Borel case. Some implications are ruled out by constructions of [7] and [13]. Several other implications are eliminated due to critical cardinality considerations.

**Problem 6.5.** Which implications can be added to the diagram in Figure 2 and to the corresponding Borel diagram?

A summary of all unsettled implications appears in [17]. As a first step towards solving Problem 6.5, one may try to answer the following.

**Problem 6.6.** What are the critical cardinalities of the remaining classes? \footnote{This problem was almost completely solved in: H. Mildenberger, S. Shelah, and B. Tsaban, The combinatorics of $\tau$-covers (see \url{http://arxiv.org/abs/math.GN/0409068}). There remains exactly one unsettled critical cardinality in the diagram.}
Part 2. Variations on the theme of $\tau$-covers

7. $\tau^*$-covers

The notion of a $\tau^*$-cover is a more flexible variant of the notion of a $\tau$-cover.

Definition 7.1. A family $Y \subseteq P_{\infty}(\mathbb{N})$ is linearly refinable if for each $y \in Y$ there exists an infinite subset $\hat{y} \subseteq y$ such that the family $\hat{Y} = \{\hat{y} : y \in Y\}$ is linearly quasiordered by $\subseteq^*$.

A cover $\mathcal{U}$ of $X$ is a $\tau^*$-cover of $X$ if it is large, and $h_{\mathcal{U}}[X]$ (where $h_{\mathcal{U}}$ is the function defined before Lemma 3.2) is linearly refinable.

For $x \in X$, we will write $x_{\mathcal{U}}$ for $h_{\mathcal{U}}(x)$, and $\hat{x}_{\mathcal{U}}$ for the infinite subset of $x_{\mathcal{U}}$ such that the sets $\hat{x}_{\mathcal{U}}$ are linearly quasiordered by $\subseteq^*$.

If $\mathcal{U}$ is a countable $\tau$-cover, then $h_{\mathcal{U}}[X]$ is linearly quasiordered by $\subseteq^*$ and in particular it is linearly refinable. Thus every countable $\tau$-cover is a $\tau^*$-cover. The converse is not necessarily true. Let $T^*$ ($\mathcal{B}_{T^*}$) denote the collection of all countable open (Borel) $\tau^*$-covers of $X$. Then $T \subseteq T^* \subseteq \Omega$ and $\mathcal{B}_T \subseteq \mathcal{B}_{T^*} \subseteq \Omega$.

Often problems which are difficult in the case of usual $\tau$-covers become solvable when shifting to $\tau^*$-covers. We will give several examples.

7.1. Refinements. One of the major tools in the analysis of selection principles is to use refinements and de-refinements of covers. In general, the de-refinement of a $\tau$-cover is not necessarily a $\tau^*$-cover.

Lemma 7.2. Assume that $\mathcal{U} \in T^*$ refines a countable open cover $\mathcal{V}$ (that is, for each $U \in \mathcal{U}$ there exists $V \in \mathcal{V}$ such that $U \subseteq V$). Then $\mathcal{V} \in T^*$.

The analogous assertion for countable Borel covers also holds.

Proof. Fix a bijective enumeration $\mathcal{U} = \{U_n\}_{n \in \mathbb{N}}$. Let $\hat{x}_{\mathcal{U}}$, $x \in X$, be as in the definition of $\tau^*$-covers. For each $n$ let $V_n \in \mathcal{V}$ be such that $U_n \subseteq V_n$. We claim that $\mathcal{W} = \{V_n : n \in \mathbb{N}\} \in T^*$. As $\mathcal{W}$ is an $\omega$-cover of $X$, it is infinite; fix a bijective enumeration $\{W_n\}_{n \in \mathbb{N}}$ of $\mathcal{W}$. For each $n$ define $S_n = \{k : U_k \subseteq W_n\}$, and $\tilde{S}_n = S_n \setminus \bigcup_{m<n} S_m$. For each $x \in X$ define $\tilde{x}_\mathcal{W}$ by:

$$n \in \tilde{x}_\mathcal{W} \iff \tilde{S}_n \cap \hat{x}_{\mathcal{U}} \neq \emptyset.$$ 

Then each $\tilde{x}_\mathcal{W}$ is a subset of $x_{\mathcal{W}}$.

Each $\tilde{x}_\mathcal{W}$ is infinite: For each $W_{n_1}, \ldots, W_{n_k}$ choose $x_i \notin W_{n_i}$, $i = 1, \ldots, k$. Then $\{x, x_1, \ldots, x_k\} \not\subseteq W_{n_i}$ for all $i = 1, \ldots, k$. As $\mathcal{U}$ is an $\tau^*$-cover of $X$, there exists $m \in \hat{x}_{\mathcal{U}}$ such that $\{x, x_1, \ldots, x_k\} \subseteq U_m$. Consider the (unique) $n$ such that $m \in \tilde{S}_n$. Then $U_m \subseteq W_n$; therefore
$W_n \not\in \{W_{n_1}, \ldots, W_{n_k}\}$, and in particular $n \not\in \{n_1, \ldots, n_k\}$. As $m \in \tilde{S}_n \cap \hat{a}_U$, we have that $n \in \hat{a}_W$.

The sets $\hat{a}_W$ are linearly quasiordered by $\subseteq^*$: Assume that $a, b \in \mathcal{X}$. We may assume that $\hat{a}_U \subseteq^* \hat{b}_U$. As \( \lim_n \min \tilde{S}_n \to \infty \), we have that $\hat{a}_U \subseteq \hat{b}_U$ for all but finitely many $n$.

This shows that $\mathcal{W}$ is a $\tau^*$-cover of $X$. Now, $\mathcal{V}$ is an extension of $\mathcal{W}$ by at most countably many elements. It is easy to see that an extension of a $\tau^*$-cover by countably many open sets is again a $\tau^*$-cover, see [18].

The first consequence of this important Lemma is that $\mathcal{S}_{\text{fin}}(\mathcal{U}, \mathcal{T}^*)$ implies $\mathcal{S}_{\text{fin}}(\mathcal{U}, \mathcal{B}_{\mathcal{T}})$), that is, the analogue of Lemma 1.1 holds.

**Corollary 7.3.** Assume that $\mathcal{U} = \bigcup_{n \in \mathbb{N}} \mathcal{F}_n$, where each $\mathcal{F}_n$ is finite, is a $\tau^*$-cover of a space $X$. Then either $\bigcup \mathcal{F}_n = X$ for some $n$, or else $\mathcal{V} = \{ \bigcup \mathcal{F}_n \}_{n \in \mathbb{N}}$ is also a $\tau^*$-cover of $X$.

**Proof.** $\mathcal{U}$ refines $\mathcal{V}$. □

**7.2. Equivalences.** All equivalences mentioned in Subsection 1.4 hold for $\tau^*$-covers as well. In particular, the analogue of Theorem 2.4 holds (with a similar proof).

**Corollary 7.4.** The following equivalences hold:

1. $\mathcal{S}_1(\mathcal{T}^*, \Gamma) = \mathcal{S}_{\text{fin}}(\mathcal{T}^*, \Gamma)$;
2. $\mathcal{S}_1(\mathcal{B}_{\mathcal{T}}^*, \mathcal{B}_\Gamma) = \mathcal{S}_{\text{fin}}(\mathcal{B}_{\mathcal{T}}^*, \mathcal{B}_\Gamma)$.

In fact, in the Borel case we get more equivalences in the case of $\tau^*$-covers than in the case of $\tau$-covers – see Subsection 7.4.

**7.3. Continuous images.** We now solve the problems mentioned in Remarks 3.9 and 3.12 in the case of $\tau^*$-covers.

**Theorem 7.5.** The following properties are equivalent for a set $X$ of reals:

1. $X$ satisfies $\bigl(\mathcal{O}_{\mathcal{T}}^*\bigr)$;
2. For each continuous image $Y$ of $X$ in $P_\infty(\mathbb{N})$, if $Y$ is centered, then $Y$ is linearly refinable.

**Proof.** $1 \Rightarrow 2$: The proof for this is similar to the proof of $1 \Rightarrow 2$ in Theorem 3.8.

$2 \Rightarrow 1$: Assume that $\mathcal{U}$ is an $\omega$-cover of $X$. Replacing each member of $\mathcal{U}$ with all finite unions of Basic clopen subsets of it, we may assume that all members of $\mathcal{U}$ are clopen (to unravel this assumption we will use the fact that $\mathcal{T}^*$ is closed under de-refinements).
Thus, $h_\mathcal{U}$ is continuous and $Y = h_\mathcal{U}[X]$ is centered. Consequently, $Y$ is linearly refinable, that is, $\mathcal{U}$ is a $\tau^*$-cover of $X$. \qed

Remark 7.6. The analogue assertion (to Theorem 7.5) for the Borel case, where open covers are replaced by Borel covers and continuous image is replaced by Borel image, also holds and can be proved similarly.

As in [14], we will use the notation
\[ [f \leq h] := \{n : f(n) \leq g(n)\}. \]
Then a subset $Y \subseteq \mathbb{N}^\mathbb{N}$ satisfies the excluded middle property if, and only if, there exists a function $h \in \mathbb{N}^\mathbb{N}$ such that the collection
\[ \{ [f \leq h] : f \in Y \} \]
is a subset of $P_\infty(\mathbb{N})$ and is linearly quasiordered by $\subseteq^*$.

Definition 7.7. We will say that a subset $Y \subseteq \mathbb{N}^\mathbb{N}$ satisfies the weak excluded middle property if there exists a function $h \in \mathbb{N}^\mathbb{N}$ such that
\[ \{ [f \leq h] : f \in Y \} \]
is linearly refinable.

Recall that $U_{\text{fin}}(\Gamma, T^*) = U_{\text{fin}}(\mathcal{O}, T^*)$.

Theorem 7.8. For a zero-dimensional set $X$ of real numbers, the following are equivalent:

1. $X$ satisfies $U_{\text{fin}}(\mathcal{O}, T^*)$;
2. Every continuous image of $X$ in $\mathbb{N}^\mathbb{N}$ satisfies the weak excluded middle property.

Proof. We make the needed changes in the corresponding proof from [19].

2 $\Rightarrow$ 1: Assume that $\mathcal{U}_n, n \in \mathbb{N}$, are open covers of $X$ which do not contain finite subcovers. For each $n$, replacing each member of $\mathcal{U}_n$ with all of its basic clopen subsets we may assume that all elements of $\mathcal{U}_n$ are clopen, and thus we may assume further that they are disjoint. For each $n$ enumerate $\mathcal{U}_n = \{U^n_m\}_{m \in \mathbb{N}}$. As we assume that the elements $U^n_m, m \in \mathbb{N}$, are disjoint, we can define a function $\Psi$ from $X$ to $\mathbb{N}^\mathbb{N}$ by
\[ \Psi(x)(n) = m \iff x \in U^n_m. \]
Then $\Psi$ is continuous. Therefore, $Y = \Psi[X]$ satisfies the weak excluded middle property. Let $h \in \mathbb{N}^\mathbb{N}$, and for each $f \in Y$, $A_f \subseteq [f \leq h]$ be such that $\{A_f : f \in Y\}$ is linearly quasiordered by $\subseteq^*$.

For each $n$ set
\[ \mathcal{F}_n = \{U^n_k : k \leq h(n)\}. \]
We claim that $\mathcal{U} = \{\cup\mathcal{F}_n\}_{n \in \mathbb{N}}$ is a $\tau^*$-cover of $X$. We will use the following property.
Let \( \{ \bigcup F_n \}_{n \in \mathbb{N}} \) be a bijective enumeration of \( \mathcal{U} \), and let \( f \in \mathbb{N}^\mathbb{N} \) be such that for each \( n \), \( \bigcup F_n = \bigcup F_{k_f(n)} \). For each \( x \in X \) set \( \hat{x}_U = f[A_{\Psi(x)}] \). We have the following.

\( \hat{x}_U \) is a subset of \( x_U \): Assume that \( f(n) \in \hat{x}_U \), where \( n \in A_{\Psi(x)} \). Then \( x \in \bigcup F_n = \bigcup F_{k_f(n)} \); therefore \( f(n) \in x_U \).

\( \hat{x}_U \) is infinite: Assume that \( f[A_{\Psi(x)}] = \{ f(n_1), \ldots, f(n_k) \} \) where \( n_1, \ldots, n_k \in A_{\Psi(x)} \). For each \( i \leq k \) choose \( x_i \notin \bigcup F_{k_f(n_i)} \), and set \( F = \{ x, x_1, \ldots, x_k \} \). Then for all \( i \leq k \) \( F \not\subseteq \bigcup F_{k_f(n_i)} \). Choose \( n \in \bigcap_{a \in F} A_{\Psi(x)} \). By property (*) \( F \subseteq \bigcup F_n = \bigcup F_{k_f(n)} \); therefore \( f(n) \notin \{ f(n_1), \ldots, f(n_k) \} \). But \( n \in A_{\Psi(x)} \), thus \( f(n) \in \hat{x}_U \), a contradiction.

As the sets \( A_{\Psi(x)} \) are linearly quasiordered by \( \subseteq^* \), so are the sets \( \hat{x}_U = f[A_{\Psi(x)}] \).

1 \( \Rightarrow \) 2: Since \( \Psi \) is continuous, \( Y = \Psi[X] \) also satisfies \( U_{f\in \mathcal{O}}(\mathcal{O}, T^*) \).

Consider the basic open covers \( \mathcal{U}_n = \{ U^m_n \}_{m \in \mathbb{N}} \) defined by \( U^m_n = \{ f \in Y : f(n) = m \} \). Then there exist finite \( \mathcal{F}_n \subseteq \mathcal{U}_n \), \( n \in \mathbb{N} \), such that either \( Y = \bigcup \mathcal{F}_n \) for some \( n \), or else \( \mathcal{V} = \{ \bigcup \mathcal{F}_n : n \in \mathbb{N} \} \) is a \( \tau^* \)-cover of \( Y \).

The first case can be split into two sub-cases: If there exists an infinite set \( A \subseteq \mathbb{N} \) such that \( Y = \bigcup \mathcal{F}_n \), then for each \( n \in A \) the set \( \{ f(n) : f \in Y \} \) is finite, and we can define

\[
h(n) = \begin{cases} 
\max \{ f(n) : f \in Y \} & n \in A \\
0 & \text{otherwise}
\end{cases}
\]

so that \( A \subseteq [f \leq h] \) for each \( f \in Y \), and we are done. Otherwise \( Y = \bigcup \mathcal{F}_n \) for only finitely many \( n \), therefore we may replace each \( \mathcal{F}_n \) satisfying \( Y = \bigcup \mathcal{F}_n \) with \( \mathcal{F}_n = \emptyset \), so we are in the second case.

The second case is the interesting one. \( \mathcal{V} = \{ \bigcup \mathcal{F}_n : n \in \mathbb{N} \} \) is a \( \tau^* \)-cover of \( Y \) – fix a bijective enumeration \( \{ \bigcup \mathcal{F}_{k_n} \}_{n \in \mathbb{N}} \) of \( \mathcal{V} \) and witnesses \( \hat{f}_\mathcal{V}, f \in Y \), for that. Define \( h(n) = \max \{ m : U^m_n \in \mathcal{F}_n \} \) for each \( n \). Then the subsets \( \{ k_n : n \in \hat{f}_\mathcal{V} \} \) of \( [f \leq h] \), \( f \in Y \), are infinite and linearly quasiordered by \( \subseteq^* \). This shows that \( Y \) is linearly refinable.

\[ \square \]

### 7.4. Borel images

Define the following notion.

\( T^* \): The set of \( X \subseteq \mathbb{R} \) such that for each linearly refinable Borel image \( Y \) of \( X \) in \( P_\infty(\mathbb{N}) \), \( Y \) has a pseudo-intersection.

By the usual method we get the following.

**Lemma 7.9.** \( T^* = (B_{T^*}^r) \).
Clearly $T^*$ implies $T$.

**Lemma 7.10.** $\text{non}(T^*) = t$.

**Proof.** It is easy to see that $\text{non}(T^*)$ is the minimal size of a linearly refinable family $Y \subseteq P_\infty(\mathbb{N})$ which has no pseudo-intersection. We will show that $t \leq \text{non}(T^*)$. Assume that $Y \subseteq P_\infty(\mathbb{N})$ is a linearly refinable family of size less than $t$, and let $\hat{Y}$ be a linear refinement of $Y$. As $|\hat{Y}| \leq |Y| < t$, $\hat{Y}$ has a pseudo-intersection, which is in particular a pseudo-intersection of $Y$. \qed

An application of Lemma 7.9 and the Cancellation Laws implies the following.

**Theorem 7.11.** $S_1(B_T^*, B_T) = T^* \cap B$.

We do not know whether $T^* = T$. In particular, we have the following (recall Theorem 3.6).

**Problem 7.12.** Is it true that every analytic set of reals satisfies $T^*$? Does $\mathbb{N}\mathbb{N} \in T^*$?

We do not know whether $S_1(B_T, B_T) = \cup_{\text{fin}}(B_T, B_T^*)$ or not.$^3$ This can be contrasted with the following result.

**Theorem 7.13.** For a set $X$ of real numbers, the following are equivalent:

1. $X$ satisfies $S_1(B_T, B_T^*)$,
2. $X$ satisfies $S_{\text{fin}}(B_T, B_T^*)$,
3. $X$ satisfies $\cup_{\text{fin}}(B_T, B_T^*)$;
4. Every Borel image of $X$ in $\mathbb{N}\mathbb{N}$ satisfies the weak excluded middle property.

**Proof.** Clearly 1 $\Rightarrow$ 2 $\Rightarrow$ 3.

3 $\Rightarrow$ 4: This can be proved like 1 $\Rightarrow$ 2 in Theorem 7.8.

4 $\Rightarrow$ 1: Assume that $U_n = \{U_{m}^n : k \in \mathbb{N}\}$, $n \in \mathbb{N}$, are Borel $\gamma$-covers of $X$. We may assume that these covers are pairwise disjoint. Define a function $\Psi : X \rightarrow \mathbb{N}\mathbb{N}$ so that for each $x$ and $n$:

$$\Psi(x)(n) = \min\{k : (\forall m \geq k) x \in U_m^n\}.$$ 

Then $\Psi$ is a Borel map, and so $Y = \Psi[X]$ satisfies the weak excluded middle property. Let $h \in \mathbb{N}\mathbb{N}$ and $A_f \subseteq [f \leq h]$, $f \in Y$, be witnesses for that. Set $U = \{U_h^n\}_{n \in \mathbb{N}}$. For each $x \in X$ set $\bar{x}U = A_{\Psi(x)}$. Then

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$^3$A negative answer follows from the results in: H. Mildenberger, S. Shelah, and B. Tsaban, *The combinatorics of $\tau$-covers* (see http://arxiv.org/abs/math.GN/0409068).
\[ \hat{x}_U \text{ is infinite and } \hat{x}_U \subseteq [\Psi(x) \leq h] \subseteq x_U \text{ for each } x \in X, \text{ and the sets } \hat{x}_U \text{ are linearly quasiordered by } \subseteq^*. \]

7.5. **Critical cardinalities.** The argument of Theorem 4.1 implies that \( \text{non}(S_{\text{fin}}(B_{T^*}, B_{\Omega})) = \text{non}(S_{\text{fin}}(T^*, \Omega)) = \emptyset. \)

**Theorem 7.14.** \( \text{non}(S_1(B_{T^*}, B_{T})) = \text{non}(S_1(T^*, \Gamma)) = t. \)

**Proof.** By Theorem 7.11, \( \text{non}(S_1(B_{T^*}, B_{T})) = \min\{ \text{non}(T^*), \text{non}(B) \} = \min\{t, b\} = t. \) On the other hand, \( S_1(T^*, \Gamma) \) implies \( S_1(T, \Gamma) \), whose critical cardinality is \( t. \) \( \square \)

Define the following properties.

- \( X \): The set of \( X \subseteq \mathbb{R} \) such that each Borel image of \( X \) in \( {}^N \mathbb{N} \) satisfies the excluded middle property.
- \( wX \): The set of \( X \subseteq \mathbb{R} \) such that each Borel image of \( X \) in \( {}^N \mathbb{N} \) satisfies the weak excluded middle property.

Recall that by Theorem 3.11, \( U_{\text{fin}}(B_{T^*}, B_{\Omega}) = X. \) In Theorem 7.13 we proved that \( S_1(B_{T^*}, B_{T^*}) = wX. \) We do not know whether \( wX = X. \)

**Problem 7.15.** Does \( \text{non}(wX) = \aleph? \)

7.6. **Finite powers.** In [7] it is observed that if \( U \) is an \( \omega \)-cover of \( X \), then for each \( k U^k = \{ U^k : U \in U \} \) is an \( \omega \)-cover of \( X^k \). Similarly, it is observed in [15] that if \( U \) is a \( \tau \)-cover of \( X \), then for each \( k U^k \) is a \( \tau \)-cover of \( X^k \). We will need the same assertion for \( \tau^* \)-covers.

**Lemma 7.16.** Assume that \( U \) is a \( \tau^* \)-cover of \( X \). Then for each \( k, U^k \) is a \( \tau^* \)-cover of \( X^k \).

**Proof.** Fix \( k. \) Let \( U = \{ U_n \}_{n \in \mathbb{N}} \) be an enumeration of \( U \), and let \( \hat{x}_U \subseteq x_U, x \in X, \) witness that \( U \) is a \( \tau^* \)-cover of \( X \). For each \( \vec{x} = (x_0, \ldots, x_{k-1}) \in X^k \) define

\[ \hat{\vec{x}}_{U^k} = \bigcap_{i < k} (\hat{x}_i)_{U^k}. \]

As the sets \( \hat{x}_U \) are infinite and linearly quasiordered by \( \subseteq^* \), the sets \( \hat{\vec{x}}_{U^k} \) are also infinite and linearly quasiordered by \( \subseteq^* \). Moreover, for each \( n \in \hat{\vec{x}}_{U^k} \) and each \( i < k, n \in (\hat{x}_i)_{U^k}, \) and therefore \( x_i \in U_n \) for each \( i < k; \) thus \( \vec{x} \in U^k_n, \) as required. \( \square \)

In [7] it is proved that the classes \( S_1(\Omega, \Gamma), S_1(\Omega, \Omega), \) and \( S_{\text{fin}}(\Omega, \Omega) \) are closed under taking finite powers, and that none of the remaining classes they considered has this property. Actually, their argument for the last assertion shows that assuming CH, there exist a Lusin set \( L \)
and a Sierpinski set $S$ such that $L \times L$ and $S \times S$ can be mapped continuously onto the Baire space $\mathbb{NN}$. Consequently, we have that none of the classes $S_1(\Gamma, T)$, $S_{fin}(\Gamma, T)$, $U_{fin}(\Gamma, T)$, $S_1(T, O)$, and their corresponding Borel versions, is closed under taking finite powers. We do not know whether the remaining 7 classes which involve $\tau$-covers are closed under taking finite powers.

**Theorem 7.17.** $S_1(\Omega, T^*)$ and $S_{fin}(\Omega, T^*)$ are closed under taking finite powers.

**Proof.** We will prove the assertion for $S_1(\Omega, T^*)$; the proof for the remaining assertion is similar. Fix $k$. In [7] it is proved that for each open $\omega$-cover $U$ of $X^k$ there exists an open $\omega$-cover $V$ of $X$ such that the $\omega$-cover $V^k$ of $X^k$ refines $U$.

Assume that $\{U_n\}_{n \in \mathbb{N}}$ is a sequence of open $\omega$-covers of $X^k$. For each $n$ choose an open $\omega$-cover $V_n$ of $X$ such that $V^k_n$ refines $U_n$. Apply $S_1(\Omega, T^*)$ to extract elements $V_n \in V_n$, $n \in \mathbb{N}$, such that $V = \{V_n\}_{n \in \mathbb{N}} \in T^*$. By Lemma 7.16, $V^k$ is a $\tau^*$-cover of $X^k$. For each $n$ choose $U_n \in U_n$ such that $V^k_n \subseteq U_n$. Then by Lemma 7.2, $\{U_n\}_{n \in \mathbb{N}}$ is a $\tau^*$-cover of $X$. \qed

### 7.7. Strong properties

Assume that $\{U_n\}_{n \in \mathbb{N}}$ is a sequence of collections of covers of a space $X$, and that $\mathcal{W}$ is a collection of covers of $X$. The following selection principle is defined in [18].

**$S_1(\{U_n\}_{n \in \mathbb{N}}, \mathcal{W})$**: For each sequence $\{U_n\}_{n \in \mathbb{N}}$ where for each $n U_n \in \mathcal{U}_n$, there is a sequence $\{U_n\}_{n \in \mathbb{N}}$ such that for each $n U_n \in \mathcal{U}_n$, and $\{U_n\}_{n \in \mathbb{N}} \in \mathcal{W}$.

The notion of **strong $\gamma$-set**, which is due to Galvin and Miller [5], is a particular instance of the new selection principle, where $\mathcal{W} = \Gamma$ and for each $n \mathcal{U}_n = \mathcal{O}_n$, the collection of open $n$-covers of $X$ (we use here the simple characterization given in [18]). It is well known that the $\gamma$-property $S_1(\Omega, \Gamma)$ does not imply the strong $\gamma$-property $S_1(\{\mathcal{O}_n\}_{n \in \mathbb{N}}, \Gamma)$. It is an open problem whether $S_1(\Omega, T)$ implies $S_1(\{\mathcal{O}_n\}_{n \in \mathbb{N}}, T)$.

The following notions are defined in [18]. A collection $\mathcal{U}$ of open covers of a space $X$ is **finitely thick** if:

1. If $\mathcal{U} \in \mathcal{U}$ and for each $U \in \mathcal{U}$ $\mathcal{F}_U$ is a finite family of open sets such that for each $V \in \mathcal{F}_U$, $U \subseteq V \neq X$, then $\bigcup_{U \in \mathcal{U}} \mathcal{F}_U \in \mathcal{U}$.
2. If $\mathcal{U} \in \mathcal{U}$ and $\mathcal{V} = \mathcal{U} \cup \mathcal{F}$ where $\mathcal{F}$ is finite and $X \not\in \mathcal{F}$, then $\mathcal{V} \in \mathcal{U}$.

A collection $\mathcal{U}$ of open covers of a space $X$ is **countably thick** if for each $\mathcal{U} \in \mathcal{U}$ and each countable family $\mathcal{V}$ of open subsets of $X$ such that $X \notin \mathcal{V}$, $\mathcal{U} \cup \mathcal{V} \in \mathcal{U}$. 
Whereas $T$ is in general not finitely thick nor countably thick, $T^*$ is both finitely and countably thick [18]. In [18] it is proved that if $\mathcal{V}$ is countably thick, then $S_1(\Omega, \mathcal{V}) = S_1(\{O_n\}_{n \in \mathbb{N}}, \mathcal{V})$. Consequently, $S_1(\Omega, T^*) = S_1(\{O_n\}_{n \in \mathbb{N}}, T^*)$.

### 7.8. Closing on the Minimal Tower problem

Clearly $S_1(T^*, \Gamma)$ implies $S_1(T, \Gamma)$, and $S_1(B_{T^*}, B_{\Gamma})$ implies $S_1(B_T, B_{\Gamma})$. So we now have new topological lower bounds on the Minimal Tower problem.

**Problem 7.18.**

1. Does $S_1(\Omega, \Gamma) = S_1(T^*, \Gamma)$?
2. Does $S_1(B_{\Omega}, B_{T^*}) = S_1(B_{T^*}, B_{\Gamma})$?

We also have a new combinatorial bound.

**Definition 7.19.** $p^*$ is the minimal size of a centered family in $P_\infty(\mathbb{N})$ which is not linearly refineable.

**Theorem 7.20.** The critical cardinalities of the properties $(\Omega_{T^*}, S_1(B_{\Omega}, B_{T^*}))$, $S_{fin}(B_{\Omega}, B_{T^*})$, $(\Omega_T)$, $S_1(\Omega, T^*)$, and $S_{fin}(\Omega, T^*)$ are all equal to is $p^*$.

**Proof.** By Theorem 7.5 and Remark 7.5, we have that the assertion holds for $(\Omega_T)$ and $(\cup_{T^*})$.

We will use the following result which is analogous to Theorem 2.7 and can be proved similarly.

**Theorem 7.21.** The following inclusions hold:

1. $(\Omega_T) \subseteq S_{fin}(\Gamma, T^*)$.
2. $(\cup_{T^*}) \subseteq S_{fin}(B_{\Gamma}, B_{T^*})$.

Consequently, $p^* \leq d$. As in Theorem 4.7, we get the remaining assertions follow from this and Corollary 2.3, as $T^*$ is closed under taking countable supersets.

The following can be proved either directly from the definitions or from the equivalence $(\Omega_T) = (\Omega_T) \cap (\Omega_{T^*})$.

**Corollary 7.22.** $p = \min\{p^*, t\}$. Thus, if $p < t$ is consistent, then $p^* < t$ is consistent as well.

We therefore have the following problem.

**Problem 7.23.** Does $p = p^*$?

### 7.9. Remaining Borel classes

We are left with Figure 3 for the Borel case.
8. Sequences of compatible $\tau$-covers

When considering sequences of $\tau$-covers, it may be convenient to have that the linear quasiorderings they define on $X$ agree, in the sense that there exists a Borel linear quasiordering $\preceq$ on $X$ which is contained in all of the induced quasiorderings. In this case, we say that the $\tau$-covers are compatible. We thus have the following new selection principle:

$S_1^\preceq(T, \mathcal{W})$: For each sequence $\{U_n\}_{n \in \mathbb{N}}$ of countable open compatible $\tau$-covers of $X$ there is a sequence $\{U_n\}_{n \in \mathbb{N}}$ such that for each $n U_n \in U_n$, and $\{U_n\}_{n \in \mathbb{N}} \in \mathcal{W}$.

The selection principle $S_{fin}^\preceq(T, \mathcal{W})$ is defined similarly. Replacing “open” by “Borel” gives the selection principles $S_1^\preceq(B_T, \mathcal{W})$ and $S_{fin}^\preceq(B_T, \mathcal{W})$.

The following implications hold:

$$S_1^\preceq(T, \mathcal{W}) \rightarrow S_{fin}^\preceq(T, \mathcal{W}) \rightarrow (T^{\mathcal{W}})$$

and similarly for the Borel case.

For $\mathcal{W} = \Gamma$ the new notions coincide with the old ones.

**Proposition 8.1.** The following equivalences hold:

1. $S_1(T, \Gamma) = S_1^\preceq(T, \Gamma) = S_{fin}(T, \Gamma) = S_{fin}^\preceq(T, \Gamma)$;
2. $S_1(B_T, B_T) = S_1^\preceq(B_T, B_T) = S_{fin}(B_T, B_T) = S_{fin}^\preceq(B_T, B_T)$.

**Proof.** We will prove (1); (2) is similar. By Theorem 2.4, we have the following implications

$$S_1^\preceq(T, \Gamma) \rightarrow S_{fin}^\preceq(T, \Gamma) \rightarrow (T^{\Gamma}) \cap S_{fin}(\Gamma, \Gamma) = S_1(T, \Gamma)$$

and similarly for the Borel case.
8.1. **The class** $S_{\text{fin}}^{\asymp}(B_T, B_T)$.

**Definition 8.2.** A $\tau$-cover of $\langle X, \preceq \rangle$ is a $\tau$-cover of $X$ such that the induced quasiordering contains $\preceq$.

**Lemma 8.3.** Let $\langle X, \preceq \rangle$ be a linearly quasiordered set of reals, and assume that every Borel image of $\preceq$ in $\mathbb{N}^\mathbb{N}$ is bounded (with respect to $\leq^*$). Assume that $U_n = \{U^n_k : k \in \mathbb{N}\}$ are Borel $\tau$-covers of $\langle X, \preceq \rangle$. Then there exist finite subsets $F_n$ of $U_n$, $n \in \mathbb{N}$, such that $\bigcup_{n \in \mathbb{N}} F_n$ is a $\tau$-cover of $\langle X, \preceq \rangle$.

**Proof.** Fix a linear quasiordering $\preceq$ of $X$, and assume that $U_n = \{U^n_k : k \in \mathbb{N}\}$ are Borel $\tau$-covers of $\langle X, \preceq \rangle$. Define a Borel function $\Psi$ from $\preceq$ to $\mathbb{N}^\mathbb{N}$ by:

$$\Psi(x, y)(n) = \min\{k : (\forall m \geq k) \ x \in U^m_n \rightarrow y \in U^m_n\}.$$ 

$\Psi[\preceq]$ is bounded, say by $g$. Now define a Borel function $\Phi$ from $X$ to $\mathbb{N}^\mathbb{N}$ by:

$$\Phi(x)(n) = \min\{k : g(n) \leq k \text{ and } x \in U^n_k\}.$$ 

Note that $\Phi[X]$ is a Borel image of $\preceq$ in $\mathbb{N}^\mathbb{N}$, thus it is bounded, say by $f$. It follows that the sequence $\{U^n_{g(n)}, \ldots, U^n_{f(n)} : g(n) \leq f(n)\}_{n \in \mathbb{N}}$ is large, and is a $\tau$-cover of $X$. 

According to [13], the property that every Borel image is bounded is equivalent to $S_1(B_T, B_T)$.

**Lemma 8.4.** Let $P$ be a collection of spaces which is closed under taking Borel subsets, continuous images (or isometries), and finite unions. Then for each set $X$ of real numbers, the following are equivalent:

1. Each Borel linear quasiordering $\preceq$ of $X$ satisfies $P$,
2. $X^2$ satisfies $P$,
3. There exists a Borel linear quasiordering $\preceq$ of $X$ satisfying $P$,

**Proof.** $1 \Rightarrow 2 \Rightarrow 3$: The set $\preceq = X^2$ is a linear quasiordering of $X$.

$3 \Rightarrow 2$: If $\preceq$ satisfies $P$, then so does its continuous image $\succeq = \{(y, x) : x \preceq y\}$. Thus, $X^2 = \succeq \cup \preceq$ satisfies $P$.

$2 \Rightarrow 1$: $P$ is closed under taking Borel subsets. 

Thus, lemma 8.3 can be restated as follows.

**Theorem 8.5.** If $X^2$ satisfies $S_1(B_T, B_T)$, then $X$ satisfies $S_{\text{fin}}^{\preceq}(B_T, B_T)$.

**Proof.** The property $S_1(B_T, B_T)$ satisfies the assumptions of Lemma 8.4. 

□
Problem 8.6. Assume that $X$ satisfies $S_{fin}^\leq(B_T, B_T)$. Is it true that $X^2$ satisfies $S_1(B_T, B_T)$?

Since $\mathfrak{b} = \text{cov}(\mathcal{N}) = \text{cof}(\mathcal{N})$ (in particular, the Continuum Hypothesis) implies that $S_1(B_T, B_T)$ is not closed under taking squares [13], a positive answer to Problem 8.6 would imply that the property $S_1(B_T, B_T)$ does not imply $S_{fin}^\leq(B_T, B_T)$.

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