STRONG CONTAINMENT OF SATURATED FORMATIONS OF
SOLUBLE LIE ALGEBRAS

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Abstract. It is shown that, if \( \mathcal{H}, \mathcal{K} \) are saturated formations of soluble Lie algebras over a field of non-zero characteristic, and \( \mathcal{H} \gg \mathcal{K} \) is a non-trivial example of strong containment, then \( \mathcal{H} = \mathcal{H}/\mathcal{N} \) and \( \mathcal{H} \) is not locally defined.

1. Introduction

The concept of strong containment for Schunck classes of finite soluble groups was introduced by Cline [8] in 1969. It is discussed extensively in Doerk and Hawkes [9].

Definition 1.1. Let \( \mathcal{H}, \mathcal{K} \) be Schunck classes of finite soluble groups. We say that \( \mathcal{H} \) strongly contains \( \mathcal{K} \), written \( \mathcal{H} \gg \mathcal{K} \) if, for every finite soluble group \( G \), every \( \mathcal{K} \)-projector of an \( \mathcal{H} \)-projector of \( G \) is a \( \mathcal{K} \)-projector of \( G \).

Much attention is given to the special case where \( \mathcal{H} \) and \( \mathcal{K} \) are formations. It is easy to give examples of strong containment of saturated formations of finite soluble groups. If \( \pi_1 \subset \pi_2 \) are sets of primes, then the class of soluble \( \pi_2 \)-groups is strongly contained in the class of soluble \( \pi_2 \)-groups since a Hall \( \pi_1 \)-subgroup of a Hall \( \pi_2 \)-subgroup of \( G \) is clearly a Hall \( \pi_1 \)-subgroup of \( G \).

I. S. Gutiérrez García has asked in a private communication to the author, if there exist non-trivial examples of strong containment of saturated formations of soluble Lie algebras.

In the following, all Lie algebras are soluble and finite-dimensional over the field \( F \) and \( \mathcal{H}, \mathcal{K} \) are Schunck classes of soluble Lie algebras over \( F \).

Definition 1.2. We say that \( \mathcal{H} \) is strongly contains \( \mathcal{K} \), written \( \mathcal{H} \gg \mathcal{K} \), if, for every soluble Lie algebra \( L \) and \( \mathcal{H} \)-projector \( H \) of \( L \), every \( \mathcal{K} \)-projector of \( H \) is a \( \mathcal{K} \)-projector of \( L \).

There are clearly three trivial cases, \( \mathcal{K} = 0 \), \( \mathcal{K} = \mathcal{H} \) and \( \mathcal{H} = \mathcal{S} \), the class of all soluble Lie algebras.

The cases of \( \text{char}(F) = 0 \) and \( \text{char}(F) \neq 0 \) are dramatically different.

Theorem 1.3. Suppose \( \text{char}(F) = 0 \). Let \( \mathcal{H} \supseteq \mathcal{K} \) be Schunck classes. Then \( \mathcal{H} \gg \mathcal{K} \).

Proof. Every soluble Lie algebra over \( F \) is completely soluble, so \( \mathcal{H} \gg \mathcal{K} \) by Barnes and Newell [7, Theorem 3.7].

For \( \text{char}(F) \neq 0 \), it is easy to produce examples of strong containment of Schunck classes, but the existence of non-trivial examples where \( \mathcal{H} \) and \( \mathcal{K} \) are formations remains unanswered.

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To avoid continual reference to trivial cases, it is always assumed in the following that $\mathfrak{g} \neq \mathfrak{h} \neq \mathfrak{r} \neq 0$. The class of all nilpotent algebras is denoted by $\mathfrak{r}$ and 0 is used to denote the zero element, the zero algebra and the class containing only the zero algebra according to context. As the case of characteristic 0 has been settled, in the following, it is assumed that $\text{char}(F) = p \neq 0$. The socle of the Lie algebra $L$ is denoted by $\text{Soc}(L)$ and the nil radical of $L$ is denoted by $\text{N}(L)$. If $V$ is an $L$-module, $C_L(V)$ denotes the centraliser of $V$ in $L$, that is, the kernel of the representation of $L$ on $V$. If $\mathfrak{g}$ is a formation, the $\mathfrak{g}$-residual of the algebra $L$ is denoted by $L_\mathfrak{g}$. This is the smallest ideal $K$ of $L$ with $L/K \in \mathfrak{g}$.

That a result is the Lie algebra analogue of a result in Doerk and Hawkes [9] is indicated by (DH, Lemma x, p. y). Proofs which are exact translations are omitted.

2. Strong containment

In this section, we investigate basic properties of strong containment of Schunck classes.

**Lemma 2.1.** Suppose $\mathfrak{h} \gg \mathfrak{r}$. Then every $\mathfrak{r}$-projector of a soluble Lie algebra $L$ is contained in some $\mathfrak{h}$-projector of $L$.

**Proof.** Let $L$ be a soluble Lie algebra of least possible dimension with a $\mathfrak{r}$-projector $K$ not contained in any $\mathfrak{h}$-projector of $L$. Let $A$ be a minimal ideal of $L$. Then $K + A/A$ is contained in some $\mathfrak{h}$-projector $H^*/A$ of $L/A$. If $H^* < L$, then there exists an $\mathfrak{h}$-projector $H$ of $H^*$ which contains $K$. But $H$ is an $\mathfrak{h}$-projector of $L$ by [6, Lemma 1.8]. Therefore $H^* = L$, and $A$ is complemented in $L$ by an $\mathfrak{h}$-projector $H$. If $B$ is a minimal ideal of $L$ contained in $H$, then $L/B \in \mathfrak{h}$ contrary to $H$ being an $\mathfrak{h}$-projector. Therefore $L$ is primitive and $H$ is faithfully represented on $A$. Let $K_1 = H \cap (K + A)$. Since $H \simeq L/A$, $K_1$ is a $\mathfrak{r}$-projector of $H$ and so also of $L$ since $\mathfrak{r} \ll \mathfrak{h}$. Thus both $K$ and $K_1$ are $\mathfrak{r}$-projectors of $K + A = K_1 + A$. By [6, Lemma 1.11], there exists $a \in A$ such that $\alpha_a(K_1) = K$ where $\alpha_a : L \to L$ is the automorphism $1 + \text{ad}_a$. Then $\alpha_a(H)$ is an $\mathfrak{h}$-projector of $L$ which contains $K$. $\Box$

**Lemma 2.2.** Let $L$ be an algebra of least possible dimension with an $\mathfrak{h}$-projector $H$ and a $\mathfrak{r}$-projector $K$ of $H$ which is not a $\mathfrak{r}$-projector of $L$. Then $L$ is primitive with $H$ complementing $\text{Soc}(L)$.

**Proof.** Clearly $H \neq L$. Let $A$ be a minimal ideal of $L$. Suppose $H + A < L$. Then $H$ is an $\mathfrak{h}$-projector of $H + A$, so $K$ is a $\mathfrak{r}$-projector of $H + A$. Also, $H + A/A$ is an $\mathfrak{h}$-projector of $L/A$ and $K + A/A$ is a $\mathfrak{r}$-projector of $H + A/A$, so $K + A/A$ is a $\mathfrak{r}$-projector of $L/A$. As $K$ is a $\mathfrak{r}$-projector of $K + A/A$, it is a $\mathfrak{r}$-projector of $L$. Therefore $H + A = L$. As this holds for every minimal ideal, there is only one minimal ideal and $L$ is primitive. $\Box$

**Definition 2.3.** The boundary of the Schunck class $\mathfrak{X}$ is the class $b(\mathfrak{X})$ of those Lie algebras not in $\mathfrak{X}$ whose proper quotients are in $\mathfrak{X}$. A class $\mathfrak{Y}$ of primitive algebras is called a boundary class if the intersection of $\mathfrak{Y}$ with the class of proper quotients of algebras in $\mathfrak{Y}$ is empty.

Clearly, $b(\mathfrak{X})$ is a boundary class. That every boundary class is the boundary of a Schunck class follows as in (DH, 2.3, p. 284).
**Definition 2.4** (DH 4.15, p. 308). The avoidance class of $\mathcal{F}$ is the class $a(\mathcal{F})$ of primitive algebras $P$ with $H \cap \text{Soc}(P) = 0$ for all $\mathcal{F}$-projectors $H$ of $P$.

Clearly, $b(\mathcal{F}) \subseteq a(\mathcal{F})$.

**Lemma 2.5.** Let $P \in a(\mathcal{R})$ and let $M$ complement $A = \text{Soc}(P)$. Then $M$ contains an $\mathcal{R}$-projector of $P$.

**Proof.** Let $U$ be a $\mathcal{R}$-projector of $P$. Then $U \cap A = 0$. Let $B/C$ be a composition factor of $A$ as $U$-module. Then $U + C/C$ is an $\mathcal{R}$-projector of $U + B/C$ and $H^1(U, B/C) = 0$. Thus $H^1(U, A) = 0$ and so, if $V$ complements $A$ in $U + A$, then there exists $a \in A$ with $\alpha_a(U) = V$ where $\alpha_a : P \to P$ is the automorphism $1 + \text{ad}_a$. In particular, for $V = M \cap (U + A)$, we have that $V = \alpha_a(U)$ is a $\mathcal{R}$-projector of $P$. \hfill \Box

**Theorem 2.6** (DH 1.5, p. 429). Let $\mathcal{F} \supset \mathcal{R}$ be Schunck classes. Then $\mathcal{F} \gg \mathcal{R}$ if and only if $b(\mathcal{F}) \subseteq a(\mathcal{R})$.

**Proof.** Suppose $\mathcal{R} \nsubseteq \mathcal{F}$. Then by Lemma 2.2, there exists a primitive algebra $L \in b(\mathcal{F}) \setminus a(\mathcal{R})$ and $b(\mathcal{F}) \nsubseteq a(\mathcal{R})$. Suppose $\mathcal{R} \ll \mathcal{F}$. Let $P \in b(\mathcal{F})$, let $K$ be a $\mathcal{F}$-projector of $P$. Then $K \leq H$ for some $\mathcal{R}$-projector $H$ of $P$. Since $H \cap \text{Soc}(P) = 0$, we have $K \cap \text{Soc}(P) = 0$ and $P \in a(\mathcal{R})$. \hfill \Box

There exist non-trivial examples of Schunck classes with $\mathcal{F} \gg \mathcal{R}$.

**Example 2.7.** Let $\mathcal{R}$ be a Schunck class and suppose that $b(\mathcal{R})$ contains more than one (isomorphism type of) primitive algebra. Let $\mathcal{X}$ be a non-empty subclass of $b(\mathcal{R})$, $\mathcal{X} \neq b(\mathcal{R})$. Let $\mathcal{F}$ be the Schunck class with boundary $\mathcal{X}$. Then $\mathcal{X} \subset a(\mathcal{R})$ and we have $\mathcal{R} \ll \mathcal{F} \neq \mathcal{S}$.

## 3. Formations

We now investigate the special case in which the Schunck classes $\mathcal{F}, \mathcal{R}$ are formations. Our investigation parallels the work of D’Arcy set out in Chapter VII of Doerk and Hawkes [9]. D’Arcy uses the formation functions $f, g$ of the canonical local definitions of the saturated formations $\mathcal{R}, \mathcal{F}$ and obtains the following necessary and sufficient condition for $\mathcal{R} \ll \mathcal{F}$.

**Theorem 3.1** (DH, VII.5.1, p. 509). Let $f, g$ be the canonical definitions of the saturated formations $\mathcal{R}, \mathcal{F}$ of finite soluble groups. Then $\mathcal{R} \ll \mathcal{F}$ if and only if, for each $H \in \mathcal{F}$ and $\mathcal{R}$-projector $K$ of $H$, we have $H_{\mathcal{R}(p)} \subseteq K_{f(p)}$ for all $p \in \text{char}(\mathcal{F})$.

A locally defined formation of soluble Lie algebras has a single defining formation, not a family as in the group case, which simplifies our analysis, as does our only needing to find a necessary condition for $\mathcal{R} \ll \mathcal{F}$. It is complicated by the fact that not all saturated formations of soluble Lie algebras are locally defined. We need a substitute for the defining formation. This is provided by the quotient formation $\mathcal{F}/\mathcal{R}$ defined as follows.

**Definition 3.2.** Let $\mathcal{F}$ be a saturated formation. We define the quotient of $\mathcal{F}$ by $\mathcal{R}$ to be

$$\mathcal{F}/\mathcal{R} = \{ L/A \mid L \in \mathcal{F}, A \triangleleft L, N(L) \leq A \}.$$
By Barnes [4, Lemma 3.2], $\mathcal{F}/\mathcal{N}$ is a formation. If $\mathcal{F}$ is locally defined by $\mathcal{F}$, then $\mathcal{F} = \mathcal{F}/\mathcal{N}$ by [4, Theorem 3.3].

Suppose that $V$ is an $L$-module, $K$ is an ideal of $L$ and that $K \in \mathcal{F}$ for some saturated formation $\mathcal{F}$. By Barnes [5, Lemma 1.1], there is an $L$-module direct decomposition $V = V(K, R^+) \oplus V(K, R^-)$ where, as $K$-modules, $V(K, R^+)$ is $\mathcal{F}$-hypercentral and $V(K, R^-)$ is $\mathcal{F}$-hypereccentric.

**Lemma 3.3.** Let $A \in \mathcal{F}$ be a subalgebra of $L$ and let $B$ be a nilpotent ideal of $L$. Suppose that $L = A + B$. Suppose that $V(A, R^+) \subseteq V(B, R^+)$ for every $L$-module $V$. Then $A \supseteq B$.

**Proof.** Let $V$ be a minimal counterexample and let $K$ be a minimal ideal of $L$. Then $L/K, A + K/K, B + K/K$ satisfy the conditions of the lemma, so $A + K \supseteq B + K$. Thus $A + K = L$. The result holds if $A = L$, so $A$ contains no minimal ideal of $L$. It follows that $L$ is primitive and $N(L) = K$, so $B = K$. Now $L$ has a faithful, completely reducible $L$-module. Since $L$ has only one minimal ideal, it follows that there exists a faithful irreducible $L$-module $V$. But $B$ acts nilpotently on the $L$-submodule $V(B, R^+)$. Since $V$ is faithful and irreducible, this implies that $V(B, R^+) = 0$. Therefore $V(A, R^+) = 0$.

Let $\eta : L \to A$ be the epimorphism $\eta(x) = (x + B) \cap A$. Since $A \cap B = 0$, we can define a new action $x \cdot v = \eta(x)v$ of $L$ on $V$. Let $W$ be $V$ with this new action. Put $X = \text{Hom}_L(W, V)$ and we have $X(B, R^+) = 0$. But for the identity function $f(v) = v$, we have $af = 0$ for all $a \in A$ and $(f)$ is the trivial $A$-module. It is $\mathcal{F}$-central, contrary to $X(A, R^+) \subseteq X(B, R^+)$. □

**Lemma 3.4.** Let $A \in \mathcal{F}$ be a subalgebra of $L$ and let $B$ be a nilpotent ideal of $L$. Suppose that for every $L$-module $V$, we have $V(A, R^+) \subseteq V(B, R^+)$. Then $A \supseteq B$.

**Proof.** Put $M = A + B$. We prove that $M, A, B$ satisfy the conditions of the lemma. Let $(L^e, [p])$ be a $p$-envelope of $L$ and let $U$ be the universal $[p]$-envelope of $L^e$. Let $U_1 \subseteq U$ be the universal $[p]$-enveloping algebra of the $[p]$-closure $M[p]$ of $M$. Let $V$ be any $M$-module. Let $W = U \otimes_{U_1} V$ be the induced $L^e$-module. Then $W$ is an $L$-module, so we have $W(A, R^+) \subseteq W(B, R^+)$. But $V_1 = \{1 \otimes v \mid v \in V\}$ is an $M$-submodule isomorphic to $V$. Since

$$V_1(A, R^+) = W(A, R^+) \cap V_1 \subseteq W(B, R^+) \cap V_1 = V_1(B, R^+),$$

we have $V(A, R^+) \subseteq V(B, R^+)$ for every $M$-module $V$. By Lemma 3.3, $A \supseteq B$. □

**Lemma 3.5.** Let $\mathcal{H}$ be a saturated formation and let $\mathcal{G} = \mathcal{F}/\mathcal{N}$. Let $H \in \mathcal{H}$ and let $V$ be an $H$-module. Then $V(H, R^+) \subseteq V(H, R^+)$.\]

**Proof.** Consider first the case where $V$ is an $\mathcal{H}$-central irreducible $H$-module. Let $C$ be the centraliser of $V$ in $H$. Since $V$ is $\mathcal{F}$-central, the split extension $X$ of $V$ by $H/C$ is in $\mathcal{F}$ and $X/V \in \mathcal{F}$. Thus $C \supseteq H_\mathcal{F}$. From this, it follows for any $V$, that $H_\mathcal{F}$ acts nilpotently on $V(H, R^+)$. Thus $V(H, R^+) \subseteq V(H, R^+)$. □

Now for the Lie algebra analogue of Theorem 3.1

**Theorem 3.6.** Suppose $\mathcal{H} \supseteq \mathcal{F}$ are saturated formations. Let $\mathcal{G} = \mathcal{H}/\mathcal{N}$. Then for each $H \in \mathcal{H}$, the $\mathcal{G}$-residual $H_\mathcal{G}$ is $\mathcal{F}$-hypercentral.

**Proof.** Let $H \in \mathcal{H}$ and let $K$ be a $\mathcal{F}$-projector of $H$. Let $V$ be an $H$-module and let $L$ be the split extension of $V$ by $H$. Put $W = V(H, R^+)$. By Lemma 3.5,
W ⊆ V^{(H \mathcal{G}, \mathcal{R}^+)}$. Now $W + H$ is the unique $\mathcal{R}$-projector of $L$ which contains $H$. Also, $X = V^{(K, \mathcal{R}^+)}$ is a $K$-submodule and $X + K$ is the unique $\mathcal{R}$-projector of $L$ which contains $K$, while $(X \cap W) + K$ is the unique $\mathcal{R}$-projector of $W + H$ which contains $K$. Since $\mathcal{R} \ll \mathcal{G}$, we must have $X \subseteq W$. By Lemma 3.4, $H_\mathcal{G} \subseteq K$ and by Barnes [3, Theorem 4], $H_\mathcal{G}$ is $\mathcal{R}$-hypercentral.

\section{4. Gutiérrez García containment}

In [10], I. Gutiérrez García introduced two weakened versions of strong containment for finite soluble groups, called G- and D-strong containment.

\textbf{Definition 4.1.} Let $\mathcal{G}$ and $\mathcal{H}$ be two saturated formations of finite soluble groups with $\mathcal{H}$ the canonical local definition of $\mathcal{G}$. Suppose $\text{char}(\mathcal{G}) \subseteq \text{char}(\mathcal{H})$. We say that $\mathcal{G}$ is G-strongly contained in $\mathcal{H}$, written $\mathcal{G} \ll_{\mathcal{G}} \mathcal{H}$, if, for each $H \in \mathcal{H}$, an $\mathcal{G}$-projector $E$ of $H$ satisfies $H_{\mathcal{G}(p)} \subseteq E$ for each $p \in \text{char}(\mathcal{G})$.

We say that $\mathcal{G}$ is D-strongly contained in $\mathcal{H}$, written $\mathcal{G} \ll_{\mathcal{D}} \mathcal{H}$, if, for each $H \in \mathcal{H}$ an $\mathcal{G}$-projector $E$ of $H$ satisfies $H_{\mathcal{G}(p)} \subseteq E$ for each $p \in \text{char}(\mathcal{G})$.

For Lie algebras, there are no considerations of different primes and we can avoid the assumption that the formations are locally defined.

\textbf{Definition 4.2.} Let $\mathcal{R} \subseteq \mathcal{L}$ be saturated formations of soluble Lie algebras. Let $\mathcal{G} = \mathcal{L}/\mathcal{R}$. We say that $\mathcal{R}$ is Gutiérrez García contained in $\mathcal{L}$, written $\mathcal{R} \ll_{\mathcal{G}} \mathcal{L}$, if for all $H \in \mathcal{L}$, $H_\mathcal{G}$ is $\mathcal{R}$-hypercentral.

This is equivalent to the condition that for each $H \in \mathcal{L}$, there exists a $\mathcal{R}$-projector $K$ of $H$ such that $H_\mathcal{G} \subseteq K$. By Theorem 3.6, if $\mathcal{R} \ll \mathcal{L}$ then $\mathcal{R} \ll_{\mathcal{G}} \mathcal{L}$.

In the following, $\mathcal{R} \ll_{\mathcal{G}} \mathcal{L}$ and $\mathcal{G} = \mathcal{L}/\mathcal{R}$.

\textbf{Lemma 4.3.} Suppose $\mathcal{H} \neq \mathcal{G}, \mathcal{G}$. Then there exists $L \notin \mathcal{G}$ with a minimal ideal $A$ such that $L/A \in \mathcal{G}$ and with an $\mathcal{G}$-central but $\mathcal{R}$-eccentric module $U$ such that $L/C_L(U) \in \mathcal{R}$.

\textit{Proof.} Take $L^1 \in \mathcal{H}, L^1 \notin \mathcal{G}$ of least possible dimension. Let $A$ be a minimal ideal of $L^1$. Take $P \in \mathcal{G}, P \notin \mathcal{R}$ of least possible dimension. Let $U = \text{Soc}(P)$ and $Q = P/U$. Then $Q \in \mathcal{G} \cap \mathcal{R}$ and $U$ is an $\mathcal{R}$-central but $\mathcal{R}$-eccentric $Q$-module. Put $L = L^1 \oplus Q$. Then $A$ is a minimal ideal of $L$ and $L/A \in \mathcal{G}$ and $U$ is an $L$-module with the required properties. \hfill \Box

\textbf{Lemma 4.4.} Let $A$ be an ideal of the Lie algebra $L$ and let $V$ be an $L$-module with $AV \neq 0$. Then there exists a section $V' = X/Y$ of $V$ such that $AV'$ is the only minimal submodule of $V'$.

\textit{Proof.} Take $X$ a submodule of $V$ of least possible dimension subject to the requirement that $AX \neq 0$. Take $Y \subset X$ of largest possible dimension subject to $Y \nsubseteq AX$. Then $V' = X/Y$ has the required properties. \hfill \Box

\textbf{Lemma 4.5.} Let $L, A, U$ be as given by Lemma 4.3. There exists an $\mathcal{G}$-hypercentral $L$-module $V$ with the following properties:

\begin{itemize}
  \item[(a)] $V$ has a unique minimal submodule $W$.
  \item[(b)] $AV = W$.
  \item[(c)] $W$ is $\mathcal{R}$-eccentric.
\end{itemize}
Lemma 4.7. Suppose that $\mathfrak{H} = L/\mathfrak{H}$, so $L$ is $\mathfrak{H}$-hypercentral. Then $\mathfrak{H} = L/\mathfrak{H}$. □

Theorem 4.6. Suppose that $\mathfrak{K} \ll G \mathfrak{H}$ and that $0 \neq \mathfrak{K} \neq \mathfrak{H} \notin \mathfrak{G}$. Then $\mathfrak{H} = \mathfrak{H}/\mathfrak{H}$. □

Proof. Let $\mathfrak{H} = \mathfrak{H}/\mathfrak{H}$. Take $L, A, U, V, W$ as given by Lemmas 4.3 and 4.5. Let $L^*_A$ be the split extension of $V$ by $L$. Then $L^*_A \in \mathfrak{H}$ since $L \in \mathfrak{H}$ and $V$ is $\mathfrak{H}$-hypercentral.

Let $X$ be any non-zero ideal of $L^*$ which is contained in $A + V$. If $X \subseteq V$, then $X \supseteq W$ since $W$ is the only minimal submodule of $V$. If $X \supsetneq V$, then there exists $a + v \in X$, $a \in A$, $v \in V$ with $a \neq 0$. The centraliser $C_A(V)$ is an ideal of $L$ and, as $A$ is minimal and acts non-trivially, $C_A(V) = 0$. Thus $aV \neq 0$. Since $X$ is an ideal of $L^*$, $aV \subseteq X \cap W$. It follows that $X \supseteq W$.

Consider $L^*_A$. As $L^*_A = A$, we have $L^*/(A + V) \in \mathfrak{H}$. Thus $L^*_A \subseteq A + V$. If $L^*_A \neq 0$, then $L^*_A \supseteq W$. But $W$ is $\mathfrak{H}$-eccentric contrary to $\mathfrak{H} \ll G \mathfrak{H}$. Therefore $L^*_A = 0$ and $L^*_A \in \mathfrak{H}$. But $L \simeq L^*/(V + M)$, so $L \in \mathfrak{H}$ contrary to the choice of $L$. Therefore $\mathfrak{H} = \mathfrak{H}/\mathfrak{H}$. □

Lemma 4.7. Suppose $\text{Loc}(\mathfrak{H}) = \mathfrak{H} \neq \mathfrak{G}$. Then $\mathfrak{H} \neq \mathfrak{H}$. □

Proof. Suppose $\mathfrak{H} = \mathfrak{H}$. Take $L \notin \mathfrak{H}$ of least possible dimension. Then $L$ is primitive. Let $A = \text{Soc}(L)$. Then $L/A \in \mathfrak{H} = \mathfrak{H}$, so $L \notin \text{Loc}(\mathfrak{H}) = \mathfrak{H}$ contrary to assumption. □

Corollary 4.8. Suppose that $\mathfrak{K} \ll \mathfrak{H}$ non-trivially. Then $\mathfrak{H} = \mathfrak{H}/\mathfrak{H}$ and $\mathfrak{H}$ is not locally defined. □

Proof. By Theorem 3.6, $\mathfrak{K} \ll G \mathfrak{H}$. By Lemma 4.7, $\mathfrak{H}$ is not locally defined. □
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