GIBBS PHENOMENON OF FRAMELET EXPANSIONS AND QUASI-PROJECTION APPROXIMATION

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ABSTRACT. Gibbs phenomenon is widely known for Fourier expansions of periodic functions and refers to the phenomenon that the nth Fourier partial sums overshoot a target function at jump discontinuities in such a way that such overshoots do not die out as n goes to infinity. The Gibbs phenomenon for wavelet expansions using (bi)orthogonal wavelets has been studied in the literature. Framelets (also called wavelet frames) generalize (bi)orthogonal wavelets. Approximation by quasi-projection operators are intrinsically linked to approximation by truncated wavelet and framelet expansions. In this paper we shall establish a key identity for quasi-projection operators and then use it to study the Gibbs phenomenon of framelet expansions and approximation by general quasi-projection operators. As a consequence, we show that the Gibbs phenomenon appears for every tight or dual framelet having at least two vanishing moments and for quasi-projection operators having at least three accuracy orders. Our results not only improve current results in the literature on the Gibbs phenomenon for (bi)orthogonal wavelet expansions but also are new for framelet expansions and approximation by quasi-projection operators.

1. INTRODUCTION

Wavelets and their generalizations such as framelets have been widely applied to many areas such as image processing and numerical algorithms with great success ([1]). It is well noticed that many wavelets and framelets suffer the visually unpleasant ringing effect near jump discontinuities, which is related to the Gibbs phenomenon of wavelet and framelet expansions. It is the purpose of this paper to study the Gibbs phenomenon of wavelet and framelet expansions as well as their associated approximation schemes by quasi-projection operators.

Let us recall the Gibbs phenomenon for Fourier expansions. Let $L_2(\mathbb{T})$ denote the space of all 2π-periodic square integrable functions equipped with the inner product $\langle f, g \rangle_{L_2(\mathbb{T})} := \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)g(x)dx$. Since $\{e^{ikx}\}_{k \in \mathbb{Z}}$ is an orthonormal basis of $L_2(\mathbb{T})$, every $f \in L_2(\mathbb{T})$ has a Fourier expansion $f = \sum_{k \in \mathbb{Z}} \hat{f}(k)e^{ikx}$ in $L_2(\mathbb{T})$ with Fourier coefficients $\hat{f}(k) := \langle f, e^{ikx} \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)e^{-ikx}dx$. That is, $\lim_{n \to \infty} \|S_n f - f\|_{L_2(\mathbb{T})} = 0$, where $[S_n f](x) := \sum_{k=-n}^{n} \hat{f}(k)e^{ikx}$ is its nth Fourier partial sum. In applications, a lot of signals are modeled by piecewise smooth/analytic functions with finitely many simple jump discontinuities. Let us consider a particular function $f \in L_2(\mathbb{T})$ defined by $f(x) = -1$ for $x \in (-\pi, 0]$ and $f(x) = 1$ for $x \in (0, \pi]$. Then $f$ is a piecewise smooth/analytic 2π-periodic function with simple jump discontinuities at $\pi\mathbb{Z}$. By a straightforward calculation, one has its nth Fourier partial sum $[S_n f](x) = \sum_{k=1}^{n} \frac{2(1-(-1)^k)}{\pi k} \sin(kx)$. At every $x_0 \notin \pi\mathbb{Z}$ (i.e., at every point where $f$ is continuous), $\lim_{n \to \infty} [S_n f](x) = f(x)$ uniformly for $x$ in some neighborhood of $x_0$. But Gibbs [4] pointed out that at the jump discontinuity 0,

$$\lim_{n \to \infty} [S_n f]\left(\frac{\pi}{n}\right) = \frac{2}{\pi} \int_{0}^{\pi} \frac{\sin x}{x} dx \approx 1.17898 > 1.$$
That is, \( S_n f \) overshots \( f \) by a fixed positive amount from the right side of the origin. Such overshots do not die out as \( n \) goes to infinity and this creates undesired visually unpleasant ringing effects near jump discontinuities in applications. This phenomenon is called the Gibbs phenomenon, which was first discovered analytically by Wilbraham [19]. There are a lot of study on the Gibbs phenomenon and how to reduce it for various expansions and bases. See [5, 9] for historical overview and recent developments on the Gibbs phenomenon for Fourier expansions. The Gibbs phenomenon for orthogonal wavelets and biorthogonal wavelets have been studied in the literature, e.g., see [9, 13, 14, 16, 17, 18] and references therein. In this paper we are particularly interested in the Gibbs phenomenon for framelet expansions and their associated quasi-projection approximation.

Every function in the square integrable function space \( L_2(\mathbb{R}) \) has a wavelet expansion. Let us first recall the definition of wavelets and framelets (e.g., see [11, 2, 7, 8, 15] and references therein). For a (vector) function \( f \) on the real line \( \mathbb{R} \), we shall adopt the notation \( f_{j,k}(x) := 2^j f(2^j x - k) \) for \( j, k \in \mathbb{Z} \). By \( f \in (L_2(\mathbb{R}))^{r \times s} \) we mean that \( f \) is an \( r \times s \) matrix of functions in \( L_2(\mathbb{R}) \) and we define
\[
\langle f, g \rangle := \int_\mathbb{R} f(x)g(x)^T \, dx, \quad f \in (L_2(\mathbb{R}))^{r \times t}, g \in (L_2(\mathbb{R}))^{s \times t}.
\]

For an \( r \times 1 \) vector function \( \phi \in (L_2(\mathbb{R}))^r \) and an \( s \times 1 \) vector function \( \psi \in (L_2(\mathbb{R}))^s \), we say that \( \{ \phi; \psi \} \) is a framelet in \( L_2(\mathbb{R}) \) if there exist positive constants \( C_1 \) and \( C_2 \) such that
\[
C_1 \| f \|^2_{L_2(\mathbb{R})} \leq \sum_{k \in \mathbb{Z}} \| \langle f, \phi(\cdot - k) \rangle \|^2_{L^2_2} + \sum_{j=0}^{\infty} \sum_{k \in \mathbb{Z}} \| \langle f, \psi_{j,k} \rangle \|^2_{L^2_2} \leq C_2 \| f \|^2_{L_2(\mathbb{R})}, \quad \forall f \in L_2(\mathbb{R}). \tag{1.1}
\]

If (1.1) holds with \( C_1 = C_2 = 1 \), then \( \{ \phi; \psi \} \) is called a (normalized) tight framelet in \( L_2(\mathbb{R}) \). If the system \( AS(\phi; \psi) := \{ \phi(\cdot - k) : k \in \mathbb{Z} \} \cup \{ \psi_{j,k} : j \in \mathbb{N} \cup \{0\}, k \in \mathbb{Z} \} \) is an orthonormal basis of \( L_2(\mathbb{R}) \), then \( \{ \phi; \psi \} \) is called an orthogonal wavelet or more precisely, an orthogonal multiwavelet in the literature. Obviously, an orthogonal wavelet must be a tight framelet. Let \( \tilde{\phi} \) be an \( r \times 1 \) vector of functions in \( L_2(\mathbb{R}) \) and \( \tilde{\psi} \) be an \( s \times 1 \) vector of functions in \( L_2(\mathbb{R}) \). We say that \( \{ \{ \tilde{\phi}; \tilde{\psi} \}, \{ \phi; \psi \} \} \) is a dual framelet in \( L_2(\mathbb{R}) \) if both \( \{ \tilde{\phi}; \tilde{\psi} \} \) and \( \{ \phi; \psi \} \) are framelets in \( L_2(\mathbb{R}) \) such that
\[
\langle f, g \rangle = \sum_{k \in \mathbb{Z}} \langle f, \tilde{\phi}(\cdot - k) \rangle \langle \tilde{\phi}(\cdot - k), g \rangle + \sum_{j=0}^{\infty} \sum_{k \in \mathbb{Z}} \langle f, \tilde{\psi}_{j,k} \rangle \langle \tilde{\psi}_{j,k}, g \rangle, \quad \forall f, g \in L_2(\mathbb{R}) \tag{1.2}
\]
with the above series converging absolutely. If in addition \( AS(\tilde{\phi}; \tilde{\psi}) \) and \( AS(\phi; \psi) \) are biorthogonal to each other, then a dual framelet \( \{ \{ \tilde{\phi}; \tilde{\psi} \}, \{ \phi; \psi \} \} \) in \( L_2(\mathbb{R}) \) is called a biorthogonal wavelet in \( L_2(\mathbb{R}) \) or more precisely, a biorthogonal multiwavelet in \( L_2(\mathbb{R}) \). It follows directly from (1.2) that every function \( f \in L_2(\mathbb{R}) \) has the following framelet/wavelet expansion:
\[
f = \sum_{k \in \mathbb{Z}} \langle f, \tilde{\phi}(\cdot - k) \rangle \phi(\cdot - k) + \sum_{j=0}^{\infty} \sum_{k \in \mathbb{Z}} \langle f, \tilde{\psi}_{j,k} \rangle \psi_{j,k} \tag{1.3}
\]
with the series converging unconditionally in \( L_2(\mathbb{R}) \). For \( n \in \mathbb{N} \cup \{0\} \), the truncated expansions of (1.3) are given by (see [3])
\[
Q_n f := \sum_{k \in \mathbb{Z}} \langle f, \tilde{\phi}(\cdot - k) \rangle \phi(\cdot - k) + \sum_{j=0}^{n-1} \sum_{k \in \mathbb{Z}} \langle f, \tilde{\psi}_{j,k} \rangle \psi_{j,k} \tag{1.4}
\]
Then \( Q_n f \in L_2(\mathbb{R}) \) and \( \lim_{n \to \infty} \| Q_n f - f \|_{L_2(\mathbb{R})} = 0 \). We shall see in Proposition 3.1 that
\[
Q_n f = \sum_{k \in \mathbb{Z}} \langle f, \tilde{\phi}_{n,k} \rangle \phi_{n,k} = \sum_{k \in \mathbb{Z}} \langle f, 2^n \tilde{\phi}(2^n \cdot - k) \rangle \phi(2^n \cdot - k), \quad n \in \mathbb{N} \cup \{0\}, f \in L_2(\mathbb{R}), \tag{1.5}
\]
which are known as the quasi-projection operators in approximation theory, e.g., see \[\text{[7,10,11,12]}\]. In particular, we define \(Q := Q_0\), that is,
\[
Qf := \sum_{k \in \mathbb{Z}} \langle f, \tilde{\phi}(\cdot - k) \rangle \phi(\cdot - k), \quad f \in L_2(\mathbb{R}).
\] (1.6)
Then \((Q_n)(2^{-n} \cdot) = Q(f(2^{-n} \cdot))\) for all \(n \in \mathbb{N}\). Recall that the sign function is defined to be
\[
\text{sgn}(x) = 1 \quad \text{if} \quad x > 0; \quad \text{sgn}(0) := 0; \quad \text{sgn}(x) := -1 \quad \text{if} \quad x < 0.
\] (1.7)
The Fourier transform in this paper is defined to be \(\hat{f}(\xi) := \int_{\mathbb{R}} f(x)e^{-ix\xi}dx\) for \(f \in L_1(\mathbb{R})\) and is naturally extended to square integrable functions.

The definition of the Gibbs phenomenon for a general approximation scheme will be stated in Definition \(2.1\). To study the Gibbs phenomenon of the truncated framelet expansions in \((1.4)\), it is necessary to study the Gibbs phenomenon of the associated quasi-projection operators in \((1.5)\). To do so, we need the following key identity in this paper on quasi-projection operators.

**Theorem 1.1.** Let \(\phi\) and \(\tilde{\phi}\) be \(r \times 1\) vectors of compactly supported functions in \(L_2(\mathbb{R})\) such that
\[
\begin{aligned}
\phi(0) \quad \text{and} \quad \tilde{\phi}(0) \
\end{aligned}
\]
\begin{align*}
\begin{array}{c}
\phi(0) = 1 \\
\tilde{\phi}(0) = 0 \\
\end{array}
\quad \forall k \in \mathbb{Z} \setminus \{0\}.
\end{align*}
(1.8)
Let \(Q\) be defined in \((1.0)\). Then \(\text{sgn} - Q\text{sgn}\) is a compactly supported function in \(L_2(\mathbb{R})\) and
\[
\langle x, \text{sgn} - Q\text{sgn} \rangle = \frac{1}{6} - \phi(0) \quad (\kappa_1 - \kappa_2) - [\hat{\phi}\hat{\phi}^T(0)] \quad [\hat{\phi}\hat{\phi}^T(0) + i[\hat{\phi}\hat{\phi}^T(0) - 2i[\hat{\phi}\hat{\phi}^T(0)] \quad \kappa_1],
\] (1.9)
where
\[
k_j := \int_0^1 \sum_{n \in \mathbb{Z}} n^j \tilde{\phi}(x - n)dx, \quad j \in \mathbb{N} \cup \{0\}.
\] (1.10)

We shall present the technical proof of Theorem \(1.1\) in Section \(4\). It is well known (also see Section 2) that \((1.8)\) is equivalent to \(Q1 = 1\). This condition is needed in order to guarantee that \(\text{sgn} - Q\text{sgn}\) is compactly supported. As we shall see in this paper, the identity \((1.9)\) in Theorem \(1.1\) plays the key role in our study of the Gibbs phenomenon of framelet expansions and quasi-projection approximation.

The structure of the paper is as follows. Using Theorem \(1.1\) we shall define and discuss in Section 2 the Gibbs phenomenon for approximation by quasi-projection operators. In particular, we show that every approximation scheme by general quasi-projection operators with at least three approximation/accuracy orders exhibits the Gibbs phenomenon. We further discuss in Section 2 about how to avoid the Gibbs phenomenon for approximation by quasi-projection operators without sacrificing the approximation orders and accuracy orders. In Section 3 we address the Gibbs phenomenon of framelet expansions. In particular, we show that every dual framelet, if both its primal and dual framelet generators have at least two vanishing moments, must exhibit the Gibbs phenomenon. We finish the paper by proving Theorem \(1.1\) in Section \(4\). Though most results in this paper can be generalized to quasi-projection operators in \(L_p(\mathbb{R})\) with \(1 \leq p \leq \infty\) and to wavelets and framelets not necessarily having compact support, for simplicity and for avoiding too much technicality, we only consider the space \(L_2(\mathbb{R})\) and framelets with compact support in this paper.

2. Gibbs Phenomenon of Approximation by Quasi-projection Operators

In this section we apply Theorem \(1.1\) to study the Gibbs phenomenon of approximation by general quasi-projection operators. Let us first give the definition of the Gibbs phenomenon of a general approximation scheme.

Generally, we often expand/represent functions in \(L_2(\mathbb{R})\) under various bases in \(L_2(\mathbb{R})\), e.g., orthogonal wavelet bases. Therefore, we approximate a function \(f \in L_2(\mathbb{R})\) by a sequence \(\{Q_nf\}_{n \in \mathbb{N}}\) of functions in \(L_2(\mathbb{R})\) using some linear operators \(Q_n\) mapping \(L_2(\mathbb{R})\) to \(L_2(\mathbb{R})\) such that \(\lim_{n \to \infty} \|Q_nf - f\| = 0\) as \(n \to \infty\).
\( f \|_{L_2(\mathbb{R})} = 0 \). For piecewise smooth/analytic functions \( f \in L_2(\mathbb{R}) \) with finitely many jump discontinuities, quite often we have \( \lim_{n \to \infty} [Q_n f](x) = f(x) \) uniformly in a neighborhood of every given point where \( f \) is continuous. Therefore, to study the Gibbs phenomenon, we only need to consider a special function with only one jump discontinuity. The definition of the Gibbs phenomenon under a general approximation scheme \( \{Q_n\}_{n \in \mathbb{N}} \) is defined as follows:

**Definition 2.1.** Let \( x_0 \in \mathbb{R} \) and \( \eta \) be a compactly supported \( C^\infty \) function on \( \mathbb{R} \) such that \( \eta(x) = 1 \) for all \( x \in [x_0 - 1, x_0 + 1] \). We say that a sequence \( \{Q_n\}_{n \in \mathbb{N}} \) of linear operators, mapping real-valued functions in \( L_2(\mathbb{R}) \) into real-valued functions in \( L_2(\mathbb{R}) \), exhibits the Gibbs phenomenon at the point \( x_0 \) if there exists a sequence \( \{c_n\}_{n \in \mathbb{N}} \) of positive numbers such that \( \lim_{n \to \infty} c_n = 0 \) and either

\[
\limsup_{n \to \infty} \text{ess-sup}_{x \in (x_0, x_0 + c_n)} [Q_n g](x) > 1,
\]

or

\[
\liminf_{n \to \infty} \text{ess-inf}_{x \in (x_0 - c_n, x_0)} [Q_n g](x) < -1,
\]

where \( g := \eta(\cdot - x_0) \text{sgn}(\cdot - x_0) \), which is smooth and continuous everywhere except at the point \( x_0 \).

The inequality in (2.1) means that the approximation \( Q_n g \) overshoots \( g \) by a fixed positive amount from the right side of \( x_0 \). Note that in the above definition we do not require that all \( Q_n g \) should be continuous functions. If all \( Q_n g \) are continuous (the most common case), then the above definition is equivalent to that there exists a sequence \( \{c_n\}_{n \in \mathbb{N}} \) of positive numbers such that \( \lim_{n \to \infty} c_n = 0 \) and either \( \limsup_{n \to \infty} [Q_n g](c_n) > 1 \) or \( \liminf_{n \to \infty} [Q_n g](-c_n) < -1 \). In this section we are particularly interested in the Gibbs phenomenon of approximation by quasi-projection operators. In this paper we only consider dyadic rational numbers \( x_0 \) in Definition 2.1. In case that \( x_0 \) is not a dyadic rational number, we consider a sequence \( \{x_j\}_{j \in \mathbb{N}} \) of dyadic rational numbers such that \( \lim_{j \to \infty} |x_j - x_0| = 0 \). Define \( g_j := \eta(\cdot - x_0) \text{sgn}(\cdot - x_j) \) and \( h_j = \text{sgn}(\cdot - x_j) - \text{sgn}(\cdot - x_0) \). Note that \( |h_j| = \chi_{(-|x_j - x_0|, |x_j - x_0|)} \). By (1.5), we have

\[
\|Q_n g - Q_n g_j\|_{L_\infty(\mathbb{R})} = \|Q_n h_j\|_{L_\infty(\mathbb{R})} \leq C\|h_j\|_{L_2(\mathbb{R})} = C \sqrt{2|x_j - x_0|},
\]

\( C := \|\sum_{k \in \mathbb{Z}} |\phi(\cdot - k)|_2 L_{\infty(\mathbb{T})} \| \sum_{k \in \mathbb{Z}} |\phi(\cdot - k)|_2 L_{\infty(\mathbb{T})} < \infty \) provided that all the entries in \( \phi \) and \( \tilde{\phi} \) are compactly supported functions in \( L_\infty(\mathbb{R}) \) (which is often the case in practice). Since all the dyadic numbers are dense in \( \mathbb{R} \), without loss of generality, we only consider dyadic rational numbers \( x_0 \) in Definition 2.1. Note that \( Q_n(f(\cdot - 2^{-n}k)) = [Q_n f](\cdot - 2^{-n}k) \) for all \( n \in \mathbb{N} \) and \( k \in \mathbb{Z} \). Hence, to study the Gibbs phenomenon of approximation by quasi-projection operators and framelet expansions, it suffices to consider the Gibbs phenomenon at the origin.

Since all involved functions in \( \phi \) and \( \tilde{\phi} \) have compact support, by the definition of quasi-projection operators in (1.5), it is easy to conclude that \( Q_n \text{sgn} \) is a well-defined function and \( Q_n \text{sgn} \) agrees with \( Q_n(\eta \text{sgn}) \) in a neighborhood of the origin for all large \( n \in \mathbb{N} \). Therefore, we only need to study \( Q_n \text{sgn} \) in a neighborhood of the origin, even though \( \text{sgn} \not\in L_2(\mathbb{R}) \).

**Lemma 2.2.** Let \( \phi \) and \( \tilde{\phi} \) be \( r \times 1 \) vectors of compactly supported real-valued functions in \( L_2(\mathbb{R}) \). Let \( Q_n, n \in \mathbb{N} \) be the quasi-projection operators defined in (1.5) and \( Q := Q_0 \). Then \( \{Q_n\}_{n \in \mathbb{N}} \) does not exhibit the Gibbs phenomenon at the origin if and only if

\[
[Q \text{sgn}](x) \leq 1 \quad \text{a.e.} \ x \in (0, \infty) \quad \text{and} \quad [Q \text{sgn}](x) \geq -1 \quad \text{a.e.} \ x \in (-\infty, 0).
\]

**Proof.** From the definition of the quasi-projection operators, we have the basic fact \( [Q_n f](2^{-n} \cdot) = Q(f(2^{-n} \cdot)) \) for all \( n \in \mathbb{N} \). In particular, by \( \text{sgn}(2^{-n} \cdot) = \text{sgn} \), we have \( [Q_n \text{sgn}](2^{-n} \cdot) = Q \text{sgn} \).

The sufficiency part is trivial, since (2.3) implies \( [Q_n \text{sgn}](x) = [Q \text{sgn}](2^n x) \leq 1 \) for almost every \( x \in (0, \infty) \) and \( [Q_n \text{sgn}](x) = [Q \text{sgn}](2^n x) \geq -1 \) for almost every \( x \in (-\infty, 0) \).

Necessity. Suppose that \( \{Q_n\}_{n \in \mathbb{N}} \) does not exhibit the Gibbs phenomenon at the origin but (2.3) fails. Without loss of generality, we assume that there is a measurable set \( E \subseteq (0, \infty) \) such that \( E \) has a positive measure and \( [Q \text{sgn}](x) > 1 \) for all \( x \in E \). Therefore, there exists \( c > 0 \) such that \( C := \text{ess-sup}_{x \in (0, c)} [Q \text{sgn}](x) > 1 \). Define \( c_n := 2^{-n} c > 0 \). Then \( \lim_{n \to \infty} c_n = 0 \) and

\[
\text{ess-sup}_{x \in (0, c_n)} [Q_n \text{sgn}](x) = \text{ess-sup}_{x \in (0, c)} [Q \text{sgn}](x) = C > 1.
\]
Therefore, \( \{Q_n\}_{n \in \mathbb{N}} \) exhibits the Gibbs phenomenon at the origin, which is a contradiction to our assumption. Consequently, (2.3) must hold and we proved the necessity part.

Before addressing the Gibbs phenomenon of approximation by general quasi-projection operators, let us recall some well known results on quasi-projection operators from approximation theory. For \( m \in \mathbb{N} \), the Sobolev space \( H^m(\mathbb{R}) \) consists of all functions \( f \in L_2(\mathbb{R}) \) such that \( f, f', \ldots, f^{(m)} \in L_2(\mathbb{R}) \) (all the derivatives are in the sense of distributions). By \( \mathbb{P}_{m-1} \) we denote the set of all polynomials having degree less than \( m \). The following result is well known in approximation theory, e.g., see [10, 11, [7, Theorem 5.4.2] and references therein.

**Theorem 2.3.** Let \( \phi \) and \( \tilde{\phi} \) be \( r \times 1 \) vectors of compactly supported functions in \( L_2(\mathbb{R}) \). Let \( Q_n, n \in \mathbb{N} \) be the quasi-projection operators defined in (1.5). For \( m \in \mathbb{N} \), \( \{Q_n\}_{n \in \mathbb{N}} \) has approximation order \( m \), that is, there exists a positive constant \( C \) such that

\[
\|Q_n f - f\|_{L_2(\mathbb{R})} \leq C 2^{-nm}\|f^{(m)}\|_{L_2(\mathbb{R})}, \quad \forall f \in H^m(\mathbb{R}), n \in \mathbb{N}
\]  

(2.4) if and only if \( Qp = p \) for all polynomials \( p \in \mathbb{P}_{m-1} \) of degree less than \( m \).

For two smooth functions \( f \) and \( g \), by \( f(\xi) = g(\xi) + \mathcal{O}(|\xi|^m) \) as \( \xi \to 0 \) we mean \( f^{(j)}(0) = g^{(j)}(0) \) for all \( j = 0, \ldots, m-1 \). By \( \delta \) we denote the Dirac sequence such that \( \delta(0) = 1 \) and \( \delta(k) = 0 \) for all \( k \in \mathbb{Z}\setminus\{0\} \). We say that \( \{Q_n\}_{n \in \mathbb{N}} \) has accuracy order \( m \) if \( Qp = p \) for all \( p \in \mathbb{P}_{m-1} \). It is also well known (e.g., see [7, Proposition 5.5.2]) that \( Qp = p \) for all \( p \in \mathbb{P}_{m-1} \) if and only if

\[
\frac{\pi}{\phi(\xi) - \tilde{\phi}(\xi + 2\pi k) - \delta(k) + \mathcal{O}(|\xi|^m)} \to 0 \quad \xi \to 0 \quad \text{for all} \quad k \in \mathbb{Z}.
\]

(2.5)

A simple sufficient condition for \( \{Q_n\}_{n \in \mathbb{N}} \) to be free of the Gibbs phenomenon is as follows.

**Proposition 2.4.** Let \( \phi \) and \( \tilde{\phi} \) be \( r \times 1 \) vectors of compactly supported real-valued functions in \( L_2(\mathbb{R}) \) such that (1.8) is satisfied, i.e., \( Q1 = 1 \), where \( Q \) is defined in (1.6).

(i) If all the entries in the vector function \( \phi \), \( \int_{-\infty}^{k} \tilde{\phi}(x)dx \) and \( \int_{k}^{\infty} \tilde{\phi}(x)dx \) are nonnegative for all \( k \in \mathbb{Z} \), then \(-1 \leq [Q \text{sgn}](x) \leq 1 \) for almost every \( x \in \mathbb{R} \) and in particular, (2.3) holds.

(ii) If all the entries in \( \phi \) and \( \tilde{\phi} \) are nonnegative, then (2.3) holds, but \( \{Q_n\}_{n \in \mathbb{N}} \) has accuracy order no more than two, where \( Q_n, n \in \mathbb{N} \) are defined in (1.5).

**Proof.** By (1.8) and \( \langle 1, \tilde{\phi}(- \cdot - k) \rangle = \tilde{\phi}(0) \), we have

\[
\frac{\pi}{\tilde{\phi}(0) - \int_{-\infty}^{k} \tilde{\phi}(x)dx} \sum_{k \in \mathbb{Z}} \langle 1, \tilde{\phi}(- \cdot - k) \rangle \phi(- \cdot - k) = 1.
\]

(2.6)

Noting that \( \tilde{\phi} \) is real-valued and \( \langle \text{sgn}, \tilde{\phi}(- \cdot - k) \rangle = \tilde{\phi}(0) - 2 \int_{-\infty}^{k} \tilde{\phi}(t)dt \) for all \( k \in \mathbb{Z} \), we have

\[
[Q \text{sgn}](x) - 1 = -2 \sum_{k \in \mathbb{Z}} \left( \int_{-\infty}^{-k} \tilde{\phi}(t)dt \right)^{T} \phi(x - k) \leq 0,
\]

for almost every \( x \in \mathbb{R} \), since by item (i) all entries in both \( \phi \) and \( \int_\infty^{-k} \tilde{\phi}(t)dt \) are nonnegative. Hence, we proved \([Q \text{sgn}](x) \leq 1 \) for almost every \( x \in \mathbb{R} \). On the other hand,

\[
[Q \text{sgn}](x) + 1 = \sum_{k \in \mathbb{Z}} 2 \left( \tilde{\phi}(0) - \int_{-\infty}^{-k} \tilde{\phi}(t)dt \right)^{T} \phi(x - k) = 2 \sum_{k \in \mathbb{Z}} \left( \int_{-\infty}^{-k} \tilde{\phi}(t)dt \right)^{T} \phi(x - k) \geq 0.
\]

Hence, \([Q \text{sgn}](x) \geq -1 \) for almost every \( x \in \mathbb{R} \). This proves item (i).

Obviously, the conditions in item (ii) imply the conditions in item (i). Therefore, (2.3) must hold. We use proof by contradiction to show that \( \{Q_n\}_{n \in \mathbb{N}} \) has accuracy order no more than two. Suppose that \( \{Q_n\}_{n \in \mathbb{N}} \) has accuracy order at least three. Define \( \eta(x) := \int_{\mathbb{R}} \tilde{\phi}(x + y) \phi(y)dy \).

By
with \( m = 3 \), we must have \( \widehat{\eta}(\xi) = \hat{\phi}(\xi) \hat{\phi}(\xi) = 1 + \mathcal{O}(|\xi|^3) \) as \( \xi \to 0 \), from which we conclude that \( \widehat{\eta}''(0) = 0 \) and \( \widehat{\eta}(0) = 1 \). By \( 0 = \widehat{\eta}''(0) = \int_{\mathbb{R}} \eta(x)(-ix)^2dx \), we must have \( \int_{\mathbb{R}} x^2 \eta(x)dx = 0 \). However, since both \( \hat{\phi} \) and \( \phi \) are nonnegative, the scalar function \( \eta \) must be nonnegative. The condition \( \int_{\mathbb{R}} x^2 \eta(x)dx = 0 \) will then force \( \eta = 0 \), a contradiction to \( \widehat{\eta}(0) = 1 \). This proves that \( \{\mathcal{Q}_n\}_{n \in \mathbb{N}} \) must have accuracy order no more than two. This proves item (ii).

Many functions in wavelet analysis and approximation theory are nonnegative. One important family of such functions are B-splines. The B-spline function \( B_m \) of order \( m \) is defined by

\[
B_1 := \chi_{(0,1]} \quad \text{and} \quad B_m := B_{m-1} * B_1 = \int_0^1 B_{m-1}(\cdot - t)dt, \quad m \in \mathbb{N}.
\]

It is easy to check that \( B_m \) is a nonnegative piecewise polynomial with support \([0,m]\) and \( \widehat{B}_m(\xi) = \frac{1 - e^{-i\xi}}{i\xi} \). Therefore, \( \widehat{B}_m(0) = 1 \) and \( \widehat{B}_m(\xi + 2\pi k) = \mathcal{O}(|\xi|^m) \) as \( \xi \to 0 \) for all \( k \in \mathbb{Z} \setminus \{0\} \).

For the Gibbs phenomenon of approximation by general quasi-projection operators, we have

**Theorem 2.5.** Let \( \phi \) and \( \tilde{\phi} \) be \( r \times 1 \) vectors of compactly supported functions in \( \mathcal{L}_2(\mathbb{R}) \). Let \( \mathcal{Q}_n \) be the quasi-projection operators defined in (1.5) and \( \mathcal{Q} := \mathcal{Q}_0 \). If (1.8) holds (i.e., \( \mathcal{Q}1 = 1 \)) and

\[
\tilde{\phi}(\xi) \phi(\xi + 2\pi k) = \delta(k) + \mathcal{O}(|\xi|^2), \quad \xi \to 0 \quad \text{for all } k \in \mathbb{Z},
\]

then

\[
\langle x, \operatorname{sgn} - \mathcal{Q} \operatorname{sgn} \rangle = -\langle \tilde{\phi}^T \mathcal{Q} \phi \rangle''(0).
\]

If in addition \( \langle \tilde{\phi}^T \mathcal{Q} \phi \rangle''(0) = 0 \), both \( \phi \) and \( \tilde{\phi} \) are real-valued, and

\[
\mathcal{Q} \operatorname{sgn} \neq \operatorname{sgn} \quad \text{on some set of positive measure},
\]

then \( \{\mathcal{Q}_n\}_{n \in \mathbb{N}} \) must exhibit the Gibbs phenomenon.

**Proof.** Since (1.8) holds, by Theorem 1.1, the identity in (1.9) must be true. Define

\[
\tilde{\mathcal{Q}} := \tilde{\mathcal{Q}}_0 \quad \text{and} \quad \tilde{\mathcal{Q}}_n f := \sum_{k \in \mathbb{Z}} \langle f, 2^n \phi(2^n \cdot - k) \rangle \phi(2^n \cdot - k), \quad f \in \mathcal{L}_2(\mathbb{R}), n \in \mathbb{N} \cup \{0\}.
\]

Then (2.8) is equivalent to saying that \( \{\tilde{\mathcal{Q}}_n\}_{n \in \mathbb{N}} \) has accuracy order two, i.e., \( \tilde{\mathcal{Q}}1 = 1 \) and \( \tilde{\mathcal{Q}}x = x \).

By calculation, we have \( \langle x, \phi(\cdot - k) \rangle = \langle x + k, \phi \rangle = -i\tilde{\phi}^T(0) + k \phi(0) \). Hence,

\[
x = \tilde{\mathcal{Q}}x = \sum_{k \in \mathbb{Z}} \langle (\cdot), \phi(\cdot - k) \rangle \tilde{\phi}(x - k) = \sum_{k \in \mathbb{Z}} \left( -i\tilde{\phi}^T(0) + k \phi(0) \right) \tilde{\phi}(x - k).
\]

Multiplying \( x \) to both sides of the above identity and taking integral on \([0,1] \), by \( x = (x - k) + k \), we have

\[
\frac{1}{3} = \int_0^1 x^2 dx = -i\tilde{\phi}^T(0) \sum_{k \in \mathbb{Z}} \int_0^1 x \tilde{\phi}(x - k)dx + \phi(0) \sum_{k \in \mathbb{Z}} \int_0^1 k x \tilde{\phi}(x - k)dx
\]

\[
= -i\tilde{\phi}^T(0) \left( \int_0^1 \sum_{k \in \mathbb{Z}} (x - k) \tilde{\phi}(x - k)dx + \int_0^1 \sum_{k \in \mathbb{Z}} k \tilde{\phi}(x - k)dx \right)
\]

\[
+ \phi(0) \left( \int_0^1 \sum_{k \in \mathbb{Z}} k(x - k) \tilde{\phi}(x - k)dx + \int_0^1 \sum_{k \in \mathbb{Z}} k^2 \tilde{\phi}(x - k)dx \right)
\]

\[
= -i\tilde{\phi}^T(0) \left( \int_{\mathbb{R}} x \tilde{\phi}(x)dx + \kappa_1 \right) + \phi(0) \left( \kappa_1 + \kappa_2 \right)
\]

\[
= \left[ \tilde{\phi}^T(0) \right] \left[ \phi^T(0) - i\tilde{\phi}^T(0) \right] \kappa_1 + \phi(0) \left( \kappa_1 + \kappa_2 \right),
\]
where $\kappa_1, \kappa_2$ are defined in (1.10), and $\kappa_* := \int_0^1 \sum_{k \in \mathbb{Z}} k(x-k) \tilde{\phi}(x-k) dx$. Since $\langle 1, \phi(-k) \rangle = \tilde{\phi}(0)$ and $\tilde{Q}1 = 1$, we have
\[
1 = \tilde{Q}1 = \sum_{k \in \mathbb{Z}} \langle 1, \phi(-k) \rangle \tilde{\phi}(-k) = \tilde{\phi}(0) \sum_{k \in \mathbb{Z}} \tilde{\phi}(x-k).
\]
Multiplying $x^2$ on both sides of the above identity and taking integral over $[0,1]$, we have
\[
\frac{1}{3} = \int_0^1 x^2 dx = \tilde{\phi}(0)^T \sum_{k \in \mathbb{Z}} \int_0^1 x^2 \tilde{\phi}(x-k) dx = \tilde{\phi}(0)^T \int_0^1 \sum_{k \in \mathbb{Z}} ((x-k)^2 + 2k(x-k) + k^2) \tilde{\phi}(x-k) dx
\]
\[
= \tilde{\phi}(0)^T \left( \int_0^1 \sum_{k \in \mathbb{Z}} (x-k)^2 \tilde{\phi}(x-k) dx + \int_0^1 \sum_{k \in \mathbb{Z}} 2k(x-k) \tilde{\phi}(x-k) dx + \int_0^1 \sum_{k \in \mathbb{Z}} k^2 \tilde{\phi}(x-k) dx \right)
\]
\[
= \tilde{\phi}(0)^T \left( \int x^2 \tilde{\phi}(x) dx + 2\kappa_* + \kappa_2 \right)
\]
\[
= -\tilde{\phi}(0)^T [\tilde{\phi}''(0) + 2\tilde{\phi}(0) \kappa_* + \tilde{\phi}(0)^T \kappa_2].
\]
That is, we proved that (2.8) implies
\[
\frac{1}{3} = [\tilde{\phi}''(0) + 2\tilde{\phi}(0) \kappa_* + \tilde{\phi}(0)^T \kappa_2].
\]
Multiplying the first identity by 2 and subtracting the second identity, we end up with
\[
\frac{1}{2} = \int_0^1 x dx = -i[\tilde{\phi}''(0)^T \tilde{\phi}(0)] - \int_0^1 \tilde{\phi}(x-k) dx + \tilde{\phi}(0)^T \int_0^1 k\tilde{\phi}(x-k) dx = -i[\tilde{\phi}''(0)^T \tilde{\phi}(0)] - \tilde{\phi}(0)^T \kappa_1,
\]
from which we have $\tilde{\phi}(0)^T \kappa_1 = \frac{1}{2} + i[\tilde{\phi}''(0)] \tilde{\phi}(0)$. Now we conclude from (1.9) in Theorem 1.1 that
\[
\langle x, \text{sgn} - \tilde{Q} \text{sgn} \rangle = \frac{1}{6} - \tilde{\phi}(0)^T (\tilde{\phi}(0) - \tilde{\phi}''(0)^T \tilde{\phi}(0)) - \frac{1}{2} - \tilde{\phi}(0)^T \tilde{\phi}(0) \kappa_* - \tilde{\phi}(0)^T \tilde{\phi}(0) \kappa_2 - \tilde{\phi}(0)^T \tilde{\phi}(0) \kappa_2 - \tilde{\phi}(0)^T \tilde{\phi}(0) \kappa_2
\]
\[
= -\frac{1}{3} + \tilde{\phi}(0)^T \kappa_* - \tilde{\phi}(0)^T \tilde{\phi}(0) \kappa_2 - \tilde{\phi}(0)^T \tilde{\phi}(0) \kappa_2
\]
\[
= -2[\tilde{\phi}''(0)] - \tilde{\phi}(0)^T \tilde{\phi}(0) \tilde{\phi}(0) - \tilde{\phi}(0)^T \tilde{\phi}(0) \tilde{\phi}(0) = -[\tilde{\phi}''(0)],
\]
where we used (2.13) in the second-to-last identity. Hence, (2.9) must be true.

If in addition $\tilde{\phi}''(0) = 0$, then it follows directly from (2.9) that $\langle x, \text{sgn} - \tilde{Q} \text{sgn} \rangle = 0$. Suppose that $\{Q_n\}_{n \in \mathbb{N}}$ does not exhibit the Gibbs phenomenon. By Lemma 2.2 and 2.3 must hold. Consequently, $x(\text{sgn} - \tilde{Q} \text{sgn})(x) \geq 0$ for almost every $x \in \mathbb{R}$. By $\langle x, \text{sgn} - \tilde{Q} \text{sgn} \rangle = 0$, we must have $x(\text{sgn} - \tilde{Q} \text{sgn})(x) = 0$ for almost every $x \in \mathbb{R}$, which is a contradiction to our assumption in (2.10). Therefore, $\{Q_n\}_{n \in \mathbb{N}}$ must exhibit the Gibbs phenomenon.

It is easy to see that (2.10) is often true. For example, if $\phi$ is continuous, then (2.10) must hold. As a direct consequence of Theorem 2.5, the following result says that quasi-projection operators having accuracy order higher than two must exhibit the Gibbs phenomenon.
Corollary 2.6. Let $\phi$ be an $r \times 1$ vector of compactly supported functions in $L_2(\mathbb{R})$. Recall that the associated quasi-projection operators are defined by
\[ Q_n f = \sum_{k \in \mathbb{Z}} \langle f, 2^n \phi(2^n \cdot - k) \rangle \phi(2^n \cdot - k), \quad f \in L_2(\mathbb{R}), n \in \mathbb{N} \cup \{0\}. \] (2.14)

If \( \{Q_n\}_{n \in \mathbb{N}} \) has accuracy order higher than two, then \( \langle x, \text{sgn} - Q \text{sgn} \rangle = 0 \), where \( Q := Q_0 \). If in addition \( 2.10 \) holds, then \( \{Q_n\}_{n \in \mathbb{N}} \) must exhibit the Gibbs phenomenon.

Proof. Since \( \{Q_n\}_{n \in \mathbb{N}} \) has accuracy order higher than two, by Theorem 2.5 we must have \( Q1 = 1, Qr = x \) and \( Qx^2 = x^2 \); or equivalently, 
\[ \frac{\phi(x)}{\phi(\xi)} \frac{\hat{\phi}(\xi + 2\pi k)}{\hat{\phi}(\xi)} = \delta(k) + O(|\xi|^3), \quad \xi \to 0, k \in \mathbb{Z}. \] (2.15)

Hence, both (1.8) and (2.8) are satisfied with \( \tilde{\phi} := \phi \). Moreover, (2.15) further implies \( \phi(x) \hat{\phi}(\xi) = 1 + O(|\xi|^3) \) as \( \xi \to 0 \), from which we have \( \hat{\phi}''(0) = 0 \). Now the conclusion follows directly from Theorem 2.5. \( \square \)

In the rest of this section we discuss how to avoid the Gibbs phenomenon while preserve accuracy order. Let \( \phi \) and \( \tilde{\phi} \) be \( r \times 1 \) vectors of compactly supported functions in \( L_2(\mathbb{R}) \). By \( S(\phi) \) we denote the space of all functions \( \sum_{k \in \mathbb{Z}} v(k) \phi(\cdot - k) \) for all sequences \( v : \mathbb{Z} \to \mathbb{C}^{1 \times r} \). If the quasi-projection operators \( Q_n \) in (1.5) have accuracy order \( m \), then \( Qp = p \) for all \( p \in P_{m-1} \). In particular, we have \( P_{m-1} \subset S(\phi) \). If \( P_{m-1} \subset S(\phi) \), then it is interesting to ask whether there exists a compactly supported vector function \( \tilde{\phi} \) such that \( \{Q_n\}_{n \in \mathbb{N}} \) in (1.5) has accuracy order \( m \) and is free of the Gibbs phenomenon. The following result partially answers this question.

Proposition 2.7. Let \( \phi \) be a nonvanishing compactly supported function in \( L_2(\mathbb{R}) \) such that \( \hat{\phi}(0) = 1 \) and \( \hat{\phi}(\xi + 2\pi k) = O(|\xi|^m) \) as \( \xi \to 0 \) for all \( k \in \mathbb{Z} \setminus \{0\} \). Then there exists a compactly supported real-valued function \( \tilde{\phi} \in L_2(\mathbb{R}) \) such that \( 2.5 \) is satisfied, \( \sum_{k \in \mathbb{Z}} \tilde{\phi}(\cdot - k) = 1 \), and \( \{Q_n\}_{k \in \mathbb{Z}} \) has accuracy order \( m \) and is free of the Gibbs phenomenon, where \( Q_n, n \in \mathbb{N} \) are defined in (1.5).

Proof. Define \( d_j := j^2 [1/\tilde{\phi}]^{(j)}(0) \) for \( j = 0, \ldots, m-1 \). Since \( \phi \) is real-valued, we can check that all \( d_j \) are real numbers. Moreover, \( d_0 = 1 \) and \( d_1 = \int_\mathbb{R} x\phi(x)dx \). Let \( N \) be the unique integer such that \( 0 \leq d_1 - N + \frac{1}{2} < 1 \). Consider \( N = x_0 < x_1 < \cdots < x_{m-1} = N + 1 \) and \( c_0, \ldots, c_{m-1} \in \mathbb{R} \). Define 
\[ \eta := \sum_{k=1}^{m-1} c_k \chi_{[x_{k-1}, x_k]} \quad \text{and} \quad \tilde{\phi} := \eta - \eta(1 + \cdot) + \chi_{(N-1, N]} \]

Then it is trivial that \( \eta \) is supported inside \([N, N+1] \), \( \tilde{\phi} \) is supported inside \([N-1, N+1] \), and \( \sum_{k \in \mathbb{Z}} \tilde{\phi}(\cdot - k) = 1 \). We now choose the unknowns \( \{c_k\}_{k=1}^{m-1} \) so that
\[ \tilde{\phi}(\xi) = \frac{1}{\phi(\xi)} + O(|\xi|^m), \quad \xi \to 0. \] (2.16)

By the definition of \( \tilde{\phi} \) and \( d_j \), we see that (2.16) becomes
\[ \int_\mathbb{R} x^j \tilde{\phi}(x)dx = \int_\mathbb{R} x^j \left( \eta(x) - \eta(x+1) + \chi_{(N-1, N]}(x) \right)dx = d_j, \quad j = 0, \ldots, m-1. \]

Note that \( \int_\mathbb{R} x^j \chi_{(N-1, N]}(x)dx = \frac{N^{j+1} - (N-1)^{j+1}}{j+1} \), \( \int_\mathbb{R} x^j \eta(x+1)dx = \int_\mathbb{R} (x-1)^j \eta(x)dx \), and
\[ \int_\mathbb{R} (x^j - (x-1)^j) \eta(x)dx = \sum_{k=1}^{m-1} c_k \int_{x_{k-1}}^{x_k} (x^j - (x-1)^j)dx. \]
Consequently, (2.16) is further equivalent to that for all \( j = 0, \ldots, m - 1 \),
\[
\sum_{k=1}^{m-1} c_k \int_{x_{k-1}}^{x_k} (x^j - (x - 1)^j)dx = d_j - \frac{N^{j+1} - (N - 1)^{j+1}}{j+1}.
\] (2.17)
For \( j = 0 \), by \( d_0 = 1 \), both sides of the above equation in (2.17) are zero. We now prove that the system of linear equations in (2.17) for \( j = 1, \ldots, m - 1 \) has a unique solution \( \{c_k\}_{k=1}^{m-1} \). In fact, we consider its dual problem: If
\[
\int_{x_{k-1}}^{x_k} p(x)dx = 0 \quad \text{for all} \quad k = 1, \ldots, m - 1 \quad \text{with} \quad p(x) := \sum_{j=1}^{m-1} b_j (x^j - (x - 1)^j),
\] (2.18)
we must prove that \( b_1 = \cdots = b_{m-1} = 0 \). Let \( q \) be the unique polynomial such that \( q' = p \) and \( q(x_0) = 0 \). Then \( \deg(q) \leq m - 1 \) by \( \deg(p) \leq m - 2 \). Since \( \int_{x_{k-1}}^{x_k} p(x)dx = q(x_k) - q(x_{k-1}) \) and \( q(x_0) = 0 \), we see that (2.18) is equivalent to \( q(x_1) = \cdots = q(x_{m-1}) = 0 \). Since \( q(x_0) = 0 \), the polynomial \( q \) has \( m \) distinct roots \( x_0, \ldots, x_{m-1} \). Then \( \deg(q) \leq m - 1 \) will force \( q = 0 \) and consequently, \( p = q' = 0 \), from which we must have \( b_1 = \cdots = b_{m-1} = 0 \). Since all \( d_j \) are real numbers, this proves that the system of linear equations in (2.17) has a unique real-valued solution \( \{c_k\}_{k=1}^{m-1} \). Therefore, (2.5) is satisfied and \( \{Q_n\}_{n \in \mathbb{Z}} \) has accuracy order \( m \).

Considering \( j = 1 \) in (2.17), we have
\[
\int_{\mathbb{R}} \eta(x)dx = \sum_{k=1}^{m-1} c_k (x_k - x_{k-1}) = d_1 - \frac{N^2 - (N - 1)^2}{2} = d_1 - N + \frac{1}{2}.
\]
Note that \( \tilde{\phi} = \eta \) on \([N, N+1]\). By our choice of \( N \in \mathbb{Z} \), we conclude that \( \int_{N}^{N+1} \tilde{\phi}(x)dx = \int_{\mathbb{R}} \eta(x)dx = d_1 - N + \frac{1}{2} \) must lie on the interval \([0, 1]\). On the other hand, since \( \tilde{\phi} = \chi_{[N-1,N]} - \eta(1+) \) on \([N-1,N]\), we have
\[
\int_{N-1}^{N} \tilde{\phi}(x)dx = 1 - \int_{\mathbb{R}} \eta(1+x)dx = 1 - \left(d_1 - N + \frac{1}{2}\right)
\]
must lie on \([0, 1]\) by \( 0 \leq d_1 - N + \frac{1}{2} < 1 \). Since \( \tilde{\phi} \) is supported inside \([N-1,N+1]\), now it follows from item (i) of Proposition 2.4 that \( \{Q_n\}_{n \in \mathbb{N}} \) is free of the Gibbs phenomenon. \( \square \)

According to Theorem 2.5 for \( m \geq 3 \), Proposition 2.7 is optimal in the sense that any desired compactly supported function \( \tilde{\phi} \) in Proposition 2.7 cannot satisfy the additional condition: \( \tilde{\phi}'(2\pi k) = 0 \) for all \( k \in \mathbb{Z} \setminus \{0\} \). Otherwise, Theorem 2.5 tells us that \( \{Q_n\}_{n \in \mathbb{N}} \) must exhibit the Gibbs phenomenon, a contradiction to the claim in Proposition 2.7.

3. Gibbs Phenomenon of Wavelet and Framelet Expansions

In this section we study the Gibbs phenomenon of wavelet and framelet expansions. As we shall see in this section, wavelets and framelets having high vanishing moments must exhibit the Gibbs phenomenon.

One of the main features of wavelets and framelets \( \{\phi, \psi\} \) is sparse representation, which is largely due to the vanishing moments of \( \psi \) (see [1]). We say that \( \psi \) has \( m \) vanishing moments if \( \int_{\mathbb{R}} x^j \psi(x)dx = 0 \) for all \( j = 0, \ldots, m - 1 \), or equivalently, \( \hat{\psi}^{(j)}(0) = 0 \) for all \( j = 0, \ldots, m - 1 \). In particular, we define \( \text{vm} (\psi) := m \) with \( m \) being the largest such nonnegative integer.

Before discussing the Gibbs phenomenon of wavelet and framelet expansions, we have the following result, which is essentially known in the literature but we shall provide a proof here for the convenience of the reader.

**Proposition 3.1.** Let \( \phi, \tilde{\phi} \) be \( r \times 1 \) vectors of compactly supported functions in \( L_2(\mathbb{R}) \) and \( \psi, \tilde{\psi} \) be \( s \times 1 \) vectors of compactly supported functions in \( L_2(\mathbb{R}) \). Suppose that \( \{\tilde{\phi}, \tilde{\psi}\}, \{\phi, \psi\} \) is a dual framelet in \( L_2(\mathbb{R}) \). Then
(i) \([1.4]\) holds, where \(Q_n\) are the operators in \([1.4]\) for the truncated framelet expansions.

(ii) \(Qp = p\) for all \(p \in \mathbb{P}_{m-1}\) with \(m : = \text{vm}(\tilde{\psi})\), where \(Qp := \sum_{k \in \mathbb{Z}} \langle p, \tilde{\phi}(\cdot - k) \rangle \phi(\cdot - k)\).

**Proof.** Item (i) is essentially known in [6]. Indeed, for \(J \in \mathbb{Z}\), replacing \(f\) and \(g\) in \([1.2]\) by \(2^{-J/2}f(2^{-J} \cdot)\) and \(2^{-J/2}g(2^{-J} \cdot)\), respectively, we have

\[
\langle f, g \rangle = \langle 2^{-J/2}f(2^{-J} \cdot), 2^{-J/2}g(2^{-J} \cdot) \rangle = \sum_{k \in \mathbb{Z}} \langle 2^{-J/2}f(2^{-J} \cdot), \tilde{\phi}(\cdot - k) \rangle \langle \phi(\cdot - k), 2^{-J/2}g(2^{-J} \cdot) \rangle
\]

\[
\quad + \sum_{j=0}^{\infty} \sum_{k \in \mathbb{Z}} \langle 2^{-J/2}f(2^{-J} \cdot), \tilde{\psi}_{j,k} \rangle \langle \psi_{j,k}, 2^{-J/2}g(2^{-J} \cdot) \rangle
\]

\[
\quad = \sum_{k \in \mathbb{Z}} \langle f, \tilde{\phi}_{j,k} \rangle \langle \phi_{j,k}, g \rangle
\]

Considering the differences between the levels \(J = n - 1\) and \(J = n\) of the above identities, we conclude that

\[
\sum_{k \in \mathbb{Z}} \langle f, \tilde{\phi}_{n-1,k} \rangle \langle \phi_{n-1,k}, g \rangle + \sum_{k \in \mathbb{Z}} \langle f, \tilde{\psi}_{n-1,k} \rangle \langle \psi_{n-1,k}, g \rangle = \sum_{k \in \mathbb{Z}} \langle f, \tilde{\phi}_{n,k} \rangle \langle \phi_{n,k}, g \rangle, \quad n \in \mathbb{Z}.
\]

(3.1)

Consequently, it follows directly from the above identities that \(\langle Q_n f, g \rangle = \langle \sum_{k \in \mathbb{Z}} \langle f, \tilde{\phi}_{n,k} \rangle \phi_{n,k}, g \rangle\) for all \(f, g \in L_2(\mathbb{R})\). Therefore, we must have \([1.3]\). This proves item (i).

We now prove item (ii). Since all the functions have compact support, without loss of generality, we assume that all \(\tilde{\phi}, \phi, \tilde{\psi}, \psi\) are supported inside \([-N_0, N_0]\). For \(N \geq N_0\), let \(\eta_N\) be a compactly supported \(C^{\infty}\) function such that \(\eta_N(x) = 1\) for \(x \in [-N, N]\). Let \(p \in \mathbb{P}_{m-1}\). Then \(\eta_N p \in L_2(\mathbb{R})\). Due to the vanishing moments of \(\tilde{\psi}\), if the support of \(\tilde{\psi}_{j,k}\) is contained inside \([-N, N]\), then \(\langle \eta_N p, \tilde{\psi}_{j,k} \rangle = \langle p, \tilde{\psi}_{j,k} \rangle = 0\). Now it follows directly from the representation in \([1.3]\) that

\[p(x) = \eta_N(x)p(x) = \sum_{k \in \mathbb{Z}} \langle p, \tilde{\phi}(\cdot - k) \rangle \phi(x - k), \quad x \in (N_0 - N, N - N_0).\]

Taking \(N \to \infty\), we conclude that \(Qp = p\). This proves item (ii). \qed

We have the following result on the Gibbs phenomenon of wavelet and framelet expansions.

**Theorem 3.2.** Let \(\{\tilde{\phi}; \tilde{\psi}\}, \{\phi; \psi\}\) be a dual framelet in \(L_2(\mathbb{R})\), where \(\phi, \tilde{\phi}\) are \(r \times 1\) vectors of compactly supported functions in \(L_2(\mathbb{R})\) and \(\psi, \tilde{\psi}\) are \(s \times 1\) vectors of compactly supported functions in \(L_2(\mathbb{R})\). Let \(Q_n, n \in \mathbb{N} \cup \{0\}\) be defined in \([1.4]\) for the truncated framelet expansions using the dual framelet \(\{\tilde{\phi}; \tilde{\psi}\}, \{\phi; \psi\}\). Define the quasi-projection operator \(Q\) as in \([1.6]\).

(i) If \(\text{vm}(\psi) \geq 2\) and \(\text{vm}(\tilde{\psi}) \geq 1\), then \(\langle x, \text{sgn} - Q \text{sgn} \rangle = 0\). If in addition \([2.10]\) holds and \(\phi, \tilde{\phi}\) are real-valued, then \(\{Q_n\}_{n \in \mathbb{N}}\) exhibits the Gibbs phenomenon.

(ii) If all the entries in \(\phi\) and \(\tilde{\phi}\) are nonnegative, then \(\{Q_n\}_{n \in \mathbb{N}}\) has no Gibbs phenomenon but \(\text{vm}(\psi) = \text{vm}(\tilde{\psi}) = 1\).

**Proof.** By item (i) of Proposition 3.1, the truncated operators \(Q_n\) in \([1.4]\) agree with the quasi-projection operators in \([1.5]\). Since \(\text{vm}(\tilde{\psi}) \geq 1\), by item (ii) of Proposition 3.1, we have \(Q1 = 1\), which is equivalent to \([1.8]\). Define \(\tilde{Q}_n\) as in \([2.11]\) and \(\tilde{Q} := \tilde{Q}_0\). Since \(\text{vm}(\psi) \geq 2\), by item (ii) of Proposition 3.1, we have \(\tilde{Q}1 = 1\) and \(\tilde{Q}x = x\), which is equivalent to \([2.8]\). Next, we prove \(\langle \phi; \tilde{\phi} \rangle^T(0) = 0\). As we have seen in the proof of Proposition 3.1 \([3.1]\) holds. Applying \([6]\) Proposition 3.1 to \([3.1]\) with \(n = 1\) and \(\lambda = 1/2\), we conclude that

\[
\phi(\xi) \tilde{\phi}(\xi) + \psi(\xi) \tilde{\psi}(\xi) = \tilde{\phi}(\xi/2) \phi(\xi/2).
\]
Because $\text{vm}(\psi) \geq 2$ and $\text{vm}(\tilde{\psi}) \geq 1$, we have $\psi(\xi)/\tilde{\psi}(\xi) = \mathcal{O}(|\xi|^3)$ as $\xi \to 0$. Therefore, we conclude from the above identity that

$$
\frac{T^{-\infty}}{\phi(\xi)} \phi(\xi) = \frac{T^{-\infty}}{\phi(\xi/2)} \phi(\xi/2) + \mathcal{O}(|\xi|^3), \quad \xi \to 0. \tag{3.2}
$$

By (2.8), we trivially have $\phi(0) = 1$. Using the Taylor expansion of $\phi$ at the origin, we deduce from (3.2) that we must have $\phi(\xi) \phi(\xi) = 1 + \mathcal{O}(|\xi|^3)$ as $\xi \to 0$. Consequently, we proved $[\phi \phi]''(0) = 0$. Now all the claims in item (i) follows directly from Theorem 2.5.

If all the entries in $\phi$ and $\tilde{\phi}$ are nonnegative, then it follows from item (ii) of Proposition 2.4 that $\{Q_n\}_{n \in \mathbb{N}}$ has no Gibbs phenomenon. We now prove $\text{vm}(\psi) = \text{vm}(\tilde{\psi}) = 1$. Since $\{\{\phi; \psi\}, \{\phi; \tilde{\psi}\}\}$ is a dual framelet in $L_2(\mathbb{R})$, it is necessary that $\text{vm}(\psi) \geq 1$ and $\text{vm}(\tilde{\psi}) \geq 1$. We use proof by contradiction to prove $\text{vm}(\psi) = \text{vm}(\tilde{\psi}) = 1$. Without loss of generality, we can assume that $\text{vm}(\psi) \geq 2$. By what has been proved a moment ago, we must have $[\phi \tilde{\phi}]''(0) = 0$. Define $\eta(x) := \int_{\mathbb{R}} \phi(x+y)^T \tilde{\phi}(y)dy$. Then $\tilde{\eta}(\xi) = \frac{T^{-\infty}}{\phi(\xi)} \phi(\xi)$. Hence, $[\phi \tilde{\phi}]''(0) = 0$ becomes $[\eta]''(0) = 0$, which is equivalent to saying that $\int_{\mathbb{R}} x^2 \eta(x)dx = 0$. Since both $\phi$ and $\tilde{\phi}$ are nonnegative, the scalar function $\eta$ is nonnegative. Consequently, $\int_{\mathbb{R}} x^2 \eta(x)dx = 0$ forces $\eta = 0$, a contradiction to $\tilde{\eta}(0) = 1$. This proves $\text{vm}(\psi) = \text{vm}(\tilde{\psi}) = 1$. This completes the proof of item (ii).

Note that an orthogonal wavelet is a special case of a tight framelet. As a direct consequence of Theorem 3.2, we have the following result on tight framelets and orthogonal wavelets.

**Corollary 3.3.** Let $\{\phi; \psi\}$ be a tight framelet in $L_2(\mathbb{R})$, where $\phi$ is an $r \times 1$ vector of compactly supported functions in $L_2(\mathbb{R})$ and $\psi$ is an $s \times 1$ vector of compactly supported functions in $L_2(\mathbb{R})$. Let $Q_n, n \in \mathbb{N}$ be the truncated operator for the truncated framelet expansions using the tight framelet $\{\phi; \psi\}$, i.e.,

$$
Q_n f := \sum_{k \in \mathbb{Z}} \langle f, \phi(\cdot - k) \rangle \phi(\cdot - k) + \sum_{j=0}^{n-1} \sum_{k \in \mathbb{Z}} \langle f, \psi_{j,k} \rangle \psi_{j,k}. \tag{3.3}
$$

If $\text{vm}(\psi) \geq 2$, then $\langle x, \text{sgn} - Q \text{sgn} \rangle = 0$, where $Q f := \sum_{k \in \mathbb{Z}} \langle f, \phi(\cdot - k) \rangle \phi(\cdot - k)$. If in addition (2.10) holds and $\phi$ is real-valued, then $\{Q_n\}_{n \in \mathbb{N}}$ must exhibit the Gibbs phenomenon.

Theorem 3.2 and Corollary 3.3 obviously cover the results in [13, 14, 16, 17, 18] as special cases on the Gibbs phenomenon of orthogonal wavelets and biorthogonal wavelets. Corollary 3.3 tells us that the truncated approximation using a tight framelet cannot avoid the Gibbs phenomenon while having at least two vanishing moments. However, high vanishing moments are of paramount importance in many applications. Therefore, to avoid the Gibbs phenomenon while having high vanishing moments, according to Theorem 3.2, the only possibility left is dual framelets $\{\{\phi; \tilde{\psi}\}, \{\phi; \psi\}\}$ (which must not be tight framelets, i.e., dual framelets with $\phi = \phi$ and $\tilde{\psi} = \psi$) such that $\text{vm}(\psi) = 1$ and $\text{vm}(\tilde{\psi}) = m$ for $m \geq 2$. By Proposition 3.1, the associated quasi-projection operators $\{Q_n\}_{n \in \mathbb{N}}$ will have accuracy order $m$. To our best knowledge, so far there is no known example of compactly supported dual framelets $\{\{\phi; \tilde{\psi}\}, \{\phi; \psi\}\}$ in $L_2(\mathbb{R})$ such that $\text{vm}(\psi) = 1$, $\text{vm}(\tilde{\psi}) \geq 2$ and its associated truncated quasi-projection operators $\{Q_n\}_{n \in \mathbb{N}}$ do not exhibit the Gibbs phenomenon. It is very interesting to construct such examples and explore their applications in practice.

4. **Proof of Theorem 1.1**

In this section we shall prove Theorem 1.1. To do so, we need some necessary definitions and auxiliary results.

By $l_0(\mathbb{Z})$ we denote the space of all finitely supported complex-valued sequences $u = \{u(k)\}_{k \in \mathbb{Z}}$ on $\mathbb{Z}$ such that $u(k) \neq 0$ for finitely many $k \in \mathbb{Z}$. For a finitely supported sequence $u = \{u(k)\}_{k \in \mathbb{Z}} \in l_0(\mathbb{Z})$, the
its Fourier series is defined to be \( \hat{u}(\xi) := \sum_{k \in \mathbb{Z}} u(k)e^{-ik\xi} \) for \( \xi \in \mathbb{R} \). Note that \( \hat{u}(0) = \sum_{k \in \mathbb{Z}} u(k) \). For two finitely supported sequences \( u \) and \( d \), their convolution is defined to be \( [u * d](n) = \sum_{k \in \mathbb{Z}} u(n-k)d(n) \) for \( n \in \mathbb{Z} \). Note that \( u * d(\xi) = \hat{u}(\xi)\hat{d}(\xi) \).

To prove Theorem 1.1 we need the following auxiliary result.

**Lemma 4.1.** Let \( c \) be a sequence on \( \mathbb{Z} \) such that \( c(k) = -c_{\infty} \) for all \( k < -N \) and \( c(k) = c_{\infty} \) for all \( k \geq N \) for some \( N \in \mathbb{N} \) and \( c_{\infty} \in \mathbb{C} \). Let \( d \in l_0(\mathbb{Z}) \) be a finitely supported sequence on \( \mathbb{Z} \) such that \( \hat{d}(0) = 0 \). Then \( c * d \in l_0(\mathbb{Z}) \) is a finitely supported sequence on \( \mathbb{Z} \) and

\[
\sum_{k \in \mathbb{Z}} [c * d](k) = c_{\infty} \sum_{k \in \mathbb{Z}} [v * d](k) = -2ic_{\infty}[\hat{d}](0), \tag{4.1}
\]

\[
\sum_{k \in \mathbb{Z}} k[c * d](k) = ic_{\infty}[\hat{d}](0) + c_{\infty}[\hat{d}'](0) + i[c - \overline{c_{\infty}}v](0)[\hat{d}](0), \tag{4.2}
\]

where \( v \) is the sequence such that \( v(k) = -1 \) for all \( k < 0 \) and \( v(k) = 1 \) for all \( k \geq 0 \).

**Proof.** By our assumption on the sequence \( c \), since \( d \) is finitely supported with \( \hat{d}(0) = \sum_{k \in \mathbb{Z}} d(k) = 0 \), it is not difficult to observe that \( c * d \) is a finitely supported sequence. Write \( c * d = (c - c_{\infty}v) * d + c_{\infty}v * d \). Since \( c - c_{\infty}v \) is a finitely supported sequence, we have \( \sum_{k \in \mathbb{Z}} [(c - c_{\infty}v) * d](k) = [c - c_{\infty}v](0)\hat{d}(0) = 0 \) by \( \hat{d}(0) = 0 \). Define a sequence \( u \) such that \( u(k) = 1 \) for all \( k \geq 0 \) and \( u(k) = 0 \) for all \( k < 0 \). Then \( v = -1 + 2u \). Since \( \hat{d}(0) = 0 \), we have \( 1 * d = 0 \) and hence \( v * d = -1 * d + 2u * d = 2u * d \). Since \( d \) is finitely supported, we can assume that \( d \) is supported inside \([-N, N]\) for some \( N \in \mathbb{N} \). Note that \( \sum_{n=-N}^{N} d(n) = \hat{d}(0) = 0 \). Then

\[
\sum_{k \in \mathbb{Z}} 2[u * d](k) = 2 \sum_{k \in \mathbb{Z}} \sum_{n \in \mathbb{Z}} u(n)d(k - n) = 2 \sum_{k \in \mathbb{Z}} \sum_{n=0}^{\infty} d(k - n) = 2 \sum_{k \in \mathbb{Z}} \sum_{n=-\infty}^{k} d(n)
\]

\[
= 2 \sum_{k \in \mathbb{Z}} \sum_{n=-N}^{N} d(n) = 2 \sum_{k=-N}^{N} \sum_{n=-N}^{N} d(n) = 2 N \sum_{n=-N}^{N} (N + 1 - n)d(n)
\]

\[
= -2 \sum_{n=-N}^{N} nd(n) = -2i[\hat{d}'](0),
\]

where we used \( [\hat{d}'](0) = \sum_{n \in \mathbb{Z}} d(n)(-\imath n) \). This proves (4.1).

By \( v * d = 2u * d \), we have \( c * d = (c - c_{\infty}v) * d + c_{\infty}v * d = 2c_{\infty}u * d + (c - c_{\infty}v) * d \). Noting that \( [u * d](k) = \sum_{n=-\infty}^{k} d(n) \), we deduce that

\[
\sum_{k \in \mathbb{Z}} k[c * d](k) = 2c_{\infty} \sum_{k \in \mathbb{Z}} \sum_{n=-\infty}^{k} kd(n) + \sum_{k \in \mathbb{Z}} k[(c - c_{\infty}v) * d](k).
\]

Since both \( c - c_{\infty}v \) and \( d \) are finitely supported sequences, by \( \hat{d}(0) = 0 \), we have

\[
\sum_{k \in \mathbb{Z}} k[(c - c_{\infty}v) * d](k) = i[(c - c_{\infty}v)\hat{d}](0) = i[c - c_{\infty}v](0)[\hat{d}](0) + i[c - c_{\infty}v]'(0)\hat{d}(0) = i[c - c_{\infty}v](0)[\hat{d}](0).
\]

On the other hand, by \( \sum_{n=-N}^{N} d(n) = \hat{d}(0) = 0 \), we have

\[
2 \sum_{k \in \mathbb{Z}} \sum_{n=-N}^{N} kd(n) = 2 \sum_{k=-N}^{N} \sum_{n=-N}^{N} kd(n) = \frac{N}{2} \sum_{n=-N}^{N} (N + 1 - n)d(n)
\]

\[
= \sum_{n=-N}^{N} (N(N + 1) + n - n^2)d(n) = \sum_{n=-N}^{N} nd(n) - \sum_{n=-N}^{N} n^2d(n) = i[\hat{d}](0) + [\hat{d}]''(0).
\]
Putting all the identities together, we proved (1.2).

We are now ready to prove Theorem 1.1.

**Proof of Theorem 1.1.** We first take a special compactly supported scalar function \( \eta \in L_2(\mathbb{R}) \) such that
\[
\hat{\eta}(\xi) = 1 + O(|\xi|^3) \quad \text{and} \quad \eta(\xi)\hat{\eta}(\xi + 2\pi k) = \delta(k) + O(|\xi|^3), \quad \xi \rightarrow 0, k \in \mathbb{Z}. \tag{4.3}
\]
Such a function can be easily constructed from B-splines in (2.7). Indeed, consider \( B_3 \) and note \( \hat{B}_3(0) = 1 \). Let \( u = \{ u(k) \}_{k \in \mathbb{Z}} \) be a finitely supported sequence such that \( \hat{u}(\xi) = \frac{1}{B_3(\xi)} + O(|\xi|^3) \) as \( \xi \rightarrow 0 \). Define \( \eta := \sum_{k \in \mathbb{Z}} u(k) B_3(\cdot - k) \). Then \( \eta \) is a compactly supported function in \( L_2(\mathbb{R}) \) and
\[
\hat{\eta}(\xi) = \hat{\eta}(\xi)\hat{B}_3(\xi) = \hat{\eta}(\xi) + 1 + O(|\xi|^3), \quad \xi \rightarrow 0.
\]
Since \( \hat{B}_3(\xi + 2\pi k) = O(|\xi|^3) \) as \( \xi \rightarrow 0 \) for all \( k \in \mathbb{Z} \setminus \{0\} \), the function \( \eta \) satisfies all the conditions in (4.3). By (4.3), we have \( \langle (\cdot)^j, \eta(\cdot - k) \rangle = k^j \) and \( x^j = \sum_{k \in \mathbb{Z}} \langle (\cdot)^j, \eta(\cdot - k) \rangle x(k) = \sum_{k \in \mathbb{Z}} k^j \eta(k) \) for all \( j = 0, 1, 2 \). That is, we have
\[
1 = \sum_{k \in \mathbb{Z}} \eta(\cdot - k), \quad x = \sum_{k \in \mathbb{Z}} k \eta(x(k)), \quad x^2 = \sum_{k \in \mathbb{Z}} k^2 \eta(x(k)). \tag{4.4}
\]
By our assumption in (1.3), we have \( Q1 = 1 \). Therefore, \( Q\text{sgn} \) agrees with \text{sgn} outside some neighborhood of the origin. Consequently, \( \text{sgn} - Q\text{sgn} \) is a compactly supported function in \( L_2(\mathbb{R}) \). We now calculate \( \langle \text{sgn} - Q\text{sgn}, x \rangle \) using (4.4). Using the expression for the polynomial \( x \) in (4.3) and noting that \( \eta \) has compact support, we have
\[
\langle \text{sgn} - Q\text{sgn}, x \rangle = \sum_{k \in \mathbb{Z}} k \langle \text{sgn} - Q\text{sgn}, \eta(\cdot - k) \rangle = \sum_{k \in \mathbb{Z}} k \left( \langle \text{sgn}, \eta(\cdot - k) \rangle - \langle Q\text{sgn}, \eta(\cdot - k) \rangle \right).
\]
On the other hand, by the definition of the quasi-projection operator \( Q \) in (1.5), we have
\[
\langle Q\text{sgn}, \eta(\cdot - k) \rangle = \sum_{n \in \mathbb{Z}} \langle \text{sgn}, \tilde{\phi}(\cdot - n) \rangle \langle \phi(\cdot - n), \eta(\cdot - k) \rangle = \sum_{n \in \mathbb{Z}} \langle \text{sgn}, \tilde{\phi}(\cdot - n) \rangle \langle \phi, \eta(\cdot - (k - n)) \rangle.
\]
For \( k \in \mathbb{Z} \), define
\[
c(k) := \langle \text{sgn}, \eta(\cdot - k) \rangle \in \mathbb{C}, \quad \tilde{b}(k) := \langle \text{sgn}, \tilde{\phi}(\cdot - n) \rangle \in \mathbb{C}^{1 \times r}, \quad b(k) := \langle \phi, \eta(\cdot - k) \rangle \in \mathbb{C}^r. \tag{4.5}
\]
Then we have
\[
\langle \text{sgn} - Q\text{sgn}, x \rangle = \sum_{k \in \mathbb{Z}} k \left( c(k) - \frac{1}{k} \tilde{b}(n)(b(k) - n) \right) = \sum_{k \in \mathbb{Z}} k(c - \tilde{b} \ast b)(k) \tag{4.6}
\]
Since both \( \phi \) and \( \eta \) have compact support, the sequence \( b \) must be finitely supported. Define \( d := b - \hat{\phi}(0) \delta \). Then \( d \in l_0(\mathbb{Z}) \). Noting that \( \tilde{b} \ast \delta = \tilde{b} \), we can write
\[
c - \tilde{b} \ast d = c - \hat{\phi}(0) \delta - \tilde{b} \ast (b - \hat{\phi}(0) \delta) = (c - \tilde{b} \hat{\phi}(0)) - \tilde{b} \ast d.
\]
Since both \( \eta \) and \( \tilde{\phi} \) have compact support, by \( \hat{\eta}(0) = 1 \), there exists a positive constant \( N \) such that
\[
c(k) = \text{sgn}(k) \quad \text{and} \quad \tilde{b}(k) = \text{sgn}(k) \tilde{\phi}(0) \quad \text{for all} \ |k| \geq N. \quad \text{Because} \ \hat{\phi}(0) \tilde{\phi}(0) = 1 \text{ by (1.8), we conclude that} \ c - \tilde{b} \hat{\phi}(0) \text{is a finitely supported sequence. By the first identity in (4.4) and the definition of the sequence} \ b, \text{we have}
\]
\[
\tilde{b}(0) = \sum_{k \in \mathbb{Z}} b(k) = \langle \phi, \sum_{k \in \mathbb{Z}} \eta(\cdot - k) \rangle = \langle \phi, 1 \rangle = \hat{\phi}(0).
\]
Hence, \( \tilde{d}(0) = \tilde{b}(0) - \hat{\phi}(0) = 0 \). Therefore, we deduce that \( \tilde{b} \ast d \) is a finitely supported sequence. By (4.6), since \( c - \tilde{b} \hat{\phi}(0) \in l_0(\mathbb{Z}) \) and \( \tilde{b} \ast d \in l_0(\mathbb{Z}) \), we conclude that
\[
\langle \text{sgn} - Q\text{sgn}, x \rangle = I_1 - I_2 \quad \text{with} \quad I_1 := \sum_{k \in \mathbb{Z}} k \left( c(k) - \tilde{b}(k) \hat{\phi}(0) \right), \quad I_2 := \sum_{k \in \mathbb{Z}} k \left( \tilde{b} \ast d \right)(k). \tag{4.7}
\]
We now calculate $I_1$ and $I_2$. To calculate $I_1$, we define
\[
\hat{c}(k) := \int_0^1 \eta(x-k)dx = \int_{-k}^{1-k} \eta(x)dx, \quad \hat{b}(k) := \int_0^1 \phi(x-k)dx = \int_{-k}^{1-k} \phi(x)dx, \quad k \in \mathbb{Z}.
\]
Then it is easy to deduce that $\sum_{k \in \mathbb{Z}} \hat{b}(k) = \hat{\phi}(0)$ and
\[
\hat{b}(k) = \langle \text{sgn}, \phi(\cdot - k) \rangle = -\hat{\phi}(0) + 2 \sum_{n=-\infty}^{k} \hat{b}(n)T.
\] (4.8)
By $\hat{\phi}(0) = 1$, the above identity leads to $\hat{b}(k) = 0 = -1 + 2 \sum_{n=-\infty}^{k} \hat{b}(n)T$. Similarly, we have $c(k) = \langle \text{sgn}, \eta(\cdot - k) \rangle = -1 + 2 \sum_{n=-\infty}^{k} \hat{c}(n)$. Since both $\hat{b}$ and $\hat{c}$ are finitely supported, we can assume that $\hat{c} - \hat{b} T \hat{\phi}(0)$ is supported inside $[-N,N]$ for some $N \in \mathbb{N}$. Then
\[
\sum_{n=-N}^{N} (\hat{c}(n) - \hat{b}(n)T\hat{\phi}(0)) = \sum_{n \in \mathbb{Z}} \hat{c}(n) - \sum_{n \in \mathbb{Z}} \hat{b}(n)T\hat{\phi}(0) = \hat{\eta}(0) - \hat{\phi}(0)\hat{\phi}(0) = 1 - 1 = 0
\]
and
\[
I_1 = \sum_{k \in \mathbb{Z}} k \left( c(k) - \hat{b}(k)\hat{\phi}(0) \right) = \sum_{k \in \mathbb{Z}} \sum_{n=-\infty}^{k} 2k \left( \hat{c}(n) - \hat{b}(n)T\hat{\phi}(0) \right) = \sum_{k=-N}^{N} \sum_{n=-N}^{k} 2k \left( \hat{c}(n) - \hat{b}(n)T\hat{\phi}(0) \right)
\]
\[
= \sum_{n=-N}^{N} \sum_{k=-N}^{N} 2k \left( \hat{c}(n) - \hat{b}(n)T\hat{\phi}(0) \right) = \sum_{n=-N}^{N} (N(N+1) + n - n^2) \left( \hat{c}(n) - \hat{b}(n)T\hat{\phi}(0) \right)
\]
\[
= \sum_{n \in \mathbb{Z}} (n-n^2) \left( \hat{c}(n) - \hat{b}(n)T\hat{\phi}(0) \right).
\]
By the definition of $\hat{c}$ and (4.3), we have
\[
\sum_{n \in \mathbb{Z}} (n-n^2)\hat{c}(n) = \sum_{n \in \mathbb{Z}} (n-n^2) \int_0^1 \eta(x-k)dx = \int_0^1 \sum_{n \in \mathbb{Z}} (n-n^2)\eta(x-n)dx = \int_0^1 (x-x^2)dx = \frac{1}{6}.
\]
Similarly, by the definition of $\hat{b}$ and the definition of $\kappa_j$ in (1.10), we have
\[
\sum_{n \in \mathbb{Z}} (n-n^2)\hat{b}(n)T = \int_0^1 \sum_{n \in \mathbb{Z}} (n-n^2)\phi(x-n)Tdx = \kappa_1 - \kappa_2.
\]
Hence, we conclude that $I_1 = \frac{1}{6} - (\kappa_1 - \kappa_2)T\hat{\phi}(0)$.

We now calculate $I_2$. Note that we already proved that $\tilde{b}(k) = \text{sgn}(k)\hat{\phi}(0)$ for all $|k| \geq N$. Define $\tilde{d} := \tilde{b} - \hat{\phi}(0) \nu$, where the sequence $\nu$ is defined in Lemma 4.4. Since $d \in l_0(\mathbb{Z})$ with $\tilde{d}(0) = 0$, by the identity (4.2) in Lemma 4.1, we have
\[
I_2 = \sum_{k \in \mathbb{Z}} k [\tilde{b} * d](k) = i \hat{\phi}(0) \left[ \tilde{d}'(0) \right] + \hat{\phi}(0) \left[ \tilde{d}''(0) + i \tilde{d}(0) \tilde{d}'(0) \right].
\]
By definition, $d = b - \hat{\phi}(0) \delta$. Hence, $\tilde{d}(\xi) = \hat{b}(\xi) - \hat{\phi}(0)$. By (4.4) and the definition of the sequence $b$, for $j = 0, 1, 2$, we have
\[
[\tilde{b}]^{(j)}(0) = (-i)^j \sum_{k \in \mathbb{Z}} k^j \langle \phi, \eta(\cdot - k) \rangle = (-i)^j \left\langle \phi, \sum_{k \in \mathbb{Z}} k^j \eta(\cdot - k) \right\rangle = (-i)^j \langle \phi, \nu^j \rangle = [\hat{\phi}]^{(j)}(0).
\]
So, \( [\hat{d}'(0) = \hat{b}'(0) = \hat{\phi}'(0) \) and \( [\hat{d}''(0) = \hat{b}''(0) = \hat{\phi}''(0) \). Hence, \( I_2 = i\hat{\phi}(0) \hat{\phi}'(0) + \hat{\phi}(0) \hat{\phi}''(0) + 
\hat{\phi}'(0) \). Note that \( \hat{d}(0) = \sum_{k \in \mathbb{Z}} (\hat{b}(k) - \hat{\phi}(0) v(k)) \). Since \( v(k) = -1 + 2 \sum_{n=-\infty}^k \delta(n) \) for all \( k \in \mathbb{Z} \)
and (4.8) holds, we deduce that
\[
\hat{d}(0) = \sum_{k \in \mathbb{Z}} \sum_{n=-\infty}^k 2\left(\overline{b(n)} - \overline{\hat{\phi}(0)} \delta(n)\right) = \sum_{n=-N}^N \sum_{k=n}^{N} 2\left(\overline{b(n)} - \overline{\hat{\phi}(0)} \delta(n)\right)
\]
\[
= \sum_{n=-N}^N (N + 1 - n)\left(\overline{b(n)} - \overline{\hat{\phi}(0)} \delta(n)\right) = \sum_{n \in \mathbb{Z}} -2n\left(\overline{b(n)} - \overline{\hat{\phi}(0)} \delta(n)\right)
\]
\[
= -2 \sum_{n \in \mathbb{Z}} nb(n) = -2 \int_0^1 \left(\sum_{n \in \mathbb{Z}} n\overline{\phi}(x - n)\right) dx = -2\kappa_1.
\]
Hence,
\[
I_2 = i\overline{\hat{\phi}(0)} \hat{\phi}'(0) + \overline{\hat{\phi}(0)} \hat{\phi}''(0) - 2i\kappa_1 \overline{\hat{\phi}'}(0).
\]
Therefore, by (4.7) and \( \langle x, \sgn - Q \sgn \rangle = \langle \sgn - Q \sgn, x \rangle \), we conclude that (1.9) holds. □

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