Josephson Current and Noise at a Superconductor-Quantum Spin Hall Insulator-Superconductor Junction

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We study junctions between superconductors mediated by the edge states of a quantum spin Hall insulator. We show that such junctions exhibit a fractional Josephson effect, in which the current phase relation has a $4\pi$ periodicity, rather than a $2\pi$ periodicity. This effect is a consequence of the conservation of fermion parity - the number of electrons modulo 2 - in a superconducting junction, and is closely related to the $\mathbb{Z}_2$ topological structure of the quantum spin Hall insulator. Inelastic processes, which violate the conservation of fermion parity, lead to telegraph noise in the equilibrium supercurrent. We predict that the low frequency noise due to these processes diverges exponentially with temperature $T$ as $T \to 0$. Possible experiments on HgCdTe quantum wells will be discussed.

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Proposals for fault tolerant topological quantum computation have motivated intense current interest in finding robust physical systems that host excitations with non-Abelian statistics [1, 2]. Recent experiments on the $\nu = 5/2$ fractional quantum Hall effect have shown encouraging indirect evidence for such excitations [3, 4], but the direct observation of non-Abelions has so far remained elusive. Recently we showed that the proximity effect between a superconductor and a three dimensional (3D) topological insulator leads to a 2D interface state that supports non-Abelian Majorana fermions [5]. A first step towards implementing this proposal would be to demonstrate experimentally the topological order responsible for Majorana fermions.

In this paper we study Josephson junctions mediated by a 2D topological insulator, also known as a quantum spin Hall insulator (QSHI) [6, 7, 8, 9, 10]. We predict that such junctions exhibit a fractional Josephson effect, whose origin is related to the presence of Majorana fermions. The signature of the fractional Josephson effect is that the current phase relation has a $4\pi$ rather than a $2\pi$ periodicity. This behavior was first predicted by Kitaev using an idealized model of a 1D spinless p wave superconductor [11]. Kwon et al. [12] proposed that a related effect can occur at junctions between unconventional 3D superconductors. They argued that it leads to Majorana fermions.

The QSHI is a time reversal invariant insulating state with a bulk energy gap generated by spin orbit interactions [3, 4]. It has recently been observed in HgCdTe quantum wells [10]. The QSHI is distinguished from an ordinary insulator by a $\mathbb{Z}_2$ topological invariant [8], which necessitates the existence of gapless edge states. The states at the edge of the QSHI form a unique 1D system that is essentially half of an ordinary spin degenerate 1D electron gas. In the simplest case it consists of a single band of right moving electrons paired via Kramers theorem with a left moving band with the opposite spin. These states are robust in the presence of disorder because time reversal symmetry prevents elastic backscattering. In the absence of inelastic scattering the transmission in the edge states is perfect.

Suppose the edge states are in intimate contact with an s wave superconductor. Due to the proximity effect, the tunneling of Cooper pairs will induce a pairing potential $\Delta = \Delta_0 e^{i\phi}$ in the edge states, which depends on the phase $\phi$ of the superconductor and the nature of the contact. Using the notation of Ref. [5] write $H = \Psi^\dagger H \Psi/2$, where $\Psi = [\psi_1, \psi_1^\dagger, \psi_2, \psi_2^\dagger]$ is express in terms of field operators $\psi_1, \psi_2$ describing the right(left) movers and

$$H = -iv\tau_z \sigma_x \partial_x - \mu \tau_z + \Delta_0 (\cos \phi \tau_x + \sin \phi \tau_y).$$

$\sigma_j$ are Pauli matrices acting in the space of right and left movers $\psi_{1,2}$ and $\tau_j$ are Pauli matrices which mix the $\psi$ and $\psi^\dagger$ blocks of $\Psi$. $v$ is the velocity of the edge states, $\mu$ is the chemical potential and we set $\hbar = 1$. The eigenstates of (1) come in pairs at $\pm E$. Due to the reducency in $\Psi$, these states are not independent, and the Bogoliubov quasiparticle operators satisfy $\Gamma_{-E} = \Gamma_E^\dagger$.

Eq. (1) is similar to Kitaev’s model of superconducting spinless electrons in 1D [11]. In Kitaev’s model there are zero energy Majorana bound states associated with the ends of the sample. In our system, the edge - which is the boundary of the 2D QSHI - can not have an end.
By breaking time reversal symmetry, however, a Zeeman field can introduce a mass term into $H$ of the form

$$V_Z = M\psi^\dagger \sigma_x \psi = M\Psi^\dagger \sigma_x \Psi/2. \quad (2)$$

When $M > \mu$, $V_Z$ opens an insulating gap in the edge state spectrum. $V_Z$ could arise either from an applied magnetic field (as in Ref. [10]) or due to proximity to a magnetic material. Zero energy Majorana bound states will exist at the interface between regions with gaps dominated by $\Delta$ and $M$. In the presence of both $\Delta$ and $M$ the gap is the smaller of $|\Delta_0 \pm M|$. When $\Delta_0 = |M|$ a single band is gapless, and for $\Delta_0 \sim |M|$ the low energy sector of [5] has the form of a Su-Schrieffer-Heeger model [13], which has a well known zero energy bound state where $\Delta_0 - |M|$ changes sign. The Bogoliubov quasiparticle operator associated with this state is a Majorana fermion, which satisfies

$$\gamma_0 \equiv \gamma^\dagger_0.$$ 

Consider a superconductor-QSHI-superconductor (S-QSHI-S) junction in which the edge states of a QSHI connect two superconductors separated by a distance $L$. Fig. 1 shows an RF SQUID geometry, in which the phase difference across the junction $\phi = (2e/h)\Phi$ is controlled by the magnetic flux $\Phi$. We also assume that the QSHI forms a Corbino disk which circles the flux. As a practical matter, this geometry is not essential, but we will see that it has considerable conceptual value. We will also consider the effect of a Zeeman term in the gap between the superconductors, which will make the connection with Majorana bound states transparent. We emphasize, however, that there will be a non trivial effect even when this term is absent. To determine the characteristics of the junction we solve the Bogoliubov de Gennes (BdG) equation with

$$\Delta(x) = \Delta_0 \left[ \theta(-x-L/2) + e^{i\phi} \theta(x-L/2) \right]$$

$$M(x) = M_0 \theta(x+L/2) \theta(-x+L/2). \quad (3)$$

By matching the solutions it is straightforward to determine the spectrum of Andreev bound states in the junction. The calculation is similar to Ref. [12], as well as the theory of superconducting quantum point contacts (SQPCs) [14, 15]. However, we shall see that there is a fundamental difference with both of those theories.

Fig. 2a shows the spectrum as a function of phase difference $\phi$ for parameters indicated in each panel. $L$ is in units of $v/\Delta_0$ and $M_0$ and $\mu$ are in units of $\Delta_0$. (a) and (c) are independent of $\mu$.

Fig. 2b shows a case where $M_0 = 0$. For $L \lesssim v/\Delta_0$ there is a single pair of bound states $E = \pm \epsilon_0(\phi)$. For $L \ll v/\Delta_0$ our model reduces to the $\delta$ function model solved in Ref. [12] where the normal state transmission probability is $D = 1/[1+(M_0 \sinh(\kappa L)/\kappa)^2]$, with $\kappa = \sqrt{M_0^2 - \mu^2}$. In that case

$$\epsilon_0(\phi) = \sqrt{D\Delta_0} \cos(\phi/2). \quad (4)$$

Fig. 2c shows a case where $M_0 = \Delta_0$, so the normal state transmission $D < 1$. When $D \ll 1$ there are two weakly coupled Majorana end states at $x = \pm L/2$. When $L > v/\Delta_0$ there will be additional Andreev bound states in the junction with a level spacing of order $v/L$. Fig. 2c shows the case where $v/L = 3$ with $M_0 = 0$, in which time reversal symmetry requires Kramers degeneracies when $\phi = 0$ or $\pi$. Fig. 2d shows the effect of finite $M_0$ and $\mu$, which lifts most of the degeneracies. However, the crossing at $E = 0$ remains, and is of special significance.

To understand the level crossing consider a low energy theory for $E \ll \Delta_0$. The two eigenvectors $\xi_{\pm}$ of [14] with energy $\pm \epsilon_0(\phi)$ define Bogoliubov operators $\Gamma_{0\pm} = \Psi^\dagger \xi_0 \pm$. Due to particle-hole symmetry, $\Gamma_{0+} = \Gamma^\dagger_{0-} \equiv \Gamma_0$. The low energy Hamiltonian is thus

$$H = \epsilon_0(\phi)(\Gamma_0^\dagger \Gamma_0 - 1/2) = 2i\epsilon_0(\phi)\gamma_1 \gamma_2, \quad (5)$$

where we introduced Majorana operators $\gamma_1 = (\Gamma + \Gamma^\dagger)/2$, $\gamma_2 = -i(\Gamma - \Gamma^\dagger)/2$. In the weak tunneling ($D \ll 1$) limit $\gamma_{1,2}$ describe the Majorana end states at $x = \pm L/2$, which are weakly coupled by tunneling of electrons. Eq. [5] describes two states distinguished by $N_0 \equiv \Gamma_0^\dagger \Gamma_0 = 0, 1$. Mixing these states requires an interaction that changes

![FIG. 1: A S-QSHI-S junction in an RF SQUID geometry where the QSHI forms a Corbino disk.](image1)

![FIG. 2: Spectrum of Andreev bound states in the junction as a function of phase difference $\phi$ for parameters indicated in each panel.](image2)
Due to the pairing term in Eq. (1), the total charge is not conserved. However, the fermion parity, defined as the number of electrons modulo 2, is conserved in Eq. (13). This forbids the coupling between the two states and protects the crossing at $\epsilon_0(\phi) = 0$.

There is a problem, however, with the fermion parity. The junction Hamiltonian (13) is invariant under a $2\pi$ phase change, but when $\phi \to \phi + 2\pi$, the system passes through a single level crossing and can only return to the initial state by a process which changes $N_0$ by 1. The fermion parity thus apparently changes when $\phi \to \phi + 2\pi$. This has to do with the unbound spectrum as $E \to -\infty$ and reflects a fermion parity anomaly similar to the $SU(2)$ anomaly in 4D field theory[10]. This anomaly is related to non-Abelian statistics. When $\phi$ advances by $2\pi \gamma_1 \to \gamma_1$ and $\gamma_2 \to -\gamma_2$, in the tunneling limit this can be interpreted as Ivanov’s rule[17] for the effect of braiding a vortex between the Majorana bound states.

The physical origin of the fermion parity anomaly lies in the topological structure of the QSHI. Consider first the Corbino disk in Fig. 1 without the superconductor. In Ref. [18] we showed that the $Z_2$ invariant characterizing the QSHI describes the change in the $Z_2$ “time reversal polarization” (TRP) when flux $h/2e$ is threaded through the hole. A nonzero TRP specifies a many body Kramers degeneracy, the fermion parity. With the superconductor present, start in the groundstate at $\Phi = 0$. When flux $\Phi = h/2e$ is threaded through the hole, $\phi$ advances by $2\pi$ and a unit of fermion parity is transferred from the inner edge of the disk to the junction on the outer edge. The anomaly in (1-3) is similar to the chiral anomaly for edge states in the quantum Hall effect, where bulk currents violate charge conservation at the edge. Note however that though (1-3) is invariant under $\phi \to \phi + 2\pi$, the global Hamiltonian, which includes the bulk QSHI, is physically distinct when $\Phi = 0$ and $h/2e$.

The local conservation of fermion parity has important consequences for the current and noise in a S-QSHI-S junction. This is most striking near the degeneracy point for $\epsilon_0 \ll \Delta_0$ and $T \ll \Delta_0$. For the remainder of the paper we will focus on that regime. We will also consider the limit $L \ll v/\Delta_0$, where there is a single Andreev bound state and Eq. (1) applies, though the results can straightforwardly be generalized to the case with multiple Andreev levels, provided $T \ll v/L$. In this case, $N_0$ distinguishes two states, with Josephson currents $I_\pm = \pm I_0$ with

$$I_0(\phi) = \frac{1}{2} \sqrt{2e} \Delta_0 \sin \phi/2.$$  

(6)

In the absence of transitions that violate local fermion parity conservation there can be no transitions between $I_+$ and $I_-$, signaling a fractional Josephson effect.

Elastic scattering processes can be incorporated into the BdG Hamiltonian from the start, and will not lead to violations of the fermion parity. However, at finite temperature, inelastic processes[21, 22] can lead to a transition between $I_+$ and $I_-$, provided an available fermion is present to switch the fermion parity. This could be either due to a thermally excited quasiparticle or due to hopping from a bulk localized state. These processes, however, will be exponentially suppressed at low temperature. On a time scale longer than the switching time $T$, the current will thermalize, with an average value $I_0(\phi) \tanh \epsilon_0(\phi)/2T$.  

$$\langle I(\phi) \rangle = I_0(\phi) \tanh \epsilon_0(\phi)/2T.$$  

(7)

On shorter times, the current will exhibit telegraph noise, as it switches between $I_\pm$.

In order to model the inelastic processes responsible for the telegraph noise we consider the interaction of the Andreev level $\Gamma_0$ with a bath of fermions $c_n$(e.g. quasiparticles) and bosons $b_m$(e.g. phonons). We thus write

$$H = \epsilon_0 \Gamma_0 |\Gamma_0 + \sum_n E_n c_n^\dagger c_n + \sum_m \omega_m b_m^\dagger b_m$$

$$+ \sum_{mn} [(V_{nm}^1 c_n^\dagger b_m + V_{nm}^2 c_n b_m^\dagger) \Gamma_0 + h.c.].$$  

(8)

Here $E_n, \omega_m > 0$, and we have ignored terms which create (or annihilate) both fermions and bosons. The transition rates $\tau^\pm_\pm (\epsilon_0, T)$ between the states $N_0$ and $N_0 \pm 1$ follow from Fermi’s golden rule. For $\epsilon_0, T \ll \Delta_0$ we find

$$\tau^\pm_\pm = e^{\mp \epsilon_0 / 2T} \left( w_1(T)e^{\epsilon_0 / 2T} + w_2(T)e^{-\epsilon_0 / 2T} \right),$$  

(9)

where

$$w_{1,2}(T) = 2\pi \sum_{n,m} e^{-E_n/T} |V_{nm}^{1,2}|^2 \delta(E_n - \omega_m).$$  

(10)

If either the Zeeman term vanishes ($M_0 = 0$) or the system is symmetric under $x \to -x$, then $w_1(T) = w_2(T) \equiv w(T)$. We will assume this below, though the results are only slightly modified otherwise. $w(T)$ depends on the dominant source of fermions, which we take to be either thermally activated quasiparticles or Mott variable range hopping from bulk localized states.

$$w(T) \propto \begin{cases} e^{-\Delta_0 / T} & \text{quasiparticles,} \\ e^{-(T_0/T)^{1/3}} & \text{hopping.} \end{cases}$$  

(11)

$T_0$ depends on the density of states and localization length, and we assume the hopping is 2D.

The transition rate is exponentially suppressed for $T \to 0$. At sufficiently low temperature the resulting telegraph noise could be observed in the time domain. At higher temperature there is a signature in the noise spectrum $S(\omega)$. We determine $S(\omega)$ semiclassically by solving a kinetic equation for the probability $p(t)$ that $N_0 = 1$. This has the form $dp/dt = -(p - \bar{p})/\tau$, where $\tau = (1 + \exp \epsilon_0/T)^{-1} \sim 4w \cosh^2 \epsilon_0/2T$. $\bar{p} = (1 + \exp \epsilon_0/T)^{-1}$ follows from the detailed balance
condition $\tau_+ / \tau_- = e^{\epsilon_0 / T}$. Temporal correlations in $I(t)$ decay exponentially on a time scale $w^{-1}$, and the noise spectrum $S(\omega) = 2 \int_{-\infty}^{\infty} e^{i \omega t} \langle I(t) I(0) \rangle$ is given by \[ S(\omega) = \frac{4 T_0^2}{\cosh^4 \epsilon_0(\phi) / 2 T} \frac{\tau}{1 + \omega^2 \tau^2}. \] (12)

In the zero frequency limit we have
\[ S(\omega \to 0) = \frac{T_0^2}{w(T) \cosh^4 \epsilon_0(\phi) / 2 T}. \] (13)

For $D = 1$, these results are similar to the theory of a SQPC\[13, 21, 22\]. However, the current in Eq. 16 is half the value of a perfect single channel SQPC. A SQPC is like two copies of a S-QSHI-S junction. This leads to a fundamental difference, because in the SQPC there is no conservation law to prevent scattering between the $\pm I_0$ states, which can occur via low energy processes that transfer an electron between the two pairs. Elastic backscattering in the SQPC leads to an avoided crossing of the states near $E = 0$, and the Andreev states carry no current at $\phi = \pi$. Even if the transmission of the SQPC is perfect, inelastic processes will couple the states. For instance, near the degeneracy point $\epsilon_0(\phi) = 0$ spin flip scattering via the nuclear hyperfine interaction will lead to a finite lifetime for transitions, even at low temperature. It is also of interest to compare with the theory of Ref. 12. In that work, multi channel junctions were considered. Independence of the different channels requires translational symmetry, so inelastic scattering will lead to the violation of the conservation fermion parity within a given channel. Thus the low temperature behavior predicted by 11 and 13 is unique to the S-QSHI-S junction, and is a signature of the fermion parity anomaly.

We now briefly consider junctions at finite voltage bias. The analysis is similar to Ref. 13. There are two cases to consider, depending on $M_0$. For $M_0 = 0$, the perfect transmission of the edge states causes the Andreev levels to merge with the continuum levels. This leads to a finite DC current, which for $eV < \Delta_0$ can be understood semiclassically in terms of multiple Andreev reflections. For $w(T) \ll eV < \Delta_0$, the current is $I(V) = (2/\pi)I_c \text{sgn} V$, where $I_c = \sqrt{DE}\Delta_0/2$. For $M_0 > 0$, there is an energy gap $\delta$ separating the Andreev levels from the continuum, as in Fig. 2(b.d). For $w(T) \ll eV \ll \delta$ there will be a fractional AC Josephson current with frequency $eV/h \chi$. For $eV \sim \delta$ Landau-Zener tunneling processes through the gap $\delta$ will lead to a damping of the AC Josephson current as well as a finite DC current.

We close by discussing the feasibility of experiments using the QSHI recently achieved in HgCdTe quantum wells[10], which has a bulk gap of order 20 meV\[23\]. The desired geometry would be similar to Ref. 24, where a 2D InAs quantum well was contacted with Nb. The proximity induced gap will depend on the contact. If optimized $\Delta_0$ could be of order the bulk gap of the superconductor. To determine the required junction size we use $v = 3.6 \text{ eV A}\chi$ and $\Delta_0 = .1 \text{ meV}$. Then, $L \lesssim v / \Delta_0 \sim 3 \mu \text{m}$ sets the scale for having a single Andreev level. The simplest experiment would be to study a single current biased junction, which is predicted to have a critical current $I_c = e\Delta_0 / 2 \sim 10 \text{nA}$, which is half the value of a perfect single channel SQPC. Measuring the equilibrium noise at $\phi \sim \pi$ requires an inductive measurement on a ring\[14\]. The physics at $M_0 \neq 0$ requires a magnetic field in the junction region. For an appropriately aligned field, a field induces a gap $B \times (3.1 \text{ meV} / T) \chi$ in the edge states, so a field of order $.03T$ could suppress the normal state transmission $D$ as well as the magnitude of the Josephson current.

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