On the coalition number of graphs

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Abstract

Let $G$ be a graph with vertex set $V$. Two disjoint sets $V_1, V_2 \subseteq V$ form a coalition in $G$ if none of them is a dominating set of $G$ but their union $V_1 \cup V_2$ is. A vertex partition $\Psi = \{V_1, \ldots, V_k\}$ of $V$ is called a coalition partition of $G$ if every set $V_i \in \Psi$ is either a dominating set of $G$ with the cardinality $|V_i| = 1$, or is not a dominating set but for some $V_j \in \Psi$, $V_i$ and $V_j$ form a coalition. The maximum cardinality of a coalition partition of $G$ is called the coalition number of $G$, denoted by $C(G)$. A $C(G)$-partition is a coalition partition of $G$ with cardinality $C(G)$. Given a coalition partition $\Psi = \{V_1, V_2, \ldots, V_r\}$ of $G$, a coalition graph $CG(G, \Psi)$ is associated on $\Psi$ such that there is a one-to-one correspondence between its vertices and the members of $\Psi$. Two vertices of $CG(G, \Psi)$ are adjacent if and only if the corresponding sets form a coalition in $G$.

In this paper, we first show that for any graph $G$ with $\delta(G) = 1$, $C(G) \leq 2\Delta(G) + 2$, where $\delta(G)$ and $\Delta(G)$ are the minimum degree and the maximum degree of $G$, respectively. Moreover, we characterize all graphs $G$ with $\delta(G) \leq 1$ and $C(G) = n$, where $n$ is the number of vertices of $G$. Furthermore, we characterize all trees $T$ with $C(T) = n$ and all trees $T$ with $C(T) = n - 1$. This solves partially one of the open problem posed in [1]. On the other hand, we theoretically and empirically determine the number of coalition graphs that can be defined by all coalition partitions of a given path $P_k$. Furthermore, we show that there is no universal coalition path, a path whose coalition partitions defines all possible coalition graphs. These solve two open problems posed by Haynes et al. [2].

Keywords: Coalition number; Domination number; Coalition partition; Coalition graphs.

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1 Introduction

Let $G$ be a graph with vertex set $V$. A set $S \subseteq V$ is called a dominating set of $G$ if any vertex of $V - S$ is adjacent to a vertex of $S$. The minimum cardinality of a dominating set of $G$ is called the domination number of $G$, denoted by $\gamma(G)$. We refer the reader to the book [4] to study an overview of dominating sets.

The term coalition is used when two or more groups of people or military forces or political parties, etc., temporarily agree with each other to achieve a common goal. There are many examples of coalitions in the real world, including countries with coalition cabinets, coalitions of economic agents, coalitions in civil societies. In particular, in parliamentary systems of government when political parties do not win a majority in general election, two or more political parties form coalitions. Turkey, Thailand, Greece, Germany, France, Austria, Australia are examples of countries that operate with a coalition cabinet. In this paper, it is assumed that there is a coalition between only two groups.

In 2020, Haynes et al. [1] presented the graph theory model of the coalition. Then, they introduced the concept of coalition in graphs. A pair of sets $V_1, V_2 \subseteq V$ is called a coalition in $G$ if none of them is a dominating set of $G$ but $V_1 \cup V_2$ is. It is called that $V_1$ and $V_2$ form a coalition. It is so-called that $V_1$ and $V_2$ are coalition partners. A vertex partition $\Psi = \{V_1, \ldots, V_k\}$ of $V$ is called a coalition partition of $G$ if every set $V_i \in \Psi$ is either a dominating set of $G$ with the cardinality $|V_i| = 1$, or is not a dominating set but for some $V_j \in \Psi$, $V_i$ and $V_j$ form a coalition. The maximum cardinality of a coalition partition of $G$ is called the coalition number of $G$, denoted by $\mathcal{C}(G)$. A coalition partition of $G$ with the cardinality $\mathcal{C}(G)$ is called a $\mathcal{C}(G)$-partition. For a coalition partition $\Psi = \{V_1, V_2, \ldots, V_r\}$ of $G$, a coalition graph $CG(G, \Psi)$ is associated there is a one-to-one correspondence between its vertices and the members of $\Psi$. Two vertices of $CG(G, \Psi)$ are adjacent if and only if the corresponding sets form a coalition in $G$. More results on the concept of coalition given by [1, 2, 3].

Let the graph $G$ has $n$ vertices. Haynes et al. [1], posed the following open problem.

**Problem 1.** When is $\mathcal{C}(G) = n$?

Let $\mathcal{F}$ be a family of graphs consisting of 18 graphs: $K_1, K_2, \overline{K_2}, K_1 \cup K_2, P_3, K_3, K_{1,3}, 2K_2, P_4, C_4, F_1, K_4 - e, P_2 \cup P_3, F_2, B, P_5, S_{1,2}$, and $S_{2,2}$. Figure 1 shows some of the graphs of $\mathcal{F}$.

![Figure 1: Some of the graphs of $\mathcal{F}$](image)

In [2], Haynes et al. introduced the family $\mathcal{F}$ of graphs and showed that if $\Psi$ is a coalition partition of a path $P_k$, then $CG(P_k, \Psi) \in \mathcal{F}$. A path is called a universal coalition path if all 18 graphs of $\mathcal{F}$. 

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can be defined by the coalition partitions of the path. In [2], Haynes et al. posed the following open problems.

**Problem 2.** Given a positive integer $k$, how many coalition graphs can be defined by the coalition partitions of path $P_k$?

**Problem 3.** Does there exist a positive integer $k$ such that all 18 graphs of $F$ can be defined by the coalition partitions of $P_k$? If so, what is the smallest universal coalition path?

In this paper, we characterize all graphs $G$ with $δ(G) = 1$ and $C(G) = n$. Moreover, we characterize all trees $T$ with $C(T) = n$ and all trees $T$ with $C(T) = n − 1$. This solves part of Problem 1. On the other hand, we solve Problem 2 and Problem 3. In particular, we theoretically and empirically determine the number of coalition graphs that can be defined by the coalition partitions of path $P_k$. Consequently, we show that there is no universal coalition path.

## 2 Trees with $C(T) = n$

Let $G$ be a graph with vertex set $V$ and $|V| = n$. For a vertex $v ∈ V$, the open neighborhood $N(v)$ of $v$ is the set of all vertices adjacent to $v$, and the closed neighborhood of $v$ is $N[v] = N(v) ∪ \{v\}$. The degree of a vertex $v$, denoted by $deg(v)$, is $|N(v)|$. If $deg(v)=1$, then $v$ is called a leaf, and the vertex adjacent to a leaf is called the support vertex of the leaf. A vertex of $G$ of degree $n−1$ is called a full vertex. In this section, we first show that the coalition number of any graph $G$ with $δ(G) ≤ 1$ and with no full vertices is at most $2Δ(G) + 2$. We show that this bound is sharp. Finally, we characterize all trees $T$ with $C(T) = n$. First, we present the following lemma given in [1].

**Lemma 1** ([1]). Let $G$ be graph with a $C(G)$-partition $Ψ$ and $S ∈ Ψ$. Then $S$ is in at most $Δ(G) + 1$ coalitions.

Now, we prove the following theorem.

**Theorem 1.** $C(G) ≤ 2Δ(G) + 2$ for any graph $G$ with $δ(G) = 1$.

**Proof.** Let $x$ be a leaf of $G$ and $y ∈ N(x)$. Let $Ψ$ be a $C(G)$-partition. Suppose first that $x, y ∈ V′$ for some $V′ ∈ Ψ$. Then, if $A ∈ Ψ$ and $B ∈ Ψ$ form a coalition, we have $A = V′$ or $B = V′$. By Lemma 9, since $V′$ is in at most $Δ(G) + 1$ coalitions, we have $C(G) ≤ Δ(G) + 1 + 1 = Δ(G) + 2$. Then, the result holds.

Now, we suppose that $x$ and $y$ belong to the different sets of $Ψ$. Let $U, W ∈ Ψ$ with $U ≠ W$ such that $x ∈ U$ and $y ∈ W$. Suppose that $U$ and $W$ form a coalition. Since $x ∈ U$ and $y ∈ W$, if $A ∈ Ψ$ and $B ∈ Ψ$ form a coalition, we must have $A ∈ \{U, W\}$ or $B ∈ \{U, W\}$. By Lemma 9, each of $U$ and $W$ is in at most $Δ(G) + 1$ coalitions. Then, we have $C(G) ≤ Δ(G) + Δ(G) + 2 = 2Δ(G) + 2$. Now, we suppose that $U$ and $W$ do not form a coalition. Then, there is a vertex $v$ of $G$ that is not dominated by $U ∪ W$. Note that $U ∪ W$ contains no vertex of $N[v]$. Let $P = \{S ∈ Ψ \mid N[v] ∩ S = \emptyset\}$. It is clear that $Ψ = \{U, W\} ∪ P$. Since $|P| ≤ Δ(G) + 1$, we have $C(G) ≤ 2 + Δ(G) + 1 = Δ(G) + 3 ≤ 2Δ(G) + 2$. This completes the proof.

□

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In [1], the authors show that for \( n \geq 10 \), \( \mathcal{C}(P_n) = 6 \). Then, clearly the path \( P_n \) with \( n \geq 10 \), achieve the bound \( 2\Delta(G) + 2 \), and therefore, the bound is sharp.

Now, by Theorem 1, we have the corollary below.

**Corollary 1.** For any tree \( T \), \( \mathcal{C}(T) \leq 2\Delta(T) + 2 \).

Now, we characterize all graphs \( G \) of order \( n \) and \( \delta(G) = 0 \) such that \( \mathcal{C}(G) = n \).

**Theorem 2.** For any graph \( G \) of order \( n \) with \( \delta(G) = 0 \) and \( \mathcal{C}(G) = n \), if and only if \( G \cong K_1 \cup K_{n-1} \).

**Proof.** Clearly, \( \mathcal{C}(K_1 \cup K_{n-1}) = n \). Conversely, suppose that \( G \) is a graph of order \( n \) with \( \delta(G) = 0 \) and \( \mathcal{C}(G) = n \). Clearly \( \{\{v\} \mid v \in V\} \) is a \( \mathcal{C}(G) \)-partition. Let \( x \) be an isolated vertex of \( G \). Then there are no vertices \( v, v' \in V(G)\setminus\{x\} \) such that \( \{v\} \) and \( \{v'\} \) form a coalition. Therefore, \( \{v\} \) and \( \{x\} \) form a coalition for each \( v \in V(G)\setminus\{x\} \). Thus, \( V(G)\setminus\{x\} \) is a clique in \( G \) of cardinality \( n-1 \). This completes the proof. \( \square \)

**Definition 1.** (The family \( \mathcal{F}_1 \)) Let \( x \) be a leaf of a graph \( G \) and \( y \in N(x) \). The family \( \mathcal{F}_1 \) contains the graph \( G \) if and only if all of the following properties hold.

1. \( V(G) = \{x, y, w, w_1, \ldots, w_k\} \), \( N(w) = \{w_1, \ldots, w_k\} \) for some integer \( k \geq 1 \) and \( y \notin N(w) \).
2. \( N(w) \) is a disjoint union of two subsets \( P \) and \( Q \) such that every vertex in \( P \) is adjacent to any other vertex in \( N(w) \), and for every \( q \in Q \), every vertex in \( N(w) \setminus \{q\} \) is adjacent to \( y \) or \( q \).

Based on the definition of \( \mathcal{F}_1 \), we can easily see that every subset \( \{v\} \), where \( v \in N[w] \) forms a coalition with \( \{x\} \) or \( \{y\} \). Then, for any \( G \in \mathcal{F}_1 \), \( \mathcal{C}(G) = n \). We record this as Observation 1 and then prove Lemma 2.

**Observation 1.** \( \mathcal{C}(G) = n \) if \( G \in \mathcal{F}_1 \).

**Lemma 2.** For any graph \( G \) with \( \delta(G) = 1 \) and with no full vertex, if \( \mathcal{C}(G) = n \), then \( G \in \mathcal{F}_1 \).

**Proof.** Let \( x \) be a leaf of \( G \) and \( y \in N(x) \). Suppose that \( \mathcal{C}(G) = n \) and let \( \Psi \) be an \( n \)-partition of \( G \). We show that \( G \in \mathcal{F}_1 \). Let \( \mathcal{A} = \{\{x\}, \{y\}\} \). We know that \( \{x\} \in \Psi \) and \( \{y\} \in \Psi \). Suppose that \( \{x\} \) and \( \{y\} \) form a coalition in \( G \). Since \( x \) is a leaf and \( N(x) = \{y\} \), all members of \( V - \{x, y\} \) must be adjacent to \( y \). Thus, the degree of \( y \) is \( n - 1 \), and therefore, \( y \) is a full vertex of \( G \) which is a contradiction. Hence, \( \{x\} \) and \( \{y\} \) do not form a coalition. Let \( w \) be a vertex which is not adjacent to \( x \) and \( y \). Let \( N(w) = \{w_1, \ldots, w_k\} \). Note that \( k \leq \Delta(G) \). Let \( \mathcal{W} = \{S \in \Psi \mid N[w] \cap S \neq \emptyset\} \). Since \( \mathcal{C}(G) = n \), \( \mathcal{W} = \{\{w\}, \{w_1\}, \ldots, \{w_k\}\} \). Now, we show that \( V = \{x, y, w, w_1, \ldots, w_k\} \). On the contrary, let \( v \in V \setminus \{x, y, w, w_1, \ldots, w_k\} \). Note that \( \{v\} \) is not in coalition with any \( \{z\} \) for \( z \in N[w] \) such that \( \deg(x) = 1 \). Moreover, if \( \{v\} \) is in coalition with \( \{x\} \) or \( \{y\} \), then \( v \in N(w) \) (as \( w \) is not dominated by \( \{x, y\} \)). This is a contradiction. Therefore, \( V = \{x, y, w, w_1, \ldots, w_k\} \). Note that since \( w \) is not dominated by \( \{x\} \cup \{y\} \), we have \( y \notin N(w) \). Since \( G \) has no full vertices, every set of \( \Psi \) must be a coalition partner of some other set of \( \Psi \). Since \( \mathcal{A} = \{\{x\}, \{y\}\} \), each member of \( \mathcal{W} \) must be in coalition with \( \{x\} \) or \( \{y\} \), and there is no coalition between the members of \( \mathcal{W} \). In particular,
\{w\} is a coalition partner of each of \{x\} and \{y\}. Let \( P \subseteq W - \{\{w\}\} \) be the collection of all sets of \( \Psi \) that are in coalition with \{x\}. Let \( Q = (W - \{\{w\}\}) - P \). Then, every member of \( Q \) is in coalition with \{y\}. For any \( \{w_i\} \in P \), since \( \{w_i\} \) is in coalition with \{x\}, each vertex in \( N(w) \setminus \{w_i\} \) is adjacent to \( w_i \). Moreover, for any \( \{w_j\} \in Q \), each vertex in \( N(w) \setminus \{w_j\} \) is adjacent to \( y \) or \( w_j \) due to the fact that \( \{w_j\} \) and \( \{y\} \) form a coalition. According to the definition of family \( \mathcal{F}_1 \), we easily see that \( G \in \mathcal{F}_1 \).

Using Observation 1 and Lemma 2, the theorem follows.

**Theorem 3.** Let \( G \) be a graph with \( \delta(G) = 1 \) and with no full vertex. Then \( \mathcal{C}(G) = n \) if and only if \( G \in \mathcal{F}_1 \).

In the following theorem, we characterize for the graphs having one full vertex.

**Theorem 4.** Let \( G \) be a graph of order \( n \) with \( \delta(G) = 1 \) and one full vertex. Then \( \mathcal{C}(G) = n \) if and only if \( G \) is obtained by attaching a leaf to a vertex of \( K_{n-1} \).

**Proof.** Clearly, the sufficiency follows. To prove the necessity, let \( \mathcal{C}(G) = n \) and \( \Psi \) be a \( \mathcal{C}(G) \)-partition. Let \( u, v \in V \) such that \( \text{deg}(u) = 1 \) and \( \text{deg}(v) = n - 1 \), respectively, and \( G' = G - v \). To complete the proof of the necessity, we need only to show that \( G' \cong K_1 \cup K_{n-2} \). Since \( v \) is a full vertex, the set \( \Psi' = \Psi - \{\{v\}\} \) is a \( \mathcal{C}(G') \)-partition, and therefore, \( \mathcal{C}(G') = n - 1 \). Since \( \delta(G') = 0 \), by Theorem 2, \( G' \cong K_1 \cup K_{n-2} \). So the necessity follows.

Now, we characterize all trees \( T \) with \( \mathcal{C}(T) = n \).

**Theorem 5.** For any tree \( T \) of order \( n \), \( \mathcal{C}(T) = n \) if and only if \( T \in \{P_i\}_{i=1}^4 \).

**Proof.** Obviously, \( \mathcal{C}(T) = n \) when \( T \in \{P_i\}_{i=1}^4 \). Conversely, let \( \mathcal{C}(T) = n \). We may assume that \( n \geq 2 \). If there is a full vertex in \( T \), then \( T \cong K_{1,n-1} \). In such a situation, it is easily observed that \( n \in \{2, 3\} \). So, \( T \in \{P_2, P_3\} \). Now let \( T \) do not have any full vertex. In particular, we have \( n \geq 4 \). We also have \( T \in \mathcal{F}_1 \) by Lemma 2. We now consider the structure of the trees in \( \mathcal{F}_1 \). If \( \text{deg}(w) = 1 \), then \( y \) is adjacent to the only neighbor of \( w \). Hence, \( T \cong P_4 \). So, we assume that \( \text{deg}(w) \geq 2 \). We have \( P = \emptyset \) because \( T \) is a tree. Since \( T \) is a tree, \( y \) is adjacent to a vertex in \( N(w) \), say \( w_1 \). If there exists a vertex \( w_1 \in N(w) \setminus \{w_1\} \), then \( w_1 \) is dominated by \( \{y, w_1\} \). This leads to the existence of a cycle, a contradiction. Therefore, \( \text{deg}(w) = 1 \), and so \( T \cong P_4 \).

## 3 Trees with \( \mathcal{C}(T) = n - 1 \)

In this section, we characterize all trees \( T \) with \( \mathcal{C}(T) = n - 1 \). Note that the only tree \( T \) with \( \mathcal{C}(T) < n \) and \( n = 4 \) is the star \( K_{1,3} \). Now, by Theorem 5, we have the following corollary.

**Corollary 2.** For any tree \( T \) of order \( n \geq 5 \), \( \mathcal{C}(T) \leq n - 1 \).
Let $T$ be a tree of order $n$ with no full vertex for which $\mathcal{C}(T) = n - 1$. Let $x$ be a leaf of $T$ and $y \in N(x)$. Let $\Psi$ be a $\mathcal{C}(T)$-partition. Since $\mathcal{C}(T) = n - 1$, the cardinality of all members of $\Psi$ is one except one member whose cardinality is two. We assume that $U$ is a member of $\Psi$ with $|U| = 2$. We assume that $\Psi = \{U, V_0, \ldots, V_{n-3}\}$ such that $|V_i| = 1$ for any $0 \leq i \leq n - 3$.

**Lemma 3.** It holds that $U \neq \{x, y\}$.

**Proof.** On contrary, assume that $U = \{x, y\}$. It is not difficult to see that every subset $\{z\}$, in which $z \not\in \{x, y\}$, forms a coalition with $\{x, y\}$. In particular, $\{z, y\}$ is a dominating set for all $z \not\in \{x, y\}$.

We set $V(G) \setminus U = A \cup B$, in which $A$ is the set of vertices not dominated by $y$ and $B = (V(G) \setminus U) \setminus A$. Note that $A \neq \emptyset$ because $y$ is not a full vertex. On the other hand, $A$ is a clique since $\{a\}$ and $\{x, y\}$ form a coalition, for every $a \in A$. Moreover, $B \neq \emptyset$ because $T$ is connected. For any $b \in B$, the subset $\{b\}$ forms a coalition with $\{x, y\}$. This implies that all vertices in $A$ are adjacent to each vertex in $B$. Therefore, $\{a, x\}$ is a dominating set for every $a \in A$. In fact, we have shown that for any vertex $v$, there exists a vertex $u$ such that $\{u, v\}$ is a dominating set in $T$. Therefore, $\mathcal{C}(T) = n$. This is a contradiction. Thus, $U \neq \{x, y\}$. \hfill $\Box$

Note that if $U \cap \{x, y\} = \emptyset$, then $\{x\} \in \Psi$ and $\{y\} \in \Psi$. Since $T$ has no full vertices, $\{x\}$ and $\{y\}$ do not form a coalition. Hence, we have the following observation.

**Observation 2.** If $U \cap \{x, y\} = \emptyset$, then $\{x\}, \{y\} \in \Psi$. Moreover, $\{x\}$ and $\{y\}$ do not form a coalition.

In Lemma 4 and Lemma 5, we assume that $w$ is a vertex of $T$ not dominated by $\{x, y\}$, and $N(w) = \{w_1, \ldots, w_k\}$.

**Lemma 4.** If $U \cap \{x, y\} = \emptyset$, and for some $i$ and $j$ with $i \neq j$, $U = \{w_i, w_j\}$, then $T = P_3$.

**Proof.** Without loss of generality, assume that $U = \{w_1, w_2\}$. Suppose that $|N(w)| \geq 3$. Then, $U$ does not form a coalition with $\{x\}$; otherwise $w_kw_1ww_k$ or $w_kw_2ww_k$ would be a cycle in $T$, a contradiction. Hence, $U$ is in coalition with $\{y\}$. Note that at least one of the sets $\{w_1\}$ and $\{w_2\}$ is not in coalition with $\{y\}$ because otherwise, we can split $U$ into two sets $\{w_1\}$ and $\{w_2\}$ that results in a new coalition partition of $T$ of size $n$ which contradicts $\mathcal{C}(T) = n - 1$. Now, consider a set $\{w_i\} \in \Psi$ with $i \geq 3$. Since $T$ has no cycle, $\{w_i\}$ and $\{x\}$ do not form a coalition. Then, $\{w_i\}$ and $\{y\}$ form a coalition. Therefore, we must have $N(w) - \{w_i\} \subseteq N(y)$. This leads to the cycle $w_1ww_2yw_1$ which is a contradiction. Then, we must have $|N(w)| = 2$. In such a situation, at least one of $\{w_1\}$ and $\{w_2\}$ is not in coalition with $\{y\}$, otherwise we have the cycle $w_1ww_2yw_1$ in $T$ which is impossible. Suppose now that none of $\{w_1\}$ and $\{w_2\}$ is in coalition with $\{y\}$. So, none of $\{y, w_1\}$ and $\{y, w_2\}$ is a dominating set. In particular, there is a vertex $z$ with no neighbor in $\{y, w_1\}$. Suppose that $z \neq w_2$. Since $\{y, w_1, w_2\}$ is a dominating set (recall that $U$ and $\{y\}$ form a coalition), $z$ is adjacent to $w_2$. Because $\deg(x) = 1$, $\{z\}$ must be in coalition with $\{x\}$ or $\{y\}$. Since $w$ is not dominated by $\{x, y\}$, $w$ must be adjacent to $z$. This leads to the cycle $zw_2wz$, a contradiction. Therefore, $z = w_2$. By symmetry, $w_1$ is the only vertex which is not dominated.
by \( \{y, w_2\} \). Therefore, \( yw_1, yw_2 \notin E(T) \). If there exists a vertex \( v \in V(T) \setminus \{x, y, w, w_1, w_2\} \), then \( \{v\} \) necessarily forms a coalition with \( \{x\} \) or \( \{y\} \). In both cases, we have the cycle \( vw_1w_1v \) in \( T \), a contradiction. Therefore, \( V(T) = \{x, y, w, w_1, w_2\} \). This shows that \( T \) is disconnected which is impossible.

So, we may assume that \( \{w_1\} \) is in coalition with \( \{y\} \) but \( \{w_2\} \) is not. So, we have the path \( P_3 : xyw_2w_1 \) in \( T \). If there is a vertex \( v \notin \{x, y, w, w_1, w_2\} \), \( \{v\} \) must be in coalition with \( \{y\} \). This leads to the triangle \( w_1vw \), a contradiction. So, we end up with \( T \cong P_3 : xyw_2w_1 \). This completes the proof. \( \square \)

Similar to the proof of Lemma 4, we can prove the following result.

**Lemma 5.** If \( U \cap \{x, y\} = \emptyset \), and for some \( i, U = \{w, w_i\} \), then \( T = P_5 \).

A double star is a tree that contains exactly two nonleaf vertices. A double star in which one support vertex is adjacent to \( p \) and the other support vertex is adjacent to \( q \) leaves is denoted by \( S_{p,q} \). Now, we prove the following lemma.

**Lemma 6.** If \( U = \{y, z\} \) with \( z \neq x \), then \( T \in \{P_5, S_{1,2}\} \).

**Proof.** If \( \{x\} \) is in coalition with \( \{y, z\} \), then \( \{x, y, z\} \) is a dominating set. Since \( y \) is the only neighbor of \( x \), it follows that \( \{y, z\} \in \Psi \) is a dominating set, a contradiction. Hence, \( \{x\} \) and \( \{y, z\} \) do not form a coalition. Let \( w \) be a vertex of \( T \) which is not dominated by \( \{x, y, z\} \). Let \( N(w) = \{w_1, \ldots, w_k\} \). Now, we claim that \( \Psi = \{\{x\}, \{y, z\}, \{w\}, \{w_1, \ldots, w_k\}\} \). In what follows, we prove the claim. Since \( \{w\} \in \Psi \), it must be in coalition with another member of \( \Psi \). If it is in coalition with a subset \( \{r\} \) such that \( r \notin \{x, y\} \), we have a contradiction to the fact that \( \deg(x) = 1 \). Therefore, \( \{w\} \) is in coalition with \( \{x\} \) or \( \{y, z\} \). If \( \{w\} \) forms a coalition with \( \{x\} \), then it forms a coalition with \( \{y, z\} \) as well. Therefore, \( \{w\} \) and \( \{y, z\} \) form a coalition. This shows that any vertex not dominated by \( \{y, z\} \) is adjacent to \( w \). If there exists a vertex \( v \in V(T) \setminus (\{x, y, z\} \cup N[w]) \), then \( \{v\} \) and \( \{y, z\} \) form a coalition. Consequently, \( w \) has a neighbor in \( \{v, y, z\} \) which is impossible. Therefore, \( V(T) = \{x, y, z\} \cup N[w] \). So, \( \Psi = \{\{x\}, \{y, z\}, \{w\}, \{w_1, \ldots, w_k\}\} \). This proves the claim. It is notable that \( k + 3 = n - 1 \). It is clear that \( \{x\} \) must be in coalition with at least one of the sets \( \{w\}, \{w_1\}, \ldots, \{w_k\} \). Since \( w \) is not dominated by \( \{y, z\} \), \( z \) is not adjacent to \( w \), and therefore, \( \{x, w\} \) is not a dominating set of \( T \). This implies that \( \{x\} \) and \( \{w\} \) do not form a coalition. Now, suppose that \( k \geq 2 \). For any \( w_i \in N(w) \), if \( \{x\} \) and \( \{w_i\} \) form a coalition, then every \( w_j \in N(w) \) with \( w_j \neq w_i \) must be adjacent to \( w_i \) that forms the cycle \( w_jw_1w_1w \) in \( T \) which is a contradiction. Then, \( \{x\} \) can not form a coalition with any sets of \( \{w\}, \{w_1\}, \ldots, \{w_k\} \) which is a contradiction. Hence, we must have \( k = 1 \). We know that \( \{x\} \) must be in coalition with \( \{w_1\} \). Now \( \{x, w_1\} \) is a dominating set of \( T \). Hence, \( z \) must be adjacent to \( w_1 \). Now, \( T \) has the following edges: \( xy, w_1w_1, zw_1 \). Since \( T \) is a connected graph, we must have exactly one of the edges of \( yw_1 \) and \( zy \). Hence, \( T = P_5 \) or \( T = S_{1,2} \). This completes the proof. \( \square \)

**Lemma 7.** If \( U = \{x, z\} \) and \( z \neq y \), then \( T \in \{P_5, P_6, S_{1,2}\} \)
Proof. To prove the claim, we consider two cases as follows:

Case 1. \( \{x, z\} \) and \( \{y\} \) form a coalition.

Suppose that \( z \notin N(y) \). Since \( \{x, z\} \) is not a dominating set of \( T \), there is a vertex \( a \) which is not dominated by \( \{x, z\} \). Then, \( a \) is not adjacent to \( z \). Since \( \{x, z\} \) and \( \{y\} \) form a coalition, the vertex \( a \) must be adjacent to \( y \). So, \( a, x \in N(y) - N(z) \). If there exists a vertex \( a' \in N(y) - N(z) \) different from \( a \) and \( x \), then we observe that \( \{a'\} \) and \( \{x, z\} \) must be in coalition, necessarily. Since \( a \) is not dominated by \( \{x, z\} \), it follows that \( aa' \in E(T) \). This leads to the triangle \( aa'y \), which is impossible. This shows that \( N(y) - N(z) = \{x, a\} \).

On the other hand, \( |N(y) \cap N(z)| \leq 1 \) because \( T \) is a tree. Suppose first that \( |N(y) \cap N(z)| = 0 \). Since \( T \) has no isolated vertices, it guarantees that \( |N(z) - N(y)| \geq 1 \). Assume that \( p, p' \in N(z) - N(y) \) are two distinct vertices. Then, any of \( \{p\} \) and \( \{p'\} \) is in coalition with \( \{y\} \) or \( \{x, z\} \). Note that \( p' \) is not dominated by \( \{p, y\} \). Therefore, \( \{p\} \) is in coalition with \( \{x, z\} \). Moreover, \( \{p'\} \) is in coalition with \( \{x, z\} \) by a similar fashion. This implies that \( a \) is adjacent to both \( p \) and \( p' \). Now, the existence of the cycle \( apzp'p/a \) is a contradiction. Since \( |N(z) - N(y)| = 1 \). Since \( \{x, z\} \) and \( \{y\} \) form a coalition, every vertex not in \( \{x, y, z\} \) is adjacent to \( y \) or \( z \). So, the above argument shows that \( n = 5 \). So, \( T = P_5: xyazp \). Suppose now that \( |N(y) \cap N(z)| = 1 \) and that \( r \in N(y) \cap N(z) \). Suppose to the contrary that there exists a vertex \( p \in N(z) - N(y) \). Since \( p \) is not dominated by \( \{r, y\} \), then \( \{r\} \) must be in coalition with \( \{x, z\} \). In such a situation, \( a \) has no neighbor in \( \{x, r, z\} \). This is impossible. Therefore, \( N(z) - N(y) = \emptyset \). Thus, \( T \cong S_{1,2} \).

Now, suppose that \( z \in N(y) \). Note that since \( T \) has no full vertices, we have \( |N(z) - N[y]| \geq 1 \). Similar to the above arguments, we can prove that there exists a vertex \( b \in N(y) - N[z] \). Since \( x \in N(y) - N[z] \), we have \( x, b \in N(y) - N[z] \). Now, let \( b' \in N(y) - N[z] \) different from \( x \) and \( b \), and let \( z' \in N(z) - N[y] \). If \( \{b'\} \) and \( \{x, z\} \) form a coalition, since \( b \) is not dominated by \( \{x, z\} \), it follows that \( bb' \in E(T) \). This leads to the triangle \( bb'yb \), which is impossible. Then, \( \{b'\} \) and \( \{y\} \) must form a coalition. Since \( z' \) is not dominated by \( \{y\} \), we must have \( z'b' \in E(T) \). This leads the cycle \( z'b'yzz' \), which is impossible. Hence, \( N(y) - N[z] = \{x, b\} \).

On the other hand, \( |N(y) \cap N(z)| = 0 \) because \( T \) is a tree and \( z \in N(y) \). Assume that \( z'' \in N(z) - N[y] \) different from \( z' \). Then, any of \( \{z''\} \) is in coalition with \( \{y\} \) or \( \{x, z\} \). Note that \( z'' \) is not dominated by \( \{z', y\} \). Therefore, \( \{z''\} \) is in coalition with \( \{x, z\} \). Moreover, \( \{z''\} \) is in coalition with \( \{x, z\} \) by a similar fashion. This implies that \( b \) is adjacent to both \( z' \) and \( z'' \). Now, the existence of the cycle \( bz'zz''b \) is a contradiction. Therefore, \( |N(z) - N[y]| = 1 \). These implies that \( T = S_{1,2} \).

Case 2. \( \{x, z\} \) and \( \{y\} \) do not form a coalition.

Therefore, there is a vertex \( w \) which is not dominated by \( \{x, y, z\} \). Let \( N(w) = \{w_1, \ldots, w_k\} \). Note that \( w \) is not adjacent to \( y \) and \( z \). Now, we claim that \( \{w\} \) is in coalition with \( \{x, z\} \). To prove the claim, suppose that \( \{w\} \) is not in coalition with \( \{x, z\} \). So, it must be in coalition with \( \{y\} \) due to the fact that \( x \) is a leaf. Since \( \{w, x, z\} \) is not a dominating set, there exists a vertex \( r \) with no neighbor in \( \{w, x, z\} \) which is dominated by \( \{w, y\} \). Since \( \{w, x, z\} \) does not dominate \( r \), it follows
that \( ry \in E(T) \) and \( rw \notin E(T) \). On the other hand, \( \{ r \} \) must form a coalition with \( \{ y \} \) or \( \{ x, z \} \) since \( \text{deg}(x) = 1 \). This contradicts the fact that \( w \) has no neighbor in \( \{ r, x, y, z \} \). Therefore, \( \{ w \} \) is in coalition with \( \{ x, z \} \). Suppose that \( k \geq 3 \). If \( \{ w_1 \} \) and \( \{ y \} \) form a coalition, since \( T \) is a tree, every \( w_j \) with \( j \neq i \) must be adjacent to \( y \). Since \( k \geq 3 \), this makes a cycle in \( T \) which is a contradiction. If \( \{ w_1 \} \) and \( \{ x, z \} \) form a coalition, every \( w_j \) with \( j \neq i \) must be adjacent to \( z \). Since \( k \geq 3 \), this makes a cycle in \( T \) which is a contradiction. Hence, we must have \( k = 2 \) or \( k = 1 \). Suppose that \( k = 2 \). If \( \{ w_1 \} \) and \( \{ y \} \) form a coalition, then \( w_2 \) is adjacent to \( y \). Hence, \( \{ w_2 \} \) must be in coalition with \( \{ x, z \} \), and therefore, \( w_1 \) is adjacent to \( z \). Hence, \( T = P_6 \). Now, suppose that \( k = 1 \). If \( \{ w_1 \} \) and \( \{ y \} \) from a coalition, \( z \) must be adjacent to \( w_1 \) or \( y \), and therefore, \( T = S_{1,2} \) or \( T = P_5 \). With the similar arguments, if \( \{ w_1 \} \) and \( \{ x, z \} \) from a coalition, we have \( T = S_{1,2} \) or \( T = P_5 \). \( \square \)

Note that if a tree \( T \) with \( n \geq 3 \) vertices contains some full vertex, then clearly \( T \) is the star \( K_{1,n-1} \). It is easy to see that \( C(K_{1,n-1}) = 3 \). Hence, by Lemmas 3, 4, 5, 6 and 7, we easily conclude the following theorem.

**Theorem 6.** For any tree \( T \) of order \( n \), \( C(T) = n - 1 \) if and only if \( T \in \{ P_5, S_{1,2}, P_6, K_{1,3} \} \).

Using Theorem 6, we easily conclude the following result.

**Corollary 3.** For any tree \( T \) of order \( n \) with \( n \geq 7 \), \( C(T) \leq n - 2 \).

## 4 Coalition graphs

In this section, we solve Problem 2 and Problem 3. In particular, we determine the number of the coalition graphs that can be defined by all coalition partitions of the path \( P_k \) for a given positive integer \( k \) (see Theorem 7) and then as a consequence of Theorem 7, we see that there is no universal coalition path.

A coalition graph corresponding to a coalition partition of a path is called a \( CP \)-graph [2]. We say a path \( P_k \) defines a \( CP \)-graph \( G \) if there is a coalition partition \( \Psi \) of \( P_k \) such that \( CG(P_k, \Psi) \) is isomorphic to \( G \). In [2], Haynes et al. proved the following lemma.

**Lemma 8 ([2]).** A graph \( G \) is a \( CP \)-graph if and only if \( G \in \mathcal{F} \).

According to Lemma 8, all 18 graphs of \( \mathcal{F} \) are \( CP \)-graph. Now, consider the \( CP \)-graph \( K_1 \). It is clear that the only path that defines the \( CP \)-graph \( K_1 \) is \( P_1 \). Hence, we have the following observation.

**Observation 3.** The \( CP \)-graph \( K_1 \) is only defined by \( P_1 \).

Now, we prove the following lemma.

**Lemma 9.** It holds that
- $P_2$ defines the CP-graph $\overline{K}_2$.
- $P_3$ defines the CP-graph $K_1 \cup K_2$.
- For any integer $k \geq 4$, $P_k$ does not define the CP-graphs $\overline{K}_2$ and $K_1 \cup K_2$.

**Proof.** For the path $P_2$, it is clear that the set $\{\{s_1\}, \{s_2\}\}$ is a coalition partition and the corresponding coalition graph is $\overline{K}_2$. For the path $P_3$, the set $\{\{s_1\}, \{s_2\}, \{s_3\}\}$ is a coalition partition such that none of the sets $\{s_1\}$ and $\{s_2\}$ is a dominating set but they form a coalition. Moreover, the set $\{s_2\}$ is a singleton dominating set. Hence, the coalition graph corresponding to $\{\{s_1\}, \{s_2\}, \{s_3\}\}$ is $K_2 \cup K_2$. Now, we prove the last item of the lemma. Since the graphs $\overline{K}_2$ and $K_1 \cup K_2$ have at least one isolated vertex, their corresponding coalition partition must contain at least one singleton dominating set. We know that for any $k \geq 4$, the domination number of $P_k$ is at least two. Therefore, every coalition partition of $P_k$ with $k \geq 4$ does not contain any singleton dominating set. Hence, the last item of lemma holds. 

In [1], Haynes et al. proved the following lemma.

**Lemma 10.** For any path $P_n$, it holds that 

$$C(P_n) = \begin{cases} n; & \text{if } n \leq 4 \\ 4; & \text{if } n = 5 \\ 5; & \text{if } 6 \leq n \leq 9 \\ 6; & \text{if } n \geq 10 \end{cases}$$

By Lemma 10, and according to the proof of Lemma 9, we easily conclude the following result.

**Corollary 4.** For any positive integer $k \leq 3$, the path $P_k$ does not define the CP-graphs $P_3, K_3, K_{1,3}, 2K_2, P_4, C_4, F_1, K_4 - e, P_2 \cup P_3, F_2, B, P_5, S_{1,2}, S_{2,2}$.

Now we prove several propositions that say about the paths and their corresponding CP-graphs.

**Proposition 1.** For any integer $k \geq 3$, $P_k$ defines the CP-graph $K_2$.

**Proof.** To prove the proposition, we provide a coalition partition $\Psi$ for $P_k$ with $k \geq 3$ that the corresponding coalition graph is $K_2$. Let $\Psi = \{A, B\}$, where $A = \{s_1, s_2, \ldots, s_{\lfloor \frac{k}{2} \rfloor}\}$ and $B = \{s_{\lfloor \frac{k}{2} \rfloor + 1}, \ldots, s_k\}$. It is easy to see that $CG(P_k, \Psi) \cong K_2$. 

**Proposition 2.** For any integer $k \geq 4$, $P_k$ defines the CP-graphs $P_3$ and $C_4$.

**Proof.** To prove the proposition, we provide two coalitions partitions $\Psi_1$ and $\Psi_2$ for $P_k$ with $k \geq 4$ that the corresponding coalition graphs are $P_3$ and $C_4$, respectively. Let $\Psi_1 = \{A, B, C\}$, where $A = \{s_1, s_2\}, B = \bigcup_{i=2}^{\lfloor \frac{k}{2} \rfloor} \{s_{2i-1}\}$ and $C = \bigcup_{i=2}^{\lfloor \frac{k}{2} \rfloor} \{s_{2i}\}$. It is easy to see that $CG(P_k, \Psi_1) \cong P_3$. Let $\Psi_2 = \{A, B, C, D\}$, where $A = \{s_1\}, B = \{s_2\}, C = \bigcup_{i=0}^{\lfloor \frac{k}{2} \rfloor - 2} \{s_{k-2i+1}\}$ and $D = \bigcup_{i=0}^{\lfloor \frac{k}{2} \rfloor - 2} \{s_{k-2i}\}$. It is easy to see that $CG(P_k, \Psi_2) \cong C_4$. 

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Proposition 3. For any integer $k \geq 5$, $P_k$ defines the CP-graph $F_1$.

Proof. To prove the proposition, we provide a coalition partition $\Psi$ for $P_k$ with $k \geq 5$ that the corresponding coalition graph is $K_2$. Let $\Psi = \{A, B, C, D\}$, where

- if $k$ is even, $A = \left(\bigcup_{i=2}^{k/2-2}\{s_{k-2i}\}\right) \cup \{s_1, s_k\}$, $B = \left(\bigcup_{i=2}^{k/2-2}\{s_{k-(2i-1)}\}\right) \cup \{s_2\}$, $C = \{s_{k-2}\}$ and $D = \{s_3, s_{k-1}\}$.

- if $k$ is odd, $A = \left(\bigcup_{i=2}^{(k-1)/2-1}\{s_{k-2i}\}\right) \cup \{s_1, s_k\}$, $B = \left(\bigcup_{i=2}^{(k-1)/2-1}\{s_{k-(2i-1)}\}\right) \cup \{s_2\}$, $C = \{s_{k-2}\}$ and $D = \{s_3, s_{k-1}\}$.

It is easy to see that $CG(P_k, \Psi) \cong F_1$.

Proposition 4. For any integer $k \geq 6$, $P_k$ defines the CP-graphs $K_{1,3}, K_3, K_4 - e, P_2 \cup P_3$.

Proof. To prove the proposition, we provide four coalition partitions $\Psi_1, \Psi_2, \Psi_3$ and $\Psi_4$ for $P_k$ with $k \geq 6$ that the corresponding coalition graphs are $K_{1,3}, K_3, K_4 - e$ and $P_2 \cup P_3$, respectively.

Let $\Psi_1 = \{A, B, C, D\}$, where $A = \{s_1, s_2, \ldots, s_{k-4}\} \cup \{s_k\}$, $B = \{s_{k-3}\}$, $C = \{s_{k-2}\}$ and $D = \{s_{k-1}\}$.

Let $\Psi_2 = \{A, B, C\}$, where $A = \left(\bigcup_{i=2}^{\lfloor k/4\rfloor-1}\{s_{3i}\}\right) \cup \{s_1, s_{k-2}\}$, $B = \left(\bigcup_{i=1}^{\lfloor k/4\rfloor-1}\{s_{k-3i+1}\}\right) \cup \{s_2, s_{k}\}$ and $C = \left(\bigcup_{i=1}^{\lfloor k/4\rfloor-2}\{s_{k-(3i+2)}\}\right) \cup \{s_{k-1}, s_{k-3}\}$.

Let $\Psi_3 = \{A, B, C, D\}$, where

- if $k$ is even, $A = \left(\bigcup_{i=0}^{k/2-3}\{s_{2i+1}\}\right) \cup \{s_{k-1}\}$, $B = \left(\bigcup_{i=1}^{k/2-2}\{s_{2i}\}\right) \cup \{s_k\}$, $C = \{s_{k-3}\}$ and $D = \{s_{k-2}\}$.

- if $k$ is odd, $A = \left(\bigcup_{i=0}^{(k-5)/2}\{s_{2i+1}\}\right) \cup \{s_{k}\}$, $B = \left(\bigcup_{i=1}^{(k-5)/2}\{s_{2i}\}\right) \cup \{s_{k-1}\}$, $C = \{s_{k-3}\}$ and $D = \{s_{k-2}\}$.

Let $\Psi_4 = \{A, B, C, D, F\}$, where

- if $k$ is even, $A = \left(\bigcup_{i=0}^{(k-6)/2}\{s_{2i+1}\}\right) \cup \{s_k\}$, $B = \{s_2, s_{k-4}\}$, $C = \{s_{k-3}\}$, $D = \{s_{k-2}\}$, $F = \left(\bigcup_{i=2}^{(k-6)/2}\{s_{2i}\}\right) \cup \{s_{k-1}\}$.

- if $k$ is odd, $A = \left(\bigcup_{i=0}^{(k-7)/2}\{s_{2i+1}\}\right) \cup \{s_{k-1}\}$, $B = \{s_2, s_{k-5}\}$, $C = \{s_{k-4}\}$, $D = \{s_{k-3}\}$, $F = \left(\bigcup_{i=2}^{(k-7)/2}\{s_{2i}\}\right) \cup \{s_{k-2}, s_k\}$.

It is easy to see that $CG(P_k, \Psi_1) \cong K_{1,3}$, $CG(P_k, \Psi_2) \cong K_3$, $CG(P_k, \Psi_3) \cong K_4 - e$ and $CG(P_k, \Psi_4) \cong P_2 \cup P_3$. 

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Proposition 5. For any integer \( k \geq 7 \), \( P_k \) defines the CP-graphs \( P_4 \) and \( P_5 \).

Proof. To prove the proposition, we provide two coalition partitions \( \Psi_1 \) and \( \Psi_2 \) for \( P_k \) with \( k \geq 7 \) that the corresponding coalition graphs are \( P_4 \) and \( P_5 \), respectively. Let \( \Psi_1 = \{A,B,C,D\} \), where \( A = \bigcup \{s_{k-2i+1}\} \cup \{s_1, s_{k-2}, s_k\} \) \( B = \bigcup \{s_{k-2i+1}\} \cup \{s_2, s_{k-3}\} \) \( C = \{s_{k-4}\} \) and \( D = \{s_{k-1}\} \). Let \( \Psi_2 = \{A,B,C,D,F\} \), where \( A = \bigcup \{s_{k-2i+1}\} \cup \{s_1\} \) \( B = \bigcup \{s_{k-2i+1}\} \cup \{s_2\} \) \( C = \{s_3\} \) \( D = \{s_4\} \) \( F = \{s_5\} \). It is easy to see that \( CG(P_k, \Psi_1) \cong P_4 \) and \( CG(P_k, \Psi_2) \cong P_5 \) \( \blacksquare \).

Proposition 6. For any integer \( k \geq 8 \), \( P_k \) defines the CP-graph \( 2K_2 \).

Proof. To prove the proposition, we provide a coalition partition \( \Psi \) for \( P_k \) with \( k \geq 8 \) that the corresponding coalition graph is \( 2K_2 \). Let \( \Psi = \{A,B,C,D\} \), where \( A = \{s_1, s_{k-1}\} \) \( B = \{s_2, s_k\} \) \( C = \bigcup \{s_{k-2i+1}\} \cup \{s_{k-4}, s_{k-5}\} \) and \( D = \bigcup \{s_{k-2i+1}\} \cup \{s_{k-2}, s_{k-3}\} \). It is easy to see that \( CG(P_k, \Psi) \cong 2K_2 \) \( \blacksquare \).

Proposition 7. For any integer \( k \geq 9 \), \( P_k \) defines the CP-graphs \( F_2, B \), and \( S_{1,2} \).

Proof. To prove the proposition, we provide four coalition partitions \( \Psi_1, \Psi_2 \) and \( \Psi_3 \) for \( P_k \) with \( k \geq 9 \) that the corresponding coalition graphs are \( F_2 \), \( B \) and \( S_{1,2} \), respectively.

Let \( \Psi_1 = \{A,B,C,D,F\} \), where \( A = \bigcup \{s_{k-2i+1}\} \cup \{s_1\} \) \( B = \bigcup \{s_{k-2i+1}\} \cup \{s_2\} \) \( C = \{s_3, s_6\} \) \( D = \{s_4, s_7\} \) \( F = \{s_5\} \).

Let \( \Psi_2 = \{A,B,C,D,F\} \), where \( A = \bigcup \{s_{k-2i+1}\} \cup \{s_1, s_4\} \) \( B = \bigcup \{s_{k-2i+1}\} \cup \{s_2, s_6\} \) \( C = \{s_3\} \) \( D = \{s_5\} \) \( F = \{s_7\} \).

Let \( \Psi_3 = \{A,B,C,D,F\} \), where \( A = \bigcup \{s_{k-2i+1}\} \cup \{s_1\} \) \( B = \bigcup \{s_{k-2i+1}\} \cup \{s_2, s_5\} \) \( C = \{s_3\} \) \( D = \{s_4\} \) \( F = \{s_7\} \).

It is easy to see that \( CG(P_k, \Psi_1) \cong F_2 \) \( CG(P_k, \Psi_2) \cong B \) and \( CG(P_k, \Psi_3) \cong S_{1,2} \) \( \blacksquare \).

Proposition 8. For any integer \( k \geq 10 \), \( P_k \) defines the CP-graph \( S_{2,2} \).

Proof. By Lemma 10, the coalition number of \( P_k \) with \( k \geq 10 \) is 6. Therefore, there is a coalition partition of \( P_k \) such that the corresponding CP-graph has 6 vertices. On the other hand, the only CP-graph with 6 vertices is \( S_{2,2} \). Hence, the path \( P_k \) with \( k \geq 10 \) defines the CP-graph \( S_{2,2} \) \( \blacksquare \).

In Propositions 1-8, we presented results that determine which CP-graphs can be defined by the path \( P_k \). But there are still cases that have not yet been identified. For example, does \( P_8 \) define the CP-graphs \( F_2 \) and \( B \) or does \( P_6 \) define the CP-graphs \( 2K_2 \) and \( P_7 \)? To this end, we empirically checked the remaining cases. Table 4 summarizes the results of Lemma 9, Corollary 4, and Propositions 1-8, and the empirical results. In the table, we used the letters \( Y, N, y \) and \( n \). Let \( T(i,j) \) be the cell of the table in row \( i \) and column \( j \). Let \( CP_i \) be the CP-graph in row \( i \), and let \( P_j \) be
Let $NC(P_i)$ be the number of CP-graphs that can be defined by $P_i$. From Table 4, we can easily obtain the following result.

**Theorem 7.** It holds that $NC(P_1) = NC(P_2) = 1$, $NC(P_3) = 2$, $NC(P_4) = 3$, $NC(P_5) = 6$, $NC(P_6) = 10$, $NC(P_7) = NC(P_8) = 12$, $NC(P_9) = 14$, and $NC(P_k) = 15$ for any integer $k \geq 10$.

Table 1: The CP-graphs that can be defined by any path $P_k$ with $k \geq 1$.

| $P_1$ | $P_2$ | $P_3$ | $P_4$ | $P_5$ | $P_6$ | $P_7$ | $P_8$ | $P_k \geq 10$ |
|-------|-------|-------|-------|-------|-------|-------|-------|---------------|
| $K_1$ | $Y$   | $N$   | $N$   | $N$   | $N$   | $N$   | $N$   | $N$           |
| $K_2$ | $N$   | $N$   | $Y$   | $Y$   | $Y$   | $Y$   | $Y$   | $Y$           |
| $K_3$ | $N$   | $N$   | $N$   | $n$   | $y$   | $Y$   | $Y$   | $Y$           |
| $K_4$ | $N$   | $N$   | $N$   | $n$   | $y$   | $Y$   | $Y$   | $Y$           |
| $K_5$ | $N$   | $N$   | $N$   | $n$   | $y$   | $Y$   | $Y$   | $Y$           |
| $K_6$ | $N$   | $N$   | $N$   | $n$   | $y$   | $Y$   | $Y$   | $Y$           |
| $K_7$ | $N$   | $N$   | $N$   | $n$   | $y$   | $Y$   | $Y$   | $Y$           |
| $K_8$ | $N$   | $N$   | $N$   | $n$   | $y$   | $Y$   | $Y$   | $Y$           |
| $K_9$ | $N$   | $N$   | $N$   | $n$   | $y$   | $Y$   | $Y$   | $Y$           |
| $K_{10}$ | $N$ | $N$ | $N$ | $n$ | $y$ | $Y$ | $Y$ | $Y$ |

the path in column $j$. When $T(i,j)$ is equal to a uppercase letter, it means that using Observation 3, Lemma 9, Corollary 4, and Propositions 1-8, we have obtained the value of $T(i,j)$, and when $T(i,j)$ is equal to a lowercase letter, it means that we empirically have obtained it. If $T(i,j) \in \{Y,y\}$, it means that the CP-graph $CP_i$ can be defined by the path $P_j$, and if $T(i,j) \in \{N,n\}$, it means that the CP-graph $CP_i$ cannot be defined by the path $P_j$. In the following, for the cases that $T(i,j) = y$, we present a coalition partition of the path $P_j$ corresponding to the CP-graph $CP_i$.

- The path $P_8$ and the CP-graph $S_{1,2}$: $\{\{1,4\}, \{2,6,8\}, \{3\}, \{5\}, \{7\}\}$.
- The path $P_7$ and the CP-graph $S_{1,2}$: $\{\{1,4\}, \{2,6\}, \{3\}, \{5\}, \{7\}\}$.
- The path $P_7$ and the CP-graph $2K_2$: $\{\{1,7\}, \{2\}, \{3, 4\}, \{5, 6\}\}$.
- The path $P_6$ and the CP-graph $P_4$: $\{\{1,4\}, \{2\}, \{3, 5\}, \{6\}\}$.
- The path $P_6$ and the CP-graph $2K_2$: $\{\{1,6\}, \{2\}, \{3, 4\}, \{5\}\}$.
- The path $P_5$ and the CP-graph $P_4$: $\{\{1\}, \{2\}, \{3, 4\}, \{5\}\}$.
- The path $P_5$ and the CP-graph $K_3$: $\{\{1,5\}, \{2\}, \{3, 4\}\}$.
By Theorem 7, there is no path $P_k$ that defines all 18 CP-graphs. Hence, we have the following result.

**Theorem 8.** There is no universal coalition path.

## 5 Conclusion

In this paper, we proved that for any graph $G$ with $\delta(G) = 1$, $C(G) \leq 2\Delta(G) + 2$. Moreover, we characterized all graphs $G$ with $\delta(G) \leq 1$ and $C(G) = n$. Furthermore, we characterized all trees $T$ with $C(T) = n$ and all trees $T$ with $C(T) = n - 1$. On the other hand, we theoretically and empirically determined the number of coalition graphs that can be defined by all coalition partitions of a given path $P_k$. Furthermore, we showed that there is no universal coalition path. The following problems remain open.

(a) For any integer $k \geq 2$, characterize all trees $T$ with $C(T) = n - k$.
(b) Is there a linear-time algorithm to compute the coalition number of a given tree?
(c) Is $C(G) \leq 2\Delta(G) + 2$ for any graph $G$?

## References

[1] T.W. Haynes, J. T. Hedetniemi, S. T. Hedetniemi, A. A. McRae, R. Mohan, Introduction to coalitions in graphs, AKCE Int. J. Graphs Combin. 17(2), (2020) 653–659.

[2] T.W. Haynes, J. T. Hedetniemi, S. T. Hedetniemi, A. A. McRae, R. Mohan, Coalition graphs of paths, cycles and trees. Discuss. Math. Graph Theory (2020) DOI:https://doi.org/10.7151/dmgt.2416.

[3] T.W. Haynes, J. T. Hedetniemi, S. T. Hedetniemi, A. A. McRae, R. Mohan, Upper bounds on the coalition number, Australas. J Comb. 80(3), 442-453 (2021)

[4] T.W. Haynes, S. T. Hedetniemi, P. J. Slater, Fundamentals of Domination in Graphs. Marcel Dekker, Inc. (1998.)