On Hydrodynamic Limits of Young Diagrams

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Joint work with I. Fatkullin, S. Sethuraman.

- Static Models of Young Diagrams
- Evolitional Models of Young Diagrams
- Main Results
- Sketch of Proof
Young diagrams are related with: combinatorics; representation theory; . . .
polymer physics; genetics; zero-temperature Ising model; . . .

2D/3D Young diagrams: static theory (statistical mechanics),
dynamical theory

We will be focusing on models of 2D Young diagrams.
Let \( p = (p_1, p_2, p_3, \ldots, p_n), \) \( p_k \geq p_{k+1}, \) be a partition of the integer

\[
M(p) = \sum_{k=1}^{n} p_k.
\]

For example, \( p = (4, 2, 2, 1) \) is a partition of

\[
9 = 4 + 2 + 2 + 1
\]

The corresponding Young diagram:

\[
p = (4, 2, 2, 1)
\]
Shape function $F(x)$:

\[
p = (4, 2, 2, 1)
\]

\[
F(x) = F(x; p)
\]

Clearly:

\[
M(p) := \sum_{k=1}^{n} p_k = \int_{0}^{\infty} F(x)dx.
\]
Size density (or configuration of particles) $\eta = (\eta(k))_{k \in \mathbb{N}}$:

\[ p = (4, 2, 2, 1) \]

\[ \eta = (1, 2, 0, 1, 0, \ldots) \]

\[ M(p) = \sum_{k=1}^{\infty} k \eta(k). \]
Relations between $F$ and $\eta$:

\[ F(x) = \sum_{k \geq x} \eta(k), \quad \eta(k) = F(k) - F(k + 1) \]

$\eta(k)$ can be viewed as negative gradient of $F$ at $k$. 
The size of a diagram grows, of course, as the $M(p)$ grows. For the limit $M \to \infty$, rescale the diagram by setting the width and height of one square by $\mu_x$ and $\mu_y$ respectively. After rescaling, the area of the Young diagram is $\mu_x \mu_y M$. 

\[ \begin{array}{c}
4 \\
3 \\
2 \\
1 \\
\end{array} \begin{array}{cccc}
1 & 2 & 3 & 4 \\
\end{array} \]

Before scaling

\[ \begin{array}{c}
4\mu_y \\
3\mu_y \\
2\mu_y \\
\mu_y \\
\end{array} \begin{array}{cccc}
\mu_x & 2\mu_x & 3\mu_x & 4\mu_x \\
\end{array} \]

After scaling
If we set $\mu_x = \mu_y = \frac{1}{\sqrt{M}}$ rescaled shape function

$$F_M(x; p) = \frac{1}{\sqrt{M}} F(x\sqrt{M}; p)$$

A classical result of A. Vershik [V]: Let $\mathcal{P}_M$ be the uniform probability on all partitions of $M$, e.g. $(4, 2, 2, 1)$ and $(5, 4)$ are equally likely. As $M \to \infty$, $F_M$ concentrate near

$$F(x) = -\frac{\sqrt{6}}{\pi} \ln \left( 1 - e^{-\pi x/\sqrt{6}} \right)$$

Precisely, for all $M > M_0(a, b, \varepsilon)$

$$\mathcal{P}_M \left\{ \sup_{x \in [a, b]} |F_M(x; p) - F(x)| > \varepsilon \right\} < \varepsilon.$$
The uniform measure $P_\mu$ can be thought of as a canonical ensemble. Similar results as above hold for other choices of measures. For example

$$P_\mu(p) = \frac{1}{Z_\mu} e^{-\mu M(p)}.$$ 

or the general Grand-canonical ensemble (cf. e.g. [EG], [FS], [V], [VY])

$$P_{\beta,\mu}(p) = \frac{1}{Z_{\beta,\mu}} e^{-\beta \sum_{k \in p} E_k - \mu M(p)}$$

Rescale $F_\mu(x; p) := \frac{1}{\mu \mathbb{E}_\mu(M)} F(x/\mu; p)$.

- $\beta = 0$: $F_\mu \to \frac{6}{\pi^2} \ln(1 - e^{-x})$

- $E_k \sim \ln k$, $0 < \beta < 1$: $F_\mu \to \frac{1}{\Gamma(2 - \beta)} \int_x^\infty u^{-\beta} e^{-u} du$

- $E_k \ll \ln k$, $\beta > 0$: $F_\mu \to e^{-x}$
For our evolutilonal models, start with the particle systems directly. Introduce generator

$$L f(\eta) = \sum_{k=1}^{\infty} \left\{ \lambda_k \left[ f(\eta_{k,k+1}^{k,x,y}) - f(\eta) \right] \chi\{\eta(k) > 0\} \right\}$$

where

$$\lambda_k = e^{-\beta(E_{k+1} - E_k) - \mu}, \quad \eta_{k,x,y} = \begin{cases} \eta(k) - 1 & k = x \\ \eta(k) + 1 & k = y \\ \eta(k) & \text{otherwise} \end{cases}.$$
Weakly asymmetric zero range process on $\mathbb{Z}^+$. 

Remember $\lambda_k = e^{-\beta(E_{k+1} - E_k) - \mu}$. 
In this example, a particle at site 2 jumps (with rate $\lambda_2$) to site 3 corresponds to creation of a square at the corner $(2, 1)$. 
Here, a particle at site 4 jumps (with rate $\lambda_4$) to site 3 corresponds to annihilation of a square at the corner (3, 0).
Remember $\eta(k) = F(k) - F(k+1)$. Since

$$F_\mu(x; p) := \frac{1}{\mu \mathbb{E}_\mu(M)} F(x/\mu; p)$$

we consider rescaled empirical measures

$$\pi^\mu_t(dx) = \pi^\mu(\eta_t, dx) = \mu \gamma_\mu \sum_{k=1}^{\infty} \eta_t(k) \delta_{k\mu}(dx).$$

where $\gamma_\mu = \frac{1}{\mu^2 \mathbb{E}_\mu(M)}$. Since $\mathbb{E}_\mu(M) \sim \mu^{-2} e^{-\beta E_1/\mu}$ (c.f. [FS])

$$\gamma_\mu = \begin{cases} 1 & \beta = 0 \\ \mu^{-\beta} & 0 < \beta < 1, E_k \sim \ln k \\ (\ln \frac{1}{\mu})^\beta & 0 < \beta, E_k \sim \ln(\ln k) \end{cases}$$
Theorem (Case $\beta = 0$)

With appropriate initial measures, for any test function $G \in C_c^\infty(0, \infty)$, for all $0 < t \leq T$, as $\mu \to 0$

$$\langle G, \pi_{t/\mu^2}^\mu \rangle \to \int_0^\infty G(x) \rho(t, x) \, dx, \quad \text{in probability}$$

where $\rho(t, x)$ is the unique weak solution of the equation

$$\begin{cases}
\partial_t \rho = \partial_x^2 \frac{\rho}{\rho + 1} + \partial_x \frac{\rho}{\rho + 1} \\
\rho(0, \cdot) = \rho_0(\cdot), \quad \int_0^\infty \rho(t, x) \, dx = \int_0^\infty \rho_0(x) \, dx \\
\rho(t, \cdot) \leq \phi(\cdot) \quad \text{for all } t \leq T
\end{cases}$$
Theorem (Case $E_k \sim \ln k$)

With appropriate initial measures, for any test function $G \in C_c^\infty(0, \infty)$, for all $0 < t \leq T$, as $\mu \to 0$

$$\langle G, \pi_{t/\mu^2}^\mu \rangle \to \int_0^\infty G(x) \rho(t, x) dx, \text{ in probability}$$

where $\rho(t, x)$ is the unique weak solution of the equation

$$
\begin{cases}
\partial_t \rho = \partial_x^2 \rho + \partial_x \left( \frac{\beta + x}{x} \rho \right) \\
\rho(0, \cdot) = \rho_0(\cdot), \quad \int_0^\infty \rho(t, x) dx = \int_0^\infty \rho_0(x) dx \quad \cdot (1) \\
\rho(t, \cdot) \leq \phi_c(\cdot) \text{ for all } t \leq T
\end{cases}
$$
Theorem (Case \( E_k \ll \ln k \))

With appropriate initial measures, for any test function \( G \in C_c^\infty(0, \infty) \), for all \( 0 < t \leq T \), as \( \mu \to 0 \)

\[
\langle G, \pi^{\mu}_{t/\mu^2} \rangle \to \int_0^\infty G(x) \rho(t, x) \, dx, \quad \text{in probability}
\]

where \( \rho(t, x) \) is the unique weak solution of the equation

\[
\begin{cases}
\partial_t \rho = \partial_x^2 \rho + \partial_x \rho \\
\rho(0, \cdot) = \rho_0(\cdot), \quad \int_0^\infty \rho(t, x) \, dx = \int_0^\infty \rho_0(x) \, dx \\
\rho(t, \cdot) \leq \phi_c(\cdot) \text{ for all } t \leq T
\end{cases}
\]
The macroscopic equations:

- $\beta = 0$:
  \[ \partial_t \rho = \partial^2_x \frac{\rho}{\rho + 1} + \partial_x \frac{\rho}{\rho + 1} \]

- $E_k \sim \ln k$:
  \[ \partial_t \rho = \partial^2_x \rho + \partial_x \left( \frac{\beta + x}{x} \rho \right) \]

- $E_k \ll \ln k$:
  \[ \partial_t \rho = \partial^2_x \rho + \partial_x \rho \]

Funaki and Sasada [FuSa] obtained the same equation, as in the case $\beta = 0$, for a different model.
Case $\beta = 0$: $\lambda_k = e^{-\mu} =: \varepsilon$

[FuSa] model: a weakly asymmetric reservoir at site 0
Invariant measures:

- model in [FuSa]:

\[ P_{\mu}(\eta) = \frac{1}{Z_\mu} e^{-\mu \sum_k k\eta(k)} = \frac{1}{Z_\mu} \prod_k \left( e^{-k\mu} \right)^{\eta(k)} \]

- Case \( \beta = 0 \): for all \( 0 < c \leq 1 \)

\[ P_{\mu,c}(\eta) = \frac{1}{Z_{\mu,c}} \prod_k \left( c e^{-k\mu} \right)^{\eta(k)} \]

Initial conditions:

- model in [FuSa]: \( \int_0^\infty \rho_0(x)dx = \infty \).

- Case \( \beta = 0 \): \( \rho_0 < \phi \), \( \int_0^\infty \rho_0(x)dx < \infty \)
Formal derivation of macroscopic equations:

\[ \langle G, \pi(\eta_t) \rangle = \langle G, \pi(\eta_0) \rangle + \int_0^t \mu^{-2} L \langle G, \pi(\eta_s) \rangle \, ds + M^G_t \]

with

\[ \mu^{-2} L \langle G, \pi_t(\eta) \rangle = \mu \sum_{k=2}^{\infty} \Delta \mu G(k\mu) \gamma \mu \chi_{\{\eta_t(k)>0\}} \]

\[ + \mu \sum_{k=2}^{\infty} \frac{\lambda_k - 1}{\mu} \nabla \mu G(k\mu) \gamma \mu \chi_{\{\eta_t(k)>0\}} \]

As \( k\mu \to x \)

\[ \frac{\lambda_k - 1}{\mu} \to \begin{cases} \beta = 0 \\ \beta + x \to x \quad E_k \sim \ln k \\ 1 \quad E_k \ll \ln k \end{cases} \]
Equilibrium measures are products of geometrics with parameters very close locally.

\[ \gamma_{\mu} \xi_{\eta}(k) \sim \gamma_{\mu} \frac{E_{\eta^e/\mu}(k)(\chi_{\eta}>0)}{1 + \eta^e/\mu}. \]

Notice that typically \( \gamma_{\mu} \eta^e/\mu(k) \to \rho^e(t, x) \) then

\[ \frac{\gamma_{\mu} \eta^e/\mu}{1 + \eta^e/\mu} \sim \frac{\rho(x)}{1 + \gamma_{\mu}^{-1} \rho(x)}. \]

- \( \beta = 0: \gamma_{\mu} = 1, \gamma_{\mu} \xi_{\eta}(k) \sim \frac{\rho(x)}{1 + \rho(x)} \)
- \( E_k \sim \ln k \) or \( E_k \ll \ln k: \gamma_{\mu} \to \infty, \gamma_{\mu} \xi_{\eta}(k) \sim \rho(x) \)
Brief sketch of proof for the case $\beta = 0$:

- **1-block estimate:**

$$
\lim_{l \to \infty} \limsup_{N \to \infty} \left| \mathbb{E}^N \frac{1}{N} \sum_{aN \leq k \leq bN} \int_0^T D_{N,k}^{G,t} \left( \chi_{\eta_{N^2 t}(k) > 0} - \frac{\eta_{N^2 t}^l(k)}{1 + \eta_{N^2 t}^l(k)} \right) dt \right| = 0.
$$

- **2-block estimate:**

$$
\lim_{l \to \infty} \limsup_{\tau \to 0} \limsup_{N \to \infty} \left| \mathbb{E}^N \frac{1}{N} \sum_{aN \leq k \leq bN} \int_0^T D_{N,k}^{G,t} \left( \frac{\eta_{N^2 t}^l(k)}{1 + \eta_{N^2 t}^l(k)} - \frac{\eta_{N^2 t}^{\tau N}(k)}{1 + \eta_{N^2 t}^{\tau N}(k)} \right) dt \right| = 0.
$$

A 1-block estimate will be sufficient for the cases when $\beta \neq 0$. 
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Thank you!