On the 2-modular reduction of the Steinberg representation of the symplectic group

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Abstract

We show that in characteristic 2, the Steinberg representation of the symplectic group $\text{Sp}_{2n}(q)$, $q$ a power of an odd prime $p$, has two irreducible constituents lying just above the socle that are isomorphic to the two Weil modules of degree $(q^n - 1)/2$.

1 Introduction

The Steinberg representation was constructed by R. Steinberg [St] in 1957 for all finite Chevalley groups $G = G(F_q)$, $q$ a power of a prime $p$. He did so by identifying a particular right ideal in the group algebra $FG$ over an arbitrary field $F$. He then showed this ideal to be minimal precisely when the characteristic of $F$ does not divide the index of the normalizer of a $p$-Sylow subgroup of $G$. In particular, the Steinberg representation is irreducible in characteristic 0.

The general problem of finding the composition factors of the Steinberg module has recently been addressed by Gow [G], who has a conjecture in the case of the general linear group. He also determined all irreducible constituents in a few examples. One composition factor that is certainly known is the socle, which is irreducible and generated as a $G$-module by the fixed points of $U$.

Our main result in this context is the following: in characteristic 2, the Steinberg representation of the symplectic group $\text{Sp}_{2n}(q)$, $q$ a power of an odd prime $p$, has two irreducible constituents lying just above the socle that are isomorphic to the two Weil modules of degree $(q^n - 1)/2$.

We start the paper by constructing a family of irreducible representations for a $p$-Sylow subgroup $U$ of $\text{Sp}_{2n}(q)$ over a field $F$ of characteristic $l \neq p$ containing a primitive $p$-th root of unity. This is done in section 3 under no restrictions on $p$. We also produce families
of irreducible $F$-modules for the normalizer $B$ of $U$, and for a maximal parabolic subgroup $P$ of $\text{Sp}_{2n}(q)$. It is then shown in section 4 that, for $p$ odd, the above irreducible modules are essentially those found in the Weil representation. All information needed in regards to the Weil representation is contained in section 4.

In section 5 we show that, provided $l$ divides $q+1$, the Steinberg module restricted to $P$ contains copies of the two Weil modules of degree $(q^n - 1)/2$ restricted to $P$, up to multiplication by an explicit linear character $P \to \{\pm 1\} \subset F^*$. Finally, in section 6 we prove that, if $l = 2$, the Steinberg module modulo the socle contains copies of the two Weil modules of degree $(q^n - 1)/2$. In order to extend the results of section 5 to those of section 6 we prove a criterion ensuring that only two identities in the symplectic group algebra $\mathbb{F}\text{Sp}_{2n}(q)$ need to be verified. The actual verification of these identities is difficult, and occupies the rest of the paper.

The most basic concepts and definitions, along with our choice of notation can be found in section 2.

I am very grateful to R. Gow for suggesting this problem.

2 Basic Notions

The Symplectic Group. Let $F_q$ be a finite field of characteristic $p$. Let $V$ be a vector space of dimension $2n$ over $F_q$ endowed with a non-degenerate alternating form $\langle \ , \ \rangle$. The symplectic group of rank $2n$ over $F_q$, denoted simply by $\text{Sp}$, is the group of all $g \in \text{GL}(V)$ satisfying

$$\langle gv, gw \rangle = \langle v, w \rangle, \quad v, w \in V.$$  

Witt decomposition of $V$ and associated subgroups of $\text{Sp}$. We fix throughout a symplectic basis of $V$. This is a basis $u_1, \ldots, u_n, v_1, \ldots, v_n$ satisfying

$$\langle u_i, v_j \rangle = \delta_{ij}, \quad \langle u_i, u_j \rangle = 0, \quad \langle v_i, v_j \rangle = 0.$$  

Write $M = \text{span} \{u_1, \ldots, u_n\}$ and $N = \text{span} \{v_1, \ldots, v_n\}$. Let $\text{Sp}^M$ be the pointwise stabilizer
of \( M \) in \( \text{Sp} \). The matrix of any \( s \in \text{Sp}^M \) has the appearance

\[
\begin{pmatrix}
1 & S \\
0 & 1
\end{pmatrix},
\]

(1)

where \( S \) is a symmetric \( n \times n \) matrix with coefficients in \( F_q \). Let \( \text{Sp}_{M,N} \) be the subgroup of \( \text{Sp} \) preserving both \( M \) and \( N \). The matrix of any \( a \in \text{Sp}_{M,N} \) has the form

\[
\begin{pmatrix}
A & 0 \\
0 & t^t A^{-1}
\end{pmatrix},
\]

(2)

where \( A \in \text{GL}_n(q) \). The group \( \text{Sp}_{M,N} \) normalizes \( \text{Sp}^M \). The semidirect product group \( \text{Sp}_M = \text{Sp}^M \rtimes \text{Sp}_{M,N} \) is the global stabilizer of \( M \) in \( \text{Sp} \). It is a maximal subgroup of \( \text{Sp} \).

**A \( p \)-Sylow subgroup of \( \text{Sp} \).** Let \( T \) be the group of all \( g \) in \( \text{Sp}_{M,N} \) whose matrix is of the form (2), where \( A \) is upper triangular and has 1’s on the main diagonal. Then \( U = T \rtimes \text{Sp}^M \) is a \( p \)-Sylow subgroup of \( \text{Sp} \).

Let \( H \) be the group of all \( g \in \text{Sp}_{M,N} \) having all \( u_i \) and \( v_j \) as eigenvalues. The group \( U \) is normalized by \( H \). The semidirect product \( B = U \rtimes H \) is the normalizer of \( U \) in \( \text{Sp} \).

**The Weyl group.** For \( 1 \leq i < n \), denote by \( w_i \) the element of \( \text{Sp} \) that preserves the given symplectic basis, and is defined by

\[
w_i = (u_i, u_{i+1})(v_i, v_{i+1}).
\]

We also define, for \( 1 \leq j \leq n \), the elements \( c_j \) of \( \text{Sp} \) by: \( c_j(u_j) = v_j, c_j(v_j) = -u_j \), while all other basis vectors remain fixed.

Let \( W_0 = \langle w_1, ..., w_{n-1} \rangle \) and \( W_1 = \langle c_1, ..., c_n \rangle \). The group \( W_1 \) is normalized by \( W_0 \). Let \( W_2 = W_1 \rtimes W_0 \). The group \( H \) is normalized by \( W_2 \), with \( W_2 \cap H = W_1 \cap H = \langle c_1^2, ..., c_n^2 \rangle \).

The Weyl group of \( \text{Sp} \) is

\[
W = W_2 H / H \cong W_2 / W_2 \cap H \cong W_1 \rtimes W_0 / W_1 \cap H \cong (W_1 / W_1 \cap H) \rtimes W_0.
\]

It is the split extension of \( C_2^n \cong \langle c_1, ..., c_n \rangle / \langle c_1^2, ..., c_n^2 \rangle \) by \( S_n \cong \langle w_1, ..., w_{n-1} \rangle \). Here the symmetric group \( S_n \) acts naturally on \( C_2^n \) by permuting the \( n \) copies of \( C_2 \).
Set $\mathcal{F} = \{w_1, \ldots, w_{n-1}, c_n\}$. Given $w \in W_2$, write $\ell(w) = \ell(wH)$ for the length of the shortest word in the letters $w_1H, \ldots, w_{n-1}H, c_nH$ which is equal to $wH$ in the Weyl group $W$. The largest value of $\ell$ is attained by $w_0 = c_1 \cdots c_n$, the element of $\text{Sp}$ defined by

$$w_0(u_i) = v_i, \quad w_0(v_i) = -u_i, \quad 1 \leq i \leq n.$$ 

Observe that $\ell(w_0) = n^2$.

For $w \in W_2$, set $U^+_w = w^{-1}Uw \cap U$, the group of all $u \in U$ whose conjugates $wuw^{-1}$ remain in $U$. Let $U^-_w$ be the the group of all $u \in U$ whose conjugates $wuw^{-1}$ belong to $w_0Uw_0^{-1}$. One has $\text{Sp} = U^+_wU^-_w$ with $U^+_w \cap U^-_w = 1$. Furthermore, $|U^-_w| = q^{n^2-\ell(w)}$ and $|U^-_w| = q^{n^2}$.

**Root subgroups.** Set $\mathcal{I} = \{(i, j) \mid 1 \leq i < j \leq 2n, \ i \leq n\}$. Let $(i, j) \in \mathcal{I}$ and take $\alpha \in F_q$. If $j \leq n$ define $x_{i,j}(\alpha)$ to be the element of $T$ whose matrix has the form (2), where $A_{i,j} = \alpha$, $A_{kk} = 1$ for all $k$, and all other entries of $A$ are equal to zero. If $j > n$ define $x_{ij}(\alpha)$ to be the element of $\text{Sp}^M$ whose matrix has the form (1), where $S_{i,j-n} = S_{j-n,i} = \alpha$, and all other entries of $S$ are equal to zero. For a fixed $(i, j) \in \mathcal{I}$ write $X_{i,j}$ for the group of all $x_{i,j}(\alpha)$ with $\alpha \in F_q$. This is the root subgroup corresponding to $(i, j)$. The fundamental root subgroups are $X_{(i,i+1)}$, for $1 \leq i < n$, and $X_{(n,2n)}$. Observe that $X_{(i,i+1)} = U^-_{w_i}$ and $X_{(n,2n)} = U^-_{c_n}$.

**Symplectic transvections.** Given $u$ in $V$ and $\alpha \in F_q$, let $\rho_{u,\alpha} \in \text{Sp}$ be the symplectic transformation defined by

$$\rho_{u,\alpha}(v) = v + \alpha(u, v)u, \quad v \in V.$$ 

**The center of $\text{Sp}$.** A symplectic transformation commuting with all symplectic transvections is necessarily a scalar operator. Thus the center $Z(\text{Sp})$ of $\text{Sp}$ is equal to $\{1, -1\}$ if $p$ is odd, and is trivial otherwise.

**Conjugation.** For group elements $g$ and $h$ we write $^{h}g = gh^{-1}$ and $^hg = h^{-1}gh$.

### 3 A representation of $\text{Sp}_M$

We fix throughout a field $F$ of characteristic $l \neq p$ possessing a non-trivial $p$-th root of unity. We also fix a non-trivial linear character $\lambda : F^+_q \to F^*$. 


For \( v \in V \), let \( \chi_v : \text{Sp} \to F^* \) be the function defined by

\[
\chi_v(g) = \lambda(\langle gv, v \rangle), \quad g \in \text{Sp}. \tag{3}
\]

We shall use the same notation for \( \chi_v \) and its restriction to various subgroups. Context will dictate what subgroup is being used at a given time.

Given \( v \in N \), let \( S_v \) be the group of all \( g \) in \( \text{Sp}_M \) satisfying \( gv \equiv v \mod M \). From the matrix representation (1) of \( \text{Sp}_M \) we see that \( \text{Sp}_M \subset S_v \). In fact, \( S_v = \text{Sp}_M \rtimes (\text{Sp}_{M,N})_v \), where \( (\text{Sp}_{M,N})_v \) is the pointwise stabilizer of \( v \) in \( \text{Sp}_{M,N} \).

3.1 Lemma Let \( v \in N \). Then \( \chi_v : S_v \to F^* \) is a linear character of \( S_v \).

Proof. Let \( g \) and \( h \) be elements of \( S_v \). Then

\[
\langle ghv, v \rangle = \langle ghv - v + v, v \rangle = \langle ghv - v, v \rangle + \langle gv, v \rangle = \langle hv - v, g^{-1}v + v \rangle + \langle gv, v \rangle.
\]

As \( h, g^{-1} \in S_v \), we have \( hv - v \in M \) and \( g^{-1}v - v \in M \), whence \( \langle hv - v, g^{-1}v - v \rangle = 0 \). It follows that

\[
\langle ghv, v \rangle = \langle hv - v, v \rangle + \langle gv, v \rangle = \langle hv, v \rangle + \langle gv, v \rangle.
\]

\( \square \)

For \( v \in N \), let \( \hat{S}_v = S_v \times Z(\text{Sp}) \), the group of all \( g \) in \( \text{Sp}_M \) satisfying \( gv \equiv \pm v \mod M \). We shall denote by \( \chi_v^\pm \) the group homomorphism \( \hat{S}_v \to F^* \) that agrees with \( \chi_v \) on \( S_v \) and maps \(-1 \in Z(\text{Sp})\) to \( \pm1 \in F^* \).

Since \( \text{Sp}_M \) is contained in \( S_v \) for all \( v \in N \), we may view \( \chi_v \) as a linear character of \( \text{Sp}_M \). A fundamental property is the following.

3.2 Lemma Let \( v, w \in N \). Then

\( \chi_v = \chi_w \) on \( \text{Sp}_M \) \( \iff v = \pm w \).
Proof. One implication follows directly from (3). Suppose conversely that
\[ \lambda(\langle gv, v \rangle) = \lambda(\langle gw, w \rangle) \]  
for all \( g \in \text{Sp}^M \). Fix any \( u \in V \). We apply (4) to \( g = \rho_{u,\alpha} \) obtaining
\[ \lambda(\alpha(\langle u, v \rangle^2 - \langle u, w \rangle^2)) = 1 \]
for all \( \alpha \in F_q \). As \( \lambda \) is non-trivial, it follows that
\[ \langle u, v \rangle^2 = \langle u, w \rangle^2 \]  
for all \( u \in V \). We infer that the linear maps \( \langle -, v \rangle \) and \( \langle -, w \rangle \) have the same kernel.

Choose \( u \in V \) satisfying \( \langle u, v \rangle = 1 \). Then (5) yields \( \langle u, v \rangle = \pm 1 \). We conclude that \( \langle -, v \rangle = \langle -, w \rangle \) or \( \langle -, v \rangle = \langle -, -w \rangle \), that is \( v = w \) or \( v = -w \), as claimed. \( \square \)

In what follows it will sometimes be convenient to think of the matrix of an arbitrary \( g \in \text{Sp}_{M,N} \) as having the form
\[
\begin{pmatrix}
\frac{t}{A^{-1}} & 0 \\
0 & A
\end{pmatrix},
\]
where \( A \in \text{GL}_n(q) \). In this case \( g \in T \) whenever \( A \) is lower triangular and has 1’s on the main diagonal. In particular, \( gv_n = v_n \) for all \( g \in T \). It follows that \( U \) is contained in \( S_{\alpha v_n} \) for all \( \alpha \in F_q \), a fact to be used repeatedly below.

3.3 Lemma. Let \( \alpha \in F_q^* \). Then the linear character \( \chi_{\alpha v_n} \) of \( U \) is trivial on \( U_{cn}^+ \) and non-trivial on \( U_{cn}^- \).

Proof. We have \( U_{cn}^+ = (\text{Sp}^M)_0 \times T \), where \( (\text{Sp}^M)_0 \) is the subgroup of \( \text{Sp}^M \) of all \( g \) whose matrix has the form (1) with \( S_{nn} = 0 \). If \( g \in T \) then \( gv_n = v_n \). Therefore (3) yields \( T \subset \ker \chi_{\alpha v_n} \). If \( g \in (\text{Sp}^M)_0 \) then \( gv_n - v_n \in \text{span}\{u_1, \ldots, u_{n-1}\} \). As \( v_n \) is orthogonal to itself and \( u_1, \ldots, u_{n-1} \), (3) yields \( (\text{Sp}^M)_0 \subset \ker \chi_{\alpha v_n} \), as well. This proves the first assertion.

As for the second, note that \( U_{cn}^- \) is the group of all \( x_n, 2n(\beta) = \rho_{u_n, \beta}, \beta \in F_q \). Now
\[ \chi_{\alpha v_n}(\rho_{u_n, \beta}) = \lambda(\langle \alpha v_n + \beta(\alpha v_n)u_n, \alpha v_n \rangle) = \lambda(\beta \alpha^2) \]
for all \( \beta \in F_q \). As \( \lambda \) is non-trivial, \( \chi_{\alpha v_n} \) is non-trivial on \( U_{cn}^- \). \( \square \)
Since the fundamental root subgroups $X_{i,i+1}$ are contained in $U_{c_n}$ we obtain the following result.

### 3.4 Corollary
The linear character $\chi_{\alpha \nu}, \alpha \in F_q^\times$, is non-trivial in exactly one fundamental root subgroup, namely $X_{(n,2n)} = U_{c_n}$.

### 3.5 Lemma
Let $v \in V$ and $g_0 \in \text{Sp}$. Then

$$\chi_v(g_0^g) = \chi_{g_0^v}(g), \quad g \in \text{Sp}.$$

**Proof.** Since $g_0$ preserves $\langle , \rangle$, we have

$$\chi_v(g_0^g) = \lambda(\langle g_0^{-1}g_0^v, v \rangle) = \lambda(\langle g_0v, g_0v \rangle) = \chi_{g_0^v}(g), \quad g \in \text{Sp}.$$ 

Let $X = X_\chi$ be a vector space over $F$ possessing a basis $(E_v)_{v \in N}$ indexed by $N$.

### 3.6 Proposition
The function $R : \text{Sp}_M \to GL(X)$ given by

$$R(sa)E_v = \lambda(\langle sav, av \rangle)E_{av} = \chi_{av}(s)E_{av}, \quad v \in N, s \in \text{Sp}^M, a \in \text{Sp}_{M,N}$$

defines an $F$-representation of $\text{Sp}_M$.

**Proof.** Suppose $s_1, s_2 \in \text{Sp}^M$, $a_1, a_2 \in \text{Sp}_{M,N}$, and $v \in N$. Then

$$R(s_1a_1s_2a_2)E_v = R(s_1a_1s_2a_2)a_1a_2E_v = \chi_{a_1a_2v}(s_1a_1s_2)a_1a_2E_v.$$  

Since $\chi_{a_1a_2v}$ is a linear character of $\text{Sp}^M$

$$\chi_{a_1a_2v}(s_1a_1s_2) = \chi_{a_1a_2v}(s_1)\chi_{a_1a_2v}(a_1s_2).$$

By Lemma 3.5 we have

$$\chi_{a_1a_2v}(a_1s_2) = \chi_{a_1a_2v}(s_2).$$

On the other hand

$$R(s_1a_1)R(s_2a_2)E_v = R(s_1a_1)\chi_{a_2v}(s_2)E_{a_2v} = \chi_{a_1a_2v}(s_1)\chi_{a_2v}(s_2)E_{a_1a_2v}.$$
This completes the proof. □

We determine the irreducible constituents of this representation, depending on whether $p$ is even or odd. Suppose first that $p$ is odd. For $v \in N \setminus \{0\}$, set $E^+ = E_v + E_{-v}$ and $E^- = E_v - E_{-v}$. We also write $X^+ = X^+_\lambda$ for the span of the $E^+_v$, and let $X^- = X^-_{\lambda}$ be the span of the $E^-_v$. When $p = 2$ we set $E^+_v = E^-_v = E_v$ for all $v \in N \setminus \{0\}$, and let $X^+ = X^- = X^+_\lambda$ be the span of the $E_v$, $v \neq 0$.

Formula (7) ensures that $X^\pm$ is an $F \mathrm{Sp}_M$-module. Its dimension is $(q^n - 1)/2$ for $p$ odd and $q^n - 1$ for $p = 2$. We write $X_0$ for the $\mathrm{Sp}_M$-fixed points of $X$, that is $X_0 = FE_0$.

For $1 \leq i \leq n$ and $\alpha \in F_q^*$, consider the following $F$-subspace of $X^\pm$:

$$X^\pm_{\alpha,i} = \text{span}\{E^\pm_{\alpha v_i + \alpha_{i+1} v_{i+1} + \cdots + \alpha_n v_n} \mid \alpha_i \in F_q\}. $$

3.7 Theorem $X^\pm_{\alpha,i}$ is an absolutely irreducible $FU$-submodule of $X^\pm$ of dimension $q^{n-i}$. Furthermore, if $w$ is the element of $W_0$ defined by

$$w = (u_i, u_n, u_{n-1}, \ldots, u_{i+1})(v_i, v_n, v_{n-1}, \ldots, v_{i+1})$$

then

$$X^+_{\alpha,i} \cong \text{ind}_{U^+_w}^U FE^+_{\alpha v_i} \cong \text{ind}_{U^-_w}^U FE^-_{\alpha v_i} \cong X^-_{\alpha,i}$$

where $FE^\pm_{\alpha v_i}$ affords the linear character $\chi_{\alpha v_i}$ of $U^+_{\alpha, i}$.

Proof. Let $E^\pm_{\alpha v_i + \alpha_{i+1} v_{i+1} + \cdots + \alpha_n v_n}$ be a typical basis vector of $X^\pm_{\alpha,i}$, and let $g \in U$. Then $g = sa$, where the matrix of $a$ has the form (6) with $A$ lower triangular with 1’s on the main diagonal, and $s \in \mathrm{Sp}_M$.

We see from (7) how $s$ and $a$ act on $X^\pm$. Indeed, $E^\pm_{\alpha v_i + \alpha_{i+1} v_{i+1} + \cdots + \alpha_n v_n}$ is an eigenvector for $s$ acting on $X^\pm$, and the above matrix description of $a$ ensures that $aE^\pm_{\alpha v_i + \alpha_{i+1} v_{i+1} + \cdots + \alpha_n v_n}$ belongs to $X^\pm_{\alpha,i}$. It follows that $g$ preserves $X^\pm_{\alpha,i}$, which is then an $FU$-module. It is clear that $\dim X^\pm_{\alpha,i} = q^{n-i}$.

Since $\text{char } F = l$ is coprime to $|U| = q^{n^2}$, the $FU$-module $X^\pm_{\alpha,i}$ is completely reducible. We proceed to show that the commuting ring of $X^\pm_{\alpha,i}$ is comprised entirely by scalar operators. This implies that $X^\pm_{\alpha,i}$ is absolutely irreducible.
Let $C$ be an $F$-endomorphism of $X^{\pm,\alpha,i}$ commuting with the action of $U$. The group $U^-_w$ consists of all $g \in \text{Sp}_{M,N}$ whose matrix has the form (6), where $A$ is a lower triangular matrix with 1’s on the diagonal and all columns different from column $i$ have zero entries below the diagonal. Let us write $0, 0, \ldots, 0, \beta_{i+1}, \ldots, \beta_n$ for the entries in the $i$-th column of such matrix $A$. Here 1 is in the $i$-th position. Then

$$g E^{\pm}_{\alpha v_i} = E^{\pm}_{\alpha v_i + \alpha \beta_{i+1} v_{i+1} + \cdots + \alpha \beta_n v_n}.$$ 

It follows that $U^-_w$ acts transitively on the basis vectors $E^{\pm}_{\alpha v_i + \alpha \beta_{i+1} v_{i+1} + \cdots + \alpha \beta_n v_n}$ of $X^{\pm,\alpha,i}$. Since the actions of $C$ and $U^-_w$ on these vectors commute, we infer that $C$ acts diagonally on them.

Given two different basis vectors $E^{\pm}_{\alpha v_i + \alpha \beta_{i+1} v_{i+1} + \cdots + \alpha \beta_n v_n}$ and $E^{\pm}_{\alpha v_i + \beta_{i+1} v_{i+1} + \cdots + \beta_n v_n}$, we have

$$\alpha v_i + \alpha_{i+1} v_{i+1} + \cdots + \alpha_n v_n \neq \pm(\alpha v_i + \beta_{i+1} v_{i+1} + \cdots + \beta_n v_n).$$

It follows from Lemma 3.2 that for some $s \in \text{Sp}_M$ one has

$$\chi_{\alpha v_i + \alpha_{i+1} v_{i+1} + \cdots + \alpha_n v_n}(s) \neq \chi_{\alpha v_i + \beta_{i+1} v_{i+1} + \cdots + \beta_n v_n}(s).$$

Since the diagonal actions of $\text{Sp}_M$ and $C$ on the basis vectors $E^{\pm}_{\alpha v_i + \alpha_{i+1} v_{i+1} + \cdots + \alpha_n v_n}$ commute, we deduce that $C$ must be a scalar operator, as claimed.

From (7) we see that $X^{\pm,\alpha,i}$ affords a monomial representation of $FU$. The stabilizer of $FE^{\pm}_{\alpha v_i}$ is the group of all $g \in U$ such that $g = sa$, $s \in \text{Sp}_M$, $a \in T$, and the $i$-th column of $A$ in the matrix representation (6) of $a$ is equal to the $i$-th column of the identity matrix. Therefore, the stabilizer of $FE^{\pm}_{\alpha v_i}$ is equal to $U^+_w$. As $U^-_w$ is a left transversal for $U^+_w$ in $U$ acting transitively on the one dimensional subspaces $FE^{\pm}_{\alpha v_i + \alpha_{i+1} v_{i+1} + \cdots + \alpha_n v_n}$, it follows that $X^{\pm,\alpha,i}_w \cong \text{ind}_{U^-_w}^{U^+_w} FE^{\pm}_{\alpha v_i}$. Furthermore, $U^+_w$ acts on $FE^{\pm}_{\alpha v_i}$ by means of the linear character $\chi_{\alpha v_i}$. This completes the proof of the theorem.

3.8 Theorem For a fixed $i$, if $\alpha, \beta \in F_\mathbb{C}^*$ satisfy $\{\alpha, -\alpha\} \neq \{\beta, -\beta\}$ then the $FU$-modules $X^{\pm,\alpha,i}_w$ and $X^{\pm,\beta,i}_w$ are not isomorphic.

Proof. The linear character $\chi_{\alpha v_i}$ of $\text{Sp}_M$ enters the restriction of $X^{\pm,\alpha,i}_w$ to $\text{Sp}_M$. The linear
characters entering the restriction of $X^\pm_{\beta,i}$ to $\text{Sp}^M$ are of the form $\chi_{\beta v_i + \beta_{i+1} v_{i+1} + \cdots + \beta_n v_n}$. If $\{\alpha, -\alpha\} \neq \{\beta, -\beta\}$ then none of them is $X^\pm_{\alpha,i}$ by Lemma 3.2.

Note that for $h \in H$ we have

$$hX^\pm_{\alpha,i} = X^\pm_{h\alpha,i},$$

where $hv_i = h_i v_i$. We shall henceforth denote by $T_q$ a transversal for the action of $\{\pm 1\}$ on $F_q$. Thus $|T_q| = (q - 1)/2$ if $p$ is odd, and $|T_q| = q - 1$ if $p = 2$. Observe that $X^\pm_{\alpha,i} = X^\pm_{-\alpha,i}$ for all $\alpha \in T_q$.

3.9 Theorem  For a fixed $i$, the direct sum $X^\pm_i = \bigoplus_{\alpha \in T_q} X^\pm_{\alpha,i}$ is an absolutely irreducible $FB$-module.

Proof. Let $Z$ be an $FB$-submodule of $X^\pm_i$. Then $Z$ is an $FU$-submodule of $X^\pm_i$. It follows from Theorems 3.7 and 3.8 that $Z$ must be the direct sum of some $X^\pm_{\alpha,i}$. Hence $Z$ is all of $X^\pm_i$ by (8). As this argument works for $F$ or any extension thereof, $X^\pm_{\alpha,i}$ is absolutely irreducible.

3.10 Theorem  The $X^\pm_i$ are not isomorphic $FB$-modules.

Proof. Observe that the dimension of $X^\pm_i$ is equal to $(q - 1)q^{n-i}/2$ if $p$ is odd and $(q - 1)q^{n-i}$ if $p = 2$. The result thus follows.

The following relation holds for all $w \in W_0$:

$$wE_v = E_{w(v)}, \quad v \in N.$$  \ (9)

3.11 Theorem  $X^\pm$ is an absolutely irreducible $\text{Sp}_M$-module. Moreover,

$$X^\pm = \text{ind}^{\text{Sp}_M}_{\tilde{S}_{v_n}} FE^\pm_{v_n},$$

where $\tilde{S}_{v_n}$ acts on the one-dimensional subspace $FE^\pm_{v_n}$ by means of $\chi^\pm_{v_n}$.

Proof. Let $Z$ be an $F\text{Sp}_M$-submodule of $X^\pm$. Then $Z$ is an $FB$-submodule of $X^\pm$. It follows from Theorems 3.9 and 3.10 that $Z$ must be the direct sum of some $X^\pm_i$. Hence at least one $E^\pm_{v_i}$ belongs to $Z$. Therefore all $E^\pm_{v_j}$ belong to $Z$, by applying (9) with all
As the $E_{v_i}^\pm$ generate $X^{\pm}_j$ as an $FB$-submodule, and the sum of all $X^{\pm}_j$ equals $X^{\pm}$, it follows that $Z = X^{\pm}$. Again, as this argument works for $F$ or any extension thereof, $X^{\pm}$ is absolutely irreducible.

Now $X^{\pm}$ affords a monomial representation of $Sp_M$; the stabilizer of $FE_{v_n}^\pm$ is equal to $\tilde{S}_{v_n}$; the linear character of $\tilde{S}_{v_n}$ afforded by $FE_{v_n}^\pm$ is equal to $\chi_{v_n}^\pm$; the index of $S_{v_n}^+$ in $Sp_M$ is equal to the number of representatives of non-zero vectors in $N$ modulo the action of $\{\pm 1\}$. This is also the number of one-dimensional subspaces $FE_{v}^\pm$ permuted by $Sp_M$. This shows that $X^{\pm} \cong \text{ind}_{S_{v_n}^-}^{Sp_M} FE_{v_n}^\pm$, and completes the proof of the theorem. \hfill $\Box$

If $l \neq 2$ we then have

$$X = X_0 \oplus X^+ \oplus X^-.$$ 

This is a decomposition of $X$ into non-isomorphic irreducible $Sp_M$-modules. If $l = 2$ then $X^+ = X^-$, which is isomorphic to the $Sp_M$-module $X/(X^+ \oplus X_0)$ via the isomorphism $E_{v_i}^+ \mapsto E_v + (X^+ \oplus X_0)$.

Given $\kappa \in F_q^*$, consider the non-trivial linear character $\chi[\kappa]$ of $F_q^+$ defined by

$$\chi[\kappa](\alpha) = \lambda(\kappa \alpha), \quad \alpha \in F_q.$$ 

By varying $\kappa$ we obtain all non-trivial linear characters of $F_q^+$.

3.12 Theorem \hfill Let $\kappa \in F_q^*$. Then

$$X^\pm_\lambda \cong X^\pm_{\lambda[\kappa]}$$ 

as $FSp_M$-modules $\iff \kappa$ is a square.

Proof. Suppose that $X^\pm_\lambda$ and $X^\pm_{\lambda[\kappa]}$ are isomorphic as $FSp_M$-modules. The linear characters entering the restriction of $X^\pm_\lambda$ to $Sp_M$ are of the form

$$g \mapsto \lambda(\langle gv, v \rangle),$$ 

where $0 \neq v \in N$. One the other hand, the linear characters entering the restriction of $X^\pm_{\lambda[\kappa]}$ to $Sp_M$ are of the form

$$g \mapsto \lambda(\kappa \langle gw, w \rangle).$$ 

where $0 \neq w \in N$. From the given hypothesis, it follows that for some $v, w$ non-zero

$$\lambda(\langle gv, v \rangle) = \lambda(\kappa \langle gw, w \rangle)$$
for all $g \in \text{Sp}^M$. Reasoning as in the proof of Lemma 3.2 we see that

$$\langle u, v \rangle^2 = \kappa \langle u, w \rangle^2$$

for all $u \in M$. Choosing $u$ so that $\langle u, v \rangle = 1$ we infer that $1 = \kappa \langle u, w \rangle^2$, that is, $\kappa$ is a square.

Suppose conversely that $\kappa = \tau^2$. Consider the $F$-isomorphism, say $f$, from $X_{\lambda}$ to $X_{\lambda[\kappa]}$ defined by

$$f(E_v) = E_{\tau^{-1}v}.$$ 

Then given any $s \in \text{Sp}^M$ and any $a \in \text{Sp}^M$ we have

$$f(saE_v) = f(\lambda(\langle sa, av \rangle)E_{av}) = \lambda(\langle sa, av \rangle)E_{\tau^{-1}av}.$$ 

On the other hand

$$saE_v = saE_{\tau^{-1}v} = \lambda(\kappa(\langle sa, av \rangle)E_{\tau^{-1}v}) = \lambda(\kappa\tau^{-1}\tau^{-1}(\langle sa, av \rangle)E_{\tau^{-1}av}$$

$$= \lambda(\langle sa, av \rangle)E_{\tau^{-1}av}.$$ 

It follows that $f$ is an isomorphism of $F\text{Sp}^M$-modules.

Now $X^+ \oplus X_0$ is the 1-eigenspace of $-1 \in Z(\text{Sp})$ acting on $X$, and $X_0$ is the fixed points of $\text{Sp}^M$. If $l \neq 2$ then $X^-$ is the $-1$-eigenspace of $-1 \in Z(\text{Sp})$ acting on $X$. Whether $l = 2$ or not, we infer that when $f$ preserves $X^\pm$, whence $X^\pm_{\lambda} \cong X^\pm_{\lambda[\kappa]}$. □

4 The Weil representation of $\text{Sp}$

We assume in this section that $p$ is odd, and rely on [Sz] as a general reference. Let $Y = Y_{\lambda}$ be an $F$-vector space having a basis $(\epsilon_v)_{v \in N}$ indexed by $N$. Let $H_0$ the group whose underlying set is $F_q \times V$, with multiplication

$$(\alpha, v)(\beta, w) = (\alpha + \beta + \langle v, w \rangle, v + w), \quad \alpha, \beta \in F_q, v, w \in V.$$ 

The function $J : H_0 \to \text{GL}(Y)$ given by

$$J(\alpha, u + v)\epsilon_w = \lambda(\alpha + \langle u, v + 2w \rangle)\epsilon_{w+v}, \quad u \in M, v, w \in N, \alpha \in F_q$$
defines an absolutely irreducible $F$-representation of $H_0$. The symplectic group acts on $H_0$ via the second coordinate: $g(\alpha, v) = (\alpha, gv)$ for $g \in \text{Sp}$, $\alpha \in F_q$, $v \in V$. For $g \in \text{Sp}$, the conjugate representation $J^g$ is similar to $J$. Moreover, there is one, and only one (except when $(n, q) = (1, 3)$ when there are three), representation $P : \text{Sp} \to \text{GL}(Y)$ satisfying:

$$P(g)J(h)P(g)^{-1} = J^g h, \quad g \in \text{Sp}, h \in H_0.$$ 

We refer to $P$ as the Weyl representation of $\text{Sp}$ over $F$ of type $\lambda$ (when $(n, q) = (1, 3)$ the Weil representation is the one given below). Consider the linear character $\theta : \text{Sp}_M \to F^*$ defined as follows for $s \in \text{Sp}^M$ and $a \in \text{Sp}_{M,N}$:

$$\theta(sa) = \theta(a) = \left( \frac{\det a|_N}{q} \right) = \begin{cases} 1 & \text{if } \det a|_N \in F_q^*, \\ -1 & \text{otherwise}. \end{cases}$$

The Weil representation can be defined on the basis vectors $\epsilon_v$ of $Y$ by means of:

$$P(sa)\epsilon_v = \theta(sa)^n \chi_{av}(s)\epsilon_{av}, \quad s \in \text{Sp}^M, a \in \text{Sp}_{M,N},$$

$$P(\rho_{v_n,-1})\epsilon_v = \left( \sum_{\alpha \in F_q} \lambda(\alpha^2) \right)^{-1} \sum_{\alpha \in F_q} \lambda(\alpha^2)\epsilon_{v+\alpha v_n}.$$  

We remark that [Sz] defines the Weil representation over a finite extension of $Q$ containing a non-trivial $p$-th root of unity. With slight modification of the arguments, one can see that (10)-(11) hold when $Q$ is replaced by any field of characteristic different from $p$. Particular care needs to be taken in the characteristic 2 case.

Having into account that $\text{Sp}_M$ is a maximal subgroup of $\text{Sp}$ and that $\rho_{v_n,-1}$ is not in $\text{Sp}_M$, the above formulae suffice to determine $P$. While explicit formulae exist for other $P(g) \notin \text{Sp}_M$, the above description of $P(\rho_{v_n,-1})$ will be enough for our purposes.

For $v \in N \setminus \{0\}$, set $\epsilon_v^\pm = \epsilon_v \pm \epsilon_{-v}$. Write $Y_0 = F\epsilon_0$, let $Y^- = Y_\lambda^-$ be the span of the $\epsilon_v^-$, and let $Y^+ = Y_\lambda^+$ be the span of the $\epsilon_v^+$. Set $G(\lambda) = \sum_{\alpha \in F_q} \lambda(\alpha^2)$.

From (11) we get

$$\rho_{v_n,-1}\epsilon_0 = G(\lambda)^{-1}(\epsilon_0 + \sum_{\alpha \in F_q} \lambda(\alpha^2)\epsilon_{\alpha v_n}^+),$$  

$$\rho_{v_n,-1}\epsilon_v = \left( \sum_{\alpha \in F_q} \lambda(\alpha^2) \right)^{-1} \sum_{\alpha \in F_q} \lambda(\alpha^2)\epsilon_{v+\alpha v_n}.$$
\[
\rho_{v_n, -1} e_v^+ = G(\lambda)^{-1}(\sum_{\alpha \in F_q} \lambda(\alpha^2)e_{v+\alpha v_n}^+), \quad v \in N, v \notin F_q v_n, \\
\rho_{v_n, -1} e_{\beta v_n}^+ = G(\lambda)^{-1}(2\lambda(\beta^2)\epsilon_0 + \sum_{\alpha \in T_q} (\lambda((\alpha + \beta)^2) + \lambda((\alpha - \beta)^2))e_{\alpha v_n}^+), \quad \beta \in T_q. 
\]

We are particularly interested in the case when \(l = 2\). In this case \(G(\lambda) = 1\) because \(\lambda(\alpha^2) + \lambda((-\alpha)^2) = 2\lambda(\alpha^2) = 0\) for all \(\alpha \in T_q\), leaving only the term \(\lambda(0) = 1\). Thus, the above equations simplify as follows when \(l = 2\):

\[
\rho_{v_n, -1} e_0 = \epsilon_0 + \sum_{\alpha \in T_q} \lambda(\alpha^2)e_{\alpha v_n}^+, \\
\rho_{v_n, -1} e_{v}^+ = \sum_{\alpha \in F_q} \lambda(\alpha^2)e_{v+\alpha v_n}^+, \quad v \in N, v \notin F_q v_n, \\
\rho_{v_n, -1} e_{\beta v_n}^+ = \sum_{\alpha \in T_q} (\lambda((\alpha + \beta)^2) + \lambda((\alpha - \beta)^2))e_{\alpha v_n}^+, \quad \beta \in T_q. 
\]

We also note that \(\theta\) is trivial when \(l = 2\). In this case (10) yields

\[
sae^+ = \chi_{av}(s)e_{av}^+, \quad s \in \text{Sp}^M, a \in \text{Sp}_{M,N}. 
\]

**4.1 Theorem** If \(l \neq 2\) then \(Y^+ \oplus Y_0\) and \(Y^-\) are absolutely irreducible \(\text{FSp}\)-modules.

If \(l = 2\) then \(Y^+ = Y^-\) is an absolutely irreducible \(\text{FSp}\)-module and \(Y/Y^+ \cong Y^+ \oplus Fy\), where \(y\) is fixed by \(\text{Sp}\). In any case, \(Y^+ \cong \theta^0 X^+\) as \(\text{FSp}_{M}\)-modules. If \(l = 2\) then actually \(Y^+ = Y^- \cong_{\text{Sp}_M} X^- = X^+\).

**Proof.** The last two assertions follow by comparing (7) with (10), and having into account that \(\theta\) is trivial when \(l = 2\).

Suppose next that \(l \neq 2\). Then \(Y^-\) and \(Y^+ \oplus Y_0\) are eigenspaces for \(-1 \in Z(\text{Sp})\) acting on \(Y\), and are therefore \(\text{Sp}\)-stable. As the restriction of \(Y^-\) to \(\text{Sp}_M\), namely \(\theta^0 X^-\), is absolutely irreducible, so must be \(Y^-\). Now \(Y^+ \oplus Y_0\) is a decomposition into non-isomorphic irreducible \(\text{Sp}_M\)-modules. From (12) we see that \(Y_0\) is not \(\text{Sp}\)-stable. By means of (14), and using \(l \neq 2\), we deduce that \(Y^+\) is not \(\text{Sp}\)-stable either. It follows that \(Y^+ \oplus Y_0\) is an irreducible \(\text{Sp}\)-module. As this argument works for \(F\) or any extension thereof, \(Y^+ \oplus Y_0\) is absolutely irreducible.
Suppose next that \( l = 2 \). Then \( Y^+ \oplus Y_0 \) is still an eigenspace for \(-1 \in \mathbb{Z}(Sp)\) and hence Sp-stable, while \( Y^- = Y^+ \). From (16) and (17) we see that now \( Y^+ \) is Sp-stable. Since \( Y^+ \cong_{Sp_M} X^+ \), the \( FSp\)-module \( Y^+ \) is absolutely irreducible. Consider the \( F\)-isomorphism \( Y/Y^+ \to Y^+ \oplus F y \) given by
\[
\epsilon_0 + Y^+ \to y, \quad \epsilon_v + Y^+ \to \epsilon_v^+, \quad v \neq 0.
\]
Using (15)-(18) we see that the action of Sp is preserved. This completes the proof. \( \square \)

We refer to \( Y^- \) (resp. \( Y^+ \oplus Y_0 \)) as the Weil module of Sp over \( F \) of type \( \lambda \) and degree \((q^n - 1)/2\) (resp. \((q^n + 1)/2\)). As our arguments are valid for any field of characteristic \( l \neq p \) containing a non-trivial \( p \)-th root of unity, the above easily yields the following result.

4.2 Theorem If \( l \neq 2, p \) then the \( l \)-modular reduction of the complex Weil modules of Sp of type \( \lambda \) and degrees \((q^n - 1)/2\) and \((q^n + 1)/2\) remain irreducible. The 2-modular reduction of the complex Weil module of Sp of type \( \lambda \) and degree \((q^n - 1)/2\) remains irreducible, and is a constituent of that of degree \((q^n + 1)/2\). The other constituent is the trivial module.

In regards to the number of different types of Weil modules we have the following.

4.3 Theorem There are exactly two isomorphsim types of Weil modules of Sp over \( F \) of degree \((q^n - 1)/2\) (resp. \((q^n + 1)/2\)).

Proof. If \( \kappa \in F_q^* \) is a square then the types \( \lambda \) and \( \lambda[\kappa] \) are isomorphic. Indeed, the isomorphism \( f \) used in the proof of Theorem 3.12 is easily seen to preserve the action of \( \rho_{v_n,-1} \). Conversely, by restricting to \( Sp_M \) and applying Theorem 3.12, we see that if the types \( \lambda \) and \( \lambda[\kappa] \) are isomorphic then \( \kappa \in F_q^* \) must be a square. \( \square \)

For future reference we record the following results.

4.4 Lemma Suppose \( l = 2 \). Then
\[
c_n \epsilon_{v_n}^+ = \sum_{\alpha \in T_q} (\lambda(-2\alpha) + \lambda(2\alpha)) \epsilon_{\alpha v_n}^+.
\]
\textbf{Proof.} Observe the identity
\[ c_n = \rho_{-1,u_n,\rho_{-1,v_n}}. \] (19)

As \( \rho_{-1,u_n} \in \text{Sp}^M \), formula (18) yields
\[ \rho_{-1,u_n,e^+_{v_n}} = \lambda(-\alpha_n^2)e^+_{v_n}, \quad v = \alpha_1 y_1 + \cdots + \alpha_n v_n. \] (20)

By (17) we also have
\[ \rho_{v_{n-1}} e^+_{v_n} = \sum_{\alpha \in T_q} (\lambda((\alpha + 1)^2) + \lambda((\alpha - 1)^2))e^+_{\alpha v_n}. \]

Therefore
\[ c_n e^+_{v_n} = \rho_{-1,u_n,\rho_{-1,v_n}} e^+_{v_n} \]
\[ = \rho_{-1,u_n,\rho_{-1,v_n}} \lambda(-1)e^+_{v_n} \]
\[ = \lambda(-1)\rho_{-1,u_n} \sum_{\alpha \in T_q} (\lambda((\alpha + 1)^2) + \lambda((\alpha - 1)^2))e^+_{\alpha v_n} \]
\[ = \sum_{\alpha \in T_q} (\lambda((\alpha + 1)^2) + \lambda((\alpha - 1)^2))\lambda(-1 - \alpha^2)e^+_{\alpha v_n} \]
\[ = \sum_{\alpha \in T_q} (\lambda(2\alpha) + \lambda(-2\alpha))e^+_{\alpha v_n}. \]

\[ \square \]

4.5 \textbf{Lemma} \quad \text{Suppose } l = 2. \text{ Then }
\[ c_{n-1} e^+_{v_n} = \sum_{\alpha \in F_q} e^+_{v_{n-1} + \alpha v_n}. \]

\textbf{Proof.} From (16) we have
\[ \rho_{-1,v_{n-1}} e^+_{v_{n-1}} = \sum_{\alpha \in F_q} \lambda(\alpha^2)e^+_{v_{n-1} + \alpha v_n}. \]
Making use of (19) and (20) we get

\[ c_n^+ \varepsilon_{v_{n-1}}^+ = \rho_{-1, u_n} \rho_{-1, v_n} \rho_{-1, u_n} \varepsilon_{v_{n-1}}^+ \]
\[ = \rho_{-1, u_n} \rho_{-1, v_n} \varepsilon_{v_{n-1}}^+ \]
\[ = \rho_{-1, u_n} \sum_{\alpha \in F_q} \lambda(\alpha^2) \varepsilon_{v_{n-1} + \alpha v_n}^+ \]
\[ = \sum_{\alpha \in F_q} \lambda(\alpha^2) \lambda(-\alpha^2) \varepsilon_{v_{n-1} + \alpha v_n}^+ \]
\[ = \sum_{\alpha \in F_q} \varepsilon_{v_{n-1} + \alpha v_n}^+. \]

Since \( c_{n-1} = w_{n-1} c_n w_{n-1} \), the above yields

\[ c_{n-1} \varepsilon_{v_n}^+ = w_{n-1} c_n w_{n-1} \varepsilon_{v_n}^+ = w_{n-1} c_n \varepsilon_{v_{n-1}}^+ = w_{n-1} \sum_{\alpha \in F_q} \varepsilon_{v_{n-1} + \alpha v_n}^+ = \sum_{\alpha \in F_q} \varepsilon_{v_n + \alpha v_{n-1}}^+. \]

\[ \square \]

5 The Steinberg representation of \( \text{Sp} \) restricted to \( \text{Sp}_M \)

For an element \( w \) of the Weyl group \( W \) of \( \text{Sp} \), let \( n_w \) be an element of \( W_2 \) satisfying \( n_w H = w \). Consider the element

\[ \bar{e} = \sum_{b \in B} \sum_{w \in W} (-1)^{f(w)} n_w \]

of the symplectic group algebra \( F\text{Sp} \). The right ideal \( \bar{I} = \bar{e}F\text{Sp} \) is a right \( F\text{Sp} \)-module, considered by Steinberg in [St]. So far we have dealt exclusively with left modules. We wish to adhere to this convention with Steinberg’s representation as well. For this purpose, consider the involution \( x \mapsto \bar{x} \) of \( F\text{Sp} \) that fixes all scalars and inverts all symplectic transformations. Set

\[ e = \left( \sum_{w \in W} (-1)^{f(w)} n_w \right) \sum_{b \in B} b, \]

and let \( I = F\text{Sp} \cdot e \). Then \( I \) is naturally a left \( F\text{Sp} \)-module. Note that \( \bar{I} \) is also a left \( F\text{Sp} \)-module under the rule

\[ g \cdot (\bar{e}x) = (\bar{e}x)\bar{g}, \quad x \in F\text{Sp}, \; g \in \text{Sp}. \]
Furthermore, $I$ and $\bar{I}$ are isomorphic as left modules, via the isomorphism $xe = \bar{e}\bar{x}$, $x \in F\text{Sp}$.

We shall henceforth work with $e$ and $I$, and refer to the latter as the (left) Steinberg module for $\text{Sp}$ over $F$. Our general reference for this section is [St].

An $F$-basis for $I$ is given by the $|U|$-elements $ue$, $u \in U$. Thus $U$ affords the regular representation of $U$. The derived quotient $U/U'$ is isomorphic to the direct product of the $n$ fundamental root subgroups, and it is therefore an elementary abelian $p$-group of order $q^n$. It follows that $F$ affords all linear characters of $U$. We infer that, given a linear character $\sigma$ of $U$, there is a unique -up to multiplication by a non-zero scalar- element $e_\sigma$ in $I$ upon which $U$ acts via $\sigma$. We may take

$$e_\sigma = \sum_{u \in U} \sigma(u)^{-1}ue.$$

Then

$$ue_\sigma = \sigma(u)e_\sigma, \quad u \in U. \quad (21)$$

Let $w(i)$ be the element of $\text{Sp}$ that preserves the given symplectic basis, and satisfies:

$$w(i) = (u_n, u_i, u_{i+1}, \ldots, u_{n-1})(v_n, v_i, v_{i+1}, \ldots, v_{n-1}).$$

For $1 \leq i \leq n$ and $\alpha \in F_q^*$, write $I_{\alpha,i}$ for the $FU$-module generated by $w(i)e_{\chi_{\alpha\nu_n}}$.

**5.1 Theorem** For $1 \leq i \leq n$ and $\alpha \in F_q^*$ we have

$$I_{\alpha,i} \cong_{FU} X^+_{\alpha,i}.$$

**Proof.** Observe first that the element $w(i)$ just defined is the inverse of the element $w$, as defined in Theorem 3.7.

Next note that for $u \in U_w^+$ we have

$$uw(i)e_{\chi_{\alpha\nu_n}} = w(i)w(i)^{-1}uw(i)e_{\chi_{\alpha\nu_n}} = w(i)uw^{-1}e_{\chi_{\alpha\nu_n}}.$$

If $u \in U_w^+$ then $wu \in U$, so (21) applies, yielding

$$uw(i)e_{\chi_{\alpha\nu_n}} = \chi_{\alpha\nu_n}(wu)w(i)e_{\chi_{\alpha\nu_n}} = \chi_{\alpha\nu_n}(u^w(i))w(i)e_{\chi_{\alpha\nu_n}} \in I_{\alpha,i}. \quad (22)$$
Since $U = U_w^+U_w^-$, it follows that $I_{\alpha,i}$ is generated by the elements $uw(i)e_{\chi_{av_n}}$, as $u$ runs through $U_w^-$. We proceed to show that these elements are linearly independent.

Let $s \in \text{Sp}^M$. For $u \in U_w^-$ we have

$$suw(i)e_{\chi_{av_n}} = uu^{-1}suw(i)e_{\chi_{av_n}},$$

where $s^u \in \text{Sp}^M \subset U_w^+$. Hence by (22) and Lemma 3.5

$$suw(i)e_{\chi_{av_n}} = u\chi_{av_n}(s^{uw(i)}w(i)e_{\chi_{av_n}} = \chi_{av_i}(s)uw(i)e_{\chi_{av_n}}.$$ 

Thus $\text{Sp}^M$ acts on $uw(i)e_{\chi_{av_n}}$, $u \in U_w^-$, via the linear character $\chi_{av_i}$. We contend these linear characters are all different. Indeed, as noted in the proof of Theorem 3.7, the group $U_w^-$ consists of all $u \in U$ whose matrix has the form (6), where $A$ is a lower triangular with 1’s on the diagonal, and all columns of $A$ different from column $i$ have zero entries below the diagonal. It follows from this matrix description that $u \in U_w^-$ is completely determined by what it does to $v_i$. In particular, we see that $u \neq v$ in $U_w^-$ implies $\alpha uv_i \neq \pm \alpha vv_i$. We deduce from Lemma 3.2 that $\chi_{av_i}$ and $\chi_{avv_i}$ are different linear characters of $\text{Sp}^M$ for $u \neq v \in U_w^-$. All in all, we get that the $uw(i)e_{\chi_{av_n}}$, $u \in U_w^-$, must be linearly independent.

As the also generate $I_{\alpha,i}$, the form a basis of $I_{\alpha,i}$.

The preceding discussion shows that $I_{\alpha,i}$ affords a monomial representation of $U$; the stabilizer of $Fw(i)e_{\chi_{av_n}}$ is $U_w^+$; $I_{\alpha,i}$ is the direct sum of left translates of $Fw(i)e_{\chi_{av_n}}$ by elements of $U_w^-$, which is a transversal for $U_w^+$ in $U$. We conclude $I_{\alpha,i} \cong \text{ind}_{U_w^-}^{U_w^+}Fw(i)e_{\chi_{av_n}}$ as $FU$-modules. By (22) and Lemma 3.5 $Fw(i)e_{\chi_{av_n}}$ affords the linear character $\chi_{av_i}$ of $U_w^+$. The result now follows from Theorem 3.7.

We next observe

$$he = e, \quad h \in H.$$ 

As $H$ normalizes $U$, for each linear character $\sigma$ of $U$ we may consider the linear character $h\sigma$ of $U$, defined by $h\sigma(u) = \sigma(u^h)$, $u \in U$. Thus, for all linear characters $\sigma$ of $U$ and all $h \in H$

$$he\sigma = h \sum_{u \in U} \sigma(u)^{-1}ue = \sum_{u \in U} \sigma(u)^{-1}(h^h)he = \sum_{u \in U} \sigma(u^h)^{-1}ue = e_{(h\sigma)}. \quad (23)$$
For $1 \leq i \leq n$, recall the element $w(i)$ defined prior to Theorem 5.1. Let $h \in H$. As $W_2$ normalizes $H$ we have $h^{w(i)} \in H$. Since $v_i$ is an eigenvector for $h$, we may write $hv_i = h_i v_i$ for some $h_i \in F_q^*$. From Lemma 3.5 we obtain the formula

$$h^{w(i)} \chi_{\alpha v} = \chi_{h_i v}.$$  

From (23) and (24) we see that for $1 \leq i \leq n$, $h \in H$, and $\alpha \in F_q^*$

$$hw(i) e_{\chi_{\alpha v}} = w(i) h^{w(i)} e_{\chi_{\alpha v}} = w(i) e_{\chi_{h_i v}}.$$  

It follows that for $1 \leq i \leq n$, $h \in H$ and $\alpha \in F_q^*$

$$hI_{h_i,i} = I_{h_i,i}.$$  

Now $\chi_{\alpha v} = \chi_{-\alpha, v}$, whence $I_{h_i,i} = I_{-h_i,i}$ for all $\alpha \in F_q^*$. As the $FU$-modules $I_{h_i,i}$, $\alpha \in T_q$, are irreducible and non-isomorphic (cf. Theorems 3.7, 3.8 and 5.1), they are in direct sum within $I$. Set $I_i = \oplus_{\alpha \in T_q} I_{h_i,i}$. From (26) we see that $I_i$ is an $FB$-module, which is clearly isomorphic to $X^+_i$.

In order to prove the next result we shall use, for the first time in the paper, a beautiful identity due Steinberg, namely formula (16) of [St].

5.2 Theorem Let $w \in \{w_1, ..., w_{n-1}\}$ and $v \in N$. Suppose $U^+_w \subset S_v$. Then

$$w(\sum_{w \in U^+_w} \sum_{u \in U^-_w} \chi_{v(u)^{-1} u u' e}) = \sum_{w \in U^+_w} \sum_{u \in U^-_w} \chi_{w(u)}(u)^{-1} u u' e - (q + 1) \sum_{w \in U^+_w} \chi_{w(v)}(u)^{-1} u e.$$  

Proof. We have

$$w(\sum_{w \in U^+_w} \sum_{u \in U^-_w} \chi_{v(u)^{-1} u u' e}) = \sum_{w \in U^+_w} \sum_{u \in U^-_w} \chi(u)^{-1}(w u) u u' e.$$  

Since $w$ has order 2, it normalizes $U^+_w$. Thus, it follows from $U^+_w \subset S_v$ that $U^+_w \subset S_{w(v)}$. All in all, $u \mapsto \chi_{v(w) u}$ is a linear character of $U^+_w$, which by Lemma 3.5 must be equal to
\(\chi_{w(v)}\). We may thus write

\[
w(\sum_{u \in U^+_w} \sum_{u' \in U^-_w} \chi_{v}(u)^{-1} uu' e) = \sum_{u \in U^+_w} \sum_{u' \in U^-_w} \chi_{v}(u)^{-1}(w u) uu' e = \sum_{u \in U^+_w} \sum_{u' \in U^-_w} \chi_{v}(w u)^{-1} uu' e = \sum_{u \in U^+_w} \sum_{u' \in U^-_w} \chi_{w(v)}(u)^{-1} uu' e.
\]

Since \(we = -e\), to \(u' = 1\) there corresponds the summand

\[-\sum_{u \in U^+_w} \chi_{w(v)}(u)^{-1} ue.\]

As \(U^-_w = X_{(i, i+1)}\) for some \(i \in \{1, ..., n-1\}\), we may write all \(u' \in U^-_w\) in the form

\[u' = x_{i, i+1}(\alpha), \quad \alpha \in F_q^*\]

By means of Steinberg’s relation (16) of [St] we may write

\[wx_{i, i+1}(\alpha)e = x_{i, i+1}(-\alpha^{-1})e - e, \quad \alpha \in F_q^*\]

It follows that \(w(\sum_{u \in U^+_w} \sum_{u' \in U^-_w} \chi_{v}(u)^{-1} uu' e)\) is equal to

\[-\sum_{u \in U^+_w} \chi_{w(v)}(u)^{-1} ue - (q - 1) \sum_{u \in U^+_w} \chi_{w(v)}(u)^{-1} u e + \sum_{u \in U^+_w} \sum_{\alpha \in F_q^*} \chi_{w(v)}(u)^{-1} u x_{i, i+1}(-\alpha^{-1})e.\]

Since \(-\alpha^{-1}\) runs through \(F_q^*\) as \(\alpha\) does, we may replace \(-\alpha^{-1}\) by \(\alpha\) in the above expression. Thus, by adding and subtracting \(\sum_{u \in U^+_w} \chi_{w(v)}(u)^{-1} u e\) we obtain the desired result. \( \Box \)

5.3 Theorem For all \(\alpha \in F_q\) we have

\[w_i e_{\chi_{\alpha \nu_n}} = e_{\chi_{\alpha \nu_n}} - (q + 1) \sum_{u \in U^+_w} \chi_{\alpha \nu_n}(u)^{-1} u e, \quad i \in \{1, ..., n - 2\},\]

whereas

\[w_{n-1} e_{\chi_{\alpha \nu_n}} = \sum_{u \in U^+_w} \sum_{u' \in U^-_w} \chi_{\alpha \nu_{n-1}}(u)^{-1} uu' e - (q + 1) \sum_{u \in U^+_w} \chi_{\alpha \nu_{n-1}}(u)^{-1} u e.\]

21
Proof. Let \( w \in \{w_1, \ldots, w_{n-1}\} \). We know that, not only \( U^+_w \subset S_{\alpha v_n} \), but in fact \( U \) is included in \( S_{\alpha v} \) and \( \chi_{\alpha v_n} \) is a linear character of \( U \). As such, its kernel contains \( U^- = X_{(i,i+1)} \) due to Corollary 3.4. Thus

\[
e_{\chi_{\alpha v_n}} = \sum_{u \in U^+_w} \sum_{u' \in U^-} \chi(uu')^{-1} uu' e = \sum_{u \in U^+_w} \sum_{u' \in U^-} \chi(u)^{-1} uu' e.
\]

Since \( w_i(\alpha v_n) = \alpha v_n \) when \( i \in \{1, \ldots, n-2\} \) and \( w_n(\alpha v_n) = \alpha v_{n-1} \), the result follows from Theorem 5.2.

5.4 Theorem Let \( g \in \text{Sp}_M \) and let \( \alpha \in F_0^* \). Then \( ge_{\chi_{\alpha v_n}} = e_{\chi_{\alpha v_n}} \) implies \( gv_n = \pm v_n \). The converse holds precisely when \( l \) divides \( q + 1 \) (or \( n = 1 \)).

Proof. Suppose \( g \in \text{Sp}_M \) satisfies \( ge_{\chi_{\alpha v_n}} = e_{\chi_{\alpha v_n}} \). Given \( s \in \text{Sp}_M \) Lemma 3.5 gives

\[
sge_{\chi_{\alpha v_n}} = gs^g e_{\chi_{\alpha v_n}} = \chi_{\alpha v_n}(s^g) e_{\chi_{\alpha v_n}} = \chi_{\alpha g v_n}(s) e_{\chi_{\alpha v_n}}.
\]

On the other hand \( se_{\chi_{\alpha v_n}} = \chi_{\alpha v_n}(s) e_{\chi_{\alpha v_n}} \). Thus the linear characters \( \chi_{\alpha g v_n} \) and \( \chi_{\alpha v_n} \) of \( \text{Sp}_M \) are equal. It follows from Lemma 3.2 that \( gv_n = \pm v_n \).

Suppose conversely that \( gv_n = \pm v_n \). We wish to analyze under what conditions \( ge_{\chi_{\alpha v_n}} = e_{\chi_{\alpha v_n}} \). For ease of notation we shall write \( \chi = \chi_{\alpha v_n} \).

As \(-1 \in Z(\text{Sp})\) acts trivially on \( I \), we may assume without loss of generality that \( gv_n = v_n \). Write \( g = sa \), where \( s \in \text{Sp}_M \) and \( a \in \text{Sp}_{M,N} \). Now

\[
v_n = gv_n = sav_n = av_n + (sav_n - av_n).
\]

Here \( av_n \in N \) and \( sav_n - av_n \in M \). As \( M \) and \( N \) intersect trivially, we infer \( av_n = v_n \) and \( sv_n = v_n \). Since \( \text{Sp}_M \) acts on \( e_{\chi} \) via \( \chi \) and \( sv_n = v_n \), it follows from (3) that \( se_{\chi} = \chi \).

We may thus assume \( g = a \).

By means of the usual BN-pair decomposition of \( \text{Sp}_{M,N} \cong \text{GL}(M) \) we may write \( a = bwu \), where \( b \in B \), \( w \in W_0 \), and \( u \in T \). Now \( wv_n = v_n \), and therefore \( we_{\chi} = e_{\chi} \), so we may dispense with \( u \). From \( bwv_n = v_n \) we easily see that \( bv_n = v_n \) and \( wv_n = v_n \). Write \( b = hv \), where \( h \in H \) and \( v \in U \). Then \( vv_n = v_n \) and \( ve_{\chi} = e_{\chi} \). Hence \( hv_n = v_n \), and therefore \( he_{\chi} = e_{\chi} \) by (25). Thus we may assume \( g = w \). As \( wv_n = v_n \), \( w \in \langle w_1, \ldots, w_{n-2} \rangle \).
We may further suppose that $w = w_i$, $i = 1, \ldots, n - 2$. By Theorem 5.3 we have

$$we\chi = e\chi - (q + 1) \sum_{u \in U_w^+} \chi(u)^{-1}ue.$$ 

It follows that $we\chi = e\chi$ if and only if $l$ divides $q + 1$. \hfill \Box

5.5 Corollary Suppose $l$ divides $q + 1$. Let $\alpha \in F_q^*$ and $w_1, w_2 \in W_0$. Then $w_1e\chi_{avn} = w_2e\chi_{avn}$ if and only if $w_1v_n = w_2v_n$.

Let $L = L_\lambda$ be the sum of all $I_i$ inside $I$. As the $I_i$ are not isomorphic $FB$-modules, the sum is direct. It follows that $\dim L = X^+ = |Sp_M : \hat{S}_{v_n}|$.

5.6 Theorem Suppose $l$ divides $q + 1$. Then $L$ is an absolutely irreducible $Sp_M$-module isomorphic to $X_\lambda^+$. 

Proof. We first show that $L$ is $Sp_{M,N}$-invariant, affording a monomial representation.

A typical basis element of $L$ is of the form $uw(i)e\chi_{avn}$ where $1 \leq i \leq n$ and $u \in U_{w(i)^{-1}}$. Multiplying this on the left by $g \in Sp_{M,N}$ yields $guw(i)\chi_{avn}$, where $guw(i)$ is in $Sp_{M,N}$, and hence is of the form $bwv$. Here $b \in B$, $w \in W_0$ and $v \in T$. We know that $v \in T$ acts trivially on $e\chi_{avn}$. By Corollary 5.5 $we\chi_{avn} = w(j)e\chi_{avn}$, where $wv_n = v_j$. Write $b = uh$, where $h \in H$ and $u \in U$. Then $hw(j)e\chi_{avn} = w(j)e\chi_{ah_ivn}$ by (25), where $hv_j = h_jv_j$. Writing $u = u_1u_2$, where $u_1 \in U_{w(j)^{-1}}$ and $u_2 \in U_{w(j)^{-1}}$, we see that $uw(j)e\chi_{ah_ivn}$, and hence $guw(i)e\chi_{avn}$, is a scalar multiple of $u_1w(j)e\chi_{ah_ivn}$. Since this another typical basis element, we have shown that $L$ is $Sp_{M,N}$-invariant and that $L$ affords a monomial representation of $Sp_{M,N}$ relative to the given basis vectors, as claimed.

As seen in the proof of Theorem 5.1, $Sp_M$ sends every typical basis element to a scalar multiple of itself. We deduce that $L$ is $Sp_M$-invariant, and permutes the one dimensional subspaces $Fuw(i)e\chi_{avn}$; the above argument shows that the stabilizer of $Fe\chi_{avn}$ is $\hat{S}_{v_n}$; we have $\dim L = |Sp_M : \hat{S}_{v_n}|$. Therefore $L \cong \ind_{\hat{S}_{v_n}}^{Sp_M} Fe\chi_{avn}$. Since the linear character of $\hat{S}_{v_n}$ afforded by $Fe\chi_{avn}$ is $\chi_{v_n}^+$, the result follows from Theorem 3.7. \hfill \Box

Suppose $l$ divides $q + 1$. If in the definition (3) of $\chi_v$ we use $\lambda[\kappa]$, $\kappa \in F_q^*$, instead of $\lambda$, then $L$ will contain a copy of $X_\lambda^+$. Now for $\kappa$ a square $L_\lambda = L_{\lambda[\kappa]}$. However if $\kappa$ is not a
square then $L_\lambda \ncong L_{\lambda[k]}$ by Theorems 3.12 and 5.6. We may combine these comments and the preceding results in the following theorem.

5.7 Theorem Suppose $l$ divides $q+1$. The Steinberg module $I$ contains an irreducible $F\text{Sp}_M$-submodule isomorphic to $X_\lambda^+$. If $p$ is odd $I$ also contains a copy of $X_{\lambda[k]}^+$, where $\kappa \in F_q^*$ is not a square. They are not isomorphic, so $I$ contains their direct sum. Moreover, $X_\lambda^+ + X_{\lambda[k]}^+$ are isomorphic to $\theta^n Y_{\lambda}^+$ and $\theta^n Y_{\lambda[k]}^+$, where $Y_{\lambda}$ and $Y_{\lambda[k]}$ are the Weil modules of types $\lambda$ and $\lambda[k]$ restricted to $\text{Sp}_M$. If $l = 2$ then $X_{\lambda}^+$ and $X_{\lambda[k]}^+$ are isomorphic to the restriction to $\text{Sp}_M$ of the Weil modules $Y_{\lambda}^-$ and $Y_{\lambda[k]}^-$ of types $\lambda$ and $\lambda[k]$ and degree $(q^n - 1)/2$.

6 The Steinberg representation of $\text{Sp}$

We assume henceforth that $l = 2$ and $p$ is odd. For the remainder of the paper we shall write $\chi = \chi_{v_n}$. Let $e_1 = \sum_{u \in U} u e$. Then $S = Fe_1$ is the socle of $F\text{Sp}$-module $I$ (cf. Theorem 4.7 of [G]), affording the trivial representation of $\text{Sp}$. Set $\tilde{L} = \tilde{L}_\lambda = L_\lambda \oplus S$ and $\bar{L} = \bar{L}_\lambda = \tilde{L}_\lambda / S$. Our main goal is to show that the $F$-subspace $\tilde{L}_\lambda$ of $I/S$ is $\text{Sp}$-stable and isomorphic to the Weil module $Y_{\lambda}^+$ of degree $(q^n - 1)/2$ as an $F\text{Sp}$-module. Our main tool will be the following criterion.

6.1 Theorem Suppose the following identities hold in $I$:

$$c_n e_{\chi} = e_1 + \sum_{\alpha \in T_q} (\lambda(-2\alpha) + \lambda(2\alpha))e_{\chi_{\alpha v_n}},$$

$$c_{n-1} e_{\chi} = \sum_{\alpha \in F_q} w_{n-1} x_{n-1, n}(-\alpha) w_{n-1} e_{\chi}. \tag{28}$$

Then $\tilde{L}$ is $\text{Sp}$-stable and $\bar{L} \cong Y^+$ as $F\text{Sp}$-modules.

Proof. As $L$ and $Y^+$ are absolutely irreducible isomorphic $\text{Sp}_M$-modules, there is a unique $F\text{Sp}_M$-isomorphism $f : L \rightarrow Y^+$ up to multiplication by a non-zero scalar. The dimension of the $F$-subspace in $L$ and $Y$ where $U$ acts via $\chi_{v_n}$ is equal to one. Thus, we may assume without loss of generality that

$$f(e_{\chi_{v_n}}) = e^+_{v_n}. \tag{29}$$

24
Let \( \tilde{f} : \tilde{L} \rightarrow Y^+ \) be the \( F\text{Sp}_M \)-isomorphism inherited from \( f \). Thus
\[
\tilde{f}(x + S) = f(x), \quad x \in L.
\]
We wish to show that
(a) \( \tilde{L} \) is \( \text{Sp} \)-stable;
(b) \( \tilde{f} \) is an \( F\text{Sp} \)-isomorphism.

We first turn our attention to (a). We know that \( \text{Sp}_M \) and \( c_n \) generate \( \text{Sp} \), as \( \text{Sp}_M \) is a maximal subgroup of \( \text{Sp} \) and \( c_n \notin \text{Sp}_M \). Hence, to see that \( \tilde{L} \) is \( \text{Sp} \)-stable, we only need to verify that \( \tilde{L} \) is stable under \( c_n \).

A typical \( F \)-generator of \( L \) is of the form \( uhwe_\chi \) where \( u \in T \), \( w \in W_0 \) and \( h \in H \). We need to show that \( c_n(uhwe_\chi) \in \hat{L} \). We divide the proof into two cases.

Case 1. \( wv_n = v_n \). In this case \( c_n \) and \( w \) commute. Since \( T \) is contained in \( U^+_{c_n} \), \( c_n \) conjugates \( u \) into \( U \). As \( H \) is normalized by \( c_n \), it follows that \( c_n(uhw) \) belongs to \( \text{Sp}_M \). Therefore \( c_n(uhw)L = L \), and a fortiori \( c_n(uhw)\hat{L} = \hat{L} \). Since \( c_ne_\chi \) belongs to \( \hat{L} \) by (27), we infer
\[
c_n(uhwe_\chi) = c_n(uhw)c_n e_\chi \in \hat{L}.
\]

Case 2. \( wv_n = v_i, i \neq n \). By Corollary 5.5 we may write \( we_\chi = w'v_{n-1}e_\chi \) where \( w' \) is an element of \( W_0 \) that fixes \( v_n \) and moves \( v_{n-1} \) to \( v_i \). By means of the identity \( w_{n-1}c_nw_{n-1} = c_{n-1} \) we may write
\[
c_n(uhwe_\chi) = c_n(uhw')c_nw_{n-1}e_\chi = c_n(uhw')w_{n-1}c_nw_{n-1}e_\chi = c_n(uhw')w_{n-1}c_{n-1}e_\chi.
\]

Now \( c_nu \in U \), \( c_nw' = w' \), and \( c_nh \in H \). Therefore \( c_n(uhw')w_{n-1} \in \text{Sp}_M \), and hence \( c_n(uhw')w_{n-1}\hat{L} = \hat{L} \). Since \( c_{n-1}e_\chi \in \hat{L} \) by (28), it follows that \( c_n(uhwe_\chi) \in \hat{L} \). This proves (a).

In order to establish (b), it suffices to see that \( \tilde{f} \) commutes with the action of \( c_n \) on \( \hat{L} \), as \( \tilde{f} \) is already an isomorphism of \( F\text{Sp}_M \)-modules.

From Lemmas 4.4 and 4.5 we have
\[
c_ne^+_n = \sum_{\alpha \in \mathcal{T}_q} (\lambda(-2\alpha) + \lambda(2\alpha))e^+_{\alpha v_n}, \quad (30)
\]
\[ c_{n-1}^+ \epsilon_{v_n} = \sum_{\alpha \in F_q} \epsilon_{v_n + \alpha v_{n-1}}^+. \]  

(31)

Let \( \alpha \in F_q^* \), and let \( h \in H \) satisfy \( hv_n = \alpha v_n \). Then \( h e_{\chi v_n} = e_{\chi v_n} \) by (25). Therefore

\[ f(e_{\chi v_n}) = f(h e_{\chi v_n}) = hf(e_{\chi v_n}) = h\epsilon_{v_n} = \epsilon_{\alpha v_n}^+, \quad \alpha \in F_q^*. \]  

(32)

It follows that

\[ \tilde{f}(c_n e_{\chi} + S) = f( \sum_{\alpha \in T_q} (\lambda(-2\alpha) + \lambda(2\alpha)) e_{\chi v_n}) \]  

by (27)

\[ = \sum_{\alpha \in T_q} (\lambda(-2\alpha) + \lambda(2\alpha)) f(e_{\chi v_n}) \]
\[ = \sum_{\alpha \in T_q} (\lambda(-2\alpha) + \lambda(2\alpha)) \epsilon_{\alpha v_n}^+ \]  

by (32)

\[ = c_n \epsilon_{v_n}^+ \]  

by (30)

\[ = c_n \tilde{f}(e_{\chi} + S). \]

We also have

\[ \tilde{f}(c_{n-1} e_{\chi} + S) = f( \sum_{\alpha \in F_q} w_{n-1} x_{n-1,n}(-\alpha) w_{n-1} e_{\chi v_n}) \]  

by (28)

\[ = \sum_{\alpha \in F_q} w_{n-1} x_{n-1,n}(-\alpha) w_{n-1} f(e_{\chi v_n}) \]
\[ = \sum_{\alpha \in F_q} w_{n-1} x_{n-1,n}(-\alpha) w_{n-1} \epsilon_{v_n}^+ \]
\[ = \sum_{\alpha \in F_q} w_{n-1} x_{n-1,n}(-\alpha) \epsilon_{v_{n-1}}^+ \]  

by (18)
\[ = \sum_{\alpha \in F_q} \epsilon_{v_{n-1} + \alpha v_n}^+ \]  

by (18)
\[ = \sum_{\alpha \in F_q} \epsilon_{v_n + \alpha v_{n-1}}^+ \]  

by (18)
\[ = c_{n-1} \epsilon_{v_n}^+ \]  

by (31)
\[ = c_{n-1} \tilde{f}(e_{\chi} + S). \]

Thus \( \tilde{f} \) commutes with the actions of \( c_n \) and \( c_{n-1} \) on the single element \( e_{\chi} \). We proceed to verify that this suffices for \( \tilde{f} \) to commute with the action of \( c_n \) on the whole \( \tilde{L} \).
Let $uhwe_\chi$ be a typical $F$-generator of $L$, where $u \in T$, $w \in W_0$ and $h \in H$. Suppose first $wv_n = v_n$. Then

$$\tilde{f}(c_n(uhwe_\chi + S)) = \tilde{f}(c_n(\alpha^{\chi}c_n e_\chi + S))$$

$$= c_n(\alpha^{\chi})\tilde{f}(c_n e_\chi + S) \quad \text{since } c_n(\alpha^{\chi}) \in \text{Sp}_M$$

$$= c_n(\alpha^{\chi})c_n \tilde{f}(e_\chi + S) \quad \text{as just proven}$$

$$= c_nuhw \tilde{f}(e_\chi + S)$$

$$= c_n \tilde{f}(uhwe_\chi + S) \quad \text{since } uhw \in \text{Sp}_M.$$

If now $wv_n = v_i$ with $i \neq n$, we may write $w = w'w_{n-1}$ as above. Then

$$\tilde{f}(c_n(uhwe_\chi + S)) = \tilde{f}(c_n(\alpha^{\chi}w_{n-1}c_n^{-1}e_\chi + S))$$

$$= c_n(\alpha^{\chi})w_{n-1} \tilde{f}(c_n^{-1}e_\chi + S) \quad \text{since } c_n(\alpha^{\chi})w_{n-1} \in \text{Sp}_M$$

$$= c_n(\alpha^{\chi})w_{n-1}c_n^{-1} \tilde{f}(e_\chi + S) \quad \text{as just proven}$$

$$= c_nuhw'w_{n-1} \tilde{f}(e_\chi + S)$$

$$= c_nuhw' \tilde{f}(e_\chi + S)$$

$$= c_n \tilde{f}(uhwe_\chi + S).$$

Thus $\tilde{f}$ commutes with the action of $c_n$ on all of $\bar{L}$. \qed

We next show that formula (27) is indeed valid.

6.2 Theorem The following identity holds in $I$:

$$c_ne_\chi = e_1 + \sum_{\beta \in T_q} (\lambda(-2\beta) + \lambda(2\beta))e_{\chi_\beta}. \quad (33)$$

Proof. We start by proving the identity

$$c_ne_\chi = \sum_{u \in U_{c_n}^+} \sum_{\alpha \in F_q^*} \lambda(\alpha^{-1})ux_{n,2n}(\alpha)e. \quad (34)$$

We know that $\chi$ is trivial on $U_{c_n}^+$ (cf. Lemma 3.3) and that $U_{c_n}^-$ is the root subgroup $X_{n,2n}$. Moreover, $c_ne = -e$. Thus

$$c_ne_\chi = c_n(\sum_{u \in U_{c_n}^+} \sum_{v \in U_{c_n}^-} \chi(v)^{-1}uve)$$

$$= \sum_{u \in U_{c_n}^+} \sum_{\alpha \in F_q^*} \chi(x_{n,2n}(\alpha))^{-1}(c_n u)c_n x_{n,2n}(\alpha)e - \sum_{u \in U_{c_n}^+} (c_n u)e.$$
In our context, Steinberg’s formula (16) of [St] reads
\[ c_n x_{n,2n}(\alpha)e = x_{n,2n}(-\alpha^{-1})e - e, \quad \alpha \in F_q^*. \]

In the proof of Lemma 3.3 we established
\[ \chi(x_{n,2n}(\alpha)) = \lambda(\alpha), \quad \alpha \in F_q. \]

Therefore \[ \sum_{u \in U_n^+} \sum_{\alpha \in F_q^*} \chi(x_{n,2n}(\alpha))^{-1}(c_n u)c_n x_{n,2n}(\alpha)e \]
is equal to
\[ \sum_{u \in U_n^+} \sum_{\alpha \in F_q^*} \lambda(-\alpha)(c_n u)x_{n,2n}(-\alpha^{-1})e - \sum_{u \in U_n^+} \sum_{\alpha \in F_q^*} \lambda(-\alpha)(c_n)u e. \]

As \( \lambda \) is a non-trivial character \( F_q^* \)
\[ \sum_{\alpha \in F_q^*} \lambda(-\alpha) = -\lambda(0) = -1. \]

Thus
\[ - \sum_{u \in U_n^+} \sum_{\alpha \in F_q^*} \lambda(-\alpha)(c_n u)e = -(\sum_{\alpha \in F_q^*} \lambda(-\alpha))(\sum_{u \in U_n^+} (c_n u)e) = \sum_{u \in U_n^+} (c_n u)e. \]

Since \( c_n^2 \in H \), conjugation by \( c_n \) is an automorphism of \( U_n^+ \). Hence
\[ \sum_{u \in U_n^+} (c_n u)e = \sum_{u \in U_n^+} u e, \]
and
\[ \sum_{u \in U_n^+} \sum_{\alpha \in F_q^*} \lambda(-\alpha) c_n u x_{n,2n}(-\alpha^{-1})e = \sum_{u \in U_n^+} \sum_{\alpha \in F_q^*} \lambda(-\alpha) u x_{n,2n}(-\alpha^{-1})e \]
\[ = \sum_{u \in U_n^+} \sum_{\alpha \in F_q^*} \lambda (\alpha^{-1}) u x_{n,2n}(\alpha)e. \]

Combining the preceding equations we obtain (34).

In order to prove that the right hand sides of (33) and (34) are equal we shall compare their coordinates relative to the basis \( (ue)_{u \in U} \) of \( I \). A typical basis element is of the form \( u x_{n,2n}(\alpha)e \), where \( u \in U_n^+ \) and \( \alpha \in F_q \). The coefficient of \( u x_{n,2n}(\alpha)e \) in the right hand side of (34) is equal to 0 if \( \alpha = 0 \), and \( \lambda(\alpha^{-1}) \) otherwise. Now the coefficient of \( u x_{n,2n}(\alpha)e \) in the right hand side of (33), say \( C \), is equal to
\[ 1 + \sum_{\beta \in F_q} (\lambda(-2\beta) + \lambda(2\beta))\chi_{\beta v_n}(u x_{n,2n}(\alpha))^{-1}. \]
Here
\[ \chi_{\beta v_n}(ux_n,2n(\alpha))^{-1} = \chi_{\beta v_n}(x_n,2n(\alpha))^{-1} = \lambda(-\alpha\beta^2). \]

Thus
\[ C = 1 + \sum_{\beta \in T_q} (\lambda(-2\beta) + \lambda(2\beta))\lambda(-\alpha\beta^2). \]

If \( \alpha = 0 \) then \( C = 1 + \sum_{\beta \in F_q^*} \lambda(2\beta) = 1 + (-1) = 0 \), since \( \lambda \) is non-trivial and \( p \) is odd.

Suppose next that \( \alpha \neq 0 \). We may write \( C \) in the form
\[ 1 + \sum_{\beta \in F_q^*} \lambda(2\beta - \alpha\beta^2). \]

Suppose \( \beta \) is different from \( 2\alpha^{-1} \) and \( \alpha^{-1} \). Then the element \( \gamma = 2\alpha^{-1} - \beta \) of \( F_q \) is different from 0 and \( \beta \), and satisfies \( 2\beta - \alpha\beta^2 = 2\gamma - \alpha\gamma^2 \). As \( F \) has characteristic 2, the summands corresponding to \( \beta \) and \( \gamma \) yield
\[ \lambda(2\beta - \alpha\beta^2) + \lambda(2\gamma - \alpha\gamma^2) = 2\lambda(2\beta - \alpha\beta^2) = 0. \]

Now when \( \beta = 2\alpha^{-1} \), we have \( \lambda(2\beta - \alpha\beta^2) = \lambda(0) = 1 \). Added to the 1 on last formula for \( C \) yields 0 in \( F \). Thus the only contributing summand for \( C \) is the one corresponding to \( \beta = a^{-1} \). It gives \( \lambda(2\alpha^{-1} - \alpha(\alpha^{-1})^2) = \lambda(\alpha^{-1}) \). This completes the proof. \( \square \)

We proceed to verify the second condition of Theorem 6.1. We shall do this by comparing the coefficient of each basis element \( ue, u \in U \), on both sides of (28). We begin by considering the right hand side.

6.3 Theorem The following identity holds for all \( \alpha \) in \( F_q \):
\[ w_{n-1}x_{n-1,n}(-\alpha)w_{n-1}e_x = \sum_{u \in U_{w_{n-1}^+}} \sum_{v \in U_{w_{n-1}^-}} \chi_{v_n+\alpha v_{n-1}}(u)^{-1}uve. \]

Proof. By virtue of Theorem 5.3 we have
\[ w_{n-1}e_x = \sum_{u \in U_{w_{n-1}^+}} \sum_{v \in U_{w_{n-1}^-}} \chi_{v_{n-1}}(u)^{-1}uve. \]

Therefore
\[ x_{n-1,n}(-\alpha)w_{n-1}e_x = \sum_{u \in U_{w_{n-1}^+}} \sum_{v \in U_{w_{n-1}^-}} \chi_{v_{n-1}}(u)^{-1}(x_{n-1,n}(-\alpha)u)x_{n-1,n}(-\alpha)ve. \]
Observe that \( X_{(n-1,n)} \) normalizes \( U_{w_{n-1}}^+ \). Thus \( u \mapsto \chi_{v_{n-1}}(u^{x_{n-1,n}(-\alpha)}) \) is a linear character of \( U_{w_{n-1}}^+ \), which by Lemma 3.5 is equal to \( \chi_{v_{n-1}+\alpha v_n} \). This makes sense, since \( U_{w_{n-1}}^+ \) fixes \( v_n \) and \( v_{n-1} \) modulo \( M \), and it is therefore contained in \( S_{v_{n-1}+\alpha v_n} \). Thus

\[
x_{n-1,n}(-\alpha)w_{n-1}e\chi = \sum_{u \in U_{w_{n-1}}^+} \sum_{v \in U_{w_{n-1}}^+} \chi_{v_{n-1}}(u^{x_{n-1,n}(-\alpha)})^{-1} u(x_{n-1,n}(-\alpha)v)e
\]

As \( v \) runs through \( U_{w_{n-1}}^- \) so does \( x_{n-1,n}(-\alpha)v \). Hence

\[
x_{n-1,n}(-\alpha)w_{n-1}e\chi = \sum_{u \in U_{w_{n-1}}^+} \sum_{v \in U_{w_{n-1}}^-} \chi_{v_{n-1}+\alpha v_n}(u)^{-1} u(x_{n-1,n}(-\alpha)v)e.
\]

We may now apply Theorem 5.2 to obtain the desired result.

We wish to transform the above identity into something that can later be compared with the left hand side of (28). For simplicity of notation we introduce the symplectic transformations \( E(b, c, d) \in \text{Sp}^M \), defined by

\[
E(b, c, d) = x_{n-1,2n-1}(b)x_{n-1,2n}(c)x_{n,2n}(d)
\]

for all \( b, c, d \in F_q \). For \( a \in F_q \), we shall also write

\[
D(a) = x_{n-1,n}(a).
\]

Observe that every element of \( U_{w_{n-1}}^+ \) can be uniquely written in the form \( uE(b, c, d) \), where \( u \in U_{c_n c_n^-1}^+ \) and \( b, c, d \in F_q \). From Lemma 3.3 we infer that for such an element

\[
\chi_{v_n+\alpha v_{n-1}}(uE(b, c, d)) = \chi_{v_n+\alpha v_{n-1}}(E(b, c, d)).
\]

It follows from Theorem 6.3 that

\[
w_{n-1}D(-\alpha)w_{n-1}e\chi = \sum_{u \in U_{c_n c_n^-1}} \sum_{d \in F_q} \sum_{c \in F_q} \sum_{b \in F_q} \sum_{a \in F_q} \chi_{v_n+\alpha v_{n-1}}(E(b, c, d))^{-1} uE(b, c, d)D(a)e.
\]

(35)
6.4 Theorem  Let \( u \in U_{c_n}^{+} \) and let \( a, b, c, d \in F_q \). Let \( C \) be the coefficient of \( uE(b,c,d)D(a)e \) in the right hand side of (28). Then

\[
C = \begin{cases} 
0 & \text{if } b = 0 \text{ and } c \neq 0, \\
\lambda(-d) & \text{if } b = 0 \text{ and } c = 0, \\
\lambda(\frac{c^2}{b} - d) & \text{if } b \neq 0.
\end{cases}
\]

Proof. By (35)

\[
C = \sum_{\alpha \in F_q} \chi_{v_{n+\alpha v_{n-1}}}(E(b,c,d))^{-1}.
\]

From (3)

\[
\chi_{v_{n+\alpha v_{n-1}}}(E(b,c,d)) = \lambda(\langle E(b,c,d)(v_{n+\alpha v_{n-1}}), v_{n+\alpha v_{n-1}} \rangle).
\]

Now

\[
\langle E(b,c,d)(v_{n+\alpha v_{n-1}}), v_{n+\alpha v_{n-1}} \rangle = \langle cu_{n-1} + du_n + \alpha u_{n-1} + \alpha c u_n, \alpha v_{n-1} + v_n \rangle
\]

\[
= ba^2 + 2ca + d.
\]

It follows that

\[
C = \sum_{\alpha \in F_q} \lambda(-(ba^2 + 2ca + d)) = \lambda(-d) \sum_{\alpha \in F_q} \lambda(-(ba^2 + 2ca)).
\]

If \( b = 0 \) and \( c = 0 \) the \( C \) is equal to \( q\lambda(-d) \), which equals \( \lambda(-d) \) in \( F \) (as \( q \) is odd and \( l = 2 \)). If \( b = 0 \) and \( c \neq 0 \) then \(-2ca\) runs through \( F_q \) as \( \alpha \) runs through \( F_q \). Since \( \lambda \) is non-trivial, \( C \) equals 0. Suppose next that \( b \neq 0 \). We inquire for which \( \alpha \) in \( F_q \) there exists \( \beta \) in \( F_q \), different from \( \alpha \), so that \( ba^2 + 2ca = b\beta^2 + 2c\beta \). This occurs precisely when \( \alpha \neq -\frac{c}{b} \). Since \( l = 2 \) it follows that the only surviving summand in the above formula for \( C \) is

\[
\lambda(-b(-\frac{c}{b})^2 + 2c(-\frac{c}{b})) = \lambda(\frac{c^2}{b}).
\]

Multiplying this by \( \lambda(-d) \) we obtain the desired result. \( \square \)

We now turn our attention to the left hand side of (28).
6.5 Theorem  

The following relation holds in $I$:

$$c_{n-1}e_{\chi} = \sum_{u \in U_{c_{n-1}}^+} \sum_{d \in F_q} \sum_{e \in F_q} \sum_{b \in F_q} \sum_{a \in F_q} \lambda(-d) u E(0, 0, d) w_{n-1} c_n E(0, c, b) D(a) e.$$ 

Proof. Since $U_{c_{n-1}}^-$ is generated by the $D(a)$ and the $E(b, c, 0)$, it follows from Lemma 3.3 that $\chi$ is trivial on $U_{c_{n-1}}^-$. Therefore

$$e_{\chi} = \sum_{u \in U_{c_{n-1}}^+} \sum_{v \in U_{c_{n-1}}^-} \chi(u^{-1}) u v e.$$ 

Hence

$$c_{n-1}e_{\chi} = \sum_{u \in U_{c_{n-1}}^+} \sum_{v \in U_{c_{n-1}}^-} \chi(u^{-1}(c_{n-1} v)(c_{n-1} v) e).$$

As $c_{n-1}$ has order 2 modulo $H$, it follows that conjugation by $c_{n-1}$ is an automorphism of $U_{c_{n-1}}^-$. Therefore

$$c_{n-1}e_{\chi} = \sum_{u \in U_{c_{n-1}}^+} \sum_{v \in U_{c_{n-1}}^-} \chi(u^{c_{n-1}}) u (c_{n-1} v) e.$$ 

But $c_{n-1} v_n = v_n$, so Lemma 3.5 gives

$$c_{n-1}e_{\chi} = \sum_{u \in U_{c_{n-1}}^+} \sum_{v \in U_{c_{n-1}}^-} \chi(u^{-1}) u (c_{n-1} v) e.$$ 

Now every element of $U_{c_{n-1}}^+$ can be uniquely written in the form $u E(0, 0, d)$, where $u \in U_{c_{n-1}}^+$ and $d \in F_q$. Also, every element of $U_{c_{n-1}}^-$ can be uniquely written in the form $E(b, c, 0) D(a)$, where $a, b, c \in F_q$. Since $\chi$ is trivial on $U_{c_{n-1}}^+$, it follows that

$$c_{n-1}e_{\chi} = \sum_{u \in U_{c_{n-1}}^+} \sum_{d \in F_q} \sum_{c \in F_q} \sum_{b \in F_q} \sum_{a \in F_q} \lambda(-d) u E(0, 0, d) (c_{n-1} E(b, c, 0) D(a)) e. \quad (36)$$

Recall that $c_{n-1} = w_{n-1} c_n w_{n-1}$. Hence

$$c_{n-1}E(b, c, 0) D(a) e = w_{n-1} c_n w_{n-1} E(b, c, 0) D(a) e = w_{n-1} c_n E(0, c, b) w_{n-1} D(a) e.$$ 

If $a = 0$ then $w_{n-1} D(a) e = -e$, whereas if $a \neq 0$ then

$$w_{n-1} D(a) e = D(-a^{-1}) e - e.$$
Summing over all $a \in F_q$, the right hand side of (36) thus yields $1 + (q - 1) = q$ terms equal to
\[
- \sum_{u \in U_{c_n-1}^+} \sum_{d \in F_q} \sum_{c \in F_q} \sum_{b \in F_q} \lambda(-d)uE(0, 0, d)w_{n-1}c_nE(0, c, b)e,
\]
and one term equal to
\[
\sum_{u \in U_{c_n-1}^+} \sum_{d \in F_q} \sum_{c \in F_q} \sum_{b \in F_q} \sum_{a \in F_q^*} \lambda(-d)uE(0, 0, d)w_{n-1}c_nE(0, c, b)D(-a^{-1})e.
\]
As $-q \equiv 1 \mod l$ and $-a^{-1}$ runs through $F_q$ when $a$ runs through $F_q$, the result follows.

In order to proceed we record a number of relations in $Sp$.
\[
D(a)E(b, c, d)D(a)^{-1} = E(b + 2ac + a^2d, c + ad, d).
\]
In particular,
\[
D(a)E(b, 0, 0)D(a)^{-1} = E(b, 0, 0),
\]
\[
D(a)E(0, c, 0)D(a)^{-1} = E(2ac, c, 0),
\]
\[
D(a)E(0, 0, d)D(a)^{-1} = E(a^2d, ad, d).
\]
We also have
\[
c_nD(a)c_n^{-1} = E(0, a, 0),
\]
\[
c_nE(0, c, 0)c_n^{-1} = D(-c),
\]
\[
c_nE(b, 0, 0)c_n^{-1} = E(b, 0, 0),
\]
\[
w_{n-1}E(b, c, d)w_{n-1}^{-1} = E(d, c, b).
\]
In regards to Steinberg’s formula (16) of [St], we have the following:
\[
c_nE(0, 0, d)e = E(0, 0, -d^{-1})e - e, \quad d \neq 0
\]
and
\[
w_{n-1}D(a)e = D(-a^{-1})e - e, \quad a \neq 0.
\]
In what follows we shall make implicit use of the above formulæ. Observe that
\[
\begin{align*}
  w_{n-1}c_n E(0, c, b) D(a)e &= w_{n-1}c_n E(0, c, 0) D(a)D(-a) E(0, 0, b) D(a)e \\
  &= w_{n-1}c_n E(0, c, 0) D(a) E(a^2 b, -ab, b)e \\
  &= w_{n-1} D(-c) E(0, a, 0)c_n E(a^2 b, -ab, b)e \\
  &= w_{n-1} D(-c) E(0, a, 0) E(a^2 b, 0, 0) D(ab)c_n E(0, 0, b)e.
\end{align*}
\]

For \( b \neq 0 \), set
\[
f_1(a, b, c, d) = \lambda(-d) E(0, 0, d) w_{n-1} D(-c) E(0, a, 0) E(a^2 b, 0, 0) D(ab) E(0, 0, -b^{-1}) e,
\]
and under no restrictions on \( b \), we write
\[
f_2(a, b, c, d) = \lambda(-d) E(0, 0, d) w_{n-1} D(-c) E(0, a, 0) E(a^2 b, 0, 0) D(ab)e.
\]

Then \( c_{n-1} \epsilon \chi \) is equal to
\[
\sum_{u \in U_{\epsilon_{n-1} c_n}} \sum_{d \in F_q} \sum_{c \in F_q} \sum_{b^* \in F_q} \sum_{a \in F_q} uf_1(a, b, c, d) + \sum_{u \in U_{\epsilon_{n-1} c_n}} \sum_{d \in F_q} \sum_{c \in F_q} \sum_{b \in F_q} \sum_{a \in F_q} uf_2(a, b, c, d).
\]

Now \( w_{n-1} D(-c) E(0, a, 0) E(a^2 b, 0, 0) D(ab) E(0, 0, -b^{-1}) e \) is equal to
\[
E(0, a, a^2 b - 2ac) w_{n-1} D(ab - c) E(0, 0, -b^{-1}) e =
\]
\[
E(0, a, a^2 b - 2ac) w_{n-1} E(-(ab - c)^2 b^{-1}, -(ab - c)b^{-1}, -b^{-1}) D(ab - c) e =
\]
\[
E(0, a, a^2 b - 2ac) E(-b^{-1}, -(ab - c)b^{-1}, -(ab - c)^2 b^{-1}) w_{n-1} D(ab - c) e =
\]
\[
E(-b^{-1}, cb^{-1}, c^2 b^{-1}) w_{n-1} D(ab - c) e
\]
while \( w_{n-1} D(-c) E(0, a, 0) E(a^2 b, 0, 0) D(ab)e \) is equal to
\[
w_{n-1} E(a^2 b - 2ac, a, 0) D(ab - c)e = E(0, a, a^2 b - 2ac) w_{n-1} D(ab - c)e.
\]

Set
\[
\begin{align*}
g_1(a, b, c, d) &= \lambda(-d) E(-b^{-1}, cb^{-1}, -c^2 b^{-1} + d) D((c - ab)^{-1}), \quad b \neq 0, c \neq ab, \\
g_2(a, b, c, d) &= \lambda(-d) E(-b^{-1}, cb^{-1}, -c^2 b^{-1} + d), \quad b \neq 0, \\
g_3(a, b, c, d) &= \lambda(-d) E(0, a, a^2 b - 2ac + d) D((c - ab)^{-1}), \quad c \neq ab,
\end{align*}
\]
\[
g_4(a, b, c, d) = \lambda(-d)E(0, a, a^2b - 2ac + d).
\]

Then \(c_{n-1}e_\chi\) is equal to
\[
\sum_{u \in U_{c_{n-1}c_n}^+} \sum_{d \in F_q} \sum_{c \neq ab} \sum_{b \neq 0} \sum_{a \in F_q} u\eta_1(a, b, c, d) + \sum_{u \in U_{c_{n-1}c_n}^+} \sum_{d \in F_q} \sum_{c \neq ab} \sum_{b \neq 0} \sum_{a \in F_q} u\eta_2(a, b, c, d) + \sum_{u \in U_{c_{n-1}c_n}^+} \sum_{d \in F_q} \sum_{c \neq ab} \sum_{b \neq 0} \sum_{a \in F_q} u\eta_3(a, b, c, d) + \sum_{u \in U_{c_{n-1}c_n}^+} \sum_{d \in F_q} \sum_{c \neq ab} \sum_{b \neq 0} \sum_{a \in F_q} u\eta_4(a, b, c, d).
\]

Let \(u \in U_{c_{n-1}c_n}^+, v \in U_{w_{n-1}}^-\) and \(a, b, c, d \in F_q\). Then

1. If \(b \neq 0\) and \(v \neq 1\) then the coefficient of \(uE(-b^{-1}, cb^{-1}, -c^2b^{-1} + d)ve\) in \(c_{n-1}e_\chi\) is equal to \(\lambda(-d)\). Throughout this process, the triple \((b, c, d)\) is transformed into the triple \((-b^{-1}, cb^{-1}, d - c^2b^{-1})\). This transformation is invertible, and \((-b^{-1}, cb^{-1}, d - c^2b^{-1})\) is the triple transformed into \((b, c, d)\). It follows that, whenever \(b \neq 0\), the coefficient of \(uE(b, c, d)ve\) in \(c_{n-1}e_\chi\) is equal to \(\lambda(c^2b^{-1} - d)\).

2. If \(b \neq 0\) then the coefficient of \(uE(-b^{-1}, cb^{-1}, -c^2b^{-1} + d)e\) in \(c_{n-1}e_\chi\) is equal to \(q\lambda(-d) = \lambda(-d)\). Reasoning as above, we see that if \(b \neq 0\) the coefficient of \(uE(b, c, d)e\) in \(c_{n-1}e_\chi\) is equal to \(\lambda(c^2b^{-1} - d)\).

3. If \(v \neq 1\) then the coefficient of \(uE(0, 0, d)v\) in \(c_{n-1}e_\chi\) is equal to \(q\lambda(-d) = \lambda(-d)\).

4. The coefficient of \(uE(0, 0, d)\) in \(c_{n-1}e_\chi\) is equal to \(q^2\lambda(-d) = \lambda(-d)\).

5. If \(a \neq 0, v \neq 1\) and \(x \in F_q\) then the coefficient of \(uE(0, a, x)v\) in \(c_{n-1}e_\chi\) is equal to \(\sum_{d \in F_q} (-d) = 0\).

6. If \(a \neq 0\) and \(x \in F_q\) the coefficient of \(uE(0, a, x)\) in \(c_{n-1}e_\chi\) is equal to \(q \sum_{d \in F_q} (-d) = 0\).

By comparing the above with Theorem 6.4 we obtain

**6.6 Theorem**  The following identity holds in \(I\):

\[
c_{n-1}e_\chi = \sum_{a \in F_q} w_{n-1}(x_{n-1}\alpha w_{-1}e_\chi).
\]

Taking into account Theorems 6.1, 6.2 and 6.6 our main result is proven. If we then replace \(\lambda\) by \(\lambda[\kappa]\), \(\kappa\) not a square, in (3), we may state our result as follows.

**6.7 Theorem**  Let \(q\) be a power of an odd prime \(p\). Let \(I\) be the Steinberg module for \(\text{Sp}_{2n}(q)\) over a field \(F\) of characteristic 2 containing a primitive \(p\)-th root of unity. Write
$S$ for the socle of $I$, affording the trivial representation of $\text{Sp}_{2n}(q)$. Then $I/S$ contains as irreducible modules the two non-isomorphic Weil modules for $\text{Sp}_{2n}(q)$ over $F$ of degree $(q^n - 1)/2$.

We wish to end the paper by locating the two composition factors just found, in terms of the filtration for the Steinberg module over $F$ introduced by Gow in [G]. Adopting his notation, we have

6.8 Theorem Suppose $q \equiv 1 \mod 4$. Let $\kappa$ be the highest power of 2 dividing $|\text{Sp}_{2n}(q) : B|$. Then $I(\kappa)/I(\kappa - 1)$ contains as irreducible modules the 2-modular reductions of the two non-isomorphic complex Weil modules for $\text{Sp}_{2n}(q)$ of degree $(q^n - 1)/2$.

Proof. It is a matter of computing the 2-valuation of $|\text{Sp}_{2n}(q) : P_J|$, where $P_J$ is the parabolic subgroup of $\text{Sp}_{2n}(q)$ associated to the subset $J$ of \{w_1, ..., w_{n-1}, c_n\} corresponding to $\chi_{v_n}$. By Corollary 3.4, $J = \{c_n\}$. Therefore $P_J = B \cup Bc_nB$, a disjoint union. Since every element of $Bc_nB$ can be written uniquely in the form $bc_nv$, where $b \in B$ and $v \in U_{c_n}$, it follows that $|P_J| = (q + 1)|B|$. Consequently, the 2-valuation of $|\text{Sp}_{2n}(q) : P_J|$ is equal to $\kappa$ minus the 2-valuation of $q + 1$. As $q \equiv 1 \mod 4$, the highest power of 2 dividing $q + 1$ is 1, as required. \hfill \square

References

[St] R. Steinberg, Prime power representations of finite linear groups II, Canad. J. Math. 9 (1959), 347-351.

[Sz] F. Szechtman, Weil representations of the symplectic group, Journal of Algebra 208 (1998), 662–686.

[G] R. Gow, The Steinberg lattice of a finite Chevalley group and its modular reduction, to appear in Proc. London Math. Soc.