Recent Results from (Full) Lattice QCD

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An overview of the Lattice technique for studies of the strong interaction is given. Recent results from the UKQCD lattice collaboration are presented. These concentrate on spectral quantities calculated using full (i.e. unquenched) QCD. A comparison with quenched results is made. Novel methods of extracting spectral properties from two-point functions are described.
1 Introduction

Lattice Gauge Theory has been applied to the study of the strong interaction in earnest for the last 20 years or so. In that time, it has grown from an fledgling, optimistic area of research to a well-developed, mature (and optimistic!) discipline. The reason for this optimism is that all the approximations involved in the technique are systematically improvable. This means that, given enough time on a computer powerful enough, we will have predictions for the QCD bound-state spectrum (for example) with arbitrary accuracy. Other approaches of studying the strong interaction, while they may have their own advantages, cannot make this claim.

In this talk I will begin by overviewing the lattice method for obtaining hadronic spectral quantities via the calculation of $n$-point functions. I will outline the caveats that exist with current lattice simulations as well as detailing some of the successes of the approach. The vexed question of performing “full” QCD calculations (i.e. without the quenched approximation) is discussed and I will explain why these simulations are even more difficult than they at first seemed. In Sec. 3 I will detail some recent results from the UKQCD collaboration, focusing on our recent full QCD work. Sec. 4 outlines a new and promising approach of obtaining physical results from the lattice by using the spectral function representation. This is an exciting area of research, and, if it reaches its potential, promises to become a standard approach in the future. I then summarise the main points raised in this talk in Sec. 5.

2 Overview of Lattice Gauge Theory

2.1 The Method

I attempt here to describe the method normally used in lattice calculations of the hadronic spectrum of QCD. There are many excellent reviews of this topic which cover the approach in more detail.

The conventional lattice approach to QCD spectrum calculations is performed in the Euclidean path integral framework. It requires the calculation of $n$-point correlation functions, $G_n(t)$, of hadronic interpolating operators, $J$, in a background sea of glue (and sea quarks in the case of full, i.e. “unquenched” calculations). A Monte Carlo approach is used to generate these background configurations with the appropriate Boltzmann factor $e^{-S}$ where $S$ is the Euclidean action. There is an obvious analogy between this Lagrangian approach to Lattice Gauge Theory, and statistical mechanics: clearly the lattice path integral corresponds to the partition function of statistical mechanics.

Note that in lattice simulations there is freedom to choose different quark fields in the interpolating operator $J$ compared to those in the Lagrangian. Hence we are able to distinguish between “valence” and “sea” quark masses, $m^{val}$ and $m^{sea}$. 
Because the QCD Lagrangian contains fermionic fields that appear quadratically, they can be integrated out analytically giving the usual determinant factor, \( \det(\mathcal{D} + m) \). Including this factor is a technical headache because it involves a huge increase in computational requirements. The usual way around this is to invoke the quenched approximation where two things are done: (i) the fermionic determinant is replaced by unity; and (ii) the gauge coupling, \( g \), is rescaled so that the physical predictions for a particular test quantity (like the rho mass, for example) is in agreement with its experimental value. The extent to which the quenched approximation reproduces physical predictions for other quantities is a measure of its success. It is a remarkable fact that for many spectral quantities, the quenched predictions agree with the experimental (i.e. the “full” QCD) values to within 10% (see [2]).

Despite the success of the quenched approximation, it is obviously essential to perform full QCD calculations in order to study the real world. Furthermore, for some quantities (such as the deconfinement temperature[3]) the quenched approximation is poor, and for others (such as the \( \eta' \) mass) the quenched approximation fails completely.

In the calculation of the two-point function, \( G_2(t) \), if the exact operator, \( J^{\text{exact}} \) for the hadron in question was used, then \( G_2(t) \) would contain information on that hadron and no others. However, since \( J^{\text{exact}} \) is not known, \( G_2(t) \) receives contributions from all hadronic states which have non-zero overlap with \( J \). It is straightforward to show that, in this case, \( G_2(t) \) has the following form

\[
G_2(t) = \sum_i Z_i e^{-M_i t}
\]

where the sum is over the hadronic states \( i \), and \( M_i \) and \( Z_i \) are the corresponding hadronic mass and overlap. Note that since the calculation is performed in Euclidean space-time, the excited states are exponentially suppressed with respect to the fundamental state (i.e. the exponentials have real arguments). This means that \( G_2(t) \) asymptotes to the two-point function of the ground state hadron as \( t \to \infty \). Obviously the parameters of the ground state, in particular the mass \( M \equiv M_0 \), can be extracted by fitting \( G_2(t) \) to an exponential, \( e^{-M t} \), for \( t \) sufficiently large.

### 2.2 Caveats

In this sub-section I explain the caveats that one must apply to any calculation using the method described above. The main point to make is that the calculations of the hadronic properties are performed with input parameter values which are \textit{not} those of the real world. Specifically, these parameters are

- valence quark mass(es), \( m^{\text{val}} \), (typically \( m^{\text{val}} \gtrsim \frac{1}{2} m^{\text{strange}} \));
- sea quark mass(es), \( m^{\text{sea}} \), (typically \( m^{\text{sea}} \gtrsim m^{\text{strange}} \));
- lattice volume, \( V \), (typically \( V \lesssim (2 \text{ fm})^3 \));
• lattice spacing, $a$, or, through dimensional transmutation, $g_0$, (typically $a \gtrsim 0.05 \text{ fm}$);

• number of dynamical fermion flavours, $N_f$. Quenching corresponds to $N_f = 0$.
  (Typically, $N_f = 0$ or 2.)

Therefore the mass, $M$, which was obtained using the procedure above is not the mass of the real world hadron, but the mass of the corresponding hadron in a world where the quark masses are the same as those input into the lattice calculation, the volume is the finite volume used in the simulation etc. So, strictly speaking, $M$ is a function of the above 5 input parameters. The final prediction of the real world value, $M^{\text{expt}}$, should be obtained by the following extrapolations:

\[
\sqrt{\sqrt{\sqrt{m^{\text{val}} \to \text{few MeV}}}} \quad \sqrt{\sqrt{m^{\text{sea}} \to \text{few MeV}}} \\
\sqrt{\sqrt{V \to \infty}} \\
\sqrt{\sqrt{a \to 0}} \\
\sqrt{N_f \to \frac{21}{2}} \text{ i.e. two light flavours for } u, d \text{ and one heavier for } s
\]

Note that the limit $m^{\text{sea}} = \infty$ corresponds to $N_f = 0$ so the $m^{\text{sea}}$ and $N_f$ extrapolations are not independent. While these extrapolations muddy the water significantly, many of them are theoretically well-understood and numerically under control. The number of $\sqrt{}$’s which appear next to the extrapolations is an indication of how well the extrapolation is under control.

As an example of the quality of the extrapolations, Fig. 1 shows an extrapolation of the nucleon and vector meson masses as a function of the lattice spacing, $a$, taken from [4]. In these plots, data points obtained with both the Wilson and various improved lattice actions (designed to have discretisation errors smaller than $O(a)$) are shown. Fits to the relevant functional forms are included in the figure and the symbols on the left of the plot are the continuum extrapolations (which clearly agreement with each other).

### 2.3 Successes

The caveats listed in the previous sub-section do not hinder the success of the Lattice technique as a means of obtaining accurate predictions from the strong interaction. To give an example of the current status of lattice calculations, Fig. 2 shows the hadronic spectrum obtained by the CP-PACS collaboration using the quenched approximation[2].

There are two features to note. The error bars in the predictions are tiny ($\lesssim 3\%$) - which is a clear measure of the success of the lattice technique. Secondly, there is a small, but statistically significant discrepancy between the lattice predictions and the experimental numbers - which is a signal that unquenching is required in order to make further progress.
While the quenched lattice calculations have clearly matured with precision estimates of many quantities of only a few percent, serious calculations involving full QCD have only recently begun. Typically the current errors in these calculations are several times that of equivalent quenched results.

3 Recent Dynamical Results from the UKQCD Collaboration

In this section I review some of the recent results from the UKQCD Collaboration’s dynamical simulations.

3.1 Dependency on Sea Quark Mass

As was outlined in Sec. 2.2 all lattice predictions are functions of $V$, $a$, $N_f$ etc. In this section, the functional dependency on $m^{sea}$ is discussed. Fig. 3 (taken from [5]) shows how the lattice spacing, obtained from the hadronic length scale $r_0$, $r_0 \approx 0.5 \text{fm}$, depends strongly on the sea quark mass, $m^{sea}$ (here expressed in terms of the hopping
parameter $\kappa^{sea}$). These calculations were performed at fixed bare coupling, $g_0$.

This effect has important consequences. Simulations at a fixed bare coupling, $g_0$, and with several values of $m^{sea}$, correspond to different physical volumes, and, furthermore, different points on the continuum extrapolation $a \to 0$. Thus finite volume and $O(a)$ systematics are mixed in this case.

UKQCD have performed two types of calculation. The first was at fixed $g_0$ for various values of $m^{sea}$ in order to calibrate this effect[5]. We then performed more sophisticated simulations at several values of the parameter pairs $(g_0, m^{sea})$ which were chosen in order to maintain fixed lattice spacing $a \approx 0.11\, fm$ (and therefore also fixed volume)[6]. This second calculation utilised the “matching” technology of [8]. In both these calculations, an improved action was used in order to reduce the effect of $O(a)$ errors.

3.2 Results

There is not space to present full details of UKQCD’s recent unquenched calculations. I discuss the results of only two quantities, and refer the reader to the original papers for full details.[5, 6]

One of the benchmark quantities of lattice calculations is the static quark potential. Fig. [1] shows UKQCD’s result for this quantity in units of $r_0$ from [5]. It can be noted that there is little immediate dependency on $m^{sea}$ in this quantity[5].

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1 A closer look however at the data points close to the origin shows a systematic effect which can
Figure 3: The inverse lattice spacing plotted against quark mass (expressed in terms of $1/\kappa_{\text{sea}}$) for different lattice sizes taken from [5]. The chiral limit is approximately at the left margin of the figure.

Fig. 5 shows the vector meson mass, $M_V$, plotted against the pseudoscalar mass squared, $M_{PS}^2$, from [5]. Again, these quantities are expressed in units of $r_0$. Quenched data which corresponds to the same lattice spacing as the unquenched simulations are included as a comparison. The experimental points corresponding to the strange mesons are also plotted.

In summary, the results of [5, 6] indicate that the effects of unquenching are small for these values of $m_{\text{sea}} \gtrsim m_{\text{strange}}$. This motivates the need to move to more physical values of $m_{\text{sea}} \approx \text{few GeV}$ in the future.

4 Lattice Spectral Functions

As outlined in Sec. 2.1, the conventional method of determining ground state properties from lattice simulations is by fitting exponentials to the tails of $n$-point functions. There have been several attempts at developing other strategies for uncovering spectral quantities from lattice data. These all revolve around the spectral function (SF) $\rho(s)$ which can be defined through

$$G_2(t) = \int_0^\infty K(t, s)\rho(s)ds,$$

where $K(t, s)$ is the kernel function - typically just $e^{-st}$ for this work. The SF contains much richer information on the channel being considered than just the ground state parameters. It also has the advantage that theoretical input can be used to guide be interpreted as different runnings of the coupling as function of $m_{\text{sea}}^\text{sea}$. 

its form for large values of $s$ where perturbation theory is valid. In fact, this was
the approach taken by \cite{9,12} who used (continuum) perturbation theory to derive a
functional form for $\rho_{PT}(s)$. This $\rho_{PT}(s)$ was then used above a certain threshold of
energy $s_0$ and a $\delta$ function used for the ground state following the approach of QCD
Sum Rules.

A very new and promising technique takes the marriage of lattice data and spectral
functions one step further.\cite{14} This approach uses the lattice data itself to determine
the SF by inverting eq.\,(2). This is a very numerically technical approach; it is in fact
an “ill-posed” problem - $G_2(t)$ is known only at a small number of discrete values of
t, whereas the aim is to determine $\rho(s)$ for a large number of values of $s$ (ideally for a
continuous range in $s$). In \cite{14} a “maximum entropy method” is employed to overcome
these hurdles.

5 Conclusion

I have given a brief overview of the current state of play of lattice gauge theory calcula-
tions of the hadronic spectrum. The problems of extrapolating the input parameters of
any lattice simulation to their physical value is emphasised. In particular I have shown
that interpreting results from naive unquenched calculations can prove difficult due to
the dependency of the lattice spacing on $m^{sea}$. A summary has been given of recent
unquenched results from the UKQCD collaboration for the static quark potential and
vector meson mass. Finally the new and interesting method of using spectral functions
in the analysis of lattice data is discussed.
Figure 5: Vector mass plotted against the pseudoscalar mass for several sets of different sea quark masses from [4]. The experimental points are denoted by * and the quenched results are labelled $\kappa_{\text{sea}} = 0$.

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