Low-Temperature Quantum Critical Behaviour of Systems with Transverse Ising-like Intrinsic Dynamics.

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Abstract

The low-temperature properties and crossover phenomena of \(d\)-dimensional transverse Ising-like systems within the influence domain of the quantum critical point are investigated solving the appropriate one-loop renormalization group equations. The phase diagram is obtained near and at \(d = 3\) and several sets of critical exponents are determined which describe different responses of a system to quantum fluctuations according to the way of approaching the quantum critical point. The results are in remarkable agreement with experiments for a wide variety of compounds exhibiting a quantum phase transition, as the ferroelectric oxides and other displacive systems.
1. Introduction

The topic of quantum phase transitions (QPT’s), where the order is destroyed solely by quantum fluctuations, is a very active field in condensed matter physics research [1]. These transitions, which occur only at temperature \( T = 0 \) varying a nonthermal control parameter, are considered to play an important role for understanding the unconventional behaviour of many quantum systems at low temperatures such as non-trivial power laws or non-Fermi liquid physics. Indeed, the existence of an influence domain around a \( T = 0 \) quantum critical point (QCP) is thought to be relevant for gaining insight into several low temperature phenomena using the reliable foundations of the modern theory of critical phenomena.

Recent theoretical studies [1-7] have shown that a lot of information can be extracted from the temperature-dependent Renormalization Group (RG) equations as an expression of the most reliable theory of thermal and quantum critical fluctuations. Along this direction, a rich low-temperature scenario was obtained in Refs. [4-6] for the case of an interacting Bose gas with the chemical potential fixed at its QCP value.

Unfortunately, the inherent simplicity of the RG equations for bosonic systems is lost when one considers other microscopic models exhibiting a QPT. Important examples are the transverse Ising model (TIM)[8] and the quantum displacive models [9,10]. These models and related ones [5,8,11] can be described in an unified way also near the QCP and the RG treatment
offers the possibility to obtain the low-temperature properties in the domain of influence of the QCP near and at $d = 3$ dimensions taking correctly into account the competing quantum and thermal fluctuations. Nevertheless, the problem does not appear so simple to be solved as for bosonic systems and some attention must be paid to extract the correct low-temperature quantum critical behaviour of the relevant thermodynamic quantities.

The aim of the present paper is to give a solution of the problem near and at $d = 3$ by solving the appropriate RG equations in the low temperature limit. This allows us to obtain asimptotically exact results so that a suitable comparison with the experiments becomes possible.

The outline of this paper is as follows. In section 2 we introduce the model together with the $T$-dependent one-loop RG equations and give a summary of the $(T = 0)$-quantum critical predictions relevant for our purposes. Section 3 is devoted to derive the solution of the RG equations in the low-temperature regime to leading order in the interaction parameter. In section 4 the global expressions of the relevant thermodynamic quantities and the phase diagram of the models in the influence domain of the QCP are obtained. The low-temperature quantum critical properties are considered in section 5 and the related crossover phenomena are studied in section 6, where a comparison with experiments is also performed. Finally, in section 7, some concluding remarks are made.
2. One-loop renormalization group equations for transverse Ising-like models.

We focus our attention on the transverse Ising-like models described by the quantum action (in convenient units):

\[ \mathcal{H}\{\vec{\psi}\} = \frac{1}{2} \int d^d x \int_0^{1/T_0} d\tau \left\{ (\nabla \vec{\psi}(\vec{x},\tau))^2 + \left( \frac{\partial \vec{\psi}(\vec{x},\tau)}{\partial \tau} \right)^2 + r_0 \vec{\psi}^2(\vec{x},\tau) + \frac{u_0}{8} \vec{\psi}^4(\vec{x},\tau) \right\} \]

(2.1)

where \( \vec{\psi}(\vec{x},\tau) \equiv \{ \psi_j(\vec{x},\tau); j = 1, \ldots, n \} \) is an \( n \)-vector ordering field and the coupling parameters \( r_0, u_0 \) depend on the quantum model under study: for the TIM and related real systems see, for instance, Refs. [8,12]; for quantum displacive systems [9,10] \( r_0 \) has to be identified as the opposite of the strength of interaction parameter \( S \) and \( u_0 > 0 \) measures the anharmonic part of the interaction.

Now, we can apply the Wilson RG procedure to the action (2.1) in the \((\vec{k},\omega_l)\)-Fourier space with \( 0 < |\vec{k}| < 1 \). Reducing the degrees of the freedom and rescaling the wave vectors \( \vec{k} \), the Matsubara frequencies \( \omega_l = 2\pi l T \) (\( l = 0 \pm 1, \pm 2, \ldots \)) and the ordering field \( \vec{\psi}(\vec{k},\omega_l) \), we obtain, to one-loop approximation, the following T-dependent RG differential equations for the renormalized parameters \( r(l), u(l), T(l) \) (see for instance Ref.[4]):

\[
\begin{align*}
\frac{dr(l)}{dl} &= 2r(l) + \frac{n + 2}{4} K_d u(l) F_1 (r(l), T(l)) \\
\frac{du(l)}{dl} &= \varepsilon u - \frac{n + 8}{4} K_d u^2(l) F_2 (r(l), T(l)) \\
\frac{dT(l)}{dl} &= z T(l)
\end{align*}
\]

(2.2)
with $\eta = 0$ for the Fisher correlation exponent and $K_d = 2^{1-d}\pi^{-d/2}/\Gamma(\frac{d}{2})$, where $\Gamma(x)$ is the gamma function. In Eqs. (2.2), $\varepsilon = 3 - d$, $z = 1$ is the dynamical exponent for the quantum model and the functions $F_i(r, T)$ ($i = 1, 2$) are given by
\begin{align*}
F_1(r, T) &= \frac{1}{2}(1 + r)^{-\frac{1}{2}} \coth \left[ \frac{(1 + r)^{\frac{1}{2}}}{2T} \right] \tag{2.3} \\
F_2(r, T) &= -\frac{\partial F_1(r, T)}{\partial r} = \frac{1}{4}(1 + r)^{-\frac{3}{2}} \coth \left[ \frac{(1 + r)^{\frac{1}{2}}}{2T} \right] + \\
&\quad + \frac{1}{8T}(1 + r)^{-1} \sinh^{-2} \left[ \frac{(1 + r)^{\frac{1}{2}}}{2T} \right]. \tag{2.4}
\end{align*}

In order to extract all the possible information about criticality, crossover phenomena and thermodynamic properties of the models here considered, one should exactly solve the system of Eqs. (2.2) with the initial conditions:
\begin{equation}
r(0) = r_0, \ u(0) = u_0, \ T(0) = T_0 \equiv T. \tag{2.5}
\end{equation}

Note that for systems whose functional representation is generated by a Hubbard-Stratonovich transformation, one expects both $r_0$ and $u_0$ to depend on temperature as, for example, in the TIM [8]. However, this $T$-dependence is usually negligible in comparison to that near criticality, so we do not consider it explicitly through this paper.

Solving the RG equations (2.2) with the initial conditions (2.5) is a very difficult task and, to obtain explicit results, one must resort to suitable approximations or treatable asymptotic regimes. In the original pioneering
treatments [11-13] the interest was essentially focused on the so called “quantum” regime at \( T = 0 \) and on the “classical” one as \( T(l) \to \infty \) by iteration of the RG transformation. Here, for future utilities we summarize the main predictions which are relevant for us.

The finite-temperature fixed points (FP’s) of Eqs.(2.2), expressed in terms of the most appropriate coupling parameter \( v = u \cdot T \), govern the critical behaviour in the classical regime, which is well known in the literature [14-16]. Indeed, in such a case, the quantum degrees of freedom are ineffective and any quantum system behaves, in terms of the critical deviation \( r_0 - r_{0c}(T) \), as a classical one with the same dimensionality and order parameter symmetry, in agreement with the universality hypothesis. Quantum effects become relevant as \( T \to 0 \). In particular, at \( T = 0 \) and in terms of the nonthermal deviation parameter \( r_0 - r_{0c} \), a critical behaviour different from the classical one occurs.

For a generic \((T = 0)\)-FP \((r^*, u^*)\), the related eigenvalues to first order in \( \varepsilon \) are [5]

\[
\begin{align*}
\lambda_r &= 2 - \frac{n + 2}{16} K_d u^* \\
\lambda_u &= \varepsilon - \frac{n + 8}{8} K_d u^*
\end{align*}
\]

(2.6)

with \( r^* = u^* = 0 \) for the Gaussian FP and \( r^* = \frac{n + 2}{n + 8} \varepsilon, \ u^* = \frac{16}{(n + 8) K_d} \varepsilon \) for the nongaussian one.

One sees that, since in any case \( \lambda_r > 0 \), the \((T=0)\)-critical surface and the related QCP \((T_c = 0, r_{0c})\) are determined by imposing that the linear
scaling field $t_r = r + bu = t_{r_0}e^{\lambda_r}$, with $b = \frac{(n+2)}{16}K_d$, is zero. This yields

$$r_{0c} = -bu_0 .$$

(2.7)

Then, in terms of variable $t_{r_0} = r_0 - r_{0c}$ driving the QPT, the critical exponents, which govern the quantum critical behaviour approaching the QCP along the $(T = 0)$-isotherm, can be easily obtained around $d = 3$. For instance, the correlation length and the susceptibility exponents $\nu_r = 1/\lambda_r$ and $\gamma_r = 2\nu_r$ (with $\eta = 0$), to first order in $\varepsilon = 3 - d$, are given by:

$$\nu_r = \frac{1}{2} + \frac{n + 2}{4(n + 8)}\varepsilon , \quad \gamma_r = 1 + \frac{n + 2}{2(n + 8)}\varepsilon , \quad d < 3 \quad (2.8)$$

$$\nu_r = \frac{1}{2} , \quad \gamma_r = 1 , \quad d > 3 \quad (2.9)$$

For $d = 3$, logarithmic corrections to the mean-field exponents occur. Notice that for $d > 3$, the remaining mean-field exponents can be obtained, as usual, taking into account that $u_0$ is a dangerous irrelevant variable [14-16].

The previous RG analysis shows that, as $T \to 0$, a dimensional crossover $d \to d + 1$ occurs with $X_{quantum}(d) = X_{classical}(d + 1)$ for a generic critical exponent $X$.

An important feature to be noted is that the temperature $T$, as well as the parameter $r_0$, is a relevant scaling field ($T(l) = Te^l$). Therefore also a small but finite value of $T$ measures a deviation from the $(T=0)$-critical surface. However, for extracting all the relevant information about the critical behaviour around the QCP, the parameter $T$ cannot be treated as $r_0$ since the functions $F_i(r, T)$ cannot be expanded in power series of $T$ and
hence, as $T \to 0$, it is not possible to follow the usual linearization scheme of the RG analysis. This unfortunate feature forces to solve the full $T$-dependent RG Eqs. (2.2), at least in the low-temperature limit. For overcoming this intrinsic difficulty of the problem we introduced in ref. [4] the “ad hoc” concept of “temperature-dependent linear scaling fields”. Although this idea allowed us to capture the essential aspects of the low-$T$ quantum criticality for a wide variety of quantum systems, it appears as a rather “nonconventional ansatz” which needs further supporting justifications. The next sections are just devoted to give such a support; moreover further physical predictions are given solving exactly Eq. (2.2) to order $O(\varepsilon, u)$ and in the low-$T$ regime, without additional artificious assumptions.

3. Solution of the RG equations in the low-temperature regime to leading order in the interaction parameter.

Our aim is to solve the general one-loop Eqs. (2.2) in the low-temperature limit to leading order in $u_0 \ll 1$ (with $u_0 \sim O(\varepsilon)$ if $\varepsilon \neq 0$). In our considerations we assume $u = O(\varepsilon) \ll 1$ and work for low-$T$ RG flow correct to the order $O(\varepsilon, u_0)$.

First, let us consider the simplified equation for $u$ (setting in the second equation of (2.2) $r = 0$ and $T = 0$):

$$\frac{du}{dl} = \varepsilon u - \frac{n + 8}{16} K_d u^2$$

(3.1)
with the initial condition \( u(0) = u_0 \). This can be solved exactly and we have:

\[
u(l) = \frac{u_0 e^{\varepsilon l}}{Q(l)} = O(\varepsilon, u_0)
\tag{3.2}
\]

where

\[
Q(l) = 1 + \frac{n + 8}{16} Kd \frac{u_0}{\varepsilon} (e^{\varepsilon l} - 1).
\tag{3.3}
\]

In the limit \( \varepsilon = 0 \), Eq.(3.2) gives:

\[
u(l) = \frac{1}{C_0(l + l_0)}
\tag{3.4}
\]

where

\[
C_0 = \frac{n + 8}{16} K_3 = \frac{n + 8}{32 \pi}
\tag{3.5}
\]

and

\[
l_0 = \frac{1}{C_0 u_0}.
\tag{3.6}
\]

It is immediate to see that

\[
Q(l) \rightarrow \begin{cases} 
1, & l \rightarrow 0 \\
\frac{n + 8}{16} Kd \frac{u_0}{\varepsilon} e^{\varepsilon l}, & \varepsilon > 0 \\
1 + \frac{n + 8}{16} Kd \frac{u_0}{|\varepsilon|}, & \varepsilon < 0
\end{cases}, \quad l \rightarrow \infty
\tag{3.7}
\]

and hence, as expected:

\[
u(l) \rightarrow \begin{cases} 
u_0, & l \rightarrow 0 \\
u^* = \begin{cases} 
\frac{16}{(n + 8) Kd}, & \varepsilon > 0 \\
0, & \varepsilon < 0
\end{cases}, & l \rightarrow \infty
\tag{3.8}
\]

The solution (3.2) of the simplified Eq. (3.1) is not, obviously, a solution of the full equation for \( u(l) \) which involves contributions arising from \( r(l) \).
and $T(l)$. We now prove that the factor $F_2(r(l), T(l))$, previously neglected in the second of Eqs. (2.2), contributes to the reduced solution only to the order $O(\varepsilon^2, u_0^2)$.

Let us assume a full solution of the form

$$u(l) = \tilde{u}(l) f(l)$$

(3.9)

where $\tilde{u}(l) = O(\varepsilon, u_0)$ denotes the reduced solution (3.2) and $f(l)$ has to be determined with the initial condition $f(0) = 1$. From Eq. (2.2) we get for $f(l)$ the equation

$$\frac{df(l)}{dl} = \frac{n + 8}{16} K_d \tilde{u}(l) f(l) \left[ 1 - 4 f(l) F_2(r(l), T(l)) \right].$$

(3.10)

Eq. (3.10) has the “Bernoulli equation” form $dy/dx + P(x)y = Q(x)y^n$ and hence we have for $f(l)$ the standard solution:

$$f(l) = e^{\frac{n+8}{16} K_d \int_0^l dl' \tilde{u}(l')} \times$$

$$\times \left\{ 1 + \frac{n + 8}{4} K_d \int_0^l dl' \tilde{u}(l') F_2(l') e^{-\frac{n+8}{16} K_d \int_0^{l'} dl'' \tilde{u}(l'')} \right\}^{-1},$$

(3.11)

where $F_2(l) \equiv F_2(r(l), T(l))$. On the other hand $\tilde{u}(l) = O(\varepsilon, u_0)$ and we are interested to obtain only results correct to the order $O(\varepsilon, u_0)$. Then, from the formal expression (3.11) it follows

$$f(l) = 1 + \frac{n + 8}{16} K_d \int_0^l dl' \tilde{u}(l') (1 - 4 F_2(l')) + O(\varepsilon^2, u_0^2)$$

$$= 1 + O(\varepsilon, u_0).$$

(3.12)
This allows us to conclude that \( u(l) = \tilde{u}(l)(1 + O(\varepsilon, u_0)) = \tilde{u}(l) + O(\varepsilon^2, u_0^2) \).

So, \( F_2(r, T) \)-contributions, and hence \( T \)-dependent terms, do not enter \( u(l) \) to \( O(\varepsilon, u_0) \) order.

The problem to integrate the system (2.2) reduces now to solve only the equation for \( r(l) \) with \( u(l) \equiv \tilde{u}(l) \) and \( T(l) = Te^l \), to the order of interest.

An explicit solution for \( r(l) \) with arbitrary \( T \) is, of course, hopeless but the problem becomes accessible working in the low-temperature regime. The solution can be formally written as

\[
  r(l) = r_0 e^{\Lambda_r(l)} h(l) \tag{3.13}
\]

with

\[
  \Lambda_r(l) = 2l - \frac{n + 2}{4} K_d \int_0^l dl' u(l') F_2^{(0)}(l') \nonumber \\
  = 2l + O(\varepsilon, u_0) \tag{3.14}
\]

\[
  h(l) = 1 + \frac{1}{r_0} \frac{n + 2}{4} K_d \int_0^l dl' e^{-\Lambda_r(l')} u(l') F_1^{(0)}(l') \tag{3.15}
\]

and \( F_i^{(0)}(l') \equiv F_i(0, T(l')) \) \( (i = 1, 2) \).

Since \( \exp(-\Lambda_r(l)) = e^{-2l} + O(\varepsilon, u_0) \) and \( du/dl = O(\varepsilon^2, u_0^2) \), \( h(l) \) with integration by parts in (3.15) becomes:

\[
  h(l) = 1 + \frac{1}{r_0} \frac{n + 2}{16} K_d \left( u_0 - e^{\Lambda_r(l)} u_l \right) + \frac{1}{r_0} \frac{n + 2}{4} K_d \int_0^l dl' \frac{e^{-2l'} u(l')}{e^{1/T(l')} - 1}. \tag{3.16}
\]
Eq. (3.16) suggests to express more simply the required solution by means of the linear combination

$$t_r(l) = r(l) + \frac{n+2}{16} K_d u(l)$$

(3.17)

and we have:

$$t_r(l) = e^{\Lambda_r(l)} \left\{ t_r(0) + \frac{n+2}{4} K d \int_0^l dl' \frac{e^{-2l'} u(l')}{e^{1/T(l')}-1} \right\}$$

(3.18)

with $t_r(0) = r_0 + \frac{n+2}{16} K_d u_0 = r_0 - r_0 c$ (see Eq. (2.7)). This combination provides the formal solution of the RG equations (2.2) to the order of interest. If we consider explicitly the additional condition $T(l) \ll 1$ as $T \to 0$, with $F_2^{(0)}(T(l)) \simeq \frac{1}{4} + O \left( \frac{1}{T e^{-1/T}} \right)$, the term $\Lambda_l(l)$ which appears in Eq. (3.18) takes the form:

$$\Lambda_l(l) \simeq 2l - \frac{n+2}{n+8} \ln \left[ 1 + \frac{n+8}{16} K_d u_0 \frac{1}{\varepsilon} (e^{\varepsilon l} - 1) \right].$$

(3.19)

In particular for $d = 3$, it reduces to:

$$\Lambda_l(l) = 2l - \frac{n+2}{n+8} \ln \left( \frac{l}{l_0} + 1 \right).$$

(3.20)

4. Relevant thermodynamic quantities and phase diagram in the influence domain of the QCP

We now focus on the flow solution (3.18) with $\Lambda_l(l)$ given by Eq. (3.19).

Since in any case $\Lambda_l(l) > 0$, $t_r(l)$ is a relevant parameter. Then, bearing in
mind the matching method [17,18], we stop the renormalization procedure until a scale \( l = l^* \gg 1 \) is reached at which \( t_r(l^*) \approx 1 \). This matching condition, in view of Eq.(3.18), provides for \( l^* \) the self-consistent equation:

\[
de^{-\Lambda_r(l^*)} \approx (r_0 - r_{0c}) + \frac{n + 2}{4} K_d u_0 T^{2-\varepsilon} I \left( \frac{e^{-l^*}}{T}, \frac{1}{T} \right). \tag{4.1}
\]

where

\[
I(x, y) = \int_x^y dx' \frac{x'^{1-\varepsilon}}{e^{x'} - 1} \left[ 1 + \frac{n + 8}{16} K_d u_0 \ln \left( \frac{1}{T x'} \right) \right]^{-1}. \tag{4.2}
\]

Eqs. (4.1)-(4.2) are valid for \( |\varepsilon| \ll 1 \) and the case \( d = 3 \) can be simply obtained taking the limit as \( \varepsilon \to 0 \).

For a solution \( l^* \gg 1 \) in the low-temperature limit, Eqs.(3.19) and (3.20) give:

\[
\Lambda_r(l^*) \approx \begin{cases} 
\lambda_r l^*, & \varepsilon \neq 0 \\
2l^* \left[ 1 - \frac{1}{2\varepsilon} \ln \left( \frac{l^*}{u_0} + 1 \right) \right], & \varepsilon = 0
\end{cases} \tag{4.3}
\]

where (see eq. (2.6)) \( \lambda_r = 2 - \frac{n + 2}{n + 8} \varepsilon \) (for \( \varepsilon > 0 \)) and \( \lambda_r = 2 \) (for \( \varepsilon < 0 \)).

At this stage, for determining \( l^* \), it is convenient to consider separately the cases \( \varepsilon \neq 0 \) and \( \varepsilon = 0 \).

(i) \( \varepsilon \neq 0 \).

For \( l^* \gg 1 \) and \( T(l^*) \ll 1 \), Eq. (4.1) can be solved by iteration yielding the solution:

\[
e^{l^*} \approx \left( (r_0 - r_{0c}) + \frac{n + 2}{4} K_d u_0 I(c, \infty) T^{2-\varepsilon} \right)^{-1/\lambda_r}, \tag{4.4}
\]

where

\[
I(c, \infty) = \int_c^\infty dx \frac{x}{e^x - 1} + O(u_0) \approx I(0, \infty) + O(u_0^{\frac{1}{2}}) \tag{4.5}
\]
with \( c = \left( \frac{n+2}{4} K_d u_0 \right)^{1/2} \) and \( I(0, \infty) = \frac{\pi^2}{16} \).

Then, to the order of interest, we can write:

\[ e^{l^*} \simeq [t_r(T)]^{-1/\lambda_r}, \quad (4.6) \]

where

\[ t_r(T) = (r_0 - r_{0c}) + \frac{n + 2}{64} \pi^2 K_d u_0 T^{2 - \varepsilon} \quad (4.7) \]

measures the low-temperature deviation from the QCP.

(ii) \( \varepsilon = 0 \ (d = 3) \).

In this case, taking into account Eq. (4.3), the self-consistent equation for \( l^* \gg 1 \) is:

\[ e^{-l^*} \simeq \left\{ (r_0 - r_{0c}) + \frac{n + 2}{8 \pi^2} u_0 T^2 I \left( e^{-l^*}, \frac{1}{T} \right) \right\}^{1/2} \frac{l^*}{l_0 + 1} \frac{n + 2}{2(n + 8)}. \quad (4.8) \]

Then, the appropriate low-\( T \) solution can be written in the form:

\[ e^{l^*} \simeq \left( \frac{1}{2l_0} \right)^{\frac{n+2}{2(n+8)}} t_r^{-\frac{1}{2}}(T) \left[ \ln t_r^{-1}(T) \right]^{\frac{n+2}{2(n+8)}} \quad (4.9) \]

where \( t_r(T) \) is given by Eq. (4.7) simply setting \( \varepsilon = 0 \):

\[ t_r(T) = (r_0 - r_{0c}) + \frac{n + 2}{128} u_0 T^2. \quad (4.10) \]

With the explicit expressions (4.6) \( (\varepsilon \neq 0) \) and (4.9) \( (\varepsilon = 0) \) for \( l^* \), we have the basic ingredients for exploring the quantum critical properties in the influence domain of the QCP. We can indeed, according to the general RG philosophy, utilize the scaling relations for the correlation length \( \xi \), the
susceptibility $\chi$ and the singular part of the free energy density. Here we focus on $\xi$ and $\chi$ for which these relation are:

$$\xi(r_0, u_0, T) \simeq \xi_0 e^{\xi}$$  \hspace{1cm} (4.11)

and (with $\eta = 0$):

$$\chi(r_0, u_0, T) \simeq \chi_0 e^{2\xi} \propto \xi^2$$  \hspace{1cm} (4.12)

where $\xi_0$ and $\chi_0$ are constants inessential for our present purposes. Then, very near the QCP we have:

$$\xi(r_0, u_0, T) \simeq \xi_0 \times \begin{cases} [t_{r_0}(T)]^{-1/\lambda_r} & , \varepsilon \neq 0 \\ \left(\frac{1}{2l_0}\right)^{\frac{2}{2(n+8)}} t_{r_0}^{-\frac{1}{2}}(T) \left[\ln t_{r_0}^{-1}(T)\right]^{\frac{n+2}{2(n+8)}} & , \varepsilon = 0 \end{cases}$$  \hspace{1cm} (4.13)

and

$$\chi(r_0, u_0, T) \simeq \chi_0 \times \begin{cases} [t_{r_0}(T)]^{-2/\lambda_r} & , \varepsilon \neq 0 \\ \left(\frac{1}{2l_0}\right)^{\frac{n+2}{n+8}} t_{r_0}^{-1}(T) \left[\ln t_{r_0}^{-1}(T)\right]^{\frac{n+2}{n+8}} & , \varepsilon = 0 \end{cases}$$  \hspace{1cm} (4.14)

where $t_{r_0}(T)$ is given by Eqs. (4.7) and (4.10) for $d \neq 3$ and $d = 3$ respectively.

The low-temperature critical line in the $(r_0, T)$-plane, merging in the QCP, is obtained setting $t_{r_0}(T) = 0$ in Eq. (4.7). We have:

$$r_{0c}(T) = r_{0c} - \frac{n + 2}{64} \pi^2 K_d u_0 T^{2-\varepsilon}$$  \hspace{1cm} (4.15)

or, equivalently, for $r_0 \leq r_{0c}$:

$$T_c(r_0) = \left(\frac{64}{\pi^2(n+2)u_0}\right)^{\frac{1}{2}} (r_{0c} - r_0)^{\frac{1}{2}(1 + \frac{2}{\varepsilon})}$$  \hspace{1cm} (4.16)

which are valid also when $\varepsilon = 0$. 

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In terms of \( r_{0c}(T) \) or \( T_c(r_0) \), the critical line deviation parameter \( t_{r_0}(T) \) can be written as:

\[
t_{r_0}(T) = r_0 - r_{0c}(T)
\]  

(4.17)

or

\[
t_{r_0}(T) = \frac{n + 2}{64} \pi^2 K_d u_0 \left( T^{2-\eps} - T_c^{2-\eps}(r_0) \right).
\]  

(4.18)

In particular this last representation yields:

\[
t_{r_0}(T) \approx \begin{cases} 
\frac{n + 2}{32} \pi^2 K_d u_0 T_c(r_0) [T - T_c(r_0)] & , \ r_0 \neq r_{0c} \ , \ (T_c(r_0) \neq 0) \\
\frac{n + 2}{64} \pi^2 K_d u_0 T^{2-\eps} & , \ r_0 = r_{0c} \ , \ (T_c(r_{0c}) = 0) 
\end{cases}
\]  

(4.19)

From the previous expressions, it is evident that different ways of approaching the critical line are possible. This is related to realistic experiments where distinct thermodynamic paths approaching a critical point may exist. Each of them is characterized by a specific set of critical exponents and different sets may be connected by appropriate relations (for instance a Fisher’s renormalization [19] when thermodynamic constraints are involved). This is just the case of quantum systems for which several paths of approaching to a critical line are possible [5]. For the transverse Ising-like models under study, we will consider the most relevant paths within the disordered phase \((t_{r_0}(T) > 0)\) in the \((r_0, T)\)-plane: \( L_{r_0} \equiv (r_0 \to r_{0c}, T = 0) \), \( L_T \equiv (r_0 = r_{0c}, T \to 0) \), \( L_{r_0-r_{0c}(T)} \equiv (r_0 \to r_{0c}^+(T), T \text{ fixed}) \) and \( L_{T-T_c(r_0)} \equiv (r_0 \text{ fixed}, T \to T_c^+(r_0)) \) with \( T_c(r_0) \to 0 \) as \( r_0 \to r_{0c} \). Correspondently, for a generic macroscopic quantity \( X \) we have the set of critical exponents \( \{x_r\} \) defined
by

\[ X \sim (r_0 - r_{0c})^{-x_r} \quad \text{or} \quad X \sim (r_0 - r_{0c}(T))^{-x_r} \]  \hspace{1cm} (4.20)

along \( L_{r_0} \) or \( L_{r_0 - r_{0c}(T)} \) and the set of critical exponents \( \{x_T\} \) defined by

\[ X \sim T^{-x_T} \quad \text{or} \quad X \sim (T - T_c(r_0))^{-x_T} \]  \hspace{1cm} (4.21)

along \( L_T \) or \( L_{T-T_c(r_0)} \). For instance, for quantum displacive systems [9,10] with \(-r_0 = S > 0\) and QCP-coordinates \((S = S_c, T = 0)\) the thermodynamic paths defined above will be denoted by \( L_S, L_{S=S_c(T)}, L_T \) and \( L_{T-T_c(S)} \).

The different paths of interest for us are schematically shown in Fig.1 for a generic transverse Ising-like system. For sake of brevity, in the following we will often speak of \( L_k \)-quantum criticality, with \( k = r_0, r_0 - r_{0c}(T), T, T - T_c(r_0) \).

5. Low-temperature quantum critical properties.

The previous arguments provide all the ingredients to derive the near-QCP properties around and at \( d = 3 \) for quantum systems with transverse Ising-like intrinsic dynamics. Here we focus on the quantum critical behaviour of the correlation length and the susceptibility along the previously defined thermodynamic paths. The near-QCP properties of the other thermodynamic quantities follow immediately from the RG picture. In the next section we will describe the related crossover phenomena of experimental
interest.

5.1. $L_{r_0}$-quantum criticality

In such a case, from Eqs. (4.7), (4.13) and (4.14) we have for $d \neq 3$ (consistently with the $(T = 0)$-results of section 2.):

$$\xi \sim (r_0 - r_{0c})^{-\nu_r}, \quad \chi \sim (r_0 - r_{0c})^{-\gamma_r} \quad (5.1)$$

where $\nu_r$ and $\gamma_r$ are given by Eqs. (2.8), (2.9) for $d < 3$ and $d > 3$, respectively. For $d = 3$, the same equations provide

$$\xi \sim (r_0 - r_{0c})^{-1/2} \left[ \ln (r_0 - r_{0c})^{-1}\right]^{n+2\over n+8}, \quad (5.2)$$

$$\chi \sim (r_0 - r_{0c})^{-1} \left[ \ln (r_0 - r_{0c})^{-1}\right]^{n+2\over n+8}. \quad (5.3)$$

These results are well known, but their independent and unified derivation signals the full consistency of our present calculations.

5.2. $L_T$ - quantum criticality

Here, the approach to QCP is controlled only by the temperature for $r_0$ fixed to its $(T = 0)$-critical value. This situation is recurrent in several, now accessible, experiments.

From the general Eqs. (4.7), (4.10), (4.13) and (4.14), we have, for $d \neq 3$

$$\xi(T) \sim T^{-\nu_T}, \quad \chi(T) \sim T^{-\gamma_T} \quad (5.4)$$
\[ \nu_T = 1 - \frac{3}{n+8} \varepsilon \quad ; \quad \gamma_T = 2 - \frac{6}{n+8} \varepsilon \quad , \quad \varepsilon > 0 \] (5.5)

and

\[ \nu_T = 1 - \frac{\varepsilon}{2} \quad ; \quad \gamma_T = 2 - \varepsilon \quad , \quad \varepsilon < 0 \] (5.6)

For the marginality case \( d = 3 \), where logarithmic corrections to MF results are expected, we have from the same equations:

\[ \xi(T) \sim T^{-1} |\ln T|^{\frac{n+2}{2(n+8)}} \sim T^{-1} |\ln T|^2 \frac{n+2}{2(n+8)} \] (5.7)

and

\[ \chi(T) \sim T^{-2} |\ln T|^{\frac{n+2}{n+8}} \sim T^{-2} |\ln T|^2 \frac{n+2}{n+8}. \] (5.8)

The previous results for \( d < 3 \) are in full agreement with the large-n limit predictions [9].

In the most interesting situation \( d = 3 \), Eqs. (5.7)-(5.8) for one-component systems yield:

\[ \xi(T) \sim T^{-1} |\ln T|^2\frac{1}{8} \] (5.9)

and

\[ \chi(T) \sim T^{-2} |\ln T|^2\frac{1}{5} \] (5.10)

which are again in full agreement with the results obtained by Schmeltzer [20-23] via a direct \((d = 3)\)-field theoretic approach, and by Caramico et al. [5] using the temperature dependent linear scaling field method.
As concerning the $T$-driven critical behaviours for $\varepsilon < 0$ ($d > 3$), it must be noted that the $L_T$-critical exponents correspond to the known $L_{r_0}$-Gaussian ones which give the leading corrections [23] to the MF behaviours. In particular, the exponents $\nu_T$, $\gamma_T$, given by Eqs. (5.6), characterize the corrections to the unknown dominant singularities for $d > 3$, corresponding to the MF ones along $L_{r_0}$. The explicit RG calculation of the exponents along $L_T$, involving a study of the role played by irrelevant dangerous variables, is beyond the purpose of the present paper.

5.3. $L_{r_0-r_{0c}(T)}$ - quantum criticality

The starting points are always the unified Eqs. (4.13), (4.14) with the representation (4.18) for $t_{r_0}(T)$. Then, it immediately follows:

(i) for $d \neq 3$:

$$\xi \sim (r_0 - r_{0c}(T))^{-\nu_r}, \chi \sim (r_0 - r_{0c}(T))^{-\gamma_r}$$

(5.11)

with the same $L_{r_0}$-critical exponents (2.8)-(2.9);

(ii) for $d = 3$:

$$\xi \sim (r_0 - r_{0c}(T))^{-\frac{2}{\nu}} \left[ \ln(r_0 - r_{0c}(T))^{-1} \right]^{\frac{n+2}{2(n+8)}}$$

(5.12)

and

$$\chi \sim (r_0 - r_{0c}(T))^{-1} \left[ \ln(r_0 - r_{0c}(T))^{-1} \right]^{\frac{n+2}{2(n+8)}}.$$  

(5.13)

These low-$T$ results in the influence domain of QCP suggest new possible experiments for realistic systems, driving the transition with a nonthermal
parameter (transverse field, elastic strength parameter, pressure, etc.) at fixed $T \neq 0$.

5.4. $L_{T-T_c(r_0)}$ - quantum criticality

Using the representation (4.18) for $t_{r_0}(T)$, in the case $\varepsilon > 0$ ($d < 3$) Eqs. (4.13), (4.14) yield:

$$\xi(T) \sim (T^{2-\varepsilon} - T_c^{2-\varepsilon}(r_0))^{-\frac{1}{\lambda r}}$$

(5.14)

and

$$\chi(T) \sim (T^{2-\varepsilon} - T_c^{2-\varepsilon}(r_0))^{-\frac{2}{\lambda r}}.$$  (5.15)

For $d > 3$ a quite similar scenario occurs and we do not consider this case explicitly.

From Eqs. (5.14), (5.15) two different behaviours governed by the temperature occur according to $r_0 \neq r_{0c}$ ($T_c'(r_0) \neq 0$) or $r_0 = r_{0c}$ ($T_c'(r_{0c}) = 0$). In the last case Eqs. (5.14) and (5.15) reduce to Eqs. (5.4), (5.5) characterizing the behaviour of $\xi$ and $\chi$ along the line $L_T$. The situation is quite different when $r_0 \neq r_{0c}$. In this case $t_{r_0}(T) \propto T - T_c(r_0)$ and Eqs. (5.14), (5.15) reduce to:

$$\xi(T) \sim (T - T_c(r_0))^{-1/\lambda r}.$$  (5.16)

and

$$\chi(T) \sim (T - T_c(r_0))^{-2/\lambda r}.$$  (5.17)
which imply
\[ \nu_T \equiv \nu_r = \frac{1}{2} \left( 1 + \frac{n + 2}{2(n + 8)} \varepsilon \right) \quad (5.18) \]
\[ \gamma_T \equiv \gamma_r = 1 + \frac{n + 2}{2(n + 8)} \varepsilon \quad , \quad (5.19) \]
respectively. Thus, for \( r_0 \to r_{0c} \), a crossover phenomenon takes place which is described by the general equations (5.14)-(5.15) (See Sect. 6 below).

For \( d = 3 \), a Gaussian-like behaviour with logarithmic corrections occurs as expected for a marginality case. Eqs. (4.13), (4.14) imply indeed:
\[ \xi(T) \sim (T^2 - T_c^2(r_0))^{-\frac{1}{2}} \left[ \ln(T^2 - T_c^2(r_0)) \right]^{-\frac{n+2}{2(n+8)}} (5.20) \]
and
\[ \chi(T) \sim (T^2 - T_c^2(r_0))^{-1} \left[ \ln(T^2 - T_c^2(r_0)) \right]^{-\frac{n+2}{n+8}} . \quad (5.21) \]
These reduce to Eqs. (5.7), (5.8) when \( r_0 = r_{0c} \) and to:
\[ \xi(T) \sim (T - T_c(r_0))^{-\frac{1}{2}} \left[ \ln(T - T_c(r_0)) \right]^{-\frac{n+2}{2(n+8)}} \quad (5.22) \]
and
\[ \chi(T) \sim (T - T_c(r_0))^{-1} \left[ \ln(T - T_c(r_0)) \right]^{-\frac{n+2}{n+8}} . \quad (5.23) \]
when \( r_0 \neq r_{0c} \). As we see, also for \( d = 3 \) a crossover phenomenon occurs for \( r_0 \to r_{0c} \) which is described by eqs. (5.20), (5.21) but now it involves logarithmic corrections to the power law.

For \( n = 1 \), the general Eqs. (5.20), (5.21) reduce to:
\[ \xi(T) \sim (T^2 - T_c^2(r_0))^{-\frac{1}{2}} \left[ \ln(T^2 - T_c^2(r_0)) \right]^{-\frac{1}{8}} \quad (5.24) \]
and
\[ \chi(T) \sim (T^2 - T_c^2(r_0))^{-1} |\ln(T^2 - T_c^2(r_0))|^{\frac{1}{\beta}} \]  
(5.25)
again in agreement with the corresponding field-theoretic results by Schmeltzer [20-22].

The near-QCP behaviours of other thermodynamic quantities can be similarly extracted from the rescaling relations of the singular part \( F_s \) of the free energy density with the help of the above expressions of the deviation parameter \( t_{r_0}(T) \). This lies on the fact that, by iteration of the RG transformation until the scale \( l^* \) is reached, one can write \( F_s \sim e^{(d+1)l^*} \) where \( e^{l^*} \) is given by Eqs. (4.6) - (4.10) and (4.18). For some explicit results we limit ourselves to the thermodynamic paths \( L_T \) and \( L_{T-T_c(r_0)} \) for \( d \leq 3 \), for which the most interesting features occur.

For \( \epsilon > 0 \), by approaching the QCP along \( L_T \) one finds

\[ F_s(T) \sim T^{(d+1)\nu_T}, \quad \frac{C_s(T)}{T} \sim T^{-\alpha_T} \]  
(5.26)
with
\[ \alpha_T = 2 - (d + 1)\nu_T = -2 + \frac{n + 20}{n + 8} \epsilon . \]  
(5.27)
In Eq. (5.26), \( C_s(T) = -T \frac{\partial^2 F_s}{\partial T^2} \) is the contribution to the specific heat arising from the singular part of the free energy density [28]. As \( T \to T_c^+(r_0) \) along \( L_{T-T_c(r_0)} \), Eq. (4.18) yields:

\[ F_s(T) \sim (T^{2-\epsilon} - T_{c-}^{2-\epsilon})^{(d+1)\nu_r} . \]  
(5.28)
Of course, from this relation, Eqs. (5.26) are reproduced as \( r_0 \to r_{0c} \) while, for \( r_0 \neq r_{0c} \) \( (T_c(r_0) \neq 0) \), one obtains \( F_s \sim (T - T_c(r_0))^{(d+1)\nu_T} \) and hence \( C_s(T)/T_c(r_0) \sim C_s(T) \sim (T - T_c(r_0))^{-\alpha_T} \) with \( \alpha_T \equiv \alpha_r = \frac{4-n}{2(n+8)} \epsilon \). Here \( \alpha_r \) is the critical exponent which characterizes the \((T = 0)\)-behaviour of the specific heat-like quantity \( C_r = -\frac{\partial^2 F_s}{\partial r_0^2} \) as \( r_0 \to r_{0c}^+ [5] \). Thus, also for \( C_s(T)/T \) a crossover occurs as \( r_0 \to r_{0c} \) (or \( T_c(r_0) \to 0 \))[28].

It is worth noting that, in any case, the \( x_T \)-exponents \( \alpha_T \) and \( \nu_T \), as \( \alpha_r \) and \( \nu_r \), satisfy the \( T \)-hyperscaling relation \( 2 - \alpha_T = (d + 1)\nu_T \).

At \( d = 3 \), as for \( \chi \) and \( \xi \), logarithmic corrections occur. Indeed, from Eqs. (4.9)-(4.10) we have

\[
F_s(T) \sim T^4 \ln T^2 \left| \ln T \right|^{-\frac{2(n+2)}{n+8}}, \quad \frac{C_s(T)}{T} \sim T^2 \ln T^2 \left| \ln T \right|^{-\frac{2(n+2)}{n+8}},
\]

(5.29)

as \( T \to 0 \) along \( L_T \). In contrast, if we approach the critical line at fixed \( r_0 \neq r_{0c} \) and \( T \to T_c^+(r_0) \) we have,

\[
F_s(T) \sim (T^2 - T_c^2(r_0))^2 \left| \ln(T^2 - T_c^2(r_0)) \right|^{\frac{2(n+2)}{n+8}} \sim (T - T_c(r_0))^2 \left| \ln(T - T_c(r_0)) \right|^{\frac{2(n+2)}{n+8}}
\]

(5.30)

and hence a mean-field behaviour in \((T - T_C(r_0))\), with logarithmic corrections for \( \frac{C_s(T)}{T} \), takes place. Then, at \( d = 3 \), a crossover occurs also for \( C_s(T)/T \) as \( r_0 \to r_{0c} \) but now logarithmic corrections are involved.
6. Crossover phenomena and comparison with experiments

In the previous subsection 5.4, we have shown that a crossover between the $L_T-T_{c}(r_0)$-critical regime and $L_T$-one occurs as $r_0 \to r_{0c}$.

For $\varepsilon \neq 0$, when logarithmic corrections are not involved, this crossover phenomenon can be conveniently described in terms of “effective exponents”. Here we focus on susceptibility and determine the appropriate effective exponents $\gamma_{T}^{\text{eff}} = 2\nu_{T}^{\text{eff}}$, where $\nu_{T}^{\text{eff}}$ denotes the effective exponent for the correlation length. Other effective exponents (as $\alpha_{T}^{\text{eff}}$, see below) can be derived similarly.

We start from the general asymptotic relation (5.15) with $2/\lambda_r = 2\nu_r = \gamma_r$ given by Eqs.(2.8)-(2.9). As clarified before, this relation, as $T \to T_{c}^{+}(r_0)$ from the disordered phase ($t_{r_0}(T) > 0$), contains both the behaviours when $T_{c}(r_{0c}) = 0$ ($\gamma_T = (2 - \varepsilon)\gamma_r$) and $T_{c}(r_0) \neq 0$ ($\gamma_T = \gamma_r$). Then, Eq. (5.15) allows us to extract an effective exponents $\gamma_{T}^{\text{eff}}$ such that

$$\gamma_r \leq \gamma_{T}^{\text{eff}} \leq (2 - \varepsilon)\gamma_r.$$  

This inequality, as $d \to 3$, is in full agreement with experimental data for quantum ferroelectric oxides [24] and other displacive systems [25]-[27] except for logarithmic corrections which can be taken into account from relations (5.22)-(5.25).

To prove the inequality (6.1), we first observe that, from the conventional
definition of “effective critical exponent” for a generic quantity $\chi \sim t^{-\gamma_{\text{eff}}}$

$$\gamma_{\text{eff}} = -\frac{\partial \ln \chi}{\partial \ln t}, \quad (6.2)$$

Eq. (5.15) leads to

$$\gamma_{\text{eff}}^T(\tau) = (2 - \epsilon)\gamma_r \frac{1 - \tau}{1 - \tau^{2-\epsilon}} \quad (6.3)$$

where $\tau = T_c(r_0)/T$ ($0 \leq \tau \leq 1$) is the appropriate crossover parameter for the problem. Then, since $1 - \tau^{2-\epsilon} = (2 - \epsilon)(1 - \tau)$ as $\tau \to 1$, it is easy to check that the effective exponent (6.3) satisfies the inequality (6.1), with $\gamma_{\text{eff}}^T(0) = (2 - \epsilon)\gamma_r$ and $\gamma_{\text{eff}}^T(1) = \gamma_r$.

With $d \to 3^+$, Eq. (6.3) becomes

$$\gamma_{\text{eff}}^T(\tau) = \frac{2}{1 + \tau} \quad (6.4)$$

satisfying the inequality $1 \leq \gamma_{\text{eff}}^T(\tau) \leq 2$.

A similar analysis can be performed to determine the effective exponent $\alpha_{\text{eff}}^T$ for $C_s(T)/T$. We find

$$\alpha_{\text{eff}}^T(\tau) = 2 - (4 - \epsilon)\nu_r f_\epsilon(\tau) \quad (6.5)$$

which crossovers between $\alpha_{\text{eff}}^T(0) = \alpha_T$ (Eq. 5.27) and $\alpha_{\text{eff}}^T(1) = \alpha_r$.

For $d = 3$, an analogous crossover between two marginal regimes (involving logarithmic corrections) occurs as $r_0 \to r_{0c}$. This crossover, as already mentioned before, can be described by means of the general relations (5.20)-(5.21) and the corresponding one for $F_s(T)$ but now effective exponents can
not be exactly defined as for $d \neq 3$ due to the presence of logarithmic corrections which are experimentally inaccessible.

All the previous RG predictions concerning the $L_k$-quantum criticality and the crossover phenomena around and at $d = 3$ are quite consistent with experiments on transverse Ising-like quantum systems [8] exhibiting a quantum phase transition. Relevant examples are the quantum displacive systems with quantum structural phase transitions [24]-[27] and, in particular, ferroelectric oxides as $K_{1-x}Na_xTaO_3$ and $KTa_{1-x}Nb_xO_3$ with fixed composition $x$ [24]. The experimental results for this class of systems, relevant for a direct comparison with our RG theoretical results, can be summarized as follows (see Fig.2 for a schematic picture of a typical phase diagrams and for related notations).

1. Approaching the QCP $(S_c, T = 0)$ along $L_S$, the susceptibility varies with the interaction parameter $S$ (proportional to the pressure) as:

$$\chi \sim (S_c - S)^{-\gamma_S}$$  \hfill (6.6)

with $\gamma_S = 1$. This agrees with our RG results (5.1)-(5.3);

2. Along $L_T$, at the quantum displacive limit $(S = S_c)$, $\chi$ varies with the temperature as:

$$\chi(T) \sim T^{-\gamma_T}$$  \hfill (6.7)

with $\gamma_T = 2$, in agreement with the theoretical results (5.4)-(5.10);

3. In the low-temperature limit, the transition temperature $T_c(S)$ as a
function of $S$ (or pressure) is expressed as:

$$T_c(S) \sim (S - S_c)^{\frac{1}{2}}$$

(6.8)

which is exactly reproduced by Eq. (4.16) with $\varepsilon = 0$;

4. Approaching the critical line along $L_{T-T_c(S)}$, one has:

$$\chi \sim (T - T_c(S))^{-\gamma_T}$$

(6.9)

where $\gamma_T$ increases from $\gamma_T = 1$, when $T_c(S) \neq 0$, to the value $\gamma_T = 2$ when $T_c(S) \to 0$ as $S \to S_c$ (see Fig. 5 by Samara [24] for $K_{1-x}Na_xTaO_3$). This signals a crossover by continuous variation of $T_c(S)$ towards zero when $S$ decreases to its quantum displacive limit $S_c$, in full agreement with our theoretical predictions (6.1)-(6.4). Of course, logarithmic corrections, expected in the marginal situation $d = 3$, are experimentally undetectable but they have to emerge from a consistent theory as it happens in the present RG calculations.
7. Concluding remarks

For a big amount of microscopic systems exhibiting a continuous QPT, the $T$-dependent RG equations are known since more than twenty years [13], a part from few next relevant modifications in a number of interesting cases [29,30].

Recent experiments, involving compounds for which unusual low-$T$ properties are observed, have motived the new tendency [1]-[6] to solve, with a minimum of reliable approximations, the very complicate RG equations with the aim to extract a lot of macroscopic information to be compared with available experimental data.

In the present paper, following this very fruitful and promising tendency, we have derived explicit low-$T$ expressions of relevant thermodynamic quantities for a wide class of quantum systems solving the related one-loop RG equations in the low-$T$ regime. Our aim was to explore the quantum critical behaviours within the domain of influence of a QCP and to compare our results with experiments, where different intriguing ways of approaching the QCP may occur. We have focused on transverse Ising-like models which are the basic starting point for exploring theoretically the thermodynamic properties of several realistic systems of experimental interest which exhibit a QPT. It is a fortunate feature that, for this class of models, the RG works just around and at $d = 3$. This allowed us to obtain practically exact low-$T$ results, so that a direct comparison with experiment data becomes suitable.
Our low-$T$ RG predictions have been shown to be in remarkable agreement with the experimental data for quantum displacive systems, as the ferroelectric oxides, for which accurate experiments have been performed around their QCP’s, where the quantum fluctuations are expected to strongly influence the response of the systems.

Finally, from a methodological point of view, we wish to underline that our RG analysis, based on a $T$-dependent solution of the flow equations which avoids “ad hoc” assumptions, provides a valid support to the so called “temperature-dependent linear scaling fields method” [5], formulated beyond the traditional schemes of the RG approaches.
FIGURE CAPTIONS

Fig.1. Schematic low-temperature phase diagram for a generic transverse-Ising-like model with the thermodynamic paths $L_{r_0}$, $L_T$, $L_{r_0-r_0c(T)}$ and $L_{T-T_c(r_0)}$ approaching the QCP.

Fig.2. Schematic low-temperature phase diagram for quantum displacive systems with the thermodynamic paths $L_S$, $L_T$ and $L_{T-T_c(S)}$ within the disordered phase in the influence domain of the QCP.
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28. In the quantum regime the exponent for specific heat can be defined in two ways since, for $T \to 0$, the additional $T$-factor in its definition reduces the singularity by one power in $T$. Then one defines $C_s \sim T^{-\tilde{\alpha}_T}$ and $C_s(T)/T \sim T^{-\alpha_T}$, where $\alpha_T = \tilde{\alpha}_T + 1$ corresponds to the usual specific heat exponent in the theory of critical phenomena.

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