Optimal regularity of open-loop mean field controls and their piecewise constant approximation

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Abstract. We consider the control of McKean-Vlasov dynamics whose coefficients have mean field interactions in the state and control. We show that for a class of linear-convex mean field control problems, the unique optimal open-loop control admits the optimal $1/2$-Hölder regularity in time. Consequently, we prove that the value function can be approximated by one with piecewise constant controls and discrete-time state processes arising from Euler-Maruyama time stepping, up to an order $1/2$ error, and the optimal control can be approximated up to an order $1/4$ error. These results are novel even for the case without mean field interaction.

Key words. Controlled McKean–Vlasov diffusion, path regularity, error estimate, piecewise constant policy, time discretization, mean field forward-backward stochastic differential equation.

AMS subject classifications. 49N80, 49N60, 60H35, 65L70

1 Introduction

In this paper, we study a class of mean field stochastic control problems where the state dynamics and cost functions depend upon the joint law of the state and the control processes. Let $T > 0$ be a given terminal time, $(\Omega, \mathcal{F}, P)$ be a complete probability space on which a $d$-dimensional Brownian motion $(W_t)_{t \in [0,T]}$ is defined, $\mathcal{F} = (\mathcal{F}_t)_{t \in [0,T]}$ be the natural filtration of $W$ augmented with an independent $\sigma$-algebra $\mathcal{F}_0$, and $\mathcal{A}$ be the set of square integrable $\mathcal{F}$-progressively measurable processes $\alpha = (\alpha_t)_{t \in [0,T]}$ taking values in a nonempty closed convex set $\mathcal{A} \subset \mathbb{R}^k$. For any initial state $\xi_0 \in L^2(\mathcal{F}_0; \mathbb{R}^n)$ and control $\alpha \in \mathcal{A}$, we consider the state process governed by the following controlled McKean–Vlasov diffusion: $X_0^\alpha = \xi_0$ and

$$dX_t^\alpha = b(t, X_t^\alpha, \alpha_t, P(X_t^\alpha, \alpha_t)) \, dt + \sigma(t, X_t^\alpha, P(X_t^\alpha, \alpha_t)) \, dW_t, \quad t \in [0,T],$$

(1.1)

where $b$ and $\sigma$ are given (possibly unbounded) Lipschitz continuous functions taking values in $\mathbb{R}^n$ and $\mathbb{R}^{n \times d}$, respectively, and $\sigma$ is possibly degenerate. The value function of the optimal control problem is defined by

$$V(\xi_0) = \inf_{\alpha \in \mathcal{A}} J(\alpha; \xi_0) \quad \text{with} \quad J(\alpha; \xi_0) = \mathbb{E}\left[ \int_0^T f(t, X_t^\alpha, \alpha_t, P(X_t^\alpha, \alpha_t)) \, dt + g(X_T^\alpha, P(X_T^\alpha)) \right],$$

(1.2)

where the running cost $f$ and terminal cost $g$ are given real valued functions of at most quadratic growth. Above and hereafter, $P_U$ stands for the law of a given random variable $U$.

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Such mean field control (MFC) problems with interactions through the joint distribution of the state and control processes have attracted an increasing interest due to the emergence of the mean field game theory and their numerous applications in various areas, including economics, biology and social interactions (see e.g. [10, 7, 3, 18, 8, 19, 1, 9, 12, 6, 16]). In particular, the solution to (1.2) describes large population equilibria of interacting individuals who obey a common policy controlled by a central planner. Equations of the type (1.2) are also motivated by control problems whose objective functions are evaluated under convex risk measures, such as the mean-variance portfolio selection problem in finance. In the case that the coefficients of the state dynamics are linear in the state, control and measure variables, and the cost functions are convex in these variables, (1.2) is called a linear-convex MFC problem and is the main focus of this paper. Moreover, if the mean field interactions enter both the controlled dynamics and cost functions through the marginal law of the state only, then (1.2) reduces to the MFC problems (without control interactions) studied in [10, 7, 8, 9].

As explicit solutions to (1.2) are rarely available, numerical schemes for solving such control problems become vital. A common strategy to obtain numerical approximations of (1.2) is to discretize the control problem on a given time grid by using piecewise constant policy timestepping. More precisely, for any given partition \( \pi = \{0 = t_0 < \cdots < t_N = T\} \) of \([0, T]\), we shall approximate the control problem (1.2) by the following discrete-time control problem:

\[
V_\pi(\xi_0) = \inf_{\alpha \in \mathcal{A}_\pi} J_\pi(\alpha; \xi_0),
\]

where \( \mathcal{A}_\pi \) is the subset of controls \( \mathcal{A} \) that are constant on each subinterval \([t_i, t_{i+1})\) in \( \pi \):

\[
\mathcal{A}_\pi := \left\{ \alpha \in \mathcal{A} : \forall \omega \in \Omega \exists a_i \in \mathcal{A}, i = 0, \ldots, N - 1, \text{ s.t. } \alpha_s(\omega) \equiv \sum_{i=0}^{N-1} a_i \mathbf{1}_{[t_i, t_{i+1})}(s) \right\},
\]

(1.4)

\( J_\pi(\alpha; \xi_0) \) is the discretized cost functional defined by

\[
J_\pi(\alpha; \xi_0) := \mathbb{E} \left[ \sum_{i=0}^{N-1} \int_{t_i}^{t_{i+1}} f(t_i, X_{t_i}^{\alpha, \pi}, \alpha_{t_i}, \mathbb{P}(X_{t_i}^{\alpha, \pi}, \alpha_{t_i})) \, dt + g(X_T^{\alpha, \pi}, \mathbb{P}(X_T^{\alpha, \pi})) \right],
\]

(1.5)

and \( X_{t_i}^{\alpha, \pi} \) is the discretized controlled process defined by the Euler-Maruyama approximation of (1.1): \( X_0^{\alpha, \pi} = \xi_0 \) and for all \( i \in \{0, \ldots, N - 1\} \), \( X_t^{\alpha, \pi} = X_{t_i}^{\alpha, \pi} \mathbf{1}_{[t_i, t_{i+1})}(t) \) and

\[
X_{t_{i+1}}^{\alpha, \pi} = X_{t_i}^{\alpha, \pi} + b(t_i, X_{t_i}^{\alpha, \pi}, \alpha_{t_i}, \mathbb{P}(X_{t_i}^{\alpha, \pi}, \alpha_{t_i}))(t_{i+1} - t_i) + \sigma(t_i, X_{t_i}^{\alpha, \pi}, \mathbb{P}(X_{t_i}^{\alpha, \pi}))(W_{t_{i+1}} - W_{t_i}).
\]

(1.6)

The computational advantage of this approach comes from the fact that over the time intervals in which the policy is constant, we only need to deal with Gaussian random variables with known mean and variance, which provides the basis for designing efficient numerical methods to solve MFC problems. We refer the reader to [21, 11, 17] for piecewise constant policy timestepping for classical control problems, and to e.g. [9, 13, 2] and references therein for numerical methods for (discrete time) mean field control problems and mean field games. Note that in practice, one can estimate the marginal laws \( \mathbb{P}(X_{t_i}^{\alpha, \pi}, \alpha_{t_i}) \) in (1.6) by a particle method (see e.g., [9]).

Motivated by the above applications, in this paper, we aim to address to what extent the continuous-time MFC problem (1.2) can be approximated by discrete-time control problems (1.3) arising from piecewise constant policy timestepping. In particular, we would answer the following two questions:

**Q1**: What is the convergence rate of \( |V_\pi(\xi_0) - V(\xi_0)| \) in terms of the stepsize \(|\pi|\)?
The approximation error of value functions was first addressed in [15] for classical control problems (with controlled diffusion coefficients but without mean field interaction), in which it is shown that the value functions of controlled diffusion processes (whose coefficients are Lipschitz continuous in space and 1/2-Hölder continuous in time) can be approximated with order 1/6 error by those with controls which are constant on uniform time intervals. The convergence rate was then improved to order 1/4 in [14] under the same regularity assumptions. The analysis in [15, 14] combines stochastic and analytic techniques, which first estimates the local error for each subinterval by controlling the generator of the controlled process, and then aggregates the local error over time by applying Itô’s lemma and a dynamic programming principle. No convergence result for optimal controls has been provided.

Unfortunately, the arguments in [15, 14] cannot be adapted to study piecewise constant policy approximation of (1.2), mainly due to the following two reasons. Firstly, controlling the generator of the state process usually involves estimating sup-norms of high-order derivatives of the value functions, which in turn requires the action set A to be compact and all coefficients of the control problem to be uniformly bounded (see [15, 14]). Here, we allow the action set A to be unbounded, the coefficients of (1.1) to be of linear growth, and the cost functions of (1.2) to be of quadratic growth, in order to include the most commonly used linear-quadratic models. Secondly, since the value function of a control problem is in general non-differentiable, [15, 14] first regularizes the (finite-dimensional) value function and then balances the regularization error and time discretization error. However, it is well-known that one has to include the marginal distribution in the state of the system to restore a dynamic programming principle of (1.2) (see e.g. [19]). This forces us to deal with an infinite-dimensional generator and an infinite-dimensional value function, for which there is no known regularization technique with quantifiable regularization error.

In fact, to the best of our knowledge, there is no published work on the accuracy of piecewise constant policy approximation for MFC problems with general open-loop controls (i.e., controls that depend on the initial condition and noise as those in A). A related work is [9], which restricts the class of admissible controls to be closed-loop controls (i.e., controls that are deterministic functions of time and state processes) and analyzes the time discretization error for special cases of linear-convex MFC problems (1.2) in which both b and f are independent of the law of controls. By assuming that the optimal feedback map is Lipschitz continuous in time and twice-differentiable in space with Lipschitz continuous derivatives (see Assumptions (B1), (C1)-(C3) in [9]), the authors show that the value functions of the discrete-time control problems converge to that of the original problem with order 1/2. We remark that establishing such a strong regularity of the feedback map is a delicate and technical issue, which usually requires to analyze the classical solutions to an infinite-dimensional partial differential equation (PDE) under the assumption that the cost functions are three-times differentiable with bounded Lipschitz continuous derivatives (see e.g. [10] for the case where the diffusion coefficient is constant and all coefficients are time-independent).

Our work. This paper studies the time discretization error of linear-convex MFC problems (1.2). The main contributions are:

- We prove under suitable conditions, which are verified for different classes of MFC problems (see Examples 2.1, 2.2 and 2.3), that (1.2) admits a unique optimal control, which can be characterized by the unique Hölder continuous solution to an associated coupled MV-FBSDE. Based on this solution characterization, we prove that the unique optimal control of (1.2) has the optimal time regularity, which is 1/2-Hölder continuous in time in the $L^p$-norm (see Theorem 3.6). We further give conditions under which the optimal control of (1.2) is
deterministic. Such time regularity results for optimal controls are novel even for the case without mean field interaction.

- We estimate the error introduced by approximating linear-convex MFC problems with piecewise constant controls and Euler-Maruyama discretizations of state processes. By using the Hölder regularity of the optimal control, we prove that the value functions of the discrete-time control problems converge to the original value function with an optimal order $1/2$, for which we merely require the cost functions to be Hölder continuous in time and Lipschitz continuously differentiable in space (see Theorems 4.1 and 4.2). We further show that the optimal controls of these discrete-time control problems converge to the optimal control of (1.2) in the $\mathcal{H}^2(\mathbb{R}^k)$-norm with an order $1/4$ (see Theorems 5.2 and 5.3), which is the first result on the convergence order of approximate controls, even for the case without mean field interaction.

Our approaches. Due to the non-Markovian nature of the controlled dynamics, instead of adapting the dynamic programming approach in [15, 14], we approach the control problem by directly characterizing the optimal control of (1.2) via the stochastic Pontryagin maximum principle. We shall investigate the solution regularity of the Pontryagin system via Malliavin Calculus, and subsequently deduce the time regularity of the open-loop optimal control. This enables us to quantify the time discretization error for MFC problems with Lipschitz differentiable cost functions, and avoids the strong regularity requirements on the optimal feedback control as in [9].

Let us briefly comment on the main difficulty encountered in analyzing the well-posedness and regularity of optimal controls to (1.2). As shown in [1, Remark 3.4], an optimal (open-loop) control $\hat{\alpha}$ of linear-convex MFC problems (1.2) can be characterized by its first-order optimality condition as follows:

$$\hat{\alpha}_t = \arg \min_{\beta} \{ \mathbb{E}[H(t, X_t, \beta, \mathbb{P}(X_t, \beta), Y_t)] \mid \beta \in L^2(\mathcal{F}_t; \mathbb{A}) \}, \quad d\mathbb{P} \otimes dt\text{-a.e.}, \quad (1.7)$$

where $H$ is an associated Hamiltonian and $Y$ is the associated adjoint process satisfying a non-Markovian forward-backward Pontryagin system. However, in contrast with the classical MFC problems without control interactions, in general one cannot express $\hat{\alpha}$ in (1.7) as $\hat{\alpha}_t = \psi(t, X_t, Y_t, \mathbb{P}(X_t, Y_t))$, with $\psi$ being the pointwise minimizer of the Hamiltonian $H$, due to the nonlinear dependence on the law of the control (see [1, Remark 4.2] for a counterexample in the linear-quadratic setting). This prevents us from simplifying the non-Markovian Pontryagin system by directly inserting the formula for the minimizer of the Hamiltonian into the forward equation as in [7, 8].

We shall overcome this difficulty by showing under various structural conditions on the running costs that, the optimality condition (1.7) can still be achieved by a Lipschitz function $\psi$ from the state and adjoint processes to the action set. The desired deterministic function $\psi$ is constructed either from a modified Hamiltonian (see Examples 2.1 and 2.2) or by solving the first-order condition explicitly (see Example 2.3). This enables us to reduce the non-Markovian Pontryagin system to a MV-FBSDE whose forward equation depends on the adjoint processes and their marginal distributions. We then prove the well-posedness of the MV-FBSDE by adapting the continuation method in [4, 7], and further establish the Hölder regularity of the solutions by extending the path regularity results for decoupled FBSDEs in [23] to the present setting. This subsequently leads to the desired Hölder continuity of optimal controls of MFC problems (see Theorem 3.6). Our proof for this time regularity of the optimal control exploits the structural properties of the control problem (e.g., the convexity of the loss functional and the uncontrolled diffusion coefficients), which allows us to establish sharper time discretization errors than existing
results for general control problems in [15, 14]. Note that our argument does not explicitly use any regularity of the value function, and the optimal feedback control is merely Lipschitz continuous in the state variable.

This work is organized as follows. Section 2 states the main assumptions of the MFC problem and derives the corresponding MV-FBSDE from the stochastic maximum principle. In Section 3, we analyze the MV-FBSDE and then establish the Hölder regularity of the optimal control of the MFC problem. We prove the order 1/2 convergence of the discrete-time approximation of the value function in Section 4 and then the order 1/4 convergence of optimizers for the discrete-time control problems in Section 5. The Appendix A is devoted to the proofs of some technical results.

**Notation.** We end this section by introducing some notations used throughout this paper. For any given \( n \in \mathbb{N} \) and \( x \in \mathbb{R}^n \), we denote by \( I_n \) the \( n \times n \) identity matrix, by \( 0_n \) the zero element of \( \mathbb{R}^n \) and by \( \delta_x \) the Dirac measure supported at \( x \). We shall denote by \( \langle \cdot, \cdot \rangle \) the usual inner product in a given Euclidean space and by \( | \cdot | \) the norm induced by \( \langle \cdot, \cdot \rangle \), which in particular satisfies for all \( n, m, d \in \mathbb{N} \) and \( \theta_1 = (x_1, y_1, z_1), \theta_2 = (x_2, y_2, z_2) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{n \times d} \) that \( \langle z_1, z_2 \rangle = \text{trace}(z_1^*z_2) \) and \( \langle \theta_1, \theta_2 \rangle = \langle x_1, x_2 \rangle + \langle y_1, y_2 \rangle + \langle z_1, z_2 \rangle \).

We then introduce several spaces: for each \( p \geq 1, k \in \mathbb{N} , t \in [0,T] \) and Euclidean space \((E,|\cdot|)\), \( L^p(\Omega; E) \) is the space of \( E \)-valued \( \mathcal{F} \)-measurable random variables \( X \) satisfying \( \|X\|_{L^p} = \mathbb{E}[|X|^p]^{1/p} < \infty \), and \( L^p(\mathcal{F}_t; E) \) is the subspace of \( L^p(\Omega; E) \) containing all \( \mathcal{F}_t \)-measurable random variables; \( S^p(t,T; E) \) is the space of \( \mathbb{F} \)-progressively measurable processes \( Y : \Omega \times [t,T] \to E \) satisfying \( \|Y\|_{S^p} = \mathbb{E}[\text{ess sup}_{s \in [t,T]} |Y_s|^p]^{1/p} < \infty \); \( \mathcal{H}^p(t,T; E) \) is the space of \( \mathbb{F} \)-progressively measurable processes \( Z : \Omega \times [t,T] \to E \) satisfying \( \|Z\|_{\mathcal{H}^p} = \mathbb{E}[\int_t^T |Z_s|^p \, ds]^{p/2} \) \( < \infty \). For notational simplicity, when \( t = 0 \), we often denote \( S^p = S^p(0,T; E) \) and \( \mathcal{H}^p = \mathcal{H}^p(0,T; E) \), if no confusion occurs.

Moreover, for every Euclidean space \((E,|\cdot|)\), we denote by \( \mathcal{P}_2(E) \) the metric space of probability measures \( \mu \) on \( E \) satisfying \( \|\mu\|_2 = (\int_E |x|^2 \, d\mu(x))^{1/2} < \infty \), endowed with the 2-Wasserstein metric defined as follows:

\[
\mathcal{W}_2(\mu_1, \mu_2) := \inf_{\kappa \in \Pi(\mu_1, \mu_2)} \left( \int_{E \times E} |x - y|^2 \, d\kappa(x,y) \right)^{1/2}, \quad \mu_1, \mu_2 \in \mathcal{P}_2(E),
\]

where \( \Pi(\mu_1, \mu_2) \) is the set of all couplings of \( \mu_1 \) and \( \mu_2 \), i.e., \( \kappa \in \Pi(\mu_1, \mu_2) \) is a probability measure on \( E \times E \) such that \( \kappa(\cdot \times E) = \mu_1 \) and \( \kappa(E \times \cdot) = \mu_2 \). For a given function \( h : \mathcal{P}_2(\mathbb{R}^n \times \mathbb{R}^k) \to \mathbb{R} \) and a measure \( \eta \in \mathcal{P}_2(\mathbb{R}^n \times \mathbb{R}^k) \) with marginals \( \mu \in \mathcal{P}_2(\mathbb{R}^n), \nu \in \mathcal{P}_2(\mathbb{R}^k) \), we denote by \( \partial_\eta h(\eta)(\cdot) : \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}^n \times \mathbb{R}^k \) the L-derivative of \( h \) at \( \eta \) and by \( (\partial_\eta h(\eta), \partial_\nu h(\eta))(\cdot) : \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}^n \times \mathbb{R}^k \) the partial L-derivatives of \( h \) with respect to the marginals; see e.g. [1, Section 2.1] or [8, Chapter 5] for detailed definitions.

Finally, we shall denote by \( C \in [0,\infty) \) a generic constant throughout this paper, which is independent of the initial condition \( \xi_0 \), though it may depend on the constants appearing in the assumptions and may take a different value at each occurrence.

## 2 MV-FBSDEs for mean field control problems

In this section, we state the main assumptions on the coefficients of the MFC problems (1.2), and then derive a coupled MV-FBSDE based on the stochastic maximum principle, which plays an essential role for our subsequent convergence analysis of piecewise constant policy approximations.
H.1. Let \( A \subset \mathbb{R}^k \) be a nonempty closed convex set and let \( b : [0, T] \times \mathbb{R}^n \times \mathbb{R}^k \times \mathcal{P}_2(\mathbb{R}^n \times \mathbb{R}^k) \to \mathbb{R}^n \), \( \sigma : [0, T] \times \mathbb{R}^n \times \mathcal{P}_2(\mathbb{R}^n) \to \mathbb{R}^{n \times d} \), \( f : [0, T] \times \mathbb{R}^n \times \mathbb{R}^k \times \mathcal{P}_2(\mathbb{R}^n \times \mathbb{R}^k) \to \mathbb{R} \) and \( g : \mathbb{R}^n \times \mathcal{P}_2(\mathbb{R}^n) \to \mathbb{R} \) be measurable functions satisfying the following properties:

1. \( b \) and \( \sigma \) are affine in \((x, a, \eta)\), i.e., there exist functions \( b_0 \in L^2(0, T; \mathbb{R}^n) \) and \((b_1, b_2, b_3, \sigma_0, \sigma_1, \sigma_2) \in L^\infty(0, T; \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times k} \times \mathbb{R}^{n \times (n+k)} \times \mathbb{R}^{n \times d} \times \mathbb{R}^{(n \times d) \times n})\) such that for all \((t, x, a, \mu, \eta) \in [0, T] \times \mathbb{R}^n \times \mathbb{R}^k \times \mathcal{P}_2(\mathbb{R}^n) \times \mathcal{P}_2(\mathbb{R}^n \times \mathbb{R}^k)\),
   
   \[
   b(t, x, a, \eta) = b_0(t) + b_1(t)x + b_2(t)a + b_3(t)\bar{\eta},
   \]
   
   \[
   \sigma(t, x, \mu) = \sigma_0(t) + \sigma_1(t)x + \sigma_2(t)\bar{\mu},
   \]

   where \( \bar{\eta} = \int (x, a) \, d\eta(x, a) \) denote the first moments of the measures \( \eta \) and \( \mu \), respectively.

2. \( f(\cdot, 0, 0, \delta_{\eta_{t\leq t'}}) \in L^\infty(0, T), \) \( f \) and \( g \) are differentiable with respect to \((x, a, \eta)\) and \((x, \mu)\), respectively, and all derivatives are of linear growth, i.e., there exists a constant \( \bar{L} \in [0, \infty) \) such that for all \( R > 0 \) and all \((t, x, a, \mu, \eta) \) with \(|x|, |a|, ||\mu||, ||\eta|| \leq R\), we have that
   
   \[
   |\partial_x f(t, x, a, \eta)| + |\partial_a f(t, x, a, \eta)| + |\partial_\mu g(t, x, a, \eta)| + |\partial_\eta g(t, x, a, \mu)| \leq \bar{L}(1 + R),
   \]

   and the \( L^2(\mathbb{R}^n \times \mathbb{R}^k, \eta) \)-norms of the maps \((x', a') \mapsto \partial_x f(t, x, a, \eta)(x', a'), (x', a') \mapsto \partial_a f(t, x, a, \eta)(x', a'), (x', a') \mapsto \partial_\mu g(t, x, a, \mu)(x', a')\) are bounded by \( \bar{L}(1 + R) \), and the \( L^2(\mathbb{R}^n, \mu) \)-norm of the map \( x' \mapsto \partial_\mu g(x, \mu)(x') \) is bounded by \( \bar{L}(1 + R) \).

3. There exists a constant \( \tilde{L} \in [0, \infty) \) such that for all \( t \in [0, T] \), the functions \( \partial_a f(t, \cdot) : \mathbb{R}^n \times A \times \mathcal{P}_2(\mathbb{R}^n \times \mathbb{R}^k) \to \mathbb{R}^n \), \( \partial_\mu f(t, \cdot) : \mathbb{R}^n \times A \times \mathcal{P}_2(\mathbb{R}^n \times \mathbb{R}^k) \to \mathbb{R}^n \) and \( \partial_\mu g(\cdot) : \mathbb{R}^n \times \mathcal{P}_2(\mathbb{R}^n) \to \mathbb{R}^n \) are \( \tilde{L} \)-Lipschitz continuous. Moreover, for any \((t, x, a, \eta, \mu) \in [0, T] \times \mathbb{R}^n \times \mathbb{R}^k \times \mathcal{P}_2(\mathbb{R}^n \times \mathbb{R}^k) \times \mathcal{P}_2(\mathbb{R}^n)\), there exist versions of \( \partial_\mu f(t, x, a, \eta)(\cdot), \partial_\mu f(t, x, a, \eta)(\cdot) \) and \( \partial_\mu g(x, \mu)(\cdot) \) such that
   
   \[
   (x, a, \eta, \mu, x', a') \in \mathbb{R}^n \times A \times \mathcal{P}_2(\mathbb{R}^n \times \mathbb{R}^k) \times \mathcal{P}_2(\mathbb{R}^n) \times \mathbb{R}^n \times \mathbb{R}^n,
   \]

   is \( \tilde{L} \)-Lipschitz continuous.

4. \( f \) is convex with respect to \((x, a, \eta)\), i.e., there exist constants \( \lambda_1, \lambda_2 \geq 0 \) satisfying \( \lambda_1 + \lambda_2 > 0 \) and for all \( t \in [0, T], (x, a, \eta), (x', a', \eta') \in \mathbb{R}^n \times A \times \mathcal{P}_2(\mathbb{R}^n \times \mathbb{R}^k)\),
   
   \[
   f(t, x', a', \eta') - f(t, x, a, \eta) - (\partial_a f(t, x, a, \eta)(x' - x)) - (\partial_\mu f(t, x, a, \eta)(\bar{a'}) - a') - \tilde{E}[(\partial_\mu f(t, x, a, \eta)(\bar{a} - \bar{a})) + (\partial_\mu f(t, x, a, \eta)(\bar{a} - \bar{a}))]
   \]
   
   \[
   \geq \lambda_1 |a' - a|^2 + \lambda_2 \bar{E}[|\bar{a}' - \bar{a}|^2],
   \]

   whenever \((\bar{X}, \bar{a}), (\bar{X}', \bar{a}') \in L^2(\tilde{\Omega}, \tilde{F}, \tilde{P}; \mathbb{R}^n \times \mathbb{R}^k)\) with distributions \( \eta \) and \( \eta' \), respectively. The function \( g \) is convex in \((x, \mu)\), i.e., we have for all \((x, \mu), (x', \mu') \in \mathbb{R}^n \times \mathcal{P}_2(\mathbb{R}^n)\) that
   
   \[
   g(x' , \mu') - g(x, \mu) - (\partial_\mu g(x, \mu)(x') - x) - \tilde{E}[(\partial_\mu g(x, \mu)(\bar{X}', \bar{X}))] \geq 0,
   \]

   whenever \( \bar{X}, \bar{X}' \in L^2(\tilde{\Omega}, \tilde{F}, \tilde{P}; \mathbb{R}^n)\) with distributions \( \mu \) and \( \mu' \), respectively. Above and hereafter, we denote by \( \tilde{E} \) the expectation on \((\tilde{\Omega}, \tilde{F}, \tilde{P})\).

Remark 2.1. (H.1) naturally extends Assumption “Control of MKV Dynamics” in [8] to the present setting with mean field interactions through controls. In particular, (H.1) allows the coefficients \((b, \sigma, f)\) to be merely measurable in time, and the cost function \( f \) to be strongly convex either in the state or in the law of the controls, which is important for the applications to
control problems whose cost function does not explicitly depend on the state of the controls (see e.g. Proposition 2.2). The assumption that the volatility coefficient is uncontrolled enables us to study the regularity of optimal controls and subsequently to quantify the time discretization error of (1.2) via a probabilistic approach (see Theorems 3.6 and 4.2).

Note that the continuous differentiability of \( f \) and the linear growth of its derivatives (see (H.1(2)/(3))) show that there exists a constant \( C \) satisfying for all \((x, a, \eta), (x', a', \eta') \in \mathbb{R}^n \times A \times \mathcal{P}_2(\mathbb{R}^n x \times \mathbb{R}^k)\) that
\[
|f(t, x, a, \eta) - f(t, x', a', \eta')| \\
\leq C(1 + |x| + |x'| + |a| + |a'| + \|\eta\|_2 + \|\eta'\|_2)((|x, a) - (x', a')| + \mathcal{W}_2(\eta, \eta')), 
\]
which together with the uniform boundedness of \( |f(t, 0, 0, \delta_{0_{n+k}})| \) implies that the function \( f \) is at most of quadratic growth with respect to \((x, a, \eta)\). Similar arguments show that the function \( g \) is locally Lipschitz continuous and at most of quadratic growth with respect to \((x, \mu)\).

It is clear that under (H.1), for any given initial state \( \xi_0 \in L^2(\mathcal{F}_0; \mathbb{R}^n) \) and admissible control \( \alpha \in \mathcal{A} \), the controlled state process \( X^\alpha \in S^2(\mathbb{R}^n) \) is well-defined by (1.1) and the cost functional \( J(\alpha; \xi_0) \) is finite since the functions \( f \) and \( g \) are at most of quadratic growth (see Remark 2.1). We now apply the stochastic maximum principle to (1.2) and characterize the optimal control by a MV-FBSDE.

Let \( H : [0, T] \times \mathbb{R}^n x \times \mathbb{R}^k \times \mathcal{P}_2(\mathbb{R}^n x \times \mathbb{R}^k) \times \mathbb{R}^n x \times \mathbb{R}^{n \times d} \rightarrow \mathbb{R} \) be the Hamiltonian of (1.2) defined as follows:
\[
H(t, x, a, \eta, y, z) := \langle b(t, x, a, \eta), y \rangle + \langle \sigma(t, x, \pi_1 \sharp \eta), z \rangle + f(t, x, a, \eta), \tag{2.1} \]
where \( \pi_1 \sharp \eta \) denotes the first marginal of the measure \( \eta \). The linearity of \( b, \sigma \) in (H.1(1)) and the convexity of \( f, g \) in (H.1(4)) ensure that the stochastic maximum principle gives a necessary and sufficient optimality condition of an optimal control of (1.2); see e.g. [1, Theorem 3.5] for the optimality condition with a bounded function \( b_0 \) in (H.1(1)), which can be easily extended to the present setting. More precisely, suppose that (H.1) holds and let \( \xi_0 \in L^2(\mathcal{F}_0; \mathbb{R}^n) \) be a given initial state. For any given admissible control \( \alpha \in \mathcal{A} \), let \( X^\alpha \) be the corresponding controlled state process satisfying (1.1), and let \((Y^\alpha, Z^\alpha) \in S^2(\mathbb{R}^n) \times \mathcal{H}^2(\mathbb{R}^{n \times d})\) be an adjoint process of \( X^\alpha \) satisfying the following MV-FBSDE: for all \( t \in [0, T]\),
\[
dY^\alpha_t = -(\partial_\alpha H(\theta^\alpha_t, Y^\alpha_t, Z^\alpha_t) + \mathbb{E}[\partial_\mu H(\theta^\alpha_t, \hat{Y}^\alpha_t, \hat{Z}^\alpha_t)(X^\alpha_t, \alpha_t)]) \, dt + Z^\alpha_t \, dW_t, \\
Y^\alpha_T = \partial_\gamma g(X^\alpha_T, \mathbb{P}X^\alpha_T) + \mathbb{E}[\partial_\mu g(\hat{X}^\alpha_T, \mathbb{P}X^\alpha_T)(X^\alpha_T)], \tag{2.2} \]
where \( \theta^\alpha_t = (t, X^\alpha_t, \alpha_t, \mathbb{P}(X^\alpha_t, \alpha_t)) \) and the tilde notation refers to an independent copy. Then the stochastic maximum principle asserts that if the following optimality condition holds:
\[
\langle \partial_\alpha H(\theta^\alpha_t, Y^\alpha_t, Z^\alpha_t) + \mathbb{E}[\partial_\mu H(\theta^\alpha_t, \hat{Y}^\alpha_t, \hat{Z}^\alpha_t)(X^\alpha_t, \alpha_t)], \alpha_t - a \rangle \leq 0, \quad \forall a \in \mathcal{A}, \; \mathbb{dP} \otimes dt \text{-a.e.}, \tag{2.3} \]
then \( \alpha \in \mathcal{A} \) is an optimal control of (1.2). Note that under (H.1), for any given control \( \alpha \in \mathcal{A} \), the adjoint process \((Y^\alpha, Z^\alpha) \in S^2(\mathbb{R}^n) \times \mathcal{H}^2(\mathbb{R}^{n \times d})\) is uniquely defined (see [1]).

One can clearly observe that the optimality condition (2.3) and the progressively measurable control process \( \alpha \) lead to a non-Markovian coupled forward-backward system (1.1), (2.2) and (2.3) with random coefficients. In the following, we shall reduce the problem into a forward-backward system with deterministic coefficients by assuming the solvability of the optimality condition (2.3).

We first observe that, by virtue of the fact that the coefficient \( \sigma \) is uncontrolled, the optimality condition (2.3) can be equivalently written as: it holds for all \( a \in \mathcal{A} \) and for \( \mathbb{dP} \otimes dt \text{-a.e.} \) that
\[
\langle \partial_\alpha H^{re}(t, X^\alpha_t, \alpha_t, \mathbb{P}(X^\alpha_t, \alpha_t), Y^\alpha_t) + \mathbb{E}[\partial_\mu H^{re}(t, \hat{X}^\alpha_t, \alpha_t, \mathbb{P}(X^\alpha_t, \alpha_t), \hat{Y}^\alpha_t)(X^\alpha_t, \alpha_t)], \alpha_t - a \rangle \leq 0, \tag{2.4} \]
where $H^{re} : [0, T] \times \mathbb{R}^n \times \mathbb{R}^k \times \mathcal{P}_2(\mathbb{R}^n \times \mathbb{R}^k) \times \mathbb{R}^n \to \mathbb{R}$ is the reduced Hamiltonian defined by:

$$H^{re}(t, x, a, \eta, y) := \langle b(t, x, a, \eta), y \rangle + f(t, x, a, \eta) \quad (2.5)$$

The following assumption then asserts that the optimality condition (2.4) can be achieved by a sufficiently regular feedback map from the state and adjoint processes to the action set, which will be verified for several MFC problems appearing in practice.

**H.2.** (1) Assume the notation of (H.1). There exists a measurable function $\hat{\alpha} : [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \times \mathcal{P}_2(\mathbb{R}^n \times \mathbb{R}^n) \to A$ and a constant $L_\alpha \in [0, \infty)$ such that for all $t \in [0, T]$, $|\hat{\alpha}(\cdot, 0, 0, \delta_{0+n})| \leq L_\alpha$, the function $\hat{\alpha}(t, \cdot) : \mathbb{R}^n \times \mathbb{R}^n \times \mathcal{P}_2(\mathbb{R}^n \times \mathbb{R}^n) \to A$ is $L_\alpha$-Lipschitz continuous, and the optimality condition (2.4) holds, i.e., for all $(x, y, \chi, a) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathcal{P}_2(\mathbb{R}^n \times \mathbb{R}^n) \times A$,

$$\langle \partial_a H^{re}(t, x, \hat{\alpha}(t, x, y, \chi), \phi(t, \chi), y) \rangle + \int_{\mathbb{R}^n \times \mathbb{R}^n} \partial_\nu H^{re}(t, \tilde{x}, \hat{\alpha}(t, \tilde{x}, \tilde{y}, \chi), \phi(t, \chi), \tilde{y})(x, \hat{\alpha}(t, x, y, \chi)) d\chi(\tilde{x}, \tilde{y}), \quad (2.6)$$

$$\hat{\alpha}(t, x, y, \chi) - a \leq 0,$$

where $\phi(t, \chi) := \chi \circ (\mathbb{R}^n \times \mathbb{R}^n \ni (x, y) \mapsto (x, \hat{\alpha}(t, x, y, \chi)) \in \mathbb{R}^n \times A)^{-1}$.

(2) The function $\hat{\alpha}$ is locally Hölder continuous in time, i.e., it holds for all $t, t' \in [0, T], (x, y, \chi) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathcal{P}_2(\mathbb{R}^n \times \mathbb{R}^n)$ that $|\hat{\alpha}(t, x, y, \chi) - \hat{\alpha}(t', x, y, \chi)| \leq L_\alpha(1 + |x| + |y| + \|\chi\|_2)|t - t'|^{1/2}$.

Roughly speaking, (H.2) ensures that there exists a deterministic function $\hat{\alpha}$ satisfying the optimality condition (2.3) pointwise, which enables us to study controls $\hat{\alpha} \in A$ of the form $\hat{\alpha}_t = \hat{\alpha}(t, X^t, Y^t, \mathbb{P}(X^t, Y^t))$, $t \in [0, T]$. In this case, it is not difficult to see that $\phi(t, \mathbb{P}(X^t, Y^t))$ is the joint law of $(X^t, \hat{\alpha})$, since it holds for any $t \in [0, T], X, Y \in L^2(\Omega; \mathbb{R}^n)$ and any Borel measurable set $A \subset \mathbb{R}^n \times \mathbb{R}^k$ that

$$\phi(t, \mathbb{P}(X, Y))(A) = \mathbb{P}(X, Y)(\langle \text{id}_{\mathbb{R}^n}, \hat{\alpha}(t, \cdot, \cdot, \mathbb{P}(X, Y)) \rangle^{-1}(A))$$

$$= \mathbb{P}(\langle X, Y \in \text{id}_{\mathbb{R}^n}, \hat{\alpha}(t, \cdot, \cdot, \mathbb{P}(X, Y)) \rangle^{-1}(A))$$

$$= \mathbb{P}(\langle X, \hat{\alpha}(t, X, Y, \mathbb{P}(X, Y)) \in A \rangle) = \mathbb{P}(X, \hat{\alpha}(t, X, Y, \mathbb{P}(X, Y)))(A). \quad (2.7)$$

Note that similar assumptions have been made in [12, Theorem 3] and [16, Assumption (A6)] to study MFC problems. Under (H.1) and (H.2), we shall establish the existence of a Hölder continuous optimal control for (1.2) in Section 3, and then analyze the convergence rate of piecewise constant policy approximation for (1.2) in Section 4.

In the following, we verify (H.2) for different classes of MFC problems appearing in practice, which are not covered by results in the existing literature. In particular, we shall give precise conditions on the functions $(b, f)$ in (1.2) to ensure the existence and regularity of the function $\hat{\alpha}$. Note that these conditions do not involve high-order derivatives of the cost functions, which enables us to quantify the time discretization error of (1.2) under much weaker assumptions than conditions (B1) and (C1)-(C3) in [9] (see the discussions above Theorem 4.2 for details). In particular, we allow merely measurable functions $(b_i, \sigma_i)$, a possibly degenerate state-dependent diffusion coefficient, and cost functions $(f, g)$ that are not necessarily twice differentiable.

**Example 2.1.** In this example, we show (H.2) is satisfied by a class of MFC problems with cost function $f$ which does not involve the law of the controls. This includes the classical MFC problem as a special case, for which the controlled dynamics is also independent of the law of the controls (see [10, 7, 8, 9]).
The proof of the following proposition is based on defining the function \( \alpha \) as the minimizer of a modified version of the reduced Hamiltonian \( H^{re} \). Note that one can adapt the arguments to verify (H.2) for more general cost functions which are affine in the law of the controls, i.e., \( f(t, x, a, \eta) = f_1(t, x, a, \pi_1^{\sharp} \eta) + \langle f_2(t, x, \pi_1^{\sharp} \eta), \pi_2^{\sharp} \eta \rangle \), but for notational simplicity, we choose to refrain from providing this level of generality without the motivation from specific applications.

**Proposition 2.1.** Suppose (H.1) holds, and for each \( (t, x, a) \in [0, T] \times \mathbb{R}^n \times \mathbb{R}^k \), the function \( \mathcal{P}_2(\mathbb{R}^n \times \mathbb{R}^k) \ni \eta \mapsto f(t, x, a, \eta) \in \mathbb{R} \) depends only on the first marginal \( \pi_1^{\sharp} \eta \) of the measure \( \eta \). Then there exists a function \( \tilde{\alpha} : [0, T] \times \mathbb{R}^n \times \mathcal{P}_2(\mathbb{R}^n) \to \mathbf{A} \) satisfying (H.2(1)).

Assume further that there exists a constant \( \tilde{K} \in [0, \infty) \) such that it holds for all \( t, t' \in [0, T] \), \( (x, a, \eta) \in \mathbb{R}^n \times \mathbf{A} \times \mathcal{P}_2(\mathbb{R}^n) \) that \( |b_2(t) - b_2(t')| + |b_3(t) - b_3(t')| \leq \tilde{K}|t - t'|^{1/2} \) and \( |\partial_a f(t, x, a, \eta) - \partial_a f(t', x, a, \eta)| \leq \tilde{K}(1 + |x| + |a| + \|\eta\|_2)|t - t'|^{1/2} \). Then there exists a function \( \tilde{\alpha} : [0, T] \times \mathbb{R}^n \times \mathcal{P}_2(\mathbb{R}^n) \to \mathbf{A} \) satisfying (H.2).

**Proof.** Observe that under the assumptions of Proposition 2.1, the reduced Hamiltonian (2.5) can be written as follows: for all \( (t, x, a, \eta, y) \in [0, T] \times \mathbb{R}^n \times \mathbb{R}^k \times \mathcal{P}_2(\mathbb{R}^n) \times \mathcal{P}_2(\mathbb{R}^n) \times \mathbb{R}^n \),

\[
H^{re}(t, x, a, \eta, y) = \langle b(t, x, a, \eta, y), \psi_1(t, x, a, \eta, y) \rangle + \tilde{f}(t, x, a, \psi_2(t, x, a, \eta, y)) + \tilde{f}(t, x, a, \eta, y),
\]

where we have \( \psi_1(t, x, a, \eta, y) := \langle b_0(t) + b_1(t)x + b_2(t)a, \psi_2(t, y, \eta, y) := \langle b_3(t) \eta, y \rangle \) and \( \tilde{f}(t, x, a, \mu) := f(t, x, a, \mu \times \delta_0) \). Moreover, we have that \( \partial_t H^{re}(t, x, a, \eta, y) = b_2(t) y + \partial_a \tilde{f}(t, x, a, \eta, y) \) and \( \partial_a H^{re}(t, x, a, \eta, y)(\cdot) = \partial_a \psi_2(t, y, \eta, y)(\cdot) = \beta(t)(y) \), where \( \beta(t) \in \mathbb{R}^{k \times n} \) is the submatrix formed by deleting the first \( n \) rows of \( b_2(t) \in \mathbb{R}^{(n+k) \times n} \).

Let us define the function \( G : [0, T] \times \mathbb{R}^n \times \mathbb{R}^k \times \mathcal{P}_2(\mathbb{R}^n) \times \mathcal{P}_2(\mathbb{R}^n) \times \mathbb{R}^n \to \mathbb{R} \) satisfying for all \( (t, x, a, \rho, y) \in [0, T] \times \mathbb{R}^n \times \mathbb{R}^k \times \mathcal{P}_2(\mathbb{R}^n) \times \mathcal{P}_2(\mathbb{R}^k) \times \mathbb{R}^n \) that

\[
G(t, x, a, \rho, y) := \psi_1(t, x, a, y) + \psi_2(t, y, \delta_a, \rho) + \tilde{f}(t, x, a, \rho)
\]

with \( \rho = \int_{\mathbb{R}^n} y \, d\rho(y) \). We further define the map \( \tilde{\alpha} : [0, T] \times \mathbb{R}^n \times \mathcal{P}_2(\mathbb{R}^n) \to \mathbf{A} \) satisfying for all \( (t, x, y, \chi) \in [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \times \mathcal{P}_2(\mathbb{R}^n) \) that

\[
\tilde{\alpha}(t, x, y, \chi) = \arg \min_{\alpha \in \mathbf{A}} G(t, x, a, \pi_1^{\sharp} \chi, \pi_2^{\sharp} \chi, y).
\]

Since the function \( f \) depends only on the first marginal \( \pi_1^{\sharp} \eta \), we see from (H.1(4)) that \( \lambda_1 > 0 \) and the map \( \mathbf{A} \ni a \mapsto \hat{f}(t, x, a, \mu) \in \mathbb{R} \) is \( \lambda_1 \)-strongly convex, which along with the linearity of the map \( \mathbf{A} \ni a \mapsto \psi_1(t, x, a, y) + \psi_2(t, \mu \times \delta_a, \rho) \in \mathbb{R} \) shows \( \mathbf{A} \ni a \mapsto G(t, x, a, \pi_1^{\sharp} \chi, \pi_2^{\sharp} \chi, y) \in \mathbb{R} \)

is \( \lambda_1 \)-strongly convex. Then by following the same argument as in [8, Lemma 3.3], we can show the above function \( \tilde{\alpha} \) is well-defined, measurable, locally bounded and Lipschitz continuous with respect to \( (x, y, \chi) \) uniformly in \( t \).

Then it remains to verify (2.6) in order to show that \( \tilde{\alpha} \) satisfies (H.2(1)). The fact that \( \tilde{\alpha} \) is a minimizer of \( G \) over \( \mathbf{A} \) and the definition of \( G \) imply for all \( (t, x, y, \chi) \in [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \times \mathcal{P}_2(\mathbb{R}^n) \), \( a \in \mathbf{A} \) that

\[
0 \geq \langle \partial_a G(t, x, a, \pi_1^{\sharp} \chi, \pi_2^{\sharp} \chi, y), \tilde{\alpha}(t, x, y, \chi) - a \rangle
= \langle b_2(t) y + \beta(t) \pi_2^{\sharp} \chi + \partial_a \tilde{f}(t, x, \pi_1^{\sharp} \chi, \pi_2^{\sharp} \chi), \tilde{\alpha}(t, x, y, \chi) - a \rangle,
\]

where \( \pi_2^{\sharp} \chi = \int_{\mathbb{R}^n} \pi_1^{\sharp} \chi(y) \). The fact that \( f(t, x, a, \eta) \) depends only on the first marginal of the measure \( \eta \) gives us that \( \partial_a \tilde{f}(t, x, a, \pi_1^{\sharp} \chi) = \partial_a \tilde{f}(t, x, a, \pi_1^{\sharp} \phi(t, \chi)) \), with the function \( \phi \) defined as
in (2.6). Hence, we can obtain from the expression of $\partial_a H^{re}$ that

$$0 \geq \langle b^*_2(t)y + \beta(t)\pi_2^X \rangle + \partial_a \tilde{f}(t, x, \alpha(t, x, y, \chi), \pi_1^X \phi(t, \chi)), \alpha(t, x, y, \chi) - a)$$

$$\quad = \langle \partial_a H^{re}(t, x, \alpha(t, x, y, \chi), \phi(t, \chi), y) + \beta(t)\pi_2^X, \alpha(t, x, y, \chi) - a)\rangle,$$

which is the optimality condition (2.6) since for all $(t, a, \eta)$,

$$\int_{\mathbb{R}^n \times \mathbb{R}^n} \partial_a H^{re}(t, \tilde{x}, \alpha, \eta, \tilde{y})(\cdot) d\chi(\tilde{x}, \tilde{y}) = \int_{\mathbb{R}^n \times \mathbb{R}^n} \beta(t)\tilde{y} d\chi(\tilde{x}, \tilde{y}) = \beta(t)\pi_2^X.$$

We now prove the time regularity of $\hat{\alpha}$ under the additional assumption on the Hölder regularity of the functions $b_2, b_3$ and $\partial_a f$. Let $t, t' \in [0, T], (x, y, \chi) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathcal{P}_2(\mathbb{R}^n \times \mathbb{R}^n)$, $\hat{\alpha} = \hat{\alpha}(t, x, y, \chi)$ and $\hat{\alpha}' = \hat{\alpha}(t', x, y, \chi)$. The optimal condition (2.9) gives us that $\langle \partial_a G(t, x, \hat{\alpha}, \pi_1^X, \pi_2^X, y), \hat{\alpha}' - \hat{\alpha}\rangle \geq 0 \geq \langle \partial_a G(t', x, \hat{\alpha}', \pi_1^X, \pi_2^X, y), \hat{\alpha}' - \hat{\alpha}\rangle$. Moreover, the $\lambda_1$-strong convexity of $A \ni a \mapsto G(t, x, a, \pi_1^X, \pi_2^X, y) \in \mathbb{R}$ shows that

$$G(t, x, \hat{\alpha}', \pi_1^X, \pi_2^X, y) - G(t, x, \hat{\alpha}, \pi_1^X, \pi_2^X, y)$$

$$\quad - \langle \partial_a G(t, x, \hat{\alpha}, \pi_1^X, \pi_2^X, y), \hat{\alpha}' - \hat{\alpha}\rangle \geq \lambda_1|\hat{\alpha}' - \hat{\alpha}|^2,$$

from which, by exchanging the role of $\hat{\alpha}'$ and $\hat{\alpha}$ in the above inequality and summing the resulting estimates, we can deduce that

$$2\lambda_1|\hat{\alpha}' - \hat{\alpha}|^2 \leq \langle \hat{\alpha}' - \hat{\alpha}, \partial_a G(t, x, \hat{\alpha}', \pi_1^X, \pi_2^X, y) - \partial_a G(t, x, \hat{\alpha}, \pi_1^X, \pi_2^X, y)\rangle$$

$$\quad \leq \langle \hat{\alpha}' - \hat{\alpha}, \partial_a G(t, x, \hat{\alpha}', \pi_1^X, \pi_2^X, y) - \partial_a G(t', x, \hat{\alpha}', \pi_1^X, \pi_2^X, y)\rangle.$$
Proposition 2.2. Suppose (H.1) holds, the function $b_2$ in (H.1(1)) satisfies $b_2(t) = 0$ for all $t \in [0, T]$, and the function $f$ is of the form $f(t, x, a, \eta) = f_1(t, x, \pi_1 \eta, \pi_2 \eta) + f_2(t, x, \pi_1 \tilde{\eta}, \pi_2 \tilde{\eta})$, where $f_1 : [0, T] \times \mathbb{R}^n \times P_2(\mathbb{R}^n) \times P_2(\mathbb{R}^k) \to \mathbb{R}$ and $f_2 : [0, T] \times \mathbb{R}^k \times P_2(\mathbb{R}^n) \times P_2(\mathbb{R}^k) \to \mathbb{R}$ are functions satisfying (H.1(2)(3)), and $\pi_1 \eta$ (resp. $\pi_2 \eta$) is the first (resp. second) marginal of the measure $\eta$. Then there exists a function $\alpha : [0, T] \times P_2(\mathbb{R}^n \times \mathbb{R}^n) \to \mathcal{A}$ satisfying (H.2(1)).

Assume further that there exists a constant $\tilde{K} \in [0, \infty)$ such that it holds for all $t, t' \in [0, T]$, $(x, a, \mu) \in \mathbb{R}^n \times \mathcal{A} \times P_2(\mathbb{R}^n)$ that $|b_3(t) - b_3(t')| \leq \tilde{K}|t - t'|^{1/2}$ and

$$
|\partial_\nu f_1(t, x, \mu, \delta)(a) - \partial_\nu f_1(t', x, \mu, \delta)(a)| + |\partial_\nu f_2(t, a, \mu, \delta)(a) - \partial_\nu f_2(t', a, \mu, \delta)(a)| + |\partial_\nu f_2(t, a, \mu, \delta)(a) - \partial_\nu f_2(t', a, \mu, \delta)(a)| \leq \tilde{K}(1 + |x| + |a| + \|\mu\|_2)|t - t'|^{1/2}.
$$

Then there exists a function $\hat{\alpha} : [0, T] \times P_2(\mathbb{R}^n \times \mathbb{R}^n) \to \mathcal{A}$ satisfying (H.2).

Proof. We shall consider the function $\hat{\alpha} : [0, T] \times P_2(\mathbb{R}^n \times \mathbb{R}^n) \to \mathcal{A}$ satisfying for all $(t, \chi) \in [0, T] \times P_2(\mathbb{R}^n \times \mathbb{R}^n)$ that

$$
\hat{\alpha}(t, \chi) = \arg \min_{a \in \mathcal{A}} h(t, \chi, a), \quad \text{with } h(t, \chi, a) := \hat{E}[H^{re}(t, \tilde{X}, a, \tilde{P}(\tilde{X}_0, \tilde{Y})],
$$

where $(\tilde{X}, \tilde{Y}) \in L^2(\Omega, \tilde{\mathcal{F}}, \tilde{P}; \mathbb{R}^n \times \mathbb{R}^n)$ has distribution $\chi$.

We first show the function $\hat{\alpha}$ is well-defined. By using the linearity of $b$ and the convexity of $f$ in (H.1), we see that the map $\mathcal{A} \ni a \mapsto h(t, \chi, a) \in \mathbb{R}$ is strongly convex with factor $\lambda_1 + \lambda_2 > 0$, which admits a unique minimizer on the nonempty closed convex set $\mathcal{A}$. The measurability of $\hat{\alpha}$ follows from [8, Lemma 3.3].

Then, we prove that the function $\hat{\alpha}$ satisfies the optimality condition (2.6). By using (H.1), we have for almost all $(t, \omega) \in [0, T] \times \tilde{\Omega}$ that the mapping $\mathbb{R}^n \ni a \mapsto H^{re}(t, \tilde{X}(\omega), a, \tilde{P}(\tilde{X}_0, \tilde{Y}(\omega)))$ is differentiable with the derivative being at most of linear growth in $(\tilde{X}(\omega), \tilde{Y}(\omega))$. Hence, Lebesgue’s differentiation theorem shows that $h$ is differentiable with respect to $a$ with the derivative

$$
\partial_\alpha h(t, \chi, a) = \hat{E}[\partial_\alpha H^{re}(t, \tilde{X}, a, \tilde{P}(\tilde{X}_0, \tilde{Y})), \tilde{Y}(\tilde{X}, a)],
$$

where $\tilde{X} \in L^2(\Omega, \tilde{\mathcal{F}}, \tilde{P}; \mathbb{R}^n)$ has distribution $\tilde{P}(\tilde{X})$.

Observe that $b_2 \equiv 0$ and the structural condition of $f$ imply that the reduced Hamiltonian (2.5) is given by $H^{re}(t, x, a, \eta, y) = \langle b_0(t) + b_1(t)x + b_3(t)\eta, y \rangle + f_1(t, x, \pi_1 \eta, \pi_2 \eta) + f_2(t, a, \pi_1 \tilde{\eta}, \pi_2 \tilde{\eta})$. Hence, for all $(t, x, y, a)$, $\partial_\alpha H^{re}(t, x, a, \tilde{P}(\tilde{X}, a), y) = \partial_\alpha f_2(t, a, \tilde{P}(\tilde{X}, a), \delta_a)$ and $\partial_\alpha H^{re}(t, x, a, \eta, y)(\cdot)$ can be chosen as a function defined only on $\mathbb{R}^k$ (not on $\mathbb{R}^n \times \mathbb{R}^k$ as in the general setting), which simplifies (2.10) into:

$$
\partial_\alpha h(t, \chi, a) = \partial_\alpha H^{re}(t, x, a, \tilde{P}(\tilde{X}_0, \tilde{Y})), \tilde{Y}(\tilde{X}, a)]
$$

for all $(t, \chi, a) \in [0, T] \times P_2(\mathbb{R}^n \times \mathbb{R}^n) \times \mathcal{A}$. Consequently, one can conclude from the fact that $\hat{\alpha}(t, \chi)$ is a minimizer and the identity that $\tilde{P}(\tilde{X}, \hat{\alpha}(t, \chi)) = \phi(t, \chi)$ (see (2.7)) that the function $\hat{\alpha}$ satisfies the optimality condition (2.6): for all $(t, \chi, a) \in [0, T] \times P_2(\mathbb{R}^n \times \mathbb{R}^n) \times \mathcal{A}$,

$$
0 \geq \langle \partial_\alpha h(t, \chi, \hat{\alpha}(t, \chi)), \hat{\alpha}(t, \chi) - a \rangle
$$

$$
= \langle \partial_\alpha H^{re}(t, x, \hat{\alpha}(t, \chi), \phi(t, \chi), y)
$$

$$
+ \hat{E}[\partial_\alpha H^{re}(t, \tilde{X}, \hat{\alpha}(t, \chi), \phi(t, \chi), \tilde{Y}(\hat{\alpha}(t, \chi))), \hat{\alpha}(t, \chi) - a],
$$

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whenever \((\bar{X}, \bar{Y}) \in L^2(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}}; \mathbb{R}^n \times \mathbb{R}^n)\) has distribution \(\chi\). Note that in the present setting (2.6) is independent of \((x, y)\) since \(\partial_a H^{\infty}(t, x, \hat{\alpha}(t, \chi), \phi(t, \chi), y) = \partial_a f_2(t, a, \bar{\mathbb{P}}_{\bar{X}}, \delta_{\alpha(t, \chi)})\).

Finally, we establish the spatial and time regularity of \(\hat{\alpha}\). Similar to [8, Lemma 3.3], by using \((\lambda_1 + \lambda_2)\)-strong convexity of \(a \mapsto h(t, \chi, a)\), we can show for all \(t \in [0, T]\) that \(|\hat{\alpha}(t, \delta_{\alpha_{0+n}}) - a_0| \leq (\lambda_1 + \lambda_2)^{-1}||\partial_a h(t, \delta_{\alpha_{0+n}}, a_0)||\), where \(a_0\) an arbitrary element in \(\mathbb{A}\). Then (2.11) and (H.1(2)) imply the estimate that 

\[
|\hat{\alpha}(t, \delta_{\alpha_{0+n}}) - a_0| \leq C \left( (|\partial_a f_2(t, a', \bar{\mathbb{P}}_{\bar{X}}, \delta_{\alpha'}| - |\partial_a f_2(t', a', \bar{\mathbb{P}}_{\bar{X}}, \delta_{\alpha'}|)\right),
\]

where the constant \(C\) is independent of \(t, t', \chi, \chi'\). Setting \(t' = t\) in the above estimate gives us that 

\[
|\hat{\alpha}(t, \chi) - \hat{\alpha}(t, \chi')| \leq CW_2(\chi, \chi'),
\]

which along with \(\|\hat{\alpha}(\cdot, \delta_{\alpha_{0+n}})||_{L^{\infty}(0, T)} < \infty\) implies \(\|\hat{\alpha}(t, \chi)||_{L^{\infty}(0, T)} < \infty\). The desired time regularity of \(\hat{\alpha}\) then follows from the additional assumptions on the time regularity of coefficients. \(\square\)

**Example 2.3.** In this example, we verify (H.2) for MFC problems whose running costs are quadratic in the control variables, which extend the commonly studied linear-quadratic models (see e.g. [1, 9, 16]) to cost functions that are convex in the state variables.

For notational simplicity, we consider a one-dimensional problem (1.2) with \(n = k = d = 1\), an action set \(\mathbb{A} = \mathbb{R}\) and a running cost of the following form

\[
f(t, x, a, \eta) = \frac{1}{2} \left( f_1(t, x, \pi_1^{\eta} \eta) + q(t) a^2 + \bar{q}(t) (a - r(t)a)^2 + 2c(t) xa \right),
\]

where \(\pi_1^{\eta} \eta\) is the first marginal of \(\eta\), \(\bar{a} = \int a \, d\eta(x, a)\), \(q, \bar{q}, r, c \in L^\infty(0, T; \mathbb{R})\), \(q \geq \lambda_1 > 0\), \(\bar{q} \geq 0\) and \(f_1 : [0, T] \times \mathbb{R} \times \mathcal{P}_2(\mathbb{R}) \to \mathbb{R}\) is a suitable function such that the running cost \(f\) satisfies (H.1). Similar arguments can be adapted to verify (H.2) for multi-dimensional running costs with a general quadratic dependence on the control variables.

In the present setting, we see that the drift coefficient of (1.1) reads as

\[
b(t, x, a, \eta) = b_0(t) + b_1(t)x + b_2(t)a + \beta(t)x + \gamma(t)a,
\]

where \(\bar{x} = \int x \, d\eta(x, a)\) and \(\beta, \gamma \in L^\infty(0, T; \mathbb{R})\) denote the first and second component of the function \(b_3\) in (H.1(1)), respectively. The definition of the reduced Hamiltonian (2.5) and the openness of the set \(\mathbb{A}\) imply that it suffices to find a function \(\hat{\alpha} : [0, T] \times \mathbb{R} \times \mathcal{P}_2(\mathbb{R} \times \mathbb{R}) \to \mathbb{A}\) such that for all \(t \in [0, T]\), \(X_t, Y_t \in L^2(\Omega; \mathbb{R})\), we have that \(\alpha_t = \hat{\alpha}(t, X_t, Y_t, \mathbb{P}(X_t, Y_t))\) satisfies

\[
b_2(t) Y_t + \gamma(t) E[Y_t] + (q(t) + \bar{q}(t)) \alpha_t + \bar{q}(t) r(t)(r(t) - 2) E[\alpha_t] + c(t) X_t = 0.
\]
Taking expectations on both sides of (2.14) gives us that

$$\mathbb{E}[\alpha_t] = \frac{-b_2(t) + \gamma(t)\mathbb{E}[Y_t] - c_t\mathbb{E}[X_t]}{q(t) + q(t)(r(t) - 1)^2},$$

(2.15)

which is well-defined since $q(t) \geq \lambda_1 > 0$ and $\bar{q}(t) \geq 0$. Then, by substituting (2.15) into (2.14), we see that it suffices to define $\hat{\alpha} : [0, T] \times \mathbb{R} \times \mathbb{R} \times \mathcal{P}_2(\mathbb{R} \times \mathbb{R}) \to \mathbb{A}$ to be the function satisfying for all $(t, x, y, z, \rho) \in [0, T] \times \mathbb{R} \times \mathbb{R} \times \mathcal{P}_2(\mathbb{R} \times \mathbb{R})$

$$\hat{\alpha}(t, x, y, \chi) = \frac{-c(t)x - b_2(t)y + \psi(t)\int_{\mathbb{R}} x \, d\chi(x, y) + (-\gamma(t) + \zeta(t))\int_{\mathbb{R}} y \, d\chi(x, y)}{q(t) + \bar{q}(t)},$$

with the coefficients

$$\psi(t) := \frac{c(t)\bar{q}(t)r(t)(r(t) - 2)}{q(t) + \bar{q}(t)(r(t) - 1)^2}, \quad \zeta(t) := \frac{(b_2(t) + \gamma(t))\bar{q}(t)r(t)(r(t) - 2)}{q(t) + \bar{q}(t)(r(t) - 1)^2}.$$ 

The fact that $q \geq \lambda_1 > 0, \bar{q} \geq 0$, and the boundedness of coefficients imply that $\hat{\alpha}$ is well-defined and satisfies (H.2(1)). By further assuming that the functions $b_2, \gamma, q, \bar{q}, r, c$ are 1/2-Hölder continuous on $[0, T]$, we can show that $\hat{\alpha}$ satisfies (H.2(2)).

Observe that in the present setting, the feedback map $\hat{\alpha}$ is independent of $(x, y)$ if and only if $b_2 \equiv c \equiv 0$. This agrees with the general condition in Proposition 2.2 under which the optimal control of (1.2) is deterministic.

With (H.2(1)) at hand, we can express the coupled MV-FBSDE (2.2) in an equivalent form that is easier to analyze. We shall seek a tuple of processes $(X_t^\alpha, Y_t^\alpha, Z_t^\alpha, \hat{\alpha}) \in \mathcal{S}^2(\mathbb{R}^n) \times \mathcal{S}^2(\mathbb{R}^n) \times \mathcal{H}^2(\mathbb{R}^{n \times d}) \times \mathbb{A}$ satisfying for all $t \in [0, T]$ that $\alpha_t = \hat{\alpha}(t, X_t^\alpha, Y_t^\alpha, \mathbb{P}_{(X_t^\alpha, Y_t^\alpha)})$ and

$$\begin{align*}
\mathrm{d}X_t^\alpha &= b(t, X_t^\alpha, \hat{\alpha}_t, \mathbb{P}_{(X_t^\alpha, \hat{\alpha}_t)}) \, \mathrm{d}t + \sigma(t, X_t^\alpha, \mathbb{P}_{X_t^\alpha}) \, \mathrm{d}W_t, \\
\mathrm{d}Y_t^\alpha &= -\left(\partial_x H(t, X_t^\alpha, \hat{\alpha}_t, \mathbb{P}_{X_t^\alpha, \hat{\alpha}_t}), Y_t^\alpha, Z_t^\alpha \right) \\
&\quad + \mathbb{E}[\partial_t H(t, X_t^\alpha, \hat{\alpha}_t, \mathbb{P}_{X_t^\alpha, \hat{\alpha}_t}), \hat{\alpha}_t)] \, \mathrm{d}t + Z_t^\alpha \, \mathrm{d}W_t, \\
X_0^\alpha &= \xi_0, \quad Y_0^\alpha = \partial_x g(X_T^\alpha, \mathbb{P}_{X_T^\alpha}) + \mathbb{E}[\partial_t g(X_T^\alpha, \mathbb{P}_{X_T^\alpha})(X_T^\alpha)],
\end{align*}$$

(2.16)

where $(\tilde{X}^\alpha, \tilde{Y}^\alpha, \tilde{Z}^\alpha, \tilde{\alpha})$ is an independent copy of $(X_t^\alpha, Y_t^\alpha, Z_t^\alpha, \hat{\alpha})$ defined on a space $L^2(\tilde{\Omega}, \tilde{\mathbb{F}}, \tilde{\mathbb{P}})$. Note that (2.16) can be equivalently formulated as follows: for all $t \in [0, T]$,

$$\begin{align*}
\mathrm{d}X_t &= \tilde{b}(t, X_t, Y_t, \mathbb{P}_{(X_t, Y_t)}) \, \mathrm{d}t + \sigma(t, X_t, \mathbb{P}_{X_t}) \, \mathrm{d}W_t, \quad X_0 = \xi_0, \\
\mathrm{d}Y_t &= -\tilde{f}(t, X_t, Y_t, Z_t, \mathbb{P}_{(X_t, Y_t, Z_t)}) \, \mathrm{d}t + Z_t \, \mathrm{d}W_t, \quad Y_T = \tilde{g}(X_T, \mathbb{P}_{X_T})
\end{align*}$$

(2.17a)

(2.17b)

with coefficients defined as follows: for all $(t, x, y, z, a, \mu, \chi, \rho) \in [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{n \times d} \times \mathbb{A} \times \mathcal{P}_2(\mathbb{R}^n) \times \mathcal{P}_2(\mathbb{R}^n \times \mathbb{R}^n) \times \mathcal{P}_2(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{n \times d})$,

$$\begin{align*}
\hat{b}(t, x, y, \chi) &= b(t, x, \hat{\alpha}(t, x, y, \chi), \phi(\chi)), \\
\hat{f}(t, x, y, z, \rho) &= \partial_x H(t, x, \hat{\alpha}(t, x, y, \pi_{1, 2}\rho), \phi(t, \pi_{1, 2}\rho), y, z) \\
&\quad + \int_E \partial_x H(t, \tilde{x}, \hat{\alpha}(t, \tilde{x}, \tilde{y}, \pi_{1, 2}\rho), \phi(t, \pi_{1, 2}\rho), \tilde{y}, \tilde{z})(x, \hat{\alpha}(t, x, y, \pi_{1, 2}\rho)) \, \mathrm{d}\rho(\tilde{x}, \tilde{y}, \tilde{z}), \\
\hat{g}(x, \mu) &= \partial_x g(x, \mu) + \int_{\mathbb{R}^n} \partial_x g(x, \mu) \, \mathrm{d}\mu(\tilde{x}),
\end{align*}$$

(2.18)
where $E := \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{n \times d}$, $\phi(t, \chi)$ is defined as in (H.2(1)) and $\pi_{1,2} \rho := \rho(\cdot \times \mathbb{R}^{n \times d})$ is the marginal of the measure $\rho$ on $\mathbb{R}^n \times \mathbb{R}^n$. In the subsequent analysis, we shall show that (2.17) (or equivalently (2.16)) admits a unique solution and then construct an optimal control for (1.2) by using the function $\hat{\alpha}$ in (H.2(1)); see Theorem 3.6 for details.

3 Regularity of mean field controls

In this section, we study the regularity of solutions to the MV-FBSDE (2.17). In particular, we shall establish that (2.17) admits a unique 1/2-Hölder continuous solution in $\mathcal{S}^2(\mathbb{R}^n) \times \mathcal{S}^2(\mathbb{R}^n) \times \mathcal{H}^2(\mathbb{R}^{n \times d})$, which subsequently enables us to show that the MFC problem admits a unique 1/2-Hölder continuous optimal control in $\mathcal{A}$.

We start by showing that the coefficients $(\hat{b}, \sigma, \hat{f}, \hat{g})$ of the MV-FBSDE (2.17) are Lipschitz continuous with respect to the spatial variables (uniformly in the time variable), and satisfy a general monotonicity condition. The detailed steps for the proofs of the following propositions can be found in Appendix A.

**Proposition 3.1.** Suppose (H.1) and (H.2(1)) hold, and let the functions $(\hat{b}, \hat{f}, \hat{g})$ be defined as in (2.18). Then there exists a constant $C > 0$ satisfying for all $t \in [0, T]$ that the functions $(\hat{b}(t, \cdot), \sigma(t, \cdot), \hat{f}(t, \cdot), \hat{g}(t, \cdot))$ are $C$-Lipschitz continuous in all variables and satisfy the estimate

$$
\|\hat{b}(\cdot, 0, 0, \delta_{0,n+})\|_{L^2(0, T)} + \|\sigma(\cdot, 0, \delta_{0,n})\|_{L^\infty(0, T)} + \|\hat{f}(\cdot, 0, 0, \delta_{0,n+}, \delta_{n+nd})\|_{L^\infty(0, T)} \leq C.
$$

**Proposition 3.2.** Suppose (H.1) and (H.2(1)) hold, and let the functions $(\hat{b}, \sigma, \hat{f}, \hat{g})$ be defined as in (2.18). Then the functions $(\hat{b}(t, \cdot), \sigma(t, \cdot), \hat{f}(t, \cdot), \hat{g}(t, \cdot))$ satisfy for all $t \in [0, T], i \in \{1, 2\}, \Theta_i := (X_i, Y_i, Z_i) \in L^2(\Omega; \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{m \times d})$ that $\Theta := (X_i, Y_i, Z_i) \in L^2(\Omega; \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{m \times d})$ that

$$
\mathbb{E}[\langle \hat{g}(X_1, \mathbb{P}_{X_1}) - \hat{g}(X_2, \mathbb{P}_{X_2}), X_1 - X_2 \rangle] \geq 0
$$

and

$$
\begin{aligned}
\mathbb{E}[\hat{b}(t, X_1, Y_1, \mathbb{P}_{X_1, Y_1}) - \hat{b}(t, X_2, Y_2, \mathbb{P}_{X_2, Y_2}), Y_1 - Y_2] \\
+ \mathbb{E}[\sigma(t, X_1, \mathbb{P}_{X_1}) - \sigma(t, X_2, \mathbb{P}_{X_2}), Z_1 - Z_2] \\
+ \mathbb{E}[\hat{f}(t, \Theta_1, \mathbb{P}_{\Theta_1}) + \hat{f}(t, \Theta_2, \mathbb{P}_{\Theta_2}), X_1 - X_2] \\
\leq -2(\lambda_1 + \lambda_2)\|\hat{\alpha}(t, X_1, Y_1, \mathbb{P}_{X_1, Y_1}) - \hat{\alpha}(t, X_2, Y_2, \mathbb{P}_{X_2, Y_2})\|_{L^2}^2,
\end{aligned}
$$

with the constants $\lambda_1, \lambda_2$ in (H.1(4)).

We then adapt the method of continuation in [4, 7] to the present setting, and establish the well-posedness and stability of (2.17). To do so, we first present a stability result for the following family of MV-FBSDEs: for $t \in [0, T],$

$$
\begin{aligned}
dx_t &= (\lambda \hat{b}(t, X_t, Y_t, \mathbb{P}_{X_t, Y_t}) + \hat{I}_t^b) \, dt + (\lambda \sigma(t, X_t, \mathbb{P}_{X_t}) + \hat{I}_t^\sigma) \, dW_t, \\
dY_t &= - (\lambda \hat{f}(t, X_t, Y_t, Z_t, \mathbb{P}_{X_t, Y_t, Z_t}) + \hat{I}_t^f) \, dt + Z_t \, dW_t, \\
X_0 &= \xi, \quad Y_T = \lambda \hat{g}(X_T, \mathbb{P}_{X_T}) + \hat{I}_T^g,
\end{aligned}
$$

where $\lambda \in [0, 1], \xi \in L^2(\mathcal{F}_0; \mathbb{R}^n), (\hat{I}_t^b, \hat{I}_t^\sigma, \hat{I}_t^f) \in \mathcal{H}^2(\mathbb{R}^n \times \mathbb{R}^{n \times d} \times \mathbb{R}^n)$ and $\hat{I}_T^g \in L^2(\mathcal{F}_T; \mathbb{R}^n)$ are given. The proof is based on Propositions 3.1 and 3.2, whose detail is presented in Appendix A.

**Lemma 3.3.** Suppose (H.1) and (H.2(1)) hold, and let the functions $(\hat{b}, \sigma, \hat{f}, \hat{g})$ be defined as in (2.18). Then, there exists a constant $C > 0$ such that, for all $\lambda_0 \in [0, 1], \forall \Theta := (X, Y, Z) \in \mathcal{S}^2(\mathbb{R}^n) \times \mathcal{S}^2(\mathbb{R}^n) \times \mathcal{H}^2(\mathbb{R}^{n \times d})$ satisfying (3.2) with $\lambda = \lambda_0$, functions $(\hat{b}, \sigma, \hat{f}, \hat{g})$ and some $(\hat{I}_t^b, \hat{I}_t^\sigma, \hat{I}_t^f) \in \mathcal{H}^2(\mathbb{R}^n \times \mathbb{R}^{n \times d} \times \mathbb{R}^n), \hat{I}_T^b \in L^2(\mathcal{F}_T; \mathbb{R}^n), \xi \in L^2(\mathcal{F}_0; \mathbb{R}^n), \forall \Theta := (X, Y, Z) \in \mathcal{S}^2(\mathbb{R}^n) \times \mathcal{S}^2(\mathbb{R}^n) \times \mathcal{H}^2(\mathbb{R}^{n \times d})$ satisfying (3.2) with $\lambda = \lambda_0$, another 4-tuple

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of Lipschitz functions \( \tilde{b}, \sigma, \tilde{f}, \tilde{g} \) and some \((\mathcal{I}^b, \mathcal{I}^\sigma, \mathcal{I}^f) \in \mathcal{H}^2(\mathbb{R}^n \times \mathbb{R}^{n \times d} \times \mathbb{R}^n), \mathcal{I}^\theta_T \in L^2(\mathcal{F}_T; \mathbb{R}^n), \xi \in L^2(\mathcal{F}_0; \mathbb{R}^n)\), we have that

\[
\|X - X\|^2_{L^2} + \|Y - Y\|^2_{L^2} + \|Z - Z\|^2_{\mathcal{H}^2}
\leq C \left\{ \|\xi - \xi\|^2_{L^2} + \|\lambda_0(\tilde{g}(X_T, \mathbb{P}_{X_T}) - \tilde{g}(X_T, \mathbb{P}_X)) + \mathcal{I}^\theta_T - \mathcal{I}^\theta_{T^2} \|^2_{\mathcal{H}^2} + \|\lambda_0(\tilde{b}(\cdot, X, Y, \mathbb{P}(X, Y)) - \tilde{b}(\cdot, X, Y, \mathbb{P}(X, Y))) + \mathcal{I}^b - \mathcal{I}^b_2 \|^2_{\mathcal{H}^2} + \|\lambda_0(\sigma(t, X, \mathbb{P}_X) - \sigma(t, X, \mathbb{P}_X)) + \mathcal{I}^\sigma - \mathcal{I}^\sigma_{T} \|^2_{\mathcal{H}^2} + \|\lambda_0(\tilde{f}(\cdot, \tilde{\Theta}, \mathbb{P}_{\tilde{\Theta}}) - \tilde{f}(\cdot, \tilde{\Theta}, \mathbb{P}_{\tilde{\Theta}})) + \mathcal{I}^f - \mathcal{I}^f_{T} \|^2_{\mathcal{H}^2} \right\},
\]

(3.3)

Now we are ready to establish the well-posedness and stability of (2.17).

**Theorem 3.4.** Suppose \((H.1)\) and \((H.2(1))\) hold. Then, for all \(t \in [0, T]\) and \(\xi \in L^2(\mathcal{F}_t; \mathbb{R}^n)\), there exists a unique triple \((X^{t, \xi}, Y^{t, \xi}, Z^{t, \xi}) \in \mathcal{S}^2(t, T; \mathbb{R}^n) \times \mathcal{S}^2(t, T; \mathbb{R}^n) \times \mathcal{H}^2(t, T; \mathbb{R}^{n \times d})\) satisfying (2.17) on \([t, T]\) with the initial condition \(X^{t, \xi} = \xi\). Moreover, there exists a constant \(C > 0\) such that it holds for all \(t \in [0, T]\) and \(\xi, \xi' \in L^2(\mathcal{F}_t; \mathbb{R}^n)\) that \(\|X^{t, \xi} - X^{t, \xi'}\|_{L^2} \leq C\|\xi - \xi\|_{L^2}\), and \(\|X^{t, \xi}\|_{L^2(t, T; \mathbb{R}^n)} + \|Y^{t, \xi}\|_{L^2(t, T; \mathbb{R}^n)} + \|Z^{t, \xi}\|_{\mathcal{H}^2(t, T; \mathbb{R}^{n \times d})} \leq C(1 + \|\xi\|_{L^2})\).

**Proof.** We shall establish the well-posedness, stability and a priori estimates for (2.17) with an initial time \(t = 0\) and initial state \(\xi_0 \in L^2(\mathcal{F}_0; \mathbb{R}^n)\) by applying Lemma 3.3. Similar arguments apply to a general initial time \(t \in [0, T]\) and initial state \(\xi \in L^2(\mathcal{F}_t; \mathbb{R}^n)\).

Let us start by proving the unique solvability of (2.17) with a given \(\xi_0 \in L^2(\mathcal{F}_0; \mathbb{R}^n)\). To simplify the notation, for every \(\lambda_0 \in [0, 1]\), we say \((\mathcal{P}_{\lambda_0})\) holds if for any \(\xi \in L^2(\mathcal{F}_0; \mathbb{R}^n), (\mathcal{I}^b, \mathcal{I}^\sigma, \mathcal{I}^f) \in \mathcal{H}^2(\mathbb{R}^n \times \mathbb{R}^{n \times d} \times \mathbb{R}^n)\) and \(\mathcal{I}^\theta_T \in L^2(\mathcal{F}_T; \mathbb{R}^n)\), (3.2) with \(\lambda = \lambda_0\) admits a unique solution in \(\mathbb{B} := \mathcal{S}^2(\mathbb{R}^n) \times \mathcal{S}^2(\mathbb{R}^n) \times \mathcal{H}^2(\mathbb{R}^{n \times d})\). It is clear that \((\mathcal{P}_0)\) holds since (3.2) is decoupled. Now we show there exists a constant \(\delta > 0\), such that if \((\mathcal{P}_{\lambda_0})\) holds for some \(\lambda_0 \in [0, 1]\), then \((\mathcal{P}_{\lambda_0})\) also holds for all \(\lambda_0' \in (\lambda_0, \lambda_0 + \delta] \cap [0, 1]\). Note that this claim along with the method of continuation implies the desired unique solvability of (2.17) (i.e., (3.2) with \(\lambda = 1\), \((\mathcal{I}^b, \mathcal{I}^\sigma, \mathcal{I}^f, \mathcal{I}^\theta_T) = 0\), \(\xi = \xi_0\)).

To establish the desired claim, let \(\lambda_0 \in [0, 1]\) be a constant for which \((\mathcal{P}_{\lambda_0})\) holds, \(\eta \in [0, 1]\) and \((\mathcal{I}^\tilde{b}, \mathcal{I}^\tilde{\sigma}, \mathcal{I}^\tilde{f}) \in \mathcal{H}^2(\mathbb{R}^n \times \mathbb{R}^{n \times d} \times \mathbb{R}^n), \mathcal{I}^\tilde{\theta}_T \in L^2(\mathcal{F}_T; \mathbb{R}^n)\), \(\xi \in L^2(\mathcal{F}_0; \mathbb{R}^n)\) be arbitrarily given functions. Then, we introduce the following mapping \(\Xi : \mathbb{B} \rightarrow \mathbb{B}\) such that for all \(\Theta = (X, Y, Z) \in \mathbb{B}, \Xi(\Theta) \in \mathbb{B}\) is the solution to (3.2) with \(\lambda = \lambda_0, \mathcal{I}^\theta_T = \eta \mathcal{I}^\theta_T, \mathcal{I}^{\tilde{b}} = \mathcal{I}^{\tilde{b}}, \mathcal{I}^{\tilde{\sigma}} = \mathcal{I}^{\tilde{\sigma}} + \mathcal{I}^{\tilde{\sigma}}\), \mathcal{I}^{\tilde{f}} \mathcal{I}^{\tilde{f}} = \mathcal{I}^{\tilde{f}} + \mathcal{I}^{\tilde{f}}\) and \(\mathcal{I}^{\tilde{\theta}_T} = \eta \mathcal{I}^{\tilde{\theta}_T}\), which is well-defined due to the fact that \(\lambda_0 \in [0, 1]\) satisfies the induction hypothesis. Observe that by setting \((\tilde{b}, \tilde{\sigma}, \tilde{f}, \tilde{g}) = (\tilde{b}, \tilde{\sigma}, \tilde{f}, \tilde{g})\) in Lemma 3.3, we see that there exists a constant \(C > 0\), independent of \(\lambda_0\), such that it holds for all \(\Theta, \Theta' \in \mathbb{B}\) that

\[
\|\Xi(\Theta) - \Xi(\Theta')\|^2_{\mathbb{B}} \leq C \left\{ \|\eta(\tilde{g}(X_T, \mathbb{P}_{X_T}) - \tilde{g}(X_T, \mathbb{P}_X))\|^2_{L^2} + \|\eta\tilde{b}(\cdot, X, Y, \mathbb{P}(X,Y)) - \tilde{b}(\cdot, X', Y', \mathbb{P}(X',Y'))\|^2_{\mathcal{H}^2} \right\},
\]

which shows that \(\Xi\) is a contraction when \(\eta\) is sufficiently small (independent of \(\lambda_0\), and subsequently leads to the desired claim due to Banach’s fixed point theorem.
For any given $\xi, \xi' \in L^2(\mathcal{F}_0; \mathbb{R}^n)$, the desired stochastic stability of (2.17) follows directly from Lemma 3.3 by setting $\lambda = 1$, $(\bar{b}, \bar{\sigma}, \bar{f}, \bar{g}) = (\bar{b}, \sigma, \tilde{f}, \tilde{g})$, $(\bar{I}^b, \bar{I}^\sigma, \bar{I}^f) = (\bar{I}^b, \bar{I}^\sigma, \bar{I}^f) = 0$, $\bar{T}_T = T_T = \bar{t}$ and $\bar{\xi} = \xi'$. Moreover, for any given $\xi \in L^2(\mathcal{F}_0; \mathbb{R}^n)$, by setting $\lambda = 1$, $(\bar{b}, \bar{\sigma}, \bar{f}, \bar{g}) = 0$, $(\bar{I}^b, \bar{I}^\sigma, \bar{I}^f) = (\bar{I}^b, \bar{I}^\sigma, \bar{I}^f) = 0$, $\bar{\xi} = 0$ and $(\bar{X}, \bar{Y}, \bar{Z}) = 0$ in Lemma 3.3, we can deduce the estimate that
\[
\|X\|_{L^2}^2 + \|Y\|_{L^2}^2 + \|Z\|_{L^2}^2 \\
\leq C \left\{ \|\xi\|_{L^2}^2 + |\bar{g}(0, \delta_{0n})|^2 + \|\bar{b}(\cdot, 0, \delta_{0n+n})\|_{L^2(0,T)}^2 + \|\sigma(\cdot, 0, \delta_{0n})\|_{L^2(0,T)}^2 \right\} \\
+ \|\bar{f}(\cdot, 0, \delta_{0n+n+n})\|_{L^2(0,T)}^2 \leq C(1 + \|\xi\|_{L^2}^2),
\]
which shows the desired moment bound of the processes $(X, Y, Z)$. 

We now give our result concerning the Hölder regularity of the solutions to (2.17).

**Theorem 3.5.** Suppose (H.1) and (H.2(1)) hold, and let $(X, Y, Z) \in \mathcal{S}^2(\mathbb{R}^n) \times \mathcal{S}^2(\mathbb{R}^n) \times \mathcal{H}^2(\mathbb{R}^{n \times d})$ be the unique solution to (2.17) with initial condition $X_0 = \xi_0 \in L^2(\mathcal{F}_0; \mathbb{R}^n)$. Moreover, for all $p \geq 2$, there exists a constant $C > 0$, depending only on $p$ and the data in (H.1) and (H.2(1)), such that $\|X\|_{L^p} + \|Y\|_{L^p} + \|Z\|_{L^p} \leq C(1 + \|\xi_0\|_{L^p})$ and $\mathbb{E} \left[ \sup_{s \leq t \leq T} |X_t - X_s|^p \right]^{1/p} \leq C(1 + \|\xi_0\|_{L^2})|t-s|^{1/2}$ for all $0 \leq s \leq t \leq T$.

**Proof.** Let $\xi_0 \in L^2(\mathcal{F}_0; \mathbb{R}^n)$ be a given initial condition and $(X, Y, Z) \in \mathcal{S}^2(\mathbb{R}^n) \times \mathcal{S}^2(\mathbb{R}^n) \times \mathcal{H}^2(\mathbb{R}^{n \times d})$ be the solution to (2.17). By using the pathwise uniqueness and the Lipschitz stability of (2.17) in Theorem 3.4, we can follow the arguments in [7, Proposition 5.7] and deduce that there exists a measurable function $v : [0, T] \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ (depending on $\xi_0$) and a constant $C > 0$ (independent of $\xi_0$) such that $\mathbb{P}(\forall t \in [0, T], Y_t = v(t, X_t)) = 1$ and it holds for all $t \in [0, T]$ and $x, x' \in \mathbb{R}^n$ that $|v(t, x) - v(t, x')| \leq C|x - x'|$ and $|v(t, 0)| \leq C(1 + \|\xi_0\|_{L^2})$.

By substituting the relation $Y_t = v(t, X_t)$ into (2.17a), we can rewrite (2.17) into the following decoupled FBSDE:
\[
\begin{align*}
\frac{dX_t}{dt} &= \bar{b}(t, X_t) dt + \bar{\sigma}(t, X_t) dW_t, \quad X_0 = \xi_0, \quad (3.4a) \\
\frac{dY_t}{dt} &= -\tilde{f}(t, X_t, Y_t, Z_t) dt + Z_t dW_t, \quad Y_T = \tilde{g}(X_T). \quad (3.4b)
\end{align*}
\]
with coefficients $\bar{b}, \bar{\sigma}, \bar{f}$ and $\tilde{g}$ defined as follows:
\[
\begin{align*}
\bar{b}(t, x) &:= \bar{b}(t, x, v(t, x), \mathbb{P}(X_t, Y_t)), \\
\bar{\sigma}(t, x) &:= \sigma(t, x, \mathbb{P}X_t), \\
\tilde{f}(t, x, y, z) &:= \tilde{f}(t, x, y, z, \mathbb{P}(X_t, Y_t, Z_t)), \\
\tilde{g}(x) &:= \tilde{g}(x, \mathbb{P}X_T).
\end{align*}
\]
By Proposition 3.1 and Theorem 3.4, these coefficients are $C$-Lipschitz continuous in the state variable with a constant $C$ independent of $\xi_0$, and satisfy the estimates $\int_0^T |\bar{b}(t, 0)|^2 dt < \infty$, $\sup_{t \in [0,T]} |\bar{\sigma}(t, 0)| < \infty$ and $\int_0^T |\tilde{f}(t, 0, 0, 0)|^2 dt < \infty$. Hence, by applying [23, Theorem 5.2.2 (i)] to (3.4), we see there exists a constant $C > 0$ such that $|Z_t| \leq C|\sigma(t, X_t, \mathbb{P}X_t)|$ for all $t \in [0, T]$. We remark that in [23] the initial state $\xi_0$ is assumed to be deterministic and the coefficients of the FBSDE are assumed to be Hölder continuous in time. However, the proof relies on expressing the process $Z$ in terms of the Malliavin derivatives of $X$ and $Y$, and hence can be extended to the present setting where $\xi_0$ is $\mathcal{F}_0$-measurable and the coefficients are measurable in time and satisfy the above estimates.
By the Lipschitz continuity of $v$, the estimate $\sup_{t\in[0,T]}|v(t, 0)| \leq C(1 + \|\xi_0\|_{L^2})$ and standard moment estimates of (3.4a), $\|X\|_{S^p} \leq C(1 + \|\xi_0\|_{L^p})$, which along with the relations $Y_t = v(t, X_t)$ and $|Z_t| \leq C|\sigma(t, X_t, \mathbb{P}_{X_t})|$ leads to $\|Y\|_{S^p} + \|Z\|_{S^p} \leq C(1 + \|\xi_0\|_{L^p})$. Moreover, by (2.17), Hölder's inequality and the Burkholder-Davis-Gundy inequality, the process $X$ satisfies for each $p \geq 2$, $t, s \in [0, T]$,

$$
\mathbb{E} \left[ \sup_{s \leq r \leq t} |X_r - X_s|^p \right] 
\leq \mathbb{E} \left[ \left( \int_s^t |\hat{b}(r, X_r, Y_r, \mathbb{P}_{(X_r, Y_r)})| \, dr \right)^p \right] + \mathbb{E} \left[ \sup_{s \leq r \leq t} \left| \int_s^r \sigma(u, X_u, \mathbb{P}_{X_u}) \, dW_u \right|^p \right] 
\leq C(p) \left\{ \left( \|\hat{b}(\cdot, 0, 0, \delta_{0.1, n})\|_{L^2(0, T)}^p + \|(X, Y)\|_{S^p}\right)t - s \right\}^{\frac{p}{2}} + \mathbb{E} \left[ \left( \int_s^t |\sigma(r, X_r, \mathbb{P}_{X_r})|^2 \, dr \right)^\frac{p}{2} \right],
$$

and the process $Y$ satisfies for each $p \geq 2$, $t, s \in [0, T]$,

$$
\mathbb{E} \left[ \sup_{s \leq r \leq t} |Y_r - Y_s|^p \right] 
\leq \mathbb{E} \left[ \left( \int_s^t |\hat{f}(r, X_r, Y_r, Z_r, \mathbb{P}_{(X_r, Y_r, Z_r)})| \, dr \right)^p \right] + \mathbb{E} \left[ \sup_{s \leq r \leq t} \left| \int_s^r Z_u \, dW_u \right|^p \right] 
\leq C(p) \left\{ \left( \|\hat{f}(\cdot, 0, 0, 0, \delta_{0.1, n+n})\|_{L^2(0, T)}^p + \|(X, Y, z)\|_{S^p}\right)t - s \right\}^{\frac{p}{2}} + \mathbb{E} \left[ \left( \int_s^t |Z_r|^2 \, dr \right)^\frac{p}{2} \right],
$$

which together with Proposition 3.1, the inequality that $\mathbb{E} \left[ (\int_s^t |Z|^2 \, dr)^\frac{p}{2} \right] \leq \|Z\|_{S^p}(t-s)\frac{p}{2}$ and the estimate of $\|(X, Y, Z)\|_{S^p}$ leads to the desired Hölder continuity of the processes $X$ and $Y$. 

The following theorem establishes the $1/2$-Hölder regularity of optimal controls to (1.2) based on the regularity results in Theorem 3.5.

**Theorem 3.6.** Suppose (H.1) and (H.2(1)) hold, and let $\xi_0 \in L^2(\mathcal{F}_0; \mathbb{R}^n)$. Then (1.2) admits a unique optimal control $\hat{\alpha} = (\hat{\alpha}_t)_{t \in [0, T]} \in \mathcal{A}$, which satisfies for all $p \geq 2$ that $\|\hat{\alpha}\|_{S^p} \leq C(1 + \|\xi_0\|_{L^p})$. If we further assume that (H.2(2)) holds, then the optimal control $\hat{\alpha}$ satisfies for all $p \geq 2, 0 \leq s \leq t \leq T$ that $\mathbb{E} \left[ \sup_{s \leq r \leq t} |\hat{\alpha}_r - \hat{\alpha}_s|^p \right]^{1/p} \leq C(1 + \|\xi_0\|_{L^p})|t - s|^{1/2}$, with a constant $C$ depending only on $p$ and the data in (H.1) and (H.2).

**Proof.** Let $\hat{\alpha} : [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \times \mathcal{P}_2(\mathbb{R}^n \times \mathbb{R}^n) \to \mathcal{A}$ be the function in (H.2(1)). We define for each $t \in [0, T]$ that $\hat{\alpha}_t = \hat{\alpha}(t, X_t, Y_t, \mathbb{P}_{(X_t, Y_t)})$, and write $\hat{\alpha} = (\hat{\alpha}_t)_{t \in [0, T]}$ with a slight abuse of notation. The local boundedness and Lipschitz continuity of the function $\hat{\alpha}$ (see (H.2(1))) show that $\|\hat{\alpha}\|_{S^p} \leq C(1 + \|\xi_0\|_{L^p})$ for all $p \geq 2$. Then, the assumption that $\xi_0 \in L^2(\mathcal{F}_0; \mathbb{R}^n)$ and the definition of the function $\hat{\alpha}$ in (H.2(1)) imply that the control $\hat{\alpha}$ is admissible (i.e., $\hat{\alpha} \in \mathcal{A}$) and satisfies (2.4) (equivalently (2.3)), which shows that $\hat{\alpha}$ is an optimal control of (1.2). The uniqueness of optimal controls of (1.2) follows from the strong convexity of the cost functional $J : \mathcal{A} \to \mathbb{R}$, which will be shown in Lemma 5.1.

Finally, for any given $0 \leq s \leq r \leq t \leq T$, we obtain from (H.2) that

$$
|\hat{\alpha}_r - \hat{\alpha}_s| = |\hat{\alpha}(r, X_r, Y_r, \mathbb{P}_{(X_r, Y_r)}) - \hat{\alpha}(s, X_s, Y_s, \mathbb{P}_{(X_s, Y_s)})| 
\leq C \left( (1 + |X_r| + |Y_r| + \|\mathbb{P}_{(X_r, Y_r)}\|_2)|r - s|^{1/2} + |X_r - X_s| + |Y_r - Y_s| + \mathcal{W}_2(\mathbb{P}_{(X_r, Y_r)}, \mathbb{P}_{(X_s, Y_s)}) \right),
$$

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which together with the moment estimates and regularity of the processes \(X,Y\) leads to

\[
E \left[ \sup_{s \leq r \leq t} |\hat{\alpha}_r - \hat{\alpha}_s|^p \right]^{\frac{1}{p}} \\
\leq C \left( (1 + \|X\|_{S^p} + \|Y\|_{S^p})|t - s|^{\frac{1}{2}} + E \left[ \sup_{s \leq r \leq t} |X_r - X_s|^p \right]^{\frac{1}{p}} + E \left[ \sup_{s \leq r \leq t} |Y_r - Y_s|^p \right]^{\frac{1}{p}} \right)
\]

\[
\leq C (1 + \|\xi_0\|_{L^p})|t - s|^{\frac{1}{2}}.
\]

This completes the proof of Theorem 3.6.

\[\square\]

**Remark 3.1.** Note that the 1/2-Hölder regularity of the optimal open-loop control \(\alpha\) in the \(S^p\)-norm and the dependence on the integrability of the initial condition \(\xi_0\) in the estimate are optimal, since it agrees with the path regularity of Brownian motions.

### 4 Error estimates of value functions for piecewise constant policy approximations

In this section, based on the regularity results of optimal controls in Theorem 3.6, we establish the convergence rate of the discrete-time control problem (1.3) in approximating the value function of (1.2).

We start with the error introduced by approximating the set \(A\) of admissible controls in (1.2) by piecewise constant controls. More precisely, let \(\pi = \{0 = t_0 < \cdots < t_N = T\}\) be a partition of \([0,T]\) with stepsize \(|\pi| = \max_{i=0,...,N-1}(t_{i+1} - t_i)\) and let \(A_{\pi}\) be the subset of piecewise constant controls defined as in (1.4). For any given initial state \(\xi_0 \in L^2(\mathcal{F}_0; \mathbb{R}^n)\), we consider the following minimization problem

\[
V_{\pi}^c(\xi_0) := \inf_{\alpha \in A_{\pi}} J(\alpha; \xi_0),
\]

where for each \(\alpha \in A_{\pi}\), \(J(\alpha; \xi_0)\) is the cost functional defined as in (1.2) with the controlled state process \(X^\alpha\) satisfying the MV-SDE (1.1).

The following theorem shows that as the stepsize \(|\pi|\) tends to zero, the value function \(V_{\pi}^c(\xi_0)\) converges from above to the value function \(V(\xi_0)\) in (1.2) with half-order accuracy.

**Theorem 4.1.** Suppose (H.1) and (H.2) hold, let the function \(V : L^2(\mathcal{F}_0; \mathbb{R}^n) \to \mathbb{R}\) be defined as in (1.2), and for each partition \(\pi\) of \([0,T]\) let the function \(V_{\pi}^c : L^2(\mathcal{F}_0; \mathbb{R}^n) \to \mathbb{R}\) be defined as in (4.1). Then there exists a constant \(C > 0\), such that it holds for all \(\xi_0 \in L^2(\mathcal{F}_0; \mathbb{R}^n)\) and for every partition \(\pi\) of \([0,T]\) with stepsize \(|\pi|\) that

\[
V(\xi_0) \leq V_{\pi}^c(\xi_0) \leq V(\xi_0) + C (1 + \|\xi_0\|_{L^2}) |\pi|^{1/2}.
\]

**Proof.** Throughout this proof, let \(\xi_0 \in L^2(\mathcal{F}_0; \mathbb{R}^n)\) be a given initial state, let \(\pi = \{0 = t_0 < \cdots < t_N = T\}\) be a given partition of \([0,T]\) with stepsize \(|\pi| = \max_{i=0,...,N-1}(t_{i+1} - t_i)\), let \(A_{\pi} \subset A\) be the associated piecewise constant controls and \(C\) be a generic constant, which is independent of the initial state \(\xi_0\) and the partition \(\pi\), and may take a different value at each occurrence.

It is clear from \(A_{\pi} \subset A\) and the definitions of \(V\) and \(V_{\pi}^c\) that \(V_{\pi}^c(\xi_0) = \inf_{\alpha \in A_{\pi}} J(\alpha; \xi_0) \geq \inf_{\alpha \in A} J(\alpha; \xi_0) = V(\xi_0)\). We now establish an upper bound of \(V_{\pi}^c(\xi_0) - V(\xi_0)\). Note that Theorem 3.6 shows that under (H.1) and (H.2), there exists an admissible control \(\hat{\alpha} \in A\) such that \(V(\xi_0) = J(\hat{\alpha}; \xi)\) and it holds for all \(0 \leq s \leq t \leq T\) that \(E \left[ \sup_{s \leq r \leq t} |\hat{\alpha}_r - \hat{\alpha}_s|^2 \right]^{1/2} \leq C (1 + \|\xi_0\|_{L^2}) |t - s|^{1/2}\). Let \(\hat{\alpha}_{\pi}\) be a piecewise constant approximation of the process \(\hat{\alpha}\) on \(\pi\) satisfying for all \(t \in [0,T)\)
that \( \hat{\alpha}_t^\pi = \sum_{i=0}^{N-1} \hat{\alpha}_t \mathbf{1}_{[t_i, t_{i+1})}(t) \). Then it is clear that \( \hat{\alpha}^\pi \in \mathcal{A}_\pi \) and it holds for all \( t \in [0, T] \) that \( t \in [t_i, t_{i+1}) \) for some \( i \in \{0, \ldots, N - 1 \} \) and

\[
\|\hat{\alpha}_t^\pi - \hat{\alpha}_t\|_{L^2} = \|\hat{\alpha}_t\|_{L^2} \leq C(1 + \|\xi_0\|_{L^2})|\pi|^{1/2}.
\]

Hence we can obtain from standard stability estimates of MV-SDEs that \( \|X^\alpha - X^\hat{\alpha}^\pi\|_{H^2} \leq C\|\hat{\alpha}^\pi - \alpha\|_{H^2} \leq C(1 + \|\xi_0\|_{L^2})|\pi| \), which together with Remark 2.1 and \( V(\xi_0) = J(\hat{\alpha}; \xi) \), gives us the estimate that

\[
V_\pi^\xi(\xi_0) - V(\xi_0) \leq J(\hat{\alpha}^\pi; \xi_0) - J(\hat{\alpha}; \xi_0)
\]

\[
\leq \mathbb{E} \left[ \int_0^T |f(t, t^\pi_t, \hat{\alpha}_t^\pi, \mathbb{P}(X^t_t, \hat{\alpha}_t^\pi)) - f(t, t^\pi_t, \hat{\alpha}_t, \mathbb{P}(X^t_t, \hat{\alpha}_t))| \, dt
\]

\[
+ |g(X^\alpha_t^\pi, \mathbb{P}(X^\alpha_t^\pi)) - g(X^\alpha_t, \mathbb{P}(X^\alpha_t))| \right]
\]

\[
\leq C \left\{ \mathbb{E} \left[ \int_0^T (1 + |X^\pi_t| + \|\mathbb{P}(X^\pi_t, \hat{\alpha}_t^\pi)\|_2 + |\hat{\alpha}_t^\pi| + |X^\alpha_t^\pi| + \|\mathbb{P}(X^\alpha_t^\pi, \hat{\alpha}_t^\pi)\|_2 + |\hat{\alpha}_t^\pi|) \right.
\]

\[
\times (|X^\pi_t - X^\alpha_t^\pi| + |\hat{\alpha}_t - \hat{\alpha}_t^\pi| + \mathcal{W}_2(\mathbb{P}(X^\pi_t^\alpha, \hat{\alpha}_t^\pi), \mathbb{P}(X^\alpha_t^\pi, \hat{\alpha}_t^\pi))) \, dt
\]

\[
+ (1 + |X^\alpha_t| + \|\mathbb{P}(X^\alpha_t)\|_2 + |X^\alpha_t^\pi| + \|\mathbb{P}(X^\alpha_t^\pi)\|_2)(|X^\alpha_t^\pi - X^\pi_t| + \mathcal{W}_2(\mathbb{P}(X^\alpha_t^\pi, \mathbb{P}(X^\alpha_t^\pi))) \right\}\}.
\]

Then, we can deduce from the above estimate and the Cauchy-Schwarz inequality that

\[
V_\pi^\xi(\xi_0) - V(\xi_0) \leq C \left\{ (1 + \|X^\pi_0\|_{L^2} + \|\hat{\alpha}^\pi\|_{H^2}) \left( \|X^\alpha - X^\hat{\alpha}^\pi\|_{H^2} + \|\hat{\alpha} - \hat{\alpha}^\pi\|_{H^2} \right)
\]

\[
+ (1 + \|X^\pi_t\|_{L^2} + \|X^\alpha_t^\pi\|_{L^2})(\|X^\alpha_t^\pi - X^\pi_t\|_{L^2}) \right\}
\]

\[
\leq C(1 + \|\xi_0\|_{L^2})|\pi|^{1/2},
\]

which completes the desired error estimate.

In practice, instead of solving (1.1) with a piecewise constant control, one can further discretize the controlled dynamics in time by the Euler-Maruyama scheme (cf. (1.3)), which allows us to only deal with Gaussian random variables with known mean and variance. To quantify the time discretization error of the controlled dynamics and the running cost, we assume the following time regularity of the coefficients:

**H.3.** Assume the notation of (H.1). The functions \( b_0, b_1, b_2, b_3, \sigma_0, \sigma_1, \sigma_2 \) in (H.1(1)) are 1/2-Hölder continuous, and there exists a constant \( K \in (0, \infty) \) satisfying for all \( t, t' \in [0, T] \), \( (x, a, \eta) \in \mathbb{R}^n \times \mathbf{A} \times \mathcal{P}_2(\mathbb{R}^n \times \mathbb{R}^k) \) that \( |f(t, x, a, \eta) - f(t', x, a, \eta)| \leq K(1 + \|x\|^2 + |a|^2 + \|\eta\|^2_2)|t - t'|^{1/2} \).

**Remark 4.1.** (H.1(1)) and (H.3) imply for all \( (x, a, \mu, \eta), (x', a', \mu', \eta') \in \mathbb{R}^k \times \mathbf{A} \times \mathcal{P}_2(\mathbb{R}^n) \times \mathcal{P}_2(\mathbb{R}^n \times \mathbb{R}^k) \),

\[
|b(r, x, a, \eta) - b(s, x', a', \eta')| \leq C \left( (1 + |x| + |a| + \|\eta\|_2)|r - s|^{1/2} + |x - x'| + |a - a'| + \mathcal{W}_2(\eta, \eta') \right),
\]

\[
|\sigma(r, x, \mu) - \sigma(s, x', \mu')| \leq C \left( (1 + |x| + \|\mu\|_2)|r - s|^{1/2} + |x - x'| + \mathcal{W}_2(\mu, \mu') \right).
\]

Under the Hölder regularity of the coefficients, we shall prove that the value function \( V_\pi(\xi_0) \) converges to the value function \( V(\xi_0) \) in (1.2) with order 1/2 as the stepsize \( |\pi| \) tends to zero, which is optimal for MFC problems with such irregular running costs \( f \).
Note that a similar convergence rate has been established in [9, Proposition 12] for the special case where both \( b \) and \( f \) are independent of the law of controls. By restricting the analysis to closed-loop (also called Markovian) controls (i.e., \( \alpha \in \mathcal{A} \) that are of the form \( \alpha_t = \phi(t, X_t) \) with \( \phi \in C^{1,2}_b((0, T] \times \mathbb{R}^n) \)), and assuming the decoupling field of (2.17) and the function \( \hat{\alpha} \) in (H.2(1)) to be twice differentiable with uniformly Lipschitz continuous derivatives in \( (t, x, y, \mu) \in [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \times \mathcal{P}_2(\mathbb{R}^n) \), the authors establish an order 1/2 convergence of \( (V_\pi(\xi_0))_\pi \) in terms of \( |\pi| \), with a constant depending on the sup-norms of the second-order derivatives of the feedback map \( \phi \) and coefficients. These conditions typically require the cost functions \( f \) and \( g \) in (1.2) to be three-times differentiable in \( (x, a, \mu) \) with bounded and Lipschitz continuous derivatives.

Here we remove these strong regularity assumptions and establish an order 1/2 convergence with general open-loop strategies and cost functions that are merely Hölder continuous in time and Lipschitz continuously differentiable in space; see Example 2.1 for precise regularity assumptions to ensure (H.2) in the setting of MFC problems.

**Theorem 4.2.** Suppose (H.1), (H.2) and (H.3) hold, let the function \( V : L^2(\mathcal{F}_0; \mathbb{R}^n) \to \mathbb{R} \) be defined as in (1.2), and for each partition \( \pi \) of \([0, T]\) let the function \( V_\pi : L^2(\mathcal{F}_0; \mathbb{R}^n) \to \mathbb{R} \) be defined as in (1.3). Then there exists a constant \( C > 0 \), such that it holds for all \( \xi_0 \in L^2(\mathcal{F}_0; \mathbb{R}^n) \) and for every partition \( \pi \) of \([0, T]\) with stepsize \(|\pi|\) that \( V_\pi(\xi_0) - V(\xi_0) \leq C(1 + \|\xi_0\|_{L_2}^2)|\pi|^{1/2} \).

If we further assume that \( \mathcal{A} \) is a compact subset of \( \mathbb{R}^k \), then it holds for all \( \xi_0 \in L^2(\mathcal{F}_0; \mathbb{R}^n) \) and for every partition \( \pi \) of \([0, T]\) with stepsize \(|\pi|\) that \( |V_\pi(\xi_0) - V(\xi_0)| \leq C(1 + \|\xi_0\|_{L_2}^2)|\pi|^{1/2} \).

**Proof.** Throughout this proof, let \( \xi_0 \in L^2(\mathcal{F}_0; \mathbb{R}^n) \) be a given initial state, let \( \pi = \{0 = t_0 < \cdots < t_N = T\} \) be a given partition of \([0, T]\) with stepsize \(|\pi| = \max_{i=0, \ldots, N-1} (t_{i+1} - t_i) \), let \( \mathcal{A}_\pi \subset \mathcal{A} \) be the associated piecewise constant controls and let \( C \) be a generic constant, which is independent of the initial state \( \xi_0 \), the partition \( \pi \) and controls \( \alpha \in \mathcal{A} \), and may take a different value at each occurrence.

**Step 1: Estimate an upper bound of** \( V_\pi(\xi_0) - V(\xi_0) \). As in the proof of Theorem 4.1, let \( \hat{\alpha} \in \mathcal{A} \) be an optimal control of (1.2) satisfying that \( \|\hat{\alpha}\|_{L_2}^2 \leq C(1 + \|\xi_0\|_{L_2}^2) \) and \( \|\hat{\alpha}_t - \bar{\alpha}_s\|_{L_2}^2 \leq C(1 + \|\xi_0\|_{L_2}^2)|t - s| \) for all \( s, t \in [0, T] \), let \( X^{\hat{\alpha}} \) be the solution to (1.1) with the control \( \hat{\alpha} \) satisfying \( \|X^{\hat{\alpha}}\|_{L_2}^2 \leq C(1 + \|\xi_0\|_{L_2}^2) \) and \( \|X^{\hat{\alpha}} - X_{\bar{\alpha}}\|_{L_2}^2 \leq C(1 + \|\xi_0\|_{L_2}^2)|t - s| \) for all \( s, t \in [0, T] \) (see Theorem 3.6), let \( \hat{\alpha}^\pi \) be a piecewise constant approximation of the process \( \hat{\alpha} \) on \( \pi \) satisfying for all \( t \in [0, T] \) that \( \|\hat{\alpha}_t^\pi - \hat{\alpha}_t\|_{L_2} = \|\hat{\alpha}_t - \bar{\alpha}_t\|_{L_2} \leq C(1 + \|\xi_0\|_{L_2})|\pi|^{1/2} \), and let \( \hat{X}^\pi \) be the solution to (1.6) with the control \( \hat{\alpha}^\pi \). Note that it is standard to show by using the Lipschitz continuity of \( (b, \sigma) \) and Gronwall’s inequality that

\[
\max_{t_i \in \pi} \|\hat{X}_{t_i}^\pi\|_{L_2}^2 \leq C(1 + \|\xi_0\|_{L_2}^2 + \max_{t_i \in \pi} \|\hat{\alpha}_{t_i}^\pi\|_{L_2}^2) \leq C(1 + \|\xi_0\|_{L_2}^2).
\]
Observe that it holds for each \(i \in \{0, \ldots, N - 1\}\) that

\[
\mathbb{E}[|X_{t_{i+1}}^{\alpha} - \hat{X}_{t_{i+1}}^\pi|^2] 
\leq C \left\{ \sum_{j=0}^{i} \int_{t_j}^{t_{j+1}} \left( b(t, X_t^{\alpha}, \hat{\alpha}_t, \mathbb{P}(X_t^{\alpha}, \hat{\alpha}_t)) - b(t, \hat{X}_t^\pi, \hat{\alpha}_t, \mathbb{P}(X_t^\pi, \hat{\alpha}_t)) \right) dt \right\}^2 
\]

\[
+ \mathbb{E} \left\{ \sum_{j=0}^{i} \int_{t_j}^{t_{j+1}} |\sigma(t, X_t^{\alpha}, \mathbb{P}(X_t^{\alpha})) - \sigma(t, \hat{X}_t^\pi, \mathbb{P}(X_t^\pi))|^2 dt \right\} 
\]

\[
\leq C \left\{ R_1 + \mathbb{E} \left[ \left( \sum_{j=0}^{i} \int_{t_j}^{t_{j+1}} \left| b(t, X_t^{\alpha}, \hat{\alpha}_t, \mathbb{P}(X_t^{\alpha}, \hat{\alpha}_t)) - b(t, \hat{X}_t^\pi, \hat{\alpha}_t, \mathbb{P}(X_t^\pi, \hat{\alpha}_t)) \right| dt \right) \right]^2 
\]

\[
+ \mathbb{E} \left\{ \sum_{j=0}^{i} \int_{t_j}^{t_{j+1}} |\sigma(t, X_t^{\alpha}, \mathbb{P}(X_t^{\alpha})) - \sigma(t, \hat{X}_t^\pi, \mathbb{P}(X_t^\pi))|^2 dt \right\} 
\]

with the residual term \(R_1\) defined as

\[
R_1 := \mathbb{E} \left[ \left( \sum_{i=0}^{N-1} \int_{t_i}^{t_{i+1}} \left| b(t, X_t^{\alpha}, \hat{\alpha}_t, \mathbb{P}(X_t^{\alpha}, \hat{\alpha}_t)) - b(t, X_t^{\alpha}, \hat{\alpha}_t, \mathbb{P}(X_t^{\alpha}, \hat{\alpha}_t)) \right| dt \right) \right]^2 
\]

\[
+ \mathbb{E} \left[ \sum_{i=0}^{N-1} \int_{t_i}^{t_{i+1}} |\sigma(t, X_t^{\alpha}, \mathbb{P}(X_t^{\alpha})) - \sigma(t, X_t^{\alpha}, \mathbb{P}(X_t^{\alpha}))|^2 dt \right] 
\]

\[
\leq \mathbb{E} \left[ \sum_{i=0}^{N-1} \int_{t_i}^{t_{i+1}} \left| f(t, \hat{X}_t^{\pi}, \hat{\alpha}_t, \mathbb{P}(\hat{X}_t^{\pi}, \hat{\alpha}_t)) - f(t, X_t^{\alpha}, \hat{\alpha}_t, \mathbb{P}(X_t^{\alpha}, \hat{\alpha}_t)) \right| dt \right] 
\]

\[
+ \mathbb{E} \left[ g(\hat{X}_T^{\pi}, \mathbb{P}(\hat{X}_T^{\pi})) - g(X_T^{\alpha}, \mathbb{P}(X_T^{\alpha})) \right] 
\]

\[
\leq C \left\{ R_2 + \mathbb{E} \left[ \sum_{i=0}^{N-1} \int_{t_i}^{t_{i+1}} \left| f(t, \hat{X}_t^{\pi}, \hat{\alpha}_t, \mathbb{P}(\hat{X}_t^{\pi}, \hat{\alpha}_t)) - f(t, X_t^{\alpha}, \hat{\alpha}_t, \mathbb{P}(X_t^{\alpha}, \hat{\alpha}_t)) \right| dt \right] 
\]

\[
+ \mathbb{E} \left[ g(\hat{X}_T^{\pi}, \mathbb{P}(\hat{X}_T^{\pi})) - g(X_T^{\alpha}, \mathbb{P}(X_T^{\alpha})) \right] \right) 
\]

which, together with the definition of \(V_\pi(\xi_0)\) and the optimality of \(\hat{\alpha}\) for (1.2), gives that

\[
V_\pi(\xi_0) - V(\xi_0) \leq J_\pi(\hat{\alpha}; \xi_0) - J(\hat{\alpha}; \xi_0) 
\]

\[
\leq \mathbb{E} \left\{ \sum_{i=0}^{N-1} \int_{t_i}^{t_{i+1}} \left| f(t, \hat{X}_t^{\pi}, \hat{\alpha}_t, \mathbb{P}(\hat{X}_t^{\pi}, \hat{\alpha}_t)) - f(t, X_t^{\alpha}, \hat{\alpha}_t, \mathbb{P}(X_t^{\alpha}, \hat{\alpha}_t)) \right| dt \right\} 
\]

\[
+ \mathbb{E} \left[ g(\hat{X}_T^{\pi}, \mathbb{P}(\hat{X}_T^{\pi})) - g(X_T^{\alpha}, \mathbb{P}(X_T^{\alpha})) \right] 
\]

\[
\leq C \left\{ R_2 + \mathbb{E} \left[ \sum_{i=0}^{N-1} \int_{t_i}^{t_{i+1}} \left| f(t, \hat{X}_t^{\pi}, \hat{\alpha}_t, \mathbb{P}(\hat{X}_t^{\pi}, \hat{\alpha}_t)) - f(t, X_t^{\alpha}, \hat{\alpha}_t, \mathbb{P}(X_t^{\alpha}, \hat{\alpha}_t)) \right| dt \right] 
\]

\[
+ \mathbb{E} \left[ g(\hat{X}_T^{\pi}, \mathbb{P}(\hat{X}_T^{\pi})) - g(X_T^{\alpha}, \mathbb{P}(X_T^{\alpha})) \right] \right) 
\]
with the residual term defined by

\[
R_2 := E \left[ \sum_{i=0}^{N-1} \int_{t_i}^{t_{i+1}} |f(t, X_t^\alpha, \hat{\alpha}_t, \mathbb{P}(X_t^\alpha, \hat{\alpha}_t)) - f(t, X_t^\alpha, \check{\alpha}_t, \mathbb{P}(X_t^\alpha, \check{\alpha}_t))| \, dt \right].
\] (4.5)

Then, by using Remark 2.1, the Cauchy-Schwarz inequality, (4.4) and the fact that \( \|\hat{\alpha}_T^\pi - \check{\alpha}_T\|_{L^2} \leq C(1 + \|\xi_0\|_{L^2})|\pi|^{1/2} \), we can deduce that

\[
V_{\pi}(\xi_0) - V(\xi_0) 
\leq C \left( R_2 + \max_{t_i \in \pi} \left( 1 + \|(X_t^\alpha, \hat{\alpha}_t, \check{\alpha}_t)^\pi \|_{L^2} \right) \left( \|X_t^\alpha - \check{\alpha}_t\|_{L^2} + \|\hat{\alpha}_t - \check{\alpha}_t\|_{L^2} \right) \right) 
\leq C \left( R_2 + \left( 1 + \|\xi_0\|_{L^2} \right)^{1/2} \left( 1 + \|\xi_0\|_{L^2} \right) \right). \quad (4.6)
\]

Hence it remains to estimate the residual terms \( R_1 \) and \( R_2 \) defined as in (4.3) and (4.5), respectively. Note that Remark 4.1 and the H"older regularity of \((X^\alpha, \hat{\alpha})\) imply that

\[
R_1 \leq C \left( 1 + \|X_t^\alpha\|_{S^2} + \|\hat{\alpha}\|_{S^2} \right) \pi + \sup_{t_i \in \pi, r \in [t_i, t_{i+1}]} \left( \|X_t^\alpha - X_r^\alpha\|_{L^2} + \|\hat{\alpha}_r - \hat{\alpha}_t\|_{L^2} \right) 
\leq C(1 + \|\xi_0\|_{L^2})|\pi|^{1/2},
\]

while Remark 2.1 and (H.3) give us that

\[
R_2 \leq C \left[ (1 + \|X_t^\alpha\|_{S^2} + \|\hat{\alpha}\|_{S^2}) \pi \right]^{1/2} 
+ \left( 1 + \|X_t^\alpha\|_{S^2} + \|\hat{\alpha}\|_{S^2} \right) \sup_{t_i \in \pi, r \in [t_i, t_{i+1}]} \left( \|X_t^\alpha - X_r^\alpha\|_{L^2} + \|\hat{\alpha}_r - \hat{\alpha}_t\|_{L^2} \right) 
\leq C(1 + \|\xi_0\|_{L^2})^{1/2}. \quad (4.6)
\]

These estimates enable us to conclude from (4.6) the upper bound that \( V_{\pi}(\xi_0) - V(\xi_0) \leq C(1 + \|\xi_0\|_{L^2})|\pi|^{1/2} \).

**Step 2:** Estimate an upper bound of \( V(\xi_0) - V_{\pi}(\xi_0) \). Note that the additional compactness assumption of \( A \) implies that there exists \( C > 0 \) such that \( \|\alpha\|_{H^2} \leq C \) for all \( \alpha \in \mathcal{A}_\pi \). Then standard moment estimates for MV-SDEs (see e.g. [20, Theorem 3.3]) shows that there exists \( C > 0 \) such that for all \( \alpha \in \mathcal{A}_\pi \), the solution to (1.1) with the control \( \alpha \) satisfies \( \|X^\alpha\|_{S^2} \leq C(1 + \|\xi_0\|_{L^2}) \). Moreover, for any \( 0 \leq s \leq r \leq t \leq T \), we can obtain from the Burkholder-Davis-Gundy inequality, Hölder’s inequality and (H.1(1)) that

\[
E \left[ \sup_{s \leq r \leq t} |X^\alpha_s - X^\alpha_r|^2 \right] 
\leq 2E \left[ \int_s^t |b(u, X_u^{\alpha, \sigma, \pi})|^2 \sigma(u, X_u^{\alpha, \sigma, \pi})^2 \, du \right] 
\leq C(\|b\|_{L^2(0,T)} + \|\sigma\|_{L^\infty(0,T)} + \|X^\alpha\|_{S^2} + \|\alpha\|_{H^2})^2 \pi(t - s) 
\leq C(1 + \|\xi_0\|_{L^2})^{1/2}(t - s). \quad (4.7)
\]

Similarly, for each \( \alpha \in \mathcal{A}_\pi \), by using the Lipschitz continuity of the coefficients \( b, \sigma \) and Gronwall’s inequality, one can show the corresponding solution \( X^{\alpha, \pi} \) to (1.6) (with control \( \alpha \)) satisfies the following moment estimate:

\[
\max_{t_i \in \pi} \|X_{t_i}^{\alpha, \pi}\|_{L^2} \leq C \left( 1 + \|\xi_0\|_{L^2}^2 + \max_{t_i \in \pi} \|\alpha_t\|_{L^2}^2 \right) \leq C(1 + \|\xi_0\|_{L^2}^2). \quad (4.8)
\]
Let $\alpha \in A_\pi$ be fixed, and let $X^\alpha$ and $X^{\alpha,\pi}$ be the solution to (1.1) and (1.6) with the control $\alpha$, respectively. Then by following similar arguments as those for (4.2) and (4.3), we have for each $i \in \{0, \ldots, N-1\}$ that

$$
E[|X^\alpha_{t_{i+1}} - X^{\alpha,\pi}_{t_{i+1}}|^2] 
\leq C \left\{ R_1^\alpha + E \left[ \left( \sum_{j=0}^{i} \int_{t_j}^{t_{j+1}} |b(t_j, X^\alpha_{t_j}, \alpha_{t_j}, P_{X^\alpha_{t_j}}) - b(t_j, X^{\alpha,\pi}_{t_j}, \alpha_{t_j}, P_{X^{\alpha,\pi}_{t_j}})| \, dt \right)^2 \right] 
+ E \left[ \sum_{j=0}^{i} \int_{t_j}^{t_{j+1}} |\sigma(t_j, X^\alpha_{t_j}, P_{X^\alpha_{t_j}}) - \sigma(t_j, X^{\alpha,\pi}_{t_j}, P_{X^{\alpha,\pi}_{t_j}})|^2 \, dt \right] \right\},
$$

with the residual term $R_1^\alpha$ defined as

$$
R_1^\alpha := E \left[ \sum_{i=0}^{N-1} \int_{t_i}^{t_{i+1}} |b(t, X^\alpha_t, \alpha_t, P_{X^\alpha_t}) - b(t, X^{\alpha,\pi}_t, \alpha_t, P_{X^{\alpha,\pi}_t})|^2 \, dt \right] 
+ E \left[ \sum_{i=0}^{N-1} \int_{t_i}^{t_{i+1}} |\sigma(t, X^\alpha_t, P_{X^\alpha_t}) - \sigma(t, X^{\alpha,\pi}_t, P_{X^{\alpha,\pi}_t})|^2 \, dt \right],
$$

which, along with the Lipschitz continuity of $b$ and $\sigma$ and Gronwall’s inequality, gives that

$$
\max_{t_i \in \pi} E[|X^\alpha_{t_i} - X^{\alpha,\pi}_{t_i}|^2] \leq CR_1^\alpha.
$$

(4.10)

Hence, we can obtain from Remark 2.1, Hölder’s inequality, the a priori estimate for $\|X^\alpha\|_{S^2}$, and the estimates (4.8) and (4.10) that

$$
V(\xi_0) - V_\pi(\xi_0) \leq \sup_{\alpha \in A_\pi} |J(\alpha; \xi_0) - J_\pi(\alpha; \xi_0)|
\leq \sup_{\alpha \in A_\pi} E \left[ \sum_{i=0}^{N-1} \int_{t_i}^{t_{i+1}} |f(t, X^\alpha_t, \alpha_t, P_{X^\alpha_t}) - f(t, X^{\alpha,\pi}_t, \alpha_t, P_{X^{\alpha,\pi}_t})| \, dt 
+ |g(X^\alpha_T, P_{X^\alpha_T}) - g(X^{\alpha,\pi}_T, P_{X^{\alpha,\pi}_T})| \right]
\leq C \sup_{\alpha \in A_\pi} \left( R_2^\alpha + E \left[ \sum_{i=0}^{N-1} \int_{t_i}^{t_{i+1}} |f(t, X^\alpha_t, \alpha_t, P_{X^\alpha_t}) - f(t, X^{\alpha,\pi}_t, \alpha_t, P_{X^{\alpha,\pi}_t})| \, dt ight] 
+ |g(X^\alpha_T, P_{X^\alpha_T}) - g(X^{\alpha,\pi}_T, P_{X^{\alpha,\pi}_T})| \right]
\leq C \sup_{\alpha \in A_\pi} \left( R_2^\alpha + (1 + \|\xi_0\|_{L^2}) \max_{t_i \in \pi} \|X^\alpha_{t_i} - X^{\alpha,\pi}_{t_i}\|_{L^2} \right)
\leq C \sup_{\alpha \in A_\pi} \left( R_2^\alpha + (1 + \|\xi_0\|_{L^2})(R_1^\alpha)^{1/2} \right)
$$

with the residual term $R_1^\alpha$ defined as in (4.9) and the residual term $R_2^\alpha$ defined by:

$$
R_2^\alpha := E \left[ \sum_{i=0}^{N-1} \int_{t_i}^{t_{i+1}} |f(t, X^\alpha_t, \alpha_t, P_{X^\alpha_t}) - f(t, X^{\alpha}_t, \alpha_t, P_{X^{\alpha}_t})| \, dt \right].
$$

(4.12)
Note that for each \( \alpha \in \mathcal{A}_\pi \), we have \( \alpha_t = \alpha_{t_i} \) for all \( t \in [t_i, t_{i+1}) \), \( i \in \{0, \ldots, N - 1\} \), which together with (H.3), Remarks 2.1 and 4.1, and the estimate (4.7) implies that

\[
R_1^\alpha \leq C \left( (1 + \|X^\alpha \|^2_{S^2} + \|\alpha\|^2_{H^2}) \pi + \sup_{t_i \leq \pi, \pi \in [t_i, t_{i+1})} \|X^\alpha_t - X_{t_i}^\alpha\|^2_{L^2} \right)
\]

\[
\leq C(1 + \|\xi_0\|^2_{L^2}) \pi,
\]

\[
R_2^\alpha \leq C \left( (1 + \|X^\alpha \|^2_{S^2} + \|\alpha\|^2_{H^2}) \pi \right)^{1/2}
\]

\[
+ (1 + \|X^\alpha \|^2_{S^2} + \|\alpha\|^2_{H^2}) \sup_{t_i \leq \pi, \pi \in [t_i, t_{i+1})} \|X^\alpha_t - X_{t_i}^\alpha\|_{L^2}
\]

\[
\leq C(1 + \|\xi_0\|^2_{L^2}) \pi^{1/2}.
\]

These estimates lead to the desired upper bound \( V(\xi_0) - V_\pi(\xi_0) \leq C(1 + \|\xi_0\|^2_{L^2}) \pi^{1/2} \).

**Remark 4.2.** The Hölder regularity of the optimal control of (1.2) is essential for quantifying the time discretization error and obtaining an upper bound of \( V_\pi(\xi_0) - V(\xi_0) \). For the lower bound of \( V_\pi(\xi_0) - V(\xi_0) \), we use the compactness of \( A \) to establish a uniform estimate for the \( H^2 \)-norms of all controls \( \alpha \in \mathcal{A}_\pi \) with any partition \( \pi \), which subsequently leads to a uniform Hölder regularity of the solution \( X^\alpha \) to (1.1) with control \( \alpha \in \mathcal{A}_\pi \) and then the desired half-order convergence; see [17, Proposition 3.1] for a similar result with controlled Itô diffusions.

A similar error bound can be established if one can obtain a uniform estimate for the \( H^2 \)-norms of minimizers of \( V_\pi(\xi_0) \) defined in (1.3). For example, for a given initial state \( \xi_0 \in L^2(\mathcal{F}_0; \mathbb{R}^n) \) and constant \( B \in [0, \infty) \), one may consider the MFC problem \( V^B(\xi_0) = \inf_{\alpha \in \mathcal{A}B} J(\alpha; \xi_0) \), with the cost functional \( J(\alpha; \xi_0) \) defined as in (1.2) and a constrained control set \( \mathcal{A}_B \subset \mathcal{A} \) consisting of all admissible controls \( \alpha \in \mathcal{A} \) satisfying the estimate \( \mathbb{E}[\int_0^T |\alpha|dt] \leq B \) (see e.g. [16]). It is clear that for a sufficiently large \( B \) (depending on the initial condition), \( V^B(\xi_0) = V(\xi_0) \) and the minimizer of (1.2) is also a minimizer of \( V^B(\xi_0) \). Hence, by following the same arguments as in Theorem 4.2, we see the value functions \( (V^B(\xi_0))_\pi \) with the corresponding piecewise constant policies \( \mathcal{A}_{B,\pi} \subset \mathcal{A}_B \) also admit a half-order convergence rate to the value function \( V^B(\xi_0) \), with a constant depending on the initial condition \( \xi_0 \).

**5. Error estimates of optimal controls for piecewise constant policy approximations**

In this section, we proceed to investigate the convergence of minimizers of the approximate control problems (4.1) and (1.3) based on the convergence of their value functions.

Before presenting our convergence analysis, let us point out that the proofs of Theorems 4.1 and 4.2 show that for every initial state \( \xi_0 \in L^2(\mathcal{F}_0; \mathbb{R}^n) \) and for every partition \( (\pi_i)_i \in \mathbb{N} \) of \([0, T] \) satisfying \( \lim_{i \to \infty} |\pi_i| = 0 \), we can find controls \( (\hat{\alpha}^{\pi_i})_i \in \mathbb{N} \) satisfying for all \( i \in \mathbb{N} \) that \( \hat{\alpha}^{\pi_i} \in \mathcal{A}_{\pi_i} \),

\[
V^\pi_{\pi_i}(\xi_0) \leq J(\hat{\alpha}^{\pi_i}; \xi_0) \leq V^\pi_{\pi_i}(\xi_0) + C(1 + \|\xi_0\|_{L^2}^2) |\pi_i|^{1/2},
\]

\[
V_{\pi_i}(\xi_0) \leq J(\hat{\alpha}^{\pi_i}; \xi_0) \leq V_{\pi_i}(\xi_0) + C(1 + \|\xi_0\|_{L^2}^2) |\pi_i|^{1/2}
\]

with a constant \( C \) independent of \( \xi_0 \) and \( \pi_i \), and \( \lim_{i \to \infty} \|\hat{\alpha}^{\pi_i} - \hat{\alpha}\|_{H^2(\mathbb{R}^k)} = 0 \), where \( \hat{\alpha} \in \mathcal{A} \) the optimal control of (1.2). In fact, such controls can be constructed based on piecewise constant approximations of the optimal control strategy \( \hat{\alpha} \in \mathcal{A} \) on \( \pi_i \). Since in practice one may not be able to exactly compute these control strategies \( (\hat{\alpha}^{\pi_i})_i \in \mathbb{N} \), in this section we shall study the convergence of any \( \varepsilon \)-optimal controls of (4.1) and (1.3). In particular, we shall establish that any \( \varepsilon \)-optimal
controls of these approximate control problems converge strongly to the optimal control of (1.2) in \(H^2(\mathbb{R}^k)\).

We start by showing several important properties of the cost functional \(J(\cdot;\xi_0) : \mathcal{A} \to \mathbb{R}\) defined as in (1.2).

**Lemma 5.1.** Suppose \((H.1)\) and \((H.2)\) hold, let \(\xi_0 \in L^2(\mathcal{F}_0; \mathbb{R}^n)\) and let \(J(\cdot;\xi_0) : \mathcal{A} \to \mathbb{R}\) be defined as in (1.2). Then \(J\) is continuous and strongly convex. More specifically, it holds for all \(\alpha, \beta \in \mathcal{A}, \tau \in [0, 1]\) that

\[
\tau J(\alpha;\xi_0) + (1 - \tau)J(\beta;\xi_0) - J(\tau \alpha + (1 - \tau)\beta;\xi_0) \geq \tau(1 - \tau)(\lambda_1 + \lambda_2)\|\alpha - \beta\|_{H^2}^2,
\]

where \(\lambda_1, \lambda_2\) are the constants appearing in \((H.1(4))\). Moreover, we have for all \(\alpha \in \mathcal{A}\) that

\[
J(\hat{\alpha};\xi_0) - J(\alpha;\xi_0) \leq -(\lambda_1 + \lambda_2)\|\hat{\alpha} - \alpha\|_{H^2}^2,
\]

where \(\hat{\alpha}\) is the unique minimizer of (1.2) defined in Theorem 3.6.

**Proof.** The continuity of \(J\) follows directly from stability results of (1.2) and the local Lipschitz continuity of functions \((f,g)\) (see \((H.1(3))\)).

We now show the strong convexity of the cost functional \(J\). Let \(\alpha, \beta \in \mathcal{A}, \tau \in [0, 1]\), and let \(X^\alpha\) (resp. \(X^\beta\)) be the solution to (1.2) with control \(\alpha\) (resp. \(\beta\)). Let \(\gamma = \tau \alpha + (1 - \tau)\beta\) and let \(X := \tau X^\alpha + (1 - \tau)X^\beta\). We first show \(X = X^\gamma\), where \(X^\gamma\) be the solution to (1.2) with control \(\gamma\). It is clear that \(X_0 = \tau X_0^\alpha + (1 - \tau)X_0^\beta = \xi_0 = X_0^\gamma\). For each \(t \in [0, T]\), we see that

\[
\mathbb{E}[(X_t, \gamma_t)] = \tau \mathbb{E}[(X_t^\alpha, \alpha_t)] + (1 - \tau)\mathbb{E}[(X_t^\beta, \beta_t)],
\]

which together with the linearity of the functions \(b, \sigma\) in \((x, a, \eta)\) (see \((H.1(1))\)) gives that

\[
b(t, X_t, \gamma_t, P_{(X_t, \gamma_t)}) = \tau b(t, X_t^\alpha, \alpha_t, P_{(X_t^\alpha, \alpha_t)}) + (1 - \tau)b(t, X_t^\beta, \beta_t, P_{(X_t^\beta, \beta_t)}),
\]

\[
\sigma(t, X_t, P_{X_t}) = \tau \sigma(t, X_t^\alpha, P_{X_t^\alpha}) + (1 - \tau)\sigma(t, X_t^\beta, P_{X_t^\beta}).
\]

Hence, we can show by using Itô’s formula that \(X\) satisfies the same MV-SDE as \(X^\gamma\), which along with the uniqueness of strong solutions shows that \(X^\gamma = X = \tau X^\alpha + (1 - \tau)X^\beta\).

Let \(\tilde{X}^\alpha_T\) and \(\tilde{X}^\beta_T\) be independent copies of \(X^\alpha_T\) and \(X^\beta_T\), respectively, defined on \(L^2(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P}; \mathbb{R}^n)\). We see that \(\tilde{X}^\gamma_T := \tau \tilde{X}^\alpha_T + (1 - \tau)\tilde{X}^\beta_T\) is an independent copy of \(X^\gamma_T\) with distribution \(\tilde{P}_{X^\gamma_T}\). Hence we can obtain from the convexity of \(g\) in \((H.1(4))\) and \(X^\gamma = \tau X^\alpha + (1 - \tau)X^\beta\) that

\[
g(X^\gamma_T, \tilde{P}_{X^\gamma_T}) - g(X^\alpha_T, \tilde{P}_{X^\alpha_T})
\]

\[
\geq \langle \partial x g(X^\gamma_T, \tilde{P}_{X^\gamma_T}), X^\alpha_T - X^\gamma_T \rangle + \tilde{E}[\langle \partial \mu g(X^\gamma_T, \mu)(\tilde{X}^\gamma_T), \tilde{X}^\alpha_T - \tilde{X}^\gamma_T \rangle]
\]

\[
= (1 - \tau)\left(\langle \partial x g(X^\gamma_T, \tilde{P}_{X^\gamma_T}), X^\beta_T - X^\gamma_T \rangle + \tilde{E}[\langle \partial \mu g(X^\gamma_T, \mu)(\tilde{X}^\gamma_T), \tilde{X}^\beta_T - \tilde{X}^\gamma_T \rangle]\right).
\]

Similarly, we can show that

\[
g(X^\beta_T, \tilde{P}_{X^\beta_T}) - g(X^\gamma_T, \tilde{P}_{X^\gamma_T})
\]

\[
\geq \tau \left(\langle \partial x g(X^\gamma_T, \tilde{P}_{X^\gamma_T}), X^\beta_T - X^\gamma_T \rangle + \tilde{E}[\langle \partial \mu g(X^\gamma_T, \mu)(\tilde{X}^\gamma_T), \tilde{X}^\beta_T - \tilde{X}^\gamma_T \rangle]\right),
\]

which implies that

\[
\tau \mathbb{E}[g(X^\alpha_T, P_{X^\alpha_T})] + (1 - \tau)\mathbb{E}[g(X^\beta_T, P_{X^\beta_T})] \geq \mathbb{E}[g(X^\gamma_T, P_{X^\gamma_T})].
\]
Now for each $t \in [0, T]$, let $(\tilde{X}_t^\alpha, \tilde{\alpha}_t)$ and $(\tilde{X}_t^\beta, \tilde{\beta}_t)$ be independent copies of $(X_t^\alpha, \alpha_t)$ and $(X_t^\beta, \beta_t)$ defined on $L^2(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}; \mathbb{R}^n \times \mathbb{R}^k)$, respectively. We see that $(\tilde{X}_t^\gamma, \tilde{\gamma}_t) := (\tilde{X}_t^\alpha, \tilde{\alpha}_t) + (1 - \tau)(\tilde{X}_t^\beta, \tilde{\beta}_t)$ is an independent copy of $(X_t^\gamma, \gamma_t)$ with distribution $\tilde{\mathbb{P}}_{(X_t^\gamma, \gamma_t)}$. Then we can obtain from the convexity of $f$ in (H.1(4)) and $(X^\gamma, \gamma) = (X^\alpha, \alpha) + (1 - \tau)(X^\beta, \beta)$ that

$$f(t, X_t^\alpha, \alpha_t, \mathbb{P}(X_t^\alpha, \alpha_t)) - f(t, X_t^\gamma, \gamma_t, \mathbb{P}(X_t^\gamma, \gamma_t)) \geq (1 - \tau)\left( \langle \partial_{(x,a)} f(t, X_t^\gamma, \gamma_t, \mathbb{P}(X_t^\gamma, \gamma_t)), (X_t^\alpha - X_t^\alpha, \alpha_t - \beta_t) \rangle \right)$$

$$+ \tilde{\mathbb{E}}[\langle \partial_\mu f(t, X_t^\gamma, \gamma_t, \mathbb{P}(X_t^\gamma, \gamma_t)), (\tilde{X}_t^\gamma, \tilde{\gamma}_t), (\tilde{X}_t^\alpha - \tilde{X}_t^\beta) \rangle]$$

$$+ \tilde{\mathbb{E}}[\langle \partial_\nu f(t, X_t^\gamma, \gamma_t, \mathbb{P}(X_t^\gamma, \gamma_t)), \tilde{\alpha}_t - \tilde{\beta}_t \rangle] + (1 - \tau)^2(\lambda_1|\alpha_t - \beta_t|^2 + \lambda_2\tilde{\mathbb{E}}[|\tilde{\alpha}_t - \tilde{\beta}_t|^2]),$$

Similarly, we can derive a lower bound of $f(t, X_t^\beta, \beta_t, \mathbb{P}(X_t^\beta, \beta_t)) - f(t, X_t^\gamma, \gamma_t, \mathbb{P}(X_t^\gamma, \gamma_t))$, which subsequently leads to the estimate that

$$\tau\mathbb{E}[f(t, X_t^\alpha, \alpha_t, \mathbb{P}(X_t^\alpha, \alpha_t))] + (1 - \tau)\mathbb{E}[f(t, X_t^\beta, \beta_t, \mathbb{P}(X_t^\beta, \beta_t))] - \mathbb{E}[f(t, X_t^\gamma, \gamma_t, \mathbb{P}(X_t^\gamma, \gamma_t))] \geq \left( \tau(1 - \tau)^2 + \tau^2(1 - \tau) \right)\left( \lambda_1\mathbb{E}[|\alpha_t - \beta_t|^2] + \lambda_2\tilde{\mathbb{E}}[|\tilde{\alpha}_t - \tilde{\beta}_t|^2] \right)$$

$$= \tau(1 - \tau)(\lambda_1 + \lambda_2)\tilde{\mathbb{E}}[|\alpha_t - \beta_t|^2].$$

Hence, we can conclude from (1.2) the desired strong convexity estimate.

We proceed to show the estimate (5.1). The linearity of $(b, \sigma)$ and the convexity of $f$ in (H.1) imply that the Hamiltonian $H$ defined as in (2.1) is convex, i.e., for all $(t, y, z) \in [0, T] \times \mathbb{R}^n \times \mathbb{R}^d$, $(x, \eta, a, (x', \eta', a')) \in \mathbb{R}^n \times \mathcal{P}_2(\mathbb{R}^n \times \mathbb{R}^k) \times \mathcal{A}$,

$$H(t, x, a, \eta, y, z) - H(t, x', a', \eta', y, z) - \langle \partial_{(x,a)} H(t, x, a, \eta, y, z), (x - x', a - a') \rangle$$

$$- \tilde{\mathbb{E}}[\langle \partial_\mu H(t, x, a, \eta, y, z)(\tilde{X}, \tilde{\alpha}), \tilde{X} - \tilde{X}' \rangle + \langle \partial_\nu H(t, x, a, \eta, y, z)(\tilde{X}, \tilde{\alpha}), \tilde{\alpha} - \tilde{\alpha}' \rangle] \leq -\lambda_1|a' - a|^2 - \lambda_2\tilde{\mathbb{E}}[|\tilde{\alpha}' - \tilde{\alpha}|^2],$$

whenever $(\tilde{X}, \tilde{\alpha}), (\tilde{X}', \tilde{\alpha}') \in L^2(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}; \mathbb{R}^n \times \mathbb{R}^k)$ with distributions $\eta$ and $\eta'$, respectively. Moreover, the same arguments as in [1, Theorem 3.5] give us that

$$J(\dot{\alpha}; \xi_0) - J(\alpha; \xi_0) \leq \mathbb{E}\left[ \int_0^T \left( H(t, X_t^\alpha, \alpha_t, \mathbb{P}(X_t^\alpha, \alpha_t), Y_t^\alpha, Z_t^\alpha) - H(t, X_t^\beta, \beta_t, \mathbb{P}(X_t^\beta, \beta_t), Y_t^\beta, Z_t^\beta) \right) dt \right]$$

$$- \mathbb{E}\left[ \int_0^T \langle \partial_\mu H(t, X_t^\alpha, \alpha_t, \mathbb{P}(X_t^\alpha, \alpha_t), Y_t^\alpha, Z_t^\alpha), Y_t^\beta, Z_t^\beta \rangle \right] dt$$

$$- \mathbb{E}\left[ \int_0^T \tilde{\mathbb{E}}[\langle \partial_\nu H(t, X_t^\alpha, \alpha_t, \mathbb{P}(X_t^\alpha, \alpha_t), Y_t^\alpha, Z_t^\alpha), \tilde{X}_t^\beta, \tilde{\alpha}_t \rangle \rangle \tilde{X}_t^\beta - \tilde{X}_t^\alpha \rangle \right] dt \right],$$

which along with (5.2) and the fact that $\dot{\alpha}$ satisfies the optimality condition (2.3) leads to

$$J(\dot{\alpha}; \xi_0) - J(\alpha; \xi_0) \leq \mathbb{E}\left[ \int_0^T \langle \partial_\mu H(\theta_t^\alpha, Y_t^\alpha, Z_t^\alpha) + \tilde{\mathbb{E}}[\partial_\nu H(\tilde{\theta}_t^\alpha, \tilde{Y}_t^\alpha, \tilde{Z}_t^\alpha)(X_t^\alpha, \alpha_t), \tilde{\alpha}_t - a_t \rangle \rangle \right]$$

$$- (\lambda_1 + \lambda_2)|\dot{\alpha} - \alpha|^2dt$$

$$\leq -(\lambda_1 + \lambda_2)|\dot{\alpha} - \alpha|^2dt.$$
Remark 5.1. It is clear that for each partition $\pi$ of $[0,T]$, $J(\cdot;\xi_0) : \mathcal{A}_\pi \to \mathbb{R}$ is continuous and strongly convex, as $\mathcal{A}_\pi \subset \mathcal{A}$ is a convex set. Moreover, similar arguments as those in Lemma 5.1 show that the discrete-time cost functional $J_\pi(\cdot;\xi_0) : \mathcal{A}_\pi \to \mathbb{R}$ defined in (1.5) is continuous and strongly convex. Then the standard theory of strongly convex minimization problems on Hilbert spaces (see e.g., [5, Lemma 2.33 (ii)]) ensures that $J(\cdot;\xi_0) : \mathcal{A}_\pi \to \mathbb{R}$ and $J_\pi(\cdot;\xi_0) : \mathcal{A}_\pi \to \mathbb{R}$ admit a unique minimizer.

We now show the strong convergence of $\varepsilon$-optimal controls of the control problem (4.1) (with piecewise constant controls but continuous-time state process).

**Theorem 5.2.** Suppose (H.1) and (H.2) hold, for every $\xi_0 \in L^2(F_0; \mathbb{R}^n)$, let $J(\cdot;\xi_0) : \mathcal{A} \to \mathbb{R}$ be defined as in (1.2) and let $\hat{\alpha} \in \mathcal{A}$ be the optimal control of (1.2), and for each partition $\pi$ of $[0,T]$ let $\mathcal{A}_\pi$ be defined as in (1.4) and $V_\pi^c(\xi_0) \in \mathbb{R}$ be defined as in (4.1). Then there exists a constant $C > 0$, such that for all $\xi_0 \in L^2(F_0; \mathbb{R}^n)$ and $\varepsilon \geq 0$, for all partitions $\pi$ of $[0,T]$ with stepsize $|\pi|$, and for all $\alpha \in \mathcal{A}_\pi$ with $J(\alpha;\xi_0) \leq V_\pi^c(\xi_0) + \varepsilon$,

$$\|\hat{\alpha} - \alpha\|_{\mathcal{H}^2} \leq C((1 + \|\xi_0\|_{L^2})|\pi|^{1/4} + \sqrt{\varepsilon}).$$

**Proof.** Recall that according to Theorem 4.1, we have $V(\xi_0) - V_\pi^c(\xi_0) \leq C(1 + \|\xi_0\|_{L^2})|\pi|^{1/2}$, for a constant $C$ independent of the initial condition and stepsize. Therefore, by applying the estimate (5.1), we have

$$\lambda_1 \lambda_2 \|\hat{\alpha} - \alpha\|_{\mathcal{H}^2}^2 \leq J(\alpha;\xi_0) - J(\hat{\alpha};\xi_0) \leq V_\pi^c(\xi_0) + \varepsilon - J(\hat{\alpha};\xi_0) \leq C(1 + \|\xi_0\|_{L^2})|\pi|^{1/2} + \varepsilon.$$

Taking the square root of both sides of the inequality yields the claim. \qed

We now establish the strong convergence of $\varepsilon$-optimal controls of the control problem (1.3) with piecewise constant controls, state processes and cost functionals. For simplicity, we only present the result for the case where $\mathcal{A}$ is a compact subset of $\mathbb{R}^k$, but refer the reader to Remark 4.2 for possible extensions to cases with non-compact $\mathcal{A}$.

**Theorem 5.3.** Suppose (H.1), (H.2) and (H.3) hold, and $\mathcal{A}$ is a compact subset of $\mathbb{R}^k$. For every $\xi_0 \in L^2(F_0; \mathbb{R}^n)$, $\hat{\alpha} \in \mathcal{A}$ be the optimal control of (1.2), and for each partition $\pi$ of $[0,T]$, let $\mathcal{A}_\pi$ be defined as in (1.4), $J_\pi(\cdot;\xi_0) : \mathcal{A}_\pi \to \mathbb{R}$ be defined as in (1.5) and $V_\pi(\xi_0) \in \mathbb{R}$ be defined as in (1.3). Then there exists a constant $C > 0$, such that for all $\xi_0 \in L^2(F_0; \mathbb{R}^n)$ and $\varepsilon \geq 0$, for all partitions $\pi$ of $[0,T]$ with stepsize $|\pi|$, and for all $\alpha \in \mathcal{A}_\pi$ with $J_\pi(\alpha;\xi_0) \leq V_\pi(\xi_0) + \varepsilon$,

$$\|\hat{\alpha} - \alpha\|_{\mathcal{H}^2} \leq C((1 + \|\xi_0\|_{L^2})|\pi|^{1/4} + \sqrt{\varepsilon}).$$

**Proof.** Recall that Step 2 of the proof of Theorem 4.2 (see (4.11)) proves that there exists a constant $C > 0$, independent of $\xi_0$ and $\pi$, such that for all $\alpha \in \mathcal{A}_\pi$, $|J(\alpha;\xi_0) - J_\pi(\alpha;\xi_0)| \leq C(1 + \|\xi_0\|_{L^2}^2)|\pi|^{1/2}$. Hence, by the estimates (5.1), for all $\alpha \in \mathcal{A}_\pi$ with $J_\pi(\alpha;\xi_0) \leq V_\pi(\xi_0) + \varepsilon$,

$$\lambda_1 \lambda_2 \|\hat{\alpha} - \alpha\|_{\mathcal{H}^2}^2 \leq J(\alpha;\xi_0) - J(\hat{\alpha};\xi_0) - J_\pi(\alpha;\xi_0) + J_\pi(\alpha;\xi_0) \\
\leq J(\alpha;\xi_0) - J(\hat{\alpha};\xi_0) - J_\pi(\alpha;\xi_0) + V_\pi(\xi_0) + \varepsilon \\
\leq C(1 + \|\xi_0\|_{L^2}^2)|\pi|^{1/2} + V_\pi(\xi_0) - J(\hat{\alpha};\xi_0) + \varepsilon \\
\leq C(1 + \|\xi_0\|_{L^2}^2)|\pi|^{1/2} + \varepsilon,$$

where the last estimate follows from Theorem 4.2. Taking the square root of both sides of the inequality yields the claim. \qed

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A Proofs of Propositions 3.1 and 3.2 and Lemma 3.3

The following Kantorovich duality theorem, which plays an important role in the following analysis, follows as a special case of [22, Theorem 5.10].

Lemma A.1. Let $(\mathcal{X}, \mu)$ and $(\mathcal{Y}, \nu)$ be two Polish probability spaces and let $\omega : \mathcal{X} \times \mathcal{Y} \to [0, \infty)$ be a continuous function. Then we have that

$$\inf_{\kappa \in \Pi(\mu, \nu)} \int_{\mathcal{X} \times \mathcal{Y}} \omega(x, y) \, d\kappa(x, y) = \sup_{(\psi, \varphi) \in C_b(\mathcal{X}) \times C_b(\mathcal{Y}), \|\psi - \varphi\|_\infty \leq 1} \left( \int_{\mathcal{Y}} \varphi(y) \, d\nu(y) - \int_{\mathcal{X}} \psi(x) \, d\mu(x) \right),$$

where $\Pi(\mu, \nu)$ is the set of all couplings of $\mu$ and $\nu$, and $C_b(\mathcal{X})$ (resp. $C_b(\mathcal{Y})$) is the space of bounded continuous functions $\mathcal{X} \to \mathbb{R}$ (resp. $\mathcal{Y} \to \mathbb{R}$).

Proof of Proposition 3.1. We first show that the functions $(\hat{b}, \sigma, \hat{f}, \hat{g})$ satisfy the Lipschitz continuity. The claim obviously holds for the function $\sigma$ due to (H.1(1)). To show the Lipschitz continuity of $\hat{g}$, for any $(x, \mu), (x', \mu') \in \mathbb{R}^n \times \mathcal{P}_2(\mathbb{R}^n)$ and any coupling $\kappa$ of $\mu$ and $\mu'$ (i.e., $\kappa \in \Pi(\mu, \mu')$), we observe from (H.1(3)) that

$$|\hat{g}(x, \mu) - \hat{g}(x', \mu')|^2$$

$$\leq 2|\partial_x g(x, \mu) - \partial_x g(x', \mu')|^2 + 2 \left( \int_{\mathbb{R}^n} |\partial_x g(x, \mu)(x) - \partial_x g(x', \mu')(x')| \, d\mu'(x') \right)^2$$

$$\leq C(|x - x'|^2 + \mathcal{W}_2^2(\mu, \mu')) + 2 \left( \int_{\mathbb{R}^n \times \mathbb{R}^n} |\partial_x g(x, \mu)(x) - \partial_x g(x', \mu')(x')| \, d\kappa(x, x') \right)^2$$

$$\leq C \left\{ |x - x'|^2 + \mathcal{W}_2^2(\mu, \mu') + \int_{\mathbb{R}^n \times \mathbb{R}^n} (|\tilde{x} - \tilde{x}'| + |x - x'| + \mathcal{W}_2(\mu, \mu')) \, d\kappa(\tilde{x}, \tilde{x}') \right\}^2,$$

where $C > 0$ depends on the Lipschitz constant in (H.1(3)). Then, by applying Jensen’s inequality to the above estimate and taking the infimum over all $\kappa \in \Pi(\mu, \mu')$, we can deduce that $|\hat{g}(x, \mu) - \hat{g}(x', \mu')| \leq C(|x - x'| + \mathcal{W}_2(\mu, \mu'))$.

Before proceeding to show the Lipschitz continuity of $\hat{b}$ and $\hat{f}$, we first establish the Lipschitz continuity of $\phi(t, \chi)$ defined as in (H.2(1)). Let $\chi, \chi' \in \mathcal{P}_2(\mathbb{R}^n \times \mathbb{R}^k)$ be given. By applying Lemma A.1 with $\mathcal{X} = \mathcal{Y} = \mathbb{R}^n \times \mathbb{R}^k$, $\nu = \phi(t, \chi), \mu = \phi(t, \chi')$ and the function $\omega((x', y'), (x, y)) := |x - x'|^2 + |y - y'|^2$ for any $(x, y) \in \mathcal{X}, (x', y') \in \mathcal{X}$, we can obtain from the definition of $\phi$ that

$$\mathcal{W}_2^2(\phi(t, \chi), \phi(t, \chi')) = \sup \left( \int_{\mathbb{R}^n \times \mathbb{R}^k} h_1(x, y) \, d\phi(t, \chi)(x, y) - \int_{\mathbb{R}^n \times \mathbb{R}^k} h_2(x', y') \, d\phi(t, \chi')(x', y') \right)$$

$$= \sup \left( \int_{\mathbb{R}^n \times \mathbb{R}^n} h_1(x, \tilde{t}(t, x, \chi)) \, d\chi(x) - \int_{\mathbb{R}^n \times \mathbb{R}^n} h_2(x', \tilde{t}(t, x, \chi')) \, d\chi'(x', y') \right),$$

where the supremum is taken over all bounded continuous functions $h_1, h_2 : \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}$ satisfying $h_1(x, y) - h_2(x', y') \leq |x - x'|^2 + |y - y'|^2$ for any $(x, y), (x', y') \in \mathbb{R}^n \times \mathbb{R}^k$. Note that for any given such functions $h_1, h_2$, the Lipschitz continuity of $\tilde{t}$ in (H.2(1)) implies that

$$h_1(x, \tilde{t}(t, x, \chi)) - h_2(x', \tilde{t}(t, x, y', \chi'))$$

$$\leq (3L^2 + 1) \left( |x - x'|^2 + |y - y'|^2 + \mathcal{W}_2^2(\chi, \chi') \right) := \omega_2((x, y), (x', y')).$$
Hence, another application of Lemma A.1 with \( \mathcal{X} = \mathcal{Y} = \mathbb{R}^n \times \mathbb{R}^n \), \( \nu = \chi \), \( \mu = \chi' \) and \( \omega = \omega_2 \) gives us that

\[
W_2^2(\phi(t, \chi), \phi(t, \chi')) \leq \sup \left( \int_{\mathbb{R}^n \times \mathbb{R}^n} \hat{h}_1(x, y) \, d\chi(x, y) - \int_{\mathbb{R}^n \times \mathbb{R}^n} \hat{h}_2(x', y') \, d\chi'(x', y') \right)
\]

\[
= \inf_{\kappa \in \Pi(\chi, \chi')} \int_{(\mathbb{R}^n \times \mathbb{R}^n) \times (\mathbb{R}^n \times \mathbb{R}^n)} \omega_2(x, y) \, d\kappa(x, y),
\]

where the supremum is taken over all bounded continuous functions \( \hat{h}_1, \hat{h}_2 : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R} \) satisfying \( \hat{h}_1 - \hat{h}_2 \leq \omega_2 \). Thus, we readily deduce from the above estimate that

\[
W_2(\phi(t, \chi), \phi(t, \chi')) \leq C W_2(\chi, \chi'),
\]

with a constant \( C > 0 \) depending only on \( L_\alpha \).

Now for any \((x, y), (x', y') \in \mathbb{R}^n \times \mathbb{R}^n \times \mathcal{P}_2(\mathbb{R}^n \times \mathbb{R}^n)\), we can obtain from (2.18), (H.1(1)) and the Lipschitz continuity of \( \hat{\alpha} \) in (H.2(1)) that

\[
|\hat{b}(t, x, y, \chi) - \hat{b}(t, x', y', \chi')|
\]

\[
= |b(t, \chi, \hat{\alpha}(t, x, y, \chi), \phi(t, \chi)) - b(t, x', \hat{\alpha}(t, x', y', \chi'), \phi(t, \chi'))|
\]

\[
\leq C \left( |x - x'| + |y - y'| + W_2(x, x') \right),
\]

which shows the Lipschitz continuity of \( \hat{b} \). Finally, we shall establish the Lipschitz continuity of \( \hat{f} \). Observe that \( \partial_x H(t, \cdot, \cdot) \) and \( \partial_\mu H(t, \cdot, \cdot, \cdot) \) are Lipschitz continuous (uniformly in \( t \)), which follows from the definition of the Hamiltonian \( H \) in (2.1) and (H.1(1)(3)). Now, for any \((x, y, \rho), (x', y', \rho') \in \mathbb{R}^n \times \mathbb{R}^n \times \mathcal{P}_2(\mathbb{R}^n \times \mathbb{R}^n)\), let \( \chi = \pi_{1,2}^\sharp \rho \) (resp. \( \chi' = \pi_{1,2}^\sharp \rho' \)) the marginal of the measure \( \rho \) (resp. \( \rho' \)) on \( \mathbb{R}^n \times \mathbb{R}^n \). Then, we can obtain from the definition of \( \hat{f} \) in (2.18) that

\[
|\hat{f}(t, x, y, z, \rho) - \hat{f}(t, x', y', z', \rho')|
\]

\[
\leq |\partial_x H(t, x, \hat{\alpha}(t, x, y, \chi), \phi(t, \chi), y, z) - \partial_x H(t, x', \hat{\alpha}(t, x', y', \chi'), \phi(t, \chi'), y', z')|
\]

\[
+ \left| \int_{\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n} \partial_\mu H(t, x, \hat{\alpha}(t, x, y, \chi), \phi(t, \chi), y, \hat{\chi}(x, \hat{\alpha}(t, x, y, \chi)) \, d\rho(x, y, \chi) \right|
\]

\[
- \int_{\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n} \partial_\mu H(t, x', \hat{\alpha}(t, x', y', \chi'), \phi(t, \chi'), y', \hat{\chi}'(x', \hat{\alpha}(t, x', y', \chi')) \, d\rho'(x', y', \chi') \right|
\]

\[
:= \Sigma_1 + \Sigma_2.
\]

Note that one can easily deduce from Lemma A.1 that \( W_2(\pi_{1,2}^\sharp \rho, \pi_{1,2}^\sharp \rho') \leq W_2(\rho, \rho') \). Then, by using the uniform Lipschitz continuity of \( \partial_x H(t, \cdot, \cdot) \), (H.2(1)) and (A.2), we have the estimate that \( \Sigma_1 \leq C (|x - x'| + |y - y'| + |z - z'| + W_2(\rho, \rho')) \). Furthermore, by using the same manipulations as in (A.1) with an arbitrary coupling of \( \rho \) and \( \rho' \), and employing the Lipschitz continuity of \( \partial_\mu H(t, \cdot, \cdot, \cdot) \) along with (H.2(1)) and (A.2), we can conclude the same upper bound for \( \Sigma_2 \), which leads to the desired Lipschitz continuity of \( \hat{f} \).

It remains to show that the functions \( (\hat{b}, \sigma, \hat{f}) \) satisfy the integrability conditions. We can clearly see from (H.1(1)) that \( \|\sigma(\cdot, 0, \delta_{0_n})\|_{L^\infty(0,T)} = \|\sigma_0\|_{L^\infty(0,T)} < \infty \). Moreover, (2.7) and (H.2(1)) imply that \( \|\hat{\phi}(t, \delta_{0_n})\|_{L^\infty(0,T)} < \infty \). Hence, we can obtain from (2.18) and (H.1(1)(3)) that \( \|\hat{b}(\cdot, 0, 0, \delta_{0_n})\|_{L^2(0,T)} + \|\hat{f}(\cdot, 0, 0, 0, \delta_{0_n})\|_{L^\infty(0,T)} < \infty \), which completes the proof.\( \square \)
Proof of Proposition 3.2. Throughout this proof, let $t \in [0, T]$, for all $i \in \{1, 2\}$ let $\Theta_i = (X_i, Y_i, Z_i) \in L^2(\Omega; \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{n \times d})$ be a given random variable and $\alpha_i = \hat{\alpha}(t, X_i, Y_i, \mathbb{P}_{(X_i, Y_i)})$.

Let $(\tilde{X}_i, \tilde{Y}_i, \tilde{Z}_i)_{i=1}^2$ be an independent copy of $(X_i, Y_i, Z_i)_{i=1}^2$ defined on the space $L^2(\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathbb{P}})$. By applying the convexity of $g$ in (II.1(4)) with $(x', \mu') = (X_1(\omega), \mathbb{P}_{X_1}), (x, \mu) = (X_2(\omega), \mathbb{P}_{X_2})$ for each $\omega$, taking the expectation with respect to the measure $\mathbb{P}$ and then exchanging the role of $X_1$ and $X_2$ in the estimates, we obtain the desired monotonicity property of $\hat{f}$ in (3.1) as follows:

$$0 \leq \mathbb{E}[(\dot{\hat{\varphi}} g(X_1, \mathbb{P}_{X_1}) - \dot{\hat{\varphi}} g(X_2, \mathbb{P}_{X_2}), X_1 - X_2)]$$
$$+ \mathbb{E}[(\dot{\hat{\varphi}} b(X_1, \mathbb{P}_{X_1}), \mathcal{F} X_1)]$$
$$+ \mathbb{E}[(\dot{\hat{\varphi}} b(X_2, \mathbb{P}_{X_2}), \mathcal{F} X_2)]$$

where for the last equality we have used Fubini’s theorem and the fact that $\mathbb{P}_{X_i} = \hat{\mathbb{P}}_{\tilde{X}_i}$ for $i = 1, 2$.

To show monotonicity of $\hat{f}$, we first deduce from the definition of $\hat{\varphi}$ (see (2.18)) and the linearity of $H$ in $(y, z)$ (see (2.1)) that

$$\langle \hat{b}(t, X_1, Y_1, \mathbb{P}_{(X_1, Y_1)}), Y_1 - Y_2 \rangle$$
$$+ \langle \sigma(t, X_1, \mathbb{P}_{X_1}), X_1 - X_2 \rangle$$

Moreover, by setting $\tilde{\alpha}_i = \hat{\alpha}(t, \tilde{X}_i, \tilde{Y}_i, \mathbb{P}_{(X_i, Y_i)})$ for all $i = 1, 2$ and using the definition of $\hat{f}$ in (2.18), we can obtain that

$$\mathbb{E}[-\hat{f}(t, \Theta_1, \mathbb{P}_{\Theta_1}) + \hat{f}(t, \Theta_2, \mathbb{P}_{\Theta_2}), X_1 - X_2]$$
$$= -\mathbb{E}[(\dot{\hat{\varphi}} H(t, X_1, \alpha_1, \mathbb{P}_{(X_1, \alpha_1)}), Y_1 - Y_2)$$
$$+ \mathbb{E}[(\dot{\hat{\varphi}} \mu H(t, X_1, \alpha_1, \mathbb{P}_{(X_1, \alpha_1)}, Y_1, Z_1), X_1 - X_2)]$$

where we have also applied Fubini’s theorem and the fact that $\mathbb{P}_{(X_i, Y_i, Z_i, \alpha_i)} = \hat{\mathbb{P}}(\tilde{X}_i, \tilde{Y}_i, \tilde{Z}_i, \tilde{\alpha}_i)$ for all $i = 1, 2$.

Therefore, we can conclude from (5.2) that

$$\mathbb{E}[-\hat{f}(t, X_1, Y_1, \mathbb{P}_{(X_1, Y_1)}))$$
$$+ \mathbb{E}[(\dot{\hat{\varphi}} H(t, X_1, \alpha_1, \mathbb{P}_{(X_1, \alpha_1)}), Y_1 - Y_2)$$
$$+ \mathbb{E}[(\dot{\hat{\varphi}} \mu H(t, X_1, \alpha_1, \mathbb{P}_{(X_1, \alpha_1)}, Y_1, Z_1), X_1 - X_2)]$$

$$\leq -2(\lambda_1 + \lambda_2)\mathbb{E}[|\alpha_1 - \alpha_2|^2],$$

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where we have applied Fubini’s theorem, (2.6), and the definitions of \((\alpha_1, \alpha_2)\) to derive the last estimate. This shows the desired monotonicity property of \(f\) and completes the proof.

**Proof of Lemma 3.3.** For ease of notation, we will write \(b, f, g\) instead of \(\hat{b}, \hat{f}, \hat{g}\). Also, throughout this proof, let \(\delta \xi = \xi - \hat{\xi}, \delta T^b_t = T^b_t - \hat{T}^b_t, g(X_T) = g(X_T, P_{X_T}), g(X_T) = g(X_T, P_{X_T})\) and \(\hat{g}(X_T) = \hat{g}(X_T, P_{\hat{X}_T})\), for each \(t \in [0, T]\) let \(\delta T^b_t = T^b_t - \hat{T}^b_t, \delta T^\sigma_t = T^\sigma_t - \hat{T}^\sigma_t, \delta T^f_t = T^f_t - \hat{T}^f_t, f(\Theta_t) = f(t, X_t, Y_t, P_{\Theta_t}), f(\hat{\Theta}_t) = f(t, \hat{X}_t, \hat{Y}_t, P_{\hat{\Theta}_t})\) and \(\hat{f}(\Theta_t) = \hat{f}(t, \hat{X}_t, \hat{Y}_t, P_{\hat{\Theta}_t})\). Similarly, we introduce the notation \(\sigma(X_t), \sigma(\hat{X}_t), \sigma(\hat{X}_t)\) and \(b(X_t, Y_t), b(\hat{X}_t, \hat{Y}_t), b(\hat{X}_t, \hat{Y}_t)\) for \(t \in [0, T]\).

By applying Itô’s formula to \(\langle Y_t - \hat{Y}_t, X_t - \hat{X}_t \rangle\), we obtain that

\[
E[\langle \lambda_0(g(X_T) - \hat{g}(X_T)) + \delta T^b_T, X_T - \hat{X}_T \rangle] = E[\langle Y_0 - \hat{Y}_0, \delta \xi \rangle]
\]

\[
\leq E\left[\int_0^T \langle \lambda_0(b(X_t, Y_t) - \hat{b}(\hat{X}_t, \hat{Y}_t)) + \delta T^b_t, Y_t - \hat{Y}_t \rangle + \langle \lambda_0(\sigma(X_t) - \sigma(\hat{X}_t)) + \delta T^\sigma_t, Z_t - \hat{Z}_t \rangle + \langle \hat{f}(\hat{\Theta}_t) - \hat{f}(\Theta_t) + \delta T^f_t \rangle, X_t - \hat{X}_t \rangle dt\right] - 2(\lambda_1 + \lambda_2)\lambda_0 \int_0^T \phi_1(t, X_t, Y_t, \hat{X}_t, \hat{Y}_t) dt,
\]

with \(\phi_1(t, X_t, Y_t, \hat{X}_t, \hat{Y}_t) := \|\hat{\alpha}(t, X_t, Y_t, \mathbb{P}_{(X_t, Y_t)}) - \hat{\alpha}(t, \hat{X}_t, \hat{Y}_t, \mathbb{P}_{(\hat{X}_t, \hat{Y}_t)})\|_{L^2}\), which together with Young’s inequality yields for each \(\varepsilon > 0\) that

\[
2(\lambda_1 + \lambda_2)\lambda_0 \int_0^T \phi_1(t, X_t, Y_t, \hat{X}_t, \hat{Y}_t) dt \leq \varepsilon(\|X_T - \hat{X}_T\|_{L^2}^2 + \|Y_0 - \hat{Y}_0\|_{L^2}^2 + \|\Theta - \hat{\Theta}\|_{H^2}^2) + C\varepsilon^{-1}\text{RHS},
\]

where RHS denotes the right-hand side of (3.3).

Now, by (A.3) and the fact that \(\lambda_1 + \lambda_2 > 0\), we have for all \(\varepsilon > 0\),

\[
\lambda_0 \int_0^T \phi_1(t, X_t, Y_t, \hat{X}_t, \hat{Y}_t) dt \leq \varepsilon(\|X - \hat{X}\|_{S^2}^2 + \|Y - \hat{Y}\|_{S^2}^2 + \|Z - \hat{Z}\|_{H^2}^2) + C\varepsilon^{-1}\text{RHS}. \tag{A.4}
\]

Then, by using the Burkholder-Davis-Gundy inequality, the definition of (3.2) and (2.18), Gronwall’s inequality and the fact that \(\lambda_0 \in [0, 1]\), we can deduce that

\[
\|X - \hat{X}\|_{S^2}^2 \leq C\left(\int_0^T \lambda_0 \phi_1(t, X_t, Y_t, \hat{X}_t, \hat{Y}_t) dt + \|\xi - \hat{\xi}\|_{L^2}^2 + \|\lambda_0(\sigma(X) - \sigma(\hat{X})) + \delta T^\sigma_t\|_{H^2}^2\right),
\]

which together with (A.4) yields for all small enough \(\varepsilon > 0\) that

\[
\|X - \hat{X}\|_{S^2}^2 \leq \varepsilon(\|Y - \hat{Y}\|_{S^2}^2 + \|Z - \hat{Z}\|_{H^2}^2) + C\varepsilon^{-1}\text{RHS}.
\]
Moreover, by standard estimates for MV-BSDEs, we can obtain that
\[
\|Y - \bar{Y}\|_{S^2}^2 + \|Z - \bar{Z}\|_{H^2}^2 \\
\leq C\left(\|X - \bar{X}\|_{S^2}^2 + \|\lambda_0(g(\bar{X}_T) - \bar{g}(\bar{X}_T)) + \delta T_T\|_{L^2}^2 + \|\lambda_0(f(\bar{\Theta}) - \bar{f}(\bar{\Theta})) + \delta I_f\|_{H^2}^2\right),
\]
which completes the desired estimate (3.3).

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