Berezin–Toeplitz Quantization on Symplectic Manifolds of Bounded Geometry

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Abstract—The theory of Berezin–Toeplitz quantization on symplectic manifolds of bounded geometry is developed. The quantization space is a suitable eigenspace of the renormalized Bochner operator associated with a neighborhood of zero. It is proved that quantization has a correct semiclassical limit.

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Dedicated to V. P. Maslov on the occasion of his 90th birthday.

1. PRELIMINARIES AND MAIN RESULTS

The quantization problem consists in constructing a quantum system corresponding to a given classical mechanical system. The correspondence is determined by a set of axioms, which was originally proposed by Dirac. The classical system is determined by a phase space, which is a symplectic manifold. The quantum system is determined by a family of operator algebras on a Hilbert space (or of abstract associative algebras with involution), depending on a parameter \( h \in (0, 1] \). In the limit as \( h \to 0 \), which is usually referred to as the semiclassical limit, the quantum system must reduce, in some sense, to the classical system. Various approaches to the quantization problem have been developed, among which we mention Weyl quantization, geometric quantization independently proposed by Kostant [1] and Souriau [2], deformation quantization introduced by Bayen, Flato, Fronsdal, Lichnerowicz, and Sternheimer [3], and asymptotic quantization suggested by Karasev and Maslov [4], [5].

Berezin–Toeplitz quantization is one of the versions of geometric quantization; it was proposed by Berezin in [6] and [7]. At present, there exist several approaches to the theory of geometric quantization and Berezin–Toeplitz quantization for compact symplectic manifolds (see, e.g., the surveys [8]–[11]). For a general compact Kähler manifold, Berezin–Toeplitz quantization was constructed by Bordemann, Meinrenken, and Schlichenmaier [12] on the basis of Boutet de Monvel and Guillemin’s theory of Toeplitz’ structures [13]. Much less is known about the quantization of noncompact symplectic manifolds.

The main purpose of this paper is to develop the theory of Berezin–Toeplitz quantization for a large class noncompact symplectic manifolds. To be more precise, we consider a symplectic manifold \( (X, \omega) \) of dimension \( 2n \) satisfying the following conditions:

(i) \( X \) is endowed with a Riemannian metric \( g \) such that \( (X, g) \) is a Riemannian manifold of bounded geometry and the symplectic form \( \omega \) is uniformly \( C^\infty \)-bounded on \( (X, g) \);

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(ii) \( \omega \) is uniformly nondegenerate on \( X \);

(iii) \((X, \omega)\) is quantizable.

In what follows, we give a more precise statement of these conditions.

Recall that a Riemannian manifold \((X, g)\) is said to be a manifold of bounded geometry if the curvature \( R^{TX} \) of the Levi-Civita connection \( \nabla^{TX} \) of the manifold \((X, g)\) and its covariant derivatives of all orders are uniformly bounded on \( X \) in norm induced by the Riemannian metric \( g \) and, moreover, the injectivity radius \( r_X \) of the manifold \((X, g)\) is positive. In particular, such a Riemannian manifold \((X, g)\) is complete. A differential form on \( X \) is uniformly \( C^\infty \)-bounded if this form and its covariant derivatives of all orders are uniformly bounded on \( X \) in norm induced by the metric \( g \).

For \( x \in X \), let \( B_x : T_x X \to T_x X \) be a skew-symmetric operator such that
\[
\omega_x(u, v) = g(B_x u, v), \quad u, v \in T_x X.
\]
The operator \( B_x^* B_x : T_x X \to T_x X \) is positive and self-adjoint. The form \( \omega \) is uniformly nondegenerate on \( X \) if
\[
\frac{(B_x^* B_x u, u)}{|u|^2} \geq \mu_0^2 > 0, \quad x \in X, \quad u \in T_x X \setminus \{0\}. \tag{1.1}
\]
Finally, the symplectic manifold \((X, \omega)\) is quantizable if there exists a Hermitian line bundle \((L, h^L)\) on \( X \) with Hermitian connection \( \nabla^L \) such that the curvature \( R^L = (\nabla^L)^2 \) of the connection \( \nabla^L \) satisfies the prequantization condition
\[
iR^L = \omega. \tag{1.2}
\]

We will also consider a Hermitian vector bundle \((E, h^E)\) on \( X \) with Hermitian connection
\[
\nabla^E : C^\infty(X, E) \to C^\infty(X, T^*X \otimes E).
\]
We assume that it has bounded geometry, i.e., the curvature \( R^E \) of the connection \( \nabla^E \) and its covariant derivatives of all orders are uniformly bounded on \( X \) in norm induced by \( g \) and \( h^E \). Note that, under the above assumptions, the bundle \( L \) has bounded geometry.

For any \( p \in \mathbb{N} \), we denote by \( L^p \) the \( p \)th tensor power of the bundle \( L \). Let
\[
\nabla^{L^p \otimes E} : C^\infty(X, L^p \otimes E) \to C^\infty(X, T^*X \otimes L^p \otimes E)
\]
be the connection on the bundle \( L^p \otimes E \) induced by the connections \( \nabla^L \) and \( \nabla^E \). We denote by \( \Delta^{L^p \otimes E} \) the induced Bochner Laplacian acting on the space \( C^\infty(X, L^p \otimes E) \) by the rule
\[
\Delta^{L^p \otimes E} = (\nabla^{L^p \otimes E})^* \nabla^{L^p \otimes E}, \tag{1.3}
\]
where \( (\nabla^{L^p \otimes E})^* : C^\infty(X, T^*X \otimes L^p \otimes E) \to C^\infty(X, L^p \otimes E) \) the operator is formally adjoint to \( \nabla^{L^p \otimes E} \). The renormalized Bochner Laplacian is the differential operator \( \Delta_p \) acting on \( C^\infty(X, L^p \otimes E) \) by the rule
\[
\Delta_p = \Delta^{L^p \otimes E} - p\tau, \tag{1.4}
\]
where \( \tau \in C^b_c(X) \) is defined by
\[
\tau(x) = \frac{1}{2} \text{Tr}(B_x^* B_x)^{1/2}, \quad x \in X. \tag{1.5}
\]
The renormalized Bochner Laplacian was introduced by Guillemin and Uribe in [14]. For a Kähler manifold \((X, \omega, g)\), it equals the doubled Kodaira Laplacian
\[
\Box^{L^p \otimes E} = \overline{\partial}^{L^p \otimes E} \overline{\partial}^{L^p \otimes E}.
\]
Since \((X, g)\) is complete, it follows that the renormalized Bochner Laplacian \( \Delta_p \) acting on the space \( C^\infty_c(X, L^p \otimes E) \) of compactly supported smooth sections of the bundle \( L^p \otimes E \) is essentially self-adjoint; see [15, Theorem 2.4]. We denote its unique self-adjoint extension by the same symbol \( \Delta_p \). According
to [15, Theorem 1.1] (see also [16, Lemma 1]), there exists a constant $C_L > 0$ such that, for any $p \in \mathbb{N}$, the spectrum $\sigma(\Delta_p)$ of the operator $\Delta_p$ in the space $L^2(X, L^p \otimes E)$ satisfies the relation

$$\sigma(\Delta_p) \subset [-C_L, C_L] \cup [2p\mu_0 - C_L, +\infty).$$

Let $\mathcal{H}_p \subset L^2(X, L^p \otimes E)$ be the spectral subspace of $\Delta_p$ corresponding to the interval $[-C_L, C_L]$, and let $P_{\mathcal{H}_p}: L^2(X, L^p \otimes E) \to \mathcal{H}_p$ be the corresponding spectral projection of $\Delta_p$ (the generalized Bergman projection). If $X$ is compact, then the spectrum of the operator $\Delta_p$ is discrete and $\mathcal{H}_p$ is the subspace generated by the eigensections of $\Delta_p$ corresponding to the eigenvalues in $[-C_L, C_L]$. The subspace $\mathcal{H}_p$ will be the Hilbert space of states for the quantization which we construct. The idea to use this space as the quantization space is due to Borthwick and Uribe [17]. Here the parameter $p$ is related to the semiclassical parameter $h$ by $h = 1/p$. Thus, $h$ has discrete range of values, and the semiclassical limit as $h \to 0$ corresponds to the limit as $p \to \infty$.

Let $(F, h^F)$ be a Hermitian vector bundle on $X$ with Hermitian connection $\nabla^F$. The Levi-Civita connection $\nabla^{TX}$ and the connection $\nabla^F$ determine the Hermitian connection

$$\nabla^F: C^\infty(X, (T^*X)^{\otimes j} \otimes F) \to C^\infty(X, (T^*X)^{\otimes(j+1)} \otimes F)$$
on the vector bundle $(T^*X)^{\otimes j} \otimes F$ for each $j \in \mathbb{N}$, which allows us to introduce the operator

$$(\nabla^F)^{\ell}: C^\infty(X, F) \to C^\infty(X, (T^*X)^{\otimes\ell} \otimes F), \ \ell \in \mathbb{N}.$$

If $F$ has bounded geometry, then by $C^k_b(X, F)$ we denote the space of sections $u \in C^k(X, F)$ with the property

$$\|u\|_{C^k_b} = \sup_{x \in X, \ell \leq k} |(\nabla^F)^{\ell} u(x)| < \infty,$$

where $| \cdot |_x$ is the norm on $(T^*_x X)^{\otimes\ell} \otimes F_x$ determined by $g$ and $h^F$.

For each $f \in C^\infty_b(X, \text{End}(E))$, the operator of multiplication by $f$ determines a bounded operator on the space $L^2(X, L^p \otimes E)$, which we denote by the same symbol $f$. We define the Berezin–Toeplitz quantization of the section $f$ as the sequence of bounded linear operators

$$T_{f,p} = P_{\mathcal{H}_p} f P_{\mathcal{H}_p}: L^2(X, L^p \otimes E) \to L^2(X, L^p \otimes E), \quad p \in \mathbb{N}.$$

**Theorem 1.** If $f, g \in C^\infty_b(X, \text{End}(E))$, then the product of the Toeplitz operators $\{T_{f,p}\}$ and $\{T_{g,p}\}$ satisfies the relation

$$T_{f,p} T_{g,p} = T_{f,g,p} + \mathcal{O}(p^{-1}), \quad p \to +\infty.$$  

Moreover, if $f, g \in C^\infty_b(X)$, then the commutator of the operators $\{T_{f,p}\}$ and $\{T_{g,p}\}$ satisfies the relation

$$[T_{f,p}, T_{g,p}] = ip^{-1} T_{\{f,g\},p} + \mathcal{O}(p^{-2}), \quad p \to +\infty,$$

where $\{f, g\}$ is the Poisson bracket on $(X, \omega)$.

In this theorem, the estimates $\mathcal{O}(p^{-1})$ and $\mathcal{O}(p^{-2})$ of the remainders can be understood as estimates in the uniform operator norm in the space $L^2(X, L^p \otimes E)$. In fact, complete asymptotic expansions in powers of $p^{-1}$ hold (see Theorem 2 below). Moreover, the estimates of the remainders can be understood as estimates in the uniform operator norm in some weighted Sobolev spaces (see Theorem 2 and its proof).

Thus, we have shown that the constructed quantization of a symplectic manifold $(X, \omega)$ has a correct semiclassical limit. In the special case where $(X, \omega)$ is a Kähler manifold of bounded geometry with complex structure $J$ endowed with the Riemannian metric $g$ induced by $\omega$ and $J$ and where $L$ and $E$ are holomorphic vector bundles, we obtain a generalization of Kähler quantization constructed in [12] to the class of noncompact manifolds under consideration.

In the case where $X$ is a compact symplectic manifold, Berezin–Toeplitz quantization with quantization space $\mathcal{H}_p$ was constructed in [17] for the almost-Kähler case and in [18] and [19] for an
arbitrary Riemannian metric. We refer the reader to [20]–[22] for various approaches to Berezin–Toeplitz quantization.

For the simplest noncompact symplectic manifold, which is the complex n-space \( \mathbb{C}^n \) with the standard symplectic form and Euclidean metric, Berezin–Toeplitz quantization is determined by Toeplitz operators on the Fock space. It was considered by Berezin in [6] and [7] and, subsequently, in [23]–[25] (see also references therein). In particular, in the paper [25], Berezin–Toeplitz quantization was constructed for functions in \( \mathcal{C} \) (see also references therein). In particular, in [23], an example was given which demonstrated that Theorem 1 is not generally true for rapidly oscillating bounded functions.

The theory of Berezin–Toeplitz quantization was developed for diverse types of domains in \( \mathbb{C}^n \) (see, e.g., [26]–[30] and references therein). In [31], Berezin–Toeplitz quantization was constructed in the general situation of symmetric spaces of compact or noncompact type, both in the real and the complex Hermitian case (see also references in [31]). In [22, Sec. 5], Ma and Marinescu considered a particular class of complete Hermitian manifolds and constructed a Berezin–Toeplitz quantization of the algebra \( C^\infty_{\text{cont}}(X) \) of smooth functions on \( X \) constant outside a compact set. The author is not aware of any publications concerned with Berezin–Toeplitz quantization for a general enough class of noncompact symplectic manifolds.

In this paper, we follow the approach to Berezin–Toeplitz quantization which was proposed by Ma and Marinescu [32], [33] in the case where the quantization space is the kernel of the spin\(^c\) Dirac operator. It is based on employing asymptotic expansions of the Bergman kernel obtained in [34]. In the case where the quantization space is \( \mathcal{H}_\mu \), similar asymptotic expansions of generalized Bergman kernels were obtained by the author in [19]. This has enabled the author to develop the theory of Berezin–Toeplitz quantization, following the approach of Ma and Marinescu (see also [18], where weaker asymptotic expansions of generalized Bergman kernels were used). Asymptotic expansions of generalized Bergman kernels on manifolds of bounded geometry were obtained in [15] (see also [35]). The main contribution of the present paper is the adaptation of the calculus of Toeplitz operators, which was developed in [18], [19], [32], and [33], to the noncompact case. To this end, we use the technique of weighted estimates developed in [19], [35], and [15].

This paper is organized as follows. In Sec. 2, we state the main results concerning the calculus of Toeplitz operators, the construction of the algebra of Toeplitz operators and the characterization of Toeplitz operators in terms of their Schwartz kernels. Theorem 1 readily follows from these results. Section 3 is devoted to the proof of the theorem on characterization.

2. CALCULUS OF TOEPLITZ OPERATORS

2.1. Algebra of Toeplitz Operators

In this section, we state the main results concerning the calculus of Toeplitz operators in the situation under consideration. First, we introduce some weighted \( L^2 \)-spaces. As usual, \( L^2(X, L^p \otimes E) \) denotes the Hilbert space of \( L^2 \)-sections of the bundle \( L^p \otimes E \) with \( L^2 \)-norm defined by

\[
\|u\|^2 = \int_X |u(x)|^2 \, dv_X(x), \quad u \in L^2(X, L^p \otimes E),
\]

where \( dv_X \) denotes the Riemannian volume form on the manifold \( (X, g) \).

Let \( d(x, y) \) be the geodesic distance on \( X \). We introduce the family \( \{d_y : y \in X\} \) of Lipschitz functions on \( X \) defined by

\[
d_y(x) = d(x, y), \quad x \in X,
\]

and the family of weighted \( L^2 \)-spaces

\[
L^2_{\alpha, y}(X, L^p \otimes E) = \{ u \in C^{-\infty}(X, L^p \otimes E) : e^{\alpha d_y} u \in L^2(X, L^p \otimes E) \}, \quad \alpha \in \mathbb{R}, \quad y \in X,
\]

with Hilbert norm

\[
\|u\|_{p, \alpha, y} = \|e^{\alpha d_y} u\|, \quad u \in L^2_{\alpha, y}(X, L^p \otimes E).
\]

Note that the space \( L^2_{\alpha, y}(X, L^p \otimes E) \) does not depend on \( y \) as a topological vector space.

The following definition is a generalization of Definition 4.1 of [33] to the case under consideration.
**Definition 1.** A Toeplitz operator is a sequence of bounded linear operators
\[ T_p : L^2(X, L^p \otimes E) \to L^2(X, L^p \otimes E), \quad p \in \mathbb{N}, \]
satisfying the following conditions:

(i) for any \( p \in \mathbb{N} \), we have
\[ T_p = P_{\mathcal{H}_p} T_p P_{\mathcal{H}_p}; \]

(ii) there exists a sequence \( g_l \in C_b^\infty(X, \text{End}(E)), \ l = 0, 1, 2, \ldots, \) for which
\[ T_p \sim P_{\mathcal{H}_p} \left( \sum_{l=0}^{\infty} p^{-l} g_l \right) P_{\mathcal{H}_p} \]
in the sense that, given any \( K \in \mathbb{Z}_+ \), there exists a \( \mu > 0 \) and a \( C > 0 \) such that, for any \( p \in \mathbb{N} \), and \( \alpha \in \mathbb{R} \) with \( |\alpha| < \mu \sqrt{p} \), and any \( y \in X \), we have
\[ \left\| T_p - P_{\mathcal{H}_p} \left( \sum_{l=0}^{K} p^{-l} g_l \right) P_{\mathcal{H}_p} : L^2_{\alpha,y}(X, L^p \otimes E) \to L^2_{\alpha,y}(X, L^p \otimes E) \right\| \leq C p^{-K-1}; \]
\[ \text{here } \| \cdot \| \text{ denotes the uniform operator norm.} \]

The following theorem asserts the existence of a complete asymptotic expansion for the composition of Toeplitz operators \( T_{f,p} \) and \( T_{g,p} \). It is a generalization of Theorem 1.1 of [33] to the case under consideration.

**Theorem 2.** Let \( f, g \in C_b^\infty(X, \text{End}(E)) \). Then the composition of the Toeplitz operators \( T_{f,p} \) and \( T_{g,p} \) is a Toeplitz operator in sense of Definition 1. Moreover, it admits the asymptotic expansion
\[ T_{f,p} T_{g,p} \sim \sum_{r=0}^{\infty} p^{-r} T_{C_r(f,g),p} \]
with \( C_r(f,g) \in C_b^\infty(X, \text{End}(E)) \). where the \( C_r \) are bidifferential operators. In particular, \( C_0(f,g) = fg \), and for \( f, g \in C_b^\infty(X) \),
\[ C_1(f,g) - C_1(g,f) = i\{f,g\}. \]

Clearly, Theorem 1 is an immediate corollary of this theorem. Theorem 2 also shows that the set of Toeplitz operators is an algebra.

An essential role in the proof of Theorem 2 is played by a characterization of Toeplitz operators in terms of the asymptotic expansions of their Schwartz kernels, which is given in the next subsection. For compact manifolds, a similar characterization was given in [33, Theorem 4.9] for the Toeplitz operators associated with the spin\(^c\) Dirac operator. It was extended to the Toeplitz operators associated with the renormalized Bochner Laplacian in [19, Theorem 6.5] (see also [18, Theorem 4.1]). It should be mentioned that, in the Kähler case, the first characterization of Toeplitz operators in terms of their Schwartz kernels was obtained by Charles in [36].

### 2.2. Criterion for Toeplitz Operators

We use normal coordinates in a neighborhood of an arbitrary point \( x_0 \in X \). By \( B^X(x_0, r) \) and \( B^{T_{x_0}X}(0, r) \) we denote the open balls of radius \( r \) centered at \( x_0 \) in \( X \) and in \( T_{x_0}X \), respectively. We identify \( B^{T_{x_0}X}(0, r_X) \) with \( B^X(x_0, r_X) \) by means of the exponential map \( \exp_{x_0}^X : T_{x_0}X \to X \). We also choose trivializations of the bundles \( L \) and \( E \) over \( B^X(x_0, r_X) \), identifying their fibers \( L_Z \) and \( E_Z \) at a point
\[ Z \in B^{T_{x_0}X}(0, r_X) \cong B^X(x_0, r_X). \]
with the spaces $L_{x_0}$ and $E_{x_0}$ by means of parallel transport with respect to the connections $\nabla^L$ and $\nabla^E$ along the curve

$$\gamma_Z: [0, 1] \ni u \mapsto \exp_{x_0}^X(uZ).$$

Let $dv_{T_{x_0}X}$ be the Riemannian volume form of the tangent space $T_{x_0}X$ endowed with the Riemannian metric $g_{T_{x_0}X}$ determined by the metric $g$. Consider the smooth positive function $\kappa_{x_0}$ on $B^{T_{x_0}X}(0, r_X) \cong B^X(x_0, r_X)$ defined by

$$dv_X(Z) = \kappa_{x_0}(Z) dv_{T_{x_0}X}(Z), \quad Z \in B^{T_{x_0}X}(0, r_X). \quad (2.3)$$

Let $\mathcal{P}_{x_0} \in C^\infty(T_{x_0}X \times T_{x_0}X)$ be a Bergman kernel in the space $T_{x_0}X$ (see [32], [33]). If $\{e_j : j = 1, \ldots, 2n\}$ is an orthonormal basis in $T_{x_0}X$ with the properties

$$B_{x_0}e_{2k-1} = a_k e_{2k}, \quad B_{x_0}e_{2k} = -a_k e_{2k-1}, \quad k = 1, \ldots, n,$

then $\mathcal{P}_{x_0}$ is given by

$$\mathcal{P}_{x_0}(Z, Z') = \frac{1}{(2\pi)^n} \prod_{j=1}^n a_j \exp \left( -\frac{1}{4} \sum_{k=1}^n a_k (|z_k|^2 + |z_k'|^2 - 2z_k z_k') \right), \quad Z, Z' \in \mathbb{R}^{2n} \cong T_{x_0}X,$

where the complex coordinates $z_k = Z_{2k-1} + iZ_{2k}, k = 1, \ldots, n$, are used.

Consider a sequence of linear operators

$$\Xi_p: L^2(X, L^p \otimes E) \to L^2(X, L^p \otimes E)$$

with smooth Schwartz kernels. We denote the projections of $X \times X$ onto the first and the second factor by $\pi_1$ and $\pi_2$, respectively. The Schwartz kernel of the operator $\Xi_p$ with respect to the Riemannian volume form $dv_X$ is a smooth section

$$\Xi_p(\cdot, \cdot) \in C^\infty(X \times X, \pi_1^*(L^p \otimes E) \otimes \pi_2^*(L^p \otimes E)^*).$$

Consider the fiber product

$$TX \times_X TX = \{(Z, Z') \in T_{x_0}X \times T_{x_0}X : x_0 \in X\}$$

endowed with the natural projection $\pi: TX \times_X TX \to X$ defined by $\pi(Z, Z') = x_0$. The kernel $\Xi_p(x, x')$ induces the smooth section $\Xi_{p,x_0}(Z, Z')$ of the vector bundle $\pi^*(\text{End}(E))$ on $TX \times_X TX$ defined for all $x_0 \in X$ and all $Z, Z' \in T_{x_0}X$ with $|Z|, |Z'| < r_X$, by the formula

$$\Xi_{p,x_0}(Z, Z') = \Xi_p(\exp_{x_0}^X(Z), \exp_{x_0}^X(Z')).$$

**Definition 2** [32], [33]. We say that the following relation holds:

$$p^{-n} \Xi_{p,x_0}(Z, Z') \preceq \sum_{r=0}^k (Q_{r,x_0} \mathcal{P}_{x_0})(\sqrt{p} Z, \sqrt{p} Z') p^{-r/2} + O(p^{-(k+1)/2}),$$

where $Q_{r,x_0} \in \text{End}(E_{x_0})[Z, Z']$ and $0 \leq r \leq k$, depend on $x_0 \in X$ smoothly and $C^\infty$-boundedly if there exists an $\varepsilon' \in (0, r_X]$, a $C_0 > 0$, and a $c_0 > 0$ satisfying the following conditions: given any $l \in \mathbb{N}$, there are a $C > 0$ and an $M > 0$ such that, for any $x_0 \in X$, $p \geq 1$, and $Z, Z' \in T_{x_0}X$ with $|Z|, |Z'| < \varepsilon'$, we have

$$p^{-n} \Xi_{p,x_0}(Z, Z') \preceq \sum_{r=0}^k (Q_{r,x_0} \mathcal{P}_{x_0})(\sqrt{p} Z, \sqrt{p} Z') p^{-r/2} \mid_{C_b^l(X)}$$

$$\leq C p^{-(k+1)/2}(1 + \sqrt{p}|Z| + \sqrt{p}|Z'|)^M e^{-C_0 \sqrt{p} |Z-Z'|} + O(e^{-c_0 \sqrt{p}}). \quad (2.5)$$

Here $C_b^l(X)$ denotes the $C_b^l$-norm with respect to the parameter $x_0 \in X$. 

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The following theorem gives a characterization of Toeplitz operators in terms of the asymptotic expansions of their Schwartz kernels.

**Theorem 3.** A family \( \{ T_p : L^2(X, L^p \otimes E) \to L^2(X, L^p \otimes E) \} \) of bounded linear operators is a Toeplitz operator in the sense of Definition 1 if and only if it satisfies the following three conditions:

(i) for any \( p \in \mathbb{N} \),
\[
T_p = P_{H_p} T_p P_{H_p};
\]

(ii) \( T_p \) has smooth Schwartz kernel \( T_p(x, x') \) with respect to \( dv_X \), and there exists a \( \mu > 0 \) such that, for any \( \epsilon_0 > 0 \),
\[
|T_p(x, x')| \leq C e^{-\mu \sqrt{p} d(x, x')}, \quad p \in \mathbb{N}, \quad x, x' \in X, \quad d(x, x') > \epsilon_0,
\]
where \( C > 0 \) depends only on \( \epsilon_0 \);

(iii) there exists a family of polynomials \( Q_{r, x_0} \in \text{End}(E_{x_0})[Z, Z'] \) smoothly and \( C^\infty \)-boundedly depending on \( x_0 \) of the same parity as \( r \in \mathbb{Z}_+ \) and an \( \epsilon' \in (0, r X/4) \) such that, for any \( k \in \mathbb{N} \), \( x_0 \in X \), and \( Z, Z' \in T_{x_0} X \) with \( |Z|, |Z'| < \epsilon' \),
\[
p^{-r} T_{p, x_0}(Z, Z') \approx \sum_{r=0}^{k} (Q_{r, x_0} P_{x_0})(\sqrt{p} Z, \sqrt{p} Z') p^{-r/2} + O(p^{-(k+1)/2}).
\]

The proof of Theorem 3 is given in the next section. After Theorem 3 is proved, it will be easy to complete the proof of Theorem 2 by using an argument in [22, Sec. 4.3], and we will omit its proof.

### 3. PROOF OF THE CHARACTERIZATION OF TOEPLITZ OPERATORS

This section is devoted to the proof of Theorem 3. In Sec. 3.1, we prove necessary conditions for a sequence \( \{ T_p \} \) to be a Toeplitz operator (the “only if” part), and in Sec. 3.2, sufficient conditions (the “if” part).

#### 3.1. Kernels of Toeplitz Operators

In this subsection, we assume that a sequence \( \{ T_p \} \) is a Toeplitz operator in the sense of Definition 1 and prove that it satisfies conditions (i)–(iii) in Theorem 3. First, we introduce appropriate Sobolev spaces (see [15] for more details).

For each integer \( m \geq 0 \), we introduce a norm \( \| \cdot \|_{p, m} \) on the space \( C^\infty_c(X, L^p \otimes E) \) by the formula
\[
\| u \|_{p, m}^2 = \sum_{\ell=0}^{m} \int_X \left( \frac{1}{\sqrt{p}} \nabla^{L^p \otimes E} \right)^\ell u(x)^2 \, dx(X), \quad u \in H^m(X, L^p \otimes E). \tag{3.1}
\]

The completion of \( C^\infty_c(X, L^p \otimes E) \) in norm \( \| \cdot \|_{p, m} \) is the Sobolev space \( H^m(X, L^p \otimes E) \) of order \( m \). For each integer \( m < 0 \), we endow the Sobolev space \( H^m(X, L^p \otimes E) \) with the norm determined by duality.

To construct weighted Sobolev spaces, we use the smoothed distance function constructed in [37, Proposition 4.1] (see also [15, Sec. 3.1]). This is a function \( \tilde{d}_p \in C^\infty(X \times X), \ p \in \mathbb{N} \), satisfying the following conditions:

(1) the following estimate holds:
\[
|\tilde{d}_p(x, y) - d(x, y)| < \frac{1}{\sqrt{p}}, \quad x, y \in X, \quad p \in \mathbb{N}; \tag{3.2}
\]
(2) for any \( k > 0 \), there exists a \( c_k > 0 \) such that
\[
\left( \frac{1}{\sqrt{p}} \right)^{k-1} |\nabla_x^k \tilde{d}_p(x,y)| < c_k, \quad x, y \in X, \quad p \in \mathbb{N}.
\] (3.3)

Consider the family \( \{ \tilde{d}_{p,y} : y \in X \} \) of functions on \( X \) defined by
\[
\tilde{d}_{p,y}(x) = \tilde{d}_p(x,y), \quad x \in X.
\] (3.4)

Let \( H^m_{\alpha,y}(X, L^p \otimes E) \) be the weighted Sobolev space
\[
H^m_{\alpha,y}(X, L^p \otimes E) = \{ u \in C^{-\infty}(X, L^p \otimes E) : e^{\alpha \tilde{d}_p,y} u \in H^m(X, L^p \otimes E) \}
\]
endowed with the Hilbert norm
\[
\| u \|_{p,m,\alpha,y} = \| e^{\alpha \tilde{d}_p,y} u \|_{p,m}.
\]

According to (3.2), we have \( H^0_{\alpha,y}(X, L^p \otimes E) = L^2_{\alpha,y}(X, L^p \otimes E) \) for any \( y \in X \) and \( \alpha \in \mathbb{R} \), and the norm \( \| u \|_{p,0,\alpha,y} \) is equivalent to the \( L^2 \)-norm \( \| u \|_{p,\alpha,y} \) uniformly in \( y \in X \) and \( \alpha \in \mathbb{R} \) with \( |\alpha| < c \sqrt{p} \) for any \( c > 0 \).

Results of [15] (see also [35]) imply the existence of a \( \mu_p > 0 \) and a \( p_0 \in \mathbb{N} \) such that, for any \( p > p_0 \), \( \alpha \in \mathbb{R} \) with \( |\alpha| < \mu_p \sqrt{p} \), \( m_1, m_2 \in \mathbb{N} \), and \( y \in X \), the operator \( P_{H^p} \) maps \( H^m_{\alpha,y}(X, L^p \otimes E) \) to \( H^m_{\alpha,y}(X, L^p \otimes E) \) with the norm estimates
\[
\| P_{H^p} : H^m_{\alpha,y}(X, L^p \otimes E) \to H^m_{\alpha,y}(X, L^p \otimes E) \| \leq C_{m_1,m_2},
\] (3.5)

where \( C_{m_1,m_2} > 0 \) does not depend on \( p, \alpha \), and \( y \).

**Proposition 1.** Let \( \{ K_p : p \in \mathbb{N} \} \) be a sequence of bounded linear operators on \( L^2(X, L^p \otimes E) \) satisfying the condition
\[
K_p = P_{H^p} K_p P_{H^p}, \quad p \in \mathbb{N}.
\] (3.6)

Suppose that there exists a \( \mu > 0 \) and a \( C > 0 \) such that, for any \( p \in \mathbb{N} \), \( \alpha \in \mathbb{R} \) with \( |\alpha| < \mu \sqrt{p} \), and \( y \in X \), \( K_p \) determines a bounded operator on \( L^2_{\alpha,y}(X, L^p \otimes E) \) with norm estimates
\[
\| K_p : L^2_{\alpha,y}(X, L^p \otimes E) \to L^2_{\alpha,y}(X, L^p \otimes E) \| \leq C,
\] (3.7)

where \( C > 0 \) does not depend on \( p, \alpha \), and \( y \). Then each operator \( K_p \) has smooth Schwartz kernel \( K_p(x,x') \), and there exists a \( \mu_0 > 0 \) such that, for any \( k \in \mathbb{N} \),
\[
|K_p(x,x')|_{C^k} \leq C_{kp}^{n+k/2} e^{-\mu_0 \sqrt{p} d(x,x')}
\] (3.8)

for all \( p > p_0 \) and \( x, x' \in X \) with a constant \( C_k > 0 \) not depending on \( p, x \), and \( x' \).

Here \( |K_p(x,x')|_{C^k} \) denotes the pointwise \( C^k \)-prenorm of the section
\[
K_p \in C^\infty(X \times X, \pi_1^*(L^p \otimes E) \otimes \pi_2^*(L^p \otimes E)^*)
\]
at the point \( (x,x') \in X \times X \), which is determined by \( h^L, h^E, g \), and the connections \( \nabla^{L^p \otimes E} \) and \( \nabla^{TX} \).

**Proof.** Using (3.5), (3.6), and (3.7), we can show that, for any \( p \in \mathbb{N} \), \( p > p_0 \), \( |\alpha| < \mu \sqrt{p} \) with \( \mu_1 := \min(\mu, \mu_p) \) and \( m_1, m_2 \in \mathbb{N} \), the operator \( K_p \) maps \( H^m_{\alpha,y}(X, L^p \otimes E) \) to \( H^m_{\alpha,y}(X, L^p \otimes E) \) with the norm estimates
\[
\| K_p : H^m_{\alpha,y}(X, L^p \otimes E) \to H^m_{\alpha,y}(X, L^p \otimes E) \| \leq C_{m_1,m_2},
\] (3.9)

where \( C_{m_1,m_2} > 0 \) does not depend on \( p, \alpha \), and \( y \).

Now, using a modified form of the Sobolev embedding theorem (as in the proof of Theorem 3.6 in [15]), we can derive estimate (3.8) with any \( \mu_0 \in (0, \mu_1) \). \( \square \)
According to Proposition 1, the sequence $\{T_p\}$ satisfies condition (ii) in Theorem 3 for some $\mu_0 > 0$. We also see that, for any $K \in \mathbb{N}$, the Schwartz kernel of the remainder

$$R_{K,p} = T_p - P_{\mathcal{H}_p} \left( \sum_{l=0}^{K} p^{-l} g_l \right) P_{\mathcal{H}_p}, \quad p \in \mathbb{N},$$

in the asymptotic expansion satisfies the estimate

$$|R_{K,p}(x, x')|_{C_b^k} \leq C_k p^{n-K-1+k/2} e^{-\mu_0 \sqrt{d(x, x')}} \quad p \in \mathbb{N}, \ x, x' \in X,$$

for any $k \geq 0$. In particular, for the local Schwartz kernel $R_{K,p,x_0}(Z, Z')$, we obtain the estimate

$$|p^{-n}R_{K,p,x_0}(Z, Z')|_{C_b^k(X)} \leq C_k p^{-M} e^{-\mu_0 \sqrt{d(Z, Z')}}$$

for any $x_0 \in X$ and $Z, Z' \in T_{x_0}X$ with $|Z|, |Z'| < \varepsilon'$, where the constant $M = K + 1 - k/2$ can be made arbitrarily large by choosing a suitable constant $K$ depending on $k$. This reduces our considerations to an operator of the form $T_{f,p}, f \in C_b^\infty(X, \text{End}(E))$.

The Schwartz kernel of the operator $T_{f,p}$ is given by

$$T_{f,p}(x, x') = \int_X P_p(x, x'') f(x'') P_p(x'', x') \, dv_X(x''),$$

where $P_p(x, x')$ is the Schwartz kernel of the operator $P_{\mathcal{H}_p}$ (the generalized Bergman kernel).

According to [15, Theorem 1.1] (see also Proposition 1), $P_p(x, x')$ satisfies condition (ii) in Theorem 3, and according to [15, Theorem 4.3], the asymptotic expansion

$$p^{-n}P_{p,x_0}(Z, Z') \approx \sum_{r=0}^{k} (F_{0,r,x_0} P_{x_0})(\sqrt{p} Z, \sqrt{p} Z') p^{-r/2} + O(p^{-(k+1)/2})$$

holds for all $k \in \mathbb{N}$. Using these facts and proceeding as in [22, Sec. 4.1], we can show that the Schwartz kernel of the operator $T_{f,p}$ satisfies condition (iii) in Theorem 3. This completes the proof of the proposition.

\[\Box\]

### 3.2. Characterization of Toeplitz Operators

In this subsection, we suppose that a sequence $\{T_p\}$ satisfies conditions (i)–(iii) in Theorem 3 and prove that it is a Toeplitz operator in the sense of Definition 1. This assertion is a generalization of Theorem 4.9 of [22], and we will follow the outline of the proof of that theorem.

Without loss of generality, we can assume that the operator $T_p$ is self-adjoint. We begin with analyzing the complete off-diagonal asymptotic expansion of the Schwartz kernel of $T_p$ given by condition (iii). It is locally and, therefore, is of virtually the same form as in the compact case. According to [22, Proposition 4.11] (see also [38] for a different proof), for the leading coefficient $Q_{0,x_0}(Z, Z')$ in this expansion, we have

$$Q_{0,x_0}(Z, Z') = Q_{0,x_0}(Z, Z') = Q_{x_0}$$

for any $x_0 \in X$ and $Z, Z' \in T_{x_0}X$ with some $Q_{x_0} \in \text{End}(E_{x_0})$. Define a section $g_0 \in C^\infty(X, \text{End}(E))$ by setting

$$g_0(x_0) = Q_{x_0}, \quad x_0 \in X.$$  

Clearly, since the family $Q_{0,x_0}$ is $C^\infty$-bounded in $x_0$, we have $g_0 \in C_b^\infty(X, \text{End}(E))$.

Using Proposition 4.17 of [22], we obtain

$$p^{-n}(T_p - T_{g_0,p})_{x_0}(Z, Z') \approx O(p^{-1})$$

for any $x_0 \in X$ and $Z, Z' \in T_{x_0}X$ with $|Z|, |Z'| < \varepsilon'$.

Now we must turn these pointwise estimates of the Schwartz kernel into operator estimate in weighted $L^2$-spaces. We use the following fact.
Proposition 2. Let \( \{K_p: C_c^\infty(X, L^p \otimes E) \to C_c^\infty(X, L^p \otimes E), p \in \mathbb{N}\} \) be a sequence of linear operators with smooth Schwartz kernels \( K_p(x, x') \). Suppose that there exists a \( \mu > 0 \) such that, for any \( \epsilon_0 > 0 \),

\[
|K_p(x, x')| \leq Ce^{-\mu \sqrt{p}d(x, x')}, \quad p \in \mathbb{N}, \ x, x' \in X, \ d(x, x') > \epsilon_0, \tag{3.10}
\]

where \( C > 0 \) does not depend on \( p, x, \) and \( x' \). Suppose also that, for any \( x_0 \in X \) and \( Z, Z' \in T_{x_0} X \) with \( |Z|, |Z'| < \epsilon' \),

\[
p^{-n}K_{p,x_0}(Z, Z') \equiv O(p^{-1}). \tag{3.11}
\]

Then there exists a \( \mu_1 > 0 \) and a \( p_0 \in \mathbb{N} \) such that, for any \( p > p_0, \ \alpha \in \mathbb{R} \) with \( |\alpha| < \mu_1 \sqrt{p} \), and \( y \in X \), the operator \( K_p \) determines a bounded linear operator on the space \( L^2_{\alpha,y}(X, L^p \otimes E) \) with norm estimate

\[
\|K_p: L^2_{\alpha,y}(X, L^p \otimes E) \to L^2_{\alpha,y}(X, L^p \otimes E)\| < C, \tag{3.12}
\]

where \( C > 0 \) does not depend on \( p, \alpha \), and \( y \).

Proof. For any \( \alpha \in \mathbb{R} \) and \( y \in X \), we have the unitary isomorphism \( L^2_{\alpha,y}(X, L^p \otimes E) \to L^2(X, L^p \otimes E) \) acting as multiplication by \( e^{\alpha d_y} \). Therefore, it suffices to prove that, for any \( p \in \mathbb{N}, \ \alpha \in \mathbb{R} \) with \( |\alpha| < \mu \sqrt{p} \), and \( y \in X \), the operator \( e^{\alpha d_y} K_p e^{-\alpha d_y} \) determines a bounded linear operator on \( L^2(X, L^p \otimes E) \) and

\[
\|e^{\alpha d_y} K_p e^{-\alpha d_y}: L^2(X, L^p \otimes E) \to L^2(X, L^p \otimes E)\| \leq C, \tag{3.13}
\]

where \( C > 0 \) does not depend on \( p, \alpha \), and \( y \).

The Schwartz kernel \( K_{p,\alpha,y}(x, x') \) of the operator \( e^{\alpha d_y} K_p e^{-\alpha d_y} \) is given by

\[
K_{p,\alpha,y}(x, x') = e^{\alpha d_y} K_p(x, x') e^{-\alpha d_y}, \quad x, x' \in X.
\]

According to Schur’s lemma, we have

\[
\|e^{\alpha d_y} K_p e^{-\alpha d_y}: L^2(X, L^p \otimes E) \to L^2(X, L^p \otimes E)\| \\
\leq \max \left( \sup_{x \in X} \int_X |K_{p,\alpha,y}(x, x')| \, dv_X(x'), \ \sup_{x' \in X} \int_X |K_{p,\alpha,y}(x, x')| \, dv_X(x) \right).
\]

Let us estimate the first integral on the right-hand side of the last formula. For the second integral, a similar estimate can be obtained. It follows from (3.10) that, for any \( p \in \mathbb{N}, \ x, x' \in X \) with \( d(x, x') > \epsilon_0 \), \( y \in X \), and \( \alpha \in \mathbb{R} \), we have

\[
|K_{p,\alpha,y}(x, x')| \leq Ce^{\alpha d_y} e^{-\mu \sqrt{p}d(x, x')} e^{-\alpha d_y} \leq Ce^{\alpha d_y} e^{-\alpha d_y} e^{-\mu \sqrt{p}d(x, x')} \leq C e^{\alpha d_y} e^{-\alpha d_y} e^{-\mu \sqrt{p}d(x, x')}
\]

if \( \alpha > 0 \) and

\[
|K_{p,\alpha,y}(x, x')| \leq Ce^{\alpha d_y} e^{-\alpha d_y} e^{-\mu \sqrt{p}d(x, x')} \leq C e^{\alpha d_y} e^{-\alpha d_y} e^{-\mu \sqrt{p}d(x, x')}
\]

if \( \alpha < 0 \). Therefore, for any \( \mu_1 \in (0, \mu) \), there exists a \( p_0 \in \mathbb{N} \) such that, for any \( p > p_0, y \in X \), and \( \alpha \in \mathbb{R} \) with \( |\alpha| < \mu_1 \sqrt{p} \), we have

\[
\sup_{x \in X} \int_{d(x, x') > \epsilon_0} |K_{p,\alpha,y}(x, x')| \, dv_X(x') < C \sup_{x \in X} \int_{d(x, x') > \epsilon_0} e^{-\mu_1 \sqrt{p}d(x, x')} \, dv_X(x') < Ce^{-\epsilon_0 \sqrt{p}},
\]

where \( C > 0 \) and \( \epsilon_0 > 0 \) do not depend on \( p, y, \) and \( \alpha \).

On the other hand, using (3.11), (2.5), and the bounded geometry assumption, we can easily prove the existence of a \( \mu_2 > 0 \) such that, for any \( p > p_0, y \in X \), and \( \alpha \in \mathbb{R} \) with \( |\alpha| < \mu_2 \sqrt{p} \), we have

\[
\sup_{x \in X} \int_{d(x, x') < \epsilon'} |K_{p,\alpha,y}(x, x')| \, dv_X(x') < Ce^{-\epsilon_0 \sqrt{p}},
\]
This completes the proof of the proposition. □

It follows from Proposition 2 that, for any $\mu_1 \in (0, \mu)$, there exists a $p_0 \in \mathbb{N}$ such that, for any $p > p_0$ and $\alpha, |\alpha| < \mu_1 \sqrt{p}$, the operator $p(T_p - P_{\mathcal{H}_p} g_0 P_{\mathcal{H}_p})$ determines a bounded linear operator in $L^2_{\alpha}(X, L^p \otimes E)$; moreover, for some $C > 0$, we have

$$\sup_{y \in X} \|p(T_p - P_{\mathcal{H}_p} g_0 P_{\mathcal{H}_p}) : L^2_{\alpha,y}(X, L^p \otimes E) \to L^2_{\alpha,y}(X, L^p \otimes E)\| < C$$

for any $p > p_0$ and $\alpha \in \mathbb{R}$ with $|\alpha| < \mu_1 \sqrt{p}$. Thus, the operator $T_p$ satisfies condition (ii) in Definition 1 for $K = 0$.

Consider the sequence of operators $p(T_p - P_{\mathcal{H}_p} g_0 P_{\mathcal{H}_p})$. It satisfies conditions (i)—(iii) in Theorem 3 with $\mu_1$. Applying the above considerations to this sequence, we see that there exists a $g_1 \in C^\infty(X, \text{End}(E))$ such that, for any $\mu_2 \in (0, \mu_1)$, $p > p_0$, and $\alpha, |\alpha| < \mu_2 \sqrt{p}$, the operator $p^2(T_p - P_{\mathcal{H}_p} g_0 P_{\mathcal{H}_p}) - P_{\mathcal{H}_p} g_0 P_{\mathcal{H}_p}$ determines a bounded linear operator on $L^2_{\alpha}(X, L^p \otimes E)$; moreover, for some $C > 0$, we have

$$\sup_{y \in X} \|p(T_p - P_{\mathcal{H}_p} g_0 P_{\mathcal{H}_p}) : L^2_{\alpha,y}(X, L^p \otimes E) \to L^2_{\alpha,y}(X, L^p \otimes E)\| < C$$

for any $p > p_0$ and $\alpha \in \mathbb{R}$ with $|\alpha| < \mu_2 \sqrt{p}$. Proceeding by induction, we conclude that $T_p$ is a Toeplitz operator in the sense of Definition 1.

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**REFERENCES**

1. B. Kostant, “Quantization and unitary representations,” Lect. Notes in Math. **170**, 87–208 (1970).
2. J.-M. Souriau, *Structure des systèmes dynamiques* (Dunod, Paris, 1970).
3. F. Bayen, M. Flato, C. Fronsdal, A. Lichnerowicz, and D. Sternheimer, “Deformation theory and quantization,” Ann. Phys. **111**, 61–151 (1978).
4. M. V. Karasev and V. P. Maslov, “Asymptotic and geometric quantization,” Russian Math. Surveys **39** (6), 133–205 (1984).
5. M. V. Karasev and V. P. Maslov, *Nonlinear Poisson Brackets. Geometry and Quantization* (Nauka, Moscow, 1991) [in Russian].
6. F. A. Berezin, “Quantization,” Izv. Math. **8** (5), 1109–1165 (1974).
7. F. A. Berezin, “General concept of quantization,” Comm. Math. Phys. **40**, 153–174 (1975).
8. S. T. Ali and M. Engliš, “Quantization methods: a guide for physicists and analysts,” Rev. Math. Phys. **17**, 391–490 (2005).
9. M. Engliš, “An excursion into Berezin–Toeplitz quantization and related topics,” in *Quantization, PDEs, and geometry, Oper. Theory Adv. Appl.* (Birkhäuser, Cham, 2016), Vol. 251, pp. 69–115.
10. X. Ma, “Geometric quantization on Kähler and symplectic manifolds,” in *Proceedings of the International Congress of Mathematicians, II* (Hindustan Book Agency, New Delhi, 2010), pp. 785–810.
11. M. Schlichenmaier, “Berezin–Toeplitz quantization for compact Kähler manifolds. A review of results,” Adv. Math. Phys., Art. ID 927280 (2010).
12. M. Bordemann, E. Meinrenken, and M. Schlichenmaier, “Toeplitz quantization of Kähler manifolds and $gl(n), n \to \infty$ limits,” Comm. Math. Phys. **165**, 281–296 (1994).
13. L. Boutet de Monvel and V. Guillemin, *The spectral theory of Toeplitz operators*, in *Ann. Math. Stud.* (Princeton Univ. Press, Princeton, NJ, 1981), Vol. 99.
14. V. Guillemin and A. Uribe, “The Laplace operator on the $n$th tensor power of a line bundle: eigenvalues which are uniformly bounded in $n$,” Asymptotic Anal. **1**, 105–113 (1988).
15. Yu. A. Kordyukov, X. Ma, and G. Marinescu, “Generalized Bergman kernels on symplectic manifolds of bounded geometry,” Comm. Partial Differential Equations **44**, 1037–1071 (2019).
16. X. Ma and G. Marinescu, “Exponential estimate for the asymptotics of Bergman kernels,” Math. Ann. 362, 1327–1347 (2015).
17. D. Borthwick and A. Uribe, “Almost complex structures and geometric quantization,” Math. Res. Lett. 3, 845–861 (1996).
18. L. Ioos, W. Lu, X. Ma, and G. Marinescu, “Berezin-Toeplitz quantization for eigenstates of the Bochner-Laplacian on symplectic manifolds,” J. Geom. Anal. 30, 2615–2646 (2020).
19. Yu. A. Kordyukov, “On asymptotic expansions of generalized Bergman kernels on symplectic manifolds,” St. Petersburg Math. J. 30 (2), 267–283 (2019).
20. L. Charles, “Quantization of compact symplectic manifolds,” J. Geom. Anal. 26, 2664–2710 (2016).
21. C.-Y. Hsiao and G. Marinescu, “Berezin–Toeplitz quantization for lower energy forms,” Comm. Partial Differential Equations 42, 895–942 (2017).
22. X. Ma and G. Marinescu, “Toeplitz operators on symplectic manifolds,” J. Geom. Anal. 18, 565–611 (2008).
23. W. Bauer and L. A. Coburn, “Uniformly continuous functions and quantization on the Fock space,” Bol. Soc. Mat. Mex. (3) 22, 669–677 (2016).
24. D. Borthwick, “Microlocal techniques for semiclassical problems in geometric quantization,” in Perspectives on Quantization, Contemp. Math. (Amer. Math. Soc., Providence, RI, 1998), Vol. 214, pp. 23–37.
25. L. A. Coburn, “Deformation estimates for Berezin–Toeplitz quantization,” Comm. Math. Phys. 149, 415–424 (1992).
26. W. Bauer, L. A. Coburn, and R. Hagger, “Toeplitz quantization on Fock space,” J. Funct. Anal. 274, 3531–3551 (2018).
27. W. Bauer, R. Hagger, and N. Vasilevski, “Uniform continuity and quantization on bounded symmetric domains,” J. London Math. Soc.(2) 96, 345–366 (2017).
28. D. Borthwick, A. Lesniewski, and H. Upmeier, “Non-perturbative deformation quantization of Cartan domains,” J. Funct. Anal. 113, 153–176 (1993).
29. M. Engliš, “Weighted Bergman kernels and quantization,” Comm. Math. Phys. 227, 211–241 (2002).
30. S. Klimek and A. Lesniewski, “Quantum Riemann surfaces I: the unit disc,” Comm. Math. Phys. 146, 103–122 (1992).
31. M. Engliš and H. Upmeier, “Asymptotic expansions for Toeplitz operators on symmetric spaces of general type,” Trans. Amer. Math. Soc. 367, 423–476 (2015).
32. X. Ma and G. Marinescu, Holomorphic Morse Inequalities and Bergman Kernels, in Progr. Math. (Birkhäuser, Basel, 2007), Vol. 254.
33. X. Ma and G. Marinescu, “Generalized Bergman kernels on symplectic manifolds,” Adv. Math. 217, 1756–1815 (2008).
34. X. Dai, K. Liu, and X. Ma, “On the asymptotic expansion of Bergman kernel,” J. Differential Geom. 72, 1–41 (2006).
35. Yu. A. Kordyukov, “Semiclassical spectral analysis of the Bochner–Schrödinger operator on symplectic manifolds of bounded geometry,” Anal. Math. Phys. 12, 22, 37 (2022).
36. L. Charles, “Berezin–Toeplitz operators, a semi-classical approach,” Comm. Math. Phys. 239, 1–28 (2003).
37. Yu. A. Kordyukov, “$L^p$–theory of elliptic differential operators on manifolds of bounded geometry,” Acta Appl. Math. 23, 223–260 (1991).
38. Yu. A. Kordyukov, “Berezin–Toeplitz quantization associated with higher Landau levels of the Bochner Laplacian,” J. Spectr. Theory 12 (1), 143–167 (2022).