INTERPOLATING SEQUENCES FOR
WEIGHTED BERGMAN SPACES OF THE BALL

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ABSTRACT. Interpolating sequences for weighted Bergman spaces $B^p_\alpha$, $0 < p \leq \infty$, $\alpha \geq -1/p$ are studied. We show that the natural inclusions between $B^p_\alpha$ for various $p$ and $\alpha$ are also verified by the corresponding spaces of interpolating sequences. We also give conditions (necessary or sufficient) for the $B^p_\alpha$-interpolating sequences. These are similar to the known conditions for the spaces $H^p$ and $A^{-\alpha}$, which in our notation correspond respectively to the particular cases $\alpha = -1/p$ and $p = \infty$.

§0. Introduction.

Let $B^p_\alpha$ be the space of $f$ holomorphic in the unit ball $\mathbb{B}^n$ of $\mathbb{C}^n$ such that $(1 - |z|^2)^\alpha f(z) \in L^p(\mathbb{B}^n)$, where $0 < p \leq \infty$, $\alpha \geq -1/p$ (weighted Bergman space). In this paper we study the interpolating sequences for various $B^p_\alpha$. The limiting cases $\alpha = -1/p$ and $p = \infty$ are respectively the Hardy spaces $H^p$ and $A^{-\alpha}$, the holomorphic functions with polynomial growth of order $\alpha$, which have generated particular interest. Note that the class of spaces we are considering is invariant under restriction to balls of lower complex dimension, which justifies the choice of those special weights.

As far as we know, for $n > 1$ the first research on this subject was carried out by Amar for the classical Bergman spaces, which in our notation correspond to the case $\alpha = 0$ ([Am]). Amar’s main result states that separated sequences (in terms of the Gleason invariant distance) can be written as a finite union of interpolating sequences for $B^p_0$.

A sufficient condition due to Berndtsson is known for the case $H^\infty$ ([Be]). Also, Varopoulos showed that if $\{a_k\}_k$ is $H^\infty$-interpolating then $\sum_k (1 - |a_k|^2)^n \delta_{a_k}$ is a Carleson measure ([Va]). Later the third author proved that the same necessary condition holds for $H^1$, and it actually characterizes the finite unions of $H^1$-interpolating sequences ([Th1]).

On the other hand, after Seip’s characterization of $A^{-\alpha}$-interpolating sequences in the unit disc ([Se], see another proof in [Be-Or]), the second author obtained some results for the case $n > 1$ ([Ma]). In particular, $\{a_k\}_k$ is a finite union of $A^{-\alpha}$-interpolating sequences for $n > 1$.

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sequences if and only if $\sum_k (1 - |a_k|^2)^{n+1}$ is a $(n+1)$-Carleson measure, or equivalently, if and only if $\{a_k\}_k$ is a finite union of separated sequences.

It is worth noting that in [Se], Seip also implicitly gives a characterization of interpolating sequences for all weighted Bergman spaces in the disk. We spell out the details for the reader’s convenience in an appendix (§5).

Here we deal with different aspects concerning $B^p_\alpha$-interpolating sequences. In §1 we first collect some definitions and well-known facts about weighted Bergman spaces and then introduce the natural interpolation problem, along with some basic properties. In §2 we describe in terms of $\alpha$ and $p$ the inclusions between $B^p_\alpha$ spaces, and in §3 we show that most of these inclusions also hold for the corresponding spaces of interpolating sequences. Unfortunately our proof does not capture the intuitive conjecture given in [Th1] to the effect that for $p' < p$ every $H^p$-interpolating sequence is also $H^{p'}$-interpolating. §4 is devoted to sufficient conditions for a sequence to be $B^p_\alpha$-interpolating, expressed in the same terms as the conditions given in [Th1] for the Hardy spaces and in [Ma] for $A^{-\alpha}$. In particular we show, under some restrictions on $\alpha$ and $p$, that finite unions of $B^p_\alpha$-interpolating sequences coincide with finite unions of separated sequences.

§1. Definitions and first properties.

1.1. Notations. For $z, w \in \mathbb{C}^n$, we set $zw := \sum_{j=1}^n z_j w_j$, $|z|^2 := zz^*$, and the unit ball $\mathbb{B}^n := \{z \in \mathbb{C}^n : |z| < 1\}$.

Given $a \in \mathbb{B}^n$, $\varphi_a$ is the involutive automorphism of the ball exchanging 0 and $a$ (see [Ru, 2.2.2]). For $a, b \in \mathbb{B}^n$, $d(a, b) := |\varphi_a(b)| = |\varphi_b(a)|$ is the invariant distance between $a$ and $b$. Recall that

$$1 - d(a, b)^2 = 1 - |\varphi_a(b)|^2 = \frac{(1 - |a|^2)(1 - |b|^2)}{|1 - ab|^2}.$$

We call hyperbolic balls the sets $E(z, r) := \{\zeta \in \mathbb{B}^n : d(z, \zeta) < r\}$.

The normalized Lebesgue measures on the ball and the sphere will be denoted by $dm$ and $d\sigma$ respectively ($dm_{2n}$ and $d\sigma_{2n-1}$ when we want to stress the dimension). The measure $d\tau(z) := (1 - |z|^2)^{-(n+1)} dm(z)$ is invariant under the automorphisms of the ball [Ru, 2.2.6]. In particular, $\tau(E(z, r))$ depends only on $r$.

Throughout this paper we will be using the following estimates:

**Lemma 1.1.** Let $a, b \in \mathbb{B}^n$, $c > 0$, $t > -1$. Then

(a) $\int_{\mathbb{B}^n} \frac{(1 - |z|^2)^t}{|1 - az|^{n+1+c-t}} dm(z) \approx (1 - |a|^2)^{-c}$.

(b) $\int_S \frac{d\sigma(\zeta)}{|1 - a\zeta|^{n+c}} dm(z) \approx (1 - |a|^2)^{-c}$. 
(c) \[ \int_{\mathbb{B}^n} \frac{(1 - |z|^2)^t}{|1 - z\bar{a}|^{n+1+c+t}|1 - z\bar{b}|^{n+1+c+t}} \, dm(z) \leq |1 - \bar{a}b|^{-(n+1+c+t)} \{\min(1 - |a|^2, 1 - |b|^2)\}^{-c}. \]

Proof. (a) and (b) are given in [Ru, 1.4.10]. To prove (c) split into the cases $|1 - z\bar{a}| \geq \frac{\sqrt{2}}{2}|1 - \bar{a}b|$ and $|1 - z\bar{a}| \leq \frac{\sqrt{2}}{2}|1 - \bar{a}b|$, which implies $|1 - z\bar{b}| \geq \frac{\sqrt{2}}{2}|1 - \bar{a}b|$ by the triangle inequality [Ru, 5.1.2(i)]; then apply (a). □

1.2. Weighted Bergman Spaces. For $p > 0$, $\alpha \in \mathbb{R}$, let $L_p^p(\mathbb{B}^n)$ be the space of all measurable complex-valued functions on $\mathbb{B}^n$ such that $(1 - |z|^2)^\alpha f(z) \in L_p$, i.e. for $p < \infty$,

\[ \|f\|_{p,\alpha}^p := \int_{\mathbb{B}^n} (1 - |z|^2)^{\alpha p} |f(z)|^p \, dm(z) < \infty, \]

and for $p = \infty$,

\[ \|f\|_{\infty,\alpha} := \sup_{z \in \mathbb{B}^n} (1 - |z|^2)^{\alpha} |f(z)| < \infty. \]

For $p > 0$, $\alpha > -1/p$, and denoting by $H(\mathbb{B}^n)$ the space of holomorphic functions in the ball, the weighted Bergman space is $B_p^p(\mathbb{B}^n) := H(\mathbb{B}^n) \cap L_p^p(\mathbb{B}^n)$.

For $\alpha \leq -1/p$, the above condition only holds for the zero function, but we sometimes will use the limiting case of the Hardy spaces:

\[ B_p^{-\frac{1}{p}} := H_p^0(\mathbb{B}^n) = \{ f \in H(\mathbb{B}^n) : \|f\|_{H^p} = \|f\|_{p, -\frac{1}{p}}^p := \sup_{0 < r < 1} \int_{\partial \mathbb{B}^n} |f(r\zeta)|^p \, d\sigma(\zeta) < \infty \}. \]

The facts presented in this section are essentially well-known, but we recap them here for the reader’s convenience, and also to write them in our notation, which differs from those of Horowitz [Ho], Coifman and Rochberg [Co-Ro], [Ro], and Seip [Se].

The following statement is an immediate consequence of [Ru, pag. 14].

**Lemma 1.2.** If $\ell \in \mathbb{Z}_+$ and $f \in B_{\alpha - \frac{\ell}{p}}^p(\mathbb{B}^{n+\ell})$, then its restriction to $\mathbb{B}^n \times \{0\}$ lies in $B^p_\alpha(\mathbb{B}^n)$; conversely whenever $g \in B^p_\alpha(\mathbb{B}^n)$, its trivial extension (constant along the vertical directions) must be in $B^{p}_{\alpha - \frac{\ell}{p}}(\mathbb{B}^{n+\ell})$.

**Lemma 1.3.** For $p > 0$, $\alpha \geq -1/p$, there exists a constant $c = c(\alpha, p, n) > 0$ such that for all $z \in \mathbb{B}^n$

\[ |f(z)| \leq c\|f\|_{p,\alpha} (1 - |z|^2)^{-\frac{n+1}{p} + \alpha}, \]

and this is the best possible exponent.

Proof. Lemma 1.2 allows us to reduce this to the case $n = 1$ by considering the disk through 0 and $z$. Then use the mean value inequality on the disk $D(z, \frac{1}{2}(1 - |z|))$. 

That the estimate is sharp can be seen by considering the functions $f_{N,a}(z) := (1 - za)^{-N}$ for $a \in \mathbb{B}^n$, $N > \frac{n+1}{p} + \alpha$. Lemma 1.1 shows that $\|f_{N,a}\|_{p,\alpha} \approx (1 - |a|^2)^{-\frac{n+1}{p} + \alpha - N}$. □

Lemma 1.3 says that $B^n_\alpha \subset B^{\infty}_{\frac{n+1}{p} + \alpha}$. A more complete catalogue of inclusions will be given in Section 2.

**Lemma 1.4.** There exists $c' = c'(\alpha, p, n) > 0$ such that for any $a, b \in \mathbb{B}^n$ with $d(a,b) < 1/2$,

$$|f(a) - f(b)| \leq c'\|f\|_{p,\alpha}(1 - |a|^2)^{-(\frac{n+1}{p}+\alpha)}d(a,b).$$

**Proof.** We apply the generalized Schwarz Lemma (see [Ru, 8.1.4]) over $E(a, \frac{1}{2})$ together with the estimate from Lemma 1.3. □

**1.3. The interpolation problem.** We’d like to call a sequence $\{a_k\}_k \subset \mathbb{B}^n$ interpolating for the space $B^n_\alpha$ if, given arbitrary values $\{v_k\}_k$ subject to some reasonable restrictions, there exists $f \in B^n_\alpha$ such that $f(a_k) = v_k$ for all $k \in \mathbb{Z}_+$. It is not so easy in general to determine what is a reasonable set of possible values for the restrictions of holomorphic functions to a sequence of points, see [Br-Ni-Oy] for the case of $H^p(\mathbb{D})$.

Here we want to impose growth restrictions only on the sequence $\{v_k\}_k$, meaning that if $\{v_k\}_k$ is in the set of possible values (required to be a Fréchet space), then so is $\{v'_k\}_k$ whenever $|v'_k| \leq |v_k|$ for all $k$. Since we’d like to allow the values $v'_k := (1 - |a_j|^2)^{-(\frac{n+1}{p}+\alpha)}f_{N,a_j}(a_k)$, for any given $j$, and since $\|(1 - |a_j|^2)^{-(\frac{n+1}{p}+\alpha)}f_{N,a_j}\|_{p,\alpha}$ does not depend on $j$, we shall adopt the following provisional definition:

**Definition.** We say that $\{a_k\}_k$ has the property (*) for $B^n_\alpha$ iff there exists $M > 0$ such that for any $j \in \mathbb{Z}_+$, there exists $f_j \in B^n_\alpha$ such that $\|f_j\|_{p,\alpha} \leq M$ and $f_j(a_k) = 0$ for $k \neq j$ and $f_j(a_j) = (1 - |a_j|^2)^{-(\frac{n+1}{p}+\alpha)}$.

The requirement on the norm could be deduced from our more general requirements if we were to assume that the sequences $\{v^N_k\}_k$ lie in the unit ball of some Banach space, which would be reasonable in the case where $p \geq 1$.

At any rate, (*) in itself leads to stronger properties, starting with a geometric restriction on the sequence $\{a_k\}_k$.

**Definition.** For $p > 0$, $\beta \in \mathbb{R}$, let

$$\ell^p_{\beta} = \ell^p_{\beta}(\{a_k\}) := \{v_k \in \mathbb{C} : \{(1 - |a_k|^2)^{\beta}v_k\}_k \in \ell^p\}$$

and $\|v\|_{p,\beta} := \sum_k[(1 - |a_k|^2)^{\beta}|v_k|^p]$, $\|v\|_{\infty,\beta} := \sup_k[(1 - |a_k|^2)^{\beta}|v_k|]$. 
Definition. We say that a sequence \( \{a_k\}_k \) is separated iff there exists \( \delta > 0 \) such that for any \( j \neq k \), \( d(a_j, a_k) \geq \delta \).

Lemma 1.5. For any \( p > 0 \), \( \alpha > -1/p \), and any separated sequence \( \{a_k\}_k \), the restriction map \( f \mapsto \{f(a_k)\}_{k \in \mathbb{Z}_+} \) is bounded from \( B^p_\alpha \) to \( \ell^p_{\frac{n+1}{p}+\alpha} \).

Proof. For \( p = \infty \), this is trivial.

For \( p < \infty \), notice that the separatedness implies that for some \( \delta > 0 \) the hyperbolic balls \( E(a_k, \delta) \) are pairwise disjoint. Thus

\[
\|f\|_{p,\alpha}^p \geq \sum_k \int_{E(a_k, \delta)} (1 - |z|^2)^{\alpha p} |f(z)|^p \ dm(z) \gtrsim \delta^{n+1} \sum_k (1 - |a_k|^2)^{\alpha p + n+1} |f(a_k)|^p,
\]

using the plurisubharmonicity of \( |f|^p \).

Assume conversely that \( p > 0 \), \( \alpha \geq -1/p \) and that \( \{a_k\}_k \) verifies (*) for \( B^p_\alpha \). Applying Lemma 1.4 to the \( f_j \) given by (*), we find that

\[
(1 - |a_j|^2)^{-\frac{n+1}{p}+\alpha} \leq c' M (1 - |a_j|^2)^{-\frac{n+1}{p}+\alpha} d(a_j, a_k),
\]

so \( \{a_k\}_k \) is separated.

This motivates the following definition:

Definition. We say that \( \{a_k\}_k \) is an interpolating sequence for \( B^p_\alpha \), denoted by \( \{a_k\}_k \in \text{Int}(B^p_\alpha) \), iff for any \( \{v_k\}_k \in \ell^p_{\frac{n+1}{p}+\alpha} \), there exists \( f \in B^p_\alpha \) such that \( f(a_k) = v_k \) for all \( k \).

This definition appeared in [Sh-Sh] in the case \( \alpha = -1/p, n = 1 \), and occurs under various guises in [Am], [Ro], [Se] and [Th1], [Th2].

Applying Baire’s Theorem to the closure of the images under the restriction map of balls of arbitrarily large radius, we see that whenever \( \{a_k\}_k \) is an interpolating sequence, there is an \( M > 0 \) such that for any \( \{v_k\}_k \in \ell^p_{\frac{n+1}{p}+\alpha} \), for any \( \varepsilon > 0 \), there exists \( f \in B^p_\alpha \), \( \|f\|_{p,\alpha} \leq M \), such that \( \|f(a_k) - v_k\|_{p,\frac{n+1}{p}+\alpha} < \varepsilon \). Thus, applying again Lemma 1.4, we have that \( \{a_k\}_k \) is separated, hence for \( \alpha > -1/p \) the restriction map is bounded. For \( \alpha = -1/p, p \geq 1 \), the proof of [Th1, Theorem 2.2] shows that (*) for \( H^p \) implies that \( \sum_k (1 - |a_k|^2)^n \delta_{a_k} \) is a Carleson measure (see [Th2] for a direct proof when \( p > 1 \)). Applying Baire’s Theorem likewise, we can correct [Th1] to get the boundedness of the restriction mapping claimed there.

We can then apply the Open Mapping Theorem.
Corollary 1.6. If \( \{a_k\}_k \) is an interpolating sequence for \( B^p_\alpha \), then \( \{a_k\}_k \) is separated, the restriction map is bounded from \( B^p_\alpha \) to \( \ell^{n+1}_p + \alpha \), and there exists a constant \( M > 0 \) so that the function \( f \) in the definition can be chosen with the additional condition \( \|f\|_{p,\alpha} \leq M \|v\|_{p, \frac{n+1}{p} + \alpha} \).

The constant \( M \) is called constant of interpolation of \( \{a_k\}_k \).

1.4. Invariance under automorphisms and restriction to subspaces. For \( \varphi \) any automorphism (holomorphic self-map) of the ball, let

\[
T_\varphi f(z) := \left( \frac{1 - |\varphi^{-1}(0)|^2}{(1 - z \varphi^{-1}(0))^2} \right)^{n+1 + \alpha} f \circ \varphi(z).
\]

Lemma 1.7.

(a) \( T_\varphi \) is an isometry of \( B^p_\alpha \).

(b) If \( \varphi \) is an automorphism of the ball, and \( \{a_k\}_k \) is an interpolating sequence for \( B^p_\alpha \), then so is \( \{\varphi(a_k)\}_k \), with the same constant of interpolation.

Proof. (a) is trivial when \( p = \infty \). Any automorphism is a composition of a map \( \varphi_a \) and a rotation, and the result is immediate in the latter case. For the former, in the case where \( p < \infty \), \( \alpha > -1/p \),

\[
\int_{\mathbb{B}^n} |T_\varphi f(z)|^p (1 - |z|^2)^{\alpha p} dm(z) = \int_{\mathbb{B}^n} (1 - |\varphi_a(z)|^2)^{n+1+\alpha p} |f(\varphi_a(z))|^p d\tau(z)
\]

\[
= \int_{\mathbb{B}^n} (1 - |\zeta|^2)^{n+1+\alpha p} |f(\zeta)|^p d\tau(\zeta) = \|f\|^p_{p,\alpha},
\]

which finishes the proof since \( (T_\varphi)^{-1} = T_{\varphi^{-1}} \), as can be seen by an elementary calculation using [Ru, 2.2.5] and the fact that \( |\varphi^{-1}(0)| = |\varphi(0)| \).

(b) Take \( v \in \ell^p_\beta(\{\varphi(a_k)\}) \). Then

\[
\left\{ \left( \frac{1 - |\varphi^{-1}(0)|^2}{(1 - a_k \varphi^{-1}(0))^2} \right)^{n+1+\alpha} v_k \right\} \in \ell^p_\beta(\{a_k\}),
\]

with the same norm, so there is \( F \) such that \( \|F\|_{p,\alpha} \leq M \|v\|_{p, \frac{n+1}{p} + \alpha} \) and

\[
F(a_k) = \left( \frac{1 - |\varphi^{-1}(0)|^2}{(1 - a_k \varphi^{-1}(0))^2} \right)^{n+1+\alpha} v_k.
\]

Then \( G := T_{\varphi^{-1}} F \) solves the original problem, with \( \|G\|_{p,\alpha} \leq M \|v\|_{p, \frac{n+1}{p} + \alpha} \).

The following lemma, which follows immediately from Lemma 1.2, has been used in [Am], and provides some necessary conditions for a sequence to be interpolating.
Lemma 1.8. Suppose \( \{a_k\} \subset \mathbb{B}^n \times \{0\} \subset \mathbb{B}^{n+\ell} \), where \( \ell \in \mathbb{Z}^+ \). Let \( \alpha \geq (\ell - 1)/p \). Then \( \{a_k\} \in \text{Int}(B^p_\alpha(\mathbb{B}^n)) \) if and only if \( \{a_k\} \in \text{Int}(B^p_{\alpha - \frac{\ell}{p}}(\mathbb{B}^{n+\ell})) \).

1.5. Stability. The proof of the following Lemma was sketched in [Lu, §6.II].

Lemma 1.9. For \( 0 < p \leq \infty \) and \( \alpha > -1/p \) or \( 1 \leq p \leq \infty \) and \( \alpha \geq -1/p \), let \( \{a_k\} \in \text{Int}(B^p_\alpha) \) and let \( \{a'_k\}_k \) be another sequence in \( \mathbb{B}^n \). There exists \( \delta > 0 \) such that if

\[
d(a_k, a'_k) < \delta \quad \forall k \in \mathbb{Z}_+
\]

then \( \{a'_k\} \in \text{Int}(B^p_\alpha) \).

Proof. Case \( \alpha > -1/p \). Let \( v \in \ell_{p+1+\alpha} \). Denote \( a = \{a_k\}_k, a' = \{a'_k\}_k \) and \( v^0 = v \).

By hypothesis there exists \( f_0 \in B^p_\alpha \) such that \( f_0(a) = v^0 \) (\( f_0(a_k) = v^0_k \) for all \( k \)) and \( \|f_0\|_{p,\alpha} \leq \|v^0\|_p \), where \( M \) denotes the constant of interpolation of \( \{a_k\}_k \). Consider now \( v^1 := v^0 - f_0(a') \).

Claim. For \( \delta \) small enough, \( \|v^1\| \leq \gamma \|v^0\| \), with \( \gamma < 1 \).

To see this we use a general estimate for holomorphic functions which is a refinement of Lemma 1.4. Let \( f \) be holomorphic and let \( z, w \in \mathbb{B}^n \) with \( d(z, w) < r < 1 \). The plurisubharmonicity of \( |f(z) - f(w)|^p \) as a function of \( z \) together with a gradient estimate shows that there exists a constant \( C = C(r) > 0 \) such that (see [Lu, Lemma 3.1] or [Th2, Lemma 2.4.4]):

\[
|f(z) - f(w)|^p \leq Cd^p(z, w) \int_{E(w, r)} |f(\zeta)|^p \, d\tau(\zeta)
\]

for any \( r > \frac{2}{3}d(z, w) \). With this estimate applied to \( f_0 \) we have, provided that \( r \) is chosen small enough so that the invariants balls \( E(a_k, r) \) are pairwise disjoint:

\[
\|v^1\|^p = \sum_k (1 - |a_k|^2)^{n+1+\alpha p} |f_0(a_k) - f_0(a'_k)|^p \\
\leq C \sum_k (1 - |a_k|^2)^{n+1+\alpha p} d^p(a_k, a'_k) \int_{E(a_k, r)} |f_0(\zeta)|^p \, d\tau(\zeta) \\
\leq C\delta^p \sum_k \int_{E(a_k, r)} |f_0(\zeta)|^p (1 - |\zeta|^2)^{\alpha p} \, dm(\zeta) \leq C\delta^p \|f_0\|^p_{p,\alpha} \leq C\delta^p M^p \|v^0\|^p .
\]

Choosing \( \delta \) so that \( \gamma^p := C\delta^p M^p < 1 \) the claim is proved.

Take now \( f_1 \in B^p_\alpha \) with \( f_1(a) = v^1 \) and \( \|f_1\|_{p,\alpha} \leq M\|v^1\| \), and define \( v^2 = v^1 - f_1(a') \). An iteration of this construction provides functions \( f_j \in B^p_\alpha \) with \( f_j(a) = v^j = v^{j-1} - f_{j-1}(a') \) and \( \|f_j\|_{p,\alpha} \leq M\|v^j\| \leq M\gamma^j\|v^0\| \). Finally the function \( f = \sum_j f_j \) solves the interpolation problem for \( \{a'_k\}_k \).
Case $\alpha = -1/p$, $p \geq 1$. We can use Luecking’s estimate, together with the fact that, when $\{a_k\} \in \text{Int}(H^p)$, $p \geq 1$, the measure $\sum_k (1 - |a_k|^2)^n \delta_{a_k}$ is a Carleson measure [Th1, Theorem 2.2], and therefore so is the measure $\sum_k (1 - |a_k|^2)^{-1} \chi_{E(a_k,r)} \, dm$. Then, applying Lemma 3.1 (see section §3) we get $\sum_k (1 - |a_k|^2)^n |f(a_k) - f(a'_k)|^p \lesssim \|f\|_{H^p}$ for any function for which the right hand side is finite. The proof then proceeds as before. $\square$

The results described in this section also hold for the corresponding interpolating sequences for the spaces $\mathcal{B}_\alpha^p$ of $\mathcal{M}$-harmonic (instead of holomorphic) functions in $L^p$. This is so because the main ingredients used above, namely Lemma 1.3 and the separatedness of the interpolating sequences, can be proven likewise in the $\mathcal{M}$-harmonic case.

§2. Inclusions for $B_\alpha^p$ and $\ell_\alpha^p$.

Our purpose is now to describe in terms of the values $\alpha$ and $p$ the relationship between $B_\alpha^p$ spaces.

**Lemma 2.1.**

(a) If $p \leq p'$, then $B_\alpha^p \subset B_{\alpha'}^{p'}$ iff $\alpha + \frac{n+1}{p} \leq \alpha' + \frac{n+1}{p'}$.

(b) If $p \geq p'$, then $\alpha + \frac{1}{p} < \alpha' + \frac{1}{p'} \Rightarrow B_\alpha^p \subset B_{\alpha'}^{p'} \Rightarrow \alpha + \frac{1}{p} \leq \alpha' + \frac{1}{p'}$.

In particular, if $\alpha < \alpha' + \min \left( (n+1)(\frac{1}{p'} - \frac{1}{p}), (\frac{1}{p'} - \frac{1}{p}) \right)$, then $B_\alpha^p \subset B_{\alpha'}^{p'}$, and if $B_\alpha^p \subset B_{\alpha'}^{p'}$, then $\alpha \leq \alpha' + \min \left( (n+1)(\frac{1}{p'} - \frac{1}{p}), (\frac{1}{p'} - \frac{1}{p}) \right)$. Those results have been obtained in the case $n = 1$ by Horowitz [Ho].

Among many other results, Coifman and Rochberg [Co-Ro, Propositions 4.2 and 4.4] prove that if $p \leq p' \leq 1$, $\alpha'$, $\alpha \geq -\frac{1}{p}$ and $\alpha + \frac{n+1}{p} = \alpha' + \frac{n+1}{p'}$, then $B_\alpha^p \subset B_{\alpha'}^{p'}$. Their proof is valid for a whole class of symmetric domains.

**Proof.** (a). Assume $\alpha + \frac{n+1}{p} \leq \alpha' + \frac{n+1}{p'}$ and $f \in B_\alpha^p$. The case $p' = \infty$ was settled by Lemma 1.3. For $p' < \infty$, one has

\[ \int_{\mathbb{B}^n} |f(z)|^{p'} (1 - |z|^2)^{\alpha'} \, dm(z) \leq c^{p'-p} \int_{\mathbb{B}^n} |f(z)|^p (1 - |z|^2)^{\left(\frac{n+1}{p'} + \alpha'\right)\left(p - p'\right) + \frac{n+1}{p'}} \, dm(z). \]

This integral is controlled by $\|f\|_{\alpha,p}^p$ whenever $(\frac{n+1}{p} + \alpha)(p - p') + \alpha' p' \geq \alpha p$, that is, when $\alpha + \frac{n+1}{p} \leq \alpha' + \frac{n+1}{p'}$.

Conversely, assume $\alpha + \frac{n+1}{p} > \alpha' + \frac{n+1}{p'}$. As in Section 1.2, let $f_{\gamma,a}(z) = (1 - z \cdot \bar{a})^{-\gamma}$. Whenever $\gamma > \alpha + \frac{n+1}{p}$, $\|f_{\gamma,a}\|_{\alpha',p'} \approx (1 - |a|^2)^{\alpha' + \frac{n+1}{p'} - \gamma}$. Choosing $\gamma > \alpha + \frac{n+1}{p}$, we see that $\|f_{\gamma,a}\|_{\alpha',p'} / \|f_{\gamma,a}\|_{\alpha,p}$ cannot be bounded as $|a|$ tends to 1.
(b). Assume $\alpha + \frac{1}{p} < \alpha' + \frac{1}{p'}$ and $f \in B^p_\alpha$. Hölder’s inequality with exponent $\frac{p}{p'} \geq 1$ yields

$$\int_{\mathbb{B}^n} |f(z)|^{p'} (1 - |z|^2)^{\alpha' p'} dm(z) \leq \left( \int_{\mathbb{B}^n} |f(z)|^p (1 - |z|^2)^{\alpha p} dm(z) \right)^{p'/p} \left( \int_{\mathbb{B}^n} (1 - |z|^2)^{(\alpha' - \alpha) \frac{p' - p}{p - p'}} dm(z) \right)^{\frac{p - p'}{p}} ,$$

and by hypothesis both integrals are finite. If $p'$ or $p$ is infinite, the analogous proof goes through even more easily.

When $\alpha + 1/p = 0$, the hypothesis becomes $f \in H^p$. Using $\int_S |f|^{p'} d\sigma \leq \int_S |f|^p d\sigma$ and integration in polar coordinates, we see that

$$\int_{\mathbb{B}^n} |f(z)|^{p'} (1 - |z|^2)^{\alpha' p'} dm(z) \preceq \|f\|_{H^p} \int_0^1 (1 - r^2)^{\alpha' p'} dr < \infty,$$

since $\alpha' p' > -1$.

When $\alpha + 1/p = \alpha' + 1/p' > 0$, $p > p'$, it is possible to find $f \in L^p_\alpha(\mathbb{B}^n) \setminus L^{p'}_{\alpha'}(\mathbb{B}^n)$. However we don’t know whether there exist holomorphic functions with that property.

When $\alpha + 1/p > \alpha' + 1/p'$, we will construct an example in the following way. For $\kappa > 1$ and for $0 < r < 1$, choose a set $\{\eta_k\}_k \subset S$ maximal for the property that the Koranyi balls $K(\eta_k, \kappa r) := \{\zeta \in S : |1 - \zeta \bar{\eta_k}| < \kappa r\}$ are disjoint. Recall that the quantity $|1 - \zeta \bar{\xi}|^{1/2}$ verifies the triangle inequality over $\mathbb{B}^n$ [Ru, 5.1.2]. For $\gamma > 0$, let

$$F_{\gamma, r}(z) := \sum_k \frac{1}{(1 - (1 - r)z\bar{\eta_k})^\gamma} .$$

**Claim:** For $\kappa$ large enough, and for all $0 < p' \leq \infty$, $\alpha' \geq -1/p'$ and $\gamma > \max(\frac{n+1}{p'} + \alpha', n)$, then $\|F_{\gamma, r}\|_{p', \alpha'} \approx r^{\frac{1}{p'} + \alpha' - \gamma}$, where the constants involved may depend on $p'$, $\alpha'$, $\gamma$, but not on $r$.

> From the claim we conclude as above, letting $r \to 0$, that $B^p_\alpha \nsubseteq B^{p'}_{\alpha'}$ when $\alpha + 1/p > \alpha' + 1/p'$.

**Proof of Claim.** We need to perform a pointwise estimate on $F_{\gamma, r}(z)$. Suppose $z = \rho \zeta$, $\zeta \in S$. Suppose first that $\zeta \in K(\eta_k, r)$. Then for any $j \neq k$,

$$\inf_{\xi \in K(\eta_j, \kappa r)} |1 - (1 - r)\rho \zeta \xi| \geq C |1 - (1 - r)\rho \zeta \bar{\eta_j}| ,$$
and using Riemann sums and Lemma 1.1,
\[
|F_{\gamma,r}(\rho \zeta)| \geq \frac{1}{|1-(1-r)|} \frac{C(kr)^{-n}}{|r|} \int_{S\backslash K(\eta_k,kr)} \frac{d\sigma(\zeta)}{|(1-(1-r)|} \geq C \max((1-\rho),r)^{-n} - Ck^{-n}r^{-n} \max((1-\rho),r)^{-n},
\]
The last quantity is bounded below by \(C_1 r^{-n}\) for \(k\) large enough and \(\rho \geq 1 - r\). The estimate from below for \(p = \infty\) is thus secured.

For \(p' < \infty\) and \(\alpha' = -1/p'\), we have that
\[
\sup_{p < 1} \left( \int_{S} |f(\rho \zeta)|^{p'} \, d\sigma(\zeta) \right)^{\frac{1}{p'}} \geq r^{-n},
\]
and for \(p' < \infty\) and \(\alpha' > -1/p'\),
\[
\int_{\mathbb{R}^n} |F_{\gamma,r}(z)|^{p'} (1 - |z|^2)^{\alpha'p'} \, dm(z) \geq \int_{1-r}^{r} r^{-\alpha'p'} (1 - \rho^2)^{\alpha'p'} \rho^{2n-1} \, d\rho \approx r^{1+(\alpha'-\gamma)p'}. \]

On the other hand, for any \(\zeta \in S\), by the same arguments,
\[
|F_{\gamma,r}(\rho \zeta)| \leq \sum_{k: \eta_k \in K(\zeta,2kr)} \frac{1}{|1-(1-r)|} + C(kr)^{-n} \int_{S\backslash K(\zeta,kr)} \frac{d\sigma(\zeta)}{|(1-(1-r)|} \leq C(n) \max((1-\rho),r)^{-n} + Ck^{-n}r^{-n} \max((1-\rho),r)^{-n}. \]

So for \(\rho \geq 1 - r\), \(|F_{\gamma,r}(\rho \zeta)| \leq C_2 r^{-n}\), and for \(\rho \leq 1 - r\), \(|F_{\gamma,r}(\rho \zeta)| \leq Cr^{-n}(1-\rho)^{-n}\).

In the case \(\alpha' > -1/p'\), assuming \(\gamma > \frac{n+1}{p'} + \alpha'\), we integrate this to get:
\[
\int_{\mathbb{R}^n} |F_{\gamma,r}(z)|^{p'} (1 - |z|^2)^{\alpha'p'} \, dm(z) \leq \int_{0}^{1-r} r^{-np'} (1 - \rho^2)^{\gamma} \rho^{2n-1} \, d\rho + \int_{1-r}^{1} r^{-\alpha'p'} (1 - \rho^2)^{\gamma} \rho^{2n-1} \, d\rho \leq r^{1+(\alpha'-\gamma)p'}. \]

The spaces of possible values we are considering verify similar inclusions.

**Lemma 2.2.** Suppose \(\{a_k\}_k\) is separated. Then
\[
(a) \text{ If } p \leq p', \text{ then } \ell_{n+1}^{p+1} + \alpha \subset \ell_{n+1}^{p+1} + \alpha' \text{ if and only if } \frac{n+1}{p} + \alpha \leq \frac{n+1}{p'} + \alpha';
\]
\[
(b) \text{ If } p \geq p' \text{ and } \frac{1}{p} + \alpha < \frac{1}{p'} + \alpha', \text{ then } \ell_{n+1}^{p+1} + \alpha \subset \ell_{n+1}^{p+1} + \alpha'.
\]
Conversely to (b),

(c) If \( p > p' \) and \( \frac{1}{p} + \alpha \geq \frac{1}{p'} + \alpha' \), then there exists a separated sequence \( \{a_k\}_k \) such that \( \ell^{n+1 \over p} + \alpha \not\subset \ell^{n+1 \over p'} + \alpha' \).

Proof. (a) Left to the reader (similar to Lemma 2.1).

(b) First observe that the separatedness of the sequence implies that for any \( \varepsilon > 0 \), \( \sum_k (1 - |a_k|)^{n+\varepsilon} < \infty \) (see Lemma 4.1). For \( p < \infty \), applying Hölder’s inequality,

\[
\sum_k (1 - |a_k|^2)^{n+1+\alpha'p'} |v_k|^{p'} \leq \left( \sum_k \left( (1 - |a_k|^2)^{\alpha p'} |v_k|^{p'} \right)^{p' \over p} (1 - |a_k|^2)^{n+1} \right)^{1 \over p'} \times \left( \sum_k \left( (1 - |a_k|^2)^{(\alpha' - \alpha)p'} \left(1 - |a_k|^2\right)^{n+1} \right)^{1 \over 1 - p' \over p} \right)^{1 \over 1 - p' \over p}.
\]

Now the exponent of \( 1 - |a_k|^2 \) in the last sum is \( \alpha' - \alpha \over p' - p \) + \( n + 1 \) > \( n \) by hypothesis, so we are done. Notice that if the sequence \( \{a_k\}_k \) was sparse, we could allow smaller values of \( \alpha' \), so that there is no hope of a general converse statement analogous to the case (a).

The reasoning in the case \( p = \infty \) is similar and simpler.

(c) Our example will be a separated sequence which is as crowded as possible (a net in the sense of [Ro] or [Lu]), i.e. having the property that all points in the ball are less than a constant invariant distance away from a point in the sequence.

Let \( r \in (0, 1) \). We pick a sequence

\[
\{a_{m,\ell}, \ m \in \mathbb{Z}_+^*, 0 \leq \ell \leq L_m\},
\]

such that for any \( \ell, a_{m,\ell} = 1 - r^m \), and for each \( m \) the set \( \{a_{m,\ell}, 0 \leq \ell \leq L_m\} \) is maximal in the sphere of radius \( 1 - r^m \) for the property that the invariant balls \( E(a_{m,\ell}, r) \) be disjoint. It is easy to check that this sequence is separated and that \( L_m \approx r^{-nm} \).

We choose \( v_{m,\ell} \) so that \( u_m := |v_{m,\ell}| \) depends only on \( m \). Now for \( 0 < p < \infty \),

\[
\sum_{m,\ell} (1 - |a_k|^2)^{n+1+\alpha p} |v_{m,\ell}|^p \approx \sum_m r^{-nm + m(n+1+\alpha p)} u_m^p,
\]

and

\[
\sum_{m,\ell} (1 - |a_k|^2)^{n+1+\alpha'p'} |v_{m,\ell}|^{p'} \approx \sum_m r^{m(1+\alpha'p')} u_m^{p'} \geq \sum_m \left( r^{m(1+\alpha p)} u_m^p \right)^{p' \over p},
\]
since $\alpha' p' \leq \alpha p' + \frac{p'}{p} - 1$. By choosing $u_m$ appropriately, we can make the last sum diverge, while $\sum_m r^{m(1+\alpha p)}u_m^p < \infty$.

In the case $p = \infty$, simply taking $u_m = r^{-m\alpha}$, we see that $\sum_m r^{m(1+\alpha' p')}u_m^{p'} \geq \sum_m 1 = \infty$. $\Box$

§3. Inclusions for $\text{Int}(B^p_\alpha)$.

In this section we show that the inclusions given in Lemma 2.1 and Lemma 2.2 are also verified by the corresponding spaces of interpolating sequences. First we recall some known facts about Carleson measures which will also be used in §4.

For any $t > 0$ and $\zeta_0 \in \mathbb{B}^n$ consider the Carleson window with centre $\zeta_0$ and radius $t$ defined by $C_t(\zeta_0) = \{z \in \mathbb{B}^n : |1 - \bar{\zeta}_0z| < t\}$. A Borel measure $\nu$ in $\mathbb{B}^n$ is a $q$-Carleson measure if

$$\nu(C_t(\zeta_0)) = O(t^q) \quad \forall t > 0 \quad \forall \zeta_0 \in \partial \mathbb{B}^n.$$ 

A $n$-Carleson measure is simply called a Carleson measure. What we call $q$-Carleson measures were studied in [Am-Bo] where they were called "Carleson measures of order $q/n"."

One of the main features of Carleson measures is the following.

**Lemma 3.1 ([Hr] [Ci-Wo]).** Let $q \geq 1 > 0$ and $\alpha \geq -1/p$. Let $\mu$ be a positive measure. Then the following are equivalent:

(a) $$(\int_{\mathbb{B}^n} |f(z)|^q d\mu(z))^{\frac{1}{q}} \leq c \|f\|_{p,\alpha} \quad \forall f \in B^p_\alpha.$$ 

(b) $$\mu(C_t(\zeta)) = O(t^{\frac{n+1}{p} + \alpha q}) \quad \forall t > 0 \quad \forall \zeta \in \partial \mathbb{B}^n.$$ 

We will be mainly interested in Carleson conditions for the measures $\sum_k (1 - |a_k|^2)^q \delta_{a_k}$, which have important relationships with the values

$$K(\{a_k\}, p, q) := \sup_{k \in \mathbb{Z}^n} \sum_{j: j \neq k} \frac{(1 - |a_k|^2)^p(1 - |a_j|^2)^q}{1 - \bar{a}_k a_j^{p+q}} \quad p, q > 0.$$ 

**Lemma 3.2.**

(a) If $\sum_k (1 - |a_k|^2)^n \delta_k$ is a Carleson measure, then for any $p > 0$, $K(\{a_k\}, p, n) < \infty$.

(b) A positive measure $\mu$ on $\mathbb{B}^n$ is a Carleson measure if and only if there exists some $\beta > n/2$ such that

$$\sup_{b \in \mathbb{B}^n} \int_{\mathbb{B}^n} \frac{(1 - |b|^2)^{2\beta-n}}{|1 - bz|^{2\beta}} d\mu(z) < \infty.$$
(a) is an immediate consequence of [Ma, Lemma 1.4]. (b) is a well-known result, which can be found in [Ma, Lemma 1.2] and ultimately goes back to [Ga, pag. 239].

We now come to the main result of this section:

**Theorem 3.3.** Let \( p, p' > 0 \) and \( \alpha \geq -1/p, \alpha' \geq -1/p' \), satisfying one of the following conditions:

1. \( p \leq p' \) and \( \frac{n+1}{p} + \alpha < \frac{n+1}{p'} + \alpha' \).
2. \( p \geq p' \) and \( \alpha + 1/p < \alpha' + 1/p' \).

Then \( \text{Int}(B_{\alpha'}^p) \subset \text{Int}(B_{\alpha'}^p). \)

In the special case where \( n = 1 \), Seip [Se] has proved a stronger result than Theorem 3.3 (b), namely \( \text{Int}(B_{\alpha}^\infty) = \text{Int}(B_{\alpha-rac{1}{p}}^{p}) \) (see Appendix). This suggests that the inequality \( \alpha + 1/p < \alpha' + 1/p' \) in (b) is critical.

In fact, if we take a sequence \( a \subset \mathbb{B}^1 \times \{0\} \subset \mathbb{B}^n \), then we see by Lemma 1.8, that \( a \in \text{Int}(B_{\alpha}^p(\mathbb{B}^n)) \) iff \( a \in \text{Int}(B_{\alpha-rac{1}{p}}^{p}(\mathbb{B}^1)) \), which, by Seip’s result, is the same as \( \text{Int}(B_{\alpha}^{\infty}(\mathbb{B}^1)) \). Since we know from [Se] that \( \text{Int}(B_{\beta}^{\infty}(\mathbb{B}^1)) \subseteq \text{Int}(B_{\beta'}^{\infty}(\mathbb{B}^1)) \) when \( \beta < \beta' \), this shows that \( \text{Int}(B_{\alpha}^{\infty}(\mathbb{B}^n)) \neq \text{Int}(B_{\alpha'}^{p}(\mathbb{B}^n)) \) when \( \frac{n+1}{p} + \alpha \neq \frac{n+1}{p'} + \alpha' \), and that \( \text{Int}(B_{\alpha}^{p}(\mathbb{B}^n)) \not\subset \text{Int}(B_{\alpha'}^{p}(\mathbb{B}^n)) \) when \( \frac{n+1}{p} + \alpha > \frac{n+1}{p'} + \alpha' \). This shows that the inclusions in parts (a) and (b) of Theorem 3.3 are strict.

**Proof.** Let \( \{a_k\}_k \) be \( B_{\alpha}^p \)-interpolating and take \( f_k \in B_{\alpha}^p \) with \( (1 - |a_j|^2)^{\frac{n+1}{p} + \alpha} f_k(a_j) = \delta_{jk} \) and \( \|f_k\|_{p, \alpha} \leq c \), for some constant \( c > 0 \) independent of \( k \). Given \( m > 0 \) define \( G_k(z) = g_k(z) \cdot f_k(z) \), where

\[
g_k(z) = \frac{(1 - |a_k|^2)^{\frac{n+1}{p} + \alpha + m}}{(1 - \bar{a}_k z)^{\frac{n+1}{p} + \alpha' + m}}.
\]

For a given \( \{\lambda_k\}_k \in \ell^{p'} \), let \( G := \sum \lambda_k G_k \). From this definition it follows immediately that \( (1 - |a_k|^2)^{\frac{n+1}{p} + \alpha'} G(a_k) = \lambda_k \), and we need to prove that \( G \in B_{\alpha'}^{p'} \).

Assume first \( \alpha > -1/p \).

Case \( 0 < p' \leq 1 \). Since \( \| \cdot \|_{p', \alpha'} \) satisfies the triangle inequality, it’s enough to show:

**Claim:** There exists \( c > 0 \) independent of \( k \) such that \( \|G_k\|_{p', \alpha'} \leq c \) for all \( k \).

**Proof:** By definition of \( G_k \), \( \|G_k\|_{p', \alpha'} \) equals

\[
I_k := \int_{\mathbb{B}^n} \frac{(1 - |a_k|^2)^{\frac{n+1}{p} + \alpha + m}p'}{|1 - \bar{a}_k z|^{\frac{n+1}{p} + \alpha' + m}p'} \left( 1 - |z|^2 \right)^{\alpha' - \alpha} \left( 1 - |z|^2 \right)^{\alpha' - \alpha} \left| f_k(z) \right|^{p'} dm(z).
\]
Case (a). Since \( p \leq p' \), estimate \(|f_k(z)|^{p'-p}\) by Lemma 1.3; we see that

\[
I_k \lesssim \|f_k\|_{p',\alpha}^{p'-p} \times \int_{\mathbb{B}^n} \frac{(1 - |a_k|^2)(\frac{n+1}{p} + \alpha + m)p' + (1 - |z|^2)(\alpha'p' + (p - p')(\frac{n+1}{p'} + \alpha) - \alpha p)}{|1 - \bar{a}_k z|^{\frac{n+1}{p'} + \alpha + m}} (1 - |z|^2)^{\alpha p} |f_k(z)|^p dm(z),
\]

which, since \((\frac{n+1}{p} + \alpha + m)p' + [\alpha'p' + (p - p')(\frac{n+1}{p'} + \alpha) - \alpha p] = (\frac{n+1}{p'} + \alpha' + m)p'\), shows that \(\|G_k\|_{p',\alpha} \lesssim \|f_k\|_{p',\alpha}\).

Case (b). We may assume \( p > p' \), and apply Hölder’s inequality with \( P = p/p' \) and \( Q = \frac{p}{p - p'} \). Since by hypothesis \((\alpha' - \alpha)p/p' > -1\), one has, for \( m \) large enough,

\[
I_k \lesssim \left[ \int_{\mathbb{B}^n} \frac{(1 - |a_k|^2)(\frac{n+1}{p} + \alpha + m)p' + (1 - |z|^2)(\alpha'p' + (p - p')(\frac{n+1}{p'} + \alpha) - \alpha p)}{|1 - \bar{a}_k z|^{\frac{n+1}{p'} + \alpha + m}} dm(z) \right]^{p/p'} \|f_k\|_{p',\alpha} \lesssim (1 - |a_k|^2)(\frac{n+1}{p} + \alpha + m)p' - (\frac{n+1}{p} + \alpha + m)p' + (\alpha'p' + (\alpha' - \alpha)p/p' + n + 1)p' \|f_k\|_{p',\alpha} \approx \|f_k\|_{p',\alpha}. \tag*{□}
\]

Case \( p' > 1 \). First we give a useful estimate:

**Lemma 3.4.** Let \( \{a_k\}_k \) and \( \{g_k\}_k \) be as above, and let \( A \) be such that \((\frac{n+1}{p} + \alpha + m)A - n - 1 > -1\). Then

\[
\sum_k |g_k(z)|^A \lesssim (1 - |z|^2)^{A[(\frac{n+1}{p} + \alpha) - (\frac{n+1}{p'} + \alpha')]}.
\]

This is a consequence of [Ma]; more precisely, it follows from Lemma 4.1(d) below with exponents \( P := A[(\frac{n+1}{p} + \alpha') - (\frac{n+1}{p'} + \alpha)] \) and \( Q := A(\frac{n+1}{p} + \alpha + m) \).

Case (a). Using in succession Hölder’s inequality (with \( 1/p' + 1/q' = 1 \)), Lemma 3.4, and Lemma 1.3 for \(|f_k(z)|\), we obtain:

\[
\|G\|_{p',\alpha} = \left[ \sum_k \lambda_k g_k(z)^{\frac{p'}{q'}} \right]^{\frac{q}{q'}} \left[ \sum_k |\lambda_k|^{\frac{q}{q'}} |f_k(z)|^{p'} \right] \left[ \sum_k |\lambda_k|^{\frac{q}{q'}} |f_k(z)|^{p'} (1 - |z|^2)^{\alpha'p'} \right] dm(z) \lesssim \left[ \sum_k |g_k(z)|^{\frac{p}{q'}} \right]^{\frac{q}{q'}} \left[ \sum_k |\lambda_k|^{\frac{q}{q'}} |f_k(z)|^{p'} (1 - |z|^2)^{\alpha'p'} \right] dm(z)
\]
which is controlled by \( \sum_k |\lambda_k|^p' \), since \( p'[\left(\frac{n+1}{p} + \alpha\right) - (\frac{n+1}{p} + \alpha')] + (p - p')(\frac{n+1}{p} + \alpha) + \alpha'p' - \alpha p = 0 \).

Case (b). Let \( 1/p + 1/q = 1 \). We first estimate

\[
|G(z)| = \left| \sum_k \lambda_k g_k(z) f_k(z) \right| \leq \left( \sum_k |\lambda_k|^{(1-\frac{p'}{p})q} |g_k(z)|^q \right)^\frac{1}{q} \left( \sum_k |\lambda_k|^{p'} |f_k(z)|^p \right)^\frac{1}{p}.
\]

Then, applying again Hölder’s inequality with \( P = p/p' \) and \( Q = \frac{p}{p-p'} \), it follows that \( \|G_k\|_{p',\alpha} \) is bounded by

\[
\int_{\mathbb{B}^n} \left[ \sum_k |\lambda_k|^{(1-\frac{p'}{p})q} |g_k(z)|^q (1-|z|^2)^q(\alpha'-\alpha) \right]^{\frac{p'}{p}} \left[ \sum_k |\lambda_k|^{p'} |f_k(z)|^p (1-|z|^2)^\alpha_p \right]^{\frac{p}{p-p'}} dm(z) \leq \int_{\mathbb{B}^n} \sum_k |\lambda_k|^{p'} |f_k(z)|^p (1-|z|^2)^\alpha_p dm(z) \leq \int_{\mathbb{B}^n} \sum_k |\lambda_k|^{p'} |f_k(z)|^p (1-|z|^2)^\alpha_p dm(z) \leq \int_{\mathbb{B}^n} \sum_k |\lambda_k|^{p'} |f_k(z)|^p (1-|z|^2)^\alpha_p dm(z).
\]

The second factor is controlled by \( \sum_k |\lambda_k|^{p'} \|f_k\|_{p',\alpha} \leq \sum_k |\lambda_k|^p' \). Taking \( \gamma, \delta \geq 1 \) with \( 1/\gamma + 1/\delta = 1 \) and applying Hölder’s inequality with exponents \( a = \frac{p'(p-1)}{p-p'} \) and \( b = \frac{p'(p-1)}{p(p-1)} \), and then Lemma 3.4, we can bound the integral appearing in the first factor by

\[
\int_{\mathbb{B}^n} \left[ \sum_k |\lambda_k|^{p'} |g_k(z)|^q (1-|z|^2)^q(\alpha'-\alpha) \right]^{\frac{p}{p-p'}} dm(z) \leq \int_{\mathbb{B}^n} \sum_k |\lambda_k|^{p'} (1-|a_k|^2)^{\frac{(n+1)}{p}+\alpha+m} \left( 1-|z|^2 \right)^{qa(\alpha'-\alpha) + \frac{qa}{\alpha} \left( \frac{n+1}{p} + \alpha - \frac{n+1}{p} - \alpha' \right)} \frac{1}{|1-a_k z|^{\frac{(n+1)}{p}+\alpha+m} \frac{qa}{\gamma}} dm(z).
\]

Since by hypothesis \( qa(\alpha'-\alpha) = \frac{p'}{p-p'} (\alpha'-\alpha) > -1 \), we can choose \( \delta > 1 \) so that \( qa(\alpha'-\alpha) + \frac{qa}{\alpha} \left( \frac{n+1}{p} + \alpha - \frac{n+1}{p} - \alpha' \right) > -1 \) and the integral is finite. Then, once more by Lemma 1.1, we see that the integral is bounded by \( \sum_k |\lambda_k|^p' \). This concludes the case \( \alpha > -1/p \).

We now turn to the case \( \alpha = -1/p \). First we handle the special situation \( p' = p \):
Lemma 3.5. For any $\alpha' > -1/p$, $\text{Int}(H^p) \subset \text{Int}(B^p_{\alpha'})$.

Accepting this, suppose $(p', \alpha')$ satisfy (a) in Theorem 3.3; then there exists $\alpha_1 > -1/p$ such that $\frac{1 + 1}{p} + \alpha_1 < \frac{1 + 1}{p'} + \alpha'$. Likewise if $(p', \alpha')$ satisfy (b) then there exists $\alpha_1 > -1/p$ such that $\frac{1}{p} + \alpha_1 < \frac{1}{p'} + \alpha'$. In either case, applying Lemma 3.5, then the case $\alpha > -1/p$ of Theorem 3.3, we get $\text{Int}(H^p) \subset \text{Int}(B^p_{\alpha_1}) \subset \text{Int}(B^p_{\alpha'})$. □

Proof of Lemma 3.5. Define $f_k, g_k, G_k$ and $G$ as before for $\{\lambda_k\}_k \in \ell^p$.

Case $p \leq 1$. It is enough to prove that $\|G_k\|_{p,\alpha'} \leq c$ for all $k$, i.e.

$$
\int_{\mathbb{B}^n} (1 - |z|^2)^{\alpha'}|g_k(z)|^p|f_k(z)|^pdm(z) \leq c \quad \text{for all } k.
$$

Applying Lemma 3.1 with $p = q$ and $\alpha = -1/p$, we see that it suffices to prove that, for an appropriate choice of the parameter $m$ in $g_k$,

$$(1 - |z|^2)^{\alpha'}|g_k(z)|^pdm(z)$$

is a Carleson measure with Carleson norm independent of $a_k$. To see this, we apply Lemma 3.2(b) with $2\beta = n + 1 + \alpha'p + mp$. By the hypotheses on $\alpha'$ and $m$ we have $2\beta > n + mp > n$. Then, by Lemma 1.1(c),

$$
\sup_{a,b \in \mathbb{B}^n} \int_{\mathbb{B}^n} \frac{(1 - |b|^2)(1 - |z|^2)^{\alpha'}(1 - |a|^2)^{mp + n}}{|1 - bz|^{2\beta}|1 - az|^{n + 1 + \alpha'p + mp}} dm(z) \leq 
$$

$$
\leq \sup_{a,b \in \mathbb{B}^n} \frac{(1 - |b|^2)(1 + \alpha'p + mp)(1 - |a|^2)^{mp + n}}{|1 - ab|^{n + 1 + \alpha'p + mp}} \{\min(1 - |a|^2, 1 - |b|^2)\}^{-mp},
$$

which is finite since $\max(1 - |a|^2, 1 - |b|^2) \leq 1 - ab$.

Case $1 < p \leq 2$. By Hölder’s inequality,

$$
\int_{\mathbb{B}^n} (1 - |z|^2)^{\alpha'}|G(z)|^pdm(z) \leq \sum_j |\lambda_j|^p \int_{\mathbb{B}^n} (1 - |z|^2)^{\alpha'} \left( \sum_k |g_k(z)|^q \right)^{\frac{p}{q}} |f_j(z)|^pdm(z),
$$

where $\frac{1}{p} + \frac{1}{q} = 1$. Again by Lemmas 3.1 and 3.2(b), it’s enough to consider

$$
(1) \quad \sup_{b \in \mathbb{B}^n} \int_{\mathbb{B}^n} \frac{(1 - |b|^2)^{2\beta - n}}{|1 - bz|^{2\beta}} \left( \sum_k |g_k(z)|^q \right)^{\frac{p}{q}} dm(z).
$$
Since \( \frac{p}{q} = p - 1 \leq 1 \), this integral is bounded by

\[
S := \sum_{k} \int_{B^n} \frac{(1 - |b|^2)^{2\beta - n}(1 - |z|^2)^{\alpha'p(1 - |a_k|^2)^{mp + n}}}{|1 - b|2^{\beta}|1 - a_k|^{n+1+\alpha'p+mp}} \, dm(z) .
\]

Choosing again \( 2\beta = n + 1 + \alpha'p + mp > n + mp > n \) and applying Lemma 1.1(c), we get \( S \leq S_1 + S_2 \), where

\[
S_1 := \sum_{k:1 - |a_k|^2 \leq 1 - |b|^2} \frac{(1 - |b|^2)^{1+\alpha'p+mp}(1 - |a_k|^2)^n}{|1 - a_k|^n+1+\alpha'p+mp},
\]

\[
S_2 := \sum_{k:1 - |a_k|^2 > 1 - |b|^2} \frac{(1 - |b|^2)^{1+\alpha'p}(1 - |a_k|^2)^{mp + n}}{|1 - a_k|^n+1+\alpha'p+mp},
\]

so that \( \sup_{b \in B^n} S_1 \leq K(\{a_k\}, 1 + \alpha'p + mp, mp + n) \), \( \sup_{b \in B^n} S_2 \leq K(\{a_k\}, 1 + \alpha'p, n) \). Since \( \{a_k\} \in \text{Int}(H^p) \), \( p > 1 \), we know from [Th1, Theorem 2.2] that \( \sum_k (1 - |a_k|^2)^n \delta_{a_k} \) is an \( n \)-Carleson measure, so Lemma 3.2(a) allows us to conclude that both quantities are finite.

Case \( 2 < p < \infty \). As before, it is enough to consider (1). We apply first Hölder’s inequality with exponents \( \frac{p-1}{p-2} \) and \( p - 1 \), then Lemma 3.4 to get

\[
\left( \sum_k |g_k(z)|^q \right)^{p-1} \leq \left( \sum_k |g_k(z)|^{p/(p-2)} \right)^{p-2} \sum_k |g_k(z)|^{p} \leq (1 - |z|^2)^{-\frac{\alpha'p+1}{2}} \sum_k |g_k(z)|^\frac{p}{2},
\]

so that in this case

\[
S \leq \sup_{b \in B^n} \sum_k \int_{B^n} \frac{(1 - |b|^2)^{2\beta - n}(1 - |z|^2)^{\alpha'p-1}(1 - |a_k|^2)^{\frac{1}{2}(mp+n)}}{|1 - b|^2\beta|1 - a_k|^\frac{1}{2}(n+1+\alpha'p+mp)} \, dm(z) .
\]

This time choose \( 2\beta = \frac{1}{2}(n + 1 + \alpha'p + mp) > \max(\frac{1}{2}(mp + n), n) \), which requires \( m > \frac{n}{p} - (\alpha' + \frac{1}{p}) \). As above, \( S \leq S_1 + S_2 \), where

\[
S_1 := \sum_{k:1 - |a_k|^2 \leq 1 - |b|^2} \frac{(1 - |b|^2)^{\frac{1}{2}(-n+1+\alpha'p+mp)}(1 - |a_k|^2)^n}{|1 - a_k|^\frac{1}{2}(n+1+\alpha'p+mp)},
\]

\[
S_2 := \sum_{k:1 - |a_k|^2 > 1 - |b|^2} \frac{(1 - |b|^2)^{\frac{1}{2}(1+\alpha'p)}(1 - |a_k|^2)^{\frac{1}{2}(mp+n)}}{|1 - a_k|^\frac{1}{2}(n+1+\alpha'p+mp)},
\]

and requiring finally \( mp \geq n \), we conclude as before.
Case $p = \infty$. We can estimate $\sup_{z \in \mathbb{B}^n} (1 - |z|^2)\alpha'|G(z)|$ by a straightforward application of Lemma 3.4. □

Remark. We have actually shown a slightly stronger result, namely that if functions $f_k$ exist with the properties mentioned at the very beginning of the proof (in the case where $\alpha = 0$, $p = \infty$, this hypothesis is sometimes called "uniform separatedness"), then $\{a_k\} \in \text{Int}(B^{p'}_{\alpha'})$ for $(\alpha', p')$ verifying (a) or (b).

Notice that the proof of Theorem 3.3 cannot be used to show the intuitive conjecture that $\text{Int}(H^p) \subset \text{Int}(H^{p'})$, for $p' < p$. It is also interesting to note that the proof only uses that $\sum_k (1 - |a_k|^2)^n \delta_{a_k}$ is an $n$-Carleson measure in the case $\{a_k\} \in \text{Int}(H^p)$, $p > 1$, which follows the arguments of [Ca-Ga] and is much easier to prove than for $p = 1$ (see [Th2, sec. 2.2], while the case $p < 1$ is not known to us for $n \geq 2$.

§4. Sufficient conditions.

In this last paragraph we give sufficient conditions for a sequence $\{a_k\}_k$ to be $B^p_{\alpha}$-interpolating in terms of the values $K(\{a_k\}, p, q)$ defined in the previous section.

Lemma 4.1 ([Ma]). The following conditions are equivalent:

(a) $\{a_k\}_k$ is the union of a finite number of separated sequences.
(b) $K(\{a_k\}, p, q) < +\infty$ $\forall q > n$ $\forall p \leq q$.
(c) $\exists q \geq n, p : K(\{a_k\}, p, q) < +\infty$.
(d) $\forall p > 0$ $\forall q > n$ $\sup_{z \in \mathbb{B}^n} \sum_k (1 - |a_k|^2)^n \delta_{a_k} < +\infty$.
(e) $\sum_k (1 - |a_k|^2)^q \delta_{a_k}$ is a $q$-Carleson measure, for $q > n$.

We will consider, given a sequence $\{a_k\}_k$, the restriction operator $T(f) = \{f(a_k)\}_k$ defined in Lemma 1.5. Notice that

$$\|T(f)\|_{\ell^{n+1}_{-\frac{p}{p+\alpha}}} = \sum_k |f(a_k)|^p (1 - |a_k|^2)^{n+1+\alpha} = \int_{\mathbb{B}^n} |f(z)|^p d\mu(z),$$

where $\mu = \sum_k (1 - |a_k|^2)^{n+1+\alpha} \delta_{a_k}$. From Lemmas 3.1 and 4.1 we deduce thus that $T$ maps $B^p_{\alpha}$ boundedly on $\ell^{n+1}_{-\frac{p}{p+\alpha}}(\{a_k\})$ if and only if $\{a_k\}_k$ is a finite union of separated sequences. This gives a partial converse to Lemma 1.5.

The first result we give in this section deals with the case $p = 1$.

Theorem 4.2. Let $\{a_k\}_k$ be a separated sequence in $\mathbb{B}^n$. If there exists $m > 0$ such that

$$K(\{a_k\}, m, n + 1 + \alpha) < 1$$

then $\{a_k\}_k$ is $B^{1}_{\alpha}$-interpolating.
Proof. Consider $T : B^1_\alpha \rightarrow \ell^1_{n+1+\alpha}(\{a_k\})$ defined above. In order to show that $T$ is onto define, given $v = \{v_k\} \in \ell^1_{n+1+\alpha}(\{a_k\})$, the "approximate extension"

$$E(v)(z) = \sum_{k=1}^{\infty} v_k \left( \frac{1 - |a_k|^2}{1 - \bar{a}_k z} \right)^{n+1+\alpha+m}.$$ 

Using Lemma 1.1 it is immediately verified that $E(v)$ is in $B^1_\alpha$:

$$\|E(v)\|_{1,\alpha} \leq \int_{\mathbb{B}^n} (1 - |z|^2)^\alpha \sum_{k=1}^{\infty} |v_k| \left( \frac{1 - |a_k|^2}{1 - \bar{a}_k z} \right)^{n+1+\alpha+m} dm(z) \leq \sum_{k=1}^{\infty} |v_k|(1 - |a_k|^2)^{n+1+\alpha+m} \int_{\mathbb{B}^n} \frac{(1 - |z|^2)^\alpha}{|1 - \bar{a}_k z|^{n+1+\alpha+m}} dm(z) \approx \|v\| .$$

On the other hand $TE - Id$, regarded as operator on $\ell^1_{n+1+\alpha}(\{a_k\})$, has norm strictly smaller than 1:

$$\|TE(v) - v\| = \sum_{k=1}^{\infty} (1 - |a_k|^2)^{n+1+\alpha} \| (TE(v))_k - v_k \| \leq \sum_{k=1}^{\infty} (1 - |a_k|^2)^{n+1+\alpha} \sum_{j, j \neq k} |v_j| \left( \frac{1 - |a_j|^2}{|1 - \bar{a}_k a_j|} \right)^{n+1+\alpha+m} = \sum_{j=1}^{\infty} (1 - |a_j|^2)^{n+1+\alpha} \| v_j \| \left( \sum_{k: k \neq j} \frac{(1 - |a_j|^2)^m (1 - |a_k|^2)^{n+1+\alpha}}{|1 - \bar{a}_k a_j|^{n+1+\alpha+m}} \right) < \|v\| .$$

Hence the series

$$Id + \sum_{k=1}^{\infty} (TE - Id)^k$$

converges and defines an inverse to $TE$. The operator $\tilde{E} = E(TE)^{-1}$ provides finally the inverse of $T$. □

Notice that by the invariance under automorphisms of the $B^n_\alpha$-interpolating sequences the hypothesis in Theorem 4.2 can be replaced by the seemingly weaker assumption of the existence of an automorphism $\varphi_\zeta$ such that $K(\{\varphi_\zeta(a_k)\}, m, n + 1 + \alpha) < 1$.

**Corollary 4.3.**

(a) Let $\{a_k\}_k$ be a separated sequence. There exists $\alpha > 0$ such that $\{a_k\}_k$ is $B^1_{\alpha}$-interpolating.

(b) Let $\alpha > -1$. There exists $\delta \in (0, 1)$ such that any sequence $\{a_k\}_k$ verifying $d_G(a_j, a_k) \geq \delta$ for any $j \neq k$ is $B^1_{\alpha}$-interpolating.
Proof. Lemma 4.1 shows that

\[ K(\{a_k\}, n + \frac{\alpha + 1}{2}, n + \frac{\alpha + 1}{2}) = \sup_k \sum_{j:j \neq k} (1 - d_G^2(a_j, a_k))^{n + \frac{\alpha + 1}{2}} < +\infty. \]

If \( \delta \) is such that \( 1 - d_G^2(a_j, a_k) < 1 - \delta^2 \) we have then:

\[ \sup_k \sum_{j:j \neq k} (1 - d_G^2(a_j, a_k))^{n + 1 + \alpha} \leq (1 - \delta^2)^{\frac{\alpha + 1}{2}} K(\{a_k\}, n + \frac{\alpha + 1}{2}, n + \frac{\alpha + 1}{2}). \]

In both cases (a) and (b) we can finish the proof by choosing respectively \( \alpha \) or \( \delta \) so that \( (1 - \delta^2)^{\frac{\alpha + 1}{2}} K(\{a_k\}, n + \frac{\alpha + 1}{2}, n + \frac{\alpha + 1}{2}) < 1 \) and applying Theorem 4.2. \( \square \)

Theorem 4.2 along with the following Mills’ lemma provides also another characterization of the sequences appearing in Lemma 4.1.

Mills’ lemma. (See [Ga] or [Th1]). Let \( A_{jk}, j, k \in \mathbb{Z}_+ \), be non-negative real numbers such that \( A_{jk} = A_{kj} \) and \( A_{jj} = 0 \) for any \( j \) and \( k \). If \( \sup_{k \in \mathbb{Z}_+} \sum_{j \in \mathbb{Z}_+} A_{jk} = M < +\infty \), there exists a partition \( \mathbb{Z}_+ = S_1 \cup S_2, \ S_1 \cap S_2 = \emptyset \) such that

\[ \sup_{k \in S_i, j \in S_i} A_{jk} \leq \frac{M}{2} \quad i = 1, 2. \]

Corollary 4.4. A sequence \( \{a_k\}_k \) is the union of a finite number of separated sequences if and only if it is the union of a finite number of \( B_1^\alpha \)-interpolating sequences.

Proof. The reverse implication is given directly by Corollary 1.6. To see the direct one we apply (b) of Lemma 4.1 with \( p = q = n + 1 + \alpha \) and Mills’ lemma with

\[ A_{jk} = \frac{(1 - |a_k|^2)^{n + 1 + \alpha}(1 - |a_j|^2)^{n + 1 + \alpha}}{|1 - a_k a_j|^{2(n + 1 + \alpha)}}. \]

For any \( N \in \mathbb{Z}_+ \) one can split \( \{a_k\}_k \) into \( 2^N \) sequences \( \{b^l_k\}_k, l = 1, \ldots, 2^N \), such that

\[ K(\{b^l_k\}, n + 1 + \alpha, n + 1 + \alpha) < \frac{1}{2^{2N}} K(\{a_k\}, n + 1 + \alpha, n + 1 + \alpha). \]

Taking \( N \) sufficiently large this term becomes smaller than 1, what by Theorem 4.2 yields the stated result. \( \square \)

With the same methods it is also possible to obtain sufficient conditions for a sequence to be \( B_p^\alpha \)-interpolating, \( p > 1 \). However, these conditions are not so well adapted to the nature of the \( B_p^\alpha \) spaces, in the sense that they are symmetrical in \( p \) and the conjugated
exponent \( q \). In the proof, which goes like [Th1, Proposition 3.2], we will use the duality between \( B^p_\alpha \) spaces. Consider the product given by
\[
< f, g > =: \int_{\mathbb{B}^n} f(z) \overline{g(z)} (1 - |z|^2)^{\alpha p} dm(z).
\]

Using Lemma 1.1 and some standard results for classical Bergman spaces (see [Am, Lemme 1.2.3] and [Ru, chap. 7]) it is easy to prove that in case \( 1 < p < \infty \), the dual space of \( B^p_\alpha \) with respect to this product is \( B^q_{\beta q} \), where \( \frac{1}{p} + \frac{1}{q} = 1 \) and \( \beta q = \alpha p \).

Furthermore, there is a reproducing kernel for \( B^p_\alpha \) functions, namely
\[
K_z(\zeta) = \frac{\Gamma(n + \alpha p + 1)}{\Gamma(n + 1)\Gamma(\alpha p + 1)} \frac{1}{(1 - \zeta \bar{\zeta})^{n+1+\alpha p}}.
\]

**Theorem 4.5.** Let \( 1 < p < \infty \) and let \( q \) be its conjugated exponent. Let \( \beta = \alpha p/q \). If there exist \( c_1, c_2 > 0 \) such that \( c_1 c_2 < 1 \) and
\[
K(\{a_k\}, \frac{n+1}{p} + \alpha, \frac{n+1}{q} + \beta) \leq c_1^p ; \quad K(\{a_k\}, \frac{n+1}{q} + \beta, \frac{n+1}{p} + \alpha) \leq c_2^p,
\]
then \( \{a_k\}_k \) is \( B^p_\alpha \)-interpolating.

**Proof.** Given \( \{v_k\} \in \ell^p_{n+1+\alpha}(\{a_k\}) \) take the approximate extension
\[
E(v)(z) = \sum_{k=1}^{\infty} v_k \left( \frac{1 - |a_k|^2}{1 - \bar{a}_k z} \right)^{n+1+\alpha p}.
\]

Using the duality described above, the reproducing kernel for \( B^p_\alpha \) and Lemma 4.1 with
\[
\mu = \sum_k (1 - |a_k|^2)^{n+1+\beta q} \delta_{a_k},
\]
one has that \( \|E(v)\|_{p,\alpha} \leq c\|v\|_{\ell^p_{n+1+\alpha}}. \)

On the other hand, if \( T \) denotes the operator on \( B^p_\alpha \) associated to \( \{a_k\}_k \), we have \( \|TE - Id\| < 1 \), since
\[
\sum_{k=1}^{\infty} (1 - |a_k|^2)^{n+1+\alpha p}\|TE(v) - v\|_k^p \leq c^p \sum_{k=1}^{\infty} \left( \sum_{j: j \neq k} \frac{(1 - |a_k|^2)^{\frac{n+1}{p} + \alpha} (1 - |a_j|^2)^{\frac{n+1}{q} + \beta}}{|1 - \bar{a}_k a_j|^{n+1+\alpha p}} \right)^{\frac{q}{p}}
\]
\[
\times \sum_{j: j \neq k} \frac{(1 - |a_k|^2)^{\frac{n+1}{p} + \alpha} (1 - |a_j|^2)^{\frac{n+1}{q} + \beta}}{|1 - \bar{a}_k a_j|^{n+1+\alpha p}} \|v_j\|^p 
\]
\[
\leq c^p \sum_{j=1}^{\infty} (1 - |a_j|^2)^{n+1+\alpha p} \|v_j\|^p \sum_{k: k \neq j} \frac{(1 - |a_k|^2)^{\frac{n+1}{q} + \beta} (1 - |a_k|^2)^{\frac{n+1}{p} + \alpha}}{|1 - \bar{a}_k a_j|^{n+1+\alpha p}} \leq (c_1 c_2)^p \|v\|_{\ell^p_{n+1+\alpha}}.
\]

This shows that \( TE \) is invertible and, as in Theorem 4.2, \( T \) is onto. \( \square \)

Similar corollaries to 4.3 and 4.4 can be derived from Theorem 4.5, with some restrictions on the values \( p \) and \( \alpha \).
Corollary 4.6. Let $p > 1$, $q$ be the conjugated exponent and denote $A(\alpha, p) = (n + 1 + \alpha p) \min(1/p, 1/q)$.

(a) If there exists $c_0 < 1$ such that

$$K(\{a_k\}, A(\alpha, p), A(\alpha, p)) \leq 2^{-(n+1+\alpha p)|1-p|/p} c_0$$

then $\{a_k\}$ is $B^p_\alpha$- interpolating.

(b) Let $\{a_k\}_k$ be separated. There exists $\alpha > 0$ such that $\{a_k\}$ is $B^p_\alpha$- interpolating.

(c) Let $A(\alpha, p) > n$. There exists $\delta > 0$ such that any sequence $\{a_k\}_k$ verifying $d_G(a_j, a_k) \geq \delta$ for all $j \neq k$ is $B^p_\alpha$- interpolating.

(d) Let $A(\alpha, p) > n$. Then $\{a_k\}_k$ is a finite union of separated sequences if and only if it is a finite union of $B^p_\alpha$- interpolating sequences.

Part (c), like (b) of Corollary 4.3, is a particular case of a theorem of Rochberg which actually shows that the result holds for any $p > 0$ and $\alpha > -1/p$ (see [Ro, pag. 231]).

Proof. (a) is the analog of [Th1, Corollary 3.3]. (b), (c) and (d) are derived from (a), like in Corollaries 4.3 and 4.4. □

In order to use Theorems 4.2 and 4.5 (and their corollaries) for a given sequence $\{a_k\}_k$ it can be useful to know whether it is enough to verify the conditions given therein for a sequence obtained from the original one by taking off a finite number of points. This is equivalent to asking whether the process of adding a finite number of points to a $B^p_\alpha$- interpolating sequence gives again a sequence that is $B^p_\alpha$- interpolating.

Theorem 4.7. The union of a $B^p_\alpha$- interpolating sequence and a finite number of points is again $B^p_\alpha$- interpolating.

Proof. It is enough to show that the union of a $B^p_\alpha$- interpolating sequence and one point is $B^p_\alpha$- interpolating, and by invariance under automorphisms of the $B^p_\alpha$- interpolating sequences, we can assume that this point is 0. Let then $\{a_k\}_k$ be the original sequence and let $\delta > 0$ be such that $|a_k| \geq \delta$, for all $k$.

We claim first that it is enough to find $f \in B^p_\alpha$ with $f(a_k) = 0$ for all $k$ and $f(0) \neq 0$. To see this let $v_k \in \ell^p_{p+1+\alpha}(\{a_k\}_k \cup 0)$, and let $g \in B^p_\alpha$ be such that $g(a_k) = v_k$. Then the function

$$F(z) = g(z) + \frac{v_0 - g(0)}{f(0)} f(z)$$

belongs to $B^p_\alpha$ and $F(0) = v_0$, $F(a_k) = v_k$ for all $k$.

Suppose that all $f \in B^p_\alpha$ with $f(a_k) = 0$ for all $k$ have $f(0) = 0$. This implies that for any $f \in B^p_\alpha$, the value $f(0)$ is determined by the values $f(a_k)$, since the difference of two functions with the same values on $\{a_k\}_k$ vanishes at 0.
Assume $1 \leq p < \infty$ and define the functional $\Lambda : \ell^p \rightarrow \mathbb{C}$ by

$$\Lambda(\{v_k\}) = f(0),$$

where $f \in B^p_\alpha$ is such that $f(a_k) = (1 - |a_k|^2)^{-(\frac{n+1}{p} + \alpha)} v_k$ for all $k$. Since $f(0)$ is determined only by these values, which are actually independent of $f$, we have that $\Lambda$ is linear. It is also continuous:

$$|\Lambda(b)| = |f(0)| \leq c\|f\|_{p, \alpha} \leq cM\|\{(1 - |a_k|^2)^{-(\frac{n+1}{p} + \alpha)} v_k\}\|_{p, \frac{n+1}{p} + \alpha} = cM\|v\|_p,$$

$M$ denoting the interpolation constant of $\{a_k\}_k$. So $\Lambda \in \ell^q$, in the sense that there exists $\{c_k\}_k \in \ell^q$ such that

$$\Lambda(v) = \sum_{k=1}^{\infty} v_k c_k \quad \forall \ v = \{v_k\} \in \ell^p.$$

Consider now the sequences $v^j = \{\delta_{jk}\}_k \in \ell^p$ and a function $f_j \in B^p_\alpha$ with $f_j(a_k) = (1 - |a_k|^2)^{-(\frac{n+1}{p} + \alpha)} \delta_{jk}$. By definition

$$\Lambda(v^j) = f_j(0) = \sum_{k=1}^{\infty} \delta_{jk} c_k = c_j.$$

Take now the functions $F_j(z) = (|a_j|^2 - \bar{a}_j z) f_j(z)$. Obviously $F_j \in B^p_\alpha$ and $F_j(a_k) = 0$ for all $k$. Therefore $F_j(0) = |a_j|^2 f_j(0) = |a_j|^2 c_j = 0$, hence, $c_j = 0$. This shows that $\Lambda \equiv 0$, which is evidently false, since there are many functions in $B^p_\alpha$ not vanishing at 0.

The case $0 < p < 1$ is solved in the same way, using that the dual of $\ell^p$ is $\ell^\infty$. For the case $p = \infty$ we can restrict the functional $\Lambda$ to the subspace $c_0 \subset \ell^\infty$ of sequences with limit 0 and apply the same argument. □

§5. Appendix.

Given a sequence $\{a_k\}_k \subset B^1$ consider its upper uniform density

$$D(\{a_k\}) = \limsup_{r \to 1} \sup_{z \in B^1} \frac{\sum_{1/2 < |\varphi_z(a_k)| < r} \log \frac{1}{|\varphi_z(a_k)|}}{\log \frac{1}{1-r}}.$$

Lemma 5.1. Let $\{a_k\}_k \subset B^1$. Then $\{a_k\} \subset \text{Int}(B^p_\alpha)$ if and only if $D(\{a_k\}) < \alpha + 1/p$.

Proof. Assume first that $D(\{a_k\}) < \alpha + 1/p$ and define $\varepsilon = 1/2(\alpha + 1/p - D(\{a_k\}))$. There exist functions $f_k \in B^\infty_{\alpha+1/p-\varepsilon}$ and $C > 0$ with $\|f_k\|_\infty, \alpha+1/p-\varepsilon \leq C$ for all $k$ and $f_k(a_j) = \delta_{jk}(1 - |a_k|^2)^{-(\alpha+1/p)+\varepsilon}$ (see [Se, pag 34]). As in Theorem 3.3 (see the remark at the end of the proof) this implies that $\{a_k\}_k$ is $B^p_{\alpha'}$-interpolating, for any $(p', \alpha')$ such that $\alpha' + 1/p' > \alpha + 1/p - \varepsilon$, and in particular for $\alpha$ and $p$.

Assume now that $\{a_k\}_k$ is $B^p_{\alpha'}$-interpolating.
Lemma 5.2. Let $z \in B^1$ such that $d(z, \{a_k\}) \geq \delta_0$. There exists $f \in B^p_\alpha$ with $f(a_k) = 0$ for all $k$, $f(z) = 1$ and $\|f\|_{p,\alpha} \leq 1 + M\delta_0^{-1}$, where $M$ denotes the constant of interpolation of $\{a_k\}$.

Proof. By invariance under automorphisms we can suppose that $z = 0$. There exists $f_0 \in B^p_\alpha$ such that $f_0(a_1) = 1/a_1$, $f_0(a_j) = 0$ for all $j \geq 2$ and $\|f_0\|_{p,\alpha} \leq M\delta_0^{-1}$. Then the function $f(z) = 1 - zf(z)$ satisfies all the requirements. □

Since $M^p_p(f, r) = \int_0^{2\pi} |f(re^{i\theta})|^p d\theta$ is an increasing function of $r$,

$$\|f\|^p_{p,\alpha} = \int_0^1 2\pi M^p_p(f, r)(1 - r^2)^{\alpha p} r \, dr \geq M^p_p(f, r_0) \frac{\pi}{\alpha p + 1}(1 - r_0^2)^{\alpha p + 1}.$$

Hence, $M^p_p(f, r) \leq \|f\|^p_{p,\alpha}(1 - r^2)^{-(\alpha p + 1)}$. By Jensen’s inequality

$$\exp \left( \int_0^{2\pi} \log |f(re^{i\theta})| \frac{d\theta}{2\pi} \right) \leq M^p_p(f, r) \leq C(1 - r^2)^{-(\alpha p + 1)},$$

and thus

$$\int_0^{2\pi} \log |f(re^{i\theta})| \frac{d\theta}{2\pi} \leq \frac{1}{p} \log C + (\alpha + \frac{1}{p}) \log \left( \frac{1}{1 - r^2} \right).$$

\(\diamondsuit\)From Jensen’s formula it follows now that

$$\sum_{1/2 < |\varphi_z(a_k)| < r} \log \frac{r}{|\varphi_z(a_k)|} \leq \frac{1}{p} \log C + (\alpha + \frac{1}{p}) \log \left( \frac{1}{1 - r^2} \right),$$

and therefore $D^+(\{a_k\}) \leq \alpha + 1/p$. To see that the inequality is strict take a sequence $\{a'_k\}_k$ and $\delta_0$ so that $d(a_k, a'_k) < \delta_0$ for all $k$ and $(1 + \delta_0)D(\{a_k\}) \leq D(\{a'_k\})$ (see the proof of [Se, Lemma 6.6]). An application of the argument above to the sequence $\{a'_k\}_k$ shows finally that $D(\{a_k\}) < \alpha + 1/p$. □

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