Eigenfunction expansions and scattering theory associated with the Dirac equation

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Abstract

The classical Lippmann-Schwinger equation (LS equation) plays an important role in the scattering theory for the non-relativistic case (Schrödinger equation). In our previous paper [9], we consider the relativistic analogue of the Lippmann-Schwinger equation (RLS equation). We represent the corresponding equation in the integral form. In the present paper, we use the corresponding integral equation and investigate the scattering problems for both stationary and dynamical cases. Our approach allows us to develop a RLS equation theory which is comparable in its completeness with the theory of the LS equation. In particular, we consider the eigenfunction expansion associated with the relativistic Dirac equation. We note that the works on the theory of the LS equation (see [5, 6, 11]) serve as a model for us.

1 Introduction

The classical integral Lippmann-Schwinger equation (LS-equation) plays an important role in the scattering theory (non-relativistic case, Schrödinger equation). The relativistic analogue of the Lippmann-Schwinger equation was formulated in the terms of the limit values of the corresponding resolvent. In our previous paper [9] we found the limit values of the resolvent in the explicit form. Using this result we represented relativistic Lippmann-Schwinger equation (RLS equation) as an integral equation. In the present
paper we use the corresponding integral equation and investigate the scattering problems, stationary and dynamical cases. Our approach allows to develop a RLS equation theory that is comparable in its completeness to the theory of LS equation. In particular we consider the eigenfunction expansion associated with relativistic Dirac equation. We note that the works on the theory of the LS equation (see [5], [6], [11]) serves as a model for us. In the paper we use RLS equation by study the stationary scattering problem, when the distance \( r \to \infty \), and dynamical scattering problem, when the time \( t \to \infty \).

Let us describe the content of the present paper. In section 2 we formulate the results of our previous paper [9]. Our approach to spectral and scattering problems is essentially based on this previous results. The RLS equation we consider now not in the \( 4 \times 1 \) vector space (see [9]) but in the \( 4 \times 4 \) matrix space. The transition to the matrix case leads to a more natural approach to the scattering and spectral problems. In section 3 we introduce the operator function \( K(\mu) \) and investigate properties of this function. Further the operator function \( K(\mu) \) plays an important role in the scattering theory. Section 4 is devoted to the construction of the Green’s function \( G(r, s, \lambda) \) and the eigenfunction \( \psi(r, \lambda) \) that corresponds to the absolutely continuous spectrum. We investigate the properties of \( G(r, s, \lambda) \) and \( \psi(r, \lambda) \). In section 5 we investigate the wave operators \( W_\pm(\mathcal{L}, \mathcal{L}_0) \) and the scattering operator \( S(\lambda) \). We prove that the wave operators \( W_\pm(\mathcal{L}, \mathcal{L}_0) \) are complete.

2 RLS equation in the integral form

1. Let us write the Dirac equation (see [1])

\[
\frac{i}{\hbar} \frac{\partial}{\partial t} u(r, t) = \mathcal{L}u(r, t),
\]

where \( u(r, t) \) is \( 4 \times 1 \) vector function, \( r = (r_1, r_2, r_3) \). The operators \( \mathcal{L} \) and \( \mathcal{L}_0 \) are defined by the relations

\[
\mathcal{L}u = [-e\nu(r)I_4 + m^2 + \alpha(p + eA(r))]u, \quad \mathcal{L}_0u = (m^2 + \alpha p)u.
\]

Here \( p = -i\nabla \), \( \nu \) is a scalar potential, \( A \) is a vector potential, \( (-e) \) is the electron charge. Now let us define \( \alpha = [\alpha_1, \alpha_2, \alpha_3] \). The matrices \( \alpha_k \) are the \( 4 \times 4 \) matrices of the forms

\[
\alpha_k = \begin{pmatrix}
0 & \sigma_k \\
\sigma_k & 0
\end{pmatrix}, \quad k = 1, 2, 3,
\]

(2.3)
where
\[ \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \] (2.4)

The matrices \( \beta \) and \( I_2 \) are defined by the relations
\[ \beta = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix}, \quad I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \] (2.5)

2. We consider separately the unperturbed Dirac equation (2.1), (2.2), when \( \nu(r) = 0 \) and \( A(r) = 0 \). The Fourier transform is defined by relation
\[ \Phi(q) = Fu(r) = (2\pi)^{-3/2} \int_{R^3} e^{iqr} u(r) dr. \] (2.6)

The inverse Fourier transform has the form
\[ u(r) = F^{-1} \Phi(q) = (2\pi)^{-3/2} \int_{Q^3} e^{-iqr} \Phi(q) dq. \] (2.7)

In the momentum space the unperturbed Dirac equation takes the form (see [1], Ch.IV):
\[ i \frac{\partial}{\partial t} \Phi(q,t) = H_0(q) \Phi(q,t), \quad q = (q_1, q_2, q_3), \] (2.8)

where \( H_0(q) \) and \( \Phi(q,t) \) are matrix functions of order \( 4 \times 4 \) and \( 4 \times 1 \) respectively. The matrix \( H_0(q) \) is defined by the relation
\[ H_0(q) = \begin{bmatrix} m & 0 & q_3 & q_1 - iq_2 \\ 0 & m & q_1 + iq_2 & -q_3 \\ q_3 & q_1 - iq_2 & -m & 0 \\ q_1 + iq_2 & -q_3 & 0 & -m \end{bmatrix}. \] (2.9)

The eigenvalues \( \lambda_k \) and the corresponding eigenvectors \( g_k \) of \( H_0(q) \) are important in our theory. We have (see [9]):
\[ \lambda_{1,2} = -\sqrt{m^2 + |q|^2}, \quad \lambda_{3,4} = \sqrt{m^2 + |q|^2} \quad (|q|^2 := q_1^2 + q_2^2 + q_3^2); \] (2.10)
\[ g_1 = \begin{bmatrix} (-q_1 + iq_2)/(m + \lambda_3) \\ q_3/(m + \lambda_3) \\ 0 \\ 1 \end{bmatrix}, \quad g_2 = \begin{bmatrix} -q_3/(m + \lambda_3) \\ (-q_1 - iq_2)/(m + \lambda_3) \\ q_1 - iq_2 \\ 1 \end{bmatrix}, \] (2.11)
\( g_3 = \begin{bmatrix} (-q_1 + iq_2)/(m - \lambda_3) \\ q_3/(m - \lambda_3) \\ 0 \\ 1 \end{bmatrix}, \quad g_4 = \begin{bmatrix} -q_3/(m - \lambda_3) \\ (-q_1 - iq_2)/(m - \lambda_3) \end{bmatrix} \). \tag{2.12}

It follows from (2.9) and (2.10) that
\[
(H_0(q) - \lambda)^{-1} = H_0^{-1}(q) + H_0^{-1}(q) \frac{\lambda^2}{\lambda^2(|q|) - \lambda^2} + \frac{\lambda}{\lambda^2(|q|) - \lambda^2}.
\tag{2.13}
\]

This equality is valid at all \( \lambda \notin E \), where \( E = (-\infty, -m] \cup [m, +\infty) \).

3. Now we will introduce a relativistic analogue of the Lippmann-Schwinger equation (RLS integral equation) which was constructed in the paper [9].

To do it we consider the expression
\[
B_+(r, \lambda) = F^{-1}[H_0(q) - (\lambda + i0)]^{-1},
\tag{2.14}
\]
where \( \lambda = \bar{\lambda} \), \( |\lambda| > m \). Let us write the following relation (see [3], formula 721).
\[
J_1(r) = F^{-1}[(m^2 + |q|^2)^{-1}] = m^{1/2} K_{1/2}(m|q|)/|r|^{1/2},
\tag{2.15}
\]
where \( K_p(z) \) is the modified Bessel function. It is known that (see [4])
\[
K_{1/2}(z) = K_{-1/2}(z) = \sqrt{\frac{\pi}{2z}} e^{-z}.
\tag{2.16}
\]

According to (2.15) and (2.16) the equality
\[
J_1(r) = \sqrt{\frac{\pi}{2}} (e^{-m|q|}/|r|)
\tag{2.17}
\]
is valid. Using (2.17) we obtain
\[
J_2(r) = F^{-1}\left(\frac{q_k}{m^2 + |q|^2}\right) = i \frac{\partial}{\partial r_k} J_1(r) = -\sqrt{\frac{\pi}{2}} e^{-m|q|} \frac{T_k}{|r|^2} (m + 1/|r|).
\tag{2.18}
\]

Let us calculate the expression
\[
J_+(r, \lambda) = F^{-1}\left[\frac{1}{\lambda^2(q) - (\lambda + i0)^2}\right], \quad \lambda \in E.
\tag{2.19}
\]

In view of (2.10) and (2.19) we have
\[
J_+(r, \lambda) = -\frac{F}{m^2 - |q|^2 - i(\text{sgn}\lambda)0}, \quad \lambda \in E.
\tag{2.20}
\]
where $k^2 = \kappa^2 = \lambda^2 - m^2$. Taking into account formulas 719 and 720 from the book [3] (table of the Fourier transformation) and relation (2.16) we obtain the equalities

$$J_+(r, \lambda) = \sqrt{\frac{\pi}{2}} e^{i\kappa |r|/|r|}, \quad \lambda > m,$$

$$J_+(r, \lambda) = \sqrt{\frac{\pi}{2}} e^{-i\kappa |r|/|r|}, \quad \lambda < -m.$$  

(2.21)

Formulas (2.17) and (2.18) imply that

$$Q(r) = F^{-1}[H_0^{-1}(q)] = \sqrt{\pi/2} e^{-m|r|} |m\beta - (m + 1/|r|)r\alpha/|r||/|r|. \quad (2.23)$$

Here $r\alpha = r_1 \alpha_1 + r_2 \alpha_2 + r_3 \alpha_3$, matrices $\alpha_k$ and $\beta$ are defined by the relations (2.3)-(2.5). Due to (2.16), (2.20) and (2.23) we have

$$B_+(r, \lambda) = Q(r) + (2\pi)^{3/2} \lambda^2 Q(r) * J_+(r, \lambda) + \lambda J_+(r, \lambda), \quad (2.24)$$

where $F(r) * G(r) = \int_{R^3} F(r-v) G(v) dv$ is the convolution of $F(r)$ and $G(r)$. Now we can write the equation

$$\phi_+(r, k, n) = e^{ikr} \hat{g}_n(k) - (2\pi)^{-3/2} \int_{R^3} B_+(r-s, \lambda) V(s) \phi_+(s, k, n) ds, \quad (2.25)$$

where

$$V(r) = -e\nu(r) I_4 + e\alpha A(r). \quad (2.26)$$

Here the vectors $g_n(k)$ are defined by the relations (2.11), (2.12) and $\hat{g}_n(k) = g_n(k)/\|g_n\|$. 

Equation (2.25) (RLS equation) is relativistic analogue of the Lippmann-Schwinger equation. 

Consider the $4\times 4$ matrices

$$Z_0(k) = [\hat{g}_1(k), \hat{g}_2(k), \hat{g}_3(k), \hat{g}_4(k)], \quad (2.27)$$

$$\hat{\phi}(r, k, \lambda) = [\hat{\phi}(r, k, \lambda, 1), \hat{\phi}(r, k, \lambda, 2), \hat{\phi}(r, k, \lambda, 3), \hat{\phi}(r, k, \lambda, 4)], \quad (2.28)$$

where

$$\hat{\phi}(r, k, \lambda, n) = \phi_+(r, k, \lambda, n)/\|g_n(k)\|. \quad (2.29)$$

RLS equation (2.25) can be rewritten in the form

$$\hat{\phi}_+(r, k) = e^{ikr} Z_0(k) - (2\pi)^{-3/2} \int_{R^3} B_+(r-s, \lambda) V(s) \hat{\phi}_+(s, k) ds. \quad (2.30)$$
4. Further we assume that the matrix $V(r)$ is self-adjoint, 
\[ V(r) = V^*(r). \]  
(2.31)

Hence $V(r)$ can be represented in the form
\[ V(r) = U(r)D(r)U^*(r), \]  
(2.32)
where $U(r)$ is an unitary matrix, $D(r)$ is a diagonal matrix
\[ D(r) = \text{diag}(d_1(r), d_2(r), d_3(r), d_4(r)). \]  
(2.33)

Let us introduce the diagonal matrices
\[ D_1(r) = \text{diag}(|d_1(r)|^{1/2}, |d_2(r)|^{1/2}, |d_3(r)|^{1/2}, |d_4(r)|^{1/2}) \]  
(2.34)
and
\[ W(r) = \text{diag}(\text{sign}d_1(r), \text{sign}d_2(r), \text{sign}d_3(r), \text{sign}d_4(r)). \]  
(2.35)

Formulas (2.32)-(2.35) imply that
\[ V(r) = V_1(r)W_1(r)V_1(r), \]  
(2.36)
where
\[ V_1(r) = U(r)D_1(r)U^*(r), \quad W_1(r) = U(r)W(r)U^*(r) \]  
(2.37)

It is easy to see that
\[ \|V_1(r)\|^2 = \|V(r)\|, \quad \|W_1(r)\| = 1. \]  
(2.38)

**Modified RLS integral equation.**

If the $4 \times 4$ matrix function $\phi_+(r, k)$ is a solution of RLS equation, then the matrix function $\psi_+(r, k, n) = V_1(r)\phi_+(r, k, n)$ is a solution of following modified RLS integral equation:
\[ \hat{\psi}_+(r, k) = e^{ikr}V_1(r)Z_0(k) - (2\pi)^{-3/2}B_+(\lambda)\hat{\psi}_+(r, k), \]  
(2.39)
where
\[ B_+(\lambda)f = \int_{R^3} V_1(r)B_+(r - s, \lambda)V_1(s)W_1(s)f(s)ds. \]  
(2.40)

We note that the operators $B_+(\lambda)$ act in the Hilbert space $L^2(R^3)$. (We say that the matrix belongs to the Hilbert space $L^2(R^3)$ if the every element of the matrix belongs to the Hilbert space $L^2(R^3)$.)

We have proved the following result [9].
Theorem 2.1 Let condition (2.31) be fulfilled. If the function \( \|V(r)\| \) is bounded and belongs to the space \( L^1(\mathbb{R}^3) \) then the operator \( B_+(\lambda) \) is compact.

5. Let us introduce the following definition [9]

Definition 2.2 We say that \( \lambda \in E \) is an exceptional value if the equation 
\[
[I + (2\pi)^{-3/2}B_+(\lambda)]\psi = 0
\]
has nontrivial solution in the space \( L^2(\mathbb{R}^3) \).

We denote by \( \mathcal{E}_+ \) the set of exceptional points and we denote by \( E_+ \) the set of such points \( \lambda \) that \( \lambda \in E, \lambda \notin \mathcal{E}_+ \).

We have [9]:

Lemma 2.3 Let conditions of Theorem 2.1 be fulfilled. If \( \lambda \in E_+ \), then equation (2.39) has one and only one solution \( \hat{\psi}_+(r, k) \) in \( L^2(\mathbb{R}^3) \).

Corollary 2.4 Let conditions of Theorem 2.1 be fulfilled. If \( \lambda \in E_+ \), then equation \( (2.30) \) has one and only one solution \( \hat{\phi}_+(r, k) \) which satisfies the condition \( V_1(r)\hat{\phi}_+(r, k) \in L^2(\mathbb{R}^3) \).

5. The following Theorem gives the connection between spectral and scattering results [9].

Theorem 2.5 Let the \( V(r) = V^*(r) \) and the function \( \|V(r)\| \) be bounded and belong to the space \( L^1(\mathbb{R}^3) \). If
\[
\|V_1(r)\| = O(|r|^{-3/2}), \quad |r| \to \infty,
\]
then the solution \( \hat{\phi}_+(r, k) \) of RLS equation (2.30) has the form
\[
\hat{\phi}_+(r, k) = e^{ikr}Z_0(k) + e^{ik|r|}|r|^{-1/2}f(\omega, k) + o(1/|r|), \quad |r| \to \infty,
\]
where \( \lambda \in E_+ \), \( k^2 = \nu^2 = \lambda^2 - m^2 \), \( \omega = r/|r| \) and
\[
f(\omega, k) = -\frac{1}{4\pi} \lambda \int_{\mathbb{R}^3} e^{-i\nu(s\omega)}V(s)\hat{\phi}_+(s, k)ds.
\]
(We note that \( k \) is a vector from the space \( \mathbb{R}^3 \), \( \nu \) is a number and \( k^2 = \nu^2 \).)

Definition 2.6 The \( 4 \times 4 \) matrix function \( f(\omega, k) \) is named the relativistic scattering amplitude.
Remark 2.7 The solution \( f(\omega, k) \) of the stationary scattering problem (\(|r| \to \infty\)) is expressed by formula (2.43) with the help of the solution \( \hat{\phi}(r, k) \) of the spectral problem.

Proposition 2.8 The matrix \( Z_0(k) \) is unitary and
\[
H_0(k) = Z_0(k)D(k)Z_0^*(k),
\]
and the diagonal matrix \( D(k) \) is defined by the relation
\[
D(k) = \text{Diag}[\lambda_1(k), \lambda_2(k), \lambda_3(k), \lambda_4(k)].
\]

6. Now we shall formulate the connection between solutions of the equation
\[
\mathcal{L}\phi = \lambda\phi
\]
and the solutions of the RLS equation (2.30) (see [9]).

Theorem 2.9 Let the matrix function \( V(r) \) satisfies the conditions of the Theorem 2.1 and let the matrix function \( \phi(r) \) satisfies the equation (2.46). We assume that the following conditions are fulfilled:
\[
\psi = V_1\phi|L^2(R^3) \text{ and } \kappa_Q\phi|L^2(R^3),
\]
where \( \lambda \in E \) and \( \kappa_Q \) is the characteristic function of a bounded domain \( Q \).
Then the matrix function \( \psi(r) \) satisfies the equation
\[
\psi(r) = -(2\pi)^{-3/2}B_+(\lambda)\psi(r), \quad \lambda \in E.
\]

Corollary 2.10 Let the conditions of the Theorem 2.1 be fulfilled. If \( \lambda \in E \) is an eigenvalue of the corresponding operator \( \mathcal{L} \), then \( \lambda \in E_+ \).

Theorem 2.11 Let the conditions of Theorem 2.1 be fulfilled and let the function \( \phi_+(r, k) \) be a solution of RLS equation (2.30) such that
\[
V_1(r)\phi_+(r, k)|L^2(R^3).
\]
If \( \lambda \in E_+ \) then the function \( \phi_+(r, k) \) is the solution of the equation (2.46) in the distributive sense.

3 Definition and properties of the holomorphic operator function \( K(\mu) \).

1. Let us consider the case \( V(r) = 0 \) separately. Taking into account (2.24) we see that \( B(r, \mu) \) is defined when \( \mu \notin E, \Im \lambda > 0, \lambda = \sqrt{\mu^2 - m^2} \):
\[
B_+(r, \mu) = Q(r) + (2\pi)^{3/2}\mu^2Q(r) \ast J_+(r, \mu) + \lambda J_+(r, \mu),
\]
where
\[ J_+(r, \mu) = \sqrt{\frac{\pi}{2}} e^{i\kappa |r|/|r|}, \quad \Im \kappa > 0. \] (3.2)

We introduce the operator
\[ B_0(\mu)f = \int_{\mathbb{R}^3} B_+(r-s, \mu)f(s)ds, \] (3.3)

where \( f(r) \) is a 4×4 matrix function with elements in the space \( L^2(\mathbb{R}^3) \). It is easy to see that
\[ \int_{\mathbb{R}^3} \|B_+(r, \mu)\|dr < \infty, \quad \mu \notin E, \quad \Im \kappa > 0. \] (3.4)

Hence the operator \( B_0(\mu) \) is bounded in the space \( L^2(\mathbb{R}^3) \). We have
\[ (\mathcal{L}_0 - \mu)^{-1}f = (2\pi)^{-3/2} \int_{\mathbb{R}^3} B_+(r-s, \mu)f(s)ds, \quad \mu \notin E, \quad \Im \kappa > 0. \] (3.5)

2. Now we consider the operators (see (2.40):
\[ B_+(\mu)f = \int_{\mathbb{R}^3} V_1(r)B_+(r-s, \mu)V_1(s)W_1(s)f(s)ds, \quad \mu \notin E, \quad \Im \kappa > 0 \] (3.6)

and
\[ K(\mu) = I + (2\pi)^{-3/2}B_+(\mu). \] (3.7)

**Lemma 3.1** Let the conditions of theorem 2.1 be fulfilled. Then the operator function \( K(\mu) \) has the following properties in the region \( \mu \notin E \):
1) The operator function \( K(\mu) - I \) is holomorphic.
2) The operator \( K(\mu) - I \) is compact.
3) For some \( \mu_0 \) the operator \( K(\mu_0) \) has a bounded inverse operator.

**Proof.** The assertion 1) follows from (3.1) and (3.5). The assertion 2) is proved in the paper ([9], Theorem 3.1) for the case when \( \lambda \in E \). The proof is correct for the case when \( \lambda \notin E, \quad \Im \kappa > 0 \) too. To prove the assertion 3) we use the following property of the self-adjoint operators
\[ \|(\mathcal{L}_0 - \mu)^{-1}\| \leq |\Im \mu|^{-1}. \] (3.8)
Then according to (3.5) - (3.8) we obtain
\[ \|B_+(\mu)\| \to 0, \quad K(\mu) \to I, \quad \Im \mu \to \infty. \] (3.9)

The assertion 3) follows directly from (3.9). The lemma is proved.

We need the following assertion (see [11], p.72).

**Lemma 3.2** Let the conditions of Theorem 2.1 be fulfilled. Then
\[ (L-\mu)^{-1} - (L_0-\mu)^{-1} = - (L_0-\mu)^{-1} V_1 W_1 [I + B_+(\mu)]^{-1} V_1 (L_0-\mu)^{-1}, \] (3.10)
where \( \mu \notin \mathcal{E} \).

Relation (3.10) and Lemma 3.1 imply the corollary:

**Corollary 3.3** Let the conditions of theorem 2.1 be fulfilled. Then
1) The operator \( L \) has only discrete spectrum in the interval \(-m < \lambda < m\).
2) The discrete spectrum \(-m < \lambda_n < m\) of the operator \( L \) has no limit points in the interval \(-m < \lambda < m\).
3) The set \( \mathcal{E}_+ \) is closed and has Lebesgue measure equal to zero.

**Proof.** Assertion 1) follows from relation (3.10) and Lemma 3.1.
Lemma 3.1 and Theorem 1 ([3], Appendix II) imply the assertion 2).
Assertion 3) follows from Lemma 3.1 in the same way as in case of classical Lippmann-Schwinger equation ([6], p.115).
It is easy to see that the following assertion is true.

**Proposition 3.4** The operator \( L_0 \) has only absolutely continuous spectrum.

## 4 Green matrix-function and eigenfunction expansion

1. Multiplying (3.10) on the left \( V_1 \) or on the right \( V_1 W_1 \) and using Theorem 2.1 we conclude.

**Corollary 4.1** Let conditions of Theorem 2.1 be fulfilled and \( \Im \mu \neq 0 \). Then the operators \( V_1 (L-\mu)^{-1} \) and \( (L-\mu)^{-1} V_1 \) are compact.

Let us consider the integral equation
\[ G(r, s, \mu) = G_0(r, s, \mu) - \int_{R^3} G_0(r, z, \mu) V(z) G(z, s, \mu) dz, \] (4.1)
where \( G_0(r, s, \mu) = (2\pi)^{-3/2} B_+(r - s, \mu), \ \Im \mu > 0 \) (see (3.1) and (3.5)).
Theorem 4.2 Let conditions of Theorem 2.1 be fulfilled and \( \Im \mu \neq 0 \). Then

1) Equation (4.1) has one and only one solution such that
\[ V_1(r)G(r, s, \mu) \in L^2(\mathbb{R}^3), \text{ where } s \text{ is fixed.} \]

2) \( G^*(r, s, \mu) = G(s, r, \mu) \)

3) The following equality
\[ (\mathcal{L} - \mu)^{-1} f(r) = \int_{\mathbb{R}^3} G(r, s, \mu) f(s) ds. \] (4.2)

is valid.

4) The Green function \( G(r, s, \mu) \) has the property \( G(r, s, \mu) \in L^1(\mathbb{R}^3) \) for almost every \( s \).

Proof. The assertion 1) follows from (3.5). The operator \( \mathcal{L}_0 \) is self-adjoint. Hence the assertion 2) is true for \( G_0(r, s, \mu) \). Using equality (4.1) we prove the assertion 2). The equality (4.1) is equivalent to equality
\[ (\mathcal{L} - \mu)^{-1} - (\mathcal{L}_0 - \mu)^{-1} = (\mathcal{L}_0 - \mu)^{-1} V(\mathcal{L} - \mu)^{-1}. \] (4.3)

Then the equality (4.2) holds, i.e. the assertion 3) is proved. To prove 4) we need to show
\[ \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \|G_0(r, z, \mu)V(z)G(z, s, \mu)\| dz dr < \infty. \] (4.4)

We note that \( \int_{\mathbb{R}^3} \|G_0(r, z, \mu)\| dr \) is finite independent of \( z \). Using inequalities
\[ \|V_1(z)\| \in L^2(\mathbb{R}^3), \quad \|V_1(z)G(z, s, \lambda)\| \in L^2(\mathbb{R}^3) \] (4.5)

we obtain
\[ \int_{\mathbb{R}^3} \|V(z)G(z, s, \mu)\| dz < \infty. \] (4.6)

Hence (4.4) is proved. Assertion 4) follows directly from (4.1). The Theorem is proved.

2. Taking into account the assertions 2) and 4) we introduce the vector function
\[ g(r, k, \mu) = (2\pi)^{-3/2} \int_{\mathbb{R}^3} G(r, s, \mu) e^{iks} ds. \] (4.7)

It follows from (2.16), (4.2) and (4.7) that
\[ g_0(r, k, \mu) = (2\pi)^{-3/2} e^{ikr} [H_0(k) - \mu]^{-1}. \] (4.8)
We apply to (4.1) the Fourier transform $F$ with respect to $s$:

$$
h(r, k, \mu) = e^{iqr} - \int_{\mathbb{R}^3} G_0(r, z, \mu) V(z) h(z, k, \mu) dz, \quad (4.9)$$

where

$$h(r, k, \mu) = (2\pi)^{3/2} g(r, k, \mu) [H_0(k) - \mu]. \quad (4.10)$$

So, we have

$$h(r, k, \mu) = e^{ikr} - \int_{\mathbb{R}^3} G_0(r, z, \mu) W_1(z) V_1(z) p(z, k, \mu) dz, \quad (4.11)$$

where $p(r, k, \mu) = V_1(r) g(r, k, \mu)$ is unique solution of equation

$$p(r, k, \mu) = V_1(r) e^{ikr} - \int_{\mathbb{R}^3} V_1(r) G_0(r, z, \mu) W_1(z) V_1(z) p(z, k, \mu) dz, \quad (4.12)$$

Let us consider the matrix functions

$$\hat{h}(r, k, \mu) = h(r, k, \mu) Z_0(k), \quad \hat{p}(r, k, \mu) = p(r, k, \mu) Z_0(k), \quad (4.13)$$

where $Z_0(k)$ is defined by the relation (2.26). Then equalities (4.12) and (4.13) can be rewritten in the forms:

$$\hat{h}(r, k, \mu) = e^{ikr} Z_0(k) - \int_{\mathbb{R}^3} G_0(r, z, \mu) W_1(z) V_1(z) \hat{p}(z, k, \mu) dz, \quad (4.14)$$

$$\hat{p}(r, k, \mu) = e^{ikr} V_1(r) Z_0(k) - (2\pi)^{-3/2} B_+(\mu) \hat{p}(r, k, \mu), \quad (4.15)$$

where

$$B_+(\mu)f = \int_{\mathbb{R}^3} V_1(r) B_+(r - s, \mu) V_1(s) W_1(s) f(s) ds. \quad (4.16)$$

We assume that $f(r) \in C_0^\infty$. Then the integral

$$\Phi(k, \mu) = (2\pi)^{-3/2} \int_{\mathbb{R}^3} \hat{h}^*(r, k, \mu) f(r) dr, \quad (4.17)$$

converges absolutely. Let us introduce the denotion

$$\tilde{f}(k, \mu) = U f(r) = (2\pi)^{-3/2} \int_{\mathbb{R}^3} \hat{\phi}^*(r, k, \mu) f(r) dr. \quad (4.18)$$
Lemma 4.3 Let conditions of Theorem 2.1 be fulfilled and $\Im \mu \neq 0$. If $m < \alpha < \beta$, $\sqrt{x^2 + m^2} \in (\alpha, \beta)$ and $[\alpha, \beta] \in E_+$, then $\Phi(k, \mu)$, $(\Re \mu = \kappa, \|k\| = \kappa)$ can be extended uniformly continuous to the region:

$$\|k\| = \kappa, \quad \sqrt{x^2 + m^2} \in (\alpha, \beta).$$

and the following equality

$$\Phi(k, \kappa) = \tilde{f}(k, \kappa), \quad (\|k\| = \kappa) \quad (4.19)$$

is valid.

Proof. If $k \in \mathbb{R}^3$ and $\|k\| = \kappa$, then the equation (4.15) is identical to the equation (2.39). Hence, in this case we have: $\hat{p}(r, k, \kappa) = \hat{\psi}(r, k)$ and $\hat{h}(r, k, \kappa) = \hat{\phi}(r, k)$. The assertion of the lemma follows directly from the last relation.

In the same way as Lemma 4.3 we obtain

Lemma 4.4 Let conditions of Theorem 2.1 be fulfilled and $\Im \mu \neq 0$. If $\alpha < \beta < -m$, $-\sqrt{x^2 + m^2} \in (\alpha, \beta)$ and $[\alpha, \beta] \in E_+$, then $\Phi(k, \mu)$, $(\Re \mu = \kappa, \|k\| = -\kappa)$ can be extended uniformly continuous to the region

$$\|k\| = -\kappa, \quad -\sqrt{x^2 + m^2} \in (\alpha, \beta) \quad (4.20)$$

and the following equality

$$\Phi(k, \kappa) = \tilde{f}(k, \kappa), \quad (\|k\| = -\kappa) \quad (4.21)$$

is valid.

Definition 4.5 We denote by $\mathcal{H}_{ac}$ the absolutely continuous invariant subspace with respect to $\mathcal{L}$ and by $P_{ac}$ orthogonal projector on $\mathcal{H}_{ac}$.

Definition 4.6 We denote by $\mathcal{H}_{\alpha}$ the maximal invariant subspace with respect to $\mathcal{L}$ such that the spectrum of $\mathcal{L}$ belongs to the set $(-\infty, \alpha]$. We denote by $P_{\alpha}$ orthogonal projector on $\mathcal{H}_{\alpha}$.

We represent the $4 \times 1$ vector function $\tilde{f}(k, \mu)$ in the form

$$\tilde{f}(k, \mu) = \tilde{f}_1(k, \mu) + \tilde{f}_2(k, \mu),$$

where $\tilde{f}_1(k, \mu) = \text{col}[\tilde{f}_{1,1}(k, \mu), \tilde{f}_{2,1}(k, \mu), 0, 0]$ and $\tilde{f}_2(k, \mu) = \text{col}[0, 0, \tilde{f}_{1,2}(k, \mu), \tilde{f}_{2,2}(k, \mu)]$. 

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Lemma 4.7 Let conditions of Theorem 2.1 be fulfilled. If \(m < \alpha < \beta, (\alpha, \beta) \in E_+\), then

\[
\| (P_\beta - P_\alpha) f \|^2 = \int_{a < |k| < b} \| \hat{f}_1(k, \mathbf{z}) \|^2 dk,
\]

where \(a = \sqrt{\alpha^2 - m^2}, \quad b = \sqrt{\beta^2 - m^2}\).

Proof. Parseval equality for Fourier transform (4.7) implies that

\[
\int_{R^3} G^*(s, z, \mu) G(r, z, \mu) dz = \int_{R^3} g^*(s, k, \mu) g(r, k, \mu) dk,
\]

where \(\mu = \lambda + i\varepsilon, \lambda = \overline{\lambda}, \varepsilon > 0\). We introduce the notations

\[
R_\mu = [H(k) - \mu]^{-1}, \quad \mathcal{D}(k) = \text{diag}[\lambda_1(k), \lambda_2(k), \lambda_3(k), \lambda_4(k)]
\]

It follows from (2.27) and (4.24) that

\[
H(k) = Z_0^{-1}(k) \mathcal{D}(k) Z_0(k).
\]

Relations (4.11), (4.23) and (4.24) imply that

\[
\int_{R^3} G^*(s, z, \mu) G(r, z, \mu) dz = (2\pi)^{-3} \int_{R^3} h^*(s, k, \lambda) R_\pi R_\mu h(r, k, \mu) dk.
\]

Using (4.13) and (4.24), (4.25), we obtain

\[
\int_{R^3} G^*(s, z, \mu) G(r, z, \mu) dz = (2\pi)^{-3} \int_{R^3} \hat{h}^*(s, k, \mu) A(k, \lambda) \hat{h}(r, k, \mu) dk.
\]

where

\[
A(k, \mu) = [\mathcal{D}(k) - \overline{\mu}]^{-1} [\mathcal{D}(k) - \mu]^{-1}.
\]

We multiply both sides (4.27) of left hand side by \(f^*(s)\) and of right hand side by \(f(r)\) and integrate:

\[
(R_\pi f, R_\pi f) = \int_{R^3} \Phi^*(k, \mu) A(k, \mu) \Phi(k, \mu) dk,
\]

where the function \(\Phi(k, \mu)\) is defined by (4.17) and \(R_\mu = (\mathcal{L} - \mu)^{-1}\). Relation (4.29) implies that

\[
([R_\mu - R_\pi] f, f) = 2i\varepsilon \int_{R^3} \Phi^*(k, \mu) A(k, \mu) \Phi(k, \mu) dk.
\]
Now we use the fundamental relation (see [13], p.183).

\[
([P_\beta + P_{\beta-0}] - (P_\alpha + P_{\alpha-0})]f, f) = \frac{1}{i\pi} \lim_{\varepsilon \to +0} \int_\alpha^\beta ([R_{\lambda+i\varepsilon} - R_{\lambda-i\varepsilon}]f, f) d\lambda, \tag{4.31}
\]

where \(\varepsilon \to +0\). Further we need the well-known relation (see [14], p.31)

\[
J(\alpha, \beta) = \frac{1}{\pi} \lim_{\varepsilon \downarrow 0} \int_\alpha^\beta \frac{\varepsilon}{(c - \lambda)^2 + \varepsilon^2} f(\lambda, \varepsilon) d\lambda = f(a, o), \quad \alpha < c < \beta. \tag{4.32}
\]

Here \(f(\lambda, \varepsilon)\) is a continuous function of \(\lambda\) and \(\varepsilon\) for \(\lambda \in [\alpha, \beta]\) and \(0 \leq \varepsilon \leq \varepsilon_0 < \infty\). It is easy to see that

\[
J(\alpha, \beta) = 0, \quad c \notin [\alpha, \beta]. \tag{4.33}
\]

Let us write one more relation

\[
\lim_{\varepsilon \downarrow 0} \int_\alpha^\beta \frac{\varepsilon}{\sqrt{(c - \lambda)^2 + \varepsilon^2}} f(\lambda, \varepsilon) d\lambda = 0. \tag{4.34}
\]

Formulas (4.28) and (4.30)-(4.34) imply

\[
([P_\beta + P_{\beta-0}] - (P_\alpha + P_{\alpha-0})]f, f) = 2 \int_{a < ||k|| < b} \|\widetilde{f}_1(k, \kappa)\|^2 dk. \tag{4.35}
\]

Letting \(\alpha \to \beta\) we obtain that

\[
P_\beta = P_{\beta-0}. \tag{4.36}
\]

We note that relation (4.36) is proved for \(f \in C_0^\infty\). Hence this relation is valid for all \(f \in L^2(R^3)\). The assertion of the lemma follows directly from (4.35) and (4.36). In the same way as Lemma 4.7 can be proved the following result.

**Lemma 4.8** Let conditions of Theorem 2.1 be fulfilled. If \(\alpha < \beta < -m\) and \((\alpha, \beta) \in E_+\), then

\[
\| (P_\beta - P_\alpha) f \|^2 = \int_{a < ||k|| < b} \|\widetilde{f}_2(k, \kappa)\|^2 dk, \tag{4.37}
\]

and

\[
P_\beta = P_{\beta-0}, \tag{4.38}
\]

where \(b = \sqrt{\alpha^2 - m^2}\), \(a = \sqrt{\beta^2 - m^2}\).
Theorem 4.9 Let conditions of Theorem 2.1 be fulfilled. Then
1) There exists no eigenvalue of $\mathcal{L}$ which belongs to $E_+$.  
2) The spectrum of $\mathcal{L}$ on the $E_+$ is absolutely continuous.  
3) The set of singular spectrum of $\mathcal{L}$ belongs to $E_+$ and has Lebesgue measure equal to zero.

Proof. The assertion 1) follows from (4.36) and (4.38). To prove assertion 2) we introduce the function $\sigma_f(\beta) = (P_\beta f, f)$. Then we have
$$\sigma_f(\beta) - \sigma_f(\alpha) = ((P_\beta - P_\alpha)f, f) = \| (P_\beta - P_\alpha)f \|^2 \quad (4.39)$$
Relations (4.22), (4.37) and (4.39) imply that the function $\sigma_f(\beta)$ is absolutely continuous when $\beta \in E_+$. Hence the assertion 2) is proved. It follows from Corollary 3.3 and assertion 2) that singular spectrum of $\mathcal{L}$ belongs to $E_+$. We have proved [9] that $E_+$ has Lebesgue measure equal to zero. Hence the assertion 3) is proved. The theorem is proved.

Let us introduce the domain $D(R, \varepsilon, k)$ such that $\|k\| < R$ and $\|k - k_0\| > \varepsilon > 0$ for all $k_0$ satisfying either the condition $\sqrt{k_0^2 + m^2} \in E_+$ or the condition $-\sqrt{k_0^2 + m^2} \in E_+$.

Proposition 4.10 Let conditions of Theorem 2.1 be fulfilled. Then
$$\| P_{ac}f \|^2 = \lim_{R \to \infty, \varepsilon \to 0} \int_{D(R, \varepsilon, k)} \| \tilde{f}(k, \varkappa) \|^2 dk, \quad \varkappa = \|k\| \quad (4.40)$$
and
$$P_{ac}f = (2\pi)^{-3/2} \lim_{R \to \infty, \varepsilon \to 0} \int_{D(R, \varepsilon, k)} \hat{\phi}(r, k, \varkappa) \tilde{f}(k, \varkappa) dk \quad (4.41)$$
where $R \to \infty, \varepsilon \to 0$.

Proof. The equality (4.40) follows from (4.22) and (4.37). To prove (4.41) we consider $g(r) \in C_0^\infty$ and $m < \alpha < \beta$, $(\alpha, \beta) \in E_+$. Using (4.22) we obtain
$$\left(g, (P_\beta - P_\alpha)f\right) = \int_{a < \|k\| < b} \tilde{g}_1(k, \varkappa) \tilde{f}_1(k, \varkappa) dk. \quad (4.42)$$
It follows from (4.18) and (4.42) that
$$\left(P_\beta - P_\alpha\right)f = \int_{a < \|k\| < b} \hat{\phi}^*(r, k, \varkappa) \tilde{f}_1(k, \varkappa) dk. \quad (4.43)$$
The same type formula as (4.43) can be deduced for the case $-m > \beta > \alpha$, $(\alpha, \beta) \in E_+$.
Hence formula (4.41) is valid. Proposition is proved.
Remark 4.11 Suppose for simplicity that the corresponding operator $L$ has no singular and discrete spectra. Then formulas (4.18) and (4.40) are expansion formulas in terms of generalized eigenfunctions $\phi(x,k,\kappa)$.

Further we shall use the following notation:

$$\int_D g(k)dk = \lim_{R \to \infty, \varepsilon \to 0} \int_{D(R,\varepsilon,k)} g(k)dk, \quad R \to \infty, \quad \varepsilon \to 0.$$ (4.44)

5 Scattering theory

1. In this section we use the constructed relativistic Lippman-Schwing equation to study the scattering problems.

We introduce the operator function

$$\Theta(t) = \exp(itL)\exp(-itL_0).$$ (5.1)

The wave operators $W_\pm(L, L_0)$ are defined by the relation (see [6]).

$$W_\pm(L, L_0) = \lim_{t \to \pm \infty} \Theta(t)P_0.$$ (5.2)

Here $P_0$ is orthogonal projector on the absolutely continuous subspace $G_0$ with respect to the operator $L_0$. The limit in (5.2) supposed to be in the sense of strong convergence.

Remark 5.1 We consider the case when the operator $L_0$ is defined by the relation (2.2). In this case we have: $G_0 = R^3$ and $P_0 = I$.

We have proved the assertion [9].

Theorem 5.2 If $V(r) = V^*(r)$, the function $\|V(r)\|$ is bounded, belongs to the space $L^1(R^3)$ and

$$\int_{-\infty}^{+\infty} \left[ \int_{|r|>\varepsilon} \|V(r)\|^2dr \right]^{1/2}dt < \infty, \quad \epsilon > 0,$$ (5.3)

then the wave operators $W_\pm(L, L_0)$ exist.

Remark 5.3 Let the condition $V(r) = V^*(r)$ be fulfilled. If the function $\|V(r)\|$ is bounded and

$$\|V(r)\| \leq \frac{M}{|r|^\alpha}, \quad |r| \geq \delta > 0, \quad \alpha > 3,$$ (5.4)

then the wave operators $W_\pm(L, L_0)$ exist.
Definition 5.4 The scattering operator $S(\mathcal{L}, \mathcal{L}_0)$ is defined by the relation

$$S(\mathcal{L}, \mathcal{L}_0) = W^*(\mathcal{L}, \mathcal{L}_0)W(\mathcal{L}, \mathcal{L}_0).$$  \hspace{1cm} (5.5)$$

2. We need the notation (compare with (4.18)):

$$\bar{g}_0(k, \lambda) = (2\pi)^{-3/2} \int_{\mathbb{R}^3} \hat{\phi}_0^*(r, k, \lambda)g(r)dr,$$  \hspace{1cm} (5.6)

where $\lambda = \sqrt{k^2 + m^2}$ and

$$\hat{\phi}_0(r, k, \lambda) = e^{ikr}Z_0(k).$$  \hspace{1cm} (5.7)

Lemma 5.5 Let $g(r) \in C_0^\infty$ then the equality

$$e^{it\lambda} \int_{\mathbb{R}^3} \hat{\phi}^*(r, k, \lambda)g(r)dr = \int_{\mathbb{R}^3} \hat{\phi}^*(r, k, \lambda)[e^{it\mathcal{L}}g(r)]dr$$  \hspace{1cm} (5.8)

is valid.

Proof. We note that the matrix function $\hat{\phi}(r, k, \lambda)$ satisfies the equation (2.46). Let us represent the left-hand side of equality (5.8) as inner product in the Hilbert space $L^2(\mathbb{R}^3)$:

$$(e^{-it\lambda}\hat{\phi}(r, k, \lambda), g(r)) = (e^{-it\mathcal{L}}\hat{\phi}(r, k, \lambda), g(r)).$$  \hspace{1cm} (5.9)

The assertion of the lemma follows directly from (5.9).

We will begin by approving:

Proposition 5.6 Let the conditions of the Theorem 5.2 be fulfilled. If the $4 \times 1$ vector-function $g(r)$ belongs to $L^2(\mathbb{R}^3)$, then the $4 \times 1$ vector-function $G(r) = W_+(\mathcal{L}, \mathcal{L}_0)g(r)$ exists and

$$\hat{G}(k, \lambda) = \hat{g}_0(k, \lambda),$$  \hspace{1cm} (5.10)

Proof. Let us consider the expression $(f, W_+(\mathcal{L}, \mathcal{L}_0)g)$. We assume in addition that $g(r) \in C_0^\infty$ and the support of $\hat{f}$ belongs to interval $(\alpha, \beta)$, where $(\alpha, \beta) \in E_+$. Using relation

$$\frac{d}{dt}[e^{it\mathcal{L}}e^{-it\mathcal{L}_0}] = i e^{it\mathcal{L}}Ve^{-it\mathcal{L}_0},$$  \hspace{1cm} (5.11)
we have
\[
(f, W_+(\mathcal{L}, \mathcal{L}_0)g) - (f, g) = i \lim_{T \to \infty} \int_0^T (f, e^{it\mathcal{L}} V e^{-it\mathcal{L}_0} g) dt.
\] (5.12)

According to Abels limits we obtain (see [6], section 6, Lemma 5)
\[
(f, W_+(\mathcal{L}, \mathcal{L}_0)g) - (f, g) = i \lim_{\delta \to +0} \int_0^\infty e^{-t\delta} (f, e^{it\mathcal{L}} V e^{-it\mathcal{L}_0} g) dt.
\] (5.13)

We introduce
\[
G(r, t, \delta) = e^{-t\delta} e^{it\mathcal{L}} V e^{-it\mathcal{L}_0} g.
\] (5.14)

Relations (4.18) and (5.8) imply
\[
\hat{G}(k, \lambda, t, \delta) = (2\pi)^{-3/2} \int_{\mathbb{R}^3} e^{-it\lambda} \hat{\phi}^*(r, k, \lambda) V(r) [e^{-it\mathcal{L}_0} g(r)] dr,
\] (5.15)

where \(\lambda = \sqrt{k^2 + m^2} \in \mathbb{E}\).

\[
e^{-it\mathcal{L}_0} g(r) = (2\pi)^{-3/2} \int_{\mathbb{R}^3} e^{-it\lambda} Z_0(q) e^{iqr} \hat{g}_0(q) dq,
\] (5.16)

where \(\lambda = \sqrt{q^2 + m^2}\). Let us consider the integral
\[
J = \int_0^\infty e^{-t\delta} (f, e^{it\mathcal{L}} V e^{-it\mathcal{L}_0} g) dt.
\] (5.17)

Taking into account (4.40), (4.43), (4.44) and (5.14) we have
\[
J = \int_0^\infty \int_D \hat{f}^*(k) \hat{G}(k, \lambda, t, \delta) dk dt,
\] (5.18)

It follows from (5.15) and (5.18) that
\[
J = -i(2\pi)^{-3/2} \int_D \hat{f}^*(k) (\hat{\phi}(r, k, \lambda), V(r) [(\mathcal{L}_0 - \lambda - i\delta)^{-1} g(r)]) dk,
\] (5.19)

Using (3.5) we obtain
\[
J = -i(2\pi)^{-3} \int_D \hat{f}^*(k) (\hat{\phi}(r, k, \lambda), V(r) [B_0(\lambda + i\delta) g(r)]) dk;
\] (5.20)
When $\varepsilon \to +0$ we receive from (5.13) and (5.20):

$$(f, W_+ g) = (f, g) + (2\pi)^{-3} \int_D \hat{f}^r(k)(\hat{\phi}(r, k, \lambda), V(r)[B_0(\lambda)g(r)])dk, \quad (5.21)$$

where $W_+ = W_+(\mathcal{L}, \mathcal{L}_0)$. In view of RLS equation (2.30) and (5.21) we have

$$(f, W_+ g) = (f, g) + (\hat{f}, \hat{g}_0) - (\hat{f}, \hat{g}) = (\hat{f}, \hat{g}_0) \quad (5.22)$$

The assertion of the proposition follows directly from (5.22).

**Definition 5.7** Suppose that the wave operators $W_{\pm}(\mathcal{L}, \mathcal{L}_0)$ exist. They are complete if

$$\text{Ran}W_{\pm}(\mathcal{L}, \mathcal{L}_0) = \mathcal{H}_{ac}.$$  \hspace{1cm} (5.23)

**Corollary 5.8** Let the conditions of the Theorem 5.2 be fulfilled. Then the corresponding wave operators $W_{\pm}(\mathcal{L}, \mathcal{L}_0)$ exist and are complete.

**Proof.** The wave operators $W_{\pm}(\mathcal{L}, \mathcal{L}_0)$ exist (see Theorem 5.2). The set of vectors $\hat{g}_0(k, \lambda)$ is dense in the space $L^2(R^3)$. Hence it follows from (4.18) and (5.10) that

$$\text{Ran}[UW_{+}(\mathcal{L}, \mathcal{L}_0)] = L^2(R^3).$$  \hspace{1cm} (5.24)

Wave operators have the property

$$\text{Ran}W_{\pm}(\mathcal{L}, \mathcal{L}_0) \subseteq \mathcal{H}_{ac}.$$  \hspace{1cm} (5.25)

According to (5.24) and (5.25) we have $\text{Ran}[W_{+}(\mathcal{L}, \mathcal{L}_0)] = \mathcal{H}_{ac}$. In the same way we obtain $\text{Ran}[W_{-}(\mathcal{L}, \mathcal{L}_0)] = \mathcal{H}_{ac}$. So the condition (5.23) is fulfilled. Hence, the wave operators $W_{\pm}(\mathcal{L}, \mathcal{L}_0)$ are complete.

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