New 5-Designs—revisited

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The main purpose of this note is to clarify some portions of, and correct some errors in, New 5-Designs, [1], hereafter “NFD”: The definitions of the quadratic-residue codes in NFD and their relationships to each other are clarified, as is their extension. The proof of the Gleason-Prange theorem is here made clearer.

Determining the sign in the extension to the “infinite” coordinate but left ambiguous in NFD led to this note.

Some examples calculated in a current work [3] applying NFD to modular representations of groups made it desirable to determine the above-mentioned sign.

Two other matters first.

Chebotarëv’s theorem. One of the results in NFD, pp. 127-128, was a new proof of the widely known Chebotarëv’s theorem on the roots of unity. At the time, the 1960s, we had no idea it was known, much less that it had its own name. (It says: for a prime \( \ell \) and a primitive \( \ell^{th} \) root of unity \( z \) over \( \mathbb{Q} \), every subdeterminant of the \( \ell \times \ell \) matrix \( (z^{ij}) \) is nonvanishing.) But we suggest our coding-theoretic proof is worth considering.

We used this result to prove “optimality”\(^1\) of all cyclic \([\ell, k]\) codes over \( \mathbb{F}_p^{\ell} \) for almost all primes \( p \). (See Theorem 2.2 in NFD.)

An article on Chebotarëv’s life and works, mentioning other proofs of his result, appears in [4].

Orbits. An example at the end of NFD has an error. The matter is the action of \( \text{PSL}_2(47) \) on the binary \([48,24,12]\) code, in particular on the

\(^1\)A word long since displaced by the abbreviation “MDS”, for a crackjaw term best left unsaid.
codewords of minimum weight. These codewords are in three orbits, but not the orbits stated in NFD. A correct account appears in [2], pp. I-26ff.

The main thing.
The reader is assumed to be familiar with proof of the Gleason-Prange theorem in NFD.
Throughout we denote by \( \ell \) an odd prime. When we speak of a polynomial, say, \( a + bx + cx^2 \), as an element of a code, we understand that as an \( n \)-tuple it is \((a, b, c, 0, \ldots, 0)\). The reverse \( f^*(x) \) of the non-0 polynomial \( f(x) \) of degree \( d \) is defined as

\[
f^*(x) := x^d f(1/x).
\]

We begin with a result on cyclic codes. We will freely represent codewords either as \( n \)-tuples over a field or as polynomials.
The setup: \( L \) is a field, and \( z \) an \( n^{th} \) root of unity over \( L \). \( K := L(z) \).

Lemma. Let \( \varphi \) be any nontrivial linear functional from \( K \) to \( L \). Let \( h(x) \in L[x] \) of degree \( k \) be the monic irreducible polynomial over \( L \) with \( z \) as root. Let \( C \) denote the cyclic code

\[
C := \{ F_z(c) := (\varphi(c), \varphi(cz), \ldots, \varphi(cz^{n-1}); c \in K\}.
\]

Then \( C \) is an \([n, k]\) cyclic code over \( L \), and \( C^\perp = (h(x)) \), the \([n, n-k]\) cyclic code generated by \( h(x) \). And

\[
C = \left( \frac{x^n - 1}{h^*(x)} \right).
\]

Proof. Let \( h(x) = h_0 + h_1 x + \cdots + h_k x^k \). It is obvious that \( v := (h_0, h_1, \ldots, h_k, 0, \ldots, 0) \) is orthogonal to \( F_z(c) \). The same is true of every cyclic shift of \( v \). As a polynomial \( v \) is \( h(x) \). Thus \( (h(x)) \subseteq C^\perp \). That is, \( \dim(C) \leq k \).

For the reverse inequality, note that \( c \mapsto F_z(c) \) is a linear map from \( K \) to \( L^n \). If \( c \neq 0 \) then \( \{c, cz, \ldots, cz^{n-1}\} \) spans \( K/L \), so \( F_z(c) \neq 0 \). Since \( [K : L] = k \), that is the dimension of \( C \). Thus \( C^\perp = (h(x)) \).

The final assertion of the Lemma follows from the well known result that when \( x^n - 1 = f(x)g(x) \), the orthogonal code of the cyclic code \((g(x))\) is \((f^*(x))\). \( \square \)
Now to NFD. The purpose of this section is to clarify the definition of the extended quadratic-residue codes and the proof of the Gleason-Prange theorem. Unexplained notations are taken from NFD.

Let us clean the slate by permanently using the factoring

\[ x^\ell - 1 = (x - 1)f(x)g(x), \]

instead of the \( g_1(x)g_2(x) \) in NFD, page 129. Now \( z \) is a primitive \( \ell \)th root of 1 over \( \mathbb{Q} \), and all takes place in \( K := \mathbb{Q}(z) \) or in its quadratic subfield \( L \). In particular, with \( R, R' \) as the set of quadratic [non]residues mod \( \ell \), if \( z \) is a root of \( f(x) \), then

\[ f(x) = \prod_{r \in R} (x - z^r); \quad g(x) = \prod_{s \in R'} (x - z^s). \]

Particular quantities are the trace-coefficients of \( f(x) \) and \( g(x) \):

\[ \eta := \sum_{r \in R} z^r; \quad \eta' := \sum_{s \in R'} z^s. \tag{1} \]

That is, with \( k := (\ell - 1)/2 \) for ease of writing, the coefficients of \( x^{k-1} \) in \( f(x) \) and \( g(x) \), respectively, are \(-\eta\) and \(-\eta'\).

Also, \( L = \mathbb{Q}(\eta) \), and \( 1 + \eta + \eta' = 0 \). In fact,

\[ (x - \eta)(x - \eta') = x^2 + x + (1 - \left(\frac{-1}{\ell}\right)\ell)/4. \]

Note also that

\[ f^*(x) = \begin{cases} -g(x) & \ell = 4N - 1 \\ f(x) & \ell = 4N + 1 \end{cases}. \]

Of course the same holds with \( f \) and \( g \) interchanged.

We focus now on the \([\ell, k]\) code \( A \) defined as that generated by \((x - 1)f(x)\). \( A^+ \) is generated by \( f(x) \). (And \( B^+ \) is generated by \( g(x) \). Orthogonality is laid out in NFD\(^2\) \( A^+ \) is mapped to \( A_\infty \subset L^{\ell+1} \) by the rule \( a \rightarrow a; a_\infty \) in which

\[ a_\infty := \gamma \sum_i a_i, \tag{2} \]

\(^2\)The codes \( A_\infty \) and \( B_\infty \) over the quadratic number-field \( L \) are conjugates of each other.
and $\gamma$ satisfies
\[ \ell \gamma^2 = \left(\frac{-1}{\ell}\right); \]
we use the Legendre symbol here. The purpose of this clarification is to determine the sign on $\gamma$. We’ll prove

**Proposition.** Define $\eta$ and $\eta'$ as in (1). The code $A^+$ generated by $f(x)$ extends to $A_\infty$ with the use of $\gamma \in L$ as specified in (2). Then
\[ \ell \gamma = -(\eta - \eta'). \]

We emphasize that $\eta$ is the trace-coefficient of $f(x)$.

This result nails down the too-vague discussion on page 132 of NFD, which was based on hazy assumptions. The proof below will be followed by a clarification of the proof of the Gleason-Prange theorem.

**Proof.** Since $x^\ell - 1 = (x - 1)f(x)g(x)$, we have

\begin{equation}
A := ((x - 1)f(x)) = \left(\frac{x^\ell - 1}{f(x)g(x)}\right) = \begin{cases} 
\ell = 4N - 1 \\
\ell = 4N + 1
\end{cases}.
\end{equation}

We first recall that $\sigma$ (1, p. 131) is this monomial transformation of $L^{\ell+1}$: with $\epsilon_i := \left(\frac{1}{\ell}\right)$ for $0 < i < \ell$ and $\epsilon_0, \epsilon_\infty \in \{1, -1\}$ to be determined,

\((a_0, \ldots, a_i, \ldots; a_\infty)\sigma := (\epsilon_0 a_\infty, \ldots, \epsilon_i a_{-1/i}, \ldots; \epsilon_\infty a_0).\)

We begin on page 131 of NFD at “CASE 1: $\ell \equiv -1 \pmod{4}.”

The proof of Case 1 has two parts. Part I is the proof that \(\langle 1, 0, 0 \rangle \sigma \in A_\infty.\) With this Part we’ll prove the Proposition.

Part II is the proof that $A\sigma \subset A_\infty$. We’ll redo it below to simplify the proof of the Gleason-Prange theorem.

Part I is OK until the top of page 132, where $z$ is not clearly specified. For $\langle 0, c \rangle = (T(cz^i))_{0 \leq i < \ell}$ to be in $A$, which we have defined to be $((x - 1)f(x))$, (3) tells us that $z$ must be a root of $f(x)$ (not of $g_1(x)$ as stated in NFD). (If $f(z) = 0$ is not clear, see Case 2 just below.) Still, the equation (3) of NFD is correct with our present definitions of $\eta$ and $\eta'$. Thus, when $\ell = 4N - 1$, and if we take $\epsilon = 1$,

\[ \ell \gamma = -(\eta - \eta'). \]
We now take up Part I of “CASE 2: $\ell \equiv +1 \pmod{4}$.” We take $\epsilon = 1$. On page 133 of NFD, it may help to note that $\langle 1, 0 \rangle$ differs from $A_\infty$ to $B_\infty$, i.e.,

$$\langle 1, 0 \rangle_A = (1, 1, \ldots, 1; \ell \gamma)$$

and

$$\langle 1, 0 \rangle_B = (1, 1, \ldots, 1; -\ell \gamma).$$

As before, the two subparts of Part I are: Ia, prove that $\langle 1, 0 \rangle_A \sigma$ is orthogonal to $\langle 1, 0 \rangle_B$; and Ib, prove that $\langle 1, 0 \rangle_A \sigma$ is orthogonal to $B$.

For Ia: The change of signs in the infinite coordinate makes this happen. For Ib: Now $A^+ = B^+ = (g(x))$. We set up

$$F_z(c) := (T(c), T(cz), \ldots, T(cz^{\ell-1}); 0)$$

as a general element of $B$. Since $B = ((x^\ell - 1)/f(x))$, and $f^*(x) = f(x)$, we see from (3) that $z$ is a root of $f(x)$.

Since

$$\langle 1, 0 \rangle_A \sigma = (\ell \gamma, \ldots, \left(\frac{i}{\ell}\right), \ldots; 1),$$

the dot product $F_z(c) \cdot \langle 1, 0 \rangle_\sigma$ is

$$T(c)(\ell \gamma + \sum_{r \in R} z^r - \sum_{s \in R'} z^s).$$

It is the same as it was in the prior case. In other words, under the setup here, $\ell \gamma + \eta - \eta' = 0$ if and only if $\langle 1, 0 \rangle_A \sigma$ is orthogonal to $B_\infty$. This proves the Proposition. \square

Now we go to Parts II of the two cases, to clarify the proof of the Gleason-Prange theorem, the burden of which is to prove that $\sigma$ is an invariance of the codes.

CASE 1: $\ell \equiv -1 \pmod{4}$. It remains to prove that $A \sigma \subset A_\infty$. For a general element of $A$ we take

$$\langle 0, c \rangle := (T(c), T(cz), \ldots, T(cz^{\ell-1}); 0),$$

where $z$ is a root of $f(x)$, as we saw earlier. We apply $\sigma$, as in NFD, p. 133, to get

$$\langle 0, c \rangle \sigma = (a_0, \ldots, a_{\ell-1}; a_\infty).$$
In NFD we first verified that $a_\infty$ is correct. (See the equation just above the line beginning “from (2) and (1).”) That calculation benefits from the Proposition just proved and states a correct result.

The only remaining hurdle is to show that as a polynomial this vector is a multiple of $f(x)$. Since

$$a_i = \epsilon_i T(cz^{-1/i}),$$

we defined (noting $a_0 = 0$) for all $c \in K$,

$$D(c) := \sum_{1 \leq i < \ell} \epsilon_i T(cz^{-1/i}) z^i.$$  

Our object now is to prove $D(c)$ is always 0.

We note that $D$ is linear from $K$ to itself. Departing from NFD’s use of the “quadratic-residue” invariance $\tau$, we therefore prove it 0 for $c = z, z^2, \ldots, z^{\ell-1}$, a spanning set for $K/L$. Thus

$$D(z^j) = \sum_{1 \leq i < \ell} \epsilon_i T(z^{j-1/i}) z^i$$

$$= \sum_{i} \epsilon_i \sum_{r \in R} z^{r(j-1/i)} z^i$$

$$= \sum_{i, r} z^{rj-r/i+i}.$$  

The rest of the proof is the same as in NFD, except that the little polynomial is now $x^2 + (rj - k)x - r$. Also, for each $k, r$ this polynomial has two distinct, or no, roots in $GF(\ell)$. If two roots, one is in $R$ and the other in $R'$. And the proof does not need that $f(z) = 0$. This settles Case 1.

**Case 2:** $\ell \equiv +1 \pmod{4}$. To prove: $A\sigma \subset A_\infty$.

We imitate Case 1, making the necessary changes.

We define the general element of $A$ as before, except that now $z$ must be a root of $g(x)$:

$$\langle 0, c \rangle := (T(c), T(cz), \ldots, T(cz^{\ell-1}); 0).$$

And, as before,

$$v := \langle 0, c \rangle \sigma = (0, \ldots, \epsilon_i T(cz^{-1/i}), \ldots; T(c)).$$
Now \( v \) is in \( A_\infty \) if and only if its finite part, as a polynomial, is a multiple of \( f(x) \). The roots of \( f(x) \) are \( z^s \) for \( s \in R' \). So: fix \( s \in R \) and see that \( v \in A_\infty \) iff
\[
D'(c) := \sum_{1 \leq i < \ell} \epsilon_i T(cz^{-1/i})z^{ni} = 0
\]
for all \( c \in K \).

Proceeding just as before, we write, for \( 0 < j < \ell \),
\[
D'(z^j) = \sum_i \epsilon_i \sum_{r \in R} z^{rj - r/i + si}.
\]
This expression is a polynomial in \( z \) in which the coefficient of \( z^k \) is the sum of the \( \epsilon_i \) for which there are \( r \in R \) such that
\[
rij - r/i + si = k.
\]
This comes down to \( si^2 + (rij - k)i - r = 0 \), i.e.,
\[
i^2 + s^{-1}(rij - k)i - s^{-1}r = 0.
\]
Again, the constant term is in \( R' \), so not only are there no double roots for \( i \), but also if it has roots, one is in \( R \) and one is in \( R' \). Thus \( D'(z^j) = 0 \). \( \square \)

A loose end. “[W]e are free to choose \( \epsilon = 1 \) or \( \epsilon = -1 \)” (page 132). Recall that \( \epsilon := \epsilon_0 \) and \( \epsilon_\infty := \left( \frac{-1}{\ell} \right) \epsilon_0 \). We have always taken \( \epsilon = 1 \). But we are not free to choose it to be \( -1 \), because we know that
\[
\langle 1, 0 \rangle \sigma = (\ell \ell \gamma, \ldots, \epsilon_i, \ldots; \left( \frac{-1}{\ell} \right) \epsilon)
\]
is in the code \( A_\infty \) when \( \epsilon = 1 \), meaning that the polynomial of its finite part is a multiple of \( f(x) \). If it were also in \( A_\infty \) when we chose \( \epsilon = -1 \), the corresponding polynomial would be the same as the first one but with the constant term, not equal 0, of opposite sign. It could not also be a multiple of \( f(x) \).

This concludes my comments on the proof of the Gleason-Prange theorem.

A red herring. In a cyclic code of length \( n \), for any polynomial \( f(x) \) dividing \( x^n - 1 \), we say that a polynomial \( a(x) \) of degree less than \( n \) is recursive for \( f(x) \) iff
\[
a(x)f(x) \equiv 0 \pmod{x^n - 1}.
\]
Thus the set of all such $a(x)$ is the cyclic code with generator polynomial $(x^n - 1)/f(x)$.

The definition of “recursive” in NFD (p. 125) used the reverse of $f(x)$ in $\mathbb{F}$. Later (p. 129) we wrote of code “A, recursive for $g_1(x)$, generated as ideal by $g_2(x)$”, so we had slipped into the definition in $\mathbb{F}$.

Except for possibly confusing readers, no damage was done, because we never used recursion in working with the codes. The best use of it, as on page 129, is that saying a code is “recursive for $f(x)$” can be more economical, even clearer, than saying it is generated by $(x^n - 1)/f(x)$.

For example, consider the [31,5] binary cyclic codes. They are recursive for the irreducible polynomials of degree 5 over $\mathbb{F}_2$. One such is $1 + x^2 + x^5$. It is easier to understand “cyclic code recursive for $1 + x^2 + x^5$” than “cyclic code generated by $(x^{31} - 1)/(1 + x^2 + x^5)$,” however one might express the latter polynomial.

Conclusions. This note clarifies some murky points and corrects some misstatements in, and simplifies, the proof of the Gleason-Prange theorem in NFD. One of these points is the sign of $\gamma$ used in the extension of the codes to length $\ell + 1$.

We emphasize that our basic datum is the factorization of $x^{\ell} - 1$ as $(x - 1)f(x)g(x)$ defined earlier. This factorization determines the values $\eta$ and $\eta'$, the trace-coefficients of the “quadratic-residue” polynomials $f(x)$ and $g(x)$, respectively. These values in turn lead us to the sign of $\gamma$.

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References

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