Symmetries of dynamical systems 
and convergent normal forms

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**Abstract.**

It is shown that, under suitable conditions, involving in particular the existence of analytic constants of motion, the presence of Lie point symmetries can ensure the convergence of the transformation taking a vector field (or dynamical system) into normal form.

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The classical technique of transforming a given vector field (describing e.g. the flow of a dynamical system (DS)) into normal form (NF) (in the sense of Poincaré and Dulac) is a well known and useful method of investigation [1-2] (see also e.g. [3] for further Ref.); it is also well known, however, that Poincaré-Dulac series are in general only formal or asymptotic series. The convergence of a normalizing transformation is in fact a quite "rare" event, and often one considers truncated series (and then approximate transformations (see e.g. [4-5] and Ref. therein)). In this note we want to propose shortly some result (details and complete proofs will be presented in a separate paper) concerning the convergence of the normalizing transformations: precisely, we will show that, under suitable conditions, the presence of some Lie symmetry [6-7] of the vector field can ensure the convergence of the normalizing transformation. As a particular case, we recover a remarkable result which has been recently obtained by Bruno and Walcher for 2-dimensional problems [8]; in the same context, see also [9] for an older result, whose proof however has been recognized to be not complete [10].

We will consider $n$--dimensional vector fields $f$ in the form of DS

$$\dot{u} = f(u) = Au + F(u) \quad u = u(t) \in \mathbb{R}^n$$

(1)

where $\dot{u} = du/dt$, $f$ is assumed to be analytic in a neighbourhood of $u = 0$, with $f(0) = 0$, and where the matrix $A \equiv (\nabla f)(0)$ is assumed to be nonzero and diagonalizable (see [3] for a discussion on the non-diagonalizable case). Introducing the usual notion of Lie-Poisson bracket

$$\{f, g\}_k = (f \cdot \nabla)g_k - (g \cdot \nabla)f_k \quad (k = 1, \ldots, n)$$

(2)

and that of "homological operator" $\mathcal{A}$ associated to the matrix $A$

$$\mathcal{A}(f) = \{Au, f\} = (Au) \cdot \nabla f - Af ,$$

(3)

it is known that a nonlinear vector function $h = h(u)$ is said to be in NF with respect to $A$ (or resonant with $A$) if

$$\mathcal{A}(h) = 0 .$$

(4)

Let us also recall the following basic theorem [2]: A DS (1) can be put into NF by means of a convergent transformation if it fulfils the two following conditions:
Condition "A": there is a coordinate transformation changing $f$ to $\hat{f}$, where $\hat{f}$ has the form

$$\hat{f} = Au + \alpha(u)Au$$

and $\alpha(u)$ is some scalar-valued power series (with $\alpha(0) = 0$);

Condition "ω": let $\omega_k = \min |(q, a)|$ (where $a_i$ are the eigenvalues of $A$, and parentheses stand for the scalar product) for all positive integers $q_i$ such that $\sum_{i=1}^{n} q_i < 2^k$ and $(q, a) \neq 0$: then the series

$$\sum_{k=1}^{\infty} 2^{-k} \ln \omega_k$$

is convergent.

While condition "ω" is weak condition, controlling the appearance of small divisors [2], condition "A" is clearly a rather strong restriction. We explicitly assume that all the DSs considered in the remaining of this paper will satisfy condition "ω", but do not satisfy condition "A".

A vector function

$$g(u) = Bu + G(u) \quad (5)$$

(not proportional to $f$) is said to be a Lie point (time independent) symmetry for the DS (1) if

$$\{f, g\} = 0 . \quad (6)$$

In terms of Lie algebras, one says that the vector field operator $g \cdot \nabla$ generates a symmetry of the DS.

We now can state our results.

**Theorem 1.** Assume that: i) the DS (1) admits an analytic symmetry (5) where either the matrix $B$ is proportional to $A$, or $B = 0$ and $G(u)$ is not proportional to $F(u)$; ii) once in NF, the DS takes the form

$$\dot{u} = h(u) = Au + \alpha(u)Au + \mu(u)Mu \quad (7)$$

where $M$ is some matrix (not proportional to $A$), $\alpha$ and $\mu$ some scalar functions, and the two linear problems $\dot{u} = Au$ and $\dot{u} = Mu$ do not admit time-independent common constants of motion.
Then, the DS can be put in NF by means of a convergent normalizing transformation.

**Sketch of the proof.** If $B = 0$, consider the new symmetry $g + f = Au + F + G$. Thanks to (4), the linear field $g_A = Au$ is a symmetry for the NF. Using hypothesis $ii$), one can show (see [11]) that the NF (7) does not admit constants of motion (i.e. functions $\kappa = \kappa(u)$ expressed by (possibly formal) series such that $h \cdot \nabla \kappa = 0$); as a consequence, its only symmetry (including nonlinear and possibly formal ones) having $Au$ as linear part is just $g_A = Au$ (let us recall that multiplying a symmetry by a constant of motion one obtains another symmetry). Then the coordinate transformation taking (1) into (7) transforms the symmetry $g$ according to

$$g = Au + G \rightarrow g_A = Au$$

This means that condition ”A” is satisfied by this transformation of the symmetry: thus there is a normalizing transformation which is convergent, and one can easily conclude that under this transformation the DS also is transformed into NF.

**Theorem 2.** Instead of $i$) in Theorem 1 assume that the DS (1) admits $\ell \,(\geq 1)$ analytic symmetries $g_j = B_j u + G_j(u)$, where the matrices $B_j(\neq 0)$ are linearly independent (and such that no linear combination is proportional to $f$), and where $\ell$ is precisely the number of the linearly independent linear symmetries admitted by the DS once in NF. Then, with the condition $ii$) as in Theorem 1, the same conclusion holds.

**Sketch of the proof.** According to a general property of NFs (see [3,12]), the linear fields $B_j u$ are (linear) symmetries for the NF. Together with $Au$, we have then $\ell + 1$ symmetries for the NF; therefore, there must be one symmetry having just $Au$ as linear part, and the argument proceeds along similar lines as in Theorem 1.

Notice that it can be easily seen that if the DS has dimension $n = 2$, the assumption $ii$) in the above Theorems is automatically satisfied, and then one reobtains the Bruno and Walcher result, namely:

**Corollary [8].** If a 2–dimensional DS admits an analytic symmetry, it can be normalized by a convergent transformation.
We now give a quite general example, which can be interesting both for its possible applications to Hamiltonian DSs, and for illustrating the role played by the presence of symmetries in ensuring the convergence of the normalizing transformations.

Let $n = 2m$ and, putting $u \equiv (x_1, \ldots, x_m, y_1, \ldots, y_m) \in R^{2m}$; assume that a Lie group $\Gamma$ acts "diagonally" on both the $m$-dimensional spaces of the vectors $x$ and $y$ through the same linear representation $\mathcal{D}$: i.e. $x \to x' = \mathcal{D}x$, $y \to y' = \mathcal{D}y$, where $\mathcal{D}$ is an absolutely irreducible representation (i.e. the only matrix commuting with $\mathcal{D}$ is a multiple of the identity). Consider then a DS of the following form

$$\dot{u} = f(u) = Au + F(u)$$

where

$$A = \begin{pmatrix} 0 & I_m \\ -I_m & 0 \end{pmatrix}$$

and assume that $F(u)$ admits the symmetries $B_i u$, where $B_i$ are the (matrix representatives in the direct sum $\mathcal{D} \oplus \mathcal{D}$ of the) Lie generators of this group $\Gamma$ (the linear part $Au$ fulfils this symmetry requirement, so that the full DS $(8)$ admits this symmetry). E.g. if the DS is

$$\dot{u} = Au + p(u)$$

the scalar function $p$ must depend only on the quantities $\rho_a = \rho_a(u)$ which are invariant under $\mathcal{D} \oplus \mathcal{D}$ (e.g., in the case $m = 3$, $\Gamma = SO(3)$ and $\mathcal{D}$ its fundamental representation, these are given by $x^2 = (x, x)$, $y^2 = (y, y)$, $x \cdot y = (x, y)$ where the parentheses stand for the scalar product in $R^3$). The NF of the DS $(8)$ or $(9)$ must admit the linear symmetries $B_i u$ and $Au$ (see [3,12]), then it is easy to see that it must take the form

$$\dot{u} = Au + \alpha Au + \mu u$$

where $\alpha$ and $\mu$ are functions of the quantities invariant under all these symmetries (e.g., of $r^2 = x^2 + y^2$ only, in the $SO(3)$ case). Assume now (cf. the example in [9]) that in the DS $(9)$ the function $p$ is a homogeneous polynomial of degree $2k$ built up with the quantities $\rho_a$: it is easy to verify that the following vector function (with vanishing linear part)

$$g = (x^2 + y^2)^k u$$

generates a nontrivial analytic symmetry for the DS $(9)$, and that there are no common constants of motion for $(10)$, as requested by $ii$) in the above Theorems. Therefore, the
convergence of the normalizing transformation is ensured by Theorem 1. It is important to notice that, if our problem would not possess the symmetry $\Gamma$, the NF (10) would contain many other terms in its r.h.s., and that it is precisely the presence of the symmetry which forces the NF to contain only $A$ and the identity, and therefore allows us to apply the argument concerning the constants of motion. This seems to confirm the conjecture [8,13] that the presence of a ”sufficient” number of symmetries may be an essential request in order to guarantee the convergence of a normalizing transformation.

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