The Linear Stability of Reissner–Nordström Spacetime: The Full Subextremal Range $|Q| < M$

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Abstract: We prove the linear stability of subextremal Reissner–Nordström spacetimes as solutions to the Einstein–Maxwell equation. We make use of a novel representation of gauge-invariant quantities which satisfy a symmetric system of coupled wave equations. This system is composed of two of the three equations derived in our previous works (Giorgi in Ann Henri Poincaré, 21: 24852580, 2020; Giorgi in Class Quantum Grav 36:205001, 2019), where the estimates required arbitrary smallness of the charge. Here, the estimates are obtained by defining a combined energy-momentum tensor for the system in terms of the symmetric structure of the right hand sides of the equations. We obtain boundedness of the energy, Morawetz estimates and decay for the full subextremal range $|Q| < M$, completely in physical space. Such decay estimates, together with the estimates for the gauge-dependent quantities of the perturbations obtained in Giorgi (Ann PDE 6:8, 2020), settle the problem of linear stability to gravitational and electromagnetic perturbations of Reissner–Nordström solution in the full subextremal range $|Q| < M$.

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1. Introduction

The problem of stability of black holes as solutions to the Einstein equation occupies a central stage in mathematical General Relativity [11]. The resolution of this problem
consists in understanding the long-time dynamics of perturbations of known stationary solutions to the Einstein equation.

There are few examples of exact solutions to the Einstein equation, the most fundamental of which is the Kerr spacetime [32], axially symmetric and stationary solution to the Einstein vacuum equation

$$\text{Ric}(g) = 0,$$

where $g$ is a Lorentzian metric in 3+1-dimensions. A particular case of the Kerr spacetime is the Schwarzschild solution [43], which is spherically symmetric and static, given in coordinates $(t, r, \theta, \phi)$ by

$$g_M = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 \left(d\theta^2 + \sin^2 \theta d\phi^2\right).$$

The parameter $M$ can be interpreted as the mass of the black hole.

In the case of the Einstein equation coupled with electromagnetic fields, the Lorentzian metric $g$ satisfies the Einstein–Maxwell equation [10], i.e.

$$\text{Ric}(g)_{\mu\nu} = 2 F_{\mu\lambda} F_\nu^\lambda - \frac{1}{2} g_{\mu\nu} F_{\alpha\beta} F^{\alpha\beta},$$

$$D_{[\alpha} F_{\beta\gamma]} = 0, \quad D^\alpha F_{\alpha\beta} = 0,$$

where $F$ is a two-form verifying the Maxwell equations (2), and $D$ is the Levi-Civita connection of $g$. In this context, the fundamental stationary and axisymmetric solution is the Kerr–Newman spacetime [39], and its spherically symmetric and static case is given by the Reissner–Nordström metric [40], given in coordinates by

$$g_{M,Q} = -\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right) dt^2 + \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)^{-1} dr^2 + r^2 \left(d\theta^2 + \sin^2 \theta d\phi^2\right).$$

The parameter $Q$ can be interpreted as the charge of the black hole for $|Q| < M$, which is referred to as the subextremal range. The case $|Q| = M$ is called the extremal case, while $|Q| > M$ corresponds to a spacetime with naked singularities. Observe that for $Q = 0$, the Reissner–Nordström metric (3) reduces to the Schwarzschild one.

The problem of stability of black hole solutions can be roughly divided into three formulations, each of increasing difficulty, from the formal mode analysis of the linearized equations to the fully non-linear perturbations, passing through the problem of linear stability.

The mode stability consists in formally separating the solutions to the linearized Einstein equation into modes at fixed frequencies, and aims at proving the lack of exponentially growing modes for all metric or curvature components. The study of the mode stability of the Schwarzschild solution was initiated by Regge, Wheeler [42] and Zerilli [47] in metric perturbations (in which the metric of the solution is perturbed), and by Bardeen and Press [5] in Newman–Penrose formalism (in which the curvature of the solution, through the Newman–Penrose scalars, is perturbed). See also Dotti [18]. Chandrasekhar [10] identified a transformation theory in mode decomposition which connects the two approaches in Schwarzschild, which is now referred to as Chandrasekhar transformation. Teukolsky [45] extended the equations in Newman-Penrose formalism to the Kerr spacetime, and Whiting [46] proved the mode stability for Kerr spacetime. The study
of mode stability of the Reissner–Nordström spacetime has been initiated by Moncrief
[36–38], who obtained the wave equations governing the perturbations in metric per-
turbations. Chandrasekhar [8,9] completed the study of the fixed mode perturbations in
Newman–Penrose formalism. See also Fernández Tío–Dotti [19].

This weak version of stability is however not sufficient to prove boundedness and
decay of the solutions even to the linearized equations. Indeed, the lack of exponentially
growing modes is still consistent with the statement that general perturbations with finite
initial energy grow unboundedly in time, because results at the level of individual modes
do not imply them for the superposition of infinitely many modes [17].

The resolution of the problem of linear stability consists in proving boundedness
and decay for the solutions to the linearized Einstein equation, which does not rely on
decomposition in modes but rather on a physical space analysis. The stability of the
Schwarzschild solution to the linearized Einstein vacuum equation has been obtained by
Dafermos–Holzegel–Rodnianski [13] by using curvature perturbations and analysis of the
Teukolsky equation. The authors introduced a physical-space version of the Chan-
drasekhar transformation and crucially used the extensive progress on boundedness and
decay results for wave equations on black hole backgrounds (for instance [15–17]). Other
proofs have followed: see Hung–Keller–Wang [28] for the proof of the linear stability
using metric perturbations, through the analysis of Regge–Wheeler and Zerilli equations.
See also Hung [29,30] for a proof in the harmonic gauge and Johnson [31] in the general-
ized harmonic gauge. There have been numerous recent results for the linear stability of
Kerr spacetime. Quantitative decay estimates for the Teukolsky equation in slowly rotat-
ing Kerr spacetime have been obtained by Ma [35] and Dafermos–Holzegel–Rodnianski
[14]. Andersson–Bäckdahl–Blue–Ma [1] used the outgoing gauge and Häfner–Hintz–
Vasy [24] used the wave gauge to prove linear stability of Kerr with small angular
momentum. Decay for solutions to the Maxwell equations in Schwarzschild spacetime
have been obtained by Blue [7] and Pasqualotto [41].

The ultimate goal of the problem of stability is the study of the dynamics of per-
turbations of solutions to the fully non-linear Einstein equation. The fully non-linear
stability of the Kerr(-Newman) family consists in showing that a small perturbation of
a Kerr(-Newman) spacetime which is a solution to the non-linear Einstein(-Maxwell)
equation converges to another member of the Kerr(-Newman) family. The only proof of
non-linear stability with no symmetry assumptions which is known at this stage, in the
asymptotically flat regime, is the global non-linear stability of Minkowski spacetime by
Christodolou-Klainerman [12], which was followed by proofs obtained through differ-
ent approaches (see [6,26,34]). See also Zipser [6] for the proof of non-linear stability of
Minkowski as solution to the Einstein–Maxwell equation. The first proof of non-linear
stability of the Schwarzschild spacetime under the class of symmetry of axially sym-
metric polarized perturbations was given by Klainerman–Szeftel [33]. In the presence
of a positive cosmological constant, the Kerr–de Sitter and the Kerr–Newman–de Sitter
family with small angular momentum have been proved to be non-linearly stable by
Hintz–Vasy [27] and by Hintz [25] respectively.

In this paper we solve the problem of linear stability of the Reissner–Nordström
spacetime (3) as solution to the linearized Einstein–Maxwell equations (1) and (2), in
the full subextremal range $|Q| < M$. More precisely, we prove boundedness and decay
statements for solutions to the linearization of the Einstein–Maxwell equation around
a Reissner–Nordström solution, and the analysis is carried out completely in physical
space. Here is a rough version of our main theorem.
Theorem 1 (Linear stability of Reissner–Nordström spacetime to gravitational and electromagnetic perturbations for \(|Q| < M\) (Rough version)). All solutions to the linearized Einstein–Maxwell equations around a Reissner–Nordström solution \(g_{M, Q}\) for \(|Q| < M\) in a certain choice of gauge\(^1\) arising from regular asymptotically flat initial data remain uniformly bounded on the exterior and decay to a linearized Kerr–Newman solution.

Theorem 1 represents the final step of a program that was initiated by the author in [20–22] to prove the linear stability of Reissner–Nordström spacetime to gravitational and electromagnetic perturbations in the full subextremal range. More precisely, the series of works [20–22] amounted to the proof of the linear stability in the case of arbitrarily small charge \(|Q| \ll M\). The main result of this paper is to extend the control of all components of the perturbation to the subextremal range \(|Q| < M\).

Remark 1. The subextremal range \(|Q| < M\) for which the linear stability holds is expected to be optimal. The estimates as hereby derived make use of the redshift vector field at the horizon, and therefore they are degenerate through the extremal case limit \(|Q| = M\). In particular, the same decay estimates obtained in [22] are not expected to hold in the extremal case due to the Aretakis instability [3,4]. Such instability causes the growth of transversal derivatives along the horizon of solutions to the non-linear wave equation [2], and it is expected that this phenomenon persists in the linearized gravity. Nevertheless, some weaker version of stability, which takes into account such degeneracy of transversal derivative along the event horizon, could hold in the extremal case, where stability and instability phenomena concur.

In what follows, we recall the main ideas from the series of works [20–22]. They consist in two parts:

- The main system of three wave equations governing the perturbations are derived and analyzed for arbitrarily small charge in [20,21]. More precisely, two wave equations of spin \(\pm 2\) (governing the gravitational perturbations) are obtained in [20], and one wave equation of spin \(\pm 1\) (governing the electromagnetic radiations) is obtained in [21]. Those are wave equations for quantities which are invariant\(^2\) to coordinate transformations at linear level, which we call gauge-invariant quantities.
- The above analysis is used in [22] to obtain control for all the components of the perturbations, upon a choice of gauge. In particular, the estimates for the gauge-invariant quantities are used to obtain estimates for the gauge-dependent ones.

In the present paper, we will make use in a fundamental way of the system of equations obtained in [20,21], and derive from them a new system to obtain control for the gauge-invariant quantities in the full subextremal range. The result in [22] will then be applied straightforwardly to obtain control for the gauge-dependent quantities from the new estimates obtained here for the gauge-invariant ones.

We now recall the main results in [20–22] which are particularly relevant for this work.

1.1. The spin \(\pm 2\) system of equations in [20]. Suppose that \((\mathcal{M}, g, F)\) is a solution to the Einstein–Maxwell equation such that the manifold \(\mathcal{M}\) can be foliated by 2-spheres

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\(^1\) The proof is obtained in Bondi gauge, see [22].

\(^2\) In a linearization of size \(\epsilon\), a quantity is called gauge invariant if it changes quadratically, i.e. by terms of the size \(\epsilon^2\), when coordinate transformations of size \(\epsilon\) are applied. See [22].
is the trace of the second null fundamental form and \( S \) relative to a null frame

\[ \alpha_{AB} = W(e_4, e_A, e_4, e_B), \quad \bar{\beta}_{AB} = \nabla^*(F)\beta_{AB} + \nabla(F)\rho \tilde{\chi}_{AB} \]

where \( W \) is the Weyl curvature, \( \nabla^*(F)\beta_{AB} = -D(A(F)\beta_B) + \frac{1}{2}g_{AB}\nabla(F)\beta \) with \( D \) the Levi-Civita connection of \( g \), \( (F)\beta_A = F(e_A, e_4) \), \( (F)\rho = \frac{1}{2}F(e_3, e_4) \) and \( \tilde{\chi} \) is the traceless part of the \( S \)-tensor \( \chi_{AB} = g(D_Ae_4, e_B) \).

The 2-tensors \( \alpha \) and \( \bar{\beta} \) are gauge-invariant, and satisfy a coupled system of Teukolsky-type equations of spin \( \pm 2 \) [20]. The estimates for the Teukolsky equations cannot be obtained directly, but rather through a Chandrasekhar transformation to obtain Regge–Wheeler-type equations. We defined the derived quantities \( q \) and \( q^F \) [20]

\[ q = \frac{1}{\ell^2} \nabla_3 \left( \frac{\ell}{2} \nabla_3 (r^3 \kappa^2 \alpha) \right), \quad q^F = \frac{1}{\ell^2} \nabla_3 (r^3 \kappa \bar{\beta}) \]

where \( \kappa := \text{tr} \chi \) is the trace of the second null fundamental form and \( \nabla_3 \) is the projection of the sphere of the covariant derivative along the incoming null direction \( D_{e_3} \).

The gauge-invariant 2-tensors \( q \) and \( q^F \) satisfy a coupled system of linear wave equations of spin \( \pm 2 \) [20] which can be schematically written as

\[ \Box_{g_{M,Q}} q + \tilde{V}_1(r) \cdot q = Q \cdot \left( b_1(r) \Delta_2 q^F + b_2(r) \partial_r q^F + b_3(r) q^F + \text{l.o.t.} \right) \quad (4) \]

\[ \Box_{g_{M,Q}} q^F + \tilde{V}_2(r) \cdot q^F = Q \cdot (c_1(r) q + \text{l.o.t.}) \quad (5) \]

where \( \Box_{g_{M,Q}} \) is the d’Alembertian of the Reissner–Nordström metric \( g_{M,Q} \) applied to 2-tensors, \( b_i, c_i, \tilde{V}_i \) are smooth functions of an area radius function \( r \), \( \Delta_2 \) denotes the Laplacian operator on 2-tensors on the sphere and l.o.t. denotes lower order terms (with respect to differentiability) for \( q \) and \( q^F \).

Estimates for this system are obtained in [20] in the case of \( |Q| \ll M \) by interpreting the right hand sides of (4) and (5) as a perturbation of zero. A careful analysis has to be done at the trapping region in order to absorb the spacetime integrals obtained from the right hand side, but the arbitrary smallness of the charge allows to absorb them into the bulk energies of the left hand side of the equations. Through transport estimates, one can then obtain control for the quantities \( \bar{\beta} \) and \( \alpha \). We refer to [20] for more details.

Observe that the first of the equations, i.e. equation (4), reduces to the Regge–Wheeler equation used in [13] in the case of Schwarzschild (with trivial right hand side).

1.2. The spin \( \pm 1 \) equation in [21]. In [21], we defined the 1-covariant \( S \)-tensor \( \tilde{\beta} \) relative to a null frame \( \{ e_3, e_4, e_A \}_{A=1,2} \) as

\[ \tilde{\beta}_A := 2(F)\rho \beta_A - 3(F)\beta_A \]

where \( (F)\beta_A = F(e_A, e_4) \), \( (F)\rho = \frac{1}{2}F(e_3, e_4) \), \( \beta_A = \frac{1}{2}W(e_A, e_4, e_3, e_4) \), \( \rho = \frac{1}{4}W(e_3, e_4, e_3, e_4) \).

3 A null frame \( \{ e_3, e_4, e_A \}_{A=1,2} \) is such that \( g(e_3, e_3) = 0, g(e_4, e_4) = 0, g(e_3, e_4) = -2 \), and \( e_A \) are orthogonal to \( e_3 \) and \( e_4 \).

4 The spin \( \pm 2 \) refers to 2-tensors on the sphere.
The 1-tensor \( \tilde{\beta} \) is a mixed curvature-electromagnetic component which is gauge-invariant and satisfies a Teukolsky-type equation of spin \( ^5 \pm 1 \) [21]. Also in this case, to obtain the estimates for this equation a Chandrasekhar transformation is applied. We defined the derived quantity \( p \) [21]

\[
p = \frac{1}{\kappa} \nabla_3 (r^5 \kappa \tilde{\beta})
\]

which is shown to satisfy a linear wave equation of spin \( \pm 1 \) [21] which can be schematically written as

\[
\Box_{g_{M,Q}} p + V_1(r) p = Q \cdot a_1(r) \text{div} q^F
\]

where \( a_1 \) and \( V_1 \) are smooth functions \( r \) and \( \text{div} \) is the divergence of a symmetric traceless 2-tensor on the sphere. Observe that Eq. (6) is coupled to equation (5) through the presence of \( q^F \).

Estimates for this equation are obtained in [21] in the case of \( |Q| \ll M \) by using the control of \( q^F \) previously obtained in [20]. By decomposing the 1-tensors \( p \) and \( \text{div} q^F \) in spherical harmonics and projecting equation (6) to the \( \ell = 1 \) harmonics, the right hand side vanishes, and the equation decouples to a single wave equation for the projection of \( p \) to the \( \ell = 1 \) mode. Standard techniques for decay of wave equations on black hole backgrounds can then be applied to control \( p_{\ell=1} \), and by transport estimates \( \tilde{\beta}_{\ell=1} \). The higher spherical harmonics of \( \tilde{\beta} \) are controlled by using a relation between the three quantities of the schematic form [21]:

\[
Q \cdot \nabla_3 \alpha = d_1(r) f + d_2(r) \mathcal{F}^2 \tilde{\beta}
\]

where \( d_i \) are smooth functions of \( r \). We refer to [21] for more details.

1.3. The proof of linear stability for small charge in [22]. The conclusions of [20,21] are the pointwise estimates for the gauge-invariant quantities \( q, q^F \) and \( p \) for \( |Q| \ll M \), and those are the starting point of [22], where such estimates are used to obtain control for all the remaining components of the perturbation. Since the remaining components are gauge-dependent, a careful choice of gauge is needed to show that \( q, q^F \) and \( p \) control the components of the perturbation, and that in addition their decay is optimal and consistent with non-linear applications.

In [22], we achieve such a proof with the choice of outgoing null geodesic, or Bondi, gauge. In this gauge, we made use of residual gauge freedom to define normalizations of scalar functions which allow to obtain integrable transport estimates. We obtain a hierarchy of transport estimates with right hand sides in terms of the known \( q, q^F \) and \( p \). By integrating along null hypersurfaces, pointwise estimates for all the remaining components can be obtained from the estimates previously obtained for \( q, q^F \) and \( p \) [22].

In particular, observe that the estimates for the gauge-dependent quantities in [22] do not make use of the smallness of the charge once the gauge-invariant quantities are controlled. Since the estimates for \( q, q^F \) and \( p \) in [20,21] are only valid for \( |Q| \ll M \), the final proof of the linear stability of Reissner–Nordström in [22] only holds for arbitrarily small charge. We refer to [22] for more details.

We stress here that, if one were able to extend the pointwise estimates for \( q, q^F \) and \( p \) to the full subextremal range \( |Q| < M \), it would be straightforward to apply

\footnote{The spin \( \pm 1 \) refers to 1-tensors on the sphere.}
the proof of linear stability in [22], which does not use smallness of the charge, to the full subextremal range. More precisely, the results on boundedness and decay for the gauge-invariant quantities in Section 8.5 of [22] could be upgraded to hold for |Q| < M, and therefore the subsequent control on the gauge-dependent quantities in the following sections of [22] would also hold in the full subextremal range.

1.4. The mixed spin ± 1 and spin ± 2 system of equations. We outline here the main ideas which allow us to extend the result of linear stability of Reissner–Nordström spacetime from very small charge |Q| ≪ M ([20–22]) to the full subextremal range |Q| < M.

The fundamental step is to introduce a system, which we denote mixed spin ± 1 and spin ± 2 Regge–Wheeler system, which governs the gravitational and electromagnetic perturbations of the Reissner–Nordström solution and has a symmetric structure, which is favorable in the derivation of the estimates.

We briefly explain how such a system is obtained from the previously mentioned equations appeared in [20,21].

Recall the relation (7) between ∇/3α, f and ℑ. By taking one derivative in the ∇/3 direction, one derives a relation between q, qF and f of the schematic form (see (31) for the exact expression):

$$Q \cdot q = d_1(r)q^F + d_2(r)\mathcal{P}_p + \text{l.o.t.} \quad (8)$$

Since Eqs. (4), (5), (6) for q, qF and p are three wave equations for three quantities which are related through the identity (8), it is clear that the above system of three equations is equivalent to a system of two equations, which is to say that one of the equations is redundant. We therefore look for a system of two equations which is equivalent to the system of three Eqs. (4), (5), (6).

Since q and qF are 2-tensors and p is a 1-tensor on the sphere, neglecting Eq. (6) for p would cause the absence of control of the projection to the ℓ = 1 spherical mode of the perturbations. For this reason, we decide to neglect one of the first two equations, more precisely (4), the equation for q, which has the most intricate right hand side.

We then substitute q through the relation (8) into the wave Eq. (5), and we obtain schematically

$$\Box_{gM,Q}q^F + V_2(r)q^F = c_1(r)\left(Q \cdot q + \text{l.o.t.}\right)$$

where the lower order terms, denoted l.o.t. cancel out in the above substitution (see Sect. 3 for the precise derivation). One then obtains

$$\Box_{gM,Q}q^F + V_2(r)q^F = a_2(r)\mathcal{P}_p \quad (9)$$

for a new potential V2 and a smooth function a2. Observe that Eq. (9) is now coupled to Eq. (6). By combining the above wave equations of spin ± 2 and spin ± 1 we obtain a system of two coupled linear wave equations of the following schematic form:

$$\begin{cases}
\Box_{gM,Q}p + V_1(r)p = Q \cdot a_1(r)\text{div}q^F \\
\Box_{gM,Q}q^F + V_2(r)q^F = a_2(r)\mathcal{P}_p.
\end{cases}$$

6 This was basically the approach of our derivation of the estimates in [20].
We call the above system the mixed spin $\pm 1$ and spin $\pm 2$ Regge–Wheeler system. Observe that since $q$ is related to $p$ and $q^F$ through the relation (8), the above system is equivalent$^7$ to the system of three equations obtained in [20,21]. The two quantities $q^F$ and $p$ therefore play the role of gravitational and electromagnetic radiation respectively for perturbations of Reissner–Nordström spacetime.

The fundamental advantage of the derived system compared to the previous one is in its symmetry: the operators $\mathcal{P}_\mu^\nu$ and $\vartheta^\nu$ appearing on the right hand sides are adjoint operators on the sphere [13]. Such symmetry is used here to define a combined energy-momentum tensor which allows for a cancellation of the highest order terms, without recurring to smallness of the charge. We are therefore able to deduce boundedness of the energy, Morawetz and $r^p$-estimates in the full subextremal range $|Q| < M$. This is in contrast with the system analyzed in [20], which has non symmetric right hand sides, and for which the analysis can be obtained for very small $Q$ only.

**Remark 2.** A similar structure in the coupling terms of a system of two wave equations has been found by Hung [29], for odd perturbations of linearized gravity of Schwarzschild in harmonic gauge. In [29], two metric components, denoted $H_1$ and $H_2$, satisfy a system of wave equations which are coupled through adjoint operators on the sphere, similarly to our mixed spin $\pm 1$ and spin $\pm 2$ Regge–Wheeler system. Hung obtains estimates for the system through a novel definition of energy-momentum tensor which makes use of the symmetric right hand side. We take a similar approach through the definition of a combined energy-momentum tensor as explained below.

### 1.5. The combined energy-momentum tensor

We now give a brief summary of the proof of boundedness and decay statements for the mixed spin $\pm 1$ and spin $\pm 2$ Regge–Wheeler system.

We define a combined energy-momentum tensor for the system, which takes into account both equations and their structure. Such combined energy-momentum tensor $\mathcal{T}_{\mu\nu}[q^F, p]$ is tailored on the specific structure of right hand side of the system. More precisely, it consists of the sum of the energy-momentum tensor associated to each equation, plus a mixed term defined in terms of the right hand side. Schematically:

$$
\mathcal{T}_{\mu\nu}[q^F, p] := \mathcal{T}_{\mu\nu}[q^F] + \mathcal{T}_{\mu\nu}[p] - Q \cdot a(r) \left( \mathcal{P}_\mu^\nu p \cdot q^F \right) g_{\mu\nu}
$$

where $\mathcal{T}_{\mu\nu}[q^F]$ and $\mathcal{T}_{\mu\nu}[p]$ are the standard energy-momentum tensor associated to the wave equations for $q^F$ and $p$ respectively. See Definition 2 for the exact expression.

Such definition of $\mathcal{T}_{\mu\nu}[q^F, p]$ is motivated by the following property: when applied with multiplier $X = \partial_t$, the associated current $P_\mu^X = \mathcal{T}_{\mu\nu}[q^F, p]X^\nu$ is divergence free. In particular, the additional term $- Q \cdot a(r) \left( \mathcal{P}_\mu^\nu p \cdot q^F \right) g_{\mu\nu}$ in the definition of the combined energy-momentum tensor is precisely the one needed to obtain cancellation of the divergence. By applying the divergence theorem to a causal domain, one only needs to prove the positivity of the modified boundary terms to obtain boundedness of the energy, which can be obtained in the full subextremal range $|Q| < M$. This is done in Sect. 6.

The derivation of Morawetz estimates is more subtle since the divergence of the current associated to $Y = f(r)\partial_r$ does not vanish, but has to be proved to be positive

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$^7$ It is interesting to observe that the one equation used in [13] to prove the linear stability of Schwarzschild can be neglected in Reissner–Nordström in favor of the two equations above (which have no correspondence in the gravitational perturbations of Schwarzschild).
definite. Because of the mixed term in the definition of $T_{\mu\nu}[q^F, p]$, we obtain a spacetime integral containing terms of the schematic form

$$c_1(r)|q^F|^2 + c_2(r)|p|^2 - c_3(r)(q^F \cdot p)$$

which has to be proved to be positive definite for a well-chosen function $f(r)$. The negativity of the discriminant of the above quadratic form ($D = c_3(r)^2 - 4c_1(r)c_2(r)$), together with the positivity of the coefficients $c_1(r)$ and $c_2(r)$, is used to conclude that a spacetime integral of the above schematic form is positive definite. This is done in Sect. 7.

Finally, the $r$-weights appearing on the right hand side of the equations in the mixed spin $\pm 1$ and spin $\pm 2$ Regge–Wheeler system are sufficiently good so that the derivation of the $r^p$-hierarchy of Dafermos–Rodnianski for the system is identical to the standard wave equation. This is done in Sect. 8.

A rough version of the result is as follows. For the precise version, see Theorem 3.

**Theorem 2** (Rough version). Solutions to the mixed spin $\pm 1$ and spin $\pm 2$ Regge–Wheeler system on Reissner–Nordström spacetime with $|Q| < M$ arising from initial data which is prescribed on a Cauchy hypersurface $\Sigma_0$ satisfy statements of energy boundedness, integrated local energy decay, and a hierarchy of $r$-weighted energy estimates.

The hierarchy of $r$-weighted estimates is such that, using a pigeonhole principle [16], one obtains in the full subextremal range $|Q| < M$, the pointwise decay estimates

$$|p| \leq C\tau^{-1+\delta}, \quad |q^F| \leq C\tau^{-1+\delta}$$

for $\delta > 0$ and a time function $\tau$, where $C$ is some constant depending on an appropriate Sobolev norm of the data.

The pointwise estimates for $p$ and $q^F$ imply estimates for $q$ in the full subextremal range through the relation (8). We are then in the condition of having extended the estimates for $q$, $q^F$ and $p$ to the full subextremal range $|Q| < M$, and therefore the proof of linear stability [22] can be applied to obtain control of all the remaining gauge-dependent quantities, as explained in Sect. 1.3.

The paper is organized as follows. In Sect. 2 we recall the main properties of Reissner–Nordström spacetime and in Sect. 3, the symmetric system used in this paper is derived from the equations obtained in [20,21]. In Sect. 4, the energy quantities are defined and the main theorem is stated. The energy-momentum tensor associated to the system is defined in Sect. 5. In Sect. 6, boundedness of the energy for the full subextremal range $|Q| < M$ is proved. Morawetz estimates for the subextremal range $|Q| < M$ are obtained in Sect. 7 and the $r^p$-estimates are derived in Sect. 8.

### 2. The Reissner–Nordström Spacetime

In this section, we introduce the Reissner–Nordström exterior metric, as well as relevant background structure. We mostly highlight the properties which are needed in this paper. For a more complete description of the Reissner–Nordström spacetime see [23].
2.1. The manifold and the metric. Define the manifold with boundary

\[ M := \mathcal{D} \times S^2 := (-\infty, 0] \times (0, \infty) \times S^2 \]  

(10)

with Kruskal coordinates \((U, V, \theta^1, \theta^2)\), as defined in Section 3 of [20]. The boundary \(\mathcal{H}^+\) will be referred to as the horizon. We denote by \(S^2_{U,V}\) the 2-sphere \(\{U, V\} \times S^2 \subset M\) in \(M\).

Fix two parameters \(M > 0\) and \(Q\), verifying \(|Q| < M\). Then the Reissner–Nordström metric \(g_{M,Q}\) with parameters \(M\) and \(Q\) is defined to be the metric:

\[ g_{M,Q} = -4 \Upsilon_K (U, V) \, dU \, dV + r^2 (U, V) \, \gamma_{AB} d\theta^A d\theta^B \]  

(11)

where

\[ \Upsilon_K (U, V) = \frac{r_- r_+}{4r(U, V)^2} \left( \frac{r(U, V) - r_-}{r_+ - r_-} \right)^{1+\left(\frac{1}{r_+}\right)} \exp \left( - \frac{r_+ - r_-}{r_+^2} r(U, V) \right) \]

and

\[ \gamma_{AB} = \text{standard metric on } S^2. \]

and

\[ r_\pm = M \pm \sqrt{M^2 - Q^2} \]  

(12)

and \(r\) is an implicit function of the coordinates \(U\) and \(V\). We denote \(r_{\mathcal{H}} = r_+ = M + \sqrt{M^2 - Q^2}\).

The Kruskal coordinates cover the entire exterior region up to the horizon. We now define another double null coordinate system that covers the interior of \(M\) (up to the event horizon), modulo the degeneration of the angular coordinates. This coordinate system, \((u, v, \theta^1, \theta^2)\), is called double null coordinates and are defined via the relations

\[ U = - \frac{2r_+^2}{r_+ - r_-} \exp \left( - \frac{r_+ - r_-}{4r_+^2} u \right) \quad \text{and} \quad V = \frac{2r_+^2}{r_+ - r_-} \exp \left( \frac{r_+ - r_-}{4r_+^2} v \right). \]  

(13)

Using (13), we obtain the Reissner–Nordström metric on the interior of \(M\) in \((u, v, \theta^1, \theta^2)\)-coordinates:

\[ g_{M,Q} = -4 \Upsilon (u, v) \, du \, dv + r^2 (u, v) \, \gamma_{AB} d\theta^A d\theta^B \]  

(14)

with

\[ \Upsilon := 1 - \frac{2M}{r} + \frac{Q^2}{r^2}. \]  

(15)

We denote by \(S_{u,v}\) the sphere \(S^2_{U,V}\) where \(U\) and \(V\) are given by (13).

Note that \(u, v\) are regular optical functions. Their corresponding null geodesic generators are

\[ L := -g^{ab} \partial_a v \partial_b = \frac{1}{\Upsilon} \partial_u, \quad L := -g^{ab} \partial_a u \partial_b = \frac{1}{\Upsilon} \partial_v. \]  

(16)

The null frame \((e_3^*, e_4^*)\) for which \(e_3\) is geodesic (which is regular towards the future along the event horizon) is given by

\[ e_3^* = L, \quad e_4^* = \Upsilon L. \]  

(17)
The null frame \((e_3, e_4)\) for which \(e_4\) is geodesic (which is regular towards null infinity) is given by
\[
e_3 = \Upsilon L, \quad e_4 = L.
\] (18)

The photon sphere of Reissner–Nordstrom corresponds to the hypersurface in which null geodesics are trapped. It is the hypersurface given by \(\{r = r_P\}\) where \(r_P\) is the largest root of the polynomial \(r^2 - 3Mr + 2Q^2\), and is given by
\[
r_P = \frac{3M + \sqrt{9M^2 - 8Q^2}}{2}.
\] (19)

The curvature and electromagnetic components which are non-vanishing are given by
\[
(F)\rho = \frac{1}{2} F(e_3, e_4) = \frac{Q}{r^2},
\]
\[
\rho = \frac{1}{4} W(e_3, e_4, e_3, e_4) = -\frac{2M}{r^3} + \frac{2Q^2}{r^4}
\] (20)
where \(F\) is the electromagnetic tensor and \(W\) is the Weyl curvature of the Reissner–Nordstrom solution.

2.2. The Killing vector fields. We define the vectorfield \(T\) to be the timelike Killing vector field \(\partial_t\) of the \((t, r)\) coordinates in (3), which in double null coordinates is given by
\[
T = \frac{1}{2} (\partial_u + \partial_v) = \frac{1}{2} (\Upsilon e_3^* + e_4^*) = \frac{1}{2} (e_3 + \Upsilon e_4).
\]

We can also define a basis of angular momentum operator \(\Omega_i, i = 1, 2, 3\) (see for example [13]). The Lie algebra of Killing vector fields of \(g_{M, Q}\) is then generated by \(T\) and \(\Omega_i\), for \(i = 1, 2, 3\).

2.3. The spherical harmonics and elliptic estimates. We collect some known definitions and properties of the Hodge decomposition of scalars, one forms and symmetric traceless two tensors in spherical harmonics. We also recall some known elliptic estimates. See Section 4.4 of [13] for more details.

We denote by \(Y^\ell_m\), with \(|m| \leq \ell\), the spherical harmonics on the sphere of radius \(r\), i.e.
\[
\Delta_0 Y^\ell_m = -\frac{1}{r^2} \ell(\ell + 1) Y^\ell_m
\] (21)
where \(\Delta_0\) denotes the laplacian on the sphere \(S_{u,v}\) of radius \(r = r(u, v)\) for scalar functions.

Definition 1. We say that a function \(f\) on \(\mathcal{M}\) is supported on \(\ell \geq 2\) if the projections
\[
\int_{S_{u,v}} f \cdot Y^\ell_m = 0
\]
vanish for \(Y^\ell_m = 1\) for \(m = -1, 0, 1\).
We recall the following angular operators on $S_{u,v}$-tensors. Let $\xi$ be an arbitrary one-form and $\theta$ an arbitrary symmetric traceless 2-tensor on $S_{u,v}$.

- $\nabla$ denotes the covariant derivative associated to the metric $g$ on $S_{u,v}$.
- $\mathcal{D}_1$ takes $\xi$ into the pair of functions $(\text{div} \xi, \text{curl} \xi)$, where
  \[ \text{div} \xi = g^{AB} \nabla_A \xi_B, \quad \text{curl} \xi = \xi^{AB} \nabla_A \xi_B. \]
- $\mathcal{D}_1^*$ is the formal $L^2$-adjoint of $\mathcal{D}_1$, and takes any pair of functions $(\rho, \sigma)$ into the one-form $-\nabla_A \rho + \xi^{AB} \nabla_B \sigma$.
- $\mathcal{D}_2$ takes $\theta$ into the one-form $\mathcal{D}_2 \theta = (\text{div} \theta)_C = g^{AB} \nabla_A \theta_{BC}$.
- $\mathcal{D}_2^*$ is the formal $L^2$-adjoint of $\mathcal{D}_2$, and takes $\xi$ into the symmetric traceless two tensor
  \[ (\mathcal{D}_2^* \xi)_{AB} = -\frac{1}{2} (\nabla_B \xi_A + \nabla_A \xi_B - (\text{div} \xi) g_{AB}). \]

We can easily check that $\mathcal{D}_k^*$ is the formal adjoint of $\mathcal{D}_k$, i.e.
\[ \int_S (\mathcal{D}_k f) g = \int_S f (\mathcal{D}_k^* g). \] (22)

Recall that an arbitrary one-form $\xi$ on $S_{u,v}$ has a unique representation $\xi = r \mathcal{D}_k^* (f, g)$, where $\mathcal{D}_k^* (f, g)$, for two uniquely defined functions $f$ and $g$ on the unit sphere, both with vanishing mean. In particular, the scalars $\text{div} \xi$ and $\text{curl} \xi$ are supported in $\ell \geq 1$.

Recall that an arbitrary symmetric traceless two-tensor $\theta$ on $S_{u,v}$ has a unique representation $\theta = r^2 \mathcal{D}_k^* \mathcal{D}_k (f, g)$ for two uniquely defined functions $f$ and $g$ on the unit sphere, both supported in $\ell \geq 2$. In particular, the scalars $\text{div} \text{div} \theta$ and $\text{curl} \text{div} \theta$ are supported in $\ell \geq 2$.

We now derive the decomposition in spherical harmonics for one-forms and for two-tensors. We denote $\Delta_1$ and $\Delta_2$ the laplacian on the sphere $S_{u,v}$ of radius $r = r(u, v)$ for one-forms and two-tensors respectively. The laplacian is related to the angular Hodge operators by the following relations [12]:
\[ \mathcal{D}_1 \mathcal{D}_1^* = -\Delta_0, \quad \mathcal{D}_1^* \mathcal{D}_1 = -\Delta_1 + K, \]
\[ \mathcal{D}_2 \mathcal{D}_2^* = -\frac{1}{2} \Delta_1 - \frac{1}{2} K, \quad \mathcal{D}_2^* \mathcal{D}_2 = -\frac{1}{2} \Delta_2 + K. \]

Using the above one can prove the following commutators (see Appendix in [20]):
\[ -\mathcal{D}_k \Delta_0 + \Delta_1 \mathcal{D}_k^* = K \mathcal{D}_k^*, \]
\[ -\mathcal{D}_2^* \Delta_1 + \Delta_2 \mathcal{D}_2^* = 3K \mathcal{D}_2^* \] (23)

where $K = \frac{1}{r^2}$ is the Gauss curvature of the sphere of radius $r$.

Let $\xi$ be a one-form supported on the spherical harmonic $\ell \geq 1$, i.e. $\xi = r \mathcal{D}_1^* (f, g)$ with $f$ and $g$ scalar functions supported on $\ell \geq 1$. We then have from (21) and (23):
\[ \Delta_1 \xi = r \Delta_1 \mathcal{D}_1^* (f, g) = r \mathcal{D}_1^* \Delta_0 (f, g) + K r \mathcal{D}_1^* (f, g) \]
\[ = -\frac{1}{r^2} \ell (\ell + 1) r \mathcal{D}_1^* (f, g) + K r \mathcal{D}_1^* (f, g) \]
\[ = -\frac{\ell (\ell + 1) - 1}{r^2} \xi. \]
Multiplying the above by $\xi$ and integrating by parts the left hand side, we obtain for a one-form $\xi$ supported on the spherical harmonics $\ell \geq 1$:

$$\int_S |\nabla \xi|^2 = \int_S \frac{\ell(\ell + 1) - 1}{r^2} |\xi|^2. \quad (24)$$

Let $\theta$ be a symmetric traceless two-tensor supported on the spherical harmonic $\ell \geq 2$, i.e. $\theta = r^2 \mathcal{P}_2 \mathcal{P}_2(f, g) = r \mathcal{P}_2 \xi$ with $\xi$ supported on $\ell \geq 2$. From (21) and (23), we have

$$\mathcal{L}_2 \theta = r \mathcal{L}_2 \mathcal{P}_2 \xi = r \mathcal{P}_2 \mathcal{L}_1 \xi + 3Kr \mathcal{P}_2 \xi$$

$$= -\frac{\ell(\ell + 1) - 1}{r^2} r \mathcal{P}_2 \xi + 3Kr \mathcal{P}_2 \xi$$

$$= -\frac{\ell(\ell + 1) - 4}{r^2} \theta.$$ 

Multiplying the above by $\theta$ and integrating by parts the left hand side, we obtain for a symmetric traceless two-tensor supported on the spherical harmonic $\ell \geq 2$:

$$\int_S |\nabla \theta|^2 = \int_S \frac{\ell(\ell + 1) - 4}{r^2} |\theta|^2. \quad (25)$$

We recall the following $L^2$ elliptic estimates.

**Proposition 1 ([12]).** Let $(S, \gamma)$ be a compact surface with Gauss curvature $K$. Then the following identities hold for 1-forms $\xi$ on $S$ and symmetric traceless 2-tensors $\theta$:

$$\int_S |\nabla \xi|^2 - K|\xi|^2 = 2\int_S |\mathcal{P}_2 \xi|^2 \quad (26)$$

$$\int_S |\nabla \theta|^2 + 2K|\theta|^2 = 2\int_S |\mathcal{P}_2 \theta|^2. \quad (27)$$

We can specialize the above elliptic estimates to tensor supported on a fixed spherical harmonic $\ell$. Using (24) and (26), we deduce for a one form supported on a fixed spherical harmonic $\ell$:

$$\int_S |\mathcal{P}_2 \xi|^2 = \int_S \frac{1}{2} |\nabla \xi|^2 - \frac{1}{2} K|\xi|^2$$

$$= \int_S \frac{\ell(\ell + 1) - 1}{2r^2} |\xi|^2 - \frac{1}{2} \frac{1}{r^2} |\xi|^2$$

$$= \int_S \frac{2\ell(\ell + 1) - 4}{4r^2} |\xi|^2.$$ 

This implies, for one form $\xi$ and two tensor $\theta$ both supported on a fixed spherical harmonic $\ell$:

$$\int_S \mathcal{P}_2 \xi \cdot \theta \geq -\int_S |\mathcal{P}_2 \xi| |\theta| = -\int_S \frac{1}{2} \frac{(2\ell(\ell + 1) - 4)^{1/2}}{r} |\xi| |\theta|. \quad (28)$$
3. The Derivation of the Mixed Spin $\pm 1$ and Spin $\pm 2$ System of Equations

In this section, we derive the mixed spin $\pm 1$ and spin $\pm 2$ system of equations from the spin $\pm 2$ equations obtained in [20] and the spin $\pm 1$ equation obtained in [21].

Recall the quantities $p$, $q^F$, and $q$ for linear perturbations of Reissner–Nordström spacetime as defined in the Introduction.

In [20], the following wave equation for $q^F$, coupled with $q$, has been derived (see Proposition 16 Appendix B.1 in [20]):

$$\Box_{g_{M,Q}} q^F + (\kappa \kappa + 3 \rho) q^F = (F) \rho \left( -\frac{1}{r} q + 4 (F) \rho \frac{r^3 \kappa f}{r^2} \right)$$ (29)

where here $\Box_{g_{M,Q}} = D^\mu D_\mu$ is the d’Alembertian of the Reissner–Nordström metric $g_{M,Q}$ applied to 2-tensors. Being an equation for the linearized quantity $q$, the coefficients of the equations are the background values in Reissner–Nordström. More precisely

- $\kappa := \text{tr} \chi$ and $\kappa := \text{tr} \chi_1$ are the trace of the second null fundamental forms and

$$\kappa \kappa = -\frac{4}{r^2} \left( 1 - \frac{2M}{r} + \frac{Q^2}{r^2} \right),$$
- $\rho = -\frac{2M}{r^3} + \frac{2Q^2}{r^4}$ and $(F) \rho = \frac{Q}{r}$ are given by (20).

In [21], the following relation between $\tilde{\beta}$, $\alpha$, and $f$ has been derived (see Lemma 6.3.1 in [21]):

$$D^\star \frac{1}{2} (r^3 \kappa \beta) = - (F) \rho \frac{1}{\kappa} \nabla_3 (r^3 \kappa^2 \alpha) - \left( 2 (F) \rho^2 + 3 \rho \right) r^3 \kappa f$$

where $\kappa$, $(F) \rho$ and $\rho$ are again the background values.

We multiply the above by $r^3$, apply $\frac{1}{\kappa} \nabla_3$, and recall that $[\nabla_3, r D^\star] = 0$ [20]. We then obtain

$$r D^\star \rho = -r^2 (F) \rho q - r^3 \left( 2 (F) \rho^2 + 3 \rho \right) q^F$$

$$- \frac{1}{\kappa} \nabla_3 \left( r^2 (F) \rho \right) r \frac{1}{\kappa} \nabla_3 (r^3 \kappa^2 \alpha) - \frac{1}{\kappa} \nabla_3 \left( r^3 (2 (F) \rho^2 + 3 \rho) \right) r^3 \kappa f$$

where we recall that $q = \frac{1}{\kappa} \nabla_3 \left( \frac{1}{\kappa} \nabla_3 (r^3 \kappa^2 \alpha) \right)$, $q^F = \frac{1}{\kappa} \nabla_3 (r^3 \kappa f)$, and $p = \frac{1}{\kappa} \nabla_3 (r^5 \tilde{\beta})$.

Using that $r^2 (F) \rho = Q$, and $r^3 (2 (F) \rho^2 + 3 \rho) = -6M + \frac{8Q^2}{r}$, we obtain

$$D^\star \rho = -r (F) \rho q - r^2 \left( 2 (F) \rho^2 + 3 \rho \right) q^F + 4Q^2 r \kappa f.$$
By writing \( Q^2 r = r^5(F) \rho^2 \), we proved that the quantities \( p, q \) and \( q^F \) are related through the following relation:

\[
\mathcal{P}_p p = -r(F) \rho q - r^2(3 \rho + 2(F) \rho^2)q^F + 4r^5(F) \rho^2 K^f. \tag{31}
\]

We use relation (31) to substitute \( q \) in (29):

\[
-\frac{1}{r} r(F) \rho q = \frac{1}{r^2} \mathcal{P}_p p - 4r^3(F) \rho^2 K^f + (3 \rho + 2(F) \rho^2)q^F.
\]

Equation (29) then becomes

\[
\Box_{gM, Q} q^F + (\kappa K + 3 \rho) q^F = -\frac{1}{r} r(F) \rho q + 4r^3(F) \rho^2 K^f + 3 \rho + 2(F) \rho^2
\]

\[
= \frac{1}{r^2} \mathcal{P}_p p - 4r^3(F) \rho^2 K^f + (3 \rho + 2(F) \rho^2)q^F + 4r^3(F) \rho^2 K^f
\]

\[
= \frac{1}{r^2} \mathcal{P}_p p + (3 \rho + 2(F) \rho^2)q^F
\]

where observe the cancellation of the term \( 4r^3(F) \rho^2 K^f \). We therefore obtain

\[
\Box_{gM, Q} q^F + (\kappa K - 2(F) \rho^2) q^F = \frac{1}{r^2} \mathcal{P}_p p.
\]

We now combine the above equation together with equation (30) for \( p \), and we obtain the following system, which we denote \textit{mixed spin} \( \pm 1 \) and \textit{spin} \( \pm 2 \) Regge–Wheeler system:

\[
\Box_{gM, Q} p - V_1(r) p = \frac{8Q^2}{r^2} \mathcal{P}_p q^F, \tag{32}
\]

\[
\Box_{gM, Q} q^F - V_2(r) q^F = \frac{1}{r^2} \mathcal{P}_p p
\]

where we wrote the divergence as \( \text{div} = \mathcal{P}_2 \). The potentials are given by

\[
V_1(r) = -\frac{1}{4} \kappa K + 5(F) \rho^2 = \frac{1}{4} \frac{4}{r^2} \left( 1 - \frac{2M}{r} + \frac{Q^2}{r^2} \right) + 5 \frac{Q^2}{r^4} = \frac{1}{r^2} \left( 1 - \frac{2M}{r} + 6 \frac{Q^2}{r^2} \right)
\]

\[
V_2(r) = -\kappa K + 2(F) \rho^2 = \frac{4}{r^2} \left( 1 - \frac{2M}{r} + \frac{Q^2}{r^2} \right) + 2 \frac{Q^2}{r^4} = \frac{4}{r^2} \left( 1 - \frac{2M}{r} + \frac{3Q^2}{2r^2} \right).
\]

In order to make the right hand sides symmetric in the presence of \( Q \), we can assume\(^8\) that \( Q \neq 0 \) and define

\[
\Phi_1 := p, \quad \Phi_2 := Q \cdot q^F
\]

with \( \Phi_1 \) a 1-tensor and \( \Phi_2 \) a symmetric traceless 2-tensor on the sphere. Then the above system becomes

\[
\Box_{gM, Q} \Phi_1 - V_1(r) \Phi_1 = \frac{8Q}{r^2} \mathcal{P}_2 \Phi_2 \tag{36}
\]

\(^8\) This case is contained in the case of \( |Q| \ll M \) treated in [20,21].
The above two equations form the symmetric system which we will analyze below.

Observe that we can restrict our attention to the case of \( \Phi_1 \) supported to the \( \ell \geq 2 \) spherical harmonics. Indeed, if \( \Phi_1 \) is supported on the \( \ell = 1 \) spherical harmonics, the two equations decouple since \( (\nabla^2 \Phi_2)_{\ell=1} = 0 \) and \( \nabla_\ell^2 (\Phi_1)_{\ell=1} = 0 \). More precisely, the first equation reduces to the main equation analyzed in [21], and the second equation reduces to one of the two equations analyzed in [20], with trivial right hand side. In what follows, we will therefore restrict to the case of \( \Phi_1 \) and \( \Phi_2 \) both supported to the \( \ell \geq 2 \) spherical harmonics.

4. Energy Quantities and Statements of the Main Theorem

We define a foliation in Reissner–Nordström spacetime \( \Sigma_\tau \) which connects the event horizon and future null infinity. We foliate \( M \) by hypersurfaces \( \Sigma_\tau \) which are:

1. Incoming null in \( \{r_H \leq r \leq \frac{11}{10} r_H\} \), with \( e_3^* \) as null incoming generator (which is regular up to horizon). We denote this portion \( \Sigma_{\text{red}} \). This is realized by a portion of \( \{v = \text{const}\} \) in the ingoing Eddington-Finkelstein coordinates.
2. Strictly spacelike in \( \{\frac{11}{10} r_H < r < R\} \) with \( R \) a fixed number \( R \gg r_P \). We denote this portion by \( \Sigma_{\text{trap}} \). This is realized by a portion of \( \tilde{\Sigma}_\tau = \{t = \tau\} \).
3. Outgoing null in \( \{r \geq R\} \) with \( e_4 \) as null outgoing generator. We denote this portion by \( \Sigma_{\text{far}} \). This is realized by a portion of \( \{u = \text{const}\} \) in the outgoing Eddington-Finkelstein coordinates.

We denote \( M(\tau_1, \tau_2) \subset M \) the spacetime region in the past of \( \Sigma(\tau_2) \) and in the future of \( \Sigma(\tau_1) \). For fixed \( R \) we denote by \( M_{\leq R} \) and \( M_{\geq R} \) the regions defined by \( r \leq R \) and \( r \geq R \). We denote by \( \Sigma_{\geq R}(\tau) \) the portion of hypersurface for \( r \geq R \). See Fig. 1.

Let \( p \) be a free parameter with \( \delta \leq p \leq 2 - \delta \), for \( \delta > 0 \), as in standard application of the \( r^p \)-method of Dafermos–Rodnianski [16].
We introduce the following weighted energies for $\Phi_1$ and $\Phi_2$.

1. Energy quantities on $\Sigma_\tau$:
   - Basic energy quantity
     
     $E[\Phi_1, \Phi_2](\tau) := \int_{\Sigma_{\text{red}}} |\nabla^3_\tau(\Phi_1)|^2 + |\nabla_\tau \Phi_1|^2 + |\nabla^3_\tau(\Phi_2)|^2 + |\nabla_\tau \Phi_2|^2 + |\Phi_1|^2 + |\Phi_2|^2$
   
   $+ \int_{\Sigma_{\text{trap}}} |\nabla_\tau(\Phi_1)|^2 + |\nabla_\tau(\Phi_2)|^2 + |\nabla_\tau \Phi_1|^2 + r^{-2}|\Phi_1|^2$
   
   $+ \int_{\Sigma_{\text{trap}}} |\nabla_\tau(\Phi_2)|^2 + |\nabla_\tau(\Phi_2)|^2 + |\nabla_\tau \Phi_2|^2 + r^{-2}|\Phi_2|^2$

   Notice that the above energy quantity is regular up to the horizon. Observe that along outgoing null hypersurfaces ($u = \text{const}$) the volume form is given by $dud\text{vol}_{S_{u,v}}$, and along ingoing null hypersurfaces ($v = \text{const}$) the volume form is $dvd\text{vol}_{S_{u,v}}$, where $d\text{vol}_{S_{u,v}}$ denotes the volume form of the sphere $S_{u,v}$.
   - Weighted energy quantity
     
     $E_p; R[\Phi_1, \Phi_2](\tau) := \int_{\Sigma_{r \geq R}(\tau)} r^p|\tilde{\nabla}_4 \Phi_1|^2 + r^p|\tilde{\nabla}_4 \Phi_2|^2$

     where $\tilde{\nabla}_4 \Psi := \nabla_4 \Psi + \frac{1}{r}\Psi$.

2. Weighted spacetime bulk energies in $\mathcal{M}(\tau_1, \tau_2)$:
   - Basic Morawetz bulk
     
     $\mathcal{M}[\Phi_1, \Phi_2](\tau_1, \tau_2) := \int_{\mathcal{M}(\tau_1, \tau_2)} \frac{1}{r^3} |\nabla_\tau(\Phi_1)|^2 + \frac{1}{r^4} |\Phi_1|^2 + \frac{r^2 - 3Mr + 2Q^2}{r^5} \left( |\nabla_\tau \Phi_1|^2 + \frac{1}{r^2} |T_\tau \Phi_1|^2 \right)$
     
     $+ \int_{\mathcal{M}(\tau_1, \tau_2)} \frac{1}{r^3} |\nabla_\tau(\Phi_2)|^2 + \frac{1}{r^4} |\Phi_2|^2 + \frac{r^2 - 3Mr + 2Q^2}{r^5} \left( |\nabla_\tau \Phi_2|^2 + \frac{1}{r^2} |T_\tau \Phi_2|^2 \right)$

     where $R = \frac{1}{2}(-\gamma e_3 + e_4^*) = \frac{1}{2}(-e_3 + \gamma e_4)$. Notice that the Morawetz bulk $\mathcal{M}[\Psi](\tau_1, \tau_2)$ is degenerate at the photon sphere ($r = r_p$).
   - Weighted bulk norm in the far away region
     
     $\mathcal{M}_{p; R}[\Phi_1, \Phi_2](\tau_1, \tau_2) := \int_{\mathcal{M}_{r \geq R}(\tau_1, \tau_2)} r^{p-1} \left( p|\tilde{\nabla}_4(\Phi_1)|^2 + (2 - p)(|\nabla_\tau \Phi_1|^2 + r^{-2}|\Phi_1|^2) \right)$
     
     $+ \int_{\mathcal{M}_{r \geq R}(\tau_1, \tau_2)} r^{p-1} \left( p|\tilde{\nabla}_4(\Phi_2)|^2 + (2 - p)(|\nabla_\tau \Phi_2|^2 + r^{-2}|\Phi_2|^2) \right)$

     Observe that for $\delta \leq p \leq 2 - \delta$, for $\delta > 0$, the above bulk norm is positive definite.
– Weighted bulk norm
\[ \mathcal{M}_p[\Phi_1, \Phi_2](\tau_1, \tau_2) := \text{Mor}[\Phi_1, \Phi_2](\tau_1, \tau_2) + \mathcal{M}_p; R[\Phi_1, \Phi_2](\tau_1, \tau_2). \]

We can now state the main theorem in terms of the above energy quantities.

**Theorem 3.** Let \( \Phi_1 \) and \( \Phi_2 \) be a 1-tensor and a symmetric traceless 2-tensor respectively, satisfying Eqs. (36) and (37) in Reissner–Nordstrom spacetime with \(|Q| < M\) and supported on the \( \ell \geq 2 \) spherical harmonics. Then,

1. for all \( \delta \leq p \leq 2 - \delta \) and for any \( \tau > 0 \), boundedness of the weighted energy holds:
   \[ E_p[\Phi_1, \Phi_2](\tau) \leq CE_p[\Phi_1, \Phi_2](0). \]  
   (41)

2. for all \( \delta \leq p \leq 2 - \delta \) and for any \( \tau > 0 \), the following integrated local energy decay estimates for \( \Phi_1 \) and \( \Phi_2 \) holds:
   \[ \mathcal{M}_p[\Phi_1, \Phi_2](0, \tau) \leq CE_p[\Phi_1, \Phi_2](0). \]  
   (42)

Observe that, as in the case of integrated local energy decay estimates for even the scalar linear wave equation on black hole backgrounds, the degeneracy at the photon sphere of the Morawetz bulk cannot be eliminated. The degeneracy is caused by the presence of orbital trapped geodesics, which are an obstruction to decay [17].

In the following we show how to obtain the boundedness of the energy in Sect. 6, the Morawetz estimates in Sect. 7 and the \( r^p \)-estimates in Sect. 8. Combining those estimates, we prove the boundedness of the the weighted energy (41) and the integrated local energy decay estimates (42), therefore proving Theorem 3.

5. The Energy-Momentum Tensor Associated to the System

In this section, we define a combined energy-momentum tensor associated to the system formed by Eqs. (36) and (37). We first recall the definition of energy-momentum tensor and current associated to a solution of a wave equation.

We define the stress-energy tensor of Eqs. (36) and (37) as:

\[ T_{\mu\nu}[\Phi_1] := D_\mu \Phi_1 \cdot D_\nu \Phi_1 - \frac{1}{2} g_{\mu\nu} \left( D_\lambda \Phi_1 \cdot D^\lambda \Phi_1 + V_1|\Phi_1|^2 \right) \]
\[ = D_\mu \Phi_1 \cdot D_\nu \Phi_1 - \frac{1}{2} g_{\mu\nu} \mathcal{L}_1[\Phi_1] \]
\[ T_{\mu\nu}[\Phi_2] := D_\mu \Phi_2 \cdot D_\nu \Phi_2 - \frac{1}{2} g_{\mu\nu} \left( D_\lambda \Phi_2 \cdot D^\lambda \Phi_2 + V_2|\Phi_2|^2 \right) \]
\[ = D_\mu \Phi_2 \cdot D_\nu \Phi_2 - \frac{1}{2} g_{\mu\nu} \mathcal{L}_2[\Phi_2] \]

where \( D \) denotes the covariant derivative of the Reissner–Nordström metric \( g = g_{M,Q} \), and \( V_1 \) and \( V_2 \) are the potentials of the equations defined in (33) and (34).

For \( X \) a vectorfield, \( w \) a scalar function and \( M \) a one form, we define the associated currents as:

\[ P_{\mu}(X,w_1,M_1)[\Phi_1] := T_{\mu\nu}[\Phi_1]X^\nu + \frac{1}{2} w_1 \Phi_1 D_\mu \Phi_1 - \frac{1}{4} \partial_\mu w_1|\Phi_1|^2 + \frac{1}{4} M_1\mu|\Phi_1|^2 \]
\[ P_{\mu}(X,w_2,M_2)[\Phi_2] := T_{\mu\nu}[\Phi_2]X^\nu + \frac{1}{2} w_2 \Phi_2 D_\mu \Phi_2 - \frac{1}{4} \partial_\mu w_2|\Phi_2|^2 + \frac{1}{4} M_2\mu|\Phi_2|^2. \]
From a standard computation (see for example [20]) we obtain that for $X = a(r)e_3 + b(r)e_4$, in a spherically symmetric spacetime, the divergence of the current is given by

$$D^\mu \mathcal{P}_\mu(X,w,M)[\Phi_1] = \frac{1}{2} T[\Phi_1] \cdot (X) \pi + \left( - \frac{1}{2} X(V_1) - \frac{1}{4} \square_g w_1 \right) |\Phi_1|^2 + \frac{1}{2} w_1 \mathcal{L}_1[\Phi_1]$$

$$+ \frac{1}{4} D^\mu (\Phi_1^2 M_\mu) + \left( X(\Phi_1) + \frac{1}{2} w_1 \Phi_1 \right) \cdot \frac{8Q}{r^2} \mathcal{P}_\Phi \Phi_2$$

and

$$D^\mu \mathcal{P}_\mu(X,w,M)[\Phi_2] = \frac{1}{2} T[\Phi_2] \cdot (X) \pi + \left( - \frac{1}{2} X(V_2) - \frac{1}{4} \square_g w_2 \right) |\Phi_2|^2 + \frac{1}{2} w_2 \mathcal{L}_2[\Phi_2]$$

$$+ \frac{1}{4} D^\mu (|\Phi_2|^2 M_\mu) + \left( X(\Phi_2) + \frac{1}{2} w_2 \Phi_2 \right) \cdot \frac{O}{r^2} \mathcal{P}_\Phi \Phi_1$$

where $(X) \pi$ is the deformation tensor of the vectorfield $X$.

Notice the presence of the right hand sides of the equations, $\frac{8Q}{r^2} \mathcal{P}_\Phi \Phi_2$ and $\frac{O}{r^2} \mathcal{P}_\Phi \Phi_1$ in (43) and (44). In particular, those right hand sides appear in the divergence of the current. Our goal is to define an energy-momentum tensor for the system which has good cancellation properties with respect to these additional terms. It turns out that the system composed by Eqs. (36) and (37) admits a conserved energy-momentum tensor.

**Definition 2.** Let $\Phi_1$ and $\Phi_2$ be a 1-tensor and a symmetric traceless 2-tensor respectively, satisfying the system of coupled wave Eqs. (36) and (37). We define the energy-momentum tensor for the system as the following symmetric two tensor $T_{\mu\nu}[\Phi_1, \Phi_2]$:

$$T_{\mu\nu}[\Phi_1, \Phi_2] := T_{\mu\nu}[\Phi_1] + 8T_{\mu\nu}[\Phi_2] - \frac{8Q}{r^2} (\mathcal{P}_\Phi \Phi_1 \cdot \Phi_2) g_{\mu\nu}$$

where $T_{\mu\nu}[\Phi_1]$ and $T_{\mu\nu}[\Phi_2]$ are the standard energy-momentum tensors associated to Eqs. (36) and (37) respectively.

We also define the associated combined current $\mathcal{P}_{\mu}(X,w,M)[\Phi_1, \Phi_2]$ for a vectorfield $X$, a pair of scalar functions $w = (w_1, w_2)$, a pair of one forms $M = (M_1, M_2)$ as

$$\mathcal{P}_{\mu}(X,w,M)[\Phi_1, \Phi_2] = \mathcal{P}_{\mu}(X,w_1,M_1)[\Phi_1] + 8\mathcal{P}_{\mu}(X,w_2,M_2)[\Phi_2] - \frac{8Q}{r^2} (\mathcal{P}_\Phi \Phi_1 \cdot \Phi_2) X_\mu$$

(46)

The above definition is motivated by the cancellation properties for the divergence of the new combined current $\mathcal{P}_{\mu}(X,w,M)[\Phi_1, \Phi_2]$, as showed in the following lemma.

**Lemma 1.** Let $\Phi_1$ and $\Phi_2$ be a 1-tensor and a symmetric traceless 2-tensor respectively, satisfying the system of coupled wave Eqs. (36) and (37). For $X = a(r)e_3 + b(r)e_4$, the divergence of the combined current $\mathcal{P}_{\mu}(X,w,M)[\Phi_1, \Phi_2]$ is given by

$$D^\mu \mathcal{P}_{\mu}(X,w,M)[\Phi_1, \Phi_2] = s \varepsilon_{(X,w,M)}[\Phi_1, \Phi_2]$$

where $s$ indicates that the equality holds upon integration on the sphere and

$$\varepsilon_{(X,w,M)}[\Phi_1, \Phi_2] := \frac{1}{2} T[\Phi_1] \cdot (X) \pi + \left( - \frac{1}{2} X(V_1) - \frac{1}{4} \square_g w_1 \right) |\Phi_1|^2 + \frac{1}{2} w_1 \mathcal{L}_1[\Phi_1] + \frac{1}{4} D^\mu (|\Phi_1|^2 M_\mu)$$

$$+ 4T[\Phi_2] \cdot (X) \pi + \left( -4X(V_2) - 2 \square_g w_2 \right) |\Phi_2|^2 + 4w_2 \mathcal{L}_2[\Phi_2] + 2D^\mu (|\Phi_2|^2 M_\mu)$$

$$+ \frac{4Q}{r^2} \left( w_1 + w_2 + \frac{4}{r} X(r) - tr(X) \right) \mathcal{P}_\Phi \Phi_1 \cdot \Phi_2 - \frac{8Q}{r^2} (X, \mathcal{P}_\Phi \Phi_1) \cdot \Phi_2.$$
Proof. We compute, using (43) and (44):

\[
D^\mu P^{(X, w, M)}_{\mu}(X_1, X_2) = \frac{1}{2} T[X_1] \cdot (X) \pi + \left( -\frac{1}{2} X(V_1) - \frac{1}{4} \Box g \psi \right) |X_1|^2 \\
+ \frac{1}{2} w_1 L_1[X_1] + \frac{1}{4} D^\mu (\Phi^2_{1} M_\mu) \\
+ \left( X(X_1) + \frac{1}{2} w_1 \Phi_1 \right) \cdot \frac{8 Q}{r^2} \cdot \langle \Phi_1, \Phi_2 \rangle \\
+ 4 T[X_2] \cdot (X) \pi \\
+ (-4 X(V_2) - 2 \Box g \psi |X_2|^2 + 4 w_2 L_2[X_2] + 2 D^\mu (|X_2|^2 M_\mu) \\
+ \left( X(X_2) + \frac{1}{2} w_2 \Phi_2 \right) \cdot \frac{8 Q}{r^2} \cdot \langle \Phi_1, \Phi_2 \rangle \\
- X \left( \frac{8 Q}{r^2} \right) (\Phi_1 \cdot \Phi_2) \\
- \frac{8 Q}{r^2} (X(\Phi_1) \cdot \Phi_2) - \frac{8 Q}{r^2} (\Phi_1 \cdot X(\Phi_2)) \\
- \frac{8 Q}{r^2} (\Phi_1 \cdot \Phi_2) D^\mu (X_\mu) \\
\]

where the last two lines are the divergence of the additional term \(-\frac{8 Q}{r^2} (\Phi_1 \cdot \Phi_2) X_\mu \) in the definition of \(P^{(X, w, M)}_{\mu}(X_1, X_2)\). Recall that \(D^\mu (X_\mu) = \frac{1}{2} \Box g \psi \). We compute

\[
- \frac{8 Q}{r^2} (X(\Phi_1) \cdot \Phi_2) = - \frac{8 Q}{r^2} (\Phi_1 \cdot X(\Phi_1)) \cdot \Phi_2 + ([X, \Phi_1] \cdot \Phi_2) \\
= - \frac{8 Q}{r^2} (X(\Phi_1) \cdot \Phi_2 + ([X, \Phi_1] \cdot \Phi_2) \\
\]

where the last equality holds upon integration on the spheres, where we used that \(\Phi_2 \) and \(\Phi_1 \cdot \Phi_2 \) are adjoint operators on the sphere as in (22). We then obtain the cancellation of the terms \(X(\Phi_2) \cdot \frac{8 Q}{r^2} \Phi_1 \) and \(X(\Phi_1) \cdot \frac{8 Q}{r^2} \Phi_2 \). This implies the lemma. \(\square\)

6. Boundedness of the Energy

In this section we prove boundedness of the energy for the mixed spin \(\pm 1\) and spin \(\pm 2\) system of equations.

To derive the energy estimates we apply Lemma 1 to the Killing vectorfield \(X = T\), with \(w = 0, M = 0\). We obtain

\[
D^\mu P^{(T, 0)}_{\mu}(X_1, X_2) = \frac{1}{2} T[X_1] \cdot (T) \pi - \frac{1}{2} T(V_1) |X_1|^2 + 4 T[X_2] \cdot (T) \pi - 4 T(V_2) |X_2|^2 \\
+ \frac{4 Q}{r^2} \left( - \frac{4}{r} T(r) - \Box g \psi \right) \Phi_1 \cdot \Phi_2 - \frac{8 Q}{r^2} ([T, \Phi_1] \cdot \Phi_2) \\
= 0 \\
\]

since \((T) \pi = 0\), \(T(r) = T(V_1) = T(V_2) = 0\) and \(T\) commutes with the angular operator.
By applying the divergence theorem to $D^\mu P^{(T,0,0)}_{\mu} [\Phi_1, \Phi_2] = 0$ in the region $\mathcal{M}(0, \tau)$, we are left to analyze the boundary terms $\int_{\Sigma_T} P^{(T,0,0)}_{\mu} [\Phi_1, \Phi_2] \cdot n_{\Sigma_T}$, where $n_{\Sigma_T}$ is the normal vector to $\Sigma_T$. Explicitly, $n_{\Sigma_T} = e^*_\Sigma$ in $\Sigma_{trap}$, $n_{\Sigma_T} = e^*_T$ in $\Sigma_{red}$, $n_{\Sigma_T} = e^*_4$ in $\Sigma_{far}$, and along $\mathcal{I}^+$, and $n_{\Sigma_T} = e^*_3$ along $\mathcal{I}^+$. In particular, $g(T, n_{\Sigma_T}) = -\mathcal{I}$. Our goal is to show that $\int_{\Sigma_T} P^{(T,0,0)}_{\mu} [\Phi_1, \Phi_2] \cdot n_{\Sigma_T}$ is positive definite, and comparable with $T_{\mu\nu} [\Phi_1] T^\mu n_{\Sigma_T} + 8 T_{\mu\nu} [\Phi_2] T^\mu n_{\Sigma_T}$, and therefore with $E [\Phi_1, \Phi_2](\tau)$. Observe that $V_1 = \frac{1}{r^2} \gamma + \frac{5Q^2}{r^4} \geq 0$, and $V_2 = \frac{4}{r^2} \gamma + \frac{2Q^2}{r^4} \geq 0$ are positive in the whole exterior region, therefore $T_{\mu\nu} [\Phi_1] T^\mu n_{\Sigma_T} + 8 T_{\mu\nu} [\Phi_2] T^\mu n_{\Sigma_T}$ is coercive.

We compute

$$P^{(T,0,0)}_{\mu} [\Phi_1, \Phi_2] \cdot n_{\Sigma_T} = T_{\mu\nu} [\Phi_1] T^\mu n^\nu_{\Sigma_T} + 8 T_{\mu\nu} [\Phi_2] T^\mu n^\nu_{\Sigma_T} - \frac{8Q}{r^2} (\nabla \Phi_1 \cdot \Phi_2) g(T, n_{\Sigma_T}).$$

In $\Sigma_{red}$, where $T = \frac{1}{2} (\gamma e^*_3 + e^*_4)$, the above boundary term becomes

$$P^{(T,0,0)}_{\mu} [\Phi_1, \Phi_2] \cdot n_{\Sigma_T} = \frac{1}{2} \gamma \mathcal{T} (e^*_3, e^*_3)[\Phi_1] + \frac{1}{2} \gamma \mathcal{T} (e^*_3, e^*_4)[\Phi_1] + 4 \mathcal{T} (e^*_3, e^*_3)[\Phi_2] + 4 \mathcal{T} (e^*_3, e^*_4)[\Phi_2] + \frac{8Q}{r^2} (\nabla \Phi_1 \cdot \Phi_2)$$

$$= \frac{\gamma}{2} [\nabla^2 \Phi_1]^2 + \frac{1}{2} [\nabla \Phi_1]^2 + \frac{1}{2} V_1 |\Phi_1|^2$$

$$+ 4 \gamma [\nabla^2 \Phi_2]^2 + 4 \gamma [\nabla \Phi_2]^2 + 4 V_2 |\Phi_2|^2$$

$$+ \frac{8Q}{r^2} \nabla \Phi_1 \cdot \Phi_2.$$

In $\Sigma_{trap}$, where $T = \frac{1}{2} (e^*_3 + \gamma e^*_4)$, the boundary term becomes

$$P^{(T,0,0)}_{\mu} [\Phi_1, \Phi_2] \cdot n_{\Sigma_T} = \frac{1}{4} \gamma \mathcal{T} \mathcal{T} [\Phi_1] + \frac{1}{2} \gamma \mathcal{T} \mathcal{T} [\Phi_1] + \frac{1}{4} \gamma^2 \mathcal{T} \mathcal{T} [\Phi_1] + \frac{2}{\gamma} \mathcal{T} \mathcal{T} [\Phi_1] + 4 \mathcal{T} \mathcal{T} [\Phi_2] + 8 Q$$

$$= \frac{1}{4} [\nabla^2 \Phi_1]^2 + \frac{1}{4} \gamma [\nabla \Phi_1]^2 + \frac{1}{2} V_1 |\Phi_1|^2$$

$$+ 2 \gamma [\nabla^2 \Phi_2]^2 + 4 \gamma [\nabla \Phi_2]^2 + 4 V_2 |\Phi_2|^2 + 8 Q \nabla \Phi_1 \cdot \Phi_2.$$

Similarly, in $\Sigma_{far}$ we have

$$P^{(T,0,0)}_{\mu} [\Phi_1, \Phi_2] \cdot n_{\Sigma_T}$$

$$= \frac{1}{2} \gamma \mathcal{T} \mathcal{T} [\Phi_1] + \frac{1}{2} \gamma \mathcal{T} \mathcal{T} [\Phi_1] + 4 \gamma \mathcal{T} \mathcal{T} [\Phi_2] + 4 \mathcal{T} \mathcal{T} [\Phi_2] + \frac{8Q}{r^2} (\nabla \Phi_1 \cdot \Phi_2)$$

$$= \frac{1}{2} \gamma [\nabla^2 \Phi_1]^2 + \frac{1}{2} [\nabla \Phi_1]^2 + \frac{1}{2} V_1 |\Phi_1|^2 + 4 \gamma [\nabla^2 \Phi_2]^2 + 4 [\nabla \Phi_2]^2 + 4 V_2 |\Phi_2|^2$$

$$+ \frac{8Q}{r^2} \nabla \Phi_1 \cdot \Phi_2.$$

The boundary terms at the event horizon and at future null infinity can be similarly analyzed. In particular, in each portion of $\Sigma_T$ we can estimate the boundary terms by

$$P^{(T,0,0)}_{\mu} [\Phi_1, \Phi_2] \cdot n_{\Sigma_T} \geq \frac{1}{2} [\nabla \Phi_1]^2 + \frac{1}{2} V_1 |\Phi_1|^2 + 4 [\nabla \Phi_2]^2.$$
+ 4 V_2 |\Phi_2|^2 + \frac{8 Q}{r^2} \Phi_2 \cdot \Phi_2
\geq \frac{1}{2} |\nabla \Phi_1|^2 + \frac{5 Q^2}{2 r^4} |\Phi_1|^2 + 4 |\nabla \Phi_2|^2
+ \frac{8 Q^2}{r^4} |\Phi_2|^2 + \frac{8 Q}{r^2} \Phi_2 \cdot \Phi_2
\]since \( V_1 = \frac{1}{r^2} \lambda + \frac{5 Q^2}{r^4} \geq \frac{5 Q^2}{r^4} \), and \( V_2 = \frac{4}{r^2} \lambda + \frac{2 Q^2}{r^4} \geq \frac{2 Q^2}{r^4} \).

We now show that the above right hand side defines a positive definite quadratic form, therefore implying that \( P(N, \Phi_1, \Phi_2) \cdot n_\mathcal{V} \) is positive definite.

Suppose that \( \Phi_1 \) and \( \Phi_2 \) are supported on the fixed \( \ell \geq 2 \) spherical harmonic. Using (24), (25) and (28) we bound
\[
\frac{1}{2} |\nabla \Phi_1|^2 + \frac{5 Q^2}{2 r^4} |\Phi_1|^2 + 4 |\nabla \Phi_2|^2 + \frac{8 Q^2}{r^4} |\Phi_2|^2 + \frac{8 Q}{r^2} \Phi_2 \cdot \Phi_2
\geq \frac{1}{2 r^2} \left( \lambda - 1 + \frac{5 Q^2}{r^2} \right) |\Phi_1|^2 + \frac{4}{r^2} \left( \lambda - 4 + \frac{2 Q^2}{r^2} \right) |\Phi_2|^2 - \frac{4 Q^2}{r^2} (2 \lambda - 4)^{1/2} |\Phi_1||\Phi_2|
\]where we denoted \( \lambda := \ell (\ell + 1) \geq 6 \).

The above is a quadratic form of the type \( a |\Phi_1|^2 + b |\Phi_2|^2 - c |\Phi_1||\Phi_2| \). Its discriminant is given by \( D = c^2 - 4 ab \), and if the discriminant is negative, then the quadratic form is positive definite. We compute the discriminant of the above quadratic form:
\[
-D = \frac{8}{r^4} \left[ \left( \lambda - 1 + \frac{5 Q^2}{r^2} \right) \left( \lambda - 4 + \frac{2 Q^2}{r^2} \right) - \frac{4 Q^2}{r^2} (\lambda - 2) \right]
\geq \frac{8}{r^4} \left[ (\lambda - 1) (\lambda - 4) + \frac{2 Q^2}{r^2} (\lambda - 1) + \frac{5 Q^2}{r^2} (\lambda - 4) - \frac{4 Q^2}{r^2} (\lambda - 2) \right]
= \frac{8}{r^4} \left[ (\lambda - 1) (\lambda - 4) + \frac{Q^2}{r^2} (3 \lambda - 14) \right].
\]Since \( \lambda \geq 6 \), we have that \( 3 \lambda - 14 > 0 \), therefore implying positivity of the above quadratic form.

To obtain the estimates for the non-degenerate energy, we make use of the celebrated redshift vectorfield, as introduced in [15]. Notice that the non-degeneracy along the horizon given by the redshift vectorfield fails exactly at the extremal case \( |Q| = M \). We have for \( X = a(r) e_3 + b(r) e_4 \) (see Lemma 5 in [20])
\[
T \cdot (X) \pi = \left( \nabla a' - b' + \left( \frac{2 M}{r^2} - \frac{2 Q^2}{r^3} \right) a \right) |\nabla \Psi|^2
+ \left( \nabla b' + \left( - \frac{2 M}{r^2} + \frac{2 Q^2}{r^3} \right) b \right) |\nabla_4 \Psi|^2 - a' |\nabla_3 \Psi|^2
+ \left( - \frac{2 \nabla r}{r} a + \frac{2 b}{r} \right) \nabla_3 \Psi \cdot \nabla_4 \Psi
+ \left( \nabla a' - b' + \left( \frac{2 M}{r^2} - \frac{2 Q^2}{r^3} + \frac{2 \nabla r}{r} \right) a - \frac{2 b}{r} \right) V_i |\Psi|^2.
\]
Consider a vector field \( N \) defined as \( N = a(r) \nabla_3 + b(r) \nabla_4 \), and such that the functions \( a(r) \) and \( b(r) \) verify
\[
a(r \mathcal{H}) = 0, \quad b(r \mathcal{H}) = -1.
\]
Clearly one can choose the functions \(a\) and \(b\) such that \(\mathcal{E}^{(N,0,0)}[\Phi_1, \Phi_2]\) is positive definite. Notice that the coefficient of \(\nabla_4\) along the horizon reduces to \(\frac{2M}{r_H^2} - \frac{2Q^2}{r_H^2}\), which degenerates to zero at the horizon of an extremal Reissner–Nordström, for which \(r_H = M = Q\). Therefore the above bulk fails to be positive definite in the extremal case.

We have therefore obtained the following.

**Proposition 2** (Boundedness of the energy). Let \(\Phi_1\) and \(\Phi_2\) be a 1-tensor and a symmetric traceless 2-tensor respectively, satisfying the system of coupled wave Eqs. (36) and (37) in Reissner–Nordström spacetime with \(|Q| < M\) and supported in \(\ell \geq 2\) spherical harmonics. Then we have

\[
E[\Phi_1, \Phi_2](\tau_2) \leq CE[\Phi_1, \Phi_2](\tau_1)
\]

for every \(\tau_1 \leq \tau_2\).

7. Morawetz Estimates

In this section we prove Morawetz estimates for the mixed spin \(\pm 1\) and spin \(\pm 2\) system of equations.

To derive the Morawetz estimates, we apply Lemma 1 to the radial vector field \(Y = f(r)R\), for \(R = \nabla_r\), and a function \(f(r)\) to be determined.

**Proposition 3.** Let \(Y = f(r)R\) and \(w = r^{-2} \nabla_r \left( r^2 f(r) \right)\). Let \(\Phi_1\) and \(\Phi_2\) be a 1-tensor and a symmetric traceless 2-tensor respectively, satisfying the system of coupled wave Eqs. (36) and (37). Then we have

\[
\mathcal{E}^{(Y,w,0)}[\Phi_1, \Phi_2] = \frac{f}{r} \left( 1 - \frac{3M}{r} + \frac{2Q^2}{r^2} \right) |\nabla \Phi_1|^2 + f' |R \Phi_1|^2
\]

\[
+ \left( -\frac{1}{2} \partial_r (\nabla V_1) f - \frac{1}{4} \nabla g w \right) |\Phi_1|^2
\]

\[
+ \frac{8f}{r} \left( 1 - \frac{3M}{r} + \frac{2Q^2}{r^2} \right) |\nabla \Phi_2|^2 + 8f' |R \Phi_2|^2
\]

\[
+ 8 \left( -\frac{1}{2} \partial_r (\nabla V_2) f - \frac{1}{4} \nabla g w \right) |\Phi_2|^2
\]

\[
+ \frac{16Q}{r^3} f \left( 1 - \frac{3M}{r} + \frac{2Q^2}{r^2} + \frac{1}{2} \nabla g \right) \Phi_1^\pi \Phi_2.
\]

**Proof.** By (48), we have

\[
\mathcal{E}^{(Y,w,0)}[\Phi_1, \Phi_2] = \frac{1}{2} T[\Phi_1] \cdot (Y, \pi) + \left( -\frac{1}{2} Y(V_1) - \frac{1}{4} \nabla g w_1 \right) |\Phi_1|^2 + \frac{1}{2} w_1 \mathcal{L}_1[\Phi_1]
\]

\[
+ 4 T[\Phi_2] \cdot (Y, \pi) + (-4 Y(V_2) - 2 \nabla g w_2) |\Phi_2|^2 + 4 w_2 \mathcal{L}_2[\Phi_2]
\]

\[
+ \frac{4Q}{r^2} \left( w_1 + w_2 + \frac{4}{r} Y(r) - tr (Y, \pi) \right) \Phi_1^\pi \Phi_2 - \frac{8Q}{r^2} ([Y, \Phi_1] \cdot \Phi_2).
\]
We have for \( Y = f(r)R \) (see Corollary 4 in [20])

\[
T[\Phi_1] \cdot \pi = 2f \left( \frac{1}{r} - \frac{3M}{r^2} + \frac{2Q^2}{r^3} \right) |\nabla \Phi_1|^2 + 2f' |R \Phi_1|^2 + \left( -\frac{2Y}{r} f - \gamma f' \right) \mathcal{L}_1[\Phi_1] \\
+ \left( -\frac{2M}{r^2} + \frac{2Q^2}{r^3} \right) f V_1 |\Phi_1|^2
\]

and similarly for \( \Phi_2 \). We therefore obtain

\[
\mathcal{E}^{(Y, w)}[\Phi_1, \Phi_2] = f \left( \frac{1}{r} - \frac{3M}{r^2} + \frac{2Q^2}{r^3} \right) |\nabla \Phi_1|^2 + 2f' |R \Phi_1|^2 \\
+ \left( \frac{1}{2} w_1 - \frac{Y}{r} f - \frac{\gamma}{2} f' \right) \mathcal{L}_1[\Phi_1] \\
+ \left( -\frac{1}{2} Y(V_1) + \left( -\frac{M}{r^2} + \frac{Q^2}{r^3} \right) f V_1 - \frac{1}{4} \Box g w_1 \right) |\Phi_1|^2 \\
+ 8f \left( \frac{1}{r} - \frac{3M}{r^2} + \frac{2Q^2}{r^3} \right) |\nabla \Phi_2|^2 + 8f' |R \Phi_2|^2 \\
+ 8 \left( \frac{1}{2} w_2 - \frac{Y}{r} f - \frac{\gamma}{2} f' \right) \mathcal{L}_2[\Phi_2] \\
+ 8 \left( -\frac{1}{2} Y(V_2) + \left( -\frac{M}{r^2} + \frac{Q^2}{r^3} \right) f V_2 - \frac{1}{4} \Box g w_2 \right) |\Phi_2|^2 \\
+ \frac{4Q}{r^2} \left( w_1 + w_2 + 4r Y(r) - \text{tr}^{(Y)} \pi \right) \nabla^2 \Phi_1 \cdot \Phi_2 \\
- \frac{8Q}{r^2} \left( \left[ Y, \nabla \Phi_2 \right] + \left[ Y, \nabla^2 \Phi_1 \right] \right) \cdot \Phi_2.
\]

With the choice \( w := w_1 = w_2 = r^{-2} Y \partial_r (r^2 f(r)) = \frac{2Y}{r} f + Y f' \) the terms involving \( \mathcal{L}_1[\Phi_1] \) and \( \mathcal{L}_2[\Phi_2] \) above cancel out. Using that (see Corollary 3 in [20])

\[
\text{tr}^{(Y)} \pi = \frac{4Y}{r} f + \left( \frac{4M}{r^2} - \frac{4Q^2}{r^3} \right) f + 2f' \gamma = \frac{4}{r} \left( 1 - \frac{M}{r} \right) f + 2f' \gamma
\]

we compute, recalling that \( R(r) = \gamma \),

\[
\left[ Y, \nabla \Phi_1 \right] = f \left[ R, \nabla \Phi_2 \right] = -\frac{\gamma}{r} f \nabla \Phi_2 \\
w_1 + w_2 + 4r Y(r) - \text{tr}^{(Y)} \pi = 2 \left( \frac{2Y}{r} f + \gamma f' \right) + 4r Y f \\
- \left( \frac{4Y}{r} f + \left( \frac{4M}{r^2} - \frac{4Q^2}{r^3} \right) f + 2f' \gamma \right) \\
= \frac{4Y}{r} f - \left( \frac{4M}{r^2} - \frac{4Q^2}{r^3} \right) f = \frac{4}{r} \left( 1 - \frac{3M}{r} + \frac{2Q^2}{r^2} \right) f.
\]
We conclude

\[ E^{(Y, w, 0)}[\Phi_1, \Phi_2] \]

\[ = f \left( \frac{1}{r} - \frac{3M}{r^2} + \frac{2Q^2}{r^3} \right) |\nabla \Phi_1|^2 + f' |R \Phi_1|^2 \]

\[ + 8 f \left( \frac{1}{r} - \frac{3M}{r^2} + \frac{2Q^2}{r^3} \right) |\nabla \Phi_2|^2 + 8 f' |R \Phi_2|^2 \]

\[ + \left( \left( - \frac{\gamma}{2} V_1' + \left( - \frac{M}{r^2} + \frac{Q^2}{r^3} \right) V_1 \right) f - \frac{1}{4} \nabla_g w \right) |\Phi_1|^2 \]

\[ + 8 \left( \left( - \frac{\gamma}{2} V_2' + \left( - \frac{M}{r^2} + \frac{Q^2}{r^3} \right) V_2 \right) f - \frac{1}{4} \nabla_g w \right) |\Phi_2|^2 \]

\[ + \frac{16Q}{r^3} \left( 1 - \frac{3M}{r} + \frac{2Q^2}{r^2} + \frac{1}{2r} \right) f \nabla^2 \Phi_1 \cdot \Phi_2. \]

Observing that \(- \frac{M}{r} + \frac{Q^2}{r^2} = - \frac{1}{2} \gamma'\), we prove the proposition. \(\square\)

For \(\Phi_1\) and \(\Phi_2\) supported in \(\ell \geq 2\) spherical harmonics, we use (24) and (25) to write

\[ E^{(Y, w, 0)}[\Phi_1, \Phi_2] \]

\[ = f' |R \Phi_1|^2 + \left( f \left( 1 - \frac{3M}{r} + \frac{2Q^2}{r^2} \right) (\lambda - 1) - \frac{1}{2} \partial_r (\gamma V_1) f - \frac{1}{4} \nabla_g w \right) |\Phi_1|^2 \]

\[ + 8 f' |R \Phi_2|^2 + 8 \left( f \left( 1 - \frac{3M}{r} + \frac{2Q^2}{r^2} \right) (\lambda - 4) - \frac{1}{2} \partial_r (\gamma V_2) f - \frac{1}{4} \nabla_g w \right) |\Phi_2|^2 \]

\[ + \frac{16Q}{r^3} f \left( 1 - \frac{3M}{r} + \frac{2Q^2}{r^2} + \frac{1}{2r} \right) \nabla^2 \Phi_1 \cdot \Phi_2 \]

for \(\lambda = \ell(\ell + 1) \geq 6\). Denote

\[ A_1 := f \left( 1 - \frac{3M}{r} + \frac{2Q^2}{r^2} \right) (\lambda - 1) - \frac{1}{2} \partial_r (\gamma V_1) f - \frac{1}{4} \nabla_g w \] (50)

\[ A_2 := f \left( 1 - \frac{3M}{r} + \frac{2Q^2}{r^2} \right) (\lambda - 4) - \frac{1}{2} \partial_r (\gamma V_2) f - \frac{1}{4} \nabla_g w \] (51)

the coefficients of \(|\Phi_1|^2\) and \(|\Phi_2|^2\) respectively.

In order to have positivity of \(E^{(Y, w, 0)}[\Phi_1, \Phi_2]\) we need the following necessary conditions in the exterior region:

- **Condition 1**: Positivity of the coefficients of the angular derivatives \(|\nabla \Phi_1|^2\) and \(|\nabla \Phi_2|^2\), i.e. \( f \left( 1 - \frac{3M}{r} + \frac{2Q^2}{r^2} \right) \geq 0\),

- **Condition 2**: Positivity of the coefficients of the radial derivatives \(|R \Phi_1|^2\) and \(|R \Phi_2|^2\), i.e. \( f' > 0\),

- **Condition 3**: Positivity of the coefficients of \(|\Phi_1|^2\) and \(|\Phi_2|^2\), i.e. \( A_1 > 0 \) and \( A_2 > 0 \)
We now consider sufficient conditions for the positivity of the quadratic form $E^{(Y, w, 0)}[\Phi_1, \Phi_2]$. Using (28) to bound the mixed term, we have

$$E^{(Y, w, 0)}[\Phi_1, \Phi_2] \geq s f'|R\Phi_1|^2 + 8 f'|R\Phi_2|^2 + A_1|\Phi_1|^2 + 8 A_2|\Phi_2|^2 - 8 Q r f \frac{1}{r^4} \left| 1 - \frac{3M}{r} + \frac{2Q^2}{r^2} + \frac{1}{2} \gamma \right| (2\lambda - 4)^{1/2} |\Phi_1||\Phi_2|.$$ 

Neglecting the terms in $R$ derivative in virtue of Condition 2, the discriminant of the quadratic terms is

$$-\frac{D_{32}}{32} = A_1 A_2 - \frac{2Q^2}{r^8} f^2 \left( 1 - \frac{3M}{r} + \frac{2Q^2}{r^2} + \frac{1}{2} \gamma \right)^2 (2\lambda - 4).$$

Writing $\lambda = (\lambda - 6) + 6$, we have

$$-\frac{D_{32}}{32} = \left( \frac{f}{r^3} \left( 1 - \frac{3M}{r} + \frac{2Q^2}{r^2} \right) (\lambda - 6) + B_1 \right)$$

$$+ \left( \frac{f}{r^3} \left( 1 - \frac{3M}{r} + \frac{2Q^2}{r^2} \right) (\lambda - 6) + B_2 \right)$$

$$- \frac{4Q^2}{r^8} f^2 \left( 1 - \frac{3M}{r} + \frac{2Q^2}{r^2} + \frac{1}{2} \gamma \right)^2 (\lambda - 6)$$

$$- \frac{16Q^2}{r^8} f^2 \left( 1 - \frac{3M}{r} + \frac{2Q^2}{r^2} + \frac{1}{2} \gamma \right)^2$$

where

$$B_1 := \frac{5f}{r^3} \left( 1 - \frac{3M}{r} + \frac{2Q^2}{r^2} \right) - \frac{1}{2} \partial_r (\gamma V_1) f - \frac{1}{4} \Box_g w \quad (52)$$

$$B_2 := \frac{2f}{r^3} \left( 1 - \frac{3M}{r} + \frac{2Q^2}{r^2} \right) - \frac{1}{2} \partial_r (\gamma V_2) f - \frac{1}{4} \Box_g w. \quad (53)$$

In particular,

$$-\frac{D_{32}}{32} = \frac{f^2}{r^6} \left( 1 - \frac{3M}{r} + \frac{2Q^2}{r^2} \right)^2 (\lambda - 6)^2$$

$$+ B_1 B_2 - \frac{16Q^2}{r^8} f^2 \left( 1 - \frac{3M}{r} + \frac{2Q^2}{r^2} + \frac{1}{2} \gamma \right)^2$$

$$+ (\lambda - 6) \left[ \frac{f}{r^3} \left( 1 - \frac{3M}{r} + \frac{2Q^2}{r^2} \right) (B_1 + B_2) \right.$$

$$\left. - \frac{4Q^2}{r^8} f^2 \left( 1 - \frac{3M}{r} + \frac{2Q^2}{r^2} + \frac{1}{2} \gamma \right)^2 \right].$$

In order to have positivity of the discriminant, we have the following sufficient conditions in the exterior region:

\[ \text{Conditions 4 and 5 are not necessary. For example, one could use the positivity of the } R \text{ derivative to absorb part of the mixed term for high spherical harmonics. Nevertheless, we prefer to have a unique approach to all frequencies.} \]
– **Condition 4**: Positivity of the third line in the above expression of the discriminant, i.e. $D_1 := f^2 \left( 1 - \frac{3M}{r} + \frac{2Q^2}{r^2} \right) (B_1 + B_2) - \frac{4Q^2}{r^2} f^2 \left( 1 - \frac{3M}{r} + \frac{2Q^2}{r^2} + \frac{1}{2} \mathcal{Y} \right)^2 \geq 0$.

– **Condition 5**: Positivity of the second line in the above expression of the discriminant, i.e. $D_2 := B_1 B_2 - \frac{16Q^2}{r^8} f^2 \left( 1 - \frac{3M}{r} + \frac{2Q^2}{r^2} + \frac{1}{2} \mathcal{Y} \right)^2 \geq 0$.

If Conditions 1, 2, 3, 4 and 5 are satisfied, then the bulk integral $\int_{\mathcal{M}(\tau_1, \tau_2)} E^{(Y, w; 0)}[\Phi_1, \Phi_2] \geq \text{Mor}[\Phi_1, \Phi_2](\tau_1, \tau_2)$. (54)

Our goal is to define functions $f$ and $w$, related by $w = r^{-2} \mathcal{Y} \partial_r \left( r^2 f(r) \right)$, which verify Conditions 1, 2, 3, 4 and 5 in the whole exterior region of the spacetime. This is done in the following subsection.

Once those conditions are proved to hold, by adding the Morawetz estimates to the energy estimates obtained in Sect. 6, i.e. considering the triplet $(X, w, M) := (Y, w, 0) + \Lambda(T, 0, 0)$ for $\Lambda$ big enough, we obtain

$$E[\Phi_1, \Phi_2](\tau) + \text{Mor}[\Phi_1, \Phi_2](0, \tau) \lesssim E[\Phi_1, \Phi_2](0) \quad \text{for any } \tau \geq 0.$$ (55)

In the remaining of this Section we will prove that Conditions 1, 2, 3, 4 and 5 are satisfied in the whole exterior region for subextremal Reissner–Nordström spacetimes.

### 7.1. Construction of the functions $w$ and $f$

Consider Reissner–Nordström spacetimes with $|Q| < M$. We collect here the following facts:

– The event horizon $r_H = M + \sqrt{M^2 - Q^2}$ ranges between $M < r_H \leq 2M$. Observe that since we are restricting our analysis in the exterior region, where $r \geq r_H$, we always have $\mathcal{Y} = 1 - \frac{2M}{r} + \frac{Q^2}{r^2} \geq 0$.

– The photon sphere $r_p = \frac{3M + \sqrt{9M^2 - 8Q^2}}{2}$ ranges between $2M < r_p \leq 3M$. Observe that $r \geq r_p$ if and only if $1 - \frac{3M}{r} + \frac{2Q^2}{r^2} \geq 0$.

We also define the following notable points which are used in the construction of the Morawetz functions.

– $r_1 := \frac{5M + \sqrt{25M^2 - 24Q^2}}{4} < r_p$, which ranges between $\frac{3M}{2} < r_1 \leq \frac{5M}{2}$. Observe that $r \geq r_1$ if and only if $1 - \frac{5M}{2r} + \frac{3Q^2}{2r^2} \geq 0$. 


- \( r_2 := 2M + \sqrt{4M^2 - 3Q^2} > r_P \), which ranges between \( 3M < r_2 \leq 4M \). Observe that \( r \geq r_2 \) if and only if \( 1 - \frac{4M}{r} + \frac{3Q^2}{r^2} \geq 0 \).

Inspired\(^{10}\) by [44] and [33], we define \( w \) as follows:

\[
\begin{cases}
\frac{2}{r^2}\gamma(r_1) := w_1 > 0 & \text{if } r < r_1 \\
\frac{2}{r^2}\gamma(r) & \text{if } r_1 \leq r \leq r_2 \\
\frac{2}{r^2}\gamma(r_2) := w_2 > 0 & \text{if } r > r_2.
\end{cases}
\] (56)

From the condition \( w = r^{-2}\gamma \partial_r \left( r^2 f \right) \) we define as in [33,44]:

\[
r^2 f = \int_{r_P}^{r} \frac{r^2}{\gamma} w.
\] (57)

Since \( w > 0 \), by (57) we notice that \( f \) changes sign at the photon sphere. Condition 1 is then always satisfied.

In what follows, we will show that Conditions 2, 3, 4 and 5 are satisfied in the whole exterior region. We separate the analysis in the three regions of the spacetime according to the definition of \( w \).

7.2. The regions \( r \leq r_1 \) and \( r \geq r_2 \). In the regions \( r \leq r_1 \) and \( r \geq r_2 \), the function \( w \) is a positive constant. In particular we have

\[
\begin{align*}
w &= w_1, & f &= \frac{w_1}{r^2} \int_{r_P}^{r} \frac{r^2}{\gamma} \quad \text{for } r \leq r_1 \\
w &= w_2, & f &= \frac{w_2}{r^2} \int_{r_P}^{r} \frac{r^2}{\gamma} \quad \text{for } r \geq r_2.
\end{align*}
\]

We start by proving that Condition 2 is satisfied in these two regions.

Lemma 2. Condition 2 is verified for \( r \leq r_1 \) and for \( r \geq r_2 \), i.e. for all subextremal Reissner–Nordström spacetimes with \( |Q| < M \) we have

\[f'(r) > 0 \quad \text{for } r \leq r_1 \text{ or } r \geq r_2.\]

Proof. Since \( f \) vanishes at \( r = r_P \) we have,

\[
f' = r^{-2}(r^2 f)' - 2r^{-3}(r^2 f) = r^{-2}(r^2 f)' - 2r^{-3} \int_{r_P}^{r} (r^2 f)'.
\]

Recall that \( w = r^{-2}\gamma (r^2 f)' = w_1 > 0 \) for \( r \leq r_1 \), and \( w = r^{-2}\gamma (r^2 f)' = w_2 > 0 \) for \( r \geq r_2 \). We deduce

\[
(r^2 f)' = \frac{r^2}{\gamma} w_1 \quad \text{for } r \leq r_1, \quad (r^2 f)' = \frac{r^2}{\gamma} w_2 \quad \text{for } r \geq r_2.
\]

\(^{10}\) Our definition of \( w \) (56) differs from [33] in that there \( w \) is defined separately in two intervals, as opposed to three, and in one of them \( w = \frac{2}{r^2}\gamma \), as opposed to \( w = \frac{2}{r^2}\gamma \). We modified it in order to obtain positivity of the bulk in the exterior region for the full subextremal range \( |Q| < M \). The definition of \( f \) in terms of \( w \) is identical to [33,44].
Consider the region \( r \leq r_1 \). We write

\[
f' = \frac{1}{\Upsilon} w_1 + 2 w_1 r^{-3} \int_r^{r_P} \frac{r^2}{\Upsilon} \, \mathrm{d}r = w_1 \left( \frac{1}{\Upsilon} + 2 r^{-3} \int_r^{r_P} \frac{r^2}{\Upsilon} \right).
\]

Observe that the function \( \frac{r^2}{\Upsilon} \) is increasing if \( r > r_P \) and decreasing otherwise. Indeed,

\[
\partial_r \left( \frac{r^2}{\Upsilon} \right) = \frac{1}{\Upsilon^2} \left( 2 r \Upsilon - r^2 \Upsilon' \right) = \frac{2 r}{\Upsilon^2} \left( 1 - \frac{3 M}{r} + \frac{2 Q^2}{r^2} \right). \tag{58}
\]

The integrand \( \frac{r^2}{\Upsilon} \) is therefore decreasing for \( r \leq r_1 < r_P \). We bound the integral of a decreasing function over an interval by the value of the function at the right end of the interval times the length of the interval. Similarly, \( \frac{1}{\Upsilon} \) is everywhere decreasing. We obtain

\[
f' > w_1 \left( \frac{1}{\Upsilon(r_P)} + 2 r_P^{-3} (r_P - r) \frac{r_P^2}{\Upsilon(r_P)} \right) = \frac{w_1}{r_P \Upsilon(r_P)} (3 r_P - 2 r)
\]

which is positive for \( r \leq r_1 < r_P < \frac{3}{2} r_P \).

Consider now the region \( r \geq r_2 \). We have

\[
f' = w_2 \left( \frac{1}{\Upsilon} - 2 r^{-3} \int_r^{r_P} \frac{r^2}{\Upsilon} \right).
\]

The integrand \( \frac{r^2}{\Upsilon} \) is increasing for \( r \geq r_2 \). We bound the integral of an increasing function from above by the value of the function at the right end of the interval times the length of the interval, i.e. \( \int_r^{r_P} \frac{r^2}{\Upsilon} \leq \frac{r^2}{\Upsilon} (r - r_P) \). This gives

\[
f' \geq w_2 \left( \frac{1}{\Upsilon} - 2 \frac{1}{r \Upsilon} (r - r_P) \right) = \frac{w_2}{r \Upsilon} (-r + 2 r_P)
\]

which is positive for \( r < 2 r_P \), which contains \( r = r_2 \). In particular, \( (r^3 f')|_{r=r_2} > 0 \).

Using (57) written as \( \partial_r (r^2 f) = \frac{r^2}{\Upsilon} w \), we have

\[
r^2 \partial_r \left( \frac{r w}{\Upsilon} \right) = r^2 \left[ r^{-1} \partial_r \left( \frac{r^2}{\Upsilon} w \right) - r^{-2} \frac{r^2}{\Upsilon} w \right] = r \partial_r \left( \frac{r^2}{\Upsilon} w \right) - \frac{r^2}{\Upsilon} w = r \partial_r (r^2 f) - \partial_r (r^2 f) = 3 r^2 f' + 3 r^3 f'' = \partial_r (r^3 f').
\]

Therefore

\[
\partial_r (r^3 f') = r^2 \partial_r \left( \frac{r w}{\Upsilon} \right) = w_2 r^2 \partial_r \left( \frac{r}{\Upsilon} \right).
\]

Since

\[
\partial_r \left( \frac{r}{\Upsilon} \right) = \frac{1}{\Upsilon^2} \left( \Upsilon - r \Upsilon' \right) = \frac{1}{\Upsilon^2} \left( 1 - \frac{4 M}{r} + \frac{3 Q^2}{r^2} \right)
\]

we have

\[
\partial_r (r^3 f') \geq 0 \quad \text{for } r \geq r_2.
\]

This implies \( r^3 f' \geq (r^3 f')|_{r=r_2} > 0 \). This concludes the proof of the lemma. \( \square \)
We now prove that Condition 3 is verified, i.e. the coefficients of the zeroth order terms are positive.

**Lemma 3.** Condition 3 is verified for \( r \leq r_1 \) and for \( r \geq r_2 \), i.e. for all subextremal Reissner–Nordström spacetimes with \( |Q| < M \) we have

\[
A_1 \geq 0, \quad A_2 \geq 0 \quad \text{for } r \leq r_1 \text{ or } r \geq r_2.
\]

**Proof.** Because of Condition 1, the first term in the definition of \( A_1 \) and \( A_2 \) in (50) and (51) is always positive. For \( r \leq r_1 \) or \( r \geq r_2 \), \( w \) is constant, therefore \( \Box_g w = 0 \). This gives

\[
A_1 \geq -\frac{1}{2} \partial_r (\Upsilon V_1) f, \quad A_2 \geq -\frac{1}{2} \partial_r (\Upsilon V_2) f.
\]

Since \( f \leq 0 \) for \( r \leq rp \) and \( f \geq 0 \) for \( r \geq rp \), we are only left to prove that for \( r \leq r_1 \) then \(-\frac{1}{2} \partial_r (\Upsilon V_1), -\frac{1}{2} \partial_r (\Upsilon V_2) \leq 0 \) and for \( r \geq r_2 \) then \(-\frac{1}{2} \partial_r (\Upsilon V_1), -\frac{1}{2} \partial_r (\Upsilon V_2) \geq 0 \).

Recall that

\[
V_1 = \frac{1}{r^2} \left( 1 - \frac{2M}{r} + \frac{6Q^2}{r^4} \right) = \frac{1}{r^2} \Upsilon + \frac{5Q^2}{r^4}, \quad V_2 = \frac{4}{r^3} \left( 1 - \frac{2M}{r} + \frac{3Q^2}{2r^2} \right) = \frac{4}{r^2} \Upsilon + \frac{2Q^2}{r^4},
\]

(59)

\[
V'_1 = -\frac{2}{r^3} \left( 1 - \frac{3M}{r} + \frac{12Q^2}{r^2} \right), \quad V'_2 = -\frac{8}{r^3} \left( 1 - \frac{3M}{r} + \frac{3Q^2}{r^2} \right).
\]

Since \( \partial_r \Upsilon = \frac{2M}{r^2} - \frac{2Q^2}{r^3} \), we compute

\[
\partial_r (\Upsilon V_1) = \left( \frac{2M}{r^2} - \frac{2Q^2}{r^3} \right) \left( \frac{1}{r^2} \Upsilon + \frac{5Q^2}{r^4} \right) - \frac{2}{r^3} \left( 1 - \frac{3M}{r} + \frac{12Q^2}{r^2} \right)
\]

\[
= -\frac{2}{r^3} \left( 1 - \frac{4M}{r} + \frac{13Q^2}{r^2} \right) \Upsilon + \frac{10Q^2}{r^4} \left( \frac{M}{r^2} - \frac{Q^2}{r^3} \right)
\]

\[
\partial_r (\Upsilon V_2) = -\frac{8}{r^3} \left( 1 - \frac{4M}{r} + \frac{4Q^2}{r^2} \right) \Upsilon + \frac{4Q^2}{r^4} \left( \frac{M}{r^2} - \frac{Q^2}{r^3} \right).
\]

Consider the region \( r \leq r_1 \). Writing in the above expressions respectively

\[
1 - \frac{4M}{r} + \frac{13Q^2}{r^2} = \left( 1 - \frac{3M}{r} + \frac{2Q^2}{r^2} \right) - \left( \frac{M}{r^2} - \frac{Q^2}{r^3} \right) + \frac{10Q^2}{r^2}
\]

\[
1 - \frac{4M}{r} + \frac{4Q^2}{r^2} = \left( 1 - \frac{3M}{r} + \frac{2Q^2}{r^2} \right) - \left( \frac{M}{r^2} - \frac{Q^2}{r^3} \right) + \frac{Q^2}{r^2}
\]

we obtain

\[
-\frac{1}{2} \partial_r (\Upsilon V_1) = \frac{1}{r^3} \left( 1 - \frac{4M}{r} + \frac{13Q^2}{r^2} \right) \Upsilon - \frac{5Q^2}{r^4} \left( \frac{M}{r^2} - \frac{Q^2}{r^3} \right)
\]

\[
= \frac{1}{r^3} \left( 1 - \frac{3M}{r} + \frac{2Q^2}{r^2} \right) \Upsilon - \frac{1}{r^3} \left( \frac{M}{r^2} - \frac{Q^2}{r^3} \right) \Upsilon + \frac{5Q^2}{r^5} \left( 2\Upsilon - \frac{M}{r} + \frac{Q^2}{r^2} \right)
\]

(60)
and

\[ -\frac{1}{2} \partial_r (\Upsilon V_2) = \frac{4}{r^3} \left( 1 - \frac{4M}{r} + \frac{4Q^2}{r^2} \right) \Upsilon - \frac{2Q^2}{r^4} \left( \frac{M}{r^2} - \frac{Q^2}{r^3} \right) \]
\[ = \frac{4}{r^3} \left( 1 - \frac{3M}{r} + \frac{2Q^2}{r^2} \right) \Upsilon - \frac{4}{r^3} \left( \frac{M}{r} - \frac{Q^2}{r^2} \right) \Upsilon + \frac{2Q^2}{r^5} \left( 2\Upsilon - \frac{M}{r} + \frac{Q^2}{r^2} \right). \tag{61} \]

The first term in both expressions is negative since \( r \leq r_1 < r_p \). The second term can be written as \(-\frac{1}{r^3} (Mr - Q^2) \Upsilon\), and since \( r_H > M \), we have \( Mr - Q^2 > M^2 - Q^2 > 0 \). This proves the negativity of the second term in both expressions. The last term is given by \( 2\Upsilon - \frac{M}{r} + \frac{Q^2}{r^2} = 2 \left( 1 - \frac{5M}{2r^2} + \frac{3Q^2}{2r^3} \right) \), which is negative since \( r \leq r_1 \).

Consider the region \( r \geq r_2 \). Writing in the expressions above respectively

\[ 1 - \frac{4M}{r} + \frac{13Q^2}{r^2} = \left( 1 - \frac{4M}{r} + \frac{3Q^2}{r^2} \right) + \frac{10Q^2}{r^2}, \]
\[ 1 - \frac{4M}{r} + \frac{4Q^2}{r^2} = \left( 1 - \frac{4M}{r} + \frac{3Q^2}{r^2} \right) + \frac{Q^2}{r^2}. \]

we obtain

\[ -\frac{1}{2} \partial_r (\Upsilon V_1) = \frac{1}{r^3} \left( 1 - \frac{4M}{r} + \frac{13Q^2}{r^2} \right) \Upsilon - \frac{5Q^2}{r^4} \left( \frac{M}{r^2} - \frac{Q^2}{r^3} \right) \]
\[ = \frac{1}{r^3} \left( 1 - \frac{4M}{r} + \frac{3Q^2}{r^2} \right) \Upsilon + \frac{5Q^2}{r^5} \left( 2\Upsilon - \frac{M}{r} + \frac{Q^2}{r^2} \right). \tag{62} \]

and

\[ -\frac{1}{2} \partial_r (\Upsilon V_2) = \frac{4}{r^3} \left( 1 - \frac{4M}{r} + \frac{4Q^2}{r^2} \right) \Upsilon - \frac{2Q^2}{r^4} \left( \frac{M}{r^2} - \frac{Q^2}{r^3} \right) \]
\[ = \frac{4}{r^3} \left( 1 - \frac{4M}{r} + \frac{3Q^2}{r^2} \right) \Upsilon + \frac{2Q^2}{r^5} \left( 2\Upsilon - \frac{M}{r} + \frac{Q^2}{r^2} \right). \tag{63} \]

The first term in the both expression is positive for \( r \geq r_2 \) and the second term is positive for \( r \geq r_2 > r_1 \). This concludes the proof of the lemma. \( \blacksquare \)

We now prove that Condition 4 is verified, i.e. that \( D_1 \) is positive.

**Lemma 4.** Condition 4 is verified for \( r \leq r_1 \) and for \( r \geq r_2 \), i.e. for all subextremal Reissner–Nordström spacetimes with \( |Q| < M \) we have

\[ D_1 \geq 0 \quad \text{for } r \leq r_1 \text{ or } r \geq r_2. \]

**Proof.** Recall the definitions (52) and (53) of \( B_1 \) and \( B_2 \). Then

\[ B_1 + B_2 = f \left( \frac{7}{r^3} \left( 1 - \frac{3M}{r} + \frac{2Q^2}{r^2} \right) - \frac{1}{2} \partial_r (\Upsilon V_1) - \frac{1}{2} \partial_r (\Upsilon V_2) \right) \]
The first term is positive because it can be bounded from below by 7

so the above becomes

and writing

the above becomes

We first consider the region \( r \leq r_1 \). In this region we use (60) and (61) to write

Factoring out the positive factor \( \frac{r^2}{r^5} \), (64) becomes

The first term is positive because it can be bounded from below by \( 7 - \frac{4Q^2}{r^2} = \frac{1}{r^5}(7r^2 - 4Q^2) \), which is always positive for \( r \geq r_H > M \). All the other terms, except the last one, are clearly positive for \( r \leq r_1 \). Observe that for \( r \leq r_1 \), we have the following bounds:

therefore the expression for \( D_1 \) reads

\[
D_1 = \frac{f^3}{r^3} \left( 1 - \frac{3M}{r} + \frac{2Q^2}{r^2} \right) (B_1 + B_2) - \frac{4Q^2}{r^8} f^2 \left( 1 - \frac{3M}{r} + \frac{2Q^2}{r^2} + \frac{1}{2} \gamma \right)^2
\]

\[
= \frac{f^2}{r^3} \left( 1 - \frac{3M}{r} + \frac{2Q^2}{r^2} \right) \left( \frac{7}{r^3} \left( 1 - \frac{3M}{r} + \frac{2Q^2}{r^2} \right) - \frac{1}{2} \partial_r (\gamma V_1) - \frac{1}{2} \partial_r (\gamma V_2) \right)
\]

\[
- \frac{4Q^2}{r^8} f^2 \left( 1 - \frac{3M}{r} + \frac{2Q^2}{r^2} + \frac{1}{2} \gamma \right)^2
\]

\[
= \frac{f^2}{r^3} \left( 1 - \frac{3M}{r} + \frac{2Q^2}{r^2} \right) \left( \frac{7}{r^3} \left( 1 - \frac{3M}{r} + \frac{2Q^2}{r^2} \right) - \frac{1}{2} \partial_r (\gamma V_1) - \frac{1}{2} \partial_r (\gamma V_2) \right)
\]

\[
- \frac{4Q^2}{r^8} f^2 \left( 1 - \frac{3M}{r} + \frac{2Q^2}{r^2} \right) \gamma - \frac{Q^2}{r^8} f^2 \gamma^2.
\]

\[
D_1 = \frac{f^2}{r^6} \left( 7 - \frac{4Q^2}{r^2} \right) \left( 1 - \frac{3M}{r} + \frac{2Q^2}{r^2} \right)^2 + \frac{f^2}{r^3} \left( 1 - \frac{3M}{r} + \frac{2Q^2}{r^2} \right)
\]

\[
- \frac{1}{2} \partial_r (\gamma V_1) - \frac{1}{2} \partial_r (\gamma V_2)
\]

\[
- \frac{4Q^2}{r^8} f^2 \left( 1 - \frac{3M}{r} + \frac{2Q^2}{r^2} \right) \gamma - \frac{Q^2}{r^8} f^2 \gamma^2.
\]

\[
- \frac{1}{2} \partial_r (\gamma V_1) - \frac{1}{2} \partial_r (\gamma V_2) = \frac{5\gamma}{r^3} \left( 1 - \frac{3M}{r} + \frac{2Q^2}{r^2} \right)
\]

\[
- \frac{5}{r^3} \left( \frac{M}{r} - \frac{Q^2}{r^2} \right) \gamma + \frac{7Q^2}{r^3} \left( 2\gamma - \frac{M}{r} + \frac{Q^2}{r^2} \right).
\]
\[-1 + \frac{3M}{r} - \frac{2Q^2}{r^2} = \left(1 + \frac{5M}{2r} - \frac{3Q^2}{2r^2}\right) + \frac{M}{2r} - \frac{Q^2}{2r^2} \geq \frac{1}{2r^2} \left(Mr - Q^2\right).
\]

(66)

The last two terms together are positive: indeed, using (65) and (66), we obtain
\[
\frac{4Q^2}{r^8} r \left(1 + \frac{3M}{r} - \frac{2Q^2}{r^2}\right) - \frac{Q^2}{r^8} r^2 \\
\geq \frac{4Q^2}{r^8} r \left(\frac{1}{2r^2} \left(Mr - Q^2\right)\right) - \frac{Q^2}{r^8} r^2 \\
= \frac{Q^2}{r^8} r \left(\frac{2}{r^2} \left(Mr - Q^2\right) - \gamma\right) \geq \frac{Q^2}{r^8} r \left(\frac{3}{2r^2} \left(Mr - Q^2\right)\right)
\]

which proves Condition 4 for \( r \leq r_1 \).

We now consider the region \( r \geq r_2 \). In this region we use (62) and (62) to write
\[-\frac{1}{2} \partial_r (\gamma V_1) - \frac{1}{2} \partial_r (\gamma V_2) = 5 \left(1 - \frac{4M}{r} + \frac{3Q^2}{r^2}\right) \gamma + \frac{7Q^2}{r^2} \left(2\gamma - \frac{M}{r} + \frac{Q^2}{r^2}\right).
\]

Factorizing out the positive factor \( \frac{r^2}{r_5^2} \), (64) becomes
\[
\left(7 - \frac{4Q^2}{r^2}\right) \left(1 - \frac{3M}{r} + \frac{2Q^2}{r^2}\right)^2 + 5\gamma \left(1 - \frac{3M}{r} + \frac{2Q^2}{r^2}\right) \left(1 - \frac{4M}{r} + \frac{3Q^2}{r^2}\right) \\
+ \frac{7Q^2}{r^2} \left(1 - \frac{3M}{r} + \frac{2Q^2}{r^2}\right) \left(2\gamma - \frac{M}{r} + \frac{Q^2}{r^2}\right) - \frac{Q^2}{r^2} \gamma^2 - \frac{4Q^2}{r^2} \left(1 - \frac{3M}{r} + \frac{2Q^2}{r^2}\right) \gamma.
\]

The first line is clearly positive. The second line can be arranged to be
\[
\frac{Q^2}{r^2} \left(1 - \frac{3M}{r} + \frac{2Q^2}{r^2}\right) \left(10\gamma - \frac{7M}{r} + \frac{7Q^2}{r^2}\right) - \frac{Q^2}{r^2} \gamma^2.
\]

Writing \( 1 - \frac{3M}{r} + \frac{2Q^2}{r^2} = \gamma - \frac{M}{r} + \frac{Q^2}{r^2} \) and factorizing out \( \frac{Q^2}{r^2} \), we have
\[
\left(\gamma - \frac{M}{r} + \frac{Q^2}{r^2}\right) \left(10\gamma - \frac{7M}{r} + \frac{7Q^2}{r^2}\right) - \gamma^2 \\
= 9\gamma^2 + 17\gamma \left(-\frac{M}{r} + \frac{Q^2}{r^2}\right) + 7 \left(-\frac{M}{r} + \frac{Q^2}{r^2}\right)^2 \\
\geq \gamma \left(9\gamma + 17\gamma \left(-\frac{M}{r} + \frac{Q^2}{r^2}\right)\right).
\]

Since for \( r \geq r_2 \), we have
\[
\gamma = 1 - \frac{2M}{r} + \frac{Q^2}{r^2} = 1 - \frac{4M}{r} + \frac{3Q^2}{r^2} + \frac{2M}{r} - \frac{2Q^2}{r^2} \geq \frac{2}{r^2} \left(Mr - Q^2\right)
\]

(67)

the above is positive: \( 9\gamma + 17\gamma \left(-\frac{M}{r} + \frac{Q^2}{r^2}\right) \geq \frac{9}{r^2} \left(Mr - Q^2\right) + 17\gamma \left(-\frac{M}{r} + \frac{Q^2}{r^2}\right) = \frac{1}{r^2} \left(Mr - Q^2\right) > 0 \). This shows that condition 4 is verified for \( r \geq r_2 \). This concludes
the proof of the lemma. \( \Box \)
We now prove that Condition 5 is verified, i.e. that $D_2$ is positive.

**Lemma 5.** Condition 5 is verified for $r \leq r_1$ and for $r \geq r_2$, i.e. for all subextremal Reissner–Nordström spacetimes with $|Q| < M$ we have

$$D_2 \geq 0 \quad \text{for } r \leq r_1 \text{ or } r \geq r_2.$$  

**Proof.** Recall the definitions (52) and (53) of $B_1$ and $B_2$. Then Condition 5 reads

$$D_2 = B_1 B_2 - \frac{16Q^2}{r^8} f^2 \left( 1 - \frac{3M}{r} + \frac{2Q^2}{r^2} + \frac{1}{2} \gamma \right)^2$$

$$= f^2 \left( \frac{5}{r^7} \left( 1 - \frac{3M}{r} \right) - \frac{1}{2} \partial_r (\gamma V_1) \right) \left( \frac{2}{r^6} \left( 1 - \frac{3M}{r} + \frac{2Q^2}{r^2} \right) - \frac{1}{2} \partial_r (\gamma V_2) \right)$$

$$- \frac{16Q^2}{r^8} f^2 \left( 1 - \frac{3M}{r} + \frac{2Q^2}{r^2} + \frac{1}{2} \gamma \right)^2. $$

The above becomes

$$D_2 = f^2 \left( \frac{10}{r^6} - \frac{16Q^2}{r^8} \right) \left( 1 - \frac{3M}{r} + \frac{2Q^2}{r^2} \right)^2$$

$$+ f^2 \frac{1}{r^3} \left( 2 \left( -\frac{1}{2} \partial_r (\gamma V_1) \right) + 5 \left( -\frac{1}{2} \partial_r (\gamma V_2) \right) \right)$$

$$\times \left( 1 - \frac{3M}{r} + \frac{2Q^2}{r^2} \right)$$

$$+ f^2 \left( -\frac{1}{2} \partial_r (\gamma V_1) \right) \left( -\frac{1}{2} \partial_r (\gamma V_2) \right) - \frac{16Q^2}{r^8} f^2 \left( 1 - \frac{3M}{r} + \frac{2Q^2}{r^2} \right) \gamma - \frac{4Q^2}{r^8} f^2 \gamma^2. $$

We first consider the region $r \leq r_1$. In this region we use (60) and (61) to write

$$-\partial_r (\gamma V_1) - \frac{5}{2} \partial_r (\gamma V_2) = \frac{22}{r^3} \left( 1 - \frac{3M}{r} + \frac{2Q^2}{r^2} \right) \gamma - \frac{22}{r^3} \left( \frac{M}{r} - \frac{Q^2}{r^2} \right)$$

$$+ \frac{20Q^2}{r^5} \left( 2\gamma - \frac{M}{r} + \frac{Q^2}{r^2} \right)$$

and

$$\left( -\frac{1}{2} \partial_r (\gamma V_1) \right) \left( -\frac{1}{2} \partial_r (\gamma V_2) \right) = \frac{4\gamma^2}{r^6} \left( 1 - \frac{3M}{r} + \frac{2Q^2}{r^2} \right)^2$$

$$- \frac{8\gamma^2}{r^6} \left( \frac{M}{r} - \frac{Q^2}{r^2} \right) \left( 1 - \frac{3M}{r} + \frac{2Q^2}{r^2} \right)$$

$$+ \frac{22Q^2 \gamma}{r^8} \left( 2\gamma - \frac{M}{r} + \frac{Q^2}{r^2} \right) \left( 1 - \frac{3M}{r} + \frac{2Q^2}{r^2} \right)$$

$$+ \frac{4\gamma^2}{r^6} \left( \frac{M}{r} - \frac{Q^2}{r^2} \right)^2 - \frac{22Q^2 \gamma}{r^8} \left( 2\gamma - \frac{M}{r} + \frac{Q^2}{r^2} \right) \left( \frac{M}{r} - \frac{Q^2}{r^2} \right)$$

$$+ \frac{10Q^4}{r^{10}} \left( 2\gamma - \frac{M}{r} + \frac{Q^2}{r^2} \right)^2.$$
Factorizing out $\frac{r^2}{r^6}$, (68) becomes
\[
\left(10 + 22\gamma + 4\gamma^2 - \frac{16Q^2}{r^2}\right)\left(-1 + \frac{3M}{r} - \frac{2Q^2}{r^2}\right)^2 + \gamma \frac{22 + 8\gamma}{r^2} \left(-1 + \frac{3M}{r} - \frac{2Q^2}{r^2}\right) (Mr - Q^2)
\]
\[
+ \frac{Q^2}{r^2} (20 + 22\gamma) \left(-2\gamma + \frac{M}{r} - \frac{Q^2}{r^2}\right) \left(-1 + \frac{3M}{r} - \frac{2Q^2}{r^2}\right) + \frac{4\gamma^2}{r^4} (Mr - Q^2)^2 - \frac{4Q^2}{r^2} \gamma^2
\]
\[
+ \frac{22Q^2}{r^4} \gamma \left(-2\gamma + \frac{M}{r} - \frac{Q^2}{r^2}\right) (Mr - Q^2) + \frac{10Q^4}{r^4} \left(-2\gamma + \frac{M}{r} - \frac{Q^2}{r^2}\right)^2 + 16Q^2 \gamma \left(-1 + \frac{3M}{r} - \frac{2Q^2}{r^2}\right).
\]

Using (66) to bound $-1 + \frac{3M}{r} - \frac{2Q^2}{r^2}$ in the above, we obtain
\[
\geq \left(\frac{5 + 33\gamma + 18\gamma^2}{2r^4} - \frac{4Q^2}{r^6}\right) (Mr - Q^2)^2 + \frac{4Q^2}{r^2} \gamma \left(\frac{2}{r^2} (Mr - Q^2) - \gamma\right)
\]
\[
+ \frac{Q^2}{r^4} (10 + 11\gamma) \left(-2\gamma + \frac{M}{r} - \frac{Q^2}{r^2}\right) (Mr - Q^2) + \frac{10Q^4}{r^4} \left(-2\gamma + \frac{M}{r} - \frac{Q^2}{r^2}\right)^2.
\]

Using (65) we can bound $\frac{2}{r^2} (Mr - Q^2) - \gamma \geq \frac{3}{2r^2} (Mr - Q^2)$, therefore
\[
\geq \left(\frac{5 + 33\gamma + 18\gamma^2}{2r^4} - \frac{4Q^2}{r^6}\right) (Mr - Q^2)^2 + \frac{6Q^2}{r^4} \gamma (Mr - Q^2)
\]
\[
+ \frac{Q^2}{r^4} (10 + 33\gamma) \left(-2\gamma + \frac{M}{r} - \frac{Q^2}{r^2}\right) (Mr - Q^2) + \frac{10Q^4}{r^4} \left(-2\gamma + \frac{M}{r} - \frac{Q^2}{r^2}\right)^2.
\]

Observe that the second line is always non negative. We write the first line as
\[
(Mr - Q^2) \left[\left(\frac{5 + 33\gamma + 18\gamma^2}{2r^4} - \frac{5Q^2}{2r^6}\right) (Mr - Q^2) + \left(-\frac{6Q^2}{r^6}\right) \left(\frac{M}{4} - \frac{Q^2}{4}\right)\right]
\]
\[
+ \frac{6Q^2}{r^4} \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)
\]
\[
= (Mr - Q^2) \left[\left(\frac{5 + 33\gamma + 18\gamma^2}{2r^4} - \frac{5Q^2}{2r^6}\right) (Mr - Q^2) + \frac{6Q^2}{r^4} \left(1 - \frac{9M}{4r} + \frac{5Q^2}{4r^2}\right)\right].
\]

Observe that the first of the above two terms is always positive since $r^2 - Q^2 \geq 0$ and the second term is positive if $\frac{9M + \sqrt{81M^2 - 80Q^2}}{8} \leq r \leq r_1$. In particular using (69) this proves that Condition 5 is verified in $\frac{9M + \sqrt{81M^2 - 80Q^2}}{8} \leq r \leq r_1$. 

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If \( r < \frac{9M+\sqrt{81M^2-80Q^2}}{8} \), we can bound the factor \(-2\gamma + \frac{M}{r} - \frac{Q^2}{r^2}\) away from zero, as

\[
-2\gamma + \frac{M}{r} - \frac{Q^2}{r^2} = -2 + \frac{5M}{r} - \frac{3Q^2}{r^2} = -2 \left( 1 - \frac{9M}{4r} + \frac{5Q^2}{4r^2} \right) + \frac{M}{2r} - \frac{Q^2}{2r^2} \geq \frac{1}{2r^2} \left( Mr - Q^2 \right).
\]

From (69), we therefore obtain

\[
\geq \left( \frac{5 + 33\gamma + 18\gamma^2}{2r^4} - \frac{4Q^2}{r^6} + \frac{Q^2}{r^2} \right) (Mr - Q^2)^2 + \frac{6Q^2}{r^4} \gamma (Mr - Q^2) + \frac{10Q^4}{4r^8} (Mr - Q^2)^2.
\]

which proves Condition 5 for all \( r \leq r_1 \).

We now consider the region \( r \geq r_2 \). In this region we use (62) and (63) to write

\[
2 \left( \frac{1}{2} \partial_r (\gamma V_1) \right) + 5 \left( \frac{1}{2} \partial_r (\gamma V_2) \right) = \frac{22}{r^3} \left( 1 - \frac{4M}{r} + \frac{3Q^2}{r^2} \right) \gamma + \frac{20Q^2}{r^5} \left( 2\gamma - \frac{M}{r} + \frac{Q^2}{r^2} \right)
\]

and

\[
\left( -\frac{1}{2} \partial_r (\gamma V_1) \right) \left( -\frac{1}{2} \partial_r (\gamma V_2) \right) = \frac{4\gamma^2}{r^6} \left( 1 - \frac{4M}{r} + \frac{3Q^2}{r^2} \right)^2 + \frac{22Q^2}{r^8} \left( 1 - \frac{4M}{r} + \frac{3Q^2}{r^2} \right) \left( 2\gamma - \frac{M}{r} + \frac{Q^2}{r^2} \right) + \frac{10Q^4}{r^{10}} \left( 2\gamma - \frac{M}{r} + \frac{Q^2}{r^2} \right)^2.
\]

Factorizing out \( \frac{Q^2}{r^2} \) and neglecting the terms \( 1 - \frac{4M}{r} + \frac{3Q^2}{r^2} \geq 0 \), (68) becomes

\[
\left( 10 - \frac{16Q^2}{r^2} \right) \left( 1 - \frac{3M}{r} + \frac{2Q^2}{r^2} \right)^2 + \frac{20Q^2}{r^2} \left( 2\gamma - \frac{M}{r} + \frac{Q^2}{r^2} \right) \left( 1 - \frac{3M}{r} + \frac{2Q^2}{r^2} \right) - \frac{16Q^2}{r^2} \left( 1 - \frac{3M}{r} + \frac{2Q^2}{r^2} \right) \gamma - \frac{4Q^2}{r^2} \gamma^2.
\]

The term \( 10 - \frac{16Q^2}{r^2} \) is positive for \( r \geq r_P \) in the subextremal range. The second line can be arranged to be, factorizing out \( \frac{4Q^2}{r^2} \),

\[
\left( 6\gamma - \frac{5M}{r} + \frac{5Q^2}{r^2} \right) \left( 1 - \frac{3M}{r} + \frac{2Q^2}{r^2} \right) - \gamma^2 = \left( 6\gamma - \frac{5M}{r} + \frac{5Q^2}{r^2} \right) \left( \gamma - \frac{M}{r} + \frac{Q^2}{r^2} \right) - \gamma^2
\]

\[
= 5\gamma^2 + 11\gamma \left( -\frac{M}{r} + \frac{Q^2}{r^2} \right) + 5 \left( -\frac{M}{r} + \frac{Q^2}{r^2} \right)^2.
\]

Using (67), we prove that the above is positive. This shows that condition 5 is verified for \( r \geq r_2 \), and concludes the proof of the lemma.
7.3. The region \( r_1 \leq r \leq r_2 \). In the region \( r_1 \leq r \leq r_2 \), the function \( w \) is defined to be \( w = \frac{2}{r^2} \mathcal{Y}(r) \). From (57) we then obtain

\[
r^2 f = \int_{r_p}^{r} \frac{r^2}{\mathcal{Y}} w = \int_{r_p}^{r} \frac{r^2}{\mathcal{Y}} \mathcal{Y} = \int_{r_p}^{r} 2 = 2r - 2r_p
\]

which gives

\[
f = \frac{2}{r} - \frac{2r_p}{r^2} = \frac{2(r - r_p)}{r^2} \quad \text{for} \quad r_1 \leq r \leq r_2.
\]

Notice that the above implies

\[
f' = -\frac{2}{r^2} + \frac{2r_p}{r^3} = -\frac{2r + 4r_p}{r^3}.
\]

Notice that if \( r \leq r_2 \leq 2r_p \), then \(-2r + 4r_p > 0\), therefore

\[
f' > 0 \quad \text{for} \quad r_1 \leq r \leq r_2.
\]

This proves that Condition 2 is verified in this region. We now check Conditions 3, 4 and 5.

**Lemma 6.** Condition 3 is verified in the region \( r_1 \leq r \leq r_2 \), i.e. for all subextremal Reissner–Nordström spacetimes with \(|Q| < M\) we have

\[
A_1 \geq 0, \quad A_2 \geq 0 \quad \text{for} \quad r_1 \leq r \leq r_2.
\]

**Proof.** Since \( \lambda \geq 6 \), we have that \( A_1 \geq B_1 \) and \( A_2 \geq B_2 \), where we recall

\[
B_1 = \frac{5f}{r^3} \left( 1 - \frac{3M}{r} + \frac{2Q^2}{r^2} \right) - \frac{1}{2} \partial_r (\mathcal{Y}V_1) f - \frac{1}{4} \Box_g w
\]

\[
B_2 = \frac{2f}{r^3} \left( 1 - \frac{3M}{r} + \frac{2Q^2}{r^2} \right) - \frac{1}{2} \partial_r (\mathcal{Y}V_2) f - \frac{1}{4} \Box_g w.
\]

We compute \( \Box_g w \). From \( w = \frac{2}{r^2} \mathcal{Y}(r) \) we obtain

\[
\partial_r w = 2 \partial_r (\frac{1}{r^2} \mathcal{Y}) = 2 \left( -2r^{-3} \mathcal{Y} + r^{-2} \mathcal{Y}' \right)
\]

\[
= 2 \left( -2r^{-3} \left( 1 - \frac{2M}{r} + \frac{Q^2}{r^2} \right) + r^{-2} \left( \frac{2M}{r^2} - \frac{2Q^2}{r^2} \right) \right) = -\frac{4}{r^3} \left( 1 - \frac{3M}{r} + \frac{2Q^2}{r^2} \right).
\]

For a radial function \( w = w(r) \) we have \( \Box_g w = r^{-2} \partial_r (r^2 \mathcal{Y} \partial_r w) \) (see Lemma 6 in [20]). We therefore compute

\[
-\partial_r \left( r^2 \mathcal{Y} \partial_r w \right) = 4 \partial_r \left( \frac{\mathcal{Y}}{r} \left( 1 - \frac{3M}{r} + \frac{2Q^2}{r^2} \right) \right)
\]

\[
= 4 \left( \frac{\mathcal{Y}'}{r} \left( 1 - \frac{3M}{r} + \frac{2Q^2}{r^2} \right) - \frac{\mathcal{Y}}{r^2} \left( 1 - \frac{3M}{r} + \frac{2Q^2}{r^2} \right) + \frac{\mathcal{Y}}{r} \left( \frac{3M}{r^2} - \frac{4Q^2}{r^3} \right) \right)
\]

\[
= 4 \left( -\frac{1}{r^2} \left( 1 - \frac{4M}{r} + \frac{3Q^2}{r^2} \right) \left( 1 - \frac{3M}{r} + \frac{2Q^2}{r^2} \right) + \frac{\mathcal{Y}}{r} \left( \frac{3M}{r^2} - \frac{4Q^2}{r^3} \right) \right).
\]
We obtain
\[-\frac{1}{4}\Box w = -\frac{1}{r^4} \left(1 - \frac{4M}{r} + \frac{3Q^2}{r^2}\right) \left(1 - \frac{3M}{r} + \frac{2Q^2}{r^2}\right) + \frac{\gamma}{r^3} \left(\frac{3M}{r^2} - \frac{4Q^2}{r^3}\right).\]

We now evaluate \(B_1\) and \(B_2\) in this region. For \(r_1 \leq r \leq r_2\), we have
\[
\gamma = \frac{c}{r^2} \left(Mr - Q^2\right) \quad \text{for} \quad \frac{1}{2} \leq c \leq 2
\]
where \(c = \frac{1}{2}\) corresponds to \(r = r_1\) and \(c = 2\) corresponds to \(r = r_2\). We write
\[
\frac{1}{4}\Box w = -\frac{1}{r^8}(c - 1)(c - 2) \left(Mr - Q^2\right)^2 + \frac{1}{r^8}c \left(Mr - Q^2\right) (3Mr - 4Q^2)
\]
\[
= \frac{1}{r^8} \left(\frac{c}{r^2} (Mr - Q^2)\right) \left(Mr - Q^2\right) + \frac{5Q^2}{r^3} \left(2Mr - Q^2\right)
\]
\[
= \frac{1}{r^7} \left((c^2 - 6c + 2)Mr + (c^2 - 7c + 2)Q^2\right) \left(Mr - Q^2\right).
\]

On the other hand, using (62), we have
\[
-\frac{1}{2}\partial_r (\gamma V_1) = \frac{1}{r^3} \left(1 - \frac{4M}{r} + \frac{3Q^2}{r^2}\right) \gamma + \frac{5Q^2}{r^3} \left(2\gamma - \frac{M}{r} + \frac{Q^2}{r^2}\right)
\]
\[
= \frac{1}{r^3} \left(c - 2\right) (Mr - Q^2) + \frac{c}{r^2} (Mr - Q^2) + \frac{5Q^2}{r^3} \left(2c - 1\right) (Mr - Q^2)
\]
\[
= \frac{1}{r^7} \left((c^2 - 2c) Mr - (c^2 - 12c + 5) Q^2\right) (Mr - Q^2)
\]
and
\[
\frac{5}{r^3} \left(1 - \frac{3M}{r} + \frac{2Q^2}{r^2}\right) = \frac{5(c-1)}{r^3} \left(Mr - Q^2\right). \quad \text{Since} \quad f = \frac{2(r-rp)}{r^2},\]
we have
\[
\frac{5}{r^3} \left(1 - \frac{3M}{r} + \frac{2Q^2}{r^2}\right) - \frac{1}{2}\partial_r (\gamma V_1) f
\]
\[
= \frac{1}{r^7} \left(5(c - 1)r^2 + (c^2 - 2c) Mr - (c^2 - 12c + 5) Q^2\right) \frac{2(r - rp)}{r^2} \left(Mr - Q^2\right)
\]
and therefore
\[
B_1 = \frac{1}{r^8} \left[5(c - 1)r^2 \frac{2(r - rp)}{r} + \left(c(c - 2) \frac{2(r - rp)}{r} - (c^2 - 6c + 2)\right) Mr
\]
\[
- \left((c^2 - 12c + 5) \frac{2(r - rp)}{r} - (c^2 - 7c + 2)\right) Q^2\right] (Mr - Q^2).
\]

Notice that the above clearly increases as \(Q\) decreases. In particular, the extremal case \(|Q| = M\) is the worst case scenario in the positivity of the above coefficient. We check that the term in parenthesis is positive at the extremal case, and that will imply that is positive for all subextremal cases \(|Q| < M\).

Notice that Eq. (72) translates into
\[
1 - \frac{(2 + c)M}{r} + \frac{(1 + c)Q^2}{r^2} = 0 \quad \text{for} \quad \frac{1}{2} \leq c \leq 2.
\]
This gives
\[ r = \frac{(2 + c)M + \sqrt{(2 + c)^2 M^2 - 4(1 + c)Q^2}}{2} \]
\[ > \frac{(2 + c)M + \sqrt{(4 + 4c + c^2)M^2 - 4(1 + c)M^2}}{2} = \frac{(2 + 2c)M}{2} = (1 + c)M. \]

In particular \( r = (1 + c)M \) at the extremal case \(|Q| = M\), where the above becomes
\[ B_1 = \frac{1}{r^8} \left[ 5(c - 1)((1 + c)M)\frac{2(c - 1)}{c + 1} \right. \]
\[ + \frac{1}{c + 1} \left( c^2 - c^3 + 27c^2 - 33c + 10 \right) M^2 \right] (Mr - Q^2) \]
\[ = \frac{M^2}{(c + 1)r^8} \left[ 10(c - 1)^2(1 + c)^2 + \left( c^4 - c^3 + 27c^2 - 33c + 10 \right) \right] (Mr - Q^2) \]
\[ = \frac{M^2}{(c + 1)r^8} \left[ 11c^4 - c^3 + 7c^2 - 33c + 20 \right] (Mr - Q^2) \]

which is a positive polynomial for all \( c \), and in particular for \( \frac{1}{2} \leq c \leq 2 \).
Similarly we compute \( B_2 \). Using (63), we have
\[ \frac{-1}{2} \partial_r \left( \gamma V_2 \right) = \frac{4}{r^3} \left( 1 - \frac{4M}{r} + \frac{3Q^2}{r^2} \right) \gamma + \frac{2Q^2}{r^5} \left( 2\gamma - \frac{M}{r} + \frac{Q^2}{r^2} \right) \]
\[ = \frac{4}{r^3} \left( \frac{c - 2}{r^2} \left( Mr - Q^2 \right) \right) + \frac{c}{r^2} \left( Mr - Q^2 \right) + \frac{2Q^2}{r^5} \left( \frac{2c - 1}{r^2} \left( Mr - Q^2 \right) \right) \]
\[ = \frac{1}{r^3} \left( 4(c - 2) \left( Mr - Q^2 \right) + 2Q^2(2c - 1) \left( Mr - Q^2 \right) \right) \]
\[ = \frac{1}{r^3} \left( (4c^2 - 8c) \left( Mr - (4c^2 - 12c + 2) Q^2 \right) \left( Mr - Q^2 \right) \right). \]

We therefore have
\[ B_2 = \frac{2}{r^3} \left( Mr - Q^2 \right) \frac{2(r - rp)}{r^2} + \frac{1}{r^8} \left[ \left( (4c^2 - 8c) \frac{2(r - rp)}{r} \right) \left( Mr - Q^2 \right) \right] \]
\[ - \left( (4c^2 - 12c + 2) \frac{2(r - rp)}{r} \right) \left( Mr - Q^2 \right). \]

At the extremal case the above becomes
\[ B_2 = \frac{1}{r^8} \left[ 2(c - 1)((1 + c)M)\frac{2(c - 1)}{c + 1} \right. \]
\[ + \frac{1}{c + 1} \left( 7c^4 - 19c^3 + 27c^2 - 15c + 4 \right) M^2 \right] (Mr - Q^2) \]
\[ = \frac{M^2}{(c + 1)r^8} \left[ 4(c - 1)^2(1 + c)^2 + 7c^4 - 19c^3 + 27c^2 - 15c + 4 \right] (Mr - Q^2) \]
\[ = \frac{M^2}{(c + 1)r^8} \left[ 11c^4 - 19c^3 + 19c^2 - 15c + 8 \right] (Mr - Q^2) \]

which is positive for all \( c \), and in particular for \( \frac{1}{2} \leq c \leq 2 \). This proves the lemma. \( \Box \)
Lemma 7. Condition 4 is verified in the region \( r_1 \leq r \leq r_2 \), i.e. for all subextremal Reissner–Nordström spacetimes with \( |Q| < M \) we have

\[
D_1 \geq 0 \quad \text{for } r_1 \leq r \leq r_2.
\]

Proof. As above, we check the positivity in the extremal case \( |Q| = M \). Using the expressions for \( B_1 \) and \( B_2 \) obtained in Lemma 6 at the extremal case, we have

\[
B_1 + B_2 = \frac{M^2}{(c+1)r^8} \left[ 22c^4 - 20c^3 + 26c^2 - 48c + 28 \right] (Mr - Q^2)
\]

and therefore

\[
\frac{f}{r^3} \left( 1 - \frac{3M}{r} + \frac{2Q^2}{r^2} \right) (B_1 + B_2)
\]

\[
= \left( \frac{c-1}{r^2} \right) \frac{M^2}{(c+1)r^9} \frac{2(r-r_P)}{r} \left[ 22c^4 - 20c^3 + 26c^2 - 48c + 28 \right] (Mr - Q^2)^2
\]

\[
= \frac{M^2}{(c+1)^2r^{14}} 2(c-1) \left[ 22c^4 - 20c^3 + 26c^2 - 48c + 28 \right] (Mr - Q^2)^2.
\]

We now consider the term \(-\frac{4Q^2}{r^8} f^2 \left( 1 - \frac{3M}{r} + \frac{2Q^2}{r^2} + \frac{1}{2} \gamma \right)^2 \). We have

\[
-\frac{4Q^2}{r^8} f^2 \left( 1 - \frac{3M}{r} + \frac{2Q^2}{r^2} + \frac{1}{2} \gamma \right)^2
\]

\[
= -\frac{4Q^2}{r^8} f^2 \left( \frac{c-1}{r^2} (Mr - Q^2) + \frac{1}{2} \frac{c}{r^2} (Mr - Q^2) \right)^2
\]

\[
= -\frac{2Q^2}{r^8} \left( \frac{2(r-r_P)}{r^2} \right)^2 \left( \frac{3c-2}{r^4} (Mr - Q^2)^2 \right)
\]

\[
= -\frac{2Q^2}{r^8} \left( \frac{2(r-r_P)}{r} \right)^2 (3c-2)^2 (Mr - Q^2)^2.
\]

At the extremal case the above becomes

\[
= -\frac{2M^2}{r^{14}} \left( \frac{2(c-1)}{c+1} \right)^2 (3c-2)^2 (Mr - Q^2)^2
\]

\[
= -\frac{8M^2}{(c+1)^2r^{14}} (c-1)^2 (3c-2)^2 (Mr - Q^2)^2.
\]

We therefore obtain the polynomial

\[
\left[ 22c^4 - 20c^3 + 26c^2 - 48c + 28 \right] - 4(3c-2)^2 = 22c^4 - 20c^3 - 10c^2 + 12
\]

which is positive for all \( c \), and in particular for \( \frac{1}{2} \leq c \leq 2 \). \( \square \)

Lemma 8. Condition 5 is verified in the region \( r_1 \leq r \leq r_2 \), i.e. for all subextremal Reissner–Nordström spacetimes with \( |Q| < M \) we have

\[
D_2 \geq 0 \quad \text{for } r_1 \leq r \leq r_2.
\]
Proof. As above, we check the positivity in the extremal case $|Q| = M$. Using the expressions for $B_1$ and $B_2$ obtained in Lemma 6 at the extremal case, we have

$$B_1B_2 = \frac{M^4}{(c + 1)^2 r^{16}} \left( 11c^4 - c^3 + 7c^2 - 33c + 20 \right) \left( 11c^4 - 19c^3 + 19c^2 - 15c + 8 \right) (Mr - Q^2)^2.$$  

We now consider the term $-\frac{16Q^2}{r^8} f^2 \left( 1 - \frac{3M}{r} + \frac{2Q^2}{r^2} + \frac{1}{2} \gamma \right)^2$. We have

$$-\frac{16Q^2}{r^8} f^2 \left( 1 - \frac{3M}{r} + \frac{2Q^2}{r^2} + \frac{1}{2} \gamma \right)^2 = -\frac{8Q^2}{r^{14}} \left( \frac{2(r - r_p)}{r} \right)^2 (3c - 2)^2 (Mr - Q^2)^2.$$  

At the extremal case the above becomes

$$=- \frac{8M^2}{r^{14}} \left( \frac{2(c - 1)}{c + 1} \right)^2 (3c - 2)^2 (Mr - Q^2)^2$$  

$$=- \frac{32M^2}{(c + 1)^2 r^{14}} (c - 1)^2 (3c - 2)^2 (Mr - Q^2)^2$$  

$$=- \frac{M^4}{(c + 1)^2 r^{16}} 32(c + 1)^2 (c - 1)^2 (3c - 2)^2 (Mr - Q^2)^2.$$  

We therefore obtain the polynomial

$$(11c^4 - c^3 + 7c^2 - 33c + 20)(11c^4 - 19c^3 + 19c^2 - 15c + 8) - 32(c + 1)^2 (c - 1)^2 (3c - 2)^2$$

$$= 121c^8 - 220c^7 + 17c^6 - 296c^5 + 1531c^4 - 1888c^3 + 899c^2 - 180c + 32$$

which is positive for all $c$, and in particular for $\frac{1}{2} < c < 2$. This concludes the proof of the lemma.  

8. $r^p$-Hierarchy Estimates

In this section, we conclude the proof of Theorem 3 by obtaining the $r^p$-hierarchy estimates for the mixed spin $\pm 1$ and spin $\pm 2$ system of equations. The equations in the system have right hand sides which present good decay in $r$, and therefore the $r^p$-estimates for the system do not differ from the standard derivation for a single wave equation as in [16].

To derive the $r^p$-estimates, we apply Lemma 1 to the vector field $Z = l(r)e_4$, for the function $l(r) = r^p$.

Proposition 5. Consider a vectorfield $Z = l(r)e_4$, a pair of scalar functions $w = (w_1, w_2)$, with $w_1 = w_2 = \frac{2\Phi_1}{r}$, and a pair of one forms $M = (M_1, M_2)$ with $M_1 = M_2 = \frac{2\Phi_1}{r} e_4 = \frac{2\Phi_1}{r^p} Z$. Let $\Phi_1$ and $\Phi_2$ be a 1-tensor and a symmetric traceless 2-tensor respectively, satisfying the system of coupled wave Eqs. (36) and (37). Then we have

$$C^{(Z, w, M)} [\Phi_1, \Phi_2] = s \left( \frac{1}{2} l' |\nabla \Phi_1|^2 + \frac{1}{2} \left( -l' + \frac{2}{r} \right) (|\nabla \Phi_1|^2 + V_1 |\Phi_1|^2) \right)$$

$$+ 4l' |\nabla \Phi_2|^2 + 4 \left( -l' + \frac{2}{r} \right) (|\nabla \Phi_2|^2 + V_2 |\Phi_2|^2) + err[\Phi_1, \Phi_2]$$
where
\[
\text{err}[\Phi_1, \Phi_2] = O \left( \frac{M + O}{r^2}, \frac{Q^2}{r^3} \right) \left( |l| + r|l'| + r^2|l''| \right) \left( |\nabla_4 \Phi_1|^2 + |\nabla_4 \Phi_2|^2 + r^{-2}(|\Phi_1|^2 + |\Phi_2|^2) \right).
\]

**Proof.** We start by computing \( \mathcal{E}^{(Z,w,0)}[\Phi_1, \Phi_2] \). Using (48) we have
\[
\mathcal{E}^{(Z,w,0)}[\Phi_1, \Phi_2] = \frac{1}{2} \mathcal{T}[\Phi_1], (Z) \pi + \left( -\frac{1}{2} Z(V_1) - \frac{1}{4} \square_g w_1 \right) |\Phi_1|^2 + \frac{1}{2} w_1 \mathcal{L}_1[\Phi_1]
+ 4 \mathcal{T}[\Phi_2], (Z) \pi + \left( -4 Z(V_2) - 2 \square_g w_2 \right) |\Phi_2|^2 + 4 w_2 \mathcal{L}_2[\Phi_2]
\]
\[
+ \frac{4Q}{r^2} \left( w_1 + w_2 + \frac{4}{r} Z(r) - \text{tr}(Z) \right) \Psi^\pi \Phi_1 \cdot \Phi_2 - \frac{8Q}{r^2} (\{Z, \Psi^\pi\} \Phi_1) \cdot \Phi_2.
\]
We have for \( Z = l(r)e_4 \) (see Corollary 4 in [20])
\[
\mathcal{T}[\Phi_1], (Z) \pi = \left( -l' + \frac{2}{r} l \right) |\nabla \Phi_1|^2 + \left( \gamma l' + \left( -\frac{2M}{r^2} + \frac{2Q^2}{r^3} \right) l \right) |\nabla_4 \Phi_1|^2
\]
and similarly for \( \Phi_2 \). We therefore obtain
\[
\mathcal{E}^{(Z,w,0)}[\Phi_1, \Phi_2] = \frac{1}{2} \left( -l' + \frac{2}{r} l \right) (|\nabla \Phi_1|^2 + V_1|\Phi_1|^2) + \frac{1}{2} \left( \gamma l' + \left( -\frac{2M}{r^2} + \frac{2Q^2}{r^3} \right) l \right) |\nabla_4 \Phi_1|^2
\]
\[
+ \left( -\frac{1}{2} Z(V_1) - \frac{1}{4} \square_g w_1 - \frac{l}{r} V_1 \right) |\Phi_1|^2 + \left( \frac{1}{2} w_1 - l \right) \mathcal{L}_1[\Phi_1]
\]
\[
+ 4 \left( -l' + \frac{2}{r} l \right) \left( |\nabla \Phi_2|^2 + V_2|\Phi_2|^2 \right) + 4 \left( \gamma l' + \left( -\frac{2M}{r^2} + \frac{2Q^2}{r^3} \right) l \right) |\nabla_4 \Phi_2|^2
\]
\[
+ \left( -4 Z(V_2) - 2 \square_g w_2 - \frac{8}{r} l V_2 \right) |\Phi_2|^2 + \left( 4 w_2 - \frac{8}{r} l \right) \mathcal{L}_2[\Phi_2]
\]
\[
+ \frac{4Q}{r^2} \left( w_1 + w_2 + \frac{4}{r} Z(r) - \text{tr}(Z) \right) \Psi^\pi \Phi_1 \cdot \Phi_2 - \frac{8Q}{r^2} (\{Z, \Psi^\pi\} \Phi_1) \cdot \Phi_2.
\]
With the choice \( w = w_1 = w_2 = \frac{2l}{r} \), the terms involving \( \mathcal{L}_1[\Phi_1] \) and \( \mathcal{L}_2[\Phi_2] \), cancel out. Using again the formula \( \square_g w = r^{-2} \partial_r (r^2 \gamma \partial_r w) \) (see Lemma 6 in [20]), we obtain
\[
\square_g w = \frac{2l''}{r} + O \left( \frac{M}{r^4}, \frac{Q^2}{r^5} \right) \left[ |l| + r|l'| + r^2|l''| \right].
\]
Therefore
\[
-\frac{1}{2} Z(V_1) - \frac{1}{4} \square_g w_1 - \frac{l}{r} V_1 = -\frac{l''}{2r} - \frac{l}{2} \left( V'_1 + \frac{2}{r} V_1 \right) + O \left( \frac{M}{r^4}, \frac{Q^2}{r^5} \right) \left[ |l| + r|l'| + r^2|l''| \right]
\]
\[
-4 Z(V_2) - 2 \square_g w_2 - \frac{8}{r} l V_2 = -8 \frac{l''}{2r} - 4l \left( V'_2 + \frac{2}{r} V_2 \right) + O \left( \frac{M}{r^4}, \frac{Q^2}{r^5} \right) \left[ |l| + r|l'| + r^2|l''| \right].
\]
Using (59), we have in both cases
\[
V'_1 + \frac{2}{r} V_1 = O \left( \frac{M}{r^4}, \frac{Q^2}{r^5} \right) \quad V'_2 + \frac{2}{r} V_2 = O \left( \frac{M}{r^4}, \frac{Q^2}{r^5} \right).
\]
Recall that $\text{tr} (Z)^{(4)} = \frac{4}{r}$ and $\text{tr} (Z)^{(r)} = 2l' + \frac{4}{r}l$ (see Corollary 3 in [20]), therefore

$$[Z, \Phi] \Phi_1 = l[e_4, \Phi] \Phi_1 = -\frac{l}{r} \Phi \Phi_1$$

$$w_1 + w_2 + \frac{4}{r} Z(r) - \text{tr} (Z)^{(r)} = \frac{4l}{r} + \frac{4}{r} - (2l' + \frac{4}{r}) = -2l' + \frac{4l}{r}.$$ 

Using (28) to write

$$\frac{4Q}{r^2} \left( w_1 + w_2 + \frac{4}{r} Z(r) - \text{tr} (Z)^{(r)} \right) \Phi \Phi_1 \cdot \Phi_2 = \frac{4Q}{r^2} \left( -2l' + \frac{6l}{r} \right) \Phi \Phi_1 \cdot \Phi_2$$

we conclude

$$\mathcal{E}^{(Z,w,0)}[\Phi_1, \Phi_2] = \frac{1}{2} \left( -l' + \frac{2}{r} l \right) \left( |\nabla \Phi_1|^2 + V_1 |\Phi_1|^2 \right) + \frac{1}{2} l' |\nabla_4 \Phi_1|^2 - \frac{l''}{2r} |\Phi_1|^2$$

$$+ 4 \left( -l' + \frac{2}{r} l \right) \left( |\nabla \Phi_2|^2 + V_2 |\Phi_2|^2 \right) + 4l' |\nabla_4 \Phi_2|^2 - \frac{l''}{r} |\Phi_2|^2 + \text{err}[\Phi_1, \Phi_2]$$

where

$$\text{err}[\Phi_1, \Phi_2] = O \left( \frac{M + Q}{r^2}, \frac{Q^2}{r^3} \right) \left( |l| + r |l'| + r^2 |l''| \right) \left( |\nabla_4 \Phi_1|^2 + |\nabla_4 \Phi_2|^2 + r^{-2} (|\Phi_1|^2 + |\Phi_2|^2) \right).$$

Using (48) we have

$$\mathcal{E}^{(Z,w,M)}[\Phi_1, \Phi_2] = \mathcal{E}^{(Z,w,0)}[\Phi_1, \Phi_2]$$

$$+ \frac{1}{4} (\text{div } M_1) |\Phi_1|^2 + \frac{1}{2} \Phi_1 \cdot M_1(\Phi_1) + 2(\text{div } M_2) |\Phi_2|^2 + 4 \Phi_2 \cdot M_2(\Phi_2).$$

Let $M := M_1 = M_2 = \frac{2l'}{r} e_4 = \frac{2l'}{r^2} Z$, we compute

$$\text{div } M = g_{\mu \nu} D_\nu \left( \frac{2l'}{r} Z_\mu \right) = l' \frac{2l'}{r} \text{tr} (Z)^{(r)} + Z \left( \frac{2l'}{r^2} \right) = \frac{l'}{r} (2l' + \frac{4l}{r}) + 2l e_4 \left( \frac{l'}{r} \right) = \frac{2l'}{r^2} + \frac{2l''}{r}.$$ 

We deduce

$$\mathcal{E}^{(Z,w,M)}[\Phi_1, \Phi_2]$$

$$= \frac{1}{2} \left( -l' + \frac{2}{r} l \right) \left( |\nabla \Phi_1|^2 + V_1 |\Phi_1|^2 \right) + \frac{1}{2} l' |\nabla_4 \Phi_1|^2 + \frac{1}{2} l' |\nabla_4 \Phi_1|^2 + \Phi_1 \cdot \frac{l'}{r} \nabla_4(\Phi_1)$$

$$+ 4 \left( -l' + \frac{2}{r} l \right) \left( |\nabla \Phi_2|^2 + V_2 |\Phi_2|^2 \right) + 4l' |\nabla_4 \Phi_2|^2 + \frac{4l''}{r} |\Phi_2|^2 + 8 \Phi_2 \cdot \frac{l'}{r} \nabla_4(\Phi_2)$$

$$+ \text{err}[\Phi_1, \Phi_2].$$

Writing

$$\frac{1}{2} l' |\nabla_4 \Phi_1|^2 + \frac{l''}{2r^2} |\Phi_1|^2 + r^{-1} l' \Phi \cdot \nabla_4(\Phi_1) = \frac{1}{2} l' (\nabla_4(\Phi_1) + r^{-1} \Phi_1)^2 = \frac{1}{2} l' |\nabla_4 \Phi_1|^2$$

where we recall that $\nabla_4 \Phi_1 := \nabla_4 \Phi_1 + \frac{1}{r} \Phi_1$, and similarly for $\Phi_2$. This concludes the proof of the proposition. □
Recalling the definition of the spacetime energy (40), we obtain
\[
E(Z, w, M)[\Phi_1, \Phi_2] = \frac{P}{2} r^{p-1} |\tilde{\Psi}_4 \Phi_1|^2 + \frac{1}{2} (2 - p) r^{p-1} (|\nabla \Phi_1|^2 + V_1 |\Phi_1|^2) + 4 p r^{p-1} (|\nabla \Phi_2|^2 + V_2 |\Phi_2|^2) + \text{err}[\Phi_1, \Phi_2].
\]

Given a fixed \( \delta > 0 \), for all \( \delta \leq p \leq 2 - \delta \) and \( R \gg \max(\frac{M+Q}{\delta}, \frac{Q^2}{R^2}) \), while integrating in \( r \geq R \), the term \( \text{err}[\Phi_1, \Phi_2] \) can be absorbed by the first two lines above. Thus, we obtain
\[
\int_{M_{\leq R}(0, r)} E(Z, w, M)[\Phi_1, \Phi_2] \geq \frac{1}{4} \int_{M_{\leq R}(\tau_1, \tau_2)} r^{p-1} (p |\tilde{\Psi}_4 \Phi_1|^2 + (2 - p) (|\nabla \Phi_1|^2 + r^{-2} |\Phi_1|^2)) + 2 \int_{M_{\leq R}(\tau_1, \tau_2)} r^{p-1} (p |\tilde{\Psi}_4 \Phi_2|^2 + (2 - p) (|\nabla \Phi_2|^2 + r^{-2} |\Phi_2|^2)).
\]

Recalling the definition of the spacetime energy (40), we obtain
\[
\int_{M_{\leq R}(0, r)} E(Z, w, M)[\Phi_1, \Phi_2] \gtrsim M_{p, R}[\Phi_1, \Phi_2](0, \tau).
\] (73)

Consider now the current \( P(Z, w, M)[\Phi_1, \Phi_2] \) associated to the vector field \( Z \). It is given by
\[
P_{\mu}^{(Z, w, M)}[\Phi_1, \Phi_2] = T_{\mu\nu}[\Phi_1] Z^\nu + 8 T_{\mu\nu}[\Phi_2] Z^\nu - \frac{8 Q}{r^2} l (\tilde{\Psi}_2 \Phi_1 \cdot \Phi_2) \Phi_2 g(e_4, e_\mu) + \frac{1}{2} w \Phi_1 D_\mu \Phi_1 - \frac{1}{4} \partial_\mu w |\Phi_1|^2 + \frac{1}{2} l' r g(e_4, e_\mu) |\Phi_1|^2 + 4 w \Phi_2 D_\mu \Phi_2 - 2 \partial_\mu w |\Phi_2|^2 + 4 l' r g(e_4, e_\mu) |\Phi_2|^2.
\]

For the boundary terms we compute
\[
P^{(Z, w, M)}[\Phi_1, \Phi_2] : e_4 = l T[\Phi_1]_{44} + \frac{l}{r} \Phi_1 \cdot \nabla_4 \Phi_1 - \frac{1}{2} e_4 (r^{-1} l) |\Phi_1|^2 + 8 l T[\Phi_2]_{44} + 8 \frac{l}{r} \Phi_2 \cdot \nabla_4 \Phi_2 - 4 e_4 (r^{-1} l) |\Phi_2|^2
\]
\[
= l |\nabla_4 \Phi_1| + \frac{l}{r} |\Phi_1|^2 - \frac{1}{2} r^{-2} e_4 (rl |\Phi_1|^2) + 8 l |\nabla_4 \Phi_2| + \frac{1}{r} |\Phi_2|^2
\]
\[
- 4 r^{-2} e_4 (rl |\Phi_2|^2)
\]
\[
= l |\tilde{\Psi}_4 \Phi_1|^2 - \frac{1}{2} r^{-2} e_4 (rl |\Phi_1|^2) + 8 l |\tilde{\Psi}_4 \Phi_2|^2 - 4 r^{-2} e_4 (rl |\Phi_2|^2)
\]

and
\[
P^{(Z, w, M)}[\Phi_1, \Phi_2] : e_3 = l T[\Phi_1]_{34} + \frac{1}{2} r^{-1} l e_3 (|\Phi_1|^2) - \frac{1}{2} e_3 (r^{-1} l) |\Phi_1|^2 - r^{-1} l' |\Phi_1|^2 + 8 l T[\Phi_2]_{34} + 4 r^{-1} l e_3 (|\Phi_2|^2) - 4 e_3 (r^{-1} l) |\Phi_2|^2 - 8 r^{-1} l' |\Phi_2|^2
\]
\[
+ \frac{16 Q}{r^2} l (\tilde{\Psi}_2 \Phi_1 \cdot \Phi_2)
\]
\[
= l (|\nabla \Phi_1|^2 + V_1 |\Phi_1|^2) + 8 l (|\nabla \Phi_2|^2 + V_2 |\Phi_2|^2)
\]
Let \( \Phi_1 \) and \( \Phi_2 \) be a one form and a symmetric traceless two tensor respectively, satisfying the system of coupled wave Eqs. (36) and (37). Consider a fixed \( \delta > 0 \) and let \( R \gg \max(\frac{M+Q}{\delta}, \frac{Q^2}{\delta^2}) \). Then for all \( \delta \leq p \leq 2 - \delta \) the following \( r^p \)-estimates hold:

\[
E_{p,R}[\Phi_1, \Phi_2](\tau) + M_{p,R}[\Phi_1, \Phi_2](0, \tau) \lesssim E_{p}[\Phi_1, \Phi_2](0). \tag{74}
\]

**Proof.** Let \( \theta = \theta(r) \) supported for \( r \geq R/2 \) with \( \theta = 1 \) for \( r \geq R \) such that \( l_p = \theta(r)r^p, Z_p = l_p e_4, w_p = \frac{2l_p}{r}, M_p = \frac{2l_p}{r}e_4 \). We apply the divergence theorem to \( \mathcal{P}_p := \mathcal{P}(Z_p, w_p, M_p) \) in the spacetime region bounded by \( \Sigma_0 \) and \( \Sigma_\tau \). Using (47), by divergence theorem we have

\[
\int_{\Sigma_\tau} \mathcal{P}_p \cdot N_\Sigma + \int_{I^+(0,\tau)} \mathcal{P}_p \cdot e_3 + \int_{\mathcal{M}(0,\tau)} \mathcal{E}(Z_p, w_p, M_p)[\Phi_1, \Phi_2] = \int_{\Sigma_0} \mathcal{P}_p \cdot N_\Sigma.
\]

Recall that \( l_p \) vanishes for \( r \leq R/2 \). We can estimate some of the terms as follows:

\[
\left| \int_{\Sigma_{R \leq 2 \leq R}(\tau)} \mathcal{P}_p \cdot N_\Sigma \right| \lesssim R^p E[\Phi_1, \Phi_2](\tau), \quad \left| \int_{\Sigma_{R \leq 2 \leq R}(0)} \mathcal{P}_p \cdot N_\Sigma \right| \lesssim R^p E[\Phi_1, \Phi_2](0),
\]

\[
\left| \int_{\mathcal{M}_{R \leq 2 \leq R}(0,\tau)} \mathcal{E}(Z_p, w_p, M_p)[\Phi_1, \Phi_2] \right| \lesssim R^{p-1} \text{Mor}[\Phi_1, \Phi_2](0, \tau),
\]

where recall that \( \text{Mor}[\Phi_1, \Phi_2](0, \tau) \) is defined by (39). Hence,

\[
\int_{\Sigma_{R \geq R}(\tau)} \mathcal{P}_p \cdot N_\Sigma + \int_{I^+(0,\tau)} \mathcal{P}_p \cdot e_3 + \int_{\mathcal{M}_{R \geq R}(0,\tau)} \mathcal{E}(Z_p, w_p, M_p)[\Phi_1, \Phi_2]
\]

\[
\lesssim \int_{\Sigma_{R \geq R}(0)} \mathcal{P}_p \cdot N_\Sigma + R^p \left( E[\Phi_1, \Phi_2](0) + E[\Phi_1, \Phi_2](\tau) + R^{-1} \text{Mor}[\Phi_1, \Phi_2](0, \tau) \right).
\]

Using the expressions for \( \mathcal{P} \cdot e_4 \) and \( \mathcal{P} \cdot e_3 \), we bound

\[
\int_{\Sigma_{R \geq R}(\tau)} \mathcal{P} \cdot N_\Sigma = \int_{\Sigma_{R \geq R}(\tau)} \mathcal{P} \cdot e_4
\]

\[
= \int_{\Sigma_{R \geq R}(\tau)} r^p |\nabla_4 \Phi_1|^2 - \frac{1}{2} r^{-2} e_4 (r^{p+1} |\Phi_1|^2) + 8 r^p |\nabla_4 \Phi_2|^2 - 4r^{-2} e_4 (r^{p+1} |\Phi_2|^2)
\]

\[
\gtrsim E_{p,R}[\Phi_1, \Phi_2](\tau)
\]
by performing the integration by parts for the second terms in the integrals, and absorbing the boundary term. Also,
\[
\int_{I^+(0, \tau)} P \cdot e_3 = \int_{I^+(0, \tau)} l(|\nabla \Phi_1|^2 + V_1|\Phi_1|^2) + 8l(|\nabla \Phi_2|^2 + V_2|\Phi_2|^2)
+ \int_{I^+(0, \tau)} \frac{1}{2} r^{-2} e_3 (rl|\Phi_1|^2) + 4r^{-2} e_3 (rl|\Phi_2|^2) + err[\Phi_1, \Phi_2]
\geq \int_{I^+(0, \tau)} r^p |\nabla \Phi_1|^2 + r^{p-2} |\Phi_1|^2 + r^p |\nabla \Phi_2|^2 + r^{p-2} |\Phi_2|^2.
\]
Using (73), we obtain
\[
E_{p, R}[\Phi_1, \Phi_2](\tau) + M_{p, R}[\Phi_1, \Phi_2](0, \tau) \lesssim E_{p}[\Phi_1, \Phi_2](0)
+ R^p \left( E[\Phi_1, \Phi_2](\tau) + R^{-1} \text{Mor}[\Phi_1, \Phi_2](0, \tau) \right).
\]
By combining the above with the boundedness of the energy in Proposition 2 and with the Morawetz estimates in (55), we prove the proposition. \(\square\)

By summing (74) and (55) we obtain the boundedness of the weighted energy (41) and the integrated weighted estimates (42), therefore proving Theorem 3.

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