Reflected BSDEs and obstacle problem for semilinear PDEs in divergence form

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Abstract

We consider the Cauchy problem for semilinear parabolic equation in divergence form with obstacle. We show that under natural conditions on the right-hand side of the equation and mild conditions on the obstacle the problem has a unique solution and we provide its stochastic representation in terms of reflected backward stochastic differential equations. We prove also regularity properties and approximation results for solutions of the problem.

1 Introduction

In the present paper we use stochastic methods based mainly on the theory of backward stochastic differential equations (BSDEs) to investigate the Cauchy problem for semilinear parabolic equation in divergence form with irregular obstacle.

Let $a : Q_T \equiv [0,T] \times \mathbb{R}^d \rightarrow \mathbb{R}^d \otimes \mathbb{R}^d, b : Q_T \rightarrow \mathbb{R}^d$ be measurable functions such that

$$
\lambda |\xi|^2 \leq \sum_{i,j=1}^{d} a_{ij}(t,x)\xi_i \xi_j \leq \Lambda |\xi|^2, \quad a_{ij} = a_{ji}, \quad |b_i| \leq \Lambda, \quad \xi \in \mathbb{R}^d
$$

for some $0 < \lambda \leq \Lambda$, and let $L_t$ be a linear differential operator of the form

$$
L_t = \frac{1}{2} \sum_{i,j=1}^{d} \frac{\partial}{\partial x_i} (a_{ij}(t,x) \frac{\partial}{\partial x_j}) + \sum_{i=1}^{d} b_i(t,x) \frac{\partial}{\partial x_i}.
$$

In the theory of variational inequalities the semilinear obstacle problem associated with $L_t$, terminal condition $\varphi \in L_{2,\text{loc}}(\mathbb{R}^d)$, generator $f$ and obstacle $h \in L_{2,\text{loc}}(Q_T)$

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consists in finding $u \in L_2(0,T; H_\varphi^1)$ such that $u \geq h$ a.e. and for every $\eta \in \mathcal{W}_\varphi$ such that $\eta(0) = 0$, $\eta \geq h$ a.e.,

$$\langle \eta - u, \frac{\partial \eta}{\partial t} \rangle_{\varphi,T} + \langle L_t u, \eta - u \rangle_{\varphi,T} + \langle f_u, \eta - u \rangle_{2,\varphi,T} \leq \frac{1}{2} \| \varphi - \eta(T) \|^2_{2,\varphi}, \quad (1.3)$$

where $f_u = f(\cdot, \cdot, u, \sigma \nabla u)$ and $\sigma \sigma^* = a$ (see, e.g., [8, 23, 26]). In this framework, $u$ is called a weak solution of the obstacle problem in the variational sense.

Roughly speaking, (1.3) means that we are looking for $u$ such that

$$\begin{cases} \min(u - h, -\frac{\partial u}{\partial t} - L_t u - f_u) = 0 & \text{in } Q_T, \\
u(T) = \varphi & \text{on } \mathbb{R}^d, \end{cases} \quad (1.4)$$

i.e. $u$ satisfies the prescribed terminal condition, takes values above the obstacle $h$, satisfies the inequality $\frac{\partial u}{\partial t} + L_t u \leq -f_u$ in $Q_T$ and the equation $\frac{\partial u}{\partial t} + L_t u = -f_u$ on the set $\{u > h\}$.

In [12] connections between viscosity solutions of (1.4) and reflected backward stochastic differential equations (RBSDEs) are investigated under natural assumptions in the theory of viscosity solutions that the data $\varphi, f, h$ are continuous and satisfy the polynomial growth condition and $L_t$ is a non-divergent operator of the form

$$L_t = \frac{1}{2} \sum_{i,j=1}^d a_{ij}(t,x) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^d b_i(t,x) \frac{\partial}{\partial x_i}$$

with coefficients ensuring existence of a unique solution of the SDE

$$dX_t^{s,x} = \sigma(t, X_t^{s,x}) dW_t + b(t, X_t^{s,x}) dt, \quad X_0^{s,x} = x \quad (\sigma \sigma^* = a).$$

In [12] it is proved that for each $(s,x) \in Q_T$ there is a unique solution $(Y^{s,x}, Z^{s,x}, K^{s,x})$ of RBSDE with forward driving process $X^{s,x}$, terminal condition $\varphi(X_T^{s,x})$, generator $f(\cdot, X^{s,x}, \cdot)$ and obstacle $h(\cdot, X^{s,x})$, and $u$ defined by the formula

$$u(s,x) = Y_s^{s,x}, \quad (s,x) \in Q_T \quad (1.5)$$

is a unique viscosity solution of (1.4).

Some attempts to give stochastic representation of solutions of obstacle problems in the variational sense are made in [4, 25, 29]. There, however, the authors consider only regular obstacles and non-divergent operators with regular coefficients, i.e. work in the set-up which is rather unnatural in the theory of variational inequalities.

In the present paper we deal with $L_t$ defined by (1.2) and we assume that $\varphi, f, h$ satisfy the following assumptions.

(H1) $\varphi \in L_{2,\varphi}(\mathbb{R}^d)$,

(H2) $f : [0,T] \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ is a measurable function satisfying the following conditions:

a) there exists $L > 0$ such that $|f(t,x,y_1,z_1) - f(t,x,y_2,z_2)| \leq L(|y_1 - y_2| + |z_1 - z_2|)$ for all $(t,x) \in Q_T$, $y_1, y_2 \in \mathbb{R}$ and $z_1, z_2 \in \mathbb{R}^d$,

b) there exist $M > 0$, $g \in L_{2,\varphi}(Q_T)$ such that $|f(t,x,y,z)| \leq g(t,x) + M(|y| + |z|)$ for all $(t,x,y,z) \in [0,T] \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d$. 

\[2\]
(H3) \( \varphi \geq h(T) \) a.e., \( h \in L_{2,\rho}(Q_T) \) and there exists a parabolic potential such that \( h^* \geq h \) a.e. (the definition of the parabolic potential is given in Section 4).

In general, if the obstacle \( h \) is irregular, a weak solution of (1.4) in the variational sense is not unique but it is known that there is a minimal solution, which of course is unique by the definition. The minimal solution is in fact the limit in \( L_{2,\rho}(\mathbb{R}^d) \) of solutions \( u_n \) of the associated penalized problems

\[
\left( \frac{\partial}{\partial t} + L_t \right) u_n = -f(u_n - n(u_n - h)^-) \quad u_n(T) = \varphi
\]  

(1.6)

(see, e.g., [8, 9, 26]).

In the present paper we propose another definition of a solution of the obstacle problem under which the problem has a unique solution. By a solution of (1.4) we mean a pair \((u, \mu)\) consisting of \( u \in L_{2,\rho}(0, T; H^1_{\rho}) \) and a positive Radon measure \( \mu \) on \( Q_T \) which vanishes on the sets of parabolic capacity zero such that \( u \geq h \) a.e., for every \( \eta \in W_{\rho} \) with \( \eta(0) = 0 \),

\[
\langle u, \frac{\partial \eta}{\partial t} \rangle_{\rho,T} - \langle L_t u, \eta \rangle_{\rho,T} = \langle \varphi, \eta(T) \rangle_{2,\rho} + \langle f u, \eta \rangle_{2,\rho,T} + \int_{Q_T} \eta g^2 \, d\mu
\]

and \( \mu \) has some minimality property saying that it acts only if \( u = h \). In case \( h \) is regular, the last condition may be expressed by the condition

\[
\int_{Q_T} (u - h) g^2 \, d\mu = 0 \quad (1.7)
\]

(see [18]).

The above definition of a solution is a counterpart to the definition of a solution of the obstacle problem for elliptic equations (see, e.g., [2, 16, 21]). For parabolic equations such definition was considered earlier in few papers (see [24] and references therein) but only in case of more regular barriers, i.e. barriers for which (1.7) is satisfied. For general barriers satisfying (H3) it appears here for the first time. The main problem in the parabolic case is to give proper meaning to (1.7) when the obstacle \( h \) is irregular. The difficulty lies in the fact that contrary to the case of elliptic equations, in the parabolic case, in general, \( u \) does not admit a quasi-continuous version. Note also that even in the elliptic case, the minimality condition for \( \mu \) is described formally only for upper quasi-continuous obstacles with respect to Newtonian capacity (see [21] and references given there).

To define properly solutions of the obstacle problem in Section 3 we refine slightly results of [40] (see also [5, 6]) on stochastic representation of solutions of the Cauchy problem and then, in Section 4 we present some elements of the parabolic potential theory for \( L_t \) and prove one-to-one correspondence between soft measures and time-inhomogeneous additive functionals of the Markov family \( X = \{(X, P_{s,x}), (s,x) \in Q_T\} \) associated with the operator \( L_t \). Let us stress that in order to encompass obstacles which in general do not have quasi-continuous versions we are forced to consider c\`adl\`ag functionals of \( X \).

In Section 5 we first provide rigorous formulation of the minimality condition for \( \mu \) and we show by example that \( \mu \) satisfying that condition need not satisfy (1.7) even
if the obstacle is upper or lower quasi-continuous. Then we prove that under (H1)–(H3) the obstacle problem has a unique solution \((u, \mu)\). In fact, its first component \(u\) coincides with the limit of \(\{u_n\}\), so our definition is consistent with the definition of weak minimal solution of \((1.3)\) in the variational sense. We show also that if \(\varphi \geq h(T)\) a.e. and \(h \in \mathbb{L}_{2, \varrho}(Q_T)\) then under (H1), (H2) the problem has a solution if and only if (H3) is satisfied, so our assumptions on \(h\) are the weakest possible.

Let us mention that in the case of linear equations another definition of solutions of the obstacle problem with irregular obstacles is given in [34]. We compare it with our definition at the end of Section 5.

In Section 5 we provide also stochastic representation of a solution of the obstacle problem. We show that under (H1)–(H3) there is a subset \(F_c \subset Q_T\) of parabolic capacity zero, which can be described explicitly in terms of \(h\) and \(g\) from condition (H2), such that for every \((s,x) \in F\) there exists a unique solution \((Y^{s,x}, Z^{s,x}, K^{s,x})\) of RBSDE with terminal condition \(\varphi(X_T)\), generator \(f(\cdot, X, \cdot, \cdot)\) and obstacle \(h(\cdot, X)\), and

\[
\begin{align*}
    u(t, X_t) &= Y^{s,x}_t, \quad t \in [s, T], \quad P_{s,x}\text{-a.s.,} \\
    \sigma \nabla u(\cdot, X_t) &= Z^{s,x}_t, \quad \lambda \otimes P_{s,x}\text{-a.s.,} \\

\end{align*}
\]

where \(\lambda\) denotes the Lebesgue measure on \([0, T]\). Hence, in particular, the first component \(u\) of the solution of the obstacle problem admits representation \((1.5)\) for quasi-every \((s,x) \in Q_T\). As for the second component \(\mu\), we show that it corresponds to \(K^{s,x}\) in the sense that for \((s,x) \in F\),

\[
E_{s,x} \int_s^T \xi(t, X_t) dK^{s,x}_t = \int_s^T \int_{\mathbb{R}^d} \xi(t, y)p(s, x, t, y) d\mu(t, y)
\]

for all \(\xi \in C_0(Q_T)\), where \(p\) stands for the transition density function of \((X, P_{s,x})\) (or, equivalently, \(p\) is the fundamental solution for \(L_t\)). Actually, one can find an additive functional of \(X\) which is equivalent under \(P_{s,x}\) to \(K^{s,x}\) for \((s,x) \in F\), so \((1.8)\) may be thought as a sort of the Revuz correspondence.

The stochastic approach to the obstacle problem allows not only to give reasonable definition of its solution and prove existence and uniqueness under minimal conditions on the obstacle but provides also useful additional information on the problem and the nature of solutions. First, we find interesting and useful that if \(\varphi \geq h(T)\) a.e. and \(h \in \mathbb{L}_{2, \varrho}(Q_T)\) then under (H1), (H2) the condition (H3), i.e. existence of a parabolic potential majorizing \(h\) is equivalent to the rather easily verifiable condition

\[
\sup_{s \in [0, T]} \int_{\mathbb{R}^d} (E_{s,x} \text{esssup}_{s \leq \tau \leq T} |h^+(t, X_t)|^2) \rho^2(x) dx < \infty.
\]

Secondly, from \((1.8), (1.9)\) it follows immediately that in the linear case for quasi-every \((s,x) \in Q_T\) (with respect to the parabolic capacity) the first component of the solution of the obstacle problem is given by

\[
\begin{align*}
    u(s, x) &= \int_{\mathbb{R}^d} \varphi(y)p(s, x, T, y) dy + \int_{Q_{sT}} f(t, y)p(s, x, t, y) dy \\
    &\quad + \int_{Q_{sT}} p(s, x, t, y) d\mu(t, y),
\end{align*}
\]
which generalizes known representation of the Cauchy problem for \( L_t \) via fundamental solutions (see [1]). Notice also that (1.9) allows one to derive some properties of \( \mu \) from those of \( K^{s,x} \) and vice versa. For instance, by analyzing \( K^{s,x} \) one can show that in some cases \( \mu \) is absolutely continuous with respect to the Lebesgue measure and moreover, calculate the corresponding density. An interesting example of such reasoning is to be found in [19]. Finally, let us mention that using the stochastic approach we prove strong convergence of gradients of solutions \( u_n \) of penalized problems (1.6) to the gradient of the solution \( u \). To be more precise, if \( h \) is quasi-continuous, then \( \nabla u_n \to \nabla u \) in \( L_{2,\varrho}(Q_T) \), while in the general case, \( \nabla u_0 \to \nabla u \) in \( L_{p,\varrho}(Q_T) \) for \( p \in [1,2] \). These results strengthen known results on convergence of \( \{u_n\} \).

Somewhat different applications of our methods is given in Section 6, where the linear Cauchy problem

\[
\frac{\partial u}{\partial t} + L_t u = -\mu, \quad u(T) = \varphi \tag{1.12}
\]

with Radon measure \( \mu \) is considered. It is shown there that if \( \mu \) is soft and satisfies some integrability condition then the unique renormalized solution of (1.12) may be represented stochastically by a unique solution of some simple BSDE. The representation makes it possible to give simple probabilistic definition of a solution of (1.12) and sheds some new light on the nature of solutions of (1.12).

In the paper we will use the following notation.

For \( t \in (0,T] \), \( Q_t = [0,t] \times \mathbb{R}^d \), \( Q_0 = (0,T] \times \mathbb{R}^d \), \( Q_{TT} = [t,T] \times \mathbb{R}^d \), \( Q_T = [0,T] \times \mathbb{R}^d \), \( \hat{Q}_T = (0,T) \times \mathbb{R}^d \), \( \nabla = (\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_d}) \).

By \( \mathcal{B}(D), \mathcal{B}_b(D), \mathcal{B}^+(D) \) we denote the set of Borel, bounded Borel, positive Borel functions on \( D \) respectively. \( C_0(D), C_0^\infty(D), C_b^\infty(D) \) are spaces of all continuous functions with compact support in \( D \), smooth functions with compact support in \( D \) and smooth functions on \( D \) with bounded derivatives, respectively. We write \( K \subset D \) if \( K \) is a compact subset of \( D \).

\( L_p^q(\mathbb{R}^d) \) (\( L_p^q(Q_T) \)) are usual Banach spaces of measurable functions on \( \mathbb{R}^d \) (on \( Q_T \)) that are \( p \)-integrable. Let \( \varrho \) be a positive function on \( \mathbb{R}^d \). By \( L_{p,\varrho}(\mathbb{R}^d) \) (\( L_{p,\varrho}(Q_T) \)) we denote the space of functions \( u \) such that \( u_{\varrho} \in L_p(\mathbb{R}^d) \) \( (u_{\varrho} \in L_p(Q_T)) \) equipped with the norm \( ||u||_{p,\varrho} = ||u_{\varrho}||_p \) \( ||u||_{p,\varrho,T} = ||u_{\varrho}||_{p,T} \). By \( \langle \cdot, \cdot \rangle_{2,\varrho} \) we denote the inner product in \( L_{2,\varrho}(\mathbb{R}^d) \). If \( \varrho \equiv 1 \), we denote it briefly by \( \langle \cdot, \cdot \rangle_2 \). By \( \langle \cdot, \cdot \rangle_{2,\varrho,T} \) we denote the inner product in \( L_{2,\varrho}(Q_T) \).

\( H^1_\varrho \) is the Banach space consisting of all elements \( u \) of \( L_{2,\varrho}(\mathbb{R}^d) \) having generalized derivatives \( \frac{\partial u}{\partial x_i}, i = 1, \ldots, d, \) in \( L_{2,\varrho}(\mathbb{R}^d) \). \( W^1_\varrho_2(\mathbb{R}^d) \) is the subspace of \( L_2(0,T;H^1_\varrho) \) consisting of all elements \( u \) such that \( \frac{\partial u}{\partial t} \in L_2(0,T;H^{-1}_{\varrho}) \) \( (\frac{\partial u}{\partial t} \in L_{2,\varrho}(Q_T)) \), where \( H^{-1}_{\varrho} \) is the dual space to \( H^1_\varrho \) (see [23] for details). By \( \langle \cdot, \cdot \rangle_{\varrho,T} \) we denote duality between \( L_2(0,T;H^{-1}_{\varrho}) \) and \( L_2(0,T;H^1_{\varrho}) \). \( \mathcal{M}(D) \) (\( \mathcal{M}^+(D) \) denotes the space of Radon measures (positive Radon measures) on \( D \). We denote \( \mathcal{M} = \mathcal{M}(Q_T), \mathcal{M}^+ = \mathcal{M}^+(Q_T) \). By \( m \) we denote the Lebesgue measure on \( \mathbb{R}^d \) and by \( m_T \) the Lebesgue measure on \( Q_T \).

By \( C \) (or \( c \)) we denote a general constant which may vary from line to line but depends only on fixed parameters. Throughout the paper \( \int _a^b \) stands for \( \int _{[a,b]} \).

## 2 Preliminary results

Let \( \Omega = C([0,T],\mathbb{R}^d) \) denote the space of continuous \( \mathbb{R}^d \)-valued functions on \([0,T]\) equipped with the topology of uniform convergence and let \( X \) be the canonical process.
on $\Omega$. It is known that given $L_t$ defined by (1.2) with $a, b$ satisfying (1.1) one can construct the weak fundamental solution $p(s, x, t, y)$ for $L_t$ and then a Markov family $\mathbb{X} = \{(X, P_{s,x}); (s, x) \in [0, T) \times \mathbb{R}^d\}$ for which $p$ is the transition density function, i.e.

$$P_{s,x}(X_t = x; 0 \leq t \leq s) = 1, \quad P_{s,x}(X_t \in \Gamma) = \int_{\Gamma} p(s, x, t, y) \, dy, \quad t \in (s, T]$$

for any $\Gamma$ in the Borel $\sigma$-field $\mathcal{B}$ of $\mathbb{R}^d$ (see [39] [48]).

In what follows by $E_{s,x}$ we denote expectation with respect to $P_{s,x}$ and by $\mathcal{R}$ the space of all measurable functions $\varrho : \mathbb{R}^d \to \mathbb{R}$ such that $\varrho(x) = (1 + |x|^2)^{-\alpha}$, $x \in \mathbb{R}^d$, for some $\alpha \geq 0$ such that $\int_{\mathbb{R}^d} \varrho(x) \, dx < \infty$.

**Proposition 2.1.** Let $\varrho \in \mathcal{R}$. Then there exist $0 < c \leq C$ depending only on $\lambda, \Lambda$ and $\varrho$ such that for any $s \in [0, T)$ and $\psi \in L_{1, \varrho}(Q_{s,T})$,

$$c \int_t^T \int_{\mathbb{R}^d} |\psi(\theta, x)| \, \varrho(x) \, d\theta \, dx \leq \int_t^T \int_{\mathbb{R}^d} E_{s,x} |\psi(\theta, X_{\theta})| \, \varrho(x) \, d\theta \, dx \leq C \int_t^T \int_{\mathbb{R}^d} |\psi(\theta, x)| \, \varrho(x) \, d\theta \, dx, \quad t \in [s, T].$$

**Proof.** Follows from Proposition 5.1 in Appendix in [3] and Aronson’s estimates (see [11] Theorem 7). \hfill $\Box$

Set $F_t^s = \sigma(X_u, u \in [s, t])$, $F_t^s = \sigma(X_u, u \in [T + s - t, T])$ and define $\mathcal{G}$ as the completion of $F_t^s$ with respect to the family $\mathcal{P} = \{P_{s, \mu} : \mu$ is a probability measure on $\mathcal{B}(\mathbb{R}^d)\}$, where $P_{s, \mu}() = \int_{\mathbb{R}^d} P_{s,x}(\cdot) \mu(dx)$, and define $G_t^s$ ($G_t^s$) as the completion of $F_t^s$ ($F_t^s$) in $\mathcal{G}$ with respect to $\mathcal{P}$. We will say that a family $A = \{A_{s,t}, 0 < s \leq t \leq T\}$ of random variables is an additive functional (AF) of $\mathbb{X}$ if $A_{s, t}$ is càdlàg $P_{s,x}$-a.s. for quasi-every $(s, x) \in Q_T$, $A_{s,t}$ is $G_t^s$-measurable for every $0 \leq s \leq t \leq T$ and $P_{s,x}(A_{s,t} = A_{s,u} + A_{u,t}, s \leq u \leq t \leq T) = 1$ for q.e. $(s, x) \in Q_T$ (for the definition of exceptional sets see Section [3]). If, in addition, $A_{s, t}$ has $P_{s,x}$-almost all continuous trajectories for q.e. $(s, x) \in Q_T$, then $A$ is called a continuous AF (CAF), and if $A_{s, t}$ is an increasing process under $P_{s,x}$ for q.e. $(s, x) \in Q_T$, it is called a positive AF (PAF). If $M$ is an AF such that for q.e. $(s, x) \in Q_T$, $E_{s,x}[M_{s,t}]^2 < \infty$ and $E_{s,x}M_{s,t} = 0$ for $t \in [s, T]$, it is called a martingale AF (MAF). Finally, we say that $A$ is an AF (CAF, increasing AF, MAF) in the strict sense if the corresponding property holds for every $(s, x) \in Q_T$.

From [40] Theorem 2.1 it follows that there exist a strict MAF $M = \{M_{s,t} : 0 \leq s \leq t \leq T\}$ of $\mathbb{X}$ and a strict CAF $A = \{A_{s,t} : 0 \leq s \leq t \leq T\}$ of $\mathbb{X}$ such that the quadratic variation $\langle A_{s, t}\rangle_T$ of $A_{s, t}$ on $[s, T)$ equals zero $P_{s,x}$-a.s. and

$$X_t - X_s = M_{s,t} + A_{s,t}, \quad t \in [s, T], \quad P_{s,x}$-a.s.$$  \hfill (2.1)

for each $(s, x) \in Q_T$. In particular, $X$ is a $(\{G_t^s\}, P_{s,x})$-Dirichlet process on $[s, T]$ for every $(s, x) \in Q_T$. Moreover, the above decomposition is unique and for each $(s, x) \in Q_T$ the co-variation process of the martingale $M_{s, t}$ is given by

$$\langle M_{s, t}^i, M_{s, t}^j\rangle_t = \int_s^t a_{ij}(\theta, X_{\theta}) \, d\theta, \quad t \in [s, T], \quad i, j = 1, \ldots, d$$

(see [40] for details).
For $0 \leq s \leq u \leq t \leq T$ and $x \in \mathbb{R}^d$ we set

$$\alpha_{u,t}^{s,x,i} = \sum_{j=1}^{d} \int_u^t \frac{1}{2} a_{ij}(\theta, X_\theta) p^{-1} \frac{\partial p}{\partial y_j}(s, x, \theta, X_\theta) d\theta, \quad \beta_{u,t}^{i} = \int_u^t b_i(\theta, X_\theta) d\theta.$$ 

From [42] it follows that for each $(s, x) \in Q_T$ the process $X$ admits under $P_{s,x}$ the following form of the Lyons-Zheng decomposition

$$X_t - X_u = \frac{1}{2} M_{u,t} + \frac{1}{2} (N_{s,T+s-t}^{s,x} - N_{s,T+s-u}^{s,x}) - \alpha_{u,t}^{s,x} + \beta_{u,t}^{i}, \quad s \leq u \leq t \leq T,$$

where $M_{s, \cdot}$ is the martingale of (2.1) and $N_{s, \cdot}^{s,x}$ is a $(\{G_t^s\}, P_{s,x})$-martingale such that

$$\langle N_{s, \cdot}^{s,x, i}, N_{s, \cdot}^{s,x, j} \rangle_t = \int_s^t a_{ij}(\theta, X_\theta) d\theta, \quad t \in [s, T], \quad i, j = 1, \ldots, d.$$

(Here and in the sequel, for a process $Y$ on $[s, T]$ and fixed measure $P_{s,x}$ we write $\hat{Y}_t = Y_{T+s-t}$ for $t \in [s, T]$).

Let $f \in (L^2(Q_T))^d$. Similarly to [31] we put

$$\int_s^t \hat{f}(\theta, X_\theta) d^s X_\theta \equiv - \int_s^t f(\theta, X_\theta)(dM_{s, \theta} + d\alpha_{s, \theta}^{s,x}) - \int_{T+s-t}^{t+s-u} \hat{f}(\theta, X_\theta) dN_{s, \theta}^{s,x}$$

for $s \leq u \leq t \leq T$ (the integrals on the right-hand side are well defined under the measure $P_{s,x}$ for a.e. $(s, x) \in Q_T$ (see [17], Proposition 7.6)).

We now give definitions of solutions of BSDEs and RBSDEs associated with $X$ and recall some known results on such equations to be used further on.

Write

$$B_{s, t} = \int_s^t \sigma^{-1}(\theta, X_\theta) dM_{s, \theta}, \quad t \in [s, T],$$

where $M$ is the MAF of (2.1). Notice that $\{B_{s, t}\}_{t \in [s, T]}$ is a Brownian motion under $P_{s,x}$.

**Definition 2.2.** A pair $(Y^{s,x}, Z^{s,x})$ of $\{G_t^s\}$-adapted stochastic processes on $[s, T]$ is a solution of BSDE$_{s,x}$($\varphi, f$) if

(i) $Y_t^{s,x} = \varphi(X_T) + \int_t^T f(\theta, X_\theta, Y_{\theta}^{s,x}, Z_{\theta}^{s,x}) d\theta - \int_t^T Z_{\theta}^{s,x} dB_{s, \theta}, \ t \in [s, T], \ P_{s,x}$-a.s.,

(ii) $E_{s,x} \sup_{s \leq t \leq T} |Y_t^{s,x}|^2 < \infty, E_{s,x} \int_s^T |Z_t^{s,x}|^2 dt < \infty.$

**Definition 2.3.** A triple $(Y^{s,x}, Z^{s,x}, K^{s,x})$ of $\{G_t^s\}$-adapted stochastic processes on $[s, T]$ is a solution of RBSDE$_{s,x}$($\varphi, f, h$) if

(i) $Y_t^{s,x} = \varphi(X_T) + \int_t^T f(\theta, X_\theta, Y_{\theta}^{s,x}, Z_{\theta}^{s,x}) d\theta + K_t^{s,x} - K_T^{s,x} - \int_t^T Z_{\theta}^{s,x} dB_{s, \theta}, \ t \in [s, T], \ P_{s,x}$-a.s.,

(ii) $E_{s,x} \sup_{s \leq t \leq T} |Y_t^{s,x}|^2 < \infty, E_{s,x} \int_s^T |Z_t^{s,x}|^2 dt < \infty,$

(iii) $Y_t^{s,x} \geq h(t, X_t), \ P_{s,x}$-a.s. for a.e. $t \in [s, T],$
(iv) $K_{s,x}^n$ is a càdlàg increasing process such that $K_{s,x}^n = 0$, $E_{s,x}|K_{s,x}^n|^2 < \infty$ and 
\[ \int_s^T (Y_{t,x}^{s,x} - H_t) \, dK_{s,x}^n = 0, \quad P_{s,x}\text{-a.s.} \]
for every càdlàg process $H$ such that $E_{s,x} \sup_{s \leq t \leq T} |H_t|^2 < \infty$ and $h(t,X_t) \leq H_t \leq Y_{t,x}^{s,x}$, $P_{s,x}$ a.s. for a.e. $t \in [s,T]$.

It is worth mentioning that the filtration $\{\mathcal{G}_t\}$ is not Brownian, nonetheless it has the representation property with respect to $B$. Namely, in [22] it is proved that if $\{M_{s,t} : t \in [s,T]\}$ is a $(\{\mathcal{G}_t\}, P_{s,x})$-square-integrable martingale for some $(s,x) \in Q_T$ then there exists a predictable square-integrable process $\{H_{s,x}^n\}_{t \in [s,T]}$ such that

\[ M_{s,t} = \int_s^T H_{s,x}^n \, dB_{s,t}, \quad t \in [s,T], \quad P_{s,x}\text{-a.s.} \]

This allows one to use results on BSDEs proved in the standard framework in which the forward diffusion process corresponds to a non-divergent form operator.

**Theorem 2.4.** Assume that (H1)–(H3) are satisfied.

(i) If for some $(s,x) \in Q_T$,

\[ E_{s,x} \text{esssup}_{s \leq t \leq T} |h^+(t,X_t)|^2 + E_{s,x} \int_s^T |g(t,X_t)|^2 \, dt < \infty, \tag{2.2} \]

then there exists a unique solution $(Y_{s,x}, Z_{s,x}, K_{s,x})$ of RBSDE$_{s,x}(\varphi, f, h)$. Moreover, if the pair $(Y_{s,x}^n, Z_{s,x}^n)$, $n \in \mathbb{N}$, is a solution of BSDE$_{s,x}(\varphi, f + n(y - h)^+)$, then $(Y_{s,x}^n)_{n \in \mathbb{N}}$ is increasing and

(a) there exists $C > 0$ depending neither on $n,m \in \mathbb{N}$ nor $s,x$ such that

\[ E_{s,x} \sup_{s \leq t \leq T} |Y_{t,x}^{s,x,n}|^2 + E_{s,x} \int_s^T |Z_{t,x}^{s,x,n}|^2 \, dt + E_{s,x} |K_{T,x}^{s,x,n}|^2 \]

\[ \leq C \left( E_{s,x} |\varphi(X_T)|^2 + E_{s,x} \sup_{s \leq t \leq T} |h^+(t,X_t)|^2 + E_{s,x} \int_s^T |g(t,X_t)|^2 \, dt \right), \]

where

\[ K_{t,x}^{s,x,n} = \int_s^t n(Y_{\theta,x}^{s,x,n} - h(\theta,X_\theta))^{-} \, d\theta, \quad t \in [s,T], \quad P_{s,x}\text{-a.s.} \]

(b) $Y_{t,x}^{s,x,n} \to Y_{t,x}^{s,x}$ for every $t \in [s,T]$, $P_{s,x}$-a.s., and for every $p \in [1,2),$

\[ E_{s,x} \int_s^T |Y_{t,x}^{s,x,n} - Y_{t,x}^{s,x}|^2 \, dt + E_{s,x} \int_s^T |Z_{t,x}^{s,x,n} - Z_{t,x}^{s,x}|^p \, dt \to 0. \]

(ii) If (2.2) is satisfied and $t \mapsto h(t,X_t)$ is continuous under $P_{s,x}$ for some $(s,x) \in Q_T$ then

\[ E_{s,x} \sup_{s \leq t \leq T} |Y_{t,x}^{s,x,n} - Y_{t,x}^{s,x}|^2 + E_{s,x} \int_s^T |Z_{t,x}^{s,x,n} - Z_{t,x}^{s,x}|^2 \, dt \]

\[ + E_{s,x} \sup_{s \leq t \leq T} |K_{t,x}^{s,x,n} - K_{t,x}^{s,x}|^2 \to 0. \tag{2.3} \]

Proof. See [32] for the proof of (i) and [12] for the proof of (ii).

□

**Corollary 2.5.** Let assumptions (H1)–(H3) hold. If (2.2) is satisfied for some $(s,x) \in Q_T$ then for every sequence $\{n\}$ there is a subsequence $\{n'\}$ such that $K_{t,x}^{s,x,n'} \to K_{t,x}^{s,x}$ for every $t \in [s,T]$, $P_{s,x}$-a.s.. In particular, $dK_{s,x}^{s,x,n'} \to dK_{s,x}^{s,x}$ weakly on $[s,T]$ in probability $P_{s,x}$.
3 Cauchy problem and BSDEs

For the purposes of Sections 4 and 5 in this section we refine slightly results of [40] on stochastic representation of solutions of the Cauchy problem.

**Definition 3.1.** The parabolic capacity of an open subset $B$ of $\hat{Q}_T$ is given by

$$
cap_L(B) = \int_0^T P_{s,m}(\{\exists t \in (s, T) : (t, X_t) \in B\}) \, ds, \quad (3.1)
$$

where $m$ is the Lebesgue measure on $\mathbb{R}^d$ and

$$
P_{s,m}(\Gamma) = \int_{\mathbb{R}^d} P_{s,x}(\Gamma) \, dx, \quad \Gamma \in \mathcal{G}.
$$

It is known (see [13, Theorem A.1.2, Lemma A.2.5, A.2.6]) that this set function can be extended to the Choquet capacity on $\mathcal{B}(\hat{Q}_T)$ in such a way that (3.1) holds for every compact set $K \subset \hat{Q}_T$. We further extend this capacity to $Q_T$ by putting $\cap_L(\{0\} \times B) = m(B)$ for every $B \in \mathcal{B}(\mathbb{R}^d)$.

From now on we say that some property is satisfied quasi-everywhere (q.e. for short) if it is satisfied except for some Borel subset of $Q_T$ of capacity $\cap_L$ zero.

**Remark 3.2.** Let $h, g : Q_T \to \mathbb{R}$ be measurable functions. Let us observe that if the condition

$$
E_{s,x} \text{ess sup}_{s \leq t \leq T} |h(t, X_t)| + E_{s,x} \int_s^T |g(t, X_t)| \, dt < \infty \quad (3.2)
$$

is satisfied for a.e. $(s, x) \in Q_T$ then it is satisfied for q.e. $(s, x) \in Q_T$. To see this, let us set $w(s, x) = E_{s,x} \text{ess sup}_{s \leq t \leq T} |h(t, X_t)|$, $\tau = \inf\{t \in (s, T) : (t, X_t) \in K\} \land T$, where $K \subset \{w = \infty\}$ is a compact set. Since $(X, P_{s,x})$ is a Feller process we conclude that $\tau$ is a stopping time and $(X, P_{s,x})$ is a strong Markov process. By the strong Markov property with random shift,

$$
P_{s,x}(\tau < T) \leq P_{s,x}(E_{\tau,X_{\tau}} \text{ess sup}_{\tau \leq t \leq T} |h(t, X_t)| = \infty, \tau < T) = P_{s,x}(E_{s,x} (\text{ess sup}_{\tau \leq t \leq T} |h(t, X_t)||\mathcal{G}^\tau_{s,x}) = \infty, \tau < T),
$$

which by the assumption equals zero for a.e. $(s, x) \in Q_T$. Thus, $\cap_L(K) = 0$ for any compact subset $K$ in $\{w = \infty\}$. Since $\cap_L$ is the Choquet capacity, it follows that $\cap_L(\{w = \infty\}) = 0$. The proof for the term involving $g$ is analogous.

**Definition 3.3.** We say that $u : Q_T \to \mathbb{R}$ is quasi-continuous (quasi-càdlàg) if it is Borel measurable and the process $t \mapsto u(t, X_t)$ has continuous (càdlàg) trajectories under the measure $P_{s,x}$ for q.e. $(s, x) \in Q_T$.

**Proposition 3.4.** If $u, \bar{u} \in L_{2,\phi}(Q_T)$ are quasi-càdlàg and $u = \bar{u}$ a.e. then $u = \bar{u}$ q.e.

**Proof.** Suppose that $\cap_L(\{u \neq \bar{u}\} \cap \hat{Q}_T) > 0$. Since $\cap_L$ is the Choquet capacity, there is $K \subset \{u \neq \bar{u}\} \cap \hat{Q}_T$ such that $K$ is compact and $\cap_L(K) > 0$. Hence there is $A \cap \hat{Q}_T$ such that $m_T(A) > 0$ and for every $(s, x) \in A$,

$$
P_{s,x}(\{\omega : \exists t \in (s, T) : (t, X_t) \in K\}) > 0.
$$
Since trajectories of the processes \( t \mapsto u(t, X_t), t \mapsto \bar{u}(t, X_t) \) are càdlàg, it follows that for every \((s, x) \in A,\)

\[
0 < E_{s,x} \int_s^T |(u - \bar{u})(t, X_t)|^2 dt = \int_s^T \int_{\mathbb{R}^d} |u - \bar{u}|^2(t, y)p(s, x, t, y) dt dy.
\]

Multiplying the above inequality by \( \frac{1}{2} \), integrating with respect to \( x \) and using Proposition 2.1 we get \( 0 < \|u - \bar{u}\|_{2, \sigma, T}^2 \), which contradicts the fact that \( u = \bar{u} \) a.e.. From the above equality with \( s = 0 \) one can conclude also that \( \text{cap}_L(\{u \neq \bar{u}\} \cap \{(0) \times \mathbb{R}^d\}) = 0 \), which completes the proof.

\[\Box\]

**Definition 3.5.** Let \( \Phi \in \mathcal{W}_\rho' \). (i) We say that \( u \in L^2(0, T; H^1_\rho) \) is a weak solution of the Cauchy problem

\[
\frac{\partial u}{\partial t} + L_t = -\Phi, \quad u(T) = \varphi
\]

(PDE(\(\varphi, \Phi\)) for short) if

\[
\langle u, \frac{\partial \eta}{\partial t} \rangle_{\sigma, T} - \langle L_t u, \eta \rangle_{\sigma, T} = \langle \varphi, \eta(T) \rangle_{2, \sigma} + \langle \Phi, \eta \rangle_{\sigma, T}
\]

for every \( \eta \in \mathcal{W}_\rho \) such that \( \eta(0) = 0 \), where

\[
\langle L_t u, \eta \rangle_{\sigma, T} = -\frac{1}{2} \langle a \nabla u, \nabla (\eta \varphi) \rangle_{2, T} + \langle b, \eta \varphi^2 \nabla u \rangle_{2, T}.
\]

(ii) \( u \in \mathcal{W}_\rho \) is a strong solution of PDE(\(\varphi, \Phi\)) if

\[
\langle \frac{\partial u}{\partial t}, \eta \rangle_{\sigma, T} + \langle L_t u, \eta \rangle_{\sigma, T} = -\langle \Phi, \eta \rangle_{\sigma, T}, \quad u(T) = \varphi
\]

for every \( \eta \in L^2(0, T; H^1_\rho) \).

It is known that for every \( \Phi \in \mathcal{W}_\rho', \varphi \in L^2(\mathbb{R}^d) \) there exists a unique weak solution of PDE(\(\varphi, \Phi\)) (see [11]).

Let \( n \in \mathbb{N} \). In the sequel we will use the symbol \( T_n \) to denote the truncation operator

\[
T_n(s) = \max\{-n, \min\{s, n\}\}, \quad s \in \mathbb{R}.
\]

**Proposition 3.6.** Assume that (H1)–(H3) are satisfied.

(i) If

\[
\forall K \subset \subset (0, T) \times \mathbb{R}^d \sup_{(s,x) \in K} E_{s,x} \int_s^T |g(t, X_t)|^2 dt < \infty
\]

then there exists a unique strong solution \( u \in \mathcal{W}_\rho \cap C(Q_T) \) of PDE(\(\varphi, f\)) and for each \((s, x) \in Q_T\) the pair

\[
(Y^{s,x}_t, Z^{s,x}_t) = (u(t, X_t), \sigma \nabla u(t, X_t)), \quad t \in [s, T]
\]

is a unique solution of BSDE_{s,x}(\(\varphi, f\)).
(ii) There exists a quasi-continuous version \( \tilde{u} \) of the unique strong solution \( u \in W_\varphi \) of PDE(\( \varphi, f \)) such that

\[
E_{s,x} \int_s^T |g(t, X_t)|^2 dt < \infty \tag{3.6}
\]

for some \((s, x) \in Q_T\) then the pair \((\bar{u}(t, X_t), \sigma \nabla \bar{u}(t, X_t))\), \(t \in [s, T]\), is a unique solution of BSDE_{s,x}(\(\varphi, f\)).

Proof. To prove (i) it suffices to repeat step by step the proof of [40, Theorem 6.1] with the usual norm in \(L_2(Q_T)\) replaced by the norm \(\| - \|_{2, \theta}\) in \(L_{2, \theta}(Q_T)\). To prove (ii), let us consider solutions \(u_{n,m}\) of the Cauchy problems

\[
\frac{\partial u_{n,m}}{\partial t} + L_t u_{n,m} = -T_n(f_u^+) + T_m(f_u^-), \quad u_{n,m}(T) = \varphi.
\]

By (i), \(u_{n,m}\) is continuous in \(Q_T\) for each \(n, m \in \mathbb{N}\), and for every \((s, x) \in Q_T\), \((u_{n,m}(t, X_t), \sigma \nabla u_{n,m}(t, X_t))\), \(t \in [s, T]\), is a solution of BSDE_{s,x}(\(\varphi, T_n(f_u^+) - T_m(f_u^-)\)). From this it follows in particular that

\[
\text{for every } (s, x) \in Q_T. \text{ Using Itô's formula and the Burkholder-Davis-Gundy inequality one can deduce from (3.7) that for any } n, k, l \in \mathbb{N},
\]

\[
E_{s,x} \sup_{s \leq t \leq T} |(u_{n,k} - u_{n,l})(t, X_t)|^2 + E_{s,x} \int_s^T |\sigma \nabla (u_{n,l} - u_{n,k})(t, X_t)|^2 dt
\]

\[
\leq CE_{s,x} \int_s^T |T_k(f_u^-) - T_l(f_u^-)|^2 dt. \tag{3.8}
\]

Moreover, since \(T_k(f_u^-) \leq T_l(f_u^-)\) a.e. if \(k \leq l\), it follows from (3.7) that for each \(n \in \mathbb{N}\) the sequence \(\{u_{n,m}\}_{m \in \mathbb{N}}\) is decreasing. Set \(F_1^{-} = F_1^{+} \cap F_1^{-}\), where

\[
F_1^{+} = \{(s, x) \in Q_T; E_{s,x} \int_s^T f_u^{+}(t, X_t) dt < \infty\},
\]

\[
F_1^{-} = \{(s, x) \in Q_T; E_{s,x} \int_s^T f_u^{-}(t, X_t) dt < \infty\},
\]

and let

\[F_2 = \{(s, x) \in Q_T; E_{s,x} \int_s^T |f_u(t, X_t)|^2 dt < \infty\}.
\]

We consider separately two cases: \(\lim_{m \to \infty} u_{n,m}(s, x) = \infty\) or \(\lim_{m \to \infty} u_{n,m}(s, x) = -\infty\). By (3.7), the last case holds true iff \((s, x) \notin F_1^{+}\). Put \(\bar{u}_n(s, x) = \lim_{m \to \infty} u_{n,m}(s, x)\) for \((s, x) \in F_1^{+}\) and \(\bar{u}_n(s, x) = 0\) for \((s, x) \notin F_1^{+}\). By (3.8), \((\bar{u}_n(t, X_t), \sigma \nabla \bar{u}_n(t, X_t))\), \(t \in [s, T]\), is a solution of BSDE_{s,x}(\(\varphi, T_n(f_u^{+}) - f_u^{-}\)) for every \((s, x) \in F_2\) and \(\bar{u}_n\) is a strong solution of PDE(\(\varphi, T_n(f_u^{+}) - f_u^{-}\)) (for the last statement see [20]). By the same method as in the case of \(\{u_{n,m}\}_{m \in \mathbb{N}}\) one can show that for every \((s, x) \in F_1^{+}\) the limit of \(\{\bar{u}_n(s, x)\}\) exists and is finite. We may therefore put \(\bar{u}(s, x) = \lim_{n \to \infty} \bar{u}_n(s, x)\) for
(s,x) ∈ F_1 and įu(s,x) = 0 for (s,x) ∉ F_1. Using once again Itô’s formula and the
Burkholder-Davis-Gundy inequality we obtain

\[ E_{s,x} \sup_{s \leq t \leq T} |(\tilde{u}_k - \tilde{u}_l)(t,X_t)|^2 + E_{s,x} \int_s^T |\sigma \nabla(\tilde{u}_l - \tilde{u}_k)(t,X_t)|^2 \, dt \]
\[ \leq CE_{s,x} \int_s^T |T_k(f^+_u) - T_l(f^+_u)|^2(t,X_t) \, dt. \]  

(3.9)

From this it follows that for every (s,x) ∈ F_2 the pair (\tilde{u}(t,X_t),\sigma \nabla \tilde{u}(t,X_t)), t ∈ [s,T],

is a solution of BSDE_{s,x}(\varphi,f). By a priori estimates for BSDEs (see, e.g., [30]), if

(3.6) is satisfied then (s,x) ∈ F_2. The fact that \tilde{u} is a strong solution of PDE(\varphi,f) is

standard (see once again [20]). Finally, from Proposition 2.1 and Remark 3.2 it follows

that cap_L(F^+_2) = 0 which shows that \tilde{u} is quasi-continuous. □

**Corollary 3.7.** The representation (3.5) holds for q.e. (s,x) ∈ Q_T.

Proof. Follows from Proposition 2.1 and Remark 3.2 □

**Remark 3.8.** An inspection of the proof of Proposition 3.6 shows that if we set

\[ \tilde{u}(T,\cdot) = \varphi, \tilde{u}(s,x) = 0 \quad \text{for} \quad (s,x) \in Q_T \setminus F_1 \]

and

\[ \tilde{u}(s,x) = E_{s,x} \varphi(X_T) + E_{s,x} \int_s^T f_u(t,X_t) \, dt \]

for (s,x) ∈ F_1 then \tilde{u} is a quasi-continuous version of a weak solution of PDE(\varphi,f).

**Remark 3.9.** Condition (3.4) is satisfied if g satisfies the polynomial growth condition

or g ∈ L^{p,q,\delta}(Q_T) with p, q ∈ (2, ∞) such that \( \frac{2}{q} + \frac{4}{p} < 1 \). The first statement

is obvious. Sufficiency of the second condition follows from Hölder’s inequality and upper

Aronson’s estimate for the transition density p (see [11]).

4 Parabolic potentials, soft measures and additive functionals

In this section we present elements of parabolic potential theory for L_t to be needed

in Section 5 and we describe correspondence between smooth measures and time-
inhomogeneous additive functionals of the Markov family X associated with L_t. Let us

mention that known results on the topic proved in the framework of Dirichlet forms
determined by L_t (see [28, 46]) are not directly applicable to our situation because

contrary to [28, 46] we consider parabolic potentials associated with the nonlinear operator

u ↦ Lu = \frac{\partial u}{\partial t} + L_t u + f_u acting on functions u : Q_T → R from L^2_2(0,T;H_0^{1,2})

which not necessarily vanish for t = 0 or t = T. As a result, potentials need not be positive.
Moreover, since L is parabolic, potentials need not have quasi-continuous versions. The last difficulty is particularly significant because forces us to go beyond

the class of continuous functionals of X.

In what follows, given a function u : Q_T → R^d we will extend it in a natural way to

the function on \([-T,2T] × R^d\), still denoted by u, by putting

\[ u(t,x) = \begin{cases} 
  u(0,x), & t \in [-T,0], \\
  u(t,x), & t \in [0,T], \\
  u(T,x), & t \in [T,2T]. 
\end{cases} \]
Let \( u_\varepsilon, \varepsilon > 0 \), denote Steklov’s mollification of \( u \) with respect to the time variable, that is

\[
u_\varepsilon(t, x) = \frac{1}{\varepsilon} \int_0^\varepsilon u(t - s, x) \, ds, \quad (t, x) \in [0, T] \times \mathbb{R}^d.
\]

Recall that if \( u \in L_2(0, T; H^1_0) \) then \( u_\varepsilon \in W^{1,1}_2(Q_T) \) and \( \nabla u_\varepsilon \to \nabla u, u_\varepsilon \to u \) in \( L_{2,q}(Q_T) \).

In what follows by \( D'(\hat{Q}_T) \) we denote the space of Schwartz distributions on \( \hat{Q}_T \).

**Lemma 4.1.** Let \( u \in L_2(0, T; H^1_0) \) and let \( \mu \) be a Radon measure on \( Q_T \). If

\[
\frac{\partial u}{\partial t} + Lu = -f_u - \mu \quad \text{in} \quad D'(\hat{Q}_T),
\]

then for every \( \varepsilon \in (0, T) \),

\[
\frac{\partial u_\varepsilon}{\partial t} + Lu_\varepsilon = -\text{div}((a\nabla u_\varepsilon - a\nabla u_\varepsilon) - ((b\nabla u_\varepsilon - b\nabla u_\varepsilon) - (f_u)_\varepsilon - \mu_\varepsilon \quad \text{in} \quad D'(\hat{Q}_T),
\]

where

\[
\mu_\varepsilon(\eta) = \frac{1}{\varepsilon} \int_0^\varepsilon \left( \int_{\varepsilon - \theta}^{T - \theta} \eta(s + \theta, x) \, d\mu(s, x) \right) \, d\theta.
\]

**Proof.** Write \( \eta_\theta(s) = \eta(s + \theta) \). By Fubini’s theorem, for every \( \eta \in C^\infty_0(\hat{Q}_T) \) we have

\[
\int_\varepsilon^T \left( \frac{\partial u_\varepsilon}{\partial s}(s, \eta_\theta(s)) \right) \, ds = -\frac{1}{\varepsilon} \int_0^\varepsilon \left( \int_{\varepsilon - \theta}^{T - \theta} (u(s), \frac{\partial \eta_\theta}{\partial s}(s)) \right) \, d\theta \nonumber
\]

\[
= -\frac{1}{\varepsilon} \int_0^\varepsilon \int_0^T (u(s), \frac{\partial \eta_\theta}{\partial s}(s)) \, d\theta = \frac{1}{\varepsilon} \int_0^\varepsilon \frac{\partial u}{\partial s}(\eta_\theta) \, d\theta
\]

\[
= \frac{1}{\varepsilon} \int_0^\varepsilon \left( \int_{\varepsilon - \theta}^{T - \theta} \frac{1}{2} (a(s)\nabla u(s), \nabla \eta_\theta(s)) \right) \, d\theta
\]

\[
- \frac{1}{\varepsilon} \int_0^\varepsilon \int_{\varepsilon - \theta}^{T - \theta} (b(s)\nabla u(s), \eta_\theta(s)) \, d\theta - \frac{1}{\varepsilon} \int_0^\varepsilon \int_{\varepsilon - \theta}^{T - \theta} (f_u(s), \eta_\theta(s)) \, d\theta
\]

\[
- \frac{1}{\varepsilon} \int_0^\varepsilon \left( \int_{\varepsilon - \theta}^{T - \theta} \int_{\mathbb{R}^d} \eta_\theta(s, x) \, d\mu(s, x) \right) \, d\theta = \frac{1}{2} \int_\varepsilon^T ((a\nabla u_\varepsilon(s), \nabla \eta(s)) \, ds
\]

\[
- \int_\varepsilon^T ((b\nabla u_\varepsilon(s), \eta(s)) \, ds - \int_\varepsilon^T ((f_u)_\varepsilon(s), \eta(s)) \, ds - \mu_\varepsilon(\eta),
\]

from which the result follows. \( \square \)

Write \( \mathcal{L}u = \frac{\partial u}{\partial t} + Lu + f_u \). We define the set of parabolic potentials associated with \( \mathcal{L} \) by

\[
\mathcal{P} = \{ u \in L_2(0, T; H^1_0) : \mathcal{L}u \leq 0 \text{ in } D'(\hat{Q}_T), \ \text{esssup}_{t \in [0, T]} \| u(t) \|_{2,q} < \infty \}
\]

and we set

\[
\| u \|_\mathcal{P} = \text{esssup}_{0 \leq t \leq T} \| u(t) \|_{2,q} + \| \nabla u \|_{2,q,T}.
\]

It is worth mentioning that \( u \in \mathcal{P} \) is not necessarily positive as it is usually assumed (see \cite{36} for linear case). Moreover, using Tanaka’s formula it is easy to check that in general \( u^+, u^- \) do not belong to \( \mathcal{P} \).
Lemma 4.2. Assume \( (H2b) \). If \( u \in \mathcal{P} \) then

\[
\int_{Q_T} \left( E_{s,x} \text{esssup}_{s \leq t \leq T} |u(t, X_t)|^2 \right) \varrho^2(x) \, ds \, dx \leq C(\|u\|_{\mathcal{F}}^2 + \|g\|_{2, \theta, T}^2).
\]

Proof. Fix \( \delta \in (0, T) \), put \( \mu = -Lu \in \mathcal{D}'(Q_T) \) and define \( \mu_n \) as in Lemma 4.1. Then \( \mu_n \equiv \mu_{1/n} \geq 0 \) and \( \mu_n \in L_2(\delta, T; H_{\theta}^{-1}) \) for \( n > \delta^{-1} \). Therefore from [17] Theorems 3.1, 5.1 it follows that for \( n > \delta^{-1} \) there exists PCAF \( K^n \) and a quasi-continuous version of \( u_n \) (still denoted by \( u_n \)) such that

\[
u_n(t, X_t) = u_n(s, x) - \frac{1}{2} \int_s^t a^{-1}(a \nabla u_n - a \nabla u_n)(\theta, X_\theta) \, d^\ast X_\theta
\]

\[
- \int_s^t (b \nabla u_n - b \nabla u_n)(\theta, X_\theta) \, d\theta - \int_t^T (f_n u_n(\theta, X_\theta) \, d\theta - K_{n,t}^n
\]

\[
+ \int_t^T \sigma \nabla u_n(\theta, X_\theta) \, dB_{s,\theta}, \quad t \in [s, T], \quad P_{s,x}-a.s.
\]

(4.1)

for a.e. \( (s, x) \in Q_{\delta T} \). By Proposition 2.1

\[
\int_{Q_T} \left( E_{s,x} \int_s^T \left( |u_n - u(t, X_t)|^2 + |\sigma \nabla (u_n - u)(t, X_t)|^2 \right) \, dt \right) \varrho^2(x) \, dx \, ds
\]

\[
+ \int_{Q_T} \left( E_{s,x} \int_s^T \left( |(b \nabla u_n - b \nabla u_n)(t, X_t)|^2 + |(f_n u_n - f_n)(t, X_t)|^2 \right) \, dt \right) \varrho^2(x) \, dx \, ds
\]

\[
\leq C(\|u_n - u\|_{2, \theta, T}^2 + \|\nabla (u_n - u)\|_{2, \theta, T}^2 + \|b \nabla u_n - b \nabla u_n\|_{2, \theta, T}^2 + \|f_n u_n - f_n\|_{2, \theta, T}^2).
\]

Hence there is a subsequence (still denoted by \( n \)) such that

\[
(u_n(\cdot, X), \sigma \nabla u_n(\cdot, X)) \to (u(\cdot, X), \sigma \nabla u(\cdot, X))
\]

in \( L_2([s, T] \times \Omega, \lambda \otimes P_{s,x}) \otimes L_2([s, T] \times \Omega, \lambda \otimes P_{s,x}) \) for a.e. \( (s, x) \in Q_{\delta T} \). Consequently, passing to the limit in (4.1) and using [17] Proposition 7.6 and properties of Steklov's mollification we conclude that for a.e. \( (s, x) \in Q_{\delta T} \) there is a process \( K^{s,x} \) such that

\[
u(t, X_t) = u(s, x) - \int_s^t f_u(\theta, X_\theta) \, d\theta - K_t^{s,x} + \int_s^t \sigma \nabla u(\theta, X_\theta) \, dB_{s,\theta}
\]

(4.2)

P_{s,x}-a.s. for a.e. \( t \in [s, T] \). Since \( \delta \in (0, T) \) can be chosen arbitrarily small, (4.2) holds true \( P_{s,x}\)-a.s. for a.e. \( (s, x) \in Q_T \) and a.e. \( t \in [s, T] \). Let \( \{Y_t^{s,x}, t \in [s, T]\} \), \( \{\tilde{K}_t^{s,x}, t \in [s, T]\} \) denote càdlàg modifications, in \( L_2([s, T] \times \Omega, \lambda \otimes P_{s,x}) \), of the processes \( t \mapsto u(t, X_t) \) and \( K^{s,x} \), respectively (existence of such modifications follows from [15] Theorem 3.13) because for a.e. \( (s, x) \in Q_T \) there is \( T_{s,x} \subset [s, T] \) such that the Lebesgue measure of the set \( [s, T] \setminus T_{s,x} \) equals zero and the process \( \{K_{t}^{s,x}, t \in T_{s,x}\} \) is a submartingale under \( P_{s,x} \), and let \( \tilde{K}_t^{s,x} = \lim_{t' \uparrow T} \tilde{K}_{t'}^{s,x}, Y_t^{s,x} = \lim_{t' \uparrow T} Y_{t'}^{s,x} \) (in both cases the convergence holds \( P_{s,x}\)-a.s.). From (4.2) we get

\[
Y_t^{s,x} = Y_{T}^{s,x} + \int_t^T f_u(\theta, X_\theta) \, d\theta + \tilde{K}_t^{s,x} - \tilde{K}_T^{s,x}
\]

\[
- \int_t^T \sigma \nabla u(\theta, X_\theta) \, dB_{s,\theta}, \quad t \in [s, T], \quad P_{s,x}-a.s.
\]

(4.3)
for a.e. \((s, x) \in Q_T\). Since \(\int_{Q_T} (E_{s,x} \int_s^T |u(t, X_t) - Y_t^{s,x}|^2 dt) \varrho^2(x) ds dx = 0\), for a.e. \(s \in [0, T]\) one can find \(\{t_n^s\}\) such that \(t_n^s \uparrow T\), \(Y_{t_n^s}^{s,x} = u(t_n^s, X_{t_n^s})\) and \((4.2)\) holds in \(t_n^s\) in place of \(t P_{s,x}\)-a.s. for a.e. \(x \in \mathbb{R}^d\). Since we can assume also that \(K_{t_n^s}^{s,x} = \tilde{K}_{t_n^s}^{s,x}\) and \(\|u(t_n^s)\|_{2, \varrho} \leq \text{esssup}_{t \in [0, T]} \|u(t)\|_{2, \varrho}\), it follows from \((4.2)\) that

\[
E_{s,x} |\tilde{K}_{T}^{s,x}|^2 = \lim_{n \to \infty} E_{s,x} |\tilde{K}_{t_n^s}^{s,x}|^2 = \lim_{n \to \infty} E_{s,x} |K_{t_n^s}^{s,x}|^2
\]

\[
\leq C \lim_{n \to \infty} \left( |u(s, x)|^2 + E_{s,x} \int_s^{t_n^s} |f_u(\theta, X_\theta)|^2 d\theta + E_{s,x} |u(t_n^s, X_{t_n^s})|^2 + E_{s,x} \int_s^{t_n^s} |\sigma \nabla u(\theta, X_\theta)|^2 d\theta \right),
\]

hence that

\[
\int_{Q_T} E_{s,x} |\tilde{K}_{T}^{s,x}|^2 \varrho^2(x) ds dx \leq C (\text{esssup}_{t \in [0, T]} \|u(t)\|_{2, \varrho}^2 + \|g\|_{2, \varrho}^2 + \|u\|_{2, \varrho, T}^2 + \|\nabla u\|_{2, \varrho, T}^2) \tag{4.4}
\]

by Fatou’s lemma and Proposition 2.1. Moreover, for a.e. \(s \in [0, T]\),

\[
\int_{\mathbb{R}^d} E_{s,x} |Y_T^{s,x}|^2 \varrho^2(x) dx \leq \lim_{n \to \infty} \int_{\mathbb{R}^d} E_{s,x} |u(t_n^s, X_{t_n^s})|^2 \varrho^2(x) dx \leq C \lim_{n \to \infty} \|u(t_n^s)\|_{2, \varrho}^2 \leq C (\text{esssup}_{t \in [0, T]} \|u(t)\|_{2, \varrho}^2) \tag{4.5}
\]

From the above, \((4.3)\), \((4.4)\) and again Proposition 2.1 we conclude that

\[
\int_{Q_T} (E_{s,x} \text{esssup}_{s \leq t \leq T} |Y_t^{s,x}|^2) \varrho^2(x) ds dx \leq (\text{esssup}_{t \in [0, T]} \|u(t)\|_{2, \varrho}^2 + \|g\|_{2, \varrho}^2 + \|u\|_{2, \varrho, T}^2 + \|\nabla u\|_{2, \varrho, T}^2). \tag{4.6}
\]

Finally,

\[
E_{s,x} \text{esssup}_{s \leq t \leq T} |Y_t^{s,x}|^2 = E_{s,x} \lim_{p \to \infty} \left( \int_s^T |Y_t^{s,x}|^{2p} dt \right)^{1/p} = \lim_{p \to \infty} E_{s,x} \left( \int_s^T |u(t, X_t)|^{2p} dt \right)^{1/p} = E_{s,x} \text{esssup}_{s \leq t \leq T} |u(t, X_t)|^2, \tag{4.7}
\]

which when combined with \((4.6)\) proves the proposition. \(\square\)

**Definition 4.3.** Let \(\mu\) be a positive Radon measure on \(Q_T\) and let \(K\) be a PAF. We say that \(\mu\) corresponds to \(K\) (or \(K\) corresponds to \(\mu\)) and we write \(\mu \sim K\) iff for quasi-every \((s, x) \in Q_T\):

\[
E_{s,x} \int_s^T f(t, X_t) dK_{s,t} = \int_{Q_T} f(t, y) p(s, x, t, y) d\mu(t, y) \tag{4.8}
\]

for all \(f \in \mathcal{B}^+(Q_T)\).
Of course, if $\mu_1 \sim K$, $\mu_2 \sim K$ then $\mu_1 = \mu_2$. Also, if $\mu \sim K^1$ and $\mu \sim K^2$ then $K^1 = K^2$ (see [38]), so the above correspondence is one-to-one.

Given a measure $\mu$ on $Q_T$ and $t \in [0,T]$ we will denote by $\mu(t)$ the measure on $\mathbb{R}^d$ defined by $\mu(t)(B) = \mu(\{t\} \times B)$ for $B \in \mathcal{B}(\mathbb{R}^d)$.

**Remark 4.4.** It is known (see, e.g., [43]) that if the Markov process $(X,Q_{s,x})$ is associated with the operator

$$A_t = \sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}(a_{ij}(t,x)\frac{\partial}{\partial x_j}),$$

(4.9)

then for every $(s,x) \in Q_T$, $\frac{dQ_{s,x}}{dP_{s,x}} = Z_T$, where the process $Z$ is a solution of the SDE

$$dZ_t = b(t,X_t)\sigma^{-1}(t,X_t)Z_t \, dB_{s,t}, \quad Z_0 = 1$$

under the measure $P_{s,x}$. It follows immediately that for every $p \geq 1$, $\sup_{(s,x) \in Q_T} E_{s,x} Z_T^p < \infty$.

**Lemma 4.5.** Let $\{T_m\} \subset (0,T)$, $T_m \nearrow T$, $\varphi_m \to \varphi$ weakly in $L^2(\mathbb{R}^d)$ and let $w_m, w \in \mathcal{W}$ be strong solutions of the Cauchy problems

$$\frac{\partial w_m}{\partial t} + A_tw_m = 0, \quad w_m(T_m) = \varphi_m$$

and

$$\frac{\partial w}{\partial t} + A_tw = 0, \quad w(T) = \varphi,$$

respectively. Then for every $s \in [0,T)$, $w_m(s) \to w(s)$ strongly in $L^2(\mathbb{R}^d)$ as $m \to +\infty$.

**Proof.** The desired result follows easily from stochastic representation of solutions $w_m, w$ (see Proposition [36]) and Aronson’s and De Giorgi-Nash’s estimates for the fundamental solution $p$ (see [1]).

**Theorem 4.6.** Let $u \in \mathcal{P}$. Then

(i) There exists $C > 0$ depending on $\lambda, \Lambda, T, M$ such that

$$\sup_{s \in [0,T]} \int_{\mathbb{R}^d} (E_{s,x} \text{esssup}_{s \leq t \leq T} |u(t,X_t)|^2) \vartheta^2(x) \, dx \leq C(\|u\|_{\mathcal{P}}^2 + \|g\|_{2,\vartheta,T}^2),$$

(4.10)

(ii) $u$ has a quasi-càdlàg version $\tilde{u}$ such that the mapping $[0,T] \ni t \to \tilde{u}(t) \in L^2_\vartheta(\mathbb{R}^d)$ is càdlàg.

(iii) For every $\varphi \in L^2_\vartheta(\mathbb{R}^d)$ such that $\varphi \leq \tilde{u}(T-)$ there exists a square-integrable PAF $K$ such that

$$\tilde{u}(t,X_t) = \varphi(X_T) + \int_t^T f_{\tilde{u}}(\theta,X_\theta) \, d\theta + K_{t,T}$$

$$- \int_t^T \sigma \nabla \tilde{u}(\theta,X_\theta) \, dB_{s,\theta}, \quad t \in [s,T], \quad P_{s,x}-a.s.$$

(4.11)

for q.e. $(s,x) \in Q_T$. 

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(iv) Set $\mu = -L\bar{u}$ in $D'(Q_T)$. Then $\mu \ll \text{cap}_L$ and $\mu$ has an extension $\bar{\mu}$ on $Q_T$ such that $\bar{\mu} \sim K$, $\mu(0) \equiv 0$ and $d\bar{\mu}(T) = (\bar{u}(T) - \varphi) \, dm$.

Proof. From the proof of Lemma 4.2 (see Eq. (4.3)) we know that for a.e. $(s, x) \in Q_T$ there exist a càdlàg process $Y^{s,x}$ and a càdlàg increasing process $K^{s,x}$ such that $Y_T^{s,x} = \lim_{t \uparrow T} Y_t^{s,x}$, $P_{s,x}$-a.s. and in $\mathbb{L}_2(P_{s,x})$,

$$E_{s,x} \int_s^T |u(t, X_t) - Y_t^{s,x}|^2 \, dt = 0, \quad (4.12)$$

and moreover,

$$Y_t^{s,x} = Y_T^{s,x} + \int_t^T f_u(\theta, X_\theta) \, d\theta + K_T^{s,x} - K_t^{s,x} - \int_t^T \sigma \nabla u(\theta, X_\theta) \, dB_{s,\theta}, \quad P_{s,x}$-a.s.

for $t \in [s, T]$. Suppose for a moment that there exists $\xi \in \mathbb{L}_{2,\varphi}(\mathbb{R}^d)$ such that $Y_T^{s,x} = \xi(X_T)$, $P_{s,x}$-a.s. for a.e. $(s, x) \in Q_T$. Then $(Y^{s,x}, \sigma \nabla u(\cdot, X), K^{s,x})$ is a solution of the RBSDE $\xi, f_u, Y_t^{s,x}$). Let $(Y^{s,x,n}, Z^{s,x,n})$ be a solution of the BSDE

$$Y_t^{s,x,n} = \xi(X_T) + \int_t^T f_u(\theta, X_\theta) \, d\theta + \int_t^T n(Y^{s,x,n}_\theta - Y^{s,x}_\theta) \, d\theta - \int_t^T Z^{s,x,n}_\theta \, dB_{s,\theta}. \quad (4.13)$$

Due to (4.12), one can replace $Y^{s,x}$ in (4.13) by $u(\cdot, X, \cdot)$. Therefore, by Proposition 3.6 for q.e. $(s, x) \in Q_T$,

$$Y_t^{s,x,n} = u_n(t, X_t), \quad t \in [s, T], \quad P_{s,x}$-a.s., \quad Z_t^{s,x,n} = \sigma \nabla u_n(t, X_t), \quad \lambda \otimes P_{s,x}$-a.s.,

where $u_n \in W_\varphi$ is a quasi-continuous version of the solution of the Cauchy problem

$$\frac{\partial u_n}{\partial t} + L_t u_n = -f_u - n(u_n - u) - \varphi, \quad u_n(T) = \xi. \quad (4.14)$$

From Proposition 2.1 and Lemma 4.2 we conclude that (2.2) is satisfied for a.e. $(s, x) \in Q_T$. Therefore from Theorem 2.4 it follows that the RBSDE $\xi, f, Y^{s,x}$ has a solution for a.e. $(s, x) \in Q_T$ and assertion (b) of Theorem 2.4 holds for a.e. $(s, x) \in Q_T$. From Proposition 2.1 and (4.12) it may be concluded now that $u_n \uparrow u$ a.e. and in $\mathbb{L}_{2,\varphi}(Q_T)$, and that $\nabla u_n \rightarrow \nabla u$ in $\mathbb{L}_{p,\varphi}(Q_T)$ for every $p \in [1, 2]$. From a priori estimates in Theorem 2.4 Proposition 2.1 Lemma 4.2 and the fact that $u_n \in C([0, T]; \mathbb{L}_{2,\varphi}(\mathbb{R}^d))$ one can deduce also that

$$\sup_{t \in [0, T]} \|u_n(t)\|_{2,\varphi} + \|\nabla u_n\|_{2,\varphi, T} \leq C(\|u\|_p + \|g\|_{2,\varphi, T}) \quad (4.15)$$

for some $C$ not depending on $n$. Let $K_{s,t}^n = \int_s^t n(u_n(\theta, X_\theta) - u(\theta, X_\theta)) \, d\theta$. By (4.13) and Ito’s isometry,

$$E_{s,x} |K_{s,T}^n|^2 \leq C(\|u_n(s, x)\|^2 + E_{s,x} \int_s^T |u(\theta, X_\theta)|^2 \, d\theta + E_{s,x} \int_s^T |\sigma \nabla u_n|^2(\theta, X_\theta) \, d\theta)$$
for q.e. \((s, x) \in Q_T\). In particular, for any fixed \(r \in [0, T)\) the above inequality holds in \((r, x)\) for a.e. \(x \in \mathbb{R}^d\). Integrating the inequality with respect to the measure \(\varrho^2(x) \, dm(x)\) and using Proposition \[2.1\] and \[4.16\] we get

\[
\int_{\mathbb{R}^d} (E_{r,x}(K^n_{s,T})^2) \varrho^2(x) \, dx \leq C(\|u\|_p^2 + \|g\|_{2,q,T}^2).
\] (4.16)

Using the BDG inequality we can deduce from \[4.13\], \[4.15\] and \[4.16\] that

\[
\sup_{s \in [0, T)} \int_{\mathbb{R}^d} (E_{s,x} \sup_{s \leq t \leq T} |u_n(t, X_t)|^2) \varrho^2(x) \, dx \leq C(\|u\|_p^2 + \|g\|_{2,q,T}^2).
\]

Since \(|u_n(\cdot, X)\) is monotone q.e., i.e. \(u_n(t, X_t) \leq u_m(t, X_t), s \leq t \leq T, P_{s,x}\)-a.s., it follows that \(u_n(t, X_t) \uparrow \bar{u}(t, X_t), t \in [s, T], P_{s,x}\)-a.s. for q.e. \((s, x) \in Q_T\), where \(\bar{u}\) is a version of \(u\). From this and the fact that the left hand-side of \(4.10\) does not depend on the version (a.e.) of \(u\) (see \[4.17\]) we get (i).

Set \(d\mu_n = n(u_n - u)^- \, dm\). Putting \(\eta \in C_0^\infty(Q_T)\) as a test function in \[4.13\] we see that \(\sup_{n \geq 1} \int_{Q_T} \eta \, d\mu_n < \infty\) for every positive \(\eta \in C_0^\infty(Q_T)\). Hence \(\{\mu_n\}\) is tight in the topology of weak convergence. Therefore choosing a subsequence if necessary we may and will assume that \(\{\mu_n\}\) converges weakly to some measure \(\mu\). We will show that \(\mu(0) = \mu(T) \equiv 0, \mu \ll \text{cap}_T\) and there exists positive additive functional associated to \(\mu\). To this end let us fix \(s \in [0, T)\). Since \(\mu(\{t\} \times \mathbb{R}^d) = 0\) for all but a countable number of \(t\)’s, we can find a sequence \(\{\delta_k\} \subset (0, T - s)\) such that \(\delta_k \downarrow 0\) and \(\mu(\{s + \delta_k\} \times \mathbb{R}^d) = 0\).

It is easy to see that for every \(f \in C_0(Q_T)\),

\[
E_{s,x} \int_s^T f(t, X_t) \, dK^n_{s,t} = \int_{\mathbb{R}^d} \int_{s+\delta_k}^T f(t, y)p(s, x, t, y) \, d\mu_n(t, y). \tag{4.17}
\]

By Corollary \[2.5\] for every \(f \in C_0(\hat{Q}_{s+\delta_k,T})\),

\[
\int_{s+\delta_k}^T f(t, X_t) \, dK^n_{s,t} \to \int_{s+\delta_k}^T f(t, X_t) \, dK^{s,x}_t, \quad P_{s,x}\text{-a.s.} \tag{4.18}
\]

Using once again Theorem \[2.4\] we get

\[
E_{s,x} \left| \int_{s+\delta_k}^T f(t, X_t) \, dK^n_{s,t} \right|^2 \leq \|f\|_\infty E_{s,x} |K^n_{s,t}|^2 \leq C,
\]

which implies that the left-hand side of \(4.18\) is uniformly integrable. Moreover, using standard arguments (see \[13\] Lemma A.3.3.) one can find PAF \(K\) such that \(P_{s,x}(\{K^{s,x}_t = K_{s,t}, t \in [s, T]\}) = 1\) for q.e. \((s, x) \in Q_T\). Therefore, letting \(n \to \infty\) in \(4.17\) and taking into account the fact that \(p(s, x, \cdot, \cdot)\) is bounded and continuous on \(Q_{s+\delta_k,T}\) shows that for q.e. \((s, x) \in Q_T\),

\[
E_{s,x} \int_{s+\delta_k}^T f(t, X_t) \, dK_{s,t} = \int_{\mathbb{R}^d} \int_{s+\delta_k}^T f(t, y)p(s, x, t, y) \, d\mu(t, y) \tag{4.19}
\]

for \(f \in C_0(Q_{s+\delta_k,T})\). Letting \(k \to \infty\) in \(4.19\) we see that for q.e. \((s, x) \in Q_T,\)

\[
E_{s,x} \int_s^T f(t, X_t) \, dK_{s,t} = \int_{\mathbb{R}^d} \int_s^T f(t, y)p(s, x, t, y) \, d\mu(t, y) \tag{4.20}
\]
for all \( f \in C_0(Q_{s+\delta_k,T}) \) and hence, by standard argument, for \( f \in C_0(Q_T) \). Now we are going to show that \( \mu \) is absolutely continuous with respect to \( \text{cap}_L \). Let \( B \in \mathcal{B}(Q_T) \) be such that \( \text{cap}_L(B) = 0 \) and let \( K \subset B \) be a compact set. By the monotone class theorem, (4.20) holds for every \( f \in B_0(Q_T) \). Let \( f = 1_K \) and let \( \delta > 0 \) be chosen so that \( K \subset Q_{\delta T} \). Then by Aronson’s estimates,

\[
\mu(K) \leq C \int_{\mathbb{R}^d} \left( \int_{Q_{\delta T}} f(t,y) p(\delta,x,t,y) \, d\mu(t,y) \right) \, dx
= \int_{\mathbb{R}^d} \left( E_{\delta,x} \int_{\delta}^{T} f(t,X_t) \, dK_{\delta,t} \right) \, dx = 0,
\]

the last equality being a consequence of the definition of \( \text{cap}_L \). Thus, \( \mu(B) = 0 \).

Repeating arguments following (4.17) with the last equality being a consequence of the definition of \( \text{cap}_L \), let \( \zeta \in f \) be such that \( \text{cap}_L(B) = 0 \) and \( \text{cap}_L(B) \) replaced by \( k \wedge p(0,x,\cdot,\cdot) \) one can show that for every \( k > 0 \),

\[
E_{0,x} \int_{[0,T]} f(t,X_t) \, dK_{0,t} \geq \int_{Q_T} f(t,y)(k \wedge p(0,x,t,y)) \, d\mu(t,y)
\]  

(4.21)

for \( f \in C_0^+(Q_T) \). From this and the fact that \( P_{s,x} \)-a.s. the process \( K_s \), does not have jumps in \( s \in [0,T) \) (the last statement follows from pointwise convergence of \( K^n \) (see Corollary 2.5)) we conclude that \( \mu(0) \equiv 0 \). The fact that \( \mu(T) \equiv 0 \) follows easily from (4.20) and the fact that \( E_{s,x} \Delta K_{s,T} = 0 \) for a.e. \( (s,x) \in Q_T \).

From Propositions 2.1 (i) and Remark 3.2 we conclude that (2.2) is satisfied for q.e. \( (s,x) \in Q_T \). Hence, by Theorem 2.4, the RBSDE\(_{s,x}(\xi,f,u)\) has a solution for q.e. \( (s,x) \in Q_T \) and assertion (b) of Theorem 2.4 holds for q.e. \( (s,x) \in Q_T \). Therefore putting \( \bar{u}(t,x) = \lim_{n \to \infty} u_n(t,x) \) if the limit exists and \( \bar{u}(t,x) = 0 \) otherwise we see that \( \bar{u} \) is a quasi-càdlàg version of \( u \). Fix \( t_0 \in [0,T) \), \( 0 \leq s \leq t_0 \) and let \( (X,Q_{s,x}) \) be a diffusion associated with operator \( A_t \). Since trajectories of \( \bar{u}(\cdot,X) \) are càdlàg, for q.e. \( (s,x) \in Q_T \),

\[
\lim_{t \to t_0^+} E_{Q_{s,x}} \bar{u}(t,X_t) \eta(X_t) = E_{Q_{s,x}} \bar{u}(t_0,X_{t_0}) \eta(X_{t_0}), \quad \eta \in C_0(Q_T).
\]  

(4.22)

From (4.22), the Lebesgue dominated convergence theorem and Remark 4.4 we get

\[
(\bar{u}(t),\eta) \to (\bar{u}(t_0),\eta), \quad \eta \in C_0(Q_T).
\]

Since \( \sup_{t \in [0,T]} \|\bar{u}(t)\|_{L^2} < \infty \), \( \bar{u}(t) \to \bar{u}(t_0) \) weakly in \( L_{2,\varrho}(\mathbb{R}^d) \) if \( t \to t_0^+ \). Let \( \bar{u}(t) = T_{k}(\bar{u}(t)) \). Then from (4.22), the fact that \( \bar{u}(\cdot,X) \) is càdlàg, boundedness of \( u_k \) and the Lebesgue dominated convergence theorem it follows that

\[
\limsup_{t \to t_0^+} \|\bar{u}(t)\|_{L^2}^2 \leq \|\bar{u}(t_0)\|_{L^2}^2.
\]

Since the sequence \( \{\|\bar{u}(t)\|_{L^2}^2\}_{k \geq 0} \) is monotone, letting \( k \to \infty \) in the above inequality we get \( \limsup_{t \to t_0^+} \|\bar{u}(t)\|_{L^2}^2 \leq \|\bar{u}(t_0)\|_{L^2}^2 \). In fact, \( \bar{u}(t) \to \bar{u}(t_0) \) in \( L_{2,\varrho}(\mathbb{R}^d) \) if \( t \to t_0^+ \) since \( L_{2,\varrho}(\mathbb{R}^d) \) is a Hilbert space. In much the same way we show that if there is \( \zeta \in L_{2,\varrho}(\mathbb{R}^d) \) such that \( \zeta(X_{t_0}) = \bar{u}_-(t_0,X_{t_0}) \equiv \lim_{t \to t_0^-} \bar{u}(t,X_t) \), \( P_{s,x} \)-a.s. for a.e. \( x \in \mathbb{R}^d \), then \( \bar{u}(t) \to \zeta \) strongly in \( L_{2,\varrho}(\mathbb{R}^d) \) if \( t \to t_0^- \). Thus, to complete the proof of
(ii) we only have to prove that $Y_{T}^{s,x} = \xi(X_T)$ and $\zeta(X_{t_0}) = \bar{u}_-(t_0, X_{t_0})$, $P_s,x$-a.s. for a.e. $x \in \mathbb{R}^d$ for some $\xi, \zeta \in \mathbb{L}_{2,\varrho}(\mathbb{R}^d)$. We shall prove the first statement. Since the proof of the second one is analogous, we omit it. By (4.12),

$$
\int_{Q_T} (E_{s,x} \int_s^T |Y_{t,x}^{s,x} - u(t, X_t)|^2 dt) \varrho^2(x) ds dx = \int_0^T \left( \int_{\mathbb{R}^d} E_{s,x} |Y_{t,x}^{s,x} - u(t, X_t)|^2 \varrho^2(x) ds dx \right) dt = 0
$$

Therefore there exists $\{t_n\} \subset [0, T]$ such that $t_n \to T^-$, $Y_{t_n}^{s,x} = u(t_n, X_{t_n})$, $P_s,x$-a.s. for a.e. $(s, x) \in Q_{t_n}$. Without loss of generality we may assume that $\|u(t_n)\|_{2,\varrho} \leq \text{esssup}_{t \in [0,T]} \|u(t)\|_{2,\varrho}$. Let us denote by $\mathcal{I} \subset [0, T]$ the set of those $s \in [0, T]$ for which there exists $n_0 \in \mathbb{N}$ such that $Y_{t_n}^{s,x} = u(t_n, X_{t_n})$, $P_s,x$-a.s., $n \geq n_0$ for a.e. $x \in \mathbb{R}^d$. Of course $\lambda([0, T] \setminus \mathcal{I}) = 0$. Let $s \in \mathcal{I}$. From the definition of $Y_{T}^{s,x}$,

$$
\lim_{n \to \infty} E_{s,x} |u(t_n, X_{t_n}) - Y_{T}^{s,x}|^2 = 0 \quad (4.23)
$$

for a.e. $x \in \mathbb{R}^d$. Let us put now $u_k = T_k(u)\phi$, where $\phi \in C_0(Q_T)$. From (4.23) it follows that for a.e. $(s, x) \in Q_{t_n}$,

$$
E_{Q_{s,x}} u_k(t_n, X_{t_n}) \eta(t_n, X_{t_n}) - E_{Q_{s,x}} u_k(t_m, X_{t_m}) \eta(t_m, X_{t_m}) \to 0, \quad \eta \in C_0(Q_T)
$$

if $n, m \to +\infty$. From the above, the Lebesgue dominated convergence theorem and Remark 4.4 it follows that

$$
|\langle u_k(t_n), \eta \rangle_{2,\varrho} - \langle u_k(t_m), \eta \rangle_{2,\varrho}| \to 0, \quad \eta \in C_0(\mathbb{R}^d).
$$

if $n, m \to +\infty$. Since $\sup_{t \in [0,T]} \|u(t)\|_{2,\varrho} < \infty$, there exists $\xi_k \in \mathbb{L}_{2,\varrho}(\mathbb{R}^d)$ such that $u_k(t_n) \to \xi_k$ weakly in $\mathbb{L}_{2,\varrho}(\mathbb{R}^d)$ if $n \to +\infty$. Because the functions $u_k$ have common compact support, the last convergence holds weakly in $\mathbb{L}_{2}(\mathbb{R}^d)$, too. Next, by the Markov property,

$$
E_{Q_{s,x}} |u_k(t_n, X_{t_n}) - u_k(t_m, X_{t_m})|^2
$$

$$
= E_{Q_{s,x}} (|u_k(t_n, X_{t_n})|^2 - 2u_k(t_n, X_{t_n})u_k(t_m, X_{t_m}) + |u_k(t_m, X_{t_m})|^2)
$$

$$
= E_{Q_{s,x}} E_{Q_{s,x}} (|u_k(t_n, X_{t_n})|^2 - 2u_k(t_n, X_{t_n})u_k(t_m, X_{t_m}) + |u_k(t_m, X_{t_m})|^2) \varrho(x) dx
$$

$$
= E_{Q_{s,x}} g(t_n, X_{t_n}),
$$

where $g(t, y) = E_{Q_{t,y}} (|u_k(t, y)|^2 - 2u_k(t, y)u_k(t_m, X_{t_m}) + |u_k(t_m, X_{t_m})|^2)$. Integrating the above identity with respect to $x$ and using symmetry of $(X, Q_{s,x})$ we get

$$
\int_{\mathbb{R}^d} E_{Q_{s,x}} |u_k(t_n, X_{t_n}) - u_k(t_m, X_{t_m})|^2 dx = \int_{\mathbb{R}^d} g(t_n, y) dy. \quad (4.24)
$$

Let $w_m, w$ be unique strong solutions of the Cauchy problems

$$
\frac{\partial w_m}{\partial t} + A_tw_m = 0, \quad w_m(t_m) = u_k(t_m)
$$

and

$$
\frac{\partial w}{\partial t} + A_tw = 0, \quad w(T) = \xi_k,
$$

where $A_t$ is the generator of $\{X_t\}_{t \geq 0}$.
An inspection of the proof of Theorem 4.6 shows that (4.11) holds for

\[ \int_{\mathbb{R}^d} E_{Q_{s,x}} |u_k(t_n, X_{t_n}) - u_k(t_m, X_{t_m})|^2 \, dx = \|u_k(t_n) - u_k(t_m)\|^2 + 2\langle u_k(t_n), u_k(t_m) - w(t_n)\rangle \]  

(4.25)

Taking limit inferior as \( m \to +\infty \) and applying Lemma 4.3 we conclude from the above inequality that

\[ \int_{\mathbb{R}^d} E_{Q_{s,x}} |u_k(t_n, X_{t_n}) - T_k(Y_T^{s,x})\phi(T, X_T)|^2 \, dx \geq \|u_k(t_n) - \xi_k\|^2 + 2\langle u_k(t_n), \xi_k - w(t_n)\rangle \]  

(4.25)

Letting \( k \to +\infty \) in the above inequality we see that \( u_k(t_n) \to \xi_k \) in \( L_2(\mathbb{R}^d) \) and in \( \mathbb{L}_{2,q}(\mathbb{R}^d) \). Therefore there exists a measurable function \( \xi \) such that for every \( k \in \mathbb{N} \) and \( \phi \in C_0(\mathbb{R}^d) \), \( u_k(t_n) \to T_k(\xi)\phi \) in \( L_2(\mathbb{R}^d) \) if \( n \to \infty \). Putting \( (T_k(\eta)(X_T) \) instead of \( u_k(t_n, X_{t_n}) \) in (4.25) we get

\[ \int_{\mathbb{R}^d} E_{Q_{s,x}} |u_k(t_n, X_{t_n}) - (T_k(\xi)\phi)(X_T)|^2 \, dx \to 0. \]

From this and (4.23) it follows that \( \xi(X_T) = Y_T^{s,x}, P_{s,x}\text{-p.n. for a.e. } x \in \mathbb{R}^d \). Since \( s \) was chosen arbitrarily from the set \( \mathbb{L} \), \( \xi(X_T) = Y_T^{s,x}, P_{s,x}\text{-a.s. for a.e. } (s, x) \in Q_{\bar{T}} \).

Hence, by (4.3) and Proposition 2.1 \( \xi \in \mathbb{L}_{2,q}(\mathbb{R}^d) \). In fact we have shown that \( \xi = \bar{u}(T^-) \). Therefore passing to the limit in (4.13) we get (4.11) and (iv) in the case where \( \varphi = \bar{u}(T^-) \) and \( \hat{K} = \bar{K} + 1_{|T|}(\bar{u}(T^-) - \varphi)(X_T) \) and \( \bar{\mu} = \mu + \mu_T \), where \( \mu_T(A) = \int_{\mathbb{R}^d} 1_A(T, x)(\bar{u}(T^-) - \varphi)(x) \, m(dx) \) for \( A \in \mathcal{B}(Q_{\bar{T}}) \), we see that \( \bar{\mu} \sim \bar{K} \) and (4.11) is satisfied with \( K \) replaced by \( \bar{K} \).

\[ \square \]

**Remark 4.7.** In the particular case where \( \mathcal{L} = \frac{D^2}{2} + \frac{1}{2}\Delta \), \( f_u \equiv 0 \), \( u \) is quasi-continuous and \( \varphi \equiv 0 \) results of Theorem 4.6 agree with those given in [24] (Theorem 2, Lemma 3), because the transition function \( p \) of the Wiener process is symmetric. For instance, integrating (4.8) with \( s = 0 \) with respect to \( m(dx) \) we get Theorem 3(v) in [24]. Furthermore, taking expectation of (4.11) with \( s = 0 \), multiplying it by \( \eta \in \mathbb{L}_{2,q}(\mathbb{R}^d) \) and then integrating with respect to \( m(dx) \) and using (4.8) we get Lemma 3 in [24].

**Remark 4.8.** An inspection of the proof of Theorem 4.6 shows that (4.11) holds for every \( (s, x) \in Q_{\bar{T}} \) such that (3.2) is satisfied with \( h, g \) replaced by \( h^2, g^2 \), respectively.

We now recall the notion of soft measures (see [11]). Let

\[ \mathcal{W}_\varphi = \{ u \in \mathbb{L}_2(0, T; H^{-1}_\varphi); \frac{\partial u}{\partial t} \in \mathbb{L}_2(0, T; H^{-1}_\varphi) \}. \]

**Definition 4.9.** Let \( V \subset \bar{Q}_{\bar{T}} \) be an open set. The parabolic capacity of \( V \) is given by

\[ \text{cap}_2(V) = \inf\{ \|u\|_{\mathcal{W}_\varphi}; u \in \mathcal{W}_\varphi, u \geq 1_V \text{ a.e.} \} \]

with the convention that \( \inf \emptyset = \infty \). The parabolic capacity of a Borel subset \( B \) of \( \bar{Q}_{\bar{T}} \) is given by

\[ \text{cap}_2(B) = \inf\{ \text{cap}_2(V); V \text{ is an open subset of } \bar{Q}_{\bar{T}}, B \subset V \}. \]
Definition 4.10. We say that a Radon measure \( \mu \) on \( \tilde{Q}_T \) is soft if \( \mu \ll \text{cap}_2 \).

By \( \mathcal{M}_0(\tilde{Q}_T) \) we denote the set of all soft measures on \( \tilde{Q}_T \) and by \( \mathcal{M}_0 \) the set of Radon measures on \( Q_T \) such that \( \mu|_Q \in \mathcal{M}_0(\tilde{Q}_T) \), \( \mu(0) = 0 \) and \( \mu(T) \ll m \).

Lemma 4.11. Capacity \( \text{cap}_L \) is equivalent to \( \text{cap}_2 \).

Proof. Follows from [28] and [36, Theorem 1]. \( \square \)

Lemma 4.12. Let \( u \in \mathcal{P} \). If for some \( \mu \in \mathcal{M}_0^+ \) and \( \varphi \in \mathbb{L}_{2,0}(\mathbb{R}^d) \),

\[
\langle u, \frac{\partial \eta}{\partial t} \rangle_{\tilde{Q}_T} - \langle L_t u, \eta \rangle_{\tilde{Q}_T} = \langle \varphi, \eta(T) \rangle_{2,0} + \langle f_u, \eta \rangle_{2,0,\tilde{T}} + \int_{Q_T} \eta \varphi^2 d\mu
\]

for every \( \eta \in \mathcal{W}_0 \), then \( \bar{u}(T-) \geq \varphi \) and \( \mu(T) = (\bar{u}(T-) - \varphi) dm \), where \( \bar{u} \) is a quasi-càdlàg version of \( u \).

Proof. Let \( \eta \in \mathcal{W}_{\theta,1}^{1,1}(Q_T) \) be positive and let \( t \in (0, T) \). Taking as a test function \( \eta^{n,t} \in \mathcal{W}_{\theta,1}^{1,1}(Q_T) \) defined by the formula

\[
\eta^{n,t}(s, x) = \begin{cases} 0, & s \in [0, t], \\ \frac{\eta(n, x)}{t_n - t}(s - t), & s \in (t_n), \\ \eta(t_n, x), & s \in [t_n, T], \end{cases}
\]

where \( \{t_n\} \subset (t, T) \) is a sequence such that \( t_n \downarrow t \), we get

\[
\frac{1}{t_n - t} \int_t^{t_n} \langle u(\theta), \eta(\theta) \rangle_{2,0} d\theta + \int_{t_n}^T \langle u(\theta), \frac{\partial \eta}{\partial t}(\theta) \rangle_{2,0} d\theta = -\langle L_t, \eta^{n,t} \rangle_{2,0,\tilde{T}} + \int_t^T \langle f_u, \eta^{n,t}(\theta) \rangle_{2,0} d\theta + \int_{Q_T} \eta^{n,t} \varphi^2 d\mu + \langle \varphi, \eta^{n,t}(T) \rangle_{2,0}.
\]

(4.26)

Letting \( n \to \infty \) and then \( t \uparrow T \), and using the fact that \( [0, T] \ni t \to \bar{u}(t) \in \mathbb{L}_{2,0}(\mathbb{R}^d) \) is càdlàg we conclude from the above that \( \langle \bar{u}(T-) - \varphi, \eta(T) \rangle_{2,0} = \int_{\mathbb{R}^d} \eta(T) \varphi^2 d\mu(T) \), which proves the lemma. \( \square \)

Proposition 4.13. For every \( \mu \in \mathcal{M}_0^+ \) there exists a unique PAF \( K \) such that \( \mu \sim K \).

Proof. Let \( \mu \in \mathcal{M}_0^+ \). Suppose that \( \mu(T) = \xi dm \) for some \( \xi \geq 0 \). From [11, Theorem 2.23] it follows that there exist \( \mu_1, \mu_2 \in \mathcal{M}_0^+((Q_T) \cap \mathcal{W}_\theta' \mathcal{W}_\theta') \) and positive \( \alpha_1, \alpha_2 \in \mathbb{L}_{1,2}((Q_T, |\mu|)) \) such that \( d\mu = \alpha_1 d\mu_1 + \alpha_2 d\mu_2 \) on \( Q_T \). Let \( u_1, u_2 \in \mathbb{L}_{2,0}(0, T; H_\theta^1) \) be such that \( Lu_i = -\mu_i, i = 1, 2 \) in \( D'(Q_T) \). Then \( u_1, u_2 \in \mathcal{P} \). Let \( \bar{u}_1, \bar{u}_2 \) be quasi-càdlàg versions of \( u_1, u_2 \) of Theorem [4.6], and let \( \varphi_i = \bar{u}_i(T-) - \frac{1}{4} \xi, i = 1, 2 \). Then, by Theorem [4.6(ii)], there exist PAFs \( K_1, K_2 \) such that \( K_1 \sim \bar{\mu}_1, K_2 \sim \bar{\mu}_2 \), where \( \bar{\mu}_1, \bar{\mu}_2 \) are extensions of \( \mu_1, \mu_2 \) on \( Q_T \) such that \( d\bar{\mu}_i(T) = (\bar{u}_i(T-) - \varphi_i) dm, i = 1, 2 \). Putting \( \tilde{\alpha}_1(T, \cdot) = 1, \tilde{\alpha}_2(0, \cdot) = 0, \tilde{\alpha}_i|Q_T = \alpha_i, i = 1, 2 \) we see that \( d\mu = \tilde{\alpha}_1 d\bar{\mu}_1 + \tilde{\alpha}_2 d\bar{\mu}_2 \) on \( Q_T \) and \( \mu \sim \tilde{\alpha}_1(\cdot, X) dK_1 + \tilde{\alpha}_2(\cdot, X) dK_2 \). \( \square \)

Definition 4.14. We say that \( dK : \Omega \times \mathcal{B}([0, T]) \to \mathbb{R} \) is a random measure if
(a) \(dK(\omega)\) is a nonnegative measure on \(\mathcal{B}([0,T])\) for every \(\omega \in \Omega\),
(b) the mapping \(\omega \to dK(\omega)\) is \((\mathcal{G},\mathcal{B}(\mathcal{M}^+([0,T])))\)-measurable,
(c) \(\int_s^t dK_\theta\) is \(\mathcal{G}^*_\theta\)-measurable for every \(0 \leq s \leq t \leq T\).

Remark 4.15. From results proved in [28] it follows that there is a Hunt process \(\{Z_t, \tilde{P}_z\}, t \geq 0, z \in \mathbb{R}^{d+1}\) associated with the operator \(\mathcal{L}\). In fact, \(Z_t = (\tau(t), X_{\tau(t)})\) and \(\tilde{P}_z\) coincides with \(P_{s,x}\) for \(z \in Q_T\), where \(\tau\) is the uniform motion to the right, i.e. \(\tau(t) = \tau(0) + t\) and \(\tau(0) = s\) under \(P_{s,x}\).

Lemma 4.16. Let \(\{dK_n\}\) be a sequence of random measures. Assume that for every \((s,x)\in F \subset Q_T\) there exists random element \(dK_{s,x}\) such that \(dK_n \to dK_{s,x}\) in \(\mathcal{M}^+([0,T])\) in probability \(P_{s,x}\) as \(n \to +\infty\). Then there exists a random measure \(dK\) such that for every \((s,x)\in F\),
\[
dK_{s,x} = dK, \quad P_{s,x^-a.s.}
\]
Proof. Let \(n_0(s,x) = 0\) and let
\[
n_k(s,x) = \inf \{m > n_{k-1}(s,x), \sup_{p,q \geq m} P_{s,x}(dM(dK^p, dK^q) > 2^{-k}) < 2^{-k}\}
\]
for \(k > 0\). By induction, for every \(k \geq 0\), \(n_k \in \mathcal{B}(Q_T)\) and hence \(dL_{s,x,k} = dK_{n_k(s,x)}\) is \(\mathcal{B}(Q_T) \otimes \mathcal{G}/\mathcal{B}(\mathcal{M}^+([0,T]))\) measurable. Put
\[
dL_{s,x}(\omega) = \begin{cases} \lim_{k \to \infty} dL_{s,x,k}(\omega) \text{ in } \mathcal{M}^+([0,T]), & \text{if the limit exists,} \\ 0, & \text{otherwise.} \end{cases}
\]
By the Borel-Cantelli lemma, for every \((s,x) \in F\) the limit in (4.27) exists \(P_{s,x^-a.s.}\) and \(dL_{s,x} = dK_{s,x}, P_{s,x^-a.s.}\). To prove the lemma it suffices now to put \(dK(\omega) = dL_{Z^0(\omega)}\), where \(Z\) is defined in Remark 4.15.

Corollary 4.17. For every \(\mu \in \mathcal{M}_0^+\) there exists a unique random measure \(dK\) such that \(K \sim \mu\), where \(K\) is PAF such that \(K_{s,t} = \int_s^t dK_\theta\).
Proof. PAFs \(K_1, K_2\) in the proof of Proposition 4.13 are limits of random measures (see Corollary 2.5 and the proof of Theorem 4.6) in the sense of Lemma 4.16 q.e.. Therefore from Lemma 4.16 we get the result.

Let \(\mu \in \mathcal{M}_0^+\). In the sequel by \(d\mu(\cdot, X_\cdot)\) we denote the unique random measure associated with \(\mu\).
5 Obstacle problem and RBSDEs

In this section we give definition of a solution of the obstacle problem in the sense of complementary system, i.e. by solution we mean a pair \((u, \mu)\), where \(\mu\) is a Radon measure satisfying a minimality condition. In the case of regular obstacles the minimality condition may be expressed by the condition \(\int (u - h) \, d\mu = 0\). The main difficulty in the case of nonregular obstacle lies in the proper and rigorous formulation of the minimality condition. In the case of linear equations M. Pierre in a series of papers (see \[34, 35\] and references given there) coped with the problem by introducing the notion of precise function, precise version and precise associated function (see \[35\]).

His theory was based on the narrower then \(\mathcal{P}\) class of potentials which forced him to decompose the obstacle problem under consideration into some parabolic equation and the obstacle problem with generator and terminal condition equal to zero (on the other hand such a decomposition was possible due to linearity of the problem). Assume for a moment that the generator and terminal condition are equal to zero. Roughly speaking, if \(\hat{u}\) is a precise version of \(u\) and \(\hat{h}\) is a precise function associated with \(h\) then in the definition given by M. Pierre the minimality condition has the form

\[
\int_{Q_T} (\hat{u} - \hat{h}) \, d\mu = 0.
\]

Due to results of Section 4 concerning the class \(\mathcal{P}\) which is wider then the class of potentials considered in \[34, 35\] we are able to give definition of the obstacle problem which allows us to deal with nonlinear problems. Instead of considering the notion of precise function we express the minimality condition via stochastic processes naturally associated with the pair \((u, \mu)\) and barrier \(h\).

It is worth pointing out that \((5.1)\) and condition \((iii)\) in the following definition are closely related because as will be shown in Proposition \[5.17\] \(\hat{u}(\cdot, X) = \bar{u}(\cdot, X)\). Moreover, our stochastic definition is a direct generalization of the definition considered in one-dimensional case in \[7\].

Put

\[
\mathcal{P}^* = \{ u \in L_2(0, T; H^1_\varrho) : u \text{ is quasi-c\`adl\`ag, } \text{esssup}_{t \in [0, T]} \| u(t) \|_{2, \varrho} < \infty \}
\]

and note that from Theorem \[4.6\] it follows that \(\mathcal{P} \subset \mathcal{P}^*\).

**Definition 5.1.** Let \((H1)–(H3)\) hold. We say that a pair \((u, \mu)\) is a solution of \(\text{OP}(\varphi, f, h)\) if \(u \in \mathcal{P}, \mu \in \mathcal{M}_0^+\) and

\[(i)\] for every \(\eta \in \mathcal{W}_\varrho\) such that \(\eta(0) \equiv 0\),

\[
\langle u, \frac{\partial \eta}{\partial t} \rangle_{\varrho, T} - \langle L_t u, \eta \rangle_{\varrho, T} = \langle \varphi, \eta(T) \rangle_{2, \varrho} + \langle f_u, \eta \rangle_{2, \varrho, T} + \int_{Q_T} \eta q^2 \, d\mu,
\]

\[(ii)\] \(u \geq h\) a.e.,

\[(iii)\] for q.e. \((s, x) \in Q_T^r\),

\[
\int_s^T (\bar{u}_-(t, X_t) - h^*_-(t, X_t)) \, d\mu(t, X_t) = 0, \quad P_{s,x} \text{-a.s.}
\]
for every \( h^* \in \mathcal{P}^* \) such that \( h \leq h^* \leq \bar{u} \) a.e., where \( \bar{u} \) is a quasi-càdlàg version of \( u \) (Here and in what follows given a measurable function \( v \) on \( QT \) we denote by \( v_-(t,X_t) \) the limit \( \lim_{s \downarrow t,s \to t} v(t,X_t) \)).

It is worth pointing out that in the above definition \( \mu \) is defined on the whole set \( QT \).

**Theorem 5.2.** Under (H1), (H2) there exists at most one solution of \( OP(\varphi,f,h) \).

**Proof.** Let \((u_1,\mu_1),(u_2,\mu_2)\) be solutions of \( OP(\varphi,f,h) \). Write \( u = u_1 - u_2, \mu = \mu_1 - \mu_2, F_u = f_{u_1} - f_{u_2} \). By Theorem 4.6 and Lemma 4.12,

\[
\begin{align*}
\bar{u}(t,X_t) &= \int_t^T \bar{F}_u(\theta,X_\theta) \, d\theta + \int_t^T \sigma \nabla \bar{u}(\theta,X_\theta) \, d\mu(\theta) \\
&= \int_t^T \sigma \nabla \bar{u}(\theta,X_\theta) \, dB_{s,\theta}, \quad t \in [s,T], \quad P_{s,x}-a.s.
\end{align*}
\]

for quasi every \((s,x) \in QT\), where \( \bar{u} = \bar{u}_1 - \bar{u}_2 \) and \( \bar{u}_1, \bar{u}_2 \) are càdlàg versions of \( u_1 \) and \( u_2 \), respectively. By Itô’s formula,

\[
E_{s,x}|\bar{u}(t,X_t)|^2 + E_{s,x} \int_t^T |\sigma \nabla \bar{u}(\theta,X_\theta)|^2 \, d\theta + E_{s,x} \sum_{t<\theta\leq T} |\Delta \mu(\theta,X_\theta)|^2
\]

\[
= 2E_{s,x} \int_t^T \bar{F}_u(\theta,X_\theta) \bar{u}(\theta,X_\theta) \, d\theta + 2E_{s,x} \int_t^T \bar{u}_-(\theta,X_\theta) \, d\mu(\theta)
\]

for \( t \in [s,T] \). Put \( h^* = \bar{u}_1 \wedge \bar{u}_2 \). Then \( h \leq h^* \leq \bar{u}_1, \bar{u}_2 \leq \bar{u}_2 \) and \( h^* \in \mathcal{P}^* \). Therefore

\[
\begin{align*}
\int_t^T \bar{u}_-(\theta,X_\theta) \, d\mu(\theta)
&= \int_t^T (\bar{u}_1-h^*)(\theta,X_\theta) \, d\mu(\theta) + \int_t^T (h^*-\bar{u}_2)(\theta,X_\theta) \, d\mu(\theta)
\end{align*}
\]

\[
= \int_t^T (\bar{u}_1-h^*)(\theta,X_\theta) \, d\mu_1(\theta,X_\theta) - d\mu_2(\theta,X_\theta))
\]

\[
+ \int_t^T (h^*-\bar{u}_2)(\theta,X_\theta) \, d\mu_1(\theta,X_\theta) - d\mu_2(\theta,X_\theta)) \leq 0.
\]

The first and fourth term on the right-hand side are equal to zero by the definition of a solution of the obstacle problem. The second and third term are negative since the integrands are negative. Consequently,

\[
E_{s,x}|\bar{u}(t,X_t)|^2 + E_{s,x} \int_t^T |\sigma \nabla \bar{u}(\theta,X_\theta)|^2 \, d\theta \leq 2E_{s,x} \int_t^T \bar{F}_u(\theta,X_\theta) \bar{u}(\theta,X_\theta) \, d\theta.
\]

Using standard arguments we deduce from the above that \( E_{s,x}|\bar{u}(t,X_t)|^2 = 0 \) for q.e. \((s,x) \in QT\), which when combined with Propositions 2.1, 3.4 shows that \( \bar{u}_1 = \bar{u}_2 \) q.e.. Hence, by condition (i) of the definition of a solution of the obstacle problem, \( \int_{QT} \eta \, d\mu_1 = \int_{QT} \eta \, d\mu_2 \) for every \( \eta \in C_0^\infty(Q_T) \) such that \( \eta(0) \equiv 0 \). Accordingly, \( \mu_1 \) coincides with \( \mu_2 \) on \((0,T] \times \mathbb{R}^d \). Since \( \mu_1(0) = \mu_2(0) = 0 \), this completes the proof. \( \square \)
Now we are going to prove existence of a solution of $OP(\varphi, f, h)$ under standard integrability assumptions on the data and condition (H3) on the barrier. It is worth pointing out that in view of Theorem 4.6, condition (H3) is necessary for existence of a solution of that problem. As we shall see, it is also sufficient.

In the proof of the following theorem we will use a priori estimates and convergence results for penalized sequence proved in [32].

**Theorem 5.3.** Let assumptions (H1)–(H3) hold.

(i) There exists a unique solution $(u, \mu)$ of $OP(\varphi, f, h)$.

(ii) Let $\bar{u}$ be a quasi-càdlàg version of $u$ and let

$$F = \{(s, x) \in Q_T : E_s x \text{esssup}_{s \leq t \leq T} |h^+(t, X_t)|^2 + E_s x \int_s^T |g(t, X_t)|^2 \, dt < \infty\}.$$

For every $(s, x) \in F$ the triple

$$(\bar{u}(t, X_t), \sigma \nabla \bar{u}(t, X_t), \int_s^t d\mu(\theta, X_\theta)), \quad t \in [s, T]$$

(5.2)

is a solution of $RBSDE_{s,x}(\varphi, f, h)$ and $\text{cap}_L(F^c) = 0$.

(iii) Let $\bar{u}_n$ be a quasi-continuous version of the solution $u_n$ of the problem

$$(\frac{\partial u_n}{\partial t} + L_t u_n = -f_n - n(u_n - h)^-, \quad u_n(T) = \varphi).$$

(5.3)

Then $\bar{u}_n \uparrow \bar{u}$ q.e. and in $L_{2, \varphi}(Q_T)$, $\nabla u_n \to \nabla u$ in $L_{p, \varphi}(Q_T)$ for $p \in [1, 2]$, and if $h$ is quasi-continuous then the last convergence holds true for $p = 2$, too.

**Proof.** The fact that $\text{cap}_L(F^c) = 0$ follows from Proposition 2.1, Theorem 4.6(i) and Remark 3.2. First we show that there exists $u$ satisfying condition (i) of the definition of a solution of $OP(\varphi, f, h)$. Let $u_n$ be a strong solution of $PDE(\varphi, f + n(y - h)^-)$. Then for every $\eta \in W_\varphi$ such that $\eta(0) \equiv 0$,

$$\langle u_n, \frac{\partial \eta}{\partial t} \rangle_{\theta, T} - \langle L_t u_n, \eta \rangle_{\theta, T} = \langle \varphi(T), \eta(T) \rangle_{2, \theta} + \langle f_n, \eta \rangle_{2, \theta, T} + \int_{Q_T} \eta \varphi^2 \, d\mu_n,$$

(5.4)

where $d\mu_n = n(u_n - h)^- \, d\mu_T$. Set

$$F_0 = \{(s, x) \in Q_T : E_s x \int_s^T (|g|^2 + |h^+|^2)(t, X_t) < \infty\}$$

and observe that $F \subset F_0$. By Proposition 3.6 $u_n$ has a quasi-continuous version of $\bar{u}_n$ such that $(\bar{u}_n(t, X_t), \sigma \nabla \bar{u}_n(t, X_t)), t \in [s, T]$, is a solution of $BSDE_{s,x}((\varphi, f + n(y - h)^-) for every $(s, x) \in F_0$. By Theorem 2.4

$$E_s x \text{esssup}_{s \leq t \leq T} |\bar{u}_n(t, X_t)|^2 + E_s x \int_s^T |\sigma \nabla u_n(t, X_t)|^2 \, dt \leq C(E_s x |\varphi(X_T)|^2 + E_s x \int_s^T |g(t, X_t)|^2 \, dt + E_s x \text{esssup}_{s \leq t \leq T} |h^+(t, X_t)|^2)$$

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for every $(s, x) \in F$. In particular, the above estimate holds for every $s \in [0, T)$ and a.e. $x \in \mathbb{R}^d$. Integrating the above inequality with respect to $x$, using Proposition 2.1 and Theorem 2.4 yields

$$
\sup_{0 \leq t \leq T} \|\bar{u}_n(t)\|_{2,\theta}^2 + \|\nabla u_n\|_{2,\theta,T}^2 \\
\leq C(\|\varphi\|_{2,\theta}^2 + \|g\|_{2,\theta,T}^2 + \sup_{s \in [0,T]} \int_{\mathbb{R}^d} E_{s,x} \text{ess su}p_{s \leq t \leq T} |h^+(t, X_t)|^2 g^2(x) \, dx). \quad (5.5)
$$

By the above and (4.10),

$$
\sup_{0 \leq t \leq T} \|\bar{u}_n(t)\|_{2,\theta}^2 + \|\nabla u_n\|_{2,\theta,T}^2 \leq C(\|\varphi\|_{2,\theta}^2 + \|g\|_{2,\theta,T}^2 + \|h^*\|_P^2). \quad (5.6)
$$

By monotonicity of $\{\bar{u}_n\}$ (see Theorem 2.4 and (5.6)), there exist a subsequence (still denoted by $n$) and $u \in L_2(0, T; H^2_{\theta})$, $\mu \in \mathcal{M}^+$ such that $\bar{u}_n \rightarrow u$ in $L_2(0, T; H^2_{\theta})$, $\nabla \bar{u}_n \rightarrow \nabla u$ weakly in $L_2(\mathbb{R}^d \times [0, T])$ and $\mu_n \Rightarrow \mu$. In fact, by Proposition 2.1 and Theorem 2.4, $\nabla \bar{u}_n \rightarrow \nabla u$ in $L_p,\theta(\mathbb{R}^d \times [0, T])$ for every $p \in [1, 2)$. Therefore passing to the limit in (5.4) we see that

$$
\langle u, \frac{\partial \eta}{\partial t}\rangle_{\theta,T} - \langle L_t u + \eta, \eta\rangle_{\theta,T} = \langle \eta(T), \varphi \rangle_{2,\theta} + \langle f_u, \eta\rangle_{2,\theta,T} + \int_{Q_T} \eta \, d\mu
$$

for every $\eta \in C^\infty_0(\mathbb{R}^d \times [0, T])$ such that $\eta(0) \equiv 0$. From Theorem 4.6 and Lemma 4.8 (see also (4.21)) it follows that $\mu \in \mathcal{M}_0$. We know that

$$
\bar{u}_n(t, X_t) = \varphi(X_T) + \int_t^T f_u(\theta, X_\theta) \, d\theta + \int_t^T d\mu_n(\theta, X_\theta)
$$

for every $(s, x) \in F$. Putting $\bar{u} = \limsup_{n \rightarrow +\infty} \bar{u}_n$ we conclude from Theorem 2.4(b) that for every $(s, x) \in F$ the triple $(u_n(\cdot, X_\cdot), \nabla \bar{u}_n(\cdot, X_\cdot), \int_{\mathbb{R}^d} d\mu_n(\theta, X_\theta))$ converges in appropriate spaces to the solution $(\bar{u}(\cdot, X_\cdot), \nabla \bar{u}(\cdot, X_\cdot), \bar{K}^s,x)$ of RBSDE$_{s,\theta}(\varphi, f, h)$. In particular this implies that $\bar{u}$ is quasi-càdlàg. An analogous calculation to that in the proof of Theorem 4.6 (see (4.17)-(4.20)) shows that $d\mu(\cdot, X_\cdot) = dK^s,x$, $P_{s,x}$-a.s. for every $(s, x) \in F$. This proves that the triple $(\bar{u}(\cdot, X_\cdot), \nabla \bar{u}(\cdot, X_\cdot), \int_{\mathbb{R}^d} d\mu(\cdot, X_\cdot))$ is a solution of RBSDE$_{s,\theta}(\varphi, f, h)$ for every $(s, x) \in F$. In particular, this implies that for every $h \leq h^* \leq u$ such that $h^* \in \mathcal{P}$,

$$
E_{s,x} \int_s^T (\bar{u}_-(t, X_t) - h^*_-(t, X_t)) \, d\mu(t, X_t) = 0
$$

for every $(s, x) \in F$. Thus, $(u, \mu)$ is a solution of OP$(\varphi, f, h)$. (iii) follows immediately from (ii) and Theorem 2.4.

**Corollary 5.4.** If $h$ is quasi-continuous then the first component $u$ of the solution of the obstacle problem has a quasi-continuous version $\bar{u}$ and

$$
\int_{Q_T} (\bar{u} - h)^2 \, d\mu = 0.
$$

Moreover, $\mu(t) = 0$ for every $t \in [0, T]$. 

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Proof. Existence of a quasi-continuous version of $u$ follows immediately from Theorems 2.4 and 5.3. Since $\bar{u}, h$ are quasi-continuous, it follows from the definition of a solution of the obstacle problem that

$$E_s,x \int_s^T (\bar{u} - h)(t, X_t) \, d\mu(t, X_t) = 0$$

for q.e. $(s, x) \in Q_T$. Hence, by Aronson’s estimate, for every $\eta \in C_0^+(Q_T)$,

$$\int_{\mathbb{R}^d} (\bar{u} - h) \eta \, d\mu \leq C \int_{Q_T} \left( E_{0,x} \int_0^T ((\bar{u} - h) \eta)(t, X_t) \, d\mu(t, X_t) \right) \, dx = 0.$$

The second assertion follows immediately from continuity of the process $\int_s^\cdot d\mu(\theta, X_\theta)$. □

Example 5.5. In general, even if $h$ is quasi-l.s.c. or u.s.c., the integral $\int_{Q_T} (u - h) \, d\mu$ may be strictly positive. Indeed, let $a > 0$, $T > 1$, and let $h(t) = 1_{[0, T-1]} e^{at} + \frac{1}{2} 1_{[T-1, T]}$. One can check that the unique solution $(u, \mu)$ of the obstacle problem

$$\frac{\partial u}{\partial t} + au = -\mu, \quad u \geq h$$

is given by

$$u(t) = 1_{[0, T-1]}(t)(c + e^{a(T-t)}) + 1_{[T-1, T]}(t)e^{a(T-t)}, \quad \mu = c\delta_{\{T-1\}},$$

where $c = h(T - 1) - e^{a(T-1)}$, and that $\int_0^T (u - h)(t) \, d\mu(t) > 0$.

It is known that solutions of obstacle problems of the form (1.3) appear as value functions of optimal stopping time problems (see [8]). In case $L_t$ is non-divergent, it is known also that the value functions correspond to solutions of some RBSDE (see [12]). The following result is an analogue of the last correspondence in case of operators of the form (1.2). For some related results we refer to [27].

Corollary 5.6. Assume that (H1)–(H3) are satisfied and $h$ is quasi-continuous. Let $\bar{u}$ be a quasi-continuous version of the first component $u$ of the solution of OP($\varphi, f, h$). Then for every $t \in [s, T]$,

$$\bar{u}(t, X_t) = \sup_{\tau \in T^s_t} E_{s,x}(\int_t^\tau f(\theta, X_\theta, u(\theta, X_\theta), \sigma \nabla u(\theta, X_\theta)) \, d\theta$$

$$+ h(\tau, X_\tau) 1_{\tau < T} + \varphi(X_T) 1_{\tau = T} |G_t^s|),$$

where $T^s_t = \{ \tau \in T^s : t \leq \tau \leq T \}$ and $T^s$ denote the set of all $\{G_t^s\}$-stopping times.

Proof. Follows from [12] Proposition 2.3 and Theorem 5.3 □

Let us recall that a measurable function $u : \bar{Q}_T \to \mathbb{R}$ is called cap$_2$-quasi continuous (lower semi-continuous) if for every $\varepsilon > 0$ there exists an open set $U_\varepsilon \subset \bar{Q}_T$ such that $u|_{\bar{Q}_T \setminus U_\varepsilon}$ is continuous (l.s.c.) and cap$_2(U_\varepsilon) < \varepsilon$.

Proposition 5.7. Let $u \in W_\varepsilon$. Then there exists a version $\bar{u}$ of $u$ such that $\bar{u}$ is cap$_2$-quasi-continuous and quasi-continuous.
Proof. Let \( \{u_n\} \subset C_0^\infty(Q_T) \) be such that \( u_n \to u \) in \( W_\theta \) (for existence of such a sequence see [11, Theorem 2.11]). By [11, Lemma 2.20] we may assume that \( \bar{u} = \limsup_{n \to \infty} u_n \) is cap2-quasi-continuous. On the other hand, by [17] Corollary 5.5,
\[
\int_{Q_T} (E_{s,x} \sup_{s \leq t \leq T} |(u_n - u_m)(t, X_t)| \mathbb{q}(x) \, dx \, ds \to 0
\]
as \( n, m \to 0 \). Hence and [17] Proposition 3.3 we may assume that
\[
\sup_{s \leq t \leq T} |u_n(t, X_t) - u_m(t, X_t)| \to 0, \quad P_{s,x,a.s.}
\]
for q.e. \((s, x) \in Q_T\), which implies that \( \bar{u} \) is quasi-continuous, too. \( \square \)

Corollary 5.8. Each \( u \in \mathcal{P} \) has a version which is quasi-càdlàg and cap2-quasi-l.s.c.

Proof. Let \( u \in \mathcal{P} \) and for \( n \in \mathbb{N} \) let \( u_n \) be a solution of (5.3) with \( h = u \). By Proposition 5.7 each \( u_n \) has a version \( \bar{u}_n \) which is quasi-continuous and cap2-quasi-continuous. Since \( u \) is a solution of \( \text{OP}(u(T^-), f, u) \), where \( \bar{u} \) is a quasi-càdlàg version of \( u \) of Theorem 4.6 it follows from Theorem 5.3 that \( \bar{u}_n \uparrow \bar{u} \) q.e., which implies that \( \bar{u} \) is cap2-quasi-l.s.c., too. \( \square \)

Corollary 5.9. Assume (H1), (H2) and that \( h \in \mathbb{L}_{2, \varphi}(Q_T), \varphi \geq h(T) \) a.e.. Then

(i) There exists a solution of \( \text{OP}(\varphi, f, h) \) iff (1.10) is satisfied.

(ii) There exists a parabolic potential \( h^* \) such that \( h^* \geq h \) a.e. iff (1.10) is satisfied.

Proof. (i) The “only if” part follows from Theorem 4.6(i). To prove the “if” part it suffices to observe that in the proof of Theorem 5.3 part (ii) of condition (H3), i.e. existence of \( h^* \in \mathcal{P} \) such that \( h^* \geq h \) is used only to ensure that the left-hand side of (5.5) is bounded uniformly in \( n \in \mathbb{N} \).

(ii) That (H3) implies (1.10) follows immediately from part (i). If (1.10) is satisfied then by part (i) there is a solution \((u, \mu)\) of \( \text{OP}(\varphi, f, h) \). In particular, \( u \geq h \) and \( u \in \mathcal{P} \), so (H3) is satisfied with \( h^* = u \). \( \square \)

Corollary 5.10. The quasi-càdlàg version \( \bar{u} \) of the first component \( u \) of the solution of the problem \( \text{OP}(\varphi, f, h) \) is given by
\[
\bar{u} = \text{quasi-essinf} \{ \bar{v} \in \mathcal{P} : \bar{v} \geq h \text{ a.e., } \bar{v}(T^-) \geq \varphi \}.
\]

Proof. Of course \( \bar{u} \in \mathcal{P} \) and \( \bar{u} \geq h \) a.e.. By Lemma 4.12 \( \bar{u}(T^-) \geq \varphi \). Let \( \bar{v} \in \mathcal{P} \) be such that \( \bar{v} \geq h \) a.e. and \( \bar{v}(T^-) \geq \varphi \). Then by Theorem 4.6 there exists PAF \( K \) such that \( P_{s,x,a.s.}, \)
\[
\bar{v}(t, X_t) = \varphi(X_T) + \int_t^T f_{\bar{v}}(\theta, X_\theta) \, d\theta + \int_t^T dK_{s,\theta} - \int_t^T \sigma \nabla \bar{v}(\theta, X_\theta) \, dB_{s,\theta}
\]
for \( t \in [s, T] \). Since \( \bar{v} \geq h \) a.e.,
\[
\bar{v}(t, X_t) = \varphi(X_T) + \int_t^T (f_{\bar{v}} + n(\bar{v} - h)^-)(\theta, X_\theta) \, d\theta + \int_t^T dK_{s,\theta} - \int_t^T \sigma \nabla \bar{v}(\theta, X_\theta) \, dB_{s,\theta}.
\]

By comparison theorem for BSDEs (see [31, Theorem 1.3]) and Theorem 5.3 \( \bar{u}_n \leq \bar{v} \) q.e., where \( \bar{u}_n \) is defined as in Theorem 5.3 which implies that \( \bar{u} \leq \bar{v} \) q.e.. \( \square \)
Corollary 5.11. Let \((u_i, \mu_i)\) be a solution of \(\text{OP}(\varphi, f_i, h_i)\), \(i = 1, 2\). If
\[
\varphi_1 \leq \varphi_2, \quad f_1(\cdot, u_1, \sigma \nabla u_1) \leq f_2(\cdot, u_1, \sigma \nabla u_1), \quad h_1 \leq h_2
\]
a.e., then
\[
\bar{u}_1 \leq \bar{u}_2, \quad \text{q.e.,}
\]
where \(\bar{u}_1, \bar{u}_2\) denote quasi-càdlàg versions of \(u_1, u_2\), respectively. If, in addition, \(h_1 = h_2\) a.e., then
\[
d\mu_1 \leq d\mu_2.
\]

Proof. Follows from Theorem 4.6 and comparison theorem [32, Theorem 4.2] applied to solutions of BSDE\((\varphi_i, f_i + n(y - h_i)^-), i = 1, 2\).

In the case of linear equations, i.e. if \(f = f(t, x)\), some definition of solutions of the obstacle problem with irregular obstacles is proposed in [34]. We close this section with comparing it with our definition of solutions.

Proposition 5.12. If \(u \in \mathcal{B}(\bar{Q}_T)\) is cap\(_2\)-quasi-continuous then it is quasi-continuous.

Proof. Let \(u\) be cap\(_2\)-quasi-continuous and let \(\{E_n\}\) be the associated nest. Then for every \(n \in \mathbb{N}\), \(t \mapsto u|_{E_n}(t, X_t)\) is a continuous process on \(E_n\) for every \((s, x) \in E_n\). Therefore the result follows from [46, Lemma 3.10] and Remark 5.13 below.

Remark 5.13. In [28, 46] capacity on \(\mathbb{R}^{d+1}\) is defined similarly to cap\(_2\) but with \(\bar{Q}_T\) replaced by \(\mathbb{R}^{d+1}\). From [33, Lemma 4] it follows that the two capacities are equivalent on \(\bar{Q}_T\).

Let us define \(P_0\) similarly to \(\mathcal{P}\) but with \(L\) replaced by \(\partial_t + L_t\), and let \(P_0^+ = \{u \in P_0 : u \geq 0\}\). Given \(u \in P_0^+\) we set
\[
\mathcal{E}(u) = \varrho^2 \bar{u}(T-) \, dm + \varrho^2 \, d\mu,
\]
where \(\mu\) is the measure of Theorem 4.6 corresponding to \(\bar{u}\) defined by (4.11) with \(\varphi = \bar{u}(T-)\), and by \(\tau^f_\varphi\) we denote a unique solution of PDE\((\varphi, f)\).

The following definition of precise functions is given in [34, 35]. Proposition 5.15 is proved in [35], while Proposition 5.16 in [34].

Definition 5.14. \(u : (0, T] \times \mathbb{R}^d \to \mathbb{R}^d\) is called precise if there exists a sequence \(\{u_n\} \subset P_0^+\) such that each \(u_n\) has a cap\(_2\)-quasi-continuous version \(\bar{u}_n\) such that \(\bar{u}_n \downarrow u\) q.e.

Let us point out that in [35] some capacity on \((0, T] \times \mathbb{R}^d\) is considered. From results in [36] it follows that the capacity defined in [35] and the notion of quasi-continuity with respect to that capacity agree with capacity cap\(_2\) on \(\bar{Q}_T\) and the notion of cap\(_2\)-quasi-continuity on \(\bar{Q}_T\).

Proposition 5.15. Let \(u \in P_0^+\).

(i) There exists a unique, up to sets of capacity zero, version \(\hat{u}\) of \(u\) such that \(\hat{u}\) is precise.
(ii) There exists a sequence \( \{u_n\} \subset \mathcal{P}_0^+ \) such that \( u_n \to u \) in \( L_2(0, T; H^1_\varphi) \), and moreover, each \( u_n \) has a capz-up quasi-continuous version \( \bar{u}_n \) such that \( \bar{u}_n \downarrow \tilde{u} \) q.e.. 

In what follows, if \( u \) has a precise version, we denote it by \( \hat{u} \).

It is worth pointing out that if \( u \) has a capz-up quasi-continuous version \( \bar{u} \) then \( u \) has a precise version and \( \hat{u} = \bar{u} \). From [35] it follows also that \( \hat{u} \) is quasi-u.s.c. and \( (0, T) \ni t \mapsto \hat{u}(t) \in L_2(\mathbb{R}^d) \) is left continuous. In particular, it follows that \( \hat{u}(t) = \bar{u}(t-) \) for every \( t \in (0, T] \). Moreover if \( u, v \in \mathcal{P}_0^+ \) or \( u \in \mathcal{P}_0^+ \) and \( v \) has quasi-continuous version \( \tilde{v} \) then \( u + v = \hat{u} + \tilde{v} \), \( u + v = \hat{u} + \tilde{v} \).

Write

\[
C = \{u \in \mathcal{W}_\varphi + \mathcal{P}_0^+; \hat{u} \geq h, \text{ q.e.}\}.
\]

**Proposition 5.16.** For every \( h \) such that \( C \neq \emptyset \) there exists a unique capz-up quasi-u.s.c. function \( \hat{h} \) such that

\[
C = \{u \in \mathcal{W}_\varphi + \mathcal{P}_0^+; \hat{u} \geq \hat{h}, \text{ q.e.}\}.
\]

Moreover, there exists a sequence \( \{\hat{h}_n\} \subset \mathcal{W}_\varphi \) such that

\[
\hat{h} = \text{quasi-essinf}\{\hat{h}_n, n \geq 1\}.
\]

**Proposition 5.17.** Let \( u \in \mathcal{P}_0^+ \). Then for q.e. \( (s, x) \in Q_T \), \( [s, T] \ni t \mapsto \hat{u}(t, X_t) \) is càglàd under \( P_{s,x} \) and

\[
\hat{u}(t, X_t) = \bar{u}_-(t, X_t), \quad t \in [s, T], \quad P_{s,x} \cdot \text{a.s.}
\]

**Proof.** Let \( \{u_n\} \) be a sequence of Proposition 5.15(ii) and let \( \bar{u}_n \) be a quasi-continuous version of \( u_n \). Using Proposition 2.1 and [17] Proposition 3.3 we conclude that for some subsequence (still denoted by \( \{n\} \)),

\[
E_{s,x} \int_s^T |\sigma \nabla (u_n - u)(t, X_t)|^2 dt \to 0
\]

for q.e. \( (s, x) \in Q_T \). By Theorem 1.6 there exists PCAF \( K^n \) such that

\[
\bar{u}_n(t, X_t) = \bar{u}_n(s, x) - K^n_{s,t} - \int_s^t \sigma \nabla u_n(\theta, X_\theta) dB_{s,\theta}, \quad t \in [s, T], \quad P_{s,x} \cdot \text{a.s.}
\]

for q.e. \( (s, x) \in Q_T \). Therefore using the fact that \( \{u_n\} \) is decreasing and repeating arguments from the proof of [31] Theorem 2.1] we show that for q.e. \( (s, x) \in Q_T \) there is a càglàd process \( Y^{s,x} \) such that for q.e. \( (s, x) \in Q_T \),

\[
\bar{u}_n(t, X_t) \to Y^{s,x}_t, \quad t \in [s, T], \quad P_{s,x} \cdot \text{a.s.}
\]

On the other hand, since \( \bar{u}_n \downarrow \hat{u} \) q.e.,

\[
Y^{s,x}_t = \hat{u}(t, X_t), \quad t \in [s, T], \quad P_{s,x} \cdot \text{a.s.}
\]

for q.e. \( (s, x) \in Q_T \). Hence, since \( \bar{u}, \hat{u} \) are versions of \( u \),

\[
\int_{Q_T} (E_{s,x} \int_s^T |(\bar{u} - \hat{u})(t, X_t)|^2 dt) \varrho^2(x) dx ds = 0.
\]
Using arguments from Remark 3.2, one can deduce from the above that

$$E_{s,x} \int_s^T |(\bar{u} - \hat{u})(t, X_t)|^2 dt = 0$$

for q.e. $(s, x) \in Q_T$. From this and the fact that $t \mapsto \bar{u}(t, X_t)$ is càdlàg and $t \mapsto \hat{u}(t, X_t)$ is càglàd, the result follows.

From now on we assume that $f : Q_T \to \mathbb{R}$, i.e., we consider the linear problem, and we assume that $h(T) \leq \varphi$ a.e.

**Lemma 5.18.** Assume that (H1)–(H3) are satisfied. Let $(u, \mu)$ be a unique solution of $\text{OP}(\varphi, f, \hat{h})$. If $\hat{h}(T) \leq \varphi$ a.e., then $\mu(T) = 0$.

**Proof.** From [14] it is known that $\Delta(f_s^T d\mu(\theta, X_\theta)) = (\hat{h}(T, X_T) - \bar{u}(T, X_T))^+$, $P_{s,x}$-a.s. for q.e. $(s, x) \in Q_T$. On the other hand, by assumptions of the lemma, $(\hat{h}(T, X_T) - \bar{u}(T, X_T))^+ = (\hat{h}(T, X_T) - \varphi(X_T))^+ = 0$, $P_{s,x}$-a.s. for q.e. $(s, x) \in Q_T$ from which we easily deduce that $\mu(T) = 0$.

**Corollary 5.19.** Assume that (H1)–(H3) are satisfied. If $\hat{h}(T) \leq \varphi$ a.e., then $\bar{u}(T-) = \varphi$ a.e..

The following definition of a solution of the obstacle problem is given in [34] (for brevity we denote the problem by $\text{OP}$).

**Definition 5.20.** We say that $u \in \tau_{\varphi}^f + \mathcal{P}_0^+$ is a solution of $\overline{\text{OP}}(\varphi, f, \hat{h})$ if

1. $\hat{u} \geq \bar{h}$ q.e., $\hat{u}(T) = \varphi$,
2. $\int_{Q_T} (\hat{u} - \hat{h}) \, d\mathcal{E}(u - \tau_{\varphi}^f) = 0$.

**Proposition 5.21.** Let $(u, \mu)$ be a unique solution of $\text{OP}(\varphi, f, \hat{h})$. Then $u$ is a unique solution of $\overline{\text{OP}}(\varphi, f, \hat{h})$.

**Proof.** Let $u$ be the first component of a solution of $\text{OP}(\varphi, f, \hat{h})$. By the definition, $u \geq \bar{h}$ a.e., so $\hat{u} \geq \hat{h}$ q.e. (see Proposition 3.2). Thus, condition (i) of the definition is satisfied. Next observe that by linearity and uniqueness arguments, $u = \omega + \tau_{\varphi}^f$, where $(\omega, \nu)$ is a unique solution of $\text{OP}(0, 0, \hat{h} - \tau_{\varphi}^f)$. By Corollary 5.19, $\mathcal{E}(\omega) = \nu$. Of course, $\omega \in \mathcal{P}_0^+$ and $\hat{u} = \hat{\omega} + \tau_{\varphi}^f$. Let $\{h_n\}$ be a sequence of Proposition 5.16. Then by the definition of a solution of $\text{OP}(0, 0, \hat{h} - \tau_{\varphi}^f)$ and Proposition 5.17,

$$\int_{Q_T} (\hat{u} - \hat{h}_n) \, d\mathcal{E}(u - \tau_{\varphi}^f) = \int_{Q_T} (\hat{\omega} - (\hat{h}_n - \tau_{\varphi}^f)) \, d\mathcal{E}(\omega) \leq C \int_{\mathbb{R}^d} (E_{0,x} \int_0^T (\hat{\omega} - (\hat{h}_n - \tau_{\varphi}^f))(\theta, X_\theta) \, d\mathcal{E}(w)(\theta, X_\theta)) \, dx$$

$$= C \int_{\mathbb{R}^d} (E_{0,x} \int_0^T (\omega - (\hat{h}_n - \tau_{\varphi}^f))(\theta, X_\theta) \, d\mathcal{E}(w)(\theta, X_\theta)) \, dx \leq 0,$$

Taking infimum over $n \in \mathbb{N}$ yields $\int_{Q_T} (\hat{u} - \hat{h}) \, d\mathcal{E}(u - \tau_{\varphi}^f) \leq 0$, which completes the proof since $\hat{u} \geq \bar{h}$ q.e..
Notice that from Proposition 5.21 it follows that solutions of the obstacle problem in the sense defined in [34] are sensitive to changes of obstacles on sets of the Lebesgue measure zero. Indeed, one can easily find $h_1, h_2 \in \mathcal{P}_0^+$ such that $h_1 = h_2$ a.e. but $\hat{h}_1, \hat{h}_2$ differ on the set of positive capacity. Consequently, solutions of $\text{OP}(\varphi, f, \hat{h}_1)$, $\text{OP}(\varphi, f, \hat{h}_2)$ are different. In other words, in [34] definition of a solution with quasi-u.s.c. obstacle $\hat{h}$ rather than with $h$ is given. The second drawback of the definition given in [34] lies in the fact that it applies only to linear equations and that it does not allow solutions to have jumps in $T$ (the last property of solutions is forced by the assumption that $\hat{h}(T) \leq \varphi$).

6 Renormalized solutions of equations with measure data and BSDEs

In this section we present some connections between solutions of parabolic differential equations with measure data and BSDEs. Since we consider solutions on unbounded domain, some integrability assumptions on the measure must be imposed. We will consider measure data from the class $\mathcal{M}_0(\varrho) = \{ \mu \in \mathcal{M}_0; \int_{Q_T} \varrho^2 d|\mu| < \infty \}$. This class is quite natural because under (H1), (H2) second components of solutions of obstacle problems considered in Section 5 belong to $\mathcal{M}_0(\varrho)$.

Recall that from [11, Theorem 2.27] it follows that $\mathcal{M}_0(\varrho) = \mathcal{W}^1_{\varrho} \cap \mathcal{M}(\varrho) + \mathbb{L}_{1,\varrho}(Q_T)$, while by [11] Lemma 2.24, for every $\Phi \in \mathcal{W}^1_{\varrho}$ there exist $g \in \mathbb{L}_2(0, T; H^1_{\varrho})$ and $G, f \in \mathbb{L}_{2,\varrho}(Q_T)$ such that

$$\Phi = g_t + \text{div}G + f,$$

where

$$\langle g_t, \eta \rangle = -\langle g, \frac{\partial \eta}{\partial t} \rangle_{\varrho,T}, \quad \eta \in \mathcal{W}_{\varrho}.$$

Let us remark that in [11] proofs of the above two facts are given in the case of bounded domains but at the expense of minor technical changes they can be adapted to the case of $Q_T$.

In the theory of partial differential equations with measure data to guarantee uniqueness of solutions the so-called renormalized solutions are considered (see, e.g., [11]).

**Definition 6.1.** A measurable function $u : Q_T \rightarrow \mathbb{R}$ is called a renormalized solution of the Cauchy problem (1.12) if

(a) for some decomposition $(g, G, f)$ of the given measure $\mu$ such that $u - g \in \mathbb{L}_\infty(0, T, \mathbb{L}_2, \varrho(\mathbb{R}^d))$ and $T_n(u - g) \in \mathbb{L}_2(0, T; H^1_{\varrho})$ for $n \in \mathbb{N}$,

$$\lim_{n \to \infty} \int_{\{n \leq |u - g| \leq n + 1\}} |\nabla u(t, x)|^2 \varrho^2(x) \, dx \, dt = 0,$$

(b) for any $S \in \mathcal{W}^2_{\infty}(\mathbb{R})$ with compact support,

$$\frac{\partial}{\partial t} (S(u - g)) + \text{div}(a \nabla u S'(u - g)) - S''(u - g) \langle a \nabla u, \nabla (u - g) \rangle_2$$

$$= -S'(u - g)f - \text{div}(GS'(u - g)) + GS''(u - g)\nabla (u - g)$$

in the sense of distributions,
such that \( Y(\cdot, s, x) \) from \([10]\) and Proposition 2.1 it follows that for q.e. \((s, x)\) the Cauchy problem denoted by \( u(s, x) \) is the solution of (1.12) in the distributional sense (see \([44]\)), but it is known that there exists a unique renormalized solution. What is interesting here is that the renormalized solution is determined uniquely by a solution of some simple BSDE.

Let \( p > 0 \). By \( M^p \) we denote the space of all progressively measurable càdlàg processes \( Y \) such that \( E(\int_0^T |Y_t|^2 dt)^{p/2} < \infty \). \( D^p (S^p) \) is the subspace of \( M^p \) consisting of all càdlàg (continuous) processes such that \( E \sup_{0 \leq t \leq T} |Y_t|^p < \infty \).

All existence and uniqueness results for PDEs considered in the following theorem and its proof follow from \([11, 37]\).

**Theorem 6.2.** Assume that \( \varphi \in L_{\mu, 2}(\mathbb{R}^d), \mu \in \mathcal{M}_0(q) \). Let \( u \) be a renormalized solution of (1.12). Then there exists a quasi-càdlàg version of \( u \) (still denoted by \( u \)) such that for q.e. \((s, x) \in Q_T, u(\cdot, X) \in D^p, \nabla u(\cdot, X) \in M^p \) for every \( p \in (0, 1) \), and

\[
  u(t, X_t) = \varphi(X_T) + \int_t^T d\mu(\theta, X_\theta) - \int_t^T \sigma \nabla u(\theta, X_\theta) dB_{s, \theta}, \quad t \in [s, T], \quad \text{\( P_{s,x} \text{-a.s.} \)}
\]

In particular, for q.e. \((s, x) \in Q_T, \)

\[
  u(s, x) = E_{s,x} \varphi(X_T) + E_{s,x} \int_t^T d\mu(\theta, X_\theta).
\]

**Proof.** Let \( \Phi \in W'_q \cap \mathcal{M}_0(q) \) and \( f \in L_{\mu, 2}(Q_T) \) be such that \( \mu = \Phi + f \). Since the problem (1.12) is linear and \( \mu \) can be decomposed into a difference of positive measures, without loss of generality we may and will assume that \( \Phi \) is positive. Let \( u \) be a solution of (1.12) and let \( u_1, u_2 \) be solutions of the Cauchy problems

\[
  \frac{\partial u_1}{\partial t} + L_t u_1 = -\Phi, \quad u_1(T) = 0, \quad \frac{\partial u_2}{\partial t} + L_t u_2 = -f, \quad u_2(T) = \varphi.
\]

Of course, \( u = u_1 + u_2 \). By Theorem 1.6 there is a quasi-càdlàg version of \( u_1 \) (still denoted by \( u_1 \)) such that for q.e. \((s, x) \in Q_T, u_1(\cdot, X) \in D^2, \sigma \nabla u_1(\cdot, X) \in M^2 \) and

\[
  u_1(t, X_t) = \int_t^T d\Phi(\theta, X_\theta) - \int_t^T \sigma \nabla u_1(\theta, X_\theta) dB_{s, \theta}, \quad t \in [s, T], \quad \text{\( P_{s,x} \text{-a.s.} \)}
\]

From \([10]\) and Proposition 2.1 it follows that for q.e. \((s, x) \in Q_T \) there exists a solution \((Y^{t, x}, Z^{t, x}) \) of the BSDE

\[
  Y^{t, x}_t = \varphi(X_T) + \int_t^T f(\theta, X_\theta) d\theta - \int_t^T Z^{s, x}_\theta dB_{s, \theta}, \quad t \in [s, T], \quad \text{\( P_{s,x} \text{-a.s.} \)}
\]

such that \((Y^{t, x}, Z^{t, x}) \in S^p \otimes M^p \) for every \( p \in (0, 1) \). Let \( u_n^0, n \in \mathbb{N} \), be a solution of the Cauchy problem

\[
  \frac{\partial u_n^0}{\partial t} + L_t u_n^0 = -T_n(f), \quad u_n^0 = T_n(\varphi).
\]
It is known that $u^n_2 \to u_2$ in $L_q(0,T; W^{1,q}_{q,0})$ for $q < \frac{d+2}{d+1}$ (see \[37\]). From Proposition 3.6 it follows that there exists a quasi-continuous version of $u^n_2$ (still denoted $u^n_2$) such that $(u^n_2(\cdot,X), \sigma \nabla u^n_2(\cdot,X)) \in S^2 \otimes M^2$ and

$$
\begin{align*}
  u^n_2(t, X_t) &= T_n(\varphi)(X_T) + \int_t^T T_n(f)(\theta, X_\theta) d\theta \\
  &- \int_t^T \sigma \nabla u^n_2(\theta, X_\theta) dB_{s,\theta}, \quad t \in [s, T], \quad P_{s,x}-a.s.
\end{align*}
$$

for q.e. $(s,x) \in Q_{\tilde{T}}$. By standard arguments (see the proof of \[10\] Proposition 6.4]), it follows that $(u^n_2(\cdot,X), \sigma \nabla u^n_2(\cdot,X)) \to (Y^{s,x}, Z^{s,x})$ in $S^p \otimes M^p$ for $p \in (0, 1)$, which completes the proof. \hfill \Box

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