Yangians and Yang-Baxter $R$-operators for ortho-symplectic superalgebras

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Abstract

Yang-Baxter relations symmetric with respect to the ortho-symplectic superalgebras are studied. We start from the formulation of graded algebras and the linear superspace carrying the vector (fundamental) representation of the ortho-symplectic supergroup. On this basis we study the analogy of the Yang-Baxter operators considered earlier for the cases of orthogonal and symplectic symmetries: the vector (fundamental) $R$ matrix, the $L$ operator defining the Yangian algebra and its first and second order evaluations. We investigate the condition for $L(u)$ in the case of the truncated expansion in inverse powers of $u$ and give examples of Lie algebra representations obeying these conditions. We construct the $R$ operator intertwining two super-spinor representations and study the fusion of $L$ operators involving the tensor product of such representations.

1 Introduction

The orthogonal and the symplectic groups have similarities, which can be traced back to the existence of an invariant bilinear form in the fundamental representation space. The similarities allow a unified treatment not only of the groups and Lie algebras but also of the Yang-Baxter relations with such symmetries. The orthogonal and symplectic groups are embedded in the ortho-symplectic ($OSp$) supergroup.

In the present paper we study Yang-Baxter relations with ortho-symplectic ($osp$) supersymmetry. In a recent paper\textsuperscript{3} Yang-Baxter relations symmetric with respect to orthogonal or symplectic groups have been considered in a unified formulation, continuing preceding studies\textsuperscript{[1]}\textsuperscript{[3]},\textsuperscript{[4]},\textsuperscript{[5]},\textsuperscript{[6]},\textsuperscript{[7]},\textsuperscript{[8]} and especially\textsuperscript{[13, 21]} where orthogonal and partially symplectic symmetries have been considered. We rely on the similarities, proceed to some extend parallel to\textsuperscript{[3]} and we follow similar motivations. The Yang-Baxter operators, in

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particular their dependence on the spectral parameter, are generally more involved as compared to the case of general linear symmetry. Compact forms of $L$ operators appear only in cases of particular representations. Such cases are of interest in physical applications, in particular for the theory of quantum integrable systems where the $L$ operators are used as building blocks. The viewpoint of the Yangian algebra and its finite order evaluations is appropriate to understand the cases of explicit solutions for $L$ operators.

The extension to the $osp$ supersymmetric case is of interest in the studies of integrable spin chains of $osp$-type [20] and exact factorizing $S$ matrices [10] for two-dimensional field theories with ortho-symplectic supersymmetries. It is also relevant to investigations of gauge field theories, in particular because of the conformal transformations form the (pseudo)orthogonal groups. From the mathematical point of view it would be interesting to extend the Drinfeld’s classification theorem for the finite-dimensional irreducible representations of the Yangians of the classical type (see [25] and references therein) to the case of the $osp$-type Yangians.

The generalization of $so$ and $sp$ cases to the supersymmetric $osp$ case is not straightforward. We shall see that it takes much attention and carefully prepared formulations to obtain in a systematic way the correct forms of all the relations starting from the theory of super-matrices of $OSp$ and its super-algebra up to the Yang-Baxter relations and the forms of the $L$ operators related to the first and second order evaluations of the Yangian $Y(osp)$. Interesting and intriguing details are related to sign factors. For deriving a particular relation one could choose the approach of starting from an ansatz up to sign and then invent a test to fix the sign. For our purpose that approach would be not sufficient, not only because of lack of elegance but mostly because it would cost more efforts then the derivations of all formulas from the first principles. The complete formulation of Lie supergroups $Osp$, their Lie superalgebras and Yangians $Y(osp)$ given in this paper provides a convenient basis for further studies and applications.

We outline the contents emphasizing the main points of the paper.

We present a systematic and detailed super-symmetric formulation of the Yang-Baxter relations and the involved operators. To set up the formulation we start in Sect. 2 with the superspace where the ortho-symplectic super-matrices act. Recalling the notion of invariant tensors we prepare the formulation of the fundamental $R$-matrix of $osp$-type. The involved invariant operators acting in the twofold tensor product of the superspace are shown to represent the Brauer centralizer algebra [2] (see also [5],[6]). In Section 2 we explain a lot of details and useful notations important in our formulation. In particular we introduce the notion of the sign operator which is useful in many relations. The action of a super-group element $U$ on a higher tensor product of the fundamental superspaces is described by the tensor product of operators $U$ dressed by these sign operators.

Whereas the braid form of the fundamental Yang-Baxter relation coincides for the supersymmetric case with the case without supersymmetry, factors of sign operators make the difference when changing to the $R$ operators including super-permutation. This is explained in Sect. 3. The modifications by factors of sign operators appear in the $RLL$ relations as well as discussed in Sect. 4.

The graded $RLL$ relation involving the $osp$ supersymmetric fundamental $R$ matrix serves as the defining relation of the $L$ operators. It defines the $osp$ Yangian algebra following the known scheme described in Sect. 4: By expanding $L(u)$ in inverse powers of the spectral parameter $u$ one obtains the Yangian algebra generators $L_{ab}^{(k)}$. Substituting this expansion in the $RLL$ relation one obtains all Yangian algebra relations. The first non-trivial term in the $u^{-1}$ expansion involves the $osp$ Lie superalgebra generators...
We formulate their symmetry relation and the Lie algebra commutation relations as derived from the $RLL$ relation. For the super traceless part $G$ of the matrix $L^{(1)}$ the commutation relations are to be compared with the ones obtained for the infinitesimal $OSp$ supermatrices formulated earlier in Sect. 2. We show that the corresponding basis sets are transformed into each other by the sign operator.

In Sect. 5 we show that in the linear evaluation of $\mathcal{Y}(osp)$ the matrix $L^{(1)}$ of generators of $osp$ algebra has to obey the additional constraint in form of a quadratic characteristic identity. We construct the infinite dimensional super spinor representation of $osp$ obeying this constraint as a generalization of the spinor representation of $so$ and the metaplectic representation of $sp$. We formulate the super spinor representation in terms of an algebra of super oscillators which generalizes the Clifford algebra and the standard multidimensional oscillator algebra.

In Sect. 6 we construct the super spinorial $R$ operator which acts in the tensor product of two super spinor representations. Our systematic supersymmetric formulation allows to extend to the $osp$ case the approach developed for the $so$ case in [13], [21] and for the $sp$ case in [3].

The graded symmetrisation of products of super oscillator operators is treated by the generating function techniques using auxiliary variables obeying a graded multiplication law.

The fusion procedure with superspinor $L$ operators is considered in Sect. 7. The projection of the tensor product of superspinor representations to the fundamental one applied to the product of such $L$ operator matrices reproduces the fundamental $R$ matrix.

In Sect. 8 we investigate the quadratic evaluation of the Yangian. We formulate the conditions implied by the $RLL$ relation with $L$ depending quadratically on the spectral parameter. The sign operator helps to formulate these conditions in a form similar to the ones in the cases of orthogonal and symplectic symmetry. We construct the solution where the second non-trivial term in $L$ is the quadratic polynomial in the generator matrix appearing in the constraint for the first order evaluation. The conditions imply constraints on the Lie superalgebra representation which are fulfilled by the Jordan-Schwinger ansatz for the generators in terms of graded Heisenberg pairs. In this case the generator matrix obeys the super anticommutator constraint which appeared in Sect. 6 in the superspinorial $RLL$ relation, and as a consequence, the generator matrix obeys a cubic characteristic relation.

2 The ortho-symplectic supergroup

2.1 The superspace, the super-group $OSp$ and its superalgebra $osp$

To fix notations we need to formulate the notion of the ortho-symplectic supergroups and their Lie superalgebras (see, e.g., [1] as an introduction to the super analysis and to the theory of Lie supergroups and superalgebras).

We denote by grad($A$) the grading of the algebraic object $A$. Let $V_{(N|M)}$ be a superspace and $z^a$ ($a = 1, \ldots, N + M$) the graded coordinates in $V_{(N|M)}$. We distinguish $N$ even and $M$ odd coordinates $z^a$, denote the grading grad($z^a$) of the coordinate $z^a$ as $[a] = 0, 1$ (mod2) and call $[a]$ the degree of the index $a$. If the coordinate $z^a$ is even then $[a] = 0$ (mod2), and if the coordinate $z^a$ is odd then $[a] = 1$ (mod2). The coordinates $z^a$
and \( w^a \) of two supervectors \( z, w \in \mathcal{V}_{(N|M)} \) commute as
\[
z^a w^b = (-1)^{|a||b|} w^b z^a . \tag{2.1}
\]
Thus the coordinates of the vectors \( z, w, \ldots \in \mathcal{V}_{(N|M)} \) can be considered as generators of a graded algebra. For two homogeneous elements \( A, B \) of this algebra we have
\[
\text{grad}(A \cdot B) = \text{grad}(A) + \text{grad}(B) , \quad A \cdot B = (-1)^{\text{grad}(A) \text{grad}(B)} B \cdot A .
\]
Further we endow the superspace \( \mathcal{V}_{(N|M)} \) with the bilinear form
\[
(z \cdot w) \equiv \varepsilon_{ab} z^a w^b = z^a w_a = z_b w_d \varepsilon^{ab} , \tag{2.2}
\]
where the super-metric \( \varepsilon_{ab} \) has the property
\[
\varepsilon_{ab} = \epsilon(-1)^{|a||b|} \varepsilon_{ba} \iff \varepsilon^{ab} = \epsilon(-1)^{|a||b|} \varepsilon^{ba} , \tag{2.3}
\]
and the matrix \( ||\varepsilon^{ab}|| \) is inverse to the matrix \( ||\varepsilon_{ab}|| \). Here \( \epsilon = \pm 1 \). Moreover, we require that the super-metric \( \varepsilon_{ab} \) is even in the sense that \( \varepsilon_{ab} \neq 0 \) iff \( |a| + |b| = 0 (\text{mod}2) \). This means that
\[
\varepsilon_{ab} = (-1)^{|a|+|b|}\varepsilon_{ab} , \tag{2.4}
\]
and therefore the properties \( \text{(2.3)} \) can be written as \( \varepsilon_{ab} = \epsilon(-1)^{|a|} \varepsilon_{ba} = \epsilon(-1)^{|b|} \varepsilon_{ba} \) (the same for \( \varepsilon^{ab} \)). Taking into account \( \text{(2.1)} \) we obtain \( (z \cdot w) = \epsilon (w \cdot z) \), i.e., the bilinear form \( \text{(2.2)} \) is symmetric for \( \epsilon = +1 \) and skew-symmetric for \( \epsilon = -1 \). Further we adopt the following rule for rising and lowering indices,
\[
z_a = \varepsilon_{ab} z^b , \quad z^a = \varepsilon^{ab} z_b . \tag{2.5}
\]
This rule will be applied also for any tensors of higher ranks. According to this rule we have
\[
\varepsilon^{ab} = \varepsilon^{ac} \varepsilon^{bd} \varepsilon_{cd} = \varepsilon^{ba} = \epsilon(-1)^{|a||b|} \varepsilon^{ab} , \tag{2.6}
\]
We see that the metric tensor with upper indices \( \varepsilon^{ab} \) does not coincide with the inverse matrix \( \varepsilon^{ab} \). Further we shall use only the inverse matrix \( \varepsilon^{ab} \) and never the metric tensor \( \varepsilon^{ab} \). So to simplify formulas below we omit the bar in the notation \( \varepsilon^{ab} \) and write simply \( \varepsilon^{ab} \) keeping in mind that this is the inverse matrix for the metric \( \varepsilon_{ab} \).

Consider a linear transformation in \( \mathcal{V}_{(N|M)} \)
\[
z^a \rightarrow z'^a = U^a_b z^b , \tag{2.7}
\]
which preserves the grading of the coordinates \( \text{grad}(z'^a) = \text{grad}(z^a) \). For the elements \( U^a_b \) of the supermatrix \( U \) from \( \text{(2.7)} \) we have \( \text{grad}(U^a_b) = |a| + |b| \). The ortho-symplectic group \( OSp \) is defined as the set of supermatrices \( U \) which preserve the bilinear form \( \text{(2.2)} \) with respect to the transformations \( \text{(2.7)} \):
\[
\varepsilon_{ab} z'^a w'^b = \varepsilon_{ab} U^a_c z^c U^b_d w^d = (-1)^{|c||b|+|d|} \varepsilon_{ab} U^a_c U^b_d \varepsilon^{cd} w^d = \varepsilon_{cd} z^c w^d \quad \Rightarrow
\]
\footnote{An alternative rule for lifting and lowering indices (instead of \( \text{(2.5)} \)) is \( z_a = \varepsilon_{ab} z^b , z^a = z_b \varepsilon^{ba} \), where \( \varepsilon^{ab} \) is the metric tensor \( \text{(2.6)} \) with upper indices. So, to lower the index we act by the matrix \( \varepsilon_{ab} \) from the left, while to lift the index we act by the matrix \( \varepsilon^{ba} \) from the right. In this case one should remember the unusual conditions of inversion: \( \varepsilon_{ab} \varepsilon^{ac} = \delta^c_b \). Note that is not the rule adopted below.}
We have to mention that the first equation in \((2.8)\) can be also rewritten by using the properties of the supermetric as
\[
( -1 )^{c([a]+[c])} \varepsilon_{ab} U^a_c U^b_d = \varepsilon_{cd}.
\] (2.8)

We represent \((2.8)\) in coordinate-free form as
\[
\varepsilon_{(12)} U_1 ( -1 )^{12} U_2 ( -1 )^{12} = \varepsilon_{12} \iff U_1 ( -1 )^{12} U_2 ( -1 )^{12} \varepsilon_{12} = \varepsilon_{12},
\] (2.9)
where we have used the concise matrix notations
\[
\varepsilon^{12) \in \mathcal{V}_{(N|M)} \otimes \mathcal{V}_{(N|M)}, \quad U_1 = U \otimes I, \quad U_2 = I \otimes U,
\]
\[
\left( ( -1 )^{a_1 a_2} b_1 b_2 = ( -1 )^{[a_1][a_2]} \delta^{a_1}_{b_1} \delta^{a_2}_{b_2}, \quad ( -1 )^{12} \in \text{End}(\mathcal{V}_{(N|M)} \otimes \mathcal{V}_{(N|M)}) \right).
\]

Here \(\otimes\) denotes the graded tensor product:
\[
(I \otimes B)(A \otimes I) = ( -1 )^{[A][B]} (A \otimes B),
\] (2.11)
where \([A] := \text{grad}(A)\) and \([B] := \text{grad}(B)\). The sign operator \((-)^{12}\) is an extremely useful tool for the \(R\)-matrix formulation of quantum supergroups and their particular cases as super-Yangians. This operator was used first in \([4]\). For more details about the graded tensor product and its consequences for the record of equations, especially the Yang-Baxter equation, see appendix \([A]\).

We also describe further details of this formalism in the next subsection.

The set of supermatrices \(U\) which satisfy equations \((2.8)\) form the supergroup \(OSp(N|M)\) for \(\epsilon = +1\) and the supergroup \(OSp(M|N)\) for \(\epsilon = -1\) with respect to the usual matrix multiplication. Let us consider the defining relations \((2.8)\) for \(OSp\) groups elements close to the unit element \(U = I + A + \ldots\). As a result we obtain the conditions for the elements \(A\) of the Lie superalgebra \(osp\) of the supergroup \(OSp\):
\[
( -1 )^{c([b]+[d])} \varepsilon_{ab} \left( \delta^a_c A^b_d + A^a_c \delta^b_d \right) = \left( ( -1 )^{[c][d]} \varepsilon_{cb} A^d_a + \varepsilon_{ad} A^a_c \right) = 0,
\] (2.12)
or equivalently
\[
A_{cd} = - \varepsilon ( -1 )^{[c][d]} \varepsilon_{ad} A^a_c .
\] (2.13)

The coordinate free form of \((2.12)\), \((2.13)\) can be obtained directly from \((2.9)\):
\[
\varepsilon_{12} (A_1 + (-)^{12} A_2 (-)^{12}) = 0 \iff (A_1 + (-)^{12} A_2 (-)^{12}) \varepsilon_{12} = 0 .
\] (2.14)

**Remark.** The set of supermatrices \(A\), which satisfy \((2.12)\), \((2.13)\), forms a vector space over \(\mathbb{C}\) denoted as \(osp\). One can check that for two supermatrices \(A, B \in osp\) the commutator
\[
[A, B] = AB - BA,
\] (2.15)
also obeys \((2.12)\), \((2.14)\) and thus belongs to the vector space \(osp\). It means that \(osp\) is an algebra. Any matrix \(A\) which satisfies \((2.12)\), \((2.14)\) can be represented as
\[
A^a_c = E^a_c - ( -1 )^{[c][d]} \varepsilon_{cb} E^b_d \varepsilon^{da} .
\] (2.16)
where \( ||E^a_c|| \) is an arbitrary matrix. Let \( \{ e_g^f \} \) be the matrix units, \( (e_g^f)^b_d = \delta_d^g \delta_f^b \). If we choose \( E = e_g^f = \varepsilon_g^{f'} e_g^f e_g^{f'} \) in (2.16) then we obtain the basis \( \{ G^f_g \} \) in the space \( osp \) of matrices (2.14):

\[
(G^f_g)^a_c \equiv (e_g^f)^a_c = - (1)^{[c]+[d]} \varepsilon_{cb}(e_g^f)^b_d = \varepsilon^{fa} \varepsilon_{gc} - \varepsilon(-1)^{[c]} \delta_f^a \delta_g^c .
\]  

(2.17)

Now any super-matrix \( A \in osp \) which satisfy (2.12), (2.14) can be expanded in this basis

\[
A^a_c = a^f_j (G^f_g)^a_c ,
\]

(2.18)

where \( a^f_j \) are super-coefficients. Since the elements \((G^f_g)^a_c \) are even, i.e., \((G^f_g)^a_c \neq 0 \) iff \([f] + [g] + [a] + [c] = 0 \) (mod2), then from grad\((A^a_c) = [a] + [c] \) we obtain that grad\((a^f_j) = [g] + [f] \). This means that the ordinary commutator (2.15) can rewritten as

\[
[A,B]^a_c = [a^f_j (G^f_g), b^k_n(G^k_n)]^a_c = a^f_j b^k_n ([G^f_g, G^k_n])^a_c ,
\]

where we define the super-commutator

\[
([G^a_{b_1}, G^a_{b_2}]^c) = (G^a_{b_1} G^a_{b_2} b_3^c - (1)^{[b_1] + [b_2]} (G^a_{b_2} G^a_{b_1} b_3) c ).
\]

(2.19)

By using the explicit representation (2.17) for \( \tilde{G}^a \) one can evaluate the supercommutator (2.19) and obtain the defining relations for the Lie superalgebra \( osp \):

\[
[\tilde{G}^a_{b_1}, \tilde{G}^a_{b_2}]^c = - (1)^{[a_1] + [a_2] + [b_1] + [b_2]} \varepsilon^{a_1 a_2} \tilde{G}_{b_1 b_2} + \varepsilon(-1)^{[b_1] + [b_2]} \delta^{a_1 a_2} \tilde{G}^a_{b_2} +

+(1)^{[a_1] + [a_2] + [b_1] + [b_2]} \varepsilon_{b_1 b_2} \tilde{G}^{a_1 a_2} - \varepsilon(-1)^{[a_1] + [a_2] + [b_1] + [b_2]} \delta^{a_1 a_2} \tilde{G}^a_{b_2}. \]

(2.20)

In component free form we write (2.20) as following

\[
[-12 \tilde{G}_{13}(-)^{12}, \tilde{G}_{23}] = [\varepsilon \mathcal{P}_{12} - \mathcal{K}_{12}, \tilde{G}_{23}] ,
\]

(2.21)

where we have introduced matrices the \( \mathcal{K}, \mathcal{P} \in \text{End}(V^{\otimes 2}_{(N|M)})\):

\[
\mathcal{K}^{a_1 a_2}_{b_1 b_2} = \varepsilon^{a_1 a_2} \varepsilon_{b_1 b_2} , \quad \mathcal{P}^{a_1 a_2}_{b_1 b_2} = (1)^{[a_1] + [a_2]} \delta^{a_1 a_2} \delta_{b_1 b_2} .
\]

(2.22)

The matrix \( \mathcal{P} \) is called the superpermutation operator since it permutes super-spaces. For example, using this matrix one can write (2.21) as \( \mathcal{P}_{cd} w^c z^d = z^a w^b \). The properties of the operators \( \mathcal{P} \) and \( \mathcal{K} \) which will be used below are listed in the appendix [B].

Note that the \( osp \) conditions (2.14) for the basis matrices \( (G^f_g)^a_c \) are represented in the form

\[
\mathcal{K}_{12}(\tilde{G}_{31} + (-)^{12} \tilde{G}_{32}(-)^{12}) = 0 , \quad \tilde{G}_{31} + (-)^{12} \tilde{G}_{32}(-)^{12}) \mathcal{K}_{12} = 0 ,
\]

(2.23)

Taking into account their property \( (G^a_{c_3})^{a_1}_{c_1} = (1)^{[a_1] [c_3]} (G^a_{c_1})^{a_2}_{c_3} (-1)^{[c_3]} \) evident from (2.17) one can rewrite (2.23) as

\[
\mathcal{K}_{12}((-)^{12} \tilde{G}_{13}(-)^{12} + \tilde{G}_{23}) = 0 , \quad ((-)^{12} \tilde{G}_{13}(-)^{12} + \tilde{G}_{23}) \mathcal{K}_{12} = 0 .
\]

(2.24)

Moreover one can check the identities (see appendix [B])

\[
\mathcal{P}_{12}((-)^{12} \tilde{G}_{13}(-)^{12}) = \tilde{G}_{23} \mathcal{P}_{12} , \quad (-)^{12} \tilde{G}_{13}(-)^{12} \mathcal{P}_{12} = \mathcal{P}_{12} \tilde{G}_{23} .
\]

(2.25)

By using (2.24) and (2.25) we can rewrite relations (2.21) as

\[
((-)^{12} \tilde{G}_{13}(-)^{12} , \tilde{G}_{23}) = [\varepsilon \mathcal{P}_{12} - \mathcal{K}_{12} , (-)^{12} \tilde{G}_{13}(-)^{12} .
\]

(2.26)

This shows that the \( osp \) defining relations (2.20) can be written in several equivalent forms.
2.2 OSp super-group invariants

Here we give the formulation of invariance of tensors with respect to actions of a supergroup. This formulation will be important for our discussion below of the invariance of the osp-type $R$-matrices which are solutions of Yang-Baxter equations.

First we recall the case of an ordinary group. Let $G$ be a matrix group which acts in the vector space $\mathcal{V}$. We say that the tensor $\mathcal{O}_{b_1\cdots b_j}$ with $k$ upper and $j$ lower indices is invariant w.r.t. a group $G$, if for all $U \in G$

$$U^{a_1}_{\cdots a_k} U^{d_1}_{\cdots d_j} \mathcal{O}_{b_1\cdots b_j} = \mathcal{O}_{d_1\cdots d_j} U^{a_1}_{\cdots a_k} U^{d_1}_{\cdots d_j}$$

(2.27)

This is represented at the level of its Lie algebra $\mathcal{G}$ as: For all $A \in \mathcal{G}$ holds that

$$\sum_{i=1}^{k} A_{c_i}^{a_i} \mathcal{O}_{b_1\cdots b_j} - \sum_{i=1}^{j} \mathcal{O}_{b_1\cdots b_j} A_{b_i}^{d_i} = 0.$$  

(2.28)

If the tensor $\mathcal{O}$ is an operator in $\mathcal{V}^{\otimes k}$, we have $j = k$ and both conditions can be written in the concise form as

$$U_1 \cdots U_k \mathcal{O}_{1\ldots k} = \mathcal{O}_{1\ldots k} U_1 \cdots U_k, \quad [\mathcal{O}, A_1 + \cdots + A_k] = 0,$$

(2.29)

where $1, \ldots, k$ are labels of super-vector spaces.

Now we extend the above formulation to the case of a supergroup $G$ which acts in a superspace $\mathcal{V}$. Let us start with looking for a compact expression describing an action of a supergroup $G$ on graded tensor products $x^{(1)}y^{(2)} \equiv x \otimes y$, where $x, y \in \mathcal{V}$. First of all, let us understand the meaning of the action on it by the tensor product of two superoperators $A_1 B_2 \equiv A \otimes B$

$$A x \otimes B y = (-1)^{|B||x|}(A \otimes B)(x \otimes y) \iff A_1 x^{(1)} B_2 y^{(2)} = A_1 (-)^{12} B_2 (-)^{12} x^{(1)} y^{(2)}.$$  

(2.30)

Here we introduce the operator $(-)^{12}$ acting on the tensor product of two homogeneous vectors as

$$(-)^{12} x^{(1)} y^{(2)} \equiv (-1)^{|x||y|} x^{(1)} y^{(2)}.$$  

(2.31)

In coordinates the operator $(-)^{12}$ has been given above in (2.10). We call it the sign operator. It satisfies

$$(-)^{12} (-)^{23} = (-)^{23} (-)^{12}, \quad (-)^{21} = (-)^{12}, \quad ((-)^{12})^2 = 1.$$  

(2.32)

The formula (2.30) can be generalized obviously for higher numbers of vector spaces in the tensor product $\mathcal{V}^{\otimes k}$. Let an operator $C$ act the $j$-th super-space in $\mathcal{V}^{\otimes k}$, then

$$x^{(1)} y^{(2)} \cdots C_j z^{(j)} \cdots w^{(k)} = C_{(1\ldots j)} x^{(1)} y^{(2)} \cdots z^{(j)} \cdots w^{(k)},$$

where we define the operator dressed by sign operators as

$$C_{(1\ldots j)} \equiv (-)^{1-j} \cdots (-)^{2-j} j \cdot C_j \cdot (-)^{1-j} \cdots (-)^{2-j} (-)^{1-j},$$  

(2.33)

and $(-)^{\ell,j} = (-)^{j \ell}$ denotes the sign operator which acts nontrivially only in $\ell$-th and $j$-th factors of the tensor product $\mathcal{V}^{\otimes k}$. Then the formula (2.30) is generalized as

$$A_1 x^{(1)} B_2 y^{(2)} \cdots C_j z^{(j)} \cdots D_k w^{(k)} = (A_1 \cdot B_{(12)} \cdots C_{(1\ldots j)} \cdots D_{(1\ldots k)}) x^{(1)} y^{(2)} \cdots z^{(j)} \cdots w^{(k)}.$$  

(2.34)
We specify now the super-operators to the elements of the super-group \( U \in G \). Taking into account (2.30) the action of \( U \) in \( \mathcal{V}^{\otimes 2} \) can be expressed using the sign operators \((-)^{12}\)

\[
x^{ab} \otimes y^{cd} = U^a_{b} x^b \otimes U^c_{d} y^d = (-1)^{[b][c]+[d][e]} (U^a_{b} \otimes U^c_{d})(x^b \otimes y^d) = \\
U_1^{\hat{R}^{[abc]de}} U_1^{\hat{R}^{[abc]de}} ((-)^{12)^{b}u_{cd} x^u y^d} \equiv x^{(1)} y^{(2)} = U_1(-)^{12} U_2(-)^{12} x^{(1)} y^{(2)} . \tag{2.35}
\]

Let \( \hat{R}_{12} \) be an operator acting in \( \mathcal{V}^{\otimes 2} \). The invariance of \( \hat{R}_{12} \) w.r.t. the action of \( G \) is expressed as

\[
\hat{R}_{12} U_1 (-)^{12} U_2 (-)^{12} = U_1 (-)^{12} U_2 (-)^{12} \hat{R}_{12} . \tag{2.36}
\]

Note that for matrix \( R_{12} = \mathcal{P}_{12} \hat{R}_{12} \), where \( \mathcal{P}_{12} \) is the superpermutation (2.22), the invariance condition modifies to

\[
R_{12} U_1 (-)^{12} U_2 (-)^{12} = (-)^{12} U_2 (-)^{12} U_1 R_{12} . \tag{2.37}
\]

The generalization of (2.36) for an operator \( O_{1...k} \) acting in \( \mathcal{V}^{\otimes k} \) for arbitrary \( k \) is obtained straightforwardly:

\[
O_{1...k} \left( \prod_{j=1}^{k} U_{\{1...j\}} \right) = \left( \prod_{j=1}^{k} U_{\{1...j\}} \right) O_{1...k} , \quad \prod_{j=1}^{k} U_{\{1...j\}} \equiv U_{\{1\}} \cdot U_{\{12\}} \cdot \cdots U_{\{1...k\}} . \tag{2.38}
\]

Then, for corresponding Lie super-algebra \( G \) we obtain \((\forall A \in G)\):

\[
\left[ O_{1...k} , \sum_{j=1}^{k} A_{\{1...j\}} \right] = 0 . \tag{2.39}
\]

The considerations above were done for operators only. They can be generalized to general tensors. With slight abuse of the notation a general tensor \( T \in \mathcal{V}^{\otimes k} \otimes \mathcal{V}^{\otimes j} \) with \( k \) upper and \( j \) lower indices can be written as \( T^{1...k}_{\{1...j\}} \). Here \( \mathcal{V} \) denotes the superspace dual to \( \mathcal{V} \). In this notation the operator \( O_{1...k} \) is written as \( O^{1...k}_{\{1...k\}} \). Then the invariance of the tensor \( T \) w.r.t. the super-group \( G \) is expressed as

\[
T^{1...k}_{\{1...j\}} \left( \prod_{i=1}^{j} U_{\{1...i\}} \right) = \left( \prod_{i=1}^{j} U_{\{1...i\}} \right) T^{1...k}_{\{1...j\}} . \tag{2.40}
\]

The infinitesimal form of (2.40) which generalizes (2.39) is

\[
T^{1...k}_{\{1...j\}} \left( \sum_{i=1}^{j} A_{\{1...i\}} \right) = \left( \sum_{i=1}^{j} A_{\{1...i\}} \right) T^{1...k}_{\{1...j\}} . \tag{2.41}
\]

We see that the relations of the super-metric invariance (2.9) and (2.14) are just special cases of the general formulas (2.40) and (2.41) for \( j = 2, k = 0 \) and for \( j = 0, k = 2 \).

### 3 The fundamental R-matrix and the graded Yang–Baxter equation

We have three \( OSp \) invariant operators in \( \mathcal{V}^{\otimes 2}_{(N|M)} \): the identity operator \( 1 \), the superpermutation operator \( \mathcal{P} \) and the tensor \( \mathcal{K} \). The super-permutation \( \mathcal{P}_{12} \) is a product of the usual permutation \( P_{12} \) and the sign operator \((-)^{12}\)

\[
\mathcal{P}_{12} = (-)^{12} P_{12} \quad \text{and in coordinates} \quad \mathcal{P}_{b_1 b_2}^{a_1 a_2} = (-1)^{[a_1][a_2]} \delta_{b_1}^{a_1} \delta_{b_2}^{a_2} . \tag{3.1}
\]
The operator $\mathcal{K}_{12}$ is defined as

$$\mathcal{K}_{12} = \varepsilon^{12} \varepsilon_{12}$$

and in coordinates

$$\mathcal{K}_{a_1 a_2}^{b_1 b_2} = \varepsilon^{a_1 a_2} \varepsilon_{b_1 b_2}. \quad (3.2)$$

In coordinates these operators $\mathcal{P}, \mathcal{K}$ have been introduced above in (2.22). Their invariance w.r.t. $OSp$ can be proved directly by applying the results of above subsection 2.2.

Using the operators $\mathcal{P}, \mathcal{K}$ one can construct the set of operators $\{s_i, e_i | i = 1, \ldots, n - 1\}$ in $V_{(N|M)}^{\otimes n}$:

$$s_i = \varepsilon \mathcal{P}_{i,i+1}, \quad e_i = \mathcal{K}_{i,i+1}, \quad i = 1, \ldots, n - 1, \quad (3.3)$$

which generate the Brauer algebra $B_n(\omega)$ \cite{2} with the parameter

$$\omega = \varepsilon^{cd} \varepsilon_{cd} = \varepsilon(N - M). \quad (3.4)$$

Here $N$ and $M$ are the numbers of even and odd coordinates, respectively. Indeed, one can check directly (see appendix 13) that the operators (3.3) satisfy the defining relations for generators of $B_n(\omega)$ (see, e.g., \cite{5}, \cite{6}, \cite{7} and references therein)

\begin{align*}
  &s_i^2 = 1, \quad e_i^2 = \omega e_i, \quad s_i e_i = e_i s_i = e_i, \quad i = 1, \ldots, n - 1, \quad (3.5) \\
  &s_i s_j = s_j s_i, \quad e_i e_j = e_j e_i, \quad s_i e_j = e_j s_i, \quad |i - j| > 1, \\
  &s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}, \quad e_i e_{i+1} e_i = e_i, \quad e_{i+1} e_i e_{i+1} = e_{i+1}, \quad (3.6) \\
  &s_i e_{i+1} e_i = s_{i+1} e_i e_i, \quad e_{i+1} e_i s_{i+1} = e_{i+1} s_i, \quad i = 1, \ldots, n - 2.
\end{align*}

Thus, one can consider eqs. (3.5) as the matrix representation $T$ of the generators of the Brauer algebra $B_n(\omega)$ in the space $V_{(N|M)}^{\otimes n}$. We note that $T$ (3.3) is the special reducible representation of $B_n(\omega)$. Irreducible representations of Brauer algebra were investigated in many papers (see, e.g. \cite{6}, \cite{7}, \cite{8} and references therein).

Let us consider the following linear combination of the generators $s_i, e_i \in B_n(\omega)$

$$\tilde{\rho}_i(u) = u(u + \beta) s_i - (u + \beta) 1 + u e_i \in B_n(\omega), \quad (3.7)$$

where $u$ is a spectral parameter and

$$\beta = 1 - \frac{\omega}{2}. \quad (3.8)$$

**Proposition 1.** The element \(3.7\) satisfies the Yang-Baxter equation

$$\tilde{\rho}_i(u) \tilde{\rho}_{i+1}(u + v) \tilde{\rho}_i(v) = \tilde{\rho}_{i+1}(v) \tilde{\rho}_i(u + v) \tilde{\rho}_{i+1}(u), \quad (3.9)$$

and the unitarity condition

$$\tilde{\rho}_i(u) \tilde{\rho}_i(-u) = (u^2 - 1)(u^2 - \beta^2) 1. \quad (3.10)$$

**Proof.** Substitute (3.7) into (3.9), (3.10). One obtains 27 terms in both sides of (3.9). The terms which do not contain the elements $e_i, e_{i+1}$ are cancelled due to the first identity in (3.6) (the remaining terms just cancel each other identically). Other terms which include the elements $e_i, e_{i+1}$ are cancelled due to the identities (3.5), (3.6). In particular to prove (3.9) identities like $e_i s_{i+1} e_i = e_i$ and $s_i e_{i+1} s_i = s_{i+1} e_{i+1} s_i$ which follow from (3.5), (3.6) are useful.

One can consider equation (3.9) as the nontrivial identity in the algebra $B_n(\omega)$. 

The matrix representation $T$ of the element is

$$R(u) \equiv \epsilon T(\hat{\rho}(u)) = u(u + \beta)\mathcal{P} - \epsilon(u + \beta)1 + cu\mathcal{K}.$$  \hspace{1cm} (3.11)

Here we suppress the index $i$ for simplicity. \[3.9\] implies (see appendix \[13\]) that $R(u)$ satisfies the braid version of the Yang–Baxter equation

$$\check{R}_{12}(u-v)\check{R}_{23}(u)\check{R}_{12}(v) = \check{R}_{23}(v)\check{R}_{12}(u)\check{R}_{23}(u-v).$$  \hspace{1cm} (3.12)

Further we use the $R$-matrix

$$R(u) = \mathcal{P}\check{R}(u) = (u - \frac{\omega}{2} + 1)(u1 - \epsilon\mathcal{P}) + u\mathcal{K} = u(u + \beta)1 - \epsilon(u + \beta)\mathcal{P} + u\mathcal{K},$$  \hspace{1cm} (3.13)

which includes the super permutation and is the image of the element \[7\]:

$$\rho_i(u) = u(u + \beta)1 - (u + \beta) s_i + u e_i \in B_n(\omega).$$

The braid version \[3.12\] of the Yang–Baxter equation has the same form in both the supersymmetric the non supersymmetric cases. However the standard matrix $R(u) = \mathcal{P}\check{R}(u) \hspace{1cm} (3.13)$ satisfies the graded version of the Yang–Baxter equation \[22\] involving extra sign factors,

$$R_{12}(u-v)(-)^{12}R_{13}(u)(-)^{12}R_{23}(v) = R_{23}(v)(-)^{12}R_{13}(u)(-)^{12}R_{12}(u-v).$$  \hspace{1cm} (3.14)

Indeed, after substituting $\check{R}_{ij}(u) = \mathcal{P}_{ij}R_{ij}(u) = (-)^{ij}P_{ij}R_{ij}(u)$ into \[3.12\] and moving all standard permutations $\mathcal{P}_{ij}$ to the left we write \[3.12\] in the form

$$R_{23}(u-v)(-)^{13}R_{13}(u)(-)^{12}R_{12}(v) = R_{12}(v)(-)^{13}R_{13}(u)(-)^{23}R_{23}(u-v).$$  \hspace{1cm} (3.15)

If $R \in \text{End}(V_{(N|M)})^{\otimes 2}$ is an even matrix, then the following condition holds

$$R^{i}_{ji} = 0 \hspace{0.5cm} \text{iff} \hspace{0.5cm} [i_1] + [i_2] + [j_1] + [j_2] = 0 \text{ (mod2).} \hspace{1cm} (3.16)$$

In particular one can easily check this property for the matrices $1, \mathcal{P}, \mathcal{K}$ out of which the operator $R(u)$ is composed. Therefore,

$$R_{ij}I_k(-)^{ik}(-)^{jk} = (-)^{ik}(-)^{jk}R_{ij}I_k.$$  \hspace{1cm} (3.17)

Using this property and the identities \[2.32\] we convert the right hand side of \[3.15\] to

$$R_{12}(-)^{13}R_{13}(-)^{23}(-)^{12}R_{23} = R_{12}(-)^{13}(-)^{23}(-)^{12}R_{13}(-)^{12}R_{23} =$$

$$(-)^{13}(-)^{23}R_{12}(-)^{12}R_{13}(-)^{12}R_{23},$$  \hspace{1cm} (3.18)

and doing the analogous transformations for the left hand side we represent \[3.15\] in the form

$$R_{23}(u-v)(-)^{12}R_{13}(u)(-)^{12}R_{12}(v) = R_{12}(v)(-)^{12}R_{13}(u)(-)^{12}R_{23}(u-v).$$  \hspace{1cm} (3.19)

After the exchange of the spectral parameters $v \rightarrow u - v, u \rightarrow u$ in \[3.19\] we obtain the graded version of Yang–Baxter equation \[3.14\]. Finally we stress that $(-)^{12}$ in \[3.14\] can
be exchanged with the sign operator \((-)^{23}\) by means of similar manipulations as in (3.18). Moreover, if \(R_{12}(u)\) solves the Yang-Baxter equation (3.14), then the twisted \(R\)-matrix
\[
\tilde{R}_{12}(u) := (-)^{12} R_{12}(u)(-)^{12},
\]
is also a solution.

**Remark.** Eqs. (3.11), (3.13) express uniformly the \(R\)-matrices which are invariant under the action of \(SO\), \(Sp\) or \(OSp\) groups. Recall that for the \(SO\) case the \(R\)-matrix was found in [10], for the \(Sp\) case it was constructed in [11, 17]. For the \(OSp\) case such \(R\)-matrices were considered in many papers (see, e.g., [23], [26], [4]). Note that the \(OSp\) type \(R\)-matrix \(R(u)\) proposed in [4] coincides with (3.13) in the case \(\epsilon = +1\) in the following way
\[
R(u) = -R_{\epsilon=+1}(-u).
\]
The \(R\)-matrix (3.13) is related to \(SO\) and \(Sp\) type \(R\)-matrix presented in [3] just by rescaling of the spectral parameter \(u \rightarrow -\epsilon u\). The parameter \(\beta = 1 - \frac{\epsilon}{2}\) is then rescaled to \(-\epsilon\beta = \frac{\epsilon\omega}{2} - \epsilon\).

### 4 The graded RLL-relation and the Yangian \(\mathcal{Y}(osp)\)

We start with the graded form of the RLL-relation (see, e.g., [22], [4])
\[
R_{12}(u - v)L_1(u)(-)^{12}L_2(v)(-)^{12} = (-)^{12}L_2(v)(-)^{12}L_1(u)R_{12}(u - v),
\]
(the different, but equivalent, choice of the RLL-relation will be discussed at the end of this Section, in Remark 3). The \(R\)-matrix is of the form (3.13) and the elements of the supermatrix \(\|L_{ab}^c(u)\|_{a,b=1}^{N+M}\) involve the generators of an associative algebra which we denote by \(\mathcal{Y}(osp)\). We specify this algebra below. Note that the sign operators in (4.1) are fixed according to the invariance condition (2.37). Consider the product \(L_1(-)^{12}L_2(-)^{23}(-)^{13}L_3\) of three \(L\)-operators and reorder it with the help of (4.1) as
\[
L_1(u)(-)^{12}L_2(v)(-)^{23}(-)^{13}L_3(w) \rightarrow L_3(w)(-)^{23}L_2(v)(-)^{13}(-)^{12}L_1(u),
\]
in two different ways in accordance with the arrangement of brackets
\[
\left( L_1(u)(-)^{12}L_2(v) \right)(-)^{23}(-)^{13}L_3(w) = L_1(u)(-)^{12}(-)^{13}\left( L_2(v)(-)^{23}L_3(w) \right).
\]
As a result we obtain (using the properties (2.32), (3.17) of the sign operators) the associativity condition for the algebra (4.1) in the form of the graded Yang-Baxter equation (3.14).

Expand the \(L\)-operator in the spectral parameter \(u\) as
\[
L_k^a(u) = 1 \delta^a_b + \sum_{k=1}^{\infty} \left( \frac{L^{(k)}}{u} \right)^a_b, \tag{4.3}
\]
where 1 denotes the unit element in \(\mathcal{Y}(osp)\). We multiply the \(R\)-matrix by \(u^{-2}v^{-2}\)
\[
\frac{R(u - v)}{uv} = \left( \frac{1}{v^2} - \frac{2\beta + \beta - \beta}{u^2} \right) - \left( \frac{\epsilon}{uv^2} - \frac{\epsilon}{u^2v^2} + \frac{\epsilon\beta}{u^2v^2} \right) \mathcal{K} + \left( \frac{1}{uv^2} - \frac{1}{u^2v} \right) \mathcal{K}, \tag{4.4}
\]
and expand the RLL-relation (4.1) in the spectral parameters $u^{-1}$ and $v^{-1}$. The coefficient at $u^{-k}v^{-j}$ in (4.1) gives the defining relations for the infinite dimensional associative algebra $\mathcal{Y}(osp)$ which is called the Yangian of the $osp$-type:

\[
\begin{align*}
\{ L_1^{(k)}(-)^{12}L_2^{(j)^{-2}} &- 2L_1^{(k-1)}(-)^{12}L_2^{(j-1)} + L_1^{(k-2)}(-)^{12}L_2^{(j)^{2}} \\
+ \beta L_1^{(k-1)}(-)^{12}L_2^{(j)^{2}} - \beta L_1^{(k-2)}(-)^{12}L_2^{(j-1)} \} (-1)^{12} \\
-(-)^{12} \{ L_2^{(j)^{2}}(-)^{12}L_1^{(k)^{-2}} &- 2L_2^{(j-1)}(-)^{12}L_1^{(k-1)} + L_2^{(j)^{2}}(-)^{12}L_1^{(k-2)} \\
+ \beta L_2^{(j)^{2}}(-)^{12}L_1^{(k)^{-2}} - \beta L_2^{(j-1)}(-)^{12}L_1^{(k-2)} \} (-1)^{12} \\
-\epsilon P_{12} \{ L_1^{(k-1)}(-)^{12}L_2^{(j)^{2}} &- L_1^{(k-2)}(-)^{12}L_2^{(j-1)} + \beta L_1^{(k-2)}(-)^{12}L_2^{(j)^{2}} \} (-1)^{12} \\
+\epsilon (-1)^{12} \{ L_1^{(j)^{2}}(-)^{12}L_2^{(k)^{-2}} &- L_2^{(j-1)}(-)^{12}L_1^{(k-2)} + \beta L_2^{(j)^{2}}(-)^{12}L_1^{(k)^{2}} \} P_{12} \\
+\mathcal{K}_{12} \{ L_1^{(k-1)}(-)^{12}L_2^{(j)^{2}} &- L_1^{(k-2)}(-)^{12}L_2^{(j-1)} \} (-1)^{12} \\
-(-)^{12} \{ L_2^{(j)^{2}}(-)^{12}L_1^{(k)^{-2}} &- L_2^{(j-1)}(-)^{12}L_1^{(k-2)} \} \mathcal{K}_{12} = 0 .
\end{align*}
\]

The super-commutator is defined as

\[
[A, B]_{\pm} \equiv AB - (-1)^{|A||B|} BA .
\]

Notice that for elements of two super-matrices $|A_0^a|, |B_0^a|$ the super-commutator has the following representation

\[
(A_1(-)^{12}B_2(-)^{12} - (-)^{12}B_2(-)^{12}A_1)_{c_1,c_2}^{a_1,a_2} = (-1)^{|c_1|(|a_2|+|c_2|)} |A_1^{a_1}_{c_1}, B_2^{a_2}_{c_2}| .
\]

This implies that the relation (4.5) for the Yangian $\mathcal{Y}(osp)$ has the following coordinate form:

\[
\begin{align*}
(-1)^{|c_1|(|a_2|+|c_2|)} \left\{ & \left[ (L^{(k)})_{c_1}^{a_1}, (L^{(j)^{2}})_{c_2}^{a_2} \right]_{\pm} - 2 \left[ (L^{(k-1)})_{c_1}^{a_1}, (L^{(j-1)^{2}})_{c_2}^{a_2} \right]_{\pm} \\
+ & \left[ (L^{(k-2)})_{c_1}^{a_1}, (L^{(j)^{2}})_{c_2}^{a_2} \right]_{\pm} + \beta \left[ (L^{(k-1)})_{c_1}^{a_1}, (L^{(j-2)^{2}})_{c_2}^{a_2} \right]_{\pm} - \beta \left[ (L^{(k-2)})_{c_1}^{a_1}, (L^{(j-1)^{2}})_{c_2}^{a_2} \right]_{\pm} \right\} \\
-\epsilon (-1)^{|a_1||a_2|+|c_1|+|c_2|} \left\{ & \left[ (L^{(k)}b_1}_{c_1}^{a_1}, (L^{(j)^{2}}b_2}_{c_2}^{a_2} \right]_{\pm} - \beta \left[ (L^{(k-2)}b_1}_{c_1}^{a_1}, (L^{(j-1)^{2}}b_2}_{c_2}^{a_2} \right]_{\pm} \\
+ & \beta \left[ (L^{(k-2)}b_1}_{c_1}^{a_1}, (L^{(j-2)^{2}}b_2}_{c_2}^{a_2} \right]_{\pm} - \beta \left[ (L^{(k-2)^{2}}b_1}_{c_1}^{a_1}, (L^{(j-1)^{2}}b_2}_{c_2}^{a_2} \right]_{\pm} \right\} \\
-\epsilon (-1)^{|a_1||a_2|+|b_1|+|b_2|} \left\{ & \left[ (L^{(k)}b_1}_{c_1}^{a_1}, (L^{(j)^{2}}b_2}_{c_2}^{a_2} \right]_{\pm} - \beta \left[ (L^{(k-2)}b_1}_{c_1}^{a_1}, (L^{(j-1)^{2}}b_2}_{c_2}^{a_2} \right]_{\pm} \\
+ & \beta \left[ (L^{(k-2)}b_1}_{c_1}^{a_1}, (L^{(j-2)^{2}}b_2}_{c_2}^{a_2} \right]_{\pm} - \beta \left[ (L^{(k-2)^{2}}b_1}_{c_1}^{a_1}, (L^{(j-1)^{2}}b_2}_{c_2}^{a_2} \right]_{\pm} \right\} = 0 .
\end{align*}
\]

**Remark 1.** We have noticed in section 3 that the relation between the $OSp$ type R-matrix (3.13) and the R-matrix in 3 is given just by rescaling $u \rightarrow -\epsilon u$. Doing the same rescaling for the L-operator we obtain $L^{(k)} \rightarrow -\epsilon L^{(k)}$ for $k$ odd and $L^{(k)} \rightarrow L^{(k)}$ for $k$ even, respectively. After that the Yangian relations in 3 appear just as a special case of the relations (4.7).

Choosing $k = 1, j = 3$ we obtain the defining relation for the Lie superalgebra $osp$

\[
L_1^{(1)}(-)^{12}L_2^{(1)}(-)^{12} = \epsilon P_{12} - \mathcal{K}_{12}, (-)^{12}L_2^{(1)}(-)^{12} .
\]
original and permuted equation we obtain the consistency condition

\[ K_{12} \left\{ L_1^{(1)} + (-)^{12} L_2^{(1)} (-)^{12} \right\} = \left\{ L_1^{(1)} + (-)^{12} L_2^{(1)} (-)^{12} \right\} K_{12} . \]  

(4.9)

Using this condition and the identities

\[ \mathcal{P}_{12} (-)^{12} L_2^{(1)} (-)^{12} = L_1^{(1)} \mathcal{P}_{12} , \quad (-)^{12} L_2^{(1)} (-)^{12} \mathcal{P}_{12} = \mathcal{P}_{12} L_1^{(1)} , \]  

(4.10)

one can simplify defining relation (4.8) of \( osp \) as

\[ L_1^{(1)} (-)^{12} L_2^{(1)} (-)^{12} - (-)^{12} L_2^{(1)} (-)^{12} L_1^{(1)} = [K_{12} - \epsilon \mathcal{P}_{12} , L_1^{(1)}] , \]  

(4.11)

which has the following component form

\[ (-1)^{|c_1|(|a_2|+|c_2|)} \left[ (L^{(1)})^{a_1}_{c_1} (L^{(1)})^{a_2}_{c_2} \right]_\pm = \epsilon (-1)^{|c_1||c_2|} \delta^{a_2}_{c_1} (L^{(1)})^{a_1}_{c_2} - \epsilon (-1)^{|a_2|} \delta^{a_1}_{c_1} (L^{(1)})^{a_2}_{c_2} - \epsilon (-1)^{|t_2|} \delta^{a_2}_{c_1} (L^{(1)})^{a_1}_{c_2} \]  

(4.12)

Note that (4.11) can be directly obtained from (4.5) with the choice \( k = 3, j = 1 \).

Multiplying both sides of (4.9) by \( K_{12} \) from the left (or from the right) and using the identities

\[ K_{12}^2 = \omega K_{12} , \quad K_{12} A_{12} = \epsilon \text{ str}(A) K_{12} , \quad K_{12} (-)^{12} A_{2} (-)^{12} K_{12} = \epsilon \text{ str}(A) K_{12} , \]  

(4.13)

we obtain

\[ K_{12} \left\{ L_1^{(1)} + (-)^{12} L_2^{(1)} (-)^{12} \right\} = \frac{2\epsilon}{\omega} \text{ str}(L^{(1)}) K_{12} = \left\{ L_1^{(1)} + (-)^{12} L_2^{(1)} (-)^{12} \right\} K_{12} , \]  

(4.14)

where \( \text{str}(A) \equiv (-1)^{|a|} A^a_a = \epsilon b \text{ Ba} \) is the supertrace of the super-matrix \( A \). One can check that the element \( \text{str}(L^{(1)}) \) belongs to the center of the Yangian \( \mathcal{Y}(osp) \) and therefore to the center of the Lie super-algebra \( osp \).

The Yangian \( \mathcal{Y}(osp) \) (as well as the \( gl \)-type Yangian \( \mathcal{Y} \) and the \( so \) and \( sp \) type Yangians \( \mathcal{Y} \)) possess the set of automorphisms which are defined by the assignments

\[ L^a_b (u) \rightarrow f(u) L^a_b (u + b_0) , \]  

(4.15)

where \( f(u) = 1 + b_1/u + b_2/u + ... \) is a scalar function and \( b_k \) are parameters (in general \( b_k \) are central elements in \( \mathcal{Y}(osp) \)). The transformations (4.15) are implied by the form of the defining relations (4.11). As shown in [3] one can use the automorphisms (4.15) to fix \( L^{(1)} \) such that \( \text{str}(L^{(1)}) = 0 \). In view of this we define the traceless generators

\[ G^a_b \equiv (L^{(1)})^a_b - \frac{\epsilon}{\omega} \text{ str}(L^{(1)}) \delta^a_b , \quad \text{str}(G) = 0 . \]  

(4.16)

These generators satisfy (since we have automorphisms (4.15) and \( \text{str}(L^{(1)}) \) is the central element) the same commutation relations (4.8) which we write as:

\[ (-)^{12} G_1 (-)^{12} G_2 - G_2 (-)^{12} G_1 (-)^{12} = [\epsilon \mathcal{P}_{12} - \overline{K}_{12} , G_2] , \]  

(4.17)

where \( \overline{K}_{12} = (-)^{12} K_{12} (-)^{12} (K_{b_1 b_2} = \epsilon^{a_1 a_2} \epsilon_{b_1 b_2}) \) and we have used \( (-)^{12} \mathcal{P}_{12} (-)^{12} = \mathcal{P}_{12} \). This is to be compared with (2.21), (2.26) if the algebra of elements \( G^a_b \) is represented in
the space $\mathcal{V}_{(N|M)}$. Note that the commutation relations (4.17) transform to the commutation relations (2.21) if we redefine the supermetric as $\varepsilon_{ab} \rightarrow \varepsilon_{ab} = \varepsilon_{ba}$ (see also discussion in Remark 3 below).

Further, for the traceless generators (4.16) $G^a_b$ from (4.14) we have (cf. (2.24))

$$K_{12}\left\{G_1 + (-)^{12}G_2(-)^{12}\right\} = 0 = \left\{G_1 + (-)^{12}G_2(-)^{12}\right\}K_{12}. \quad (4.18)$$

In components this reads as the condition (2.13) for the generators of the Lie superalgebra $osp$:

$$G_{ab} + \varepsilon_{(1)}[a][b] + \varepsilon_{(2)}[a][b] = 0. \quad (4.19)$$

**Remark 2.** Comparing the $RLL$-relations (4.1) and the graded Yang-Baxter equation (3.14) one finds that the $L$-operator is represented (as an operator in $\mathcal{V}_{(N|M)} \otimes \mathcal{V}_{(N|M)}$) in the form of the twisted solution (3.20) of the Yang-Baxter equation:

$$L(u) = \frac{1}{u^2}(-)^{12}R_{12}(u)(-)^{12} = 1 + \frac{1}{u} (1\beta + (\tilde{K} - \varepsilon P)) - \frac{\varepsilon \beta}{u^2}P. \quad (4.20)$$

The operators $L_1(u)$ and $L_2(v)$ in (4.1) should be understood as $\frac{1}{u^2}(-)^{12}R_{13}(u)(-)^{13}$ and $\frac{1}{v^2}(-)^{23}R_{23}(u)(-)^{23}$, respectively. According to the above consideration of the Yangian $\mathcal{Y}(osp)$ the coefficient $(1\beta + (\tilde{K} - \varepsilon P))$ of $u^{-1}$ in (4.20) is a representation of the element $L^{(1)}$ of the Yang-Baxter equation:

$$T^{a_2}_{c_2}(L^{(1)})^{a_1}_{c_1} = \beta^{a_1}_{c_1} \delta^{a_2}_{c_2} + (\tilde{K}^{a_1}_{c_1} \delta^{a_2}_{c_2} - \varepsilon P^{a_1 a_2}_{c_1 c_2}). \quad (4.21)$$

Since the $L$-operator (4.20) satisfies the $RLL$-relations (4.1) the operator (4.21) should obey commutation relations (4.8) and conditions (4.14). Taking into account (4.21) we represent the traceless part (4.16) as

$$G^{a_1 a_2}_{c_1 c_2} \equiv T^{a_2}_{c_2}(G^{a_1}_{c_1}) = (\tilde{K}^{a_1 c_1}_{c_1 c_2} - \varepsilon P^{a_1 a_2}_{c_1 c_2}) = (-1)^{[a_1] + [c_1]}\varepsilon^{a_1 a_2} \beta^{c_1}_{c_2} - \varepsilon(-1)^{[a_1][a_2]} \delta^{a_1}_{c_1} \delta^{a_2}_{c_2}. \quad (4.22)$$

This formula defines the fundamental representation $T$ of the $osp$-generators $G^a_b$ which satisfy commutation relations (4.17) and conditions (4.18), (4.19).

**Remark 3.** We note that the choice of the basis of $osp$ in (4.22) differs from the choice of the basis of $osp$ in (2.17) by sign factors

$$T^{a_2}_{c_2}(G^{a_1}_{c_1}) = (-1)^{[a_1][a_2] + [c_1][c_2]}(\tilde{G}^{a_1}_{c_1})^{a_2}_{c_2}. \quad (4.23)$$

So we have

$$\tilde{G}_{12} = G_{21} = (-)^{12}G_{12}(-)^{12},$$

where $G_{12}$ and $\tilde{G}_{12}$ are defined in (4.22) and (2.17) (compare eqs. (2.24), (2.24) with (4.8), (4.18)). However one can start from the different form of the graded RLL-relation (cf. (4.1))

$$R_{12}(u - v)(-)^{12}\tilde{L}_1(u)(-)^{12}\tilde{L}_2(v) = \tilde{L}_2(v)(-)^{12}\tilde{L}_1(u)(-)^{12}R_{12}(u - v), \quad (4.23)$$

which yields the equivalent definition of the Yangian $\mathcal{Y}(osp)$. In this case the $R$-matrix (fundamental) representation of the Yangian (4.23) is given by the formula (cf. (4.20))

$$\tilde{L}(u) = \frac{1}{u^2}R_{12}(u) = 1 + \frac{1}{u} (1\beta + (K - \varepsilon P)) - \frac{\varepsilon \beta}{u^2}P. \quad (4.24)$$

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which leads to the following fundamental representations of the Yangian generators $\tilde{L}^{(1)}$ and their traceless $\text{osp}$ generators $\tilde{G}$ (cf. (4.24), (4.22)):

\[
T^{a_2}_{c_2}((\tilde{L}^{(1)})^a_{c_1}) = \beta \delta^{a_1}_{c_1} \delta^{a_2}_{c_2} + (K^{a_1a_2}_{c_1c_2} - \epsilon P^{a_1a_2}_{c_1c_2}) , \quad (4.25)
\]

\[
\tilde{G}^{a_1a_2}_{c_1c_2} = T^{a_2}_{c_2}(\tilde{G}^a_{c_1}) = (K^{a_1a_2}_{c_1c_2} - \epsilon P^{a_1a_2}_{c_1c_2}) = \epsilon^{a_1a_2} \varepsilon_{c_1c_2} - \epsilon (-1)^{[a_1][a_2]} \delta^{a_1}_{c_2} \delta^{a_2}_{c_1} . \quad (4.26)
\]

We see that the representation of the basis elements of $\text{osp}$ in (4.26) coincides with the basis of $\text{osp}$ proposed in (2.17) and, thus, the consideration of Subsection 2.1 is relevant to the definition of the Yangian $\mathcal{Y}(\text{osp})$ given in (4.23). The next point which we would like to stress here is that $RLL$ relations (4.11) can be rewritten in the form of (4.23):

\[
\tilde{R}_{12}(u - v)(-)^{12}L_1(u)(-)^{12}L_2(v) = L_2(v)(-)^{12}L_1(u)(-)^{12}\tilde{R}_{12}(u - v) , \quad (4.27)
\]

where $\tilde{R}_{12}(u)$ is the twisted solution (3.20) of the Yang-Baxter equation (3.14). The twisted matrix $\tilde{R}_{12}(u)$ can be obtained from $R$-matrix (3.13) (compare (4.20) and (4.24)) by the substitution $\varepsilon_{ab} \rightarrow \epsilon(-1)^a \varepsilon_{ab} = \varepsilon_{ba}$. It means that all formulas which we obtain below for the Yangian (4.11) can be easily transformed to the formulas for the Yangian (4.23) by the simple transformation of the supermetric $\varepsilon_{ab} \rightarrow \varepsilon_{ba}$.

5 The linear evaluation of the Yangian $\mathcal{Y}(\text{osp})$

5.1 The conditions for linear evaluation

Let us suppose that the L-operator expansion (4.30) terminates after the first term, i.e.,

\[
L(u) = u1 + L^{(1)} . \quad (5.1)
\]

Writing the RLL-relation (4.11) with this form of the L-operator and expanding it in $u$ and $v$ we obtain a set of conditions imposed on $L^{(1)}$.

All terms in the RLL-relation (4.11) proportional to $u^k v^\ell$ for $(k + \ell) \geq 3$ give trivial conditions which are automatically satisfied. The coefficients at $u^2$, $v^2$ and $uv$ give the defining relations (4.11) for generators $L^{(1)} \in \text{osp}$ and the condition (4.9). The condition appearing at first powers of $u$ and $v$ is

\[
\mathcal{K}_{12} \left( L^{(1)}_{12}(-)^{12}L^{(1)}_{12}(-)^{12} + \beta L^{(1)}_{11} \right) = \left( (-)^{12}L^{(1)}_{22}(-)^{12}L^{(1)}_{11} + \beta L^{(1)}_{11} \right) \mathcal{K}_{12} . \quad (5.2)
\]

Multiplying it by $P_{12}$ from both sides and using (4.10), (4.9) one represents it in the equivalent form

\[
\mathcal{K}_{12} \left( (-)^{12}L^{(1)}_{22}(-)^{12}L^{(1)}_{11} - \beta L^{(1)}_{11} \right) = \left( L^{(1)}_{11}(-)^{12}L^{(1)}_{22}(-)^{12} - \beta L^{(1)}_{11} \right) \mathcal{K}_{12} . \quad (5.3)
\]

These two equations can be obtained directly from (4.5) taking $j = 2, k = 3$ or $j = 3, k = 2$. Finally the condition at zero power of $u$ and $v$ is trivial in view of identities (4.10).

Thus we come to the following statement.

Proposition 2. The L-operator

\[
L^a_b(u) = (u + \alpha) 1^a_b + (L^{(1)})^a_b ,
\]

where $\alpha$ is an arbitrary constant, solves the RLL-relation (4.11) iff the elements $(L^{(1)})^a_b$ generate the osp-algebra with the defining relations (4.11) and satisfy the conditions (4.9), (5.3).
In the case of linear evaluation of $L$-operator (5.1), in addition to the defining relations (4.11) and the condition (4.9) we obtain only one non-trivial constraint (5.3) on the generators $L^{(1)}$ of $osp$. Using (4.14) we write (5.3) as the quadratic relation in $L^{(1)}$:

$$[\mathcal{K}_{12}, (L^{(1)}_1)^2 + \beta' L^{(1)}_1] = 0$$

(5.4)

where $\beta' = \beta - \frac{2}{\omega} \text{str}(L^{(1)})$. Multiplying it by $\mathcal{K}_{12}$ from one side and using (4.13) we arrive at the quadratic characteristic equation imposed on $L^{(1)}$

$$(L^{(1)})^2 + \beta' L^{(1)} - \frac{\epsilon}{\omega} \left\{ \text{str} \left( (L^{(1)})^2 \right) + \beta' \text{str} \left( L^{(1)} \right) \right\} 1 = 0.$$  

(5.5)

For the generators $G^a_b$ defined in (4.16) with vanishing supertrace, $\text{str}(G) = 0$, this condition simplifies to

$$G^2 + \beta G - \frac{\epsilon}{\omega} \text{str}(G^2) 1 = 0,$$

(5.6)

where $\beta = 1 - \omega/2$. We arrive at the following statement.

**Proposition 3.** The linear evaluation (5.7) of the $L$-operator

$$L^a_b(u) = (u + \alpha) 1^a_b + G^a_b,$$

where $\alpha$ is an arbitrary constant, solves the RLL-relation (4.1) if $G^a_b$ is a traceless matrix of generators of $osp$, which satisfy eqs. (4.17), (4.18), and in addition obeys the quadratic characteristic identity (5.6).

### 5.2 The super-spinor representation

In this subsection we intend to construct an explicit representation of $\mathcal{Y}(osp)$ where the generators of $osp \subset \mathcal{Y}(osp)$ satisfy the quadratic characteristic equation (5.6) required for the linear evaluation (5.1).

We look for a generalization of the metaplectic or spinor representations of the $Sp(n)$ or $SO(n)$ groups which can be formulated according to [3] based on algebras of bosonic and fermionic oscillators with the defining relation invariant under the group action. We introduce the algebra $\mathcal{A}$ of super-oscillators involving both bosonic and fermionic oscillators.

Consider the super-oscillators $c^a$ ($a = 1, 2, \ldots, N + M$) as generators of an associative algebra $\mathcal{A}$ with the defining relation

$$[c^a, c^b]_\epsilon = c^a c^b + \epsilon(-1)^{[a][b]} c^b c^a = \epsilon^{ab},$$

(5.7)

where $\epsilon^{ab}$ is the super-metric defined in (2.3) and (2.4). The super-oscillators $c^a$ with $[a] = 0 \mod 2$ are bosonic and with $[a] = 1 \mod 2$ are fermionic. It is important that the defining relations (5.7) are invariant under the action $c^a \rightarrow c^a = U^a_c c^c$ of the super-group $OSp$ as can be easily shown:

$$[c^a, c^b]_\epsilon = [U^a_c c^c, U^b_d c^d]_\epsilon = U^a_c c^c U^b_d c^d + \epsilon(-1)^{[a][b]} U^b_d c^d U^a_c c^c$$

$$= (-1)^{[c][b] + [c][d]} U^a_c U^b_d (c^c c^d + \epsilon(-1)^{[c][d]} c^d c^c) = (-1)^{[c][b] + [c][d]} U^a_c U^b_d \epsilon^{cd} = \epsilon^{ab},$$

(5.8)

where we have used the condition (2.8) for the elements $U \in Osp$. 

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With the help of the convention \((2.5)\) for lowering indices one can write the relations \((5.7)\) in the equivalent forms
\[
[c_a, c_b^\epsilon] \equiv c_ac_b + \epsilon(-1)^{[a][b]}c_b^\epsilon c_a = \epsilon_{ba} \Leftrightarrow c_ac_b^\epsilon + \epsilon(-1)^{[a][b]}c_b^\epsilon c_a = \delta_a^b .
\] (5.9)

The super-oscillators satisfy the following contraction identity:
\[
c^a c_a = \epsilon(-1)^{[a]}c_a c^a = \frac{1}{2} \varepsilon^{ab}(c_b c_a + \epsilon(-1)^{[a]}c_a c_b) = \frac{1}{2} \varepsilon^{ab} \varepsilon_{ab} = \frac{\omega}{2} .
\] (5.10)

Further we need the super-symmetrised product of two super-oscillators:
\[
c^{(a}c^{b)} := \frac{1}{2}(c^a c^b - \epsilon(-1)^{[a][b]}c^b c^a) \in \mathcal{A} ,
\] (5.11)

and define the operators
\[
F^{ab} \equiv \epsilon(-1)^{b}c^{(a}c^b) , \quad F^a_b = \varepsilon_{bc}F^{ac} = \epsilon(-1)^{b}c^a c_b - \frac{1}{2} \delta^a_b .
\] (5.12)

**Proposition 4.** The operators \(F^{ab} \in \mathcal{A}\) are traceless, \(\text{str}(F) = (-1)^{[a]}F^a_a = 0\), and possess the symmetry property (cf. \((4.19)\))
\[
F^{ab} = -\epsilon(-1)^{[a][b]+[a]+[b]}F^{ba} .
\] (5.13)

They satisfy the supercommutation relations \((4.12)\) for generators of \(\text{osp}\)
\[
[F^{a_1}_{b_1}, F^{a_2}_{b_2}] \pm = -\epsilon(-1)^{[a_1][a_2]+[b_1][b_2]+[a_2]} \delta^{a_1}_{b_2} F^{a_2}_{b_1} - \epsilon(-1)^{[a_2]+[b_1][b_2]+[a_1]} \delta^{a_2}_{b_1} F^{a_1}_{b_2} + \epsilon(-1)^{[b_1][b_2]+[a_2]} \delta^{a_1}_{b_1} F^{a_2}_{b_2} + \epsilon(-1)^{[b_1]} \delta^{a_2}_{b_2} F^{a_1}_{b_1} ,
\] (5.14)

and obey the quadratic characteristic identity \((5.6)\):
\[
F^{a}_b F^{b}_c + \beta F^a_c - \frac{c}{\omega} \text{str}(F^2) \delta^a_c = 0 ,
\] (5.15)

where \(\beta = 1 - \omega/2\).

**Proof.** The property \((5.13)\) follows from the definition \((5.12)\) of \(F^{ab}\). The traceless property follows from the identity \((5.10)\). To prove \((5.14)\) we need the following relation:
\[
[c^a c^b, c^e c^f] \pm = -\epsilon(-1)^{[b][e]+[a][e]+[b][f]} \varepsilon^{af} c^e c^b
\]
\[
+ (-1)^{[b][e]+[a][e]} \varepsilon^{af} c^a c^f - \epsilon(-1)^{[b][e]} \varepsilon^{ae} c^b c^e + \varepsilon^{be} c^a c^f ,
\] (5.16)

which implies for the supercommutator of two supersymmetrized quadratic operators \(c^{(a}c^{b)}\):
\[
[c^{(a_1}c^{e_1)}, c^{(a_2}c^{e_2)}] \pm = (-1)^{[a_2][c_2]+[e_1][c_2]} \varepsilon^{a_1c_2} c^{(a1}c^{e_2)} + (-1)^{[a_2][c_1]} \varepsilon^{a_1c_2} c^{(a_2}c^{e_1)}
\]
\[
- \epsilon(-1)^{[c_1][a_2]} \varepsilon^{a_1a_2} c^{(c_1}c^{e_2)} + \varepsilon^{c_1a_2} c^{(a_1}c^{e_2)} .
\] (5.17)

Then using the definition \((5.12)\) of \(F^{ab}\) we obtain
\[
[F^{a_1}_{b_1}, F^{a_2}_{b_2}] \pm = \epsilon^2 (-1)^{[b_1][+][b_2]} \varepsilon_{b_1c_1} \varepsilon_{b_2c_2} [c^{(a_1}c^{e_1)}, c^{(a_2}c^{e_2)}] \pm
\]
\[
= (-1)^{[a_2][b_2]+[b_1][b_2]+[a_1]} \delta^{a_1}_{b_2} F^{a_2}_{b_1} + \epsilon(-1)^{[b_1][a_1]} \delta^{a_1}_{b_1} F^{a_2}_{b_2} + \epsilon(-1)^{[b_1][a_2]} \delta^{a_1}_{b_1} F^{a_2}_{b_2} .
\] (5.18)
Applying the properties of the supermetric \( \varepsilon^{ab} \) and using the symmetry \((5.13)\) one can show that this relation is equivalent to \((5.14)\). From the contraction identity \((5.10)\) we obtain that

\[
F^a_b F^b_d = \left( \frac{\omega}{2} - 1 \right) F^a_d + \frac{1}{4} \delta^a_d = - \beta F^a_d + \frac{1}{4} \delta^a_d, \quad (5.19)
\]

\[
\text{str}(F^2) = (-1)^a F^a_b F^b_a = \frac{1}{4} (-1)^a \delta^a_a = \frac{\epsilon \omega}{4}, \quad (5.20)
\]

which proves \((5.15)\).

Thus the elements \( F^{ab} \in \mathcal{A} \) form a set of traceless generators of \( \text{osp} \). Indeed, the elements \( F^{ab} \) satisfy the supercommutation relation \((4.12)\) and the symmetry condition \((4.19)\). Moreover they satisfy the quadratic characteristic identity \((5.6)\) for the linear evaluation representation \((5.1)\). It means (see Proposition 4) that the \( L \)-operator which solves \( \text{RLL-eq} \) equation \((4.1)\) has the form

\[
L^a_b (u + \alpha) = (u + \alpha) \delta^a_b + F^a_b, \quad (5.21)
\]

where \( F^a_b \) is defined in \((5.12)\) and \( \alpha \) is an arbitrary constant. Note that the appearance of the parameter \( \alpha \) in the solution \((5.21)\) is explained by the invariance of the \( \text{RLL-eq} \) equations \((4.1)\) under the shift of the spectral parameters \( u \to u + \alpha, v \to v + \alpha \).

**Remark.** At the end of this subsection we note that for every super-matrix \( ||A_{ba}|| \) we have

\[
\left[ \frac{1}{2} A_{ba} F^{ab}, c^d \right] = \epsilon \frac{1}{2} (-1)^{[b]} A_{ba} \left[ c^{(a} c^{b)}, c^d \right] = A^d_b c^b, \quad (5.22)
\]

where we applied the supercommutation relations between the symmetrized quadratic product \( c^{(a} c^{b)} \) and the super-oscillator \( c^d \):

\[
\left[ c^{(a} c^{b)}, c^d \right] = - \epsilon (-1)^{[b]} \omega_{bc} c^d + \epsilon^{bd} c^a + (-1)^{[a][b]+[a][d]} \omega_{bc} c^a - \epsilon (-1)^{[a][b]} \omega^{ad} c^b. \quad (5.23)
\]

Equation \((5.22)\) demonstrates that the operators \( F^{ab} \) generate any linear transformation of generators \( c^a \in \mathcal{A} \) under the adjoint action. Let us consider the graded tensor products \( \mathcal{A} \otimes \mathcal{A} \) of two algebras of the super-oscillators. It is useful to denote the generators of \( \mathcal{A} \otimes \mathcal{A} \) as \( c^a \otimes e = c^a_1 \) and \( e \otimes c^a = c^a_2 \) where \( e \) is the unit element of \( \mathcal{A} \). Then formula \((5.22)\) for the adjoint action is generalized to the case of \( \mathcal{A} \otimes \mathcal{A} \) as follows.

\[
\left[ \frac{1}{2} A_{ba} (F_1^{ab} + F_2^{ab}), c_1^{1...r} c_2^{r+1,...k} \right] = \sum_{i=1}^{k} A_{\{i\}_a} c_1^{1...r} c_2^{r+1,...k}, \quad (5.24)
\]

where \( c_\ell^{1...r} = c^{a_1}_\ell \cdots c^{a_r}_\ell \) for \( \ell = 1, 2, F_\ell^{ab} = \epsilon (-1)^{[b]} c^{(a}_\ell c^{b)}_\ell \) and the dressed supermatrices \( A_{\{i\}_a} \) have been defined in \((2.33)\). Comparing this formula with \((2.41)\) we find that the invariance condition for any function \( f(c^a_1, c^b_2) \in \mathcal{A} \otimes \mathcal{A} \) is written in the form

\[
\left[ \frac{1}{2} A_{ba} (F_1^{ab} + F_2^{ab}), f(c^a_1, c^b_2) \right] = 0.
\]

If \( \text{grad}(f) = 0 \), this invariance condition is equivalent to

\[
\left[ (F_1^{ab} + F_2^{ab}), f(c^a_1, c^b_2) \right] = 0. \quad (5.25)
\]

We shall use this condition in Section 6.
6 The super-spinorial R-operator

We shall construct the $R$ operator intertwining in the RLL relation two super-spinor representations formulated in terms of super-oscillators. We follow here the approach developed for the $so$-case in [13, 21] and then extended for the $sp$-case in [3]. We define the L-operator as

$$L(u) \equiv u1 - \frac{1}{2}F^{ab} \otimes G_{ba} \in A \otimes \mathcal{Y}(osp),$$

where $G_{ba}$ are generators of $osp$ and $F^{ab} = (-1)^{bc}c^{(a}c^{b)} \equiv \tau(G^{ab}) \in A$ are elements $G^{ab}$ in the super-spinor representation $\tau$ (see proposition [4]). We shall construct the R-operator $\mathcal{R}_{12}(u) \in A \otimes A$ intertwining the $L$-operators (6.1) via the following RLL-relation

$$\mathcal{R}_{12}(u)L_1(u + v)L_2(v) = L_1(v)L_2(u + v)\mathcal{R}_{12}(u) \in A \otimes A \otimes \mathcal{Y}(osp).$$

The conditions restricting the $R$-operator are obtained from the expansion of the RLL-relation $\mathcal{R}_{12}$ in the parameter $v$. The condition at $v^2$ is trivial. At $v^1$ we obtain the invariance condition (5.25) w.r.t. the adjoint action of $osp$

$$\left[ \mathcal{R}(u), F_1^{ab} + F_2^{ab} \right] = 0.$$  

(6.3)

The condition appearing at $v^0$ is

$$u \left[ \mathcal{R}(u)F_2^{ab} - F_1^{ab}\mathcal{R}_{12}(u) \right] \otimes G_{ba} - \frac{1}{2}(-1)^{|[b]+[c]|(|d|+[a])} \left[ \mathcal{R}(u)F_1^{cb}F_2^{ad} - F_1^{cb}F_2^{ad}\mathcal{R}_{12}(u) \right] \otimes G_{bc}G_{da} = 0.  

(6.4)

The product of two generators can be rewritten via the supercommutator (4.6) and the superanticommutator

$$G_{bc}G_{da} = \frac{1}{2} \left\{ [G_{bc}, G_{da}]_\pm + \{ G_{bc}, G_{da} \}_\mp \right\}.  

(6.5)

The superanticommutator is defined as

$$\{ A, B \}_\mp \equiv AB + (-1)^{|A||B|} BA.  

(6.6)

We introduce the following notation

$$X^{(cb)(ad)} \equiv (-1)^{|[b]+[c]|(|d|+[a])} \left[ \mathcal{R}(u)F_1^{cb}F_2^{ad} - F_1^{cb}F_2^{ad}\mathcal{R}_{12}(u) \right]$$

and use the supercommutation relations for $osp$ (4.8) to write (6.4) as

$$\left\{ u \left[ \mathcal{R}(u)F_2^{ab} - F_1^{ab}\mathcal{R}_{12}(u) \right] - \varepsilon_{cd}X^{(cb)(ad)} \right\} \otimes G_{ba} = \frac{1}{4}X^{(cb)(ad)} \otimes \{ G_{bc}, G_{da} \}_\mp.  

(6.8)
This condition is fulfilled only if both sides vanish separately. This becomes the key point of the construction of the R-operator $\hat{R}(u)$.

$$\{u \left[ \hat{R}(u) F_{12}^{ab} - F_{12}^{ab}(u) \right] - \varepsilon_{cd} X^{(cb)(ad)} \} \otimes G_{ba} = 0,$$

$$(6.9)$$

$$X^{(cb)(ad)} \otimes \{G_{bc}, G_{da}\} = 0.$$  

$$\hfill (6.10)$$

### 6.2 Auxiliary variables

We have to deal with the supersymmetrization of the product of super-oscillators generalizing (5.11),

$$c(a_1 c a_2 \ldots c a_k) \equiv \frac{1}{k!} \sum_{\sigma \in S_k} (-\varepsilon)^p(\sigma)(-1)^\vartheta c^{\sigma_1} \ldots c^{\sigma_k}$$  

$$(6.11)$$

where $p(\sigma)$ denotes the parity of the permutation $\sigma$. Let us explain what we mean by $\vartheta$. We denote the basic transposition as $\sigma_j \equiv \sigma_{j+1}$ permuting the $j$-th and $(j + 1)$-st site. For the basic transposition we define $\sigma_j = [a_j][a_{j+1}]$. For a general permutation $\sigma$ with a given decomposition into the basic transpositions $\sigma = \sigma_j \sigma_j \ldots \sigma_{j_{k-1}} \sigma_{j_k}$, we define

$$\hat{\sigma} = [a_{j_k}][a_{j_k+1}] + [a_{j_k+1}][a_{j_k}] + \cdots + [a_{j_{k-2} \sigma_{j_{k-1}} \sigma_{j_k}}][a_{j_{k-1}} \sigma_{j_{k-2}} \sigma_{j_k}].$$

$$(6.12)$$

Thus, the factor $(-1)^\vartheta$ in (6.11) is needed to take into account the graded properties of the super-oscillators $c^a$ (the example of (6.11) for $k = 2$ is given in (5.11)).

It is useful to introduce a set of auxiliary variables $\kappa, \kappa'$ with the following properties

$$\kappa_a = \varepsilon_{ab} \kappa^b,$$  

$$(6.13)$$

$$\kappa^a \kappa^b = -\varepsilon(-1)^{|a| |b|} \kappa^b \kappa^a, \quad \kappa^a \kappa^b = -\varepsilon(-1)^{|a| |b|} \kappa^b \kappa^a,$$

$$\kappa^a c^b = -\varepsilon(-1)^{|a| |b|} c^b \kappa^a, \quad \kappa^a c^b = -\varepsilon(-1)^{|a| |b|} c^b \kappa^a.$$

with the derivatives $\partial^a \equiv \frac{\partial}{\partial \kappa_a}$ satisfying (cf. (5.7))

$$[\partial^a, \kappa^b] = \partial^b \kappa^a + \varepsilon(-1)^{|a| |b|} \kappa^b \partial^a = \varepsilon^{ba},$$

$$\partial^a \partial^b + \varepsilon(-1)^{|a| |b|} \partial^b \partial^a = 0, \quad \partial^a c^b + \varepsilon(-1)^{|a| |b|} c^b \partial^a = 0.$$  

$$(6.14)$$

The scalar product is defined by the supermetric $(\kappa \cdot \kappa') \equiv \varepsilon_{ba} \kappa^a \kappa'^b = \kappa_b \kappa'^b$. This product is skew-symmetric $(\kappa \cdot \kappa') = -(\kappa' \cdot \kappa)$. It is easy to show that

$$[\partial^a, (\kappa \cdot c)] = c^a$$  

$$(6.15)$$

and using this property we deduce that

$$\frac{1}{k!} \partial^{a_1} \ldots \partial^{a_k} (\kappa \cdot c)^k = c^{(a_1 \ldots c^{a_k})}$$  

$$(6.16)$$

or equivalently

$$\partial^{a_1} \ldots \partial^{a_k} c^{(\kappa \cdot c)} \bigg|_{\kappa = 0} = c^{(a_1 \ldots c^{a_k})}.$$  

$$(6.17)$$

Thus we represent the supersymmetrized product $c^{(a_1 \ldots c^{a_k})}$ of super-oscillators (6.11) with nontrivial commutation relations (5.7) by the ordinary product $\partial^{a_1} \ldots \partial^{a_k}$ of $\kappa$-derivatives which obey homogeneous commutation relations (6.14). The derivative $\partial^a$ w.r.t. the variable $\kappa_a$ commutes with the product $(\kappa \cdot c)$. We can also show that

$$[(\kappa \cdot c), (\kappa' \cdot c)] = -(\kappa \cdot \kappa')_1 = (\kappa' \cdot \kappa)_1$$  

$$(6.18)$$
where we introduce the second type of a scalar product
\[ (\kappa \cdot \kappa')(1) \equiv \kappa^a \kappa'^b = \varepsilon_{ab} \kappa^a \kappa'^b = \epsilon(-1)^{[b]} \kappa_b \kappa'^b. \] (6.19)

Using the Baker-Campbell-Hausdorff formula we calculate the product of two symmetrized factors
\[
c^{(a_1 \ldots a_k)}c^{(a \ b)} = \partial a_1 \ldots \partial a_k e^{(\kappa \cdot c)} \partial^a \partial^b e^{(\kappa' \cdot c)} \bigg|_{\kappa, \kappa' = 0} = \partial a_1 \ldots \partial a_k e^{(\kappa' \cdot c)} \bigg|_{\kappa, \kappa' = 0} =
\]
and use \([ (\kappa \cdot c), (\kappa' \cdot \kappa) ] = 0 \) and \([ \partial^a, (\kappa' \cdot \kappa) ] = \epsilon(-1)^a \kappa^a \) to obtain
\[
\partial a_1 \ldots \partial a_k \left[ \partial^a \partial^b e^{(\kappa \cdot c) + \frac{1}{4}(\kappa' \cdot \kappa) 1} \right]_{\kappa = 0} = \partial a_1 \ldots \partial a_k \left[ \right. \\
+ \left. \right]_{\kappa = 0},
\]
(6.20)

Here we introduce the concise notation
\[
[+]_{ab} = \partial^a \partial^b + \frac{1}{2} \left( \epsilon(-1)^{[a]} \kappa^a \partial^b - (-1)^{[a][b]} \kappa^b \partial^a \right) + \frac{1}{4}(-1)^{[a][b]} \kappa^a \kappa^b . \] (6.21)

Similarly we obtain
\[
c^{(a \ b)}c^{(a_1 \ldots a_k)} = (-1)^{([a]+[b])([a_1]+\ldots+[a_k])} \partial a_1 \ldots \partial a_k \partial^a \partial^b e^{(\kappa' \cdot c) - \frac{1}{4}(\kappa' \cdot \kappa) 1} \bigg|_{\kappa, \kappa' = 0} =
\]
\[
(-1)^{([a]+[b])([a_1]+\ldots+[a_k])} \partial a_1 \ldots \partial a_k \left[ [-]_{ab} e^{(\kappa \cdot c)} \right]_{\kappa = 0},
\]
(6.22)

where (cf. (6.21))
\[
[-]_{ab} = \partial^a \partial^b - \frac{1}{2} \left( \epsilon(-1)^{[a]} \kappa^a \partial^b - (-1)^{[a][b]} \kappa^b \partial^a \right) + \frac{1}{4}(-1)^{[a][b]} \kappa^a \kappa^b . \] (6.23)

Hence using (6.20) and (6.22) we write the supercommutator \([11.6]\) of \(c^{(a_1 \ldots a_k)}\) and \(c^{(a \ b)}\) as
\[
[c^{(a_1 \ldots a_k)}, c^{(a \ b)}] = \partial a_1 \ldots \partial a_k \left[ \epsilon(-1)^{[a]} \kappa^a \partial^b - (-1)^{[a][b]} \kappa^b \partial^a \right) e^{(\kappa \cdot c)} \bigg|_{\kappa = 0}. \] (6.24)

**Proposition 5.** The elements
\[
\varepsilon_{a_1 b_1} \ldots \varepsilon_{a_k b_k} c^{(a_1 \ldots a_k)} c^{(b_k \ldots b_1)} \in \mathcal{A} \otimes \mathcal{A}
\]
(6.25)

are invariant under the action \([2.7]\) of the supergroup OSp:
\[
c^a \rightarrow U^a_b c^b . \] (6.26)
It means that the elements (6.25) are invariant under the action of the Lie superalgebra osp and satisfy the infinitesimal form (5.25) of the invariance condition

\[ [\varepsilon_{a_1b_1} \ldots \varepsilon_{a_kb_k} c^{(a_1} \ldots c^{a_k)} c_1^{b_k} \ldots c_2^b, F_{\bar{1}} + F_{\bar{2}}] = 0, \tag{6.27} \]

where

\[ F_{\bar{1}}^a \equiv \epsilon(-1)^{b} c_1^a c_1^b, \quad F_{\bar{2}}^a \equiv \epsilon(-1)^{b} c_2^a c_2^b \tag{6.28} \]

are generators of osp (see proposition 4).

**Proof.** According to (6.26) the element \( U \in OSp \) acts on the product \( c^{b_k} \ldots c^{b_1} \) as (see (2.34)):

\[ c^{(k)} \ldots c^{(l)} \rightarrow U_k U_{\{k,k-1\}} \ldots U_{\{k,\ldots,1\}} c^{(k)} \ldots c^{(l)}, \tag{6.29} \]

where \( U_{\{k,\ldots,j\}} = (-)^{k,k-1} \ldots (-)^{k,j} U_j (-)^{k,k} \ldots (-)^{k,k-1} \). Let the oscillators \( c^a \) commute as in (5.7), where in the right hand side we put \( \varepsilon^{ab} = 0 \). Then we have \( c^{(b_k} \ldots c^{b_1)} = c^{b_k} \ldots c^{b_1} \) and (6.29) gives

\[ c^{(k)} \ldots c^{(l)}) \rightarrow U_k U_{\{k,k-1\}} \ldots U_{\{k,\ldots,1\}} c^{(k)} \ldots c^{(l)}) \], \tag{6.30} \]

where the parentheses (\ldots) denote the supersymmetrization. Since the commutation relations of elements \( U_{\bar{a}} \) and \( c^d \) are independent of the right hand side of (5.7), the transformation rule (6.30) will be the same for the algebra of super-oscillators (5.7). From (6.26) and in view of the invariance of the bilinear form (2.2) we have the transformation rule for new variables \( \bar{c}_a \equiv \epsilon_{da} c^d; \)

\[ \bar{c}_a \rightarrow \bar{c}_b (U^{-1})^b_a \iff \bar{c}_{ij} \rightarrow \bar{c}_{ij} U_j^{-1}, \tag{6.31} \]

where \( \bar{c}_a = \epsilon(-1)^{[a]} c_a \) and \( j \) denotes the label of the superspace. Arguing as above we obtain the transformation rule for the supersymmetrized product of the super-oscillators \( \bar{c}_a \):

\[ \bar{c}_a (1 \cdot \bar{c}_{(k)}) \rightarrow \bar{c}_a (1 \cdot \bar{c}_{(k)}) U_{\{k,\ldots,1\}}^{-1} \ldots U_{\{k,k-1\}}^{-1} U_k^{-1}. \tag{6.32} \]

From eqs. (6.30) and (6.32) we immediately see that the element

\[ \varepsilon_{a_1 b_1} \ldots \varepsilon_{a_k b_k} c_1^{(a_1} \ldots c_1^{a_k)} c_2^{b_k} \ldots c_2^b \equiv \bar{c}_1 (1 \cdot \bar{c}_{1\{k\}} c_2^{(k)} \ldots c_2^b) \in \mathcal{A}, \]

is invariant under the action (6.26) of the supergroup OSp. Considering now the infinitesimal form of this action \( U = I + A + \ldots \) and taking into account eqs. (5.24) we deduce the condition (6.27).

We present the direct proof of (6.27) in appendix C giving an alternative of the above proof.

### 6.3 The construction of the R-operator

Having introduced generating functions as an effective formulation of the supersymmetrization of super-oscillators, we are prepared to solve the conditions (6.3) and (6.8) imposed on the R-operator \( \tilde{R}_{12}(u) \).

The condition (6.3) says that the R-operator has to be invariant w.r.t. the Lie superalgebra osp. Therefore, it has to be a sum of osp-invariants (6.25)

\[ \tilde{R}_{12}(u) = \sum_k \frac{r_k(u)}{k!} \bar{c}_a \bar{c}_b \bar{c}_1^{(a_1} \ldots c_1^{a_k)} c_2^{(b_k} \ldots c_2^{b_1)}, \tag{6.33} \]

...
where we use the concise notation

\[ \varepsilon_{a,b} = \varepsilon_{a_1 b_1} \ldots \varepsilon_{a_k b_k}, \quad c^{(a_1 \ldots a_k)}_1 := c^{(a_1 \ldots a_k)}_1 = c^{(a_1 \ldots a_k)}_2 = c^{(b_k \ldots b_1)}_2. \]

Inserting this ansatz into the condition (6.9), we obtain

\[
\sum_k \frac{r_k(u)}{k!} \varepsilon_{a,b} \partial_1^{a_1} \ldots \partial_2^{a_k} \partial_2^{b_k} \ldots \partial_2^{b_1} \left\{ \varepsilon u(-1)^{|b|} \left[ (+)_{12}^{ab} - (-)_{12}^{ab} \right] - \right.
\]

\[
- (1)^{(|b|+|c|)}([a]+[d])+|b|+|d| \varepsilon_{cd} \left[ (+)_{12}^{ab} (+)_{12}^{cd} - (+)_{12}^{cd} (+)_{12}^{ab} \right] \varepsilon_{12}^{(\kappa_1 \cdot c_1) \varepsilon_{12}^{(\kappa_2 \cdot c_2)}}\right|_{\kappa_1,\kappa_2=0} = 0. \quad (6.34)
\]

We use now the advantage of the generating function formulation developed in the last subsection (in particular we apply relations (6.20), (6.22)) and rewrite the equation as

\[
\sum_{k} \frac{r_k(u)}{k!} \varepsilon_{a,b} \partial_1^{a_1} \ldots \partial_1^{a_k} \partial_1^{b_k} \ldots \partial_1^{b_1} \left\{ \varepsilon u(-1)^{|b|} \left[ (+)_{12}^{ab} - (-)_{12}^{ab} \right] - \right.
\]

\[
- (1)^{(|b|+|c|)}([a]+[d])+|b|+|d| \varepsilon_{cd} \left[ (+)_{12}^{ab} (+)_{12}^{cd} - (+)_{12}^{cd} (+)_{12}^{ab} \right] \varepsilon_{12}^{(\kappa_1 \cdot c_1) \varepsilon_{12}^{(\kappa_2 \cdot c_2)}}\right|_{\kappa_1,\kappa_2=0} = 0. \quad (6.35)
\]

where the notation \([\pm]^{ab}\) was introduced in (6.21), (6.23). We also see that

\[ \varepsilon_{a,b} \partial_1^{a_1} \ldots \partial_1^{a_k} \partial_2^{b_k} \ldots \partial_2^{b_1} = (\partial_1 \cdot \partial_2)^k |_{\lambda=0}, \]

and obtain

\[
\sum_{k} \frac{r_k(u)}{k!} \left( \partial_1 \cdot \partial_2 \right)_1^k \varepsilon_{a,b} \partial_1^{a_1} \ldots \partial_2^{a_k} \partial_2^{b_k} \ldots \partial_2^{b_1} \left\{ \varepsilon u(-1)^{|b|} \left[ (+)_{12}^{ab} - (-)_{12}^{ab} \right] - \right.
\]

\[
- (1)^{(|b|+|c|)}([a]+[d])+|b|+|d| \varepsilon_{cd} \left[ (+)_{12}^{ab} (+)_{12}^{cd} - (+)_{12}^{cd} (+)_{12}^{ab} \right] \varepsilon_{12}^{(\kappa_1 \cdot c_1) \varepsilon_{12}^{(\kappa_2 \cdot c_2)}}\right|_{\kappa_1,\kappa_2=0} = 0. \quad (6.36)
\]

We want to commute all the partial derivatives \(\partial_1, \partial_2\) to the right and the variables \(\kappa_1, \kappa_2\) to the left and then apply \(\kappa_1 = 0, \kappa_2 = 0\). For this purpose we need to know how the operator \(e^{\lambda(\partial_1 \cdot \partial_2)}\) acts on the variables \(\kappa_1, \kappa_2\)

\[
e^{\lambda(\partial_1 \cdot \partial_2)} \kappa_1^a = (\kappa_1^a - \lambda e(-1)^{|a|} \partial_2^a) e^{\lambda(\partial_1 \cdot \partial_2)}, \quad (6.37)
\]

\[
e^{\lambda(\partial_1 \cdot \partial_2)} \kappa_2^a = (\kappa_2^a + \lambda e(-1)^{|a|} \partial_1^a) e^{\lambda(\partial_1 \cdot \partial_2)}. \quad (6.38)
\]

First of all

\[
e^{\lambda(\partial_1 \cdot \partial_2)} \left[ (+)_{12}^{ab} - (-)_{12}^{ab} \right] |_{\kappa_1,\kappa_2=0} = \left( \frac{\lambda^2}{4} - 1 \right) \left( \partial_1^a \partial_1^b - \partial_2^a \partial_2^b \right) e^{\lambda(\partial_1 \cdot \partial_2)}. \quad (6.39)
\]

Let us denote

\[
Y^{(cb)(ad)} = (1)^{(|b|+|c|)}([a]+[d])+|b|+|d| \left[ (+)_{12}^{cb} (+)_{12}^{ad} - (+)_{12}^{ad} (+)_{12}^{cb} \right] = (1)^{(|b|+|c|)}([a]+[d])+|b|+|d|
\]

\[
\times \left\{ \left( \partial_1^c \partial_1^b + \frac{1}{4} (-1)^{|c|+|b|} \kappa_1 c_1^b \right) \left( e(-1)^{|a|} \kappa_2^a \partial_2^d - (-1)^{|a|+|d|} \kappa_2^a \partial_2^d \right) \right. \left( \partial_2^d \partial_2^d + \frac{1}{4} (-1)^{|a|+|d|} \kappa_2^a \kappa_2^d \right) \}
\]

\[
+ \left. \left( e(-1)^{|c|} \kappa_1 c_1^a \partial_1^a - (-1)^{|c|+|b|} \kappa_1 c_1^a \partial_1^a \right) \left( \partial_1^a \partial_1^a + \frac{1}{4} (-1)^{|a|+|d|} \kappa_2^a \kappa_2^d \right) \right\}. \quad (6.40)
\]
Commuting $e^{\lambda(\partial_1 \cdot \partial_2)}$ through the operator $Y^{(cb)(ad)}$ and imposing $\kappa_1, \kappa_2 = 0$ we obtain

\[
e^{\lambda(\partial_1 \cdot \partial_2)} Y^{(cb)(ad)} \bigg|_{\kappa_1, \kappa_2 = 0} = (-1)^{|b| + |c| + |a| + |d|} + [b + d]
\]

\[
\times \left\{ \lambda \left[ \left( \partial_1^a \partial_2^d + \frac{\lambda^2}{4} \partial_2^a \partial_1^d \right) \left( \partial_1^a \partial_2^d - \epsilon(-1)^{|a|d} \partial_1^a \partial_2^d \right) - \left( \partial_1^a \partial_2^d - \epsilon(-1)^{|b|c} \partial_2^a \partial_1^d \right) \left( \partial_2^a \partial_1^d + \frac{\lambda^2}{4} \partial_1^a \partial_2^d \right) \right] + \right.
\]

\[
+ \epsilon^2 \left[ \left( (-1)^{|a|} \xi_{ab} \partial_2^d - (-1)^{|a|}[b] \right) \xi_{ac} \partial_2^d \partial_2^d
\]

\[
- (-1)^{|a|d} \xi_{ab} \partial_2^a \partial_2^a + (-1) \left( (a + b) + [d][d] \xi_{ac} \partial_2^d \partial_2^d \right) \right) - \left( 2 \rightarrow 1 \right) \right\} e^{\lambda(\partial_1 \cdot \partial_2)}.
\]

Therefore

\[
\epsilon_{cd} e^{\lambda(\partial_1 \cdot \partial_2)} Y^{(cb)(ad)} = (-1)^{|a|b + [b]}
\]

\[
\left[ \lambda \left( \frac{\lambda^2}{4} + 1 \right) \left( \partial_1 \cdot \partial_2 \right) - \frac{\lambda^2}{4} (\omega - 2) \right]
\]

\[
\times \left( \partial_1^a \partial_1^a - \partial_2^a \partial_2^a \right) \right) e^{\lambda(\partial_1 \cdot \partial_2)} \right|_{\lambda = 0} = 0.
\]

By means of the general formula

\[
(\partial_\lambda)^k \lambda^r = \sum_{i \geq 0} \frac{r!k!}{i!(r-i)!(k-i)!} \lambda^{r-i} \partial_\lambda^{k-i},
\]

we commute the derivatives w.r.t. $\lambda$ to the right and obtain

\[
\sum_k \frac{r_k(u)}{k!} \left\{ (k - u)(\partial_\lambda)^k + \frac{k(k - 1)}{4}(k + u - \omega)(\partial_\lambda)^{k-2} \right\} e^{\lambda(\partial_1 \cdot \partial_2)} \right|_{\lambda = 0}
\]

\[
\times \left( \partial_1^a \partial_1^a - \partial_2^a \partial_2^a \right) \left( \kappa_1 \cdot \kappa_2 \right) \right) \right|_{\kappa_1, \kappa_2 = 0} = 0.
\]

Finally we deduce the recurrence relation for $r_k(u)$

\[
r_{k+2}(u) = \frac{4(u - k)}{k + 2 + u - \omega} r_k(u)
\]

which is solved in terms of the $\Gamma$-functions:

\[
r_{2m}(u) = (-4)^m \frac{\Gamma(m - \frac{1}{2})}{\Gamma(m + 1 + \frac{1}{2})} A(u),
\]

\[
r_{2m+1}(u) = (-4)^m \frac{\Gamma(m - \frac{3}{2})}{\Gamma(m + 1 + \frac{1}{2})} B(u),
\]

\[24\]
where $\omega = \epsilon(N-M)$ (see (3.41)), and $A(u), B(u)$ are arbitrary functions of $u$. Substitution of (6.47) in (6.33) gives the expression for the $osp$-invariant $R$-matrix.

This expression for the $osp$-invariant $R$-matrix generalizes the formulas for the $so$-type $R$-matrices obtained in [14], [21] (see also [15], [16], [18], [20], [3]). The $so$- and $sp$-invariant $R$-matrices are obtained easily by restriction to the corresponding Lie subalgebras of $osp$.

The bosonic part of $osp(N|M)$ (in the case $\epsilon = 1$) corresponds to the embedded subalgebra $so(N)$. Similarly the fermionic part (in the case $\epsilon = -1$) corresponds to the embedded Lie subalgebra $so(M)$. Hence, restricting ourselves to $so \subset osp$ in (6.46) we obtain the recurrence relations for the coefficients $r_k(u)$ of the $so(d)$-symmetric $R$-operator

$$r_{k+2}(u) = \frac{4(u-k)}{k+2+u-d} r_k(u)$$

(6.48)

with the solution

$$r_{2m}(u) = (-4)^m \frac{\Gamma(m-\frac{3}{2})}{\Gamma(m+1+\frac{3}{2})} A(u),$$

$$r_{2m+1}(u) = (-4)^m \frac{\Gamma(m-\frac{1}{2})}{\Gamma(m+1+\frac{1}{2})} B(u).$$

(6.49)

Moreover, in such a restriction the supersymmetrizers (6.11) appearing in the ansatz (6.33) transfer to the antisymmetrizers. This result coincides with the results obtained in [14], [21]. Indeed, after the rescaling of the spectral parameter $u \rightarrow -u$ and of the generators $c^a \rightarrow \sqrt{2} c^a$ one can directly see the coincidence with [21]. The rescaling $c^a \rightarrow \sqrt{2} c^a$ gives the standard Clifford algebra $c^a c^b + c^b c^a = 2\delta^{ab}$ for $so \subset osp$ which was used in [14], [21] instead of the algebra $A (5.7)$ used in this text. Moreover, the generators $F^{ab}$ of $so$ (5.12) used in our text differ by the factor $-\epsilon(-1)^{[b]} = -1$ from their equivalents in [21]. This is the reason that here and in the left hand side of (6.8) of the spectral parameter is to be rescaled as $u \rightarrow -u$.

Similar considerations can be done for the Lie subalgebra $sp \subset osp$. The fermionic part of $osp(N|M)$ (for $\epsilon = 1$) corresponds to $sp(M) \subset osp(N|M)$. The bosonic part of $osp(N|M)$ (for $\epsilon = -1$) corresponds to $sp(N) \subset osp(N|M)$. Restricting (6.46) to $sp \subset osp$ we obtain the recurrence relation for the $(sp(d))$-symmetric $R$-operator

$$r_{k+2}(u) = \frac{4(u-k)}{k+2+u+d} r_k(u)$$

(6.50)

with the solution

$$r_{2m}(u) = (-4)^m \frac{\Gamma(m-\frac{3}{2})}{\Gamma(m+1+\frac{3}{2})} A(u),$$

$$r_{2m+1}(u) = (-4)^m \frac{\Gamma(m-\frac{1}{2})}{\Gamma(m+1+\frac{1}{2})} B(u).$$

(6.51)

The supersymmetrizers (6.11) appearing in the ansatz (6.33) transfer to the symmetrizers.

### 6.4 The condition on the generators $G$

We intend to prove here that from the condition (6.10) follows that $\{G_{[bc]}, G_{[da]}\} = 0$. We study $X^{(cb)(ad)}$ defined in (6.7). It possess obviously the following two symmetries:

$$X^{(cb)(ad)} = -\epsilon(-1)^{|c||b|+|c|+|b|} X^{(bc)(ad)}, \quad X^{(cb)(ad)} = -\epsilon(-1)^{|a||d|+|a|+|d|} X^{(bc)(ad)}. \quad (6.52)$$

They are the same symmetries as of the generators $F^{cb}, F^{ad}$. 

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Further, we see that from the properties of the superanticommutator \( \{ G_{bc}, G_{da} \} \) and (6.10) follows

\[
\left( X^{(cb)(ad)} + (-1)^{(b' + [c])(a' + [d])} X^{(ad)(cb)} \right) \left\{ G_{bc}, G_{da} \right\} = 0. \tag{6.53}
\]

Using the results of section [6.3] we see that this equation can be rewritten as

\[
\sum_k \frac{r_k(u)}{k!} \partial_k^A \left( Z^{(cb)(ad)} + (-1)^{(b' + [c])(a' + [d])} Z^{(ad)(cb)} \right) e^{\lambda \partial_i \partial_2} e^{(\kappa_1 \cdot c_1) e^{(\kappa_2 \cdot c_2)}} \bigg|_{\lambda = \kappa_1 = \kappa_2 = 0} = 0
\]

(6.54)

where

\[
Z^{(cb)(ad)} = (-1)^{(b + [c])(a + [d])} Z^{(ad)(cb)}
\]

\begin{align*}
&\times \left\{ \lambda \left[ \left( \partial_1^A \partial_2^A + \frac{\lambda^2}{4} \partial_1^A \partial_2^A \right) \left( \partial_1^A \partial_2^A - \epsilon(-1)^{[a][d]} \partial_1^A \partial_2^A \right) \\
&\quad - \left( \partial_2^A \partial_1^A - \epsilon(-1)^{[b][c]} \partial_2^A \partial_1^A \right) \left( \partial_2^A \partial_1^A + \frac{\lambda^2}{4} \partial_1^A \partial_2^A \right) \right] + \\
&\quad + \frac{\epsilon \lambda^2}{4} \left[ \left( (-1)^{[a][d]} \partial_2^A \partial_1^A - (-1)^{[a][b]} \partial_1^A \partial_2^A \right) - \left( (-1)^{[b][c]} \partial_2^A \partial_1^A + (-1)^{[a][b][d]} \partial_1^A \partial_2^A \right) \right] \right\}.
\end{align*}

(6.55)

Let us investigate all the terms appearing in \( Z^{(cb)(ad)} + (-1)^{(b'[c])(a'[d])} Z^{(ad)(cb)} \). It is a third order polynomial in \( \lambda \):

\[
Z^{(cb)(ad)} + (-1)^{(b + [c])(a + [d])} Z^{(ad)(cb)} = \lambda \cdot A^{(cb)(ad)} + \frac{\epsilon \lambda^2}{4} \cdot B^{(cb)(ad)} + \frac{\lambda^3}{4} \cdot C^{(cb)(ad)}.
\tag{6.56}
\]

The coefficient \( A^{(cb)(ad)} \) separates into two parts. The first part contains terms with the structure \( \partial_1^A \partial_2^A \) whereas the second part contains terms with the structure \( \partial_1^A \partial_2^2 \). We describe here only the first part, the second one is analysed in the same way. The first part of \( A^{(cb)(ad)} \) is:

\[
(-1)^{(b + [c])(a + [d])} + (-1)^{(b'[c])(a'[d])} \left\{ \partial_1^A \partial_2^A - \epsilon(-1)^{[a][d]} \partial_1^A \partial_2^A + \\
\quad + \left( \partial_1^A \partial_2^A - \epsilon(-1)^{[b][c]} \partial_1^A \partial_2^A \right) \right\} = \\
= (-1)^{(b + [c])(a + [d])} + (-1)^{(b'[c])(a'[d])} \left\{ \partial_1^A \partial_2^2 + \\
\quad + \left( \partial_1^A \partial_2^2 + \epsilon(-1)^{[b][c]} \partial_2^A \partial_1^A \right) \right\}.
\tag{6.57}
\]

Hence, we see the following symmetry:

\[
A^{(bd)(ac)} = (-1)^{(b + [d])(c + [a]) + (b + [c])} A^{(cb)(ad)}
\tag{6.58}
\]

which can be regarded as the supercyclic symmetry in three indices \( bdc \to cbd \). The coefficient \( C^{(cb)(ad)} \) is analysed in the same way and possess the same symmetry as \( A^{(cb)(ad)} \):

\[
C^{(bd)(ac)} = (-1)^{(b + [d])(c + [a]) + (b + [c])} C^{(cb)(ad)}.
\tag{6.59}
\]
Moreover, it is not difficult to see that \( B^{(cb)(ad)} = \mathbf{0} \). This implies that \( X^{(cb)(ad)} + (-1)^{(|b|+|c|)(|a|+|d|)}X^{(ad)(cb)} \) possess the supercyclic symmetry too. From this fact and the properties (6.52) we conclude that \( X^{(cb)(ad)} + (-1)^{(|b|+|c|)(|a|+|d|)}X^{(ad)(cb)} \) is supersymmetric w.r.t. the cyclic permutation of any three of its four indices.

It follows from the above considerations and equation (6.54) that

\[
\{G_{bc}, G_{da}\} = 0
\]

where \((bcd)\) denotes the supersymmetrization over the indices \(b,c,d\). Let us remark that this supersymmetrization differs from the supersymmetrization of the super-oscillators (5.11). The corresponding symmetrizer is defined like in (6.11) with the replacement \( \hat{\sigma} \rightarrow \tilde{\sigma} \), where for the elementary permutation of adjacent sites \( j, j+1 \) \( \tilde{\sigma}_j = \hat{\sigma}_j + [a_j] + [a_{j+1}] \).

We summarize the results of this section in the

**Proposition 6.** The \( L \)-operator

\[
L(u) \equiv u \cdot e \otimes 1 - \frac{1}{2} F^{ab} \otimes G_{ba}
\]

constructed from the super-oscillator \( osp \) generators (5.12) and \( osp \) generators \( G_{ba} \), which solve the additional constraint

\[
\{G_{bc}, G_{da}\} = 0
\]

obeys the spinorial RLL-relation

\[
\mathcal{R}_{12}(u) L_1(u + v)L_2(v) = L_1(v)L_2(u + v)\mathcal{R}_{12}(u),
\]

where super-spinorial \( R \)-operator \( \mathcal{R}_{12}(u) \in \mathcal{A} \otimes \mathcal{A} \) is

\[
\mathcal{R}_{12}(u) = \sum_k r_k(u) \frac{1}{k!} \varepsilon_{a_1 ... a_k} c_1^{(a_1 ... a_k)} c_2^{(b_k ... b_1)},
\]

\[
r_{2m}(u) = (-4)^m \frac{\Gamma(m - \frac{u}{2})}{\Gamma(m + 1 + \frac{u-\omega+1}{2})} A(u), \quad r_{2m+1}(u) = (-4)^m \frac{\Gamma(m - \frac{u-1}{2})}{\Gamma(m + 1 + \frac{u-\omega+1}{2})} B(u).
\]

Here \( \omega = \epsilon(N - M) \) (see (3.34)), and \( A(u), B(u) \) are arbitrary functions of \( u \).

### 7 The fusion of super-spinor \( L \) operators

It was shown in [18] (Theorem 3) that the so-type \( L \)-operator (i.e. the spinor-vector so-type \( R \)-matrix) can be obtained by fusion of two spinor-spinor so-type \( R \)-matrices \( \mathcal{R} \). The main result of Sect. 6 about the generalization of the matrix \( \mathcal{R} \) to the case of the Lie superalgebra \( osp \) is given in Proposition 6 (see eq. (6.39) at the end of Subsection 6.3 for the explanation of reducing to the so-case). The vector-vector so-type \( R \)-matrix (analog of the osp-type \( R \)-matrix (8.13)) was obtained in [18] (Theorem 5) by the fusion of two spinor-vector so-type \( R \)-matrices. The standard fusion procedure [19] applied in [18] requires the use of the projector operators \( V_s \otimes V_s \rightarrow V_f \) which are not simple objects, so that the fusion procedure of [18] is technically non-trivial. In [3], for the cases of the so and sp Lie algebras, we found that the vector-vector \( R \)-matrix can be constructed as the

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7 Here "spinor-vector" (or "spinor-spinor") means that the \( R \)-matrix acts in the space of the tensor product \( V_s \otimes V_f \) (or \( V_s \otimes V_s \)), where \( V_s \) and \( V_f \) are spinor and vector (fundamental) representation spaces.
fusion of two so- and sp-type \( L \)-operators using instead of those projectors the intertwining operators \( V_s \otimes V_s \to V_f \) which are realized respectively in terms of gamma-matrices and generators of the oscillator algebra. In this Section we generalize the fusion procedure of the paper [3] to the case of the \( osp \) Lie superalgebra.

The RLL relation (7.1) has the following component form,

\[
(-1)^{(l_2]+[c_2])[c_1]} R^{a_1a_2}_{b_1b_2}(u-v) L^{b_1}(u-\lambda)L^{b_2}(v-\lambda) = \\
= (-1)^{(l_2]+[b_2])[a_1]} L^{a_1}(v-\lambda)L^{a_2}(u-\lambda) R^{b_1b_2}(u-v). \quad (7.1)
\]

It is convenient to introduce the supertensor product \( \otimes_s \) modifying the graded tensor products as

\[
(A \otimes_s B)^{a_1a_2}_{b_1b_2} = (-1)^{(l_2]+[b_2])[b_1]} A^{a_1}_{b_1} B^{a_2}_{b_2}. \quad (7.2)
\]

It has the important property of associativity,

\[
(A \otimes_s B)(C \otimes_s D) = AC \otimes_s BD. \quad (7.3)
\]

This is checked by the following calculation

\[
(A \otimes_s B)^{a_1a_2}_{b_1b_2} (C \otimes_s D)^{b_1b_2}_{c_1c_2} = (-1)^{(l_2]+[b_2])[b_1]} A^{a_1}_{b_1} B^{a_2}_{b_2} (-1)^{(l_2]+[c_2])[c_1]} C^{b_1}_{c_1} D^{b_2}_{c_2} = \\
= (-1)^{(l_2]+[c_2])[c_1]} A^{a_1}_{b_1} C^{b_1}_{c_1} B^{a_2}_{b_2} D^{b_2}_{c_2} = (AC \otimes_s BD)^{a_1a_2}_{c_1c_2}.
\]

Using the supertensor product one can represent the graded RLL relation (7.1) in the form

\[
R(u-v)\left( L(u-\lambda) \otimes_s L(v-\lambda) \right) = \left( L(v-\lambda) \otimes_s L(u-\lambda) \right) R(u-v). \quad (7.4)
\]

Consider two different \( L \)-operators \( L(v-\mu) \) and \( L'(u-\lambda) \) which commute up to the standard sign factor according to grading:

\[
L^{a_2}_{b_2}(u-\mu)L^{b_1}_{c_1}(u-\lambda) = (-1)^{(l_1+c_1)(a_2+b_2)} L^{b_1}_{c_1}(u-\mu)L^{a_2}_{b_2}(u-\lambda).
\]

This means that for the supertensor products (7.2) we have

\[
(L_1 \otimes_s L_2)(L'_1 \otimes_s L'_2) = (L_1 L'_1 \otimes_s L_2 L'_2). \quad (7.5)
\]

We assume the RLL relation (7.1), (7.4) to hold for \( L \) replaced by \( L' \). The property (7.5) allows to apply the "train argument" [24] in the fusion procedure as in the non-supersymmetric case without extra signs responsible for the grading. Thus the RLL relation (7.1), (7.4) holds for the matrix product

\[
T(u) = L(u+\lambda)L'(u+\mu),
\]

where \( \lambda, \mu \) are any shifts of the spectral parameter. In components this matrix product reads as

\[
(T^{b}_{d}(u))^{a_1a_2}_{\beta_1\beta_2} = (L^{b}_{c}(u+\lambda))^{a_1}_{\beta_1} (L'^{c}_{d}(u+\mu))^{a_2}_{\beta_2}, \quad (7.6)
\]

One can define another supertensor product (cf. (7.2)) \( (A \otimes_s B)^{a_1a_2}_{b_1b_2} = (-1)^{(l_1]+[b_1])[a_2]} A^{a_1}_{b_1} B^{a_2}_{b_2} \) which respects the property (7.2) as well. For this supertensor product the RLL-relation (7.4) is equivalent to (7.6).
We show now that the fusion expression (7.8) is related to the fundamental operators in the space \( V \) where (7.6) with two intertwiners \((c^\alpha)_{\beta} \) are generators of the \( \text{osp} \) Lie superalgebra. Indeed one can check directly that they satisfy the graded commutation relations (4.12), (5.13). The elements \( F' \) define a representation of \( \text{osp} \) which is contra-gradient to the representation given by elements (5.12). Further, the generators \((L^{(1)})_c = F_c \) satisfy the conditions of Proposition (2) by construction. This implies that the generators \( F'_c \) obey the conditions of Proposition (2) as well. Thus, both operators \( L' \) and \( T \) in (7.6) and (7.7) satisfy the \( RLL \) relations (7.1).

The projection of the elements \( T^{b_f}(u) \) which are operators in the space \( V \) to the operators in the space \( V_f \) gives us the desirable fusion of two \( L \) operators to the vector-vector \( R \)-matrix. This projection can be done by the invariant contraction of the matrices (7.6) with two intertwiners \((c_{d_2})_{\beta_1}\beta_2 = (c_{d_2})_{\beta_1}\delta_{\beta_2}^{\gamma} D_{\gamma\beta'} \delta'_{\beta'} \) and \((c^b)_{\alpha_2}\alpha_1 = (c^{b_2})_{\alpha_1}\delta'_{\beta'} D_{\alpha_2}\alpha_1 \) as follows:

\[
(T^{b_f}(u))_{\alpha_1\alpha_2}^{\beta_1\beta_2} (c_{d_2})_{\beta_1\beta_2} (c^b)_{\alpha_2\alpha_1} =
= (-1)^{(c_1+d_1)d_2} Tr \left( L^{b_1} c_1 (u+\lambda) c_{d_2} L^{c_1} (u+\mu) c^b \right) = T^{b_1 b_2}_{d_1 d_2} (u),
\]

where

\[
(\bar{L}^{c_1}_1)_{\alpha_1}^{\beta'} = D_{\alpha_1}\alpha_2 (L^{c_1}_1 (u+\mu))_{\alpha_2}^{\beta_2} D_{\beta_2\beta'} = (u+\mu) \delta_{\alpha_1}^{\gamma} \delta_{\beta'}^{\gamma} - (F^{c_1}_1)_{\alpha_1}^{\beta'}. \]

We show now that the fusion expression (7.8) is related to the fundamental \( R \) matrix up to a multiplication by certain sign factors which will be fixed at the end of this Section. Traces \( Tr \) of products of super-oscillators with definite grading are fixed by the symmetry arguments. In the cases of \( so \), \((\epsilon = +1)\) and \( sp \), \((\epsilon = -1)\) we had in (3)

\[
Tr(c^a c^b) = \epsilon^{ab} Tr 1, \quad Tr(c^a c^b c^c) = \frac{1}{2} (\epsilon^{ab} \epsilon^{cd} - \epsilon^{ac} \epsilon^{bd} + \epsilon^{ad} \epsilon^{bc}) Tr 1,
\]

where \( Tr 1 \) is a normalization constant which is not important here. In the supersymmetric case of the \( \text{osp} \) algebra this is modified as follows:

\[
Tr(c^a c^b) = \epsilon^{ab} Tr 1, \quad Tr(c^a c^b c^c) = \frac{1}{2} (\epsilon^{ab} \epsilon^{cd} - \epsilon(-1)^a b c d + \epsilon^{ad} \epsilon^{bc}) Tr 1. \quad (7.9)
\]

To simplify formulas here and below in this Section we write the gradings \([a],[b],\ldots\) in sign factors as \( a,b,\ldots \). Now we calculate the projection (7.8):

\[
T^{b_1 b_2}_{d_1 d_2} (u) = (-1)^{(c_1+d_1)d_2} Tr \left( L^{b_1} c_1 (u+\lambda) c_{d_2} \bar{L}^{c_1} (u+\mu) c^b \right) =
= (-1)^{(c_1+d_1)d_2} Tr \left( (u+\lambda - \frac{1}{2}) \delta_{c_1}^{b_1} + \epsilon(-1)^{c_1} c_1 c_{c_1} [c_{d_2} (u+\mu + \frac{1}{2}) \delta_{d_1}^{c_1} - \epsilon(-1)^{d_1} c^c c_{d_1}] c^b \right) =
\]

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\[
= (-1)^{(c_1+d_1)}d_2(u + \lambda - \frac{1}{2})(u + \mu + \frac{1}{2}) \text{Tr}\left(\delta_{c_1}^{b_1}c_{d_2}\delta_{d_1}^{c_2}c_{b_2}\right) - \\
(1)^{(c_1+d_1)}d_2(u + \lambda - \frac{1}{2})\text{Tr}\left(\delta_{c_1}^{b_1}c_{d_2}\delta_{d_1}^{c_2}c_{b_2}\right) + \\
+ (-1)^{(c_1+d_1)}d_2+c_1\epsilon(u + \mu + \frac{1}{2}) \text{Tr}\left(\epsilon_{c_1}^{b_1}c_{d_2}\delta_{d_1}^{c_2}c_{b_2}\right) + \\
- (-1)^{(c_1+d_1)}d_2+c_1d_1\text{Tr}\left(\epsilon_{c_1}^{b_1}c_{d_2}\delta_{d_1}^{c_2}c_{b_2}\right).
\]

(7.11)

In the last line we commute \(c_{d_2}\) with \(c_{b_2}\) and use the identity (5.10) which leads to

\[
(1)^{(c_1+d_2+c_1+d_1)}d_2\text{Tr}\left(\epsilon_{c_1}^{b_1}c_{d_2}\delta_{d_1}^{c_2}c_{b_2}\right) = (-1)^{(d_1)}d_1\left(1 - \frac{\omega}{2}\right)\text{Tr}\left(\epsilon_{c_1}^{b_1}c_{d_2}\delta_{d_1}^{c_2}c_{b_2}\right).
\]

Then applying formulas for traces (7.9) we write (7.11) as

\[
= (u + \lambda - \frac{1}{2})(u + \mu + \frac{1}{2}) \delta_{d_1}^{b_2} \delta_{d_2}^{b_1} \text{Tr}(1) - \\
\frac{1}{2}(u + \lambda - \frac{1}{2}) \left(\delta_{d_1}^{b_1}\delta_{d_2}^{b_2} + \epsilon(1)^{d_1+d_2+d_1d_2}\delta_{d_1}^{b_1}\delta_{d_2}^{b_2} - \delta_{d_1}^{b_1}\delta_{d_2}^{b_2}\right) \text{Tr}(1) + \\
\frac{1}{2}(u + \mu + \frac{1}{2}) \left(\delta_{d_1}^{b_1}\delta_{d_2}^{b_2} - \epsilon(1)^{d_1+d_2+d_1d_2}\delta_{d_1}^{b_1}\delta_{d_2}^{b_2} + \delta_{d_1}^{b_1}\delta_{d_2}^{b_2}\right) \text{Tr}(1) + \\
\frac{1}{2}(1 - \frac{\omega}{2}) \left(\delta_{d_1}^{b_1}\delta_{d_2}^{b_2} - \epsilon(1)^{d_1+d_2+d_1d_2}\delta_{d_1}^{b_1}\delta_{d_2}^{b_2} - \delta_{d_1}^{b_1}\delta_{d_2}^{b_2}\right) \text{Tr}(1) =
\]

\[
= \left((u + \lambda)(u + \mu) + \frac{3}{4} - \frac{\omega}{4}\right) \delta_{d_1}^{b_1}\delta_{d_2}^{b_2} \text{Tr}(1) - \\
- \epsilon\left((u + \lambda)(u + \mu) + 1 - \frac{\omega}{2}\right) \left(1\right)^{d_1+d_2+d_1d_2}\delta_{d_1}^{b_1}\delta_{d_2}^{b_2} \text{Tr}(1) + \\
+ \left((u + \frac{1}{2}(\lambda + \mu) - 1 + \frac{\omega}{2}\right) \epsilon\delta_{d_1}^{b_2}\delta_{d_2}^{b_1} \text{Tr}(1).
\]

(7.12)

Let arbitrary parameters \(\lambda\) and \(\mu\) be expressed via one parameter \(\kappa\) as following

\[
\mu = \kappa - \frac{1}{2}, \quad \lambda = \kappa + \frac{3 - \omega}{2}.
\]

For this choice of the parameters we finally obtain:

\[
\mathcal{T}_{d_1d_2}^{b_1b_2}(u') = \left(u'(u' + \beta)\delta_{d_1}^{b_1}\delta_{d_2}^{b_2} - (u' + \beta)\left(1\right)^{d_1+d_2+d_1d_2}\delta_{d_1}^{b_1}\delta_{d_2}^{b_2} + u'\epsilon\delta_{d_1}^{b_2}\delta_{d_2}^{b_1}\right) \text{Tr}(1),
\]

where \(u' = u + \kappa\) and as usual we denote \(\beta = 1 - \frac{\omega}{2}\). So we see that the projection (7.8) leads to the result that the fusion of two conjugated super-oscillator \(\mathcal{L}\) operators decorated by sign factors coincides with the vector-vector (fundamental) \(\text{osp}\ \mathcal{R}\) matrix (3.13) and with the twisted \(\mathcal{R}\)-matrix \((-)^{12}\mathcal{R}(u)\mathcal{R}(u)^{-12}\) (7.14)

\[
(1)^{b_1d_2}\mathcal{T}_{d_1d_2}^{b_1b_2}(u)(-1)^{b_1d_2} = \mathcal{R}_{d_1d_2}^{b_1b_2}(u),
\]

\[
(1)^{b_1d_2}\mathcal{T}_{d_1d_2}^{b_1b_2}(u)(-1)^{b_1d_2} = (1)^{b_1d_2}\mathcal{R}_{d_1d_2}^{b_1b_2}(u)(-1)^{b_1d_2}.
\]

(7.14)

Recall that the twisted \(\mathcal{R}\)-matrix \((-)^{12}\mathcal{R}(u)\mathcal{R}(u)^{-12}\) defines the vector (fundamental) representation of \(\mathcal{L}\)-operator (4.20).

Remark. The Yang-Baxter equation (3.14) for the vector-vector \(\mathcal{R}\)-matrix follows from the \(\mathcal{R}\mathcal{L}\) relations (7.11) for the matrices \(\mathcal{T}_{d_1d_2}^{b_1b_2}(u)\) defined in (7.10). Indeed, this statement is based on the remarkable identity

\[
(1)^{[b][b][c]}L_{c}^{a}\left(v + \beta + \frac{1}{2}\right) c_{p}(\tilde{L}^{c'})_{b}\left(v - \frac{1}{2}\right) = (-1)^{[a][b]}R_{b}^{c}_{d}(v)(-1)^{[b][c]}c_{c}.
\]

(7.15)

which generalizes the relations (7.14) and justifies the use of the super-oscillator generators as intertwiners.
8 The quadratic evaluation of the Yangian $\mathcal{Y}(osp)$

8.1 The conditions for the quadratic evaluation

We derive the conditions on the terms of a quadratic evaluated L-operator following from the RLL-relation. We investigate a particular solution for the second term.

As above we denote the operator $A$ acting non-trivially only in the first space of a tensor product of vector spaces as $A_1$ and the operator $B$ acting non-trivially only in the second space of the tensor product as $B_2$. We introduce a new symbol $\bar{B}_2$ for the following object

$$\bar{B}_2 \equiv (-)^{12} B_2 (-)^{12}$$

(8.1)

where $(-)^{12}$ is the sign operator (2.31) introduced in the first section. It is a particular case of a sign operator dressed operator (2.33).

Let us solve the graded RLL-relation (4.1)

$$R_{12}(u-v)L_1(u)(-)^{12}L_2(v)(-)^{12} = (-)^{12}L_2(v)(-)^{12}L_1(u)R_{12}(u-v)$$

for a quadratic evaluation of the L-operator

$$L(u) = u^2 \cdot 1 + u \cdot G + N$$

(8.2)

with the $osp$-invariant R-matrix (3.13). Expanding in $u,v$, we obtain the following set of six equations. The rest of equations is linearly dependent on these six.

A. $[G_1, \tilde{G}_2] = [\epsilon P - K, \tilde{G}_2]$,  
B. $[G_1, \tilde{N}_2] = [\epsilon P - K, \tilde{N}_2]$,  
C. $[N_1, \tilde{G}_2] - 2[G_1, \tilde{N}_2] + \beta[G_1, \tilde{G}_2] = [K - \epsilon P, \tilde{N}_2] + \beta[\epsilon P, \tilde{G}_2] - K G_1 \tilde{G}_2 + \tilde{G}_2 G_1 K$,  
D. $[N_1, \tilde{N}_2] + \beta[G_1, \tilde{N}_2] = (\epsilon P - K) G_1 \tilde{N}_2 - \tilde{N}_2 G_1 (\epsilon P - K) + \beta[\epsilon P, \tilde{N}_2]$,  
E. $-2[N_1, \tilde{N}_2] - \beta[G_1, \tilde{N}_2] + \beta[N_1, \tilde{G}_2] = \epsilon P (N_1 \tilde{G}_2 - G_2 \tilde{N}_2) - (\tilde{G}_2 N_1 - \tilde{N}_2 G_1) \epsilon P$,  
F. $\beta[N_1, \tilde{N}_2] = \beta(\epsilon P G_1 \tilde{N}_2 - \tilde{N}_2 G_1 \epsilon P) - K N_1 \tilde{N}_2 + \tilde{N}_2 N_1 K$.

Equation A. says that at the first level appear the generators of $osp$. For details, please, see section 6. We discussed there that the generators can be arranged supertraceless and that they satisfy

$$K(G_1 + \tilde{G}_2) = 0 = (G_1 + \tilde{G}_2) K.$$  

(8.4)

This can be arranged also here as a consequence of equation A.

Equation B. is fulfilled if the second level operator $N$ is a linear combination of powers of $G$,

$$N = \sum_{j=0}^{\infty} b_j G^j$$

(8.5)

where $b_j$ commute with $G^k$ for all $j,k$.

Equation C. can be rearranged in the following way:

$$[K, N_1 + \tilde{N}_2] = \beta[K, \tilde{G}_2] + K \tilde{G}_2 - \tilde{G}_2 K$$

(8.6)
Multiplying this equation from both sides by the super-permutation $P$ we obtain an equivalent equation

$$[K, N_1 + \tilde{N}_2] = \beta[K, G_1] + KG_1^2 - G_1^2K.$$

(8.7)

Adding these two equations, multiplying them by $K$ and using the identities (4.13) we obtain

$$N_1 + \tilde{N}_2 - \frac{1}{2}(\beta G_1 + G_1^2 + \beta \tilde{G}_2 + \tilde{G}_2^2) = \frac{\epsilon}{\omega}[2\text{str}(N) - \text{str}(G^2)].$$

(8.8)

which is obviously solved by

$$N = \frac{\beta}{2}G + \frac{1}{2}G^2.$$

(8.9)

One can easily show that

$$KN_1 = K\tilde{N}_2, \quad N_1K = \tilde{N}_2K.$$

(8.10)

### 8.2 Generators of $osp$ in Jordan-Schwinger form

We introduce a set of graded canonical pairs, variables $x_a$ and the corresponding partial derivatives $\partial_a$, such that (cf. (2.1))

$$x_a x_b = \epsilon (-1)^{|a||b|} x_b x_a, \quad \partial_a \partial_b = \epsilon (-1)^{|a||b|} \partial_b \partial_a,$$

$$\partial_a x_b - \epsilon (-1)^{|a||b|} x_b \partial_a = \epsilon_{ab}.$$

(8.11)

For $\epsilon = +1$ the variables $\{x_a, \partial_a\}$ can be identified with the superspace coordinates and derivatives with the degree $|a|$ as introduced in sect. 2. According to (8.11), for $\epsilon = -1$ and $|a| = 0$ (or $|b| = 0$), these variables anticommute while for $|a| = |b| = 1$ they are commutative. For $\epsilon = -1$ these variables behave like the graded differential forms. (8.11) implies that the invariant bilinear form $(x, y) = \epsilon^{ba} x_a y_b$ is always symmetric $(x, y) = (y, x)$.

One can directly prove that the elements of this graded Heisenberg algebra

$$M_{ab} \equiv x_a \partial_b - \epsilon (-1)^{|a||b|+|a|+|b|} x_b \partial_a$$

satisfy the supercommutation relations of $osp$ (4.8) and the symmetry condition (4.19). Therefore, they compose a set of generators of $osp$. One can easily check that the matrix of generators (8.12) is supertraceless

$$\text{str}(M) = \epsilon (-1)^{|a|} M_a^a = \epsilon \epsilon^{ba} M_{ba} = 0.$$

(8.13)

**Proposition 7.** The matrix (8.12) of the generators of the Lie superalgebra $osp$ satisfies the following cubic characteristic condition

$$M^3 = (\omega - 1)M^2 + \left(\frac{\epsilon}{2}\text{str}(M^2) - \omega + 2\right)M - \frac{\epsilon}{2}\text{str}(M^2)\epsilon.$$

(8.14)

**Proof.** It is useful to introduce the following operator

$$H \equiv \epsilon^{bc} x_b \partial_c = x_b \partial^b$$

(8.15)

with the properties

$$H x_a = x_a (H + 1), \quad H \partial_a = \partial_a (H - 1), \quad [H, x_b \partial_c] = 0.$$

(8.16)
The square of $M$ is then expressed as
\[
(M^2)_{ad} = \varepsilon^{bc} M_{ab} M_{cd} = (2H + \omega - 4)x_a \partial_d + \\
+ (1 - H) M_{ad} - \epsilon (-1)^{|d|}x_a x_d \partial^2 - \epsilon (-1)^{|a|} x^2 \partial_a \partial_d + H \varepsilon_{ad},
\]
(8.17)
where we remind $\omega \equiv \varepsilon^{ef} \varepsilon_{ef}$ and introduce concise notation $x^2 = \varepsilon^{ba} x_a x_b = x^b x_b$ and $\partial^2 = \partial^b \partial_b$. The supertrace of $M^2$ is
\[
\text{str}(M^2) = \epsilon \left[ (2H + 2\omega - 4)H - 2x^2 \partial^2 \right].
\]
(8.18)
It is useful to use also the following identities
\[
\partial^2 x_b = x_b \partial^2 + 2\epsilon (-1)^{|b|} \partial_b, \quad \partial_b x^2 = x^2 \partial_b + 2\epsilon (-1)^{|b|} x_b, \quad \partial^2 M_{ac} = M_{ac} \partial^2.
\]
(8.19)
After a lengthy calculation we obtain (8.14).

The $osp$ representation generated by $M_{ab}$ satisfies the condition (6.60) for the linear L-operators intertwining the super-oscillator with the Jordan-Schwinger type representation.

**Proposition 8.** The matrix (8.12) of the generators of the Lie superalgebra $osp$ satisfies the condition
\[
\{M_{bc}, M_{d}^a\} = 0
\]
and the L-operator
\[
L(u) = u I - \frac{1}{2} F^{ab} M_{ba},
\]
with $F^{ab}$ (5.12) generating the super-oscillator representation obeys the RLL-relation with the super-spinorial R-operator (6.33) constructed in section (6).

**Proof.** The super-anticommutator condition can be checked by direct calculation using the expression for $M_{ab}$ (8.12). The super-spinorial RLL-relation is fulfilled by the proposition (8).

The above cubic characteristic condition of (8.14) follows from the condition (6.60) written in terms of $M$. This is the consequence of the following:

**Proposition 9.** If the generators $G^a_b$ of $osp$ obey the condition (6.60)
\[
\{G^a_{(a_1 a_2), G^b_{(c_1 c_2)}}\} = 0,
\]
(8.20)
then the matrix $G = ||G^a_b||$ obeys the cubic characteristic identity (8.14) written in terms of $G$ as
\[
G^3 = (\omega - 1) G^2 + \left( \frac{\epsilon}{2} \text{str}(G^2) + 2 - \omega \right) G - \frac{\epsilon}{2} \text{str}(G^2) \varepsilon.
\]
(8.21)

**Proof.** The condition (8.20) can be rewritten as
\[
|G_{a_1 a_2}, G_{c_1 c_2}|^2 = 2(-1)^{|(a_1)|+|a_2|}|(c_1)|+|c_2| G_{c_1 c_2} G_{a_1 a_2}
+ (-1)^{|c_2|} G_{a_1 a_2} G_{c_1 c_2} G_{a_1 a_2} + 2(-1)^{|c_2|} G_{a_1 a_2} G_{c_1 c_2} G_{a_1 a_2}
\]
(8.22)
and using the super-commutation relations (4.12) and the symmetry condition (4.19), one obtains

\[
(-1)^{[a_1]+[a_2]}[c_1][c_2] G_{c_1 c_2} G_{a_1 a_2} + (-1)^{[c_1][a_1]+[c_2][c_2] + [a_1][c_2][a_2] + [a_1][a_2]} G_{a_2 c_2} G_{c_1 a_1} + (-1)^{[a_2][c_2][c_1][c_2] + [a_2][c_2][a_1][a_2] + [a_1][a_2]} G_{a_2 c_2} G_{c_1 a_2} + 
\]

\[
\epsilon (-1)^{[c_1][a_1] + [a_1][a_2] + [a_1][c_2] + [a_2][c_2]} \varepsilon_{a_1 c_2} G_{c_1 a_2} + \epsilon (-1)^{[c_1][a_1] + [a_1][a_2] + [a_1][c_2] + [a_2][c_2]} \varepsilon_{c_1 c_2} G_{a_2 a_1}.
\]

(8.23)

Further we use that \( G \) is super-traceless \( \text{str}(G) = 0 \). Multiplying (8.23) from the right by \((-1)^{[a_1]+[a_2]}([c_1]+[c_2]) G^{a_1 a_2}\) and summing over \( a_1, a_2 \), the left hand-side is equal to

\[
\text{Left} = \epsilon G_{c_1 c_2} \text{str}(G^2) + 2(-1)^{[c_1][c_2]+[c_1]+[c_2]+[a_1][a_2]+[a_1][a_2] G_{c_2 a_1} G_{a_2 c_1} G^{a_1 a_2}.
\]

(8.24)

Super-commuting \( G_{a_2 c_1} \) with \( G^{a_1 a_2} \), this can be further rewritten to

\[
\text{Left} = (-1)^{[c_1][c_2]+[c_1]+[c_2]} \left\{ -G_{c_2 c_1} \text{str}(G^2) + 2\epsilon \left( (G^3)_{c_2 c_1} + (2 - \omega) (G^2)_{c_2 c_1} \right) \right\}.
\]

(8.25)

The right hand-side is after multiplication from the right by \((-1)^{[a_1]+[a_2]([c_1]+[c_2]) G^{a_2 a_1}\) and summation over \( a_1, a_2 \) of the form

\[
\text{Right} = 2(G^2)_{c_1 c_2} - \epsilon \cdot \text{str}(G^2) \varepsilon_{c_1 c_2}.
\]

(8.26)

If we use the properties of the super-metric \( \varepsilon_{c_1 c_2} \) and the transposition rule for \( G^2 \)

\[
(G^2)_{c_1 c_2} = \epsilon (-1)^{[c_1][c_2]+[c_1]+[c_2]} [(G^2)_{c_2 c_1} + (2 - \omega) G_{c_2 c_1}]
\]

(8.27)

the right hand-side can be further rewritten to

\[
\text{Right} = (-1)^{[c_1][c_2]+[c_1]+[c_2]} \left\{ 2\epsilon (G^2)_{c_2 c_1} + 2\epsilon (2 - \omega) G_{c_2 c_1} - \text{str}(G^2) \varepsilon_{c_2 c_1} \right\}.
\]

(8.28)

Comparing the left and right hand-side, we arrive at the statement of the proposition. ■

The fusion of \( L(u) = u I - \frac{1}{2} F^{ab} M_{ba} \) and \( \bar{L}(u) = u I + \frac{1}{2} (F^t)^{ab} M_{ba} \) with respect to the super-spinor representations generated by \( F, F^t \) results in an \( L \) operator obeying the RLL relation with the vector (fundamental) \( R \) matrix (4.1). It is quadratic in \( u \) and equivalent to the form (8.22) with \( N \) of the form (8.29),

\[
L(u) = u^2 \cdot 1 + u \cdot M + N, \quad N = \frac{1}{2} (M^2 + \beta M),
\]

shown above to obey the conditions A–C. It obeys also the remaining conditions D–F. The proof can be done by direct calculations using the relations (6.60) (8.14).

9 Discussion

Yang-Baxter relations with orthosymplectic supersymmetry, in particular the ones involving the fundamental \( R \) matrix, can be written in a similar form like the ones with orthogonal or symplectic symmetry. The formulation presented in this paper provides a systematic treatment and displays explicitly the features distinguishing the \( osp \) case from the \( so \) and \( sp \) cases.
We have pointed out that the invariant tensors appearing in the fundamental $R$ matrix represent the Brauer algebra. $L$ operators can have a simple form in distinguished representations. We have identified the superspinor representation resulting in an $L$ operator linear in the spectral parameter being the generalization of the spinor representation of the $so$ case and the metaplectic representation in the $sp$ case.

The super spinorial $R$ operator which intertwines super-spinor representations (see Proposition 6) has been constructed by the generalizing the methods developed in [13], [21], [3] for the $so$ and $sp$ cases.

The superspinorial $RLL$ relation holds for $L$ operators linear in the spectral parameter and acting in the spinor and the vector (fundamental) representations. It also holds for generalized $L$ operators where the vector (fundamental) representation is replaced by another one obeying a constraint (6.60) represented in the form of a super anticommutator of the elements $G_{ab}$ of the matrix of generators. All these results were summarized in the Proposition 6.

We have investigated the case of the second order Yangian evaluation, in particular the solution for the $L$ operators with all terms expressed as function of the Lie algebra generator matrix $G$. Its second non-trivial term is proportional to the supertraceless part of $G^2$. The Lie algebra representation generated by the matrix elements of $G$ is constraint in such a way that $G$ obeys a condition in terms of a cubic characteristic polynomial. The latter condition is related to the super anticommutator condition appearing in connection with the spinorial Yang Baxter relation. The class of Lie algebra representations constructed by the Jordan-Schwinger ansatz based on graded Heisenberg pairs obeys these constraints.

Acknowledgment. We thank S.Derkachov for valuable discussions.

The work of J.F. was supported by the Grant Agency of the Czech Technical University in Prague, grant No. SGS15/215/OHK4/3T/14 and by the Grant of the Plenipotentiary of the Czech Republic at JINR, Dubna.

The work of A.P.I. was supported by Russian Science Foundation grant 14-11-00598 (Sections 1-5) and by RFBR grants 16-01-00562-a, 15-52-05022 Arm-a (Sections 6-8).

The work of D.K. was partially supported by the Armenian State Committee of Science grant SCS 15RF-039. It was done within programs of the ICTP Network NET68 and of the Regional Training Network on Theoretical Physics sponsored by Volkswagenstiftung Contract nr. 86 260.

Our collaboration was also supported by JINR (Dubna) via the programs Heisenberg-Landau (J.F. and R.K.) and Smorodinski-Ter-Antonyan (D.K.).
A The graded tensor product and Yang-Baxter relations

There appear different conventions in the literature regarding the R-matrices and Yang-Baxter equations. In this section we intend to relate the Yang-Baxter equation used, e.g., in [26] to the Yang-Baxter equation (3.11).

Let \( \mathcal{V} \) be a vector superspace \( \mathcal{V} \) with the basis \( \{ |e_1|, \ldots, |e_{N+M}| \} \), where \( \{ |e_1|, \ldots, |e_N| \} \) are the basis vectors of the even part of \( \mathcal{V} \) and \( \{ |e_{N+1}|, \ldots, |e_{N+M}| \} \) are the basis vectors of the odd part of \( \mathcal{V} \). The basis of the dual superspace \( \mathcal{V}' \) is \( \{ \langle e_1 |, \ldots, \langle e_{N+M} | \} \) with the even part \( \{ \langle e_1 |, \ldots, \langle e_N | \} \) and the odd part \( \{ \langle e_{N+1} |, \ldots, \langle e_{N+M} | \} \). We demand that these two bases are dual in the following sense:

\[
\langle a | b \rangle = \delta_a^b, \quad \forall a, b = 1, \ldots, N + M. \tag{A.1}
\]

Unlike the formulation used in the main part of this paper the gradation is now carried by the basis vectors, i.e., \( \text{grad}(|e_a|) = \text{grad}(|e^a|) = [a] \), whereas the coordinates are ordinary numbers from the field \( \mathbb{F} \) over which the superspace \( \mathcal{V} \) is constructed (compare with the approach introduced in this article, especially in section 2). The matrix units and the identity operator on \( \mathcal{V} \) can be expressed as:

\[
E_a^b = |e_a\rangle \langle e_b|, \quad I = \sum_{a=1}^{N+M} |e_a\rangle \langle e_a|. \tag{A.2}
\]

The gradation of \( E_a^b \) is \([a] + [b]\) and \( I \) is the even operator.

The matrix elements of the operator \( A : \mathcal{V} \to \mathcal{V} \) are

\[
A_a^b = \langle e^b | A | e_a \rangle \tag{A.3}
\]

and one can immediately check that

\[
A = \sum_{a,b} |e_a\rangle A_a^b \langle e_b| = \sum_{a,b} A_a^a E_a^b. \tag{A.4}
\]

We introduce the graded tensor product of the superspaces \( \mathcal{V} \otimes \mathcal{V} \) [2111]. The basis of \( \mathcal{V} \otimes \mathcal{V} \) is \( \{ |e_{b_1} \otimes e_{b_2}| \}_{b_1,b_2=1}^{N+M} \). Its dual basis in \( \mathcal{V}' \otimes \mathcal{V}' \) is \( \{ (-1)^{|a_1||a_2|} \langle e_{a_1} | \otimes \langle e_{a_2} | \}_{a_1,a_2=1}^{N+M} \) as can be easily seen,

\[
(-1)^{|a_1||a_2|} (\langle e_{a_1} | \otimes \langle e_{a_2} | ) (|e_{b_1} \rangle \otimes |e_{b_2} \rangle) = (-1)^{|a_1||a_2|+|a_2||b_1|} \langle e_{a_1} | e_{b_1} \rangle \otimes \langle e_{a_2} | e_{b_2} \rangle = \delta_{b_1}^{a_1} \delta_{b_2}^{a_2}. \tag{A.5}
\]

The operator \( R \) acting in \( \mathcal{V} \otimes \mathcal{V} \) has the components w.r.t. the above basis of the form

\[
R_{b_1b_2}^{a_1a_2} = (-1)^{|a_1||a_2|} (\langle e_{a_1} | \otimes \langle e_{a_2} | ) R (|e_{b_1} \rangle \otimes |e_{b_2} \rangle) \tag{A.6}
\]

and satisfies

\[
R = \sum_{b_1b_2} R_{b_1b_2}^{a_1a_2} (|e_{a_1} \rangle \otimes |e_{a_2} \rangle) (-1)^{|b_1||b_2|} (\langle e_{b_1} | \otimes \langle e_{b_2} | ) = \sum_{b_1b_2} R_{b_1b_2}^{a_1a_2} (-1)^{|b_1||b_2|+|a_2||b_1|} E_{a_1}^{b_1} \otimes E_{a_2}^{b_2}. \tag{A.7}
\]

The graded permutation is defined as

\[
P |e_a \rangle \otimes |e_b \rangle = (-1)^{|a||b|} |e_b \rangle \otimes |e_a \rangle. \tag{A.8}
\]

In view of the above considerations, it has the following components

\[
P_{b_1b_2}^{a_1a_2} = (-1)^{|a_1||a_2|} \delta_{b_2}^{a_1} \delta_{b_1}^{a_2}. \tag{A.9}
\]
as expected (compare with (3.1)). It can be expressed using the matrix units as
\[
P = \sum_{a,b} (-1)^{|b|} E_a^b \otimes E_b^a.
\] (A.10)

The identity operator on \( V \otimes V \) is
\[
I = \sum_{a,b} E_a \otimes E_b = \sum_{a,b} (-1)^{|a||b|} (|e_a\rangle \otimes |e_b\rangle)\langle e_a| \otimes \langle e_b|).
\] (A.11)

This formalism can be obviously extended to \( V^\otimes n \) for arbitrary \( n \). Due to the Yang-Baxter relation we need to discuss the situation \( V^\otimes 3 \). Its basis is \( \{|e_{b_1}\rangle \otimes |e_{b_2}\rangle \otimes |e_{b_3}\rangle\}_{b_1,b_2,b_3=1}^{N+M} \) and the corresponding basis of the dual superspace \( V^\otimes 3 \) is
\[
\left\{ (-1)^{|a_1||a_2|+|a_1||a_3|+|a_2||a_3|} \langle e_{a_1}| \otimes \langle e_{a_2}| \otimes \langle e_{a_3}| \right\}_{a_1,a_2,a_3=1}^{N+M}.
\]

Let us remark that the identity operator in \( V^\otimes 3 \) is
\[
I = \sum_{a,b,c} E_a^a \otimes E_b^b \otimes E_c^c = \sum_{a,b,c} (-1)^{|a||b|+|a||c|+|b||c|} (|e_a\rangle \otimes |e_b\rangle \otimes |e_c\rangle)\langle e_a| \otimes \langle e_b| \otimes \langle e_c|.
\] (A.12)

It is useful to use the shorthand notation
\[
\langle e_{a_1}e_{a_2}e_{a_3} | = \langle e_{a_1}| \otimes \langle e_{a_2}| \otimes \langle e_{a_3}|, \quad |e_{b_1}e_{b_2}e_{b_3}| = |e_{b_1}\rangle \otimes |e_{b_2}\rangle \otimes |e_{b_3}\rangle.
\] (A.13)

The YB equation appearing, e.g., in [26] is of the form
\[
R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}.
\] (A.14)

We show here that if we write it in components, we obtain our form of the Yang-Baxter relation (3.11) provided that the R-matrix is even (see the definition of the even R-matrix (3.16)). The left hand side of (A.14) has the component form
\[
(R_{12}R_{13}R_{23})_{b_1b_2b_3}^{a_1a_2a_3} = (-1)^{|a_1||a_2|+|a_1||a_3|+|a_2||a_3|} \langle e_{a_1}e_{a_2}e_{a_3}| \langle e_{c_1}e_{c_2}e_{c_3}| \langle e_{b_1}e_{b_2}e_{b_3}|.
\] (A.15)

and we embed the identity operator (A.12) in \( V^\otimes 3 \) between \( R_{12} \) and \( R_{13} \)
\[
\langle e_{c_1}e_{c_2}e_{c_3}| \langle e_{b_1}e_{b_2}e_{b_3}| = (-1)^{|a_1||a_2|+|a_1||a_3|+|a_2||a_3|} \langle e_{c_1}e_{c_2}| \langle e_{c_3}| \langle e_{b_1}e_{b_2}e_{b_3}|.
\]

where we used the properties of the graded tensor product (2.11). We embed the identity operator (A.12) between \( R_{13} \) and \( R_{23} \) and obtain
\[
(R_{12}R_{13}R_{23})_{b_1b_2b_3}^{a_1a_2a_3} = (-1)^{|a_1||a_3|+|a_2||a_3|+|c_1||c_2|+|d_1||d_2|+|d_1||d_3|+|d_2||d_3|} \langle e_{a_1}e_{a_2}| \langle e_{c_1}e_{c_2}| \langle e_{d_1}d_2d_3| \langle e_{b_1}e_{b_2}e_{b_3}|.
\]

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For the even R-matrix we obtain

\[
(R_{12}R_{13}R_{23})^{a_1 a_2 a_3}_{b_1 b_2 b_3} = (-1)^3[c_1 || c_2 + d_1 || d_3]^{[a_1]}[c_2 || d_3]^{[a_2]} [c_1 c_2 d_3]^{[a_3]} R^{c_1 a_2}_{d_1 c_2} R^{c_1 a_3}_{d_1 c_2} R^{c_2 d_3}_{b_1 b_2} R^{c_3 d_4}_{b_1 b_3} R^{c_4 d_1}_{b_1 b_3},
\]

which is exactly the left hand side of (3.13). Similarly, the right hand side of (A.14) has the component form

\[
(R_{23}R_{13}R_{12})^{a_1 a_2 a_3}_{b_1 b_2 b_3} = R^{a_2 a_3}_{c_2 c_3} (-1)^{[a_1]}[c_2]^{[a_2]} [c_1 c_3 d_1]^{[a_3]} R^{c_2 d_3}_{b_1 b_2} R^{c_3 d_1}_{b_1 b_2}
\]

which coincides with the right hand side of (3.14). Thus, the equivalence of (A.14) and (3.14) is established. We recall that this equivalence holds due to the R-matrix being even.

**B Properties of operators \( P, K \)**

We use here the concise matrix notation introduced in sections 2, 3 (in bosonic case this notation was proposed in [12]). Matrices (2.22) satisfy identities

\[
P_{12} = P_{21}, \quad K_{12} = (-)^{12}K_{21} (-)^{12}, \quad (-)^1 K_{12} = (-)^2 K_{12}, \quad K_{12}(-)^1 = K_{12}(-)^2, \quad P_{12}P_{12} = 1, \quad K_{12}K_{12} = \omega K_{12}, \quad K_{12}P_{12} = P_{12}K_{12} = \epsilon K_{12}, \quad \omega = \epsilon(N - M), \quad \epsilon = (-)^{i} = (-)^{[i]} \delta_{bi}^{0} \text{ is the matrix of super-trace in the } i\text{-th superspace } \mathcal{V}_{(N|M)}. \]

Then we have

\[
P_{12}P_{12} = P_{12}(-)^2, \quad P_{12}P_{23} = (-)^{-12}P_{23}P_{12} = P_{23}(-)^{-12}P_{12}, \quad P_{12}K_{13} = (-)^{-12}K_{23}(-)^{-12}P_{12}, \quad P_{12}K_{13}(-)^{-12} = K_{23}P_{12},
\]

\[
\epsilon K_{12}P_{31} = K_{12}(-)^{-12}K_{32}(-)^{-12}, \quad \epsilon P_{31}K_{12} = (-)^{-12}K_{32}(-)^{-12}K_{12}, \quad K_{12}K_{31} = \epsilon K_{12}(-)^{-12}P_{32}(-)^{-12}, \quad K_{31}K_{12} = \epsilon(-)^{-12}P_{32}(-)^{-12}K_{12},
\]

Identities (B.2) follow from the representation (5.1): \( P_{12} = (-)^{12}P_{12} = P_{12}(-)^{12} \), where \( P_{12} \) is the usual permutation operator. Identities (B.3) follow from the definitions (2.22), (3.1), (3.2) of the operators \( P \) and \( K \). We prove only the last equality in (B.3) since the other identities in (B.3) can be proved in the same way. We denote incoming matrix indices by \( a_1, a_2, a_3 \) and outcoming indices by \( c_1, c_2, c_3 \) while dummy indices are \( b_1 \) and \( d_i \). Then we have

\[
(K_{31}K_{12})^{a_1 a_2 a_3}_{c_1 c_2 c_3} = \epsilon^{a_3 a_1} e_{c_3 b_1} e^{b_1 a_2} e_{c_1 c_2} = \epsilon^{a_3 a_1} \delta^{a_2 b_2} e_{c_1 c_2} = \delta^{a_2 b_2} \delta^{a_3 a_1} \epsilon(-)^{[a_1][b_2]} \epsilon^{a_1 b_2} e_{c_1 c_2} = \epsilon(-)^{[a_2][b_2]} (P_{23})^{a_2 a_3}_{b_2 c_3} (-1)^{[a_1][b_2]} (K_{12})^{a_1 b_2}_{c_1 c_2} = \epsilon(-)^{23}P_{23}(-)^{-12}K_{12}(-)^{-12}K_{31},
\]

in view of the relation \((-)^{23}K_{31} = (-)^{-12}K_{31} \) which follows from (2.4) we obtain the last formula in (B.3).

By means of the relations (B.2), (B.3) one can immediately check eqs. (2.24), (2.25) and also deduce

\[
P_{12}P_{23}P_{12} = P_{23}P_{12}P_{23}, \quad K_{12}K_{23}K_{12} = K_{12}, \quad K_{23}K_{12}K_{23} = K_{23}, \quad P_{12}K_{23}K_{12} = P_{23}K_{12}, \quad K_{12}K_{23}P_{12} = K_{12}P_{23},
\]

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Indeed, if we act from the left on both sides of the first relation in (B.6) by the first relation in (B.7) the second relation in (B.8). Now we act on both sides of (B.6) identity (3.9). Thus, it follows from proposition that the matrix (3.11) is the image of the element (3.7) and the Yang-Baxter equation (3.12) is the image of the defining relations (3.6) for the Brauer algebra in the representation (3.3). The second relation in (B.5) can be proved in the same way. Then we prove the first equation in (B.5). For the left hand side of (B.6) one has:

\[ (\mathcal{P}_{12}\mathcal{K}_{23}\mathcal{K}_{12})_{a_1a_2a_3} = \varepsilon^{a_1a_2} \varepsilon_{b_1b_2} \varepsilon^{b_3} \varepsilon_{c_2c_3} \varepsilon^{b_1d_2} \varepsilon_{c_1c_2} = \varepsilon^{a_1a_2} \delta^{b_1}_{b_3} \delta^{b_2}_{c_3} \varepsilon_{c_1c_2} = \mathcal{K}^{a_1a_2} \delta^{a_3}_{c_3}. \]

The identity (B.4) follows from the relations in the first line of (B.2). We consider few examples in (B.5) – (B.7) in details. We start to prove the first relation in (B.5):

\[ (\mathcal{P}_{12}\mathcal{K}_{23}\mathcal{K}_{12})_{a_1a_2a_3} = (-1)^{|(a_1||a_2||a_3)|} \delta_{b_1}^{a_1} \delta_{b_2}^{a_2} \delta_{b_3}^{a_3} \varepsilon_{d_2c_3} \varepsilon^{b_1d_2} \varepsilon_{c_1c_2} = (-1)^{|(a_1||a_2||a_3)|} \delta_{c_3}^{a_1} \delta_{c_2}^{a_2} \delta_{c_1}^{a_3} \varepsilon_{d_2c_3} \varepsilon^{b_1d_2} \varepsilon_{c_1c_2} = \mathcal{K}^{a_1a_2a_3}, \]

and similarly one deduces other relations in (B.6) and (B.7). From the identities (B.4) – (B.7) we also deduce the following relations

\[ \mathcal{K}_{12}\mathcal{P}_{23}\mathcal{K}_{12} = \epsilon\mathcal{K}_{12}, \quad \mathcal{K}_{23}\mathcal{P}_{12}\mathcal{K}_{23} = \epsilon\mathcal{K}_{23}. \]

\[ \mathcal{P}_{12}\mathcal{K}_{23}\mathcal{P}_{12} = \mathcal{K}_{23}\mathcal{P}_{12}, \quad \mathcal{P}_{23}\mathcal{K}_{12}\mathcal{P}_{23} = \mathcal{K}_{12}\mathcal{P}_{12}. \]

Indeed, if we act from the left on both sides of the first relation in (B.6) by \( \mathcal{K}_{12} \) and use (B.1), (B.5) we obtain the first relation in (B.8). In the same way one can deduce from the first relation in (B.7) the second relation in (B.8). Now we act on both sides of (B.6) by \( \mathcal{P}_{23} \) from the right and use the last equation in (B.7). As a result we arrive at the identity (B.9). Finally the relations (B.10) trivially follow from eq. (B.9).

At the end of this appendix we stress that identities (B.1), (B.4) – (B.7) are images of the defining relations (3.6) for the Brauer algebra in the representation (3.3). The \( R \)-matrix (3.11) is the image of the element (3.7) and the Yang-Baxter equation (3.12) is the image of the identity (3.9). Thus, it follows from proposition 1 that the \( R \)-matrix (3.11) is a solution of the braided version of the Yang-Baxter equation (3.12).

\section*{C \quad Direct proof of proposition 5}

We shall use the advantage of the generating functions language developed in subsection 6.2 and shall work with two sets of auxiliary variables \( \kappa^a, \kappa^b \) with the corresponding derivatives \( \partial^a, \partial^b \). Then

\[ [\varepsilon_{a_1b_1} \ldots \varepsilon_{a_kb_k} c_1^{(a_1)} \ldots c_1^{(a_k)} c_2^{(b_1)} \ldots c_2^{(b_k)} , c_1^{(a)} c_1^{(b)} + c_2^{(a)} c_2^{(b)}] = \]

\[ = \varepsilon_{a_1b_1} \ldots \varepsilon_{a_kb_k} \left\{ (-1)[|a|+|b|](|a|+\ldots+|b|) \left[ c_1^{(a_1)} \ldots c_1^{(a_k)} c_1^{(a)} + c_2^{(a)} c_2^{(a)} \right] \right. \]

\[ + \left. c_1^{(a_1)} \ldots c_1^{(a_k)} \left[ c_2^{(b_1)} \ldots c_2^{(b_k)} + c_2^{(b)} c_2^{(b)} \right] \right\} = \varepsilon_{a_1b_1} \ldots \varepsilon_{a_kb_k} \times \]

\[ \times \left\{ (-1)[|a|+|b|](|a|+\ldots+|b|) \partial_{a_1} \ldots \partial_{a_k} \left( \varepsilon_{a_1} |a| \kappa_{\partial^a} \kappa_{\partial^b} - (-1)^{|a||b|} |b| \kappa_{\partial^b} \partial_{a_1} \right) \right. \]

\[ \left. + \partial_{a_1} \ldots \partial_{a_k} \partial_{b_1} \ldots \partial_{b_k} \left( \varepsilon_{a_1} |a| \kappa_{\partial^a} \partial_{b_1} - (-1)^{|a||b|} |b| \kappa_{\partial^b} \partial_{a_1} \right) \right\} \varepsilon^{(\kappa_{\partial^a} \kappa_{\partial^b})}_{c_1} \varepsilon^{(\kappa_{\partial^a} \kappa_{\partial^b})}_{c_2} \big|_{\kappa, \kappa'=0}, \]

\[ \mathcal{P}_{23}\mathcal{K}_{12}\mathcal{K}_{23} = \mathcal{P}_{12}\mathcal{K}_{23}, \quad \mathcal{K}_{23}\mathcal{K}_{12}\mathcal{P}_{23} = \mathcal{K}_{12}\mathcal{P}_{12}. \]
where we used the supercommutation relations \((6.24)\). We need two identities:

\[
\begin{aligned}
\partial^{a_1} \ldots \partial^{a_k} \kappa^a &= \sum_{j=1}^{k} (-\epsilon)^{k-j} (-1)^{\sum_{i=j+1}^{k} [a_i][a_j]} \varepsilon^{aa_i} \partial^{a_1} \ldots \partial^{a_{j-1}} \partial^{a_{j+1}} \ldots \partial^{a_k} + \\
&\quad + (-\epsilon)^{k} (-1)^{[a]} \sum_{i=1}^{k} [a_i] \kappa^a \partial^{a_1} \ldots \partial^{a_k},
\end{aligned}
\]

\(\text{(C.2)}\)

\[
\begin{aligned}
\partial^{[b]} \ldots \partial^{[b]} K^b &= \sum_{j=1}^{k} (-\epsilon)^{j-1} (-1)^{\sum_{i=1}^{j-1} [b_i][b_j]} \varepsilon^{bb_j} \partial^{[b]} \ldots \partial^{[b]}_{j-1} \partial^{[b]}_{j+1} \ldots \partial^{[b]}, + \\
&\quad + (-\epsilon)^{k} (-1)^{[b]} \sum_{i=1}^{k} [b_i] \kappa^b \partial^{[b]} \ldots \partial^{[b]},
\end{aligned}
\]

\(\text{(C.3)}\)

We show now that the two underlined terms cancel. We write them here without the factor \(\epsilon^{(\kappa-\epsilon_1)} \epsilon^{(\kappa-\epsilon_2)} |_{\kappa, \kappa'=0}\). The first underlined term is

\[
\begin{aligned}
\epsilon^{(-1)^{[a]} \varepsilon^{a_1 b_1} \ldots \varepsilon^{a_k b_k} (-1)^{([a]+[b])([b_1]+\ldots+[b_k])}} \partial^{a_1} \ldots \partial^{a_k} \kappa^a \partial^{b_1} \ldots \partial^{b_k} = \\
= \epsilon^{(-1)^{[a]} \varepsilon^{a_1 b_1} \ldots \varepsilon^{a_k b_k} (-1)^{([a]+[b])([b_1]+\ldots+[b_k])}} \times \\
\sum_{j=1}^{k} (-\epsilon)^{k-j} (-1)^{\sum_{i=j}^{k} [a_i][a_j]} \varepsilon^{a_j b_1} \ldots \varepsilon^{a_k b_k} \partial^{a_1} \ldots \partial^{a_{j-1}} \partial^{a_{j+1}} \ldots \partial^{a_k} \partial^{b_1} \ldots \partial^{b_k} + Z_1 = \\
= \epsilon^{(-1)^{[a]} \sum_{j=1}^{k} (-1)^{([a]+[b])([a_1]+\ldots+[a_j-1])} \varepsilon^{a_1 b_1} \ldots \varepsilon^{a_{j-1} b_j} \varepsilon^{a_{j+1} b_j+1} \ldots \varepsilon^{a_k b_k} \times \\
\partial^{a_1} \ldots \partial^{a_{j-1}} \partial^{b_j} \partial^{a_{j+1}} \ldots \partial^{a_k} \partial^{b_k} \partial^{b_1} \ldots \partial^{b_k} \partial^{b_1} + Z_1.
\end{aligned}
\]

\(\text{(C.4)}\)

The second underlined term is

\[
\begin{aligned}
- (-1)^{[a][b]+[b]} \varepsilon^{a_1 b_1} \ldots \varepsilon^{a_k b_k} \partial^{a_1} \ldots \partial^{a_k} \partial^{b_1} \ldots \partial^{b_k} K^a = \\
= - (-1)^{[a][b]+[b]} \epsilon^{a_1 b_1} \ldots \epsilon^{a_k b_k} \partial^{a_1} \ldots \partial^{a_k} \times \\
\sum_{j=1}^{k} (-\epsilon)^{j-1} (-1)^{\sum_{i=1}^{j-1} [b_i][b_j]} \varepsilon^{b_j} \partial^{b_k} \ldots \partial^{b_{j-1}} \partial^{b_j} \partial^{b_1} + Z_2 = \\
= - \epsilon^{(-1)^{[a][b]} \sum_{j=1}^{k} (-1)^{([a]+[b])([b_1]+\ldots+[b_{j-1}])} \varepsilon^{a_1 b_1} \ldots \varepsilon^{a_{j-1} b_{j-1}} \varepsilon^{a_{j+1} b_{j+1}} \ldots \varepsilon^{a_k b_k} \times \\
\partial^{a_1} \ldots \partial^{a_{j-1}} \partial^{b_j} \partial^{a_{j+1}} \ldots \partial^{a_k} \partial^{b_k} \partial^{b_1} \ldots \partial^{b_k} \partial^{b_1} + Z_2.
\end{aligned}
\]

\(\text{(C.5)}\)

Let us remark that \(Z_1, Z_2\) are proportional to \(\kappa, \kappa'\) respectively and, therefore, vanish. As we see, the two underlined terms really cancel. The two non-underlined terms cancel, too.

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