ON THE SYMMETRY OF LOCAL CONSTANTS FOR GL$_n$

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Abstract. Let $K$ be a non-archimedean local field and let $G = \text{GL}_n(K)$. We have shown in previous work that the smooth dual $\text{Irr}(G)$ admits a complex structure: it is the disjoint union of smooth algebraic varieties, each of which is the quotient of a complex torus by a product of symmetric groups. In this article we show how the local constants interface with this complex structure. The local constants, up to a constant term, factor as characters through the corresponding complex tori. For the arithmetically unramified smooth dual of $\text{GL}_n$, the smooth varieties form a single extended quotient, namely $T/W$ where $T$ is a maximal torus in the complex Langlands dual $\text{GL}_n(\mathbb{C})$, and $W$ is the Weyl group. In this case, we have explicit formulas for the local constants.

1. Introduction

Let $K$ be a non-archimedean local field and let $G = \text{GL}_n(K)$. We have shown in previous work that the smooth dual $\text{Irr}(G)$ admits a complex structure: it is the disjoint union of smooth algebraic varieties, each of which is the quotient of a complex torus by a product of symmetric groups. In this article we show how the local constants interface with this complex structure.

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We continue with some background on the local constants, which are also called epsilon factors [L, D]. The epsilon factors are very important and central in the theory of Artin $L$-functions. If $E$ denotes a global field, the completed $L$-function $L(s, V)$ of a representation $W_E \to \text{GL}(V)$ of the Weil group $W_E$ of the global field $E$ defines a meromorphic function in the complex plane satisfying the functional equation

$$L(s, V) = \varepsilon(s, V)L(1 - s, V^*)$$

where $V^*$ is the dual of the representation $V$ of $W_E$, and the epsilon factor $\varepsilon(s, V)$ is defined by the product

$$\varepsilon(s, V) = \prod \varepsilon_{E_\nu}(s, V_\nu, \psi_\nu).$$

Here, $\psi_\nu$ is the local component at a place $\nu$ of a non-trivial additive character $\psi$ of $A^+_K$ trivial on $E$, so that $\psi_\nu$ is a non-trivial additive character of the local field $E_\nu$. See [D, 5.11].

From now on, we will focus on the non-archimedean places of the global field $E$, and we will write $K$ for the non-archimedean local field $E_\nu$. An elementary substitution, see §2, allows on to replace the epsilon factor $\varepsilon_K(s, V, \psi)$ with three variables, by the epsilon factor $\varepsilon_K(V, \psi)$ with two variables. From now on, we will be concerned with the epsilon factor $\varepsilon_K(V, \psi)$.

From the point of view of the local Langlands correspondence for $\text{GL}_n$, the relevant representations are the Weil-Deligne representations, see section 2. The set $\mathcal{G}_n(K)$ of equivalence classes of $n$-dimensional Weil-Deligne representations can be organised as a disjoint union of complex algebraic varieties:

$$\mathcal{G}_n(K) = \bigsqcup \mathfrak{X}$$

Each variety $\mathfrak{X}$ arises in the following way. Let $\rho'$ denote a Weil-Deligne representation of the Weil group $W_K$, and let $m$ denote the number of indecomposable summands in $\rho'$. Then $\mathfrak{X}$ is the quotient of a complex torus $\mathfrak{T}$ of dimension $m$ by a certain finite group $\mathfrak{S}$:

$$\mathfrak{T} = \mathfrak{X}/\mathfrak{S}.$$  

By a rational character, or algebraic character, or simply character, we shall mean a morphism of algebraic groups

$$\mathfrak{T} \to \mathbb{C}^\times.$$  

Such a character has the form

$$(z_1, \ldots, z_m) \mapsto z_1^{\beta_1} \ldots z_m^{\beta_m}$$

where the $\beta_j$ are all integers.

**Theorem 1.1.** Up to a constant $\varepsilon(\mathfrak{X}, \psi)$, each local constant factors through a rational character $\chi(\mathfrak{X}, \psi)$ of $\mathfrak{T}$. Quite specifically, we have

$$\chi(\mathfrak{T}, \chi) \mapsto (z_1^{\beta_1}, \ldots, z_k^{\beta_k})$$

(2)
where the $z_j$ are torus coordinates, $\mathfrak{X}$ is the orbit of the Weil-Deligne representation

$$V_1 \otimes \text{Sp}(d_1) \oplus \cdots \oplus V_k \otimes \text{Sp}(d_k)$$

and

$$\beta_j = (d_j - 1) \dim V_j + d_j [a(V_j) + n(\psi) \dim(V_j)]$$

where $a(V_j)$ denotes the Artin conductor exponent of $V_j$, $n(\psi)$ denotes the conductor of $\psi$, and $I$ denotes the inertia subgroup of the local Weil group $W_K$.

Since the conductors are integers, the number $\beta_j$ is an integer. So the map (2) is a rational character of $T$.

The rather lengthy formula for the constant $e(\mathfrak{X}, \psi)$ appears later in this article, see (19) and (20).

We emphasize that the character $\chi(\mathfrak{X}, \psi)$ and the constant $e(\mathfrak{X}, \psi)$ depend only on the irreducible component $\mathfrak{X}$, once the additive character $\psi$ has been chosen and fixed.

Let $T$ be a maximal torus in the Langlands dual group $\text{GL}_n(\mathbb{C})$. We recall the idea, familiar from noncommutative geometry [Kh a, p.77], of the noncommutative quotient algebra

$$\mathcal{O}(T) \rtimes W.$$ 

Within periodic cyclic homology (a noncommutative version of de Rham theory) there is a canonical isomorphism

$$\text{HP}_*(\mathcal{O}(T) \rtimes W) \simeq \text{H}^*(T//W; \mathbb{C})$$

where

$$T//W$$

denotes the extended quotient of $T$ by $W$, see §4. From our point of view, the extended quotient $T//W$, a complex algebraic variety, is a more concrete version of the noncommutative quotient algebra $\mathcal{O}(T) \rtimes W$.

In §4, we focus on a part of the smooth dual of $\text{GL}_n(K)$, namely the arithmetically unramified smooth dual. The extended quotient $T//W$ is a model for this part of the dual, and we calculate explicitly the local constants.

We note that $\varepsilon_K(V, \psi)$ is denoted $\varepsilon_{\text{Langlands}}^K(V, \psi)$ in [Ikeda] and $\varepsilon_L(V, \psi)$ in [Tate 3.6].

In writing this article, we were greatly influenced by the preprint of Ikeda [Ikeda]. We thank Paul Baum for several valuable conversations, which led to major changes in the exposition of this Note. The main background reference is Deligne [D], but we prefer to use the notation in Langlands [L].

2. Weil-Deligne Representations

We need to recall some material, following closely the exposition in [BP]. Let $K$ be a non-archimedean local field. The Weil group $W_K$ fits into a short exact sequence

$$0 \to I_K \to W_K \xrightarrow{d} \mathbb{Z}$$
where $I_K$ is the inertia group of $K$. A Weil-Deligne representation is a pair $(\rho, N)$ consisting of a continuous representation $\rho : W_K \to \GL_n(V)$, $\dim_{\mathbb{C}}(V) = n$, together with a nilpotent endomorphism $N \in \End(V)$ such that

$$\rho(w)N\rho(w)^{-1} = ||w||N.$$  

For any $n \geq 1$, the representation $\Sp(n)$ is defined by

$$V = \mathbb{C}^n = \mathbb{C}e_0 + \cdots + \mathbb{C}e_{n-1}$$

with $\rho(e_i) = ||w||^i e_i$ and $Ne_i = e_{i+1}$ ($0 \leq i \leq n-1$), $Ne_{n-1} = 0$.

Let $\mathcal{G}_n(K)$ be the set of equivalence classes of semisimple $n$-dimensional Weil-Deligne representations. Let $\text{Irr}(\GL_n(K))$ be the set of equivalence classes of irreducible smooth representations of $\GL_n(K)$.

We recall the local Langlands correspondence

$$\text{rec}_K : \text{Irr}(\GL_n(K)) \to \mathcal{G}_n(K)$$

which is unique subject to the conditions listed in [HT, p.2].

We identify the elements of the set $\mathcal{G}_1(K)$, the quasicharacters of $W_K$, with quasicharacters of $K^\times$ via the local Artin reciprocity map

$$\text{Art}_K : W_K \to K^\times$$

The local Langlands correspondence is compatible with twisting by quasicharacters [HT, p.2].

A quasicharacter $\psi : W_K \to \mathbb{C}^\times$ is (arithmetically) unramified if $\psi$ is trivial on the inertia group $I_K$. In that case we have $\psi(w) = z^{d(w)}$ with $z \in \mathbb{C}^\times$. The group of unramified quasicharacters of $W_K$ is denoted $\Psi(W_K)$. Let $\Phi = \Phi_K$ denote a geometric Frobenius element in $W_K$. The isomorphism $\Psi(W_K) \simeq \mathbb{C}^\times$ is secured by the map $\psi \mapsto \psi(\Phi_K)$.

Let now

$$\rho' = \rho_1 \otimes \Sp(r_1) \oplus \cdots \oplus \rho_m \otimes \Sp(r_m)$$

be a Weil-Deligne representation. The set

$$\{\psi_1 \otimes \Sp(r_1) \oplus \cdots \oplus \psi_m \rho_m \otimes \Sp(r_m) : \psi_1, \ldots, \psi_m \in \Psi(W_K)\}$$

will be called the orbit of $\rho'$ under the action of

$$\Psi(W_K) \times \cdots \times \Psi(W_K)$$

($m$ factors). This orbit will be denoted $\mathcal{O}(\rho')$. The orbits create a partition of $\mathcal{G}_n(K)$. The set $\mathcal{G}_n(K)$ is a disjoint union of orbits:

$$\mathcal{G}_n(K) = \bigsqcup \mathcal{O}(\rho')$$

We note that $\Psi(W_K)^m \simeq (\mathbb{C}^\times)^m$, a complex torus. To determine the structure of each orbit, we have to pay attention to the torsion numbers of $\rho_1, \ldots, \rho_m$ and to the action of $\GL_n(\mathbb{C})$ by conjugation. In this way, the set $\mathcal{G}_n(K)$ acquires (locally) the structure of complex algebraic variety. Each irreducible component in this variety is the quotient of a complex torus by a product of symmetric groups.
We shall view each orbit \( O(\rho') \) as a pointed set, by choosing a Galois representative for each irreducible representation of \( W_K \). We recall that, given an irreducible representation \( V \) of \( W_K \), there exists an irreducible representation \( V^{\text{Gal}} \) of Galois type such that \( V = V^{\text{Gal}} \otimes \omega_s \) for some \( s \in \mathbb{C} \), see [Tate (2.2.1)].

We will view each orbit \( O(\rho') \) as a pointed set, with base point

\[
\rho' = \rho_1 \otimes \text{Sp}(t_1) \oplus \cdots \oplus \rho_m \otimes \text{Sp}(t_m).
\]

In the notation of [Ikeda], we will choose \( \rho_j \) to be of Galois type, \( \rho_j = V_j^{\text{Gal}} \).

3. The formulas

The elementary substitution referred to in the Introduction is as follows. Let

\[\varepsilon_K(s, V, \psi) = \varepsilon_K(V \otimes \omega_{s-1/2}, \psi)\]

for all \( s \in \mathbb{C} \), see [Tate (3.6.4)], [L, p.6]. For \( s \in \mathbb{C} \), \( \omega_s : W_K \to \mathbb{C}^\times \) is the quasicharacter defined by \( \omega_s(w) = ||w||_K^s \) for all \( w \in W_K \).

If \( V \) is a 1-dimensional continuous complex representation of \( W_K \), and \( \chi : W_K \to \mathbb{C}^\times \) is the corresponding quasicharacter, then \( \varepsilon_K(\chi, \psi) \) is the abelian local constant of Tate, see [Tate (3.6.3)].

We recall that, if \( (V, N) \) is any \( \Phi \)-semisimple Weil-Deligne representation, then we have a finite direct sum decomposition of \( (V, N) \) into indecomposable Weil-Deligne representations as follows:

\[(V, N) = V_1 \otimes \text{Sp}(d_1) \oplus \cdots \oplus V_m \otimes \text{Sp}(d_m)\]

We will write

\[V_j = V_j^{\text{Gal}} \otimes \omega_{s_j} \]

**Lemma 3.1.** Let \( a(V) \) denote the Artin conductor exponent of \( V \). Then we have \( a(V \otimes \omega_s) = a(V) \).

**Proof.** The definition is

\[a(V) = \dim V - \dim V^I + \sum_{k \geq 1} \frac{1}{[I : I_k]} \cdot \dim V/V^{I_k}\]

where \( I = I_0 \supset I_1 \supset \cdots \supset I_k \supset \cdots \) are the ramification subgroups of the inertia group \( I \).

We have

\[\dim(V \otimes \omega_s) = \dim V\]

Now \( \omega_s \) is an unramified quasi-character of \( W_K \):

\[\omega_s(I) = ||\text{Art}_K(I)||^s = ||U_K||^s = 1\]

and so

\[(V \otimes \omega_s)^{I_k} = V^{I_k}\]

for all \( k \geq 0 \). The result now follows from (4). \( \square \)
In particular, we have
\[ a(V_j) = a(V_j^{\text{Gal}}). \]

We need the following three items in order to compute epsilon factors.

3.1. **Additivity.** Additivity with respect to \( V \), see [Tate 3.4.2], [L Lemma 22.4]:
\[ \epsilon_K(V_1 \oplus \cdots \oplus V_k, \psi) = \epsilon_K(V_1, \psi) \cdots \epsilon_K(V_k, \psi) \] (5)

3.2. **Unramified twist.** Behaviour under unramified twist, see [Tate 3.4.5], [L, Lemma 22.4]:
\[ \epsilon_K(V \otimes \omega_s, \psi) = \epsilon_K(V, \psi) q^{-s[a(V) + n(\psi) \dim V]} \] (6)

where \( a(V) \) is the Artin conductor exponent of \( V \), and \( n(\psi) \) is the conductor of \( \psi \).

3.3. **The extension formula.** The extension to Weil-Deligne representations is as follows [Tate 4.1.6]:
\[ \epsilon_K((V, N), \psi) := \epsilon_K(V, \psi) \det(-\Phi|V^I/N^I) \] (7)

3.4. **The term \( \epsilon_K(V, \psi) \).** A typical direct summand in (3) is
\[ V_j^{\text{Gal}} \otimes \omega_{s_j} \otimes \omega_k \]
with \( 1 \leq j \leq m, \ 0 \leq k \leq d_j - 1 \). We have
\[ V_j^{\text{Gal}} \otimes \omega_{s_j} \otimes \omega_k = V_{j+s_j+k}^{\text{Gal}} \]
For this summand, we have by (6)
\[ \epsilon_K(V_j^{\text{Gal}} \otimes \omega_{s_j} \otimes \omega_k, \psi) = \epsilon_K(V_j^{\text{Gal}}, \psi) q^{-[a(V_j^{\text{Gal}}) + n(\psi) \dim V_j^{\text{Gal}}]} \] (8)

and then the formula for \( \epsilon_K(V, \psi) \) follows from (5). Applying Lemma 3.1, we obtain
\[ \epsilon_K(V, \psi) = \prod_{j=1}^{m} \left( \epsilon_K(V_j^{\text{Gal}}, \psi) \right)^{d_j} \cdot q^{-[s_j d_j + (d_j-1)d_j/2][a(V_j) + n(\psi) \dim(V_j)]} \] (9)

Note that Ikeda succeeds in describing the numbers \( \epsilon_K(V_j^{\text{Gal}}, \psi) \) in terms of the non-abelian local class field theory of \( K \), see [Ikeda Theorem 5.4].

3.5. **The determinant.** The determinant is additive
\[ \det(A \oplus B) = (\det A)(\det B) \]
and so it suffices to consider a typical factor in (7), namely
\[ \det(-\Phi|E_j^I/(E_j^I)_{N_j}) \] (10)
where $E_j$ is the $W_K$-module given by

$$E_j = V_j \otimes (\omega_0 \oplus \omega_1 \oplus \cdots \oplus \omega_{d_j-1})$$

$$= (V_j^{\text{Gal}} \otimes \omega_{s_j}) \oplus (\omega_0 \oplus \omega_1 \oplus \cdots \oplus \omega_{d_j-1})$$

$$= V_j^{\text{Gal}} \otimes (\omega_{s_j} \oplus \omega_{s_j+1} \oplus \cdots \oplus \omega_{s_j+d_j-1})$$

We note that $V_j^I = (V_j^{\text{Gal}})^I$. Then the $W_K$-submodule fixed by the inertia group $I$ is

$$E_j^I := V_j^I \otimes (\omega_{s_j} \oplus \omega_{s_j+1} \oplus \cdots \oplus \omega_{s_j+d_j-1})$$

The $W_K$-submodule of $E_j$ annihilated by $N_j$ is

$$(E_j)_N = V_j^{\text{Gal}} \otimes \omega_{s_j+d_j-1}$$

from which it follows that

$$(E_j)_N^I = V_j^I \otimes \omega_{s_j+d_j-1}$$

For the quotient we have the following $W_K$-module:

$$(E_j)_{N_j} = V_j^I \otimes (\omega_{s_j} \oplus \cdots \oplus \omega_{s_j+d_j-2})$$

Recall that

$$\omega_{s}(\Phi) = ||\varpi_K||^s = q_K^{-s}$$

It is enough to compute the action of $-\Phi$ on the $W_K$-module $V_j^I \otimes \omega_{s_j} \otimes \omega_k$ with $0 \leq k \leq d_j - 1$. On this $W_K$-module, $-\Phi$ will act as

$$q^{-(s_j+k)}(-\Phi|V_j^I)$$

and the determinant will be

$$q^{-(s_j+k)\dim V_j^I} \det(-\Phi|V_j^I)$$

There are $d_j - 1$ direct summands in (11) so the resulting determinant will be the product

$$\prod_{k=0}^{d_j-2} q^{-(s_j+k)\dim V_j^I} \cdot \det(-\rho_j(\Phi)|V_j^I)$$

provided that $d_j \geq 3$. By inspection, this formula is also valid for $d_j = 1$ or 2.
3.6. **The term** $\varepsilon_K((V,N),\psi)$. From the extension formula (7) we infer that $\varepsilon((V,N,\psi)$ is the product of (9) and (12).

We wish to isolate the term which is dependent on the variables $s_1, \ldots, s_m$. This term is of exponential type as follows:

$$\text{const} \cdot q^{\sum_{j=1}^{m} - s_j \beta_j}$$

where

$$\beta_j = (d_j - 1) \dim V_j^I + d_j[a(V_j) + n(\psi) \dim(V_j)]$$

for all $1 \leq j \leq m$. Note that $\beta_j$ is an integer: $\beta_j \in \mathbb{Z}$.

The constant can be read off from (9) and (12). The formula (15) is intricate, but, from our point of view, it has a simple form, namely

$$\text{const} \cdot z_1^{\beta_1} \cdots z_m^{\beta_m}$$

where

$$z_j = q^{-s_j}$$

Apart from the constant term, the formula (15) for the epsilon factor is a rational character of the complex torus $(\mathbb{C}^\times)^m$, i.e. the morphism

$$(\mathbb{C}^\times)^m \to \mathbb{C}^\times, \quad (z_1, \ldots, z_m) \mapsto z_1^{\beta_1} \cdots z_m^{\beta_m}$$

of algebraic groups.

Consider the following set:

$$\{ \omega_1 \otimes V^\text{Gal}_1 \otimes \text{Sp}(d_1) \oplus \cdots \oplus \omega_m \otimes V^\text{Gal}_m \otimes \text{Sp}(d_m) : s_1, \ldots, s_m \in \mathbb{C} \}.$$ (17)

After allowing for conjugacy in the Langlands dual group $\text{GL}_n(\mathbb{C})$, this set has the structure of a complex algebraic variety $\mathfrak{X}$ in $\mathcal{G}_n$. In fact $\mathfrak{X}$ is an irreducible component in $\mathcal{G}_n(K)$:

$$\mathfrak{X} \subset \mathcal{G}_n(K)$$

Applying the local Langlands correspondence, we have, by transport of structure, an irreducible component in the smooth dual:

$$\text{rec}^{-1}_K(\mathfrak{X}) \subset \text{Irr}(\text{GL}_n(K))$$

Looking carefully at the formulas (9) and (12), we see that the constant in (15) depends on the variety $\mathfrak{X}$, and on the additive character $\psi$. We will denote this constant by $e(\mathfrak{X}, \psi)$, so that (15) can be re-written

$$e(\mathfrak{X}, \psi) \cdot q^{-s_1 \beta_1} \cdots q^{-s_m \beta_m}$$

which, up to the constant $e(\mathfrak{X}, \psi)$, factors as a rational character through $\mathfrak{X}$. The constant $e(\mathfrak{X}, \psi)$ is itself the product of

$$\prod_{j=1}^{m} \left( \varepsilon_K(V^\text{Gal}_j, \psi) \right)^{d_j} \cdot q^{-\frac{1}{2}(d_j - 1)d_j[a(V_j) + n(\psi) \dim(V_j)]}$$

(19)
with

\[ \prod_{j=1}^{m} \det(-\rho_j(\Phi)|V_j^F)^{d_j-1} \cdot q^{-\frac{1}{2}(d_j-2)(d_j-1)} \dim V_j \]

The terms \( \varepsilon_K(V_j^\text{Gal}, \psi) \) are the epsilon factors attached to irreducible representations of the local absolute Galois group \( G_K \). These terms are defined in \( \text{Ikeda, p.15} \). There is one case where they are readily computed.

**Lemma 3.2.** Let \( \psi \) be an additive character \( K \to \mathbb{C}^\times \). Then we have \( \varepsilon_K(1, \psi) = 1 \).

**Proof.** We start with the classical formula in \( \text{Tate, 3.6.3} \):

\[ \varepsilon_K(\chi, \psi) = \chi(c) \frac{\int_{O^\times} \chi^{-1}(u)\psi(u/c) \, du}{\int_{O^\times} \psi(u/c) \, du} \]

where \( c \) is an element of \( K^\times \) of valuation \( a(\chi) + n(\psi) \). Now set \( \chi = 1 \) and let \( n(\psi) = k \). Then we take \( c = \varpi^k \). Then \( u \in O^\times \implies u/c \in \varpi^{-k}O^\times \). But we have \( \psi(\varpi^{-k}O) = 1 \) since \( \psi \) has conductor \( k \). Therefore we have

\[ \varepsilon_K(1, \psi) = \frac{\int_{O^\times} \psi(u/c) \, du}{\int_{O^\times} \psi(u/c) \, du} = 1 \]

\[ \square \]

4. **The arithmetically unramified representations of \( \text{GL}_n(K) \)**

Here, the underlying representation of the Weil group is the trivial \( n \)-dimensional representation \( \rho : W_K \to \text{GL}_n(\mathbb{C}) \). So we have \( V_j^\text{Gal} = 1, 1 \leq j \leq n \).

Let \( W \) be the Weyl group \( \mathfrak{S}_n \). The arithmetically unramified representations of \( \text{GL}_n(K) \) have, by definition, the following set of Langlands parameters (Weil-Deligne representations):

\[ \{ \omega_{s_1} \otimes \text{Sp}(d_1) \oplus \cdots \oplus \omega_{s_k} \otimes \text{Sp}(d_k) : s_j \in \mathbb{C} \} \]

where \( d_1 + \cdots + d_k = n \). This set determines a complex algebraic variety \( \mathfrak{X} \) in \( \mathcal{G}_n(K) \).

We choose \( \psi \) to have conductor 0, and now apply Lemma 3.2. In this case \( \beta_j = d_j - 1 \). We have

\[ \varepsilon_K((V, N, \psi) = e(\mathfrak{X}, \psi) \prod_{j=1}^{m} q^{-(d_j-1)s_j} \]

\[ = e(\mathfrak{X}, \psi) \prod_{j=1}^{m} z_j^{d_j-1} \]
where
\[ e(X, \psi) = \prod_{j=1}^{m} (-1)^{d_j-1} q^{-(d_j-1)(d_j-2)/2} \]
and \( z_j := q^{-s_j} \).

The epsilon factor records the dimensions \( d_j \) of the special representations \( \text{Sp}(d_j) \) which occur in the Weil-Deligne representation \( (V, N) \).

We will now re-organise the partition \( d_1 + \cdots + d_k = n \). Suppose that this partition has distinct parts \( t_1, \ldots, t_m \) with \( t_1 < t_2 < \cdots < t_m \) and that \( t_j \) is repeated \( r_j \) times so that \( r_1 t_1 + \cdots + r_m t_m = n \).

Then, as a function on the complex torus \( (\mathbb{C} \times)^{r_1 + \cdots + r_m} \), the epsilon factor is (invariant under the following product of symmetric groups):
\[ \mathfrak{S}_{r_1} \times \mathfrak{S}_{r_2} \times \cdots \times \mathfrak{S}_{r_m} \]
and therefore factors through the following quotient variety
\[ (\mathbb{C}^x)^{r_1} / \mathfrak{S}_{r_1} \times \cdots \times (\mathbb{C}^x)^{r_m} / \mathfrak{S}_{r_m} \]
which is precisely an irreducible component of the extended quotient \( T//W \).

We recall the extended quotient. Let the finite group \( \Gamma \) act on the complex algebraic variety \( X \). Let \( \tilde{X} = \{(x, \gamma) : \gamma x = x \} \), let \( \Gamma \) act on \( \tilde{X} \) by \( \gamma_1(x, \gamma) = (\gamma_1 x, \gamma_1 \gamma \gamma_1^{-1}) \). Then the extended quotient of \( X \) by \( \Gamma \) is
\[ X//\Gamma := \tilde{X}/\Gamma. \]
Let \( X^\gamma \) denote the \( \gamma \)-fixed set, and let \( Z(\gamma) \) be the \( \Gamma \)-centralizer of \( \gamma \). Choose one \( \gamma \) in each \( \Gamma \)-conjugacy class, then we have
\[ X//\Gamma = \bigsqcup X^\gamma / Z(\gamma). \]
This is reminiscent of, but distinct from, the stack quotient \([X/\Gamma]\). The stack quotient \([X/\Gamma]\) is the category whose objects are the points of \( X \) and for which a morphism \( x \to x' \) is given by an element \( \gamma \in \Gamma \) such that \( \gamma x = x' \), see \([\mathfrak{O}], \text{p.1}\).

Returning to the extended quotient \( T//W \), every irreducible component is accounted for in this way. The epsilon factors have precisely the amount of symmetry required to factor through these quotient varieties.

**Example.** Here, we consider the following Weil-Deligne representation of \( \text{GL}_{19}(K) \):
\[ \omega_{s_1} \text{Sp}(2) \oplus \omega_{s_2} \text{Sp}(2) \oplus \omega_{s_3} \text{Sp}(2) \oplus \omega_{s_4} \text{Sp}(3) \oplus \omega_{s_5} \text{Sp}(3) \oplus \omega_{s_6} \text{Sp}(7) \]
The epsilon factor of this representation is
\[ \text{const} \cdot z_1 z_2 z_3 z_4 z_5 z_6 \]
which will factor through the following irreducible component of the extended quotient \( T//W \):
\[ \text{Sym}^3(\mathbb{C}^x) \times \text{Sym}^2(\mathbb{C}^x) \times \mathbb{C}^x \]
This perfectly illustrates the symmetry properties of the epsilon factors. Each epsilon factor has precisely the symmetry, \textit{neither more nor less}, of the corresponding irreducible component in the extended quotient $T//W$. Each epsilon factor will therefore factor through the corresponding irreducible component in $T//W$.

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