Exotic plasma as classical Hall Liquid

C. Duval
Centre de Physique Théorique, CNRS
Luminy, Case 907
F-13 288 MARSEILLE Cedex 9 (France)

Z. Horváth
Institute for Theoretical Physics, Eötvös University
Pázmány P. sétány 1/A
H-1117 BUDAPEST (Hungary)

and

P. A. Horváthy
Laboratoire de Mathématiques et de Physique Théorique
Université de Tours
Parc de Grandmont
F-37 200 TOURS (France)

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Abstract

A non-relativistic plasma model endowed with an “exotic” structure associated with the two-parameter central extension of the planar Galilei group is constructed. Introducing a Chern-Simons statistical gauge field provides us with a self-consistent system; when the magnetic field takes a critical value determined by the extension parameters, the fluid becomes incompressible and moves collectively, according to the Hall law.

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1 Introduction

In a recent paper [1], the ground states of the Fractional Quantum Hall Effect (FQHE), represented by the “Laughlin” wave functions [2, 3], were derived by taking advantage of the two-fold “exotic” extension [4] of the planar Galilei group. Our clue has been that the two extension parameters $m$ and $k$ combine with the magnetic field into an effective mass,

$$m^* = m - \frac{ekB}{m}.
(1.1)$$

The main result in [1] says that for vanishing effective (rather than real [5]) mass, $m^* = 0$, i.e., when the magnetic field takes the (constant) critical value

$$B = B_{\text{crit}} \equiv \frac{m^2}{ek},
(1.2)$$

the consistency of the equations of motion requires that the particle move with the Hall velocity $\dot{q}_i = v_i^\text{Hall} \equiv \epsilon_{ij} E_j / B$, with $i, j = 1, 2$. Intuitively, in uniform electric and magnetic fields, the
cyclotronic motion of an ordinary charged particle can, for a specific initial velocity, degenerate to a straight line \[3\]. For an exotic particle with zero effective mass this is the only allowed motion.

The generalization to \(N\) particles interacting through dynamical gauge fields \[6\] is readily seen to be inconsistent with our vanishing effective mass condition (1.2), though. Fortunately, the ground state of the FQHE is actually described, however, by a self-consistent, incompressible quantum liquid, presented as a novel state of matter \[2, 7, 8, 9, 10\], rather than by a single particle moving in an external field. Below we construct, starting with the 1-particle model of \[1\] and following the general principles of plasma physics \[11\], an “exotic” plasma model. When the magnetic field takes the critical value (1.2), our plasma reduces to an incompressible fluid which moves according to the Hall law. It can be viewed, hence, as the classical counterpart of the quantum liquid in \[2, 7, 8, 9, 10\]. Requiring that the statistical gauge field have a Chern-Simons dynamics yields finally self-consistent solutions, which represent the ground state of the Hall fluid.

2 Exotic particle – gauge field system

The planar Galilei group has long been known to admit a non-trivial two-parameter central extension \[4\]. One of the extension parameters, present in any dimension, is conventional: it appears in the commutator of translations and boosts and is interpreted as the mass, \(m\). The other, “exotic” one, denoted by \(k\), only appears in two space dimensions; it comes from the commutator of the Galilean boosts. Our fundamental assumption is to view both parameters as physical.

In \[1\] we found that minimal coupling to an arbitrary planar electromagnetic field \(\vec{E}\) and \(B\) yields the equations of motion

\[
\begin{align*}
    m^* \dot{q}_i &= p_i - \frac{ek}{m} \varepsilon_{ij} E_j, \\
    \dot{p}_i &= e (E_i + B \varepsilon_{ij} \dot{q}_j).
\end{align*}
\]

(2.1)

The fields \(\vec{E}\) and \(B\) here satisfy the homogeneous Maxwell equation, implying that they derive from potentials \(V\) and \(\vec{A}\), respectively. Note that \(\vec{p}\) and \(m\dot{\vec{q}}\) are different. Eliminating \(\vec{p}\) in favor of \(\dot{\vec{q}}\) allows us to present (2.1) as

\[
    m^* \ddot{q}_i = e \left( E_i + \varepsilon_{ij} B_j - \frac{ek}{m} \varepsilon_{ij} \left( \dot{q}_k \partial_k E_j + \partial_i E_j + \varepsilon_{jk} \dot{q}_k (\dot{q}_\ell \partial_\ell B + \partial_i B) \right) \right),
\]

(2.2)

which shows that the “exotic” structure results in modifying the Lorentz force.

In \[1\] we analyzed our system in Souriau’s symplectic framework \[12\] (actually equivalent to “Faddeev-Jackiw” reduction \[13\]). Let us explain our results using Poisson brackets. Setting \(\xi = (\vec{q}, \vec{p})\), the equations of motion (2.1) can indeed be written in the Hamiltonian form

\[
\dot{\xi} = \{\xi, h\}, \quad h = \frac{\vec{p}^2}{2m} + eV(\vec{q}, t),
\]

(2.3)
the “exotic” Poisson bracket being given by

\[ \{ f, g \} = \frac{m}{m^*} \sum_{i=1}^{2} \partial_{q_i} f \partial_{p_i} g - \partial_{q_i} g \partial_{p_i} f \]

\[ + \frac{k}{mm^*} [\partial_{q_1} f \partial_{q_2} g - \partial_{q_1} g \partial_{q_2} f] + eB \frac{m}{m^*} [\partial_{p_1} f \partial_{p_2} g - \partial_{p_1} g \partial_{p_2} f] \]

(2.4)

The first term here is the conventional one; the second one combines the “exotic” structure and the magnetic field. Note that the plane became consequently non-commutative: the coordinates satisfy

\[ \{ q_1, q_2 \} = \frac{k}{mm^*} \]

(2.5)

rather than commute. Let us emphasize that the non-commutativity of the plane here arises even in the absence of any gauge field, and follows rather directly from the assumed “exotic” structure.

Let us emphasize that these formulæ are valid for any planar field. In particular, the “exotic” Poisson bracket (2.4) satisfies, despite the presence of the a priori position-dependent quantity \( m^* \), the Jacobi identity as long as the electromagnetic field satisfies the homogeneous Maxwell equation, as it can be verified by a tedious calculation. (A quicker proof is obtained using the associated symplectic structure.)

Further insight is gained by observing that, when \( B \) is constant such that \( m^* \neq 0 \),

\[
\begin{cases}
Q_i = q_i + \frac{1}{eB} \left( 1 - \sqrt{\frac{m^*}{m}} \right) \varepsilon_{ij} p_j \\
P_i = \sqrt{\frac{m^*}{m}} p_i - \frac{1}{2} eB \varepsilon_{ij} Q_j
\end{cases}
\]

(2.6)

are canonical coordinates on the 1-particle phase space. The “exotic” Poisson bracket (2.4) becomes

\[ \{ F, G \} = \sum_{i=1}^{2} \partial_{Q_i} F \partial_{P_i} G - \partial_{Q_i} G \partial_{P_i} F, \]

(2.7)

so that the 4D volume element reads \( dQ_1 \wedge dQ_2 \wedge dP_1 \wedge dP_2 \).

In these coordinates, the canonical structure retains hence the standard form, while the Hamiltonian becomes, however, rather complicated.

For vanishing effective mass, \( m^* = 0 \), the coordinates \( Q_i \) and momenta, \( P_i = -(eB/2)\varepsilon_{ij} Q_j \), are no more independent and the Poisson bracket (2.4) becomes singular. Then symplectic reduction yields a 2-dimensional reduced phase space with canonical coordinates \( \bar{Q}_i \)

\[ Q_i = q_i - \frac{mE_i}{eB^2_{\text{crit}}}, \]

(2.8)

The reduced Hamiltonian and Poisson bracket are

\[ H \equiv H_{\text{red}} = eV(\bar{Q}), \quad \{ F, G \}_{\text{red}} = -\frac{1}{eB_{\text{crit}}} (\partial_{Q_1} F \partial_{Q_2} G - \partial_{Q_1} G \partial_{Q_2} F), \]

(2.9)

respectively. The new coordinates satisfy now

\[ \{ Q_1, Q_2 \}_{\text{red}} = -\frac{1}{eB_{\text{crit}}}, \]

(2.10)
and the equations of motion, 
\[ \dot{\vec{Q}} = \{\vec{Q}, H\}_{\mathrm{red}}, \]  
(2.11)

become, by (2.9), 
\[ \dot{Q}^i = v^i_{\mathrm{Hall}} \equiv \varepsilon_{ij} E_j / B_{\mathrm{crit}}, \]
i.e., the Hall law. Note that the condition \( m^* = 0 \) plainly requires a constant magnetic field \( B = B_{\mathrm{crit}} \), whereas \( \vec{E} \) is an otherwise arbitrary curl-free electric field. Also observe that the reduced Hamiltonian is just the potential expressed in terms of the non-commuting coordinates \( Q_i \): this is the so-called “Peierls substitution” [5, 1].

Note for further reference that the 4D volume element became also degenerate; the reduced volume element is 
\[ e B_{\mathrm{crit}} dQ_1 \wedge dQ_2, \]
where \( m^* \) come from the exotic structure. It is worth noting that the magnetic term \( e B dq_1 \wedge dq_2 \) does not contribute to the volume element, since it drops out from the square of the symplectic form. (Similarly, when replacing the mechanical momenta, \( p_i \), by the canonical momenta, \( p_i - eA_i \), the gauge potentials would drop out by the same reason.)

Let us consider a distribution function \( f(\vec{q}_a, \vec{p}_a, t) \) on phase space. According to Liouville’s theorem, the volume element (3.1) is invariant w.r.t. the classical dynamics, and \( df / dt = 0 \). Using the equations of motion, this means that
\[ \partial_t f + \sum_a m_a \left[ \frac{p_a}{m_a} - \frac{e k}{m_a} \varepsilon_{ij} E_j^a \right] \partial_q f + e \left( E_i^a + B^a \varepsilon_{ij} \frac{p_j^a}{m} \right) \partial_p f = 0, \]
(3.2)
where \( E_j^a = E_j(\vec{q}_a) \). It is worth mentioning that (3.2) is indeed
\[ \partial_t f + \{ f, h \} = 0, \]
(3.3)
where \( h = \sum_a (\vec{p}_a)^2 / 2m + eV(\vec{q}_a) \) is the \( N \)-particle Hamiltonian, and \( \{ \cdot, \cdot \} \) denotes the \( N \)-particle Poisson bracket \( \{ f, g \} = \sum_a \{ f, g \}_a \).

Let us first assume that the effective mass does not vanish, \( m^* = m^*_1 \neq 0 \). Following the “regressive (BBGKY) method” [11], we integrate over the last \( (N - 1) \)-particle phase space and
define the 1-particle distribution $\phi$ as

$$\phi = \frac{m}{m^*} \int \prod_{a=2}^{N} \frac{m^*}{m} d\tilde{p}_a d\tilde{q}_a. \quad (3.4)$$

Integrating over the last $(N-1)$ particles and suppressing the particle label $a = 1$ allows us to infer (see App. 8.1 of Ref. [11]), the novel (Boltzmann) transport equation

$$\frac{\partial \phi}{\partial t} + \frac{1}{m^*} \left( p_i - \frac{ek}{m} \varepsilon_{ij} E_j \phi \right) \frac{\partial \phi}{\partial q_i} + \frac{m}{m^*} \varepsilon_{ij} \left( E_i + \frac{B}{m} \varepsilon_{ij} P_j \right) \frac{\partial \phi}{\partial p_i} + \frac{ek}{mm^*} \dot{B} \phi = 0, \quad (3.5)$$

where $\dot{B} = \partial_t B + \vec{q} \cdot \vec{\nabla} B$ is the material (or convective) derivative. In (3.5) a complicated expression called the collision integral, representing the two and more particle interactions [11], has been put to zero. This is justified since the collisions of our particles can indeed be neglected, owing to their infinitely short-range $\delta$-type interactions (see (4.5) below).

Our final step is to consider the mean matter density, the mean velocity, and the mean current by averaging over the last remaining momentum $\vec{p} \equiv \vec{p}_1$, namely

$$\bar{\rho} = \int \phi d\vec{p}, \quad \vec{v} = \frac{1}{\bar{\rho}} \int \vec{q} \phi d\vec{p}, \quad \vec{f} = \bar{\rho} \vec{v}. \quad (3.6)$$

Then (3.5) yields the hydrodynamical equations

$$\partial_t \rho + \vec{\nabla} \cdot (\rho \vec{v}) = 0, \quad (3.7)$$

$$\rho (\partial_t \vec{v} + \vec{v} \cdot \vec{\nabla} \vec{v}) = \vec{f} - \vec{\nabla} \cdot \sigma, \quad (3.8)$$

where $\vec{f} = \int \vec{q} \phi d\vec{p}$ is the mean value of the force on the r.h.s. of (2.2), and $\sigma = (\sigma_{ij})$ is the kinetic stress tensor [11]. Owing to the infinitely short-range forces, the inter-particle pressure can be neglected, and $\sigma$ retains, hence, the form

$$\sigma_{ij} = \int (\dot{q}_i - v_i)(\dot{q}_j - v_j) \phi d\vec{p}. \quad (3.9)$$

This statement follows from the general discussion in [11], Chap. 9. Reassuringly, it can also be shown directly: firstly, the continuity equation comes from the transport equation, using Stokes’ theorem and the homogeneous Maxwell equation $\partial_t B + \vec{\nabla} \times \vec{E} = 0$.

The exotic structure only enters the force. This latter is indeed found, using (2.2), to be

$$f_i = \frac{e \rho}{m^*} \left[ E_i + \varepsilon_{ij} v_j B - \frac{k}{m} \varepsilon_{ij} \left( \dot{E}_j + \varepsilon_{jk} v_k \dot{B} \right) \right] + \frac{ek}{mm^*} \sigma_{ij} \partial_j B, \quad (3.10)$$

where $\dot{E}_j = v_k \partial_k E_j + \partial_t E_j$ and $\dot{B} = v_k \partial_k B + \partial_t B$. Then, the Euler equation (3.8) follows from the modified force law (2.2) [or from (3.10)] by a tedious calculation.

A look at the $N$–particle transport equation (3.3) shows now that, in the limit $m^* \to 0$, the consistency requires that the coefficients of $1/m^*$ vanish:

$$p_i - \frac{k}{m} \varepsilon_{ij} E_j = 0, \quad E_i + \frac{B}{m} \varepsilon_{ij} P_j = 0, \quad (3.11)$$

yielding the Hall law. This same condition can also be obtained from the hydrodynamical equations. A tedious calculation yields in fact that the Hall law is necessary for the consistency of (3.8).
Further insight is gained by rewriting, for constant $B$ and nonvanishing $m^*$, (3.5) in terms of the twisted position coordinates $\vec{Q}$ and the original momenta, $\vec{p}$, as

$$\partial_t \phi + \dot{\vec{Q}}_i \frac{\partial \phi}{\partial Q_i} + \varepsilon_{ij} \frac{eB}{m^*} \left( p_j - m\varepsilon_{jk} \frac{E_k}{B} \right) \frac{\partial \phi}{\partial p_i} = 0,$$

(3.12)

where

$$\dot{Q}_i = \varepsilon_{ij} \frac{E_j}{B} + \frac{1}{\sqrt{mm^*}} \left( p_i - m\varepsilon_{ij} \frac{E_j}{B} \right).$$

(3.13)

It follows, as in (3.11), that in the limit $m^* \to 0$ the vector $\vec{p}$ and hence also $\vec{Q}$ have to satisfy the Hall constraint, namely

$$p_i = m\dot{Q}_i = m\varepsilon_{ij} \frac{E_j}{B}.$$  

(3.14)

Next, for $m^* \to 0$, the 4$N$-dimensional phase space “shrinks” to a 2$N$-dimensional reduced phase space, and the very definition (3.4) of the 1-particle distribution $\phi$ becomes meaningless. The reduced quantities can not be obtained by setting simply $m^* = 0$: the physical quantities may not behave continuously as $m^* \to 0$ [5]. The whole construction of Section 2 has to be repeated therefore once again, using the reduced structures. Let us hence consider a distribution function $F \equiv F_{\text{red}}(\vec{Q}, t)$ on reduced phase space. Then Liouville’s equation (3.2) is replaced, using the reduced Hamiltonian structure (2.9), by

$$\partial_t F + \{F, H\}_{\text{red}} = \partial_t F - \vec{E} \times \vec{\nabla} F \frac{B_{\text{crit}}}{B} = 0.$$  

(3.15)

The reduced 1-particle distribution on 2D phase space,

$$\Phi \equiv \Phi_{\text{red}}(\vec{Q}) = \int F \prod_{a=2}^N eB_{\text{crit}} dQ_1^a dQ_2^a$$  

(3.16)

satisfies therefore

$$\partial_t \Phi + \varepsilon_{ij} \frac{E_j}{B_{\text{crit}}} \frac{\partial \Phi}{\partial Q_i} = 0,$$

(3.17)

which replaces, for $m^* = 0$, the fundamental equation (3.12) by fixing the velocity $\dot{\vec{Q}}$ and putting the term proportional to $1/m^*$ to zero.

The mean matter density $\rho$ is in fact $\Phi$ in (3.16). Since all particles are frozen in a collective Hall motion, integrating out the momenta in (3.6) amounts to restricting the currents to the 2D surface in 4D phase space, defined by the Hall constraint (3.14). In fact, lifting $\Phi$ to the original phase space as $\phi(t, \vec{Q}, \vec{p}) = \Phi(t, \vec{Q}) \delta(\vec{p} - m\vec{v}_{\text{Hall}})$, the mean charge and velocity in (3.6) become

$$\rho = \Phi, \quad \vec{v} = \vec{v}_{\text{Hall}}.$$  

(3.18)

The density hence satisfies

$$\partial_t \rho + \varepsilon_{ij} \frac{E_j}{B_{\text{crit}}} \frac{\partial \rho}{\partial Q_i} = 0,$$

(3.19)

which is clearly is a Hamiltonian equation w.r.t. the reduced structure.

Note that our equation (3.19) is, indeed, consistent with the continuity equation (3.7) for the current $\vec{j} = \vec{v}\rho$, because $\vec{\nabla} \cdot \vec{v} = (1/B) \vec{\nabla} \times \vec{E} = 0$ thanks to the homogeneous Maxwell
equation $\partial_t B + \nabla \times \vec{E} = 0$ with $B = B_{\text{crit}}$. The fluid is therefore incompressible. It thus admits, as any incompressible fluid in the plane, the infinite dimensional symmetry of area-preserving diffeomorphisms [16], found above for a single particle.

Let us observe that no Euler equation analogous to (3.8) is obtained here, since the mean velocity is entirely determined by the Hall law. (This is somehow analogous to the drop in the phase space dimension.)

In conclusion, our results obtained so far say that for vanishing effective mass the consistency requires that the fluid move according to the Hall law, with the velocity determined by the gauge field $B = B_{\text{crit}}$ and $\vec{E}$ felt by the particle, whatever is the origin of this latter.

## 4 Coupled Chern-Simons – matter system, and the Hall states

The dynamics of the gauge field has not been specified so far. To this end, let us consider a coupled matter-gauge field system described by an action $S = \int \mathcal{L}_{\text{matter}} + \mathcal{L}_{\text{gauge-field}}$. To be consistent with the fundamental galilean symmetry of our approach, we choose for $\mathcal{L}_{\text{gauge-field}}$ the Chern-Simons Lagrangian [17], also including external magnetic and electric fields, $\vec{E}_{\text{ext}}$ and $B_{\text{ext}}$, respectively. Generalizing the matter Lagrangian of [6], we add $N$ “exotic” terms (actually equivalent to the second-order Lagrangian of Lukierski et al. [18], and is consistent with the symplectic form used in [1]). Thus, we describe our $N$ identical exotic particles with mass $m$, exotic structure $k$ and charge $e$, minimally coupled to a Chern-Simons gauge field $(A_\mu) = (A_t, \vec{A})$, by the action

$$S = \sum_{a=1}^{N} \int (\vec{p}_a - e \vec{A}) \cdot d\vec{q}_a - \left[ \frac{(\vec{p}_a)^2}{2m} + eA_t \right] dt + \frac{k}{2m^2} \varepsilon_{ij} p_i^a dp_j^a$$

$$+ \kappa \int \left\{ \frac{1}{2} \varepsilon_{\alpha\beta\gamma} F_{\alpha\beta} A_{\gamma} - B_{\text{ext}} A_t - \vec{A} \times \vec{E}_{\text{ext}} \right\} d\vec{q} dt,$$

where $\kappa$ is a new (Chern-Simons) coupling constant. Variation w.r.t. the particle coordinates yields $N$ “exotic” matter equations (2.1), and variation w.r.t. the gauge field yields the Chern-Simons field equations

$$\kappa B = -e \varrho_{\text{tot}}, \quad \kappa \varepsilon_{ij} E_j = -e j_i^{\text{tot}},$$

the external fields being hidden here in the total density and current, $(j_\mu^{\text{tot}}) = (\varrho_{\text{tot}}, \vec{j}_{\text{tot}})$, defined as $j_\mu^{\text{tot}} = \delta S/\delta A_\mu$, viz.

$$e \varrho_{\text{tot}} = e \varrho - \kappa B_{\text{ext}}, \quad e j_i^{\text{tot}} = e j_i - \kappa \varepsilon_{ij} E_j^{\text{ext}}.$$

Decomposing the total fields into the sum of the external and “statistical” quantities $(b, \vec{e})$, $B = b + B_{\text{ext}}$ and $\vec{E} = \vec{e} + \vec{E}_{\text{ext}}$ respectively, we see that the gauge-field part of (1.1) is actually equivalent to $\frac{1}{2} \int \varepsilon_{\alpha\beta\gamma} f_{\alpha\beta} a_{\gamma} d\vec{q} dt$. Thus, while the particle feels the total gauge field, the statistical field itself obeys the Chern-Simons dynamics with the particle current as source. From (1.2) we infer in fact

$$\kappa b = -e \varrho, \quad \kappa \varepsilon_{ij} e_j = -e j_i.$$

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The $\delta$-functions in the particle current represent point-like vortices in the effective-field theory approach [19], and correspond to Laughlin’s quasiparticles [3]. It is now clear that, owing precisely to these vortices, $\rho$ (and hence $B$) can never be a constant. Thus, while the coupled Chern-Simons gauge field system associated with (4.1) may admit (even interesting) solutions, our reduction trick, which would require $B = \text{const}$, can not work. Hence the necessity to “smear out” these point-like singularities, and use, instead, the continuum model constructed in the previous Section. But, before doing this, let us remember that the gauge fields can be eliminated by solving the Chern-Simons equations [6],

$$
a_i(\vec{q}, t) = \frac{e}{2 \pi \kappa} \int \varepsilon_{ij} \frac{q_j - y_j}{|\vec{q} - \vec{y}|^2} \rho(\vec{y}, t) \, d\vec{y}, \quad a_i(\vec{q}, t) = \frac{e}{2 \pi \kappa} \int \frac{\langle \vec{q} - \vec{y} \rangle \times \vec{\jmath}(\vec{y}, t)}{|\vec{q} - \vec{y}|^2} d\vec{y}. \quad (4.5)
$$

It follows that the inter-particle forces are infinitely short-range, as anticipated when deriving our plasma model.

Let us now turn to considering continuum matter with Chern-Simons coupling, i.e., replace the $N$-particle system with our “exotic” plasma. Owing to the well-known difficulties encountered in the action formulation of fluid dynamics [20], we do not insist on a variational approach, and work, instead, directly with the equations of motion. Thus, we posit the Chern-Simons equations (4.4), coupled to the fluid dynamical equations (3.7-3.8), for $m^* \neq 0$, and (3.19) for $m^* = 0$, respectively.

We now observe that these equations are consistent with collective–motion Ansatz

$$
\vec{j}_{\text{tot}} = \vec{v} \, \rho_{\text{tot}}, \quad (4.6)
$$

provided the velocity is $\vec{v} = \vec{v}_{\text{Hall}}$. Note that the “external sources” $\rho_{\text{ext}} = -(\kappa/e) B_{\text{ext}}$ and $j_i^{\text{ext}} = -(\kappa/e) \varepsilon_{ij} E_j^{\text{ext}}$ can also be viewed [23] as background charge/current densities, which also satisfy the Hall law.

The general coupled system (3.7-3.8)-(4.2), will be studied elsewhere.

Here our point is that when we restrict ourselves to the case $m^* = 0$, i.e., when the magnetic field felt by the fluid takes the critical value $B = B_{\text{crit}}$, the collective-motion Ansatz (4.6) becomes mandatory for the reduced system, yielding $E_j / B_{\text{crit}} = E_j^{\text{ext}} / B_{\text{ext}}$. The velocity determined by the fields felt by the particle is, hence, also given by the external field alone,

$$
v_i = \varepsilon_{ij} \frac{E_j}{B_{\text{crit}}} = \varepsilon_{ij} \frac{E_j^{\text{ext}}}{B_{\text{ext}}} \equiv v_i^{\text{Hall}}. \quad (4.7)
$$

As seen above, the total flow is incompressible; when the external field $B_{\text{ext}}$ is also uniform, the matter density $\rho$ becomes also constant, and the matter flow is also incompressible.

In the critical case the Chern-Simons equations require hence

$$
\rho = \frac{\kappa}{e} (B_{\text{ext}} - B_{\text{crit}}), \quad \vec{j} = \rho \, \vec{v}_{\text{Hall}}, \quad (4.8)
$$

with $\vec{v}$ obeying the Hall law (1.7). Equation (1.8) represents hence the ground state of the Hall fluid. If the electric field is, in addition, divergence-free, $\vec{\nabla} \cdot \vec{E} = 0$, (e.g., if the external fields are uniform), then the flow becomes also irrotational. When $B_{\text{ext}} = B_{\text{crit}}$, the particle density vanishes, $\rho = 0$. Hence there is no statistical field, merely a uniform background charge $\rho_{\text{ext}}$, which moves according to the Hall law. When the external field is moved out from the critical
value, excitations are created: the quantity \( \rho = \rho_{\text{tot}} - \rho_{\text{ext}} \) describes in fact the deviation from the background density. If the external fields are uniform, so is \( \rho \): the excitations condensate into collective modes. The sign of \( \rho \) is positive or negative depending on that of \( (B_{\text{ext}} - B_{\text{crit}}) \), corresponding to quasiparticles and quasiholes, respectively [3, 7, 11].

According to the Chern-Simons equations (4.2) \( \rho \) and \( \vec{j} \) are the sources of the statistical field, whose rôle is to maintain the total magnetic field at the critical value \( B_{\text{crit}} \) (and to create an electric field such that the “external Hall law” (4.7) holds).

5 Conclusion

In this paper we have derived an exact, self-consistent solution, (4.8), of the coupled exotic matter–Chern-Simons gauge field system. Our solution is associated with vanishing effective mass, \( m^* = 0 \), i.e., with the magnetic field taking the critical value \( B_{\text{crit}} = m^2/ek \). Intuitively, we want to view the limit \( m^* \to 0 \) as “condensation into the collective ground state”; in other words, a kind of “phase transition” into this strongly correlated “novel state of matter” [3, 7, 8, 10] we identify with the FQH ground state. We are not in the position to prove, within our classical context, that \( m^* = 0 \) would be mandatory. Our investigations imply, however, that when this condition holds, then the Hall motions are the only consistent ones. It is worth to be mentioned, however, that our reduced fluid dynamical equation (3.19) has been proposed to describe the chiral bosons of edge currents [8], which indicates that our vanishing effective mass condition \( m^* = 0 \) may be physically relevant.

Quantum fluids, non-commutative structures, and Chern-Simons theory have already been considered in this context by many authors; see, e.g., [2, 5, 7, 9, 10, 21, 22]. The approach closest to ours would be that of Stone [7], who derives the Euler equations of fluid dynamics using the “Madelung” transcription of the effective “Landau-Ginzburg” theory [19]. Our approach here is, however, rather different: it is entirely based on the classical model associated with the “exotic” structure of the planar Galilei group. In particular, the non-commutativity of the plane follows from this structure unlike in other approaches [23].

It has been noticed before [5] that a reduced model leading to the ground states of the QHE can be obtained by letting the ordinary mass, \( m \), go to zero. Our “exotic” model allows us, however, to avoid taking such an unphysical limit: our vanishing effective mass condition, \( m^* = 0 \), only requires to fine-tune the magnetic field to its critical value determined by the parameters \( m \) and \( k \) (assumed here physical). Note, however, that in our approach the “good” coordinates are the “twisted coordinates” \( Q_i \), and not the physical coordinates \( q_i \).

Our fluid model is derived straightforwardly from the modified force law (2.2), following the general principles of plasma physics [1]. Galilean invariance would actually allow to add a magnetic (but no electric) Maxwell term [23], which would contribute a term \( \varepsilon_{ij} \partial_j B \) in the second Chern-Simons equation (4.2). This would not change our conclusions, though, since the new term would drop out since \( B = B_{\text{crit}} = \text{const} \).

Our results are consistent with some of the essential properties of Hall fluids and constitute therefore a strong argument in support of the physical relevance of the “exotic” Galilean structure. Let us insist that, despite its physical dimension \( [k] = [\hbar/c^2] \), our “exotic parameter” is a classical object. (Remember that the phase space symplectic form \( dp_i \wedge dq_i \) has also dimension
[\hbar]. It is, just like anyonic spin, unquantized. As pointed out by Jackiw and Nair \[24\], \( k \) can be viewed as a kind of non-relativistic “shadow” of relativistic spin.

At last, our derivation shows indeed some similarity with that of Martinez and Stone \[15\], who obtain the Hall law as the first approximation to their second-quantized equation of motion. This latter corresponds in fact to the quantized version of our classical equation of motion \( (3.15) \), their Moyal bracket being the quantum deformation of our classical Poisson bracket. Similarly, the classical symmetry of area-preserving transformations, \( w_\infty \) is replaced, under quantization, by its quantum version \( W_\infty \) \[14, 15\]. Our theory might hence be viewed as a classical counterpart of that of Martinez and Stone.

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