Replication and Its Application to Weak Convergence

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Abstract

Herein, a methodology is developed to replicate functions, measures and stochastic processes onto a compact metric space. Many results are easily established for the replica objects and then transferred back to the original ones. Two problems are solved within to demonstrate the method: (1) Finite-dimensional convergence for processes living on general topological spaces. (2) New tightness and relative compactness criteria are given for the Skorokhod space $D(\mathbb{R}^+; E)$ with $E$ being a general Tychonoff space. The methods herein are also used in companion papers to establish the: (3) existence of, uniqueness of and convergence to martingale problem solutions, (4) classical Fujisaki-Kallianpur-Kunita and Duncan-Mortensen-Zakai filtering equations and stationary filters, (5) finite-dimensional convergence to stationary signal-filter pairs, (6) invariant measures of Markov processes, and (7) Ray-Knight theory all in general settings.
Frequently Used Notations

Set, numbers and mappings.

$\triangleq$ “Being defined by”.
$\emptyset$ Empty set.
$I$ Non-empty index set.
$\mathbb{N}$, $\mathbb{N}_0$ Positive and nonnegative integers respectively.
$\mathbb{Q}$, $\mathbb{Q}^+$ Rational and nonnegative rational numbers respectively.
$\mathbb{R}$, $\mathbb{R}^+$ Real and nonnegative real numbers respectively.
$\mathbb{R}^+$ Nonnegative real numbers.
$(\mathbb{R}^k, |\cdot|)$ $k$-dimensional ($k \in \mathbb{N}$) Euclidean space with Euclidean norm $|\cdot|$.
$\uparrow$, $\downarrow$ Non-decreasing and non-increasing convergence of real numbers (including convergence to $\infty$ and $-\infty$) respectively.
$
\subseteq, \supseteq$
Containment of sets including equalities.
$\aleph(E)$ Cardinality of set $E$.
$\mathcal{P}_0(E)$ All finite non-empty subsets of $E$.
$\circ$ Composition of mappings.
$1_A$ Indicator function of set $A$.
$f|_A$ Restriction of mapping $f$ to subset $A$ of its domain.
$\mathcal{D}|_A$ Restrictions of the members of mapping family $\mathcal{D}$ to $A$ (see (2.1.2)).
$\bigotimes \mathcal{D}$ Joint mapping of members of $\mathcal{D}$ (see (2.1.3)).
$\var$ Variant of mapping (see Notation 4.1.1).

Measurable and measure spaces.
| Symbol | Description |
|--------|-------------|
| $\mathcal{U}|_A$ | Concentration of $\sigma$-algebra $\mathcal{U}$ on $A$ (see (2.1.4)). |
| $\sigma(D)$ | $\sigma$-algebra induced by mapping family $D$ (see (2.1.5)). |
| $\delta_x$ | Dirac measure at $x$. |
| $\mathfrak{M}^+(E, \mathcal{U}), \mathfrak{P}(E, \mathcal{U})$ | Non-trivial finite and probability measures on measurable space $(E, \mathcal{U})$ respectively. |
| $\mathcal{N}(\mu)$ | Subsets of measure space $(E, \mathcal{U}, \mu)$ with zero outer measure (see §2.1.2). |
| $\nu|_E$ | Expansion of $\nu \in \mathfrak{M}^+(A, \mathcal{U}|_A)$ onto superspace $E$ (see (2.1.6)). |
| $\mu|_A$ | Concentration of $\mu \in \mathfrak{M}^+(E, \mathcal{U})$ on subset $A$ (see (2.1.7)). |
| $\mu \circ f^{-1}$ | Push-forward measure of $\mu$ by mapping $f$ (see (2.1.8)). |

**Topological spaces.**

| Symbol | Description |
|--------|-------------|
| $\mathcal{O}(E), \mathcal{C}(E)$ | Open and closed subsets of topological space $E$ respectively. |
| $\mathcal{K}(E), \mathcal{K}^m(E)$ | Compact and metrizable compact subsets of $E$ respectively. |
| $\mathcal{K}_\sigma(E), \mathcal{K}^{m\sigma}(E)$ | $\sigma$-compact and $\sigma$-metrizable compact subsets of $E$. |
| $\mathcal{B}(E)$ | Borel subsets of $E$. |
| $\mathcal{O}_E(A)$ | Subspace topology of $A$ induced from $E$ (see (2.1.9)). |
| $\mathcal{B}_E(A)$ | Subspace Borel $\sigma$-algebra of $A$ induced from $E$ (see (2.1.10)). |
| $A^\epsilon$ | $\epsilon$-envelope of subset $A$ of a metric space (see (2.1.12)). |
| $\mathcal{O}_D(A)$ | Topology induced by mapping family $D$ on $A$ (see (2.1.13)). |
| $\mathcal{B}_D(A)$ | Borel $\sigma$-algebra induced by $D$ on $A$, i.e. $\sigma[\mathcal{O}_D(A)]$. |

**Product spaces.**

| Symbol | Description |
|--------|-------------|
| $\times$ | Cartesian product. |
| $\prod_{i \in I} S_i$ | Cartesian product of $\{S_i\}_{i \in I}$. |
| $E^I$ | Cartesian power of $E$ with respect to index set $I$; in particular, $E^\infty$ abbreviates $E^I$ when $\aleph(I) = \aleph(\mathbb{N})$ and $E^d$ abbreviates $E^I$ when $\aleph(I) = d \in \mathbb{N}$. |
| $p_{I_0}$ | Projection mapping on $\prod_{i \in I} S_i$ for non-empty sub-index-set $I_0 \subseteq I$; in particular, $p_j$ abbreviates $p_{(j)}$. |
FREQUENTLY USED NOTATIONS

$,\otimes_i \mathcal{A}_i$ Product of $\sigma$-algebras (or topologies).

Product $\sigma$-algebra (or product topology) on the Cartesian product $\prod_{i \in I} S_i$ of measurable (or topological) spaces $\{ (S_i, \mathcal{A}_i) \}_{i \in I}$; in particular, $\mathcal{A} \otimes \mathcal{A}$ abbreviates $\otimes_{i \in I} \mathcal{A}_i$ when all $\mathcal{A}_i = \mathcal{A}$, $\mathcal{A} \otimes \mathcal{A}$ abbreviates $\otimes_{i \in I} \mathcal{A}_i$ when $\aleph(I) = \aleph(\mathbb{N})$, and $\mathcal{A} \otimes d$ abbreviates $\otimes_{i \in I} \mathcal{A}_i$ when $\aleph(I) = d \in \mathbb{N}$.

Spaces of general mappings.

$\varpi_1(f)$ Associated path mapping of mapping $f \in \mathcal{S}(E)$ (see (2.2.1)); in particular, $\varpi(f)$, $\varpi_T(f)$ and $\varpi_{a,b}(f)$ abbreviate $\varpi_{R^+}(f)$, $\varpi_{[0,T]}(f)$ and $\varpi_{[a,b]}(f)$ respectively.

$\varpi_1(D)$ Joint path mapping of mapping family $D \subset \mathcal{S}(E)$ (see (2.2.2)).

$M(S; E)$ Measurable mappings from measurable space $S$ to measurable space $E$.

$C(S; E)$ Continuous mappings from topological space $S$ to topological space $E$.

$\text{hom}(S; E)$, $\text{biso}(S; E)$ Homeomorphisms and Borel isomorphisms between $S$ and $E$ respectively.

$\text{imb}(S; E)$ Imbeddings from $S$ to $E$.

$\mathcal{D}(\mathcal{L}), \mathcal{R}(\mathcal{L})$ Domain and range of single-valued linear operator $\mathcal{L}$ respectively (see (2.2.23) and (2.2.24)).

$L|_D$ Restriction of $\mathcal{L}$ to $D \subset \mathcal{D}(\mathcal{L})$ (see (2.2.25)).

Skorokhod $\mathcal{J}$-spaces.

$w_{t,d,r}(x)$ r-modulus of continuity of $x \in \mathbb{ER}^+$ (see (2.2.3)).

$J(x)$ Set of left-jump times of $x \in \mathbb{ER}^+$ (see (2.2.5)).

$\text{TC}(\mathbb{R}^+), \text{TC}([a,b])$ All time-changes on $\mathbb{R}^+$ and $[a,b]$ respectively (see (2.2.2)).

$\varrho(x, y)$ $\varrho_{[a,b]}(x, y)$ Skorokhod pseudometric for $x, y \in \mathbb{ER}^+$ and $E^{[a,b]}$ induced by $r$ respectively (see (2.2.3), (2.2.7) and (2.2.4)).

$(D(\mathbb{R}^+; E), \mathcal{J}(E))$ Skorokhod $\mathcal{J}$-space of all càdlàg members of $\mathbb{ER}^+$ (see (2.2.2)).

$(D([a,b]; E), \mathcal{J}_{a,b}(E))$ Skorokhod $\mathcal{J}$-space of all càdlàg members of $E^{[a,b]}$ (see (2.2.2)).
**FREQUENTLY USED NOTATIONS**

\[ J(\mu) \] Set of fixed left-jump times of a finite measure \( \mu \) on \( D(\mathbb{R}^+; E) \) equipped with the restriction of \( \mathcal{B}(E)^{\otimes \mathbb{R}^+} \) (see 2.3.12).

**Spaces of functions.**

\[ f \vee g(x), f \wedge g(x) \] \( \max\{f(x), g(x)\} \) and \( \min\{f(x), g(x)\} \) for \( \{f, g\} \subset \mathbb{R}^E \) respectively.

\[ f^+(x), f^-(x) \] \( \max\{f(x), 0\} \) and \( \max\{-f(x), 0\} \) respectively.

\[ ae(D), ac(D), mc(D) \] Additive expansion, additive closure and multiplicative closure of \( \mathbb{R}^k \)-valued function family \( D \) respectively (see (2.2.9), (2.2.10), (2.2.11)).

\[ ag_Q(D), ag(D) \] \( \mathbb{Q} \)-algebra and algebra of \( D \) respectively (see (2.2.12), (2.2.13)).

\[ \Pi^I(D) \] Product functions of \( \mathbb{R} \)-valued function family \( D \) on \( E^I \) for finite \( I \); in particular, \( \Pi^d(D) \) abbreviates \( \Pi^I(D) \) with \( d = 8(I) \) (see (2.2.14)).

\[ u \rightarrow \] Uniform convergence of \( \mathbb{R}^k \)-valued functions.

\[ \| \cdot \|_\infty \] Supremum norm of bounded functions.

\[ cl(D), ca(D) \] Closure of and closed algebra generated by bounded \( \mathbb{R}^k \)-valued function family \( D \) under \( \| \cdot \|_\infty \) respectively (see (2.2.15)).

\[ M_b(E; \mathbb{R}^k) \] Banach space of all bounded members of \( M(E; \mathbb{R}^k) \) equipped with \( \| \cdot \|_\infty \).

\[ C_b(E; \mathbb{R}^k) \] Banach space over scalar field \( \mathbb{R} \) of all bounded members of \( C(E; \mathbb{R}^k) \) equipped with \( \| \cdot \|_\infty \).

\[ C_c(E; \mathbb{R}^k) \] Compact-supported continuous functions from \( E \) to \( \mathbb{R}^k \).

\[ C_0(E; \mathbb{R}^k) \] Subspace of all \( f \in C(E; \mathbb{R}^k) \) such that for any \( \epsilon > 0 \), there exists a \( K_\epsilon \in \mathcal{X}(E) \) satisfying \( \| f|_{E \setminus K_\epsilon} \|_\infty < \epsilon \).

**Finite Borel measures.**

\[ \mathfrak{b}e(\mu) \] All Borel extension(s) of \( \mu \) (if any) (see §2.3).

\[ \mathcal{M}^+(E), \mathcal{P}(E) \] Weak topological spaces of all finite and probability Borel measures on topological space \( E \) respectively.

\[ f^* \] Integral functional of \( f \in M_b(E; \mathbb{R}^k) \) (see (2.3.2)).

\[ \mathcal{D}^* \] Integral functionals of members of function family \( \mathcal{D} \subset M_b(E; \mathbb{R}) \) (see (2.3.3)).
Weak convergence of finite Borel measures.

\[ \text{w-lim}_{n \to \infty} \mu_n \]

Weak limit (see (2.3.5)).

Random variables and stochastic processes.

\((\Omega, \mathcal{F}, \mathbb{P})\)

Probability space.

\(\mathbb{E}\)

Expectation operator of \((\Omega, \mathcal{F}, \mathbb{P})\).

\(\text{pd}(X)\)

Process distribution of stochastic process \(X\) (see §2.5).

\(X_{T_0}\)

\(\mathbb{P}_{T_0} \circ X\), the section of stochastic process \(X\) for \(T_0 \in \mathcal{P}_0(\mathbb{R}^+)\) (see §2.5).

\(J(X)\)

Set of fixed left-jump times of stochastic process \(X\).

\(\mathcal{F}_t^X\)

Augmented natural filtration of \(X\) (see §2.5.1).

\(\mathcal{D}(T)\rightarrow\)

Finite-dimensional convergence along \(T \subset \mathbb{R}^+\) (see Definition 6.2.4).

\(\mathfrak{f}_{T}(\{X^n\}_{n \in \mathbb{N}})\)

Finite-dimensional limit of process sequence \(\{X^n\}_{n \in \mathbb{N}}\) along \(T \subset \mathbb{R}^+\) (see Definition 6.2.4).

\(\mathfrak{f}_{\mathbb{P}}(\{X^i\}_{i \in I})\)

Family of finite-dimensional limit points of process family \(\{X^i\}_{i \in I}\) along \(T \subset \mathbb{R}^+\) (see Definition 6.2.4).

Replication.

\((E_0, \mathcal{F}; \widehat{E}, \widehat{\mathcal{F}})\)

Replication base over \(E\) (see Definition 3.1.1).

\(\overline{f}\)

Abbreviation of \(\text{var}(f; \widehat{E}, E_0, 0)\) (see Notation 4.1.5).

\(\widehat{f}\)

Replica function of \(f\) (see Definition 4.1.3 and Notation 4.1.5).

\(\tilde{f}\)

\(f|_{E_0^d}\) for base \((E_0, \mathcal{F}; \widehat{E}, \widehat{\mathcal{F}})\) and \(\mathbb{R}^k\)-valued function \(f\) on \(E^d\) (see Notation 4.2.3).

\(\overline{\mathcal{F}}\)

\(\mathcal{F}|_{E_0}\) for base \((E_0, \mathcal{F}; \widehat{E}, \widehat{\mathcal{F}})\) (see Notation 4.2.3).

\(\mathcal{L}_0, \mathcal{L}_1\)

\(\mathcal{L}|_{\text{aff}(\mathcal{F})}\) and \(\mathcal{L} \cap (\text{ca}(\mathcal{F}) \times \text{ca}(\mathcal{F}))\) respectively given base \((E_0, \mathcal{F}; \widehat{E}, \widehat{\mathcal{F}})\) and linear operator \(\mathcal{L}\) (see Notation 4.2.3).

\(\mathcal{L}_i\)

\(\{(\tilde{f}, \tilde{g}) : (f, g) \in \mathcal{L}_i\}\) for each \(i \in \{0, 1\}\) (see Notation 4.2.3).

\(\widehat{\mathcal{L}}_0, \widehat{\mathcal{L}}_1\)

core and extended replicas of linear operator \(\mathcal{L}\) respectively (see Definition 4.2.2).

\(\mathfrak{m}\)

Replica measure of \(\mu\) (see Definition 5.2.1 and Notation 5.2.3).
\text{rep}(X; E_0, \mathcal{F}) \quad \text{replica processes of stochastic process } X \text{ with respect to base } (E_0, \mathcal{F}; \hat{E}, \hat{\mathcal{F}}); \text{ in particular, } \text{rep}_m(X; E_0, \mathcal{F}), \text{rep}_p(X; E_0, \mathcal{F}) \text{ and rep}_c(X; E_0, \mathcal{F}) \text{ denotes measurable, progressive and càdlàg members of rep}(X; E_0, \mathcal{F}) \text{ respectively (see Definition 6.1.3).}
CHAPTER 1

Introduction

Traditionally, researchers (e.g. [Kun71], [Mit83], [EK86], [BK93b], [Jak97a], [BBK00] and [KL08]) have focused on stochastic processes living on “good” topological spaces with some metrizable, separable, completely metrizable or (locally) compact properties. However, there are many settings of interest that violate these assumptions. For example, [HS79] and [Mit83] considered probability measures on the Skorokhod $\mathcal{J}_1$-space of tempered distributions. [Szp76] considered the nonlinear filtering problem for càdlàg signals living on Lusin spaces. [MZ84] considered tightness in the space of all càdlàg functions from the non-negative real numbers $\mathbb{R}^+$ to the real line $\mathbb{R}$ equipped with the pseudo-path topology, which was further discussed by [Str85] and [Kur91]. [Fit88] constructed Markov branching processes whose values are finite Borel measures on a Lusin space. [Jak97a] extended the Skorokhod Representation Theorem to tight sequences of probability measures on non-metrizable spaces. [Jak97b] considered a sequentially defined topology on the space of all càdlàg functions from the compact interval $[0,T]$ to $\mathbb{R}$. [DZ98] considered the space of all Borel probability measures on a Polish space equipped with the strong topology. [Jak86] and [Kou16] considered probability measures on $D([a,b]; E)$, the Skorokhod $\mathcal{J}_1$-space of all càdlàg mappings from a compact interval $[a,b]$ to a Tychonoff space $E$. [Lyo94, Lyo98, FV10] and [FH14] worked with non-separable Banach spaces of rough paths equipped with homogeneous $p$-variation or $1/p$-Hölder norms. None of these spaces are necessarily Polish nor compact. Some are not even metrizable or separable.

Figure 1. The idea of replication

The point of our work, as illustrated in Figure 1 above, is that Borel measurable functions, finite measures and stochastic processes living on a general topological space $E$ often can be replicated as replica functions, measures or processes living on some compact metric space $\hat{E}$. Non-càdlàg processes can have càdlàg replicas.
These replica objects are more easily analyzed on \( \hat{E} \) than the original objects on \( E \), and many results about the replica objects are transferrable back to the original ones by proper (often indistinguishable) redefinitions.

The idea of replication was motivated in part by [EK86], [BK93b], [BK10] and [Kou16] which exploit the use of imbedding and compactification techniques in various aspects of probability theory. One could extend results to various generalized settings in a case-by-case manner. However, replication is probably an easier and more unified approach of extending results from compact or Polish spaces to a large category of exotic spaces simultaneously. This approach is believed to have merit in areas such as weak convergence, martingale problems, nonlinear filtering, large deviations, Markov processes etc., where compactness or metric completeness can play a big role. Indeed, even a Polish space can be improved by adding compactness.

The contributions herein are:

**Theme 1** The methodology of replication and its applications to the Radon-Riesz Representation Theorem and Skorokhod Representation Theorem (Chapter 3 - 6).

**Theme 2** Finite-dimensional convergence of possibly non-c\’adl\’ag processes living on general spaces (§6.2 and Chapter 7).

**Theme 3** Tightness in Skorokhod \( \mathcal{J}_1 \)-spaces, the relationship between weak convergence on Skorokhod \( \mathcal{J}_1 \)-spaces and finite-dimensional convergence, and relative compactness in Skorokhod \( \mathcal{J}_1 \)-spaces (§6.4 and Chapter 8).

**Theme 1** tells when and how one can perform replication and serves as the theoretical foundation of all our developments. The question what replication can do is partially answered by **Theme 2** and **Theme 3** herein. Further motivation and illustration of the utility of replication can be found in the companion works of this article (see [DK20b, DK20a, DK21]) and other related papers (see e.g. [KK20]).

**Theme 2** grew out of the convergence of a stochastic evolution system to its stationary distribution(s) or solution(s) over the long term, which has been the central topic of many classical works on both theory and application ends. For example, [EK86] §10.2 and §10.4 considered the existence of stationary distributions for diffusion approximations of the Wright-Fisher model. For a Fleming-Viot process \( X \), [EK93] and [DK99] considered the existence of a stationary distribution \( \mu \), and [EK98] established the pointwise ergodic theorem

\[
\lim_{T \to \infty} \frac{1}{T} \int_0^T f(X_t) \, dt = \int_E f(x) \mu(dx), \text{ a.s.}
\]

(1.0.1)

In nonlinear filtering, [Kun71] Theorem 4.1] considered the “asymptotic mean square filtering error”

\[
\lim_{T \to \infty} \frac{1}{T} \int_0^T \mathbb{E} \left[ \left( \pi^\mu_t(f) - f(X^\mu_t) \right)^2 \right] \, dt,
\]

(1.0.2)

where \( X^\mu \) is the signal with initial distribution \( \mu \) and \( \pi^\mu_t(f) \) is the optimal filter for function \( f \) of \( X^\mu \). [BK99] (2.6) considered weak limit points of the “pathwise
average error”

\[
\frac{1}{T} \int_0^T (\pi_{t}^\mu(f) - f(X_t^\mu))^2 \, dt \text{ as } T \to \infty.
\]

\[\textbf{(1.0.3)}\]

\[\textbf{[Bud01 (1.2)]} \text{ studied the “}(\mu,\mu')\text{-stability”}
\]

\[
\lim_{T \to \infty} \frac{1}{T} \int_0^T \mathbb{E} \left[ (\pi_{t}^\mu(f) - \pi_{t}^{\mu'}(f))^2 \right] \, dt
\]

of \(\pi_{t}^\mu(f)\), where \(\pi_{t}^{\mu'}\) represents an approximate filter with incorrect initiation \(\mu'\). In other areas, \[\textbf{[CMP10]} \text{ and } \textbf{[CDP13]} \text{ considered non-trivial stationary solutions for the Lotka-Volterra model and those for perturbations of the voter model. [CG83] and [Cox88] established (1.0.1) for a basic voter process } X \text{ and an invariant measure } \mu \text{ of } X. \text{ All the works (and many others) above were based on separable compact Hausdorff spaces, Polish spaces or compact metric spaces. The following question, considering a weak and abstract form of long-time-average limits like (1.0.1), (1.0.2), (1.0.3) and (1.0.4), still remains unanswered:}
\]

\[\textbf{Q1} \text{ Let } E \text{ be a non-Polish, non-compact or even non-metrizable space, and } X = \{X_t\}_{t \geq 0} \text{ be an } E\text{-valued, non-c}^\Delta \text{, measurable process. Then, is there an } E\text{-valued stationary process } X^\infty \text{ such that the long-time-averaged distributions}
\]

\[
\frac{1}{T_n} \int_0^{T_n} \mathbb{P} \circ (X_{\tau+t_1},...,X_{\tau+t_d})^{-1} \, d\tau
\]

\[\text{converge weakly to the distribution of } (X^\infty_{\tau+t_1},...,X^\infty_{\tau+t_d}) \text{ as } T_n \to \infty \text{ for } \text{almost all finite subsets } \{t_1,...,t_d\} \text{ of } \mathbb{R}^+?\]

Weak convergence of finite-dimensional distributions of \(E\)-valued càdlàg processes is implied by their weak convergence as \(D(\mathbb{R}^+; E)\)-valued random variables when \(E\) is a separable metric space. Here, \(D(\mathbb{R}^+; E)\), \(\mathbb{N}\) and “⇒” denotes the Skorokhod \(\mathcal{J}_1\)-space of all càdlàg mappings from \(\mathbb{R}^+\) to \(E\), the positive integers and weak convergence respectively. For processes \(\{X^n\}_{n \in \mathbb{N}}\) and \(X\) with paths in \(D(\mathbb{R}^+; E)\),

\[
X_n \Rightarrow X \text{ as } n \to \infty \text{ on } D(\mathbb{R}^+; E)
\]

has two implications:

(1) \(\{X^n_{t_1},...,X^n_{t_d}\}_{n \in \mathbb{N}}\) converge weakly to \((X_{t_1},...,X_{t_d})\) as \(n \to \infty\) for all finite collection \(\{t_1,...,t_d\}\) in a dense subset of \(\mathbb{R}^+\).

(2) \(\{X^n\}_{n \in \mathbb{N}}\) is relatively compact in \(D(\mathbb{R}^+; E)\).

Often (2) requires strong or difficult-to-verify conditions in practice. By contrast, (1) or a weaker form of it is believed to be establishable for possibly non-càdlàg processes under much milder conditions than those for weak convergence on \(D(\mathbb{R}^+; E)\). For instance, \[\textbf{[BK93b]} \text{ discussed (1) with } X \text{ being a progressive martingale problem solution and } \{X^n\}_{n \in \mathbb{N}} \text{ progressive approximating processes, none of which is necessarily càdlàg. Their development was based on a Polish space } E \text{ and furthered that of } [\textbf{EK86}, \S 4.8]. \text{ Herein, we address the following general questions on a more general } E \text{ without martingale problem setting:}
\]

\[\textbf{Q2} \text{ When will a subsequence of } E\text{-valued processes } \{X_i\}_{i \in I} \text{ converge finite-dimensionally to an } E\text{-valued process with general paths?}\]
1. INTRODUCTION

Q3 When will a subsequence of $E$-valued processes $\{X_i\}_{i \in I}$ converge finite-dimensionally to an $E$-valued progressive process?

Either of these two questions may be answered in an individual way, but replication helps to handle $Q2$, $Q3$, and the weak convergence of càdlàg processes on path spaces in one framework. We shall establish several relatively mild and explicitly verifiable criteria for uniqueness and existence of the limit processes in $Q2$ and $Q3$ above. These criteria will be used to deduce (1.0.5) and answer $Q1$ that motivates Theme 2.

Theme 3 is concerned with two basic problems on a Tychonoff space $E$:

Q4 When is a family of $E$-valued càdlàg processes bijectively indistinguishable from a tight family of $D(R^+;E)$-valued random variables?

Q5 More generally, when is a family of $E$-valued càdlàg processes bijectively indistinguishable from a relatively compact family of $D(R^+;E)$-valued random variables?

The main importance of tightness is that it implies relative compactness for Borel probability measures on Hausdorff spaces. However, the verification of tightness can be challenging. [Kur75, Mit83, Jak86, Daw93, BK93b, KX95] and [Kou16], to name a few, all spent considerable efforts establishing tightness of càdlàg processes on exotic spaces. In particular, [Jak86] developed systematic tightness criteria for probability measures on both $D([0,1];E)$ and $D(R^+;E)$, which extended several results of [Mit83] and [EK86 §3.7 - 3.9] from the Polish to the possibly non-metrizable Tychonoff case. [Kou16] generalized the results of [Jak86] for $D([a,b];E)$ by loosening [Jak86, Theorem 3.1, (3.4)] to the milder Weak Modulus of Continuity Condition (see [Kou16 §6]). As a continuation of [Kou16] on infinite time horizon, we answer $Q4$ by establishing the equivalence among:

- Indistinguishability from a tight family of $D(R^+;E)$-valued random variables;
- Metrizable Compact Containment plus Weak Modulus of Continuity;
- Metrizable Compact Containment plus Modulus of Continuity; and
- Mild Pointwise Containment plus Modulus of Continuity for $\tau$ when $(E,\tau)$ is a complete metric space.

As previously mentioned, weak convergence on $D(R^+;E)$ is commonly thought to be composed of finite-dimensional convergence along densely many times plus relative compactness in $D(R^+;E)$. Herein, we give a more precise interpretation by showing that:

- Relative compactness in $D(R^+;E)$ implies the Modulus of Continuity Condition for any Tychonoff space $E$.
- When $E$ is a baseable space, weak convergence on $D(R^+;E)$ implies finite-dimensional convergence along densely many times. Baseable spaces, defined in the sequel, are more general than metrizable and separable spaces.
- When $E$ is metrizable and separable, finite-dimensional convergence along densely many times plus the Modulus of Continuity Condition (weaker than relative compactness) are sufficient for weak convergence on $D(R^+;E)$.

Based on the results above, we answer $Q5$ on metrizable spaces by showing that:
1. INTRODUCTION

- When $E$ is metrizable and separable, relative compactness in $D(\mathbb{R}^+; E)$ is equivalent to the Modulus of Continuity Condition plus relative compactness for finite-dimensional convergence to $E$-valued càdlàg processes along densely many times.

- When $(E, \tau)$ is a metric space, the combination of the Modulus of Continuity Condition for $\tau$, Mild Pointwise Containment Condition and relative compactness for finite-dimensional convergence to $D(\mathbb{R}^+; E)$-valued random variables along densely many times is sufficient for relative compactness in $D(\mathbb{R}^+; E)$.

Relative compactness can be weaker than tightness and require milder conditions in non-Polish settings. The results of Theme 3 demonstrate the Compact Containment Condition frequently used in verifying tightness is superfluous for relative compactness in Skorokhod $J^1$-spaces. This is also why the second approach of [BK93b] was well received. While their work was restricted to a martingale problem setting, our results are general.

The methods and results herein are also used in solving more concrete problems. For martingale problems in general settings:

- [BK93b] gave a classical framework of establishing finite-dimensional convergence of not-necessarily-càdlàg $\{X_n\}_{n \in \mathbb{N}}$ to not-necessarily-càdlàg solution $X$. [DK20b] §3.2.1 extended such convergence results to non-Polish spaces and milder separability and tightness conditions (see [DK20b] Remark 3.24 for comparison). As a byproduct, [DK20b] §3.3 also give conditions for long-time typical behaviors of martingale solutions to be stationary solutions.

- Finite-dimensional convergence of càdlàg $\{X_n\}_{n \in \mathbb{N}}$ to càdlàg solution $X$ is known to imply their weak convergence on the path space $D(\mathbb{R}^+; E)$, which is generally stronger, when $E$ is a Polish space and $\{X_n\}_{n \in \mathbb{N}}$ is relatively compact (see e.g. [EK86] §4.8 and [BK93b]). [DK20b] §3.2.2 shows that the completeness of $E$ can be exempted and relative compactness can be reduced to weaker conditions for such implication (see [DK20b] Remark 3.37 for comparison). When $E$ is a Tychonoff space, [DK20b] establishes similar implication by the tightness results of Theme 3.

- [KK20] shows existence of solution to an infinite-dimensional system of stochastic differential equations by constructing a class of weighted empirical processes and establishing their weak convergence to càdlàg solutions to the associated measure-valued martingale problem. They used one of the mild conditions of modulus of continuity herein to show tightness, which is generally weaker than the martingale-problem-type of conditions in [EK86] §4.8 and [BK93b].

For the classical nonlinear filtering problem in general settings:
The Fujisaki-Kallianpur-Kunita (FKK) and Duncan-Mortensen-Zakai (DMZ) equations are known to be (infinite) equation systems that uniquely identify the (measure-valued) normalized and unnormalized filters respectively (see e.g. [Szp76], [KO88] and [BKK95]) for Polish or compact Hausdorff state spaces. Assuming a more general state space than those of [Szp76], [KL08], [KO88] and [BKK95], [DK20a] §2.3 uses the replication method to establish uniqueness of general solution to the FKK and DMZ equations systems. This uniqueness result generalizes that of [Szp76] §V as it works for any usual filtration without imposing the càdlàg property of the solution (see [DK20a], Remarks 31 and 35).

[DK20a] §2.4] extends the stationary filter construction of [Kun71] and [BBK00] to non-Polish or non-compact spaces. With the aid of replication, [DK20a] reduces the problem in a general setting to the case of [Kun71] and [BBK00] and then transforms the known stationary filter back to the original setting.

The remainder of this manuscript is organized as follows. Chapter 2 serves a preliminary collection of notations, terminologies and facts for the three themes of this article. Moreover, this chapter provides several examples of non-Polish settings of interest to probabilists and convergence results that goes beyond the regular setting in e.g. [EK86], Chapter 4 and [BK93b]. Theme 1 occupies four chapters: Chapter 3 develops the space change method of replication. Chapter 4 focuses on the replication of function and linear operator. Chapter 5 discusses weak convergence on general topological spaces by replication of measure and weak convergence of replica measures. Chapter 6 is devoted to the replication of stochastic process and the associated convergence problems. Chapters 7 and 8 correspond to Theme 2 and Theme 3 respectively. We provide background content in Appendix 9 and miscellaneous results in Appendix 10 for self-containment and referral ease, especially for readers’ convenience.

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1 This uniqueness has various senses.
CHAPTER 2

Preliminaries

This chapter contains preparatory materials for Chapter 3 - 8, introduce our general notations, terminologies and background. Further background materials are provided in Appendix 9. § 2.7 motivates the replication approach by examples of “defective” settings and results established by space change in probability theory.

2.1. Basic concepts

2.1.1. Numbers, sets and mappings. “∅” denotes the empty set. \( \mathbb{N} \) denotes the positive integers and \( \mathbb{N}_0 = \mathbb{N} \cup \{0\} \). \( \mathbb{Q} \) denotes the rational numbers and \( \mathbb{Q}^+ = \{q \in \mathbb{Q} : q \geq 0\} \). “↑” and “↓” denote the non-decreasing and non-increasing convergence of real numbers (including convergence to ±∞) respectively.

“⊂” and “⊃” denote the containment of sets including equalities. Let \( A \subset E \) be non-empty sets. \( \mathcal{N}(E) \) denotes the cardinality of \( E \). \( \mathcal{P}_0(E) \) denotes all finite non-empty subsets of \( E \). A is a countable subset of \( E \) if \( E \setminus A \) is a countable set. Empty, finite and countably infinite sets are all considered as countable sets.

“×” denotes the Cartesian product of non-empty sets. Let \( \prod_{i \in I} S_i \) denote the Cartesian product of non-empty \( \{S_i\}_{i \in I} \). When \( S_i = E \) for all \( i \in I \), \( \prod_{i \in I} S_i \) is often denoted by \( E^I \), or by \( E^\infty \) if \( \mathcal{N}(I) = \mathcal{N}(\mathbb{N}) \), or by \( E^d \) if \( \mathcal{N}(I) = d \in \mathbb{N} \). The projection on \( \prod_{i \in I} S_i \) for non-empty sub-index-set \( I_0 \subset I \) is defined by

\[
\pi_{I_0} : \prod_{i \in I} S_i \to \prod_{i \in I_0} S_i, \quad x \mapsto \prod_{i \in I_0} \{x(i)\}.
\]

In particular, \( \pi_j = \pi_{\{j\}} \) is called the one-dimensional projection on \( \prod_{i \in I} S_i \) for \( j \in I \). “◦” denotes the composition of mappings. \( 1_A \) denotes the indicator function of \( A \). For a mapping \( f \) defined on \( E \), \( f|A \) denotes the restriction of \( f \) to \( A \). Given \( f_i : E \to S_i \) a mapping for each \( i \in I \) and \( \mathcal{D} = \{f_i\}_{i \in I} \), we define

\[
\mathcal{D}|_A = \{f|_A : f \in \mathcal{D}\}
\]

and

\[
\bigotimes \mathcal{D} = \bigotimes_{i \in I} f_i : E \to \prod_{i \in I} S_i, \quad x \mapsto \prod_{i \in I} \{f_i(x)\}.
\]

\(^1\)“↓” means “being defined by”.

\(^2\)So, “countable” is indifferent than “at most countable.”
2.1.2. Measurable space and measure space. Let \((E, \mathcal{U})\) and \((S, \mathcal{A})\) be measurable spaces and \(A \subset E\) be non-empty. The concentration of \(\mathcal{U}\) on \(A\) is
\[
\mathcal{U}|_A \doteq \{ B \cap A : B \in \mathcal{U} \},
\]
which is apparently a \(\sigma\)-algebra on \(A\). For a family of mappings \(D\) from \(E\) to \(S\), the \(\sigma\)-algebra induced by \(D\) is
\[
\sigma(D) \doteq \sigma \left( \{ f^{-1}(B) : B \in \mathcal{A}, f \in D \} \right).
\]
\(\delta_x\) denotes the Dirac measure at \(x \in E\) so \(\delta_x(B) = 1\) when \(B \in \mathcal{U}\) contains \(x\).
\(\mathcal{M}^+(E, \mathcal{U})\) (resp. \(\mathcal{P}(E, \mathcal{U})\)) denotes non-trivial finite measures (resp. probability measures) on \((E, \mathcal{U})\). Non-triviality of a measure \(\mu\) on \((E, \mathcal{U})\) means \(\mu(E)\) is non-zero. All measures are non-negative and countably additive.

Let \((E, \mathcal{U}, \mu)\) be a measure space. \(\mathcal{N}(\mu)\) denotes \(\mu\)-null subsets of \(E\), i.e. each member of \(\mathcal{N}(\mu)\) has zero outer measure induced by \(\mu\) (see [Dud02, p.89]). Their complements are called \(\mu\)-conull sets. If \(\mathcal{N}(\mu) \subset \mathcal{U}\), then \(\mathcal{U}\) is called \(\mu\)-complete and \((E, \mathcal{U}, \mu)\) is called complete.

\(A \subset E\) is a support\(^3\) of \(\mu\) (or \(\mu\) is supported on \(A\)) if \(\mathcal{N}(\mu)\). The expansion of \(\nu \in \mathcal{M}^+(A, \mathcal{U}|_A)\) onto \(E\) is defined by
\[
\nu^E(B) \doteq \nu(A \cap B), \ \forall B \in \mathcal{U}(E).
\]
When \(A \in \mathcal{U}\), the concentration of \(\mu\) on \(A\) is defined by
\[
\mu|_A \doteq \mu(B), \ \forall B \in \mathcal{U}|_A \subset \mathcal{U}.
\]
The following facts are well-known and we omit the proof for brevity.

**Fact 2.1.1.** Let \((E, \mathcal{U})\) be a measurable space, \(A \subset E\) be non-empty, \(\mu \in \mathcal{M}^+(E, \mathcal{U})\) and \(\nu \in \mathcal{M}^+(A, \mathcal{U}|_A)\). Then:

(a) If \(A \in \mathcal{U}\), then \(2.1.7\) well defines \(\mu|_A \in \mathcal{M}^+(A, \mathcal{U}|_A)\). If, in addition, \(\mu \in \mathcal{P}(E, \mathcal{U})\), then \(\mu|_A \in \mathcal{P}(A, \mathcal{U}|_A)\) precisely when \(\mu(A) = 1\).

(b) \(2.1.6\) well defines \(\nu^E \in \mathcal{M}^+(E, \mathcal{U})\) and \(\nu^E \in \mathcal{P}(E, \mathcal{U})\) precisely when \(\nu(A) = 1\).

(c) If \(A \in \mathcal{U}\), then \(\nu = (\nu^E)|_A\). If, in addition, \(\mu(E\setminus A) = 0\), then \(\mu = (\mu|_A)|^E\).

For a measurable mapping \(f : E \rightarrow S\), the push-forward measure of \(\mu\) by \(f\) is \(\mu \circ f^{-1} \in \mathcal{M}^+(S, \mathcal{A})\) defined by
\[
\mu \circ f^{-1}(B) \doteq \mu \left[ f^{-1}(B) \right], \ \forall B \in \mathcal{A}.
\]
Let \(\mathcal{V}\) be another \(\sigma\)-algebra on \(E\). If \(\mathcal{V} \subset \mathcal{U}\) and \(\nu\) is the restriction of \(\mu\) to \(\mathcal{V}\), then \(\nu \in \mathcal{M}^+(E, \mathcal{V})\) is called the restriction of \(\mu\) to \(\mathcal{V}\), an extension of \(\nu\) to \(\mathcal{U}\), and \((E, \mathcal{U}, \mu)\) an extension of \((E, \mathcal{V}, \nu)\). \((E, \mathcal{U}, \mu)\) is the completion of \((E, \mathcal{V}, \nu)\) if: (1) \((E, \mathcal{V}, \mu)\) is a complete extension of \((E, \mathcal{V}, \nu)\), and (2) any complete extension \((E, \mathcal{V}', \mu')\) of \((E, \mathcal{V}, \nu)\) is also an extension of \((E, \mathcal{U}, \mu)\).

---

\(^3\)“resp.” abbreviates “respectively”.

\(^4\)By our definition, a measure may have more than one supports.
2.1.3. Topological space. Hereafter, we will often exclude the \( \sigma \)-algebra (resp. topology) in a simplified notation of a measurable (resp. topological) space.

Let \( E \) be a topological space. \( \mathcal{O}(E), \mathcal{C}(E), \mathcal{K}(E), \mathcal{K}^m(E), \mathcal{X}(E), \mathcal{X}^m(E) \) and \( \mathcal{B}(E) \triangleq \sigma(\mathcal{O}(E)) \) denote the families of open, closed, compact (resp. topological) space. Also,

\[
\mathcal{O}_E(A) \triangleq \{ O \cap A : O \in \mathcal{O}(E) \}
\]

denotes the subspace topology of \( A \) induced from \( E \), and

\[
\mathcal{B}_E(A) \triangleq \sigma(\mathcal{O}_E(A)) = \mathcal{B}(E)|_{A}
\]

denotes the subspace \( \sigma \)-algebra of \( A \) induced from \( E \) for non-empty \( A \subset E \).

Let \( \mathcal{R} \) be a family of pseudometrics \(^{[2.1.10]}\) on \( E \). The topology induced by \( \mathcal{R} \) is generated by the topological basis \(^{[2.1.11]}\) on \( E \).

\[
\mathcal{R}_0(\mathcal{R}) = \left\{ \bigcap_{r \in \mathcal{R}_0} \{ y \in E : r(x,y) < 2^{-p} \} : x \in E, p \in \mathbb{N}, \mathcal{R}_0 \in \mathcal{R}_0(\mathcal{R}) \right\}.
\]

This topology is the metric topology of \((E, \mathcal{R})\) when \( \mathcal{R} \) is the singleton of a metric \( r \) on \( E \). When \((E, \mathcal{R})\) is a metric space, we define

\[
A^\epsilon \triangleq \{ x \in E : r(x,y) < \epsilon \text{ for some } y \in A \}, \forall \epsilon \in (0, \infty).
\]

Let \( S \) be a topological space and \( \mathcal{D} \) be a family of mappings from \( E \) to \( S \). The topology generated by the topological basis

\[
\mathcal{B}_\mathcal{D}(A) \triangleq \sigma(\mathcal{B}_\mathcal{D}(A)) = \mathcal{B}(E)|_{A}
\]

denotes the subspace \( \sigma \)-algebra of \( A \) induced from \( E \) for non-empty \( A \subset E \).

Let \( \mathcal{W} \) be another topology on \( E \). If \( \mathcal{W} \subset \mathcal{O}(E) \), then the topological space \((E, \mathcal{W})\) is called a topological coarsening of \( E \), or equivalently, \( E \) is called a topological refinement of \((E, \mathcal{W})\).

Hereafter, by “\( x_n \to x \) as \( n \uparrow \infty \) in \( E \)” we mean that: (1) \( \{ x_n \}_{n \in \mathbb{N}} \) and \( x \) are members of \( E \), and (2) \( \{ x_n \}_{n \in \mathbb{N}} \) converges to \( x \) with respect to the topology of \( E \).

2.1.4. Morphisms. Let \( E \) and \( S \) be topological spaces. A mapping \( f : E \to S \) is a homeomorphism between \( E \) and \( S \) if \( f \) is bijective and both \( f \) and \( f^{-1} \) are continuous. \( E \) and \( S \) are homeomorphic to and homeomorphs of each other if a homeomorphism between them exists. \( f \) is an imbedding from \( E \) into \( S \) if \( f \) is a homeomorphism between \( E \) and \( (f(E), \mathcal{O}_S(f(E))) \). The less common notions of standard Borel subset and Borel isomorphism are critical for our developments.

Definition 2.1.2. Let \( E \) and \( S \) be topological space,\(^5\) and \( A \subset E \) be non-empty.

\(^5\)Standard Borel property can be defined for general measurable spaces. Herein, we focus on the topological space case. There are other common equivalent definitions to ours, which are discussed in\(^{[9.5]}\).
A mapping \( f : E \to S \) is a \textbf{Borel isomorphism between \( E \) and \( S \)} if \( f \) is bijective and both \( f \) and \( f^{-1} \) are measurable with respect to \( \mathcal{B}(E) \) and \( \mathcal{B}(S) \).

\( E \) and \( S \) are \textbf{Borel isomorphic to} and \textbf{Borel isomorphs of each other} if there exists a Borel isomorphism between them.

\((A, \mathcal{O}_E(A))\) is a \textbf{Borel subspace of \( E \)} if \( A \in \mathcal{B}(E) \).

\( E \) is a \textbf{standard Borel space} if \( E \) is Borel isomorphic to a Borel subspace of some \textit{Polish} (see p.151) space.

\( A \) is a \textbf{standard Borel subset of \( E \)} if \((A, \mathcal{O}_E(A))\) is a standard Borel space. \( \mathcal{B}^*(E) \) denotes the family of all standard Borel subsets of \( E \).

Standard Borel spaces inherit many nice properties of Borel \( \sigma \)-algebras of Polish spaces. A brief review of standard Borel spaces/subsets is provided in \S 9.5.

\subsection{2.1.5. Product space.} \textit{“⊗”} denotes product of \( \sigma \)-algebras. Given measurable spaces \( \{(S_i, \mathcal{A}_i)\}_{i \in I} \), the product \( \sigma \)-algebra of \( \{\mathcal{A}_i\}_{i \in I} \) on \( \prod_{i \in I} S_i \) refers to

\begin{equation}
\bigotimes_{i \in I} \mathcal{A}_i \doteq \sigma (\{p_i\}_{i \in I}).
\end{equation}

When \((S_i, \mathcal{A}_i) = (S, \mathcal{A})\) for all \( i \in I \), \( \bigotimes_{i \in I} \mathcal{A}_i \) is often denoted by \( \mathcal{A}^\otimes I \), or by \( \mathcal{A}^\otimes \infty \) if \( \aleph(I) = \aleph(\mathbb{N}) \), or by \( \mathcal{A}^\otimes d \) if \( \aleph(I) = d \in \mathbb{N} \). The following facts are well-known.

\textbf{FACT 2.1.3.} Let \((E, \mathcal{U})\) and \(\{(S_i, \mathcal{A}_i)\}_{i \in I}\) be measurable spaces, \( I_0 \subset I \) be non-empty, \( S = \prod_{i \in I} S_i \), \( \mathcal{A} \doteq \bigotimes_{i \in I} \mathcal{A}_i \), \( S_{I_0} = \prod_{i \in I_0} S_i \), \( \mathcal{A}_{I_0} = \bigotimes_{i \in I_0} \mathcal{A}_i \) and \( f : E \to S_i \) be a mapping for each \( i \in I \). Then:

\( a) \ p_{I_0} \) is a measurable mapping from \((S, \mathcal{A})\) to \((S_{I_0}, \mathcal{A}_{I_0})\).

\( b) \ \bigotimes_{i \in I} f_i : (E, \mathcal{U}) \to (S, \mathcal{A}) \) is measurable if and only if \( f_i : (E, \mathcal{U}) \to (S_i, \mathcal{A}_i) \) is measurable for all \( i \in I \).

\textit{“⊗”} also denotes the product of topologies. Given topological spaces \( \{S_i\}_{i \in I} \), the product topology of \( \{\mathcal{O}(S_i)\}_{i \in I} \) on \( \prod_{i \in I} S_i \) refers to

\begin{equation}
\bigotimes_{i \in I} \mathcal{O}(S_i) \doteq \mathcal{O}(\{p_i\}_{i \in I} \left( \prod_{i \in I} S_i \right)).
\end{equation}

When \( S_i = E \) for all \( i \in I \), \( \bigotimes_{i \in I} \mathcal{O}(S_i) \) is often denoted by \( \mathcal{O}(E)^I \), or by \( \mathcal{O}(E)^\infty \) if \( \aleph(I) = \aleph(\mathbb{N}) \), or by \( \mathcal{O}(E)^d \) if \( \aleph(I) = d \in \mathbb{N} \). The following facts are well-known.

\textbf{FACT 2.1.4.} Let \( E \) and \(\{(S_i, \mathcal{A}_i)\}_{i \in I}\) be topological spaces, \( I_0 \subset I \) be non-empty, \( S = \prod_{i \in I} S_i \), \( \mathcal{U} = \bigotimes_{i \in I} \mathcal{O}(E_i) \), \( S_{I_0} = \prod_{i \in I_0} S_i \), \( \mathcal{U}_{I_0} = \bigotimes_{i \in I_0} \mathcal{O}_i(E) \) and \( f : E \to S_i \) be a mapping for each \( i \in I \). Then:

\( a) \ p_{I_0} \) is a continuous mapping from \((S, \mathcal{U})\) to \((S_{I_0}, \mathcal{U}_{I_0})\).

\( b) \ \bigotimes_{i \in I} f_i : E \to (S, \mathcal{U}) \) is continuous if and only if \( f_i : E \to S_i \) is continuous for all \( i \in I \).

\( c) \ \bigotimes_{i \in I} f_i : E \to (S, \mathcal{U}) \) is continuous at \( x \in E \) (see [Mun00] p.104) if and only if \( f_i : E \to S_i \) is continuous at \( x \in E \) for all \( i \in I \).
Standard discussions about product topological spaces can be found in e.g. [Mun00 §15 and §19] and [Bog07] Vol.II, §6.4]. Herein, we remind the readers of one basic but indispensable fact: For general topological space \( \{ S_i \}_{i \in I} \), the Borel \( \sigma \)-algebra \( \sigma(\bigotimes_{i \in I} \mathcal{O}(S_i)) \) generated by their product topology is likely to differ from \( \bigotimes_{i \in I} \mathcal{O}(S_i) \), the product of their individual Borel \( \sigma \)-algebras. Such difference happens even in the two-dimensional case (see [Bog07] Vol.II, Example 6.4.3]). This is why we use different notations for product \( \sigma \)-algebra and product topologies. Avoidance of the difference above needs additional countability of the product topology \( \bigotimes_{i \in I} \mathcal{O}(S_i) \) (see Proposition 10.2.4).

Hereafter, \( \mathbb{R}^k \) (with \( k \in \mathbb{N} \)) denotes the \( k \)-dimensional Euclidean space equipped with the usual norm "\( |\cdot|\)" and the \( k \)-dimensional Lebesgue measure. Conull subsets of \( \mathbb{R}^k \) are in the Lebesgue sense. \( |\cdot| \) also denotes the norm metric on \( \mathbb{R}^k \).

2.2. Spaces of mappings

2.2.1. General mappings. Let \( I, E \) and \( S \) be non-empty sets. The Cartesian power \( E^I \) is the family of all mappings from \( I \) to \( E \). When \( I \) has certain index meaning (e.g. time, order), a member of \( E^I \) is often considered as a "path indexed by \( I \)". So, we define the associated path mapping of \( f \in S^E \) by\(^6\)

\[
\varpi_I(f) = \bigotimes_{i \in I} f \circ p_i \in (S^I)^{E^I}.
\]

This mapping sends every \( E \)-valued path \( x \) indexed by \( I \) to the \( S \)-valued path \( f \circ x \) indexed by \( I \). We define the associated joint path mapping of \( D \subset S^E \) by

\[
\varpi_I(D) = \bigotimes \{ \varpi_I(f) \in (S^I)^{E^I} : f \in D \} \subseteq [(S^I)^{E^I}].
\]

For simplicity, \( \varpi_I(f) \) is often denoted by \( \varpi(f) \) if \( I = [0, T] \), or by \( \varpi_T(f) \) if \( I = [a, b] \). Similar notations apply to \( \varpi_I(D) \).

Remark 2.2.1. \( \varpi_I(D) \) and \( \varpi_I(\bigotimes D) \) differ. The latter maps \( E^I \) to \( (S^D)^I \).

Let \( \delta, T \in (0, \infty) \), \( [a, b] \subset \mathbb{R}^+ \) and \( \tau \) be a pseudometric on \( E \). We define the \( \tau \)-modulus of continuity

\[
w^{\tau}_{[a,b],T}(x) = \inf \left\{ \max_{1 \leq i \leq n} \sup_{s,t \in [t_{i-1},t_i]} \tau(x(t), x(s)) : 0 \leq t_0 < \ldots < T < t_n, \inf_{i \leq n} (t_i - t_{i-1}) > \delta, n \in \mathbb{N} \right\}
\]

for each \( x \in E^{\mathbb{R}^+} \), define

\[
\tau_{[a,b]}(x,y) = \sup_{t \in [a,b]} \tau(x(t), y(t))
\]

for each \( x, y \in E^{[a,b]} \) or \( E^{\mathbb{R}^+} \), and let

\[
J(x) = \left\{ t \in \mathbb{R}^+: x(t) \neq \lim_{s \to t^-} x(s) \in E \right\}
\]

denotes the set of left-jump times of \( x \in E^{\mathbb{R}^+} \).

\(^6\) \( \varpi \) is the calligraphical form of the greek letter \( \pi \).
2.2.2. Measurable, càdlàg and continuous mappings. When $E$ and $S$ are measurable, $M(S; E)$ denotes measurable mappings from $S$ to $E$. When $E$ or $S$ is a topological space, $M(S; E)$ abbreviates $M(S; E, \mathcal{B}(E))$ or $M(S, \mathcal{B}(S); E)$ respectively. When $E$ and $S$ are both topological spaces, $C(S; E), \text{hom}(S; E), \text{im}(S; E)$ and $\text{biso}(S; E)$ denote continuous mappings, homeomorphisms, imbeddings and Borel isomorphisms from $S$ to $E$, respectively.

$\text{TC}(\mathbb{R}^+)$ (resp. $\text{TC}([a, b])$) denotes the family of all time-changes on $\mathbb{R}^+$ (resp. $[a, b] \subset \mathbb{R}^+$). That is, each $\lambda \in \text{TC}(\mathbb{R}^+)$ (resp. $\lambda \in \text{TC}([a, b])$) is a strictly increasing homeomorphism from $\mathbb{R}^+$ (resp. $[a, b]$) to itself and satisfies

$$
(2.2.6) \quad ||\lambda|| \triangleq \sup_{t>s} \left| \ln \frac{\lambda(t) - \lambda(s)}{t-s} \right| < \infty.
$$

Then, we define

$$
(2.2.7) \quad \varrho^\#: \mathbb{R}^+ \rightarrow \mathbb{R}^+
$$

for each $x, y \in E^{[a, b]}$, and define

$$
(2.2.8) \quad \varrho^\#: \mathbb{R}^+ \rightarrow \mathbb{R}^+
$$

for each $x, y \in E^{\mathbb{R}^+}$.

When $E$ is a topological space, $x \in E^{\mathbb{R}^+}$ is càdlàg (i.e. right-continuous and left-limited) if for every $t \in \mathbb{R}^+$, there exists a unique $y^t \in E$ such that $x(u_n) \to y^t$ as $n \to \infty$ in $E$ for all $u_n \to t$ and $x(v_n) \to x(t)$ as $n \to \infty$ in $E$ for all $v_n \to t$. When $E$ is a Tychonoff space\footnote{We use the terminologies “Tychonoff space” instead of “completely regular space” since the latter sometimes is used in a non-Hausdorff context.} (see p.154), Proposition 9.3.1 to follow shows that the topology of $E$ is induced by a family $\mathcal{R}$ of pseudometrics on $E$. Then, by $D(\mathbb{R}^+; E)$ (resp. $D([a, b]; E)$) we denote the space of all càdlàg members of $E^{\mathbb{R}^+}$ (resp. $E^{[a, b]}$) equipped with the Skorokhod $\mathcal{J}_1$-topology $\mathcal{J}(E)$ (resp. $\mathcal{J}_{a,b}(E)$), that is, the topology induced by pseudometrics $\{\varrho^\#: \mathbb{R}^+ \rightarrow \mathbb{R}^+\}$ (resp. $\{\varrho^\#: [a, b] \rightarrow \mathbb{R}^+\}$). $\mathcal{J}(E)$ and $\mathcal{J}_{a,b}(E)$ turn out to be independent of the choice of the pseudometrics $\mathcal{R}$.

More information about Skorokhod $\mathcal{J}_1$-spaces can be found in §9.6.

2.2.3. $\mathbb{R}^k$-valued functions. Let $E$ be a non-empty set and $\{f, g\} \subset \mathbb{R}^E$. We define $f \lor g(x) \triangleq \max\{f(x), g(x)\}$, $f \land g(x) \triangleq \min\{f(x), g(x)\}$, $f^+(x) \triangleq \max\{f(x), 0\}$ and $f^-(x) \triangleq \max\{-f(x), 0\}$ for all $x \in E$. A subset of $\mathbb{R}^E$ is a function lattice if it is closed under “$\land$” and “$\lor$”.

Let $k \in \mathbb{N}$ and $D \subset (\mathbb{R}^k)^E$. The additive expansion of $D$ is defined by

$$
(2.2.9) \quad \text{ac}(D) \triangleq D \cup \{f + g : f, g \in D\},
$$

and the additive closure of $D$ is defined by

$$
(2.2.10) \quad \text{ac}(D) \triangleq \left\{ \sum_{f \in D_0} f : D_0 \in \mathcal{P}_0(D) \right\}.
$$

When $k = 1$, the multiplicative closure of $D$ is defined by

$$
(2.2.11) \quad \text{mc}(D) \triangleq \left\{ \prod_{f \in D_0} f : D_0 \in \mathcal{P}_0(D) \right\},
$$
the \textbf{Q-algebra generated by $D$} is defined by

\begin{equation}
\ag_Q(D) \doteq \text{ac} \left( \{af : f \in \text{mc}(D), a \in \mathbb{Q} \} \right),
\end{equation}

and, for a finite index set $I$, we define\footnote{The function class in (2.2.14) is formed similarly to the function class in [EK88] §4.4, (4.15).}

\begin{equation}
\Pi^I(D) \doteq \left\{ g \in \mathbb{R}^{|I|} : g = \prod_{j=1}^{i} f_j \circ p_j, f_j \in D, 1 \leq i \leq \aleph(I) \right\}.
\end{equation}

Hereafter, $\Pi^I(D)$ is often denoted by $\Pi^d(D)$ with $d \doteq \aleph(I)$. The enlargements of $D$ above are often used to construct a rich but countable collection of functions that includes $D$. Some of their basic properties are specified in §9.2 and §10.1 - §10.2.

\textit{"Un"} denotes uniform convergence of $\mathbb{R}^k$-valued functions. When the members of $D \subset (\mathbb{R}^k)^E$ are bounded\footnote{\textit{f} is bounded if $\|f\|_{\infty} \in \mathbb{R}^+$.}, $\text{cl}(D)$ denotes the closure of $D$ under the supremum norm $\| \cdot \|_{\infty}$ and, if $k = 1$, we define

\begin{equation}
\text{ca}(D) \doteq \text{cl} \left[ \text{ag}(D) \right] = \text{cl} \left[ \text{ag}_Q(D) \right].
\end{equation}

The second equality above is immediate by the denseness of $\mathbb{Q}$ in $\mathbb{R}$ and properties of uniform convergence.

$M_b(E; \mathbb{R}^k)$ (resp. $C_b(E; \mathbb{R}^k)$) denotes the Banach space over scalar field $\mathbb{R}$ of all bounded members of $M(E; \mathbb{R}^k)$ (resp. $C(E; \mathbb{R}^k)$) equipped with $\| \cdot \|_{\infty}$. $C_c(E; \mathbb{R}^k)$ denotes the subspace of all members of $C(E; \mathbb{R}^k)$ that have compact supports, i.e. the closure of $E \backslash f^{-1}(\{0\})$ is compact. $C_0(E; \mathbb{R}^k)$ is the subspace of all $f \in C(E; \mathbb{R}^k)$ such that given $\epsilon > 0$, there exists a $K_\epsilon \in \mathcal{K}(E)$ satisfying $\|f|_{E \backslash K_\epsilon}\|_{\infty} < \epsilon$.

\subsection*{2.2.4. Functions and separation of points}

Let $E$ and $A \subset E$ be non-empty sets and $D \subset \mathbb{R}^E$. \textbf{$D$ separates points on $A$} if $\otimes D$ is injective, or equivalently, $f(x) = f(y)$ for all $f \in \mathbb{D}$ implies $x = y$ in $A$. Suppose $E$ is a topological space. Then, $D$ \textbf{strongly separates points on $A$} if $\partial_E(A) \subset \partial_D(A)$. $D$ \textbf{determines point convergence on $A$} if $\otimes D(x_n) \to \otimes D(x)$ as $n \uparrow \infty$\footnote{$\otimes D(x_n) \to \otimes D(x)$ as $n \uparrow \infty$ is equivalent to $\lim_{n \to \infty} f(x_n) = f(x)$ for all $f \in \mathbb{D}$ by Fact \ref{10.1.11}.} in $(\mathbb{R}^E, \partial(\mathbb{R}^E))$ implies $x_n \to x$ as $n \uparrow \infty$ in $(A, \partial_E(A))$.

\textbf{Note 2.2.2.} The point separability, strong point separability or point convergence determining of $D \subset \mathbb{R}^E$ on $A \subset E$ is inherited by any $D' \subset \mathbb{R}^E$ with $D \subset D'$.

The following are several examples with these point-separation properties.

\textbf{Example 2.2.3.}

(I) \textbf{Let $C([0,1]; \mathbb{R})$ be alternatively equipped with the product topology $\partial(\mathbb{R})^{[0,1]}$.} For each $x \in [0,1]$, the one-dimensional projection $p_x$ is...
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continuous on $C([0,1]; \mathbb{R})$ since the convergence under product topology means pointwise convergence (see [Mun00 Theorem 46.1]). Note that

$$\bigotimes_{x \in \mathbb{Q} \cap [0,1]} p_x(f) = \bigotimes_{x \in \mathbb{Q} \cap [0,1]} p_x(g)$$

implies $f = g$ by the denseness of $\mathbb{Q}$ in $\mathbb{R}$ and the continuity of $f$ and $g$. Hence, $\{p_x\}_{x \in \mathbb{Q} \cap [0,1]}$ is a countable collection of $\mathbb{R}$-valued continuous functions that separates points on $C([0,1]; \mathbb{R})$.

(II) Let $(E, r)$ be a metric space and define

$$g_{y,k}(x) \doteq [1 - kr(x, y)] \vee 0, \forall x, y \in E, k \in \mathbb{N},$$

by Proposition [BK10 (4)] showed that $\{g_{y,k}\}_{y \in E, k \in \mathbb{N}}$ strongly separates points on $E$. It separates points and determines point convergence on $E$ by Proposition [9.2.1 (a, b)] to follow.

(III) Let $(E, \tau)$ be a metric space. For each $x \in E$,

$$g_x(y) \doteq r(y, x), \forall y \in E$$

is a Lipschitz function by triangular inequality and $g_x(y) > 0$ for any $x \neq y$ in $E$. Hence, $D \doteq \{g_x\}_{x \in E}$ separates points on $E$. For each $x \in E$, $g_x(x_n) \rightarrow g_x(x)$ as $n \uparrow \infty$ in $\mathbb{R}$ implies

$$\lim_{n \rightarrow \infty} r(x_n, x) = \lim_{n \rightarrow \infty} |g_x(x_n)| = \lim_{n \rightarrow \infty} |g_x(x_n) - g_x(x)| = 0$$

and so $x_n \rightarrow x$ as $n \uparrow \infty$ in $E$. Thus, $D$ determines point convergence and strongly separates points on $E$ by (2.2.19) and Proposition [9.2.1 (b)] to follow. The family of all Lipschitz functions on $E$ has the same point-separation properties as $D$ by Note [2.2.2].

(IV) For each $n \in \mathbb{N}$,

$$f_n(x) \doteq \begin{cases} 
1, & \text{if } x \in [0, \frac{1}{n}], \\
\frac{1}{n^2} x - \frac{1}{n^2} x^2, & \text{if } x \in \left(\frac{1}{n}, 1\right), \\
0, & \text{if } x \in [n, \infty) 
\end{cases}$$

defines a bounded continuous function on $\mathbb{R}^+$ which is strictly decreasing on its compact support $[0, n]$. One immediately observes that $D \doteq \{f_n\}_{n \in \mathbb{N}}$ separates points and determines point convergence on $\mathbb{R}^+$. $D$ strongly separates points on $\mathbb{R}^+$ by Proposition [9.2.1 (b)] to follow. $C_c(\mathbb{R}^+; \mathbb{R})$ has the same point-separation properties as $D$ by Note [2.2.2].

Note 2.2.4. $C(E; \mathbb{R})$ and $C_b(E; \mathbb{R})$ separate and strongly separate points on $E$ when $E$ is a Tychonoff space (see Proposition [9.3.1]). For more general $E$, however, even $C(E; \mathbb{R})$ does not necessarily separate points on $E$ (see Example [9.4.6]).

For $D = \{f_j\}_{j \in \mathbb{N}} \subset \mathbb{R}^E$ and $d \in \mathbb{N}$, the pseudometric

$$\rho_D(x_1, x_2) \doteq \sum_{j=1}^{\infty} 2^{-j+1} (|f_j(x_1) - f_j(x_2)| \wedge 1), \forall x_1, x_2 \in E$$

on $E$ induces $\mathcal{E}_D(E)$ (see Proposition [9.2.1 (d)]), and the pseudometric

$$\rho_D^d(y_1, y_2) \doteq \max_{1 \leq i \leq d} \rho_D\left(p_i(y_1), p_i(y_2)\right), \forall y_1, y_2 \in E^d$$. 
2.2.5. Linear Operator Review. Let \((S, \| \|)\) be a Banach space over scalar field \(\mathbb{R}\) (like \((C_b(E, \mathbb{R}), \| \cdot \|_\infty)\)). By a single-valued linear operator \(L \) on \(S\) we refer to a linear subspace \(L \subset S \times S\) such that for each \(f \in S, \{g \in S : (f, g) \in L\}\) is either \(\emptyset\) or a singleton \(\{Lf\}\). The domain \(D(L)\) and range \(R(L)\) of \(L\) are

\[
D(L) = \{ f \in S : L \cap \{ f \} \times S \neq \emptyset \}
\]

and

\[
R(L) = \{ g \in S : L \cap (S \times \{ g \}) \neq \emptyset \},
\]

which are well-known to be linear subspaces of \(S\).

\(L\) is closed if \(L = \text{cl}(L)\), where \(\text{cl}(L)\) is the closure of \(L\) in \(S \times S\). \(\text{cl}(L)\) is single-valued if \(L\) is. The restriction of \(L\) to \(D \subset D(L)\) is

\[
L|_D = \{ (f, Lf) : L \subset f \in D\}.
\]

\(L\) is dissipative if

\[
\beta \| f \| \leq \|\beta f - Lf\|, \quad \forall f \in D(L), \beta \in (0, \infty).
\]

\(L\) satisfies positive maximum principle if \(\sup_{x \in E} f(x) = f(x_0) \geq 0\) implies \(Lf(x_0) \leq 0\) for all \(f \in D(L)\). \(L\) is a strong generator on \(S\) if \(\text{cl}(L)\) is the infinitesimal generator (see [Yos80, p.231]) of a strongly continuous contraction semigroup on \(S\). When \(E\) is a locally compact (see p.152) Hausdorff (see p.149) space and \(S = (C_0(E, \mathbb{R}), \| \cdot \|_\infty)\), \(L\) is a Feller generator on \(S\) if \(\text{cl}(L)\) is the infinitesimal generator of a Feller semigroup on \(S\). There are multiple Feller semigroup definitions in the literature. We follow [EK86, §4.2, p.166] and define Feller semigroups to be strongly continuous, positive, contraction semigroup with conservative infinitesimal generators.

More details about operators on Banach spaces can be found in standard texts like [Yos80] Chapter VIII and Chapter IX and [EK86] Chapter I.

2.3. Spaces of finite Borel measures

Let \(E\) be a topological space. The members of \(\mathfrak{M}^+(E, \mathcal{B}(E))\) and \(\mathfrak{P}(E, \mathcal{B}(E))\) are called finite Borel measures and Borel probability measures respectively.

**Definition 2.3.1.** Let \(E\) be a topological space and \(\mathcal{B}\) be a sub-\(\sigma\)-algebra of \(\mathcal{B}(E)\). Then, any extension of \(\mu \in \mathfrak{M}^+(E, \mathcal{B})\) to \(\mathcal{B}(E)\) is said to be a Borel extension of \(\mu\).

Hereafter, \(\mathcal{B}(\mu)\) denotes the family of all Borel extension(s) of \(\mu\) (if any). If \(\mu'\) is the unique member of \(\mathcal{B}(\mu)\), then we specially denote \(\mu' = \mathcal{B}(\mu)\). Any \(\mu' \in \mathcal{B}(\mu)\) has the same total mass as \(\mu\) since \(E \in \mathcal{B} \subset \mathcal{B}(E)\).

\(\mathcal{M}^+(E)\) denotes the space of all finite Borel measures on \(E\) equipped with the weak topology. \(\mathcal{P}(E)\) denotes the subspace of all probability members of \(\mathcal{M}^+(E)\). To be specific, the weak topology of \(\mathcal{M}^+(E)\) is defined by

\[
\sigma[\mathcal{M}^+(E)] = \sigma_{\mathcal{C}_b(E, \mathbb{R})^*}[\mathcal{M}^+(E)],
\]

\[
\sigma_{\mathcal{D}(E)} = \sigma_{\mathcal{C}_b(E, \mathbb{R})^*}[\mathcal{M}^+(E)]\]

\[
\sigma_{\mathcal{D}(E)} = \sigma_{\mathcal{C}_b(E, \mathbb{R})^*}[\mathcal{M}^+(E)]\]

\[
\sigma_{\mathcal{D}(E)} = \sigma_{\mathcal{C}_b(E, \mathbb{R})^*}[\mathcal{M}^+(E)]\]

As mentioned in §2.1.3 and §2.1.5, \(\mathcal{D}(E)^d\) is the product topology on \(E^d\) with \(E\) being equipped with the topology \(\sigma_{\mathcal{D}(E)}\).
where $f^*$ denotes the linear functional
\begin{equation}
  f^* : \mathcal{M}^+(E) \rightarrow \mathbb{R},
\end{equation}
for each $f \in M_b(E; \mathbb{R})$, and
\begin{equation}
  D^* \doteq \{ f^* : f \in D \}
\end{equation}
for each $D \subset M_b(E; \mathbb{R})$. Given $f \in M_b(E; \mathbb{R})$, we define $f^* \doteq \bigotimes_{i=1}^{k}(p_i \circ f)^*$.

Remark 2.3.2. The weak topology of $\mathcal{M}^+(E)$, sometimes called the “narrow topology”, generally differs from the standard weak-$*$ topology induced by the dual space $E'$. They become equal when $E$ is a locally compact Hausdorff space (see [Mal95, Chapter II, §6.5 - 6.7]). Hereafter, we use “$f^*$” to denote the linear functional in (2.3.2). $D^*$ is not necessarily a dual space of $D$ even when $D$ is a linear space.

Weak convergence is one of the central interests of this work. As specified in §2.1.3 for general topological spaces, the statement
\begin{equation}
  \mu_n \implies \mu \text{ as } n \uparrow \infty \text{ in } \mathcal{M}^+(E)
\end{equation}
means that: (1) $\{\mu_n\}_{n \in \mathbb{N}} \cup \{\mu\}$ are members of $\mathcal{M}^+(E)$, and (2) $\{\mu_n\}_{n \in \mathbb{N}}$ converges to $\mu$ with respect to the weak topology of $\mathcal{M}^+(E)$ (converge weakly to $\mu$ for short). Similar terminology and notation apply to $\mathcal{P}(E)$.

$\mu \in \mathcal{M}^+(E)$ is a weak limit point of $\Gamma \subset \mathcal{M}^+(E)$ if there exist $\{\mu_n\}_{n \in \mathbb{N}} \subset \Gamma$ satisfying (2.3.4). We denote the weak limit $\mu$ of $\{\mu_n\}_{n \in \mathbb{N}} \subset \mathcal{M}^+(E)$ by
\begin{equation}
  \lim_{n \rightarrow \infty}^{w-} \mu_n = \mu
\end{equation}
if (2.3.4) holds and $\mu$ is the unique weak limit point of $\{\mu_n\}_{n \in \mathbb{N}}$. $\Gamma \subset \mathcal{M}^+(E)$ is relatively compact if any infinite subset of $\Gamma$ has a weak limit point in $\mathcal{M}^+(E)$.

Remark 2.3.3. (2.3.4) does not necessarily imply (2.3.5) since $\mathcal{M}^+(E)$ in general is not guaranteed to be a Hausdorff space so $\mu$ in (2.3.4) might not be unique.

2.3.1. Weak topology in sequential view. In this work, we consider weak convergence as the topological convergence of the weak topology defined by (2.3.1). However, (2.3.4) is equivalent to the integral test
\begin{equation}
  \lim_{n \rightarrow \infty} \int_E f(x)\mu_n(dx) = \int_E f(x)\mu(dx), \forall f \in C_b(E; \mathbb{R}).
\end{equation}
One often defines weak convergence by (2.3.6) first and then defines a topology sequentially by weak convergence, which is called the topology of weak convergence herein. We look at the relationship between the weak topology and the topology of weak convergence.

Every topology determines a sense of topological convergence. Conversely, convergence is definable without any topological structure. Weak convergence defined by (2.3.6) is one example since it does not formally involve any topology of $\mathcal{M}^+(E)$. Two other examples are almost-sure convergence and bounded pointwise convergence.

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13This weak limit point need not belong to $\Gamma$. 

2.3. SPACES OF FINITE BOREL MEASURES

(see §3.4].) Suppose a sense of convergence is given, which is called convergence a priori. Then, one defines closedness of a set to be the containment of limits of all convergent a priori sequences. A topology is generated by these “sequentially closed” sets, which is called the topology of convergence a priori.

Remark 2.3.4. The topology of convergence a priori has its own topological convergence, which is called convergence a posteriori. Convergence a priori and convergence a posteriori need not be the same in general (see §5 for further details). Examples of convergence a priori that is stronger than convergence a posteriori include almost-sure convergence, bounded pointwise convergence and the $S$-convergence introduced by (see §2.7.2 for a short glance).

When convergence a priori is the topological convergence of some given topology, the topology of convergence a priori and the original topology are not necessarily the same. One example is the weak topology and the topology of weak convergence. A weak limit point is always a weak limit point. A weak limit point is always a weak limit point. A weak limit point is always a weak limit point.

Example 2.3.5. explained that the product space $E = (R^{[0,1]}, \sigma(R^{[0,1]}))$ is not first-countable (see p.150). More specifically, the constant function 0 lies in the closure of the set

$$A = \left\{ x \in R^{[0,1]} : x(i) = \begin{cases} 0, & \text{if } i \in I, \\ 1, & \text{otherwise,} \end{cases} \right\},$$

but no sequence in $A$ converges to 0. Let $\Gamma = \{ \delta_x : x \in A\}$, $\Gamma_1$ be the closure of $\Gamma$ with respect to the weak topology of $M^+(E)$, and $\Gamma_2$ be the closure of $\Gamma$ with respect to the sequential topology induced by weak convergence. We show $\Gamma_1 \neq \Gamma_2$ by verifying $\delta_0 \in \Gamma_1 \setminus \Gamma_2$. For any $\epsilon \in (0, \infty)$, $m \in N$ and $\{f_1, \ldots, f_m\} \subset C_0(E; R)$, there exists an $O_\epsilon \in \sigma(R^{[0,1]})$ such that $0 \in O_\epsilon$ and

$$\max_{1 \leq i \leq m} |f_i^*(\delta_x) - f_i^*(\delta_0)| = \max_{1 \leq i \leq m} |f_i(x) - f_i(0)| < \epsilon, \forall x \in O_\epsilon.$$

Since 0 lies in the closure of $A$, there exists some $x_\epsilon \in (A \cap O_\epsilon) \setminus \{0\}$ such that $\delta_{x_\epsilon} \in \Gamma$ and

$$\max_{1 \leq i \leq m} |f_i^*(\delta_{x_\epsilon}) - f_i^*(\delta_0)| < \epsilon.$$

This proves $\delta_0$ is a limit point of $\Gamma$ with respect to the weak topology, so $\delta_0 \in \Gamma_1$. Meanwhile, $E$ is a Tychonoff space by Proposition 9.3.2(c). According to Lemma 10.2.16(a, b), no sequence in $\Gamma$ may converge weakly to $\delta_0$, since no sequence in $A$ converges to 0. Thus, $\delta_0 \notin \Gamma_2$.

Relative compactness of finite Borel measures is defined in terms of weak convergence (see §2.3.6) not the weak topology. Consequently, relative compactness of $\Gamma \subset M^+(E)$ can be different than: (1) $\Gamma$ having a compact closure in $M^+(E)$ with the weak topology, the usual interpretation of “relative compactness” (see p.152) of $\Gamma$, or (2) $\Gamma$ having a limit point compact (see p.152) closure with the weak topology, or (3) $\Gamma$ having a sequentially compact (see p.152) closure with the weak topology.

When $E$ is a metrizable space, however, $M^+(E)$ is a metrizable space by Proposition 9.4.11, the weak topology is the same as the topology of weak convergence.
2.3.2. Separation of finite Borel measures by functions. The measure-
separation properties of $D \subset M_b(E; \mathbb{R})$ (i.e., point-separation properties of $D^*$) are vital for studying weak convergence and $\mathcal{M}^+(E)$-valued or $\mathcal{P}(E)$-valued processes (e.g., filters, measure-valued diffusions, non-Markov branching particle systems). The following terminologies are adapted from [EK86 §3.4]: $D \subset M_b(E; \mathbb{R})$ is separating or convergence determining on $E$ if $D^*$ separates points or determines point convergence on $\mathcal{M}^+(E)$ respectively.

Note 2.3.6. $C_b(E; \mathbb{R})^*$ by definition strongly separates points and so determines point convergence on $\mathcal{M}^+(E)$ by Proposition 9.2.1 (b). Hence, $C_b(E; \mathbb{R})$ is convergence determining on $\mathcal{M}^+(E)$.

More details on the point-separation and other topological properties of $\mathcal{M}^+(E)$ and $\mathcal{P}(E)$ can be found in [§ 9.4 Top70 Part II] and [Bog07 Vol. II, Chapter 8].

2.3.3. Portmanteau’s Theorem. One way of establishing (2.3.4) is establishing $\lim_{n \to \infty} f^*(\mu_n) = f^*(\mu)$ for all $f$ from a convergence determining collection. A useful alternative is the Portmanteau’s Theorem. This useful tool was commonly established on metric spaces (see [KX95 Lemma 2.2.2]). [Top70] p.XII and p.40 - 41 gave the following partial generalization to Hausdorff spaces.

**Theorem 2.3.7 (Portmanteau’s Theorem, Top70 Theorem 8.1).** Let $E$ be a Hausdorff space. Consider the following statements:
(a) [2.3.4] holds.
(b) $\limsup_{n \to \infty} \mu_n(F) \leq \mu(F)$ for all $F \in \mathcal{C}(E)$.
(c) $\liminf_{n \to \infty} \mu_n(O) \geq \mu(O)$ for all $O \in \mathcal{O}(E)$.
Then, (b) and (c) are equivalent and each of them implies (a). If, in addition, $E$ is a Tychonoff space, then (a) - (c) are equivalent.

2.3.4. Tightness. Tightness is often more easily verified than relative compactness. Compact subsets are not necessarily Borel subsets in non-Hausdorff spaces. At the same time, they can be in the domain of possibly non-Borel measures (see §3.3.4). So, we slightly adjust the ordinary definition of and extend tightness to general finite measures.

**Definition 2.3.8.** Let $(E, \mathcal{U})$ be a measurable space, $S$ be a topological space and $\mathcal{A}$ be a $\sigma$-algebra on $S$.
- When $S \subset E$, $\Gamma \subset \mathcal{M}^+(E, \mathcal{U})$ is **tight in** $S$ (resp. **m-tight in** $S$) if for any $\epsilon \in (0, \infty)$, there exists a $K_\epsilon \in \mathcal{H}(S)$ (resp. $K_\epsilon \in \mathcal{K}(S)$) such that $K_\epsilon \subset E$ and $\sup_{\mu \in \Gamma} \mu(E \setminus K_\epsilon) \leq \epsilon$.
- $\Gamma \subset \mathcal{M}^+(S, \mathcal{A})$ is **tight in** $A \subset S$ (resp. **m-tight in** $A$) if $A$ is non-empty and $\Gamma$ is tight (resp. m-tight) in $(A, \mathcal{O}_S(A))$.
- $\Gamma \subset \mathcal{M}^+(S, \mathcal{A})$ is **tight** (resp. **m-tight**) if it is tight (resp. m-tight) in $S$.

**Note 2.3.9.** Tightness of a measure $\mu$ refers to that of the singleton $\{\mu\}$.

**Remark 2.3.10.** m-tightness is stronger than tightness, and they are the same if every compact subset of the underlying space is metrizable. We refer the readers to [§3.3.4] for specific discussion about metrizable compact subsets.
The classical Ulam’s Theorem (see [Bil68, Theorem 1.4]), showing tightness of every finite set of Borel probability measures on a Polish space $E$, has the following stronger form about $m$-tightness.

**Theorem 2.3.11 (Ulam’s Theorem, [Bog07, Vol.II, Theorem 7.4.3]).** If $E$ is a Souslin space (see p. 151), especially if $E$ is a Lusin (see p. 151) or Polish space, then any finite subset of $\mathcal{M}^+(E)$ is $m$-tight.

The Prokhorov’s Theorem is a fundamental result connecting relative compactness and tightness of finite Borel measures. Part (a) below, adapted from [Bog07, Vol.II, Theorem 8.6.2], gives one direction of Prokhorov’s Theorem. Part (b), extending the other direction from Polish to Hausdorff spaces, is adapted from [KX95, Theorem 2.2.1].

**Theorem 2.3.12 (Prokhorov’s Theorem).**

(a) If $E$ is a Polish space, then relative compactness implies tightness for any subset of $\mathcal{M}^+(E)$.

(b) If $E$ is a Hausdorff space, then tightness implies relative compactness for any subset of $\mathcal{P}(E)$.

**2.3.5. Finite Borel measures on $D(R^+; E)$.** When $E$ is a Tychonoff space, the Skorokhod $\mathcal{F}_1$-space $D(R^+; E)$ always satisfies

\[(2.3.10) \quad \mathcal{B} \left[ D(R^+; E) \right] = \sigma \left( \mathcal{F}(E) \right) \subset \mathcal{B}(E)^{\otimes R^+}_{\left| D(R^+; E) \right.}
\]

and

\[(2.3.11) \quad \mathcal{M}^+ \left[ D(R^+; E) \right] \subset \mathcal{M}^+ \left( D(R^+; E), \mathcal{B}(E)^{\otimes R^+}_{\left| D(R^+; E) \right.} \right).
\]

However, equality in (2.3.10) or (2.3.11) may not hold in general. The set of fixed left-jump times of $\mu \in \mathcal{M}^+ (D(R^+; E), \mathcal{B}(E)^{\otimes R^+}_{\left| D(R^+; E) \right.})$ refers to

\[(2.3.12) \quad J(\mu) := \{ t \in R^+ : \mu (\{ x \in D(R^+; E) : t \in J(x) \}) > 0 \}
\]

if it is well-defined. (2.3.10), (2.3.11) and (2.3.12) are further discussed in §9.6.

**2.4. Random variable**

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space (i.e. $\mathbb{P} \in \Psi(\Omega, \mathcal{F}))$, $(E, \mathcal{U})$ be a measurable space and $S$ be a topological space. Any $X \in M(\Omega, \mathcal{F}; E, \mathcal{U})$ is said to be an $(E, \mathcal{U})$-valued random variable. $\mathbb{P} \circ X^{-1} \in \Psi(E, \mathcal{U})$, the push-forward measure of $\mathbb{P}$ by $X$, is called the distribution of $X$. $S$-valued random variables refer to $(S, \mathcal{B}(S))$-valued random variables if not otherwise specified. Hereafter, we let $(\Omega, \mathcal{F}, \mathbb{P}, X)$ denote a random variable $X$ defined on probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

Any type of tightness in Definition 2.3.8 is defined for random variables by referring to the corresponding property of their distributions. “$X_n \Rightarrow X$ as $n \uparrow \infty$ on $S$” means the distributions of $S$-valued random variables $(X_n)_{n \in \mathbb{N}}$ converge weakly to that of $S$-valued random variable $X$ as $n \uparrow \infty$ in $\mathcal{P}(S)$. Similar interpretations apply to the statements “$X$ is the weak limit of $(X_n)_{n \in \mathbb{N}}$ on $S$”, “$X$ is a weak limit point of $(X_i)_{i \in I}$ on $S$” and “$(X_i)_{i \in I}$ is relatively compact in $S$”.

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14“converging weakly” was defined in [2.3.3] as converging with respect to the weak topology.
2. Preliminaries

2.5. Stochastic process

The stochastic processes treated in this work are indexed by time horizon \( \mathbb{R}^+ \) and take values in topological spaces. Throughout this section, we let \((\Omega, \mathcal{F}, \mathbb{P})\) be a probability space, \(E\) be a topological space and \(X \in (\mathbb{R}^+)^\Omega\).

2.5.1. Definition. \(X\) is an \(E\)-valued (stochastic) process if \(\mathcal{B}(E)^{\mathbb{R}^+}\) is a sub-\(\sigma\)-algebra of

\[
\forall_X = \left\{ B \subseteq E^{\mathbb{R}^+} : X^{-1}(B) \in \mathcal{F} \right\},
\]

or equivalently,

\[
X \in M(\Omega, \mathcal{F}; E^{\mathbb{R}^+}, \mathcal{B}(E)^{\mathbb{R}^+}).
\]

Remark 2.5.1. The \(\forall_X\) in (2.5.1) is often called the “push-forward \(\sigma\)-algebra of \(X\)”. In any case, \(X \in M(\Omega, \mathcal{F}; E^{\mathbb{R}^+}, \forall_X)\).

Let \((\Omega, \mathcal{F}, \mathbb{P}, X)\) be an \(E\)-valued process. For each \(\omega \in \Omega\), \(X(\omega) \in E^{\mathbb{R}^+}\) is called a (realization) path of \(X\). The process distribution of an \(E\)-valued process \(X\) refers to the push-forward measure of \(\mathbb{P}\) by \(X : (\Omega, \mathcal{F}) \to (E^{\mathbb{R}^+}, \forall_X)\) and is denoted by \(\text{pd}(X) \in \mathcal{P}(E^{\mathbb{R}^+}; \forall_X)\). For each \(t \in \mathbb{R}^+\), \(X_t = p_t \circ X\) denotes the (one-dimensional) section of \(X\) for \(t\). For each \(T_0 \in \mathcal{P}_0(\mathbb{R}^+)\), the section of \(X\) for \(T_0\) refers to the \(E^{T_0}\)-valued mapping \(X_{T_0} = p_{T_0} \circ X\), and the finite-dimensional distribution of \(X\) for \(T_0\) refers to \(\text{pd}(X) \circ p_{T_0}^{-1}\). From Fact 2.1.3, we immediately observe that:

Fact 2.5.2. Let \((\Omega, \mathcal{F}, \mathbb{P})\) be a probability space and \(E\) be a topological space. Then:

(a) \(X \in (E^{\mathbb{R}^+})^\Omega\) is an \(E\)-valued process if and only if \(p_t \circ X \in M(\Omega, \mathcal{F}; E)\) for all \(t \in \mathbb{R}^+\).

(b) If \(\zeta_t \in M(\Omega, \mathcal{F}; E)\) for all \(t \in \mathbb{R}^+\), then

\[
X(\omega)(t) = \zeta_t(\omega), \forall t \in \mathbb{R}^+, \omega \in \Omega,
\]

well defines an \(E\)-valued process \(X\) satisfying \(\zeta_t = p_t \circ X\) for all \(t \in \mathbb{R}^+\).

(c) The section of an \(E\)-valued process \((\Omega, \mathcal{F}, \mathbb{P}, X)\) for each \(T_0 \in \mathcal{P}_0(\mathbb{R}^+)\) is a member of \(M(\Omega, \mathcal{F}; E^{T_0}, \mathcal{B}(E)^{\mathbb{R}^+})\).

(d) The finite-dimensional distribution of an \(E\)-valued process \(X\) for each \(T_0 \in \mathcal{P}_0(\mathbb{R}^+)\) is the distribution of \(X_{T_0}\) and belongs to \(\mathcal{P}(E^{T_0}, \mathcal{B}(E)^{\mathbb{R}^+})\). In particular, the distribution of \(X_t\) is a member of \(\mathcal{P}(E)\) for all \(t \in \mathbb{R}^+\).

Remark 2.5.3. Given an \(E\)-valued process \(X\) and a general \(T_0 \in \mathcal{P}_0(\mathbb{R}^+)\), \(X\) (resp. \(X_{T_0}\)) need not be an \((E^{\mathbb{R}^+}, \mathcal{B}(E^{\mathbb{R}^+}))\)-valued (resp. \((E^{T_0}, \mathcal{B}(E^{T_0}))\)-valued) random variable, nor is the process distribution of \(X\) (resp. the finite-dimensional distribution of \(X\) for \(T_0\)) necessarily a Borel measure. This is due to the possible difference between the Borel \(\sigma\)-algebra generated by product topology and product of Borel \(\sigma\)-algebras on each individual dimension, which was mentioned in \S\textsuperscript{2.1.5}.

\footnote{Stochastic processes are definable on measurable spaces without any topological structure.}

\footnote{An \(E\)-valued process is an \((E^{\mathbb{R}^+}, \mathcal{B}(E)^{\mathbb{R}^+})\)-valued random variable, hence it is consistent for \((\Omega, \mathcal{F}, \mathbb{P}, X)\) to denote an \(E\)-valued process \(X\) defined on \((\Omega, \mathcal{F}, \mathbb{P})\).}
Hereafter, an $E$-valued process $X$ is also denoted by $X = \{X_t\}_{t \geq 0}$ or just by $\{X_t\}_{t \geq 0}$: its section $X_{T_0}$ for $T_0 = \{t_1, ..., t_d\}$ is also denoted by $(X_{t_1}, ..., X_{t_d})$.

Let $S$ be a topological space, $X$ be an $E$-valued process and $f \in M(E; S)$. The process $\{f \circ X_t\}_{t \geq 0}$ is the mapping $\omega \mapsto f(X(\omega))$ of $S$-valued path $\omega \in \Omega$ to the $S$-valued path $f(X(\omega))$. A popular notation of this process is $f \circ X$.

### 2.5.2. Càdlàg process.

$X \in M(\Omega, \mathcal{F}; E^{R^+})$ is an $E$-valued càdlàg process if

\[
\omega \in \Omega : X(\omega) \text{ is not a càdlàg member of } E^{R^+} \in \mathcal{F} \cap \mathcal{N}(\mathbb{P}).
\]

When $E$ is a Tychonoff space, the **path space** of an $E$-valued càdlàg process is thought to be in $D(R^+; E)$. From (2.3.10) we have:

**Fact 2.5.4.** Let $E$ be a Tychonoff space. Then,

(a) Every $E$-valued càdlàg process $(\Omega, \mathcal{F}, \mathbb{P}; X)$ satisfies that

\[
\Omega \setminus X^{-1} \left[ D(R^+; E) \right] \in \mathcal{F} \cap \mathcal{N}(\mathbb{P}),
\]

and

\[
X^{-1}(A) \in \mathcal{F}, \forall A \in \mathcal{B}(E) \otimes R^+ \Big|_{D(R^+; E)},
\]

(b) Every member of $M(\Omega, \mathcal{F}; D(R^+; E))$ is an $E$-valued càdlàg process defined on $(\Omega, \mathcal{F}, \mathbb{P})$.

**Remark 2.5.5.** As $\mathcal{B}[D(R^+; E)]$ is generically larger than $\mathcal{B}(E) \otimes R^+ |_{D(R^+; E)}$, an $E$-valued càdlàg process is not necessarily a $D(R^+; E)$-valued random variable. More details about càdlàg processes are presented in §9.7.

### 2.5.3. Stochastic process terminologies.

Let $X$ and $Y$ be $E$-valued processes defined on $(\Omega, \mathcal{F}, \mathbb{P})$. The **set of fixed left-jump times** of $X$ refers to

\[
J(X) \triangleq \left\{ t \in R^+ : P \left( \lim_{s \to t^-} X_s \text{ exists and equals } X_t \right) < 1 \right\}
\]

if it is well-defined. $X$ is a **stationary process** if

\[
P \circ X_{T_0}^{-1} = P \circ X_{T_0+c}^{-1}, \forall T_0 \in \mathcal{T}_0(R^+), c \in (0, \infty),
\]

where

\[
T_0 + c \triangleq \left\{ t + c : t \in T_0 \right\}.
\]

$X$ and $Y$ are **(pathwisely) indistinguishable** if $\{X \neq Y\} \in \mathcal{N}(\mathbb{P}) \cap \mathcal{F}$. $X$ and $Y$ are **modifications of each other** if $\{X_t \neq Y_t\} \in \mathcal{N}(\mathbb{P}) \cap \mathcal{F}$ for all $t \in R^+$.

A **filtration** (see [Dud02] p.453) $\{\mathcal{F}_t\}_{t \geq 0}$ on $(\Omega, \mathcal{F}, \mathbb{P})$ is **$\mathbb{P}$-complete** if $\mathcal{F}_t$ is $\mathbb{P}$-complete for all $t \geq 0$. We call $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$ a **stochastic basis** if both $\mathcal{F}$ and $\{\mathcal{F}_t\}_{t \geq 0}$ are $\mathbb{P}$-complete. $X$ is $\mathcal{F}_t$-**adapted** if $\mathcal{F}_t^X \subset \mathcal{F}_t$ for all $t \geq 0$, where

\[
\mathcal{F}_t^X \triangleq \sigma \left\{ X_u : u \in [0, t] \right\} \cup \mathcal{N}(\mathbb{P}), \forall t \geq 0.
\]

$M(\Omega, \mathcal{F}; D(R^+; E))$ denotes the $D(R^+; E)$-valued random variables defined on $(\Omega, \mathcal{F}, \mathbb{P})$. 

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17 $M(\Omega, \mathcal{F}; D(R^+; E))$ denotes the $D(R^+; E)$-valued random variables defined on $(\Omega, \mathcal{F}, \mathbb{P})$. 

\( \mathcal{F}_t^X \triangleq \{ \mathcal{F}_t^X \}_{t \geq 0} \) is called the augmented natural filtration of \( X \). Let \( \xi(t, \omega) \triangleq X_t(\omega) \) for each \( \omega \in \Omega \) and \( t \in \mathbb{R}^+ \). Then, \( X \) is a measurable process if
\[
(2.5.12) \quad \xi \in M \left( \mathbb{R}^+ \times \Omega, \mathcal{B}(\mathbb{R}^+) \otimes \mathcal{F}_t; E, \mathcal{B}(E) \right).
\]
\( X \) is a \( \mathcal{G}_t \)-progressive process if
\[
(2.5.13) \quad \xi|_{[0, t] \times \Omega} \in M \left( [0, t] \times \Omega, \mathcal{B}([0, t]) \otimes \mathcal{G}_t; E, \mathcal{B}(E) \right), \forall t \in \mathbb{R}^+.
\]
\( X \) is a progressive process if it is \( \mathcal{F}_t^X \)-progressive.

### 2.6. Conventions

The following conventions hold hereafter if not otherwise specified:

- \( I \) is a non-empty index set.
- Subsets are non-empty.
- Measures are non-trivial.
- Subsets of topological spaces are equipped with their subspace topologies.
- Topological spaces are equipped with their Borel \( \sigma \)-algebras.
- Any Cartesian product of topological spaces is equipped with the product topology and, hence, is equipped with the Borel \( \sigma \)-algebra generated by the product topology.
- Linear spaces are over the scalar field \( \mathbb{R} \).
- Linear operators are single-valued.
- \( (\Omega, \mathcal{F}, \mathbb{P}), \{(\Omega^n, \mathcal{F}_n^n, \mathbb{P}^n)\}_{n \in \mathbb{N}_0} \) and \( \{(\Omega^i, \mathcal{F}_i^i, \mathbb{P}^i)\}_{i \in I} \) are complete probability spaces with expectation operators \( E \), \( \{E^n\}_{n \in \mathbb{N}_0} \) and \( \{E^i\}_{i \in I} \), respectively.

### 2.7. Motivating examples

The Introduction motivates the general use of replication. Herein, we outline nine specific examples. \( \{2.7.1\} - \{2.7.6\} \) contain brief reviews of the pseudo-path topology of càdlàg functions, \( \mathcal{S} \)-topology of càdlàg functions, strong topology of Borel probability measures, strong dual of nuclear Fréchet space, Banach spaces of finite \( p \)-variation or \( 1/p \)-Hölder continuous paths, and Banach spaces of rough paths. These spaces fail the traditional assumptions of metric completeness (see [Mun00 §43, Definition, p.264]), compactness, separability and/or metrizability. In \( \{2.7.7\} \) we review a version of Kolmogorov’s Extension Theorem for standard Borel spaces to illustrate of boosting results by space change. Moreover, \( \{2.7.8\} \) and \( \{2.7.9\} \) refer to two examples of the convenience of our convergence results in Chapter 8.

#### 2.7.1. Pseudo-path topology.

In [MZ84], the pseudo-path topology (also known as “Meyer-Zheng topology”) was used to characterize tightness of càdlàg semimartingales with respect to the topology of convergence in measure. This is an example of a non-Polish metrizable Lusin space.

**Example 2.7.1.** Let \( D^{pp}(\mathbb{R}^+; \mathbb{R}) \) denote the càdlàg members of \( \mathbb{R}^{\mathbb{R}^+} \) equipped with the pseudo-path topology\(^1\). This topology is induced by the mapping \( \psi^{pp} \)

---

\(^1\)Pseudo-path topology can be defined similarly on the family of all càdlàg members of \( E^{\mathbb{R}^+} \) when \( E \) is a Polish space. In that case, \( K \) will be a metrizable compactification of \( \mathbb{R}^+ \times E \).
associating each \( x \in D^{\text{pp}}(\mathbb{R}^+; \mathbb{R}) \) to its \( \lambda' \)-almost everywhere unique pseudo-path \( \psi^{\text{pp}}(x) \in \mathcal{P}(K) \), where

\[
(2.7.1) \quad \lambda'(A) \doteq \int_A e^{-t} \lambda(dt), \quad \forall A \in \mathcal{B}(\mathbb{R}^+),
\]

\( \lambda \) is the Lebesgue measure on \( \mathbb{R}^+ \), \( K \doteq [0, \infty] \times [-\infty, \infty] \), and

\[
(2.7.2) \quad \psi^{\text{pp}}(x)(B) \doteq \lambda' \left( \{ t \in \mathbb{R}^+ : (t, x(t)) \in B \} \right), \quad \forall B \in \mathcal{B}(K).
\]

\[ \text{[MZ84] Theorem 2} \] showed that

\[
(2.7.3) \quad \psi^{\text{pp}} \in \text{imb} \left( D^{\text{pp}}(\mathbb{R}^+; \mathbb{R}); \mathcal{P}(K) \right)
\]

and

\[
(2.7.4) \quad \psi^{\text{pp}} \left[ D^{\text{pp}}(\mathbb{R}^+; \mathbb{R}) \right] \in \mathcal{B}[\mathcal{P}(K)].
\]

\( K \) is a Polish space by Proposition \[9.1.12\] (d). \( \mathcal{P}(K) \) is a Polish space by Theorem \[9.4.10\] (b). Hence, \( \psi^{\text{pp}} \left[ D^{\text{pp}}(\mathbb{R}^+; \mathbb{R}) \right] \) is a metrizable Lusin space by Proposition \[9.5.2\] (a, d), and so is its homeomorph \( D^{\text{pp}}(\mathbb{R}^+; \mathbb{R}) \). However, \[MZ84\] p.355 - 356 pointed out that \( \psi^{\text{pp}} \left[ D^{\text{pp}}(\mathbb{R}^+; \mathbb{R}) \right] \) and \( D^{\text{pp}}(\mathbb{R}^+; \mathbb{R}) \) are not Polish spaces.

### 2.7.2. \( S \)-topology

\[ \text{[Jak97b]} \] defined the \( S \)-topology by introducing the \( S \)-convergence of càdlàg functions from \( [0, T] \subset \mathbb{R}^+ \) to \( \mathbb{R} \), which is related to the pseudo-path topology. The tightness conditions proposed by \[Str85\] for the pseudo-path topology turns out to be superfluous (see \[Kur91\]) but serves as a tightness condition and motivation for the \( S \)-topology (see \[Jak97b\] and \[Jak12\]).

**Example 2.7.2.** We define the total variation of \( x \in \mathbb{R}^{[0, T]} \) by

\[
(2.7.5) \quad \|x\|_{1, \text{var}, [0, T]} \doteq |x(0)| + \sup_{0 \leq t_0 < \ldots < t_n \leq T, n \in \mathbb{N}} \sum_{i=1}^{n} |x(t_i) - x(t_{i-1})|,
\]

and put

\[
(2.7.6) \quad V \doteq \left\{ x \in \mathbb{R}^{[0, T]} : x \text{ is càdlàg, } \|x\|_{1, \text{var}, [0, T]} < \infty \right\}.
\]

Càdlàg functions \( \{ x_n \}_{n \in \mathbb{N}} \subset \mathbb{R}^{[0, T]} \) \( S \)-converge to càdlàg function \( x_0 \in \mathbb{R}^{[0, T]} \) if for any \( \epsilon \in (0, \infty) \), there exist \( \{ v_n \}_{n \in \mathbb{N}_0} \subset V \) such that

\[
(2.7.7) \quad \sup_{n \in \mathbb{N}_0} \|x_n - v_n\|_{1, \text{var}, [0, T]} < \epsilon
\]

and

\[
(2.7.8) \quad \lim_{n \to \infty} \int_{[0, T]} f(t) dv_n(t) = \int_{[0, T]} f(t) dv_0(t), \quad \forall f \in C([0, T]; \mathbb{R}).
\]

Considering \( S \)-convergence as convergence a priori, the \( S \)-topology is the topology of convergence a priori on the càdlàg members of \( \mathbb{R}^{[0, T]} \). This \( S \)-topological space is a coarsening of the Skorokhod \( \mathcal{J}_1 \)-space \( D([0, T]; \mathbb{R}) \) (see \[Jak12\] p.5) that is neither necessarily a Tychonoff space, nor known to be a topological vector space (see \[Jak97b\] and \[Jak12\] p.5). The corresponding convergence a posteriori, called \( S^* \)-convergence, is different than \( S \)-convergence (see \[Jak12\] p.3 - 4). Moreover, \( S^* \)-convergence is also the topological convergence of some coarsening of the \( S \)-topology. The equality of these two topologies is an open question (see \[Jak12\] p.4).
2.7.3. Strong topology of Borel probability measures. [DZ98] used the strong topology of Borel probability measures on a Polish space in large deviations. This gives a non-metrizable and non-separable (see p.150) Tychonoff space.

**Example 2.7.3.** Let $E$ be a Polish space and $\mathcal{P}_E$ be the space of all Borel probability measures on $E$ equipped with the strong topology

\[(2.7.9) \quad \mathcal{O}[\mathcal{P}_E] = \mathcal{O}_{M_b(E;\mathbb{R})^*}[\mathcal{P}(E)].\]

Then, $\mathcal{P}_E$ is a topological refinement of $\mathcal{P}(E)$,

\[(2.7.10) \quad M_b(E;\mathbb{R})^* \subset C_b(\mathcal{P}_E;\mathbb{R}),\]

and $M_b(E;\mathbb{R})^*$ strongly separates points on $\mathcal{P}_E$. Furthermore, from the fact

\[(2.7.11) \quad \{(1_A)^*: A \in \mathcal{B}(E)\} \subset M_b(E;\mathbb{R})^*\]

it follows that $M_b(E;\mathbb{R})^*$ separates points on $\mathcal{P}_E$. Hence, $\mathcal{P}_E$ is a Tychonoff space by Proposition 9.3.1(a, b). However, [DZ98] p.263 explained that $\mathcal{P}_E$ is neither metrizable nor separable.

2.7.4. Strong dual of nuclear Frechét space. [Jak86] §5.II discussed tightness of probability measures on the Skorokhod $\mathcal{F}_1$-space $D([0,1];E)$ with $E$ being the strong dual of a general nuclear Frechét space. This is an example of a possibly non-metrizable, Tychonoff topological vector space.

**Example 2.7.4.** Let $E$ be the strong dual of some infinite-dimensional, unnormable, nuclear Frechét space. $E$ is not metrizable by [GK12] §29.1, (7), p.394, nor is it necessarily separable. However, $E$ is a nuclear space by [SW99] IV.9.6, Theorem, p.172, the topology of $E$ is induced by a family of Hilbertian semi-norms (see [SH12] Definition A.4), and $E$ is a Tychonoff space by [KX95] Theorem 2.1.1. Hence, $D([0,1];E)$ is a non-metrizable, possibly non-separable, Tychonoff space by [Jak86] Proposition 1.6 ii) - iii)] and Proposition 9.6.1(c).

2.7.5. Spaces of finite-variation or Hölder continuous functions. The spaces of $\mathbb{R}^d$-valued continuous functions with finite $p$-variation or $\mathbb{R}^d$-valued 1/p-Hölder continuous functions are frequently used in stochastic differential equations driven by non-classical noises. They are examples of non-separable Banach spaces.

**Example 2.7.5.** Let $d, N \in \mathbb{N}$, $p \in [1,\infty)$ and $T \in (0,\infty)$. A path $x \in (\mathbb{R}^d)^{[0,T]}$ has finite $p$-variation if the homogeneous $p$-variation norm of $x$ defined by

\[(2.7.12) \quad \|x\|_{p\text{-var},[0,T]} \overset{\Delta}{=} |x(0)| + \sup_{0 \leq t_0 < \ldots < t_n \leq T, n \in \mathbb{N}} \left( \sum_{i=1}^n |x(t_i) - x(t_{i-1})|^p \right)^{1/p},\]

or is 1/p-Hölder continuous if the 1/p-Hölder norm of $x$ defined by

\[(2.7.13) \quad \|x\|_{\frac{1}{p}\text{-Hölder},[0,T]} \overset{\Delta}{=} \sup_{0 \leq s < t \leq T} \frac{|x(t) - x(s)|}{|t - s|^{\frac{1}{p}}},\]

is finite respectively. The normed spaces

\[(2.7.14) \quad \{ x \in C([0,T];\mathbb{R}^d) : \|x\|_{p\text{-var},[0,T]} < \infty \}\]

and

\[(2.7.15) \quad \{ x \in (\mathbb{R}^d)^{[0,T]} : \|x\|_{\frac{1}{p}\text{-Hölder},[0,T]} < \infty \}\]

are non-separable Banach spaces (see [FV10] Theorem 5.27).
2.7.6. Space of rough paths. The rough path approach, initiated by the pioneering works of [Lyo94] and [Lyo98], is important to generalizing stochastic differential equations like

\[ dY_t = \alpha(t, Y_t) dt + \sigma(t, Y_t) dX_t, \]

to the case where the driving noise \( X \) is not necessarily a semimartingale. By this approach, the original noise \( X \) is enhanced to a random rough path \( X \) (see [FV10, §9.1]) and the Stratonovich solution of \( 2.7.16 \) is closely linked to the solution of

\[ dY_t = \alpha(t, Y_t) dt + \sigma(t, Y_t) dX_t, \]

where \( 2.7.17 \) is considered as rough differential equations driven by the realization paths of \( X \) as a process (see [FV10, §10.3, §10.4, §17.1 and §17.2]). [FV10] and [FH14] considered the following spaces for the paths of \( X \), which are also examples of non-separable Banach spaces.

Example 2.7.6. Let \( d, N \in \mathbb{N}, p \in [1, \infty) \) and \( T \in (0, \infty) \). A rough path is often considered as a mapping from \([0, T]\) to \( G^N(\mathbb{R}^d) \), the free nilpotent group of Step \( N \) over \( \mathbb{R}^d \) (see [FV10, p.142-143]). As a Lie group, \( G^N(\mathbb{R}^d) \) is equipped with the usual addition \( + \) of functions and the Carnot-Caratheodory norm \( \|\cdot\|_{cc} \) (see [FV10, Theorem 7.32]). Similar to \( \mathbb{R}^d \)-valued paths, a path \( x \in G^N(\mathbb{R}^d) \) has finite \( p \)-variation or is \( 1/p \)-Hölder continuous if the homogeneous \( p \)-variation \( cc \)-norm of \( x \) defined by

\[ \|x\|_{cc,p-\text{var},[0,T]} \triangleq \sup_{0 \leq t_0 < \ldots < t_n \leq T, n \in \mathbb{N}} \left( \sum_{i=1}^{n} \|x(t_i) - x(t_{i-1})\|_{cc}^p \right)^{1/p} \]

or the homogeneous \( 1/p \)-Hölder \( cc \)-norm of \( x \) defined by

\[ \|x\|_{cc,1/p-\text{Höld},[0,T]} \triangleq \sup_{0 \leq s < t \leq T} \frac{\|x(t) - x(s)\|_{cc}}{|t-s|^{1/p}} \]

is finite respectively. The random rough path \( X \) in \( 2.7.17 \) may have paths in

\[ \{ x \in C ([0, T]; G^N(\mathbb{R}^d)) : \|x\|_{cc,p-\text{var},[0,T]} < \infty \} \]

or

\[ \{ x \in G^N(\mathbb{R}^d)^{[0,T]} : \|x\|_{cc,1/p-\text{Höld},[0,T]} < \infty \}. \]

By [FV10] Theorem 8.13, these normed spaces are non-separable Banach spaces.

2.7.7. Kolmogorov’s Extension Theorem. The Kolmogorov’s Extension Theorem (see [AB06, §15.6]) is a cornerstone of probability theory that depends purely on the relevant \( \sigma \)-algebras. Hence, existence of Kolmogorov extension should be transferrable from a “nice”, typically Polish topological space to “defective”, typically standard Borel topological space which are “indifferent” as measurable spaces.

Let \( \{S_i\}_{i \in I} \) be a family of standard Borel spaces,

\[ (S, \mathcal{A}) \triangleq \left( \prod_{i \in I} S_i, \bigotimes_{i \in I} \mathcal{B}(S_i) \right), \]
For each $i \in I$, Proposition 9.5.5 (a, d) allows us to change the topology of $S_i$ to a possibly different one $\mathcal{U}_i$ such that $(S_i, \mathcal{U}_i)$ is a Polish space and the Borel sets $\mathcal{B}(S_i) = \mathcal{B}(S_i, \mathcal{U}_i)$ remain unchanged. So, any $\mu_{I_0} \in \mathfrak{P}(S_{I_0}, \mathcal{A}_{I_0})$ can be viewed as a probability measure on $(S_{I_0}, \bigotimes_{i \in I_0} \mathcal{B}(S_i, \mathcal{U}_i))$ for each $I_0 \in \mathcal{P}_0(I)$, and any Kolmogorov extension of $\{\mu_{I_0}\}_{I_0 \in \mathcal{P}_0(I)}$ on $(S, \bigotimes_{i \in I} \mathcal{B}(S_i, \mathcal{U}_i))$ would be a desired Kolmogorov extension of them on $(S, \mathcal{A})$. Therefore, the well-known version of Kolmogorov’s Extension Theorem for Polish spaces extends immediately to the standard Borel case.

**Theorem 2.7.7** (Kolmogorov’s Extension Theorem, Kal97 Theorem 5.16). Let $\{S_i\}_{i \in I}$ be standard Borel spaces, $(S, \mathcal{A})$ be as in (2.7.23), $\{(S_{I_0}, \mathcal{A}_{I_0})\}_{I_0 \in \mathcal{P}_0(I)}$ be as in (2.7.23) and $\mu_{I_0} \in \mathfrak{P}(S_{I_0}, \mathcal{A}_{I_0})$ for each $I_0 \in \mathcal{P}_0(I)$. Suppose in addition that for each $I_0, I_2 \in \mathcal{P}_0(I)$ with $I_1 \subset I_2$, $\mu_{I_2}$ is the push-forward measure of $\mu_{I_1}$ by the projection from $S_{I_2}$ to $S_{I_1}$. Then, there exists a $\mu \in \mathfrak{P}(S, \mathcal{A})$ such that for each $I_0 \in \mathcal{P}_0(I)$, $\mu_{I_0}$ is the push-forward measure of $\mu$ by the projection from $S$ to $S_{I_0}$.

**2.7.8. Approximation by Markov Empirical Processes.** A traditional way of constructing measure-valued processes is to show convergence of weighted empirical processes to it. Kou16 confirmed the general availability of such approximation on a Polish underlying space. Kou16 employed similar compactification technique to that of this paper and reduced tightness to the verification of Modulus of Continuity Condition (see §6.4.[1]). Unlike the convergence results in BK93b §2, Kou16 neither assumed pointwise tightness condition, for either the limit or pre-limit processes, nor required a martingale problem set-up. More importantly, an inspection into the development of Kou16 Theorem 2 and Theorem 3 shows that the Polish space imposition was absent in verifying finite-dimensional convergence or Modulus of Continuity Condition. Instead, metric completeness was only used to compactify the space of measures and establish tightness. By Corollary 8.2.11 to follow, one only needs Modulus of Continuity Condition rather than tightness for deriving weak convergence of càdlàg processes from their finite-dimensional convergence. Therefore, the following generalization of Kou16 Theorem 3 is immediate from Corollary 8.2.11 and Corollary 9.4.9.

**Theorem 2.7.8.** Let $E$ be a metrizable and separable space and $V = \{V_t\}_{t \geq 0}$ be a process with $D(R^+; \mathcal{P}(E))$-valued paths. Then, there exists $m_N \uparrow \infty$ and conditionally i.i.d. $E$-valued càdlàg Markov processes $\{\xi^1, \ldots, \xi^{m_N} \} \in \mathbb{N}$ such that the empirical processes $V^N \defeq \frac{1}{m_N} \sum_{i=1}^{m_N} \delta_{\xi^i}$ converge weakly to $V$ almost surely on $D(R^+; \mathcal{P}(E))$.

**2.7.9. General Existence of Infinite Systems of Stochastic Differential Equations.** Infinite-dimensional systems of stochastic differential equations are important for modelling many vast and complicated real systems and for analysis of stochastic partial differential equations. Several authors, e.g., Fri87, Sko99 Sko01 and ABGP06, have studied specific systems, like stochastic gradient equations or Ornstein-Uhlenbeck perturbations, in detail. However, recently
2.7. MOTIVATING EXAMPLES

[KK20] showed weak existence for a relatively general system of the form:

\[
X^i_t = X^i_0 + \int_0^t \sigma \left( X^i_s, N_s \right) dB^i_s + \int_0^t b \left( X^i_s, N_s \right) ds + \int_0^t \alpha \left( X^i_s, N_s \right) dW_s, \quad i \in \mathbb{Z},
\]

where \( \mathbb{Z} \) is the integers and \( N_t = \sum_{i \in \mathbb{Z}} \delta_{X^i_t} \) is the configuration at time \( t \geq 0 \).

Let \( \mathcal{M}_\beta(\mathbb{R}^d) \) for \( \beta > 0 \) denote the family of all Borel measure \( \mu \) on \( \mathbb{R}^d \) satisfying

\[
\int_{\mathbb{R}^d} (1 + |x|^2)^{-\beta} \mu(dx) < \infty.
\]

Then, the coefficient functions \( b(x, \nu), \sigma(x, \nu) \) and \( \alpha(x, \nu) \) and initial configuration are regulated by the following mild conditions:

(I) There exist \( \beta, \delta \in (0, \infty) \) such that

\[
\mathbb{E} \left[ \sum_{i \in \mathbb{Z}} \left( 1 + |X^i_0|^2 \right)^{-\beta} \right]^{2+\delta} < \infty.
\]

(LG) \( b(x, \nu), \sigma(x, \nu) \) and \( \alpha(x, \nu) \) satisfy for \( \beta \) in (I) that

\[
\sup_{x \in \mathbb{R}^d, \nu \in \mathcal{M}_\beta(\mathbb{R}^d)} \frac{|b(x, \nu)|}{1 + |x|} < \infty,
\]

\[
\sup_{x \in \mathbb{R}^d, \nu \in \mathcal{M}_\beta(\mathbb{R}^d)} \frac{|\sigma(x, \nu)|}{1 + |x|} < \infty,
\]

\[
\sup_{x \in \mathbb{R}^d, \nu \in \mathcal{M}_\beta(\mathbb{R}^d)} \frac{|\alpha(x, \nu)|}{1 + |x|} < \infty.
\]

(JC) There is a \( \overline{\beta} > \beta \) in (I) such that \( b \) and both \( \sigma, \alpha \) are continuous functions from \( \mathbb{R}^d \times \mathcal{M}_\beta(\mathbb{R}^d) \) to \( \mathbb{R}^d \) and to \( \mathbb{R}^d \times \mathbb{R}^d \) respectively.

Their main result, establishing both existence of solutions to the infinite system (2.7.24) and weak convergence of weighted-empirical-measures into the set of all solutions, is stated below:

**Theorem 2.7.9 ([KK20] Theorem 2).** Suppose that \( \{LG\}, \{I\} \) hold. Then:

(a) The weighted-empirical-measures

\[
N^n_t = \sum_{i \in \mathbb{Z}} \Phi^{i,n}_t \delta_{X^i_{n,t}}, \quad t \in \mathbb{R}^+, \quad n \in \mathbb{N}
\]

constructed in [KK20] p.6 are bijectively indistinguishable from a tight sequence of \( D(\mathbb{R}^+; \mathcal{M}^+(\mathbb{R}^d)) \)-valued random variables, where \( \Phi^{i,n}_t \) is a discretization of \( (1 + |X^{i,n}_t|^2)^{-\overline{\beta}} \).

(b) If, in addition, \( \{JC\} \) holds, then every weak limit point of the tight sequence in (a) is a weak solution \( N \) to (2.7.24) via density \( (1 + |x|^2)^{\overline{\beta}} \).

Let \( b, \sigma, \alpha \) could be unbounded and non-Lipschitz under the conditions above. The approximations \( \{X^{i,n}_t\}_{n \in \mathbb{N}} \) of \( X^i_t \) and the weight processes \( \{\Phi^{i,n}_t\}_{n \in \mathbb{N}} \) of particle \( \delta_{X^i_{n,t}} \) in (2.7.28) are constructed naturally in Euler-type form (see [KK20] (2.11)-(2.14)) and so computer implementable. [KK20] §4 - 5] showed their convergence to a solution to the martingale problem form of (2.7.24). Our Theorem 8.1.2 to
follow reduced the desired tightness to Weak Modulus of Continuity Condition (see §6.4.1) regarding a simple family of rational functions. [BK93b] Theorems 2.1 and Theorem 2.3 did not apply as the well-posedness of the martingale problem was not imposed and little was required on the coefficients.
CHAPTER 3

Space Change in Replication

A topological space $E$ needs no enhancement when it is compact and metrizable. Otherwise, a problem on $E$ might be simplified if it is translated onto such a “perfect” space. Replication is a method of space change for this purpose.

The current chapter discusses the space change aspect of replication. §3.1 introduces the notion of base as our core platform to implement space change and other goals of replication. §3.2 and §3.3 explain the existence and various properties of baseable spaces or baseable subsets with which one can construct the desired bases.

3.1. Base

The goal of space change in replication is to create a compact metric space $\hat{E}$ related to the original space $E$. As illustrated by the following figure, the most natural way is to establish a metrizable compactification\(^1\) (see p.155) of $E$ itself or, more generally, a Borel subset $E_0$ of $E$.

![Diagram of space change in replication]

**Figure 1.** Space change in replication

---

\(^1\)In this work, we always consider any compactification to be a Hausdorff space.
3.1.1. Definition. A base is a foundational notion of replication that concretizes the space change idea mentioned above.

**Definition 3.1.1.** Let \( E \) be a topological space. The quadruple \((E_0, \mathcal{F}; \hat{E}, \hat{\mathcal{F}})\) is a **replication base over** \( E \) (a base over \( E \) or a base for short) if:
- \( E_0 \) is a non-empty Borel subset of \( E \).
- \( \mathcal{F} \subset C_b(E; \mathbb{R}) \) is countable and contains the constant function 1.
- \( \hat{E} \) is a topological space containing \( E_0 \).
- \( \hat{\mathcal{F}} \subset \mathbb{R}^{\hat{E}} \) is a countable collection, separates points on \( \hat{E} \) and satisfies
  \[
  \otimes \hat{\mathcal{F}}|_{E_0} \neq \otimes \hat{\mathcal{F}}|_{E_0}.
  \]
  (3.1.1)

and

\[
\otimes \hat{\mathcal{F}}(\hat{E}) \text{ is the closure of } \otimes \mathcal{F}(E_0) \text{ in } \mathbb{R}^\infty.
\]

**Remark 3.1.2.** In general, \( E \) need not be a subset of \( \hat{E} \).

The following lemma shows a base establishes the compactification in Figure 1.

**Lemma 3.1.3.** Let \( E \) be a topological space, \((E_0, \mathcal{F}; \hat{E}, \hat{\mathcal{F}})\) be a base over \( E \) and \( A \subset E_0 \). Then:
- \( \hat{\mathcal{F}} \subset C(\hat{E}; \mathbb{R}) \) is countable, contains the constant function 1 and strongly separates points on \( \hat{E} \). In particular,
  \[
  (\hat{\mathcal{F}}(\hat{E})) \in \text{imb}(\hat{E}; \mathbb{R}^\infty).
  \]
  (3.1.3)

- \( \hat{\mathcal{F}}(\hat{E}) \) is a compactification of \( \otimes \mathcal{F}(E_0) \) and \( \hat{E} \) is a compactification of \((E_0, \mathcal{O}_F(E_0))\).
- \( \hat{E} \) is a Polish space and is completely metrized by \( \rho_{\hat{\mathcal{F}}} \).
- \( \otimes \mathcal{F}|_A \in \text{imb}(A, \mathcal{O}_E(\hat{E}); \mathbb{R}^\infty) \). Moreover, \((A, \mathcal{O}_E(A))\) is a metrizable and separable topological coarsening of \((A, \mathcal{O}_E(A))\).
- \( \otimes \hat{\mathcal{F}} \subset C(E; \mathbb{R}^\infty) \) is injective on \( A \). Moreover, \( \mathcal{F} \) separates points on the Hausdorff space \((A, \mathcal{O}_E(A))\).

**Proof.** (a) The members of \( \hat{\mathcal{F}} \) are continuous and \( \hat{\mathcal{F}} \) strongly separates points on \( \hat{E} \) by (3.1.2). \( \hat{\mathcal{F}} \) is countable and contains 1 by (3.1.1) and 1 \( \in \mathcal{F} \). Moreover, (3.1.3) follows by Lemma 9.3.4 (a, c) (with \( E = \hat{E} \) and \( D = \hat{\mathcal{F}} \)).

(b) \( \mathbb{R}^\infty \) is a Polish space by Proposition 9.1.11 (f). It follows by the Tychonoff Theorem (Proposition 9.1.12 (b)) and Proposition 9.1.12 (a) that

\[
K_F = \prod_{f \in \mathcal{F}} [-\|f\|_{\infty}, \|f\|_{\infty}] \subset \mathcal{C}(\mathbb{R}^\infty) \subset \mathcal{O}(\mathbb{R}^\infty).
\]

\[\text{Remark 3.1.2.} \quad \text{In general, } E \text{ need not be a subset of } \hat{E}.\]

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\[\text{Remark 3.1.2.} \quad \text{In general, } E \text{ need not be a subset of } \hat{E}.\]
\( \bigotimes \hat{F}(E) \) is the closure of \( \bigotimes F(E_0) \) in \( \mathbb{R}^\infty \) by definition. So,
\[
(3.1.5) \quad \bigotimes \hat{F}(E) \in \mathcal{C}(K_{\mathcal{F}}; \mathcal{O}_{\mathbb{R}^\infty}(K_{\mathcal{F}})) \subset \mathcal{K}(\mathbb{R}^\infty) \subset \mathcal{O}(\mathbb{R}^\infty)
\]
and \( \hat{E} \) is compact by (3.1.4), (3.1.3) and Proposition 9.1.12 (a, e).
Moreover, it follows by (a) and (3.1.1) that
\[
(3.1.6) \quad \mathcal{O}_{\hat{E}}(E_0) = \mathcal{O}_{\hat{F}}(E_0) = \mathcal{O}_F(E_0).
\]
\( \bigotimes F(E_0) \) is dense in \( \bigotimes \hat{F}(E) \) by definition, so \( E_0 \) is a dense subset of \( \hat{E} \) by (3.1.3). \( \hat{E} \) is a Hausdorff space by (3.1.2) and Proposition 9.2.1 (c) (with \( E = A = \hat{E} \) and \( D = \hat{F} \)). Hence, \( \hat{E} \) is a compactification of \( (E_0, \mathcal{O}_F(E_0)) \).

(c) \( \rho_{\hat{F}} \) metrizes \( \hat{E} \) by (a) and Proposition 9.2.1 (d) (with \( E = \hat{E} \) and \( D = \hat{F} \)).
\( \bigotimes F \) is an isometry (see p. 150) between \( \bigotimes \hat{F}(E), d^\infty \), where\(^4\)
\[
(3.1.7) \quad d^\infty(x, y) := \sum_{n=1}^{\infty} 2^{-(n+1)} \left( |p_n(x) - p_n(y)| \wedge 1 \right), \quad \forall x, y \in \mathbb{R}^\infty
\]
completely metrizes \( \mathbb{R}^\infty \) by Proposition 9.1.7 (b) (with \( (S_i, v_i) = \mathbb{R} \)). \( \bigotimes \hat{F}(E), d^\infty \) is complete by its compactness and Proposition 9.1.12 (c). Thus, \( \hat{E}, \rho_{\hat{F}} \) is a complete metric space by Proposition 9.1.6 (a).

(d) The first statement of (d) follows by (3.1.3) and (3.1.1), \( (A, \mathcal{O}_{\hat{E}}(A)) \) is metrizable and separable by (c) and Proposition 9.1.11 (c). Moreover, one finds by (3.1.6) and \( \mathcal{F} \subset C(E; \mathbb{R}) \) that
\[
(3.1.8) \quad \mathcal{O}_{\hat{E}}(A) = \mathcal{O}_F(A) \subset \mathcal{O}_E(A).
\]
The second part follows by Proposition 9.2.1 (e) \( (A, D = \mathcal{F}) \).

**Corollary 3.1.4.** Let \( E \) be a topological space. If \( \{(E_0, \mathcal{O}; \hat{E}_i, \hat{F}_i)\}_{i=1,2} \) are bases over \( E \), then \( \hat{E}_1 \) and \( \hat{E}_2 \) are isometric hence homeomorphic.

**Proof.** Let \( \mathcal{F} = \{f_n\}_{n \in \mathbb{N}} \). By (3.1.1) and Lemma 3.1.3 (a) (with \( E = \hat{E}_i \) and \( \hat{F} = \hat{F}_i \) ), \( \hat{F}_i \subset C(E_i; \mathbb{R}) \) can be written as \( \hat{F}_i = \{f_i^1\}_{n \in \mathbb{N}} \) for each \( i \in \{1,2\} \) such that \( f_i^1|_{E_0} = f_n|_{E_0} = f_i^2|_{E_0} \) for all \( n \in \mathbb{N} \). Then, \( (E_0, \rho_{\hat{F}_i}) \) and \( (E_0, \rho_{\hat{F}_2}) \) are identical metric spaces. Now, the corollary follows by Lemma 3.1.3 (c) (with \( E = \hat{E}_1 \) and \( \hat{F} = \hat{F}_1 \) ) and Proposition 9.1.6 (a) (with \( E = \hat{E}_1 \) and \( S = \hat{E}_2 \)). \( \Box \)

**Remark 3.1.5.** Compactification determined by extending\(^5\) continuous functions was used in e.g. [BK86], [BK93b], [BK10] and [Kou16] to imbed stochastic processes into compact metric spaces. The well-known Stone-Čech compactification (see p. 156) exists for any compactifiable, or equivalently, Tychonoff space and is determined by the continuous extension of all bounded continuous function. The compactification \( \hat{E} \) of \( (E_0, \mathcal{O}_F(E_0)) \) can be thought of as a “possibly smaller” compactification which might not extend all of \( C_b(E_0, \mathcal{O}_F(E_0); \mathbb{R}) \).

---

\(^4\) \( p_n \) denotes the one-dimensional projection on \( \mathbb{R}^\infty \) for \( n \in \mathbb{N} \).

\(^5\) Recall that mapping \( g \) is a continuous extension of mapping \( f \) if \( g \) is continuous, the domain of \( g \) contains that of \( f \) and \( g = f \) restricted to \( f \)'s domain.
REMARK 3.1.6. As a cost of metrizability, \( \hat{E} \) compactifies (see p.155) \( E_0 \) with respect to a possibly coarser topology than its natural subspace topology induced from \( E \). In fact, neither the original space \( E \) nor the subspace \( (E_0, \mathcal{O}_E(E_0)) \) is necessarily a Tychonoff space, hence need not have compactification.

REMARK 3.1.7. One-point compactifications (see p.155) exist for locally compact Hausdorff spaces (see Proposition 9.3.7). We do not presume \( E_0 \) to be a locally compact subspace of \( \hat{E} \), and \( \hat{E} \) is not necessarily a one-point compactification. Nonetheless, even Stone-Čech compactifications are sometimes one-point compactifications. Corollary 4.1.8 to follow later illustrates when the compactification establishing a base is of one-point compactification type.

The following theorem gives a partial converse of Lemma 3.1.3 (e) and answers the question when a base exists.

THEOREM 3.1.8. Let \( E \) be a topological space, \( E_0 \in \mathcal{B}(E) \) and \( \mathcal{F} \ni 1 \) be a countable subset of \( C_0(E; \mathbb{R}) \). Then, there exists a base \( (E_0, \mathcal{F}; \hat{E}, \hat{\mathcal{F}}) \) over \( E \) if and only if \( \mathcal{F} \) separates points on \( E_0 \).

PROOF. Necessity follows by Lemma 3.1.3 (e). We prove sufficiency. There exist a compactification \( \hat{E} \) of \( (E_0, \mathcal{O}_E(E_0)) \) and an extension \( \varphi \in \text{imb}(\hat{E}; \mathbb{R}^\infty) \) of \( \bigotimes \mathcal{F}|_{E_0} \) by Lemma 9.3.4 (a, b) (with \( E = (E_0, \mathcal{O}_E(E_0)) \) and \( \mathcal{D} = \mathcal{F}|_{E_0} \)). Then, \( (E_0, \mathcal{F}; \hat{E}, \hat{\mathcal{F}}) \) is base over \( E \) with \( \hat{\mathcal{F}} \doteq \{p_n \circ \varphi\}_{n \in \mathbb{N}} \). \( \square \)

3.1.2. Properties. The following four results specify basic finite-dimensional properties of bases.

LEMMA 3.1.9. Let \( E \) be a topological space, \( (E_0, \mathcal{F}; \hat{E}, \hat{\mathcal{F}}) \) be a base over \( E \), \( d \in \mathbb{N} \) and \( A \subset E_0^d \). Then:

(a) \( (E_0^d, \Pi^d(\mathcal{F}); \hat{E}^d, \Pi^d(\hat{\mathcal{F}})) \) is a base over \( E^d \).

(b) \( \Pi^d(\hat{\mathcal{F}}) \subset C(\hat{E}^d; \mathbb{R}) \) contains the constant function 1, separates points and strongly separates points on \( \hat{E}^d \). In particular,

\[
\bigotimes \Pi^d(\hat{\mathcal{F}}) \in \text{imb} \left( \hat{E}^d; \mathbb{R}^\infty \right).
\]

(c) \( \hat{E}^d \) is a compactification of \( (E_0^d, \mathcal{O}_E(E_0)^d) \). Moreover, \( \hat{E}^d \) is a Polish space and is completely metrized by \( \rho^d_E \).

(d) \( \bigotimes \Pi^d(\mathcal{F})|_A \in \text{imb}(A, \mathcal{O}_{\hat{E}^d}(A); \mathbb{R}^\infty) \). Moreover, \( (A, \mathcal{O}_{\hat{E}^d}(A)) \) is a metrizable and separable coarsening of \( (A, \mathcal{O}_E(A)) \).

(e) \( \Pi^d(\mathcal{F}\{1\}) \) separates points on the Hausdorff space \( (A, \mathcal{O}_{\hat{E}^d}(A)) \).

PROOF. (a) We verify the four properties of Definition 3.1.1 in four steps:

Step 1. We have by \( E_0 \in \mathcal{B}(E) \) and Fact 2.1.3 (a) that

\[(3.1.10) \quad E_0^d = \bigcap_{i=1}^d p_i^{-1}(E_0) \in \mathcal{B}(E)^{\otimes d}, \quad \forall d \in \mathbb{N}.
\]

\( \square \)
Step 2. We have by $1 \in \mathcal{F} \subset C_b(\mathbb{E}; \mathbb{R})$, Fact 9.1.15 (a) (with $\mathcal{D} = \mathcal{F}$) and Proposition 9.2.5 (a) (with $\mathcal{D} = \mathcal{F}$) that $\Pi^d(\mathcal{F})$ is a countable collection and $1 \in \mathfrak{a} \{ \Pi^d(\mathcal{F}) \} \subset C_b(\mathbb{E}^d, \mathcal{O}_{\mathbb{F}}(\mathbb{E}^d, \mathbb{R})) \subset C_b(\mathbb{E}^d; \mathbb{R})$.

Step 3. We have by (3.1.1) that

\begin{equation}
\bigotimes \Pi^d(\mathcal{F}) \bigg|_{E_0^d} = \bigotimes \Pi^d(\tilde{\mathcal{F}}) \bigg|_{E_0^d}.
\end{equation}

and Proposition 9.2.5 (a) (with $E = \tilde{\mathcal{E}}$ and $\mathcal{D} = \tilde{\mathcal{F}}$) imply that

\begin{equation}
\Pi^d(\tilde{\mathcal{F}}) \subset C(\mathbb{E}^d; \mathbb{R})
\end{equation}

Then, $\Pi^d(\tilde{\mathcal{F}})$ separates point on $\mathbb{E}^d$, strongly separates points on $\mathbb{E}^d$ and satisfies

\begin{equation}
\mathcal{O}_{\Pi^d(\tilde{\mathcal{F}})}(\mathbb{E}^d) = \mathcal{O}(\mathbb{E}^d) = \mathcal{O}_{\tilde{\mathcal{F}}}(\mathbb{E}^d)
\end{equation}

by Lemma 3.1.3 (a), Proposition 9.2.5 (b) (with $E = \tilde{\mathcal{E}}$ and $\mathcal{D} = \tilde{\mathcal{F}}$), (3.1.13) and Proposition 9.2.1 (b, c) (with $E = A = \tilde{\mathcal{E}}^d$ and $\mathcal{D} = \Pi^d(\tilde{\mathcal{F}})$).

Step 4. $E_0^d$ is dense in $\mathbb{E}^d$ by the denseness of $E_0$ in $\tilde{\mathcal{E}}$ and the product topology definition. $\mathbb{E}^d$ is compact by Lemma 3.1.3 (b) and the Tychonoff Theorem (see Proposition 9.1.12 (b)). $\bigotimes \Pi^d(\tilde{\mathcal{F}}) \subset C(\mathbb{E}^d; \mathbb{R}^\infty)$ by (3.1.13) and Fact 2.1.4 (b).

\begin{equation}
\bigotimes \Pi^d(\tilde{\mathcal{F}})(\mathbb{E}^d) \subset \mathcal{K}(\mathbb{R}^\infty) \subset \mathcal{C}(\mathbb{R}^\infty)
\end{equation}

by Proposition 9.1.12 (a, c). So, $\bigotimes \Pi^d(\tilde{\mathcal{F}})(\mathbb{E}^d)$ is the closure of $\bigotimes \Pi^d(\mathcal{F})(E_0^d)$ in $\mathbb{R}^\infty$ by (3.1.12) and [Mun00] Theorem 18.1 (a, b)]]

(b) follows by (a) and Lemma 3.1.3 (a).

(c) The first statement follows by (a) and Lemma 3.1.3 (b). $\tilde{\mathbb{E}}^d$ is a Polish space by (a) and Lemma 3.1.3 (c). Moreover, $\rho^d_{\tilde{\mathbb{E}}}$ metrizes $\mathbb{E}^d$ by Lemma 3.1.3 (c) and Proposition 9.1.7 (a) (with $\mathcal{I} = \{1, \ldots, d\}$ and $(\mathcal{E}_i, \tau_i) = (\tilde{\mathbb{E}}, \rho_{\tilde{\mathbb{E}}})$).

(d) follows by (a) and Lemma 3.1.3 (d).

(e) $\Pi^d(\mathcal{F}\setminus\{1\})$ separates points on $E_0^d$ by Lemma 3.1.3 (c) and Proposition 9.2.5 (b) (with $\mathcal{D} = \mathcal{F}\setminus\{1\}$). The rest of (e) follows by (a) and Lemma 3.1.3 (c).

**Corollary 3.1.10.** Let $E$ be a topological space, $(E_0, \mathcal{F}, \tilde{\mathcal{E}}, \tilde{\mathcal{F}})$ be a base over $E$ and $d \in \mathbb{N}$. Then\[9\]

\begin{equation}
C_\ast(\mathbb{E}^d; \mathbb{R}) = C_0(\mathbb{E}^d; \mathbb{R}) = C_b(\mathbb{E}^d; \mathbb{R}) = C(\tilde{\mathbb{E}}^d; \mathbb{R}) = C(\tilde{\mathbb{E}}^d; \mathbb{R}) = \bigcl \left[ a_{\mathbb{Q}} \left( \Pi^d(\tilde{\mathcal{F}}) \right) \right].
\end{equation}

and

\begin{equation}
\bigcl \left[ a_{\mathbb{Q}} \left( \Pi^d(\mathcal{F}|E_0) \right) \right] = C(\tilde{\mathbb{E}}^d; \mathbb{R}) \bigg|_{E_0^d}.
\end{equation}

**Proof.** $a_{\mathbb{Q}}(\Pi^d(\tilde{\mathcal{F}}))$ is uniformly dense\[10\] in $C(\mathbb{E}^d; \mathbb{R})$ by Lemma 3.1.9 (b, c) and the Stone-Weierstrass Theorem (see [Dud02] Theorem 2.4.11). Thus, (3.1.16) follows by Lemma 3.1.9 (c), Fact 10.2.1 (with $E = \mathbb{E}$ and $k = 1$) and 2.2.15 (with $\mathcal{D} = \Pi^d(\tilde{\mathcal{F}})$). (3.1.17) follows by (3.1.16), the denseness of $E_0^d$ in $\mathbb{E}^d$ and properties of uniform convergence.

---

\[9\] In contrast, $\mathcal{O}(\mathbb{E}^d)$ is not necessarily the same as $\mathcal{O}_{\mathbb{F}}(\mathbb{E}^d)$.

\[10\] The notation “$a(\cdot)$”, “$a_{\mathbb{Q}}(\cdot)$” and “$a_{\mathbb{Q}}(\cdot)$” were defined in 2.2.5.

\[11\] “Uniformly dense” means dense with respect to the topology induced by the supremum norm.
Corollary 3.1.11. Let $E$ be a topological space, $(E_0, \mathcal{F}; \hat{E}, \widehat{\mathcal{F}})$ be a base over $E$, $d \in \mathbb{N}$, $A \subset \hat{E}^d$ and $\mathcal{D} = \text{mc}[\Pi^d(\hat{\mathcal{F}} \setminus \{1\})]$. Then:

(a) $\mathcal{D}|_A^{13}$ separates and strongly separates points on $\mathcal{P}(A, \mathcal{E}_d(A))$.

(b) $\mathcal{D}|_A \cup \{1\}$ (especially $\text{mc}[\Pi^d(\hat{\mathcal{F}})]|_A$) is separating and convergence determining on $(A, \mathcal{E}_d(A))$.

(c) $\mathcal{M}^+(\hat{E}^d)$ and $\mathcal{P}(\hat{E}^d)$ are Polish spaces and, in particular, $\mathcal{P}(\hat{E}^d)$ is compact.

Proof. (a) follows by Lemma 3.1.3 (a) and Lemma 9.4.3 (b) (with $E = (A, \mathcal{E}_d(A))$).

(b) follows by (a), Fact 10.1.20 and the fact $(\mathcal{D} \cup \{1\}) \subset \text{mc}[\Pi^d(\hat{\mathcal{F}})]$.

(c) follows by Lemma 3.1.9 (c) and Theorem 9.4.10 (with $E = \hat{E}^d$).

Fact 3.1.12. Let $E$ be a topological space, $(E_0, \mathcal{F}; \hat{E}, \widehat{\mathcal{F}})$ be a base over $E$, $d \in \mathbb{N}$ and $A \subset \hat{E}^d$. Then, $\mathcal{B}_E(A) = \mathcal{B}(\hat{E})^{\otimes d}|_A^{14}$.

Proof. This fact follows immediately by Lemma 3.1.3 (c) and Proposition 10.2.4 (d) (with $I = \{1, \ldots, d\}$ and $S_i = \hat{E}$).

Corollary 3.1.13. Let $E$ be a topological space, $(E_0, \mathcal{F}; \hat{E}, \widehat{\mathcal{F}})$ be a base over $E$ and $d \in \mathbb{N}$. Then:

(a) Any $A \subset E_0^d$ satisfies $\mathcal{B}(E_0^d, \mathcal{E}(\mathcal{F})^d)(A) = \mathcal{B}(\hat{E})^{\otimes d}|_A^{15} = \mathcal{B}_E(A) = \mathcal{B}_{\Pi^d(\mathcal{F})}(A) = \mathcal{B}_{\mathcal{F}}(E)^{\otimes d}|_A$

In particular, $E_0^d$ satisfies

$$\mathcal{B}_E(E_0^d) = \mathcal{B}_{\mathcal{F}}(E_0^d) \subset \mathcal{B}_E(E_0^d)^{\otimes d} \subset \mathcal{B}(E)^{\otimes d} \subset \mathcal{B}(E^d)$$

(b) $\mathcal{F}$ satisfies

$$\mathcal{C}(\Pi^d(\mathcal{F})) \subset \mathcal{M}_b(E^d, \mathcal{B}_{\mathcal{F}}(E)^{\otimes d}; \mathbb{R}) \quad \text{and} \quad \mathcal{C}(\Pi^d(\mathcal{F})) \subset \mathcal{M}_b(E^d, \mathcal{B}(E)^{\otimes d}; \mathbb{R}) \subset \mathcal{M}_b(E^d; \mathbb{R}).$$

Proof. (a) We have that

$$\mathcal{B}(E_0^d, \mathcal{E}(\mathcal{F})^d)(A) = \mathcal{B}(E_0^d, \mathcal{E}(\mathcal{F})^d)(A)$$

$$= \mathcal{B}(E_0^d, \mathcal{E}(\mathcal{F})^d)(A) = \mathcal{B}_E(A) = \mathcal{B}_E(A) = \mathcal{B}_{\Pi^d(\mathcal{F})}(A) = \mathcal{B}_{\mathcal{F}}(E)^{\otimes d}|_A$$

by 3.1.6, Fact 3.1.12 and the fact $A \subset E_0^d \subset (E^d \cap \hat{E}^d)$. Then, the first line of (3.1.18) follows by (3.1.21), (3.1.12) and Lemma 3.1.9 (b). The second line of (3.1.18) follows by $\mathcal{F} \subset C(E; \mathbb{R})$ and Lemma 10.2.4 (a). Now, (a) follows by (3.1.10).

---

12 The notation “mc(·)” was defined in § 2.2.3.
13 The notation “$\mathcal{D}^*$” and the terminologies “separating”, “convergence determining” were defined in § 2.3.
14 The notations “$\mathcal{B}(\hat{E})^{\otimes d}$” and “$\mathcal{B}(E)^{\otimes d}$” were defined in § 2.1.5.
15 The notation “$\mathcal{B}_{\mathcal{F}}(E)$” was defined in § 2.1.3.
Hereafter, we frequently use the latter notation.

\[ A \in \mathcal{A} \]

Then,

\[ B \]

and

\[ (3.1.23) \]

injective.

\[ f \]

\[ B \]

and by Corollary 3.1.13 (a) (with \( d = N \)) and 3.1.19.

\[ \mathcal{B}_{\hat{E}}(A) = \mathcal{B}_{\hat{E}}(A) \text{ and } A \in \mathcal{B}(\hat{E}^d). \]

(b) follows by \( \mathcal{F} \subset C_b(E; \mathbf{R}) \), Proposition 9.2.5 (a) (with \( D = \mathcal{F} \)) and 3.1.19.

\[ E_0 \]

is not necessarily a Borel subset of \( \hat{E} \) and \( \hat{E} \) does not endow \( E_0 \) with the same Borel sets as \( E \) unless \( E_0 \) is a standard Borel subset of \( E \).

**Lemma 3.1.14.** Let \( E \) be a topological space, \((E_0, \mathcal{F}; \hat{E}, \hat{\mathcal{F}})\) be a base over \( E \), \( d \in \mathbf{N} \) and \( A \subset E_0^d \). Then:

(a) \( A \in \mathcal{B}^*(E^d) \) if and only if \( \hat{E} \)

\[ (3.1.22) \]

(b) \( \mathcal{B}_{\hat{E}}(A) = \mathcal{B}_{\hat{E}}(A) \text{ and } A \in \mathcal{B}(\hat{E}^d). \)

(c) \( A \subset \mathcal{B}^*(E^d) \)

\[ (3.1.23) \]

Proposition (a - Necessity) Let \( f \) be the identity map on \( A \), which is certainly injective. \( f \in \mathcal{M}(A, \mathcal{B}_{E}(A); \hat{E}^d) \) by 3.1.18. \( A \in \mathcal{B}^*(E^d) \) and \( A \subset E_0^d \) imply \( A \in \mathcal{B}^*(A, \mathcal{O}_{E}(A)) \). \( \hat{E}^d \) is a Polish space by Lemma 3.1.9 (c) so \( A \in \mathcal{B}^*(\hat{E}^d) \) and \( \mathcal{B}_{\hat{E}}(A) = \mathcal{B}_{\hat{E}}(A) \) by Proposition 9.5.9 with \( E = A \) and \( S = \hat{E}^d \). Then, \( A \in \mathcal{B}(\hat{E}^d) = \mathcal{B}(\hat{E}^d) \) by Proposition 9.5.8 (b) (with \( E = \hat{E}^d \)).

(a - Sufficiency) follows by 3.1.22 and Fact 9.5.1 (a).

(b) \( A \in \mathcal{B}_{\hat{E}}(E_0^d) \) by the fact \( A \subset E_0^d \) and 3.1.22. \( \mathcal{B}_{\hat{E}}(A) \subset \mathcal{B}(\hat{E}^d) \) by 3.1.19. Now, (b) follows by (a) and 3.1.18.

(c) We find by (a) (with \( A = A_n \)) that

\[ (3.1.25) \]

Then, \( A \) satisfies 3.1.22 by 3.1.25, Fact 10.1.1 (with \( E = A \), \( \mathcal{U}_1 = \mathcal{B}_{\hat{E}}(A) \) and \( \mathcal{U}_2 = \mathcal{B}_{\hat{E}}(A) \)) and 3.1.18. Hence, \( A \in \mathcal{B}^*(\hat{E}^d) \) by (a).

(d) \( A \subset E_0^d \) implies \( A_n = p_n(A) \subset E_0 \) for all \( 1 \leq n \leq d \). We have that

\[ (3.1.26) \]

by (a, b) (with \( d = 1 \) and \( A = A_n \)). It then follows by 3.1.26 that

\[ A = \bigcap_{n=1}^{d} p_n^{-1}(A_n) \subset \mathcal{B}(E) \cap \mathcal{B}(\hat{E}), \]

and by Corollary 3.1.13 (a) (with \( d = 1 \) and \( A = A_n \)) that

\[ (3.1.28) \]

\[ \mathcal{B}_{\hat{E}}(A) = \mathcal{B}_{\hat{E}^d} \bigg|_A = \bigotimes_{n=1}^d \mathcal{B}_{\hat{E}}(A_n) = \bigotimes_{n=1}^d \mathcal{B}_{\hat{E}}(A_n) = \mathcal{B}(E)^{\otimes d} \bigg|_A. \]

\[ \mathcal{B}^*(E^d) \]

denotes the family of all standard Borel subsets of \( E^d \).

\[ \mathcal{B}_{\hat{E}^d}(A) = \mathcal{B}_{\hat{E}^d}(A) \text{ plus } A \in \mathcal{B}(E^d) \text{ is equivalent to } \mathcal{B}_{\hat{E}^d}(A) = \mathcal{B}_{\hat{E}^d}(A) \subset \mathcal{B}(\hat{E}^d). \]

Hereafter, we frequently use the latter notation.
COROLLARY 3.1.15. Let $E$ be a topological space, $(E_0, \mathcal{F}; \hat{E}, \hat{\mathcal{F}})$ be a base over $E$, $d \in \mathbb{N}$ and $A \subset E_0^d$. Then:

(a) If $A \in \mathcal{K}(E_0^d, \mathcal{O}_E(E_0)^d)$, then

$$A \subset \mathcal{K}(E_0^d) \cap \mathcal{K}(E_0^d) \cap \mathcal{O}_E(E_0)^d).$$

(b) If $A \in \mathcal{K}(E_0^d, \mathcal{O}_E(E_0)^d)$, then

$$A \in \mathcal{K}(E_0^d) \cap \mathcal{K}(E_0^d) \cap \mathcal{O}_E(E_0)^d).$$

PROOF. (a) follows by Lemma 3.1.13 (b, c, e) and Fact 10.2.9 (b) (with $E = E^d$, $\mathcal{D} = \Pi^d(\mathcal{F})$ and $K = \hat{A}$).

(b) $A \in \mathcal{K}(E_0^d, \mathcal{O}_E(E_0)^d)$ by (3.1.30). $\mathcal{K}(E_0^d, \mathcal{O}_E(E_0)^d) \subset \mathcal{K}(E_0^d)$ by (a) and Lemma 3.1.14 (a). $A \in \mathcal{K}(E_0^d)$ by Lemma 3.1.14 (c). Moreover, $A \in \mathcal{O}(E)\hat{\mathcal{K}}d$ by Lemma 3.1.14 (b).

Note 3.1.16. Given a base $(E_0, \mathcal{F}; \hat{E}, \hat{\mathcal{F}})$ over $E$ and $d \in \mathbb{N}$, Corollary 3.1.15 (b) shows that $\mathcal{K}(E_0^d, \mathcal{O}_E(E_0)^d)$ lies in the domain of any $\mu \in \mathcal{M}^+(E^d, \mathcal{O}(E)\hat{\mathcal{K}}d)$.

COROLLARY 3.1.17. Let $E$ be a topological space, $(E_0, \mathcal{F}; \hat{E}, \hat{\mathcal{F}})$ be a base over $E$, $d \in \mathbb{N}$ and $\mu \in \mathcal{M}^+(E^d, \mathcal{O}(E)\hat{\mathcal{K}}d)$.

(a) If $\mu$ is supported on $A \subset E_0^d$ and $A \in \mathcal{K}(E_0^d)$, then $\mathcal{B}(\mu)$ is a singleton.

(b) If $\mu$ is tight in $(E_0^d, \mathcal{O}_E(E_0)^d)$, then there exists a $\mu' = \mathcal{B}(\mu)$ which is tight in $(E_0^d, \mathcal{O}_E(E_0)^d)$.

Note 3.1.18. Any $\mu' \in \mathcal{B}(\mu)$ is not an expansion of $\mu$ to a superspace but rather an extension of $\mu$ to all Borel sets. Any support of $\mu$ is also that of $\mu'$.

PROOF of COROLLARY 3.1.17 (a) One finds by Lemma 3.1.14 (b) that $A \in \mathcal{B}(E)\hat{\mathcal{K}}d$ and $\mathcal{B}_E(A) = \mathcal{B}(E)\hat{\mathcal{K}}d|_A$. Hence, (a) follows by Lemma 10.2.6 (c) (with $I = \{1, \ldots, d\}$, $S_1 = E$, $S_2 = E$ and $\mathcal{A} = \mathcal{B}(E)\hat{\mathcal{K}}d$).

(b) $\mu$ is supported on some $A \in \mathcal{K}(E_0^d, \mathcal{O}_E(E_0)^d)$ by its tightness. $A \in \mathcal{B}(E^d)$ by Corollary 3.1.15 (b). Hence, (b) follows by (a).

The standard Borel property of $A \subset E_0^d$ also yields useful properties of the weak topological space $\mathcal{M}^+(A, \mathcal{O}_E(E_0)(A))$.

COROLLARY 3.1.19. Let $E$ be a topological space, $(E_0, \mathcal{F}; \hat{E}, \hat{\mathcal{F}})$ be a base over $E$, $d \in \mathbb{N}$, $A \subset E_0^d$ with $A \subset E_0^d$ and $\mathcal{D} = \mathcal{M}^d(\mathcal{F}\setminus\{1\})$. Then:

(a) $\mathcal{D}^*_A$ separates points on $\mathcal{P}(A, \mathcal{O}_E(E_0)(A))$. Moreover, $\mathcal{D}^*_A \cup \{1^*_A\}$ (especially $\mathcal{M}^d(\mathcal{F}|\{1\})$ is separating on $(A, \mathcal{O}_E(E_0)(A))$.

(b) $\mathcal{M}^+(A, \mathcal{O}_E(E_0)(A))$ are Tychonoff spaces.

PROOF. (a) $\mathcal{D}^*_A$ separates points on $\mathcal{P}(A, \mathcal{O}_E(E_0)(A)) = \mathcal{P}(A, \mathcal{O}_E(E_0)(A))$ by (3.1.1), Corollary 3.1.11 (a) and Lemma 3.1.14 (a). Now, (a) follows by Fact 10.1.20 (a) and the fact $(\mathcal{D} \cup \{1\}) \subset \mathcal{M}^d(\mathcal{F}|\{1\})$.

(b) follows by (a) and Proposition 9.4.1 (a, c) (with $E = (A, \mathcal{O}_E(E_0)(A))$).

\footnote{The notations “$\mathcal{C}(\cdot)$”, “$\mathcal{K}(\cdot)$” and “$\mathcal{H}(\cdot)$”, defined in 2.1.3, denote the families of closed, compact and metrizable compact subsets, respectively.}

\footnote{Support of measure was specified in 2.1.2.}

\footnote{$\mathcal{B}(\mu)$ as defined in 2.3 denotes the family of all Borel extensions of $\mu$.}
3.2. Baseable space

Theorem 3.1.8 shows establishing a base is equivalent to separating points of a subset of \( E \) by countably many bounded continuous functions. Boundedness is unnecessary.

**Lemma 3.2.1.** Let \( E \) be a topological space, \( E_0 \in \mathcal{B}(E) \) and \( D \subset C(E; \mathbb{R}) \) be countable and separate points on \( E \). Then, there exists a base \((E_0, F; \hat{E}, \hat{F})\) over \( E \) satisfying the following properties:

(a) \( \theta_D(E) \subset \theta_F(E) \).

(b) \( (D \cap C_b(E; \mathbb{R})) \subset F \).

(c) \( F \) can be taken to equal \( D \cup \{1\} \) whenever \( D \subset C_b(E; \mathbb{R}) \).

**Proof.** \( \{\langle f \wedge n \rangle \vee (-n)\}_{n \in \mathbb{N}, f \in D \cup \{1\}} \subset C_b(E; \mathbb{R}) \) so by Lemma 10.2.10 (with \( G = D \cup \{1\} \) and \( H = C_b(E; \mathbb{R}) \)), there exists a countable \( F \subset C_b(E; \mathbb{R}) \) that satisfies (a) - (c). \((E_0, \theta_D(E_0))\) is a Hausdorff space by Proposition 9.2.1 (c) (with \( A = E_0 \)), hence \((E_0, \theta_F(E_0))\) is also by (a) and Fact 9.1.1 \( F \) separates points by Proposition 9.2.1 (c) (with \( A = E_0 \) and \( D = F \)). The result follows by Theorem 3.1.8. \( \Box \)

We use “baseable” and “baseability” to describe the ability to create bases.

**Definition 3.2.2.**

- \( E \) is a **\( D \)-baseable space** if \( D \subset C(E; \mathbb{R}) \) has a countable subset separating points on \( E \).
- \( A \) is a **\( D \)-baseable subset of \( E \)** if \( A \in \mathcal{B}(E) \), \( D \subset C(E; \mathbb{R}) \) and \( (A, \theta_E(A)) \) is a \( D|_A \)-baseable space.
- \( E \) is a **baseable space** if \( E \) is a \( C(E; \mathbb{R}) \)-baseable space.
- \( A \) is a **baseable subset of \( E \)** if \( A \) is a \( C(E; \mathbb{R}) \)-baseable subset.

**Remark 3.2.3.** “being a baseable subset” equals “being a baseable subspace” plus Borel measurability. Moreover, the “\( D|_A \)-baseable space” above is a proper statement since

\[
D|_A \subset C(E; \mathbb{R})|_A \subset C(A, \theta_E(A); \mathbb{R}).
\]

Baseable spaces or their analogues have appeared in many works such as [EK86, Chapter 3], [Jak86], [KO88], [Jak97], [Bog07, Chapter 6], [BK10], [KS17] and [Kou16] etc. Herein, we are the first to characterize baseable topological spaces. The next section will treat baseable subsets.

**3.2.1. Characterization.** Baseable spaces are a broad category of spaces.

**Theorem 3.2.4.** Baseable spaces are precisely the topological refinements of metrizable and separable spaces.

The theorem above follows by two straightforward observations. First, we note that baseable spaces sit between Hausdorff spaces and separable metric spaces.

**Fact 3.2.5.** The following statements are true:

(a) If \( E \) is a baseable space, then \( E \) is a Hausdorff space.
(b) If $E$ is a metrizable and separable space, then $E$ is a $\mathcal{D}$-baseable space for some countable $\mathcal{D} \subset C_b(E; \mathbb{R})$ that strongly separates points on $E$.

**Proof.** (a) follows by Proposition 9.2.1(e) (with $A = E$).
(b) follows by Corollary 9.3.6(a, b). \qed

**Corollary 3.2.6.** Metrizable Souslin spaces, metrizable Lusin spaces and metrizable standard Borel spaces are all baseable spaces.

**Proof.** Souslin and Lusin spaces are separable by Proposition 9.1.11(d).
Metrizable standard Borel spaces are Lusin spaces by Proposition 9.5.6(a, b).
Now, the result follows by Fact 3.2.5(b). \qed

Refining the topology of a metrizable and separable space $E$ may cause a function class $\mathcal{D} \subset C(E; \mathbb{R})$ to forfeit strongly separation of points. However, it does not affect the $\mathcal{D}$-baseability of $E$.

**Fact 3.2.7.** If $E$ is $\mathcal{D}$-baseable, then any topological refinement of $E$ is also.

**Proof.** Note that $\mathcal{D} \subset C(E; \mathbb{R}) \subset C(E, \mathcal{U}; \mathbb{R})$ if $\mathcal{U}$ refines $\mathcal{O}(E)$. \qed

**Proof of Theorem 3.2.4.** (Necessity) If $E$ is a baseable space and $\mathcal{D} \subset C(E; \mathbb{R})$ is countable and separate points on $E$, then $(E, \mathcal{O}_\mathcal{D}(E))$ coarsens $E$ and is a metrizable and separable space by Proposition 9.2.1(d).
(Sufficiency) follows by Fact 3.2.5(b) and Fact 3.2.7 \qed

**3.2.2. Examples of baseable spaces.** The following figure illustrates the relationship of baseable spaces and other major categories of topological spaces.
3.2. BASEABLE SPACE

Topological refinements of metrizable and separable spaces, i.e. so baseable spaces range from Polish spaces to even non-$T_3^{21}$ (see p.149) Hausdorff spaces as in the following examples.

**Example 3.2.8.**

(I) **Baseable, metrizable, non-Polish Lusin space**: Example 2.7.1 mentioned that the pseudo-path topological space $D^{pp}(\mathbb{R}^+; \mathbb{R})$ is a metrizable but non-Polish Lusin space. Lusin spaces are separable by Proposition 9.1.11 (d). Hence, $D^{pp}(\mathbb{R}^+; \mathbb{R})$ is a baseable space.

(II) **Baseable, non-separable Banach space - I**: Let $l^\infty$ be the space of all bounded $\mathbb{R}$-valued sequences equipped with the supremum norm, i.e.

\[ l^\infty \doteq \left\{ x \in \mathbb{R}^\infty : \|x\|_\infty = \sup_{n \in \mathbb{N}} |x_n| < \infty \right\}. \]

$(l^\infty, \mathcal{O}_R(l^\infty))$ is metrizable and separable by Proposition 9.1.11 (c, f) $(l^\infty, \|\cdot\|_\infty)$ is a Banach refinement of $(l^\infty, \mathcal{O}_R(l^\infty))$ by Mun00 Theorem 43.5 and Theorem 20.4, so $(l^\infty, \|\cdot\|_\infty)$ is a baseable space. However, $(l^\infty, \|\cdot\|_\infty)$ is non-separable by BN12 Example 6.6, p.83.

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$^{21}$Herein, we use the terminologies “$T_3$” and “$T_4$” instead of “regular” and “normal” since the latter sometimes are used in a non-Hausdorff context.
(III) **Baseable, non-separable Banach space - 2**: Consider the non-separable Banach spaces mentioned in Examples 2.7.5 and 2.7.6. [FV10, Corollary 7.50] showed that $G^N(R^d)$, the free nilpotent group of Step $N$ over $R^d$ is a separable Banach space with the Carnot-Caratheodory norm $\| \cdot \|_{cc}$. $C([0,T]; R^d)$ with $\| \cdot \|_\infty$ and $C([0,T]; G^N(R^d))$ equipped with the supremum $cc$-norm

\[ \| x \|_{\infty,cc} = \sup_{t \in [0,T]} \| x(t) \|_{cc} \]

are Polish spaces by [Sri98, Theorem 2.4.3]. Then, the spaces in (2.7.14) and (2.7.15) equipped with $\| \cdot \|_\infty$, and those in (2.7.20) and (2.7.21) equipped with $\| \cdot \|_{\infty,cc}$ are all metrizable and separable spaces by Proposition 9.1.11 (c). It is known that the norms in (2.7.12) and (2.7.13) both induce finer topologies than $\| \cdot \|_\infty$, while the norms in (2.7.18) and (2.7.19) both induce finer topologies than $\| \cdot \|_{\infty,cc}$ (see [BF13, p.262 and Remark 3.6]). Thus, all the non-separable Banach spaces in Example 2.7.5 and Example 2.7.6 are baseable spaces.

(IV) **A baseable, non-second-countable (see p.150), separable, Lindel"of and non-metrizable T4 space**: The Sorgenfrey Line $R_l$ refers to the space of all real numbers equipped with the lower limit topology, which is generated by the topological basis

\[ \{(a,b) : a,b \in R, a < b \} . \]

$R_l$ as a topological refinement of $R$ is a baseable space. $R_l$ is separable, Lindel"of but not second-countable by [Mun00, §30, Example 3]. So, $R_l$ is non-metrizable by Proposition 9.1.4 (c). Nonetheless, $R_l$ is a T4 space (see p.149) by [Mun00, §31, Example 2].

(V) **A baseable, non-Lindel"of, separable and non-T4 Tychonoff space**: The Sorgenfrey Plane $R^2_l$ is a topological refinement of $R^2$ hence baseable. Since $R_l$ is a separable Tychonoff space, $R^2_l$ is also by Proposition 9.1.3 (c) and Proposition 9.3.2 (c). However, $R^2_l$ is neither a Lindel"of space nor a T4 space by [Mun00, §30, Example 4 and §31, Example 3].

(VI) **A baseable, non-separable and non-metrizable Tychonoff space**: When $E$ is a Polish space, $\mathcal{P}(E)$ is also by Theorem 9.4.10 (b). Example 2.7.3 explained that the strong topological space $\mathcal{P}_S(E)$ of all Borel probability measures on $E$ is a non-metrizable, non-separable, Tychonoff refinement of $\mathcal{P}(E)$, so $\mathcal{P}_S(E)$ is a baseable space.

(VII) **A baseable, second-countable and non-T3 space**: Let $R_K$ denote the space of all real numbers equipped with the $K$-topology which is generated by the countable topological basis

\[ \{(a,b) : a,b \in Q, a < b \} \cup \{ (a,b) \setminus \{1/n\}_{n \in N} : a,b \in Q, a < b \} . \]

So, $R_K$ is a second-countable topological refinement of $R$ and hence is baseable. However, [Mun00, §31, Example 1] explained that $R_K$ is not a T3 space, nor is it a Tychonoff space by Proposition 9.3.2 (a).

A baseable space has no more points than $R$ since its points are distinguished by a countable function class.
3.3. BASEABLE SUBSET

FACT 3.2.9. The cardinality of a baseable space is no greater than $\aleph(\mathbb{R})$.

PROOF. The cardinality of a metrizable and separable space never exceeds $\aleph(\mathbb{R}^\infty) = \aleph(\mathbb{R})$ by Corollary 9.3.6 (a, c), nor can their topological refinements. □

In general, Tychonoff spaces, metrizable spaces or separable spaces are not necessarily “small” enough to be baseable spaces.

EXAMPLE 3.2.10. $\mathbb{R}^{[0,1]}$ equipped with the product topology is a Tychonoff space by Proposition 9.3.2 (c) and is separable by [Mun00, §30, Exercise 16 (a)]. $(\mathbb{R}^{[0,1]}, \|\cdot\|_\infty)$ is a Banach space by [Mun00, Theorem 43.5]. However, $\mathbb{R}^{[0,1]}$ is not baseable with any topology since its cardinality is strictly greater than $\aleph(\mathbb{R})$.

It is worth mentioning that some of the baseable spaces in Example 3.2.8 are also examples of non-Polish, non-separable or non-metrizable standard Borel spaces.

EXAMPLE 3.2.11. Every metrizable Lusin space is a standard Borel space by Proposition 9.5.6. Indeed, the pseudo-path topological space $\mathcal{D}^p(\mathbb{R}^+; \mathbb{R})$ is an example of a baseable, non-Polish, metrizable, separable, standard Borel space by Example 3.2.8 (I).

By Proposition 9.5.5 (a, d), a topological space is standard Borel if its Borel $\sigma$-algebra can be generated by some Polish topology. The following are examples:

EXAMPLE 3.2.12.

(I) The Sorgenfrey Line $\mathbb{R}_l$ is a baseable, separable, non-metrizable, standard Borel space. Example 3.2.8 (IV) notes $\mathcal{B}(\mathbb{R}_l) \supset \mathcal{B}(\mathbb{R})$. By definition, $\mathbb{R}_l$ is a baseable space.

(II) For a Polish space $E$, the strong topological space $\mathcal{P}_S(E)$ has the same Borel $\sigma$-algebra as the Polish space $\mathcal{P}(E)$ by Lemma 9.4.13. According to Example 3.2.8 (VI), $\mathcal{P}_S(E)$ is a baseable, non-separable, non-metrizable, standard Borel space.

(III) The $K$-topological space $\mathbb{R}_K$ is a baseable, second-countable, non-T3, standard Borel space. By Example 3.2.8 (VII), $\mathcal{B}(\mathbb{R}_K) \supset \mathcal{B}(\mathbb{R})$. By the definition of $\mathbb{R}_K$ and [Mun00, Lemma 13.1], any $O \in \mathcal{O}(\mathbb{R}_l)$ satisfies

$$O = \bigcup_{i \in I}[a_i, b_i] \in \mathcal{B}(\mathbb{R})$$

for some $\{(a_i)_{i \in I} \cup \{b_i\}_{i \in I}\} \subset \mathbb{R}$, thus proving $\mathcal{B}(\mathbb{R}_l) \subset \mathcal{B}(\mathbb{R})$.

(III) For a Polish space $E$, the strong topological space $\mathcal{P}_S(E)$ has the same Borel $\sigma$-algebra as the Polish space $\mathcal{P}(E)$ by Lemma 9.4.13. According to Example 3.2.8 (VI), $\mathcal{P}_S(E)$ is a baseable, non-separable, non-metrizable, standard Borel space.

(III) The $K$-topological space $\mathbb{R}_K$ is a baseable, second-countable, non-T3, standard Borel space. By Example 3.2.8 (VII), $\mathcal{B}(\mathbb{R}_K) \supset \mathcal{B}(\mathbb{R})$. By the definition of $\mathbb{R}_K$ and [Mun00, Lemma 13.1], any $O \in \mathcal{O}(\mathbb{R}_K)$ satisfies

$$O = \left(\bigcup_{i \in I}[a_i, b_i]\right) \setminus \{1/n\} \in \mathcal{B}(\mathbb{R})$$

for some $\{(a_i)_{i \in I} \cup \{b_i\}_{i \in I}\} \subset \mathbb{Q}$, thus proving $\mathcal{B}(\mathbb{R}_K) \subset \mathcal{B}(\mathbb{R})$.

REMARK 3.2.13. The baseable but non-separable Banach spaces in Example 3.2.8 (II, III) can not be standard Borel, since Lemma 9.5.7 (b) shows metrizable standard Borel spaces must be separable. The lack of the standard-Borel property increases the complexity of random rough paths and their distributions.

3.3. Baseable subset

When a topological space is not necessarily baseable, its baseable subsets are often used as “blocks” for building baseable subspaces.
3.3.1. Properties. The following three facts describe baseable subsets.

**Fact 3.3.1.** Let $E$ be a topological space and $A \subset E$. Consider the statements:

(a) $A$ is a baseable subset of $E$.
(b) There exists a base $(A, \mathcal{F} ; \tilde{E}, \tilde{F})$ over $E$.
(c) $A$ is a $C_b(E; \mathbb{R})$-baseable subset of $E$.
(d) $(A, \mathcal{O}_E(A))$ is a baseable space.
(e) There exists a base $(A, \mathcal{F} ; \tilde{E}, \tilde{F})$ over $(A, \mathcal{O}_E(A))$.

Then, (a) - (c) are equivalent, so are (d) and (e). Moreover, (a) implies (d).

**Proof.** $(a) \to (b)$ follows by Lemma 3.2.1 (with $E_0 = A$). $(d) \to (e)$ follows by (a, b) (with $E = (A, \mathcal{O}_E(A))$). The other parts are immediate by definition. □

**Fact 3.3.2.** Let $E$ be a topological space, $A \subset E$ and $D \subset C(E; \mathbb{R})$. Consider the statements:

(a) $(A, \mathcal{O}_E(A))$ is a $D|A$-baseable space.
(b) $A$ is a $D$-baseable subset of $E$.
(c) $A$ is a $D_0$-baseable subset of $E$ for some countable $D_0 \subset D$.
(d) $A$ is a $D'$-baseable subset of $E$ for any $D' \subset C(E; \mathbb{R})$ that includes $D$.
(e) Any $B \in \mathcal{B}_E(A)$ is a $D$-baseable subset of any topological refinement of $E$.

Then, (b) - (e) are equivalent and any of them implies (a). Moreover, if $A \in \mathcal{B}(E)$, then (a) - (c) are all equivalent.

**Proof.** To show $(b) \to (e)$, we note that if $(E, \mathcal{U})$ is a topological refinement of $E$, then $A \in \mathcal{B}(E) \subset \sigma(\mathcal{U})$, $\mathcal{B}_E(A) \subset \mathcal{B}(E, \mathcal{U})$ and $C(E; \mathbb{R}) \subset C(E, \mathcal{U}; \mathbb{R})$. The other implications follow from definition. □

**Fact 3.3.3.** Let $E$ be a topological space. Then, the following statements are equivalent:

(a) $E$ is a $D$-baseable space (resp. baseable space).
(b) Every $A \subset E$ is a $D|A$-baseable subspace (resp. baseable subspace).
(c) All members of $\mathcal{B}(E)$ are $D$-baseable subsets (resp. baseable subsets) of any topological refinement of $E$.

**Proof.** $(a) \to (b)$ follows by 3.2.1. $(b) \to (c)$ follows by Fact 3.3.2(a, e) (with $A = E$ and $D = D$ or $C(E; \mathbb{R})$). $(c) \to (a)$ is automatic. □

Next, we describe countable unions and products of baseable subsets.

**Fact 3.3.4.** Let $E$ be a topological space and $A_n$ be a $D_n$-baseable subset of $E$ for each $n \in \mathbb{N}$. If $\{A_n\}_n \subset \mathbb{N}$ is nested (i.e. any $A_{n_1}$ and $A_{n_2}$ admit a common superset $A_{n_3}$), then $\bigcup_{n \in \mathbb{N}} A_n$ is a $\bigcup_{n \in \mathbb{N}} D_n$-baseable subset of $E$.

**Proof.** This result is immediate by Fact 10.1.17. □

**Proposition 3.3.5.** Let $A_n$ be a $D_n$-baseable subset of topological space $S_n$ for each $n \in \mathbb{N}$. Then:
(a) $\prod_{n=1}^{m} A_n$ is a $D^m$-baseable subset of $\prod_{n=1}^{m} S_n$ with $D^m \doteq \{ f \circ p_n : 1 \leq n \leq m, f \in \mathcal{D}_n \}$ for all $m \in \mathbb{N}$.

(b) $\prod_{n \in \mathbb{N}} A_n$ is a $\bigcup_{n \in \mathbb{N}} D^m$-baseable subset of $\prod_{n \in \mathbb{N}} S_n$.

**Proof.** (a) $\prod_{n=1}^{m} A_n = \bigcap_{n=1}^{m} p_n^{-1}(A_n) \in \mathcal{B}(E)^{\otimes m} \subset \mathcal{B}(E^m)$ by Proposition 10.2.4 (a). Let $\{f_{n,k} : k \in \mathbb{N}\} \subset \mathcal{D}_n$ separate points on $A_n$ for each $n \in \mathbb{N}$ and $D_n^m \doteq \{ f_{n,k} \circ p_n \}_{1 \leq n \leq m, k \in \mathbb{N}}$. Then, $\bigotimes D_n^m(x) = \bigotimes D_n^m(y)$ in $\mathbb{R}^{D_n^m}$ implies $x = \bigotimes_{n=1}^{m} p_n(x) = \bigotimes_{n=1}^{m} p_n(y) = y$ in $\prod_{n=1}^{m} S_n$. Thus, $D_n^m$ is a countable subset of $D^m$ that separates points on $\prod_{n=1}^{m} A_n$.

(b) follows by an argument similar to (a).

**Corollary 3.3.6.** Let $E$ be a topological space, $\{A_n\}_{n \in \mathbb{N}}$ be $\mathcal{D}$-baseable subsets of $E$ and $d \in \mathbb{N}$. Then, $\prod_{n=1}^{d} A_n$ is a $\Pi^d(\mathcal{D})$-baseable subset of $E^d$.

**Proof.** This corollary follows by Proposition 3.3.5 (a) (with $m = d$ and $\mathcal{D}_n = \mathcal{D}$), the definition of $\Pi^d(\mathcal{D})$ and Fact 3.3.2 (b, d).

### 3.3.2. Selection of point-separating functions.

When using $\mathcal{D}$-baseable subsets to construct a base, one can include desired, up to countably many bounded members of $\mathcal{D}$ within the base. This is useful in applications because one may include a desirable set of functions such as subdomains of operators, observation functions of nonlinear filters and test functions for measure-valued processes, etc.

**Lemma 3.3.7.** Let $E$ be a topological space, $E_0$ be a $\mathcal{D}$-baseable subset of $E$ and $\mathcal{D}_0 \subset C_b(E; \mathbb{R})$ be countable. Then:

(a) There exists a base $(E_0, \mathcal{F}; \hat{E}, \hat{\mathcal{F}})$ over $E$ with $\mathcal{D}_0 \subset \mathcal{F}$.

(b) If $\mathcal{D}$ is countable, then the $\mathcal{F}$ in (a) can be taken to contain $\mathcal{D} \cap C_b(E; \mathbb{R})$.

If, in addition, $\mathcal{D} \subset C_b(E; \mathbb{R})$, then $\mathcal{F}$ can be taken to equal $\mathcal{D} \cup \{1\}$.

(c) If $\mathcal{D}_0 \subset \mathcal{D} \subset C_b(E; \mathbb{R})$, then the $\mathcal{F}$ in (a) can be taken within $\mathcal{D} \cup \{1\}$.

If, in addition, $\mathcal{D}$ is closed under addition or multiplication, then $\mathcal{F}$ can be taken to have the same closedness.

**Proof.** Let $\mathcal{D}' \subset \mathcal{D}$ be countable and separate points on $E_0$. Then, (a) and (b) follow by Lemma 3.2.1 (with $\mathcal{D} = \mathcal{D}' \cup \mathcal{D}_0$).

For (c), we define

$$3.3.1 \quad \mathcal{D}' \doteq \begin{cases} 
D_0 \cup \mathcal{D}' \cup \{1\}, & \text{in general,} \\
\mathcal{ac}(D_0 \cup \mathcal{D}' \cup \{1\}), & \text{if } \mathcal{D} \ni 1 \text{ is closed under addition,} \\
\mathcal{mc}(D_0 \cup \mathcal{D}' \cup \{1\}), & \text{if } \mathcal{D} \ni 1 \text{ is closed under multiplication,} \\
\mathcal{ac}(\mathcal{mc}(D_0 \cup \mathcal{D}' \cup \{1\})), & \text{if } \mathcal{D} \ni 1 \text{ is closed under both.}
\end{cases}$$

In any case above, $\mathcal{D}'$ is a countable subset of $\mathcal{D} \cup \{1\}$, contains $\{1\} \cup D_0$ and separates points on $E_0$. Now, (c) follows by (b) (with $\mathcal{D} = \mathcal{D}'$).

Often $\mathcal{D} \subset C(E; \mathbb{R})$ is uncountable but known to separate points on $E$, and one desires a subset $A \subset E$ is $\mathcal{D}$-baseable. In other words, one desires reducing this specific $\mathcal{D}$ to a countable subcollection separating points on $A$. One sufficient condition for such reduction is the *hereditary Lindelöf property* (see p.150) of $A$.

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22The notation “$\mathcal{ac}(\cdot)$” was defined in 2.2.3
PROPOSITION 3.3.8. Let $E$ be a topological space, $A \in \mathcal{B}(E)$ and $\mathcal{D} \subset C(E; \mathbb{R})$ separate points on $A$. If $\{(x, y) : x \in A\}$ is a Lindelöf subspace of $E \times E$, especially if $A$ is a Souslin or second-countable subspace of $E$, then $A$ is a $\mathcal{D}$-baseable subset of $E$.

PROOF. If $A$ is a Souslin or second-countable subspace of $E$, then $A \times A$ is a hereditary Lindelöf subspace of $E \times E$ by Proposition 9.1.11 (d, f) and Proposition 9.1.3 (b, c), which implies $\{(x, y) : x \in A\}$ is a Lindelöf subspace of $E \times E$. Now, the result follows by Proposition 9.2.8 (a) (with $E = (A, \sigma_E(A))$ and $\mathcal{D} = \mathcal{D}|_A$).

REMARK 3.3.9. Separating points is usually weaker than strongly separating points. Compared to Proposition 9.2.8 (b), the selection result above uses hereditary Lindelöf property, which is weaker than second-countability.

REMARK 3.3.10. When $E$ is a Tychonoff space, $C(E; \mathbb{R})$ separates points on $E$ by Proposition 3.3.1 (a, b). So, Proposition 3.3.8 (with $\mathcal{D} = C(E; \mathbb{R})$) slightly generalizes [Bog07] Vol. II, Theorem 6.7.7 (ii).

Point-separating functions can be selected from a uniformly dense collection.

PROPOSITION 3.3.11. Let $E$ be a topological space and $A$ be a $\mathcal{D}$-baseable subset. If $\mathcal{D}_0 \subset C_b(E; \mathbb{R})$ satisfies $\mathcal{D} \subset \sigma(\mathcal{D}_0)$, then $E$ is a $\mathcal{D}_0$-baseable subset.

PROOF. Let $\{f_n\}_{n \in \mathbb{N}} \subset \mathcal{D}$ separate points on $A$. $\mathcal{D}$ and its superset $\sigma(\mathcal{D}_0)$ both lie in the Banach space $(M_0(E; \mathbb{R}), \|\cdot\|_{\infty})$, so there exist $\{f_{n,k}\}_{n,k \in \mathbb{N}} \subset \mathcal{D}_0$ such that $f_{n,k} \to f_n$ as $k \uparrow \infty$ for each $n \in \mathbb{N}$ by Fact 9.1.9. Hence, $\{f_n\}_{n \in \mathbb{N}} \subset \sigma(\{f_{n,k}\}_{n,k \in \mathbb{N}})$ and $\{f_{n,k}\}_{n,k \in \mathbb{N}}$ separates points on $A$ by Corollary 9.2.3 (with $\mathcal{D} = \{f_n\}_{n \in \mathbb{N}}$ and $\mathcal{D}_0 = \{f_{n,k}\}_{n,k \in \mathbb{N}}$).

3.3.3. Baseable standard Borel subsets. The standard Borel subsets and Borel subsets coincide for a baseable standard Borel subspace. The following result generalizes its classical version on metrizable spaces (see Proposition 9.5.8 (b)).

PROPOSITION 3.3.12. Let $E$ be a topological space and $A \in \mathcal{B}^*(E)$. Then:
(a) If $(A, \sigma_E(A))$ is a baseable space, then $\mathcal{B}^*(A, \sigma_E(A)) = \mathcal{B}_E(A) \subset \mathcal{B}^*(E)$.
(b) If $E$ is a baseable space, then $\mathcal{B}^*(E) \subset \mathcal{B}(E)$.
(c) If $E$ is a baseable standard Borel space, then $\mathcal{B}(E) = \mathcal{B}^*(E)$.

PROOF. (a) $\mathcal{B}_E(A) \subset \mathcal{B}^*(A, \sigma_E(A)) \subset \mathcal{B}^*(E)$ by Proposition 9.5.8 (a). Now, let $B \in \mathcal{B}^*(A, \sigma_E(A))$. There exists a base $(A, \mathcal{F}; \widehat{E}, \widehat{\mathcal{F}})$ over $(A, \sigma_E(A))$ by Fact 3.3.1 (d, e). Then, $B \in \mathcal{B}_E(A)$ by Lemma 3.1.14 (b) (with $d = 1$, $E = E_0 = (A, \sigma_E(A))$ and $A = B$).

(b) There exists a base $(E, \mathcal{F}; \widehat{E}, \widehat{\mathcal{F}})$ over $E$ by Fact 3.3.1 (d, e) (with $A = E$). Then, (b) follows by Lemma 3.1.14 (b) (with $d = 1$ and $E_0 = E$).

(c) follows immediately by (a) (with $A = E$).

For $\mathcal{D}$-baseable standard Borel subsets, the function class $\mathcal{D}$ not only separates their points but also determines their subspace Borel $\sigma$-algebras.

PROPOSITION 3.3.13. Let $E$ be a topological space and $A \in \mathcal{B}^*(E)$. Then, $A$ is a $\mathcal{D}$-baseable subset of $E$ if and only if $A \in \mathcal{B}(E)$ and there exists a countable $\mathcal{D}_0 \subset \mathcal{D}$ such that $\mathcal{B}_E(A) = \mathcal{B}_{\mathcal{D}_0}(A) = \sigma(\mathcal{D}_0)|_A$.

\footnote{The $\sigma$-algebra $\sigma(\mathcal{D}_0)$ was defined in \ref{2.1.2}. Recall that $\sigma(\mathcal{D}_0)|_A$ is generally smaller than $\mathcal{B}_{\mathcal{D}_0}(A)$.}
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Proof. The $\mathcal{D}$-baseability of $A$ implies $A \in \mathcal{B}(E)$ and a countable $\mathcal{D}_0 \subset \mathcal{D} \subset C(E; \mathbb{R})$ that separates points on $A$. Then, the result follows by Proposition 9.5.11 (b, c) (with $E = (A, \mathcal{O}_E(A))$ and $\mathcal{D} = \mathcal{D}_0|_A$).

Baseability facilitates transferability of the standard Borel property.

**Proposition 3.3.14.** Let $E$ be a topological space, $I$ be a countable set, \{\(A_i\)\}_{i \in I} \subset \mathcal{B}^*(E)$ and $A = \bigcup_{i \in I} A_i$. If $(A, \mathcal{O}_E(A))$ is a baseable space, then $A \in \mathcal{B}^*(E)$.

Proof. There exists a base $(A, \mathcal{F}; \hat{E}, \hat{F})$ over $(A, \mathcal{O}_E(A))$ by Fact 3.3.1 (d, e). Then, $A \in \mathcal{B}^*(E)$ by Lemma 3.3.1 (c) (with $E = E_0 = (A, \mathcal{O}_E(A))$ and $d = 1$).

Baseable standard Borel support often implies unique Borel extension.

**Proposition 3.3.15.** Let $E$ be a topological space, $A$ be a baseable subset of $E$, $d \in \mathbb{N}$ and $\mu \in \mathcal{M}^+(E^d, \mathcal{B}(E)^{\otimes d})$. If $\mu$ is supported on $B \in \mathcal{B}^*(E^d)$ and $B \subset A^d$, then $\mathfrak{b}(\mu)$ is a singleton.

Proof. The baseability of $A$ implies $A \in \mathcal{B}(E)$ and $A^d \in \mathcal{B}(E)^{\otimes d}$. One finds by Fact 2.1.1 (a) (with $\mathcal{B} = \mathcal{B}(E)^{\otimes d}$ and $A = A^d$) that $\mu|_{A^d} \in \mathcal{M}^+(A^d, \mathcal{B}(E)^{\otimes d}|_{A^d})$. There exists a base $(A, \mathcal{F}; \hat{E}, \hat{F})$ over $(A, \mathcal{O}_E(A))$ by Fact 3.3.1 (d, e). $\mathfrak{b}(\mu|_{A^d})$ is a singleton by Corollary 3.1.17 (a) (with $E = E_0 = (A, \mathcal{O}_E(A))$, $A = B$ and $\mu = \mu|_{A^d}$). Now, $\mathfrak{b}(\mu)$ is a singleton by the fact $A^d \in \mathcal{B}(E)^{\otimes d}$ and Lemma 10.2.6 (b) (with $I = \{1, \ldots, d\}$, $S_i = E$, $S = E^d$, $\mathcal{D} = \mathcal{B}(E)^{\otimes d}$ and $A = A^d$).

The following three results relate the baseability of a standard Borel space $E$ and that of $\mathcal{P}(E)$.

**Proposition 3.3.16.** Let $E$ be a topological space, $\mathcal{D} \subset C_b(E; \mathbb{R})$ and $A \in \mathcal{B}^*(E)$. If $(A, \mathcal{O}_E(A))$ is a $\mathcal{D}|_A$-baseable space, then $\mathcal{P}(A, \mathcal{O}_E(A))$ is an $\mathfrak{m}(\mathcal{D}|_A)^*$-baseable space.

Proof. There exists a base $(A, \mathcal{F}; \hat{E}, \hat{F})$ over $(A, \mathcal{O}_E(A))$ with $\mathcal{F}\{1\} \subset \mathcal{D}|_A$ by Lemma 3.3.7 (c) (with $E = E_0 = (A, \mathcal{O}_E(A))$ and $\mathcal{D} = \mathcal{D}|_A$). Then, $\mathfrak{m}(\mathcal{F}\{1\})$ is a countable subset of $\mathfrak{m}(\mathcal{D}|_A)^*$ by Fact 10.1.14 and separates points on $\mathcal{P}(A, \mathcal{O}_E(A))$ by Corollary 3.1.19 (a) (with $E = (A, \mathcal{O}_E(A))$ and $d = 1$).

**Corollary 3.3.17.** Let $E$ be a baseable standard Borel space. Then, $\mathcal{M}^+(E)$ and $\mathcal{P}(E)$ are baseable spaces.

Proof. The result follows by Fact 3.3.1 (a, c) (with $A = E$), Proposition 3.3.16 (with $\mathcal{D} = C_b(E; \mathbb{R})$ and $A = E$) and Fact 10.1.20 (a) (with $\mathcal{D} = C_b(E; \mathbb{R})$).

**Proposition 3.3.18.** Let $E$ be a first-countable space and $\{\{x\} : x \in E\} \subset \mathcal{B}(E)$.

Proof. We suppose $\{g_n\}_{n \in \mathbb{N}} \subset C(\mathcal{P}(E); \mathbb{R})$ separates points on $\mathcal{P}(E)$ and define $f_n(x) = g_n(\delta_x)$ for all $x \in E$ and $n \in \mathbb{N}$. For distinct $x, y \in E$, $\delta_x \neq \delta_y$ by Proposition 9.4.7 (a) and so $\bigotimes_{n \in \mathbb{N}} f_n(x) = \bigotimes_{n \in \mathbb{N}} g_n(\delta_x) \neq \bigotimes_{n \in \mathbb{N}} g_n(\delta_y) = \bigotimes_{n \in \mathbb{N}} f_n(y)$. Hence, $\{f_n\}_{n \in \mathbb{N}} \subset \mathcal{R}^E$ separates points on $E$. We show each $f_n \in C(E; \mathbb{R})$. If $x_k \to x$ as $k \uparrow \infty$ in $E$, then $\delta_{x_k} \Rightarrow \delta_x$ as $k \uparrow \infty$ in $\mathcal{P}(E)$ by Fact 9.1.20 (a).

\[^{24}\text{The property that singletons are Borel sets is milder than the Hausdorff property or the T1 axiom (see }\text{[Mmm00]}\text{, §17, p.99).}

\[^{25}\text{\(\delta_x\) denotes the Dirac measure at }x.\]
It follows by the continuity of $g_n$ that $\lim_{k \to \infty} f_n(x_k) = \lim_{k \to \infty} g_n(\delta x_k) = g_n(\delta x) = f_n(x)$. Now, the continuity of $f_n$ follows by the first-countability of $E$ and [Mun00] Theorem 30.1 (b). □

### 3.3.4. Metrizable compact subsets.
Metrizable compact subsets form an essential class of hereditary Lindelöf, standard Borel, baseable subsets for replication and weak convergence. The following proposition gives several equivalent forms of metrizable compact subsets.

**Proposition 3.3.19.** Let $E$ be a topological space, $K \in \mathcal{K}(E)$ and $\mathcal{D} \subset C(E; \mathbb{R})$. Consider the following statements:

(a) $K$ is a $\mathcal{D}$-baseable subset of $E$.

(b) $K$ is a Souslin subspace of $E$.

(c) $K$ is a Hausdorff subspace of $E$ and $\{(x, x) : x \in K\}$ is a Lindelöf subspace of $E \times E$.

(d) $(K, \mathcal{O}_E(K))$ is a baseable space.

(e) $K$ is a Hausdorff and second-countable subspace of $E$.

(f) $K \in \mathcal{X}^m(E)$.

(g) $K \in \mathcal{B}^s(E)$.

Then, (b) - (f) are equivalent and implied by (a). (f) implies (g). Moreover, if $\mathcal{D}$ separates points on $E$, then (a) - (f) are all equivalent.

**Proof.** (b) $\to$ (c) $K$ is a Hausdorff subspace by the definition of Souslin spaces. $K \times K$ is a Souslin subspace of $E \times E$ by Proposition 9.1.11 (f). Now, (c) follows by Proposition 9.1.11 (d).

(c) $\to$ (d) $(K, \mathcal{O}_E(K))$ is Tychonoff by Proposition 9.1.12 (d) and Proposition 9.3.2 (a). Then, (d) follows by Proposition 9.3.1 (a, b) (with $E = (K, \mathcal{O}_E(K))$) and Proposition 3.3.8 (with $E = A = (K, \mathcal{O}_E(K))$ and $\mathcal{D} = C(K, \mathcal{O}_E(K); \mathbb{R})$).

(d) $\to$ (e, f) $(K, \mathcal{O}_E(K))$ is Hausdorff by Fact 9.2.5 (a). Let $\mathcal{D} \subset C_b(K, \mathcal{O}_E(K); \mathbb{R})$ be countable and separate points on $(K, \mathcal{O}_E(K))$. $\mathcal{D}$ strongly separates points on $(K, \mathcal{O}_E(K))$ by Lemma 9.2.4. Both (e) and (f) follow by Proposition 9.2.1 (d).

(e) $\to$ (c) $K \times K$ is a second-countable subspace of $E \times E$ by Proposition 9.1.3 (c). Then, (c) follows by Proposition 9.1.3 (b).

(f) $\to$ (b, g) $K \in \mathcal{B}^s(E)$. (d) follows by Proposition 9.1.12 (d), Proposition 9.1.11 (a) and Fact 9.5.1 (a).

(a) $\to$ (d) follows by Fact 3.3.2 (b, d) (with $\mathcal{D}' = C(E; \mathbb{R})$) and Fact 3.3.1 (a, d) (with $A = K$).

Moreover, if $\mathcal{D}$ separates points on $E$, then $K \in \mathcal{B}(E)$ by Proposition 9.2.1 (e) (with $A = E$) and Proposition 9.1.12 (a), and (c) implies (a) by Proposition 3.3.8 (with $A = K$).

Baseable spaces, Lusin spaces and Souslin spaces are not necessarily metrizable, but all of them have metrizable compact subsets.

**Corollary 3.3.20.** Let $E$ be a topological space and $K \in \mathcal{X}(E)$. Then:

(a) If $E$ is a baseable space, then $K$ is a metrizable standard Borel subspace and is a baseable subset of $E$. 
(b) If $E$ is a Souslin or Lusin space, then $K$ is a metrizable, baseable, standard Borel subspace of $E$. If, in addition, $C(E; \mathcal{R})$ separates points on $E$, then $K$ is a baseable subset of $E$.

**Proof.** (a) follows by Fact 3.3.3 and Proposition 3.3.19 (a, f, g) (with $D = C(E; \mathcal{R})$). Next, Proposition 9.1.12 (a), Proposition 9.1.11 (a, b) and Proposition 9.1.2 (c) imply that Lusin (resp. Souslin) spaces are Souslin (resp. Hausdorff) spaces and compact subsets of a Souslin space are closed, Souslin, Hausdorff subspaces. Then, (b) follows by Proposition 3.3.19 (a, b, d, f, g) (with $D = C(E; \mathcal{R})$). □

The two results above indicated that many non-metrizable topological spaces like those in Example 3.2.8 have metrizable compact subsets. Still, having metrizable compact subsets is a strictly milder property than baseability.

**Example 3.3.21.** $([0,1][0,1], \| \cdot \|_{\infty})$ is a Banach space so it certainly has metrizable compact subsets. However, it is not baseable as in Example 3.2.10.

Here are several constructive properties of metrizable compact subsets.

**Lemma 3.3.22.** Let $E$ be a topological space, $m \in \mathbb{N}$ and $\{A_i\}_{1 \leq i \leq m} \subset \mathcal{K}^m (E)$. If $A = \bigcup_{i=1}^m A_i$ is a Hausdorff subspace of $E$, then $A \in \mathcal{K}^m (E)$.

**Proof.** $A \subset \mathcal{K}^m (E)$ by Proposition 9.1.12 (b). $\{A_i\}_{1 \leq i \leq m}$ are Souslin subspaces of $E$ by Proposition 3.3.19 (b, f). $A$ is a Souslin subspace of $E$ by Proposition 9.1.11 (g). Now, the result follows by Proposition 3.3.19 (b, f). □

**Lemma 3.3.23.** Let $I$ be a countable index set, $\{S_i\}_{i \in I}$ be topological spaces and $S \doteq \prod_{i \in I} S_i$. Then:

(a) If $A_i \subset \mathcal{K}^m (S_i)$ for all $i \in I$, then $\prod_{i \in I} A_i \subset \mathcal{K}^m (S)$.

(b) If $A \subset \mathcal{K}^m (S)$ and $p_i (A)$ is a Hausdorff subspace of $S_i$ for some $i \in I$, then $p_i (A) \subset \mathcal{K}^m (S_i)$.

**Proof.** (a) follows by Proposition 9.1.12 (b) and Proposition 9.1.8 (b) $\{(x, x) : x \in A\}$ is a Lindelöf subspace of $S \times S$ by Proposition 3.3.19 (c, f) (with $E = S$). It follows that $p_i (A) \subset \mathcal{K}^m (S_i)$ and

\[
\{(y, y) : y \in p_i (A)\} = \{(p_i (x), p_i (x)) : x \in A\}
\]

is a Lindelöf subspace of $S_i \times S_i$ by Fact 2.1.4 (a), Proposition 9.1.12 (e) and Proposition 9.1.3 (d). Thus, $p_i (A) \subset \mathcal{K}^m (S_i)$ by its Hausdorff property and Proposition 3.3.19 (c, f). □

To handle $m$-tightness of non-Borel measures, Definition 2.3.8 required a collection of metrizable compact sets lying in their domains. The next lemma shows that if $E$ is a Hausdorff space, then this requirement is automatically satisfied by the members of $\mathfrak{M}^+ (E^d, \mathcal{B} (E)^{\otimes d})$.

**Lemma 3.3.24.** Let $I$ be a countable index set, $\{S_i\}_{i \in I}$ be topological spaces, $(S, \mathcal{A})$ be as in (2.7.22) and $A \subset \mathcal{K}^m (S)$. If $B_i \subset \mathcal{B} (S_i)$ is a Hausdorff subspace of $S_i$ and contains $p_i (A)$ for all $i \in I$, then $A \subset \mathcal{A}$ and $\mathcal{B} (A) = \mathcal{A} \cap A$.

\[\text{We remind the readers that: (i) the Borel } \sigma\text{-algebra of the product topological space } E^d \text{ can be different than the product } \sigma\text{-algebra } \mathfrak{B} (E)^{\otimes d}, \text{ and (ii) the notation } \mathfrak{M}^+ (\cdot) \text{ means the family of finite measures.} \]
herit many nice properties from its metrizable compact components.

Lemma 3.3.24 (with \(\bigcup\)) are equivalent.

\[ A \subset F = \prod_{i \in I} p_i(A) \in \bigotimes_{i \in I} \mathcal{B}_S(B_i) \subset \mathcal{A} \]

by Corollary \[9.1.13\] (a) (with \(S_i = B_i\)). \(F \in \mathcal{K}^m(S)\) by Lemma \[3.3.23\] (a). \(F\) is a second-countable subspace of \(S\) by Proposition \[3.3.19\] (e, f).

\[ \mathcal{B}_S(F) = \mathcal{A}_F \subset \mathcal{A} \]

by Proposition \[10.2.4\] (c) (with \(S_i = p_i(A)\)) and \[3.3.3\]. This implies \(\mathcal{B}_S(A) = \mathcal{A}|_A\) since \(A \subset F\). Moreover, \(F\) is a Hausdorff subspace of \(S\) by Proposition \[9.1.2\] (d). Hence, \(A \in \mathcal{B}_S(F) \subset \mathcal{A} \) by Proposition \[9.1.12\] (a) and \[3.3.4\].

\(\mathcal{m}\)-tightness ensures a unique and tight Borel extension on product space.

Proposition 3.3.25. Let \(I\) be a countable index set, \(\{S_i\}_{i \in I}\) be topological spaces, \((S, \mathcal{A})\) be as in \[2.7.22\], \(\Gamma \subset \mathcal{K}^\infty(S, \mathcal{A})\) and \(A \subset S\). Suppose in addition that \(p_i(A) \in \mathcal{B}(S_i)\) is a Hausdorff subspace of \(S_i\) for all \(i \in I\). Then, \(\Gamma\) is \(\mathcal{m}\)-tight in \(A\) if and only if \(\{\mu' = \mathfrak{b}(\mu)\}_{\mu \in \Gamma}^{m^\mathcal{A}}\) is \(\mathcal{m}\)-tight in \(A\).

Proof. We first show \(\mathcal{m}\)-tightness in \(A\) implies the existence of \(\mu' = \mathfrak{b}(\mu)\) for each \(\mu \in \Gamma\). Given such tightness, \(\mu\) is supported on some \(B \in \mathcal{K}^m(\mathcal{A}, \mathcal{O}_S(\mathcal{A}))\).

\(\mathcal{B}_S(B) = \mathcal{A}_B\) by Lemma \[3.3.24\] (with \(B_i = p_i(A)\)). Then, the unique existence of \(\mu' \) follows by Lemma \[10.2.4\] (c) (with \(A = B\)). Now, \(\mathcal{K}^m(\mathcal{A}, \mathcal{O}_S(\mathcal{A})) \subset \mathcal{A}\) by Lemma \[3.3.24\] (with \(B_i = p_i(A)\)). So, the \(\mathcal{m}\)-tightness of \(\Gamma\) and that of \(\{\mu' \}_{\mu \in \Gamma}\) (if any) are equivalent.

\[3.3.5\] \(\sigma\)-metrizable compact subsets. \(\sigma\)-metrizable compact subsets inherit many nice properties from its metrizable compact components.

Proposition 3.3.26. Let \(E\) be a topological space, \(\{K_n\}_{n \in \mathbb{N}} \subset \mathcal{K}(E)\), \(A = \bigcup_{n \in \mathbb{N}} K_n\) and \(D \subset C(E; \mathbb{R})\). Consider the following statements:

\(a\) \((A, \mathcal{O}_E(A))\) is a Souslin space.

\(b\) \(\{K_n\}_{n \in \mathbb{N}} \subset \mathcal{K}^m(E)\) (hence \(A \in \mathcal{K}_\sigma^m(E)\)).

\(c\) \((K_n, \mathcal{O}_E(K_n))\) is a baseable space for all \(n \in \mathbb{N}\).

\(d\) \((A, \mathcal{O}_E(A))\) is a baseable space.

\(e\) \(A\) is a \(D\)-baseable subset of \(E\).

\(f\) \(A \in \mathcal{B}(E)\).

Then, \(a\) - \(e\) are successively stronger. \(b\) implies \(f\). Moreover, if \(D\) separates points on \(E\), then \(a\) - \(e\) are all equivalent.

Proof. ((b) \(\rightarrow\) (a)) Each \((K_n, \mathcal{O}_E(K_n))\) is Souslin by Proposition \[3.3.19\] (b, f).

Hence, (a) follows by Proposition \[9.1.11\] (g).

((c) \(\rightarrow\) (b)) follows by Proposition \[3.3.19\] (d, f) (with \(K = K_n\)).

((d) \(\rightarrow\) (c)) follows by Fact \[3.3.3\] (a, b) (with \(E = (A, \mathcal{O}_E(A))\) and \(A = K_n\)).

((e) \(\rightarrow\) (d)) is automatic by definition.

((b) \(\rightarrow\) (f)) follows by Proposition \[3.3.19\] (f, g) and Proposition \[3.3.14\].

When \(D\) separates points on \(E\), \(A \in \mathcal{B}(E)\) by Proposition \[9.2.1\] (e) (with \(A = E\)) and Proposition \[9.1.12\] (a), and (a) implies (e) by Proposition \[3.3.8\].

\[27\] The notation “\(\mu' = \mathfrak{b}(\mu)\)” defined in \[2.3\] means \(\mu'\) is the unique Borel extension of \(\mu\).
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Below are several constructive properties of \(\sigma\)-metrizable compact subsets.

**Lemma 3.3.27.** Let \(E\) be a topological space and \(\{A_n\}_{n\in\mathbb{N}} \subset \mathcal{K}^m(E)\). If 
\(A = \bigcup_{n\in\mathbb{N}} A_n\) is a Hausdorff subspace of \(E\), then there exist \(\{K_q\}_{q\in\mathbb{N}} \subset \mathcal{K}^m(E)\) such that 
\(A = \bigcup_{q\in\mathbb{N}} K_q\) and \(K_q \subset K_{q+1}\) for all \(q \in \mathbb{N}\).

**Proof.** Let \(A_n = \bigcup_{p\in\mathbb{N}} K_{p,n}\) with \(\{K_{p,n}\}_{p\in\mathbb{N}} \subset \mathcal{K}^m(E)\) for each \(n \in \mathbb{N}\). Define 
\(K_q = \bigcup_{p=1}^q \bigcup_{n=1}^p K_{p,n}\) so \(K_q \subset K_{q+1}\) for all \(q \in \mathbb{N}\) and 
\(A = \bigcup_{q\in\mathbb{N}} K_q\). Each \((K_q, \mathcal{O}_E(K_q))\) is Hausdorff by Proposition 9.1.2 (c) and hence metrizable by Lemma 3.3.22.

**Lemma 3.3.28.** Let \(I\) be a countable index set, \(\{S_i\}_{i\in I}\) be topological spaces and 
\(S = \prod_{i\in I} S_i\). Then:

(a) If \(I\) is finite and \(A_i \in \mathcal{K}^m(S_i)\) for all \(i \in I\), then 
\(\prod_{i\in I} A_i \in \mathcal{K}^m(S)\).

(b) If \(A \in \mathcal{K}^m(S)\) and \(p_i(A)\) is a Hausdorff subspace of \(S_i\) for all \(i \in I\), then 
\(p_i(A) \in \mathcal{K}^m(S_i)\) for all \(i \in I\).

**Proof.** (a) Without loss of generality, we suppose \(I = \{1, \ldots, d\}\) and let 
\(A_i = \bigcup_{p\in\mathbb{N}} K_{p,i}\) with \(\{K_{p,i}\}_{p\in\mathbb{N}} \subset \mathcal{K}^m(S_i)\) for each \(1 \leq i \leq d\). We have that 
\[
\prod_{i=1}^d A_i = \bigcup_{(p_1, \ldots, p_d) \in \mathbb{N}^d} K_{p_1,1} \times \ldots \times K_{p_d,d},
\]
by Lemma 3.3.23 (a). For any \(x \in \prod_{i=1}^d A_i\), there exist \(p_1, \ldots, p_d \in \mathbb{N}\) such that 
\[
p_i(x) \in K_{p_i,i}, \quad \forall 1 \leq i \leq d.
\]
It then follows by (3.3.5) that 
\[
\prod_{i=1}^d A_i = \bigcup_{(p_1, \ldots, p_d) \in \mathbb{N}^d} F_{p_1, \ldots, p_d} \in \mathcal{K}^m(S).
\]

(b) Let \(A = \bigcup_{p\in\mathbb{N}} K_p\) with \(\{K_p\}_{p\in\mathbb{N}} \subset \mathcal{K}^m(S)\) so 
\(p_i(K_p) \in \mathcal{K}^m(S_i)\) for all \(p \in \mathbb{N}\) and \(i \in I\) by the fact \(p_i(K_p) \subset p_i(A)\), the Hausdorff property of \(p_i(A)\), 
Proposition 9.1.2 (c) and Lemma 3.3.23 (b). Hence, 
\(p_i(A) = p_i(\bigcup_{p\in\mathbb{N}} K_p) = \bigcup_{p\in\mathbb{N}} p_i(K_p) \in \mathcal{K}^m(S_i)\) for all \(i \in I\).

3.3.6. Baseability about Skorokhod \(\mathcal{F}_1\)-space. When \(E\) is a Tychonoff space, the associated Skorokhod \(\mathcal{F}_1\)-space \(D(\mathbb{R}^+; E)\) is also (see Proposition 9.6.1 (c)). The following proposition shows that baseability of \(E\) passes to \(D(\mathbb{R}^+; E)\).

**Proposition 3.3.29.** Let \(E\) be a Tychonoff space. Then:

(a) If \(E\) is a \(D\)-baseable space with \(D \subset C_b(E; \mathbb{R})\), then 
\(D(\mathbb{R}^+; E)\) is a \(\{\alpha_{t,n}^f : f \in D, t \in \mathbb{Q}^+, n \in \mathbb{N}\}\)-baseable space with 
\[
\alpha_{t,n}^f(x) = n \int_t^{t+1/n} f(x(s))\, ds
\]
for each \(f \in D, t \in \mathbb{Q}^+\) and \(n \in \mathbb{N}\).

(b) If \(E\) is a baseable space, then \(D(\mathbb{R}^+; E)\) is also a baseable space and 
\(J(x) \subset (0, \infty)\) is at most countable for all \(x \in D(\mathbb{R}^+; E)\).

\(^{28}\) The Skorokhod \(\mathcal{F}_1\)-space \(D(\mathbb{R}^+; E)\) was defined in 2.2.2.

\(^{29}\) The notation "\(J(x)\)" was defined in 4.2.1.
PROOF. (a) Without loss of generality, we suppose \( D \) is countable. Then, (a) follows immediately by Proposition 9.6.1(b).

(b) There exists a countable \( D \subseteq C_b(E; \mathbb{R}) \) separating points on \( E \) by Fact 3.3.1(a, b) (with \( A = E \)) so \( \varphi \equiv \bigotimes D \) is injective. \( \varphi \in C(E; \mathbb{R}^D) \) by Fact 2.1.4(b) and \( D(\mathbb{R}^+; E) \) is baseable by (a). \( \mathbb{R}^D \) is Polish by Proposition 9.1.11(f). Therefore,

\[
J(x) = J[\varphi(x)], \forall x \in D(\mathbb{R}^+; E)
\]

by Proposition 9.6.1(d) (with \( S = E, E = \mathbb{R}^D \) and \( f = \varphi \)) and \( J[\varphi(x)] \) is a countable subset of \((0, \infty)\) by [EK08] §3.5, Lemma 5.1. \( \square \)

Metrizability of compact subsets of \( E \) also passes to \( D(\mathbb{R}^+; E) \).

PROPOSITION 3.3.30. If \( E \) is a Tychonoff space with \( \mathcal{K}(E) = \mathcal{K}^m(E) \), then \( \mathcal{K}(D(\mathbb{R}^+; E)) = \mathcal{K}^m(D(\mathbb{R}^+; E)) \).

PROOF. Let \( K \in \mathcal{K}(D(\mathbb{R}^+; E)) \). By Proposition 9.6.5, there exist \( \{A_n\}_{n \in \mathbb{N}} \subseteq \mathcal{K}(E) = \mathcal{K}^m(E) \) such that \( K \subseteq D(\mathbb{R}^+; A) \) with \( A \subseteq \bigcup_{n \in \mathbb{N}} A_n \). \( A \) is a baseable space by Proposition 9.3.1(a, c) and Proposition 3.3.26(b, d) (with \( K_n = A_n \) and \( D = C_b(E; \mathbb{R}) \)). \( D(\mathbb{R}^+; A) \) is a baseable space by Proposition 3.3.29(b) (with \( E = A \)). \( K \) is a baseable subspace of \( D(\mathbb{R}^+; E) \) by Fact 3.3.3(a, b) (with \( E = A \)) and Corollary 9.6.3. Hence, \( K \) is metrizable by Proposition 3.3.19(d, f) (with \( E = D(\mathbb{R}^+; E) \)). \( \square \)

The countability of an \( E \)-valued càdlàg process’s fixed left-jump times is well-known when \( E \) is metrizable and separable. We extend this to baseable Tychonoff spaces.

PROPOSITION 3.3.31. Let \( E \) be a baseable Tychonoff space. Then:

(a) For any \( \mu \in \mathcal{M}^+(D(\mathbb{R}^+; E), \mathcal{B}(E)^{\otimes \mathbb{R}^+}|_{D(\mathbb{R}^+; E)}) \) especially \( \mu \in \mathcal{M}^+(D(\mathbb{R}^+; E)) \),

\[
J(\mu) \text{ is a well-defined countable subset of } (0, \infty).
\]

(b) For any \( E \)-valued càdlàg process \( X \), \( J(X) \) is a well-defined countable subset of \((0, \infty)\).

In particular, (a) and (b) hold when \( E \) is a metrizable and separable space or a Polish space.

PROOF. (a) Metrizable and separable spaces are baseable spaces by Fact 3.2.5(b). As \( E \) is a baseable space, there exists a countable \( D \subseteq C_b(E; \mathbb{R}) \) separating points on \( E \) by Fact 3.3.1(a, b) (with \( A = E \)) so \( \varphi \equiv \bigotimes D \) is injective. \( \varphi \in C(E; \mathbb{R}^D) \) by Fact 2.1.4(b).

\[
(3.3.10) \quad \varphi(\varphi) \in C \left( D(\mathbb{R}^+; E); D(\mathbb{R}^+; \mathbb{R}^D) \right)
\]

by Proposition 9.6.1(d) (with \( S = E, E = \mathbb{R}^D \) and \( f = \varphi \)), which implies (3.3.9). As \( \varphi(\varphi) \) maps \( D(\mathbb{R}^+; E) \) into \( D(\mathbb{R}^+; \mathbb{R}^D) \), we have that

\[
(3.3.11) \quad \varphi(\varphi) \in M \left( D(\mathbb{R}^+; E), \mathcal{B}(E)^{\otimes \mathbb{R}^+}|_{D(\mathbb{R}^+; E)}; D(\mathbb{R}^+; \mathbb{R}^D), \mathcal{B}(\mathbb{R}^D)^{\otimes \mathbb{R}^+}|_{D(\mathbb{R}^+; \mathbb{R}^D)} \right)
\]

30\( D(\mathbb{R}^+; \bigcup_{n \in \mathbb{N}} A_n) \) is well-defined by Corollary 9.6.3.

31Recall that \( \mathcal{B}(E)^{\otimes \mathbb{R}^+}|_{D(\mathbb{R}^+; E)} \) is generally smaller than \( \mathcal{B}(D(\mathbb{R}^+; E)) \).

32\( J(\mu) \), the set of fixed left-jump times of \( \mu \) was defined in (2.3.12).

33\( J(X) \), the set of fixed left-jump times of \( X \) was defined in (2.5.8).
by Fact 10.1.10 (b) (with \( f = \varphi \)).

(3.3.12) \( \nu \asymp \mu \circ \varpi(\varphi)^{-1} \in \mathfrak{M}^+ \left( D(\mathbb{R}^+; \mathbb{R}^D), \mathcal{B}(\mathbb{R}^D)^{\otimes \mathbb{R}^+} \big|_{D(\mathbb{R}^+; \mathbb{R}^D)} \right) \)

by (3.3.11). \( \mathbb{R}^D \) is a Polish space, so (3.3.9) implies

(3.3.13) \[
\begin{align*}
\mu \left( \{ x \in D(\mathbb{R}^+; E) : t \in J(x) \} \right) &= \mu \left( \{ x \in D(\mathbb{R}^+; E) : t \in J(\varpi(\varphi)(x)) \} \right) \\
&= \nu \left( \{ y \in D(\mathbb{R}^+; \mathbb{R}^D) : t \in J(y) \} \right), \quad \forall t \in \mathbb{R}^+,
\end{align*}
\]

while the equalities in (3.3.13) as well as \( J(\mu) \) and \( J(\nu) \) are well-defined by Fact 9.6.9. Hence, we have \( J(\mu) = J(\nu) \) by (3.3.13) and this set is a countable subset of \( (0, \infty) \) by [EK86, §3.7, Lemma 7.7].

(b) follows immediately by Fact 2.5.4 (a), the definition of \( J(X) \) and (a) (with \( \mu = \text{pd}(X)|_{D(\mathbb{R}^+; E)}^{[34]} \)).

\[\square\]

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34 \( \text{pd}(X) \) denotes the process distribution of \( X \) and was specified in §2.5.
CHAPTER 4

Replication of Function and Operator

The previous chapter introduced space change through a base \((E_0, \mathcal{F}; \hat{E}, \hat{F})\) over topological space \(E\) and the notions of baseable spaces and subsets. Now, we discuss the replication of objects from \(E\) onto \(\hat{E}\) and the association of the original and replica objects. \S 4.1 introduces the replicas of continuous functions. \S 4.2 introduces the replicas of linear operators on \(C_b(E; \mathbb{R})\). The replica operators constructed in \S 4.2.4 are strong generators of semigroups on \(C(\hat{E}; \mathbb{R})\).

4.1. Replica function

Given a base \((E_0, \mathcal{F}; \hat{E}, \hat{F})\) over topological space \(E\), replicating a function \(f \in M(E; \mathbb{R})\) onto \(\hat{E}\) basically means extending \(f|_{E_0}\) onto \(\hat{E}\). A naive approach preserves the values of \(f\) on \(E_0\) and assigns \(0\) on \(\hat{E}\setminus E_0\). We make the following general notation for simplicity.

**Notation 4.1.1.** Let \(E, S_1\) and \(S_2\) be non-empty sets, \(A\) be an arbitrary subset of \(S_1 \cap S_2\), \(y_0 \in E\) and \(f \in E^{S_1}\). By \(\text{var}(f; S_2, A, y_0)\) we denote the mapping

\[
\text{var}(f; S_2, A, y_0)(x) = \begin{cases} f(x), & \text{if } x \in A, \\ y_0, & \text{otherwise, } \forall x \in S_2 \end{cases}
\]

from \(S_2\) to \(E\).

**Note 4.1.2.** \(\text{var}(f; S_2, f^{-1}(\{y_0\}), y_0) = \text{var}(f; S_2, \emptyset, y_0)\) maps all \(x \in S_2\) to \(y_0\).

\(\text{var}(f; \hat{E}, E_0, 0)\) realizes the naive idea of replicating \(f\) above, but it may not preserve the Borel measurability of \(f\) if \(E_0\) is not a standard Borel set. Noticing that \(\S 3.1.1\) links members of \(\mathcal{F}\) bijectively to a member of \(\hat{F} \subset C(\hat{E}; \mathbb{R})\), we can define the replica of a suitable continuous function on \(E\) as a continuous function on \(\hat{E}\).

**Definition 4.1.3.** Let \(E\) be a topological space, \((E_0, \mathcal{F}; \hat{E}, \hat{F})\) be a base over \(E\) and \(d, k \in \mathbb{N}\). The replica of \(f \in C(E^d; \mathbb{R}^k)\) with respect to \((E_0, \mathcal{F}; \hat{E}, \hat{F})\) (if any) refers to the continuous extension \(\hat{f}\) of \(f|_{E_0^d}\) on \(\hat{E}^d\).

**Remark 4.1.4.** Remark \(3.1.5\) stated that the compactification inducing \(\hat{E}\) does not necessarily extend every member of \(C_b(E_0, \mathcal{O}_{\mathcal{F}}(E_0); \mathbb{R})\) continuously onto \(\hat{E}\). So, a general \(f \in C_b(E^d; \mathbb{R}^k)\) need not have a replica.

**Notation 4.1.5.** Let \((E_0, \mathcal{F}; \hat{E}, \hat{F})\) be a base over \(E\) and \(d, k \in \mathbb{N}\). Hereafter, we will always let \(\hat{f}\) denote \(\text{var}(f; \hat{E}, E_0, 0)\) for \(f \in (\mathbb{R}^k)^{E^d}\) and \(\hat{f}\) denote the replica of \(f \in C(E^d; \mathbb{R}^k)\) if no confusion is caused.

\(^1\) "\text{var}" is "var" in fraktur font which stands for "variant".
Below are several basic properties of $\mathcal{T}$ and $\hat{f}$.

**Proposition 4.1.6.** Let $E$ be a topological space, $(E_0, \mathcal{F}, \hat{E}, \hat{\mathcal{F}})$ be a base over $E$ and $a, k \in \mathbb{N}$. Then:

(a) If $f \in (\mathbb{R}^k)^E$ is bounded, then $\mathcal{T}$ is also bounded.

(b) If $f \in M(E^d; \mathbb{R}^k)$ and $f|_{E^d_0 \setminus A} = 0$ for some $A \in \mathcal{B}(E^d)$ with $A \subset E^d_0$, then $\mathcal{T} \in M(\hat{E}^d; \mathbb{R}^k)$. In particular, this is true if $E^d_0 \in \mathcal{B}(E^d)$.

(c) The replica of $f \in C(E^d; \mathbb{R}^k)$ (if any) is unique.

(d) If $f_1, f_2 \in C(E^d; \mathbb{R}^k)$ have replicas, then $a \hat{f}_1 + b \hat{f}_2$ (resp. $\hat{f}_1 \hat{f}_2$ when $k = 1$) is the replica of $af_1 + bf_2$ for all $a, b \in \mathbb{R}$ (resp. $f_1f_2$).

(e) $\hat{\mathcal{F}} = \{\hat{f} : f \in \mathcal{F}\}$, $\hat{a}(\hat{\mathcal{F}}) = \{\hat{f} : f \in a(\mathcal{F})\}$ and $\hat{a}(\mathcal{F}) = \{\hat{f} : f \in a(\mathcal{F})\}$.

(f) $f \in C(E^d; \mathbb{R}^k)$ admits a replica if and only if

\[
\text{Proposition 4.1.6 (f) for its existence.}
\]

In particular, every $f \in \mathcal{G}[\mathcal{F}]$ admits a replica.

**Proof.** (a) The definition of $\mathcal{T}$ implies $\|\mathcal{T}\|_\infty \leq \|f\|_\infty$.

(b) $\mathcal{T}_A = h|_A \in M(A, \mathcal{G}_A(A); \mathbb{R}^k)$ by Lemma 3.1.14 (a). So, $\mathcal{T} = \mathcal{T}_A \in M(\hat{E}^d; \mathbb{R}^k)$ by Fact 10.1.2 (with $E = \hat{E}^d$, $\mathcal{B} = \mathcal{B}(E^d)$ and $f = \mathcal{T}$).

(c) follows since $E^d_0$ is dense in $\hat{E}^d$, $f|_{E^d_0} = \hat{f}|_{E^d_0}$ and $f$ and $\mathcal{T}$ are continuous.

(d) follows by the fact that $(af_1 + bf_2)|_{E^d_0} = (a\hat{f}_1 + b\hat{f}_2)|_{E^d_0}$ and $f_1f_2|_{E^d_0} = \hat{f}_1\hat{f}_2|_{E^d_0}$, that $af_1 + bf_2 \in C(E^d; \mathbb{R}^k)$ and that $\hat{f}_1\hat{f}_2 \in C(\hat{E}^d; \mathbb{R})$ when $k = 1$.

(e) We let $\mathcal{F} = \{f_n\}_{n \in \mathbb{N}}$ and find $\hat{\mathcal{F}} = \{\hat{f}_n\}_{n \in \mathbb{N}}$ by (3.1.1) and Lemma 3.1.3 (a). Let $1 \leq l \leq d, n_1, \ldots, n_l \in \mathbb{N}$, $f = \prod_{i=1}^l f_{n_i} \circ p_i$ and $g = \prod_{i=1}^l \hat{f}_{n_i} \circ p_i$. Then,

\[
f|_{E^d_0} = \prod_{i=1}^l f_{n_i}|_{E^d_0} \circ p_i = \prod_{i=1}^l \hat{f}_{n_i}|_{E^d_0} \circ p_i = g|_{E^d_0}.
\]

and (3.1.13) imply $g = \hat{f}$. The members of $\hat{a}(\mathcal{F})$ (resp. $\hat{a}(\mathcal{F})$) correspond bijectively to those of $\hat{a}(\mathcal{F})$ (resp. $\hat{a}(\mathcal{F})$). Hence, (e) follows by (d).

(f) - Necessity If $\hat{f}$ exists, then $\{p_i \circ \hat{f}\}_{1 \leq i \leq k} \subset C(\hat{E}^d; \mathbb{R})$ by Fact 2.1.4 (a). Hence, (4.1.2) follows by (3.1.17).

(f) - Sufficiency If (4.1.2) holds, then $\{p_i \circ f\}_{1 \leq i \leq k}$ exists by Corollary 3.1.10. Hence, $\bigotimes_{i=1}^k p_i \circ f = \hat{f}$ by Fact 2.1.4 (b) and the fact

\[
f|_{E^d_0} = \bigotimes_{i=1}^k p_i \circ f|_{E^d_0} = \bigotimes_{i=1}^k p_i \circ f|_{E^d_0}.
\]

\[\square\]

**Note 4.1.7.** For the sake of brevity, hereafter we may use the replica of $f \in \mathcal{G}[\mathcal{F}]$ without referring to Proposition 4.1.6 (f) for its existence.

We next show a nice property of locally compact baseable spaces which recovers \textsuperscript{2}Sri98 Corollary 2.3.32. This is also an example where $\mathcal{T}$ and $\hat{f}$ coincide.

\textsuperscript{2}1_A denotes the indicator function of $A$. 
Proposition 4.1.8. Let $E$ be a locally compact space and $D \subset C_0(E; \mathbb{R})$.

Consider the following statements:

(a) $E$ is a $D$-baseable space.

(b) There exists a base $(E, F; \widehat{E}, \widehat{F})$ over $E$ such that $\widehat{E}$ is a one-point compactification of $E$, $(F \setminus \{1\}) \subset D \subset C_0(E; \mathbb{R}) \subset ca(F)$ and $F$ strongly separates points on $E$.

(c) $E$ is a Polish space.

(d) $E$ is a metrizable and separable space.

(e) $E$ is a $C_0(E; \mathbb{R})$-baseable space.

Then, (a) - (e) are successively weaker. Moreover, (e) implies (a) when $D$ is uniformly dense in $C_0(E; \mathbb{R})$.

Proof. ((a) $\rightarrow$ (b)) By (a), there exists a countable $F \subset (D \cup \{1\}) \subset C_b(E; \mathbb{R})$ that separates points on $E$. $E$ is a Hausdorff space by Fact 3.2.5 (a) and admits a one-point compactification $\widehat{E}$ by Proposition 9.3.7. It follows by Lemma 9.3.8 (with $D = F$) that $F$ strongly separates points on $E$ and

(4.1.5) $\widehat{F} \doteq \{ \text{var}(f : \widehat{E}, E, 0) : f \in F \setminus \{1\} \} \cup \{1\} \subset C(\widehat{E}; \mathbb{R})$

separates and strongly separates points on $\widehat{E}$. Hence, $(E, F; \widehat{E}, \widehat{F})$ by definition is a base over $E$. Moreover, we get $C_0(E; \mathbb{R}) \subset ca(F)$ by Corollary 3.1.10 (with $d = 1$ and $E_0 = E$) and Fact 10.2.1.

((b) $\rightarrow$ (c)) $\theta(E) = \theta_F(E)$ by (b), so $E$ is an open subspace of the Polish space $\widehat{E}$ by Proposition 9.1.2 (a) and Lemma 3.1.3 (b, c). Hence, (c) follows by Proposition 9.1.11 (b).

((c) $\rightarrow$ (d)) follows by Proposition 9.1.11 (c).

((d) $\rightarrow$ (e)) $C_0(E; \mathbb{R})$ separates points on $E$ by Proposition 9.1.4 (a) and Proposition 9.3.9 (a, d). Then, (e) follows by Proposition 9.1.4 (c) and Proposition 3.3.8 (with $A = E$ and $D = C_0(E; \mathbb{R})$).

Moreover, if $C_0(E; \mathbb{R}) \subset d(D)$, then (e) implies (a) by Proposition 3.3.11 (with $A = E$, $D = C_0(E; \mathbb{R})$ and $D_0 = D$).

\[\square\]

4.2. Replica operator

We now focus on replicating a linear operator $L$ on $C_b(E; \mathbb{R})$ as a linear operator on $C(\widehat{E}; \mathbb{R})$. Many general concepts about linear operators used below were reviewed in §2.2.5 and, as noted in §2.6, we always consider single-valued operators.

4.2.1. Definition. Replicating $L$ from $C_b(E; \mathbb{R})$ onto $C(\widehat{E}; \mathbb{R})$ means constructing a linear operator on $C(\widehat{E}; \mathbb{R})$ whose domain and range are formed by the replicas of the member of $D(L)$ and $\mathfrak{R}(L)$ respectively. Below is a mild sufficient condition for its existence.

\[^3\] $C_0(E; \mathbb{R})$, the family of all $\mathbb{R}$-valued continuous functions on $E$ that vanishes at infinity, was defined in §2.2.3.
4. REPLICA FUNCTION AND OPERATOR

**Proposition 4.2.1.** Let $E$ be a topological space, $(E_0, \mathcal{F}; \hat{E}, \hat{F})$ be a base over $E$ and $\mathcal{L}$ be a linear operator on $C_b(E; \mathbb{R})$ such that

\begin{align*}
\text{(4.2.1)} \quad & \text{mc}(\mathcal{F}) \subset \mathcal{D}(\mathcal{L}), \\
& \mathcal{R}(\mathcal{L}|_{\text{mc}(\mathcal{F})}) \subset \text{ca}(\mathcal{F}).
\end{align*}

Then:

(a) There exists a unique linear operator $\hat{\mathcal{L}}_0$ on $C(\hat{E}; \mathbb{R})$ such that

\begin{align*}
\text{(4.2.2)} \quad & \mathcal{D}(\hat{\mathcal{L}}_0) = \text{ag}(\hat{\mathcal{F}}) \\
\text{and}
\end{align*}

\begin{align*}
\text{(4.2.3)} \quad & \hat{\mathcal{L}}_0 \hat{f} = \hat{\mathcal{F}} f, \forall \hat{f} \in \text{ag}(\hat{\mathcal{F}}).
\end{align*}

(b) There exists a unique linear operator $\hat{\mathcal{L}}_1$ on $C(\hat{E}; \mathbb{R})$ such that

\begin{align*}
\text{(4.2.4)} \quad & \mathcal{D}(\hat{\mathcal{L}}_1) = \{ \hat{f} : (f, \mathcal{L} f) \in \mathcal{L} \cap (\text{ca}(\mathcal{F}) \times \text{ca}(\mathcal{F})) \} \\
\text{and}
\end{align*}

\begin{align*}
\text{(4.2.5)} \quad & \hat{\mathcal{L}}_1 \hat{f} = \hat{\mathcal{F}} f, \forall \hat{f} \in \mathcal{D}(\hat{\mathcal{L}}_1).
\end{align*}

**Proof.** We have that

\begin{align*}
\text{(4.2.6)} \quad & \text{ag}(\mathcal{F}) = \text{ag}(\{a f : f \in \text{mc}(\mathcal{F}), a \in \mathbb{R}\}) \subset \mathcal{D}(\mathcal{L}) \\
\text{and}
\end{align*}

\begin{align*}
\text{(4.2.7)} \quad & \mathcal{R}(\mathcal{L}|_{\text{mc}(\mathcal{F})}) = \{ \mathcal{L} h : h \in \text{ag}(\{a f : f \in \text{mc}(\mathcal{F}), a \in \mathbb{R}\}) \} \\
& = \text{ag}(\{\text{ag}(\mathcal{L}|_{\text{mc}(\mathcal{F})}) : a \in \mathbb{R}\}) \\
& = \text{ag}[\mathcal{R}(\mathcal{L}|_{\text{mc}(\mathcal{F})})] \subset \text{ca}(\mathcal{F})
\end{align*}

by (4.2.1) and the linear space properties of $\mathcal{L}$, $\mathcal{D}(\mathcal{L})$ and $\text{ca}(\mathcal{F})$. Thus,

\begin{align*}
\text{(4.2.8)} \quad & \hat{\mathcal{L}}_0 \triangleq \{ (\hat{f}, \hat{\mathcal{L}}_0 \hat{f}) : f \in \text{ag}(\mathcal{F}) \} \\
\text{and}
\end{align*}

\begin{align*}
\text{(4.2.9)} \quad & \hat{\mathcal{L}}_1 \triangleq \{ (\hat{f}, \hat{\mathcal{L}}_1 \hat{g}) : (f, g) \in \mathcal{L} \cap (\text{ca}(\mathcal{F}) \times \text{ca}(\mathcal{F})) \}
\end{align*}

satisfy (4.2.6), (4.2.7) and Proposition 4.1.6 (d, f) (with $d = k = 1$). \hfill \square

The $\hat{\mathcal{L}}_0$ and $\hat{\mathcal{L}}_1$ above are defined as two (possibly) different replicas of $\mathcal{L}$.

**Definition 4.2.2.** Let $E$ be a topological space, $(E_0, \mathcal{F}; \hat{E}, \hat{F})$ be a base over $E$ and $\mathcal{L}$ be a linear operator on $C_b(E; \mathbb{R})$.

- $(E_0, \mathcal{F}; \hat{E}, \hat{F})$ is said to be a base for $\mathcal{L}$ if (4.2.1) holds.
- When $(E_0, \mathcal{F}; \hat{E}, \hat{F})$ is a base for $\mathcal{L}$, the operator $\hat{\mathcal{L}}_0$ in (4.2.8) and the operator $\hat{\mathcal{L}}_1$ in (4.2.9) are called the core replica and the extended replica of $\mathcal{L}$ respectively.

Hereafter, we use the following notations for brevity if no confusion is caused.

**Notation 4.2.3.** Let $E$ be a topological space, $(E_0, \mathcal{F}; \hat{E}, \hat{F})$ be a base over $E$, $\mathcal{L}$ be a linear operator on $C_b(E; \mathbb{R})$ and $\beta \in \mathbb{R}$.

- We define $\bar{f} \triangleq f|_{E_0^d}$ for each $d, k \in \mathbb{N}$ and $f \in M(E^d; \mathbb{R}^k)$. Moreover, $\bar{\mathcal{F}} \triangleq \mathcal{F}|_{E_0} = \{ \bar{f} : f \in \mathcal{F} \}$. 

○ When \((E_0, \mathcal{F}; \hat{E}, \hat{F})\) is a base for \(\mathcal{L}\), we define
\[
\mathcal{L}_0 = \mathcal{L}_{\operatorname{ag}(\mathcal{F})},
\]
\[
\mathcal{L}_1 = \mathcal{L} \cap (\operatorname{ca}(\mathcal{F}) \times \operatorname{ca}(\mathcal{F})),
\]
\[
\tilde{\mathcal{L}}_i = \left\{ (\tilde{f}, \tilde{g}) : (f, g) \in \mathcal{L}_i \right\}, \forall i = 0, 1.
\]

○ The operator \(\beta - \mathcal{L}\) is defined by
\[
(\beta - \mathcal{L})f = \beta f - \mathcal{L}f, \forall f \in \mathcal{D}(\mathcal{L}).
\]

Similar notations apply to \(\mathcal{L}_i, \tilde{\mathcal{L}}_i\) and \(\tilde{\mathcal{L}}_i\) for each \(i = 0, 1\).

The domain and range of the operators above have the following properties.

**Proposition 4.2.4.** Let \(E\) be a topological space, \(\mathcal{L}\) be a linear operator on \(C_0(E; \mathbb{R})\) and \((E_0, \mathcal{F}; \hat{E}, \hat{F})\) be a base over \(E\) for \(\mathcal{L}\). Then:

(a) The linear operators \(\mathcal{L}_0\) and \(\mathcal{L}_1\) satisfy \(\mathcal{L}_0 = \mathcal{L}_{\operatorname{ag}(\mathcal{F})}\) and
\[
\mathcal{R}(\mathcal{L}_i) \subset \operatorname{cl}(\mathcal{D}(\mathcal{L}_i)) = \operatorname{ca}(\mathcal{F}), \forall i = 0, 1.
\]

(b) The linear operators \(\hat{\mathcal{L}}_0\) and \(\hat{\mathcal{L}}_1\) satisfy \(\hat{\mathcal{L}}_0 = \hat{\mathcal{L}}_{\operatorname{ag}(\hat{\mathcal{F}})}\) and
\[
\mathcal{R}(\hat{\mathcal{L}}_i) \subset \operatorname{cl}(\mathcal{D}(\hat{\mathcal{L}}_i)) = \operatorname{ca}(\hat{\mathcal{F}}), \forall i = 0, 1.
\]

(c) The linear operators \(\hat{\mathcal{L}}_0\) and \(\hat{\mathcal{L}}_1\) satisfy \(\hat{\mathcal{L}}_0 = \hat{\mathcal{L}}_{\operatorname{ag}(\hat{\mathcal{F}})}\) and
\[
\mathcal{R}(\hat{\mathcal{L}}_i) \subset \operatorname{cl}(\mathcal{D}(\hat{\mathcal{L}}_i)) = \operatorname{cl}(\mathcal{D}(\hat{\mathcal{L}}_i)) \subset \operatorname{ca}(\hat{\mathcal{F}}), \forall i = 0, 1.
\]

(d) If \(\mathcal{L}1 = 0\), then \(\hat{\mathcal{L}}_01 = 0\) and \(\hat{\mathcal{L}}_11 = 0\).

**Proof.** (a) The linearities of \(\mathcal{L}_0\) and \(\mathcal{L}_1\) follow by that of \(\mathcal{L}\). It follows by (4.2.10) and (4.2.7) that
\[
\mathcal{R}(\mathcal{L}_0) = \mathcal{R}(\mathcal{L}_{\operatorname{ag}(\mathcal{F})}) \subset \operatorname{ca}(\mathcal{F}) = \operatorname{cl}(\operatorname{ag}(\mathcal{F})) = \operatorname{cl}(\mathcal{D}(\mathcal{L}_0)).
\]

It follows by (4.2.10) and (4.2.15) that
\[
\mathcal{R}(\mathcal{L}_1) \subset \operatorname{ca}(\mathcal{F}) = \operatorname{cl}(\mathcal{D}(\mathcal{L}_0)) \subset \operatorname{cl}(\mathcal{D}(\mathcal{L}_1)) \subset \operatorname{ca}(\mathcal{F}).
\]

Now, (a) follows by (4.2.15) and (4.2.16).

(b) The linearity of \(\mathcal{L}_0\) (resp. \(\mathcal{L}_1\)) follows by that of \(\mathcal{L}_0\) (resp. \(\mathcal{L}_1\)). It follows by (4.2.10), (4.2.7) and properties of uniform convergence that
\[
\mathcal{R}(\hat{\mathcal{L}}_0) = \left\{ \hat{\tilde{f}} : f \in \operatorname{ag}(\mathcal{F}) \right\} = \mathcal{R}(\mathcal{L}_{\operatorname{ag}(\mathcal{F})})|_{E_0}
\]
\[
\subset \operatorname{ca}(\mathcal{F})|_{E_0} \subset \operatorname{ca}(\hat{\mathcal{F}}) = \operatorname{cl}(\operatorname{ag}(\mathcal{F})) = \operatorname{cl}(\mathcal{D}(\hat{\mathcal{L}}_0)).
\]

It follows by (4.2.10) and (4.2.17) that
\[
\mathcal{R}(\hat{\mathcal{L}}_1) = \left\{ \hat{g} : g \in \mathcal{R}(\mathcal{L}_1) \right\} \subset \operatorname{ca}(\mathcal{F})|_{E_0}
\]
\[
\subset \operatorname{ca}(\hat{\mathcal{F}}) = \operatorname{cl}(\mathcal{D}(\hat{\mathcal{L}}_0)) \subset \operatorname{cl}(\mathcal{D}(\hat{\mathcal{L}}_1)) \subset \operatorname{ca}(\hat{\mathcal{F}}).
\]

Now, (b) follows by (4.2.17) and (4.2.18).

(c) follows by Proposition 4.2.1 and Corollary 3.1.10 (with \(d = 1\)).

(d) \(\tilde{1} = 1 \in \tilde{\mathcal{F}} \subset \mathcal{D}(\mathcal{L}_0)\) and \(\mathcal{L}_01 = \tilde{L}1 = 0 = 0\) by the fact \(1 \in \mathcal{F}\), (4.2.2), (4.2.3) and the denseness of \(E_0\) in \(\hat{E}\). □
4.2.2. Markov-generator properties. The replica operators $\hat{L}_0$ and $\hat{L}_1$ may inherit or refine properties of the original operator $L$ by the compactness of $\hat{E}$ and the association of the original and replica functions. [DK20b, DK20a, DK21] use the following Markov-generator-type properties of replica operators:

**Property.**

$P1$ $\hat{L}_1 = \mathcal{d}(\hat{L}_0)$.

$P2$ $\hat{L}_0$ satisfies the positive maximum principle.

$P3$ $\hat{L}_1$ satisfies the positive maximum principle.

$P4$ $\hat{L}_0$ is a strong generator on $C(\hat{E};\mathbb{R})$.

$P5$ $\hat{L}_0$ is a Feller generator on $C(\hat{E};\mathbb{R})$.

The following lemma gives a sufficient condition for $P1$ and explains why we call $\hat{L}_0$ and $\hat{L}_1$ the core and extended replica of $L$.

**Lemma 4.2.5.** Let $E$ be a topological space, $L$ be a linear operator on $C_0(E;\mathbb{R})$ and $(E_0, F; \hat{E}, \hat{F})$ be a base over $E$ for $L$. Then:

(a) If $P2$ (resp. $P3$) holds, then $\hat{L}_0$ (resp. $\hat{L}_1$) is dissipative.

(b) If $P4$ holds, and $\hat{L}_1$ is dissipative (especially $P3$ holds), then $P1$ holds.

**Proof.** $\hat{E}$ is a compact Polish space by Lemma 3.1.3 (b, c). $C_0(\hat{E};\mathbb{R}) = C(\hat{E};\mathbb{R})$ by (3.1.16) (with $d = 1$). Then, the result follows by Proposition 4.2.4 (c) and [EK86] §4.2, Lemma 3.1.3 (b, c).

**Remark 4.2.6.** $\hat{L}_1$ is a linear superspace of $\hat{L}_0$, so we call $\hat{L}_1$ the extended replica. “core replica” comes from the fact that $\mathcal{D}(\hat{L}_0)$ is a core (see [EK86] §1.3, p.17) of $\hat{L}_1$ in the setting of Lemma 4.2.5 (b).

The following lemma specifies when $P2$ - $P5$ hold.

**Lemma 4.2.7.** Let $E$ be a topological space, $L$ be a linear operator on $C_0(E;\mathbb{R})$ and $(E_0, F; \hat{E}, \hat{F})$ be a base over $E$ for $L$. Then:

(a) $\hat{L}_0$ (resp. $\hat{L}_1$) is dissipative if and only if $\hat{L}_0$ (resp. $\hat{L}_1$) is dissipative.

(b) $P2$ (resp. $P3$) holds if and only if for any $\epsilon \in (0, \infty)$ and $f \in \text{ag}(F)$ (resp. $f \in \mathcal{D}(\hat{L}_1)$), there exists an $n^f_\epsilon \in \mathbb{N}$ such that

\[
\sup_{x \in E_0} \left[ Lf(x) - n^f_\epsilon \left( \| \hat{f}^+ \|_\infty - f(x) \right) \right] \leq \epsilon.
\]

(c) $P4$ holds if and only if (1) $\hat{L}_0$ is dissipative, and (2) there exists a $\beta \in (0, \infty)$ such that

\[
\hat{F} \subset \text{ca} \left\{ \left( \beta - \hat{L} \right) \hat{f} : f \in \text{mc}(F) \right\}.
\]

(d) $P5$ holds if and only if (1) $L1 = 0$, (2) there exists a $\beta \in (0, \infty)$ such that

\[
\hat{L}_1 \subseteq \hat{L} \hat{F} \subset \text{ca} \left\{ \left( \beta - \hat{L} \right) \hat{f} : f \in \text{mc}(F) \right\},
\]

\[
\hat{F} \subset \text{ca} \left\{ (\beta - \hat{L}) \hat{f} : f \in \text{mc}(F) \right\}.
\]

\[
(4.2.20)
\]

$\hat{F} \subset \text{ca} \left\{ (\beta - \hat{L}) \hat{f} : f \in \text{mc}(F) \right\}.
\]

$P4$ was defined in 4.2.5 and $f^+$ was defined in Notation 4.2.3.

$P5$ The operator $\lambda - L$ was defined in Notation 4.2.3.
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Proof. (a) We have by Fact 10.3.1 (a) (with \( d = k = 1 \)) that
\[
\|\hat{f}\|_\infty = \|\hat{f}\|_\infty, \quad \forall f \in \mathfrak{a}(\mathcal{F}).
\]
Letting \( g \doteq (\beta - \mathcal{L})f \), we have by (4.2.5) and Proposition 4.1.6 (d) that
\[
\left\| \beta \hat{f} - \mathcal{L}\hat{f} \right\| _\infty = \left\| \hat{g} \right\|_\infty = \left\| \beta \hat{f} - \mathcal{L}\hat{f} \right\|_\infty, \quad \forall f \in \mathfrak{D} (\mathcal{L}_1), \beta \in (0, \infty).
\]
Now, (a) follows by (4.2.21), (4.2.22) and the fact that \( \mathcal{L}_0 \subset \mathcal{L}_1 \) and \( \mathcal{L}_0 \subset \hat{\mathcal{L}}_1 \).

(b - Sufficiency) We state the proof for \( \mathcal{L}_0 \). \( \hat{\mathcal{L}}_1 \) follows similarly. Let \( \epsilon \in (0, \infty) \), \( f \in \mathfrak{a}(\mathcal{F}) \) and \( n_\ell' \in \mathbb{N} \) satisfy (4.2.19). We have by Fact 10.3.1 (with \( d = k = 1 \)) that
\[
\hat{f}^+ = \hat{f}^+ \text{ and } \|\hat{f}^+\|_\infty = \|\hat{f}^+\|_\infty.
\]
Then, there exists an \( x_0 \in \hat{E} \) such that
\[
\left\| \hat{f}^+ \right\|_\infty = \|\hat{f}^+\|_\infty = \hat{f}(x_0)
\]
by (4.2.23), the compactness of \( \hat{E} \), the continuity of \( \hat{f}^+ \) and [Mun00 Theorem 27.4]. \( \hat{E} \) is metrizable by Lemma 3.1.3 (c), so there exist \( \{x_k\}_{k \in \mathbb{N}} \subset E_0 \) such that
\[
x_k \longrightarrow x_0 \text{ as } k \uparrow \infty \text{ in } \hat{E}
\]
by \( E_0 \)'s denseness in \( \hat{E} \) and Fact 9.1.9 (with \( E = \hat{E} \) and \( A = E_0 \)). It follows that
\[
\hat{\mathcal{L}}f(x_k) = \mathcal{L}f(x_k) \leq n_\ell' \left( \|\hat{f}^+\|_\infty - f(x_k) \right) + \epsilon
\]
\[
\text{by (4.2.3), (4.2.19) and (4.2.24). Hence, we have that}
\]
\[
\hat{\mathcal{L}}f(x_0) = \lim_{k \to \infty} \hat{\mathcal{L}}f(x_k)
\]
\[
\leq \lim_{k \to \infty} n_\ell' \left( \hat{f}(x_0) - \hat{f}(x_k) \right) + \epsilon
\]
\[
= n_\ell' \left( \hat{f}(x_0) - \lim_{k \to \infty} \hat{f}(x_k) \right) + \epsilon = \epsilon
\]
by the continuities of \( \hat{f} \) and \( \hat{\mathcal{L}}f \), (4.2.26) and the independence of \( n_\ell' \) and \( \{x_k\}_{k \in \mathbb{N}} \). Letting \( \epsilon \downarrow 0 \) in (4.2.27), we get \( \mathcal{L}f(x_0) \leq 0 \).

(b - Necessity) Fix \( \epsilon \in (0, \infty) \) and \( f \in \mathfrak{a}(\mathcal{F}) \). Then, \( f \) satisfies (4.2.23) by Fact 10.3.1 (with \( d = k = 1 \)). \( \mathcal{L}_0 \) is dissipative by P2 and Lemma 4.2.5 (a). By (4.2.3), (4.2.21), the compactness of \( \hat{E} \), [EK86 §4.5, Lemmas 5.3 and EK86 §4.5, (5.15)], there exist an \( n_\ell' \in \mathbb{N} \) and a positive contraction (see [EK86 §1.1, p.6 and §4.2, p.165]) \( S_{1/n_\ell'} \) on \( C(\hat{E}; \mathbb{R}) \) such that
\[
\mathcal{L}f(x) = \hat{\mathcal{L}}f(x) \leq n_\ell' \left( S_{1/n_\ell'} \hat{f}(x) - \hat{f}(x) \right) + \epsilon
\]
\[
\leq n_\ell' g(x) - S_{1/n_\ell'} g(x) + \epsilon \leq n_\ell' \left( \|\hat{f}^+\|_\infty - f(x) \right) + \epsilon
\]
for all \( x \in E_0 \subset \hat{E} \), where \( g \doteq \|\hat{f}^+\|_\infty - f \geq 0 \) satisfies \( S_{1/n_\ell'} g(x) \geq 0 \) by the positiveness of \( S_{1/n_\ell'} \).
(c) It follows by [4.2.13], [4.2.20], the linearity of $\tilde{\mathcal{L}}_0$ and [4.2.10] that
\[
\text{ct} \left[ \mathcal{D}(\tilde{\mathcal{L}}_0) \right] = \text{ca}(\mathcal{F})
\]
\[
\subset \text{ca} \left[ \mathcal{R} \left( \beta - \tilde{\mathcal{L}}_0 \right) \right]
\]
\[
\subset \text{ca} \left[ \mathcal{R} \left( \beta - \tilde{\mathcal{L}}_0 \right) \right] = \text{cl} \left[ \text{ag} \left( \mathcal{R} \left( \beta - \tilde{\mathcal{L}}_0 \right) \right) \right]
\]
\[
(4.2.29)
\]
thus proving the equivalence between [4.2.20] and
\[
(4.2.30) \quad \text{ct} \left[ \mathcal{D}(\tilde{\mathcal{L}}_0) \right] = \text{ct} \left[ \mathcal{R} \left( \beta - \tilde{\mathcal{L}}_0 \right) \right].
\]
Next, we find by [4.2.10], [4.2.2], Proposition [4.1.6] (d, e) and [4.2.3] that
\[
\mathcal{D}(\tilde{\mathcal{L}}_0) = \text{ag}(\mathcal{F}) = \text{ag}(\tilde{\mathcal{F}}) \bigg|_{E_0} = \mathcal{D}(\tilde{\mathcal{L}}_0) \bigg|_{E_0}
\]
and
\[
(4.2.31) \quad \mathcal{R} \left( \beta - \tilde{\mathcal{L}}_0 \right) = \mathcal{R} \left( \beta - \tilde{\mathcal{L}}_0 \right) \bigg|_{E_0}.
\]
Then, [4.2.30] is equivalent to
\[
(4.2.33) \quad \text{ct} \left[ \mathcal{D}(\tilde{\mathcal{L}}_0) \right] = C(\tilde{E}; \mathbb{R}) = \text{ct} \left[ \mathcal{R} \left( \beta - \tilde{\mathcal{L}}_0 \right) \right]
\]
by [4.2.31], [4.2.32], the denseness of $E_0$ in $\tilde{E}$, properties of uniform convergence and [4.2.14]. So far, we have shown the equivalence of [4.2.20], [4.2.30] and [4.2.33]. Now, (c) follows by (a) and the Lumer-Phillips Theorem (see [Yos80 §IX.8]).

(d) follows by (b), Proposition [4.2.4] (d), [EK86 §4.2, Theorem 2.2] and the equivalence between [4.2.20] and [4.2.33].

4.2.3. Conditions on operator. We consider typical conditions on $\mathcal{L}$ under which: a) One can construct bases in either of the following two forms, and b) The associated replica operators $\tilde{\mathcal{L}}_0$ and $\tilde{\mathcal{L}}_1$ satisfy some of $P_1,P_5$

**Property.** Suppose $\mathcal{D}_0 \subset \mathcal{D}(\mathcal{L})$ must be replicated. Then:

**P6** There exists a base $(E, \mathcal{F}; \tilde{E}, \tilde{F})$ over $E$ for $\mathcal{L}$ with $\mathcal{D}_0 \subset \mathcal{F} = \text{ag}_\mathbb{Q}(\mathcal{F})$.

**P7** There exists a base $(E_0, \mathcal{F}; \tilde{E}, \tilde{F})$ over $E$ for $\mathcal{L}$ such that $A \subset E_0$, $E_0 \in \mathbb{H}_m(E)$ and $\mathcal{D}_0 \subset \mathcal{F} = \text{ag}_\mathbb{Q}(\mathcal{F})$.

**Remark 4.2.8.** $\mathcal{D}_0$ also appeared in Lemma [3.3.7] In martingale problems, $\mathcal{D}_0$ can be a rich collection that approximates both the domain and range of $\mathcal{L}$ (see e.g. [DK20b] §3.1]). In filtering, $\mathcal{D}_0$ may approximate the given sensor function (see e.g. [DK20a]). $\tilde{A}$ above stands for a desired subset of $E$ to be contained in $E_0$. In martingale problems, the set $A$ could be a support of the given initial distribution (see e.g. [DK20b] §A.1]).

Our regularity conditions consist of four types. The first type is about the denseness of the domain and range of $\mathcal{L}$. 
Condition \textbf{(Denseness)}. The operator $\mathcal{L}$ satisfies:

- **D1** $\mathcal{R}(\mathcal{L}) \subset \text{cl}(\mathcal{D}(\mathcal{L}))$.
- **D2** $\mathcal{D}(\mathcal{L}) \subset \text{cl}(\mathcal{R}(\beta - \mathcal{L}))$ for some $\beta \in (0, \infty)$.

Remark 4.2.9. **D1** and **D2** are true for any strong generator $\mathcal{L}$ by the Lumer-Phillips Theorem.

The second type is about the point-separability of the domain of $\mathcal{L}$.

Condition \textbf{(Separability)}. The operator $\mathcal{L}$ satisfies:

- **S1** $\mathcal{D}(\mathcal{L})$ contains the constant function 1 and separates points on $E$.
- **S2** $\mathcal{D}(\mathcal{L})$ contains the constant function 1 and $E$ is a $\mathcal{D}(\mathcal{L})$-baseable space.

**S2** is stronger than **S1**. They are mild requirements about the richness of $\mathcal{D}(\mathcal{L})$ by our discussions in §3.2, and the next example further illustrates their generality.

Example 4.2.10.

(I) For martingale problems and nonlinear filtering problems, a common setting (see [Szp76], [BKK95] and [BBK00]) is that $E$ is a metrizable Lusin (especially Polish) space and the domain $\mathcal{D}(\mathcal{L})$ of $\mathcal{L}$ contains 1 and separates points on $E$. Then, $E$ is a second-countable space by Proposition 9.1.11 (d) and Proposition 9.1.4 (c). So, $\mathcal{L}$ satisfies **S2** by Proposition 3.3.8 (with $A = E$ and $\mathcal{D} = \mathcal{D}(\mathcal{L})$).

(II) Another classical setting for martingale problems is that $E$ is a locally compact separable metric space with one-point compactification $E \cup \{\Delta\}$, $\mathcal{L}$ is a linear operator on $C_0(E; \mathbb{R})$ and its domain $\mathcal{D}(\mathcal{L})$ is uniformly dense in $C_0(E; \mathbb{R})$ (see [EK86, Chapter 4] and [KO88]). In this case, one can simply extend $\mathcal{L}$ to a linear operator $\mathcal{L}^*$ on $C_b(E; \mathbb{R})$ by defining $\mathcal{D}(\mathcal{L}^*)$ as the linear span of $\mathcal{D}(\mathcal{L}) \cup \{1\}$ and defining

\[ \mathcal{L}^*(af + b) = a\mathcal{L}f, \quad \forall f \in \mathcal{D}(\mathcal{L}), a, b \in \mathbb{R}. \]

By Proposition 4.1.8 (a, b, d, e) (with $\mathcal{D} = \mathcal{D}(\mathcal{L})$), there exists a countable $\mathcal{F} \subset C_b(E; \mathbb{R})$ such that $\mathcal{F} \setminus \{1\} \subset \mathcal{D}(\mathcal{L})$ strongly separates points on $E$ and $C_0(E; \mathbb{R}) \subset \text{ca}(\mathcal{F})$. Thus, $\mathcal{L}^*$ satisfies **S2** by Proposition 9.2.1 (a). Moreover, this $\mathcal{L}^*$ satisfies **D1**.

(III) Suppose that $E$ is a (possibly non-metrizable) Tychonoff space and $\mathcal{L}$ is a strong generator on $C_b(E; \mathbb{R})$. Without loss of generality, one can consider $1 \in \mathcal{D}(\mathcal{L})$. Otherwise, we extend $\mathcal{L}$ to $\mathcal{L}^*$ as in (II). $\mathcal{D}(\mathcal{L})$ is uniformly dense in $C_b(E; \mathbb{R})$ by the Lumer-Phillips Theorem, so **D1** holds. $C_b(E; \mathbb{R})$ separates points on $E$ by Proposition 9.3.1 (a, c), so **S1** holds by Corollary 9.2.3.

The third type of regularity conditions includes several analogues of dissipativeness and positive maximum principle.

Condition \textbf{(Generator)}. The operator $\mathcal{L}$ satisfies:

- **G1** $\mathcal{L}$ is dissipative.
G2 For any $\epsilon, \beta \in (0, \infty)$ and $x \in E$, there exist $\{K_{n,\epsilon}^{x,\beta}\}_{n \in \mathbb{N}} \subset \mathcal{X}^m(E)$ (independent of $f$) such that each $f \in \mathcal{D}(\mathcal{L})$ satisfies
\[
\beta |f(x)| - \left\| (\beta f - \mathcal{L}f)|_{K_{n,\epsilon}^{x,\beta}} \right\|_\infty 
\leq (\beta\|f\|_\infty + \|\mathcal{L}f\|_\infty + 1) \epsilon, \forall f \in \mathcal{D}(\mathcal{L})
\]
for some $n = n_{f,x,\beta}^f \in \mathbb{N}$.

G3 For any $f \in \mathcal{D}(\mathcal{L})$,
\[
\lim \sup_{n \to \infty} \sup_{x \in E} ([\mathcal{L}f(x) - n\|f\|_\infty - f(x))] \leq 0.
\]

G4 For any $\epsilon \in (0, \infty), x \in E$ and $f \in \mathcal{D}(\mathcal{L})$, there exist $\{K_{n,\epsilon}^{x}\}_{n \in \mathbb{N}} \subset \mathcal{X}^m(E)$ independent of $f$ and an $n_{f,x,\beta}^f \in \mathbb{N}$ independent of $x$ such that
\[
\mathcal{L}f(x) - n_{f,x,\beta}^f \left( \left\| f^+|_{K_{n,\epsilon}^{x}} \right\|_\infty - f(x) \right) \leq \epsilon.
\]

The next example shows that G2, G3 and G4 are not unnatural.

**Example 4.2.11.** Let $E$, $\mathcal{L}$ and $\mathcal{L}^*$ be as in Example 4.2.10 (II) and $\epsilon \in (0, \infty)$.

(I) If $\mathcal{L}$ satisfies positive maximum principle, then $\mathcal{L}^*$ does also. Consequently, G3 is satisfied by both $\mathcal{L}$ and $\mathcal{L}^*$ by an argument similar to the proof of Lemma 4.2.7 (b - “only if”).

(II) When $\mathcal{L}$ is a Feller generator, the Feller semigroup $\{\mathcal{S}_t\}_{t \geq 0}$ generated by $\mathcal{L}(\mathcal{D})$ on $C_0(E; \mathbb{R})$ is often given by a transition function (see [EK86 §4.1, p.156]) $\kappa : \mathbb{R}^+ \times E \times \mathcal{B}(E) \to [0,1]$. In the remainder of the example, we fix $x \in E$.

- Fix $\beta \in (0, \infty)$ and let $Q = \{q_n\}_{n \in \mathbb{N}}$. $E$ is a Polish space by Proposition 4.1.8 (c, d). So for each $n \in \mathbb{N}$, $\kappa(q_n, x, \cdot)$ is tight in $E$ by Ulam’s Theorem (Theorem 2.3.11), i.e. there exist $\{K_{n,\epsilon}^{x,\beta}\}_{n \in \mathbb{N}} \subset \mathcal{X}(E)$ such that
\[
\kappa(q_n, x, E\setminus K_{n,\epsilon}^{x,\beta}) \leq \epsilon, \forall n \in \mathbb{N}.
\]

One finds by [EK86 §1.2, (2.1) and (2.6)], change of variable and Jensen’s Inequality that
\[
\beta |f(x)| = \left| \int_0^\infty \beta e^{-\beta t} \mathcal{S}_t(\beta - \mathcal{L})f(x) dt \right|
\leq \int_0^{1} \left| \mathcal{S}_{\frac{1}{\beta}}(\beta - \mathcal{L})f(x) \right| du, \forall f \in \mathcal{D}(\mathcal{L}).
\]

Then, there exist $\{f_{f,x,\beta}\}_{f \in \mathcal{D}(\mathcal{L})} \subset (0,1)$ such that
\[
\beta |f(x)| \leq |\mathcal{S}_{f_{f,x,\beta}}(\beta - \mathcal{L})f(x)|, \forall f \in \mathcal{D}(\mathcal{L})
\]
by \(4.2.39\), Mean-Value Theorem and Jensen’s Inequality. For each fixed $f \in \mathcal{D}(\mathcal{L})$, there exists an $n = n_{f,x,\beta}^f \in \mathbb{N}$ such that
\[
\left| \mathcal{S}_{q_n}(\beta - \mathcal{L})f - \mathcal{S}_{f_{f,x,\beta}}(\beta - \mathcal{L})f \right| < \epsilon
\]
by the strong continuity of \( \{S_t\}_{t \geq 0} \). From (4.2.40), (4.2.38) and (4.2.41) it follows that

\[
\beta |f(x)| \leq \int_E |(\beta - \mathcal{L})f(y)| \kappa(q_n, x, dy) + \epsilon
\]

(4.2.42)

\[
\leq \int_{K^{x, \beta}_{n, \epsilon}} |(\beta f - \mathcal{L}f)(y)| \kappa(q_n, x, dy)
+ (\|\beta f\|_\infty + \|\mathcal{L}f\|_\infty) \epsilon + \epsilon
\]

\[
\leq \left\| (\beta f - \mathcal{L}f)\right\|_{K^{x, \beta}_{n, \epsilon}} + (\|\beta f\|_\infty + \|\mathcal{L}f\|_\infty + 1) \epsilon.
\]

Thus, \( \mathcal{L} \) satisfies \( G_2 \) as \( \{K^{x, \beta}_{n, \epsilon}\}_{n \in \mathbb{N}} \) does not involve any \( f \).

\begin{itemize}
  \item For each \( n \in \mathbb{N} \), the tightness of \( \kappa(n^{-1}, x, \cdot) \) implies a \( K^{x, \beta}_{n, \epsilon} \in \mathcal{K}(E) \) satisfying

\[
\kappa(n^{-1}, x, E \setminus K^{x, \beta}_{n, \epsilon}) \leq \frac{\epsilon}{2n^2}.
\]

Meanwhile, we fix \( f \in \mathcal{D}(\mathcal{L}) \). \( \mathcal{L}f \) is the infinitesimal generator of \( \{S_t\}_{t \geq 0} \), so there is a sufficiently large \( n = n_f \in \mathbb{N} \) such that

\[
n \geq \|f\|_\infty
\]

and

\[
\sup_{z \in E} |\mathcal{L}f(z) - n \left[ \int_E f(y) \kappa(n^{-1}, z, dy) - f(z) \right]| \leq \frac{\epsilon}{2}.
\]

The sequence of compact sets \( \{K^{x, \beta}_{n, \epsilon}\}_{n \in \mathbb{N}} \) is determined by \( x \) and the transition function \( \kappa \), which is independent of \( f \). The convergence rate \( n_0 = n_f \) is an intrinsic parameter of \( f \) and is unrelated to \( x \). From (4.2.43), (4.2.44) and (4.2.45) it follows that

\[
\mathcal{L}f(x) \leq n_0 \left[ \int_{K^{x, \beta}_{n_0, \epsilon}} f^+(y) \kappa(n_0^{-1}, x, dy) + \frac{\|f\|_\infty n_0^2 \epsilon}{2} - f(x) \right] + \frac{\epsilon}{2}
\]

(4.2.46)

\[
\leq n_0 \left( \left\| f^+|_{K^{x, \beta}_{n_0, \epsilon}} \right\|_\infty - f(x) \right) + \epsilon.
\]

Thus, \( \mathcal{L} \) satisfies \( G_4 \).

The fourth condition type is a equivalent to assuming \( \mathcal{D}(\mathcal{L}) \) is closed under multiplication since \( \mathcal{L} \) and \( \mathcal{D}(\mathcal{L}) \) are linear spaces.

**CONDITION (DA).** The domain of the operator \( \mathcal{L} \) is a subalgebra of \( C_b(E; \mathbb{R}) \).

**4.2.4. Existence of Markov-generator-type replica operator.** Now, we give four constructions of Markov-generator-type replica operators under the aforementioned conditions. The first two assume \( S_2 \) and construct bases satisfying \( P_6 \).

**Lemma 4.2.12.** Let \( E \) be a topological space, \( \mathcal{L} \) be a linear operator on \( C_b(E; \mathbb{R}) \) and \( \mathcal{D}_0 \subset \mathcal{D}(\mathcal{L}) \) be countable. Then, \( S_2, D_1, G_3 \) plus \( DA \) imply \( P_6 \).

**Proof.** We use induction to construct the \( \mathcal{F} \) in \( P_6 \). By \( S_2 \), there exists a countable \( \mathcal{D} \subset \mathcal{D}(\mathcal{L}) \) that separates on \( E \). For \( k = 0 \),

\[
\mathcal{F}_0 \triangleq (\mathcal{D}_0 \cup \mathcal{D} \cup \{1\}) \subset \mathcal{D}(\mathcal{L})
\]
is countable, contains 1 and separates points on \( E \). For \( k \in \mathbb{N} \), we assume \( \mathcal{F}_0 \subset \mathcal{F}_{k-1} \subset \mathcal{D}(\mathcal{L}) \) and find by \( \text{DA} \) that
\[ \text{ag}_Q(\mathcal{F}_{k-1}) \subset \mathcal{D}(\mathcal{L}). \]

Then, we define
\[ \mathcal{F}_k := \bigcup_{f \in \text{ag}_Q(\mathcal{F}_{k-1}), q \in \mathbb{N}} \{ f, g_{f,k}^q \} \subset \mathcal{D}(\mathcal{L}), \]
where each \( g_{f,k}^q \in \mathcal{D}(\mathcal{L}) \) is chosen by \( \text{D1} \) to satisfy
\[ \| g_{f,k}^q - \mathcal{L}f \|_\infty \leq 2^{-q}. \]

It follows immediately that
\[ \text{ag}_Q(\mathcal{F}_{k-1}) \subset \mathcal{F}_k \]
and
\[ \mathcal{R}(\mathcal{L}|\text{ag}_Q(\mathcal{F}_{k-1})) \subset \text{cl}(\mathcal{F}_k). \]

Based on the construction above\(^6\),
\[ \mathcal{F} := \bigcup_{k \in \mathbb{N}_0} \mathcal{F}_k \]
satisfies
\[ \mathcal{D}_0 \cup \{1\} \cup \mathcal{D} = \mathcal{F}_0 \subset \mathcal{F} = \text{ag}_Q(\mathcal{F}) \subset \mathcal{D}(\mathcal{L}) \]
and \( \text{DA} \). So, \( \mathcal{F} \) separates points on \( E \) as \( \mathcal{D} \) does. Now, \( \text{P6} \) follows by Lemma 3.3.7 (b) (with \( E_0 = E \) and \( \mathcal{D} = \mathcal{F} \)) and \( \text{4.2.54} \). \( \text{P2} \) and \( \text{P3} \) follow by \( \text{G3} \) and Lemma 4.2.7 (b) (with \( E_0 = E \), \( \mathcal{L}_0 = \mathcal{L} \) and \( \mathcal{L}_1 = \mathcal{L}_1 \)).

**Lemma 4.2.13.** Let \( E \) be a topological space, \( \mathcal{L} \) be a linear operator on \( C_b(E; \mathbb{R}) \) such that \( \text{S2}, \text{D1}, \text{D2} \) and \( \text{DA} \) hold, and \( \mathcal{D}_0 \subset \mathcal{D}(\mathcal{L}) \) be countable. Then:
(a) If \( \text{G1} \) holds, then \( \text{P6}, \text{P1}, \text{P4} \) hold.
(b) If \( \mathcal{L}1 = 0 \) and \( \text{G3} \) holds, then \( \text{P6}, \text{P1}, \text{P3}, \text{P5} \) hold.

**Proof.** (a) We choose \( \beta \in (0, \infty) \) so \( \text{D2} \) is satisfied. For each \( f \in \mathcal{D}(\mathcal{L}) \), we choose each \( g_{f,k}^q \in \mathcal{D}(\mathcal{L}) \) by \( \text{D1} \) so (4.2.50) is satisfied and choose each \( h_{f,k}^q \in \mathcal{D}(\mathcal{L}) \) by \( \text{D2} \) so
\[ \| (\beta - \mathcal{L})h_{f,k}^q - f \|_\infty < 2^{-q}. \]

Now, we follow the proof of Lemma 4.2.12 to establish the \( \mathcal{F} = \bigcup_k \mathcal{F}_k \) in \( \text{P6} \) where
\[ \mathcal{F}_k := \bigcup_{f \in \text{ag}_Q(\mathcal{F}_{k-1}), q \in \mathbb{N}} \{ f, g_{f,k}^q, h_{f,k}^q \} \subset \mathcal{D}(\mathcal{L}). \]

Consequently, \( \mathcal{F}_k \) defined in (4.2.56) satisfies not only (4.2.51) and (4.2.52) but also
\[ \text{ag}_Q(\mathcal{F}_{k-1}) \subset \text{cl} \left[ \mathcal{R}(\beta - \mathcal{L}|\mathcal{F}_k) \right]. \]

\( \mathcal{F} \) defined in (4.2.53) not only satisfies (4.2.54) and (4.2.1) but also satisfies
(4.2.58) \( \mathcal{F} \subset \text{cl} \left[ \mathcal{R}(\beta - \mathcal{L}|\mathcal{F}) \right] \subset \text{cl} \left[ \mathcal{R}(\beta - \mathcal{L}|\text{mc}(\mathcal{F})) \right] \).

\(^6\)\( \mathbb{N}_0 = \{0\} \cup \mathbb{N} \) denotes the non-negative integers.
Now, $P6$ follows by Lemma 3.3.7 (b) (with $E_0 = E$ and $D = F$) and (4.2.54). Both $L_0$ and $L_1$ are dissipative by $G1$ so $P4$ and the dissipativeness of $L_1$ follow by (4.2.58) and Lemma 4.2.7 (a, c) (with $E_0 = E$, $L_0 = L_0$ and $L_1 = L_1$). Moreover, $P1$ follows by Lemma 4.2.5 (b).

(b) Let $F$ be constructed and $P6$ verified as in (a). $P5$ and $P3$ follow by $G3$ (4.2.58) and Lemma 4.2.7 (b, d) (with $E_0 = E$, $L_0 = L_0$ and $L_1 = L_1$). $P1$ follows by Lemma 4.2.5 (b).

□

The third method alternatively assumes $S1$ and uses an arbitrary $A \in \mathcal{X}_\sigma^m(E)$ and metrizable compacts provided by $G4$ to construct a base satisfying $P7$.

**Lemma 4.2.14.** Let $E$ be a topological space, $A \in \mathcal{X}_\sigma^m(E)$, $L$ be a linear operator on $C_b(E; R)$ satisfy $S1$ $G4$ and $DA$ hold, and $D_0 \subset \mathcal{D}(L)$ be countable. Then:

(a) If $D1$ holds, then $P7$ $P2$ and $P3$ hold.
(b) If $L1 = 0$, and if $D1$ and $D2$ hold, then $P7$ $P1$ $P3$ and $P5$ hold.

**Proof.** (a) We construct the $(E_0, F)$ in $P7$ by induction. For $k = 0$, we define

$$A_0 = A \in \mathcal{X}_\sigma^m(E)$$

and

$$F_0 \doteq (D_0 \cup \{1\}) \subset \mathcal{D}(L).$$

For $k \in \mathbb{N}$, we assume $A_{k-1} \in \mathcal{X}_\sigma^m(E)$ and $F_0 \subset F_{k-1} \subset \mathcal{D}(L)$. $(A_{k-1}, \mathcal{O}_E(A_{k-1}))$ is a separable space by Proposition 3.3.26 (a, b) and Proposition 9.1.12 (d), so it has a countable dense subset $\{x_j^{k-1}\}_{j \in \mathbb{N}}$. For each $i \in \mathbb{N}$ and $x \in A_{k-1}$, one finds by $G4$ a sequence of metrizable compact sets $\{K_{n,j}^x\}_{n \in \mathbb{N}} \subset \mathcal{X}_\sigma^m(E)$ such that each $f \in \mathcal{D}(L)$ satisfies

$$\mathcal{L}f(x) - n \left( \| f^+|_{K_{n,j}} \|_\infty - f(x) \right) \leq 2^{-i}$$

for some sufficiently large $n = n_i \in \mathbb{N}$ independent of $x$. Then, we redefine

$$A_k = A_{k-1} \cup \bigcup_{i,j,n \in \mathbb{N}} K_{n,j}^{x_{n,j}} \in \mathcal{X}_\sigma^m(E).$$

By $S1$ and Proposition 3.3.26 (b, e) (with $A = A_k$ and $D = \mathcal{D}(L)$), there exists a countable $\mathcal{J}_k \subset \mathcal{D}(L)$ that separates points on $A_k$. $DA$ implies

$$\mathcal{O}_Q (F_{k-1} \cup J_k) \subset \mathcal{D}(L).$$

Then, we define

$$F_k \doteq \bigcup_{f \in \mathcal{O}_Q (F_{k-1} \cup J_k), q \in \mathbb{N}} \{ f, g_q^f \} \subset \mathcal{D}(L),$$

where each $g_q^f$ is chosen by $D1$ to satisfy (4.2.50).

By the construction above, $\{x_{n,j}^{k-1}\}_{j,k \in \mathbb{N}}$ is a countable dense subset of

$$E_0 = \bigcup_{k \in \mathbb{N}_0} A_k \in \mathcal{X}_\sigma^m(E)$$

under the subspace topology $\mathcal{O}_E(E_0)$. $E$ is a Hausdorff space by $S1$ and Proposition 9.2.1 (e) (with $A = E$ and $D = \mathcal{D}(L)$), so $E_0 \in \mathcal{B}(E)$ by Proposition 9.1.12 (a).
\( \mathcal{F} \) defined by (4.2.53) satisfies (4.2.54) and (4.2.1). \( \mathcal{F} \) contains \( J_k \) and separates points on \( A_k \) for all \( k \in \mathbb{N} \). \( \{A_k\}_{k \in \mathbb{N}} \) are nested by (4.2.62), so \( \mathcal{F} \) separates points on \( E_0 \) by Fact 10.1.17. Fixing \( i \in \mathbb{N} \) and \( f \in \mathfrak{D}(\mathcal{L}) \), we have by (4.2.61) that

\[
\mathcal{L} f(x_j^{k-1}) - n \left( \left\| \tilde{f}^+ \right\|_\infty - f(x_j^{k-1}) \right) 
\leq \mathcal{L} f((x_j^{k-1}) - n \left( \left\| f^+ \right\|_{K_{n,i}^j} \right) - f(x_j^{k-1}) \right) \leq 2^{-i}
\]

for all \( j, k \in \mathbb{N} \) and a sufficiently large \( n = n_i \in \mathbb{N} \) independent of any \( x_j^{k-1} \). Therefore, it follows that

\[
\mathcal{L} f(x) - n \left( \left\| f^+ \right\|_\infty - f(x) \right) \leq 2^{-i}, \forall x \in E_0
\]

by the denseness of \( \{x_j^{k-1}\}_{j,k \in \mathbb{N}} \) in \((E_0, \mathcal{O}_E(E_0))\) and the continuities of \( f \) and \( \mathcal{L} f \).

Now, \( \mathbf{P7} \) follows by Lemma 3.3.7 (b) (with \( \mathcal{D} = \mathcal{F} \)). \( \mathbf{P2} \) and \( \mathbf{P3} \) follow by (4.2.67) and Lemma 4.2.7 (b).

(b) We follow the proof of (a) to establish \((E_0, \mathcal{F})\) except for reconstructing

\[
\mathcal{F}_k \doteq \bigcup_{f \in \mathbb{R}_Q(\mathcal{F}_{k-1} \cup J_k), q \in \mathbb{N}} \{f, g_{q}^{f,k}, h_{q}^{f,k}\} \subset \mathfrak{D}(\mathcal{L}).
\]

Here, we choose each \( g_{q}^{f,k} \in \mathfrak{D}(\mathcal{L}) \) by \( \mathbf{D1} \) to satisfy (4.2.50). We find a constant \( \beta \in (0, \infty) \) and choose each \( h_{q}^{f,k} \in \mathfrak{D}(\mathcal{L}) \) by \( \mathbf{D2} \) to satisfy (4.2.55). Consequently, \( \mathcal{F} \) not only satisfies (4.2.54) and (4.2.1) but also satisfies (4.2.58).

Now, \( \mathbf{P7} \) follows by the same argument of (a). (4.2.20) follows by (4.2.58) and properties of uniform convergence. Hence, \( \mathbf{P5} \) follows by (4.2.67), Proposition 4.2.4 (d) and Lemma 4.2.7 (b, d). Moreover, \( \mathbf{P1} \) follows by Lemma 4.2.5 (b). \( \square \)

Our fourth construction is similar to the third with \( \mathbf{G4} \) replaced by \( \mathbf{G2} \).

**Lemma 4.2.15.** Let \( E \) be a topological space, \( A \in \mathcal{K}_m^m(E) \), \( \mathcal{L} \) be a linear operator on \( C_0(E; \mathbb{R}) \) and \( \mathcal{D}_0 \subset \mathfrak{D}(\mathcal{L}) \) be countable. If \( \mathbf{S1} \), \( \mathbf{D1} \), \( \mathbf{D2} \), \( \mathbf{G2} \) and \( \mathbf{DA} \) hold, then \( \mathbf{P7} \), \( \mathbf{P1} \) and \( \mathbf{P4} \) hold.

**Proof.** The proof of Lemma 4.2.14 (a) establishes the \((E_0, \mathcal{F})\) in \( \mathbf{P7} \) except \( E_0 = \bigcup_{k \in \mathbb{N}} A_k \in \mathcal{K}_m^m(E) \) is constructed as follows. For \( k = 0 \), we still define \( A_0 \) as in (4.2.59). For each \( i \in \mathbb{N}, x \in A_{k-1} \) and \( \beta \in \mathbb{Q}_+ \) one finds by \( \mathbf{G2} \) a sequence of metrizable compact sets \( \{K_{n,i}^{x,\beta}\}_{n \in \mathbb{N}} \subset \mathcal{K}_m^m(E) \) such that each \( f \in \mathfrak{D}(\mathcal{L}) \) satisfies

\[
\beta \left| f(x) \right| - \left\| (\beta f - \mathcal{L} f) \right\|_{K_{n,i}^{x,\beta}} \leq (\beta \left| f \right|_\infty + \left\| \mathcal{L} f \right\|_\infty + 1) 2^{-i}
\]

for some \( n = n_i^{x,\beta} \in \mathbb{N} \).

Now, we take a countable dense subset \( \{x_j^{k-1}\}_{j \in \mathbb{N}} \) of \((A_{k-1}, \mathcal{O}_E(A_{k-1}))\) and define

\[
A_k \doteq A_{k-1} \cup \left( \bigcup_{\beta \in \mathbb{Q}_+} \bigcup_{i,j,n \in \mathbb{N}} K_{n,i}^{x_j^{k-1},\beta} \right) \subset \mathcal{K}_m^m(E).
\]

\(^7\) The terminology “nested” was explained in Fact 3.3.4.

\(^8\) \( \mathbb{Q}_+ \) denotes the non-negative rational numbers.
4.2. REPLICA OPERATOR 73

By the reconstruction above, \((E_0, \mathcal{F})\) has almost the same properties as in the proof of Lemma 4.2.14 (b) except for (4.2.61). Instead, fixing \(f \in \mathcal{D}(\mathcal{L}), \beta \in \mathbb{Q}^+\) and \(i \in \mathbb{N}\), we have by (4.2.69) and (4.2.70) that

\[
\beta |f(x_j^{k-1})| - \left\| \beta \tilde{f} - \tilde{L}f \right\|_{\infty} \\
\leq \beta |f(x_j^{k-1})| - \left(\left\| \beta f - \mathcal{L}f \right\|_{K_{n,i}^{x_j^{k-1},\beta}}\right)_{\infty} \\
\leq (\beta \left\| f \right\|_{\infty} + \left\| \mathcal{L}f \right\|_{\infty} + 1) 2^{-i}
\]

(4.2.71)

for all \(j, k \in \mathbb{N}\) and some \(n = n_i f_{x_j^{k-1},\beta} \in \mathbb{N}\). Therefore, it follows that

\[
\beta \left\| \tilde{f} \right\|_{\infty} \leq \left\| \beta \tilde{f} - \tilde{L}f \right\|_{\infty}, \forall f \in \mathcal{D}(\mathcal{L}).
\]

(4.2.72)

by the denseness of \(\{x_j^{k-1}\}_{j,k \in \mathbb{N}}\) in \((E_0, \mathcal{C}_E(E_0))\) and the continuities of \(f\). Letting \(i \uparrow \infty\) in (4.2.72), we obtain

\[
\beta \left\| f \right\|_{\infty} \leq \left\| \beta \tilde{f} - \tilde{L}f \right\|_{\infty}, \forall f \in \mathcal{D}(\mathcal{L}).
\]

(4.2.73)

Next, we let \(\beta \in (0, \infty)\), take \(\{\beta_m\}_{m \in \mathbb{N}} \subset \mathbb{Q}^+ \cap (0, \infty)\) with \(\lim_{m \to \infty} \beta_m = \beta\) and find that

\[
\beta \left\| f \right\|_{\infty} \leq \lim_{m \to \infty} \beta_m \left\| \tilde{f} \right\|_{\infty} \\
\leq \left\| \beta \tilde{f} - \tilde{L}f \right\|_{\infty} + \lim_{m \to \infty} \left\| f \right\|_{\infty} |\beta_m - \beta| = \left\| \beta \tilde{f} - \tilde{L}f \right\|_{\infty}
\]

(4.2.74)

by (4.2.73) (with \(\beta = \beta_m\)), thus proving the dissipativeness of \(\tilde{L}_0\) and \(\tilde{L}_1\).

Now, \(\mathbf{P7}\) follows by Lemma 3.3.7 (b) (with \(D = \mathcal{F}\)). (4.2.58) holds as in the proof of Lemma 4.2.14 (a) and implies (4.2.20). Then, \(\mathbf{P4}\) and the dissipativeness of \(\tilde{L}_1\) follow by Lemma 4.2.7 (a, c). Moreover, \(\mathbf{P1}\) follows by Lemma 4.2.5 (b). \(\square\)

Moreover, it is worth noting that replica operators will exist under much less regularity of \(\mathcal{L}\) if no Markov-generator-type property is required.

**Proposition 4.2.16.** Let \(E\) be a topological space, \(A \in \mathcal{H}^m(E), \mathcal{D}_0 \subset \mathcal{D}(\mathcal{L})\) be countable and \(\mathcal{L}\) be a linear operator on \(C_b(E; \mathbb{R})\) satisfying \(\mathbf{D1}\) and \(\mathbf{DA}\). Then:

(a) If \(\mathbf{S2}\) holds, then \(\mathbf{P6}\) holds.

(b) If \(\mathbf{S1}\) holds, then \(\mathbf{P7}\) holds.

**Proof.** The construction of the desired bases for (a) and (b) are already contained in the proofs of Lemma 4.2.12 and Lemma 4.2.14 (a) respectively. \(\square\)
CHAPTER 5

Weak Convergence and Replication of Measure

Replication has an important role in weak convergence. Our baseable space and baseable subset results are used in §5.1 to establish mild conditions for the uniqueness, existence and consistency of weak limit points on the finite-dimensional Cartesian power $E^d$ of a topological space $E$. §5.2.1 introduces the (Borel) replica of possibly non-Borel measure on $E^d$. §5.2.2 relates weak convergence of the replica measures to that of the original ones, which will be a basic tool for our developments in Theme 2 and Theme 3. Using replica functions and measures, we extend two fundamental theorems in probability theory to non-classical settings. In §5.3.1, we establish a version of the Radon-Riesz Representation Theorem on a non-locally-compact and even non-Tychonoff space. In §5.3.2, we establish the Skorokhod Representation Theorem under mild conditions.

5.1. Tightness and weak convergence

Given a general topological space $E$, tightness and $m$-tightness unsurprisingly play key roles in establishing weak convergence on $E^d$. Existence of a tight subsequence usually implies existence of a weak limit point. Uniqueness requires slightly stronger tightness.

DEFINITION 5.1.1. Let $(E, \mathcal{U})$ be a measurable space, $S$ be a topological space and $\mathcal{A}$ be a $\sigma$-algebra on $S$.

- When $S \subset E$, $\Gamma \subset M^+(E, \mathcal{U})$ is sequentially tight (resp. sequentially $m$-tight) in $S$ if: (1) $\Gamma$ is an infinite set, and (2) Any infinite subset of $\Gamma$ admits a subsequence being tight (resp. $m$-tight) in $S$.

- $\Gamma \subset M^+(S, \mathcal{A})$ is sequentially tight (resp. sequentially $m$-tight) in $A \subset S$ if: (1) $\Gamma$ is an infinite set and $A$ is non-empty, and (2) Any infinite subset of $\Gamma$ admits a subsequence being tight (resp. $m$-tight) in $(A, \mathcal{O}_S(A))$.

- $\Gamma \subset M^+(S, \mathcal{A})$ is sequentially tight (resp. $m$-tight) if $\Gamma$ is sequentially tight (resp. $m$-tight) in $S$.

NOTE 5.1.2. Any type of sequential tightness in Definition 5.1.1 is defined for random variables as the corresponding property of their distributions.

The following sequential version of Proposition 3.3.25 relates sequential $m$-tightness and unique existence of Borel extension.

PROPOSITION 5.1.3. Let $I$ be a countable index set, $\{S_i\}_{i \in I}$ be topological spaces, $(S, \mathcal{A})$ be as in §2.7.22, $\Gamma \subset M^+(S, \mathcal{A})$ and $A \subset S$. Suppose in addition that $p_i(A) \in \mathcal{B}(S_i)$ is a Hausdorff subspace of $S_i$ for all $i \in I$. Then, $\Gamma$ is sequentially...
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\[ \text{m-tight in } A \text{ if and only if there exists a } \Gamma_0 \in \mathcal{P}_0(\Gamma) \text{ such that } \{\mu' = b\epsilon(\mu)\}_{\mu \in \Gamma \setminus \Gamma_0} \text{ is sequentially m-tight in } A. \]

**Proof.** Suppose \( \Gamma' \subset \Gamma \) is an infinite set and \( \mathcal{N}(b\epsilon(\mu)) \neq 1 \) for each \( \mu \in \Gamma' \). Given \( \Gamma' \) 's sequential m-tightness, there exists an m-tight subsequence \( \{\mu_n\}_{n \in \mathbb{N}} \subset \Gamma' \) and, by Proposition 3.3.25, \( b\epsilon(\mu_n) \) is a singleton for all \( n \in \mathbb{N} \). Contradiction! Hence, there exists \( \{\mu' = b\epsilon(\mu)\}_{\mu \in \Gamma \setminus \Gamma_0} \) for some \( \Gamma_0 \in \mathcal{P}_0(\Gamma) \). Finally, the m-tightness of \( \{\mu_n\}_{n \in \mathbb{N}} \setminus \Gamma_0 \) in \( \Gamma \) is equivalent to that of \( \{\mu' : \mu \in \{\mu_n\}_{n \in \mathbb{N}} \setminus \Gamma_0\} \) (if any) by Proposition 3.3.25.

We then give our conditions for Borel extensions of finite measures on the product measurable space \((E^d, \mathcal{B}(E)^{\otimes d})\) to have a unique weak limit point.

**Theorem 5.1.4.** Let \( E \) be a topological space, \( d \in \mathbb{N}, \Gamma \subset \mathcal{M}^+(E^d, \mathcal{B}(E)^{\otimes d}), \mathcal{D} \subset C_b(E; \mathbb{R}) \) and \( \mathcal{G} \triangleq mc[\Pi^d(\mathcal{D})] \). Suppose that:

(i) \( \Gamma \) is sequentially m-tight.

(ii) \( \{\int_E f(x)\mu(dx)\}_{\mu \in \Gamma} \) has at most one\(^1\) limit point in \( \mathbb{R} \) for all \( f \in \mathcal{G} \cup \{1\} \).

(iii) \( \mathcal{D} \) separates points on \( E \).

Then:

(a) \( \Gamma' \equiv \{\mu' = b\epsilon(\mu)\}_{\mu \in \Gamma \setminus \Gamma_0} \) exists for some \( \Gamma_0 \in \mathcal{P}_0(\Gamma) \) and is sequentially m-tight.

(b) \( \Gamma' \) has at most one weak limit point in \( \mathcal{M}^+(E^d) \).

(c) If, in addition, \( \{\mu(E^d)\}_{\mu \in \Gamma} \subset [a, b] \) for some \( 0 < a < b \), then \( \Gamma' \) has a unique weak limit point \( \nu \) in \( \mathcal{M}^+(E^d) \). Moreover, \( \nu \) is an m-tight measure with total mass\(^2\) in \([a, b]\) and satisfies

\[ \text{w- lim}_{n \to \infty} \mu'_n = \nu, \forall \{\mu_n\}_{n \in \mathbb{N}} \subset \Gamma \setminus \Gamma_0. \]

**Note 5.1.5.** The condition (iii) above is true for a wide subclass of Hausdorff spaces which need neither be Tychonoff nor baseable.

**Note 5.1.6.** Any \( \mathcal{D} \subset M_b(E; \mathbb{R}) \) satisfies

\[ \text{ca } [\Pi^d(\mathcal{D})] \subset M_b(E^d, \mathcal{B}(E)^{\otimes d}; \mathbb{R}) \]

and any \( \mathcal{D} \subset C_b(E; \mathbb{R}) \) satisfies

\[ \text{ca } [\Pi^d(\mathcal{D})] \subset C_b(E^d, \mathcal{B}(E)^{\otimes d}; \mathbb{R}) \]

by Proposition 9.2.5 (a) and properties of uniform convergence. So, \( \int_{E^d} f(x)\mu(dx) \) is well-defined for all \( f \in \text{ca}[\Pi^d(M_b(E; \mathbb{R}))] \) and \( \mu \in \mathcal{M}^+(E^d, \mathcal{B}(E)^{\otimes d}) \).

Before proving Theorem 5.1.4, we give a Portmanteau-type lemma for compact sets.

**Lemma 5.1.7.** Let \( C(E; \mathbb{R}) \) separate points on topological space \( E \). Then:

---

\(^1\)\( \mathcal{P}_0(\Gamma) \) is the family of all finite subsets of \( \Gamma \).

\(^2\)\{\int_{E^d} f(x)\mu(dx)\}_{\mu \in \Gamma} \) lies in \([-\|f\|_{\infty}E^d, \|f\|_{\infty}E^d]\) and has at least one limit point in \( \mathbb{R} \) by the Bolzano-Weierstrass Theorem. So, we are effectively assuming it has a unique limit point.

\(^3\)The notion of total mass was specified in 2.1.2. The notation \( \text{w-lim}_{n \to \infty} \mu_n = \nu \) introduced in 2.3 means that \( \nu \) is the weak limit of \( \{\mu_n\}_{n \in \mathbb{N}} \). In other words, it means \( \mu_n \Rightarrow \nu \) as \( n \uparrow \infty \) and \( \nu \) is the unique weak limit point of \( \{\mu_n\}_{n \in \mathbb{N}} \).
(a) [2.3.4] implies \( \mu(K) \geq \limsup_{n \to \infty} \mu_n(K) \) for all \( K \in \mathcal{K}(E) \).

(b) If \( \mu \) is a weak limit point of \( \Gamma \) in \( \mathcal{M}^+(E) \), and if \( \Gamma \) is tight (resp. \( m \)-tight) in \( A \subset E \), then \( \Gamma \cup \{ \mu \} \) is tight (resp. \( m \)-tight) in \( A \).

Note 5.1.8. \( C(E; \mathbb{R}) \) separating points (resp. strongly separating points) on \( E \) is equivalent to \( C_b(E; \mathbb{R}) \) separating points (resp. strongly separating points) on \( E \) (see Corollary 10.2.11).

Remark 5.1.9. The classical Portmanteau’s Theorem asserts that the mass of a weakly convergent sequence does not escape any closed set. The mass may no longer be confined by a general closed subset of the possibly non-Tychonoff space \( E \) (see Theorem 2.3.7). Nonetheless, Lemma 5.1.7 confirms that the subclass \( \mathcal{K}(E) \) of \( \mathcal{G}(E) \) with (5.1.4) still maintains this property.

Proof of Lemma 5.1.7 (a) \( (E, \mathcal{C}_E(E; \mathbb{R}))(E) \) is a Tychonoff coarsening of \( E \) by Proposition 3.3.1 (a, b). \( K \in \mathcal{K}(E, \mathcal{C}_E(E; \mathbb{R}))(E) \subset \mathcal{G}(E, \mathcal{C}_E(E; \mathbb{R}))(E) \) by Fact 10.2.9 (b) \( (D = C(E; \mathbb{R})) \).

(5.1.4) \( \mu_n \to \mu \) as \( n \to \infty \) in \( \mathcal{M}^+(E, \mathcal{C}_E(E; \mathbb{R}))(E) \)

by Fact 10.1.24 (b) \( (\mathcal{G} = E, \mathcal{C}_E(E; \mathbb{R}))(E) \). Now, (a) follows by Theorem 2.3.7 (a, b) \( (E, \mathcal{C}_E(E; \mathbb{R}))(E) \).

(b) Let \( \{ \mu_n \}_{n \in \mathbb{N}} \subset \Gamma \) satisfy [2.3.4]. By tightness (resp. \( m \)-tightness) of \( \Gamma \) in \( A \) and (a), there exist \( \{ K_p \}_{p \in \mathbb{N}} \subset \mathcal{K}(A, \mathcal{C}_E(A)) \) (resp. \( \mathcal{G}^m(A, \mathcal{C}_E(A)) \)) such that

(5.1.5) \( \mu(E \setminus K_p) \leq \liminf_{n \to \infty} \mu_n(E \setminus K_p) \leq \sup \mu(E \setminus K_p) \leq 2^{-p}, \forall p \in \mathbb{N} \).

□

Proof of Theorem 5.1.3 (a) \( E \) is a Hausdorff space by Proposition 9.2.1 (e) \( (A = E) \). By Proposition 5.1.3 (with \( I = \{ 1, \ldots, d \}, S_i = E \) and \( A = E^d \)), there exists a \( \Gamma_0 \subset \mathcal{P}_0(\Gamma) \) such that (a) holds.

(b) Suppose \( \{ \mu_{i,n} : i = 1,2, n \in \mathbb{N} \} \subset \Gamma' \) satisfy

(5.1.6) \( \mu_{i,n} \to \mu_i \) as \( n \to \infty \) in \( \mathcal{M}^+(E^d), \forall i = 1,2 \).

The sequential \( m \)-tightness of \( \Gamma' \) implies an \( m \)-tight subsequence \( \{ \mu_{i,n_k} \}_{k \in \mathbb{N}} \) for each \( i = 1,2 \). \( \mathcal{G} \) separates points on \( E^d \) by Proposition 9.2.5 (b), so does \( C(E^d, \mathbb{R}) \) by \( \{ 5.1.3 \} \). Then, \( \{ \mu_{i,n_k} \}_{k \in \mathbb{N}, i=1,2} \cup \{ \mu_1 \} \cup \{ \mu_2 \} \) is \( m \)-tight by Lemma 5.1.7 (b) \( (E = A = E^d \) and \( \Gamma = \{ \mu_{i,n_k} \}_{k \in \mathbb{N}, i=1,2} \) \).

It follows that

\[
\int_{E^d} f(x) \mu_1(dx) = \lim_{n \to \infty} f^*(\mu_{1,n})
\]

\[
= \lim_{n \to \infty} f^*(\mu_{2,n}) = \int_{E^d} f(x) \mu_2(dx) \forall f \in \mathcal{G} \cup \{ 1 \}
\]

by \( \{ 5.1.6 \}, \{ 5.1.3 \} \) and the fact that \( \{ \int_{E^d} f(x) \mu_{i,n}(dx) \}_{n \in \mathbb{N}, i=1,2} \) has at most one limit point in \( \mathbb{R} \) for all \( f \in \mathcal{G} \cup \{ 1 \} \). Now, \( \mu_1 = \mu_2 \) by Lemma 10.2.17 (a) \( (E = E^d \) and \( D = \mathcal{G} \)).

(c) \( E^d \) is a Hausdorff space by Proposition 9.1.2 (d). So, \( \Gamma' \) has a unique weak limit point \( \nu \) in \( \mathcal{M}^+(E^d) \) with \( \nu(E^d) \in [a,b] \) by (b) and Lemma 9.4.12 (with \( E = E^d \)).

\(^4\)The notation \( f^* \) was defined in (2.3.2).
and \( \Gamma = \Gamma' \). \( \nu \) is \( m \)-tight by Lemma 5.1.7 (b) (with \( E = A = E^d \) and \( \Gamma = \Gamma' \)). Furthermore, \( (5.1.11) \) follows by the sequential \( m \)-tightness of \( \Gamma' \), the fact
\[
\mu'(E^d) = \mu(E^d) \in [a, b] \subset (0, \infty), \forall \mu \in \Gamma \backslash \Gamma_0
\]
and Corollary 10.2.15 (with \( E = A = E^d \), \( \mu = \nu \), \( \Gamma = \Gamma' \) and \( \mu_n = \mu_n' \)). \( \square \)

For finite measures on infinite-dimensional Cartesian power of \( E \), Theorem 5.1.4 shows unique existence of weak limit point for their finite-dimensional distributions under suitable conditions. The following theorem considers the Kolmogorov Extension of these finite-dimensional weak limit points.

**Theorem 5.1.10.** Let \( E \) be a topological space, \( \Gamma \subset \mathcal{M}^+(E^1, \mathcal{B}(E)^{\otimes 1}) \) and \( D \subset C_b(E; \mathbb{R}) \). Suppose that:

(i) \( \{ \mu \circ p_{I_0}^{-1} \}_{\mu \in \Gamma} \) is sequentially \( m \)-tight for all \( I_0 \in \mathcal{P}_0(I) \).

(ii) \( \{ \int_{E} f(x) \mu \circ p_{I_0}^{-1}(dx) \}_{\mu \in \Gamma} \) has at most one limit point in \( \mathbb{R} \) for all \( f \in mc[\Pi^0(D)] \cup \{1\} \) for all \( I_0 \in \mathcal{P}_0(I) \).

(iii) \( D \) separates points on \( E \).

(iv) \( \{ \mu(E^1) \}_{\mu \in \Gamma} \subset [a, b] \) for some \( 0 < a < b \).

Then, there exist a unique \( \mu^\infty \in \mathcal{M}^+(E^1, \mathcal{B}(E)^{\otimes 1}) \) such that:

(a) The total mass of \( \mu^\infty \) lies in \( [a, b] \).

(b) \( \mu^\infty \circ p_{I_0}^{-1} \in \mathcal{M}^+(E^1) \) is \( m \)-tight for all \( I_0 \in \mathcal{P}_0(I) \).

(c) For each \( I_0 \in \mathcal{P}_0(I) \), there is some \( \Gamma^\infty_0 \in \mathcal{P}_0(\Gamma) \) such that \( \mu^\infty \circ p_{I_0}^{-1} \) is the weak limit of any subsequence of \( \{ \mu^\infty_0 = \beta \epsilon(\mu \circ p_{I_0}^{-1}) \}_{\mu \in \Gamma \backslash \Gamma^\infty_0} \) in \( \mathcal{M}^+(E^1) \).

**Proof of Theorem 5.1.10.** We fix \( I_1, I_2 \in \mathcal{P}_0(I) \) with \( I_1 \subset I_2 \), let \( p_{I_j} \) denote the projection from \( E^J \) to \( E^{I_j} \) for each \( j = 1, 2 \), use \( \bar{p} \) to specially denote the projection from \( E^J \) to \( E^{I_1} \) and observe \( \mu \circ p_{I_1}^{-1}(E^{I_2}) = \mu(E^1) \in [a, b] \) for all \( \mu \in \Gamma \) and \( j = 1, 2 \).

By Theorem 5.1.4 (a, c) (with \( d = \mathcal{N}(I_j) \) and \( \Gamma = \{ \mu \circ p_{I_j}^{-1} \}_{\mu \in \Gamma} \)), there exist \( \mu^\infty_{I_j} \in \mathcal{M}^+(E^1) \) and \( \Gamma^\infty_{I_j} \in \mathcal{P}_0(\Gamma) \) for each \( j = 1, 2 \) such that \( \mu^\infty_{I_j} \) is an \( m \)-tight measure with total mass in \( [a, b] \) and is the weak limit of any subsequence of
\[
\Gamma^\infty_{I_j} = \left\{ \mu^\infty_{I_j} = \beta \epsilon(\mu \circ p_{I_j}^{-1}) \right\}_{\mu \in \Gamma \backslash \Gamma^\infty_{I_j}} \subset \mathcal{M}^+(E^1).
\]

Suppose that \( \{ \mu_n \}_{n \in \mathbb{N}} \subset \Gamma \backslash (\Gamma^\infty_{I_1} \cup \Gamma^\infty_{I_2}) \) satisfies
\[
\text{w-} \lim_{n \to \infty} \mu^\infty_{I_j} = \mu^\infty_{I_j}, \forall j = 1, 2,
\]
where \( \mu^\infty_{I_j} = \beta \epsilon(\mu_n \circ p_{I_j}^{-1}) \in \Gamma^\infty_{I_j} \) for each \( n \in \mathbb{N} \) and \( j = 1, 2 \). \( \bar{p} \in C(E^{I_2}; E^{I_1}) \) by Fact 2.1.4 (a). It follows that
\[
\mu^\infty_{I_1, I_2} = \mu_n \circ p_{I_2}^{-1} \circ \bar{p}^{-1} \Rightarrow \mu^\infty_{I_2} \circ \bar{p}^{-1} \text{ as } n \to \infty \text{ in } \mathcal{M}^+(E^{I_1})
\]
by (5.1.10) and the Continuous Mapping Theorem (Theorem 10.1.23 (a)). So, \( \mu^\infty_{I_1} \) = \( \mu^\infty_{I_2} \circ p_{I_1}^{-1} \).

\[\text{5}\] The notation “\( \Pi^0(D) \)” was defined in §2.3 \( \Pi^0(D) = \Pi^d(D) \) with \( d = \mathcal{N}(I_0) \).

\[\text{6}\] The notion of weak limit was specified in §2.3.
By the argument above, $\Gamma$ uniquely determines \{\(\mu^\infty_{I_0} \in \mathcal{M}^+(E^{I_0})\)\}_{I_0 \in \mathcal{P}_0(I)}$ such that: (1) \{\(\mu^\infty_{I_0} \in \mathcal{M}^+(E^{I_0})\)\}_{I_0 \in \mathcal{P}_0(I)} satisfies the Kolmogorov consistency and admits a common total mass \(c \in [a, b] \), and (2) each \(\mu^\infty_{I_0}\) is \(m\)-tight and is the weak limit of any subsequence of \{\(\mu^\infty_{I_0} \in \mathcal{M}^+(E^{I_0})\)\}_{I_0 \in \mathcal{P}_0(I)} for some \(\Gamma^0_0 \in \mathcal{P}_0(\Gamma)\). \{\(E^{I_0}\)\}_{I_0 \in \mathcal{P}_0(I)} are all Hausdorff spaces by (5.1.3) (with \(d = \aleph(I_0)\)) and Proposition 9.2.1 (c) (with \(E = A = E^{I_0}\) and \(\mathcal{D} = C(E^{I_0}; \mathbb{R})\)). Now, the unique existence of \(\mu^\infty \in \mathcal{M}^+(E^1, \mathcal{B}(E)^{\otimes 1})\) satisfying \(\mu^\infty(E^1) = c\)

By the argument above, $\Gamma$ uniquely determines \{\(\mu^\infty_{I_0} \in \mathcal{M}^+(E^{I_0})\)\}_{I_0 \in \mathcal{P}_0(I)} such that: (1) \{\(\mu^\infty_{I_0} \in \mathcal{M}^+(E^{I_0})\)\}_{I_0 \in \mathcal{P}_0(I)} satisfies the Kolmogorov consistency and admits a common total mass \(c \in [a, b] \), and (2) each \(\mu^\infty_{I_0}\) is \(m\)-tight and is the weak limit of any subsequence of \{\(\mu^\infty_{I_0} \in \mathcal{M}^+(E^{I_0})\)\}_{I_0 \in \mathcal{P}_0(I)} for some \(\Gamma^0_0 \in \mathcal{P}_0(\Gamma)\). \{\(E^{I_0}\)\}_{I_0 \in \mathcal{P}_0(I)} are all Hausdorff spaces by (5.1.3) (with \(d = \aleph(I_0)\)) and Proposition 9.2.1 (c) (with \(E = A = E^{I_0}\) and \(\mathcal{D} = C(E^{I_0}; \mathbb{R})\)). Now, the unique existence of \(\mu^\infty \in \mathcal{M}^+(E^1, \mathcal{B}(E)^{\otimes 1})\) satisfying \(\mu^\infty(E^1) = c\)

Moreover,

\begin{equation}
\mathcal{M}^+(E^1, \mathcal{B}(E)^{\otimes 1}) \subset \mathcal{M}^+(E^d, \mathcal{B}(E)^{\otimes d})
\end{equation}

Moreover,

\begin{equation}
\mathcal{M}^+(E^d, \mathcal{B}(E)^{\otimes d}) \subset \mathcal{M}^+(E^d, \mathcal{B}(E)^{\otimes d})
\end{equation}

\begin{equation}
\mathcal{M}^+(E^1, \mathcal{B}(E)^{\otimes 1}) \subset \mathcal{M}^+(E^d, \mathcal{B}(E)^{\otimes d})
\end{equation}

\begin{proof}
The first line of (5.2.2) follows by (3.1.10) and Fact 2.1.1 (a) (with \(E = E^d\), \(\mathcal{D} = \mathcal{B}(E)^{\otimes d}\) and \(A = E^{I_0}\)). The second line of (5.2.2) and (5.2.3) follow by (3.1.10).
\end{proof}

\begin{notation}
If no confusion is caused, we will let $\overline{\pi}$ denote the replica of $\mu$ with respect to the underlying base and will not make special mention.
\end{notation}

\begin{proposition}
Let $E$ be a topological space, \((E_0, \mathcal{F}; \widehat{E}, \widetilde{\mathcal{F}})\) be a base over $E$ and $d \in \mathbb{N}$. Then:

(a) (5.2.1) well defines $\overline{\pi} \in \mathcal{M}^+(\widehat{E}^d)$. Moreover,

\begin{equation}
\overline{\pi}(A) = \mu \left( A \cap E_0^d \right), \forall A \in \mathcal{B}(\widehat{E}^d)
\end{equation}

(b) $\overline{\pi} \in \mathcal{P}(\widehat{E}^d)$ if and only if $\mu(E_0^d) = 1$.

(c) Any $\nu \in \mathcal{B}(\mu)$ satisfies $\overline{\nu} = \overline{\pi}$.
\end{proposition}

\begin{proof}
\end{proof}

\begin{addendum}
\item [\(\overline{\mu}|_A\)] was defined in (2.1.2)
\end{addendum}
(5.2.6) Thus \( \nu \) satisfies (5.2.10).

Now, (5.2.5) is well-defined and follows by (5.2.7) and (a).

(5.2.6) is well-defined and follows by (5.2.5) and (a).

The next proposition gives a sufficient condition for a Borel measure on \( \tilde{E}^d \) to be the replica of some Borel measure on \( E^d \).

**Proposition 5.2.5.** Let \( E \) be a topological space, \( (E_0, F, \tilde{E}, \tilde{F}) \) be a base over \( E \) and \( d \) in \( \mathbb{N} \). If \( \nu \in \mathcal{M}^+(\tilde{E}^d) \) is supported on \( A \subset E_0^d \) and \( A \in \mathcal{B}(E^d) \), then

\[
\mu = \frac{\nu}{|A|} \in \mathcal{M}^{+}(E^d)
\]

satisfies \( \mu \big| A = \nu \big| A \in \mathcal{M}^+(A, \mathcal{O}_{E^d}(A)) \) and \( \nu = \tilde{\nu} \).

**Proof.** We have by \( A \in \mathcal{B}(E^d) \) and Lemma 3.1.14 (a, b) that

\[
\mathcal{B}_{E^d}(A) = \mathcal{B}_{\tilde{E}^d}(A) \subset \mathcal{B}(E^d) \cap \mathcal{B}(\tilde{E}^d).
\]

By Fact 2.1.1 (a) (with \( \mu = \nu \) and \( (E, \mathcal{U}) = (\tilde{E}^d, \mathcal{B}(\tilde{E}^d)) \) and (5.2.11),

(5.2.12) \( \nu \big| A \in \mathcal{M}^+(A, \mathcal{O}_{E^d}(A)) = \mathcal{M}^+(A, \mathcal{O}_{E^d}(A)) \).

\( \tilde{\nu} \) was defined in Notation 4.1.5.
Then, \( \mu \in \mathcal{M}^+(E^d) \) by \((5.2.12)\) and Fact \((2.1.1)\) (with \( E = E^d, \mathcal{U} = \mathcal{B}(E^d) \) and \( \nu = \nu|_A \)). It follows by Fact \((2.1.1)\) (with \( E = E^d, \mathcal{U} = \mathcal{B}(E^d) \) and \( \nu = \nu|_A \)) and \((5.2.12)\) that
\[
(5.2.13) \quad \mu|_A = \nu|_A \in \mathcal{M}^+(A, \mathcal{O}_E(A)).
\]
It follows by the fact \( \nu(\widetilde{E}^d, A) = 0 \), \((5.2.13)\) and Fact \((2.1.1)\) (with \( E = \widetilde{E}^d, \mathcal{U} = \mathcal{B}(\widetilde{E}^d) \) and \( \mu = \nu \)) that
\[
(5.2.14) \quad \nu = (\nu|_A)|_{\widetilde{E}^d} = (\mu|_A)|_{\widetilde{E}^d} = \overline{\mu}.
\]

5.2.2. Weak convergence of replica measures. We now consider the association of weak convergence of Borel extensions on \( E^d \) and that of replica measures on \( \widetilde{E}^d \). The next proposition discusses the direction from \( E^d \) to \( \widetilde{E}^d \).

**Proposition 5.2.6.** Let \( E \) be a topological space, \((E_0, \mathcal{F}, \widetilde{E}, \mathcal{G})\) be a base over \( E, d \in \mathbb{N}, \mathcal{G} \triangleq \mathcal{mc}([\Pi^d(\mathcal{F}\setminus \{1\})]) \) and \( \{\mu_n\}_{n \in \mathbb{N}} \cup \{\mu\} \subset \mathcal{M}^+(E^d, \mathcal{B}(E^d)^d) \). Consider the following statements:

(a) The replica measures \( \{\overline{\mu}_n\}_{n \in \mathbb{N}} \) and \( \overline{\mu} \) satisfy
\[
(5.2.15) \quad \lim_{n \to \infty} \overline{\mu}_n = \overline{\mu} \text{ in } \mathcal{M}^+(\widetilde{E}^d).
\]

(b) The concentrated measures \( \{\mu_n|_{E^d}\}_{n \in \mathbb{N}} \) and \( \mu|_{E^d} \) satisfy
\[
(5.2.16) \quad \lim_{n \to \infty} \mu_n|_{E^d} = \mu|_{E^d} \text{ in } \mathcal{M}^+(E^d, \mathcal{O}_E(E^d)^d).
\]

(c) The original measures \( \{\mu_n\}_{n \in \mathbb{N}} \) and \( \mu \) satisfy
\[
(5.2.17) \quad \lim_{n \to \infty} \int_{E^d} f(x) \mathbf{1}_{E^d}(x) \mu_n(dx) = \int_{E^d} f(x) \mathbf{1}_{E^d}(x) \mu(dx), \forall f \in \mathcal{G} \cup \{1\}.
\]

(d) \( E^d_0 \) is a common support of \( \{\mu_n\}_{n \in \mathbb{N}} \cup \{\mu\} \). Moreover,
\[
(5.2.18) \quad \lim_{n \to \infty} \int_{E^d} f(x) \mu_n(dx) = \int_{E^d} f(x) \mu(dx), \forall f \in \mathcal{G} \cup \{1\}.
\]

(e) \( E^d_0 \) is a common support of \( \{\mu_n\}_{n \in \mathbb{N}} \cup \{\mu\} \). Moreover, there exist \( \{\mu'_n\}_{n \in \mathbb{N}} \in \mathcal{B}(\mu_n) \) such that
\[
(5.2.19) \quad \mu_n \Rightarrow \mu' \text{ as } n \uparrow \infty \text{ in } \mathcal{M}^+(E^d).
\]

(f) \( E^d_0 \) is a common support of \( \{\mu_n\}_{n \in \mathbb{N}} \cup \{\mu\} \). Moreover, there exist \( \{\mu'_n\}_{n \in \mathbb{N}} \in \mathcal{B}(\mu_n) \) such that
\[
(5.2.20) \quad \mu'_n|_{E^d_0} \Rightarrow \mu'|_{E^d_0} \text{ as } n \uparrow \infty \text{ in } \mathcal{M}^+(E^d_0, \mathcal{O}_E(E^d_0)^d).
\]

Then, (a) - (c) are equivalent. (c) - (f) are successively stronger. Moreover, (e) and (f) are equivalent when \( E^d \) is a Tychonoff space.

**Proof.** (a) \( \Rightarrow \) (b) \( \widetilde{E}^d \) is a Tychonoff space by Lemma \( 3.1.9 \) (c) and Proposition \( 9.3.2 \) (a). \((E^d_0, \mathcal{O}_E(E^d_0)^d)\) is a metrizable and separable space by Lemma \( 3.1.9 \) (d) (with \( A = E^d_0 \)) and is a Tychonoff space by Proposition \( 9.3.2 \) (a). Now, (b)

\[\text{□}\]

\(^9\)The integrals in \( 5.2.17 \) are well-defined by the proof of Proposition \( 5.2.4 \) (d, e). Those in \( 5.2.18 \) are well-defined by Note \( 5.1.6 \) (with \( D = \mathcal{F} \)).
follows by (5.2.15), (5.2.1), Lemma 10.2.13 (with $E = \hat{E}^d$, $A = E_0^d$, $\nu_n = \mu_n|_{E_0^d}$ and $\nu = \mu|_{E_0^d}$) and the Tychonoff property of $(E_0^d, \mathcal{E}(E_0)^d)$.

((b) $\to$ (c)) We have by Lemma 3.1.9 (b, d) (with $A = E_0^d$) that $\mathcal{G}|_{E_0^d} \subset C_b(E_0^d, \mathcal{E}(E_0)^d; \mathbb{R})$. Hence, we have by (5.2.16) and (5.2.2) that

$$
\lim_{n \to \infty} \int_{E_0^d} f(x) 1_{E_0^d}(x) \mu_n(dx) = \lim_{n \to \infty} \int_{E_0^d} (E_0^d, \mathcal{E}(E_0)^d) f|_{E_0^d}(x) \mu_n|_{E_0^d}(dx)
$$

(5.2.21)

$$
= \int_{(E_0^d, \mathcal{E}(E_0)^d)} f|_{E_0^d}(x) \mu|_{E_0^d}(dx)
$$

$$
= \int_{E_0^d} f(x) 1_{E_0^d}(x) \mu(dx), \ \forall f \in \mathcal{G} \cup \{1\}.
$$

((c) $\to$ (a)) It follows by Proposition 5.2.4 (e) (with $\mu = \mu_n$ and $\mu$ that

$$
\lim_{n \to \infty} \hat{f}^\ast(\pi_n) = \lim_{n \to \infty} \int_{E_0^d} f(x) 1_{E_0^d}(x) \mu_n(dx)
$$

(5.2.22)

$$
= \int_{E_0^d} f(x) 1_{E_0^d}(x) \mu(dx) = \hat{f}^\ast(\pi), \ \forall f \in \mathcal{G} \cup \{1\}.
$$

$\mathcal{M}^+(\hat{E}^d)$ is a metrizable space by Corollary 3.1.11 (c). Now, (a) follows by (5.2.22), Corollary 3.1.11 (b) (with $A = \hat{E}^d$) and the Hausdorff property of $\mathcal{M}^+(\hat{E}^d)$.

((d) $\to$ (c)) Note that if $E_0^d$ is a common support of $\{\mu_n\}_{n \in \mathbb{N}} \cup \{\mu\}$, then

$$
\int_{E_0^d} f(x) 1_{E_0^d}(x) \mu_n(dx) - \int_{E_0^d} f(x) \mu_n(dx)
$$

(5.2.23)

$$
= \int_{E_0^d} f(x) 1_{E_0^d}(x) \mu(dx) - \int_{E_0^d} f(x) \mu(dx) = 0, \ \forall n \in \mathbb{N}.
$$

((e) $\to$ (d)) follows by $\mathcal{F} \subset C_b(E; \mathbb{R})$ and Fact 10.2.12 (with $\mu = \mu_n$ or $\mu$).

In both (e) and (f), $E_0^d$ is a common support of $\{\mu_n\}_{n \in \mathbb{N}} \cup \{\mu^\prime\}$ and so $\mu_n = (\mu_n|_{E_0^d})|_{E_0^d}$ for all $n \in \mathbb{N}$ and $\mu^\prime = (\mu^\prime|_{E_0^d})|_{E_0^d}$ by Fact 2.1.1 (c) (with $E = E^d$, $\mathcal{F} = \mathcal{E}(E_0)$, $A = E_0^d$ and $\mu = \mu_n$ or $\mu^\prime$). It then follows by Lemma 10.2.13 (with $E = E^d$, $A = E_0^d$, $\mu_n = \mu_n|_{E_0^d}$ and $\mu = \mu^\prime|_{E_0^d}$) that (f) implies (e) in general, and (e) implies (f) when $E_0^d$ is a Tychonoff space.

The following corollary specializes Proposition 5.2.6 to probability measures.

**Corollary 5.2.7.** Let $E$ be a topological space, $(E_0, \mathcal{F}; \hat{E}, \hat{\mathcal{F}})$ be a base over $E$, $d \in \mathbb{N}$, $\mathcal{G} \triangleq \text{mc}[\Pi^d(\mathcal{F}\backslash\{1\})]$ and $\{\mu_n\}_{n \in \mathbb{N}} \cup \{\mu\} \subset \mathfrak{P}(E^d, \mathcal{B}(E)^{\otimes d})$. Then, the following statements are equivalent:

(a) The replica measures $\{\pi_n\}_{n \in \mathbb{N}}$ and $\pi$ satisfy

(5.2.24)

$$
\pi_n \Rightarrow \pi \text{ as } n \uparrow \infty \text{ in } \mathfrak{P}(E^d).
$$

(b) The concentrated measures $(\mu_n|_{E_0^d})_{n \in \mathbb{N}}$ and $\mu|_{E_0^d}$ satisfy

(5.2.25)

$$
\mu_n|_{E_0^d} \Rightarrow \mu|_{E_0^d} \text{ as } n \uparrow \infty \text{ in } \mathfrak{P}(E_0^d, \mathcal{E}(E_0)^d).
$$

(c) $E_0^d$ is a common support of $(\mu_n)_{n \in \mathbb{N}} \cup \{\mu\}$ and (5.2.18) holds.
Then, there exist \( \mu \in \mathcal{M}^+(E^d) \) and \( N \in \mathbb{N} \) such that:

(a) \( \mu \) is \( m \)-tight in \( E_0^d \) and \( \{ \mu'_n = \delta(\mu_n) \}_{n>N} \) exists.

(b) \( \{ \mu'_n \}_{n>N} \) satisfies

\[
(5.2.26) \quad \text{w-} \lim_{n \to \infty} \mu'_n|_{E_0^d} = \mu|_{E_0^d} \text{ in } \mathcal{M}^+(E_0^d, \mathcal{O}_E(E_0^d)) \\

\text{and}

(5.2.27) \quad \mu'_n \Rightarrow \mu \text{ as } n \uparrow \infty \text{ in } \mathcal{M}^+(E^d).
\]

**Proof.** \( \{ \mu_n \}_{n \in \mathbb{N}} \) is sequentially \( m \)-tight in \( (E_0^d, \mathcal{O}_E(E_0^d)) \) by Corollary 3.1.15 (a). There exists an \( N_1 \in \mathbb{N} \) such that \( \{ \mu_n \}_{n>N_1} \) are all supported on \( E_0^d \) by Fact 10.1.26 (with \((E, \mathcal{O}) = (E_0^d, \mathcal{O}_E(E_0^d))\), \( A = E_0^d \) and \( \Gamma = \{ \mu_n \}_{n \in \mathbb{N}} \)). There exist \( \nu \in \mathcal{M}^+(E_0^d, \mathcal{O}_E(E_0^d)) \) and \( N_2 \in \mathbb{N} \) such that \( \nu \) is \( m \)-tight in \( (E_0^d, \mathcal{O}_E(E_0^d)) \), \( \{ \nu'_n = \delta(\mu_n) \}_{n>N_2} \) exists and

\[
(5.2.28) \quad \text{w-} \lim_{n \to \infty} \nu'_n = \nu \text{ in } \mathcal{M}^+(E_0^d, \mathcal{O}_E(E_0^d))
\]

by Lemma 3.1.9 (e) (with \( A = E_0^d \)) and Theorem 5.1.4 (a, c) (with \( E = (E_0, \mathcal{O}_E(E_0)) \)), \( \Gamma = \{ \mu_n|_{E_0^d} \}_{n \in \mathbb{N}} \) and \( \mathcal{D} = \mathcal{F}\setminus\{1\} \).

\( \mu \triangleq \nu|_{E^d} \) satisfies \( \mu|_{E_0^d} = \nu \) by (3.1.10) and Fact 2.1.1 (c) (with \( (E, \mathcal{O}) = (E_0^d, \mathcal{O}_E(E_0^d)) \)).

\[
(5.2.29) \quad \nu'_n|_{E_0^d} = \delta\left(\left\{\mu_n|_{E_0^d}\right\}\right)^{E_0^d} = \delta(\mu_n) = \mu'_n, \forall n > N \triangleq N_1 \vee N_2
\]

by (3.1.10), Fact 2.1.1 (c) (with \( (E, \mathcal{O}) = (E_0^d, \mathcal{O}_E(E_0^d)) \)), \( A = E_0^d \) and \( \nu = \nu_n \) and Lemma 10.2.6 (b) (with \( I = \{1, \ldots, d\} \), \( S_i = E \), \( A = E_0^d \), \( \mu = \mu_n \) and \( \delta(\mu|_A) = \nu_n \)).

Hence, (5.2.26) follows by (5.2.28) and Fact 2.1.1 (c) (with \( E = E^d \), \( \mathcal{U} = \mathcal{B}(E^d) \)) and \( A = E_0^d \). (5.2.27) follows by (5.2.26) and Lemma 10.2.13 (with \( E = E^d \), \( A = E_0^d \), \( \mu = \nu_n \) and \( \mu = \nu \)).

**Corollary 5.2.9.** Let \((E_0, \mathcal{F}; \hat{E}, \hat{\mathcal{F}})\) be a base over topological space \( E \) and \( d \in \mathbb{N} \). If \( \{ \mu_n \}_{n \in \mathbb{N}} \subset \mathcal{M}^+(E^d, \mathcal{B}(E)^{\otimes d}) \) is sequentially tight in \( (E_0^d, \mathcal{O}_E(E_0^d)) \), and if their replicas satisfy

\[
(5.2.30) \quad \mu_n \Rightarrow \nu \text{ as } n \uparrow \infty \text{ in } \mathcal{M}^+(E^d),
\]

then there exist \( \mu \in \mathcal{M}^+(E^d) \) and \( N \in \mathbb{N} \) satisfying Proposition 5.2.8 (a, b) and, in particular, \( \nu = \mu \).
WEAK CONVERGENCE AND REPLICA MEASURE

Proof. There exists an $N_1 \in \mathbb{N}$ such that $\{\mu_n\}_{n > N_1}$ are all supported on $E_0^d$ by Fact 10.1.26 (with $(E, \mathcal{U}) = (E^d, \mathcal{B}(E)^{\otimes d})$, $A = E_0^d$ and $\Gamma = \{\mu_n\}_{n \in \mathbb{N}}$). It follows by (5.2.30) and Proposition 5.2.4 (e) (with $\mu = \mu_n$) that

\[ \lim_{n \to \infty} \int_{E_0^d} f(x) \mu_n(dx) = \lim_{n \to \infty} \hat{f}^*(\mu_n) = \hat{f}^*(\nu), \quad \forall f \in \mathcal{M} \left[ \Pi^d(F) \right]. \]

$\nu \in \mathcal{M}^+(\hat{E}^d)$ means $\nu(\hat{E}^d) > 0$ by our convention in §2.1.2. It then follows by the fact 1 $\in \Pi^d(F)$ and (5.2.31) that

\[ \mu_n(E^d) \in \left( \frac{\nu(\hat{E}^d)}{2}, \frac{3\nu(\hat{E}^d)}{2} \right) \subset (0, \infty), \quad \forall n > N_2 \]

for some $N_2 \in \mathbb{N} \cap (N_1, \infty)$. Now, we obtain the desired $\mu$ and $N$ by (5.2.31), (5.2.32) and Proposition 5.2.8 (with $n = N_2 + n$, $a = \nu(\hat{E})/2$ and $b = 3a$). \(\{\mu'_n = b \varepsilon(\mu_n)\}_{n \in \mathbb{N}}\) satisfies \[5.2.32\] by Proposition 5.2.4 (c) (with $\mu = \mu_n$ and $\nu = \mu'_n$) and Proposition 5.2.6 (a, e).

□

5.3. Generalization of two fundamental results

5.3.1. Integral representation of linear functional. The celebrated Riesz-Radon Representation Theorem was established for positive linear functionals on $C_0(E; \mathbb{R})$ \[11\] with $E$ being a locally compact Hausdorff space. This result is now extended to baseable spaces by replication, avoiding the local compactness assumption which is violated by many infinite-dimensional spaces. As mentioned in §3.2.2, baseable spaces need not be locally compact nor Tychonoff.

Theorem 5.3.1. Let $E$ be a $C_c(E; \mathbb{R})$-baseable space, $\varphi$ be a linear functional on $C_c(E; \mathbb{R})$ and

\[ \mathcal{B} \doteq \{ g \in C_c(E; \mathbb{R}) : 0 < ||g||_\infty \leq 1 \}. \]

Then, the following statements are equivalent:

(a) There exists a positive linear functional $\Lambda$ on $C_b(E; \mathbb{R})$ such that

\[ \varphi(g) \leq \Lambda(g), \quad \forall g \in C_c(E; \mathbb{R}) \]

and

\[ \lambda_0 \doteq \sup_{g \in \mathcal{B}} \varphi(g) = \Lambda(1) < \infty. \]

(b) There exists unique $\mathfrak{m}$-tight $\mu \in \mathcal{M}^+(E)$ such that

\[ \varphi(g) = g^*(\mu), \quad \forall g \in C_c(E; \mathbb{R}) \]

and

\[ \mu(E) = \sup_{g \in \mathcal{B}} g^*(\mu). \]

\[\text{10\mu'n} denote the replica of $\mu'_n$. \]

\[\text{11\text{Positiveness of a functional on} C_0(E; \mathbb{R}) \text{ means it maps non-negative functions into} \mathbb{R}^+.}\]
5.3. Generalization of Two Fundamental Results

Remark 5.3.2. In the theorem above, $E$ is Hausdorff by Fact 3.2.5(a). $C_c(E;\mathbb{R})$ is a possibly non-unit subalgebra of $C_b(E;\mathbb{R})$ and is a function lattice by Proposition 10.2.2(a). $C_c(E;\mathbb{R}) \neq \{0\}$ since $C_c(E;\mathbb{R})$ separates points on $E$, so $\mathcal{B} \neq \emptyset$ and the supremum in (5.3.3) is well-defined.

Proof of Theorem 5.3.1. ((a) → (b)) We divide our proof into six steps.

Step 1: Extend $\varphi$ to a positive linear functional on $C_b(E;\mathbb{R})$. $\varphi(g) \leq \Lambda(g) \leq 0$ for all non-positive $g \in C_c(E;\mathbb{R})$ by (5.3.16) and the positiveness of $\Lambda$, so $\varphi$ is also a positive linear functional. Then, there exists a positive linear functional $\Phi$ on $C_b(E;\mathbb{R})$ satisfying

(5.3.6) $\varphi = \Phi|_{C_c(E;\mathbb{R})}$

and

(5.3.7) $\Phi(g) \leq \Lambda(g), \ \forall g \in C_b(E;\mathbb{R})$

by a suitable version of the Hahn-Banach Theorem (see [AB06, Theorem 8.31]). In particular,

(5.3.8) $\lambda_0 = \Lambda(1) \geq \Phi(1) \geq \sup_{g \in C_c(E;\mathbb{R}) \setminus \{0\}} \Phi\left(\frac{g}{\|g\|_{\infty}}\right) \geq \sup_{g \in \mathcal{B}} \Phi(g) = \lambda_0$

by (5.3.3), (5.3.7), the positiveness of $\Phi$ and the fact

(5.3.9) $g \leq \|g\|_{\infty} \leq 1, \ \forall g \in \mathcal{B}$.

Step 2: Construct a suitable base. Letting

(5.3.10) $f_{g,a,b} = ag + b, \ \forall g \in C_c(E;\mathbb{R}), a, b \in \mathbb{R},$

we have that

(5.3.11) $\mathcal{D} \doteq \{ag : g \in C_c(E;\mathbb{R}) \cup \{1\}\} = \{f_{g,a,b} : g \in C_c(E;\mathbb{R}), a, b \in \mathbb{R}\}.$

$E$ is a $\mathcal{D}$-baseable space by Fact 3.3.2(d) (with $A = E$, $\mathcal{D} = C_c(E;\mathbb{R})$ and $\mathcal{D}' = \mathcal{D}$).

There exist $\{g_p\}_{p \in \mathbb{N}} \subset \mathcal{B}$ satisfying

(5.3.12) $\Phi(1) = \lambda_0 = \lim_{p \to \infty} \varphi(g_p) = \lim_{p \to \infty} \Phi(g_p)$

by (5.3.8), (5.3.3) and (5.3.6). We then find a base $(E, \mathcal{F}; \widehat{E}, \widehat{\mathcal{F}})$ over $E$ satisfying

(5.3.13) $\{g_p\}_{p \in \mathbb{N}} \subset (\mathcal{F} \cap \mathcal{B}) \subset (\mathcal{F} \setminus \{1\}) \subset C_c(E;\mathbb{R}) \subset \mathcal{D}$

by Lemma 3.3.7(c) (with $E_0 = E$ and $\mathcal{D}_0 = \{g_p\}_{p \in \mathbb{N}}$).

Step 3: Construct a replica positive linear functional $\widehat{\Phi}$ on $C(\widehat{E};\mathbb{R})$ satisfying

(5.3.14) $\widehat{\Phi}(h) = \Phi(h|E), \ \forall h \in C(\widehat{E};\mathbb{R}).$

$E_0 = E$ here, so

(5.3.15) $\widehat{f}_{g,a,b} = a\widehat{g} + b = a\widehat{g} + b, \ \forall f_{g,a,b} \in \mathcal{D}$

by Proposition 1.1.6(d) (with $d = k = 1$ and $E_0 = E$), Lemma 3.3.3(c) and Fact 10.2.3(b) (with $E = \widehat{E}$, $A = E$ and $f = \widehat{g}$).

(5.3.16) $ag(\widehat{\mathcal{F}}) = \left\{\widehat{f}_{g,a,b} : f_{g,a,b} \in ag(\mathcal{F})\right\}$. 

\(^{12}\)non-unit means excluding the constant function 1.
\(^{13}\)The terminology “function lattice” was specified in 2.2.3.
\(^{14}\)We noted in Notation 4.1.3 that $g \doteq \varphi(g; \widehat{E}, E_0, 0)$. 
is a linear subspace of $C(\hat{E}; \mathbb{R})$ on which
\begin{align}
\hat{\Phi}(f_{g,a,b}) = a \varphi(g) + b \lambda_0, \forall f_{g,a,b} \in \mathfrak{a}g(\hat{F})
\end{align}
defines a positive linear functional. Moreover,
\begin{align}
\hat{\Phi}(f_{g,a,b}) = \Phi(f_{g,a,b}) \leq \lambda_0 \| f_{g,a,b} \|_{\infty} = \lambda_0 \| f_{g,a,b} \|_{\infty}, \forall f_{g,a,b} \in \mathfrak{a}g(F)
\end{align}
by (5.3.6), the first equality of (5.3.12) and Fact 10.3.1 (a) (with $d = k = 1, E_0 = E$ and $f = f_{g,a,b}$). Hence, $\hat{\Phi}$ extends linearly onto $C(\hat{E}; \mathbb{R})$ and satisfies
\begin{align}
\lambda_0 = \Phi(1) = \hat{\Phi}(1) = \sup_{h \in C(\hat{E}; \mathbb{R})} \frac{\hat{\Phi}(h)}{\| h \|_{\infty}} < \infty
\end{align}
by (5.3.17), (5.3.18) (with $a = 0$ and $b = 1$) and the classical Hahn-Banach Theorem (see [Dud02 Theorem 6.1.4]).

$\mathfrak{a}g(\hat{F})$ is uniformly dense in $C(\hat{E}; \mathbb{R})$ by Corollary 3.1.10 (with $d = 1$ and $E_0 = E$). For each fixed $h \in C(\hat{E}; \mathbb{R})$, there exist $\{f_n\} \subset \mathfrak{a}g(F)$ such that
\begin{align}
\lim_{n \to \infty} \| h|_E - f_n \|_{\infty} = \lim_{n \to \infty} \| h|_E - \hat{f}_n \|_{\infty} \leq \lim_{n \to \infty} \| h - \hat{f}_n \|_{\infty} = 0.
\end{align}
$\Phi$ and $\hat{\Phi}$ are continuous functionals by (5.3.8), (5.3.19) and [Dud02 Theorem 6.1.2]. So, (5.3.18) and (5.3.20) imply
\begin{align}
\hat{\Phi}(h) = \lim_{n \to \infty} \hat{\Phi}(f_n) = \lim_{n \to \infty} \Phi(f_n) = \Phi(h|_E).
\end{align}
(5.3.21) verifies (5.3.14) and implies $\hat{\Phi}(h) \geq 0$ for all non-negative $h \in C(\hat{E}; \mathbb{R})$ since $\Phi$ is positive, thus proving the positiveness of $\hat{\Phi}$.

**Step 4: Establish integral representation of the replica functional.** Since $\hat{E}$ is a compact Polish space, we apply the classical Riesz Representation Theorem (see [KX95 Theorem 2.1.5]) to $\hat{\Phi}$ and obtain a $\nu \in \mathcal{M}^+(\hat{E})$ satisfying
\begin{align}
\hat{\Phi}(h) = h^*(\nu), \forall h \in C(\hat{E}; \mathbb{R}).
\end{align}
It follows by (5.3.14) and (5.3.22) that
\begin{align}
\Phi(h|_E) = h^*(\nu), \forall h \in C(\hat{E}; \mathbb{R}).
\end{align}
Moreover, it follows by (5.3.23), (5.3.12), and (5.3.22) that
\begin{align}
\nu(\hat{E}) = \Phi(1) = \lim_{p \to \infty} \Phi(g_p) = \lim_{p \to \infty} \hat{\Phi}(\hat{g}_p) = \lim_{p \to \infty} \hat{g}_p(\nu).
\end{align}

**Step 5: Establish the desired measure $\mu$.** We define
\begin{align}
A = \{ g \in C_c(E; \mathbb{R}) : f_{g,a,b} \in \mathfrak{a}g_Q(F) \text{ for some } a, b \in Q \text{ with } ab \neq 0 \},
\end{align}
let $K_g \in \mathcal{K}(E)$ denote the closure of $E \setminus g^{-1}(\{0\})$ in $E$ for each $g \in A$, and have by Corollary 3.1.15 (a) (with $d = 1$ and $E_0 = E$) that
\begin{align}
\{K_g\}_{g \in A} \subset \mathcal{K}(\hat{E}) \subset \mathcal{B}(\hat{E}).
\end{align}
$\mathfrak{a}g_Q(F)$ is a countable collection by Fact 10.1.14, so $A$ is also countable and
\begin{align}
A = \bigcup_{g \in A} K_g \in \mathcal{K}_{\sigma}^m(E) \cap \mathcal{B}(\hat{E})
\end{align}
by Corollary 3.1.15 (b) (with $d = 1$ and $E_0 = E$). We have $\{g_p\}_{p \in \mathbb{N}} \subset A$ and
\begin{align}
\hat{g}_p = \hat{g}_p 1_{K_{g_p}} \leq 1, \forall p \in \mathbb{N}
\end{align}
by (5.3.13), (5.3.15) (with $g = g_p$, $a = 1$ and $b = 0$) and (5.3.9),
\[
\nu(\hat{E}) \geq \nu(A) \geq \lim_{p \to \infty} \nu(K_{gp})
\]
(5.3.29)
\[
\geq \lim_{p \to \infty} \left( \hat{g}_p 1_{K_{gp}} \right)^*(\nu) = \lim_{p \to \infty} \hat{g}_p^*(\nu) = \nu(\hat{E})
\]
by (5.3.26), (5.3.27), (5.3.28), (5.3.29) and (5.3.31) (with $g = g_p$). Hence, we have by (5.3.29) and Proposition 5.2.5 that $\nu$ is the replica of $\mu \equiv (\nu|A)|^E$ and
\[
\mu(E) = \lambda_0 = \lim_{p \to \infty} \nu(\hat{g}_p) = \lim_{p \to \infty} \hat{g}_p^*(\mu)
\]
by (5.3.30), (5.3.12) and (5.3.31) (with $g = g_p$). Then, (5.3.5) follows by (5.3.32) and (5.3.9).

(b) $\to$ (a) $\Lambda(g) \equiv g^*(\mu)$ for each $g \in C_b(E; \mathbb{R})$ defines a positive linear functional with $\Lambda|_{C_c(E; \mathbb{R})} = \varphi$. (5.3.3) follows by (5.3.5). \hfill \Box

**Corollary 5.3.3.** Let $E$ be a $C_c(E; \mathbb{R})$-baseable space, $\varphi$ be a linear functional on $C_0(E; \mathbb{R})$ and $B$ be as in (5.3.1). Then, the following statements are equivalent:

(a) $\varphi$ is continuous and there exists a positive linear functional $\Lambda$ on $C_b(E; \mathbb{R})$ satisfying (5.3.2) and (5.3.3).

(b) There exists an $m$-tight $\mu \in \mathcal{M}^+(E)$ satisfying (5.3.3) and
\[
\varphi(g) = g^*(\mu), \ \forall g \in C_0(E; \mathbb{R}).
\]

**Proof.** ((a) $\to$ (b)) There exists an $m$-tight $\mu \in \mathcal{M}^+(E)$ satisfying (5.3.3) and (5.3.4) by Theorem 5.3.1. $C_c(E; \mathbb{R})$ is uniformly dense in $C_0(E; \mathbb{R})$ by Fact 3.2.5 (a) and Proposition 10.2.2 (b). Hence, (5.3.3) follows by (5.3.4), the continuity of $\varphi$ and the Dominated Convergence Theorem.

((b) $\to$ (a)) The functional defined by $\Lambda(g) \equiv g^*(\mu)$ for each $g \in C_b(E; \mathbb{R})$ satisfies $\Lambda|_{C_0(E; \mathbb{R})} = \varphi$, has linearity and is continuous by the Dominated Convergence Theorem. Moreover, (5.3.3) is immediate by (5.3.5). \hfill \Box

**5.3.2. Almost-sure representation of weak convergence.** We now generalize the Skorokhod Representation Theorem in [Jak97a]. Commonly, the Skorokhod Representation Theorem is established on separable metric spaces. [Jak97a] Theorem 2 extended this result to sequences of tight probability measures on baseable spaces. \footnote{While [Jak97a] did not use the term “baseable”, he did assume point-separability by countably many continuous functions.} m-tightness is equivalent to tightness in a baseable space $E$ by Corollary 5.3.20 (a). Hence, the conditions of the following theorem are strictly milder than those in [Jak97a].
THEOREM 5.3.4. Let $E$ be a topological space, $C(E;\mathbb{R})$ separate points on $E$, 
(5.3.34) $\mu_n \implies \mu_0$ as $n \uparrow \infty$ in $\mathcal{P}(E)$,
and $\{\mu_n\}_{n \in \mathbb{N}}$ be m-tight. Then, there exist E-valued random variables $\{\xi_n\}_{n \in \mathbb{N}_0}$
defined on the same probability space such that $\xi_n$ has distribution $\mu_n$ for all $n \in \mathbb{N}_0$
and $\{\xi_n\}_{n \in \mathbb{N}}$ converges to $\xi_0$ as $n \uparrow \infty$ almost surely.

PROOF. $\{\mu_n\}_{n \in \mathbb{N}_0}$ is m-tight by Lemma 5.1.7 (b) (with $\Gamma = \{\mu_n\}_{n \in \mathbb{N}}$). There
exists a base $(E_0, \mathcal{F}, \hat{E}, \hat{\mathcal{F}})$ such that $\{\mu_n\}_{n \in \mathbb{N}_0}$ is tight in
(5.3.35) $E_0 \in \mathcal{A}^m(E) \cap \mathcal{B}(E) \cap \mathcal{B}(\hat{E})$
by Lemma 10.3.2 (with $\Gamma_i = \{\mu_n\}_{n \in \mathbb{N}_0}$ and $D = C(E;\mathbb{R})$) and Corollary 3.1.15
(b).
(5.3.36) $\inf_{n \in \mathbb{N}_0} \mu_n(E_0) = \inf_{n \in \mathbb{N}_0} \bar{\mu}_n(E_0) = 1$
by the tightness of $\{\mu_n\}_{n \in \mathbb{N}_0}$ in $E_0$ and Proposition 5.2.4 (a). Furthermore,
(5.3.37) $\bar{\mu}_n \implies \bar{\mu}_0$ as $k \uparrow \infty$ in $\mathcal{P}(\hat{E})$
by (5.3.34) and Proposition 5.2.6 (a) (with $d = 1$, $\mu'_n = \mu_n$ and $\mu' = \mu_0$).

$E$ is a Polish space by Lemma 3.1.3 (c), so the classical Skorokhod Representation Theorem (see [Dud92]
Theorem 11.7.2) is applicable to $\{\bar{\mu}_n\}_{n \in \mathbb{N}_0}$, yielding random variables $\{\bar{\xi}_n\}_{n \in \mathbb{N}_0} \subset M(\Omega, \mathcal{F}; \mathcal{P}, \hat{E})$
that satisfy $\mathcal{P} \circ \bar{\xi}_0^{-1} = \bar{\mu}_n$ for all $n \in \mathbb{N}_0$
and
(5.3.38) $\mathcal{P}(\bar{\xi}_n \rightarrow \bar{\xi}_0$ as $n \uparrow \infty) = 1$.

Singletons are Borel sets in $\hat{E}$ by Proposition 9.1.2 (a, b). Hence, there exist
(5.3.39) $\xi_n \in M(\Omega, \mathcal{F}; E_0, \mathcal{B}(\hat{E}(E_0)))$, $\forall n \in \mathbb{N}_0$
such that
(5.3.40) $\inf_{n \in \mathbb{N}_0} \mathcal{P}(\xi_n = \bar{\xi}_n) \geq \inf_{n \in \mathbb{N}_0} \mathcal{P}(\bar{\xi}_n \in E_0) = \inf_{n \in \mathbb{N}_0} \bar{\mu}_n(E_0) = 1$
by (5.3.36) and Fact 10.1.3 (b) (with $(S, \mathcal{S}) = (\Omega, \mathcal{F})$, $(E, \mathcal{B}) = (\hat{E}, \mathcal{B}(\hat{E}))$, $A = E_0$
and $f = \xi_n$).

Now, we have by (5.3.38) and (5.3.40) that
(5.3.41) $\mathcal{P}(\xi_n \rightarrow \xi_0$ as $n \uparrow \infty) \geq \mathcal{P}(\bar{\xi}_n \rightarrow \bar{\xi}_0$ as $n \uparrow \infty) = 1$.
(5.3.35) gives $E_0 \in \mathcal{B}(\hat{E})$. It then follows by Lemma 3.1.14 (a) (with $d = 1$ and $A = E_0$)
and (5.3.39) that
(5.3.42) $\xi_n \in M(\Omega, \mathcal{F}; E_0, \mathcal{B}(\hat{E}(E_0))) \subset M(\Omega, \mathcal{F}; E)$, $\forall n \in \mathbb{N}_0$.

$\square$

REMARK 5.3.5. $([0, 1], ||\cdot||_\infty)$ mentioned in Example 3.3.21 is non-baseable.
Compact subsets of this normed space are automatically metrizable. This space is
Tychoff by Proposition 9.3.2 (a) and so its points are separated by $C([0, 1], ||\cdot||_\infty)$
by Proposition 9.3.1 (a, b). Theorem 5.3.4 applies in this case whereas
[Jak97a] Theorem 2 does not.

CHAPTER 6

Replication of Stochastic Process

This chapter is devoted to the replication of $E$-valued stochastic process via a base $(E_0, \mathcal{F}; \hat{E}, \hat{\mathcal{F}})$ over $E$. §6.1 introduces and discusses the basic properties of replica process. §6.3 focuses on the special case of càdlàg replica. Properties like tightness and relative compactness are simple to verify or even automatic on the compact Polish space $\hat{E}$. §6.2 associates the finite-dimensional convergence of general processes to that of their general replicas. §6.4 discusses tightness and weak convergence of càdlàg replicas as path-space-valued random variables. Finally, §6.5 considers when a family of processes can be contained in a baseable set to perform the desired replication. If necessary, the readers are referred to §2.5 where we specify our terminologies and notations about stochastic processes.

6.1. Introduction to replica process

6.1.1. Definition. Given a base $(E_0, \mathcal{F}; \hat{E}, \hat{\mathcal{F}})$ over topological space $E$, a replica of $E$-valued process $X$ is a related process that takes values in the compact Polish space $\hat{E}$. Since $X$ and its replicas may live in different spaces, they are related by the mappings $\otimes \mathcal{F}$ and $\otimes \hat{\mathcal{F}}$ rather than their own values.

Definition 6.1.1. Let $E$ be a topological space and $(\Omega, \mathcal{F}, P; X)$ be an $E$-valued process. With respect to a base $(E_0, \mathcal{F}; \hat{E}, \hat{\mathcal{F}})$ over $E$, an $\hat{E}$-valued process $(\Omega, \mathcal{F}, P; \hat{X})$ is said to be a replica of $X$ if

\[
P\left(\otimes \hat{\mathcal{F}} \circ \hat{X}_t = \otimes \mathcal{F} \circ X_t\right) \geq P\left(\otimes \mathcal{F} \circ X_t \in \otimes \hat{\mathcal{F}}(\hat{E})\right), \quad \forall t \in \mathbb{R}^+.
\]

Note 6.1.2. An $E$-valued process $X$ may have multiple replicas, divisible into equivalence classes by indistinguishability\(^2\).

We make the following notations for simplicity.

Notation 6.1.3. Let $E$ be a topological space, $(\Omega, \mathcal{F}, P; X)$ be an $E$-valued process.

- $\text{rep}(X; E_0, \mathcal{F})$ denotes the family of all equivalence classes of $X$’s replicas with respect to $(E_0, \mathcal{F}; \hat{E}, \hat{\mathcal{F}})$ under the equivalence relation of indistinguishability.
- $\text{rep}_m(X; E_0, \mathcal{F}), \text{rep}_p(X; E_0, \mathcal{F})$ and $\text{rep}_c(X; E_0, \mathcal{F})$ denote the measurable\(^4\), progressive and càdlàg members of $\text{rep}(X; E_0, \mathcal{F})$, respectively.

\(^1\)“$(\Omega, \mathcal{F}, P; X)$” as defined in §2.4 means an $E$-valued random variable or process $X$ defined on probability space $(\Omega, \mathcal{F}, P)$. We imposed in §2.6 that the probability space $(\Omega, \mathcal{F}, P)$ is complete.

\(^2\)The terminology “indistinguishability” was explained in §2.5.

\(^3\)“rep” is “rep” in fraktur font which stands for “replica”.

\(^4\)The notions of measurable process and progressive process were reviewed in §2.5.
\[ \hat{X} = \text{rep}(X; E_0, \mathcal{F}) \text{ means } \text{rep}(X; E_0, \mathcal{F}) \text{ is the single equivalence class } \{ \hat{X} \}. \]

Similar notations apply to the above-mentioned subfamilies of \( \text{rep}(X; E_0, \mathcal{F}) \).

**Remark 6.1.4.** The notation \( \text{"rep}(X; E_0, \mathcal{F})" \) merely specifies the first two components \((E_0, \mathcal{F})\) of the base \((E_0, \mathcal{F}, \hat{E}, \hat{\mathcal{F}})\) since Corollary 3.1.4 and Theorem 3.1.8 showed that this base is totally determined by \((E_0, \mathcal{F})\).

**Note 6.1.5.**

\( \hat{E}, \hat{\mathcal{F}} \) and \( \hat{\mathcal{F}}(\hat{E}) \) (as a subspace of \( R^\infty \)) are Polish spaces by Proposition 9.1.11 (f), Lemma 3.1.3 (c) and (3.1.3).

\( D(R^+; R), D(R^+; R^\infty), D(R^+; \hat{E}) \) and \( D(R^+; \hat{\mathcal{F}}(\hat{E})) \) are well-defined Polish spaces by Proposition 9.6.10 (d).

\( B(\hat{E}^d) = B(\hat{E}) \otimes^d \) for all \( d \in \mathbb{N} \) by Fact 3.1.12, so finite-dimensional distributions of any \( \hat{E} \)-valued process (especially any replica process) are all Borel probability measures.

The following proposition justifies the general existence of replica processes.

**Proposition 6.1.6.** Let \( E \) be a topological space, \((E_0, \mathcal{F}; \hat{E}, \hat{\mathcal{F}})\) be a base over \( E \) and \((\Omega, \mathcal{F}, \mathbb{P}; X)\) be an \( E \)-valued process. Then:

(a) \( \text{rep}(X; E_0, \mathcal{F}) \) is non-empty.

(b) If \( X \) is a measurable process, then \( \text{rep}_m(X; E_0, \mathcal{F}) \) is non-empty.

**Proof.** (a) We fix \( x_0 \in E_0 \), let \( \varphi \) be the identity mapping on \( R^\infty \) and define

\[
\varphi_{x_0} \triangleq \text{var} \left( \varphi; R^\infty, \hat{\mathcal{F}}(\hat{E}), \mathcal{F}(x_0) \right).
\]

\( \otimes \hat{\mathcal{F}}(\hat{E}) \in \mathcal{B}(R^\infty) \) by (3.1.5). \( R^\infty \) is a Polish space, so \( \{ \otimes \mathcal{F}(x_0) \} \in \mathcal{B}(R^\infty) \) by Proposition 9.1.2 (a, b). We then have that

\[
\varphi_{x_0}(y) = y, \forall y \in \otimes \hat{\mathcal{F}}(\hat{E})
\]

and

\[
\varphi_{x_0} \in M \left( R^\infty; \otimes \hat{\mathcal{F}}(\hat{E}) \right)
\]

by Fact 10.1.3 (b) (with \((S, \mathcal{A}) = (E, \mathcal{A}) = (R^\infty, \mathcal{B}(R^\infty)), A = \otimes \hat{\mathcal{F}}(\hat{E}), f = \varphi \) and \( y_0 = \otimes \mathcal{F}(x_0) \)). It follows that

\[
(\otimes \hat{\mathcal{F}})^{-1} \circ \varphi_{x_0} \circ \otimes \mathcal{F} \in M(E; \hat{E})
\]

by (3.1.3), (6.1.4) and Lemma 3.1.3 (e). Hence,

\[
\hat{X} \triangleq \text{var} \left[ (\otimes \hat{\mathcal{F}})^{-1} \circ \varphi_{x_0} \circ \otimes \mathcal{F} \right] \circ X
\]

\[ \text{While the finite-dimensional distributions of any Polish-space-valued process are Borel, this is not necessarily true for general processes (see §2.5).} \]

\[ \text{"var(\cdot)" was introduced in Notation 4.1.1.} \]
well defines an \( \hat{E} \)-valued process \((\Omega, \mathcal{F}, \hat{P}; \hat{X})\) by Fact 10.1.29 (a) (with \( S = \hat{E} \) and \( f = (\otimes \hat{\mathcal{F}})^{-1} \circ \varphi_{x_0} \circ (\otimes \mathcal{F}) \)). It follows by (6.1.10) and (6.1.11) that\footnote{By (6.1.12), \( \varphi_{x_0} \circ (\otimes \mathcal{F}) \circ X_t \in (\otimes \hat{\mathcal{F}}) \) might not imply \( \otimes \hat{\mathcal{F}} \circ X_t \in (\otimes \hat{\mathcal{F}}) \) when \( \otimes \mathcal{F}(x_0) \in (\otimes \hat{\mathcal{F}}) \).}

\[
\mathbb{P} \left( \bigotimes \mathcal{F} \circ X_t \in \bigotimes \hat{\mathcal{F}}(\hat{E}) \right) \leq \mathbb{P} \left( \bigotimes \hat{\mathcal{F}} \circ \left( \bigotimes \hat{\mathcal{F}} \right)^{-1} \circ \varphi_{x_0} \circ \bigotimes \mathcal{F} \circ X_t \in \bigotimes \hat{\mathcal{F}}(\hat{E}) \right) \leq \mathbb{P} \left( \bigotimes \hat{\mathcal{F}} \circ \hat{X}_t = \bigotimes \mathcal{F} \circ X_t \right), \quad \forall t \in \mathbb{R}^+,
\]

thus proving \( \hat{X} \in \text{rep}(X; E_0, \mathcal{F}) \) by (6.1.11).

(b) Let \( \hat{X} \) be as above and define \( \xi(t, \omega) \equiv X_t(\omega) \) and \( \hat{\xi}(t, \omega) \equiv \hat{X}_t(\omega) \) for each \( (t, \omega) \in \mathbb{R}^+ \times \Omega \). If \( X \) is a measurable process, then

\[
\hat{\xi} = \left( (\bigotimes \hat{\mathcal{F}})^{-1} \circ \varphi_{x_0} \circ \bigotimes \mathcal{F} \circ \xi \right) \in M \left( \mathbb{R}^+ \times \Omega, \mathcal{B}(\mathbb{R}^+) \otimes \mathcal{F}; \hat{E}, \mathcal{B}(\hat{E}) \right)
\]

by (6.1.10), thus proving \( \hat{X} \in \text{rep}_m(X; E_0, \mathcal{F}) \).

6.1.2. Association with the original process. The next two propositions expose the connection between the original and replica processes using properties of the base.

PROPOSITION 6.1.7. Let \( E \) be a topological space, \((E_0, \mathcal{F}; \hat{E}, \hat{\mathcal{F}})\) be a base over \( E \), \( T \subset \mathbb{R}^+ \), \((\Omega, \mathcal{F}, \hat{P}; X)\) be an \( E \)-valued process satisfying

\[
\inf_{t \in T} \mathbb{P} \left( X_t \in E_0 \right) = 1
\]

and \( \hat{X}, \hat{X}_1, \hat{X}_2 \in \text{rep}(X; E_0, \mathcal{F}) \). Then:

(a) \( \hat{X} \) satisfies

\[
\inf_{t \in T} \mathbb{P} \left( X_t = \hat{X}_t \in E_0 \right) = 1.
\]

Moreover, \( \hat{X}_1 \) and \( \hat{X}_2 \) satisfy

\[
\inf_{t \in T} \mathbb{P} \left( \hat{X}_1^t = \hat{X}_2^t \in E_0 \right) = 1.
\]

(b) \( \mathbb{P} \circ \hat{X}_T^{-1} \) is the replica measure of \( \mathbb{P} \circ X_T^{-1} \) for all \( T_0 \in \mathcal{P}(T_0) \).

PROOF. (a) (6.1.10) follows by (6.1.11) and Fact 10.3.3 (with \( Y = \hat{X} \)). (6.1.11) is immediate by (6.1.10) (with \( \hat{X} = X_1 \) or \( X_2 \)).

(b) Note 6.1.5 implies that \( \mathbb{P} \circ \hat{X}_T^{-1} \in \mathcal{P}(\hat{E}T_0) \). Moreover, we have by (a) that

\[
\mathbb{P} \left( \hat{X}_{T_0} \in A \right) = \mathbb{P} \left( X_{T_0} \in A \cap E_0^{T_0} \right), \quad \forall A \in \mathcal{B}(\hat{E}T_0)
\]

and

\[
\mathbb{P} \left( X_{T_0} = \hat{X}_{T_0} \in E_0^{T_0} \right) = 1.
\]

\[\square\]
Proposition 6.1.8. Let $E$ be a topological space, $(E_0, F; \hat{E}, \hat{F})$ be a base over $E$, $T \subset \mathbb{R}^+$ and $(\Omega, \mathcal{F}, \mathbb{P}; X)$ be an $E$-valued process satisfying

\[(6.1.14) \quad \inf_{t \in T} \mathbb{P} \left( \bigotimes F \circ X_t \in \bigotimes \hat{F}(\hat{E}) \right) = 1. \]

Then:

(a) Any $\hat{X} \in \text{rep}(X; E_0, F)$ satisfies

\[(6.1.15) \quad \mathbb{P} \left( f \circ X_{T_0} = \hat{f} \circ \hat{X}_{T_0} \right) = 1, \quad \forall f \in \text{ca} [\Pi^{T_0}(F)], T_0 \in \mathcal{P}_0(T) \]

and

\[(6.1.16) \quad \mathbb{E} \left[ f \circ X_{T_0} \right] = \mathbb{E} \left[ \hat{f} \circ \hat{X}_{T_0} \right], \quad \forall f \in \text{ca} [\Pi^{T_0}(F)], T_0 \in \mathcal{P}_0(T). \]

(b) If $T \subset \mathbb{R}^+$ is dense, then $\text{rep}_c(X; E_0, F)$ is at most a singleton.

Proof. (a) Any $\hat{X} \in \text{rep}(X; E_0, F)$ satisfies

\[(6.1.17) \quad \inf_{t \in T} \mathbb{P} \left( \bigotimes F \circ \hat{X}_t = \bigotimes F \circ X_t \right) \geq \inf_{t \in T} \mathbb{P} \left( \bigotimes F \circ X_t \in \bigotimes \hat{F}(\hat{E}) \right) = 1 \]

by \[6.1.1\] and \[6.1.14\]. Now, \[6.1.15\] follows by \[6.1.17\] and properties of uniform convergence. \[6.1.16\] is immediate by \[6.1.15\].

(b) $T$ must have a countable subset $T_0$ being dense in $\mathbb{R}^+$. $\varpi(\bigotimes \hat{F})$ is injective on $D(\mathbb{R}^+; \hat{E})$ by Lemma \[3.1.3\] (a) and Fact \[10.1.18\] (with $E = A = \hat{E}$ and $D = \hat{F}$). Given any $\hat{X}_1, \hat{X}_2 \in \text{rep}_c(X; E_0, F)$, $\{\varpi(\bigotimes \hat{F}) \circ \hat{X}\}_{i=1,2}$ are càdlàg processes by \[3.1.3\] and Fact \[10.1.31\] (a) (with $E = \hat{E}$, $S = \mathbb{R}^\infty$ and $f = \bigotimes \hat{F}$), and

\[(6.1.18) \quad \mathbb{P} \left( \hat{X}_1 = \hat{X}_2 \right) = \mathbb{P} \left( \varpi(\bigotimes \hat{F}) \circ \hat{X}_1 = \varpi(\bigotimes \hat{F}) \circ \hat{X}_2 \right) \geq \mathbb{P} \left( \bigotimes \hat{F} \circ \hat{X}_1 = \bigotimes \hat{F} \circ \hat{X}_2, \forall t \in T_0 \right) = 1 \]

by (a) (with $\hat{X} = \hat{X}_i$), Proposition \[10.1.30\] (g) and the injectiveness of $\varpi(\bigotimes \hat{F})$. \qed

The following consequence of \[3.1.1\] is apparent but indispensable.

Fact 6.1.9. \[6.1.9\] is stronger than \[6.1.14\].

6.1.3. Application to replicating measure-valued processes.

Lemma 6.1.10. Let $E$ be a topological space, $(E_0, F; \hat{E}, \hat{F})$ be a base over $E$, $\varphi \cong \bigotimes \text{mc}(F)^{10}$, $\tilde{\varphi} \cong \bigotimes \text{mc}(\hat{F})^*$, $S_0 \in \mathcal{B}(\mathbb{R}^\infty)$ be contained in $\mathcal{D}[\mathcal{M}^+(\hat{E})]$, $y_0 \in S_0$, $\phi$ be the identity mapping on $\mathbb{R}^\infty$ and $X \in M(\Omega, \mathcal{F}; \mathcal{M}^+(E))^\dagger$ satisfy

\[(6.1.19) \quad \mathbb{P} (\varphi \circ X \in S_0) = 1. \]

Then:

(a) $\Psi \cong \tilde{\varphi}^{-1} \circ \text{var}(\phi; \mathbb{R}^\infty, S_0, y_0)$ satisfies

\[(6.1.20) \quad \Psi \in M \left[ R^\infty; \tilde{\varphi}^{-1}(S_0), \mathcal{O}_{\mathcal{M}^+(E)} (\tilde{\varphi}^{-1}(S_0)) \right]. \]

9The notations “$\varpi(f)$” and “$\varpi(\bigotimes \hat{F})$” were defined in \[3.2.1\.

10The notation “$\text{mc}(F)^*$” was specified in \[2.3\.

11$X \in M(\Omega, \mathcal{F}; \mathcal{M}^+(E))$ means $X$ is a finite Borel random measure on $E$.\]
(b) $Y \overset{\bullet}{=} \Psi \circ \varphi \circ X \in \mathcal{M}(\Omega, \mathcal{F}; \mathcal{M}^+(\mathcal{E}))$ satisfies

\[ \mathbb{P} \left( f^* \circ X = \hat{f}^* \circ Y \right) = 1, \quad \forall f \in \mathfrak{ca}(\mathcal{F}). \]

(c) If

\[ \{ \omega \in \Omega : X(\omega)(E \setminus E_0) > 0 \} \in \mathcal{N}(\mathbb{P}), \]

then $Y(\omega)$ equals the replica (measure) of $X(\omega)$ for $\mathbb{P}$ almost all $\omega \in \Omega$.

(d) If $A \in \mathcal{B}^s(E)$ satisfies $A \subset E_0$ and

\[ \{ \omega \in \Omega : X(\omega)(E \setminus A) > 0 \} \in \mathcal{N}(\mathbb{P}), \]

then $(hf)^* \circ X$ and $(\overline{h}A\hat{f})^* \circ Y$ belong to $\mathcal{M}(\Omega, \mathcal{F}; \mathbb{R}^k)$ and satisfy

\[ \mathbb{P} \left( (hf)^* \circ X = (\overline{h}A\hat{f})^* \circ Y \right) = 1 \]

for all $f \in \mathfrak{ca}(\mathcal{F})$, $h \in M_b(E; \mathbb{R}^k)$ and $k \in \mathbb{N}$.

**Remark 6.1.11.** Every $f \in C_b(E; \mathbb{R})$ satisfies $f^* \in C(\mathcal{M}^+(E); \mathbb{R})$ by the definition of weak topology and so $f^* \circ X \in \mathcal{M}(\Omega, \mathcal{F}; \mathbb{R})$. For $f \in M_b(E; \mathbb{R})$, however, $f^*$ does not necessarily belong to $\mathcal{M}(\mathcal{M}^+(E); \mathbb{R})$ in general, nor is $f^* \circ X$ always a random variable (see e.g. Example [9.4.6]).

**Proof of Lemma 6.1.10.** (a) $\mathfrak{mc}(\mathcal{F})^*$ (resp. $\mathfrak{mc}(\mathcal{F}^*)^*$) is a countable subset of $C(\mathcal{M}^+(E); \mathbb{R})$ (resp. $C(\mathcal{M}^+(\mathcal{E}); \mathbb{R})$) by Fact 10.1.14 (with $E = E$ or $\mathcal{E} = \mathcal{F}$ or $\mathcal{F}$ and $d = k = 1$), Definition 3.1.1 and Lemma 3.1.3 (a). Then, we have by Fact 2.1.4 (b) that

\[ \varphi \in C(\mathcal{M}^+(E); \mathbb{R}^\infty). \]

At the same time, we have that

\[ \hat{\varphi} \in \mathfrak{im} \left( \mathcal{M}^+(\mathcal{E}); \mathbb{R}^\infty \right) \]

by Corollary 3.1.11 (b) (with $d = 1$ and $A = \mathcal{E}$) and Lemma 10.1.7 (b) (with $E = \mathcal{M}^+(\mathcal{E})$, $S = \mathbb{R}^\infty$ and $\mathcal{D} = \mathfrak{mc}(\mathcal{F})^*$). Furthermore,

\[ \mathfrak{var} (\varphi; \mathbb{R}^\infty; S_0, y_0) \in \mathcal{M}(\mathbb{R}^\infty; S_0) \]

by the fact $S_0 \in \mathcal{B}(\mathbb{R}^\infty)$ and Fact 10.1.3 (b) (with $(S, \omega') = (E, \mathcal{F}) = (\mathbb{R}^\infty, \mathcal{B}(\mathbb{R}^\infty))$, $A = S_0$ and $f = \phi$). Now, (a) follows by (6.1.26) and (6.1.27).

(b) $Y \in \mathcal{M}(\Omega, \mathcal{F}; \mathcal{M}^+(\mathcal{E}))$ by (6.1.25) and (a). $\Psi_{|S_0} = \hat{\varphi}^{-1}|_{S_0}$, hence

\[ \mathbb{P} (\varphi \circ X = \hat{\varphi} \circ Y \in S_0) \geq \mathbb{P} (\varphi \circ X \in S_0) = 1 \]

by (6.1.19), which implies

\[ \mathbb{P} (g^* \circ X = \hat{g}^* \circ Y, \forall g \in \mathfrak{ag}(\mathcal{F})) = 1 \]

by linearity of integral. Fixing $f \in \mathfrak{ca}(\mathcal{F})$ and $g \in \mathfrak{ag}(\mathcal{F})$, we find that

\[ \left| f^* \circ X - \hat{f}^* \circ Y \right| (\omega) \leq \| f - g \|_\infty + \| \hat{f} - \hat{g} \|_\infty + \| g^* \circ X - \hat{g}^* \circ Y \| (\omega) \]

\[ \leq 2\| f - g \|_\infty + \| g^* \circ X - \hat{g}^* \circ Y \| (\omega), \quad \forall \omega \in \Omega \]

\[ \hat{\varphi} \] denotes the function $\hat{\varphi} = \mathfrak{var}(h1_A; E, A, 0)$.
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by \([6.1.29]\), Proposition \(4.1.6\) (d) (with \(a = 1\) and \(b = -1\)) and Fact \(10.3.1\) (a) (with \(f = f - g\)). Now, (b) follows by \([6.1.29]\), \([6.1.30]\) and \([2.2.15]\) (with \(D = \hat{E}\)).

(c) The replica \(\nu^\omega\) of \(X(\omega)\) exists for each \(\omega \in \Omega\) by Proposition \(5.2.4\) (a) (with \(d = 1\), \(\mu = X(\omega)\) and \(\bar{\pi} = \nu^\omega\)).

\[
\Omega \setminus \left\{ \omega \in \Omega : \varphi \circ X(\omega) = \hat{\varphi}(\nu^\omega) \in \hat{\varphi} \left[ \mathcal{M}^+(\hat{E}) \right] \right\}
= \left\{ \omega \in \Omega : X(\omega)(E \setminus E_0) > 0 \right\} \in \mathcal{N}(\mathbb{P})
\]

by the countability of \(\mathcal{M}(\mathcal{F})\), Proposition \(5.2.4\) (e) (with \(d = 1\), \(\mu = X(\omega)\) and \(\bar{\pi} = \nu^\omega\)) and \([6.1.22]\). Since \(S_0\) is contained in the closure of \(\hat{\varphi}^*\left[ \mathcal{M}^+(\hat{E}) \right]\), it follows by \([6.1.28]\), \([6.1.31]\) and the completeness of \((\Omega, \mathcal{F}, \mathbb{P})\) that

\[
\mathbb{P} \left( \{ \omega \in \Omega : \hat{\varphi} \circ Y(\omega) = \hat{\varphi}(\nu^\omega) \} \right) = 1.
\]

Hence, (c) follows by \([6.1.26]\) and \([6.1.32]\).

(d) We fix \(f \in \mathfrak{c}(\mathcal{F})\) and \(h \in M_0(E; \mathbb{R}^k)\), get \(\{ \overline{h\mathbf{1}_A}, h\mathbf{1}_A \} \subset M_0(\hat{E}; \mathbb{R}^k)\) from Proposition \(4.1.6\) (b) (with \(d = 1\) and \(f = h\mathbf{1}_A\) or \(h\mathbf{1}_A\)), and find

\[
\left\{ \omega \in \Omega : (hf)^* \circ X(\omega) = \left(h\mathbf{1}_A\right)^* \circ X(\omega) = (hf)^* \circ (\mathbf{1}_A) = (h\mathbf{1}_A \hat{f})^* \nu^\omega \right\}
\supset \left\{ \omega \in \Omega : X(\omega)(E \setminus A) = 0 \right\}
\]

by Proposition \(5.2.4\) (d) (with \(d = 1\)), the fact \(\overline{h\mathbf{1}_A | E_0} = \overline{h\mathbf{1}_A \hat{f} | E_0}\) and the definition of \(\nu^\omega\). Thus, \([6.1.24]\) follows by \([6.1.33]\) and (c). Moreover, we have \((h\mathbf{1}_A \hat{f})^* \circ Y \in M(\Omega, \mathcal{F}; \mathbb{R}^k)\) by the fact \(h\mathbf{1}_A \hat{f} \in M_b(\hat{E}; \mathbb{R}^k)\), Lemma \(3.1.3\) (c) and Proposition \(10.2.29\) (ii) (with \(E = \hat{E}\), \(f = h\mathbf{1}_A \hat{f}\) and \(\xi = Y\)). Hence, \((hf)^* \circ X \in M(\Omega, \mathcal{F}; \mathbb{R}^k)\) by Lemma \(10.1.28\) (a) (with \(E = S = \mathbb{R}^k\), \(\mathcal{F} = \mathbb{B}(\mathbb{R}^k)\), \(X = (hf)^* \circ X\) and \(Z = (h\mathbf{1}_A \hat{f})^* \circ Y\)).

Proposition 6.1.12. Let \(E\) be a topological space, \((E_0, \mathcal{F}, \hat{E}, \hat{\mathcal{F}})\) be a base over \(E\), and \((\Omega, \mathcal{F}, \mathbb{P}; X)\) be an \(\mathcal{M}^+(E)\)-valued process satisfying

\[
\{ \omega \in \Omega : X_t(\omega)(E \setminus E_0) > 0 \} \in \mathcal{N}(\mathbb{P}), \forall t \in \mathbb{R}^+.
\]

Then, there exists an \(\mathcal{M}^+(\hat{E})\)-valued \(\mathcal{F}_t\)-adapted process \((\Omega, \mathcal{F}, \mathbb{P}; Y)\) satisfying the following properties:

(a) \(\varpi(\hat{f}^*) \circ Y\) is a modification of \(\varpi(f^*) \circ X\) for all \(f \in \mathfrak{c}(\mathcal{F})\).

(b) If \(X\) satisfies

\[
\inf_{t \in \mathbb{R}^+} \mathbb{P}(X_t \in \mathcal{P}(E)) = 1,
\]

then \(Y\) can be a \(\mathcal{P}(\hat{E})\)-valued process.

(c) For each \(t \in \mathbb{R}^+\), there exists an \(\Omega_t \in \mathcal{N}(\mathbb{P})\) such that \(Y_t(\omega)\) equals the replica (measure) of \(X_t(\omega)\) for all \(\omega \in \Omega \setminus \Omega_t\).

(d) If \(A \in \mathbb{B}(\mathbb{R}^+; E_0)\) satisfies

\[
\{ \omega \in \Omega : X_t(\omega)(E \setminus A) > 0 \} \in \mathcal{N}(\mathbb{P}), \forall t \in \mathbb{R}^+,
\]

then \(\varpi((h\mathbf{1}_A \hat{f})^* \circ Y)\) is a modification of \(\varpi((hf)^* \circ X)\) for all \(f \in \mathfrak{c}(\mathcal{F})\), \(h \in M_b(E; \mathbb{R}^k)\) and \(k \in \mathbb{N}\).
6.2. Finite-dimensional convergence about replica process

6.2.1. Definitions. Given a general space $E$, the finite-dimensional convergence of $E$-valued processes is about the Borel extensions of their possibly non-Borel finite-dimensional distributions.

**Definition 6.2.1.** Let $E$ be a topological space and $\{\{(\Omega^i, \mathcal{F}^i, \mathbb{P}^i; X^i)\}_{i \in I}^{14}\}$ be $E$-valued processes. Then $\{X^i\}_{i \in I}^{14}$ converges finite-dimensionally to $X$ along $T$ if:

1. $T \subset \mathbb{R}^+$ is non-empty,
2. For each $T_0 \in \mathcal{P}_0(T)$, there exist $N_{T_0} \in \mathbb{N}$, $\mu_n \in \mathcal{M}^+(\hat{E})$ and $\mu \in \mathcal{M}^+(\hat{E})$ such that $\mu_n \Rightarrow \mu$ as $n \to \infty$ in $\mathcal{P}(E_{T_0})$.

**Remark 6.1.13.** An implication of the statements “$\varpi(f^\ast) \circ Y$ is a modification of $\varpi(f^\ast) \circ X$” in (a) and “$\varpi((h \hat{f})^\ast) \circ Y$ is a modification of $\varpi((h \hat{f})^\ast) \circ X$” in (d) is that $\varpi(f^\ast) \circ Y$ is a modification of $\varpi((h \hat{f})^\ast) \circ Y$ and $\varpi((h \hat{f})^\ast) \circ Y$ are processes. For (a), we know $f \in \mathcal{C}(E, \mathbb{R})$ and $\hat{f} \in C(E, \mathbb{R})$, so $\varpi(f^\ast) \circ X$ and $\varpi(f^\ast) \circ Y$ are processes by Proposition 10.2.29(a). For (d), $\varpi((h \hat{f})^\ast) \circ Y$ and $\varpi((h \hat{f})^\ast) \circ X$ are processes by Lemma 6.1.10(d) (with $X = X_t$ and $Y = Y_t$).

**Proof.** We set $\varphi, \tilde{\varphi}, y_0$ and $\Psi$ as in Lemma 6.1.10. If (6.1.35) holds, we let $S_0 = \tilde{\varphi}[\mathcal{P}(E)]$ or else we let $S_0 = \tilde{\varphi}[\mathcal{M}^+(\hat{E})]$. Recall that $\tilde{\varphi}$ satisfies (6.1.26). $\mathcal{M}^+(\hat{E}), \mathcal{P}(\hat{E})$ and $\mathbb{R}^\infty$ are Polish spaces by Corollary 3.1.11(c) (with $d = 1$) and Note 6.1.5. Consequently, $S_0 \in \mathcal{B}(\mathbb{R}^\infty)$ in both cases by Proposition 9.5.9 (with $E = A = \mathcal{M}^+(\hat{E})$ or $\mathcal{P}(\hat{E})$). $f = \varphi$ and $S = \mathbb{R}^\infty$ in Proposition 9.5.8(b) (with $E = \mathbb{R}^\infty$). Hence, $\hat{f} = \varphi \circ \varphi \circ X$ is the desired process by (6.1.25), Lemma 6.1.10 (with $X = p_t \circ X$ and $Y = p_t \circ Y$) and Fact 10.1.29(a) (with $E = \mathcal{M}^+(E)$, $\hat{f} = \Psi \circ \varphi$ and $\mathcal{F}_t = \mathcal{F}_t^X$).

14$\{\{(\Omega^i, \mathcal{F}^i, \mathbb{P}^i; X^i)\}_{i \in I}^{14}\}$ and $\{\{(\Omega^n, \mathcal{F}^n, \mathbb{P}^n; X^n)\}_{n \in \mathbb{N}}\}$ were assumed in 2.6 to be complete probability spaces. Completeness of measure space was specified in 2.1.2.

15Hereafter, “$(T, D)$-FDC” and “$(T, D)$-AS” also stand for “$(T, D)$-finite-dimensional convergence” and “$(T, D)$-asymptotical stationarity”.

16$\hat{\varphi}$ denotes the expectation operator of $(\Omega^t, \mathcal{F}^t, \mathbb{P}^t)$.

17In the definitions of $(T, D)$-FDC and $(T, D)$-AS, $\{\mathbb{E}[(f \circ X_{T_0}^i)_{i \in I}]\}$ and $\{\mathbb{E}[(f \circ X_{T_0}^i - f \circ X_{T_0+i}^i)_{i \in I}]\}$ both lie in $[-2||f||_{\infty}, 2||f||_{\infty}]$. Each of them has at least one limit point in $\mathbb{R}$ by the Bolzano-Weierstrass Theorem, so it is enough to assume “at most one limit point”.

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○ \(\{X^i\}_{i \in I}\) is asymptotically stationary along \(T\) under \(D\) ((\(T, D\))-AS for short) if: (1) \(I\) is infinite, (2) \(T \subset \mathbb{R}^+\) and \(D \subset M_b(E; \mathbb{R})\) are non-empty, and (3) The unique limit point of \(|\{E[f \circ X_{T_0}^i - f \circ X_{T_0}^0]\}_{i \in I}\}|_{i \in I}\) in \(\mathbb{R}\) is 0 for all \(c \in (0, \infty)\), \(f \in \text{mc}[\Pi^{T_0}(D)]\) and \(T_0 \in \mathcal{P}_0(T)\).

Note 6.2.2. Given \(E\)-valued process \(X\), the expectation \(E[f \circ X_{T_0}]\) is well-defined for any \(f \in \text{ca}[\Pi^d(M_b(E; \mathbb{R}))]\) by Fact 2.5.2 (d) and Note 5.1.6 (with \(\mu = \mathbb{P} \circ X_{T_0}^{-1}\)).

Note 6.2.3. Let \(X\) and \(Y\) be \(E\)-valued processes and \(T \subset \mathbb{R}^+\). We relate \(X \sim Y\) if \(X_{T_0}\) and \(Y_{T_0}\) have the same distribution for all \(T_0 \in \mathcal{P}_0(T)\). This \(\sim\), which Definition 6.2.1 uses to define equivalence of finite-dimensional limit points, is an equivalence relation among \(E\)-valued stochastic processes.

We make the following notations for simplicity.

Notation 6.2.4. Let \(X, \{X^i\}_{i \in I}\) and \(\{X^n\}_{n \in \mathbb{N}}\) be \(E\)-valued processes. \(\{X^n\}_{n \in \mathbb{N}}\) converging finite-dimensionally to \(X\) along \(T\) is denoted by

\[
X^n \xrightarrow{D(T)} X \text{ as } n \uparrow \infty.
\]

○ By \(X = \mathcal{f}_{T}(\{X^n\}_{n \in \mathbb{N}})\) we mean \(X\) is the finite-dimensional limit of \(\{X^n\}_{n \in \mathbb{N}}\) along \(T\).

○ By \(\mathcal{f}_{T}(\{X^i\}_{i \in I})\) we denote the family of all equivalence classes (see Note 6.2.3) of finite-dimensional limit points of \(\{X^i\}_{i \in I}\) along \(T\).

○ By \(X = \mathcal{f}_{T}(\{X^i\}_{i \in I})\) we mean \(X\) is the unique member of \(\mathcal{f}_{T}(\{X^i\}_{i \in I})\).

Remark 6.2.5. In general, \(X = \mathcal{f}_{T}(\{X^n\}_{n \in \mathbb{N}})\) is stronger than 6.2.1 because:

1. Each of the finite-dimensional distributions of \(\{X^n\}_{n \in \mathbb{N}}\) may have multiple Borel extensions.
2. \(\mathcal{P}(E^{T_0})\) is not necessarily a Hausdorff space and a weakly convergent sequence may have multiple limits.

The following is a straightforward property of finite-dimensional convergence.

Fact 6.2.6. Let \(E\) be a topological space, \(T \subset \mathbb{R}^+\) and \(\{(\Omega^n, \mathcal{F}^n, \mathbb{P}^n; X^n)\}_{n \in \mathbb{N}}\) be \(E\)-valued processes. If 6.2.1 holds, then

\[
\lim_{n \to \infty} E^n[f \circ X_{T_0}^n] = E[f \circ X_{T_0}]
\]

for all \(f \in \text{mc}[\Pi^{T_0}(C_b(E; \mathbb{R}))]\) and \(T_0 \in \mathcal{P}_0(T)\). As a consequence, \(\{X^n\}_{n \in \mathbb{N}}\) is \((T, C_b(E; \mathbb{R}))-\text{FDC}\).

Proof. Fixing \(f \in \text{mc}[\Pi^{T_0}(C_b(E; \mathbb{R}))]\) and \(T_0 \in \mathcal{P}_0(T)\), it follows by Fact 10.2.12 (with \(d = \#(T_0)\) and \(X = X_{T_0}^n\) or \(X_{T_0}\)), 5.1.3 (with \(d = \#(T_0)\) and \(D = C_b(E; \mathbb{R})\)) and 6.2.1 that

\[
\lim_{n \to \infty} E^n[f \circ X_{T_0}^n] = \lim_{n \to \infty} f^*(\mu_n) = f^*(\mu) = E[f \circ X_{T_0}]
\]

for some \(\{\mu_n \in \text{be}(\mathbb{P} \circ (X_{T_0}^{-1}))\}_{n > N_{T_0}}\) with \(N_{T_0} \in \mathbb{N}\) and \(\mu \in \text{be}(\mathbb{P} \circ X_{T_0}^{-1})\). □

---

18. The notation “\(T_0 + c\)” was defined in 2.5.10.

19. “\(\mathcal{f}\)” and “\(\mathcal{f}_{T}\)” are “\(\mathcal{f}\)” and “\(\mathcal{f}_{T}\)” in fraktur font which stand for “finite-dimensional limit” and “finite-dimensional limit point” respectively. Members of \(\mathcal{f}_{T}(\cdot)\) or \(\mathcal{f}_{T}(\cdot)\) are processes with time horizon \(\mathbb{R}^+\) no matter \(T = \mathbb{R}^+\) or not.

20. \(E^n\) denotes the expectation operator of \((\Omega^n, \mathcal{F}^n, \mathbb{P}^n)\).
6.2.2. Transformation of finite-dimensional convergence. The following theorem is our main tool for transforming a finite-dimensional limit point of replica processes back into that of original processes.

Theorem 6.2.7. Let $E$ be a topological space, $\{(\Omega^n, \mathcal{F}^n, \mathbb{P}^n; X^n)\}_{n \in \mathbb{N}}$ be $E$-valued processes, $(E_0, \mathcal{F}; \mathbb{P}, \mathcal{F})$ be a base over $E$, $x_0 \in E_0$, $T \subset \mathbb{R}^+$ and

$$X_t = \begin{cases} \operatorname{var}(Y_t; \Omega, Y_t^{-1}(E_0), x_0), & \text{if } t \in T, \\ \operatorname{var}(Y_t; \Omega, Y_t^{-1}(\{x_0\}), x_0), & \text{if } t \in \mathbb{R}^+ \backslash T, \end{cases}$$

where $(\Omega, \mathcal{F}, \mathbb{P}; Y)$ is an $E$-valued process. Suppose that:

(i) $\{X^n_t\}_{n \in \mathbb{N}}$ is sequentially tight in $E_0$ for all $t \in T$.

(ii) $\{X^n\}_{n \in \mathbb{N}}$ and $Y$ satisfy

$$\lim_{n \to \infty} E^n [f \circ X^n_{T_0}] = E [f \circ Y_{T_0}]$$

for all $f \in \mathcal{M}([\prod^0_{T_0}(\mathcal{F}(\{1\})])$ and $T_0 \in \mathcal{P}_0(T)$.

Then:

(a) $X = \{X_t\}_{t \geq 0}$ is an $E$-valued process with paths in $E^R_0$ and for each $T_0 \in \mathcal{P}_0(T)$, there exist

$$\mu_{T_0, n} = \mathbb{E} \left[ \mathbb{P} \circ (X^n_{T_0})^{-1} \right] \in \mathcal{P}(E^{T_0}), \forall n > N_{T_0}$$

for some $N_{T_0} \in \mathbb{N}$ and

$$\mu_{T_0} = \mathbb{E} \left[ \mathbb{P} \circ (X_{T_0})^{-1} \right] \in \mathcal{P}(E^{T_0})$$

such that

$$\lim_{n \to \infty} \mu_{T_0, n} |_{E^R_0} = \mu_{T_0} |_{E^R_0}$$

in $\mathcal{P}(E^{T_0}, \mathcal{O}(E_0)^{T_0})$.

Moreover, $X_{T_0}$ is $\mathbf{m}$-tight in $E^{T_0}_0$ for all $T_0 \in \mathcal{P}_0(T)$,

(b) If $T = \mathbb{R}^+$ and $\{X^n\}_{n \in \mathbb{N}}$ is $(\mathbb{R}^+, \mathcal{F}(\{1\}))$-AS, then $X$ and $Y$ are both stationary processes

(c) If $C(E; \mathbb{R})$ separates points on $E$ then $X = \hat{f}_T(\{X_n\}_{n \in \mathbb{N}})$.

---

21Note 4.1.2 mentioned that $\mathbb{P}(Y_t; \Omega, Y_t^{-1}(\{x_0\}), x_0)$ is the constant mapping that sends every $\omega \in \Omega$ to $x_0$. We do not use $x_0$ to denote this mapping for clarity.

22Sequential tightness of random variables was specified in Note 5.1.2.

23“with paths in $E^R_{0+}$” means all paths of the process lying in $E^R_0$. Of course, an $E$-valued process with paths in $E^R_{0+}$ is equivalent to an $(E_0, \mathcal{O}(E_0))$-valued process.

24The notation “$\mu_{T_0, n} = \mathbb{E}(\mathbb{P} \circ (X^n_{T_0})^{-1})$” as defined in 3.1.3 means $\mu_{T_0, n}$ is the unique Borel extension of $\mathbb{P} \circ (X^n_{T_0})^{-1}$.

25The notation “$w$-lim”, introduced in 2.3, means weak limit of a sequence of finite Borel measures.

26The notion of stationary process was specified in 2.5.

27Note 5.1.8 showed that $C(E; \mathbb{R})$ separating points is equivalent to $C(E; \mathbb{R})$ separating points on $E$. 
Remark 6.2.8. The condition (i) above will ensure the following two facts for \( T_0 \in \mathcal{P}_0(T) \) with finite exception of \( n \in \mathbb{N} \): (1) \( \mathbb{P}^n \circ (X^n_{T_0})^{-1} \) admits a unique Borel extension, and (2) \( \mathbb{P}^n \circ (\hat{X}^n_{T_0})^{-1} \) is the replica measure of \( \mathbb{P}^n \circ (X^n_{T_0})^{-1} \). Hence, transforming finite-dimensional convergence from the replicas \( \{\hat{X}^n\}_{n \in \mathbb{N}} \) to the original processes \( \{X^n\}_{n \in \mathbb{N}} \) comes down to transforming weak convergence from replica measures to Borel extensions of original measures.

Proof of Theorem 6.2.7 (a) We divide the proof of (a) into four steps.

Step 1: Constructing \( \mu_{T_0} \) for each \( T_0 \in \mathcal{P}_0(T) \). We know from (i) and Fact 10.3.4 (with \( I = N \)) that \( \{\mu_{T_{0,n}} \circ \mathbb{P}^n \circ (X^n_{T_0})^{-1}\}_{n \in \mathbb{N}} \) is sequentially tight in \( E_{T_0} \) and there exists an \( N_{T_0} \in \mathbb{N} \) such that

\[
\inf_{n > N_{T_0}, t \in T_0} \mathbb{P}^n (X^n_t \in E_0) = 1
\]

and the \( \{\mu^t_{T_{0,n}} \circ \mathbb{P}^n \circ (\hat{X}^n_{T_0})^{-1}\}_{n > N_{T_0}} \) in \( (6.2.6) \) exist. We then have

\[
\inf_{n > N_{T_0}, t \in T_0} \mathbb{P}^n (\bigotimes \mathcal{F} \circ X^n_t \in \bigotimes \hat{\mathcal{F}}(E)) = 1
\]

by \( (6.2.10) \) and Fact 6.1.9 (with \( X = X^n \)). From \( (6.2.11) \), the condition (ii) above and Lemma 10.3.6 (c, e) it follows that

\[
\hat{X}^n \xrightarrow{D(T)} Y \text{ as } n \uparrow \infty.
\]

Therefore, \( \mathbb{P}^n \circ (X^n_{T_0})^{-1} \) is a process and satisfies \( (6.2.4) \) for each \( t \in T_0 \) defined by \( \mathbb{P} \circ (X^n_{T_0})^{-1} \) and \( \nu_{T_0} \).

Step 2: Verify \( X = \{X_t\}_{t \geq 0} \) defined by \( (6.2.4) \) is a process and satisfies \( (6.2.9) \).

For each \( t \in T_0 \), we let \( \mu_{\{t\}} \in \mathcal{P}(\mathcal{E}) \) be the measure constructed in Step 1 with \( T_0 = \{t\} \). By our argument above, each \( \mu_{\{t\}} \) is tight in \( (E_0, \mathcal{O}_E(E_0)) \) and so is supported on some \( S_t \in \mathcal{B}(E_0) \). \( S_t \in \mathcal{B}(E) \) and \( \mathcal{B}_E(S_t) = \mathcal{B}_E(S_t) \) by Corollary 3.1.14 (b) and Lemma 3.1.14 (a). Let \( \nu_{\{t\}} = \mathbb{P} \circ Y_{\{t\}}^{-1} = \mathcal{P}_{\{t\}} \) be defined as in \( (6.2.13) \) with \( T_0 = \{t\} \). It follows by Proposition 5.2.4 (a) that

\[
\mathbb{P} (Y_t \in S_t) = \nu_{\{t\}}(S_t) = \mu_{\{t\}}(S_t) = 1, \forall t \in T_0
\]

Hence, \( X_t \in M(\Omega, \mathcal{F}; E_0, \mathcal{B}_E(E_0)), \forall t \in T \)

satisfy \( (6.2.9) \) by Lemma 10.1.28 (b, c) (with \( (E, \mathcal{U}) = (\hat{E}, \mathcal{B}(\hat{E})), S_0 = S_t, (S, \mathcal{U}') = (E_0, \mathcal{B}_E(E_0)), X = Y_t \) and \( Y = X_t \)). Furthermore, we have

\[
\{x_0\} \in \mathcal{B}(\hat{E}) \cap \mathcal{B}(E_0, \mathcal{O}_E(E_0)) \cap \mathcal{B}(E),
\]

by Lemma 3.1.3 (c, e), the fact \( E_0 \in \mathcal{B}_E(E) \) and Proposition 9.1.2 (a), which implies

\[
X_t \in M(\Omega, \mathcal{F}; E_0, \mathcal{O}_E(E_0)), \forall t \in \mathbb{R}^+ \setminus T.
\]

\( \mathbb{P}_{T_{0,n}}, \) as specified in Notation 5.2.3, denotes the replica measure of \( \mu_{T_{0,n}} \).
by (6.2.4). Now, $X$ is an $(E_0, \mathcal{O}_E(E_0))$-valued process by Fact 2.5.2 (b).

**Step 3:** Verify the $m$-tightness of $X_{T_0}$ in $E_{T_0}$ and (6.2.7) for each $T_0 \in \mathcal{R}_0(T)$.

Letting $\{S_t\}_{t \in T}$ be as in Step 2, we have that

\[(6.2.19) \quad S_{T_0} \triangleq \prod_{t \in T_0} S_t \in \mathcal{K}_\sigma \left( E_{T_0}^0, \mathcal{O}_E(E_0)^T_0 \right) \]

by Corollary 9.1.14 (b) (with $I = T_0$ and $S_t = (E_0, \mathcal{O}_E(E_0))$). It follows that

\[(6.2.20) \quad S_{T_0} \in \mathcal{B}(E^{T_0}) \cap \mathcal{B}(E)_{T_0} \cap \mathcal{K}_\sigma^m(E^{T_0}) \]

and

\[(6.2.21) \quad \mathcal{B}(E_{T_0})(S_{T_0}) = \mathcal{B}(E_{T_0})(S_{T_0}) \]

by Corollary 3.1.16 (b) and Lemma 3.1.14 (a). Now,

\[(6.2.22) \quad \nu_{T_0}(S_{T_0}) = \mathbb{P}(Y_{T_0} \in S_{T_0}) = 1 \]

by (6.2.20) and (6.2.15). Moreover,

\[(6.2.23) \quad \nu_{T_0}(A \cap S_{T_0}) = \mu_{T_0}(A \cap S_{T_0}) = \mu_{T_0}(A), \quad \forall A \in \mathcal{B}(E^{T_0}) \]

by (6.2.20), (6.2.21), the fact $\nu_{T_0} = \mu_{T_0} \cap (6.2.22)$ and Proposition 5.2.4 (a) (with $\mu = \mu_{T_0}$). We established (6.2.9) in Step 2. It follows that

\[(6.2.24) \quad \mathbb{P}(X_{T_0} \in A) = \mathbb{P}(Y_{T_0} \in A) \]

by (6.2.23) and (6.2.24). Thus, $X_{T_0}$ is $m$-tight in $E_{T_0}$ by (6.2.19), (6.2.20), (6.2.22) and (6.2.24). It follows by (6.2.24) and Proposition 3.3.25 (with $\Gamma = \{X_{T_0}^{-1}\}$).

**Step 4:** Verify (6.2.1). The desired convergence follows from (6.2.14) and (6.2.7).

(b) We have by (a) (with $T = \mathbb{R}^+$) that

\[(6.2.25) \quad \inf_{t \in \mathbb{R}^+} \mathbb{P}(X_t = Y_t \in E_0) = 1 \]

and

\[(6.2.26) \quad X^n \xrightarrow{D(\mathbb{R}^+)} X \text{ as } n \uparrow \infty. \]

It then follows that

\[(6.2.27) \quad \mathbb{E}\left[ \hat{f} \circ Y_{T_0} - \hat{f} \circ Y_{T_0+c} \right] = \mathbb{E}\left[ f \circ X_{T_0} - f \circ X_{T_0+c} \right] = \lim_{n \to \infty} \mathbb{E}^n\left[ f \circ X_{T_0} - f \circ X_{T_0+c} \right] = 0 \]

for all $c > 0$, $T_0 \in \mathcal{R}_0(\mathbb{R}^+)$ and $f \in \text{me}[\Pi_{T_0}(\mathcal{F}\{1\})]$ by (6.2.26), Fact 10.1.32 (b) (with $T = \mathbb{R}^+$), Fact 6.2.4 (with $T = \mathbb{R}^+$), (6.2.25) and Lemma 10.3.5 (a) (with $T = \mathbb{R}^+$). Hence, the stationarity of $Y$ follows by Corollary 3.1.11 (a) (with $d = \mathbb{R}(T_0)$ and $A = E_0$). $\mathbb{P} \circ X_{T_0}^{-1}$ is $m$-tight in $E_0$ by (a), so there exists an $A \in \mathcal{K}_\sigma(E_0, \mathcal{O}_E(E_0))$ such that

\[(6.2.28) \quad \inf_{t \in \mathbb{R}^+} \mathbb{P}(X_t = Y_t \in A) = 1, \]

by (6.2.25) and the stationarity of $Y$. Now, $X$ is stationary by Lemma 10.3.5 (e).

\[29 \mathbb{P} \circ X_{T_0}^{-1} \text{ and each } \mathbb{P}^n \circ (X_{T_0}^{-1})^{-1} \text{ are Borel probability measures as mentioned in Note } 6.1.5 \]
(c) We fix $T_0 \in \mathcal{P}_0(T)$ and let each $\mu_{T_0,n}$, $\mu'_{T_0,n}$ and $\mu_{T_0}$ be as in (a). It follows by (6.2.14), (5.1.3) (with $D = C_b(E;\mathbb{R})$) and Fact 10.2.12 (with $d = \aleph(T_0)$), $\mu = \mu_{T_0,n}$ and $\nu_1 = \mu'_{T_0,n}$) that

\[
\lim_{n \to \infty} \int_{E^{T_0}} f(x) \mu_{T_0,n}(dx) = \lim_{n \to \infty} f^*(\mu'_{T_0,n}) = f^*(\mu_{T_0})
\]

for all $f \in \text{mc}[\Pi^{T_0}(C_b(E;\mathbb{R}))]$. Hence, (6.2.14) implies

\[
\text{w- lim}_{n \to \infty} \mu'_{T_0,n} = \mu_{T_0}
\]

by Theorem 5.1.4 (a, b) (with $d = \aleph(T_0)$, $\Gamma = \{\mu_{T_0,n}\}_{n \in \mathbb{N}}$ and $D = C_b(E;\mathbb{R})$). Now, (c) follows by (6.2.6), (6.2.30) and Fact 10.1.33 (with $I = \mathbb{N}$). \qed

Remark 6.2.9. As mentioned in Note 6.1.5, Polish-space-valued processes (especially replica processes) have Borel finite-dimensional distributions and their finite-dimensional convergence refers exactly to the weak convergence of their finite-dimensional distributions. Moreover, $\mathcal{P}^*(E)$ is a Polish space by Corollary 3.1.11 (c). Hence, (6.2.12) is equivalent to $Y = \mathbb{H}_T(\{\hat{X}^n\}_{n \in \mathbb{N}})$.

The next corollary leverages Theorem 6.2.7 to establish finite-dimensional convergence to a given limit process.

Corollary 6.2.10. Let $E$ be a topological space, $\{(\Omega^n, \mathcal{F}^n, \mathbb{P}^n; X^n)\}_{n \in \mathbb{N}}$ be $E$-valued processes, $(E_0, \mathcal{F}; \hat{E}, \hat{\mathcal{F}})$ be a base over $E$ and $T \subset \mathbb{R}^+$. Suppose that:

(i) $\{X^n_t\}_{n \in \mathbb{N}}$ is sequentially tight in $E_0$ for all $t \in T$.

(ii) (6.2.2) holds for all $f \in \text{mc}[\Pi^{T_0}(\mathcal{F}\{\{1\}\})]$ and $T_0 \in \mathcal{P}_0(T)$.

(iii) $E$-valued process $(\Omega, \mathcal{F}, \mathbb{P}; X)$ satisfies (6.1.14) (or (6.1.9)).

Then:

(a) (6.2.6), (6.2.7), (6.2.8) and (6.2.9) hold for some $\{N_{T_0}\}_{T_0 \in \mathcal{P}_0(T)} \subset \mathbb{N}$. Moreover, $X_{T_0}$ is $\mathfrak{m}$-tight for all $T_0 \in \mathcal{P}_0(T)$.

(b) If $T = \mathbb{R}^+$ and $\{X^n\}_{n \in \mathbb{N}}$ is $(\mathbb{R}^+, \mathcal{F}\{\{1\}\})$-AS, then $X$ is stationary.

(c) If $C_b(E;\mathbb{R})$ separates points on $E$, then $X = \mathbb{H}_T(\{X^n\}_{n \in \mathbb{N}})$.

Proof. Letting $Y = \hat{X}$, we obtain (6.2.5) for all $f \in \text{mc}[\Pi^{T_0}(\mathcal{F}\{\{1\}\})]$ and $T_0 \in \mathcal{P}_0(T)$ by the conditions (ii) and (iii) above, Fact 6.1.9 and Proposition 6.1.8 (a). Now, the result follows by Theorem 6.2.7 immediately. \qed

6.3. Càdlàg replica

Properties like relative compactness, tightness, well-posedness of martingale problems and convergence of nonlinear filters are often easily established for càdlàg replicas. As Polish-space-valued càdlàg processes, they have the following nice measurability.
6.3. Càdlàg Replica

**Fact 6.3.1.** Let $E$ be a topological space, $(E_0, \mathcal{F}; \hat{E}, \hat{F})$ be a base over $E$ and $(\Omega, \mathcal{F}, \mathbb{P}; X)$ be an $E$-valued process. Then, $\text{rep}_c(X; E_0, \mathcal{F}) \subset M(\Omega, \mathcal{F}; D(\mathbb{R}^+; \hat{E}))$ and $\text{rep}_c(X; E_0, \mathcal{F}) \subset \text{rep}_p(X; E_0, \mathcal{F}) \subset \text{rep}_m(X; E_0, \mathcal{F})$.

**Proof.** The first statement follows by Fact 9.7.2 (b) (with $E = \hat{E}$ and $X = \hat{X} \in \text{rep}_c(X; E_0, \mathcal{F})$) and Fact 9.7.1 (a) (with $E = \hat{E}$). The second statement follows by Proposition 10.1.30 (a, c).

Due to the topological difference of $E$ and $\hat{E}$, non-càdlàg $E$-valued processes can have càdlàg replicas if they are almost càdlàg on the functions in $\mathcal{F}$.

**Definition 6.3.2.** Let $E$ be a topological space and $(\Omega, \mathcal{F}, \mathbb{P}; X)$ be an $E$-valued process. $X$ is said to be **weakly càdlàg along $T$ under $D$** ((T, D)-càdlàg for short) if: (1) $T \subset \mathbb{R}^+$ and $D \subset M(\mathbb{R}; E)$ are non-empty, and (2) There exist $\mathbb{R}$-valued càdlàg processes $\{(\Omega, \mathcal{F}, \mathbb{P}; \zeta_f')\}_{f \in T}$ such that

\[
\inf_{f \in D, t \in T} \mathbb{P} \left( f \circ X_t = \zeta_f' \right) = 1.
\]

**Note 6.3.3.** $X$ is $(\mathbb{R}^+, D)$-càdlàg if $\{\varpi(f) \circ X\}_{f \in D}$ are all càdlàg processes, especially if $X$ is càdlàg and $D \subset C(E; \mathbb{R})$ by Fact 10.1.31 (a) (with $S = \mathbb{R}$). Apparently, $(\mathbb{R}^+, D)$-càdlàg property is transitive between modifications.

**Remark 6.3.4.** A special case of an $(\mathbb{R}^+, D)$-càdlàg $X$ is when $\varpi(f) \circ X$ is càdlàg for all $f \in D$.

Here are three typical sufficient conditions for unique existence of càdlàg replica.

**Proposition 6.3.5.** Let $E$ be a topological space, $(E_0, \mathcal{F}; \hat{E}, \hat{F})$ be a base over $E$, $(\Omega, \mathcal{F}, \mathbb{P}; X)$ be an $E$-valued process and $T \subset \mathbb{R}^+$ be dense. Then:

(a) If $X$ is $(\mathbb{R}^+, \mathcal{F})$-càdlàg and \([6.1.14]\) holds, then $\hat{X} = \text{rep}_c(X; E_0, \mathcal{F})$ exists and $\varpi(\hat{X}) \circ X$ (resp. $\varpi(\hat{f}) \circ X$) is the unique càdlàg modification\(33\) of $\varpi(\hat{X}) \circ X$ (resp. $\varpi(\hat{f}) \circ X$ for each $f \in \mathcal{F}$) up to indistinguishability.

(b) If $\{\varpi(f) \circ X\}_{f \in \mathcal{F}}$ are all càdlàg and \([6.1.14]\) holds, then $\hat{X} = \text{rep}_p(X; E_0, \mathcal{F})$ exists and\(35\)

\[
\mathbb{P} \left( \varpi(f) \circ X = \varpi(\hat{f}) \circ \hat{X}, \forall f \in \text{ca}(\mathcal{F}) \right) = 1.
\]

(c) If $X$ is càdlàg and satisfies \([6.1.9]\), then $\hat{X} = \text{rep}_c(X; E_0, \mathcal{F})$ exists and\(36\)

\[
\mathbb{P} \left( \varpi(f) \circ X = \varpi(\hat{f}) \circ \hat{X}, \forall f \in C(E; \mathbb{R}) \right) = 1.
\]

\(30\) $M(\Omega, \mathcal{F}; D(\mathbb{R}^+; \hat{E}))$ denotes the family of all $D(\mathbb{R}^+; \hat{E})$-valued random variables defined on measurable space $(\Omega, \mathcal{F})$.

\(31\) $\text{rep}_m(X; E_0, \mathcal{F})$, $\text{rep}_p(X; E_0, \mathcal{F})$ and $\text{rep}_c(X; E_0, \mathcal{F})$, introduced in Notation 6.1.3, are all equivalence classes of measurable, progressive and càdlàg replicas of $X$ with respect to $(E_0, \mathcal{F}; \hat{E}, \hat{F})$.

\(32\) Please be reminded that all processes in this article are indexed by $\mathbb{R}^+$.

\(33\) We specified in Notation 6.1.3 that “$\hat{X} = \text{rep}_p(X; E_0, \mathcal{F})$” means $\hat{X}$ is the unique càdlàg replica of $X$ up to indistinguishability. “Unique up to indistinguishability” means any two processes with the relevant property is indistinguishable.

\(34\) The terminology “modification” was explained in \([2.5]\).

\(35\) Please be reminded that $\hat{f}$ denotes the continuous replica of $f$.

\(36\) The replica of functions was discussed in \([4.1]\).
Remark 6.3.6. The functional indistinguishability of \( X \) and \( \tilde{X} \) in (6.3.2) is a valuable property of càdlàg replica. Corollary 3.1.10 showed \( C(\tilde{E}; R) = ca(\mathcal{F}) = \{ \hat{f} : f \in ca(\mathcal{F}) \} \), so (6.3.2) allows many properties to be transferred between \( X \) and \( \tilde{X} \).

Our construction of \( \tilde{X} \) is based on the following technical lemma.

Lemma 6.3.7. Let \( E \) be a topological space, \( (E_0; \mathcal{F}; \tilde{E}; \hat{\mathcal{F}}) \) be a base over \( E \), \( (\Omega, \mathcal{F}, P; X) \) be an \( E \)-valued process satisfying (6.1.14) for some dense \( T \subset R^+ \) and \( T \subset S \subset R^+ \). Then, the following statements are equivalent:

(a) \( X \) is \((S, \mathcal{F})\)-càdlàg.

(b) There exists an \( R^\infty \)-valued càdlàg process \((\Omega, \mathcal{F}, P; \zeta)\) such that

\[
\inf_{t \in S} P \left( \bigotimes \mathcal{F} \circ X_t = \zeta \right) = 1. \tag{6.3.4}
\]

(c) There exists an \( \tilde{X} \in M(\Omega, \mathcal{F}; D(R^+; \tilde{E})) \) such that

\[
\inf_{t \in S} P \left( \bigotimes \mathcal{F} \circ X_t = \bigotimes \hat{\mathcal{F}} \circ \tilde{X}_t \right) = 1. \tag{6.3.5}
\]

Proof. ((a) \( \rightarrow \) (b)) is immediate by Fact 10.1.34 (with \( D = \mathcal{F} \) and \( T = S \)).

((b) \( \rightarrow \) (c)) Let \( T_0 \subset T \) be countable and dense in \( R^+ \). \( \bigotimes \hat{\mathcal{F}}(\tilde{E}) \) is a closed subspace of \( R^\infty \) by (3.1.5).

\[
P \left[ \zeta \in D \left( R^+; \bigotimes \hat{\mathcal{F}}(\tilde{E}) \right) \right] \geq P \left( \zeta_t = \bigotimes \mathcal{F} \circ X_t \in \bigotimes \hat{\mathcal{F}}(\tilde{E}), \forall t \in T_0 \right) = 1
\]

by (6.3.4), \( T_0 \subset S \), (6.1.14), the càdlàg property of \( \zeta \) and the closedness of \( \bigotimes \hat{\mathcal{F}}(\tilde{E}) \). Then, there exists a \( \zeta' \in M \left[ \Omega, \mathcal{F}; D \left( R^+; \bigotimes \hat{\mathcal{F}}(\tilde{E}) \right) \right] \) satisfying

\[
P \left( \zeta = \zeta' \right) = 1 \tag{6.3.6}
\]

by (6.3.6), Proposition 9.6.10 (b) (with \( E = R^\infty \)) and Lemma 10.2.28 (b) (with \( E = R^\infty \), \( E_0 = \bigotimes \hat{\mathcal{F}}(\tilde{E}) \), \( S_0 = D(R^+; \bigotimes \hat{\mathcal{F}}(\tilde{E})) \) and \( X = \zeta \)).

It follows by by (3.1.3) and Proposition 9.6.1 (d) (with \( S = \bigotimes \hat{\mathcal{F}}(\tilde{E}) \), \( E = \tilde{E} \) and \( f = \left( \bigotimes \hat{\mathcal{F}} \right)^{-1} \)) that

\[
\varpi \left[ \left( \bigotimes \hat{\mathcal{F}} \right)^{-1} \right] \in C \left[ D \left( R^+; \bigotimes \hat{\mathcal{F}}(\tilde{E}) \right) ; D(R^+; \tilde{E}) \right]. \tag{6.3.7}
\]

It follows by (6.3.7) and (6.3.9) that

\[
\tilde{X} \vartriangleq \varpi \left[ \left( \bigotimes \hat{\mathcal{F}} \right)^{-1} \right] \circ \zeta' \in M \left( \Omega, \mathcal{F}; D(R^+; \tilde{E}) \right). \tag{6.3.10}
\]

It follows by (6.3.8), (6.3.10) and (3.1.3) that

\[
P \left( \zeta = \varpi \left( \bigotimes \hat{\mathcal{F}} \right) \circ \varpi \left[ \left( \bigotimes \hat{\mathcal{F}} \right)^{-1} \right] \circ \zeta' = \varpi \left( \bigotimes \hat{\mathcal{F}} \right) \circ \tilde{X} \right) = 1. \tag{6.3.11}
\]

Now, (6.3.5) follows by (6.3.4) and (6.3.11).

((c) \( \rightarrow \) (a)) is automatic. \( \square \)
6.3. Càdlàg Replica

Proof of Proposition 6.3.5
(a) By Lemma 6.3.7 (a - c) (with \( S = \mathbb{R}^+ \)), there exists an \( \hat{X} \in M(\Omega, \mathcal{F}; D(\mathbb{R}^+; \hat{E})) \) such that
\[
(6.3.12) \quad \inf_{t \in \mathbb{R}^+} P \left( \bigotimes \mathcal{F} \circ X_t = \bigotimes \hat{F} \circ \hat{X}_t \in \bigotimes \hat{F}(\hat{E}) \right) = 1.
\]
\( \varpi(\bigotimes \hat{F}) \circ \hat{X} \) (resp. \( \varpi(\hat{f}) \circ \hat{X} \)) is a càdlàg modification of \( \varpi(\bigotimes \mathcal{F}) \circ X \) (resp. \( \varpi(f) \circ X \) for each \( f \in \mathcal{F} \)) by (3.1.3), the fact \( \hat{F} \subset C(\hat{E}; \mathbb{R}) \), (6.3.12) and Fact 10.1.31 (a)
(with \( E = \hat{E}, X = \hat{X} \) and \( f = \hat{f} \) or \( \bigotimes \hat{F} \)). Now, (a) follows by Proposition 6.1.8
(b) and Proposition 10.1.30 (h).

(b) Given any \( \hat{X} \in \text{rep}_c(X; E_0, \mathcal{F}) \), one finds that
\[
(6.3.13) \quad \left\{ \omega \in \Omega : \varpi(f) \circ X(\omega) = \varpi(\hat{f}) \circ \hat{X}(\omega), \forall f \in \text{ca}(\mathcal{F}) \right\}
\]
by properties of uniform convergence. Then, (b) follows by (6.3.13) and (a).

(c) Let \( T_0 \subset T \) be countable and dense in \( \mathbb{R}^+ \). Given a càdlàg \( X, \hat{X} \in \text{rep}_c(X; E_0, \mathcal{F}) \) exists by (a), and \( \varpi(f) \circ X \) and \( \varpi(\hat{f}) \circ \hat{X} \) are càdlàg process for all \( f \in C(E; \mathbb{R}) \) by Fact 10.1.31 (a).
\[
(6.3.14) \quad \left\{ \omega \in \Omega : \varpi(f) \circ X(\omega) = \varpi(\hat{f}) \circ \hat{X}(\omega), \forall f \in \text{ca}(E) \right\}
\]
\( \supset \left\{ \omega \in \Omega : X_t(\omega) = \hat{X}_t(\omega) \in E_0, \forall t \in T_0 \right\} \)
by the fact \( f|_{E_0} = \hat{f}|_{E_0} \) and Proposition 10.1.30 (g). Now, (c) follows by Fact 6.1.9
(b), Proposition 6.1.7 (a) and (6.3.14).

Remark 6.3.8. The càdlàg property of \( \varpi(\bigotimes \mathcal{F}) \circ X(\omega) \) in \( (\mathbb{R}^\infty)^{\mathbb{R}^+} \) does not guarantee that of \( X(\omega) \) in \( E^{\mathbb{R}^+} \) since \( \bigotimes \mathcal{F} \) is not necessarily an imbedding on \( E \).

If \( E_0 \) is large enough for \( X \) to almost surely live in \( E_0^{\mathbb{R}^+} \), then the following result shows one can modify merely a \( \mathbb{P} \)-negligible amount of paths of \( X \) and obtain an indistinguishable replica of \( X \).

Proposition 6.3.9. Let \( E \) be a topological space, \( E_0 \in \mathcal{B}(E), S_0 \subset E_0^{\mathbb{R}^+} \) and \( (\Omega, \mathcal{F}, \mathbb{P}; X) \) be an \( E \)-valued process satisfying
\[
(6.3.15) \quad \mathbb{P}(X \in S_0) = 1.
\]
Then, there exists an \( \hat{X} \in S_0^{\mathbb{R}^+} \) satisfying the following properties:

(a) \( \hat{X} \) is an \((E_0, \mathcal{E}(E_0))\)-valued process and \( \mathbb{P}(X = \hat{X} \in S_0) = 1 \).

(b) \( \hat{X} \in \text{rep}_c(X; E_0, \mathcal{F}) \) for any base \((E_0, \mathcal{F}; \hat{E}, \hat{F})\) over \( E \).

(c) If every element of \( S_0 \) is a càdlàg member of \((E_0^{\mathbb{R}^+}, \mathcal{E}(E_0))^{\mathbb{R}^+} \), then \( \hat{X} \) is a càdlàg member of \((E_0^{\mathbb{R}^+}, \mathcal{E}(E_0))^{\mathbb{R}^+} \) over \( E \).

Proof. We fix \( \eta_0 \in S_0 \) and define \( \hat{X} = \text{bar}(X; \Omega, X^{-1}(S_0), \eta_0) \). Then, (a) follows by Lemma 10.1.28 (b, c) (with \( (E, \mathcal{E}) = (E^{\mathbb{R}^+}, \mathcal{B}(E)^{\mathbb{R}^+}) \), \( S = S_0 \), \( \mathcal{E} = \mathcal{E}|_{S_0} \) and \( Y = \hat{X} \)). Given any base \((E_0, \mathcal{F}; \hat{E}, \hat{F})\), \((E_0, \mathcal{E}(E_0))\) is coarser than
\[\text{bar}(\cdot) \text{ was introduced in Notation 4.1.3}\]

\[\text{bar}(\cdot) \text{ was introduced in Notation 4.1.3}\]
\((E_0, \mathcal{O}_E(E_0))\) by Lemma 3.1.3 (d) and so \(\hat{X}\) is an \(\hat{E}\)-valued process. We have by \(\text{Lemma 3.1.1}\) that
\[
\inf_{t \in \mathbb{R}^+} \mathbb{P} \left( \bigotimes_{t \in \mathbb{R}^+} \mathcal{F} \circ X_t = \bigotimes_{t \in \mathbb{R}^+} \mathcal{F} \circ \hat{X}_t \in \bigotimes_{t \in \mathbb{R}^+} \mathcal{F}(\hat{E}) \right) 
\geq \mathbb{P} \left( X_t = \hat{X}_t \in E_0, \forall t \in \mathbb{R}^+ \right) \geq \mathbb{P} \left( X = \hat{X} \in S_0 \right) = 1,
\]
thus proving (b). The càdlàg property of \(\hat{X}(\omega) : \mathbb{R}^+ \to (E_0, \mathcal{O}_E(E_0))\) implies \(\hat{X}(\omega) \in D(\mathbb{R}^+; \hat{E})\) for all \(\omega \in \Omega^{E_0}\) by Fact 10.1.13 (b) (with \(E = (E_0, \mathcal{O}_E(E_0))\), \(S = (E_0, \mathcal{O}_E(E_0))\) and \(f\) being the identity mapping on \(E_0\)). Hence, (c) follows by (b), \(6.3.16\) and Proposition 6.1.8 (b) (with \(T = \mathbb{R}^+\)).

**Remark 6.3.10.** In general, many paths of a càdlàg replica could live outside \(E_0^{\mathbb{R}^+}\). Such a càdlàg replica is not necessarily an \(E\)-valued process, nor is it (necessarily) indistinguishable from \(X\). Specifically, if \(X\) is càdlàg and satisfies
\[
\inf_{t \in \mathbb{R}^+} \mathbb{P}(X_t \in E_0) = 1,
\]
then \(\hat{X} = \text{rep}_\psi(X; E_0, \mathcal{F})\) satisfies
\[
\inf_{t \in \mathbb{R}^+} \mathbb{P}(X_t = \hat{X}_t \in E_0) = 1
\]
by Proposition 6.3.5 (c) (with \(T = \mathbb{R}^+\)) and Proposition 6.1.7 (a) (with \(T = \mathbb{R}^+\)). However, this does not necessarily imply \(\mathbb{P}(X \in E_0^{\mathbb{R}^+}) = 1\) nor \(\mathbb{P}(X = \hat{X} \in E_0^{\mathbb{R}^+}) = 1\) since \(E_0\) might not be a closed subspace of \(E\) or \(\hat{E}\).

Moreover, we transform \(\mathcal{M}^+(E)\)-valued weakly càdlàg processes into \(\mathcal{P}(\hat{E})\)-valued càdlàg processes by similar construction techniques for càdlàg replica, which furthers Proposition 6.1.12.

**Lemma 6.3.11.** Let \(E\) be a topological space, \((E_0, \mathcal{F}; \hat{E}, \hat{\mathcal{F}})\) be a base over \(E\), \((\Omega, \mathcal{F}, \mathbb{P}, X)\) be an \(\mathcal{M}^+(E)\)-valued \((\mathbb{R}^+, \mathcal{M}(\mathcal{F})^*)\)-càdlàg process satisfying \(6.1.34\) and \(6.1.35\). Then, there exists an \(\mathcal{F}_X^X\)-adapted \(D(\mathbb{R}^+; \mathcal{P}(\hat{E}))\)-valued random variable \((\Omega, \mathcal{F}, \mathbb{P}; Y)\) satisfying Proposition 6.1.12 (a, c, d).

**Remark 6.3.12.** In the proof below, we let \(\varphi, \hat{\varphi}, y_0\) and \(\Psi\) be as in Lemma 6.1.10 and set \(S_0 = \hat{\varphi}[\mathcal{P}(\hat{E})]\). Recall that \(\hat{\varphi}\) satisfies (6.1.26). \(\mathcal{P}(\hat{E})\) is a compact Polish space by Corollary 3.1.11 (c) (with \(d = 1\)). Hence, \(S_0 \in \mathcal{C}(\mathbb{R}^\infty)\) is a Polish subspace of \(\mathbb{R}^\infty\) by Proposition 9.1.12 (a, e) and Proposition 9.1.11 (b, f). \(D(\mathbb{R}^+; \mathcal{P}(\hat{E}))\) and \(D(\mathbb{R}^+; S_0, \mathcal{O}_{R^\infty}(S_0))\) are Polish spaces by Proposition 9.6.10 (d) (with \(E = \mathcal{P}(\hat{E})\) or \(S_0, \mathcal{O}_{R^\infty}(S_0)\)). Therefore, \(D(\mathbb{R}^+; \mathcal{P}(\hat{E}))\)-valued and \(D(\mathbb{R}^+; S_0, \mathcal{O}_{R^\infty}(S_0))\)-valued random variables are càdlàg processes by Fact 9.7.1 (a), for which \(\mathcal{F}_X^X\)-adaptedness is a proper concept.

**Proof of Lemma 6.3.11.** The proof of Lemma 6.1.10 mentioned that \(\mathcal{M}(\mathcal{F})\) is countable and \(\varphi\) satisfies (6.1.25), so \(\varphi \circ X\) has an \(\mathbb{R}^\infty\)-valued càdlàg modification \(\zeta\) by the converse part of Fact 10.1.34 (with \(E = \mathcal{M}^+(E)\), \(D = \mathcal{M}(\mathcal{F})^*\) and \(T = \mathbb{R}^+\) and \(\zeta\) is \(\mathcal{F}_X^X\)-adapted by Proposition 10.1.30 (e).
Let \( \nu_{t} \) be the replica of \( X_t(\omega) \) for each fix \( \omega \in \Omega \) and \( t \in \mathbb{R}^+ \). Similar to (6.1.31), we have that

\[
\inf_{t \in \mathbb{R}^+} \mathbb{P}(\zeta_t = \varphi \circ X_t) = \inf_{t \in \mathbb{R}^+} \mathbb{P}(\{\omega \in \Omega : \varphi \circ X_t(\omega) = \hat{\varphi}(\nu_{t}) \in S_0\}) = \inf_{t \in \mathbb{R}^+} \mathbb{P}(\{\omega \in \Omega : X_t(\omega)(E) = X_t(\omega)(E_0) = 1\}) = 1
\]

by the countability of \( \text{mc}(\mathcal{F}) \), Proposition 5.2.4 (a, b, e) (with \( d = 1 \), \( \mu = X_t(\omega) \) and \( \mathbf{m} = \nu^\omega \)), (6.1.34) and (6.1.35). It follows that

\[
\mathbb{P}(\zeta \in D(\mathbb{R}^+; S_0, \mathcal{O}_{\mathbb{R}^\infty}(S_0))) = 1
\]

by (6.3.19), the closedness of \( S_0 \) and the c\( \ddot{a} \)dl\( \ddot{a} \)g property of \( \zeta \). Then, there exists a \( \zeta' \in M[\Omega, \mathcal{F}; D(\mathbb{R}^+; S_0, \mathcal{O}_{\mathbb{R}^\infty}(S_0))] \) satisfying (6.3.8) by (6.3.20), Proposition 9.6.10 (b) (with \( E = \mathbb{R}^\infty \)) and Lemma 10.2.28 (b) (with \( E = E_0 = \mathbb{R}^\infty, E_0 = S_0, S_0 = D(\mathbb{R}^+; S_0, \mathcal{O}_{\mathbb{R}^\infty}(S_0)) \) and \( X = \zeta \)). As \( \zeta \) is \( \mathcal{F}_i^X \)-adapted, \( \zeta' \) is \( \mathcal{F}_i^X \)-adapted by (6.3.8) and Proposition 10.1.30 (e). Furthermore,

\[
\inf_{t \in \mathbb{R}^+} \mathbb{P}(\zeta'_t = \varphi \circ X_t) = 1.
\]

by (6.3.8) and (6.3.19).

The proof of Lemma 6.1.10 mentioned that \( \hat{\varphi} \) satisfies (6.1.26) and \( \Psi \) equals \( \hat{\varphi}^{-1} \) restricted to \( S_0 \). Hence, we have that:

\[
Y := \varphi(\Psi) \circ \zeta' = \varphi(\hat{\varphi}^{-1}) \circ \zeta' \in M[\Omega, \mathcal{F}; D(\mathbb{R}^+; \mathcal{P})]
\]

by (6.3.23) and Proposition 9.6.1 (d) (with \( S = (S_0, \mathcal{O}_{\mathbb{R}^\infty}(S_0)), E = \mathcal{P}(\hat{E}) \) and \( f = \hat{\varphi}^{-1} \)), (2) \( Y \) is \( \mathcal{F}_i^X \)-adapted by (6.1.26), the \( \mathcal{F}_i^X \)-adaptedness of \( \zeta' \) and Fact 10.1.20 (a) (with \( E = (S_0, \mathcal{O}_{\mathbb{R}^\infty}(S_0)), S = \mathcal{P}(\hat{E}), f = \hat{\varphi}^{-1} \) and \( X = \zeta' \)), and (3)

\[
\inf_{t \in \mathbb{R}^+} \mathbb{P}(\varphi \circ X_t = \zeta'_t = \hat{\varphi} \circ Y_t) = 1
\]

by (6.3.23) and (6.3.22).

Now, the result follows by Proposition 6.1.12 (a) - (d), (6.3.24) and (6.1.26). □

6.4. Weak convergence about c\( \ddot{a} \)dl\( \ddot{a} \)g replica

6.4.1. Regularity conditions about processes. Before discussing weak convergence of c\( \ddot{a} \)dl\( \ddot{a} \)g replicas, we give some regularity conditions about stochastic processes.

**Definition 6.4.1.** Let \( E \) be a topological space and \( \{(\Omega^i, \mathcal{F}^i, \mathbb{P}^i, X^i)\}_{i \in I} \) be \( E \)-valued processes.

- When \((E, \tau)\) is a metric space, \( \{X^i\}_{i \in I} \) satisfies **Mild Pointwise Containment Condition for \( T \)** (T-MPCC for short) if \( T \subset \mathbb{R}^+ \) is non-empty and for any \( \epsilon \in (0, \infty) \) and \( t \in T \), there exists a **totally bounded** (see p. 151) set \( A_{t,\epsilon} \) satisfying

\[
\inf_{i \in I} \mathbb{P}^i(X^i_t \in A_{t,\epsilon}^i) \geq 1 - \epsilon.
\]

\footnote{The notation “\( A^\prime \)” was defined in 7.2.1.3}
\( \{X^i\}_{i \in I} \) satisfies **T-Pointwise m-Tightness Condition** or **T-Pointwise Sequential m-Tightness Condition** in \( A \subset E \) (T-PMTC or T-PSMTC in \( A \) for short) if \( T \subset \mathbb{R}^+ \) is non-empty and \( \{X^i\}_{i \in I} \) is m-tight or sequentially m-tight in \( A \) for all \( t \in T \), respectively. Moreover, we say \( \{X^i\}_{i \in I} \) satisfies T-PMTC (resp. T-PSMTC) if it satisfies T-PMTC (resp. T-PSMTC) in \( E \).

\( \{X^i\}_{i \in I} \) satisfies **Metrizable Compact Containment Condition** in \( A \) (MCC in \( A \) for short) if for each \( \epsilon, T \in (0, \infty) \), there exists a \( K_{\epsilon, T} \in \mathscr{F}^m(E) \) such that \( K_{\epsilon, T} \subset A \),

\[
\bigcap_{t \in [0, T]} (X^i_t)^{-1}(K_{\epsilon, T}) \in \mathcal{F}, \forall i \in I
\]

and

\[
\inf_{i \in I} P^i \left( X^i_t \in K_{\epsilon, T}, \forall t \in [0, T] \right) \geq 1 - \epsilon.
\]

Moreover, by “\( \{X^i\}_{i \in I} \) satisfies MCC” we mean it satisfies MCC in \( E \).

\( \{X^i\}_{i \in I} \) are measurable processes, \( \{X^i\}_{i \in I} \) is said to satisfy **Long-time-average m-Tightness Condition** in \( A \) for \( \{T_k\}_{k \in \mathbb{N}} \) (Tk-LMTC in \( A \) for short) if \( T_k \uparrow \infty \) and

\[
\left\{ \frac{1}{T_k} \int_0^{T_k} P^i \circ (X^i_t)^{-1} dt \right\}_{k \in \mathbb{N}, i \in I} \subset \mathcal{P}(E)
\]

is m-tight in \( A \). Moreover, by “\( \{X^i\}_{i \in I} \) satisfies Tk-LMTC” we mean it satisfies Tk-LMTC in \( E \).

\( \{X^i\}_{i \in I} \) satisfies **Modulus of Continuity Condition** for \( \tau \) (\( \tau \)-MCC for short) if: (1) \( \tau \) is a pseudometric on \( E \), and (2) For any \( \epsilon, T \in (0, \infty) \), there exists a \( \delta_{\epsilon, T} \in (0, \infty) \) such that\(^{43}\)

\[
\left\{ \omega \in \Omega : w'_{\tau, \delta, \epsilon, T} \circ X^i(\omega) \geq \epsilon \right\} \in \mathcal{F}, \forall i \in I
\]

and

\[
\sup_{i \in I} P^i \left( w'_{\tau, \delta, \epsilon, T} \circ X^i \geq \epsilon \right) \leq \epsilon.
\]

\( \{X^i\}_{i \in I} \) satisfies **Modulus of Continuity Condition** (MCC for short) if there exist a family of pseudometrics \( \mathcal{R} \) that induces \( \mathcal{O}(E) \) and \( \{X^i\}_{i \in I} \) satisfies \( \tau \)-MCC for all \( \tau \in \mathcal{R} \).

\( \{X^i\}_{i \in I} \) satisfies **Functional Modulus of Continuity Condition** for \( \mathcal{D} \) (DFMCC for short) if: (1) \( \varpi(f) \circ X^i \) admits a càdlàg modification \( \zeta^{f,i} \) for each \( f \in \mathcal{D} \subset M(E; \mathbb{R}) \) and \( i \in I \), and (2) \( \{\zeta^{f,i}\}_{i \in I} \) satisfies \( \|\cdot\|\text{-MCC} \) for all \( f \in \mathcal{O}(\mathcal{D}) \).

\( \{X^i\}_{i \in I} \) satisfies **Weak Modulus of Continuity Condition** (WMCC for short) if: (1) There exists a \( \mathcal{D} \subset C(E; \mathbb{R}) \) separating points on \( E \), and (2) \( \{X^i\}_{i \in I} \) satisfies \( \mathcal{D}\text{-FMCC} \).

\(^{42}\)“\( T_k \uparrow \infty \)” as usual denotes a non-decreasing sequence \( \{T_k\}_{k \in \mathbb{N}} \subset \mathbb{R} \) that converges to \( \infty \).

\(^{43}\)“\( w'_{\tau, \delta, \epsilon, T} \)” is defined by \(^{2.2.3}\) with \( E = \mathbb{R} \) and \( \tau = \|\cdot\| \).

\(^{44}\)The meaning of \( \mathcal{R} \) inducing \( \mathcal{O}(E) \) was explained in \(^{2.1.3}\).

\(^{45}\)\( \|\cdot\|\text{-MCC} \) means MCC for Euclidean metric \( \|\cdot\| \). The notation “\( \mathcal{O}(\mathcal{D}) \)” was defined in \(^{2.2.3}\).
6.4. WEAK CONVERGENCE ABOUT C\(\text{\c{a}dl\'ag}\) REPLICA

**Note 6.4.2.** An \(E\)-valued process \(X\) is said to satisfy any of the properties above except \(T\)-PSMTC (in \(A\)) if the singleton \(\{X\}\) does.

**Note 6.4.3.** If \(\{X^i\}_{i\in I}\) and \(\{Y^i\}_{i\in I}\) are two bijectively indistinguishable families of \(E\)-valued processes (i.e. \(X^i\) and \(Y^i\) are indistinguishable for all \(i\in I\)), then each of the conditions above is transitive between \(\{X^i\}_{i\in I}\) and \(\{Y^i\}_{i\in I}\). Moreover, \(\mathcal{D}\)-FMCC and WMCC are transitive between \(\{X^i\}_{i\in I}\) and \(\{Y^i\}_{i\in I}\) if \(Y^i\) is a modification of \(X^i\) for all \(i\in I\).

**Remark 6.4.4.**
- Assuming total boundedness in lieu of compactness for each \(A_{\epsilon,t}\), \(R^+\)-MPCC weakens the Pointwise Containment Property in \([EK86]\) §3.7, Theorem 3.7.2] and \([Kou16]\ §5).
- MCCC is a variant of the standard Compact Containment Condition (see \([Jak86]\ §4, (4.8)] and \([EK86]\ §3.7, (7.9)]) using \(m\)-tightness, which becomes standard if \(E\) has metrizable compact sets. \(R^+\)-PMTC is a similar variant of the Pointwise Tight Condition in \([Kou16]\ §5 and \([EK86]\ §3.7, (7.7)]\).
- \(T_{\epsilon,K}\)-LMTC often appears in constructing stationary distributions (see \([Kun71]\) and \([BBK00]\)). The measures in \((6.4.4)\) are well-defined by properties of measurable process and Fubini’s Theorem.

**Remark 6.4.5.**
- MCC was used in \([Jak86]\ and \([Kou16]\) (in its finite-time-horizon form) for general Tychonoff spaces. As long as \(E\) is Hausdorff, the assumption of pseudometrics \(\mathcal{R}\) inducing \(\mathcal{O}(E)\) in MCC implies \(E\) is Tychonoff (see Proposition 9.3.1 (a, d)).
- WMCC is a special case of \(\mathcal{D}\)-FMCC. Both of them are generically milder than MCC as \(\mathcal{D}\) need not strongly separate points on \(E\).

**Note 6.4.6.** If \(\{X^i\}_{i\in I}\) satisfy \(\mathcal{D}\)-FMCC, then they are apparently \((R^+, \mathcal{D})\)-\(\text{\c{a}dl\'ag}\) processes.

**Remark 6.4.7.** In standard texts like \([Bil68]\ and \([EK86]\), \(r\)-MCC and MCCC are fundamental criteria for establishing tightness or relative compactness in Skorokhod \(\mathcal{J}_1\)-spaces. \((E, r)\) is usually a separable metric space and our measurability conditions \((6.4.2)\) and \((6.4.5)\) are automatically true. We consider general spaces so we have to specify \((6.4.2)\) and \((6.4.5)\) as part of MCCC and \(r\)-MCC. Given a \(\text{\c{a}dl\'ag}\) \(X^i\), Lemma 9.7.3 justifies \((6.4.2)\) under very mild conditions about \(E\) and \(K_{\epsilon,T}\), and Lemma 9.7.4 justifies \((6.4.5)\) for the following four cases:

1. \((E, r)\) is a metric space and \(X^i\) is a \(D(R^+; E)\)-valued random variable.
2. \((E, r)\) is a separable metric space.
3. \(r = \rho(f)\) with \(f \in C(E; R)\).
4. \(r = \rho_D\) with \(D \subset C(E; R)\) being a countable point-separating collection (hence \(E\) is baseable).

\(^{46}\) Note that sequential \(m\)-tightness is for infinite collections of measures or random variables.

\(^{47}\) The pseudometric \(\rho(f)\) is defined by \([2.2.2]\) with \(D = \{f\}\).
Consequently, $\mathcal{D}$-FMCC never incurs measurability issue like (6.4.5) by case (2) above (with $E = \mathbb{R}$), nor does $\rho_{\{f\}}$-MCC (resp. $\rho_{\mathcal{D}}$-MCC) for càdlàg processes and $\mathcal{D}$ consistent with case (3) (resp. case (4)) above.

Besides the measurability conditions (6.4.2) and (6.4.5), §9.7 of Appendix C also provides several results about the relationship among $\mathfrak{c}$-MCC, MCC, $\mathcal{D}$-FMCC and WMCC. The above-mentioned containment or tightness conditions will be further discussed in §6.5.

### 6.4.2. Tightness of càdlàg replicas

Given $E$-valued processes $\{X^i\}_{i \in I}$, we consider tightness of their càdlàg replicas $\{\tilde{X}^i \in \text{rep}_c(X^i; E_0, \mathcal{F})\}_{i \in I}$ in $D(\mathbb{R}^+; \tilde{E})$.

**Remark 6.4.8.** Càdlàg replicas are always $D(\mathbb{R}^+; \tilde{E})$-valued random variables (see Fact 6.3.1) and their tightness in $D(\mathbb{R}^+; \tilde{E})$ has the usual meaning (compared to our general interpretation in §2.4).

**Note 6.4.9.** Thanks to the compactness of $\tilde{E}$ (see Lemma 3.1.3(b)), the stringent MCCC is automatic for $\tilde{E}$-valued processes.

Given MCCC, tightness of $\{\tilde{X}^i\}_{i \in I}$ in $D(\mathbb{R}^+; \tilde{E})$ is reduced to $\mathcal{F}$-FMCC.

**Proposition 6.4.10.** Let $E$ be a topological space, $(E_0, \mathcal{F}; \tilde{E}, \tilde{\mathcal{F}})$ be a base over $E$ and $\{(\mathfrak{P}, \mathfrak{F}, \mathfrak{P}^i; X^i)\}_{i \in I}$ be $E$-valued processes satisfying

\[(6.4.7)\]

\[\inf_{i \in I} E^i \left( \bigotimes \mathcal{F} \circ X^i \in \bigotimes \tilde{\mathcal{F}}(\tilde{E}) \right) = 1\]

for some dense $T \subset \mathbb{R}^+$. Then:

(a) If $\{X^i\}_{i \in I}$ satisfies $\mathcal{F}$-FMCC, then $\{\tilde{X}^i \in \text{rep}_c(X^i; E_0, \mathcal{F})\}_{i \in I}$ exists, satisfies $\tilde{\mathcal{F}}$-FMCC, satisfies $\rho_{\mathfrak{P}}$-MCC and is tight in $D(\mathbb{R}^+; \tilde{E})$.

(b) The converse of (a) is true when $T = \mathbb{R}^+$ or $\{X^i\}_{i \in I}$ are all càdlàg processes.

(c) If $I$ is an infinite set and any subsequence of $\{X^i\}_{i \in I}$ has a sub-subsequence satisfying $\mathcal{F}$-FMCC, then $\{\tilde{X}^i \in \text{rep}_c(X^i; E_0, \mathcal{F})\}_{i \in I_{0, \mathcal{I}}} \subset \mathcal{F}$ exists and is sequentially tight in $D(\mathbb{R}^+; \tilde{E})$ for some $I_{0, \mathcal{I}}$.

**Proof.** (a) Suppose $\zeta^{f,i} \in \text{rep}_c(X^i; E_0, \mathcal{F})$ for each $f \in \mathfrak{ac}(\mathcal{F})$ and $i \in I$ (see Note 6.4.6) and $\{\zeta^{f,i}\}_{i \in I}$ satisfies $|\cdot|$-MCC for all $f \in \mathfrak{ac}(\mathcal{F})$ (see Note 6.4.3). It follows by (6.4.7) and Proposition 6.3.5(a) (with $X = X^i$) that $\{\tilde{X}^i \in \text{rep}_c(X^i; E_0, \mathcal{F})\}_{i \in I}$ exists and satisfies

\[(6.4.8)\]

\[\inf_{f \in \mathfrak{ac}(\mathcal{F}), i \in I} E^i \left( \tilde{\mathcal{F}}(\tilde{E}) \cap \tilde{\mathcal{F}}(\tilde{E}) \right) = 1,\]

thus proving $\{\tilde{X}^i\}_{i \in I}$ satisfies $\tilde{\mathcal{F}}$-FMCC. $\{\tilde{X}^i\}_{i \in I}$ satisfies $\rho_{\mathfrak{P}}$-MCC by Lemma 3.1.3(a) and Proposition 9.7.8 (with $E = \tilde{E}$ and $\mathcal{D} = \tilde{\mathcal{F}}$). Now, (a) follows by Note 6.4.9 Lemma 3.1.3(c) and Theorem 9.7.12 (with $(E, \mathfrak{r}) = (\tilde{E}, \rho_{\mathfrak{P}})$ and $X^i = \tilde{X}^i$).

(b) Given tightness of $\{\tilde{X}^i\}_{i \in I}$ in $D(\mathbb{R}^+; \tilde{E})$, $\{\tilde{\mathcal{F}}(\tilde{E}) \cap \tilde{\mathcal{F}}(\tilde{E})\}_{i \in I}$ is tight in $D(\mathbb{R}^+; \mathbb{R})$ for all $f \in \mathfrak{ac}(\tilde{\mathcal{F}})$ by Proposition 9.6.1(d) (with $E = \tilde{E}$ and $\mathcal{D} = \tilde{\mathcal{F}}$) and Fact 10.2.18(a) (with $E = A = D(\mathbb{R}^+; \tilde{E})$, $\mathcal{S} = D(\mathbb{R}^+; \mathbb{R})$, $f = \tilde{\mathcal{F}}(\tilde{E}) \cap \tilde{\mathcal{F}}(\tilde{E})$ and $\mathcal{I} = \{E^i \circ (\tilde{X}^i)^{-1}\}_{i \in I}$). Then, $\{\tilde{\mathcal{F}}(\tilde{E}) \cap \tilde{\mathcal{F}}(\tilde{E})\}_{i \in I}$ satisfies $|\cdot|$-MCC for all $f \in \mathfrak{ac}(\tilde{\mathcal{F}})$ by Theorem 9.7.12 (with $(E, \mathfrak{r}) = (\mathbb{R}, |\cdot|)$ and $X^i = \tilde{\mathcal{F}}(\tilde{E}) \cap \tilde{\mathcal{F}}(\tilde{E})$).
We have that
\[
\inf_{i \in T, i \in I, f \in \alpha(\mathcal{F})} \mathbb{P}^i \left( f \circ X_i = \hat{f} \circ \hat{X}_i \right) \geq \inf_{i \in T, i \in I} \mathbb{P}^i \left( \bigotimes\mathcal{F} \circ X_i \in \bigotimes\hat{\mathcal{F}}(\hat{E}) \right) = 1
\]
by (6.4.7) and (6.1.1) (with \( X = X_i \) and \( \hat{X} = \hat{X}_i \)). If \( T \neq \mathbb{R}^+ \) and \( \{X_i\}_{i \in I} \) are all càdlàg, then \( \varpi(f) \circ \hat{X}_i \) is indistinguishable from \( \varpi(f) \circ X_i \) for all \( i \in I \) and \( f \in \alpha(\mathcal{F}) \) by Proposition 6.3.5(c). If \( T = \mathbb{R}^+ \), (6.4.9) shows \( \varpi(\hat{f}) \circ \hat{X}_i \) is a càdlàg modification of \( \varpi(f) \circ X_i \). Therefore, \( \{X_i\}_{i \in I} \) satisfies \( \mathcal{F} \)-FMCC in either case.

(c) It follows by (a) and the given condition that any infinite subset of \( \{X_i\}_{i \in I} \) admits a sub-subsequence \( \{X_{i_n}\}_{n \in \mathbb{N}} \) such that \( \{\hat{X}_{i_n} = \text{rep}_c(X_{i_n}; E_0, \mathcal{F})\}_{n \in \mathbb{N}} \) exists and is tight in \( D(\mathbb{R}^+; \hat{E}) \). Therefore, \( \hat{X}_i = \text{rep}_c(X_i; E_0, \mathcal{F}) \) must exist for all \( i \in I \) with only finite exceptions and (c) follows by the definition of sequential tightness.

Next, we consider tightness of the indistinguishable càdlàg replicas constructed by Proposition 6.3.9 in \( D(\mathbb{R}^+; E_0, \mathcal{O}_E(E_0)) \) or \( D(\mathbb{R}^+; E) \).

**Proposition 6.4.11.** Let \( E \) be a topological space, \((E_0, \mathcal{F}; \hat{E}, \hat{\mathcal{F}})\) be a base over \( E \) and \( \{\omega_i, \mathcal{F}_i; \mathbb{P}_i; X_i\}_{i \in I} \) be \( E \)-valued càdlàg processes. Suppose that:

(i) \((E_0, \mathcal{O}_E(E_0))\) is a Tychonoff space.

(ii) \( \{X_i\}_{i \in I} \) satisfies MCC in \( E_0 \).

Then, there exists an \( S_0 \subset \mathbb{D}_0 \hexarrowdown{\subset} D(\mathbb{R}^+; E_0, \mathcal{O}_E(E_0)) \) such that:

(a) \( S_0 \) and \( \mathbb{D}_0 \) satisfy\(^{48}\)
\[
\mathcal{B}(\mathbb{D}_0)|_{S_0} = \mathcal{B}(E)^{\otimes \mathbb{R}^+}|_{S_0} \subset \mathcal{B}(E)^{\otimes \mathbb{R}^+}|_{\mathbb{D}_0} \subset \mathcal{B}(\mathbb{D}_0).
\]

(b) \( \{\hat{X}_i = \text{rep}_c(X_i; E_0, \mathcal{F})\}_{i \in I} \) satisfies
\[
\hat{X}_i \in M[\Omega_i, \mathcal{F}_i; S_0, \mathcal{O}_{\mathbb{D}_0}(S_0)], \forall i \in I
\]
and
\[
\inf_{i \in I} \mathbb{P}_i^i \left( X_i = \hat{X}_i \in S_0 \right) = 1.
\]

(c) \( \{\hat{X}_i\}_{i \in I} \) is \( \mathbb{m} \)-tight in \( S_0 \) as \( \mathbb{D}_0 \)-valued random variables if and only if \( \{X_i\}_{i \in I} \) satisfies \( \mathcal{F} \)-FMCC.

**Remark 6.4.12.** The càdlàg replicas \( \{\hat{X}_i\}_{i \in I} \) above are \( \mathbb{D}_0 \)-valued random variables and their tightness in \( S_0 \subset \mathbb{D}_0 \) has the usual meaning. We noted in §2.3.5 that \( \mathbb{D}_0 \) and \( S_0 \) always satisfy\(^{49}\)
\[
\mathcal{B}(\mathbb{D}_0)|_{S_0} \supset \mathcal{B}(E)^{\otimes \mathbb{R}^+}|_{S_0}
\]
but not necessarily the equality in (6.4.10). So, \( \mathbb{D}_0 \)-valued random variable (like \( \hat{X}_i \) in (6.4.11)) is generally a stronger concept than \((E_0, \mathcal{O}_E(E))\)-valued càdlàg process.

\(^{48}\)\(\mathcal{B}(\mathbb{D}_0)\) is generated by the Skorokhod \( \mathcal{J}_1 \)-topology \( \mathcal{J}(E_0, \mathcal{O}_E(E_0)) \).

\(^{49}\)\(\mathcal{B}(\mathbb{D}_0)\) is generated by the Skorokhod \( \mathcal{J}_1 \)-topology of \( D(\mathbb{R}^+; E_0, \mathcal{O}_E(E_0)) \).
In fact, the developments of Proposition 6.4.11 do not require a base. We establish the following more general result without imposing the boundedness of the point-separating functions.

**Theorem 6.4.13.** Let $E$ be a topological space, $\{(\Omega^i, \mathcal{F}^i, \mathbb{P}^i; X^i)\}_{i \in I}$ be $E$-valued càdlàg processes, $E_0 \in \mathcal{B}(E)$ and $\mathcal{D} \subset C(E; \mathbb{R})$. Suppose that:

(i) $\mathcal{D}$ separates points on $E_0$.

(ii) $(E_0, \mathcal{G}(E_0))$ is a Tychonoff space.

(iii) $\{X^i\}_{i \in I}$ satisfies MCCC in $E_0$.

Then, there exist $S_0 \subset \mathbb{D}_0 \doteq \mathbb{D}(\mathbb{R}^+; E_0, \mathcal{G}(E_0))$ and $\{\hat{X}^i \in S_0^0\}_{i \in I}$ such that:

(a) (6.4.10) holds.

(b) $\{\hat{X}^i\}_{i \in I}$ satisfies (6.4.11) and (6.4.12).

(c) $\{\hat{X}^i\}_{i \in I}$ is $m$-tight in $S_0$ as $\mathbb{D}_0$-valued random variables if and only if $\{X^i\}_{i \in I}$ satisfies $\mathcal{D}$-FMCC.

**Proof.** We divide the proof into five steps. We equip $E_0$ with the subspace topology $\mathcal{G}(E_0)$ throughout the proof, which we make implicit for convenience.

**Step 1:** Construct $S_0$. By the condition (iii) above,

\begin{equation}
\inf_{i \in I} \mathbb{P}^i(\bigcup_{i=1}^q X^i_t \in A_{p,q}, \forall t \in [0, q]) \geq 1 - 2^{-p-q}, \forall p, q \in \mathbb{N}.
\end{equation}

holds for some $\{A_{p,q}\}_{p,q \in \mathbb{N}} \subset \mathcal{M}(E_0)$. It follows that

\begin{equation}
K_{p,q} = \bigcup_{i=1}^q A_{p,i} \in \mathcal{M}(E_0) \subset \mathcal{C}(E_0), \forall p, q \in \mathbb{N}
\end{equation}

by the Hausdorff property of $E_0$, Proposition 9.1.12 (c), Lemma 3.3.22 and Proposition 9.1.12 (a). From (6.4.14) and (6.4.15) we obtain that

\begin{equation}
K_{p,q} \subset K_{p,q+1}, \forall p, q \in \mathbb{N}
\end{equation}

and

\begin{equation}
\inf_{i \in I} \mathbb{P}^i(\bigcup_{i=1}^q X^i_t \in K_{p,q}, \forall t \in [0, q]) \geq 1 - 2^{-p-q}, \forall p, q \in \mathbb{N}.
\end{equation}

Letting $50$

\begin{equation}
V_p = \bigcap_{q \in \mathbb{N}} \left\{ x \in \mathbb{D}_0 : x|_{[0,q]} \in K_{p,q}^{0,q} \right\}, \forall p \in \mathbb{N},
\end{equation}

one finds by the fact $E_0 \in \mathcal{B}(E)$, Lemma 10.2.8 (b) (with $E = E_0$, $A = K_{p,q}$ and $T = q$) and and Lemma 9.6.6 (b) (with $E = E_0$) that

\begin{equation}
V_p \in \mathcal{B}(E_0)^{\mathbb{R}^+} \big|_{\mathbb{D}_0} \subset \mathcal{B}(E)^{\mathbb{R}^+} \big|_{\mathbb{D}_0} \subset \mathcal{B}(\mathbb{D}_0), \forall p \in \mathbb{N},
\end{equation}

which immediately implies $S_0 \in \mathcal{B}(\mathbb{D}_0)$. Consequently,

\begin{equation}
S_0 = \bigcup_{p \in \mathbb{N}} V_p \subset \mathcal{B}(E)^{\mathbb{R}^+} \big|_{\mathbb{D}_0} \subset \mathcal{B}(\mathbb{D}_0).
\end{equation}

**Step 2:** Verify (a). Each of $\{K_{p,q}\}_{p,q \in \mathbb{N}}$ is a $\mathcal{D}|_{E_0}$-baseable subset of $E_0$ by (6.4.15) and Proposition 3.3.19 (a, f) (with $E = E_0$, $K = K_{p,q}$ and $\mathcal{D} = \mathcal{D}|_{E_0}$). So, $50 K_{p,q}^{0,q}$ in (6.4.18) means the Cartesian power of $K_{p,q}$ for the index set $[0, q)$.
\( \mathcal{D} \) has a countable subset that separates and strongly separates points on each of \( \{K_{p,q}\}_{p,q\in\mathbb{N}} \) by Lemma 9.2.4. For simplicity, we assume \( \mathcal{D} \) is countable in Step 2 - 4 of the proof.

Letting \( \Psi \triangleq \varpi[\alpha(\mathcal{D})] \), we have

\[
\Psi|_{V_p} \in \text{im} \left( V_p, \mathcal{G}_0(V_p); D(\mathbb{R}^+; \mathbb{R})^{\alpha(\mathcal{D})} \right), \quad \forall p \in \mathbb{N}
\]

and

\[
\mathcal{B}(\Omega_0)|_{V_p} = \mathcal{B}(E)^{\mathbb{R}^+}|_{V_p}, \quad \forall p \in \mathbb{N}
\]

by Lemma 10.2.23 (with \( E = E_0, V = V_p, p = q \) and \( A_p = K_{p,q} \)). One then finds by (6.4.19), (6.4.22) and Fact 10.1.1 (with \( E = S_0, n = p, A_n = V_p, \mathcal{H}_1 = \mathcal{B}(\Omega_0)|_{S_0} \)) and \( \mathcal{H}_2 = \mathcal{B}(E)^{\mathbb{R}^+}|_{S_0} \) that

\[
\mathcal{B}(\Omega_0)|_{S_0} \subset \mathcal{B}(E)^{\mathbb{R}^+}|_{S_0}.
\]

Now, (a) follows by (6.4.19), (6.4.22) and (6.4.23).

**Step 3: Construct \( \{\hat{X}^i\}_{i\in I} \) and verify (b).** It follows by (6.4.16) and (6.4.17) that

\[
\inf_{i \in I} \mathbb{P}^i \left( X^i \in V_p \right) \geq 1 - \sup_{i \in I} \sum_{q \in \mathbb{N}} \left[ 1 - \mathbb{P}^i \left( X^i \in K_{p,q}, \forall t \in [0,q] \right) \right]
\]

\[
\geq 1 - 2^{-p}, \quad \forall p \in \mathbb{N}.
\]

Then, (6.4.20) and (6.4.24) imply

\[
\inf_{i \in I} \mathbb{P}^i \left( X^i \in S_0 \subset E_0^{\mathbb{R}^+} \right) = 1.
\]

By Proposition 6.3.9 (a) (with \( X = X^i \)), there exist

\[
\hat{X}^i \in M \left( \Omega^i, \mathcal{F}^i; S_0, \mathcal{B}(E)^{\mathbb{R}^+}|_{S_0} \right), \quad \forall i \in I
\]

satisfying (6.4.12). Now, (b) follows by (6.4.26) and (a).

**Step 4: Verify sufficiency of (c).** We have that \( ^{51} \)

\[
\inf_{f \in \alpha(\mathcal{D}), i \in I} \mathbb{P}^i \left( \varpi(f) \circ X^i = \varpi(f) \circ \hat{X}^i \right) = 1
\]

and

\[
\varpi(f) \circ \hat{X}^i \in M \left( \Omega^i, \mathcal{F}^i; D(\mathbb{R}^+; \mathbb{R}) \right), \quad \forall f \in \alpha(\mathcal{D}), i \in I
\]

by (6.4.12) and Proposition 9.6.1 (d) (with \( S = E_0 \) and \( E = \mathbb{R} \)). Fixing \( f \in \alpha(\mathcal{D}) \), \( \varpi(f) \circ \hat{X}^i \) is the unique càdlàg modification of \( \varpi(f) \circ X^i \) up to indistinguishability for all \( i \in I \) by (6.4.27) and Proposition 10.1.30 (h, i). \( \{ \varpi(f) \circ \hat{X}^i \}_{i \in I} \) satisfies \( \| - \| \) MCC by (6.4.27) and \( \{ X^i \}_{i \in I} \) satisfying \( \mathcal{D} \)-FMCC. \( \{ \varpi(f) \circ \hat{X}^i \}_{i \in I} \) satisfies MCCC by the boundedness of \( f \). Hence, \( \{ \varpi(f) \circ \hat{X}^i \}_{i \in I} \) is tight in \( D(\mathbb{R}^+; \mathbb{R}) \) by Theorem 9.7.12 (with \( (E, \tau) = (\mathbb{R}, \| - \|) \) and \( X^i = \varpi(f) \circ \hat{X}^i \)).

Letting \( \Psi = \varpi[\alpha(\mathcal{D})] \) again, \( \{ \Psi \circ \hat{X}^i \}_{i \in I} \) is tight in \( D(\mathbb{R}^+; \mathbb{R})^{\alpha(\mathcal{D})} \) by tightness of \( \{ \varpi(f) \circ \hat{X}^i \}_{i \in I} \) in \( D(\mathbb{R}^+; \mathbb{R}) \) and Proposition 10.2.21 (a) (with \( E = E_0, \mathcal{D} = \alpha(\mathcal{D}) \) and \( \mu^i = \mathbb{P}^i \circ (\hat{X}^i)^{-1} \in \mathcal{P}(\Omega_0) \)). \( \alpha(\mathcal{D}) \) is countable by Fact 10.1.14 and the

\(^{51}\) Here, the replica process \( \hat{X}^i \) is an \( E_0 \)-valued process and so \( f \circ \hat{X}^i \) is well-defined.
fact \( D \) is countable in step 2 - 4, so \( R^{a(D)} \) is a Polish space by Proposition 9.1.11 (f).

(6.4.29) \[ \varphi \equiv \bigotimes \text{ac}(D) \in C \left( E; R^{a(D)} \right) \]
by Fact 2.1.4 (b). Letting \( \{K_{p,q}\}_{p,q \in \mathbb{N}} \) be as in (6.4.15), we have that

(6.4.30) \[ \bigotimes \text{ac}(D)(K_{p,q}) \in \mathcal{X} \left( R^{a(D)} \right) \subset \mathcal{C} \left( R^{a(D)} \right) \]
by (6.4.29) and Proposition 9.1.12 (a, e). It follows by Lemma 10.2.24 (with (6.4.31) \( \Psi(V) \).

Step 5: Verify necessity of (c). We no longer take \( D \) to be countable. Suppose \( \{\hat{X}^i\}_{i \in I} \) in \( S_0 \). For each fixed \( f \in \text{ac}(D) \), \( \{\varpi(f) \circ \hat{X}^i\}_{i \in I} \) is tight in \( D(R^+; R) \) by Proposition 9.6.1 (d) (with \( E = E_0 \) and \( S = R \)) and Fact 10.2.18 (with \( E = \mathbb{D}_0 \), \( S = D(R^+; R) \), \( f = \varpi(f), \mu^i = \mathbb{P}^i \circ (\hat{X}^i)^{-1} \in \mathcal{P}(\mathbb{D}_0) \) and \( \Gamma = \{\mu^i\}_{i \in I} \). \( \{\varpi(f) \circ \hat{X}^i\}_{i \in I} \) satisfies \( \ldots \) MCC by Theorem 9.7.12 (with \( (E, \tau) = (R, \ldots) \)) and \( X^i = \varpi(f) \circ \hat{X}^i \). Hence, \( \{X^i\}_{i \in I} \) satisfies \( \mathcal{D} \) FMCC by (6.4.27) and Proposition 10.1.30 (b).

PROOF OF PROPOSITION 6.4.11 This result follows immediately by Theorem 6.4.13 (with \( D = \mathcal{F} \)) and Proposition 6.3.9 (with \( X = X^i \)).

6.4.3. Weak convergence of càdlàg replicas. The following proposition connects the weak convergence of càdlàg replicas on path space and consequence of their finite-dimensional convergence.

PROPOSITION 6.4.14. Let \( E \) be a topological space, \( \{(\Omega^n, \mathcal{F}^n, \mathbb{P}^n; X^n)\}_{n \in \mathbb{N}} \) be \( E \)-valued processes, \( (E_0, \mathcal{F}; \hat{E}, \hat{\mathcal{F}}) \) be a base over \( E, \hat{X}^n \in \text{rep}_c(X^n; E_0, \mathcal{F}) \) exist for each \( n \in \mathbb{N} \) and \( (\Omega, \mathcal{F}, \mathbb{P}; Y) \) be a \( D(R^+; \hat{E}) \)-valued random variable. Then:

(a) If \( \{\hat{X}^n\}_{n \in \mathbb{N}} \) and \( Y \) satisfy

(6.4.32) \[ \hat{X}^n \Rightarrow Y \text{ as } n \uparrow \infty \text{ on } D(R^+; \hat{E}), \]

then

(6.4.33) \[ \hat{X}^n \overset{D(R^+ \setminus Y)}{\Rightarrow} Y \text{ as } n \uparrow \infty. \]

\[ ^{52} \text{Several conditions for the existence of càdlàg replicas were given in Proposition 6.3.5.} \]

\[ ^{53} \text{The meaning of } (6.4.32) \text{ is explained in } 3.2.4. \text{ The notation } “J(Y)” \text{ is defined in } (2.5.8). \]
(b) If (6.4.32) holds, for some dense \( T \subset \mathbb{R}^+ \), and if \( \{X^n\}_{n \in \mathbb{N}} \) satisfies
\[
\inf_{t \in T, n \in \mathbb{N}} \mathbb{P}^n \left( \bigotimes \mathcal{F} \circ X^n_t \in \bigotimes \widehat{\mathcal{F}}(E) \right) = 1
\]
and \( \mathcal{F} \)-FMCC, then (6.4.33) holds.

(c) If (6.4.35) holds, \( \{X^n\}_{n \in \mathbb{N}} \) is \((T, \mathcal{F}\{1\})\)-AS\(^{55}\) and (6.4.34) holds for some countable \( T \subset \mathbb{R}^+ \), then \( Y \) is an \( \hat{E} \)-valued stationary process.

Note 6.4.15. If \( Y \) is an \( \hat{E} \)-valued càdlàg process (especially a càdlàg replica), then \( J(Y) \subset (0, \infty) \) is countable by Lemma 3.1.3 (c) and Proposition 3.3.31. In other words, \( \mathbb{R}^+ \setminus J(Y) \) is a co-countable\(^{56}\) subset of \( \mathbb{R}^+ \).

**Proof of Proposition 6.4.14.** (a) follows by Lemma 3.1.3 (c) and Theorem 9.7.11 (a) (with \( T = \hat{E} \), \( X^n = \hat{X}^n \) and \( X = Y \)).

(b) (6.4.34) is a version of (6.4.7) with \( I = \mathbb{N} \). Given dense \( T \) and \( \mathcal{F} \)-FMCC, \( \{\hat{X}^n\}_{n \in \mathbb{N}} \) as the unique càdlàg replicas of \( \{X^n\}_{n \in \mathbb{N}} \) is tight in the Polish space \( D(\mathbb{R}^+; \hat{E}) \) by Proposition 6.4.11 (a) (with \( I = \mathbb{N} \)). It is relatively compact in \( D(\mathbb{R}^+; \hat{E}\{1\}) \) by the Prokhorov's Theorem (Theorem 2.3.12 (b)). Now, (b) follows by Theorem 9.7.11 (b) (with \( T = \hat{E} \), \( X^n = \hat{X}^n \) and \( X = Y \)).

(c) \( T \setminus J(Y) \) is a conull set, so
\[
\mathbb{P}_{\mathbf{Y};\mathbf{T}_0} = \bigcap_{t \in \mathbf{T}_0} \{c \in (0, \infty) : t + c \in \mathbf{T} \setminus J(Y)\}
\]
is a conull hence dense subset of \( \mathbb{R}^+ \) for any \( \mathbf{T}_0 \in \mathcal{P}_0(\mathbf{T} \setminus J(Y)) \). Fixing \( \mathbf{T}_0 \in \mathcal{P}_0(\mathbf{T} \setminus J(Y)) \) and \( f \in \text{mc}[\Pi_{\mathbf{T}_0}(\mathcal{F}\{1\})] \), we have
\[
\mathbb{E} \left[ \hat{f} \circ Y_{\mathbf{T}_0} \right] = \mathbb{E} \left[ \hat{f} \circ Y_{\mathbf{T}_0+c} \right]
\]
for all \( c \in \mathbb{S}_{\mathbf{Y};\mathbf{T}_0} \) by (a) and Lemma 10.3.6 (b, e) (with \( \mathbf{T} = \mathbf{T} \setminus J(Y) \) and \( \mathbf{S}_{\mathbf{T}_0} = \mathbb{S}_{\mathbf{Y};\mathbf{T}_0} \)). \( \{Y_{t+c}\}_{c \geq 0} \) is a càdlàg process for all \( t \in \mathbf{T}_0 \) since \( Y \) is càdlàg. \( \zeta \subseteq \{\varpi(\hat{f} \circ Y_{\mathbf{T}_0+c})_{c \geq 0} \} \) is also a càdlàg process by Fact 10.1.31 (a, b) (with \( \mathbf{I} = \mathbf{T}_0 \), \( i = t \), \( X^i = Y_{t+c} \), \( X = \{Y_{t+c}\}_{c \geq 0} \) and \( f = \hat{f} \)). Then, (6.4.36) extends to all \( c \in (0, \infty) \) by the denseness of \( \mathbb{S}_{\mathbf{T}_0;\mathbf{Y}} \) in \( \mathbb{R}^+ \), the càdlàg property of \( \zeta \) and the Dominated Convergence Theorem. Now, (c) follows by Corollary 3.1.11 (a) (with \( d = \mathbb{N}(\mathbf{T}_0) \) and \( A = \hat{E}^d \)) and Fact 9.7.11 (c) (with \( E = \hat{E} \)).

The next proposition connects weak convergence of càdlàg replicas on \( D(\mathbb{R}^+; \hat{E}) \) and that on the restricted path space \( D(\mathbb{R}^+; \mathcal{E}_0; \mathcal{G}_E(\mathcal{E}_0)) \) (if well-defined).

**Proposition 6.4.16.** Let \( E \) be a topological space, \( (E_0, \mathcal{E}; \hat{E}, \hat{\mathcal{F}}) \) be a base over \( E \), \( (\Omega, \mathcal{F}, \hat{\mathcal{F}}; X) \) and \( \{\Omega^n, \mathcal{F}^n, \hat{\mathcal{F}}^n; X^n\}_{n \in \mathbb{N}} \) be \( E \)-valued càdlàg processes, \( \hat{X} \in \text{rep}_E(X; E_0, \mathcal{E}) \) and \( \hat{X}^n \in \text{rep}_E(X^n; E_0, \mathcal{F}) \) exist for each \( n \in \mathbb{N} \). In addition, suppose \( (E_0, \mathcal{G}_E(\mathcal{E}_0)) \) is a Tychonoff space. Then:

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\(^{54}\)The notion of \((\mathbf{T}, \mathcal{F}\{1\})\)-AS was introduced in Definition 6.2.1.

\(^{55}\)Conull set was specified in 21.1.5. Conul subset of \( \mathbb{R} \) is in the Lebesgue sense.

\(^{56}\)The notion of cocountable set was defined in 21.1.1.

\(^{57}\)Relative compactness of \( D(\mathbb{R}^+; \hat{E}) \)-valued random variables \( \{\hat{X}^n\}_{n \in \mathbb{N}} \) follows our interpretation in 2.4.
(a) If $\hat{X}$ and $\{\hat{X}^n\}_{n \in \mathbb{N}}$ satisfy

\[(6.4.37)\quad \hat{X}^n \Rightarrow \hat{X} \text{ as } n \uparrow \infty \text{ on } D(\mathbb{R}^+; \mathcal{E}_E(E_0)) ;
\]

then they satisfy

\[(6.4.38)\quad \hat{X}^n \Rightarrow \hat{X} \text{ as } n \uparrow \infty \text{ on } D(\mathbb{R}^+; \hat{\mathcal{E}}).
\]

(b) If there exists an $S_0 \subset E_0^\mathbb{R}^+$ satisfying \[6.3.15\] and

\[(6.4.39)\quad \inf_{n \in \mathbb{N}} P^n(X^n \in S_0) = 1,
\]

and if $\mathcal{F}$ strongly separates points on $E_0$, then \[(6.4.38)\] implies \[(6.4.37)\].

**Proof.** (a) For ease of notation, we let $(\Omega^0, \mathcal{F}^0, P^0, X^0) \triangleq (\Omega, \mathcal{F}, P, X)$, $D_0 \triangleq D(\mathbb{R}^+; E_0, \mathcal{E}_E(E_0))$ and $\hat{D} \triangleq D(\mathbb{R}^+; \hat{\mathcal{E}})$, $(E_0, \mathcal{E}_E(E_0))$ is a topological refinement of $(E_0, \mathcal{E}_\hat{E}(E_0))$ by Lemma \[3.1.3\](d). $D(\mathbb{R}^+; E_0, \mathcal{E}_E(E_0))$ is a subspace of $\hat{D}$ by Corollary \[9.6.3\] (with $E = \hat{E}$ and $A = E_0$). It then follows by \[(6.4.37)\] and Proposition \[6.3.9\](c) (with $E = (E_0, \mathcal{E}_E(E_0))$ and $S = (E_0, \mathcal{E}_E(E_0))$) that $D_0 \subset \hat{D}$, $\hat{D}_0 \triangleq (\hat{D}_0, \mathcal{E}_\hat{D}(\hat{D}_0))$ is a topological coarsening of $D_0$ and

\[(6.4.40)\quad \hat{X}^n \in M(\Omega^n, \mathcal{F}^n; \hat{D}_0) \subset M(\Omega^n, \mathcal{F}^n; \hat{D}_0), \quad \forall n \in \mathbb{N}.
\]

Let $\mu_n$, $\hat{\nu}_n$ and $\nu_n$ denote the distribution of $\hat{X}^n$ as $D_0$-valued, $\hat{D}$-valued and $\hat{D}_0$-valued random variables for each $n \in \mathbb{N}$. It follows by \[(6.4.37)\]

\[(6.4.41)\quad \nu_n \Rightarrow \nu_0 \text{ as } n \uparrow \infty \text{ in } \mathcal{P}(\hat{D}_0).
\]

It follows by \[(6.4.41)\] and Lemma \[10.2.13\] (with $E = \hat{D}$, $A = \hat{D}_0$, $\mu_n = \nu_n$ and $\mu = \nu_0$) that

\[(6.4.42)\quad \hat{\nu}_n = \nu_n|_{\hat{D}} \Rightarrow \nu_0|_{\hat{D}} = \nu_0 \text{ as } n \uparrow \infty \text{ in } \mathcal{P}(\hat{D}),
\]

which proves \[(6.4.38)\].

(b) The given conditions imply

\[(6.4.43)\quad \inf_{n \in \mathbb{N}_0} P^n(X^n \in D_0) \geq \inf_{n \in \mathbb{N}_0} P^n(X^n \in S_0 \cap D_0) = 1.
\]

We have $\mathcal{E}_E(E_0) = \mathcal{E}_\hat{E}(E_0)$ by \[(3.1.1)\], \[(3.1.2)\] and $\mathcal{F}$ strongly separating points on $E_0$, which implies $D_0 = \hat{D}_0$. According to Proposition \[6.3.9\](c) (with $S_0 = D_0$ and $X = X^n$ or $X$), one can take

\[(6.4.44)\quad \hat{X}^n = \text{rep}_{\mathcal{F}}(X^n; E_0, \mathcal{F})
\]

\[(6.4.45)\quad \mu_n = \nu_n \in \mathcal{P}(\hat{D}_0) = \mathcal{P}(D_0), \quad \forall n \in \mathbb{N}_0.
\]

\[(6.4.38)\] implies \[(6.4.42)\]. As $D(\mathbb{R}^+; \hat{E})$ is a Polish space, \[(6.4.42)\] implies \[(6.4.41)\] by Lemma \[10.2.13\] (with $E = \hat{D}$, $A = \hat{D}_0$, $\mu_n = \nu_n$ and $\mu = \nu_0$). Now, \[(6.4.37)\] follows by \[(6.4.44)\], \[(6.4.41)\] and \[(6.4.45)\].

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58As specified in \[2.4\], \[(6.4.37)\] abbreviates the statement that $\{\hat{X}^n\}_{n \in \mathbb{N}}$ and $\hat{X}$ are $D_0$-valued random variables and the distributions of $\{\hat{X}^n\}_{n \in \mathbb{N}}$ converge weakly to that of $\hat{X}$ in $\mathcal{P}(D_0)$.
6.5. Containment in large baseable subsets

Given $E$-valued processes $\{(\Omega^i, \mathcal{F}^i, P^i; X^i)\}_{i \in \mathbf{I}}$ and a base $(E_0, \mathcal{F}; \hat{E}, \hat{\mathcal{F}})$ over $E$, most of the developments of §6.1, §6.2 and §6.3 require $E_0$ to have containment properties like (6.4.7),

\[
(6.5.1) \quad \inf_{i \in I} P^i \left( X^i_{t} \in E_0 \right) = 1
\]

or (6.4.25) for $\{X^i\}_{i \in \mathbf{I}}$. In practice, one usually constructs a baseable set $E_0$ satisfying the non-functional conditions (6.5.1) or (6.4.25) first, and then selects proper functions to establish the base $(E_0, \mathcal{F}; \hat{E}, \hat{\mathcal{F}})$. From Fact 6.1.9 we immediately observe that:

**Fact 6.5.1.** \((6.4.7), (6.5.1)\) and \((6.4.25)\) are successively stronger for any index set $\mathbf{I}$ and $T \subset \mathbb{R}^+$.\[59\]

The simplest case is when $E$ itself is a baseable space. Then, one easily obtains a base $(E, \mathcal{F}; \hat{E}, \hat{\mathcal{F}})$ by Lemma 3.3.7 and the containment properties in Fact 6.5.1 are automatic. When $E$ is non-baseable, one can use $T$-MPCC, MCCC, $T$-PMTC or $T_k$-LMTC introduced in §6.4.1 to construct the desired $E_0$ in (6.4.25) or (6.5.1).

When $\{X^i\}_{i \in \mathbf{I}}$ are all càdlàg, the following proposition uses $T$-MPCC and $\tau$-MCC to construct an $E_0$ satisfying (6.4.25).

**Proposition 6.5.2.** Let $(E, \tau)$ be a metric space, $\mathcal{D} \subset C(E; \mathbb{R})$ separate points on $E$, $T$ be a dense subset of $\mathbb{R}^+$ and $\{(\Omega^i, \mathcal{F}^i, P^i; X^i)\}_{i \in \mathbf{I}}$ be $E$-valued càdlàg processes satisfying $T$-MPCC and $\tau$-MCC. Then, there exist $\{A_{p,q}\}_{p,q \in \mathbf{N}} \subset \mathcal{C}(E)$ satisfying the following properties:

(a) $\{A_{p,q}\}_{p,q \in \mathbf{N}}$ are totally bounded and satisfy

\[
(6.5.2) \quad A_{p,q} \subset A_{p,q+1}, \forall p, q \in \mathbf{N}
\]

and

\[
(6.5.3) \quad \inf_{i \in \mathbf{I}} P^i \left( X^i_{t} \in A_{p,q}, \forall t \in \left[0, q\right] \right) \geq 1 - 2^{-p-q}, \forall p, q \in \mathbf{N}.
\]

(b) $E_0 \doteq \bigcup_{p,q \in \mathbf{N}} A_{p,q}$ is a second-countable subspace and is a $\mathcal{D}$-baseable subset of $E$.

(c) $E_0$ and $S_0 \doteq \bigcup_{p \in \mathbf{N}} V_p$ satisfy (6.4.25), where\[60\]

\[
(6.5.4) \quad V_p \doteq \bigcap_{q \in \mathbf{N}} \left\{ x \in \mathbb{D}_0 : x|_{\left[0, q\right]} \in A_{p,q}^{\left[0, q\right]} \right\}, \forall p \in \mathbf{N}.
\]

(d) If $(E, \tau)$ is complete, then $\{A_{p,q}\}_{p,q \in \mathbf{N}} \subset \mathcal{K}(E)$ and $\{X^i\}_{i \in \mathbf{I}}$ satisfies MCCC in $E_0$.

As noted in §3.3.1, metrizable compact subsets provided by MCCC, $T$-PMTC and $T_k$-LMTC are nice baseable “blocks” for building $E_0$. In metric spaces, totally bounded subsets provided by $T$-MPCC form another category of such blocks.

**Fact 6.5.3.** Let $(E, \tau)$ be a metric space, $\mathcal{D} \subset C(E; \mathbb{R})$ separate points on $E$ and $\{A_n\}_{n \in \mathbf{N}}$ be totally bounded Borel subsets of $E$. Then, $\bigcup_{n \in \mathbf{N}} A_n$ is a second-countable subspace and, in particular, is a $\mathcal{D}$-baseable subset of $E$.\[59\]

\[60\] by contrast is a functional condition depending on the choice of $\mathcal{F}$.

\[59\] $A_{p,q}$ in (6.5.4) below means the Cartesian power of $A_{p,q}$ for the index set $[0, q)$.\[60\]
Proof. $A \doteq \bigcup_{n \in \mathbb{N}} A_n$ is a separable subspace of $E$ by Proposition 9.1.10 (a) and Proposition 9.1.3 (e). Now, the result follows by Proposition 9.1.4 (c) and Proposition 3.3.8. \hfill \square

Proof of Proposition 6.5.2. (a) An inspection of the proof of [Kou16 Theorem 17] shows that $\mathbf{T}$-MPCC is enough for their developments. So, one follows [Kou16] to construct totally bounded $\{A_{p,q}\}_{p,q \in \mathbb{N}} \subset \mathcal{C}(E)$ satisfying (a).

(b) follows by (a) and Fact 6.5.3.

(c) One finds by (a) that

$$\inf_{i \in I} \mathbb{P}^i \left( X^i \in V_p \right) \geq 1 - \sup_{i \in I} \sum_{q \in \mathbb{N}} \left[ 1 - \mathbb{P}^i \left( X^i_q \in A_{p,q} \forall t \in [0,q] \right) \right] \geq 1 - 2^{-p}, \forall p \in \mathbb{N}.$$  

(6.5.5)

(d) Each $(A_{p,q}, \tau)$ is complete by the fact $A_{p,q} \subset \mathcal{C}(E)$ and Proposition 9.1.6.

(c). Then, (d) follows by Proposition 9.1.15. \hfill \square

The next proposition uses MCCC to construct an $E_0$ satisfying (6.4.25).  

Proposition 6.5.4. Let $E$ be a topological space, $\mathcal{D} \subset C(E; \mathbb{R})$ separate points on $E$ and $\{ (\Omega^i, \mathcal{F}^i, \mathbb{P}^i, X^i) \}_{i \in I}$ be $E$-valued processes satisfying MCCC in $A \subset E$. Then, there exist $\{ K_{p,q} \}_{p,q \in \mathbb{N}} \subset \mathcal{K}^m(E)$ satisfying the following properties:

(a) (6.4.16) and (6.4.17) hold.

(b) $E_0 \doteq \bigcup_{p,q \in \mathbb{N}} K_{p,q} \subset A$ is a $\mathcal{D}$-baseable subset of $E$. Moreover, $\{ X^i \}_{i \in I}$ satisfies MCCC in $E_0$.

(c) $E_0$ and $S_0 \doteq \bigcup_{p \in \mathbb{N}} V_p$ satisfy (6.4.25), where $\{ V_p \}_{p \in \mathbb{N}}$ are defined as in (6.4.18).

Proof. (a) We pick $\{ A_{p,q} \}_{p,q \in \mathbb{N}} \subset \mathcal{K}^m(E)$ satisfying $A_{p,q} \subset A$ for all $p, q \in \mathbb{N}$ and (6.4.14). $E$ is a Hausdorff space by Proposition 9.2.1 (e) (with $A = E$). Then,

$$K_{p,q} \doteq \bigcup_{i=1}^{q} A_{p,i} \in \mathcal{K}^m(E) \subset \mathcal{C}(E) \subset \mathcal{B}(E), \forall p, q \in \mathbb{N}$$  

(6.5.6)

by Proposition 9.1.2 (c), Lemma 3.3.22 and Proposition 9.1.12 (a). Now, (6.4.16) and (6.4.17) follow by (6.4.14) and (6.5.6).

(b) follows by (6.5.6) (a) and Proposition 3.3.26 (b, e).

(c) $E_0$ and $S$ satisfy (6.4.24) by (a), which implies (6.4.25) immediately. \hfill \square

The following fact gives an $E_0$ satisfying (6.5.1) for countable $T \subset \mathbb{R}^+$ by T-PMTC.

Fact 6.5.5. Let $E$ be a topological space, $A \subset E$ and $\{ (\Omega^i, \mathcal{F}^i, \mathbb{P}^i, X^i) \}_{i \in I}$ be $E$-valued processes. Then:

(a) If $I$ is an infinite set and $\{ X^i \}_{i \in I}$ satisfies T-PMTC in $A$, then $\{ X^i \}_{i \in I}$ satisfies T-PSMTC in $A$.

(b) $\{ X^i \}_{i \in I}$ satisfies T-PMTC in $A$ if and only if $\{ X^i_{T_0}, \}_{i \in I}$ is m-tight in $A^{T_0}$ for all $T_0 \in \mathcal{P}(T)$.

(c) If $\{ X^i \}_{i \in I}$ satisfies T-PSMTC in $A$, then $\{ X^i_{T_0}, \}_{i \in I}$ is sequentially m-tight in $A^{T_0}$ for all $T_0 \in \mathcal{P}(T)$. 

Proof.
(d) If \( \{ X^i \}_{i \in I} \) satisfies \( T \)-PMTC in \( A \) for a countable \( T \subset \mathbb{R}^+ \) and \( D \subset C(E; \mathbb{R}) \) separates points on \( E \), then there exists a \( D \)-baseable subset \( E_0 \in \mathcal{K}^\sigma_m(E) \) such that \( \{ X^i \}_{i \in I} \) satisfies \( T \)-PMTC in \( E_0 \subset A \).

(e) When \( (E, \tau) \) is a metric space, \( \{ X^i \}_{i \in I} \) satisfying \( T \)-PMTC implies \( \{ X^i \}_{i \in I} \) satisfying \( T \)-MPCC and the converse is true if \( (E, \tau) \) is complete.

(f) If \( \{ X^i \}_{i \in I} \) satisfies MCCC in \( A \), then \( \{ X^i \}_{i \in I} \) satisfies \( \mathbb{R}^+ \)-PMTC in \( A \).

**Proof.** (a) and (f) are automatic by definition. (b) and (c) follow by Lemma \[10.2.19\] (with \( I = T_0, S_i = E, A_i = A \) and \( \Gamma = \{ F^\tau \circ (X^i_{T_0})^{-1} \}_{i \in I} \)). (d) follows by Lemma \[10.3.2\] (with \( I = T, i = t \) and \( \Gamma_i = \{ F^\tau \circ (X^i)^{-1} \}_{i \in I} \)).

(e) The first statement is immediate by Proposition 9.1.15. Then, we suppose \( \{ X^i \}_{i \in I} \) satisfies \( T \)-MPCC and \( (E, \tau) \) is complete. Fixing \( t \in T \) and \( \epsilon \in (0, \infty) \), we pick a totally bounded \( A_{\epsilon, p, t} \subset E \) for each \( p \in \mathbb{N} \) such that

\[
\inf_{i \in I} \mathbb{P}^T \left( X^i_t \in A_{\epsilon, p, t} \right) \geq 1 - \epsilon^{2^{-p}}, \forall p \in \mathbb{N}.
\]

Letting \( K_{\epsilon, t} \) be the closure of \( \bigcap_{p \in \mathbb{N}} A_{\epsilon, p, t} \), one finds that

\[
\inf_{i \in I} \mathbb{P}^T \left( X^i_t \in K_{\epsilon, t} \right) \geq 1 - \epsilon, \forall p \in \mathbb{N}.
\]

\( \bigcap_{p \in \mathbb{N}} A_{\epsilon, p, t} \) is totally bounded by definition and so is \( K_{\epsilon, t} \) by Proposition 9.1.10 (c). \( (K_{\epsilon, t}, \tau) \) is complete by Proposition 9.1.16 (c). Hence, \( K_{\epsilon, t} \in \mathcal{K}^\tau(E) \) by Proposition 9.1.15 and (e) follows by (6.5.8). \( \square \)

Given countably many processes satisfying \( T_k \)-LMTC, the next proposition constructs an \( E_0 \) satisfying (6.5.1) for a conull \( T \subset \mathbb{R}^+ \).

**Proposition 6.5.6.** Let \( E \) be a topological space, \( D \subset C(E; \mathbb{R}) \) separate points on \( E \) and \( \{(\Omega^n, \mathcal{F}^n, \mathbb{P}^n; X^n)\}_{n \in \mathbb{N}} \) be \( E \)-valued measurable processes satisfying \( T_k \)-LMTC\(^{\small 61}\) in \( A \subset E \). Then, there exists a \( D \)-baseable subset \( E_0 \in \mathcal{K}^\sigma_m(E) \) with \( A \supset E_0 \) and a conull \( T \subset \mathbb{R}^+ \) such that

\[
\inf_{t \in T, n \in \mathbb{N}} \mathbb{P}^n (X^n_t \in E_0) = 1
\]

and \( \{ X^n \}_{n \in \mathbb{N}} \) satisfies \( T_k \)-LMTC in \( E_0 \).

**Proof.** We take \( \{ K_p \}_{p \in \mathbb{N}} \subset \mathcal{K}^\sigma_m(A, \mathcal{O}_E(A)) \) satisfying

\[
\inf_{k, n \in \mathbb{N}} \frac{1}{T_k} \int_0^{T_k} \mathbb{P}^n (X^n_t \in K_p) \, d\tau \geq 1 - 2^{-p}, \forall p \in \mathbb{N}
\]

and let \( E_0 = \bigcup_{p \in \mathbb{N}} K_p \in \mathcal{K}^\sigma_m(A, \mathcal{O}_E(A)) \). It follows that

\[
\sup_{k, n \in \mathbb{N}} \int_0^{T_k} \mathbb{P}^n (X^n_t \notin E_0) \, d\tau = 0
\]

by (6.5.10) and continuity of measure. Hence, (6.5.9) holds for the conull set

\[
T = \mathbb{R}^+ \setminus \bigcup_{k, n \in \mathbb{N}} \{ t \in [0, T_k] : \mathbb{P}^n (X^n_t \notin E_0) > 0 \}.
\]

Now, the result follows by (6.5.10) and Proposition 3.3.26 (b, e) (with \( A = E_0 \)). \( \square \)

\(^{61}\)Recall that the definition of \( T_k \)-LMTC includes \( T_k \uparrow \infty \).
The relationship among MCCC, T-MPCC, T-PMTC and $T_k$-LMTC is illustrated in Figure 1 below, where green solid arrows mean definite implication, blue dashed arrow means conditional implication and red crossed arrow means false converse.

**Figure 1.** *The relationship among tightness/containment conditions*

**Remark 6.5.7.** All the unlabelled arrows in Figure 1 are immediate. Below is some explanation for the labelled ones:

- (I) was justified in Proposition 6.5.2 (a, d) for càdlàg processes living on a complete (but not necessarily separable) metric space $(E, r)$ and satisfying $r$-MCC. This is a generalization of [Kou16, Theorem 17] on infinite time horizon since $T$-MPCC with a dense $T$ is weaker than the Pointwise Containment Property in [Kou16, §5].

- By Fact 6.5.5 (e), (II) is true on arbitrary metric spaces and (III) is true on complete metric spaces.

- (IV) was justified in Proposition 6.5.6 for a countable collection of measurable processes.

- (i) is not true because $T_k$-LMTC will not be affected by changing the distributions of $\{X_t^i\}_{i \in I}$ to a non-tight family for each $t \in Q^+$. 

- (ii) and (iii) are disproved by the constant process $\{t\}_{t \geq 0}$.

- (iv) and (v) are disproved by Example 6.5.8 below, where we construct a non-stationary càdlàg process that satisfies $T_k$-LMTC and $R^+$-PMTC but violates MCCC. (iv) was also disproved by [Kou16, Example 2].
Example 6.5.8. Let \( \mu \) be the uniform distribution on \((0,1)\) and

\[
\eta_t(\omega) = \begin{cases} 
1 - \omega + t, & \text{if } t \in [0, \omega), \\
\frac{1}{2}, & \text{if } t \in [\omega, \infty),
\end{cases} \quad \forall \omega \in (0,1), \ t \in \mathbb{R}^+.
\]

\( \eta = \{\eta_t\}_{t \geq 0} \) satisfies \( \mathbb{R}^+\text{-PMTC} \) since \((0,1)\) is \( \sigma \)-compact. However, \( \eta \) violates MCCC because for any \( a, b \in (0,1) \),

\[
\mu(\eta_t \in [a, b], \forall t \in [0,1]) 
\leq 1 - \mu(\{\omega \in (0,1) : 0 \leq \omega - t < 1 - b, \exists t \in [0,\omega)\}) = 0.
\]

For each \( \tau > 0 \) and \( \epsilon \in (0,1/2) \),

\[
\{\eta_t \in [\epsilon, 1 - \epsilon]\} = ((\tau \wedge 1) \vee (\epsilon + \tau), 1 \wedge (1 + \tau - \epsilon)) \cup (0, \tau \wedge 1).
\]

Letting \( T > 1/\epsilon \), one finds by (6.5.15) that

\[
\frac{1}{T} \int_0^T \mu(\eta_t \in [\epsilon, 1 - \epsilon]) \, d\tau \geq \frac{1}{T} \int_1^T 1 \, d\tau \geq 1 - \epsilon.
\]

Hence, \( \eta \) satisfies \( T_k\text{-LMTC} \) for any \( T_k \uparrow \infty \). Moreover, \( \eta \) is non-stationary since \( \eta_0 \) and \( \eta_{1/2} \) have distinct expectations.
CHAPTER 7

Application to Finite-Dimensional Convergence

The previous four chapters elaborate Theme 1 of this work. With the help of replication, we have developed in §6.2 several tool results for Theme 2 the finite-dimensional convergence of possibly non-càdlàg processes. Now, we are going to answer the target questions Q1, Q2, and Q3 of Theme 2 in the following three sections. §7.1 establishing finite-dimensional convergence to processes with general paths, answers Q2 §7.2 establishing finite-dimensional convergence of weakly càdlàg processes to weakly càdlàg or progressive limit processes, provides answers to both Q2 and Q3. In §7.3 we answer Q1 by establishing finite-dimensional convergence to long-time typical behaviors of a given measurable process.

7.1. Convergence of process with general paths

Let \( \{X^i\}_{i \in I} \) be infinitely many \( E \)-valued processes and \( S \subset \mathbb{R}^+ \). We give in this section a set of sufficient conditions for the unique existence of \( X \in \mathbb{F}_S(\{X^i\}_{i \in I}) \) establishing a Kolmogorov extension of weak limit points of the finite-dimensional distributions of \( \{X^i\}_{i \in I} \) for each \( T_0 \in \mathcal{P}_0(S) \). Hence, our goal can be achieved by directly applying Theorem 5.1.10 established in §5.1.

Theorem 7.1.1. Let \( E \) be a topological space, \( \{\{\Omega^i, \mathcal{F}^i, [\mathbb{P}^i; X^i]\}_{i \in I}\} \) be \( E \)-valued processes, \( D \subset C_b(E; \mathbb{R}) \) separate points on \( E \) and \( S \subset \mathbb{R}^+ \). Then:

(a) If \( \{X^i\}_{i \in I} \) satisfies S-PSMTC in \( A \subset E \), then any \( X \in \mathbb{F}_S(\{X^i\}_{i \in I}) \) satisfies S-PMTC in \( A \).

(b) If \( \{X^i\}_{i \in I} \) is \( (S, D) \)-FDC and satisfies S-PSMTC, then there exists an \( X = \mathbb{F}_S(\{X^i\}_{i \in I}) \) satisfying S-PMTC and \( X = \mathbb{F}_S(\{X^i\}_{i \in I}) \) for any \( \{i_n\}_{n \in \mathbb{N}} \subset \mathcal{I} \).

(c) If \( \{X^i\}_{i \in I} \) is \( (\mathbb{R}^+, D) \)-FDC, is \( (\mathbb{R}^+, D) \)-AS and satisfies \( \mathbb{R}^+ \)-PSMTC, then there exists a stationary \( X = \mathbb{F}_{\mathbb{R}^+}(\{X^i\}_{i \in I}) \) satisfying \( \mathbb{R}^+ \)-PMTC and \( X = \mathbb{F}_{\mathbb{R}^+}(\{X^i\}_{i \in I}) \) for any \( \{i_n\}_{n \in \mathbb{N}} \subset \mathcal{I} \).

Remark 7.1.2. We pointed out in Fact 6.5.9 (b) (with \( T = S \)) that \( X \) satisfying S-PMTC in \( A \) is equivalent to \( X_{T_0} \) being \( m \)-tight in \( A_{T_0} \) for every \( T_0 \in \mathcal{P}_0(S) \).

---

1The readers are referred to §6.2 for definitions and notations about finite-dimensional convergence, finite-dimensional limit point and finite-dimensional limit.

2As mentioned in Note 5.1.3, the assumption of \( D \subset C_b(E; \mathbb{R}) \) separating points on \( E \) below does not require \( E \) to be a Tychonoff or baseable space.

3S-PMTC (in \( A \)) and S-PSMTC (in \( A \)) were introduced in §6.4.1. As specified in Note 6.4.2 that \( X \) satisfies S-PMTC in \( A \) means the singleton \( \{X\} \) satisfies S-PMTC in \( A \).

4The notions of \( (S, D) \)-FDC and \( (T, D) \)-AS was introduced in §6.2.
In particular, $X$ satisfying $R^+\text{-PMTC}$ is equivalent to all finite-dimensional distributions of $X$ being $m$-tight.

**Proof of Theorem 7.1.1.** (a) Suppose $X$ is defined on $(\Omega, \mathcal{F}, \mathbb{P})$ and

(7.1.1) \[ X^{i_n} \xrightarrow{D(S)} X \quad \text{as} \quad n \uparrow \infty. \]

It follows by (7.1.1) and Fact 2.5.2(d) that

(7.1.2) \[ \mathbb{P}^{i_n} \circ (X^{i_n})^{-1} \xrightarrow{} \mathbb{P} \circ X_t^{-1} \quad \text{as} \quad n \uparrow \infty \quad \text{in} \quad \mathcal{P}(E), \quad \forall t \in S. \]

$\{X^{i_n}_t\}_{n \in \mathbb{N}}$ is sequentially $m$-tight in $A$ for all $t \in S$ as $\{X^i\}_{i \in I}$ satisfies $S\text{-PSMTC}$ in $A$. Hence, (a) follows by (7.1.2) and Lemma 5.1.7(b) (with $\Gamma = \{\mathbb{P}^{i_n} \circ (X^{i_n})^{-1}\}_{n \in \mathbb{N}}$ and $\mu = \mathbb{P} \circ X^{-1}$).

(b) For each $i \in I$, we let $\mu_i$ be the restriction of $\mathbb{P}(X)$ to $\mathcal{B}(E)^{\otimes R^+}$.

(7.1.3) \[ \mu_i \circ p^{-1}_{T_0} = \mathbb{P} \circ (X^{i})^{-1} \in \mathcal{P}(E^{T_0}, \mathcal{B}(E)^{\otimes T_0}), \quad \forall i \in I \]

form a sequentially $m$-tight family by Fact 6.5.5(c) (with $A = E$). For each $T_0 \in \mathcal{P}_0(S)$ and $f \in \mathcal{M}([\Pi^{T_0}(D)] \cup \{1\}$, the integrals

(7.1.4) \[ \int_{E^{T_0}} f(x) \mu_i \circ p^{-1}_{T_0}(dx) = \mathbb{E}^i \left[ f \circ X^{i}_{T_0} \right], \quad \forall i \in I \]

admit at most one limit point in $R$ since $\{X^i\}_{i \in I}$ is $(S, D)$-FDC. Hence, it follows by Theorem 5.1.10 (with $\Gamma = \{\mu_i\}_{i \in I}, I = S, I_0 = T_0$ and $a = b = 1$) that there exists a unique $\mu \in \mathcal{P}(E^S, \mathcal{B}(E)^{\otimes S})$ and some $\{I_{t_0} \in \mathcal{P}_0(I)\}_{T_0 \in \mathcal{P}_0(S)}$ such that $\mu \circ p^{-1}_{T_0} \in \mathcal{P}(E^{T_0})$ is the weak limit of any subsequence of and, hence, is the unique weak limit point of $\{\mu_{I_{T_0}, i} = \mathbb{E}^i (\mu_i \circ p^{-1}_{T_0})\}_{i \in I, T_0}$ for all $T_0 \in \mathcal{P}_0(S)$.

We fix $t_0 \in S$ and define

(7.1.5) \[ X_t = \begin{cases} p_t, & \text{if} \ t \in S, \\ p_{t_0}, & \text{if} \ t \in R^+ \setminus S, \end{cases} \quad \forall t \in R^+. \]

By Fact 2.1.3(a) and Fact 2.5.2(b), $X = \{X_t\}_{t \geq 0}$ well-defines an $E$-valued process on the probability space $(E^S, \mathcal{B}(E)^{\otimes S}, \mu)$ that satisfies

(7.1.6) \[ \mu \circ X^{-1}_{T_0} = \mu \circ p^{-1}_{T_0}, \quad \forall T_0 \in \mathcal{P}_0(S). \]

Now, (b) follows by (a) and Fact 10.1.33 with $T = S$.

(c) One obtains by (b) (with $S = R^+$) an $X = \mathbb{E} \{p_{R^+}(\{X^i\}_{i \in I})$ satisfying all conclusions of (c) except for stationarity. Suppose $X$ is defined on $(\Omega, \mathcal{F}, \mathbb{P})$ and

(7.1.7) \[ X^{i_n} \xrightarrow{D(R^+)} X \quad \text{as} \quad n \uparrow \infty. \]

Fixing $c \in (0, \infty)$ and $T_0 \in \mathcal{P}(R^+)$, it follows by (7.1.7), Fact 10.1.32(b) (with $n = i_n$) and Fact 6.2.6 (with $n = i_n$) that

(7.1.8) \[ \mathbb{E} \left[ f \circ X^{T_0}_t \right] - \mathbb{E} \left[ f \circ X^{T_0}_{t_0} \right] = \lim_{n \to \infty} \mathbb{E}^n \left[ f \circ X^{n}_{T_0} - f \circ X^{n}_{T_0+c} \right] = 0 \]

for all $f \in \mathcal{M}([\Pi^{T_0}(D)] \cup \{1\})$. Hence, $S \circ X^{-1}_{T_0} \equiv S \circ X^{-1}_{T_0+c}$ by their $m$-tightness and Lemma 6.2.17(b) (with $d = \mathbb{E}(T_0$)).
Theorem 7.1.1 can be used to identify a given $E$-valued limit process as the unique finite-dimensional limit point.

**Corollary 7.1.3.** Let $E$ be a topological space, $S \subset \mathbb{R}^+$ and $\{(\Omega^i, \mathcal{F}^i, \mathbb{P}^i; X^i)\}_{i \in I}$ and $(\Omega, \mathcal{F}, \mathbb{P}; X)$ be $E$-valued processes. Suppose that:

(i) $\mathcal{D} \subset C_b(\Omega; \mathbb{R})$ separates points on $\Omega$.
(ii) $\{X^i\}_{i \in I}$ satisfies $S$-PSMTC.
(iii) $X$ satisfies $S$-PMTCT.
(iv) $E[f \circ X_{T_0}]$ is the unique limit point of $\{E[f \circ X^i_{T_0}]\}_{i \in I}$ in $\mathbb{R}$ for all $f \in \mathcal{M}[\Pi^{T_0}(\mathcal{D})]$ and $T_0 \in \mathcal{P}_0(S)$.

Then:

(a) $X = \mathfrak{f}_{\mathbb{S}}(\{X^i\}_{i \in I})$ and $X = \mathfrak{f}_{\mathbb{S}}(\{X^{i_n}\}_{n \in \mathbb{N}})$ for any $\{i_n\}_{n \in \mathbb{N}} \subset I$.
(b) If $S = \mathbb{R}^+$ and $\{X^i\}_{i \in I}$ is $(\mathbb{R}^+, \mathcal{D})$-ASM, then $X = \mathfrak{f}_{\mathbb{S}}(\{X^i\}_{i \in I})$ is a stationary process and $X = \mathfrak{f}_{\mathbb{S}}(\{X^{i_n}\}_{n \in \mathbb{N}})$ for any $\{i_n\}_{n \in \mathbb{N}} \subset I$.

**Proof.**

(a) $\{X^i\}_{i \in I}$ is $(S, \mathcal{D})$-FDC by the condition (iv) above. By Theorem 7.1.4, there exists a $Z = \mathfrak{f}_{\mathbb{S}}(\{X^i\}_{i \in I})$ such that $Z$ satisfies $S$-PMTCT and $Z = \mathfrak{f}_{\mathbb{S}}(\{X^{i_n}\}_{n \in \mathbb{N}})$ for any $\{i_n\}_{n \in \mathbb{N}} \subset I$. We take $Z$ defined on $(\Omega, \mathcal{F}, \mathbb{P})$ for simplicity, fix $\{i_n\}_{n \in \mathbb{N}} \subset I$ and show $\mathbb{P} \circ X^{-1}_{T_0} = \mathbb{P} \circ Z^{-1}_{T_0}$ for all $T_0 \in \mathcal{P}_0(S)$. Since

$$X^{i_n} \overset{D(S)}{\to} Z \text{ as } n \to \infty,$$

we have by (iv) and Fact 6.2.6 (with $X = Z$) that

$$E[f \circ X_{T_0}] = E[f \circ Z_{T_0}]$$

for all $f \in \mathcal{M}[\Pi^{T_0}(\mathcal{D})] \cup \{1\}$. $X_{T_0}$ and $Z_{T_0}$ are $\mathfrak{m}$-tight by the condition (iii) above and Remark 7.1.2 (with $X = X$ or $Z$). Hence, $\mathbb{P} \circ X^{-1}_{T_0} = \mathbb{P} \circ Z^{-1}_{T_0}$ by Lemma 10.2.17 (b) (with $d = \mathbb{N}(T_0)$).

(b) follows by (a) (with $S = \mathbb{R}^+$) and Theorem 7.1.1 (c). $\square$

**Remark 7.1.4.** A variant of Theorem 7.1.1 will be given in Proposition 7.2.11 that relies heavily on our results herein.

### 7.2. Convergence of weakly càdlàg processes

This section deals with finite-dimensional convergence of $E$-valued processes satisfying $\mathcal{D}$-$\text{FMCC}$\footnote{X satisfying $S$-PMTCT means the singleton $\{X\}$ satisfies $S$-PMTCT.} Such processes are $(\mathbb{R}^+, \mathcal{D})$-càdlàg according to Note 6.4.6.

Given $\{X^n\}_{n \in \mathbb{N}}$ satisfying $\mathcal{D}$-$\text{DMCC}$, part (a) of the next theorem establishes an $(S, \mathcal{D})$-càdlàg $X \in \mathfrak{f}_{\mathbb{S}}(\{X^n\}_{n \in \mathbb{N}})$ and gives an alternative answer to $\mathcal{Q}_2$ in Introduction. Part (b) further imposes the standard Borel property and establishes a progressive member of $\mathfrak{f}_{\mathbb{S}}(\{X^n\}_{n \in \mathbb{N}})$. In lieu of a standard Borel assumption, part (c) assumes the $(T, \mathcal{D})$-AS of $\{X^n\}_{n \in \mathbb{N}}$ for a conull $T \supset S$ and establishes a stationary and progressive member of $\mathfrak{f}_{\mathbb{S}}(\{X^n\}_{n \in \mathbb{N}})$. These two parts provide answers to $\mathcal{Q}_3$ in Introduction.

**Theorem 7.2.1.** Let $E$ be a topological space, $\{(\Omega^n, \mathcal{F}^n, \mathbb{P}^n; X^n)\}_{n \in \mathbb{N}}$ be $E$-valued processes and $S \subset T \subset \mathbb{R}^+$ with $S$ being dense. Suppose that:

\footnote{\text{The notion of $D$-$\text{FMCC}$ was introduced in Definition 6.4.1.}}

\footnote{\text{The notion of $(S, F)$-càdlàg process was introduced in Definition 6.3.2.}}
(i) \( C_0(E; \mathbb{R}) \) separates points on \( E \).

(ii) \( \mathcal{D} \subset C_0(E; \mathbb{R}) \) is countable and \( E_0 \) is a \( \mathcal{D} \)-baseable subset of \( E \).

(iii) \( \{X^n\}_{n \in \mathbb{N}} \) satisfies \( \mathcal{S} \)-PSMTC in \( E_0 \) and \( \mathcal{D} \)-FMCC.

(iv) \( \{X^n\}_{n \in \mathbb{N}} \) satisfies \( \mathcal{S} \)-PSMTC in \( E_0 \) and \( \mathcal{D} \)-FMCC.

(v) \( \mathfrak{P}_\mathcal{S}(\{\varphi \circ X^n\}_{n \in \mathbb{N}}) \) has at least one càdlàg member with \( \varphi \triangleq \bigotimes \). Then, there exist a stochastic basis \( \mathcal{S}(\Omega, \mathcal{F}, \{\mathcal{G}_t\}_{t \geq 0}, \mathcal{P}) \), some \( \{n_k\}_{k \in \mathbb{N}} \subset \mathbb{N} \) and an \( X \in (E_0^{\mathbb{R}^+})^\Omega \) such that:

(a) \( X = \mathfrak{f}_\mathcal{S}(\{X^{n_k}\}_{k \in \mathbb{N}}) \) is an \( \mathcal{S}, \mathcal{D} \)-càdlàg process and satisfies \( \mathcal{S} \)-PMTC in \( E_0 \).

(b) If \( E_0 \in \mathcal{B}(E) \), then \( X = \mathfrak{f}_\mathcal{S}(\{X^{n_k}\}_{k \in \mathbb{N}}) \) can be chosen to be an \( \mathcal{E}, \mathcal{D} \)-càdlàg, \( \mathcal{G}_t \)-progressive\[10\] process that satisfies \( \mathcal{S} \)-PMTC in \( E_0 \) and admits an \( \mathcal{S}, \mathcal{D} \)-càdlàg progressive modification with paths in \( E_0^{\mathbb{R}^+} \).

(c) If \( \mathcal{T} \) is conull and \( \{X^{n_k}\}_{k \in \mathbb{N}} \in (\mathcal{T}, \mathcal{D})-\text{AS} \), then \( X = \mathfrak{f}_\mathcal{S}(\{X^{n_k}\}_{k \in \mathbb{N}}) \) can be chosen to be an \( \mathcal{E}, \mathcal{D} \)-càdlàg, \( \mathcal{G}_t \)-progressive\[10\] process that satisfies \( \mathcal{R}^+ \)-PMTC in \( E_0 \) and admits an \( \mathcal{R}^+, \mathcal{D} \)-càdlàg progressive modification with paths in \( E_0^{\mathbb{R}^+} \).

Remark 7.2.2. If \( E_0 \) is a \( \mathcal{D} \)-baseable subset with \( \mathcal{D} \subset C(E; \mathbb{R}) \), then \( E_0 \) is \( \mathcal{D} \)-baseable for some countable \( \mathcal{D}_0 \subset \mathcal{D} \) (see Fact \[3.3.2\] (c)) and \( \mathcal{D}_0 \)-FMCC is a weaker assumption than \( \mathcal{D} \)-FMCC. Hence, it is no less general to make \( \mathcal{D} \) a countable collection in the theorem above.

Remark 7.2.3. Any compact subset contained in a baseable set \( E_0 \) is metrizable by Corollary \[3.3.20\] (a). Thus, the m-tightness within \( \mathcal{S} \)-PSMTC in \( E_0 \) is reduced to ordinary tightness.

Remark 7.2.4. The proof of Theorem 7.2.1 relies on Theorem \[6.2.7\] in which the limit processes \( X \) was defined by a collection of \( E_0 \)-valued mappings \( \{X_t\}_{t \geq 0} \). This is equivalent to describing the limit process as an \( E_0^{\mathbb{R}^+} \)-valued mapping (like \( X \) and \( X' \) in Theorem 7.2.1). Moreover, both Theorem \[6.2.7\] and Theorem 7.2.1 consider the limit processes as \( \mathcal{E} \)-valued with paths in \( E_0^{\mathbb{R}^+} \) for finite-dimensional convergence.

We use the next lemma to establish progressiveness in Theorem 7.2.1 (b, c).

Lemma 7.2.5. Let \( E \) be a topological space, \( x_0 \in E_0 \subset E \), \( \mathcal{T} \subset \mathbb{R}^+ \) and \( (\Omega, \mathcal{F}, \mathcal{P}, Y) \) be an \( \mathcal{E} \)-valued process satisfying

\[
\inf_{t \in \mathcal{T}} \mathbb{P}(Y_t \in E_0) = 1.
\]

Then, the mapping

\[
X = \bigotimes_{t \in \mathbb{R}^+} \varphi(Y_t; \Omega, Y_t^{-1}(E_0), x_0) \in \left( E_0^{\mathbb{R}^+} \right)^\Omega
\]

satisfies the following statements:

\[9\] The notion of stochastic basis was reviewed in \[2.5\]

\[10\] The notion of \( \mathcal{G}_t \)-progressive processes was specified in \[2.5\]

\[11\] Given \( E_0 \subset E \), an \( \mathcal{E} \)-valued process with paths in \( E_0^{\mathbb{R}^+} \) is equivalent to an \( (E_0, \mathcal{G}_t(E_0)) \)-valued process.
(a) $X_t \doteq p_t \circ X \in M(\Omega, \mathcal{F}; E_0, \mathcal{B}_E(E_0))$ for all $t \in T$ and \[(6.2.25)\] holds.

(b) If $T = \mathbb{R}^+$, then $X$ is an $E$-valued process with paths in $E_0^{\mathbb{R}^+}$, satisfies \[(6.2.25)\] and is a modification of $Y$.

(c) If $E_0 \in \mathcal{B}(E)$, then $X$ is an $(E_0, \mathcal{O}_E(E_0))$-valued $\mathcal{F}_t^Y$-adapted process. If, in addition, $Y$ is a measurable or progressive process, then $X$ is measurable or $\mathcal{F}_t^Y$-progressive respectively.

(d) If $E_0 \in \mathcal{B}(E) \cap \mathcal{B}(E)$, and if $Y$ is a measurable process, then $X$ is an $(E_0, \mathcal{O}_E(E_0))$-valued, measurable, $\mathcal{F}_t^Y$-adapted process and admits an $\mathcal{F}_t^Y$-progressive modification.

**Proof.** (a) follows by Lemma \[10.1.28\] (b, c) (with $\mathcal{U} = \mathcal{B}(E)$, $S = S_0 = E_0$, $\mathcal{W}' = \mathcal{B}_E(E_0)$, $X = Y$, and $S = S_0 = E_0$).

(b) follows by (a) (with $T = \mathbb{R}^+$) and Fact 2.5.2 (b) (with $E = (E_0, \mathcal{O}_E(E_0))$).

(c) Let $\varphi$ denote the identity mapping on $E$. We find that

\[(7.2.3)\] $\varphi' \doteq \vartheta \circ \varphi; E_0, x_0) \in M(E; E_0, \mathcal{O}_E(E_0))$

by $E_0 \in \mathcal{B}(E)$ and Fact 10.1.29 (b) (with $(S, \mathcal{A}) = (E, \mathcal{B}(E))$, $(E, \mathcal{F}) = (E_0, \mathcal{B}_E(E_0))$, $A = E_0$, $F = \varphi'$ and $y_0 = x_0$). Then, (c) follows by \[(7.2.3)\] the fact $X = \vartheta(\varphi') \circ Y$ and Fact 10.1.29 (a) (with $S = (E_0, \mathcal{O}_E(E_0))$, $F = \varphi'$, $X = Y$ and $\mathcal{G} = \mathcal{F}_t^Y$).

(d) $\hat{X}$ is an $(E_0, \mathcal{O}_E(E_0))$-valued, measurable, $\mathcal{F}_t^Y$-adapted process by (c). Let $\varphi_0$ denote the identity mapping on $E_0$. By $E_0 \in \mathcal{B}(E)$ and Proposition 9.5.9 (a, d), there exists a topology $\mathcal{U}$ on $E_0$ such that $(E_0, \mathcal{U})$ is a Polish space and $\varphi_0 \in \text{biso}(E_0, E_0; E_0, \mathcal{U})$. Then, $(E_0, \mathcal{O}_E(E_0))$-valued measurable (resp. $\mathcal{F}_t^Y$-progressive) processes are equivalent to $(E_0, \mathcal{U})$-valued measurable (resp. $\mathcal{F}_t^Y$-progressive) processes by Fact 10.1.29 (a) (with $E$ (or $S$) being $(E_0, \mathcal{O}_E(E_0))$, $S$ (or $E$) being $(E_0, \mathcal{U})$, $F = \varphi_0$ and $\mathcal{G} = \mathcal{F}_t^Y$). \cite{OS13} Theorem 0.1] established that every Polish-space-valued, measurable, $\mathcal{F}_t^Y$-adapted process (like $X$) admits an $\mathcal{F}_t^Y$-progressive modification. Thus (d) follows immediately. \hfill $\Box$

**Corollary 7.2.6.** Let $E$ be a topological space, $E_0 \in \mathcal{B}(E) \cap \mathcal{B}(E)$, and $(\Omega, \mathcal{F}, \mathbb{P}; X)$ be an $E$-valued measurable process satisfying \[(6.3.17)\]. Then, $X$ has a progressive modification with paths in $E_0^{\mathbb{R}^+}$.

**Proof.** This corollary follows by Lemma 7.2.5 (b, d) (with $Y = X$ and $X = Y$) and Proposition 10.1.30 (e). \hfill $\Box$

**Corollary 7.2.7.** Let $E$ be a topological space, $\mathcal{D} \subset C_0(E; \mathbb{R})$, $E_0$ be a $\mathcal{D}$-baseable standard Borel subset of $E$ and $(\Omega, \mathcal{F}, \mathbb{P}; X)$ be an $E$-valued $(\mathbb{R}^+, \mathcal{D})$-càdlàg process satisfying \[(6.3.17)\]. Then, $X$ has a progressive modification with paths in $E_0^{\mathbb{R}^+}$.

**Proof.** There exists a base $(E_0, \mathcal{F}; \hat{E}, \hat{\mathcal{F}})$ over $E$ with $\mathcal{F} \subset (\mathcal{D} \cup \{1\})$ by Lemma 3.3.7 (c). $\hat{X} = \varphi_{\mathbb{R}^+}(X; E_0, \mathcal{F})$ exists by the fact $(\mathcal{F}; \{1\}) \subset \mathcal{D}$ and Proposition 6.3.5 (a). It follows by Fact 6.3.1 and Proposition 6.1.7 (a) (with $T = \mathbb{R}^+$) that $X$ and $\hat{X}$ satisfy \[(6.3.18)\] and $X$ is a progressive process. $\mathcal{F}_t^X = \mathcal{F}_{\hat{X}}^X$ by Lemma 10.3.5 (e) (with $A = E_0$ and $Y = \hat{X}$), so $\hat{X} \in \mathcal{F}_t^X$-progressive. Furthermore, we have

\[(7.2.4)\] $\mathcal{B}_E(E_0) = \mathcal{B}_{\hat{E}}(E_0) \subset \mathcal{B}(\hat{E})$

\[\text{The notation } \mathcal{F}_t^X \text{ as defined in 2.5 means the augmented natural filtration of } X.\]
by Lemma 3.1.14 (a) (with \(d = 1\) and \(A = E_0\)). \(\hat{X}\) has an \(\mathcal{F}^X\)-progressive modification \(Z\) with paths in \(E_0^R\) by (6.3.18), (7.2.4) and Lemma 7.2.5 (b, c) (with \(E = \hat{E}\) and \(Y = \hat{X}\)). \(Z\) is an \((E_0, \mathcal{F}(E_0))\)-valued process by (7.2.4). \(Z\) is a modification of \(X\) by (6.3.18). So, \(Z\) is progressive by Proposition 10.1.30 (e).

\[\square\]

**Remark 7.2.8.** A special case of Corollary 7.2.6 and Corollary 7.2.7 is when \(E = E_0\) is a \(\mathcal{D}\)-baseable standard Borel space and (6.3.17) becomes automatic.

The next result shows that the condition (v) of Theorem 7.2.1 is realizable.

**Proposition 7.2.9.** Let \(E\) be a topological space, \(\mathcal{D} \subset C_b(E; \mathbb{R})\) be countable, \(E_0\) be a \(\mathcal{D}\)-baseable subset of \(E\) and \(I\) be an infinite index set. If \(E\)-valued processes \(\{(\Omega^i, \mathcal{F}^i, \mathbb{P}^i, X^i)\}_{i \in I}\) satisfy (6.4.7) and \(\mathcal{D}\)-FMCC, then there exists a countable \(J \subset (0, \infty)\) such that \(\mathbf{fP}(\mathcal{R}^\infty \setminus J (\{\mathcal{C}\mathcal{D} \circ X^i\})_{i \in I})\) has at least one \(\mathcal{C}\mathcal{D}\) member.

**Proof.** There exists a base \((E_0, \mathcal{F}; \hat{E}, \hat{\mathcal{F}})\) over \(E\) with \(\mathcal{F} = \mathcal{D} \cup \{1\}\) by Lemma 3.3.7 (b), so \(\{X^n\}_{n \in \mathbb{N}}\) satisfies \(\mathcal{F}\)-FMCC. It follows by Proposition 6.4.10 (a) and Note 6.1.5 that \(\{\hat{X}^i = \mathbf{rD}(X^i; E_0, \mathcal{F})\}_{i \in I}\) is tight in the Polish space \(D(\mathbb{R}^+; \hat{E})\). \(\{\hat{X}^i\}_{i \in I}\) admits at least one weak limit point \(Y\) on \(D(\mathbb{R}^+; \hat{E})\) by the Prokhorov’s Theorem (Theorem 2.3.12 (b)). \(J = J(Y) \subset (0, \infty)\) is countable by Note 6.4.15. Now, the result follows by Proposition 6.4.14 (a) and Lemma 10.3.8.

We now prove the main theorem of this section.

**Proof of Theorem 7.2.1.** The proof is divided into six steps.

**Step 1:** Establish a base \((E_0, \mathcal{F}; \hat{E}, \hat{\mathcal{F}})\) and \(\mathcal{C}\mathcal{D}\) replicas \(\{\hat{X}^n\}_{n \in \mathbb{N}}\). There exists a base \((E_0, \mathcal{F}; \hat{E}, \hat{\mathcal{F}})\) over \(E\) with \(\mathcal{F} = \mathcal{D} \cup \{1\}\) by the condition (ii) above and Lemma 3.3.7 (b). (6.4.34) holds by the condition (iii) above and Fact 6.5.1 (with \(I = \mathbb{N}\)). \(\{X^n\}_{n \in \mathbb{N}}\) satisfies \(\mathcal{F}\)-FMCC by the condition (iv) above and the fact \(\mathcal{F}\setminus\{1\} = \mathcal{D}\), so they are \((\mathbb{R}^+, \mathcal{F})\)-cadlag. It then follows by Proposition 6.4.10 (a) (with \(I = \mathbb{N}\)), Proposition 6.3.5 (a) (with \(X = X^n\)) and Note 6.1.5 that \(\{X^n = \mathbf{rD}(X^n; E_0, \mathcal{F})\}_{n \in \mathbb{N}}\) is tight in the Polish space \(D(\mathbb{R}^+; \hat{E})\) and satisfies

\[(7.2.5) \quad \inf_{n \in \mathbb{N}} \mathbb{P}^n (\varphi \circ X^n_t = \hat{\varphi} \circ \hat{X}^n_t) = 1\]

with \(\hat{\varphi} \triangleq \mathbf{D}(\hat{\mathcal{F}}\setminus\{1\})\).

**Step 2:** Establish \(\{n_k\}_{k \in \mathbb{N}}\) and a \(D(\mathbb{R}^+; \hat{E})\)-valued random variable \(Y\) such that

\[(7.2.6) \quad \hat{X}^{n_k} \xrightarrow{\text{D}(\mathbb{S})} Y \quad \text{as} \quad k \uparrow \infty.\]

By the condition (v) above, the tightness of \(\{\hat{X}^n\}_{n \in \mathbb{N}}\) in \(D(\mathbb{R}^+; \hat{E})\) and Prokhorov’s Theorem (Theorem 2.3.12 (b)), there exist \(\{n_k\}_{k \in \mathbb{N}} \subset \mathbb{N}\), a \(D(\mathbb{R}^+; \hat{E})\)-valued random variable \(Y\) (see Remark 7.2.10 below for an explicit construction) and an \(\mathbb{R}^2\)-valued cadlag process \(Z\) such that

\[(7.2.7) \quad \hat{X}^{n_k} \implies Y \quad \text{as} \quad k \uparrow \infty \quad \text{on} \quad D(\mathbb{R}^+; \hat{E})\]

and

\[(7.2.8) \quad \mathcal{C}D(\varphi \circ X^{n_k}) \xrightarrow{\text{D}(\mathbb{S})} Z \quad \text{as} \quad k \uparrow \infty.\]

\[\text{Please be noted that members of } \mathbf{fP}(\mathcal{R}^\infty \setminus J (\{\mathcal{C}\mathcal{D} \circ X^i\})_{i \in I}) \text{ are processes with time horizon } \mathbb{R}^+ \text{ no matter } J = \emptyset \text{ or not.}\]
Without loss of generality, we suppose $Y$ and $Z$ are both defined on $(\Omega, \mathcal{F}, \mathbb{F})$.

Since $\mathcal{D}$ is countable, $\mathbb{R}^{\mathcal{D}}$ and $D(\mathbb{R}^+; \mathbb{R}^{\mathcal{D}})$ are Polish spaces as mentioned in Note 6.1.5. So, $Z$ can be considered as a $D(\mathbb{R}^+; \mathbb{R}^{\mathcal{D}})$-valued random variable by Fact 9.7.2 (b) (with $E = \mathbb{R}^{\mathcal{D}}$). $\hat{\mathcal{F}} \setminus \{1\}$ separates points on $\hat{E}$ by Definition 3.1.1. $\hat{\mathcal{F}} \setminus \{1\}$ strongly separates points on $\hat{E}$ and

$$\tag{7.2.9} \hat{\varphi} \in \text{imb}(\hat{E}; \mathbb{R}^{\mathcal{D}})$$

by Lemma 3.1.3 (a).

$$\tag{7.2.10} \varpi(\hat{\varphi}) \in C \left( D(\mathbb{R}^+; \hat{E}); D(\mathbb{R}^+; \mathbb{R}) \right)$$

by Proposition 9.6.1 (d) (with $S = \hat{E}$, $E = \mathbb{R}^{\mathcal{D}}$ and $f = \hat{\varphi}$).

It follows by (7.2.7), (7.2.10) and Continuous Mapping Theorem (Theorem 10.1.23 (a)) that

$$\tag{7.2.11} \varpi(\hat{\varphi}) \circ \hat{X}^{n_k} \Rightarrow \varpi(\hat{\varphi}) \circ Y \text{ as } k \uparrow \infty \text{ on } D(\mathbb{R}^+; \mathbb{R}^{\mathcal{D}}).$$

by the Prokhorov’s Theorem (Theorem 2.3.12 (b)), the denseness of $\mathcal{S}$ and Theorem 9.7.11 (b) (with $E = \mathbb{R}^{\mathcal{D}}$, $X^n = \varpi(\hat{\varphi}) \circ X^{n_k}$, $X = Z$ and $T = \mathcal{S}$). $\mathcal{P}(D(\mathbb{R}^+; \mathbb{R}^{\mathcal{D}}))$ is a Polish space by Theorem 9.4.10 (b) (with $E = D(\mathbb{R}^+; \mathbb{R}^{\mathcal{D}})$) in which a sequence converges weakly to at most one point by [Mum00 Theorem 17.10], Proposition 9.1.11 (c) and Proposition 9.1.4 (a). So,

$$\tag{7.2.12} \varpi(\hat{\varphi}) \circ \hat{X}^{n_k} \overset{D(\mathcal{S})}{\longrightarrow} Z \text{ as } k \uparrow \infty.$$

We fix $T_0 \in \mathcal{P}_0(\mathcal{S})$ and put $d = \mathbb{N}(T_0)$. $\{\hat{X}^{n_k}\}_{k \in \mathbb{N}}$, $\{\varpi(\hat{\varphi}) \circ \hat{X}^{n_k}\}_{k \in \mathbb{N}}$, $Y$ and $\varpi(\hat{\varphi}) \circ Y$ all have Borel finite-dimensional distributions as mentioned in Note 6.1.5 so

$$\tag{7.2.15} \varpi(\hat{\varphi}) \circ \hat{X}^{n_k} \overset{D(\mathcal{S})}{\longrightarrow} \varpi(\hat{\varphi}) \circ Y \text{ as } k \uparrow \infty.$$

One finds that

$$\tag{7.2.16} \left( \bigotimes_{t \in T_0} \hat{\varphi} \circ p_t \right) \circ \hat{X}^{n_k} \Rightarrow \left( \bigotimes_{t \in T_0} \hat{\varphi} \circ p_t \right) \circ Y \text{ as } k \uparrow \infty \text{ on } \mathbb{R}^d.$$
by (7.2.9) and Fact 2.1.4 (a, b). Hence, it follows by (7.2.17), (7.2.16) and Continuous Mapping Theorem (Theorem 10.1.23 (a)) that

\[ \hat{X}^{n_k}_{T_0} = \Psi \circ \left( \bigotimes_{t \in T_0} \varphi \circ \rho_t \right) \circ \hat{X}^{n_k} \]

(7.2.18)

\[ \implies \Psi \circ \left( \bigotimes_{t \in T_0} \varphi \circ \rho_t \right) \circ Y = Y_{T_0} \text{ as } k \to \infty \text{ on } \hat{E}^{T_0}. \]

**Step 3: Construct** \((\Omega, \mathcal{F}, \{\mathcal{G}_t\}_{t \geq 0}, \mathbb{P})\) and \(X\). We fix an arbitrary \(x_0 \in E_0\) and set \(\mathcal{G}_t = \mathcal{F}_t^Y\). For (a), we define \(X\) by (6.2.4) with \(T = S\). For (b, c), we shall need an \(X\) with different paths to meet the measurability requirements. For this purpose, we define the process \(X\)' by (7.2.2) with \(X\) replaced by \(X'\).

**Step 4: Verify (a).** It follows by the conditions (i, iv) above, (7.2.6), Lemma 10.3.6 (c, e) (with \(n = n_k\)) and Theorem 6.2.7 (a, c) (with \(n = n_k\) and \(T = S\)) that the \(X\) defined in Step 3 is an \(E\)-valued process with paths in \(E_0^{R^+}\) that satisfies:

1. \(S\)-PMTC in \(E_0\), (2)

\[ \inf_{t \in S} \mathbb{P}(X_t = Y_t \in E_0) = 1, \]

and (3) \(X = \hat{f}_S(\{X^{n_k}\}_{k \in \mathbb{N}})\). Hence, (a) follows by the fact \(D \subset \mathcal{F}, (7.2.19)\) and Lemma 10.3.5 (b) (with \(T = S\)).

**Step 5: Verify (b).** \(Y\) is càdlàg hence \(\mathcal{G}_t\)-progressive by Proposition 10.1.30 (a). Given \(E_0 \in \mathcal{B}(E), (7.2.4)\) holds by Lemma 3.1.14 (a) (with \(d = 1\) and \(A = E_0\)). Hence, the \(X\)' defined in Step 3 is an \((E_0, \mathcal{O}_E(E_0))\)-valued \(\mathcal{G}_t\)-progressive process satisfying

\[ \inf_{t \in S} \mathbb{P}(X_t = Y_t = X'_t \in E_0) = 1 \]

by (7.2.19), (7.2.4) and Lemma 7.2.5 (a, c) (with \(E = \hat{E}\) and \(T = S\) and \(X = X'\)). Then, \(X' = \hat{f}_S(\{X^{n_k}\}_{K \in \mathbb{N}})\) by (a) and (7.2.20). \(X'\) is \((S, D)\)-càdlàg by the fact \(D \subset \mathcal{F}, (7.2.20)\) and Lemma 10.3.5 (b) (with \(T = S\) and \(X = X'\)). \(X'\) is a measurable process by Proposition 10.1.30 (c). Now, (b) with \(X = X'\) follows by the fact \(E_0 \in \mathcal{B}(E) \cap \mathcal{B}(E), (7.2.6)\) (with \(X = X'\)) and Note 6.3.3.

**Step 6: Verify (c).** \(\{X^n\}_{n \in \mathbb{N}}\) is \((T, \mathcal{F}_1)\)-AS since \(D = \mathcal{F}_1\). Then, \(Y\) is a stationary process by (7.2.7) and Proposition 6.4.14 (c) (with \(n = n_k\)). We know from (a) that \(X\) satisfies \(S\)-PMTC in \(E_0\) and (7.2.19). By \(S\)-PMTC in \(E_0\) there exists an \(A \in \mathcal{X}_0^{\mathcal{M}}(E_0, \mathcal{O}_E(E_0))\) and some \(t_0 \in S\) such that \(\mathbb{P}(X_{t_0} \in A) = 1\).

(7.2.19) implies \(\mathbb{P}(Y_{t_0} \in A) = 1\). The stationarity of \(Y\) implies

\[ \inf_{t \in \mathbb{R}^+} \mathbb{P}(Y_t \in A \cap E_0) = 1. \]

A is a \(D\)-baseable standard Borel subset of \(E\) and satisfies

(7.2.22)

\[ \mathcal{B}_E(A) = \mathcal{B}_E(A) \subset \mathcal{B}(\hat{E}) \]

by Corollary 3.1.15 (b) (with \(d = 1\)), Lemma 3.1.14 (a) (with \(d = 1\)), the fact \(\mathcal{F}_1 = \mathcal{D}\) and Fact 3.3.2 (a, b). Hence, the \(X'\) defined in Step 3 is an \((E_0, \mathcal{O}_E(E_0))\)-valued process satisfying both (7.2.20) and

(7.2.23)

\[ \inf_{t \in \mathbb{R}^+} \mathbb{P}(X'_t = Y_t \in A \subset E_0) = 1 \]
by (7.2.21), (7.2.22), Lemma 7.2.5 (b) (with $E = \hat{E}$, $E_0 = A$ and $X = X'$) and (7.2.19). $X'$ is an $(E_0, \mathcal{E}_E(E_0))$-valued process and satisfies $\mathbb{R}^+$-PMTC in $E_0$ by (7.2.22), Lemma 10.1.28 (b, c) (with $E = \hat{E}$, $S_0 = A$, $S = E_0$, $\mathcal{W} = \mathcal{B}_E(E_0)$, $X = Y_t$ and $Y = X'_t$) and Fact 2.5.2 (b). Thus, $X'_t = \mathcal{f}_S(\{X^{n_k}_t\}_{k\in\mathbb{N}})$ by (a) and (7.2.20). $X'$ is stationary by (7.2.23) and Lemma 10.3.5 (e) (with $X = X'$). $X'$ is $(\mathbb{R}^+, D)$-càdlàg by the fact $D \subset \mathcal{F}$, (7.2.23) and Lemma 10.3.5 (b) (with $T = \mathbb{R}^+$ and $X = X'$). Finally, (c) (with $X = X'$) follows by Corollary 7.2.7 (with $E_0 = A$ and $X = X'$) and Note 6.3.3.

Remark 7.2.10. Let $\{\hat{X}^{n_k}\}_{k\in\mathbb{N}}$ be as in the proof of Theorem 7.2.1. One can realize (7.2.7) by letting $\Omega = D(\mathbb{R}^+; \hat{E})$, $\mathbb{P}$ be the weak limit of the distributions of $\{\hat{X}^{n_k}\}_{k\in\mathbb{N}}$ in $\mathcal{P}(D(\mathbb{R}^+; \hat{E}))$, $\mathcal{F}$ be the completion$^{14}$ of $\mathcal{B}(\Omega)$ with respect to $\mathbb{P}$ and $Y$ be the identity mapping on $D(\mathbb{R}^+; \hat{E})$. This process $(\Omega, \mathcal{F}, \mathbb{P}; Y)$ is often called the coordinate process or canonical process on $D(\mathbb{R}^+; \hat{E})$.

With the help of Lemma 7.2.5 we give a variant of Theorem 7.1.1 (b) that can be used to show uniqueness in the settings of Theorem 7.2.1.

Proposition 7.2.11. Let $E$ be a topological space, $\{(\Omega^i, \mathcal{F}^i, \mathbb{P}^i, X^i)\}_{i\in I}$ be $E$-valued processes and $S \subset T \subset \mathbb{R}^+$. Suppose that:

(i) $C_b(E; \mathbb{R})$ separates points on $E$.

(ii) $D \subset C_b(E; \mathbb{R})$ separates points on $E_0 \in \mathcal{B}(E)$.

(iii) (6.5.1) holds.

(iv) $\{X^i\}_{i\in I}$ is $(S, D)$-FDC and satisfies $S$-PSMTC in $E_0$.

Then, there exists an $X = \mathcal{f}_S(\{X^i\}_{i\in I})$ with paths in $E^0_{\mathbb{R}^+}$ and satisfying $S$-PMTC in $E_0$. Moreover, $X = \mathcal{f}_S(\{X^{n_k}\}_{n\in\mathbb{N}})$ for any $\{n_k\}_{k\in\mathbb{N}} \subset I$.

Proof. We let $\hat{f} \doteq f|_{E_0}$

for any $f \in C_b(E^d; \mathbb{R})$ and any $d \in \mathbb{N}$, put $\hat{D} \doteq D|_{E_0} = \{\hat{f} : f \in D\}$, fix $x_0 \in E_0$ and define $\{Z^i_t\}_{t \geq 0} \subset E^0_t$ for each $i \in I$ by (6.2.4) with $X_t$, $Y_t$, $\Omega$ replaced by $Z^i_t$, $X^i_t$, $\mathbb{P}^i$ respectively.

It follows by (6.5.1) and Lemma 7.2.5 (a) (with $(\Omega, \mathcal{F}, \mathbb{P}; Y) = (\Omega^i, \mathcal{F}^i, \mathbb{P}^i, X^i)$) and $X_t = Z^i_t$ that

$Z^i_t \in M \left(\Omega^i, \mathcal{F}^i; E_0, \mathcal{B}_E(E_0)\right), \forall t \in T, i \in I$

and

$\inf_{t \in T, i \in I} \mathbb{P}^i(X^i_t = Z^i_t \in E_0) = 1$.

$E$ is a Hausdorff space by Proposition 9.2.1 (c) (with $A = E$ and $D = C_b(E; \mathbb{R})$). So, $(E_0, \mathcal{G}_E(E_0))$ is a Hausdorff subspace and $x_0 \in \mathcal{B}(E)$ by Proposition 9.1.2 (a, c) and the fact $E_0 \in \mathcal{B}(E)$. This immediately implies

$Z^i_t \in M \left(\Omega^i, \mathcal{F}^i; E_0, \mathcal{G}_E(E_0)\right), \forall t \in \mathbb{R}^+, i \in I$.

Hence, $Z^i_t \doteq \{Z^i_t\}_{t \geq 0}$ is an $(E_0, \mathcal{G}_E(E_0))$-valued process for all $i \in I$ by Fact 2.5.2 (b) (with $E = (E_0, \mathcal{G}_E(E_0))$).

\textsuperscript{14}Completion of measure space was specified in 2.1.2

\textsuperscript{15}Similar notations were used in Notation 1.2.3
\(\{ Z^i \}_{i \in I} \) satisfies S-PSMTC by (7.2.25) and the condition (iv) above. At the same time, we observe by (7.2.25) that

\[
\mathbb{E}^i \left[ f \circ Z^i_{T_0} \right] = \mathbb{E}^i \left[ f \circ X^i_{T_0} \right]
\]

for all \( f \) in \( \text{mc}^{-1}[\Pi^0(C_b(E; \mathbb{R}))] \), \( T_0 \in \mathcal{P}_0(T) \) and \( i \in I \), so \( \{ Z^i \}_{i \in I} \) is (S, \( \tilde{D} \))-FDC.

Now, we apply Theorem 7.1.1 (b) (with \( E = (E, \mathcal{E}_E(E_0)) \), \( D = \tilde{D} \), \( X^i = Z^i \) and \( X = Z \)) and obtain an \( (E_0, \mathcal{E}_E(E_0)) \)-valued process \( (\Omega, \mathcal{F}, \mathbb{P}; Z) \) satisfying: (1) S-PMTG (in \( E_0 \)), and (2) \( Z = \mathcal{f}_S(\{Z^i_n\}_{n \in \mathbb{N}}) \) for any \( \{ i_n \}_{n \in \mathbb{N}} \subset I \).

Considering \( Z \) as an \( E \)-valued process with paths in \( E_0^{\mathbb{R}^+} \), it follows by \( Z \)'s property (2) above, Fact 6.2.6 (with \( X^n = Z^{i_n} \) and \( X = Z \)) and (7.2.27) that \( \mathbb{E}[f \circ Z_{T_0}] \) is the unique limit point of \( \{ \mathbb{E}^i[f \circ X^i_{T_0}] \}_{i \in I} \) for all \( f \) in \( \text{mc}^{-1}[\Pi^0(C_b(E; \mathbb{R}))] \) and \( T_0 \in \mathcal{P}_0(S) \). Now, the result follows with \( X = Z \) by \( Z \)'s property (1) and Corollary 7.1.3 (a) (with \( X = Z \) and \( D = C_b(E; \mathbb{R}) \)). \( \square \)

**Remark 7.2.12.** Proposition 7.2.11 does not require \( D \) to separate points on the entire space \( E \) as in Theorem 7.1.1 (b).

**Corollary 7.2.13.** Let \( E \) be a topological space, \( S \subset T \subset \mathbb{R}^+ \) and \( (\Omega, \mathcal{F}, \mathbb{P}; X) \) and \( \{ (\Omega', \mathcal{F}', \mathbb{P}', X^i) \}_{i \in I} \) be \( E \)-valued processes. Suppose that:

(i) \( C_b(E; \mathbb{R}) \) separates points on \( E \).

(ii) \( D \subset C_b(E; \mathbb{R}) \) separates points on \( E_0 \in \mathcal{B}(E) \).

(iii) (6.5.1) holds.

(iv) \( \{ X^i \}_{i \in I} \) satisfies S-PSMTC in \( E_0 \).

(v) \( X \) satisfies S-PMTC.

(vi) \( \mathbb{E}[f \circ X_{T_0}] \) is the unique limit point of \( \{ \mathbb{E}^i[f \circ X^i_{T_0}] \}_{i \in I} \) in \( \mathbb{R} \) for all \( f \) in \( \text{mc}^{-1}[\Pi^0(D)] \) and \( T_0 \in \mathcal{P}_0(S) \).

Then, \( X = \mathcal{f}_S(\{X^i\}_{i \in I}) \) and \( X = \mathcal{f}_S(\{X^i_n\}_{n \in \mathbb{N}}) \) for any \( \{ i_n \}_{n \in \mathbb{N}} \subset I \).

**Proof.** This corollary follows immediately by Proposition 7.2.11 and a similar argument to the proof of Corollary 7.1.3 (a). \( \square \)

### 7.3. Stationary long-time typical behavior

**Q1** in Introduction motivates our interest in finite-dimensional convergence of stochastic processes. In order to utilize our results in §7.1 and §7.2 we introduce the randomly advanced processes of a given measurable process \( X \) whose finite-dimensional distributions are the long-time-averaged distributions in (1.0.5).

**Definition 7.3.1.** Let \( E \) be a topological space and \((\Omega, \mathcal{F}, \mathbb{P}; X)\) be an \( E \)-valued measurable process.

1. For each \( T \in (0, \infty) \), by \((\tilde{\Omega}, \mathcal{F}, \mathbb{P}; X_T) = \text{rap}_T(X)^{16} \) (\( X_T = \text{rap}_T(X) \) for short) we denote that \( \tilde{\Omega} \cong \mathbb{R}^+ \times \Omega, \mathcal{F} \cong \mathcal{B}(\mathbb{R}^+) \otimes \mathcal{F}, \)

\[
\mathbb{P}^T(A) = \frac{1}{T} \int_0^T \int_\Omega 1_A(\tau, \omega) \mathbb{P}(d\omega) d\tau, \forall A \in \mathcal{F},
\]

\(^{16} \)“rap” is “rap” in fraktur font which stands for randomly advanced process.
and
\[(7.3.2)\quad X^T(\tau, \omega)(t) = X_{\tau+t}(\omega), \forall t \in \mathbb{R}^+, (\tau, \omega) \in \tilde{\Omega}.\]

\(X^T \in (E^{\mathbb{R}^+})^{\tilde{\Omega}}\) defined by (7.3.2) is called the \(T\)-randomly advanced process of \(X\).

- A long-time typical behavior of \(X\) along \(T\) refers to a member of \(\mathfrak{F}_T(\{X^{T_k}\}_{k \in \mathbb{N}})\) with \(T_k \uparrow \infty, X^{T_k} = \text{rap}_{T_k}(X)\) for each \(k \in \mathbb{N}\) and \(\mathbb{R}^+\backslash T\) being a countable subset of \((0, \infty)\)\(^\text{17}\).

**Remark 7.3.2.** As its name implies, the \(T\)-randomly advanced process of \(X\) is defined by advancing \(X\) to start at a random time \((\tau, \omega) \mapsto \tau\) defined on \((\tilde{\Omega}, \tilde{\mathcal{F}})\).

Below is a justification of our definition of randomly advanced process.

**Proposition 7.3.3.** Let \(E\) be a topological space, \((\Omega, \mathcal{F}, P; X)\) be an \(E\)-valued measurable process and \(T \in (0, \infty)\). Then, \((\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P}; X^T) = \text{rap}_T(X)\) is an \(E\)-valued measurable process.

**Proof.** (7.3.1) well defines \(P^T \in \mathfrak{F}(\tilde{\Omega}, \tilde{\mathcal{F}})\) by Fubini’s Theorem. Let \(\xi(t, \omega) = X_t(\omega), \xi^T(t, (\tau, \omega)) = X_{\tau+t}(\omega)\) and \(\varphi(t, (\tau, \omega)) = (\tau + t, \omega)\) for each \(t \in \mathbb{R}^+\) and \((\tau, \omega) \in \tilde{\Omega}\). It is well-known that
\[(7.3.3)\quad \varphi \in M \left(\mathbb{R}^+ \times \tilde{\Omega}, \mathbb{B}(\mathbb{R}^+) \otimes \tilde{\mathcal{F}}; \tilde{\Omega}, \tilde{\mathcal{F}} \right).\]

\(X\) being a measurable process implies \(\xi \in M(\tilde{\Omega}, \tilde{\mathcal{F}}; E)\) and
\[(7.3.4)\quad \xi^T = \xi \circ \varphi \in M \left(\mathbb{R}^+ \times \tilde{\Omega}, \mathbb{B}(\mathbb{R}^+) \otimes \tilde{\mathcal{F}}; E \right),\]
thus proving \(X^T\) is a measurable process. \(\square\)

We present several further properties of randomly advanced process in §10.1 of Appendix 10. Now, we give our answer to Q1.

**Theorem 7.3.4.** Let \(E\) be a topological space, \((\Omega, \mathcal{F}, P; X)\) be an \(E\)-valued measurable process satisfying \(T_k\)-LMTC in \(A \in E^{\text{18}}\) and \(\mathcal{D} \subset C_b(E; \mathbb{R})\) separate points on \(E\). Then:

(a) If \(\{\frac{1}{T_k} \int_0^{T_k} f \circ X_{T_k} \, d\tau\}_{k \in \mathbb{N}}\) is convergent in \(\mathbb{R}\) for all \(f \in \mathfrak{m}[\mathbb{P}^{T_k}(\mathcal{D})]\) and \(T_0 \in \mathfrak{F}_0(\mathbb{R}^+)\) with \(0 \in T_0\), then \(X\) has a stationary long-time typical behavior along \(\mathbb{R}^+\).

(b) If \(\mathcal{D}\) is countable and \(\{X^{T_k}\}_{k \in \mathbb{N}}\) satisfies \(\mathcal{D}\)-FMCC, then there exist a countable \(S \subset \mathbb{R}^+\), an \(E_0 \in \mathfrak{K}_0^m(E)\) such that \(\{\frac{1}{T_k} \int_0^{T_k} P \circ X_{-1}^{-1} \, d\tau\}_{k \in \mathbb{N}}\) is \(\mathfrak{m}\)-tight in \(E_0 \subset A\), and a stationary long-time typical behavior of \(X\) along \(S\) which is an \(E\)-valued, \((\mathbb{R}^+, \mathcal{D})\)-càdlàg, progressive process with paths in \(E_0^{\mathbb{R}^+}\).

**Proof.** (a) Let \(X^{T_k} = \text{rap}_{T_k}(X)\) for each \(k \in \mathbb{N}\). By Proposition 6.5.6 (with \(\{X^n\}_{n \in \mathbb{N}} \equiv \{X\}\)), there exists a \(\mathcal{D}\)-baseable subset \(E_0 \subset \mathfrak{K}_0^m(E)\) such that \(E_0 \subset A\) and 6.1.9 holds for some conull \(T \subset \mathbb{R}^+\) and \(\{X^{T_k}_0\}_{k \in \mathbb{N}}\) is \(\mathfrak{m}\)-tight in \(E_0\). By

\(^{17}\text{This means } T \text{ is cocountable.}\)

\(^{18}\text{The terminology “X satisfying } T_k\text{-LMTC in } A\text{” was defined in Definition 6.4.1 and Note 6.4.2.}\)
Lemma 10.1.37 (b, c, d) (with $A = E_0$), \{$X^{T_k}\}_{k \in \mathbb{N}}$ satisfies $R^+$-PSMTC in $E_0$, is $(R^+, D)$-AS and is $(R^+, D)$-FDC. Now, (a) follows by Theorem 7.1.1 (c) (with $X_i = X^{T_i}$).

(b) Let $E_0$ be as above. As (6.1.9) holds for the conull set $T$, we have that
\[
\inf_{t \in R^+, k \in \mathbb{N}} P^{T_k} (X_i^{T_k} \in E_0) = 1.
\]
by Lemma 10.3.7 (a) (with $T = T_k$). Now, (b) follows by (7.3.5), Proposition 7.2.9 (with $i = T_k$) and Theorem 7.2.1 (c) (with $n = T_k$, $T = R^+$ and $S = R^+ \setminus J$). □
CHAPTER 8

Application to Weak Convergence on Path Space

The current chapter addresses the target problems $Q_4$ and $Q_5$ of Theme 3 using the replication tools developed in §6.3. Throughout this chapter, we consider càdlàg processes taking values in a (at least) Tychonoff space $E$, whose common path space is the Skorokhod $J_1$-space $D(\mathbb{R}^+; E)$. If necessary, the readers can look back at §2.2.2, §2.4 and §2.5 for our terminologies and notations about the Skorokhod $J_1$-space and càdlàg process. Also, §9.6 of Appendix 9 together with §10.2 of Appendix 10 provide a short, almost self-contained review of Skorokhod $J_1$-space.

The results of this chapter are sketched in Figure 1 below.
Given MCCC, §8.1 establishes the equivalence among tightness in $D(\mathbb{R}^+; E)$ and the MCC-type conditions introduced in §6.4.2, which answers Q4. §8.2 looks into the relationship between weak convergence on $D(\mathbb{R}^+; E)$ and finite-dimensional convergence. §8.3 establishes several results connecting finite-dimensional convergence and relative compactness in $D(\mathbb{R}^+; E)$, which answers Q5.

Prior to the formal discussion, we recall several basic but essential facts for this chapter. Let $E$ be a Tychonoff space, $\mu \in \mathcal{M}^+(D(\mathbb{R}^+; E))$, $\mathcal{T}_0 \in \mathcal{P}_0(\mathbb{R}^+)$ and $(\Omega, \mathcal{F}, \mathbb{P}; X)$ be an $E$-valued càdlàg process. Then:

- $D(\mathbb{R}^+; E)$ is a Tychonoff space as mentioned in §3.3.6.
- $\mu \circ p_T^{-1}$ lies in $\mathcal{M}^+(E^{\mathcal{T}_0}, \mathcal{B}(E)^{\mathcal{T}_0})$ (see Corollary 9.6.7).

---

\textsuperscript{1}Herein, $p_{T_0}$ denotes the projection on $E^{\mathbb{R}^+}$ for $T_0 \subset \mathbb{R}^+$ restricted to $D(\mathbb{R}^+; E)$. 

---

*Figure 1. Tightness and relative compactness in $D(\mathbb{R}^+; E)$*
8.1. Tightness

Our treatment of tightness in Skorokhod $\mathcal{J}_1$-space continues [Kou16] in the infinite time horizon setting. Tightness in $D(R^+; E)$ is stronger than the Compact Containment Condition in [Jak86] and [EK86] (or MCCC if $E$ has metrizable compact subsets).

**Fact 8.1.1.** Let $E$ be a Tychonoff space and $\{(\Omega^i, \mathcal{F}^i, \mathbb{P}^i, X^i)\}_{i \in I}$ be a tight family of $D(R^+; E)$-valued random variables. Then, $\{X^i\}_{i \in I}$ satisfies the Compact Containment Condition in [Jak86] §4, (4.8)]. In particular, $\{X^i\}_{i \in I}$ satisfies MCCC when $\mathcal{K}(E) = \mathcal{K}^m(E)$.

**Proof.** This fact follows by Proposition 9.6.5 and Proposition 3.3.30.

The following theorem is a version of [Kou16] Theorem 20 on infinite time horizon. This result plus Fact 8.1.1 answer [Q4] in Introduction.

**Theorem 8.1.2.** Let $E$ be a Tychonoff space and $\{(\Omega^i, \mathcal{F}^i, \mathbb{P}^i, X^i)\}_{i \in I}$ be $E$-valued càdlàg processes. Consider the following statements:

(a) $X^i$ is indistinguishable from some $\tilde{X}^i \in M(\Omega^i, \mathcal{F}^i; D(R^+; E))$ for all $i \in I$, and $\{\tilde{X}^i\}_{i \in I}$ is m-tight in $D(R^+; E)$.

(b) $\{X^i\}_{i \in I}$ satisfies $D$-FMCC for some $D \subset C(E; R)$ and the closure of $D$ under the topology of compact convergence (see [Mun00] §46, Definition, p.283]) contains $C_b(E; R)$.

(c) $\{X^i\}_{i \in I}$ satisfies MCC.

(d) $\{X^i\}_{i \in I}$ satisfies WMCC.

Then, (a) implies (b), (c) implies (d), and (a) - (d) are all equivalent when $\{X^i\}_{i \in I}$ satisfies MCCC.

**Note 8.1.3.** Part (a) of the theorem above addresses stronger m-tightness than tightness of the $D(R^+; E)$-valued random variables $\{\tilde{X}^i\}_{i \in I}$ above in the usual sense.

\[\text{Footnotes:}\]

2 $J(\mu)$ was defined in (2.3.12).
3 $\mathcal{F}(X)$ was defined in (2.5.3).
4 $\mathcal{F}(E)$ denotes the Skorokhod $\mathcal{J}_1$-topology of $D(R^+; E)$. Restriction of measure to sub-$\sigma$-algebra and $X$’s process distribution $\text{pd}(X)$ were specified in (2.1.2) and (2.3.9) respectively.
5 The $E$-valued processes $\{X^i\}_{i \in I}$ in Theorem 8.1.2 are $(E^+$, $\mathcal{B}(E)^{\mathbb{R}^+})$-valued but not necessarily $D(R^+; E)$-valued random variables. In general, $D(R^+; E)$ as a subset may not inherit...
Remark 8.1.4. The condition in Theorem 8.1.2 (b) was used in Kur75 p.628-629 to show tightness in $D(R^+; E)$ with $E$ being a locally compact Polish space. For general Polish spaces, it appeared in EK86 §3.7.

Remark 8.1.5. When $(E, r)$ is a metric space, the standard combination of $r$-MCC plus MCC was used as a sufficient condition for relative compactness in $D(R^+; E)$ by EK86 §3.7, Theorem 7.6. Its necessity was treated in EK86 §3.7, Theorem 7.2, Remark 7.3 with $E$ being a Polish space. Theorem 8.1.2 refines these two results as well as EK86 §3.9, Theorem 9.1 and a few other analogues in the following four aspects:

- We establish tightness which is often stronger than relative compactness.
- The $E$ herein need not be metrizable nor separable.
- We allow unbounded functions in $D$, which can be handy when working with algebras of polynomials for random measures as in Daw93 §2.1.
- $\{\varpi(f) \circ X^i\}_{i \in I}$ satisfying $|\cdot|$-MCC is milder than $\{\varpi(f) \circ X^i\}_{i \in I}$ being relatively compact if $f$ is not necessarily bounded. So, WMCC is weaker than the analogous condition in Jak86 Theorem 4.6, (4.9) which was shown very useful for establishing tightness of measure-valued processes in Daw93 §3.7 and Per02 §II.4.

Proof of Theorem 8.1.2. ((a) $\Rightarrow$ (b)) For each fixed $f \in D \triangleq C(E; R)$, $\{\varpi(f) \circ X^i\}_{i \in I}$ is tight in $D(R^+; R)$ by Proposition 9.6.1 (d) and Fact 10.2.18 (with $E = D(R^+; E)$, $S = D(R^+; R)$, $f = \varpi(f)$ and $T = \{\varpi^i \circ (X^i)^{-1}\}_{i \in I}$).

$\{\varpi(f) \circ X^i\}_{i \in I}$ satisfies $|\cdot|$-MCC by Theorem 9.7.12 (a) (with $E = R$). Now, (b) follows by the bijective indistinguishability of $\{\varpi(f) \circ X^i\}_{i \in I}$ and $\{\varpi(f) \circ X^i\}_{i \in I}$.

$((c) \Rightarrow (d))$ is proved in Fact 9.7.9. $((b) \Rightarrow (e)$ given MCC). For each $g \in C_b(E; R)$, $\epsilon \in (0, 1/2)$ and $T \in (0, \infty)$, there exist $K_{\epsilon, T} \in \mathcal{X}^m(E)$ and $f_{g, \epsilon, T} \in D$ such that

$$\|f_{g, \epsilon, T}|_{K_{\epsilon, T}} - g|_{K_{\epsilon, T}}\|_{\infty} \leq \epsilon < 1 - \epsilon$$

(8.1.1)

which implies that

$$\sup_{i \in I} \left( \sup_{t \in [0, T]} |f_{g, \epsilon, T} \circ X^i_t - g \circ X^i_t| > \epsilon \right)$$

(8.1.2)

$$\leq 1 - \inf_{i \in I} \mathbb{P}^i \left( X^i_t \in K_{\epsilon, T}, \forall t \in [0, T] \right) < \epsilon.$$ 

Then, $\{X^i\}_{i \in I}$ satisfies MCC by 8.1.2, Proposition 9.3.1 (a, c) and Proposition 9.7.9 (a, b) (with $D_1 = D$ and $D_2 = C_b(E; R)$).

$((d) \Rightarrow (e)$ given MCC) follows by Theorem 6.4.13 (with $E_0 = E$). \hfill \square

Note 6.4.3. Mentioned the terminology “bijective indistinguishability” and the transitivity of $|\cdot|$-MCC between two bijectively indistinguishable families of processes.
When \((E, \tau)\) is a complete (but not necessarily separable) metric space and \(\tau\)-MCC is given, we have shown in Proposition 6.5.2 the equivalence between MCCC and \(\tau\)-MPCC with a dense \(\tau\). This gives us one more tightness criterion.

**Proposition 8.1.6.** Let \((E, \tau)\) be a metric space, \(T\) be a dense subset of \(\mathbb{R}^+\) and \(E\)-valued cádlág processes \(\{X^i\}_{i \in I}\) satisfy \(\tau\)-MCC and \(T\)-MPCC. Then:

(a) There exists an \(E_0 \in \mathcal{B}(E)\) such that \(E_0\) is a separable subspace of \(E\) and \(X^i\) is indistinguishable from a \(D(\mathbb{R}^+; E)\)-valued random variable \(\tilde{X}^i\) with paths in \(E_0^{\mathbb{R}^+}\) for all \(i \in I\).

(b) If \((E, \tau)\) is complete, then the \(\{\tilde{X}^i\}_{i \in I}\) in (a) is tight in \(D(\mathbb{R}^+; E)\).

**Remark 8.1.7.**

- Compared to [EK86 §3.7, Theorem 7.2], part (b) above applies to non-separable spaces, loses compact containment to totally bounded containment and improves relative compactness into tightness in \(D(\mathbb{R}^+; E)\).

- Compared to [EK86 §3.7, Lemma 7.5], part (a) above replaces MCCC by \(\tau\)-MCC plus \(T\)-MPCC with a dense \(\tau\). \(T\)-MPCC is weaker than MCCC for any \(T \subset \mathbb{R}^+\) on metric spaces. In practice, \(\tau\)-MCC is usually no more difficult than MCCC to verify.

**Proof of Proposition 8.1.6.** (a) \(C(\mathbb{R}; \mathbb{R})\) separates points on \(E\) by Proposition 9.3.2(a) and Proposition 9.3.1(a, c). By Proposition 6.5.2(a, b, c) (with \(\mathcal{D} = C(\mathbb{R}; \mathbb{R})\)) and Proposition 6.3.9(a) (with \(X = X^i\) and \(S_0 = D(\mathbb{R}^+; E_0, \mathcal{O}_E(E_0))\)), there exists an \(E_0 \in \mathcal{B}(E)\) such that \((E_0, \mathcal{O}_E(E_0))\) is a separable subspace and \(X^i\) is indistinguishable from an \(E\)-valued process \(\tilde{X}^i\) with paths in \(D(\mathbb{R}^+; E_0, \mathcal{O}_E(E_0))\)\(^7\) for all \(i \in I\). These \(\{\tilde{X}^i\}_{i \in I}\) are \(D(\mathbb{R}^+; E_0, \mathcal{O}_E(E_0))\)-valued variables by Fact 9.7.2(a) (with \(E = (E_0, \mathcal{O}_E(E_0))\)). They are \(D(\mathbb{R}^+; E)\)-valued random variables by Corollary 9.6.3 (with \(A = E_0\)).

(b) When \((E, \tau)\) is complete, the \(\{\tilde{X}^i\}_{i \in I}\) above satisfies MCCC in \(E_0\) by Proposition 6.5.2(d). Then, (b) follows by Corollary 9.7.10(a) and Theorem 8.1.2(a, c). \(\blacksquare\)

## 8.2. Weak convergence and finite-dimensional convergence

We discuss in this section the relationship of the following properties of \(E\)-valued cádlág processes \(\{(\Omega^n, \mathcal{F}^n, \mathbb{P}^n, X^n)\}_{n \in \mathbb{N}_0}\).

**Property.**

\(P8\) There exists a dense subset \(S\) of \(\mathbb{R}^+\), a subset \(\mathcal{D}\) of \(C_b(\mathbb{R}; \mathbb{R})\) such that\(^8\)

\[
\lim_{n \to \infty} \mathbb{E}^n [f \circ X^n_{\tau_0}] = \mathbb{E}^0 [f \circ X^0_{\tau_0}]
\]

for all \(f \in \text{mc}[\Pi^{\mathcal{T}_0(\mathcal{D})}]\) and \(\tau_0 \in \mathcal{P}_0(S)\).

\(P9\) \(P8\) holds with \(\mathcal{D}\) separating points on \(E\).

\(P10\) \(P8\) holds with \(\mathcal{D}\) strongly separating points on \(E\).

---

\(^7\)"with paths in \(D(\mathbb{R}^+; E_0, \mathcal{O}_E(E_0))\)" means all paths of the process lie in \(D(\mathbb{R}^+; E_0, \mathcal{O}_E(E_0))\).

\(^8\)Note 6.2.2 argued that the expectations in 6.2.2 are well-defined.
8. WEAK CONVERGENCE ON PATH SPACE

**P11** \( S \subset \mathbb{R}^+ \) is dense and \( \{X^n\}_{n \in \mathbb{N}_0} \) satisfies

\[
(8.2.2) \quad X^n \xrightarrow{D(S)} X^0 \text{ as } n \uparrow \infty.
\]

**P12** There exist \( \{\hat{X}^n \in M(\Omega^n, \mathcal{F}^n; D(\mathbb{R}^+; E))\}_{n \in \mathbb{N}_0} \) such that \( X^n \) and \( \hat{X}^n \)
are indistinguishable for all \( n \in \mathbb{N}_0 \) and

\[
(8.2.3) \quad \hat{X}^n \Rightarrow \hat{X}^0 \text{ as } n \uparrow \infty \text{ on } D(\mathbb{R}^+; E).
\]

**Remark 8.2.1.** Suppose \( E \) is a Tychonoff space. Then, \( \mathcal{P}(D(\mathbb{R}^+; E)) \) is a Tychonoff space by Proposition 9.4.2 and so (8.2.3) is equivalent to that \( \hat{X}^0 \) is the weak limit of \( \{\hat{X}^n\}_{n \in \mathbb{N}} \) on \( D(\mathbb{R}^+; E) \).

Below are several immediate observations about \( \text{P8-P11} \).

**Fact 8.2.2.** Let \( E \) be a topological space and \( \{\Omega^n, \mathcal{F}^n, P^n; X^n\}_{n \in \mathbb{N}_0} \) be \( E \)-valued processes. Then, \( \text{P11} \) implies \( \text{P8} \) with \( D = C_b(E; \mathbb{R}) \). Moreover, if \( E \) is a Hausdorff (Tychonoff) space, then \( \text{P10} \) (resp. \( \text{P11} \)) implies \( \text{P9} \) (resp. \( \text{P10} \)) with \( D = C_b(E; \mathbb{R}) \).

**Proof.** Let \( D = C_b(E; \mathbb{R}) \). \( \text{P11} \) implies \( \text{P8} \) by Fact 6.2.6 (with \( X = X^0 \) and \( T = S \)). If \( E \) is a Hausdorff space, \( \text{P10} \) implies \( \text{P9} \) by Proposition 9.2.1 (a) (with \( A = E \)). If \( E \) is a Tychonoff space, \( \text{P11} \) implies \( \text{P10} \) since the weaker property \( \text{P8} \) with \( D = C_b(E; \mathbb{R}) \) implies \( \text{P10} \) by Proposition 9.3.1 (a, c).

When \( E \) is a metrizable and separable space, weak convergence on \( D(\mathbb{R}^+; E) \) is well-known to imply finite-dimensional convergence along all time points with no fixed left-jumps (see Theorem 9.7.11 (a)). Recall that every metrizable and separable space is baseable (see Fact 3.2.5 (b)), so the result below generalizes Theorem 9.7.11 (a).

**Theorem 8.2.3.** Let \( E \) be a Tychonoff space and \( \{\Omega^n, \mathcal{F}^n, P^n; X^n\}_{n \in \mathbb{N}_0} \) be \( E \)-valued càdlàg processes. Then:

(a) If \( M(E; \mathbb{R}) \) has a countable subset separating points on \( E^{10} \) and if \( S \supseteq \mathbb{R}^+ \setminus J(X^0) \neq \emptyset \), then \( \text{P12} \) implies (8.2.2).

(b) If \( E \) is a baseable space, then \( \text{P12} \) implies \( \text{P11} \) with \( S \supseteq \mathbb{R}^+ \setminus J(X^0) \).

We prove the more general result below, and Theorem 8.2.3 then follows.

**Lemma 8.2.4.** Let \( E \) be a Tychonoff space and \( \{\mu_n\}_{n \in \mathbb{N}_0} \subset \mathcal{M}^+(D(\mathbb{R}^+; E)) \) satisfy

\[
(8.2.4) \quad \mu_n \Rightarrow \mu_0 \text{ as } n \uparrow \infty \text{ in } \mathcal{M}^+(D(\mathbb{R}^+; E)).
\]

If \( M(E; \mathbb{R}) \) has a countable subset separating points on \( E^{11} \) and \( \mathbb{R}^+ \setminus J(\mu_0) \) is non-empty, especially if \( E \) is a baseable space, then there exist \( \{\nu_{T_0,n} \in \text{be}(\mu_n \circ p_{T_0}^{-1})\}_{n \in \mathbb{N}_0} \) such that

\[
(8.2.5) \quad \nu_{T_0,n} \Rightarrow \nu_{T_0,0} \text{ as } n \uparrow \infty \text{ in } \mathcal{M}^+(E^{10}).
\]

\(^9\)Weak convergence, weak limit point and weak limit of random variables were interpreted in 8.2.4.

\(^{10}\)This condition ensures \( J(X^0) \) is well-defined.

\(^{11}\)This condition ensures \( J(\mu_0) \) is well-defined.
and \[ \lim_{n \to \infty} \int_{E T_0} f(x) \mu_n \circ p_{T_0}^{-1}(dx) = \int_{E T_0} f(x) \mu_0 \circ p_{T_0}^{-1}(dx) \]

for all \( f \in \mathcal{M}[\Pi T_0^r(C_0(E; R))] \) and \( T_0 \in \mathcal{P}_0(\mathbb{R}^+ \setminus J(\mu_0)) \).

**Proof.** The introductory part of this chapter noted that \( \mathbb{R}^+ \setminus J(\mu_0) \neq \emptyset \) when \( E \) is baseable. Let \( D = D(\mathbb{R}^+; E) \) and \( (D, \mathcal{S}_n, \nu_n) \) be the completion of \( (D, \mathcal{B}(D), \mu_n) \) for each \( n \in \mathbb{N}_0 \). It follows by Lemma 10.2.20 (with \( (\mathcal{S}_n, \mu_n, \nu_n) \)) that

\[ \nu_{T_0, n} = \nu_n \circ p_{T_0}^{-1} \in \mathcal{B} \left( \mu_n \circ p_{T_0}^{-1} \right) , \forall n \in \mathbb{N}_0. \]

It follows by (8.2.4) and Fact 10.1.25 (with \( E = D \) and \( \mathcal{S}_n = \mathcal{S}_n \)) that

\[ \nu_n \Rightarrow \nu_0 \text{ as } n \uparrow \infty \text{ in } \mathcal{M}^+(D). \]

The set of discontinuity points of \( p_{T_0} \) has zero measure under \( \mu_0 \) (hence under \( \nu_0 \)) by the definition of \( J(\mu_0) \) and Lemma 9.6.6 (c). Now, (8.2.5) follows by (8.2.8) and the Continuous Mapping Theorem (Theorem 10.1.23 (b) with \( E = D, S = E T_0, \mu_n = \nu_n, \mu = \nu_0 \) and \( f = p_{T_0} \)). (8.2.6) follows by (8.2.5) and Fact 10.2.12 (with \( d = \mathbb{N}(T_0), \mu = \mu_n \circ p_{T_0}^{-1} \) and \( \nu_1 = \nu_{T_0, n} \)). \( \square \)

**Proof of Theorem 8.2.3.** Let \( \{X^n\}_{n \in \mathbb{N}_0} \) be as in (P12). Then, (a) follows by Lemma 8.2.4 (with \( \mu_n = \mu_n \circ (\tilde{X}^n)^{-1} \)) and the indistinguishability of \( X^n \) and \( \tilde{X}^n \). When \( E \) is baseable, \( S \) is a dense subset of \( \mathbb{R}^+ \). Thus (b) follows by (a). \( \square \)

The remainder of this section is about the converse of Theorem 8.2.3. First of all, we establish a result about the converse of Lemma 8.2.4.

**Lemma 8.2.5.** Let \( E \) be a baseable Tychonoff space, \( \{\mu_n\}_{n \in \mathbb{N}_0} \subset \mathcal{M}^+(D(\mathbb{R}^+; E)) \)

and \( D \subset C_0(E; R) \). Suppose that:

(i) \( S \) is a dense subset of \( \mathbb{R}^+ \) and (8.2.6) holds for all \( f \in \mathcal{M}[\Pi T_0^r(D)] \cup 1 \) and \( T_0 \in \mathcal{P}_0(S) \).

(ii) There exists an \( S_0 \in \mathcal{B}(D(\mathbb{R}^+; E)) \) such that \( \mu_0 \) is supported on \( S_0 \) and

\[ \mathcal{B}_D(\mathbb{R}^+; E)(S_0) = \mathcal{B}(E) \circ \mathbb{R}^+ \bigg|_{S_0}. \]

(iii) There exist \( \{V_p\}_{p \in N} \subset \mathcal{C}(D(\mathbb{R}^+; E)) \) such that \( \forall p \in N \)

\[ \liminf_{n \to \infty} \mu_n \left( D(\mathbb{R}^+; E) \setminus V_p \right) \leq 2^{-p}, \forall p \in \mathbb{N}. \]

(iv) \( \{\mu_n\}_{n \in \mathbb{N}} \) is relatively compact.

Then:

(a) If \( D \) strongly separates points on \( E \), then (8.2.4) holds.

(b) If \( D \) separates points on \( E \), \( \mu_0 \circ p_{T_0}^{-1} \) is tight and \( \{\mu_n \circ p_{T_0}^{-1}\}_{n \in \mathbb{N}} \) is sequentially tight for all \( t \) in a conull \( T \subset \mathbb{R}^+ \), then (8.2.4) holds.

---

\[ \mu_n \circ p_{T_0}^{-1} \text{ and } \mu_0 \circ p_{T_0}^{-1} \text{ are members of } \mathcal{M}^+(E T_0, \mathcal{B}(E) T_0), \text{ so the integrals in (8.2.6) are well-defined by Note 5.1.4}. \]
Proof. Let \( \gamma^1 \equiv \mu_0 \) and \( \mathbb{D} \equiv D(\mathbb{R}^+; E) \). By the condition (iv) above and Fact 10.1.6 (8.2.4) follows if we show \( \gamma^1 = \gamma^2 \) for any weak limit point \( \gamma^2 \) of \( \{\mu_n\}_{n \in \mathbb{N}} \) in \( \mathcal{M}^+(\mathbb{D}) \). By passing to a subsequence if necessary, we suppose \( \gamma_n \rightarrow \gamma^2 \) as \( n \uparrow \infty \) in \( \mathcal{M}^+(\mathbb{D}) \) and let \( S^1 \equiv S \) and \( S^2 \equiv J(\gamma^2) \). The rest of the proof is divided into three steps.

Step 1: Verify

(8.2.12) \[
\lim_{n \rightarrow \infty} \int_{E_{\Gamma_0}} f(x) \mu_n \circ p_{T_0}^{-1}(dx) = \int_{E_{\Gamma_0}} f(x) \gamma^1 \circ p_{T_0}^{-1}(dx)
\]

for each \( f \in mc[\Pi^{T_0}(\mathbb{D})] \cup 1 \), \( T_0 \in \mathcal{R}_0(S^\prime) \) and \( i = 1, 2 \). For \( i = 1 \), (8.2.12) is given by the condition (i) above. For \( i = 2 \), (8.2.12) follows by (8.2.11) and Lemma 8.2.4 (with \( \mu_0 = \gamma^2 \)).

Step 2: Verify

(8.2.13) \[
\gamma^1 \circ p_{T_0}^{-1} = \gamma^2 \circ p_{T_0}^{-1} \text{ in } \mathcal{M}^+(E^{T_0}, \mathcal{B}(E) \otimes T_0), \forall T_0 \in \mathcal{R}_0(\mathbb{R}^+).
\]

Under the conditions of (a), (8.2.13) follows immediately by (2.3.11), Step 1 and Lemma 10.2.27 (a).

It takes a bit more work to show (8.2.13) for (b). \( E \) is baseable, so \( S^2 \) is conull. \( T \) is conull by the hypothesis of (b), so \( S^2 \cap T \) is a conull hence dense subset of \( \mathbb{R}^+ \). Fixing \( t \in S^2 \cap T \), we find from Step 1 that

(8.2.14) \[
\lim_{n \rightarrow \infty} \int_{E} f(x) \mu_n \circ p_{t}^{-1}(dx) = \int_{E} f(x) \gamma^2 \circ p_{t}^{-1}(dx)
\]

for each \( f \in mc(\mathcal{D}) \cup 1 \). Letting \( f = 1 \) in (8.2.14), we find that \( \{\mu_n \circ p_{T_0}^{-1}(E)\}_{n \in \mathbb{N}} \) must be contained in a compact sub-interval of \( [0, \infty) \). \( \gamma^1 \circ p_{T_0}^{-1} \) is \( m \)-tight and \( \{\mu_n \circ p_{T_0}^{-1}\}_{n \in \mathbb{N}} \) is sequentially \( m \)-tight by the hypothesis of (b), the baseability of \( E \) and Corollary 3.3.20 (a). It then follows by (8.2.14) and Theorem 5.1.4 (c) (with \( d = 1 \) and \( \Gamma = \{\mu_n\}_{n \in \mathbb{N}} \)) that

(8.2.15) \[
\text{w- lim}_{n \rightarrow \infty} \mu_n \circ p_{t}^{-1} = \gamma^2 \circ p_{t}^{-1}
\]

and \( \gamma^2 \circ p_{t}^{-1} \) is \( m \)-tight.

For each \( T_0 \in \mathcal{R}(S^2 \cap T) \), both \( \gamma^1 \circ p_{T_0}^{-1} \) and \( \gamma^2 \circ p_{T_0}^{-1} \) are \( m \)-tight by Lemma 10.2.19 (a) (with \( I = T_0 \), \( S_i = A_i = E \), \( A = E^{T_0} \) and \( \Gamma = \{\gamma^1 \circ p_{T_0}^{-1}\} \) or \( \{\gamma^2 \circ p_{T_0}^{-1}\} \)). Thus, (8.2.13) follows by (2.3.11) and Lemma 10.2.27 (b) (with \( S = S^2 \cap T \)).

Step 3: Verify \( \gamma^1 = \gamma^2 \) in \( \mathcal{M}^+(\mathbb{D}) \). As \( \mathbb{D} \) is a Tychonoff space, we have that

(8.2.16) \[
\gamma^1(\mathbb{D} \setminus V_p) \leq \liminf_{n \rightarrow \infty} \mu_n(\mathbb{D} \setminus V_p) \leq 2^{-p}, \forall p \in \mathbb{N}
\]

by \( \{V_p\}_{p \in \mathbb{N}} \subseteq \mathcal{C}(\mathbb{D}) \), (8.2.10) and the Portmanteau’s Theorem (Theorem 2.3.7 (a, c) with \( E = \mathbb{D} \)). As \( S_0 \in \mathcal{R}(\mathbb{D}) \) contains every \( V_p \), we have by (8.2.16) and (ii) that

(8.2.17) \[
\gamma^1(\mathbb{D} \setminus S_0) = \gamma^2(\mathbb{D} \setminus S_0) = 0.
\]

It follows by Step 2, the definition of \( \mathcal{B}(E) \otimes \mathbb{R}^+ \) and (8.2.17) that

(8.2.18) \[
\gamma^1(A \cap S_0) = \gamma^2(A \cap S_0), \forall A \in \mathcal{B}(E) \otimes \mathbb{R}^+ \mid_{\mathbb{D}}.
\]

It follows by (8.2.18) and (8.2.9) that

(8.2.19) \[
\gamma^1|_{S_0} = \gamma^2|_{S_0} \text{ in } \mathcal{M}^+(S_0, \mathcal{O}_0(S_0)).
\]
It then follows that
\[(8.2.20)\quad \gamma^1 = \gamma^1|_{S_0} = \gamma^2|_{S_0} = \gamma^2\text{ in } M^+(\mathbb{D})\]

by \(8.2.17, 8.2.19\) and Fact 2.1.1 (c) (with \(E = \mathbb{D}, \mathcal{U} = \mathcal{B}(E), A = S_0\) and \(\mu = \gamma^1\) or \(\gamma^2\)).

The following proposition treats a typical case of Lemma 8.2.5 where each \(\mu_n\) is the distribution of \(D(\mathbb{R}^+; E)\)-valued random variable \(X^n\) and the \(\{V_p\}_{p \in \mathbb{N}}\) in condition (iii) are compact sets provided by tightness.

**Proposition 8.2.6.** Let \(E\) be a baseable Tychonoff space and \(X^n \in M(\Omega^n, \mathcal{F}^n, \mathbb{P}^n; D(\mathbb{R}^+; E))\) for each \(n \in \mathbb{N}_0\). Suppose that:

(i) There is an \(S_0 \in \mathcal{B}(D(\mathbb{R}^+; E))\) satisfying \(\mathbb{P}^0(X^0 \in S_0) = 1\) and \(8.2.9\).

(ii) \(\{X^n\}_{n \in \mathbb{N}}\) is tight in \(S_0\).

(iii) \(T \subset \mathbb{R}^+\) is conull and \(X^0\) satisfies \(T\)-PMTC.

Then, \(P9\) implies \(P12\)

**Remark 8.2.7.** In Proposition 8.2.6, tightness in \(S_0\) is different than tightness in \(D(\mathbb{R}^+; E)\) since \(D(\mathbb{R}^+; E)\) does not necessarily satisfy \(8.2.9\) with \(S_0 = D(\mathbb{R}^+; E)\) when \(E\) is not separable.

**Proof of Proposition 8.2.6.** Let \(\mu_n = \mathbb{P}^n \circ (X^n)^{-1} \in \mathcal{P}(D(\mathbb{R}^+; E))\) for each \(n \in \mathbb{N}_0\). \(P9\) implies a dense subset \(S\) of \(\mathbb{R}^+\) and \(8.2.6\) holds for \(f \in \text{mc}[\Pi^{T_0}(\mathbb{D})] \cup \{1\}\) and \(T_0 \in \mathcal{P}_0(S)\). The condition (i) above implies \(\mu_0(S_0) = 1\). By the condition (ii) above, there exist \(\{V_p\}_{p \in \mathbb{N}} \subset \mathcal{X}(D(\mathbb{R}^+; E))\) such that \(V_p \subset S_0\) for all \(p \in \mathbb{N}\) and \(\inf_{n \in \mathbb{N}} \mu_n(V_p) \geq 1 - 2^{-p}\). As \(D(\mathbb{R}^+; E)\) is a Tychonoff space, \(\{V_p\}_{p \in \mathbb{N}} \subset \mathcal{C}(D(\mathbb{R}^+; E))\) by Proposition 9.1.12 (a) and the tight sequence \(\{\mu_n\}_{n \in \mathbb{N}}\) is relatively compact by the Prokhorov’s Theorem (Theorem 2.3.12 (b)). As \(E\) is a baseable space, the tight sequence \(\{X^n\}_{n \in \mathbb{N}}\) satisfies MCCC by Fact 8.1.1 and Corollary 3.1.15 (a). \(\{X^n\}_{n \in \mathbb{N}}\) satisfies \(\mathbb{R}^+\)-PMTC by Fact 6.5.5 (f) (with \(I = \mathbb{N}, i = n\) and \(A = E\)) so \(\mu_n \circ (X^n)^{-1}\) is \(\mathbb{m}\)-tight for all \(t \in \mathbb{R}^+\). Moreover, \(\mu_0 \circ (X^n)^{-1}\) is \(\mathbb{m}\)-tight for all \(t \in T\) by the condition (iii) above. So far, we have justified all conditions of Lemma 8.2.5 (b) for \(\{\mu_n\}_{n \in \mathbb{N}_0}\), thus \(8.2.4\) and \(P12\) hold.

The following proposition uses our tightness criteria established in 8.1 to realize the “tightness in \(S_0\)” desired by Proposition 8.2.6.

**Proposition 8.2.8.** Let \(E\) be a Tychonoff space and \(\{\Omega^n, \mathcal{F}^n, \mathbb{P}^n, X^n\}_{n \in \mathbb{N}_0}\) be \(E\)-valued càdlàg processes. If \(P9\) holds, \(\{X^n\}_{n \in \mathbb{N}_0}\) satisfies \(D\)-FMCC for the \(D\) in \(P9\) and \(\{X^n\}_{n \in \mathbb{N}_0}\), satisfies MCCC, then \(P12\) holds.

**Proof.** The proof is divided into three steps.

**Step 1: Construct a suitable base.** By Proposition 6.5.4 (b) (with \(I = \mathbb{N}_0\) and \(i = n\)), there exists a \(D\)-baseable subset \(E_0\) of \(E\) such that \(\{X^n\}_{n \in \mathbb{N}_0}\) satisfies MCCC in \(E_0\). By Lemma 3.3.7 (c) (with \(D_0 = \{1\}\)), there exists a base \((E_0, \mathcal{F}; \mathcal{F}; \mathcal{F})\) with \(\mathcal{F} \subset (D \cup \{1\})\).

**Step 2: Construct \(\{\hat{X}^n\}_{n \in \mathbb{N}_0}\).** \(E_0\) is a Tychonoff subspace of \(E\) by Proposition 9.3.2 (b), \(D_0 \sqsupset D(R^+; E_0, \mathcal{O}_E(E_0))\) is a Tychonoff subspace of \(D(R^+; E)\) by Corollary 9.6.3 (with \(A = E_0\)). \(\{X^n\}_{n \in \mathbb{N}}\) satisfies \(\mathcal{F}\)-FMCC since \((\mathcal{F}\setminus\{1\}) \subset \mathcal{D}\).
By Proposition 6.4.11 (with \( \mathbf{I} = \mathbb{N}_0 \)), there exists an \( S_0 \subset \mathbb{D}_0 \) such that \( 6.4.10 \) holds, there exist
\[
\hat{X}^n = \text{rep}_p(X^n; E_0, \mathcal{F}) \in M(\Omega^n, \mathcal{F}^n; S_0, \mathcal{D}_0(S_0))
\subset M(\Omega^n, \mathcal{F}^n; D_0) \subset M(\Omega^n, \mathcal{F}^n; D(\mathbb{R}^+; E)), \ \forall n \in \mathbb{N}_0
\]
satisfying
\[
\inf_{n \in \mathbb{N}_0} \mathbb{P}^n \left( X^n = \hat{X}^n \in S_0 \right) = 1,
\]
and \( \{\hat{X}^n\}_{n \in \mathbb{N}} \) is \( m \)-tight in \( S_0 \). Then, \( S_0 \in \mathcal{B}(E) \circ \mathbb{R}^+_d | D_0 \) and \( 8.2.9 \) holds by \( 6.4.10 \) and Corollary 9.6.3 (with \( A = E_0 \)).

Step 3: Show \( 8.2.3 \). It follows by \( 8.2.22 \), Fact 6.5.1 (with \( \mathbf{I} = \mathbb{N}_0 \) and \( i = n \)), the fact \( \{\mathcal{F} \setminus \{1\}\} \subset \mathcal{D} \) and Lemma 10.3.6 (d) (with \( X = X^0 \)) that
\[
\lim_{n \to \infty} \mathbb{E}^n \left[ f \circ \hat{X}^n_{T_0} \right] = \mathbb{E}^0 \left[ f \circ \hat{X}^0_{T_0} \right]
\]
for all \( f \in \mathcal{M}(\mathbb{P}^0(\mathcal{F} \setminus \{1\})) \) and \( T_0 \in \mathcal{D}_0(S) \). \( X^0 \) satisfies MCC, so it satisfies \( \mathbb{R}^+\text{-PMTC} \) by Fact 6.5.5 (f) (with \( A = E \) and \( \mathbf{I} = \{0\} \)). It then follows that
\[
\hat{X}^n \Rightarrow \hat{X}^0 \) as \( n \uparrow \infty \) on \( \mathbb{D}_0 \)
by \( 8.2.21 \) and Proposition 8.2.6 (with \( E = E_0, \mathcal{D} = \mathcal{F}|_{E_0} \setminus \{1\}, X^n = \hat{X}^n \) and \( T = \mathbb{R}^+ \)). Now, \( 8.2.23 \) follows by \( 8.2.21 \), \( 8.2.24 \) and Lemma 10.2.13 (with \( E = D(\mathbb{R}^+; E), A = \mathbb{D}_0, \mu_n = \mathbb{P}^n \circ (\hat{X}^n)^{-1} \in \mathcal{P}(\mathbb{D}_0) \)) and \( \mu = \mathbb{P}^0 \circ (\hat{X}^0)^{-1} \in \mathcal{P}(\mathbb{D}_0) \). □

Another typical case of Lemma 8.2.5 is when \( E \) has a metrizable and separable subspace \( E_0 \), and the \( S_0 \) in condition (ii) and the \( \{V_p\}_{p \in \mathbb{N}} \) in condition (iii) of Lemma 8.2.5 are all taken to be \( D(\mathbb{R}^+; E_0, \mathcal{D}_0(E_0)) \). Then, the assumption of relative compactness in Lemma 8.2.5 (iv) can be loosened to \( D \)-FMCC.

**Theorem 8.2.9.** Let \( E \) be a Tychonoff space and \( \{(\Omega^n, \mathcal{F}^n, \mathbb{P}^n, X^n)\}_{n \in \mathbb{N}_0} \) be \( E \)-valued c
dal\( \) processes. Suppose that:

(i) \( \textbf{P8} \) holds with \( \mathcal{D} \subset C_b(E; \mathbb{R}) \) being countable and strongly separating points on some \( E_0 \in \mathcal{B}(E) \).

(ii) \( \{X_n\}_{n \in \mathbb{N}_0} \) satisfies
\[
\inf_{n \in \mathbb{N}_0} \mathbb{P}^n \left( X^n \in E_0^{R^+} \right) = 1.
\]

(iii) \( \{X^n\}_{n \in \mathbb{N}} \) satisfies \( D \)-FMCC for the \( \mathcal{D} \) above.

Then, \( \textbf{P12} \) holds.

**Remark 8.2.10.** The condition (i) above implies \( E_0 \) is a second-countable subspace of \( E \) by Proposition 9.2.1 (d) (with \( A = E_0 \)). Given such \( E_0 \), the condition (i, iii) above is weaker than relative compactness by Theorem 8.3.1 (a) to follow, Corollary 9.7.7 and Proposition 9.2.8 (b).

**Proof of Theorem 8.2.9.** The proof is divided into four steps.

*Step 1: Construct a suitable base by \( E_0 \) and \( \mathcal{D} \). \( E_0 \) is a Tychonoff subspace and is a \( D \)-baseable subset of \( E \) by Proposition 9.3.2 (b) and Proposition 9.2.1 (a) (with

13It is the subsequence \( \{X^n\}_{n \in \mathbb{N}} \) that satisfies \( \mathcal{F} \)-FMCC, so Proposition 6.4.11 (c) is just applied to \( \{X^n\}_{n \in \mathbb{N}} \) with \( X^0 \) removed.
A = E_0). As 𝒟 is countable, there exists a base (E_0, 𝒢; Ê, 𝔓) with 𝒢 = 𝒟 ∪ {1} strongly separating points on E_0 by Proposition 8.3.7 (b).

Step 2: Construct \( \{\tilde{X}_n\}_{n \in \mathbb{N}_0} \). Let \( \mathbb{D} \equiv D(\mathbb{R}^+; E) \), \( S_0 \equiv D(\mathbb{R}^+; E, \vartheta_E(E_0)) \) and \( \widehat{\mathbb{D}} \equiv D(\mathbb{R}^+; \hat{E}) \). \( S_0 \) is a subspace of \( \mathbb{D} \) by Corollary 8.6.3 (with \( A = E_0 \)). \( (E_0, \vartheta_E(E_0)) \) is metrizable and separable by Corollary 9.3.6 (a, b) (with \( E = E_0 \) and \( \mathcal{D} = \mathcal{F}[E_0] \)). Thus, \( S_0 \) satisfies (8.2.23) by Proposition 9.6.10 (b) (with \( E = (E_0, \vartheta_E(E_0)) \)). Then, there exist

\[
\tilde{X}^n = \text{rep}_c(X^n; E_0, \mathcal{F})
\]

(8.2.26) satisfying (8.2.22) by (8.2.25), Proposition 6.3.9 (with \( \hat{E} \) and strongly separating points on \( E = E_0 \)).

Step 3: Show \( \tilde{X}^0 \) is the weak limit of \( \{\tilde{X}_n\}_{n \in \mathbb{N}_0} \) on \( \widehat{\mathbb{D}} \). In this step, we consider \( \{\tilde{X}_n\}_{n \in \mathbb{N}_0} \) as \( \mathbb{D} \)-valued random variables. As mentioned in Note 6.1.5, \( \hat{E} \) is a compact Polish space, so \( \{\tilde{X}_n\}_{n \in \mathbb{N}_0} \) automatically satisfies MCC by Note 6.4.9; \( \{\tilde{X}_n\}_{n \in \mathbb{N}} \) satisfies \( \mathcal{F} \)-FMCC since \( \mathcal{F}\{1\} = \mathcal{D} \), so \( \{\tilde{X}_n\}_{n \in \mathbb{N}_0} \) satisfies \( \mathcal{F} \)-FMCC by Proposition 6.4.10 (a). (8.2.23) holds for all \( \tilde{f} \in \text{mc}[\Pi\mathbb{T}_0(\hat{\mathbb{F}}\{1\})] \) and \( \mathbb{T}_0 \in \mathcal{P}_0(S) \) by (8.2.25), Fact 6.5.1 (with \( \tilde{f} = \mathbb{T}_0 \) and \( \mathbb{T}_0 \equiv \mathcal{D} \)). Hence \( \mathcal{F}\{1\} = \mathcal{D} \) and Lemma 10.3.6 (d) (with \( X = X^0 \)). \( \hat{\mathcal{F}}\{1\} \) is a subset of \( C_b(\hat{E}; \mathbb{R}) \) by Corollary 3.1.10 (a) and separates points on \( \hat{E} \) by definition of base. Now, the conclusion of Step 3 follows by Proposition 8.2.8 (with \( E = \hat{E} \), \( X^n = \tilde{X}^n \) and \( \mathcal{D} = \hat{\mathcal{F}}\{1\} \)).

Step 4: Show (8.2.3). \( \{\tilde{X}_n\}_{n \in \mathbb{N}_0} \) as \( \mathbb{D} \)-valued random variables satisfies (8.2.24) with \( \mathbb{D}_0 = S_0 \) by Step 3 and Proposition 6.4.16 (b). \( \{\tilde{X}_n\}_{n \in \mathbb{N}_0} \) as \( \mathbb{D} \)-valued random variables satisfies (8.2.3) by (8.2.26), (8.2.24) and Lemma 10.2.13 (with \( E = \mathbb{D} \), \( A = S_0 \), \( \mu_n = \mathbb{P}^\circ (\tilde{X}^n)^{-1} \in \mathcal{P}(S_0) \) and \( \mu = \mathbb{P}^0 \circ (\tilde{X}^0)^{-1} \in \mathcal{P}(S_0) \)). □

If \( E \) itself is a metrizable and separable space, then the \( E_0 \) in Theorem 8.2.9 can be taken to equal \( E \).

**Corollary 8.2.11.** Let \( E \) be a metrizable and separable space and \( \{(\Omega^n, \mathcal{F}^n, \mathbb{P}^n; X^n)\}_{n \in \mathbb{N}_0} \) be \( E \)-valued càdlàg processes. Then, the following statements are successively weaker:

(a) \( \{X^n\}_{n \in \mathbb{N}} \) satisfies MCC and \( P11 \)

(b) \( \{X^n\}_{n \in \mathbb{N}} \) satisfies \( \mathcal{D} \)-FMCC and \( P10 \) for some \( \mathcal{D} \subset C_b(E; \mathbb{R}) \).

(c) \( \{X^n\}_{n \in \mathbb{N}} \) satisfies \( \mathcal{D} \)-FMCC and \( P10 \) for some countable \( \mathcal{D} \subset C_b(E; \mathbb{R}) \).

(d) \( \{X^n\}_{n \in \mathbb{N}} \) satisfies \( P12 \)

**Proof.** (a) \( \Rightarrow \) (b) follows by Fact 8.2.2 and Corollary 9.7.10 (b). (b) \( \Rightarrow \) (c) follows by Proposition 9.1.4 (c) and Proposition 9.2.8 (b). (c) \( \Rightarrow \) (d) follows by Theorem 8.2.9 (with \( E_0 = E \)). □

**Remark 8.2.12.** Compared to Theorem 9.7.11 (b), Corollary 8.2.11 (a, d) reduces relative compactness to MCC which is a weaker condition by Theorem 8.3.1 (a) to follow.

When \( E \) is a non-separable metric space, one can obtain the \( E_0 \) in Theorem 8.2.9 by \( r \)-MCC and \( T \)-MPCC.

**Proposition 8.2.13.** Let \( (E, \mathfrak{r}) \) be a metric space and \( \{(\Omega^n, \mathcal{F}^n, \mathbb{P}^n; X^n)\}_{n \in \mathbb{N}_0} \) be \( E \)-valued càdlàg processes. Then, the following statements are successively weaker:
(a) \( \{X^n\}_{n \in \mathbb{N}} \) satisfies \( \tau \)-MCC and \( T \)-MPCC with a dense \( T \subset \mathbb{R}^+ \). \( X^0 \) satisfies \( \tau \)-MCC. Moreover, \( P11 \) holds.

(b) \( \{X^n\}_{n \in \mathbb{N}} \) satisfies \( \tau \)-MCC, \( D \)-FMCC with \( D \subset C_b(E; \mathbb{R}) \) and \( T_1 \)-MPCC with a dense \( T_1 \subset \mathbb{R}^+ \). \( X^0 \) satisfies \( \tau \)-MCC and \( T_2 \)-MPCC with a dense \( T_2 \subset \mathbb{R}^+ \). Moreover, \( P10 \) holds.

(c) \( \{X^n\}_{n \in \mathbb{N}_0} \) satisfies \( P12 \).

**Proof.** (a) \( \Rightarrow \) (b) By Corollary 9.7.10 (a), \( \{X^n\}_{n \in \mathbb{N}} \) satisfies MCC. By Fact 8.2.2 and Corollary 9.7.7 (a, b), there exists a \( D \subset C_b(E; \mathbb{R}) \) such that \( \{X^n\}_{n \in \mathbb{N}} \) satisfies \( D \)-FMCC and \( P10 \) holds. By Proposition 6.5.2 (a), there exist \( \{A_{p,q}\}_{p,q \in \mathbb{N}} \subset \Theta(E) \) such that each \( A_{p,q} \) is a totally bounded set and

\[
\inf_{n \in \mathbb{N}} P^n (X^n_t \in A_{p,q}) \geq 1 - 2^{-p}, \quad \forall t \in [0, q], \quad p, q \in \mathbb{N}. \tag{8.2.27}
\]

Hence, \( \{X^n\}_{n \in \mathbb{N}} \) satisfies \( \mathbb{R}^+ \)-MPCC. For each \( t \) in the \( S \) of \( P11 \) and \( n \in \mathbb{N}_0 \), \( P^n \circ (X^n_t)^{-1} \in \mathcal{P}(E) \) by Fact 2.5.2 (d) and so \( P11 \) implies

\[
X^n_t \implies X^0_t \quad \text{as} \quad n \uparrow \infty \quad \text{on} \quad E. \tag{8.2.28}
\]

As \( E \) is a Tychonoff space, it follows by \( [8.2.27], [8.2.28] \), the closedness of each \( A_{p,q} \) and the Portmannean’s Theorem (Theorem 2.3.7 (a, b)) that

\[
P^0 (X^0_t \in A_{p,q}) \geq \inf_{n \in \mathbb{N}} P^n (X^n_t \in A_{p,q}) \geq 1 - 2^{-p}, \quad \forall t \in S \cap [0, q], \quad p, q \in \mathbb{N},
\]

thus proving \( X^0 \) satisfies \( S \)-MPCC. Now, (b) follows by letting \( T_1 = T \) and \( T_2 = S \).

(b) \( \Rightarrow \) (c) The union of two second-countable subspaces of \( E \) is still second-countable by Proposition 9.1.4 (c) and Proposition 9.1.3 (b, e). So, we apply Proposition 6.5.2 (a - c) to \( \{X^n\}_{n \in \mathbb{N}} \) and the singleton \( \{X^0\} \) respectively and find a second-countable subspace \( E_0 \) of \( E \) satisfying \( [8.2.25] \). There exists a countable \( D_0 \subset D \) strongly separating points on \( E_0 \) by \( P10 \) and Proposition 9.2.8 (b). \( P10 \) implies \( P8 \) so (c) follows by Theorem 8.2.9 (with \( D = D_0 \)).

### 8.3. Relative compactness and finite-dimensional convergence

When \( (E, \tau) \) is a separable metric space, relative compactness in \( D(\mathbb{R}^+; E) \) implies \( \tau \)-MCC (see e.g. [EK86 §3.7, Theorem 7.2]). For a general Tychonoff space \( E \), we now justify the sufficiency of relative compactness in \( D(\mathbb{R}^+; E) \) for MCC. If \( E \) is also baseable, we leverage Theorem 8.2.3 and establish the sufficiency of relative compactness in \( D(\mathbb{R}^+; E) \) for “relative compactness” under finite-dimensional convergence.

**Theorem 8.3.1.** Let \( E \) be a Tychonoff space, \( I \) be an infinite index set and \( \{(\Omega^i, \mathcal{F}^i, P^i, X^i)\}_{i \in I} \) be a relatively compact\(^{14} \) family of \( D(\mathbb{R}^+; E) \)-valued random variables. Then:

(a) \( \{X^i\}_{i \in I} \) satisfies \( C(E; \mathbb{R}) \)-FMCC and MCC.

(b) If \( E \) is baseable, then any infinite subset of \( \{X^i\}_{i \in I} \) has a subsequence that converges finite-dimensionally to some \( D(\mathbb{R}^+; E) \)-valued random variable \( X \) along \( \mathbb{R}^+ \setminus J(X) \).

\(^{14}\)Relative compactness of random variables was discussed in §2.3.
PROOF. By the relative compactness of \( \{X^i\}_{i \in I} \) in \( D(\mathbb{R}^+; E) \), any infinite sub
set \( J \) of \( I \) contains a sequence \( \{i_n\}_{n \in \mathbb{N}} \subset J \subset I \) such that
\[(8.3.1) \quad \mathbb{P}^n \circ (X^{i_n})^{-1} \Rightarrow \mu \text{ as } n \uparrow \infty \text{ in } \mathbb{P}(D(\mathbb{R}^+; E)) \]
for some \( \mu \in \mathbb{P}(D(\mathbb{R}^+; E)) \). Let \( \Omega \overset{8.3.2}{=} D(\mathbb{R}^+; E), \mathcal{F} \overset{8.3.2}{=} \mathcal{B}(\Omega), \mathbb{P} \overset{8.3.2}{=} \mu \) and \( X \) be
the identity mapping on \( \Omega \). Then, \( (\Omega, \mathcal{F}, \mathbb{P}, X) \) is a \( D(\mathbb{R}^+; E) \)-valued random
variable and satisfies
\[(8.3.2) \quad X^{i_n} \Rightarrow X \text{ as } n \uparrow \infty \text{ on } D(\mathbb{R}^+; E). \]
For each \( f \in C(E; \mathbb{R}) \), it follows that
\[(8.3.3) \quad \varpi(f) \circ X^{i_n} \Rightarrow \varpi(f) \circ X \text{ as } n \uparrow \infty \text{ on } D(\mathbb{R}^+; \mathbb{R}) \]
by \[8.3.2\], Proposition \[9.6.1\](d) (with \( S = \mathbb{R} \)) and the Continuous Mapping Theorem
(Theorem \[10.1.23\](a) with \( E = D(\mathbb{R}^+; E), S = D(\mathbb{R}^+; \mathbb{R}) \) and \( f = \varpi(f) \)). The
argument above proves the relative compactness of \( \{\varpi(f) \circ X^i\}_{i \in I} \) in \( D(\mathbb{R}^+; \mathbb{R}) \).
\( D(\mathbb{R}^+; \mathbb{R}) \), as mentioned in Note \[6.1.5\], is a Polish space, so \( \{\varpi(f) \circ X^i\}_{i \in I} \) is tight in
\( D(\mathbb{R}^+; \mathbb{R}) \) by the Prokhorov’s Theorem (Theorem \[2.3.12\](a)) and satisfies \(|\cdot|\)-MCC
by Theorem \[9.7.12\](with \( (E, \tau) = (\mathbb{R}, |\cdot|) \)). \( C(E; \mathbb{R}) \) strongly separates points on \( E \)
by Proposition \[9.3.1\](a, b). Now, (a) follows by Fact \[9.7.5\](b) (with \( D = C(E; \mathbb{R}) \)) and
Corollary \[9.7.7\](a, d) (with \( D = C(E; \mathbb{R}) \)). If \( E \) is also baseable, then we have
by Theorem \[8.2.3\](b) (with \( n = i_n \)) that
\[(8.3.4) \quad X^{i_n} \overset{D(\mathbb{R}^+\backslash J(\hat{X})))}{\to} X \text{ as } n \uparrow \infty, \]
thus proving (b).

We then consider the converse of Theorem \[8.3.1\]

**Theorem 8.3.2.** Let \( E \) be a Tychonoff space, \( I \) be an infinite index set and
\( \{(\Omega^i, \mathcal{F}^i, \mathbb{P}^i, X^i)\}_{i \in I} \) be \( E \)-valued càdlàg processes. Suppose that for each infinite
\( \mathcal{T}^* \subset I \), there exists a subsequence \( \mathcal{T} \subset \{i_n\}_{n \in \mathbb{N}} \subset \mathcal{T}^* \), an \( E_{0, \mathcal{T}} \in \mathcal{B}(E) \), a \( \mathcal{D}_\mathcal{T} \subset C_0(E; \mathbb{R}) \), an \( S_\mathcal{T} \subset \mathbb{R}^+ \) and an \( E \)-valued càdlàg process
(\( \Omega, \mathcal{F}, \mathbb{P}, X_\mathcal{T} \)) such that:

(i) \( \mathcal{D}_\mathcal{T} \) is countable and strongly separates points on \( E_{0,\mathcal{T}} \).

(ii) \( \{X^{i_n}\}_{n \in \mathbb{N}} \) and \( X_\mathcal{T} \) satisfy
\[(8.3.5) \quad \inf_{n \in \mathbb{N}} \mathbb{P}^{i_n} \left( X^{i_n} \in E_{0,\mathcal{T}}^+ \right) = \mathbb{P} \left( X_\mathcal{T} \in E_{0,\mathcal{T}}^+ \right) = 1. \]

(iii) \( \{X^{i_n}\}_{n \in \mathbb{N}} \) satisfies \( \mathcal{D}_\mathcal{T} \)-FMCC.

(iv) \( S_\mathcal{T} \) is dense in \( \mathbb{R}^+ \) and
\[(8.3.6) \quad \lim_{n \to \infty} \mathbb{E}^{i_n} \left[ f \circ X^{i_n}_{T_0} \right] = \mathbb{E} \left[ f \circ X^0_{T_0} \right]
\quad \text{for all } f \in \text{mc}[\Pi^0_0(D_{\mathcal{T}})] \quad \text{and} \quad T_0 \in \mathcal{P}_0(S_\mathcal{T}). \]

Then, there exist an \( \mathbf{I}_0 \subset \mathcal{P}_0(I) \) and \( D(\mathbb{R}^+; E) \)-valued random variables
\( \{\hat{X}^i\}_{i \in \mathbf{I}_0} \) such that \( \hat{X}^i \) is indistinguishable from \( X^i \) for all \( i \in \mathbf{I}_0 \) and \( \{\hat{X}^i\}_{i \in \mathbf{I}_0} \) is relatively
compact in \( D(\mathbb{R}^+; E) \).

\[\text{15Such } X \text{ is known as the coordinate process on } D(\mathbb{R}^+; E).\]
PROOF. Let \( D = D(R^+; E) \) and \( D_0 = D(R^+; E_{0,T}, \mathcal{O}_E(E_{0,T})) \). It follows by (8.3.5) and the càdlàg properties of \( \{X^i\}_{i \in I} \) and \( X^T \) that
\[
\inf_{n \in \mathbb{N}} \mathbb{P}^{\pi_n} \left( X^{i_n} \in D_0 \right) = \mathbb{P} \left( X^T \in D_0 \right) = 1. \tag{8.3.7}
\]
By Proposition 6.3.9 (a) (with \( S_0 = D_0 \) and \( X = X^{i_n} \) or \( X^T \)) and Corollary 9.6.3 (with \( A = E_{0,T} \)), there exist
\[
\left( \left\{ \hat{X}^{i_n} \right\}_{n \in \mathbb{N}} \cup \left\{ \hat{X}^T \right\} \right) \subset M \left( \Omega^{i_n}, \mathcal{F}^{i_n}; D_0 \right) \subset M \left( \Omega^{i_n}, \mathcal{F}^{i_n}; D \right)
\]
satisfying
\[
\inf_{n \in \mathbb{N}} \mathbb{P}^{\pi_n} \left( X^{i_n} = \hat{X}^{i_n} \in D_0 \right) = \mathbb{P} \left( X^T = \hat{X}^T \in D_0 \right) = 1. \tag{8.3.8}
\]
It follows by (8.3.9) and the condition (iv) above that
\[
\lim_{n \to \infty} \mathbb{E}^{\pi_n} \left[ f \circ \hat{X}^{i_n}_{T_0} \right] = \mathbb{E} \left[ f \circ \hat{X}^T_{T_0} \right] \tag{8.3.10}
\]
for each \( f \in \text{mc}[\Pi^{T_0}(D_T)] \) and \( T_0 \in \mathcal{P}_0(S_T) \). It then follows by Theorem 8.2.9 (with \( X^n = \hat{X}^{i_n}, X^0 = \hat{X}^T, E_0 = E_{0,T}, D = D_T \) and \( S = S_T \)) that
\[
\hat{X}^{i_n} \Rightarrow \hat{X}^T \quad \text{as} \quad n \uparrow \infty \quad \text{on} \quad D. \tag{8.3.11}
\]
From the argument above we draw two conclusions: (1) There would be at most finite members of \( \{X^i\}_{i \in I} \) which may not admit an indistinguishable \( D \)-valued copy. Let \( I_0 \in \mathcal{P}_0(I) \) be the indices of these exceptions. For each \( i \in I \setminus I_0 \), different \( \{i_n\}_{n \in \mathbb{N}} \subset I \) that contains \( i \) may induce different \( D \)-valued copies of \( X^i \). However, such copy can be thought of as a unique one up to indistinguishability, which we denote by \( \hat{X}^i \). (2) For any infinite \( I^* \subset (I \setminus I_0) \), there exist a subsequence \( \{i_n\}_{n \in \mathbb{N}} \subset I^* \) and a \( \mathbb{D} \)-valued random variable \( \hat{X}^T \) such that (8.3.11) holds. In other words, \( \{\hat{X}^i\}_{i \in I^* \setminus I_0} \) is relatively compact in \( \mathbb{D} \).

The following special cases of Theorem 8.3.1 correspond to the settings of Corollary 8.2.11 and Proposition 8.2.13 respectively.

**Corollary 8.3.3.** Let \( E \) be a metrizable and separable space, \( I \) be an infinite index set and \( \{\Omega^i, \mathcal{F}^i, \mathbb{P}^i, X^i\}_{i \in I} \) be \( E \)-valued càdlàg processes. Then, the following statements are successively weaker:

(a) Any infinite subset of \( \{X^i\}_{i \in I} \) has a subsequence that satisfies MCC and converges finite-dimensionally to some \( E \)-valued càdlàg process along a dense subset of \( R^+ \).

(b) For any infinite \( I^* \subset I \), there exist a subsequence \( I \doteq \{i_n\}_{n \in \mathbb{N}} \subset I^* \), a \( D_T \subset C_0(E; R) \), a dense \( S_T \subset R^+ \) and an \( E \)-valued càdlàg process \( \Omega, \mathcal{F}, \mathbb{P}, X^T \) such that: (i) \( D_T \) is countable and strongly separates points on \( E \), (ii) \( \{X^{i_n}\}_{n \in \mathbb{N}} \) satisfies \( D_T \)-FMCC, and (iii) (8.3.6) holds for all \( f \in \text{mc}[\Pi^{T_0}(D_T)] \) and \( T_0 \in \mathcal{P}_0(S_T) \).

(c) There exist an \( I_0 \in \mathcal{P}_0(I) \) and \( D(R^+; E) \)-valued random variables \( \{\hat{X}^i\}_{i \in I \setminus I_0} \) such that \( \hat{X}^i \) is indistinguishable from \( X^i \) for all \( i \in I \setminus I_0 \) and \( \{\hat{X}^i\}_{i \in I \setminus I_0} \) is relatively compact in \( D(R^+; E) \).

**Proof.** (a) \( \Rightarrow \) (b) (i) and (ii) follow by Corollary 9.7.10 (b). (iii) follows by (a) and Fact 8.2.2 (with \( X^n = X^{i_n}, X = X^T, S = S_T \) and \( D_T \subset D = C_0(E; R) \)). (b) \( \Rightarrow \) (c) follows by Theorem 8.3.2 (with \( E_{0,T} = E \)). \( \square \)
Proposition 8.3.4. Let \((E, r)\) be a metric space, \(I\) be an infinite index set and \(\{(\Omega^i, \mathcal{F}^i, \mathbb{P}^i, X^i)\}_{i \in I}\) be \(E\)-valued càdlàg processes. Then, the following statements are successively weaker:

(a) For any infinite \(I^* \subset I\), there exist a subsequence \(I \equiv \{i_n\}_{n \in \mathbb{N}} \subset I^*\), an \(E\)-valued càdlàg process \((\Omega, \mathcal{F}, \mathbb{P}; X^I)\) and dense subsets \(T^I, S^I\) of \(\mathbb{R}^+\) such that: (i) \(\{X^{i_n}\}_{n \in \mathbb{N}}\) satisfies \(r\)-MCC and \(T^I\)-MPCC, (ii) \(X^I\) satisfies \(r\)-MCC, and (iii) (8.3.12) \(X^{i_n} \xrightarrow{D[S^I]} X^I\) as \(n \uparrow \infty\).

(b) For any infinite \(I^* \subset I\), there exist a sub-subsequence \(I \equiv \{i_n\}_{n \in \mathbb{N}} \subset I^*\), a \(D_I \subset C^b(E; \mathbb{R})\), an \(E\)-valued càdlàg process \((\Omega, \mathcal{F}, \mathbb{P}; X^I)\) and dense sub-

sets \(T^I_1, T^I_2, S^I\) of \(\mathbb{R}^+\) such that: (i) \(D_I\) strongly separates points on \(E\), (ii) \(\{X^{i_n}\}_{n \in \mathbb{N}}\) satisfies \(r\)-MCC, \(D_I\)-FMCC and \(T^I_1\)-MPCC, (iii) \(X^I\) satisfies \(r\)-MCC and \(T^I_2\)-MPCC, and (iv) (8.3.6) holds for all \(f \in \text{mc}[\pi^T_0(D_I)]\) and \(T_0 \in \mathcal{P}_0(S^I)\).

(c) There exist an \(I^*_0 \in \mathcal{P}_0(I)\) and \(D(\mathbb{R}^+; E)\)-valued random variables \(\{\tilde{X}^i\}_{i \in I \setminus I^*_0}\) such that \(\tilde{X}^i\) is indistinguishable from \(X^i\) for all \(i \in I \setminus I^*_0\) and \(\{\tilde{X}^i\}_{i \in I \setminus I^*_0}\) is relatively compact in \(D(\mathbb{R}^+; E)\).

Proof. ((a) \(\rightarrow\) (b)) follows by Proposition 8.2.13 (a, b) (with \(X^n = X^{i_n}\), \(X^0 = X^I\), \(T = T^I\) and the \(\mathcal{S}\) in \(\mathcal{P}^{11}\) being \(S^I\)). ((b) \(\rightarrow\) (c)) follows by Proposition 8.2.13 (b, c) (with \(X^n = X^{i_n}\), \(X^0 = X^I\), \(T_1 = T^I_1\) and \(T_2 = T^I_2\)) and our argument about the finite exception set \(I^*_0\) in the proof of Theorem 8.3.2. \(\square\)
CHAPTER 9

Background

This appendix presents a series of background results used in Chapters 3 - 8. We limit our discussion to the most necessary material. §9.1, §9.2 and §9.3 are about the point-set topology. More details are found in [Mun00, Chapter 1 - 7], [Bog07, Vol. II, Chapter 6], [EK86 §3.4] and [BK10]. §9.4 deals with weak topology of Borel measures in the spirit of [EK86 §3.1 - 3.4], [KX95 Chapter 1] and [BK10]. §9.5 discusses standard Borel property of topological spaces and their subsets, where we refer the readers to [Sri98 §3.3] and [Bog07 Vol. II, Chapter 6] for further materials. §9.6 gives a short review of Skorokhod $J_1$-spaces. Excellent treatments of this topic are available in [EK86 §3.5 - 3.10], [Jak86], [BK10] and [Kou16]. §9.7 recalls several basic properties of càdlàg processes.

This appendix compiles with all our notations, terminologies and conventions introduced before. Several general technicalities used herein are provided in §10.1 of Appendix 10. A collection of miscellaneous results about the topics above are presented in §10.2 of Appendix 10.

9.1. Point-set topology

In this appendix, $E$ denotes a topological space if not otherwise specified.

9.1.1. Separability. $E$ is a Hausdorff space if for any distinct $x, y \in E$, there exist disjoint $O_x, O_y \in \mathcal{O}(E)^1$ such that $x \in O_x$ and $y \in O_y$. From this definition one immediately observes that:

FACT 9.1.1. Any topological refinement\(^2\) of a Hausdorff space is also Hausdorff.

$E$ is a T3 space if $E$ is a Hausdorff space and for any $x \in E$ and $F \in \mathcal{C}(E)$ excluding $x$, there exist disjoint $O_x, O_F \in \mathcal{O}(E)$ such that $x \in O_x$ and $F \subset O_F$. $E$ is a T4 space if $E$ is a Hausdorff space and for any disjoint $F_1, F_2 \in \mathcal{C}(E)$, there exist disjoint $O_1, O_2 \in \mathcal{O}(E)$ such that $F_1 \subset O_1$ and $F_2 \subset O_2$. Below are several basic properties of Hausdorff, T3 and T4 spaces.

PROPOSITION 9.1.2. The following statements are true:

(a) Any finite subset of a Hausdorff space is closed.

(b) The families of T4, T3 and Hausdorff spaces are successively larger.

(c) Subspaces of a T3 or Hausdorff space are T3 or Hausdorff spaces respectively. Moreover, closed subsets of a T4 space are T4 subspaces.

\(^1\) \(\mathcal{O}(E)\) and \(\mathcal{C}(E)\) denotes the family of all open and closed subsets of $E$ respectively.

\(^2\) The terminology “topological refinement” was introduced in §2.1.3.
Any product space of T3 or Hausdorff spaces is a T3 or Hausdorff space respectively.

9.1.2. Countability. $E$ is first-countable if for each $x \in E$, there exists a countable collection $O_x \subset \mathcal{O}(E)$ such that $O \ni x$ for all $O \in O_x$ and any $U \in \mathcal{O}(E)$ containing $x$ is the superset of some member of $O_x$. $E$ is separable if $E$ has a countable dense subset. $E$ is a Lindelöf space if any $\{O_i\}_{i \in I} \subset \mathcal{O}(E)$ satisfying $E = \bigcup_{i \in I} O_i$ admits a countable subset $\{O_{i_n}\}_{n \in \mathbb{N}}$ satisfying $E = \bigcup_{n \in \mathbb{N}} O_{i_n}$. $E$ is a second-countable space if it admits a countable topological basis.

Proposition 9.1.3. The following statements are true:

(a) Subspaces of a first-countable, second-countable or hereditary Lindelöf space are first-countable, second-countable or hereditary Lindelöf spaces, respectively. Moreover, closed subsets of a Lindelöf space are Lindelöf subspaces.

(b) Every second-countable space is first-countable, separable and hereditary Lindelöf simultaneously.

(c) The product space of countably many first-countable, second-countable or separable spaces is first-countable, second-countable or separable, respectively.

(d) The image of a separable, Lindelöf or hereditary Lindelöf space under a continuous mapping is separable, Lindelöf or hereditary Lindelöf, respectively.

(e) The union of countably many separable, Lindelöf or hereditary Lindelöf subspaces is a separable, Lindelöf or hereditary Lindelöf subspace, respectively.

9.1.3. Metrizability. If the topology of $E$ is the same as the metric topology induced by some metric $r$ on $E$, then $E$ is said to be metrizable, $r$ is said to metrize $E$ and $(E, r)$ is called a metrization of $E$.

Proposition 9.1.4. The following statements are true:

(a) Metrizable spaces are T4 (hence T3 and Hausdorff) spaces.

(b) Subspaces of a metrizable space are metrizable.

(c) Subspaces of a metrizable and separable space are metrizable and second-countable (hence separable and hereditary Lindelöf).

(d) Homeomorphs of metrizable spaces are metrizable.

Let $(E, r)$ and $(S, d)$ be metric spaces. $f \in S^E$ is an isometry between $(E, r)$ and $(S, d)$ if $f$ is a surjective and $r(x, y) = d(f(x), f(y))$ for all $x, y \in E$. $(E, r)$ and $(S, d)$ are isometric if there exists an isometry between them. $(S, d)$ is a completion of $(E, r)$ if $(S, d)$ is complete and $E$ is isometric to a dense subspace of $S$.

Note 9.1.5. Without loss of generality, a metric space $(E, r)$ can always be treated as a dense subset of its completion $(S, d)$ such that $d = r$ restricted to $E \times E$.

Proposition 9.1.6. Let $(E, r)$ and $(S, d)$ be metric spaces. Then:

(a) If $(E, r)$ and $(S, d)$ are isometric, then they are homeomorphic. In particular, $(E, r)$ is complete precisely when $(S, d)$ is complete.

3The notion of topological basis was mentioned in §2.1.3.
(b) There exists a unique completion of \((E, \tau)\) up to isometry.

(c) If \((E, \tau)\) is complete, then the closure of \(A \subset E\) equipped with (the restricted metric of) \(\tau\) is the completion of \(A\).

The next two results are about metrizability of countable Cartesian products.

**Proposition 9.1.7.** Let \(\{(S_i, \tau_i)\}_{i \in I}\) be metric spaces and \(S = \prod_{i \in I} S_i\). Then:

(a) When \(I = \{1, \ldots, d\}\), \(S\) is metrized by

\[
\tau^d(x, y) = \max_{1 \leq i \leq d} \tau_i(p_i(x), p_i(y)), \forall x, y \in S.
\]

(9.1.1) If \(\{(S_i, \tau_i)\}_{1 \leq i \leq d}\) are all complete, then \((S, \tau^d)\) is also.

(b) When \(I = \mathbb{N}\), \(S\) is metrized by

\[
\tau^\infty_1(x, y) = \sup_{i \in \mathbb{N}} \tau_i(p_i(x), p_i(y)) \wedge 1, \forall x, y \in S,
\]

or alternatively by

\[
\tau^\infty_2(x, y) = \sum_{i=1}^{\infty} 2^{-i+1} \tau_i(p_i(x), p_i(y)) \wedge 1, \forall x, y \in S.
\]

(9.1.2) If \(\{(S_i, \tau_i)\}_{i \in \mathbb{N}}\) are all complete, then \((S, \tau^\infty_1)\) and \((S, \tau^\infty_2)\) are also.

**Proposition 9.1.8.** Let \(I\) be a countable index set, \(\{S_i\}_{i \in I}\) be topological spaces and \(S = \prod_{i \in I} S_i\). Then, \(A\) is a metrizable subspace of \(S\) if and only if \(p_i(A)\) is a metrizable subspace of \(S_i\) for all \(i \in I\).

The following property of first-countable and metrizable spaces is indispensable.

**Fact 9.1.9.** Let \(E\) be a topological space, \(x \in E\) and \(A \subset E\). Then:

(a) If \(E\) is metrizable, then \(E\) is first-countable.

(b) If there exist \(\{x_n\}_{n \in \mathbb{N}} \subset A\) converging to \(x\) in \(E\), then \(x\) is a limit point of \(A\). The converse is true when \(E\) is first-countable space.

A subset \(A\) of metric space \((E, \tau)\) is said to be totally bounded if for any \(\epsilon \in (0, \infty)\), there exists an \(A_\epsilon \in \mathcal{P}_0(E)\) such that \(E = \bigcup_{x \in A_\epsilon} \{y \in E : \tau(x, y) < \epsilon\}\).

**Proposition 9.1.10.** Let \((E, \tau)\) be a metric space. Then:

(a) If \(A \subset E\) is totally bounded, \((A, \tau|_A)\) is a second-countable space.

(b) The union of finitely many totally bounded subsets of \(E\) is totally bounded.

(c) If \(A \subset E\) is totally bounded, then its closure is also.

**9.1.4. Polish, Lusin and Souslin spaces.** Polish, Lusin and Souslin spaces are topological variations of complete separable metric spaces. Homeomorphs of complete separable metric spaces are called Polish spaces. \(E\) is a Lusin space (resp. Souslin space) if \(E\) is a Hausdorff space and there exists a bijective (resp. surjective) \(f \in C(S; E)\) with \(S\) being a Polish space.

**Proposition 9.1.11.** The following statements are true:

\[
\text{\footnote{The notation \(\mathcal{P}_0(E)\) denotes one-dimensional projection on \(S\) for \(i \in I\).}}
\]

\[
\text{\footnote{The notion of limit point was mentioned in p.23.}}
\]

\[
\text{\footnote{\(\mathcal{P}_0(E)\) denotes the family of all finite subsets of \(E\).}}
\]
(a) Every Polish (resp. Lusin) space is a Lusin (resp. Souslin) space.
(b) Open and closed subsets of a Polish, Lusin or Souslin space are Polish, Lusin or Souslin subspaces, respectively.
(c) Subspaces of a Polish space are metrizable and second-countable.
(d) Subspaces of a Polish, Lusin or Souslin space are separable and hereditary Lindelöf.
(e) A metric space is separable if and only if its completion is a Polish space.
(f) The product space of countably many Polish, Lusin or Souslin spaces is a Polish, Lusin or Souslin space, respectively. In particular, $\mathbb{R}^\infty$ and its subspace $\mathbb{N}^\infty$ are Polish spaces.
(g) The intersection or union of countably many Souslin subspaces is a Souslin subspace.

9.1.5. Compactness. $E$ is compact if any $\{O_i\}_{i \in I} \subset \mathcal{O}(E)$ satisfying $E = \bigcup_{i \in I} O_i$ admits a finite subset $\{O_{i_1}, \ldots, O_{i_n}\}$ satisfying $E = \bigcup_{j=1}^n O_{i_j}$. $E$ is sequentially compact (resp. limit point compact) if any infinite subset of $E$ has a convergent subsequence (resp. a limit point). Let $A \subset E$ be non-empty. $A$ is a compact, sequentially compact or limit point compact subset of $E$ if $A$'s closure is compact. $E$ is locally compact if for any $x \in E$, there exist $K_x \in \mathcal{K}(E)$ and $O_x \in \mathcal{O}(E)$ such that $x \in O_x \subset K_x$.

Proposition 9.1.12. The following statements are true:
(a) Closed subsets of a compact space are compact. Moreover, compact subsets of a Hausdorff space are closed and hence Borel subsets.
(b) The union of finitely many compact subsets is compact. Moreover, any product space of compact spaces is a compact space.
(c) Compact metric spaces are complete.
(d) Hausdorff (resp. metrizable) compact spaces are $T_\delta$ (resp. Polish) spaces.
(e) Compact spaces are Lindelöf spaces. Moreover, the image of a compact space under any continuous mapping is a compact space.
(f) Compactness implies limit point compactness. Moreover, compactness, sequential compactness and limit point compactness are equivalent in metrizable spaces.

Corollary 9.1.13. Let $\{S_i\}_{i \in I}$ be topological spaces and $(S, \mathcal{A})$ be as in 2.7.22. Then:
(a) If $A_i \in \mathcal{K}(S_i)$ for all $i \in I$, then $\prod_{i \in I} A_i \in \mathcal{K}(S)$. If, in addition, $I$ is countable and $\{S_i\}_{i \in I}$ are all Hausdorff spaces, then $A_i \in \mathcal{B}(S_i)$ for all $i \in I$ and $\prod_{i \in I} A_i \in \mathcal{A}$.

7Having a compact closure is commonly known as relative compactness. Herein, we use the alternative terminology “precompactness” to distinguish this notion from the relative compactness about finite Borel measures.
8$\mathcal{K}(E)$ denotes the family of all compact subsets of $E$. 
(b) If \( A \in \mathcal{X}(S) \), then \( p_i(A) \in \mathcal{X}(S_i) \) for all \( i \in I \). If, in addition, \( \{S_i\}_{i \in I} \) are all Hausdorff spaces, then \( p_i(A) \in \mathcal{B}(S_i) \) for all \( i \in I \).

**Corollary 9.1.14.** Let \( I \) be a finite index set, \( \{S_i\}_{i \in I} \) be topological spaces, \( (S, O) \) be as in \(^{2.7.22}\) and \( A_i \in \mathcal{X}_{\sigma}(S_i) \) for all \( i \in I \). Then, \( \prod_{i \in I} A_i \in \mathcal{X}_{\sigma}(S) \).

We used the following connection of total boundedness and compactness in \(^{\S 6.5}\).

**Proposition 9.1.15.** Compactness of a metric space is equivalent to total boundedness plus completeness.

### 9.2. Point-separation properties of functions

\(^{2.2.4}\) introduced three functional separabilities of points: separating points, strongly separating points and determining point convergence. The following proposition specifies the relationship among them.

**Proposition 9.2.1.** Let \( E \) be a topological space, \( A \subset E \) be non-empty, \( D \subset \mathbb{R}^E \) and \( d \in \mathbb{N} \). Then:

(a) If \( \{x\} : x \in A \} \subset \mathcal{C}(E) \), especially if \( A \) is a Hausdorff subspace of \( E \), then \( D \) strongly separating points on \( A \) implies \( D \) separating points on \( A \).

(b) If \( D \) strongly separates points on \( A \), then \( D \) determines point convergence on \( A \). The converse is true when \( (A, \mathcal{E}_E(A)) \) is a Hausdorff space.

(c) \( D \) separates points on \( A \) if and only if \( (A, \mathcal{E}_D(A); \mathbb{R}) \) is a Hausdorff space.

(d) \( \mathcal{E}_D(A) \) is induced by pseudometrics \( \{p_f\}_{f \in \mathbb{R}} \). If \( D \) is countable, then \( (A, \mathcal{E}_D(A)) \) is a second-countable space pseudometrized by \( p_D \). If, in addition, \( D \) separates points on \( A \), then \( \mathcal{E}_D(A) \) is metrized by \( p_D \).

(e) If \( D|A \subset C(A, \mathcal{E}_E(A); \mathbb{R}) \) separates points (resp. strongly separates points) on \( A \), then \( (A, \mathcal{E}_E(A)) \) is a Hausdorff space (resp. \( \mathcal{E}_E(A) = \mathcal{E}_D(A) \)).

**Corollary 9.2.2.** Let \( E \) be a topological space, \( D \subset \mathbb{R}^E \) be countable and \( d \in \mathbb{N} \). Then, \( (\mathbb{R}^d, \mathcal{E}_D(\mathbb{R})^d) \) is a second-countable space pseudometrized by \( \rho_d \). If, in addition, \( D \) separates points on \( E \), then \( \mathcal{E}_D(\mathbb{R})^d \) is metrized by \( \rho_d \).

**Corollary 9.2.3.** Let \( E \) be a topological space and the members of \( D_0 \subset \mathbb{R}^E \) and \( D \subset \mathbb{R}^E \) are bounded. If \( D \subset \mathcal{D}(D_0) \), then \( \mathcal{E}_D(E) \subset \mathcal{E}_{D_0}(E) \). In particular, if \( D \) separates points or strongly separates points on \( E \), then \( D_0 \) does also.

The following property of compact spaces is important for this work.

**Lemma 9.2.4.** Let \( E \) be a compact space and \( D \subset C(E; \mathbb{R}) \). Then, \( E \) is a Hausdorff space and \( D \) strongly separates points on \( E \) if and only if \( D \) separates points on \( E \).

Below are two useful properties of the function class \( \Pi^d(D) \) introduced in \(^{2.2.3}\).

**Proposition 9.2.5.** Let \( E \) be a topological space and \( d \in \mathbb{N} \). Then:

\(^{2.3}\) The notation \( \mathcal{E}_D(A) \) was introduced in \(^{2.1.3}\).

\(^{2.4}\) The pseudometric \( \rho_D \) was defined in \(^{2.2.4}\) and \( \rho_{\{f\}} \) refers to \( \rho_D \) with \( D = \{f\} \). The meaning of \( \{p_f\} \) inducing \( \mathcal{E}_D(A) \) was explained in \(^{2.1.3}\).

\(^{2.5}\) The pseudometric \( \rho_D \) was defined in \(^{2.2.4}\).

\(^{2.6}\) Recall that \( \mathcal{C}(E, \mathbb{R}) \) refers to closure under supremum norm.
(a) Any $\mathcal{D} \subset \mathbb{R}^E$ satisfies 13
\[
\text{ag} \left[ \Pi^d(\mathcal{D}) \right] \subset \left[ C \left( E^d, \sigma(\mathcal{D})^d; \mathbb{R} \right) \cap M \left( E^d, \sigma(\mathcal{D})^d; \mathbb{R} \right) \right].
\]
If, in addition, the members of $\mathcal{D}$ are bounded, then,
\[
\text{ca} \left[ \Pi^d(\mathcal{D}) \right] \subset \left[ C_b \left( E^d, \sigma(\mathcal{D})^d; \mathbb{R} \right) \cap M_b \left( E^d, \sigma(\mathcal{D})^d; \mathbb{R} \right) \right].
\]
(b) If $\mathcal{D}$ separates points (resp. determines point convergence) on $E$, then $\Pi^d(\mathcal{D})$ separates points (resp. determines point convergence) on $E^d$.

Remark 9.2.6. Please be reminded that the $\sigma$-algebra $\sigma(\mathcal{D})$ induced by $\mathcal{D}$ is possibly smaller than the Borel $\sigma$-algebra $\mathcal{B}(E)$ induced by $\mathcal{D}$ (see Fact 10.1.5).

Remark 9.2.7. $\Pi^d(\mathcal{D})$ is defined in a way that we do not need $1 \in \mathcal{D}$ in Proposition 9.2.5 (b).

The following are two typical cases where one can select countably many (strongly) point-separating functions.

Proposition 9.2.8. Let $E$ be a topological space and $\mathcal{D} \subset C(E; \mathbb{R})$. Then:
(a) If $\{(x, x) : x \in E\}$ is a Lindelöf subspace of $E \times E$ and $\mathcal{D}$ separates points on $E$, then there exists a countable $\mathcal{D}_0 \subset \mathcal{D}$ that separates points on $E$.
(b) If $E$ is a second-countable space and $\mathcal{D}$ strongly separates points on $E$, then there exists a countable $\mathcal{D}_0 \subset \mathcal{D}$ that strongly separates points on $E$.

9.3. Tychonoff space and compactification

$E$ is a Tychonoff (or T3 1/2) space if $E$ is a Hausdorff space and for any $x \in E$ and $F \in \mathcal{C}(E)$ that excludes $x$, there exists an $f_{x,F} \in C(E; [0, 1])$ such that $f_{x,F}(x) = 0$ and (the image) $f_{x,F}(F) = \{1\}$. Besides the functional separability of points and closed sets above, Tychonoff spaces are also defined as the spaces whose topology is induced by some family of $\mathbb{R}$-valued functions, or alternatively by some family of pseudometrics.

Proposition 9.3.1. Let $E$ be a topological space. Then, the following statements are equivalent:
(a) $E$ is a Tychonoff space.
(b) $C(E; \mathbb{R})$ separates and strongly separates points on $E$.
(c) $C_b(E; \mathbb{R})$ separates and strongly separates points on $E$.
(d) $E$ is a Hausdorff space and $\mathcal{C}(E)$ is induced by a family of pseudometrics.
(e) $E$ is a Hausdorff space and $\mathcal{C}(E) = \mathcal{O}_D(E)$ for some $D \subset \mathbb{R}^E$.

Below are a few more properties of Tychonoff spaces.

Proposition 9.3.2. The following statements are true:
(a) T4 spaces, especially metrizable spaces, are Tychonoff spaces. Moreover, Tychonoff spaces are T3 spaces.
(b) Subspaces of a Tychonoff space are Tychonoff spaces.
(c) Any product space of Tychonoff spaces is a Tychonoff space.

13The notations “ag(·)” and “ca(·)” were defined in §2.2.3.
14The definition of $\Pi^d(\mathcal{D})$ refers to §2.2.14.
Tychonoff space has close link to compactification. $S$ is called a compactification of $E$ (or $S$ compactifies $E$) if $S$ is a compact Hausdorff space and $E$ is a dense subspace of $S$. $S$ is the Stone-Čech compactification of $E$ if $S$ compactifies $E$ and $\otimes C_b(E; \mathbb{R})$ extends to a member of $\text{imb}(S; R^{C_b(E; \mathbb{R})})$. The following proposition shows the equivalence of Tychonoff property, general compactifiability and Stone-Čech compactifiability.

**Proposition 9.3.3.** Let $E$ be a topological space. Then, the following statements are equivalent:

(a) $E$ has a compactification.

(b) $E$ is a Tychonoff space.

(c) $E$ has a unique Stone-Čech compactification up to homeomorphism\(^{16}\).

We prove the proposition above by the following compactification result, of which the Stone-Čech compactification is a special case.

**Lemma 9.3.4.** Let $E$ be a topological space and $D \subseteq R^E$ be a collection of bounded functions. Then, the following statements are equivalent:

(a) $D \subseteq C_b(E; \mathbb{R})$ separates and strongly separates points on $E$.

(b) $E$ admits a unique compactification $S$ up to homeomorphism such that $\otimes D$ extends to a homeomorphism between $S$ and the closure of $\otimes D(E)$ in $R^D$.

(c) $\otimes D \in \text{imb}(E; R^D)$.

**Remark 9.3.5.** If the $D$ above is countable, then the induced compactification has a homeomorph in $R^\infty$ and hence is metrizable. This is the foundation of the replication bases.

**Corollary 9.3.6.** Let $E$ be a topological space. Then, the following statements are equivalent:

(a) $E$ is metrizable and separable.

(b) There exists a countable subset of $D \subseteq C_b(E; \mathbb{R})$ that separates and strongly separates points on $E$.

(c) $E$ has a compactification that is homeomorphic to a compact subset of $R^\infty$.

(d) $E$ admits a metrizable compactification.

(e) $E$ is a dense subspace of some Polish space.

$S$ is the one-point compactification of $E$ if $S$ compactifies $E$ and $S \setminus E$ is a singleton. Every locally compact Hausdorff space is well-known to have a unique one-point compactifiable up to Homeomorphism.

**Proposition 9.3.7.** Let $E$ be a locally compact space. Then, the following statements are equivalent:

(a) $E$ is a Hausdorff space.

(b) $E$ has a unique one-point compactification up to homeomorphism.

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\(^{15}\)As noted in §3.1, this work only considers Hausdorff compactification.

\(^{16}\)"Unique up to homeomorphism" means any two spaces with the relevant property are homeomorphic.
(c) \( E \) is a Tychonoff space.

The next lemma is an analogue of Lemma 9.2.4 for locally compact spaces.

**Lemma 9.3.8.** Let \( E \) be a locally compact space and \( D \subset C_0(E; \mathbb{R}) \). Then, the following statements are equivalent:

(a) \( D \) separates points on \( E \).

(b) \( D^\Delta \doteq \{f^\Delta \}_{f \in D} \cup \{1\} \) is a subset of \( C(E^\Delta; \mathbb{R}) \) that separates and strongly separates points on \( E^\Delta \), where \( E^\Delta \) is a one-point compactification of \( E \) and \( f^\Delta \doteq \text{var}(f; E^\Delta, E, 0) \) for each \( f \in D \).

(c) \( E \) is a Hausdorff space and \( D \) strongly separates points on \( E \).

The local compactness of \( E \) leads to the following point-separability of \( C_0(E; \mathbb{R}) \).

**Proposition 9.3.9.** Let \( E \) be a locally compact space. Then, the following statements are equivalent:

(a) \( E \) is a Hausdorff space.

(b) \( C_c(E; \mathbb{R}) \) separates points on \( E \).

(c) \( C_c(E; \mathbb{R}) \) separates and strongly separates points on \( E \).

(d) \( C_0(E; \mathbb{R}) \) separates and strongly separates points on \( E \).

(e) \( E \) is a Tychonoff space.

### 9.4. Weak topology of finite Borel measures

The weak topology of \( \mathcal{M}^+(E) \) is induced by \( C_b(E; \mathbb{R})^* \). Hence, the Tychonoff properties of \( \mathcal{M}^+(E) \) and \( \mathcal{P}(E) \) are reduced to their Hausdorff properties.

**Proposition 9.4.1.** Let \( E \) be a topological space. Then, the following statements are equivalent:

(a) \( \mathcal{M}^+(E) \) and \( \mathcal{P}(E) \) are Tychonoff spaces.

(b) \( \mathcal{P}(E) \) is a Hausdorff space.

(c) \( C_b(E; \mathbb{R})^* \) separates points on \( \mathcal{P}(E) \).

We now investigate the connection between the Tychonoff property of \( E \) and those of \( \mathcal{M}^+(E) \) and \( \mathcal{P}(E) \). On one hand, we establish \cite{KX95} Theorem 2.1.4] without the restriction to Radon measures.

**Proposition 9.4.2.** Let \( E \) be a Tychonoff space and \( d \in \mathbb{N} \). Then, \( \mathcal{M}^+(E^d) \) and \( \mathcal{P}(E^d) \) are Tychonoff spaces and \( \text{mc}[\Pi^d(C_b(E; \mathbb{R}))] \) is separating on \( E^d \).

To show the proposition above, we prepare a lemma that relates \( E \)'s functional separabilities of points to those of \( \mathcal{M}^+(E) \) and \( \mathcal{P}(E) \).

**Lemma 9.4.3.** Let \( E \) be a topological space, \( D \subset M_0(E; \mathbb{R}) \), \( d \in \mathbb{N} \) and \( G \doteq \text{mc}[\Pi^d(D)] \). Then:

---

17 “\( \text{var}(\cdot) \)” was defined in Notation 4.1.1.
18 The notation “\( C_b(E; \mathbb{R})^* \)” was specified in §2.3.
19 The terminology “separating” was introduced in §2.3.
20 As mentioned in Remark 9.2.7, we need not impose \( 1 \in D \) in Lemma 9.4.3 by the definition of \( \Pi^d(D) \).

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(a) If \( D \subseteq C_b(E; \mathbb{R}) \) strongly separates points on \( E \), then \( \mathcal{G}^* \) separates points on \( \mathcal{P}(E^d) \) and \( \mathcal{G} \cup \{1\} \) is separating on \( E^d \).

(b) If \( D \) is countable and strongly separates points on \( E \), then \( \mathcal{G}^* \) separates and strongly separates points on \( \mathcal{P}(E^d) \), and \( \mathcal{G} \cup \{1\} \) is separating and convergence determining on \( E^d \).

On the other hand, we give an explicit example showing that the converse of Proposition 9.4.2 is not true.

Example 9.4.4. Example 3.2.8 (VII) and Example 3.2.12 (III) mentioned that \( R_K \) is a non-T3 (hence non-Tychonoff) topological refinement of \( R \) with \( \mathcal{B}(R_K) = \mathcal{B}(R) \). \( \mathcal{P}(R) \) is a Tychonoff space by Proposition 9.4.2. \( \mathcal{P}(R_K) \) is a Hausdorff topological refinement of \( \mathcal{P}(R) \) by Fact 10.1.24 (a) and Fact 9.1.1. Thus, \( \mathcal{P}(R_K) \) and \( \mathcal{M}^+(R_K) \) are Tychonoff spaces by Proposition 9.4.1 (a, b).

The two examples below illustrate that the Hausdorff property of \( E \) and those of \( \mathcal{M}^+(E) \) and \( \mathcal{P}(E) \) do not imply each other, where the Borel measurability of singletons and the distinctiveness of Dirac measures at distinct points play essential roles.

Example 9.4.5. Let \( E = \{1, 2, 3, 4\} \), \( A \doteq \{1, 2\} \), \( B \doteq \{3, 4\} \) and equip \( E \) with the topology \( \mathcal{O}(E) \doteq \{\varnothing, A, B, E\} \). Then, \( \mathcal{B}(E) = \mathcal{O}(E) = \mathcal{O}(E) \) and singletons in \( E \) are neither closed nor Borel. So, \( E \) is non-Hausdorff by Proposition 9.1.2 (a). Letting \( a_1 = \frac{\delta_1 + \delta_2}{2} \) and \( a_2 = \frac{\delta_3 + \delta_4}{2} \), we observe that \( C_b(E; \mathbb{R}) = \{a_1A + b_1B : a, b \in \mathbb{R}\} \) and \( \mathcal{P}(E) = \{a_1 = a_2 = \mu_a + (1 - a) \mu_b : a \in [0, 1]\} \). \( C_b(E; \mathbb{R})^* \) separates points on \( \mathcal{P}(E) \) since for any \( a_1, a_2 \in [0, 1] \),

\[
\int_E 1_A(x) \mu_{a_1}(dx) = a_1 = a_2 = \int_E 1_A(dx) \mu_{a_2}(dx)
\]

implies \( \mu_{a_1} = \mu_{a_2} \). Thus, \( \mathcal{P}(E) \) and \( \mathcal{M}^+(E) \) are Hausdorff by Proposition 9.4.1.

Example 9.4.6. Due to the limit of space, we refer the readers to [Mun90, §33, Exercise 11] for the non-trivial construction of a topological space \( E \) satisfying:

1. \( E \) is a T3 (hence Hausdorff) but non-Tychonoff space, and
2. there exist \( a \neq b \) in \( E \) such that \( f^*(\delta_a) = f(a) = f(b) = f^*(\delta_b) \) for all \( f \in C(E; \mathbb{R}) \).

\( \delta_a \) and \( \delta_b \) are distinct measures by the Hausdorff property of \( E \) and Proposition 9.4.7 (a) below, but they can not be separated by \( C_b(E; \mathbb{R})^* \). So, neither \( \mathcal{P}(E) \) nor \( \mathcal{M}^+(E) \) is Hausdorff by Proposition 9.4.1.

As long as the extreme non-Borel singletons are avoided, the Hausdorff property of (the usually more complicated space) \( \mathcal{P}(E) \) implies that of \( E \).

Proposition 9.4.7. Let \( E \) be a topological space satisfying \( \{\{x\} : x \in E\} \subseteq \mathcal{B}(E) \) and \( D \subseteq M_b(E; \mathbb{R}) \). Then:

(a) \( \delta_x \neq \delta_y \) for any distinct \( x, y \in E \).

(b) If \( D^* \) separates points on \( \mathcal{P}(E) \), then \( D \) separates points on \( E \).

(c) If \( \mathcal{P}(E) \) is a Hausdorff space, then \( C_b(E; \mathbb{R}) \) separates points on \( E \) and \( E \) is a Hausdorff space.

\( \delta_x \) denotes the Dirac measure at \( x \).

\( f^* \) was specified in §2.3.
Corollary 9.4.8. Let $E$ be a metrizable and separable space. Then, there exists a countable $\mathcal{D} \subset C_b(E; \mathbb{R})$ satisfying the following:
(a) $\mathcal{D}$ is closed under addition and multiplication.
(b) $\mathcal{D}$ separates and strongly separates points on $E$.
(c) $\mathcal{D}$ is separating and convergence determining on $E$.

Corollary 9.4.9. The following statements are equivalent:
(a) $E$ is a Tychonoff, and $\mathcal{M}^+(E)$ and $\mathcal{P}(E)$ are metrizable and separable spaces.
(b) $E$ is a metrizable and separable space.

The properties of $\mathcal{M}^+$ and $\mathcal{P}(E)$ below are vital.

Theorem 9.4.10. The following statements are true:
(a) If $E$ is a compact Hausdorff space, then $\mathcal{P}(E)$ is also.
(b) If $E$ is a Polish space, then $\mathcal{M}^+(E)$ and $\mathcal{P}(E)$ are also.

As noted in p.23, the sequential concepts “weak limit point” and “relative compactness” in $\mathcal{M}^+(E)$ may cause ambiguity in general, but one can get rid of that when $E$ is a metrizable space.

Proposition 9.4.11. Let $E$ be a topological space and $\Gamma \subset \mathcal{M}^+(E)$. Then:
(a) If $E$ is metrizable, then $\mathcal{M}^+(E)$ and $\mathcal{P}(E)$ are metrizable by the same metric.
(b) If $\nu$ is a weak limit point of $\Gamma$, then $\nu$ is a limit point of $\Gamma$ with respect to weak topology. The converse is true when $E$ is metrizable.
(c) If $E$ is metrizable, then the relative compactness of $\Gamma$ is equivalent to its precompactness with respect to weak topology.

The next lemma extends Theorem 2.3.12 (b) to the non-probabilistic case.

Lemma 9.4.12. Let $E$ be a Hausdorff space, $\Gamma \subset \mathcal{M}^+(E)$ be sequentially tight and $0 < a < b$ satisfy $\{\mu(E)\}_{\mu \in \Gamma} \subset [a, b]$. Then, $\Gamma$ is relatively compact and the total mass of any weak limit point of $\Gamma$ lies in $[a, b]$.

Moreover, given a perfectly normal (see e.g. [Mun00] §33, Exercise 6) space $E$, we equate the Borel sets of $\mathcal{M}^+(E)$ generated by its strong and weak topologies, which generalizes [BS89] Lemma 2.1.

Lemma 9.4.13. Let $E$ be a perfectly normal (especially metrizable or Polish) space. Then, $\mathcal{B}_{\mathcal{M}^+(E; \mathbb{R})}^*(\mathcal{M}^+(E)) = \mathcal{B}(\mathcal{M}^+(E))$.

9.5. Standard Borel space

This section reviews a few fundamental properties of standard Borel spaces and standard Borel subsets.
9.5. STANDARD BOREL SPACE

Fact 9.5.1. The following statements are true:

(a) Borel isomorphs of standard Borel spaces are still standard Borel spaces. In particular, Polish spaces, their Borel isomorphs and their Borel subspaces are standard Borel spaces.

(b) The cardinality of a standard Borel space can never exceed $\aleph(\mathbb{R})$.

Standard Borel spaces are Borel isomorphic to Borel subsets of Polish spaces. The latter turns out to be precisely the metrizable Lusin spaces.

Proposition 9.5.2. Let $E$ be a metrizable space. Then, the following statements are equivalent:

(a) $E$ is a Lusin space.

(b) $E$ has a Polish topological refinement $(E, \mathcal{U})$ with $\mathcal{B}(E) = \sigma(\mathcal{U})$.

(c) $E$ is separable and for any metrization $(E, \tau)$ of $E$, $E$ is a Borel subset of the completion of $(E, \tau)$.

(d) $E$ is a Borel subspace of some Polish space.

(e) There exist an $S \in C(\mathbb{N}^\infty)$ and a bijective $f \in C(S; E)$.

The key to prove the equivalence above is the preservation of Borel sets under bijective Borel measurable mappings. Here is a standard result about this.

Lemma 9.5.3. Let $E$ be a Lusin space, $S$ be a Polish space, $f \in M(S; E)$ and

(9.5.1) $\mathcal{U}_f = \{ O \subset E : f^{-1}(O) \in \mathcal{G}(S) \}$.

Then:

(a) If $f$ is continuous and bijective, then $f \in \text{hom}(S; E, \mathcal{U}_f)$ and $(E, \mathcal{U}_f)$ is a Polish topological refinement of $E$.

(b) If $E$ is metrizable (especially a Polish space) and $f$ is injective, then $f(B) \in \mathcal{B}(E)$ for all $B \in \mathcal{B}(S)$.

(c) If $E$ is metrizable (especially a Polish space) and $f$ is bijective, then $f \in \text{biso}(S; E)$.

Corollary 9.5.4. Lusin spaces (resp. Souslin spaces) are precisely the Hausdorff topological coarsenings of Polish spaces (resp. Lusin spaces).

From above we observe that a general (resp. metrizable) Lusin space coarsens some Polish space topologically but does not necessarily preserve (resp. does preserve) its Borel $\sigma$-algebra. By contrast, the next proposition shows that a general standard Borel space is a topological variant (not necessarily a coarsening or refinement) of some Polish space that preserves its Borel $\sigma$-algebra.

Proposition 9.5.5. Let $E$ be a topological space. Then, the following statements are equivalent:

(a) $E$ is a standard Borel space.

---

26 The notion of Borel subspace was introduced in Definition 2.1.2.

27 $\mathcal{U}_f$ is known as the “push-forward topology of $f$”. In any case, $f \in C(S; E, \mathcal{U}_f)$.

28 The notation “biso” was defined in §2.2.2.

29 The terminology “topological coarsening” was specified in §2.1.3.
(b) $E$ is Borel isomorphic to some metrizable Lusin space.

c) There exists a topology $\mathcal{U}_1$ on $E$ such that $(E, \mathcal{U}_1)$ is a metrizable Lusin space and $\mathcal{B}(E) = \sigma(\mathcal{U}_1)$.

d) There exists a topology $\mathcal{U}_2$ on $E$ such that $(E, \mathcal{U}_2)$ is a Polish space and $\mathcal{B}(E) = \sigma(\mathcal{U}_2)$.

e) $E$ is Borel isomorphic to some Polish space.

Given metrizability, standard Borel and Lusin properties becomes indifferent.

**Proposition 9.5.6.** Let $E$ be a metrizable space. Then, the following statements are equivalent:

(a) $E$ is a standard Borel space.

(b) $E$ is a Lusin space.

c) $E$ admits a Polish topological refinement $(E, \mathcal{U})$ satisfying $\mathcal{B}(E) = \sigma(\mathcal{U})$.

Our proof is based on the following interesting result which illustrates that a Borel measurable mapping may preserve some topological properties.

**Lemma 9.5.7.** Let $E$ be a standard Borel space and $S$ be a metrizable space. Then:

(a) If there is a bijective $f \in M(E; S)$, then $S$ is separable.

(b) If $E$ is metrizable, then $E$ is separable (hence second-countable).

The next proposition compares standard Borel and Borel subsets which are likely to be different in general topological spaces.

**Proposition 9.5.8.** Let $E$ be a topological space. Then:

(a) If $A \in B_s(E)$, then $B_E(A) \subset B^s(E)$. In particular, if $E$ is a standard Borel space, then $\mathcal{B}(E) \subset \mathcal{B}^s(E)$.

(b) If $E$ is a metrizable standard Borel space, especially if $E$ is a Polish space, then $\mathcal{B}(E) = \mathcal{B}^s(E)$.

If $E$ is compact and $S$ is Hausdorff, then any bijective $f \in C(E; S)$ belongs to $\text{hom}(E; S)$ and $S$ is also compact (see [Mun00, Theorem 26.6]). One of Kuratowski’s theorems provides a similar identification for bijective Borel measurable mappings from standard Borel spaces to metrizable spaces. Herein, we give a short proof for integrity.

**Proposition 9.5.9.** Let $E$ be a topological space, $S$ be a metrizable space, $f \in M(E; S)$ be injective and $A \in \mathcal{B}^s(E)$. Then, $f(A) \in \mathcal{B}^s(S)$ and $f|_A \in \text{biso}(A; f(A))$.

**Remark 9.5.10.** The $E$ in Lemma 9.5.3 (b, c) is standard Borel by Proposition 9.5.6 (a, b) and the Polish space $S$ in Lemma 9.5.3 satisfies $\mathcal{B}(S) = \mathcal{B}^s(S)$ by Proposition 9.5.8 (b). Hence, Proposition 9.5.9 generalizes Lemma 9.5.3 (b, c).

The next proposition is about the functional separabilities of points and probability measures on standard Borel spaces.

**Proposition 9.5.11.** Let $E$ be a standard Borel space. Then:

(a) There exists a countable subset of $M_b(E; \mathbb{R})$ that separates points on $E$. 

(b) If \( D \subset \mathbb{R}^E \) satisfies \( \mathcal{B}_D(E) = \mathcal{B}(E) \), then \( D \) separates points on \( E \).

(c) If \( D \subset M(E; \mathbb{R}) \) is countable and separates points on \( E \), then \( \sigma(D) = \mathcal{B}_D(E) = \mathcal{B}(E) \).

9.6. Skorokhod \( \mathcal{J}_1 \)-space

We start with several most essential properties of \( D(\mathbb{R}^+; E) \).

**Proposition 9.6.1.** Let \( E \) and \( S \) be Tychonoff spaces. Then:

(a) If \( D \subset C(E; \mathbb{R}) \) strongly separates points on \( E \) (especially \( D = C(E; \mathbb{R}) \)), then \( \{ \varpi(f) : f \in \alpha(D) \} \subset D(\mathbb{R}^+; \mathbb{R})^{D(\mathbb{R}^+; E)} \) satisfies \( \varpi(\alpha(D)) \in \text{imb} \left( D(\mathbb{R}^+; E); D(\mathbb{R}^+; \mathbb{R})^{\alpha(D)} \right) \) and

\[
\mathcal{J}(E) = \mathcal{O}_{\{ \varpi(f) : f \in \alpha(D) \}} \left( D(\mathbb{R}^+; E) \right).
\]

(b) If \( D \subset C_b(E; \mathbb{R}) \) separates points on \( E \), then \( \{ \alpha_{t,n}^f : f \in D, t \in \mathbb{Q}^+, n \in \mathbb{N} \} \) is a subset of \( C(D(\mathbb{R}^+; E); \mathbb{R}) \) separating points on \( D(\mathbb{R}^+; E) \) with each \( \alpha_{t,n}^f \) defined as in \( 3.3.8 \).

(c) \( D(\mathbb{R}^+; E) \) is a Tychonoff space.

(d) If \( f \in D \subset C(S; E) \), then

\[
\varpi(f) \in C \left( D(\mathbb{R}^+; S); D(\mathbb{R}^+; E) \right)
\]

and

\[
\varpi(D) \subset C \left( D(\mathbb{R}^+; S); D(\mathbb{R}^+; E)^D \right).
\]

(e) If \( E \) is a topological coarsening of \( S \), then \( D(\mathbb{R}^+; S) \subset D(\mathbb{R}^+; E) \) and \( \mathcal{J}(S) \supset \mathcal{O}_{D(\mathbb{R}^+; E)} \left( D(\mathbb{R}^+; S) \right) \).

**Remark 9.6.2.** By Proposition 9.6.1 (a), \( \mathcal{J}(E) \) is uniquely determined by \( \mathcal{O}(E) \) and does not depend on the choice of the pseudometrics in its definition.

**Corollary 9.6.3.** Let \( E \) be a Tychonoff space. Then, \( D(\mathbb{R}^+; A, \mathcal{O}_E(A)) \) is a topological subspace of \( D(\mathbb{R}^+; E) \) for any non-empty \( A \subset E \).

**Corollary 9.6.4.** The one-dimensional projections \( \mathcal{J} = \{ p_i \}_{i \in \mathbb{I}} \) on \( \mathbb{R}^I \) satisfy

\[
\varpi \left[ \alpha(\mathcal{J}) \right] \in \text{imb} \left[ D(\mathbb{R}^+; \mathbb{R}^I); D(\mathbb{R}^+; \mathbb{R})^{\alpha(\mathcal{J})} \right]
\]

and \( \mathcal{J} \)

\[
\varpi \left[ {\bigotimes} \alpha(\mathcal{J}) \right] \in C \left[ D(\mathbb{R}^+; \mathbb{R}^I); D \left( \mathbb{R}^+; \mathbb{R}^{\alpha(\mathcal{J})} \right) \right].
\]

The property below is well-known for compact subsets of \( D(\mathbb{R}^+; E) \).

---

30 “\( \varpi(f) \)” and “\( \varpi(D) \)” were defined in §2.2.1 while “\( \alpha(D) \)” was defined in §2.2.3
31 \( \mathcal{J}(E) \) denotes the Skorokhod \( \mathcal{J}_1 \)-topology of \( D(\mathbb{R}^+; E) \).
32 \( \mathbb{Q}^+ \) denotes non-negative rational numbers.
33 \( D(\mathbb{R}^+; \mathcal{J}) \) is a Tychonoff space by Proposition 9.3.2 (c), so \( D(\mathbb{R}^+; \mathbb{R}^{\alpha(\mathcal{J})}) \) satisfies our definition of Skorokhod \( \mathcal{J}_1 \)-spaces in §2.2.2.
Proposition 9.6.5. Let \( E \) be a Tychonoff space and \( K \in \mathcal{K}(D(\mathbb{R}^+; E)) \). Then, there exist \( \{K_n\}_{n \in \mathbb{N}} \subset \mathcal{K}(E) \) such that
\[
K \subset \bigcap_{n \in \mathbb{N}} \{ x \in D(\mathbb{R}^+; E) : x(t) \in K_n, \forall t \in [0, n) \}.
\]

The next lemma is about the finite-dimensional projections on \( D(\mathbb{R}^+; E) \).

Lemma 9.6.6. Let \( E \) be a Tychonoff space and \( T_0 \in \mathcal{P}_0(\mathbb{R}^+) \). Then:
(a) \( p_{T_0} \in M(D(\mathbb{R}^+; E); E^{T_0}, \mathcal{B}(E) \otimes T_0)^{2.3.10} \)
(b) \( 2.3.10 \) and \( 2.3.11 \) hold.
(c) \( p_{T_0} \) is continuous at \( x \in D(\mathbb{R}^+; E) \) whenever \( T_0 \subset \mathbb{R}^+ \setminus J(x) \).

Corollary 9.6.7. Let \( E \) be a Tychonoff space. Then, \( \mu \circ p_{T_0}^{-1} \) is a member of \( \mathcal{M}^+(E^{T_0}, \mathcal{B}(E) \otimes T_0) \) for all \( \mu \in \mathcal{M}^+(D(\mathbb{R}^+; E), \mathcal{B}(E) \otimes \mathbb{R}^+ | D(\mathbb{R}^+; E)) \) (especially \( \mu \in \mathcal{M}^+(D(\mathbb{R}^+; E)) \)) and \( T_0 \in \mathcal{P}_0(\mathbb{R}^+) \).

The following result is about the measurability of \( w_{t, \delta, T}^\prime \) on \( D(\mathbb{R}^+; E) \).

Proposition 9.6.8. Let \( E \) be a Tychonoff space and \( \delta, T \in (0, \infty) \). Then:
(a) \( w_{t, \delta, T}^\prime \in M(D(\mathbb{R}^+; E); \mathbb{R})^{35} \) if \( E \) allows a metrization \((E, \tau)\).
(b) \( w_{\rho(f, \delta, T)}^\prime \in M(D(\mathbb{R}^+; E); \mathbb{R}) \) for all \( f \in C(E; \mathbb{R}) \).

The next fact discusses the measurability issue in \( 2.3.12 \).

Fact 9.6.9. Let \( E \) be a Tychonoff space. If \( M(E; \mathbb{R}) \) has a countable subset separating points on \( E \), then
\[
\{ x \in D(\mathbb{R}^+; E) : t \in J(x) \} \in \mathcal{B}(E) \otimes \mathbb{R}^+ | D(\mathbb{R}^+; E), \forall t \in \mathbb{R}^+
\]
and \( J(\mu)^{36} \) is well-defined for all \( \mu \in \mathcal{M}^+(D(\mathbb{R}^+; E), \mathcal{B}(E) \otimes \mathbb{R}^+ | D(\mathbb{R}^+; E)) \) (especially \( \mu \in \mathcal{M}^+(D(\mathbb{R}^+; E)) \)).

The next proposition discusses the metrizability of \( D(\mathbb{R}^+; E) \).

Proposition 9.6.10. Let \( E \) be a metrizable space. Then:
(a) If \((E, \tau)\) is a metrization of \( E \), then \( D(\mathbb{R}^+; E) \) is metrized by \( \varphi^\tau \).
(b) If \( E \) is separable (especially a Polish space), then \( D(\mathbb{R}^+; E) \) is also separable and \( \mathcal{B}(D(\mathbb{R}^+; E)) = \mathcal{B}(E) \otimes \mathbb{R}^+ | D(\mathbb{R}^+; E) \).
(c) If a metric \( \tau \) completely metrizes \( E \), then \( \varphi^\tau \) completely metrizes \( D(\mathbb{R}^+; E) \).
(d) If \( E \) is a Polish space, then \( D(\mathbb{R}^+; E) \) is also.

The following three results relate convergence in \( D(\mathbb{R}^+; E) \) to that in the finite time horizon case. These results are contained explicitly or implicitly in standard texts like [JS03] and [Jak86]. Herein, we reestablish them for readers’ convenience.

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34 Herein, \( p_{T_0} \) denotes the projection on \( E^{R^+} \) for \( T_0 \subset \mathbb{R}^+ \) restricted to \( D(\mathbb{R}^+; E) \).
35 The notation “\( w_{t, \delta, T}^\prime \)” was defined in \( 2.2.1 \). The notation “\( w_{\rho(f, \delta, T)}^\prime \)” is defined by \( 2.2.3 \) with \( \tau = \rho(f) \) (resp. \( E = \mathbb{R} \) and \( \tau = |t| \)).
36 \( J(\mu) \), the set of fixed left-jump times of \( \mu \), was defined in \( 2.3.12 \).
Proposition 9.6.11. Let $E$ be metrizable, $\{y_k\}_{k \in \mathbb{N}_0} \subset D(\mathbb{R}^+; E)$ and \( L^\vartheta \).

(9.6.9) \( y_k^u \doteq \text{var}(y_k; [0, u + 1], [0, u], y_k(u)), \forall k \in \mathbb{N}_0, u \in (0, \infty). \)

Then for each $u \in \mathbb{R}^+ \setminus J(y_0)$,

(9.6.10) \( y_k \to y_0 \) as $k \uparrow \infty$ in $D(\mathbb{R}^+; E)$ implies \( L^\vartheta \).

(9.6.11) \( y_k^u \to y_0^u \) as $k \uparrow \infty$ in $D([0, u + 1]; E)$.

Lemma 9.6.12. Let $E$ be a Tychonoff space, $D \subset C(E; \mathbb{R})$ strongly separate points on $E$, $\Psi \doteq \mathcal{w}[\text{ac}(D)]$, \( \{y_k\}_{k \in \mathbb{N}_0} \subset D(\mathbb{R}^+; E)$, $u \in \mathbb{R}^+ \setminus J(y_0)$ and \( \{y_k^u\}_{k \in \mathbb{N}_0} \) be as in (9.6.9). Then,

(9.6.12) \( \Psi(y_k) \to \Psi(y_0) \) as $k \uparrow \infty$ in $D(\mathbb{R}^+; \mathbb{R})^{\text{ac}(D)}$ implies (9.6.11). In particular, (9.6.10) implies (9.6.11).

Lemma 9.6.13. Let $E$ be a Tychonoff space, \( \{y_k\}_{k \in \mathbb{N}_0} \subset D(\mathbb{R}^+; E)$ and \( y_k^u \) be as in (9.6.9) for each $k \in \mathbb{N}_0$ and $u \in (0, \infty)$. If $\mathbb{R}^+ \setminus J(y_0)$ is dense in $\mathbb{R}^+$, and if (9.6.11) holds for all $u \in \mathbb{R}^+ \setminus J(y_0)$, then (9.6.10) holds.

9.7. Càdlàg process

The following two facts compare $E$-valued càdlàg processes and $D(\mathbb{R}^+; E)$-valued random variables.

Fact 9.7.1. Let $E$ be a Tychonoff space and $\mu \in \mathcal{M}^+(D(\mathbb{R}^+; E))$ be the distribution of $D(\mathbb{R}^+; E)$-valued random variable $X$. Then:

(a) $X$ is an $E$-valued càdlàg process

(b) $\mu$ equals the restriction of $\text{pd}(X)|_{D(\mathbb{R}^+; E)}$ to $\sigma(\mathcal{F}(E))$.

(c) The finite-dimensional distribution of $X$ for each $T_0 \in \mathcal{P}(\mathbb{R}^+)$ is $\mu \circ p_T^{-1}$.

Fact 9.7.2. Let $E$ be a metrizable and separable space and $(\Omega, \mathcal{F}, \mathbb{P}; X)$ be an $E$-valued process. Then:

(a) If all paths of $X$ lie in $D(\mathbb{R}^+; E)$, then $X \in \mathcal{M}(\Omega, \mathcal{F}; D(\mathbb{R}^+; E))$.

(b) If $X$ is càdlàg, then there exists a $Y \in \mathcal{M}(\Omega, \mathcal{F}; D(\mathbb{R}^+; E))$ that is indistinguishable from $X$.

In particular, the statements above are true when $E$ is a Polish space.

The next lemma solves the measurability issue in (6.4.2) under mild conditions.

---

\(^{37}\mathbb{N}_0\) denotes the non-negative integers.

\(^{38}\)Our notation $y_k^u \doteq \text{var}(x; [0, u + 1], [0, u], x(u))$ represents the piecewise function defined by $y_k^u(t) \doteq y_k(t)$ for all $t \in [0, u]$ and $y_k^u(t) \doteq y_k(u)$ for all $t \in (u, u + 1]$.

\(^{39}\)As mentioned in (2.2.2), $D([0, u]; E)$ denotes the Skorokhod $\mathcal{J}_1$-space of all càdlàg mappings from $[0, u]$ to $E$.

\(^{40}\)Restriction of measure to sub-$\sigma$-algebra and $X$’s process distribution $\text{pd}(X)$ were specified in (2.1.2) and (2.5) respectively.

\(^{41}\)We arranged in 2.6 that $(\Omega, \mathcal{F}, \mathbb{P})$ and $(\Omega^i, \mathcal{F}^i, \mathbb{P}^i)_{i \in I}$ are complete probability spaces. Completeness of measure space was specified in (2.1.2)
9. BACKGROUND

LEMMA 9.7.3. Let $E$ be a Hausdorff space, $V$ be the family of all càdlàg members of $L^2(\Omega, \mathcal{F}, \mathbb{P}, X)$ be an $E$-valued càdlàg process and $T \in (0, \infty)$. Then, $\bigcap_{t \in [0,T]} X_t^{-1}(A) \in \mathcal{F}$ for all $A \in \mathcal{B}(E)$, especially for all $A \in \mathcal{F}(E)$ when $E$ is a Hausdorff space.

The next lemma treats the measurability issue in \cite{Sato09}.

LEMMA 9.7.4. Let $E$ be a topological space, $(\Omega, \mathcal{F}, \mathbb{P}, X)$ be an $E$-valued càdlàg process, $\tau$ be a pseudometric on $E$ and $\delta, T \in (0, \infty)$. Then, $u_{t, \delta, T}(\omega, X) \in M(\Omega, \mathcal{F}; R)$ in each of the following settings:

(a) $(E, \tau)$ is a metric space and $X \in M(\Omega, \mathcal{F}; D(R^+; E))$.

(b) $(E, \tau)$ is a separable metric space.

(c) $\tau = \rho(f)$ with $f \in C(\Omega; E)$.

(d) $\tau = \rho_D$ with $D \subset C(\Omega; E)$ being countable and separating points on $E$.

The following five results discuss the relationship among $\tau$-MCC\cite{Kou16} MCC, $D$-FMCC and WMCC for càdlàg processes.

FACT 9.7.5. Let $E$ be a topological space, $D \subset M(E; R)$ and $\{X^i\}_{i \in I}$ be $E$-valued processes such that $\{\varpi(f) \circ X^i\}_{f \in D, i \in I}$ are all càdlàg. Then:

(a) $\{X^i\}_{i \in I}$ satisfies $\rho(f)$-$\text{MCC}$ for all $f \in D$ if and only if $\{\varpi(f) \circ X^i\}_{i \in I}$ satisfies $|\cdot|$-$\text{MCC}$ for all $f \in D$.

(b) $\{X^i\}_{i \in I}$ satisfies $D$-FMCC if and only if $\{\varpi(f) \circ X^i\}_{i \in I}$ satisfies $|\cdot|$-$\text{MCC}$ for all $f \in \mathcal{A}(D)$.

In particular, the two statements above are true when $D \subset C(\Omega; E)$ and $\{X^i\}_{i \in I}$ are all càdlàg.

The next result is a version of \cite{Kou16} Proposition 14 for infinite time horizon.

PROPOSITION 9.7.6. Let $E$ be a Hausdorff space and $\{(\Omega^i, \mathcal{F}^i, \mathbb{P}^i; X^i)\}_{i \in I}$ be $E$-valued càdlàg processes. Then, the following statements are equivalent:

(a) $\{X^i\}_{i \in I}$ satisfies MCC.

(b) There exist a $D_1 \subset C(\Omega; R)$ and a $D_2 \subset C_0(\Omega; R)$ such that: (1) $\{X^i\}_{i \in I}$ satisfies $D_1$-FMCC, (2) $D_2 = \mathcal{A}(D_2)$ strongly separates points on $E$, and (3) for any $g \in D_2$ and $\epsilon, T > 0$, there exists an $f_{g, \epsilon, T} \in D_1$ satisfying

\begin{equation}
\sup_{i \in I} \mathbb{P}^i \left( \sup_{t \in [0,T]} |f_{g, \epsilon, T} \circ X^i_t - g \circ X^i_t| \geq \epsilon \right) \leq \epsilon.
\end{equation}

(c) There exist $D \subset C_0(\Omega; R)$ and $\{(\Omega^i, \mathcal{F}^i, \mathbb{P}^i; \zeta^i_{f, \epsilon, T})\}_{i \in I, f \in D, \epsilon, T > 0}$ such that:

(1) $D = \mathcal{A}(D)$ strongly separates points on $E$, and (2) for each $f \in \mathcal{A}(D)$ and $\epsilon, T > 0$, $R$-valued processes $\{\zeta^i_{f, \epsilon, T}\}_{i \in I}$ satisfy $|\cdot|$-$\text{MCC}$\cite{Kou16} and

\begin{equation}
\sup_{i \in I} \mathbb{P}^i \left( \sup_{t \in [0,T]} |f \circ X^i_t - \zeta^i_{f, \epsilon, T}| \geq \epsilon \right) \leq \epsilon.
\end{equation}

---

42$E$ need not be a Tychonoff space, so we avoid the notation $D(R^+; E)$ for clarity.

43The $E$ in (d) is a $\mathcal{D}$-baseable space.

44$\tau$-MCC, MCC, $D$-FMCC and WMCC were introduced in Definition 6.4.1.

45$D_2 = \mathcal{A}(D_2)$ means $D_2$ is closed under multiplication.

46$|\cdot|$-$\text{MCC}$ means MCC for the Euclidean norm metric $|\cdot|$. 
(d) There exists an \( D \subset C(E; \mathbb{R}) \) such that: (1) \( D \) strongly separates points on \( E \), and (2) \( \{ \varpi(\mathcal{D}_0) \circ X^i \}_{i \in I} \) satisfies \(|\cdot|\)-MCC and MCC\(^{[47]}\) for all \( \mathcal{D}_0 \in \mathcal{P}(D) \).

(e) There exists a \( D \subset C(E; \mathbb{R}) \) such that: (1) \( D \) strongly separates points on \( E \), and (2) \( \{ \varpi(g) \circ X^i \}_{i \in I} \) satisfies \(|\cdot|\)-MCC and MCC for all \( g \in \mathcal{ac}\{\{af : f \in D, a \in \mathbb{R}\}\} \)\(^{[48]}\).

**Corollary 9.7.7.** Let \( E \) be a Hausdorff space and \( \{(\Omega^i, \mathcal{F}^i, \mathbb{P}^i; X^i)\}_{i \in I} \) be \( E \)-valued càdlàg processes. Then, the following statements are equivalent:

(a) \( \{X^i\}_{i \in I} \) satisfies MCC.

(b) \( \{X^i\}_{i \in I} \) satisfies \( D \)-FMCC for some \( D = \mathcal{ac}(D) \subset C_b(E; \mathbb{R}) \) and \( D \) strongly separates points on \( E \).

(c) \( \{X^i\}_{i \in I} \) satisfies \( D \)-FMCC for some \( D = \mathcal{ac}(D) \subset C(E; \mathbb{R}) \) and \( D \) strongly separates points on \( E \).

(d) \( \{X^i\}_{i \in I} \) satisfies \( D \)-FMCC for some \( D \subset C(E; \mathbb{R}) \) and \( D \) strongly separates points on \( E \).

**Proposition 9.7.8.** Let \( E \) be a topological space, \( \{X^i\}_{i \in I} \) be \( E \)-valued processes and \( D \subset M(E; \mathbb{R}) \) be countable and separate points on \( E \). If \( \{\varpi(f) \circ X^i\}_{f \in D, i \in I} \) are all càdlàg, especially if \( \{X^i\}_{i \in I} \) are all càdlàg and \( D \subset C(E; \mathbb{R}) \), then \( \{X^i\}_{i \in I} \) satisfying \( \mathcal{p}_D \)-MCC is equivalent to \( \{\varpi(f) \circ X^i\}_{i \in I} \) satisfying \(|\cdot|\)-MCC for all \( f \in D \).

**Fact 9.7.9.** Let \( E \) be a Hausdorff space. If \( E \)-valued càdlàg processes \( \{X^i\}_{i \in I} \) satisfy MCC, then \( \{X^i\}_{i \in I} \) satisfies WMCC.

**Corollary 9.7.10.** Let \( E \) be a metrizable space and \( \{X^i\}_{i \in I} \) be \( E \)-valued càdlàg processes. Then:

(a) If \( (E, \tau) \) is a metrization of \( E \) and \( \{X^i\}_{i \in I} \) satisfies \( \tau \)-MCC, then \( \{X^i\}_{i \in I} \) satisfies MCC.

(b) If \( E \) is separable and \( \{X^i\}_{i \in I} \) satisfies MCC, then \( \{X^i\}_{i \in I} \) satisfies \( D \)-FMCC for some countable \( D = \mathcal{ac}(D) \subset C_b(E; \mathbb{R}) \) and \( D \) strongly separates points on \( E \). Moreover, \( \{X^i\}_{i \in I} \) satisfies \( \tau \)-MCC for some metrization \( (E, \tau) \) of \( E \).

The classical results below are quoted from \[EK86\] for ease of reference.

**Theorem 9.7.11** ([EK86] §3.7, Theorem 7.8]). Let \( E \) be a metrizable and separable space and \( \{X^i\}_{i \in I} \cup \{X\} \) be \( D(\mathbb{R}^+; E) \)-valued random variables. Then:

(a) \[1.0.6\] implies \[50\]

\[
X^n \overset{D(\mathbb{R}^+ \setminus J(X))}{\longrightarrow} X \quad \text{as} \quad n \uparrow \infty.
\]

(b) If \( \{X^n\}_{n \in \mathbb{N}} \) is relatively compact in \( D(\mathbb{R}^+; E) \) and \( \{6.2.1\} \) holds for some dense \( T \subset \mathbb{R}^+ \), then \[1.0.6\] holds.

---

\(^{[47]}\)The notion of MCC was specified in Definition 6.4.1.

\(^{[48]}\)The notation \( \mathcal{ac}(\cdot) \) was defined in 2.2.3.

\(^{[49]}\)Weak convergence and relative compactness of random variables were specified in §2.4.

\(^{[50]}\)\( J(X) \), the set of fixed left-jump times of \( X \) was defined in (2.5.8).
Theorem 9.7.12. Let $(E, \tau)$ be a complete separable metric space and $\{X^i\}_{i \in I}$ be $D(\mathbb{R}^+; E)$-valued random variables. Then, $\{X^i\}_{i \in I}$ is tight in $D(\mathbb{R}^+; E)$ if and only if $\{X^i\}_{i \in I}$ satisfies MCCC and $\tau$-MCC.

\[\text{ Tightness of random variables was specified in §2.4.}\]
CHAPTER 10

Miscellaneous

This chapter consists of auxiliary results related to this work. §10.1 contains a set of basic and general technicalities. §10.2 supplements the topics of Appendix §10.3 houses several auxiliary lemmas about replication which are used in Chapter 3. All notations, terminologies and conventions introduced before apply to this appendix. Proofs of many relatively simple results are omitted.

10.1. General technicalities

\textbf{Fact 10.1.1.} Let $E$ be the union of non-empty sets $\{A_n\}_{n \in \mathbb{N}}$. If $\sigma$-algebras $\mathcal{U}_1$ and $\mathcal{U}_2$ on $E$ satisfy $A_n \in \mathcal{U}_2$ and $\mathcal{U}_1|_{A_n} = \mathcal{U}_2|_{A_n}$ for all $n \in \mathbb{N}$, then $\mathcal{U}_1 \subset \mathcal{U}_2$.

\textbf{Proof.} We observe that

\[ \mathcal{U}_1 \supseteq \left\{ \bigcup_{n \in \mathbb{N}} B \cap A_n : B \in \mathcal{U}_1 \right\} \supset \left\{ \bigcup_{n \in \mathbb{N}} B_n : B_n \in \mathcal{U}_1|_{A_n}, \forall n \in \mathbb{N} \right\} \]

(10.1.1)

\[ = \left\{ \bigcup_{n \in \mathbb{N}} B_n : B_n \in \mathcal{U}_2|_{A_n}, \forall n \in \mathbb{N} \right\} \supset \sigma \left( \bigcup_{n \in \mathbb{N}} \mathcal{U}_2|_{A_n} \right) \subset \mathcal{U}_2. \]

\[ \square \]

\textbf{Fact 10.1.2.} Let $(E, \mathcal{U})$ be a measurable space, $A \in \mathcal{U}$ and $k \in \mathbb{N}$. If $f \in \mathfrak{R}^k$ satisfies $f|_A \in M(A, \mathcal{U}|_A; \mathfrak{R}^k)$, then $f1_A \in M(E, \mathcal{U}; \mathfrak{R}^k)$.\footnote{1}$f1_A$ denotes the indicator function of $A$.\footnote{2}Completeness of measure space and the notation “$\mathcal{N}(\mu)$” were specified in §2.1.2

\textbf{Fact 10.1.3.} Let $E$ and $S$ be non-empty sets, $y_0 \in A \subset E$, $f \in \mathcal{F}$ and $g \doteq \text{var}(f; S, f^{-1}(A), y_0)$. Then:

(a) $\{ x \in S : f(x) = g(x) \} \supset f^{-1}(A)$.

(b) If $(E, \mathcal{U})$ and $(S, \mathcal{A})$ are measurable spaces, $f \in M(S, \mathcal{A}; E, \mathcal{U})$, $A \in \mathcal{U}$ and $\{y_0\} \in \mathcal{U}$, then $g \in M(S, \mathcal{A}; A, \mathcal{U}|_A) \subset M(S, \mathcal{A}; E, \mathcal{U})$.

\textbf{Fact 10.1.4.} Let $E$ and $S$ be topological space, $\mu \in \mathcal{M}^+(E)$ and $A$ denote the set of discontinuity points of $f \in \mathcal{L}$. If $(E, \mathcal{B}, \nu)$ is the completion\footnote{2} of $(E, \mathcal{B}(E), \mu)$ and $A \in \mathcal{N}(\mu)$, then $f \in M(E, \mathcal{B}, S)$.

\textbf{Proof.} Fixing $O \in \mathcal{B}(S)$, we have that

\[ (f^{-1}(O) \setminus A) = (f|_{E\setminus A})^{-1}(O) \in \mathcal{B}(E\setminus A) \subset \mathcal{B}(E\setminus A) \subset \mathcal{U}. \]

Next, we recall the fact $A \in \mathcal{N}(\mu) \subset \mathcal{U}$, so $f^{-1}(O) \in \mathcal{U}$ and get by the continuity of $f|_{E\setminus A}$ that

\[ f^{-1}(O) \cap A \in \mathcal{N}(\mu) \subset \mathcal{U}. \]
\[
\sigma(C(E; \mathbb{R})), \text{ the Baire } \sigma\text{-algebra on } E \text{ is generally smaller than } \mathcal{B}(E).
\]

**FACT 10.1.5.** Let \( E \) be a topological space, \( S \) be a non-empty set and \( A \subset S \). Then, \( \sigma(\mathcal{D})|_A \subset \mathcal{B}_\mathcal{D}(A) \) for any \( \mathcal{D} \subset E^S \) and the equality holds if \( E \) is a second-countable space and \( \mathcal{D} \) is countable.

**Proof.** We have that
\[
\mathcal{B}_\mathcal{D}(A) \supset \sigma \left( \bigcup_{f \in \mathcal{D}} \{ f^{-1}(O) \cap A : O \in \mathcal{G}(E) \} \right)
\]
(10.1.4)
\[
= \sigma \left[ \bigcup_{f \in \mathcal{D}} \sigma \left( \{ f^{-1}(O) : O \in \mathcal{G}(E) \} \right) \right]
\]
\[
= \sigma \left( \{ f^{-1}(B) : B \in \sigma(\mathcal{G}(E)) = \mathcal{B}(E), f \in \mathcal{D} \} \right) = \sigma(\mathcal{D})|_A.
\]

If \( \{O_n\}_{n \in \mathbb{N}} \) is a countable topological basis of \( E \) and \( \mathcal{D} \) is countable, then
\[
\left\{ \bigcap_{f \in \mathcal{D}_n} f^{-1}(O_n) \cap A : n \in \mathbb{N}, \mathcal{D}_0 \in \mathcal{B}_0(\mathcal{D}) \right\}
\]
(10.1.5)
is a countable basis for \( \mathcal{B}_\mathcal{D}(A) \) by [Mun00] Lemma 13.1. Consequently,
\[
\mathcal{B}_\mathcal{D}(A) \subset \sigma \left( \left\{ \bigcap_{f \in \mathcal{D}_n} f^{-1}(O_n) \cap A : n \in \mathbb{N}, \mathcal{D}_0 \in \mathcal{B}_0(\mathcal{D}) \right\} \right)
\]
(10.1.6)
\[
\subset \sigma \left( \{ f^{-1}(B) : B \in \mathcal{B}(E), f \in \mathcal{D} \} \right) = \sigma(\mathcal{D})|_A.
\]
\]

**FACT 10.1.6.** Let \( E \) be a topological space and \( \{x_n\}_{n \in \mathbb{N}} \subset E \). If every convergent subsequence of \( \{x_n\}_{n \in \mathbb{N}} \) must converge to \( x \) as \( n \uparrow \infty \), and if any infinite subset of \( \{x_n\}_{n \in \mathbb{N}} \) has a convergent subsequence, then \( x_n \to x \) as \( n \uparrow \infty \) in \( E \).

**Lemma 10.1.7.** Let \( E \) and \( S \) be topological spaces, \( \mathcal{D} \subset E^S \) and equip \( V = \bigotimes \mathcal{D}(E) \) with the subspace topology \( \mathcal{O}_{\mathcal{D}}(V) \). Then:

(a) \( (\bigotimes \mathcal{D})^{-1} \in C(V; E) \) if and only if \( \mathcal{O}(E) \subset \mathcal{O}_{\mathcal{D}}(E) \) and \( \bigotimes \mathcal{D} \) is injective.

(b) \( \bigotimes \mathcal{D} \in \text{hom}(E; V) \) if and only if \( \mathcal{O}(E) = \mathcal{O}_{\mathcal{D}}(E) \) and \( \bigotimes \mathcal{D} \) is injective.

**Proof.** (a) We find by Fact 2.1.4 (b) that \( \bigotimes \mathcal{D} \in \text{imb}(E, \mathcal{O}_{\mathcal{D}}(E); V) \) if and only if \( \bigotimes \mathcal{D} \) is injective. Given the injectiveness of \( \bigotimes \mathcal{D}, \ (\bigotimes \mathcal{D})^{-1} \in C(V; E) \) precisely when \( \mathcal{O}(E) \) is coarser than \( \mathcal{O}_{\mathcal{D}}(E) \).

(b) is immediate by (a).

**Lemma 10.1.8.** Let \( E \) and \( S \) be topological spaces, \( f \in E^S \) and \( \mathcal{U}_f = \{ O \subset E : f^{-1}(O) \in \mathcal{O}(S) \} \). Then:

(a) If \( f \) is bijective, then \( f \in \text{hom}(S; (E, \mathcal{U}_f)) \).

(b) If \( f \in \text{biso}(S; E) \), then \( \mathcal{B}(E) = \sigma(\mathcal{U}_f) \).
PROOF. (a) \(\U_s\) is a topology, \(f \in C(S;E,\U_s)\) and
\[
\U_s = \{ f(B) : B \in \mathcal{O}(S) \}
\]
by the bijectiveness of \(f\), thus proving \(f^{-1} \in C(E,\U_s;S)\).
(b) \(f \in \text{biso}(S;E)\) satisfies (10.1.7) and further satisfies
\[
\mathcal{B}(E) = \{ f(B) : B \in \mathcal{B}(S) \} = \sigma \{ f(B) : B \in \mathcal{O}(S) \} = \sigma(\U_s).
\]
\(
\square
\)

Remark 10.1.9. The lemma above shows a Borel isomorphism can be turned into a homeomorphism by changing the topology generating the Borel \(\sigma\)-algebra.

Fact 10.1.10. Let \(I, E\) and \(S\) be non-empty sets and \(f \in S^E\). Then:
(a) If \(f\) is injective or surjective, then \(\varpi(I)\) is also.
(b) If \((E,\U)\) and \((S,\mathcal{A})\) are measurable spaces and \(f \in M(E,\U;S,\mathcal{A})\), then \(\varpi(f) \in M(E^I,\U^\otimes I;S^I,\mathcal{A}^\otimes I)\).
(c) If \(E\) and \(S\) are topological spaces and \(f \in C(E;S)\), then \(\varpi(f) \in C(E^I;S^I)\).

Proof. (a), (b) and (c) are immediate by definition, Fact 2.1.3 (b) and Fact 2.1.4 (b) respectively.

Fact 10.1.11. Let \(\{S_i\}_{i \in I}\) be topological spaces. Then, \(x_k \to x\) as \(k \uparrow \infty\) in \(\prod_{i \in I} S_i\) if and only if \(p_i(x_k) \to p_i(x)\) as \(k \uparrow \infty\) in \(S_i\) for all \(i \in I\).

Proof. This fact was justified in [Mun00 §19, Exercise 6].

Fact 10.1.12. Let \(I\) be an arbitrary index set and \(\{a_i, b_i\} \subset \mathbb{R}\) satisfy \(a_i < b_i\) for all \(i \in I\). Then, \(\bigcup_{i \in I} [a_i, b_i] \in \mathcal{B}(\mathbb{R})\).

Proof. For each \(\{i_1, i_2\} \subset I\), we define \(i_1 \sim i_2\) if there exist some \(I_0 \subset I\) such that \(\bigcup_{i \in I_0 \cup \{i_1, i_2\}} [a_i, b_i]\) is an interval. It is not difficult to see \(\sim\) defines an equivalence relation on \(I\). Let \(\{I_j : j \in J\}\) be the \(\sim\) equivalence classes of the members of \(I\) and \(A_j = \bigcup_{i \in I} [a_i, b_i]\) for each \(j \in J\). These \(\{A_j\}_{j \in J}\) are pairwise disjoint intervals by the definition of \(\sim\), so \(J\) is countable by [Mun00 §30, Exercise 13]. Hence, \(\bigcup_{i \in I} [a_i, b_i] = \bigcup_{j \in J} A_j \in \mathcal{B}(\mathbb{R})\).

Fact 10.1.13. Let \(E, S\) and \(\{S_i\}_{i \in I}\) be topological spaces and \(f \in C(E;S)\). Then:
(a) If \(x \in E^{R^+}\) is right-continuous, then \(x \in M(R^+;E)\).
(b) If \(x \in E^{R^+}\) is càdlàg and \(f \in C(E;S)\), then \(\varpi(f)(x) \in S^{R^+}\) is also càdlàg.
(c) \(\bigotimes_{i \in I} f_i : E \to S^i\) is càdlàg if and only if \(f_i : E \to S_i\) is càdlàg for all \(i \in I\).

Proof. (a) Note that
\[
x_n = \sum_{k=1}^{2^n} x \left( \frac{i}{2^n} \right) 1_{[\frac{i-1}{2^n}, \frac{i}{2^n}]} + x(n) 1_{[n, \infty)} \in M(R^+;E), \forall n \in \mathbb{N}
\]
and \(x_n \to x\) as \(n \uparrow \infty\) in \(E^{R^+}\). Then, \(x \in M(R^+;E)\) as pointwise convergence preserves measurability.
(b) and (c) are immediate by the definitions of \(\varpi(f), \bigotimes_{i \in I} \mathcal{O}(S_i)\) and product topology.
\(\square\)
FACT 10.1.14. Let $E$ be a non-empty set, $d, k \in \mathbb{N}$ and $\mathcal{D} \subset (\mathbb{R}^k)^E$ be a countable collection. Then, $\text{ac}(\mathcal{D})$ and $\text{mc}(\mathcal{D})$ are countable collections. When $k = 1$, $\text{mc}(\mathcal{D})$ and $\text{ag}_{\mathbb{Q}}(\mathcal{D})$ are also countable collections.

**Proof.** Let $\mathcal{D} = \{f_n\}_{n \in \mathbb{N}}$. For each $m \in \mathbb{N}$, we observe that $\sum_{i=1}^m f_{n_i} \mapsto (n_1, \ldots, n_m)$ defines an injective mapping from $\mathcal{D}_m \equiv \{\sum_{i=1}^m f_{n_i} : n_1, \ldots, n_m \in \mathbb{N}\}$ to the countable set $\mathbb{N}^m$, so $\mathcal{D}_m$ is countable. As a result, $\text{ac}(\mathcal{D}) = \mathcal{D}_1 \cup \mathcal{D}_2$ and $\text{ac}(\mathcal{D}) = \bigcup_{m \in \mathbb{N}} \mathcal{D}_m$ are both countable.

Next, we let $k = 1$ and observe that $\prod_{i=1}^m f_{n_i} \mapsto (n_1, \ldots, n_m)$ defines an injective mapping from $\mathcal{D}'_m \equiv \{\prod_{i=1}^m f_{n_i} : n_1, \ldots, n_m \in \mathbb{N}\}$ to the countable set $\mathbb{N}^m$, so $\mathcal{D}'_m$ is countable. As a result, $\text{mc}(\mathcal{D}) = \bigcup_{m \in \mathbb{N}} \mathcal{D}'_m$ is also countable.

Furthermore, we index $\text{mc}(\mathcal{D})$ by $\mathbb{N}$ as $\{g_j\}_{j \in \mathbb{N}}$ and observe that $ag_j \mapsto (j, a)$ defines an injective mapping from $\mathcal{D}_{\mathbb{Q}} \equiv \{ag_j : j \in \mathbb{N}, a \in \mathbb{Q}\}$ to the countable set $\mathbb{N} \times \mathbb{Q}$, so $\mathcal{D}_{\mathbb{Q}}$ is countable. As a result, $\text{ag}_{\mathbb{Q}}(\mathcal{D}) = \text{ac}(\mathcal{D}_{\mathbb{Q}})$ is also countable. □

FACT 10.1.15. Let $E$ be a non-empty set, $d \in \mathbb{N}$ and $\mathcal{D} \subset \mathbb{R}^E$. Then:

(a) $\Pi^d(\mathcal{D})$ is a countable collection whenever $\mathcal{D}$ is. Moreover,

$$
\Pi^d(\text{ac}(\mathcal{D})) \subset \text{ac}(\Pi^d(\mathcal{D})),
$$

$$
\Pi^d(\text{mc}(\mathcal{D})) = \text{mc}(\Pi^d(\mathcal{D})),
$$

(10.1.10)

$$
\Pi^d(\text{ag}_{\mathbb{Q}}(\mathcal{D})) \subset \text{ag}_{\mathbb{Q}}(\Pi^d(\mathcal{D})),
$$

$$
\Pi^d(\text{ag}(\mathcal{D})) \subset \text{ag}(\Pi^d(\mathcal{D})).
$$

(b) If the members of $\mathcal{D}$ are bounded, then those of $\Pi^d(\mathcal{D})$ are also. Moreover,

$$
\Pi^d(\text{cl}(\mathcal{D})) \subset \text{cl}(\Pi^d(\mathcal{D})),
$$

(10.1.11)

$$
\Pi^d(\text{ca}(\mathcal{D})) \subset \text{ca}(\Pi^d(\mathcal{D})).
$$

**Proof.** (a) If $\mathcal{D} = \{f_n\}_{n \in \mathbb{N}}$, then $\prod_{i=1}^k f_{p_i} \mapsto (n_1, \ldots, n_k)$ defines an injective mapping from $\mathcal{D}_k \equiv \{\prod_{i=1}^k f_{p_i} : n_i \in \mathbb{N}\} \subset \mathbb{R}^{kd}$ to $\mathbb{N}^k$ for each $k \in \{1, \ldots, d\}$, so $\mathcal{D}_k$ is countable. Hence, $\Pi^d(\mathcal{D}) = \bigcup_{k=1}^d \mathcal{D}_k$ is countable.

Letting $k \in \{1, \ldots, d\}$, $n_1, \ldots, n_k \in \mathbb{N}$ and $\{f_{i,j}\}_{1 \leq j \leq n_i, 1 \leq i \leq k} \subset \mathcal{D}$, one notes

$$
\sum_{i=1}^k \left( \sum_{j=1}^{n_i} f_{i,j} \right) \circ p_i = \sum_{j=1}^{n_1} \ldots \sum_{j_k=1}^{n_k} \left( \prod_{i=1}^k f_{i,j_i} \circ p_i \right) \in \text{ac}(\Pi^d(\mathcal{D})),
$$

(10.1.12)

and

$$
\prod_{i=1}^k \left( \prod_{j=1}^{n_i} f_{i,j} \right) \circ p_i = \prod_{1 \leq j \leq n_i, 1 \leq i \leq k} f_{i,j} \circ p_i \in \text{mc}(\Pi^d(\mathcal{D})).
$$

(10.1.13)

Letting $N \in \mathbb{N}$, $k_1, \ldots, k_N \in \{1, \ldots, d\}$ and $\{f_{i,j}\}_{1 \leq i \leq k_i, 1 \leq j \leq N} \subset \mathcal{D}$, we observe that

$$
\prod_{j=1}^{N} \left( \prod_{i=1}^{k_j} f_{i,j} \circ p_i \right) = \prod_{i=1}^{k^*} \left( \prod_{j \in J_i} f_{i,j} \right) \circ p_i \in \Pi^d(\text{mc}(\mathcal{D})),
$$

(10.1.14)

where $k^* \equiv \max\{k_1, \ldots, k_N\}$ and $J_i \equiv \{j \in \{1, \ldots, n\} : k_j \geq i\}$ for each $1 \leq i \leq k^*$.

Then, the first two lines of (10.1.10) follow by (10.1.12), (10.1.13) and (10.1.14).
Using the second line of (10.1.10), we have that
\[
\Pi^d(\{af : f \in \mathrm{mc}(\mathcal{D}), a \in Q\}) = \{af : f \in \Pi^d(\mathrm{mc}(\mathcal{D})), a \in Q\}
\]
(10.1.15)
\[\Pi^d(\mathcal{Q}(\mathcal{D})) = \Pi^d[\mathrm{ac}(\{af : f \in \mathrm{mc}(\mathcal{D}), a \in Q\})]
\]
(10.1.16)
which proves the third line of (10.1.10). The fourth line of (10.1.10) follows by a
similar argument with $Q$ replaced by $\mathcal{R}$.

(b) We fix $1 \leq k \leq d$. Boundedness is immediate. Suppose \{\(f_1,...,f_k\) \} \(\subset \) \(\mathrm{cl}(\mathcal{D})\). By Fact 9.1.9 (with $E = (\mathrm{cl}(\mathcal{D}), \|\cdot\|_\infty)$ and $A = \mathcal{D}$), there exist \{\(f_{i,n}\)\}_{1 \leq i \leq k, n \in \mathbb{N}} \(\subset \) \(\mathcal{D}\) such that $f_{i,n} \to f$ as $n \to \infty$ for all $1 \leq i \leq k$. We let $c = (\sup_{1 \leq i \leq k} \|f_i\|_\infty)^{k-1}$ and find that
\[
\lim_{n \to \infty} \left\| \prod_{i=1}^k f_i \circ p_i - \prod_{i=1}^k f_{i,n} \circ p_i \right\| \leq c \lim_{n \to \infty} \sum_{i=1}^k \|f_i - f_{i,n}\|_\infty = 0,
\]
thus proving the first line of (10.1.11). The second line of (10.1.11) is immediate by the first line (with $\mathcal{D} = \mathcal{Q}(\mathcal{D})$).

**Lemma 10.1.16.** Let $E$ be an open subspace of $S$ and $f \in C(E; \mathbb{R}^k)$. If for any $\epsilon \in (0, \infty)$, there exists an $A_\epsilon \subset E$ such that $A_\epsilon \in \mathcal{C}(S)$ and $\|f|_{E \setminus A_\epsilon}\|_\infty < \epsilon$, then $g = \var{f; E, 0}$ is a continuous extension of $f$ on $S$.

**Proof.** We need only prove the case of $k = 1$ and the general result follows by Fact 2.1.4(b). Let $\epsilon \in \mathbb{R}\setminus\{0\}$. From the facts
\[
g^{-1}(-\infty, \epsilon]\setminus A_\epsilon = \begin{cases} S\setminus A_\epsilon & \text{if } \epsilon > 0, \\ \emptyset & \text{if } \epsilon < 0 \end{cases}
\]
(10.1.18) and $E \in \mathcal{O}(S)$ it follows that
\[
g^{-1}(-\infty, \epsilon) = \begin{cases} f^{-1}(-\infty, \epsilon) \cup (S\setminus A_\epsilon) & \text{if } \epsilon > 0, \\ f^{-1}(-\infty, \epsilon) & \text{if } \epsilon < 0 \end{cases}
\]
(10.1.19) thus proving the continuity of $g$.

**Fact 10.1.17.** Let \(\{A_n\}_{n \in \mathbb{N}}\) be nested\(^3\) non-empty subsets of $E$ and $\mathcal{D}_n \subset \mathbb{R}^E$ separate points on $A_n$ for each $n \in \mathbb{N}$. Then, $\bigcup_{n \in \mathbb{N}} \mathcal{D}_n$ separates points on $\bigcup_{n \in \mathbb{N}} A_n$.

**Fact 10.1.18.** Let $A$ be a non-empty subset of $E$ and $\mathcal{D} \subset \mathbb{R}^E$ separate points on $A \subset E$. Then, $\var{\bigotimes \mathcal{D}}$ and $\var{\mathcal{D}}$ are both injective restricted to $\mathbb{A}$.\(^4\)

**Proof.** The injectiveness of $\var{\bigotimes \mathcal{D}}$ on $\mathbb{A}$ is immediate by Fact 10.1.10 (a) (with $E = A$, $I = \mathbb{R}^+$, $S = \mathbb{R}^E$ and $f = \bigotimes \mathcal{D}|_A$). Furthermore, we note that $\var{\mathcal{D}}(x) = \var{\mathcal{D}}(y)$ in $(\mathbb{A})^\mathcal{D}$ implies $\bigotimes \mathcal{D}[x(t)] = \bigotimes \mathcal{D}[y(t)]$ for all $t \in \mathbb{R}^+$. This indicates $x(t) = y(t)$ for all $t \in \mathbb{R}^+$, i.e. $x = y$.\(\square\)

\(^3\)"nested" was defined in Notation 4.1.1.

\(^4\)We explained the meaning of "nested" in Fact 3.3.4.
FACT 10.1.19. Let $E$ be a topological space. Then:
(a) $\mu_1 = \mu_2$ in $\mathcal{M}^+(E)$ if and only if $\mu_1/\mu_2(E) = \mu_2/\mu_2(E)$ in $\mathcal{P}(E)$ and $\mu_1(E) = \mu_2(E)$.
(b) \[ \text{[2.3.4]} \] holds if and only if $\lim_{n \to \infty} \mu_n(E) = \mu(E)$ and
\[ (10.1.20) \quad \frac{\mu_n}{\mu_n(E)} \Rightarrow \frac{\mu}{\mu(E)} \text{ as } n \uparrow \infty \text{ in } \mathcal{P}(E). \]

FACT 10.1.20. Let $E$ be a topological space and $1 \in \mathcal{D} \subset \mathcal{M}_b(E; \mathbb{R})$. Then:
(a) $\mathcal{D}$ is separating on $E$ if and only if $\mathcal{D}^*$ separates points on $\mathcal{P}(E)$.
(b) $\mathcal{D}$ is convergence determining on $E$ if and only if $\mathcal{D}^*$ determines point convergence on $\mathcal{P}(E)$.

PROOF. This result is immediate by Fact 10.1.19. \[ \square \]

FACT 10.1.21. Let $E$ be a topological space. Then, $\mathcal{P}(E) \in \mathcal{C}[\mathcal{M}^+(E)]$.

PROOF. Let $\mu$ be a limit point of $\mathcal{P}(E)$ in $\mathcal{M}^+(E)$. Then, there exist $\{\mu_p\}_{p \in \mathbb{N}} \subset \mathcal{P}(E)$ such that $\lim_{p \to \infty} |\mu(E) - 1| = \lim_{p \to \infty} |\mu(E) - \mu_p(E)| = 0$. \[ \square \]

FACT 10.1.22. If $x_n \to x$ as $n \uparrow \infty$ in topological space $E$, then $\delta_x \Rightarrow \delta_x$ as $n \uparrow \infty$ in $\mathcal{P}(E)$.

The generalized Portmanteau’s Theorem helps to establish the Continuous Mapping Theorem on general topological spaces.

THEOREM 10.1.23 (Continuous Mapping Theorem). Let $E$ and $S$ be topological spaces. Then:
(a) If $f \in C(E; S)$, then \[ \text{[2.3.4]} \] implies
\[ (10.1.21) \quad \mu \circ f^{-1} \Rightarrow \mu \circ f^{-1} \text{ as } n \uparrow \infty \text{ in } \mathcal{M}^+(S). \]
(b) If $E$ is a Tychonoff space and the set of discontinuity points of $f \in M(E; S)$ belongs to $\mathcal{N}(\mu)$, then \[ \text{[2.3.4]} \] implies (10.1.21).

PROOF. (a) follows by the fact that
\[ (10.1.22) \quad \lim_{n \to \infty} g^* (\mu_n \circ f^{-1}) = \lim_{n \to \infty} (g \circ f)^*(\mu_n) \]
\[ = (g \circ f)^*(\mu) = g^* (\mu \circ f^{-1}), \quad \forall g \in C_b(S; \mathbb{R}). \]

(b) Let $O \in \mathcal{E}(S)$, $A \subset E$ be the set of discontinuity points of $f$, $U$ be the interior of $f^{-1}(O)$ and $\nu$ be the completion of $\mu$. $\mu(U) = \mu \circ f^{-1}(O)$ since $(f^{-1}(O) \setminus A) \subset U \subset f^{-1}(O)$ and $A \in \mathcal{N}(\mu)$. It follows by the Tychonoff property of $E$ and Theorem 2.3.7 (a, c) that
\[ (10.1.23) \quad \mu \circ f^{-1}(O) = \mu(U) \leq \liminf_{n \to \infty} \mu_n(U) \leq \liminf_{n \to \infty} \mu_n \circ f^{-1}(O). \]

Now, (b) follows by (10.1.23) and Theorem 2.3.7 (a, c). \[ \square \]

FACT 10.1.24. Let $E$ be a topological space, $(E, \mathcal{U})$ be a topological coarsening of $E$ and $S \supseteq \mathcal{M}^+(E, \mathcal{U})$. Then:

\[ ^{\text{5We mentioned in 2.6 that any measure in this work have positive total mass.}} \]

\[ ^{\text{6The terminologies "separating" and "convergence determining" were introduced in 2.3}} \]

\[ ^{\text{7} \mathcal{M}^+(E) \text{ as aforementioned is not necessarily first-countable. So, } \mu \text{ being a limit point of } \mathcal{P}(E) \text{ does not necessarily imply a subsequence of } \mathcal{P}(E) \text{ converging weakly to } \mu.} \]
(a) \( (\mathcal{M}^+(E), \mathcal{O}_S[\mathcal{M}^+(E)]) \) and \( (\mathcal{P}(E), \mathcal{O}_S[\mathcal{P}(E)]) \) are topological coarsenings of \( \mathcal{M}^+(E) \) and \( \mathcal{P}(E) \) respectively.

(b) If \( \mu_n \Rightarrow \mu \) as \( n \uparrow \infty \) in \( \mathcal{M}^+(E) \), then \( \mu_n \Rightarrow \mu \) as \( n \uparrow \infty \) in \( S \).

**Proof.** (a) \( \mathcal{U} \subset \mathcal{O}(E) \) implies \( \mathcal{B}(E, \mathcal{U}) \subset \mathcal{B}(E) \), so every \( \mu \in \mathcal{M}^+(E) \) is naturally a member of \( S \). \( \mathcal{U} \subset \mathcal{O}(E) \) implies \( C_0(E, \mathcal{U}, \mathbb{R}) \subset C_0(E, \mathbb{R}) \). Then, (a) follows by the fact that

\[
\mathcal{O}[\mathcal{M}^+(E)] = \mathcal{O}_S(\mathcal{M}^+(E)) \quad \mathcal{M}^+(E) \subset C_0(E, \mathbb{R})
\]

and

\[
\mathcal{O}_S[\mathcal{M}^+(E)] = \mathcal{O}_S[\mathcal{M}^+(E)]
\]

(b) is immediate by (a). \( \square \)

**Fact 10.1.25.** Let \( E \) be a topological space, \( \mu_n \Rightarrow \mu_0 \) as \( n \uparrow \infty \) in \( \mathcal{M}^+(E) \) and \( (E, \mathcal{U}_n, \nu_n) \) be the completion of \( (E, \mathcal{B}(E), \mu_n) \) for each \( n \in \mathbb{N} \). Then, \( \nu_n \Rightarrow \nu_0 \) as \( n \uparrow \infty \) in \( \mathcal{M}^+(E) \).

**Fact 10.1.26.** Let \( E \) be a topological space and \( \mathcal{U} \) be a \( \sigma \)-algebra on \( E \). If \( \Gamma \subset \mathcal{M}^+(E, \mathcal{U}) \) is sequentially tight in \( A \subset \mathcal{P}(E) \), then there exists a \( \Gamma_0 \in \mathcal{B}(\Gamma) \) such that \( A \) is a common support of all members of \( \Gamma \). \( \square \)

**Proof.** Suppose none of \( \{\mu_n\}_{n \in \mathbb{N}} \subset \Gamma \) is supported on \( A \). The sequential tightness of \( \Gamma \) implies a subsequence \( \{\mu_{n_k}\}_{k \in \mathbb{N}} \) being tight in \( A \). In other words, \( \{\mu_{n_k}\}_{k \in \mathbb{N}} \) are all supported on some \( B \in \mathcal{K}(E) \) with \( B \subset \mathcal{A} \). Contradiction! \( \square \)

**Fact 10.1.27.** Let \( \mathcal{U} \) and \( \mathcal{A} \) be \( \sigma \)-algebras on topological spaces \( E \) and \( S \) respectively. If \( \Gamma \subset \mathcal{M}^+(E, \mathcal{U}) \) is tight in \( A \subset \mathcal{P}(E) \) and if \( f \in \mathcal{M}(E, \mathcal{U}; \mathcal{A}) \) satisfies \( f(K) \in \mathcal{K}(S) \) \( \forall K \in \mathcal{K}(E) \), then \( \{f^{-1}\mu \}_{\mu \in \Gamma} \) is tight (resp. \( \mathcal{m} \)-tight) in \( f(A) \). This implication is also true if tightness, \( \mathcal{K}(S) \) and \( \mathcal{K}(S) \) are replaced by \( \mathcal{m} \)-tightness, \( \mathcal{K}(E) \) and \( \mathcal{K}(S) \), respectively.

**Lemma 10.1.28.** Let \( (E, \mathcal{U}) \) be a measurable space, \( S_0 \subset S \subset E \), \( y_0 \in S \), \( X \) be a mapping from \( (\Omega, \mathcal{F}, \mathbb{P}) \) to \( E \) and \( Y = \inf(X) \). Then:

(a) If \( \mathbb{P}(X = Z) = 1 \) for some \( Z \in \mathcal{M}(\Omega, \mathcal{F}; S, \mathcal{U}|_S) \), then \( X \in \mathcal{M}(\Omega, \mathcal{F}; E, \mathcal{U}) \).

(b) If \( X \in \mathcal{M}(\Omega, \mathcal{F}; E, \mathcal{U}) \) satisfies \( \mathbb{P}(X) = 1 \), then \( \mathbb{P}(Y) = 1 \).

(c) If, in addition to the condition of (b), \( (S, \mathcal{U}) \) is a measurable space satisfying \( \mathcal{U}|_{S_0} \subset \mathcal{A} \), then \( Y \in \mathcal{M}(\Omega, \mathcal{F}; S, \mathcal{U}) \).

**Proof.** (a) Let \( \Omega_0 = \{\omega \in \Omega : X(\omega) = Z(\omega)\} \). It follows by \( \mathbb{P}(Z) = 1 \) and the completeness of \( (\Omega, \mathcal{F}, \mathbb{P}) \) that \( \Omega_0 \in \mathcal{F} \) and \( X^{-1}(A) \subset \Omega_0 \in \mathcal{F} \) for all \( A \in \mathcal{F} \). Hence, we have that

\[
X^{-1}(A) = [Z^{-1}(A \cap S) \cap \Omega_0] \cup (X^{-1}(A) \setminus \Omega_0) \in \mathcal{F}, \forall A \in \mathcal{U}.
\]

(b) We find \( \mathbb{P}(X = Y) = 0 \) by Fact 10.1.3 (a) (with \( (S, \mathcal{A}) = (\Omega, \mathcal{F}) \) and \( A = S \)) and the completeness of \( (\Omega, \mathcal{F}, \mathbb{P}) \).

---

8 Recall that \( \{\nu_n \} \rightarrow \nu_0 \) means \( \{\nu_n \} \rightarrow \nu_0 \) converges weakly to \( \nu_0 \) as members of \( \mathcal{M}^+(E) \), which is well-defined since \( \mathcal{U}_n \supset \mathcal{B}(E) \) and \( f^*(\nu_n) = f^*(\mu_n) \) for all \( n \in \mathbb{N} \) and \( f \in C_0(E, \mathbb{R}) \).

9 The notion of sequential tightness was introduced in Section 10.1.3.1.

10 The definition of tightness and \( \mathcal{m} \)-tightness for possibly non-Borel measures are in Definition 2.3.3.

11 We mentioned in 2.3.2 that \( (\Omega, \mathcal{F}, \mathbb{P}) \) denotes a complete probability space. Completeness of measure space was specified in 2.1.2.
(c) We fix $A \in \mathcal{U}'$ and find $A \cap S_0 \in \mathcal{U}|_{S_0}$ by $\mathcal{U}'|_{S_0} = \mathcal{U}|_{S_0}$. It follows by $\mathbb{P}(X \in S_0) = 1$ and the completeness of $(\Omega, \mathcal{F}, \mathbb{P})$ that $X^{-1}(S_0) \in \mathcal{F}$ and $Y^{-1}(A \cap S) \setminus X^{-1}(S_0) \in \mathcal{M}(\mathbb{P}) \subset \mathcal{F}$. Hence, we have that

\[
(10.1.26) \quad Y^{-1}(A \cap S) = [X^{-1}(A) \cap X^{-1}(S_0)] \cup [Y^{-1}(A \cap S) \setminus X^{-1}(S_0)] \in \mathcal{F}.
\]

\[\Box\]

**FACT 10.1.29.** Let $E$ and $S$ be topological spaces, $\{X^i\}_{i \in \mathcal{I}}$ and $X$ be $E$-valued processes defined on stochastic basis $(\Omega, \mathcal{F}, \{\mathcal{G}_t\}_{t \geq 0}, \mathbb{P})$ and $f \in M(E; S)$. Then:

(a) If $X$ is a general, $\mathcal{G}_t$-adapted, measurable or $\mathcal{G}_t$-progressive process, then $\varphi(f) \circ X$ is an $S$-valued process with the corresponding measurability.

(b) If $\{X^i\}_{i \in \mathcal{I}}$ are general, $\mathcal{G}_t$-adapted, measurable or $\mathcal{G}_t$-progressive processes, then $\{\bigotimes_{i \in \mathcal{I}} X^i_t\}_{t \geq 0}$ is an $E^1$-valued process with the corresponding measurability.

**PROOF.** This result follows by Fact \[10.1.10\] (b), Fact \[2.1.3\] (b), Fact \[2.5.2\] (b) and the definitions of measurable processes, $\mathcal{G}_t$-progressive processes and product topology. \[\Box\]

**PROPOSITION 10.1.30.** Let $E$ be a topological space and $X$ and $Y$ be $E$-valued processes defined on stochastic basis $(\Omega, \mathcal{F}, \{\mathcal{G}_t\}_{t \geq 0}, \mathbb{P})$. Then:

(a) If $X$ has càdlàg paths (resp. is a càdlàg process), then it is (resp. is indistinguishable from) an $E$-valued progressive process.

(b) If $X$ is progressive and $\mathcal{G}_t$-adapted, then it is $\mathcal{G}_t$-progressive.

(c) If $X$ is $\mathcal{G}_t$-progressive, then it is $\mathcal{G}_t$-adapted and measurable.

(d) If $X$ is measurable, then $X(\omega) \in M(\mathbb{R}^+; E)$ for all $\omega \in \Omega$.

(e) If $X$ and $Y$ are modifications of each other, then $\mathcal{F}^X = \mathcal{F}^Y$.

(f) If $X$ and $Y$ are indistinguishable, then they are modifications of each other.

If, in addition, $X$ is a measurable, $\mathcal{G}_t$-progressive, progressive or càdlàg process, then $Y$ is also.

(g) If $\inf_{t \in T} \mathbb{P}(X_t = Y_t) = 1$ for some dense $T \subset \mathbb{R}^+$, and if $X$ and $Y$ are càdlàg, then $X$ and $Y$ are indistinguishable.

(h) If $X$ is càdlàg, then it is indistinguishable from any of its càdlàg modifications and such modification is at most unique up to indistinguishability.

**PROOF.** The well-known facts above are treated in standard texts like \[EK86\] Chapter 2, \[Pro90\] Chapter 1 and \[Nik06\] for $E$ being a Euclidean or metric space. An inspection into their proofs shows that there is no problem to make $E$ a general topological space. \[\Box\]

**FACT 10.1.31.** Let $E$ and $S$ be topological spaces, $\{X^i\}_{i \in \mathcal{I}}$ and $X$ be $E$-valued càdlàg processes defined on $(\Omega, \mathcal{F}, \mathbb{P})$ and $f \in C(E; S)$. Then:

(a) $\varphi(f) \circ X$ is an $S$-valued càdlàg process.

(b) If $\mathcal{I}$ is countable, then $\{\bigotimes_{i \in \mathcal{I}} X^i_t\}_{t \geq 0}$ is an $E^1$-valued càdlàg process.
(c) If $S$ is a topological coarsening of $E$, then $X$ is an $S$-valued càdlàg process.

Proof. This result is immediate by Fact \ref{10.1.29} and Fact \ref{10.1.13} (b, c). □

Fact 10.1.32. Let $E$ be a topological space, $T \subset \mathbb{R}^+$ and $\{(\Omega^n, \mathcal{F}^n, \mathbb{P}^n, X^n)\}_{n \in \mathbb{N}}$ and $(\Omega, \mathcal{F}, \mathbb{P}, X)$ be $E$-valued processes. Then:

(a) If $\{X^n\}_{n \in \mathbb{N}}$ is $(T, D)$-FD, then $\{\mathbb{E}^n[f \circ X^n]_T\}_{n \in \mathbb{N}}$ is a convergent sequence in $\mathbb{R}$ for all $f \in \text{mc}[\Pi^{T_0}(D)]$ and $T_0 \in \mathcal{P}_0(T)$.

(b) If $\{X^n\}_{n \in \mathbb{N}}$ is $(T, D)$-AS, then $\lim_{n \to \infty} \mathbb{E}^n[f \circ X^n_0 - f \circ X^n_{T_0+c}] = 0$ for all $c \in (0, \infty)$, $f \in \text{mc}[\Pi^{T_0}(D)]$ and $T_0 \in \mathcal{P}_0(T)$.

Proof. This result is immediate by the Bolzano-Weierstrass Theorem and Fact \ref{10.1.6} (with $E = \mathbb{R}$ and $x_n = \mathbb{E}^n[f \circ X^n_0]$ or $\mathbb{E}^n[f \circ X^n_0 - f \circ X^n_{T_0+c}]$). □

Fact 10.1.33. Let $E$ be a topological space, $T \subset \mathbb{R}^+$ and $\{(\Omega^i, \mathcal{F}^i, \mathbb{P}^i, X^i)\}_{i \in I}$ be $E$-valued processes. If for each $T_0 \in \mathcal{P}_0(T)$, there exists some $I_{T_0} \in \mathcal{P}_0(I)$ such that $\mu_{T_0,i} = \mathbb{P}^i(\{X^i_{T_0} - 1\})_i \in \nu_I T_{\mathbb{R}}$ has at most one weak limit point, then $\mathbb{P}^i(X^i)_{i \in I}$ is at most a singleton.

Proof. Suppose $(\Omega, \mathcal{F}, \mathbb{P}; Y) \in \mathbb{P}_\mathbb{R}((X^i)_{i \in I})$ for each $j = 1, 2$. Fixing $T_0 \in \mathcal{P}_0(T)$, there exist $\nu_1, \nu_2$ such that $\nu_1$ and $\nu_2$ are both weak limit points of $(\mu_{T_0,i})_i \in \nu_I T_{\mathbb{R}}$. So, $\nu_1 = \nu_2$ and hence $\mathbb{P}^i(Y_{T_0}^i) = \mathbb{P}^i(Y_{T_0}^j)$. □

Fact 10.1.34. Let $E$ be a topological space, $(\Omega, \mathcal{F}, \mathbb{P}; X)$ be an $E$-valued process and $T \subset \mathbb{R}^+$. If there exists an $R^D$-valued càdlàg process $(\Omega, \mathcal{F}, \mathbb{P}; \zeta)$ such that

\[ \inf_{i \in I} \mathbb{P}\left(D \circ X_i = \zeta_i\right) = 1, \]

then $X$ is $(T, D)$-càdlàg. The converse is true when $D$ is a countable collection.

Proof. $(\zeta^f_1, \zeta^f_1)$ are $\mathbb{R}$-valued càdlàg processes satisfying

\[ \inf_{i \in I} \mathbb{P}\left(p_f \circ \zeta_i = \zeta^f_1 = f \circ X_i = p_f \circ D \circ X_i, \forall f \in D\right) \]

\[ = \inf_{i \in I} \mathbb{P}\left(\zeta_i = D \circ X_i\right) = 1 \]

by Fact \ref{2.1.4} (a) and Fact \ref{10.1.31} (a) (with $E = R^D$ and $f = p_f$).

Conversely, we suppose $D$ is countable and $R^D$-valued càdlàg processes $\{\zeta^f_i\}_{i \in D}$ satisfy (6.3.1). Letting $\zeta_t = \bigotimes_{f \in D} \zeta^f_t$ for each $t \in [0, \infty)$, we find that $\{\zeta_t\}_{t \geq 0}$ is an $R^\infty$-valued càdlàg process satisfying (10.1.28) by the countability of $D$, Fact \ref{10.1.31} (b) (with $I = D$, $i = f$ and $X_i = \zeta^f_i$) and (6.3.1). □

Lemma 10.1.35. Let $E$ be a topological space, $\{(\Omega^i, \mathcal{F}^i, \mathbb{P}^i; X^i)\}_{i \in I}$ be $E$-valued measurable processes, $T_k \uparrow \infty$, $\{A_p\}_{p \in \mathbb{N}} \subset \mathcal{B}(E)$ and $\{(\Omega^i, \mathcal{F}^i, \mathbb{P}^i; X^i; T_k) = \text{r} \mathbb{P}^i T_k (X^i)^{17}\}$ for each $i \in I$ and $k \in \mathbb{N}$. If

\[ \inf_{i \in I, k \in \mathbb{N}} \mathbb{P}^i T_k \left(X_{i,T_k}^i \in A_p\right) \geq 1 - 2^{-p}, \forall p \in \mathbb{N}, \]

\footnotetext{14}{The notions of $(T, D)$-FDC and $(T, D)$-AS were introduced in \textbf{Definition 6.2.6}.}

\footnotetext{15}{The notation $\mathbb{P}^i(X^n)_{n \in \mathbb{N}}^\omega$ was introduced in \textbf{6.2.2} and stands for the family of all equivalence classes of finite-dimensional limit points of $\{X^n\}_{n \in \mathbb{N}}$ along $T$.}

\footnotetext{16}{Recall that $p_f$ denotes the projection on $\mathbb{R}^D$ for $f \in D$.}

\footnotetext{17}{Randomly advanced process and related notations were introduced in \textbf{7.3}.
then for each $T_0 \in \mathcal{P}_0(\mathbb{R}^+)$ with $d \geq \mathbb{N}(T_0)$, there are \( \{N_{T_0,p}\}_{p \in \mathbb{N}} \subset \mathbb{N} \) such that

\[
(10.1.30) \quad \inf_{i \in I, k > N_{T_0,p}} \mathbb{P}^i T_k \left( X_{i,T_k}^A d \right) \geq 1 - (d + 1)2^{-p}, \forall p \in \mathbb{N}.
\]

**Proof.** Let $T_0 = \{ t_1, \ldots, t_d \}$, $t = \sum_{j=1}^d t_j$, $N_{T_0,0} = 0$ and $T_{N_{T_0,0}} = 0$. Define \( \{N_{T_0,p}\}_{p \in \mathbb{N}} \) inductively by

\[
(10.1.31) \quad N_{T_0,p} = \min \{ k \in \mathbb{N} : T_k > (2^{p+1}t_d) \lor T_{N_{T_0,p-1}} \}, \forall p \in \mathbb{N}.
\]

For each $p \in \mathbb{N}$, it follows by \( (10.1.29) \) and \( (10.1.31) \) that

\[
(10.1.32) \quad \inf_{i \in I, k > N_{T_0,p}} \mathbb{P}^i T_k \left( X_{i,T_k}^A d \right)
\]

\[
geq 1 - \sup_{i \in I, k > N_{T_0,p}} \sum_{j=1}^d \mathbb{P}^i T_k \left( X_{t_j}^A \right)
\]

\[
\geq 1 - d \sup_{i \in I, k > N_{T_0,p}} \mathbb{P}^i T_k \left( X_{0}^A \right)
\]

\[
- \sup_{i \in I, k > N_{T_0,p}} \frac{d}{t_k} \int_{[0,t] \cup [T_k, T_k + t]} \mathbb{E} \left[ X_i^2 \right] \mu_i \, d\tau
\]

\[
\geq 1 - d2^{-p} - \frac{2td}{2^{p+1}td} = 1 - (d + 1)2^{-p}.
\]

\[\Box\]

**Lemma 10.1.36.** Let $E$ be a topological space, $(\Omega, \mathcal{F}, \mathbb{P}; X)$ be an $E$-valued càdlàg process, $\epsilon, \delta, T, c \in (0, \infty)$ and $(\Omega, \mathcal{F}, \mathbb{P}^T; X^T) = \text{rap}_T(X)$. Then:

(a) $\xi^T = \{ X_{\tau+1} \}_{\tau \geq 0}$ well defines an $E$-valued càdlàg process for all $\tau \in \mathbb{R}^+$.

(b) $X^T$ is an $E$-valued càdlàg process.

(c) If $(E, \tau)$ is a separable metric space, then

\[
(10.1.33) \quad \mathbb{P} \left( \left\{ w_{t,\delta,c}^T \circ \xi^T \geq \epsilon \right\} \right) = \mathbb{P}^T \left( w_{t,\delta,c}^T \circ X^T \geq \epsilon \right).
\]

**Proof.** \( \{ \xi^T \}_{\tau \in \mathbb{R}^+} \) are $E$-valued processes by Fact 2.5.2 (b). Letting $\Omega_0 = \{ \omega \in \Omega : X(\omega) \text{ is càdlàg} \}$, we find that

\[
(10.1.34) \quad \{ \omega \in \Omega : \xi^T(\omega) \text{ is càdlàg} \} \supset \Omega_0, \forall \tau \in \mathbb{R}^+
\]

and

\[
(10.1.35) \quad \left\{ (\tau, \omega) \in \Omega \times \mathbb{R}^+ \right. \text{is càdlàg} \supset \mathbb{R}^+ \times \Omega_0.
\]

Then, (a, b) follows by \( (10.1.34), (10.1.35) \) and the fact $\mathbb{P}^T(\mathbb{R}^+ \times \Omega_0) = \mathbb{P}(\Omega_0) = 1$.

(c) It follows by (a), (b) and Lemma 9.7.4 (b) (with $X = \xi^T$ or $X^T$) that $w_{t,\delta,c}^T \circ \xi^T \in M(\Omega, \mathcal{F}; R)$ for all $\tau \in \mathbb{R}^+$ and $w_{t,\delta,c}^T \circ X^T \in M(\Omega, \mathcal{F}; R)$. Hence, both sides of \( (10.1.33) \) are all well-defined and \( (10.1.33) \) is true since $\xi^T(\omega) = X^T(\tau, \omega)$ for all $(\tau, \omega) \in \Omega$.

\[\Box\]

**Lemma 10.1.37.** Let $E$ be a topological space, $A \subset E$, $D \subset M_b(E; R)$ and $(\Omega, \mathcal{F}, \mathbb{P}; X)$ be an $E$-valued measurable process, $T_k \uparrow \infty$ and $(\Omega, \mathcal{F}, \mathbb{P}^T_k; X^T_k) = \text{rap}_T(X)$ for each $k \in \mathbb{N}$. Then:
10.2. Supplementary results for Appendix ??

(a) $X$ satisfies $T_k$-LMT$C$ in $A^{18}$ if and only if $\{X^{T_k}\}_{n \in \mathbb{N}}$ is $m$-tight in $A$.
(b) If $\{X^{T_k}\}_{n \in \mathbb{N}}$ is $m$-tight in $A$, then $\{X^{T_k}\}_{k \in \mathbb{N}}$ satisfies $R^+\text{-PSMT$C$}$ in $A^{19}$
(c) $\{X^{T_k}\}_{k \in \mathbb{N}}$ is $(R^+, M_b(E; R))$-AS.
(d) If $\{X^{T_k}\}_{k \in \mathbb{N}}$ is $(T, D)$-FDC, then it is $(T + c, D)$-FDC for all $c \in (0, \infty)$.

Proof. (a) is automatic by the definition of $\{X^{T_k}\}_{k \in \mathbb{N}}$.
(b) follows by (a) and Lemma 10.1.35 with $\{X^i\}_{i \in I} = \{X\}$.
(c) and (d) follow immediately by the fact that

$$\lim_{k \to \infty} \left| \mathbb{E}^{T_k} \left[ f \circ X^{T_k} - f \circ X^{T_k+c} \right] \right|$$

(10.1.36)

$$\leq \lim_{k \to \infty} \frac{1}{T_k} \int_0^c \mathbb{E} \left[ \left| f \circ X^{T_k} - f \circ X^{T_k+c} \right| \right] d\tau \leq \lim_{k \to \infty} \frac{2c\|f\|_{\infty}}{T_k} = 0$$

for all $c \in (0, \infty)$, $f \in \Pi^{T_k}(M_b(E; R))$ and $T_0 \in \mathscr{T}_0(R^+)$, where $\mathbb{E}^{T_k}$ denotes the expectation operator of $(\Omega, \mathcal{F}, \mathbb{P}^{T_k})$ for each $k \in \mathbb{N}$.

10.2. Supplementary results for Appendix 9

Fact 10.2.1. Let $E$ be a topological space and $k \in \mathbb{N}$. Then, $C_c(E; R^k) \subset C_0(E; R^k) \subset C_b(E; R^k)$ and they are indifferent if $E$ is compact.

Proof. This result follows by [Mun00 Theorem 27.4].

Proposition 10.2.2. Let $E$ be a Hausdorff space. Then:
(a) $C_c(E; R)$ is a subalgebra of $C_b(E; R)$ and is a function lattice.
(b) $C_c(E; R) \subset C_0(E; R) \subset \mathfrak{d}(C_c(E; R))$.

Proof. It is straightforward to show $C_c(E; R)$ is an algebra. Observing that

$$|f| \in C_c(E; R), \forall f \in C_c(E; R),$$

we have that

$$f \vee g = \frac{1}{2} (f + g) + \frac{1}{2} |f - g| \in C_c(E; R), \forall f, g \in C_c(E; R),$$

and that

$$f \wedge g = \frac{1}{2} (f + g) - \frac{1}{2} |f - g| \in C_c(E; R), \forall f, g \in C_c(E; R),$$

thus proving $C_c(E; R)$ is a function lattice.

(b) The first inclusion is immediate. We prove the second one. We fix $f \in C_0(E; R)$, $p \in \mathbb{N}$ and a $K_p \in \mathscr{K}(E)$ such that $\|f|_{E\setminus K_p}\|_{\infty} < 2^{-p}$. The case where $K_p = E$ is trivial. Otherwise, we define $A \equiv f^{-1}((2^{-p}, \infty))$, $B \equiv f^{-1}([-\infty, -2^{-p}])$ and

$$f_p \equiv (f^+(x) - 2^{-p})^+ - (f^-(x) - 2^{-p})^+.$$
A and B are disjoint subsets of $K_\mu$ such that $A \cup B = E \setminus f^{-1}\{\{0\}\}$. Letting $F$ be the closure of $A \cup B$ in $E$, we have by Proposition 9.1.12 (a) that $K_\mu \in \mathcal{C}(E)$, $F \subset K_\mu$ and $F \in \mathcal{K}(E)$, thus proving $f_\mu \in C_c(E; \mathbb{R})$. Furthermore, from the fact

\[(10.2.5) \quad f_\mu(x) - f(x) = \begin{cases} \end{cases}
\]  

\[-2^{-p}, \quad \text{if } x \in A,\]
\[2^{-p}, \quad \text{if } x \in B,\]
\[-f(x) \in (-2^{-p}, 2^{-p}), \quad \text{if } x \in E \setminus (A \cup B)\]

it follows that $\|f_\mu - f\|_\infty \leq 2^{-p}$. \hfill \Box

**FACT 10.2.3.** Let $E$ be a topological space, $f \in C(E; \mathbb{R})$ and $A$ be a dense subset of $E$ with $E \setminus A \neq \emptyset$. Then:

(a) If $E$ is a first-countable space and $(A \setminus B) \subset f^{-1}\{\{0\}\}$ for some $B \in \mathcal{C}(E)$ with $B \subset A$, then $f|_{E \setminus A} = 0$.

(b) If $E$ is a metrizable space and $f \in C_c(A, \mathcal{C}(A))$, then $f|_{E \setminus A} = 0$.

**PROOF.** (a) For each $x \in E \setminus A$, the first-countability of $E$ implies a sequence $(x_n)_{n \in \mathbb{N}} \subset A$ converging to $x$ as $n \uparrow \infty$. As $x \in E \setminus B$ and $E \setminus B \in \mathcal{C}(E)$, there exists an $N \in \mathbb{N}$ such that $x_n \in A \setminus B$ and $f(x_n) = 0$ for all $n > N$. Hence, the continuity of $f$ implies $f(x) = \lim_{n \to \infty} f(x_n) = 0$.

(b) If $E$ is a metrizable space, then it is first-countable by Fact 9.1.9. If, in addition, $f \in C_c(A, \mathcal{C}(A))$, then we let $B$ be the closure of $A \setminus f^{-1}(\{0\})$ and $B \in \mathcal{C}(E)$ by Proposition 9.1.2 (a) and Proposition 9.1.12 (a). \hfill \Box

**Proposition 10.2.4.** Let $(S_i)_{i \in I}$ be topological spaces and $(\mathcal{A}, \mathcal{I})$ be defined as in (2.7.22). Then:

(a) $\mathcal{B}(S) \supset \mathcal{I}$.

(b) If $I$ is countable and $S$ is hereditarily Lindelöf, then $\mathcal{B}(S) = \mathcal{I}$.

(c) If $I$ is countable and $(S_i)_{i \in I}$ are all second-countable, then $\mathcal{B}(S) = \mathcal{I}$.

(d) If $I$ is countable and $(S_i)_{i \in I}$ are all metrizable and separable spaces (especially Polish spaces), then $\mathcal{B}(S) = \mathcal{I}$.

**Note 10.2.5.** As arranged in (2.6), the Cartesian product $S = \prod_{i \in I} S_i$ above is equipped with the product topology $\mathcal{O}(S) = \bigotimes_{i \in I} \mathcal{O}(S_i)$ and its Borel $\sigma$-algebra is $\mathcal{B}(S) = \sigma[\mathcal{O}(S)]$.

**Proof of Proposition 10.2.3.** (a) follows by the argument establishing [Bog07, Vol. II, Lemma 6.4.1].

(b) and (c) were proved in [Bog07, Vol. II, Lemma 6.4.2 (ii)].

(d) follows by (c), Proposition 9.1.4 (c) and Proposition 9.1.11 (c). \hfill \Box

**Lemma 10.2.6.** Let $(S_i)_{i \in I}$ be topological spaces, $(\mathcal{A}, \mathcal{I})$ be as in (2.7.22), $A \in \mathcal{I}$, $(A_n)_{n \in \mathbb{N}} \subset \mathcal{A}$ and $\mu \in \mathfrak{M}^+(S, \mathcal{A})$. Then:

(a) If $S = \bigcup_{n \in \mathbb{N}} A_n$ and $\mathcal{A}|_{A_n} = \mathcal{B}_S(A_n)$ for all $n \in \mathbb{N}$, then $\mathcal{B}(S) = \mathcal{I}$.

(b) If $\mu$ is supported on $A$ and $\mathfrak{C}(\mu|_A)$ is a singleton, then $\mathfrak{C}(\mu)$ is a singleton.

(c) If $\mu$ is supported on $A$ and $\mathcal{B}_S(A) = \mathcal{I}|_{A}$, then $\mathfrak{C}(\mu)$ is a singleton.

\[^{20}\text{“}\mu|_A\text{” and }\nu|_E\text{ denote the concentration of }\mu\text{ on }A\text{ and the expansion of }\nu\text{ onto }E.\text{ “}\mathfrak{C}(\mu|_A)\text{” denotes the Borel extension(s) of }\mu.\]
(a) follows by Proposition 10.2.4 (a) and Fact 10.1.1 (with \( E = S, \mathcal{B}_1 = \mathcal{B}(S) \) and \( \mathcal{B}_2 = \mathcal{A} \)).

(b) Let \( \nu = \text{be}(\mu|_A) \) and \( \mu_1 \equiv \nu|_S \in \mathcal{M}^+(S) \) by Fact 2.1.1 (b) (with \( E = S \) and \( \mathcal{U} = \mathcal{B}(E) \)). Since \( A \in \mathcal{A} \), we have that

\[
B \cap A \in \mathcal{A}|_A \subset \mathcal{B}(A), \forall B \in \mathcal{A}.
\]

\( \nu \) and \( \mu|_A \) are identical restricted to \( \mathcal{A}|_A \). It then follows by (10.2.6) that

\[
\mu|_A(B \cap A) = \nu(B \cap A) = \mu_1(B), \forall B \in \mathcal{A}.
\]

It follows by the fact \( \mu(A) = 1 \), the fact \( A \in \mathcal{A} \) and (10.2.7) that

\[
\mu(B) = \mu(B \cap A) = \mu|_A(B \cap A) = \mu_1(B), \forall B \in \mathcal{A},
\]

thus proving \( \mu_1 \in \text{be}(\mu) \). If \( \mu_2 \in \text{be}(\mu) \), then we have that

\[
\mu_2|_A = \text{be}(\mu|_A) = \nu \in \mathcal{M}^+(A, \mathcal{O}_S(A)).
\]

It follows that

\[
\mu_2 = (\mu_2|_A)|^S = \nu|^S = \mu_1
\]

by (10.2.9) and Fact 2.1.1 (a, c) (with \( E = S, \mathcal{U} = \mathcal{B}(E) \) and \( \mu = \mu_2 \)).

(c) \( \mathcal{B}_S(A) = \mathcal{A}|_A \) implies \( \mu|_A = \text{be}(\mu|_A) \). Then, (c) follows by (b).

\[\square\]

**Lemma 10.2.7.** Let \( E \) and \( S \) be measurable spaces and \( f, g \in M(S; E) \). If there exists a countable subset of \( M(E; \mathbb{R}) \) separating points on \( E \), then \{ \( x \in S : f(x) = g(x) \) \} is a measurable subset of \( S \). In particular, this is true when \( E \) is baseable.

**Proof.** Let \( \{h_n\}_{n \in \mathbb{N}} \subset M(E; \mathbb{R}) \) separate points on \( E \). Then,

\[
\{x \in S : f(x) = g(x)\} = \{x \in S : h_n \circ f(x) = h_n \circ g(x)\}
\]

is a measurable subset of \( S \).

\[\square\]

**Lemma 10.2.8.** Let \( E \) be a topological space, \( V \) be the family of all càdlàg members of \( E^{\mathbb{R}^+} \) in \( \mathbb{R}^+ \) and \( T \in (0, \infty) \). Then:

(a) If \( M(E; \mathbb{R}) \) has a countable subset separating points on \( E \), then \{ \( x \in V : t \in J(x) \) \} \( \in \mathcal{B}(E)^{\otimes \mathbb{R}^+}|_V \).

(b) \{ \( x \in V : x|_{[0,T]} \in A^{[0,T]} \) \} \( \in \mathcal{B}(E)^{\otimes \mathbb{R}^+}|_V \) for all \( A \in \mathcal{G}(E) \), especially for all \( A \in \mathcal{H}(E) \) when \( E \) is a Hausdorff space.

**Proof.** (a) We fix \( t \in \mathbb{R}^+ \), let \( p_{t-} \) denote the mapping associating each \( x \in V \) to its left limit at \( t \) and find by Fact 2.1.3 (a) that

\[
p_{t-}^{-1}(A) = \bigcap_{p \in \mathbb{Q}^+ \cap (0,t)} \bigcup_{q \in \mathbb{Q}^+ \cap (p,t)} p_q^{-1}(A) \in \mathcal{B}(E)^{\otimes \mathbb{R}^+}|_V, \forall A \in \mathcal{G}(E),
\]

so \( p_{t-} \in M(V, \mathcal{B}(E)^{\otimes \mathbb{R}^+}|_V; E) \). It then follows by Lemma 10.2.7 (with \( S = V, f = p_{t-} \) and \( g = p_t \)) that

\[
\{x \in V : t \in J(x)\} = \{x \in V : p_{t-}(x) = p_t(x)\} \in \mathcal{B}(E)^{\otimes \mathbb{R}^+}|_V.
\]
(b) When $E$ is Hausdorff, $\mathcal{K}(E) \subset \mathcal{C}(E)$ by Proposition 9.1.12 (a). It follows by the closedness of $A$ and the right-continuity of each $x$ that

\[(10.2.14) \quad \{ x \in V : x|_{[0,T]} \in A^{[0,T]} \} = \bigcap_{t \in \mathbb{Q} \cap [0,T)} V \cap p_t^{-1}(A). \]

It follows by the fact $A \in \mathcal{C}(E) \subset \mathcal{B}(E)$ and Fact 2.1.3 (a) that

\[(10.2.15) \quad V \cap p_t^{-1}(A) \in \mathcal{B}(E)^{\otimes \mathbb{R}^+} |_{V}, \forall t \in \mathbb{R}^+. \]

Now, (b) follows by $(10.2.14)$, $(10.2.15)$ and the countability of $\mathbb{Q} \cap [0, T)$. □

**FACT 10.2.9.** Let $E$ be a topological space and $K \in \mathcal{K}(E)$. Then:

(a) $K \in \mathcal{K}(E, \mathcal{U})$ for any topological coarsening $(E, \mathcal{U})$ of $E$.

(b) If $\mathcal{D} \subset C(E; \mathbb{R})$ separate points on $K$, then $\mathcal{O}_E(K) = \mathcal{O}_\mathcal{D}(K)$ and $K \in \mathcal{K}(E, \mathcal{O}_\mathcal{D}(E))$.

**PROOF.** (a) is immediate by the definition of compactness.

(b) $\mathcal{O}_E(K) = \mathcal{O}_\mathcal{D}(K)$ is a Hausdorff topology by Lemma 9.2.4 with $E = K$ and $\mathcal{D} = \mathcal{D}_E(K)$ and Proposition 9.2.1 (c) (with $A = E$). Now, (b) follows by Proposition 9.1.12 (a). □

**LEMMA 10.2.10.** Let $E$ be a non-empty set, $\mathcal{G} \subset \mathbb{R}^E$ and $\mathcal{H} \subset \mathbb{R}^E$. Suppose that for any $g \in \mathcal{G}$ and $n \in \mathbb{N}$, there exists a bounded function $f_{g,n} \in \mathcal{H}$ such that

\[(10.2.16) \quad A_{g,n} \triangleq \{ x \in E : |g(x)| < n \} = \{ x \in E : |f_{g,n}(x)| < n \}\]

and

\[(10.2.17) \quad g^1_{A_{g,n}} = f_{g,n}^1_{A_{g,n}}. \]

Then, there exists a subset $\mathcal{F} \subset \mathcal{H}$ such that:

(a) The members of $\mathcal{F}$ are all bounded and include all the bounded members of $\mathcal{G}$. In particular, $\mathcal{F} = \mathcal{G}$ when the members of $\mathcal{G}$ are all bounded.

(b) $\mathcal{F}$ is countable if $\mathcal{G}$ is.

(c) $\mathcal{O}_\mathcal{G}(E) \subset \mathcal{O}_\mathcal{F}(E)$. Moreover, if $\mathcal{G}$ separates points on $E$, or if $E$ is a topological space and $\mathcal{G}$ strongly separates points on $E$, then $\mathcal{F}$ has the same property.

**PROOF.** (a, b) are immediate.

(c) It follows by $(10.2.16)$ and $(10.2.17)$ that

\[(10.2.18) \quad \{ x \in E : g(x) < a \} = \bigcup_{n \geq a} \{ x \in E : g(x) < a, |g(x)| < n \} = \bigcup_{n \geq a} \{ x \in E : f_{g,n}(x) < a, |f_{g,n}(x)| < n \} = \bigcup_{n \geq a} \{ x \in E : -n < f_{g,n}(x) < a \} \in \mathcal{O}_\mathcal{F}(E), \forall a \in \mathbb{R}, g \in \mathcal{G}, \]

thus proving $\mathcal{O}_\mathcal{G}(E) \subset \mathcal{O}_\mathcal{F}(E)$.

The Hausdorff property of $\mathcal{O}_\mathcal{G}(E)$ implies that of $\mathcal{O}_\mathcal{F}(E)$ by Fact 9.1.1. So, $\mathcal{G}$ separating points on $E$ implies $\mathcal{F}$ separating points on $E$ by Proposition 9.2.1 (c).

If $\mathcal{G}$ strongly separates points on topological space $E$, then $\mathcal{O}(E) \subset \mathcal{O}_\mathcal{G}(E) \subset \mathcal{O}_\mathcal{F}(E)$ and so $\mathcal{F}$ strongly separates points on $E$. □
COROLLARY 10.2.11. Let $E$ be a topological space. Then, $C(E; \mathbb{R})$ separates points (resp. strongly separates points) on $E$ if and only if $C_b(E; \mathbb{R})$ does.

FACT 10.2.12. Let $E$ be a topological space, $\mu \in \mathfrak{M}^+(E^d; \mathcal{B}(E)^{\otimes d})$, $\nu_1 \in \mathfrak{b}(\mu)$, $X \in \mathcal{M}(\Omega, \mathfrak{F}, \mathcal{P}; E^d, \mathcal{B}(E)^{\otimes d})$ and $\nu_2 \in \mathfrak{b}(\mathcal{P} \circ X^{-1})$. Then, $\int_{E^d} f(x) \mu(dx) = f^*(\nu_1)$ and $E[f \circ X] = f^*(\nu_2)$ for all $f \in \mathfrak{ca}[\Pi^d(M_b(E; \mathbb{R}))]$.

PROOF. This result follows by Proposition 9.2.5 (a) (with $\mathcal{D} = M_b(E; \mathbb{R})$) and the fact that $\nu_1$ (resp. $\nu_2$) and $\mu$ (resp. $\mathcal{P} \circ X^{-1}$) are the same measures on $(E^d, \mathcal{B}(E)^{\otimes d})$. □

LEMMA 10.2.13. Let $E$ be a topological space and $A \subset E$. Then,

$$(10.2.19) \quad \mu_n \Rightarrow \mu \text{ as } n \uparrow \infty \text{ in } \mathcal{M}^+(A, \mathcal{E}_E(A))$$

implies

$$(10.2.20) \quad \mu_n|_E \Rightarrow \mu|_E \text{ as } n \uparrow \infty \text{ in } \mathcal{M}^+(E).$$

The converse is true when $E$ is a Tychonoff space.

PROOF. It follows by (10.2.19) and $C_b(E; \mathbb{R})|_A \subset C_b(A, \mathcal{E}_E(A); \mathbb{R})$ that

$$\lim_{n \to \infty} \int_{E} f(x) \mu_n|_E(dx) = \lim_{n \to \infty} \int_{A} f|_A(x) \mu_n(dx) = \int_{A} f|_A(x) \mu(dx)$$

(10.2.21)

$$= \int_{E} f(x) \mu|_E(dx), \forall f \in C_b(E; \mathbb{R}),$$

proving (10.2.20). Conversely, if $E$ is Tychonoff, then (10.2.20) implies

$$(10.2.22) \quad \limsup_{n \to \infty} \mu_n(F \cap A) = \limsup_{n \to \infty} \mu_n|_E(F) \leq \mu|_E(F) = \mu(F \cap A), \forall F \in \mathcal{E}(E)$$

by Theorem 2.3.7 (a, b) (with $\mu_n = \mu_n|_E$ and $\mu = \mu|_E$). $(A, \mathcal{E}_E(A))$ is a Hausdorff subspace by Proposition 9.3.2 (b). Now, (10.2.19) follows by (10.2.22) and Theorem 2.3.7 (a, b). □

COROLLARY 10.2.14. Let $E$ be a topological space and $\Gamma \subset \mathcal{M}^+(A, \mathcal{E}_E(A))$ with $A \subset E$. Then, $\{\mu|_E\}_{\mu \in \Gamma}$ is relatively compact in $\mathcal{M}^+(E)$ whenever $\Gamma$ is.

COROLLARY 10.2.15. Let $E$ be a topological space, $A \in \mathcal{B}(E)$ be a Hausdorff subspace, $\{\mu_n\}_{n \in \mathbb{N}} \subset \mathcal{M}^+(E)$ be sequentially tight in $A$ and $\{\mu_n(E)\}_{n \in \mathbb{N}} \subset [a, b]$ for some $0 < a < b$. If $\mu$ is the unique weak limit point of $\{\mu_n\}_{n \in \mathbb{N}}$ in $\mathcal{M}^+(E)$, then (2.3.3) holds.

PROOF. $\{\mu_n\}_{n \in \mathbb{N}}$ are all supported on $A$ with finite exception by Fact 10.1.26 (with $\mathcal{W} = \mathcal{B}(E)$ and $\Gamma = \{\mu_n\}_{n \in \mathbb{N}}$) and $\{\mu_n|_A\}_{n \in \mathbb{N}}$ is relatively compact in $\mathcal{M}^+(A, \mathcal{E}_E(A))$ by Lemma 9.4.12 (with $E = (A, \mathcal{E}_E(A))$ and $\Gamma = \{\mu_n|_A\}_{n \in \mathbb{N}}$). Then, $\{\mu_n\}_{n \in \mathbb{N}}$ is relatively compact in $\mathcal{M}^+(E)$ by Fact 2.1.1 (c) (with $\mathcal{W} = \mathcal{B}(E)$ and $\nu = \mu|_A$) and Corollary 10.2.14. Now, the corollary follows by Fact 10.1.6 (with $(E, x_n, x) = (\mathcal{M}^+(E), \mu_n, \mu)$). □

LEMMA 10.2.16. Let $E$ be a Hausdorff space. Then, the following statements are equivalent:

(a) $E$ is a Tychonoff space.

(b) $\delta_{x_n} \Rightarrow \delta_x$ as $n \uparrow \infty$ in $\mathcal{P}(E)$ implies $x_n \to x$ as $n \uparrow \infty$ in $E$.

(c) Convergence determining implies determining point convergence on $E$. 
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Proof. ((a) → (b)) \( \delta_{x_n} \to \delta_x \) as \( n \to \infty \) in \( \mathcal{P}(E) \) implies

\[
\text{(10.2.23)} \quad \lim_{n \to \infty} f(x_n) = \lim_{n \to \infty} f^*(\delta_{x_n}) = f(\delta_x) = f(x)
\]

for all \( f \in C_b(E; \mathbb{R}) \). \( C_b(E; \mathbb{R}) \) determines point convergence on \( E \) by Proposition 9.3.1 (a, c) and Proposition 9.2.1 (b). Hence, \( \text{(10.2.23)} \) implies \( x_n \to x \) as \( n \to \infty \).

If \( D \subset M_b(E; \mathbb{R}) \) satisfies \( \bigotimes \mathcal{D}(x_n) \to \bigotimes \mathcal{D}(x) \) as \( n \to \infty \), then \( \text{(10.2.23)} \) holds for all \( f \in D \). This implies \( \delta_{x_n} \to \delta_x \) as \( n \to \infty \) since \( D \) is convergence determining on \( E \). Now, we have \( x_n \to x \) as \( n \to \infty \) by (b).

((c) → (a)) \( C_b(E; \mathbb{R}) \) determines point convergence on \( E \) by (c). It strongly separates points and separates points on \( E \) by Proposition 9.2.1 (a, b). Now, (a) follows by Proposition 9.3.1 (a, c).

Lemma 10.2.17. Let \( E \) be a topological space, \( D \subset C_b(E; \mathbb{R}) \) separate points on \( E \) and \( d \in \mathbb{N} \). Then:

(a) If each of \( \mu_1, \mu_2 \in \mathcal{M}^+(E) \) is tight and \( D \) is closed under multiplication, then \( f^*(\mu_1) = f^*(\mu_2) \) for all \( f \in D \cup \{1\} \) implies \( \mu_1 = \mu_2 \).

(b) If each of \( \mu_1, \mu_2 \in \mathcal{M}^+(E^d, \mathcal{B}(E)^{\otimes d}) \) is \( \mathfrak{m} \)-tight, then \( f^*(\mu_1) = f^*(\mu_2) \) for all \( f \in \text{mc}[\Pi^d(D)] \cup \{1\} \) implies \( \mu_1 = \mu_2 \).

Proof. (a) Let \( a \equiv \mu_1(E) = \mu_2(E) > 0 \) and \( \nu_i \equiv \mu_i/a \) for each \( i \in \{1, 2\} \). Each of \( \nu_1 \) and \( \nu_2 \) is a tight member of \( \mathcal{P}(E) \) and they satisfy \( f^*(\nu_1) = f^*(\nu_2) \) for all \( f \in D \). Then, \( \nu_1 = \nu_2 \) by [BK10, Theorem 11 (d)] and so \( \mu_1 = \mu_2 \).

(b) \( E \) is a Hausdorff space by Proposition 9.2.1 (e) (with \( A = E \)). For each \( j \in \{1, 2\} \), there exists an \( \mathfrak{m} \)-tight \( \mu_j' \equiv \mathcal{B}(\mu_j) \) by Proposition 3.3.25 (with \( I = \{1, \ldots, d\} \), \( S_i = E, A = E^d \) and \( \Gamma = \{\mu_j\} \)). \( \text{mc}[\Pi^d(D)] \) separates points on \( E^d \) by Proposition 9.2.5 (b). Now, (b) follows by (a) (with \( E = E^d, D = \text{mc}[\Pi^d(D)] \) and \( \mu_j = \mu_j' \)).

Fact 10.2.18. Let \( E \) be a topological space and \( S \) a Hausdorff space. If \( \Gamma \subset \mathcal{M}^+(E) \) is tight in \( A \subset E \) and \( f \in \mathcal{C}(E; S) \), then \( \{\mu \circ f^{-1} : \mu \in \Gamma\} \) is tight in \( f(A) \).

Proof. \( f(K) \in \mathcal{K}(S) \subset \mathcal{B}(S) \) for all \( K \in \mathcal{K}(E) \) by Proposition 9.1.12 (a, e). Now, the result follows by Fact 10.1.27 (with \( \mathcal{U} = \mathcal{B}(E) \) and \( \mathcal{A} = \mathcal{B}(S) \)).

Lemma 10.2.19. Let \( I \) be a countable index set, \( \{S_i \}_{i \in I} \) be topological spaces, \( (S, \mathcal{A}) \) be as in (2.7.22), \( \Gamma \subset \mathcal{M}^+(S, \mathcal{A}) \), \( A_i \subset S_i \) for each \( i \in I \) and \( A \equiv \prod_{i \in I} A_i \). Then:

(a) If \( \{\mu \circ p_i^{-1} : \mu \in \Gamma\} \) is tight (resp. \( \mathfrak{m} \)-tight) in \( A_i \) for all \( i \in I \), then \( \Gamma \) is tight (resp. \( \mathfrak{m} \)-tight) in \( A \). The converse is true when \( (A_i, \mathcal{O}_{S_i}(A_i)) \) is a Hausdorff subspace of \( S_i \) and \( A_i \in \mathcal{B}(S_i) \).

(b) If \( \{\mu \circ p_i^{-1} : \mu \in \Gamma\} \) is sequentially tight (resp. \( \mathfrak{m} \)-tight) in \( A_i \) for all \( i \in I \), then \( \Gamma \) is sequentially tight (resp. \( \mathfrak{m} \)-tight) in \( A \). The converse is true when \( (A_i, \mathcal{O}_{S_i}(A_i)) \) is a Hausdorff subspace of \( S_i \) and \( A_i \in \mathcal{B}(S_i) \).

Proof. (a) Without loss of generality, we suppose \( I = \mathbb{N} \). Each \( A_i \) is equipped with the subspace topology \( \mathcal{O}_{S_i}(A_i) \) throughout the proof. If \( \{\mu \circ p_i^{-1} : \mu \in \Gamma\} \) is tight in \( A_i \) for all \( i \in I \), then there exist

\[
(10.2.24) \quad A_i \supseteq K_{p, i} \in \mathcal{K}(S_i) \cap \mathcal{B}(S_i), \quad \forall i, p \in \mathbb{N}
\]

---

23We mentioned in [Z4] that any measure in this work has positive total mass.
such that
\[(10.2.25) \sup_{\mu \in \Gamma} \mu \circ p_i^{-1} (S_i \setminus p_i(K_p)) \leq 2^{-p-i}, \forall i, p \in \mathbb{N}.\]

It follows that
\[(10.2.26) A \supset \prod_{i \in \mathbb{N}} K_{p,i} \in \mathcal{X}(S) \cap \mathcal{A}, \forall p \in \mathbb{N}\]
by Proposition 9.1.12 (b), Fact 2.1.4 (a) and the fact \(\prod_{i \in I} K_{p,i} = \bigcap_{i \in I} p_i^{-1}(K_{p,i})\).

Now, we conclude the tightness of \(\Gamma\) in \(A\) by observing that
\[(10.2.27) \sup_{\mu \in \Gamma} \left( S \setminus \prod_{i \in \mathbb{N}} K_{p,i} \right) \leq \sum_{i=1}^{\infty} \sup_{\mu \in \Gamma} (S_i \setminus K_{p,i}) \leq 2^{-p}, \forall p \in \mathbb{N}.\]

If \(\{\mu \circ p_i^{-1}\}_{\mu \in \Gamma}\) is \(m\)-tight in \(A_i\) for all \(i \in I\), then we reiterate each \(K_{p,i}\) above from \(\mathcal{X}^m(S_i) \cap \mathcal{B}(S_i)\), find \(\prod_{i \in I} K_{p,i} \in \mathcal{X}^m(S)\) by Lemma 3.3.23 (a) (with \(A_i = K_{p,i}\)) and verify the \(m\)-tightness of \(\Gamma\) by a similar argument.

Next, we suppose \(A_i \in \mathcal{B}(S_i)\) is a Hausdorff subspace for all \(i \in I\) and justify the converse statement. We have that
\[(10.2.28) \quad p_i(K) \in \mathcal{X}(A_i) \subset \mathcal{B}(A_i) \subset \mathcal{B}(S_i), \forall K \in \mathcal{X}(S), i \in I\]
and
\[(10.2.29) \quad p_i(K) \in \mathcal{X}^m(A_i) \subset \mathcal{B}(A_i) \subset \mathcal{B}(S_i), \forall K \in \mathcal{X}^m(S), i \in I\]
by Corollary 9.1.13 (b) (with \(A = K\) and \(S_i = A_i\)), Lemma 3.3.23 (b) (with \(A = K\) and \(S_i = A_i\)) and the fact \(A_i \in \mathcal{B}(S_i)\). If \(\Gamma\) is tight (resp. \(m\)-tight) in \(A\), then for each \(i \in I\), the tightness (resp. \(m\)-tightness) of \(\{\mu \circ p_i^{-1}\}_{\mu \in \Gamma}\) in \(A_i\) follows by \(10.2.28\), \(10.2.29\), the fact \(A_i = p_i(A)\), Fact 2.1.3 (a) and Fact 10.1.27 (with \((E, \mathcal{A}) = (S, \mathcal{A}), (S, \mathcal{A}) = (S_i, \mathcal{B}(S_i))\) and \(f = p_i\)).

(b) follows immediately by (a) and a triangular array argument. \(\square\)

**Lemma 10.2.20.** Let \(E\) be a Tychonoff space and \((D(R^+; E), \mathcal{F}, \nu)\) be the completion of \((D(R^+; E), \mathcal{F}, \sigma(\mathcal{F})), \mu\), \(24\) if \(M(E; R)\) has a countable subset separating points on \(E\), especially if \(E\) is baseable, then \(\nu \circ p_{T_0}^{-1} \in \mathcal{M}^+(E^{T_0})\) is a Borel extension of \(\mu \circ p_{T_0}^{-1} \in \mathcal{M}^+(E^{T_0}, \mathcal{B}(E)^{T_0})\) for all non-empty \(T_0 \in \mathcal{Z}(R^+ \setminus J(\mu))\), \(25\)

**Proof.** \(\mu \circ p_{T_0}^{-1}\) is a member of \(\mathcal{M}^+(E^{T_0}, \mathcal{B}(E)^{T_0})\) by Lemma 9.6.6 (a). \(\nu \circ p_{T_0}^{-1} \in \mathcal{M}^+(E^{T_0}, \mathcal{B}(E)^{T_0})\) by Lemma 9.6.6 (c), the definition of \(J(\mu)\) and Fact 10.1.4 (with \(E = D(R^+; E), \mathcal{F} = \mathcal{F}, S = E^{T_0}\) and \(f = p_{T_0}\)). Hence, \(\nu \circ p_{T_0}^{-1} \in \mathcal{M}^+(E^{T_0})\) as \(\nu\) is an extension of \(\mu\) to \(\mathcal{F}\). \(\square\)

**Lemma 10.2.21.** Let \(E\) be a Tychonoff space, \(D \subset C(E; R)\) be countable and \({\mu_i}\}_{i \in I} \subset \mathcal{M}^+(D(R^+; E)).\) Then:

(a) If \(\{\mu_i \circ \varpi(f)^{-1}\}_{i \in I}\) is tight in \(D(R^+; R)\) for all \(f \in D\), then \({\mu_i \circ \varpi(D)^{-1}\}_{i \in I}\) is tight in \(D(R^+; R)^D\).

\(24\) \(\mathcal{M}^+(D(R^+; E)) \setminus \sigma(\mathcal{F})), \sigma(\mathcal{F})), \mu\) so the measure space notation “\((D(R^+; E), \sigma(\mathcal{F})), \mu\)” implies \(\mu \in \mathcal{M}^+(D(R^+; E)).\)

\(25\) Herein, we show the domain of \(\nu \circ p_{T_0}^{-1}\) contains \(\mathcal{B}(E^{T_0})\) so \(\nu \circ p_{T_0}^{-1}\) can be viewed as a member of \(\mathcal{M}^+(E^{T_0})\).

\(26\) The \(J(\mu)\) herein is well-defined by Lemma 9.6.6 (b) and Fact 9.6.9

\(27\) Extension of measure was specified in 2.1.2
(b) If $D$ strongly separates points on $E$ and $\{\mu_i \circ \varpi(f)^{-1}\}_{i \in I}$ is tight in $D(R^+; R)$ for all $f \in \varpi(D)$, then $\{\mu_i \circ \varpi(\varpi(D))^{-1}\}_{i \in I}$ is tight in $D(R^+; \mathbb{R}^{\varpi(D)})$.

Note 10.2.22. When $D$ is countable, $\varpi(D)$ is also by Fact 10.1.14. Then, $\mathbb{R}^{\varpi(D)}$, $D(R^+; \mathbb{R})$, $D(R^+; \mathbb{R}^{\varpi(D)})$ and $D(R^+; \mathbb{R}^{\varpi(D)})$ are all Polish spaces by Proposition 9.1.11 and Proposition 9.6.10 (d).

Proof of Lemma 10.2.21. (a) follows by the fact (10.2.30) 
\[(\mu_i \circ \varpi(D)^{-1}) \circ p_f^{-1} = \mu_i \circ \varpi(f)^{-1} \in M^+(D(R^+; R)), \forall f \in D\]
and Lemma 10.2.19 (a) (with $I = D$, $S_i = D(R^+; R)$ and $\Gamma = \{\mu_i \circ \varpi(D)^{-1}\}_{i \in I}$). Here, $p_f$ denotes the one-dimensional projection on $D(R^+; \mathbb{R})^D$ for $f \in D$.

(b) Letting $J = \{p_f\}_{f \in D}$ be the one-dimensional projections on $\mathbb{R}^D$, we have by Corollary 9.6.4 (with $I = D$) that $\varphi_i \oplus \varpi(\varpi(D))^{-1}$ is tight in $D(R^+; \mathbb{R}^{\varpi(D)})$ by (a) (with $D = \varpi(D)$). Observing that (10.2.34) 
\[\mu_i \circ \varphi_i^{-1} \in C \left[D(R^+; \mathbb{R}^{\varpi(D)}); D \left(R^+; \mathbb{R}^{\varpi(D)}\right)\right],\]
we have the desired tightness of $\{\mu_i \circ \varphi_i^{-1}\}_{i \in I}$ by (10.2.33), (10.2.34) and Fact 10.2.18 (with $E = D(R^+; \mathbb{R}^{\varpi(D)})$, $S = D(R^+; \mathbb{R}^{\varpi(D)})$ and $f = \varphi_i \circ \varphi_i^{-1}$).

Lemma 10.2.23. Let $E$ be a Tychonoff space, $D \subset C(E; R)$ be countable, $\Psi \equiv \varpi(\varpi(D))$, $V \subset D(R^+; E)$ and $\{A_p\}_{p \in \mathbb{N}} \subset \mathcal{B}(E)$. If $A_p \subset A_{p+1}$, $D$ strongly separates points on $A_p$ and $x\|0_p \in A_{p}^0$ for all $x \in V$ and $p \in \mathbb{N}$, then
\[(10.2.35) \Psi|_V \equiv \varpi(D(R^+; E)(V); D(R^+; \mathbb{R}^{\varpi(D)}))\]
and
\[(10.2.36) \mathcal{B}_D(R^+; E)(V) = \mathcal{B}(E) \circ \mathbb{R}^+\bigg|_V.\]

Proof. Step 1: Show $\Psi|_V$ is injective. $E_0 \equiv \bigcup_{p \in \mathbb{N}} A_p \in \mathcal{B}(E)$ satisfies (10.2.37) 
\[V \subset D \left(R^+; E_0, \varpi(\varpi(E_0))\right).\]

$\varpi(D)$ separates points on $E_0$ by the Hausdorff property of $E$, Proposition 9.2.1 (a) and Fact 10.1.17 (with $n = p$ and $D_n = \varpi(D)$). So, $\Psi|_V$ is injective by (10.2.37) and Fact 10.1.18 (with $A = E_0$ and $D = \varpi(D)$).

Step 2: Show the continuity of $\Psi|_V$. We have by Fact 10.1.14 and Proposition 9.2.1 (d) (with $A = E_0$) that $(E_0, \rho_D)$ is a separable metric space, $\varpi(D)$ is a

\[\footnote{Remark 2.2.1 noted that $\varphi_1$ and $\varphi_2$ are different.} \]
countable subset of \(C(E_0, \rho_D; \mathbb{R})\) and \(\mathfrak{a}(D)\) strongly separates points on \((E_0, \rho_D)\). It follows that

\[
\Psi|_D(\mathbb{R}^+; E_0, \rho_D) \in \text{imb} \left(D \left(\mathbb{R}^+; E_0, \rho_D\right); D(\mathbb{R}^+; \mathbb{R})^{\mathfrak{a}(D)}\right)
\]

by Proposition 9.6.1 (a) (with \(D = \mathfrak{a}(D)\)). \((E_0, \rho_D) = (E_0, \mathcal{O}_D(E_0))\) is a topological coarsening of \((E_0, \mathcal{O}_E(E_0))\) since \(D \subset C(E; \mathbb{R}^+)\). So,

\[
\Psi|_D(\mathbb{R}^+; E_0, \mathcal{O}_E(E_0)) \in C \left(D \left(\mathbb{R}^+; E_0, \mathcal{O}_E(E_0)\right); D(\mathbb{R}^+; \mathbb{R})^{\mathfrak{a}(D)}\right)
\]

by \((10.2.38)\) and Proposition 9.6.1 (c) (with \(E = (E_0, \rho_D)\) and \(S = (E_0, \mathcal{O}_E(E_0))\)). Hence, we have by \((10.2.37)\) and \((10.2.39)\) that

\[
\Psi|_V \in C \left(V, \mathcal{O}_{D(\mathbb{R}^+; E)}(V); D(\mathbb{R}^+; \mathbb{R})^{\mathfrak{a}(D)}\right).
\]

**Step 3:** Show the continuity of \((\Psi|_V)^{-1}\). \(D(\mathbb{R}^+; \mathbb{R})^{\mathfrak{a}(D)}\) is a Polish space, so its subspace \(\Psi(V)\) is metrizable by Proposition 9.1.4 (b). According to [Mun00 Theorem 21.3], showing the continuity of \((\Psi|_V)^{-1}\) is equivalent to showing that

\[
(9.6.12)\]

implies \((9.6.10)\) for all \(\{y_k\}_{k \in \mathbb{N}} \subset V\).

We suppose \((9.6.12)\) holds, fix \(u \in \mathbb{R}^+, J(y_0)\), define \(p_u \equiv \min\{p \in \mathbb{N} : p > u + 1\}\) and let \(\{y_k\}_{k \in \mathbb{N}}\) be as in \((9.6.9)\). Observing that

\[
\{y_k[0, u+1]\}_{k \in \mathbb{N}} \cup \{y[0, u+1]\} \subset D([0, u+1]; A_{p_u}, \mathcal{O}_E(A_{p_u}))
\]

we have

\[
y_k^u \rightarrow y_0^u \text{ as } k \uparrow \infty \text{ in } D([0, u+1]; A_{p_u}, \mathcal{O}_E(A_{p_u}))
\]

by Lemma 9.6.12 with \(E = (A_{p_u}, \mathcal{O}_E(A_{p_u}))\) and \(D = D_{A_{p_u}}\). This implies

\[
y_k^u \rightarrow y_0^u \text{ as } k \uparrow \infty \text{ in } D([0, u+1]; E_0, \mathcal{O}_E(E_0))
\]

by Corollary 9.6.3 (with \(E = (E_0, \mathcal{O}_E(E_0))\) and \(A = A_{p_u}\)). The countable collection \(D \subset C(E; \mathbb{R})\) separates points on \((E_0, (E_0, \mathcal{O}_E(E_0)))\) is a baseable space. Hence, it follows by \((10.2.44)\) and Lemma 9.6.13 (with \(E = (E_0, \mathcal{O}_E(E_0))\) that

\[
y_k \rightarrow y_0 \text{ as } k \uparrow \infty \text{ in } D(D(\mathbb{R}^+; E_0), \mathcal{O}_E(E_0))\).
\]

Now, the desired \((9.6.10)\) follows by Corollary 9.6.3 (with \(A = E_0\)).

**Step 4:** Show \((10.2.36)\). The three steps above established \((10.2.35)\).

\[
\varphi(f) \in M \left(E^{\mathbb{R}^+}, \mathcal{B}(E)^{\mathfrak{a}(D)}; \mathbb{R}^{\mathbb{R}^+}, \mathcal{B}(\mathbb{R})^{\sigma(\mathbb{R})}\right), \forall f \in \mathfrak{a}(D)
\]

by \(D \subset C(E; \mathbb{R})\) and Fact 10.1.10 (b) (with \(I = \mathbb{R}^+\)).

\[
\left[\mathcal{B}(\mathbb{R})^{\mathfrak{a}(D)} \big|_{D(\mathbb{R}^+; \mathbb{R})}\right]^{\sigma(\mathfrak{a}(D))} = \sigma \left[\mathcal{B}(\mathbb{R})^{\mathfrak{a}(D)}\right] = \sigma \left[\mathcal{B}(\mathbb{R})^{\mathfrak{a}(D)}\right]
\]

by Proposition 9.6.10 (b) (with \(E = \mathbb{R}\)), the fact that \(D(\mathbb{R}^+; \mathbb{R})\) is a Polish space and Proposition 10.2.4 (d) (with \(S_i = D(\mathbb{R}^+; \mathbb{R})\)).

\[
\Psi|_V \in M \left(V, \mathcal{B}(E)^{\mathfrak{a}(D)} \big|_V ; \Psi(V), \sigma \left[\mathcal{B}(\mathbb{R})^{\mathfrak{a}(D)}\right] \big|_{\Psi(V)}\right)
\]

by \((10.2.45)\), Fact 2.1.3 (b) and \((10.2.46)\).

\[
\mathcal{B}_{D(\mathbb{R}^+; E)}(V) = \sigma \left(\left\{\Psi|_V^{-1}(O) : O \in \mathcal{B}(\mathbb{R})^{\mathfrak{a}(D)}\right\}\right)
\]

\[
= \{\Psi|_V^{-1}(B) : B \in \sigma \left(\mathcal{B}(\mathbb{R})^{\mathfrak{a}(D)}\right)\} \subset \mathcal{B}(E)^{\mathfrak{a}(D)} \big|_V
\]
by \[10.2.35\] and \[10.2.47\]. Now, \[10.2.36\] follows by Lemma \[9.6.6\] (b).

\[\Box\]

**Lemma 10.2.24.** Let \( E \) be a Tychonoff space, \( \mathcal{D} \subset C(E; \mathbb{R}) \) be countable, \( \Psi \models \mathbb{R}[\mathfrak{a}(\mathcal{D})], \varphi \models \otimes \mathfrak{a}(\mathcal{D}), \{ A_p \}_{p \in \mathbb{N}} \subset \mathcal{B}(E) \) and

\[
(10.2.49) \quad V \doteq \bigcap_{p \in \mathbb{N}} \left\{ x \in D(\mathbb{R}^+; E) : x|_{[0,p)} \in A_p^{(0,p)} \right\}.
\]

If \( A_p \subset A_{p+1}, \mathcal{D} \) strongly separates points on \( A_p \) and \( \varphi(A_p) \in \mathcal{C}(\mathbb{R}^{\mathfrak{a}(\mathcal{D})}) \) for all \( p \in \mathbb{N} \), then \( \Psi(V) \in \mathcal{C}(D(\mathbb{R}^+; \mathbb{R}^{\mathfrak{a}(\mathcal{D})})) \).

\[\text{Proof.} \quad D(\mathbb{R}^+; \mathbb{R}^{\mathfrak{a}(\mathcal{D})}) \text{ is a Polish space, so } \Psi(V) \text{ as a subspace is metrizable by Proposition \[9.1.4\] (b). Hence, showing the closeness of } \Psi(V) \text{ is reduced by Fact \[9.1.9\] (with } E = D(\mathbb{R}^+; \mathbb{R}^{\mathfrak{a}(\mathcal{D})}) \text{ and } A = \Psi(V) \text{) to showing that}
\]

\[
(10.2.50) \quad \Psi(y_k) \to z \text{ as } k \to \infty \text{ in } D(\mathbb{R}^+; \mathbb{R}^{\mathfrak{a}(\mathcal{D})})
\]

imply \( z \in \Psi(V) \) for any \( \{ y_k \}_{k \in \mathbb{N}} \subset V \).

Let \( \{ p_f \}_{f \in \mathcal{D}} \) be the one-dimensional projections on \( D(\mathbb{R}^+; \mathbb{R}^{\mathfrak{a}(\mathcal{D})}) \).

\[
(10.2.51) \quad z'(t) \doteq \bigotimes_{f \in \mathfrak{a}(\mathcal{D})} p_f(z)(t), \forall t \in \mathbb{R}^+
\]
defines a member of \( D(\mathbb{R}^+; \mathbb{R}^{\mathfrak{a}(\mathcal{D})}) \) by Fact \[10.1.13\] (c).

\[
(10.2.52) \quad T \doteq \bigcap_{f \in \mathfrak{a}(\mathcal{D})} \mathbb{R}^+ \setminus J(p_f(z))
\]
is countable by Proposition \[3.3.29\] (b) (with \( E = \mathbb{R} \) and \( x = p_f(z) \)).

\[
(10.2.53) \quad \varphi \circ y_k(t) \to z'(t) \text{ as } k \to \infty \text{ in } \mathbb{R}^{\mathfrak{a}(\mathcal{D})}, \forall t \in T
\]
by \[10.2.50\], Fact \[10.1.11\] and Lemma \[9.6.6\] (c). It then follows that

\[
(10.2.54) \quad z'(t) \in \varphi(A_p), \forall t \in [0,p), p \in \mathbb{N}
\]
by \[10.2.53\], the closedness of each \( \varphi(A_p) \) in \( \mathbb{R}^{\mathfrak{a}(\mathcal{D})} \), the denseness of \( T \) in \( \mathbb{R}^+ \) and the right-continuity of \( z' \).

\[
(10.2.55) \quad \varphi|_{A_p} \in \text{im}(A_p, \mathcal{O}_E(A_p); \mathbb{R}^{\mathfrak{a}(\mathcal{D})}), \forall p \in \mathbb{N}
\]
by Lemma \[9.3.4\] (a, c). So, \( (\varphi|_{A_p})^{-1} \circ z'|_{[0,p)} \) is a càdlàg mapping from \([0,p)\) to \( (A_p, \mathcal{O}_E(A_p)) \) for all \( p \in \mathbb{N} \) and, hence,

\[
(10.2.56) \quad y(t) \doteq (\varphi|_{A_p})^{-1}(z'(t)), \forall t \in [0,p), p \in \mathbb{N}
\]
well defines a member of \( V \). Now, one observes \( \Psi(y) = z \) from \[10.2.51\], \[10.2.56\] and the definitions of \( \Psi \) and \( \varphi \).

\[\Box\]

**Lemma 10.2.25.** Let \( E \) and \( S \) be topological spaces, \( \{ A_p \}_{p \in \mathbb{N}} \subset \mathcal{B}(E) \) and \( f \in S^E \) satisfy \( f(A_p) \}_{p \in \mathbb{N}} \subset \mathcal{C}(S) \) and

\[
(10.2.57) \quad f|_{A_p} \in \text{hom}(A_p, \mathcal{O}_E(A_p); f(A_p), \mathcal{O}_S(f(A_p))), \forall p \in \mathbb{N},
\]
\( E_0 \doteq \bigcup_{p \in \mathbb{N}} A_p \) satisfy \( f \in M(E_0, \mathcal{O}_E(E_0); S) \) and \( \{ \mu_i \}_{i \in \mathbb{I}} \subset \mathcal{P}(E) \) satisfy

\[
(10.2.58) \quad \inf_{i \in \mathbb{I}} \mu_i(A_p) \geq 1 - 2^{-p}, \forall p \in \mathbb{N}.
\]

Then, tightness of \( \{ \mu_i \circ f^{-1} \}_{i \in \mathbb{I}} \) implies that of \( \{ \mu_i \}_{i \in \mathbb{I}} \) in \( E_0 \).
LEMMA 10.2.26. Let $E$ be a Tychonoff space, $T \subset \mathbb{R}^+$ be dense, $d \in \mathbb{N}$ and $f \in C_0(E^d; \mathbb{R})$. Then:

(a) For each $T_0 = \{t_1, \ldots, t_d\} \in \mathcal{P}_0(\mathbb{R}^+)$, there exists a $T_p = \{t_{p,1}, \ldots, t_{p,d}\} \in \mathcal{P}_0(T)$ for each $p \in \mathbb{N}$ such that

$$\lim_{p \to \infty} \int_{E_{T_0}} f(x) \mu \circ p_{T_p}^{-1}(dx) = \int_{E_{T_0}} f(x) \mu \circ p_{T_0}^{-1}(dx).$$

(10.2.59)

for all $\mu \in \mathfrak{M}^+(D(\mathbb{R}^+; E), \mathcal{B}(E) \otimes \mathbb{R}^+ |_{D(\mathbb{R}^+; E)})$.

(b) If $\gamma^1, \gamma^2 \in \mathfrak{M}^+(D(\mathbb{R}^+; E), \mathcal{B}(E) \otimes \mathbb{R}^+ |_{D(\mathbb{R}^+; E)})$ satisfy [8.2.12] for all $T_0 = \{t_1, \ldots, t_d\} \in \mathcal{P}_0(T)$, then they satisfy

$$\int_{E_{T_0}} f(x) \gamma^1 \circ p_{T_0}^{-1}(dx) = \int_{E_{T_0}} f(x) \gamma^2 \circ p_{T_0}^{-1}(dx)$$

(10.2.60)

for all $T_0 = \{t_1, \ldots, t_d\} \in \mathcal{P}_0(\mathbb{R}^+)$.              

PROOF. (a) For ease of notation, we define $\phi_S = f \circ p_S$ for each $S = \{s_1, \ldots, s_d\} \in \mathcal{P}_0(\mathbb{R}^+)$. The denseness of $T$ in $\mathbb{R}^+$ allows us to take $T_p = \{t_{p,1}, \ldots, t_{p,d}\} \in \mathcal{P}_0(T)$ for each $p \in \mathbb{N}$ such that

$$\lim_{p \to \infty} \sup_{1 \leq i \leq d} |t_i - t_{p,i}| = 0.$$

(10.2.61)

It follows that

$$\lim_{p \to \infty} |\phi_{T_0}(x) - \phi_{T_p}(x)| = 0, \forall x \in D(\mathbb{R}^+; E)$$

by [10.2.61], the right-continuity of $x \in S$, Fact [10.1.1] and the continuity of $f$. The boundedness of $f$ implies

$$\sup_{p \in \mathbb{N}} \|\phi_{T_p}\|_{\infty} \leq \|f\|_{\infty} < \infty.$$  

(10.2.63)

Now, we have by [10.2.62], [10.2.63] and the Dominated Convergence Theorem that

$$\lim_{n \to \infty} \int_{E_{T_0}} f(x) \mu \circ p_{T_p}^{-1}(dx) = \lim_{n \to \infty} \int_S \phi_{T_p}(y) \mu(dy)$$

(10.2.64)

$$= \int_S \phi_{T_0}(y) \mu(dy) = \int_{E_{T_0}} f(x) \mu \circ p_{T_0}^{-1}(dx).$$

(b) follows immediately by (a) (with $\mu = \gamma^1$ or $\gamma^2$). \hfill \Box

LEMMA 10.2.27. Let $E$ be a Tychonoff space, $S^1, S^2$ and $S$ be dense subsets of $\mathbb{R}^+$, $D \subset C_0(E; \mathbb{R})$ and $\{\mu_n\}_{n \in \mathbb{N}} \cup \{\gamma^1, \gamma^2\} \subset \mathfrak{M}^+(D(\mathbb{R}^+; E), \mathcal{B}(E) \otimes \mathbb{R}^+ |_{D(\mathbb{R}^+; E)})$ satisfy [8.2.12] for each $f \in mc(\Pi_{T_0}(D)) \cup \{1\}$, $T_0 \in \mathcal{P}_0(S^i)$ and $i = 1, 2$. Then:

(a) If $D$ strongly separates points on $E$, then [8.2.13] holds.

(b) If $D$ separates points on $E$, and if each of $\gamma^1 \circ p_{T_0}^{-1}$ and $\gamma^2 \circ p_{T_0}^{-1}$ is $m$-tight for all $T_0 \in \mathcal{P}_0(S)$, then [8.2.13] holds.

\footnote{Recall that Corollary 9.6.7 verified $\mu \circ p_{T_0}^{-1} \in \mathfrak{M}^+(E^{T_0}, \mathcal{B}(E) \otimes T_0)$ for all $T_0 \in \mathcal{P}_0(\mathbb{R}^+)$ and $\mu \in \mathfrak{M}^+(D(\mathbb{R}^+; E), \mathcal{B}(E) \otimes \mathbb{R}^+ |_{D(\mathbb{R}^+; E)}).$}
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Proof. For ease of notation, we define \( \phi_T \triangleq f \circ p_T \) for each \( T \in \mathcal{P}_0(\mathbb{R}^+) \) and \( f \in M_b(E^T; \mathbb{R}) \). The proof is divided into four steps.

Step 1: Show \[10.2.60\] for each \( f \in mc(\Pi^D) \cup \{1\} \) and \( T_0 = \{t_1, \ldots, t_d\} \in \mathcal{P}_0(\mathbb{S}^1) \). Note \[5.1.6\] argued that \( f \in C_b(E^d; \mathbb{R}) \). By Lemma \[10.2.26\] (a) (with \( T = S^2 \)), there exists a \( T_p = \{t_{p,1}, \ldots, t_{p,d}\} \in \mathcal{P}_0(\mathbb{S}^2) \) for each \( p \in \mathbb{N} \) such that

\[
\lim_{p \to \infty} \int_{D(R^+:E)} (\phi_{T_p} - \phi_{T_0})^2(x) dx = \lim_{p \to \infty} \int_{D(R^+:E)} (\phi_{T_p} - \phi_{T_0})^2(x) \mu_n dx = 0, \ \forall n \in \mathbb{N}.
\]

From \[8.2.12\] we get

\[
\lim_{n \to \infty} \int_{D(R^+:E)} \phi_{T_0}(x) \mu_n dx = \int_{D(R^+:E)} \phi_{T_0}(x) \gamma^1 dx
\]
and

\[
\lim_{n \to \infty} \int_{D(R^+:E)} \phi_{T_p}(x) \mu_n dx = \int_{D(R^+:E)} \phi_{T_p}(x) \gamma^2 dx, \ \forall p \in \mathbb{N}.
\]

Let \( \epsilon \in (0, \infty) \) be arbitrary and \( n_0 = 1 \). By \[10.2.66\] and \[10.2.67\], we inductively choose an \( n_p \in \mathbb{N} \cap (n_{p-1} - 1, \infty) \) for each \( p \in \mathbb{N} \) such that

\[
\left| \int_{D(R^+:E)} \phi_{T_0}(x) \mu_{n_p} dx - \int_{D(R^+:E)} \phi_{T_0}(x) \gamma^1 dx \right|< \epsilon.
\]

From Triangle Inequality and \[10.2.68\] it follows that

\[
\left| \int_{E_{T_0}} f(x) \gamma^1 \circ p_{T_0}^{-1}(dx) - \int_{E_{T_0}} f(x) \gamma^2 \circ p_{T_0}^{-1}(dx) \right|
= \left| \int_{D(R^+:E)} \phi_{T_0}(x) \gamma^1(dx) - \int_{D(R^+:E)} \phi_{T_0}(x) \gamma^2(dx) \right|
\leq \left| \int_{D(R^+:E)} \left( \phi_{T_0} - \phi_{T_p} \right)(x) \gamma^2(dx) \right|
+ \epsilon, \ \forall p \in \mathbb{N}.
\]

Now, \[10.2.60\] follows by \[10.2.65\], letting \( p \uparrow \infty \) in \[10.2.69\] and then letting \( \epsilon \downarrow 0 \). For each \( f \in mc(\Pi^D) \cup \{1\} \) and \( T_0 = \{t_1, \ldots, t_d\} \in \mathcal{P}_0(\mathbb{R}^+) \). This step follows by Step 1, the denseness of \( S^1 \) in \( \mathbb{R}^+ \) and Lemma \[10.2.26\] (b) (with \( T = S^1 \)).

Step 3: Verify \( \gamma^1 \circ p_{T_0}^{-1} = \gamma^2 \circ p_{T_0}^{-1} \) for each \( T_0 \in \mathcal{P}_0(\mathbb{R}^+) \) in (a). For each \( i = 1, 2 \), we let \( (D(R^+:E), \mathcal{M}_{n_0}, \nu^i) \) be the completion of \( (D(R^+:E), \gamma^i) \) and find by Lemma \[10.2.26\] (with \( \mu = \gamma^i \) and \( \nu = \nu^i \)) that \( \nu^i \circ p_{T_0}^{-1} \) is a Borel

\[30\] Herein, the domain of \( \nu^i \circ p_{T_0}^{-1} \) is larger than \( \mathcal{B}(E_{T_0}) \), so we consider \( \nu^i \circ p_{T_0}^{-1} \) as a member of \( \mathcal{M}^+(E_{T_0}) \).
extension of $\gamma^i \circ p^{-1}_T$. It follows that

$$
(10.2.70) \int_{E^T_0} f(x)\nu^1 \circ p^{-1}_T(dx) = \int_{E^T_0} f(x)\nu^2 \circ p^{-1}_T(dx)
$$

for all $f \in mc[\Pi^{T_0}(D)] \cup \{1\}$ by Step 2 and Fact 10.2.12 (with $d = \kappa(T_0)$) and $\nu_1 = \nu^i \circ p^{-1}_T$ by Lemma 9.4.3 (a) (with $d = \kappa(T_0)$) and Fact 10.1.20 (a) (with $E = E^{T_0}$ and $D = mc[\Pi^{T_0}(D)] \cup \{1\}$), which of course implies $\gamma^i \circ p^{-1}_T = \gamma^2 \circ p^{-1}_T$.

**Step 4:** Verify $\gamma^1 \circ p^{-1}_T = \gamma^2 \circ p^{-1}_T$ for each $T_0 \in \mathcal{P}_0(R^+)$ in (b). When $T_0 \in \mathcal{P}_0(S)$, $\gamma^1 \circ p^{-1}_T = \gamma^2 \circ p^{-1}_T$ by Step 2 and Lemma 10.2.17 (b) (with $d = \kappa(T_0)$) and so (10.2.60) holds for all $f \in mc[\Pi^{T_0}(C_b(E; R))]$. For general $T_0 \in \mathcal{P}_0(R^+)$, the key equality (10.2.60) holds for all $f \in mc[\Pi^{T_0}(C_b(E; R))]$ by (5.1.3) (with $D = C_b(E; R)$ and $d = \kappa(T_0)$), the denseness of $S$ in $R^+$ and Lemma 10.2.26 (b) (with $T = S$). $C_b(E; R)$ strongly separates points on Proposition 9.3.1 (a, c). Then, one follows the argument of Step 3 (with $D = C_b(E; R)$) to show $\gamma \circ p^{-1}_T = \gamma^2 \circ p^{-1}_T$. □

**LEMMA 10.2.28.** Let $E$ be a topological space, $(E_0, \mathcal{S}_E(E_0))$ be a Tychonoff subspace of $E$, $y_0 \in S_0 \subset \mathbb{D}_0 = D(R^+; E_0, \mathcal{S}_E(E_0))$, $\mathcal{Y} = \mathcal{B}(E)^{\otimes R^+}$ and $X$ be a mapping from $(\Omega, \mathcal{F}, \mathbb{P})$ to $E^{R^+}$. Then:

(a) If $\Omega \setminus \{X = Z\} \in \mathcal{N}(\mathbb{P})$ for some $Z \in M(\Omega, \mathcal{F}; \mathbb{D}_0)$, then $X$ is an $E$-valued càdlàg process.

(b) If $X$ is an $E$-valued process, $\mathbb{P}(X \in S_0) = 1$ and $S_0$ satisfies $\mathcal{B}(\mathbb{D}_0)|_{S_0} = \mathcal{Y}|_{S_0}$, then

$$
(10.2.71) \quad Y = \text{var}(X; \Omega, X^{-1}(S_0), y_0) \in M(\Omega, \mathcal{F}; S_0, \mathcal{D}_0(S_0))
$$

and $\mathbb{P}(X = Y \in S_0) = 1$.

**Proof.** (a) follows by Lemma 9.6.6 (b) (with $E = (E_0, \mathcal{S}_E(E_0))$) and Lemma 10.1.28 (a) (with $E = E^{R^+}$ and $S = \mathbb{D}_0$).

(b) follows by Lemma 10.1.28 (b, c) (with $(E, S, \mathcal{Y}) = (E^{R^+}, S_0, \mathcal{D}_0(S_0))$). □

**PROPOSITION 10.2.29.** Let $E$ be a topological space, $(\Omega, \mathcal{F}, \{\mathcal{G}_t\}_{t \geq 0}, \mathbb{P})$ be a stochastic basis, $k \in \mathbb{N}$, $\xi \in M(\Omega, \mathcal{F}; \mathcal{M}^+(E))$ and $(\Omega, \mathcal{F}, X)$ be an $\mathcal{M}^+(E)$-valued process. In addition, suppose either of the following hypotheses is true:

(i) $f \in C_b(E; R^k)$.

(ii) $E$ is a perfectly normal space\(^{31}\) (especially metrizable or Polish) space and $f \in M_b(E; R^k)$.

Then, $f^* \circ \xi \in M(\Omega, \mathcal{F}; R^k)$ and $\varpi(f^* \circ \xi)$ is an $R^k$-valued process. If, in addition, $X$ is a $\mathcal{G}_t$-adapted, measurable or $\mathcal{G}_t$-progressive process, then $\varpi(f^*) \circ X$ also has the corresponding measurability.

**Proof.** Under (i), $f^* \in C_b(\mathcal{M}^+(E); R^k)$ by the definition of weak topology and Fact 2.1.4 (b). Under (ii), $f^* \in M_b(\mathcal{M}^+(E); R^k)$ by Lemma 9.4.13 and Fact 2.1.4 (b). The result now follows by Fact 10.1.29 (a) (with $E = \mathcal{M}^+(E)$, $S = R^k$ and $f = f^*$).

\(^{31}\)The notion of perfectly normal space was mentioned in 9.4.13.
10.3. Auxiliary results about replication

FACT 10.3.1. Let $E$ be a topological space, $(E_0, \mathcal{F}; \tilde{E}, \tilde{\mathcal{F}})$ be a base over $E$ and $d, k \in \mathbb{N}$. If $\tilde{f} \in C(E^d; \mathbb{R}^k)$ has a replica $f$, then:

(a) $\|\tilde{f}\|_\infty = \|f|_{E^d_0}\|_\infty \leq \|f\|_\infty$.

(b) $\tilde{f}^+ = \tilde{f}^+$ and $\tilde{f}^- = \tilde{f}^-$.

PROOF. (a) follows by the fact $f|_{E^d_0} = \tilde{f}|_{E^d_0}$, the denseness of $E_0$ in $\tilde{E}$ and the continuities of $f$ and $\tilde{f}$.

(b) follows by the facts $\tilde{f}^+ = \tilde{f}^+|_{E^d_0}$, $\tilde{f}^- = \tilde{f}^+|_{E^d_0}$ and the continuities of $\tilde{f}^+$, $\tilde{f}^-$.

□

LEMMA 10.3.2. Let $E$ be a topological space, $\mathcal{D} \subset C(E; \mathbb{R})$ separate points on $E$, $d \in \mathbb{N}$, $I$ be a countable index set and $\Gamma_i \subset 2^{\mathbb{N}}(E, \mathcal{B}(E)^{d})$ be $\mathcal{m}$-tight for each $i \in I$. Then, there exists a base $(E_0, \mathcal{F}; \tilde{E}, \tilde{\mathcal{F}})$ over $E$ such that $E_0 \in \mathcal{K}_\sigma^m(E)$ and $\Gamma_i$ is tight in $(E_0^d, \mathcal{G}_E(E_0)^d)$ for all $i \in I$. In particular, $\mathcal{F}$ can be taken within $\mathcal{D} \cup \{1\}$ when $\mathcal{D} \subset C(E; \mathbb{R})$.

PROOF. Without loss of generality, we let $I = \mathbb{N}$. By the $\mathcal{m}$-tighness of each $\Gamma_i$, there exist \{\(K_{p,i}\)\}_{p,i \in \mathbb{N}} \subset \mathcal{K}_\sigma^m(E^d)$ satisfying

\[
\sup_{\mu \in \Gamma_i} \mu(E^d \setminus K_{p,i}) \geq 1 - 2^{-p}, \forall p, i \in \mathbb{N}.
\]

$E$ is a Hausdorff space by Proposition 9.2.1 (e) (with $A = E$).

\[
\{K_{p,i,j} \equiv p_j(K_{p,i}) : 1 \leq j \leq d, p, i \in \mathbb{N}\} \subset \mathcal{K}_\sigma^m(E) \subset \mathcal{B}(E)
\]

by Proposition 9.1.2 (c) and Lemma 3.3.23 (b) (with $A = K_{p,i}$). So,

\[
E_0 \equiv \bigcup_{i \in \mathbb{N}} \bigcup_{p \in \mathbb{N}} \bigcup_{j=1}^{d} K_{p,i,j} \in \mathcal{K}_\sigma^m(E).
\]

We have by Corollary 9.1.13 (a) and (10.3.1) that

\[
\prod_{j=1}^{d} K_{p,i,j} \subset \mathcal{K} \left( E_0^d, \mathcal{G}_E(E_0)^d \right), \forall p, i \in \mathbb{N}
\]

and

\[
\mu(E_0^d) \geq \mu \left( \prod_{j=1}^{d} K_{p,i,j} \right) \geq \mu(K_{p,i}) \geq 1 - 2^{-p}, \forall \mu \in \Gamma_i, p, i \in \mathbb{N}.
\]

thus proving the tightness of each $\Gamma_i$ in $(E_0^d, \mathcal{G}_E(E_0)^d)$. $E_0$ is a $\mathcal{D}$-baseable subset of $E$ by (10.3.3) and Proposition 3.3.26 (b, c) (with $A = E_0$). Now, the result follows by Lemma 3.3.7 (a, c) (with $D_0 = \emptyset$).

□

FACT 10.3.3. Let $E$ be a topological space, $(E_0, \mathcal{F}; \tilde{E}, \tilde{\mathcal{F}})$ be a base over $E$, $(\Omega, \mathcal{F}, \mathbb{P}; X)$ be an $E$-valued process, $(\Omega, \mathcal{F}, \mathbb{P}; Y)$ be an $\tilde{E}$-valued process and $\mathcal{T} \subset \mathbb{R}^+$. If $X$ satisfies (6.1.9), and if

\[
\mathbb{P} \left( \bigotimes \mathcal{F} \circ X_t = \bigotimes \tilde{\mathcal{F}} \circ Y_t \right) \geq \mathbb{P} \left( \bigotimes \mathcal{F} \circ X_t \in \bigotimes \tilde{\mathcal{F}}(\tilde{E}) \right) , \forall t \in \mathcal{T},
\]

then $X$ and $Y$ satisfy (6.2.9).
PROOF. It follows by \[\text{(10.3.6)}, \text{(3.1.1)}\] and \[\text{(3.1.3)}\] that
\[
1 = \mathbb{P}(X_t \in E_0)
\]
\[= \mathbb{P}(\bigotimes \widehat{F} \circ Y_t = \bigotimes F \circ X_t \in \bigotimes \widehat{F}(E), X_t \in E_0)
\]
\[\leq \mathbb{P}(Y_t = \left(\bigotimes \widehat{F}\right)^{-1} \circ \bigotimes F \circ X_t = X_t \in E_0), \forall t \in T.
\]
\[\square
\]

FACT 10.3.4. Let \(E\) be a topological space, \((E_0, \mathcal{F}; \widehat{E}, \widehat{F})\) be a base over \(E\) and \(\{(\Omega^i, \mathcal{F}^i, \mathbb{P}^i, X^i)\}_{i \in I}\) be \(E\)-valued processes satisfying \(T\)-PSMTC in \(E_0^{T_0}\). Then for each \(T_0 \in \mathcal{P}_{\mathbb{P}}(T)\):

(a) \([X^i_{T_0}]_{i \in I}\) is sequentially \(m\)-tight in \(E_0^{T_0}\) and \(\mathbb{P}(X^i_{T_0} \in E_0^{T_0}) = 1\).

(b) \([\mu^i = \mathbb{E}(\mathbb{P}^i \circ X^{-1}_0)]_{i \in I} \subseteq \mathcal{P}(E_0)\) exists for some \(I_{T_0} \in \mathcal{P}_{\mathbb{P}}(I)\).

PROOF. Let \(I \triangleq \{\mathbb{P}^i \circ X^{-1}_0\}_{i \in I}\) and \(A \triangleq E_0^{T_0}\). Then, (a) follows by Lemma 10.2.19(b) (with \(I = T_0, S_i = E\) and \(A_i = E_0\)) and Fact 10.1.26 (with \(E = E_0^{T_0}\) and \(\mathcal{F} = \mathcal{B}(E)^{\otimes_{T_0}}\)). (b) follows by Lemma 3.1.3(e) (with \(A = E_0\)) and Proposition 5.1.3 (with \(I = T_0\) and \(S_i = E\)).

\[\square
\]

LEMMA 10.3.5. Let \(E\) be a topological space, \((E_0, \mathcal{F}; \widehat{E}, \widehat{F})\) be a base over \(E\), \(T \subset \mathbb{R}^+\), \((\Omega, \mathcal{F}, \mathbb{P}; X)\) be \(E\)-valued process and \((\Omega, \mathcal{F}, \mathbb{P}; Y)\) be an \(\widehat{E}\)-valued process. Then:

(a) If
\[
\inf_{f \in \mathcal{F}, t \in T} \mathbb{P}\left(f \circ X_t = \widehat{f} \circ Y_t\right) = 1,
\]
then
\[
\mathbb{P}\left(f \circ X_{T_0} = \widehat{f} \circ Y_{T_0}\right) = 1, \forall f \in \mathcal{C}(\Pi_{T_0}(\mathcal{F})), T_0 \in \mathcal{P}_{\mathbb{P}}(T).
\]

Moreover, \[\text{(6.2.9)}\] implies \[\text{(10.3.8)}\].

(b) If \[\text{(10.3.8)}\] holds (especially \[\text{(6.2.9)}\] holds) and \(Y\) is càdlàg, then \(X\) is \((T, \mathcal{F})\)-càdlàg.

(c) If \(\mathbb{E}[f \circ X_{T_0}] = \mathbb{E}[\widehat{f} \circ Y_{T_0}]\) for all \(f \in \mathcal{C}(\Pi_{T_0}(\mathcal{F}))\) and \(T_0 \in \mathcal{P}_{\mathbb{P}}(\mathbb{R}^+)\) (especially \[\text{(10.3.8)}\] or \[\text{(6.2.9)}\] holds) and \(X\) is stationary, then \(Y\) is stationary.

(d) If \(\mathbb{E}[f \circ X_{T_0}] = \mathbb{E}[\widehat{f} \circ Y_{T_0}]\) for all \(f \in \mathcal{C}(\Pi_{T_0}(\mathcal{F}))\) and \(T_0 \in \mathcal{P}_{\mathbb{P}}(T)\) (especially \[\text{(10.3.8)}\] or \[\text{(6.2.9)}\] holds), \(T\) is conull, \(X\) is stationary and \(Y\) is càdlàg, then \(Y\) is stationary.

(e) If \(A \in \mathcal{B}^*(E)\) (especially \(A \in \mathcal{K}_0^*(E)\)) satisfies \(A \subset E_0\) and \[\text{(6.2.28)}\] holds, then \(\mathcal{F}^X = \mathcal{F}^Y\). If, in addition, \(Y\) is stationary, then \(X\) is stationary.

(f) If \[\text{(6.2.9)}\] holds, and if \(f \in \mathcal{M}_0(E_0^{T_0}; \mathbb{R})\) and \(T_0 \in \mathcal{P}_{\mathbb{P}}(T)\) satisfy \(\overline{f} \in \mathcal{M}_0(E_0^{T_0}; \mathbb{R})\) (especially if \(E_0^{T_0} \in \mathcal{B}^*(E_0^{T_0})\)), then
\[
\mathbb{P}\left(f \circ X_{T_0} = \overline{f} \circ Y_{T_0}\right) = 1.
\]

\[32\text{The notion of } T\text{-PSMTC was introduced in Definition } 6.4.1\]
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Proof. (a) \[ \text{(10.3.8)} \] implies \[ \text{(10.3.9)} \] by properties of uniform convergence. \[ \text{(10.2.9)} \] implies \[ \text{(10.3.8)} \] by \text{3.1.11}.

(b) \((\pi(f) \circ Y)_{f \in F}\) are all càdlàg processes by Fact \text{10.1.31} (a) (with \(E = \hat{E}\), \(S = R\) and \(X = Y\)). Then, (b) follows by (a).

(c) One finds by (a) (with \(T = R^\ast\)) and the stationarity of \(X\) that

\[
E \left[ \hat{f} \circ Y_{T_0} - \hat{f} \circ Y_{T_0+c} \right] = E \left[ f \circ X_{T_0} - f \circ X_{T_0+c} \right] = 0
\]

for all \(c \in (0, \infty)\) and \(T_0 \in \mathcal{P}_0(R^\ast)\). Then, (c) follows by Corollary \text{3.1.11} (a) (with \(d = N(T_0)\) and \(A = \hat{E}^d\)).

(d) Fixing \(T_0 \in \mathcal{P}_0(R^\ast)\), one finds by (a) and the stationarity of \(X\) that \[ \text{(10.3.11)} \] holds for all \(c \in \text{conull\ set} \).

\[
\mathbf{S}_{T_0} = \bigcap_{t \in T_0} \{ c \in (0, \infty) : t + c \in T \}.
\]

Then, (d) follows by a similar argument to the proof of Proposition \text{6.4.14} (c).

(e) Any \(A \in \mathcal{X}_\sigma(E)\) satisfying \(A \subset E_0\) belongs to \(\mathcal{B}^a(E)\) by Corollary \text{3.1.15} (b) (with \(d = 1\)). For each fixed \(t \in R^\ast\), we let \(\Omega^0_t = \{ \omega \in \Omega : X_t(\omega) = Y_t(\omega) \in A \}\) and find by \text{6.2.28}, the \(P\)-completeness of \(\mathcal{B}^a\) and Lemma \text{3.1.14} (a) (with \(d = 1\)) that \(\Omega \cap \Omega^0_t \in \mathcal{M}(\mathcal{P}) \subset \mathcal{F}\),

\[
X_t^{-1} (B) \cap \Omega^0_t = X_t^{-1} (B \cap A) \cap \Omega^0_t,
\]

\[
\forall B \in \mathcal{B}(E)
\]

and

\[
Y_t^{-1} (V) \cap \Omega^0_t = Y_t^{-1} (V \cap A) \cap \Omega^0_t,
\]

\[
\forall V \in \mathcal{B} (\hat{E}).
\]

Thus, \(\mathcal{F}^X = \mathcal{F}^Y\) by their \(P\)-completeness. When \(Y\) is stationary, we fix \(T_0 \in \mathcal{P}_0(R^\ast)\) and find by Lemma \text{3.1.14} (d) (with \(d = \delta(T_0)\)) and \text{6.2.28} that

\[
P (X_{T_0} \in B) = P (Y_{T_0} \in B \cap A_{T_0}^s)
\]

\[
P (Y_{T_0+c} \in B \cap A_{T_0}^s) = P (X_{T_0+c} \in B)
\]

for all \(B \in \mathcal{B}(E) \otimes T_0\) and \(c \in (0, \infty)\), which gives the stationarity of \(X\).

(f) follows by the definition of \(\tilde{f}\) and Proposition \text{4.1.6} (b) (with \(d = \delta(T_0)\)).

Lemma 10.3.6. Let \(E\) be a topological space, \((E_0, F; \hat{E}, \hat{F})\) be a base over \(E\), \(T \subset R^\ast\), \(\mathcal{G}_T = \text{mc}[\Pi T_0(F \setminus \{1\})]\) for each \(T_0 \in \mathcal{P}_0(T)\), \{(\Omega^n, \mathcal{F}^n, P^n; X^n)\}_{n \in N}\) be \(E\)-valued processes satisfying \text{6.4.34}, \(\hat{X}_n \in \text{rep}(X^n; E_0; F)\) for each \(n \in N\), \((\Omega, \mathcal{F}, P; Y)\) be an \(\hat{E}\)-valued process, \((\Omega, \mathcal{F}, P; X)\) be an \(E\)-valued process satisfying \text{6.1.14} and \(\hat{X} \in \text{rep}(X; E_0, \mathcal{F})\). Then:

(a) \{(X^n)\}_{n \in N}\ is \((T, F \setminus \{1\})\)-FDC if and only if \{(\hat{X}^n)\}_{n \in N}\ is \((T, \hat{F} \setminus \{1\})\)-FDC.

(b) If \{(X^n)\}_{n \in N}\ is \((T, F \setminus \{1\})\)-AS, then

\[
\lim_{t \to \infty} E^n \left[ \hat{f} \circ \hat{X}_{T_0} - \hat{f} \circ \hat{X}_{T_0+c} \right] = 0
\]

\[\text{\[\text{10.3.16}}\]\]

\[\text{\[\text{3.1.11}}\]\] Completeness of measure space was specified in \text{2.1.2} Completeness of filtration was specified in \text{4.1.6}.
for all $f \in G_{T_0}$, $T_0 \in \mathcal{P}_0(T)$ and $c$ (if any) in the set $S_{T_0}$ defined in 10.3.12.

(c) \((6.2.9)\) is equivalent to
\begin{equation}
\lim_{n \to \infty} E^n \left[ \hat{f} \left( \hat{X}^n_{T_0} \right) \right] = E \left[ \hat{f} \left( Y_{T_0} \right) \right]
\end{equation}
for all $f \in G_{T_0}$ and $T_0 \in \mathcal{P}_0(T)$.

(d) If \((10.3.9)\) holds (especially \((6.2.9)\) holds), then \((6.2.2)\) is equivalent to
\begin{equation}
\lim_{n \to \infty} E^n \left[ \hat{f} \left( \hat{X}^n_{T_0} \right) \right] = E \left[ \hat{f} \left( \hat{X}_{T_0} \right) \right]
\end{equation}
for all $f \in G_{T_0}$ and $T_0 \in \mathcal{P}_0(T)$.

(e) \((6.2.12)\) holds if and only if \((10.3.17)\) holds for all $f \in G_{T_0}$ and $T_0 \in \mathcal{P}_0(T)$.

(f) \((6.2.2)\) holds for all $f \in G_{T_0}$ and $T_0 \in \mathcal{P}_0(T)$ if and only if
\begin{equation}
\hat{X}^n \xrightarrow{D(T)} \hat{X} \text{ as } n \uparrow \infty.
\end{equation}

(g) \((6.2.1)\) implies \((10.3.19)\).

In particular, the conclusions above are when if \(\{X^n\}_{n \in \mathbb{N}}\) (resp. $X$) satisfies the stronger condition\(^{34}\) \((6.5.9)\) (resp. \((6.1.9)\)) than \((6.4.34)\) (resp. \((6.1.14)\)).

**Proof.** (a) - (c) follow by Fact 10.1.32 (b) and Proposition 6.1.8 (a) (with $X = X^n$).

(d) \((6.2.9)\) implies \((10.3.9)\) by Lemma 10.3.5 (a). \((10.3.9)\) implies \((6.1.14)\). Hence, (d) follows by (c) and Proposition 6.1.8 (a).

(e) follows by \((3.1.16)\) and Corollary 3.1.11 (a) (with $(d, A) = (\mathcal{N}(T_0), \hat{E}^d)$).

(f) follows by Proposition 6.1.8 (a) and (d, e) (with $Y = \hat{X}$).

(g) follows by $\mathcal{F} \subset C_0(E; \mathbb{R})$, Fact 6.2.6 and (f).

**Lemma 10.3.7.** Let $E$ be a topological space, $(\Omega, \mathcal{F}, \mathbb{P}; X)$ be an $E$-valued measurable process, $(E_0, \mathcal{F}; \hat{E}, \hat{\mathcal{F}})$ be a base over $E$, $T \in (0, \infty)$ and $(\Omega, \mathcal{F}, \mathbb{P}^T; X^T) = \text{rap}_T(X)$. Then:

(a) If \((6.1.14)\) or \((6.1.9)\) holds for some conull $T \subset \mathbb{R}^+$, then $X^T$ satisfies
\begin{equation}
\inf_{t \in \mathbb{R}^+} \mathbb{P}^T \left( \bigotimes_{t \in \mathbb{R}^+} \mathcal{F} \circ X^T_t \in \bigotimes_{t \in \mathbb{R}^+} \hat{\mathcal{F}}(\hat{E}) \right) = 1
\end{equation}
or
\begin{equation}
\inf_{t \in \mathbb{R}^+} \mathbb{P}^T \left( X^T_t \in E_0 \right) = 1.
\end{equation}
respectively.

(b) If \((6.3.15)\) holds for $S_0 \subset E^{R^+}_0$, then
\begin{equation}
\mathbb{P}^T \left( X^T \in S_0 \right) = 1.
\end{equation}

(c) If $\hat{X} \in \text{rep}_m(X; E_0, \mathcal{F})$, then $\hat{X}^T = \text{rap}_T(X) \in \text{rep}_m(X^T; E_0, \mathcal{F})$\(^{35}\)

(d) If $\hat{X} \in \text{rep}_c(X; E_0, \mathcal{F})$, then $\hat{X}^T = \text{rap}_T(X) \in \text{rep}_c(X^T; E_0, \mathcal{F})$.

\(^{34}\)We compared these conditions in Fact 6.1.9 and Fact 6.5.1.

\(^{35}\)The notations \(\text{"{r}e}_m(\cdot; \cdot)\) and \(\text{"{r}e}_c(\cdot; \cdot)\) were specified in Notation 6.1.3.
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PROOF. (a) One finds by the conullity of $T$ that
\[
\inf_{t \in \mathbb{R}^+} \mathbb{P}^T \left( \bigotimes \mathcal{F} \circ X_t^T \in \bigotimes \mathcal{F}(\hat{E}) \right) \\
\geq \frac{1}{T} \int_{[t,T+t[} \mathbb{P} \left( \bigotimes \mathcal{F} \circ X_{\tau} \in \bigotimes \mathcal{F}(\hat{E}) \right) d\tau = 1
\]
(10.3.23)
and
\[
\inf_{t \in \mathbb{R}^+} \mathbb{P}^T (X_t^T \in \mathcal{E}_0) \geq \frac{1}{T} \int_{[t,T+t[} \mathbb{P} (X_{\tau} \in \mathcal{E}_0) d\tau = 1.
\]
(10.3.24)
(b) follows by the fact that $(X^T)^{-1}(S_0) \supset \mathbb{R}^+ \times X^{-1}(S_0)$.
(c) follows by Proposition 6.3.3 (with $X = \hat{X}$), (6.1.1) and the fact that
\[
\frac{1}{T} \int_0^T \mathbb{P} \left( \bigotimes \mathcal{F} \circ \hat{X}_{t+t} = \bigotimes \hat{\mathcal{F}} \circ \hat{X}_{t+t} \right) d\tau
\]
(10.3.25)
\[
\geq \frac{1}{T} \int_0^T \mathbb{P} \left( \bigotimes \mathcal{F} \circ X_{t+t} \in \bigotimes \mathcal{F}(\hat{E}) \right) d\tau, \forall t \in \mathbb{R}^+.
\]
(d) follows by (c), Fact 6.3.1 and Lemma 10.1.36 (b) (with $X = \hat{X}$). \qed

LEMMA 10.3.8. Let $E$ be a topological space, $(E_0, \mathcal{F}, \hat{E}, \mathcal{F})$ be a base over $E$, $T \subset \mathbb{R}^+$, $\{(\Omega^n, \mathcal{F}_n, \mathcal{P}^n; X^n)\}_{n \in \mathbb{N}}$ be $E$-valued processes satisfying (6.4.34) (especially (6.5.9)), $\hat{X}_n \in \text{rep}(X^n; E_0, \mathcal{F})$ for each $n \in \mathbb{N}$ and $(\Omega, \mathcal{F}, \mathcal{P}; Y)$ be an $\hat{E}$-valued process. Then, (6.4.34) implies $\hat{\mathcal{F}} \circ Y = \mathcal{F}_T(\{(\hat{\mathcal{F}} \circ X^n)_{n \in \mathbb{N}}\})$.

PROOF. (6.4.34) is weaker than (6.5.9) by Fact 6.5.1 (with $I = \mathbb{N}$). Define $Z \triangleq \bigotimes \mathcal{F} \circ Y$, $Z^n \triangleq \bigotimes \mathcal{F} \circ X^n$ and $\xi^n \triangleq \bigotimes \mathcal{F} \circ X^n$ for each $n \in \mathbb{N}$. One finds that
\[
\inf_{t \in T \cap \mathbb{N}} \mathbb{P}^T (\xi_t^n = Z_t^n) = 1
\]
(10.3.26)
by Proposition 6.1.8 (a). We fix $T_0 \in \mathcal{A}(T)$ and put $d \triangleq \mathbb{N}(T_0)$. As mentioned in Note 6.1.5, $\{\hat{X}_{n \in \mathbb{N}}, \{\xi_t^n\}_{n \in \mathbb{N}}, \{Z^n\}_{n \in \mathbb{N}}, Y$ and $Z$ all have Borel finite-dimensional distributions, so (6.2.12) implies
\[
\hat{X}_{n \in \mathbb{N}} \Rightarrow Y_{T_0} \text{ as } k \uparrow \infty \text{ on } \mathbb{R}^d.
\]
(10.3.27)
One finds that
\[
\varphi = \bigotimes_{t \in T_0} \left( \bigotimes \hat{\mathcal{F}} \right) \circ p_t \in C \left[ \hat{E}^d; (\mathbb{R}^\infty)^d \right]
\]
(10.3.28)
by (3.1.3) and Fact 2.1.4 (a, b). Hence, it follows by (10.3.28), (10.3.27) and Continuous Mapping Theorem (Theorem 10.1.23 (a)) that
\[
\mathbb{Z}_{T_0}^d = \varphi \circ \hat{X}_{T_0}^d = \varphi \circ Y_{T_0} = Z_{T_0} \text{ as } k \uparrow \infty \text{ on } (\mathbb{R}^\infty)^d.
\]
(10.3.29)
$(\mathbb{R}^\infty)^d$ and $\mathcal{P}((\mathbb{R}^\infty)^d)$ are Polish spaces by Proposition 9.1.11 (f) and Theorem 9.4.10 (b) (with $E = (\mathbb{R}^\infty)^d$). Now, the result follows by (10.3.29), (10.3.26) and Fact 10.1.33 (with $E = \mathbb{R}^\infty$ and $X^i = \xi^n$). \qed
Bibliography

[AB06] Charalambos D. Aliprantis and Kim C. Border, *Infinite dimensional analysis*, third ed., Springer, Berlin, 2006, A hitchhiker’s guide. MR 2378491

[ABGP06] Siva R. Athreya, Richard F. Bass, Maria Gordina, and Edwin A. Perkins, *Infinite dimensional stochastic differential equations of ornstein–uhlenbeck type*, Stochastic Processes and their Applications 116 (2006), no. 3, 381 – 406.

[BBK00] A. G. Bhatt, A. Budhiraja, and R. L. Karandikar, *Markov property and ergodicity of the nonlinear filter*, SIAM J. Control Optim. 39 (2000), no. 3, 928–949. MR 1786337 (2001m:93122)

[BFG13] Christian Bayer and Peter K. Friz, *Cubature on Wiener space: pathwise convergence*, Appl. Math. Optim. 67 (2013), no. 2, 261–278. MR 3027591

[Bil68] Patrick Billingsley, *Convergence of probability measures*, John Wiley & Sons, Inc., New York-London-Sydney, 1968. MR 0233396 (38 #1718)

[BK93a] Abhay G. Bhatt and Rajeeva L. Karandikar, *Invariant measures and evolution equations for Markov processes characterized via martingale problems*, Ann. Probab. 21 (1993), no. 4, 2246–2268. MR 1245309 (95d:60120)

[BK93b] ---, *Weak convergence to a Markov process: the martingale approach*, Probab. Theory Related Fields 96 (1993), no. 3, 335–351. MR 1231928 (94i:60047)

[BK99] Amarjit Budhiraja and Harold J. Kushner, *Approximation and limit results for nonlinear filters over an infinite time interval*, SIAM J. Control Optim. 37 (1999), no. 6, 1946–1979. MR 1720146 (2000i:93080)

[BK10] Douglas Blount and Michael A. Kouritzin, *On convergence determining and separating classes of functions*, Stochastic Process. Appl. 120 (2010), no. 10, 1898–1907. MR 2673979 (2012e:60008)

[BKK95] Abhay G. Bhatt, G. Kallianpur, and Rajeeva L. Karandikar, *Uniqueness and robustness of solution of measure-valued equations of nonlinear filtering*, Ann. Probab. 23 (1995), no. 4, 1895–1938. MR 1379173

[BN12] G. Bachman and L. Narici, *Functional analysis*, Dover Books on Mathematics, Dover Publications, 2012.

[Bog07] V. I. Bogachev, *Measure theory. Vol. I, II*, Springer-Verlag, Berlin, 2007. MR 2267655 (2008g:28002)

[BS89] Erwin Bolthausen and Uwe Schmuck, *On the maximum entropy principle for uniformly ergodic Markov chains*, Stochastic Process. Appl. 33 (1989), no. 1, 1–27. MR 1027105

[Bud01] A. Budhiraja, *Ergodic properties of the nonlinear filter*, Stochastic Process. Appl. 95 (2001), no. 1, 1–24. MR 1847089 (2002e:60065)

[CDP13] J. Theodore Cox, Richard Durrett, and Edwin A. Perkins, *Voter model perturbations and reaction diffusion equations*, Astérisque (2013), no. 349, vi+113. MR 3075759

[CG83] J. Theodore Cox and David Griffeath, *Occupation time limit theorems for the voter model*, Ann. Probab. 11 (1983), no. 4, 876–893. MR 714952 (85b:60096)

[CMP10] J. Theodore Cox, Mathieu Merle, and Edwin Perkins, *Coexistence in a two-dimensional Lotka-Volterra model*, Electron. J. Probab. 15 (2010), no. 38, 1190–1266. MR 2678390 (2011k:60318)

[Cox88] J. T. Cox, *Some limit theorems for voter model occupation times*, Ann. Probab. 16 (1988), no. 4, 1559–1569. MR 958202 (89m:60251)

[Daw93] Donald A. Dawson, *Measure-valued Markov processes*, École d’Eté de Probabilités de Saint-Flour XXI—1991, Lecture Notes in Math., vol. 1541, Springer, Berlin, 1993, pp. 1–260. MR 1242575
[DK99] Peter Donnelly and Thomas G. Kurtz, *Genealogical processes for Fleming-Viot models with selection and recombination*, Ann. Appl. Probab. 9 (1999), no. 4, 1091–1148. MR 1728556 (2001h:92029)

[DK20a] Chi Dong and Michael A. Kouritzin, *On filtering equations and stationary solutions*.

[DK20b] ———, *Solving martingale problems in general settings*.

[DK21] ———, *On convergence of approximate filters to stationary filters*.

[Dud02] R. M. Dudley, *Real analysis and probability*, Cambridge Studies in Advanced Mathematics, vol. 74, Cambridge University Press, Cambridge, 2002, Revised reprint of the 1989 original. MR 1932358 (2003k:60001)

[DZ98] Amir Dembo and Ofer Zeitouni, *Large deviations techniques and applications*, second ed., Applications of Mathematics (New York), vol. 38, Springer-Verlag, New York, 1998. MR 1619036 (99d:60030)

[EK86] Stewart N. Ethier and Thomas G. Kurtz, *Markov processes: Characterization and convergence*, Wiley Series in Probability and Mathematical Statistics: Probability and Mathematical Statistics, John Wiley & Sons, Inc., New York, 1986. MR 838085 (88a:60130)

[EK93] S. N. Ethier and Thomas G. Kurtz, *Fleming-Viot processes in population genetics*, SIAM J. Control Optim. 31 (1993), no. 2, 345–386. MR 1205982 (94d:60131)

[EK98] ———, *Coupling and ergodic theorems for Fleming-Viot processes*, Ann. Probab. 26 (1998), no. 2, 533–561. MR 1626158 (99f:60074)

[FH14] Peter K. Friz and Martin Hairer, *A course on rough paths*, Universitext, Springer, Cham, 2014, With an introduction to regularity structures. MR 3289027

[Fit88] P. J. Fitzsimmons, *Construction and regularity of measure-valued Markov branching processes*, Israel J. Math. 64 (1988), no. 3, 337–361 (1989). MR 955575

[Fri87] J. Fritz, *Gradient dynamics of infinite point systems*, Ann. Probab. 15 (1987), no. 2, 478–514.

[FV10] Peter K. Friz and Nicolas B. Victoir, *Multidimensional stochastic processes as rough paths*, Cambridge Studies in Advanced Mathematics, vol. 120, Cambridge University Press, Cambridge, 2010, Theory and applications. MR 2604669

[GK12] D.J.H. Garling and G. Köthe, *Topological vector spaces i*, Grundlehren der mathematischen Wissenschaften, Springer Berlin Heidelberg, 2012.

[HS79] R. Holley and D. W. Stroock, *Central limit phenomena of various interacting systems*, Ann. of Math. (2) 110 (1979), no. 2, 333–393. MR 549491 (82e:60163)

[Jak97a] A. Jakubowski, *The almost sure Skorokhod representation for subsequences in non-metric spaces*, Teor. Veroyatnost. i Primenen. 42 (1997), no. 1, 209–216. MR 1453342

[Jak97b] Adam Jakubowski, *A non-Skorohod topology on the skorohod space*, Electron. J. Probab. 2 (1997), no. 4, pp. 21. MR 1473862

[Jak12] ———, *New characterizations of the s topology on the skorohod space*, Electron. Commun. Probab. 0 (2012), no. 0, 1–11.

[JS03] Jean Jacod and Albert N. Shiryaev, *Limit theorems for stochastic processes*, second ed., Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 288, Springer-Verlag, Berlin, 2003. MR 1943877 (2003j:60001)

[Kal97] Olav Kallenberg, *Foundations of modern probability*, Probability and its Applications (New York), Springer-Verlag, New York, 1997. MR 1464694

[KK20] Michael A. Kouritzin and Thomas G. Kurtz, *Weak existence and approximation in systems of stochastic differential equations*.

[KL08] Michael A. Kouritzin and Hongwei Long, *On extending classical filtering equations*, Statist. Probab. Lett. 78 (2008), no. 18, 3195–3202. MR 2479478 (2010d:60105)

[KO88] T. G. Kurtz and D. L. Ocone, *Unique characterization of conditional distributions in nonlinear filtering*, Ann. Probab. 16 (1988), no. 1, 80–107. MR 920257 (88m:93146)

[Kou16] Michael A. Kouritzin, *On tightness of probability measures on Skorokhod spaces*, Trans. Amer. Math. Soc. 368 (2016), no. 8, 5675–5700. MR 3458395

[KS17] Michael A. Kouritzin and Wei Sun, *Weak laplace principles on topological spaces*.

[Kun71] Hiroshi Kunita, *Asymptotic behavior of the nonlinear filtering errors of Markov processes*, J. Multivariate Anal. 1 (1971), 365–393. MR 0301812 (46 #967)
BIBLIOGRAPHY

[Kur75] Thomas G. Kurtz, *Semigroups of conditioned shifts and approximation of Markov processes*, Ann. Probability 3 (1975), no. 4, 618–642. MR 0383544

[Kur91] _____, *Random time changes and convergence in distribution under the Meyer-Zheng conditions*, Ann. Probab. 19 (1991), no. 3, 1010–1034. MR 1112405

[KX95] Gopinath Kallianpur and Jie Xiong, *Stochastic differential equations in infinite-dimensional spaces*, Institute of Mathematical Statistics Lecture Notes—Monograph Series, 26, Institute of Mathematical Statistics, Hayward, CA, 1995, Expanded version of the lectures delivered as part of the 1993 Barrett Lectures at the University of Tennessee, Knoxville, TN, March 25-27, 1993, With a foreword by Balram S. Rajput and Jan Rosinski. MR 1465436 (98h:60001)

[Lyo94] Terry Lyons, *Differential equations driven by rough signals. I. An extension of an inequality of L. C. Young*, Math. Res. Lett. 1 (1994), no. 4, 451–464. MR 1302388

[Lyo98] Terry J. Lyons, *Differential equations driven by rough signals*, Rev. Mat. Iberoamericana 14 (1998), no. 2, 215–310. MR 1654527

[Mal95] Paul Malliavin, *Integration and probability*, Graduate Texts in Mathematics, vol. 157, Springer-Verlag, New York, 1995, With the collaboration of Hélène Airault, Leslie Kay and Gérard Letac, Edited and translated from the French by Kay, With a foreword by Mark Pinsky. MR 1335234

[Mit83] Itaru Mitoma, *Tightness of probabilities on $C([0,1];S')$ and $D([0,1];S')$*, Ann. Probab. 11 (1983), no. 4, 989–999. MR 714961 (85f:60008)

[Mun00] J.R. Munkres, *Topology*, Featured Titles for Topology Series, Prentice Hall, Incorporated, 2000.

[MZ84] P.-A. Meyer and W. A. Zheng, *Tightness criteria for laws of semimartingales*, Ann. Inst. H. Poincaré Probab. Statist. 20 (1984), no. 4, 353–372. MR 771895 (86c:60008)

[Nik06] Ashkan Nikeghbali, *An essay on the general theory of stochastic processes*, Probab. Surv. 3 (2006), 345–412. MR 2280298

[OS13] Martin Ondrejat and Jan Seidler, *On existence of progressively measurable modifications*, Electron. Commun. Probab. 18 (2013), no. 20, 1–6.

[Per02] Edwin Perkins, *Dawson-Watanabe superprocesses and measure-valued diffusions*, Lectures on probability theory and statistics (Saint-Flour, 1999), Lecture Notes in Math., vol. 1781, Springer, Berlin, 2002, pp. 125–324. MR 1915445

[Pro90] Philip Protter, *Stochastic integration and differential equations*, Applications of Mathematics (New York), vol. 21, Springer-Verlag, Berlin, 1990, A new approach. MR 1037262

[SH12] T.P. Speed and T. Hida, *Brownian motion*, Stochastic Modelling and Applied Probability, Springer New York, 2012.

[Sko76] A. Skorokhod, *On infinite systems of stochastic differential equations*, Methods Funct. Anal. Topology 5 (1999), no. 4, 54–61.

[Sko01] A. Skorokhod, *On stochastic differential equations in a configuration space*, Georgian Mathematical Journal 8 (2001), no. 2.

[Sri98] S. M. Srivastava, *A course on Borel sets*, Graduate Texts in Mathematics, vol. 180, Springer-Verlag, New York, 1998. MR 1619545 (99d:04002)

[Str85] Christophe Stricker, *Lois de semimartingales et critères de compacité*, Séminaire de probabilités, XIX, 1983/84, Lecture Notes in Mathematics, vol. 1163, Springer, Berlin, 1985, pp. 209–217. MR 889478

[SW99] H. H. Schaefer and M. P. Wolff, *Topological vector spaces*, second ed., Graduate Texts in Mathematics, vol. 3, Springer-Verlag, New York, 1999. MR 1741419

[Szp76] Jacques Szpirglas, *Sur des équations différentielles stochastiques intervenant dans le filtrage non linéaire*, C. R. Acad. Sci. Paris Sér. A-B 283 (1976), no. 8, Av, A671–A674. MR 0423506 (54 #11482)

[Top70] Flemming Topsoe, *Topology and measure*, Lecture Notes in Mathematics, Vol. 133, Springer-Verlag, Berlin-New York, 1970. MR 0422560

[Yos80] Kōsaku Yosida, *Functional analysis*, sixth ed., Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 123, Springer-Verlag, Berlin-New York, 1980. MR 617913