Determinantal divisors of products of matrices over Dedekind domains

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Abstract
Given three lists of ideals of a Dedekind domain, the question is raised whether there exist two matrices \( A \) and \( B \) with entries in the given Dedekind domain, such that the given lists of ideals are the determinantal divisors of \( A \), \( B \), and \( AB \), respectively. To answer this question, necessary and sufficient conditions are developed in this article.

Key words: Determinantal divisor, Dedekind domain

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1. Introduction

When dealing with Hecke algebras with respect to a group \( G \) and a subgroup \( U \), one is interested in the characterisation of \( k \in UgU \) and \( k \in UgUhU \) for \( g, h, k \in G \), since these conditions appear in formulas for the calculation of products in the algebra (\[1\] I (4.4)). In the case that \( U \) is the unimodular group \( \text{GL}_n(\mathfrak{o}) \) over a Dedekind domain \( \mathfrak{o} \), the condition \( k \in UgU \) can be characterised by the comparison of determinantal divisors (\[2\] Satz 11). Considering the condition \( k \in UgUhU \), one has to deal with determinantal divisors of products of matrices. Since the obtained results are of interest not only in the field of Hecke algebras (see e. g. \[3\]), in this text they are developed and presented without any means and notation from Hecke theory; applications to Hecke theory will be published in another article named “Hecke algebras related to unimodular groups over Dedekind domains”.

2. Preliminaries and notation

Denote by \( \mathfrak{o} \) a Dedekind domain, by \( K \) its field of fractions, and by \( \mathfrak{o}^\times \) its group of unities. For every integer \( n \) let \( I_n \) be the set of \((n \times n)\) matrices with entries in \( \mathfrak{o} \) and non-zero determinant; furthermore, denote by \( U_n \) the set of matrices in \( I_n \) with determinant in \( \mathfrak{o}^\times \) (in other words \( U_n = \text{GL}_n(\mathfrak{o}) \) and \( I_n = \text{GL}_n(K) \cap \mathfrak{o}^{\times n} \)). Following \[4\], for every \( k, n \in \mathbb{N} \) with \( k \leq n \) fix an enumeration \((M_{n,k,l})_l\) of the subsets of \([1, \ldots, n]\) containing exactly \( k \) elements, and for \( 1 \leq i, j \leq \binom{n}{k} \) denote by \( A_{i,j} \) the submatrix of \( A \) obtained from an \( A \in \mathfrak{o}^{\times n} \) by deleting all but the rows numbered by elements of

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M_{n,k,j} and all but the columns numbered by elements of M_{n,k,j}. Then the k-th derivative of A is the matrix A^{[k]} := (\tilde{a}_{ij}) with \tilde{a}_{ij} = \det A_{ij} for all 1 \leq i, j \leq \binom{n}{k}. With this notation, the k-th determinantal divisor of A is the g. c. d. (in an ideal theoretic sense) of all entries of A^{[k]} – is denoted by \delta_k(A). For convenience let \delta_0(A) = 0 for all k \leq 0 and \delta_k(A) = \{0\} for all k > n. Then define the k-th elementary divisor \epsilon_k(A) by 
\epsilon_k(A) = \delta_k(A)\delta_{k-1}(A)^{-1}, if \delta_k(A) \neq \{0\}, and \epsilon_k(A) = \{0\} otherwise. (From the general results in \cite{5} one can conclude that all \epsilon_k(A) are ideals in \sigma.) By these definitions it is possible to exchange determinantal divisors for elementary divisors without loss of information. With this notation, the rank of A is biggest r \in \mathbb{N}_0 satisfying \delta_r(A) \neq \{0\}, and the column class \mathcal{C}(A) of A is the ideal class of the g. c. d. of a nonzero column of A^{[r]} (which is independent of the choice of the column).

In order to give handy formulations of the developed theorems, in this article the triple ((a_1, \ldots, a_n), (b_1, \ldots, b_n), (c_1, \ldots, c_n)) is called realisable, if there exist matrices A, B \in I_n such that \delta_k(A) = a_k and \delta_k(B) = b_k as well as \delta_k(AB) = \epsilon_k hold for all 1 \leq k \leq n.

Before dealing with the existence of products in the next sections, it is sensible to characterise the existence of single matrices with prescribed determinantal divisors or elementary divisors first. One can cite the following slight variation of \cite{5} Satz 7.

\textbf{2.1 Theorem.} Let n \in \mathbb{N} with n \geq 2 and b_1, \ldots, b_n be ideals in \sigma. Then there exists a matrix A \in \sigma^{\max} satisfying \epsilon_k(A) = b_k for every 1 \leq k \leq n, if and only if \ epsilon_k(b_k) for all 1 \leq k < n and b_1 \cdot \cdots \cdot b_n is a principal ideal (including \{0\}).

To end this section of preliminaries, two known auxiliary results that are needed in the rest of this article are stated. The first is part of \cite{2} Satz 11.

\textbf{2.2 Theorem.} Let n \in \mathbb{N} and A, B \in \sigma^{\max}. Then there exist P, Q \in U_n satisfying B = PAQ if and only if \delta_k(A) = \delta_k(B) for all 1 \leq k \leq n and \mathcal{C}(A) = \mathcal{C}(B). (In the case A, B \in I_n, the condition \mathcal{C}(A) = \mathcal{C}(B) is always true.)

A slight variation of the normal form constructed in the considerations preceding \cite{2} Satz 11 directly yields the following theorem.

\textbf{2.3 Theorem.} Let A \in I_n and A' = (\begin{array}{c|c}
A & 0
\end{array}) \in \sigma^{2n \times 2n}. Then there exist A_1, \ldots, A_n \in \sigma^{2 \times 2} and P, Q \in U_{2n}, such that A_k = (\begin{array}{c|c}
\epsilon_k & 0
\end{array}) and \delta_1(A_k) = \epsilon_k(A) for all 1 \leq k \leq n and

\[ PA'Q = \begin{pmatrix}
A_1 & & \\\[ \vdots & \ddots & \\
& & A_n
\end{pmatrix} \]

\section{Necessary conditions for realisability}

In this section, some results on determinantal divisors of products of given matrices are shown. These results are then translated into propositions on realisability.

\textbf{3.1 Theorem.} For every A, B \in I_n one has \delta_k(A)\delta_k(B) | \delta_k(AB) for all 1 \leq k \leq n.
Proof. By definition, \( \mathcal{d}_k(A) = \mathcal{d}_1(A^{[k]}) \) holds. Cauchy-Binet's formula (6.39) then yields \( \mathcal{d}_k(AB) = \mathcal{d}_1((AB)^{[k]}) = \mathcal{d}_1(A^{[k]}B^{[k]}) \), and since \( \mathcal{d}_1(C)\mathcal{d}_1(D) = \mathcal{d}_1(CD) \) holds for all matrices in \( \sigma^{mxm} \), one obtains \( \mathcal{d}_k(A)\mathcal{d}_k(B) = \mathcal{d}_1(A^{[k]}\mathcal{d}_1(B^{[k]})) = \mathcal{d}_1(A^{[k]}B^{[k]}) = \mathcal{d}_k(AB) \).

The just proven theorem can be seen as a “lower bound” for \( \mathcal{d}_k(AB) \). An upper bound, which generalises an unpublished result of Koecher ([7] Thm. I.7.1), can also be given.

3.2 Theorem. For every \( A, B \in \mathbb{I}_n \) and \( 1 \leq k \leq n \) one has

\[
\mathcal{d}_k(AB) \mid \mathcal{d}_k(A)\mathcal{d}_n(B)\mathcal{d}_{n-k}(B)^{-1} + \mathcal{d}_k(B)\mathcal{d}_n(A)\mathcal{d}_{n-k}(A)^{-1}.
\]

Proof. First assume that \( \sigma \) is a principal ideal domain. According to the Smith normal form ([8] Thm. II.9) there exist \( a_1, \ldots, a_n, b_1, \ldots, b_n \in \sigma \) and \( P, Q_1, Q_2, R \in \mathbb{U}_n \) such that

\[
A' = \begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{pmatrix} \quad \text{and} \quad B' = \begin{pmatrix} b_1 & & \\ & \ddots & \\ & & b_n \end{pmatrix}
\]

satisfy \( A = PA'Q_1 \) and \( B = Q_2B'R \). Then

\[
\mathcal{d}_k(AB) = \mathcal{d}_k(PA'Q_1Q_2B'R) = \mathcal{d}_k(A'Q_1Q_2B')
\]

since determinantal divisors of a matrix are invariant under multiplication with elements of \( U_2 \) ([4] 10). Let \( 1 \leq r \leq \binom{n}{k} \) such that \( M_{n,k,r} = \{1, \ldots, k\} \). By the definition of determinantal divisors,

\[
\mathcal{d}_k(AB) = \mathcal{d}_k(A'Q_1Q_2B') \mid \det(A'Q_1Q_2B')(r)
\]

holds for all \( 1 \leq s \leq \binom{n}{k} \), and since

\[
\det(A'Q_1Q_2B')(r) = a_1 \cdots a_k \cdot \det(Q_1Q_2B')(r) + a_1 \cdots a_n \cdot \det(Q_1Q_2)(r) \cdot b_{n-k+1} \cdots b_n
\]

is true for all \( 1 \leq s \leq \binom{n}{k} \), one obtains

\[
\mathcal{d}_k(AB) \mid a_1 \cdots a_k \cdot b_{n-k+1} \cdots b_n \cdot \sum_{1 \leq s \leq \binom{n}{k}} \det(Q_1Q_2)(r) \cdot a = a_1 \cdots a_k \cdot b_{n-k+1} \cdots b_n
\]

since the g. c. d. of the \( \det(Q_1Q_2)(r) \cdot a \) equals \( \sigma \) (otherwise Leibniz’ determinant formula would yield \( \det(Q_1Q_2) \not\in \sigma^* \), which contradicts \( Q_1Q_2 \in \mathbb{U}_n \)). This shows the relation \( \mathcal{d}_k(AB) \mid \mathcal{d}_k(A)\mathcal{d}_n(B)\mathcal{d}_{n-k}(B)^{-1} \), and \( \mathcal{d}_k(AB) \mid \mathcal{d}_k(B)\mathcal{d}_n(A)\mathcal{d}_{n-k}(A)^{-1} \) can be proven analogously, which completes the proof of the proposition for the case that \( \sigma \) is a principal ideal domain.

Now consider an arbitrary Dedekind domain \( \sigma \) and the localisation \( \sigma_p \) of \( \sigma \) at a prime ideal \( p \) of \( \sigma \). Denote by \( a_1, \ldots, a_n, b_1, \ldots, b_n, c_1, \ldots, c_n \) the determinantal divisors of \( A \) and \( B \) as well as \( AB \), where the matrices are considered to be elements of \( \sigma_p^{uxu} \). Since \( \sigma_p \) is a principal ideal domain, the already proven part yields \( e_k \mid a_k b^{-1}_{n-k} + b_k a^{-1}_n \) and thus \( a_{n-k} b_{n-k} c_k \mid a_{n-k} b_k + b_{n-k} a_k \). Using

\[
v_p(a_{n-k} b_{n-k} c_k \cap \sigma) = v_p(a_{n-k} \cap \sigma) + v_p(b_{n-k} \cap \sigma) + v_p(c_k \cap \sigma) = v_p(\mathcal{d}_{n-k}(A)\mathcal{d}_{n-k}(B)\mathcal{d}_k(AB))
\]
and the similarly obtained relation
\[ v_p((a_{n-k}b_n + b_{n-k}a_n) \cap \sigma) = v_p(\mathfrak{d}_{n-k}(A)\mathfrak{d}_k(A)\mathfrak{d}_n(B) + \mathfrak{d}_{n-k}(B)\mathfrak{d}_k(B)\mathfrak{d}_n(A)), \]
from \( a_{n-k}b_n \mid a_{n-k}a_n + b_{n-k}b_n \) follows
\[ v_p(\mathfrak{d}_{n-k}(A)\mathfrak{d}_n(B)\mathfrak{d}_k(B)) \leq v_p(\mathfrak{d}_{n-k}(A)\mathfrak{d}_k(A)\mathfrak{d}_n(B) + \mathfrak{d}_{n-k}(B)\mathfrak{d}_k(B)\mathfrak{d}_n(A)), \]
and since this is valid for every prime ideal \( p \) of \( \sigma \), this yields that \( \mathfrak{d}_{n-k}(A)\mathfrak{d}_n(B)\mathfrak{d}_k(B) \) divides \( \mathfrak{d}_{n-k}(A)\mathfrak{d}_k(A)\mathfrak{d}_n(B) + \mathfrak{d}_{n-k}(B)\mathfrak{d}_k(B)\mathfrak{d}_n(A) \) and thus proves the proposition. \( \square \)

Now the achieved results are put together.

3.3 Corollary. The triple \((a_1, \ldots, a_n), (b_1, \ldots, b_n), (c_1, \ldots, c_n)\) is not realisable, if one of the following conditions is violated:

1. \( a_n, b_n, \) and \( c_n \) are principal ideals.
2. \( a_2 \mid a_1 \) and \( b_2 \mid b_2 \) as well as \( c_2 \mid c_1 \) hold.
3. \( a_{k-1}^{-1}a_k \mid a_k \) and \( b_{k-1}^{-1}b_k \mid b_k \) as well as \( c_{k-1}^{-1}c_k \mid c_k \) hold for all \( 3 \leq k \leq n \).
4. \( a_nb_n = c_n \).
5. \( a_kb_k \mid c_k \) holds for all \( 1 \leq k < n \).
6. \( c_k \mid a_kb_k(a_n^{-1}a_{n-k}^{-1} + b_n^{-1}b_{n-k}^{-1}) \) holds for all \( 1 \leq k < n \).

Proof. (1)–(3) come from 2.1, (4) is the multiplicativity of the determinant, (5) comes from 3.1, and (6) from 3.2. \( \square \)

4. Sufficient conditions for realisability

In this section, it is shown that in certain circumstances the realisability of determinantal divisors of a product can be assured. Therefore first an auxiliary result has to be stated.

4.1 Lemma. Let \( n \in \mathbb{N} \) and \( A_1, \ldots, A_n \in \sigma^{2 \times 2} \) be matrices of rank 1 satisfying the divisibility condition \( \mathfrak{d}_1(A_1) \mid \mathfrak{d}_1(A_2) \mid \cdots \mid \mathfrak{d}_1(A_n) \). Let
\[ A = \begin{pmatrix} A_1 & & \\ & \ddots & \\ & & A_n \end{pmatrix}. \]
Then \( c_k(A) = \mathfrak{d}_1(A_k) \) for all \( 1 \leq k \leq n \), and the column class of \( A \) is the product of the column classes of \( A_1, \ldots, A_n \).

Proof. This result is obtained from the definitions of column class and elementary divisors by elementary considerations using the special structure of \( A \). \( \square \)

The next theorem is the main result of this section.
4.2 Theorem. Let $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ be ideals in $\mathfrak{a}$ such that there exist matrices $A, B \in I_n$ with $a_k(A) = a_k$ and $a_k(B) = b_k$ for all $1 \leq k \leq n$. Then one can find matrices $\tilde{A}, \tilde{B} \in I_n$ satisfying $\tilde{a}_k(\tilde{A}) = a_k$ and $\tilde{a}_k(\tilde{B}) = b_k$ as well as $\tilde{a}_k(\tilde{A}\tilde{B}) = a_k b_k$ for all $1 \leq k \leq n$.

Proof. Let

$$E' = \begin{pmatrix} E_n \\ 0 \end{pmatrix} \in \mathfrak{a}^{2n \times n}$$

and

$$\tilde{E} = \begin{pmatrix} E_n \\ 0 \end{pmatrix} \in \mathfrak{a}^{n \times 2n}$$

where $E_n$ denotes the $(n \times n)$ identity matrix, and let $A' = E'AE = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}$ as well as $B' = E'B\tilde{E} = \begin{pmatrix} B & 0 \\ 0 & 0 \end{pmatrix}$. Denote by $A'$ and $B'$ the normal forms in the sense of 2.3 of $A'$ and $B'$, respectively, and let $C' = A'(B')^T$ (where $(B')^T$ denotes the transpose of $B'$). Furthermore, choose a matrix $C \in \mathfrak{a}^{n \times n}$ satisfying $\varepsilon_k(C) = \varepsilon_k(A)\varepsilon_k(B)$ and thus $\varepsilon_k(C) = \varepsilon_k(A)\varepsilon_k(B)$ for all $1 \leq k \leq n$ according to 2.2.

Let $C' = E'CE = \begin{pmatrix} C & 0 \\ 0 & 0 \end{pmatrix}$ and show (using 2.2) that there exist $P, Q \in U_{2n}$ satisfying $C' = PC^*Q$. By definition, $C^*$ is a block diagonal matrix, where the $(2 \times 2)$ blocks $C_1, \ldots, C_n$ on the diagonal are of rank 1 and satisfy $\tilde{a}_1(C_k) = \varepsilon_k(A)\varepsilon_k(B)$ as well as $\mathfrak{c}(C_k) = \tilde{a}_1(A)\tilde{a}_1(B)$ for all $1 \leq k \leq n$, where $\mathfrak{c}$ denotes the principal ideal class of $K$. Since $\tilde{a}_1(C_{k-1}) = \varepsilon_k(A)\varepsilon_k(B)$ holds for all $1 \leq k \leq n$, 4.1 implies $\varepsilon_k(C') = \varepsilon_k(A)\varepsilon_k(B) = \varepsilon_k(C) = \varepsilon_k(C')$ for all $1 \leq k \leq n$ and

$$\mathfrak{c}(C') = \tilde{a}_1(C_1) \cdots \tilde{a}_1(C_n) = \varepsilon_1(C_1) \cdots \varepsilon_1(C_n) = \varepsilon_n(C)\mathfrak{c} = \mathfrak{c}(C).$$

Since furthermore $\varepsilon_k(C') = \{0\}$ holds for all $n < k \leq 2n$, 2.2 implies the existence of $P, Q \in U_{2n}$ such that $C' = PC^*Q$.

According to 2.3 there exist $P_1, Q_1, P_2, Q_2 \in U_{2n}$ such that $A' = P_1A'Q_1$ and $B' = P_2B'Q_2$. Using $C = E'CE'$ as well as the definitions of $A'$ and $B'$, one obtains

$$C = \begin{array}{cccc}
EPP_1 & E & EQ_1 & E^T \tilde{B}^T \tilde{E}^T P_2 Q_2 E' \\
R_1 & = & R_2 & = R_3
\end{array}$$

with $R_1, R_2, R_3 \in U_n$. Thus, $C \in U_nU_nB_nU_n$ follows, and since $U_nB_nU_n = U_nU_n$ (deducible from 2.2), one has $C \in U_nA_nU_n$ and thus can find $\tilde{A} \in U_nA_nU_n$ and $\tilde{B} \in BU_n$ satisfying $\tilde{A}\tilde{B} = C$. Since 4.10 implies $\tilde{a}_k(\tilde{A}) = \varepsilon_k(A)$ and $\tilde{a}_k(\tilde{B}) = \varepsilon_k(B)$ for all $1 \leq k \leq n$, the already proven assertion $\tilde{a}_k(C) = \varepsilon_k(A)\varepsilon_k(B)$ for all $1 \leq k \leq n$ yields the desired result.

The translation of the last theorem into the language of realisability yields the following corollary.

4.3 Corollary. The triple $((a_1, \ldots, a_n), (b_1, \ldots, b_n), (\varepsilon_1, \ldots, \varepsilon_n))$ is realisable, if the conditions (1)–(3) of 3.4 are satisfied and $\varepsilon_k = a_k b_k$ holds for all $1 \leq k \leq n$.

To close this article, a characterisation of realisability in the special case that $\mathfrak{a}$ is a principal ideal domain and $n = 2$ is investigated.

4.4 Theorem. Let $\mathfrak{a}$ be a principal ideal domain. The triple $((a_1, a_2), (b_1, b_2), (\varepsilon_1, \varepsilon_2))$ is realisable, if and only if the following conditions are satisfied:
(1) $a_1^2 \mid a_2$ and $b_1^2 \mid b_2$ as well as $c_1^2 \mid c_2$ hold.

(2) $a_2b_2 = c_2$.

(3) $a_1b_1 \mid c_1 \mid a_1b_1(a_1^{-2}a_2 + b_1^{-2}b_2)$.

Proof. It can be derived from [3,3] that the given conditions are necessary for realizability, so it remains to show that (1)–(3) imply realizability. Let (1)–(3) be satisfied. Denote by $a_1, a_2, b_1, b_2, c_1, c_2 \in \sigma$ generators of $a_1, a_2, b_1, b_2, c_1, c_2$, respectively. Let $d = c_1a_1^{-1}b_1^{-1}$. Then $d \in \sigma$ according to (3). Furthermore, let

$$A = \begin{pmatrix} a_1d & a_1 \\
 a_2a_1^{-1} & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} b_1 & 0 \\
 0 & b_2b_1^{-1} \end{pmatrix}.$$ 

Then $\delta_1(A) = a_1\sigma$, since $a_1 \mid a_2a_1^{-1}$ according to (1); analogously, $\delta_1(B) = b_1\sigma$ holds. Moreover, $\delta_2(A) = a_2\sigma$ and $\delta_2(B) = b_2\sigma$ as well as $\delta_2(AB) = c_2\sigma$ are satisfied (the latter following from (2)), so it remains to prove $\delta_1(AB) = c_1\sigma$. For the first determinantal divisor one obtains

$$\delta_1(AB) = \delta_1\left( \begin{pmatrix} a_1db_1 & a_1b_2b_1^{-1} \\
 a_2a_1^{-1}b_1 & 0 \end{pmatrix} \right) = a_1b_1(d\sigma + b_2b_1^{-2}\sigma + a_2a_1^{-2}\sigma),$$

and since (3) implies $d = c_1a_1^{-1}b_1^{-1} \mid a_2a_1^{-2}\sigma + b_2b_1^{-2}\sigma$, the above stated equation yields $\delta_1(AB) = a_1b_1d\sigma = c_1\sigma$, which completes the proof. 

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