KAM THEORY FOR LOWER DIMENSIONAL TORI
WITHIN THE REVERSIBLE CONTEXT 2

MIKHAIL B. SEVRYUK

To the memory of Vladimir Igorevich Arnold who is so unexpectedly gone

Abstract. The reversible context 2 in KAM theory refers to the situation where \( \dim \text{Fix} G < \frac{1}{2} \text{codim} T \), here \( \text{Fix} G \) is the fixed point manifold of the reversing involution \( G \) and \( T \) is the invariant torus one deals with. Up to now, the persistence of invariant tori in the reversible context 2 has been only explored in the extreme particular case where \( \dim \text{Fix} G = 0 \) [M. B. Sevryuk, Regul. Chaotic Dyn. 16 (2011), no. 1–2, 24–38]. We obtain a KAM-type result for the reversible context 2 in the general situation where the dimension of \( \text{Fix} G \) is arbitrary. As in the case where \( \dim \text{Fix} G = 0 \), the main technical tool is J. Moser’s modifying terms theorem of 1967.

1. Introduction

1.1. Reversible Systems. Any diffeomorphism \( G : \mathcal{M} \to \mathcal{M} \) of a manifold \( \mathcal{M} \) induces the inner automorphism \( \text{Ad}_G \) of the Lie algebra of vector fields on \( \mathcal{M} \) according to the formula \( \text{Ad}_G V = TG(V \circ G^{-1}) \). We will always assume the phase space \( \mathcal{M} \) to be finite dimensional and connected.

Definition 1. A vector field \( V \) on \( \mathcal{M} \) is said to be equivariant with respect to \( G \) (or \( G \)-equivariant) if \( \text{Ad}_G V = V \) and is said to be weakly reversible with respect to \( G \) (or weakly \( G \)-reversible) if \( \text{Ad}_G V = -V \). If the diffeomorphism \( G \) is an involution (i.e., its square is the identity transformation), then the word “weakly” is usually omitted.

For instance, the Newtonian equations of motion \( \ddot{\mathbf{r}} = \mathcal{F}(\mathbf{r}, \dot{\mathbf{r}}), \mathbf{r} \in \mathbb{R}^K \), are reversible with respect to the phase space involution \( G : (\mathbf{r}, \dot{\mathbf{r}}) \mapsto (\mathbf{r}, -\dot{\mathbf{r}}) \) if and only if the forces \( \mathcal{F} \) are even in the velocities \( \dot{\mathbf{r}} \) (e.g., are independent of \( \dot{\mathbf{r}} \)). This is probably the best known example of a reversible vector field. The papers [17,25] present general surveys of the theory of finite dimensional reversible systems with extensive bibliographies. There exists a remarkable deep similarity between reversible and Hamiltonian dynamics [1,3,17,25,27,28]. In particular, a great deal of the Hamiltonian KAM theory can be carried over to the reversible realm [3,7–11,22,23,27,31–35]. Of the three great founders of KAM theory (A. N. Kolmogorov, V. I. Arnold, and J. K. Moser, all in blessed remembrance now), the last two—Arnold [1,3] and Moser [20,
—made fundamental contributions not only to the Hamiltonian version of this theory but also to its reversible counterpart. The reversible KAM theory started with Moser’s paper [20]. Arnold [1] suggested that KAM theory for reversible systems can be generalized to weakly reversible systems, and this idea was soon realized [3, 27]. A brief survey of the reversible KAM theory as it stood in 1997 is presented in the review [32]. Of subsequent works, one may mention e.g. the references [7, 8, 18, 33–35, 37, 40–45].

1.2. Invariant Tori of Reversible Flows. The key object of KAM theory is an invariant torus carrying quasi-periodic motions. In the case of weakly reversible systems, such a torus is always assumed to be invariant not only under the dynamical system itself but also under the corresponding reversing diffeomorphism [3, 27, 32, 37]. So, consider an $n$-torus $T$ ($n \geq 1$) invariant under both the flow of a weakly reversible vector field and its reversing diffeomorphism $G$. Suppose that $T$ carries conditionally periodic motions, i.e., in some angular coordinates $\varphi \in T^n = (\mathbb{R} / 2\pi \mathbb{Z})^n$ in $T$, the dynamics on $T$ takes the form $\dot{\varphi} = \omega$ with some constant $\omega \in \mathbb{R}^n$.

Lemma 1 ([10, 11, 27]). If the motions on $T$ are quasi-periodic, i.e., the components of $\omega$ are rationally independent, then the diffeomorphism $G|_T$ has the form $\varphi \mapsto \Delta - \varphi$ with some constant $\Delta \in T^n$ and is therefore an involution. The coordinate shift $x = \varphi - \Delta / 2$ puts this involution into the standard form $G|_T : x \mapsto -x$ while the dynamics is still given by the equation $\dot{x} = \omega$.

Proof. The function $t \mapsto \varphi(t) = \omega t$ is a solution of the equation $\dot{\varphi} = \omega$. The function $t \mapsto G(\varphi(-t))$ is also a solution (due to weak reversibility), so that $G(-\omega t) = G(0) + \omega t$ for any $t \in \mathbb{R}$. Since the orbit $\{\omega t \mid t \in \mathbb{R}\}$ is dense in $T^n$, we conclude that $G(\varphi) = G(0) - \varphi$ for any $\varphi \in T^n$.

The incommensurability condition on $\omega_1, \ldots, \omega_n$ in Lemma 1 is essential. For instance, the zero vector field ($\omega = 0$) on a torus is reversed by any diffeomorphism.

Despite its triviality, Lemma 1 has three very important consequences. First, it suggests that a passage from reversible systems to weakly reversible ones cannot affect drastically KAM-type theorems (and this is indeed so [3, 27, 32]). Second, under the hypotheses of Lemma 1, the fixed point set $\text{Fix}(G|_T)$ is a collection of $2^n = 2^\dim T$ isolated points:

$$\text{Fix}(G|_T) = (\text{Fix} G) \cap T = \{0; \pi\}^n = \{x \in T^n \mid x_\ell = 0 \text{ or } x_\ell = \pi, \ 1 \leq \ell \leq n\}. \quad (1)$$

Third, if the fixed point set $\text{Fix} G$ is a smooth submanifold of the phase space $\mathcal{M}$ and all the connected components of $\text{Fix} G$ are of the same dimension (so that $\dim \text{Fix} G$ is well defined), then $\dim \text{Fix} G \leq \text{codim} T$ (\text{codim} $T$ being the codimension of $T$ in $\mathcal{M}$). Indeed, at each point $w \in (\text{Fix} G) \cap T$, the number $-1$ is an eigenvalue of $T_w G$ of multiplicity at least $n = \dim T$.

If $G : \mathcal{M} \to \mathcal{M}$ is an involution, then $\text{Fix} G = \{w \in \mathcal{M} \mid G(w) = w\}$ is always a smooth submanifold of $\mathcal{M}$ of the same smoothness class as $G$ itself (a particular case of the Bochner theorem [6, 19]). This submanifold can well be empty or consist of several connected components.
of different dimensions. Extensive information on the fixed point submanifolds of involutions of various manifolds is presented in the books [6, 14], see also the articles [24, 30, 37]. In the sequel, while speaking of reversing involutions (or, as they are sometimes called, reversing symmetries or reversers) $G$, we will always suppose that $\text{Fix} G \neq \emptyset$ and all the connected components of $\text{Fix} G$ are of the same dimension, so that $\dim \text{Fix} G$ is well defined (this is the case for almost all the reversible systems encountered in practice).

The ubiquity of invariant tori carrying conditionally periodic motions stems, in the long run, from the fact that any finite-dimensional connected compact Abelian Lie group is a torus.

1.3. Reversible Contexts 1 and 2. Let a $G$-reversible system admit an invariant $n$-torus $T$ carrying quasi-periodic motions. As we saw above, $(\text{Fix} G) \cap T$ is a collection of $2^n$ points and $\dim \text{Fix} G \leq \text{codim} T$.

**Definition 2** ([10, 11, 36, 37]). The situation where $\frac{1}{2} \text{codim} T \leq \dim \text{Fix} G \leq \text{codim} T$ is called the reversible context 1, whereas the opposite situation where $0 \leq \dim \text{Fix} G < \frac{1}{2} \text{codim} T$ is called the reversible context 2.

An extensive discussion on the differences between reversible contexts 1 and 2 is presented in [37] and will not be repeated here (the paper [37] also contains an example where the reversible context 2 appears quite naturally). We will confine ourselves with some brief observations concerning integrable cases of the extreme reversible contexts 1 and 2.

Let the phase space variables be $x \in \mathbb{T}^n$ and $y \in \mathcal{Y} \subset \mathbb{R}^m$ where $\mathcal{Y}$ is an open and connected domain ($n \geq 1$, $m \geq 1$). On the phase space $\mathbb{T}^n \times \mathcal{Y}$, consider vector fields reversible with respect to one of the two involutions $G : (x, y) \mapsto (-x, \delta y)$, $\delta = \pm 1$ (in the case of $\delta = -1$, the domain $\mathcal{Y}$ is assumed to contain the origin 0 and to be symmetric with respect to the origin: $-y \in \mathcal{Y}$ whenever $y \in \mathcal{Y}$). Suppose that these vector fields are integrable [equivariant with respect to the torus translations $(x, y) \mapsto (x + \text{const}, y)$] and depend on a parameter $\nu \in \mathcal{N} \subset \mathbb{R}^s$ where $\mathcal{N}$ is an open and connected domain ($s \geq 0$). We are interested in invariant $n$-tori of such reversible systems of the form $T = \{y = \text{const}\}$.

If $\delta = 1$ then $\text{Fix} G = \{0; \pi\}^n \times \mathcal{Y}$ [cf. (1)], and we have the extreme reversible context 1: $\dim \text{Fix} G = m = \text{codim} T$. One easily sees that integrable $G$-reversible systems in this case have the form

$$\dot{x} = H(y, \nu), \quad \dot{y} = 0$$

with an arbitrary function $H$. For any value of $\nu$, the phase space is foliated into $n$-tori $\{y = \text{const}\}$ invariant under both the flow itself and the reversing involution $G$ and carrying conditionally periodic motions with frequency vectors $H(y, \nu)$. This picture is very similar to the dynamics of completely integrable Hamiltonian systems (in the Liouville–Arnold sense [2]), $(y, x)$ playing the roles of action-angle variables. Under suitable nondegeneracy conditions on the unperturbed frequency map $(y, \nu) \mapsto H(y, \nu)$, one can prove various KAM-type theorems for systems (2) [3, 9–11, 22, 23, 27, 31–33].
If \( \delta = -1 \) then \( \text{Fix } G = \{0; \pi\}^n \times \{0\} \) [cf. (1)], and we have the extreme reversible context 2: \( \dim \text{Fix } G = 0 \) (in fact, \( \text{Fix } G \) consists of \( 2^n \) isolated points). It is easy to verify that integrable \( G \)-reversible systems in this case have the form

\[
\dot{x} = H(y, \nu), \quad \dot{y} = P(y, \nu)
\]

with arbitrary functions \( H \) and \( P \) even in \( y \). The only \( n \)-torus of the form \( \{y = \text{const}\} \) invariant under the reversing involution \( G \) is \( \{y = 0\} \), and this torus is invariant under the flow of (3) if and only if \( P(0, \nu) = 0 \). Generically, the latter equation has no solutions for \( s < m \) and determines an \( (s - m) \)-dimensional surface in \( \mathcal{N} \) for \( s \geq m \).

Thus, the extreme reversible contexts 1 and 2 are drastically different [37]. In the extreme reversible context 2, the setup is not local with respect to \( y \) (and \( y = 0 \) is a special value), \( y \) is not an analog of the action variable (one may call \( y \) an anti-action variable), and even integrable systems with less than \( m \) external parameters (e.g. individual integrable systems) have generically no invariant tori.

By now, the reversible KAM theory in context 1 is nearly as developed as the Hamiltonian KAM theory [1, 3, 5, 7–11, 18, 20–24, 26–28, 30–35, 40–45]. On the other hand, to the best of my knowledge, the first and only result on the reversible context 2 was published no earlier than in 2011 [37]. In the paper [37], we studied the extreme reversible context 2 (\( \dim \text{Fix } G = 0 \)) and proved a KAM-type theorem for small \( G \)-reversible perturbations of systems (3) for sufficiently many (at least \( n + m \)) external parameters. The main result of [37] was obtained as an almost immediate corollary of J. Moser’s modifying terms theory [22] of 1967. At the end of [37], we listed ten tentative topics and directions for further research. The fourth topic was the non-extreme reversible context 2 (\( \dim \text{Fix } G > 0 \)).

The goal of the present paper is to prove a first theorem in the general reversible context 2 for an arbitrary dimension of \( \text{Fix } G \) (provided that \( \dim \text{Fix } G < \frac{1}{2} \text{codim } \mathcal{T} \)). As the principal technical tool, we again use Moser’s modifying terms theory [22] which turns out to be very powerful.

Invariant tori \( \mathcal{T} \) in the non-extreme reversible context 1 (\( \frac{1}{2} \text{codim } \mathcal{T} \leq \dim \text{Fix } G < \text{codim } \mathcal{T} \)) are often said to be lower dimensional [5, 7–11, 18, 26, 28, 31–35, 40–45]. Similarly, in the Hamiltonian KAM theory, isotropic invariant tori whose dimension is less than the number of degrees of freedom (for general references, see e.g. [2, 11–13]) are also said to be lower dimensional. Following this pattern, we call invariant tori \( \mathcal{T} \) in the non-extreme reversible context 2 (\( 0 < \dim \text{Fix } G < \frac{1}{2} \text{codim } \mathcal{T} \)) lower dimensional as well.

The task of developing the reversible KAM theory in context 2 was listed (as problem 9) among the ten problems of the classical KAM theory in the note [36].

1.4. **Structure of the Paper.** The paper is organized as follows. A precise formulation of the particular case of Moser’s theorem we need is given in Section 2. In Section 3, we explain
why Theorem 1 of Section 2 can be applied to reversible systems. Our main result for the non-extreme reversible context 2 and its detailed proof are presented in Section 4. For comparison, the parallel result for the non-extreme reversible context 1 is formulated in Section 5.

1.5. Dedication. This paper is dedicated to the memory of V. I. Arnold, one of the most creative, deep, distinctive, versatile, prolific, and influential scholars in the history of mathematics. Under his supervision and with his generous help, I mastered KAM theory and the theory of reversible systems in 1983–87 as a student and post-graduate student at the Moscow State University, wrote the book [27], and defended my PhD thesis “Reversible dynamical systems” in 1988. Many subsequent important events in my life would not have occurred without Arnold either. His untimely death is a bereavement for the world scientific and educational community.

2. Moser’s Theorem

2.1. Notation: Part 1. In this section, we present the particular case of Moser’s modifying terms theorem [22] suitable for our purposes. In fact, the modifying terms (in a somewhat different form) first appeared in the review [21]. Some generalizations of Moser’s modifying terms theorem are obtained in the works [4, 5, 15, 26, 37, 39], see also a discussion in the memoir [12, Part I, \S 7b].

In the sequel, \( \mathcal{O}_K(w) \) will denote an unspecified neighborhood of a point \( w \in \mathbb{R}^K \). We will also use the notation \( \partial_w = \partial/\partial w \) for any variable \( w \) and \( |w| = |w_1| + \cdots + |w_K| \) for any variable or vector \( w \in \mathbb{R}^K \). The standard inner product of real vectors will be denoted by \( \langle \cdot, \cdot \rangle \) while the Poisson bracket of vector fields, by \( [\cdot, \cdot] \). For \( w \in \mathbb{R} \), we will employ the notation \( |w| = \max\{\ell \in \mathbb{Z} \mid \ell \leq w\} \). For \( w \in \mathcal{O}_K(0) \), we will write \( O(w) \) instead of \( O(|w|) \) and \( O_\ell(w) \) instead of \( O\left(|w|^{\ell}\right) \) for \( \ell \geq 2 \). Similarly, \( O\left(|w'| + |w''|\right) \) will mean \( O\left(|w'|^{\ell} + |w''|^{\ell}\right) \) for \( \ell \geq 2 \). Finally, \( I_K \) and \( 0_{K \times L} \) will denote the identity \( K \times K \) matrix and the zero \( K \times L \) matrix, respectively.

The phase space variables in this section and the next one are \( x \in \mathbb{T}^n \) and \( X \in \mathcal{O}_N(0) \), whereas \( \nu \in \mathcal{N} \subset \mathbb{R}^s \) is an external parameter and \( \varepsilon \in \mathcal{O}_1(0) \) is the perturbation parameter \( (n \geq 1, N \geq 0, s \geq 0) \), \( \mathcal{N} \) being an open and connected domain for \( s \geq 1 \). Boldface letters will denote \( N \times N \) matrices and matrix-valued functions. We will keep most of Moser’s original notation [22].

2.2. Vector \( \omega \) and Matrices \( \Omega(\nu) \). Fix a vector \( \omega \in \mathbb{R}^n \) and an analytic matrix-valued function \( \Omega : \mathcal{N} \to \mathfrak{gl}(N, \mathbb{R}) \) satisfying the following hypothesis.

**Condition \( \Omega \) (Diagonalizability, imaginary constancy, and Diophantine property).** The matrix \( \Omega(\nu) \) is diagonalizable over \( \mathbb{C} \) for any \( \nu \in \mathcal{N} \) and the multiplicities of its eigenvalues do not depend on \( \nu \). Moreover, the imaginary parts of the eigenvalues of \( \Omega(\nu) \) do not depend on \( \nu \) either. If \( \beta_1, \ldots, \beta_d \ (0 \leq d \leq \lfloor N/2 \rfloor) \) are the positive imaginary parts (counting multiplicities)
of the eigenvalues of $\Omega(\nu)$, then there exist numbers $\tau > n - 1$ and $\gamma > 0$ such that
\begin{equation}
\left| \langle j, \omega \rangle + \langle J, \beta \rangle \right| \geq \gamma |j|^{-\tau}
\end{equation}
for all $j \in \mathbb{Z}^n \setminus \{0\}$ and $J \in \mathbb{Z}^d$, $|J| \leq 2$.

The inequalities (4) imply, in particular, that $\omega$ is Diophantine in the usual sense. Clearly, these inequalities (and even inequalities $\langle j, \omega \rangle + \langle J, \beta \rangle \neq 0$) cannot be valid for all $\nu \in \mathcal{N}$ in the case of a non-constant continuous function $\beta : \mathcal{N} \to \mathbb{R}^d$ (provided that $n \geq 2$) because the set $\{ \langle j, \omega \rangle \mid j \in \mathbb{Z}^n \}$ is dense in $\mathbb{R}$ for rationally independent $\omega_1, \ldots, \omega_n$.

2.3. Sets of Vector Fields and Diffeomorphisms. Now introduce the following spaces of analytic vector fields and groups of analytic diffeomorphisms (their meaning is explained in Moser’s paper [22] and in Subsection 3.2 below):

- $\mathfrak{L}$, the Lie algebra of all the analytic vector fields on $\mathbb{T}^n \times \mathcal{O}_N(0)$;
- $\mathcal{G}$, the corresponding Lie group of analytic diffeomorphisms;
- $\mathfrak{L}_1 \subset \mathfrak{L}$, the Lie subalgebra of all the vector fields of the form
  \[ a(x) \partial_x + [b(x) + c(x)X] \partial_X \]
  (it is very easy to verify that the space of such vector fields is indeed closed with respect to the Poisson bracket);
- $\mathcal{G}_1 \subset \mathcal{G}$, the corresponding Lie subgroup of analytic diffeomorphisms of the form
  \[ (x, X) \mapsto (A(x), B(x) + C(x)X) \]
  where $A$ is a diffeomorphism of $\mathbb{T}^n$ and $\det C(x) \neq 0$ for any $x \in \mathbb{T}^n$;
- $\hat{\mathfrak{L}} \subset \mathfrak{L}$, a certain Lie subalgebra of vector fields;
- $\hat{\mathcal{G}} \subset \mathcal{G}$, the corresponding Lie subgroup of diffeomorphisms;
- $\mathfrak{M} \subset \mathfrak{L}$, a certain vector subspace of $\mathfrak{L}$ (not necessarily a Lie subalgebra).

The vector fields
\[ D_\nu = \omega \partial_x + \Omega(\nu)X \partial_X \in \mathfrak{L}_1 \]
will be treated as the unperturbed ones. Condition $\Omega$ implies that the adjoint operators
\[ \vartheta_\nu = \text{ad} D_\nu : \mathfrak{L}_1 \to \mathfrak{L}_1, \quad \vartheta_\nu V = [D_\nu, V] \]
are semisimple [22, Section 2b]): for each $\nu \in \mathcal{N}$, the algebra $\mathfrak{L}_1$ is decomposed as
\begin{equation}
\mathfrak{L}_1 = \mathfrak{R}_\nu \oplus \mathcal{R}_\nu, \quad \mathfrak{R}_\nu = \ker \vartheta_\nu, \quad \mathcal{R}_\nu = \text{Image} \vartheta_\nu = \vartheta_\nu(\mathfrak{L}_1),
\end{equation}
where the nullspace $\mathfrak{R}_\nu$ of $\vartheta_\nu$ is finite dimensional [22, Section 2b]). Moreover, this decomposition depends analytically on $\nu$, and so does the operator $\vartheta_\nu^{-1} : \mathcal{R}_\nu \to \mathfrak{R}_\nu$. In particular, $\text{dim} \mathfrak{R}_\nu$ does not depend on $\nu$. One straightforwardly verifies that
\[ \vartheta_\nu (a \partial_x + (b + cX) \partial_X) = a_x \omega \partial_x + [b_x \omega - \Omega(\nu)b + (c_x \omega + c \Omega(\nu) - \Omega(\nu)c)X] \partial_X, \]
where \( a = a(x), \ a_x \omega = \omega_1 \partial_x a + \cdots + \omega_n \partial_{x_n} a, \) and the same notation is used for \( b \) and \( c. \)

Invoking Condition \( \Omega \) again, we conclude that \([22, \text{Section 2b})\]

\[
\mathfrak{R}_\nu = \left\{ a_0 \partial_x + (b_0 + c_0 X) \partial_X \mid a_0 \in \mathbb{R}^n, \ b_0 \in \mathbb{R}^N, \ \text{and} \ c_0 \in \mathfrak{gl}(N, \mathbb{R}) \right\} \text{ are constants such that } \Omega(\nu)b_0 = 0 \ \text{and} \ \Omega(\nu)c_0 = c_0\Omega(\nu). \quad (6)
\]

The Lie algebra \( \hat{\mathfrak{L}} \), Lie group \( \hat{\mathfrak{G}} \), space \( \mathfrak{M} \), and vector fields \( D_\nu \) are assumed to satisfy the following requirements (cf. \([37]\)):

(a) The space \( \mathfrak{M} \) is invariant under the action of \( \hat{\mathfrak{G}} \) by inner automorphisms: \( \text{Ad}_W V \in \mathfrak{M} \) whenever \( W \in \hat{\mathfrak{G}} \) and \( V \in \mathfrak{M} \).

(b) The spaces \( \hat{\mathfrak{L}} \) and \( \mathfrak{M} \) are closed with respect to “linearizations”: if \( u(x, X) \partial_x + v(x, X) \partial_X \in \hat{\mathfrak{L}} \) (or \( \in \mathfrak{M} \)),

\[
\text{then } u(x, 0) \partial_x + \left[ v(x, 0) + \frac{\partial v(x, 0)}{\partial X} X \right] \partial_X \in \hat{\mathfrak{L}} \ (\text{respectively } \in \mathfrak{M}).
\]

(c) \( D_\nu \in \mathfrak{M} \) for any \( \nu \in \mathcal{N}. \)

(d) The decomposition \( \mathfrak{L}_1 = \mathfrak{R}_\nu \oplus \mathfrak{R}_\nu \) “persists” under the intersection with \( \mathfrak{M} \) for any \( \nu \in \mathcal{N} \), i.e.,

\[
\mathfrak{L}_1 \cap \mathfrak{M} = (\mathfrak{R}_\nu \cap \mathfrak{M}) \oplus (\mathfrak{R}_\nu \cap \mathfrak{M}),
\]

and both the spaces \( \mathfrak{R}_\nu \cap \mathfrak{M} \) and \( \mathfrak{R}_\nu \cap \mathfrak{M} \) depend analytically on \( \nu. \)

(e) \( V \in \hat{\mathfrak{L}} \) whenever \( \nu \in \mathcal{N}, \ V \in \mathfrak{R}_\nu \), and \( \partial_\nu V \in (\mathfrak{R}_\nu \cap \mathfrak{M}). \)

2.4. Formulation of the Theorem. Under Condition \( \Omega \) and conditions (a)–(e) just presented, the following statement holds.

Theorem 1 (Moser \([22, \text{Section 5d})\]), see also a discussion in \([37]\)). Consider a family of systems of differential equations

\[
\dot{x} = \omega + \varepsilon f(x, X, \nu, \varepsilon), \quad \dot{X} = \Omega(\nu)X + \varepsilon F(x, X, \nu, \varepsilon),
\]

where the perturbation terms \( f \) and \( F \) are analytic in all their arguments and

\[
f(x, X, \nu, \varepsilon) \partial_x + F(x, X, \nu, \varepsilon) \partial_X \in \mathfrak{M}
\]

for any \( \nu \) and \( \varepsilon. \) Then, for \( \varepsilon \) sufficiently small, there exist unique analytic functions

\[
\lambda : (\nu, \varepsilon) \mapsto \lambda(\nu, \varepsilon) \in \mathbb{R}^n,
\]

\[
\mu : (\nu, \varepsilon) \mapsto \mu(\nu, \varepsilon) \in \mathbb{R}^N,
\]

\[
M : (\nu, \varepsilon) \mapsto M(\nu, \varepsilon) \in \mathfrak{gl}(N, \mathbb{R}) \quad (7)
\]

such that

\[
\lambda(\nu, \varepsilon) \partial_x + \left[ \mu(\nu, \varepsilon) + M(\nu, \varepsilon) X \right] \partial_X \in (\mathfrak{R}_\nu \cap \mathfrak{M}) \quad (8)
\]
for any \( \nu \) and \( \varepsilon \) and the following holds. For any \( \nu \) and any sufficiently small \( \varepsilon \), there is a coordinate transformation in \( \mathcal{G}_1 \cap \hat{\mathcal{G}} \) of the form

\[
\begin{align*}
    x &= \xi + \varepsilon A(\xi, \nu, \varepsilon), \quad X = \Xi + \varepsilon B(\xi, \nu, \varepsilon) + \varepsilon C(\xi, \nu, \varepsilon)\Xi \\
    [\xi \in \mathbb{T}^n \text{ and } \Xi \in \mathcal{O}_N(0) \text{ being new phase space variables}] \text{ that casts the modified system}
\end{align*}
\]

\[
\begin{align*}
    \dot{x} &= \omega + \varepsilon f(x, X, \nu, \varepsilon) + \varepsilon \lambda(\nu, \varepsilon), \\
    \dot{X} &= \Omega(\nu)X + \varepsilon F(x, X, \nu, \varepsilon) + \varepsilon \mu(\nu, \varepsilon) + \varepsilon M(\nu, \varepsilon)X
\end{align*}
\]

into a system of the form

\[
\begin{align*}
    \dot{\xi} &= \omega + \varepsilon O(\Xi), \quad \dot{\Xi} = \Omega(\nu)\Xi + \varepsilon O_2(\Xi). \\
    \end{align*}
\]

The coefficients \( A, B, \) and \( C \) in (9) are analytic in \( \xi, \nu, \) and \( \varepsilon \).

The required smallness of \( \varepsilon \) in Theorem 1 is uniform in \( \nu \) ranging in any fixed compact subset of \( \mathcal{N} \). The functions (7) are usually called Moser’s modifying terms. For any \( \nu \) and \( \varepsilon \), the modifying vector field (8) lies in the finite dimensional space \( \mathfrak{R}_\nu \cap \mathfrak{M} \).

Generally speaking, a coordinate transformation (9) is not unique. Suppose that a transformation

\[
\begin{align*}
    \xi^{\text{new}} &= \xi + \varepsilon \Delta(\nu, \varepsilon), \quad \Xi^{\text{new}} = \Xi + \varepsilon S(\nu, \varepsilon)\Xi \\
\end{align*}
\]

(with analytic coefficients \( \Delta \) and \( S \)) lies in \( \hat{\mathcal{G}} \). This additional coordinate transformation retains the form (11) of the system (10) provided that the matrix \( I_N + \varepsilon S(\nu, \varepsilon) \) commutes with \( \Omega(\nu) \) for any \( \nu \) and \( \varepsilon \).

2.5. Reducible Invariant Tori. In the lower dimensional KAM theory, the following concepts are of principal importance.

**Definition 3** ([2, 10, 11, 21, 22, 32]). Let an invariant \( n \)-torus \( \mathcal{T} \) of some flow on an \( (n + N) \)-dimensional manifold carry conditionally periodic motions with frequency vector \( \omega \in \mathbb{R}^n \). This torus is said to be reducible (or Floquet) if in a neighborhood of \( \mathcal{T} \), there exist coordinates \( (x \in \mathbb{T}^n, X \in \mathcal{O}_N(0)) \) in which the torus \( \mathcal{T} \) itself is given by the equation \( \{ X = 0 \} \) and the dynamical system takes the Floquet form \( \dot{x} = \omega + O(X), \dot{X} = \Omega X + O_2(X) \) with an \( x \)-independent matrix \( \Omega \in \mathfrak{gl}(N, \mathbb{R}) \). This matrix (not determined uniquely) is called the Floquet matrix of the torus \( \mathcal{T} \), and its eigenvalues are called the Floquet exponents of \( \mathcal{T} \).

In other words, an invariant torus is reducible if the variational equation along this torus can be reduced to a form with constant coefficients. One sees that the modified system (10) in Theorem 1 possesses a reducible invariant \( n \)-torus \( \{ \Xi = 0 \} \) with frequency vector \( \omega \) and Floquet matrix \( \Omega(\nu) \).

The version of Moser’s modifying terms theorem we have just presented differs from the version in our previous paper [37] in the only point: in Theorem 1 above, the matrix \( \Omega \) is allowed to depend on the external parameter \( \nu \). In fact, in Moser’s original paper [22], all versions of the modifying terms theorem were given without an external parameter whatever.
But Moser himself pointed out [22, p. 170] that an extension to differential equations depending analytically on several parameters is obvious. While examining lower dimensional invariant tori with real Floquet exponents in Hamiltonian systems [22, Section 6d)], Moser treated positive Floquet exponents as external parameters, so that the corresponding Floquet matrix turned out to be parameter-dependent. He wrote [22, p. 174]: “We do not have to keep the Ωμ [the eigenvalues of Ω] fixed but only their imaginary part. Then the real parts of these eigenvalues can be considered as additional parameters.”

3. Applications to Reversible Systems

3.1. Preliminaries. Before proceeding to applications of Moser’s Theorem 1, recall some simple general facts about the action of diffeomorphisms on vector fields. Consider an arbitrary diffeomorphism \( G : \mathcal{M} \to \mathcal{M} \) of a manifold \( \mathcal{M} \).

**Lemma 2.** If \( \text{Ad}_G V = \delta V \) (\( \delta = \pm 1 \)), then \( \text{Ad}_G[V, U] = \delta[V, \text{Ad}_G U] \) for any vector field \( U \). If \( \text{Ad}_G V = 1 \) and \( \text{Ad}_G U = \delta_2 U \) (\( \delta_1 = \pm 1, \delta_2 = \pm 1 \)), then \( \text{Ad}_G[V, U] = \delta_1 \delta_2[V, U] \).

**Proof.** Indeed, \( \text{Ad}_G \) is an automorphism of the Lie algebra of vector fields on \( \mathcal{M} \). \( \Box \)

**Lemma 3.** If a diffeomorphism \( W : \mathcal{M} \to \mathcal{M} \) commutes with \( G \), then \( \text{Ad}_W \) casts \( G \)-equivariant vector fields to \( G \)-equivariant ones and weakly \( G \)-reversible vector fields to weakly \( G \)-reversible ones.

**Proof.** The correspondence \( G \mapsto \text{Ad}_G \) is a representation of the group of diffeomorphisms of \( \mathcal{M} \). Hence, if \( \text{Ad}_G V = \delta V \) (\( \delta = \pm 1 \)), then \( \text{Ad}_G \text{Ad}_W V = \text{Ad}_G \text{Ad}_W V = \text{Ad}_W \text{Ad}_G V = \delta \text{Ad}_W V \).

Let \( \mathfrak{H} \) be a vector subspace of vector fields on \( \mathcal{M} \). If \( \mathfrak{H} \) is invariant under \( \text{Ad}_G \), we will write

\[
\mathfrak{H}^+ = \{ V \in \mathfrak{H} \mid \text{Ad}_G V = V \}, \quad \mathfrak{H}^- = \{ V \in \mathfrak{H} \mid \text{Ad}_G V = -V \}.
\]

**Lemma 4.** If \( G \) is an involution, then \( \mathfrak{H} = \mathfrak{H}^+ \oplus \mathfrak{H}^- \).

**Proof.** For any \( V \in \mathfrak{H} \), we have the decomposition \( V = \frac{1}{2}(V + \text{Ad}_G V) + \frac{1}{2}(V - \text{Ad}_G V) \). Since \( G^2 \) is the identity transformation, \( \frac{1}{2}(V + \text{Ad}_G V) \in \mathfrak{H}^+ \) and \( \frac{1}{2}(V - \text{Ad}_G V) \in \mathfrak{H}^- \). \( \Box \)

The involutivity condition on \( G \) in Lemma 4 is essential. For instance, let \( \mathcal{M} = \mathbb{R} \) and let \( G \) be linear: \( G(w) = aw \), \( a \neq 0 \). Set \( \mathfrak{H} \) to be the space of all smooth vector fields on \( \mathbb{R} \). If \( |a| \neq 1 \) (so that \( G \) is not an involution), then \( \mathfrak{H}^+ \) is the subspace of linear vector fields \( bw \partial_w \) and \( \mathfrak{H}^- = \{0\} \). Thus, \( \mathfrak{H}^+ \oplus \mathfrak{H}^- \) is far from coinciding with \( \mathfrak{H} \).

3.2. The Main Setup. In the notation of Section 2, consider an involution \( G : (x, X) \mapsto (-x, LX) \), where \( L \in \text{GL}(N, \mathbb{R}) \) is an involutive matrix (\( L^2 = I_N \)) and the neighborhood \( \mathcal{O}_x(0) \ni X \) is invariant under the linear involution \( X \mapsto LX \). The action of \( \text{Ad}_G \) on vector fields in \( \mathfrak{L} \) and \( \mathfrak{L}_1 \) is described by the formulas

\[
\text{Ad}_G(u(x, X)\partial_x + v(x, X)\partial_X) = -u(-x, LX)\partial_x + L v(-x, LX)\partial_X, \quad (12)
\]
Ad_G(a(x)\partial_x + [b(x) + c(x)X]\partial_X) = -a(-x)\partial_x + [Lb(-x) + Lc(-x)LX]\partial_X. \quad (13)

Consequently, the subalgebra \( \mathfrak{L}_1 \) is invariant under \( \text{Ad}_G \). According to Lemma 4, \( \mathfrak{L}_1 = \mathfrak{L}_1^+G \oplus \mathfrak{L}_1^-G \).

Given a vector \( \omega \in \mathbb{R}^n \) and an analytic matrix-valued function \( \Omega : \mathcal{N} \to gl(N, \mathbb{R}) \) satisfying Condition \( \Omega \) of Subsection 2.2 and such that \( \Omega(\nu)L \equiv -L\Omega(\nu) \), we will apply Theorem 1 to the situations where

- \( \widehat{\mathfrak{L}} = \mathfrak{L}_1^+G \subset \mathfrak{L} \) is the Lie algebra of \( G \)-equivariant vector fields;
- \( \widehat{\mathfrak{G}} \subset \mathfrak{G} \) is the corresponding Lie group of diffeomorphisms commuting with \( G \);
- \( \mathfrak{M} = \mathfrak{L}_1^-G \subset \mathfrak{L} \) is the space of vector fields reversible with respect to \( G \).

Note that \( \mathfrak{M} \) here is not a Lie algebra: according to Lemma 2, the Poisson bracket of two \( G \)-reversible vector fields is a \( G \)-equivariant vector field rather than a \( G \)-reversible one.

The following result seems to be new. In the papers [22, 37], similar statements were verified for \( L = \pm I_N \) only. The paper [26] considered only the case where the dimension of the \((-1)\)-eigenspace of \( L \) does not exceed \([N/2]\) which essentially corresponds to the reversible context 1.

**Lemma 5.** For any involutive matrix \( L \) and the choice of \( \omega, \Omega(\nu), \widehat{\mathfrak{L}}, \widehat{\mathfrak{G}}, \) and \( \mathfrak{M} \) just described, all the conditions (a)–(e) of Subsection 2.3 are satisfied.

**Proof.** Property (a) follows immediately from Lemma 3. Property (b) can be verified by a very easy computation using (12). Property (c) is obvious because the matrices \( \Omega(\nu) \) and \( L \) anti-commute for any \( \nu \in \mathcal{N} \), and therefore \( D_\nu \in \mathfrak{L}_1^-G \).

Since \( \text{Ad}_G D_\nu = -D_\nu \), Lemma 2 implies that \( \partial_\nu (\text{Ad}_G V) = -\text{Ad}_G (\partial_\nu V) \) for any \( \nu \in \mathcal{N} \) and \( V \in \mathfrak{L} \). Consequently, both the spaces \( \mathfrak{R}_\nu \) and \( \mathfrak{R}_\nu \) in the decomposition (5) are invariant under \( \text{Ad}_G \) for any \( \nu \in \mathcal{N} \). According to Lemma 4, \( \mathfrak{R}_\nu = \mathfrak{R}_\nu^+G \oplus \mathfrak{R}_\nu^-G \) with \( \mathfrak{R}_\nu^+G = \mathfrak{R}_\nu \cap \widehat{\mathfrak{L}}, \mathfrak{R}_\nu^-G = \mathfrak{R}_\nu \cap \mathfrak{M} \) and \( \mathfrak{R}_\nu = \mathfrak{R}_\nu^+G \oplus \mathfrak{R}_\nu^-G \) with \( \mathfrak{R}_\nu^+G = \mathfrak{R}_\nu \cap \widehat{\mathfrak{L}}, \mathfrak{R}_\nu^-G = \mathfrak{R}_\nu \cap \mathfrak{M} \). Thus,

\[ \mathfrak{L}_1 = \mathfrak{R}_\nu \oplus \mathfrak{R}_\nu = \mathfrak{R}_\nu^+G \oplus \mathfrak{R}_\nu^-G \oplus \mathfrak{R}_\nu^+G \oplus \mathfrak{R}_\nu^-G, \]

and \( \mathfrak{L}_1 \cap \widehat{\mathfrak{L}} = \mathfrak{L}_1^+G = \mathfrak{R}_\nu^+G \oplus \mathfrak{R}_\nu^-G \), \( \mathfrak{L}_1 \cap \mathfrak{M} = \mathfrak{L}_1^-G = \mathfrak{R}_\nu^-G \oplus \mathfrak{R}_\nu^-G \). Each of the four subspaces \( \mathfrak{R}_\nu^+G, \mathfrak{R}_\nu^-G, \mathfrak{R}_\nu^+G, \) and \( \mathfrak{R}_\nu^-G \) depends analytically on \( \nu \) because the spaces \( \mathfrak{R}_\nu \) and \( \mathfrak{R}_\nu \) depend analytically on \( \nu \) and are invariant under \( \text{Ad}_G \). We have verified property (d).

Finally, \( \partial_\nu (\mathfrak{R}_\nu^+G) = \mathfrak{R}_\nu^-G \) and \( \partial_\nu (\mathfrak{R}_\nu^-G) = \mathfrak{R}_\nu^+G \) for any \( \nu \in \mathcal{N} \) according to Lemma 2. Consequently, if \( V \in \mathfrak{R}_\nu \) for some \( \nu \) and \( \partial_\nu V \in \mathfrak{R}_\nu^+G \), then \( V \in \mathfrak{R}_\nu^+G \) (similarly, if \( \partial_\nu V \in \mathfrak{R}_\nu^-G \), then \( V \in \mathfrak{R}_\nu^-G \)). Thus, property (e) is also valid. \( \square \)

Taking (6) and (13) into account, we conclude that for \( \mathfrak{M} = \mathfrak{L}_1^-G \), the spaces \( \mathfrak{R}_\nu \cap \mathfrak{M} = \mathfrak{R}_\nu^-G \) where Moser’s modifying vector fields (8) lie are

\[ \mathfrak{R}_\nu^-G = \left\{ a_0\partial_x + (b_0 + c_0X)\partial_X \mid a_0 \in \mathbb{R}^n, b_0 \in \mathbb{R}^N, \text{ and } c_0 \in gl(N, \mathbb{R}) \right\} \]

such that \( \Omega(\nu)b_0 = 0, Lb_0 = -b_0, \Omega(\nu)c_0 = c_0\Omega(\nu), \) and \( c_0L = -Lc_0 \).

(14)
3.3. Information from Linear Algebra. In the rest of this paper, we will explore the case where

- \( N = m + 2p \) \((m \geq 0, p \geq 1)\),
- the matrix \( L \) is block diagonal with blocks \( \delta I_m \) and \( K \), where \( \delta = \pm 1 \) and \( K \in \text{GL}(2p, \mathbb{R}) \) is an involutive matrix \((K^2 = I_{2p})\) with eigenvalues 1 and \(-1\) of multiplicity \( p \) each,
- the matrices \( \Omega(\nu) \) are block diagonal for any \( \nu \in \mathcal{N} \) with blocks \( 0_{m \times m} \) and \( \Lambda(\nu) \), where the spectrum of the matrix \( \Lambda(\nu) \in \text{gl}(2p, \mathbb{R}) \) is simple for any \( \nu \in \mathcal{N} \) and \( \Lambda(\nu)K = -KA(\nu) \).

The equality \( \Lambda(\nu)K = -KA(\nu) \) implies that the eigenvalues of \( \Lambda(\nu) \) come in pairs \((w, -w)\), \( w \in \mathbb{C} [16, 27, 29, 38]\). Since the spectrum of \( \Lambda(\nu) \) is simple, it does not contain 0 and consists of real pairs \((\alpha, -\alpha)\), purely imaginary pairs \((i\beta, -i\beta)\), and complex quadruplets \( \pm \alpha \pm i\beta \). In the sequel, we will use the following notation.

**Notation \( \mathfrak{T} \).** Let \( \alpha_1, \ldots, \alpha_{d_1+d_3} \) and \( \beta_1, \ldots, \beta_{d_2+d_3} \) be arbitrary real numbers \((d_1, d_2, \text{ and } d_3 \text{ being non-negative integers such that } d_1 + d_2 + 2d_3 = p)\). Then \( \mathfrak{T}(d_1, d_2, d_3; \alpha, \beta) \in \text{gl}(2p, \mathbb{R}) \) denotes the block diagonal matrix with the \( d_1 + d_2 + d_3 \) blocks

\[
\begin{pmatrix}
0 & \alpha_k \\
\alpha_k & 0
\end{pmatrix}, \quad 1 \leq k \leq d_1, \quad \begin{pmatrix}
0 & \beta_l \\
-\beta_l & 0
\end{pmatrix}, \quad 1 \leq l \leq d_2,
\]

\[
\begin{pmatrix}
0 & \alpha_{d_1+i} & 0 & \beta_{d_2+i} \\
\alpha_{d_1+i} & 0 & \beta_{d_2+i} & 0 \\
0 & -\beta_{d_2+i} & 0 & \alpha_{d_1+i} \\
-\beta_{d_2+i} & 0 & \alpha_{d_1+i} & 0
\end{pmatrix}, \quad 1 \leq i \leq d_3.
\]

It is easy to see that the matrix \( \mathfrak{T}(d_1, d_2, d_3; \alpha, \beta) \) is diagonalizable over \( \mathbb{C} \) for any \( \alpha \in \mathbb{R}^{d_1+d_3} \) and \( \beta \in \mathbb{R}^{d_2+d_3} \), and its eigenvalues are

\[
\pm \alpha_1, \ldots, \pm \alpha_{d_1}, \quad \pm i \beta_1, \ldots, \pm i \beta_{d_2}, \quad \pm \alpha_{d_1+1}, \pm i \beta_{d_2+1}, \ldots, \pm \alpha_{d_1+d_3}, \pm i \beta_{d_2+d_3}.
\]

The theory of normal forms and versal unfoldings of matrices anti-commuting with a fixed involutive matrix (of infinitesimally reversible matrices in the terminology of [27, 29]) has been developed in e.g. the papers [16, 29, 38]. One of the simplest results of this theory is the following statement.

**Lemma 6 ([16, 29, 38]).** Let linear operators \( K \) and \( \Lambda \) in \( \text{gl}(2p, \mathbb{R}) \) satisfy the conditions \( K^2 = I_{2p} \) and \( \Lambda K = -KA \). Suppose that the spectrum of \( \Lambda \) is simple and has the form (15) with \( \alpha_k > 0 \) \((1 \leq k \leq d_1 + d_3)\) and \( \beta_l > 0 \) \((1 \leq l \leq d_2 + d_3)\). Then each of the two eigenvalues 1 and \(-1\) of \( K \) is of multiplicity \( p \), and in a suitable basis of \( \mathbb{R}^{2p} \), the matrices of \( K \) and \( \Lambda \) have the form \( K = \text{diag}(1, -1, 1, -1, \ldots, 1, -1) \) and \( \Lambda = \mathfrak{T}(d_1, d_2, d_3; \alpha, \beta) \). Moreover, this basis can be chosen to depend analytically on \( K \) and \( \Lambda \).

Returning to the matrices

\[
L = \begin{pmatrix}
\delta I_m & 0_{m \times 2p} \\
0_{2p \times m} & K
\end{pmatrix}, \quad \Omega(\nu) = \begin{pmatrix}
0_{m \times m} & 0_{m \times 2p} \\
0_{2p \times m} & \Lambda(\nu)
\end{pmatrix},
\]
one concludes from Lemma 6 that without loss of generality, we can set
\[ K = \text{diag}(1, -1, \ldots, 1, -1), \quad \Lambda(\nu) = \mathfrak{z}(d_1, d_2, d_3; \alpha(\nu), \beta), \]
where \(d_1, d_2, d_3\) are non-negative integers independent of \(\nu\) (and such that \(d_1 + d_2 + 2d_3 = p\)), \(\alpha_k(\nu) > 0\) \((\nu \in \mathcal{N}, 1 \leq k \leq d_1 + d_3)\), and \(\beta_l > 0\) \((1 \leq l \leq d_2 + d_3)\). Recall that the imaginary parts of the eigenvalues of \(\Omega(\nu)\) do not depend on \(\nu\) (see Subsection 2.2).

**Lemma 7.** For this choice of \(L\) and \(\Omega(\nu)\), under the condition that the spectrum of \(\Lambda(\nu)\) is simple for any \(\nu \in \mathcal{N}\), the space \(\mathfrak{A}_\nu \cap \mathfrak{M} = \mathfrak{A}_\nu^{-G} \subset \mathcal{L}_1\) for any \(\nu \in \mathcal{N}\) consists of vector fields
\[
a_0 \partial_x + (b_0 + c_0 X) \partial_X \quad \text{where}
\]
- \(a_0 \in \mathbb{R}^n\) are arbitrary,
- \(b_0 = 0\) for \(\delta = 1\) and
\[ b_0 = (b_{01}, \ldots, b_{0m}, 0, \ldots, 0) \]
\[ c_0 = \begin{pmatrix} 0_{m \times m} & \nu \times 2p \\ 0_{2p \times m} & \mathfrak{z}(2d_1, 2d_3; q, r) \end{pmatrix} \]
with arbitrary real \(b_{01}, \ldots, b_{0m}\) for \(\delta = -1\),
- with arbitrary \(q \in \mathbb{R}^{d_1 + d_3}\) and \(r \in \mathbb{R}^{d_2 + d_3}\).

**Proof.** This lemma easily follows from (14). The key point is to determine the centralizer \(\{c_0 \in \mathfrak{gl}(m, \mathbb{R}) \mid \Omega(\nu)c_0 = c_0\Omega(\nu)\}\) of \(\Omega(\nu)\). One can verify that this centralizer is the space of all the block diagonal matrices \(c_0\) with \(1 + d_1 + d_2 + d_3\) blocks of the form
\[
\Gamma \in \mathfrak{gl}(m, \mathbb{R}), \quad \begin{pmatrix} w_k & q_k & w_k \\ q_k & w_k & q_k \\ w_k & q_k & w_k \end{pmatrix}, \quad 1 \leq k \leq d_1, \quad \begin{pmatrix} t_l & r_l & t_l \\ -r_l & t_l & -r_l \end{pmatrix}, \quad 1 \leq l \leq d_2,
\]
\[ \begin{pmatrix} \eta & \eta & \eta \\ \eta & \eta & \eta \\ \eta & \eta & \eta \end{pmatrix}, \quad 1 \leq l \leq d_3. \]
For both values of \(\delta\), the condition \(c_0 L = -Lc_0\) is tantamount to that \(\Gamma = 0, w_k = 0 (1 \leq k \leq d_1 + d_3)\), and \(t_l = 0 (1 \leq l \leq d_2 + d_3)\).

The centralizer of \(\Omega(\nu)\) would be larger if the spectrum of \(\Lambda(\nu)\) were not simple.

### 3.4. Notation: Part 2.
From now on, we will consider reversible systems with phase space variables \(x \in \mathbb{T}^n, y \in \mathcal{Y} \subset \mathbb{R}^m\) \((\mathcal{Y}\) being an open domain), and \(z \in \mathcal{Z} = \mathcal{O}_{2p}(0)\) where \(n \geq 1\), \(m \geq 0\), and \(p \geq 1\). These reversible systems will be supposed to depend on an external parameter \(\nu \in \mathcal{N} \subset \mathbb{R}^s\) \((\mathcal{N}\) being an open domain), \(s \geq 0\), and on a small perturbation parameter \(\varepsilon \geq 0\). The reversing involution will be \(G : (x, y, z) \mapsto (-x, \delta y, Kz)\) with \(\delta = \pm 1\) and \(K = \text{diag}(1, -1, \ldots, 1, -1)\). The neighborhood \(\mathcal{Z}\) of \(0 \in \mathbb{R}^{2p}\) is assumed to be invariant under the linear involution \(z \mapsto Kz\). Similarly, for \(\delta = -1\), the domain \(\mathcal{Y}\) is assumed to contain the origin \(0\) and to be symmetric with respect to the origin.
We will look for reducible invariant $n$-tori $\mathcal{T}$ (see Definition 3) of such systems. Since $\text{codim} \mathcal{T} = m + 2p$ and
\[
\text{dim Fix } G = m + p \text{ (for } \delta = 1) \quad \text{or} \quad \text{dim Fix } G = p \text{ (for } \delta = -1),
\]
we are within the non-extreme reversible context 1 ($\frac{1}{2} \text{codim} \mathcal{T} \leq \text{dim Fix } G < \text{codim} \mathcal{T}$) for $\delta = 1$ and within the non-extreme reversible context 2 ($0 < \text{dim Fix } G < \frac{1}{2} \text{codim} \mathcal{T}$) for $\delta = -1$ and $m \geq 1$.

4. The Reversible Context 2

Now we are in the position to formulate and prove the main result of this paper. In the notation of Subsection 3.4, let $m \geq 1$ and $\delta = -1$, so that the reversing involution is $G : (x, y, z) \mapsto (-x, -y, Kz)$. Consider a family of $G$-reversible systems on $\mathbb{T}^n \times \mathcal{Y} \times \mathcal{Z}$ of the form
\[
\begin{align*}
\dot{x} &= H(y, \nu) + f^\sharp(x, y, z, \nu) + \varepsilon f(x, y, z, \nu, \varepsilon), \\
\dot{y} &= P(y, \nu) + g^\sharp(x, y, z, \nu) + \varepsilon g(x, y, z, \nu, \varepsilon), \\
\dot{z} &= Q(y, \nu)z + h^\sharp(x, y, z, \nu) + \varepsilon h(x, y, z, \nu, \varepsilon)
\end{align*}
\tag{16}
\]
(with $2p \times 2p$ matrix-valued function $Q$), where $f^\sharp = O(z)$, $g^\sharp = O_2(z)$, and $h^\sharp = O_2(z)$ [cf. (3)]. Reversibility of (16) with respect to $G$ means that
\[
H(-y, \nu) = H(y, \nu), \quad P(-y, \nu) = P(y, \nu), \\
Q(-y, \nu)K = -KQ(y, \nu)
\]
and
\[
\begin{align*}
f^\sharp(-x, -y, Kz, \nu) &= f^\sharp(x, y, z, \nu), \\
g^\sharp(-x, -y, Kz, \nu) &= g^\sharp(x, y, z, \nu), \\
h^\sharp(-x, -y, Kz, \nu) &= -Kh^\sharp(x, y, z, \nu),
\end{align*}
\]
for all $1 \leq k \leq d_1 + d_3$, $\nu \in \mathcal{N}$, and $\beta_l(\nu) > 0$ for all $1 \leq l \leq d_2 + d_3$, $\nu \in \mathcal{N}$. Introduce the notation $d_2 + d_3 = d$.

Fix an arbitrary (possibly, empty) subset of indices $3 \subset \{1; 2; \ldots; d_1 + d_3\}$ consisting of $\kappa$ elements ($0 \leq \kappa \leq d_1 + d_3$). We will write
\[
\alpha_+ = (\alpha_k \mid k \in 3), \quad \alpha_- = (\alpha_k \mid k \notin 3).
\]
Below, we will also use similar notation without special mention for vector quantities in \( \mathbb{R}^{d_1 + d_3} \) denoted by \( \alpha^0, \alpha', \alpha^\vee, \alpha^*, \) and \( q. \)

**Theorem 2.** Suppose that

- \( s \geq n + m + d + \kappa, \)
- \( P(0, \nu^0) = 0 \) for some \( \nu^0 \in \mathcal{N}, \)
- the vectors \( \omega = H(0, \nu^0) \in \mathbb{R}^n \) and \( \beta^0 = \beta(\nu^0) \in \mathbb{R}^d \) satisfy the following Diophantine condition: there exist constants \( \tau > n - 1 \) and \( \gamma > 0 \) such that
  \[
  |\langle j, \omega \rangle + \langle J, \beta^0 \rangle| \geq \gamma |j|^{-\tau}
  \]
  for all \( j \in \mathbb{Z}^n \setminus \{0\} \) and \( J \in \mathbb{Z}^d, |J| \leq 2 \) [cf. (4)],
- the mapping
  \[ \nu \mapsto (H(0, \nu), P(0, \nu), \beta(\nu), \alpha_+(\nu)) \]
  is submersive at point \( \nu^0 \), i.e.
  \[
  \text{rank} \left. \frac{\partial(H(0, \nu), P(0, \nu), \beta(\nu), \alpha_+(\nu))}{\partial \nu} \right|_{\nu=\nu^0} = n + m + d + \kappa.
  \]

Then for sufficiently small \( \varepsilon \) there exists an \( (s - n - m - d - \kappa) \)-dimensional analytic surface \( \mathcal{S}_\varepsilon \subset \mathcal{N} \) such that for any \( \nu \in \mathcal{S}_\varepsilon \), system (16) admits an analytic reducible invariant \( n \)-torus carrying quasi-periodic motions with frequency vector \( \omega \). The Floquet exponents of this torus are

\[
0, \ldots, 0, \quad \pm \alpha'_1(\nu, \varepsilon), \ldots, \pm \alpha'_{d_1}(\nu, \varepsilon), \quad \pm i \beta'_1, \ldots, \pm i \beta'_d,
\]

\[
\pm \alpha'_{d_1 + 1}(\nu, \varepsilon), \quad \pm i \beta'_{d_2 + 1}, \ldots, \pm \alpha'_{d_1 + d_3}(\nu, \varepsilon), \quad \pm i \beta'_{d_2 + d_3}
\]

[cf. (15)], where \( \alpha'_k(\nu, \varepsilon) > 0 \) for all \( 1 \leq k \leq d_1 + d_3, \nu \in \mathcal{S}_\varepsilon \), and \( \alpha'_+(\nu, \varepsilon) \equiv \alpha^0_+ \), i.e.,

\( \alpha'_k(\nu, \varepsilon) \equiv \alpha_k(\nu^0) \) for \( k \in \mathbb{Z} \) [here and henceforth, \( \alpha^0 = \alpha(\nu^0) \in \mathbb{R}^{d_1 + d_3} \)]. These tori and the numbers \( \alpha'_k(\nu, \varepsilon), k \notin \mathbb{Z} \), depend analytically on \( \nu \in \mathcal{S}_\varepsilon \) and on \( \sqrt{\varepsilon} \). At \( \varepsilon = 0 \), the surface \( \mathcal{S}_0 \) contains \( \nu^0 \) and all the tori are \( \{y = 0, z = 0\} \).

**Proof.** As in the case of the extreme reversible context 2 [37], to deduce Theorem 2 from Theorem 1, it in fact suffices to invoke the implicit function theorem twice.

By virtue of (18), one can introduce in \( \mathcal{N} \) near \( \nu^0 \) a new coordinate system

\[
(\sigma \in \mathcal{O}_n(0), \psi \in \mathcal{O}_m(0), \rho \in \mathcal{O}_d(0), \phi \in \mathcal{O}_\kappa(0), \chi \in \mathcal{O}_{s - n - m - d - \kappa}(0))
\]
such that \( \nu \) depends analytically on \( (\sigma, \psi, \rho, \phi, \chi) \), the point \( \nu = \nu^0 \) corresponds to \( \sigma = 0, \psi = 0, \rho = 0, \phi = 0, \chi = 0 \), and

\[
H(0, \nu(\sigma, \psi, \rho, \phi, \chi)) \equiv \omega + \sigma, \quad P(0, \nu(\sigma, \psi, \rho, \phi, \chi)) \equiv \psi,
\]

\[
\beta(\nu(\sigma, \psi, \rho, \phi, \chi)) \equiv \beta^0 + \rho, \quad \alpha_+(\nu(\sigma, \psi, \rho, \phi, \chi)) \equiv \alpha^0_+ + \phi.
\]

Since the functions \( H \) and \( P \) are even in \( y \), we have

\[
H(y, \nu(\sigma, \psi, \rho, \phi, \chi)) - \omega - \sigma = O_2(y), \quad P(y, \nu(\sigma, \psi, \rho, \phi, \chi)) - \psi = O_2(y).
\]
Following Moser’s simple but very efficient trick [22, Section 6b]), set for \( \varepsilon > 0 \)
\[
y = \sqrt{\varepsilon} Y, \quad z = \sqrt{\varepsilon} Z, \quad \psi = \sqrt{\varepsilon} \Psi,
\]
where the variable \( Y \) ranges in a certain fixed (\( \varepsilon \)-independent) domain in \( \mathbb{R}^m \) containing 0 and symmetric with respect to 0, the variable \( Z \) ranges in a certain fixed (\( \varepsilon \)-independent) domain in \( \mathbb{R}^{2p} \) containing 0 and invariant under the linear involution \( Z \mapsto KZ \), and the parameter \( \Psi \) ranges in a certain fixed (\( \varepsilon \)-independent) neighborhood of 0 in \( \mathbb{R}^m \). It is not hard to verify using the relation (20) for \( P \) that in the variables \((x, Y, Z)\), systems (16) take the form
\[
\dot{x} = \omega + \sigma + \sqrt{\varepsilon} \tilde{f}(x, Y, Z, \sigma, \sqrt{\varepsilon} \Psi, \rho, \phi, \chi, \sqrt{\varepsilon}),
\]
\[
\dot{Y} = \Psi + \sqrt{\varepsilon} \tilde{g}(x, Y, Z, \sigma, \sqrt{\varepsilon} \Psi, \rho, \phi, \chi, \sqrt{\varepsilon}),
\]
\[
\dot{Z} = Q(0, \nu(\sigma, 0, \rho, \phi, \chi)) Z + \sqrt{\varepsilon} \tilde{h}(x, Y, Z, \sigma, \Psi, \rho, \phi, \chi, \sqrt{\varepsilon}) \tag{21}
\]
with functions \( \tilde{f}, \tilde{g}, \) and \( \tilde{h} \) analytic in all their arguments for \( \varepsilon \geq 0 \) (not merely for \( \varepsilon > 0 \)) sufficiently small. One cannot write \( \tilde{h}(x, Y, Z, \sigma, \sqrt{\varepsilon} \Psi, \rho, \phi, \chi, \sqrt{\varepsilon}) \) because \( \sqrt{\varepsilon} \tilde{h} \) incorporates the term \( Q(0, \nu(\sigma, \sqrt{\varepsilon} \Psi, \rho, \phi, \chi)) Z - Q(0, \nu(\sigma, 0, \rho, \phi, \chi)) Z \). Systems (21) are reversible with respect to the involution \( G: (x, Y, Z) \mapsto (-x, -Y, KZ) \).

Obviously,
\[
Q(0, \nu(\sigma, 0, \rho, \phi, \chi)) = \mathfrak{T}(d_1, d_2, d_3; \alpha^+, \beta^0 + \rho)
\]
where \( \alpha^+ = \alpha^0_+ + \phi \) and \( \alpha^- = \alpha^-_-(\nu(\sigma, 0, \rho, \phi, \chi)) \). Introduce a new parameter \( \Phi \in \mathcal{O}_{d_1+d_3-\kappa}(0) \) where the neighborhood \( \mathcal{O}_{d_1+d_3-\kappa}(0) \) does not depend on \( \varepsilon \). Let
\[
\Lambda(\Phi) = \mathfrak{T}(d_1, d_2, d_3; \alpha^*, \beta^0)
\]
with \( \alpha^*_+ = \alpha^0_+ \) and \( \alpha^*_- = \alpha^0_- + \Phi \).

Taking Lemmas 5 and 7 into account, apply Moser’s Theorem 1 to the family of systems
\[
\dot{x} = \omega + \sqrt{\varepsilon} \tilde{f}(x, Y, Z, \sigma, \sqrt{\varepsilon} \Psi, \rho, \phi, \chi, \sqrt{\varepsilon}),
\]
\[
\dot{Y} = \sqrt{\varepsilon} \tilde{g}(x, Y, Z, \sigma, \sqrt{\varepsilon} \Psi, \rho, \phi, \chi, \sqrt{\varepsilon}),
\]
\[
\dot{Z} = \Lambda(\Phi) Z + \sqrt{\varepsilon} \tilde{h}(x, Y, Z, \sigma, \Psi, \rho, \phi, \chi, \sqrt{\varepsilon}) \tag{22}
\]
reversible with respect to the involution \( G: (x, Y, Z) \mapsto (-x, -Y, KZ) \) and depending on the parameters \( (\sigma, \Psi, \rho, \phi, \chi, \Phi) \in \mathcal{O}_{s+d_1+d_3-\kappa}(0) \). For \( \varepsilon \) sufficiently small, we obtain modified systems
\[
\dot{x} = \omega + \sqrt{\varepsilon} \tilde{f}(x, Y, Z, \sigma, \sqrt{\varepsilon} \Psi, \rho, \phi, \chi, \sqrt{\varepsilon}) + \sqrt{\varepsilon} \lambda(\sigma, \Psi, \rho, \phi, \chi, \Phi, \sqrt{\varepsilon}),
\]
\[
\dot{Y} = \sqrt{\varepsilon} \tilde{g}(x, Y, Z, \sigma, \sqrt{\varepsilon} \Psi, \rho, \phi, \chi, \sqrt{\varepsilon}) + \sqrt{\varepsilon} \mu(\sigma, \Psi, \rho, \phi, \chi, \Phi, \sqrt{\varepsilon}),
\]
\[
\dot{Z} = \Lambda(\Phi) Z + \sqrt{\varepsilon} \tilde{h}(x, Y, Z, \sigma, \Psi, \rho, \phi, \chi, \sqrt{\varepsilon}) + \sqrt{\varepsilon} M(\sigma, \Psi, \rho, \phi, \chi, \Phi, \sqrt{\varepsilon}) Z \tag{23}
\]
where
\[
M = \mathfrak{T}(d_1, d_2, d_3; q, r)
\]
with
\[
q = q(\sigma, \Psi, \rho, \phi, \chi, \Phi, \sqrt{\varepsilon}), \quad r = r(\sigma, \Psi, \rho, \phi, \chi, \Phi, \sqrt{\varepsilon}).
\]
Here the functions $\lambda, \mu, q, r$ are determined uniquely and analytic in all their arguments, the values of these functions ranging in $\mathbb{R}^n$, $\mathbb{R}^m$, $\mathbb{R}^{d_1+d_3}$, $\mathbb{R}^d$, respectively. After the coordinate change

$$x = \xi + \sqrt{\varepsilon} A(\xi, \sigma, \Psi, \rho, \phi, \chi, \Phi, \sqrt{\varepsilon}),$$

$$\begin{pmatrix} Y \\ Z \end{pmatrix} = \begin{pmatrix} \eta \\ \zeta \end{pmatrix} + \sqrt{\varepsilon} B(\xi, \sigma, \Psi, \rho, \phi, \chi, \Phi, \sqrt{\varepsilon}) + \sqrt{\varepsilon} C(\xi, \sigma, \Psi, \rho, \phi, \chi, \Phi, \sqrt{\varepsilon}) \begin{pmatrix} \eta \\ \zeta \end{pmatrix}$$

$[$\xi \in \mathbb{T}^n$, $\eta \in \mathcal{O}_m(0)$, and $\zeta \in \mathcal{O}_{2p}(0)$ being new phase space variables$]$, system (23) takes the form

$$\dot{\xi} = \omega + \sqrt{\varepsilon} O(\eta, \zeta),$$

$$\dot{\eta} = \sqrt{\varepsilon} O_2(\eta, \zeta),$$

$$\dot{\zeta} = \Lambda(\Phi) \zeta + \sqrt{\varepsilon} O_2(\eta, \zeta) \tag{24}$$

with the right-hand sides analytic in $\xi, \eta, \zeta, \sigma, \Psi, \rho, \phi, \chi, \Phi$, and $\sqrt{\varepsilon}$. This coordinate change commutes with the involution $G$. The functions $A, B, C$ are analytic in all their arguments, the values of these functions ranging in $\mathbb{R}^n$, $\mathbb{R}^m + 2p$, and $\mathfrak{gl}(m + 2p, \mathbb{R})$, respectively.

Compared with the notation of Section 2, here

- $N = m + 2p$,
- $(Y, Z)$ plays the role of $X$,
- $(\eta, \zeta)$ plays the role of $\Xi$,
- $s + d_1 + d_3 - \kappa$ plays the role of $s$ and $\mathcal{O}_{s+d_1+d_3-\kappa}(0)$ plays the role of $\mathcal{N}$,
- $(\sigma, \Psi, \rho, \phi, \chi, \Phi)$ plays the role of $\nu$,
- $\sqrt{\varepsilon}$ plays the role of $\varepsilon$,
- $\tilde{f}$ plays the role of $f$,
- $(\tilde{g}, \tilde{h})$ plays the role of $F$,
- the block diagonal matrix with blocks $0_{m \times m}$ and $\Lambda(\Phi)$ plays the role of $\Omega(\nu)$,
- $\beta^0$ plays the role of $\beta$,
- $(\mu, 0, \ldots, 0)$ plays the role of $\mu$,
- the block diagonal matrix with blocks $0_{m \times m}$ and $M$ plays the role of $M$.

Now one can use $\nu$ and $\alpha_-$ to “compensate” for the modifying terms $\lambda, \mu, M$. Observe that the systems (21) and (23) coincide if $\sqrt{\varepsilon} \lambda = \sigma$, $\sqrt{\varepsilon} \mu = \Psi$, and $\Lambda(\Phi) + \sqrt{\varepsilon} M = Q(0, \nu(\sigma, 0, \rho, \phi, \chi))$, i.e.

$$\sqrt{\varepsilon} \lambda(\sigma, \Psi, \rho, \phi, \chi, \Phi, \sqrt{\varepsilon}) = \sigma,$$

$$\sqrt{\varepsilon} \mu(\sigma, \Psi, \rho, \phi, \chi, \Phi, \sqrt{\varepsilon}) = \Psi,$$

$$\sqrt{\varepsilon} r(\sigma, \Psi, \rho, \phi, \chi, \Phi, \sqrt{\varepsilon}) = \rho,$$

$$\sqrt{\varepsilon} q_+(\sigma, \Psi, \rho, \phi, \chi, \Phi, \sqrt{\varepsilon}) = \phi,$$

$$\alpha^0_+ + \Phi + \sqrt{\varepsilon} q_- (\sigma, \Psi, \rho, \phi, \chi, \Phi, \sqrt{\varepsilon}) = \alpha_-(\nu(\sigma, 0, \rho, \phi, \chi)).$$
According to the implicit function theorem, for sufficiently small $\varepsilon$ and any $\chi$, one can solve this system of equations with respect to $\sigma$, $\Psi$, $\rho$, $\phi$, and $\Phi$:

$$
\begin{align*}
\sigma &= \sqrt{\varepsilon} \Sigma^1(\chi, \sqrt{\varepsilon}), \\
\Psi &= \sqrt{\varepsilon} \Sigma^2(\chi, \sqrt{\varepsilon}), \\
\rho &= \sqrt{\varepsilon} \Sigma^3(\chi, \sqrt{\varepsilon}), \\
\phi &= \sqrt{\varepsilon} \Sigma^4(\chi, \sqrt{\varepsilon}), \\
\Phi &= \alpha_-(\nu(0, 0, 0, \chi)) - \alpha^0_0 + \sqrt{\varepsilon} \Pi(\chi, \sqrt{\varepsilon})
\end{align*}
$$

with analytic functions $\Sigma^1, \Sigma^2, \Sigma^3, \Sigma^4$, and $\Pi$.

We conclude that for $\varepsilon > 0$ sufficiently small and any $\chi \in \mathcal{O}_{n-n-m-d-k}(0)$ with

$$
\begin{align*}
\sigma &= \sqrt{\varepsilon} \Sigma^1(\chi, \sqrt{\varepsilon}), \\
\psi &= \varepsilon \Sigma^2(\chi, \sqrt{\varepsilon}), \\
\rho &= \sqrt{\varepsilon} \Sigma^3(\chi, \sqrt{\varepsilon}), \\
\phi &= \sqrt{\varepsilon} \Sigma^4(\chi, \sqrt{\varepsilon}), \\
\Phi &= \alpha_-(\nu(0, 0, 0, \chi)) - \alpha^0_0 + \sqrt{\varepsilon} \Pi(\chi, \sqrt{\varepsilon})
\end{align*}
$$

(25)

the coordinate transformation

$$
\begin{align*}
x &= \xi + \sqrt{\varepsilon} A(\xi, \sigma, \Psi, \rho, \phi, \chi, \Phi, \sqrt{\varepsilon}), \\
\begin{pmatrix} y \\ z \end{pmatrix} &= \sqrt{\varepsilon} \begin{pmatrix} \eta \\ \zeta \end{pmatrix} + \varepsilon B(\xi, \sigma, \Psi, \rho, \phi, \chi, \Phi, \sqrt{\varepsilon}) + \varepsilon C(\xi, \sigma, \Psi, \rho, \phi, \chi, \Phi, \sqrt{\varepsilon}) \begin{pmatrix} \eta \\ \zeta \end{pmatrix}
\end{align*}
$$

(26)

with

$$
\Psi = \psi/\sqrt{\varepsilon}, \quad \Phi = \alpha_-(\nu(0, 0, 0, \chi)) - \alpha^0_0 + \sqrt{\varepsilon} \Pi(\chi, \sqrt{\varepsilon})
$$

casts the original system (16) into a system of the form (24) with the right-hand sides analytic in $\xi$, $\eta$, $\zeta$, $\sigma$, $\Psi$, $\rho$, $\phi$, $\chi$, $\Phi$, and $\sqrt{\varepsilon}$. Clearly, the $n$-torus $\{\eta = 0, \zeta = 0\}$ is invariant under the flow of system (24), and the dynamics on this torus determined by the equation $\dot{\xi} = \omega$ is quasi-periodic with frequency vector $\omega$. This torus is also reducible with the Floquet matrix

$$
\begin{pmatrix}
0_{m \times m} & 0_{m \times 2p} \\
0_{2p \times m} & \Lambda(\Phi)
\end{pmatrix}
$$

whose eigenvalues have the form (19) with

$$
\alpha'_{+} = \alpha^0_+, \quad \alpha'_- = \alpha^0_- + \Phi = \alpha_-(\nu(0, 0, 0, \chi)) + \sqrt{\varepsilon} \Pi(\chi, \sqrt{\varepsilon}).
$$

Since the coordinate change (26) commutes with the involution $G$, the torus $\{\eta = 0, \zeta = 0\}$ is invariant under this involution as well, and the restriction of $G$ to this torus has the form $\xi \mapsto -\xi$.

In the original coordinates $(x, y, z)$, the torus $\{\eta = 0, \zeta = 0\}$ is given by the equations

$$
\begin{align*}
x &= \xi + \sqrt{\varepsilon} A(\xi, \sigma, \Psi, \rho, \phi, \chi, \Phi, \sqrt{\varepsilon}), \\
\begin{pmatrix} y \\ z \end{pmatrix} &= \varepsilon B(\xi, \sigma, \Psi, \rho, \phi, \chi, \Phi, \sqrt{\varepsilon})
\end{align*}
$$

(\xi \in \mathbb{T}^n) and depends analytically on $\chi$ and $\sqrt{\varepsilon}$ for $\varepsilon \geq 0$ (not merely for $\varepsilon > 0$) sufficiently small. Equations (25) determine the desired surface $S_\varepsilon$. The surface $S_0$ has the form $\{\sigma = 0, \psi = 0, \rho = 0, \phi = 0\}$ and passes through the point $\nu^0$ corresponding to $\chi = 0$. For any $\chi$, the torus $\{\eta = 0, \zeta = 0\}$ for $\varepsilon = 0$ coincides with $\{y = 0, z = 0\}$. \Box
Most probably, the invariant $n$-tori in the setting of Theorem 2 in fact depend analytically on $\varepsilon$, not merely on $\sqrt{\varepsilon}$, but it would be hardly possible to prove that within Moser’s approach. Theorem 2 describes the persistence of the unperturbed reducible invariant torus \( \{ y = 0, z = 0, \nu = \nu^0 \} \) with the preservation of

- the frequencies $\omega_1, \ldots, \omega_n$,
- all the imaginary parts $\pm \beta_1^0, \ldots, \pm \beta_d^0$ of the Floquet exponents,
- an arbitrary subcollection (of length $\kappa$) of the pairs of the real parts $\pm \alpha_1^0, \ldots, \pm \alpha_{d_1+d_3}$ of the Floquet exponents.

The situation resembles the so-called partial preservation of Floquet exponents [35] in the “well developed” contexts of KAM theory (namely, the reversible context 1, the Hamiltonian context, the volume preserving context, and the dissipative context). The preservation of $\omega, \beta^0, \text{and } \alpha^0_+$ requires at least $n + m + d + \kappa$ external parameters. If $\kappa = 0$ (one is not going to have any control over the real parts of the Floquet exponents), then the minimal number of external parameters is equal to $n + m + d = M - \dim \text{Fix } G - d_1 - d_3$ where $M = n + m + 2p$ is the phase space dimension. Indeed, since $\dim \text{Fix } G = p$,

$$M - \dim \text{Fix } G - d_1 - d_3 = n + m + p - d_1 - d_3 = n + m + d_2 + d_3 = n + m + d.$$  

If $\kappa = d_1 + d_3$ (all the Floquet exponents are to be completely preserved), then the minimal number of external parameters is equal to $n + m + d + d_1 + d_3 = M - \dim \text{Fix } G$. In the latter case, it suffices to use Moser’s theorem with the matrix $\Omega$ independent of $\nu$. Indeed, the parameter $\Phi$ in (22) is absent for $\kappa = d_1 + d_3$. In Section 2, we allowed $\Omega$ to depend on $\nu$ in order to relax the preservation requirement for the real parts of the Floquet exponents (cf. [22, Section 6d]) and accordingly to reduce the necessary number of external parameters.

5. The Reversible Context 1

We will also formulate the counterpart of Theorem 2 for the reversible context 1. In the notation of Subsection 3.4, let $\delta = 1$, so that the reversing involution is $G : (x, y, z) \mapsto (-x, y, Kz)$. Consider a family of $G$-reversible systems on $\mathbb{T}^n \times \mathcal{Y} \times \mathcal{Z}$ of the form

$$\begin{align*}
\dot{x} &= H(y, \nu) + f^z(x, y, z, \nu) + \varepsilon f(x, y, z, \nu, \varepsilon), \\
\dot{y} &= g^z(x, y, z, \nu) + \varepsilon g(x, y, z, \nu, \varepsilon), \\
\dot{z} &= Q(y, \nu)z + h^z(x, y, z, \nu) + \varepsilon h(x, y, z, \nu, \varepsilon)
\end{align*}$$  

(with $2p \times 2p$ matrix-valued function $Q$), where $f^z = O(\varepsilon), g^z = O_2(\varepsilon), \text{and } h^z = O_2(\varepsilon)$. Reversibility of (27) with respect to $G$ means that

$$Q(y, \nu)K \equiv -KQ(y, \nu)$$  

and

$$\begin{align*}
f^z(-x, y, Kz, \nu) &\equiv f^z(x, y, z, \nu), & f(-x, y, Kz, \nu, \varepsilon) &\equiv f(x, y, z, \nu, \varepsilon), \\
g^z(-x, y, Kz, \nu) &\equiv -g^z(x, y, z, \nu), & g(-x, y, Kz, \nu, \varepsilon) &\equiv -g(x, y, z, \nu, \varepsilon),
\end{align*}$$
Theorem 3. Suppose that there are known much deeper results concerning invariant parallelism in applications of Theorem 1 to the non-extreme reversible contexts 1 and 2.\[5, 7–11, 18, 26, 31–35, 40–45]\]. We have given the formulation of Theorem 3 just to emphasize use similar notation for vector quantities in \(\mathbb{R}^d\) where the numbers \(\{y, \nu\}\) is to use \(\nu \in \mathbb{N}\) All the functions \(\{y, \nu\}\) for some \(m\) • \(s\) • \(\kappa\), \(\beta_0 = \beta(y^0, \nu^0) \in \mathbb{R}^d\) satisfy the following Diophantine condition: there exist constants \(\tau > n – 1\) and \(\gamma > 0\) such that the inequalities (17) hold for all \(j \in \mathbb{Z}^n \setminus \{0\}\) and \(J \in \mathbb{Z}^d, |J| \leq 2\), \(\gamma\) the mapping \((y, \nu) \rightarrow (H(y, \nu), \beta(y, \nu), \alpha(y, \nu))\) is submersive at point \((y^0, \nu^0)\), i.e.\[
\text{rank} \frac{\partial(H(y, \nu), \beta(y, \nu), \alpha(y, \nu))}{\partial(y, \nu)} \bigg|_{y=y^0, \nu=\nu^0} = n + d + \kappa.
\]Then for sufficiently small \(\varepsilon\), in \(\mathbb{T}^n \times \mathcal{Y} \times \mathbb{Z} \times \mathcal{N}\), there exists an \((m + s – n – d – \kappa)\)-parameter analytic family of analytic reducible invariant \(n\)-tori of systems (27). These tori carry quasi-periodic motions with frequency vector \(\omega\). The Floquet exponents of these tori have the form \[
0, \ldots, 0, \quad \pm \alpha_1', \ldots, \pm \alpha_d', \quad \pm i \beta_1^0, \ldots, \pm i \beta_d^0
\]For each torus, \(\alpha_k' > 0\) for all \(1 \leq k \leq d_1 + d_3\) and \(\alpha_+ = \alpha_+^0\), i.e., \(\alpha_k' = \alpha_k(y^0, \nu^0)\) for \(k \in \mathcal{J}\) [here \(\alpha_0 = \alpha(y^0, \nu^0) \in \mathbb{R}^{d_1+d_3}\)]. These tori and the numbers \(\alpha_k'\), \(k \notin \mathcal{J}\) (which vary along the family of the tori), depend analytically on \(\sqrt{\varepsilon}\). At \(\varepsilon = 0\), all the tori have the form \(\{y = \text{const}, z = 0, \nu = \text{const}\}\) and among them, there is the torus \(\{y = y^0, z = 0, \nu = \nu^0\}\).

The proof of this theorem is analogous to that of Theorem 2. The main idea of the proof is to use \(y, \nu\), and \(\alpha_+\) to “compensate” for the suitable modifying terms. Of course, by now there are known much deeper results concerning invariant \(n\)-tori of systems similar to (27) \[5, 7–11, 18, 26, 31–35, 40–45\]. We have given the formulation of Theorem 3 just to emphasize parallelism in applications of Theorem 1 to the non-extreme reversible contexts 1 and 2.
References

[1] V. I. Arnold, *Reversible systems*, Nonlinear and turbulent processes in physics, vol. 3 (Kiev, 1983), Harwood Academic Publ., Chur, 1984, pp. 1161–1174. MR 0824779

[2] V. I. Arnold, V. V. Kozlov, and A. I. Neishtadt, *Mathematical aspects of classical and celestial mechanics*, 3rd ed., Encyclopaedia of Mathematical Sciences, vol. 3, Springer-Verlag, Berlin, 2006. MR 2269239

[3] V. I. Arnold and M. B. Sevryuk, *Oscillations and bifurcations in reversible systems*, Nonlinear phenomena in plasma physics and hydrodynamics, Mir Publishers, Moscow, 1986, pp. 31–64.

[4] Yu. N. Bibikov, *A sharpening of a theorem of Moser*, Dokl. Akad. Nauk SSSR 213 (1973), no. 4, 766–769 (Russian). MR 0333359. English translation: Soviet Math. Dokl. 14 (1973), no. 6, 1769–1773.

[5] Yu. N. Bibikov, *Multifrequency nonlinear oscillations and their bifurcations*, Leningrad Univ. Press, Leningrad, 1991 (Russian). MR 112680

[6] G. E. Bredon, *Introduction to compact transformation groups*, Pure and Applied Mathematics, vol. 46, Academic Press, New York, 1972. MR 0413144

[7] H. W. Broer, M. C. Ciocci, H. Hanßmann, and A. Vanderbauwhede, *Quasi-periodic stability of normally resonant tori*, Phys. D 238 (2009), no. 3, 309–318. MR 2590451

[8] H. W. Broer, J. Hoo, and V. Naudot, *Normal linear stability of quasi-periodic tori*, J. Differential Equations 232 (2007), no. 2, 355–418. MR 2286385

[9] H. W. Broer and G. B. Huitema, *Unfoldings of quasi-periodic tori in reversible systems*, J. Dynam. Differential Equations 7 (1995), no. 1, 191–212. MR 1321710

[10] H. W. Broer, G. B. Huitema, and M. B. Sevryuk, *Families of quasi-periodic motions in dynamical systems depending on parameters*, Nonlinear dynamical systems and chaos (Groningen, 1995), Progr. Nonlinear Differential Equations Appl., vol. 19, Birkhäuser, Basel, 1996, pp. 171–211. MR 1391497

[11] H. W. Broer, G. B. Huitema, and M. B. Sevryuk, *Quasi-periodic motions in families of dynamical systems. Order amidst chaos*, Lecture Notes in Mathematics, vol. 1645, Springer-Verlag, Berlin, 1996. MR 1484969

[12] H. W. Broer, G. B. Huitema, F. Takens, and B. L. J. Braaksma, *Unfoldings and bifurcations of quasi-periodic tori*, Mem. Amer. Math. Soc. 83 (1990), no. 421, viii + 175 pp. MR 1041003

[13] H. W. Broer and M. B. Sevryuk, *KAM theory: quasi-periodicity in dynamical systems*, Handbook of Dynamical Systems, vol. 3, Elsevier B.V., Amsterdam, 2010, chapter 6, pp. 249–344.

[14] P. E. Conner and E. E. Floyd, *Differentiable periodic maps*, Ergebnisse der Mathematik und ihrer Grenzgebiete, N. F., Band 33, Academic Press Inc., New York; Springer-Verlag, Berlin, 1964. MR 0176478

[15] L. Corsi, G. Gentile, and M. Procesi, *KAM theory in configuration space and cancellations in the Lindstedt series*, Comm. Math. Phys. 302 (2011), no. 2, 359–402. MR 2770017

[16] I. Hoveijn, *Versal deformations and normal forms for reversible and Hamiltonian linear systems*, J. Differential Equations 126 (1996), no. 2, 408–442. MR 1383984

[17] J. S. W. Lamb and J. A. G. Roberts, *Time-reversal symmetry in dynamical systems: a survey*, Phys. D 112 (1998), no. 1–2, 1–39. MR 1605826

[18] B. Liu, *On lower dimensional invariant tori in reversible systems*, J. Differential Equations 176 (2001), no. 1, 158–194. MR 1861186

[19] D. Montgomery and L. Zippin, *Topological transformation groups*, Robert E. Krieger Publishing Co., Huntington, NY, 1974. MR 0379739. Reprint of the 1955 original.

[20] J. Moser, *Combination tones for Duffing's equation*, Comm. Pure Appl. Math. 18 (1965), no. 1–2, 167–181. MR 0179430

[21] J. Moser, *On the theory of quasiperiodic motions*, SIAM Rev. 8 (1966), no. 2, 145–172. MR 0203160

[22] J. Moser, *Convergent series expansions for quasi-periodic motions*, Math. Ann. 169 (1967), no. 1, 136–176. MR 0208078

[23] J. Moser, *Stable and random motions in dynamical systems. With special emphasis on celestial mechanics*, Princeton Landmarks in Mathematics, Princeton Univ. Press, Princeton, NJ, 2001. MR 1829194. Reprint of the 1973 original.

[24] G. R. W. Quispel and M. B. Sevryuk, *KAM theorems for the product of two involutions of different types*, Chaos 3 (1993), no. 4, 757–769. MR 1256318

[25] J. A. G. Roberts and G. R. W. Quispel, *Chaos and time-reversal symmetry. Order and chaos in reversible dynamical systems*, Phys. Rep. 216 (1992), no. 2–3, 63–177. MR 1173588

[26] J. Scheurle, *Bifurcation of quasi-periodic solutions from equilibrium points of reversible dynamical systems*, Arch. Rational Mech. Anal. 97 (1987), no. 2, 103–139. MR 0860303
M. B. Sevryuk, Reversible systems, Lecture Notes in Mathematics, vol. 1211, Springer-Verlag, Berlin, 1986. MR 0871875

M. B. Sevryuk, Lower-dimensional tori in reversible systems, Chaos 1 (1991), no. 2, 160–167. MR 1135903

M. B. Sevryuk, Linear reversible systems and their versal deformations, Trudy Sem. Petrovsk. no. 15 (1991), 33–54 (Russian). MR 1294389. English translation: J. Soviet Math. 60 (1992), no. 5, 1663–1680. MR 1181098

M. B. Sevryuk, Some problems in KAM theory: conditionally periodic motions in typical systems, Uspekhi Mat. Nauk 50 (1995), no. 2, 111–124 (Russian). MR 1339266. English translation: Russian Math. Surveys 50 (1995), no. 2, 341–353.

M. B. Sevryuk, The iteration-approximation decoupling in the reversible KAM theory, Chaos 5 (1995), no. 3, 552–565. MR 1350564

M. B. Sevryuk, The finite-dimensional reversible KAM theory, Phys. D 112 (1998), no. 1–2, 132–147. MR 1605834

M. B. Sevryuk, Partial preservation of frequencies in KAM theory, Nonlinearity 19 (2006), no. 5, 1099–1140. MR 2221801

M. B. Sevryuk, Invariant tori in quasi-periodic non-autonomous dynamical systems via Herman’s method, Discrete Contin. Dyn. Syst. 18 (2007), no. 2–3, 569–595. MR 2291912

M. B. Sevryuk, Partial preservation of frequencies and Floquet exponents in KAM theory, Trudy Mat. Inst. Steklova 259 (2007), 174–202 (Russian). MR 2433684. English translation: Proc. Steklov Inst. Math. 259 (2007), 167–195.

M. B. Sevryuk, KAM tori: persistence and smoothness, Nonlinearity 21 (2008), no. 10, T177–T185. MR 2439472

M. B. Sevryuk, The reversible context 2 in KAM theory: the first steps, Regul. Chaotic Dyn. 16 (2011), no. 1–2, 24–38. MR 2774376

Ch.-W. Shih, Normal forms and versal deformations of linear involutive dynamical systems, Chinese J. Math. 21 (1993), no. 4, 333–347. MR 1247555

F. Wagener, A parametrised version of Moser’s modifying terms theorem, Discrete Contin. Dyn. Syst. Ser. S 3 (2010), no. 4, 719–768. MR 2684071

X. Wang and J. Xu, Gevrey-smoothness of invariant tori for analytic reversible systems under Rüssmann’s non-degeneracy condition, Discrete Contin. Dyn. Syst. 25 (2009), no. 2, 701–718. MR 2525200

X. Wang, J. Xu, and D. Zhang, Persistence of lower dimensional elliptic invariant tori for a class of nearly integrable reversible systems, Discrete Contin. Dyn. Syst. Ser. B 14 (2010), no. 3, 1237–1249. MR 2670193

X. Wang, D. Zhang, and J. Xu, Persistence of lower dimensional tori for a class of nearly integrable reversible systems, Acta Appl. Math. 115 (2011), no. 2, 193–207.

B. Wei, Perturbations of lower dimensional tori in the resonant zone for reversible systems, J. Math. Anal. Appl. 253 (2001), no. 2, 558–577. MR 1808153

J. Xu, Normal form of reversible systems and persistence of lower dimensional tori under weaker nonresonance conditions, SIAM J. Math. Anal. 36 (2004), no. 1, 233–255. MR 2083860

J. Zhang, On lower dimensional invariant tori in $C^d$ reversible systems, Chin. Ann. Math. Ser. B 29 (2008), no. 5, 459–486. MR 2447481

Institute of Energy Problems of Chemical Physics, The Russia Academy of Sciences, Leninski prospect 38, Bldg. 2, Moscow 119334, Russia
E-mail address: sevryuk@mccme.ru