Declarative Combinatorics: Boolean Functions, Circuit Synthesis and BDDs in Haskell
– unpublished draft –

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Abstract

We describe Haskell implementations of interesting combinatorial generation algorithms with focus on boolean functions and logic circuit representations.

First, a complete exact combinational logic circuit synthesizer is described as a combination of catamorphisms and anamorphisms.

Using pairing and unpairing functions on natural number representations of truth tables, we derive an encoding for Binary Decision Diagrams (BDDs) with the unique property that its boolean evaluation faithfully mimics its structural conversion to a natural number through recursive application of a matching pairing function.

We then use this result to derive ranking and unranking functions for BDDs and reduced BDDs.

Finally, a generalization of the encoding techniques to Multi-Terminal BDDs is provided.

The paper is organized as a self-contained literate Haskell program, available at [http://logic.csci.unt.edu/tarau/research/2008/fBDD.zip](http://logic.csci.unt.edu/tarau/research/2008/fBDD.zip).

Keywords exact combinational logic synthesis, binary decision diagrams, encodings of boolean functions, pairing/unpairing functions, ranking/unranking functions for BDDs and MTBDDs, declarative combinatorics in Haskell

1. Introduction

This paper is an exploration with functional programming tools of ranking and unranking problems on Binary Decision Diagrams. The practical expressiveness of functional programming languages (in particular Haskell) are put to test in the process. The paper is part of a larger effort to cover in a declarative programming paradigm, arguably more elegantly, some fundamental combinatorial generation algorithms along the lines of [Knuth 2006].

The paper is organized as follows:

Sections 2 and 3 overview efficient evaluation of boolean formulae in Haskell using bitvectors represented as arbitrary length integers and Binary Decision Diagrams (BDDs).

Section 4 describes an exact combinational circuit synthesizer.

Section 5 discusses classic pairing and unpairing operations and introduces new pairing/unpairing functions acting directly on bitlists.

Section 6 introduces a novel BDD encoding (based on our unpairing functions) and discusses the surprising equivalence between boolean evaluation of BDDs and the inverse of our encoding, the main result of the paper.

Section 7 describes ranking and unranking functions for BDDs and reduced BDDs.

Section 8 extends our results to Multi-Terminal BDDs.

Sections 9 and 10 discuss related work, future work and conclusions.

The code in the paper, embedded in a literate programming LaTeX file, is entirely self contained and has been tested under GHC 6.4.3.

2. Evaluation of Boolean Functions with Bitvector Operations

Evaluation of a boolean function can be performed one bit at a time as in the function if_then_else

\[
\text{if_then_else } 0 \_ z = z \\
\text{if_then_else } 1 y \_ = y
\]

resulting in

\[
> \text{[(x,y,z),if_then_else x y z]} \\
\text{x\leftarrow[0,1],y\leftarrow[0,1],z\leftarrow[0,1]} \\
\text{[(0,0,0),0,]},
\]
(0,0,1),
(0,1,0),
(0,1,1),
(1,0,0),
(1,0,1),
(1,1,0),
(1,1,1)]

Clearly, this does not take advantage of the ability of modern hardware to perform such operations one word at a time - with the instant benefit of a speed-up proportional to the word size. An alternate representation, adapted from (Knuth 2006), uses integer encodings of \(2^n\) bits for each boolean variable \(x_0, \ldots, x_{n-1}\). Bitvector operations are used to evaluate all value combinations at once.

**Proposition 1.** Let \(x_k\) be a variable for \(0 \leq k < n\) where \(n\) is the number of distinct variables in a boolean expression. Then column \(k\) of the truth table represents, as a bitstring, the natural number:

\[
x_k = \frac{2^{2^n} - 1}{(2^{2^n} - 1) + 1}
\]

For instance, if \(n = 2\), the formula computes \(x_0 = 3 = [0, 0, 1, 1]\) and \(x_1 = 5 = [0, 1, 0, 1]\).

The following functions, working with arbitrary length bitstrings are used to evaluate the \([0..n-1]\) variables \(x_k\) with formula \(1\) and map the constant 1 to the bitstring of length \(2^n\).

\[
\begin{align*}
\text{var}_n (n) k &= \text{var}_m (\text{bigone} n) n k \\
\text{var}_m n k &= \text{var}_m \text{mask} n k \\
\text{var}_n n k &= \text{var}_n \text{mask} n k
\end{align*}
\]

-- represents constant 1 as 11...1

\[
\text{bigone} nvars = 2^2^n - 1
\]

3. **Exact Combinational Circuit Synthesis**

A first application of these variable encodings is combinational circuit synthesis, known to be intractable for anything beyond a few input variables. Clearly, a speed-up by a factor proportional to the machine's wordsize matters in this case.

3.1 **Encoding the Primary Inputs**

First, let us extend the encoding to cover constants 1 and 0, that we will represent as "variables" \(n\) and \(n+1\) and encode as vectors of \(n\) zeros or \(n\) ones (i.e. \(2^n - 1\), passed as the precomputed parameter \(m\) to avoid costly recomputation).

\[
\begin{align*}
\text{decode}_\text{var} nvars v &= \text{v} = (\text{bigone} nvars) = nvars \\
\text{decode}_\text{var} nvars 0 &= nvars + 1 \\
\text{decode}_\text{var} nvars v &= \text{head} \\
\text{mask} k &= [0..nvars-1], (\text{encode}_\text{var} m nvars k) = v \\
\text{where} m &= \text{bigone} nvars
\end{align*}
\]

Next we can precompute all the inputs knowing the number \(n\) of primary inputs for the circuit we want to synthesize:

\[
\text{init}_\text{inputs} n = \\
\{0:m:(\text{map}\ (\text{encode}_\text{var} m n))\ [0..n-1]\}\ 
\]

Given that inputs have all distinct encodings, we can decode them back - this function will be needed after the circuit is found.

\[
\begin{align*}
\text{map}\ (\text{decode}_\text{var} 2)\ (\text{init}_\text{inputs} 2) &= [3, 2, 0, 1] \\
\text{map}\ (\text{decode}_\text{var} 3)\ (\text{init}_\text{inputs} 3) &= [4, 3, 0, 1, 2]
\end{align*}
\]

We can now connect the inputs to their future occurrences as leaves in the tree representing the circuit. This means simply finding all the functions from the set of inputs to the set of occurrences, represented as a list (with possibly repeated) values of the inputs.

\[
\begin{align*}
\text{bindings}\ 0\ us &= [[]] \\
\text{bindings}\ n\ us &= \text{[zs|ys=bindings (n-1) us,zs=map (:ys) us]}
\end{align*}
\]

For fast lookup, we place the precomputed value combinations in a list of arrays.

\[
\text{bindings}\ 2\ [0,3,5] \\
\text{bindings}\ 3\ [0,3,5]
\]

For fast lookup, we place the precomputed value combinations in a list of arrays.

\[
\begin{align*}
\text{generateVarMap}\ occs\ vs &= \text{map}\ (\text{listArray}\ (0,\text{occ}-1))\ (\text{bindings}\ occs\ vs)
\end{align*}
\]
3.2 The Folds and the Unfolds

We are ready now to generate trees with library operations marking internal nodes of type \( F \) and primary inputs marking the leaves of type \( V \).

```hs
data T a = V a | F a (T a) (T a) deriving (Show, Eq)
```

Generating all trees is a variant of an unfold operation (anamorphism).

```hs
generateT lib n = unfoldT lib n 0
```

For later use, we will also define the dual fold operation (catamorphism) parameterized by a function \( f \) describing action on the leaves and a function \( g \) describing action on the internal nodes.

```hs
foldT _ g (V i) = g i
foldT f g (F i l r) = f i (foldT f g l) (foldT f g r)
```

The following example shows the action of the decoder:

```hs
> decodeV 2 (array (0,1) [(0,1)]) 0
V 1
> decodeV 2 (array (0,1) [(0,5),(1,3)]) 1
> decodeResult 2 (F 1 (V 0) (V 1)),
(array (0,1) [(0,5),(1,3)]), 4)
  F 1 (V 1) (V 0)
```

The following function uses foldT to generate a human readable string representation of the result (using the opname function given in Appendix):

```hs
showT nvars t = foldT f g t where
g i =
  if i<nvars
  then "x"++(show i)
  else show (nvars++"-i")
f i l r = (opname i)++("++l++","++r++")
> showT 2 (F 4 (V 0) (F 1 (V 1) (V 0)))
"xor(x0,nor(x1,x0))"
```

3.3 Assembling the Circuit Synthesizer

A Leaf-DAG generalizes an ordered tree by fusing together equal leaves. Leaf equality in our case means sharing a primary input variable or a constant.

In the next function we build candidate Leaf-DAGs by combining two generators: the inputs-to-occurrences generator `generateVarMap` and the expression tree generator `generateT`. Then we compute their bitstring value with a foldT based boolean formula evaluator. The function is parameterized by a library of logic gates `lib`, the number of primary inputs `nvars` and the maximum number of leaves it can use `maxleaves`:

```hs
buildAndEvalLeafDAG lib nvars maxleaves = [
  (leafDAG, varMap, foldT (opcode mask) (varMap!) leafDAG) |
    k ←[1..maxleaves],
    varMap ← generateVarMap k vs,
    leafDAG ← generateT lib k
] where
  mask = bigone nvars
  vs = init_inputs nvars
```

We are now ready to test if the candidate matches the specification given by the truth table of \( n \) variables \( ttn \).

```hs
findFirstGood lib nvars maxleaves ttn =
  head [r |
    r ← buildAndEvalLeafDAG lib nvars maxleaves,
    testspec ttn r
  ] where
    testspec spec (_,_,v) = spec==v
```

The final steps of the circuit synthesizer consist in converting to a human readable form the successful first candidate (guaranteed to be minimal as they have been generated by increasing order of nodes).
The following example shows a minimal circuit for the
2
6

synall lib nvars = -- for functions with nvars inputs
    map (syn lib nvars) [0..(bigone nvars)]

The following examples show circuits synthesized for 3 ar-

f

6

synmixops 3 83
"83:xor(x1,less(xor(x2,x1),x0))"

The following examples show circuits synthesized for 3 arg-

nand operator

2

syn asymops 3 83
"83:nand(impl(x2,x0),nand(x1,x0))"

The construction is known as Shannon expansion (Shan-

10

syn symops 3 83
"83:nor(nor(x2,x0),nor(x1,nor(x0,0)))"

The following examples show circuits synthetized for 3 ar-

2

syn mixops 3 83
"83:impl(impl(x2,x0),less(x1,impl(x0,0)))"

We refer to (Cégielski and Richard 2001) for a typical use

map (syn lib nvars) [0..(bigone nvars)]

in terms of a few different libraries. As this function is the building block of boolean
circuit representations like Binary Decision Diagrams, having perfect minimal circuits for it in terms of a given library
has clearly practical value. The reader might notice that it is quite unlikely to come up intuitively with some of these
synthesized circuits.

> syn [0] 2 6
"6:nand(nand(x0,nand(x1,1)),nand(x1,nand(x0,1)))"

We refer to the Appendix for a few details, related to the

bitvector operations on various boolean functions used in the
libraries, as well as a few tests.

4. Binary Decision Diagrams

We have seen that Natural Numbers in [0..2^n − 1] can be
used as representations of truth tables defining n-variable
boolean functions. A binary decision diagram (BDD) (Bryant
1986) is an ordered binary tree obtained from a boolean
function, by assigning its variables, one at a time, to 0 (left
branch) and 1 (right branch).

The construction is known as Shannon expansion (Shan-
non 1933), and is expressed as a decomposition of a function
in two cofactors, f[x ← 0] and f[x ← 1]

\[ f(x) = (\bar{x} \land f[x ← 0]) \lor (x \land f[x ← 1]) \] (2)

where f[x ← a] is computed by uniformly substituting a for
x in f. Note that by using the more familiar boolean
if-the-else function, the Shannon expansion can also be
expressed as:

\[ f(x) = \text{if } f[x ← 0] \text{ then } f[x ← 0] \text{ else } f[x ← 1] \] (3)

Alternatively, we observe that the Shannon expansion can
be directly derived from a 2^n size truth table, using bitstring
operations on encodings of its n variables. Assuming that the
first column of a truth table corresponds to variable x, x = 0
and x = 1 mask out, respectively, the upper and lower half of
the truth table.

\[ \text{Seeen as an operation on bitvectors, the Shannon expan-
}\]
\[ \text{sion (for a fixed number of variables) defines a bijection as-
}\]
\[ \text{sociating a pair of natural numbers (the cofactors’s truth}
\]
\[ \text{tables) to a natural number (the function’s truth table), i.e.}
\]
\[ \text{it works as a pairing function.}
\]

5. Pairing Functions

Definition 1. A pairing function is a bijection \( f : \text{Nat} \times \text{Nat} \rightarrow \text{Nat} \). An unpairing function is a bijection \( g : \text{Nat} \rightarrow \text{Nat} \times \text{Nat} \).

5.1 Classic Pairing Functions

Following Julia Robinson’s notation (Robinson 1950), given
a pairing function \( J \), its left and right inverses \( K \) and \( L \) are such that

\[ J(K(z), L(z)) = z \] (4)

\[ K(J(x, y)) = x \] (5)

\[ L(J(x, y)) = y \] (6)

We refer to (Cégielski and Richard 2001) for a typical use
in the foundations of mathematics and to (Rosenberg 2002)
for an extensive study of various pairing functions and their
computational properties.

Starting from Cantor’s pairing function

\[ f(x, y) = (x + y) \ast (x + y + 1)/2 + y \] (7)

and the Pepis-Kalmar-Robinson function

\[ f(x, y) = 2^x \ast (2 \ast y + 1) - 1 \] (8)

bijections from \( \text{Nat} \times \text{Nat} \) to \( \text{Nat} \) have been used for
various proofs and constructions of mathematical objects
(Pepis 1938; Kalmar 1939; Robinson 1950, 1955, 1968; Cégielski and Richard 2001).

5.2 Pairing/Unpairing operations acting directly on
bitlists

We will introduce here a pairing function, expressed as sim-
ple bitlist transformations. This unusually simple pairing
function (that we have found out recently as being the same
as the one in defined in Steven Pigeon’s PhD thesis on
Data Compression (Pigeon 2001), page 114), provides com-
plete representations for various constructs involving ordered
pairs.

The function bitmerge_pair implements a bijection
from \( \text{Nat} \times \text{Nat} \) to \( \text{Nat} \) that works by splitting a number’s big endian bitstring representation into odd and even
bits, while its inverse bitmerge_unpair blends the odd and
even bits back together. The helper functions nat2set and set2nat, given in the Appendix, convert from/to natural numbers to sets of nonzero bit positions.

\[
\text{bitmerge\_pair}\ (i,j) =
\text{set2nat}\ ((\text{evens}\ i) ++ (\text{odds}\ j))\ \\
\text{evens}\ x = \text{map}\ (2+)\ (\text{nat2set}\ x)\ \\
\text{odds}\ y = \text{map}\ \text{\texttt{\('div' 2\)}}\ (\text{evens}\ y)
\]

The transformation of the bitlists is shown in the following example with bitstrings aligned:

\[
\begin{align*}
\text{bitmerge\_unpair}\ 2008 & =
\begin{cases}
(60,26) & \text{-- 2008:}[0, 0, 0, 1, 1, 0, 1, 1, 1, 1, 1] \\
(60,26) & \text{-- 60:}[0, 0, 1, 1, 1, 1, 1, 1, 1, 1, 0] \\
(26,1) & \text{-- 26:}[0, 0, 1, 0, 1, 1, 1]
\end{cases}
\end{align*}
\]

**Proposition 2.** The following function equivalences hold:

\[\text{bitmerge\_pair} \circ \text{bitmerge\_unpair} \equiv \text{id}\ 
\]

\[\text{bitmerge\_unpair} \circ \text{bitmerge\_pair} \equiv \text{id}\]

6. **Pairing Functions and Encodings of Binary Decision Diagrams**

We will build a BDD by applying \text{bitmerge\_unpair} recursively to a Natural Number \(tt\), seen as an \(n\)-variable \(2^n\) bit truth table. This results in a complete binary tree of depth \(n\). As we will show later, this binary tree represents a BDD that returns \(tt\) when evaluated applying its boolean operations.

We represent a BDD in Haskell as a binary tree BT with constants 0 and 1 as leaves, marked with the function symbol C. Internal nodes representing if-then-else decision points, marked with D, are controlled by variables, ordered identically in each branch, as first arguments of D. The two other arguments are subtrees representing the THEN and ELSE branches. Note that, in practice, reduced, canonical DAG representations are used instead of binary tree representations.

The constructor BDD wraps together the number of variables of a binary decision diagram and the binary tree representation it.

```
data BT a = C a | D a (BT a) (BT a) deriving (Eq, Show)
```

The following functions apply \text{bitmerge\_unpair} recursively, on a Natural Number \(tt\), seen as an \(n\)-variable \(2^n\) bit truth table, to build a complete binary tree of depth \(n\), that we will represent using the BDD data type.

```
data BDD a = BDD a (BT a) deriving (Eq, Show)
```

The following functions apply \text{bitmerge\_unpair} recursively, on a Natural Number \(tt\), seen as an \(n\)-variable \(2^n\) bit truth table, to build a complete binary tree of depth \(n\), that we will represent using the BDD data type.

```
data BDD a = BDD a (BT a) deriving (Eq, Show)
```

-- n-number of variables, tt=a truth table
plain\_bdd\ n \ tt = \text{BDD} \ n \ \text{bt}

\begin{align*}
\text{bt} & = \text{if} \ tt < \text{max} \ \text{then} \ \text{shf} \ \text{bitmerge\_unpair} \ n \ tt \\
& \text{else} \ \text{error}
\end{align*}

\begin{align*}
\text{("plain\_bdd: last arg "}++\text{show} tt \text{\")+}
& \text{" should be } \text{\textless} \text{\text{++}}\text{show} \text{\max}
\end{align*}

\begin{align*}
\text{where} \ & \text{max} = 2^{2^n}
\end{align*}

```
-- recurses to depth n, splitting tt into pairs
\text{shf} \ n \ tt | n<1 = C tt

\text{shf} \ n \ tt = D k \ (\text{ashf} f k \ tt1) \ (\text{ashf} f k \ tt2) \text{where}

\begin{align*}
k = \text{pred} \ n \\
(tt1,tt2) = f tt
\end{align*}
```

The following examples show the results returned by \text{plain\_bdd} for all \(2^n\) truth tables associated to \(n\) variables for \(n=2\), with help from printing function \text{print\_plain} given in Appendix.

```
\begin{align*}
\text{print\_plain} \ 2
\end{align*}
```

We will build a BDD by applying \text{bitmerge\_unpair} recursively to a Natural Number \(tt\), seen as an \(n\)-variable \(2^n\) bit truth table. This results in a complete binary tree of depth \(n\). As we will show later, this binary tree represents a BDD that returns \(tt\) when evaluated applying its boolean operations.

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The constructor BDD wraps together the number of variables of a binary decision diagram and the binary tree representation it.

```
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```

The following functions apply \text{bitmerge\_unpair} recursively, on a Natural Number \(tt\), seen as an \(n\)-variable \(2^n\) bit truth table, to build a complete binary tree of depth \(n\), that we will represent using the BDD data type.

```
data BDD a = BDD a (BT a) deriving (Eq, Show)
```

The following functions apply \text{bitmerge\_unpair} recursively, on a Natural Number \(tt\), seen as an \(n\)-variable \(2^n\) bit truth table, to build a complete binary tree of depth \(n\), that we will represent using the BDD data type.

```
data BDD a = BDD a (BT a) deriving (Eq, Show)
```

The following examples show the results returned by \text{plain\_bdd} for all \(2^n\) truth tables associated to \(n\) variables for \(n=2\), with help from printing function \text{print\_plain} given in Appendix.

```
\begin{align*}
\text{print\_plain} \ 2
\end{align*}
```

The function \text{bdd\_reduce} reduces a BDD by collapsing identical left and right subtrees, and the function \text{bdd} associates this reduced form to \(n \in \text{Nat}\).

```
\text{bdd\_reduce} \ (BDD \ n \ \text{bt}) = \ (BDD \ n \ \text{reduce} \ \text{bt}) \ \\
\text{reduce} \ (C \ b) = C b

\text{reduce} \ (D \ _ \ l \ r) | l == r = \text{reduce} \ l

\text{reduce} \ (D \ v \ l \ r) = D v \ (\text{reduce} \ l) \ (\text{reduce} \ r)
```

```
\text{bdd} \ n = \text{bdd\_reduce} \ . \ \text{plain\_bdd} \ n
```

Note that we omit here the reduction step consisting in sharing common subtrees, as it is obtained easily by replacing trees with DAGs. The process is facilitated by the fact that our unique encoding provides a perfect hashing key for each subtree.

The following examples show the results returned by \text{bdd} for \(n=2\), with help from printing function \text{print\_reduced} given in Appendix.

```
\begin{align*}
\text{print\_reduced} \ 2
\end{align*}
```
6.2 From BDDs to Natural Numbers

One can “evaluate back” the binary tree representing the BDD, by using the pairing function `bitmerge_pair`. The inverse of `plain_bdd` is implemented as follows:

```plaintext
code
plain_inverse_bdd (BDD _ bt) =
  rshf bitmerge_pair bt

rshf rf (C tt) = tt
rshf rf (D _ l r) = rf ((rshf rf l),(rshf rf r))
```

The main result of this subsection can now be summarized as follows:

**Proposition 3.** The complete binary tree of depth \( n \), obtained by recursive applications of `bitmerge_pair_unpair` on a truth table \( tt \) computes an (unreduced) BDD, that, when evaluated, returns the truth table, i.e.:

\[
plain_inverse_bdd (plain_bdd n tt) \equiv id
\]  
(11)

\[
ev n (plain_bdd n tt) \equiv id
\]  
(12)

Moreover, \( ev \) also acts as a left inverse of `bdd`, i.e.

\[
ev n (bdd n tt) \equiv id
\]  
(13)

**Proof sketch:** The function `plain_bdd` builds a binary tree by splitting the bitstring \( tt \in [0..2^n-1] \) up to depth \( n \). Observe that this corresponds to the Shannon expansion [Shannon1993] of the formula associated to the truth table, using variable order \([n-1, ..., 0]\). Observe that the effect of `bitstring_unpair` is the same as

- the effect of `var_mn` \( n \rightarrow (n-1) \) acting as a mask selecting the left branch, and
- the effect of its complement, acting as a mask selecting the right branch.

Given that \( 2^n \) is the double of \( 2^{n-1} \), the same invariant holds at each step, as the bitstring length of the truth table reduces to half. On the other hand, it is clear that \( ev \) reverses the action of both `plain_bdd` and `bdd`, as BDDs and reduced BDDs represent the same boolean function [Bryant1986].

This result can be seen as yet another intriguing isomorphism between boolean, arithmetic and symbolic computations.

7. Ranking and Unranking of BDDs

One more step is needed to extend the mapping between BDDs with \( n \) variables to a bijective mapping from/to \( \text{Nat} \):
we will have to “shift towards infinity” the starting point of each new block of BDDs in Nat as BDDs of larger and larger sizes are enumerated.

First, we need to know by how much - so we will count the number of boolean functions with up to \( n \) variables.

\[
\begin{align*}
\text{bsum } 0 &= 0 \\
\text{bsum } n &\mid n>0 = \text{bsum1 } (n-1) \\
\text{bsum1 } 0 &= 2 \\
\text{bsum1 } n &\mid n>0 = \text{bsum1 } (n-1) + 2^2^n \\
\end{align*}
\]

The stream of all such sums can now be generated as usual\[1\]

\[
\begin{align*}
\text{bsums } &= \text{map bsum } [0..] \\
\text{genericTake 7 bsums} &= [0, 2, 6, 22, 78, 65814, 4295033110] \\
\end{align*}
\]

What we are really interested into, is decomposing \( n \) into the distance \( n-m \) to the last \( \text{bsum } m \) smaller than \( n \), and the index that generates the sum, \( k \).

\[
\begin{align*}
\text{to_bsum } n &= (k, n-m) \text{ where } \\
&k = \text{pred } (\text{head } [x | x \leftarrow [0..], \text{bsum } x > n]) \\
&m = \text{bsum } k \\
\end{align*}
\]

Unranking of an arbitrary BDD is now easy - the index \( k \) determines the number of variables and \( n-m \) determines the rank. Together they select the right BDD with \( \text{plain_bdd} \) and \( \text{bdd} \).

\[
\begin{align*}
\text{nat2plain_bdd } n &= \text{plain_bdd } k \ n_m \text{ where } (k, n_m) = \text{to_bsum } n \\
\text{nat2bdd } n &= \text{bdd } k \ n_m \text{ where } (k, n_m) = \text{to_bsum } n \\
\end{align*}
\]

Ranking of a BDD is even easier: we shift its rank within the set of BDDs with \( \text{nv} \) variables, by the value \( (\text{bsum } \text{nv}) \) that counts the ranks previously assigned.

\[
\begin{align*}
\text{plain_bdd2nat } \text{bdd@}(\text{bdd } _) &= (\text{bsum } \text{nv}) + (\text{plain_inverse_bdd } \text{bdd}) \\
\text{bdd2nat } \text{bdd@}(\text{bdd } _) &= (\text{bsum } \text{nv}) + (\text{ev } \text{bdd}) \\
\end{align*}
\]

As the following example shows, \( \text{nat2plain_bdd} \) and \( \text{plain_bdd2nat} \) implement inverse functions.

\[
\begin{align*}
\text{nat2plain_bdd } 42 &= \text{bdd } 3 \ 	ext{BD}\ 
\text{D } 2 \\
&\ (\text{D } 1 \ (\text{D } 0 \ (\text{C } 0) \ (\text{C } 1)) \\
&\ (\text{D } 0 \ (\text{C } 1) \ (\text{C } 0))) \\
\text{plain_bdd2nat } \text{it } &= 42 \\
\end{align*}
\]

The same applies to \( \text{nat2bdd} \) and its inverse \( \text{bdd2nat} \).

\[
\begin{align*}
\text{nat2bdd } 42 &= \text{bdd } 3 \ 	ext{BD}\ 
\text{D } 2 \\
&\ (\text{D } 1 \ (\text{D } 0 \ (\text{C } 0) \ (\text{C } 1)) \\
&\ (\text{D } 0 \ (\text{C } 0) \ (\text{C } 0))) \\
\text{bd2nat } \text{it } &= 42 \\
\end{align*}
\]

We can now generate infinite streams of BDDs as follows:

\[
\begin{align*}
\text{plain_bdds } &= \text{map } \text{nat2plain_bdd } [0..] \\
\text{bdds } &= \text{map } \text{nat2bdd } [0..] \\
\text{genericTake 4 plain_bdds} &= [\text{BDD } 0 \ (\text{C } 0), \text{BDD } 0 \ (\text{C } 1), \text{BDD } 1 \ (\text{C } 0), \text{BDD } 1 \ (\text{D } 0 \ (\text{C } 0) \ (\text{C } 1))] \\
\text{genericTake 6 bdds} &= [\text{BDD } 0 \ (\text{C } 0), \text{BDD } 0 \ (\text{C } 1), \text{BDD } 1 \ (\text{C } 0), \text{BDD } 1 \ (\text{D } 0 \ (\text{C } 1) \ (\text{C } 0)), \text{BDD } 1 \ (\text{D } 0 \ (\text{C } 0) \ (\text{C } 1)), \text{BDD } 1 \ (\text{C } 1)] \\
\end{align*}
\]

8. Multi-Terminal Binary Decision Diagrams (MTBDD)

MTBDDs [Fujita et al. 1997] [Ciesinski et al. 2008] are a natural generalization of BDDs allowing non-binary values as leaves. Such values are typically bitstrings representing the outputs of a multi-terminal boolean function, encoded as unsigned integers.

We shall now describe an encoding of \( \text{MTBDDs} \) that can be extended to ranking/unranking functions, in a way similar to \( \text{BDDs} \) as shown in section\[7\]

Our MTBDD data type is a binary tree like the one used for \( \text{BDDs} \), parameterized by two integers \( m \) and \( n \), indicating that an MTBDD represents a function from \( [0..\text{nv} - 1] \) to \( [0..\text{mv} - 1] \), or equivalently, an \( n \)-input/\( m \)-output boolean function.

\[
\begin{align*}
\text{data MTBDD a } &= \text{MTBDD a a (BT a) deriving (Show, Eq)} \\
\text{The function to_mtbdd creates, from a natural number \( tt \) representing a truth table, an MTBDD representing functions of type } N \rightarrow M \text{ with } M = [0..2^m - 1], N = [0..2^n - 1]. \text{ Similarly to a BDD, it is represented as binary tree of } n \text{ levels, except that its leaves are in } [0..2^n - 1].
\end{align*}
\]

\[1\] bsums is sequence A060803 in The On-Line Encyclopedia of Integer Sequences, http://www.research.att.com/~njas/sequences
to_mtbdd m n tt = MTBDD m n r where
  mlimit = 2^m
  nlimit = 2^n
  ttlimit = mlimit * nlimit
  r = if tt < ttlimit
   then (to_mtbdd mlimit n tt)
   else error
       "bt: last arg " ++ (show tt) ++
       " should be < " ++ (show ttlimit))

Given that correctness of the range of tt has been checked,
the function to_mtbdd applies bitmerge_unpair recursively up to depth n,
where leaves in range [0..mlimit − 1] are created.

to_mtbdd mlimit n tt | (n < 1) && (tt < mlimit) = C tt
  | (n < 1) && (tt < nlimit) = D k l r
  | (tt < ttlimit) = (D k l r)

Converting back from MTBDDs to natural numbers is
basically the same thing as for BDDs, except that assertions
about the range of leaf data are enforced.

from_mtbdd (MTBDD m n b) = from_mtbdd mlimit n b

from_mtbdd mlimit n (C tt) | (n < 1) && (tt < mlimit) = tt
  | (n < 1) && (tt < nlimit) = (D _ l r)
  | (tt < ttlimit) = tt

The following examples show that to_mtbdd and from_mtbdd are indeed inverses values in [0..2^n − 1] × [0..2^m − 1].

>to_mtbdd 3 3 2008
MTBDD 3 3
   (D 2
      (D 1
         (D 0 (C 2) (C 1))
         (D 0 (C 2) (C 1)))
      (D 1
         (D 0 (C 2) (C 0))
         (D 0 (C 1) (C 1))))

>from_mtbdd it
2008

>mprint (to_mtbdd 2 2) [0..3]
MTBDD 2 2
   (D 1
      (D 0 (C 0) (C 0))
      (D 0 (C 0) (C 0)))
   (D 2
      (D 1
         (D 0 (C 1) (C 0))
         (D 0 (C 0) (C 0)))
      (D 0 (C 1) (C 0))
      (D 0 (C 1) (C 0))))

9. Related work

Pairing functions have been used for work on decision problems as early as (Pepis 1938; Kalmar 1939; Robinson 1950).

BDDs are the dominant boolean function representation in the field of circuit design automation (Meinel and Theobald 1999; Drechsler et al. 2004).

Besides their uses in circuit design automation, MTBDDs have been used in model-checking and verification of arithmetic circuits (Fujita et al. 1997; Ciesinski et al. 2008).

MTBDDs have also been used in a Genetic Programming context (Sakanashi et al. 1996; Rothlauf et al. 2006; Chen et al. 2004) as a representation of evolving individuals subject to crossovers and mutations expressed as structural transformations.

10. Conclusion and Future Work

Our new pairing/unpairing functions and their surprising connection to BDDs, have been the indirect result of implementation work on a number of practical applications. Our initial interest has been triggered by applications of the encodings to combinational circuit synthesis (Tarau and Ludermann 2008).

We have found them also interesting as uniform blocks for Genetic Programming applications. In a Genetic Programming context (Koza 1992; Poli et al.), the bijections between bitvectors/natural numbers on one side, and trees/graphs representing BDDs on the other side, suggest exploring the mapping and its action on various transformations as a phenotype-genotype connection. Given the connection between BDDs to boolean and finite domain constraint solvers it would be interesting to explore in that context, efficient succinct data representations derived from our BDD encodings.

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**Appendix**

To make the code in the paper fully self contained, we list here some auxiliary functions.

**Bitvector Boolean Operation Definitions**

```
type Nat = Integer
nand_ :: Nat -> Nat -> Nat
nand_ m = nand_ m l
 nor_ :: Nat -> Nat -> Nat
 nor_ m l = nor_ m l
 impl_ :: Nat -> Nat -> Nat
 impl_ m l = impl_ m l
 less_ :: Nat -> Nat -> Nat
 less_ m l = less_ m l

nand_ mask x y = mask .&. (complement (x .&. y))
nor_ mask x y = mask .&. (complement (x .|. y))
impl_ mask x y = (mask .&. (complement x)) .|. y
less_ _ x y = x .&. (complement y)
```

**Boolean Operation Encodings and Names**

```
-- operation codes
opcode m 0 = nand_ m
opcode m 1 = nor_ m
opcode m 2 = impl_ m
opcode m 3 = less_ m
opcode _ 4 = xor
opcode _ n = error ("unexpected opcode:"++(show n))

-- operation names
opname 0 = "nand"
opname 1 = "nor"
opname 2 = "impl"
opname 3 = "less"
opname 4 = "xor"
opname n = error ("no such opcode:"++(show n))
```

**A Few Interesting Libraries**

```
mixops = [0,2]
symops = [0,1]
asymops = [2,3]
```

**Tests for the Circuit Synthesizer**

```
t0 = findFirstGood symops 3 8 71
t1 = syn asymops 3 71
t2 = mapM_ print (synall mixops 2)
t3 = syn asymops 3 83 -- ite
  t4 = syn asymops 3 83
  t5 = syn [0..4] 3 83 -- ite with all ops
    -- x xor y xor z -- cpu intensive
  t6 = syn asymops 3 105
```

**Bit crunching functions**

This function splits a natural number in a set of natural numbers indicating the positions of its 1 bits in its right to left binary representation.

```
  nat2set n = nat2exps n 0 where
    nat2exps 0 _ = []
    nat2exps n x =
      if (even n) then xs else (x:xs) where
        xs = nat2exps (div n 2) (succ x)
This function aggregates a set of natural numbers indicating positions of 1 bits into the corresponding natural number.

\[
\text{set2nat } ns = \text{sum} \ (\text{map} \ (2^\cdot) \ ns)
\]

**I/O functions**

These functions print out the BDDs of all the \(2^k\) truth tables associated to \(k\) variables.

\[
\text{print\_plain } k = \text{mapM}_\cdot \\
\quad (\text{print} \ . \ (\text{plain\_bdd } k)) \ [0..(\text{bigone } k)]
\]

\[
\text{print\_reduced } k = \text{mapM}_\cdot \\
\quad (\text{print} \ . \ (\text{bdd } k)) \ [0..(\text{bigone } k)]
\]

This function applies \(f\) to a list of objects and prints the results on successive lines.

\[
\text{mprint } f = (\text{mapM}_\cdot \ \text{print}) \ . \ (\text{map } f)
\]