DONALDSON–THOMAS INVARIANTS VERSUS INTERSECTION COHOMOLOGY OF QUIVER MODULI

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Abstract. The paper has two aims. The main aim is to prove that the Hodge theoretic Donaldson–Thomas invariant for a quiver with zero potential and a generic King stability condition agrees with the compactly supported intersection cohomology of the closure of the stable locus inside the associated coarse moduli space of semistable quiver representations. In fact, we prove an even stronger result for the intersection complex. The second aim of the paper is to show that the integrality conjecture in Donaldson–Thomas theory holds for all abelian categories of homological dimension one. The result will be a direct consequence of a stronger version for Donaldson–Thomas “sheaves” on moduli spaces.

Contents

1. Introduction 1
2. Moduli spaces of quiver representations 4
2.1. Quiver representations 4
2.2. Moduli spaces 5
3. Intersection complex 6
3.1. From perverse sheaves to mixed Hodge modules 6
3.2. Intersection complex 7
3.3. Schur functors 7
4. DT invariants and intersection complexes 10
4.1. Donaldson–Thomas invariants 10
4.2. The main result 10
4.3. Application to matrix invariants 12
5. Proof of Theorem 2.1 13
5.1. The stack of nilpotent quiver representations 13
5.2. Virtual smallness of the Hilbert–Chow map 14
6. Motivic DT-theory and the integrality conjecture 15
6.1. Stack functions 15
6.2. λ-ring structures 16
6.3. The setting 16
6.4. Donaldson–Thomas invariants 17
6.5. The integrality conjecture 18
References 20

1. Introduction

The theory of Donaldson–Thomas invariants started around 2000 with the seminal work of R. Thomas [33]; associating numerical invariants, that is, numbers, to moduli spaces in the absence of strictly semistable objects. Six years later D. Joyce [12, 13, 14, 15, 16, 17] and Y. Song [18] extended the theory, producing numbers
even in the presence of semistable objects which is the generic situation. Around the same time, M. Kontsevich and Y. Soibelman [23], [25], [24] independently proposed a theory producing motives instead of simple numbers, also in the presence of semistable objects. The technical difficulties occurring in their approach disappear in the special situation of representations of quivers (with zero potential). This case has been intensively studied by the second author in a series of papers [28], [29], [30]. Despite some computations of motivic or even numerical Donaldson–Thomas invariants for quivers with or without potential (see [1], [6], [7], [26]), the true nature of Donaldson–Thomas invariants still remains mysterious.

This paper is a first step to disclose the secret by showing that the Donaldson–Thomas invariants for quiver representations compute the compactly supported intersection cohomology of the closure of the stable locus inside the associated coarse moduli space of semistable representations. While trying to prove this result, the authors observed the importance of the integrality conjecture, which was the reason to extend the paper by a second part containing its proof.

We will actually prove an even stronger version by defining a Donaldson–Thomas “sheaf” on the coarse moduli space \( M^{ss} \). Strictly speaking, this sheaf is not a (perverse) sheaf but a class in a suitably extended Grothendieck group of mixed Hodge modules. The cohomology with compact support of that sheaf is the usual Hodge theoretic Donaldson–Thomas invariant - a class in the Grothendieck group of mixed Hodge structures. Our main result is the following (we refer to the following sections for precise notation):

\[ \text{Theorem 1.1.} \text{ Let us fix a generic King stability condition. Then the Donaldson–Thomas sheaf and the intersection complex of the closure of the stable locus inside the coarse moduli space } M^{st} \text{ agree in the Grothendieck group of mixed Hodge modules. In particular, by taking cohomology with compact support, we obtain for every dimension vector } d \]

\[ DT_d = \begin{cases} IC_c(M^{ss}_d, \mathbb{Q}) = IC(M^{ss}_d, \mathbb{Q})^\vee & \text{if } M^{st}_d \neq \emptyset, \\ 0 & \text{otherwise} \end{cases} \]

\[ \text{in the Grothendieck ring of (polarizable) mixed Hodge structures.} \]

As Donaldson–Thomas invariants for quiver representations can be computed with computer power quite effectively, this theorem provides a quick algorithm to determine intersection Hodge numbers. The previous algorithm to do that goes back to extensive work of F. Kirwan around 1985 (see [19], [20], [21], [22]) and is impracticable. Moreover, using wall-crossing formulas, we are now able to understand the change of intersection Hodge numbers under variations of stability conditions.

**Corollary 1.2 (Positivity).** For a generic (King) stability condition the (motivic) Donaldson–Thomas invariant is a polynomial in the Lefschetz motive with positive coefficients.

Indeed, the coefficients are the dimensions of the intersection cohomology groups.

Another corollary is obtained by using the fact that the moduli space of semistable quiver representations admits a proper map to the affine, connected moduli space of semisimple representations of the same dimension vector. If the quiver is acyclic, there is only one, thus the moduli space must be compact and we can apply the Hard Lefschetz theorem to intersection cohomology.

**Corollary 1.3 (Unimodularity).** If \( Q \) is acyclic, the Donaldson–Thomas invariant for a generic stability condition is a unimodular polynomial in the Lefschetz motive.

The next result is a direct consequence of our main theorem and Proposition 6.4.
Corollary 1.4 (Locality). Fix a generic stability condition and a point \( V = \oplus_{k \in K} U_k^m \) in the moduli space of representations, that is, a polystable object with stable summands \( U_k \). If the moduli space also contains stable representations, then the fiber at \( V \) of the intersection complex of the moduli space is given by a certain Donaldson–Thomas invariant for the \( \text{Ext} \)-quiver of the collection \( (U_k)_{k \in K} \) up to a duality operation interchanging \( L_1/2 \) with \( L_{-1/2} \).

Finally, we will give, in Theorem 4.8, an explicit formula for the intersection Betti numbers of the classical spaces of matrix invariants (that is, the quotient of tuples of linear operators by simultaneous conjugation), using the explicit formula for motivic DT invariants for loop quivers in [30].

The paper is organized as follows. Section 2 provides some background on quivers and their representations. The main purpose is to fix the notation. Although we will not use it, subsection 2.1 also contains a quick link to 3-Calabi–Yau categories - the natural environment of Donaldson–Thomas theory. The most important result of section 2 is Theorem 2.1, stating that the so-called Hilbert–Chow morphism from the moduli space \( M_{\text{ss}}^{d,f} \) of framed representations to the moduli space \( M_{\text{ss}}^{d} \) of unframed representations is what we will call virtually small.

Theorem 1.5. Let \( \mu \) be the slope of a dimension vector \( d \) with respect to a King stability condition \( \theta \). If \( \theta \) is \( \mu \)-generic, the morphism \( \pi : M_{\text{ss}}^{d,f} \to M_{\text{ss}}^{d} \) is projective and virtually small, that is, there is a finite stratification \( M_{\text{ss}}^{d} = \bigsqcup \lambda S_{\lambda} \) with empty or dense stratum \( S_0 = M_{\text{ss}}^{d} \) such that \( \pi^{-1}(S_{\lambda}) \to \lambda \) is étale locally trivial and

\[
\dim \pi^{-1}(x_{\lambda}) - \dim \mathbb{P}f^{d-1} \leq \frac{1}{2} \text{codim} S_{\lambda}
\]

for every \( x_{\lambda} \in S_{\lambda} \) with equality only for \( S_{\lambda} = S_0 \neq \emptyset \) with fiber \( \pi^{-1}(x_0) \cong \mathbb{P}f^{d-1} \).

The proof of this important technical result is postponed to section 5 to keep the introduction short.

Section 3 is devoted to intersection complexes and the Schur functor formalism. As we need the notion of a weight filtration, restricting to perverse sheaves is not sufficient. Hence, we have to consider mixed Hodge modules, but there is no reason to be worried about that. We only need that the Grothendieck group is freely generated (as a group) by some sort of intersection complexes, and also the Decomposition Theorem of Beilinson, Bernstein, Deligne and Gabber will be used sometimes.

Taking direct sums of representations induces a symmetric monoidal tensor product on the category of mixed Hodge modules by convolution. Using some general machinery (see [8]), one can introduce Schur (endo)functors. Among them the symmetric and alternating powers are the most famous ones, and we finally obtain a \( \lambda \)-ring structure on the Grothendieck group of mixed Hodge structures.

The latter is used in Section 4 to define the Donaldson–Thomas sheaves. A well-known proposition connecting Donaldson–Thomas invariants with framed quiver representations generalizes literally to our situation of mixed Hodge modules. Using this, the virtual smallness of the Hilbert–Chow morphism, and the Decomposition Theorem, we finally deliver the proof of our main theorem by comparing degrees of the weight filtration.

While proving our main result in section 4, we will observe that a certain finiteness condition is crucial. It turns out that this condition is a sheaf version of the famous integrality conjecture in Donaldson–Thomas theory. Fortunately, we can
give a proof in our situation of quiver representations by reducing the problem to a result of Efimov (see [9], Theorem 1.1). In fact, the arguments are rather general, and thus we will generalize the setting to all abelian categories of homological dimension one and with good moduli problems. Here is the main result of the second part of our paper, that is, of section 6, which holds for all abelian categories of homological dimension one.

**Theorem 1.6** (integrality conjecture, sheaf version). For a \( \mu \)-generic stability condition the motivic Donaldson–Thomas sheaf \( DT^\text{mot}_\mu \) is in the image of the natural map

\[
\hat{K}(\text{Var}/\mathcal{M}^{ss}_\mu)[L^{-1/2}] \longrightarrow \hat{K}(\text{Var}/\mathcal{M}^{ss}_\mu)[L^{-1/2}, (L^n - 1)^{-1}]: n \in \mathbb{N}.
\]

By “integrating” over the moduli space of objects in a given class \( d \) of arbitrary slope \( \mu \), we obtain a proof of the famous integrality conjecture.

**Corollary 1.7** (integrality conjecture). For a \( \mu \)-generic stability condition the motivic Donaldson–Thomas invariant \( DT_d^{\text{mot}} \) is in the image of the natural map

\[
\hat{K}(\text{Var}/k)[L^{-1/2}] \longrightarrow \hat{K}(\text{Var}/k)[L^{-1/2}, (L^n - 1)^{-1}]: n \in \mathbb{N}.
\]

This result has been obtained by Efimov for representations of symmetric quivers and trivial stability condition (see [9], Theorem 1.1). A very complicated prove of the integrality conjecture even for quivers with potential was given by Kontsevich and Soibelman (see [23], Theorem 10).

The authors expect the methods developed in the present paper to be applicable to other situations where the computation of the intersection cohomology of moduli spaces and of motivic Donaldson–Thomas invariant is desirable, in particular to the moduli spaces of vector bundles on smooth projective curves (in preparation).

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2. Moduli spaces of quiver representations

2.1. Quiver representations. We fix a field \( \mathbb{K} \) which might either be our ground field \( k \) or, as in section 6, a not necessarily algebraic extension of the latter. Let \( Q = (Q_0, Q_1) \) be a quiver consisting of a finite set \( Q_0 \) of vertices and a finite set \( Q_1 \) of arrows. To any quiver we associate its path algebra \( \mathbb{K}Q \). The underlying \( \mathbb{K} \)-vector space is spanned by paths of arbitrary length with a path of length zero attached to every vertex. Multiplication on \( \mathbb{K}Q \) is given by \( \mathbb{K} \)-linear extension of concatenation. Equivalently, one could think of \( \mathbb{K}Q \) as a \( \mathbb{K} \)-linear category with set of objects \( Q_0 \) and Hom\(_{\mathbb{K}Q}(i,j) \) being the \( \mathbb{K} \)-vector space generated by all paths from \( i \) to \( j \). Again, composition is induced by \( \mathbb{K} \)-linear extension of concatenation.

There is a second (dg-)algebra associated to \( Q \), namely its Ginzburg algebra \( \Gamma_\mathbb{K}Q \). The underlying algebra is the path algebra \( \mathbb{K}Q^{\text{ex}} \) associated to the extended quiver \( Q^{\text{ex}} = (Q_0, Q_1 \cup Q_1^{(p)} \cup Q_0) \) obtained from \( Q \) by adding to every arrow \( \alpha : i \to j \) of \( Q \) another arrow \( \alpha^* : j \to i \) with opposite orientation, and a loop \( l_i : i \to i \) for every vertex \( i \in Q_0 \). We make \( \Gamma_\mathbb{K}Q \) into a dg-algebra by introducing a grading such that \( \deg(\alpha) = 0, \deg(\alpha^*) = -1, \) and \( \deg(l_i) = -2 \). The differential is uniquely determined by putting

\[
da = da^* = 0 \quad \text{and} \quad dl_i = \sum_{\alpha: i \to j} \alpha^* \alpha - \sum_{\alpha: j \to i} \alpha \alpha^*.
\]
Again, we can think of $\Gamma \mathbb{K}Q$ as a dg-category with set of objects being $Q_0$. Moreover, $H^0(\Gamma \mathbb{K}Q) \cong \mathbb{K}Q$ can be interpreted as a dg-category with zero grading and trivial differential.

By looking at dg-functors $V : \mathbb{K}Q \rightarrow \text{dg-Vect}_\mathbb{K}$ and $W : \Gamma \mathbb{K}Q \rightarrow \text{dg-Vect}_\mathbb{K}$ into the category of dg-vector spaces with finite dimensional total cohomology, we get two dg-categories with model structures and associated triangulated homotopy $(\mathcal{A}_\infty)$-categories $D^b(\mathbb{K}Q - \text{Rep})$ and $D^b(\Gamma \mathbb{K}Q - \text{Rep})$. Each has a bounded t-structure with heart $\mathbb{K}Q - \text{Rep}$ being the abelian category of quiver representations, that is, of functors $V : \mathbb{K}Q \rightarrow \text{Vect}_\mathbb{K}$ into the category of finite dimensional $\mathbb{K}$-vector spaces. In particular,

$$K_0(D^b(\mathbb{K}Q - \text{Rep})) \cong K_0(D^b(\Gamma \mathbb{K}Q - \text{Rep})) \cong K_0(\mathbb{K}Q - \text{Rep}).$$

There is a group homomorphism $\text{dim} : K_0(\mathbb{K}Q - \text{Rep}) \rightarrow \mathbb{Z}^{Q_0}$ associating to every representation resp. functor $V$ the tuple $(\dim V_i)_{i \in Q_0}$ of dimensions of the vector spaces $V_i := V(i)$. There are two pairings on $\mathbb{Z}^{Q_0}$ defined by

$$(d, e) := \sum_{i \in Q_0} d_i e_i - \sum_{Q_1 \ni i \rightarrow j} d_i e_j$$

and

$$(d, e) := (d, e) - (e, d)$$

such that the pull-back of these pairings via $\text{dim}$ is just the Euler pairing induced by $D^b(\mathbb{K}Q - \text{Rep})$ resp. $D^b(\Gamma \mathbb{K}Q - \text{Rep})$. The skew-symmetry of the latter reflects the fact that $D^b(\Gamma \mathbb{K}Q - \text{Rep})$ is a 3-Calabi–Yau category, that is, the triple shift functor $[3]$ is a Serre functor. This provides the link to Donaldson–Thomas theory.

2.2. Moduli spaces. The stack of $Q$-representations, that is, of objects in $\mathbb{K}Q - \text{Rep}$, can be described quite easily. For this, fix a dimension vector $d = (d_i) \in \mathbb{N}^{Q_0}$ and note that $G_d := \prod_{i \in Q_0} \text{Aut}(\mathbb{K}^d_i)$ acts on $R_d := \prod_{i \in Q_0} \text{Hom}(\mathbb{K}^d_i, \mathbb{K}^d_i)$ in a canonical way. The stack of $Q$-representations of dimension $d$ is just the quotient stack $\mathfrak{M}_d = R_d/G_d$. There are also derived (higher) stacks of objects in $D^b(\mathbb{K}Q - \text{Rep})$ resp. $D^b(\Gamma \mathbb{K}Q - \text{Rep})$ containing $\mathfrak{M}_d$ as a substack, but we are not going into this direction.

Instead, we want to study semistable representations of $Q$. As the radical of the Euler pairing contains the kernel of $\text{dim} : K_0(\mathbb{K}Q - \text{Rep}) \rightarrow \mathbb{Z}^{Q_0}$, every tuple $z = (z_i)_{i \in Q_0} \in \{r \exp(i \pi \phi) \in \mathbb{C} \mid r > 0, 0 < \phi \leq 1\}^{Q_0} \subset \mathbb{C}^{Q_0}$ provides a numerical Bridgeland stability condition on $D^b(\mathbb{K}Q - \text{Rep})$ and on $D^b(\Gamma \mathbb{K}Q - \text{Rep})$ with central charge $Z(V) = z \cdot \dim V := \sum_{i \in Q_0} z_i \dim V_i$ of slope $\mu(V) := -\Re Z(V)/3mZ(V)$ and standard t-structure. Hence we get an open substack $\mathfrak{M}^{ss}_d = R_d^{ss}/G_d$ of semistable $Q$-representations. For every $\mu \in (-\infty, +\infty]$ let $\Lambda_\mu \subset \mathbb{N}^{Q_0}$ be the monoid of dimension vectors $d$ (including $d = 0$) such that $z \cdot d = \sum_{i \in Q_0} z_i d_i \in \mathbb{K}$ has slope $\mu$. We call $z$ $\mu$-generic if $(d, e) = 0$ for all $d, e \in \Lambda_\mu$, and generic if that holds for all $\mu$. The non-generic “stability conditions” $z$ lie on a countable but locally finite union of walls in $\{r \exp(i \pi \phi) \in \mathbb{C} \mid r > 0, 0 < \phi \leq 1\}^{Q_0}$ of real codimension one. Obviously every stability for a symmetric quiver is generic. Another important class is given by complete bipartite quivers and the maximally symmetric stabilities used in [31] to construct a correspondence between the cohomology of quiver moduli and the GW invariants of [11].

As we wish to form moduli schemes, we should restrict ourselves to King stability conditions $z = (-\theta_i + \sqrt{-1})_{i \in Q_0}$ for some $\theta = (\theta_i) \in \mathbb{Z}^{Q_0}$, giving rise to a linearization of the $G_d$ action on $R_d$ with semistable points $R_d^{ss}$. Let us denote the GIT quotient by $\mathcal{M}^{ss}_d = R_d^{ss}/G_d$. The points in $\mathcal{M}^{ss}_d$ correspond to polystable representations $V = \oplus_i V_i$, and the obvious morphism $p : \mathfrak{M}^{ss}_d \rightarrow \mathcal{M}^{ss}_d$ maps a semistable representation to the direct sum of its stable factors. We also have the open substack $\mathfrak{M}^{st}_d \subset \mathfrak{M}^{ss}_d$ of stable representations which is a $\mathbb{K}^*$-gerbe over the
open subscheme $M_{st}^d \subset M_{ss}^d$ of stable representations. Note that $M_d, M_{ss}^d, M_{st}^d$, and $M_{d, st}$ are smooth while $M_{d, ss}$ is not. Moreover, $M_{d, st}$ is either dense in $M_{d, ss}$ or empty. We call $\theta (\mu)$-generic if $z = (-\theta_i + \sqrt{-1})_{i \in Q_0}$ is $(\mu)$-generic in the previous sense.

For later applications we also need framed $Q$-representations (see [10]). For this we fix a framing vector $f \in \mathbb{N}^{Q_0}$ and consider representations of a new quiver $Q_f = (Q_0 \sqcup \{\infty\}, Q_1 \sqcup \{\beta_i : \infty \to i \mid i \in Q_0, 1 \leq i \leq f_i\}$ with dimension vector $d'$ obtained by extending $d$ via $d_\infty = 1$. We also extend $\theta$ appropriately (see [10]) and get a King stability condition $\theta'$ for $Q_f$. Let $M_{st, f}$ be the moduli space of $\theta'$-semistable $Q_f$-representations of dimension vector $d'$. It turns out that $M_{st, f} = M_{d', f}$, and thus $M_{st, f}$ is smooth. There is an obvious morphism $\pi : M_{st, f} \to M_{d, f}$, obtained by restricting a $\theta'$-(semi)stable representation of $Q_f$ to the subquiver $Q$ which turns out to be $\theta$-semistable. The following theorem will we crucial for proving our main theorem. To keep the introduction short, we will postpone its proof to section 5.

Theorem 2.1. Let $\mu$ be the slope of a dimension vector $d$ with respect to a King stability condition $\theta$. If $\theta$ is $\mu$-generic, the morphism $\pi : M_{d, f} \to M_{d, ss}$ is projective and virtually small, that is, there is a finite stratification $M_{d, ss} = \sqcup_i S_i$ with empty or dense stratum $S_0 = M_{d, ss}$ such that $\pi^{-1}(S_0) \to S_0$ is étale locally trivial and

$$\dim \pi^{-1}(x_\lambda) - \dim \mathbb{P}^f d - 1 \leq \frac{1}{2} \operatorname{codim} S_\lambda$$

for every $x_\lambda \in S_\lambda$ with equality only for $S_\lambda = S_0 \neq \emptyset$ with fiber $\pi^{-1}(x_0) \cong \mathbb{P}^f d - 1$.

Let us finish this section with a simple but important observation. Given a slope $\mu \in (-\infty, +\infty]$, the moduli stack $M_{d, ss}^\mu := \sqcup_{d \in \Lambda_\mu} M_{d, ss}^\mu$ resp. the moduli space $M_{d, ss}^\mu := \sqcup_{d \in \Lambda_\mu} M_{d, ss}^\mu$, is a commutative monoid in the category of stacks, resp. schemes, with respect to direct sums of representations. The unit is given by the zero-dimensional representation which is considered to be semistable with any slope. Obviously, the morphisms $p : M_{d, ss}^\mu \to M_{d, ss}$ and $\dim : M_{d, ss}^\mu \to \Lambda_\mu$ mapping every polystable representation to its dimension vector are monoid homomorphism. Let us also introduce the smooth schemes $M_{d, f}^\mu := \sqcup_{d \in \Lambda_\mu} M_{d, f}^\mu$ and $M_{d, ss}^\mu := \sqcup_{d \notin \Lambda_\mu} M_{d, ss}^\mu$, none of which is a monoid.

3. Intersection complex

3.1. From perverse sheaves to mixed Hodge modules. The ground field in the next two sections will be $k = \mathbb{C}$. In this section we recall some standard facts about perverse sheaves, intersection complexes and Schur functors. The interested reader will find more details in [5] and [32]. Let $X$ be a variety, locally of finite type over $\mathbb{C}$. We denote by $\operatorname{Perv}(X)$ resp. $\operatorname{MHM}(X)$ the abelian categories of perverse sheaves resp. mixed Hodge modules on $X$. There is a natural functor $\operatorname{rat} : \operatorname{MHM}(X) \to \operatorname{Perv}(X)$ associating to every mixed Hodge module its underlying perverse sheaf. For a morphism $f : X \to Y$ of finite type we get two pairs $(f^*, f_*), (f_!, f_{!*})$ of adjoint triangulated functors $f_*, f_! : D^b(X) \to D^b(Y)$ and $f^!, f^* : D^b(Y) \to D^b(X)$, and similarly for mixed Hodge modules, satisfying Grothendieck’s axioms of the four functor formalism. Moreover, the functor $\operatorname{rat}$ is compatible with these functors in the obvious way. We also mention that each of the categories $\operatorname{Perv}(X)$ and $\operatorname{MHM}(X)$ is of finite length and has a duality functor relating $f_*$ with $f_!$ and $f^!$ with $f^*$. Moreover, there is an exact equivalence $\mathbb{T}$ of $\operatorname{MHM}(X)$, called the Tate twist, commuting with all four functors and satisfying $\operatorname{rat} \circ \mathbb{T} = \operatorname{rat}$. In practice, $\mathbb{T}$ usually acts by means of the composition $\mathbb{T} := [-2] \circ \mathbb{T}$ on $D^b(\operatorname{MHM}(X))$. In our applications, both $\operatorname{Perv}(X)$
and MHM(X) are equipped with an exact symmetric monoidal tensor product with unit object 1 and \(\mathbb{T}(-) \equiv \mathbb{T}(1) \otimes (-)\). By abuse of language, we will also denote \(\mathbb{T}(1)\), resp. \(L(1)\), by \(\mathbb{T}\), resp. \(L\). The actions of \(\mathbb{T}\) and \(L\) on \(K_0(\mathrm{MHM}(X))\) coincide, making it into a \(\mathbb{Z}[\mathbb{L}^{\pm 1}]\)-module. We denote by \(K_0(\mathrm{MHM}(X))\) \(\mathbb{L}^{-1/2}\) the \(\mathbb{Z}[\mathbb{L}^{\pm 1}]\)-module obtained by adjoining a square root of \(\mathbb{L}\). One can also categorify this, giving rise to an exact equivalence \(\mathbb{T}^{1/2}\) on an enlarged abelian category of mixed Hodge motives. Then, \(\mathbb{L}^{-1/2}\) is given by \([1] \circ \mathbb{T}^{-1/2}\), and should be seen as a refinement of the shift functor \([1]\) on \(\text{Perv}(X)\).

3.2. Intersection complex

Given a closed equidimensional subvariety \(Z \subset X\) and a local system on a dense open subset \(Z^\circ\) of the regular part \(Z_{\text{reg}}\) of \(Z\), there is canonical perverse sheaf \(\mathcal{IC}_{Z}(L)\) on \(X\), called the \(L\)-twisted intersection complex of \(Z\), such that \(\mathcal{IC}_{Z}(L)|_{Z^\circ} = L[\text{dim}\, Z]\). If \(Z\) and \(L\) are irreducible, \(\mathcal{IC}_{Z}(L)\) is an irreducible object of \(\text{Perv}(X)\), and all irreducible objects are obtained in this way. For \(\text{MHM}(X)\), there is a similar construction, with \(L\) replaced by a polarizable variation \(V\) of mixed Hodge structures. We will, however, use the slightly non-standard convention \(\mathcal{IC}_{Z}(V)|_{Z^\circ} = L^{-\text{dim}\, Z/2}(V) = \mathbb{T}^{-\text{dim}\, Z/2}(V)[\text{dim}\, Z]\). Note that an irreducible variation of mixed Hodge structures is pure, and application of \(\mathbb{T}^{-1/2}\) reduces the weight by one. If \(Z\) has several connected components of different dimension, the construction of \(\mathcal{IC}_{Z}(L)\) resp. \(\mathcal{IC}_{Z}(V)\) generalizes accordingly. Applying this to the trivial variation \(\mathbb{Q}\) of pure Hodge structures of type \((0,0)\) on \(Z_{\text{reg}}\), we obtain a distinguished intersection complex \(\mathcal{IC}_{Z}(\mathbb{Q})\).

3.3. Schur functors

Let us now specialize to \(X = M_{\mu}^{ss}\), although everything in this section not involving infinite sums remains true for arbitrary commutative monoids \((X, \odot, 0)\) in the category of varieties (locally of finite type) such that \(\odot: X \times X \to X\) is quasi-finite. Due to the last property, the higher derived direct images \(R^i\odot_\ast\) vanish, and we obtain a symmetric monoidal tensor product

\[ \odot: \mathrm{MHM}(M_{\mu}^{ss}) \times \mathrm{MHM}(M_{\mu}^{ss}) \to \mathrm{MHM}(M_{\mu}^{ss}), \quad E \otimes F := \odot_\ast(E \boxtimes F), \]

and similarly for \(\text{Perv}(M_{\mu}^{ss})\). The unit \(1\) is given by \(\mathcal{IC}_{M_{\mu}^{ss}}(\mathbb{Q})\), which is a skyscraper sheaf of rank one supported at the zero-dimensional representation \(0\). We drop the \(\odot\)-sign when dealing with the associated Grothendieck groups \(K_0(\text{Perv}(M_{\mu}^{ss}))\) and \(K_0(\mathrm{MHM}(M_{\mu}^{ss}))\), respectively.

Given \(E \in \mathrm{MHM}(M_{\mu}^{ss})\) and \(n \in \mathbb{N}\), the mixed Hodge module \(E^{\otimes n}\) carries a natural action of the symmetric group \(S_n\). By general arguments (see [3]), we obtain a decomposition

\[ E^{\otimes n} = \bigoplus_{\lambda \vdash n} W_\lambda \otimes S^\lambda(E) \]

for certain mixed Hodge modules \(S^\lambda(E)\), where \(W_\lambda\) denotes the irreducible representation of \(S_n\) associated to the partition \(\lambda\) of \(n\). The tensor product used on the right hand side can be defined for every additive category, and should not be confused with the tensor product explained above. However, after identifying vector spaces \(W\) with trivial variations of pure Hodge structures of type \((0,0)\) over \(M_{\mu}^{ss}\), both tensor products agree. The decomposition is functorial, giving rise to Schur functors \(S^\lambda: \mathrm{MHM}(M_{\mu}^{ss}) \to \mathrm{MHM}(M_{\mu}^{ss})\) for every partition \(\lambda\). The same construction also applies to \(\text{Perv}(M_{\mu}^{ss})\), and \(\text{rat}: \mathrm{MHM}(M_{\mu}^{ss}) \to \text{Perv}(M_{\mu}^{ss})\) “commutes” with Schur functors of the same type.

Example 3.1.

1. For \(\lambda = (n)\), the representation \(W_\lambda\) is the trivial representation of \(S_n\) and \(S^n(E) = \text{Sym}^n(E)\). If \(E|M_{\mu}^{ss} = 0\), we get \(\text{Sym}^n(E)|_{M_{\mu}^{ss}} = 0\) for every \(d \in X_\mu\) provided \(n \gg 0\). In particular, \(\text{Sym}(E) = \bigoplus_n \text{Sym}^n(E)\) is well-defined.
(2) For \( \lambda = (1, \ldots, 1) \), the representation \( W_\lambda \) is the sign representation of \( S_n \) and \( S^\lambda(E) =: \text{Alt}^n(E) \). As before \( \text{Alt}(E) = \oplus_n \text{Alt}^n(E) \) is well-defined provided \( E|_{M^n_\mu} = 0 \).

The following proposition is a standard result.

**Proposition 3.2.** Let \( E, F \) be in \( \text{MHM}(M^n_\mu) \) or in \( \text{Perv}(M^n_\mu) \) such that \( E|_{M^n_\mu} = F|_{M^n_\mu} = 0 \). Denote by \( \mathcal{P} \) be the set of all partitions of arbitrary size. Then

\[
\text{Sym}(E \oplus F) = \text{Sym}(E) \otimes \text{Sym}(F), \quad \text{in particular}
\]

\[\text{Sym}^n(E \oplus F) = \bigoplus_{i+j=n} \text{Sym}^i(E) \otimes \text{Sym}^j(F), \quad \text{in particular}\]

\[\text{Sym}(E) = \bigoplus_{\lambda \in \mathcal{P}} S^\lambda(E) \otimes S^\lambda(F), \quad \text{in particular}\]

\[\text{Sym}^n(E \otimes F) = \bigoplus_{\lambda \vdash n} S^\lambda(E) \otimes S^\lambda(F).\]

Equations (1) and (2) are of course also true without the additional assumptions on \( E \) and \( F \). The next result is also well-known.

**Proposition 3.3.** The Schur functors \( S^\lambda \) induce well defined operations on the Grothendieck groups \( K_0(\text{Perv}(M^n_\mu)) \) and \( K_0(\text{MHM}(M^n_\mu)) \), respectively, satisfying the analogs of equation (1) and (2). In particular, both Grothendieck groups carry the structure of a (special) \( \lambda \)-ring.

It is worth to mention the following technical detail. Although \( \text{Sym}(E) = \oplus_n \text{Sym}^n(E) \) by definition, this equation cannot hold on the level of Grothendieck groups as we do not have infinite sums. To define these, we need to complete the Grothendieck groups as follows. Let \( F^p \subset K_0(\text{Perv}(M^n_\mu)) \) be the subcategory generated by all perverse sheaves \( E \) such that \( E|_{M^n_\mu} = 0 \) if \( d \) cannot be written as a sum of \( p \) nonzero dimension vectors, i.e., \( |d| := \sum_{i \in Q_\mu} d_i < p \). It is easy to see that \( F^p F^q \subset F^{p+q} \) and \( S^\lambda(F^p) \subset F^{np} \) for all \( \lambda \vdash n \) and all \( n, p, q \in \mathbb{N} \). Hence, the \( F^p \) provide a \( \lambda \)-ring filtration, and the corresponding completion \( \hat{K}_0(\text{Perv}(M^n_\mu)) = \prod_{d \in \Lambda_\mu} K_0(\text{Perv}(M^n_d)) \) has a well-defined ring structure and action of \( S^\lambda \). Moreover, \( \sum_n \text{Sym}^n(E) \) is well-defined and agrees with the class of \( \text{Sym}(E) \) for \( E \in F^1 \). The completion of \( K_0(\text{MHM}(M^n_\mu)) \) is done in the same way.

As \( T = L \) in \( K_0(\text{MHM}(M^n_\mu)) \) and \( \text{Sym}^n(T^{\pm 1}) = T^{\pm n} \), the \( \lambda \)-ring structure of Proposition 3.3 does also extend to \( K_0(\text{MHM}(M^n_\mu))[L^{-1/2}] \), and even to

\[
\hat{K}_0(\text{MHM}(M^n_\mu))[L^{-1/2}, (L^n - 1)^{-1} : n \in \mathbb{N}] =
\]

\[
= K_0(\text{MHM}(M^n_\mu)) \otimes_{\mathbb{Z}[L^{\pm 1}]} \mathbb{Z}[L^{-1/2}, (L^n - 1)^{-1} : n \in \mathbb{N}]
\]

by putting

\[
S^\lambda(T^{\pm 1/2}) = \begin{cases} T^{\pm n/2} & \text{for } \lambda = (n), \\ 0 & \text{otherwise} \end{cases}
\]

and extending continuously. Note that \( S^\lambda(L^{\pm 1/2}) = S^\lambda(-L^{\pm 1/2}) \) satisfies more complicated equations.

Again, we consider the filtration \( F^p[L^{-1/2}] \), resp. \( F^p[L^{-1/2}, (L^n - 1)^{-1} : n \in \mathbb{N}] \), defined accordingly and perform a completion as before. By abusing notation let us denote the resulting \( \lambda \)-ring by

\[
\hat{K}_0(\text{MHM}(M^n_\mu))[L^{-1/2}, (L^n - 1)^{-1} : n \in \mathbb{N}] :=
\]
\[
\prod_{d \in \Lambda_n} \left( K_0(\text{MHM}(\mathcal{M}_d^{ss})) \otimes_{\mathbb{Z}[L^1]} [L^{-1/2}, (L^n - 1)^{-1} : n \in \mathbb{N}] \right)
\]

which should not be confused with

\[
\prod_{d \in \Lambda_n} K_0(\text{MHM}(\mathcal{M}_d^{ss})) \otimes_{\mathbb{Z}[L^1]} \mathbb{Z}[L^{-1/2}, (L^n - 1)^{-1} : n \in \mathbb{N}].
\]

**Remark 3.4.** The reason for adjoining \(L^{\pm 1/2}\) to the Grothendieck group is given by our non-standard convention for the intersection complex involving powers of \(L^{1/2}\). The various completions are needed in the next section when we pass to stacks and define Donaldson–Thomas invariants.

The following result illustrates the nice behavior of intersection complexes with respect to Schur functors.

**Proposition 3.5.** Given a dimension vector \(d\) and a natural number \(n\), let us denote by \(\Delta\) resp. \(\tilde{\Delta}\) the big diagonal in \(\text{Sym}^n \mathcal{M}_d \subset \mathcal{M}_d^{ss}\) resp. \((\mathcal{M}_d^{ss})^n\). For an irreducible representation \(W_\lambda\) of \(S_n\) denote by \(\overline{W}_\lambda\) the variation of Hodge structure of type \((0,0)\) on \(\text{Sym}^n \mathcal{M}_d \setminus \Delta\) given by \(((\mathcal{M}_d^{ss})^n \setminus \tilde{\Delta}) \times_{S_n} W_\lambda\). Then

\[
S^n(\overline{\text{IC}_{\mathcal{M}_d^{ss}}(\mathbb{Q})}) = \text{IC}_{\text{Sym}^n \mathcal{M}_d}^{\overline{W}_\lambda}.
\]

**Proof.** We may assume \(\mathcal{M}_d^{ss} \neq \emptyset\) so that \(\overline{\mathcal{M}}_d^{ss} = \mathcal{M}_d^{ss}\), and we may replace \(\mathcal{M}_d^{ss}\) with \(\text{Sym}^n \mathcal{M}_d^{ss}\). Since \(\oplus : (\mathcal{M}_d^{ss})^n \to \text{Sym}^n \mathcal{M}_d^{ss}\) is finite, the Decomposition Theorem of Beilinson, Bernstein, Deligne and Gabber tells us that \(\text{IC}_{\mathcal{M}_d^{ss}}(\mathbb{Q})^{\oplus n} = \oplus_\ast (\text{IC}_{\mathcal{M}_d^{ss}}(\mathbb{Q}))^{\oplus n} = \text{IC}_{\text{Sym}^n \mathcal{M}_d}^{\overline{W}_\lambda}(V)\) for a suitable variation of Hodge structures \(V\) on the smooth locus \(\text{Sym}^n \mathcal{M}_d \setminus \Delta\). As the restriction of \(\oplus\) to \((\mathcal{M}_d^{ss})^n \setminus \tilde{\Delta}\) is a left \(S_n\)-principal bundle over \(\text{Sym}^n \mathcal{M}_d^{ss} \setminus \Delta\), we can trivialize it étale locally as \(U \times S_n\), showing that the fiber of \(V\) is just \(H^0(S_n, \mathbb{Q})\). The natural \(S_n\)-action on \(\text{IC}_{\mathcal{M}_d^{ss}}(\mathbb{Q})^{\oplus n}\) is induced by the left multiplication with \(S_n\) on \(U \times S_n\), while the right multiplication induces the transitions between different trivializations. The \(S_n\)-bimodule \(H^0(S_n, \mathbb{Q})\) decomposes as \(\oplus_{\lambda \in \Lambda_n} W_\lambda \otimes W_\lambda\) with the left factor, resp. the right factor, corresponding to the left action, resp. to the right action. Hence, \(V = \oplus_{\lambda \in \Lambda_n} W_\lambda \otimes W_\lambda\), completing the proof.

As mentioned at the beginning of this section, we can also replace \(\mathcal{M}_d^{ss}\) by \(\Lambda_\mu\) considered as a zero-dimensional monoid in the category of varieties locally of finite type. All of the constructions above go through with \(\mathcal{M}_0^{ss}\) replaced with \(\{0\} \subset \Lambda_\mu\). It is not difficult to see that

\[
K_0(\text{MHM}(\Lambda_\mu))[[L^{-1/2}, (L^n - 1)^{-1} : n \in \mathbb{N}]] = \prod_{d \in \Lambda_\mu} \left( K_0(\text{MHM}(\mathcal{C}))[[L^{-1/2}, (L^n - 1)^{-1} : n \in \mathbb{N}]] \right).
\]

Since \(\text{dim} : \mathcal{M}_d^{ss} \to \Lambda_\mu\) is a homomorphism of monoids locally of finite type, \(\text{dim}_\ast\) and \(\text{dim} \circ \text{dim}_\ast\) define triangulated tensor functors \(D^b(\text{MHM}(\mathcal{M}_d^{ss})) \to D^b(\text{MHM}(\Lambda_\mu))\) commuting with Schur functors of the same type. In particular, we get \(\lambda\)-ring homomorphisms \(\text{dim}_\ast\) and \(\text{dim} \circ \text{dim}_\ast\) from

\[
K_0(\text{MHM}(\mathcal{M}_d^{ss}))[[L^{-1/2}, (L^n - 1)^{-1} : n \in \mathbb{N}]]
\]

to

\[
K_0(\text{MHM}(\Lambda_\mu))[[L^{-1/2}, (L^n - 1)^{-1} : n \in \mathbb{N}]]
\]

commuting with the Schur operators, and similarly for perverse sheaves.
4. DT invariants and intersection complexes

4.1. Donaldson–Thomas invariants. We will now introduce a generalization of Donaldson–Thomas invariants using the notation of the previous sections. Let us fix a slope $\mu \in (-\infty, +\infty]$ and consider the morphism $p : \mathcal{M}_\mu^{ss} \to \mathcal{M}_d^{ss}$. Our first object is $\tilde{K}_0(\text{MMH}(\mathcal{M}_d^{ss}))|L^{-1/2}, (L^n - 1)^{-1} : n \in \mathbb{N}|$. To define it properly, we should develop a theory of mixed Hodge modules on Artin stacks along with a four functor formalism. However, in our situation of smooth quotient stacks we will use a more direct approach avoiding complicated machinery. First of all, $\mathcal{M}_\mu^{ss}$ is smooth, motivating $\mathcal{IC}_{\mathcal{M}_\mu^{ss}}(Q) = \mathbb{L}^{-\dim \mathcal{M}_\mu^{ss}}(Q) = \mathbb{L}^{(d,d)/2}(Q)$. Recall that $q : \mathcal{M}_d^{ss} \to \mathcal{M}_d^{ss}$ is a $G_d$-principal bundle for every dimension vector $d$. By means of the projection formula we would expect a formula like

$$H^*_c(G_d, \mathbb{Q}) \mathcal{IC}_{\mathcal{M}_d^{ss}}(\mathbb{Q}) = q_! q^* \mathcal{IC}_{\mathcal{M}_d^{ss}}(\mathbb{Q}) = \mathbb{L}^{\dim G_d/2} q_! \mathcal{IC}_{\mathcal{M}_d^{ss}}(\mathbb{Q}) = \mathbb{L}^{(d,d)/2} q_! \mathbb{Q}$$

in $\hat{K}_0(\text{MMH}(\mathcal{M}_d^{ss}))|L^{-1/2}, (L^n - 1)^{-1} : n \in \mathbb{N}|$. Hence, we will define $p_! \mathcal{IC}_{\mathcal{M}_\mu^{ss}}(\mathbb{Q})$ as the product in $\hat{K}_0(\text{MMH}(\mathcal{M}_d^{ss}))|L^{-1/2}, (L^n - 1)^{-1} : n \in \mathbb{N}|$ of $\mathbb{L}^{(d,d)/2} p_! q_! \mathbb{Q}$ with the inverse of the class $\prod_{i \in \mathbb{Q}_0} L^{(\mathbb{Q})} \prod_{n=1}^{d}(L^n - 1) \in \mathbb{Z}[\mathbb{L}]$ of $H^*_c(G_d, \mathbb{Q})$. Summing over $d \in \Lambda_\mu$ gives $p_! \mathcal{IC}_{\mathcal{M}_\mu^{ss}}(\mathbb{Q})$. The following lemma is a standard fact in the theory of (filtered) $\lambda$-rings.

**Lemma 4.1.** There is an element $\mathcal{DT}_\mu \in \hat{K}_0(\text{MMH}(\mathcal{M}_d^{ss}))|L^{-1/2}, (L^n - 1)^{-1} : n \in \mathbb{N}|$ with $\mathcal{DT}_\mu|_{\mathcal{M}_d^{ss}} = 0$ such that

$$p_! \mathcal{IC}_{\mathcal{M}_\mu^{ss}}(\mathbb{Q}) = \text{Sym} \left( \frac{1}{\mathbb{L}^{1/2} - \mathbb{L}^{-1/2}} \mathcal{DT}_\mu \right).$$

**Definition 4.2.** We call $\mathcal{DT}_d := \mathcal{DT}_\mu|_{\mathcal{M}_d^{ss}} \in \hat{K}_0(\text{MMH}(\mathcal{M}_d^{ss}))|L^{-1/2}, (L^n - 1)^{-1} : n \in \mathbb{N}|$ the Donaldson–Thomas “sheaf” and $\mathcal{DT}_d := \dim \mathcal{DT}_d = H^*_c(\mathcal{M}_d^{ss}, \mathcal{DT}_d) \in \hat{K}_0(\text{MMH}(\mathbb{C}))|L^{-1/2}, (L^n - 1)^{-1} : n \in \mathbb{N}|$ the Donaldson–Thomas invariant of dimension vector $d$ with respect to the given stability condition $\theta$.

As $\dim$ is a $\lambda$-ring homomorphism, our definition of Donaldson–Thomas invariants agrees with the usual one \cite{25}. The proof of the following result is literally the same as the one for Donaldson–Thomas invariants given at various places using Ringel–Hall algebra techniques. The key observation is that $p : \mathcal{M}_\mu^{ss} \to \mathcal{M}_d^{ss}$ maps all non-trivial extensions of two representations to the trivial one.

**Proposition 4.3.** Given a slope $\mu \in (-\infty, +\infty]$ and a framing vector $f \in \mathbb{N}^Q_0$ such that $2|f_i$ for all $i \in Q_0$, we obtain the following formula

$$\pi_* \mathcal{IC}_{\mathcal{M}_\mu^{ss}, f}(\mathbb{Q}) = \pi_* \mathcal{IC}_{\mathcal{M}_\mu^{ss}, f}(\mathbb{Q}) = \text{Sym} \left( \sum_{0 \neq d \in \Lambda_\mu} [p^f d^{-1}]_{\text{vir}, \mathcal{DT}_d} \right)$$

in $\hat{K}_0(\text{MMH}(\mathcal{M}_\mu^{ss}))|L^{-1/2}, (L^n - 1)^{-1} : n \in \mathbb{N}|$, using the shorthand $[p^f d^{-1}]_{\text{vir}} := \frac{1}{L^{d/2} - L^{-d/2}}$. Here $\pi : \mathcal{M}_\mu^{ss, f} \to \mathcal{M}_\mu^{ss}$ is the morphism forgetting the framing.

The parity assumption on the framing vector is made to avoid typical “sign problems”.

4.2. The main result. Recall that our King stability condition $\theta \in \mathbb{Z}^Q_0$ was called $\mu$-generic if $\langle d, e \rangle = 0$ for all $d, e \in \Lambda_\mu$, and generic if that holds for all $\mu \in (-\infty, +\infty]$. We also need to assume the following result proven in section 6.

\footnote{Note that $p_!$ is the derived direct image with compact support, while $p_*$ is the usual derived direct image.}
Theorem 4.4. If \( \theta \) is \( \mu \)-generic, the Donaldson–Thomas sheaf \( \mathcal{D}T_\mu \) is in the image of the natural map

\[ K_0(\text{MHM}(\mathcal{M}^a ))[L^{-1/2}] \longrightarrow K_0(\text{MHM}(\mathcal{M}^a ))[L^{-1/2}, (L^n - 1)^{-1} : n \in \mathbb{N}] . \]

Remark 4.5. Note that \( K_0(\text{MHM}(\mathcal{M}^a )) \) is free over \( \mathbb{Z}[L^{\pm 1}] \). Indeed, the set of all intersection complexes \( \mathcal{IC} \) holds in \( K \) intersection complexes, and denote by \( \text{deg}(\mathcal{IC}) \) the maximal degree of its summands. If \( L \) running through all irreducible closed subvarieties of \( \mathcal{M}^a \) and \( \lambda \) running through all irreducible variations of pure Hodge structures with weight filtration by two, we require that \( \text{deg}(\mathcal{IC}) \) holds in \( K \).

Theorem 4.6. Assume that \( \hat{K}_0(\text{MHM}(\mathcal{M}^a ))[L^{-1/2}] \) holds in \( K_0(\text{MHM}(\mathcal{M}^a ))[L^{-1/2}] \) for every dimension vectors \( d \in \mathbb{N}Q_0 \).

Proof. To prove the theorem we make use of the weight filtration of a mixed Hodge module. Given a mixed Hodge module \( \mathcal{E} \) on a variety \( X \) with irreducible support \( Z \), the stalk (shifted by \( - \dim Z \)) at the generic point \( \eta \in Z \) is a mixed Hodge structure with weight filtration

\[ 0 = W_{m-1} \subseteq W_m \subset \ldots \subset W_{n-1} \subseteq W_n = \mathcal{E}_\eta. \]

Define \( \deg \mathcal{E} := \max(|n_1|, |n_2|) + \dim Z \). As the action of \( T \) shifts the weight filtration by two, we require that \( T^{1/2} \) increases the indices by one. In particular, \( \deg \mathcal{IC}_Z(\mathbb{Q}) = 0 \) by our convention. More generally, if \( \mathcal{E} \) is a class in \( K_0(\text{MHM}(\mathcal{X}))[L^{-1/2}] \), we write it uniquely as a linear combination of irreducible intersection complexes, and denote by \( \deg(E) \) the maximal degree of its summands. If \( P \) is a Laurent polynomial in \( L^{1/2} \) invariant under \( L^{-1/2} \leftrightarrow L^{-1/2} \), we get \( \deg(PE) = \deg(P) \). In particular, if \( \lambda \) is a partition of \( N \) and \( r \in \mathbb{N} \), then \( \deg(S^\lambda[\mathbb{P}^r], \mathcal{E}) \leq \deg(\mathcal{E}) + Nr \), with equality only for \( \lambda = (N) \).

As before, \( \mathcal{P} \) denotes the set of all partitions of arbitrary size. We fix a framing vector \( f \in \mathbb{N}Q_0 \) such that \( 2|f| \) for all \( i \in Q_0 \) and rewrite equation (3) using (1) and (2):

\[ \pi_* \mathcal{IC}_{\mathcal{M}^a } = \sum_{\lambda, \mu \rightarrow\mathcal{P}} \prod_{e \in \mu} S^{\lambda_e} [f^e - 1] \cdot S^{\lambda} \cdot \mathcal{D}T_e. \]

By induction over \( |d| = \sum_{i \in Q_0} d_i \), we conclude using equation (3) that

\[ \pi_* \mathcal{IC}_{\mathcal{M}^a, d} = [f^d - 1] \cdot \mathcal{D}T_d + \sum_{\lambda, \mu \rightarrow\mathcal{P}} \prod_{e \in \mu} S^{\lambda_e} [f^e - 1] \cdot \mathcal{IC}_{\text{Sym}^d \mathcal{M}^a}(\mathbb{W}_\lambda) \]

\[ = [f^d - 1] \cdot \mathcal{D}T_d + \sum_{\lambda, \mu \rightarrow\mathcal{P}} \prod_{e \in \mu} S^{\lambda_e} [f^e - 1] \cdot \mathcal{IC}_{\text{Sym}^d \mathcal{M}^a}(\mathbb{W}_\lambda), \]
where we used the shorthands \( \text{Sym}^\lambda \mathcal{M}^{st}_{\mu} := \prod_e \text{Sym}^{\lambda_e} | \mathcal{M}^{st}_e \subset \mathcal{M}^{ss}_d \) and \( \mathbb{W}_\lambda := \mathbb{V}_{\lambda} \). Moreover, \( \delta_d : \Lambda_\mu \to \mathcal{P} \) maps \( d \) to (1) and any other dimension vector to the zero partition.

Note that if \( DT_d \neq 0 \), the first summand in (5) has degree\(^2\) at least \( f \cdot d - 1 \). In contrast to this, the summands for \( \lambda \neq \delta_d \) have degree at most \( f \cdot d - \sum_e |\lambda_e| \) which is less than \( f \cdot d - 1 \) by assumption on \( \lambda \).

On the other hand, we can use the Decomposition Theorem of Beilinson, Bernstein, Deligne and Gabber and the fact that \( \pi \) is virtually small (see Theorem 2.1) to obtain

\[
\pi_* \mathcal{IC}_{\mathcal{M}^{ss}_{\mu,f}} = [P^{f-d-1}]_{vir} \mathcal{IC}_{\mathcal{M}^{st}_d}(\mathbb{Q}) + \sum_{\text{lower strata}} \sum_{\text{irred. VHS}} m_{S,L} \mathcal{IC}_S(L)
\]

for certain \( m_{S,L} \in \mathbb{Z} \). Moreover, \( \deg(\mathcal{IC}_S(L)) < f \cdot d - 1 \), and if \( \mathcal{M}^{st}_d \neq \emptyset \), the first summand has degree at least \( f \cdot d - 1 \).

Using the integrality condition 4.4, we make the Ansatz

\[
DT_d = \sum_{\text{strata}} \sum_{\text{irred. VHS}} u_{S,L} \mathcal{IC}_S(L)
\]

for certain unique \( u_{S,L} \in \mathbb{Z} \) and put this into equation (6). By comparing the degrees of the resulting expression with equation (6), we finally conclude

\[
u_{S,L} = \begin{cases} 1 & \text{if } S = \mathcal{M}^{st}_d \neq \emptyset, L = \mathbb{Q}, \\ 0 & \text{otherwise} \end{cases}
\]

denoting the number of \( \mathbb{Z} \)-valued affine spaces.

**Remark 4.7.** We could drop reference to Theorem 4.4 in the above proof if we were able to prove Theorem 2.1 along with the following stronger version of equation (6)

\[
\pi_* \mathcal{IC}_{\mathcal{M}^{ss}_{\mu,f}} = [P^{f-d-1}]_{vir} \mathcal{IC}_{\mathcal{M}^{st}_d}(\mathbb{Q}) + \sum_{\lambda : \Lambda_\mu \to \mathcal{P}} \left( \prod_{e \in \Lambda_\mu} S^{\lambda_e}[P^{f-e-1}]_{vir} \right) \mathcal{IC}_{\text{Sym}^\lambda \mathcal{M}^{st}_d}(\mathbb{W}_\lambda).
\]

Theorem 4.4 would be a consequence of such a result. But we can only prove this equation up to contributions from even lower strata than \( \text{Sym}^\lambda \mathcal{M}^{st}_d \).

### 4.3. Application to matrix invariants

Since the motivic DT invariants of \( m \)-loop quivers are computed explicitly in [30], our main result allows us to give an explicit formula for the Poincaré polynomial in (compact support) intersection cohomology of the corresponding moduli spaces, which are the classical spaces of matrix invariants.

So let \( Q^{(m)} \) be the quiver with a single vertex and \( m \geq 2 \) loops (in the case of no loop, or of one loop, the non-empty moduli spaces reduce to affine spaces). We consider the trivial stability and a positive integer \( d \), and fix an \( d \)-dimensional \( \mathbb{C} \)-vector space \( V \). Then the moduli space \( \mathcal{M}^{ss}_d(Q^{(m)}) \) equals the invariant theoretic quotient \( \mathcal{M}^{(m)}_d := \text{End}_\mathbb{C}(V)^m \sslash \text{GL}(V) \) of \( m \)-tuples of linear operators up to simultaneous conjugation. This is an irreducible normal affine variety of dimension \( (m-1)d^2 + 1 \), singular except in case \( d = 1 \) or \( m = d = 2 \).

To formulate the explicit formula for the compactly supported intersection Betti numbers of \( \mathcal{M}^{(m)}_d \), we need some combinatorial notions from [30]. Let \( U_d \) be the

\[\text{deg}(DT_d)\text{ is well-defined by the integrality condition 5.3}\]
set of sequences \((a_1, \ldots, a_d)\) of natural numbers summing up to \((m-1)d\), on which the cyclic group \(G_d\) of order \(d\) acts by cyclic rotation. We call a sequence \(a\), primitive if it is different from all its cyclic rotations, and almost primitive if it is either primitive, or \(m\) is even, \(d \equiv 2 \mod 4\), and the sequence equals twice a primitive sequence of length \(d/2\). We define the degree of the sequence as \(\sum_{i=1}^{d} (d-i) a_i\), and we define the degree of a cyclic class of sequences as the minimal degree of sequences in this class. Let \(\mathcal{U}_d^m/G_d\) we the set of cyclic classes of almost primitive sequences. Combining our main result with the formula for DT invariants in \([30]\), we arrive at:

**Theorem 4.8.** For all \(d \geq 1\) and \(m \geq 2\), we have

\[
\sum_p \dim \text{IC}^p(\mathcal{M}_d^{(m)}, \mathbb{Q})v^p = v^{(m-1)d^2+1} \frac{1 - v^{-2}}{1 - v^{-2d}} \sum_{C \in \mathcal{U}_d^m/G_d} v^{-2\deg C}.
\]

5. **Proof of Theorem 2.1**

5.1. **The stack of nilpotent quiver representations.** As before, let \(Q\) be a finite quiver and \(d \in \mathbb{N}\) a dimension vector for \(d\), and consider the action of the linear algebraic group \(G_d\) on the vector space \(R_d\). Let \(p : R_d \to R_d//G_d\) be the invariant-theoretic quotient; in other words, \(R_d//G_d\) is the spectrum of the ring of \(G_d\)-invariants in \(R_d\), which, by \([4]\), is generated by traces along oriented cycles in \(Q\). We consider the nullcone of the representation of \(G_d\) on \(R_d\), that is,

\[
N_d := p^{-1}(p(0)).
\]

By a standard application of the Hilbert criterion (see \([3]\) Chapter 6) for a much finer analysis of the geometry of \(N_d\) using the Hesselink stratification) we can characterize points in \(N_d\) either as those representations such that every cycle is represented by a nilpotent operator, or as those representation admitting a composition series by the one-dimensional irreducible representations \(S_i\) concentrated at a single vertex \(i \in Q_0\) (and with all loops at \(i\) represented by 0).

The main observation of this section is that, under the assumption of \(Q\) being symmetric, there is an effective estimate for the dimension of \(N_d\).

**Theorem 5.1.** If \(Q\) is symmetric, we have

\[
\dim N_d - \dim G_d \leq -\frac{1}{2}(d, d) + \frac{1}{2} \sum_{i \in Q_0} (i, i) d_i - \dim d.
\]

**Proof.** For a decomposition \(d = d^1 + \ldots + d^s\), denoted by \(d^*\), we consider the closed subvariety \(R_{d^*}\) of \(R_d\) consisting of representations \(V\) admitting a filtration \(0 = V_0 \subset V_1 \subset \ldots \subset V_s = V\) by subrepresentations, such that \(V_k/V_{k-1}\) equals the zero representation of dimension vector \(d^k\) for all \(k = 1, \ldots, s\). This subvariety being the collapsing of a homogeneous bundle over a variety of partial flags in \(\bigoplus_{i \in Q_0} \mathbb{K}^{d_i}\), its dimension is easily estimated as

\[
\dim R_{d^*} \leq \dim G_d - \sum_{k<l} (d^l, d^k) - \sum_{i \in Q_0} \sum_k (d^k_i)^2.
\]

The above characterization of \(N_d\) allows us to write \(N_d\) as the union of all \(R_{d^*}\) for decompositions \(d^*\) which are thin, that is, all of whose parts are one-dimensional (one-dimensionality is obscured by the notation to avoid multiple indexing and to make the argument more transparent). Thus \(\dim N_d - \dim G_d\) is bounded above by the maximum of the values

\[
-\sum_{k<l} (d^l, d^k) - \sum_{i \in Q_0} \sum_k (d^k_i)^2
\]
over all thin decompositions. Since $Q$ is symmetric, we can rewrite
\[
\sum_{k<l} (d^i, d^k) = \frac{1}{2} (d, d) - \frac{1}{2} \sum_k (d^k, d^k).
\]
All $d^k$ being one-dimensional, we can easily rewrite
\[
\sum_{i \in Q_0} \sum_k (d^i)^2 = \dim d_i, \quad \sum_k (d^k, d^k) = \sum_{i \in Q_0} (i, i) d_i.
\]
All terms now being independent of the chosen thin decomposition, we arrive at the required estimate. \hfill \Box

5.2. Virtual smallness of the Hilbert–Chow map. We consider again the Hilbert–Chow map $\pi : M_{d, f}^{ss} \to M_{d}^{ss}$ forgetting the framing datum; our aim is to prove a strong dimension estimate for its fibers when the stability is $\mu$-generic (cf. section 2.2) for $\mu$ being the slope of $d$.

We consider the Luna stratification of $M_{d}^{ss}$: a decomposition type $\xi$ for $d$ consists of a sequence $((d^1, m_1), \ldots, (d^n, m_n))$ in $\Lambda_\mu \times \mathbb{N}$ such that $\sum_k m_k d^k = d$. Inside the moduli space $M_{d}^{ss}$ parametrizing isomorphism classes of polystable representations of dimension vector $d$, we can consider the subset $S_\xi$ of representations of the form $\bigoplus_k U^\mu_k$ for pairwise non-isomorphic stable representations $U^\mu_k$ of dimension vector $d^k$ and slope $\mu$. We thus have
\[
\dim S_\xi = \sum_k \dim M_{d^k}^{ss}(Q) = s - \sum_k (d^k, d^k).
\]
By [10], $S_\xi$ is locally closed, and the map $\pi$ is étale locally trivial over $S_\xi$. We fix a point $V \in S_\xi$. This stratum being nonempty, $M_{d^k}^{ss}(Q)$ is nonempty, and thus $(d^k, d^k) = 1 - \dim M_{d^k}^{ss}(Q) \leq 1$ for all $k$. The fiber $\pi^{-1}(V)$ over a point $V \in S_\xi$ can be described as follows:

Define the local quiver $Q_\xi$ with vertices $i_1, \ldots, i_s$ and $\delta_{kl} - (d^k, d^l)$ arrows from $i_k$ to $i_l$. Define a local dimension vector $d_\xi$ for $Q_\xi$ by $(d_\xi)_{i_k} = m_k$, and a local framing datum $f_\xi$ by $(f_\xi)_{i_k} = f \cdot d^k$. We consider the trivial stability on $Q_\xi$. Then we have a local Hilbert–Chow map
\[
\pi_\xi : M_{d_\xi, f_\xi}^{0-ss}(Q_\xi) \to M_{d_\xi}^{0-ss}(Q_\xi) = R_{d_\xi} \parallel G_{d_\xi}.
\]
We denote the fiber over the class of the zero representation by $M_{d_\xi, f_\xi}^{nilp}(Q_\xi)$. Then, by [10], we have
\[
\pi^{-1}(V) \simeq M_{d_\xi, f_\xi}^{nilp}(Q_\xi).
\]
By construction, we have
\[
\dim M_{d_\xi, f_\xi}^{nilp}(Q_\xi) = \dim N_{d_\xi} - \dim G_{d_\xi} + f_\xi \cdot d_\xi.
\]
Now assume $\theta$ to be $\mu$-generic, thus $Q_\xi$ is symmetric, and Theorem 5.1 estimates the dimension of the fiber $\pi^{-1}(V)$ as
\[
\dim \pi^{-1}(V) = \dim M_{d_\xi, f_\xi}^{nilp}(Q_\xi) = \dim N_{d_\xi} - \dim G_{d_\xi} + f_\xi \cdot d_\xi \leq
\]
\[
\leq -\frac{1}{2} (d_\xi, d_\xi)_{Q_\xi} + \frac{1}{2} \sum_k (i_k, i_k)_{Q_\xi} (d_\xi)_{i_k} + \dim d_\xi + f_\xi \cdot d_\xi.
\]
Using the definition of $Q_\xi$, $d_\xi$ and $f_\xi$, this simplifies to
\[
\dim \pi^{-1}(V) \leq -\frac{1}{2} (d, d) + \frac{1}{2} (d^k, d^k) m_k - \sum_k m_k + f \cdot d.
\]
On the other hand, we can rewrite the dimension formula for $S_\xi$ as
\[
\text{codim}S_\xi = -(d, d) + \sum_k (d^k, d^k) + 1 - s.
\]
The inequality
\[
\dim \pi^{-1}(V) - (f \cdot d - 1) \leq \frac{1}{2} \text{codim}S_\xi
\]
(with equality only if $\xi = ((d, 1)))$ claimed in Theorem 2.1 can thus be rewritten as
\[
-\frac{1}{2}(d, d) + \frac{1}{2} \sum_k (d^k, d^k)m_k + 1 \leq -\frac{1}{2}(d, d) + \frac{1}{2} \sum_k (d^k, d^k) + \frac{1}{2}(1 - s).
\]
This is easily simplified to
\[
\frac{1}{2} \sum_k ((d^k, d^k) - 2)(m_k - 1) \leq \frac{1}{2}(s - 1).
\]
Since $(d^k, d^k) \leq 1$, the left hand side is nonpositive, whereas the right hand side is nonnegative. Equality holds if both sides are zero, thus $s = 1$, proving virtual smallness.

6. Motivic DT-theory and the integrality conjecture

We prove a stronger version of Theorem 4.4 involving stack functions instead of mixed Hodge modules. The reader not familiar with stack functions might have a look at [16], but we will also recall the main definitions below. In fact, there is no difference between the two versions of Donaldson–Thomas invariants, as long as we work with representations of quivers without potential, since all computations will take place in the ring of Tate motives. However, we also aim at proving the integrality conjecture for all moduli problems related to abelian categories of homological dimension one. We will make extensive use of restriction to generic points. Since it is not so clear how to deal with mixed Hodge modules on varieties defined over arbitrary (function) fields, we will work in the motivic world. The machinery used to define Donaldson–Thomas sheaves will also work in this more general context, and one ends up with the motivic Donaldson–Thomas invariants respectively Donaldson–Thomas sheaves. In fact, we will not construct any motivic sheaf. Instead, we define an element in some ring $\hat{K}(\text{Var}/\mathcal{M}_{\mu}^{ss})[\mathbb{L}^{-1/2}, (\mathbb{L}^n - 1)^{-1} : n \in \mathbb{N}]$, which is a simplified version of the Grothendieck group of the category of motivic sheaves. There is a $\lambda$-ring homomorphism from
\[
\hat{K}(\text{Var}/\mathcal{M}_{\mu}^{ss})[\mathbb{L}^{-1/2}, (\mathbb{L}^n - 1)^{-1} : n \in \mathbb{N}]
\]
to
\[
\hat{K}_0(\text{MHM}(\mathcal{M}_{\mu}^{ss})[\mathbb{L}^{-1/2}, (\mathbb{L}^n - 1)^{-1} : n \in \mathbb{N}],
\]
induced by $[X \xrightarrow{\phi} \mathcal{M}] \mapsto \phi^*\mathcal{Q}$, giving rise to corresponding results for mixed Hodge modules.

6.1. Stack functions. Given an arbitrary Artin stack or scheme $\mathcal{B}$ over $k$, we define the Grothendieck group $K(\text{Var}/\mathcal{B})$ to be the free abelian group generated by isomorphism classes $[X \to \mathcal{B}]$ of representable morphisms, locally of finite type, subject to the cut and paste relation
\[
[X \to \mathcal{B}] = [Z \to \mathcal{B}] + [X \setminus Z \to \mathcal{B}]
\]
where $Z \subset X$ is a closed substack. The fiber product defines a ring structure on $K(\text{Var}/\mathcal{B})$ and a $K(\text{Var}/\mathcal{B})$-module structure on $K(\text{Var}/\mathcal{B})$. Let us also introduce the module
\[
K(\text{Var}/\mathcal{B})[\mathbb{L}^{-1/2}, (\mathbb{L}^n - 1)^{-1} : n \in \mathbb{N}] := K(\text{Var}/\mathcal{B})\otimes_{\mathbb{Z}[\mathbb{L}]}[\mathbb{L}^{-1/2}, (\mathbb{L}^n - 1)^{-1} : n \in \mathbb{N}]
\]
with $L$ denoting the Lefschetz motive $L := [A^1_k] \in K(\text{Var}/k)$.

We will also assume that for a generator $[X \rightarrow B]$, the total space has a stratification by quotient stacks $X_i = X_i/\text{GL}_{n_i}$. That brings us to the second class of relations

$$[X/\text{GL}_{n_i} \rightarrow B] = [X \rightarrow B]/[\text{GL}_{n_i}]$$

for every $\text{GL}_{n}$-action on a scheme $X$. Here, $[\text{GL}_{n}] = L_{(2)} \prod_{p=1}^{n} (L^p - 1)$. In particular, $K(\text{Var}/B)[L^{-1/2}, (L^n - 1)^{-1} : n \in \mathbb{N}]$ is generated as a $\mathbb{Z}[L^{-1/2}, (L^n - 1)^{-1} : n \in \mathbb{N}]$-module by morphisms $[X \rightarrow B]$, with $X$ being a scheme. Because of this, the proper push forward $\phi$ along morphisms $\phi : B \rightarrow B'$ locally of finite type is well defined by composition $\phi((X \rightarrow B)) = [X \rightarrow B']$. The restriction of stack functions, that is, of elements in $K(\text{Var}/B)[L^{-1/2}, (L^n - 1)^{-1} : n \in \mathbb{N}]$, to substacks $U \hookrightarrow B$ is given by the fiber product $[U \times_B X \rightarrow U]$ for generators $[X \rightarrow B]$. We will identify two stack functions on $B$ if their restrictions to any substack of finite type agree, and denote the resulting group by $\hat{K}(\text{Var}/M)[L^{-1/2}, (L^n - 1)^{-1} : n \in \mathbb{N}]$.

### 6.2. $\lambda$-ring structures.

If the base stack $B$ has an additional structure of a commutative monoid with zero $\text{Spec } k \rightarrow B$ and sum $\oplus : B \times_k B \rightarrow B$, then $K(\text{Var}/B)$ can be equipped with the structure of a $\lambda$-ring by putting

$$[X \rightarrow B] : [Y \rightarrow B] := [X \times_k Y \rightarrow B \times_k B \rightarrow B]$$

and

$$\sigma^n([X \rightarrow B]) := [X^n/\Sigma_n \rightarrow B^n/\Sigma_n \oplus B],$$

where $S_n$ denotes the symmetric group. Due to a result of Kresch, the quotient by $S_n$ can be taken either in the stacky or in the naive sense if $X$, respectively, $B$, is a scheme. On can extend the $\lambda$-ring structure to $K(\text{Var}/B)[L^{-1/2}, (L^n - 1)^{-1} : n \in \mathbb{N}]$ by defining $-L^{1/2}$ to be a line element, that is, $\sigma^n(-L^{1/2}) := (-L^{1/2})^n$. Moreover, the $\lambda$-ring structure extends to $\hat{K}(\text{Var}/B)[L^{-1/2}, (L^n - 1)^{-1} : n \in \mathbb{N}]$.

Given a stack function $f \in \hat{K}(\text{Var}/B)[L^{-1/2}, (L^n - 1)^{-1} : n \in \mathbb{N}]$ such that $\sigma^n(f)|_{U}$ vanishes for all but finitely many $n \in \mathbb{N}$ and all substacks $U \subset B$ of finite type, the sum

$$\text{Sym}(f) := \sum_{n \geq 0} \sigma^n(f)$$

is well defined in $\hat{K}(\text{Var}/B)[L^{-1/2}, (L^n - 1)^{-1} : n \in \mathbb{N}]$ and satisfies $\text{Sym}(0) = 1 = [\text{Spec } k \rightarrow B]$ as well as $\text{Sym}(f + g) = \text{Sym}(f) \cdot \text{Sym}(g)$.

### 6.3. The setting.

In this subsection we fix terminology and provide the setting which will be used throughout this section. First of all we choose a ground field $k$ of characteristic zero and let $\mathcal{A}$ be a $k$-linear abelian category of homological dimension one. We also assume that the $k$-dimension of all Hom- and Ext- groups is finite, which allows us to define the Euler pairing $\text{dim}_k \text{Hom}(V, W) - \text{dim}_k \text{Ext}^1(V, W)$ of two objects $V, W$ in $\mathcal{A}$. It descends to the Grothendieck group $K_0(\mathcal{A})$ of $\mathcal{A}$. A class of examples is given by representations of quivers or of smooth algebras. Compactely supported sheaves on algebraic curves provide another useful class of examples. Moreover, we assume the existence of a lattice $\Lambda$, equipped with a pairing $(-, -)$ and a group homomorphism $\text{dim} : K_0(\mathcal{A}) \rightarrow \Lambda$, such that $\text{dim}^*(-, -)$ is the Euler pairing. In the case of quiver representations, this has been given in section 2.1. For compactely supported sheaves we have $\Lambda = \mathbb{Z}^2$, and $\text{dim}$ maps (a class of) a sheaf to its rank and degree. Finally, we require the existence of an Artin moduli stack, locally of finite type over $k$. One can show that such a stack admits a locally finite stratification by quotient stacks. Due to our assumption on

---

3For $B = \text{Spec } k$, we simplify the notation by suppressing the structure morphism to $\text{Spec } k$.

4An Artin stack with an atlas of finite type will be called of finite type.
the homological dimension, \( \mathfrak{M} \) is a smooth Artin stack.

A good (Bridgeland) stability condition on \( \mathcal{A} \) gives rise to an open substack \( \mathfrak{M}^{ss}_{d} \) such that \( \mathfrak{M}^{ss}_{d} \) is of finite type for every \( d \in \Lambda \), where \( \mathfrak{M}^{ss}_{d} \) is defined in the obvious way. A good stability condition is called integral if a moduli space \( \mathfrak{M}^{ss} \) along with a morphism \( p : \mathfrak{M}^{ss} \to \mathfrak{M}^{ss}_{d} \) exists. Note that the moduli space \( \mathfrak{M}^{ss} \) will be singular in general, despite the smoothness of \( \mathfrak{M} \). The points of \( \mathfrak{M}^{ss} \) should parametrize polystable objects, and the morphism maps a semistable object to the direct sum of its stable Jordan–Hölder factors. The objects \( \mathfrak{M}^{ss}_{\mu}, \mathfrak{M}^{ss} \) and \( \Lambda_{\mu} \) are defined as in section 2.2 for any slope \( \mu \in (-\infty, +\infty] \). We think of \( \Lambda_{\mu} \) as a variety defined over \( k \) and denote the obvious morphism \( \mathfrak{M}^{ss}_{\mu} \to \Lambda_{\mu} \) by dim abusing notation.

The stability condition is called \( \mu \)-generic if for all \( d, e \in \Lambda_{\mu} \), the equality \( (d, e) = (e, d) \) holds, and generic if it is \( \mu \)-generic for all \( \mu \in (-\infty, +\infty] \). Every stability condition on a symmetric quiver is generic. Moreover, there are generic stability conditions for bipartite quivers. Mumford stability for sheaves on proper algebraic curves is also generic.

Formation of direct sums of polystable representations, respectively of sums of dimension vectors, provides \( \mathcal{M}^{ss}_{\mu} \), respectively \( \Lambda_{\mu} \), with the structure of a commutative monoid, inducing a \( \lambda \)-ring structure on \( \hat{K}(\text{Var}/\mathcal{M}^{ss}_{\mu})[L^{-1/2}, (L^n - 1)^{-1} : n \in \mathbb{N}] \), respectively on \( \hat{K}(\text{Var}/\Lambda_{\mu})[L^{-1/2}, (L^n - 1)^{-1} : n \in \mathbb{N}] \). If a stack function \( f \) on \( \mathcal{M}^{ss}_{\mu} \), respectively on \( \Lambda_{\mu} \), is supported away from the zero element, the infinite sum \( \text{Sym}(f) \) is well defined.

6.4. Donaldson–Thomas invariants. The following definition of Donaldson–Thomas invariants applies only to our situation of abelian categories \( \mathcal{A} \) of homological dimension one. There is a more general and much more complicated version which can be applied to triangulated 3-Calabi–Yau \( \mathcal{A}_{\infty} \)-categories. If \( \mathcal{A} \) is the category of quiver representations, we can embed \( \mathcal{A} \) into the 3-Calabi–Yau \( \mathcal{A}_{\infty} \)-category \( D^{b}(\Gamma_{\infty} \mathcal{Q} - \text{Rep}) \) introduced in section 2.1, and the general version simplifies to the one given here.

The motivic version of the intersection complex \( IC^{mot}_{\mathfrak{M}^{ss}_{\mu}} \) is defined by the stack function
\[
IC^{mot}_{\mathfrak{M}^{ss}_{\mu}} := \sum_{d \in \Lambda_{\mu}} \mathbb{L}^{(d, d)/2}[\mathfrak{M}^{ss}_{d} \hookrightarrow \mathfrak{M}^{ss}_{\mu}].
\]
Here \( \mathbb{L}^{(d, d)/2} \) is the analog of the “perverse shift” since \( \text{dim} \mathfrak{M}^{ss}_{d} = -(d, d) \). Taking the proper push forward along the morphisms \( p : \mathfrak{M}^{ss}_{d} \to \mathcal{M}^{ss}_{\mu} \) and \( \text{dim} : \mathcal{M}^{ss}_{\mu} \to \Lambda_{\mu} \), respectively, we can define the motivic Donaldson–Thomas sheaf \( DT^{mot}_{\mu} \in \hat{K}(\text{Var}/\mathcal{M}^{ss}_{\mu})[L^{-1/2}, (L^n - 1)^{-1} : n \in \mathbb{N}] \) and the motivic Donaldson–Thomas invariants \( DT^{mot}_{\mu} \in \hat{K}(\text{Var}/\Lambda_{\mu})[L^{-1/2}, (L^n - 1)^{-1} : n \in \mathbb{N}] \), both vanishing at the zero element, by means of
\[
pt IC^{mot}_{\mathfrak{M}^{ss}_{\mu}} = \text{Sym} \left( \frac{1}{\mathbb{L}^{1/2} - \mathbb{L}^{-1/2}} DT^{mot}_{\mu} \right),
\]
\[
dim pt IC^{mot}_{\mathfrak{M}^{ss}_{\mu}} = \text{Sym} \left( \frac{1}{\mathbb{L}^{1/2} - \mathbb{L}^{-1/2}} \text{dim} DT^{mot}_{\mu} \right).
\]
For the last equation we use the fact that the monoid homomorphism \( \dim \) induces a \( \lambda \)-ring homomorphism \( \text{dim} : \hat{K}(\text{Var}/\mathcal{M}^{ss}_{\mu})[L^{-1/2}, (L^n - 1)^{-1} : n \in \mathbb{N}] \to \text{Sym} \left( \frac{1}{\mathbb{L}^{1/2} - \mathbb{L}^{-1/2}} \right) \).
The integrality conjecture. The following rather technical conjecture plays a fundamental role in Donaldson–Thomas theory. A proof has been sketched in [28]. A relative version, saying that whenever the conjecture holds for one stability condition, it also holds for any other, has been given in [18] (see also [29]). Our proof is different from the very complicated one given by Kontsevich and Soibelman. In fact, we reduce the general situation of abelian categories of homological one to a special situation for which the integrality conjecture has been proven by Efimov [9].

In simple terms the integrality conjecture says that the Donaldson–Thomas invariants look like (a linear combination of) motives of varieties rather than Artin stacks. Actually, we prove a sheaf theoretic version of this, from which the original integrality conjecture can be deduced by applying the proper push forward dimens
tions.

Theorem 6.1 (integrality conjecture, sheaf version). For a \( \mu \)-generic stability condition \( \theta \) the Donaldson–Thomas sheaf \( DT^\mu_{\text{mot}} \) is in the image of the natural map

\[
\hat{K}(\text{Var}/\Lambda_m)[L^{-1/2}, (L^n - 1)^{-1} : n \in \mathbb{N}] \longrightarrow \hat{K}(\text{Var}/\Lambda_m^s)[L^{-1/2}, (L^n - 1)^{-1} : n \in \mathbb{N}].
\]

Corollary 6.2 (integrality conjecture). For a \( \mu \)-generic stability condition \( \theta \) the Donaldson–Thomas invariant \( DT^\mu_{\text{mot}} \) is in the image of the natural map

\[
\hat{K}(\text{Var}/\Lambda_m)[L^{-1/2}] \longrightarrow \hat{K}(\text{Var}/\Lambda_m)[L^{-1/2}, (L^n - 1)^{-1} : n \in \mathbb{N}].
\]

In order to prove Theorem 6.1 it suffices to show that the stalk of \( DT^\mu_{\text{mot}} \) at any necessary closed points \( V = \bigoplus_{k \in K} U^m_k \) of \( \mathcal{M}^s_\mu \) with residue field \( \mathbb{k} \) does not involve denominators apart from powers of \( L^{1/2} \). Now if a denominator is canceled after restriction to the generic point of \( \mathcal{M}^s_\mu \), the relations used to show this will also exist over an open subscheme of \( \mathcal{M}^s_\mu \). By looking at its complement in \( \mathcal{M}^s_\mu \), we can reduce the problem of canceling denominators to lower dimensions. In fact, we will show the following:

Consider the embedding \( \iota_U : K^K \hookrightarrow \mathcal{M}^s_\mu \) of locally finite schemes mapping Spec \( \mathbb{k} \) indexed by \( \{m_k\}_{k \in K} \) to \( \bigoplus_{k \in K} U^m_k \). Note that \( \hat{K}(\text{Var}/K)[L^{-1/2}, (L^n - 1)^{-1} : n \in \mathbb{N}] \) can be identified with the ring \( K_0(\text{Var}/K)[L^{-1/2}, (L^n - 1)^{-1} : n \in \mathbb{N}][[t_1, \ldots, t_s]] \) of power series in \( s := |K| \) variables. We will prove that \( \iota_U^* DT^\mu_{\text{mot}} \) lies in the image of \( K_0(\text{Var}/K)[L^{-1/2}] \longrightarrow \hat{K}_0(\text{Var}/K)[L^{-1/2}, (L^n - 1)^{-1} : n \in \mathbb{N}] \).

Let us form the following fiber product:

\[
\begin{array}{ccc}
\mathcal{M}_U & \xrightarrow{\iota_U} & \mathcal{M}^s_\mu \\
\mathcal{M}_U & \xrightarrow{\mathcal{I}_U} & \mathcal{M}^s_\mu \\
\end{array}
\]

\( \mathcal{M}_U = \sqcup_{m \in \mathbb{N}^K} \mathcal{M}_{U,m} \) can be seen as the substack of representations having a decomposition series with factors in the collection \( U = \{U_k\}_{k \in K} \). Since \( \rho \) commutes with base change and \( \iota_U : K^K \hookrightarrow \mathcal{M}^s_\mu \) is a monoid homomorphism, we get

\[
\rho_!(\iota_U^* \mathcal{I}_U^{\text{mot}}) = \text{Sym} \left( \frac{1}{[L^{1/2} - L^{-1/2}]^2} \iota_U^* DT^\mu_{\text{mot}} \right).
\]

In particular, the stable summands \( U_k \) are only defined over the field \( \mathbb{k} \) containing \( k \).
Note that $\tilde{i}_U^*\mathcal{IC}^{\text{mot}}_{\text{Ext}^1}$ restricted to $\mathcal{M}_{U,n}$ is just $L^{(d(n), d(n)) / 2} [\mathcal{M}_{U,n} \xrightarrow{id} \mathcal{M}_{U,n}]$, where $d(n) := \sum_{k \in K} n_k \dim U_k$ is the dimension vector of $\bigoplus_{k \in K} U_k^{n_k}$.

Let us recall the notion of the Ext-quiver $Q_U$ of the collection $(U_k)_{k \in K}$. Its vertex set is $K$, and the number of arrows from $k$ to $l$ is given by the $k$-dimension of $\text{Ext}^1(U_k, U_l)$. For a dimension vector $n \in \mathbb{N}^K$ of $Q_U$, we denote by $R_{U,n} \cong A_{K}^{\sum_{k \in K} n_k}$ the affine space of all possible assignments of matrices with coefficients in $K$ to the arrows of $Q_U$. Recall that $R_{U,n} / G_n$ is the stack of $n$-dimensional $Q_U$-representations.

Using $\dim \text{Ext}^1(U_k, U_l) = -(\dim U_k, \dim U_l)$ for $k \neq l$, the quiver $Q_U$ is symmetric if $\theta$ is $\mu$-generic, and we can apply the following result of Efimov rephrased in our language to the quiver $Q_U$.

**Theorem 6.3** ([9], Theorem 1.1). Given any quiver $Q$ with vertex set $K$, we define for every $n \in \mathbb{N}^K \setminus \{0\}$ the motivic Donaldson–Thomas invariants $\text{DT}^\text{mot}_n \in \mathbb{Z}[L^{-1/2}, (L^n - 1)^{-1} : n \in \mathbb{N}]$ of $Q$ with respect to the trivial stability condition $\theta = 0$ by means of

$$
\sum_{n \in \mathbb{N}^K} L^{(n,n)/2} \frac{[R_n]}{[G_n]} t^n := \text{Sym} \left( \frac{1}{L^{1/2} - L^{-1/2}} \sum_{n \in \mathbb{N}^K \setminus \{0\}} \text{DT}^\text{mot}_n t^n \right).
$$

If the quiver $Q$ is symmetric, the invariant $\text{DT}^\text{mot}_n$ is contained in the Laurent subring $\mathbb{Z}[L^{1/2}]$ of $\mathbb{Z}[L^{-1/2}, (L^n - 1)^{-1} : n \in \mathbb{N}]$.

When we apply Efimov’s Theorem to $Q_U$, we use the notation $(-, -)_U, R_U, \text{DT}^\text{mot}_n$ to distinguish the objects from their counterparts for $Q$. Theorem 6.1 is then a direct consequence of the following result and the remarks made at the beginning of this section.

**Proposition 6.4.** Denote by $\text{DT}^\text{mot}_U|_{L^{1/2} \rightarrow L^{-1/2}}$ the series in $\mathbb{Z}[L^{1/2}][[t_1, \ldots, t_s]]$ obtained by the indicated substitution. Then $\text{DT}^\text{mot}_U|_{L^{1/2} \rightarrow L^{-1/2}} = \tilde{i}_U^* \text{DT}^\text{mot}_\mu$. In particular, $\tilde{i}_U^* \text{DT}^\text{mot}_\mu$ is an element of the subring $\mathbb{Z}[L^{1/2}][[t_1, \ldots, t_s]]$, respectively $K_0(\text{Var} / K)[L^{1/2}][[t_1, \ldots, t_s]]$, of $K_0(\text{Var} / K)[L^{-1/2}, (L^n - 1)^{-1} : n \in \mathbb{N}][[t_1, \ldots, t_s]]$.

**Remark 6.5.** The substitution $L^{1/2} \mapsto L^{-1/2}$ has an intrinsic meaning. For any base $\mathcal{B}$ there is a duality operation on $K(\text{Var} / \mathcal{B})[L^{-1/2}, (L^n - 1)^{-1} : n \in \mathbb{N}]$ which can be seen as a motivic version of (relative) Poincaré duality. See [2] for more details on this.

**Proof.** As the substitution $L^{1/2} \mapsto L^{-1/2}$ is compatible with the $\lambda$-ring structure of $K[\mathcal{L}^{-1/2}, (\mathcal{L}^n - 1)^{-1} : n \in \mathbb{N}][[t_1, \ldots, t_s]]$, which contains $K[\mathcal{L}^{1/2}][[t_1, \ldots, t_s]]$ as a $\lambda$-subring, it suffices to show the identity

$$
\left( \sum_{n \in \mathbb{N}^K} L^{(n,n)/2} \frac{[R_n]}{[G_n]} t^n \right)|_{L^{1/2} \rightarrow L^{-1/2}} = \left( \sum_{m \in \mathbb{N}^K} L^{(d(m), d(m))/2} [\mathcal{M}_{U,m}] [m]^m \right) = 1
$$

in $K_0(\text{Var} / K)[L^{-1/2}, (L^n - 1)^{-1} : n \in \mathbb{N}][[t_1, \ldots, t_s]]$. Recall that the expression $d(m) := \sum_{k \in K} m_k \dim U_k$ is the dimension vector of $\bigoplus_{k \in K} U_k^{m_k}$. Indeed, the factor on the left hand side is by definition

$$
\text{Sym} \left( \frac{\text{DT}^\text{mot}_U|_{L^{1/2} \rightarrow L^{-1/2}}}{L^{1/2} - L^{-1/2}} \right)|_{L^{1/2} \rightarrow L^{-1/2}} = \text{Sym} \left( \frac{- \text{DT}^\text{mot}_\mu|_{L^{1/2} \rightarrow L^{-1/2}}}{L^{1/2} - L^{-1/2}} \right).
$$

On the other hand, the factor on the right hand side is nothing else than

$$
\tilde{p}^*_U(\tilde{i}_U^* \mathcal{IC}^{\text{mot}}_{\text{Ext}^1}) = \text{Sym} \left( \frac{\tilde{i}_U^* \text{DT}^\text{mot}_\mu|_{L^{1/2} \rightarrow L^{-1/2}}}{L^{1/2} - L^{-1/2}} \right).
$$
Consider the following two stack functions on \(M^ss_\mu\)

\[
f := \sum_{n \in \mathbb{N}^K} (-1)^{|n|} L^{\sum_{k \in K} \binom{g}{2}} [\text{Spec } \mathbb{K}/G_n \rightarrow M^ss_\mu] \quad \text{and} \quad g := [M_U \rightarrow M^ss_\mu],
\]

where for \(n \in \mathbb{N}^K\) the quotient stack \(\text{Spec } \mathbb{K}/G_n\) maps to the representation \(\bigoplus_{k \in K} U_k^{n_k}\) of dimension vector \(d(n)\) and its automorphism group. In particular, the morphisms used to define \(f\) and \(g\) correspond to substacks of \(M^ss_\mu\). We compute the convolution product \(f * g\) by means of the following diagram

\[
\begin{array}{ccc}
\mathbb{K}/G_n \times M_U, d(m) & \xrightarrow{\pi_1 \times \pi_3} & M^ss_{d(n)+d(m)} \\
\downarrow & & \downarrow \\
\mathbb{K}/G_n \times M_U, d(m) & \xrightarrow{\pi_2} & M^ss_{d(n)+d(m)}
\end{array}
\]

where the left square is cartesian, and \(\mathfrak{E}ract_{d(n), d(m)}\) denotes the stack of short exact sequences with prescribed dimension vectors for the first and third object in the sequence. The morphisms \(\pi_1, \pi_2, \text{ and } \pi_3\) map a sequence to the corresponding entries. Since \(\pi_2\) is representable, \(\mathbb{K}/G_n \rightarrow M^ss_{d(n)+d(m)}\) is representable, too. In fact, \(Z_{d(n), d(m)}\) is the substack of \(Q\)-representations \(V\) that are extensions of a representation with dimension vector \(d(m)\) and Jordan–Hölder factors among the \((U_k)_{k \in K}\) by the representation \(\bigoplus_{k \in K} U_k^{n_k}\). In particular, the Jordan–Hölder factors of \(V\) are also among \((U_k)_{k \in K}\), and \(\bigoplus_{k \in K} U_k^{n_k}\) must embed into the socle \(\bigoplus_{k \in K} U_k^{N_k}\) of \(V\) for certain integers \(N_k\) depending on \(V\). The space of such embeddings, that is, the fiber of the map \(Z_{d(n), d(m)} \rightarrow M^ss_{d(n)+d(m)}\) over \(V\), is given by the product of finite Grassmannians \(\prod_{k \in K} \text{Gr}_{N_k}^{V_k}(\mathbb{K})\). Hence, the convolution product \(f * g\) restricted to \(V \in M^ss_\mu\) is

\[
\sum_{0 \leq n_k \leq N_k} (-1)^{n_k} L^{\binom{n_k}{2}} \binom{N_k}{n_k},
\]

the \(L\)-binomial coefficient \(\binom{N_k}{n_k}\) being the Lefschetz motive of the Grassmannian \(\text{Gr}_{N_k}^{V_k}(\mathbb{K})\). A standard identity for \(L\)-binomial coefficients then shows that this sum vanishes as soon as \(N_k \neq 0\) for some \(k \in K\), that is, for every non-zero \(V\). Since \(V\) was arbitrary, this can only happen if the stack function \(f * g\) is concentrated on the zero representation, and a direct computation shows \(f * g = [\text{Spec } \mathbb{K} \rightarrow M^ss_\mu] = 1\).

To continue, we use the following lemma.

**Lemma 6.6** (see [27], [34]). Since \(\theta\) is \(\mu\)-generic, there is an algebra homomorphism from the ring of stack functions supported on \(M^ss_\mu\) with convolution product to the ring \(K_0(\text{Var} / \Lambda_\mu)[L^{-1/2}, (L^n - 1)^{-1}] : n \in \mathbb{N}[[t_1, \ldots, t_s]]\) mapping a stack function of the form \([X \rightarrow M^ss_\mu]\) to \(L^{(d, d)/2} [X]_{e_d}\) with \(e_d = [\text{Spec } \mathbb{K} \rightarrow \Lambda_\mu]\).

We apply the algebra homomorphism to \(f, g \text{ and } f * g\). Note that the resulting stack function on \(\Lambda_\mu\) are actually the proper push forward of stack functions on \(\mathbb{N}^K\) under the morphism \(\mathbb{N}^K \rightarrow \Lambda_\mu\) mapping \(\text{Spec } \mathbb{K}\) indexed by \(n\) to \(\text{Spec } \mathbb{K}\) indexed by \(d(n)\). Using \([R_{U,n}] = L^{-\sum_{k \in K} n_k^2} + \sum_{k \in K} n_k^2 = L^{-\sum \binom{n_k}{2}} [\text{Spec } \mathbb{K}]_{e_d}\), finally yields equation (7) as one can easily check.

□

**References**

[1] K. Behrend, J. Bryan, and B. Szendrői. Motivic degree zero Donaldson–Thomas invariants. *Invent. Math.*, 192, 2013. arXiv:0909.5088.
[2] F. Bittner. The universal euler characteristic for varieties of characteristic zero. Comp. Math., 140:1011–1032, 2004.

[3] L. Le Bruyn. Noncommutative geometry and Cayley-smooth orders, volume 290 of Pure and Applied Mathematics (Boca Raton). Chapman & Hall/CRC, Boca Raton, FL, 2008.

[4] L. Le Bruyn and C. Procesi. Semisimple representations of quivers. Trans. Amer. Math. Soc., 317(2):585–598, 1990.

[5] M. A. Cataldo and L. Migliorini. The decomposition theorem, perverse sheaves and the topology of algebraic maps. Bull. of the Am. Math. Soc., 46, no. 4:535–633, 2009.

[6] B. Davison and S. Meinhardt. Motivic DT-invariants for the one loop quiver with potential. to appear in Geometry and Topology, 2011. arXiv:1108.5956.

[7] B. Davison and S. Meinhardt. The motivic Donaldson-Thomas invariants of (-2) curves. 2012. arXiv:1208.2462.

[8] P. Deligne. Catégories tensorielles. Mosc. Math. J., 2, no.:227–248, 2002. Dedicated to Yuri I. Manin on the occasion of his 65th birthday.

[9] A. Efimov. Cohomological Hall algebra of a symmetric quiver. Comp. Math., 148, no. 4:1133–1146, 2012.

[10] J. Engel and M. Reineke. Smooth models of quiver moduli. Math. Z., 262, no. 4:817–848, 2009. arXiv:0706.4306.

[11] M. Gross, R. Pandharipande, and B. Siebert. The tropical vertex. Duke Math. J., 153(2):297–362, 2010.

[12] D. Joyce. Configurations in abelian categories. I. Basic properties and moduli stacks. Advances in Mathematics, 203:194–255, 2006. math.AG/0312190.

[13] D. Joyce. Constrictable functions on Artin stacks. J. L.M.S., 74, 2006. math.AG/0403305.

[14] D. Joyce. Configurations in abelian categories. II. Ringel–Hall algebras. Advances in Mathematics, 210:635–706, 2007. math.AG/0503029.

[15] D. Joyce. Configurations in abelian categories. III. Stability conditions and identities. Advances in Mathematics, 215:153–219, 2007. math.AG/0410267.

[16] D. Joyce. Motivic invariants of Artin stacks and ‘stack functions’. Quarterly Journal of Mathematics, 58, 2007. math.AG/0509722.

[17] D. Joyce. Configurations in abelian categories. IV. Invariants and changing stability conditions. Advances in Mathematics, 217:125–204, 2008. math.AG/0503029.

[18] D. Joyce and Y. Song. A theory of generalized Donaldson–Thomas invariants. Mem.Amer. Math. Soc., 217(1020), 2012. math.AG/08105645.

[19] F. Kirwan. Cohomology of quotients in symplectic and algebraic geometry. 31.

[20] F. Kirwan. Partial desingularisations of quotients of nonsingular varieties and their Betti numbers. Ann. of Math., 122, no. 2:41–85, 1985.

[21] F. Kirwan. Rational intersection cohomology of quotient varieties. Inventiones mathematicae, 86, no. 3:471–505, 1986.

[22] F. Kirwan. Rational intersection cohomology of quotient varieties. II. Inventiones mathematicae, 90, no. 1:153–167, 1987.

[23] M. Kontsevich and Y. Soibelman. Stability structures, motive Donaldson–Thomas invariants and cluster transformations. 2008. math.AG/08112435.

[24] M. Kontsevich and Y. Soibelman. Motivic Donaldson–Thomas invariants: summary of results. In Mirror symmetry and tropical geometry, volume 527 of Contemp. Math., pages 55–89. Amer. Math. Soc., Providence, RI, 2010.

[25] M. Kontsevich and Y. Soibelman. Cohomological Hall algebra, exponential Hodge structures and motivic Donaldson–Thomas invariants. Commun. Number Theory Phys., 5, 2011. arXiv:1006.2706.

[26] A. Morrison, S. Mozgovoy, K. Nagao, and B. Szendrői. Motivic Donaldson–Thomas invariants of the conifold and the refined topological vertex. Adv. Math., 230, 2012.

[27] M. Reineke. The Harder-Narasimhan system in quantum groups and cohomology of quiver moduli. Invent. Math., 152(2):349–368, 2003.

[28] M. Reineke. Poisson automorphisms and quiver moduli. J. Inst. Math. Jussieu, 9, no. 3:653–667, 2010. arXiv:0804.3214.

[29] M. Reineke. Cohomology of quiver moduli, functional equations, and integrality of Donaldson-Thomas type invariants. Comp. Math., 147, no. 3:943–964, 2011. arXiv:0903.0261.

[30] M. Reineke. Degenerate Cohomological Hall algebra and quantized Donaldson-Thomas invariants for m-loop quivers. Doc. Math., 17:1–22, 2012. arXiv:1102.2878.

[31] M. Reineke and T. Weist. Refined gw/kronecker correspondence. Math. Ann.

[32] M. Saito. Introduction to mixed Hodge modules. Astérisque, 179-180:145–162, 1989.

[33] R.P. Thomas. A holomorphic casson invariant for Calabi–Yau 3-folds, and bundles on K3 fibrations. J. Diff. Geom., 54:367–438, 2000. math.AG/9806111.
