Succinct Certificates for the Solvability of Binary Quadratic Diophantine Equations

J. C. Lagarias
Department of Mathematics
University of Michigan
Ann Arbor, MI 48109-1043
lagarias@umich.edu

(July 1, 2009 revision)

Abstract

Binary quadratic Diophantine equations are of interest from the viewpoint of computational complexity theory. This class of equations includes as special cases many of the known examples of natural problems apparently occupying intermediate stages in the $P - NP$ hierarchy, i.e., problems not known to be solvable in polynomial time nor to be NP-complete, for example the problem of factoring integers.

Let $L(F)$ denote the length of the binary encoding of the binary quadratic Diophantine equation $F$ given by $ax_1^2 + bx_1x_2 + cx_2^2 + dx_1 + ex_2 + f = 0$. Suppose $F$ is such an equation having a nonnegative integer solution. This paper shows that there is a proof (i.e., “certificate”) that $F$ has such a solution which can be checked in $O(L(F)^5 \log L(F) \log \log L(F))$ bit operations.

A corollary of this result is that the set $\Sigma = \{F : F$ has a nonnegative integer solution} is in the complexity class $NP$. The result that $\Sigma$ is in $NP$ is interesting because it is known that there are binary quadratic Diophantine equations whose smallest nonnegative integer solution is so large that it requires time exponential in $L(F)$ just to write this solution down in the usual binary representation.

1. Introduction

There has been considerable interest in bounding the computational complexity of various number-theoretic problems. A particular motivation is the Rivest-Shamir-Adleman enciphering scheme [51] whose resistance to cryptanalysis depends on the apparent difficulty of factoring large integers. Many of these number-theoretic problems can be formulated as one of two types of problems involving Diophantine equations.

(i) Deciding whether a given Diophantine equation has an admissible solution or not.

(ii) Exhibiting an admissible solution to such an equation when it has one.

Here an admissible solution denotes an integer solution which may also be required to satisfy some side conditions characteristic of the particular problem. The side conditions that arise are generally of the following two types:

(i) Nonnegativity. Certain variables are required to be nonnegative.
(ii) **Congruence.** Certain variables \( x_i \) are required to satisfy congruence restrictions \( x_i \equiv \alpha_i \pmod{\Gamma} \) where the \( \alpha_i \) and \( \Gamma \) are given as input.

In this framework, for example, an integer \( N \) is composite if and only if the binary quadratic Diophantine equation
\[
(x + 2)(y + 2) = N \tag{1.1}
\]
has a solution in nonnegative integers \( x, y \in \mathbb{N} \). The problem of factoring \( N \) involves exhibiting nonnegative solutions to a series of equations (1.1), and showing that certain other equations of the form (1.1) are solvable.

There is a close relation between Diophantine equations and the theory of computation. The methods developed by Davis, Putman, Robinson and Matijasevic in their solution to Hilbert’s 10th problem established that for any recursively enumerable set \( A \) of natural numbers there is a Diophantine equation \( P(x_1, \ldots, x_n) = 0 \) such that
\[
x \in A \iff \exists \text{ nonnegative } x_2, \ldots, x_n \text{ such that } P(x, x_2, \ldots, x_n) = 0
\]
(see in particular [22], [23], [40].) Adleman and Manders [1], [2] used these methods to establish a computational complexity theory based on the notion of recognizing sets for which a given Diophantine equation has solutions of size bounded by a given complexity function \( \Phi \). More precisely, they considered sets \( S \) given
\[
x \in S \iff \exists \text{ nonnegative } x_2, \ldots, x_n \text{ with } L(x_2), \ldots, L(x_n) \leq \Phi(L(x))
\]
such that
\[
P(x, x_2, \ldots, x_n) = 0
\]
where \( P(x_1, \ldots, x_n) \in \mathbb{Z}[x_1, \ldots, x_n] \) is a fixed Diophantine equation, \( L(x) \) denotes the length of the binary integer \( x \), and \( \Phi(t) \) is a complexity measure which is an increasing function of \( t \). They introduced a complexity class \( D \) which is a Diophantine analogue of the complexity class \( NP \). It consists of all relations \( R \subseteq \mathbb{N}^m \) specified by a Diophantine equation \( P(x_1, \ldots, x_{m+n}) = 0 \) with \( P \in \mathbb{Z}[x_1, \ldots, x_{m+n}] \) and a polynomial \( q(t) \) as follows:
\[
< x_1, \ldots, x_m > \in R \iff \exists \text{ nonnegative } y_1, \ldots, y_n \\
\text{ such that } MAX L(y_i) \leq q(L(x_1) + \ldots + L(x_m)) \\
\text{ and } P(x_1, \ldots, x_m, y_1, \ldots, y_n) = 0. \tag{1.2}
\]

It is immediately clear that \( D \subseteq NP \). It is an open problem whether or not \( D = NP \); this is an important problem in determining the relative computing power of Diophantine equations as compared to that of nondeterministic Turing machines.

### 1.1. Binary Quadratic Diophantine Equations

The class of binary quadratic Diophantine equations (BQDE’s)
\[
a x_1^2 + b x_1 x_2 + c x_2^2 + d x_1 + e x_2 + f = 0 \tag{1.3}
\]
is of special interest from the viewpoint of both number theory and complexity theory. From the viewpoint of number theory, this class of equations can be used to encode the problem of factorization, the problem of solving quadratic congruences

\[ x^2 = f \pmod{e} \]

(which corresponds to \( x_1^2 - ex_2 - f = 0 \)), of solving Pell’s equation

\[ x_1^2 - dx_2^2 = 1, \]

and problems in representation and equivalence of binary quadratic forms. From the viewpoint of complexity theory this class of equations seems to represent the borderline between tractable and intractable computational problems. It includes as special cases most of the known examples of natural problems apparently occupying intermediate stages in the \( P - NP \) hierarchy (i.e., problems not known to be in \( P \) nor to be \( NP \)-complete) as well as \( NP \)-complete problems. For example, it known that:

(i) \( S = \{p|p \text{ is prime}\} \in P \) (Agrawal, Kayal and Saxena [6]).

(ii) \( \{\alpha, \beta, \gamma \in \mathbb{N} | \exists \text{ nonnegative } x_1, x_2 \text{ such that } ax_1^2 + \beta x_2 - \gamma = 0\} \) is \( NP \)-complete. (Manders and Adleman [38]).

(iii) \( \{a, c \in \mathbb{N} | \exists x_1, x_2 \text{ such that } ax_1 x_2 + x_2 = c\} \in NP \setminus co-NP \) provided \( NP \neq co-NP \). This problem is \( \gamma \)-complete (Adleman and Manders [3]).

(iv) \( \{a, c \in \mathbb{N} | \exists x_1, x_2 \text{ such that } x_1^2 - a^2 x_2^2 = c\} \) is unfaithfully random complete (Adleman and Manders [1]. [5]).

All of the sets (i)–(iv) are in \( D \), and hence are certainly in \( NP \). Note that the existence of \( NP \)-complete sets in \( D \) does not establish \( D = NP \). Indeed, Adleman and Manders [2] exhibit a set in \( P \) not known to be in \( D \).

This paper treats the general problem of recognizing those binary quadratic Diophantine equations which have nonnegative solutions, which may also be required to satisfy given congruence side conditions. This problem appears to be fundamentally harder computationally than any of the special subclasses of binary quadratic Diophantine equations considered up to now (i.e. including (i)–(iv) above) as indicated by the following example.

**Example.** (Anti-Pellian Equation) Consider the set of equations

\[ x^2 - dy^2 = -1 \]  \hspace{1cm} (1.4)

where \( d \) is given in its binary representation as input. The equations (1.4) are often called the non-Pellian or anti-Pellian equations. For the subset \( d = 5^{2n+1} \), the input requires no more than \( 7n \) bits. In Lagarias [35, Appendix A], it is shown that for \( d = 5^{2n+1} \) this equation has solutions for \( n = 1, 2, 3, \ldots \) and that the solution \( (t_1, u_1) \) to this equation with minimal binary lengths \( L(t_1), L(u_1) \) is given by

\[ t_1 + u_1 \sqrt{5} = (2 + \sqrt{5})^{5^n}. \]  \hspace{1cm} (1.5)
This implies that the length of any solution \( x \) to (1.4) expressed in binary for these \( d \) satisfies
\[
L(x) > \frac{1}{3}5^n. \quad \Box
\]

This example shows that there are some binary quadratic Diophantine equations whose solutions are so large that it requires exponential space (in terms of the length of the coefficients of the equation) to store any such solution in binary. In addition this shows the set
\[
\Pi = \{ d \in \mathbb{N} | x^2 - dy^2 = -1 \text{ is solvable in integers} \}
\]
(1.6)
cannot be established to be in the complexity class \( D \) by using the Diophantine equation \( x^2 - dy^2 + 1 = 0 \) in a relation (1.2). Indeed we cannot hope to show the set (1.6) is in \( NP \) by guessing a solution \( x, y \) in binary and verifying it is a solution by substitution in (1.4), because this potentially requires exponential time to check. These same restrictions apply to the general quadratic Diophantine equation (1.3), because the problem of recognizing the subclass (1.4) is clearly in \( P \).

1.2. Main Results

The major result of this paper is that there exist short certificates of the solvability of all binary quadratic Diophantine equations which have solutions. By the preceding example, these certificates must sometimes verify that solutions exist without exhibiting these solutions written in binary. The certificates actually contain in a compact form enough information to exactly calculate an admissible solution.

In order to state the main result, we need two definitions. We consider a binary quadratic Diophantine equation (BQDE) \( F(x_1, x_2) = 0 \) where
\[
F(x_1, x_2) = ax_1^2 + bx_1x_2 + cx_2^2 + dx_1 + ex_2 + f,
\]
together with a congruence side condition
\[
x_1 \equiv \alpha_1 \pmod{\Gamma} \]
\[
x_2 \equiv \alpha_2 \pmod{\Gamma}
\]
with \( 0 \leq \alpha_1, \alpha_2 < \Gamma \). We define the length \( L(F) \) of the input to be
\[
L(F) := L(a) + L(b) + L(c) + L(d) + L(e) + L(f) + 3L(\Gamma), \quad (1.7)
\]
in which
\[
L(a) := 2 + \log_2(|a| + 1) \quad (1.8)
\]
is a measure of the binary length of an integer \( a \), allowing one extra bit for its sign. The other definition concerns the measurement of the running time of a program. We shall measure running time in terms of elementary operations, which consists of a Boolean operation on a bit or pair of bits, and an input or shift of a single bit. Our main result is the following, concerning nonnegative solutions to the system above.
Theorem 1.1. Let $F(x_1, x_2) = 0$ be a binary quadratic Diophantine equation, where
\[ F(x_1, x_2) = ax_1^2 + bx_1x_2 + cx_2^2 + dx_1 + ex_2 + f , \]
which has an integer solution $(x_1, x_2)$ satisfying the congruence condition
\[ x_1 \equiv \alpha_1 \pmod{\Gamma} \]
\[ x_2 \equiv \alpha_2 \pmod{\Gamma} . \]
and the nonnegativity condition
\[ x_1 \geq 0, \quad x_2 \geq 0. \]
Then there exists a certificate showing that $F(x_1, x_2) = 0$ has such an admissible solution which requires $O(L(F)^5 \log L(F) \log \log L(F))$ elementary operations to verify.

This result gives certificates imposing two side conditions on the solutions: a congruence condition and a nonnegativity condition. This theorem is formulated to impose a nonnegativity side condition, in order for it to provide a result compatible with the framework of Hilbert’s 10-th problem, and also with the Diophantine complexity theory of Adleman and Manders [3], [4] which requires nonnegative variables.

An immediate consequence of the form of the certificates produced by Theorem 1.1 is the following result.

Theorem 1.2. The following sets $\Sigma_i$ are all in $NP$.

(i) $\Sigma_1 = \{a, b, c, d, e, f, \alpha_1, \alpha_2, \Gamma \in \mathbb{Z} | \exists$ nonnegative integers $x_1, x_2$ with $x_1 \equiv \alpha_1 \pmod{\Gamma}, x_2 \equiv \alpha_2 \pmod{\Gamma}$ and $ax_1^2 + bx_1x_2 + cx_2^2 + dx_1 + ex_2 + f = 0\}$

(ii) $\Sigma_2 = \{a, b, c, d, e, f \in \mathbb{Z} | \exists$ nonnegative integers $x_1, x_2$ with $ax_1^2 + bx_1x_2 + cx_2^2 + dx_1 + ex_2 + f = 0\}$

(iii) $\Sigma_3 = \{a, b, c, d, e, f \in \mathbb{Z} | \exists$ integers $x_1, x_2$ with $ax_1^2 + bx_1x_2 + cx_2^2 + dx_1 + ex_2 + f = 0\}$.

Since one can tell in polynomial time whether or not a binary quadratic Diophantine equation (1.3) is of the special form
\[ ax_1^2 + ex_2 + f = 0 \]
and since the set of equations of the form (1.9) which have nonnegative integral solutions is $NP$-complete [38], we conclude that: testing membership in the sets $\Sigma_1$ and $\Sigma_2$ in Theorem 1.2 are each $NP$-complete. We are unable to decide whether or not any of the sets $\Sigma_i$ above are in the Diophantine complexity class $D$.

The proofs of Theorems 1.1 and 1.2 use the theory of binary quadratic forms in the form developed by Gauss in Disquisitiones Arithmeticae [27]. A treatment of this theory can be found in Buell [17]. The certificates are based on Gauss’ operation of composition of forms, and
crucially use an idea of Shanks [55], which he called the “infrastructure” of quadratic forms. Shanks did not give detailed proofs of his “infrastructure” method, but the “infrastructure” method was put on a rigorous footing by Lenstra [37] in 1980, in the framework of quadratic number fields. This paper gives an alternate justification of the infrastructure method in the framework of composition of forms (Lemma 6.2). The proof of Theorem 1.1 is outlined in Section 2, and the details appear in the following sections.

In addition to the main theorem, we show that whenever two integer binary quadratic forms are equivalent there exist succinct certificates verifying this equivalence (Theorem 7.1).

1.3. Related Work

We add some remarks on related work. The following result is a direct consequence of Lagarias [35, Theorem 1.1].

**Theorem 1.3.** The set

\[ \Pi_{AP} = \{d \mid \exists \text{ integers } x, y \text{ with } x^2 - dy^2 = -1\} \]

is in \(NP \cap co-NP\).

The problem of characterizing the set \(\Pi_{AP}\) of solvable anti-Pellian equations has been extensively studied in algebraic number theory, see Narkiewicz [42, pp. 124–126], and Redei [50]. For recent work, see Williams [65], Jacobson and Williams [31] and Fouvry and Klüners [24].

In another direction Gurari and Ibarra [28] consider a class of Diophantine equations containing (1.9) as a special case, but not containing (1.3). They show that the subclass of such equations having nonnegative solutions is in \(D\), i.e., when such equations have a nonnegative solution, they have one that is small enough to serve as a certificate. Their \(NP\)-completeness result then follows from [38].

1.4. Retrospective: Smale’s Problem 5

These certificates given in Theorem 1.1 are also relevant to Problem 5 of the mathematical problems for the twenty-first century formulated by S. Smale [56, p. 275], which concerns height bounds for solutions to Diophantine equations.

**Smale’s Problem 5.** Can one decide if a single Diophantine equation \(f(x, y) = 0\) with in two variables of exact total degree \(d\), and of genus at least one, has an integer solution, in time \(O(2^{cs})\) for some universal constant \(c\), where \(c\) is a universal constant? That is, can the problem be decided in exponential time?
In this problem
\[ f(u, v) = \sum_{\alpha_1 + \alpha_2 \leq d} a_{\alpha_1, \alpha_2} u^{\alpha_1} v^{\alpha_2} \in \mathbb{Z}[u, v], \]
with \( f(u, v) \) having some nonvanishing term of total degree \( \alpha_1 + \alpha_2 = d \), and
\[ s = s(f) := \sum_{\alpha_1 + \alpha_2 \leq d} \max(\log |a_{\alpha_1, \alpha_2}|, 1). \]
is a measure of the “height” of \( f \).

Here we have the following result, which affirmatively answers Problem 5 for polynomials of total degree \( d \leq 2 \).

**Theorem 1.4.** Let \( F(x_1, x_2) = 0 \) be a binary quadratic Diophantine equation, given by
\[ F(x_1, x_2) = ax_1^2 + bx_1 x_2 + cx_2^2 + dx_1 + ex_2 + f, \]
given with coefficients encoded in binary. Then there exists a deterministic exponential time algorithm which decides whether or not \( F(x, y) = 0 \) has an integer solution. This algorithm uses at most \( O(2^{c_1 L(F)}) \) elementary operations, where \( c_1 \) is an absolute constant.

Theorem 1.4 shows we may take the universal constant \( c = 1 \) in Smale’s problem 5, when the input is restricted to bivariate polynomials \( f(u, v) \) of total degree \( d \leq 2 \). The anti-Pellian example above shows that to write down a minimal solution in binary may sometimes require at least \( \Omega(2^{c_2 L(F)}) \) bits.

Theorem 1.4 is proved using a complexity analysis of the classical algorithmic approach of Gauss for finding solutions of binary quadratic Diophantine equations. It is given in Section 5.6 and does not require use of the succinct certificates found in Theorem 1.1.

A more general version of Theorem 1.4, which works for testing for admissible solutions to a BQDE that also satisfy a congruence side condition and a positivity side condition, can be proved directly from Theorem 1.1. This (inefficient) algorithm used sequentially tests all possible candidate certificates produced in Theorem 1.1. If none of them work, then the system has no solution. The number of candidate certificates can be shown to be \( O(2^{c_3 L(F)^3}) \) and leads to a running time bound \( O\left(2^{c_3 L(F)^3}\right) \). We omit details.

In connection with his Problem 5, Smale [56, p. 276] also put forward the following hypothesis, for dealing with curves of genus one or larger:

**Height bound hypothesis:** If the curve \( f \), of positive genus, has any integer solution, then it has a solution \((a, b)\) satisfying the estimate: \( \log \max(|a|, |b|) \) is polynomially bounded by \( s(f) \).

The truth of this (unproved) hypothesis would solve Smale’s Problem 5 affirmatively in case the genus is 1 or larger. Such a height bound does not hold for genus 0 curves, as shown...
by the anti-Pell equation example above (given in Lagarias [35]). Effective bounds on size of solutions are known for a large class of curves covered by Runge’s method, see for example Walsh [63], and these may imply the height bound in these cases. There are also very large effective bounds for size of integer solutions for genus one curves, based on Baker’s method, see Baker and Coates [7] and Schmidt [52]. This method extends to curves given using Galois coverings, for which see Bilu [8]. Note that the height bound hypothesis, if true, would provide certificates falling in the Diophantine complexity class D, and this provides renewed motivation for studying the complexity class D.

Concerning the remaining case of genus 0 curves, Theorem 1.4 solves problem 5 affirmatively for quadratic Diophantine equations, which form a restricted class of genus 0 plane curves, without invoking the height bound hypothesis. General bounds on the integer solutions of genus zero curves were given in Bilu and Poulakis BP93 and improved in Poulakis ([44], [46], [47], [48]). However these bounds depend in part on Baker’s method, and it appears that they are not strong enough to resolve Smale’s Problem 5 for all genus 0 plane curves.

1.5. Retrospective: Infrastructure Method

The results of this paper were announced in preliminary report form in the 1979 FOCS conference proceedings [33]. Detailed proofs were given in a 1981 Bell Laboratories technical report ([36]); this work was contemporaneous with that of Lenstra [37]. Renewed motivation to publish this work, after a long delay, came from its relevance to Smale’s problem 5. The present paper is a slightly revised version of [36], which adds the application to Smale’s problem 5, and corrects errata.

The basic idea of this paper exploits the Shanks infrastructure method, which is here presented in the language of integral binary quadratic forms, and composition of forms. Since 1982 there has been extensive development of the infrastructure method, mostly given in the language of algebraic number fields and ideals. It is now a workhorse in computational number theory, and is implemented in PARI. We now review these developments.

In 1982 H. W. Lenstra, Jr. [37] gave a rigorous analysis of the infrastructure method of Shanks for the purpose of computing regulators (units in quadratic fields) and class numbers.

In 1989 the “infrastructure” method was used by Buchmann and Williams [15] to give succinct certificates for class numbers and approximate representation of regulators of quadratic number fields, under the assumption of the generalized Riemann hypothesis. Further work was done by Buchmann, Thiel and Williams [13]. In 1994 Theil [59] showed that verifying the value of the class number falls in the class NP \cap co-NP, assuming the truth of the generalized Riemann hypothesis. Recently the infrastructure has been given a more precise theoretical formulation in terms of the Arakelov class group of a number field, by Schoof [53].

The infrastructure method is well known to be computationally effective in practice, as described in Chapter 5 of Cohen [20]. It is used in computations of class numbers and regulators of quadratic and cubic number fields and function fields. For a recent survey on the computation of solutions of Pell’s equation, see Williams [65].
1.6. Acknowledgments

Acknowledgments. This work was revised while the author was supported by NSF grants DMS-0500555 and DMS-0801029.

2. Outline of the Proof

In this section we describe the main ideas of the proof and establish some notational conventions.

For the proofs we deal throughout with a system \( F \) consisting of a binary quadratic Diophantine equation (BQDE) having the Gauss standard form:

\[
ax_1^2 + 2bx_1x_2 + cx_2^2 + 2dx_1 + 2ex_2 + f = 0,
\]

with side conditions

\[
x_i \equiv \alpha_i \pmod{\Gamma}, \quad i = 1, 2, \quad (2.2)
\]

\[
x_i \geq 0, \quad i = 1, 2, \quad (2.3)
\]

The requirement that the coefficients of \( x_1x_2 \) and \( x_1 \) and \( x_2 \) be even integers is imposed for compatibility with Gauss’ formulation of this problem. Any system can be brought to this form by multiplying \( (1.3) \) by 2. A solution to \( (2.1) \)–\( (2.3) \) will be called admissible.

We follow the approach of Gauss to finding solutions of such equations, which is outlined in G. B. Mathews [39, Chap. IX] and H. J. S. Smith [57, Arts. 93-97].

Binary quadratic Diophantine equations \( (2.1) \) are classified as definite, indefinite or degenerate according to the value of the determinant

\[
D = b^2 - ac \quad (2.4)
\]

being negative, positive and not a square, or a perfect square, respectively. (The determinant \( D \) is just \( \frac{1}{4} \text{Disc}(f_Q) \), where \( f_Q \) is the polynomial \( f_Q(x) = Q(x, 1) = ax^2 + bx + c \).) This classification is useful because the sets of solutions to these three types of equations have qualitatively different behaviors. In particular, definite and degenerate binary quadratic Diophantine equations with an admissible solution always have admissible solutions small enough to serve directly as certificates. (Lemma \( 3.2 \)). The crucial part of the proof concerns the case of indefinite binary quadratic Diophantine equations.

Gauss [27, Art. 216–221] gave a method to determine whether \( (2.1) \) has any integer solutions and if so to give a complete parametric description of all solutions. This method is based on his theory of integral binary quadratic forms, and in particular on determining the equivalence or inequivalence of such forms. Gauss’ method easily extends to include the congruential side
condition (2.2), but the positivity side condition (2.3) adds new complications. We follow the outline of Gauss’s method in reducing the problem to that of recognizing the equivalence of two quadratic forms. In Section 3 we transform the problem to that of studying the generalized Pell equation
\[ x_1^2 - Dy_2^2 = g, \]
with \((x_1, x_2)\) satisfying side conditions on their signs and congruence conditions to some modulus. In Section 4 (primitive) binary quadratic forms are introduced and the problem is transformed to that of demonstrating that the reduced identity form \(\tilde{I}\) of determinant \(D\) is equivalent to a particular reduced form \(Q_{\text{red}}\) via an equivalence matrix \(W\) having certain properties. (See Section 4 for definitions.) The proofs are complicated by the need to bound the size of the least admissible solution and to keep track of the nonnegativity condition (2.3) under these transformations.

These reductions have not yet addressed the main difficulty in finding certificates of solvability, which is the possible exponentially large size (number of binary bits) of the least admissible solution. This difficulty is here transformed into the possibly equally large size of the entries of the equivalence matrix \(W\) appearing in Lemma 4.2. We show that in order to verify admissibility we need only know (1) that \(W\) gives an equivalence, that (2) the entries of \(W\) satisfy certain congruence side conditions, and (3) the entries of \(W\) are known in floating point to sufficient accuracy to check a certain sign condition.

The remainder of the proof is devoted to a detailed study of those matrices \(W\) that demonstrate the equivalence of the reduced identity form \(\tilde{I}\) and any particular reduced form \(Q_{\text{red}}\). In Section 5 we describe results of Gauss. Gauss defined a notion of two reduced forms being neighbors. If we form a graph in which the reduced forms are vertices, and edges correspond to two reduced forms being neighbors, then Gauss showed that this graph is a union of disjoint cycles. Furthermore the cycle including \(\tilde{I}\) contains exactly the reduced forms equivalent to \(\tilde{I}\), which we call the principal cycle. These results imply that the associated equivalence matrices have a very special form, which is related to the ordinary continued fraction algorithm. However this form by itself is insufficient to produce succinct certificates. However it is sufficient to yield an exponential time algorithm for determining if a BQDE has an integer solution, and we prove Theorem 1.3.

In Section 6 we come to the main idea of the succinct certificate proof. This uses another set of relations between these equivalence matrices, which comes from Gauss’ operation of composition of binary quadratic forms (Lemma 6.1). The idea of using the action of composition of forms on the principal cycle is due to Shanks [55], who called it the “infrastructure”. The “infrastructure” asserts that composition is a kind of doubling of distance on the graph of the set of forms. Shanks did not give detailed proofs, but a rigorous justification of the “infrastructure” was later given by Lenstra [37], in the language of ideals. In this paper we give an alternate rigorous justification in the language of composition of forms, in Lemma 6.2. The action of composition can be combined with Gauss’s reduction steps to find a short sequence of composition formulae that prove the equivalence of any two given forms in the principal cycle (Lemma 6.3). In effect each composition causes a squaring, so that if one multiplied out all the compositions to write down the matrix giving the equivalence, the resulting entries would
potentially have exponentially many digits, in terms of the input size. However the correctness of the composition steps can be verified for each formula separately, avoiding this potential exponential blowup.

In Section 7 we further apply these composition formulae to give succinct certificates for the equivalence of two (equivalent) indefinite binary quadratic forms. We remark that the equivalence or inequivalence of two definite or degenerate quadratic forms can be decided in polynomial time (34.)

In Section 8 we complete the proof of Theorems 1.1 and then deduce Theorem 1.2 and Theorem 1.3 from it. This is done by showing that the formulae of Lemma 6.2 can be used to check that the entries of W satisfy a given side congruence condition, and to evaluate W using floating-point computations to enough accuracy to verify a sign condition on the solutions. This is of interest if one wishes to recognize positive solutions, as are studied in Hilbert’s 10-th problem. It requires significant extra work to establish these extra side condition properties. In general it is difficult to rigorously prove results that the number of significant digits present after a sequence of floating-point operations, because there is the possibility of losing all significant figures when adding two nearly equal floating-point number of opposite signs. We are able to show this potential cancellation effect cannot occur here, using a priori information about the magnitudes of the quantities being computed at all intermediate steps of the computation.

Appendix A gives bounds on the period lengths (mod M) of solutions to certain second-order linear recurrences. Appendix B gives needed results on floating point computation, concerning bounds on the loss of accuracy in floating-point operations.

2.1. Notations and Conventions.

The length $L(a)$ of an integer $a$ is defined by

$$L(a) = 1 + \log_2(|a| + 1).$$

(2.5)

This measures its binary length, plus one bit for its sign.

The size $||F||$ of the BQDE system (2.1) - (2.3) is given by

$$||F|| := MAX(|a|, |b|, |c|, |d|, |e|, |f|, |G|).$$

(2.6)

The size $||F||$ is related to the length $L(F)$ in (1.7) by

$$\frac{1}{2} \log ||F|| \leq L(F) \leq 9 \log ||F|| + 18,$$

(2.7)

where $\log x$ denotes the natural logarithm.

We also need an analogous notion of size $||M||$ of a matrix $M = [m_{ij}]$ given by

$$||M|| := MAX_{i,j}|m_{ij}|.$$

(2.8)

If $M, N$ are $m \times k$ and $k \times n$ matrices, respectively, then we have the trivial bound

$$||MN|| \leq k \ ||M|| \ ||N||.$$
When counting elementary operations (bit operations) we will sometimes use the function

\[ M(n) := n(\log n)(\log \log n) \]  

(2.10)

arising from the Schönhage-Strassen bound \( O(M(n)) \) for the multiplication of two \( n \) bit binary integers. The \( O \)-symbol has the usual meaning, that \( O(f(n)) \) means \( \leq c|f(n)| \), where \( c \) is an absolute, effectively computable positive constant, which may differ at each occurrence of the \( O \)-symbol.

3. Bounding the Size of the Least Admissible Solution

In this section we bound the size of the least admissible solution to definite and degenerate binary quadratic Diophantine equations, and show that this solution itself may serve as a certificate. The difficult case is that of indefinite binary quadratic Diophantine equations. In this case we obtain an exponential upper bound for the size of the minimal admissible solution.

3.1. Transformation to Generalized Pell Equation \( y_1^2 - Dy_2^2 = g \).

We start with a standard form binary quadratic Diophantine equation

\[ ax_1^2 + 2bx_1x_2 + cx_2^2 + 2dx_1 + 2ex_2 + f = 0 , \]  

(3.1)

with side conditions

\[ x_i \equiv \alpha_i(\text{mod } \Gamma), \ i = 1,2, \]  

(3.2)

\[ x_i \geq 0, \ i = 1,2, \]  

(3.3)

We can immediately simplify (3.1) by an invertible variable change provided \( D \neq 0, c \neq 0 \), where \( D = b^2 - ac \). Following Mathews ([39], 258–260) we introduce new variables

\[ \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} D & 0 \\ b & c \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} be - cd \\ e \end{bmatrix} \]  

(3.4)

We have the identity

\[ y_1^2 - Dy_2^2 = -cD(ax_1^2 + 2bx_1x_2 + cx_2^2 + 2dx_1 + 2ex_2 + f) + g \]

where

\[ g = -c \begin{vmatrix} a & b & d \\ b & c & e \\ d & e & f \end{vmatrix} = -c(2bde - cd^2 - ae^2). \]  

(3.5)

Thus we obtain that when (3.1) holds, then

\[ y_1^2 - Dy_2^2 = g , \]  

(3.6)

and we note the bound

\[ |g| \leq 4||F||^4 . \]  

(3.7)
Inverting the system (3.4) yields

\[
\begin{bmatrix}
cDx_1 \\
cDx_2
\end{bmatrix} = \begin{bmatrix} c & 0 \\ -b & D \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \begin{bmatrix} c(cd - be) \\ c(ae - bd) \end{bmatrix}.
\] (3.8)

This yields the following result.

**Lemma 3.1.** Given a BQDE (3.1) having \( cD \neq 0 \). Let \((y_1, y_2)\) be an integral solution to (3.6), and let a modulus \( \Gamma \) be given. Then \((x_1, x_2)\) given by (3.8) is a rational solution to (3.1). The congruence class of \((y_1, y_2) \mod cD\Gamma\) determines whether \(x_1, x_2\) is integral, and if so specifies \((x_1, x_2) \mod \Gamma\).

### 3.2. Certificates for Definite and Degenerate BQDE’s

We now bound the size of solutions to definite and degenerate binary quadratic Diophantine equations. These bounds are strong enough that the smallest admissible solution will directly serve as a polynomial time certificate.

**Lemma 3.2.** Suppose that a given binary quadratic Diophantine equation system (3.1)-(3.6) is either definite or degenerate. If it has any admissible solutions at all, then it has an admissible solution \((x_1, x_2)\) with

\[
\text{MAX}(|x_1|, |x_2|) \leq 8||F||^4.
\] (3.9)

In particular, this solution satisfies

\[
\text{MAX}(L(x_1), L(x_2)) \leq 8 \log ||F|| + 8.
\] (3.10)

**Proof.** It suffices to prove (3.9) since (3.10) follows on taking logarithms. We treat several cases, of which the first is the generic case.

**Case 1.** \( D \neq 0, \ c \neq 0 \). Then the change of variables (3.4) takes integral solutions of (2.1) to integral solutions of

\[
y_1^2 - Dy_2^2 = g.
\] (3.11)

In the definite case \( D < 0 \), all integer solutions to (3.11) have

\[
\text{MAX}(|y_1|, |y_2|) \leq \sqrt{|g|}.
\]

Using (3.7) and (3.8), we see that all integer solutions to (2.1) must have

\[
\text{MAX}(|x_1|, |x_2|) \leq \sqrt{6}(|F|^3 + ||F||^2) + 2||F||^2 \leq 6||F||^4.
\]

This implies (3.9).

In the degenerate case \( D = h^2 \) is a perfect square and (3.11) becomes

\[
(y_1 + hy_2)(y_1 - hy_2) = g.
\] (3.12)
Hence each integer solution \((y_1, y_2)\) gives rise to a factorization \(g = g_1g_2\) where
\[
\begin{align*}
y_1 + hy_2 &= g_1 \\
y_1 - hy_2 &= g_2.
\end{align*}
\]  
(3.13)

Solving (3.13) for \((y_1, y_2)\) we obtain
\[
\text{MAX}(|y_1|, |y_2|) < |g|
\]
using (3.8) again, this implies (3.10).

**Case 2.** \(D \neq 0, c = 0, a \neq 0\). Interchange \(x_1\) and \(x_2\), proceed as in Case 1.

**Case 3.** \(D \neq 0, a = 0, c = 0\). Then (2.1) multiplied by \(b\) factorizes as
\[
2(bx_1 + e)(bx_2 + d) = 2de - bf.
\]
A similar argument to (3.12), (3.13) yields
\[
\text{MAX}(|x_1|, |x_2|) \leq 3||F||^2
\]
in this case, implying (3.9).

**Case 4.** \(D = 0\). Then
\[
ax_1^2 + 2bx_1x_2 + cx_2^2 = m(\alpha x_1 + \beta x_2)^2
\]
where \(m, \alpha, \beta\) are integers, \(m \neq 0\). Let
\[
z = \alpha x_1 + \beta x_2.
\]  
(3.14)

Suppose first \(\alpha \beta \neq 0\). Substituting \(x_2 = \frac{z - \alpha x_1}{\beta}\) in (2.1) we obtain
\[
m\beta z^2 + 2ez + 2(d\beta - e\alpha)x_1 + f\beta = 0
\]
if \(e\alpha - d\beta = 0\), then we obtain
\[
m\beta z^2 + 2ez + f\beta = 0.
\]
There are two solutions to this quadratic equation, both having
\[
|z| \leq 3||F||^{\frac{3}{2}},
\]
and if (3.14) has any solution it has one with
\[
\text{MAX}(|x_1|, |x_2|) \leq 6||F||^2.
\]
If \(e\alpha - d\beta \neq 0\) then we obtain
\[
x_1 = \frac{m\beta z^2 + 2ez + f\beta}{2(e\alpha - d\beta)} \quad (3.15)
\]
and $x_2 = \frac{z-\alpha x_1}{\beta}$ yields
\[ x_2 = -\left(\frac{\alpha z^2 + 2dz + f\alpha}{2(e\alpha - d\beta)}\right). \tag{3.16} \]

Now the congruence class of $z \,(\text{mod } 2(e\alpha - d\beta)\Gamma)$ determines whether the quantities $(x_1, x_2)$ given by (3.15), (3.16) are integral, and if so specifies their congruence class (mod $\Gamma$). Hence any $(x_1, x_2)$ (mod $\Gamma$) that can occur is given by some $z$ in any block of $2(e\alpha - d\beta)\Gamma$ consecutive values of $z$. Next, we consider what signs of $(x_1, x_2)$ can occur. The sign of $x_1$ changes when the numerator of the right side of (3.15) changes sign, and the sign change occurs at some $z$ with $|z| < 6||F||^2$. A similar result holds for the sign of $x_2$ via (3.16). We conclude that if there is an admissible solution, there will be one with
\[ |z| < 6||F||^2 + |2(e\alpha - d\beta)\Gamma| < 8||F||^3. \tag{3.17} \]

Using (3.15) and (3.16) then gives (3.9).

Finally suppose $\alpha\beta = 0$. If $\alpha = \beta = 0$ then (2.1) is linear and (3.9) is easily verified. If $\alpha = 0$ and $\beta \neq 0$ then use (3.15) and replace (3.16) with
\[ x_2 = \frac{z}{\beta}. \]

The same argument as in the case $\alpha\beta \neq 0$ now proves (3.9). The case $\alpha \neq 0, \beta = 0$ is treated similarly. $\blacksquare$

### 3.3. Indefinite BQDE’s: Standard Form $y_1^2 - Dy_2^2 = g$ with side conditions

Lemma 3.2 produces polynomial size certificates for definite and degenerate BQDE’s. In the sequel it remains to consider indefinite binary quadratic Diophantine equations.

In the indefinite case a standard form equation has $cD \neq 0$, hence the variable change (3.4) is invertible, and necessarily $D \geq 2$. We now reduce the problem of finding admissible solutions to (2.1) to that of finding an admissible solution (suitably defined) to a generalized Pell equation $y_1^2 - Dy_2^2 = g$.

**Lemma 3.3.** Suppose that the BQDE system (2.1)–(2.3) has an admissible solution $x = (x_1, x_2)$. Then one of the following holds.

(i) The solution $x$ satisfies
\[ ||x|| < 200||F||^6. \tag{3.17} \]

(ii) The equation
\[ y_1^2 - Dy_2^2 = g \tag{3.18} \]

with $g$ given by (3.5) has a solution $(y_1, y_2)$ such that
\[ y_1 > 0 \tag{3.19} \]

and one of
\[ y_2 > 0 \text{ and } c(-b + \sqrt{D}) > 0, \tag{3.20} \]
\( y_2 < 0 \) and \( c(b + \sqrt{D}) > 0 \), \hspace{1cm} (3.21)

holds. In addition \((y_1, y_2) \pmod{cD\Gamma}\) satisfies

\[
cy_1 + c(cd - be) \equiv cD\alpha_1 \pmod{cD\Gamma}, \hspace{1cm} (3.22)
\]

\[-by_1 + Dy_2 + c(ac - bd) \equiv cD\alpha_2 \pmod{cD\Gamma}. \hspace{1cm} (3.23)
\]

Conversely, if the system \((3.18)-(3.23)\) has a solution, then \((2.1)\) has an admissible solution.

**Remark.** The lemma shows that either (i) the solution is small enough to write down as a certificate or (ii) else we get definite information on the sign conditions of the solution values \((y_1, y_2)\) of the transformed equation \((3.18)\).

**Proof.** Suppose that the admissible solution \(x\) has

\[ ||x|| \geq 200||F||^6. \hspace{1cm} (3.24) \]

We show that the solution \((y_1, y_2)\) to \((3.18)\) given by \((3.4)\) satisfies \((3.19)-(3.17)\). Now \((3.8)\) shows that the congruence conditions \((3.22), (3.17)\) hold.

We first show \((3.24)\) implies \(|y_1|\) and \(|y_2|\) are large. If \(||y|| < 90||F||^5\), then absolute value estimates in \((3.8)\) yield

\[ ||x|| < 190||F||^6 \]

contradicting \((3.19)\). So \(||y|| \geq 90||F||^5\). If \(|y_1| < 90||F||^5\) then \(|y_2| \geq 90||F||^5\) and

\[ y_1^2 \geq D(y_2)^2 - |g| \]

implies

\[ |y_1| > 89||F||^5, \hspace{1cm} (3.25) \]

so this holds in all cases. Then

\[ y_2^2 \geq \frac{y_1^2 - |g|}{D} \]

implies

\[ |y_2| \geq \frac{88}{\sqrt{D}}||F||^5. \hspace{1cm} (3.26) \]

To prove \((3.19)\) holds, suppose for contradiction that \(y_1 \leq 0\). Then by \((3.4)\)

\[ y_1 - (be - cd) = Dx_1 \geq 0 \]

so

\[ |y_1| \leq |be - cd| < 2||F||^2, \]

contradicting \((3.25)\). Hence \((3.19)\) holds.

To prove one of \((3.20)\) or \((3.21)\) holds, note \((3.18)\) yields

\[ |y_1 + \sqrt{D}y_2| \leq 6||F||^4. \hspace{1cm} (3.27) \]
Since
\[ y_1 + \sqrt{D} y_2 = (y_1 - \sqrt{D} y_2) + 2\sqrt{D} y_2 , \]
(3.20) implies that
\[ \text{MAX}(|y_1 + \sqrt{D} y_2|, |y_1 - \sqrt{D} y_2|) \geq 88||F||^5 . \]
Hence (3.21) yields
\[ \text{MIN}(|y_1 + \sqrt{D} y_2|, |y_1 - \sqrt{D} y_2|) \leq \frac{6}{88}||F||^{-1} < ||F||^{-1} . \] (3.28)

Consequently \( y_1 \) is very close to one of \( \pm \sqrt{D} y_2 \).

We consider first the case that
\[ |y_1 - \sqrt{D} y_2| < ||F||^{-1} . \] (3.29)
Necessarily \( y_2 > 0 \), and (3.8) yields
\[ cD x_2 = -by_1 + Dy_2 + c(ae - bd) \]
\[ = \sqrt{D}(-b + \sqrt{D})y_2 + \xi \] (3.30)
where
\[ |\xi| \leq |b|||F||^{-1} + 2||F||^3 < 9||F||^4 . \] (3.31)

We claim
\[ | - b + c\sqrt{D}| \geq \frac{1}{3}||F||^{-1} . \] (3.32)
Indeed, suppose \( | - b + \sqrt{D}| < 1 \). If so, then \( | - b - \sqrt{D}| > 1 \). So
\[ | - b + \sqrt{D}| \geq \frac{|b^2 - D|}{| - b - \sqrt{D}|} \geq \frac{1}{3||F||} , \]
using \( D < 2||F||^2 \). Now (3.26) and (3.32) imply
\[ |\sqrt{D}(-b + \sqrt{D})y_2| > 29||F||^4 . \] (3.33)

Thus \( \xi \) is too small to change the sign of the two terms on opposite sides of (??), so that
\[ \text{sign} (cD x_2) = \text{sign} (\sqrt{D}(-b + \sqrt{D})y_2) . \] (3.34)
Since \( x_2 \geq 0 \), this yields
\[ \text{sign} (y_2) = \text{sign} (c(-b + \sqrt{D})) \]
which proves (3.20) holds in this case. Next we consider the case
\[ |y_1 + \sqrt{D} y_2| < ||F||^{-1} . \]
Necessarily \( y_2 < 0 \) in this case, and an analysis similar to the previous case shows that
\[ \text{sign} (y_2) = - \text{sign} (c(b + \sqrt{D})) \]
which shows (3.21) holds in this case.

To prove the converse in (ii), suppose first that we have a solution \((y_1, y_2)\) to conditions (3.18)–(3.23) which satisfies (3.20). Let \((t_1, u_1)\) be the minimal positive solution to Pell’s equation

\[
t^2 - Du^2 = 1 .
\]

Let

\[
t_k + u_k \sqrt{D} = (t_1 + u_1 \sqrt{D})^k .
\]

It is well-known that \((t_k, u_k)\) satisfy (3.35) and that for any modulus \(M\) there exists an integer \(P(M)\), the period modulo \(M\), such that

\[
t_k \equiv 1 \pmod{M} \\
u_k \equiv 0 \pmod{M}
\]

whenever

\[
P(M) \mid k .
\]

(See Appendix A.) Certainly \(t_k, u_k \to \infty\) as \(k \to \infty\). We now set

\[
y_1^* + y_2^* \sqrt{D} = (y_1 + y_2 \sqrt{D})(t_k + u_k \sqrt{D})
\]

where \(P(cd\Gamma) \mid k\). Note that since \(y_1, y_2, t_k, u_k\) are all positive, \(y_1^* \geq t_k, y_2^* \geq u_k\). By picking \(k\) large enough, we may guarantee

\[
\text{MIN} (y_1^*, y_2^*) > 88||F||^5 .
\]

Furthermore (3.31) applied with \(M = cD\Gamma\) guarantees that

\[
y_i^* \equiv y_i (\text{mod } cd\Gamma) \quad i = 1, 2 .
\]

Also (3.38) and (3.35) guarantee that \((y_1^*, y_2^*)\) satisfies the generalized Pell equation (3.18). Now let \((x_1^*, x_2^*)\) be the rational solution to (2.1) associated to \((y_1^*, y_2^*)\) by (3.8). The congruence condition above implies that \((y_1^*, y_2^*)\) satisfies the congruences (3.22), (3.23) hence \((x_1^*, x_2^*)\) is an integer solution and

\[
x_i^* \equiv \alpha_i (\text{mod } \Gamma) .
\]

We claim that \((x_1^*, x_2^*)\) is nonnegative. If so \((x_1^*, x_2^*)\) is the desired admissible solution. Using (3.8) and (3.39), we obtain

\[
Dx_1^* \geq y_1^* - |be - cd| > 86||F||^5 ,
\]

hence \(x_1^* > 0\). The bound (3.39) implies that the argument (3.27)–(3.28) is valid, and since \(y_1^* > 0, y_2^* > 0\) we obtain

\[
|y_1^* - \sqrt{D} y_2^*| < ||F||^{-1} .
\]

The argument (3.24)–(3.28) assumed only the truth of (3.26), so it is valid here as well and we obtain

\[
\text{sign} (cDx_2) = \text{sign} (\sqrt{D}(-b + c\sqrt{D})y_2) .
\]

18
We are given \( y_2 > 0 \) by (3.20), and \( c(-b + c \sqrt{D}) > 0 \), hence \( x_2 > 0 \) follows in this case.

Now assume a solution exists to (3.18)–(3.23) which satisfies (3.21). In this case set
\[
y_1^* + y_2^* \sqrt{D} = (y_1 + y_2 \sqrt{D})(t_k - u_k \sqrt{D})
\]
where \( P(cD \Gamma) \mid k \). Now \( y_1^* > 0, y_2^* < 0 \), and since \( t_k > 0, u_k > 0 \), we obtain \( y_1^* > 0, y_2^* < 0 \) and \( y_1^* \geq t_k, |y_2^*| \geq |u_k| \). By picking \( k \) large enough, we ensure that
\[
\text{MIN}(|y_1^*|, |y_2^*|) > 88 ||F||^5 . \tag{3.41}
\]
Also \( y_i^* \equiv y_i \pmod{cD \Gamma} \), and \((y_1^*, y_2^*)\) satisfies (3.18). An analogous argument to the case treated above now shows that \((x_1^*, x_2^*)\) associated to this \((y_1^*, y_2^*)\) is an admissible solution to (2.1).

\[\blacksquare\]

3.4. Generalized Pell Equation: Exponential upper bound

We next need an upper bound on the size of admissible solutions to the generalized Pell equation
\[
y_1^2 - Dy_2^2 = g . \tag{3.42}
\]
The set of solutions to this equation has a simple form, related to solutions of the Pell equation
\[
t^2 - Du^2 = 1 .
\]
Let \((t_1, u_1)\) denote the minimal positive solution to Pell’s equation, the fundamental solution, and write
\[
\epsilon = t_1 + u_1 \sqrt{D} . \tag{3.43}
\]
This is a unit in the real quadratic field \( \mathbb{Q}(\sqrt{D}) \), which is the fundamental unit in the real quadratic order \( O_D = \mathbb{Z}[1, \sqrt{D}] \). Note that \( \bar{\epsilon} := t_1 - u_1 \sqrt{D} = \epsilon^{-1} \) satisfies \( 0 < \bar{\epsilon} < 1 \). We recall the following upper bound on the size of the fundamental solution.

**Proposition 3.1.** (Hua [30]) Let \((t_1, u_1)\) be the minimal positive solution to Pell’s equation \( t^2 - Du^2 = 1 \). If \( \epsilon = t_1 + u_1 \sqrt{D} \) then
\[
\epsilon < D^{\sqrt{D}} . \tag{3.44}
\]

Now we call a solution \((y_1, y_2)\) to (3.42) basic provided \( \eta = y_1 + y_2 \sqrt{D} \) has
\[
1 \leq |\eta| < \epsilon . \tag{3.45}
\]
the generalized Pell equation states \( \eta \bar{\eta} = g \). We have the following finiteness result.

**Lemma 3.4.** For a positive squarefree \( D \) the complete set of solutions to
\[
y_1^2 - Dy_2^2 = g
\]
is given by \((y_1, y_2) = (y_{1,k}, y_{2,k})\) with
\[
y_{1,k} + y_{2,k} \sqrt{D} = \eta \epsilon^k \tag{3.46}
\]
for some basic solution \( \eta \) and some integer \( k \). There are only a finite number of basic solutions.
Proof. Suppose \((y_1, y_2)\) is a solution to (3.42). Then for some integer \(k\),
\[
e^k \leq |y_1 + y_2\sqrt{D}| < e^{k+1}.
\]
Consequently for the correct choice of sign
\[
\eta = \epsilon^{-k}(x_1 + x_2\sqrt{D})
\]
(3.47) is a basic solution.

There are only a finite number of basic solutions since
\[
\bar{\eta} \equiv y_1 - y_2\sqrt{D} = g/\eta
\]
using (3.42). Hence
\[
|y_1| \leq |\eta| + |\bar{\eta}| < \epsilon + |g|
\]
(3.48)
\[
|y_2| \leq \frac{\epsilon + |g|}{\sqrt{D}},
\]
as required. \(\square\)

We apply Lemma 3.4 to establish an exponential upper bound on the bit complexity of writing down the least admissible solution to a generalized Pell equation, if one exists, in binary.

Lemma 3.5. Suppose that the equation
\[
y_1^2 - Dy_2^2 = g
\]
(3.49) has an integral solution \((y_1, y_2)\) with
\[
y_i \equiv \alpha_i \pmod{M}, \ i = 1, 2,
\]
and with \((y_1, y_2)\) having prescribed signs. Then it has such a solution with
\[
MAX(\log |y_1|, \log |y_2|) \leq 9||E||^{3/2}(\log ||E||)^2
\]
(3.50)
where \(||E|| = MAX(|D|, |g|, M)).

Proof. Consider the set of solutions \((y_{1,k}, y_{2,k})\) to (3.49) where
\[
y_{1,k} + y_{2,k}\sqrt{D} = \eta\epsilon^k,
\]
(3.51)
k runs through the integers, and
\[
\eta = y_{1,0} + y_{2,0}\sqrt{D},
\]
where \((y_{1,0}, y_{2,0})\) is a fixed basic solution of (3.49). We consider these solutions from the viewpoint of their sign patterns and congruence classes (mod \(M\)).
For sign patterns, we will show that the signs of \((y_{1,k}, y_{2,k})\) become constant for all sufficiently large positive \(k\), and also constant for negative \(k\) with \(|k|\) sufficiently large. We show sign \((y_{1,k})\) is constant for all \(k\) with
\[
k \geq \log |g| ,
\]
and is constant for all \(k\) with
\[
k \leq -\log |g| - 2 .
\]
The same holds for \(y_{2,k}\). To do this, we use the standard notation \(\bar{\alpha} = a - b\sqrt{D}\) for the algebraic conjugate of \(\alpha = a + b\sqrt{D}\). Suppose (3.51) holds. Then
\[
y_{1,k} = \frac{1}{2}(\eta \epsilon^k + \bar{\eta} \bar{\epsilon}^k) .
\]
Now \(\bar{\eta} = g/\eta\) by (3.49) and \(\bar{\epsilon} = \epsilon^{-1}\). Now suppose
\[
k \geq \log |g| \geq (\log \epsilon)^{-1} \log |g|
\]
since the smallest \(\epsilon\) that occurs is \(\epsilon = 2 + \sqrt{3}\) for \(D = 3\), and so \(\epsilon > \epsilon\). Then since \(1 < \eta \leq \epsilon\), \(0 < \bar{\epsilon} < 1\),
\[
|\eta \epsilon^k| \geq |g\epsilon^{-1}| = |\bar{\eta} \bar{\epsilon}^k| .
\]
In this case \(y_{1,k}\) has the same sign as \(\eta\), and is constant. Similarly when (3.53) holds we find \(y_{1,k}\) has the same sign as \(\bar{\eta}\). (Use the fact \(1 \leq |\eta| \leq \epsilon\).) Similar arguments apply to \(y_{2,k}\) using
\[
y_{2,k} = \frac{1}{2\sqrt{D}}(\eta \epsilon^k - \bar{\eta} \bar{\epsilon}^k) .
\]

For congruence conditions, in Appendix A we show that for
\[
t_k + u_k \sqrt{D} = \epsilon^k
\]
and any modulus \(M\), the sequences \(\{t_k\}, \{u_k\}\) formed when \(k\) varies are both periodic (mod \(M\), and that the minimal positive period \(P(M)\) of both series jointly has
\[
P(M) \leq 2M(\log M + 1) .
\]
Using (3.51) this guarantees that the sequences \(\{y_{1,k}\}, \{y_{2,k}\}\) are both periodic (mod \(M\) with period \(P(M)\).

Combining these results, we find that all possible combinations of sign patterns and congruence conditions (mod \(M\) that occur for \((y_{1,k}, y_{2,k})\) in the sequence (3.51) occur for some \(k\) with
\[
|k| \leq 4M(\log M) + \log g + 2 .
\]
In this circumstance
\[
|y_{1,k}| = \frac{1}{2}|\eta \epsilon^k + \bar{\eta} \bar{\epsilon}^k|
\]
\[
\leq \frac{1}{2}(\epsilon^{k+1} + |g|) \leq |g|\epsilon^{k+1}
\]
using (3.45), \(|\tilde{\eta}| \leq |g|\), and \(\bar{\epsilon} < 1\). Hence

\[
\log |y_{1,k}| \leq \log g + (k + 1) \log \epsilon \\
\leq (4M \log M + 2 \log g + 3) \sqrt{D} \log D
\]

(3.59)

using Proposition 3.1. Using

\[
|y_{2,k}| = \frac{1}{2\sqrt{D}} |\eta \epsilon^k - \tilde{\eta} \bar{\epsilon}^k|
\]

we obtain the same bound (3.53) for \(|y_{2,k}|\). The bound (3.53) implies

\[
\text{MAX}(\log |y_1|, \log |y_2|) \leq 9||E||^{3/2}(\log ||E||)^2.
\]

By Lemma 3.4 all integer solutions to (3.43) fall in one of the sequences (3.51), and this proves (3.50). \(\square\)

3.5. Indefinite BQDE: Exponential upper bound

Using Lemma 3.5, we obtain an upper bound on the bit complexity of an admissible solution \((x_1, x_2)\) to a general indefinite BQDE. In terms of the number of binary digits \(\log |x_1| + \log |x_2|\) needed to write down this solution, it is singly exponential in terms of the input size \(L(F)\), since \(L(F)\) is proportional to \(\log ||F||\). However, as a bound on the solution size \(\max(|x_1|, |x_2|)\) it is \textit{double exponential} in terms of the input size \(L(F)\).

Lemma 3.6. \textit{Any indefinite binary quadratic Diophantine equation that has an admissible solution has such a solution \((x_1, x_2)\) with

\[
\text{MAX}(\log |x_1|, \log |x_2|) \leq 210||F||^6(\log ||F||)^2.
\]

(3.60)

Proof. We apply the results of Lemma 3.3 and Lemma 3.5. Using (3.41) a solution \((y_1^*, y_2^*)\) of (3.49) will correspond to an admissible solution \((x_1^*, x_2^*)\) of (2.1) provided (ii) of Lemma 3.3 holds and

\[
\text{MIN}(|y_1^*|, |y_2^*|) \geq 88||F||^5.
\]

By Lemma 3.4 we may write \((y_1^*, y_2^*) = (y_{1,k}, y_{2,k})\) for some \(\eta\) and \(k\) in (3.51). But for

\[
k \geq 7 \log ||F|| + 11
\]

we have

\[
|y_{1,k}| \geq \frac{1}{2}(|\eta \epsilon^k| - |\tilde{\eta} \bar{\epsilon}^k|)
\]

(3.61)

\[
\geq \frac{1}{2} [\epsilon^k - |g|]
\]

(3.62)

\[
\geq e^7 \log ||F|| + 10 - |g| \geq 88||F||^5.
\]

(3.63)
A similar bound holds for $|y_{2,k}|$. We obtain the same bound for

$$k \leq -7 \log ||F|| - 13$$

using

$$|y_{1,k}| \geq \frac{1}{2}(||\overline{\eta}e^k| - |\eta e^k||)$$

$$\geq \frac{1}{2}|e^{k-1} - 1|.$$ 

and similarly for $y_{2,k}$. Combining these inequalities with the argument of Lemma 3.5 we find that if (3.1) has an admissible solution $(x_1, x_2)$ it has one whose corresponding solution $(y_{1,k}, y_{2,k})$ to (3.49) (for some $\eta$) has

$$|k| \leq 4|cD\Gamma| \log cD \Gamma \log ||F|| + 20$$

$$\leq 90||F||^4 \log ||F||.$$ 

The same argument as Lemma 3.5 then gives

$$\text{MAX}(\log y_{1,k}, \log y_{2,k}) \leq 100||F||^5 \log ||F||^2$$

and (3.8) then gives (3.60). \(\square\)

4. Integral Binary Quadratic Forms

Section 3 essentially reduces the problem of finding succinct certificates in the indefinite case to that of finding such certificates for admissible solutions of the generalized Pell equation

$$y_1^2 - Dy_2^2 = g.$$ 

The method of Gauss now relates solutions of this equation to the theory of integral binary forms; a solution to the equation will show the equivalence of two particular binary forms, as given in section §4.1. Our problem will then be transformed to finding succinct certificates establishing such equivalence.

A final step before applying the theory of integral binary forms is to reduce to the special case in which $y_1, y_2$ are relatively prime; such $(y_1, y_2)$ are called primitive solutions to (4.1). If $y_1, y_2$ is a solution to (4.1) and $(y_1, y_2) = h$, set

$$z_1 = \frac{y_1}{h}, \quad z_2 = \frac{y_2}{h}, \quad G = \frac{g}{h^2}$$

and obtain the equation

$$z_1^2 - Dz_2^2 = G$$

$$(z_1, z_2) = 1.$$  

(4.3)
If we specify congruence conditions on \( z_1, z_2 \) (mod \( cD \Gamma \)) then we certainly know \( y_1, y_2 \) (mod \( cD \Gamma \)).

An *integral binary quadratic form* \( Q = [a, 2b, c] \) is given by

\[
Q(x_1, x_2) = ax_1^2 + 2bx_1x_2 + cx_2^2 = x^T Q x.
\]

Here

\[
x^T = [x_1, x_2]
\]

and we call

\[
Q = \begin{bmatrix} a & b \\ b & c \end{bmatrix}
\]

the symmetric matrix *associated* to the form \( Q \). The *determinant* \( D \) of a form \( Q = [a, 2b, c] \) is given by

\[
D = b^2 - ac = - \det(Q).
\]

(4.4)

A form \( Q \) is *primitive* if the greatest common divisor \( (a, b, c) = 1 \). Primitive forms subdivide into *properly primitive forms* which have \( (a, 2b, c) = 1 \) and it *improperly primitive forms* which have \( (a, b, c) = 1 \) but \( (a, 2b, c) = 2 \). We shall mainly deal with properly primitive forms in the rest of this paper.

We say a form \( Q \) *primitively represents* an integer \( G \) provided

\[
Q(z_1, z_2) = G
\]

for two relatively prime integers \( z_1, z_2 \). The *identity form* \( I = ID \) is \([1, 0, -D]\), and it is properly primitive. In this terminology \([1, 3]\) asserts that the identity form primitively represents \( G \).

### 4.1. Equivalence of Indefinite Binary Quadratic Forms

Gauss transformed the question of (i) primitive representation of an integer by a form to that of (ii) determining the equivalence of two forms. A form \( Q_1 \) is *(properly) equivalent* to a form \( Q_2 \) if there is a \( 2 \times 2 \) integer matrix \( S \in SL(2, \mathbb{Z}) \) (i.e. \( \det(S) = 1 \)) such that

\[
S^T Q_1 S = Q_2.
\]

(4.6)

This is an equivalence relation, and we denote it by \( Q_1 \sim Q_2 \), via \( S \). This equivalence relation preserves the determinant \( D \), the property of being a properly primitive form, and the property of primitively representing a given integer \( G \).

**Lemma 4.1.** Let \( z_1, z_2 \) be a primitive integer solution to the generalized Pell equation

\[
z_1^2 - Dz_2^2 = G.
\]

where \( D \) is arbitrary. Then there exists \( z_3, z_4 \) giving a (proper) reduction matrix

\[
Z = \begin{bmatrix} z_1 & z_3 \\ z_2 & z_4 \end{bmatrix} \in SL(2, \mathbb{Z}) \text{ whose first column is } (z_1, z_2)^T \text{ which shows the identity form}
\]

\( I_D = [1, 0, -D] \) of determinant \( D \) is properly equivalent to a form

\[
Q_0 = [G, 2B, C],
\]

(4.7)
i.e \( I \sim Q_0 \), whose coefficients satisfy the bound

\[
\text{MAX}(2|B|, |C|) \leq |D| + 4G^2.
\]

For any choice of \( B, C, z_3, z_4 \) satisfying (4.8) we have

\[
\text{MAX}(|z_i|) \leq 6(|z_1| + |D| + G^2).
\]

\[
\text{MIN}(|z_i|) \geq \frac{1}{3\sqrt{D}}(|z_1| - 5|D| - 5G^2).
\]

**Proof.** Choose \( z^*_3, z^*_4 \) so that \( z_1z^*_4 - z_2z^*_3 = 1 \). Then \( S^* = \begin{bmatrix} z_1 & z^*_3 \\ z_2 & z^*_4 \end{bmatrix} \) shows \( I_D \sim Q^* \) where

\[
I_D = \begin{bmatrix} 1 & 0 \\ 0 & -D \end{bmatrix}
\]

and

\[
Q^* = [G, 2B^*, C^*],
\]

that is

\[
Q^* := \begin{bmatrix} G & B^* \\ B^* & C^* \end{bmatrix} = (S^*)^t \begin{bmatrix} 1 & 0 \\ 0 & -D \end{bmatrix} S^*
\]

Now select \( \lambda \) so that \( S_1 := \begin{bmatrix} 1 & \lambda \\ 0 & 1 \end{bmatrix} \) shows \( Q^* \approx Q_0 \) where

\[
Q_0 = [G, 2B, C], \quad 0 < B < |G|.
\]

Then

\[
C = \frac{D - B^2}{G}
\]

so

\[
|C| \leq |D| + G^2,
\]

whence (4.7) and (4.8) hold. Also \( I \sim Q_0 \) via

\[
Z := S^*S_1 \equiv \begin{bmatrix} z_1 & z_3 \\ z_2 & z_4 \end{bmatrix},
\]

where we have \( z_3 = z^*_3 + \lambda z_1, z_4 = z^*_4 + \lambda z_2 \).

Now suppose that \( Z \) is chosen to satisfy (4.7), (4.8) with \( I \sim Q_0 \). To bound the sizes of \( z_2, z_3, z_4 \) we observe that

\[
G = z^2_1 - Dz^2_2
\]

\[
B = z_1z_3 - Dz_2z_4
\]

\[
C = z^2_3 - Dz^2_4
\]

Then (4.12) gives

\[
||z_1| - \sqrt{D}|z_2|| = \frac{|G|}{|z_1| + \sqrt{D}|z_2|} \leq |G|.
\]

Hence

\[
\frac{1}{\sqrt{D}}(|z_1| - |G|) < |z_2| < \frac{1}{\sqrt{D}}(|z_1| + |G|).
\]
Similar arguments using (4.11) and (4.14) show
\[ ||z_3| - \sqrt{D}|z_4|| \leq \frac{|C|}{|z_3| + \sqrt{D}|z_4|} \leq |C| \leq |D| + G^2, \] (4.17)
yielding
\[ \frac{1}{\sqrt{D}}(|z_3| - |D| - G^2) \leq |z_4| \leq \frac{1}{\sqrt{D}}(|z_3| + |D| + G^2) . \] (4.18)

Next note that
\[ B = \frac{1}{2}\{(z_1 + \sqrt{D}z_2)(z_3 - \sqrt{D}z_4) + (z_1 - \sqrt{D}z_2)(z_3 + \sqrt{D}z_4)\} \]
\[ = \frac{1}{2}G \left\{ \frac{z_1 + \sqrt{D}z_2}{z_3 + \sqrt{D}z_4} + \frac{z_3 + \sqrt{D}z_4}{z_1 + \sqrt{D}z_2} \right\}. \] (4.19)

Viewing this as \( B = \frac{1}{2}G(x + \frac{1}{x}) \), then \( 0 < B < |G| \) gives
\[ \frac{1}{3} < |x| < 3 . \]

If \( z_3, z_4 \) have the same sign, then these bounds yield
\[ \frac{1}{3}(|z_1| + \sqrt{D}|z_2|) < |z_3| + \sqrt{D}|z_4| < 3(|z_1| + \sqrt{D}|z_2|). \] (4.20)

Then using (4.16) we obtain
\[ |z_3| < 6(|z_1| + |G|) \] (4.21)
and
\[ |z_4| < \frac{6}{\sqrt{D}}(|z_1| + |G|) . \] (4.22)

Combining (4.17) and (4.20), we obtain
\[ 2|z_3| > \frac{1}{3}(|z_1| + \sqrt{D}|z_2|) - (|D| + G^2) \]

Now we can apply (4.15) to obtain
\[ 2|z_3| > \frac{1}{3}(|z_1| - 3|D| - 3G^2) + \frac{1}{3}(|z_1| - |G|) \geq \frac{1}{3}(2|z_1| - 3|D| - 4G^2). \] (4.23)

Substituting this bound in the first inequality in (4.18) yields
\[ |z_4| > \frac{1}{3\sqrt{D}}(2|z_1| - \frac{9}{2}|D| - 5G^2) . \] (4.24)

If \( z_3, z_4 \) have opposite signs, we use
\[ B = \frac{1}{2}G \left\{ \frac{z_1 - \sqrt{D}z_2}{z_3 - \sqrt{D}z_4} + \frac{z_3 - \sqrt{D}z_4}{z_1 - \sqrt{D}z_2} \right\} \]
and again conclude the bounds (4.20)–(4.24) hold by similar arguments. \( \square \)
4.2. Reduction of Indefinite Binary Quadratic Forms

The problem of equivalence of indefinite forms, to determine if \( I_D \sim Q_0 \) given by Lemma 4.1 may be simplified further.

Gauss introduced a notion of reduced indefinite form, whose coefficients are bounded in absolute value by \( 2\sqrt{D} \), cf. (4.26) below. He gave a reduction algorithm which shows that each indefinite form is properly equivalent to some reduced form. This algorithm runs in polynomial time and is similar to the ordinary continued fraction algorithm. Application of this reduction algorithm permits the problem of equivalence of indefinite forms to be simplified to determining equivalence of reduced indefinite forms.

**Definition 4.1.** An indefinite form \( Q = [a, 2b, c] \) is *reduced* when its coefficients satisfy the bounds

\[
0 < b < \sqrt{D} \\
\sqrt{D} - b < |a| < \sqrt{D} + b.
\]

(4.25)

The reduction inequalities (4.25) imply that any (indefinite reduced form \( Q_{\text{red}} \) satisfies

\[
\sqrt{D} - b < |c| < \sqrt{D} + b
\]

so that

\[
|Q_{\text{red}}| < 2\sqrt{D}.
\]

(4.26)

There are in general many different reduced forms equivalent to a given form; this is the subject of §5.

Gauss’s algorithm for reducing an indefinite form runs in polynomial time, as given by the following bound.

**Proposition 4.1.** (Indefinite BQF Reduction Bound) *Given any indefinite form \( Q \), there exists a reduced form \( Q_{\text{red}} \) and a reduction matrix \( S_1 \in SL(2, \mathbb{Z}) \) such that \( Q \sim Q_{\text{red}} \) via \( S_1 \), and \( S_1 \) satisfies

\[
\log ||S_1|| = O(\log ||Q||).
\]

(4.27)

There is a reduction procedure which, when given \( Q \) will obtain \( Q_{\text{red}} \) and \( S_1 \) which takes at most \( O(\log ||Q|| M(\log( ||Q|| )) \) elementary operations.

**Proof.** This bound is obtained in Lagarias [34, Theorem 4.1].

We note that the (indefinite) identity form \( I_D = [1, 0, -D] \) is not reduced in the sense above.

**Definition 4.2.** The *reduced identity form* \( \bar{I} := \bar{I}_D \) of positive, nonsquare determinant \( D > 0 \) is given by

\[
\bar{I}_D = [1, 2\lambda, \mu]
\]

(4.28)
by \( I_D \sim \tilde{I}_D \) via
\[
S^* = \begin{bmatrix} 1 & \lambda \\ 0 & 1 \end{bmatrix}, \quad \text{with} \quad \lambda = \lfloor \sqrt{D} \rfloor,
\]
that is, \((S^*)^T Q I S^* = Q \tilde{I}\) and \(\mu = \lambda^2 - D\).

The reduced identity form \( \tilde{I}_D \) is a reduced form in the sense above, and is obtained by one step of the reduction algorithm in Proposition 4.1 applied to the identity form \( I_D \).

4.3. Admissible Solutions of Indefinite BQDE’s and Reduced Forms

The results obtained so far are summarized in the following lemma, which will provide one part of the certificates. This lemma shows an equivalence between (i) existence of an admissible solution to an indefinite BQDE (3.1), and (ii) equivalence of the reduced identity form \( \tilde{I}_D \) to a particular reduced form \( Q_{\text{red}} \) of determinant \( D \), constructed using this BQDE.

Lemma 4.2. Consider the generalized Pell equation \( E(y_1, y_2) = 0 \) given by
\[
y_1^2 - D y_2^2 = g,
\]
with \( D > 0 \) not a perfect square. This equation has a solution \((y_1, y_2)\) satisfying
\[
y_i \equiv \alpha_i \pmod{M}, \quad i = 1, 2,
\]
and with specified sign conditions
\[
\text{sign}(y_i) = \text{sign}(i), \quad i = 1, 2,
\]
with \( \text{sign}(1), \text{sign}(2) \) given signs, if and only if there exist integers \( h, B, C \) and \( 2 \times 2 \) matrices \( S, W \in SL(2, \mathbb{Z}) \) having the following properties.

(i) \( h \) is a positive integer and \( G = g/h^2 \) is an integer.
(ii) The quadratic form \( Q_0 = [G, 2B, C] \) is properly primitive of determinant \( D \).
(iii) The matrix \( S \in SL(2, \mathbb{Z}) \) shows
\[
Q_0 \sim Q_{\text{red}} \quad \text{via} \quad S
\]
where \( Q_{\text{red}} \) is a reduced form.
(iv) The matrix \( W \in SL(2, \mathbb{Z}) \) shows
\[
\tilde{I}_D \sim Q_{\text{red}} \quad \text{via} \quad W
\]
where \( \tilde{I}_D \) is the reduced identity form and \( Q_{\text{red}} \) is given by (4.33).
(v) Define \( U = \begin{bmatrix} u_1 & u_3 \\ u_2 & u_4 \end{bmatrix} \in SL(2, \mathbb{Z}) \) by
\[
U := \begin{bmatrix} 1 & -\lambda \\ 0 & 1 \end{bmatrix} W S^{-1}
\]

28
where $\lambda = \lceil \sqrt{D} \rceil$. The congruence class of $W (\mod M)$ is such that

$$hu_i \equiv \alpha_i (\mod M), \ i = 1, 2.$$  \hspace{1cm} (4.36)

In addition

$$\text{sign}(u_i) = \text{sign}(i), \ i = 1, 2.$$ \hspace{1cm} (4.37)

In fact $y_1 = hu_1, y_2 = hu_2$ then satisfy (4.30)–(4.32). If such an admissible solution exists, and we set

$$||E|| := \text{MAX}(D, |g|, M)$$

then there exist integers $h, B, C$ and $2 \times 2$ matrices $S, W$ having properties (i)–(v) and satisfying the bounds:

(vi) \hspace{2cm} \log h = O(\log ||E||) \hspace{1cm} (4.38)

(vii) \hspace{2cm} \text{MAX}(|B|, |C|) \leq D + 4g^2, \hspace{1cm} (4.39)

(viii) \hspace{2cm} \log ||S|| = O(\log ||E||), \hspace{1cm} (4.40)

(ix) \hspace{2cm} \log ||W|| = O(||E||^{3/2}(\log ||E||)^2). \hspace{1cm} (4.41)

Proof. Suppose that properties (i)–(v) above hold. We check that $y_1 = hu_1, y_2 = hu_2$ satisfy (4.30)–(4.32). The congruence and sign conditions hold by (4.36), (4.37), since $h > 0$ by (i).

To show (4.30) holds, we observe that

$$I_D \sim Q_0 \text{ via } U$$

where $I_D = [1, 0, -D]$ and $U$ is given by (4.38). For

$$(S^*)^{-1} = \begin{bmatrix} 1 & -\lambda \\ 0 & 1 \end{bmatrix}$$

where $S^*$, given in (4.29), shows $I_D \sim \tilde{I}_D, W$ shows $\tilde{I}_D \sim Q_{\text{red}}$ by (iv), and $S^{-1}$ shows $Q_{\text{red}} \sim Q_0$ by (iii). Thus

$$u^T \begin{bmatrix} 1 & 0 \\ 0 & -D \end{bmatrix} U = \begin{bmatrix} G & B \\ B & C \end{bmatrix}. \hspace{1cm} (4.42)$$

Examining the upper left corner of this identity gives

$$u_1^2 - Du_2^2 = G.$$ \hspace{1cm} (4.42)

Using $G = g/h^2$ by (i), (4.30) follows.

Now suppose an admissible solution $(y_1, y_2)$ satisfying (4.30)–(4.32) exists. Using Lemma 3.5 we may suppose

$$\text{MAX}(\log |y_1|, \log |y_2|) < 9||E||^{3/2}(\log ||E||)^2. \hspace{1cm} (4.43)$$
Set 
\[ h = \text{g.c.d.} (y_1, y_2) \]
and \( G = g/h^2 \), establishing (i). Since \( h \) divides \( g \),
\[ \log h = O(\log ||E||) \]
giving the bound (vi). Letting \( z_1 = \frac{y_1}{h}, z_2 = \frac{y_2}{h} \), we may apply Lemma 4.1 to produce \( B, C \) satisfying (ii) and the bound(vii), and a matrix \( Z \in SL(2, \mathbb{Z}) \) showing \( I_D \sim Q_0 \), and the lemma gives bounds \((4.5)\) yielding
\[ \log ||Q_0|| = O(\log ||E||) \] (4.44)
and the bound \((4.9)\) gives
\[ \log ||Z|| = O(||E||^{3/2} (\log ||E||)^2) \] (4.45)
using \((4.43)\), since \( z_1 \) divides \( y_1 \). Proposition \((4.4)\) and \((4.4)\) produces an \( S \in SL(2, \mathbb{Z}) \) satisfying (iii), (viii). Take \( U = Z \) in \((4.35)\) and use this equation to define \( W \), namely
\[ W := \begin{bmatrix} 1 & \lambda \\ 0 & 1 \end{bmatrix} ZS. \] (4.46)
The \( u_1 = z_1, u_2 = z_2 \) so (v) holds. Also \( \begin{bmatrix} 1 & \lambda \\ 0 & 1 \end{bmatrix} \) shows \( \tilde{I}_D \sim I_D, Z \) shows \( I_D \sim Q_0 \) and \( S \) shows \( Q_0 \approx Q_{\text{red}} \), hence \((4.40)\) shows (iv) holds. Finally \((4.46)\) gives
\[ ||W|| \leq 8||S^*|| ||Z|| ||S|| \]
where \( S^* = \begin{bmatrix} 1 & \lambda \\ 0 & 1 \end{bmatrix} \) with \( \lambda = [\sqrt{D}] \). Hence
\[ \log ||W|| = O(||E||^{3/2} (\log ||E||)^2) \]
using \((4.45)\) and the already established (viii). \( \square \)

5. Equivalence of Reduced Indefinite Binary Quadratic Forms

The problem has now been simplified to that of finding a particular matrix \( W \) which demonstrates the equivalence of the reduced identity form \( \tilde{I}_D \) and a reduced form \( Q_{\text{red}} \). Gauss [11, Arts. 183–205] showed that the set of all reduced indefinite forms of determinant \( D \) has a simple structure, which we describe below.

5.1. Cycles of Indefinite Reduced Forms and the Principal Cycle

An (indefinite) reduced form \( Q_1 = [a_1, 2b_1, c_1] \) is said to have as a right neighbor the reduced form \( Q_2 = [a_2, 2b_2, c_2] \) provided \( a_2 = c_1 \). In this case \( Q_2 \) is unique and \( Q_1 \sim Q_2 \) via \( S \) where
\[ S = \begin{bmatrix} 0 & 1 \\ -1 & \lambda \end{bmatrix}, \] (5.1)
in which \( \lambda \) is specified by
\[
-\sqrt{D} - b_1 < \lambda c_1 < -\sqrt{D} - b_1 + |c_1|.
\]

(5.2)

Travelling to right neighbors results in traversing a cycle of reduced forms. The collection of all reduced indefinite primitive forms of determinant \( D \) (which is finite) partitions into a finite set of cycles of possibly different lengths under the right-neighbor relation.

The cycle containing the reduced principal form \( \tilde{I}_D \) is called the principal cycle. Let \( Q^{(1)} \) denote the right-neighbor of \( \tilde{I}_D \), and \( Q^{(j)} \) the right-neighbor of \( Q^{(j-1)} \). Let \( S^{(j)} \) denote the matrix given by (5.1), (5.2) taking \( Q^{(j-1)} \) to \( Q^{(j)} \). The set of \( Q^{(j)} \) form a closed cycle of even period \( 2p \), i.e., there exists some \( Q^{(k)} = \tilde{I} \) and the smallest \( k = 2p \). For \( 1 \leq j \leq 2p \), \( \tilde{I} \sim Q^{(j)} \) via \( L_j \).

\[
L_j = S^{(1)} \ldots S^{(j)}.
\]

(5.3)

We call \( L_j \) a simple equivalence matrix. The matrix \( U = L_{2p} \) is called the fundamental automorph. If we set
\[
U = \begin{bmatrix}
u & w \\
t & v
\end{bmatrix}
\]
the condition \( \tilde{I} \sim \tilde{I} \) via \( U \) shows that \( t, u \) satisfies Pell’s equation
\[
t^2 - Du^2 = 1.
\]

(5.4)

In fact \((|t|, |u|)\) is the least strictly positive solution to (5.4), the fundamental solution, and
\[
U = \begin{bmatrix}
u & -t \\
t & -Du
\end{bmatrix}.
\]

(5.5)

We may consistently extend the definition of \( L_j \) to apply for all integers \( j \) by first defining \( S^{(j)} \) for negative \( j \) by
\[
S^{(j)} = [S^{(j_0)}]^{-1}
\]
where \( j \equiv j_0 \pmod{2p} \) and \( 0 < j_0 \leq 2p \), using (5.3) for all positive \( j \) and using
\[
L_{-j} = S^{(-j)} \ldots S^{(-1)}
\]
for \( j > 0 \). In that case, for any integer \( k \) we have
\[
L_{j+2kp} = U^k L_j.
\]

Gauss proved the following result (see Mathews [39, Arts. 76, 88], Venkov [62]).

**Proposition 5.1.** (Gauss) Let \( \tilde{I} \sim Q \) via \( T \) where \( Q \) is reduced. Then there is some \( j \) with \( 1 \leq j \leq 2p \) such that \( Q = Q^{(j)} \). Furthermore there is an integer \( k \) such that
\[
T = \pm U^k L_j = \pm L_{j+2kp}.
\]

(5.6)
5.2. Sign Patterns of Equivalence Matrices $L_j$

To handle the nonnegativity conditions in Theorem 1.1 we will need detailed information about the signs of entries in the equivalence matrices $L_j$. We first introduce the notation that if a matrix $M = \begin{bmatrix} m_{ij} \end{bmatrix}$, then

$$|M| = \begin{bmatrix} |m_{ij}| \end{bmatrix}.$$  \hspace{1cm} (5.7)

Lemma 5.1. The equivalence matrices $L_j$ have the following properties.

(i) For $j > 0$ the entries of $L_j$ have the sign patterns $\begin{bmatrix} ++ & ++ \end{bmatrix}$, $\begin{bmatrix} + & -+ \end{bmatrix}$, $\begin{bmatrix} - & -\end{bmatrix}$, $\begin{bmatrix} + & - \end{bmatrix}$ according as $j = 0, 1, 2$ or $3$ (mod 4).

(ii) For $j > 0$,

$$|L_j| = |S^{(1)}| \cdots |S^{(j)}|$$  \hspace{1cm} (5.8)

and

$$|L_{-j}| = |S^{(-j)}| \cdots |S^{(-1)}|.$$  \hspace{1cm} (5.9)

(iii) The four entries of $L_j = (l_{ij})$ are all about the same size in the sense that for any $|j| \geq 2$,

$$1 \leq \frac{\max |l_{ij}|}{\min |l_{ij}|} \leq 4(D + \sqrt{D}).$$  \hspace{1cm} (5.10)

Proof. We first observe that a reduced form $Q = [a, 2b, c]$ has by definition (4.20) $|b| < \sqrt{D}$ hence

$$ac < 0.$$  \hspace{1cm} (5.11)

The reduced forms $Q^{(i)} = [a_i, 2b_i, c_i]$ in the fundamental cycle have $a_{i+1} = c_i$. Noting $Q^{(0)} = \tilde{I}$ so $a_0 = 1$, by induction using (5.11) we obtain

$$( -1)^i a_i > 0.$$  \hspace{1cm} (5.12)

Now (4.25) and (5.2) imply that

$$\lambda_i c_{i-1} < 0,$$

so that (5.11), (5.12) yield

$$(-1)^{i+1} \lambda_i > 0.$$  \hspace{1cm} (5.13)

To prove (i) and (ii), note (5.13) implies $S^{(i)}$ for $i > 0$ has the sign patterns $\begin{bmatrix} - & + \end{bmatrix}$ when $i$ is odd, $\begin{bmatrix} + & - \end{bmatrix}$ when $i$ is even. Then it is easy to establish by induction on $i > 0$ that the entries of $L_i$, have the sign patterns $\begin{bmatrix} + & ++ \end{bmatrix}$, $\begin{bmatrix} + & + \end{bmatrix}$, $\begin{bmatrix} - & - \end{bmatrix}$, $\begin{bmatrix} + & + \end{bmatrix}$ according as $i = 0, 1, 2, 3$ (mod 4). Another induction on $i > 0$ shows no cancellation occurs in multiplying the entries of $L_i$ by $S^{(i+1)}$ and (5.8) follows. For $i < 0$, we observe first that

$$\begin{bmatrix} 0 & 1 \\ -1 & \lambda \end{bmatrix}^{-1} = \begin{bmatrix} \lambda & -1 \\ 1 & 0 \end{bmatrix}.$$
Then note $\lambda_i = \lambda_{i-2p}$ so (5.13) holds for $i < 0$ as well. This implies that for $i < 0$, $S^{(i)}$ has the sign patterns $[ ++ ]$ if $i$ is odd, $[ -+ ]$ if $i$ is odd, $[ ++ ]$ if $i$ is even. Another induction shows for $i < 0$ that the entries of $L_i$ have the sign patterns $[ ++ ]$, $[ -+ ]$, $[ ++ ]$, $[ ++ ]$ according as $i \equiv 0, 1, 2$ or $3$ (mod 4). Then (5.9) follows by induction.

To prove (iii), consider first the case $j > 0$. Using (5.8), we need only bound the entries $|L_j| = \left| \begin{array}{cc} p_{j-1} & p_j \\ q_{j-1} & q_j \end{array} \right|$

The formulae for $|L_j|$ is exactly that of the ordinary continued fraction algorithm, where

$|L_j| = \left| \begin{array}{cc} p_{j-1} & p_j \\ q_{j-1} & q_j \end{array} \right|$ (5.15)

and $\frac{p_j}{q_j}$ is the $j$th convergent to $\theta = [0, |\lambda_1|, |\lambda_2|, \ldots]$. In particular, for any $j \geq 2$ we have

$\frac{|\lambda_2|}{|\lambda_1| |\lambda_2| + 1} = \frac{p_2}{q_2} \leq \frac{p_j}{q_j} \leq \frac{p_2}{q_2} = \frac{1}{|\lambda_1|}.$ (5.16)

Now (5.2) implies

$1 \leq |\lambda_i| < 2\sqrt{D}$ (5.17)

so (5.15) yields

$\frac{1}{2\sqrt{D} + 1} \leq \frac{p_j}{q_j} < 1.$ (5.18)

In addition

$q_{j+1} = |\lambda_j| q_j + q_j - 1 \leq (|\lambda_j| + 1) q_j.$ (5.19)

Combining (5.16), (5.17), we obtain

$p_j \leq p_{j+1} \leq q_j - 1 \leq q_{j+1}.$ (5.20)

Finally

$q_{j+1} \leq (2\sqrt{D} + 1) q_j \leq (2\sqrt{D} + 1)^2 p_j$ (5.21)

and since $p_j \geq 1$ for $j \geq 2$ by (5.15) this implies (5.10) on this range. The case $j < 0$ is treated analogously to $j > 0$. In this case

$|L_{-j}| = \left| \begin{array}{cc} q_{j+1} & q_j \\ p_{j+1} & p_j \end{array} \right|,$

however. □
Remark. There is a close connection between the $\lambda_i$ and the ordinary continued fraction (OCF) expansion of $\sqrt{D}$. It is known that the OCF expansion has the form

$$\sqrt{D} = [\mu_0, \mu_1, \ldots, \mu_n]$$

in which $[\mu_1, \ldots, \mu_n]$ is the purely periodic part of the expansion and $n$ is the shortest period. (Stark [58, Sec. 7.7]). The $2 \times 2$ matrices in the continued fraction expansion have determinant $-1$ (see Stark [58, Sec. 7.6]) while the matrices in visiting neighboring forms in the Gaussian reduction procedure have determinant $+1$, but the entries of the resulting two expansions are simply related up to signs, and one obtains that $n = 2p$ if $n$ is even, and $n = p$ otherwise. In either case $\mu_i = |\lambda_i|$ for $1 \leq i \leq 2p$.

5.3. Bounds on Sizes of Equivalence Matrices $||L_j||$

Our next step is to estimate the size of the entries of $L_j$ in relation to $j$.

Lemma 5.2. For all $j > 0$,

$$\log ||L_{j+2}|| \geq \log ||L_j|| + 1 \quad (5.22)$$

and

$$\log ||L_j|| \leq \log ||L_{j+1}|| \leq \log ||L_j|| + \log D + 2 \quad (5.23)$$

Proof. For the bound (5.22), we use (5.14), (5.15) to obtain

$$q_{j+2} = (|\lambda_{j+1}\lambda_j| + 1)q_j + |\lambda_{j+1}|q_{j-1} \geq 2q_j .$$

The left side of (5.23) follows from (5.21). Finally (5.21) implies

$$\log ||L_{j+1}|| \leq \log ||L_j|| + \log(2\sqrt{D} + 1)$$

from which the right side of (5.23) follows.

Analogous inequalities to (5.22), (5.23) hold for $j < 0$. □

Using Lemma 5.2 it is easy to prove, by induction that for $j \geq 1$, that we have

$$\frac{1}{2}|j| \leq \log ||L_j|| \leq |j|(\log D + 2) \quad (5.24)$$

The same holds for $j \leq -1$.

5.4. Recursion Formula for $W$ in Terms of $L_j$’s

We can use the preceding results to determine a matrix formula for the equivalence matrix $W$ of Lemma 4.2.
Lemma 5.3. Suppose that the indefinite BQDE $y_1^2 - Dy_2^2 = g$ in Lemma 4.2 has an admissible solution. Let $W$ be the equivalence matrix guaranteed to exist in Lemma 4.2, satisfying (i)–(ix) of that lemma, and certifying an admissible solution. Then

$$W = \pm (L_{2p})^k L_j$$

for some $j$ with $1 \leq j \leq 2p$ and an integer $k$ satisfying

$$|k| = O(\|E\|^{3/2}(\log \|E\|)^2),$$

with $\|E\| = \text{MAX}(D, |g|, M)$.

Proof. This follows immediately on combining Proposition 5.1, Lemma 5.2 with the size bound (4.41) on $W$. The role of the extra power $k$ is to meet the side congruence conditions. \[\square\]

5.5. Upper Bound on Length of the Principal Cycle.

We next give an upper bound for the length $2p$ of the principal cycle. This upper bound holds more generally to all cycles of reduced forms of determinant $D$.

Proposition 5.2. The period $2p$ of the fundamental cycle of reduced forms of positive non-square determinant $D$ satisfies

$$p < (\sqrt{D} + 1) \log D.$$  

(5.27)

Proof. The result of Hua \[30\] given in Proposition 3.1 asserts that if $(t_0, u_0)$ is the fundamental positive solution to $x^2 - Dy^2 = 1$ then

$$\frac{t_0 + u_0 \sqrt{D}}{2} < D^{\sqrt{D}}.$$  

Using (5.5) we obtain

$$\|L_{2p}\| \leq D^{\sqrt{D} + 1}.  

(5.28)$$

Combining (5.28) with (5.24) gives

$$p \leq \log \|L_{2p}\| \leq (\sqrt{D} + 1) \log D. \quad \square$$

Remark. The examples $D = 5^{2n+1}$ with period $p = 5^n$ mentioned in the introduction show that periods $p > \frac{1}{3} \sqrt{D}$ do occur.

5.6. Exponential Time Algorithm for Solving a BQDE

We now consider the general binary quadratic Diophantine equation (3.1) in standard form,

$$ax_1^2 + 2bx_1x_2 + cx_2^2 + 2dx_1 + 2ex_2 + f = 0,$$

but with no side conditions imposed. We give an exponential time running bound for determining if the equation has an integer solution. The algorithm analyzed is a variant of the method of Gauss to find an integer solution. The algorithm is simplified since our object is only to obtain a bound of form $O(2^{c_1 L_F})$, without optimizing the constant $c_1$. 

35
Proof of Theorem 1.4. We reduce the Diophantine equation in Theorem 1.4 to the standard form by multiplying its coefficients by 2 if necessary. We consider the following algorithm. If the BQDE is definite or degenerate, it suffices by Lemma 3.2 to sequentially test all integer vectors \((x_1, x_2)\) with \(||x|| \leq 8||F||^4\) to see if they satisfy the equation. This takes at most \(O(||F||^4M(\log ||F||)) = O(2^{c_0^*L(F)})\) elementary operations. If the BQDE is indefinite, we reduce it in polynomial time to a generalized Pell equation \(E(y_1, y_2) = 0\) with \(E(y_1, y_2) = y_1^2 - Dy_2^2 - g\), as in Lemma 3.2, noting that \(\log ||E|| = O(L(F))\).

We solve the indefinite case by checking all possible certificates for solutions that are of the form given by Lemma 4.2 (i)-(ix). To do this we first find all square divisors \(h^2\) of \(g\) by exhaustive search, set \(G = g h^2\) and determine all properly primitive candidate forms \(Q_0 = [G, B, C]\) of determinant \(D\) having \(\text{MAX}(|B|, |C|) \leq D + 4g^2\). This can be done by enumeration in \(O((D + 4g^2)g) = O(2^{c_1 L(F)})\) elementary operations. For each such quadratic form \(Q_0\), an integer solution to the BQDE will exist if it is equivalent to \(\widetilde{I}_D\). If none of the forms \(Q_0\) are equivalent to the reduced identity form \(\widetilde{I}_D\), then the BQDE has no integer solution.

To test if a given indefinite form \(Q_0\) is equivalent to \(\widetilde{I}_D\), we first reduce \(Q_0\) to an indefinite reduced form \(Q_{\text{red}}\), which by Proposition 4.1 takes at most \(O(\log ||Q_0||M(\log ||Q_0||)) = O(\log ||E||^3)\) bit operations. Next one tests if \(Q_{\text{red}}\) is one of the \(2p\) forms in the principal cycle. To do this it suffices to step through all forms in the principal cycle, starting with \(\widetilde{I}_d\) and see if there is a match. By Proposition 5.2 there are at most \(\sqrt{D + 1}\log D\) forms in the cycle, and this test takes at most \(O\left(\sqrt{D}(\log D)^3\right)\) bit operations. We conclude that a single such test takes at most \(O(2^{c_1 L(F)})\) elementary operations. Combining all these tests, we can (wastefully) take the constant \(c_1 = c_0^* + c_1^* + c_2^*\) to get a running time bound \(O(2^{c_1 L(F)})\) elementary operations on the algorithm. \(\Box\)

6. Composition of Binary Quadratic Forms and Infrastructure

The certificates we construct use the operation of composition of binary quadratic forms introduced by Gauss [27], in particular the action of composition on the fundamental cycle of reduced forms. Our treatment of composition of forms is based on Mathews [39], Venkov [62] and Lagarias [34]. One may also consult Buell [17], Shanks [54], and Smith [57, Arts. 105–113].

The idea of analyzing the action of composition on the cycle of reduced forms equivalent to \(\widetilde{I}\) is due to D. Shanks [55], whose called it the “infrastructure”. We give bounds on the infrastructure in terms of composition of forms.

6.1. Composition of Binary Quadratic Forms

The simplest example of composition of two binary quadratic forms is the identity

\[
(x_1^2 + x_2^2)(y_1^2 + y_2^2) = (x_1 y_1 - x_2 y_2)^2 + (x_1 y_2 + x_2 y_1)^2 \tag{6.1}
\]
noted by Fermat. This identity shows that the product of two numbers which are the sum of two squares is itself the sum of two squares. We can rewrite (6.1) in the form

\[ Q_1(x_1, x_2)Q_1(y_1, y_2) = Q_1(x_1y_1 - x_2y_2, x_1y_2 + x_2y_1) \]  

where \( Q_1 = [1, 0, 1] \), and in matrix terms as

\[ (x^T Q_1 x)(y^T Q_1 y) = z^T B^T Q_1 B z \]  

(6.3)

where

\[ x^T = [x_1, x_2], \quad y^T = [y_1, y_2] \]
\[ z^T = [x_1y_1, x_1y_2, x_2y_1, x_2y_2] \]  

(6.4)

and

\[ B = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \end{bmatrix}. \]  

(6.5)

In this case we say \( Q_1 \) is composed of \( Q_1 \) and \( Q_1 \) via the bilinear matrix \( B \) of (6.3).

In the general case we say a quadratic form \( Q_3 = [a_3, 2b, c] \) is composed of forms \( Q_1 = [a_1, 2b_1, c_1] \) and \( Q_2 = [a_2, 2b_2, c_2] \) via a bilinear matrix \( B \) provided the matrix equation

\[ x^T Q_1 x y^T Q_2 y = z^T B^T Q_3 B z \]  

holds, where \( x, y, z \) are given by (6.4), the \( x_i \) and \( y_j \) are indeterminates. Here \( B \) is a \( 2 \times 4 \) integer matrix (“bilinear matrix”), which is required to be unimodular and oriented (terms defined below). We write \( Q_3 = Q_1 \circ Q_2 \) to indicate composition of forms, with the associated bilinear matrix \( B \) being omitted from the notation.

**Definition 6.1.** A \( 2 \times 4 \) integer matrix \( B = [b_{ij}] \) is said to be:

(i) unimodular provided the six cofactors

\[ \Delta_{ij} = \begin{bmatrix} b_{1i} & b_{1j} \\ b_{2i} & b_{2j} \end{bmatrix}, \quad 1 \leq i < j \leq 4 \]

have greatest common divisor 1.

(ii) oriented provided \( a_1 \Delta_{12} > 0 \) and \( a_2 \Delta_{13} > 0 \).

Recall that a form \( Q = [a, 2b, c] \) is properly primitive if \( \text{GCD}(a, 2b, c) = 1 \). If \( Q_3 \) is the composition of two properly primitive forms \( Q_1 \) and \( Q_2 \) of determinant \( D \), then \( Q_3 \) itself is properly primitive of determinant \( D \). This is a consequence of the unimodularity property of \( B \).

In the rest of this section we deal only with properly primitive indefinite forms. We use the following result on composition.
Proposition 6.1. Given any two properly primitive reduced forms $Q_1, Q_2$ of determinant $D$ there is a properly primitive reduced form $Q_3$ of determinant $D$ and a bilinear matrix $B$ such that $Q_3 = Q_1 \circ Q_2$ via $B$ and
\[ \log ||B|| = O(\log D) . \] (6.7)

There is an algorithm which when given as input $Q_1, Q_2$ in binary will determine $Q_3$ and matrix $B$ in binary, which runs in at most $O((\log D)M(\log D))$ bit operations.

Proof. This is shown in Lagarias [34, Theorem 5.5]. \[\square\]

6.2. Infrastructure Bounds

The key result facilitating the use of composition to create short certificates is the following lemma. Before stating it, we recall that the Kronecker product $S \otimes T$ of an $m \times n$ matrix $S = [s_{ij}]$ and a $k \times l$ matrix $G$ is a $km \times ln$ matrix
\[
S \otimes T = \begin{bmatrix} s_{11}T & \cdots & s_{1n}T \\
\vdots & & \vdots \\
s_{m1}T & \cdots & s_{mn}T \end{bmatrix}
\]
given in block matrix form.

Lemma 6.1. Let $\tilde{I}_D \sim Q_1$ via $S_1$ and $\tilde{I}_D \sim Q_2$ via $S_2$. If $Q_3 = Q_1 \circ Q_2$ via $B$, then $\tilde{I}_D \sim Q_3$ via $S_3$ where $S_3$ satisfies the matrix equation
\[ S_3B = B_0(S_1 \otimes S_2) \] (6.8)
where
\[ B_0 = \begin{bmatrix} 1 & 0 & 0 & D - \lambda^2 \\
0 & 1 & 1 & 2\lambda \end{bmatrix} \] (6.9)
and $\lambda = [\sqrt{D}]$.

Proof. It is straightforward to check that the identity form $I_D = I_D \circ I_D$ via
\[ B = \begin{bmatrix} 1 & 0 & 0 & D \\
0 & 1 & 1 & 0 \end{bmatrix} . \]

Using Lagarias [34] Lemma 5.1 (i), since $I_D \sim \tilde{I}_D$ via $\begin{bmatrix} 1 & \lambda \\
0 & 1 \end{bmatrix}$ we obtain that the reduced identity form has $\tilde{I}_D = I_D \circ I_D$ via $B_0$. Using the same [34] Lemma 5.1 (i), we next conclude $\tilde{I}_D = Q_1 \circ Q_2$ via $B_0(S_1 \oplus S_2)$. Then using [34] Lemma 5.1 (ii), we conclude there exists an integer matrix $S_3 \in SL(2, \mathbb{Z})$ such that
\[ S_3B = B_0(S_1 \otimes S_2) , \]
the desired result. \[\square\]
We note that $S_3$ is uniquely determined by equation \((6.8)\), since $B$ contains an invertible $2 \times 2$ submatrix by the unimodularity condition.

Now suppose $Q_1$ and $Q_2$ are forms in the principal cycle. Lemma \([6.1]\) shows that if $Q_3 = Q_1 \circ Q_2$ and $Q_3$ is reduced, then $Q_3$ is also in the principal cycle. By Proposition \([5.1]\) there are integers $k_1$, $k_2$, and $k_3$ such that $S_i = \pm L_{k_i}$ for $1 \leq i \leq 3$. What is the relation among the $k_i$’s? We do not determine this exactly, but show instead the following approximate additive relation among the $\log ||L_{k_i}||$’s.

\textbf{Lemma 6.2 (Infrastructure Bounds)} Let $Q_1, Q_2, Q_3$ be in the principal cycle and suppose $\tilde{I}_D \approx Q_1$ via $\pm L_{k_1}$, $\tilde{I}_D \approx Q_2$ via $\pm L_{k_2}$, where $k_1, k_2 \geq 0$. Suppose $Q_3 = Q_1 \circ Q_2$ via $B$ and that \(\log ||B|| \leq c_1 \log D\). \((6.10)\)

Let $S_3$ be defined by

$$S_3B = B_0(S_0(S_1 \otimes S_2)) .$$

If $S_i = \pm L_{k_i}$ and $\xi$ is defined by

$$\xi := \log ||L_{k_1}|| + \log ||L_{k_2}|| - \log ||L_{k_3}||,$$  \((6.11)\)

then we have the bound

$$|\xi| \leq (c_1 + 4) \log D .$$  \((6.12)\)

\textbf{Proof.} By \((6.8)\) we have

$$||S_3B|| = ||B_0(S_1 \otimes S_2)|| .$$  \((6.13)\)

Now

$$||S_0(S_1 \otimes S_2)|| \leq 4||B_0|| ||S_1 \otimes S_2||$$

$$= 4||B_0|| ||S_1|| ||S_2|| .$$  \((6.14)\)

We next note that $B_0$ is nonnegative and that $S_1 \otimes S_2$ has constant sign on columns by Lemma \([5.1](i)\). This implies

$$||B_0(S_1 \otimes S_2)|| \geq ||S_1 \otimes S_2|| = ||S_1|| ||S_2|| .$$  \((6.15)\)

On the other hand

$$||S_3B|| \leq 2||B|| ||S_3|| .$$  \((6.16)\)

Using orientability the first two columns of $B$ form an invertible $2 \times 2$ submatrix $B_1$ and we obtain

$$||S_3B|| \geq ||S_3B_1|| \geq \frac{||S_3||}{2||B_1||} \geq \frac{||S_3||}{2||B||} \geq \frac{||S_3||}{2||B||} ,$$  \((6.17)\)

where the center inequality is deduced from

$$||S_3|| \leq 2||S_3B_1|| ||B_1^{-1}||$$
and
\[ \|B_1^{-1}\| = (\det B)^{-1}\|B\| \leq \|B\|. \]

Now (6.13), (6.15), (6.16) yield
\[ \|S_3\| \geq \frac{2}{\|B\|}(\|S_1\| \|S_2\|), \quad (6.18) \]
while (6.13), (6.13), (6.17) yield
\[ \|S_3\| \leq 8\|B_0\| \|B\| (\|S_1\| \ |S_2|), \quad (6.19) \]

Using
\[ \log \|B_0\| \leq \log D \]
and the hypothesis (6.10), the inequalities (6.18) and (6.19) establish (6.12). \(\Box\)

### 6.3. Infrastructure Composition Chains for Equivalence on Principal Cycle

Lemma 6.2 shows that, for \( j \geq 1 \), \( \log \|L_j\| \) provides a measure of the size of the subscript \( j \); it shows these quantities are approximately additive under composition, up to an error (6.11), (6.12). In particular, composing a form \( Q^{(j)} \) with itself essentially doubles this size. By repeatedly doubling the size we can rapidly move to forms far apart in the principal cycle. This allows us to find "chains" of composition steps going from \( I_D \) to any reduced form in the principal cycle, of length at most \( O((\log D)^2) \) (polynomial in the input size), as given in the following result. These chains play a role analogous to "addition chains" in straight-line programming.

**Lemma 6.3 (Infrastructure Composition Chain)** For any \( L_j \) with \( 1 \leq j \leq 2p \) there is a sequence of equivalence matrices \( V_k \), and reduced forms \( Q_k \) of length \( K \) with
\[ K = O((\log D)^2) . \quad (6.20) \]

having the following properties.

(i) \( \hat{Q}_0 = I_D \), \( V_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \).

(ii) Each pair \( (\hat{Q}_{k+1}, V_{k+1}) \) is obtained from the preceding \( (\hat{Q}_k, V_k) \) by a transformation of either Type I or Type II, where:

Type I. \( Q_{k+1} \) is the right-neighbor of \( Q_k \) so that
\[ \hat{Q}_{k+1} = S_{k+1}^T \hat{Q}_k S_{k+1} \; , \quad (6.21) \]
\[ V_{k+1} = V_k S_{k+1} \; , \quad (6.22) \]

and
\[ \log \|S_k\| \leq \frac{1}{2}(\log D) \; . \quad (6.23) \]
Type II. \( \tilde{Q}_{k+1} = \tilde{Q}_{k_1} \circ \tilde{Q}_{k_2} \) via \( B_{k+1} \) for some \( 0 \leq k_1, k_2 \leq k \) so that
\[
X^T \tilde{Q}_i x y^T \tilde{Q}_j y = z^T B_{k+1}^T \tilde{Q}_{k+1} B_{k+1} z ,
\]
and where
\[
\log ||B_{k+1}|| = O(\log D) .
\]

(iii) \( Q_K = Q^{(j)} \) and \( V_K = L_j \).

Proof. We suppose that the composition of reduced forms is done as in Proposition 6.1, so that (6.26) is satisfied. We let \( c_1 \) denote the constant implied by the \( O \)-symbol in (6.26). Let \( \sigma_j \) denote the minimal number of type I and type II transformations sequentially applied to get from \( \tilde{I}_D \) to \( Q^{(j)} \) via \( L_j \). First note
\[
\sigma_j \leq j
\]
by using type I transformations only. We will prove by induction on \( j \) that for
\[
2(c_1 + 6) \log D \leq j \leq 2p
\]
we have
\[
\sigma_j \leq (5 + 4(c_1 + 4) \log D)(\log ||L_j||) .
\]
Suppose (6.29) holds. Take \( j_1 \) to be some \( l \) such that
\[
-(c_1 + 5) \log D - 2 < \log ||L_l|| - \frac{1}{2} \log ||L_j|| \leq -(c_1 + 4) \log D .
\]
At least one such \( l \) exists by (5.23) and \( 1 \leq l < j \). (Note (5.24) shows \( \frac{1}{2} \log ||L_j|| - (c_1 + 4) \log D \geq 2 \). Hence we can obtain \( \tilde{Q}_k = Q^{(j_1)} \), \( V_k = L_{j_1} \) where \( k = \sigma_{j_1} \) satisfies (6.30) by the induction hypothesis. Now apply a type II transformation, using \( \tilde{Q}_k \circ \tilde{Q}_k \), obtaining \( \tilde{Q}_{k+1} = Q^{j_2} \) and \( V_{k+1} = L_{j_2} \). Using Lemma 6.3 and (6.31) we have
\[
2(c_1 + 4) \log D + 2 \geq \log ||L_{j_2}| - \log ||L_{j_2}|| \geq 0 .
\]
Then Lemma 5.3 implies
\[
0 \leq j - j_2 \leq 4(1 + (c_1 + 4) \log D) .
\]
Hence \( 4(1 + (c_1 + 4) \log D) \) type I transformations will take us to \( Q^{(j)}, L_j \). Hence
\[
\sigma_j \leq \sigma_{j_1} + 4((c_1 + 4) \log D) + 5 .
\]
But the right side inequality of (6.31) gives
\[
\log ||L_{j_1}|| \leq \frac{1}{2} \log ||L_j|| \leq \log ||L_j|| - 1 .
\]
Substituting (6.30) for \( j_1 \) into (6.32) and using (6.33) establishes (6.30) for \( j \) and completes the induction step. \( \square \)

Remark. By more detailed argument, the bound (6.20) can be sharpened to
\[
K = O(\log D) .
\]
7. Certificates for Equivalence of Two Indefinite Binary Quadratic Forms

Lemma 6.3 can immediately be used to provide certificates for the equivalence of two indefinite binary quadratic forms.

**Theorem 7.1.** Let $Q_1$ and $Q_2$ be two indefinite integer binary quadratic forms with the same discriminant. If $Q_1$ is properly equivalent to $Q_2$, then there is a certificate of this equivalence requiring at most

$$O(\log ||Q_1|| + \log ||Q_2|| + (\log D)^2 M(\log D))$$

(7.1)

elementary operations to verify.

**Proof.** A necessary condition for the equivalence of two forms $Q_1 = [a_1, 2b_1, c_1]$ and $Q_2 = [a_2, 2b_2, c_2]$ is that

$$G.C.D.(a_1, b_1, c_1) = G.C.D.(a_2, b_2, c_2) = \sigma_1$$

and

$$G.C.D.(a_1, 2b_1, c_2) = G.C.D.(a_2, 2b_2, c_2) = \sigma_2 .$$

By removing $\sigma_1$ from the coefficients of both $Q_1$ and $Q_2$ we need only consider the case $\sigma_1 = 1$. In that case the forms are proper primitive if $\sigma_2 = 1$ and improperly primitive if $\sigma_2 = 2$.

Suppose first that the forms are properly primitive. Replace $Q_2 = [a_2, 2b_2, c_2]$ by $\tilde{Q}_2 = [a_2 - 2b_2, c_2]$, its inverse form. Reduce $Q_1$ and $\tilde{Q}_2$, obtaining $Q_1^*, \tilde{Q}_2^*$. This requires $O(\log ||Q_1|| + \log ||Q_2||)$ operations by Proposition 4.1. Compose $Q_1^*$ and $\tilde{Q}_2^*$ to obtain a reduced form $\tilde{Q}_3^*$. By Proposition 6.1 this can be done in $O(M(\log D))$ operations.

Now $Q_1 \sim Q_2$, if and only if $\tilde{Q}_3^* \sim \tilde{I}_D$. This follows from the well-known facts that: (i) composition of forms induces the structure of an abelian group on equivalence classes $[Q]$ of properly primitive forms $Q$, that (ii) $\tilde{I}_D$ is the identity element of this group, and that (iii) $[Q]^{-1} = [\tilde{Q}]$. (e.g. see Mathews [39, Arts. 141, 145].)

We now take the sequence of reduced forms $\tilde{Q}_k$ showing $\tilde{Q}_3^* \sim \tilde{I}_D$ that are guaranteed to exist by Lemma 6.3 together with the matrices $S_k$ and $B_k$ involved in the corresponding type I or II transformation. For each transformation we verify either (6.21) or (6.24), and this requires $O(M(\log D))$ elementary operations. We obtain a total of $O((\log D)^2 M(\log D))$ elementary operations in all, by (6.20).

Finally, we verify by induction on $k$ that checking (6.21), (6.24) at each step guarantees that all $\tilde{Q}_k = \tilde{I}_D$. Certainly $\tilde{Q}_0 \sim \tilde{I}_D$. If a type I transformation is used, then $\tilde{Q}_{k+1} \sim \tilde{Q}_k \sim \tilde{I}_D$ by definition of equivalence. If a type II transformation is used, then $\tilde{Q}_i \sim \tilde{I}_D$ and $\tilde{Q}_j \sim \tilde{I}_D$ guarantees $\tilde{Q}_{k+1} = \tilde{Q}_1 \circ \tilde{Q}_j \sim \tilde{I}_D$ by Lemma 6.1. This completes the proof in the properly primitive case.
We treat the improperly primitive case by reducing it to the properly primitive case by the following method given in Mathews [39, Art. 153]. We first note that improperly primitive forms have $D \equiv 1 \pmod{4}$. Let

\[ Q \begin{bmatrix} a & b \\ c & d \end{bmatrix} \equiv Q(ax + by, cx + dy). \]

If $D \equiv 1 \pmod{8}$ and $Q$ is improperly primitive, then $Q \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} = 2Q^*$ where $Q$ is properly primitive. Furthermore if $Q_1, Q_2$ are two such improperly primitive forms then $Q_1 \sim Q_2$ if and only if $Q_1^* \sim Q_2^*$. We may find a certificate for this as above. If $D \equiv 5 \pmod{8}$ and $Q$ is improperly primitive, then

\[ Q \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} = 2Q^{(1)}, \quad Q \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix} = 2Q^{(2)} \quad \text{and} \quad Q \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} = 2Q^{(i)} \]

where the $Q^{(i)}$ are all properly primitive. Furthermore if $Q_1, Q_2$ are two such improperly primitive forms then $Q_1 \sim Q_2$ if and only if one of $Q_1^{(i)} \sim Q_2^{(j)}$ for $1 \leq i \leq 3$. We may find a certificate for this as above. In order to get the bound (7.1) we first reduce the improperly primitive forms and then apply the procedure above. This reduction uses only $O(\log D)$ additional operations. \[\square\]

**Remark.** Since $\|Q\| > \frac{1}{2} \sqrt{D}$ for any form $Q$, (7.1) gives a bound polynomial in the length of the input $\log \|Q_1\| + \log \|Q_2\|$.

8. **Succinct Certificates for BQDE’s**

We now prove the main results, Theorem 1.1 and Theorem 1.2.

**Proof of Theorem 1.1.** If (2.1) has an admissible solution with $\|x\| < 256\|F\|^8$, then it serves as the certificate, and only $O(M(\log |F|))$ operations are needed to verify it is one. By Lemma 3.2 this is always the case for definite or degenerate binary quadratic Diophantine equations.

Now suppose (2.1) is indefinite, and has admissible solutions, but none with $\|x\| < 256\|F\|^8$. Then by part (ii) of Lemma 3.3, there exists $\beta_1, \beta_2$ such that the

\[ y_1^2 - Dy_2^2 = g. \]

has a solution with

\[ y_i \equiv \beta_i (\mod cD\Gamma) \]

satisfying (3.22), (3.23), and $y_1 > 0$ and the sign of $y_2$ is specified and satisfies one of (3.20), (3.21). Call the system (8.1), (8.2) with the given sign conditions $E$, and observe that

\[ \|E\| \leq \text{MAX}(D, |g|, |cD\Gamma|) \leq 6\|F\|^4 \]

using (3.7). By Lemma 3.3 it now suffices to give a certificate for this equation, to guarantee (8.1) has an admissible solution. Note that it takes only $O(M(\log D)\log D)$ operations to check the conditions of (ii) of Lemma 3.3 hold, in particular $O(M(\log D)\log D)$ operations to compute $\sqrt{D}$ to one digit past the decimal point, for application to test the inequalities (3.20).
Lemma 4.2 shows that to show the system $E$ has an admissible solution it suffices to produce certificates showing there exist integers $h, B, C$ and $2 \times 2$ matrices $S, W$ such that (i)–(v) of that lemma hold. The rest of the proof will accomplish this.

Lemma 4.2 also shows that there exist integers $h, B, C$ and $2 \times 2$ matrices $S, W$ such that (i)–(ix) of that lemma hold. In the rest of the proof we shall fix this particular choice of $h, B, C, S,$ and $W$, as well as

$$Q_{\text{red}} = [a_0, 2b_0, c_0]$$

arising in (iii) of that lemma. In that case (i), (ii) of Lemma 4.2 can be verified in $O(M(\log ||E||))$ operations by (vi) of that Lemma and (8.3). To verify (iii) of Lemma 4.2 we note that it asserts that

$$\begin{bmatrix} a_0 & b_0 \\ b_0 & c_0 \end{bmatrix} = S^T \begin{bmatrix} G & B & C \end{bmatrix} S.$$  

(8.5)

Using the bound (4.26) for a reduced form, (vii), (viii) of Lemma 4.2 and (8.3), all entries in (8.5) are $O(\log ||E||)$ so (8.5) can be verified in $O(M(\log ||E||))$ operations.

The essential difficulty in producing the certificates is the possible large size of the entries of $W$, evidenced by the bound (4.41), so that we cannot afford to keep track of these entries as fixed point binary integers. Consequently (iv) and (v) of Lemma 4.2 must be verified indirectly.

The certificates verifying (iv) and (v) are based on two kinds of formulae, which we call short and long. The short formulae can be evaluated using fixed-point integer arithmetic. We will use these to verify (iv). The long formulae involve integers with too many binary digits to allow direct evaluation. We use these to verify (v), by evaluating them (mod $cD\Gamma$) to verify (4.36), and by evaluating them using floating-point arithmetic to enough accuracy to verify (4.37).

The formulae are those guaranteed to exist by Lemma 5.3 and Lemma 6.3. By Lemma 5.3 the $W$ of Lemma 4.2 can be written in the form

$$W = (-1)^m(L_{2p})^K L_j$$

(8.6)

for some $j$ with $1 \leq j \leq 2p$, for some $m = 0$ or $1$, and $K$ is bounded by

$$|K| = O \left(||E||^{\frac{3}{2}}(\log ||E||)^2 \right) = O \left(||F||^{17}(\log ||F||)^2 \right).$$  

(8.7)

Assuming that $L_{2p}$ is known, we obtain $(L_{2p})^K$ by an exponential addition chain of $O(\log ||F||)$ squarings and multiplications of powers of $L_{2p}$. Then we obtain $W$ by combining this with $L_j$ using (8.6). Here (8.6) and the exponential addition chain formulas are all long formulas.
Next, by Lemma 6.3 for each \( L_j \) there exists a chain of reduced forms \( \{ \tilde{Q}_k : 1 \leq k \leq K_j \} \) with corresponding reduction matrices \( S_k \) and equivalence matrices \( V_k \) having the properties (6.21)–(6.28). Recall that the type I and II reduction and composition formulas

\[
\tilde{Q}_{j+1} = S_{j+1}^T \tilde{Q}_j S_{j+1}
\]

and

\[
x^T \tilde{Q}_j x y^T \tilde{Q}_j y = z^T B_{k+1}^T \tilde{Q}_{k+1} B_{k+1} z
\]

are short formulas, while the type I and II update formulas

\[
V_{k+1} = V_k S_{k+1}
\]

and

\[
V_{k+1} B_{k+1} = B_0 (V_{k_1} \otimes V_{k_2})
\]

are long formulas.

Consider the short formulas used in computing \( L_j \) and \( L_{2p} \). Lemma 6.3 gives that all entries in \( B_k \) and \( S_k \) have \( O(\log D) \) binary digits. The size bounds (4.26) on \( Q_k \) with these bounds imply that each formula can be evaluated exactly using fixed-point integer arithmetic with \( O(\log ||E||) \) binary digits. Each evaluation takes \( O(M(\log ||E||) \log ||E||)^2 \) bit operations used in evaluating all the short formulas. In addition we must verify that the bilinear matrices \( B_i \) used in short formulae are unimodular and oriented. Using the Euclidean algorithm to check unimodularity takes \( O(M(\log ||E||) \log ||E||) \) operations for each \( B_k \), by [34, Prop. 3.3], for a total of \( O(M(\log ||E||) \log ||E||)^3 \) operations in all. Checking orientability requires \( O(M(\log ||E||) \log ||E||) \) operations in all.

We now verify that the certificate satisfies property (iv) of Lemma 4.2. Since \( \tilde{I}_D \sim \tilde{I}_D \) via \( L_{2p} \) and \( \tilde{I}_D \sim Q^{(j)} \) via \( L_j \), (8.6) implies that

\[
\tilde{I}_D \sim Q^{(j)} \text{ via } W.
\]

In order to verify (iv) it suffices to check that

\[
Q^{(j)} = Q_{\text{red}},
\]

where \( Q_{\text{red}} \) is as in (4.33), and \( Q^{(j)} \) denotes the \( Q_k \) produced in (6.27) for \( L_j \). Checking that (8.8) holds takes another \( O(\log ||E||) \) operations.

We now describe certificates for (v) of Lemma 4.2. We first must verify

\[
hu_i \equiv \alpha_i \pmod{cD\Gamma} \quad i = 1, 2
\]

where

\[
\begin{bmatrix} u_1 & u_3 \\ u_2 & u_4 \end{bmatrix} = \begin{bmatrix} 1 & -\lambda \\ 0 & 1 \end{bmatrix} WS^{-1}.
\]
We define $W$ to be given by (8.6), and the $L_{2p}, L_j$ are defined by the long formulae of Lemma 6.3. We evaluate all these long formulae as congruences (mod $cD\Gamma$). Since
\[
\log cD\Gamma = O(\log ||E||),
\]
so we can use binary numbers with $O(\log ||E||)$ digits throughout. The long formulae for $V_k$ in Lemma 6.3 are evaluated successively. Evaluating each formula (6.22) (mod $cD\Gamma$) takes $O(M(\log ||E||)\log ||E||)$ operations. We next must check in (6.25) that given $B_0, B_{k+1}, V_i$ and $V_j$ (mod $cD\Gamma$) we can calculate $v_{k+1}$ (mod $cD\Gamma$). It is straightforward to calculate $B_0(V_k \otimes V\Gamma)$. We use the unimodularity condition of the matrix $B_{k+1}$, that the greatest common divisor of its $2 \times 2$ submatrices $\Delta_{ij} = \begin{bmatrix} b_{1i} & b_{1j} \\ b_{2i} & b_{2j} \end{bmatrix}$ is 1. By an algorithm similar to step 1 of Lagarias [34], Theorem 5.4], repeatedly using the Euclidean algorithm with the $\det(\Delta_{ij})$ we can find a factorization
\[
cD\Gamma = m_{12}m_{13}m_{14}m_{23}m_{24}m_{34}
\]
with the $m_{ij}$ pairwise relatively prime and with
\[
(m_{ij}, \det(\Delta_{ij})) = 1.
\]
for all $i, j$. This takes $O(M(\log ||E||)\log ||E||)$ operations. (Alternatively we can guess a set of $m_{ij}$ and check that they have the required properties.) Then
\[
(\Delta_{ij})^{-1} \equiv (\det(\Delta_{ij}))^{-1} \begin{bmatrix} b_{ij} & -b_{1j} \\ -b_{2i} & b_{1i} \end{bmatrix} \pmod{m_{ij}}
\]
and $(\det \Delta_{ij})^{-1}$ (mod $m_{ij}$) is calculated in $O(M(\log ||E||)\log ||E||)$ operations using [34, Corollary 3.4]. Hence
\[
V_{k+1} \equiv (\Delta_{ij})^{-1}[B_0(V_{k1} \otimes V_{k2})]_{ij} \pmod{m_{ij}},
\]
where $[M]_{ij}$ denotes the submatrix obtained taking columns $i$ and $j$, yields $V_{k+1}$ (mod $m_{ij}$). Finally we use the Chinese reminder theorem on each entry of $V_{k+1}$ separately to obtain $V_{k+1}$ (mod $cD\Gamma$) in $O(M(\log ||E||)\log ||E||)$ operations, by [34, Prop. 3.6]. Thus we may at last obtain $L_{2p}, L_j$ (mod $cD\Gamma$) in $O(M(\log ||E||)\log ||E||)^2$ operations, by (6.20). Next we calculate $(L_{2p})^2, (L_{2p})^4$ etc. by successive squarings and reductions (mod $cD\Gamma$), and use the binary expansion of $k$ to evaluate $W$ (mod $cD\Gamma$) using formula (8.6) in $O(M(\log ||E||)\log ||E||)^2$ operations, noting the bound (8.7). Finally (8.10) is evaluated (mod $cD\Gamma$) and then (8.9) verified in a further $O(M(\log ||E||))$ operations. Thus the congruence conditions (1.36) are verified in $O(M(\log ||E||)\log ||E||)^3$ elementary operations.

Finally we check that the sign conditions (1.37) of Lemma 4.2 (v) hold. These can be verified by evaluating the long formulae using floating-point arithmetic with floating-point integers maintaining $c_0(\log D)^3 = O((\log ||E||)^3)$ binary digits in both the exponent and fraction parts, where $c_0$ is a sufficiently large absolute constant fixed once and for all as described below. Basic terminology and error estimates for floating-point computations are given in Appendix B. We say that a normalized floating-point number $\bar{x} = f2^e$ with $\frac{1}{2} \leq f < 1$ approximates $x$ to accuracy $s$ significant figures if
\[
|\bar{x} - x| < 2^{-s}.
\]
(Here $(e, f)$ is the representation of $\bar{x}$ used in the calculation.) We wish to show $u_1$ and $u_2$ are computed to accuracy at least 1 significant figure, which permits determination of their signs. Assuming for the moment this accuracy is proved, it is straightforward to estimate the total number of elementary operations involved in evaluating all the long formulae to be $O(M(\log ||E||)^3)(\log ||E||)^2)$ which is $O(M(\log ||E||)(\log ||E||)^4)$. Note here that in evaluating $V_{k+1}$ by (6.25) that we merely pick an invertible $\Delta_{ij}$, and use

$$V_{k+1} = (\Delta_{ij})^{-1}[B_0 V_{k_1} \otimes V_{k_2}]_{ij}$$  \hspace{1cm} (8.16)$$
evaluated in floating-point, noting that

$$\log(\det(\Delta_{ij})) = O(\log ||E||)$$  \hspace{1cm} (8.17)$$
using (6.26).

It remains to estimate the loss of significant figures during the floating-point computations. The sources of loss of accuracy in floating-point computations are roundoff error, exponent overflow, exponent underflow in multiplication, and loss of accuracy in addition to two nearly equal numbers of opposite signs (e.g. this includes exponent underflow during addition as a special case).

By using $O(\log ||E||)^3)$ digits in the exponent part, we guarantee that exponent overflow never occurs. Indeed, only $O(\log ||E||)$ binary digits are needed to represent the exponent part $e$ of any entry of $W$, since

$$e = O(||E||^3/2(\log ||E||)^2)$$  \hspace{1cm} (8.18)$$
by (4.11). It is easy to check that the bound (8.18) applies to any exponent of every element occurring in the long formulae, since the $V_j$’s are just various $L_k$ with $1 \leq k \leq p$, to which the bounds (5.23), (5.27) apply. Now as long as the floating point calculations agree with the two entries of the long formulae to one significant figure, their exponents must agree within $\pm 1$ and these calculated exponents will then satisfy (8.18) and exponent overflow cannot occur. This demonstrates that exponent overflow cannot occur unless all significant digits have first been lost due to the other three sources of error.

We next show that exponent underflow during multiplications can never occur unless all significant digits have first been lost due to the remaining two sources of error. Indeed the entries of the matrices $V_j$ in Lemma (5.3) are known a priori to be nonzero integers by Lemma (5.1(iii)), except for $L_i$ with $|i| < 2$ (and if these occur they may be placed at the beginning of the computation, which is done in fixed point as explained below). The entries of $W$ are nonzero integers since $W = \pm L_j$ for some $|j| \geq 2$. We may suppose the entries of $U$ are nonzero integers, for if some $u_j = 0$ then since $U = Z$ satisfies the hypotheses of Lemma (4.1) the inequalities (4.9), (4.10) would imply the $u_j$ are small enough that they could be calculated directly in fixed point as certificates in $O(M(\log ||E||))$ operations to verify (iv), (v) of Lemma (4.2). Since these entries are nonzero integers, the exponents of their floating-point approximations must be $\geq 0$, and exponent underflow during multiplication cannot occur by Lemma B-1 in Appendix B. (We note that some multiplications by zero may occur, but these are exact using (B-16), (B-17) of
We must now bound the effects of roundoff error and that of addition of nearly equal quantities of opposite signs. We start with \( p = c_0 (\log ||E||)^3 \) significant digits of accuracy. We first consider the calculation of the \( V_k \) in Lemma 6.2. The entries of \( S_k, B_0, B_{k+1} \) are known to \( p \) significant digits by the bounds (6.23), (6.26). We will use Lemma B–2 to bound roundoff error, and Corollary B–4 to bound addition of nearly equal quantities. Evaluating \( V_{k+1} \) by the long formula (6.22) involves a loss of at most 5 significant digits by Lemma B–1, since each entry of \( V_{k+1} \) uses two floating-point multiplications and one addition, and the quantities added always have the same sign by Lemma 5.1 (i), (ii). The crucial step lies in showing that evaluating \( V_{k+1} \) by the long formula (6.25) (actually by (8.16) above) involves a loss of at most \( O(\log D) \) significant digits accuracy. Indeed \( V_{k_1} \otimes V_{k_2} \) can be evaluated losing at most 3 significant digits accuracy by Lemma B–1, as only multiplications are involved. Now the bound (6.18) applies to show that

\[
||V_k|| \geq \frac{2}{||B_k||} (||V_{k_1}|| ||V_{k_2}||). \tag{8.19}
\]

hence

\[
\log ||V_k|| \geq \log ||V_{k_1}|| + \log ||V_{k_2}|| - c_1 \log D - 1 \tag{8.20}
\]

using (6.10). But all entries of \( V_k \) have about the same size by Lemma 5.1 (iii), hence the nearest floating-point approximations to each entry of \( V_k \) must have exponents \( e \) satisfying

\[
e \geq \log ||V_{k_1}|| + \log ||V_{k_2}|| - (c_1 + 2) \log D - 3 . \tag{8.21}
\]

On the other hand, each entry of \( V_{k_1} \otimes V_{k_2} \) has exponent

\[
e \leq \log ||V_{k_1}|| + \log ||V_{k_2}|| . \tag{8.22}
\]

We now evaluate the entries of (8.16) doing all multiplications first, followed by additions. The multiplications lose at most 6 significant digits each, and the resulting exponents satisfy

\[
e \leq \log ||V_{k_1}|| + \log ||V_{k_2}|| + 2c_1 (\log D) + 3 \tag{8.23}
\]

using (8.22). Then the additions producing a given entry of \( V_{k+1} \) lose at most

\[
(3c_1 + 2) \log D + 11
\]

significant digits accuracy, using Corollary B–4, using (8.23) as an upper bound on \( e \) and (8.21) as a lower bound on \( e-A \). Thus at most \( (8c_1 + 2) \log D + 17 \) significant digits are lost in evaluating \( V_{k+1} \) using the long formula (6.25), and thus at most \( O((\log ||E||)^3) \) significant digits are lost in evaluating \( L_2p \) and \( L_j \) using Lemma 6.2. Next, we note that the calculation of \( (L_{2p})^k \) in formula (8.6) involves a loss of \( O(\log D) \) significant digits, because \( O(\log D) \) matrix multiplications are involved in computing \( (L_{2p})^2, (L_{2p})^4 \) etc., and the bounds of Lemma B–2 apply because all numbers added have the same sign. Calculating \( W \) using the long formula (8.6) loses another 5 significant digits; again all quantities added have the same sign. Finally we evaluate

\[
U = \begin{bmatrix} 1 & -\lambda \\ 0 & 1 \end{bmatrix} WS^{-1} . \tag{8.24}
\]
where $\lambda = \lceil \sqrt{D} \rceil$. Now

$$ W = \begin{bmatrix} 1 & \lambda \\ 0 & 1 \end{bmatrix} US $$

so

$$ ||W|| \leq 2\sqrt{D}||U|| ||S|| $$

yields

$$ \log ||U|| \geq \log ||W|| - c_2 \log ||E|| . \quad (8.25) $$

for some absolute constant $c_2$, using (4.40). The exponents $e$ of the individual entries of $U$ all satisfy

$$ e \geq \log ||W|| - c_3 \log ||E|| $$

using the inequality (4.10) of Lemma 4.1 (which applies since $U = Z$). Now evaluate the right side of (8.24), doing all multiplications first, and then additions. The resulting multiplied quantities all have exponents

$$ e \leq \log ||W|| + (c_1 + 2) \log ||E|| + 3 . \quad (8.27) $$

Then Corollary B–4 guarantees we can evaluate $U$ with a loss of at most $(c_1 + c_3 + 2) \log ||E|| + 7$ significant digits accuracy. We have shown at most $O((\log ||E||)^3)$ significant digits accuracy can be lost due to roundoff and adding of nearly equal quantities of opposite sign in evaluating $U$. Choosing $c_0$ large enough once and for all, we guarantee preservation of a positive number of significant digits to the end of the computation, and Theorem 1.1 is proved. $\Box$

**Proof of Theorem 1.2.** This essentially follows from Theorem 1.1. The only additional fact that needs to be checked is that the certificates of Theorem 1.1 can be “guessed” in polynomial time. The bounds (vi)–(ix) of Lemma 4.2, the bounds on the the size of the power $k$ in $W = \pm (L_{2p})^k L_j$ in Lemma 5.3 and on the $S_{k+1}, B_{k+1}$ in Lemma 6.3 demonstrate that this can be done. $\Box$
Appendix A. Period lengths (mod m) of certain linear recurrences.

Let \((t_1, u_1)\) be the least strictly positive solution to Pell’s equation
\[
X^2 - DY^2 = 1 \tag{A.1}
\]
and set
\[
\epsilon = t_1 + u_1 \sqrt{D} . \tag{A.2}
\]
In this appendix we show the sequences \(\{t_k\}, \{u_k\}\) defined by
\[
(\epsilon)^k = t_k + u_k \sqrt{D} \tag{A.3}
\]
are periodic (mod m) and we bound the length of the minimal period \(P(m)\) for which
\[
t_{k+P(m)} \equiv t_k \pmod{m} \tag{A.4}
\]
\[
u_{k+P(m)} \equiv u_k \pmod{m} \tag{A.5}
\]
both hold.

The sequences \(\{t_k\}, \{u_k\}\) both satisfy the second order linear recurrence.
\[
w_k = t_1 w_{k-1} - w_{k-2} . \tag{A.6}
\]
Periodicity of solutions to this recurrence (mod m) is closely related to divisibility of \(u_k\) by \(m\). Carmichael [18], [19] studied divisibility properties of a class of sequences which includes \(\{t_k\}, \{u_k\}\) as special cases. Periodicity properties for general linear recurrences were considered by Engstrom [26], Ward [64] and other authors.

**Lemma A–1.** For each \(m \geq 1\), the period \(P(m)\) is finite.

**Proof.** Let \(\bar{\epsilon} = t_1 - u_1 \sqrt{D}\) so that
\[
t_k = \frac{1}{2}(\epsilon^k + \epsilon^{-k}) \tag{A.7}
\]
\[
u_k = \frac{1}{2\sqrt{D}}(\epsilon^k - \epsilon^{-k}) . \tag{A.8}
\]
Pell’s equation asserts that
\[
\epsilon \bar{\epsilon} = 1 . \tag{A.9}
\]
Thus \(\epsilon, \bar{\epsilon}\) are units in the ring of integers \(\mathcal{O}_D\) of \(Q(\sqrt{D})\). For any ideal \(\mathcal{A}\) in \(\theta_D\) let \(S(\mathcal{A})\) denote the smallest \(k\) such that
\[
\epsilon^k \equiv \bar{\epsilon}^{-k} \equiv 1 \pmod{\mathcal{A}} \tag{A.10}
\]
over \(\mathcal{O}_D\), such \(S(\sigma)\) existing since \(\epsilon, \bar{\epsilon}\) are invertible (mod \(\mathcal{A}\)). It’s easy to check that
\[
t_{k+R} \equiv t_k \pmod{m} \tag{A.4}
\]
\[
u_{k+R} \equiv u_k \pmod{m} \tag{A.5}
\]
where $R = S((2\sqrt{D}m))$. Hence
\[ P(m)|S((2\sqrt{D}m)) \quad (A.11) \]
exists. □

**Lemma A-2.** If $(m, n) = 1$ then
\[ P(mn) = \text{l.c.m.}\{P(m), P(n)\} . \quad (A.12) \]

**Proof.** This follows from the definition (A-4) and the Chinese Remainder Theorem. □

It thus suffices to calculate $P(p^a)$ for prime powers $p^a$.

**Lemma A-3.** For all primes $p$ and $a \geq 1$,
\[ P(p^{a+1})|P(p^a) . \quad (A.13) \]

**Proof.** For $R = P(p^a)$ we have
\[
\begin{align*}
t_R &= 1 + p^as_1 \\
u_R &= p^as_2
\end{align*} \quad (A.14)
\]
for some $s_1, s_2$. Since \[ t_{pR} + u_{pR}\sqrt{D} = (t_R + u_R\sqrt{D})^p \]
we have, for $p$ an odd prime,
\[
\begin{align*}
t_{pR} &= \sum_{j=0}^{p-1/2} \binom{p}{2j} (t_R)^{p-2j}(u_R)^{2j}D^j \\
u_{pR} &= \sum_{j=0}^{p-1/2} \binom{p}{2j} (t_R)^{2j}(u_R)^{p-2j}D^{p-1-j}
\end{align*} \quad (A.15, A.16)
\]
Since $p|\binom{p}{j}$ for $1 \leq j \leq p - 1$, these equations and (A.14) yield
\[
\begin{align*}
t_{pR} &\equiv (t_R)^p \equiv 1 \pmod{p^{a+1}} \\
u_{pR} &\equiv 0 \pmod{p^{a+1}} .
\end{align*}
\]
and (A.13) follows. For the remaining case $p = 2$ we have
\[
\begin{align*}
t_{2R} &= t_R^2 + Du_R^2 \equiv 1 \pmod{2^{a+1}} \\
u_{2R} &= 2t_Ru_R \equiv 1 \pmod{2^{a+1}},
\end{align*}
\]
giving (A.13) in this case. □

In order to bound $P(p)$, let $\left( \frac{D}{p} \right)$ denote the Legendre symbol.

**Lemma A-4.** Let $p$ be an odd prime.
If \((\frac{D}{p}) = 1\), then
\[ P(p)|p - 1 \] (A.17)

(ii) If \((\frac{D}{p}) = -1\), then
\[ P(p)|2(p + 1) \]. (A.18)

(iii) If \(p|D\), then
\[ P(p)|2p \]. (A.19)

(iv) \(P(2) = 1\) or \(2\).

**Proof.** Suppose \(p \nmid 2D\) so \((\frac{D}{p}) = \pm 1\). Then examination of (A.7)–(A.10) shows that (A.11) can be sharpened to
\[ P(p)|S(pO_D) \]. (A.20)

(i) If \((\frac{D}{p}) = 1\), then \((p)\) factors as \((p) = \mathcal{P}_1\mathcal{P}_2\) the product of two distinct conjugate prime ideals in \(O_D\). Then \(O_D/\mathcal{P}_i \cong GF(p)\). Since \(x^{p-1} = 1\) in \(GF(p)\) when \(x \neq 0\), we have
\[ \varepsilon^{p-1} \equiv \bar{\varepsilon}^{p-1} \equiv 1 \pmod{\mathcal{P}_i} \]
for \(i = 1, 2\). Thus
\[ \varepsilon^{p-1} \equiv \bar{\varepsilon}^{p-1} \equiv 1 \pmod{pO_D} \]
so \(S(pO_D)|p - 1\). Then (A.20) proves (A.17).

(ii) If \((\frac{D}{p}) = -1\), then \((p)O_D\) is inert, and \(O_D/(p) \cong GF(p^2)\). Now \(x^{p+1} \in GF(p)\) for all \(x \in GF(p^2)\) hence
\[ \varepsilon^{p+1} \equiv a \pmod{pO_D} \]
for some \(a \in \mathbb{Z}\). (Note \(GF(p) \cong \mathbb{Z}/p\mathbb{Z} \subseteq O_D/(p)\).) Applying the conjugation automorphism, we have
\[ \bar{\varepsilon}^{p+1} \equiv a \pmod{pO_D} \].
But \(\varepsilon \bar{\varepsilon} = 1\) hence
\[ \bar{a}^2 = 1 \pmod{pO_D} \].
Hence
\[ \varepsilon^{2(p+1)} \equiv \bar{\varepsilon}^{2(p+1)} \equiv 1 \pmod{pO_D} \]
and \(S(pO_D)|2(p + 1)\). Then (A.20) implies (A.18).

(iii) If \(p|D\) then
\[ t_2 = t_1^2 - Du_1^2 \equiv t_1^2 \equiv 1 \pmod{p} \]
since \(t_1^2 = 1 + Du_1^2\). Then (A.15), (A.16) applied with \(R = 2\) show
\[ t_{2p} \equiv (t_2)^p \equiv 1 \pmod{p} \]
\[ u_{2p} \equiv 0 \pmod{p} \].
Hence \(P(p)|2p\).
Lemma A-5. For any $m \geq 2$,

$$P(m) \leq 2m(1 + \log m)$$

Proof. Lemmas A-2 through A-4 imply that if $m = \prod p_j^{a_j}$ then

$$P(m) \mid R(m) := 2 \prod_j \left( p_j^{a_j-1} \left( p_j - \left( \frac{D}{p_j} \right) \right) \right).$$

Now

$$R(m) \leq 2m \prod_j \left( 1 + \frac{1}{p_j} \right),$$

and

$$\prod_j \left( 1 + \frac{1}{p_j} \right) \leq \sum_{j=1}^m \frac{1}{j} < 1 + \log m,$$

so the lemma follows. \(\Box\)

Remark. By more careful argument one can obtain the improved bound $P(m) = O(m \log \log m)$.

Appendix B. Floating-Point Computations.

This appendix gives upper bounds on the magnitude of errors accumulated in floating-point computations. We use the conventions and notation of Knuth [14, Sect. 4.2], to which we refer for greater detail.

We use normalized floating-point numbers with base $2$, excess $0$, with $p$ digits. Such a number will be denoted $(e, f)$ where

$$(e, f) = f2^e.$$  \hfill (B.1)

Here $e$ is an integer satisfying

$$|e| < N$$  \hfill (B.2)

and $f$ is a signed fraction such that $2^p f$ is an integer and satisfying the normalization condition

$$\frac{1}{2} \leq |f| < 1.$$  \hfill (B.3)

provided $f \neq 0$. By convention 0 is $(0,0)$.

We introduce a notation to distinguish general real numbers from floating-point numbers, which are just real numbers satisfying (B.1)–(B.3). To this end we always denote floating-point
numbers with a bar, i.e., \( \bar{x} \) is a floating-point number (to be thought of as an approximation to the real number \( x \)).

To define the floating-point operations of addition, subtraction, multiplication and division, we use the function “Round to \( p \) significant figures” defined by

\[
\text{Round } (x, p) = \begin{cases} 
2^{e-p}[2^{p-e}x + \frac{1}{2}], & 2^{e-1} \leq x \leq 2^e \\
0, & x = 0 \\
2^{e-p}[2^{p-e}x - \frac{1}{2}], & 2^{e-1} \leq -x < 2^e .
\end{cases}
\]

We define floating-point addition \( \oplus \) by

\[
\bar{x} \oplus \bar{y} = \begin{cases} 
0, & |\bar{x} + \bar{y}| < 2^{-N} \\
\text{Round } (x + y, p), & 2^{-E} \leq |\bar{x} + \bar{y}| < 2^N .
\end{cases}
\]

Exponent overflow occurs if \( |\bar{x}| \geq 2^N \) and \( \bar{x} \oplus \bar{y} \) is left undefined. We define floating-point subtraction of \( \bar{x} \) as floating-point addition of \( -\bar{x} \). We define floating-point multiplication \( \otimes \) by

\[
\bar{x} \otimes \bar{y} = \begin{cases} 
0, & |\bar{x}\bar{y}| < 2^{-N} \\
\text{Round } (\bar{x}\bar{y}, p), & 2^{-E} \leq |\bar{x}\bar{y}| < 2^N .
\end{cases}
\]

Exponent overflow occurs if \( |\bar{x}\bar{y}| \geq 2^N \) and \( \bar{x} \otimes \bar{y} \) is left undefined. Floating-point division \( \phi \) is defined similarly to multiplication, but we will not need it. Note that these operations are well-defined even when exponent underflow occurs.

Let \( \bar{x} \) be a floating point number approximating a nonzero real number \( x \). Let

\[
2^e \leq x < 2^{e+1} .
\]

We say \( \bar{x} \) approximates \( x \) to \( s \) significant digits if

\[
|\bar{x} - x| < 2^{e-s-1} .
\]

There are four sources of loss of significant digits in floating-point operations.

1. roundoff error,
2. exponent overflow,
3. exponent underflow in multiplication,
4. addition of two nearly equal quantities of opposite signs (includes exponent underflow).
We deal with these sources separately.

Exponent overflow, and exponent underflow in multiplication are the easiest to handle, by giving sufficient conditions that they do not occur. By convention multiplication by zero does not count as exponent underflow.

**Lemma B-1.** Let $\bar{x} = (e_1, f_1)$, and $\bar{y} = (e_2, f_2)$ be two floating-point numbers. If

$$-N + 2 \leq e_1 + e_2 \leq N - 1$$

(B.9)

then $\bar{x} \otimes \bar{y}$ does not involve exponent overflow or underflow. If

$$\text{MAX}(e_1, e_2) \leq N - 2$$

(B.10)

then $\bar{x} \oplus \bar{y}$ does not involve exponent overflow.

**Proof.** Immediate. $\square$

In order to analyze roundoff error, we note that when

$$2^{e-1} \leq |x| < 2^e$$

(B.11)

we have the bound

$$|\text{Round}(x, p) - x| < 2^{e-p-1}.$$  

(B.12)

**Lemma B-2.** Let $\bar{x} - \bar{y}$ be two floating-point numbers, both having $s$ significant digits.

(i) If $\bar{x}, \bar{y}$ have the same sign, then at most 2 significant digits are lost in computing $\bar{x} \oplus \bar{y}$.

(ii) If exponent underflow does not occur, then at most 3 significant digits are lost in computing $\bar{x} \otimes \bar{y}$.

**Proof.** (i) Since $\bar{x}, \bar{y}$ have the same sign, underflow cannot occur. Then

$$\bar{x} \oplus \bar{y} = \text{Round} \ (\bar{x} + \bar{y}, p).$$  

(B.13)

Let $\bar{x}, \bar{y}$ have exponents $e_1, e_2$. Then the exponent $e_3$ of $\bar{x} \oplus \bar{y}$ is at least $\text{MAX}(e_1, e_2)$. But

$$|\bar{x} - x| < 2^{e_1-s-1}$$

$$|\bar{y} - y| < 2^{e_2-s-1}.$$  

(B.14)

Note $s \leq p$. Then

$$|\bar{x} \oplus \bar{y} - (x + y)| \leq |\bar{x} \oplus \bar{y} - (\bar{x} + \bar{y})| + |\bar{x} - x| + |\bar{y} - y|$$

$$\leq 2^{e_3-p-1} + 2^{e_2-s-1} + 2^{e_1-s-1} \leq 2^{e_3-s+1}$$  

(B.15)
(ii) Since underflow does not occur, we have

\[ \bar{x} \otimes \bar{y} = \text{Round} (\overline{xy}, p) . \]  

(B.16)

If \( e_4 \) is the exponent of \( \bar{x} \otimes \bar{y} \), then

\[ e_4 \geq e_1 + e_2 - 1 . \]

Now

\[ |x \bar{y} - xy| \leq |x - x| |\bar{y} - y| |x| \leq 2^{e_1 + e_2 - s} \]

using (B.14). Hence

\[ |\bar{x} \otimes \bar{y} - xy| \leq |\bar{x} \otimes \bar{y} - \overline{xy}| + |\overline{xy} - xy| \leq 2^{e_3 - p - 1} + 2^{e_1 + e_2 - s} \leq 5 \cdot 2^{e_3 - s - 1} , \]

using (B.12), (B.16). \( \square \)

We remark that Lemma B-2 (i) also holds when \( \bar{y} = 0 \) and

\[ |\bar{y} - y| < 2^{e_1 - s - 1} , \]  

(B.17)

and that

\[ \bar{x} \otimes \bar{y} = xy = 0 \]  

(B.18)

where \( y = \bar{y} = 0 \).

We next consider the bounds for addition.

\textbf{Lemma B-3.} Let \( \bar{x}_1, \ldots, \bar{x}_j \) be floating-point numbers such that all \( \bar{x}_i \) have exponents \( \leq e \). Suppose that

\[ |\bar{x}_i - x_i| < 2^{e - s - 1} , \quad 1 \leq i \leq j , \]  

(B.19)

and suppose that \( e - s \geq -N \). Let

\[ v_j = x_1 + \ldots + x_j , \]

(B.20)

and define \( \bar{v}_1 = \bar{x}_1 \) and

\[ \bar{v}_{i+1} = \bar{v}_i \oplus \bar{x}_{i+1} , \quad 2 \leq i \leq j - 1 . \]

(B.21)

Let \( j \leq 2^k \) and \( k \leq p \). Then

\[ |\bar{v}_j - v_j| < 2^{3 + 2k + 3 - s} . \]  

(B.22)
Proof. We have

\[ |\bar{x}_i| \leq 2^e - 2^{e-p}, \]

from which it is easy to establish

\[ \bar{v}_i \leq i(2^e - 2^{e-p})(1 + i2^{-p}) < 2^{e+k+2}. \] (B.23)

(The term \(i2^{-p}\) is a roundoff bound.) Now we have

\[ |\bar{v}_i - v_i| \leq |\bar{v}_i - (\bar{v}_{i-1} + \bar{x}_i)| + |\bar{x}_i - x_i| + |\bar{v}_{i-1} - v_{i-1}|. \] (B.24)

If we let \(e_i\) be the exponent of \(\bar{v}_i\) then (B.22) gives

\[ e_i \leq e + k + 2. \] (B.25)

But

\[ |\bar{v}_i - (\bar{s}_{i-1} - \bar{x}_i)| \leq \text{MAX}(2^{e_i-p-1}, 2^{-N}), \] (B.26)

the bound \(2^{-N}\) occurring in the case of underflow. Then apply (B-18) and (B-25) to (B-22) and sum over \(i\) to obtain

\[ |\bar{v}_j - v_j| < \sum_{i=1}^{j} [2^{e_j-p-1} + 2^{-N} + 2^{e-s-1}]. \] (B.27)

using (B-24) gives

\[ |\bar{v}_j - v_j| < 2^{e+2k+1-p} + 2^N + 2^{e+k-s-1} \]
\[ < 2^{e+wk+3-s}, \]

the desired bound. \(\square\)

Lemma B-3 allows one to show that if one knows “a priori” that a sum \(\sum_i x_i\) is not too small with respect to its largest term, then the loss of significant digits in calculating a floating-point approximation to this sum cannot be large.

Corollary B-4. Let \(\bar{x}_1, \ldots, \bar{x}_j\) be floating point numbers approximating \(x_1, \ldots, x_j\) to \(s\) significant digits, with the largest \(|\bar{x}_j|\) having exponent \(e \geq -N + s\). Let

\[ s_j = x_1 + \ldots + x_j, \]
\[ \bar{s}_j = \bar{s}_{j-1} \oplus \bar{x}_j, \bar{s}_1 = \bar{x}_1, \]

Suppose \(j \leq 4\) and that

\[ |s_j| \geq 2^{e-A}. \]

Then \(\bar{s}_j\) approximates \(s_j\) to at least \(s - A - 8\) significant digits. \(\square\)
References

[1] L. Adleman, *Number Theoretic Aspects of Computational Complexity*, Thesis, Univ. of California, Berkeley, 1976.

[2] L. Adleman and K. Manders, Diophantine Complexity, Proc. 17th IEEE Annual Symp. on Foundations of Computer Science (1976), 81–88.

[3] L. Adleman and K. Manders, Reducibility, Randomness and Intractability, Proc. 9th Annual ACM Symposium on Theory of Computing (1977), 151–163.

[4] L. Adleman and K. Manders, Intractability Proofs and the Computational Complexity of Binary Quadratics, U. C. Berkeley, College of Engineering Technical Report No. UCB/ERL M78/30 (1978).

[5] L. Adleman and K. Manders, Reductions that Lie, Proc. 20th IEEE Symp. on Foundations of Computer Science (1979), 397–410.

[6] M. Agrawal, N. Kayal and N. Saxena, PRIMES is in P, Annals of Math. 160 (2004), 781–793.

[7] A. Baker and J. Coates, Integer points on curves of genus 1, Proc. Camb. Phil. Soc. 67 (1970), 595-602.

[8] Y. Bilu, Quantitative Siegel’s theorem for Galois coverings, Compositio Math. 106 (1997), 125–158.

[9] Y. Bilu and D. Poulakis, Points entiers sur les courbes de genre 0, Colloq. Math. 66 (1993), 1–7.

[10] J. Buchmann, On the computation of units and class numbers by a generalization of Lagrange’s algorithm, J. Number Theory 26 (1987), 8–30.

[11] J. Buchmann, A subexponential algorithm for the determination of class numbers and regulators of algebraic number fields, pp. 27–41 in *Séminaire de Théorie des Nombres* (Paris 1988-1989), C. Goldstein, Ed., Progress in Math. Vol. 91, Birkhäuser: Boston 1990.

[12] J. Buchmann and S Dülmann, A probabilistic class group and regulator algorithm and its implementation, pp. 53–72 in: *Computational Number Theory* (Debrecen 1989), A. Pethö, Ed., de Gruyter: Berlin 1991.

[13] J. Buchmann, C. Theil, and H. C. Williams, Short representations of quadratic integers, in: *Computational Algebra and Number Theory (Sydney 1992)*, 159–185, Math. Appl. Vol. 325, Kluwer: Dordrecht 1992.

[14] J. Buchmann and H. C. Williams, On the infrastructure of the principal ideal class of an algebraic number field of unit rank one, Math. Comp. 50 (1988), 569–579.
[15] J. Buchmann and H. C. Williams, On the existence of a short proof for the value of the class number and the regulator of a real quadratic field, in: *Number Theory and Applications* (Banff, AB 1988), 327–345, NATO ASI Series C Math. Phys. Sci, No. 265, Kluwer Academic Publ., Dordrecht 1989.

[16] J. Buchmann and H. C. Williams, On the computation of the class number of an algebraic number field, Math. Comp. 53 (1989), 679–688.

[17] D. Buell, *Binary Quadratic Forms. Classical theory and modern computations.* Springer-Verlag, New York 1989.

[18] R. D. Carmichael, On the numerical factors of the arithmetic forms $\alpha^n \pm \beta^n$, Annals of Math. 15 (1913), 30–70.

[19] R. D. Carmichael, A Simple Principle of Unification in the Elementary Theory of Numbers, Amer. Math. Monthly 36 (1929), 132–143.

[20] H. Cohen, *A Course in Computational Algebraic Number Theory*, Graduate Texts in Mathematics 138, Springer-Verlag: New York 1993.

[21] J. H. Conway, *The (Sensual) Quadratic Form. With the assistance of Francis Y. C. Fung* Carus Mathematical Monographs, 26. MAA: Washington D.C. 1997.

[22] M. Davis, Hilbert’s Tenth Problem in Unsolvable, Amer. Math. Monthly 80 (1973), 233-269.

[23] M. Davis, H. Putnam and J. Robinson, The Decision Problem for Exponential Diophantine Equations, Annals of Math. 74 (1961), 425–436.

[24] E. Fouvyry and J. Kl"uners, On the negative Pell equation, Annals of Math., to appear.

[25] R. de Haan, M. J. Jacobson, Jr., and H. C. Williams, A fast, rigorous techniques for computing the regulator of a real quadratic field, Math. Comp. 76 (2007), 2139–2160.

[26] H. T. Engstrom, On Sequences Defined by Linear Recurrence Relations, Trans. Amer. Math. Soc. 33 (1931), 210–218.

[27] C. F. Gauss, *Disquisitiones Arithmeticae*, Leipzig. 1801 (English translation: Springer-Verlag: New York 1986.)

[28] E. M. Gurari and O. H. Ibarra, An $NP$-Complete Number-Theoretic Problem, J. ACM 26 (1979), 567–581.

[29] J. L. Hafner and K. S. McCurley, A rigorous subexponential algorithm for computation of class groups, J. Amer. Math. Soc. 2 (1989), 837–850.

[30] L. K. Hua, On the least solution to Pell’s equation, Bull. Amer. Math. Soc. 48 (1942), 731–735.

[31] M. J. Jacobson, Jr. and H. C. Williams, *Solving the Pell equation*, CMS Books in Mathematics, Springer: New York 2009.
[32] D. E. Knuth, *The Art of Computer Programming, Vol. 2, Seminumerical Algorithms*, Addison-Wesley Publ. Co., Reading, Mass. 1969.

[33] J. C. Lagarias, Succinct Certificates for the Solvability of Binary Quadratic Diophantine Equations (Extended Abstract), Proc. 20th IEEE Symp. on Foundations of Computer Science (1979), 47–54.

[34] J. C. Lagarias, Worst-case complexity bounds for algorithms in the theory of integral quadratic forms, J. of Algorithms 1 (1980), 42–86.

[35] J. C. Lagarias, On the computational complexity of determining the solvability or unsolvability of the equation \( x^2 - Dy^2 = -1 \), Trans. Amer. Math. Soc. 260 (1980), 485–508.

[36] J. C. Lagarias, Succinct Certificates for the Solvability of Binary quadratic Diophantine Equations, Bell Labs Technical Memorandum 81-11216-54, Sept. 28, 1981, 61 pages.

[37] H. W. Lenstra, Jr, On the calculation of regulators and class numbers of quadratic fields, in: *Number theory days, 1980 (Exeter 1980)*, pp. 123–150, London Math. Soc. Lecture Notes Series 56, Cambridge University Press, Cambridge 1982.

[38] K. Manders and L. Adleman, NP-complete decision problems for binary quadratics, J. Comp. Sys. Sci. 16 (1978), 168–184.

[39] G. B. Mathews, *Theory of Numbers*, , 2nd Edition, Chelsea: New York 1961. (Original: London: G. Bell and Sons. 1892.)

[40] Y. Matijasevic, Enumerable Sets are Diophantine, Dokl. Akad. Nauk SSSR 191 (1970), 279–282.

[41] G. Miller, Riemann’s Hypothesis and Tests for Primality, J. Computer and Systems Science 13 (1976), 300–317.

[42] W. Narkiewicz, *Elementary and Analytic Theory of Algebraic Numbers*, Polish Scientific Publishers, Warsaw, 1974.

[43] I. Niven, H. S. Zuckerman and H. L. Montgomery, *An Introduction to the Theory of Numbers*, John Wiley & Sons, Inc., New York 1991.

[44] D. Poulakis, Integer points on algebraic curves with exceptional units, J. Australian Math. Soc. A 63 (1997), 145-164.

[45] D. Poulakis, Polynomial bounds for the solutions of a class of Diophantine equations, J. Number Theory 66 (1997), 271–281.

[46] D. Poulakis, Bounds for the minimal solution of genus zero Diophantine equations, Acta Arith. 86 (1998), 51–90.

[47] D. Poulakis, Bounds for the size of integral points on curves of genus zero, Acta Math. Hungar. 93 (2001), 327–346.
[48] D. Poulakis, Bounds for the smallest integer point of a rational curve, Acta Arith. 107 (2003), 251–268.

[49] V. Pratt, Every Prime has a Succinct Certificate, SIAM J. Computing 4 (1975), 214–220.

[50] L. Redei, Die 2-Ringklassengruppe des Quadratischen Zahlkörperws und die theorie der Pellschen Gleichung, Acta. Math. Acad. Sci. Hung. 4 (1953), 31–87.

[51] R. L. Rivest, A. Shamir and L. Adleman, A method for obtaining digital signatures and public key cryptosystems, Comm. ACM 21 (1978), 120–126.

[52] W. M. Schmitdt, Integer points on curves of genus 1, Compositio Math.. 81 (1992), 33–59.

[53] R. Schoof, Computing Arakelov class groups, in: Surveys in Algorithmic Number Theory, Cambridge Univ Press: Cambridge 2008, pp. 447–495.

[54] D. Shanks, Class number, a theory of factorization and genera, in: 1969 Number Theory Institute, Proc. Symp. Pure Math. XX , American Math. Society: Providence, 1971, pp. 415–440.

[55] D. Shanks, The Infrastructure of a Real Quadratic Field and Its Applications, in: Proc. 1972 Number Theory Conference, U. of Colorado, Boulder, Colorado, 1972, pp. 217–224.

[56] S. Smale, Mathematical problems for the next century, in: Mathematics: frontiers and perspectives, pp. 271–294, Amer. Math. Soc., Providence RI 2000.

[57] H. J. S. Smith, Report on the Theory of Numbers, Chelsea Publ. Co., New York 1965. (Reprint).

[58] H. M. Stark, Introduction to Number Theory, Markham: Chicago 1970. (Reprint: MIT Press, Cambridge, MA 1978.)

[59] C. Thiel, Under the assumption of the generalized Riemann hypothesis, verifying the class number belongs to NP ∩ co – NP, in: Algorithmic number theory (Ithaca, NY 1994), pp. 234–247, Lecture Notes in Computer Science 877, Springer-Verlag, Berlin 1994.

[60] C. Thiel, Short proofs using compact representations of algebraic integers, J. Complexity 11 (1995), 310–329.

[61] C. Thiel, On the complexity of some problems in algorithmic algebraic number theory, Ph. D. Thesis, Universität des Saarlandes, Saarbrücken 1995.

[62] B. A. Venkov, Elementary Number Theory, Wolters-Nordhoff Publ. Co., Groningen, The Netherlands, 1970.

[63] P. G. Walsh, A quantitative version of Runge’s theorem on diophantine equations, Acta Arith. 62 (1992), 157–172.

[64] M. Ward, The arithmetical theory of linear recurring series, Trans. Amer. Math. Soc. 35 (1933) 600–628.
[65] H. C. Williams, Solving the Pell equation, in :Number theory for the millennium, III (Urbana, IL 2000), 397–435, A K Peters: Natick, MA, 2002.

[66] H. C. Williams, G. W. Dueck and B. K. Schmid, A rapid method of evaluating the regulator and class number of a pure cubic field, Math. Comp. 41 (1983), 235–286.

[67] H. C. Williams and D. Shanks, A note on class number one in pure cubic fields, Math. Comp. 33 (1979) 1317–1320.