The temperature influence on quantum tunneling in the spin-boson model

Zhi-De Chen, Ying-Hua Huang, and Hang Wong
Department of Physics, Jinan University, Guangzhou 510632, China

The existing studies of the temperature effect of the spin-boson model show that the tunneling splitting will increase with temperature, no matter how weak the couplings are between the bath and the spin. However, the small polaron theory said that in the weak coupling and low temperature regime, the tunneling is, in fact, dominated by the diagonal transitions whereas this diagonal contribution to the tunneling should be faded away with increasing temperature. Taking advantage of the analysis originated from the Feynman-Vernon’s influence functional theory, the influence on the tunneling by the phonon bath can be expressed as a product of the effects of every single phonon mode, which can be studied by numerical exact diagonalization. We find that, in the weak coupling and low temperature regime, all the spin-single-mode systems exhibit the same decreasing dependence of tunneling, on increasing temperature. In turn, with the conclusion of influence functional theory, the tunneling of the spin-boson model should decrease and it is independent of the bath structure. In the strong coupling regime, however, the temperature effect reverses from suppressing to enhancing the tunneling with the increase of temperature. Discrepancies between the old theories and the small polaron theory are also explained.

PACS numbers: 03.65.Yz, 03.65.Xp, 73.40.Gk

I. INTRODUCTION

The spin-boson model which consists of a two-state system (TSS) coupled linearly to a phonon bath is a paradigm model of quantum dissipative systems. The study on this model has a long history and it still attracts a great deal of interest.1−19 The basic question is how thermal fluctuations from the environment influence quantum tunneling of the TSS since all the real systems are at finite temperature. The answer to this question is important for the observation of macroscopic quantum phenomena and also for the realization of the quantum computer where the stability of a “qubit” against thermal fluctuations is critical.20 Up to now, this issue has been addressed by various treatments,2,3,5,11,12,15 including numerical calculation,2 and it seems that all got the same conclusion, i.e., quantum tunneling is enhanced by the temperature in low temperature regions. However, besides the different quantitative results, the preconditions to the conclusion are also different. Using a perturbation approach combined with a renormalization group (RG) treatment based on path integral, Bray and Moore firstly found that quantum tunneling increases with temperature, presumed the coupling to the bath exceeds some critical value.2 Numerical calculation based on path-integral also showed that the same conclusion can be found in the strong coupling regime but it was also stated that the situation in weak coupling regime is qualitatively different from that in the strong coupling regime.2 On the other hand, the flow equation analysis12 and the variational calculation9,15 suggested the enhancement of quantum tunneling happens in the weak coupling and the low temperature regions. Notably a quantitative result was derived and physics analysis on the enhancing mechanism was also provided in the variational calculation for the sub-Ohmic case.12 In fact, the quantitative result found in the sub-Ohmic case is in agreement with that found by Weiss2 such a $T^{1+\delta}$ dependence was also considered as the universality in dissipative two-state system and was declared to be valid for all the coupling strength before the localization happens.21 Nevertheless, physics analysis from the well-known small polaron theory tells a different story.

It is known that the zero-biased spin-boson model is just a simple example of the polaron-phonon (or exciton-phonon) system, i.e., the two-site problem.21 According to the small polaron theory,22 the contribution to tunneling has two parts at finite temperature, say, the diagonal and non-diagonal contribution. In the weak coupling regime, the diagonal part makes the main contribution to quantum tunneling and hence quantum tunneling should decrease with increasing temperature in the low temperature regions, a result that is in confliction with some analysis of the spin-boson model mentioned above. The discrepancy between various treatments indicates that the consensus of temperature effect is still lacking and this is the main interest of the present work.

In this paper, the temperature effect on quantum tunneling in the spin-boson model is studied by analysis originated from the influence functional theory developed by Feynman and Vernon.23 One important conclusion of this theory is that the contribution of the whole bath can be expressed as a weighted integration over the contribution of single phonon modes; hence the temperature effect on quantum tunneling can be qualitatively known by studying the temperature effect in the single mode case which can be exactly done by numerical calculation. It is found that, in the weak coupling regime, all the phonon modes contribute the same temperature effect, i.e., quantum tunneling is suppressed by temperature, and the enhancing mechanisms by both the variational calculation15 and flow equation analysis12 do not exist in a real system. In the strong coupling regime, however, the temperature effect is reversed, in agreement with the
known results. The organization for the rest of the paper is as follows. In the next section, the model and the explanation of our calculation method are presented. Numerical results for both $T = 0$ and $T \neq 0$ cases are given in Sec. III. Conclusion and discussion are presented in the last section.

II. THE MODEL AND THE EXPLANATION

The Hamiltonian of the spin-boson model is given by (setting $\hbar = 1$)

$$\hat{H} = -\frac{\Delta_0}{2} \sigma_x + \sum_k \omega_k \hat{b}^\dagger_k \hat{b}_k + \sigma_z \sum_k g_k (\hat{b}^\dagger_k + \hat{b}_k),$$

(1)

where $\sigma_i (i = x, y, z)$ is the Pauli matrix, $\hat{b}_k (\hat{b}^\dagger_k)$ is the annihilation (creation) operator of the $k$th phonon mode with energy $\omega_k$ and $g_k$ is the coupling parameter. The solution of this model depends on the so-called bath spectral function (density) which is defined as

$$J(\omega) = \pi \sum_k g_k^2 \delta(\omega - \omega_k).$$

For frequencies below a high-energy cutoff $\omega_c$, the spectral function has the power law form

$$J(\omega) = \frac{\pi}{2} \alpha \omega^s/\omega_c^{s-1}, \quad 0 < \omega \leq \omega_c,$$

(2)

where $\alpha$ is a dimensionless coupling strength which characterizes the dissipation strength and the parameter $s$ characterizes the property of the bath, i.e., $0 < s < 1$, $s = 1$, and $s > 1$ represent, respectively, the sub-Ohmic, Ohmic, and super-Ohmic cases. It should be noted that the solution of the model (1) is merely determined by the spectral function $J(\omega)$ comes from the fact that the bath degree of freedom can be integrated out as Gaussian integrals, an ideas originated from the influence functional theory. The phonon bath can be considered as a distribution of single modes and the single modes can be treated as dynamically and statistically independent subsystems. An illustration of how the phonon bath with frequency lies in $\{0, \omega_c\}$ is divided into a series of subsystems, i.e., the single-mode systems, as shown in Fig. 1.

According to the influence functional theory, the total influence functional of the bath can be expressed as a product of that of each subsystem

$$\mathcal{F} = \prod_k \mathcal{F}_k,$$

(3)

where $\mathcal{F}_k$ is the influence functional of the $k$th mode. The validity of the above result is based on the assumption that the correlation between different modes can be ignored even when the interaction between the TSS and phonon modes is considered. This implies that the ground state of the whole system can be expressed as

$$|\Phi_0\rangle = \frac{1}{\sqrt{Z}} \sum_{\pm} |\pm\rangle \prod_k \langle \phi^\pm_k (0) | \phi^\pm_k (0) \rangle = \delta_{kk'},$$

(4)

where $|\pm\rangle$ is the eigen-state of $\sigma_z$ and $|\phi^\pm_k (0)\rangle$ is the eigenstate of the $k$th mode when it interacts with the TSS individually. Accordingly, one can find out the tunneling splitting

$$\Delta/\Delta_0 = \langle \Phi_0 | \sigma_x | \Phi_0 \rangle = \prod_k d_k (0),$$

(5)

where $d_k (0) = \langle \phi^+_k (0) | \phi^-_k (0) \rangle$ is the dressing factor contributed by the $k$th mode.

The above result can be generalized to the $T \neq 0$ case by assuming that all the subsystems are still statistically independent when the interactions with the TSS are considered. In the influence functional theory, a trace over each phonon mode is made in Eq. (3). In the present case, the generalization can be done in two steps. Firstly, all the excited states of the whole system are suggested to have the same form as the ground state given in Eq. (4), that is, the $n$th excited state with eigenvalue $E_n$ can be expressed as

$$|\Phi_n\rangle = \frac{1}{\sqrt{Z}} \sum_{\pm} |\pm\rangle \prod_k |\phi^\pm_k (n)\rangle,$$

(6)

where $\langle \phi^+_k (n) | \phi^+_k (n') \rangle = \delta_{kk'} \delta_{nn'}$, then the tunneling splitting at $T \neq 0$ can be found by

$$\Delta_T/\Delta_0 = \frac{1}{Z} \sum_n e^{-\beta E_n} \langle \Phi_n | \sigma_x | \Phi_n \rangle = \frac{1}{Z} \sum_n e^{-\beta E_n} \prod_k d_k (n),$$

(7)

where

$$Z = \sum_n e^{-\beta E_n}, \quad d_k (n) = \langle \phi^+_k (n) | \phi^+_k (n) \rangle.$$  

(8)

The assumption that each subsystem is still statistically independent means

$$Z = \prod_k z_k,$$

(9)
where $z_k = \sum_n e^{-\beta \epsilon_n(k)}$ is the partition function of the $k$th mode interacting with the TSS individually. Therefore, one can find

$$\Delta T / \Delta_0 = \prod_k D_k(T), \quad D_k(T) = \frac{1}{z_k} \sum_n e^{-\beta \epsilon_n(k)} d_k(n).$$

(10)

The above analysis shows that, by Eqs. (5) and (10), the contribution of the phonon bath to the tunneling splitting is a product of the contribution from every single mode. This is the starting point of our calculation. In the next section, the validity of such a simple product assumption (i.e., Eqs. (5) and (10)) will be verified numerically in a simple way.

Although other analytical treatments, such as the variational calculation and flow equation method, are not based on the influence functional theory, the expression of tunneling splitting found have exactly the same forms given in Eqs. (5) and (10). In fact, the dressing factor of a single mode can be expressed as the form of influence functional given in Ref. [2].

$$D_k(T) = \exp\{-\psi_k(T)\},$$

(11)

where $\psi_k(T)$ can be considered as the dressing phase of the $k$th mode. Since the solution should be determined by the spectral function only, a natural choice of the dressing phase is

$$\psi_k(T) = g_k^2 f(\omega_k, T),$$

(12)

which is just the form found by other analytical treatments, including the perturbation calculation, the variational calculation, and the flow equation method in spite of their self-consistent forms [3,2,12,16]. This implies that the ideas of influence functional theory has been tacitly adopted by analytical treatments generally. It should be noted that, in the present work, the dressing factor $D_k(T)$ is calculated numerically, and hence one can only get qualitative conclusion and no quantitative result can be found.

III. TEMPERATURE EFFECT ON QUANTUM TUNNELING

According to the above analysis, the influence of the bath on quantum tunneling can be known by solving the following single-mode system

$$\hat{h} = -\frac{\Delta_0}{2} \sigma_x + \omega \hat{b}^\dagger \hat{b} + \sigma_z \lambda (\hat{b}^\dagger + \hat{b}),$$

(13)

where $\lambda / \omega$ is a dimensionless coupling parameter. The above Hamiltonian is solved by numerical diagonalization using the basis $|\pm\rangle \otimes \{|n\rangle\}$, here $|n\rangle$ is the eigenstate of $\hat{b}^\dagger \hat{b}$ and $n = 0, 1, 2, \cdots$. To find out the dressing factor $D_k(T)$ in Eq. (10), we shall take the first $N$ eigenvalues $\{\epsilon_n\}$ and the corresponding eigenstates $\{|\varphi_n\rangle\}$ for calculation in practice, here $N$ is determined by

$$\beta(\epsilon_N - \epsilon_1) \geq L,$$

(14)

and $L$ is a fixed number to control the calculation error. It is found that the result becomes $N$-independent when $L \geq 20$. Since the sign of $\langle \varphi_n | \sigma_z | \varphi_n \rangle$ is not relevant for the calculation of the total tunneling splitting, we have

$$D_k(T) \approx \frac{1}{z} \sum_{n=1}^N e^{-\beta \epsilon_n} |\langle \varphi_n | \sigma_z | \varphi_n \rangle|,$$

(15)

where $z \approx \sum_{n=1}^N e^{-\beta \epsilon_n}$. We have compared numerical result with other approach using a different basis [23] and the result is the same within the calculation error. A detailed discussion on numerical diagonalization of the single mode Hamiltonian can be found in Ref. [24].

In the following, we shall present a simple verification of Eqs. (5) and (10) by numerical calculation. This is done in the following way. Firstly, the numerical solutions for two given modes with frequencies $\omega_1$ and $\omega_2$ and coupling parameters $\lambda_1 / \omega_1$ and $\lambda_2 / \omega_2$ are found separately, and the tunneling splitting of the ground state $\Delta_{1,2}$ and the temperature dependence $D_1,2(T)$ can be found by Eq. (15). Secondly, the numerical solution for a two-mode system with the same frequencies and coupling parameters are done in the same way, and the tunneling splitting of the ground state $\Delta_{1+2}$ and the temperature dependence $\Delta_{1+2}(T)$ can be found. According to Eqs. (5) and (10), one should have

$$\Delta_{1+2} = \Delta_1 \Delta_2, \quad \text{and} \quad \Delta_{1+2}(T) = \Delta_1(T) \Delta_2(T).$$

(16)

Typical result for the ground state (i.e., the case of $T = 0$) is shown in Fig. 2. It is seen that the validity of the above equation depends on parameters $\omega_1 / \omega_2$, $\omega_1,2 / \Delta_0$, and $\lambda / \omega_{1,2}$. However, one can find that the above equation holds quite well within the calculation errors when $\lambda / \omega_{1,2} \leq 0.1$. The result for the case of $T \neq 0$ is shown in Fig. 3. One can see that, in this case, the validity of the above equation also depends on the temperature region. The $\Delta_{1+2}(T) = \Delta_1(T) \Delta_2(T)$ holds for $\lambda / \omega_{1,2} < 0.1$ in the temperature regions with $k_B T / \Delta_0 < 0.1$. Since in a real thermodynamical system, the coupling parameter of each single mode should scale with $1/N$ and $N \to \infty$, one can conclude that Eqs. (5) and (10) hold in the low temperature region with $k_B T / \Delta_0 < 0.1$.

The above analysis shows that numerical solutions to the single-mode systems can serve as the solid basis for understanding the effect of the whole bath. As we have shown, tunneling splitting found by analytical treatments with various approximations have the form given in Eq. (11), this implies that numerical solution of the single-mode system can be used as a touchstone to test qualitatively the analytical results. Let us first see the result of the ground state. Figure 4 shows the dependence of the tunneling splitting of the ground state on the coupling
parameter for various frequencies. For comparison, we also present the result by adiabatic approximation, i.e., \( \Delta_n/\Delta_0 = |0\pm|0\rangle \), where \(|0\pm\rangle = \exp\{\mp(\lambda/\omega)(\hat{b}^\dagger - \hat{b})\}|0\rangle \) is the displaced-oscillator ground state. It shows that the numerical result agrees with the adiabatic approximation for \( \Delta_0/\omega \leq 0.1 \), but discrepancy becomes obvious when \( \Delta_0/\omega \geq 1/3 \) and the lower the frequency, the larger the deviation. The result shown in Fig. 4 provides a quantitative explanation for the fact that the usual Debye-Waller factor tells that the low-frequency modes lead to no effect on the tunneling splitting, while the Debye-Waller factor increases but both \( \Delta_{1+2} \) and \( \Delta_1\Delta_2 \) show similar \( \lambda/\omega_{1,2} \)-dependence. In the weak coupling regime with \( \lambda/\omega_{1,2} \leq 0.1 \), it is found that \( \Delta_{1+2} = \Delta_1\Delta_2 \) holds well within calculation error.

\[
\psi_k(T = 0) = -2\frac{g_k^2}{(\omega_k + \Delta)^2}. \tag{18}
\]

Comparing with the usual Debye-Waller factor, i.e., \( \psi_k(T = 0) = -2g_k^2/\omega_k^2 \), the above result effectively shifts the frequency origin from \( \omega = 0 \) to \( \omega = \Delta \), viz, the contribution of the low frequency modes with frequency \( \omega < \Delta \) is ignored, an ideas similar to adiabatic renormalization. As one can see in Fig. 4, such a treatment is reasonable since the low frequency modes have less effect on tunneling splitting in the weak coupling regime. This explains the fact that the variational ground state is a good approximation to the true ground state in the weak coupling regime. However, the variational calculation fails to trace the phonon-induced localization in the strong coupling regime.

Now we turn to see the temperature effect on tunneling splitting. Temperature dependence on the dressing factor for the single-mode system is calculated by Eq. (15) and some typical results are shown in Fig. 5. In the weak coupling regime, we have \( d\Delta_k(T)/dT < 0 \) for all the phonon modes. For a given coupling strength with

![Figure 2: Numerical verification of the simple product assumption in the case of \( T = 0 \) and \( \Delta_0/\omega_1 = 1/3 \).]
(\lambda/\omega) \leq 0.1$, the lower the frequency (i.e., the larger \(\Delta_0/\omega\)), the lower temperature where tunneling splitting starts to decrease. This is easy to understand since the contribution of the excited states appears at a higher temperature for a larger frequency. In the low temperature region with \(k_B T/\Delta_0 \leq 0.1\), the phonon modes with frequency \(\Delta_0/\omega < 0.1\) show almost no observable temperature effect.

FIG. 4: Dependence of the tunneling splitting of the ground state on the coupling parameter \((\lambda/\omega)\) for various frequencies. For comparison, the result by adiabatic approximation is shown in solid line. The numerical result agrees with the result by adiabatic approximation only when \(\Delta_0/\omega \leq 0.1\), and the lower the frequency the larger the discrepancy.

FIG. 3: Numerical verification of the simple product assumption in the case of \(T \neq 0\) with \(\lambda/\omega_{1,2} = 0.1\) and \(\Delta_0/\omega_1 = 1/3\). The deviation between \(\Delta_{1+2}(T)\) and \(\Delta_1(T)\Delta_2(T)\) becomes obvious as temperature increases but both \(\Delta_{1+2}(T)\) and \(\Delta_1(T)\Delta_2(T)\) always show similar temperature dependence. In the low temperature regions of \(k_B T/\Delta_0 \leq 0.1\), it is found that \(\Delta_{1+2}(T) = \Delta_1(T)\Delta_2(T)\) holds well in the weak coupling regime with \(\lambda/\omega_{1,2} < 0.1\).

FIG. 5: Temperature dependence on the dressing factor for the single-mode system in the case of \((\lambda/\omega) = 0.1\) for various frequencies. We have \(d\Delta_0(T)/dT < 0\) for all the phonon modes. In the low temperature region with \(k_B T/\Delta_0 \leq 0.1\), the phonon modes with frequency \(\Delta_0/\omega < 0.1\) show almost no observable temperature effect.
all the modes have the same temperature dependence, the result of whole bath according to Eq. (10) is qualitatively independent of the weight of different modes, namely, it holds for the sub-Ohmic, Ohmic, and super-Ohmic cases. Obviously, this conclusion is in conflict with some known results and explanations will be given below.

According to the variational calculation, temperature enhancing mechanism comes from the result that the dressing factors for frequency modes with frequency $k_B T < \omega < \Delta$ decrease with increasing temperature. We have searched for these enhancing modes in numerical calculation, however, no evidence for the existence of such enhancing modes can be found in the weak coupling regime. The searching work is done in the following way. We perform a variational calculation for a single-mode system given in Eq. (13) by the same way as done in Ref. 13, then a self-consistent equation for the tunneling splitting of the single-mode system can be found

$$\Delta_e(T)/\Delta_0 = \exp\{-(\lambda/\omega)^2 f_1(T,\omega)\},$$

where

$$f_1(T,\omega) = \frac{\coth(\beta\omega/2)}{\{\omega + \Delta_e(T)\tanh[\beta\Delta_e(T)/2] \coth(\beta\omega/2)\}^2},$$

from which the temperature dependence on tunneling splitting can be obtained numerically. It can be verified that, in the weak coupling regime ($\lambda/\omega \leq 0.1$), the variational ground state is a good approximation to the true ground state and the temperature-enhancing modes do exist as predicted in multi-mode case, that is, we have $d\Delta_e(T)/dT > 0$ when $k_B T < \omega < \Delta_e$. Some typical results are shown in Fig. 6, where one can see that, for the high frequency modes with $\Delta_e/\omega \leq 1$, the temperature dependence is qualitatively the same for both the variational calculation and numerical calculation that tunneling splitting decreases with increasing temperature. However, in the low frequency region with $\Delta_e/\omega > 1$, the result by variational calculation shows opposite temperature dependence to numerical result. This result indicates that the enhancing modes could be an artifact of the variational calculation in the case of $T \neq 0$. It should also be noted that another related prediction, the re-entrance phenomena has been shown to be impossible in a real system.

In the flow equation analysis, the enhancement of tunneling splitting by temperature comes from the existence of a frequency boundary $\omega^* = \Delta$. As one can see from Eq. (15), those modes with frequency $\omega < \omega^* = \Delta$ always enhance the tunneling splitting, while in the low temperature region with $k_B T < \Delta_0$, the main contribution of the whole bath comes from the modes with frequency $0 < \omega < \Delta$, this leads to the enhancement of tunneling splitting by temperature as shown in Ref. 12. However, as we have shown, there is no evidence for the existence of $\omega^*$, hence the enhancing mechanism in this case does not exist. We believe that, in the case of $T = 0$, the enhancement of the low frequency modes by Eq. (17) can compensate for the over-estimation of the usual Debye-Waller reduction of the high frequency modes, which leads to a good final result. In the case of $T \neq 0$, however, the fake enhancing mechanism can lead to a wrong temperature dependence in the low temperature regions.

In the above analysis, temperature enhancing mechanisms in the weak coupling regime by both the variational calculation and flow equation analysis have been ruled out. Now we turn to see what happens in the strong coupling regime. In this case, it is worth noting that the analysis in the strong coupling regime may not be as precisely as that in the weak coupling regime. This is because, in the strong coupling regime, the assumption of simple product (e.g., Eqs. (5) and (10)) does not hold exactly. However, our numerical result shows that, in the whole coupling range we studied, $\Delta_{1+2}(T)$ always shows similar temperature dependence as $\Delta_1(T)\Delta_2(T)$, and furthermore, to the first order approximation (i.e., by omitting the correlation between different modes), the effect of the whole bath can be considered as a weighted-integration over different modes, hence the temperature effect in the strong coupling regime can be approximately traced in the same way as that in the weak coupling regime. In the following, it will be shown that the temperature enhancing mechanism appears in the strong coupling regime.

According to Eq. (15), if the tunneling splittings of the first several excited states are larger than that of the ground state, the resulted $\Delta_k(T)$ will increase with temperature, otherwise $\Delta_k(T)$ will decreases with increasing temperature. In other words, one can see the temperature dependence by examining the spectrum of tunneling splitting. In the weak coupling regime and for all the frequency ranges we studied, $|\langle \phi_n | \hat{\sigma}_x | \phi_n \rangle|$ decreases with increasing $n$ for $n < 5$, leading to a decreasing tunneling splitting with temperature; however, as the coupling parameter $\lambda/\omega$ increases to some critical value, which is frequency-dependent, $|\langle \phi_n | \hat{\sigma}_x | \phi_n \rangle|$ turns to increase with $n$ and, of course, the temperature dependence reverses. Typical results are shown in Fig. 7, where one can see the variation of the spectrum of tunneling splitting and the reversed temperature dependence as the coupling strength exceeds some critical value. Since the reversed temperature dependence in the strong coupling regime is found for all the frequency range we studied, one can conclude that tunneling splitting increases with temperature in the spin-boson model provided that the coupling to the bath is strong enough, confirming the results having been found before. Such an evolution of temperature dependence with coupling to the bath is in agreement with physics analysis of the small polaron theory. According to the small polaron theory, the contribution to the tunneling has two parts: the diagonal and non-diagonal contribution. The diagonal contribution decreases with increasing temperature and can only survive in the weak coupling regime. On the other hand, the non-diagonal
contribution increases with the temperature and only becomes important when the coupling strength is not too low. In the weak coupling regime and low temperature region, the main contribution to the tunneling is the diagonal part, which decreases with increasing temperature. In the strong coupling regime, the diagonal contribution is suppressed and non-diagonal part makes the main contribution to the tunneling, hence tunneling splitting will increase with temperature. In the medium coupling range, a coherent-incoherent transition can happen when temperature increases as shown in Ref. 24, where one can also find a detail analytical analysis on temperature dependence of tunneling splitting for single-mode system by using the small polaron theory. The result shown in Fig. 7 provides an alternative explanation for the results found in Refs. 5 and 7.

IV. CONCLUSION AND DISCUSSION

In the previous sections, the temperature effect on tunneling splitting of the TSS is studied by a theory originated from the influence functional theory. It is shown that the effect of the whole bath can be expressed as a simple product of every single mode, then the single-mode system is solved by numerical diagonalization, from which the effect of the whole bath can be found. In the weak coupling regime, all the phonon modes have the same effect on tunneling splitting of the TSS, i.e., coupling to any phonon modes suppresses the tunneling splitting, and the suppressing effect increases with temperature. The result indicates that tunneling splitting will decrease with increasing temperature for the sub-Ohmic, Ohmic, and super-Ohmic cases. Temperature enhancing mechanisms by both variational calculation and the flow equation analysis have been ruled out by numerical analysis. The reversed temperature effect on tunneling splitting in the strong coupling regime is also explained. Our analysis shows that the temperature effect in the spin-boson model can be understood based on the small polaron theory. The present work can help to resolve the discrepancy between various treatments. However, in the present stage, it is not totally clear about the conclusion found in Refs. 2 and 11. Here we just want to emphasize two points. First of all, the temperature effect on quantum tunneling in the spin-boson model is hard to deal with since the usual Debye-Waller factor tends to zero even at $T = 0$ for both the Ohmic and sub-Ohmic case. This implies that the diagonal contribution to tunneling dies out by the small polaron theory and hence calculations done along this line is impossible to include the diagonal contribution. As we known, temperature dependence on the diagonal contribution is opposite to that of the non-diagonal contribution. Accordingly, it is possible to get a correct temperature dependence only when the calculation is made by a better approximation than the small polaron theory. Unfortunately most analytical calculations are done in similar way to the small polaron theory. By comparing the main result (see Eq. (9) in Ref. 11 and Eq. (3.36) in Ref. 5), it is seen that the result found in Ref. 11 is similar to that found along the line of small polaron theory. This seems...
to suggest that the diagonal contribution might be omitted in the calculation of Ref. 11. The second point is the difference between the TSS system and the double-well system. In a double-well system, there are excited states inside each well and the tunneling splitting increases exponentially with the excited state.\textsuperscript{25,26} The temperature dependence on tunneling splitting can be known without knowing the bath effect on tunneling splitting of each level. As the temperature increases, the occupation probability of the excited states increases, which makes the tunneling splitting increases with temperature.\textsuperscript{25,26} On the other hand, for a spin 1/2 system, there is not excited state for each direction. The temperature dependence on quantum tunneling depends on the spectrum of tunneling splitting when coupling to the bath. Unfortunately, even the bath effect on quantum tunneling in ground state is not exactly known in the spin-boson model. Some discussion on temperature effect given in Ref. 2 is for the double-well system, but not for the spin-boson model.

Acknowledgments

This work was supported by a grant from the Natural Science Foundation of China under Grant No. 10575045.

\textsuperscript{1} A. J. Leggett, S. Chakravarty, A. T. Dorsey, M. P. A. Fisher, A. Garg, and W. Zwerger, Rev. Mod. Phys. 59, 1 (1987) and references there in.

\textsuperscript{2} U. Weiss, Quantum Dissipative Systems, (World Scientific,
Singapore 1999).
3 S. Takagi, *Macroscopic Quantum Tunneling*, (Cambridge University Press, Cambridge, 2002).
4 R. P. Feynman and F. L. Vernon, Jr., Annals of Physics 281, 547 (2000) reprinted from Annals of Physics 24, 118 (1963).
5 A. J. Bray and M. A. Moore, Phys. Rev. Lett. 49, 1546 (1982).
6 J. P. Sethna, Phys. Rev. B 24, 698 (1982).
7 B. De Raedt and H. De Raedt, Phys. Rev. Lett. 50, 1926 (1983).
8 W. Zwerger, Z. Phys. B 53, 53 (1983).
9 R. Silbey and R. A. Harris, J. Chem. Phys. 80, 2615 (1984).
10 S. Chakravarty and S. Kivelson, Phys. Rev. B 32, 76 (1985).
11 M. Sassetti and U. Weiss, Phys. Rev. Lett. 65, 2262 (1990).
12 S. K. Kehrein and A. Mielke, Phys. Lett. A 219, 313 (1996).
13 R. Bulla, N.-H. Tong, and M. Vojta, Phys. Rev. Lett. 91, 170601 (2003).
14 R. Bulla, H.-J. Lee, N.-H. Tong, and M. Vojta, Phys. Rev. B 71, 045122 (2005).
15 A. Chin and M. Turlakov, Phys. Rev. B 73, 075311 (2006).
16 Z. Lü and H. Zheng, Phys. Rev. B 75, 054302 (2007).
17 H. Wong and Z.-D. Chen, Phys. Rev. B 76, 077301 (2007).
18 H. Wong and Z.-D. Chen, Phys. Rev. B 77, 174305 (2008).
19 Z.-D. Chen and H. Wong, Phys. Rev. B 78, 064308 (2008).
20 Y. Makhlin, G. Schön, and A. Shnirman, Rev. Mod. Phys. 73, 357 (2001).
21 H. B. Shore and L. M. Sander, Phys. Rev. B 7, 4537 (1973).
22 G. D. Mahan, *Many-Particle Physics*, (Plenum Press, New York 1990).
23 E. K. Irish, J. Gea-Banacloche, I. Martin, and K. C. Schwab, Phys. Rev. B 72, 195410 (2005).
24 Y.-H. Huang, H. Wong, and Z.-D. Chen, arXiv:0807.3475.
25 I. Affleck, Phys. Rev. Lett. 46, 388 (1981).
26 E. M. Chudnovsky, Phys. Rev. A 46 8011 (1992).