TRUNCATED TOEPLITZ OPERATORS OF FINITE RANK

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Abstract. We give a complete description of the finite-rank truncated Toeplitz operators.

1. Introduction

Truncated Toeplitz operators are compressions of multiplication operators on $L^2$ to model subspaces $K_\theta = H^2 \ominus \theta H^2$ of the Hardy class $H^2$, where $\theta$ is an inner function. In [1] D. Sarason gave a characterization of all truncated Toeplitz operators of rank one. Moreover, he constructed a class of finite-rank truncated Toeplitz operators and asked whether this class exhausts all truncated Toeplitz operators having finite rank. Our main result, Theorem 1, answers this question in the affirmative.

1.1. Definitions. As usual, we identify the Hardy class $H^2$ in the unit disk $\mathbb{D}$ with the subspace of the space $L^2$ on the unit circle $\mathbb{T}$ in a non-tangential boundary values. A function $\theta \in \bar{H}^2$ is called inner if $|\theta(z)| = 1$ almost everywhere with respect to the Lebesgue measure on $\mathbb{T}$. Denote by $P_\theta$ the orthogonal projection from $L^2$ onto $K_\theta$. The truncated Toeplitz operator $A_\varphi$ with symbol $\varphi \in L^2$ is the mapping

$$A_\varphi : f \mapsto P_\theta(\varphi f), \quad f \in K_\theta \cap L^\infty.$$  

We deal only with bounded truncated Toeplitz operators. For $\lambda \in \mathbb{D}$ define

$$k_\lambda = \frac{1 - \overline{\theta(\lambda)} \theta}{1 - \overline{\lambda} z}, \quad \tilde{k}_\lambda = \frac{\theta - \theta(\lambda)}{z - \lambda}.$$  

The function $k_\lambda$ is the reproducing kernel of the space $K_\theta$ at the point $\lambda$; $\tilde{k}_\lambda$ is the conjugate kernel at $\lambda$ (see Section 2 for details). For an integer $n \geq 0$, denote by $\Omega_n = \Omega(\theta, n)$ the set of all points $\lambda \in \mathbb{T}$ such that every function from $K_\theta$ and its derivatives up to order $n$ have non-tangential limits at $\lambda$. A description of $\Omega_n$ is given by P. R. Ahern and D. N. Clark in [2]; we discuss it in Section 2. In particular, for $\lambda \in \mathbb{T}$, we have $\lambda \in \Omega_0$ if and only if $k_\lambda, \tilde{k}_\lambda \in K_\theta$.  

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The only compact Toeplitz operator on the Hardy space $H^2$ is the zero operator \([3]\). The situation is different for Toeplitz operators on $K_0$. It was proved in \([1]\) that the general rank-one truncated Toeplitz operator on $K_0$ is a scalar multiple of

\[
k_\lambda \otimes \tilde{k}_\lambda \quad \text{or} \quad \tilde{k}_\lambda \otimes k_\lambda,
\]

for some $\lambda \in \mathbb{D} \cup \Omega_0$, where we use the standard notation for rank-one operators in the Hilbert space: $x \otimes y : h \mapsto (h, y)x$. Some finite-rank truncated Toeplitz operators can be obtained from the rank-one operators \([3]\) by differentiation. Consider the analytic mapping $\Phi : \lambda \mapsto \tilde{k}_\lambda \otimes k_\lambda$ from the unit disk $\mathbb{D}$ into the space of all bounded operators on $K_0$. Take an integer $n \geq 1$. Denote by $D^n[\tilde{k}_\lambda \otimes k_\lambda]$ the value $\Phi^{(n)}(\lambda)$ of the $n$-th derivative of $\Phi$ at the point $\lambda \in \mathbb{D}$. For $\lambda \in \Omega_n$ let $D^n[\tilde{k}_\lambda \otimes k_\lambda]$ denote the $n$-th angular derivative of $\Phi$ at $\lambda$. Existence of this derivative follows from results of P. R. Ahern and D. N. Clark and from representation \([11]\) of $D^n[\tilde{k}_\lambda \otimes k_\lambda]$ in terms of derivatives of reproducing kernels; see Section 2.2 for details. Since the set of all truncated Toeplitz operators is linear space closed in the weak operator topology \([1]\), the derivative $D^n[\tilde{k}_\lambda \otimes k_\lambda]$ determines the truncated Toeplitz operator of rank $n$ for every $\lambda \in \mathbb{D} \cup \Omega_n$. Parallel arguments work for the adjoint operator $D^n[k_\lambda \otimes \tilde{k}_\lambda]$; it is the derivative of $k_\lambda \otimes \tilde{k}_\lambda$ of order $n$ with respect to $\lambda$. Operators of the form

\[
D^n[\tilde{k}_\lambda \otimes k_\lambda], \quad D^n[\tilde{k}_\lambda \otimes k_\lambda], \quad n \geq 0, \quad \lambda \in \mathbb{D} \cup \Omega_n,
\]

were originally constructed in \([1]\) as examples of finite-rank truncated Toeplitz operators that are not linear combinations of the rank-one operators \([3]\).

1.2. The main result.

**Theorem 1.** The general finite-rank truncated Toeplitz operator on $K_0$ is a finite linear combination of the operators in \([4]\).

The key step in proving Theorem \([1]\) is the identification of the range of a general finite-rank truncated Toeplitz operator. Set

\[
F(\lambda, n) = \begin{cases} 
\text{Ran } D^n[\tilde{k}_\lambda \otimes k_\lambda], & \text{if } \lambda \in \mathbb{D} \cup \Omega_n, \\
\text{Ran } D^n[\tilde{k}_\lambda^* \otimes k_\lambda^*], & \text{if } \lambda \in \mathbb{D}_c,
\end{cases}
\]

where $\mathbb{D}_c = \{ z : |z| > 1 \} \cup \{ \infty \}$, $\lambda^* = 1/\bar{\lambda}$ and $\infty^* = 0$. It will be shown in Section 2.3 that a subspace $F \subset K_0$ is the range of an operator in \([4]\) if and only if we have $F = F(\lambda, n)$ for some $n \geq 0$ and $\lambda \in \mathbb{D} \cup \Omega_n \cup \mathbb{D}_c$. Theorem \([1]\) will be proved as soon as we establish the following three results.

**Lemma 1.1.** Suppose $A$ is a finite-rank truncated Toeplitz operator on $K_0$. Then there exists a finite collection of points $\lambda_k \in \mathbb{D} \cup \Omega_{n_k} \cup \mathbb{D}_c$ such that

\[
\text{Ran } A = F(\lambda_1, n_1) + \ldots + F(\lambda_s, n_s).
\]

**Lemma 1.2.** Suppose $A$ is a truncated Toeplitz operator on $K_0$ with range of the form \([5]\). Then $A$ is a sum of $s$ truncated Toeplitz operators $A_k$ such that $\text{Ran } A_k = F(\lambda_k, n_k)$.

**Lemma 1.3.** Suppose $A$ is a truncated Toeplitz operator on $K_0$ with the range $\text{Ran } A = F(\lambda, n)$, where $\lambda \in \mathbb{D} \cup \Omega_n \cup \mathbb{D}_c$. Then $A$ is a finite linear combination of the operators in \([4]\).
In Section 2 we collect some standard results about the space $K_\theta$. Section 3 concerns a description of the range of a general bounded truncated Toeplitz operator. Lemmas 1.1, 1.2, and 1.3 will be proved in Section 4.

1.3. A duality approach. The rank-one truncated Toeplitz operators can be regarded as a point evaluation in a special Banach space of analytic functions. Given an inner function $\theta$, define the space $X_a$ by

$$X_a = \{ \sum_0^\infty x_k y_k : x_k, y_k \in K_\theta \text{ and } \sum_0^\infty \|x_k\| \cdot \|y_k\| < \infty \}.$$ 

This space was introduced in [4], where the fact that $X_a$ is the predual of the space of all bounded truncated Toeplitz operators was established. The pairing is given by

$$\langle A, \sum x_k y_k \rangle = \sum (Ax_k, y_k).$$

It was shown in [4] that $\langle \tilde{k}_\lambda \otimes k_\lambda, xy \rangle = (x\tilde{y})(\lambda)$, where $\tilde{y} = \bar{z}\theta \bar{y} \in K_\theta$. This yields the nice formula

$$\langle D^n[\tilde{k}_\lambda \otimes k_\lambda], \sum x_k y_k \rangle = \left( \sum x_k \tilde{y}_k \right)(\lambda).$$

A similar relation holds for the operators $D^n[\tilde{k}_\lambda \otimes \tilde{k}_\lambda]$. In the sense of the pairing (6), the operator $k_\lambda \otimes \tilde{k}_\lambda$ is the point evaluation at $\lambda^* = 1/\bar{\lambda}$ of functions from $X_a$ up to the scalar $(\lambda^*/\theta(\lambda^*))^2$, and $D^n[k_\lambda \otimes \tilde{k}_\lambda]$ is its derivative with respect to $\lambda$. Since $|\lambda^*| \geq 1$, the point evaluation should be understood in terms of pseudocontinuations of functions from $K_\theta$ to the exterior of the unit disk (see Lecture II in [5]).

1.4. Notation.

$\mathbb{Z}_+$ is the set of non-negative integers;
$\mathcal{T}_\theta$ is the linear space of all bounded truncated Toeplitz operators on $K_\theta$;
$\text{Ker } A$ is the kernel of a bounded operator $A$;
$\text{Ran } A$ is the range of a bounded operator $A$;
$\overline{\text{Ran } A}$ is the closure of $\text{Ran } A$;
$\langle f_1, \ldots, f_n \rangle = \text{span}\{f_j, j = 1, \ldots, n\}$;
$H^\perp$ is the orthogonal complement to a subspace $H$;
$H_1 + H_2$ is the linear span of the union $H_1 \cup H_2$;
$H_1 \oplus H_2$ is the direct sum of two subspaces $H_1, H_2$.

2. Preliminaries

This section contains the preliminary information concerning spaces $K_\theta$ and truncated Toeplitz operators: reproducing kernels, Clark unitary perturbations, and Sarason’s characterization of truncated Toeplitz operators. A more detailed discussion is available in [1], [5], [6].

2.1. The conjugation. The space $K_\theta$ is closed under the conjugation

$C : x \mapsto \bar{z}\theta \bar{x}$.

Truncated Toeplitz operators are complex symmetric with respect to $C$, which means $CA = A^*C$; see [1]. Hence the ranges of $A$ and $A^*$ are mutually conjugate for every operator $A \in \mathcal{T}_\theta$: $\text{Ran } A^* = C \text{Ran } A$. For more information on this property of truncated Toeplitz operators see [7], [8].
2.1. Reproducing kernels and their derivatives. For \( \lambda \in \mathbb{D} \), set
\[
(8) \quad k_\lambda = \frac{1 - \bar{\theta}(\lambda)\theta}{1 - \lambda \bar{z}}, \quad \tilde{k}_\lambda = \frac{\theta - \theta(\lambda)}{z - \lambda}.
\]
Note that \( \tilde{k}_\lambda = Ck_\lambda \), where the conjugation \( C \) is defined by (7). The function \( k_\lambda \) (respectively, \( \tilde{k}_\lambda \)) is the reproducing kernel (respectively, conjugate reproducing kernel) of the space \( K_\theta \) at the point \( \lambda \):
\[
(9) \quad (f, k_\lambda) = f(\lambda), \quad (f, \tilde{k}_\lambda) = (Cf)(\lambda).
\]
Differentiating (9), we obtain
\[
(10) \quad (f, \partial^\alpha k_\lambda) = f^{(\alpha)}(\lambda), \quad (f, \partial^\alpha \tilde{k}_\lambda) = (Cf)^{(\alpha)}(\lambda).
\]
In what follows, the symbols \( \partial^\alpha k_\lambda \) and \( \partial^\alpha \tilde{k}_\lambda \) denote the \( n \)-th derivatives of \( k_\lambda, \tilde{k}_\lambda \) with respect to \( \lambda, \lambda \), respectively. For example, in the case \( n = 1 \) we have
\[
\partial k_\lambda = \lim_{\mu \to \lambda} \frac{k_\lambda - k_\mu}{\lambda - \mu}, \quad \partial \tilde{k}_\lambda = \lim_{\mu \to \lambda} \frac{\tilde{k}_\lambda - \tilde{k}_\mu}{\lambda - \mu}.
\]
Let \( \theta = B_\Lambda S_\nu \) be an inner function with the set of zeroes \( \Lambda = (a_k)_{k=1}^N \), repeated according to multiplicity, and the singular part \( S_\nu \) that corresponds to a singular measure \( \nu \) on the unit circle \( \mathbb{T} \). Take a point \( \lambda \in \mathbb{T} \) and an integer \( n \in \mathbb{Z}_+ \). The following result is in [2]; see also [9].

**Theorem 2** (P. R. Ahern and D. N. Clark). The following are equivalent:

(a) functions \( f, f', \ldots, f^{(n)} \) have non-tangential limits at \( \lambda \) for every \( f \in K_\theta \);
(b) the non-tangential limits \( \partial^j k_\lambda = \lim_{\mu \to \lambda} \partial^j k_\mu \) and \( \partial^j \tilde{k}_\lambda = \lim_{\mu \to \lambda} \partial^j \tilde{k}_\mu \) exist in norm of \( K_\theta \) for every \( j = 0, \ldots, n \);
(c) \( \sum_{k=1}^N \frac{1 - |a_k|^2}{|1 - \lambda a_k|^{n+2}} < \infty \) and \( \int_\mathbb{T} \frac{d\nu(z)}{|1 - \lambda z|^{n+2}} < \infty \).

Denote by \( \Omega_n = \Omega(\theta, n) \) the set of all points \( \lambda \in \mathbb{T} \) that satisfy the conditions of Theorem 2. For \( \lambda \in \mathbb{D} \) we have
\[
(11) \quad A_{\varphi} = \mathbb{D}^n[k_\lambda \otimes \tilde{k}_\lambda] = \sum_{k=0}^n C_n^k \left( \partial^k k_\lambda \otimes \partial^{n-k} \tilde{k}_\lambda \right), \quad \varphi(z) = \frac{n! \cdot \overline{\theta(z)}}{(z - \lambda)^{n+1}},
\]
by using the Leibniz formula for derivatives of a bilinear expression. It follows from Theorem 2 that formulas (8)-(11) hold in the sense of non-tangential boundary values for every \( \lambda \in \Omega_n \). Thus, the operators in (11) are exactly the operators in (11) for \( n \in \mathbb{Z}_+ \) and \( \lambda \in \mathbb{D} \cup \Omega_n \).

2.3. The restricted shift. Consider the operator \( S_\theta : f \mapsto P_\theta(zf) \). We have
\[
(12) \quad S_\theta k_\lambda = (k_\lambda - k_0)/\lambda, \quad S_\theta \tilde{k}_\lambda = \lambda \tilde{k}_\lambda - \theta(\lambda)k_0,
\]
where \( \lambda \neq 0 \) is a point from \( \mathbb{D} \cup \Omega_0 \). In particular, if \( \theta(0) = 0 \), then \( \tilde{k}_0 \in \text{Ker} S_\theta \). Actually, we have \( \text{Ker} S_\theta = \langle k_0 \rangle \) in that case.

**Proposition 2.1.** We have \( S_\theta^n k_0 = \frac{1}{n!} \partial^\alpha k_0 \) and \( S_\theta \partial^\alpha \tilde{k}_0 = n \partial^{n-1} \tilde{k}_0 - \theta(0)k_0 \) for every integer \( n \geq 1 \).
Proof. Take a function \( f \in K_\theta \) and consider \((f, S^n_\theta k_0) = ((S^n_\theta)^n f, k_0)\). The space \( K_\theta \) is invariant under the backward shift operator \( S^*: f \mapsto (f - f(0))/z \); see \[5\]. Hence \( S^*|_{K_\theta} = S^n_\theta \) and \( ((S^n_\theta)^n f, k_0) = \frac{1}{n!} f(n)(0) \). It follows from \[\ref{10}\] that \( f^{(n)}(0) = (f, \tilde{\partial}^n k_0) \), and therefore \((f, S^n_\theta k_0) = \frac{1}{n!} f(n)(0)\). Since this equality holds for every function \( f \in K_\theta \), we obtain \( S^n_\theta k_0 = \frac{1}{n!} \tilde{\partial}^n k_0 \). Differentiating the identity \( S_\theta \tilde{k}_\lambda = \lambda \tilde{k}_\lambda - \theta(\lambda) k_\lambda \) with respect to \( \lambda \) at the point \( \lambda = 0 \), we get the formula \( S_\theta \tilde{k}_\lambda = n \partial^{n-1} \tilde{k}_0 - \theta^{(n)}(0) k_0 \). \( \square \)

Consider the subspaces \( F(\lambda, n) \) defined in Section \[1.2\]. It follows from formula \[\ref{11}\] that

\[
F(\lambda, n) = \begin{cases} 
\text{span}\{\tilde{\partial}^j k_\lambda, \ j = 0, \ldots, n\}, & \text{if } \lambda \in \mathbb{D} \cup \Omega_n; \\
\text{span}\{\partial^j \tilde{k}_\lambda, \ j = 0, \ldots, n\}, & \text{if } \lambda \in \mathbb{D}_e.
\end{cases}
\]

For each point \( \lambda \in \Omega_0 \) we have \( \lambda^* = \lambda \) and \( \tilde{k}_\lambda = \tilde{\lambda} \theta(\lambda) k_\lambda \). Therefore,

\[
\text{span}\{\tilde{\partial}^j k_\lambda, \ j = 0, \ldots, n\} = \text{span}\{\partial^j \tilde{k}_\lambda, \ j = 0, \ldots, n\}
\]

for all \( \lambda \in \Omega_n \) and \( n \in \mathbb{Z}_+ \). We now see from \[\ref{11}\] that the range of the operator \( \mathbb{D}^n[k_\lambda \otimes \tilde{k}_\lambda] \) coincides with the range of the operator \( \mathbb{D}^n[\tilde{k}_\lambda \otimes k_\lambda] \) if \( \lambda \in \Omega_n \). Therefore, a subspace \( F \subset K_\theta \) is the range of an operator in \[\ref{14}\] if and only if we have \( F = F(\lambda, n) \) for some \( n \geq 0 \) and \( \lambda \in \mathbb{D} \cup \Omega_n \cup \mathbb{D}_e \), as claimed in Section \[1.2\].

**Proposition 2.2.** We have

\[
S_\theta F(\lambda, n) \subset F(\lambda, n) + \langle k_0 \rangle, \text{ if } \lambda \neq 0;
\]

\[
S_\theta F(0, n) \subset F(0, n) + \langle \tilde{\partial}^{n+1} k_0 \rangle.
\]

**Proof.** The first formula in \[\ref{14}\] can be obtained from \[\ref{12}\] by differentiation. The second one follows from Proposition \[2.1\]. \( \square \)

2.4. **The Frostman shift.** Let \( \theta \) be an inner function. The Frostman shift of \( \theta \) corresponding to the point \( \theta(0) \) is the inner function \( \Theta = \frac{\theta - \theta(0)}{1 - \theta(0) \theta} \). We have \( \Theta(0) = 0 \). Define the unitary operator \( J : K_\theta \to K_\Theta \) by

\[
J : f \mapsto \frac{\sqrt{1 - |\theta(0)|^2}}{1 - \theta(0) \theta} f.
\]

**Proposition 2.3.** For \( \lambda \in \mathbb{D} \) and \( n \in \mathbb{Z}_+ \) we have

\[
J \text{span}\{\tilde{\partial}^j k_\lambda, \ j = 0, \ldots, n\} = \text{span}\{\tilde{\partial}^j k_\lambda^\Theta, \ j = 0, \ldots, n\}.
\]

where \( k_\lambda^\Theta \) is the reproducing kernel of the space \( K_\Theta \) at the point \( \lambda \).

**Proof.** The formula \( J k_\lambda = (1 - \theta(0) \theta(\lambda))/\left(\sqrt{1 - |\theta(0)|^2}\right) k_\lambda^\Theta \) follows from the definition of \( J \) and implies \[\ref{15}\] by differentiation; see details in Section 13 of \[1\]. \( \square \)

The following fact is a particular case of Theorem 13.2 from \[1\].

**Proposition 2.4.** A bounded operator \( A \) is a truncated Toeplitz operator on \( K_\theta \) if and only if the operator \( J A J^{-1} \) is a truncated Toeplitz operator on \( K_\Theta \).
2.5. Clark’s unitary perturbations. In [9] Clark described one-dimensional unitary perturbations of $S_\theta$. Given a number $\alpha \in \mathbb{T}$, define

$$
U_\alpha = S_\theta + c_\alpha k_0 \otimes \bar{k}_0,
$$

where $k_0, \bar{k}_0$ are the reproducing kernels of $S$ at the origin. The operators $U_\alpha$ are unitary and cyclic; every one-dimensional unitary perturbation of $S_\theta$ is $U_\alpha$ for an appropriate number $\alpha \in \mathbb{T}$; see [9]. It is shown in [9] that the spectral measure $\sigma_\alpha$ of the unitary operator $U_\alpha$ can be chosen so that

$$
U_\alpha = V_\alpha^{-1} M_\alpha V_\alpha,
$$

where $V_\alpha : K_\theta \to L^2(\sigma_\alpha)$ is the unitary operator that sends functions from a dense subset of $K_\theta$ to their boundary values on $\mathbb{T}$; $M_\alpha$ is the operator of multiplication by $\alpha$ on $L^2(\sigma_\alpha)$.

A. G. Poltoratski [10] established the existence of non-tangential boundary values $\sigma_\alpha$-almost everywhere for all functions from $K_\theta$. Thus, the operator $V_\alpha$ is the well-defined unitary embedding $K_\theta \to L^2(\sigma_\alpha)$. For every function $f \in K_\theta$ we have

$$
f(z) = \int_\mathbb{T} (V_\alpha f)(\xi) \frac{1 - \bar{\alpha}\theta(z)}{1 - \bar{\xi}z} d\sigma_\alpha(\xi), \quad z \in \mathbb{D}.
$$

The following fact is due to Clark [9]; see also [6].

**Proposition 2.5** (D. N. Clark). We have $\sigma_\alpha\{\xi \in \mathbb{T} : \theta(\xi) = \alpha\} = \sigma_\alpha(\mathbb{T})$ for each $\alpha \in \mathbb{T}$. The measure $\sigma_\alpha$ has an atom at a point $\lambda \in \mathbb{T}$ if and only if $\lambda \in \Omega_0$ and $\theta(\lambda) = \alpha$. In that case we have $V_\alpha^{-1} \mathbb{I}_{\{\lambda\}} = \sigma_\alpha(\{\lambda\}) \cdot k_\lambda$, where $\mathbb{I}_{\{\lambda\}}$ denotes the indicator of the singleton $\{\lambda\}$.

**Proposition 2.6.** If $(z - \lambda)^{-n-1} \in L^2(\sigma_\alpha)$ for some $\lambda \in \mathbb{T}$ and $n \in \mathbb{Z}_+$, then $\lambda \in \Omega_n$ and $\theta(\lambda) \neq \alpha$.

**Proof.** At first, let $\theta(0) = 0$. In this case the constants lie in the space $K_\theta$. Formula (18) with $f \equiv 1$ gives us

$$
\frac{1}{1 - \bar{\alpha}\theta(z)} = \int_\mathbb{T} d\sigma_\alpha(\xi) \frac{1}{1 - \bar{\xi}z}, \quad z \in \mathbb{D}.
$$

We have

$$
\frac{1}{|1 - \bar{\alpha}\theta(z)|} \leq \frac{c}{|1 - \bar{\xi}\lambda|}, \quad \xi \in \mathbb{T},
$$

for some constant $c$, as $z$ tends non-tangentially to $\lambda$. Since $(1 - \bar{\xi}\lambda)^{-1} \in L^2(\sigma_\alpha)$, we see from (19) and (20) that the function $\theta$ has a non-tangential limit at $\lambda$ and $\theta(\lambda) \neq \alpha$. Similarly, differentiating (19) with respect to $z$, one can prove that $\theta'$, $\theta''$, $\ldots$, $\theta^{(n)}$ also have non-tangential limits at $\lambda$. In the case $\theta(0) \neq 0$ this fact follows from the consideration of the Frostman shift of the function $\theta$.

Take $f \in K_\theta$ and consider $f^{(j)}$, where $0 \leq j \leq n$ is integer. It follows from (18) that

$$
f^{(j)}(z) = \int_\mathbb{T} (V_\alpha f)(\xi) \left( \frac{1 - \bar{\alpha}\theta(z)}{1 - \bar{\xi}z} \right)^{(j)} d\sigma_\alpha(\xi), \quad z \in \mathbb{D}.
$$

Since $(1 - \bar{\lambda}z)^{-j-1} \in L^2(\sigma_\alpha)$, we see from (20) and (21) that $f^{(j)}$ has the non-tangential limit at the point $\lambda$ for every $j = 0, \ldots, n$. Thus, we have $\lambda \in \Omega_n$. □
We now describe boundary values of functions from subspaces $F(\lambda, n)$ in (13).

**Proposition 2.7.** Let $\alpha \in T$, $n \in \mathbb{Z}_+$ and let $\lambda \in \mathbb{D} \cup \Omega_n \cup \mathbb{D}_e$, where in the case $|\lambda| = 1$ we assume that $\theta(\lambda) \neq \alpha$. We have

\begin{equation}
V_\alpha F(\lambda, n) = \text{span}\{ (z - \lambda^*)^{-j}, \ j = 1, \ldots, n + 1 \}, \quad \lambda \neq 0, \ \lambda^* = 1/\bar{\lambda},
\end{equation}

\begin{equation}
V_\alpha F(0, n) = \text{span}\{ z^j, \ j = 0, \ldots, n \}.
\end{equation}

**Proof.** In the case $|\lambda| \neq 1$ formula (22) follows from the definition of $V_\alpha$. Formula (23) is a consequence of Proposition 2.7. Take a point $\lambda = \lambda^*$ from $\Omega_0$ such that $\theta(\lambda) \neq \alpha$. By Proposition 2.5 we have $\sigma_\alpha(\{ \lambda \}) = 0$ and

$$V_\alpha k_\lambda = \frac{1 - \theta(\lambda)\alpha}{1 - \bar{\lambda}z}$$

$s_\alpha$–almost everywhere. Since $V_\alpha k_\lambda \in L^2(\sigma_\alpha)$, we have $(z - \lambda)^{-1} \in L^2(\sigma_\alpha)$, and therefore (22) holds in the case $n = 0$. For $n = 1$ and $\lambda \in \Omega_1$, consider

$$V_\alpha \partial_k k_\lambda = \frac{-\bar{\theta}(\lambda)\alpha}{1 - \lambda z} + \frac{z(1 - \theta(\lambda)\alpha)}{(1 - \bar{\lambda}z)^2}.$$ 

Since $V \partial k_\lambda \in L^2(\sigma_\alpha)$ and $(z - \lambda)^{-1} \in L^2(\sigma_\alpha)$, we have $(z - \lambda)^{-2} \in L^2(\sigma_\alpha)$. Hence formula (22) holds in the case $n = 1$. Arguing as above, we prove (22) for all $\lambda \in \Omega_n$ and $n \in \mathbb{Z}_+$. □

**Proposition 2.8.** Suppose that an inner function $\theta$ is not a finite Blashke product. Then every finite collection of the functions $\partial^{s_k} k_{\lambda_k}, \partial^{l_k} k_{\mu_k}$, where $\lambda_k \in \mathbb{D} \cup \Omega_{s_k}$ and $\mu_k \in \mathbb{D} \cup \Omega_{l_k}$, is linearly independent in $K_\theta$.

**Proof.** Since $\theta$ is not a finite Blashke product, the space $K_\theta$ has infinite dimension; see Lecture II in [5]. Hence the space $L^2(\sigma_\alpha), |\alpha| = 1$, has infinite dimension as well. The result now follows from Proposition 2.7. □

2.6. **A characterization.** In what follows we will often use the following characterization of truncated Toeplitz operators.

**Theorem 3** (D. Sarason, [11]). A bounded operator $A$ on $K_\theta$ is a truncated Toeplitz operator if and only if there exist functions $\psi, \chi$ in $K_\theta$ such that

\begin{equation}
A - S_\theta S_\theta^* = \psi \otimes k_0 + k_0 \otimes \chi,
\end{equation}

in which case $A = A_{\psi + \bar{\chi}}$.

**Remark 2.9.** Suppose $\theta(0) = 0$; then $\psi = Ak_0 - \bar{\chi}(0)k_0$.

**Proof.** Apply both sides of (24) to the vector $k_0 \equiv 1$ and use the relation $S_\theta^* 1 = 0$. □

3. **The range of a bounded truncated Toeplitz operator**

In this section we prove the following result.

**Proposition 3.1.** Let $A \in T_\theta$, and assume that $\text{Ran} A \neq K_\theta$. Then

\begin{equation}
S_\theta \text{Ran} A \subset \text{Ran} A + (\partial^j k_0),
\end{equation}

where $n \in \mathbb{Z}_+$ is the maximal integer such that $\partial^j k_0 \in \text{Ran} A$ for every $0 \leq j < n$.

For the proof we need two lemmas.
Lemma 3.2. Let $\theta(0) = 0$, $A \in \mathcal{T}_\theta$. Then:

1. $S_\theta A \left[ \langle \hat{k}_0 \rangle^\perp \right] \subset \text{Ran} \overline{A} + \langle k_0 \rangle$;
2. $S_\theta \text{Ran} \overline{A} \subset \text{Ran} \overline{A} + \langle k_0, g \rangle$ for some $g \in K_\theta$.

Proof. Indeed, by Theorem 3.1 and Remark 2.4 we have $S_\theta A^* h \in \text{Ran} \overline{A} + \langle k_0 \rangle$ for every vector $S_\theta h$. \hfill \Box

Lemma 3.3. Let $\theta(0) = 0$, $A \in \mathcal{T}_\theta$. Assume that $k_0 \notin \text{Ran} \overline{A}$. Then

$S_\theta \text{Ran} \overline{A} \subset \text{Ran} \overline{A} + \langle k_0 \rangle$.

Proof. Denote by $P_1, P_2$ the orthogonal projections on $K_\theta$ with the ranges $\text{Ran} \overline{A}^*$ and $\langle \hat{k}_0 \rangle$, respectively. Note that $A = AP_1$. For every $h \in K_\theta$ we have

$S_\theta Ah = S_\theta AP_1 h = S_\theta AP_2 P_1 h + f_1$,

where $f_1 = S_\theta AP_2^* P_1 h$. It follows from assertion (1) of Lemma 3.2 that $f_1 \in \text{Ran} \overline{A} + \langle k_0 \rangle$. Proceeding inductively, we obtain the relation

(26) $S_\theta Ah = S_\theta A(P_2 P_1)^n h + f_n$

for some $f_n \in \text{Ran} \overline{A} + \langle k_0 \rangle$. Since $\text{Ran} A^* = C \text{Ran} A$ (see Section 2.1) and $k_0 \notin \text{Ran} \overline{A}$, we have $h \notin \text{Ran} \overline{A}$, and therefore $\|P_2 P_1\| < 1$. Passing to the limit in (26), we see that $S_\theta Ah \in \text{Ran} \overline{A} + \langle k_0 \rangle$ and the result follows. \hfill \Box

Proof of Proposition 3.1. At first, let $\theta(0) = 0$. In the case $k_0 \notin \text{Ran} \overline{A}$, Lemma 3.3 gives us formula (25) with $n = 0$. Now suppose that $k_0 \in \text{Ran} \overline{A}$. Since the family $\{\partial^j k_0\}_{j=0}^\infty$ is complete in $K_\theta$ (see formula (10)), one can choose the maximal integer $n \geq 1$ such that $\partial^j k_0 \in \text{Ran} \overline{A}$ for every $0 \leq j < n$ and $\partial^n k_0 \notin \text{Ran} \overline{A}$. It follows from Proposition 2.4 and Lemma 3.2 that

(27) $\partial^n k_0 = S_\theta \partial^{n-1} k_0 \in S_\theta \text{Ran} \overline{A} \subset \text{Ran} \overline{A} + \langle k_0, g \rangle = \text{Ran} \overline{A} + \langle g \rangle$

for some $g \in K_\theta$ such that $S_\theta \text{Ran} \overline{A} \subset \text{Ran} \overline{A} + \langle k_0, g \rangle$. By the construction we have $\partial^n k_0 \notin \text{Ran} \overline{A}$. Comparing this with (27), we get $g \in \text{Ran} \overline{A} + \langle \partial^n k_0 \rangle$, and thus

$S_\theta \text{Ran} \overline{A} \subset \text{Ran} \overline{A} + \langle k_0, g \rangle \subset \text{Ran} \overline{A} + \langle \partial^n k_0 \rangle$.

Hence the proposition is proved in the case $\theta(0) = 0$. The general situation can be reduced to this case by using Propositions 2.3 and 2.4. \hfill \Box

Remark. Truncated Toeplitz operators are symmetric with respect to the conjugation $C$; see Section 2.1. Using this fact, one can obtain an analogue of Proposition 3.1 for $\text{Ran} \overline{A}$ with $S_\theta^*$ and $\partial^n \hat{k}_0$ in place of $S_\theta$ and $\partial^n k_0$.

Let $U_\alpha = S_\theta + c_\alpha \hat{k}_0 \otimes \hat{k}_0$ be the Clark unitary perturbation of $S_\theta$. Consider the embedding $V_\alpha : K_\theta \to L^2(\sigma_\alpha)$ from Section 2.5.

Proposition 3.4. Let $A \in \mathcal{T}_\theta$, and assume that $\text{Ran} \overline{A} \neq K_\theta$. Set $F = V_\alpha \text{Ran} \overline{A}$. Then $z F \subset F + \langle z^n \rangle$, where $n \in \mathbb{Z}_+$ is the maximal integer such that $z^j \in F$ for every $0 \leq j < n$.

Proof. By Proposition 3.1 we have $S_\theta \text{Ran} \overline{A} \subset \text{Ran} \overline{A} + \langle \partial^n k_0 \rangle$, where $n \in \mathbb{Z}_+$ is the maximal integer such that $\partial^j k_0 \in \text{Ran} \overline{A}$ for every $0 \leq j < n$. Hence $U_\alpha \text{Ran} \overline{A} \subset \text{Ran} \overline{A} + \langle \partial^n k_0 \rangle$. It remains to apply the operator $V_\alpha$ to both sides of this inclusion and use Proposition 2.7. \hfill \Box
Proposition. Given a finite Borel measure \( \nu \) supported on the unit circle \( \mathbb{T} \), it is difficult to describe all subspaces \( F \subset L^2(\mathbb{T}, \nu) \) such that \( zF \subset F + \langle 1 \rangle \).

In the next section we treat the finite-dimensional case of this problem.

4. Proof of Theorem \( 1 \)

Hereinafter we assume that the space \( K_\theta \) has infinite dimension (equivalently, the inner function \( \theta \) is not a finite Blaschke product). The case \( \dim K_\theta < \infty \) is considered in \( [1] \), where the following fact is proved: every truncated Toeplitz operator on \( K_\theta \), \( \dim K_\theta < \infty \), is a finite linear combination of the rank-one operators in \( [3] \).

Theorem \( 1 \) follows from Lemmas \( 1.1 \) \( 1.2 \) and \( 1.3 \). We now turn to proving these results.

4.1. Proof of Lemma \( 1.1 \). The proof is based on the following proposition.

**Proposition 4.1.** Let \( \sigma \) be a finite Borel measure supported on the unit circle \( \mathbb{T} \). Suppose that \( F \subset L^2(\sigma) \) is a finite-dimensional subspace satisfying \( zF \subset F + \langle 1 \rangle \).

If \( F \) does not contain indicators of singletons, then there exists a finite collection of points \( \lambda_k \in \mathbb{C} \) such that \( F = Q(\lambda_1, p_1) + Q(\lambda_2, p_2) + \ldots + Q(\lambda_s, p_s) \), where \( Q(\lambda_k, p_k) = \text{span}\{(z - \lambda_k)^{-j}, j = 1, \ldots, p_k\}, k = 1, \ldots, s \).

**Proof.** Denote by \( \mathcal{P} \) the non-orthogonal projection on \( F + \langle 1 \rangle \) with the range \( F \) and the kernel \( \langle 1 \rangle \). Let \( M_z \) be the operator of multiplication by the independent variable on \( L^2(\sigma) \). The finite-rank operator \( T = \mathcal{P}M_z : F \to F \) has a complete family of root vectors. Consider the root subspace \( G_\lambda = \text{Ker}(T - \lambda I)^p, G_\lambda \neq \text{Ker}(T - \lambda I)^{p-1} \). The proposition will be proved as soon as we show that \( G_\lambda = Q(\lambda, p) \).

**Case 1.** \( \lambda \in \mathbb{T}, \sigma(\{\lambda\}) > 0 \).

Take a vector \( f \in \text{Ker}(T - \lambda I) \subset G_\lambda \). We have \( (z - \lambda)f = c \) for a constant \( c \in \mathbb{C} \). Therefore, \( f \) is a scalar multiple of the indicator of the singleton \( \{\lambda\} \). But this contradicts the hypothesis. Hence this case does not arise.

**Case 2.** \( \lambda \in \mathbb{C}, \sigma(\{\lambda\}) = 0 \).

Take a vector \( f \in G_\lambda \) such that \( f_1 = (T - \lambda I)^{p-1}f \neq 0 \). We have \( (T - \lambda I)f_1 = 0 \) by the construction. On the other hand, we have \( (T - \lambda I)f_1 = (z - \lambda)f_1 - c_1 \) for some constant \( c_1 \in \mathbb{C} \). Taking into account \( \sigma(\{\lambda\}) = 0 \), we see that

\[
f_1 = \frac{c_1}{z - \lambda}.
\]

If \( p = 1 \), we stop the procedure. Otherwise put \( f_2 = (T - \lambda I)^{p-2}f \) and consider \( (T - \lambda I)f_2 = (z - \lambda)f_2 - c_2 \). Since \( (T - \lambda I)f_2 = f_1 \), we have

\[
f_2 = \frac{c_1}{(z - \lambda)^2} + \frac{c_2}{z - \lambda}.
\]

Continuing this procedure, we obtain

\[
f = f_p = \frac{c_1}{(z - \lambda)^p} + \frac{c_2}{(z - \lambda)^{p-1}} + \ldots + \frac{c_p}{z - \lambda}.
\]

Since \( c_1 \neq 0 \) and \( f_j \in G_\lambda \) for every \( j = 1, \ldots, p \), we get \( (z - \lambda)^{-j} \in G_\lambda \), and hence \( Q(\lambda, p) \subset G_\lambda \). Now take an arbitrary vector \( f \in G_\lambda \) and find a number \( r, 1 \leq r < p \), such that \( f \in \text{Ker}(T - \lambda I)^r \) but \( f_1 = (T - \lambda I)^{r-1}f \neq 0 \). Arguing as above, we see that \( f \in Q(\lambda, r) \subset Q(\lambda, p) \), and thus \( G_\lambda \subset Q(\lambda, p) \).
Proof of Lemma 1.1. Let $A$ be a finite-rank truncated Toeplitz operator on $K_{\theta}$. Since the subspace $\text{Ran} A$ has finite dimension, it cannot contain an infinite system of reproducing kernels; see Proposition 2.8. Therefore, by Proposition 2.3, we can choose the Clark measure $\sigma_\alpha$ so that the subspace $F = V_\alpha \text{Ran} A$ of the space $L^2(\sigma_\alpha)$ does not contain indicators of singletons. It follows from Proposition 3.4 that $zF \subset F + \langle z^n \rangle$, where $n \in \mathbb{Z}_+$ is the maximal integer such that $z^j \in F$ for every $0 \leq j < n$. The subspace $z^n F$ satisfies the assumptions of Proposition 4.1. We have

$$F = z^n(Q(\lambda_1, p_1) + Q(\lambda_2, p_2) + \ldots + Q(\lambda_s, p_s)).$$

In the case $n = 0$, the application of Propositions 2.6 and 2.7 concludes the proof. Now assume that $n \geq 1$. Since $z^j \in F$ for every integer $0 \leq j < n$, we necessarily have $\lambda_{k_0} = 0$, $p_{k_0} = n$ for some $1 \leq k_0 \leq s$. Renumber the sequence $\{\lambda_k\}$ so that $\lambda_1 = 0$. The subspace $z^n Q(0, n)$ is the set of polynomials of degree at most $n - 1$. A simple algebra gives us

$$F = z^n Q(0, n) + Q(\lambda_2, p_2) + \ldots + Q(\lambda_s, p_s).$$

Now the result follows from Propositions 2.6 and 2.7. □

4.2. Proof of Lemma 1.2. Before the proof we need the following general result.

Proposition 4.2. Let $n \in \mathbb{Z}_+$, $A \in T_{\theta}$. Suppose that $\text{Ran} A = F_1 + F_2$, where $F_1$ and $F_2$ are subspaces of $K_{\theta}$ such that

$$(28)\quad S_\theta F_1 \subset F_1 + \langle k_0 \rangle; \quad S_\theta F_2 \subset F_2 + \langle \bar{\partial}^n k_0 \rangle, \quad \bar{\partial}^j k_0 \in F_2, \quad 0 \leq j < n.$$ 

Also let $\bar{\partial}^n k_0 \notin \text{Ran} A$. Then $A = A_1 + A_2$, where $A_k \in T_{\theta}$ and $\text{Ran} A_k = F_k$, $k = 1, 2$. 

Proof. Denote by $P$ the non-orthogonal projection on $F_1 + F_2 + \langle \bar{\partial}^n k_0 \rangle$ with the range $F_2 + \langle \bar{\partial}^n k_0 \rangle$ and the kernel $F_1$. We want to show that $A_2 = PA \in T_{\theta}$. By Theorem 3 we need to check that

$$(29)\quad A_2 - S_\theta A_2 S_\theta^* = \psi \otimes k_0 + k_0 \otimes \chi$$

for some $\psi, \chi \in K_{\theta}$. We have

$$A_2 - S_\theta A_2 S_\theta^* = P(A - S_\theta AS_\theta^*) + (PS_\theta - S_\theta P)AS_\theta^*.$$ 

Since $A$ is a truncated Toeplitz operator, it satisfies (24) with $\psi_1, \chi_1 \in K_{\theta}$. Hence,

$$P(A - S_\theta AS_\theta^*) = (P\psi_1) \otimes k_0 + (P k_0) \otimes \chi_1 = (P\psi_1) \otimes k_0 + k_0 \otimes \chi_1.$$ 

Next, the operator $PS_\theta - S_\theta P$ vanishes on $F_2$ and maps $F_1$ to $\langle k_0 \rangle$. Therefore, we have $\text{Ran} (PS_\theta - S_\theta P)AS_\theta^* \subset \langle k_0 \rangle$. This proves (29). Now put $A_1 = A - A_2$ and obtain the required representation. □

Proof of Lemma 1.2. It follows from Proposition 2.2 that any splitting of the sum in (5) into two summands gives us subspaces $F_1, F_2$ with property (28). Consequently, applying Proposition 4.2, we obtain the required representation. □
4.3. Proof of Lemma 1.3. Let $A$ be a truncated Toeplitz operator on $K_\theta$ with the range $\text{Ran} \ A = F(\lambda, n)$, where $\lambda \in D \cup \Omega_n \cup D$. Truncated Toeplitz operators are complex symmetric with respect to the conjugation $C$; see Section 2.1. Hence

$$
\text{Ran} \ A = \text{span}\{\partial^j k_\lambda, \ j = 0, \ldots, n\} \quad \text{in the case } \lambda \in D,
$$

$$
\text{Ran} \ A^* = \text{span}\{\bar{\partial}^j k_\lambda, \ j = 0, \ldots, n\} \quad \text{in the case } \lambda \in D \cup \Omega_n.
$$

Passing if necessary to the adjoint operator, we can assume that

$$
\text{Ran} \ A = \text{span}\{\partial^j k_\mu, \ j = 0, \ldots, n\}
$$

for some point $\mu \in D \cup \Omega_n$. Then $\text{Ran} \ A^* = \text{span}\{\bar{\partial}^j k_\mu, \ j = 0, \ldots, n\}$. Every such operator has the form

$$
A = \sum_{0 \leq p, q \leq n} a_{p,q} \left( \partial^p \tilde{k}_\mu \otimes \bar{\partial}^q k_\mu \right)
$$

for some coefficients $a_{p,q} \in \mathbb{C}$. We claim that $A = \sum_{s=0}^n a_{0,s} \cdot D^s[\tilde{k}_\mu \otimes k_\mu]$. Consider first the case $\mu \neq 0$. Set $T_{pq} = \partial^p \tilde{k}_\mu \otimes \bar{\partial}^q k_\mu$. For $1 \leq p, q \leq n$ we have

$$
T_{pq} - S_\theta T_{pq} S^*_\theta = \partial^p \tilde{k}_\mu \otimes \bar{\partial}^q k_\mu - \partial^p (S_\theta \tilde{k}_\mu) \otimes \bar{\partial}^q (S_\theta k_\mu)
$$

$$
= \partial^p \tilde{k}_\mu \otimes \bar{\partial}^q k_\mu - \partial^p (\mu \tilde{k}_\mu - \bar{\partial}^q (k_\mu / \bar{\mu}))
$$

$$
= \partial^p \tilde{k}_\mu \otimes \bar{\partial}^q k_\mu - \partial^p (\mu \tilde{k}_\mu) \otimes \bar{\partial}^q (k_\mu / \bar{\mu}) + Z_{pq},
$$

where $Z_{pq}$ is an operator of the form $\psi_{pq} \otimes k_0 + k_0 \otimes \chi_{pq}$. Using the identity $(z \mu)^{(p)} = pf^{(p-1)} + zf^{(p)}$, we get

$$
\partial^p (\mu \tilde{k}_\mu) = p \partial^p - 1 \tilde{k}_\mu + \mu \partial^p \tilde{k}_\mu, \quad \bar{\partial}^q k_\mu = \bar{\partial}^q (k_\mu / \bar{\mu}) = q \bar{\partial}^q - 1 (k_\mu / \bar{\mu}) + \bar{\mu} \bar{\partial}^q (k_\mu / \bar{\mu}).
$$

Substituting this into (31), we obtain

$$
(32) \quad T_{pq} - S_\theta T_{pq} S^*_\theta = q \left( \partial^p \tilde{k}_\mu \otimes \bar{\partial}^q (k_\mu / \bar{\mu}) \right) - p \left( \partial^p - 1 \tilde{k}_\mu \otimes \bar{\partial}^q (k_\mu / \bar{\mu}) \right) + Z_{pq}.
$$

For the operators $T_{00}$, $T_{01}$ and $T_{10}$ we have

$$
(33) \quad T_{00} - S_\theta T_{00} S^*_\theta = Z_{00},
$$

$$
T_{01} - S_\theta T_{01} S^*_\theta = \tilde{k}_\mu \otimes (k_\mu / \bar{\mu}) + Z_{01},
$$

$$
T_{10} - S_\theta T_{10} S^*_\theta = -\tilde{k}_\mu \otimes (k_\mu / \bar{\mu}) + Z_{10}.
$$

Since $A$ is a truncated Toeplitz operator, it satisfies (21) with some $\psi, \chi \in K_\theta$. Combining (24), (32) and (33), we obtain

$$
(34) \quad \Psi \otimes k_0 + k_0 \otimes \Phi = \sum_{0 \leq p, q \leq n} \left( (q+1)a_{p,q+1} - (p+1)a_{p+1,q} \right) \left( \partial^p \tilde{k}_\mu \otimes \bar{\partial}^q (k_\mu / \bar{\mu}) \right),
$$

where $\Psi, \Phi \in K_\theta$ and $a_{n+1,q} = a_{p,n+1} = 0$ for all $0 \leq p, q \leq n$. It follows from Proposition 2.8 that

$$
(35) \quad (q+1)a_{p,q+1} - (p+1)a_{p+1,q} = 0, \quad 0 \leq p, q \leq n.
$$
In the above formula there is no restriction on $a_{0,0}$, which agrees well with the fact that $k_{\mu} \otimes k_{\mu} \in \mathcal{F}_\theta$. For each $1 \leq s \leq n$, from (35) we get the following system:

\[
\begin{align*}
 sa_{0,s} - a_{1,s} & = 0 \\
 (s - 1)a_{1,s} - 2a_{2,s} & = 0 \\
 & \quad \vdots \\
 2a_{s-2,1} - (s - 1)a_{s-1,1} & = 0 \\
 a_{s-1,1} - sa_{s,0} & = 0
\end{align*}
\]

(36)

Solving this system, we obtain

\[
a_{t,s-t} = a_{0,s}C_{s-t}^t, \quad C_{s-t}^t = \frac{s!}{t!(s-t)!}, \quad 0 \leq t \leq s.
\]

It follows from (35) that $na_{n,n} - (n + 1)a_{n+1,n-1} = 0$ and thus $a_{n,n} = 0$. By induction, we have $a_{p,q} = 0$ for all indexes $p, q$ such that $p + q > n$. Now we get the required representation from formulas (11) and (30):

\[
A = \sum_{s=0}^{n} \sum_{t=0}^{s} a_{t,s-t} \left( \partial^t k_{\mu} \otimes \partial^{s-t} k_{\mu} \right) = \sum_{s=0}^{n} a_{0,s} \cdot D^s[\tilde{k}_{\mu} \otimes k_{\mu}].
\]

(37)

In the case $\mu = 0$, put $T_{pq} = \partial^p \tilde{k}_0 \otimes \partial^q k_0$. It follows from Proposition 2.1 that

\[
T_{pq} - S_\theta T_{pq}S^*_\theta = \partial^p \tilde{k}_0 \otimes \partial^q k_0 - \frac{p}{q+1} \partial^{p-1} \tilde{k}_0 \otimes \partial^{q+1} k_0 + Z_{pq}, \quad p \geq 1;
\]

\[
T_{pq} - S_\theta T_{pq}S^*_\theta = \partial^p \tilde{k}_0 \otimes \partial^q k_0 + Z_{pq}, \quad p = 0.
\]

Proceeding as in the case $\mu \neq 0$, we obtain the system

\[
a_{p,q} - \frac{p+1}{q} a_{p+1,q-1} = 0, \quad 0 \leq p \leq n, \quad 1 \leq q \leq n + 1,
\]

where $a_{n+1,q} = 0$ for all $q$ and $a_{p,-1} = 0$ for all $p$. This system has the same solution as the system in (35). Hence we have representation (37) in the case $\mu = 0$ as well. □

References

[1] Donald Sarason, *Algebraic properties of truncated Toeplitz operators*, Oper. Matrices 1 (2007), no. 4, 491–526, DOI 10.7153/oam-01-29. MR2363975 (2008j:47060)

[2] P. R. Ahern and D. N. Clark, *Radial limits and invariant subspaces*, Amer. J. Math. 92 (1970), 332–342. MR0262511 (41 #7117)

[3] Arlen Brown and P. R. Halmos, *Algebraic properties of Toeplitz operators*, J. Reine Angew. Math. 213 (1963/1964), 89–102. MR0160136 (35 #3350)

[4] Anton Baranov, Roman Bessonov, and Vladimir Kapustin, *Symbols of truncated Toeplitz operators*, J. Funct. Anal. 261 (2011), no. 12, 3437–3456, DOI 10.1016/j.jfa.2011.08.005. MR2838030 (2012j:47041)

[5] N. K. Nikol’skii, *Treatise on the shift operator*, Spectral function theory, with an appendix by S. V. Hrusc’ev [S. V. Khruschëv] and V. V. Peller, translated from the Russian by Jaak Peetre. Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 273, Springer-Verlag, Berlin, 1986. MR827233 (87j:47042)

[6] Joseph A. Cima, Alec L. Matheson, and William T. Ross, *The Cauchy transform*, Mathematical Surveys and Monographs, vol. 125, American Mathematical Society, Providence, RI, 2006. MR2215991 (2006m:30003)

[7] Stephan Ramon Garcia and Mihai Putinar, *Complex symmetric operators and applications*, Trans. Amer. Math. Soc. 358 (2006), no. 3, 1285–1315 (electronic), DOI 10.1090/S0002-9947-05-03742-6. MR2187654 (2006j:47036)
[8] Stephan Ramon Garcia and Mihai Putinar, *Complex symmetric operators and applications. II*, Trans. Amer. Math. Soc. 359 (2007), no. 8, 3913–3931 (electronic), DOI 10.1090/S0002-9947-07-04213-4. MR2302518 (2008b:47005)

[9] Douglas N. Clark, *One dimensional perturbations of restricted shifts*, J. Analyse Math. 25 (1972), 169–191. MR0301534 (46 #692)

[10] A. G. Poltoratski˘ı, *Boundary behavior of pseudocontinuable functions* (Russian, with Russian summary), Algebra i Analiz 5 (1993), no. 2, 189–210; English transl., St. Petersburg Math. J. 5 (1994), no. 2, 389–406. MR1223178 (94k:30090)

[11] Anton Baranov, Isabelle Chalendar, Emmanuel Fricain, Javad Mashreghi, and Dan Timotin, *Bounded symbols and reproducing kernel thesis for truncated Toeplitz operators*, J. Funct. Anal. 259 (2010), no. 10, 2673–2701, DOI 10.1016/j.jfa.2010.05.005. MR2679022 (2011h:47051)

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