A DIRECT METHOD OF MOVING PLANES FOR FULLY NONLINEAR NONLOCAL OPERATORS AND APPLICATIONS

YUXIA GUO*
Department of Mathematics, Tsinghua University
Beijing, 100084, China

SHAOLONG PENG
Department of Mathematics, Tsinghua University
Beijing, 100084, China

Abstract. In this paper, we are concerned with the following generalized fully nonlinear nonlocal operators:

\[ F_{s,m}(u) = c_{N,s}m \frac{N}{2} + s \, P.V. \int_{\mathbb{R}^N} \frac{G(u(x) - u(y))}{|x - y|^{N + s}} K_{N + s}(m|x - y|) \, dy + m^{2s}u(x), \]

where \( s \in (0, 1) \) and mass \( m > 0 \). By establishing various maximal principle and using the direct method of moving plane, we prove the monotonicity, symmetry and uniqueness for solutions to fully nonlinear nonlocal equation in unit ball, \( \mathbb{R}^N \), \( \mathbb{R}^N_+ \) and a coercive epigraph domain \( \Omega \) in \( \mathbb{R}^N \) respectively.

1. Introduction. In this paper, we are concerned with the following generalized fully nonlinear nonlocal equation:

\[ F_{s,m}(u(x)) = f(u(x)) \quad \text{in} \quad \Omega, \quad (1) \]

where \( s \) is any real number between 0 to 1 and mass \( m > 0 \), \( \Omega \) can be unit ball, whole space \( \mathbb{R}^N \), half space \( \mathbb{R}^N_+ \), or a coercive epigraph domain \( \Omega \) in \( \mathbb{R}^N \).

The fully nonlinear nonlocal operator \( F_{s,m} \) is given by

\[ F_{s,m}(u(x)) = c_{N,s}m \frac{N}{2} + s \, P.V. \int_{\mathbb{R}^N \setminus B_1(0)} \frac{G(u(x) - u(y))}{|x - y|^{N + s}} K_{N + s}(m|x - y|) \, dy + m^{2s}u(x), \quad (2) \]

where \( P.V. \) stands for the Cauchy principal value of the integral, and

\[ c_{N,s} = 2^{1-\frac{N}{2} + s} \pi^{-\frac{N}{2}} s(1-s) \frac{\Gamma(2-s)}{\Gamma(2-s)}, \quad (3) \]

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* Corresponding author: Yuxia Guo.
and $K_{\nu}$ denotes the modified Bessel function with order $\nu$ and satisfies the equation
\[ r^2 K''_{\nu} + r K'_{\nu} - (r^2 + \nu^2) K_{\nu} = 0. \] (4)

Note that $K_{\nu}(r)$ satisfy the following integral representation
\[ K_{\nu}(r) = \int_0^\infty e^{-r \cosh t} \cosh(\nu t) dt. \]

It is easy to verify that $K_{\nu}(r)$ is a real and positive function satisfying $K_{\nu}(r) < 0$ for all $r > 0$, $K_{\nu} = K_{-\nu}$ for $\nu < 0$. Moreover, for $\nu > 0$,
\[ K_{\nu}(r) \sim \frac{\Gamma(\nu)}{2} \left( \frac{r}{2} \right)^{-\nu}, \] (5)
as $r \to 0$ (see [34, 35, 36]). Here "~" means,
\[ \lim_{r \to 0} \frac{K_{\nu}(r)}{\frac{\Gamma(\nu)}{2} \left( \frac{r}{2} \right)^{-\nu}} = 1. \]

Hence there exists a small $r_0 > 0$ and two constants $C_0 > c_0 > 0$ such that
\[ \frac{c_0}{r^\nu} \leq K_{\nu}(r) \leq \frac{C_0}{r^\nu}, \] (6)
for all $r \leq r_0$. Next, for $\nu > 0$, we can derive that
\[ K_{\nu}(r) \sim \frac{\sqrt{\pi}}{\sqrt{2}} r^{-\frac{\nu}{2}} e^{-r}, \] (7)
as $r \to \infty$. Hence there exists a large $R_\infty > 0$ and two constants $C_\infty > c_\infty > 0$ such that
\[ \frac{c_\infty}{r^\nu e^r} \leq K_{\nu}(r) \leq \frac{C_\infty}{r^\nu e^r}, \] (8)
for all $r \geq R_\infty$.

Throughout this paper, $G$ is a local Lipschitz continuous function and the following conditions.

A1: $G \in C^1(\mathbb{R})$, $G(-t) = -G(t)$ and $G(0) = 0$.
A2: $\exists a_0 > 0$ such that $G'(t) \geq a_0 > 0$, that is $G(t)$ is strict increasing.
A3: $\exists a_1 > 0$ such that $G'(t) \leq a_1 |t|^p$, where $p > 2s - 2$.

We consider the following function spaces
\[ \mathcal{L}_s(\mathbb{R}^N) := \left\{ u : \mathbb{R}^N \to \mathbb{R} \mid \int_{\mathbb{R}^N} e^{-|x|} |G(1 + |u(x)|)| dx < +\infty \right\}. \] (9)

Then the integral on the right hand side of the definition (2) is well-defined, and by asymptotic properties of $K_{\nu}(r)$ as $r \to 0$ and $+\infty$, it is easy to verify that $F_{s,m}(u(x))$ make sense for any $u \in \mathcal{L}_s(\mathbb{R}^N) \cap C_{loc}^{1,1}(\mathbb{R}^N)$.

In the special case when $G(t) = t$ is an identity map, $F_{s,m}$ reduces to the general pseudo-relativistic operators $(-\Delta + m^2)^{s/4} (0 < s < 1)$, which is a nonlocal pseudo-differential operator defined by (refer to e.g. [1, 25, 35, 36, 40])
\[ (-\Delta + m^2)^{s/4} u(x) = c_{N,s} m^{\frac{s}{2} + s} P.V. \int_{\mathbb{R}^N} \frac{u(x) - u(y)}{|x - y|^{\frac{N+1}{2}+s}} K_{\frac{N}{2}+s}(m|x - y|) dy + m^{2s}u(x). \] (10)
The general pseudo-relativistic operators \((-\Delta + m^2)^s\) \((0 < s < 1)\) is well-defined for any \(u \in C^{1,1}_{loc}(\mathbb{R}^N) \cap \dot{L}_s(\mathbb{R}^N)\) with the function spaces

\[
\dot{L}_s(\mathbb{R}^N) := \left\{ u: \mathbb{R}^N \to \mathbb{R} \mid \int_{\mathbb{R}^N} \frac{e^{-|x|}|u(x)|}{1 + |x|^\frac{N}{2} + s} \, dx < +\infty \right\}.
\]

Furthermore, the generalized pseudo-relativistic operators \((-\Delta + m^2)^s\) become the fractional Laplacian \((-\Delta)^s\) as \(m \to 0^+\), which is defined by (see e.g. \([10, 13, 14, 17, 19, 30]\))

\[
(-\Delta)^s u(x) = C_{N,s} \, P.V. \int_{\mathbb{R}^N} \frac{u(x) - u(y)}{|x-y|^{N+2s}} \, dy,
\]

where \(0 < s < 1\), fractional Laplacian \((-\Delta)^s\) is well-defined for any \(u \in C^{1,1}_{loc}(\mathbb{R}^N) \cap \dot{L}_s(\mathbb{R}^N)\) with the function spaces

\[
\dot{L}_s(\mathbb{R}^N) := \left\{ u: \mathbb{R}^N \to \mathbb{R} \mid \int_{\mathbb{R}^N} \frac{|u(x)|}{1 + |x|^{N+2s}} \, dx < +\infty \right\}.
\]

It is worth to be pointed out that the definition of \((-\Delta + m^2)^s\) and \((-\Delta)^s\) can be also defined equivalently through Caffarelli and Silvestre’s extension method (refer to \([4, 20, 21, 35, 36, 45]\]).

In recent years, due to its various applications in fluid mechanics, molecular dynamics, relativistic quantum mechanics of stars (see e.g. \([18, 22]\)), in conformal geometry (see e.g. \([14]\)) and in probability and Langevin (see \([5, 21]\)). Problems involving fractional order operators have attracted more and more attentions. The main difficulty in studying the problem involving these operator lie in the nonlocal nature of the operators. To overcome this difficulty, there are two basically approaches in the literature. One approach is to define the fully nonlinear nonlocal operators \(F_{s,m}\) via extension method by Caffarelli and Silvestre in \([20]\), and reduce the nonlocal problem into a local one in higher dimensions. Another way is to use the integral representation formulae of solutions \([14, 16]\). By establishing the equivalence between the fractional order equation and its corresponding integral equation (with Bessel kernel if \(m > 0\), Riesz kernel if \(m = 0\)), one can apply the method of moving planes (spheres) in integral forms. In fact, these two methods have been applied successfully to study equations involving nonlocal fractional operators \((-\Delta)^s(0 < s < 1)\), and a series of fruitful results have been achieved (see \([4, 16, 20, 21, 26, 27, 28, 37, 43, 45]\) and the references therein). There also got a series of fruitful results for fractional elliptic systems (see \([11, 39, 41, 42, 46]\)) by moving planes. The disadvantage of the above mentioned approaches is that one need to impose some additional conditions on the solutions. Therefore, it is desirable for us to develop direct methods without going through the above mentioned approaches. Direct moving planes (spheres) method have been introduced for fractional Laplacian \((-\Delta)^s\), \((-\Delta)^s_p\) (see \([13, 12]\) and the references therein). Very recently, Dai, Qin and Wu established a direct method for the general pseudo-relativistic Schrödinger equations and derive various applications in \([32]\).

It should be mentioned that the symmetry and monotonicity of solutions play crucial roles in the analysis of nonlinear PDEs. For any other related to results for elliptic equations involving nonlocal operators, and a number of approaches have been established to study the qualitative properties of the nonlocal operator, we refer the reader to the method of moving planes \([4, 15, 16, 20, 21, 26, 27, 39]\), the
method of moving sphere \([16, 17, 27]\), the method of scaling spheres \([15, 29, 31, 44]\), and the sliding methods \([2, 3, 7, 8, 9, 23, 32]\).

Motivated by their ideas, our main goal in the present paper is to extend the direct method of moving planes for general fully nonlinear nonlocal operators \(F_{s,m}\). For this purpose, we first establish a narrow region principle and a decay theorem at infinity. Then combining these results with the method of moving plane, we derive the symmetry and monotonicity of solutions to the general fully nonlinear nonlocal equation (1) in various domains including unit ball, \(\mathbb{R}^N\), \(\mathbb{R}^N_+\) and an epigraph domain \(\Omega\) in \(\mathbb{R}^N\). Next we shall establish a maximum principles for anti-symmetric functions in unbounded domains. Then combining maximum principles in unbounded domains with the direct method of moving plane, we derive the Liouville-type theorem about equation (1) in \(\mathbb{R}^N\).

More precisely, we first consider the following Dirichlet problem on a unit ball:

\[
\begin{aligned}
F_{s,m}(u(x)) &= f(u(x)), \quad \forall x \in B_1(0), \\
u(x) &> 0, \quad \forall x \in B_1(0), \\
u(x) &= 0, \quad \forall x \in \mathbb{R}^N \setminus B_1(0).
\end{aligned}
\tag{12}
\]

We have

**Theorem 1.1.** Assume that \(u \in L_s(\mathbb{R}^N) \cap C^{1,1}_{loc}(B_1(0)) \cap C(\overline{B_1(0)})\) is a positive solution to (12) with \(f(\cdot)\) being Lipschitz continuous. Then \(u\) must be radially symmetric and strictly monotone decreasing about the origin 0.

Next, we will consider the general fully nonlinear nonlocal equation (1) in whole space \(\mathbb{R}^N\). And we have

**Theorem 1.2.** Assume that \(u \in L_s(\mathbb{R}^N) \cap C^{1,1}_{loc}(\mathbb{R}^N)\) is a nonnegative solution of

\[
F_{s,m}(u(x)) = f(u(x)), \quad \forall x \in \mathbb{R}^N.
\tag{13}
\]

Suppose that either

- \(u\) is bounded from above and
  \[
  \sup_{t_1, t_2 \in [\inf u, \sup u]} \frac{f(t_1) - f(t_2)}{t_1 - t_2} < m^{2s},
  \tag{14}
  \]
  or
- \[
  \lim_{|x| \to +\infty} u(x) = 0 \quad \text{and} \quad f'(t) \leq m^{2s}, \quad \forall t \text{ sufficiently small}.
  \tag{15}
  \]

Then \(u\) must be radially symmetric and monotone decreasing about some point in \(\mathbb{R}^N\).

Then we consider general fully nonlinear nonlocal equation (1) in a coercive epigraph \(\Omega\). We say a domain \(\Omega \subseteq \mathbb{R}^N\) is coercive epigraph if there exists a continuous function \(\varphi : \mathbb{R}^{N-1} \to \mathbb{R}\) satisfying

\[
\lim_{|x'| \to +\infty} \varphi(x') = +\infty,
\tag{16}
\]

such that \(\Omega = \{x = (x', x_N) \in \mathbb{R}^N | x_N > \varphi(x')\}\).

Our main result in a coercive epigraph \(\Omega\) is as following.

**Theorem 1.3.** Assume that \(u \in L_s(\mathbb{R}^N) \cap C^{1,1}_{loc}(\Omega) \cap C(\overline{\Omega})\) be a solution to

\[
\begin{aligned}
F_{s,m}(u(x)) &= f(u(x)) \quad \text{in} \ \Omega, \\
u(x) &> 0 \quad \text{in} \ \Omega, \\
u(x) &= 0 \quad \text{in} \ \mathbb{R}^N \setminus \Omega,
\end{aligned}
\tag{17}
\]
where $\Omega$ is a coercive epigraph in $\mathbb{R}^N$, and $f(u)$ being Lipschitz continuous in the range of $u$. Then $u$ is strictly monotone increasing in $x_N$.

We also consider the corresponding Dirichlet problem in a half space $R^+_N$.

$$R^+_N = \{ x = (x', x_N) \in \mathbb{R}^N | x_N > 0 \}.$$ 

We prove the following result.

**Theorem 1.4.** Assume that $u \in L^s(\mathbb{R}^N) \cap C^{1,1}_{loc}(\mathbb{R}^N)$ be a nonnegative solution of

$$\begin{align*}
F_{s,m}(u(x)) &= f(u(x)), \quad \forall x \in R^+_N, \\
u &= 0, \quad \forall x \notin R^+_N,
\end{align*} \quad (18)$$

Suppose that

$$\lim_{|x| \to \infty} u(x) = 0, \quad (19)$$

and $f(0) = 0$. Then we have $u \equiv 0$ in $R^+_N$.

In the rest part of the paper, we shall give a Maximum Principles for anti-symmetric functions in unbounded domains. This maximum principle will be powerful in carrying out the method of moving plane on unbounded domains. Our result is as follows.

**Theorem 1.5** (Maximum Principles for anti-symmetric functions in unbounded domains). Let $T$ be any given hyper-plane in $\mathbb{R}^N$ and $\Sigma$ be the half space on one side of the plane. We denote

$$w(x) = u(x^\lambda) - u(x),$$

with $x^\lambda$ be the reflection of $x$ about the plane $T$. Assume that $w \in L^s(\mathbb{R}^N) \cap C^{1,1}_{loc}(\Sigma)$ is bounded from above and $w(x^\lambda) = -w(x)$ in $\Sigma$, where $\tilde{x}$ is the reflection of $x$ with respect to $T$. Suppose that, at any points $x \in \Sigma$ such that $w(x) > 0$, $w$ satisfies

$$F_{s,m}(u(x^\lambda)) - F_{s,m}(u(x)) + c(x)w(x) \leq 0, \quad (20)$$

where $c(x)$ satisfies

$$\inf_{\{x \in \Sigma | w(x) > 0\}} c(x) > -m^{2s}. \quad (21)$$

Then

$$w(x) \leq 0, \quad \forall x \in \Sigma. \quad (22)$$

Furthermore, assume that

$$F_{s,m}(u(x^\lambda)) - F_{s,m}(u(x)) + c(x)w(x) \leq 0 \quad (23)$$

at the points $x \in \Sigma$ where $w(x) = 0$. Then if $w = 0$ at some point in $\Sigma$, we have

$$w = 0 \quad \text{almost everywhere in } \mathbb{R}^N. \quad (24)$$

As an application of Theorem 1.5, we can obtain the following Liouville Theorem about generalized fully nonlinear nonlocal equation 1 in $\mathbb{R}^N$.

**Theorem 1.6.** (Liouville Theorem) Assume that $u \in L^s(\mathbb{R}^N) \cap C^{1,1}_{loc}(\mathbb{R}^N)$ is bounded, if

$$F_{s,m}(u(x)) = f(u(x)), \quad \forall x \in \mathbb{R}^N, \quad (24)$$

where the function $f(u)$ satisfies

$$\sup_{u_1, u_2 \in [\inf u, \sup u]} \frac{f(u_1) - f(u_2)}{u_1 - u_2} < m^{2s}. \quad (25)$$
Then, we have
\[ u = C \quad \text{with} \quad C \quad \text{satisfying} \quad f(C) = m^{2s}C. \]

Especially, if \( f(u(x)) = 0 \), then we have \( u \equiv 0 \).

As another application of Theorem 1.5, we will be able to derive the following monotonicity result on (24).

**Theorem 1.7.** Suppose \( u \in L^s_\ast(R^N) \cap C^{1,1}_{loc}(R^N) \) is a solution of the equation (24) and satisfy that
\[ |u(x)| \leq 1, \quad \forall x \in R^N, \]
\[ \lim_{x_N \to \pm \infty} u(x', x_N) = \pm 1 \quad \text{uniformly with respect to} \quad x' = (x_1, \ldots, x_{n-1}) \in R^{N-1}. \]

(26)

Assume there exists a \( \delta > 0 \) such that
\[ \sup_{-1 \leq u_1 < u_2 \leq -1+\delta} \frac{f(u_2) - f(u_1)}{u_2 - u_1} < m^{2s}, \]

(27)

then there exists \( M > 0 \) such that
\[ \frac{\partial u}{\partial x_n} > 0, \quad \text{for all} \ x \text{ provided that} \ |x_n| > M. \]

**Remark 1.8.** We point out that the condition on \( f \) is satisfied for \( f(u) = u - u^3 \) as given in the De Giorgi Conjecture[38]. Our assumption is weaker than those condition for \((-\Delta)^s \) (see e.g. [6, 23, 33]), which need \( f \) to be non-increasing for \(|t| \) sufficiently close to 1.

Our paper is organized as follows. In Section 2, we first establish the maximum principles for anti-symmetric functions and narrow region principle for the generalized fully nonlinear nonlocal equation (1). Then we obtain the decay estimates for the solutions at infinity. As applications, we prove the symmetric and monotone for the generalized fully nonlinear nonlocal equation (1) in unit ball, whole space \( R^N \), a coercive epigraph domain \( \Omega \) and the half space \( R^N_+ \) (Theorem (1.1)-Theorem (1.4)) respectively in Section 3. Section 4 is devoted to the proof of Theorem 1.5. In the last section, we study the Liouville type result(Theorem 1.6) and Theorem 1.7 for the generalized fully nonlinear nonlocal equation (1) respectively.

2. Various maximum principles for anti-symmetric functions. In this section, we shall establish various maximum principles which are key ingredients in applying the direct method of moving planes for \( F_{s,m} \).

We begin with the following notations. Let \( T \) be any given hyper-plane in \( R^N \) and \( \Sigma \) be the half space on one side of the hyper-plane \( T \) hereafter. Denote the reflection of a point \( x \) with respect to \( T \) by \( x^\lambda \), \( u_\lambda(x) = u(x^\lambda) \) and \( w(x) = u(x^\lambda) - u(x) \).

From now on and in the following of the paper, we always use the same \( C \) denotes a constant whose value may be different from line to line, and only the relevant dependence is specified.

First, we shall give the following maximum principle for anti-symmetric functions in bounded open sets.
Proposition 1 (Maximum principle for anti-symmetric functions). Let $\Omega$ be a bounded open set in $\Sigma$. Assume that $w \in L^p(\mathbb{R}^N) \cap C^{1,1}_{loc}(\Omega)$ and is lower semi-continuous on $\overline{\Omega}$. If
\[
\begin{cases}
F_{s,m}(u_m(x)) - F_{s,m}(u(x)) + c(x)w(x) \geq 0 & x \in \Omega, \\
w(x) \geq 0 & \text{in } \Sigma \setminus \Omega, \\
w(x^+) = -w(x) & \text{in } \Sigma,
\end{cases}
\]  
(28)
where $c(x) \geq -m^{2s}$ for any $x \in \{x \in \Omega \mid w(x) < 0\}$. Then we have
\[w(x) \geq 0 \quad \text{in } \Omega.\]
Furthermore, assume that
\[F_{s,m}(u_m(x)) - F_{s,m}(u(x)) \geq 0 \quad \text{at points } x \in \Omega \text{ where } w(x) = 0, \quad (29)\]
then if $w = 0$ at some point in $\Omega$, we must have
\[w = 0 \quad \text{almost everywhere in } \mathbb{R}^N.\]
These conclusions hold for unbounded open set $\Omega$ if we further assume that
\[\liminf_{|x| \to \infty} w(x) \geq 0.\]

Proof. If $w$ is not nonnegative, then the lower semi-continuity of $w$ on $\overline{\Omega}$ indicates that there exists a $\hat{x} \in \overline{\Omega}$ such that
\[w(\hat{x}) = \min_{\overline{\Omega}} w < 0.\]
One can further deduce from (28) that $\hat{x}$ is in the interior of $\Omega$. It follows that
\[F_{s,m}(u_m(\hat{x})) - F_{s,m}(u(\hat{x})) + c(\hat{x})w(\hat{x}) = c_{N,s,m}\frac{s}{2} + s P.V. \int_{\mathbb{R}^N} G(u_m(\hat{x}) - u_m(y)) - G(u(\hat{x}) - u(y)) |\hat{x} - y|^\frac{s}{2} + s K_{\frac{s}{2} + s} (m |\hat{x} - y|) \, dy + (c(\hat{x}) + m^{2s}) w(\hat{x}) = c_{N,s,m}\frac{s}{2} + s P.V. \left\{ \int_{\Sigma} \frac{G(u_m(\hat{x}) - u_m(y)) - G(u(\hat{x}) - u(y))}{|\hat{x} - y|^\frac{s}{2} + s} K_{\frac{s}{2} + s} (m |\hat{x} - y|) \, dy \right\} + (c(\hat{x}) + m^{2s}) w(\hat{x}) \leq c_{N,s,m}\frac{s}{2} + s P.V. \left( \int_{\Sigma} \frac{1}{|\hat{x} - y|^\frac{s}{2} + s} - \frac{1}{|\hat{x} - y|^\frac{s}{2} + s} \right) (G(u_m(\hat{x}) - u_m(y)) - G(u(\hat{x}) - u(y)) K_{\frac{s}{2} + s} (m |\hat{x} - y|) \, dy + c_{N,s,m}\frac{s}{2} + s P.V. \int_{\Sigma} \frac{1}{|\hat{x} - y|^\frac{s}{2} + s} (G(u_m(\hat{x}) - u_m(y)) - G(u(\hat{x}) - u(y)) + G(u_m(\hat{x}) - u(y)) - G(u(\hat{x}) - u(y)) K_{\frac{s}{2} + s} (m |\hat{x} - y|) \, dy \right) = : I_1 + I_2. \quad (30)\]
The first inequality is true by using the fact of $c(x) \geq -m^{2s}$. 
Next, we will estimate $I_1$ and $I_2$. To estimate $I_1$, we first notice that
\[G(u_m(\hat{x}) - u_m(y)) - G(u(\hat{x}) - u(y)) = G'(\xi) \left[w(\hat{x}) - w(y)\right] \leq 0,\]
where \( \eta_1 \) be the value between of \( u_\lambda(\hat{x}) - u_\lambda(y) \) and \( u(\hat{x}) - u(y) \). So we have
\[
I_1 \leq 0.
\]
To estimate \( I_2 \), by a direct calculation, we have,
\[
I_2 = c_{N,s}m^{\frac{N}{2} + s}P.V. \int_\Sigma \frac{1}{|\hat{x} - y|^{\frac{N}{2} + s}}[G(u_\lambda(\hat{x}) - u_\lambda(y)) - G(u(\hat{x}) - u(y))]
\]
\[
+ G(u_\lambda(\hat{x}) - u(y)) - G(u(\hat{x}) - u(y))]K_{\frac{N}{2} + s}(m|\hat{x} - y|) dy
\]
\[
= c_{N,s}m^{\frac{N}{2} + s}P.V. \int_\Sigma \frac{G'(\eta_1)w(\hat{x}) + G'(\eta_2)w(\hat{x})}{|\hat{x} - y|^{\frac{N}{2} + s}}K_{\frac{N}{2} + s}(m|\hat{x} - y|) dy
\]
\[
\leq c_{N,s}m^{\frac{N}{2} + s}w(\hat{x}) \int_\Sigma \frac{G'(\eta_1) + G'(\eta_2)}{|\hat{x} - y|^{\frac{N}{2} + s}}K_{\frac{N}{2} + s}(m|\hat{x} - y|) dy
\]
\[
\leq 2c_{N,s}m^{\frac{N}{2} + s}w(\hat{x}) \int_\Sigma \frac{1}{|\hat{x} - y|^{\frac{N}{2} + s}}K_{\frac{N}{2} + s}(m|\hat{x} - y|) dy
\]
\[< 0,
\]
where \( \eta_1 \) be the value between of \( u_\lambda(\hat{x}) - u_\lambda(y) \) and \( u(\hat{x}) - u(y) \), \( \eta_2 \) be the value between of \( u_\lambda(\hat{x}) - u(y) \) and \( u(\hat{x}) - u(y) \). Then we have
\[
I_1 + I_2 < 0,
\]
which contradicts (28), hence we must have
\[
w(x) \geq 0 \quad \text{in} \quad \Omega. \tag{32}
\]
Combine (28) and (32), we have proved that \( w(x) \geq 0 \) in \( \Sigma \). If there is some point \( \bar{x} \in \Omega \) such that \( w(\bar{x}) = 0 \), then we must have
\[
w = 0 \quad \text{almost everywhere in} \quad \mathbb{R}^N. \tag{33}
\]
To this end, we re-estimate \( I_1 \) and \( I_2 \) at \( x = \bar{x} \). From the previous arguments, we have
\[
I_1 = c_{N,s}m^{\frac{N}{2} + s}P.V. \int_\Sigma \left( \frac{1}{|\bar{x} - y|^{\frac{N}{2} + s}} - \frac{1}{|\bar{x} - y|^{\frac{N}{2} + s}} \right) [G(u_\lambda(\bar{x}) - u_\lambda(y))
\]
\[- G(u(\bar{x}) - u(y))]K_{\frac{N}{2} + s}(m|\bar{x} - y|) dy
\]
\[
= c_{N,s}m^{\frac{N}{2} + s}P.V. \int_\Sigma \left( \frac{1}{|\bar{x} - y|^{\frac{N}{2} + s}} - \frac{1}{|\bar{x} - y|^{\frac{N}{2} + s}} \right) G'(\xi_2)(-w(y)) dy
\]
\[
\leq -c_{N,s}m^{\frac{N}{2} + s} \int_\Sigma \left( \frac{1}{|\bar{x} - y|^{\frac{N}{2} + s}} - \frac{1}{|\bar{x} - y|^{\frac{N}{2} + s}} \right) G'(\xi_2)w(y) dy,
\]
where \( \xi_2 \) be the value between of \( u_\lambda(\bar{x}) - u_\lambda(y) \) and \( u(\bar{x}) - u(y) \). If \( w \neq 0 \) in \( \Sigma \), then we have
\[
I_1 < 0.
\]
Obviously, we also have \( I_2 = 0 \), hence
\[
F_{s,m}(u_\lambda(\bar{x})) - F_{s,m}(u(\bar{x})) + c(\bar{x})w(\bar{x}) < 0.
\]
That is impossible! Therefore, we must have
\[ w(x) = 0, \quad \forall x \in \Sigma, \]
and consequently, by symmetry, we immediately arrive at (33).

This completes the proof of Proposition 1. \( \square \)

In order to carry on the direct method of moving planes in \( \mathbb{R}^N \), we need the following Narrow Region Principle.

**Proposition 2** (Narrow Region Principle). Let \( \Omega \) be a bounded open set in \( \Sigma \) which can be contained in the region between \( T \) and \( T_\Omega \), where \( T_\Omega \) is a hyper-plane that is parallel to \( T \). Let \( d(\Omega) := \text{dist}(T, T_\Omega) \). Suppose that \( w \in L^s(\mathbb{R}^N) \cap C^{1,1}_{loc}(\Omega) \) and is lower semi-continuous on \( \Omega \), if \( c(x) \) is uniformly bounded from below (with respect to \( d(\Omega) \)) in \( \{ x \in \Omega | w(x) < 0 \} \) and satisfies
\[
\begin{aligned}
& F_{s,m}(u_\lambda(x)) - F_{s,m}(u(x)) + c(x)w(x) \geq 0, \quad \text{in } \{ x \in \Omega | w(x) > 0 \}, \\
& w(x) \geq 0, \quad \text{in } \Sigma \setminus \Omega, \\
& w(\lambda x) = -w(x), \quad \text{in } \Sigma.
\end{aligned}
\]
(34)

We assume \( \Omega \) is narrow in the sense that, \( d(\Omega) \leq \frac{r_0}{4m} \), and there exists a constant \( C_{N,s} > 0 \) such that
\[
\left( -\inf_{\{ x \in \Omega | w(x) < 0 \}} c(x) - m^{2s} \right) d(\Omega)^{2s} < C_{N,s}.
\]
(35)

Then, we have
\[ w(x) \geq 0 \text{ in } \Omega. \]

Furthermore, assume that
\[
F_{s,m}(u_\lambda(x)) - F_{s,m}(u(x)) \geq 0 \quad \text{at points } x \in \Omega \text{ where } w(x) = 0,
\]
(36)
then if \( w = 0 \) at some point in \( \Omega \), we have
\[ w = 0 \quad \text{almost everywhere in } \mathbb{R}^N. \]

These conclusions hold for unbounded open set \( \Omega \) if we further assume that
\[
\lim_{|x| \to \infty} w(x) \geq 0.
\]

**Proof.** Without loss of generalities, we may assume that
\[ T = \{ x \in \mathbb{R}^N | x_1 = 0 \} \quad \text{and} \quad \Sigma = \{ x \in \mathbb{R}^N | x_1 < 0 \}, \]
and hence \( \Omega \subseteq \{ x \in \mathbb{R}^N | -d(\Omega) < x_1 < 0 \}. \)

If \( w \) is not nonnegative in \( \Omega \), then the lower semi-continuity of \( w \) on \( \overline{\Omega} \) indicates that, there exists a \( \bar{x} \in \Omega \) such that
\[ w(\bar{x}) = \min_{\overline{\Omega}} w < 0. \]

One can further deduce from (34) that \( \bar{x} \) is in the interior of \( \Omega \). By a similar calculation, we have
\[
\begin{aligned}
& F_{s,m}(u_\lambda(\bar{x})) - F_{s,m}(u(\bar{x})) + c(\bar{x})w(\bar{x}) \\
\leq & 2c_{N,s} c_0 m^{2+s} w(\bar{x}) \int_{\Sigma} \frac{1}{|\bar{x} - y|^{n+s}} K_N |m|^{2+s} (m|\bar{x} - y|) dy \\
+ & (c(\bar{x}) + m^{2s}) w(\bar{x}).
\end{aligned}
\]
(37)
Let
\[ D := \left\{ y = (y_1, y') \in \mathbb{R}^N \mid d(\Omega) < y_1 - (\bar{x})_1 < 2d(\Omega), \left| y' - (\bar{x})' \right| < \frac{r_0}{2m} \right\}, \]
then we have
\[ m|y - \bar{x}| < r_0 \quad \text{for all} \quad y \in D. \]
Now we denote \( t := y_1 - (\bar{x})_1, \tau := |y' - (\bar{x})'|. \) By using (6), we have
\[
\begin{align*}
m^2 \int_{d(\Omega)} & \int_{\left| \bar{x} - y' \right|^{-\frac{N}{2} + s}} 1 \frac{1}{\left| \bar{x} - y' \right|^\frac{N}{2} + s} K_{\frac{N}{2} + s} \left( m |\bar{x} - y'| \right) dy \\
\geq m^2 \int_{D} & \int_{\left| \bar{x} - y \right|^{-\frac{N}{2} + s}} 1 \frac{1}{\left| \bar{x} - y \right|^\frac{N}{2} + s} K_{\frac{N}{2} + s} \left( m |\bar{x} - y| \right) dy \\
\geq c_0 \int_{d(\Omega)} & \int_{0}^{\frac{r_0}{t}} \frac{\sigma_{N-1} N^{-2} d\tau}{(t^2 + \tau^2)^\frac{N}{2} + s} dt = c_0 \int_{d(\Omega)} \int_{0}^{\frac{r_0}{s}} \frac{\sigma_{N-1} (t^2 + 1)^{\frac{N-2}{2}} d\rho}{(1 + \rho^2)^\frac{N}{2} + s} dt \\
\geq c_0 \int_{d(\Omega)} & \int_{0}^{\frac{r_0}{t}} \frac{1}{t^{1+2s}} \int_{0}^{1} \frac{\sigma_{N-1} \rho N^{-2} d\rho}{(1 + \rho^2)^\frac{N}{2} + s} dt \\
\geq C_{N,s} \int_{d(\Omega)} & \int_{0}^{\frac{r_0}{t}} \frac{1}{t^{1+2s}} dt = \frac{C_{N,s}}{d(\Omega)^{2s}},
\end{align*}
\]
where we have used the substitution \( \rho := \tau/t \) and \( \sigma_{N-1} \) to denote the area of the unit sphere in \( \mathbb{R}^{N-1} \). Since \( c(x) \) is uniformly bounded from below (with respect to \( d(\Omega) \)) in \( \{ x \in \Omega \mid w(x) < 0 \} \), then, from (35), (37) and (38), we get
\[
F_{s,m}(u_\lambda(x)) - F_{s,m}(u(\bar{x})) + c(\bar{x}) w(\bar{x}) \\
\leq \left[ \frac{C_{N,s}}{d(\Omega)^{2s}} + \inf_{\{ x \in \Omega \mid w(x) < 0 \}} c(x) + m^{2s} \right] w(\bar{x}) < 0,
\]
which contradicts (34) and hence proves \( w(x) \geq 0 \) in \( \Omega \). The proof of the rest of this Proposition is similar to Proposition (1), so we omit it here. This finishes the proof of Proposition (2).

**Remark 2.1.** The assumptions “\( w \) is lower semi-continuous on \( \partial \Omega \)” in Proposition 1 and Proposition 2 can be weaken into: “if \( w < 0 \) somewhere in \( \Sigma_\lambda \), then the negative minimum \( \inf_{\Sigma_\lambda} w_\lambda(x) \) can be attained in \( \partial \Omega \)” , the same conclusions are still true.

Now we give the estimate of the decay for the solutions to generalized fully nonlinear local equation at infinity.

**Proposition 3 (Decay at Infinity ).** Let \( \Omega \) be an unbounded open set in \( \Sigma \). Assume that \( w \in L^1_\lambda(\mathbb{R}^N) \cap C^{1,1}_{loc}(\Omega) \) is a solution of
\[
\begin{align*}
F_{s,m}(u_\lambda(x)) - F_{s,m}(u(x)) + c(x) w(x) & \geq 0, \quad \text{in} \quad \{ x \in \Omega \mid w(x) < 0 \}, \\
w(x) & \geq 0, \\
w(x^\lambda) = -w(x),
\end{align*}
\]
with
\[
\liminf_{x \in \Sigma, w(x) < 0} \left( c(x) + m^{2s} \right) \geq 0,
\]
\[
\liminf_{|x| \to +\infty} \left( c(x) + m^{2s} \right) \geq 0,
\]
\[
\liminf_{|x| \to +\infty} \left( c(x) + m^{2s} \right) \geq 0,
\]
then there exists a constant $R_0 > 0$ (depending only on $c(x)$, $m$, $N$ and $s$, but independent of $w$ and $\Sigma$) such that, if $\hat{x} \in \Omega$ satisfying
\[
  w(\hat{x}) = \min_\Omega w(x) < 0.
\]
Then we have $|\hat{x}| \leq R_0$.

Proof. Without loss of generalities, we may assume that, for some $\lambda \leq 0$,
\[
  T = \{ x \in \mathbb{R}^N | x_1 = \lambda \} \quad \text{and} \quad \Sigma = \{ x \in \mathbb{R}^N | x_1 < \lambda \}.
\]
Since $w \in L_\ast(\mathbb{R}^N) \cap C^{1,1}_{\text{loc}}(\Omega)$ and $\hat{x} \in \Omega$ satisfying
\[
  w(\hat{x}) = \min_\Omega w(x) < 0.
\]
By using the similar calculations as (30), we deduce
\[
  F_{s,m}(u_\Omega(\hat{x})) - F_{s,m}(u(\hat{x})) + c(\hat{x})w(\hat{x})
  \leq 2c_{N,s}c_0 m^{\frac{s}{2} + s} w(\hat{x}) \int_{\Sigma} \frac{1}{|\hat{x} - y^\lambda|^{\frac{N-s}{2} + 1}} K_{\frac{N}{2} + s}(m|\hat{x} - y^\lambda|) dy + (c(\hat{x}) + m^2) w(\hat{x}).
\] (41)

Note that $\lambda \leq 0$ and $\hat{x} \in \Omega$, it follows that $B_{|\hat{x}|}(\hat{x}) \subset \{ x \in \mathbb{R}^N | x_1 > \lambda \}$, where $\bar{x} := (2|\hat{x}| + (\hat{x})_1, (\hat{x})')$. Thus we derive that, if $|\hat{x}| \geq \frac{R_0}{3m}$,
\[
  m_{\frac{N}{2} + s} \int_{\Sigma} \frac{K_{\frac{N}{2} + s}(m|\hat{x} - y^\lambda|)}{|\hat{x} - y^\lambda|^{\frac{N-s}{2} + 1}} dy
  \geq m_{\frac{N}{2} + s} \int_{B_{|\hat{x}|}(\hat{x})} \frac{K_{\frac{N}{2} + s}(m|\hat{x} - y|)}{|\hat{x} - y|^{\frac{N-s}{2} + 1}} dy
  \geq m_{\frac{N}{2} + s} \int_{B_{|\hat{x}|}(\hat{x})} \frac{K_{\frac{N}{2} + s}(3m|\hat{x}|)}{3^{\frac{N}{2} + s}|\hat{x}|^{\frac{N-s}{2} + 1}} dy
  \geq \frac{c_\infty m^{\frac{N-1}{2} + s} \omega_N}{3^{\frac{N+1}{2} + s}|\hat{x}|^{\frac{s+1}{2} - \frac{N}{2} - \frac{s}{2}} e^{3m|\hat{x}|}},
\] (42)

where $\omega_N := |B_1(0)|$ be the volume of unit ball in $\mathbb{R}^N$. Then from (39), (41) and (42), we have
\[
  0 \leq F_{s,m}(u_\Omega(\hat{x})) - F_{s,m}(u(\hat{x})) + c(\hat{x})w(\hat{x})
  \leq \left[ \frac{2c_{N,s}c_0c_\infty \omega_N m^{\frac{N-1}{2} + s}}{3^{\frac{N+1}{2} + s}|\hat{x}|^{\frac{s+1}{2} - \frac{N}{2} - \frac{s}{2}} e^{3m|\hat{x}|}} + (c(\hat{x}) + m^2) \right] w(\hat{x}).
\] (43)

It follows from $w(\hat{x}) < 0$ and (43) that
\[
  |\hat{x}|^{s+1 - \frac{s}{2} - \frac{N}{2}} e^{3m|\hat{x}|} \left[ (c(\hat{x}) + m^2) \right] \leq - \frac{2c_{N,s}c_0c_\infty \omega_N m^{\frac{N-1}{2} + s}}{3^{\frac{N+1}{2} + s}}.
\] (44)

On the other hand, from (40), we infer that there exists a $R_1$ sufficiently large such that, for any $|x| \geq R_1$,
\[
  |x|^{s+1 - \frac{s}{2} - \frac{N}{2}} e^{3m|x|} \left[ (c(x) + m^2) \right] > - \frac{2c_{N,s}c_\infty \omega_N m^{\frac{N-1}{2} + s}}{3^{\frac{N+1}{2} + s}}.
\] (45)

Combining (44) and (45), we arrive at
\[
  |\hat{x}| \leq R_1.
\]
Therefore, take \( R_0 := \max\{ \frac{R_{\infty}}{3}, R_1 \} \), we must have \( |\hat{x}| \leq R_0 \).

This completes the proof of Proposition 3.

3. Method of moving planes and its applications. In this section, as applications of various maximum principle established in Section 2 and the direct method of moving planes, we shall prove the monotonicity and symmetry of solutions to generalized fully nonlinear nonlocal equations (1) in different type of regions, including unit ball (Theorem 1.1), whole space \( \mathbb{R}^N \) (Theorem 1.2), a coercive epigraph domain \( \Omega \) (Theorem 1.3) and the half space \( \mathbb{R}^N_+ \) (Theorem 1.4).

3.1. Symmetry and monotone of solutions in a unit ball.

In this subsection, we shall prove the monotonicity and symmetry of positive solutions to generalized fully nonlinear nonlocal equations (1) in unit ball, which is Theorem 1.1.

Proof. For arbitrary \( \lambda \in (-1, 1) \), choose an arbitrary direction to be the \( x_1 \)-direction, let \( T_\lambda := \{ x \in \mathbb{R}^N | x_1 = \lambda \} \), and

\[
\Sigma_\lambda := \{ x \in B_1(0) | x_1 < \lambda \}.
\]

We denote \( x^\lambda \) be the reflection of \( x \) with respect to \( T_\lambda \) and \( w_\lambda(x) = u(x^\lambda) - u(x) \).

Our goal is to show that \( w_\lambda \geq 0 \) in \( \Sigma_\lambda \) for any \( \lambda \in (-1, 0) \).

First, it is easy to verify that, for any \( \lambda \in (-1, 0) \),

\[
F_{s,m}(u_\lambda(x)) - F_{s,m}(u(x)) + c_\lambda(x)w_\lambda(x) = 0, \quad \forall x \in \Sigma_\lambda,
\]

with

\[
c_\lambda(x) := \begin{cases} 
-\frac{f(u(x^\lambda)) - f(u(x))}{u(x^\lambda) - u(x)}, & \text{if } u(x) \neq u(x^\lambda), \\
0, & \text{if } u(x) = u(x^\lambda).
\end{cases}
\]

The Lipschitz continuity assumption on \( f \) guarantees that \( c_\lambda(x) \) is uniformly bounded from above, which is

\[
\|c_\lambda\|_{L^\infty(\Sigma_\lambda)} \leq L,
\]

where \( L \) (independent of \( \lambda \)) is the Lipschitz constant for the function \( f(u(x)) \).

Next, we will carry out the moving plane procedure in two steps.

**Step 1.** Start moving the plane \( T_\lambda \) from near \( \lambda = -1 \) to the right along \( x_1 \)-axis. Let

\[
H_\lambda := \{ x \in \mathbb{R}^N | x_1 < \lambda \}.
\]

It is easy to see \( \Sigma_\lambda \) is a narrow region for \( \lambda \) close enough to \(-1\) and

\[
w_\lambda(x) \geq 0, \quad \forall x \in H_\lambda \setminus \Sigma_\lambda.
\]

Thus we can apply the narrow region principle Proposition 2 and the fact of \( u > 0 \) in \( B_1(0) \) to conclude that for \( \lambda \) close to \(-1\), we have

\[
w_\lambda(x) > 0, \quad \forall x \in \Sigma_\lambda.
\]

**Step 2.** Move the plane to continuously to the right until its limiting position. Define

\[
\lambda_0 := \sup \{ \lambda \in (-1, 0] | w_\mu > 0, \quad x \in \Sigma_\mu, \quad \forall -1 < \mu < \lambda \}.
\]

Then we will show that \( \lambda_0 = 0 \) via contradiction arguments.
Suppose on the contrary that \( \lambda_0 < 0 \), then we will be able to move \( T_\lambda \) to the right a little bit further while (49) still holds, which contradicts the definition (50) of \( \lambda_0 \). Indeed, due to \( \lambda_0 < 0 \) and systems (12), we have

\[
w_{\lambda_0} > 0 \quad \text{in} \quad ((B_1(0))^{\lambda_0} \cap H_{\lambda_0}) \setminus \Sigma_{\lambda_0},
\]

where \(((B_1(0))^{\lambda_0} \cap H_{\lambda_0})\) denotes the reflection of \( B_1(0) \) about \( T_\lambda \). So \( w_{\lambda_0}(x) \) is not identically zero. By using Maximum principle Proposition 1, we get

\[
w_{\lambda_0}(x) > 0, \quad \forall \ x \in \Sigma_{\lambda_0}.
\]  

(51)

Now select 0 < \( \epsilon_1 < \min\{-\lambda_0, \lambda_0 + 1\} \) small enough such that \( \Sigma_{\lambda_0+\epsilon_1} \setminus \Sigma_{\lambda_0-\epsilon_1} \) is a narrow region. Since \( w_\lambda > 0 \) in \( (B_1(0))^{\lambda_0} \cap H_{\lambda_0} \), so there exists a constant \( c > 0 \) such that

\[
w_{\lambda_0}(x) \geq c > 0, \quad \forall \ x \in \Sigma_{\lambda_0-\epsilon_1}.
\]  

(52)

Due to the continuity of \( w_\lambda \) with respect to \( \lambda \), so there exists a sufficiently small 0 < \( \epsilon_2 < \epsilon_1 \) such that, for any \( \lambda \in [\lambda_0, \lambda_0 + \epsilon_2] \),

\[
w_{\lambda}(x) > 0, \quad \forall \ x \in \Sigma_{\lambda_0-\epsilon_1}.
\]  

(53)

For any \( \lambda \in [\lambda_0, \lambda_0 + \epsilon_2] \), note that \( \Sigma_\lambda \setminus \Sigma_{\lambda_0-\epsilon_1} \) is a narrow region, we can deduce from the narrow region principle Proposition 2 that

\[
w_{\lambda}(x) > 0, \quad \forall \ x \in \Sigma_\lambda \setminus \Sigma_{\lambda_0-\epsilon_1}.
\]  

(54)

This together with (53), we have,

\[
w_{\lambda}(x) > 0, \quad \forall \ x \in \Sigma_\lambda, \ \forall \lambda \in [\lambda_0, \lambda_0 + \epsilon_2].
\]  

(55)

This contradicts with the definition of \( \lambda_0 \) (50). Thus we must have \( \lambda_0 = 0 \), or more apparently,

\[
u(-x_1, x_2, \cdots, x_N) \geq u(x_1, x_2, \cdots, x_N), \quad \forall \ x \in \Sigma_0.
\]  

(56)

Through similar arguments, we can also move the plane \( T_\lambda \) from near \( \lambda = 1 \) to the left and derive

\[
u(-x_1, x_2, \cdots, x_N) \leq u(x_1, x_2, \cdots, x_N), \quad \forall \ x \in B_1(0) \text{ with } x_1 < 0.
\]  

(57)

Combining (56) with (57), we derive that \( u(x) \) is symmetric about the plane \( T_0 \). Since the \( x_1 \)-direction is arbitrarily chosen, \( u(x) \) is radially symmetric with respect to the origin 0. The strict monotonicity is a consequence of the fact that

\[
w_{\lambda}(x) > 0, \quad \forall \ x \in \Sigma_\lambda
\]

holds for all \( -1 < \lambda \leq 0 \).

This is finish the proof of Theorem 1.1. \( \square \)

3.2. Symmetry and monotone of solutions in whole space \( \mathbb{R}^N \).

In this subsection, we shall prove the monotonicity and symmetry of positive solutions to generalized fully nonlinear nonlocal equations (1) in whole space \( \mathbb{R}^N \), which is Theorem 1.2.

Proof. Choose an arbitrary direction to be the \( x_1 \)-direction and keeping the notations \( T_\lambda, \Sigma_\lambda, x^\lambda, w_\lambda(x) \) defined in Subsection 3.1.

Assume that \( u \) is a nonnegative solution of the equations (13). Then, for any \( \lambda \in \mathbb{R} \), at points \( x \in \Sigma_\lambda \) where \( w_\lambda(x) < 0 \), we have

\[
F_{s,m}(w_\lambda(x)) - F_{s,m}(u(x)) + c(x)w_\lambda(x) \geq 0,
\]  

(58)
where \( c(x) := -\frac{f(u(x)) - f(u(x))}{u(x) - u(x)} \). From the assumption (14) or the assumption (15), we derive that, for any \( \lambda \in \mathbb{R} \),

\[
\lim_{\substack{x \in \Sigma_{\lambda, w_{\lambda}(x) < 0} \rightarrow +\infty}} c(x) > -m^{2s}. \tag{59}
\]

Next, we will carry out the moving planes procedure in two steps.

**Step 1.** Start moving the plane \( T_{\lambda} \) along the \( x_1 \)-axis from near \(-\infty\) to the right. We need to show that for sufficiently negative \( \lambda \),

\[
w_{\lambda}(x) \geq 0, \quad \forall \ x \in \Sigma_{\lambda}. \tag{60}
\]

If (60) does not hold, then there exist a \( \hat{x} \in \Sigma_{\lambda} \) satisfying

\[
w_{\lambda}(\hat{x}) = \inf_{\Sigma_{\lambda}} w_{\lambda}(x) < 0.
\]

By (59) and (60), we can deduce from Proposition 3 that, there exist a \( R_0 > 0 \) large enough such that

\[
\hat{x} \leq R_0.
\]

Now take \( \Omega = \Sigma_{\lambda} \), by using Proposition 3, it is easy to conclude that \( \lambda \leq -R_0 \), so (60) must holds. This completes the preparation for the moving of planes.

**Step 2.** Step 1 provides a starting point, keep moving the plane to the limiting position as long as (60) holds.

To this end, let us define

\[
\lambda_0 := \sup \{ \lambda \in \mathbb{R} \mid w_{\mu} \geq 0 \text{ in } \Sigma_{\mu}, \ \forall \mu \leq \lambda \} . \tag{61}
\]

It follows from Step 1 that \(-R_0 \leq \lambda_0 < +\infty\). One can easily verify that

\[
w_{\lambda_0}(x) \geq 0, \quad \forall \ x \in \Sigma_{\lambda_0}. \tag{62}
\]

Next, we shall show that

\[
w_{\lambda_0}(x) \equiv 0, \quad \forall \ x \in \Sigma_{\lambda_0}. \tag{63}
\]

Suppose (63) is false, which is

\[
w_{\lambda_0} \geq 0 \text{ but } w_{\lambda_0} \neq 0 \text{ in } \Sigma_{\lambda_0}, \tag{64}
\]

then we must have

\[
w_{\lambda_0}(x) > 0, \quad \forall \ x \in \Sigma_{\lambda_0}. \tag{65}
\]

In fact, if (65) is violated, then there exists a point \( \hat{x} \in \Sigma_{\lambda_0} \) such that

\[
w_{\lambda_0}(\hat{x}) = \min_{\Sigma_{\lambda_0}} w_{\lambda_0} = 0.
\]

By (13), we get

\[
F_{s,m}(u_{\lambda}(\hat{x})) - F_{s,m}(u(\hat{x})) = 0,
\]

Using the maximum principle for anti-symmetric functions Proposition 28, we have

\[
w_{\lambda_0}(x) \equiv 0, \quad \forall \ x \in \Sigma_{\lambda_0},
\]

which contradicts (64). Hence equation (65) must holds.

Then we will show that the plane \( T_{\lambda} \) can be moved a little bit further from \( T_{\lambda_0} \) to the right. More precisely, there exists an \( \delta > 0 \), such that for any \( \lambda \in [\lambda_0, \lambda_0 + \delta] \), we have

\[
w_{\lambda}(x) \geq 0, \quad \forall \ x \in \Sigma_{\lambda}. \tag{66}
\]
First, let us introduce the following notations

\[ w_{\lambda}(x) \geq c_0 > 0, \quad \forall \ x \in \Sigma_{\lambda_0 - \delta_1} \cap B_{R_0}(0), \]  

(67)

where \( R_0 \) is defined in Step 1. Due to the continuity of \( w_{\lambda} \) about \( \lambda \), there exists a \( 0 < \delta_2 < \delta_1 \) sufficiently small such that,

\[ w_{\lambda}(x) > 0, \quad \forall \ x \in \Sigma_{\lambda_0 - \delta_1} \cap B_{R_0}(0), \quad \text{for any} \ \lambda \in [\lambda_0, \lambda_0 + \delta_2]. \]  

(68)

For any \( \lambda \in [\lambda_0, \lambda_0 + \delta_2] \), if we suppose that \( \inf_{\Sigma_{\lambda}} w_{\lambda}(x) < 0 \), then there exist a \( x_0 \) such that

\[ w_{\lambda}(x_0) = \inf_{\Sigma_{\lambda}} w_{\lambda}(x) < 0. \]

Then, from (68), we infer that

\[ x_0 \in \left( \Sigma_{\lambda} \setminus \Sigma_{\lambda_0 - \delta_1} \right) \cap B_{R_0}(0). \]  

(69)

Notice that \( \left( \Sigma_{\lambda} \setminus \Sigma_{\lambda_0 - \delta_1} \right) \cap B_{R_0}(0) \) is a narrow region and \( c(x) \) is uniformly bounded from below. By Narrow region principle Proposition 2, we have

\[ w_{\lambda}(x) > 0, \quad \forall \ x \in \left( \Sigma_{\lambda} \setminus \Sigma_{\lambda_0 - \delta_1} \right) \cap B_{R_0}(0), \quad \text{for any} \ \lambda \in [\lambda_0, \lambda_0 + \delta_2], \]  

(70)

which contradicts (69), and hence

\[ w_{\lambda}(x) \geq 0, \quad \forall \ x \in \Sigma_{\lambda}. \]  

(71)

Thus (66) holds, which contradicts the definition (61) of \( \lambda_0 \). Hence (63) must be valid.

Since the \( x_1 \)-direction can be chosen arbitrarily, equation (63) implies \( u(x) \) is radial symmetry and monotonicity about some point \( x_0 \in \mathbb{R}^N \). This completes the proof of Theorem 1.2.

3.3. Monotone of solutions in a coercive epigraph \( \Omega \).

In this subsection, we shall prove the monotonicity solutions to generalized equations involving fully nonlinear nonlocal operators (1) in a coercive epigraph \( \Omega \), which is Theorem 1.3.

\textbf{Proof.} Without loss of generality, we assume

\[ \inf_{x \in \Omega} x_N = 0. \]

First, let us introduce the following notations

\[ T_{\lambda} := \{ x \in \mathbb{R}^N_+ | x_N = \lambda, \text{ for some } \lambda \in \mathbb{R}_+ \}, \]

\[ \Sigma_{\lambda} := \{ x \in \mathbb{R}_+^N | x_N < \lambda \}, \]

and denoting the reflection of \( x \) about \( T_{\lambda} \) by \( x^\lambda = (x_1, \ldots, 2\lambda - x_N) \).

Assume that \( u \) is a solution to problem (17). By a direct calculation, we have

\[ F_{s,m}(u_{\lambda}(x)) - F_{s,m}(u(x)) = -c_{\lambda}(x)w_{\lambda}(x), \]  

(72)

with

\[ c_{\lambda}(x) := \begin{cases} \frac{f(u(x)) - f(u_{\lambda}(x))}{w_{\lambda}(x) - w(x)} , & \text{if } u_{\lambda}(x) \neq u(x), \\ 0, & \text{if } u_{\lambda}(x) = u(x). \end{cases} \]

Next, We carry out the moving planes procedure in two steps.

\textit{Step 1.} First, We will first show that, for \( \lambda > 0 \) sufficiently close to 0,

\[ w_{\lambda}(x) > 0, \quad x \in \Sigma_{\lambda} \cap \Omega. \]  

(73)
Due to the Lipschitz continuity of $f$ and the local boundedness of $u$, for any $\lambda$, there exist a $C > 0$ depending on $\lambda$ such that
\[
\|c_\lambda\|_{L^\infty(\Sigma_\lambda \cap \Omega)} \leq C \quad \text{for all } 0 < \lambda < \bar{\lambda}.
\]
Then we also have
\[
w_\lambda(x) \geq 0 \quad \text{in } \Sigma_\lambda \setminus \Omega.
\] Notice that, $\Omega$ is a coercive epigraph, $\Sigma_\lambda \cap \Omega$ is always bounded for every $\lambda > 0$.

**Step 2.** We continue to move the plane $T_\lambda$ along the $x_N$-axis until to its limiting position as long as (73) holds. More precisely, let us define
\[
\lambda_0 := \sup \{ \lambda > 0 \mid w_\mu > 0 \text{ in } x \in \Sigma_\mu \cap \Omega, \quad \forall \ 0 < \mu < \lambda \}.
\] We shall show that
\[
\lambda_0 = +\infty.
\]
Otherwise, suppose on the contrary that $0 < \lambda_0 < +\infty$, we shall show that the plane $T_{\lambda_0}$ can be moved upward a little bit more, that is, there exists an $\varepsilon > 0$ small enough such that
\[
w_\lambda(x) > 0, \quad \forall \ x \in \Sigma_\lambda \cap \Omega, \quad \forall \ \lambda_0 \leq \lambda \leq \lambda_0 + \varepsilon,
\] which contradicts the definition (75) of $\lambda_0$.

In fact, by the definition of $\lambda_0$, we have
\[
w_{\lambda_0} \geq 0 \text{ in } \Sigma_{\lambda_0} \cap \Omega.
\] From (17), we have
\[
w_{\lambda_0}(x) > 0 \text{ for any } x \in \Omega^{\lambda_0} \setminus \Omega,
\] where the notation $\Omega^{\lambda_0}$ denotes the reflection of $\Omega$ about the plane $T_{\lambda_0}$. Then, we obtain from Proposition 1 that
\[
w_{\lambda_0} > 0 \text{ in } \Sigma_{\lambda_0} \cap \Omega.
\]
Then we choose $\varepsilon_1 > 0$ sufficiently small such that $(\Sigma_{\lambda_0 + \varepsilon_1} \setminus \Sigma_{\lambda_0 - \varepsilon_1}) \cap \Omega$ is a bounded narrow region. By the fact that $w_{\lambda_0}(x) > 0$ for $x \in \Omega^{\lambda_0} \cap \Sigma_{\lambda_0}$ and the continuity of $w_{\lambda_0}$, there exists $c_0 > 0$ such that
\[
w_{\lambda_0}(x) > c_0, \quad \forall \ x \in \Sigma_{\lambda_0 - \varepsilon_1} \cap \Omega.
\] Therefore, for every $\lambda_0 \leq \lambda \leq \lambda_0 + \varepsilon_2$, we can choose $0 < \varepsilon_2 < \varepsilon_1$ sufficiently small such that
\[
w_\lambda(x) > \frac{c_0}{2} > 0, \quad \forall \ x \in \Sigma_{\lambda_0 - \varepsilon_1} \cap \Omega.
\] Since $(\Sigma_{\lambda_0 + \varepsilon_2} \setminus \Sigma_{\lambda_0 - \varepsilon_1}) \cap \Omega$ is also a bounded narrow region, we deduce from (74), (77) and the narrow region principle Proposition 2 that
\[
w_\lambda > 0 \quad \text{in } \Sigma_\lambda \cap \Omega, \quad \forall \ \lambda_0 \leq \lambda \leq \lambda_0 + \varepsilon_2,
\] which contradicts the definition (75) of $\lambda_0$. Thus, we must have $\lambda_0 = +\infty$. This completes the proof of Theorem 1.3. \[\Box\]
3.4. Nonexistence of positive solutions on a half space $R_N^+$.

In this subsection, we shall prove nonexistence of positive solutions to generalized equations involving fully nonlinear nonlocal operators (1) on a half space $R_N^+$, which is Theorem 1.4.

**Proof.** We first show that either $u(x) > 0$ or $u \equiv 0$, $\forall x \in R_N^+$.

Indeed, if there exists $x_0 \in R_N^+$ such that $u(x_0) = 0$. Then we must have $u(x) = 0$ in $R_N^+$.

If $u(x) \neq 0$, we have

$$
F_{s,m}(u(x)) = \epsilon_{N,s} m^{\frac{N}{2} + s} P.V. \int_{R_N} \frac{G(u(x) - u(y))}{|x - y|^{\frac{N}{2} + s}} K_{\frac{N}{2} + s}(m|x - y|)dy
$$

$$
= \epsilon_{N,s} m^{\frac{N}{2} + s} P.V. \int_{R_N} \frac{G(-u(y))}{|x - y|^{\frac{N}{2} + s}} K_{\frac{N}{2} + s}(m|x - y|)dy
$$

$$
= - \epsilon_{N,s} m^{\frac{N}{2} + s} P.V. \int_{R_N^+} \frac{G(u(y))}{|x - y|^{\frac{N}{2} + s}} K_{\frac{N}{2} + s}(m|x - y|)dy
$$

$$
< 0,
$$

where use the assumption of $A_1$. On the other hand, we have

$$
F_{s,m}(u(x)) = f(0) = 0.
$$

This is a contradiction, which implies that $u(x) \equiv 0$ in $R_N^+$.

So equation (78) must hold.

In the following, we always assume that $u(x) > 0$. Then, we carry out the moving planes procedure in two steps.

**Step 1.** We shall show that, for $\lambda > 0$ sufficiently close to 0,

$$
w_\lambda(x) \geq 0, \quad x \in \Sigma_\lambda,
$$

where $\Sigma_\lambda = \{x \in R_N^+ | 0 < x_N < \lambda \}$. Due to $\lim_{|x| \to \infty} u(x) = 0$ and $u > 0$, we have

$$
\lim_{|x| \to \infty} w_\lambda(x) \geq 0.
$$

Then, applying Proposition 2, we get (81).

**Step 2.** We continue to move the plane $T_\lambda$ along the $x_N$-axis until to its limiting position as long as (81) holds. More precisely, let

$$
\lambda_0 := \sup \{ \lambda > 0 | w_\mu(x) > 0, \lambda \leq \lambda \}.
$$

By the similar argument as proof of Theorem 1.3, we are able to show that

$$
\lambda_0 = +\infty.
$$

It reveals that

$$
u(x_1, x_2, \ldots, x_{N-1}, 2\lambda_0) = u(x_1, x_2, \ldots, x_{N-1}, 0) = 0.
$$

That is a contradictive with $u > 0$. So (83) must holds.
Therefore, \( u \) is increasing with respect to the \( x_N \)-axis. In terms of the assumption (19) in Theorem 1.4, we know that is impossible. So \( u(x) \equiv 0 \) in \( \mathbb{R}^N \).

This completes the proof of Theorem 1.4. \( \square \)

4. Maximum principles in unbounded domains. In this section, we shall prove the maximum principle for generalized fully nonlinear nonlocal equation (1) on unbounded domains (Theorem (1.5)). In section 2, we have derived the maximum principle for anti-symmetric functions in bounded open set. The maximum principle for generalized fully nonlinear nonlocal equation (1) on unbounded domains without assuming that the function vanishes near infinity.

In the proof process of Theorem (1.5), we need the following Lemma.

**Lemma 4.1.** Assume that \( u \in \mathcal{L}_s(\mathbb{R}^N) \cap C^{1,1}_{loc}(\mathbb{R}^N) \), for any function \( \psi \in C^\infty_0(\mathbb{R}^N) \) and any small \( \delta > 0 \). Then we have

\[
F_{s,m}(u + \varepsilon \psi)(x) - F_{s,m}(u)(x) \leq \varepsilon C_2 + C_2 \delta^{p+2-2s} + m^{2s} \varepsilon \psi(x), \quad \forall \, x \in \mathbb{R}^N,
\]

where \( C_2 \) is independent of \( \varepsilon \).

**Proof.** For simplicity, we denote

\[
v(x) = u(x) + \varepsilon \psi(x).
\]

A direct calculation yields

\[
F_{s,m}(u + \varepsilon \psi)(x) - F_{s,m}(u)(x) = c_{N,s} m^{\frac{s}{2} + s} P.V. \int_{\mathbb{R}^N} \left( \frac{G(v(x) - v(y)) - G(u(x) - u(y))}{|x - y|^{\frac{N}{2} + s}} - \frac{G(u(x) - u(y))}{|x - y|^{\frac{N}{2} + s}} \right) K_{\frac{N}{2} + s}(m|x - y|) \, dy \\
+ m^{2s}(v(x) - u(x)) \tag{84}
\]

\[
= c_{N,s} m^{\frac{s}{2} + s} P.V. \int_{B_\delta(x)} \left( \frac{G(v(x) - v(y)) - G(u(x) - u(y))}{|x - y|^{\frac{N}{2} + s}} - \frac{G(u(x) - u(y))}{|x - y|^{\frac{N}{2} + s}} \right) K_{\frac{N}{2} + s}(m|x - y|) \, dy \\
+ \int_{B_\delta^c(x)} \left( G(v(x) - v(y)) - G(u(x) - u(y)) \right) K_{\frac{N}{2} + s}(m|x - y|) \, dy \\
+ m^{2s}(v(x) - u(x)) \\
= : H_1 + H_2 + m^{2s} \varepsilon \psi(x).
\]

Next, we shall estimate \( H_1 \) and \( H_2 \). From (6) and (8), one can easily infer that, for any \( r > 0 \) and \( \nu > 0 \),

\[
0 < K_\nu(r) \leq C_{\frac{r}{\nu}}^\nu.
\]

First, we estimate \( H_1 \) in \( B_\delta \). By Taylor expansion, we get

\[
v(x) - v(y) = \nabla v(x) \cdot (x - y) + O(|x - y|^2).
\]

Combine the anti-symmetry of \( \nabla v(x) \cdot (x - y) \) for \( y \in B_\delta(x) \) with the assumption \( A_1 \), we have

\[
c_{N,s} m^{\frac{s}{2} + s} P.V. \int_{B_\delta(x)} \left( \frac{G(\nabla v(x) \cdot (x - y))}{|x - y|^{\frac{N}{2} + s}} \right) K_{\frac{N}{2} + s}(m|x - y|) \, dy = 0.
\]
We deduce that

\[
|c_{N,s}m^{\frac{N}{2}+s} \int_{B_\delta(x)} \left( \frac{G(v(x) - v(y))}{|x-y|^{\frac{N}{2}+s}} \right) K_{\frac{N}{2}+s} (m|x-y|) \, dy | \leq c_{N,s}m^{\frac{N}{2}+s} \int_{B_\delta(x)} \left( \frac{G(v(x) - v(y))}{|x-y|^{\frac{N}{2}+s}} \right) K_{\frac{N}{2}+s} (m|x-y|) \, dy \\
= a_1 c_{N,s} \int_{B_\delta(x)} \frac{O(|x-y|^{p+2})}{|x-y|^{N+2s}} \, dy \\
\leq C_1 \delta^{p+2-2s}.
\]

where \( \xi_1 \) be the value between \( \nabla v(x) \cdot (x-y) \) and \( v(x) - v(y) \), the first inequality is true because of assumption \( A_3 \), \( C_1 \) is independent of \( \varepsilon \), since for any fixed \( x \), we have

\[
\nabla v(x) = \nabla (u(x) + \varepsilon \psi(x)) \leq |\nabla u(x)| + \varepsilon |\nabla \psi(x)| \leq |\nabla u(x)| + |\nabla \psi(x)|, \quad \text{if} \ \varepsilon \leq 1.
\]

Similarly, we have

\[
\left| c_{N,s}m^{\frac{N}{2}+s} \int_{B_\delta(x)} \left( \frac{G(u(x) - u(y))}{|x-y|^{\frac{N}{2}+s}} \right) K_{\frac{N}{2}+s} (m|x-y|) \, dy \right| \leq C_1 \delta^{p+2-2s}.
\]

In summary, we have

\[
|H_1| \leq C_2 \delta^{p+2-2s},
\]

where \( C_2 \) is independent of \( \varepsilon \).

Second, we estimate \( H_2 \) in \( B_\delta^c \). Note that

\[
G(v(x) - v(y)) - G(u(x) - u(y)) \\
= \varepsilon (\psi(x) - \psi(y)) \int_0^1 G'(u(x) - u(y) + t\varepsilon (\psi(x) - \psi(y))) \, dt \\
=: \varepsilon L(x, y).
\]
Since \( u \in \mathcal{L}_{s}(\mathbb{R}^{N}) \cap C^{1,1}_{\text{loc}}(\mathbb{R}^{N}) \) and \( \psi \in C_{0}^{\infty}(\mathbb{R}^{N}) \), we have
\[
|H_{2}| = \left| c_{N,s,m}^{N} \int_{B_{\delta}(x)} \frac{\varepsilon L(x,y)}{|x-y|^{N+2s}} K_{N,s}^{N} (m|x-y|) \, dy \right|
\]
\[
\leq c_{N,s,m}^{N} P.V. \int_{B_{\delta}(x)} \frac{\varepsilon L(x,y)}{|x-y|^{N+2s}} C_{N,s} \left( m|x-y| \right) \, dy
\]
\[
= c_{N,s} P.V. \int_{B_{\delta}(x)} \frac{\varepsilon C_{N,s} L(x,y)}{|x-y|^{N+2s}} \, dy
\]
\[
\leq \varepsilon C_{\delta}.
\]

Combine (88) and (90), we get
\[
F_{s,m}(u + \varepsilon \psi)(x) - F_{s,m}(u)(x) \leq \varepsilon C_{\delta} + C_{2} \delta^{p+2-2s} + m^{2s} \varepsilon \psi(x), \quad \forall \ x \in \mathbb{R}^{N},
\]
where \( C_{2} \) is independent of \( \varepsilon \). This completes the proof of Lemma 4.1. \( \square \)

Next we shall prove Theorem 1.5 by using the contradiction arguments.

**The proof of Theorem 1.5**

**Proof.** Suppose that (22) is not valid, then we have
\[
\sup_{x \in \Sigma} w(x) := M > 0.
\]
If the supremum can be attained, the next proof of Theorem 1.5 is similar to Proposition 1. Now the main problem is that \( \Sigma \) is an unbounded domain, and hence \( \sup_{\Sigma} w(x) \) may not be attained. Hence, there exists sequences \( x^{k} \in \Sigma \) and \( 0 < \beta_{k} < 1 \) with \( \beta_{k} \to 1 \) as \( k \to \infty \) such that
\[
w(x^{k}) \geq \beta_{k} M. \quad (91)
\]
Let \( x_{1} \) be any given direction in \( \mathbb{R}^{N} \). We may assume that
\[
T = \{ x \in \mathbb{R}^{N} | x_{1} = 0 \},
\]
and
\[
\Sigma = \{ x \in \mathbb{R}^{N} | x_{1} < 0 \}.
\]
Then the reflection of the point \( x \) about the plane \( T \) is
\[
\tilde{x} = (-x_{1}, x_{2}, \cdots, x_{N}).
\]
We denote by
\[
\tilde{u}(x) = u(\tilde{x}), \quad w(x) = \tilde{u}(x) - u(x),
\]
and
\[
d_{k} := \text{dist}(x^{k}, T).
\]
Define the function
\[
\psi(x) = \begin{cases} e^{\frac{|x|^{2}}{|x|^{2} - 1}}, & |x| < 1, \\ 0, & |x| \geq 1. \end{cases}
\]
Notice that
\[
\psi(0) = \max_{\mathbb{R}} \psi(x) = 1.
\]
It is well known that \( \psi \in C_{0}^{\infty}(\mathbb{R}^{N}) \), thus
\[
|F_{s,m}(\psi(x))| \leq C, \quad \forall \ x \in \mathbb{R}^{N}.
\]
Moreover, \( F_{s,m}(\psi(x)) \) is monotone decreasing with respect to \( |x| \).
Next, we denote
\[ \psi_k(x) := \psi \left( \frac{x - \tilde{x}_k}{d_k} \right) \]
and
\[ \tilde{\psi}_k(x) = \psi_k(\tilde{x}) = \psi \left( \frac{x - x^k}{d_k} \right). \]

Then \( \tilde{\psi}_k - \psi_k \) is anti-symmetric with respect to \( T \).

Now pick \( \varepsilon_k = (1 - \beta_k)M \), then we have
\[ w(x^k) + \varepsilon_k[\tilde{\psi}_k - \psi_k](x^k) \geq M. \]

Set
\[ w_k(x) := w(x) + \varepsilon_k[\tilde{\psi}_k - \psi_k](x). \]

Then \( w_k \) is also anti-symmetric with respect to \( T \).

Note that, for any \( x \in \Sigma \setminus B_{d_k}(x^k) \), \( w(x) \leq M \) and \( \tilde{\psi}_k(x) = \psi_k(x) = 0 \), we conclude
\[ w_k(x^k) \geq w_k(x), \quad \forall x \in \Sigma \setminus B_{d_k}(x^k). \]

Hence the supremum of \( w_k(x^k) \) in \( \Sigma \) is achieved in \( B_{d_k}(x^k) \). Consequently, there exists a point \( \bar{x}^k \in B_{d_k}(x^k) \) such that
\[ w_k(\bar{x}^k) = \sup_{x \in \Sigma} w_k(x) \geq M. \]

By the choice of \( \varepsilon_k \), it is easy to verify that \( w(\bar{x}^k) \geq \beta_k M > 0 \).

Now we estimate the upper bound and the lower bound of
\[ F_{s,m}(\bar{u} + \varepsilon_k \tilde{\psi}_k)(\bar{x}^k) - F_{s,m}(u + \varepsilon_k \psi_k)(\bar{x}^k) \]
respectively, to derive a contradiction.

We first estimate the upper bound of \( F_{s,m}(\bar{u} + \varepsilon_k \tilde{\psi}_k)(\bar{x}^k) - F_{s,m}(u + \varepsilon_k \psi_k)(\bar{x}^k) \).

By using Lemma (4.1), we get
\[ F_{s,m}(\bar{u} + \varepsilon_k \tilde{\psi}_k)(\bar{x}^k) - F_{s,m}(u + \varepsilon_k \psi_k)(\bar{x}^k) \]
\[ = F_{s,m}(\bar{u} + \varepsilon_k \tilde{\psi}_k)(\bar{x}^k) - F_{s,m}(\bar{u})(\bar{x}^k) + F_{s,m}(\bar{u})(\bar{x}^k) - F_{s,m}(u)(\bar{x}^k) \]
\[ + F_{s,m}(u)(\bar{x}^k) - F_{s,m}(u + \varepsilon_k \psi_k)(\bar{x}^k) \]
\[ \leq 2\varepsilon_k C_\delta + 2C_2 \delta^p + 2^{-2s} + m^{2s} \varepsilon_k \tilde{\psi}_k(\bar{x}^k) + m^{2s} \varepsilon_k \psi_k(\bar{x}^k) - c(\bar{x}^k) w(\bar{x}^k) \]
\[ \leq 2\varepsilon_k C_\delta + 2C_2 \delta^p + 2^{-2s} + 2m^{2s} \varepsilon_k - c(\bar{x}^k) w(\bar{x}^k), \]
where we can choose \( \delta \) sufficiently small such that \( \delta < d_k \). The first inequality is applied to (20) and Lemma (4.1). The last inequalities using the fact that \( |\psi(x)| \leq 1. \)
Next, we estimate the lower bound of $F_{s,m}(\tilde{u} + \varepsilon_k \hat{\psi}_k)\(\tilde{x}^k\) - F_{s,m}(u + \varepsilon_k \psi_k)\(\tilde{x}^k\)$ by a direct calculations.

\[ F_{s,m}(\tilde{u} + \varepsilon_k \hat{\psi}_k)\(\tilde{x}^k\) - F_{s,m}(u + \varepsilon_k \psi_k)\(\tilde{x}^k\) = c_{N,s}m^{\frac{N}{2} + s} P.V. \int_{\Sigma_N} \frac{1}{|\tilde{x}^{k} - y|^{\frac{N}{2} + s}} \left[ G(\tilde{u}\(\tilde{x}^k\) + \varepsilon_k \hat{\psi}_k\(\tilde{x}^k\)) - \tilde{u}(y) - \varepsilon_k \hat{\psi}_k(y) \right] dy + m^{2s} \omega_k(\tilde{x}^k) \]

As a consequence of (92), we get

\[ = c_{N,s}m^{\frac{N}{2} + s} P.V. \int_{\Sigma_N} \frac{1}{|\tilde{x}^{k} - y|^{\frac{N}{2} + s}} \left[ G(\tilde{u}\(\tilde{x}^k\) + \varepsilon_k \hat{\psi}_k\(\tilde{x}^k\)) - \tilde{u}(y) - \varepsilon_k \hat{\psi}_k(y) \right] dy + m^{2s} \omega_k(\tilde{x}^k) \]

In the following we will estimate the above terms one by one. Note that

\[ \frac{1}{|\tilde{x}^{k} - y|^{\frac{N}{2} + s}} - \frac{1}{|\tilde{x}^{k} - \tilde{y}|^{\frac{N}{2} + s}} > 0, \quad \forall \ y \in \Sigma. \quad (95) \]

As a consequence of (92), we get

\[ \left( \tilde{u}\(\tilde{x}^k\) + \varepsilon_k \hat{\psi}_k\(\tilde{x}^k\) - \tilde{u}(y) - \varepsilon_k \hat{\psi}_k(y) \right) - \left( u\(\tilde{x}^k\) + \varepsilon_k \psi_k\(\tilde{x}^k\) - u(y) - \varepsilon_k \psi_k(y) \right) \]

\[ = w_k(\tilde{x}^k) - w_k(y) \geq 0. \]
By using the assumption $A_2$, we have
\[ G(\tilde{u}(\bar{x}^k) + \varepsilon_k \tilde{\psi}_k(\bar{x}^k)) - \tilde{u}(y) - \varepsilon_k \tilde{\psi}_k(y)) - G(u(\bar{x}^k) + \varepsilon_k \tilde{\psi}_k(\bar{x}^k) - u(y) - \varepsilon_k \tilde{\psi}_k(y)) \geq 0. \]
Consequently,
\[ I_1 = c_{N,s,m}^{\frac{N}{2} + s} P.V. \int_{\Omega} \left( \frac{1}{|\bar{x}^k - y|^\frac{N}{2} + s} - \frac{1}{|\bar{x}^k - \tilde{y}|^\frac{N}{2} + s} \right) [G(\tilde{u}(\bar{x}^k) + \varepsilon_k \tilde{\psi}_k(\bar{x}^k)) - u(y) - \varepsilon_k \tilde{\psi}_k(y))] dy \geq 0. \]
Next we estimate the term $I_2$. For any $y \in \Sigma$, we have
\[ I_2 = c_{N,s,m}^{\frac{N}{2} + s} P.V. \int_{\Omega} \left( \frac{1}{|\bar{x}^k - y|^\frac{N}{2} + s} G(\tilde{u}(\bar{x}^k) + \varepsilon_k \tilde{\psi}_k(\bar{x}^k) - u(y) - \varepsilon_k \tilde{\psi}_k(y)) \right) \]
\[ = \frac{1}{|\bar{x}^k - y|^\frac{N}{2} + s} G(\tilde{u}(\bar{x}^k) + \varepsilon_k \tilde{\psi}_k(\bar{x}^k) - u(y) - \varepsilon_k \tilde{\psi}_k(y)) \]
\[ = w_k(\bar{x}^k) \geq M, \]
and
\[ \left( \tilde{u}(\bar{x}^k) + \varepsilon_k \tilde{\psi}_k(\bar{x}^k) - \tilde{u}(y) - \varepsilon_k \tilde{\psi}_k(y)) - (u(\bar{x}^k) + \varepsilon_k \tilde{\psi}_k(\bar{x}^k) - \tilde{u}(y) - \varepsilon_k \tilde{\psi}_k(y)) \right) \]
\[ = w_k(\bar{x}^k) \geq M. \]
As a consequence of the assumption $A_2$ and (6), we derive that
\[ I_2 = c_{N,s,m}^{\frac{N}{2} + s} P.V. \int_{\Omega} \frac{1}{|\bar{x}^k - y|^\frac{N}{2} + s} \left( G(\tilde{u}(\bar{x}^k) + \varepsilon_k \tilde{\psi}_k(\bar{x}^k) - u(y) - \varepsilon_k \tilde{\psi}_k(y)) \right) \]
\[ = c_{N,s,m}^{\frac{N}{2} + s} P.V. \int_{\Omega} \frac{G'(\xi) + G'(\eta)}{|\bar{x}^k - y|^\frac{N}{2} + s} \left( w_k(\bar{x}^k) \tilde{\psi}_k(\bar{x}^k) - \varepsilon_k \tilde{\psi}_k(\bar{x}^k) \right) \]
\[ = c_{N,s,m}^{\frac{N}{2} + s} P.V. \int_{\Omega} \frac{G'(\xi) + G'(\eta)}{|\bar{x}^k - y|^\frac{N}{2} + s} \left( w_k(\bar{x}^k) \right) dy \]
\[ = c_{N,s,m}^{\frac{N}{2} + s} P.V. \int_{\Omega} \frac{G'(\xi) + G'(\eta)}{|\bar{x}^k - y|^\frac{N}{2} + s} \left( w_k(\bar{x}^k) K_{\frac{N}{2} + s} (m|\bar{x}^k - \tilde{y}|) dy \right) \]
\[ = c_{N,s,m}^{\frac{N}{2} + s} P.V. \int_{\Omega} \frac{G'(\xi) + G'(\eta)}{|\bar{x}^k - y|^\frac{N}{2} + s} w_k(\bar{x}^k) K_{\frac{N}{2} + s} (m|\bar{x}^k - \tilde{y}|) dy \]
\[ = 2M_{0}c_{N,s,m}^{\frac{N}{2} + s} P.V. \int_{\Omega} \frac{G'(\xi) + G'(\eta)}{|\bar{x}^k - y|^\frac{N}{2} + s} w_k(\bar{x}^k) K_{\frac{N}{2} + s} (m|\bar{x}^k - \tilde{y}|) dy \]
\[ \geq 2M_{0}c_{N,s,m}^{\frac{N}{2} + s} P.V. \int_{\Omega} \frac{1}{|\bar{x}^k - y|^\frac{N}{2} + s} K_{\frac{N}{2} + s} (m|\bar{x}^k - \tilde{y}|) dy \]
\[ \geq 2M_{0}c_{N,s,m}^{\frac{N}{2} + s} P.V. \int_{\Omega} \frac{1}{|\bar{x}^k - y|^\frac{N}{2} + s} dy \]
\[ \geq 2M_{0}c_{N,s,m}^{\frac{N}{2} + s} P.V. \int_{\Omega} \frac{1}{|\bar{x}^k - y|^\frac{N}{2} + s} dy \]
\[ \geq 2M_{0}c_{N,s,m}^{\frac{N}{2} + s} P.V. \int_{\Omega} \frac{1}{|\bar{x}^k - y|^\frac{N}{2} + s} dy \]
\[ \geq C_{1} \]
\[ F_{s,m}(\tilde{u} + \varepsilon_k \tilde{\psi}_k(\bar{x}^k)) - F_{s,m}(u + \varepsilon_k \tilde{\psi}_k)(\bar{x}^k) \geq m^2 w_k(\bar{x}^k) + C_{1} M. \]
Combining (93) and (98), we derive
\[ C_1 M + m^{2s} w(\bar{x}) \leq C_1 M + m^{2s} w_k(\bar{x}) \]
\[ \leq F_{s,m}(\bar{u} + \varepsilon_k \bar{\psi}_k)(\bar{x}) - F_{s,m}(u + \varepsilon_k \psi_k)(\bar{x}) \]
\[ \leq 2\varepsilon_k C_\delta + 2C_2 \delta^{p+2-2s} + 2m^{2s} \varepsilon_k - \inf_{\{x \in \Sigma | w(x) > 0\}} c(x) w(\bar{x}), \]
which implies that
\[ \left( \inf_{\{x \in \Sigma | w(x) > 0\}} c(x) + m^{2s} \right) w(\bar{x}) + C_1 M \leq 2\varepsilon_k C_\delta + 2C_2 \delta^{p+2-2s} + 2m^{2s} \varepsilon_k \]
\[ \leq (2C_\delta + 2m^{2s})(1 - \beta_k)M + 2C_2 \delta^{p+2-2s}. \]

We first choose \( \delta \) sufficiently small such that
\[ C_2 \delta^{p+2-2s} \leq \frac{1}{2} C_1 M, \] (100)
where \( p + 2 - 2s > 0 \). Substitute equation (100) into (99), we have
\[ \left( \inf_{\{x \in \Sigma | w(x) > 0\}} c(x) + m^{2s} \right) w(\bar{x}) \leq (2C_\delta + 2m^{2s})(1 - \beta_k)M. \] (101)

Notice that \( w(\bar{x}) \geq \beta_k M \), divide both sides of the equation (99) by \( M \). Then we get
\[ \left( \inf_{\{x \in \Sigma | w(x) > 0\}} c(x) + m^{2s} \right) \beta_k \leq (2C_\delta + m^{2s})(1 - \beta_k). \]
This will lead to a contradiction if we take the limit \( k \to \infty \).

We completes the proof of Theorem 1.5.

\[ \square \]

5. Applications of the maximum principle. In this section, we shall prove Liouville Theorem 1.6 and Theorem 1.7 by using Maximum Principle Theorem 1.5 and the direct method of moving planes.

The proof of Theorem 1.6

Proof. We first show that \( u \) is symmetric with respect to any hyper-plane \( T \). Let \( \Sigma \) be the half space on one side of the plane \( T \) and \( x_\alpha \) be any given direction in \( \mathbb{R}^N \). Set
\[ w(x) = u(\bar{x}) - u(x) \quad \text{for all} \quad x \in \Sigma, \]
where \( \bar{x} \) is the reflection of \( x \) with respect to \( T \). Since \( u \in \mathcal{L}_s(\mathbb{R}^N) \cap C^{1,1}_{\text{loc}}(\mathbb{R}^N) \) is bounded, we have \( w \in \mathcal{L}_s(\mathbb{R}^N) \cap C^{1,1}_{\text{loc}}(\mathbb{R}^N) \) is bounded, and at any points \( x \in \Sigma \) where \( w(x) > 0 \), one has
\[ F_{s,m}(u(\bar{x})) - F_{s,m}(u(x)) = f(u(\bar{x})) - f(u(x)) \]
\[ = \frac{f(u(\bar{x})) - f(u(x))}{u(\bar{x}) - u(x)} \cdot w(x) \]
\[ =: c(x) w(x), \]
where \( c(x) := \frac{f(u(\bar{x})) - f(u(x))}{u(\bar{x}) - u(x)} \) satisfies
\[ \sup_{\{x \in \Sigma | w(x) > 0\}} c(x) < m^{2s}. \]
Therefore, applying Theorem 1.5, we arrive immediately
\[ w \leq 0 \text{ in } \Sigma. \]
In a similar way, we can also prove that
\[ w \geq 0 \text{ in } \Sigma. \]
Hence, we get
\[ w \equiv 0 \text{ in } \mathbb{R}^N. \] (102)
That implies \( u \) is symmetric with respect to any hyper-plane \( T \). Since the \( x_n \)-direction can be chosen arbitrarily, (102) implies \( u \) is radially symmetric about any point, so we must have
\[ u \equiv C \text{ in } \mathbb{R}^N. \]
By the definition of \( F_{s,m} \), one has
\[ F_{s,m}(u(x)) = F_{s,m}(C) = c_{N,s,m} \frac{N}{2} + P.V. \int_{\mathbb{R}^N} \frac{G(C-C')}{|x-y|^{N+2s}} \left( m|x-y| \right) dy + m^{2s}C \]
\[ = m^{2s}C = f(u(x)) = f(C), \]
thus \( C \) is determined by the equation \( f(C) = m^{2s}C \). Under special condition \( f(u(x)) = 0 \), then we have \( u \equiv C = 0 \).
This finishes the proof of Liouville Theorem 1.6.

**The proof of Theorem 1.7**

*Proof.* For arbitrary \( \lambda \in \mathbb{R} \), let \( x_n \) be any given direction in \( \mathbb{R}^N \). We assume that
\[ T_{\lambda} := \{ x \in \mathbb{R}^N | x_n = \lambda \}, \quad \Sigma_{\lambda} := \{ x \in \mathbb{R}^N | x_n > \lambda \}, \]
and
\[ x^\lambda := (x_1, x_2, \ldots, 2\lambda - x_n). \]
We need only to prove that
\[ w_{\lambda}(x) := u(x^\lambda) - u(x) \leq 0 \text{ in } \Sigma_{\lambda} \text{ for any } |\lambda| \text{ sufficiently large.} \]
By the assumption (26), there exists \( M > 0 \) such that
\[ u(x) \in [-1, -1 + \delta] \cup [1 - \delta, 1] \text{ for any } x \text{ with } |x_N| > M. \]
Consequently, for any \( |\lambda| > M \), at any point \( x \in \Sigma_{\lambda} \) where \( w_{\lambda}(x) = u(x^\lambda) - u(x) > 0 \), we infer from assumption (27) that
\[ F_{s,m}(u(x^\lambda)) - F_{s,m}(u(x)) = f(u(x^\lambda)) - f(u(x)) = c_{\lambda}(x)w_{\lambda}(x), \]
where \( c_{\lambda}(x) := \frac{f(u(x^\lambda)) - f(u(x))}{u(x^\lambda) - u(x)} \), and satisfying
\[ \sup_{\{x \in \Sigma_{\lambda} | w_{\lambda}(x) > 0\}} c_{\lambda}(x) < m^{2s}. \]
Therefore, we deduce from Theorem 1.5 that \( w(x) \leq 0 \) in \( \Sigma_{\lambda} \) for all \( \lambda \) with \( |\lambda| > M \).
Now, we prove that
\[ w_{\lambda}(x) := u(x^\lambda) - u(x) < 0 \text{ in } \Sigma_{\lambda} \text{ for all } |\lambda| > M. \] (105)
Suppose, by contradiction that there exists a \( \lambda \in (-\infty, -M) \cup (M, +\infty) \) and a point \( \hat{x} \in \Sigma_{\lambda} \) such that
\[ u(\hat{x}) = 0. \]
Then, as consequence of (104), we have
\[
F_{s,m}(u((\hat{x})^{\lambda})) - F_{s,m}(u(\hat{x})) = f(u((\hat{x})^{\lambda})) - f(u(\hat{x})) = 0,
\]
and hence we can derive from Proposition 1 immediately that
\[
w_{\lambda}(x) = 0 \text{ almost everywhere in } \mathbb{R}^N,
\]
which contradicts assumption (26). Thus equation (105) must holds.

This completes the proof of Theorem 1.7.

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