A MOSER TYPE INEQUALITY FOR BESSEL LAPLACE EQUATIONS AND APPLICATIONS

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Abstract. In this paper, we study Bessel operators and Bessel Laplace equations studied by Weinstein, Huber, and related the harmonic function theory introduced by Muckenhoupt–Stein. We establish the Moser type inequality for these harmonic functions, which is missing in this setting before. We then apply it to give a direct proof for the equivalence of characterizations of the Hardy spaces associated to Bessel operator via non-tangential maximal function and radial maximal function defined in terms of the Poisson semigroup.

1. Introduction and statement of main results

The study of harmonic functions in different spaces and domains has played an important role in harmonic analysis and partial differential equations. For harmonic functions, one can list the maximum principle, Harnack’s inequality among their important properties. Another useful feature is the so-called Moser inequality that we remind the reader now. Suppose $u(t, x)$ is a harmonic function on $\mathbb{R}^{n+1}_+$, i.e.,

$$\left(\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2}\right)u(t, x) = 0, \quad x := (x_1, \ldots, x_n) \in \mathbb{R}^n, \ t > 0,$$

then we have that for any $0 < p < \infty$, there exists a positive constant $C_{n,p}$ depending only on $n$ and $p$ such that

$$|u(t_0, x_0)| \leq C_{n,p} \left(\frac{1}{|B|} \int_B |u(t, x)|^p \, dt \, dx\right)^{\frac{1}{p}}$$

for any $(t_0, x_0) \in \mathbb{R}^{n+1}_+$ and any ball $B \subset \mathbb{R}^{n+1}_+$ centered at $(t_0, x_0)$. C. Fefferman and E. Stein [FS Section 9, Lemma 2] first proved this inequality for harmonic functions, using the Poisson representation on the sphere (note that they also pointed out that this result is essentially due to Hardy and Littlewood [HaLi]). Later, Han and Lin [HL] reproved this Moser inequality on $\mathbb{R}^n$ for $n \geq 3$, where they used the properties of the fundamental solution of the Laplace equation and the construction of suitable test functions to obtain a version of this type of inequality for $p = 2$, and then, by the standard iteration approach, they proved it for general $p \in (0, \infty)$.

In 1965, B. Muckenhoupt and E. Stein in [MS] introduced the harmonic function theory associated with Bessel operator $\Delta_\lambda$, defined by setting for suitable functions $f$,

$$\Delta_\lambda f(x) := \frac{d^2}{dx^2} f(x) + \frac{2\lambda}{x} \frac{d}{dx} f(x), \quad \lambda > 0, \quad x \in \mathbb{R}_+ := (0, \infty).$$

The associated Bessel Laplace equation given by

$$\Delta_{t,x}(u) := \partial^2_t u + \partial^2_x u + \frac{2\lambda}{x} \partial_x u = 0$$

was studied by A. Weinstein [W], and A. Huber [Hu] in higher dimension. In these works, they considered the generalised auxially symmetric potentials, and obtained the properties of the...
solutions of this equation, such as the extension, the uniqueness theorem, and the boundary value problem for certain domains.

If $u \in C^2(\mathbb{R}_+ \times \mathbb{R}_+)$ is a solution of (1.1) then $u$ is said to be $\lambda$-harmonic. The function $u$ and its conjugate (denoted by $\tilde{v}$) satisfy the following Cauchy–Riemann type equations

$$
\partial_x u = -\partial_t \tilde{v} \quad \text{and} \quad \partial_t u = \partial_x \tilde{v} + \frac{2\lambda}{x} v \quad \text{in} \quad \mathbb{R}_+ \times \mathbb{R}_+.
$$

In [MSI] they developed a theory of functions in the setting of $\triangle_\lambda$ which parallels the classical one associated to the standard Laplacian, where results on $L^p(\mathbb{R}_+, dm_\lambda)$-boundedness of conjugate functions and fractional integrals associated with $\triangle_\lambda$ were obtained for $p \in [1, \infty)$ and $d\rho_\lambda(x) := x^{2\lambda} \, dx$. We also point out that Haimo [H] studied the Hankel convolution transforms $\varphi^\sharp f$ associated with the Hankel transform in the Bessel setting systematically, which provides a parallel theory to the classical convolution and Fourier transforms. Also note that the Poisson integral of $f$ studied in [MSI] is the Hankel convolution of Poisson kernel with $f$, see [BDT].

Since then, many problems based on the Bessel context were studied, such as the boundedness of Bessel Riesz transform, Littlewood–Paley $g$-functions, Hardy and BMO spaces associated with Bessel operators, $A_p$ weights associated with Bessel operators (see, for example, [K, AK, BFBMT, V, BFS, BHNW, BCFR, YY, DLWY, DLMWY, DLWY2] and the references therein).

The aim of this paper is to give a positive answer to the open question whether the Moser inequality is true for $\lambda$-harmonic functions. We will prove the Moser inequality in Section 2, then apply it to show the equivalence of characterizations of the Hardy spaces associated to Bessel operator via non-tangential maximal function and radial maximal function.

To be more precise, for any $(t_0, x_0) \in (\mathbb{R}_+ \cup \{0\}) \times (\mathbb{R}_+ \cup \{0\})$ and $R > 0$, we define the ball $B((t_0, x_0), R)$ as follows

$$
(1.3) \quad B((t_0, x_0), R) := \{ (t, x) \in \mathbb{R}_+ \times \mathbb{R}_+ : (t - t_0)^2 + (x - x_0)^2 < R^2 \}.
$$

We also define the measure of these balls as $\tilde{m}_\lambda(B((t_0, x_0), R)) := \int_{B((t_0, x_0), R)} 1 \, dx \, x^{2\lambda} \, dx$.

Our main result is the following.

**Theorem 1.1.** Suppose $u \in C^2(\mathbb{R}_+ \times \mathbb{R}_+)$ and $u$ is a solution of (1.1). Let $p \in (0, \infty)$. Then there exists a positive constant $C_{p, \lambda}$ such that for any $(t_0, x_0) \in \mathbb{R}_+ \times \mathbb{R}_+$ and $R > 0$, we have

$$
(1.4) \quad \sup_{(t, x) \in B((t_0, x_0), R)} |u(t, x)| \leq \left[ \frac{C_{p, \lambda}}{\tilde{m}_\lambda(B((t_0, x_0), 12R))} \right] \left[ \int_{B((t_0, x_0), 12R)} |u(t, x)|^p \, x^{2\lambda} \, dx \, dt \right]^{1/p}.
$$

As we mentioned earlier, in the classical case, there are two different approaches to prove the Moser inequality for harmonic functions: (i) via Poisson representation [FS], (ii) via fundamental solutions of the Laplace equation and the iteration (1.1).

However, we point out that in this Bessel setting, none of these two methods can be applied directly. One of the main difficulties here is that for a given ball $B((t_0, x_0), R)$, the measure $\tilde{m}_\lambda(B((t_0, x_0), R))$ depends on both its radius and center, i.e., $\tilde{m}_\lambda$ is not translation invariant with respect to $x$. Thus, for the first approach, note that for the Poisson representation for the $\lambda$-harmonic functions $u$ (i.e., $u$ satisfies (1.1)), it is only known for the points $(0, 0)$ or $(t, 0)$, but not known when $x \neq 0$. Hence, this approach is not applicable when we consider the points $(t, x)$ with $x > t$. For the second approach, we first point out that the “homogeneous dimension” with respect to the measure $\tilde{m}_\lambda$ is $2\lambda + 2$, which is not the standard one. Moreover, the idea of constructing some suitable Schwarz functions does not work in this setting since the measure $\tilde{m}_\lambda$ depends on both the radius and the center of the ball.

To prove our main theorem, we need to combine some of the ideas of these two approaches, together with our new observation of applying the Sobolev embedding theorems. To be more precise, we observe that the equation (1.1) is translation invariant with respect to $t$ but not $x$. Thus, it is natural to compare the radius $R$ with the coordinate $x_0$ of the centre. We will consider the following two cases: Case (i): $R \leq x_0/4$, and Case (ii): $R > x_0/4$. 


To handle Case (i), we first establish the Caccioppoli inequality for the \( \lambda \)-harmonic function \( u \), then establish a suitable version of Sobolev embedding theorem in this setting to obtain the required estimates for the partial derivatives of \( u \). Combining these estimates together, we obtain an \( L^2 \) version of the Moser type inequality for \( u \), which implies \( (1.4) \) by using the standard iteration approach.

To prove Case (ii), noting that in this case, \( R > x_0/4 \), we can consider a larger ball centered at \((t_0, 0)\) with radius \( 5R \), which contains the ball \( B((t_0, x_0), R) \) as defined in \((1.3)\) and with comparable measures. Then we apply the Poisson representation for the \( \lambda \)-harmonic functions, and follows the idea of C. Fefferman and E. Stein \([FS]\) Section 9, Lemma 2] to obtain \((1.4)\).

As one of the applications, we can prove directly that the \( L^p \) norms of the non-tangential maximal function and radial maximal function of the Poisson integral of \( f \) are equivalent. For more details, we refer the reader to Section 3.

Throughout the paper, for every interval \( I \subset \mathbb{R}_+ \), we denote it by \( I := I(x, t) := (x - t, x + t) \cap \mathbb{R}_+ \). The measure of \( I \) is defined as \( m_\lambda(I(x, t)) := \int_{I(x, t)} x^{2\lambda} dx \).

2. Proof of Theorem 1.1

To begin with, we consider two cases: Case (i): \( R \leq x_0/4 \), and Case (ii): \( R > x_0/4 \).

We now consider Case (i). We claim that when \( R \leq x_0/4 \), we have

\[
\sup_{(t, x) \in B((t_0, x_0), R)} |u(t, x)| \leq \left[ \frac{C_{p, \lambda}}{m_\lambda(B((t_0, x_0), 2R))} \int_{B((t_0, x_0), 2R)} |u(t, x)|^p x^{2\lambda} dx dt \right]^{1/p}.
\]

Note that in this case, we have \( 2R \leq x_0/2 \), then for \((t, x) \in B((t_0, x_0), 2R)\) we have \( x \sim x_0 \) and thus

\[
m_\lambda(B((t_0, x_0), 2R)) \sim x_0^{2\lambda} R^2.
\]

To handle this case, we will establish the Caccioppoli inequality, Sobolev embedding theorem to obtain an \( L^2 \) version of the Moser type inequality for \( p = 2 \), and then using an iteration approach (see for example \([HL]\)) to obtain the general case for \( p \in (0, \infty) \).

We first establish the Caccioppoli inequality in this Bessel setting as follows.

**Lemma 2.1** (Caccioppoli inequality). Let \((t_0, x_0) \in \mathbb{R}_+ \times \mathbb{R}_+ \) and \( R \in (0, \infty) \). Then there exists a positive constant \( C \), independent of \((t_0, x_0), R \) and \( u \), such that

\[
\int_{B((t_0, x_0), R)} |\nabla_{t, x} u(t, x)|^2 \, dm_\lambda(x) \, dt \leq \frac{C}{R^2} \int_{B((t_0, x_0), 2R)} |u(t, x)|^2 \, dm_\lambda(x) \, dt.
\]

**Proof.** Choose a function \( \eta \in C^\infty_c(\mathbb{R}_+ \times \mathbb{R}_+) \) with \( \text{supp} \( \eta \) \subset B((t_0, x_0), 2R) \), \( 0 \leq \eta \leq 1 \), \( \eta \equiv 1 \) on \( B((t_0, x_0), R) \) and \( |\nabla_{t, x} \eta| \leq R^{-1} \). Define \( \varphi := \eta^2 u \). By \((1.1)\), we see that

\[
0 = \int_{B((t_0, x_0), 2R)} \left[ \partial_t^2 u(t, x) + \partial_x^2 u(t, x) + \frac{2\lambda}{x} \partial_x u(t, x) \right] \varphi(t, x) x^{2\lambda} dx dt
\]

\[
= \int_{B((t_0, x_0), 2R)} \left[ \partial_t^2 u(t, x) + \frac{2\lambda}{x} \partial_x u(t, x) \right] \eta^2(t, x) u(t, x) x^{2\lambda} dx dt
\]

\[
+ \int_{B((t_0, x_0), 2R)} \partial_x^2 u(t, x) \eta^2(t, x) u(t, x) x^{2\lambda} dx dt
\]

\[
= - \int_{B((t_0, x_0), 2R)} \partial_x u(t, x) \partial_x u(t, x) \eta^2 + 2 \partial_t u(t, x) \eta \partial_x \eta x^{2\lambda} dx dt
\]

\[
- \int_{B((t_0, x_0), 2R)} \partial_u(t, x) \partial_t u(t, x) \eta^2 + 2 \partial_t u(t, x) \eta \partial_x \eta x^{2\lambda} dx dt.
\]
Applying the Cauchy-Schwarz inequality to each term in the right-hand side above, and then adding them up, we obtain that
\[
\iint_{B((t_0, x_0), 2R)} |\nabla_{t,x}u(t,x)|^2 \eta^2 x^{2\lambda} \, dx \, dt \leq 2 \iint_{B((t_0, x_0), 2R)} \left[ \frac{1}{4} |\nabla_{t,x}u(t,x)|^2 \eta^2 + 4|u(t,x)|^2 |\nabla_{t,x}\eta|^2 \right] x^{2\lambda} \, dx \, dt.
\]
Combining this with the property \( \eta \equiv 1 \) on \( B((t_0, x_0), R) \), we further deduce that
\[
\iint_{B((t_0, x_0), R)} |\nabla_{t,x}u(t,x)|^2 x^{2\lambda} \, dx \, dt \lesssim \iint_{B((t_0, x_0), 2R)} |u(t,x)|^2 |\nabla_{t,x}\eta|^2 x^{2\lambda} \, dx \, dt \lesssim \frac{1}{R^2} \iint_{B((t_0, x_0), 2R)} |u(t,x)|^2 x^{2\lambda} \, dx \, dt.
\]
This finishes the proof of Lemma 2.1. \( \Box \)

**Lemma 2.2** (Sobolev Embedding). Let \((t_0, x_0) \in \mathbb{R}_+ \times \mathbb{R}_+ \) and \( R \in (0, \infty) \). Then for the ball \( B((t_0, x_0), R) \), and for \( f \in C^2(\mathbb{R}^+ \times \mathbb{R}_+) \) we have
\[
\|f\|_{L^\infty(B((t_0, x_0), R))} \lesssim \left( \frac{1}{R^2} \iint_{B((t_0, x_0), R)} |f(t,x)|^2 \, dt \, dx \right)^{\frac{1}{2}} + \left( \iint_{B((t_0, x_0), R)} |\nabla_{t,x}f(t,x)|^2 \, dt \, dx \right)^{\frac{1}{2}}
+ \left( R^2 \iint_{B((t_0, x_0), R)} |\nabla_{t,x}^2f(t,x)|^2 \, dt \, dx \right)^{\frac{1}{2}}.
\]

Proof. For \( R = 1 \), this is a consequence of the standard Sobolev embedding, see for example [A]. That is,
\[(2.2) \quad \|f\|_{L^\infty(B((t_0, x_0), 1))} \lesssim \left( \iint_{B((t_0, x_0), 1)} |f(t,x)|^2 \, dt \, dx \right)^{\frac{1}{2}} + \left( \iint_{B((t_0, x_0), 1)} |\nabla_{t,x}f(t,x)|^2 \, dt \, dx \right)^{\frac{1}{2}}
+ \left( \iint_{B((t_0, x_0), 1)} |\nabla_{t,x}^2f(t,x)|^2 \, dt \, dx \right)^{\frac{1}{2}}.
\]

For general case, we use the rescaling. Namely, for any \( R \in (0, \infty) \), consider the function \( f(t,x) \) on the ball \( B((t_0, x_0), R) \). Let \( \tilde{t} = \frac{1}{R}(t-t_0) \) and \( \tilde{x} = \frac{x-x_0}{R} \). Then it is obvious that \((\tilde{t}, \tilde{x}) \in B((t_0, x_0), 1) \). Now define
\[
g(\tilde{t}, \tilde{x}) = f(R\tilde{t}+t_0, R\tilde{x}+x_0).
\]
Then, by applying (2.2) to \( g(\tilde{t}, \tilde{x}) \) and changing of variables, we obtain our version of the embedding result. \( \Box \)

We now apply the Caccioppoli inequality, the Sobolev embedding theorem to prove the Moser type inequality in \( L^2 \). To be more precise, we claim that for all \( \alpha \in (0,1) \), \( r \in (0,2R] \) and all \((t,x) \in B((t_0, x_0), \alpha r) \),
\[(2.3) \quad |u(t,x)| \lesssim \left\{ \frac{1}{m_\lambda(B((t_0, x_0), (1-\alpha)r))} \iint_{B((t_0, x_0), r)} |u(s,y)|^2 y^{2\lambda} \, dy \, ds \right\}^{1/2},
\]
where the implicant constant is independent of \( u, \alpha, t_0 \) and \( x_0 \).

To see this, consider the ball \( B := B((t,x),(1-\alpha)r/2) \). Observe that \( x \sim x_0 \), and moreover, for any \((s,y) \in B \), we have \( \lambda \sim x_0 \).

In this case, we first note that \( \partial_t u \) is also a solution of (1.1) since \( u \) is a solution. Hence, applying the Caccioppoli inequality in Lemma 2.1 to \( \partial_t u \), we get that
\[(2.4) \quad \iint_{\mathbb{B}} |\nabla_{t,x} \partial_t u(t,x)|^2 \, dm_\lambda(x) \, dt \leq \frac{C}{(1-\alpha)^2 r^2} \iint_{\mathbb{B}} |\partial_t u(t,x)|^2 \, dm_\lambda(x) \, dt.
\]
\[
\leq \frac{C}{(1-\alpha)^2 r^2} \int_B |\nabla_{t,x} u(t,x)|^2 \, dm_\lambda(x) \, dt
\]
\[
\leq \frac{C}{(1-\alpha)^4 r^4} \int_B |u(t,x)|^2 \, dm_\lambda(x) \, dt.
\]

Again, since \(u\) is a solution of (1.1), we have
\[
\int_B |\partial_x^2 u(t,x)|^2 \, dm_\lambda(x) \, dt \leq \int_B |\partial_x^2 u(t,x)|^2 \, dm_\lambda(x) \, dt
\]
\[
+ \int_B \left( \frac{2\lambda}{x} |\partial_x u(t,x)| \right)^2 \, dm_\lambda(x) \, dt =: I + II.
\]

Note that the term \(I\) is bounded by the left-hand side of (2.4). We obtain that
\begin{equation}
I \leq \frac{C}{(1-\alpha)^2 r^2} \int_B |u(t,x)|^2 \, dm_\lambda(x) \, dt.
\end{equation}

For the term \(II\), using the Caccioppoli inequality in Lemma 2.1 and the fact that \((1-\alpha)r < x/2\), we get
\begin{equation}
II \leq \frac{C}{(1-\alpha)^4 r^4} \int_B |u(t,x)|^2 \, dm_\lambda(x) \, dt.
\end{equation}

Combining the estimates of \(I\) and \(II\), we obtain that
\begin{equation}
\int_B |\partial_x^2 u(t,x)|^2 \, dm_\lambda(x) \, dt \leq \frac{C}{(1-\alpha)^4 r^4} \int_B |u(t,x)|^2 \, dm_\lambda(x) \, dt.
\end{equation}

Then we further have
\begin{equation}
\int_B |\nabla_{t,x}^2 u(t,x)|^2 \, dm_\lambda(x) \, dt
\end{equation}
\[
= \int_B \left( \partial_t^2 u(t,x) + \partial_x^2 u(t,x) \right)^2 \, dm_\lambda(x) \, dt
\]
\[
\leq 2 \int_B (\partial_t^2 u(t,x))^2 \, dm_\lambda(x) \, dt + 2 \int_B (\partial_x^2 u(t,x))^2 \, dm_\lambda(x) \, dt
\]
\[
\leq \frac{C}{(1-\alpha)^4 r^4} \int_B |u(t,x)|^2 \, dm_\lambda(x) \, dt,
\]

where the last inequality follows from (2.6) and the estimate of the term \(I\) in (2.5).

Now applying the Sobolev embedding in Lemma 2.2, we have
\[
\|u\|_{L^\infty(B)} \lesssim \frac{1}{(1-\alpha)^2 r^2} \int_B |u(t,x)|^2 \, dt \, dx \quad + \quad \left( \int_B |\nabla_{t,x} u(t,x)|^2 \, dt \, dx \right)^{1/2}
\]
\[
+ \left( (1-\alpha)^2 r^2 \int_B |\nabla_{t,x} u(t,x)|^2 \, dt \, dx \right)^{1/2}
\]
\[
\lesssim \left( \frac{1}{(1-\alpha)^2 r^2 x_0^2} \int_B |u(t,x)|^2 \, dt \, dm_\lambda(x) \right)^{1/2} \quad + \quad \left( \frac{1}{x_0^2} \int_B |\nabla_{t,x} u(t,x)|^2 \, dt \, dm_\lambda(x) \right)^{1/2}
\]
\[
+ \left( (1-\alpha)^2 r^2 \frac{1}{x_0^2} \int_B |\nabla_{t,x} u(t,x)|^2 \, dt \, dm_\lambda(x) \right)^{1/2}
\]
\[
\lesssim \left( \frac{1}{(1-\alpha)^2 r^2 x_0^2} \int_B |u(t,x)|^2 \, dt \, dm_\lambda(x) \right)^{1/2},
\]

where the second inequality follows from the fact that \(x \sim x_0\), and the last inequality follows from the Caccioppoli inequality in Lemma 2.1 and the inequality (2.7).

This implies that the claim (2.3) holds in this case.

By Hölder’s inequality and (2.3), we further obtain that (2.3) holds with 2 replaced by \(p\) for all \(p \in [2, \infty)\). Then (2.1) for \(p \in [2, \infty)\) follows from (2.3) with \(\alpha := 1/2\) and \(r := 2R\).
We now use the technique of iteration (see for example [HL]) to establish the Moser-type inequality for the Bessel harmonic function $u(t, x)$ for $p \in (0, 2)$ as follows. Observe that for all $a, b \in (0, \infty)$ and $q \in (1, \infty)$, $ab < a^{\frac{q'}{q}} + b^{\frac{q'}{q}}$, where $q' := \frac{q}{q-1}$. Then by this with $q := \frac{1}{1-p/2}$ and (2.3), there exists positive constant $C$ such that for all $r \in (0, 2R], \tilde{r} \in (0, r)$ and almost all $(t, x) \in B((t_0, x_0), \tilde{r})$,

$$|u(t, x)| \lesssim \|u\|_{L^{\infty}(B((t_0, x_0), r), \tilde{r})} \tilde{m}_\lambda(B((t_0, x_0), r, \tilde{r}))^{-\frac{1}{p}} \|u\|_{L^p(B((t_0, x_0), 2R), d\tilde{m}_\lambda)} \lesssim \frac{1}{2} \|u\|_{L^{\infty}(B((t_0, x_0), r), \tilde{r})} + C \tilde{m}_\lambda(B((t_0, x_0), r, \tilde{r}))^{-\frac{1}{p}} \|u\|_{L^p(B((t_0, x_0), 2R), d\tilde{m}_\lambda)}.$$  

We note that in this case, we have $\|u\|_{L^{\infty}(B((t_0, x_0), \tilde{r}))} \lesssim \|u\|_{L^p(B((t_0, x_0), 2R), \tilde{m}_\lambda)}$. An application of (2.8) gives us that for all $r, \tilde{r}$ such that $0 < \tilde{r} < r \leq 2R$,

$$f(\tilde{r}) \leq \frac{1}{2} f(r) + C \tilde{m}_\lambda(B((t_0, x_0), r, \tilde{r}))^{-\frac{1}{p}} \|u\|_{L^p(B((t_0, x_0), 2R), d\tilde{m}_\lambda)}.$$  

Now fix $\tilde{r}, r$ and write $r_0 := \tilde{r}, r_\infty := r$, and $r_j := j(1 - \tau)^j(r - \tilde{r})$ for $j \in \mathbb{Z}_+$ := $\mathbb{N} \cup \{0\}$, where $\tau \in (0, 1)$ such that $2^{\tau(2k+1)/p} > 1$. An application of (2.8) gives us that

$$f(r_j) \leq \frac{1}{2} f(r_{j+1}) + C \tilde{m}_\lambda(B((t_0, x_0), r_{j+1} - r_j))^{-\frac{1}{p}} \|u\|_{L^p(B((t_0, x_0), 2R), d\tilde{m}_\lambda)}$$  

for all $j \in \mathbb{Z}_+$. This in turn implies that, by iteration of $k$ steps,

$$f(\tilde{r}) = f(r_0) \leq \frac{1}{2} f(r_1) + C \tilde{m}_\lambda(B((t_0, x_0), r_1 - r_0))^{-\frac{1}{p}} \|u\|_{L^p(B((t_0, x_0), 2R), d\tilde{m}_\lambda)} \leq \frac{1}{2} f(r_k) + C \sum_{j=0}^{k-1} 2^{-j} \tau^{-j(2\lambda+1)/p} \tilde{m}_\lambda(B((t_0, x_0), (1 - \tau)(r - \tilde{r})))^{-\frac{1}{p}} \|u\|_{L^p(B((t_0, x_0), 2R), d\tilde{m}_\lambda)}.$$  

We note that $u \in C^2(\mathbb{R}^n \times \mathbb{R}_+)$, then $u \in L^{\infty}(B((t_0, x_0), 2R))$. Letting $k \to \infty$ and using $f(r_j) \leq f(2R) < \infty$ for all $j \in \mathbb{Z}_+$, we have that

$$f(\tilde{r}) \lesssim \frac{\|u\|_{L^p(B((t_0, x_0), 2R), d\tilde{m}_\lambda)}}{\tilde{m}_\lambda(B((t_0, x_0), (1 - \tau)(r - \tilde{r})))^{\frac{1}{p}}}.$$  

Taking $r := 2R$ and $\tilde{r} := R$, we obtain that

$$\|u\|_{L^{\infty}(B((t_0, x_0), R))} \lesssim \frac{\|u\|_{L^p(B((t_0, x_0), 2R), d\tilde{m}_\lambda)}}{\tilde{m}_\lambda(B((t_0, x_0), R))^{\frac{1}{p}}}.$$  

This finishes the proof of (2.1) for $p \in (0, 2)$ and hence, the proof of case (i).

We now consider Case (ii). We claim that when $R > x_0/4$, we have

$$\sup_{(t, x) \in B((t_0, x_0), R)} |u(t, x)| \lesssim \left[ \frac{C_{p, \lambda}}{\tilde{m}_\lambda(B((t_0, x_0), 12R))} \int_{B((t_0, x_0), 12R)} |u(t, x)|^p \tilde{m}_\lambda(B((t_0, x_0), 12R)) \right]^{1/p}.$$  

Note that in this case, we have $12R \geq 3x_0$, then $B((t_0, x_0), R) \subset B((t_0, x_0), 5R) \subset B((t_0, x_0), 12R)$. We have $\tilde{m}_\lambda(B((t_0, x_0), 12R)) \sim R^{2\lambda+2} \sim \tilde{m}_\lambda(B((t_0, x_0), 5R))$. Thus, to prove (2.9), it suffices to show that

$$\sup_{(t, x) \in B((t_0, 0), R)} |u(t, x)| \lesssim \left[ \frac{C_{p, \lambda}}{\tilde{m}_\lambda(B((t_0, 0), 5R))} \int_{B((t_0, 0), 5R)} |u(t, x)|^p \tilde{m}_\lambda(B((t_0, 0), 5R)) \right]^{1/p}.$$  


We point out that the equation (1.1) is translation invariant under the variable \( t \). Thus, to prove (2.10), it suffices to prove that

\[
\sup_{(t,x) \in B((0,0),R)} |u(t,x)| \leq \left[ \frac{C_{p,\lambda}}{m_\lambda(B((0,0),5R))} \int_{B((0,0),5R)} |u(t,x)|^p x^{2\lambda}dxight]^{1/p}.
\]

To obtain (2.11), we use the Poisson representation of the harmonic function \( u \). To begin with, following Muckenhoupt and Stein [MSt], we consider the even extension of (2.11) and

\[
\alpha \quad \text{where} \quad (2.16)
\]

\[
\text{which implies (2.11).}
\]

We now prove (2.12). To begin with, we can assume that for all \( r \in (R,5R) \), we have

\[
m_\infty(r_0) \leq H.
\]

In fact, assume (2.12) at the moment. Then by the maximal principle (Theorem 1 in [MSt]), we obtain that for any \( (t,x) \in B((0,0), R) \), we have

\[
|u(t,x)| \leq m_\infty(r_0) \leq H,
\]

which implies (2.11).

To prove (2.11), it suffices to prove that there exists \( r_0 \in (R,5R) \) such that

\[
m_\infty(r_0) \leq H.
\]

since otherwise (2.12) holds.

To continue, we first note that

\[
m_1(r) = \int_0^\pi |u(r \cos \theta, r \sin \theta)| (\sin \theta)^{2\lambda} d\theta
\]

\[
\leq m_\infty(r)^\alpha \int_0^\pi |u(r \cos \theta, r \sin \theta)|^{1-\alpha} (\sin \theta)^{2\lambda} d\theta
\]

\[
= m_\infty(r)^\alpha m_p(r)^p,
\]

where \( \alpha := 1 - p \).

Next we recall the Poisson representation (see [MSt, P. 25]) as follows.

\[
u(\rho \cos \theta, \rho \sin \theta) = \int_0^\pi P\left(\frac{2}{r}, \theta, \phi\right) u(r \cos \phi, r \sin \phi) (\sin \phi)^{2\lambda} d\phi,
\]

where \( \rho < r \), and \( P\left(\frac{2}{r}, \theta, \phi\right) \) is the Poisson kernel defined as

\[
P\left(\frac{2}{r}, \theta, \phi\right) := \frac{\lambda^\left(1 - \frac{\rho^2}{r^2}\right)}{\pi} \int_0^\pi \frac{(\sin \beta)^{2\lambda-1}}{\left[(t - \xi)^2 + (x - \eta)^2 + 2\eta(1 - \cos \beta)\right]^{\lambda+1}} d\beta,
\]

with \( t := \frac{\rho}{r} \cos \theta, x := \frac{\rho}{r} \sin \theta, \xi := \cos \phi \) and \( \eta := \sin \phi \).
From the definition, it is direct that
\[ \left\| P\left( \frac{\rho}{r}, \theta, \phi \right) \right\|_\infty \lesssim \left( 1 - \frac{\rho^2}{r^2} \right) \frac{1}{\left( 1 - \frac{\rho}{r} \right)^{2\lambda + 2}} \approx \frac{1}{\left( 1 - \frac{\rho}{r} \right)^{2\lambda + 1}}. \]

Then, from (2.15) we obtain that
\[ |u(t, x)| = |u(\rho \cos \theta, \rho \sin \theta)| \leq \int_0^\pi |u(r \cos \phi, r \sin \phi)| (\sin \phi)^{2\lambda} \, d\phi \left\| P\left( \frac{\rho}{r}, \theta, \phi \right) \right\|_\infty \approx \frac{1}{\left( 1 - \frac{\rho}{r} \right)^{2\lambda + 1}} m_1(r), \]
which implies that
\[ (2.17) \quad m_\infty(\rho) \lesssim \frac{1}{\left( 1 - \frac{\rho}{r} \right)^{2\lambda + 1}} m_1(r). \]

This, together with (2.14), gives
\[ m_\infty(\rho) \leq C_p \frac{1}{\left( 1 - \frac{\rho}{r} \right)^{2\lambda + 1}} m_\infty(r)^{\alpha} m_p(r)^p, \]
and hence
\[ (2.18) \quad \frac{m_\infty(\rho)}{H^p} \leq C_p \frac{1}{\left( 1 - \frac{\rho}{r} \right)^{2\lambda + 1}} \frac{m_\infty(r)^{\alpha} m_p(r)^p}{H^p} H. \]

To continue, we consider the following 3 cases.

Case 1: \( R > 1. \)

Let \( a < 1 \) and \( \rho = r^a \). We now take the logarithm on both side of (2.18) and integrate on both side from \( R \) to \( 5R \). Then we have
\[ (2.19) \quad \int_R^{5R} \log \left( \frac{m_\infty(r)^{\alpha}}{H^p} \right) \frac{dr}{r} \leq \int_R^{5R} \log \left( \frac{C_p}{\left( 1 - \frac{\rho}{r} \right)^{2\lambda + 1}} \frac{m_\infty(r)^{\alpha} m_p(r)^p}{H^p} H \right) \frac{dr}{r} \]
\[ \leq \int_R^{5R} \log \left( \frac{C_p}{\left( 1 - \frac{\rho}{r} \right)^{2\lambda + 1}} \right) \frac{dr}{r} \]
\[ + \int_R^{5R} \log \left( \frac{m_\infty(r)^{\alpha}}{H^p} \right) \frac{dr}{r} \]
\[ + \int_R^{5R} \log \left( \frac{m_p(r)^p}{H} \right) \frac{dr}{r}. \]

Next we claim that
\[ (2.20) \quad \int_R^{5R} m_p(r)^p \frac{dr}{r} \lesssim H. \]

To see this, note that
\[ H := \frac{1}{m_\lambda(B((0, 0), 5R))} \int_0^\pi \int_0^\pi |u(r \cos \theta, r \sin \theta)|^p (r \sin \theta)^{2\lambda} \, d\theta \, dr \]
\[ \int_R^{5R} \log \left( \frac{m_\infty(r^a)}{H^\frac{a}{p}} \right) \frac{dr}{r} \leq \tilde{C}_p + \alpha \int_R^{5R} \log \left( \frac{m_\infty(r)}{H^\frac{1}{p}} \right) \frac{dr}{r}, \]

which gives
\[ \int_0^{(5R)^a} \log \left( \frac{m_\infty(r)}{H^\frac{1}{p}} \right) \frac{dr}{r} + \frac{1}{a} \int_0^{(5R)^a} \log \left( \frac{m_\infty(r)}{H^\frac{1}{p}} \right) \frac{dr}{r} \leq \tilde{C}_p. \]

Now for arbitrary small \( \epsilon > 0 \), by choosing \( a \) less than 1 but sufficiently close to 1, we obtain that \( \frac{1}{a} - \alpha > 0 \) and that
\[ \alpha \int_0^{(5R)^a} \log \left( \frac{m_\infty(r)}{H^\frac{1}{p}} \right) \frac{dr}{r} < \epsilon, \]
which implies that
\[ \int_R^{(5R)^a} \log \left( \frac{m_\infty(r)}{H^\frac{1}{p}} \right) \frac{dr}{r} \leq \tilde{C}_p. \]

Thus, there exists \( r_0 \) such that
\[ m_\infty(r_0) \leq \tilde{C}_p. \]

Case 2: \( 5R < 1 \).

Now let \( a > 1 \) and \( \rho = r^a \). Then repeating again the steps in Case 1, we obtain again (2.24) for a > 1. Hence, for arbitrary small \( \epsilon > 0 \), by choosing \( a \) greater than 1 but sufficiently close to 1, we obtain that \( \frac{1}{a} - \alpha > 0 \) and that
\[ \alpha \int_0^{(5R)^a} \log \left( \frac{m_\infty(r)}{H^\frac{1}{p}} \right) \frac{dr}{r} < \epsilon, \]
which implies that

\[ \int_R^{(5R)^a} \log \left( \frac{m_\infty(r)}{H_\frac{r}{p}} \right) \frac{dr}{r} \leq \tilde{C}_p. \]

Thus, there exists \( r_0 \) such that

\[ m_\infty(r_0) \leq \tilde{C}_p. \]

Case 3: \( 1/5 \leq R \leq 1 \).

Now let \( a > 1 \) and \( \rho = r^a \). Then repeating again the steps in Case 1, we obtain again (2.23) for \( a > 1 \).

\[ \frac{1}{a} \int_{R^a} \log \left( \frac{m_\infty(r)}{H_\frac{r}{p}} \right) \frac{dr}{r} + \left( \frac{1}{a} - \alpha \right) \int_{R^a} \log \left( \frac{m_\infty(r)}{H_\frac{r}{p}} \right) \frac{dr}{r} - \alpha \int_{5R}^{(5R)^a} \log \left( \frac{m_\infty(r)}{H_\frac{r}{p}} \right) \frac{dr}{r} \leq \tilde{C}_p, \]

which, again, implies that there exists \( r_0 \) such that

\[ m_\infty(r_0) \leq \tilde{C}_p. \]

Combining the three cases above, we obtain that (2.12) holds, which implies (2.11). And hence, we obtain that (2.9) holds.

In the end, combining the estimates in Case (i) and Case (ii), we obtain that (1.4) holds, which finishes the proof of Theorem 1.1.

3. Applications

As an application, we can obtain a direct proof of equivalent characterizations of the Hardy spaces associated to Bessel operator \( \Delta_\lambda \) via non-tangential maximal function and radial maximal functions defined in terms of the Poisson semigroup.

Recall that in [BDT], they introduced the Hardy spaces associated with \( \Delta_\lambda \) via the Riesz transforms, radial maximal functions, and showed that they are equivalent. To be more specific, consider the following spaces (see Definition 1.1 in [BDT]):

(a) \( H^1_{\Delta_\lambda,Riesz}(\mathbb{R}_+, dm_\lambda) := \{ f \in L^1(\mathbb{R}_+, dm_\lambda) : R_{\Delta_\lambda}(f) \in L^1(\mathbb{R}_+, dm_\lambda) \} \) with the norm

\[ \| f \|_{H^1_{\Delta_\lambda,Riesz}(\mathbb{R}_+, dm_\lambda)} := \| f \|_{L^1(\mathbb{R}_+, dm_\lambda)} + \| R_{\Delta_\lambda}(f) \|_{L^1(\mathbb{R}_+, dm_\lambda)} \]

(b) \( H^1_{\Delta_\lambda,max}(\mathbb{R}_+, dm_\lambda) := \{ f \in L^1(\mathbb{R}_+, dm_\lambda) : \mathcal{R}(f) \in L^1(\mathbb{R}_+, dm_\lambda) \} \) with the norm

\[ \| f \|_{H^1_{\Delta_\lambda,max}(\mathbb{R}_+, dm_\lambda)} := \| f \|_{L^1(\mathbb{R}_+, dm_\lambda)} + \| \mathcal{R}(f) \|_{L^1(\mathbb{R}_+, dm_\lambda)}, \]

where

\[ \mathcal{R}(f)(x) := \sup_{t>0} e^{-t\sqrt{\lambda}} |f| \]

The following result was proved [BDT] Theorem 1.7): let \( \lambda > 0 \) and \( f \in L^1(\mathbb{R}_+, dm_\lambda) \). Then the following assertions are equivalent:

(i) \( f \in H^1_{\Delta_\lambda,Riesz}(\mathbb{R}_+, dm_\lambda) \);

(ii) \( f \in H^1_{\Delta_\lambda,max}(\mathbb{R}_+, dm_\lambda) \).

Moreover, the corresponding norms are equivalent.

We now consider the Hardy spaces associated to Bessel operator \( \Delta_\lambda \) via non-tangential maximal function as follows.
Definition 3.1. Suppose $\lambda > 0$. Define

$$H_{\Delta, n, \text{max}}^{1}(\mathbb{R}^+, dm_{\lambda}) := \left\{ f \in L^{1}(\mathbb{R}^+, dm_{\lambda}) : \mathcal{N}(f) \in L^{1}(\mathbb{R}^+, dm_{\lambda}) \right\}$$

with the norm

$$\| f \|_{H_{\Delta, n, \text{max}}^{1}(\mathbb{R}^+, dm_{\lambda})} := \| f \|_{L^{1}(\mathbb{R}^+, dm_{\lambda})} + \| \mathcal{N}(f) \|_{L^{1}(\mathbb{R}^+, dm_{\lambda})},$$

where

$$\mathcal{N}(f)(x) := \sup_{|x-y| < t} \left| e^{-t\sqrt{\lambda}} f(y) \right|.$$ 

Then, based on our main result, the Moser inequality, we obtain that

Theorem 3.2. Let $\lambda > 0$ and $f \in L^{1}(\mathbb{R}^+, dm_{\lambda})$. Then the following assertions are equivalent:

(i) $f \in H_{\Delta, \text{max}}^{1}(\mathbb{R}^+, dm_{\lambda})$;

(ii) $f \in H_{\Delta, n, \text{max}}^{1}(\mathbb{R}^+, dm_{\lambda})$;

(iii) $f \in H_{\Delta, n, \text{max}}^{1}(\mathbb{R}^+, dm_{\lambda})$.

Moreover, the corresponding norms are equivalent.

Proof of Theorem 3.2. Suppose $\lambda > 0$ and $f \in H_{\Delta, n, \text{max}}^{1}(\mathbb{R}^+, dm_{\lambda})$. It is obvious that for every $x \in \mathbb{R}^+$,

$$\mathcal{R}(f)(x) \leq \mathcal{N}(f)(x),$$

which implies that

$$\| f \|_{H_{\Delta, \text{max}}^{1}(\mathbb{R}^+, dm_{\lambda})} \leq \| f \|_{H_{\Delta, n, \text{max}}^{1}(\mathbb{R}^+, dm_{\lambda})},$$

i.e., we have $f \in H_{\Delta, \text{max}}^{1}(\mathbb{R}^+, dm_{\lambda})$. Thus, we obtain that

$$H_{\Delta, n, \text{max}}^{1}(\mathbb{R}^+, dm_{\lambda}) \subset H_{\Delta, \text{max}}^{1}(\mathbb{R}^+, dm_{\lambda}).$$

Conversely, Suppose $f \in H_{\Delta, \text{max}}^{1}(\mathbb{R}^+, dm_{\lambda})$. Let $u(t, x) := e^{-t\sqrt{\lambda}} f(x) = P_{t}^{[\lambda]} f(x)$. Then $u(t, x)$ is $\lambda$-harmonic, i.e., $u(t, x)$ satisfies \((\ref{1.1})\).

For any $q \in (0, 1)$, for all $y, t \in \mathbb{R}^+$ with $|y-x| < t$, from Theorem \((\ref{1.1})\) with $R := t$, we deduce that

$$|u(t, y)|^{q} \lesssim \frac{1}{m_{\lambda}(B((t, x), 12t))} \int_{B((t, x), 12t)} |u(s, z)|^{q} z^{2\lambda} dz ds$$

$$\lesssim \frac{1}{m_{\lambda}(B((t, x), 12t))} \int_{B((t, x), 12t)} \mathcal{R}(f)(z)^{q} z^{2\lambda} dz ds$$

$$\lesssim \frac{1}{m_{\lambda}(12t)} \int_{12t} \mathcal{R}(f)(z)^{q} z^{2\lambda} dz$$

$$\lesssim \mathcal{M}(\mathcal{R}(f)^{q})(x),$$

where $I := I(x, t)$ and $\mathcal{M}$ is the Hardy–Littlewood maximal function.

This implies that

$$\mathcal{N}(f)(x) \lesssim \{\mathcal{M}(\mathcal{R}(f)^{q})(x)\}^{\frac{1}{q}}.$$

(3.1)
By taking the $L^1$ norm on both side of the inequality above and using the boundedness of the Hardy–Littlewood maximal function, we obtain that
\[
\|N(f)\|_{L^1(\mathbb{R}^+, dm_\lambda(x))} \lesssim \|R(f)\|_{L^1(\mathbb{R}^+, dm_\lambda(x))},
\]
which implies that
\[
\|f\|_{H^1_{\Delta, n, \text{max}}(\mathbb{R}^+, dm_\lambda)} \lesssim \|f\|_{H^1_{\Delta, \text{max}}(\mathbb{R}^+, dm_\lambda)},
\]
i.e., we have $f \in H^1_{\Delta, n, \text{max}}(\mathbb{R}^+, dm_\lambda)$. Thus, we obtain that
\[
H^1_{\Delta, \text{max}}(\mathbb{R}^+, dm_\lambda) \subset H^1_{\Delta, n, \text{max}}(\mathbb{R}^+, dm_\lambda).
\]
This completes the proof of Theorem 3.2. \qed

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