Brauer algebras of type $F_4$

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Abstract

We present an algebra related to the Coxeter group of type $F_4$ which can be viewed as the Brauer algebra of type $F_4$ and is obtained as a subalgebra of the Brauer algebra of type $E_6$. We also describe some properties of this algebra.

1 Introduction

When studying tensor decompositions for orthogonal groups, Brauer (\cite{2}) introduce algebras which we now call Brauer algebras of type A. Cohen, Frenk and Wales (\cite{4}) extended the definition to simply laced types, including type $E_6$. Tits (\cite{18}) described how to obtain the Coxeter group of type $F_4$ as the fixed subgroup of the Coxeter group of type $E_6$ under a diagram automorphism (also seen in \cite{3}). For this, M"uhlherr gave a more general way by admissible partitions to obtain Coxeter groups as subgroups in Coxeter groups in \cite{16}. Here we will apply a similar method to the Brauer algebra $Br(E_6)$. This is a part of a project to define Brauer algebras of spherical types (\cite{6}, \cite{7}). It turns out that the presentation by generators and relations obtainable from the Dynkin diagram of type $F_4$ in the same way as was done for type $B_n$ (\cite{7}) and $C_n$ (\cite{6}).

First we give the definition of $Br(F_4)$ using a presentation. Let $\delta$ be the generator of the infinite cyclic group.

Definition 1.1. The Brauer algebra of type $F_4$, denoted by $Br(F_4)$, is a unital associative $\mathbb{Z}[\delta^{\pm 1}]$-algebra generated by $\{r_i, e_i\}_{i=1}^4$, subject to the fol-
lowing relations.

\[ r_i^2 = 1 \quad \text{for any } i \] (1.1)
\[ r_i e_i = e_i r_i = e_i \quad \text{for any } i \] (1.2)
\[ e_i^2 = \delta e_i \quad \text{for } i > 2 \] (1.3)
\[ e_i^2 = \delta^2 e_i \quad \text{for } i < 3 \] (1.4)
\[ r_i r_j = r_j r_i, \quad \text{for } i \sim j \] (1.5)
\[ e_i r_j = r_j e_i, \quad \text{for } i \sim j \] (1.6)
\[ e_i e_j = e_j e_i, \quad \text{for } i \sim j \] (1.7)
\[ r_i r_j r_i = r_j r_i r_j, \quad \text{for } i \sim j \] (1.8)
\[ r_j r_i e_j = e_j e_i, \quad \text{for } i \sim j \] (1.9)
\[ e_j r_i = r_i e_j, \quad \text{for } i \sim j \] (1.10)

and for \( \circ \rightarrow 2 \rightarrow 3 \),

\[ r_2 r_3 r_2 r_3 = r_3 r_2 r_3 r_2 \] (1.11)
\[ r_2 r_3 e_2 = r_3 e_2 \] (1.12)
\[ r_2 e_3 r_2 e_3 = e_3 e_2 e_3 \] (1.13)
\[ (r_2 r_3 r_2) e_3 = e_3 (r_2 r_3 r_2) \] (1.14)
\[ e_2 r_3 e_2 = \delta e_2 \] (1.15)
\[ e_2 e_3 e_2 = \delta e_2 \] (1.16)
\[ e_2 r_3 e_2 = e_2 r_3 \] (1.17)
\[ e_2 e_3 r_2 = e_2 e_3 \] (1.18)

Here \( i \sim j \) means that \( i \) and \( j \) are connected by a simple bond and \( i \sim j \) means that there is no bond (simple or multiple) between \( i \) and \( j \) in the Dynkin Diagram of type \( F_4 \) depicted in the Figure 1. The submonoid of the multiplicative monoid of \( \text{Br}(F_4) \) generated by \( \delta, \{r_i, e_i\}_{i=1}^4 \) is denoted by \( \text{BrM}(F_4) \). This is the monoid of monomials in \( \text{Br}(F_4) \).

The defining relations (1.11)—(1.18) can be found in \( \text{Br}(C_2) \) in [6] and \( \text{Br}(B_2) \) in [7] by renumbering indices. Note that these relations are not symmetric for 2 and 3. Their relations are fully determined by the Dynkin diagram in the sense that all relations depend only on the vertices and bonds of the Dynkin diagram and the lengths of their roots.

It is well known that the Coxeter group \( W(F_4) \) of type \( F_4 \), can be obtained as the subgroup from the Coxeter group \( W(E_6) \) of type \( E_6 \), of elements invariant under the automorphism of \( W(E_6) \) determined by the diagram automorphism \( \sigma = (1, 6)(3, 5) \) indicated as a permutation on the generators of \( W(E_6) \) whose Dynkin diagram are labeled and presented in Figure 1.
The action $\sigma$ can be extended to an automorphism of the Brauer algebra of type $E_6$ by acting on the Temperley-Lieb generators $E_i$ ([17]) by the same permutation as for Weyl group generators. We denote by $SBr(E_6)$ the subalgebra generated by $\sigma$-invariant elements in $BrM(E_6)$. The main theorem of this paper is the following. In order to avoid confusion with the above generators, the generators of $Br(E_6)$ have been capitalized.

**Theorem 1.2.** There is an algebra isomorphism

$$\phi : Br(F_4) \longrightarrow SBr(E_6)$$

determined by $\phi(r_1) = R_1R_6$, $\phi(r_2) = R_3R_5$, $\phi(r_3) = R_4$, $\phi(r_4) = R_2$, and $\phi(e_1) = E_1E_6$, $\phi(e_2) = E_3E_5$, $\phi(e_3) = E_4$, $\phi(e_4) = E_2$. Furthermore, the algebra $Br(F_4)$ is free over $\mathbb{Z}[\delta^{\pm 1}]$ of rank 14985.

The proof of the theorem is finished in Section 5. Moreover, in the last section we will prove that the algebra $Br(F_4) \otimes R$ for a field $R$ is cellularly stratified. This paper is included as chapter 4 in the author’s PhD thesis ([15]).
2 Basic properties of Br(F₄)

By the properties of Br(B₃) in [7] or Br(C₃) in [6], we have more relations between \{r₂, r₃, e₂, e₃\} Such as those of [6, Lemma 4.1]. Just as [6, Remark 3.5] for type C, there is an anti-involution on Br(F₄).

**Proposition 2.1.** There is a unique anti-involution on Br(F₄) that fixes the generators \(r_i, e_i\ (1 \leq i \leq 4)\).

**Definition 2.2.** Let \(Q\) be a graph. The Brauer monoid BrM(\(Q\)) is the monoid generated by the symbols \(R_i\) and \(E_i\) for each node \(i\) of \(Q\) and \(δ, δ^{-1}\) subject to the relation (1.1)–(1.3) and (1.5)–(1.10). The Brauer algebra Br(\(Q\)) is the free \(\mathbb{Z}[δ^{\pm1}]\)-algebra for Brauer monoid BrM(\(Q\)).

**Proposition 2.3.** The map \(φ\) determined on generators in Theorem 1.2 induces an algebra homomorphism from Br(F₄) to Br(E₆).

**Proof.** It suffices to prove that the relations in Definition 1.1 still holds when the generators are replaced by their images under \(φ\). The difficult ones are (1.11)–(1.18); which have been proved in the meanwhile of the homomorphism from Br(C₂) to Br(A₃) in [6].

To distinguish them from the generators of Br(F₄), we denote the generators of Br(B₃) \( (\overbrace{\varepsilon \varepsilon \varepsilon}^2 \overbrace{\sigma \sigma}^1 \overbrace{\delta \delta}^0) \) in [7] by \(\{r''_i, e''_i\}_{i=0}^2\) and the generators of Br(B₃) \( (\overbrace{\varepsilon \varepsilon \sigma}^2 \overbrace{\sigma \delta}^1 \overbrace{\delta \delta}^0) \) in [7] by \(\{r'_i, e'_i\}_{i=0}^2\). By checking their defining relations, we have the following proposition.

**Proposition 2.4.** There are injective algebra homomorphisms

\[
\begin{align*}
φ_1 : \ & Br(C₃) \to Br(F₄) \\
φ_2 : \ & Br(B₃) \to Br(F₄),
\end{align*}
\]

defined on generators as follows.

\[
\begin{align*}
φ_1(r'_i) &= r_{3-i}, \ & φ_1(e'_i) = e_{3-i}, \ & \text{for} \ & 0 \leq i \leq 2, \\
φ_2(r''_i) &= r_{2+i}, \ & φ_2(e''_i) = e_{2+i}, \ & \text{for} \ & 0 \leq i \leq 2.
\end{align*}
\]

**Proof.** Just checking the defining relations of two algebras, we find that \(φ_1\) and \(φ_2\) are algebra morphisms. We see that \(φφ_1(\text{Br}(C₃))\) is contained in the subalgebra of Br(\(E₆\)) generated by \(\{R_i, E_i\} \cup \{R_i, E_i\}_{i=3}^6\), which is isomorphic to Br(A₅) according to [8]. By the main theorem of [6] and the behavior of \(φφ_1\) on generators, the homomorphism \(φφ_1\) coincides with the embedding of Br(C₃) in Br(A₅). Similarly, with [8] and the main theorem of [7], the homomorphism \(φφ_2\) coincides with the embedding of Br(B₃) in Br(D₄); by application of these two theorems, it follows that the homomorphisms \(φ_1\) and \(φ_2\) are injective. \(\square\)
3 Admissible root sets

In this section, we give the definition and description for admissible sets of type $F_4$ and study some of their basic properties.

Let $\{\beta_i\}_{i=1}^4$ be simple roots of $W(F_4)$. They can be realized in $\mathbb{R}^4$ as

$$
\beta_1 = \frac{\epsilon_1 - \epsilon_2 - \epsilon_3 - \epsilon_4}{2}, \quad \beta_2 = \epsilon_2,
\beta_3 = \epsilon_3 - \epsilon_2, \quad \beta_4 = \epsilon_4 - \epsilon_3,
$$

with $\{\epsilon_i\}_{i=1}^4$ being the standard orthonormal basis of $\mathbb{R}^4$. The set

$$
\Psi^+ = \left\{ \frac{\epsilon_1 \pm \epsilon_2 \pm \epsilon_3 \pm \epsilon_4}{2} \right\} \cup \{\epsilon_i\}_{i=1}^4 \cup \{\epsilon_j \pm \epsilon_i\}_{1 \leq i < j \leq 4}
$$

of cardinality 24 is the set of positive roots of a root system $\Psi$ of $W(F_4)$ having $\beta_1, \ldots, \beta_4$ as simple roots. We call a vector $\beta \in \Psi^+$ a short root if its Euclidean length is 1, a long root if its Euclidean length is $\sqrt{2}$.

Let $\{\alpha_i\}_{i=1}^6$ be simple roots of $W(E_6)$. The $\{\alpha_i\}_{i=1}^6$ span a linear space over $\mathbb{R}$ of dimension 6. We define a linear map $p : \mathbb{R}^6 \to \mathbb{R}^4$ by specifying its images on the given basis:

$$
p(\alpha_1) = \beta_1, \quad p(\alpha_6) = \beta_1, \quad p(\alpha_3) = \beta_2, \quad p(\alpha_5) = \beta_2, \quad p(\alpha_4) = \beta_3, \quad p(\alpha_2) = \beta_4.
$$

Now $p(\mathbb{R}^6)$ is the $\sigma$-invariant space of $\mathbb{R}^6$, where $\sigma$ is the linear transformation of $\mathbb{R}^6$ determined by:

$$
\sigma(\alpha_1) = \alpha_6, \quad \sigma(\alpha_6) = \alpha_1, \quad \sigma(\alpha_3) = \alpha_5, \quad \sigma(\alpha_5) = \alpha_3, \quad \sigma(\alpha_4) = \alpha_4, \quad \sigma(\alpha_2) = \alpha_2.
$$

Let $\Phi \subset \mathbb{R}^6$ be the root system of $E_6$ with simple roots $\{\alpha_i\}_{i=1}^6$, and $\Phi^+$ the positive roots of $\Phi$. In [4], the admissible root sets for simply-laced type are given. Now we will define the admissible for type $F_4$.

**Definition 3.1.** Let $X \subset \Psi^+$ be a set of mutually orthogonal roots. The set $X$ is called admissible if $p^{-1}(X) \cap \Phi^+$ is an admissible set.

**Proposition 3.2.** There is a one-to-one correspondence between $\sigma$-invariant admissible root sets of type $E_6$ and the admissible root sets of type $F_4$. Collections of all admissible sets can be partitioned into six $W(F_4)$-orbits given by the following representatives.

**(I)** $\emptyset$,

**(II)** $\{\epsilon_4 - \epsilon_3\}$,

**(III)** $\{\frac{\epsilon_1 - \epsilon_2 - \epsilon_3 - \epsilon_4}{2}\}$,

**(IV)** $\{\epsilon_3 - \epsilon_2, \epsilon_3 + \epsilon_2\}$,

**(V)** $\{\frac{\epsilon_1 - \epsilon_2 - \epsilon_3 - \epsilon_4}{2}, \epsilon_3 - \epsilon_2, \epsilon_4 + \epsilon_1\}$,
\((VI)\) \(\{\epsilon_3 - \epsilon_2, \epsilon_3 + \epsilon_2, \epsilon_4 + \epsilon_1, \epsilon_4 - \epsilon_1\}\).

Furthermore, their cardinalities are respectively, 1, 12, 12, 18, 36, 3.

**Proof.** In the admissible root sets of type \(E_6\), there are four \(W(E_6)\)-orbits, which are the orbits of \(\emptyset\), \(\{\alpha_2, \alpha_3, \alpha_5\}\), \(\{\alpha_2, \alpha_3, \alpha_5, 2\alpha_4 + \alpha_2 + \alpha_3 + \alpha_5\}\) of respective sizes. The \(W(E_6)\)-orbits are easily seen to possess 1 \((I)\), 12 \((II)\), 30 \((III), (IV)\), 39 \((V), (VI)\) \(\sigma\)-invariant admissible root sets which can be decomposed into \(W(F_4)\)-orbits with representatives \((I)\), \((II)\), \((III), (IV)\), \((V), (VI)\) as listed in the proposition. \(\square\)

As in [4], [6], and [7], we have the following lemmas.

**Lemma 3.3.** Let \(i\) and \(j\) be nodes of the Dynkin diagram \(F_4\). If \(w \in W(F_4)\) satisfies \(w\beta_i = \beta_j\), then \(we_iw^{-1} = e_j\).

Consider a positive root \(\beta\) and a node \(i\) of type \(F_4\). If there exists \(w \in W\) such that \(w\beta_i = \beta\), then due to Lemma 3.3 we can define the element \(e_\beta\) in \(BrM(F_4)\) by

\[e_\beta = we_iw^{-1}.\]

In general,

\[we_\beta w^{-1} = e_{w\beta},\]

for \(w \in W(F_4)\) and \(\beta\) a root of \(W(F_4)\), and here we just consider the natural action of \(W(F_4)\) on \(\Psi^+\) by negating negative roots. Analogous to the argument in [7, Lemma 4.5], the following lemma can be obtained by checking case by case listed in Proposition 3.2.

**Lemma 3.4.** Let \(\gamma_1, \gamma_2 \in \Psi^+\) and \(\gamma_1\) orthogonal to \(\gamma_2\) such that \(\{\gamma_1, \gamma_2\}\) can be a subset of some admissible root set, then

\[e_{\gamma_1}e_{\gamma_2} = e_{\gamma_2}e_{\gamma_1}.\]

If \(X \subset \Psi^+\) is a subset of some admissible root set, then by the lemma we can define

\[e_X = \prod_{\beta \in X} e_\beta.\]  \hspace{1cm} (3.1)

**Definition 3.5.** Suppose that \(X \subset \Psi^+\) is a mutually orthogonal root set. If \(X\) can be contained in some admissible root set, then the minimal admissible set containing \(X\) is called the *admissible closure* of \(X\), denoted by \(X^{\text{cl}}\).

Thanks to an argument similar to [7, Lemma 4.7], the following lemma holds.

**Lemma 3.6.** Let \(X \subset \Psi^+\) be a mutually orthogonal root set. If \(X^{\text{cl}}\) exists, then

\[e_{X^{\text{cl}}} = \delta^{\#(X^{\text{cl}} \setminus X)} e_X.\]
4 An upper bound for the rank

We introduce notation for the following admissible sets of type $F_4$ corresponding to those listed in Proposition 3.2.

\[
X_0 = \emptyset
\]
\[
X_1 = \{\beta_4\}
\]
\[
X_2 = \{\beta_1\}
\]
\[
X_3 = \{\beta_3, \beta_3 + 2\beta_2\}
\]
\[
X_4 = \{\beta_1, \beta_3\}^{cl}
\]
\[
X_5 = \{\beta_3, \beta_3 + 2\beta_2, \beta_3 + 2\beta_2 + 2\beta_1\}^{cl}
\]

Let

(I) $N_0 = W(F_4)$, $A_0 = \{1\}$, $C_0 = N_0$,

(II) $N_1 = \langle r_1, r_2, r_{\epsilon_4+\epsilon_3}, r_4 \rangle$, $C_1 = \langle r_1, r_2, r_{\epsilon_4+\epsilon_3} \rangle$, $A_1 = \langle r_4 \rangle$,

(III) $N_2 = \langle r_1, r_3, r_4, r_{\epsilon_1+\epsilon_2+\epsilon_3-\epsilon_4}/2 \rangle$, $C_2 = \langle r_3, r_4 \rangle$, $A_2 = \langle r_1, r_{\epsilon_1+\epsilon_2+\epsilon_3-\epsilon_4}/2, r_{\epsilon_1+\epsilon_2-\epsilon_3+\epsilon_4}/2, r_{\epsilon_1-\epsilon_2+\epsilon_3+\epsilon_4}/2 \rangle$,

(IV) $N_3 = \langle r_2, r_3, r_{\epsilon_4}, r_{\epsilon_4-\epsilon_1} \rangle$, $C_3 = \langle r_{\epsilon_4-\epsilon_1} \rangle$, $A_3 = \langle r_2, r_3, r_{\epsilon_4}, r_{\epsilon_1} \rangle$,

(V) $N_4 = \langle r_3, r_1, r_{\epsilon_1+\epsilon_2+\epsilon_3-\epsilon_4}/2, r_{\epsilon_1-\epsilon_2+\epsilon_3+\epsilon_4}/2 \rangle$, $C_4 = \{1\}$, $A_4 = N_4$,

(VI) $N_5 = \langle r_1, r_2, r_3, r_{\epsilon_4}, r_{\epsilon_1} \rangle$, $C_5 = \{1\}$, $A_5 = N_5$.

The structure of these groups can be determined below.

**Lemma 4.1.** For $N_i, A_i, C_i, i = 1, \ldots, 5$, the following holds,

(I) $N_0 \cong W(F_4)$, $A_0 \cong \{1\}$, $C_0 \cong N_0$,

(II) $N_1 \cong W(B_3) \times W(A_1)$, $C_1 \cong W(B_3)$, $A_2 \cong W(A_1)$,

(III) $N_2 \cong W(B_3) \times W(A_1)$, $C_2 \cong W(A_2)$, $A_2 \cong W(A_1)^4$,

(IV) $N_3 \cong W(B_2)^2$, $C_3 \cong W(A_1)$, $A_3 \cong W(B_2) \times W(A_1)^2$,

(V) $A_4 = N_4 \cong W(B_2) \times W(A_1)^2$, $C_4 = \{1\}$,

(VI) $A_5 = N_5 \cong W(B_3) \times W(B_2)$, $C_5 = \{1\}$.

**Proof.** We do not give the full proof but restrict to the case $i = 1$. It can be checked that $\langle \epsilon_3 + \epsilon_4, \beta_1 \rangle = -1$ and $\langle \epsilon_3 + \epsilon_4, \beta_2 \rangle = 0$, and $\{\epsilon_3 + \epsilon_4, \beta_1, \beta_2\}$ are linearly independent, hence $C_1 \cong W(B_3)$. Since each element of $\{\epsilon_3 + \epsilon_4, \beta_1, \beta_2\}$ is orthogonal to $\beta_4$, so we get that $N_1 \cong W(B_3) \times W(A_1)$. □
When we consider these groups in $\text{BrM}(F_4)$, the following lemma can be obtained.

**Lemma 4.2.** For $i = 0, \ldots, 5$, the following holds.

(I) The group $N_i$ is the normalizer of $X_i$ in $W(F_4)$.

(II) The group $N_i$ is the semidirect product of $A_i$ and $C_i$, with $C_i$ normalized by $A_i$.

(III) For $x \in A_i$, we have $xe_{X_i} = e_{X_i}$.

(IV) For $x \in C_i$, we have $xe_{X_i} = e_{X_i}x$.

**Proof.** Clearly, $N_i$ normalizes $X_i$, so $N_i \leq N(X_i)$ (the normalizer of $X_i$ in $W(F_4)$), and the equality follows from Lagrange’s Theorem by verification in the table below. Here $\#N_i$ is known from Lemma 4.1 and the lengths of $W(F_4)$-orbits are given in Proposition 3.2. Therefore the first claim hold. The proof of the left conclusions is as arguments in [6, Section 6] and [7, Section 6].

Suppose $D_i$ is a set of left coset representatives of $N_i$ in $W(F_4)$. We have the table below. In the table the product of the three entries in each row is equal to 1152.

| $i$ | $\#D_i$ | $\#C_i$ | $\#A_i$ |
|-----|----------|----------|----------|
| 0   | 1        | 1152     | 1        |
| 1   | 12       | 48       | 2        |
| 2   | 12       | 6        | 16       |
| 3   | 18       | 2        | 32       |
| 4   | 36       | 1        | 32       |
| 5   | 3        | 1        | 384      |

We find that for the Brauer monoid action of some monomial $\phi(a)$ for $a \in \text{BrM}(F_4)$, the admissible root sets $p(\phi(a)\emptyset)$ and $p(\phi(a)^{op}\emptyset)$ belong to different $W(F_4)$-orbits; for example $p(\phi(e_2e_3)\emptyset) = \{\beta_2\}$, and $p(\phi(e_2e_3)^{op}\emptyset) = \{\beta_3, \beta_2 + \beta_3\}$. Some more groups as follows are needed. Let

$$N_6^L = \langle r_2, r_3, r_{\epsilon_4}, r_{\epsilon_4 - \epsilon_1} \rangle,$$  \hspace{1cm} (4.1)

$$N_6^R = \langle r_2, r_{\epsilon_3}, r_{\epsilon_4}, r_{\epsilon_4 - \epsilon_1} \rangle,$$  \hspace{1cm} (4.2)

$$C_6 = \langle r_{\epsilon_4 - \epsilon_1} \rangle,$$  \hspace{1cm} (4.3)

$$N_8 = \langle r_2, r_4, r_{\epsilon_1}, r_{\epsilon_3} \rangle.$$  \hspace{1cm} (4.4)

Additionally, we choose $D_6^L$, $D_6^R$, and $D_8$ to be sets of left coset representatives of $N_6^L$, $N_6^R$, and $N_8$ in $W(F_4)$, respectively.
Let $N^L_7 = N^R_6, N^R_7 = N^L_6, D^L_7 = D^R_6, D^R_7 = D^L_6, C_7 = C_6, N^L_9 = N_5, N^R_9 = N_4, D^L_9 = D_5, D^R_9 = D_4, N^L_{10} = N_4, N^R_{10} = N_5, D^L_{10} = D_4, D^R_{10} = D_5$.

In view of [6] and [7], the following lemma holds.

**Lemma 4.3.** For above groups, we have

(I) $N^L_6 \cong W(B_2) \times W(B_2), N^R_6 \cong W(B_2) \times W(A_1)^2, C_6 \cong W(A_1), N_8 \cong W(B_2) \times W(A_1)^2$.

(II) For each $a \in N^L_6$ ($b \in N^R_6$), there exists some $c \in C_6$ such that $ae_3e_2 = e_3e_2c (e_3e_2b = e_3e_2e_4)$.

(III) For each $a \in N_8$ we have that $ae_4r_3e_2e_3e_4 = e_4r_3e_2e_3e_4$.

(IV) For each $a \in N^L_9$ and $b \in N^R_9$, we have $ae_3e_2e_1e_3 = e_3e_2e_1e_3$ and $e_3e_2e_1e_3b = e_3e_2e_1e_3$.

**Theorem 4.4.** Up to some power of $\delta$, each monomial in $BrM(F_4)$ can be written in one of the following forms.

(I) $ue_{X_i}vw, u \in D_i, w \in D_i^{-1}, v \in C_i, 0 \leq i \leq 5$.

(II) $ue_3e_2vw, u \in D_6^L, w \in (D_6^R)^{-1}, v \in C_6$.

(III) $ue_2e_3vw, u \in D_7^L, w \in (D_7^R)^{-1}, v \in C_7$.

(IV) $ue_4r_3e_2e_3e_4w, u \in D_8, w \in D_8^{-1}$.

(V) $ue_3e_2e_1e_3w, u \in D_9^L, w \in (D_9^R)^{-1}$.

(VI) $ue_3e_1e_2e_3w, u \in D_{10}^L, w \in (D_{10}^R)^{-1}$.

**Proof.** From [7], $(e_4r_3e_2e_3e_4)^{op} = e_4r_3e_2e_3e_4$. In view of Proposition 2.1 it suffices to prove that the claim that the result of a left multiplication by each $r_i$ and $e_\beta$ for $\beta \in \Psi^+$ at the left of each element of $S$ can be written as in (I)-(VI), where

$$S = \{e_{X_i} \}_{i=0}^5 \cup \{e_3e_2, e_2e_3, e_4r_3e_2e_3e_4, e_3e_2e_1e_3, e_3e_1e_2e_3\}.$$

According to 4.2 and Lemma 4.3 the above holds for $\{r_i\}_{i=1}^4$.

By Proposition 2.4 and special cases $Br(C_3)$ in [6] and $Br(B_3)$ in [7], we have that

$$\phi_1(\text{Br}(C_3)) = \bigoplus_{s \in S_1} \mathbb{Z}[\delta^{\pm 1}]W(C_3)sW(C_3),$$

$$\phi_2(\text{Br}(B_3)) = \bigoplus_{s \in S_2} \mathbb{Z}[\delta^{\pm 1}]W(B_3)sW(B_3),$$
Where
\[ S_1 = \{ 1, e_3, e_{X_2}, e_3e_2, e_2e_3, e_{X_4}, e_3e_2e_1e_3, e_1e_3e_2e_3, e_{X_5} \} \]
\[ S_2 = \{ 1, e_{X_1}, e_2, e_3e_2, e_2e_3, e_{X_3}, e_4e_4^*, e_4e_2, e_4r_3e_2e_3e_4 \} \]
with \( e_4^* = r_3r_2r_3e_4r_3r_2r_3 \). It can be seen that each element of \( S_1 \) and \( S_2 \) is in \( S \) or conjugate to some element of \( S \) under \( W(F_4) \). Therefore the proof is reduced to cases of \( \text{Br}(B_3) \) and \( \text{Br}(C_3) \), which can be found in [6] and [7].

As a consequence, we obtain some information about rank of \( \text{Br}(F_4) \) over \( \mathbb{Z}[\delta^\pm 1] \).

**Corollary 4.5.** As a \( \mathbb{Z}[\delta^\pm 1] \)-algebra, the algebra \( \text{Br}(F_4) \) is spanned by 14985 elements.

**Proof.** For \( i = 6, \ldots, 10 \), the following holds.

| set | cardinality | set | cardinality |
|-----|-------------|-----|-------------|
| \( D_6^L, D_7^R \) | 18 | \( D_6^R, D_7^L \) | 36 |
| \( C_6, C_7 \) | 2 | \( D_8 \) | 36 |
| \( D_9^L, D_{10}^R \) | 3 | \( D_9^R, D_{10}^L \) | 36 |

By Theorem 4.4 and numerical information from the above two tables, the algebra \( \text{Br}(F_4) \) over \( \mathbb{Z}[\delta^\pm 1] \) has rank at most

\[
\sum_{i=0}^{5} (\#D_i)^2 \#C_i + 2\#D_6^L \#D_6^R \#C_6 + \#D_8^2 + 2\#D_9^L \#D_9^R = 14985.
\]

---

**5 \( \phi(\text{Br}(F_4)) \) in \( \text{Br}(E_6) \)**

We keep notation as in [8] Section 2] and first introduce some basic concepts. Let \( M \) be the diagram of a connected finite simply laced Coxeter group (type \( A, D, E_6, E_7, E_8 \)). \( \text{BrM}(M) \) is the associated Brauer monoid as Definition 2.2. An element \( a \in \text{BrM}(M) \) is said to be of height \( t \) if the minimal number of \( R_i \) occurring in an expression of \( a \) is \( t \), denoted by \( \text{ht}(a) \). By \( B_Y \) we denote the admissible closure (\[4\]) of \( \{ \alpha_i | i \in Y \} \), where \( Y \) is a coclique of \( M \). The set \( B_Y \) is a minimal element in the \( W(M) \)-orbit of \( B_Y \) which is endowed with a poset structure induced by the partial ordering \( < \) defined on \( W(E_6) \)-orbits in \( A \) (the set of all admissible sets) in [5]. If \( d \) is the Hasse diagram distance for \( W(M)B_Y \) from \( B_Y \) to the unique maximal element ([5 Corollary 3.6]), then for \( B \in W(M)B_Y \) the height of \( B \), already used in Definition notation
ht(B), is \(d - l\), where \(l\) is the distance in the Hasse diagram from \(B\) to the maximal element.

In [4], a Brauer monoid action is defined as follows. For any mutually orthogonal positive root set \(B\), we define \(B^{\text{cl}}\) to be the admissible closure of \(B\), already used in Lemma 3.4 and Definition 3.5 for a more difficult type, namely the minimal admissible root set ([4]) containing \(B\). The generator \(R_i\) acts by the natural action of Coxeter group elements on its root sets, where negative roots are negated so as to obtain positive roots, the element \(\delta\) acts as the identity, and the action of \(\{E_i\}_{i=1}^{n+1}\) is defined below.

\[
E_i B := \begin{cases} 
B & \text{if } \alpha_i \in B, \\
(B \cup \{\alpha_i\})^{\text{cl}} & \text{if } \alpha_i \perp B, \\
R_\beta R_i B & \text{if } \beta \in B \setminus \alpha_i^\perp. 
\end{cases}
\] (5.1)

By the natural involution, we can define a right monoid action of \(\text{BrM}(M)\) on \(A\).

Considering our \(\sigma\) and table 3 in [8], let

\[Y \in \mathcal{Y} = \{\emptyset, \{2\}, \{1, 6\}, \{2, 3, 5\}\}.
\]

Obviously, each element of \(\mathcal{Y}\) is \(\sigma\)-invariant. From [8, Theorem 2.7], it is known that each monomial \(a\) in \(\text{BrM}(E_6)\) can be uniquely written as \(\delta^i a_B \hat{\epsilon}_Y h a_B^{\text{op}}\) for some \(i \in \mathbb{Z}\) and \(h \in W(M_Y)\) in [8, table 3], where \(B = a\emptyset, B' = a_B \in \text{BrM}(E_6), a_B^{\text{op}} \in \text{BrM}(E_6)\) and

(i) \(a\emptyset = a_B \emptyset = a_B B_Y, \emptyset a = \emptyset a_B^{\text{op}} = B_Y a_B^{\text{op}}\),

(ii) \(\text{ht}(B) = \text{ht}(a_B), \text{ht}(B') = \text{ht}(a_B^{\text{op}})\).

Now we can apply this to obtain the following corollary immediately.

**Corollary 5.1.** Let \(B, Y, B'\) be as above. Then \(a = a_B \hat{\epsilon}_Y h a_B^{\text{op}}\) is \(\sigma\)-invariant monomial in \(\text{BrM}(E_6)\) if and only if

(i) the sets \(B, B'\) are \(\sigma\)-invariant,

(ii) the element \(h \in W(M_Y)\) is \(\sigma\)-invariant.

In type \(E_6\), we see that \(E_2 E_4 E_5 \{\alpha_6\} = \alpha_2, E_1 E_3 \{\alpha_4, \alpha_6\} = \{\alpha_1, \alpha_6\}\). Then

\[
W(M_{(2)}) = E_2 E_4 E_5 W(M_{(6)}) E_5 E_4 E_2,
\]

\[
W(M_{(1,6)}) = E_1 E_3 W(M_{(4,6)}) E_3 E_1.
\]

Let \(\hat{E}_i = \delta^{-1} E_i\). By the 6th column list of [8, table 3], the group \(W(M_{(2)}) \cong W(A_5)\) has generators \(R_1 \hat{E}_2, R_3 \hat{E}_2, R_5 \hat{E}_2, R_6 \hat{E}_2, E_2 E_4 R_3 E_5 E_4 \hat{E}_2\) and their relation is corresponding to subdiagram of \(E_6\) by deleting the node 2 and the
edge between 2 and 4 with \( E_2E_4R_3E_5E_4E_2 \) corresponding to node 4, \( R_i\hat{E}_2 \) corresponding to node \( i \) for \( i = 1, 3, 5, 6 \); the group \( W(M_{(1,6)}) \cong W(A_2) \) has generators \( R_2\hat{E}_1\hat{E}_6, R_4\hat{E}_1\hat{E}_6 \) and whose relation is the subdiagram of \( E_6 \) of nodes 2, 4 and the edge between them. When \( \sigma \) acts on the generators of \( W(M_{(2)}) \) and \( W(M_{(1,6)}) \), we have that

\[
\begin{align*}
\sigma(E_2E_4R_3E_5E_4E_2) &= E_2E_4R_3E_5E_4E_2, \\
\sigma(R_i\hat{E}_2) &= R_{\sigma(i)}\hat{E}_2, \quad (i = 1, 3, 5, 6,), \\
\sigma(R_j\hat{E}_1\hat{E}_6) &= R_j\hat{E}_1\hat{E}_6, \quad (j = 2, 4,)
\end{align*}
\]

which implies that the \( \sigma \)-action on those two groups is determined the \( \sigma \)-action on the subdigrams of \( E_6 \) described in the above. Hence \( W(M_{(2)})^\sigma \cong W(B_3) \), and \( W(M_{(1,6)})^\sigma = W(M_{(1,6)}) \cong W(A_2) \). This conclusion can be summarized in the table below with the second column from our GAP [9] code.

| \( Y \) | \#((\( W(E_6)B_Y \))^\sigma) | \( M_Y \) | \( M_Y^\sigma \) |
|---|---|---|---|
| \( \emptyset \) | 1 | \( E_6 \) | \( F_4 \) |
| 2 | 12 | \( A_5 \) | \( B_3 \) |
| 1, 6 | 30 | \( A_2 \) | \( A_2 \) |
| 2, 3, 5 | 39 | \( \emptyset \) | \( \emptyset \) |

**Corollary 5.2.** The algebra \( \text{Br}(E_6)^\sigma \) is free over \( \mathbb{Z}[\delta^{\pm 1}] \) with rank

\[
1152 + 12^2 \times 48 + 30^2 \times 6 + 39^2 = 14985.
\]

**Proof.** By Corollary 5.1 we have

\[
\text{rk}(\text{Br}(E_6)^\sigma) = \Sigma_{Y \in \mathcal{Y}}(\#((\text{Br}(E_6)B_Y)^\sigma))^2 \#W(M_Y)^\sigma.
\]

Hence the corollary holds. \(\square\)

The following can be checked by computation and application of Theorem 4.4.

**Lemma 5.3.** Let \( K_Y = \{a \in \text{BrM}(E_6) \mid \sigma(a) = a, a\emptyset \in W(E_6)B_Y\} \) for \( Y \in \mathcal{Y} \). Then \( K_Y = \{\phi(b) \mid b \in \text{BrM}(F_4), \phi(b)\emptyset \in W(E_6)B_Y\} \).

**Proof.** If \( Y = \{2\} \), then \( \{\phi(b) \mid b \in \text{BrM}(F_4), \phi(b)\emptyset \in W(E_6)B_Y\} \) are the elements in (I) for \( i = 1 \) in Theorem 4.4 which has cardinality \( 12^2 \times 48 \), and \( \hat{E}_2\phi(C_1)\hat{E}_2 = W(M_Y)^\sigma \) (in the proof of Corollary 5.1), and the \( \phi(W(F_4))\)-orbit of \( \{\alpha_2\} \) is \( \sigma \)-invariant elements in \( W(E_6)B_Y \). Hence the lemma holds for \( Y = \{2\} \).

If \( Y = \{1, 6\} \), then \( \{\phi(b) \mid b \in \text{BrM}(F_4), \phi(b)\emptyset \in W(E_6)B_Y\} \) are those monomials in (I) for \( i = 2 \) and \( i = 3 \), (II), (III) and (IV) in Theorem 4.4. Analogous
to the above argument for \( Y = \{2\} \), we see that the image of those monomials under \( \phi \) correspond to different normal forms for \( \text{BrM}(E_6) \) up to some powers of \( \delta \), and those monomials of in (I) for \( i = 2 \) and \( i = 3 \), (II), (III) and (IV) in Theorem 4.4 have cardinality \( 12^2 \times 6 + 18^2 \times 2 + 18 \times 36 \times 2 + 18 \times 36 \times 2 + 36^2 = 30^2 \times 6 \). Hence the lemma holds for \( Y = \{1, 6\} \).

If \( Y = \{2, 3, 5\} \), then \( \{ \phi(b) \mid b \in \text{BrM}(F_4), \phi(b)\emptyset \in W(E_6)B_Y \} \) are those monomials in (I) for \( i = 4 \) and \( i = 5 \), (V), and (VI) in Theorem 4.4. Similarly, we see those monomials are corresponding to different normal forms for \( \text{BrM}(E_6) \) up to some powers of \( \delta \), and they have cardinality \( 36^2 + 3^2 + 3^6 \times 3 + 3^6 \times 3 = 39^2 \). Hence the lemma holds for \( Y = \{2, 3, 5\} \).

Now, we can give the proof of our main Theorem 1.2.

**Proof.** Proposition 2.3 implies \( \phi \) is a homomorphism. Corollary 4.5 indicates that the \( \phi(\text{Br}(F_4)) \) has rank at most 14985. Corollary 5.2 implies that the \( \text{Br}(E_6)^* \) has rank 14985. Lemma 5.3 indicates that \( \phi \) has image \( S\text{Br}(E_6) \); therefore \( \phi \) is surjective. The homomorphism \( \phi \) is an isomorphism and \( \text{Br}(F_4) \) is free over \( \mathbb{Z}[\delta^{\pm 1}] \) of rank 14985 because of the freeness of \( S\text{Br}(E_6) \).

### 6 Cellularity

Recall from [11] and [12] that an associative algebra \( A \overcommutative ring R \) is cellular if there is a quadruple \( (\Lambda, T, C, \ast) \) satisfying the following three conditions.

(C1) \( \Lambda \) is a finite partially ordered set. Associated to each \( \lambda \in \Lambda \), there is a finite set \( T(\lambda) \). Also, \( C \) is an injective map

\[
\prod_{\lambda \in \Lambda} T(\lambda) \times T(\lambda) \to A
\]

whose image is an \( R \)-basis of \( A \).

(C2) The map \( \ast : A \to A \) is an \( R \)-linear anti-involution such that \( C(x, y)^\ast = C(y, x) \) whenever \( x, y \in T(\lambda) \) for some \( \lambda \in \Lambda \).

(C3) If \( \lambda \in \Lambda \) and \( x, y \in T(\lambda) \), then, for any element \( a \in A \),

\[
aC(x, y) \equiv \sum_{u \in T(\lambda)} r_a(u, x)C(u, y) \mod A_{<\lambda},
\]

where \( r_a(u, x) \in R \) is independent of \( y \) and where \( A_{<\lambda} \) is the \( R \)-submodule of \( A \) spanned by \( \{ C(x', y') \mid x', y' \in T(\mu) \text{ for } \mu < \lambda \} \).
Such a quadruple \((\Lambda, T, C, *)\) is called a cell datum for \(A\).

There is also an equivalent definition due to König and Xi.

**Definition 6.1.** Let \(A\) be \(R\)-algebra. Assume there is an anti-automorphism \(i\) on \(A\) with \(i^2 = id\). A two sided ideal \(J\) in \(A\) is called cellular if and only if \(i(J) = J\) and there exists a left ideal \(\Delta \subset J\) such that \(\Delta\) has finite rank and there is an isomorphism of \(A\)-bimodules \(\alpha: J \simeq \Delta \otimes R i(\Delta)\) making the following diagram commutative:

\[
\begin{array}{ccc}
J & \stackrel{\alpha}{\longrightarrow} & \Delta \otimes R i(\Delta) \\
\downarrow{i} & & \downarrow{(y \otimes x) \mapsto i(y) \otimes i(x)} \\
J & \stackrel{\alpha}{\longrightarrow} & \Delta \otimes R i(\Delta)
\end{array}
\]

The algebra \(A\) is called cellular if there is a vector space decomposition \(A = J_1 \oplus \cdots \oplus J_n\) with \(i(J_j) = J_j'\) for each \(j\) and such that setting \(J_j = \bigoplus_{k=1}^{j} J_j'\) gives a chain of two sided ideals of \(A\) such that for each \(j\) the quotient \(J_j' = J_j/J_{j-1}\) is a cellular ideal of \(A/J_{j-1}\).

Also recall definitions of iterated inflations from [14] and cellularly stratified algebra from [13] in Definition 6.4. Given an \(R\)-algebra \(B\), a finitely generated free \(R\)-module \(V\), and a bilinear form \(\varphi: V \otimes_R V \longrightarrow B\) with values in \(B\), we define an associative algebra (possibly without unit) \(A(B, V, \varphi)\) as follows: as an \(R\)-module, \(A(B, V, \varphi)\) equals \(V \otimes_R V \otimes_R B\). The multiplication is defined on basis element as follows:

\[(a \otimes b \otimes x)(c \otimes d \otimes y) := a \otimes d \otimes x \varphi(b, c)y.\]

Assume that there is an involution \(i\) on \(B\). Assume, moreover, that \(i(\varphi(v, w)) = \varphi(w, v)\). If we can extend this involution \(i\) to \(A(B, V, \varphi)\) by defining \(i(a \otimes b \otimes x) = b \otimes a \otimes i(x)\). Then We call \(A(B, V, \varphi)\) is an inflation of \(B\) along \(V\). Let \(B\) be an inflated algebra (possibly without unit) and \(C\) be an algebra with unit. We define an algebra structure in such a way that \(B\) is a two-sided ideal and \(A/B = C\). We require that \(B\) is an ideal, the multiplication is associative, and that there exists a unit element of \(A\) which maps onto the unit of the quotient \(C\). The necessary conditions are outlined in [14] Section 3]. Then we call \(A\) an inflation of \(C\) along \(B\), or iterated inflation of \(C\) along \(B\). We present Proposition 3.5 and Theorem 4.1 of [14].

**Proposition 6.2.** An inflation of a cellular algebra is cellular again. In particular, an iterated inflation of \(n\) copies of \(R\) is cellular, with a cell chain of length \(n\) as in Definition 6.1.
More precisely, the second statement has the following meaning. Start with $C$ a full matrix ring over $R$ and $B$ an inflation of $R$ along a free $R$-module, and from a new $A$ which is an inflation of the old $A$ along the new $B$, and continue this operation. Then after $n$ steps we have produced a cellular algebra $A$ with a cell chain of length $n$.

**Theorem 6.3.** Any cellular algebra over $R$ is the iterated inflation of finitely many copies of $R$. Conversely, any iterated inflation of finitely many copies of $R$ is cellular.

Let $A$ be cellular (with identity) which can be realized as an iterated inflation of cellular algebras $B_l$ along vector spaces $V_l$ for $l = 1, \ldots, n$. This implies that as a vector space

$$A = \bigoplus_{l=1}^n V_l \otimes V_l \otimes B_l,$$

and $A$ is cellular with a chain of two sided ideals $0 = J_0 \subset J_1 \cdots \subset J_n = A$, which can be refined to a cell chain, and each quotient $J_l/J_{l-1}$ equals $V_l \otimes V_l \otimes B_l$ as an algebra without unit. The involution $i$ of $A$ is defined through the involution $i_l$ of the algebra $B_l$ where $i(a \otimes b \otimes x) = b \otimes a \otimes j_l(x)$.

The multiplication rule of a layer $V_l \oplus V_l \oplus B_l$ is indicated by

$$(a \otimes b \otimes x)(c \otimes d \otimes y) := a \otimes d \otimes x\varphi(b, c)y + \text{lower terms}.$$  

Here lower terms refers to element in lower layers $V_h \otimes V_h \otimes B_h$ for $h < l$. Let $1_{B_l}$ be the identity of the algebra $B_l$.

Let that $R$ is a field.

**Definition 6.4.** A finite dimensional associative algebra $A$ is called cellularly stratified with stratification data $(B_1, V_1, \ldots, B_n, V_n)$ if and only if the following conditions are satisfied:

1. The algebra is an iterated inflation of cellular algebra $B_l$ along vector spaces $V_l$ for $l = 1, \ldots, n$.

2. For each $l = 1, \ldots, n$, there exist $u_l$, $v_l$ such that $e_l = u_l \otimes v_l \otimes 1_{B_l}$ is an idempotent.

3. If $l > m$, then $e_le_m = e_m = e_me_l$.

As [1], the following theorem can be obtained.

**Theorem 6.5.** Let $R$ be a field with 2 and 3 being invertible in $R$ and containing $\mathbb{Z}[\delta \pm 1]$ as a subring. Then the algebra $Br(R, F_4) = Br(F_4) \otimes_{\mathbb{Z}[\delta \pm 1]} R$ is a cellularly stratified algebra over $R$. 

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Proof. To prove this, it suffices to prove it for \( \text{Br}(R, E_6) = \text{SBr}(E_6) \otimes_{Z[\delta \in \pm 1]} R \) because of Theorem [1,2]. Let \( Z_0 \subset Z_1 \subset Z_2 \subset Z_3 \) be a \( \sigma \)-invariant and admissible root set sequence of type \( E_4 \), where \( Z_0 = \emptyset \), \( Z_1 = \{ \alpha_2 \} \), \( Z_2 = \{ \alpha_2, \alpha_3 + \alpha_5 + \alpha_6 + 2\alpha_4 \} \), \( Z_3 = \{ \alpha_2, \alpha_3, \alpha_5, \alpha_6 + \alpha_5 + \alpha_6 + 2\alpha_4 \} \). As \( E_2 E_4 E_5 E_3 \{ \alpha_1, \alpha_6 \} = Z_2 \), we have \( \text{ht}(Z_2) = 0 \).

For \( 0 \leq i \leq 3 \), let \( B_{Z_i} \) be the group algebras of \( W_{Z_0} = W(M_{\emptyset}^\sigma) \), \( W_{Z_1} = W(M_{\{2\}}^\sigma) \), \( W_{Z_2} = E_2 E_4 E_5 E_3 W(M_{\{1,6\}}^\sigma) E_5 E_4 E_2 \), \( W_{Z_3} = W(M_{\{2,3,5\}}^\sigma) \) over \( R \), respectively, whose group rings over \( R \) are cellular algebras due to \([10, \text{Theorem 1.1}]\).

Then each monomial \( a \) in \( \text{BrM}(E_6)^\sigma \) can be uniquely written as \( \delta^i a_{Z_i,B} \hat{\psi}_Y h a_{Z_{i+1},B'}^\text{op} \) for some \( i \in \{0, 1, 2, 3\} \) and \( h \) is from the above four groups, where \( B = \emptyset \), \( B' = \emptyset a \) being \( \sigma \)-invariant, \( a_{Z_i,B} \in \text{BrM}(E_6)^\sigma \), \( e_{Z_i,B}^\text{op} \in \text{BrM}(E_6)^\sigma \) and

1. \( a \emptyset = a_{Z_i,B} \emptyset = a_{B_i} \emptyset, \emptyset a = \emptyset a_{Z_{i},B'}^\text{op} = Z_i a_{Z_i,B}^\text{op}, \)
2. \( \text{ht}(B) = \text{ht}(a_{Z_i,B}), \text{ht}(B') = \text{ht}(a_{Z_i,B'}^\text{op}). \)

For each \( Z_i \), let \( V_{Z_i} \) be a linear space over \( R \) with basis \( u_{Z_i,B} \) where \( B \in W(E_6) Z_i \) and \( \sigma(B) = B \). and let \( \varphi_{Z_i} \) be a bilinear map defined as

\[
V_{Z_i} \otimes_R V_{Z_i} \longrightarrow B_{Z_i}
\]

\[
\varphi_{Z_i}(u_{Z_i,B}, u_{Z_i,B'}) = a_{Z_i,B}^\text{op} a_{Z_i,B'}, \text{ if } Z_i = a_{Z_i,B}^\text{op} a_{Z_i,B} \emptyset,
\]

\[
\varphi_{Z_i}(u_{Z_i,B}, u_{Z_i,B'}) = 0, \text{ if } Z_i \subset \emptyset a_{Z_i,B}^\text{op} a_{Z_i,B'} \emptyset.
\]

We first prove that \( \varphi \) is well defined. As \( a_{Z_i,B}^\text{op} a_{Z_i,B} \emptyset = Z_i \), we find \( Z_i \subset a_{Z_i,B}^\text{op} a_{Z_i,B} \emptyset \), similarly \( Z_i \subset \emptyset a_{Z_i,B}^\text{op} a_{Z_i,B'} \). If \( Z_i = a_{Z_i,B}^\text{op} a_{Z_i,B'} \emptyset \), this indicates that \( Z_i = \emptyset a_{Z_i,B}^\text{op} a_{Z_i,B'} \) and that \( a_{Z_i,B}^\text{op} a_{Z_i,B'} \) will be in \( W_{Z_i} \) up to some power of \( \delta \). Therefore our \( \varphi_{Z_i} \) is well defined. Observe that

\[
(a_{Z_i,B}^\text{op} a_{Z_i,B'})^\text{op} = a_{Z_i,B'}^\text{op} (a_{Z_i,B})^\text{op} = a_{Z_i,B}^\text{op} a_{Z_i,B},
\]

so \( (\varphi_{Z_i}(u_{Z_i,B}, u_{Z_i,B'}))^\text{op} = \varphi_{Z_i}(u_{Z_i,B'}, u_{Z_i,B}) \). By linear extension, we find \( (\varphi_{Z_i}(u, v))^\text{op} = \varphi_{Z_i}(v, u) \), for \( u, v \in V_{Z_i} \). By the proof of Lemma [5,3] the algebra \( \text{SBr}(R, E_6) \otimes_{Z[\delta \in \pm 1]} \) is an iterated inflation of cellular algebra \( B_{Z_i} \) along vector space \( V_{Z_i} \) for \( Z_0, \ldots, Z_3 \), namely \( \text{SBr}(R, E_6) \otimes_{Z[\delta \in \pm 1]} \) satisfies (1) of cellular stratified algebra. We take \( e_{Z_i} = u_{Z_i,Z_i} \otimes u_{Z_i,Z_i} \otimes 1_{B_{Z_i}} \), where \( 1_{B_{Z_i}} = \delta^{-\#Z_i} E_{Z_i} \). Because \( E_{Z_i} E_{Z_i} = \delta^{\#Z_i} E_{Z_i} \) for \( Z_i \subset Z_j \), hence the condition (2) and (3) follows since that that \( Z_i > Z_j \) means \( Z_i \nsubset Z_j \). Finally, \( \text{SBr}(R, E_6) \otimes_{Z[\delta \in \pm 1]} \) is a cellularly stratified algebra. \( \square \)

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