Bifurcation study on fractional non-smooth oscillator containing clearance constraints

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Abstract
Bifurcation characteristics of a fractional non-smooth oscillator containing clearance constraints under sinusoidal excitation are investigated. First, the bifurcation response equation of the fractional non-smooth system is obtained via the K–B method. Second, the stability of the bifurcation response equation is analyzed, and parametric conditions for stability are acquired. The bifurcation characteristics of the fractional non-smooth system are then studied using singularity theory, and the transition set and bifurcation diagram under six different constrained parameters are acquired. Finally, the analysis of the influence of fractional terms on the dynamic characteristics of the system is emphasized through numerical simulation. Local bifurcation diagrams of the system under different fractional coefficients and orders verify that the system will present various motions, such as periodic motion, multiple periodic motion, and chaos, with the change in fractional coefficient and order. This manifestation indicates that fractional parameters have a direct effect on the motion form of this non-smooth system. Thus, these results provide a theoretical reference for investigating and repressing oscillation problems of similar systems.

Keywords
Bifurcation, fractional non-smooth oscillator, clearance constraint, singularity theory

Introduction
It has been more than 300 years since the concept of fractional calculus was first proposed. In recent years, the fractional derivatives have been widely concerned. He¹ defined a new fractal derive and two examples are used to illustrate the use of fractal derivatives to establish governing equations and how to solve these equations. He² introduced the fractal-Cantorian spacetime and fractional calculus, then He³ studied further fractal calculus and its geometrical explanation. What’s more Wang and Wang⁴ considered the fractional heat transfer equations and obtained the approximate analytical solutions. Fan et al.⁵ applied this theory to wool fiber, indicating that wool fiber has an excellent warm retention property. Wang et al.⁶ used Adomian’s decomposition method and variational iteration method to study the fractional KdV–Burgers–Kuramo equation. Wang et al.⁷ established a fractal modification of the telegraph equation with fractal derivatives and the approximate analytical solution is obtained by two-scale transform method and He’s homotopy perturbation method. He⁸ analyzed a generalized KdV–Burgers equation with fractal derivatives and provides conservation laws in an energy form and possible solution structures. He and Ain⁹ clarified the properties of fractional calculus and revealed the relationship between the fractal calculus and traditional calculus using the two-scale transform.
Analytical method is an important means to study nonlinear systems, many methods such as the Hamiltonian approach method, He’s frequency formulation method, K–B method, multi-scale method, harmonic balance method, variational iteration method, and homotopy analysis method are adopted. Forsat\textsuperscript{10} studied the nonlinear control equation based on beam theory using Hamiltonian method, and the nonlinear vibration frequencies are obtained. He’s frequency method comes from ancient Chinese mathematics, by using He’s amplitude–frequency formula. Zhang\textsuperscript{11} obtained the analytical solution of nonlinear oscillator with discontinuity by using He’s amplitude–frequency formula, which has a high accuracy. Some methods are also extended to solve the fractional-order systems. Niu et al.\textsuperscript{12} studied the approximate analytical solution of a fractional-order single degree-of-freedom oscillator with clearance by using the KBM asymptotic method. Xu et al.\textsuperscript{13} proposed a new method to solve the second-order approximate solution of fractional Duffing oscillator by combining L–P method and multi-scale method. Xiao et al.\textsuperscript{14} established the approximate expression of the analytical solution of fractional Vander Pol oscillator by harmonic balance method. At the same time, in 1998, the variational iteration method was an effective tool for factional calculus.\textsuperscript{2} In 2007, Momani and Odibat adopted the homotopy perturbation method for fractional differential equations with great success.\textsuperscript{15}

Most engineering materials are not ideal solids or liquids but viscous–elastic. Therefore, the model, which is established by only considering elastic properties or damping characteristics, fails to reflect the material essence completely. Nearly, all physical systems can be expressed by a fractional model, and related studies also show that the model established using fractional calculus is accurate and can effectively reflect constitutive relations. Therefore, fractional calculus has been extensively applied in different engineering fields.\textsuperscript{16–19} Sun et al.\textsuperscript{20} introduced fractional derivative to establish a model for oil and gas suspension. They proved that the fractional model could more accurately describe the characteristics of the oil and gas suspension than the integer-order model through numerical and experimental verifications. Lewandowski and Lasecka-Plura\textsuperscript{21} described a viscoelastic damper through a fractional model, conducted a theoretical analysis of design sensitivity of this damper, and verified their results through illustration. Li et al.\textsuperscript{22} introduced an FKV-based constitutive fractional vibration attenuation model for viscoelastic suspension and found that this viscoelastic suspension response was of global correlation and memorability. Wu and Sangguan\textsuperscript{23} proved that the model containing fractional derivative could effectively predict the dynamic characteristics of rubber vibration isolators. They also described that the correlation between dynamic characteristics and frequency of the rubber vibration isolator through the fractional derivative was reasonable and accurate.

Non-smooth motion is extensively used in many oscillatory systems. However, dynamic behaviors and mechanisms of some important non-smooth systems remain unclear. Particularly, dynamic behaviors, such as bifurcation and chaos, which aggravate machines, have imposed potential safety hazards on machine operation. Therefore, conducting an in-depth analysis of non-smooth systems is necessary. Scholars have already investigated non-smooth systems. Zhong and Chen\textsuperscript{24} used singularity theory to study the bifurcation of resonant solutions of clearance-containing suspension system models. Wei et al.\textsuperscript{25} analyzed the transformation of single-degree-of-freedom piecewise smooth systems from $n - 1$ periodic motion into chaotic behaviors through period-doubling bifurcation and then investigated the control of system chaotic motions. Zhang et al.\textsuperscript{26} established a generalized piecewise linear model of Chua’s circuit and studied codimension-1 unconventional fold bifurcation generated during the passage of the system trajectory through the interface. They also analyzed the resulting bursting oscillation phenomenon. Zhang et al.\textsuperscript{27} used Melnikov theory for the oscillation model of clearance-containing gear systems to investigate global bifurcation conditions for the heteroclinic orbit and the stability of its periodic motion. Li and Zhao\textsuperscript{28} obtained Melnikov function for the subharmonic orbit of piecewise smooth systems to investigate the existence of subharmonic orbits. Huang and Xu\textsuperscript{29} acquired frequency response and stability conditions for a controlled piecewise smooth system with negative stiffness and explored symmetrical breaking bifurcation and chaotic motion in this system. Hou et al.\textsuperscript{30} established a roller model for rolling mills containing nonlinear constraints and analyzed bifurcation characteristics of the system under autonomous and non-autonomous conditions via stability systems of singular points and singularity theory.

Studies regarding fractional non-smooth systems containing clearance constraints are currently limited. Dynamic characteristics are complex due to the strong nonlinearity and singularity of such systems, and most investigations have adopted numerical methods.\textsuperscript{31,32} Numerical and analytical methods are the main research methods for fractional systems, wherein the former generally only provide solutions under specific parameters, while the latter can acquire definite relational expressions between system characteristics and parameters to realize quantitative analysis. Hence, analytical and numerical methods are combined in this study to investigate fractional non-smooth oscillators containing clearance constraints. This paper is organized as follows: first is the Introduction part; the next section introduces the physical model of the system and the motion equations; then the
next section obtains the bifurcation response equation of the system and its stability through the analytical method and examines the bifurcations of the system under different parameters using singularity theory; following section presents the analysis of chaotic motions of the system under fractional coefficient and order via the numerical method; and the last section draws the main conclusions.

System model and motion equation

The fractional non-smooth system model containing clearance constraints is shown in Figure 1. This model comprises a mass, a damper, a linear spring, a nonlinear spring, and the nonlinear constraints with clearance symmetrically, where \( m, c, k_1, k_2, \) and \( k_3 \) are the system mass, linear damping coefficient, linear stiffness coefficient, nonlinear stiffness coefficient, and nonlinear constraint stiffness coefficient, respectively. \( K^{p}[x(t)]B_0\cos(\omega t) \) is the \( p \)-order derivative of \( x(t) \) to \( t \) with the fractional coefficient \( K \). The non-smooth system is subject to a force excitation.

The nonlinearity \( k_2 \) and fractional derivative may be caused by the physical properties of the material; the nonlinear spring \( k_3 \) may be caused by the restraint. When the displacement of \( m \) is less than clearance \( \Delta \), then only spring \( k_1, k_2 \) and damping \( c \) are involved, if the displacement of \( m \) exceeds clearance \( \Delta \), the nonlinear spring \( k_3 \) is engaged. The differential equation of the system motion is presented as follows

\[
m\ddot{x} + k_1x + k_2x^3 + c\dot{x} + f(x) + K^{p}[x(t)] = B_0\cos(\omega t)
\]

where

\[
f(x) = \begin{cases} 
  k_3(x + \Delta)^3 & x < -\Delta \\
  0 & -\Delta \leq x \leq \Delta \\
  k_3(x - \Delta)^3 & x > \Delta 
\end{cases}
\]

The following parameters are inputted into equation (1)

\[
\omega_0 = \sqrt{\frac{k_1}{m}}, \quad \varepsilon \lambda_1 = \frac{k_2}{m}, \quad \varepsilon \lambda_2 = \frac{k_3}{m}, \quad 2\varepsilon \mu_1 = \frac{c_1}{m}, \quad 2\varepsilon \mu_2 = \frac{c_2}{m}, \\
2\varepsilon \mu_3 = \frac{c_3}{m}, \quad \varepsilon B = \frac{B_0}{m}, \quad \varepsilon K_1 = \frac{K}{m}
\]

Equation (1) can be transformed into

\[
\ddot{x} + \omega_0^2x + \varepsilon \lambda_1 x^3 + 2\varepsilon \mu \dot{x} + eg(x) + \varepsilon K_1 D^{p}[x(t)] = \varepsilon B\cos(\omega t)
\]

\[
g(x) = \begin{cases} 
  \dot{k}_2(x + \Delta)^3 & x < -\Delta \\
  0 & -\Delta \leq x \leq \Delta \\
  \dot{k}_2(x - \Delta)^3 & x > \Delta 
\end{cases}
\]

Solution of the system bifurcation response

Solution of the bifurcation response equation through the K–B method

Based on the K–B method, which means the excitation frequency is close to the natural one, that is, \( \omega_0 \approx \omega \), a detuning parameter \( \sigma \) is introduced to illustrate the proximate degree

\[
\omega^2 = \omega_0^2 + \varepsilon \delta
\]
Then, equation (2) is transformed into the following formula

$$\ddot{x} + \omega^2 x = \varepsilon [f \cos(\omega t) + \delta x - \lambda_1 x^3 - 2 \mu \dot{x} - g(x) - K_1 D^p[x]]$$

Equation (4) is assumed to satisfy the following conditions

$$\begin{cases} x = a(t) \cos \phi \\ \dot{x} = -a(t) \omega \sin \phi \end{cases}$$

where $\phi = \omega t + \theta(t)$. $a(t)$ and $\theta(t)$ are slowly varying functions of time $t$.

Equation (5) is substituted into equation (4) to obtain the following

$$\begin{cases} \dot{a} = -\frac{\varepsilon}{\omega} \{ f \cos(\omega t) + \sigma a \cos \phi + 2 \mu a \omega \sin \phi - \lambda_1 a^3 \cos^3 \phi - f_3(a \cos \phi) - \varepsilon K_1 D^p[a \cos \phi] \} \sin \phi \\ a \dot{\theta} = -\frac{\varepsilon}{\omega} \{ f \cos(\omega t) + \sigma a \cos \phi + 2 \mu a \omega \sin \phi - \lambda_1 a^3 \cos^3 \phi - f_3(a \cos \phi) - \varepsilon K_1 D^p[a \cos \phi] \} \cos \phi \end{cases}$$

Integrals are taken for $\dot{a}$ and $a \dot{\theta}$, and averaging values are taken as follows

$$\begin{cases} \ddot{a} = -\frac{\varepsilon}{2 \pi \omega} \int_0^{2\pi} \{ f \cos(\omega t) + \sigma a \cos \phi + 2 \mu a \omega \sin \phi - \lambda_1 a^3 \cos^3 \phi - f_3(a \cos \phi) \} \sin \phi \, d\phi \\ -\lim_{T \to \infty} \frac{\varepsilon}{2 \pi \omega} \int_0^T K_1 D^p[a \cos \phi] \sin \phi \, d\phi, \\ a \ddot{\theta} = -\frac{\varepsilon}{2 \pi \omega} \int_0^{2\pi} \{ f \cos(\omega t) + \sigma a \cos \phi + 2 \mu a \omega \sin \phi - \lambda_1 a^3 \cos^3 \phi - f_3(a \cos \phi) \} \cos \phi \, d\phi \\ -\lim_{T \to \infty} \frac{\varepsilon}{2 \pi \omega} \int_0^T K_1 D^p[a \cos \phi] \cos \phi \, d\phi \end{cases}$$

The fractional differential is defined as Caputo, and the integral of the fractional part can be obtained through residue theorem and contour integral shown as below

$$\begin{align*} -\lim_{T \to \infty} \frac{\varepsilon}{2 \pi \omega} \int_0^T K_1 D^p[a \cos \phi] \sin \phi \, d\phi &= -\frac{\alpha \cos^{p-1} \sin(p \pi/2)}{2}, \\ -\lim_{T \to \infty} \frac{\varepsilon}{2 \pi \omega} \int_0^T K_1 D^p[a \cos \phi] \cos \phi \, d\phi &= \frac{\alpha \cos^{p-1} \cos(p \pi/2)}{2} \end{align*}$$
The solved integrals are substituted into the original parameters to obtain the following
\[
\begin{align*}
\dot{a} &= -\frac{B_0\sin\theta}{2m\omega} - \frac{a}{2m}(c + c_1)
\quad (9) \\
\dot{\theta} &= -\frac{a\omega}{2} - \frac{B_0\cos\theta}{2m\omega} + \frac{a}{2m\omega} \left( k_1 + \frac{3a^2}{4}k_2 + k'_3 + k \right) 
\end{align*}
\]
where \(k'_3\) is the piecewise equivalent stiffness
\[
k'_3 = \frac{a^2k_3}{4\pi} \left( 18\phi_0 + 12\phi_0\cos2\phi_0 - 14\sin2\phi_0 - \frac{1}{2}\sin4\phi_0 \right),
\]
\(\phi_0 = \arccos\frac{a}{\Delta}
\]
c is fractional equivalent damping
\[
c = K\phi^{-1}\sin\left(\rho\pi/2\right)
\]
and \(k\) is fractional equivalent stiffness
\[
k = K\phi\cos\left(\rho\pi/2\right)
\]
Let \(cc = c_1 + c, kk = k_1 + \frac{3a^2}{4}(k_2 + k'_3 + k)\) then equation (9) can be rewritten as
\[
\begin{align*}
\dot{a} &= -\frac{B_0\sin\theta}{2m\omega} - \frac{a}{2m}cc \\
\dot{\theta} &= -\frac{a\omega}{2} - \frac{B_0\cos\theta}{2m\omega} + \frac{a}{2m\omega}kk
\end{align*}
\quad (10)
\]
Let \(\dot{a} = a\dot{\theta} = 0\). After eliminating \(\dot{\theta}\) (phase of the steady-state response) from equation (10), the amplitude–frequency equation of the bifurcation response can be obtained as follows
\[
\bar{a}^2 \left[ \omega^2 cc^2 + (m\omega^2 - kk)^2 \right] - B_0^2 = 0 
\quad (11)
\]
where \(\bar{a}\) is amplitude of the steady-state response.

**Stability analysis of system response equation**

The system is stable when \(a \leq \Delta\). Thus, only the situation under \(a > \Delta\) is considered.
\(a = \bar{a} + \Delta a\) and \(\theta = \theta + \Delta \theta\) are substituted into equation (1) to investigate the stability of the response equation, and then
\[
\begin{align*}
\frac{d\Delta a}{dt} &= -\left( \frac{cc}{2m} + \frac{\bar{a}c_2cc'\bar{a}}{2m} \right) \Delta a - \frac{B_0\cos\theta}{2m\omega} \Delta \theta \\
\frac{d\Delta \theta}{dt} &= \left[ \frac{F\cos\theta}{2a^2m\omega} + \frac{k_2kk'\bar{a}}{2m\omega} \right] \Delta \bar{a} + \frac{F\sin\theta}{2m\omega} \Delta \bar{\theta} 
\end{align*}
\quad (12)
\]
\[
\det \begin{bmatrix}
-\frac{cc}{2m} - \frac{\bar{a}c_2cc'}{2m} & -\frac{\bar{a}c_2cc'}{2m} & \frac{\omega \bar{a}}{2} - \frac{\bar{a}}{2m\omega} kk \\
\frac{\omega}{2} & \frac{\omega}{2} + \frac{k_2kk'}{2m\omega} & \frac{cc}{2m} - \frac{\omega}{2} - \frac{\bar{a}}{2m\omega} kk \\
-\frac{2a}{2a} - \frac{k_2kk'}{2m\omega} + \frac{\omega}{2} + \frac{\omega}{2} & \frac{\omega}{2} - \frac{\omega}{2} - \frac{\bar{a}}{2m\omega} kk & \frac{cc}{2m} - \frac{\omega}{2} - \frac{\bar{a}}{2m\omega} kk
\end{bmatrix} = 0 
\quad (13)
\]
The following characteristic equations are obtained according to equation (13)

\[\lambda^2 + \left(\frac{\ddot{c}c' + 2cc}{2m}\right)\lambda + \left(\frac{cc^2 + cc\ddot{c}c'}{4m}\right) - \frac{2kk + \ddot{a}k_0kk'}{4m} + \frac{cc^2\omega^2 + (kk - m\omega^2)^2 + \ddot{a}c_2c\omega^2c'}{4m^2\omega^2} + \ddot{a}k_0(kk - m\omega^2)kk' = 0\]  

(14)

where

\[
cc' = \frac{c_3\omega^2}{a^2\pi} \left(\Delta^2 + 2\Delta^3\right) \left[1 - \frac{\Delta^2}{a^2} - 3\Delta \phi_0\right],
\]

\[
kk' = \frac{\Delta k_3 \left(33 - 28\Delta^2 + 8\Delta^4\right) + 42ak_3\phi_0}{4\pi \left[1 - \frac{\Delta^2}{a^2}\right]}.
\]

Defining

\[P = \frac{\ddot{a}c_2cc' + 2cc}{2m}\]

and

\[Q = \frac{cc^2 + cc\ddot{c}c'}{4m} - \frac{2kk + \ddot{a}k_0kk'}{4m} + \frac{cc^2\omega^2 + (kk - m\omega^2)^2 + \ddot{a}c_2c\omega^2c' + \ddot{a}k_0(kk - m\omega^2)kk'}{4m^2\omega^2}\]

the stability conditions for the bifurcation response equation are as follows:

\[P > 0 \text{ and } Q > 0\]

**Bifurcation characteristic analysis**

The bifurcation response equation of the system is analyzed using singularity theory. The bifurcation response equation is transformed into the following

\[b_6a^6 + b_5a^5 + b_4a^4 + b_3a^3 + b_2a^2 + b_1a + b_0 = 0\]  

(16)

where

\[
b_0 = \frac{\Delta^2 k_3}{\pi^2} - B_0^2, \quad b_1 = \frac{\Delta^3 k_3}{\pi^2} \left(2k_1\pi - \Delta^2k_3 - 2m\pi \omega^2 + 2\pi K\omega'^{-1}\cos\frac{p\pi}{2}\right),
\]

\[
b_2 = k_1^2 + \omega^2(m^2\pi^2 + c_1^2 - 2k_1m) + \frac{1}{2} K\omega^{2p-2}(1 - \omega^2)(1 + \cos p\pi) - \frac{63\Delta^4 k_3^2}{4\pi^2},
\]

\[
+ \frac{1}{\pi} \left(\Delta^2 k_1k_3 + 6\Delta^4 k_3^2 + \Delta^2 k_3m\omega^2\right) + K_0\omega'^{-1}\cos\frac{p\pi}{2} \left(k_1 - m_1\omega^2 - \frac{\Delta^2 k_3}{\pi}\right) + c_1 K_0\omega'^{-1}\sin\frac{p\pi}{2} + K_0(3\pi - 8)(k_1 - m_\omega^2)\omega'^{-1}\cos\frac{p\pi}{2},
\]

\[
b_3 = \frac{\Delta k_3}{2\pi^2} \left[\Delta^2 (16k_3 + 3k_2 - 3\pi k_3) + 4K_0(3\pi - 8)(k_1 - m\omega^2)\omega'^{-1}\cos\frac{p\pi}{2}\right],
\]

\[
b_4 = \Delta^2 k_3 \left[\frac{64}{\pi} k_3 - \frac{3}{4} (k_2 + 65k_3) + 9k_3\right] + \frac{3}{2} (k_2 + k_3) \left(k_1 - m\omega^2\right) + \frac{3}{2} K_0(k_2 + k_3)\omega'^{-1}\sin\frac{p\pi}{2},
\]

\[
b_5 = \frac{3}{2} \Delta k_3(k_2 + k_3)(3\pi - 8), \quad b_6 = \frac{9}{16} (k_2 + k_3)^2
\]
Let \( a = y - \frac{b}{b_6} \). Then, equation (16) is transformed into the following form

\[
y^6 + B_4 y^4 + B_3 y^3 + B_2 y^2 + B_1 y + \mu = 0
\]

where

\[
B_1 = \frac{b_1}{b_6} + \frac{b_5 (b_5^4 - 6b_4 b_5^2 b_6 + 27b_3 b_5 b_6^2 - 108b_2 b_6^3)}{324b_6^5}
\]
\[
B_2 = \frac{b_2}{b_6} - \frac{b_5 (5b_5^3 - 24b_4 b_5 b_6 + 72b_3 b_6^2)}{144b_6^4}
\]
\[
B_3 = \frac{b_3}{b_6} + \frac{5b_5^3 - 18b_4 b_5 b_6}{27b_6^3}
\]
\[
B_4 = \frac{b_4}{b_6} - \frac{5b_5^2}{12b_6^2}
\]
\[
\mu = \frac{b_0}{b_6} + \frac{b_5 (1296b_5 b_6^3 - 216b_3 b_5^2 b_6^2 - 7776b_1 b_6^4 + 36b_4 b_5^3 b_6 - 5b_5^5)}{46656b_6^5}
\]

Singularity theory indicates that equation (17) is the universal unfolding of form \( y^6 + \mu = 0 \). The durability of this universal unfolding is thus analyzed. This analysis is conducted by constraining the parameters due to numerous constraints.

(1) Constrained parameters \( B_1 \) and \( B_4 \)
- Bifurcation point set, \( B_4^4 + \frac{256}{27} (B_2)^3 = 0 \)
- Hysteretic point set, \( \frac{4096b_6^3}{729} = B_4^4 \cup B_2 = 0 \)
- Double-limit point set, \( B_3^4 + \frac{256}{27} (B_2)^3 = 0 \)

The transition set and bifurcation diagram under constrained parameters \( B_1 \) and \( B_4 \) are shown in Figure 2.

(2) Constrained parameters \( B_2 \) and \( B_4 \)
- Hysteretic point set is

\[
\left\{ \left( \frac{5B_1}{9} \right)^3 = -\frac{B_5^5}{25} \right\} \cup \{ B_1 = 0 \}
\]
and double-limit point set is

\[ \frac{5}{27} B_1^3 = -\frac{4}{25} B_3^5 \]

The transition set and bifurcation diagram under constrained parameters \( B_2 \) and \( B_4 \) are shown in Figure 3.
(3) Constrained parameters $B_1$ and $B_2$
Hysteretic point set, \( \frac{64}{27} B_4^3 + \frac{B_2}{2} = 0, B_4 < 0 \)
Double-limit point set, \( \Phi \)
The transition set and bifurcation diagram under constrained parameters $B_1$ and $B_2$ are shown in Figure 4.

(4) Constrained parameters $B_3$ and $B_4$
Hysteretic point set, \( 3125B_1^2 + 512B_4^5 = 0 \)
Double-limit point set, \( \frac{3125}{4} B_1^2 + 27B_4^5 = 0 \)
The transition set and bifurcation diagram under constrained parameters $B_2$ and $B_3$ are shown in Figure 5.
(5) Constrained parameters $B_3$ and $B_4$

Hysteretic point set, $\frac{625}{100} B_1^4 + B_2 = 0, B_2 < 0$

Double-limit point set, $\frac{625}{100} B_1^4 + \frac{1}{2} B_2^2 = 0$

The transition set and bifurcation diagram under constrained parameters $B_3$ and $B_4$ are shown in Figure 6.

(6) Constrained parameters $B_1$ and $B_3$

Hysteretic point set, $\{3 B_2 - B_1^2 = 0, B_1 < 0\} \cup \{B_2 = 0\}$

Double-limit point set, $B_2 = \frac{1}{4} B_1^2$

The transition set and bifurcation diagram under constrained parameters $B_1$ and $B_3$ are shown in Figure 7.

Figure 7. Transition set and bifurcation diagram under constrained parameters $B_1$ and $B_3$.

Figure 8. Local bifurcation diagram of the system under varying fractional coefficient $K$. 

Figure 9. System (a) phase diagram, (b) displacement time history diagram, and (c) Poincare section under $K = 0.7$ (single-periodic motion).

Figure 10. System (a) phase diagram, (b) displacement time history diagram, and (c) Poincare section under $K = 0.74$ (double-periodic motion).
Figure 9. System (a) phase diagram, (b) displacement time history diagram, and (c) Poincare section under $K = -0.7$ (single-periodic motion).

Figure 10. System (a) phase diagram, (b) displacement time history diagram, and (c) Poincare section under $K = -0.74$ (double-periodic motion).
Numerical analysis

System bifurcation and path to chaos under the influence of fractional parameter $K$

The following system parameters are selected: $m = 1$, $c = 0.3$, $F = 0.5$, $k_1 = 0.5$, $k_2 = 0.5$, $k_3 = 0.1$, $\delta = 0.2$, and $p = 0.1$. Under different values of fractional parameter $K$, the local bifurcation diagram of the system is shown in Figure 8.

Figure 8 shows that the path for the system to chaos under varying fractional coefficient $K$ is as follows: chaotic motion $\rightarrow$ alternation of multiperiodic and chaotic motions $\rightarrow$ chaotic motion $\rightarrow$ from period-doubling motion to single-periodic state.

Figures 9–12 demonstrate that the system presents different motion states under different values of fractional $K$. Under $K = -0.7$, system phase diagram, displacement time history diagram, and Poincare section are shown in Figure 9. This figure reveals that the phase trajectory of the system is an ellipse and the Poincare section is a dot. Meanwhile, the displacement time history diagram shows that the system is under stable single-periodic motion. Under $K = -0.74$, the phase trajectory of the system is turned into two ellipses: the Poincare section is also turned into two dots, and the time history diagram also displays regular two-periodic motions. Therefore, the system is under a double-periodic motion state. Figure 11 shows that the phase trajectory of the system under $K = -0.752$ becomes four ellipses. The system is also under a four-periodic motion state based on the Poincare section and the time history diagram. When $K = -1$, the phase trajectories are intertwined without overlapping, the Poincare section is disorderly and unsystematic, and the displacement time history diagram also displays a disorderly state. These findings verify the chaotic motion state of the system.

Figure 11. System (a) phase diagram, (b) displacement time history diagram, and (c) Poincare section under $K = -0.752$ (four-periodic motion).
The following system parameters are selected: $m = 1$, $c = 0.3$, $F = 0.3$, $k_1 = 0.5$, $k_2 = 0.5$, $k_3 = 0.1$, $\delta = 0.2$, and $K = -0.6$. The local bifurcation diagram, which varies with fractional order $p$, is shown in Figure 13. Paroxysmal bifurcation and chaos are generated in the system with the change in fractional order $p$. As $p$ turns from 0 to 0.0045, the system enters a periodic motion state from the chaotic motion state; when $p$ becomes 0.007, the system again enters a chaotic motion state from periodic motion. Under $p = 0.01$, the system is

**Figure 12.** System (a) phase diagram, (b) displacement time history diagram, and (c) Poincare section under $K = -1$ (chaotic motion).

**Figure 13.** Local bifurcation diagram under varying fractional order $p$. 

**System bifurcation and path to chaos under the influence of fractional order $p$**

The following system parameters are selected: $m = 1$, $c = 0.3$, $F = 0.3$, $k_1 = 0.5$, $k_2 = 0.5$, $k_3 = 0.1$, $\delta = 0.2$, and $K = -0.6$. The local bifurcation diagram, which varies with fractional order $p$, is shown in Figure 13. Paroxysmal bifurcation and chaos are generated in the system with the change in fractional order $p$. As $p$ turns from 0 to 0.0045, the system enters a periodic motion state from the chaotic motion state; when $p$ becomes 0.007, the system again enters a chaotic motion state from periodic motion. Under $p = 0.01$, the system is
Figure 14. System (a) phase diagram, (b) displacement time history diagram, and (c) Poincare section under $p = -0.004$ (chaotic motion).

Figure 15. System (a) phase diagram, (b) displacement time history diagram, and (c) Poincare section under $p = -0.0045$ (four-periodic motion).
transformed from a chaotic motion state into a periodic motion state once again. With the change in fractional order $p$, this fractional non-smooth system presents alternate motion states between periodic and chaotic motions. The system demonstrates different motion states under different values of fractional order $p$. Figure 14 shows the phase diagram, displacement time history diagram, and Poincare section of the system under $p = 0.004$. The phase trajectory of the system comprises numerous ellipses, which are disorderly, irregular, and are not overlapping. Thus, the system is under a chaotic motion state. The phase diagram, displacement time history diagram, and Poincare section under $p = 0.0045$ are presented in Figure 15. The amplified phase diagram shows that the phase trajectory comprises four ellipses, and the Poincare section comprises four independent dots. Meanwhile, the system is under a four-periodic motion state according to the displacement time history diagram. When $p = 0.012$ (Figure 16), the amplified phase diagram indicates the transformation of the phase trajectory into five ellipses and the Poincare section into five independent dots. Thus, the system is under a five-periodic motion state from the displacement time history diagram.

Conclusions
The fractional non-smooth oscillator containing clearance constraints was taken as the study object, and the nonlinear kinetic equation of the system was established. The dynamic behaviors of the system, such as bifurcation and chaos, were also investigated through analytical and numerical methods. The main conclusions are presented as follows.

1. The system bifurcation equation was analyzed via singularity theory, and the transition set and bifurcation diagram of the system under six different constraint parameters are obtained. On each bifurcation diagram, the transition set divides the system into multiple sub-regions, and the bifurcation curves of different sub-regions represent different bifurcation characteristics of the system under different parameter conditions.
The bifurcation of the system in each region is persistent. On the dividing line, the bifurcation of the system is not persistent, and the bifurcation graphs determined by parameters in each subregion are topologically equivalent. Therefore, dynamic behaviors of the system can be changed by varying its parameters to provide a theoretical reference for the reasonable selection of system parameters.

(2) The numerical results show that besides the route to chaos through period-doubling with the change of fractional order \( p \), with the change of fractional-order coefficient \( K \), the system still exists Paroxysmal bifurcation, in which the periodic motion and chaotic motion of the system alternately appear. At the same time, the above bifurcation and chaos phenomena are further confirmed by the displacement time history diagram, phase portrait, and Poincare section. Hence, the dynamic behaviors of the fractional non-smooth system containing clearance constraints can be changed by varying fractional coefficient and order, thus a foundation for repressing oscillations of similar systems has been established.

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