On an Application of Higher Energies to Sidon Sets

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Abstract
We show that for any finite set \( A \) and an arbitrary \( \varepsilon > 0 \) there exists \( k = k(\varepsilon) \) such that the higher energy \( E_k(A) \) is at most \( |A|^{k+\varepsilon} \) unless \( A \) has a very specific structure. As an application we obtain that any finite subset \( A \) of the real numbers or the prime field either contains an additive Sidon-type subset of size \( |A|^{1/2+c} \) or a multiplicative Sidon-type subset of size \( |A|^{1/2+c} \).

Keywords Sidon sets · Higher energies · Additive combinatorics · Sum–product phenomenon

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1 Introduction
Sidon set is a classical object of Combinatorial Number theory that was introduced by Sidon [26]. A subset \( S \) of an abelian group \( G \) is a Sidon set if and only if all its non-zero differences are distinct. Being “more random than random” this interesting class of sets was extensively studied by various authors see, e.g., [1–10], [14–20]. A detailed survey about Sidon sets can be found [17].

Let \( \text{Sid}(A) \) be the size of a maximal (by cardinality) Sidon subset of a set \( A \subseteq G \). We specify the group operation by writing \( \text{Sid}^+(A) \) or \( \text{Sid}^-(A) \). In [12] (also, see paper [22]) it was proved that for any \( A \subseteq \mathbb{R} \) one has

\[
\text{Sid}(A) \geq c\sqrt{|A|},
\]

(1)

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where \( c > 0 \) is an absolute constant. Of course, this result is tight (just consider \( A \) being a segment of integers to see that \( \text{Sid}^+ (A) \ll \sqrt{|A|} \)). Oleksiy Klurman and Cosmin Pohoata asked in [11] the following sum–product-type question (see more on the sum–product phenomenon in [27, Section 8]): is it true that bound (1) can be improved either for \( \text{Sid}^+ (A) \) or for \( \text{Sid}^x (A) \), where \( A \) is any finite subset of the real numbers?

Let \( \text{Sid}^k (A) \) stand for the size of a maximal subset of \( A \) having at most \( k \) representations of any non-zero element as a difference. Thus, \( \text{Sid}^1 (A) = \text{Sid} (A) \) and hence (1) cannot be improved for the quantity \( \text{Sid}^k (A) \) (again consider \( A \) being a segment of integers). Our main result is the following.

**Theorem 1** Let \( A \subseteq \mathbb{F} \) be a set, where \( \mathbb{F} = \mathbb{R} \) or \( \mathbb{F} = \mathbb{F}_p \) (in the latter case suppose, in addition, that \( |A| < \sqrt{p} \)). Then there are some absolute constants \( c > 0, K \geq 1 \) such that

\[
\max\{\text{Sid}_K^+ (A), \text{Sid}_K^x (A)\} \gg |A|^{1/2 + c}.
\]

On the other hand, for any integer \( k \geq 1 \) there exists \( A \subseteq \mathbb{F} \) with

\[
\max\{\text{Sid}_k^+ (A), \text{Sid}_k^x (A)\} \ll k^{1/2} |A|^{2/3}.
\]

In [19] Oliver Roche–Newton and Audie Warren obtained a bound similar to (3) and another estimate of the same form that was obtained by Green–Peluse, see [11] (also, see [6, page 57]). Our construction is different from these counterexamples and we give our own proof of (3) at the end of Sect. 3 for the sake of completeness.

Actually, Theorem 1 is a consequence of a more general fact about the so-called higher energies, see [21] and Sect. 2 for all the required definitions. Theorem 2 below is interesting in its own right and could potentially find further applications in Additive Combinatorics.

**Theorem 2** Let \( A \subseteq G \) be a set, \( \delta, \varepsilon \in (0, 1] \) be parameters. Then there exists \( k = k(\delta, \varepsilon) \) such that either \( E_k (A) \leq |A|^{k + \delta} \) or there is a set \( H \subseteq G \), \( |H| \gg |A|^\delta (1 - \varepsilon) \), \( |H + H| \ll |A|^{\varepsilon} |H| \) and there exists \( Z \subseteq G \), \( |Z||H| \ll |A|^{1+\varepsilon} \) with \( |(H + Z) \cap A| \gg |A|^{1-\varepsilon} \).

In other words, one can always choose \( k \) large enough to make \( E_k (A) \) as small as possible unless the set \( A \) has a very rigid structure. It is easy to see (see [23, Theorem 22] for an explanation) that Theorem 2 is actually a criterion.

Finally, in the last section of the present paper we study \( k \)-Sidon sets (as well as its generalizations), that is, sets with elements having at most \( k \) representations of any non-zero element as a difference. This class of sets was introduced by Erdős [6] and it is strongly connected to the higher energies. We show that such sets are even more natural than the usual \( B_2 [g] \)-sets, see [17] for the definitions. In particular, the size of such sets can be estimated relatively easily (unlike to \( B_2 [g] \)-sets), they have heritability properties as well as a natural reinterpretation in terms of its Cayley graph and so on.
I would like to thank Oliver Roche–Newton who communicated the question of Oleksiy Klurman and Cosmin Pohoata to the author. Also, I thank him for very useful comments, discussions and remarks. Finally, I deeply thank all the reviewers for their valuable feedback while completing this manuscript.

2 Definitions

Let $G$ be an abelian group. We specify where it is necessary the group operation by writing $+$ or $\times$ in the considered quantities (such as the energy, the representation function and so on, see below). Let $\mathbb{F}$ be the field $\mathbb{R}$ or $\mathbb{F} = \mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$ for a prime $p$.

We use the same capital letter to denote a set $A \subseteq \mathbb{F}$ and its characteristic function $A : \mathbb{F} \to \{0, 1\}$. Given two sets $A, B \subset G$, define the sumset of $A$ and $B$ as

$$A + B := \{a + b : a \in A, b \in B\}.$$  

In a similar way we define the difference set and higher sumsets, e.g., $2A - A$ is $A + A - A$. We write $\oplus$ for a direct sum, i.e., $|A \oplus B| = |A||B|$. For an abelian group $G$ the Plünnecke–Ruzsa inequality (see, e.g., [27]) states

$$|nA - mA| \leq \left(\frac{|A + A|}{|A|}\right)^{n+m} |A|,$$

where $n, m$ are any positive integers. Throughout the paper we use the representation functions notations such as $r_{A+B}(x)$ or $r_{A-B}(x)$ and so on, which counts the number of ways $x \in G$ can be expressed as a sum $a + b$ or $a - b$ with $a \in A, b \in B$, respectively. For example, $|A| = r_{A-A}(0)$.

For any two sets $A, B \subseteq G$ the additive energy of $A$ and $B$ is defined as

$$E(A, B) = E^+(A, B) = |\{(a_1, a_2, b_1, b_2) \in A \times A \times B \times B : a_1 + b_1 = a_2 + b_2\}|.$$  

When $A = B$ we simply write $E(A)$ for $E(A, A)$. For $k \geq 2$ set

$$E_k(A) = \sum_x r_{A-A}^k(x) = \sum_{\alpha_1, \ldots, \alpha_{k-1}} |A \cap (A + \alpha_1) \cap \cdots \cap (A + \alpha_{k-1})|^2.$$  

Clearly, $|A|^k \leq E_k(A) \leq |A|^{k+1}$. We also write $\hat{E}_k(A) = \sum_x r_{A+A}^k(x)$. It was proved in [24, Lemma 13] that for any sets $A_1, \ldots, A_k, B_1, \ldots, B_k \subseteq G$ one has

$$\sum_x r_{A_1+B_1}(x) \ldots r_{A_k+B_k}(x) \leq \prod_{j=1}^k E_k^{1/2k}(A_j) E_k^{1/2k}(B_j).$$

The signs $\ll$ and $\gg$ are the usual Vinogradov symbols. When the constants in the signs depend on a parameter $M$, we write $\ll_M$ and $\gg_M$. All logarithms are to base 2.
For a set $A$, then we write $a \lesssim b$ or $b \gtrsim a$ if $a = O(b \cdot \log^c |A|)$, $c > 0$. Finally, let \([n]\) be the set \(\{1, 2, \ldots, n\}\) and $K_{s,t}$ denote the complete bipartite graph ($s$ and $t$ are sizes of its parts).

### 3 Proof of the Main Result

First, we present a probability lemma that was known before in the case $k = 2$, see \cite{1}. Roughly speaking, ignoring the presence of the parameter $k$ in $\text{Sid}_{O(k)}$ from formula (7) below, Lemma 3 works better than inequality (1) if $E_k(A) \leq |A|^{k + \omega}$ for a certain $\omega < 1/2$.

**Lemma 3** Let $A \subseteq G$ be a set. Then for any $k \geq 2$ one has

$$\text{Sid}_{3k-3}(A) \gg \left( \frac{|A|^{2k}}{E_k(A)} \right)^{1/(2k-1)} \quad \text{and} \quad \text{Sid}_{2k-2}(A) \gg \left( \frac{|A|^{2k}}{E_k(A)} \right)^{1/(2k-1)}.$$  

(7)

**Proof** Form a random set $A_\ast \subseteq A$ by picking elements of $A_\ast$ from $A$ independently at random with probability $q$. Then the expectation of the number of solutions to the equation

$$E_k'(A) := |\{x_1 - x'_1 = \cdots = x_k - x'_k : x_j, x'_j \in A, \ x_j, x'_j \text{ are different}\}|$$  

(8)

is $q^{2k} E_k'(A) \leq q^{2k} E_k(A)$. If $q^{2k} E_k(A) \leq q|A|/2$, we can delete roughly half of the elements of $A_\ast$ to find a subset $B$ of $A_\ast$, $|B| \gg |A_\ast|$ that has no solutions to equation (8). We next consider an arbitrary $z \neq 0$ and show that $r_{B-B}(z) < 3k - 2 := g$.

Suppose that for some $x_j, x'_j \in B$ the following holds

$$0 \neq z = x_1 - x'_1 = \cdots = x_g - x'_g.$$  

(9)

Clearly, any $x_j$ or $x'_j$ presents in (9) in at most two equations and hence a pair $(x_j, x'_j)$ presents in at most three equations. Also, by the definition of the set $B$ any $k$ equations from (9) must have some equal variables. It follows that $r_{B-B}(z) \leq 3 \cdot (k - 1)$ as required. Finally, taking $q = (|A|E_k^{-1}(A)/2)^{1/(2k-1)}$, we obtain our result. The second bound in (7) can be obtained similarly. In this case the corresponding analogue of (9) is

$$x_1 + x'_1 = \cdots = x_g + x'_g$$  

(10)

and it can be clearly seen that a pair $(x_j, x'_j)$ presents in at most two equations from (10). Again by the definition of the set $B$ any $k$ equations from (10) must have some equal variables. It follows that $r_{B+B}(z) \leq 2k - 2$ as required. \(\square\)
Next we establish our driving tool, namely, Theorem 4 (Theorem 2 from the introduction is a particular case of it). The proof is in the spirit of [23, Theorem 21].

**Theorem 4** Let \( A \subseteq G \) be a set, \( \delta, \varepsilon \in (0, 1) \) be some parameters with \( \varepsilon \leq \delta \).

1. Then there is \( k = k(\delta, \varepsilon) = \exp(O(\varepsilon^{-1} \log(1/\delta))) \) such that either \( E_k(A) \leq |A|^{k+\delta} \)
   or there is \( H \subseteq G, |H| \gtrsim |A|^\delta(1-\varepsilon), |H+H| \ll |A|^\varepsilon |H| \) and there exists \( Z \subseteq G, |Z||H| \ll |A|^{1+\varepsilon} \) with

   \[
   |(H+Z) \cap A| \gg |A|^{1-\varepsilon}.
   \]

2. Similarly, either there is a set \( A' \subseteq A, |A'| \gg |A|^{1-\varepsilon} \) and \( P \subseteq G, |P| \gtrsim |A|^\delta \)
   such that for all \( x \in A' \) one has \( r_{A-P}(x) \gg |P||A|^{-\varepsilon} \) or \( E_k(A) \leq |A|^{k+\delta} \) with \( k \ll 1/\varepsilon \).

**Proof** Let \( E_l = E_l(A) := |A|^{l+k_l} \), where \( k_l \in [0, 1] \) and \( l \geq 2 \) be any integer satisfying \( k_l \geq \delta \). By the pigeon-hole principle there is a number \( \Delta > 0 \) and a set \( P \subseteq G \) such that \( P = \{ x : \Delta \leq r_{A-A}(x) \leq 2\Delta \} \) and \( E_{l+1} \gtrsim \Delta^{l+1}|P| \). In particular,

\[
|P| \gtrsim E_{l+1}|A|^{-(l+1)} = |A|^{k_l+1} \gg |A|^\delta.
\]

Let \( M \) be a parameter that we set later in an appropriate way in each case 1 and 2 of Theorem 4. Suppose that

\[
E_{l+1} \geq \frac{|A|E_l}{M}.
\]

Then

\[
\frac{|A|\Delta|P|}{M} \leq \frac{|A|E_l}{M} \leq E_{l+1} \lesssim \Delta^{l+1}|P|.
\]

Hence

\[
\frac{|A||P|}{M} \lesssim \sum_{x \in P} r_{A-A}(x) = \sum_{a \in A} |A \cap (P+a)|,
\]

and the inequality \( \Delta \gtrsim |A|/M \) follows from (13). Consider \( A' = \{ a \in A : |A \cap (P+a)| \gtrsim |P|/CM \} \) with a sufficiently large constant \( C \). Then from (14), we obtain \( |A'| \gtrsim |A|/M \) and by the definition of the set \( A' \), we have \( r_{A-A}(x) \gtrsim |P|/M \) for all \( x \in A' \). To prove the second claim of the theorem we set \( M = |A|^\varepsilon/2 \) and hence we are done under assumption (12). Now suppose that \( E_{l+1} \leq |A|E_{l-1}/M \). We then apply the previous argument to \( E_l \) instead of \( E_{l+1} \). Again, if \( E_l \gtrsim |A|E_{l-1}/M \), then we are done, otherwise \( E_l \leq |A|E_{l-1}/M \) and one repeats the argument. Clearly, after at most \( k \ll 1/\varepsilon \) steps our algorithm terminates with \( E_k(A) \leq |A|^{k+\delta} \).
It remains to obtain the first claim. Returning to (14) and using the Hölder inequality several times, we get

\[
\frac{|A||P|^2}{M^2} \lesssim E(A, P) = \sum_x r_{A-A}(x)r_{P-P}(x),
\]

and hence

\[
\left(\frac{|A||P|^2}{M^2}\right)^{l+1} \lesssim E_{l+1}\left(\sum_x r_{P-P}^{1+1/l}(x)\right)^l \lesssim |P|^l \cdot E(P)|P|^{2l-2}.
\]

In other words,

\[
E(P) \gtrsim \frac{|P|^3}{M^{2l+2}} : = \frac{|P|^3}{M_*}.
\]

By the Balog-Szemerédi-Gowers Theorem (see [27]), we find \( H \subseteq P, |H| \gg |P|/M_*^C, |H + H| \ll M_*^C |H| \), where \( C > 0 \) is an absolute constant that can change from line to line. As above

\[
|A||H|/M \lesssim |H| \Delta \ll \sum_x r_{A-A}(x) = \sum_a |A \cap (H + a)|.
\]

Again, we define \( W \subseteq A \) similarly to the set \( A' \) and obtain in particular, that \( |W| \gtrsim |A|/M. \) Also, let \( Z \subseteq W \) be a set such that the sets \( \{H + z\}_{z \in Z} \) form a maximal system of disjoint sets. By maximality, \( W \subseteq Z + H - H \) and hence by the Plünnecke inequality (4), we get

\[
|A|/M \lesssim |W| \leq |Z||H - H| \ll M_*^C |Z||H|
\]

and thus

\[
|(H + Z) \cap A| = \sum_{z \in Z} |A \cap (H + z)| \gtrsim |Z||H|/M \gtrsim |A|/M_*^C.
\]

In particular,

\[
M|A| \gtrsim |Z||H|.
\]

We now put \( M = |A|^{\varepsilon/2} \), where \( C_* > 0 \) is a sufficiently large constant. Hence \( |H + H| \ll |A|^\varepsilon |H| \) and \( |(H + Z) \cap A| \gg |A|^{1-\varepsilon} \). Also, in view of (11) one has

\[
|H| \gg |P|/M_*^C \gtrsim |A|^{\delta} / M_*^C \gtrsim |A|^{\delta(1-\varepsilon)}.
\]
as required. Now suppose that inequality (12) fails, i.e., $E_{l+1} \leq |A|E_l/M$. It gives us $\kappa_{l+1} \leq \kappa_l(1 - \varepsilon^{1 - \delta})$ and on iterating that, we see that our algorithm stops after at most $\exp(O(e^{-1} \log(1/\delta)))$ steps. This completes the proof. \hfill $\square$

**Remark 5** Notice that Lemma 3 as well as the first part of Theorem 4 work for both sums and differences. Thus the first part of Theorem 1 can be obtained for the quantity $\text{Sid}^+_k$ defined with sums instead of differences.

As a consequence of Theorem 4 we present Corollary 6 below, which is a straightforward result with no the sum–product flavour. Roughly speaking, it says that basic estimate (1) can be easily improved for a wide class of sets once the presence of the parameter $K$ in (15) is ignored.

**Corollary 6** Let $A \subseteq G$ be a set, $\delta \in (0, 1/2)$ be a parameter. Then either for some $K = K(\delta), c = c(\delta)$ we have

$$\text{Sid}^+_K(A) \gg |A|^{1/2 + c}$$

(15)

or there is a set of shifts $T, |T| \gg |A|^{\delta/8}$ such that for any $t \in T$ one has $|A \cap (A + t)| \gg |A|^{1-\delta}$.

**Proof** We apply the first part of Theorem 4 with $\delta = \delta/2$ and $\varepsilon = \delta/16$. If the first alternative holds, then we are done in view of Lemma 3 as $E^+_k(A) \leq |A|^{k+\delta}$ implies (15). Otherwise there is a set $H \subseteq G$, $|H| \gtrsim |A|^{\delta(1-\varepsilon)/2}$, $|H + H| \ll |A|^\delta |H|$ and there exists $Z \subseteq G$, $|Z||H| \ll |A|^{1+\varepsilon}$ with $|(H + Z) \cap A| \gg |A|^{1-\varepsilon}$. For any $z \in Z$ put $A_z = A \cap (H + z) \subseteq A$. Applying the Cauchy–Schwarz inequality several times, we find

$$\sum_x r_{A-A}(x)r_{H-H}(x) = E(A, H) \geq \sum_{z \in Z} E(A_z, H) \gg \sum_{z \in Z} \frac{|A_z|^2 |H|^2}{|A|^\delta |H|}$$

(16)

$$\gg (|H||Z|)^{-1}|A|^{2-3\delta} |H|^2 \gg |A|^{1-4\delta} |H|^2.$$ 

On setting $T = \{t \in G : |A \cap (A + t)| \gg |A|^{1-4\delta}\}$, we obtain from (16) that

$$|T| \gg |H||A|^{-4\delta} \gg |A|^{\delta/8}.$$ 

This completes the proof. \hfill $\square$

Now we are ready to obtain bound (2) of Theorem 1. We provide even two proofs using both statements of our driving Theorem 4.

**Proof** Take any $\delta < 1/2$, e.g., $\delta = 1/4$ and let $\varepsilon \leq \delta/4$ be a parameter that is to be set later. In view of Lemma 3 we see that $E^+_k(A) \leq |A|^{k+\delta}$ implies

$$\text{Sid}^+_k(A) \gg |A|^{1/2 + \frac{1-\delta}{4k-1}} = |A|^{1/2 + \frac{1}{4k+1}},$$

(17)
hence we are done (here \( k = k(\epsilon) \)). Otherwise there is a set \( H \subseteq \mathbb{F}, |H| \gtrsim |A|^{\delta(1-\epsilon)} \) \( \geq |A|^\delta/2, |H + H| \ll |A|^{\delta/2} |H| \) and there exists \( Z \subseteq \mathbb{F}, |Z||H| \ll |A|^{1+\epsilon} \) such that \( |(H + Z) \cap A| \gg |A|^{1-\epsilon} \). Define \( A_\ast = (H + Z) \cap A, |A_\ast| \gg |A|^{1-\epsilon} \). We would like to estimate \( E_{l+1}^\times(A_\ast) \) or \( \hat{E}_{l+1}^\times(A_\ast) \) for large \( l \). Once a good upper bound for \( E_{l+1}^\times(A_\ast) \) or \( \hat{E}_{l+1}^\times(A_\ast) \) is obtained, we apply Lemma 3 again to find large multiplicative Sidon subset of \( A_\ast \).

First of all, notice that \( |A_\ast + H| \leq |H + H||Z| \ll |A|^\delta |H||Z| \ll |A|^{1+2\epsilon} \). In other words, the set \( A_\ast \) almost does not grow after the summation with \( H \). Let \( Q = A_\ast + H \), \(|Q| \ll |A|^{1+2\epsilon} \). Secondly, fix any \( \lambda \neq 0 \). Then the number of the solutions to the equation \( a_1a_2 = \lambda, \) where \( a_1, a_2 \in A_\ast \) does not exceed

\[
\sigma_\lambda := |H|^{-2} |[h_1, h_2 \in H, q_1, q_2 \in Q : (h_1 - q_1)(h_2 - q_2) = \lambda]|.
\]

The last equation can be interpreted as a question about the number of incidences between points and modular hyperbolas. Namely, by [25, Theorem 22] for any sets \( A, B, C, D \subseteq \mathbb{F} \) with \(|B||C| \geq (|A||D|)^{\epsilon} \) and \( \lambda \in \mathbb{F}^\times \) one has

\[
|\{(a, b, c, d) \in A \times B \times C \times D : (a + b)(c + d) = \lambda\}| \lesssim \frac{|A||B||C||D|}{|\mathbb{F}|} \lesssim (|A||D|)^{1/2} (|B||C|)^{1-\kappa_\ast(\zeta)}, \tag{18}
\]

where \( \kappa_\ast(\zeta) > 0 \) depends on \( \log(|B||C|)/\log(|A||B|) \) and hence it is a function of \( \zeta \) only. Hence for each non-zero \( \lambda \) the quantity \( \sigma_\lambda \) can be bounded as

\[
\sigma_\lambda \lesssim |H|^{-2} |Q||H|^{2-\kappa} \ll |A|^{1+2\epsilon} |H|^{-\kappa},
\]

(see [15] in the case \( \mathbb{F} = \mathbb{R} \) and formula (18) for both fields with the convention \( |\mathbb{R}| = \infty \)). Here \( \kappa = \kappa(\delta) > 0 \). Recalling that \( |H| \gg |A|^{\delta/2}, |A_\ast| \gg |A|^{1-\epsilon} \) and taking any \( \epsilon \leq \delta \kappa/100 \), we obtain after some calculations that \( \sigma_\lambda \ll |A_\ast|^{1-\delta \kappa/4} \).

Hence choosing sufficiently large \( l \gg (\delta \kappa)^{-1} \), we derive

\[
\hat{E}_{l+1}^\times(A_\ast) = \sum_{\lambda} r_{A_\ast A_\ast}(\lambda) \ll |A_\ast|^{l+1} + |A_\ast|^{1-\delta \kappa/2} |A_\ast|^2 \ll |A_\ast|^{l+1} + |A|^{l+2-\delta kl/2} \ll |A_\ast|^{l+1}.
\]

Applying Lemma 3 and choosing \( \epsilon \ll l^{-1} \), one readily sees that

\[
\hat{\text{Sid}}_{2l}^\times(A) \geq \hat{\text{Sid}}_{2l}^\times(A_\ast) \gg |A_\ast|^{l+1+2\epsilon} \gg |A|^{-\epsilon l(l+1)/2} = |A|^{1/2 + \frac{1-2\epsilon(l+1)}{2(l+1)}}, \gg |A|^{1/2+c},
\]

where \( c = c(\delta) > 0 \) is an absolute constant.

Now we present the second proof via applying the last part of Theorem 4. The argument is essentially the same, however we estimate \( \hat{E}_{l+1}^\times(A') \). Again, the number of solutions to the equation \( a_1a_2 = \lambda \) with \( a_1, a_2 \in A' \) does not exceed

\[
\sigma_\lambda := (|P||A|^{-\epsilon})^{-2} \cdot |\{(a_1, a_2 \in A, p_1, p_2 \in P : (a_1 - p_1)(a_2 - p_2) = \lambda)|.
\]
On applying results of [15], [25] (see formula (18)) one has

\[ \sigma_{\lambda} \ll \left( |P||A|^{-\varepsilon} \right)^{-2} \cdot |A||P|^{2-\kappa} \ll |A|^{1+2\varepsilon}|P|^{-\kappa}, \]

where \( \kappa = \kappa(\delta) > 0 \). Again \( |P| \gg |A|^{\delta} \) and we can use the arguments as above. This concludes the proof. \( \square \)

**Remark 7** Let us talk a bit about the dependence of \( K \) on \( c \). The main loss is in formula (18), where the dependence has the form \( \kappa_*(\zeta) = \exp(O(1/\zeta)) \). Despite of the fact that this exponential loss is typical for the sum–product phenomenon, the author thinks that in this concrete problem it is possible to avoid such a big loss. More precisely, one can show that for our small absolute constant \( c > 0 \) one has \( K = O(1/c) \).

To complete the proof of Theorem 1 we need upper bounds for the sizes of Sidon sets in sumsets.

**Lemma 8** Let \( A \subseteq G \) be a set, \( A = B + C \), and \( k \geq 1 \) be an integer. Then

\[ \text{Sid}_k(A) \leq \min\{|C|\sqrt{k}|B| + |B|, |B|\sqrt{k}|C| + |C|\}. \]

More generally, if \( r_{B+C}(a) \geq \sigma \) for any \( a \in A \), then

\[ \text{Sid}_k(A) \leq \sigma^{-1}\min\{|C|\sqrt{k}|B| + |B|, |B|\sqrt{k}|C| + |C|\}. \]

**Proof** We give two proofs. The first proof uses graph theory and provides a clear view on how \( \text{Sid}_k \)-sets are naturally connected with Cayley graphs.

Let \( \Lambda \) be an arbitrary subset of \( A \) such that \( r_{\Lambda-A}(x) \leq k \) for any \( x \neq 0 \). Consider the graph \( G = G(V, E) \), where \( V \) is the disjoint union of \( B \) and \( C \), the edge \( (b, c) \in E \) if and only if \( b + c \in \Lambda \). Moreover, ignoring those elements of \( \Lambda \) that have several representations as \( b + c \) we assume that \( |E| = |\Lambda| \). Using the Cauchy–Schwarz inequality, we obtain

\[ |\Lambda|^2 \leq |B| \sum_{b \in B} \sum_{c, c' \in C} E(b, c) E(b, c') \]

\[ = |B| \sum_{c, c' \in C} \sum_{b \in B} E(b, c) E(b, c') = |B||\Lambda| + |B| \sum_{c, c' \in C, c \neq c'} \sum_{b \in B} E(b, c) E(b, c'). \]

If the last sum over \( b \) is at least \( k + 1 \), then we find a complete bipartite subgraph \( K_{2,k+1} \) in \( G \) and hence there are different elements \( \lambda_1, \lambda'_1, \ldots, \lambda_{k+1}, \lambda'_{k+1} \in \Lambda \) such that \( \lambda_1 - \lambda'_1 = \cdots = \lambda_{k+1} - \lambda'_{k+1} \). The last fact contradicts the assumption that \( r_{\Lambda-A}(x) \leq k \) for any \( x \neq 0 \). Hence

\[ |\Lambda|^2 \leq |B||\Lambda| + k|B||C|^2 \]

as required.
To obtain the second part of our lemma we use a slightly different method. Again, let \( \Lambda \) be the set as before. Then by the Cauchy–Schwarz inequality, we get

\[
(\sigma |\Lambda|)^2 \leq S^2 := \left( \sum_{x \in \Lambda} r_{B+C}(x) \right)^2 = \left( \sum_{b \in B} r_{\Lambda-C}(b) \right)^2
\]

\[
\leq |B| \sum_{x, y \in \Lambda} |B \cap (x - C) \cap (y - C)|
\]

\[
\leq |B| |S + |B| \sum_{x \neq y \in \Lambda} |(x - C) \cap (y - C)| \leq |B| |S + k|B||C|^2.
\]

This completes the proof. \( \square \)

Now we can easily obtain a non-trivial upper bound for size of maximal Sidon set in any difference set or sumset.

**Corollary 9** Let \( A \subseteq G \) be a set and \( D = A - A, S = A + A \). Then for any positive integer \( k \) one has \( \text{Sid}_k(D) \ll \sqrt{k} \min\{|A|^{3/2}, |D|^3|A|^{-1}\} \) and

\[
\text{Sid}_k(S) \ll \sqrt{k} \min\{|A|^{3/2}, |A|^{-1} \min\{|D|\sqrt{|S|} + |S|, |S|\sqrt{|D|} + |D|\}\}.
\]

**Proof** The bound \( \text{Sid}_k(D), \text{Sid}_k(S) \ll \sqrt{k}|A|^{3/2} \) follows immediately from the first part of Lemma 8. Further, it is easy to see (one can consult [21]) that for any \( d \in D \) one has \( r_{D-d}(d) \geq |A| \). Applying the second part of Lemma 8 with \( A = B = C = D \) and \( \sigma = |A| \), we obtain \( \text{Sid}_k(D) \ll \sqrt{k}|D|^3|A|^{-1} \). To prove the second part of our lemma notice that for any \( s \in S \) the following holds \( r_{D+S}(s) \geq |A| \). This concludes the proof. \( \square \)

To complete the proof of Theorem 1 we have to obtain upper bound (3). In the case of \( \mathbb{F} = \mathbb{R} \) we put \( B = \Gamma, C = H\Gamma \), where \( \Gamma = \{1, g, \ldots, g^n\}, g \geq 2 \) is an integer, \( \bar{\Gamma} = \{g^{-n}, \ldots, g^{-1}, 1, g, \ldots, g^n\}, H = \{g^{n+1}, g^{2(n+1)}, \ldots, g^{n(n+1)}\} \). Then \( A = B + C = \Gamma + H\Gamma \) is contained in \( \Gamma (1 + \bar{\Gamma}) \) and in view of Lemma 8 any additive/multiplicative \( k \)-Sidon subset of \( A \) has the size of at most \( O(\sqrt{k}|\Gamma|^2) = O(\sqrt{k}|A|^{2/3}) \) because as one can easily see \( |A| = |\Gamma|^3 \). Similarly, in the case \( \mathbb{F} = \mathbb{F}_p \) we apply Lemma 8 with \( B = \Gamma, C = H\Gamma \), where \( \Gamma \leq \mathbb{F}_p^*, H \subseteq \mathbb{F}_p^*/\Gamma \) and \( |H| = |\Gamma| \) is sufficiently small relatively to \( p \). Then \( A := B + C = \Gamma (1 + \Gamma H) \) and hence by Lemma 8 any additive/multiplicative \( k \)-Sidon subset of \( A \) has the size of at most \( 2\sqrt{k}|\Gamma|^2 \). To obtain the required bound \( |A| = |\Gamma + H\Gamma| \gg |\Gamma|^2 |H| = |\Gamma|^3 \) for an appropriate \( H \) one can use the random choice (we leave the details to the interested reader).

### 4 On \( B_2^g[g] \)-Sets

In the previous section we have obtained some results about the family of sets \( S \) with

\[
r_{S-S}(x) \leq g, \quad \forall x \in G, \quad x \neq 0.
\]
This class of sets were introduced by Erdős in [6, page 57] (see as well [7]). We denote this family as $B_2^{(k)}$ [Erdős used the notation $B^{(k)}_2$]. The following was written in [6]:

V.T. Sós and I considered $B_2^{(k)}$ sequences... We could not decide whether there is a $B_2^{(k)}$ sequence which is not the union of a finite number of Sidon sequences.

According to the author’s knowledge this paper of Erdős and Sós was not published. The question from [6] is a nice problem of Erdős that is open and if it has a negative answer, then the original question of Klurman–Pohoata would be closed thanks to our Theorem 1. Let us underline it one more time that it is possible to construct sets $S$ with bounded $r_{S+S}(x)$, which are not the union of a finite number of Sidon sequences (see [1, 6]). It seems like condition (19) has another nature and that is why we devote this section studying some further properties of $B_2^{(g)}$-sets. To see that this family is truly special, notice that, for example, the condition $r_{S-S}(x) \ll 1$, $x \neq 0$ has an interpretation in terms of the Cayley graph of $S$ but $r_{S+S}(x) \ll 1$ cannot be expressed in terms of any Cayley graph.

First of all, notice that if $S$ is a random subset of $[N]$, which was obtained by picking elements from $[N]$ independently at random with probability $q \sim N^{-1/2}$, then one has $r_{S-S}(x) \ll \log N$ with probability $1-o(1)$, see, e.g., [5, Lemma 4.3] and a similar lower bound for the function $r_{S-S}(x)$ takes place. Thus for any fixed $g$ subsets of $B_2^{(g)}$ are far from being random.

Secondly, as it was noted in the proof of Lemma 8 a set $S$ belongs to the family $B_2^{(g)}$ if and only if its Cayley graph Cay$(A, G)$ has no complete bipartite subgraphs $K_{2,g+1}$. Recall that the vertex set of Cay$(A, G)$ is $G$ and $(x, y)$ is an edge of Cay$(A, G)$ iff $x - y \in A$. Another equivalent interpretation of (19) is

$$|S \cap (S+x_1) \cap \cdots \cap (S+x_g)| \leq 1 \quad \text{for all distinct and non-zero } x_1, \ldots, x_g \in G.$$  

(20)

This reinterpretation of $B_2^{(g)}$-sets says that the considered family is naturally connected with the higher energies [21] (see as well the second formula in definition (5)).

Also, on writing for an arbitrary $A \subseteq G$

$$\Delta_g(A) := \{(a, \ldots, a) \in A^g : a \in A\} \subseteq G^g,$$

we readily see that (20) means that the sum of $S^g$ and $\Delta_g(S)$ is direct. In other words, $S^g$ and $\Delta_g(S)$ form a co-Sidon pair, see [5] (indeed, if $(s_1, \ldots, s_g) + (s, \ldots, s) = (s'_1, \ldots, s'_g) + (s', \ldots, s')$, then $s_1 - s'_1 = \cdots = s_g - s'_g = s' - s = x \neq 0$ and hence $r_{S-S}(x) \geq g + 1$, another direction of that inequality can be obtained in a similar way). Further, formula (20) suggests the following definition for $B_2^{(g)}$ sets

$$|S \cap (S+x_1) \cap \cdots \cap (S+x_g)| < k \quad \text{for all distinct and non-zero } x_1, \ldots, x_g \in G.$$  

(21)

Similarly, $S \in B_2^{(g)}$ if and only if Cay$(S, G)$ does not contain $K_{k,g+1}$ or, in other words, $S$ does not contain any sumsets $X + Y$, where $|X| = g + 1$, $Y = k$. Since the
number of edges in $\text{Cay}(S, G)$ equals $|S||G|$ for any finite group $G$, it follows that to estimate size of $S$ it is enough to bound the number of edges in $K_{k,g+1}$-free graphs. Such results are discussed in detail for example in survey [9]. Of course there is a direct approach to estimating the cardinality of $S \in B_2^o[g]$. Namely, if $S \in B_2^o[g]$ belongs to a group $G$ of size $N$, then, obviously,

$$|S|^2 = \sum_x r_{S-S}(x) \leq |S| + g(N - 1)$$

and hence

$$|S| < \sqrt{gN + 1}. \quad (22)$$

Similarly, if $S \in B_k^o[g]$ and $S$ belongs to a group $G$ of size $N$, then

$$|S|^{g+1} = \sum_{x_1, \ldots, x_g} |S \cap (S + x_1) \cap \cdots \cap (S + x_g)| < kN^g + \left(\frac{g+1}{2}\right)|S|^g$$

and thus $|S| < k^{\frac{1}{g+1}} N^{\frac{g}{g+1}} + (\frac{g+1}{2}).$ Of course in the case when $S \subseteq [N]$ there are no such good upper bounds for the size of $S$ even if $S$ is a classical Sidon set. Nevertheless, we easily obtain a generalization of Linström’s result [14] for $B_2^o[g]$-sets in the segment (as well as for $B_k^o[g]$-sets but it is not the main topic of our paper, see [18] for better bounds).

**Proposition 10** Let $S \subseteq [N]$ belong to the family $B_2^o[g]$. Then

$$|S| < \sqrt{gN} + (gN)^{1/4} + 1. \quad (23)$$

More generally, if $S \subseteq B_k^o[g]$, then

$$|S| < k^{\frac{1}{g+1}} N^{\frac{g}{g+1}} + \left(\frac{g+1}{2}\right)^{\frac{1}{g+1}} k^{\frac{g}{(g+1)^2}} N^{\frac{g^2}{(g+1)^2}} + 1. \quad (24)$$

**Proof** We apply the technique of [10], which is a modern form of the classical argument [8]. Let $u = \lceil N^{3/4} g^{-1/4} \rceil$ be a parameter and $I = [u]$. Embed $S$ into $\mathbb{Z}/(N + u)\mathbb{Z}$. It is easy to check that for any $x \in [-u, u] \setminus \{0\}$ one has $r_{S-S}(x) \leq g$, where now $x$ runs over $\mathbb{Z}/(N + u)\mathbb{Z}$. On applying the Cauchy–Schwarz inequality to estimate the common energy $E(S, I)$, we get

$$\frac{|S|^2 u^2}{N + u} \leq E(S, I) = \sum_x r_{S-S}(x)r_{I-I}(x) < |S|u + gu^2.$$
or, in other words,

\[ gu^2 + u(gN + |S| - |S|^2) + |S|N > 0. \]

Substituting \( u = [N^{3/4}g^{-1/4}] < N^{3/4}g^{-1/4} \) and \(|S| = \sqrt{gN} + (gN)^{1/4} + C \), we find after some calculations that the condition \( C \leq 1 \) is enough. To obtain (24) we use a similar argument (with another parameter \( u \), of course) to estimate an analogue of the higher common energy of \( S \) and \( I \), namely,

\[
\frac{(|S|u)^{g+1}}{(N + u)^g} \leq \sum_x r_{S-I}^{g+1}(x) = \sum_{x_1, \ldots, x_g} |S \cap (S + x_1) \cap \cdots \cap (S + x_g)| |I \cap (I + x_1) \cap \cdots \cap (I + x_g)| < ku^{g+1} + \left( \frac{g+1}{2} \right) \sum_x r_{S-I}^{g}(x) \leq ku^{g+1} + \left( \frac{g+1}{2} \right) |S|^g u := ku^{g+1} + C_g |S|^g u.
\]

We have used in (25) that all the variables \( x_j \) belong to \([-u, u]\). Finally, in the case of \( B^2_k[g] \)-sets an appropriate choice of the parameter \( u \) is \( u = \lfloor C_g^{1/(g+1)}k^{-1/(g+1)^2}N^{1-g/(g+1)} \rfloor \) and after some calculations we obtain (24). We estimated the third term in the formula above roughly by one and actually. We also did not aim at optimizing the constant in the middle term of (24) and took the one that is simplest for checking. This completes the proof. □

As for the lower bounds on the size of maximal subsets of \( B^2_k[g] \), again one can consult [9] (see as well [13, 18]) to find the corresponding lower bound for the number of edges in graphs having no \( K_{s,t} \). Our graphs must be Cayley graphs and such constructions are known for \( K_{2,2}, K_{3,3} \) (Brown’s construction, see [2]) and for \( K_{s,t}, t \geq s!+1 \) (so-called, norm-graphs, see [9]). As for \( K_{2,t}, t > 2 \) one can obtain a result similar to [4, Theorem 1.6]. Namely, define for any \( g \) the quantity

\[ \alpha_g = \limsup_{N \to \infty} \max\{|S| : S \subseteq \mathbb{Z}/N\mathbb{Z}, S \in B^2_k[g]\}/\sqrt{N}. \]

**Theorem 11** We have

\[ \alpha_g = \sqrt{g} + O(g^{3/10}). \]

**Proof** Actually, our argument almost coincides with the approach of the proof [4, Theorem 1.6], so we omit some details.

By the method of [4, section 4] it is enough to construct a set \( A \subseteq (\mathbb{Z}/p\mathbb{Z})^2 \) (here \( p \) is a prime number) such that \( A \in B^2_k[g] \) with \( g = k^2 + O(k^{3/2}) \) and \(|A| = kp + O(k)\). We put \( A = \bigcup_{u \in U} A_u \), where \( U \) is an appropriate arithmetic progression, \( U = t + [k] \)
and for any non-zero \( u \in \mathbb{Z}/p\mathbb{Z} \) we define
\[
A_u = \{(x, x^2/u) : x \in \mathbb{Z}/p\mathbb{Z}\} \subset (\mathbb{Z}/p\mathbb{Z})^2.
\]

Clearly, \(|A| = kp - k + 1\). Let \( r_{u,v}(x) = r_{A_u,A_v}(x) \). Our task is to estimate for any \( x \) the sum
\[
r_{A-A}(x) \leq \sum_{u,v \in U} r_{u,v}(x).
\]

By [4, Lemma 3.2] one has \( r_{u,v}(x) + r_{u',v'}(x) = 2 \), provided that \( u + v = u' + v' \) and \( \left(\frac{uu'v'v}{p}\right) = -1 \). Using this lemma and acting exactly as on pages 2794–2795 of [4], we find
\[
r_{A-A}(x) \leq k^2 + \sum_{|l|<k} \left| \sum_{i+j=k+1+l} \left( \frac{(t+i)(t+j)}{p} \right) \right|.
\]

After that taking the summation over \( t \) (to find an appropriate \( U \)) and applying the Cauchy–Schwarz inequality and Weil’s bound for the sum of Legendre symbols, we obtain the required estimate (see the rest of the argument from [4]). This completes the proof. \( \square \)

We continue this section by considering heritability properties of \( B^g_2 \)-sets. For different \( x_1, \ldots, x_s \) and any set \( A \subseteq G \) put \( A_X := (A + x_1) \cap \cdots \cap (A + x_s) \), where 
\( X = \{x_1, \ldots, x_s\} \). Taking the same set \( X_1 = \cdots = X_l = X \) in inequality (26) below, one can see that any \( B^g_2 \)-set generates \( B^g_2 \)-sets with a smaller \( g \). In particular, in Brown’s construction [2] of the set \( S \in B^2_2[2] \), this set \( S \) is a (almost disjoint) union of classical Sidon sets \( S_w = S \cap (S + w), w \in (S - S) \setminus \{0\} \).

**Proposition 12** Let \( S \in B^g_2 \), \( l \geq 2 \) and take any sets \( X_1, \ldots, X_l \) with \(|X_1| + \cdots + |X_l| \geq g + \binom{l}{2} + 1 \) and such that any \((X_i, X_j)\) forms a co-Sidon pair. Then
\[
|S_{X_1} \cap (S_{X_2} + z_1) \cap \cdots \cap (S_{X_l} + z_{l-1})| < k \quad \text{for any different} \quad z_1, \ldots, z_{l-1} \neq 0.
\]

(26)

In particular, if \( S \in B^2_2[2] \) and \( G \) has no elements of order two, then for any non-zero \( w \) the set \( S_w = S \cap (S + w) \) is a Sidon set.

**Proof** Let \( X_j = \{x_1^{(j)}, \ldots, x_{s_j}^{(j)}\}, j \in [l] \). For any different nonzero \( z_1, \ldots, z_{l-1} \in G \), we have
\[
S := S_{X_1} \cap (S_{X_2} + z_1) \cap \cdots \cap (S_{X_l} + z_{l-1})
\]
\[
= (S + x_1^{(1)}) \cap \cdots \cap (S + x_1^{(1)}) \cap (S + x_1^{(2)} + z_1) \cap \cdots \cap (S + x_2^{(2)} + z_1) \cap \cdots \cap (S + x_l^{(l)} + z_{l-1}) \cap \cdots \cap (S + x_l^{(l)} + z_{l-1}).
\]
Since \( \sum_{j=1}^{l} s_j \geq g + \binom{l}{2} + 1 \), then either \( |S'| < k \) by (20), (21) or we have for some indices \( x_i^{(j+1)} + z_i = x_j^{(j+1)} + z_j \) and \( x_i^{(j+1)} + z_i = x_j^{(j+1)} + z_j \). Here we put \( z_0 = 0 \).

The last alternative implies that \( x_i^{(j+1)} - x_j^{(j+1)} = x_j^{(j+1)} - x_j^{(j+1)} \) but \( X_{j+1} \) and \( X_{j+1} \) form a co-Sidon pair by the assumption and hence this is impossible.

In the case \( S_w = S \cap (S + w) \) our ground set \( X \) is \( \{0, w\} \) and it is easy to see that this is a Sidon set. This completes the proof. \( \square \)

In Proposition 12 we assume that each \((X_j, X_j)\) forms a co-Sidon pair. Again (and this is in the spirit of this section) one can make more general assumptions on the intersections of some shifts of the sets \( X_j \) as in (20), (21) to obtain higher order Sidon sets.

We finish this section by discussing the tightness of Lemma 3. In the next proposition we show that any set \( A \) contains rather large (in terms of its energy) subset with controllable size of the maximal \( B_k^g \) subset.

**Proposition 13** Let \( A \subseteq G \) be a set. Then there is \( A_* \subseteq A \) such that \( E_{g+1}(A_*) \gg_g E_{g+1}(A) \) and any \( B_k^g \) subset of \( A_* \) has size \( O_g(k^{1/(g+1)}|A|^{2}E_{g+1}^{-1/(g+1)}(A)) \).

In particular, for any \( n \) one has \( \text{Sid}_n(A_*) \ll n^{1/2}|A|^{2}E^{-1/2}(A) \).

**Proof** We have

\[
E_{g+1}(A) = \sum_{y} r_{A-A}^{A}(y) = \sum_{a \in A} \sum_{x \in A} r_{A-A}(x-a).
\]

Put \( A_* = \{a \in A : \sum_{x \in A} r_{A-A}(x-a) \geq E_{g+1}(A)/(2|A|)\} \). Then

\[
E_{g+1}(A) \leq 2 \sum_{a \in A_*} \sum_{x \in A} r_{A-A}(x-a) = 2 \sum_{y} r_{A-A}^{A}(y)r_{A-A_*}(y)
\]

and using the Hölder inequality several times (or just applying estimate (6)), we obtain

\[
E_{g+1}(A) \leq 2^{g+1} \sum_{y} r_{A-A_*}^{A}(y) \leq 2^{g+1}(E_{g+1}(A_*)E_{g+1}^{2}(A))^{1/(2g+2)} \tag{27}
\]

or, in other words, \( E_{g+1}(A_*) \gg_g E_{g+1}(A) \).

Now let \( \Lambda \) be any \( B_k^g \) subset of \( A_* \). Then by the definition of the set \( A_* \), one has

\[
2^{-1} \Lambda |E_{g+1}(A)|^{-1} \leq \sum_{y} r_{A-A}^{A}(y)r_{A-A}(y). \tag{28}
\]

Hence as in (27) we obtain

\[
2^{-(g+1)} \Lambda |E_{g+1}(A)|^{-(g+1)} \leq \sum_{y} r_{A-A}^{A}(y)
\]

\[
= \sum_{x_1, \ldots, x_g} |A \cap (A + x_1) \cap \cdots \cap (A + x_g)| \Lambda \cap (A + x_1) \cap \cdots \cap (A + x_g)|.
\]
Applying the fact that $\Lambda \in B_k^g[\mathbb{g}]$, we get

$$2^{-(g+1)}|\Lambda|^{g+1}E_{g+1}(A)|A|^{-(g+1)} \leq k|A|^{g+1} + \left(\frac{g+1}{2}\right)|A||\Lambda|^g.$$  

On noticing that the second term in the last formula is negligible, the required result follows. \(\square\)

**Remark 14** As can be seen from (28) the conclusion of Proposition 13 remains true for a wider family of sets $\Lambda$, namely, for $\Lambda$ with $E_{g+1}(\Lambda) \ll |\Lambda|^{g+1}$. This class of sets and its connection with Sidon sets were discussed in [19, Section 3.2].

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