SPECTRA FOR COMPACT QUANTUM GROUP COACTIONS
AND CROSSED PRODUCTS

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Abstract. We present definitions of both the Connes spectrum and the strong
Connes spectrum for actions of compact quantum groups on $C^*$-algebras and
obtain necessary and sufficient conditions for a crossed product to be a prime
or a simple $C^*$-algebra. Our results extend to the case of compact quantum ac-
tions, namely the results of Gootman, Lazar and Peligrad, which in turn, gen-
eralize results by Connes, Olesen, Pedersen and Kishimoto for abelian group
actions. We prove in addition that the Connes spectra are closed under tensor
products. These results are new for compact nonabelian groups as well.

1. Introduction

In his fundamental paper [4], Connes defines the invariant $\Gamma$ called, in his name,
the Connes spectrum, for abelian group actions on von Neumann algebras. Among
other results, he obtained conditions for a crossed product to be a factor. Subse-
quently, Olesen and Pedersen [11] have defined the Connes spectrum for abelian
group actions on $C^*$-algebras. They have extended the results of Connes to the case
of crossed products of $C^*$-algebras by abelian group actions, obtaining conditions
that such a crossed product be a prime $C^*$-algebra. However, the similar result
for simple crossed products using the Olesen-Pedersen version of the Connes spec-
trum is false. In [9], Kishimoto has shown that the result is true for simple crossed
products if his “strong Connes spectrum” is used instead of Olesen and Pedersen’s
Connes spectrum. Rieffel [15] and Landstad [10] have put the problem of finding
a “good” definition of the Connes spectrum for compact, not necessarily abelian,
group actions on $C^*$-algebras. In [10], Landstad remarks that a “good” definition
of the Connes spectrum should lead to a result that generalizes the Olesen-Pedersen
characterization of prime crossed products to the case of nonabelian compact group
actions. Gootman, Lazar, and Peligrad [8] have defined the Connes spectrum and
the strong Connes spectrum for compact, not necessarily abelian, group actions on
$C^*$-algebras. In the case of abelian groups, these notions coincide with the previous
ones. Moreover, in [8], Gootman, Lazar, and Peligrad prove the characterizations of
the primeness and simplicity of crossed products using their definitions. In this
paper we present definitions of both the Connes spectrum (Definition 3.1) and the
strong Connes spectrum (Definition 4.1) in the case of compact quantum groups
and prove the corresponding characterizations of primeness and simplicity of crossed
products (Theorems 3.4 and 4.4). In addition, we prove that the Connes spectra
are closed under tensor products (Propositions 5.4 and 5.6). This result is new
for nonabelian compact groups as well. We will use the techniques developed by
Woronowicz [16, 17], Boca [2], and the authors in [5, 7]. In addition, this paper contains new methods for the study of the hereditary C*-subalgebras that are invariant under a compact quantum group coaction (Section 3) and for the proof of the key Lemma 2.2.

2. Preliminaries

Let $G = (A, \Delta)$ be a compact quantum group (see [17]) and let $(B, G, \delta)$ be a quantum dynamical system, where $B$ is a C*-algebra and $\delta$ is a coaction of $A$ on $B$ (see [2] or [14]). Denote by $\hat{G}$ the set of all equivalence classes of irreducible representations of $G$ ([16], section 4).

For each $\alpha \in \hat{G}$, $u^\alpha = \sum_{i,j=1}^{d_{\alpha}} m_{ij}^\alpha \otimes u_{ij}^\alpha$ denotes a representative of each class. Then the linear space generated by $\{u_{ij}^\alpha | \alpha \in \hat{G}, 1 \leq i, j \leq d_{\alpha}\}$ is a $\ast$-algebra $A$, called the Woronowicz-Hopf algebra ([17], Section 5). For $\alpha \in \hat{G}$ and $u^\alpha \in A$ a unitary representative, denote $\overline{u^\alpha} = \sum_{i,j=1}^{d_{\alpha}} m_{ij}^\alpha \otimes u_{ij}^\alpha$. Then $\overline{u^\alpha}$ is a (not necessarily unitary) representation of $G$ called the adjoint of $u^\alpha$. We will denote by $\overline{\alpha}$ the equivalence class of $\overline{u^\alpha}$.

Set $\chi_\alpha = \sum_{i=1}^{d_{\alpha}} u_{i1}^\alpha$. We use $F_\alpha$ to denote the unique positive, invertible operator in $B(H_\alpha)$ that intertwines $u^\alpha$ with its double contragradient representation $(u^\alpha)^{\ast c}$ such that $tr(F_\alpha) = tr(F_\alpha^{-1})$. Set $M_\alpha = tr(F_\alpha) ([16], Theorem 5.4).

Since every positive matrix is unitarily equivalent to a diagonal matrix, we may assume that the matrices $F_\alpha$ are diagonal: $F_\alpha = \text{diag}\{f_1^\alpha, \dots, f_{d_{\alpha}}^\alpha\}$. The formula $f_i^\alpha(u_{mn}^\alpha) = \delta_{nm} f_i^\alpha$ defines a linear form on $A$. The above assumption implies that all $u_{ij}^\alpha$ are mutually orthogonal in $H_\alpha$ and therefore

\begin{align}
(1) \quad h(u_{ij}^\alpha u_{mn}^\alpha) &= \frac{1}{M_\alpha} \frac{1}{f_i^\alpha} \delta_{im} \delta_{jn} \\
(2) \quad h(u_{mk}^\alpha u_{nl}^\alpha) &= \frac{f_l^\alpha}{M_\alpha} \delta_{mn} \delta_{lk},
\end{align}

where $h$ is the Haar state on $G$ and $\delta_{rs}$ are the Kronecker $\delta$'s ([16], Theorem 5.7).

For each $\alpha \in \hat{G}$, denote by $B^\alpha_\delta$ the associated spectral subspace defined by (see [7] or [2]):

$$B^\alpha_\delta = \{P_\alpha(x) \mid x \in B\},$$

where $P_\alpha(x) = (\iota \otimes h_\alpha) (\delta(x))$ and $h_\alpha = M_\alpha h \cdot (\chi_\alpha \ast f_1^\alpha)^\ast$. Recall that for all $a, b \in A$, $h \cdot a(b) = h(ba)$ and for all linear functionals $\xi$ on $A$, $a \ast \xi = (\xi \otimes \iota)(\Delta(a))$ (see [17], relation 1.14, or [5]). In particular, for $\alpha = \iota$, $P_\iota$ is the projection of $B$ onto the fixed point algebra $B^\iota$.

Let $c_{ij} = M_\alpha(u_{ij}^\alpha \ast f_1^\alpha)^\ast$. Note that, since $F_\alpha$ is a diagonal matrix we obtain $u_{ij}^\alpha \ast f_1^\alpha = f_i^\alpha u_{ij}^\alpha$ and hence $c_{ij} = M_\alpha f_i^\alpha u_{ij}^\alpha$.

Define the mapping $P_{ij}^{\alpha} : B \rightarrow B$ by

$$P_{ij}^{\alpha}(x) = (id \otimes h \cdot c_{ij}^\alpha)(\delta(x)),$$

for all $x \in B$. Note that $P_{ij}^{\alpha} P_{kl}^{\alpha} = \delta_{ik} P_{kj}^{\alpha}$.

For each $\alpha \in \hat{G}$, define

$$B_2^\alpha(u^\alpha) = \{[P_{ij}^{\alpha}(x)]_{ij} \mid x \in B\} \subseteq B \otimes M_{d_\alpha},$$
where \([P_{ij}^\alpha(x)]_{ij} = \sum_{i,j=1}^{d_\alpha} P_{ij}^\alpha(x) \otimes m_{ij}^\alpha\) with \(\{m_{ij}^\alpha\}_{1 \leq i, j \leq d_\alpha}\) the set of elementary matrices in the algebra \(\mathcal{M}_{d_\alpha}\) of scalar matrices of order \(d_\alpha \times d_\alpha\).

Notice that \(B_2^\delta(u^\alpha)\) depends on the representative \(u^\alpha\), not only on the equivalence class \(\alpha \in \hat{G}\). However, for two equivalent representations \(u_1^\alpha\) and \(u_2^\alpha\), the corresponding \(B_2^\delta\) are spatially isomorphic.

The proofs of the following remarks are straightforward from the definitions.

**Remark 2.1.**

1. If \(u^\alpha\) is an irreducible unitary representation of \(G\), then \[\delta(P_{ij}^\alpha(x)) = \sum_{k=1}^{d_\alpha} P_{ik}^\alpha(x) \otimes u_{kj}^\alpha.\]

2. \(P_{ij}^\alpha = \text{linspan}\{P_{ij}^\alpha(x) | x \in B, i, j = 1, 2, \ldots, d_\alpha\}.\)

3. For \(x \in B\), let \(X = [P_{ij}^\alpha(x)]_{ij} = \sum_{i,j=1}^{d_\alpha} P_{ij}^\alpha(x) \otimes m_{ij}^\alpha\). Then \(X \in B_2^\delta(u^\alpha)\) and \(\delta_{13}(X) = (X \otimes 1_A)(1_B \otimes u^\alpha)\), where \(1_A\) is the unit of \(A\) and \(1_B\) is the unit of the multiplier algebra of \(B\). The leg numbering notation used here is the standard one [14] and [16]. Also, \(B_2^\delta(u^\alpha)\) is isomorphic as a Banach space to \(B_\alpha^H\) through the mapping \(X \to \sum_{i=1}^{d_\alpha} P_{ii}^\alpha(x)\). Therefore:

\[B_2^\delta(u^\alpha) = \{X \in B \otimes \mathcal{M}_{d_\alpha} | \delta_{13}(X) = (X \otimes 1_A)(1_B \otimes u^\alpha)\}.\]

4. Let \(x \in B\) and fix \(x_{i_0,j_0} = P_{i_0,j_0}^\alpha(x)\). Then \(x_{i_0,j_0} \in B\) and \([P_{ij}^\alpha(x_{i_0,j_0})]_{ij} \in B_2^\delta(u^\alpha)\) is a matrix whose only nonzero row is the \(j_0\)-row and whose \(j_0\)-entry is given by \(P_{i_0,j}^\alpha(x)\), for each \(j = 1, 2, \ldots, d_\alpha\). Furthermore,

\[B_2^\delta(u^\alpha) = \text{linspan}\{[P_{ij}^\alpha(x_{rs})]_{ij} | r, s = 1, 2, \ldots, d_\alpha\}.\]

With \(\alpha \in \hat{G}\) and \(v\) the right regular representation of \(G\), we will use the following notation (see [14], relation 3.2 and [5]):

\[\mathcal{F}_v(a) = (id \otimes ha)(v^*).\]

In particular, for \(a = (\chi_\alpha * f_\alpha^*\) we denote

\[p_\alpha = \mathcal{F}_v((\chi_\alpha * f_\alpha^*))^* = (id \otimes h_\alpha)(v^*).\]

Denote by \(\tilde{A}\) the norm closure of the set of all operators of the form \(\mathcal{F}_v(a)\), where \(a \in A\).

Recall that the crossed product \(B \times_\delta G\) is defined to be the \(C^*\)-algebra generated by all elements of the form \((\pi_u \times \pi_b)(\delta(b))(1 \otimes \mathcal{F}_v(a))\), where \(a \in A\), \(b \in B\), \(\pi_u : B \to B(H_u)\) is the universal representation of the \(C^*\)-algebra \(B\) and \(\pi_h : A \to B(H_h)\) is the GNS representation of \(A\) associated to the Haar state \(h\).

Furthermore, if \(\alpha_1, \alpha_2 \in \hat{G}\), define

\[S_{\alpha_1,\alpha_2} = (1 \otimes p_{\pi_\alpha})(B \times_\delta G)(1 \otimes p_{\pi_\alpha}),\]

\[S_\alpha S_\alpha = S_\alpha.\]

For properties of \(S_{\alpha_1,\alpha_2}\), see [5], Lemma 3.1 and Proposition 3.2. It is straightforward to check that

\[S_\alpha S_\alpha = \text{linspan}\{(1 \otimes p_{\pi_\alpha})\delta(b^*)(1 \otimes p_\delta) | b \in B_\alpha\}.\]
Lemma 2.2. Q discussed above. We will use this result in Section 3.

Let the map \( \Psi : (H \otimes \xi_h) \) considered in Section 2.3, Lemma 2.10 to the case of compact quantum groups. The proof uses the matricial representation of the elements of \( S_{\alpha,t} \) discussed above. We will use this result in Section 4.

Lemma 2.2. \( Q(S_{\pi,\tau} S_{\tau,\pi}) = \Psi(B^2_\alpha(u^\alpha) \ast B^2_\alpha(u^\alpha)) \).

Proof. Let \( b \in B \) and \( b_{ij} = P^\alpha_{ij}(b) \). Then, if \( \eta \in H_u \) and \( \xi_h \) is the cyclic vector in \( H_h \), for \( i_0, j_0 = 1, 2, ..., d_a \) we have:

\[
(1 \otimes p_a) \delta(h_{i_0 j_0}^* \otimes (1 \otimes p_i)) (\eta \otimes \xi_h) = (1 \otimes p_a) (\sum_{l=1}^{d_a} b_{i_0 l}^* \otimes u_{j_0 \ast}^\alpha) (1 \otimes p_i) (\eta \otimes \xi_h)
\]

\[
= \sum_{l=1}^{d_a} (b_{i_0 l}^* \otimes (p_a u_{j_0 \ast}^\alpha p_i)) (\eta \otimes \xi_h)
\]

\[
= \sum_{l=1}^{d_a} (b_{i_0 l}^* \otimes \mathcal{F}(a^\alpha)) u_{j_0 \ast}^\alpha \mathcal{F}(1) (\eta \otimes \xi_h)
\]

\[
= \sum_{l=1}^{d_a} (b_{i_0 l}^* \otimes (a^\alpha h \ast u_{j_0 \ast}^\alpha)) (\eta \otimes \xi_h)
\]

\[
= \sum_{l,r,m,n} (b_{i_0 l}^* \otimes M_a f_1(u^\alpha_{nm}) u_{ir}^\ast h(u_{rj_0}^\ast u_{mn}^\alpha)) (\eta \otimes \xi_h)
\]

\[
= \sum_{l,r,m,n} M_a f_1(u^\alpha_{nm}) h(u_{rj_0}^\ast u_{mn}^\alpha) (b_{i_0 l}^* \otimes u_{ir}^\ast) (\eta \otimes \xi_h)
\]

\[
= \sum_{l,r,m,n} \delta_{j_0 \ast} f_1(u^\alpha_{nm}) f_{-1} (u^\alpha_{mr}) (b_{i_0 l}^* \otimes u_{ir}^\ast) (\eta \otimes \xi_h)
\]

\[
= \sum_{l=1}^{d_a} b_{i_0 l}^* \otimes u_{j_0 \ast}^\alpha \xi_h.
\]
Consequently, the matrix of \((1 \otimes p_α)\delta(b_{iαj0})(1 \otimes p_α)\) viewed as an operator from \(H_a \otimes C_b\) to \(H_a \otimes p_aH_b\) is the \(d_a^2 \times 1\) column matrix whose entry \([(k - 1)d α + j_0] \times 1\) is \(b_{iαk}\) and all the other entries are 0. Now let \(c \in B\) and \(c_{rασ0} = P_{rασ0}^α(c)\). Then, similarly, \((1 \otimes p_α)\delta(c_{rασ0})(1 \otimes p_α)\) can be represented by a \(1 \times d_b^2\) row matrix whose entry 1 \times [(k - 1)d α + s_0] is \(c_{rαk}\) and all the other entries are 0.

Therefore, the product \((1 \otimes p_α)\delta(b_{iαj0})(1 \otimes p_α)\delta(c_{rασ0})(1 \otimes p_α)\) is represented by a \(d_a^2 \times d_b^2\) matrix \(X\), partitioned into \(d_a^2\) blocks \(X_{ij}\), where each block \(X_{ij}\) has the entry \(j_0s_0\) equal to \(b_{iαj}c_{rαj0} \otimes m_{ij} \otimes m_{j_0s_0}\).

Hence \(X = \sum_{i,j} b_{iαj}^*c_{rαj0} \otimes m_{ij} \otimes m_{j_0s_0}\).

On the other hand, by \([6]\), proof of Proposition 9), \(v(p_α \otimes 1) = \sum I_{da} \otimes u^α = \sum_{p,q} I_{da} \otimes m_{pq} \otimes u_{pq}^α\). Hence:

\[
v_{23}(X \otimes 1)v_{23}^* = \sum_{i,j,p,q,k,l} b_{ij}^*c_{rαj0} \otimes m_{ij} \otimes m_{pq}m_{j_0s_0}m_{kl} \otimes u_{pq}^αu_{kl}^α
\]

Applying \(id \otimes h\) to the above expression and using Formula 2 above, we get:

\[
Q(X) = (\sum_{i,j,p,k} b_{iαj}^*c_{rαj0} \otimes m_{ij} \otimes m_{pk})h(u_{p0}^αu_{0ks0}^α)
\]

\[
= \frac{1}{M_{α}} f_1(u_{j_0s_0}) \sum_{i,j,p,k} δ_{pk}b_{iαj}^*c_{rαj0} \otimes m_{ij} \otimes m_{pk}
\]

\[
= \frac{1}{M_{α}} \sum_{i,j,p} b_{iαj}^*c_{rαj0} \otimes m_{ij} \otimes m_{pp}
\]

\[
= \frac{1}{M_{α}} \sum_{i,j} b_{iαj}^*c_{rαj0} \otimes m_{ij} \otimes I_{da}.
\]

Hence, if \(j_0 = s_0\), we have:

\[
Q(X) = c\sum_{i,j} b_{iαj}^*c_{rαj0} \otimes m_{ij} \otimes I_{da},
\]

where \(c = \frac{f_{α0}^α}{M_{α}} > 0\).

But this is exactly \(Ψ(M^αN)\), where \(M ∈ B_2^α(u^α)\) is the matrix whose \(j_0\) row is \([c_{rαj}]\) and the other entries are 0 and \(N ∈ B_2^α(u^α)\) is the matrix whose \(s_0\) row is \([c_{rαj}]\) and the other entries are 0. If \(j_0 \neq s_0\), then \(Q(X) = 0\) but, as can be easily checked, also \(M^αN = 0\) and \(Ψ(M^αN) = 0\).

Now let \((B, G, δ)\) be a quantum dynamical system. We say that a \(C^*\)-subalgebra \(C \subset B\) is \(δ\)-invariant if the following two conditions hold:

1. \(δ(C) ⊆ C \otimes A\),
2. \(δ(C)(1 \otimes A) = C \otimes A\).

In other words, \(C\) is called \(δ\)-invariant if the restriction of \(δ\) to \(C\) is a coaction. The set of all hereditary, \(δ\)-invariant \(C^*\)-subalgebras of \(B\) will be denoted by \(H_δ(B)\).
A $C^*$-algebra $B$ is called $G$-prime if the product of two nonzero $\delta$-invariant ideals is nonzero.

A $C^*$-algebra $B$ is called $G$-simple if $B$ has no nontrivial $\delta$-invariant two-sided ideals.

We will need the following remarks. Their proofs are straightforward.

**Remark 2.3.** (1) $\mathcal{S}_\iota = B^\delta \otimes 1$.

(2) Using the proof of Proposition 3.2 in [5], one can show that for $a_0, a_1 \in B^\delta$ and $\alpha \in \hat{G}$, then $a_1 B^\delta a_0 = (0)$ if and only if $(a_0 \otimes 1) S_{\alpha, \iota} (a_1 \otimes 1) = (0)$.

(3) If $C \in \mathcal{H}^\delta(B)$, then $C \times_\delta G$ is a hereditary subalgebra of $B \times_\delta G$.

(4) If $J \subseteq B^\delta$ is a two-sided ideal, then $D = JBJ \in \mathcal{H}^\delta(B)$.

The next lemma and its corollary will be used in Section 4.

**Lemma 2.4.** Let $\alpha \in \hat{G}$. Then $S^\iota = \overline{linspan\{S_{\alpha, \beta} S_{\beta, \iota} S_{\iota, \alpha} : \beta \in \hat{G}\}}$. 

**Proof.** Since $\sum p_\beta = 1$ in the strict topology of $\hat{A}$ we have 

$$(1 \otimes p_\alpha)(B \times_\delta G)(1 \otimes p_\alpha) = (1 \otimes p_\alpha)(B \times_\delta G) \sum_{\beta \in \hat{G}} (1 \otimes p_\beta)(B \times_\delta G)(1 \otimes p_\alpha),$$

and the claim follows. \qed

**Corollary 2.5.** Let $J \subseteq B^\delta$ be a two-sided ideal. Then $C = BJB \in \mathcal{H}^\delta(B)$ and 

$C^\delta \otimes 1 = \overline{linspan\{S_{\iota, \beta} (J \otimes 1) S_{\beta, \iota} : \beta \in \hat{G}\}}$. 

**Proof.** Clearly, $\delta(C) \subseteq C \otimes A$, since $\delta(J) = J \otimes 1$. The fact that $\delta(C)(1 \otimes A)$ is dense in $C \otimes A$ follows from the definition of $C$.

The equality $C^\delta \otimes 1 = \overline{linspan\{S_{\iota, \beta} (J \otimes 1) S_{\beta, \iota} : \beta \in \hat{G}\}}$ follows from Lemma \[2.4\] and Remark \[2.3\] (1). \qed

### 3. Connes spectrum and prime crossed products

A notion of a spectrum of an action $\delta$ of a compact group $G$ on a $C^*$-algebra $B$ was given in [8] by Gootman, Lazar, and Peligrad. They used the spectral subspaces $B^\alpha_{2}(\alpha)$ to define the Arveson and Connes spectra and proved that the conjugate $\tau$ belongs to the Arveson spectrum $Sp(\delta)$ if and only if the closure of the ideal $S_{\tau, \iota} S_{\iota, \alpha}$ is essential in $S_{\alpha}$ ([8], Proposition 1.3). We are going to use this correspondence rather than the direct definition given in [8] to define the spectra for coactions of a compact quantum group on a $C^*$-algebra $B$.

**Definition 3.1.**

(1) $Sp(\delta) = \{\alpha \in \hat{G} | S_{\tau, \iota} S_{\iota, \tau} S_{\iota, \alpha} \text{ is an essential ideal of } S_{\iota, \alpha}\}$.

(2) $\Gamma(\delta) = \bigcap_{C \in \mathcal{H}^\delta(B)} Sp(\delta|C)$.

The connection to the definition in [8] is made by the following lemma.

**Lemma 3.2.** Let $\alpha \in \hat{G}$. Then $\alpha \in Sp(\delta)$ if and only if $B^\delta_{2}(u^\alpha)^* B^\delta_{2}(u^\alpha)$ is an essential ideal of $(B \otimes \mathcal{M}_{\tau, \alpha}(C))^\delta_{\iota, \alpha}$. 

Proof. Let $\alpha \in Sp(\delta)$ and assume to the contrary that $B^3_\alpha(u^\alpha)^*B^3_\alpha(u^\alpha)$ is not an essential ideal of $(B \otimes M_{d_{\alpha}}(C))^\delta\alpha$.

Using Lemma 2.2, there exists a positive, nonzero element $c \in \mathcal{I}_\tau$, such that $cP(S_{\tau,1}S_{\tau})c = 0$. Since, in particular, $c \in S_{\tau}$, then $P(c(S_{\tau,1}S_{\tau}))c = 0$. The faithfulness of $P$ now implies that $c(S_{\tau,1}S_{\tau})c = 0$, which is a contradiction with $\alpha \in Sp(\delta)$.

Conversely, assume that $B^3_\alpha(u^\alpha)^*B^3_\alpha(u^\alpha)$ is an essential ideal of $(B \otimes M_{d_{\alpha}}(C))^\delta\alpha$. By Lemma 2.2, $P(S_{\tau,1}S_{\tau})$ is an essential ideal in $\mathcal{I}_\tau$. Using the same lemma,

$$P(S_{\tau,1}S_{\tau}) = \mathcal{I}_\tau \cap (S_{\tau,1}S_{\tau}).$$

By Remark 3.5 in [3], $S_{\tau}$ is isomorphic to $\mathcal{I}(\tau) \otimes \mathcal{I}_{\tau}$, where $\mathcal{I}(\tau) = \widehat{\mathcal{P}_\tau}$. It is easy to check that the image of $\mathcal{I}_{\tau} \subset S_{\tau}$ by this isomorphism is $\chi_{\tau} \otimes \{\text{diag}(x, x, \ldots, x)\} | x \in \mathcal{I}_{\tau}$, where $\text{diag}(x, x, \ldots, x)$ is the $d_{\alpha} \times d_{\alpha}$ matrix with all the diagonal elements equal to $x$ and all the others equal to 0. Thus $\{\text{diag}(y, y, \ldots, y) | y \in \mathcal{I}_{\tau} \cap S_{\tau,1}S_{\tau}\}$ is essential in $\{\text{diag}(x, x, \ldots, x) | x \in \mathcal{I}_{\tau}\}$. This implies that $S_{\tau,1}S_{\tau}$ is essential in $S_{\tau}$. \[\square\]

Proposition 3.3. If $B$ is $G$-prime and $\Gamma(\delta) = \hat{G}$, then $B^\delta$ is prime.

Proof. Assume, to the contrary, that $B^\delta$ is not prime. Then there exist two non-zero positive elements $a_0, a_1 \in B^\delta$ such that $a_1B^\delta a_0 = (0)$. Since $B$ is $G$-prime, $a_1B_{\alpha}a_0 \neq (0)$. On the other hand, since $B$ is the closure of the linear span of its spectral subspaces $B^\alpha|\alpha \in \hat{G}$, then there exists $a_0 \in \hat{G}$ such that

$$a_1B_{\alpha}a_0 \neq (0).$$

Since $a_1B^\delta a_0 = (0)$, Remark 2.3(1) above implies that

$$(1 \otimes p_\alpha)((a_1 \otimes 1)(B \times \delta G))(1 \otimes p_\alpha)(B \times \delta G)(a_0 \otimes 1)(1 \otimes p_\alpha) = (0),$$

that is,

$$(a_1 \otimes 1)S_{\tau,\tau}S_{\tau,\tau}(a_0 \otimes 1) = (0).$$

Therefore, since $S_{\tau,\tau}(a_0 \otimes 1)S_{\tau,\tau} \subset S_{\tau,\tau}S_{\tau,\tau}$, then

$$(a_1 \otimes 1)S_{\tau,\tau}(a_0 \otimes 1)S_{\tau,\tau}(a_0 \otimes 1) = (0).$$

Multiply the above equation to the left by $(a_0 \otimes 1)S_{\tau,\tau}(a_1 \otimes 1)$ and to the right by $(a_0 \otimes 1)S_{\tau,\tau}(a_0 \otimes 1)$. We get:

$$(a_0 \otimes 1)S_{\tau,\tau}(a_1^2 \otimes 1)S_{\tau,\tau}(a_0 \otimes 1)S_{\tau,\tau}(a_0 \otimes 1)(a_0 \otimes 1) = (0).$$

$$(6) \quad [(a_0 \otimes 1)S_{\tau,\tau}(a_1^2 \otimes 1)S_{\tau,\tau}(a_0 \otimes 1)](a_0 \otimes 1)S_{\tau,\tau}(a_0 \otimes 1) = (0).$$

Let $C = a_0Ba_{\alpha}$. Clearly, $C \in \mathcal{H}(B)$. The second factor on the left-hand side of equation (6) is just $S_{\tau,\tau}^\alpha S_{\tau,\tau}^\beta$, where $S_{\alpha,\beta}^\tau$ denotes the corresponding subspace of the crossed product $C \times \delta G$.

Since $\Gamma(\delta) = \hat{G}$, then $a_0 \in \Gamma(\delta)$. Therefore $S_{\tau,\tau}^\alpha S_{\tau,\tau}^\beta$ is an essential ideal of $S_{\tau,\tau}^\alpha$. Since the first factor in equation (6) is included in $S_{\tau,\tau}^\alpha$, it follows that it equals (0). In particular,

$$(a_0 \otimes 1)S_{\tau,\tau}(a_1 \otimes 1) = (0).$$

Using Remark 2.3(2), this means that $a_1B_{\alpha}a_0 = (0)$, which is a contradiction to relation (6). \[\square\]
Next we will prove the main result of this section. The result is a generalization of [8], Theorem 2.2.

**Theorem 3.4.** The following are equivalent:

1. $B \times_\delta G$ is prime.
2. $B$ is $G$-prime and $\Gamma(\delta) = \hat{G}$.

**Proof.** Assume that $B \times_\delta G$ is prime. Since for every $\delta$-invariant ideal $J \subset B$, $J \times_\delta G$ is an ideal of $B \times_\delta G$, the fact that $B$ is $G$-prime is immediate. Next we will show that $\Gamma(\delta) = \hat{G}$.

Let $C \in \mathcal{H}^g(B)$ and $\alpha \in \hat{G}$. By Remark 2.3 above, $C \times_\delta G$ is a hereditary subalgebra of $B \times_\delta G$ and is therefore prime. Using [5], Corollary 4.9, $(C \otimes \mathcal{M}_d)^{B \delta,\alpha}$ is prime and $C_2^g(\alpha) \neq (0)$ (since $C_2^g(\alpha) \neq (0)$). Thus the ideal $C_2^g(\alpha)^S(\alpha)$ is essential in $(C \otimes \mathcal{M}_d)^{B \delta,\alpha}$. Therefore $\alpha \in \Gamma(\delta)$ and so $\Gamma(\delta) = \hat{G}$.

Conversely, assume that $B$ is $G$-prime and $\Gamma(\delta) = \hat{G}$.

For each $\alpha \in \hat{G}$, the $C^*$-algebras $S_{\alpha,\delta}S_{\tau,\tau}$ and $S_{\tau,\tau}S_{\alpha,\delta}$ are strongly Morita equivalent ($S_{\alpha,\delta}$ being the imprimitivity bimodule). By Proposition 3.3, $B^\delta$ is prime and therefore, by Remark 2.3.1, $S_\delta$ is prime. Since $S_\delta$ is prime, so is the ideal $S_{\alpha,\delta}S_{\alpha,\delta}$ and the Morita equivalent algebra $S_{\alpha,\delta}S_{\alpha,\delta}$. By the definition of $\Gamma(\delta)$, $S_{\alpha,\delta}S_{\alpha,\delta}$ is an essential ideal of $S_\alpha$ and thus $S_\alpha$ is prime also. The implication follows now from [8], Corollary 4.9. \qed

4. **Strong Connes spectrum and simple crossed products**

We begin by defining the strong Arveson and Connes spectra for compact quantum group coactions.

**Definition 4.1.**

1. Strong Arveson spectrum $\tilde{\text{Sp}}(\delta) = \{\alpha \in \hat{G}|\overline{S_{\alpha,\alpha}S_{\alpha,\alpha}}\} = S_\alpha$.
2. Strong Connes spectrum $\tilde{\Gamma}(\delta) = \bigcap_{\alpha \in H^g(B)} \tilde{\text{Sp}}(\delta|\alpha)$.

Using similar arguments as in Lemma 3.2 we obtain the following result.

**Lemma 4.2.** Let $\alpha \in \hat{G}$. Then $\alpha \in \tilde{\text{Sp}}(\delta)$ if and only if $B_2^g(\alpha)^S \cap B_2^g(\alpha) = B \otimes \mathcal{M}_d$.

The following result makes a connection between the strong Connes spectrum and the simplicity of the fixed point algebra $B^\delta$.

**Proposition 4.3.** If $B$ is $G$-simple and $\tilde{\Gamma}(\delta) = \hat{G}$, then $B^\delta$ is simple.

**Proof.** Let $J \subset B^\delta$ be a nonzero two-sided ideal. We will prove that $J = B^\delta$ and thus $B^\delta$ is simple. To this end we will show that $S_{\alpha,\delta}(J \otimes 1) \subset J \otimes 1$, for any $\alpha \in \hat{G}$. The claim will then follow from Corollary 2.5

Let $D = JBJ$ and let $\alpha \in \hat{G}$. Then $D \in \mathcal{H}^g(B)$. Since $\alpha \in \hat{G} = \Gamma(\delta)$, we have

$$S_{\alpha,\delta}D_{\alpha,\delta} = S_{\alpha,\delta},$$

where $S_{\alpha,\delta}$ and $S_{\alpha,\delta}$ are the corresponding subspaces of $D \times_\delta G$.

By the definition of $D$, it obviously follows that $S_{\alpha,\delta} = (J \otimes 1)S_{\alpha,\delta}(J \otimes 1)$ and $S_{\alpha} = (J \otimes 1)S_{\alpha}(J \otimes 1)$. 

Therefore,
\[(J \otimes 1)S_{\alpha,\iota}(J \otimes 1)S_{\alpha,\iota}(J \otimes 1) = (J \otimes 1)S_\alpha(J \otimes 1)\].

Multiplying equation (7) to the left by \(S_{\alpha,\iota}\) and to the right by \(S_{\alpha,\iota}\) we get
\[(8) \quad S_{\alpha,\iota}(J \otimes 1)S_{\alpha,\iota}(J \otimes 1)S_{\alpha,\iota} = S_{\alpha,\iota}(J \otimes 1)S_\alpha(J \otimes 1)S_{\alpha,\iota}.

By Remark 2.3(1), \(S_{\alpha,\iota},S_{\alpha,\iota} \subseteq S_t = B^\delta \otimes 1\) and since \(J \subseteq B^\delta\), the left-hand side of equation (8) is included in \(J \otimes 1\). Therefore, the right-hand side of equation (8) is included in \(J \otimes 1\):
\[(9) \quad S_{\alpha,\iota}(J \otimes 1)S_\alpha(J \otimes 1)S_{\alpha,\iota} \subseteq J \otimes 1.

Since \(S_{\alpha,\iota}(J \otimes 1)S_{\alpha,\iota} \subseteq S_{\alpha,\iota}(J \otimes 1)S_\alpha(J \otimes 1)S_{\alpha,\iota}\), from equation (9) it follows that
\[(10) \quad S_{\alpha,\iota}(J \otimes 1)S_{\alpha,\iota} \subseteq J \otimes 1\]
and we are done. \(\square\)

We can now prove:

**Theorem 4.4.** The following are equivalent:

1. \(B \times_\delta G\) is simple.
2. \(B\) is \(G\)-simple and \(\hat{\Gamma}(\delta) = \hat{G}\).

**Proof.** First assume that \(B \times_\delta G\) is simple. That \(B\) is \(G\)-simple follows easily since for every nontrivial ideal \(J \in \mathcal{H}^\delta(B)\), \(J \times_\delta G\) is a nontrivial ideal of \(B \times_\delta G\).

Now let \(\alpha \in \hat{G}\) and \(C \in \mathcal{H}^\delta(B)\). Then \(C \times_\delta G\) is a hereditary subalgebra of \(B \times_\delta G\) by Remark 2.3(3) and hence it is simple. By \([5]\), Corollary 4.9, \(S_{\alpha,\iota} \neq 0\) and \(S_\alpha\) is simple. Hence \(S_{\alpha,\iota}S_{\alpha,\iota} = S_\alpha\) and \(\alpha \in \hat{\Gamma}(\delta)\).

Conversely, assume that \(B\) is \(G\)-simple and \(\hat{\Gamma}(\delta) = \hat{G}\). By Proposition 18 \(B^\delta\) is simple. Now, for every \(\alpha \in \hat{G} = \hat{\Gamma}(\delta)\), the nonzero ideal \(S_{\alpha,\iota}S_{\alpha,\iota} \subseteq S_\alpha = B^\delta \otimes 1\) is simple and so is the Morita equivalent algebra \(S_{\alpha,\iota}S_{\alpha,\iota} = S_\alpha\). The conclusion follows now from \([5]\), Corollary 4.9. \(\square\)

### 5. Spectra are closed under tensor products

In order to prove the results about the stability of the Connes spectrum to tensor products, we need to make some notation. If \(\alpha \in \hat{G}\) and \(\beta \in \hat{G}\) and \(u^\alpha \in \alpha, u^\beta \in \beta\), denote by \(u^\alpha \otimes u^\beta = \sum_{p,q,r,s} m_{pq}^{\alpha} \otimes m_{rs}^{\beta} \otimes u_{pq}^\alpha u_{rs}^\beta\) the Kronecker tensor product of \(u^\alpha\) and \(u^\beta\), which is a representation of \(A\) (see \([17]\), Section 6). Then \(u^\alpha \otimes u^\beta\) is unitary if both \(u^\alpha\) and \(u^\beta\) are unitary. Moreover, \(u^\alpha \otimes u^\beta\) is equivalent to a direct sum of irreducible representations, \(u^\iota \otimes u^\iota \equiv \sum_{i} u_{\iota}^i \rho_i \in \hat{G}\). The equivalence and \(\rho_i \in \hat{G}\) are unitary if both \(u^\alpha\) and \(u^\beta\) are unitary \([17]\).

**Definition 5.1.** Let \(\Pi \subseteq \hat{G}\) be a subset. We say that \(\Pi\) is closed under tensor products if for every \(\alpha \in \Pi, \beta \in \Pi\) and \(u^\alpha \in \alpha, u^\beta \in \beta\) it follows that every irreducible component of \(u^\alpha \otimes u^\beta\) belongs to \(\Pi\).

If \(X \in B_2^\delta(u^\alpha)\) and \(Y \in B_2^\delta(u^\beta)\), we denote \(X \circ Y = \sum_{i,k,l,j} X_{ik}Y_{lj} \otimes m_{ik}^{\alpha} \otimes m_{lj}^{\beta}\) (for the case of groups this notation was used in \([12]\)). Standard calculations show that \(X \circ Y \in B_2^\delta(u^\alpha \otimes u^\beta)\). Furthermore, \(X \circ Y\) can be viewed as the matrix of
order $d_αd_3 \times d_αd_3$ partitioned in $d_3^2$ blocks of order $d_α \times d_α$ as follows: $X \odot Y = [X \text{diag}(Y_{ij})]$, where $\text{diag}(Y_{ij})$ is the $d_α \times d_α$ matrix with all the diagonal entries equal to $Y_{ij}$ and all the others equal to 0.

**Remark 5.2.** If $u^α \odot u^β \equiv \sum_1^d u^ρ_i, \rho_i \in \hat{G}$, then:

1. $(B \otimes \mathcal{M}_{d_αd_3})^{β,u^α \odot u^β}$ is spatially isomorphic to $\sum_i^d (B \otimes \mathcal{M}_{d_α})^{β,u^ρ_i}$ ($\ast$-isomorphic if both $u^α$ and $u^β$ are unitary) and
2. $B_2^β(u^α \odot u^β)$ is spatially isomorphic to $\sum_i^d B_2^β(\rho_i)$.

The proof of the above remark follows immediately using a change of basis in $\mathcal{M}_{d_αd_3}$.

First we prove

**Lemma 5.3.** $\hat{Sp}(δ|C)$ is closed under tensor products for every $C \in \mathcal{H}^δ(B)$.

**Proof.** We have to prove that if $α, β \in \hat{Sp}(δ|C), C \in \mathcal{H}^δ(B)$, then every irreducible component of $u^α \odot u^β$ belongs to $\hat{Sp}(δ|C)$.

It is enough to prove the above claim for $C = B$. We first show that if $α \in \hat{Sp}(δ)$ and $β \in \hat{Sp}(δ)$, then $B_2^β(u^α \odot u^β)$ is a dense ideal of $B \otimes \mathcal{M}_{d_αd_3}$.

Indeed, by (9, Theorem 2.1), $(B \otimes \mathcal{M}_{d_α})^{δ} \mathcal{E}$ has an approximate identity $\{E_λ\}$ of the form $E_λ = \sum_1^d \gamma X_λ^1X_λ^2, X_i^1 \in B_2^α(u^α), i = 1, 2, ..., n_λ$. By (17, Lemma 2.7), $\{E_λ\}$ is an approximate identity of $B \otimes \mathcal{M}_{d_α}$. Hence $(Y_1 \otimes I_{d_3})^*(Y_2 \otimes I_{d_3}) \in B_2^β(u^α \odot u^β)$, for all $Y_1, Y_2 \in B_2^β(u^α)$. Since $β \in \hat{Sp}(δ)$, $B_2^β(u^β)$ is a dense ideal of $(B \otimes \mathcal{M}_{d_β})^{δ}β$. Using an approximate identity of $(B \otimes \mathcal{M}_{d_β})^{δ}β$, we obtain that $B_2^β(u^α \odot u^β)$ is a dense ideal of $(B \otimes \mathcal{M}_{d_αd_3})^{δ}u^α \odot u^β$.

On the other hand, since $u^α \odot u^β$ is equivalent to a direct sum of irreducible representations, $u^α \odot u^β \equiv \sum_1^d u^ρ_i, \rho_i \in \hat{G}$, by Remark 5.2(1), $(B \otimes \mathcal{M}_{d_αd_3})^{δ}u^α \odot u^β$ is spatially $\ast$-isomorphic to $\sum_i^d (B \otimes \mathcal{M}_{d_α})^{δ,u^ρ_i}$. Thus, since by Remark 5.2(2), $B_2^β(u^α \odot u^β)$ is spatially isomorphic to $\sum_i^d B_2^β(\rho_i)$, it follows that $B_2^β(\rho_i) \ast B_2^β(\rho_i)$ is dense in $(B \otimes \mathcal{M}_{d_αd_3})^{δ,ρ_i}$ for all $i$. Therefore $\rho_i \in \hat{Sp}(δ)$ for every $i$. Thus $\hat{Sp}(δ|C)$ is closed under tensor products for every $C \in \mathcal{H}^δ(B)$.

Therefore:

**Proposition 5.4.** $\hat{Γ}(δ)$ is closed under tensor products.

**Proof.** This is obvious since $\hat{Γ}(δ) = \bigcap_{C \in \mathcal{H}^δ(B)} \hat{Sp}(δ|C)$. 

Next we will prove that the Connes spectrum is closed under tensor products. As in the case of the strong Connes spectrum, first we will show that our Arveson spectrum $\hat{Sp}(δ|C)$ is closed under tensor products for every $C \in \mathcal{H}^δ(B)$.

Let $α \in \hat{G}$ and $β \in \hat{G}$ and $u^α \in α, u^β \in β$. If $u^α$ and $u^β$ are unitary, then, as noticed above, $u^α \odot u^β$ is a unitary representation. If $u^α$ is a representation in the class $α$, not necessarily unitary, then there exists an invertible matrix $S \in \mathcal{M}_d$, such that $u^α_1 = (S^{-1} \odot 1)u^α(S \odot 1)$. Notice that $B_2^β(u^α_1) = \{(1_B \otimes S^{-1})X(1_B \otimes S)\}|X \in B_2^β(u^α_1)}$.
Lemma 5.5. \( \text{Sp}(\delta_{C}) \) is closed under tensor products for every \( C \in \mathcal{H}^{3}(B) \).

**Proof.** We may assume that \( C = B \). Let \( \alpha, \beta \in \text{Sp}(\delta) \) and \( u^{\alpha} \in \alpha, u^{\beta} \in \beta \) be unitary representatives of \( \alpha \) and \( \beta \). First we will show that

\[
\text{linspan}\{(X \otimes Y)^{*}(X \otimes Y) | X \in B^{2}\delta(u^{\alpha}), Y \in B^{2}\delta(u^{\beta})\}
\]

is an essential ideal of \( (B \otimes \mathcal{M}_{d_{\alpha},d_{\beta}})^{\delta_{\alpha} \otimes \delta_{\beta}} \). It then follows immediately that each irreducible component of \( u^{\alpha} \otimes u^{\beta} \) belongs to \( \text{Sp}(\delta) \).

Let \( Z \in (B \otimes \mathcal{M}_{d_{\alpha},d_{\beta}})^{\delta_{\alpha} \otimes \delta_{\beta}}, Z \geq 0 \). Assume that \((X \otimes Y)Z = 0\), for every \( X \in B^{2}\delta(u^{\alpha}), Y \in B^{2}\delta(u^{\beta}) \). Let \( Z \) be partitioned into blocks as follows: \( Z = \sum_{i,k=1}^{d_{\beta}} Z_{i,k} \otimes m_{ik}^{\beta} \), where \( Z_{i,k} \) are \( d_{\alpha} \times d_{\alpha} \) matrices with entries in \( B \). Since \((X \otimes Y)Z = 0\), for every \( X \in B^{2}\delta(u^{\alpha}), Y \in B^{2}\delta(u^{\beta}) \), it follows that \((X \otimes Y)(I_{d_{\alpha}} \otimes Y^{*}) = 0\), for every such \( X, Y \). In particular, if \( Y \) is as in Remark 2.1(4), that is, \( Y \) has only one nonzero row consisting of \( y_{1}, y_{2}, \ldots, y_{d_{\beta}} \), we have

\[
X \sum_{i,j=1}^{d_{\beta}} y_{i}Z_{ij}y_{j}^{*} = 0,
\]

for every \( X \in B^{2}\delta(u^{\alpha}) \) and \( Y \in B^{2}\delta(u^{\beta}) \) as chosen, where the multiplication \( y_{i}Z_{ij}y_{j}^{*} \) is the multiplication in \( B \) of \( y_{i}, y_{j}^{*} \) with each entry of \( Z_{ij} \). First we prove the following:

\[
\sum y_{i}Z_{ij}y_{j}^{*} \in (B \otimes \mathcal{M}_{d_{\alpha}})^{\delta_{\alpha}}, \tag{11}
\]

for every \( Y \in B^{2}\delta(u^{\beta}) \) as chosen (i.e. with only one nonzero row).

In the following leg numbering notation, there are four places in the following order: \( B, \mathcal{M}_{d_{\alpha}}, \mathcal{M}_{d_{\beta}}, A \).

Since \( Z \in (B \otimes \mathcal{M}_{d_{\alpha},d_{\beta}})^{\delta_{\alpha} \otimes \delta_{\beta}} \), we have

\[
\delta_{14}(Z) = (1_{B} \otimes (u^{\alpha} \otimes u^{\beta})^{*})(Z \otimes 1_{A})(1_{B} \otimes (u^{\alpha} \otimes u^{\beta})). \tag{12}
\]

By the definition of \( u^{\alpha} \otimes u^{\beta} \), we have

\[
\delta_{14}(\sum Z_{ij} \otimes m_{ij}^{\beta}) = (\sum 1_{B} \otimes m_{\alpha}^{\alpha} \otimes m_{rs}^{\beta} \otimes u_{\alpha}^{\alpha} \otimes u_{\alpha}^{\alpha})(\sum Z_{ij} \otimes m_{ij}^{\beta} \otimes 1_{A})
\times (\sum 1_{B} \otimes m_{\alpha}^{\alpha} \otimes m_{\alpha}^{\alpha} \otimes u_{\alpha}^{\alpha} \otimes u_{\alpha}^{\alpha}). \tag{13}
\]

On the other hand, taking into account that \( Y \in (B \otimes \mathcal{M}_{d_{\alpha}})^{\delta_{\alpha} \otimes \delta_{\beta}} \), it follows that

\[
\delta_{14}(Y_{13}) = (1_{B} \otimes (u^{\beta})_{34}^{*})(Y_{13} \otimes 1_{A})(1_{B} \otimes (u^{\beta})_{34}), \tag{14}
\]

where \((u^{\beta})_{34} = \sum 1_{B} \otimes I_{d_{\alpha}} \otimes m_{ik}^{\beta} \otimes u_{ik}^{\beta} \).

By combining Formulas \( 12 \) and \( 14 \) and taking into account that \( u^{\alpha} \) and \( u^{\beta} \) are unitary, we get Formula \( 14 \). Therefore, since \( \alpha \in \text{Sp}(\delta) \), it follows that \( \sum y_{i}Z_{ij}y_{j}^{*} = 0 \) for every such \( Y \).

Let \( u_{1}^{\alpha} \in \alpha \) be a not necessarily unitary representation, but such that \( u_{1}^{\alpha} \) is unitary. Then, since \( u_{1}^{\alpha} \) and \( u_{1}^{\beta} \) are equivalent, there is an invertible matrix \( S \in \mathcal{M}_{d_{\alpha}} \) such that \( u_{1}^{\alpha} = (S^{-1} \otimes 1)u_{1}^{\alpha}(S \otimes 1) \). Notice that

\[
B^{2}\delta(u_{1}^{\alpha}) = \{(1_{B} \otimes S^{-1})X(1_{B} \otimes S)|X \in B^{2}\delta(u^{\alpha})\}.
\]
Denote $V_{ij} = (1_B \otimes S^*)Z_{ij}(1_B \otimes S)$ for all $i, j = 1, 2, ..., d_\beta$. Thus, since $\sum y_iZ_{ij}y_j^* = 0$, it immediately follows that

$$\sum y_iV_{ij}y_j^* = 0,$$

for every $Y$ as chosen.

In particular, $\sum y_iV_{ij}^p q_yj^* = 0$, for all $p, q = 1, 2, ..., d_\alpha$, where $V_{ij}^p q_yj^*$ is the entry $pq$ of the $d_\alpha \times d_\alpha$ matrix $V_{ij}$. Hence, $\sum y_i(\sum_{p=1}^{d_\alpha} V_{ij}^p q_yj^*)y_j^* = 0$. Let $d_{ij} = \sum_{p=1}^{d_\alpha} V_{ij}^p q_yj^*$. Therefore, if $Y = B_{ij}^\delta(u^\beta)$ is as before and $D = \sum_{i,j=1}^{d_\delta} d_{ij} \otimes m_{ij}^\delta$, we have $YDY^* = 0$.

By Remark 2.1(4), the matrices $Y \in B_{ij}^\delta(u^\beta)$ that have only one nonzero row span $B_{ij}^\delta(u^\beta)$ linearly. Therefore, $YDY^* = 0$ for every $Y \in B_{ij}^\delta(u^\beta)$. Since $Z \geq 0$ it follows that $V \geq 0$ and so $D \geq 0$. Therefore $YD = 0$ for every $Y \in B_{ij}^\delta(u^\beta)$. Notice that $V = \sum_{i,j} V_{ij} \otimes m_{ij}^\delta$ satisfies formula (12) with $u^\alpha$ replaced by $u_i^\alpha$. This fact will be used in the proof of the next claim.

**Claim.** $D \in (B \otimes \mathcal{M}_{d_\alpha})^{\delta,u^\alpha}$.

The proof of the claim will be achieved in two steps:

**Step 1.** We prove that

$$d_{ij} \otimes 1_A = \sum_{p=1}^{d_\alpha} [(1_B \otimes u_i^\alpha)^*(V_{ij} \otimes 1_A)(1_B \otimes u_i^\alpha)]_{qq},$$

where $[(1_B \otimes u_i^\alpha)^*(V_{ij} \otimes 1_A)(1_B \otimes u_i^\alpha)]_{qq}$ denotes the entry $qq$ of the matrix $(1_B \otimes u_i^\alpha)^*(V_{ij} \otimes 1_A)(1_B \otimes u_i^\alpha)$, $q = 1, 2, ..., d_\alpha$. Tedious but straightforward calculations show that the right hand side of the above formula is

$$\sum_{q=1}^{d_\alpha} [(1_B \otimes u_i^\alpha)^*(V_{ij} \otimes 1_A)(1_B \otimes u_i^\alpha)]_{qq} = \sum_{p,n,r,s} V_{ij}^{pn} \otimes m_{r,s}^\alpha \otimes m_{p,n}^\alpha \otimes (u_i^\alpha)^* \otimes (u_i^\alpha)^*_{pp} \otimes (u_i^\alpha)^* \otimes (u_i^\alpha)^*_{qq}$$

$$= \sum_{q=1}^{d_\alpha} [(1_B \otimes u_i^\alpha)^*(V_{ij} \otimes 1_A)(1_B \otimes u_i^\alpha)]_{qq} = \sum_{q} \delta_{q,p} \delta_{p,n} \delta_{r,s} \delta_{m_{pq}^\alpha m_{rq}^\alpha m_{sk}^\alpha m_{lk}^\alpha (u_i^\alpha)^*_{pp} (u_i^\alpha)^*_{qq}.$$
Step 2 (Proof of claim). We have to prove that

\[(16) \quad \delta_{13}(\sum_{i,j} d_{ij} \otimes m_{ij}^\beta) = (1_B \otimes u^\beta)^*(\sum_{i,j} d_{ij} \otimes m_{ij}^\beta \otimes 1_A)(1_B \otimes u^\beta).\]

We will evaluate separately the right and left hand sides of formula (16) and show that they are the same. First, the right hand side:

\[(17) \quad (1_B \otimes u^\beta)^*(\sum_{i,j} d_{ij} \otimes m_{ij}^\beta \otimes 1_A)(1_B \otimes u^\beta)\]

Next we will calculate the left hand side of formula (16). As noticed above, by multiplying formula (12) above by \(1_B \otimes S^* \otimes I_{d_B} \otimes 1_A\) to the left and by \(1_B \otimes S \otimes I_{d_B} \otimes 1_A\) to the right, if we denote \(V = \sum V_{ij} \otimes m_{ij}^\beta\), we get

\[\delta_{14}(V) = (1_B \otimes (u_1^\alpha \otimes u^\beta)^*)(V \otimes 1_A)(1_B \otimes (u_1^\alpha \otimes u^\beta)).\]

Therefore:

\[\delta_{14}(V) = \sum \left(1_B \otimes m_{r,s}^\alpha \otimes m_{k,l}^\beta \otimes u_{ik}^*(u_1^\alpha)^*_{sr}((V^pq \otimes m_{pq}^\alpha \otimes m_{ij}^\beta \otimes 1_A)\right)\times \left(1_B \otimes m_{tu}^\alpha \otimes m_{vw}^\beta \otimes (u_1^\alpha)^*_{tw}u_{vw}^\beta\right)\]

Hence, if \(k = i_0\) and \(w = j_0\) we get:

\[\delta_{14}(V_{i_0,j_0} \otimes m_{i_0,j_0}^\beta) = \sum V^pq_{ij} \otimes m_{tu}^\alpha \otimes m_{i_0,j_0}^\beta \otimes u_{i_0}^*(u_1^\alpha)^*_{pq}((u_1^\alpha)^*_{tu}u_{i_0}^\beta)\]

and if \(r = u = l\),

\[\delta_{14}(V^{l}_{l_0,j_0} \otimes m_{l_0,j_0}^\beta) = \sum V^pq_{ij} \otimes m_{l_0}^\alpha \otimes m_{l_0,j_0}^\beta \otimes u_{i_0}^*(u_1^\alpha)^*_{pq}((u_1^\alpha)^*_{l_0}u_{l_0}^\beta).\]

Therefore:

\[\delta_{13}(d_{i_0,j_0} \otimes m_{i_0,j_0}^\beta) = \sum V^pq_{ij} \otimes m_{i_0,j_0}^\beta \otimes u_{i_0}^*(\sum_{l=1}^{d_{st}}(u_1^\alpha)^*_{st}u_{l}^{\beta})q_{l}u_{j_0}^{\beta}.\]

Since \(u_1^\alpha\) is a unitary representation, we have \(\sum_{l=1}^{d_{st}}(u_1^\alpha)^*_{st}u_{l}^{\beta}q_{l} = \delta_{pq}\), where, as usual, \(\delta_{pq}\) is the Kronecker symbol. Hence:

\[\delta_{13}(d_{i_0,j_0} \otimes m_{i_0,j_0}^\beta) = \sum_{i,j} d_{ij} \otimes m_{i_0,j_0}^\beta \otimes u_{i_0}^*(u_{j_0}^\beta).\]

Thus:

\[(18) \quad \delta_{13}(D) = \sum_{i,j} d_{ij} \otimes m_{i_0,j_0}^\beta \otimes u_{i_0}^*(u_{j_0}^\beta).\]
Formulas (18) and (17) show that the claim is true.

Since \( \beta \in \text{Sp}(\delta) \), \( D \in (B \otimes M_{d\beta})^{\delta}\alpha\beta \) and \( YD = 0 \) for every \( Y \in B_2^2(u^\delta) \), it follows that \( D = 0 \). This means in particular that all the diagonal entries of the matrix \( V \) are equal to 0. Since \( V \geq 0 \), it follows that \( V = 0 \) and thus \( Z = 0 \). Therefore \( \text{linspan}( (X \otimes Y)^* (X \otimes Y) | X \in B_2^2(u^\alpha), Y \in B_2^2(u^\beta) ) \) is an essential ideal of \( (B \otimes M_{d\beta,d\alpha})^{\delta\alpha\otimes\delta\beta} \), as claimed.

Now let \( u^\alpha \otimes u^\beta = \sum_i u^\alpha_i \otimes u^\beta_i \), where the \( \rho_i \) are irreducible. Then, by Remark 5.2(1) above, it follows that \( (B \otimes M_{d\beta,d\alpha})^{\delta\alpha\otimes\delta\beta} \) is spatially \( * \)-isomorphic to \( \sum^\oplus (B \otimes M_{d\beta_i})^{\delta\beta_i} \). Thus, since \( B_2^2(u^\alpha \otimes u^\beta) \) is spatially isomorphic to \( \sum^\oplus B_2^2(\rho_i) \) (Remark 5.2(2) above), it follows that \( B_2^2(\rho_i)^* B_2^2(\rho_i) \) is an essential ideal of \( (B \otimes M_{d\beta_i})^{\delta\beta_i} \), for all \( i \). Therefore \( \rho_i \in \text{Sp}(\delta) \), for every \( i \). Thus \( \text{Sp}(\delta|_C) \) is closed under tensor products for every \( C \in \mathcal{H}^2(B) \) and the lemma is proven. \( \square \)

We can now state:

**Proposition 5.6.** The Connes spectrum, \( \Gamma(\delta) \), is closed under tensor products.

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