SOLVABILITY IN GEVREY CLASSES OF SOME LINEAR FUNCTIONAL EQUATIONS

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This modest work is dedicated to the memory of our beloved master Ahmed Intissar
(1951-2017), a distinguished professor, a brilliant mathematician, a man with a golden heart.

Abstract. In this paper, we associate to each positive number $k$ a new class
of endomorphisms of the sheaf of germs of holomorphic functions on $[-1,1]$ and
prove the solvability in the Gevrey class $G_k([-1,1])$ of some linear functional
equations related to these linear endomorphisms.

1. Introduction

The functional equations have been the subject of intensive studies because of
their relation to applied and social sciences. The extreme variety of areas where
functional equations are found only enhance their attractiveness. In the study of
such equations there are different approaches and various research directions (cf for
example ([1])-([3]), ([5])-([6]), ([8])). However in our opinion there are a few studies
on their solvability in Gevrey classes. In this paper, we associate to each number
$k > 0$ a new class of endomorphisms of the sheaf of germs of holomorphic functions
on $[-1,1]$ and prove the solvability in a Gevrey class of linear functional equations
related to these endomorphisms. We apply then the result obtained to prove the
solvability in the Gevrey class $G_k([-1,1])$ of some linear functional equations.

2. Notations, definitions and preliminaries

Let $S$ be a nonempty subsets of $\mathbb{C}$ and $f : S \to \mathbb{C}$ a bounded function. $\|f\|_{\infty,S}$
denotes the quantity :

$$\|f\|_{\infty,S} = \sup_{z \in S} |f(z)|$$

For $z \in \mathbb{C}$ we set $\rho(z, S) := \inf_{u \in S} |z - u|$.

$O(S)$ denotes the set of holomorphic functions on some neighborhood of $S$.

For $z \in \mathbb{C}$ and $h > 0$, $B(z, h)$ is the open ball in $\mathbb{C}$ with center $z$ and radius $h$.

For $r > 0$, $k > 0$, $A > 0$ and $n \in \mathbb{N}^*$, we set :

$$\begin{align*}
[-1,1]_r &= [-1,1] + B(0,r) \\
[-1,1]_{k,A,n} &= [-1,1] + B(0; An^{-\frac{k}{n}})
\end{align*}$$

Thus we have :

$$\begin{align*}
[-1,1]_r &= \{ z \in \mathbb{C} : \rho(z,[-1,1]) < r \} \\
[-1,1]_{k,A,n} &= \{ z \in C : \rho(z;[-1,1]) < An^{\frac{1}{n}} \}
\end{align*}$$

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Let $E$ be a non empty set and $g: E \to E$ a mapping. For every $n \in \mathbb{N}$, $g^{\leq n}$ denotes the iterate of order $n$ of $g$ for the composition of mappings.

Through this paper the real number $k > 0$ will be a fixed constant real number.

The Gevrey class $G_k([−1, 1])$ is the set of all functions $f: [−1, 1] \to \mathbb{C}$ of class $C^\infty$ on $[−1, 1]$ such that there exist some constants $C > 0, A > 0$ verifying the following inequalities:

$$\|f^{(n)}\|_{\infty,[−1,1]} \leq CA^nn^{(1+\frac{k}{n})}, \quad n \in \mathbb{N}$$

with the convention that $0^0 = 1$.

Let $\psi$ be a holomorphic function on a neighborhood of $[−1, 1]$ such that $\psi([−1, 1]) \subset [−1, 1]$. We set:

$$\lambda(\psi) := \sup_{n \geq 1} \left( \frac{\|\psi(n)\|_{\infty,[−1,1]}}{n!} \right)^\frac{1}{n}$$

Let us observe that:

$$\lambda(\psi) < +\infty$$

A sequence $(f_n)_{n \geq 1}$ of germs of holomorphic functions on $[−1, 1]$. $(f_n)_{n \geq 1}$ is called $k$—sequence if there exists $B > 0$ such that the following conditions hold for every $n \geq 1$:

$$\left\{ \begin{array}{l} f_n \in O([−1,1]_{k,B,n}) \\ \|f_n\|_{\infty,[−1,1]_{k,A,n}} \leq C\theta^n \end{array} \right.$$  

where $C > 0$ and $\theta \in ]0,1]$ are constants.

The following result which is a direct consequence of a theorem stated in (4), page 223, point out the link between the set of $k$—sequences and the Gevrey class $G_k([−1, 1])$.

**Theorem 1.** A function $F$ of class $C^\infty$ on $[−1; 1]$ belongs to $G_k([−1; 1])$ if and only if there exists a $k$—sequence $(g_n)_{n \geq 1}$ we have that:

$$F = \sum_{n=1}^{+\infty} g_n|_{[−1;1]}$$

Let $F := (\varphi_p)_{p \in \mathbb{N}}$ be a sequence of functions $\varphi_p: [−1, 1]_\sigma \to \mathbb{C}$ $(\sigma > 0)$. We say that $F$ verifies the $E(k)$ property if there exists a constant $\tau \in ]0,\sigma]$ depending only on $F$ such that for all $A$ in $]0,\tau]$ there exists an integer $M(A)$ depending only on $A$ such that we have:

$$(2.1) \quad \varphi_p([−1, 1]_{k,A,n+1}) \subset [−1,1]_{k,A,n} \quad n \geq M(A), \quad p \in \mathbb{N}$$

The real $\tau$ is then called a $k$—threshold for the family $F$.

**Remark 2.1.** If the family $F$ verify the $A(k)$ property then

$$\varphi_p([−1, 1]) \subset [−1, 1], \quad p \in \mathbb{N}$$

An endomorphism $T$ of the sheaf $O([−1, 1])$ of holomorphic functions on $[−1, 1]$ is said to verify the $A(k)$ property if there exist two constants $\tau > 0, \rho \in ]0,1]$ depending only on $T$ such that for all $A$ in $]0,\tau]$ there exists an integer $N(A)$ depending only on $A$ satisfying the following properties:

$$(2.2) \quad T(O([−1, 1]_{k,A,n})) \subset O([−1, 1]_{k,n+1,A}, \quad n \geq N(A)$$

$$(2.3) \quad \|T(f)\|_{\infty,[−1,1]_{k,n+1,A}} \leq \rho\|f\|_{\infty,[−1,1]_{k,n,A}} \quad n \geq N(A), \quad f \in O([−1, 1]_{k,n,A})$$
\[(2.4) \quad \|T(f)\|_{[-1,1]} \leq C\|f\|_{[-1,1]}\]  

The real \(\tau\) is then called a k-threshold for the endomorphism \(T\).

An endomorphism \(T\) of the sheaf \(O([-1,1])\) which verifies the \(\mathfrak{A}(k)\) property has the following fundamental property.

**Proposition 2.** Let \(T\) be an endomorphism of the sheaf \(O([-1,1])\) which verify the \(\mathfrak{A}(k)\) property. Then \(T\) induces a unique endomorphism \(\tilde{T}\) of the Gevrey class \(G_k([-1,1])\) such that for every \(k\)-sequence \((\Phi_n)_{n\geq 1}\) the sequence \((T(\Phi_n))_{n\geq 1}\) is also a \(k\)-sequence and the following condition holds:

\[(2.5) \quad \tilde{T}\left(\sum_{n=1}^{+\infty} f_n\right) = \sum_{n=1}^{+\infty} T(f_n)\]

**Proof.** Let \(f \in G_k([-1,1])\). Let \((g_n)_{n\geq 1}\) and \((h_n)_{n\geq 1}\) be \(k\)-sequences such that:

\(f = \sum_{n=1}^{+\infty} g_n\) and \(h_n = \sum_{n=1}^{+\infty} h_n\)

Then there exist \(A_1 > 0\), \(C_1 > 0\), \(0 < \theta_1 < 1\) such that the following conditions hold for all \(n \geq 1\):

\[
\left\{ \begin{array}{l}
g_n, h_n \in O([-1,1]; 1, n) \\
The \max \left(\|g_n\|_{\infty, [-1,1]; 1, n}, \|h_n\|_{\infty, [-1,1]; 1, n}\right) \leq C_1 \theta_1^n
\end{array} \right.
\]

Let us set for all \(n \geq 1\):

\[
\begin{align*}
w_n & := g_n - h_n \\
\Phi_n & := \sum_{j=1}^{n} w_j
\end{align*}
\]

We have for all \(n \geq 1\):

\[
\begin{align*}
T(\Phi_n) & = \sum_{j=1}^{n} T(w_j) \\
& = \sum_{j=1}^{n} T(g_j) - \sum_{j=1}^{n} T(h_j)
\end{align*}
\]

Since the sequence of functions \((\Phi_n)_{n\geq 1}\) is uniformly convergent on \([-1,1]\) to the null function, it follows from the condition \((2.4)\) that the function series \(\sum T(g_j)\) and \(\sum T(h_j)\) are uniformly convergent on \([-1,1]\) to the same function which we denote by \(\tilde{T}(f)\). The mapping \(\tilde{T}\) is well defined and linear on the Gevrey class \(G_k([-1,1])\). Furthermore \(\tilde{T}\) is the unique endomorphism on \(G_k([-1,1])\) which satisfies the condition \((2.5)\).

\[\square\]

3. **Statement of the main result and of its corollary**

**Theorem 3.** Let \(T\) an endomorphism on the sheaf \(O([-1,1])\) which verify the \(\mathfrak{A}(k)\) property. Then for every \(u \in O([-1,1])\) the function series \(\sum T(\Phi_n)\) is uniformly convergent on \([-1,1]\) and its sum \(\Phi\) is a solution of the linear functional equation

\[(3.1) \quad \Phi - \tilde{T}(\Phi) = u\]
Corollary 4. Let \( a := (a_n)_{n \geq 0} \) be a sequence of holomorphic functions on \([-1, 1]_{\sigma} \) \((\sigma > 0)\) and \( \varphi := (\varphi_n)_{n \geq 0} \) a sequence which verify the \( E(k) \) property. Assume that :
\[
\sum_{n=0}^{+\infty} ||a_n||_{\infty, [-1, 1]_{\sigma}} < 1
\]

Then the linear functional equation :
\[
\Phi(x) - \sum_{n=0}^{+\infty} a_n(x) \Phi(\varphi_n(x)) = u(x)
\]
has a unique solution which furthermore belongs to the Gevrey class \( G_k([-1, 1]) \).

4. Proof of the main result and of its corollary

4.1. Proof of the main result. Let \( \tau \) be a \( k \)-threshold for the endomorphism \( T \). Let \( r \in [0, \tau] \) such that \( u \in O([-1, 1], \tau) \). Consider the sequence of functions \((T^{(n)}(u))_{n \geq 0}\). Then there exists, thanks to the conditions (2.1) and (2.2) an integer \( N \) such that :
\[
\begin{align*}
T(O([-1, 1]_{k,n,r}) & \subset O([-1, 1]_{k,n+1,r}, n \geq N) \\
||T(f)||_{\infty, [-1, 1]_{k,n,r}} & \leq ||f||_{\infty, [-1, 1]_{k,n,r}}, n \geq N, f \in O([-1, 1]_{k,n,r})
\end{align*}
\]

It is then clear that we have for all \( n \geq N \):
\[
\begin{align*}
T^{(n)}(u) & \in O([-1, 1]_{k,n,r}) \\
||T^{(n)}(u)||_{\infty, [-1, 1]_{k,n,r}} & \leq ||T(u)||_{\infty, [-1, 1]_{k,n,r}} \rho^n
\end{align*}
\]

It follows, by virtue of theorem 1, that the function series \( \sum T^{(n)}(u)\) is uniformly convergent to a function \( w \in G_k([-1, 1]) \) which is a solution of the equation (3.2). \( \square \)

4.2. Proof of the corollary. Let \( f \in O([-1, 1]) \), then there exists \( \alpha \in [0, \sigma] \) such that \( f \in O([-1, 1], \alpha) \) where \( \sigma \) is a \( k \)-threshold of the sequence \( \varphi \). Then there exists \( M \in \mathbb{N}^* \) such that :
\[
\varphi_p([-1, 1]_{k,\alpha,n+1}) \subset [-1, 1]_{k,\alpha,n}, n \geq M, p \in \mathbb{N}
\]

It follows that :
\[
\begin{align*}
f \circ \varphi_p & \in O([-1, 1]_{k,M+1,\alpha}), p \in \mathbb{N} \\
|a_p(z)f(\varphi_p(z))| & \leq ||a_p||_{\infty, [-1, 1]_{\sigma}} ||f||_{\infty, [-1, 1]_{k,M+1,\alpha}}, z \in [-1, 1]_{k,M+1,\alpha}
\end{align*}
\]

Thence from the condition (3.2) entails that the function series \( \sum a_n.(f \circ \varphi_n) \) is uniformly convergent on \([-1, 1]_{k,\alpha,M+1}\). Thence if we set :
\[
T_1(f) := \sum_{n=1}^{+\infty} a_n.(f \circ \varphi_n)
\]

we define an endomorphism \( T_1 \) of \( O([-1, 1]) \). Let us prove that \( T_1 \) verify the \( \mathbb{A}(k) \) property. Let \( A \in [0, \sigma] \), then there exists an integer \( M(A) \) depending only on \( A \) satisfying the following properties :
\[
\varphi_p([-1, 1]_{k,m+1,A}) \subset O([-1, 1]_{k,m,A}), m \geq M(A)
\]

which belongs to the Gevrey class \( G_k([-1, 1]) \).
Let \( v \in O([-1, 1]_{k,n,A}) \) where \( n \) is an integer such that \( n \geq M(A) \). Then :
\[
\begin{cases}
    \forall v \circ \varphi_p \in O([-1, 1]_{k,n+1,A}), \ p \in \mathbb{N} \\leq ||a_p||_{\infty,[−1,1]_\sigma} \leq ||v||_{\infty,[−1,1]_\sigma} \leq \forall u \in O([-1, 1]_{k,n,A}, \ p \in \mathbb{N}
\end{cases}
\]

It follows that the following facts hold for each integer \( n \geq M(A) \):
\[
\begin{align*}
    T_1(v) & \in O([-1, 1]_{k,n+1,A}) \\
    ||T_1(v)||_{\infty,[−1,1]_{k,n+1,A}} & \leq \left( \sum_{p=0}^{+\infty} ||a_p||_{\infty,[−1,1]_{\sigma}} \right) ||v||_{\infty,[−1,1]}
\end{align*}
\]

On the other hand we have for every \( x \in [-1, 1] \):
\[
|T_1(v)(x)| \leq \left( \sum_{p=0}^{+\infty} ||a_p||_{\infty,[−1,1]_{\sigma}} \right) |v(\varphi_p(x))|
\]

It follows, from the remark 1, that :
\[
||T_1(v)||_{−1,1} \leq \left( \sum_{p=0}^{+\infty} ||a_p||_{\infty,[−1,1]_{\sigma}} \right) ||v||_{−1,1}
\]

Consequently since \( \sum_{p=0}^{+\infty} ||a_p||_{\infty,[−1,1]_{\sigma}} < 1 \), we conclude that the endomorphism \( T_1 \) satisfy the \( \mathfrak{A}(k) \) property. Thence thanks to the main result the linear functional equation (3.3) has for every \( u \in O([-1, 1]) \) a unique solution which furthermore belongs to the Gevrey class \( G_k([-1, 1]) \).

\( \square \)

5. Some examples

We need first to prove some useful propositions.

**Proposition 5.** Let \( \psi \) be a holomorphic function on a neighborhood of \([-1,1]\) such that \( \psi([-1,1]) \subset [−1,1] \). Assume that \( \lambda(\psi) \leq 1 \). Then the function \( \psi \) verify the \( E(1) \) property.

**Proof.** Let \( \sigma > 0 \) be such that \( \psi \in O([-1,1], \sigma) \). Let \( A \in \mathbb{N}^\ast \) and \( z \in [-1,1] \). Let \( \hat{z} \) the closest point of \([-1,1]\) to \( z \). We have the following inequalities :
\[
\begin{align*}
\psi(z) \leq [\psi(z) - \psi(\hat{z})] \\
& \leq \sum_{j=1}^{\infty} \frac{\psi^{(j)}(\hat{z})}{j!} |z - \hat{z}|^j \\
& \leq \sum_{j=1}^{\infty} |z - \hat{z}|^j \\
& \leq \sum_{j=1}^{\infty} \rho(z, [-1,1])^j \\
& \leq \frac{A}{p + 1 - A} \\
& < \frac{A}{p}
\end{align*}
\]
It follows that:

\[ \psi([-1,1]_{1,p+1,A}) \subset [-1,1]_{1,p,A} \quad p \in \mathbb{N}^* \]

Thence the function \( \psi \) has the \( E(1) \) property. \( \square \)

**Proposition 6.** Let \( g \) be an entire function such that

\[
\left\{
\begin{array}{l}
g([-1,1]) \subset [-1,1] \\
\lambda(g) \leq 1
\end{array}
\right.
\]

Let \((P_n)_{n \geq 1}\) a sequence of holomorphic functions on \([-1;1]_\sigma(\sigma > 0)\) such that we have for every \( n \in \mathbb{N}^* \)

\[
P_n([-1,1]) \subset [-1,1] \\
\lambda(P_n) \leq 1
\]

\((g_n)_{n \geq 1}\) is the sequence of functions \( g_n : \mathbb{C} \to \mathbb{C} \) defined by the formula

\[
g_n(z) := g^{(n)} \left( \frac{z}{2n-1} \right)
\]

The sequences of functions \((g \circ P_n)_{n \geq 1}\) and \((g_n)_{n \geq 1}\) have the \( E(1) \) property.

**Proof.** 1- For every \( n \in \mathbb{N}^* \) the function \( g \circ P_n \in O([-1,1]_\sigma). \) Let \( A \in [0, \min(\frac{1}{2}, \sigma)] \), \( p \in \mathbb{N}^* \) and \( z \in [-1,1]_{1,p+1,A}. \) Let \( \hat{z} \) be the closest point of \([-1,1]\) to \( z. \) We have the following inequalities:

\[
g(g \circ P_n(z), [-1,1]) \\
\leq |g \circ P_n(z) - g \circ P_n(\hat{z})| \\
\leq \sum_{j=1}^{+\infty} |g^{(j)}(P_n(\hat{z}))||P_n(z) - P_n(\hat{z})|^j \\
\leq \sum_{j=1}^{+\infty} \left( \sum_{m=1}^{+\infty} \left| P_n^{(m)}(\hat{z}) \right| |z - \hat{z}|^m \right)^j \\
\leq \sum_{j=1}^{+\infty} \left( \sum_{m=1}^{+\infty} g(z, [-1,1])^m \right)^j \\
\leq \frac{A}{p + 1 - 2A} \\
< \frac{A}{p}
\]

It follows that:

\[ g \circ P_n([-1,1]_{1,p+1,A}) \subset [-1,1]_{1,p,A} \quad p \in \mathbb{N}^* , \ n \in \mathbb{N}^* \]

Thence the sequence \((g \circ P_n)_{n \geq 1}\) has the \( E(1) \) property.

2- For every \( n \in \mathbb{N}^* , g_n \) is an entire function such that \( g_n([-1,1]) \subset [-1,1]. \) Let us show by induction that \( \lambda(g_n) \leq 1, \) for every \( n \in \mathbb{N}^* . \) We have \( \lambda(g_1) = \lambda(g) \leq 1. \) Assume that \( \lambda(g_n) \leq 1 \) for a certain \( n \geq 1. \) Then by virtue of Faa-di-Bruno formula
we have for every \( x \in [-1, 1] \) and \( p \in \mathbb{N}^* \):

\[
\frac{|g_n^{(p)}(x)|}{p!} \leq \frac{1}{2^p} \sum_{j_1+j_2+\ldots+j_p=p} \frac{(j_1+j_2+\ldots+j_p)!}{j_1!j_2!\ldots j_p!} \frac{|g^{(j_1+j_2+\ldots+j_p)}(g_n(x))|}{(j_1+j_2+\ldots+j_p)!} \prod_{s=1}^{p} \left( \frac{|g^{(s)}(x)|}{s!} \right)^{j_s}
\]

\[
\leq \frac{1}{2^p} \sum_{j_1+j_2+\ldots+j_p=p} \frac{(j_1+j_2+\ldots+j_p)!}{j_1!j_2!\ldots j_p!} \leq 1
\]

It follows that \( \lambda(g_{n+1}) \leq 1 \). Thence according to the previous part of this proposition the sequence \((g_n)_{n \geq 1}\) verify the \( E(1) \) property.

The proof of the proposition is then complete. \( \square \)

**Example 1.**

Direct computations show that \( \lambda(\sin) = 1 \). It follows that the linear functional equation:

\[
\Phi(x) - \frac{1}{2} \Phi(\sin x) = -x
\]

has a unique solution which belongs to the Gevrey class \( G_1([-1, 1]) \).

**Example 2.**

Let \((P_n)_{n \geq 1}\) be a sequence of holomorphic functions on \([-1, 1]_{\sigma}(\sigma > 0)\) such that we have for every \( n \in \mathbb{N}^* \):

\[
\begin{align*}
\{ & P_n([-1, 1]) \subset [-1, 1] \\
& \lambda(P_n) \leq 1 
\end{align*}
\]

Let \( g \) be an entire function such that:

\[
\begin{align*}
\{ & g([-1, 1]) \subset [-1, 1] \\
& \lambda(g) \leq 1 
\end{align*}
\]

Let \((a_n)_{n \geq 1}\) be a sequence of holomorphic functions on \([-1, 1]_{\sigma}\) such that:

\[
\sum_{n=1}^{+\infty} ||a_n||_{\infty, [-1, 1]_{\sigma}} < 1
\]

Then it follows from corollary 4 and proposition 6 that the functional equation:

\[
\Phi(x) - \sum_{n=1}^{+\infty} a_n(x) \Phi(g(P_n(x))) = u(x)
\]

has for every \( u \in O([-1, 1]) \) a unique solution which belongs to the Gevrey class \( G_1([-1, 1]) \).

**Example 3.**

Let \((\alpha_n)_{n \geq 1}\) be a sequence of real numbers. \( f_n \) denotes for every \( n \in \mathbb{N}^* \) the entire function defined by:

\[
f_n(z) := \sin(z - \alpha_n), \quad z \in \mathbb{C}
\]
Direct computations show that:

$$\lambda(f_n) \leq 1, \ n \in \mathbb{N}^*$$

Thence it follows from the previous example that the linear functional equation:

$$\Phi(x) - \sum_{n=1}^{\infty} \frac{x^2}{2n+1(x^2+1)} \Phi(\sin(x - \alpha_n))) = u(x)$$

has for every $$u \in O([-1,1])$$ a unique solution which belongs to the Gevrey class $$G_1([-1,1])$$.

**Example 4.**

According to the proposition 6 above and to the fact that $$\lambda(\sin) = 1$$, it follows that the sequence of functions $$(g_n)_{n \geq 1}$$ defined by:

$$g_n(z) := \sin^{(n)}\left(\frac{x}{2n-1}\right), \ z \in \mathbb{C}$$

verify the $$E(1)$$ property. Consequently the linear equation:

$$\Phi(x) - \sum_{n=1}^{\infty} \frac{\cos(\varepsilon_n x)}{2n+1} \Phi\left(\sin^{(n)}\left(\frac{x}{2n-1}\right)\right) = u(x)$$

has for every $$u \in O([-1,1])$$ and every bounded sequences $$(\varepsilon_n)_{n \geq 1}$$ of real numbers a unique solution which belongs to the Gevrey class $$G_1([-1,1])$$.

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