The Runge-Lenz vector for quantum Kepler problem in the space of positive constant curvature and complex parabolic coordinates

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Abstract

By analogy with the Lobachevsky space $H_3$, generalized parabolic coordinates $(t_1, t_2, \phi)$ are introduced in Riemannian space model of positive constant curvature $S_3$. In this case parabolic coordinates turn out to be complex-valued and obey additional restrictions involving the complex conjugation. In that complex coordinate system, the quantum-mechanical Coulomb problem is studied: separation of variables is carried out and the wave solutions in terms of hypergeometric functions are obtained. At separating the variables, two parameters $k_1$ and $k_2$ are introduced, and an operator $B$ with the eigenvalues $(k_1 + k_2)$ is found, which is related to third component of the known Runge-Lenz vector in space $S_3$ as follows: $iB = A_3 + i\vec{L}^2$, whereas in the Lobachevsky space as $B = A_3 + \vec{L}^2$. General aspects of the possibility to employ complex coordinate systems in the real space model $S_3$ are discussed.

1 Introduction

In Euclidean 3-dimension space $E_3$ there exist 11 coordinate systems [1-3], allowing for the complete separation of variables in the Helmholtz equation

$$\left[ \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^\alpha} \sqrt{g} g^{\alpha\beta} \frac{\partial}{\partial x^\beta} + \text{const} \right] \Phi(x^1, x^2, x^3) = 0 \ ;$$

(1)

g^{\alpha\beta}(x)$ stands for the metric tensor of space $E_3$ specified for curvilinear coordinates $(x^1, x^2, x^3)$. Solution of the same problem for spaces of constant positive and negative curvature, Riemannian $S_3$ and Lobachevsky $H_3$ models was given by Olevsky in [4], see also [5]. It was established that there exist 34 such coordinate systems for hyperbolic space $H_3$, whereas in the case of spherical model $S_3$ the number of those systems is only 6.

This result may seem rather unexpected and even intriguing by some reasons. Indeed, there must exist a limiting procedure from both curved space models $H_3$ and $S_3$ to flat space:

$$H_3, S_3 \rightarrow E_3$$

and it is natural to expect the reduction $34 \rightarrow 11$, however one can hardly perform the reduction $6 \rightarrow 11$.

The above asymmetry between $H_3$ and $S_3$ may be seen as even more strange if one calls the known relations of these models through the analytical continuation:

$$S_3 \quad u_0^2 + u_1^2 + u_2^2 + u_3^2 = +R^2 \ ,$$

$$H_3 \quad u_0^2 - u_1^2 - u_2^2 - u_3^2 = +R^2 \ ;$$

(2)

where $R$ stands for a curvature radius for $S_3$ and $H_3$. 
The asymmetry of the models $H_3$ and $S_3$ with respect to coordinate systems finds its logical corollary when turning to the study of the quantum mechanical model for a hydrogen atom on the background of a curved space. Firstly, such a model was considered in [6,7,8] where the wave function in spherical coordinates and energy spectrum were established. In particular, an additional degeneracy like in the case of flat space was observed, which presumes existence of a hidden symmetry in the (curved space) problem. In [9-12] the symmetry operators accounting for such additional degeneracy in Kepler problem on curved space background were found for both model $H_3$ and $S_3$, and an analog of the conventional Runge-Lenz vector in the flat space was constructed.

Connection between the Runge-Lenz operator $\vec{A}$ in the quantum Kepler problem and parabolic coordinates in Euclidean space is well known: by solving the Schrödinger equation in these coordinates the eigenfunctions of the third component $A_3$ arise [13]. Analogous situation exists in the hyperbolic space $H_3$ but not in in the spherical $S_3$ [14]. In the Lobachewsky space, among 34 coordinates established by Olevsky one may select one special case, parabolic system of coordinates in $H_3$, in which the Schrödinger equation allows the separation of variables and the wave functions arisen turn out to be eigenfunctions of the operator $B = A_3 + L^2$. Among six coordinate systems mentioned in [4] an analog of parabolic coordinates is not encountered.

If one looks at 34 and 6 systems in $H_3$ and $S_3$ respectively, one can note that all six ones from $S_3$ have their counterparts in $H_3$. The main purpose of the present paper consists is the search of some counterparts of remaining $34 - 6 = 28$ systems. It turns out that such 28 systems in $S_3$ can be constructed, but they should be complex-valued; to preserve real nature of the geometrical space one must impose additional restrictions including complex conjugation.

In particular, the complex analog for parabolic coordinates in space of the positive curvature $S_3$ can be introduced and used in studying the quantum mechanical Kepler problem in this space.

### 2 Complex parabolic coordinates in real space $S_3$

Let us start with the following fact: from the the metric in Lobachevsky space

$$dl^2 = R^2 \left[ d\chi^2 + \sinh^2 \chi \left( d\theta^2 + \sin^2 \theta \, d\phi^2 \right) \right], \quad \chi \in [0, +\infty)$$

(3)

by means of the formal change $\chi \rightarrow i\chi$, $\sinh \chi \rightarrow i\sin \chi$ one can obtain the corresponding metric of the Riemannian space

$$dl^2 = -R^2 \left[ d\chi^2 + \sin^2 \chi \left( d\theta^2 + \sin^2 \theta \, d\phi^2 \right) \right], \quad \chi \in [0, \pi] .$$

(4)

This simple observation on $H_3 - S_3$ connection leads us to interesting consequences. Indeed, let us compare, for instance, wave functions and spectra for hydrogen atom in spaces of negative and constant curvature [14]:

$$\Psi_{nlm}(\chi, \theta, \phi) = N \, S(\chi) \, Y_{lm}(\theta, \phi) ,$$

$$S(\chi) = \sinh^l \chi \exp\left[ (n - l - 1 - \frac{e}{n}) \chi \right] \times$$

$$F\left( \frac{e}{n} + l + 1, l - n + 1, 2l + 2; 1 - e^{-2\chi} \right),$$

$$\epsilon_n = -\frac{e^2}{2n^2} - \frac{1}{2} \frac{2}{n^2} \left( n^2 - 1 \right);$$

(5)
\[ S_{3}, \quad \Psi_{nlm}(\chi, \theta, \phi) = K S(\chi) Y_{lm}(\theta, \phi), \]
\[ S(\chi) = \sin' \chi \exp[(i(n - l - 1) - \frac{e}{n})\chi] \times \]
\[ F(-i\frac{e}{n} + l + 1, l - n + 1, 2l + 2; 1 - e^{-2i\chi}), \quad (7) \]
\[ \epsilon_n = -\frac{e^2}{2n^2} + \frac{1}{2} (n^2 - 1), \quad e = \frac{\alpha}{R} \frac{M_{h}^2}{R^2}; \quad (8) \]

quantity \((M_{h}^2/R^2)\) provides us with natural unit for energy, \(e\) is a dimensionless parameter characterizing intensity of the Coulomb interaction. One may readily note that these two solutions turn into each other by the following formal substitutions:

\[ \chi \rightarrow i\chi, \quad e \rightarrow -i e, \quad \epsilon \rightarrow -\epsilon. \quad (9) \]

This example indicates that the relation between \(H_3\) and \(S_3\) reflected by substitution \(\chi \rightarrow i\chi\) is meaningful. In the context of the described above situation with coordinate systems in \(H_3\) and \(S_3\), let us make use of this correspondence \((\chi \rightarrow i\chi)\) as follows:

Let in the Lobachevsky space \(H_3\) be chosen a coordinate system \((\rho_1, \rho_2, \rho_3)\) (one of those 34 found by Olevsky), then as a first step one has to establish connection of such a system with spherical one:

\[ \rho_k = f_k(\chi, \theta, \phi), \quad (10) \]

and a second step is to introduce a corresponding coordinate system in the space \(S_3\) through the formal change \(\chi \rightarrow i\chi:\)

\[ \rho_k = f_k(i\chi, \theta, \phi). \quad (11) \]

With help of this prescription one can determine 34 coordinate systems in space \(S_3\) in comparison to six ones given in [4]. It turns out that 28 new (added) coordinate systems are complex-valued and therefore additional restrictions should be imposed which involve complex conjugation. All these extra coordinate systems permit the full separation of variables in the Helmholtz equation on the sphere \(S_3\).

Below, only one example of such coordinates, analog of the parabolic ones in space \(H_3\), will be examined in detail and applied to the study of the quantum-mechanical Kepler problem on the sphere \(S_3\).

### 3 Complex parabolic coordinates in \(S_3\) and the hydrogen atom

In [4] Olevsky had given the following coordinate system (the case XXV) in Lobachevsky space:

\[ dl^2 = R^2 \left[ \frac{(\rho_1 - \rho_2)}{4(\rho_1 - a)(\rho_1 - b)^2} \rho_1^2 + \frac{(\rho_2 - \rho_1)}{4(\rho_2 - a)(\rho_2 - b)^2} \rho_2^2 - (\rho_1 - a)(\rho_2 - a)\rho_3^2 \right], \quad (12) \]

where \((\rho_1, \rho_2, \rho_3)\) are connected with the four “Cartesian” (dimensionless) coordinates \((x_0, x_1, x_2, x_3)\) by the formulas

\[ x_0^2 - x_1^2 - x_2^2 - x_3^2 = 1, \quad x_0 > +1, \]
\[ \frac{x_2}{x_1} = \tan[(a - b)\rho_3], \quad b < \rho_1 < a < \rho_2, \]
\[ \frac{x_1^2 + x_2^2}{\rho_1 - a} + \frac{x_3^2 - x_0^2}{\rho_1 - b} + \frac{(x_3 - x_0)^2}{(\rho_1 - b)^2} = 0, \quad (i = 1, 2). \quad (13) \]
With \(a = +1\), \(b = 0\) and notations \(x_1^2 + x_2^2 = \sigma^2\), \(x_3 - x_0 = U\), \(x_3 + x_0 = V\), eqs. (13) give

\[
\sigma^2 + UV = -1, \quad x_1 = \sigma \cos \rho, \quad x_2 = \sigma \sin \rho, \\
\frac{\sigma^2}{\rho_1 - 1} + \frac{UV}{\rho_1} + \frac{U^2}{\rho_1^2} = 0, \quad \frac{\sigma^2}{\rho_2 - 1} + \frac{UV}{\rho_2} + \frac{U^2}{\rho_2^2} = 0,
\]

Combining two last relations one obtains

\[
\left(\frac{\rho_1}{\rho_1 - 1} - \frac{\rho_2}{\rho_2 - 1}\right) \sigma^2 + \left(\frac{1}{\rho_1} - \frac{1}{\rho_2}\right) U^2 = 0, \quad \left(\frac{\rho_1^2}{\rho_1 - 1} - \frac{\rho_2^2}{\rho_2 - 1}\right) \sigma^2 + (\rho_1 - \rho_2) UV = 0,
\]

whence having in mind \(\sigma^2 + UV = -1\), one gets

\[
U = \frac{\rho_1 \rho_2}{\rho_1 \rho_2 - \rho_1 - \rho_2}, \quad UV = \rho_1 \rho_2 - \rho_1 - \rho_2,
\]

and further

\[
U^2 = \rho_1 \rho_2, \quad V = U \frac{\rho_1^2 - \rho_1 - \rho_2}{\rho_1 \rho_2}.
\]

As a result, for \(U, V, \sigma\) we arrive at (take notice that the Lobachevsky model is realized on the upper sheet of hyperboloid \(x_0 > +1\), and therefore \(x_3 - x_0 \leq 0\))

\[
u = x_3 - x_0 = -\sqrt{\rho_1 \rho_2}, \quad V = x_3 + x_0 = \frac{\rho_1 + \rho_2 - \rho_1 \rho_2}{\sqrt{\rho_1 \rho_2}}, \quad \sigma = \sqrt{-1 - UV} = \sqrt{-(1 - \rho_1)(1 - \rho_2)}.
\]

Thus, explicit formulas relating \(\rho_1, \rho_2, \rho_3\) with Cartesian coordinates \((0, x_1)\) look as

\[
x_1 = \sqrt{-(1 - \rho_1)(1 - \rho_2)} \cos \rho_3, \quad x_2 = \sqrt{-(1 - \rho_1)(1 - \rho_2)} \sin \rho_3, \quad x_3 = \frac{\rho_1 + \rho_2 - 2 \rho_1 \rho_2}{2 \sqrt{\rho_1 \rho_2}}, \quad x_0 = \frac{\rho_1 + \rho_2}{2 \sqrt{\rho_1 \rho_2}}; \tag{14}
\]

and the inverse formulas are

\[
\rho_1 = \frac{x_0 - x_3}{x_0 + x}, \quad \rho_2 = \frac{x_0 - x_3}{x_0 - x}, \quad \rho_3 = \arctan \frac{x_2}{x_1}, \quad x = \sqrt{x_1^2 + x_2^2 + x_3^2}. \tag{15}
\]

Now, instead of the introduced \(\rho_1, \rho_2, \rho_3\) one can define other coordinates which behave simply in the limit \(R \to \infty\) (the curvature vanishes). Such a limiting procedure for spherical coordinates of the hyperbolic space \(H_3\) with metric

\[
dl^2 = \rho^2 \left[ d\chi^2 + \sinh^2 \chi \left( d\theta^2 + \sin^2 \theta \, d\phi^2 \right) \right]
\]

going over into spherical ones of the flat space \(E_3\)

\[
dl^2 = \left[ dr^2 + r^2 \left( d\theta^2 + \sin^2 \theta \, d\phi^2 \right) \right],
\]

looks as follows:

\[
\lim_{R \to \infty} (R \chi) = r, \quad \lim_{R \to \infty} (R \sinh \chi) = r. \tag{16}
\]

Eliminating \(x_0\) through \(q_1\):

\[
q_1 = \frac{x_1}{x_0} = \frac{x_1}{\sqrt{1 + x^2}}, \quad q_1 = \tanh \chi \, n_l,
\]

\[
n_l = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta), \quad q = \sqrt{q_1^2 + q_2^2 + q_3^2} = \tan \chi ;
\]

\[4\]
we can readily see that when $R \to \infty$ the coordinates $q_l$ will reduce to

$$
\lim_{R \to \infty} (Rq_l) = \lim_{R \to \infty} (R \tanh \chi n_l) = rn_l.
$$

(17)

So, to have coordinates with the known and understandable behavior in the limit $R \to \infty$ we define coordinates $t_1, t_2, \phi$:

$$
t_1 = 1 - \rho_1 = \frac{q_3 + q}{1 + q}, \quad t_2 = 1 - \rho_2 = \frac{q_3 - q}{1 - q}, \quad \phi = \rho_3 = \arctan \frac{q_2}{q_1};
$$

(18)

in the limit of the flat space they provide us with the known parabolic coordinates $(\xi, \eta, \phi)$ (see in [13]):

$$
\lim_{R \to \infty} (Rt_1) = z + r = \xi, \quad \lim_{R \to \infty} (Rt_2) = z - r = -\eta.
$$

(19)

The metric (12) in coordinates $(t_1, t_2, \phi)$ takes the form

$$
dl^2 = R^2 \left[ \frac{t_1 - t_2}{4t_1(1 - t_1)} dt_1^2 + \frac{t_2 - t_1}{4t_2(1 - t_2)} dt_2^2 - t_1t_2 d\phi^2 \right],
$$

$$
0 \leq t_1 \leq 1, \quad t_2 \leq 0, \quad 0 \leq \phi \leq 2\pi.
$$

(20)

Now, with the help of the rules (10)-(11) one has to define corresponding parabolic coordinates $t_1, t_2$ on the sphere $S_3$. To this end, coordinates $(t_1, t_2)$ in $H_3$ must be expressed in term of spherical ones $(\chi, \theta)$:

$$
t_1 = (1 + \cos \theta) \frac{\tanh \chi}{1 + \tanh \chi}, \quad t_2 = (1 - \cos \theta) \frac{-\tanh \chi}{1 - \tanh \chi}.
$$

(21)

From (21) we get defining relations for corresponding coordinates in $S_3$

$$
t_1 = (1 + \cos \theta) \frac{i \tan \chi}{1 + i \tan \chi}, \quad t_2 = (1 - \cos \theta) \frac{-i \tan \chi}{1 - i \tan \chi}.
$$

(22)

Take special notice that $(t_1$ and $t_2$) in (22) are complex-valued expressed through two real $(\chi, \theta)$. The inverse formulas are readily found:

$$
1 + \cos \theta = t_1(1 + \frac{1}{i \tan \chi}), \quad 1 - \cos \theta = t_1(1 - \frac{1}{i \tan \chi}),
$$

and therefore

$$
\cos \theta = \frac{t_1 - t_2 - 2t_1t_2}{t_1 - t_2}, \quad i \tan \chi = \frac{t_1 - t_2}{2 - t_1 - t_2}.
$$

(23)

So defined parametrization of $S_3$ by coordinates $t_1, t_2$ can be additionally detailed by the formulas:

$$
t_1 = (1 + \cos \theta) \varphi(\chi), \quad t_2 = (1 - \cos \theta) \varphi^*(\chi),
$$

$$
\varphi(\chi) = \sin^2 \chi + i \sin \chi \cos \chi = \sin \chi \exp[i(\frac{\pi}{2} - \chi)].
$$

(24)

From (24) one can derive the relationship between $t_1$ and $t_2$:

$$
t_1t_1^* = t_1 - t_2 - t_1t_2.
$$

(25)
its existence may evidently be referred to the real nature of the space $S_3$. The values of $\theta = 0$ and $\theta = \pi$ are peculiar:

\[
\begin{align*}
\theta = 0, \quad &\implies t_1 = 2 \varphi(\chi), \ t_2 = 0; \\
\theta = \pi, \quad &\implies t_1 = 0; \ t_2 = 2 \varphi^*(\chi).
\end{align*}
\] (26)

In the following, so defined coordinates $(t_1, t_2, \phi)$ are called parabolic coordinates on the sphere $S_3$. In the limit of the flat space, they reduce to the ordinary parabolic coordinates $[\ldots]$ in accordance with

\[
\lim_{\rho \to \infty} (-i\rho t_1) = \xi, \quad \lim_{\rho \to \infty} (-i\rho t_2) = -\eta.
\] (27)

Now, we transform the metric of the space $S_3$ in spherical coordinates

\[
dl^2 = -R^2 \left[ d\chi^2 + \sin^2 \chi \left( d\theta^2 + \sin^2 \theta d\phi^2 \right) \right],
\]
to complex parabolic $t_1, t_2, \phi$. As a first step, with the help of

\[
\sin^2 \theta = t_2 \left( \frac{1 - iq}{-iq} \right) t_1 \left( \frac{1 + iq}{iq} \right) = t_1 t_2 \left( \frac{1 + g^2}{q^2} \right), \quad \sin^2 \chi = \frac{q^2}{1 + q^2},
\]

we get

\[
\sin^2 \chi \sin^2 \theta d\phi^2 = t_1 t_2 d\phi^2.
\]

As a second step, we have

\[
(d\theta)^2 = \frac{1}{\sin^2 \theta} \left( d\cos \theta \right)^2 = \frac{1}{t_1 t_2} \frac{q^2}{1 + q^2} \left[ d\left( \frac{t_1 + t_2 - 2t_1 t_2}{t_1 - t_2} \right) \right]^2,
\]
or

\[
\sin^2 \chi (d\theta)^2 = \frac{4}{(1 + q^2)^2 t_1 t_2 (t_1 - t_2)^2} \left[ t_2(t_2 - 1) dt_1 - t_1(t_1 - 1) dt_2 \right]^2;
\]

so that

\[
\sin^2 \chi (d\theta)^2 = \frac{1}{4(1 - t_1)^2(1 - t_2)^2 t_1 t_2} \left[ t_2(t_2 - 1) dt_1 - t_1(t_1 - 1) dt_2 \right]^2.
\]

Finally, taking into account relations

\[
i \tan \chi = \frac{t_1 - t_2}{2 - t_1 - t_2}, \quad \implies \frac{id\chi}{\cos^2 \chi} = \left[ dt_1 \frac{2(1 - t_2)}{(2 - t_1 - t_2)^2} - dt_2 \frac{2(1 - t_1)}{(2 - t_1 - t_2)^2} \right],
\]
and

\[
\cos^2 \chi = \frac{1}{1 + \tan^2 \chi} = \frac{(2 - t_1 - t_2)^2}{4(1 - t_1)(1 - t_2)},
\]

we get

\[
(d\chi)^2 = \frac{-1}{4(1 - t_1)^2(1 - t_2)^2} \left[ (1 - t_2)dt_1 - (1 - t_1)dt_2 \right]^2.
\]
Therefore, for the metric in parabolic coordinates in $S_3$ we have arrived at the form

$$dl^2 = R^2 \left[ \frac{t_2 - t_1}{4t_1(1 - t_1)^2} dt_1^2 + \frac{t_1 - t_2}{4t_2(1 - t_2)^2} dt_2^2 + t_1 t_2 d\phi^2 \right] .$$

(28)

Formally, this formula differs from its counterpart in the space $H_3$ only by presence of $(-1)$ in the expression for $dl^2$.

The Schrödinger Hamiltonian for the Kepler problem

$$H = -\frac{1}{2} \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^\alpha} \sqrt{g} g^{\alpha\beta} \frac{\partial}{\partial x^\beta} - \frac{\epsilon}{q} ,$$

(29)

will take the following explicit form

$$H = 2 \frac{1 - t_1}{t_1 - t_2} \frac{\partial}{\partial t_1} t_1(1 - t_1) \frac{\partial}{\partial t_1} + 2 \frac{1 - t_2}{t_2 - t_1} \frac{\partial}{\partial t_2} t_2(1 - t_2) \frac{\partial}{\partial t_2} -$$

$$- \frac{1}{2t_1 t_2} \frac{\partial^2}{\partial \phi^2} - ie \frac{2 - t_1 - t_2}{t_1 - t_2} .$$

(30)

This Hamiltonian may be referred to analogous one in $H_3$ with the help of formal changes: $\epsilon \rightarrow -ie$ and $H \rightarrow -H$.

Now, acting in the way used for Lobachevsky space [14], one can perform the separation of variables in the Schrödinger equation in the spherical space $S_3$:

$$\Psi(t_1, t_2, \phi) = f_1(t_1) f_2(t_2) e^{i m \phi} .$$

(31)

From the equation $H \Psi = \epsilon \Psi$ it follows

$$f_2 \frac{2(1 - t_1)}{t_1 - t_2} \frac{d}{dt_1} t_1(1 - t_1) \frac{d}{dt_1} f_1 + f_1 \frac{2(1 - t_2)}{t_2 - t_1} \frac{d}{dt_2} t_2(1 - t_2) \frac{d}{dt_2} f_2$$

$$+ \frac{m^2}{2t_1 t_2} f_1 f_2 - ie \frac{2 - t_1 - t_2}{t_1 - t_2} f_1 f_2 = \epsilon f_1 f_2 .$$

(32)

or

$$\frac{1}{f_1} (1 - t_1) \frac{d}{dt_1} t_1(1 - t_1) \frac{d}{dt_1} f_1 - \frac{m^2}{4t_1} f_1 + \frac{i \epsilon}{2} t_1 + \frac{\epsilon}{2} t_1 - \frac{m^2}{4t_1} t_2 + (k_1 - k_2) = 0 ;$$

(33)

where two separation constants $k_1$ and $k_2$ are introduced:

$$k_1 - k_2 = -i \epsilon .$$

(34)

As a result, we arrive at the system of two 2-order ordinary differential equations

$$(1 - t_1) \frac{d}{dt_1} t_1(1 - t_1) \frac{d}{dt_1} f_1 + (\frac{i \epsilon - \epsilon}{2} t_1 - \frac{m^2}{4t_1} + k_1) f_1 = 0 ,$$

$$(1 - t_2) \frac{d}{dt_2} t_2(1 - t_2) \frac{d}{dt_2} f_2 + (\frac{-i \epsilon - \epsilon}{2} t_2 - \frac{m^2}{4t_1} + k_2) f_2 = 0 .$$

(35)

Analogous system of equation in the Lobachevsky space has the form [14]

$$(1 - t_1) \frac{d}{dt_1} t_1(1 - t_1) \frac{d}{dt_1} f_1 + (\frac{-\epsilon + \epsilon}{2} t_1 - \frac{m^2}{4t_1} + k_1) f_1 = 0 ,$$

$$(1 - t_2) \frac{d}{dt_2} t_2(1 - t_2) \frac{d}{dt_2} f_2 + (\frac{\epsilon + \epsilon}{2} t_2 - \frac{m^2}{4t_1} + k_2) f_2 = 0 .$$

(36)
Solutions of eqs. (35) and (36) can be searched for with the help of substitution
\[ f_1 = t_1^{a_1} (1 - t_1)^{b_1} S_1(t_1), \quad f_2 = t_2^{a_2} (1 - t_2)^{b_2} S_2(t_2). \] (37)

Below, all calculations will be done for the case (35); at any point, transition to \( H_3 \) space is accomplished by the formal changes \( \epsilon \rightarrow -\epsilon, \quad -i\epsilon \rightarrow \epsilon \). It suffices to consider in detail only the first equation for \( f_1(t_1) \) (index 1 is omitted below)
\[

t(1 - t) S'' + S' [2a(1 - t) - 2bt + (1 - 2t)] \\
+ [a(a - 1)(\frac{1}{t} - 1) - 2ab + b(b - 1)(\frac{1}{t} - 1) + a(\frac{1}{t} - 2) - b(2 - \frac{1}{1 - t})] \\
+ \frac{ie - \epsilon}{2} (\frac{1}{1 - t} - 1) - \frac{m^2}{4} (\frac{1}{t} + \frac{1}{1 - t}) + k \frac{1}{1 - t} \] \( S = 0 \). (38)

Both terms proportional to \( t^{-1} \) and \( (1 - t)^{-1} \) may be eliminated from the equation by adding the requirements:
\[ a^2 - \frac{m^2}{4} = 0, \quad b^2 + \frac{ie - \epsilon}{2} - \frac{m^2}{4} + k = 0; \] (39)
then eq. (38) results in
\[
t(1 - t) S'' + S' [2a + (1 - 2a + 2b + 2)t] \\
- [a(a + 1) + 2ab + b(b + 1) + \frac{ie - \epsilon}{2}] S = 0. \] (40)

That is, \( S(t) \) turns out to be a hypergeometric function \( S(t) = F(\alpha, \beta, \gamma; t) \) whose parameters are determined by
\[
\alpha + \beta + 1 = 2a + 2b + 2, \quad \\
\alpha \beta = a(a + 1) + 2ab + b(b + 1) + \frac{ie - \epsilon}{2}, \quad \gamma = 2a + 1, \] (41)
which implies
\[
\alpha = a + b + \frac{1}{2} + \sqrt{\frac{1}{4} + \frac{ie - \epsilon}{2}}, \quad \beta = a + b + \frac{1}{2} - \sqrt{\frac{1}{4} + \frac{ie - \epsilon}{2}}, \quad \gamma = 2a + 1. \] (42)

Thus, the quantum mechanical Kepler problem in complex parabolic coordinates \( t_1, t_2, \phi \) has been solved in hypergeometric functions. The separation constants \( k_1 \) and \( k_2 \) are connected by the relation (33):
\[
f_1 = t_1^{a_1} (1 - t_1)^{b_1} S_1, \quad f_2 = t_2^{a_2} (1 - t_2)^{b_2} S_2; \]
\[ S_1 = F(\alpha_1, \beta_1, \gamma_1; t_1), \quad S_2 = F(\alpha_2, \beta_2, \gamma_2; t_2); \]
\[ a_1^2 = \frac{m_1^2}{4}, \quad a_2^2 = \frac{m_2^2}{4}; \]
\[ b_1^2 = \frac{\epsilon - \epsilon}{2} + \frac{m_1^2}{4} - k_1, \quad b_2^2 = \frac{\epsilon + \epsilon}{2} + \frac{m_2^2}{4} - k_2; \]
\[ \alpha_1 = a_1 + b_1 + \frac{1}{2} + \sqrt{\frac{1}{4} + \frac{\epsilon - \epsilon}{2}}, \quad \alpha_2 = a_2 + b_2 + \frac{1}{2} + \sqrt{\frac{1}{4} + \frac{\epsilon + \epsilon}{2}}; \]
\[ \beta_1 = a_1 + b_1 + \frac{1}{2} - \sqrt{\frac{1}{4} + \frac{\epsilon - \epsilon}{2}}, \quad \beta_2 = a_2 + b_2 + \frac{1}{2} - \sqrt{\frac{1}{4} + \frac{\epsilon + \epsilon}{2}}; \]
\[ \gamma_1 = 2a_1 + 1; \quad \gamma_2 = 2a_2 + 1. \] (43)

Some additional study is required to obtain the physical wave solutions of the Schrödinger equation.
4 The Runge-Lenz vector in $S_3$ and complex parabolic coordinates

At separating the variables in Schrödinger equation two constants were introduced $k_1 \ k_2$; the problem is to find an operator that is diagonalized on wave functions (31) with eigenvalues $(k_1 + k_2)$:

$$\hat{B} f_1 f_2 e^{im\phi} = (k_1 + k_2) f_1 f_2 e^{im\phi}.$$ (44)

Taking into account (36), one can obtain for the operator $\hat{B}$ the following representation:

$$\hat{B} = -(1 - t_1) \frac{\partial}{\partial t_1} t_1(1 - t_1) \frac{\partial}{\partial t_1} - t_1 \frac{(-H + ie)}{2} \frac{1}{4t_1} \frac{\partial^2}{\partial \phi^2}$$

$$- (1 - t_2) \frac{\partial}{\partial t_2} t_2(1 - t_2) \frac{\partial}{\partial t_2} - t_2 \frac{(-H - ie)}{2} \frac{1}{4t_2} \frac{\partial^2}{\partial \phi^2},$$ (45)

or after substituting the expression for $H$ from (30)

$$\hat{B} = -ie \frac{t_1 + t_2 - 2t_1 t_2}{t_1 - t_2} + \frac{2t_2(1-t_1)(1-2t_1)}{t_1 - t_2} \frac{\partial}{\partial t_1} + \frac{2t_1(1-t_2)(1-2t_2)}{t_1 - t_2} \frac{\partial}{\partial t_2}$$

$$+ \frac{2t_1 t_2(1-t_1)^2}{t_1 - t_2} \frac{\partial^2}{\partial t_1^2} + \frac{2t_2 t_1(1-t_2)^2}{t_2 - t_1} \frac{\partial^2}{\partial t_2^2} - \frac{t_1 + t_2}{2t_1 t_2} \frac{\partial^2}{\partial \phi^2},$$ (46)

note the identity

$$-ie \frac{t_1 + t_2 - 2t_1 t_2}{t_1 - t_2} = -ie \cos \theta = -ie \frac{q_3}{q}. $$ (47)

In $H_3$ and $S_3$ the quantum mechanical Runge-Lenz operator is constructed in term of momentum and orbital momentum by the formula [11,12.]

$$\vec{A} = \vec{q} + \frac{1}{2} (\vec{L} \vec{P} - [\vec{P} \vec{L}]),$$ (48)

where

$$\vec{P} = (P_i), \ P_i = -i(\delta_{ij} \mp q_i q_j) \frac{\partial}{\partial q_j}, \ \vec{L} = [\vec{q} \vec{P}],$$ (49)

upper sign corresponds to the model $H_3$, lower corresponds to $S_3$ model; operators and $\vec{L}, \vec{P}$ are measured in units $\hbar$ and $\hbar/\rho$ respectively. In correspondence with symmetry of space models, the components of $\vec{P}, \vec{L}$ obey the commutation relations of Lie algebras so(3.1) and so(4):

$$[L_a, L_b] = i \epsilon_{abc} L_c, \ [L_a, P_b] = i \epsilon_{abc} P_c, \ [P_a, P_b] = \pm i \epsilon_{abc} L_c.$$ 

As in the above expression for $\hat{B}$, specific term $-ie q_3/q$ is presented, in the model $H_3$ we see the term $e q_3/q$, it is natural to look for certain relationship between $\hat{B}$ and $A_3$ – they look as follows (all details are omitted here)

$$\hat{B} = (A + \vec{L}^2),$$

in $H_3$, \ \ \ \ $\hat{B} = (A + i \vec{L}^2),$ (50)
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