Super-Golden-Gates for \( PU(2) \)

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\textit{To David Kazhdan with admiration.}

Abstract

To each of the symmetry groups of the Platonic solids we adjoin a carefully designed involution yielding topological generators of \( PU(2) \) which have optimal covering properties as well as efficient navigation. These are a consequence of optimal strong approximation for integral quadratic forms associated with certain special quaternion algebras and their arithmetic groups. The generators give super efficient 1-qubit quantum gates and are natural building blocks for the design of universal quantum gates.

1 Introduction

The \( n \)-qubit circuits used for quantum computation are unitaries in \( U \left( (\mathbb{C}^2)^\otimes n \right) = U(2^n) \) which are products of elementary unitaries, each of which operates on a fixed (typically at most 3) number of qubits. The standard universal gate set for quantum computing consists of all 1-qubit unitaries i.e. \( U(2) \) (which can be applied to any of the single qubits) and the 2-bit XOR gate, together these generate \( U(2^n) \) ([NC11]). In a classical computer the only operations on a single bit are to leave it or flip it. In the quantum setting we can rotate by these \( 2 \times 2 \) unitaries.

To further reduce to a finite universal gate set one has to settle for a topologically dense set, and since overall phases do not matter it suffices to find “good” topological generators of \( G = PU(2) \). The Solovay-Kitaev algorithm [NC11] ensures that any fixed topological generators of \( G \) have reasonably short words (i.e. circuits) to approximate any \( x \in G \) (with respect to the bi-invariant metric \( d^2(x,y) = 1 - \frac{\text{trace}(x^* y)}{2} \)). From the point of view of general polynomial type complexity classes this is sufficient, however there is much interest ([NC11, KMM13b, KMM13a]) both theoretical and practical, to optimize the choice of such generators.

Golden-Gates ([Sar15d]) which correspond to special arithmetic subgroups of unit quaternions and which were introduced as optimal generating rotations in [LPS87], yield variants of optimal generators. A particular case is the “Clifford plus T” gates described below and which appear in most textbooks. The Clifford gates form a finite subgroup \( C_{24} \) of order 24 in \( G \) and to make the set universal one needs to add an extra element of \( G \). The popular choice is the order 8 element \( T = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{pmatrix} \). That these generate an \( S \)-arithmetic group was shown in [KMM13b], see also [Sar15d].

Considerations of fault-tolerance when applying these to make circuits in \( U(2^n) \), require among other things that the universal gate set consists of elements of finite order. Moreover for the Clifford plus T gates, the applications of the \( c \) gates with \( c \in C_{24} \) in a circuit are considered to
be of small cost compared to $T$ ([BK05, BS12]). This leads to the “T-count” being the measure of complexity of a word and to the problem that we consider in this note:

To find universal gate sets (i.e. ones that are topological generators of $G$) which are of the form a finite group $C$ in $G$ together with an extra element $T$, which we take to be an involution, so that $C$ plus $T$ is optimal with respect to covering $G$ with a small T-count, and at the same time to be a able to navigate $G$ efficiently with these gates.

The key feature that was needed for a Golden-Gate construction was that the corresponding $S$-arithmetic unit quaternion group act transitively on the vertices of the corresponding $(q + 1)$-regular tree (here $q$ is a prime power). The extra “miracle” that is needed here is that the group act transitively on the edges. With this extra requirement there are only finitely many such “Super-Golden-Gates”, see Section 3. We list some of them; in each case the finite group $C$ is naturally a subgroup of the symmetries of a platonic solid:

(1) Pauli plus $T$ (Cube)

$$C_4 = \left\langle \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\rangle,$$

the 4-group of Pauli matrices

$$T_4 = \begin{pmatrix} 1 & 1-i \\ 1+i & -1 \end{pmatrix}.$$

(2) Minimal Clifford plus $T$ (Octahedron)

$$C_3 = \left\langle \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & -i \end{pmatrix} \right\rangle,$$

which is a subgroup of the Clifford group,

$$T_3 = \begin{pmatrix} 0 & \sqrt{2} \\ 1+i & 0 \end{pmatrix}.$$

(Note: These generate a finite index subgroup of the usual “Clifford plus T” group but the latter has redundancies in that the circuits for the exactly synthesizable elements are not unique. For our choices, it is.)

(3) Hurwitz group plus $T$ (Tetrahedron)

$$C_{12} = \left\langle \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & -i \end{pmatrix} \right\rangle,$$

$$T_{12} = \begin{pmatrix} 3 & 1-i \\ 1+i & -3 \end{pmatrix}.$$

(4) Clifford plus $T$ (Octahedron)

$$C_{24} = \left\langle \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \right\rangle,$$

the Clifford group

$$T_{24} = \begin{pmatrix} -1 - \sqrt{2} & 2 - \sqrt{2} + i \\ 2 - \sqrt{2} - i & 1 + \sqrt{2} \end{pmatrix}.$$

(5) Klein’s Icosahedral group plus $T$ (Icosahedron)

$$C_{60} = \left\langle \begin{pmatrix} 1 & 1 \\ i & -1 \end{pmatrix}, \begin{pmatrix} 1 & \varphi - i/\varphi \\ \varphi + i/\varphi & -1 \end{pmatrix} \right\rangle,$$

$$\varphi = \frac{1 + \sqrt{5}}{2};$$

$$T_{60} = \begin{pmatrix} 2 + \varphi & 1-i \\ 1+i & -2-\varphi \end{pmatrix}.$$
These Super-Golden-Gate sets all enjoy the same relative optimal distribution and navigation properties that we describe next. The only difference between them is the number \( N(t) \) below, and from the point of view of this count, example (5) is the best and presumably it is the optimal absolute Super-Golden-Gate set.

Let \( N(t) \) be the number of circuits in the elements of \( C \) and \( T \), and of \( T\)-count \( t \). That is words of the form \( c_0Tc_1T\ldots Tc_t \) where the \( c_j \)'s are not 1 except possibly at the ends. Clearly

\[
N(t) = |C|^2 (|C| - 1)^{t-1}, \quad t \geq 1
\]  

(1.1)

We want these \( N(t) \) circuits to represent distinct elements in \( G \). In fact, for any \( k \), we want the circuits of length at most \( k \) to realize distinct element in \( G \) (these elements are often referred to as the “exactly synthesized” gates). This requirement is equivalent to the subgroup of \( G \) generated by \( C \) and \( T \) being isomorphic to \( C \ast (\mathbb{Z}/2\mathbb{Z}) \). This is the case for the super-gates.

Next we want these \( N(t) \) elements to almost cover \( G \) optimally. That is almost all balls \( B \) in \( G \) (with respect to the bi-invariant metric) of volume \( V \), where \( V \cdot N \gg (\log N)^2 \), should contain at least one of the \( N = N(t) \) points. In order to cover almost all balls, clearly \( V \cdot N \) must be large, so that for an almost all covering the above is essentially best possible, that is almost all \( x \in G \) have a circuit of essentially the shortest possible \( T\)-count approximating \( x \). The Super-Golden-Gates enjoy this optimal almost covering property (see Section 3). As with Golden-Gates (see [Sar15d]) these super-gates do not cover all balls of this smallest size. There are rare balls with volume \( V \approx N^{-3/4} \) which are free of the \( N = N(t) \) exactly synthesized elements, while every ball \( B \) of volume at least \( N^{-1/2} \) does contain one of the \( N(t) \) points. Whether there exist gate sets which do not have this “big hole” feature is an interesting open problem ([Sar15d, RS17]).

The final requirement for these super-gates is that we can find the short circuits (whose existence is ensured from the discussion above) efficiently. The task is given a ball \( B \) in \( G \) and a \( k \), to find (if it exists) a circuit in the gates of length at most \( k \) and which lies in \( B \). The problem is clearly in NP. The algorithm introduced in [R-S] can be executed for our super-gate sets, and it leads under the assumption that one can factor integers quickly, to a heuristic algorithm, which for \( B \) whose center is a diagonal matrix \( y \in G \), resolves the above task in poly \((\log 1/\sqrt{\nu(B)})\) steps. In particular for such a \( y \) it finds for a given \( k \), (one of) the circuit of length at most \( k \), which best approximates \( y \). On the other hand, the Diophantine problem (see below) that is at the heart of the above algorithm for \( y \) diagonal, is NP-complete when \( y \) is replaced by a general \( x \in G \). Thus finding the shortest circuit approximating a general \( x \in G \) is apparently a genuinely hard complexity problem. This does not preclude there being an efficient algorithm which gives a good approximation to the shortest circuit. By factoring the general \( x \in G \) into a product \( y_1y_2y_3 \) where the \( y_j \)'s are in different diagonal subgroups (see Section 3) one can produce circuits with \( T\)-count which is three times longer than the optimally short circuit. Removing this factor of 3 remains a basic open problem concerning Golden Gates.

The Diophantine problem that underlies the analysis of these Golden Gates is “strong approximation for sums of four squares”. For simplicity we restrict here to \( \mathbb{Z} \) and \( n \in \mathbb{N} \) odd (see Sections 2 and 3):

\[
x_1^2 + x_2^2 + x_3^2 + x_4^2 = n.
\]  

(1.2)

Let \( S(n) \) be the set of integer solutions \( x = (x_1, x_2, x_3, x_4) \) to (1.2). To each \( x \in S(n) \) let \( \bar{x} = \frac{x}{\sqrt{n}} \in S^3 \), the unit sphere in \( \mathbb{R}^4 \) with its round metric and normalized volume \( \mu \). In Section

\(^{(1)}\)Without the efficient factorization assumption one can find efficiently a circuit in \( B \) whose length is \((1 + o(1))\) times longer then the optimal one, see (3.13).
we show that these \(|S(n)|\) points \(\bar{x}\) almost cover \(S^3\) optimally. More precisely, if \(d(n)\) is the number of divisors of \(n\) then if

\[
\frac{V \cdot |S(n)|}{d(n)} \to \infty \quad \text{as} \quad n \to \infty,
\]

then

\[
\mu \left( S^3 \setminus \left( \bigcup_{x \in S(n)} B_V(\bar{x}) \right) \right) \to 0.
\]  (1.3)

Here \(B_V(\xi)\) is the ball in \(S^3\) centered at \(\xi\) and of volume \(V\). Among the ingredients in the proof of (1.3) are the Ramanujan Conjectures, which are theorems ([Del74]) for the cases at hand. The computational complexity problem associated with strong approximation for (1.2) is (see Section 2):

**Task I.** Given \(n, \xi \in S^3\) and \(V\) to find \(x \in S(n)\) such that \(\bar{x} \in B_V(\xi)\).

This task is clearly in NP and Theorem 2.5 shows that it is NP-complete, at least under a randomized reduction. On the other hand if \(\xi\) is of the special form, \(\xi = (\xi_1, \xi_2, 0, 0)\) then an adaption of [RS15] is given in Section 2 which resolves the above task efficiently (i.e. in \(\text{poly}(\log n)\) steps). The algorithm assumes that one has an efficient algorithm to factor \(m\)'s in \(\mathbb{N}\), and for its running time it relies on some heuristics (see Section 2.2).

We note that the analogous problem of an optimal topological generator for \(U(1) = \mathbb{R}/\mathbb{Z}\) has a golden solution. If \(R_\alpha\) is the rotation \(x \mapsto x + \alpha\) (one could start with two reflections whose composition is such a rotation) then the covering volume \(V(n, \alpha)\) if the words of length at most \(n\) (i.e. \(S_\alpha(n) = \{R_\alpha^j; j = 1, \ldots, n\}\)) is

\[
V(n, \alpha) = \sup_{I \text{ an interval}} |I|.
\]

In [GVL68] it is shown that

\[
\lim_{n \to \infty} V(n, \alpha)|S_\alpha(n)| \geq 1 + \frac{2}{\sqrt{5}}
\]

with equality iff \(\alpha = \frac{a \varphi + b}{c \varphi + d}\) with \(\varphi\) the golden ratio and \(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in GL_2(\mathbb{Z})\). Moreover, the continued fraction algorithm allows one to find efficiently the \(j \leq n\) such that \(R_{\xi_j}\) best approximates a given \(\xi \in U(1)\).

We end the Introduction with a brief outline of the paper. In Section 2 the computational complexity results connected with (1.2) and Task I are established. In Section 3.1 the optimal covering properties of solutions to (1.2) and its generalizations to number rings are proven. Section 3.2 applies these results to unit groups of quaternions verifying the advertised properties of Golden and Super-Golden gate sets. Section 4 is devoted to the construction of such gate sets. Finally in Section 5 we examine some semigroups and asymmetric random walks associated with Super-Golden gates, and use the examples of Section 4 to construct Ramanujan Cayley digraphs.

**Acknowledgement.** The authors thank O. Regev and J. Vondrák for for illuminating discussions on integer programming and NP-completeness.
2 Sums of squares and complexity

2.1 Sums of two squares

The algorithm for navigating $G$ using Golden Gates is based on solving the simplest quadratic Diophantine inequalities. We begin with the setting of the integers $\mathbb{Z}$. Let $n = p_1^{e_1} \ldots p_k^{e_k}$ be a sum of two squares:

$$n = x^2 + y^2, \quad (2.1)$$

iff each odd prime factor $p_j$ of $n$ with $e_j$ odd is congruent to 1 (mod 4).

**Task II.** To solve (2.1) efficiently.

By efficiently we mean in polynomial time in the input. In this case the input is $n \in \mathbb{N}$ and specifying $n$ requires $\log n$ bits, so the Task is to find $x$ and $y$ satisfying (2.1) in $(\log n)^c$ steps, for some fixed $c$. More generally denote by $h(\alpha)$ the height of a rational number $\alpha = \frac{a}{b}$, that is $\max \{ \log |a|, \log |b| \}$, so that $h(\alpha)$ measures the number of bits in $\alpha$. As discussed in the introduction we assume throughout that factoring an $n \in \mathbb{N}$ can be done efficiently. The following is due to Schoof [Sch85].

**Theorem 2.1.** Task II has an efficient resolution.

**Proof.** First factor $n = 2^t p_1^{e_1} \ldots p_k^{e_k}$ (note $k$ and $e_j$ are $O(\log n)$) and for each odd $e_j$ with $p_j \equiv 1 \pmod{4}$, solve

$$x_j^2 + y_j^2 = p_j \quad (2.2)$$

[Sch85] gives an $O\left((\log p_j)^9\right)$ algorithm to solve (2.2). One could also proceed instead with various random (running time) algorithms for solving (2.2) which run even faster and work very well in practice [Coh93, §1.5.1]. With $x_j + iy_j \in \mathbb{Z}[\sqrt{-1}]$ at hand we can find a solution to (2.1) by taking

$$x + iy := (1 + i)^t \prod_{e_j \text{ odd}} (x_j + iy_j)^{e_j} \prod_{e_j \text{ even}} p_j^{e_j/2},$$

as $N(x + iy) := (x + iy)(x - iy) = n$.

If we modify Task II by adding an inequality, the complexity of the problem changes dramatically.

**Task III.** Given $n \in \mathbb{N}; \alpha, \beta \in \mathbb{Q}$ find a solution to (2.1) with $\alpha \leq \frac{x}{y} \leq \beta$.

Task III is plainly in NP, that is to say one can recognize a solution efficiently if one is presented with $x$ and $y$.

**Theorem 2.2.** Task III is NP-complete under a randomized reduction.

What this says is that under a randomized reduction (see below) if Task III has an efficient solution then $P=NP$. So for us the upshot is that unlike factoring, Task III is genuinely hard.

**Proof.** The Diophantine approximation condition in Task III can be replaced by III': find $x+iy \in \mathbb{Z}[\sqrt{-1}]$, $N(x+iy) = n$ and $\alpha' \leq \arg(x+iy) \leq \beta'$. We show that if III' can be resolved efficiently then so can the long known to by NP-complete subsum (or ‘Knapsack’ as it is called in [Kar72]) problem:
**Task IV.** Given \(t_1, t_2, \ldots, t_n, t \in \mathbb{N}\), are there \(\varepsilon_j \in \{0, 1\}, j = 1, \ldots, k\) such that

\[
\sum_{j=1}^{k} \varepsilon_j t_j = t.
\]

Note that the bit content in Task IV is \(H = \sum_{j=1}^{k} \log t_j\) and so we seek a \(O(H^c)\) steps algorithm (\(c\)-fixed). Let \(M = [100k \max_j \{t_j, t\}]\) and \(\tau = \sum_{j=1}^{k} t_j\). We seek primes \(P_j\) in the ring \(\mathbb{Z} \left[ \sqrt{-1} \right]\) with \(N(P_j) = X\) and \(\arg(P_j)\) near \(t_j/M, j = 1, \ldots, k\), and \(X\) to be chosen. According to [Kub55] the number of primes \(P\) with \(N(P) \leq X\) and \(|\arg(P) - \beta| \leq X^{-\frac{1}{10}}\) is asymptotic to

\[
\frac{X \log X}{X^{\frac{1}{10}} \pi} \text{ as } X \to \infty,
\]

uniformly for any \(\beta\). Hence for each \(j = 1, \ldots, k\) there is \(P_j\) with \(N(P_j) \leq (100kM)^{10}\) and for which

\[
\left| \arg(P_j) - \frac{t_j}{M} \right| \leq \frac{1}{100kM},
\]

(where \(\arg(z) \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)\)).

We discuss how to find \(P_j\) efficiently below, this leading to the randomized reduction part. For \(\varepsilon_j \in \{0, 1\}\) and \(\eta_j = 2\varepsilon_j - 1 \in \{-1, 1\}\) we have

\[
2 \left[ \sum_{j=1}^{k} \frac{\varepsilon_j t_j}{M} - \frac{t}{M} \right] = \sum_{j=1}^{k} \eta_j \arg(P_j) + \frac{\tau - 2t}{M} + \sum_{j=1}^{k} \eta_j \left[ \frac{t_j}{M} - \arg(P_j) \right]
= \sum_{j=1}^{k} \eta_j \arg(P_j) + \frac{\tau - 2t}{M} + (\Theta)
\]

where

\[
|\Theta| \leq \frac{1}{100M}.
\]

Hence

\[
\sum_{j=1}^{k} \varepsilon_j t_j = 1 \iff \left| \sum_{j=1}^{k} \eta_j \arg(P_j) - \frac{2t - \tau}{M} \right| \leq \frac{1}{100M}.
\]

Let

\[
n = N(P_1)N(P_2)\ldots N(P_k)
\]

and

\[
P_j = x_j + iy_j \text{ with } x_j > 0.
\]

Then \(z = x + iy\) solves Task III’ with

\[
\frac{2t - \tau}{M} - \frac{1}{100M} \leq \arg z \leq \frac{2t - \tau}{M} + \frac{1}{100M}
\]

iff \(z = P_1^{\sigma(\eta_1)} \ldots P_k^{\sigma(\eta_k)}\) and \(\sum_{j=1}^{k} \varepsilon_j t_j = t\), where \(\sigma(\eta_j) = \text{id}\) if \(\eta_j = 1\) and \(\sigma(\eta_j)\) is complex conjugation if \(\eta_j = -1\).
Note that \( h(M) = O(H) \) and

\[
h(n) \leq \sum_{j=1}^{k} h(N(P_j)) \leq k \log (100kM)^{10} \ll H^2,
\]

so that the input for Task III' is polynomial in terms of that of Task III. It remains to find \( P_1, \ldots, P_k \) efficiently. That is to find a prime \( P \) in the sector \( N(P) \leq X \) and \( \arg(P) \in S_X \) where \( |S_X| = X - \frac{X}{10} \) and \( h(X) = H \). If we choose a random \( \beta \in \mathbb{Z} \) in this section the probability that it is prime is \( \frac{1}{\log X} \). Thus sampling such \( \beta \)'s and checking if they are prime will produce the requisite \( P \) in polynomial \( H \) steps. In this way we produce the \( P_j \)'s using a randomized polynomial time procedure. Once the \( P_j \)'s are determined in polynomial \( H \) steps then so is \( n \). Thus Task III is reduced to Task III' albeit by a randomized reduction.

Remark 2.3. A similar NP-complete problem is discussed in [Sud10]. It asserts that the task: Given \( n, \alpha, \beta \) to find integers \( x, y \) such that \( xy = n \) and \( \alpha \leq x \leq \beta \); is NP-complete under a randomized reduction (see discussion [Cad11]).

2.2 Sums of four squares

The Diophantine approximation problem that is directly connected to our navigation of \( G \) is that of four squares,

\[
x_1^2 + x_2^2 + x_3^2 + x_4^2 = n. \tag{2.3}
\]

It is well known [Lag70] that for every \( n \geq 0 \) (2.3) has integral solutions. The question of finding a solution efficiently is discussed in [RS86]. A randomized efficient algorithm to do this is to choose \( x_3 \) and \( x_4 \) with \( x_3^2 + x_4^2 \leq n \) at random and to check if \( n - (x_3^2 + x_4^2) \) is a prime \( p \equiv 1 \pmod{4} \). The last can be done efficiently ([AKS04]) and if yes, then Schoof gives \( x_1 \) and \( x_2 \) and we have a solution to (2.3). If \( n - (x_3^2 + x_4^2) \) is not such a prime we repeat with another choice of \( x_3, x_4 \). The probability of success is \( \frac{1}{\log n} \) in view of the density of primes, and this leads to a randomized algorithm.

For the approximation problem we project the solutions \( x = (x_1, x_2, x_3, x_4) \) of (2.3) onto \( S^3 = \{ \xi \in \mathbb{R}^4 : ||\xi|| = 1 \} \) by sending \( x \) to

\[
\bar{x} = \frac{x}{\sqrt{n}}. \tag{2.4}
\]

\( S^3 \) comes with its round metric, relative to which all measurements are made. So for example \( B_r(\xi) \) is the ball centered at \( \xi \) of radius \( r \).

Task V. Given \( n, \xi, \varepsilon \) to find a solution of (2.3) with \( \bar{x} \in B_\varepsilon(\xi) \).

Remark 2.4. In terms of the bit content or height \( h \) we take \( \varepsilon \) and \( \xi \) to be rational and \( h(\xi) = \max \{ h(\xi_j) | j = 1, 2, 3, 4 \} \).

Task V is clearly in NP. We use Theorem 2.2 to show:

Theorem 2.5. Task V is NP-complete under a randomized reduction.

Proof. While our interest is four squares, our proof of this is inductive. We formulate Task V for a sum of \( k \)-squares. For \( k = 2 \) Theorem 2.2 asserts what is claimed here. If Theorem 2.5 is true for \( k \) it is true for \( k + 1 \). We explain the case from 2 to 3, the general case is similar. Let \( n, \alpha', \beta' \) be the input for Task III'. We show how to resolve it efficiently assuming Task V
for \( x_1^2 + x_2^2 + x_3^2 = m \) can be done efficiently. Choose \( \frac{n}{2} < y_3 < n \) and let \( m = y_3^2 + n \), so \( y_3^2 < m < (y_3 + 1)^2 \). Hence for any \( x = (x_1, x_2, x_3) \) with \( x_1^2 + x_2^2 + x_3^2 = m \)

\[
d^2(\bar{x}, (0,0,1)) = \frac{2(\sqrt{m} - x_3)}{m} \geq \frac{2(\sqrt{m} - y_3)}{m}
\]

with equality iff \( x_3 = y_3 \). In particular those \( \bar{x} \)'s which are all the closest solutions to \((0,0,1)\), correspond exactly to solutions of

\[
x_1^2 + x_2^2 = n \quad \text{and} \quad x_3 = y_3.
\]

So if we seek the solution \( x = (x_1,x_2) \in S^1 \) to (2.1) with \( \bar{x} \) closest to \( \alpha = (\alpha_1,\alpha_2) \in S^1 \), we can do so by determining the \( x = (x_1, x_2, x_3) \) with \( x_1^2 + x_2^2 + x_3^2 = m \) and for which \( \bar{x} \) is closest to \( \eta (\alpha_1,\alpha_2,0) + (0,0,1) \), for \( \eta \) small enough and positive. One checks that this determination is efficient and hence it follows that an efficient solution of Task V for \( k = 3 \), yields one for \( k = 2 \).

While Theorem 2.5 limits what one can do efficiently in general, the good news is that special cases of this task can be done. For \( \xi = (\xi_1,\xi_2,0,0) \), Ross and Selinger [RS15] give an algorithm (see also [Ros15, BBG15]).

**Theorem 2.6** (Ross-Selinger). There is a heuristic efficient algorithm to solve Task V for \( \xi = (\xi_1,\xi_2,0,0) \).

**Discussion.** The precise meaning of heuristic will be clarified in what follows. An important extra feature is that the algorithm can and has, been implemented ([RS15]) and that it runs and terminates quickly. A similar algorithm has been devised and implemented in the \( p \)-adic setting, namely that of navigating related Ramanujan graphs in [PLQ08] and [Sar17].

The key property as far as these special \( \xi \)'s go is that for \( x \) solving (2.3)

\[
d^2(\xi, \bar{x}) = 2 \left[ 1 - \frac{\xi_1 x_1 + \xi_2 x_2}{\sqrt{n}} \right],
\]

and this is a linear constraint depending on \( x_1 \) and \( x_2 \) only. Hence \( \bar{x} \in B_{2^z} (\xi) \) is equivalent to

(A) \[
\frac{\xi_1 x_1 + \xi_2 x_2}{\sqrt{n}} > 1 - \frac{\varepsilon}{2} ; \quad x_1^2 + x_2^2 \leq n
\]

and

(B) \[
n - (x_1^2 + x_2^2) = x_3^2 + x_4^2.
\]

In this way tasks (A) and (B) are decoupled, and we proceed by first finding candidate solution to (A) and then solving for (B). Now (A) is a problem of finding integer lattice points \( (x_1, x_2) \) in \( \mathbb{Z}^2 \) which lie in the convex 'miniscus' region defined by the inequalities in (A). Listing the solutions one at a time (which is done with polynomial cost) is in P. The algorithm to do this is due to Lenstra [LJ83] and applies to such integer convex programming problems for \( \mathbb{Z}^k \) in any fixed dimension \( k \) (see [Lov86]). His key input being Minkowski reduction of bases of lattices. If the input complexity for Task V is \( H \), then we examine \( O(H^c) \) of the solutions to (A) (in some geometric ordering or randomly) and for each \( (x_1, x_2) \) we check if (B) has a solution by running the efficient solution to Task (II). If we arrive at a solution \( (x_1, x_2, x_3, x_4) \) then we have
resolved Task V. If our $O(H^c)$ (here $c$ is a fixed number such as 10) steps cover all the solutions to (A) and no $(x_3, x_4)$ is found, then we output that Task V has no solution, which is the case. The only stumbling block to the algorithm terminating efficiently is that there are more than a polynomial in $H$ number of solutions to (A) and our $O(H^c)$ inspections produce no solutions to (B), in particular this would happen if in fact there are no solutions to (B) among these very many solutions to (A). The heuristic aspect of the algorithm is that this last scenario will not happen. The density of numbers $t$ in $[X, 2X]$ which are sums of two squares is $1/\sqrt{\log X}$, so that one expects that (B) will have a solution with probability $1/\sqrt{\pi}$. Thus the probability of hitting this stumbling block after $O(H^c)$ tries is very small, and possibly never arises. This completes the analysis and meaning of Theorem 2.6.

**Remark 2.7.** In Theorem 2.6 and elsewhere we have assumed that one can factor efficiently. Without appealing to such a factoring algorithm we can seek an approximation to $\xi$ as above but in (B) we require that $n - (x_1^2 + x_2^2)$ be a prime $p \equiv 1 (4)$. The density of such special solutions is a bit smaller ($1/\log x$ rather than $1/\sqrt{\log X}$) and we can find these solutions efficiently.

### 2.3 Sums of squares in number rings

The coordinates of the Golden Gates Sets lie in the ring of $S$-integers of certain number fields. This leads us to the examination of the tasks discussed in Sections 2.1 and 2.2 with $\mathbb{Z}$ replaced by the ring of integers $\mathcal{O}_K$, of number fields such as $K = \mathbb{Q}(\sqrt{2})$ and $K = \mathbb{Q}(\sqrt{5})$. These are unique factorization domains and in fact even Euclidean domains. This allows us to extend the results of the previous sections to these rings. We explicate what we need for later. Task II is to unique factorization domains and in fact even Euclidean domains. This allows us to extend the results of the previous sections to these rings.

With this we can address Task V over $\mathcal{O}_K$ in the form of Theorem 2.6. That is to solve

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 = n \quad \text{with} \quad \bar{x} \in B_c(\xi), \xi = (\xi_1, \xi_2, 0, 0).$$

(2.5)

Here $n$ is totally positive, i.e. $n > 0$ and $n' > 0$ where $\prime$ is the Galois conjugate of $K/\mathbb{Q}$, and $\bar{x} = \frac{x}{\sqrt{n}} \in S^3$.

As before this decouples into

(A') \quad x_1, x_2 \in \mathcal{O}_K, \quad \frac{\xi_1 x_1 + \xi_2 x_2}{\sqrt{n}} > 1 - \frac{1}{2}; \quad x_1^2 + x_2^2 \leq n \quad (x_1')^2 + (x_2')^2 \leq n'

and

(B') \quad n - (x_1^2 + x_2^2) = x_3^2 + x_4^2 \quad \text{with} \quad x_3, x_4 \in \mathcal{O}_K.

For (A’) the points $(x_1, x_1', x_2, x_2')$ with $x_1, x_2 \in \mathcal{O}_K$ form a rank 4 lattice $L$ in $\mathbb{R}^4$ and the inequalities defining the set in which they lie is a convex (compact) set. Lenstra’s algorithm allows us to find these points efficiently. As in the setting with (A), (B), each instance of (B’) with a given $x_1, x_2 \in \mathcal{O}_K$ is Task II over $\mathcal{O}_K$, and so has an efficient solution. The rest of the analysis is as before. We have shown that Theorem 2.6 is valid for $\mathcal{O}_K$. 

9
3 Strong approximation for integer points on spheres

3.1 Spectral gap and strong approximation

We are interested in how well the points $x$ satisfying (2.3) or (2.5) cover $S^3$. A powerful method to address this makes use of these points corresponding to orbits defining Hecke operators, and their eigenvalues may be estimated (optimally) using the Ramanujan Conjectures (see [LPS87]).

We formulate a covering estimate in terms of a spectral gap, for a general compact topological group $L$ as it clarifies the roles of the ingredients (we will apply it for $G = L = PU(2)$). Let $\mu$ be Haar measure on $L$ normalized to be a probability measure. Let $S$ be a finite subset of $L$ and $|S|$ its cardinality. For $B \subseteq L$ with $\mu(B) > 0$, we are interested in $\bigcup_{s \in S} (Bs)$. If these right $S$-translates of $B$ cover all of $L$ we say that the pair $B, S$ covers. If $\mu(L \setminus \bigcup_{s \in S} Bs) = o(1)$ as $|S| \to \infty$ (our interest is $|S| \to \infty$ and we allow everything including $L$ to change) we say that $B, S$ is almost covering. It is clear that to be almost covering we must have $|S| \mu(B) \geq 1 + o(1)$.

If $S$ is chosen at random (that is i.i.d. w.r.t. $\mu$) then $|S| \mu(B) \to \infty$ (3.1) is necessary and sufficient for almost covering (1).

Let $a_s \geq 0$ with $\sum_{s \in S} a_s = 1$ and let $\nu_s$ be the probability measure on $L$ given by

$$\nu_{S,a} := \sum_{s \in S} a_s \delta_s$$

($\delta_x$ is a point mass at $x$). $\nu_s$ defines a right convolution operator $T_{S,a}$ on $L^2(L)$ by

$$T_{S,a}f(x) = \sum_{s \in S} a_s f(xs^{-1}).$$

Clearly $T_{S,a}1 = 1$; $\|T_{S,a}\| = 1$.

The orthogonal space to $1$, $L^2_0(L)$, is $T_{S,a}$ invariant. Let

$$W = W_{S,a} := \|T_{S,a}|_{L^2_0(L)}\|.$$  

$W \leq 1$ and we estimate the covering properties of $S$ in terms of the spectral norm $W$.

**Proposition 3.1.** For $B$ and $S$ as above $\mu(L \setminus \bigcup_{s \in S} Bs) \leq \frac{W^2}{\mu(B)}$, in particular if $\frac{W^2}{\mu(B)} = o(1)$, then $B, S$ is almost covering.

**Proof.** Let $I_B(x)$ be the indicator function of the set $B$. $I_B(x) - \mu(B)$ is in $L^2_0(L)$ and $\int_L (I_B(x) - \mu(B))^2 \, d\mu(x) = \mu(B)(1 - \mu(B))$. Hence $\int_L [T_{S,a}I_B(x) - \mu(B)]^2 \, d\mu(x) \leq W^2 \mu(B)(1 - \mu(B))$, that is

$$\int_L \left[ \sum_{s \in S} a_s I_B(xs^{-1}) - \mu(B) \right]^2 \, d\mu(x) \leq W^2 \mu(B)(1 - \mu(B)).$$

(1) This being a well known property of the Coupon Collector’s problem.
If \( x \not\in Bs \) for any \( s \in S \) then \( xs^{-1} \not\in B \) and the bracketed expression in (3.3) is \( |\mu(B)|^2 \), and hence \( \mu(L \setminus \bigcup_{s \in S} Bs) \mu(B)^2 \leq W^2 \mu(B) \), which gives the claimed inequality. \( \Box \)

If \( L \) is a continuous group then we can choose \( B \) to satisfy \( \mu(B) = \frac{1}{2|S|} \), so that \( \mu(L \setminus \bigcup_{s \in S} Bs) \geq \frac{1}{2} \) and hence from Proposition 3.1, \( W^2 \geq \frac{1}{4|S|} \) or \( W \geq \frac{1}{2|S|^{1/2}} \). In our applications \( W \) will be of this size, namely \( 1/\sqrt{|S|} \), and in this case \( W^2/\mu(B) \) is \( o(1) \) iff \( \mu(B) |S| \to \infty \), matching (3.1). Hence in such cases \( B, S \) is essentially optimal in its almost covering property.

To conclude that \( B, S \) is actually covering we need to assume more about the shape of \( B \). Specifically we take \( B \) to be a ball which is the case of interest to us. Let \( d(x,y) \) be a \( L \)-bi-invariant metric on \( L \) (if \( L \) is finite then \( d \) is the discrete metric \( d(x,y) = 0 \) if \( x = y \), \( d(x,y) = \infty \) if \( x \neq y \)), and let \( B \) be a ball centered at \( e \) of volume \( \mu(B) = v \). Then \( B = B^{-1} \) and \( sB = Bs \) is the ball centered at \( s \) and of volume \( v \). \( B, S \) is covering iff every ball in \( L \) of volume \( v \) contains an \( s \in S \).

**Corollary 3.2.** If \( B \) is a ball in \( L \) with center \( e \), radius \( r \geq 0 \) and volume \( v > W \) then \( B_{2r}(e) \), \( S \) is covering (where \( B_r(x) \) is the ball centered at \( x \) and of radius \( r \)).

**Proof.** If there is \( x \in L \) such that no \( s \in S \) meets \( B_{2r}(x) \), then for each \( y \in B_r(x) \), \( B_r(y) \cap S = \emptyset \). Hence each such \( y \) lies in \( L \setminus \bigcup_{s \in S} Bs \), so that the measure of this set is at least \( v \). On the other hand by Proposition 3.1 we have \( v \leq \frac{W^2}{v} \) or \( v \leq W \), which contradicts our assumption. \( \Box \)

**Remarks.**

1. In the continuous group case when \( W \approx \frac{1}{\sqrt{|S|}} \) is as small as it can possibly be, Corollary 3.2 yields that every ball of volume \( |S|^{-1/2} \) contains a point from \( S \). While in some cases this covering volume bound might be far from the truth for \( S \), there are very natural cases (see below) where this is sharp.

2. If \( L \) is finite then the only balls are singletons and \( L \) itself. In this case if \( B = \{e\} \), \( \mu(B) = \frac{1}{|L|} \) and the upper bound in Proposition 3.1 gives a lower bound for \( \bigcup_{s \in S} Bs = S \) (almost covering asserts that \( |S|/|L| \to 1 \)). The covering Proposition 3.1 asserts that \( S = L \) if \( W < \frac{1}{|L|} \).

3. There are variants of Proposition 3.1 for symmetric spaces (see [BSR12]) and for Cayley graphs ([Sar15a, LP16]). We formulated the main variance inequality (3.3) in terms of the spectral norm \( W \) alone. In some important cases one can exploit a more explicit version.

If \( T_{S,a} \) is normal and hence diagonalizable with an o.n.b. \( \phi_j \), \( j = 1, 2, \ldots \) of \( L_0^2(L) \), then the l.h.s. of (3.3) is equal to

\[
\sum_{j=1}^{\infty} |\langle I_B, \phi_j \rangle|^2 |\lambda_j|^2 (T_{S,a})
\]  

(3.4)

where \( \lambda_j \) is the eigenvalue corresponding to \( \phi_j \). (3.3) then follows by invoking \( |\lambda_j| \leq W \). In the form (3.4) one can allow exceptions to this as long as there aren’t too many such \( j \)’s and one can also remove very high frequency \( j \)’s (in the continuous setting) since \( |\langle I_B, \phi_j \rangle|^2 \) is very small for these. Such an analysis is carried out in [Sar15c] in a different setting.

We first apply Proposition 3.1 to the question of the covering properties of the arguments of Gaussian primes \( P \) in \( \mathbb{Z} \left[ \sqrt{-1} \right] \), as discussed in Section 2.
Proposition 3.1, hence Proposition 3.1 and its Corollary. For each $P$ we give a weight $a_P = \log N(P) \left(1 - \frac{N(P)}{X}\right)$.

Hence

$$\nu = \frac{1}{\Psi_1(X)} \sum_{N(P) \leq X} a_P \delta_{\theta_P}.$$ 

$$\Psi_1(X) = \sum_{N(P) \leq X} \log N(P) \left(1 - \frac{N(P)}{X}\right) \sim c_1 X, \ c_1 > 0.$$ 

The point about these weights is that assuming the generalized Riemann Hypothesis for the Hecke $L$-functions, $L(s, \lambda^m)$ where $\lambda^m(\alpha) = \left(\frac{\alpha}{m}\right)^{4m}$ we have (see [Sar85]) that

$$\sum_{N(P) \leq X} \lambda^m(\theta_P) a_P \ll (\log |m| + 1) X^{1/2}.$$ 

Hence for the o.n.b. $\lambda^m, m \in \mathbb{Z}$ of $L^2[0, 2\pi]$

$$\nu_{S,a} * \lambda^m = \hat{\nu}_{S,a}(m) \cdot \lambda^m$$

where

$$|\hat{\nu}_{S,a}(m)| \ll X^{-1/2} (\log |m| + 1).$$

This is an example where the high frequencies can be estimated easily, and we find that (3.4) is

$$\ll X^{-1} \sum_{m \neq 0} (\log |m| + 1)^2 |\langle I_B, \lambda^m \rangle|^2 \ll X^{-1} (\log X)^2 \mu(B).$$

Hence for $\frac{(\log X)^2}{X \mu(B)} = o(1)$ or $\frac{\mu(B)|S_X|}{\log X} \to \infty$; $B, S_X$ is almost covering. That is almost all intervals of length $h$ with $\frac{|S_X|}{\log X} \to \infty$, have a $\theta_P$ with $P \in S_X$. So except for the extra $\log X$ factor this is sharp. This has direct bearing to the randomized reduction in Section 2. Namely to find the $P_j$’s there in the requisite sector, we can choose a random angle $\theta$ therein and then according to the above there will be a prime in a much smaller sector of area $(\log X)^2$. We can check efficiently each of the few integer lattice points in this sector to see if they are prime. Of course this is still a randomized reduction.

Since the almost covering length of the $\theta_P$’s is optimally small (under GRH) a natural question is how small is the covering length? Proposition 3.1 coupled with GRH which gives (essentially) that $W \leq \frac{1}{|S_X|^{1/2}} (\log X)^{1/2}$, shows that the covering length is at most $\sqrt{\log X/|S_X|}$. It turns out that this upper bound is very close to optimal. Let $\alpha = x_1 + iy_1, \beta = x_2 + iy_2$ be two distinct primitive elements of $\mathbb{Z}[\sqrt{-1}]$ with arguments $\theta_\alpha, \theta_\beta$ in $[0, \pi/4)$. Then

$$|\tan \theta_\alpha - \tan \theta_\beta| = \frac{|y_1 - y_2|}{|x_1 x_2|} \geq \frac{1}{x_1 x_2} \geq \frac{1}{|\alpha||\beta|}.$$ 

Hence if $\beta$ is fixed of small norm and $N(\alpha) \leq X$, then

$$|\tan \theta_\alpha - \tan \theta_\beta| \gg \frac{1}{\sqrt{X}}.$$
so there is an interval of length \( \frac{1}{\sqrt{X}} \) free of \( \theta_\alpha \)'s for \( \alpha \) primitive and hence certainly free of \( \theta_P \)'s with \( P \) a prime. Thus our upper bounds of \( \frac{\Omega(X) \log X}{|S_X|} \) with \( g \to \infty \), for the almost covering length, and of \( \sqrt{\log X / |S_X|} \) for the covering length, are sharp up to the log factors.

### 3.2 Quaternions and Ramanujan

Let \( H(\mathbb{R}) \) denote the Hamilton quaternions \( \alpha = x_0 + x_1 i + x_2 j + x_3 k, x_j \in \mathbb{R} \). The projection \( \alpha \mapsto \tilde{\alpha} = \alpha / |\alpha| \) is a morphism of \( H^\times(\mathbb{R}) \) onto \( H^1(\mathbb{R}) \), the quaternions of norm 1. Moreover \( s \) given by

\[
\alpha \mapsto s(\alpha) = \begin{bmatrix} x_0 + ix_1 & x_2 + ix_3 \\ -x_2 + ix_3 & x_0 - ix_1 \end{bmatrix}
\]

is an isomorphism of \( H^1(\mathbb{R}) \) with \( SU(2) \), and both with their bi-invariant metrics are isometric to \( S^3 \). In this way the solutions to (2.3) which we denote by \( S(n) \) give points in \( G = SU(2) \simeq S^3 \). In this way the solutions to (2.3) which we denote by \( S(n) \) give points in \( G = SU(2) \simeq S^3 \) and we can apply the results in Section 3.1 to study their covering properties w.r.t. balls. We assume that \( n \) is odd. Jacobi showed that (see [DSV03])

\[
|S(n)| = 8 \sum_{d \mid n} d.
\]

For the convolution operator on \( L^2(SU(2)) \) we take \( a_s = \frac{1}{|S(n)|} \), \( s \in S(n) \) so that

\[
Tf(x) = \frac{1}{|S(n)|} \sum_{\alpha \in S(n)} f(xs^{-1}(\alpha)).
\]

The Ramanujan Conjectures (Deligne’s Theorem [Del74]) imply that (see [LPS87] and also the discussion below)

\[
W = W_{S(n)} \leq \frac{n^{1/2} \sum_{d\mid n} 1}{|S(n)|}.
\]

Hence

\[
\frac{W^2}{|S| \mu(B)} \leq \frac{\sum_{d\mid n} 1}{8 \sum_{d\mid n} \frac{1}{d}} \cdot \frac{1}{|S| \mu(B)},
\]

and if

\[
\frac{|S(n)| \mu(B)}{\sum_{d\mid n} \frac{1}{d}} \to \infty,
\]

then \( S(n), B \) is almost covering \( SU(2) \). Two special cases of interest are: \( n \) a prime; in which \( |S(n)| \mu(B) \to \infty \) suffices and matches (3.1), while for the case \( n = p^k \) for \( p > 2 \) a fixed prime, \( \frac{\mu(B)/|S(n)|}{\log|S(n)|} \to \infty \) suffices to almost cover. The last is the case of interest for the covering properties of Golden Gates and up to the log factor it is optimal.

As far as the covering properties of \( SU(2) \) by \( S(n) \), we apply (3.6) together with Corollary 3.2 to get that

\[
\mu(B) > \left( \frac{\sum_{d\mid n} 1}{\sum_{d\mid n} d^{-1/2}} \right) |S(n)|^{-1/2}
\]

suffices for \( S(n) \) to cover \( SU(2) \) with balls of volume \( \mu(B) \).
A very interesting question is to determine the covering volume for \( S(n) \), or at least its exponent. An elementary argument using repulsion of the projections of \( H(\mathbb{Z}) \) points onto \( S^3 \) (see [Sar15d]) shows that there are balls \( B \) of volume \( |S(n)|^{-3/4} \) which are free of any points from \( S(n) \). Hence the exponent for the covering volume lies in the interval \( [-\frac{3}{4}, -\frac{1}{2}] \) and it is probably equal to \(-\frac{3}{4}\). One can establish (3.7) with exponent \( |S(n)|^{-1/2+\varepsilon} \) using the 'Kloosterman circle method', see [Sar15b]. In this approach Kloosterman sums and estimates associated with them play a central role and assuming a natural conjecture about cancellations of sums involving these, one can establish that the volume covering exponent is \(-\frac{3}{4}\) [BKS16]. That there are such points which are badly approximable by \( S(n) \) leads to a rich metric diophantine approximation theory in this and much more general contexts, see [GGN14].

We turn to these questions for integers in a number field. Let \( K \) be a totally real number field of degree \( k \) and denote by \( \sigma_1, \ldots, \sigma_k \) its Galois embeddings into \( \mathbb{R} \) with \( \sigma_1 = \text{identity} \). For a definite quadratic form \( F(x_1, x_2, x_3, x_4) \) such as the sum of four squares we can use Hilbert modular form theory as above to study the distribution of the \( S(m) \) solutions to

\[
F(x_1, x_2, x_3, x_4) = x_1^2 + x_2^2 + x_3^2 + x_4^2 = m \quad \text{with} \quad x_j \in \mathcal{O}_K \quad \text{and} \quad m \in \mathcal{O}_K.
\]

Let \( U \) denote the group of units of \( \mathcal{O}_K \). Now \( x = (x_1, x_2, x_3, x_4) \in S(m) \) iff \( \varepsilon x \in S(\varepsilon^2 m) \) for any \( \varepsilon \in U \). Thus any properties of \( S(m) \) that are of interest to us are the same for \( S(m) \) and \( S(\varepsilon^2 m) \).

Except for very special \( K \)'s (see [KV10]) the class number of the form \( F \) above is not one so that the exact number of solutions to (3.8) is not a simple divisor function of \( (m) \). However the asymptotic behavior of \( |S(m)| \) for \( (m) \) (the principal ideal generated by \( m \)) which does not have prime factors in the finitely many ramified primes of \( F \) is given by a divisor sum ([SP04]):

\[
|S(m)| \sim C \sum_{a|(m)} N(a),
\]

here \( C \) is a positive constant related to the finite group \( \text{Aut}_\mathcal{O}(F) \) and \( N(a) \) is the norm of the ideal \( a \).

To each \( x \in S(m) \) and \( j = 1, \ldots, k \), \( \sigma_j(x)/\sqrt{\sigma_j(m)} \) (note \( \sigma_j(m) > 0 \) if \( S(m) \neq \emptyset \)) lies in \( S^3 \), which we have identified with \( SU(2) \). Hence via the diagonal embedding each \( x \in S(m) \) gives a point in \( L \cong (S^3)^k \). The analysis at the beginning of this Section and the Ramanujan Conjectures which are valid here as well ([HT01]) show that

\[
W_{S(m)} \leq \frac{N((m))^{1/2} \sum_{a|(m)} 1}{|S(m)|^1}.
\]

Applying Proposition 3.1 to \( L \) we deduce that if

\[
\frac{\mu(B) |S(m)|}{\sum_{a|(m)} 1} \to \infty \quad \text{for} \quad \frac{\mu(B) |S(m)|}{\sum_{a|(m)} 1} \to \infty
\]

then \( B, S(m) \) almost covers \( L \).

As before this is essentially an optimal covering. Two two extreme cases of interest are if \( B \) is a ball about \( e \in L \) of radius \( r \) so that \( \mu(B) \sim cr^{3k} \), then we require \( N(m) \) to be a little bigger than \( r^{-3k} \) in order for almost all balls of this radius to contain a point from \( S(m) \). The other case which is of most interest to us is to fixate on the first factor \( \sigma_1 \). Take \( B \) to be \( B_1 \times S^3 \times \cdots \times S^3 \)
with $B_1$ a ball of radius $r$ about $e_1$ in $SU(2)$. So $\mu(B) = \mu_1(B_1)$ and as long as $\mu(B_1)N(m)$ goes to infinity as required then $\sigma_1(x), x \in S(m)$ almost covers $SU(2)$. So in this case we need $N(m)$ to be a little larger than $r^{-9}$.

The above, when applied to special cases, suffices for the analysis of the optimal covering properties of Golden Gate circuits. For the purpose of navigation and exact circuit length we need to exploit special quadratic forms of class number one and which arise as norm forms of quaternion algebras. We give a different treatment of (3.9) from the point of view of the local and global arithmetic of special quaternion algebras over $K$. This also brings out the important special finite subgroups $C$ of $PU(2)$ which are used to define our Super-Golden Gates.

Let $K$ be as above and $K_j$ the real completion of $K$ at $\sigma_j$. For $P$ a prime (ideal) in $\mathcal{O}_K$ we let $K_P$ be the completion of $K$ at $P$. Let $D$ be a totally definite quaternion algebra over $K$, that is $D \otimes K_j \cong H(\mathbb{R})$ for all $j$, where $H$ is the standard Hamilton quaternion algebra. The primes at which $D$ is unramified are those for which $D \otimes K_P \cong Mat_2(K_P)$. The ramified primes for $D$ are even in number (if one include the $K_j$’s in the count). Choose an order $\mathcal{O}$, which we usually take to be maximal, in $D$ (for the basic properties and definitions see [Vig80]). Let $D(\mathfrak{A})$ denote the corresponding adele ring of $D$ w.r.t. the order $\mathcal{O}$. For $P$ unramified and $S = \{P\}$ we are interested in the $S$-arithmetic group

$$\Gamma_S = \{\gamma \in D(K) | \gamma \text{ is } \mathcal{O}\text{-integral outside } S\}.$$

Now $D^{\times}(K_{\mathcal{O}})/_{center} \cong PGL_2(K_P)$, and via this identification, $\Gamma_S$ projects onto a lattice $\Gamma$ in $PGL_2(K_P)$ (see [Vig80]). Also $H^{\times}(\mathbb{R})/_{center} \cong PU(2)$, so using the embeddings $\sigma_j$, $j = 1, 2, \ldots, k$ we get a diagonal embedding of $\Gamma = \Gamma_{\mathcal{O}/_{center}}$ into $L \times PGL_2(K_P)$, where $L = (PU(2))^k$.

The quotient space $X_P = PGL_2(K_P)/_{PGL_2(\mathcal{O}_P)}$ is a $N(p) + 1$ regular tree ([Ser80]) on which $\Gamma$ acts isometrically. Let $L^2(\Gamma \setminus (L \times X_P))$ be the space of $L^2$ (w.r.t. $dg \times dg_P$) $\Gamma$-periodic functions:

$$f(\gamma g, \gamma x) = f(g, x) \text{ for } \gamma \in \Gamma, g \in L, x \in X_P.$$

The Hecke operators $T_t, t > 1$ acting on the second variable are defined by

$$T_t f(g, x) = \sum_{d(y, x) = t} f(g, y).$$

$T_k$ preserves $L^2(\Gamma \setminus (L \times X_P))$ and also $L^2_0(\Gamma \setminus (L \times X_P))$; the orthocomplement of the constant function. It is self adjoint and its eigenvalues $\lambda$ on $L^2_0(\Gamma \setminus (L \times X_P))$ satisfy

$$|\lambda| \leq (t + 1) N(P)^{1/2}. \quad (3.10)$$

One way to derive (3.10) is to use automorphic representations. The eigenfunctions in $L^2_0(\Gamma \setminus (L \times X_P))$ can, after a further diagonalization of Hecke operators at other primes, be embedded as a vector in an (non one-dimensional) irreducible representation $\pi$ of $D(\mathfrak{A})$ occurring in its right regular representation on $L^2(D(\mathfrak{A}) \setminus D(\mathfrak{A}))$. The Ramanujan Conjectures, which are theorems in this context - thanks to the Jacquet-Langlands correspondence and Deligne’s theorem (see [JL72, Del74]), assert that such a $\pi \cong \bigotimes_v \pi_v$ with $v$ ranging over all places $v$ of $K$, has each $\pi_v$ tempered ([Sat66]). In particular, this applies to the spherical representation $\pi_P$ of $PGL_2(F_P)$ and this in turn implies that the eigenvalue $\lambda$ satisfies (3.10).

To connect this with circuit lengths of elements $\gamma \in \Gamma$ we impose a strong condition, namely that the action of $\Gamma$ on $X_P$ is transitive. For this to happen for $P$ outside a finite set of primes,
the class number of the order \( \mathcal{O} \) needs to be 1. In this definite quaternion algebra setting this happens only for finitely many \( D \)'s and \( K \)'s which have been enumerated in [KV10].

In this transitive case \( L^2(\Gamma \setminus (G \times X_P)) \) may be identified with functions \( h \) on \( G \) which are \( U_T \) invariant where \( U_T = \{ \gamma \in \Gamma \mid \gamma e = e \} \), \( e \) the identity coset \( PGL_2(\mathcal{O}_P) \) in \( X_P \). More explicitly for \( f \in L^2(\Gamma \setminus (G \times X_P)) \)

\[
h(g) = g(g,e),
\]

and \( h(\delta g) = h(g) \) for \( \delta \in U_T \).

Such \( h \) has a unique \( \Gamma \) extension to \( G \times X_P \). The action of \( T_k \) on \( L^2(U_T \setminus G) \) becomes

\[
T_k h(g) = \sum_{\gamma \in \Gamma \setminus U_T, d(\gamma e, e) = k} h(g\gamma)
\]

and according to (3.10) \( T_k \) is the kind of operator considered in (3.2), with a sharp spectral bound.

Finally in this \( X_P \) transitive setting we relate the distance moved by \( \gamma \) on the tree to circuit (word) length for generators of \( \Gamma \). Let \( \delta_1, \ldots, \delta_r \in \Gamma \) with \( r = N(P) + 1 \), be elements that take \( v_0 \) to its \( r \) immediate neighbors \( S_1 = \{ \xi \mid d(\xi, v_0) = 1 \} \) in \( X_P \). If \( y \in X_P \) and \( y = \beta v_0 \) with \( \beta \in \Gamma \) then \( \beta \delta_j v_0 \) for \( j = 1, \ldots, r \) are the \( r \) immediate neighbors of \( y \). Hence if \( \mu \in \Gamma \) and \( d(\mu v_0, v_0) = d(\mu^{-1} v_0, v_0) = k \geq 1 \), then there is a unique \( j_1 \) such that \( d(\mu^{-1} \delta_{j_1} v_0, v_0) = k - 1 \).

If \( k - 1 \geq 1 \) we repeat this to find \( \delta_{j_2} \) such that \( d(\mu^{-1} \delta_{j_1} \delta_{j_2} v_0, v_0) = k - 2 \). Repeat this until we arrive at \( d(\mu^{-1} \delta_{j_1} \ldots \delta_{j_k} v_0, v_0) = 0 \), and hence

\[
\mu^{-1} \delta_{j_1} \ldots \delta_{j_k} = u \in U, \quad \text{or} \quad \mu = \delta_{j_1} \ldots \delta_{j_k} u^{-1}.
\]

(3.11)

Since at every step the choices are determined we have that any \( \mu \in \Gamma \) with \( d(\mu v_0, v_0) = k \) has a unique expression in the form (3.11).

Thus the distance moved by \( \gamma \) on the tree corresponds to the word length in generators \( \delta_1, \ldots, \delta_r \) and the “Hecke orbit” corresponds to a circuit of a given length. Moreover if we are given \( \gamma \in \Gamma \) and want to express \( \gamma \) in the form (3.11) then this can be done efficiently by navigating the tree, as described above. This is an important feature that will be exploited in the navigation algorithms below.

In the case that \( U_T = \{ 1 \} \), that is when \( \Gamma \) acts simply transitively on \( X_P \), the set \( S = \{ \delta_1, \ldots, \delta_r \} \) is invariant under \( s \mapsto s^{-1} \). Hence \( S = \{ \delta_1, \ldots, \delta_t, \mu_1, \mu_1^{-1}, \ldots, \mu_v, \mu_v^{-1} \} \) with \( \delta_j = \delta_j^{-1}, \mu_j \neq \mu_j^{-1} \) and \( t + 2v = r \). (3.11) then shows that \( X_P \) is the Cayley graph of \( \Gamma \) with respect to the generators \( S \), the \( \mu_i \)'s being of infinite order and \( \Gamma \cong \langle \mu_1 \rangle \cdots \langle \mu_v \rangle \ast \langle \delta_1 \rangle \ast \cdots \ast \langle \delta_t \rangle \). The distance on the tree \( X_P \) corresponding to the word (circuit) length w.r.t. the symmetric set of generators \( S \). This is the case of Golden-Gate sets which as we have noted can arise from only a finite number of \( D \)'s and \( K \)'s; see Section 4 for some examples (by varying \( P \) there are infinitely many such Golden Gate sets).

For our super Golden Gate sets we require further that \( \Gamma \) acts transitively on the edges of \( X_P \). In fact we assume that \( U_T \) acts simply transitively on \( S_1 \) and that among the \( \delta_j \)'s there is an involution \( T \). In this case the set \( \delta_1, \ldots, \delta_r \) can be taken as \( u^{-1} Tu, \mu \in U_T \) (note \( |U_T| = r \)). (3.11) now asserts that \( \Gamma \cong U_T \ast \langle T \rangle \) and \( d(\gamma v_0, v_0) \) is exactly the \( T \)-count in the representation of \( \gamma \in \Gamma \) as a product of \( T \) and \( u \)'s for \( u \in U \). This gives rise to our super-golden gate sets \( C \) plus \( T \), with \( C = U_T \). There are only finitely many of these since the set of \( D \)'s and \( K \)'s lie in a
finite list as does the set of orders $O$. Hence $U_L$ which is the unit group of the order is limited, and this limits the primes $P$ for which this group can act transitively on $S_1$. We conclude that there are only finitely many possibilities for super Golden Gate sets. There are a number of very interesting examples where all of this happens, and these are described in the next section.

We end this Section by describing the relation between word length of elements of $\Gamma$ in terms of the generators $\mu_1, \ldots, \mu_v, \delta_1, \ldots, \delta_t$ and strong approximation for (2.3). We explain this for the case that $D = H = (1, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ the Hamilton quaternions over $\mathbb{Q}$, this being the case that was highlighted in [LPS86] and is the basic example of what we are calling Golden-Gates. Let $H (\mathbb{Z})$ be the order $\mathbb{Z} + \mathbb{Z}_4 + \mathbb{Z}_2 + \mathbb{Z}_2$ in $H$. The group $U$ of invertible elements in $H (\mathbb{Z})$ is $\{ \pm 1, \pm \frac{1}{2}, \pm j, \pm k \}$. The spheres $S(m) = \{ \alpha \in H (\mathbb{Z}) : \.InputStreamReader (\alpha) := a\alpha = m \}$ correspond to the solutions to (2.3). These spheres are acted on by $U \times U$ by left and right multiplication

$$\alpha \mapsto u_1 \alpha u_2.$$  

For $p$ and odd prime $|S (p)| = 8 (p + 1)$ and after multiplying on the right by $u \in U$ there are $(p + 1)$ solutions. If $p = 1 (4)$ exactly one coordinate of an $\alpha \in S (p)$ is odd and we use $U$ to make it the first coordinate and also ensure that it is positive. This gives us $\mu_1, \mu_2, \mu_3, \mu_4, \ldots, \mu_v$ with $2v = p + 1$ as the solutions (see [LPS87, DSV03]). If $p = 3 (4)$ then exactly one of the coordinates of an $\alpha \in S (p)$ is even and we can use $U$ to arrange that this coordinate is the first one, and again we can act by $\pm 1$ to choose the sign. In this way the $(p + 1)$ solutions obtained become $\mu_1, \mu_2, \ldots, \mu_v \mu_j, \delta_1, \ldots, \delta_t$ with $p + 1 = 2v + t$ and $\delta_j = -p$ for $j = 1, \ldots, t$ (see [DSV03]).

Now $H$ is unramified at each odd prime $p$ so that the group $\Gamma$ generated by these $p + 1$ elements can be realized as a subgroup of $\text{PGL}_2 (\overline{\mathbb{Q}}_p)$. These generators take the identity coset $v_0 = \text{PGL}_2 (\mathbb{Z}_p) \in X_p = \text{PGL}_2 (\mathbb{Q}_p)/\text{PGL}_2 (\mathbb{Z}_p)$ to its $(p + 1)$ neighbors, and $\Gamma$ acts simply transitively on $X_p$. Moreover $\Gamma \cong \langle \mu_1 \cdots \mu_v, \delta_1 \cdots \delta_t \rangle$ with $\langle \mu_j \rangle \cong \mathbb{Z}$ and $\langle \delta_j \rangle \cong \mathbb{Z}/2\mathbb{Z}$, and $X_p$ is the Cayley graph of $\Gamma$ w.r.t. $\mu_1, \mu_2, \ldots, \mu_v, \delta_1, \ldots, \delta_t$. Now a reduced word of length $\ell$ in these generators yields a unique (up to right multiplication by $U$) $\alpha$ in $S (p^\ell)$, which is primitive, namely, $\gcd (x_1, \ldots, x_4) = 1$. Furthermore, every primitive $\alpha$ in $S (p^\ell)$ is achieved this way. This identifies the distance $d_{X_p} (v_0, \alpha v_0)$ with the word length of $\alpha$ in our generators. The splitting of $\overline{\mathbb{R}}/\mathbb{Z}$ by the isomorphism (3.5) to $G = \text{PU} (2)$, yields the subgroup, also denoted by $\Gamma$, generated by our $p + 1$ elements. The question of how well the words of length $\ell, \ell \leq L \text{ cov} \text{er} G$ is equivalent to the same question for the $\alpha \in S (p^\ell)$ covering $S^4, \ell \leq L$. That is the problem discussed at the beginning of this Section. It follows from these that the circuits of length $\ell \leq L$ almost cover $G$ optimally. Precisely, the number of circuits of length $\ell \leq L$ in $G$ is $\approx p^L$ (here $p$ is fixed and $L \rightarrow \infty$) and as long as

$$\frac{V p^L}{L} \rightarrow \infty,$$

then almost all balls $B_V (\xi)$ in $G$ are covered.

Finally as far as navigating $G$ with these generators, again the problem reduces to one of strong approximation, namely Task V of Section 2, with $n = p^\ell$. The task needs to be resolved in poly $(L)$ steps. So to determine if there is a circuit of length $\ell \leq L$ in $B_V (\xi)$, we can do so for each $\ell \leq L$ separately. The question for a given such $\ell$ decouples into two steps, firstly finding an $\alpha \in S (p^\ell)$ which is in $B_V (\xi)$. According to Theorem 2.6 this can be done efficiently at least if $\xi$ is special in $S^4$, which amounts to $\xi \in G$ being diagonal. Once we have found $\alpha$ (or determined that none exists) we can express $\alpha$ as a circuit of length $\ell$ in the generators by viewing $\alpha \in \text{PGL}_2 (\overline{\mathbb{Q}}_p)/\text{PGL}_2 (\mathbb{Z}_p)$. There is a unique generator which will move $\alpha$ one step closer to $v_0$. Repeating this $\ell$ times gives an efficient factorization of $\alpha$. 

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To summarize, given $\xi \in G$ diagonal and $B_V(\xi)$, then assuming that we can factor efficiently and the heuristic in the discussion in Theorem 2.6, we can find efficiently a circuit of least word length in our generators which lies in $B_V(\xi)$.

Without an efficient factorization algorithm we settle for finding special solutions to the strong approximation problem as described in remark 2.7 and otherwise proceed as above. This secures a circuit which lies in whose word length is:

$$(1 + o(1)) \text{ times longer than the optimally short circuit.} \quad (3.13)$$

The above allows us to navigate to diagonal $\xi$'s in $G$ optimally and efficiently. As far as an optimal navigation to a general $\xi$ via strong approximation, the problem at least in the form of Task I is apparently hard. We can navigate efficiently to a $\xi = (\xi_1, \xi_2, \xi_3, \xi_4) \in S^3$ which has at least two coordinate equal to 0. In particular to elements $x \in G$ of the form $\mu_1\xi\mu_2$ with $\mu_1, \mu_2 \in C_4$ the Pauli group and $\xi$ diagonal. Now any element of $G$ can be expressed as a product of three such $\mu_1\xi\mu_2$'s and efficiently, hence we can navigate efficiently to $x$ with a circuit whose length is 3-times longer then the shortest circuit is for the generic $x$. The efficient navigation with Super Golden Gates and using $T$-counts is carried out similarly. This completes the theoretical diophantine and algorithmic analysis concerning Golden and Super-Golden gates that was announced in the Introduction. We end this Section with two explicit examples.

(A) **V-gates:** We take $p = 5$. Then $p + 1 = 6$ and the generators are $1 \pm 2i, 1 \pm 2j, 1 \pm 2k$; which yield $S_1 = \left( \begin{array}{cc} 1 & 2i \\ 0 & 1 + 2i \end{array} \right), S_2 = \left( \begin{array}{cc} 1 & 2j \\ -2 & 1 \end{array} \right), S_3 = \left( \begin{array}{cc} 1 & 2k \\ -2 & 1 \end{array} \right)$ (as $\big-generated by \langle S_1, S_2, S_3 \rangle \cong \mathbb{Z} \ast \mathbb{Z} \ast \mathbb{Z}$) as a Golden Gate set $S_1^{\pm 1}, S_2^{\pm 1}, S_3^{\pm 1}$ called V-gates.

(B) $p = 3, v = 0, t = 4$: $\delta_1 = \hat{i} + j + k, \delta_2 = \hat{i} - j + k, \delta_3 = \hat{i} + j - k, \delta_4 = \hat{i} - j - k$. These yield the four involutions $S_j$ in $G$ given by $S_1 = \left( \begin{array}{cc} 1 & i \\ -1 & 1 + i \end{array} \right), S_2 = \left( \begin{array}{cc} i & -1 + i \\ -1 & i \end{array} \right), S_3 = \left( \begin{array}{cc} i & 1 - i \\ 1 & i \end{array} \right), S_4 = \left( \begin{array}{cc} i & 1 - i \\ 1 & -i \end{array} \right)$ (as $\big-generated by \langle S_1 \rangle \ast \langle S_2 \rangle \ast \langle S_3 \rangle \ast \langle S_4 \rangle$) and are a Golden Gate set.

The action of $U \times U$ on $S(3)$ given by (3.12) is transitive and this leads to our first example of a super gate to which we now turn.

### 4 Super Golden Gates

Each Super-Golden-Gate set is composed of a finite group $C$ and an involution $T$, which lie in a $\{P\}$-arithmetic group for $P$ a prime ideal of the integers $\mathcal{O}_K$ of a totally definite quaternion algebra $D$, over a totally real number field $K$. We require that:

1. $C$ acts simply transitively on the neighbors of the origin in $X_P$.

2. $T$ is an involution which takes the origin to one of the neighbors.

Since $T$ is an involution, it inverts some edge $e_0$ whose origin is $v_0$. Denote by $\Gamma$ the group generated by $C$ and $T$, and by $\Delta$ the group generated by $\{cTc^{-1} | c \in C\}$. The next Proposition follows by Bass-Serre theory from the assumptions (1), (2) above.

**Proposition 4.1.**  
(1) $\Gamma$ acts simply transitively on the directed edges of $X_P$.

(2) The $T$-count of $\gamma \in \Gamma$ is $d_{X_P}(v_0, \text{orig}(\gamma e_0)) = d_{X_P}(v_0, \gamma v_0)$, and in particular there are $|C|^2 \left( |C| - 1 \right)^{t-1}$ elements of $T$-count $t$. 

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(3) $\Gamma$ is the free product of $C$ and $(T) \simeq \mathbb{Z}/2\mathbb{Z}$.

(4) $\Delta$ is a Golden-Gate set; namely, it acts simply transitively on the vertices of $X_P$, and thus is a free product of $|C|$ copies of $\mathbb{Z}/2\mathbb{Z}$.

(5) $\Gamma = \Delta \rtimes C$.

One can navigate in $\Gamma$ as follows: If the origin of $\gamma c_0$ is $v_0$, then $\gamma \in C$. In not, there is a unique $c_0 \in C$ for which $\gamma^{-1}c_0$ is pointing towards $v_0$, since $\{\gamma cc_0 | c \in C\}$ are the rotations of $\gamma c_0$ around its origin. Then,

$$d_{X_P}(v_0, \text{orig } (\gamma c_0^{-1}Tc_0)) = d_{X_P}(v_0, \text{term } (\gamma c_0^{-1}c_0))$$

Continuing in this manner one finds $c_0, \ldots, c_{\ell-1} \in C$ such that $\text{orig } (\gamma c_0^{-1}T \ldots c_{\ell-1}^{-1}Tc_0) = v_0$, hence $c_\ell := \gamma c_0^{-1}T \ldots c_{\ell-1}^{-1}T$ is in $C$, and $\gamma = c_\ell Tc_{\ell-1}T \ldots Tc_1Tc_0$. 

### 4.1 Examples

In the examples of Super-Golden-Gate sets that follow we explicate the definite quaternion algebra $D$ over the totally real field $K$ and order $O$ as well as the prime $P$ at which we allow denominators. Both $K$ and $O$ have class number 1 and we refer to the corresponding entries in Table 8.2 of [KV10].

#### 4.1.1 Lipschitz quaternions and Pauli matrices

The simplest example occurs for $D$ being the Hamilton quaternions over $\mathbb{Q}$, and $O$ the Lipschitz order $\mathbb{Z} + \mathbb{Z}i + \mathbb{Z}j + \mathbb{Z}k$ which was used to construct the golden gates in the previous Section. If one takes $P = (3)$, then an explicit isomorphism from $(D \otimes \mathbb{Q}_3)^{\times} / \mathbb{Q}_3^{\times}$ to $\text{PGL}_2(\mathbb{Q}_3)$ is given by

$$a + bi + cj + dk \mapsto \begin{pmatrix} a + c + \rho d & b + \rho c - d \\ -b + \rho c - d & a - c - \rho d \end{pmatrix}, \quad (\rho = \sqrt{-2}), \quad (4.1)$$

where $\sqrt{-2} = \ldots 200211 \in \mathbb{Q}_3$. The unit group $O^{\times}/\mathbb{Z}^{\times}$ is $\{1, i, j, k\}$, which corresponds in $\text{PU}(2)$ to the Pauli matrices

$$C_4 := \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \right\},$$

and it is mapped under (4.1) to $\{(1 \ 0 \ 0), (0 \ 1 \ 0), (0 \ 0 \ 1), (0 \ 0 \ 0)\}$. These indeed act simply transitively on the neighbors of $v_0 \in X_3 = \text{PGL}_2(\mathbb{Q}_3)/\text{PGL}_2(\mathbb{Z}_3)$, as one can verify by multiplying them on the right with $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and computing the Iwasawa decomposition of the result. The element $i + j + k \in O$ is an involution in $D^{\times}/\mathbb{Q}_3^{\times}$, it inverts the edge $e_0 = [v_0 \rightarrow (3 \ 1) v_0]$ (again using (4.1)), and it corresponds in $\text{PU}(2)$ to

$$T_4 := \begin{pmatrix} 1 & 1-i \\ 1+i & -1 \end{pmatrix}.$$
4.1.2 Hurwitz quaternions and tetrahedral gates

Keeping $D = \left( \frac{-1,-1}{\mathbb{Q}} \right)$, we consider the maximal order of Hurwitz quaternions
\[ \mathcal{O} = \mathbb{Z} \left[ i, \frac{1+i+j+k}{2} \right] = \mathbb{Z} \oplus \mathbb{Z} \cdot i \oplus \mathbb{Z} \cdot j \oplus \mathbb{Z} \cdot \frac{1+i+j+k}{2}, \]
which corresponds to the entry 1 1 2 1 in [KV10, Tab. 8.2]. The unit group $\mathcal{O}^\times / \mathcal{O}_K^\times$ is the Platonic tetrahedral group, which is isomorphic to $\text{Alt}_4$. In $PU(2)$ it corresponds to
\[ C_{12} := \left\langle \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} \right\rangle. \]

For $P = (11)$, the same splitting as in (4.1) can be used (with $\rho = \sqrt{-2} \in \mathbb{Q}_{11}$), and $C_{12}$ acts simply transitively on the neighbors of the origin of $X_P$. As an involution one can take $3i+j+k$, which maps to
\[ T_{12} := \begin{pmatrix} 3 & 1 - i \\ 1 + i & -3 \end{pmatrix} \]
in $PU(2)$. Here and in the examples which follow we do not write down the explicit action of $C_k$ and $T_k$ on the tree $X_P$ (where $k = N(P) + 1$). In all the examples the splitting of $D \otimes K_P$ is chosen so that $C_k$ fixes the vertex $PGL_2(\mathcal{O}_{K_P}) \in X_P$, and $T_k$ invert an edge which is incident to it.

4.1.3 Octahedral gates

We now take $D$ to be the Hamilton quaternions over $K = \mathbb{Q} \left( \sqrt{2} \right)$, which has $\mathcal{O}_K = \mathbb{Z} \left[ \sqrt{2} \right]$ and $U_K = \mathcal{O}_K^\times = \left\{ \pm \left(1 + \sqrt{2}\right)^m \mid m \in \mathbb{Z} \right\}$. A maximal order in $D$ is given by
\[ \mathcal{O} = \mathcal{O}_K \oplus \mathcal{O}_K \cdot \frac{1+i}{\sqrt{2}} \oplus \mathcal{O}_K \cdot \frac{1+j}{\sqrt{2}} \oplus \mathcal{O}_K \cdot \frac{1+i+j+k}{2}, \]
which is entry 2 8 1 1 in [KV10, Tab. 8.2] (see also [Vig80, §5]). The unit group of $\mathcal{O}$ modulo scalars is the Platonic octahedral group
\[ \mathcal{O}^\times / \mathcal{O}_K^\times = \left( \frac{1+i}{\sqrt{2}}, \frac{1+i+j+k}{2} \right) \cong \text{Sym}_4, \]
which corresponds in $PU(2)$ to the Clifford group
\[ C_{24} := \left\langle \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} \right\rangle. \]

For $P = (5 - \sqrt{2})$, there is a (unique) continuous isomorphism $K_P = \mathbb{Q} \left( \sqrt{2} \right)_{5 - \sqrt{2}} \cong \mathbb{Q}_{23}$, which is given by sending $\sqrt{2}$ to a square root of 2 in $\mathbb{Q}_{23}$, chosen so that $P$ maps to a uniformizer. Then, a splitting $(D \otimes K_P)^\times / K_P^\times \cong PGL_2(\mathbb{Q}_{23})$ is given by
\[ a + bi + cj + dk \mapsto \begin{pmatrix} a + 2c + \rho d & b + \rho c - 2d \\ -b + \rho c - 2d & a - 2c - \rho d \end{pmatrix}, \quad (\rho = \sqrt{-5} \in \mathbb{Q}_{23}). \]

Adding the involution $\left(1 + \frac{1}{\sqrt{2}}\right)i + \frac{j}{\sqrt{2}} + (1 - \sqrt{2})k$, which corresponds to
\[ T_{24} := \begin{pmatrix} -1 - \sqrt{2} & 2 - \sqrt{2} + i \\ 2 - \sqrt{2} - i & 1 + \sqrt{2} \end{pmatrix} \in PU(2), \]

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gives the super-golden-gate set $\Gamma = \langle C_{24}, T_{24} \rangle$, which is the full $\{P\}$-arithmetic group in $\mathcal{O}$.

Some subgroups of Sym$_4$ give other super-gate-sets:

**8-gates.** For $P = (3 + \sqrt{2})$, we have $(D \otimes K_P)^\times / K_P^\times \sim PGL_2(\mathbb{Q}_7)$ by

$$a + bi + cj + dk \mapsto \begin{pmatrix} a + \frac{(1+\rho)+d(1-\rho)}{2} & b + \frac{(1-\rho)-d(1+\rho)}{2} \\ -b + \frac{(1-\rho)-d(1+\rho)}{2} & a - \frac{c(1+\rho)+d(1-\rho)}{2} \end{pmatrix}, \quad (\rho = \sqrt{-3}),$$

and we obtain the gates

$$C_8 := \left\langle \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} \right\rangle, \quad T_8 := \begin{pmatrix} \sqrt{2} - 1 & 1 - \sqrt{2}i \\ 1 + \sqrt{2}i & 1 - \sqrt{2} \end{pmatrix},$$

which correspond to Dih$_4 \leq$ Sym$_4$ and the involution $i + j + \frac{k-i}{\sqrt{2}}$.

**3-gates (ramified $P$).** For the prime $P = (\sqrt{2})$, one has $K_P \cong \mathbb{Q}_2(\sqrt{2})$, and a splitting $(D \otimes K_P)^\times / K_P^\times \sim PGL_2(\mathbb{Q}_2(\sqrt{2}))$ is given by

$$a + bi + cj + dk \mapsto \begin{pmatrix} a + \beta (\alpha - \beta) + \frac{d(\alpha + \beta)}{2} & \frac{(\sqrt{2} - 1)(b + \beta c + d\alpha)}{2} \\ -\sqrt{2}(b + \beta c + d\alpha) & a - \beta (\alpha - \beta) - \frac{d(\alpha + \beta)}{2} \end{pmatrix},$$

where $\alpha = \frac{2 + \sqrt{17}}{3}$ and $\beta = \frac{2\sqrt{2} - \sqrt{17}}{3}$. The gate set obtained is

$$C_3 := \left\langle \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} \right\rangle, \quad T_3 := \begin{pmatrix} 0 & \sqrt{2} \\ 1 + i & 0 \end{pmatrix},$$

which correspond to $\left\langle \frac{1+i+j+k}{2} \right\rangle \leq$ Sym$_4$, and $(1 - \frac{1}{\sqrt{2}})j + \frac{1}{\sqrt{2}}k$.

**V-gates (hybrid example).** Taking the subgroup $\left\langle \frac{1+i+j+k}{2}, \frac{i-j}{\sqrt{2}} \right\rangle \cong$ Sym$_3$ of the octahedral group, one can scale its generators to $1+i+j+k$ and $i-j$, which lie in the Lipschitz quaternions. There they generate an infinite group, which projects onto a copy of Sym$_3$ in $\left(\frac{-1+\sqrt{5}}{2}\right)^\times / \mathbb{Q}^\times$. Together with the involution $j + 2k$ this gives the gate set

$$C_6 := \left\langle \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}, \begin{pmatrix} 0 & i \\ 1 & 0 \end{pmatrix} \right\rangle, \quad T_6 := \begin{pmatrix} 0 & 2 - i \\ 2 + i & 0 \end{pmatrix}.$$

### 4.1.4 Icosahedral gates

We move to the golden field $K = \mathbb{Q}(\sqrt{5})$, for which $\mathcal{O}_K = \mathbb{Z}[\varphi]$ ($\varphi = \frac{1+\sqrt{5}}{2}$), and $U_K = \mathcal{O}_K^\times = \{\pm \varphi^m \mid m \in \mathbb{Z}\}$. A maximal order in $D := (\frac{-1-\sqrt{5}}{2})$ is given by the ring of icosians

$$\mathcal{O} = \left\{ \frac{1}{2} \left( \begin{pmatrix} a + b\sqrt{5} \\ + \right) \begin{pmatrix} c + d\sqrt{5} \\ 0 \end{pmatrix} \begin{pmatrix} i \\ j \end{pmatrix} \begin{pmatrix} + \end{pmatrix} \begin{pmatrix} g + h\sqrt{5} \\ k \end{pmatrix} \begin{pmatrix} \right) \begin{pmatrix} a + c + e + g \equiv b + d + f + h \equiv 0 \mod 2 \end{pmatrix} \right\},$$

or

$$\begin{pmatrix} b, d, f \equiv 0 \mod 2 \end{pmatrix}, \quad \text{or} \quad \begin{pmatrix} (b, d, f) + (1, 1, 1) \mod 2 \end{pmatrix},$$

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which is 2 5 1 1 in [KV10, Tab. 8.2], and which forms, with respect to a non-standard quadratic form, a copy of the $E_8$ lattice (c.f. [Tit80] or [CS99, §8]). The unit group of $\mathcal{O}$ modulo scalars is the Platonic icosahedral group

$$\frac{u / u_K = \mathcal{O}^\times / \mathcal{O}_P^\times}{2} = \left\langle \frac{1 + i + j + k}{2}, \frac{i + \varphi^{-1}j + \varphi k}{2} \right\rangle \cong \text{Alt}_5,$$

which corresponds to

$$C_{60} := \left\langle \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}, \begin{pmatrix} 1 & i/\varphi \\ \varphi + i/\varphi & -1 \end{pmatrix} \right\rangle.$$

For $P = (7 + 5\varphi)$ one has $K_P \cong \mathbb{Q}_{25}$, and the splitting (4.1) can be used again. As an involution one can take $(2 + \varphi) i + j + k$, which gives

$$T_{60} := \left(\begin{array}{cc} 2 + \varphi & 1 - i \\ 1 + i & 2 - \varphi \end{array}\right),$$

and the generated group $\Gamma = \langle C_{60}, T_{60} \rangle$ is the full $\{7 + 5\varphi\}$-arithmetic group of $\mathcal{O}$.

**Icosahedral 12-gates.** Taking $P = (4 - \varphi)$, one has $(D \otimes K_P)^\times / K_P^\times \twoheadrightarrow \text{PGL}_2(\mathbb{Q}_{11})$ using (4.1). This gives another set of gates acting on a 12-regular tree:

$$C_{12} := \left\langle \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}, \begin{pmatrix} 1 & i/\varphi \\ \varphi + i/\varphi & -1 \end{pmatrix} \right\rangle, \quad T_{12} := \left(\begin{array}{cc} \varphi - 1 & 1 - i \\ 1 + i & 1 - \varphi \end{array}\right),$$

corresponding to $\langle 1 + i + j + k, \frac{i - \varphi^{-1}j + \varphi k}{2} \rangle \cong \text{Alt}_4 \leq \text{Alt}_5$ and the involution $(\varphi - 1) i + j + k$.

**5-gates (inert $P$).** Taking $P = (2)$, the completion $K_P$ has a residue field of size four, and a splitting $(D \otimes K_P)^\times / K_P^\times \xrightarrow{\sim} \text{PGL}_2(K_P)$ is given by

$$a + bi + cj + dk \mapsto \left(\begin{array}{cc} a + c\alpha + d\beta & b + c\beta - d\alpha \\ -b + c\beta - d\alpha & a - c\alpha - d\beta \end{array}\right), \quad \left(\begin{array}{c} \alpha = \frac{\sqrt{5} - \sqrt{2}}{2} \\ \beta = \frac{\sqrt{5} + \sqrt{2}}{2} \end{array}\right).$$

The cyclic group $\langle 2 + i/\varphi + k \rangle \leq \text{Alt}_5$ and the involution $j + k$ give the gate set

$$C_5 := \left\langle \begin{pmatrix} 1 + \varphi & i \\ -\varphi & 1 + \varphi - i \end{pmatrix} \right\rangle, \quad T_5 := \left(\begin{array}{cc} 0 & 1 \\ i & 0 \end{array}\right).$$

**Nonexamples.** For the ramified and inert primes $P = (\sqrt{5})$ and $P = (3)$ of $\mathbb{Z}[\varphi]$, the corresponding $\{P\}$-arithmetic groups act on a 6-regular and a 10-regular tree, respectively. While the involutions $T_6 = i + \varphi j$ and $T_{10} = i + j + k$ invert an edge at the origin, the groups $\text{Sym}_3, \text{Dih}_5 \leq \text{Alt}_5$ do not act transitively on its neighbors.

**Remark 4.2.** Our focus throughout has been on optimal covering exponent and navigation. If one relaxes the class number 1 assumption or that $S$ consists of a single prime, then one loses these optimal features. However the gate sets that arise from such general $S$-arithmetic groups coming from definite quaternion algebras still have reasonably good covering exponents and navigation properties. These have been studied in [KBRY15].
5 Iwahori-Hecke operators

In this section we study asymmetric Hecke operators, both on $PU(2)$ and on the tree $X_p$, whose spectral properties coincides with that of the non-backtracking random walk (NBRW) on a Ramanujan graph, as explored in [LP16]. This corresponds to words (circuits) in free semigroups generated by Golden Gate sets.

5.1 Ramanujan semigroups in $PU(2)$

For a Super-Golden-Gate set $\Gamma = (C, T)$, and an edge $e_0$ flipped by $T$ (as in Section 4), denote by $\mathfrak{T}$ the “sector” descended from $e_0$, namely, all vertices $v$ for which the shortest path from $v_0$ to $v$ begins with $e_0$. Let

$$S = \{Tc \mid 1 \neq c \in C\} \quad \text{and} \quad S^r = \{s_1 \cdots s_r \mid s_i \in S\},$$

and observe $\Sigma := \bigcup_{r=0}^{\infty} S^r$, the semigroup generated by $S$. Assuming that $e_0$ is leading away from the origin, the map $e_0 \mapsto \sigma e_0$ gives a correspondence between $S^r$ and the edges which lead from the $r$-th to the $(r+1)$-th level in $\mathfrak{T}$. In particular, $\Sigma$ is a free semigroup on $S$, and $|S^r| = k^r$, where $k = |S| = |C| - 1$. The spectrum of $T_S = T_{S,a}$ (with $a_s \equiv 1/k$, in the notation of Section 3.1) is particularly nice:

$$\text{Spec} \left( T_S \big|_{L^2_0(\mathfrak{F}(2))} \right) \subseteq \left\{ \lambda \in \mathbb{C} \mid |\lambda| = \frac{1}{\sqrt{k}} \right\} \cup \left\{ \frac{1}{k} \right\}. \quad (5.2)$$

This should be compared with [LPS86, Thm. 1.3], which asserts that for any symmetric set $S' \subseteq PU(2)$ of size $k$

$$\max \text{Spec} \left( T_{S'} \big|_{L^2_0(\mathfrak{F}(2))} \right) \geq \frac{2\sqrt{k-1}}{k}.$$ 

While the latter bound corresponds to the spectral radius of the random walk on the Cayley graph of a free group, the bound $\frac{1}{\sqrt{k}}$ in (5.2) is the spectral radius of the random walk on a free semigroup. Furthermore, since $T_{S'}$ is precisely $T_{S}$, we obtain $|\lambda| \leq k^{-r/2}$ for every $\lambda \in \text{Spec} \left( T_{S'} \big|_{L^2_0(\mathfrak{F}(2))} \right)$.

One should note, however, that $T_{S}$ is not normal, so that (5.2) is not enough to determine the spectral norm $W_S$. In fact, it turns out that $W_S = 1$. However,

$$W_{S'} = \sqrt{\frac{1}{2k^{r+1}} \left( (k-1) r \sqrt{r^2(k-1)^2 + 4k + r^2(k-1)^2 + 2k} \right) = O \left( \frac{r}{k^{r/2}} \right).} \quad (5.3)$$

This, as well as (5.2), is obtained by considering the action of $T_S$ on the tree $X_p$ and applying strong approximation and the Ramanujan conjectures. We describe the case which corresponds to the standard Hamilton quaternions for simplicity. Since $\Gamma$ acts simply transitively on the edges of $X_p$, we have

$$L^2(\mathfrak{F}(2)) \cong \Gamma \backslash (G_{\mathbb{R}} \times G_p) / B_p, \quad (5.4)$$

where $G = D^\times / \mathbb{Z} (D^\times)$, and $B_p$ is the Iwahori subgroup of $G_p := G_{\mathbb{Q}_p}$, namely, $B_p = \text{Stab}_{G_p} (e_0)$. Via the isomorphism (5.4), $L^2(\mathfrak{F}(2))$ is a representation space for the Iwahori-Hecke algebra of $G_p$. This is the algebra of compactly supported bi-$B_p$-invariant functions on $G_p$, which acts on $L^2(G_p/B_p) \cong \text{Edges}(X_p)$ by convolution, and $T_S$ acts by an element in this algebra.
The Ramanujan conjectures imply that every irreducible, infinite-dimensional Iwahori-spherical representation which appears in $L^2 (PU (2))$ is tempered, so that the matrix coefficients of its Iwahori-fixed vectors are in $L^{2+\varepsilon}$ (Edges $(X_P)$) for all $\varepsilon > 0$. The operator $T_S$ corresponds to NBRW on $X_P$, and its spectrum is contained in that of the NBRW on a $(k+1)$-regular tree, which is $\{ \lambda \in \mathbb{C} | |\lambda| = 1/\sqrt{k} \} \cup \{ \pm 1/k \}$. Furthermore, every tempered, Iwahori-spherical, unitary irreducible representation $V$ of $G_P \cong PGL_2 (\mathbb{Q}_p)$ is either (1) a twist of the Steinberg representation, in which case $\lambda \in \mathbb{C}$, or satisfies $|\lambda| = \pm 1/k$; or (2) a principal series representation induced from

$$\chi : \left( \begin{array}{cc} s & * \\ 0 & * \end{array} \right) \to \mathbb{C}^\times, \quad \chi \left( \left( \begin{array}{cc} a & b \\ 0 & d \end{array} \right) \right) = \left( \frac{s}{\sqrt{p}} \right)^{ord_p(a/d)}$$

for some $s \in \mathbb{C}$ of norm 1. In this case $\dim V^{B_P} = 2$, and in an appropriate orthonormal basis for $V^{B_P}$

$$T_S|_{V^{B_P}} = \left( \begin{array}{cc} \frac{s}{\sqrt{p}} & 0 \\ \frac{(p-1)s}{p} & \frac{1}{\sqrt{p}} \end{array} \right),$$

from which (5.3) follows. We remark that this decomposition, in the graph setting, is the key ingredient in [LP16].

### 5.2 Ramanujan digraphs

The golden-gate sets of [LPS86, LPS87] were used in [LPS88] to construct explicit Ramanujan graphs, which are finite $k$-regular graphs whose nontrivial spectrum is contained within that of the $k$-regular tree. Namely, every eigenvalue $\lambda$ of their adjacency operator is either trivial ($\lambda = \pm k$) or satisfies $|\lambda| \leq 2\sqrt{k} - 1$.

In a similar manner, the set $S$ defined in (5.1) can be used to construct Ramanujan digraphs; we say that a $k$-regular digraph $G$ is Ramanujan if every $\lambda \in \text{Spec}(\text{Adj}_G)$ satisfies $\lambda = |k|$ or $|\lambda| \leq \sqrt{k}$. In other words, its nontrivial spectrum is contained within that of the Cayley digraph of the free semigroup on $k$ generators. For more on Ramanujan digraphs, see [LLP17].

For $K, \mathcal{O}, P, C_k, T_k$ as in one of the examples in Section 4.1, let $S_k = \{ T_k \cdot c \mid 1 \neq c \in C_k \}$. The Cayley graph of $\Gamma = (C_k, T_k)$ with respect to $S_k$ is naturally identified with the directed line graph of the tree $X_P$. Namely, its vertices correspond to directed edges in $X_P$, and its edges to non-backtracking steps in $X_P$, so that the adjacency operator on Cay $(\Gamma, S_k)$ describes NBRW on a tree. For an ideal $Q \neq P$ in $\mathcal{O}_K$, let $\Gamma_Q = \text{ker}(\Gamma \to G_{\mathcal{O}_K / Q})$, and let $Y^{P,Q}$ be the Cayley graph of $\Gamma/\Gamma_Q$ with respect to the set of generators $S_k$. Again, $L^2 (\text{Verts}(Y^{P,Q}))$ is a representation of the Iwahori-Hecke algebra of $G_P$, and the Ramanujan conjectures imply that every nontrivial eigenvalue $\lambda$ of $\text{Adj}(Y^{P,Q})$ corresponds to an Iwahori-fixed vector in a tempered representation, giving $\lambda = \pm 1$ or $|\lambda| = \sqrt{k}$, as in Section 5.1. In addition, we have the trivial eigenvalue $k$ (which arises from the trivial representation), but unlike the continuous case here we also obtain the eigenvalue $-k$ whenever $P$ is a quadratic non-residue in $\mathcal{O}_K/Q$.

Finally, we note that the groups $\Gamma/\Gamma_Q$ can be identified concretely: denoting $\mathbb{F}_q = \mathcal{O}_K/Q$, we have by Wedderburn’s theorem

$$G_{\mathbb{F}_q} = (D \otimes \mathbb{F}_q)^\times / \mathbb{F}_q^\times \cong \text{Mat}_2 (\mathbb{F}_q)^\times / \mathbb{F}_q^\times = PGL_2 (\mathbb{F}_q),$$

and the image of $\Gamma/\Gamma_Q$ in $G_{\mathbb{F}_q}$ is either $PGL_2 (\mathbb{F}_q)$ or $PSL_2 (\mathbb{F}_q)$, according to the Legendre symbol $(\frac{q}{p})$. For example, taking $C_4 = \{ i, j, k \}$, $T_1 = i + j + k$ (see Section 4.1.1) and $Q = (23)$.
one obtains $\Gamma / T_Q \cong PSL_2(\mathbb{F}_{23})$ by $i \mapsto (0 \ 11), j \mapsto (15 \ 1)$. This gives $T_4 \cdot i \mapsto (16 \ 5)$, and similarly for $T_4 \cdot j, T_4 \cdot k$. The spectrum of the Ramanujan digraph obtained is shown in Figure 5.1.

![Graph](image)

Figure 5.1: The adjacency spectrum of $Y^{3,23}$, the Cayley graph of $PSL_2(\mathbb{F}_{23})$ w.r.t. $S = \{(16 \ 5), (20 \ 8), (7 \ 10)\}$.

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