An Ising model having permutation spin and double Eulerian interaction energy

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Dedicated to Einar Steingrímsson on the occasion of his retirement.

Abstract

In this paper we define a variant of the Ising model in which spins are replaced with permutations and the energy between two spins is a function of the relative disorder of one spin, a permutation, to the other. To measure the relative disorder, we use a symmetrized version of the descent permutation statistic that has appeared in the works of Chatterjee & Diaconis and Petersen. The classical Ising model corresponds to the length-2 permutation case. We consider in detail the 1D version of this model on a ring in which spins are length-3 permutations. Exact solutions are given for certain restricted permutation spin (permaspin) sets that are defined in terms of pattern avoidance. We also consider the low-temperature approximation to, and mean-field version of, this 1D model.

1 Introduction

Permutations and permutation statistics have become a popular feature in combinatorial physics over the last few years. Simple transformations of well established results have led to their appearance in some new and unlikely ways. One recent example is the body of work on the combinatorics of the Abelian sandpile model. This explored connections between recurrent configurations of the sandpile model on different classes of graphs and other combinatorial objects. Permutations resulting from canonical toppling orders for Dhar’s burning algorithm have yielded proven links between recurrent configurations of the sandpile model and parallelogram polyominoes [4], tiered trees and EW tableaux [13, 5, 6], Motzkin paths and tiered parking functions [7], amongst others.

A separate direction to this line of research is the study of labelled chip-firing on the infinite-line graph in which the toppling rule is replaced with a refinement [8]. The resulting process was shown to satisfy a confluence property. An outcome of this property is that if the labelled chips are dropped on the origin, then the stabilization of this process sorts the original permutation of labelled chips. Extensions to this work are considered in [10]. Another model in statistical physics in which permutations are a main feature is the partially asymmetric exclusion process (PASEP). This is a model of particle-hopping on a line and its analysis has resulted in rich connections to permutation tableaux [14, 9].

The Ising model [9], a well-established model of magnetism, has witnessed several variants since its inception (see e.g. Baxter [11]). In this paper we introduce a toy model that is a variant of the Ising model that we term the permaspin model. The key feature of this model that
distinguishes it from other Ising-like models is that a site’s spin is some permutation of the set \{1, \ldots, k\}. The interaction energy between two neighbouring sites will be a measure of how unordered the state, a permutation, of one site is with respect to the state of the other site. The external field acting on a site will be a measure of the disorder of that site’s spin, in and of itself.

Let us define the permaspin model on a general graph. The length \(k\) of a permutation representing the spin state of a site is a parameter of the model and we will signify this by using the term \(k\)-permaspin. Let \(G = G(V, E)\) be a simple graph with vertex set \(V = \{v_1, \ldots, v_n\}\) and edge set \(E\). Let \(S_k\) be the set of all permutations of the set \(\{1, \ldots, k\}\) and let \(id\) be the identity permutation. It will be useful to use the notation \(S^a_k = S_k \times \cdots \times S_k\) for the \(n\)-fold Cartesian product that will be the state space of the system. Given a configuration of permaspins \(\bar{\pi} := (\pi^{(1)}, \ldots, \pi^{(n)}) \in S^a_k\) on \(G\) where \(\pi^{(i)}\) is the permaspin of vertex \(v_i\), let us define the energy of this configuration through the Hamiltonian

\[
\mathcal{H}(\bar{\pi}) = -J \sum_{(v_i, v_j) \in E} \phi(\pi^{(i)}, \pi^{(j)}) - H \sum_{v_i \in V} \phi(\pi^{(i)}),
\]

for some yet to be specified interaction energy \(\phi\). We consider \(J\) to be positive so that the model is ferromagnetic. The partition function for the permaspin model is

\[
Z(\beta) = \sum_{\bar{\pi} \in S^a_k} \exp(-\beta \mathcal{H}(\bar{\pi})),
\]

where \(\mathcal{H}(\bar{\pi})\) is given in equation (1). The free energy is

\[
f(\beta) = -\lim_{n \to \infty} \frac{1}{\beta n} \log Z(\beta).
\]

The configuration probability that the state of the system \(\sigma\) takes a particular permaspin configuration is given by the Boltzmann distribution

\[
P_{\beta}(\sigma = \bar{\pi}) = \frac{\exp(-\beta \mathcal{H}(\bar{\pi}))}{Z(\beta)}.
\]

In this paper we will take some first steps in considering simple instances of the permaspin model. In Section 2 we will discuss the interaction energy for this model and make a choice for the interaction energy based on both interaction symmetry and simplicity. Our choice of interaction energy yields the classical Ising model as the 2-permaspin model. In Section 3 we consider the 1D \(k\)-permaspin model in the absence of an external field, and see how closed forms for both the partition function and free energy can be given as simple transformations of the generating function for the interaction energy.

In Section 4 we build up to the case of considering the 1D 3-permaspin model in an external field by first looking at two ‘subcases’ whereby the set of permaspins is slightly restricted. For the 1D 3-permaspin case in which the permaspins 123 and 321 are forbidden, we are able to give exact expressions for the partition function and free energy. Following this, in Section 4.2 we then consider the case when only the permaspin 123 is forbidden. Again, we are able to given exact expressions for both the partition function and free energy. For both of the restrictions in Sections 4.1 and 4.2 there is no phase transition for \(\beta > 0\).

In Section 4.3 we now allow the permaspin 123 and, in doing so, are considering the general 1D 3-permaspin model. It is not possible to give closed form expressions for the six eigenvalues associated with this system. However, we are able to rule out several eigenvalues from being largest and see that the characteristic polynomial for the associated transition matrix factors.
as a cubic and three linear terms (two of which are equal). Experimentally, it seems the largest
eigenvalue is one of the cubic roots and we note some properties in relation to it. We further
note that one of the eigenvalues with an explicit expression could be a potentially good lower
bound for the largest eigenvalue.

In Section 5 we given a low-temperature approximation for the free energy of the 1D 3-
permaspin model while in Section 6 we consider the mean-field 1D 3-permaspin model that
results in a sum for the partition function. Finally, we conclude with a discussion in Section 7.

2 A double Eulerian interaction energy

For two neighbouring permaspins, \( \sigma \) and \( \pi \), we would like the interaction energy \( \phi \) to be a
measure of how much one differs from the other. There are several ways to model this, but
for the purposes of this paper we will consider this interaction energy to be a function of the
permutation \( \tau \) that is required to sort \( \sigma \) into \( \pi \),

\[
\tau = \sigma^{-1} \pi.
\]

If \( \sigma \) and \( \pi \) are the same then \( \tau = id \) whereas if \( \sigma \) is the reverse of \( \pi \) then \( \tau \) will be id reversed.
For permaspins \( \alpha, \beta \in S_k \) and \( \tau := \alpha^{-1} \beta \), suppose the generic permutation statistic \( \text{stat}(\tau) \)
takes values in the set \( \{0, 1, \ldots, s_{\text{max}}\} \). To normalize the interaction energy we define

\[
\phi(\alpha, \beta) := 1 - \frac{2}{s_{\text{max}}} \text{stat}(\alpha^{-1} \beta),
\]

that takes values in the closed interval \([-1, 1] \subset \mathbb{R} \). A further requirement is one of symmetry:
the interaction energy between \( \alpha \) and \( \beta \) must be the same as the interaction energy between \( \beta \)
and \( \alpha \), i.e. \( \phi(\alpha, \beta) = \phi(\beta, \alpha) \). This is equivalent to the statistic \( \text{stat} \) satisfying

\[
\text{stat}(\pi) = \text{stat}(\pi^{-1}) \quad \text{for all permutations} \ \pi. \tag{4}
\]

There are several permutation statistics that might be used to measure how unordered a
given permutation is. These include, but are not limited to, the permutation statistics number of descents (des), number of inversions (inv), number of exceedance, and number of weak exceedances. For example, the number of descents in a permutation \( \pi = (\pi_1, \ldots, \pi_n) \) is the number of indices \( i \) for which \( \pi_i > \pi_{i+1} \). A permutation statistic that is equidistributed with the descent statistic on permutations is called Eulerian whereas one that is equidistributed with the major index permutation statistic is called Mahonian. Many of the more common permutation statistics are either Eulerian or Mahonian. Different statistics will of course lead to different calculation challenges, with those for Mahonian statistics being more involved due mainly to the range of values the statistic may take.

The inversions statistic, being the only one mentioned above to satisfy the interaction symmetry property for permaspins (Equation 4) would seem like the natural choice. However, even for the 3-permaspin case, the calculation and comparative analysis of eigenvalues for the inv statistic case (to be discussed in the last section) appears difficult. The reason for this is the larger range of values that the inversion statistic can take. With this point in mind, in this paper we choose the symmetrized statistic \( \text{cddes} \) whereby:

\[
\text{cddes}(\pi) := \text{des}(\pi) + \text{des}(\pi^{-1}). \tag{5}
\]

Note that \( s_{\text{max}} \) for \( \text{cddes} \) on \( S_n \) is \( 2(n - 1) \). This statistic was investigated in Chatterjee & Diaconis [2] and Petersen [12]. The coefficients of the generating function for this statistic have
become known as the double Eulerian numbers \[ A298248 \]. See Table 1 for the first few double Eulerian numbers.

The generating function for this statistic is (see [12, Theorem 2])

$$\text{CDdes}_n(u) = \sum_{\pi \in S_n} u^{\text{cdedes}(\pi)} = (1 - u)^{2n+2} \sum_{i,j \geq 1} \binom{i+j+n-1}{n} u^{i+j-2}. \quad (6)$$

We will write $\phi_{\text{cdedes}}$ to signify the use of Equation (5) in the interaction energy. With the interaction energy defined using the cdedes statistic, the 2-permaspin model is the classical Ising model where the spins take values in $\{-1, +1\}$. This is easily seen via the identification of the permutations $(1, 2)$ and $(2, 1)$ with the classical spin $+1, -1$, respectively. If $\alpha, \beta \in S_2$ with corresponding classical spins $s_\alpha, s_\beta$, then one finds $\phi(\alpha, \beta) = s_\alpha s_\beta$. In some places will will present results in some generality so that the results for other choices of statistic are easily achieved, such as in Section 3.

One outstanding matter is the term in the Hamiltonian that represents the energy interaction of a permaspin with a hypothesised external field. The term in the Hamiltonian (Equation 1) that corresponds to this is $-H\phi(\pi)$, even though $\phi(\pi)$ as defined above is a function of two permutations. We will assume that $\phi(\pi) := 1 - \frac{2}{s_{\text{max}} \text{stat}(\pi)}$, which is precisely the same value one gets from assuming this is the energy of the permaspin $\pi$ with another permaspin that is the identity permutation, i.e. $\phi(\pi) = \phi(\text{id}, \pi) = \phi(\pi, \text{id})$. In other words, one can consider the hypothesised external field to have permaspin id.

### 3 The 1D permaspin model without external field

Let $G$ be the closed path graph with vertices $V(G) = \{v_1, \ldots, v_n\}$, edges $E(G) = \{(v_i, v_{i+1}) : 1 \leq i \leq n\}$, and the convention $v_{n+1} := v_1$. We will sometimes refer to this closed path graph as a ring. The partition function for this 1D model is easily calculated since the partition function can be written as a deformation of the generating function for the stat statistic.

**Theorem 1.** The partition function for the closed path graph is

$$Z_n^{(\text{stat})}(\beta) = k! e^{\beta J} \left( \exp(\beta J) \text{Stat}_k \left( \exp \left( \frac{-2\beta J}{s_{\text{max}}} \right) \right) \right)^{n-1},$$

where $\text{Stat}_k(x) := \sum_{\pi \in S_k} x^{\text{stat}(\pi)}$ and the free energy is

$$f^{(\text{stat})}(\beta) = -\frac{1}{\beta} \ln \left( \exp(\beta J) \text{Stat}_k \left( \exp \left( \frac{-2\beta J}{s_{\text{max}}} \right) \right) \right).$$
Proof. As the edges are pairs \((v_i, v_{i+1})\) the partition function is
\[
Z_n^{\text{(stat)}}(\beta) = \sum_{\pi \in S_k^n} \exp \left( \beta J \sum_{i=1}^n \phi(\pi(i), \pi(i+1)) \right)
= \sum_{\pi \in S_k^n} \exp \left( \beta J \sum_{i=1}^n \left( 1 - \frac{2}{s_{\max}} \text{stat}\((\pi(i))^{-1}\pi(i+1)) \right) \right).
\]

Unity from the internal sum can be extracted and moved outside, and the internal exponent containing a sum can be written as a product of exponents:
\[
Z_n^{\text{(stat)}}(\beta) = \exp(\beta J n) \sum_{\pi \in S_k^n} \exp \left( -\frac{2\beta J}{s_{\max}} \sum_{i=1}^n \text{stat}\((\pi(i))^{-1}\pi(i+1)) \right)
= \exp(\beta J n) \sum_{\pi \in S_k^n} \prod_{i=1}^n \exp \left( -\frac{2\beta J}{s_{\max}} \text{stat}\((\pi(i))^{-1}\pi(i+1)) \right).
\]

Fix \(\pi^{(1)}\) and set \(\sigma^{(1)} = (\pi^{(1)})^{-1}\pi^{(2)}\). Instead of summing over all \(\pi^{(2)} \in S_k\), we can exploit the form in the exponent to sum over all \(\sigma^{(1)} \in S_k\), and extend this to \(\sigma^{(2)}, \ldots, \sigma^{(n-1)}\). The above expression becomes:
\[
Z_n^{\text{(stat)}}(\beta) = \exp(\beta J n) \sum_{\pi^{(1)} \in S_k} \sum_{\sigma^{(1)} \in S_k} \frac{-2\beta J}{s_{\max}} \text{stat}(\sigma^{(1)}) \sum_{\sigma^{(2)} \in S_k} \frac{-2\beta J}{s_{\max}} \text{stat}(\sigma^{(2)}) \cdots \sum_{\sigma^{(n-1)} \in S_k} \frac{-2\beta J}{s_{\max}} \text{stat}(\sigma^{(n-1)})
= \exp(\beta J n) k! \left( \text{Stat}_k \left( \exp \left( \frac{-2\beta J}{s_{\max}} \right) \right) \right)^{n-1}
= k! \exp(\beta J) \text{Stat}_k \left( \exp \left( \frac{-2\beta J}{s_{\max}} \right) \right)^{n-1},
\]
where \(\text{Stat}_k(x) := \sum_{\pi \in S_k} x^{\text{stat}(\pi)}\). The free energy is therefore
\[
f^{\text{(stat)}}(\beta) = -\lim_{n \to \infty} \frac{1}{\beta n} \ln Z_n^{\text{(stat)}}(\beta) = -\frac{1}{\beta} \ln \left( \exp(\beta J) \text{Stat}_k \left( \exp \left( \frac{-2\beta J}{s_{\max}} \right) \right) \right).
\]

When the permutation statistic \(\text{stat}\) is \(\text{cddes}\), we have

**Corollary 2.** The partition function
\[
Z_n^{\text{(cddes)}}(\beta) = k! \exp(\beta J) \left( \exp \left( \frac{-\beta J}{k-1} \right) \right)^{n-1},
\]
and the free energy is
\[
f^{\text{(cddes)}}(\beta) = -\frac{1}{\beta} \ln \left( \exp(\beta J) \text{CDdes}_k \left( \exp \left( \frac{-\beta J}{k-1} \right) \right) \right),
\]
where \(\text{CDdes}_k(x)\) is given in Equation 6.

As mentioned before, the \(k = 2\) case of the \(\text{cddes}\) interaction energy corresponds to the classical 1D Ising model and, using \(\text{CDdes}_2(x) = 1 + x^2\), the equations in this case are well known:
\[
Z_n^{\text{(cddes)}}(\beta) = 2 \exp(\beta J) \left( \exp(\beta J) + \exp(-\beta J) \right)^{n-1}
\]
\[
f^{\text{(cddes)}}(\beta) = -\frac{1}{\beta} \ln \left( \exp(\beta J) + e^{-\beta J} \right).
\]
4 The 1D 3-permaspin model in an external field

In this section we will consider the 1D 3-permaspin model in an external field. As the 2-permaspin model on a general graph corresponds to the classical Ising model on that same graph, any properties relating to the latter are also properties of the 2-permaspin model. The 3-permaspin model is the simplest one to first consider and for which results are not known.

The set of permaspins for the 3-permaspin model is \( \{123, 132, 213, 231, 312, 321\} \). In order to calculate the partition function and free energy for the 3-permaspin model on a ring, one will be at the mercy of calculating eigenvalues of an order 6 matrix. This will be the topic of Subsection 4.3.

First let us define a slightly generalized version of the 1D \( k \)-permaspin model where the set of allowed permaspins is some subset, \( P \), of all possible permaspins \( S_k \). Following that we will consider three different \( P \subseteq S_k \) for the 1D 3-permaspin model. The partition function for the general 1D \( k \)-permaspin model with restricted permaspin set \( P \) is:

\[
Z_n^{(\text{stat})}(\beta) = \sum_{\pi \in P} \exp \left( -\beta \left( -J \sum_{(i, j) \text{nnn}} \phi(\pi(i), \pi(j)) - H \sum_v \phi(\pi(v)) \right) \right)
\]

\[
= \sum_{\pi \in P} \exp \left( \beta J \sum_{i=1}^n \phi(\pi(i), \pi(i+1)) + \beta H \sum_{i=1}^n \phi(\pi(i)) \right)
\]

\[
= \sum_{\pi \in P} \exp \left( \beta J \sum_{i=1}^n \left( 1 - \frac{2\text{stat}(\pi(i)-1\pi(i+1))}{s_{\text{max}}} \right) + \beta H \sum_{i=1}^n \left( 1 - \frac{2\text{stat}(\pi(i))}{s_{\text{max}}} \right) \right)
\]

\[
= e^{\beta(J+H)n} \sum_{\pi \in P} \exp \left( -2\frac{\beta J}{s_{\text{max}}} \sum_{i=1}^n \left( \text{stat}(\pi(i)-1\pi(i+1)) \right) - 2\frac{\beta H}{s_{\text{max}}} \sum_{i=1}^n \left( \text{stat}(\pi(i)) \right) \right).
\]

Let us now set \( J' = -2\beta J/s_{\text{max}} \) and \( H' = -2\beta H/s_{\text{max}} \), so that the previous expression becomes:

\[
Z_n^{(\text{stat})}(\beta) = e^{\beta(J+H)n} \sum_{\pi \in P} \exp \left( J' \sum_{i=1}^n \left( \text{stat}(\pi(i)-1\pi(i+1)) \right) + H' \sum_{i=1}^n \left( \text{stat}(\pi(i)) \right) \right)
\]

\[
= e^{\beta(J+H)n} \prod_{\pi \in P} \exp \left( \frac{H'\text{stat}(\pi(i))}{2} + J'\text{stat}(\pi(i)-1\pi(i+1)) + \frac{H'\text{stat}(\pi(i+1))}{2} \right)
\]

\[
= e^{\beta(J+H)n} \prod_{\pi \in P} \exp \left( \frac{H'(\text{stat}(\pi(i)) + \text{stat}(\pi(i+1)))}{2} + J'\text{stat}(\pi(i)-1\pi(i+1)) \right). \tag{7}
\]

Define the \( |P| \times |P| \) matrix \( A \) whereby the \( i \)th row corresponds to the \( i \)th permutation of \( P \) when \( P \) is listed in standard lexicographic order, and the same labelling convention for the columns. Define entry \( A_{\pi, \sigma} \) to be

\[
A_{\pi, \sigma} := \exp \left( \frac{H'}{2} (\text{stat}(\pi) + \text{stat}(\sigma)) + J'\text{stat}(\pi^{-1}\sigma) \right)
\]

\[
= a^{\text{stat}(\pi) + \text{stat}(\sigma)} b^{\text{stat}(\pi^{-1}\sigma)},
\]

where we have used the substitution \( a = e^{\frac{1}{2}H'} \) and \( b = e^{J'} \). Using this matrix formalization, we can now write

\[
Z_n^{(\text{stat})}(\beta) = e^{\beta(J+H)n} \text{Tr}(A^n) = e^{\beta(J+H)n} \left( \lambda_1^n + \ldots + \lambda_k^n \right), \tag{8}
\]
where \( \lambda_1, \ldots, \lambda_k \) are the eigenvalues of \( A \). Since we assume the statistic \text{stat} \) to satisfy the interaction symmetry condition (Equation 4) the matrix \( A \) will be symmetric. That \( A \) is symmetric guarantees all eigenvalues of \( A \) are real.

### 4.1 A restricted 1D 3-permaspin model in an external field

In this subsection we will consider the 3-permaspin model whose permaspins are restricted to the set \( P = \{132, 213, 231, 312\} \). In pattern avoidance literature, the set \( S_k(\tau) \) is the of length \( k \) permutations that avoid the pattern \( \tau \). This notation extends to sets of patterns so the set of allowed permaspins we consider here is \( S_3(123, 321) \). The \( A \) matrix for this case is

\[
A = \begin{pmatrix}
  a^4 & a^4b^2 & a^4b^4 & a^4b^2 \\
  a^4b^2 & a^4 & a^4b^2 & a^4b^4 \\
  a^4b^4 & a^4b^2 & a^4 & a^4b^2 \\
  a^4b^2 & a^4b^4 & a^4b^2 & a^4 \\
\end{pmatrix}.
\]  

(9)

This matrix has characteristic polynomial

\[
c_A(\lambda) = (a^4b^4 + 2a^4b^2 + a^4 - \lambda)(a^4b^4 - 2a^4b^2 + a^4 - \lambda)(a^4b^4 - a^4 + \lambda)^2,
\]

and the eigenvalues of \( A \) are

\[
\lambda_1 = a^4(1 - b^2)(1 + b^2), \quad \lambda_2 = a^4(b^2 - 1)^2, \quad \text{and} \quad \lambda_3 = a^4(b^2 + 1)^2,
\]

where \( \lambda_1 \) is a double eigenvalue. This allows us to write the partition function for this case:

\[
Z_n^{(\text{cddes})}(\beta) = e^{\beta(1+H)n}a^{4n} \left( 2(1 - b^2)n(1 + b^2)^n + (1 - b^2)^{2n} + (1 + b^2)^{2n} \right)
\]

\[
= e^{\beta(1+H)n}e^{2Hn} \left( 2(1 + e^{2\beta J})^n + (1 - e^{2\beta J})^n \right)
\]

\[
= e^{\beta Jn} \left( 2(1 - e^{-\beta J})^n(1 + e^{-\beta J}) + (1 - e^{-\beta J})^2n + (1 + e^{-\beta J})^2n \right).
\]

(10)

(11)

Recall that \( H' = -2\beta H/s_{\text{max}} = -2\beta H/4 = -\beta H/2 \) and \( J' = -2\beta J/s_{\text{max}} = -\beta J/2 \), so the previous expression becomes

\[
Z_n^{(\text{cddes})}(\beta) = e^{\beta(J+H)n}e^{-n\beta H} \left( 2(1 - e^{-\beta J})^n(1 + e^{-\beta J})^n + (1 - e^{-\beta J})^2n + (1 + e^{-\beta J})^2n \right)
\]

\[
= e^{\beta Jn} \left( 2(1 - e^{-\beta J})^n(1 + e^{-\beta J})^n + (1 - e^{-\beta J})^2n + (1 + e^{-\beta J})^2n \right)
\]

(12)

The variables \( a \) and \( b \) may take only strictly positive values as they are exponentials and so \( \lambda_3 \) will always be the largest. The free energy is therefore

\[
f^{(\text{cddes})}(\beta) = -\lim_{n \to \infty} \frac{1}{\beta n} \ln Z_n^{(\text{stat})}(\beta)
\]

\[
= -\frac{1}{\beta} \ln \left( e^{\beta J} \left( 1 + e^{-\beta J} \right)^2 \right)
\]

\[
= -J - \frac{2}{\beta} \ln \left( 1 + e^{-\beta J} \right).
\]

\[
\text{Theorem 3. The partition function for the restricted } P = S_3(123, 321) \text{ 1D 3-permaspin model is}
\]

\[
Z_n^{(\text{cddes})}(\beta) = e^{\beta Jn} \left( 2(1 - e^{-\beta J})^2n + (1 + e^{-\beta J})^2n \right),
\]

and the free energy is

\[
f^{(\text{cddes})}(\beta) = -J - \frac{2}{\beta} \ln \left( 1 + e^{-\beta J} \right).
\]
We notice that \( H \) does not feature in \( Z_{n}^{\text{cddes}}(\beta) \) nor in \( f^{\text{cddes}}(\beta) \). The reason for this is that the spins that were forbidden were precisely those permaspins that contributed non-zero values to the Hamiltonian, since
\[
cddes(123) = -cddes(321) = 1 \quad \text{and} \quad cddes(132) = cddes(213) = cddes(231) = cddes(312) = 0.
\]
The expression for the free energy is a continuous function of \( \beta \) for \( \beta > 0 \) so there will be no phase transition.

### 4.2 A less restricted 1D 3-permaspin model in an external field

We next consider the case for permaspins \( P = S_{3}(123) \). It enlarges the allowed permaspin set of the previous subsection but forbids permaspin 123 at any of the sites (although the external field will still be treated as having permaspin 123 for reasons mentioned earlier). For this case we have
\[
A = \begin{pmatrix}
  a^4 & a^4b^2 & a^4b^4 & a^4b^2 & a^6b^2 \\
  a^4b^2 & a^4 & a^4b^2 & a^4b^4 & a^6b^2 \\
  a^4b^4 & a^4b^2 & a^4 & a^4b^2 & a^6b^2 \\
  a^4b^2 & a^4b^4 & a^4b^2 & a^4 & a^6b^2 \\
  a^6b^2 & a^6b^2 & a^6b^2 & a^6b^2 & a^8
\end{pmatrix}.
\]

The characteristic polynomial of \( A \) is
\[
c_A(\lambda) = (-a^2(3b^2 + 1)(b^2 - 1) - a^4(a^4 + (b^2 + 1)^2)\lambda + \lambda^2)(a^4(1 - b^2)^2 - \lambda)(a^4b^4 - a^4 + \lambda)^2.
\]
This yields the following eigenvalues of \( A \):
\[
\begin{align*}
\lambda_1 & = \frac{a^4}{2} \left( a^4 + (1 + b^2)^2 - \sqrt{(a^4 + (1 + b^2)^2)^2 + 4a^4(1 + 3b^2)(b^2 - 1)} \right) \\
\lambda_2 & = \frac{a^4}{2} \left( a^4 + (1 + b^2)^2 + \sqrt{(a^4 + (1 + b^2)^2)^2 + 4a^4(1 + 3b^2)(b^2 - 1)} \right) \\
\lambda_3 & = a^4(1 - b^4) \\
\lambda_4 & = a^4(1 - b^2)^2,
\end{align*}
\]
where \( \lambda_3 \) is a double eigenvalue. Notice that if \( b \in [1, +\infty) \) then \( \lambda_4 \geq \lambda_3 \), whereas if \( b \in (0, 1) \) then \( \lambda_4 < \lambda_3 \). It is always true that the eigenvalue \( \lambda_2 \geq \lambda_1 \). To determine which is the largest eigenvalue we must consider two cases:

**Case** \( b \in [1, +\infty) \): For this case notice that the discriminant in \( \lambda_2 \) is such that the right additive term \( 4a^4(3b^2 + 1)(b^2 - 1) \) is always non-negative. The discriminant is therefore bounded below by \( (a^4 + (1 + b^2)^2)^2 \), and so
\[
\begin{align*}
\lambda_2 & \geq \frac{a^4}{2} \left( a^4 + (1 + b^2)^2 + \sqrt{(a^4 + (1 + b^2)^2)^2} \right) \\
& = a^4(a^4 + (1 + b^2)^2) \\
& = a^8 + a^4(1 + b^2)^2 \\
& > 0 + a^4(1 - b^2)^2 \\
& = \lambda_4.
\end{align*}
\]
Therefore the largest eigenvalue for this case is \( \lambda_2 \).
**Case** $b \in (0, 1)$: For this case, we know from above that $\lambda_2 > \lambda_1$ and $\lambda_3 > \lambda_4$. It remains to examine which of $\lambda_2$ and $\lambda_3$ is larger. Consider when $\lambda_2 \geq \lambda_3$:

$$\frac{a^4}{2} (a^4 + (1 + b^2)^2 + \sqrt{(a^4 + (1 + b^2)^2)^2 + 4a^4(1 + 3b^2)(b^2 - 1)}) \geq a^4(1 - b^4),$$

which, given $a, b \geq 0$, is equivalent to

$$a^4 + (1 + b^2)^2 + \sqrt{(a^4 + (1 + b^2)^2)^2 + 4a^4(1 + 3b^2)(b^2 - 1)} \geq 2(1 - b^4).$$

Let us set $c = a^4$ and $d = b^2$ so that the previous inequality is equivalent to

$$c + (1 + d)^2 + \sqrt{c + (1 + d)^2 - 4c(1 + 3d)(1 - d)} \geq 2(1 - d^2). \quad (14)$$

Consider the two subcases:

**Subcase** $0 \leq a \leq \sqrt{1 + b^2}$: In terms of $c$ and $d$, this corresponds to $0 \leq c \leq (1 + d)^2$. In (14) notice that since $\lambda_2$ is a real number, the LHS of that inequality is necessarily real. Since that is the case, the quantity

$$c + (1 + d)^2 - \sqrt{(c + (1 + d)^2)^2 - 4c(1 + 3d)(1 - d)}$$

is also real and the quadratic

$$
\begin{align*}
(x - (c + (1 + d)^2 + \sqrt{(c + (1 + d)^2)^2 - 4c(1 + 3d)(1 - d)})) \\
(x - (c + (1 + d)^2 - \sqrt{(c + (1 + d)^2)^2 - 4c(1 + 3d)(1 - d)})) \\
= x^2 - 2 (c + (1 + d)^2) x + 4c(1 + 3d)(1 - d)
\end{align*}
$$

is concave and has real roots. If setting $x = 2(1 - d^2)$ in this quadratic gives a quantity that is $\leq 0$ then we are done since it means $2(1 - d^2)$ is weakly to the left of the larger root, i.e. what is stated in (14). Setting $x = 2(1 - d^2)$ yields:

$$2^2(1 - d^2) - 2(c + (1 + d)^2) \cdot 2(1 - d^2) + 4c(1 + 3d)(1 - d)$$

$$= 4(1 - d^2)^2 - 4c(1 - d^2) - 4(1 + d)^2(1 - d^2) + 4c(1 + 2d - 3d^2)$$

$$= 4(1 - d^2) (1 - d^2 - (1 + d)^2) + 4c(2d - 2d^2)$$

$$= 4(1 - d^2) (-2d - 2d^2) + 4c(2d - 2d^2)$$

$$= 8d(1 - d)(c - (1 + d)^2)$$

$$\leq 0,$$

which holds true since both $d$ and $1 - d \geq 0$ but $c \leq (1 + d)^2$.

**Subcase** $\sqrt{1 + b^2} \leq a$: This case corresponds to $c \geq (1 + d)^2$. On inspecting the LHS of (14), we have

$$c + (1 + d)^2 + \sqrt{(c + (1 + d)^2)^2 - 4c(1 + 3d)(1 - d)} \geq c + (1 + d)^2,$$

where the (real) square root term has been removed. By assumption, as $c \geq (1 + d)^2$ we have

$$c + (1 + d)^2 \geq 2(1 + d)^2.$$

Now since $b \in (0, 1)$ we have $d \in (0, 1)$ and so $2(1 + d)^2 \geq 2 \geq 2(1 - d)^2$. Combining these inequalities yields (14).
The eigenvalue $\lambda_2$ is therefore always the largest eigenvalue. Since

$$Z_n^{(cddes)}(\beta) = e^{\beta(J+H)n} \left( \lambda_1^n + \lambda_2^n + 2\lambda_3^n + \lambda_4^n \right),$$

$$f^{(cddes)}(\beta) = -\lim_{n \to \infty} \frac{1}{\beta n} \ln Z_n^{(cddes)}(\beta) = -\frac{1}{\beta} (\beta(J+H) + \ln \lambda_2),$$

and $a = e^{-\beta H/2}$, $b = e^{-\beta J}$, we have:

**Theorem 4.** The partition function for the restricted 1D 3-permaspin model with permaspin set $S_3(123)$ is

$$Z_n^{(cddes)}(\beta) = e^{\beta(J-H)n} \times \left[ \begin{array}{c} \left( e^{-2\beta H} \left( 1 + e^{-2\beta J} \right) \right)^n \\ \frac{e^{-2\beta H} \left( 1 + e^{-2\beta J} \right)^2 - \sqrt{e^{-2\beta H} \left( 1 + e^{-2\beta J} \right)^2 + 4e^{-2\beta H} \left( 1 + 3e^{-2\beta J} \right) \left( e^{-2\beta J} - 1 \right)}}{2} \\ \frac{e^{-2\beta H} \left( 1 + e^{-2\beta J} \right)^2 + \sqrt{e^{-2\beta H} \left( 1 + e^{-2\beta J} \right)^2 + 4e^{-2\beta H} \left( 1 + 3e^{-2\beta J} \right) \left( e^{-2\beta J} - 1 \right)}}{2} \\ + \left( 1 - e^{-2\beta J} \right)^n \left( 2 \left( 1 + e^{-2\beta J} \right)^n + \left( 1 - e^{-2\beta J} \right)^n \right) \end{array} \right]$$

and the free energy is

$$f^{(cddes)}(\beta) = - (J - H) + \frac{\ln 2}{\beta} - \frac{1}{\beta} \ln \left( e^{-2\beta H} \left( 1 + e^{-2\beta J} \right)^2 \right)$$

$$+ \sqrt{e^{-2\beta H} \left( 1 + e^{-2\beta J} \right)^2 + 4e^{-2\beta H} \left( 1 + 3e^{-2\beta J} \right) \left( e^{-2\beta J} - 1 \right)}. $$

The expression for the free energy is a continuous function of $\beta$ for $\beta > 0$ so again there will be no phase transition.

### 4.3 The 1D 3-permaspin model in an external field

In this subsection we consider the case for permaspins $P = S_3$, i.e. the general 3-permaspin model on the closed path graph. It can be seen to build on the previous subsection by now allowing the permaspin 123 as a site spin. Here we have

$$A = \begin{pmatrix} 1 & a^2b^2 & a^2b^2 & a^2b^2 & a^2b^2 & a^2b^2 & a^2b^2 & a^2b^2 & a^4b^4 \\ a^2b^2 & a^4 & a^4b^2 & a^4b^2 & a^4b^2 & a^4b^2 & a^4b^2 & a^4b^2 & a^4b^2 \\ a^2b^2 & a^4b^2 & a^4 & a^4b^2 & a^4b^2 & a^4b^2 & a^4b^2 & a^4b^2 & a^4b^2 \\ a^2b^2 & a^4b^2 & a^4b^2 & a^4 & a^4b^2 & a^4b^2 & a^4b^2 & a^4b^2 & a^4b^2 \\ a^2b^2 & a^4b^2 & a^4b^2 & a^4b^2 & a^4 & a^4b^2 & a^4b^2 & a^4b^2 & a^4b^2 \\ a^2b^2 & a^4b^2 & a^4b^2 & a^4b^2 & a^4b^2 & a^4 & a^4b^2 & a^4b^2 & a^4b^2 \\ a^2b^2 & a^4b^2 & a^4b^2 & a^4b^2 & a^4b^2 & a^4b^2 & a^4 & a^4b^2 & a^4b^2 \\ a^2b^2 & a^4b^2 & a^4b^2 & a^4b^2 & a^4b^2 & a^4b^2 & a^4b^2 & a^4 & a^4b^2 \\ a^2b^2 & a^4b^2 & a^4b^2 & a^4b^2 & a^4b^2 & a^4b^2 & a^4b^2 & a^4b^2 & a & \\ \end{pmatrix}. \quad (15)$$

The characteristic polynomial of $A$ is

$$c_A(\lambda) = - \left( a^{12}(b^{12} + 2b^{10} - 7b^8 + 7b^4 - 2b^2 - 1) \right)$$

$$+ a^4\lambda(1 - 3a^8b^4 - a^4b^8 + 2a^8b^2 + a^8 + a^4 - 3b^4 + 2b^2)$$

$$+ \lambda^2(-a^4 - a^8 - a^4b^4 - 2a^4b^2 - 1) + \lambda^3$$

$$\left( a^4b^4 - 2a^4b^2 + a^4 - \lambda \right) \left( a^4b^4 - a^4 + \lambda \right).$$

10
Let us substitute $c$ for $a^4$ and $d$ for $b^2$ to write
\[
c_A(\lambda) = -(d^2 + 4d + 1)c^3(d + 1)(d - 1)^3 - \lambda c (cd^3 + 3c^2d + cd^2 + c^2 + cd + c + 3d + 1)(d - 1) \]
\[
- \lambda^2 (cd^2 + c^2 + 2cd + c + 1) + \lambda^3 \right) (cd^2 - 2cd + c - \lambda) (cd^2 - c + \lambda).
\]
This yields the eigenvalues of $A$:
\[
\lambda_4 = c(1 - d)^2 \\
\lambda_5 = c(1 - d^2)
\]
and, since $A$ is a real symmetric matrix all its eigenvalues will be real, a further three real eigenvalues $\lambda_1, \lambda_2, \lambda_3$ as solutions to $t(\lambda) = 0$ where
\[
t(\lambda) := \lambda^3 - (cd^2 + c^2 + 2cd + c + 1)\lambda^2 - c (cd^3 + 3c^2d + cd^2 + c^2 + cd + c + 3d + 1)(d - 1) \lambda \\
+ (d^2 + 4d + 1)c^3(d + 1)(d - 1)^3 \\
= A^{3}\lambda^3 + B^{3}\lambda^2 + C^{3}\lambda + D^{3}.
\]
Let $\lambda^*$ be the largest root of $t(\lambda) = 0$. It remains to see which of $\lambda^*, \lambda_4,$ and $\lambda_5$ is largest. Although it will not be possible to derive a closed formula for $\lambda^*$, we will use some computational observations in conjunction with an analysis of the roots to provide a route to what we think to be the answer for this case.

Let us deal with this in a systematic way to see how to determine the largest root $\lambda^*$ of the cubic. The general cubic equation formula is as follows. Form
\[
\Delta_0 = B^{2} - 3A^{2}C'
\]
\[
\Delta_1 = 2B^{3} - 9A^{1}B^{1}C' + 27A^{2}D'
\]
\[
\Delta_2 = \Delta_{1}^{2} - 4\Delta_{0}^{3}.
\]
For non-negative values of $c, d$, it seems that $\Delta_0$ is non-negative (see Figure 1a), $\Delta_1$ is bounded above by approximately 0.08 (see Figure 1b), and $\Delta_2 \leq 0$ (see Figure 1c).

We next form
\[
E = \sqrt{\frac{\Delta_1 + \sqrt{\Delta_2}}{2}}.
\]
Our observation that $\Delta_2 \leq 0$ allows us to write
\[
E = \sqrt[3]{\frac{\Delta_1 + i\sqrt{-\Delta_2}}{2}},
\]
where we can be certain that $\sqrt{-\Delta_2} \in \mathbb{R}$. Note that from the observed values of $\Delta_1$ and $\Delta_2$, the complex number $\Delta_1 + i\sqrt{-\Delta_2}$ will necessarily be in the upper half of the complex plane and is equal to $re^{i\theta}$ for some $\theta \in [0, \pi]$. This means the three cube roots of $(\Delta_1 + \sqrt{\Delta_2})/2$ will have the form:

\[
E_1 = re^{i\theta_1}, \; \theta_1 \in [0, \pi/3], \quad E_2 = re^{i\theta_2}, \; \theta_2 \in [2\pi/3, \pi], \quad E_3 = re^{i\theta_3}, \; \theta_3 \in [4\pi/3, 5\pi/3].
\]
The quantities $\theta_2 = \theta_1 + 2\pi/3$ and $\theta_3 = \theta_1 + 4\pi/3$. Notably the root $E_1$ will be in the first quadrant. Let us refer to these as
\[
E_1 = E = u + iv, \quad E_2 = E(-1 + i\sqrt{3})/2 = u_2 + iv_2, \quad \text{and} \quad E_3 = E(-1 - i\sqrt{3})/2 = u_3 + iv_3.
\]
The three roots of \( t(\lambda) = 0 \) are

\[
\lambda_i = -\frac{1}{3} \left( B' + E_i + \frac{\Delta_0}{E_i} \right), \text{ for } i = 1, 2, 3. \tag{16}
\]

As \( \lambda_1 \) is real it must be the case that \( E_1 + \Delta_0/E_1 \) is also real where \( E_1 = u + iv \). The quantity \( E_1 + \Delta_0/E_1 \) is

\[
E_1 + \Delta_0/E_1 = u + iv + \frac{\Delta_0}{u + iv} = u + \frac{u\Delta_0}{u^2 - v^2} + i \left( v - \frac{v\Delta_0}{u^2 - v^2} \right), \tag{17}
\]

thereby giving \( \Delta_0 = u^2 - v^2 = r^2 \), from which it follows that

\[
E_1 + \Delta_0/E_1 = 2u = 2r \cos \theta_1 = 2\text{Re}(E_1).
\]

Since \( \lambda_2 \) is also real, we must have that \( E_2 + \Delta_0/E_2 \) is real where

\[
E_2 = E \left( \frac{-1 + i\sqrt{3}}{2} \right) = \frac{u - v\sqrt{3} + i(u\sqrt{3} - v)}{2} = u_2 + iv_2.
\]

The quantity \( E_2 + \Delta_0/E_2 \) is

\[
u_2 + iv_2 + \frac{\Delta_0}{u_2 + iv_2} = u_2 + \frac{u_2\Delta_0}{u_2^2 - v_2^2} + i \left( v_2 - \frac{v_2\Delta_0}{u_2^2 - v_2^2} \right), \tag{18}
\]

thereby giving \( \Delta_0 = u_2^2 - v_2^2 = r^2 \), from which it follows that

\[
E_2 + \Delta_0/E_2 = 2u_2 = u - v\sqrt{3} = 2r \cos \theta_2 = 2\text{Re}(E_2).
\]

Similarly, for \( E_3 \) we find

\[
E_3 + \frac{\Delta_0}{E_3} = 2u_3 = u + v\sqrt{3} = 2r \cos \theta_3 = 2\text{Re}(E_3).
\]
Let us bring these observations together. We wish to determine the largest eigenvalue amongst $\lambda_1, \lambda_2, \lambda_3$ where

$$\lambda_i = -\frac{1}{3} \left( B' + E_i + \frac{\Delta_0}{E_i} \right) \quad (19)$$

and the $E_i$'s are as stated above. Due to the minus sign, this corresponds to finding the $i$ that minimizes $E_i + \frac{\Delta_0}{E_i} = 2 \text{Re}(E_i)$.

Consider again the range of values of $\theta_1$ that determines $E_1$ and the rotations through $2\pi/3$ and $4\pi/3$ radians that give $E_2$ and $E_3$, respectively. The point in the complex plane that will have the most negative real part will be that for $E_2$. (A simple graphical consideration shows this.) Therefore the minimum is attained at $i = 2$, and so the largest amongst $\lambda_1, \lambda_2, \lambda_3$ will be

$$\lambda^* = \lambda_2 = -\frac{1}{3} \left( B' + E_2 + \frac{\Delta_0}{E_2} \right)$$

$$= -\frac{1}{3} \left( B' + 2 \text{Re}(E_2) \right)$$

$$= -\frac{1}{3} \left( -(cd^2 + c^2 + 2cd + c + 1) + u - v\sqrt{3} \right)$$

$$= \frac{1}{3} \left( (c^2d + c^2 + 2cd + c + 1) - u + v\sqrt{3} \right),$$

where $c = a^4 = e^{-2\beta H}$ and $d = b^2 = e^{-2\beta J}$. Since we do not have closed form expressions for $u$ and $v$, proving inequalities relating to $\lambda^*$ is a difficult task. Through experimenting, however, we can see which of the eigenvalues amongst $\lambda^*, \lambda_4$, and $\lambda_5$ is the largest. It turns out to always be $\lambda^*$ which corresponds to the blue surface in Figures 2a and 2b. The orange surface is $\lambda_4$ and the green surface is $\lambda_5$. We summarise the results of the above experimental observations and analysis as follows:

**Conjecture 5.** The free energy for the 1D 3-permaspin model is

$$f^{(cddes)}(\beta) = - (J + H) - \frac{1}{\beta} \ln \left( (c^2d + c^2 + 2cd + c + 1) - 2 \text{Re}(z) \right)$$

where $z$ is the unique root of the cubic

$$\lambda^3 - (cd^2 + c^2 + 2cd + c + 1)\lambda^2$$

$$- c(cd^3 + 3cd^2 + cd^2 + c^2 + cd + c + 3d + 1)(d - 1)\lambda + (d^2 + 4d + 1)c^3(d + 1)(d - 1)^3 = 0.$$

for which $\text{arg}(z) \in [2\pi/3, \pi]$ and where $c = e^{-2\beta H}$ and $d = e^{-2\beta J}$.

For larger values of $c$ and $d$ it seems that that $\lambda_4$ serves as a close lower bound to the $\lambda^*$. This suggests that free energy for the 1D 3-permaspin model at low-temperature has behaviour:

$$f^{(cddes)}(\beta) \approx - \lim_{n} \frac{1}{\beta n} \ln(e^{\beta(J+H)}\lambda_4^n) = -\frac{1}{\beta} (\beta(J + H) + \ln \lambda_4)$$

$$= -(J + H) - \frac{1}{\beta} \ln c(1 - d)^2$$

$$= -(J + H) - \frac{1}{\beta} \ln a^4(1 - b^2)^2.$$
As \( a = e^{-\beta H/2} \) and \( b = e^{-\beta J} \) this becomes

\[
f^{(cddes)}(\beta) \simeq -(J + H) - \frac{1}{\beta} \ln(e^{-\beta H/2})^4 (1 - (e^{-\beta J})^2)^2
\]

\[
= -(J + H) - \frac{1}{\beta} \ln e^{-2\beta H} (1 - e^{-2\beta J})^2
\]

\[
= -(J + H) + 2H - \frac{2}{\beta} \ln(1 - e^{-2\beta J}).
\]

While not as satisfying as the previous two sections, we record this computational observation regarding the free energy as a comment.

**Comment 6.** The free energy for the 1D 3-permaspin model at low-temperature seems to have the behaviour:

\[
f^{(cddes)}(\beta) \sim -(J - H) - \frac{2}{\beta} \ln(1 - e^{-2\beta J}) \approx -(J - H) - \frac{2}{\beta} \ln \beta.
\]

Notice that as \( \beta \to \infty \) in the above expression, \( -(J - H) - \frac{2}{\beta} \ln \beta \to -(J - H) \).

## 5 Low-temperature approximation of the 1D 3-permaspin model

Let us consider two configurations that represent the ‘largest’ terms in the partition function for low temperatures, i.e. when \( \beta \) is large. The first configuration is that in which all permaspins are the same. The contribution to the partition function \( Z \) for configurations of this type is

\[
\sum_{\pi \in S_3} \exp(n_3 J \phi(\pi, \pi) + n_3 H \phi(\pi, \text{id})) = e^{n_3 J} \sum_{\pi \in S_3} \exp(n_3 H \phi(\pi, \text{id}))
\]

\[
= e^{n_3 J} \left(e^{-n_3 H} + 4 + e^{n_3 H}\right). \quad (20)
\]

In the event that

- \( H = 0 \) the expression \((20)\) is

\[
6e^{n_3 J}. \quad (21)
\]
\( H > 0 \) the expression is dominated by the term
\[
e^{n\beta J} \left( 4 + e^{n\beta H} \right).
\]

\( H < 0 \) the expression is dominated by the term
\[
e^{n\beta J} \left( e^{-n\beta H} + 4 \right).
\]

The second configuration we will consider is when \( n - k \) consecutive spins are the same, \( \pi \) say, and the remaining \( k \) consecutive spins are \( \pi' \neq \pi \). The contribution to \( Z \) will be
\[
\frac{n}{2} \sum_{k=1}^{n-1} \sum_{\pi, \pi' \in S_3 \atop \pi \neq \pi'} e^{\beta J((n-k-1)\phi(\pi,\pi)+(k-1)\phi(\pi',\pi')+2\phi(\pi,\pi')) + \beta H((n-k)\phi(\pi,\text{id})+k\phi(\pi',\text{id}))}
\]
\[
= \frac{n}{2} e^{\beta J(n-2)} \sum_{k=1}^{n-1} \sum_{\pi, \pi' \in S_3 \atop \pi \neq \pi'} \exp \left( 2\beta J \phi(\pi, \pi') + \beta H((n-k)\phi(\pi, \text{id}) + k\phi(\pi', \text{id})) \right)
\]
\[
= 4 \exp(\beta H((n-k)(1) + k(0))) + \exp(-2\beta J + \beta H((n-k)(1) + k(-1)))
\]
\[
= 4 e^{(n-k)\beta H} + e^{-2\beta J + (n-2k)\beta H}.
\]

When \( \pi = 123 \), the contribution to this sum is
\[
\sum_{\pi' \in S_3 \atop \pi \neq 123} \exp \left( 2\beta J \phi(123, \pi') + \beta H((n-k)\phi(123, \text{id}) + k\phi(\pi', \text{id})) \right)
\]
\[
= 4 e^{(n-k)\beta H} + e^{-2\beta J + (n-2k)\beta H}.
\]

When \( \pi = 132 \), the contribution to this sum is
\[
\sum_{\pi' \in S_3 \atop \pi \neq 132} \exp \left( 2\beta J \phi(132, \pi') + \beta H((n-k)\phi(132, \text{id}) + k\phi(\pi', \text{id})) \right)
\]
\[
= 2 + e^{k\beta H} + e^{-k\beta H} + e^{-2\beta J}.
\]

The same expression appears for \( \pi = 213, 231, 312 \), whereas for \( \pi = 321 \) we have the contribution
In the event that \( H > 0 \) then the dominant terms in expression \( 27 \) are

\[
\frac{n e^{\beta J(n-2)}}{2} \left( (8 + 4e^{-2\beta J}) (n - 1) + 8(e^{(n-1)\beta H} + 2e^{-2\beta J} e^{(n-2)\beta H}) \right).
\]

For large \( \beta \) we can approximate this as \( Z \approx 6e^{n\beta J} \).

If \( H < 0 \) then the dominant terms in expression \( 27 \) are

\[
\frac{n e^{\beta J(n-2)}}{2} \left( (8 + 4e^{-2\beta J}) (n - 1) + 8e^{(n-1)\beta H} + 2e^{-2\beta J} e^{-(n-2)\beta H} \right).
\]

For large \( \beta \) we can approximate this as \( Z \approx e^{n\beta J} (4 + e^{n\beta H}) \).
If $H < 0$ then precisely the same reasoning (i.e. by summing 23 and 30) will give us the approximation

$$Z \approx e^{n\beta J} \left( e^{-n\beta H} + 4 \right).$$

Using this to see what picture it paints regarding the free energy, one finds that these crude approximations seem to imply the following low-temperature behavior for the free energy.

**Comment 7.** A lower bound for free energy for the 1D 3-permaspin model at low-temperature seems to have the behaviour:

$$f^{(\text{cddes})}(\beta) \approx -(J + |H|).$$

This approximation coincides with the prediction of Comment 6 for $H \leq 0$, but differs from it significantly for $H > 0$. We wonder which of the two is more representative of the actual low-temperature free energy for the $H > 0$ case.

### 6 A mean-field 3-permaspin model

Let us suppose that for a 3-permaspin model, each of the $n$ permaspins has $q$ nearest-neighbors, so that the 1D model corresponds to $q = 2$ whereas the 2D model corresponds to $q = 4$. In this section we will consider a mean-field 3-permaspin model. For a mean-field model, the total field acting on site $v_i$ is

$$H \phi(\pi^{(i)}, \text{id}) + \frac{qJ}{n-1} \sum_{j \neq i} \phi(\pi^{(i)}, \pi^{(j)}).$$

Here, the sum over all the interactions of site $i$ with the $n-1$ sites that are not $i$ is averaged out by dividing the sum by $n-1$. The $q$ in the numerator reflects the multiplicity of this quantity with respect to the number of neighbours of site $i$. This is equivalent to replacing the Hamiltonian with

$$H^{(\text{mean})}(\pi) = -\frac{qJ}{n-1} \sum_{\{v_i, v_j\} \text{n.n.}} \phi(\pi^{(i)}, \pi^{(j)}) - H \sum_{i=1}^{n} \phi(\pi^{(i)}, \text{id}).$$

Note that the first sum is over all $\binom{n}{2}$ distinct pairs of vertices $\{v_i, v_j\}$.

Since every permaspin interacts with every other permaspin, we can simplify the above sum by considering the distribution of permaspins in $\tilde{\pi}$. Let us write $n_\sigma$ for the number of $\pi^{(i)}$ that equal $\sigma$. Then

$$n_{123} + n_{132} + n_{213} + n_{231} + n_{312} + n_{321} = n.$$  

Table 2(a) illustrates $\text{cddes}(\sigma^{-1}\pi)$ for all $\sigma$ and $\pi$, and the internal energy corresponding to a nearest neighbour pairing is given in Table 2(b).

So if $\pi$ is a configuration with $n_\sigma$ permaspins equal to $\sigma$, then the Hamiltonian $H^{(\text{mean})}(\pi)$
The partition function is
\[ H = S \]
and columns are indexed with the permutations from \( S_3 \) (symmetric) matrix illustrating \( S \)

Table 2: (a) the (symmetric) matrix illustrating cddes(\( \sigma^{-1} \pi \)) for all \( \sigma, \pi \in S_3 \) where the rows and columns are indexed with the permutations from \( S_3 \) in lexicographic order. (b) the (symmetric) matrix illustrating \( \phi(\sigma, \pi) := 1 - \text{cddes}(\sigma^{-1} \pi)/2 \) for all \( \sigma, \pi \in S_3 \) where the rows and columns are indexed with the permutations from \( S_3 \) in lexicographic order.

\[
\begin{pmatrix}
0 & 2 & 2 & 2 & 2 & 4 \\
2 & 0 & 2 & 4 & 2 & 2 \\
2 & 2 & 0 & 2 & 4 & 2 \\
2 & 4 & 2 & 0 & 2 & 2 \\
2 & 2 & 4 & 2 & 0 & 2 \\
4 & 2 & 2 & 2 & 2 & 0
\end{pmatrix}
\]

(a) cddes(\( \sigma^{-1} \pi \))

\[
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & -1 \\
0 & 1 & 0 & -1 & 0 & 0 \\
0 & 0 & 1 & 0 & -1 & 0 \\
0 & -1 & 0 & 1 & 0 & 0 \\
0 & 0 & -1 & 0 & 1 & 0 \\
-1 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
\]

(b) \( \phi(\sigma, \pi) \)

equals \( H^{(\text{mean})}(n_{123}, \ldots, n_{321}) \) where

\[
H^{(\text{mean})}(n_{123}, \ldots, n_{321})
= - \frac{qJ}{n-1} \left( n_{\sigma} \phi(\sigma, \sigma) + \sum_{\sigma < \tau \in S_3} n_{\sigma} n_{\tau} \phi(\sigma, \tau) \right) - H \sum_{\sigma \in S_3} n_{\sigma} \phi(\sigma, \text{id})
\]

\[
= - \frac{qJ}{n-1} \left( \sum_{\sigma \in S_3} \left( n_{\sigma} \phi(\sigma, \sigma) + \sum_{\sigma < \tau \in S_3} n_{\sigma} n_{\tau} \phi(\sigma, \tau) \right) \right) - H \left( n_{123}(1) + (n_{132} + n_{213} + n_{231} + n_{312})(0) + n_{321}(-1) \right)
\]

\[
= - \frac{qJ}{2(n-1)} \left( (n_{123} - n_{321})^2 + (n_{132} - n_{231})^2 + (n_{213} - n_{312})^2 - (n_{123} + \ldots + n_{321}) \right)
\]

\[
= - \frac{qJ}{2(n-1)} \left( (n_{123} - n_{321})^2 + (n_{132} - n_{231})^2 + (n_{213} - n_{312})^2 - n \right) - H \left( n_{123} - n_{321} \right)
\]

\[
= - \frac{qJ}{2(n-1)} \left( (n_{123} - n_{321})^2 + (n_{132} - n_{231})^2 + (n_{213} - n_{312})^2 - n \right) - \frac{2qJ}{n-1} \left( (n_{123} - n_{321})^2 + (n_{132} - n_{231})^2 + (n_{213} - n_{312})^2 \right)
\]

The partition function is

\[
Z_n(\beta) = \sum_{n_{123} + \ldots + n_{321} = n} \left( n_{123}, n_{132}, \ldots, n_{321} \right) \exp \left( -\beta H^{(\text{mean})}(n_{123}, \ldots, n_{321}) \right).
\]

Using the expression for the mean field Hamiltonian above, we can now write the partition function as a reduced sum:

**Theorem 8.** The mean-field partition function is

\[
Z_n(\beta) = e^{-\frac{\beta}{2} \frac{qJ}{n-1} + \frac{H^2(n-1)}{qJ}} \sum_{a+b+c=n} \left( n_{a, b, c} \right) G_a \left( H(n-1)/qJ, e^{\frac{\beta qJ}{2(n-1)}} \right) G_b \left( 0; e^{\frac{\beta qJ}{2(n-1)}} \right) G_c \left( 0; e^{\frac{\beta qJ}{2(n-1)}} \right)
\]
where
\[ G_m(\ell; x) := \sum_{i+j=m} \binom{m}{i} x^{(i-j+\ell)^2} = \sum_i \binom{m}{i} x^{(2i-m+\ell)^2}. \]

Given the quadratic term in the exponent of \( x \), there is no closed formula for \( G_m(\ell, x) \) that can help us achieve a closed form for \( Z_n(\beta) \).

This does not rule out the possibility of approximating \( Z_n(\beta) \). The form of \( Z_n(\beta) \) is of an exponential times a triple convolution. Moreover, two of the series in the convolution are the same. Another perhaps useful remark in this regard is that the squared behaviour in the exponent suggests some continuous approximation of the sum as an integral of a deformation of the normal distribution might be a useful avenue to investigate.

In attempting to do this from a purely discrete viewpoint, we make the following observations. As the \( \ell \) we consider is positive and the only non-zero one is proportional to \( n \), and since the \( x \) in each case is \( e^{\beta q J/2(n-1)} \), the dominant term in each \( G \) is the one corresponding to \( i = j = m/2 \). With the product of the three \( G \) terms being replaced with a product of central binomial coefficients while the powers stay the same within each term, the maximum ought to occur when \( a = b = c = n/3 \). This analysis points to the following term approximating the dominant one:

\[
e^{-\frac{\alpha q J}{2(n-1)}} \left( \frac{\beta q J}{2(n-1)} \right)^{\frac{1}{2}} \left( \frac{H(n-1)}{q J} \right)^2 \approx e^{-\frac{\alpha q J}{2(n-1)}} \left( \frac{\beta q J}{2(n-1)} \right)^{\frac{1}{2}} \left( \frac{H(n-1)}{q J} \right)^2 \]

The conjectured dominating term seems to have the unfortunate effect of removing the external field, and also does not lend itself well to the limit in the free energy. Using this approximation it would appear that the free energy is independent of both \( J \) and \( H \) and is \( f(\text{mean})(\beta) = -\frac{1}{\beta} \ln 6 \). If one now includes in the approximation for \( G_m(\ell; x) \) a further two terms corresponding to \( i = m/2 - 1 \) and \( i = m/2 \), then we find an expression that contains \( H \) although it suffers from the same drawback that the limit in the free energy will remove such terms. It seems likely that another approach is required in order to gather information from our mean-field sum (Theorem 8) for the partition function.

**7 Conclusion**

In this paper we introduced a variant of the Ising model in which permutations and permutation statistics play a leading role. While the 2-permaspin model corresponds exactly to the classical Ising model, we explored the next non-trivial case on the cycle graph, the graph that the Ising model was first considered upon. We have seen that the calculations involved in the derivations of the partition function and free energy are challenging and non-trivial. By restricting the permaspins to two restricted sets we were able to give exact expressions for the partition function and free energy for the 3-permaspin model on the cycle graph. Unfortunately, we were not able to do this for the general 3-permaspin model on the cycle graph but our analysis (Conjecture 5) was able to indicate which of the eigenvalues will govern the asymptotic behaviour in that case. We were also able to see the approximate nature of the free energy at low temperatures (Comments 7 and 6) and provide (in Theorem 8) a sum for the mean-field partition function.

In our discussion in Section 2, we noted that the number of inversions permutation statistic could be another natural choice to replace cddes in the calculation of the free energy. The
reason for this is that the inv statistic satisfies the symmetry property given in equation [4] without modification. In order to consider the 1D 3-permaspin model using the inv statistic in place of cddes, one will find the transition matrix is now:

\[
A^{(\text{inv})} = \begin{pmatrix}
1 & ab & ab & a^2b^2 & a^2b^2 & a^3b^3 \\
ab & a^2 & a^2b^2 & a^3b^3 & a^3b & a^4b^2 \\
ab & a^2b^2 & a^2 & a^3b & a^3b^3 & a^4b^2 \\
a^2b^2 & a^3b^3 & a^3b & a^4 & a^4b^2 & a^5b \\
a^3b^2 & a^3b & a^3b^3 & a^4b^2 & a^4 & a^5b \\
a^3b^3 & a^4b^2 & a^4b^2 & a^5b & a^5b & a^6
\end{pmatrix}.
\]

Let us also mention that the characteristic polynomial for this matrix is

\[
c_{A^{(\text{inv})}}(\lambda) = \left( (b^2 + b + 1)(b^2 - b + 1)a^{12}(b + 1)^4(b - 1)^4 \
- \lambda(a^4b^2 - a^2b^4 + a^4 + b^2 + 1)(a^2 + 1)a^6(b + 1)^2(b - 1)^2 \
- \lambda^2(a^8 + 2a^6b^2 + 2a^4b^4 + a^6 + 3a^4b^2 + 2a^4 + 2a^2b^2 + a^2 + 1)a^2(b^2 - 1) \
+ \lambda^3(-(a^4 + a^2b^2 + 1)(a^2 + 1)) + \lambda^4 \
\right) \left( a^6(1 + b)^3(1 - b)^3 + \lambda(a^2 + 1)a^2(b + 1)(b - 1) + \lambda^2 \right).
\]

There remain some outstanding questions that warrant further investigation and we mention these here. The first of these is whether it is possible to exactly solve the 1D 3-permaspin model on the cycle graph. Secondly, is the choice of permutation statistic in the calculation of the energy something which can affect the existence of a phase transition? A first step in this direction would be to ask whether there exists some choice of permutation statistic such that a phase transition appears for a 1D \( k \)-permaspin model for some \( k \geq 3 \).

In Section 4 our analysis of the 1D 3-permaspin model was hierarchical in that, although it was not possible to exactly solve the general case, we considered the model on two proper subsets of spins that were ‘close’ to the model, i.e. the permaspin sets

\[
\{132, 213, 231, 312\} \subset \{132, 213, 231, 321, 312\} \subset \{123, 132, 213, 231, 321, 312\}.
\]

We saw that the models for the first two permaspin sets were exactly solvable, and contained no phase transition. While it was not possible to do this for the final (more general) case, we noticed that from numerical investigations there was one dominant eigenvalue that appeared to have another eigenvalue as a good lower bound. There is no phase transition for that lower eigenvalue, and from these facts we would be inclined to think there is no phase transition for the general 1D 3-permaspin model. Can an argument along these lines be placed on a more formal footing?

Finally, a main goal when dealing with permutation statistics is to derive their generating function since it encodes the distribution of the statistic for arbitrary length permutations. In this paper we have seen that information about such a generating function is useful for the zero-field case, but not so for the case of an external field. Instead, determining the eigenvalues of a (transition) matrix whose exponents are encoded by the permutation statistic is the main goal. Might these eigenvalues have other uses, perhaps in some spectral theory of permutations that has yet to be formalised?

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