Andreev reflection between a normal metal and the FFLO superconductor

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We consider a process of the Andreev reflection between a normal metal and the s-wave superconductor in the FFLO state. It is shown that the process takes place if the energy of the incoming electron is bound within the finite interval called the Andreev window. The position of the window determines the value of the non-zero total momentum of Cooper pairs and the value of the gap.

I. INTRODUCTION

During the last two decades one has faced a renaissance of the interest in non-uniform superconductivity, where the spatial symmetry is broken by a non-zero total momentum \( \mathbf{q} \) of Cooper pairs \(^1\). This has happened mainly due to the experimental discoveries of the possible candidates for the superconductors in the FFLO (Fulde-Ferrel-Larkin-Ovchinnikov) state \(^2\). Recently, strong evidence for the existence of the FFLO state has been brought forward for the organic superconductor \(^3\) and for the CeCoIn\(_5\) superconductor \(^4\). Additionally a new interest has arisen in the field of strong interaction physics where the new state of matter called color superconductivity was suggested \(^5\).

In condensed matter physics the FFLO state requires an applied external magnetic field which leads to the Zeeman splitting of the Fermi surface of conduction electrons, while in QCD the Fermi surfaces of quarks are already split because of the different masses of different flavor quarks \(^6\). It is also a good place to point out that the non-uniform condensates were also considered in the context of chiral symmetry breaking \(^7\), the subject which is still under the current debate \(^8\).

The Andreev reflection \(^9\) between a normal metal and an anisotropic superconductor with a directionally dependent gap was discussed in \(^10\) with the main interest concentrated on the d-wave superconductors. In this paper we consider the junction between a normal metal and the s-wave superconductor in the simplest possible FFLO state where the gap parameter has an oscillating phase \( \Delta(\mathbf{r}) \sim \exp (i \mathbf{q} \cdot \mathbf{r}) \). All the calculations are performed in the pure Pauli limit.

We show that the Andreev reflection can be used, at least in principle, to detect the existence of the non-zero total momentum of Cooper pairs. Despite the clear result still a lot of work must be done to bridge between the ideal case we discuss in this paper and the description of real materials used in experiments.

In the first section we discuss the Bogolubov - de Gennes equations and the general properties of their solutions. In the second section an exact, numerical solution is given for some generic parameters. In the last section there is a short discussion of the possible further work.

II. BOGOLUBOV - DE GENNES EQUATIONS FOR THE 1-DIM LOFF PHASE

Bogolubov - de Gennes equation for a non-relativistic superconductor in one dimension takes the form:

\[
E f(z) = - \left( \frac{1}{2m} \frac{d^2}{dz^2} + \mu \right) f(z) + \Delta(z) g(z) \\
E g(z) = \left( \frac{1}{2m} \frac{d^2}{dz^2} + \mu \right) g(z) + \Delta^*(z) f(z),
\]

where the FFLO gap function is chosen as \( \Delta(z) = \Delta_0 e^{i q z} \) and the FFLO momentum \( q \) is assumed to be non-negative.

The plane wave ansatz \( f(z) = f_{k+q} e^{i(k+q)z} \) and \( g(z) = g_{k-q} e^{i(k-q)z} \) lead to the matrix equation

\[
\begin{pmatrix}
E - \epsilon_{k+q} & \Delta_0 \\
\Delta^*_0 & E + \epsilon_{k-q}
\end{pmatrix}
\begin{pmatrix}
f_{k+q} \\
g_{k-q}
\end{pmatrix}
= 0,
\]

where \( \epsilon_k = k^2/2m - \mu \). The nontrivial solution is possible when

\[
(E - \epsilon_{k+q})(E + \epsilon_{k-q}) - |\Delta_0|^2 = 0
\]
which gives the dispersion relations

\[
E_\pm = \frac{1}{2} \left( \epsilon_{k+q} - \epsilon_{k-q} \pm \sqrt{(\epsilon_{k-q} + \epsilon_{k+q})^2 + 4|\Delta_0|^2} \right).
\]  

(4)

In Fig 1, examples of the dispersion relation (4) are given for \( q = 0 \) (left panel) and \( q \neq 0 \) (right panel). It can be immediately seen that the non-zero value of the total momentum \( q \) breaks the parity symmetry \( k \rightarrow -k \). The exact shape of the curves depends on the size of the total momentum \( q \). For the values of \( 0 \leq E_q < \frac{|\Delta_0|^2}{4m} \), where \( E_q = \frac{q^2}{2m} \), there are two positive minima of \( E_+ \) of different depths at the points that are not related by the parity transformation. The curve \( E_- \) resides below zero. However, for \( E_q > \frac{|\Delta_0|^2}{4m} \) one of the minima of \( E_+ \) descents below zero. Simultaneously a part of the curve \( E_- \) emerges above zero on the opposite side of the momentum axis. This phenomenon leads to more complicated dispersion relations for hole-like and particle-like excitations.

Let us consider the process of the Andreev reflection in a simple one dimensional geometry configuration where the FFLO superconductor resides at \( z > 0 \) with the gap parameter \( \Delta(z) = \Theta(z)\Delta_0 e^{2i\eta_0^2} \). The normal metal is at \( z < 0 \) with a junction to superconductor at \( z = 0 \). An electron of the mass \( m \) and the energy \( E \) above the Fermi energy arrives from the left at the junction.

![FIG. 1: The dispersion relations (4) with the upper curve \( E_+ \) and the lower curve \( E_- \). In the left panel \( q = 0 \) and in the right panel \( q \neq 0 \). The lines \( a, b \) represents the energy \( E \) of the incoming electron. The crossing points \( k_{1,2}^\pm \) describes the solutions of the equations \( E = E_\pm \).](image)

Then one needs to solve the equations (2) for \( f_{k+q} \) and \( g_{k-q} \) with the sewing conditions \( \psi_{<}(0) = \psi_{>}(0), \psi'_{<}(0) = \psi'_{>}(0) \) imposed at \( z = 0 \). The incoming electron of given energy \( E \) can excite quasiparticles inside the superconductor with real or complex momenta. Quasiparticles with the real momenta can propagate inside the superconductor freely whereas those with the complex momenta penetrate only the region close to the junction. The solutions with real momenta exist if there are non-zero real solutions of the equations \( E_\pm = E \). Let us first consider the case of normal superconductor with \( q = 0 \). These solutions geometrically are placed at the crossing of the horizontal lines \( a, b \) and the curves \( E_\pm \) (lines \( a, b \) in the left panel of Fig. 1). For the s-wave BCS superconductor there are four possible real solutions for \( E > \Delta \) (line \( b \)) among which \( k_{1}^\pm \) describes hole-like and \( k_{2}^\pm \) particle-like excitations propagating from the left to the right whereas \( k_{1}^-, k_{2}^- \) are those which propagate in the opposite direction. Only two of these solutions are compatible with the boundary conditions at infinity (determined by the direction of the incoming electron plane wave). Let us consider the case where the electron is incoming from the left. Then solutions \( k_{1,2}^- \) are rejected and the only left are \( k_{1,2}^+ \). It is worth to mention that the incoming electron mainly excites the electron-like quasiparticle in the superconductor, which follows from the kinematics of the reflection process. In the case of \( E < \Delta \) there are no real solutions (no crossing of the line \( a \) with the curve \( E_+ \) in Fig. 1, left panel). The lack of the freely propagating excitations in the superconductor results in the process called the Andreev reflection.

In the case of the FFLO superconductor one can immediately infer from Fig. 1 (right panel) that there are always real solutions of the equations \( E_\pm = E \). Let us consider first the solutions described by the horizontal line \( a \). This corresponds to the case of \( E > \Delta \) for the uniform BCS superconductor. There are two solutions \( k_{1,2}^+ \) of possible four that are compliant with the boundary conditions. Similarly to the uniform case the electron-like excitation \( k_{2}^+ \) dominates over the hole-like \( k_{1}^+ \) quasiparticle. In the case described by the line \( b \) there are two possible solutions \( k_{1}^+, k_{2}^- \) and only one solution \( k_{1}^- \) is compatible with the boundary condition. This is exactly the one which describes
the hole-like quasiparticle propagating from the left to the right. However, this solution is strongly suppressed by the kinematics of the reflection. As a result the quasi-Andreev process takes place when the energy of the incoming electron is within the gap between the minimum of the upper \( E_+ \) and the maximum of the lower \( E_- \). Approximately this is in a range

\[
\frac{q}{m} \sqrt{2m\mu - q^2} - |\Delta_0| < E < \frac{q}{m} \sqrt{2m\mu - q^2} + |\Delta_0|
\]

(5)

with the corrections of the order of \( \frac{q^2}{m\mu} \) to the limits. These inequalities are valid as long as \( E_q < E_F \).

The inequalities (5) in principle can be used for the determination of the important superconductor parameters. The difference between the upper and the lower bound gives the gap parameter \( |\Delta_0| \) whereas the sum determines the value of the total pair momentum \( q \) when the Fermi energy \( \mu \) and the mass of the charge carrier are given.

III. NUMERICAL RESULTS

We consider an electron that is injected from the conductor side of the junction. In this case the wave function takes the form

\[
\psi_<(z) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{ikz} + B \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-ikz} + C \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{ipz}, \quad z < 0,
\]

(6)

\[
\psi_>(z) = \begin{pmatrix} \psi_1 \\ \psi_1 \end{pmatrix} = F \begin{pmatrix} f_1 e^{iqz} \\ g_1 e^{-iqz} \end{pmatrix} e^{ikz} z + J \begin{pmatrix} f_2 e^{iqz} \\ g_2 e^{-iqz} \end{pmatrix} e^{ikz} z, \quad z > 0.
\]

(7)

The total probability current obeys a continuity equation

\[
\frac{\partial}{\partial t} (|\psi_1|^2 + |\psi_1|^2) + \frac{\partial}{\partial z} j_p = 0,
\]

\[
j_p = \frac{1}{2mi} \left( \psi_1 \frac{\partial}{\partial z} \psi_1 - \psi_1 \frac{\partial}{\partial z} \psi_1 + \psi_1 \frac{\partial}{\partial z} \psi_1 - \psi_1 \frac{\partial}{\partial z} \psi_1 \right).
\]

(8)

One can decompose the current \( j_p \) on both sides of the junction into parts connected with (quasi)hole and (quasi)particle excitations. These are the incident probability current generated by the incoming electron \( j_i^< = \frac{1}{m} k \),

the probability current connected with the reflected hole \( j_{th}^< = -\frac{1}{m} k |B|^2 \),

the probability current connected with the transmitted quasihole

\[
j_{th}^> = \frac{1}{m} |F|^2 \left( |f_1|^2 - |g_1|^2 \right) \Re|k_+| + q \right).
\]

(10)

For the presentation of our numerical results we set the typical values of the superconducting gap \( |\Delta_0| = 0.001 \text{ eV} \) and the Fermi energy \( \mu = 1 \text{ eV} \). The value of the momentum \( q \) corresponds to the wavelength of several interatomic distances \( \sim 10 \text{ nm} \) which gives \( q \approx 10 \text{ eV} \). The quasi-Andreev reflection, \( T_{\text{quasi}} \approx 0 \) and \( R_{\text{hole}} \approx 1 \), is expected to occur within the energy range \( E \leq 0.0187826 \text{ eV} \leq E \leq 0.0207826 \text{ eV} \). As can be seen in Fig. 2, this expectation is very well confirmed by the numerical findings. For the incoming electron of the energy outside the Andreev window \( \Delta_0 \) there are solutions of the propagating quasiparticles in the FFLO superconductor, and they are responsible for the non-zero transmission coefficient \( T_{\text{quasi}} \). It is worth to mention that in the opposition to the situation with \( q = 0 \), there are always non-evanescenting waves in the FFLO. Even within the Andreev window \( \Delta_0 \) the solutions with the real momentum \( k \) exists. The amplitude \( F \) (responsible for the propagation of quasiholes
in the FFLO) is non zero in this region and the total transmission coefficient $T_{\text{total}} = |j_{\psi e}^+ + j_{\psi e}^-|/|j_{\psi e}^e|$ is different from zero. However, the value of $|F|^2$ is negligibly small and is not visible in Fig. 4.

Another interesting quantity is a charge transport on the superconductor side of the junction. The total charge current obeys the continuity equation \[11\]

$$e\frac{\partial}{\partial t}(|\psi_1|^2 - |\psi_1|^2) + \frac{\partial}{\partial z} j_{\psi e}^e = 4e\Im[\Delta\psi_1^e\psi_1]$$

where

$$j_{\psi e}^e = \frac{e}{m}(\Im[\psi_1^e \nabla \psi_1] + \Im[\psi_1^e \nabla \psi_1])$$

and the charge current carried by the condensate $j_{\psi e}^c$ take the form:

$$j_{\psi e}^c = \frac{e}{m}(|F|^2 (\Re[k_1^e] + q(|f_1|^2 - |g_1|^2)) + |J|^2 (\Re[k_2^e] + q(|f_2|^2 - |g_2|^2))e^{-2\Im(k_2^e)z}$$

$$+ \Re[J F^* f_1^e (k_1^e + k_2^e + 2q)e^{i(k_2^e - k_1^e)z}] + \Re[J g_2 F^* g_1^e (k_1^e + k_2^e - 2q)e^{i(k_2^e - k_1^e)z}],$$

$$j_{\psi e}^c = \frac{4e\Delta_0\Im[|J|^2 f_2^* g_2]}{\Re(k_2^e)} (1 - e^{-2\Im(k_2^e)z}) + i \frac{J^* f_2^* F g_1}{(k_1^e + k_2^e)} (1 - e^{i(k_2^e - k_1^e)z})$$

$$+ i \frac{F^* f_1^e J g_2}{(k_2^e - k_1^e)} (1 - e^{i(k_2^e - k_1^e)z}).$$

The main contribution to the charge current originates from the part of the equation \[13\] that is proportional to the coefficient $|J|^2$, that is:

$$j_{\psi e}^c \approx \frac{2e\Delta_0 |J|^2 \Im(f_2^* g_2)}{\Re(k_2^e)} \left(1 - e^{-2\Im(k_2^e)z}\right).$$

Considering the limit $q \to 0$ the charge current has a simple form

$$j_{\psi e}^c(q = 0) = 2e v_F \left(1 - e^{-2\Im(k_2^e)z}\right)$$

where $v_F = \sqrt{2m\mu}$ is the Fermi velocity. For $q \neq 0$ one can rewrite equation \[16\] in the approximate form:

$$j_{\psi e}^c \approx 2e v_F \left[1 - \exp \left(-\frac{2\Im(k_2^e)z}{\sqrt{v_F^2 - \frac{q^2}{m^2}}}\right)\right].$$
Here one introduces a standard parameter $\xi$ that describes the penetration depth of the evanescent currents

$$\xi(q = 0) = \frac{v_F}{2\sqrt{|\Delta|^2 - E^2}}, \quad \xi(q) = \frac{\sqrt{v_F^2 - \frac{q^2}{m^2}}}{2\sqrt{|\Delta|^2 - (E - \frac{q}{m\sqrt{2}}\sqrt{2m\mu - q^2})^2}}. \quad (19)$$

In Fig. 3 we present the penetration depth $\xi$ as a function of energy of the incoming electron. We keep the values of $m$, $\Delta$ and $\mu$ constant. As one could expect, when $q \neq 0$ penetration distance approaches towards infinity as $E$ approaches towards the lower or higher limit at the Andreev window. The minimum value is at $E = \frac{q}{m\sqrt{2}}\sqrt{2m\mu - q^2}$. On the contrary, for $q = 0$, the maximum penetration distance is at $E = \Delta$ and the minimum is reached at zero energy. The dependence of $\xi$ on the energy is also seen in Fig. 4, that presents the conversion of the normal charge current into the supercurrent at the FFLO phase.

**IV. CONCLUSIONS**

In this paper we have considered a process of the Andreev reflection between a normal metal and s-wave superconductor in the FFLO state with a single plane wave in the pure Pauli limit. We have found that the Andreev process
takes place only for electrons with the energy located within the Andreev window given by equation \[ 5 \]. This is clearly visible in the dependence of transmission and reflection coefficients associated with the energy of the incoming electrons. The other interesting quantities are the charge currents flowing through the junction and the penetration depth of the excitation inside superconductor. This last parameter depends on the energy of the incoming electron in very different way compared to the s-wave BCS superconductor (Fig. 3). In conclusion the Andreev process, in principle, can serve as a good probe to look for the FFLO state. Obviously the model given here is a far reaching idealization and additional works must be done to bring the results closer to experimental conditions. However, the results presented in this paper are robust in a sense that they are the simple consequences of the existence of the non-zero total momentum of Cooper pairs.

The next interesting step is analysis of the FFLO state with an oscillating value of the gap parameter \( \Delta \sim \cos(\mathbf{q} \cdot \mathbf{x}) \). Another important point is to adopt the external magnetic field in the picture and try to go off the pure Pauli limit. One can also extend the analysis of the Andreev reflection in the color superconductors \[12\] for the FFLO states.

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[1] P. Fulde and R. A. Ferrel, Phys. Rev. 135 (1964) A550; A. I. Larkin and Yu. N. Ovchinnikov, Zh. Eksp. Teor. Fiz 47 (1964) 1136 (Sov. Phys. JETP 28 (1965) 762).
[2] K. Gloos et al., Phys. Rev. Lett. 70 (1993) 501; G. Yin and K. Maki, Phys. Rev. B48 (1993) 650; M. Tachiki et al. Z. Phys. B100 (1996) 369; A. I. Buzdin and H. Kachkachi, Phys. lett. A225 (1997) 341; see also review Y. Matsuda and H. Shimahara, J. Phys. Soc. Jpn. 76 (2007) 051005.
[3] J Singleton et al., J. Phys.: Condens. Matter 12 (2000) L641; R. Lortz et al., Phys. Rev. Lett. 99 (2007) 187002.
[4] K. Kumagai et al., Physica B: Condensed Matter 378-380 (2006) 347-350.
[5] D. Bailin and A. Love, Nucl. Phys. B190[FS3] (1981) 175; B205[FS5] (1982) 119; Phys. Rep. 107 (1984) 325; M. Alford, K. Rajagopal and F. Wilczek, Phys. Lett. B422 (1998) 247; R. Rapp, T. Schafer and E. V. Shuryak, Phys. Rev. Lett. 81 (1998) 53.
[6] M. Alford, J. A. Bowers and K. Rajagopal, Phys. Rev. D63 (2001) 074016; J. A. Bowers, J. Kundu, K. Rajagopal and E. Shuster, Phys. Rev. D64 (2001) 014024.
[7] F. Dautry, E. M. Nyman, Nucl. Phys. A319 (1979) 323; M. Kutscher, W. Broniowski and A. Kotlorz, Nucl. Phys. A516 (1990) 566; M. Sadzikowski and W. Broniowski, Phys. Lett. B488 (2000) 63.
[8] E. Nakano and T. Tatsumi, Phys. Rev D71 (2005) 114006; M. Sadzikowski, Phys. Lett. B642 (2006) 238, T. L. Partyka and M. Sadzikowski, J. Phys. G36 (2009) 025004.
[9] A. F. Andreev, Zh. Eksp. Teor. Fiz. 46 (1964) 1823 (Sov. Phys. JETP 19 (1964) 1228).
[10] Chr. Bruder, Phys. Rev. B41 (1990) 4017; C. Hu, Phys. Rev. Lett. 72 (1994) 1526.
[11] G. E. Blonder, M. Tinkham and T. M. Klapwijk, Phys. Rev. B25. 4515.
[12] M. Sadzikowski, Acta Phys. Polon. B33 (2002) 1601; M. Sadzikowski and M. Tachibana, Phys. Rev. D66, (2002) 045024.