There are no structurally stable
diffeomorphisms of odd-dimensional manifolds
with codimension one non-orientable
expanding attractors

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Dedicated to Carlos Gutierrez on his 60th birthday

Abstract

We prove that a structurally stable diffeomorphism of closed \((2m + 1)\)-manifold, \(m \geq 1\), has no codimension one non-orientable expanding attractors.

1 Introduction

Structurally stable diffeomorphisms exist on any closed manifold (say a diffeomorphism \(f\) structurally stable if all diffeomorphisms \(C^1\)-close to \(f\) are conjugate to \(f\)). It is natural to study the question of existence of such diffeomorphisms with some additional conditions. The condition we consider here is the presence of a codimension one non-orientable expanding attractor. Due to well known example of Plykin [7], the answer is YES for 2-manifolds. Medvedev and Zhuzhoma [6] proved that for 3-manifolds the answer is NO. In the paper, we generalize the result of [6] proving that there are no structurally stable diffeomorphisms with a codimension one non-orientable expanding attractor on closed odd-dimensional manifolds. The proof is shorter than [6] and includes \(d = 3\). As to orientable attractors, the answer is YES for any \(d \geq 2\). Namely, starting with a codimension one Anosov diffeomorphism of the \(d\)-torus \(T^d\), \(d \geq 2\), a structurally stable diffeomorphism of \(T^d\) with an
orientable codimension one expanding attractor can be obtained by Smale’s surgery \[11\], so-called DA-diffeomorphism (see also \[4, 8, 10\]).

Before the formulation of exact result, we give necessary definitions and notions. Let \( f : M \to M \) be a diffeomorphism of a closed \( d \)-manifold \( M \), \( d = \dim M \geq 2 \), endowed with some Riemann metric \( \rho \) (all definitions in this section can be found in \[4\] and \[10\], unless otherwise indicated). A point \( x \in M \) is called an expanding attractor if for any neighborhood \( U \) of \( x \), \( f^n(U) \cap U \neq \emptyset \) for infinitely many integers \( n \). Then the non-wandering set \( NW(f) \), defined as the set of all non-wandering points, is an \( f \)-invariant and closed. A closed invariant set \( \Lambda \subset M \) is hyperbolic if there is a continuous \( f \)-invariant splitting of the tangent bundle \( T_\Lambda M \) into stable and unstable bundles \( E^s_\Lambda \oplus E^u_\Lambda \) with

\[
\|df^n(v)\| \leq C\lambda^n\|v\|, \quad \|df^{-n}(w)\| \leq C\lambda^n\|w\|, \quad \forall v \in E^s_\Lambda, \forall w \in E^u_\Lambda, \forall n \in \mathbb{N},
\]

for some fixed \( C > 0 \) and \( \lambda < 1 \). For each \( x \in \Lambda \), the sets \( W^s(x) = \{ y \in M : \lim_{j \to \infty} \rho(f^j(x), f^j(y)) \to 0 \} \), \( W^u(x) = \{ y \in M : \lim_{j \to \infty} \rho(f^{-j}(x), f^{-j}(y)) \to 0 \} \) are smooth, injective immersions of \( E^s_x \) and \( E^u_x \) that are tangent to \( E^s_x, E^u_x \) respectively. \( W^s(x) \), \( W^u(x) \) are called stable and unstable manifolds at \( x \).

For a diffeomorphism \( f : M \to M \), Smale \[11\] introduced the Axiom A: \( NW(f) \) is hyperbolic and the periodic points are dense in \( NW(f) \). A diffeomorphism satisfying the Axiom A is called A-diffeomorphism. According to Spectral Decomposition Theorem, \( NW(f) \) of an A-diffeomorphism \( f \) is decomposed into finitely many disjoint so-called basic sets \( B_1, \ldots, B_k \) such that each \( B_i \) is closed, \( f \)-invariant and contains a dense orbit.

A basic set \( \Omega \) is called an expanding attractor if there is a closed neighborhood \( U \) of \( \Omega \) such that \( f(U) \subset \text{int } U, \cap_{j \geq 0} f^j(U) = \Omega \), and the topological dimension \( \dim \Omega \) of \( \Omega \) is equal to the dimension \( \dim(E^u_\Omega) \) of the unstable splitting \( E^u_\Omega \) (the name is suggested in \[12, 13\]). \( \Omega \) is codimension one if \( \dim \Omega = \dim M - 1 \). It is well known that a codimension one expanding attractor consists of the \((d-1)\)-dimensional unstable manifolds \( W^u(x) \), \( x \in \Omega \), and is locally homeomorphic to the product of \((d-1)\)-dimensional Euclidean space and a Cantor set. \( W^s(x) \) is homeomorphic to \( \mathbb{R}^{d-1} \) and can be endowed with some orientation. \( W^u(x) \) is homeomorphic to \( \mathbb{R}^{d-1} \) and can be endowed with some normal orientation (even if \( M \) is non-orientable). Due to hyperbolic structure, any \( W^s(x) \) intersects \( W^u(x) \) transversally, \( x \in \Omega \).

Following \[1\], say that \( \Omega \) is orientable if for every \( x \in \Omega \) the index of the intersection \( W^s(x) \cap W^u(x) \) does not depend on a point of this intersection (it is either \(+1\) or \(-1\)). The main result is the following theorem.
**Theorem 1** Let \( f : M \to M \) be a structurally stable diffeomorphism of a closed \((2m + 1)\)-manifold \( M \), \( m \geq 1 \). Then the spectral decomposition of \( f \) does not contain codimension one non-orientable expanding attractors.

Our proof does not work for even-dimensional manifolds for which the existence of codimension one non-orientable expanding attractors stay open question (except \( d = 2 \)).

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## 2 Proof of the main theorem

Later on, \( \Omega \) is a codimension one non-orientable expanding attractor of diffeomorphism \( f : M \to M \). A point \( p \in \Omega \) is called boundary if at least one component of \( W^s(p) - p \) does not intersect \( \Omega \). Boundary points exist and satisfy to the following conditions [1], [7]:

- There are finitely many boundary points and each is periodic.
- Given a boundary point \( p \in \Omega \), there is a unique component of \( W^s(p) - p \) denoted by \( W^s_{\emptyset}(p) \) which does not intersect \( \Omega \).
- Given a point \( x \in W^u(p) - p \), there is a unique arc \((x, y)^s \subset W^s(x)\) denoted by \((x, y)^s_{\emptyset} \) such that \((x, y)^s \cap \Omega = \emptyset \) and \( y \in \Omega \).

An unstable manifold \( W^u(p) \) containing a boundary point is called a boundary unstable manifold. Due to [1] and [5], the accessible boundary of \( M - \Omega \) from \( M - \Omega \) is a finite union of boundary unstable manifolds that splits into so-called bunches defined as follows. The family \( W^u(p_1), \ldots, W^u(p_k) \) is said to be a \( k \)-bunch if there are points \( x_i \in W^u(p_i) \) and arcs \((x_i, y_i)^s_{\emptyset}, y_i \in W^u(p_{i+1})\), \( 1 \leq i \leq k \), where \( p_{k+1} = p_1, y_k \in W^u(p_1) \), and there are no \((k + 1)\)-bunches containing the given one.

**Lemma 1** Let \( f : M \to M \) be an \( A \)-diffeomorphism of a closed \((2m + 1)\)-manifold \( M \), \( m \geq 1 \). If the spectral decomposition of \( f \) contains a codimension one non-orientable expanding attractor, then \( M \) is non-orientable.
Proof. The non-orientability of \( \Omega \) implies that \( \Omega \) has at least one 1-bunch, say \( W^u(p) \) \[2\]. Therefore, given any point \( x \in W^u(p) - p \), there is a unique point \( y \in W^u(p) - p \) such that \( (x, y)^\theta = (x, y)^\theta_0 \), and vice versa. Let the map \( \phi : W^u(p) - p \to W^u(p) - p \) be given by \( \phi(x) = y \) whenever \( (x, y)^\theta = (x, y)^\theta_0 \). Then \( \phi \) is an involution, \( \phi^2 = id \).

Let \( r \) be the period of \( p \). Since the stable (as well as unstable) manifolds are \( f \)-invariant, \( f^r \circ \phi |_{W^u(p)} = \phi \circ f^r |_{W^u(p)} \). Since the restriction \( f^r |_{W^u(p)} \) is an expansion map with the unique hyperbolic fixed point \( p \), \( \phi \) can be extended homeomorphically to \( W^u(p) \) putting \( \phi(p) = p \). By theorem 2.7 and lemma 2.1 \[3\], \( \phi \) is conjugate to the antipodal involution, i.e. there exist a homeomorphism \( h : W^u(p) \to \mathbb{R}^{d-1} \) (in the intrinsic topology of \( W^u(p) \)) and the involution \( \theta : \mathbb{R}^{d-1} \to \mathbb{R}^{d-1} \) of the type \( \vec{v} \mapsto -\vec{v} \) such that \( \theta \circ h = h \circ \phi \).

This implies that there is the \((d - 1)\)-dimensional ball \( B^{d-1} \subset W^u(p) \) such that \( p \in B^{d-1} \), the boundary \( \partial B^{d-1} \) is tamely embedded in \( W^u(p) \), and \( S^{d-2} \) is \( \phi \)-invariant. Moreover, there is the annulus \( S^{d-2} \times [0, 1] \subset W^u(p) \) foliated by \( S^{d-2}_t = S^{d-2} \times \{t\}, t \in [0, 1] \), \( S^{d-2} = S^{d-2}_0 \), such that every \( S^{d-2}_t \) is \( \phi \)-invariant and bounds the \((d-1)\)-dimensional ball \( B^{d-1}_t \subset W^u(p) \) containing \( p \). Since \( \phi^2 = id \), the set

\[
B^{d-1}_t \bigcup_{x \in S^{d-2}} [x, \phi(x)] = P_t
\]

is homeomorphic to the projective space \( \mathbb{R}P^{d-1} \) for every \( t \in [0, 1] \). Since \( d - 1 = 2m \) is even, \( P_t \) is non-orientable. For any \( x \in S^{d-2}_{t_1} \) and \( y \in S^{d-2}_{t_2} \) with \( t_1 \neq t_2 \), \( [x, \phi(x)] \cap [y, \phi(y)] = \emptyset \). Hence the set

\[
\bigcup_{x \in S^{d-2} \times [0, 1]} [x, \phi(x)] \subset M
\]

is homeomorphic to \( \mathbb{R}P^{d-1} \times [0, 1] \). Since \( \mathbb{R}P^{d-1} \times [0, 1] \) is a non-orientable \( d \)-manifold, \( M \) is non-orientable. \( \square \)

Proof of theorem \[7\] Assume the converse. Then the spectral decomposition of \( f \) contains a codimension one non-orientable expanding attractor, say \( \Omega \). According to lemma \[8\] \( M \) is non-orientable. Let \( \overline{M} \) be an orientable manifold such that \( \pi : \overline{M} \to M \) is a (nonbranched) double covering for \( M \). Then there exists a diffeomorphism \( \overline{f} : \overline{M} \to \overline{M} \) that cover \( f \), i.e., \( f \circ \pi = \pi \circ \overline{f} \).

It is easy to see that \( \overline{f} \) is an \( A \)-diffeomorphism with a codimension one expanding attractor \( \overline{\Omega} \subset \pi^{-1}(\Omega) \). It follows from lemma \[9\] and orientability of \( \overline{M} \) that \( \overline{\Omega} \) is orientable.
Because of $f$ is a structurally stable diffeomorphism, $f$ satisfies to the strong transversality condition [5] which is a local condition. Since $\pi$ is a local diffeomorphism, $\overline{f}$ satisfies to the strong transversality condition as well. Hence, $\overline{f}$ is structurally stable [9].

Take a periodic point $p \in \Omega$ on the boundary unstable manifold $W^u(p)$ that is a 1-bunch. Then the preimage $\pi^{-1}(W^u(p))$ is a 2-bunch of $\overline{\Omega}$ consisting of unstable manifolds $W^u(p_1), W^u(p_2)$, where $\{p_1, p_2\} = \pi^{-1}(p)$ are boundary periodic points of $\overline{f}$. It was proved in [2], [3] that $W^s_\emptyset(p_1)$ and $W^s_\emptyset(p_2)$ belong to the unstable manifolds $W^s(\alpha_1)$ and $W^s(\alpha_2)$ respectively of the repelling periodic points $\alpha_1, \alpha'$ (possibly, $\alpha_1 = \alpha'$). Moreover, there are repelling periodic points $\alpha_1, \ldots, \alpha_{k+1} = \alpha'$ and saddle periodic points $P_1 = p_1, P_2, \ldots, P_{k+1}, P_{k+2} = p_2, k \geq 0$, of index $d - 1$ such that the following conditions hold:

1. The set
   $$l = P_1 \cup W^s_\emptyset(P_1) \cup \alpha_1 \cup W^s(P_2) \cup \ldots \cup \alpha_{k+1} \cup W^s_\emptyset(P_{k+2}) \cup P_{k+2}$$
   is homeomorphic to an arc with no self-intersections whose endpoints are $P_1$ and $P_{k+2}$.

2. $l - (P_1 \cup P_{k+2}) \subset \overline{M} - \overline{\Omega}$.

3. The repelling periodic points $\alpha_i$ alternate with saddle periodic points $P_i$ on $l$.

It follows from $f \circ \pi = \pi \circ \overline{f}$ that $\pi$ maps the stable and unstable manifolds of $\overline{f}$ into the stable and unstable manifolds respectively of $f$. Since $\pi(P_1) = \pi(P_{k+2}) = p$,

$$\pi(W^s_\emptyset(P_1)) = \pi(W^s_\emptyset(P_2)), \quad \pi(\alpha_1) = \pi(\alpha_{k+1}).$$

Hence (if $k \geq 1$),

$$\pi(W^s(P_2)) = \pi(W^s(P_{k+1})), \quad \pi(P_2) = \pi(P_{k+1}), \quad \pi(\alpha_2) = \pi(\alpha_k), \quad \ldots,$$

Due to item (3) above, the number of all periodic points on $l$ equals $2k+3$ that is odd. As a consequence, there is either a periodic point $\alpha_i$ with $\pi(W^s(P_i)) = \pi(W^s(P_{i+1}))$ or a periodic point $P_i$ with $\pi(W^s_1(P_i)) = \pi(W^s_2(P_i))$, where $\pi(W^s(P_i)), \pi(W^s_1(P_i)) = \pi(W^s_2(P_i))$ are different components of $W^s(P_i) - P_i$. In both cases, there is a point ($\alpha_i$ or $P_i$) at which $\pi$ is not a local homeomorphism. This contradiction concludes the proof. □
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