The adjoint algebra for 2-categories

Martín Mombelli

CIEM- FAMAF, Universidad Nacional de Córdoba, Argentina

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Based on:

N. Bortolussi and M. Mombelli, *The adjoint algebra for 2-categories*, Preprint arXiv:2005.05271

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Problem we want to attack:

For a finite group $G$, let $\mathcal{D} = \bigoplus_{g \in G} \mathcal{C}_g$ be a $G$-graded tensor category

$$\mathcal{D} \text{ is a } G\text{-extension of } \mathcal{C}_1$$

- Compute Shimizu’s adjoint algebra $Ad_M \in \mathcal{Z}(\mathcal{D})$ for any left $\mathcal{D}$-module category $\mathcal{M}$.

- Might be useful to study nilpotent, solvable fusion categories (?)

$\mathcal{C}$ is nilpotent: $\text{vect}_k \subseteq \mathcal{C}_1 \subseteq \cdots \subseteq \mathcal{C}_n = \mathcal{C}$, $\mathcal{C}_i$ is a $G_i$-extension of $\mathcal{C}_{i-1}$. 
$\mathcal{C}$ a finite tensor category.

- $\mathcal{C} \text{Mod}$ is the 2-category of left $\mathcal{C}$-module categories.
- $\mathcal{C} \text{Mod}_e$ is the 2-category of exact indecomposable $\mathcal{C}$-module categories.
If $\mathcal{B}, \mathcal{B}'$ are 2-categories

- For any pseudofunctor $F : \mathcal{B} \to \mathcal{B}'$: $PseudoNat(F, F)$ is the monoidal category of pseudo-natural transformations;

- The **center** of $\mathcal{B}$ is $Z(\mathcal{B}) = PseudoNat(\text{Id}_{\mathcal{B}}, \text{Id}_{\mathcal{B}})$. 
Objects in $Z(B)$ are pairs $(V, \sigma)$:

$$V = \{ V_A \in B(A, A) \text{ 1-cells, } A \in B \text{ a 0-cell } \},$$

$$\sigma = \{ \sigma_X : V_B \circ X \to X \circ V_A \}, \text{ the half-braiding}$$

where, for any $X \in B(A, B)$, $\sigma_X$ is a natural isomorphism 2-cell such that

$$\sigma_{I_A} = \text{id}_{V_A}, \quad \sigma_X \circ Y = (\text{id}_X \circ \sigma_Y)(\sigma_X \circ \text{id}_Y),$$

$\forall$ 0-cells $A, B, C \in B$, and any pair of 1-cells $X \in B(A, B), Y \in B(C, B)$.
If $\mathcal{C}$ is a tensor category. There is a monoidal equivalence:

$$\mathcal{Z}(\mathcal{C} \text{Mod}) \simeq \otimes \mathcal{Z}(\mathcal{C})$$

$$((V_M, \sigma) \mapsto V_C(1))$$

$V_M : \mathcal{M} \to \mathcal{M}$ is a $\mathcal{C}$-module functor.

If $\mathcal{D} = \bigoplus_{g \in G} \mathcal{C}_g$, $\Phi : \mathcal{D} \text{Mod} \to \mathcal{C}_1 \text{Mod}$ is the restriction 2-functor, then

$$\text{PseudoNat}(\Phi, \Phi) \simeq \otimes \mathcal{Z}_{\mathcal{C}_1}(\mathcal{D})$$

Is the relative center.

If $F : \mathcal{B} \to \mathcal{B}'$ is a 2-equivalence $\sim \sim$ induces a monoidal equivalence

$$\hat{F} : \mathcal{Z}(\mathcal{B}) \to \mathcal{Z}(\mathcal{B})$$
Let \( \mathcal{C}, \mathcal{D} \) be finite tensor categories. \( \mathcal{N} \) be an invertible \((\mathcal{C}, \mathcal{D})\)-bimodule category.

\[ \theta^\mathcal{N} : \mathcal{C}\text{-Mod} \to \mathcal{D}\text{-Mod} \]

\[ \theta^\mathcal{N}(M) = \text{Fun}_\mathcal{C}(\mathcal{N}, M) \]

Identifying \( \mathcal{D} = \text{End}_\mathcal{C}(\mathcal{N}) \)

\[ \theta(X)(N) = X \triangleright N \]

\( X \in \mathcal{C}, \ N \in \mathcal{N} \).
Idea to solve the problem

Translate known constructions, problems to the realm of 2-categories
G a finite group, B a 2-category.

In [E. Bernaschini, C. Galindo and M. Mombelli, Group actions on 2-categories, Manuscripta Math. 159, No. 1-2, 81–115 (2019).]

We defined:
- An action of $G$ on $B$,
- The equivariantized 2-category $B^G$
- There is a forgetful 2-functor $\Phi : B^G \to B$, such that
  - $G$ acts on $\text{PseudoNat}(\Phi)$, and
  - $\mathcal{Z}(B^G) \simeq \otimes PseudoNat(\Phi)^G$. 
Theorem

If $\mathcal{D} = \bigoplus_{g \in G} \mathcal{C}_g$ is a $G$-graded tensor category. Then

- $G$ acts on $\mathcal{C}_1 \text{Mod}_e$
- There exists a 2-equivalence $(\mathcal{C}_1 \text{Mod}_e)^G \simeq \mathcal{D} \text{Mod}_e$
Define an analogue of Shimizu’s adjoint algebra for 2-categories.

Prove that when applied to $\mathcal{C}_{\text{Mod}}$ it coincides with Shimizu’s version.

How our adjoint algebra behaves under 2-equivalences.

Apply all the above to $\mathcal{D} = \bigoplus_{g \in G} \mathcal{C}_g$, $\mathcal{D}_{\text{Mod}}$ and use $(\mathcal{C}_1_{\text{Mod}})^G \simeq \mathcal{D}_{\text{Mod}}$

We did only the first 3 steps.
Let $\mathcal{B}$ be a 2-category such that any 1-cell has right and left duals.

For any 0-cell $B \in \mathcal{B} \rightsquigarrow Ad_B \in \mathcal{Z}(\mathcal{B})$

$$(Ad_B)_A = \int_{X \in \mathcal{B}(A,B)} X \circ X \in \mathcal{B}(A, A)$$
For any $X \in \mathcal{B}(A, B)$ we shall denote by

$$
\pi_X^{(A,B)} : \int_{X \in \mathcal{B}(A,B)} \ast X \circ X \longrightarrow \ast X \circ X
$$

$$(\int_{X \in \mathcal{B}(A,B)} \ast X \circ X) \circ Z \xrightarrow{\sigma_B^Z} Z \circ (\int_{X \in \mathcal{B}(C,B)} \ast X \circ X) \xrightarrow{\text{id}_Z \circ \pi_{(C,B)}^{(A,B)} \circ \text{ev}_Z \circ \text{id}_{\ast Y \circ Y \circ X}} Z \circ \ast Z \circ \ast Y \circ Y \circ Z, $$
Results:

- For any 0-cell $B \in \mathcal{B}$, $\text{Ad}_B \in \mathcal{Z}(\mathcal{B})$ is an algebra.

- For equivalent 0-cells $A, B$ the corresponding algebras $\text{Ad}_A, \text{Ad}_B$ are isomorphic.

Theorem

$\mathcal{C}$ a tensor category. $\mathcal{M}$ an exact indecomp. $\mathcal{C}$-module category:

$$\Phi : \mathcal{Z}(\mathcal{C} \text{Mod}) \xrightarrow{\sim} \mathcal{Z}(\mathcal{C})$$

an algebra isomorphism $\Phi(\text{Ad}_\mathcal{M}) \simeq A_\mathcal{M}(\leftarrow \text{Shimizu's algebra})$
Theorem

If \( F : \mathcal{B} \to \tilde{\mathcal{B}} \) is a 2-equivalence, then

- \( F \) induces a monoidal equivalence \( \hat{F} : \mathcal{Z}(\mathcal{B}) \to \mathcal{Z}(\tilde{\mathcal{B}}) \)
- There is an algebra isomorphism \( \hat{F}(\text{Ad}_B) \simeq \text{Ad}_{F(B)} \).
If $\mathcal{C}, \mathcal{D}$ are finite tensor categories. $\mathcal{N}$ is an invertible $(\mathcal{D}, \mathcal{C})$-bimodule category.

$$\theta^\mathcal{N}: \mathcal{D}\text{Mod} \to \mathcal{C}\text{Mod}$$

$$\theta^\mathcal{N}(\mathcal{M}) = \text{Fun}_\mathcal{D}(\mathcal{N}, \mathcal{M})$$

There is an isomorphism of algebras

$$\hat{\theta}^\mathcal{N}(\text{Ad}_\mathcal{M}) \simeq \text{Ad}_{\text{Fun}_\mathcal{C}(\mathcal{N}, \mathcal{M})}.$$  

Here $\hat{\theta}^\mathcal{N}: \mathcal{Z}(\mathcal{C}) \to \mathcal{Z}(\mathcal{D})$ is the induced monoidal equivalence.