ESTIMATES OF THE CARATHÉODORY METRIC ON THE SYMMETRIZED POLYDISC

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Abstract. Estimates for the Carathéodory metric on the symmetrized polydisc are obtained. It is also shown that the Carathéodory and Kobayashi distances of the symmetrized three-disc do not coincide.

1. Introduction

A consequence of the fundamental Lempert theorem (see [9]) is the fact that the Carathéodory distance and the Lempert function coincide on any domain \( D \subset \mathbb{C}^n \) with the following property (*) (cf. [4]):

\((*) D \text{ can be exhausted by domains which are biholomorphic to convex domains.}\)

For more than 20 years it has been an open question whether the converse of the above result is true in some reasonable class of domains (e.g. in the class of bounded pseudoconvex domains). In other words, does the equality between the Carathéodory distance \( c_D \) and the Lempert function \( \tilde{k}_D \) of a bounded pseudoconvex domain \( D \) imply that \( D \) satisfies property (*).

The first counterexample, the so-called symmetrized bidisc \( G_2 \), has been recently discovered and discussed in a series of papers (see [1], [2], [3] and [5], see also [4]).

In fact, it was proved that \( c_{G_2} \) and \( \tilde{k}_{G_2} \) coincide with a natural distance \( p_{G_2} \) related to (the geometry of) \( G_2 \).

The symmetrized polydisc \( G_n \) \((n \geq 3)\) can also be endowed with a similar distance \( p_{G_n} \) which does not exceed \( c_{G_n} \). Using \( p_{G_n} \), three of the authors have recently shown that \( \tilde{k}_{G_n} \) is not a distance (see [12]); in particular, \( G_n \) does not satisfy property (*) (for a direct proof of this

2000 Mathematics Subject Classification. 32F45.

Key words and phrases. symmetrized polydisc, Carathéodory distance and metric, Kobayashi distance and metric, Lempert function.

The first and second named authors were supported by grants from DFG (DFG-Projekt 227/8); the fourth-named author was supported by the KBN Research Grant No. 1 PO3A 005 28 and Alexander von Humboldt Foundation.
They have also proved that the Kobayashi distance of $G_n$ does not coincide with $p_{G_n}$.

In the present paper we improve this result showing that $c_{G_n}(0; \cdot) \neq p_{G_n}(0; \cdot)$. The proof is based on the comparison of the infinitesimal version of these distances at the origin, $\gamma_{G_n}(0; \cdot)$ and $\rho_n$, where $\gamma_{G_n}$ is the Carathéodory-Reiffen metric of $G_n$. We also give lower and upper bounds for $\gamma_{G_2n+1}(0; e_2)$ (where $e_2$ is the second basis vector). The bounds give an asymptotic estimate for $\gamma_{G_2n+1}(0; e_2)$ with an error of the form $o(n^{-3})$. Finally, estimating more precisely the value of $\gamma_{G_2}$ at the point $(0; e_2) \in G_3 \times C^3$, we obtain that it is smaller than the infinitesimal version of the Kobayashi distance at the same point which implies that the Kobayashi distance does not coincide with the Carathéodory distance on $G_3$.

Acknowledgements. We thank Dr. Pencho Marinov for the computer programmes helping us to obtain the estimates in the last section of the paper.

2. Background

Let $D$ be the unit disc in $C$. Let $\sigma_n = (\sigma_{n,1}, \ldots, \sigma_{n,n}) : C^n \to C^n$ be defined as follows:

$$\sigma_{n,k}(z_1, \ldots, z_n) = \sum_{1 \leq j_1 < \cdots < j_k \leq n} z_{j_1} \cdots z_{j_k}, \quad 1 \leq k \leq n.$$ 

The domain $G_n = \sigma_n(D^n)$ is called the symmetrized $n$-disc.

Recall now the definitions of the Carathéodory pseudodistance, the Carathéodory-Reiffen pseudometric, the Lempert function and the Kobayashi-Royden pseudometric of a domain $D \subset C^n$ (cf. [4]):

$$c_D(z, w) := \sup\{|f'(w)| : f \in \mathcal{O}(D, \mathbb{D}), f(z) = 0\},$$

$$\gamma_D(z; X) := \sup\{|f'(z)X| : f \in \mathcal{O}(D, \mathbb{D}), f(z) = 0\},$$

$$\tilde{k}_D(z, w) := \inf\{\tanh^{-1}|\alpha| : \exists \varphi \in \mathcal{O}(\mathbb{D}, D) : \varphi(0) = z, \varphi(\alpha) = w\},$$

$$\kappa_D(z; X) := \inf\{\alpha \geq 0 : \exists \varphi \in \mathcal{O}(\mathbb{D}, D) : \varphi(0) = z, \alpha \varphi'(0) = X\},$$

where $z, w \in D, X \in C^n$. The Kobayashi pseudodistance $k_D$ (respectively, the Kobayashi–Buseman pseudometric $\tilde{k}_D$) is the largest pseudodistance (respectively, pseudonorm) which does not exceed $\tilde{k}_D$ (respectively, $\kappa_D$).

It is well-know that $c_D \leq k_D \leq \tilde{k}_D, \gamma_D \leq \tilde{k}_D \leq \kappa_D$, and

$$\gamma_D(z; X) = \lim_{C, 3t \to 0} \frac{c_D(z, z + tX)}{t} \quad (\text{cf. [4]}).$$
Moreover, if $D$ is taut, then

$$\kappa_D(z; X) = \lim_{t \to 0} \frac{\bar{k}_D(z, z + tX)}{t} \quad (\text{see } \text{[13]}),$$

$$\hat{\kappa}_D(z; X) = \lim_{t \to 0} \frac{k_D(z, z + tX)}{t} \quad (\text{see } \text{[8]}).$$

Note that $\mathbb{G}_n$ is a hyperconvex domain (see [6]) and, therefore, a taut domain.

In the proofs below we shall need some mappings defined on $\mathbb{G}_n$.

For $\lambda \in \overline{D}$, $n \geq 2$, one may define the rational mapping $p_{n, \lambda}$ as follows

$$p_{n, \lambda}(z) := (\tilde{z}_1(\lambda), \ldots, \tilde{z}_{n-1}(\lambda)) = \tilde{z}(\lambda) \in \mathbb{C}^{n-1}, \quad z \in \mathbb{C}^n, \ n + \lambda z_1 \neq 0,$$

where $\tilde{z}_j(\lambda) = \frac{(n - j)z_j + \lambda(j + 1)z_{j+1}}{n + \lambda z_1}$, $1 \leq j \leq n - 1$. Then $z \in \mathbb{G}_n$ if and only if $\tilde{z}(\lambda) \in \mathbb{G}_{n-1}$, $n + \lambda z_1 \neq 0$ for any $\lambda \in \overline{D}$ (see Corollary 3.4 in [4]).

We may also define for $\lambda_1, \ldots, \lambda_{n-1} \in \overline{D}$ the rational function

$$f_{\lambda_1, \ldots, \lambda_{n-1}} := p_{2, \lambda_1} \circ \ldots \circ p_{n, \lambda_{n-1}}.$$

Observe that

$$f_{\lambda}(z) := f_{\lambda, \ldots, \lambda}(z) = \frac{\sum_{j=1}^n jz_j \lambda^{j-1}}{n + \sum_{j=1}^{n-1} (n - j)z_j \lambda^j}.$$

By Theorem 3.2 in [4], $z \in \mathbb{G}_n$ if and only if $\sup_{\lambda \in \overline{D}} |f_{\lambda}(z)| < 1$. In fact, by Theorem 3.5 in [4], if $z \in \mathbb{G}_n$, then the last supremum is equal to $\sup_{\lambda_1, \ldots, \lambda_{n-1} \in \overline{D}} |f_{\lambda_1, \ldots, \lambda_{n-1}}(z)|$.

It follows that

$$c_{\mathbb{G}_n}(z, w) \geq p_{\mathbb{G}_n}(z, w) := \max_{\lambda_1, \ldots, \lambda_{n-1} \in T} |p_D(f_{\lambda_1, \ldots, \lambda_{n-1}}(z), f_{\lambda_1, \ldots, \lambda_{n-1}}(w))|,$$

where $T = \partial \overline{D}$ and $p_D$ is the Poincaré distance. Observe that $p_{\mathbb{G}_n}$ is a distance on $\mathbb{G}_n$.

Let $e_1, \ldots, e_n$ be the standard basis of $\mathbb{C}^n$ and $X = \sum_{j=1}^n X_j e_j$. Set

$$\bar{f}_{\lambda}(X) = \frac{\sum_{j=1}^n jX_j \lambda^{j-1}}{n} \quad \text{and} \quad \rho_n(X) := \max_{\lambda \in \overline{T}} |\bar{f}_{\lambda}(X)|.$$

Then the last inequality above implies that

$$\gamma_{\mathbb{G}_n}(z; X) \geq \lim_{t \to 0} \frac{p_{\mathbb{G}_n}(0, tX)}{|t|} = \rho_n(X).$$
Let \( L_{k,l} \) be the span of \( e_k \) and \( e_l \). Note that if \( X \in L_{k,l} \), \( k \neq l \), then
\[
\rho_n(X) = \frac{k|X_k| + l|X_l|}{n}.
\]

For \( n = 2 \) one has equalities \( k_{G_2} = c_{G_2} = p_{G_2} \) (see [1], [2]). On the other hand, we have the following (see [12]).

**Proposition 1.** Let \( n \geq 3 \).

(a) If \( k \) divides \( n \), then \( \kappa_{G_n}(0; e_k) = \rho_n(e_k) \). Therefore, if \( l \) also divides \( n \), then \( \kappa_{G_n}(0; X) = \rho_n(X) \) for any \( X \in L_{k,l} \).

(b) If \( X \in L_{1,n} \setminus (L_{1,1} \cup L_{n,n}) \), then \( \kappa_{G_n}(0; X) > \rho_n(X) \).

(c) If \( k \) does not divide \( n \), then \( \kappa_{G_n}(0; e_k) > \rho_n(e_k) \).

In particular, \( \kappa_{G_n}(0, \cdot) \neq \kappa_{G_n}(0, \cdot) \) and \( \kappa_{G_n}(0, \cdot) \neq p_{G_n}(0, \cdot) \).

In the next section we shall prove a stronger inequality than that in Proposition 1 (c).

3. If \( k \) does not divide \( n \), then \( \gamma_{G_n}(0; e_k) > \rho_{G_n}(e_k) \)

Our first aim is the proof of a result, which implies the inequality between \( c_{G_n} \) and \( p_{G_n} \), \( n \geq 3 \).

**Proposition 2.** If \( k \) does not divide \( n \geq 3 \), then \( \gamma_{G_n}(0; e_k) > \rho_n(e_k) \).

In particular, \( c_{G_n}(0, \cdot) \neq p_{G_n}(0, \cdot) \).

**Proof.** Let \( \sqrt{1} = \{\xi_1, \ldots, \xi_k\} \). For \( z \in \mathbb{G}_n \) and \( \lambda \in \mathbb{B} \), such that the denominator in the formula below does not vanish, set
\[
g_z(\lambda) := \lambda f_{\lambda}(z) = \frac{\sum_{j=1}^{n} jz_j \lambda^j}{n + \sum_{j=1}^{n-1} (n - j)z_j \lambda^j}
\]
and
\[
g_{z,k}(\lambda) = \frac{\sum_{j=1}^{k} g_z(\xi_j \lambda)}{k \lambda^k}.
\]
The equalities \( \sum_{j=1}^{k} \xi_j^m = 0 \), \( m = 1, \ldots, k-1 \), and the Taylor expansion of \( g_{z,k} \) show that this function can be extended at 0 as \( g_{z,k}(0) = P_k(z) \), where \( P_k \) is a polynomial with \( \frac{\partial P_k}{\partial z_k |_{z=0} = \frac{k}{n}} \) and
\[
P_k(tw_1, t^2 w_2, \ldots, t^m w_n) = t^k P(w), \ w = (w_1, w_2, \ldots, w_n) \in \mathbb{C}^n, \ t \in \mathbb{C}.
\]
It follows by the maximum principle that \( g_{z,k} \in \mathcal{O}(\mathbb{D}, \overline{\mathbb{D}}) \). In particular, \( |P_k(z)| \leq 1 \). To prove the desired inequality, it is enough to show that \( |P_k(z)| < 1 \) for any \( z \in \mathbb{G}_n \). Assume the contrary, that is, \( P_k(z) = e^{i\theta} \) for some \( \theta \in \mathbb{R} \) and some \( z \in \mathbb{G}_n \). Then the maximum principle and
the triangle inequality implies that $g_z(\xi_j \lambda) = e^{i\theta} \lambda^k$, $\lambda \in \mathbb{T}$, $1 \leq j \leq k$. In particular, $g_z(\lambda) = e^{i\theta} \lambda^k$, that is

$$
\sum_{j=1}^{n} j z_j \lambda^j = e^{i\theta} (n \lambda^k + \sum_{j=1}^{n-1} (n-j) z_j \lambda^{k+j}).
$$

Comparing the corresponding coefficients of these two polynomials of $\lambda$, we get that $z_k = e^{i\theta} \frac{n}{k}$, $z_{n+1-k} = \cdots = z_{n-1} = 0$ and

$$(k+j)z_{k+j} = e^{i\theta} (n-j)z_j, \quad 1 \leq j \leq n-k.$$ 

The last relations imply that $z_{kl} = e^{i\theta} \left( \frac{n}{k} \right), \quad 1 \leq l \leq \lfloor n/k \rfloor$. On the other hand, since $k$ does not divide $n$, then $n-k < k \lfloor n/k \rfloor < n$ and hence $z_{k\lfloor n/k \rfloor} = 0$ – a contradiction. □

Remarks. It will be interesting to know whether $\hat{\kappa}_{G_n}(0; \cdot) \neq \gamma_{G_n}(0; \cdot)$ and hence $k_{G_n}(0; \cdot) \neq c_{G_n}(0; \cdot)$ for any $n \geq 4$. In the last section we shall prove these inequalities for $n = 3$.

4. Estimates for $\gamma_{G_{2n+1}}(0; e_2)$

Let $n$ and $k$ be positive integers, $k \leq n$. Note that

$$\kappa_{G_n}(0; e_k) \leq \kappa_{G_{k\lfloor n/k \rfloor}}(0; e_k) = \frac{1}{\lfloor n/k \rfloor}.$$ 

Thus,

$$\frac{k}{n} \leq \gamma_{G_n}(0; e_k) \leq \kappa_{G_n}(0; e_k) \leq \frac{1}{\lfloor n/k \rfloor}.$$ 

Therefore, one has that

$$\lim_{n \to \infty} n \gamma_{G_n}(0; e_k) = \lim_{n \to \infty} n \kappa_{G_n}(0; e_k) = k.$$ 

Let now $n \geq 3$ be odd. It follows that

$$\gamma_{G_n}(0; e_k) \leq \kappa_{G_n}(0; e_k) \leq \frac{2}{n-1}.$$ 

On the other hand, $\frac{2}{n} < \gamma_{G_n}(0; e_2)$ by Proposition 1. The aim of this section is to improve both estimates.

To obtain a more precise upper bound, we shall need the following definition. Let $k_1 \leq \cdots \leq k_n$ be positive integers. For $\lambda \in \mathbb{C}$, define the mapping

$$\pi_{\lambda} : \mathbb{C}^n \ni (z_1, \ldots, z_n) \to (\lambda^{k_1} z_1, \ldots, \lambda^{k_n} z_n) \in \mathbb{C}^n.$$ 

We shall say that a domain $D \subset \mathbb{C}^n$ is $(k_1, \ldots, k_n)$-balanced if $\pi_{\lambda}(z) \in D$ for any $\lambda \in \overline{B}$ and any $z \in D$. For such a domain $D$ and any
$j = 1, \ldots, n$, denote by $\mathcal{P}_j$ the set of polynomials $P$ with $\sup_{D} |P| \leq 1$ and $P \circ \pi_{\lambda} = \lambda^k P$, and by $\mathcal{L}_j$ the span of the vectors $e_j, \ldots, e_l$, where $l \geq j$ is the maximal integer with $k_l = k_j$. The proof of Proposition 1 implies the following result.

**Proposition 3.** If $D \subset \mathbb{C}^n$ is a $(k_1, \ldots, k_n)$-balanced domain and $X \in \mathcal{L}_j$, $1 \leq j \leq n$, then $\gamma_D(0; X) = \sup\{|P'(0)X| : P \in \mathcal{P}_j\}$.

**Remarks.** (i) One can obtain a similar description for any Reiffen pseudometric of higher order (for the definition see the next section).

(ii) A consequence of Proposition 3 is the well-know fact that if $D$ is a balanced domain, that is, $k_1 = \cdots = k_n = 1$, then $\gamma_D(0; X) = \hat{h}_D(X)$, where $\hat{h}_D$ is the Minkowski function of the convex hull of $D$.

(iii) Another consequence of Proposition 3 is the formula

\[
(1) \quad \gamma_{G_n}(0; e_2) = \frac{1}{\inf_{c \in \mathbb{C}} \max_{z \in \partial G_n} |z_2 + cz_1|}.
\]

Despite of (1), it is difficult to find explicitly $\gamma_{G_n}(0; e_2)$ for odd $n \geq 3$ (see the last section).

(iv) Note that in the case of an even $n$ the extremal polynomials for $\gamma_{G_n}(0; e_2) = \frac{2}{n}$ are not unique up to a rotation. Namely, the proof of Proposition 2 delivers the polynomial $\frac{2}{n} z_2 - \frac{n-1}{n^2} z_1^2$, but $\frac{2}{n} z_2 - \frac{1}{n^2} z_1^2$ is also an extremal polynomial.

**Proposition 4.** If $n \geq 3$ is odd, then

\[
\frac{2}{n} \left(1 + \frac{2}{(n-1)(n+2)}\right) < \gamma_{G_n}(0; e_2) < \frac{2}{n} \left(1 + \frac{2}{(n-1)(n+1)}\right).
\]

**Proof.** The lower bound: First, we shall see that for the polynomial $P_n(z) := \frac{n-1}{2(n+1)} z_2^2 - z_2$ one gets the equality $\max_{\partial G_n} |P_n| = M_n := \frac{(n-1)(n+2)}{2(n+1)}$. This means that if

\[
g_n(t) := \frac{1}{2} \sum_{j=1}^{n} t_j^2 - \frac{1}{n+1} (\sum_{j=1}^{n} t_j)^2, \quad t \in \mathbb{C}^n,
\]

then $\max_{T^n}|g_n| = M_n$. Indeed, let $M_n^* = \max_{T^n} |g_n|$. Since $g_n(e^{i\theta}t) = e^{2i\theta} g_n(t)$ for any $\theta \in \mathbb{R}$, $t \in \mathbb{C}^2$, there exists a point $u \in T^n$ such that $g_n(u) = M_n^*$. Setting $u_j = x_j + iy_j$, $x_j, y_j \in \mathbb{R}$, $1 \leq j \leq n$, it
follows that

\[ M^* = \text{Re}(g_n(u)) = \frac{1}{2} \sum_{j=1}^{n} (x_j^2 - y_j^2) + \frac{1}{n+1} (\sum_{j=1}^{n} y_j)^2 - (\sum_{j=1}^{n} x_j)^2 \]

\[ \leq \frac{1}{2} \sum_{j=1}^{n} (x_j^2 - y_j^2) + \frac{1}{n+1} (\sum_{j=1}^{n} y_j^2 - (\sum_{j=1}^{n} x_j)^2) \]

\[ = \frac{(n-1)n}{2(n+1)} + \frac{1}{n+1} (\sum_{j=1}^{n} x_j^2 - (\sum_{j=1}^{n} x_j)^2) \]

by the Cauchy-Schwarz inequality and the equalities \( y_1^2 = 1 - x_1^2, \ldots, y_n^2 = 1 - x_n^2 \). The last term is a linear function in any \( x_j \). Hence it attains maximum at \( \pm 1 \).

Since \( n \) is odd, then

\[ M^* = \frac{(n-1)n}{2(n+1)} + \frac{n-1}{n+1} = M_n \]

and the maximum is attained at \( t \in \mathbb{T}^n \) if and only \([n/2]\) or \([n/2] + 1\) of the \( t_j \)'s are equal to some \( t_0 \in \mathbb{T} \) and the other ones to \(-t_0\).

Using this last fact, it is not difficult to see that if \( \varepsilon > 0 \) is small and

\[ g_{n,\varepsilon}(t) = g_n(t) + \varepsilon \sum_{j=1}^{n} t_j^2 - \varepsilon(n+1)(\sum_{j=1}^{n} t_j)^2, \ t \in \mathbb{C}^n, \]

then \( \max_{\mathbb{T}^n} |g_{n,\varepsilon}| < M_n \). Therefore, for

\[ P_{n,\varepsilon} = \frac{n - 1 - 2n(n+1)\varepsilon}{2(n+1)} z_1 - (1 + 2\varepsilon) z_2 \]

one has the inequality \( \max_{\partial G_n} |P_{n,\varepsilon}| < M_n \) which implies that

\[ \gamma_{G_n}(0; e_2) > \frac{1}{M_n} = \frac{2}{n} \left( 1 + \frac{2}{(n-1)(n+2)} \right). \]

The upper bound: In virtue of (1), we have to show that if \( c \in \mathbb{C} \), then

\[ m_{n,c} := \max_{z \in \partial G_n} |z_2 + cz_1^2| > \frac{n(n^2 - 1)}{2(n^2 + 1)}. \]

The coefficients of the polynomials \((t - 1)^n\) and \((t - 1)(t^2 - 1) \frac{n-1}{2}\) give points \( z \in \partial G_n \) with \( z_1 = n, z_2 = \frac{n(n-1)}{2} \) and \( z_1 = 1, z_2 = \frac{1-n}{2} \), respectively. Then

\[ 2m_{n,c} \geq \max\{|n - 1 - 2c|, |n(n-1) + 2cn^2|\} \]
and hence
\[
2(n^2 + 1)m_{n,c} \geq |n^2(n - 1) - 2cn^2| + |n(n - 1) + 2cn^2|
\]
\[
\geq n^2(n - 1) + n(n - 1) = n(n^2 - 1).
\]
This implies that \( m_{n,c} \geq \frac{n(n^2 - 1)}{2(n^2 + 1)} \). Assume that the equality holds.

Then \( c = -\frac{(n - 1)^2}{2(n^2 + 1)} \). On the other hand, the coefficients of the polynomial \((t - i)(t - 1)^{n-1}\) give a point \( z \in \partial G_n \) with \( z_1 = n - 1 + i, z_2 = \frac{(n - 1)(n - 2)}{2} + (n - 1)i \), for which
\[
|z_2 - \frac{(n - 1)^2}{2(n^2 + 1)}z_1| > \frac{n(n^2 - 1)}{2(n^2 + 1)},
\]
a contradiction. \(\square\)

5. The proof of the inequality \( \hat{\gamma}_{G_n}^{(2)}(0; e_2) > \gamma_{G_n}(0; e_2) \)

Let \( D \) be a domain in \( \mathbb{C}^n \) and \( k \in \mathbb{N} \). Recall that the \( k \)-th Reiffen pseudometric is defined as (see [7])

\[
\gamma_{D}^{(k)}(z; X) := \sup \left\{ \left| \frac{f^{(k)}(z)X}{k!} \right|^\frac{1}{k} : f \in \mathcal{O}(D, \mathbb{D}), \ord f \geq k \right\}.
\]

Note that \( \gamma_D \leq \gamma_{D}^{(k)} \leq \kappa_D \). Denote by \( \hat{\gamma}_{D}^{(k)} \) the largest pseudonorm which does not exceed \( \gamma_{D}^{(k)} \). Since \( \gamma_D(z; \cdot) \) is a pseudonorm, it follows that \( \gamma_D \leq \hat{\gamma}_{D}^{(k)} \leq \kappa_D \). We also point out that the family \( \mathcal{O}(G_3, \mathbb{D}) \) is normal and then the argument as in [11] shows that there are \( m \) \((m \leq 2n - 1) \) \( \mathbb{R} \)-linearly independent vectors \( X_1, \ldots, X_m \in \mathbb{C}^n \) with the sum \( X \) such that

\[
\hat{\gamma}_{D}^{(k)}(z; X) = \sum_{j=1}^{m} \gamma_{D}^{(k)}(z; X).
\]

The purpose of this section is to show the following

**Proposition 5.** \( \hat{\gamma}_{G_3}^{(2)}(0; e_2) > \gamma_{G_3}(0; e_2) \). In particular, \( \hat{\kappa}_{G_3}(0; e_2) > \gamma_{G_3}(0; e_2) \) and hence \( \kappa_{G_3}(0, \cdot) \neq c_{G_3}(0, \cdot) \).

**Remark.** We believe that the idea of the proof below works for \( G_n \) for any \( n \geq 3 \).

Proposition 5 is a consequence of the next two lemmas.

**Lemma 6.** \( \gamma_{G_3}(0; e_2) \leq C_0 := \sqrt{\frac{8}{13\sqrt{13} - 35}} = 0,8208 \ldots . \)

**Lemma 7.** \( \hat{\gamma}_{G_3}^{(2)}(0; e_2) \geq C_1 = \sqrt{0,675} = 0,8215 \ldots . \)
Proof of Lemma 6. By (1), we have to show that for any \( c \in \mathbb{C} \) one has
\[
\max_{z \in \partial G_3} |z_2 - cz_1^2|^2 \geq \frac{1}{C_0^2}.
\]
First, observe that it is enough to prove this inequality in the case, when \( c \in \mathbb{R} \). Indeed, for any \( z \in \partial G_3 \) one has that \( \overline{z} \in \partial G_3 \) and therefore
\[
2 \max_{z \in \partial G_3} |z_2 - cz_1^2| \geq \max_{z \in \partial G_3} (|z_2 - cz_1^2| + |\overline{z}_2 - c\overline{z}_1^2|)
\]
\[
\geq \max_{z \in \partial G_3} |2z_2 - (c + \overline{c})z_1^2| = 2 \max_{z \in \partial G_3} |z_2 - \text{Re}(c)z_1^2|.
\]
Let now \( c \in \mathbb{R} \). Then
\[
\max_{z \in \partial G_3} |z_2 - cz_1^2|^2 \geq \max_{\varphi \in [0,2\pi)} |1 + 2e^{i\varphi} - c(2 + e^{i\varphi})|^2
\]
\[
= \max_{\varphi \in [0,2\pi)} (4c(4c - 1)\cos^2 \varphi + 4(10c^2 - 7c + 1)\cos \varphi + 25c^2 - 22c + 5).
\]
Set
\[
f_c(x) := 4c(4c - 1)x^2 + 4(2c - 1)(5c - 1)x + 25c^2 - 22c + 5, \ x \in [-1,1].
\]
If \( c \not\in \Delta := \left(\frac{1}{6}, \frac{5 - \sqrt{17}}{4}\right) \), then
\[
\max_{x \in [-1,1]} f_c(x) = \max\{f_c(-1), f_c(1)\} \geq \left(\frac{9 - \sqrt{17}}{4}\right)^2 > \frac{1}{C_0^2}.
\]
Otherwise,
\[
\max_{x \in [-1,1]} f_c(x) = f_c\left(\frac{10c^2 - 7c + 1}{2c(1 - 4c)}\right) = \frac{(3c - 1)^3}{c(4c - 1)} =: g(c)
\]
and it remains to check that \( \min_{c \in \Delta} g(c) = g\left(\frac{\sqrt{13} - 1}{12}\right) = \frac{1}{C_0^2} \). \( \Box \)

Remark. Set \( c_0 = \frac{\sqrt{13} - 1}{12} \) and \( M := \max_{z \in \partial G_3} |z_2 - c_0z_1^2| \). As in the proof of Proposition 4 we have that
\[
M = \max_{z \in \partial G_3} \text{Re}(z_2 - c_0z_1^2) = \max_{\alpha, \beta, \gamma \in \mathbb{R}} h(\alpha, \beta, \gamma),
\]
where
\[
h(\alpha, \beta, \gamma) = (1 - 2c_0)(\cos(\alpha + \beta) + \cos(\beta + \gamma) + \cos(\gamma + \alpha))
\]
\[-c_0(\cos 2\alpha + \cos 2\beta + \cos 2\gamma).
\]
Computer calculations show that the critical points of \( h \) (up to permutations of the variables) are of the form \((k\pi, l\pi, m\pi)\) or \((\pm \alpha_0 + j\pi/2 + \)
Proposition 8. If $X_1, X_n \in \mathbb{C}$, then

$$
\gamma_{G_n}^2(0; X_1 e_1 + X_n e_n) \geq \sqrt{\frac{n+1}{2} \gamma_{G_n}(0; e_2)|X_1 X_n|}.
$$
In particular, since $\gamma_{G^3}(0; e_2) > \frac{2}{3}$ and $\gamma_{G^n}(0; e_n) \geq \frac{2}{n}$ then

$$\gamma_{G^n}^{(2)}(0; ne_1 + e_n) > 2 = \kappa_{G^n}(0; ne_1 + e_n) = \gamma_{G^n}^{(2)}(0; ne_1) + \gamma_{G^n}^{(2)}(0; e_n), n \geq 3.$$ 

Proof. Let $t_1, \ldots, t_n \in \mathbb{D}$. Consider $\sum_{k=1}^{n} \frac{t_{k}^{n+1}}{n}$ as a function $f_n$ of $z_1, \ldots, z_n$.

Then $f_n \in O(G_n, \mathbb{D})$, ord$_0 f_n = 2$ and by the Waring formula (cf. [14]) the coefficient at $z_1 z_n$ equals $(-1)^{n-1} \frac{n + 1}{n}$. Hence

$$\gamma_{G^n}^{(2)}(0; X_1 e_1 + X_n e_n) \geq \sqrt{\frac{n + 1}{n} \gamma_{G^n}(0; e_2)} |X_1 X_n|.$$ 

Since $\gamma_{G^n}(0; e_2) = \frac{2}{n}$ for even $n$, we are done for such $n$.

On the other hand, we know by Proposition 3 that there is $c_n$ such that $P$ with $P(z) := 2C_n z^2 - c_n z^2_1$ is an extremal function for $\gamma_{G^n}(0; e_2) = 2C_n$. For $n = 2k - 1$ replace $t_1, \ldots, t_n$ by $t_{1}^{k}, \ldots, t_{n}^{k}$. Then we obtain the function

$$g_n(\sigma_n(t)) := \tilde{g}_n(t) = (C_n - c_n) \left( \sum_{j=1}^{n} t_{j}^{k} \right)^2 - C_n \sum_{j=1}^{n} t_{j}^{2k}.$$ 

Then $g_n \in O(G_n, \mathbb{D})$, ord$_0 g_n = 2$, and the coefficient at $z_1 z_n$ equals $-(n+1)C_n$. Now, it is enough to take $g_n$ as a competitor for $\gamma_{G^n}^{(2)}(0; X_1 e_1 + X_n e_n)$. □

REFERENCES

[1] J. Agler, N. J. Young, The hyperbolic geometry of the symmetrized bidisc, J. Geom. Anal. 14 (2004), 375–403.
[2] C. Costara, Dissertation, Université Laval (2004).
[3] C. Costara, The symmetrized bidisc and Lempert’s theorem, Bull. London Math. Soc. 36 (2004), 656–662.
[4] C. Costara, On the spectral Nevanlinna–Pick problem, Studia Math. 170 (2005), 23–55.
[5] A. Edigarian, A note on Costara’s paper, Ann. Polon. Math. 83 (2004), 189–191.
[6] A. Edigarian, W. Zwonek, Geometry of the symmetrized polydisc, Arch. Math. (Basel) 84 (2005), 364–374.
[7] M. Jarnicki, P. Pflug, Invariant distances and metrics in complex analysis–revisited, Diss. Math. 430 (2005), 1–192.
[8] M. Kobayashi, On the convexity of the Kobayashi metric on a taut complex manifold, Pacific J. Math. 194 (2000), 117–128.
[9] L. Lempert, La métrique de Kobayashi et la représentation des domaines sur la boule, Bull. Soc. Math. France 109 (1981), 427–474.
[10] N. Nikolov, The symmetrized polydisc cannot be exhausted by domains biholomorphic to convex domains, Ann. Polon. Math., 88 (2006), 279-283.
[11] N. Nikolov, P. Pflug, *On the definition of the Kobayashi-Buseman metric*, Internat. J. Math., to appear (arXiv:math.CV/0603020).

[12] N. Nikolov, P. Pflug, W. Zwonek, *The Lempert function of the symmetrized polydisc in higher dimensions is not a distance*, Proc. Amer. Math. Soc., to appear (arXiv:math.CV/0601367).

[13] M.-Y. Pang, *On infinitesimal behavior of the Kobayashi distance*, Pacific J. Math. 162 (1994), 121–141.

[14] B. L. van der Waerden, *Algebra, erster Teil*, Springer Verlag, Berlin-Göttingen-Heidelberg (1964).

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