ON EULER CHARACTERISTICS OF SELMER GROUPS FOR
ABELIAN VARIETIES OVER GLOBAL FUNCTION FIELDS

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Abstract. Let $F$ be a global function field of characteristic $p > 0$, $K/F$ an
$\ell$-adic Lie extension ($\ell \neq p$) and $A/F$ an abelian variety. We provide Euler
characteristic formulas for the $\text{Gal}(K/F)$-module $\text{Sel}_A(K)_\ell$.

1. Introduction

Let $\ell \in \mathbb{Z}$ be a prime and let $G$ be a profinite $\ell$-adic Lie group of finite dimension
$d \geq 1$. Let $M$ be a $G$-module and consider the following properties
1. $H^i(G, M)$ is finite for any $i \geq 0$;
2. $H^i(G, M) = 0$ for all but finitely many $i$.

Definition 1.1. If a $G$-module $M$ verifies 1 and 2, we define the Euler characteristic
of $M$ as
\[
\chi(G, M) := \prod_{i \geq 0} |H^i(G, M)|^{(-1)^i}.
\]

Let $A/F$ be an abelian variety and, for any extension $L/F$, let $\text{Sel}_A(L)_\ell$ be the
$\ell$-power part of the Selmer group of $A$ over $L$ (for a precise definition see Section
2). When $G$ is the Galois group of a field extension $L/F$, the study of the Euler
characteristic $\chi(G, \text{Sel}_A(L)_\ell)$ is a first step towards understanding the relation,
predicted by the Iwasawa Main Conjecture, between a characteristic element for
$\text{Sel}_A(L)_\ell$ and a suitable $\ell$-adic $L$-function.

In the number field setting, various papers have considered Euler characteristic
formulas for the Selmer groups. Among these works, we mention that of Coates
and Howson [7], where they considered the extension generated by the $p$-power
torsion points of an elliptic curve $E/F$. Interesting generalizations can be found in
the papers of Van Order [16] and Zerbes [17] and [18].

The aim of this paper is to provide Euler characteristic formulas for the Selmer
group in the function field case. Let $F$ be a global function field of characteristic
$p > 0$ and let $K/F$ be an $\ell$-adic Lie extension ($\ell \neq p$) with Galois group $G$ and
unramified outside a finite and nonempty set $S$ of primes of $F$. The case $\ell = p$,
which requires a few more technical tools related to flat cohomology, will be treated
in a different paper [6].

Here is a summary of the present work. In Section 2 we provide two formulations
for $\chi(G, \text{Sel}_A(K)_\ell)$. The first (see Theorem 2.3) mainly depends on the cohomol-
y and Euler characteristic of torsion points while the second (see Theorem 2.5)
involves more directly the Tate-Shafarevich group $\Sha(A/F)$ (hinting at a connection with values of $L$-functions and the Birch and Swinnerton-Dyer conjecture). In Section $\ref{sec:elliptic}$ we specialize our formulas to the case of elliptic curves (all notation are canonical and will be explained in the next section or soon as they appear) and obtain (see Theorem $\ref{thm:main}$)

**Theorem 1.2.** Assume $\Sel_{E}(F)_{\ell}$ is finite, $H^{2}(F_{S}/K, E[\ell^{\infty}]) = 0$, $\chi(G, E(K)[\ell^{\infty}])$ and $\chi(G_{v}, E(K_{w})[\ell^{\infty}])$ are well defined for any $w|v \in S$ and the map $\psi_{K}$ in the sequence

\[
\psi_{K} : \Sel_{E}(K)_{\ell} \rightarrow H^{1}(F_{S}/K, E[\ell^{\infty}]) \rightarrow \prod_{v \in S} \Coind^{G_{v}}_{E} H^{1}(K_{w}, E[\ell^{\infty}])
\]

is surjective. If $\ell \geq 5$ we have

\[
\chi(G, \Sel_{E}(K)_{\ell}) = \chi(G, E(K)[\ell^{\infty}]) \frac{\Sha(E/F)[\ell^{\infty}]}{|E(F)[\ell^{\infty}]|^{2}} \cdot \prod_{v \in S} \chi(G_{v}, E(K_{w})[\ell^{\infty}])
\]

\[
\cdot \prod_{v \in S, \text{ cd}_{\ell}(K_{w}) = 0} \left| L_{v}(E, 1) \right|_{\ell} \cdot \prod_{v \in S, \text{ cd}_{\ell}(K_{w}) = 0} \left| L_{v}(E, 1) \right|_{\ell}
\]

1.1. **Setting and notations.** Before moving on we recall the main objects and the setting we shall work with.

Let $F$ be a global function field of characteristic $p > 0$ with finite constant field $\mathbb{F}_{q}$ ($q = p^{r}$ for some $r \in \mathbb{N}$). For any place $v$ of $F$, let $F_{v}$ be the completion of $F$ at $v$ and its residue field. For any Galois extension $L/F$ and any place $v$ of $F$, fix a place $w$ of $L$ lying above $v$ and let $G_{v} := \text{Gal}(L_{w}/F_{v})$ be the associated decomposition group. We also fix an embedding of $F$ (a separable algebraic closure of $F$) into $\overline{F}$ in order to get a restriction map $\text{Gal}(F_{L}/F_{v}) \rightarrow \text{Gal}(\overline{F}/F) = : G_{F}$.

Let $K/F$ be an $\ell$-adic Lie extension unramified outside a finite and nonempty set of places $S$ of $F$. We write $G := \text{Gal}(K/F)$ and assume it has finite dimension $d$ (as $\ell$-adic Lie group) and no elements of order $\ell$ (then the $\ell$-cohomological dimension of $G$ is $\text{cd}_{\ell}(G) = d$ by [12 Corollaire (1) p. 413]).

Let $F_{S}$ be the maximal (separable) extension of $F$ unramified outside $S$, so that $K \subseteq F_{S}$.

Recall that $\text{cd}_{\ell}(\text{Gal}(F_{S}/F)) = 2$ ([10 Corollary 10.1.3 (iii)])

Let $A/F$ be an abelian variety of dimension $g$. We denote by $A^{\ell}$ its dual abelian variety and, as usual, $A[\ell^{\infty}]$ will be the scheme of $\ell^{\infty}$-torsion points of $A$, with $A[\ell^{\infty}] := \lim_{\longrightarrow} A[\ell^{n}] = \bigcup_{\ell} A[\ell^{n}]$.

For any $\ell$-adic Lie group $G$ we denote by

\[
\Lambda(G) = \mathbb{Z}[G][[\ell]] := \lim_{\longrightarrow} \mathbb{Z}[G/U]
\]

the associated *Iwasawa algebra*, where the limit is taken on the open normal subgroups of $G$.

It is well known that in our setting $\Lambda(G)$ is a Noetherian and (if $G$ is pro-$\ell$ and has no elements of order $\ell$) integral domain.
Let $H$ be a closed subgroup of $G$. For every $\Lambda(H)$-module $N$ we consider the $\Lambda(G)$-modules

$$\text{Coind}^H_G(N) := \text{Map}_{\Lambda(H)}(\Lambda(G), N)$$

and

$$\text{Ind}^G_H(N) := \Lambda(G) \otimes_{\Lambda(H)} N.$$  

For a $\Lambda(G)$-module $M$, we denote by $M^\vee := \text{Hom}_{\text{cont}}(M, \mathbb{Q}_\ell / \mathbb{Z}_\ell)$ its Pontrjagin dual, which has a natural structure of $\Lambda(G)$-module.

We enlarge our set $S$ so that it contains all primes ramified in $K/F$ and all places of bad reduction for $A$. Then, the extension $F(A[\ell^\infty])/F$ is contained in $F_S/F$ and $A[\ell^\infty]$ is an unramified $G_F$-module for every $v \not\in S$. For any Galois extension $L/F$ such that $L \subseteq F_S$, let us consider the map

$$\rho : H^1(F^s/L, A[\ell^\infty]) \to \prod_{v \not\in S} \text{Coind}^G_{G_v} H^1(F^s_{v^w}/L_w, A[\ell^\infty]).$$

Direct computations on local Galois cohomology groups give (for more details see [15, Proposition 1.4.4])

$$Ker(\rho) \simeq H^1(F_S/L, A[\ell^\infty]) \cap Ker(\eta).$$

Let us consider the map:

$$\eta : H^1(F^s/L, A[\ell^\infty]) \to \prod_{v \in S} \text{Coind}^G_{G_v} H^1(F^s_{v^w}/L_w, A[\ell^\infty]).$$

Thanks to the fact that for $\ell \neq p$ the image of the Kummer maps is trivial (see [1, Proposition 3.3]), we have

$$\text{Sel}_A(L) \ell = Ker(\rho) \cap Ker(\eta) \simeq H^1(F_S/L, A[\ell^\infty]) \cap Ker(\eta).$$

Then, we can use the following definition (already employed in [4]) for the Selmer group.

**Definition 1.3.** For any finite extension $L$ of $F$, the $\ell$-part of the Selmer group of $A$ over $L$ is

$$\text{Sel}_A(L) \ell = Ker \left( H^1(F_S/L, A[\ell^\infty]) \to \prod_{v \not\in S} \text{Coind}^G_{G_v} H^1(F^s_{v^w}/L_w, A[\ell^\infty]) \right)$$

where $w$ is a fixed place of $L$ lying above $v$.

The Tate-Shafarevich group $\text{III}(A/L)$ is the group that fits into the exact sequence

$$A(L) \otimes \mathbb{Q}_\ell / \mathbb{Z}_\ell \to \text{Sel}_A(L) \ell \to \text{III}(A/L)[\ell^\infty].$$

For infinite extensions we define the Selmer groups by taking direct limits on the finite subextensions. In particular, $\text{Sel}_A(K) \ell$ is a $\Lambda(G)$-module whose structure has been studied in [6].

If $L/F$ is a finite extension the group $\text{Sel}_A(L) \ell$ is a cofinitely generated $\mathbb{Z}_\ell$-module (see, e.g. [5 III.8 and III.9]). Whenever we assume that $\text{Sel}_A(L) \ell$ is finite, we have that the $\mathbb{Z}$-rank of $A(L)$ is 0, hence

$$A(L) \otimes \mathbb{Q}_\ell / \mathbb{Z}_\ell = 0 \quad \text{and} \quad |\text{Sel}_A(L) \ell| = |\text{III}(A/L)[\ell^\infty]|.$$

\footnote{Some texts, e.g. [10], switch the definitions of $\text{Ind}^H_G(N)$ and $\text{Coind}^H_G(N)$.}
2. EULER CHARACTERISTIC FOR ABELIAN VARIETIES

2.1. Cohomological lemmas. We list some useful results on the cohomology of the $\ell$-power torsion points.

Lemma 2.1. Let $L$ be a finite extension of $F$ contained in $F_S$. If $\text{Sel}_{A^i}(L)$ is finite, then

$$H^2(F_S/L,A[\ell^\infty]) = 0.$$  

Proof. See (the proof of) [10] Proposition 4.4. □

Lemma 2.2. If $\text{Sel}_{A^i}(F)$ is finite and $H^2(F_S/K,A[\ell^\infty]) = 0$ we have

1. $H^1(G, H^1(F_S/K, A[\ell^\infty])) \simeq H^1(G, A(K)[\ell^\infty]) = 0$ for all $i \geq 1$.

Moreover, let $w$ be any prime of $K$ such that $w \mid v \in S$. Then

2. $H^i(G_v, H^1(K_w, A[\ell^\infty])) \simeq H^i \big( H^1(K_w, A(K)[\ell^\infty]) \big) = 0$ for all $i \geq 1$.

Proof. 1. By [10] Corollary 10.1.3 (iii) and Proposition 3.3.5 we have that $\text{cd}_v(\text{Gal}(F_S/K)) \leq 2$. Therefore, our hypothesis on the cohomology group $H^2(F_S/K,A[\ell^\infty])$ yields

$$H^i(F_S/K,A[\ell^\infty]) = 0 \quad \forall \ i \geq 2.$$

So, from the Hochschild-Serre spectral sequence, we have

$$H^1(G, A(K)[\ell^\infty]) \longrightarrow H^1(F_S/F, A[\ell^\infty]) \longrightarrow H^0(G, H^1(F_S/K, A[\ell^\infty])) \rightarrow \cdots$$

$$H^2(G, A(K)[\ell^\infty]) \longrightarrow H^2(F_S/F, A[\ell^\infty]) \longrightarrow H^1(G, H^1(F_S/K, A[\ell^\infty])) \rightarrow \cdots$$

(by [10] Lemma 2.1.3). Since $\text{cd}_v(\text{Gal}(F_S/F)) = 2$ and, by Lemma 2.1, the group $H^2(F_S/F,A[\ell^\infty])$ is zero, one gets

$$H^i(G, H^1(F_S/K, A[\ell^\infty])) \simeq H^i+2(G, A(K)[\ell^\infty]) \quad \forall \ i \geq 1.\]$$

Moreover, by [10] Lemma 2.1.4, we have the following isomorphisms

$$H^i(G, H^1(F_S/K, A[\ell^\infty])) = E_2^{i1} \simeq E^{i1} = H^i+1(F_S/F, A[\ell^\infty]) \quad \forall \ i \geq 1.$$  

Then

$$H^i(G, H^1(F_S/K, A[\ell^\infty])) = 0 \quad \forall \ i \geq 1.$$  

2. The argument is the same of part 1. In order to get $H^2(K_w, A[\ell^\infty]) = 0$, just use [10] Theorem 7.2.6 or [10] Theorem 7.1.8 (i) according to $K_w$ being a local field or not (so according to $K/F$ being a finite extension or not). □

2.2. Euler characteristic: Selmer groups and torsion points. Now we give a first formula for the Euler characteristic of $\text{Sel}_A(K)$ which relates it to the (local and global) Euler characteristics of the $\ell^\infty$-torsion points in $K$ and $K_w$.

Theorem 2.3. With notations as above, assume that $\text{Sel}_A(F)$ and $\text{Sel}_{A^i}(F)$ are finite, $H^2(F_S/K,A[\ell^\infty]) = 0$, $\chi(G,A(K)[\ell^\infty])$ and $\chi(G_v,A(K_w)[\ell^\infty])$ are well defined for any $w \mid v \in S$ and the map $\psi_K$ in the sequence

$$\text{Sel}_A(K) \hookrightarrow H^1(F_S/K,A[\ell^\infty]) \xrightarrow{\psi_K} \bigoplus_{v \in S} \text{Coind}^G_{K_v} H^1(K_w,A[\ell^\infty]).$$
is surjective. Then $H^i(G, \text{Sel}_A(K)_\ell) = 0$ for any $i \geq 2$, the Euler characteristic of $\text{Sel}_A(K)_\ell$ is well defined and

\[
\chi(G, \text{Sel}_A(K)_\ell) = \chi(G, A(K)[\mathbb{F}_\ell]) \frac{|H^1(F_S/F, A[F])|}{|A(F)[\mathbb{F}_\ell]|} \cdot \prod_{v \in S} \chi(G_v, A(K_v)[\mathbb{F}_\ell]) |H^1(F_v, A[F])|^{-1}. (5)
\]

where, for every $v \in S$, $w$ is a fixed place of $K$ dividing $v$.

**Proof.** Let us consider the sequence (4) and take its cohomology with respect to $\psi$. Recalling that, by Lemma 2.2, $H^i(K_v, A[F])$ is surjective. Then, using Shapiro’s Lemma ([10, Proposition 1.6.4]), Lemma 2.2 and the finiteness of the cohomological dimension of $G$, we obtain

\[
\text{Sel}_A(K)_\ell \xrightarrow{\psi} H^1(F_S/F, A[F]) \xrightarrow{\chi} H^1(G, \text{Sel}_A(K)_\ell)
\]

and $H^i(G, \text{Sel}_A(K)_\ell) = 0$ for any $i \geq 2$. Therefore

\[
\chi(G, \text{Sel}_A(K)_\ell) = \frac{[\text{Sel}_A(K)_\ell]}{|H^1(G, \text{Sel}_A(K)_\ell)|} = \frac{|H^1(F_S/K, A[F])|}{\prod_{v \in S} |H^1(K_v, A[F])|}. (6)
\]

The Hochschild-Serre spectral sequence yields

\[
H^1(G, A(K)[\mathbb{F}_\ell]) \xrightarrow{\psi} H^1(F_S/F, A[F]) \xrightarrow{|\cdot|} H^2(G, A(K)[\mathbb{F}_\ell])
\]

Recalling that, by Lemma 2.2, $H^i(G, A(K)[\mathbb{F}_\ell]) = 0$ for any $i \geq 3$, we have

\[
|H^1(F_S/K, A[F])| = \frac{|H^1(F_S/F, A[F])| |H^2(G, A(K)[\mathbb{F}_\ell])|}{|H^1(G, A(K)[\mathbb{F}_\ell])|} = \frac{|H^1(F_S/F, A[F])| |\chi(G, A(K)[\mathbb{F}_\ell])|}{|H^0(G, A(K)[\mathbb{F}_\ell])|} = \frac{|H^1(F_S/F, A[F])| |\chi(G, A(K)[\mathbb{F}_\ell])|}{|A(F)[\mathbb{F}_\ell]|}. (7)
\]

The local computations are similar and, substituting in (7), we get (5). \[ \Box \]

**Example 1.** Let us consider $K = F(A[\mathbb{F}_\ell])$. By [11, Proposition 4.5] we know $H^2(F_S/K, A[F]) = 0$. Moreover, it is easy to see that $\chi(G, A[\mathbb{F}_\ell])$ is well defined thanks to [13, Théorème 2]. Besides, if all primes in $S$ are of split multiplicative reduction, then the Mumford parametrization and the form of the $\ell$-power torsion points yields $\text{cd}_\ell(K_w) = 0$. So, the Hochschild-Serre spectral sequence provides isomorphisms

\[
H^n(G_v, A(K_w)[\mathbb{F}_\ell]) \cong H^n(F_v, A[\mathbb{F}_\ell]) \quad \forall \ n \geq 0.
\]
Then
\[ \chi(G_v, A[\ell^\infty]) = \frac{|A[\ell^\infty](F_v)|}{|H^1(F_v, A[\ell^\infty])|} \]
is well defined thanks to [10] Theorem 7.1.8 and the fact that Tate local duality ([10] Theorem 7.2.6) yields \( H^2(F_v, A[\ell^\infty]) = 0 \). It follows that if both \( Sel_A(F) \) and \( Sel_A(F) \) are finite and the map \( \psi_K \) in [10] is surjective
\[ \chi(G, Sel_A(K)) = \chi(G, A[\ell^\infty]) \frac{|H^1(F_S/F, A[\ell^\infty])|}{|A(F)[\ell^\infty]|}. \]

2.3. Euler Characteristic II: descent diagrams. Now we look for a slightly different formulation for the global factor, which is more closely related, especially in the case of elliptic curves, to special values of \( L \)-functions. We consider the classical descent diagram

\[
\begin{array}{c}
Sel_A(F)_\ell \ar[r]^\alpha & H^1(F_S/F, A[\ell^\infty]) \ar[d]^{\beta'} \ar[r]^{\psi'} & \text{Im}(\psi_F) \\
Sel_A(K)^G_\ell \ar[r] & H^1(F_S/K, A[\ell^\infty])^G \ar[d]^{\beta} \ar[r]^{\psi_K} & \text{Im}(\psi_K^G) \\
\end{array}
\]

where \( \text{Im}(\psi_F) \) and \( \text{Im}(\psi_K^G) \) lie in the diagram

\[
\begin{array}{c}
\text{Im}(\psi_F) \ar[r] & \prod_{v \in S} H^1(F_v, A[\ell^\infty]) \ar[r] & \text{Coker}(\psi_F) \\
\downarrow^{\beta'} & & \downarrow^{\gamma = \oplus \gamma_v} \\
\text{Im}(\psi_K^G) \ar[r] & \prod_{v \in S} \text{Coint}^G_{\ell^G} H^1(K_w, A[\ell^\infty])^G \ar[r] & \text{Coker}(\psi_K^G) \\
\end{array}
\]

\[ \text{Proposition 2.4. Assume } Sel_A(F)_\ell \text{ is finite. Then, } \]
\[ \text{Ker}(\beta) = H^1(G, A(K)[\ell^\infty]) \text{ and } \text{Coker}(\beta) = H^2(G, A(K)[\ell^\infty]). \]

Moreover,
\[ \text{Ker}(\gamma) = \prod_{v \in S, \text{cd}(K_w) = 1} H^1(G_v, A(K_w)[\ell^\infty]) \cdot \prod_{v \in S, \text{cd}(K_w) = 0} H^1(F_v, A[\ell^\infty]) \]
\[ \text{Coker}(\gamma) = \prod_{v \in S, \text{cd}(K_w) = 1} H^2(G_v, A(K_w)[\ell^\infty]). \]

\[ \text{Proof. For the map } \beta \text{ just use the Hochschild-Serre five term exact sequence and Lemma [10].} \]

For the map \( \gamma \), by Shapiro’s Lemma we can rewrite every \( \gamma_v \) as
\[ \gamma_v : H^1(F_v, A[\ell^\infty]) \rightarrow H^1(K_w, A[\ell^\infty])^G_v \]
for a fixed place \( w \) of \( K \) dividing \( v \). Using again the five term exact sequence and the fact that \( H^2(F_v, A[\ell^\infty]) = 0 \), one has
\[ \text{Ker}(\gamma_v) \cong H^1(G_v, A(K_w)[\ell^\infty]) \text{ and } \text{Coker}(\gamma_v) \cong H^2(G_v, A(K_w)[\ell^\infty]). \]

If \( v \) is totally split we have that \( G_v = 0 \). In this case \( \gamma_v \) is an isomorphism. If \( v \) is inert or ramified, by [10] Theorem 7.1.8 (i) \( \text{cd}(K_w) \leq 1 \). This implies that
when \( cd_t(K_w) = 0 \), \( \gamma_v \) is the zero-map and we have \( \text{Ker}(\gamma_v) \simeq H^1(F_v, A[\ell\infty]) \) and \( \text{Coker}(\gamma_v) = 0 \). The claim follows.

**Theorem 2.5.** With hypotheses as in Theorem 2.3 one has

\[
\chi(G, Sel_A(K)_\ell) = \chi(G, A(K)[\ell\infty]) \cdot \frac{\prod (A/F)[\ell\infty]}{|A(F)[\ell\infty]| |A^t(F)[\ell\infty]|} \cdot \prod_{v \in S, |cd_t(K_w)| = 1} |A(F_v, A(K)[\ell\infty])| \cdot \prod_{v \in S, |cd_t(K_w)| = 0} |H^1(F_v, A[\ell\infty])|.
\]

**Proof.** Since we are assuming that \( Sel_A(F)_\ell \) is finite and \( \psi_K \) is surjective, then equation (6) shows that \( \text{Coker}(\psi_K^G) \simeq H^1(G, Sel_A(K)_\ell) \). Therefore

\[
\chi(G, Sel_A(K)_\ell) = \frac{|Sel_A(K)_\ell|}{|\text{Coker}(\psi_K^G)|}.
\]

The snake lemma sequence of diagram (9) yields

\[
|\text{Coker}(\psi_K^G)| = \frac{|\text{Coker}(\gamma')| |\text{Coker}(\psi_F)|}{|\text{Ker}(\gamma')|} = \frac{|\text{Coker}(\psi_F)| |\text{Ker}(\beta')| |\text{Coker}(\gamma)|}{|\text{Coker}(\beta')| |\text{Ker}(\gamma)|}.
\]

Using the snake lemma sequence of diagram (9) one gets

\[
\frac{|\text{Ker}(\beta')|}{|\text{Coker}(\beta')|} = \frac{|\text{Ker}(\beta)|}{|\text{Coker}(\alpha)|} = \frac{|\text{Coker}(\beta)|}{|\text{Ker}(\alpha)|} \cdot \frac{|\text{Sel}_A(K)^G_\ell|}{|H^1(G, A(K)[\ell\infty])|} \cdot \frac{|\text{Sel}_A(F)_\ell|}{|\text{Sel}_A(A)[\ell\infty]|} = \chi(G, A(K)[\ell\infty])^{-1} \frac{|A(F)[\ell\infty]| |\text{Sel}_A(K)^G_\ell|}{\prod |A/F[\ell\infty]|}.
\]

Substituting in \( \text{Coker}(\psi_K^G) \) one gets

\[
|\text{Coker}(\psi_K^G)| = \frac{|\text{Coker}(\psi_F)| |\text{Sel}_A(K)^G_\ell| |A(F)[\ell\infty]| |\text{Coker}(\gamma)|}{\chi(G, A(K)[\ell\infty]) |\prod |A/F[\ell\infty]| |\text{Ker}(\gamma)|}.
\]

and

\[
\chi(G, Sel_A(K)_\ell) = \chi(G, A(K)[\ell\infty]) \frac{|\prod |A/F[\ell\infty]| |\text{Ker}(\gamma)|}{|\text{Coker}(\psi_F)| |A(F)[\ell\infty]| |\text{Coker}(\gamma)|}.
\]

The cardinality of \( \text{Ker}(\gamma) \) and \( \text{Coker}(\gamma) \) can be taken from Proposition 2.4. To conclude observe that \( Sel_A(F)_\ell \) finite yields

\[
\text{Coker}(\psi_F) \simeq (A^t(F)^\ast)^\vee
\]

(where the \( \ast \) denotes the \( \ell \)-adic completion, see, e.g., Proposition 4.4). Hence, since \( A^t(F) \) is finite by hypothesis,

\[
|\text{Coker}(\psi_F)| = |(A^t(F)^\ast)^\vee| = \lim_{n} |A^t(F)/\ell^n| = |A^t(F)[\ell\infty]|.
\]

**Example 2.** Let \( K = F(A[\ell\infty]) \) as in the example after Theorem 2.3. Suppose that all hypotheses of Theorem 2.3 are verified and that all primes in \( S \) are of split multiplicative reduction. In this case, the formula for the Euler characteristic of \( Sel_A(K)_\ell \) is the following:

\[
\chi(G, Sel_A(K)_\ell) = \chi(G, A[\ell\infty]) \frac{|\prod |A/F[\ell\infty]| |A^t(F)[\ell\infty]|}{|A(F)[\ell\infty]| |A^t(F)[\ell\infty]|} \cdot \prod_{v \in S} |H^1(F_v, A[\ell\infty])|.
\]
In order to observe that this formula coincides with that of the previous example, just note that we have the following exact sequence

$$Sel_A(F) \hookrightarrow H^1(F_S/F, A[\ell^\infty]) \xrightarrow{\psi_F} \prod_{v \in S} H^1(F_v, A[\ell^\infty]) \twoheadrightarrow \text{Coker}(\psi_F)$$

with $|Sel_A(F)| = |\text{III}(A/F)[\ell^\infty]|$ and $|\text{Coker}(\psi_F)| = |A^1(F)[\ell^\infty]|$.

3. Euler Characteristic for elliptic curves

When $A = E$ is an elliptic curve we can find an explicit connection between Euler characteristic and the Hasse-Weil $L$-function. Replacing, if needed, $F$ by a finite extension, we can (and will) assume that the places of multiplicative reductions are of split multiplicative reduction.

Let $\tilde{E}_v$ be the image of $E$ under the usual reduction map at any prime $v$. We denote by $\tilde{E}_{v,\text{ns}}$ the group of non singular points of $\tilde{E}_v$. Moreover, we define two subset of $E(F_v)$ as follows:

$$E_0(F_v) = \{ P \in E(F_v) : \tilde{P} \in \tilde{E}_{v,\text{ns}}(\mathbb{F}_v) \}, \ E_1(F_v) = \{ P \in E(F_v) : \tilde{P} = O \}.$$  

Finally, let $c_v(E) = |E(F_v)/E_0(F_v)|$ be the local Tamagawa factor of $E$ at $v$ and $L_v(E, s)$ the Euler factor at $v$ of the Hasse-Weil $L$-function $L(E, s)$.

**Proposition 3.1.** The group $H^1(F_v, E[\ell^\infty])$ is finite and has order $\left| \frac{L_v(E, 1)}{c_v(E)} \right|_{\ell}$ (where $| \cdot |_{\ell}$ denotes the normalized $\ell$-adic absolute value, i.e., with $|\ell|_\ell = \ell^{-1}$). Moreover, if $v \in S$ is of additive reduction and $\ell \geq 5$, then $|H^1(F_v, E[\ell^\infty])| = 1$ (in particular $|\text{Ker}(\gamma_v)| = 1$ for those primes).

**Proof.** From [9] Remark I.3.6 we have the following isomorphisms:

$$E(F_v)^* \cong H^1(F_v, E[\ell^\infty])^\vee,$$

where $E(F_v)^* \cong \varprojlim_n E(F_v)/\ell^nE(F_v)$. Consider the exact sequence

$$E_1(F_v) \hookrightarrow E_0(F_v) \twoheadrightarrow \tilde{E}_{v,\text{ns}}(\mathbb{F}_v).$$

Taking inverse limits of appropriate quotients we obtain

$$E_1(F_v)^* \hookrightarrow E_0(F_v)^* \twoheadrightarrow (\tilde{E}_{v,\text{ns}}(\mathbb{F}_v))^* = \tilde{E}_{v,\text{ns}}(\mathbb{F}_v)[\ell^\infty],$$

because $\tilde{E}_{v,\text{ns}}(\mathbb{F}_v)$ is finite. Since $E_1(F_v)$ has no points of order $\ell$ ([13] Proposition VII.3.1)), the first term in the previous sequence is trivial and

$$E_0(F_v)^* \cong (\tilde{E}_{v,\text{ns}}(\mathbb{F}_v))^*.$$

By the exact sequence

$$E_0(F_v)^* \hookrightarrow E(F_v)^* \twoheadrightarrow (E(F_v)/E_0(F_v))^*,$$

we deduce that the order of $E(F_v)^*$ is the exact power of $\ell$ dividing the factor $c_v(E)|\tilde{E}_{v,\text{ns}}(\mathbb{F}_v)|$. By [14] Appendix C

$$|\tilde{E}_{v,\text{ns}}(\mathbb{F}_v)| = |\mathbb{F}_v|L_v(E, 1)^{-1}.$$ 

Since $|\mathbb{F}_v|$ is a power of $p$ the first claim follows.

If $v \in S$ is of additive reduction, by [14] VII, Theorem 6.1 we have that $E(F_v)/E_0(F_v)$
has order at most 4 and (by [11] VII, Proposition 5.1 (c)) \( \tilde{E}_{v,n,s}(F_v) \) is a \( p \)-group. Thus, if \( \ell \geq 5 \), then \( \ell \) does not divide \( c_v[\tilde{E}_{v,n,s}(F_v)[\ell^\infty]] \) and it follows that

\[ H^1(F_v, E[\ell^\infty]) = 1. \]

**Theorem 3.2.** With hypotheses as in Theorem 2.3 if \( \ell \geq 5 \) we have

\[
\chi(G, Sel_E(K)_\ell) = \chi(G, E(K)[\ell^\infty]) \frac{\prod \left| \frac{\ell}{E(F)[\ell^\infty]} \right| \prod_{\nu \in S} \frac{|E(F_\nu)[\ell^\infty]|}{\chi(G, E(K)[\ell^\infty])}}{\prod_{\nu \in S} \frac{|E(F_\nu)[\ell^\infty]|}{\chi(G, E(K)[\ell^\infty])}}.
\]

(10)

**Proof.** Just adjust the formula of Theorem 2.5 using Proposition 3.1 and recall that an elliptic curve is self dual, i.e., \( E \equiv E \).

**Example 3.** Suppose that \( G \cong \mathbb{Z}_\ell \), i.e., \( K/F \) is the arithmetic \( \mathbb{Z}_\ell \)-extension of \( F \) (the only one available here, see [2] Proposition 4.3). Since there is no ramification, all elements of \( S \) are of bad reduction and \( cd_{\ell}(K_\nu) = 0 \) cannot happen. By [3] Theorem 4.2 we know that \( E(K)[\ell^\infty] \) is a finite group, then \( |H^0(G, E(K)[\ell^\infty])| = |H^1(G, E(K)[\ell^\infty])| = |H^i(G, E(K)[\ell^\infty])| = 0 \) for any \( i \geq 2 \). Hence

\[
\chi(G, E(K)[\ell^\infty]) = 1,
\]

and with hypotheses as in Theorem 2.3, we have

\[
\chi(G, Sel_E(K)_\ell) = \frac{\prod \left| \frac{\ell}{E(F)[\ell^\infty]} \right| \prod_{\nu \in S} \frac{|E(F_\nu)[\ell^\infty]|}{\chi(G, E(K)[\ell^\infty])}}{\prod_{\nu \in S} \frac{|E(F_\nu)[\ell^\infty]|}{\chi(G, E(K)[\ell^\infty])}}.
\]

**Example 4.** Let \( F \) be of characteristic \( p > 3 \); consider \( K = F(E[\ell^\infty]) \) with \( \ell \geq 5 \) and assume \( Sel_E(F)_\ell \) is finite. In this setting the only primes in \( S \) are the ones of bad reduction. Moreover, using the Tate parametrization it is not hard to check that primes in \( S \) such that \( cd_{\ell}(K_\nu) = 0 \) are exactly the split multiplicative reduction places and those in \( S \) with \( cd_{\ell}(K_\nu) = 1 \) the additive reduction ones. Moreover

\[
\chi(G, E(K)[\ell^\infty]) = 1
\]

because of [8] Theorem 1. Since in this case \( \psi_K \) is surjective (see [11] Theorem III.27), one has

\[
\chi(G, Sel_E(K)_\ell) = \frac{\prod \left| \frac{\ell}{E(F)[\ell^\infty]} \right| \prod_{\nu \in S} \frac{|E(F_\nu)[\ell^\infty]|}{\chi(G, E(K)[\ell^\infty])}}{\prod_{\nu \in S} \frac{|E(F_\nu)[\ell^\infty]|}{\chi(G, E(K)[\ell^\infty])}}.
\]

For more details on this case see the (unpublished) thesis [11].

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