BEREZIN SYMBOLS ON LIE GROUPS

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Abstract. In this paper we present a general framework for Berezin covariant symbols, and we discuss a few basic properties of the corresponding symbol map, with emphasis on its injectivity in connection with some problems in representation theory of nilpotent Lie groups.

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1. Introduction

Let \( V \) be a finite-dimensional complex Hilbert space and \( N \) be a second countable smooth manifold with a fixed Radon measure \( \mu \). We denote by \( L^2(N; V; \mu) \) the complex Hilbert space of (equivalence classes of) \( V \)-valued functions \( \mu \)-measurable on \( N \) that are absolutely square integrable with respect to \( \mu \). We also endow the space of smooth functions \( C^\infty(N; V) \) with the Fréchet topology of uniform convergence on compact sets together with their derivatives of arbitrarily high degree.

If \( \mathcal{H} \subseteq L^2(N, V) \) is a closed linear subspace with \( \mathcal{H} \subseteq C^\infty(N, V) \), then the inclusion map \( \mathcal{H} \hookrightarrow C^\infty(N, V) \) is continuous, hence for every \( x \in \mathcal{H} \) the evaluation map \( K_x : \mathcal{H} \rightarrow V, f \mapsto f(x) \), is continuous. The map

\[
K : N \times N \rightarrow B(V), \quad K(x, y) := K_x K_y^*
\]

is called the reproducing kernel of the Hilbert space \( \mathcal{H} \). Then for every linear operator \( A \in B(\mathcal{H}) \) we define its full symbol as

\[
K^A : N \times N \rightarrow B(V), \quad K^A(x, y) := K_x A K_y^* : V \rightarrow V
\]

and \( K^A \in C^\infty(N \times N, B(V)) \). See [Ne00, §1.2] for a detailed discussion of this construction, which goes back to [Be74] and [Be75].

Main problem. In the above setting, the full symbol map

\[
B(\mathcal{H}) \rightarrow C^\infty(N \times N, B(V)), \quad A \mapsto K^A
\]

is injective, as easily checked (see also Proposition 2.12 below). Therefore it is interesting to find sufficient conditions on a continuous map \( \iota : \Gamma \rightarrow N \times N \), ensuring that the corresponding \( \iota \)-restricted symbol map

\[
S^\iota : B(\mathcal{H}) \rightarrow C(\Gamma, B(V)), \quad A \mapsto K^A \circ \iota
\]

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is still injective. The case of the diagonal embedding \( \iota: \Gamma = N \cong N \times N, x \mapsto (x, x) \), is particularly important and in this case the \( \iota \)-restricted symbol map is called the (non-normalized) Berezin covariant symbol map and is denoted simply by \( S \), hence

\[
S: B(\mathcal{H}) \to C^\infty(N, B(\mathcal{V})), \quad (S(A))(x) := K_x A K_x^* : \mathcal{V} \to \mathcal{V}.
\]

In the present paper we will discuss the above problem and we will briefly sketch an approach to that problem based on results from our forthcoming paper \[\text{BBC16}\]. This approach blends some techniques of reproducing kernels and some basic ideas of linear partial differential equations, in order to address a problem motivated by representation theory of Lie groups (see \[\text{C09}, \text{C10}, \text{C13}, \text{C14}\]). This problem is also related to some representations of infinite-dimensional Lie groups that occur in the study of magnetic fields (see \[\text{BB11}a\] and \[\text{BB12}\]). Let us also mention that linear differential operators associated to reproducing kernels have been earlier used in the literature (see for instance \[\text{BG14}\]).

2. Basic properties of the Berezin covariant symbol map

In the following we denote by \( \mathfrak{S}_p(\bullet) \) the Schatten ideals of compact operators on Hilbert spaces for \( 1 \leq p < \infty \).

**Proposition 2.1.** In the above setting, if \( A \in B(\mathcal{H}) \), then one has:

1. If \( A \geq 0 \), then \( S(A) \geq 0 \), and moreover \( S(A) = 0 \) if and only if \( A = 0 \).
2. For all \( f \in \mathcal{H} \) and \( x \in N \) one has

\[
(Af)(x) = \int_N K^A(x, y) f(y) d\mu(y).
\]

3. If \( \{e_j\}_{j \in J} \) is an orthonormal basis of \( \mathcal{H} \), then for all \( x, y \in N \) one has

\[
K^A(x, y) = \sum_{j \in J} K_x e_j \otimes K_y A^* e_{j} = \sum_{j \in J} e_j(x) \otimes (A^* e_{j})(y) \in B(\mathcal{V}),
\]

where for any \( v, w \in \mathcal{V} \) we define their corresponding rank-one operator

\[
v \otimes w := (\cdot | w)v \in B(\mathcal{V}).
\]

4. If \( A \in \mathfrak{S}_2(\mathcal{H}) \), then

\[
\|A\|_{\mathfrak{S}_2(\mathcal{H})}^2 = \int_{N \times N} \|K^A(x, y)\|_{\mathfrak{S}_2(\mathcal{V})}^2 d\mu(x) d\mu(y)
\]

and if \( A \in \mathfrak{S}_1(\mathcal{H}) \), then

\[
\text{Tr } A = \int_N \text{Tr } K^A(x, x) d\mu(x).
\]

**Proof.** See \[\text{BBC16}\] for more general versions of these assertions, in which in particular the Hilbert space \( \mathcal{V} \) is infinite-dimensional. Assertion (2) is a generalization of \[\text{Ne00} \text{ Ex. I.2.3(c)}\], Assertion (3) is a generalization of \[\text{Ne00} \text{ Prop. I.1.8(b)}\], while Assertion (4) is a generalization of \[\text{Ne00} \text{ Cor. A.1.12}\]. \( \square \)
3. Examples of Berezin symbols and specific applications

Here we specialize to the following setting:

1. \( G \) is a connected, simply connected, nilpotent Lie group with its Lie algebra \( \mathfrak{g} \), whose center is denoted by \( \mathfrak{z} \), and \( \mathfrak{g}^* \) is the linear dual space of \( \mathfrak{g} \), with the corresponding duality pairing \( \langle \cdot, \cdot \rangle : \mathfrak{g}^* \times \mathfrak{g} \to \mathbb{R} \).
2. \( \pi : G \to \mathcal{B}(\mathcal{H}) \) be a unitary irreducible representation associated with the coadjoint orbit \( O \subseteq \mathfrak{g}^* \).

The group \( G \) will be identified with \( \mathfrak{g} \) via the exponential map, so that \( G = (\mathfrak{g}, \cdot_G) \), where \( \cdot_G \) is the Baker-Campbell-Hausdorff multiplication.

We use the notation \( \mathcal{H}_\infty = \mathcal{H}_\infty(\pi) \) for the nuclear Fréchet space of smooth vectors of \( \pi \). Let then \( \mathcal{H}_{-\infty} \) be the space of antilinear continuous functionals on \( \mathcal{H}_\infty \). \( \mathcal{B}(\mathcal{H}_\infty, \mathcal{H}_{-\infty}) \) be the space of continuous linear operators between the above space (these operators are thought of as possibly unbounded linear operators in \( \mathcal{H} \)), and \( S(\bullet) \) and \( S'(\bullet) \) for the spaces of Schwartz functions and tempered distributions, respectively. Then we have that

\[ \mathcal{H}_\infty \hookrightarrow \mathcal{H} \hookrightarrow \mathcal{H}_{-\infty}. \]

Let \( X_1, \ldots, X_m \) be a Jordan-Hölder basis in \( \mathfrak{g} \) and \( e \subseteq \{1, \ldots, m\} \) be the set of jump indices of the coadjoint orbit \( O \). Select \( \xi_0 \in O \) and let \( \mathfrak{g} = \mathfrak{g}_{\xi_0} + \mathfrak{g}_e \) be its corresponding direct sum decomposition, where \( \mathfrak{g}_e \) is the linear span of \( \{X_j \mid j \in e\} \) and \( \mathfrak{g}_{\xi_0} := \{x \in \mathfrak{g} \mid [x, \mathfrak{g}] \subseteq \text{Ker} \xi_0\} \).

We need the notation for the Fourier transform. For \( a \in S(O) \) we set

\[ \hat{a}(x) = \int_{O} e^{-i\langle \xi, x \rangle} a(\xi) d\xi, \]

where on \( O \) we consider the Liouville measure normalized such that the Fourier transform is unitary when extended to \( L^2(O) \to L^2(\mathfrak{g}_e) \). We denote by \( \hat{F} \) the inverse Fourier transform of \( F \in L^2(\mathfrak{g}_{\xi_0}). \)

**Definition 3.1.**

1. For \( f \in \mathcal{H} \) and \( \phi \in \mathcal{H} \), or \( f \in \mathcal{H}_{-\infty} \) and \( \phi \in \mathcal{H}_\infty \), let \( \mathcal{A} \in C(\mathfrak{g}_e) \cap S'(\mathfrak{g}_e) \) be the coefficient mapping for \( \pi \), defined by

\[ \mathcal{A}_\phi f(x) = \mathcal{A}(f, \phi)(x) := (f \mid \pi(x)\phi), \ x \in \mathfrak{g}_e. \]

2. For \( f \in \mathcal{H} \) and \( \phi \in \mathcal{H} \), or \( f \in \mathcal{H}_{-\infty} \) and \( \phi \in \mathcal{H}_\infty \), the cross-Wigner distribution \( \mathcal{W}(f, \phi) \in S'(O) \) is defined by the formula

\[ \mathcal{W}(f, \phi) = \mathcal{A}_\phi f. \]

**Proposition 3.2.** For \( f, \phi \in \mathcal{H} \) we have that \( \mathcal{A}(f, \phi) \in L^2(\mathfrak{g}_{\xi_0}), \mathcal{W}(f, \phi) \in L^2(O) \). Moreover

\[ (\mathcal{A}(f_1, \phi_1) \mid \mathcal{A}(f_2, \phi_2))_{L^2(\mathfrak{g}_{\xi_0})} = (f_1 \mid f_2)(\phi_1 \mid \phi_2), \]

\[ (\mathcal{W}(f_1, \phi_1) \mid \mathcal{W}(f_2, \phi_2))_{L^2(O)} = (f_1 \mid f_2)(\phi_1 \mid \phi_2) \]

for all \( f_1, f_2, \phi_1, \phi_2 \in \mathcal{H} \).

**Proof:** This follows from \[\text{[Berezin 1987]}\text{ Prop. 2.8(i)}.\]

From now on we assume that

\[ \phi \in \mathcal{H}_\infty \text{ with } \|\phi\| = 1 \text{ is fixed.} \]
We let $V : \mathcal{H} \to L^2(\mathfrak{g}_e)$ be the isometry defined by
$$(V f)(x) := (f \mid \phi_x) \text{ for all } x \in \mathfrak{g}_e$$
where $\phi_x := \pi(x)\phi$. We denote
$$\mathcal{K} := \text{Ran } V \subset L^2(\mathfrak{g}_0).$$
Then $\mathcal{K}$ is a reproducing kernel Hilbert space of smooth functions, with inner product equal to the $L^2(\mathfrak{g}_0)$-inner product, so the present construction is a special instance of the general framework of Section III with $\mathcal{V} = \mathbb{C}$.

The reproducing kernel of $\mathcal{K}$ is given by
$$K(x, y) = (\pi(x)\phi \mid \pi(y)\phi) = (\phi_x \mid \phi_y),$$
and $K_y(\cdot) := K(\cdot, y) \in \text{Ran } V$, for all $y \in \mathfrak{g}_0$. We also note that
$$(\forall x \in \mathfrak{g}_0) \quad K_x = V\phi_x.$$

The Berezin covariant symbol of an operator $T \in \mathcal{B}(\mathcal{K})$ is then the bounded continuous function
$$S(T) : \mathfrak{g}_e \to \mathbb{C}, \quad S(T)(x) = (TK_x \mid K_x)_{\mathcal{K}}.$$ One thus obtains a well-defined bounded linear operator
$$S : \mathcal{B}(\mathcal{K}) \to C^\infty(\mathfrak{g}_e) \cap L^\infty(\mathfrak{g}_e)$$
which also gives by restriction a bounded linear operator
$$S : \mathcal{S}(\mathcal{K}) \to L^2(\mathfrak{g}_0).$$

To find accurate descriptions of the kernels of the above operators is a very important problem for many reasons, as explained in [C09], [C10], [C13], and [C14] also for other classes of Lie groups than the nilpotent ones.

The case of flat coadjoint orbits of nilpotent Lie groups. We now assume that the coadjoint orbit $\mathcal{O}$ is flat, hence its corresponding representation $\pi$ is square integrable modulo the center of $G$.

Remark 3.3. Consider the representation $\rho : G \to \mathcal{B}(\mathcal{K})$,
$$\rho(g) = V\pi(g)V^*,$$
that is a unitary representation of $G$ equivalent to $\pi$, thus it corresponds to the same coadjoint orbit $\mathcal{O}$. We denote by $\text{Op}_\rho$ the Weyl calculus corresponding to this representation. The following then holds:

(1) For $a \in S'(\mathcal{O})$ one has $\text{Op}_\rho(a) = V\text{Op}(a)V^* = T_a$.
(2) For $T \in \mathcal{B}(\mathcal{K})$ and $X \in \mathfrak{g}_0$, one has
$$S(\rho(x)^{-1}T\rho(x))(z) = S(T)(x \cdot z), \quad \text{for all } z \in \mathfrak{g}_0. \quad (3.1)$$

Theorem 3.4. Assume that in the constructions above,
$$\phi \in \mathcal{H}_\infty \quad \text{is such that } \mathcal{W}(\phi, \phi) \text{ is a cyclic vector for } \alpha. \quad (3.2)$$
Then $S : \mathcal{S}(\mathcal{K}) \to L^2(\mathfrak{g}_0)$ is injective.

Proof. The method of proof is based on specific properties of the Weyl-Pedersen calculus from [BB11b]. □

We refer to [BBC16] for a more complete discussion and for proofs of the above assertions in a much more general setting. To conclude this paper we will just briefly discuss an important example.
The special case of the Heisenberg groups. Let $G$ be the Heisenberg group of dimension $2n + 1$ and $H$ be the center of $G$. Let \( \{X_1, \ldots, X_n, Y_1, \ldots, Y_n, Z\} \) be a basis of $\mathfrak{g}$ in which the only non trivial brackets are \( [X_k, Y_k] = Z, 1 \leq k \leq n \) and let \( \{X^*_1, \ldots, X^*_n, Y^*_1, \ldots, Y^*_n, Z^*\} \) be the corresponding dual basis of $\mathfrak{g}^*$.

For $a = (a_1, a_2, \ldots, a_n) \in \mathbb{R}^n$, $b = (b_1, b_2, \ldots, b_n) \in \mathbb{R}^n$ and $c \in \mathbb{R}$, we denote by \([a, b, c]\) the element $\exp_G(\sum_{k=1}^n a_k X_k + \sum_{k=1}^n b_k Y_k + cZ)$ of $G$. Then the multiplication of $G$ is given by

\[
[a, b, c][a', b', c'] = [a + a', b + b', c + c' + \frac{1}{2}(ab' - a'b)]
\]

and $H$ consists of all elements of the form $[0, 0, c]$ with $c \in \mathbb{R}$.

The coadjoint action of $G$ is then given by

\[
\text{Ad}^*([a, b, c])(\sum_{k=1}^n \alpha_k X^*_k + \sum_{k=1}^n \beta_k Y^*_k + \gamma Z^*) = \sum_{k=1}^n (\alpha_k + \gamma b_k) X^*_k + \sum_{k=1}^n (\beta_k - \gamma a_k) Y^*_k + \gamma Z^*.
\]

Fix a real number $\lambda > 0$. By the Stone-von Neumann theorem, there exists a unique (up to unitary equivalence) unitary irreducible representation $\pi_0$ of $G$ whose restriction to $H$ is the character $\chi : [0, 0, c] \rightarrow e^{i\lambda c}$. This representation is realized on $\mathcal{H}_0 = L^2(\mathbb{R}^n)$ as

\[
\pi_0([a, b, c])(f)(x) = e^{i\lambda(c - bx + \frac{1}{2}ab)}f(x - a).
\]

Here we take $\phi$ to be the function $\phi(x) = (\frac{\lambda}{2})^{1/4} e^{-\lambda x^2/2}$. Then we have $\|\phi\|_2 = 1$.

Theorem 3.4 gives a new proof of the following known fact:

**Corollary 3.5.** The map $S$ is a bounded linear operator from $\mathcal{S}(\mathcal{H}_0)$ to $L^2(\mathbb{R}^{2n})$ which is one-to-one and has dense range.

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