Patterson-Sullivan theory for groups with a strongly contracting element

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Abstract

Using Patterson-Sullivan measures we investigate growth problems for groups acting on a metric space with a strongly contracting element.

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1 Introduction

Let $G$ be a group acting properly by isometries on a proper geodesic space $(X,d)$. Its exponential growth rate measures the size of its orbits and is defined as

$$\omega(G,X) = \limsup_{\ell \to \infty} \frac{1}{\ell} |\{ g \in G : d(o,go) \leq \ell \}|.$$

This number does depend on the space $X$. Nevertheless, if the context is clear, we simply write $\omega_G$ instead of $\omega(G,X)$. It is also the critical exponent of the Poincaré series of $G$ defined by

$$P_G(s) = \sum_{g \in G} e^{-sd(o,go)};$$

that is $P_G(s)$ diverges (respectively converges) whenever $s < \omega_G$ (respectively $s > \omega_G$). If $G$ is the fundamental group of a hyperbolic manifold $M$ acting on the universal cover $X = \tilde{M}$, then $\omega_G$ has numerous interpretations: it is the entropy of the geodesic flow, the Hausdorff dimension of the radial limit set of $G$, etc. In this context, the exponential growth rate is a central object connecting geometry, group theory, dynamical systems, etc.

Growth spectrum. In this article we are interested in the (normal) subgroup growth spectrum of $G$, i.e. the set

$$\text{Spec}(G,X) = \{ \omega(N,X) : N \triangleleft G \}.$$

Note that Spec$(G,X)$ is contained in $[0,\omega_G]$. In particular, $\omega_N = 0$ (respectively $\omega_N = \omega_G$) if $N$ is finite (respectively has finite index in $G$). A natural question, which has received much attention, is to understand more precisely the extremal values of this set. This problem is rather well understood if $G$ is a group acting properly, co-compactly by isometries on a hyperbolic metric space $X$. For instance, any infinite normal subgroup $N \triangleleft G$ satisfies

$$\frac{1}{2} \omega(G,X) < \omega(N,X) \leq \omega(G,X).$$

see Matsuzaki-Yabuki-Jaerisch [34]. Moreover, the second inequality is an equality if and only if $G/N$ is amenable. Similarly, if $G$ has Kazhdan property (T), then $\omega_N$ cannot be arbitrarily close to $\omega_G$, unless $N$ has finite index in $G$. See Coulon-Dougall-Schapira-Tapie [16] and the references therein.
Main results. The goal of this article is to investigate the extremal values of the subgroup growth spectrum in the context of group actions admitting a contracting element. Some of our results refine existing statements in the literature. In particular, we answer most of the questions raised by Arzhantseva and Cashen in [2]. Our main contribution though is the method that we use: we extend to this context the construction of Patterson-Sullivan measures (see below).

When it comes to counting problems, the behavior of the Poincaré series of $G$ at the critical exponent plays a major role. This motivates the following definition. The action of $G$ on $X$ is divergent (respectively convergent) if the Poincaré series $P_G(s)$ of $G$ diverges (respectively converges) at $s = \omega_G$. Our first statement deals with the bottom of the subgroup growth spectrum.

**Theorem 1.1** (see Corollary 4.29 and Proposition 5.24). Let $X$ be a proper geodesic metric space. Let $G$ be a group acting properly, by isometries on $X$ with a contracting element. Let $N$ be an infinite normal subgroup of $G$. Then

$$\omega(N, X) + \frac{1}{2} \omega(G/N, X/N) \geq \omega(G, X).$$

Assume in addition that $G$ is not virtually cyclic and the action of $G$ is divergent. Then

$$\omega(N, X) > \frac{1}{2} \omega(G, X).$$

Remark. The first inequality was proved by Matsuzaki and Jaerisch when $G$ is a finitely generated free group acting on its Cayley graph with respect to a free basis [26]. Their method involves fine estimates of the Cheeger constant and the spectral radius of the random walk in $G/N$. To the best of our knowledge this result is new, even if $G$ is a hyperbolic group.

The second inequality is well known in the context of hyperbolic spaces, see Roblin [40] and Matsuzaki-Yabuki-Jaerisch [34]. For groups acting with a contracting element, it was proved by Arzhantseva and Cashen under the stronger assumption that $G$ has pure exponential growth, that is when the map

$$\ell \mapsto \{g \in G : d(o, go) \leq \ell \} e^{-\omega_G \ell}$$

is bounded from above and away from zero [2]. Note that even if $X$ is Gromov hyperbolic, there are groups $G$ acting on $X$, which are divergent but do not have pure exponential growth.

The next two results focus on the top of the subgroup growth spectrum. Let $H$ be a subgroup of $G$. Denote by $Y = G/H$ the set of left cosets of $H$. The left action of $G$ on $Y$ induces an action of $G$ on $\ell^\infty(Y)$. The subgroup $H$ is co-amenable in $G$ if there exists a $G$-invariant mean $\ell^\infty(Y) \to \mathbb{R}$. If $N$ is a normal subgroup, then $N$ is co-amenable in $G$ if and only if $G/N$ is amenable.

**Theorem 1.2** (see Corollary 4.27). Let $X$ be a proper, geodesic, metric space. Let $G$ be a group acting properly, by isometries on $X$ with a contracting element. Let $H$ be a subgroup of $G$. If $H$ is co-amenable in $G$, then $\omega(H, X) = \omega(G, X)$. 

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Remark. This type of results has a long history. Assume that $G$ is the fundamental group of a compact hyperbolic manifold and $X$ the universal cover of $M$. Let $N$ be a normal subgroup of $G$. Brooks proved that $\omega_N = \omega_G$ if and only if $G/N$ is amenable \cite{Brooks} – Brooks’ result is actually stated in terms of the bottom spectra of certain Laplace’s operators, but they can be related to the growth rates of the groups via Sullivan’s formula \cite{Sullivan}. A similar statement was obtained independently by Grigorchuk and Cohen when $G$ is a free group acting on its Cayley graph $X$ with respect to a free basis \cite{Grigorchuk, Cohen}. The “easy direction” stated above was generalized by Roblin to the settings of CAT(-1) spaces \cite{Roblin}. However those results assume that $H$ is a normal subgroup of $G$. If $H$ is not normal, the proofs existing in the literature require that $X$ is hyperbolic and the action of $G$ is either co-compact or strongly positively recurrent, see for instance \cite{Matsuzaki, Yabuki}.

Recall that a subgroup $H$ of $G$ is commensurated, if for every $g \in G$, the intersection $gHg^{-1} \cap H$ has finite index in $H$. The class of commensurated subgroups contains all normal subgroups of $G$ and is stable by taking finite index subgroups.

Theorem 1.3 (see Theorem 5.26). Let $X$ be a proper, geodesic, metric space. Let $G$ be a group acting properly, by isometries on $X$ with a contracting element. Let $H$ be a commensurated subgroup of $G$. If the action of $H$ on $X$ is divergent, then $\omega(H, X) = \omega(G, X)$ and the action of $G$ on $X$ is divergent.

Remark. To the best of our knowledge the statement in the literature only cover the case where $H$ is normal. With this stronger assumption, it was proved by Matsuzaki and Yabuki, if $G$ is a kleinian group, and generalized by Matsuzaki, Yabuki and Jaerisch when $X$ is Gromov hyperbolic \cite{MatsuzakiYabuki, YabukiJaerisch}.

Patterson-Sullivan theory. Assume that $G$ is the fundamental group of a closed riemannian manifold $M$ with negative sectional curvature. In this context dynamical systems – first and foremost the study of the geodesic flow on the unit tangent bundle of $M$ – provide efficient tools to tackle counting problems. For instance, using the dynamics of the geodesic flow, Margulis proved that the number $c(\ell)$ of simple closed geodesics on $M$ of length at most $\ell$ behaves like

$$c(\ell) \sim \frac{e^{\omega_G \ell}}{\omega_G \ell}.$$  

See \cite{Margulis}. Fix a base point $o \in X$. Denote by $X = \tilde{M}$ the universal cover of $M$ and $\partial X$ its visual boundary. In this topic, measures on the boundary play a prominent role. Recall that a $G$-invariant, $\omega_G$-conformal density is a collection $\nu = (\nu_x)_{x \in X}$ of non-zero finite measures on $X \cup \partial X$, all in the same measure class, satisfying the following properties: for all $g \in G$, for all $x, y \in X$, we have

- $g_*\nu_x = \nu_{gx}$ (invariance),
- $\frac{d\nu_x}{d\nu_y}(\xi) = e^{-\omega_G d(x,y)} \nu$-almost everywhere (conformality),
where $b_\xi$ stands for the Buseman cocycle at $\xi \in \partial X$. In particular $\omega_G$ can be interpreted as the dimension of the measure $\nu_o$. Patterson’s construction provides examples of such densities which are supported on $\partial X$. These measures are designed so that the action of $G$ on $(\partial X, \nu_o)$ captures many properties of the geodesic flow on $M$. The theory can be generalized for groups acting on a Gromov hyperbolic space, see for instance Coornaert [11] and Bader-Furman [3]. For such groups, Theorem 1.1, 1.2 and 1.3 can be proved using invariant conformal densities.

In the past years, growth problems in groups with a contracting element have been investigated by various people, see for instance [47, 1, 18, 48, 49, 2, 30]. Since no Patterson-Sullivan theory existed in this context, each time the authors developed ad hoc methods. Actually they often make a point of avoiding “fairly sophisticated” results about Patterson-Sullivan “machinery”. We adopt here an opposite point of view. For us, these results witnessed the fact that a Patterson-Sullivan theory should exist. Building this “missing” theory is the purpose of this work. If the ambient space $X$ is CAT(0) this task has been achieved by Link [31] extending the work of Knieper [27, 28]. Our approach does not require any CAT(0) assumption though. Our goal is to stress that this construction is particularly robust and requires very little hypotheses, beside the existence of a contracting element. Once the basic properties of invariant conformal densities have been established, they provide a unified framework for solving various growth problems. We believed that these tool can be used for many other applications inspired by non positive curvature.

**Strategy.** We would like to understand the behavior of certain densities supported on the “boundary at infinity” of $X$. Thus, the first task is to build an appropriate compactification of $X$ to carry these measures. There have been many attempts to build an analogue of the Gromov boundary for groups with a contracting element: the contracting and Morse boundaries [9, 36, 14], the sublinearly Morse boundary [38, 39], etc. However these boundaries are sometimes “too small” (for instance the Morse boundary cannot be used as a topological model of the Poisson boundary) and often not compact. This can be a difficulty to build Patterson-Sullivan measures. Instead, we choose to work with the horocompactification. In short, it is the “smallest” compactification $\bar{X}$ of $X$ such that the map

$$X \times X \times X \to \mathbb{R}$$

$$(x, y, z) \mapsto d(x, z) - d(y, z)$$

extends continuously to a map $X \times X \times \bar{X} \to \mathbb{R}$. The horoboundary of $X$ is $\partial X = \bar{X} \setminus X$. A point in the horoboundary is a cocycle $c: X \times X \to \mathbb{R}$, playing the role of a Buseman cocycle. Hence this choice is natural to give a rigorous sense to conformal densities. If $X$ is CAT(0), then the horoboundary coincides with the visual boundary. In general, this boundary is slightly too large though for invariant conformal densities to behave as expected. Let us illustrate this fact with an example.
Example. Consider a group $G$ acting properly, co-compactly, by isometries on a CAT(0) space $X_0$. Build a new space $X = X_0 \times [0,1]$ endowed with the $L^1$-metric. Let $G$ act trivially on $[0,1]$ and consider the diagonal action of $G$ on $X$. This action is still proper and co-compact and $\omega(G, X) = \omega(G_0, X)$. The horoboundary of $X$ is homeomorphic to $\partial X = \partial X_0 \times [0,1]$. To carry the analogy with negatively curved manifold, we would like that if $\mu = (\mu_x)$ is a $G$-invariant, $\omega_G$-conformal density supported on $\partial X$ then the action of $G$ on $(\partial X, \mu_o)$ is ergodic. However, in this example we can choose a $G$-invariant, $\omega_G$-conformal density $\nu = (\nu_x)$ on $\partial X_0$ and form the average $\mu = (\nu^0 + \nu^i)/2$, where $\nu^i$ is a copy of $\nu$ supported on $\partial X_0 \times \{i\}$. Then the action of $G$ on $(\partial X, \mu_o)$ is not ergodic.

This issue already arises if $X$ is Gromov hyperbolic. In this context, it can be fixed by passing to the reduced horoboundary. Endow the horoboundary $\partial X$ with the equivalence relation $\sim$ defined as follows: two cocycles $c,c' \in \partial X$ are equivalent, if $\|c - c'\|_\infty < \infty$. The reduced horoboundary is the quotient $\partial X/\sim$. If $X$ is hyperbolic, then it coincides with the Gromov boundary. Moreover the projection $\pi: \partial X \to \partial X/\sim$ is very well understood, see Coornaert-Papadopoulos [13]. Pushing forward in $\partial X/\sim$, the densities built in $\partial X$ provide well behaved measures.

However, in general the reduced horoboundary $\partial X/\sim$ is a rather nasty topological space. For instance, if $X = \mathbb{R}^2$ is endowed with the taxicab metric, then $\partial X/\sim$ is not even Hausdorff. To bypass this difficulty we adopt a measure theoretic point of view. Denote by $\mathfrak{R}$ the $\sigma$-algebra that consists of all Borel sets which are saturated for the equivalence relation $\sim$. We make an abuse of vocabulary and call the measurable space $(\partial X, \mathfrak{R})$ the reduced horoboundary. When restricted to the reduced horoboundary, the invariant, conformal densities are well behaved. For instance, we prove the following partial form of the Hopf-Tsuji-Sullivan dichotomy (we refer the reader to Section 4.5 for the definition of the radial limit set).

**Theorem 1.4** (see Corollary 4.25 and Corollary 5.20). Let $X$ be a proper geodesic metric space and $o \in X$. Let $G$ be a group acting properly, by isometries on $X$ with a contracting element. Suppose that $G$ is not virtually cyclic. Let $\mu = (\mu_x)$ be the restriction to the reduced horoboundary $(\partial X, \mathfrak{R})$ of a $G$-invariant, $\omega_G$-conformal density. The following are equivalent.

(i) The action of $G$ on $X$ is divergent.

(ii) $\mu_o$ gives positive measure to the radial limit set.

(iii) $\mu_o$ gives full measure to the radial limit set.

Remark. In a forthcoming work, we plan to complete the Hopf-Tsuji-Sullivan dichotomy by investigating the ergodicity of the geodesic flow in this context and its consequences for growth problems.

If the action of $G$ on $X$ is divergent, we prove that invariant, conformal densities are ergodic and essentially unique, when restricted to the reduced horoboundary.
Theorem 1.5 (see Proposition 5.23). Let $X$ be a proper, geodesic, metric space and $o \in X$. Let $G$ be a non virtually cyclic group acting properly, by isometries on $X$ with a contracting element. Assume that the action of $G$ on $X$ is divergent. Let $\mu = (\mu_x)$ be the restriction to the reduced horoboundary $(\partial X, \mathcal{R})$ of a $G$-invariant, $\omega_G$-conformal density. Then

(i) $\mu_o$ is ergodic;

(ii) $\mu_o$ is non-atomic;

(iii) $\mu$ is almost unique in the following sense: there is $C \in \mathbb{R}^*_+$, such that if $\mu' = (\mu'_x)$ is the restriction to the reduced horoboundary of another $G$-invariant, $\omega_G$-conformal density, then for every $x \in X$, we have $\mu'_x \leq C \mu_x$.

Finally, we complete Theorem 1.3 as follows.

Theorem 1.6 (see Theorem 5.26). Let $X$ be a proper, geodesic, metric space. Let $G$ be a group acting properly, by isometries on $X$ with a contracting element. Suppose that $G$ is not virtually cyclic. Let $H$ be a commensurated subgroup of $G$. If the action of $H$ on $X$ is divergent, then any $H$-invariant, $\omega_H$-conformal density is $G$-almost invariant when restricted to the reduced horoboundary $(\partial X, \mathcal{R})$.

Remark. Other applications can be found in Sections 4.5 and 5.5. In this article we focused on growth problems. Nevertheless, we believe that the tools we introduced can be used for other purposes, e.g. to generalize the “no proper conjugation” property of divergent subgroups exhibited by Matsuzaki, Yabuki and Jaerisch [34].

Strongly positively recurrent actions. As we mentioned before, divergent actions play an important role in counting problems. Any proper and co-compact action is divergent. In particular, if $G$ acts properly on $X$ with a quasi-convex orbit, then its action is divergent. This framework has been generalized independently by Schapira-Tapie [42] and Yang [48] under the names strongly positively recurrent action (SPR) and statistically convex co-compact action (SCC) respectively – the idea also implicitly appears in the work of Arzhantseva-Cashen-Tao [1]. The notion has an independent dynamical origin as well, see for instance [25, 41]. Roughly speaking the idea is to ask that the elements of $g \in G$ which “violate” the quasi-convexity of $G$ are statistically very rare. It was proved by Yang that such actions are divergent. In Appendix A we provide an alternative proof of this fact in the spirit of Schapira-Tapie [42].

Remark. Since obtaining the results in this article, we have learned that Wenyuan Yang independently investigated conformal measures in the same context [50]. The techniques used by Wenyuan Yang are slightly different. For instance, his proof of Theorem 1.5 (partial form of the Hopf-Tsuji-Sullivan dichotomy) relies on projection complexes introduced by Bestvina, Bromberg, and Fujiwara [4]. In contrast, we tried to use whenever possible low-tech arguments (both of geometric and measure theoretic nature).
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2 Groups with a contracting element

Notations and vocabulary. In this article $(X,d)$ is a proper, geodesic, metric space. For every $x \in X$ and $r \in \mathbb{R}_+$, we denote by $B(x,r)$ the open ball of radius $r$ centered at $x$. Let $Y$ be a closed subset of $X$. Given $x \in X$, a point $y \in Y$ is a (nearest point) projection of $x$ on $Y$ if $d(x,y) = d(x,Y)$. The projection of a subset $Z \subset X$ onto $Y$ is

$$\pi_Y(Z) = \{ y \in Y : y \text{ is the projection of a point } z \in Z \} .$$

Let $I \subset \mathbb{R}$ be a closed interval and $\gamma : I \to X$ a continuous path intersecting $Y$. The entry point and exit point of $\gamma$ in $Y$ are the points $\gamma(t) = \inf \{ s \in I : \gamma(s) \in Y \}$ and $t' = \sup \{ s \in I : \gamma(s) \in Y \}$.

If $I$ is bounded, such points always exist (the subset $Y$ is closed). Given $d \in \mathbb{R}_+$, we denote by $N_d(Y)$ the $d$-neighborhood of $Y$, that is the set of points $x \in X$ such that $d(x,Y) \leq d$. The distance between two subsets $Y, Y'$ of $X$ is

$$d(Y,Y') = \inf_{(y,y') \in Y \times Y'} d(y,y').$$

Contracting set.

Definition 2.1 (Contracting set). Let $\alpha \in \mathbb{R}_+$. A closed subset $Y \subset X$ is $\alpha$-contracting if for any geodesic $\gamma$ with $d(\gamma,Y) \geq \alpha$, we have diam $\pi_Y(\gamma) \leq \alpha$. The set $Y$ is contracting if $Y$ is $\alpha$-contracting for some $\alpha \in \mathbb{R}_+$.

The next statements are direct consequences of the definition. Their proofs are left to the reader.

Lemma 2.2. Let $Y$ be an $\alpha$-contracting subset. If $\gamma$ is a geodesic joining two points of $N_\alpha(Y)$, then $\gamma$ lies in the $3\alpha$-neighborhood of $Y$.

Lemma 2.3. Let $Y$ be an $\alpha$-contracting subset. Let $x, z \in X$ and $\gamma$ be a geodesic from $x$ to $z$. Let $p$ and $q$ be respective projections of $x$ and $z$ onto $Y$. If $d(x,Y) < \alpha$ or $d(p,q) > \alpha$, then the following holds:

(i) $d(\gamma,Y) < \alpha$;
(ii) the entry point (respectively exit point) of $\gamma$ in $N_\alpha(Y)$ is $2\alpha$-closed to $p$ (respectively $q$);

(iii) $d(x, z) \geq d(x, p) + d(p, q) + d(q, z) - 8\alpha$.

**Remark 2.4.** It follows from the above statement that the nearest point projection onto $Y$ is large-scale 1-Lipschitz. More precisely, for every subset $Z \subset X$, we have

$$\text{diam}(\pi_Y(Z)) \leq \text{diam}(Z) + 4\alpha.$$  

**Lemma 2.5.** For every $\alpha, d \in \mathbb{R}_+$, there exists $\beta \in \mathbb{R}_+$ with the following property. Let $Y$ and $Z$ be two closed subsets of $X$. Assume that the Hausdorff distance between them is at most $d$. If $Y$ is $\alpha$-contracting, then $Z$ is $\beta$-contracting.

**Contracting element.** Consider now a group $G$ acting properly, by isometries on $X$.

**Definition 2.6 (Contracting element).** Let $y \in X$. An element $g \in G$ is contracting, for its action on $X$, if the orbit map $Z \to X$, sending $n$ to $g^ny$ is a quasi-isometric embedding and its image is contracting.

Note that the definition does not depend on the point $y$. The next statement is a reformulation of Yang’s Lemma 3.3 (and its proof) in [49].

**Lemma 2.7.** Let $h \in G$ be a contracting element. For every $d \in \mathbb{R}_+$ and $z \in X$, there is $\alpha \in \mathbb{R}_+$ with the following property. Let $p, q \in Z$ with $p \leq q$. Let $x, y \in X$ such that $d(x, h^pz) \leq d$ and $d(y, h^qz) \leq d$. Let $\gamma$ be a geodesic joining $x$ to $y$. Then any subpath of $\gamma$ is $\alpha$-contracting. Moreover for every integer $n \in \llbracket p, q \rrbracket$, the point $h^nz$ is $\alpha$-close to $\gamma$.

Let $g \in G$ be a contracting element and $A$ be the $\langle g \rangle$-orbit of a point $y \in X$. Define $E(g)$ as the set of elements $u \in G$ such that the Hausdorff distance between $A$ and $uA$ is finite. It follows from the definition that $E(g)$ is a subgroup of $G$ that does not depend on $y$. It is the maximal virtually cyclic subgroup of $G$ containing $\langle g \rangle$. Moreover $E(g)$ is almost-malnormal, that is $uE(g)u^{-1} \cap E(g)$ is finite, for every $u \in G \setminus E(g)$, see Yang [48, Lemma 2.11].

**Lemma 2.8.** Assume that $G$ is not virtually cyclic and contains a contracting element. Let $H \subset G$ be a commensurated subgroup. If $H$ is infinite, then $H$ is not virtually cyclic and contains a contracting element.

**Proof.** We only sketch the proof. For more details we refer the reader to Arzhantseva-Cashen-Tao [1, Section 3] where similar arguments are given. We first claim that for every contracting element $g \in G$, the group $H$ is not contained in $E(g)$. Assume on the contrary that $H \subset E(g)$. Since $G$ is not virtually cyclic, there is $u \in G \setminus E(g)$. By malnormality, $uHu^{-1} \cap H$ is finite and cannot have finite index in $H$, which contradicts the fact that $H$ is commensurated.
We now fix once and for all a contracting element \( g \in G \). We denote by \( A \) an orbit of \( \langle g \rangle \). It is \( \alpha \)-contracting for some \( \alpha \in \mathbb{R}_+ \). Moreover there is \( C \in \mathbb{R}_+ \) such that for every \( u \in G \setminus E(g) \), we have \( \text{diam}(\pi_A(uA)) \leq C \) see Yang [48, Lemma 2.11]. We choose \( n \in \mathbb{N} \) such that \( d(o, g^na) \) is very large compare to both \( \alpha \) and \( C \).

We claim that there is an element \( h \in H \setminus E(g) \) such that both \( g^n hg^{-n} \) and \( g^{-n} h g^n \) belong to \( H \). Since \( H \) is commensurated, both \( g^{-n} H g^n \cap H \) and \( g^n H g^{-n} \cap H \) have finite index in \( H \). Thus

\[
H_0 = (g^{-n} H g^n) \cap (g^n H g^{-n}) \cap H
\]

has finite index in \( H \). It follows that \( H_0 \) is not contained in \( E(g) \). Indeed otherwise \( E(g) \) should also contain \( H \), contradicting our previous claim. Any element \( h \in H_0 \setminus E(g) \) satisfies the conclusions of our second claim.

Consider now the element

\[
f = (g^n h g^{-n}) (g^{-n} h g^n)^{-1} = g^n (h g^{-2n} h^{-1}) g^n.
\]

Note that \( f \) belongs to \( H \). We claim that \( f \) is contracting. For simplicity we let \( g_0 = g^n \) and \( g_1 = h g^{-2n} h^{-1} \) so that \( f = g_0 g_1 g_0 \). They respectively act “by translation” on the \( \alpha \)-contracting sets \( A_0 = A \) and \( A_1 = hA \). Fix a point \( x \in \pi_A(hA) \). Since \( \pi_A(hA) \) has diameter at most \( C \), any geodesic \( \gamma: [0, T] \to X \) from \( x \) to \( f x \) fellow-travels for a long times with \( A_0 = A \), \( g_0 A_1 = g_0 h A \) and \( g_0 g_1 A_0 = f A \) (see Figure 1). Note that the construction has been designed so

**Figure 1:** The geodesic from \( x \) to \( f x \). The gray shapes are “axis” of conjugates of \( g \).

that the end of \( \gamma \) and the beginning of \( f \gamma \) both fellow-travel with \( f A \) in the “same direction”. Consequently, the concatenation of \( \gamma \) and \( f \gamma \) cannot backtrack much. We extend \( \gamma \) to a bi-infinite \( (f) \)-invariant path, still denoted by \( \gamma: \mathbb{R} \to X \), which is characterized as follows: \( \gamma(t + kT) = f^k \gamma(t) \) for every \( t \in \mathbb{R} \) and \( k \in \mathbb{Z} \). One proves that \( \gamma \) is a quasi-geodesic which fellow-travels for a long time with \( f^k A \), for every \( k \in \mathbb{Z} \) (see Figure 2). This suffices to show that \( f \) is contracting.

**Figure 2:** The “axis” of \( f \).

We are left to prove that \( H \) is not virtually cyclic. If \( H \) was virtually cyclic, it would be contained in \( E(f) \). This contradicts our first claim. \( \square \)
3 Compactification of $X$

3.1 Horocompactification

Let $C(X)$ be the set of all real valued, continuous functions on $X$ endowed with the topology of convergence on every compact subset. We denote by $C^*(X)$ the quotient of $C(X)$ by the subspace consisting of all constant functions, and endowed with the quotient topology. Given a base point $o \in X$, one can think of $C^*(X)$ as the set of continuous functions that vanish at $o$. Alternatively $C^*(X)$ is the set of continuous cocycles $c: X \times X \to \mathbb{R}$. By cocycle we mean that $c(x, z) = c(x, y) + c(y, z), \quad \forall x, y, z \in X.$

For example, given $z \in X$, we define the cocycle $b_z: X \times X \to \mathbb{R}$ by

$$b_z(x, y) = d(x, z) - d(y, z), \quad \forall x, y \in X.$$ 

Since $X$ is geodesic, the map $\iota: X \to C^*(X)$ $z \mapsto b_z$ is a homeomorphism from $X$ onto its image.

**Definition 3.1 (Horoboundary).** The horocompactification $\bar{X}$ of $X$ is the closure of $\iota(X)$ in $C^*(X)$. The horoboundary of $X$ is the set $\partial X = \bar{X} \setminus \iota(X)$.

From now on; we identify $X$ with its image under the map $\iota: X \to C^*(X)$. By construction, every cocycle $c \in \bar{X}$ is 1-Lipschitz, or equivalently $|c(x, x')| \leq d(x, x')$, for every $x, x' \in X$. It is a consequence of the Azerla-Ascoli theorem, that the horocompactification $\bar{X}$ is indeed a compact set. We denote by $\mathcal{B}$ the Borel $\sigma$-algebra on $\bar{X}$. In the remainder of the article we make an abuse of notations, and write $(\partial X, \mathcal{B})$ to denote the horoboundary endowed with the $\sigma$-algebra $\mathcal{B}$ restricted to $\partial X$.

**Definition 3.2.** Let $c \in \bar{X}$ and $\varepsilon \in \mathbb{R}_+$. An $\varepsilon$-quasi-gradient arc for $c$ is a path $\gamma: I \to X$ parametrized by arc length such that

$$(t - s) - \varepsilon \leq c(\gamma(s), \gamma(t)) \leq t - s, \quad \forall s, t \in I.$$ 

A gradient arc for $c$ is a 0-quasi-gradient arc for $c$. If $I = \mathbb{R}_+$, we call $\gamma$ a (quasi-)gradient ray.

**Remark 3.3.** The following observations follow from the definition and/or the triangle inequality.

- Since cocycles in $\bar{X}$ are 1-Lipschitz, a gradient arc is always geodesic.

- Conversely, let $x, y \in X$ and $\varepsilon \in \mathbb{R}_+$ such that $c(x, y) \geq d(x, y) - \varepsilon$. Any geodesic from $x$ to $y$ is an $\varepsilon$-quasi-gradient arc for $c$.
• Let \( \gamma_1: [a_1, b_1] \to X \) and \( \gamma_2: [a_2, b_2] \to X \) be two paths such that \( \gamma_i \) is an \( \varepsilon_i \)-quasi-gradient arc for \( c \). If \( \gamma_1(b_1) = \gamma_2(a_2) \) then the concatenation of \( \gamma_1 \) and \( \gamma_2 \) is an \( (\varepsilon_1 + \varepsilon_2) \)-quasi-gradient arc for \( c \).

The existence of gradient rays is given by the next statement.

**Lemma 3.4.** Let \( c \in \partial X \). For every \( x \in X \), there exists a gradient ray \( \gamma: \mathbb{R}_+ \to X \) for \( c \) such that \( \gamma(0) = x \).

**Proof.** Let \( (z_n) \) be a sequence of points of \( X \) converging to \( c \). For every \( n \in \mathbb{N} \), we let \( b_n = \iota(z_n) \) and denote by \( \gamma_n: [0, \ell_n] \to X \) a geodesic from \( x \) to \( z_n \). Since \( X \) is proper, \( (\gamma_n) \) converges, up to passing to a subsequence, to a geodesic ray \( \gamma: \mathbb{R}_+ \to X \) starting at \( x \). As \( (b_n) \) converges uniformly on every compact subset to \( c \), we check that \( \gamma \) is a gradient ray for \( c \).

Given \( c \in \partial X \) and a gradient ray \( \gamma: \mathbb{R}_+ \to X \) for \( c \), we think of \( \gamma \) as a geodesic from \( \gamma(0) \) to \( c \). The next definition is designed to handle simultaneously the cocycles corresponding to points in \( X \) and the ones in \( \partial X \).

**Definition 3.5.** Let \( c \in \bar{X} \) and \( x \in X \). A complete gradient arc for \( c \) starting at \( x \) is

- any geodesic from \( x \) to \( z \), if \( c = \iota(z) \) for some \( z \in X \),
- any gradient ray for \( c \) starting at \( x \), if \( c \) belongs to \( \partial X \).

### 3.2 Reduced horoboundary

Given a cocycle \( c \in C^*(X) \) we write \( \|c\|_{\infty} \) for its uniform norm, i.e.

\[
\|c\|_{\infty} = \sup_{x, x' \in X} |c(x, x')|.
\]

Note that \( \|c\|_{\infty} \) can be infinite. If \( K \subset X \) is compact, then \( \|c\|_{K} \) is the uniform norm of \( c \) restricted to \( K \times K \). We endow \( \bar{X} \) with a binary relation: two cocycles \( c, c' \in \bar{X} \) are equivalent, and we write \( c \sim c' \), if one of the following holds:

- either \( c \) and \( c' \) lie in the image of \( \iota: X \to C^*(X) \) and \( c = c' \),
- or \( c, c' \in \partial X \) and \( \|c - c'\|_{\infty} < \infty \).

Given a subset \( B \subset \bar{X} \), the saturation of \( B \), denoted by \( B^+ \), is the union of all equivalence classes intersecting \( B \). We say that \( B \) is saturated if it is a union of equivalence classes, or equivalently if \( B^+ = B \). Note that the collection of saturated subsets is closed under complement as well as (uncountable) union and intersection. The reduced algebra, denoted by \( \mathfrak{R} \), is the sub-\( \sigma \)-algebra of \( \mathfrak{B} \) which consists of all saturated Borel subsets.

**Definition 3.6.** The reduced horocompactification and reduced horoboundary of \( X \) are respectively the pairs \( (\bar{X}, \mathfrak{R}) \) and \( (\partial X, \mathfrak{R}) \).
3.3 Boundary at infinity of a contracting set

The goal of this section is to understand how cocycles at infinity interact with contracting subsets of \( X \).

**Definition 3.7.** Let \( Y \) be a closed subset of \( X \). Let \( c \in \bar{X} \). A projection of \( c \) on \( Y \) is a point \( q \in Y \) such that for every \( y \in Y \), we have \( c(q,y) \leq 0 \).

Given \( z \in X \), the projection of \( b = \iota(z) \) on \( Y \) coincides with the definition of the nearest point projection. If \( c \) is a point in \( \partial X \), a projection of \( c \) on \( Y \) may exist or not. This leads to the following definition.

**Definition 3.8.** Let \( Y \) be a closed subset of \( X \). The boundary at infinity of \( Y \), denoted by \( \partial^+ Y \), is the set of all cocycles \( c \in \partial X \) for which there is no projection of \( c \) on \( Y \).

We give several equivalent characterizations of the boundary at infinity of a contracting subset.

**Proposition 3.9.** Let \( \alpha \in \mathbb{R}_+ \). Let \( Y \) be an \( \alpha \)-contracting subset of \( X \). Let \((z_n)\) be a sequence of points in \( X \) which converges to a cocycle \( c \in \partial X \). For every \( n \in \mathbb{N} \), denote by \( q_n \) a projection of \( z_n \) onto \( Y \). Let \( \gamma : \mathbb{R}_+ \rightarrow X \) a gradient ray for \( c \). Define \( T \in \mathbb{R}_+ \cup \{\infty\} \) by

\[
T = \sup \{ t \in \mathbb{R}_+ : d(\gamma(t),Y) \leq \alpha \}
\]

with the convention that \( T = 0 \), whenever \( \gamma \) does not intersect \( N_\alpha(Y) \). The following are equivalent.

(i) \( c \notin \partial^+ Y \).

(ii) For every \( x \in X \), the map \( Y \rightarrow \mathbb{R} \) sending \( y \) to \( c(x,y) \) is bounded from above.

(iii) The ray \( \gamma \) does not stay in a neighborhood of \( Y \).

(iv) The projection \( \pi_Y(\gamma) \) is bounded.

(v) \( T < \infty \).

(vi) The sequence \((q_n)\) is bounded.

Moreover, in this situation,

- the diameter of the set \( Q = \pi_Y(\gamma|_{[T,\infty)}) \) is at most \( \alpha \).

- any accumulation point \( q^* \) of \((q_n)\) is a projection of \( c \) on \( Y \) which lies in the \( \alpha \)-neighborhood of \( Q \).

**Remark.** It follows from (ii) that \( \partial^+ Y \) is saturated (as the notation suggested).
Proof. The equivalences (iii) \(\iff\) (iv) and (iv) \(\iff\) (v) are standard properties of contracting sets, which do not use the fact that \(\gamma\) is a gradient ray. Note that \(c(x, y) = c(x, x') + c(x', y)\), for every \(x, x' \in X\) and \(y \in Y\). The implication (i) \(\imp\) (ii) follows from this observation. The proof of (ii) \(\imp\) (iii) is by contraposition. Suppose that there exists \(d \in \mathbb{R}_+\) such that \(\gamma\) lies \(\mathcal{N}_d(Y)\). Let \(x = \gamma(0)\). Let \(t \in \mathbb{R}_+\). We denote by \(q_t\) a projection of \(\gamma(t)\) onto \(Y\). Using the fact that \(c\) is 1-Lipschitz, we get

\[
 c(x, q_t) \geq c(\gamma(0), \gamma(t)) - d(\gamma(t), q_t) \geq t - d.
\]

This inequality holds for every \(t \in \mathbb{R}_+\), hence the map \(Y \to \mathbb{R}\), sending \(y\) to \(c(x, y)\) is not bounded from above.

We now focus on (vi) \(\imp\) (vi). For every \(n \in \mathbb{N}\), we let \(b_n = s(z_n)\). Assume that \(T < \infty\). Let \(Q\) be the projection onto \(Y\) of \(\gamma\) restricted to \([T, \infty)\). Since \(Y\) is contracting, the diameter of \(Q\) is at most \(\alpha\). Let \(q \in Q\) be a projection of \(\gamma(T)\) on \(Y\). We claim that there is \(N \in \mathbb{N}\), such that for every \(n \geq N\), the point \(q_n\) stays at a distance at most \(\alpha\) of \(Q\). Assume on the contrary that it is not the case. Up to passing to a subsequence, \(d(q_n, Q) > \alpha\), for every \(n \in \mathbb{N}\). Using Lemma 2.3, we observe that for every \(t \geq T, \) for every \(n \in \mathbb{N}\),

\[
 b_n(\gamma(t), q) \geq d(q, \gamma(t)) - 10\alpha.
\]

After passing to the limit we get \(c(\gamma(t), q) \geq d(q, \gamma(t)) - 10\alpha\), for every \(t \geq T\). In particular, \(t \mapsto c(\gamma(t), q)\) diverges to infinity as \(t\) tends to infinity, which contradicts the fact that \(\gamma\) is a gradient line for \(c\). This completes the proof of our claim and thus implies (vi).

We finish the proof with (vi) \(\imp\) (i). Assume now that \((q_n)\) is bounded. Let \(q^*\) be an accumulation point of \((q_n)\). As \(Y\) is closed \(q^*\) belongs to \(Y\). Observe that for every \(y \in Y\), for every \(n \in \mathbb{N}\), we have

\[
 b_n(q^*, y) \leq b_n(q_n, y) + d(q^*, q_n) \leq d(q^*, q_n).
\]

By construction \(b_n\) converges to \(c\) on every compact subset, hence \(c(q^*, y) \leq 0\) for every \(y \in Y\). Thus \(q^*\) is a projection of \(c\) onto \(Y\). In particular, \(c \not\in \partial^+ Y\). □

The next two statements extend Lemma 2.3 for a gradient ray joining a point \(x \in X\) to a cocycle \(c \in \partial X\). We distinguish two cases depending whether \(c\) belongs to \(\partial^+ Y\) or not.

**Corollary 3.10.** Let \(\alpha \in \mathbb{R}_+^*\). Let \(Y\) be an \(\alpha\)-contracting set and \(c \in \bar{X} \setminus \partial^+ Y\). Let \(q\) be a projection of \(c\) on \(Y\). Let \(\gamma\) be a complete gradient arc for \(c\) starting at a point \(x \in X\) and \(p\) a projection of \(x\) on \(Y\). If \(d(x, Y) < \alpha\) or \(d(p, q) > 6\alpha\), then the following holds:

- \(d(\gamma, Y) < \alpha\);
- the entry point (respectively exit point) of \(\gamma\) in \(\mathcal{N}_\alpha(Y)\) is \(2\alpha\)-closed (respectively \(7\alpha\)-closed) to \(p\) (respectively \(q\)).
Lemma 2.3. Assume now that $T_a$ particular case of projection onto $Y$ point of equality.  

Corollary 3.11.  

The conclusion follows from ($\gamma$) where $\gamma$ lies in the discussion above applies. The conclusion follows from ($\gamma$, $Y$). In particular, $\gamma(T)$ is the exit point of $\gamma$ from $\mathcal{N}_\alpha(Y)$. Let $p'$ (respectively $q'$) be a projection of the entry (respectively exit) point of $\gamma$ in $\mathcal{N}_\alpha(Y)$ – note that if $d(x, Y) < \alpha$, then $x$ is the entry point of $\gamma$ in $Y$ so we can choose $p' = p$. Using again the contraction of $Y$, we see that $d(p, p') \leq \alpha$. Moreover $q'$ belongs to $Q$, thus $d(q', q^*) \leq 2\alpha$. In addition, 

$$d(x, \gamma(T)) \geq d(x, p') + d(p', q') + d(q', \gamma(T)) - 4\alpha.$$ 

The path $\gamma$ is a gradient line, thus $c(x, \gamma(T)) = d(x, \gamma(T))$, hence 

$$c(x, \gamma(T)) \geq d(x, p') + d(p', q') + d(q', \gamma(T)) - 4\alpha.$$ 

Combined with the fact that $c$ is 1-Lipschitz, we get 

$$c(x, q') \geq c(x, \gamma(T)) - d(q', \gamma(T)) \geq d(x, p') + d(p', q') - 4\alpha. \quad (1)$$ 

Remark. Note that if $x$ belongs to $Y$, then $x = p = p'$. So we get the following sharper estimate $c(x, q') \geq d(x, q') - 2\alpha$. Recall that $q$ is a projection of $c$ onto $Y$, so that $c(q, y) \leq 0$, for every $y \in Y$. If we apply the above argument with $x = q$, we get $d(q, q') \leq 2\alpha$. However $q'$ belongs to $Q$, hence $d(q, q^*) \leq 4\alpha$.

Suppose now that $d(p, q) \geq 6\alpha$. In particular, $d(p, q^*) \geq 2\alpha$. Hence the discussion above applies. The conclusion follows from (1) and the triangle inequality. 

\begin{proof}

Suppose now that $d(p, q) \geq 6\alpha$. In particular, $d(p, q^*) \geq 2\alpha$. Hence the discussion above applies. The conclusion follows from (1) and the triangle inequality. 

\end{proof}
Proof. According to Proposition 3.9 (iv) the set $\pi_Y(\gamma)$ is unbounded. Thus there exists $s \in \mathbb{R}_+$ and a projection $q$ of $\gamma(s)$ onto $Y$ such that $d(p, q) > \alpha$. It follows that $d(\gamma, Y) < \alpha$. Let $\gamma(t)$ be the entry point of $\gamma$ in $N_\alpha(Y)$. Using again the contraction of $Y$, we get $d(p, \gamma(t)) \leq 2\alpha$. Since $\gamma$ is a gradient line we have $c(x, \gamma(t)) = d(x, \gamma(t))$. Combined with the triangle inequality and the fact that cocycles are $1$-Lipschitz we get $c(x, p) \geq d(x, p) - 4\alpha$.

### Proposition 3.12

Assume that $G$ is not virtually cyclic. Let $g$ be a contracting element and $A$ an orbit of $\langle g \rangle$. There exists $u \in G$ such that $\partial^+ A \cap \partial^+ (uA) = \emptyset$.

Proof. Consider an element $u \in G$ such that $\partial^+ A \cap \partial^+ (uA)$ is not empty. Let $c$ be a cocycle in this intersection and $\gamma : \mathbb{R}_+ \to X$ a gradient ray for $c$. By Proposition 3.9 (iii) there exist $d, T \in \mathbb{R}_+$ such that $\gamma$ restricted to $[T, \infty)$ is contained in $N_d(A) \cap N_d(uA)$. In particular, the diameter of this intersection is infinite. It follows that $u \in E(g)$, see Yang [48, Lemma 2.12]. Recall that $E(g)$, unlike $G$, is virtually cyclic. Thus there exists $u \in G \setminus E(g)$. It follows from the above discussion that $\partial^+ A \cap \partial^+ (uA) = \emptyset$.

## 4 Conformal densities

As previously, $(X, d)$ is a proper, geodesic, metric space, while $G$ is a group acting properly, by isometries on $X$.

### 4.1 Definition and existence

If $\mu$ is a finite measure on $\tilde{X}$, we denote by $\|\mu\|$ its total mass.

**Definition 4.1** (Density). Let $\omega \in \mathbb{R}_+$. Let $\mathfrak{A}$ be a $G$-invariant sub-$\sigma$-algebra of the Borel $\sigma$-algebra $\mathfrak{B}$. A density on $(\tilde{X}, \mathfrak{A})$ is a collection $\nu = (\nu_x)$ of positive finite measures on $(\tilde{X}, \mathfrak{A})$ indexed by $X$ such that $\nu_x \ll \nu_y$, for every $x, y \in X$, and normalized by $\|\nu_o\| = 1$. Such a density is

(i) **$G$-invariant**, if $g_* \nu_x = \nu_{gx}$, for every $g \in G$ and $x \in X$.

(ii) **$\omega$-conformal**, if for every $x, y \in X$,

$$\frac{d\nu_x}{d\nu_y}(c) = e^{-\omega c(x, y)}, \quad \nu_y\text{-a.e.}$$

(iii) **$\omega$-quasi-conformal**, if there is $C \in \mathbb{R}_+^*$ such that for every $x, y \in X$,

$$\frac{1}{C} e^{-\omega c(x, y)} \leq \frac{d\nu_x}{d\nu_y}(c) \leq C e^{-\omega c(x, y)}, \quad \nu_y\text{-a.e.}$$

**Vocabulary.** Let $\nu = (\nu_x)$ be a density on $(\tilde{X}, \mathfrak{A})$. We make an abuse of vocabulary and say that a property on $(\tilde{X}, \mathfrak{A})$ holds $\nu$-almost everywhere if it holds $\nu_x$-almost everywhere for some (hence every) $x \in X$. 

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Remark 4.2. Let \( \nu = (\nu_x) \) be an \( \omega \)-conformal density on \( \bar{X}, \mathfrak{A} \). Recall that every cocycle in \( \bar{X} \) is 1-Lipschitz. It follows that

\[
\mu_x(A) \leq e^{\omega d(x,y)} \mu_y(A), \quad \forall x, y \in X, \forall A \in \mathfrak{A}.
\]

This observation will be useful many times later.

In practice, we will consider only two \( \sigma \)-algebras on \( \bar{X} \): the Borel \( \sigma \)-algebra \( \mathcal{B} \) and the reduced \( \sigma \)-algebra \( \mathfrak{R} \) (see Section 3.2). If \( \nu \) is a conformal density on \( (\bar{X}, \mathcal{B}) \), then its restriction to the reduced \( \sigma \)-algebra \( \mathfrak{R} \) is not necessarily quasi-conformal. We will see later that this pathology can be avoided if the action of \( G \) on \( X \) is divergent.

**Topology.** We denote by \( D(\omega) \) the set of all \( \omega \)-conformal densities on the horocompactification \( (\bar{X}, \mathcal{B}) \). We endow \( D(\omega) \) with the following topology: a sequence \( \nu^n = (\nu^n_x) \) of densities converges to \( \nu = (\nu_x) \) if for every \( x \in X \), the measure \( \nu^n_x \) converges to \( \nu_x \) for the weak-* topology. Let \( \mathcal{P}(\bar{X}) \) be the set of all Borel probability measures on \( \bar{X} \) (endowed with the weak-* topology). An \( \omega \)-conformal density \( \nu \in D(\omega) \) is entirely determined by the measure \( \nu_o \). More precisely the map

\[
D(\omega) \to \mathcal{P}(\bar{X})
\]

is a homeomorphism. We denote by \( D(G, \omega) \) the convex closed subspace of \( D(\omega) \) consisting of all \( G \)-invariant, \( \omega \)-conformal densities on \( (\bar{X}, \mathcal{B}) \). The densities \( \nu = (\nu_x) \) in \( D(G, \omega) \) for which the action of \( G \) on \( (\bar{X}, \mathcal{B}, \nu_o) \) is ergodic are exactly the extremal points of \( D(G, \omega) \).

**Patterson’s construction.** We now prove the existence of invariant conformal densities supported on the horoboundary. Actually we focus on a slightly more general settings that will be useful for our applications. A map \( \chi : G \to \mathbb{R} \) is a quasi-morphism if there exists \( C \in \mathbb{R}^+ \) such that for every \( g, g' \in G \), we have

\[
|\chi(g) + \chi(g') - \chi(gg')| \leq C.
\]

To such a quasi-morphism \( \chi \), we associate a twisted Poincaré series defined as

\[
\mathcal{P}_\chi(s) = \sum_{g \in G} e^{\chi(g)} e^{-sd(o,go)},
\]

and write \( \omega_\chi \) for its critical exponent. It follows from (4) that

\[
e^{-2C} \mathcal{P}_\chi(s) \leq \mathcal{P}_{-\chi}(s) \leq e^{2C} \mathcal{P}_\chi(s), \quad \forall s \in \mathbb{R}^+.
\]

Hence \( \omega_{-\chi} = \omega_\chi \). Note also that

\[
\frac{1}{2} [\mathcal{P}_\chi(s) + \mathcal{P}_{-\chi}(s)] \geq \sum_{g \in G} \text{ch}(\chi(g)) e^{-sd(o,go)} \geq \mathcal{P}_G(s).
\]

Thus \( \omega_\chi \geq \omega_G \).
Proposition 4.3. Let $H$ be a subgroup of $G$. Let $\chi : G \to \mathbb{R}$ be a quasi-morphism such that $\chi(hg) = \chi(g)$, for all $h \in H$ and $g \in G$. There is an $H$-invariant, $\omega_\chi$-conformal density $\nu = (\nu_x)$ on $(\bar{X}, \mathcal{B})$ with the following properties:

- $\nu$ is supported on $\partial X$;
- there is $C \in \mathbb{R}_+^*$ such that for every $g \in G$, and $x \in X$, we have
  \[ \frac{1}{C} \nu_x \leq e^{-\chi(g)} g^{-1} \nu_{gx} \leq C \nu_x. \]

Remark. If $H = G$ and $\chi$ is the trivial morphism, then the proposition says that there exists a $G$-invariant, $\omega_G$-conformal density supported on $\partial X$.

Proof. The proof follows Patterson’s strategy [37]. As Burger and Mozes already observed, this construction can be carried without difficulty in the horocompactification of $X$ [7]. We only review here its main steps. Note that the twisted Poincaré series $P_\chi(s)$ may converge at the critical exponent $s = \omega_\chi$. Using Patterson’s idea, one produces a “slowly growing” function $\theta : \mathbb{R}_+ \to \mathbb{R}_+$ with the following properties – see Roblin [40, Lemme 2.1.1].

(P1) For every $\varepsilon > 0$, there exists $t_0 \geq 0$, such that for every $t \geq t_0$ and $u \geq 0$, we have
  \[ \theta(t + u) \leq e^{\varepsilon u} \theta(t). \]

(P2) The weighted twisted Poincaré series, defined by
  \[ Q(s) = \sum_{g \in G} \theta(d(o, go)) e^{\chi(g)} e^{-sd(go, o)} \]
  is divergent whenever $s \leq \omega_\chi$, and convergent otherwise. In particular, $Q(s)$ diverges to infinity as $s$ approaches $\omega_\chi$ (from above).

For every $x \in X$ and $s > \omega_\chi$, we define a measure on $\bar{X}$ by
  \[ \nu_x^s = \frac{1}{Q(s)} \sum_{g \in G} \theta(d(x, go)) e^{\chi(g)} e^{-sd(x, go)} \text{Dirac}(go). \tag{5} \]

Since $\bar{X}$ is compact, the space of probability measures on $\bar{X}$ is compact for the weak-* topology. Consequently there exists a sequence of real numbers $(s_n)$ converging to $\omega_\chi$ from above, and such that for every $x \in X$, the measure $\nu_x^{s_n}$ converges to a measure on $\bar{X}$, that we denote by $\nu_x$. Note that $\nu^s = (\nu_x^s)$ is $H$-invariant, for every $s > \omega_\chi$. Moreover, since $\chi$ is a quasi-morphism, there exists $C \in \mathbb{R}_+^*$ such that for every $s > \omega_\chi$, $g \in G$ and $x \in X$, we have
  \[ \frac{1}{C} \nu_x^s \leq e^{-\chi(g)} g^{-1} \nu_{sx}^s \leq C \nu_x^s. \]
Hence the same properties hold for \( \nu \). The horocompactification is precisely designed so that the map

\[
X \times X \times X \to \mathbb{R}
\]

\[
(x, y, z) \mapsto d(x, z) - d(y, z)
\]

extends continuously to a map \( X \times X \times \bar{X} \to \mathbb{R} \). Taking advantage of this fact, one checks that \( \nu \) is \( \omega_\chi \)-conformal. Since \( Q(s) \) diverges when \( s \) approaches \( \omega_\chi \), the density \( \nu \) is supported on \( \partial X \).

4.2 Group action on the space of density

**Group action.** The action of \( G \) on \( X \) induces a right action of \( G \) on the set of densities. Let \( \mathfrak{A} \) be a \( G \)-invariant sub-\( \sigma \)-algebra of \( \mathfrak{B} \). Given a density \( \nu = (\nu_x) \) on \( (X, \mathfrak{A}) \) and \( g \in G \), we define a new density \( \nu^g \) as follows

\[
\nu^g_x = \frac{1}{\|\nu^g_{go}\|} g^{-1} \nu_{gx}, \quad \forall x \in X.
\]

We make the following observations.

(i) If \( \nu \) is \( \omega \)-conformal, then the same holds for \( \nu^g \).

(ii) If \( \nu \) is \( H \)-invariant, for some subgroup \( H \subset G \), then \( \mu^g \) is \( H^g \)-invariant, where \( H^g = g^{-1} H g \).

In particular, if \( N \) is a normal subgroup of \( G \), then the map

\[
\mathcal{D}(N, \omega) \times G \to \mathcal{D}(N, \omega)
\]

\[
(\nu, g) \mapsto \nu^g
\]

defines a right action of \( G \) on \( \mathcal{D}(N, \omega) \) which is trivial when restricted to \( N \).

**Fixed point properties.** For our study, we need to distinguish several fixed point properties for the action of \( G \) on the space of densities. Recall that a density \( \nu = (\nu_x) \) is

(i) **G-invariant**, if \( g_* \nu_x = \nu_{gx} \) for every \( g \in G \) and \( x \in X \).

We say that \( \nu \) is

(ii) **G-almost invariant**, if there exists \( C \in \mathbb{R}^*_+ \) such that for every \( g \in G \) and \( x \in X \),

\[
\frac{1}{C} \nu_{gx} \leq g_* \nu_x \leq C \nu_{gx},
\]

(iii) **fixed by \( G \)**, if \( \nu^g = \nu \), for every \( g \in G \),

(iv) **almost-fixed by \( G \)**, if there exists \( C \in \mathbb{R}^*_+ \) such that for every \( g \in G \) and \( x \in X \),

\[
\frac{1}{C} \nu_x \leq \nu^g_x \leq C \nu_x,
\]

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(v) $G$-quasi-invariant, if $g_\ast \nu_o \ll \nu_o$, for every $g \in G$.

If we want to emphasize the constant $C$ in (iv), we will say that $\nu$ is $C$-almost-fixed by $G$. These properties are related as follows.

\[
\begin{array}{c}
(i) \leftrightarrow (ii) \quad (i) \rightarrow (iv) \rightarrow (v).
\end{array}
\]

The reverse implications do not hold in general. The next statement highlights the role of quasi-morphisms in our study.

**Lemma 4.4.** Let $\nu = (\nu_x)$ be a density. Assume that $\nu$ is fixed (respectively almost-fixed) by $G$. Then the map $\chi : G \to \mathbb{R}$ sending $g$ to $\ln \|\nu_{go}\|$ is a morphism (respectively quasi-morphism).

**Proof.** Assume that $\nu$ is almost-fixed by $G$ (if $\nu$ is fixed by $G$ the proof works in the exact same way). There exists $C \in \mathbb{R}_+^*$ such that for every $g \in G$ and $x \in X$, we have

\[ \frac{1}{C} \nu_x \leq \nu_{go} \leq C \nu_x. \]

Let $g, g' \in G$. Comparing the total masses of the above measures for $x = g'o$, we get

\[ \frac{1}{C} \|\nu_{g'o}\| \leq \frac{\|\nu_{go}\|}{\|\nu_g\|} \leq C \|\nu_{g'o}\|. \]

**Lemma 4.5.** Let $H$ be a subgroup of $G$. Let $\mu = (\mu_x)$ be an $H$-invariant, $\omega$-quasi-conformal density on the reduced horocompactification $(\bar{X}, \mathcal{R})$. Assume that for every $g \in G$, the action of $H^0 \cap H$ on $(\bar{X}, \mathcal{R}, \mu_o)$ is ergodic. If $\mu$ is $G$-quasi-invariant, then $\mu$ is almost-fixed by $G$.

**Proof.** Since $\mu$ is $G$-quasi-invariant, we can define the following map

\[ F : G \times \bar{X} \to \mathbb{R}_+^* \]

\[ (g, c) \mapsto \frac{d (g^{-1} \ast \mu_{go})}{d \mu_o}(c). \]

**Claim 4.6.** There exists $C \in \mathbb{R}_+^*$ such that for every $g_1, g_2 \in G$, we have

\[ \frac{1}{C} \frac{F(g_1g_2, c)}{F(g_2, c)} \leq F(g_1, g_2c) \leq C \frac{F(g_1g_2, c)}{F(g_2, c)}, \quad \mu\text{-a.e.} \]

Let $g_1, g_2 \in G$. The computation gives

\[ F(g_1g_2, c) = \frac{d (g_1^{-1} \ast \mu_{g_1g_2o})}{d \mu_{g_2o}}(g_2c) \frac{d (g_2^{-1} \ast \mu_{g_2o})}{d \mu_o}(c), \quad \mu\text{-a.e.} \quad (6) \]
Since $\mu$ is $\omega$-quasi-conformal, there exists $C \in \mathbb{R}_+$ such that for every $x, y \in X$, for every $g \in G$, we have

$$\frac{1}{C} \frac{d\mu_x}{d\mu_y} \leq \frac{d\mu_{gx}}{d\mu_{gy}} \circ g \leq C \frac{d\mu_x}{d\mu_y}.$$ 

Hence

$$\frac{1}{C} \frac{d\left(g^{-1}_* \mu_{gy}\right)}{d\mu_y} \leq \frac{d\left(g^{-1}_* \mu_{gx}\right)}{d\mu_x} \leq C \frac{d\left(g^{-1}_* \mu_{gy}\right)}{d\mu_y}.$$ 

It follows that the first factor in the right-hand side of (6) is $F(g_1, g_2 c)$ – up to a multiplicative error that does not depend on $g_1$ or $g_2$ – while the second factor is exactly $F(g_2, c)$. This completes the proof of our claim.

Let $g \in G$ and set $H_0 = H^g \cap H$. Let $h \in H_0$. According to our claim we have

$$\frac{1}{C} F(g h, c) \leq F(g, h c) \leq C F(g h, c), \quad \mu\text{-a.e.}$$

Recall that $\mu$ is $H$-invariant, thus

$$F(h, c) = 1, \quad \text{and} \quad F(g h, c) = F(g h g^{-1} g, c) = F(g, c), \quad \mu\text{-a.e.}$$

Our previous inequalities becomes

$$\frac{1}{C} F(g, c) \leq F(g, h c) \leq C F(g, c), \quad \mu\text{-a.e.}$$

We now define an auxiliary function $F_g : \bar{X} \to \mathbb{R}_+$ by

$$F_g(c) = \inf_{h \in H_0} F(g, h c).$$

By construction $F_g$ is $H_0$-invariant. Since the action of $H_0$ on $(\bar{X}, \mathcal{R}, \mu_o)$ is ergodic, $F_g$ is constant. From now on, we denote by $F_g$ its essential value. It follows from our previous observation that

$$F_g \leq F(g, c) \leq C F_g, \quad \mu\text{-a.e.}$$

Coming back to the definition of $F$ this means that

$$F_g \leq \frac{d\left(g^{-1}_* \mu_{go}\right)}{d\mu_o} \leq C F_g, \quad \mu\text{-a.e.}$$

Integrating these inequalities, we see that $F_g \leq \|\mu_{go}\| \leq C F_g$. Hence

$$\frac{1}{C} \|\mu_{go}\| \leq \frac{d\left(g^{-1}_* \mu_{go}\right)}{d\mu_o} \leq C \|\mu_{go}\|.$$ 

Recall that $C$ does not depend on $g$. We have proved that there exists $C \in \mathbb{R}_+$ such that for every $g \in G$,

$$\frac{1}{C} \mu_o \leq \mu_{go} \leq C \mu_o.$$ 

Using the quasi-conformality of $\mu$, we conclude that $\mu$ is almost-fixed by $G$. \qed

Remark. The same argument shows that if $\mu$ is $\omega$-conformal (instead of $\omega$-quasi-conformal) then $\mu$ is fixed by $G$. 21
4.3 The Shadow Principle

Given $x, y \in X$ and $c \in \bar{X}$, we define the following Gromov product

$$\langle x, c \rangle_y = \frac{1}{2} [d(x, y) + c(y, x)].$$

**Remark.** If $c = \iota(z)$ for some $z \in X$, then the above formula coincides with the usual definition of the Gromov product.

Since cocycles in $\bar{X}$ are 1-Lipschitz, we always have $0 \leq \langle x, c \rangle_y \leq d(x, y)$.

We also observe that

$$\langle x, c \rangle_y - \langle x', c \rangle_{y'} \leq d(x, x') + d(y, y'), \quad \forall x, x', y, y' \in X. \quad (7)$$

**Definition 4.7.** Let $x, y \in X$. Let $r \in \mathbb{R}_+$. The $r$-shadow of $y$ seen from $x$, is the set

$$\mathcal{O}_x(y, r) = \{ c \in \bar{X} : \langle x, c \rangle_y \leq r \}.$$

By construction, $\mathcal{O}_x(y, r)$ is a closed subset of $\bar{X}$. It follows from (7) that for every $x, x', y, y' \in X$ and $r \in \mathbb{R}_+$,

$$\mathcal{O}_x(y, r) \subset \mathcal{O}_{x'}(y', r'), \quad \text{where } r' = r + d(x, x') + d(y, y'). \quad (8)$$

**Remark.** A more intuitive definition of shadows could have been the following: a cocycle $c \in \bar{X}$ belongs to $\mathcal{O}_x(y, r)$ if some complete gradient arc for $c$ starting at $x$ passes at a distance at most $r$ from $y$. Nevertheless, unlike our approach, this definition is very sensitive to the change of point $x$.

Following Roblin with small variations we define the Shadow Principle [40].

**Definition 4.8.** Let $\omega \in \mathbb{R}_+$ and $(\varepsilon, r_0) \in \mathbb{R}_+^* \times \mathbb{R}_+^*$. Let $\nu = (\nu_x)$ be an $\omega$-conformal density on $(\bar{X}, \mathcal{B})$. We say that $(G, \nu)$ satisfies the Shadow Principle with parameters $(\varepsilon, r_0)$ if for every $g \in G$ and $r \geq r_0$, we have

$$\nu_{o}(\mathcal{O}_o(go, r)) \geq \varepsilon\|\nu_{go}\|e^{-\omega d(o, go)}. \quad (9)$$

We say that $(G, \nu)$ satisfies the Shadow Principle, if there are $(\varepsilon, r_0) \in \mathbb{R}_+^* \times \mathbb{R}_+$ such that $(G, \nu)$ satisfies the Shadow Principle with parameters $(\varepsilon, r_0)$.

**Remark.** If the inequality (9) holds for $r = r_0$, then it automatically holds for every $r \geq r_0$. We will see later that a similar upper bound is always satisfied without any additional assumption.

Our next task is to adapt Sullivan’s celebrated Shadow Lemma (Corollary 4.10). It states that $(G, \nu)$ satisfies the Shadow Principle whenever $\nu$ is an $\omega$-conformal density which is $N$-invariant for some infinite normal subgroup $N \lhd G$.

**Proposition 4.9.** Assume that $G$ is not virtually cyclic and contains a contracting element. Let $\omega, C \in \mathbb{R}_+$. There exists $(\varepsilon, r_0) \in \mathbb{R}_+^* \times \mathbb{R}_+$ such that for every $r \geq r_0$, the following holds. Let $\nu = (\nu_x)$ be an $\omega$-conformal density. If $\nu$ is $C$-almost fixed by $G$, then for every $z \in X$, we have

$$\nu_{o}(\mathcal{O}_z(o, r)) \geq \varepsilon.$$
Proof. Assume that our claim fails. We can find a sequence \((r_n)\) diverging to infinity, a sequence \((z_n)\) of points in \(X\), and a sequence \(\nu^n = (\nu^n_o)\) of \(\omega\)-conformal densities, which are \(C\)-almost fixed by \(G\) and such that

\[
\nu^n_o (\mathcal{O}_{z_n}(o, r_n))
\]

converges to zero. Being \(C\)-almost fixed by \(G\) is a closed condition for the weak-* topology. Up to passing to a subsequence, we may assume that \(\nu^n\) converges to an \(\omega\)-conformal density \(\nu\), which is \(C\)-almost fixed by \(G\). Note that for every \(x, y \in X\) and \(r \in \mathbb{R}_+\), we have \(\mathcal{O}_x(y, r) = \bar{X}\), whenever \(d(x, y) \leq r\). Hence \(d(o, z_n)\) necessarily diverges to infinity. Up to passing again to a subsequence, we can assume that \(z_n\) converges to \(c \in \partial X\).

By assumption \(G\) is not virtually cyclic and contains a contracting element. According to Proposition 3.12, there exists a contracting element \(h \in G\), such that \(c \notin \partial^+ A\), where \(A = \langle h \rangle_o\) is \(\alpha\)-contracting for some \(\alpha \in \mathbb{R}_+^\ast\). For every \(n \in \mathbb{N}\), we write \(p_n\) for a projection of \(z_n\) onto \(A\). Up to passing to a subsequence \((p_n)\) converges to a point \(p \in A\) which is a projection of \(c\) onto \(A\) (Proposition 3.9). We introduce the following closed subset of \(\bar{X}\)

\[
F = \{ b \in \bar{X} : b(p, y) \leq 6\alpha, \forall y \in Y \}.
\]

We are going to prove that \(\nu_o(F) = 1\).

Let \(b \in \bar{X} \setminus F\). Suppose first that \(b \notin \partial^+ A\). Let \(q\) be a projection of \(b\) onto \(A\). Since \(b\) is 1-Lipschitz, we have

\[
b(p, y) \leq b(q, y) + d(p, q) \leq d(p, q), \quad \forall y \in A.
\]

As \(b\) does not belong to \(F\), necessarily \(d(p, q) > 6\alpha\). In particular, there exists \(N_0 \in \mathbb{N}\), such that for every \(n \geq N_0\), we have \(d(p_n, q) > 6\alpha\). According to Corollary 3.10, \(\langle z_n, b \rangle_{p_n} \leq 9\alpha\). Note that the conclusion still holds if \(b \in \partial^+ A\).

This is indeed a consequence of Corollary 3.11. Hence in all cases, we have

\[
\langle z_n, b \rangle_o \leq \langle z_n, b \rangle_{p_n} + d(o, p_n) \leq 9\alpha + d(o, p_n).
\]

Recall that \((p_n)\) is bounded. Consequently, there is \(N_1 \geq N_0\) such that for every \(n \geq N_1\), the set \(\bar{X} \setminus F\) is contained in \(\mathcal{O}_{z_n}(o, r_n)\). In particular, \(\nu^n_o(\bar{X} \setminus F)\) converges to zero. Since \(\bar{X} \setminus F\) is an open subset of \(\bar{X}\), we deduce that \(\nu_o(X \setminus F) = 0\), i.e. \(\nu_o(F) = 1\).

The group \(\langle h \rangle\) has unbounded orbits. Thus there is \(q \in \langle h \rangle\) such that \(d(gp, p) > 48\alpha\). We claim that \(F \cap gF\) is empty. Let \(b \in F\). Remember that \(b\) does not belong to \(\partial^+ A\) by Proposition 3.9 (ii). Choose a projection \(q\) of \(b\) onto \(A\). It follows from Corollary 3.10 that

\[
6\alpha \geq b(p, q) \geq d(p, q) - 18\alpha.
\]

Hence \(d(p, q) \leq 24\alpha\). Suppose now that contrary to our claim \(b\) also belongs to \(gF\). In particular, \(g^{-1}q\) is a projection of \(g^{-1}b\) onto \(A\). Following the same
argument as above, we get \(d(p, g^{-1}q) \leq 24\alpha\). Consequently \(d(p, gp) \leq 48\alpha\), a contradiction. Observe that
\[
\nu_o(gF) = g^{-1}\nu_o(F) = \|\nu_{go}\|\nu_{g^{-1}o}(F).
\]
Since \(\nu\) is \(C\)-almost-fixed by \(G\) and \(\omega\)-conformal, we get
\[
\nu_o(gF) \geq \frac{1}{C} \|\nu_{go}\|\nu_{g^{-1}o}(F) \geq \frac{1}{C} \|\nu_{go}\|e^{-\omega d(o, go)}\nu_o(F),
\]
where, the second inequality follows from (2). Consequently,
\[
\nu_o(F \cup gF) = \nu_o(F) + \nu_o(gF) \geq \left[1 + \frac{1}{C} \|\nu_{go}\|e^{-\omega d(o, go)}\right] \nu_o(F) > 1.
\]
This contradicts the fact that \(\nu_o\) is a probability measure.

**Corollary 4.10.** Assume that \(G\) is not virtually cyclic and has a contracting element. Let \(N\) be an infinite normal subgroup of \(G\). Let \(\omega \in \mathbb{R}_+\). There exists \((\varepsilon, r_0) \in \mathbb{R}_+^* \times \mathbb{R}_+\) such that for every \(r \geq r_0\), for every \(N\)-invariant, \(\omega\)-conformal density \(\nu = (\nu_x)\), for every \(g \in G\), we have
\[
\varepsilon \|\nu_{go}\|e^{-\omega d(o, go)} \leq \nu_o(\mathcal{O}_o(go, r)) \leq e^{2\omega r} \|\nu_{go}\|e^{-\omega d(o, go)}.
\]
In particular, \((G, \nu)\) satisfies the Shadow Principle with parameters \((\varepsilon, r_0)\).

**Proof.** Let \(\nu = (\nu_x)\) be an \(\omega\)-conformal density. A classical computation shows that for every \(g \in G\) and \(r \in \mathbb{R}_+\), we have
\[
\nu_o(\mathcal{O}_o(go, r)) = \|\nu_{go}\| \int \mathbb{1}_{\mathcal{O}_{g^{-1}o}(o, r)}(c)e^{-\omega c(g^{-1}o, o)}d\nu_{go}(c).
\]
Shadows have been designed so that for every \(c \in \mathcal{O}_{g^{-1}o}(o, r)\), we have
\[
d(o, go) - 2r \leq c(g^{-1}o, o) \leq d(o, go).
\]
Consequently,
\[
\nu_{go}^0(\mathcal{O}_{g^{-1}o}(o, r)) \leq e^{\omega d(go, o)} \frac{1}{\|\nu_{go}\|} \nu_o(\mathcal{O}_o(go, r)) \leq e^{2\omega r} \nu_{go}^0(\mathcal{O}_{g^{-1}o}(o, r)).
\]
The upper bound in (10) follows from the fact that \(\nu_{go}^0\) is a probability measure. Let us focus on the lower bound. To that end, we assume now that \(\nu\) is \(N\)-invariant. As \(N\) is a normal subgroup of \(G\), the density \(\nu_{go}^0\) is \(N\)-invariant as well. Since \(G\) contains a contracting element, and is not virtually cyclic, the same holds for \(N\) (Lemma 2.8). The result now follows from Proposition 4.9 applied with the group \(N\).

**Remark 4.11.** Note that the upper bound in (10) was proved without assuming any invariance for \(\nu\). Moreover it works for any \(r \in \mathbb{R}_+\).
Here is another variation of the Shadow Lemma.

**Corollary 4.12.** Assume that $G$ is not virtually cyclic and contains a contracting element. Let $\omega \in \mathbb{R}_+$ and $\nu = (\nu_x)$ be an $\omega$-conformal density. If $\nu$ is almost fixed by $G$, then $(G, \nu)$ satisfies the Shadow Principle.

**Proof.** Let $g \in G$ and $r \in \mathbb{R}_+$. Reasoning as in the proof of Corollary 4.10, we see that

$$\nu_o(\mathcal{O}_o(go, r)) \geq \|\nu_{go}\| e^{-\omega d(o, go)} \nu^g_o(\mathcal{O}_{g^{-1}}(o, r)).$$

Since $\nu$ is almost-fixed by $G$, there is $\varepsilon \in \mathbb{R}_+^*$, which does not depend on $g$ or $r$, such that $\nu_{go}^g \geq \varepsilon \nu_o$. Consequently,

$$\nu_o(\mathcal{O}_o(go, r)) \geq \varepsilon \|\nu_{go}\| e^{-\omega d(o, go)} \nu_o(\mathcal{O}_{g^{-1}}(o, r)).$$

The result now follows from Proposition 4.9.

### 4.4 Contracting tails

In order to take full advantage of the Shadow Lemma, we consider a particular kind of shadows, namely shadows of the form $\mathcal{O}_x(y, r)$ where $x$ is joined to $y$ by a geodesic whose tail is contracting. The next definition quantifies the “contraction strength” of this tail.

**Definition 4.13.** Let $\alpha \in \mathbb{R}_+^*$ and $L \in \mathbb{R}_+$. Let $x, y \in X$. The pair $(x, y)$ has an $(\alpha, L)$-contracting tail, if there exists an $\alpha$-contracting geodesic $\tau$ of ending at $y$ and a projection $p$ of $x$ on $\tau$ satisfying $d(p, y) \geq L$. The path $\tau$ is called a (contracting) tail of $(x, y)$ — the contracting strength the length of the tail should be clear from the context.

**Notations 4.1.** Given $\alpha, L \in \mathbb{R}_+$, we denote by $T(\alpha, L)$ the set of all elements $g \in G$ such that the pair $(o, go)$ has an $(\alpha, L)$-contracting tail.

The next statement is essentially a reformulation of Corollary 3.10. It states that if $(x, y)$ has a sufficiently long contracting tail, then the shadow of $y$ seen from $x$ behaves according to our intuition coming from hyperbolic geometry.

**Lemma 4.14.** Let $\alpha \in \mathbb{R}_+^*$ and $r, L \in \mathbb{R}_+$ with $L > r + 17\alpha$. Let $x, y \in X$. Assume that $(x, y)$ has an $(\alpha, L)$-contracting tail, say $\tau$. Let $p$ be a projection of $x$ on $\tau$. Let $c \in \mathcal{O}_x(y, r)$. Let $\gamma$ be a complete gradient arc for $c$ starting at $x$. Let $q$ be a projection of $c$ onto $\tau$. Then the following holds.

(i) $d(y, q) \leq r + 9\alpha$.

(ii) $\gamma$ intersects $N_\alpha(\tau)$.

(iii) The entry point of $\gamma$ in $N_\alpha(\tau)$ is $2\alpha$-close to $p$.

(iv) The exit point of $\gamma$ from $N_\alpha(\tau)$ is $7\alpha$-close to $q$.
Proof. According to Corollary 3.10, we know that $d(y, q) - 18\alpha \leq c(y, q)$, while $c(x, q) \leq d(x, q)$. Hence

$$c(x, y) \leq c(x, q) + c(q, y) \leq d(x, q) - d(y, q) + 18\alpha \leq d(x, y) - 2d(y, q) + 18\alpha.$$ 

Since $c$ belongs to $O_x(y, r)$, we have $c(x, y) \geq d(x, y) - 2r$. Hence $d(y, q) \leq r - 9\alpha$. Since $\tau$ is an contracting tail, there is a projection $p'$ of $x$ on $\tau$ such that $d(p', y) \geq L$. However $p$ is another projection of $x$ on $\tau$, which is $\alpha$-contracting, thus $d(p, p') \leq 2\alpha$. Consequently,

$$d(p, q) \geq d(p', y) - d(p, p') - d(y, q) \geq L - (r + 11\alpha) > 6\alpha.$$

According to Corollary 3.10, $\gamma$ intersects $N_q(\tau)$. Moreover the entry (respectively exit) point of $\gamma$ in $N_{\alpha}(\tau)$ is $2\alpha$-close to $p$ (respectively $7\alpha$-closed to $q$). Hence the result.

Lemma 4.15. Let $\alpha \in \mathbb{R}_+^*$ and $r, L \in \mathbb{R}_+$ with $L > r + 17\alpha$. Let $x, y_1, y_2 \in X$ such that $(x, y_1)$ has an $(\alpha, L)$-contracting tail, say $\tau_i$. Let $p_i$ be a projection of $x$ on $\tau_i$. If $O_x(y_1, r) \cap O_x(y_2, r)$ is non empty, then

(i) $d(y_1, y_2) \leq |d(x, y_1) - d(x, y_2)| + 4r + 64\alpha$, and

(ii) $d(p_1, p_2) \leq |d(x, p_1) - d(x, p_2)| + 8\alpha$.

Proof. Let $c$ be a cocycle in the intersection $O_x(y_1, r) \cap O_x(y_2, r)$. Let $\gamma$ be a complete gradient arc for $c$ starting at $x$. According to Lemma 4.14 the points $y_1$ and $y_2$ lie in the $(r + 16\alpha)$-neighborhood of $\gamma$, while $p_1$ and $p_2$ are $2\alpha$-closed to $\gamma$. The result follows.

In order to state the next lemma, we need a notion of spheres in $G$ (for the metric induced by $X$). This is the purpose of the following notation. Given $\ell, a \in \mathbb{R}_+$, we let

$$S(\ell, a) = \{ g \in G : \ell - a \leq d(o, go) < \ell + a \}.$$ 

Lemma 4.16. Assume that $G$ has a contracting element. There is $\alpha \in \mathbb{R}_+^*$ such that for every $L \in \mathbb{R}_+$, for every $g \in G$, there exist $u \in S(L, \alpha)$ and a geodesic $\gamma : [0, T] \to X$ from $o$ to $gu$ such that

(i) the path $\gamma$ restricted to $[T - L, T]$, which we denote by $\tau$, is an $(\alpha, L)$-contracting tail for $(o, guo)$

(ii) $guo$ is $\alpha$-close to the initial point of $\tau$ (hence $\alpha$-close to $\gamma$).

Proof. Let $h \in G$ be a contracting element. Denote by $A$ the $(h)$-orbit of $o$. It is $\beta$-contracting for some $\beta \in \mathbb{R}_+^*$. Let $a_0 \in \mathbb{R}_+$ be the parameter given by Lemma 2.7 applied with the element $h$, the point $z = o$ and $d = 2\beta$. Up to increasing the value of $a_0$ we can assume that $a_0 \geq 2\beta + d(o, ho)$.

Let $L \in \mathbb{R}_+$. Since $h$ is contracting, the orbit map $Z \to X$ sending $n$ to $h^n o$ is a quasi-isometric embedding. Thus there is $N \in \mathbb{N}$, such that for every
Lemma 2.3, and our choice of \( k \) prove that other case works in the exact same way. Let \( \gamma \) property. In particular, Observe that for a fixed \( \ell \), if \( (\gamma, t, r) \in X \) \((\gamma, t, r) \in \mathcal{N}_3(A)\). By our choice of \( N \), we have

\[
d (\gamma(t), h^{-N}o) \geq d (h^k o, h^{-N}o) - 2\beta \geq d (o, h^{N+k}o) - 2\beta \geq L.
\]

In particular \( t \leq T - L \). Denote by \( \tau \) the restriction of \( \gamma \) to \([T - L, L]\). It follows from Lemma 2.7 and our choice of \( \alpha_0 \) that

- \( \tau \) is \( \alpha_0 \)-contracting,
- there is \( s \in [t, T] \) such that \( d(\gamma(s), o) \leq \alpha_0 \).

Moreover \( \gamma(T - L) \) (which is the initial point of \( \tau \)) is the unique projection of \( g^{-1}o \) on \( \tau \). Using the triangle inequality we observe that

\[
|d(\gamma(s), h^{-N}o) - d(o, h^{-N}o)| \leq d(\gamma(s), o) \leq \alpha_0.
\]

On the one hand \( d(\gamma(s), h^{-N}o) = T - s \). On the other hand \( h^{-N} \in S(L, \alpha_0) \). Thus \( |(T - L) - s| \leq 2\alpha_0 \). Consequently the triangle inequality yields

\[
d(o, \gamma(T - L)) \leq d(o, \gamma(s)) + d(\gamma(s), \gamma(T - L)) \leq 3\alpha_0.
\]

Observe now that the path \( g\gamma \) satisfies the conclusion of the lemma with the parameter \( \alpha = 3\alpha_0 \).

Given \( \alpha, r, L, \ell \in \mathbb{R}_+ \), we consider the following set

\[
A_{\ell}(\alpha, r, L) = \bigcup_{g \in S(\ell, r) \cap T(\alpha, L)} \mathcal{O}_o(go, r).
\]

Observe that for a fixed \( \ell \in \mathbb{R} \), the set \( A_{\ell}(\alpha, r, L) \) is a non-decreasing function of \( r \) (respectively \( \alpha \)) and a non-increasing function of \( L \).

**Proposition 4.17.** Assume that \( G \) contains a contracting element. There is \( \alpha \in \mathbb{R}_+ \), such that for every \( \omega, a \in \mathbb{R}_+ \) and \( (\varepsilon, r_0) \in \mathbb{R}_+ \times \mathbb{R}_+ \), there exist \( r_1, C \in \mathbb{R}_+ \), with the following property. Let \( \nu = (\nu_x) \) be an \( \omega \)-conformal density. If \((G, \nu)\) satisfies the Shadow Principle with parameters \((\varepsilon, r_0)\), then for every \( r \geq r_1, L > r + 17\alpha, \) and \( \ell \in \mathbb{R}_+ \),

\[
\sum_{g \in S(\ell, a)} \|\nu_{go}\| e^{-\omega d(o, go)} \leq C e^{2\omega L} \nu_o(A_{\ell+L}(\alpha, r, L)).
\]
Lemma 4.16. Let $\omega, a \in \mathbb{R}_+$ and $(\varepsilon, r_0) \in \mathbb{R}_+^* \times \mathbb{R}_+$. The action of $G$ on $X$ is proper, so there is $M \in \mathbb{N}$ such that

$$\left|\{g \in G : d(o, go) \leq 2a + 12\alpha\}\right| \leq M.$$  

In addition, we set

$$r_1 = \max\{r_0, a + 3\alpha\}.$$  

Let $\nu = (\nu_x)$ be an $\omega$-conformal density such that $(G, \nu)$ satisfies the Shadow Principle with parameters $(\varepsilon, r_0)$. Let $r \geq r_1$, $L > r + 17\alpha$, and $\ell \in \mathbb{R}_+$. According to Lemma 4.16, for every $g \in S(\ell, a)$, there is $u_g \in S(L, \alpha)$ such that $g_u$ belongs to $T(\alpha, L)$. Moreover $go$ is $\alpha$-close to the initial point, say $z_g$, of a contracting tail of length exactly $L$ for the pair $(o, gu)$. In particular,

$$|d(o, go) + d(o, u_g o) - d(o, gu o)| \leq 2\alpha.$$  

Since $g \in S(\ell, a)$, we get $gu_g \in S(\ell + L, a + 3\alpha)$. Hence $O_o(gu_o, r)$ is contained in $A_{\ell + L}(\alpha, r, L)$.

Consider now $g, g' \in S(\ell, a)$ such that $O_o(gu_o, r) \cap O_o(g'u'_o, r)$ is non empty. By construction, $go$ and $g'o$ are $\alpha$-close to $z_g$ and $z_g'$. Using Lemma 4.15 and the triangle inequality, we observe that

$$d(go, g'o) \leq d(z_g, z_g') + 2\alpha \leq |d(o, z_g) - d(o, z_g')| + 10\alpha \leq |d(o, go) - d(o, g'o)| + 12\alpha \leq 2a + 12\alpha.$$  

It follows from our choice of $M$ that any cocycle $c \in \bar{X}$ belongs to a most $M$ shadows for the form $O_o(gu_o, r)$, where $g \in S(\ell, a)$. Consequently

$$\sum_{g \in S(\ell, a)} \nu_o(O_o(gu_o, r)) \leq M \nu_o(A_{\ell + L}(\alpha, r, L)) .$$  

Recall that $(G, \nu)$ satisfies the Shadow Principle with parameters $(\varepsilon, r_0)$. Hence for every $g \in S(\ell, a)$, we have

$$\nu_o(O_o(gu_o, r)) \geq \varepsilon \nu_o(e^{-\omega d(o, gu_o)}) \geq \varepsilon e^{-2\omega(L + \alpha)} \nu_o(e^{-\omega d(o, go)}).$$  

The second inequality follows from (2). Consequently (11) becomes

$$\sum_{g \in S(\ell, a)} \|\nu_o\| e^{-\omega d(o, go)} \leq \left(\frac{Me^{2\alpha\omega}}{\varepsilon}\right) e^{2\omega L} \nu_o(A_{\ell + L}(\alpha, r, L)).$$  

\hfill \Box

Corollary 4.18. Assume that $G$ contains a contracting element. For every $\omega, a \in \mathbb{R}_+$ and $(\varepsilon, r_0) \in \mathbb{R}_+^* \times \mathbb{R}_+$, there exists $C \in \mathbb{R}_+^*$, with the following property. Let $\nu = (\nu_x)$ be an $\omega$-conformal density. Assume that $(G, \nu)$ satisfies the Shadow Principle with parameters $(\varepsilon, r_0)$. For every $l \in \mathbb{R}_+$,

$$\sum_{g \in S(\ell, a)} \|\nu_o\| e^{-\omega d(o, go)} \leq C \nu_o(\bar{X} \setminus B(a, \ell)).$$  

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Proof. Denote by \( \alpha, r_1, C \) the parameters given by Proposition 4.17 applied with \( \omega, a, \) and \( (\varepsilon, r_0) \). We choose \( r \geq r_1 \) and \( L > \max \{ 2r, r + 17a \} \). Let \( \nu = (\nu_x) \) be an \( \omega \)-conformal density such that \( (G, \nu) \) satisfies the Shadow Principle with parameters \( (\varepsilon, r_0) \). By Proposition 4.17, we have

\[
\sum_{g \in S(\ell, a)} \| \nu_g \| e^{-\omega d(o, go)} \leq C e^{2\omega L} \nu_o (A_{\ell + L}(\alpha, r, L)), \quad \forall \ell \in \mathbb{R}_+.
\]

Consider now \( x, y, z \in X \). By the very definition of shadows, if \( z \in O_x(y, r) \), then

\[
d(x, y) - d(x, z) \leq \langle x, z \rangle_y \leq r.
\]

Hence for every \( \ell \in \mathbb{R}_+ \), the set \( A_{\ell + L}(\alpha, r, L) \) lies in \( \bar{X} \setminus B(o, \ell + L - 2r) \). The result follows from the fact that \( L \geq 2r \).

4.5 First applications

For our first applications, we assume that \( G \) is a group acting properly, by isometries on \( X \) with a contracting element. In addition, we suppose that \( G \) is not virtually cyclic.

**Proposition 4.19.** There exists \( C \in \mathbb{R}_+ \) such that for every \( \ell \in \mathbb{R}_+ \),

\[
| \{ g \in G : d(o, go) \leq \ell \} | \leq C e^{\omega \alpha \ell}.
\]

**Remark.** An alternative proof of this fact can be found in Yang [48].

**Proof.** According to Proposition 4.3 there exists a \( G \)-invariant, \( \omega_G \)-conformal density \( \nu = (\nu_x) \). By Corollary 4.10, the pair \( (G, \nu) \) satisfies the Shadow Principle for some parameters \( (\varepsilon, r_0) \in \mathbb{R}_+^* \times \mathbb{R}_+^* \). Fix \( a \in \mathbb{R}_+^* \). Applying Corollary 4.18, there exists \( C \in \mathbb{R}_+ \) such that for every \( \ell \in \mathbb{R}_+^* \),

\[
\sum_{g \in S(\ell, a)} \| \nu_g \| e^{-\omega_G d(o, go)} \leq C \nu_o (\bar{X} \setminus B(o, \ell)) \leq C.
\]

However, since \( \nu \) is \( G \)-invariant, \( \| \nu_g \| = \| \nu_o \| = 1 \), for every \( g \in G \). Hence we get

\[
|S(\ell, a)| \leq C e^{\omega_G a \omega \alpha \ell}, \quad \forall \ell \in \mathbb{R}_+^*.
\]

The result follows by summing this inequality over \( \ell \in a\mathbb{N} \).

Before stating our next application, we define the radial limit set for the action of \( G \) on \( X \).

**Definition 4.20.** Let \( r \in \mathbb{R}_+ \) and \( x \in X \). The set \( \Lambda_{\text{rad}}(G, x, r) \) consists of all cocycles \( c \in \partial X \) with the following property: for every \( T \geq 0 \), there exists \( g \in G \) with \( d(x, go) \geq T \) such that \( c \in O_x(go, r) \). The radial limit set of \( G \) is the union

\[
\Lambda_{\text{rad}}(G) = \bigcup_{r \geq 0} G\Lambda_{\text{rad}}(G, o, r).
\]

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Remark 4.21. Note that $\Lambda_{\text{rad}}(G, x, r)$ is a non-decreasing function of $r$. The radial limit set $\Lambda_{\text{rad}}(G)$ is $G$-invariant. It follows from (8) that

$$\Lambda_{\text{rad}}(G) = \bigcup_{r \geq 0} \Lambda_{\text{rad}}(G, o, r).$$

Lemma 4.22. The radial limit set is saturated.

Proof. Let $x, y \in X$ and $r \in \mathbb{R}_+$. Let $c, c' \in \partial X$ such that $c \sim c'$. Note that if $c$ belongs to $O_x(y, r)$ then $c'$ belongs to $O_x(y, r')$ where $r' = r + \|c - c'\|_{\infty}$. The result follows from this observation. $\square$

Proposition 4.23. Let $\omega \in \mathbb{R}_+$ and $\nu = (\nu_x)$ be an $\omega$-conformal density. If $(G, \nu)$ satisfies the Shadow Principle then the series

$$\sum_{g \in G} \|\nu_{go}\| e^{-\omega d(o, go)}$$

converges whenever $s > \omega$. If, in addition, $\nu_o$ gives positive measure the radial limit set $\Lambda_{\text{rad}}(G)$, then the series diverges at $s = \omega$. In particular its critical exponent is exactly $\omega$.

Proof. Fix $a \in \mathbb{R}_+^*$. According to Corollary 4.18 there is $C \in \mathbb{R}_+^*$ such that for every $\ell \in \mathbb{R}_+$,

$$\sum_{g \in S(\ell, a)} \|\nu_{go}\| e^{-\omega d(o, go)} \leq C \nu_o \left( \bar{X} \setminus B(o, \ell) \right) \leq C.$$

Hence the series (12) converges whenever $s > \omega$. The second part of the proposition is proved by contraposition. Suppose that the series (12) converges at $s = \omega$. Let $r \in \mathbb{R}_+$. For simplicity we write $\Lambda = \Lambda_{\text{rad}}(G, o, r)$. Observe that for every $T \in \mathbb{R}_+$,

$$\Lambda \subset \bigcup_{g \in G} O_o(go, r).$$

It follows from Remark 4.11, that

$$\nu_o(\Lambda) \leq \sum_{g \in G, d(o, go) \geq T} \nu_o(O_o(go, r)) \leq e^{2\omega r} \sum_{g \in G, d(o, go) \geq T} \|\nu_{go}\| e^{-\omega d(o, go)}.$$

The right-hand side of the inequality is the remainder of the series we are interested in. Since this series converges at $s = \omega$, we get $\nu_o(\Lambda) = 0$. We observed in Remark 4.21 that

$$\Lambda_{\text{rad}}(G) = \bigcup_{r \geq 0} \Lambda_{\text{rad}}(G, o, r).$$

Hence $\nu_o(\Lambda_{\text{rad}}(G)) = 0$. $\square$
Remark 4.24. Note that the proof of the second assertion (the series diverges at $s = \omega$ whenever $\nu_o$ gives positive measure to the radial limit set) only uses the upper estimate from the Shadow lemma. Hence it holds even if $G$ is virtually cyclic or does not contain a contracting element.

Corollary 4.25. Let $\omega \in \mathbb{R}_+$. Let $\nu = (\nu_x)$ be a $G$-invariant, $\omega$-conformal density. Then $\omega \geq \omega_G$. Moreover if $\nu_o$ gives positive measure to the radial limit set $\Lambda_{\text{rad}}(G)$, then $\omega = \omega_G$ and the action of $G$ on $X$ is divergent.

Remark. If the action of $G$ on $X$ is proper and co-compact, one checks that the radial limit set is actually $\partial X$. Hence if $\nu_o$ is a $G$-invariant, $\omega$-conformal density supported on $\partial X$, then $\omega = \omega_G$.

Proof. By Corollary 4.10, the pair $(G, \nu)$ satisfies the Shadow Principle. Since the density $\nu$ is $G$-invariant, the series (12) which appears in Proposition 4.23 is exactly the Poincaré series of $G$. The result follows.

Corollary 4.26. Let $\omega \in \mathbb{R}_+$. Let $\nu = (\nu_x)$ be an $\omega$-conformal density and $\mu = (\mu_x)$ its restriction to the reduced horo-compactification $(\bar{X}, \mathfrak{R})$. Assume that $(G, \nu)$ satisfies the Shadow Principle. If $\mu$ is almost-fixed by $G$, then $\omega \geq \omega_G$.

Proof. According to Lemma 4.4 the map $\chi: G \to \mathbb{R}_+$ sending $g$ to $\ln \|\nu_{go}\|$ is a quasi-morphism. Note that the exponent of the series

$$\sum_{g \in G} \|\nu_{go}\| e^{-s d(o,go)} = \sum_{g \in G} e^{\chi(g)} e^{-s d(o,go)}$$

is exactly $\omega_\chi$. We observed earlier that $\omega_\chi \geq \omega_G$. The result follows from Proposition 4.23.

Corollary 4.27. Let $H$ be a subgroup of $G$ such that $H$ is co-amenable in $G$. Then $\omega_H = \omega_G$.

The proof relies on a fixed point characterization of co-amenability due to Eymard.

Proposition 4.28 (Eymard [20, Exposé 1, §2]). Let $H$ be a subgroup of $G$ such that $H$ is co-amenable in $G$. Let $V$ be a locally convex, topological vector space, endowed with a continuous affine action of $G$. Let $K$ be a $G$-invariant, compact, convex subset of $V$. If $H$ fixes a point in $K$, then $G$ fixes a point in $K$.

Proof of Corollary 4.27. The proof is a variation of Roblin’s argument [40]. Recall that the space $D(\omega_H)$ of all $\omega_H$-conformal densities is homeomorphic to the space of probability measures $\mathcal{P}(\bar{X})$. In particular, $D(\omega_H)$ is a convex, compact subspace of the locally convex, topological space $V$ which consists of all signed finite measures on $\bar{X}$. It follows from Patterson’s construction that there exists an $H$-invariant, $\omega_H$-conformal density (Proposition 4.3). In particular, $H$ fixes a point in $D(\omega_H)$. According to Proposition 4.28, there is an $\omega_H$-conformal density $\nu = (\nu_x)$ which is fixed by $G$. Note that $\nu$ is not necessarily $H$-invariant.
Corollary 4.29. Let \( N \) be an infinite, normal subgroup of \( G \). Then
\[
\omega(N, X) + \frac{1}{2} \omega(G/N, X/N) \geq \omega(G, X).
\]

Proof. Let \( Q = G/N \). Denote by \( \pi: G \to Q \) and \( \zeta: X \to X/N \) the canonical projections. For simplicity we write \( \omega_N \) and \( \omega_G \) for the growth rates of \( N \) and \( G \) acting on \( X \), while \( \omega_Q \) stands for the growth rate of \( Q \) acting on \( X/N \). We denote by \( H \) the Hilbert space \( H = \ell^2(Q) \). Given \( s, t \in \mathbb{R}^+ \) we consider the following maps
\[
\phi_s: Q \to \mathbb{R} \quad \text{and} \quad \psi_t: Q \to \mathbb{R}
\]
\[
q \mapsto \sum_{g \in \pi^{-1}(q)} e^{-sd(o,go)} \quad \text{and} \quad q \mapsto e^{-td(\zeta(o),q\zeta(o))}
\]

One checks that \( \psi_t \in H \) (respectively \( \psi_t \notin H \)) whenever \( 2t > \omega_Q \) (respectively \( 2t < \omega_Q \)). Similarly \( \phi_s \notin H \), whenever \( s < \omega_N \). We now prove that the converse essentially holds true.

Claim 4.30. If \( s > \omega_N \), then \( \phi_s \in H \).

Let \( s > \omega_N \). Consider the \( N \)-invariant, \( s \)-conformal density \( \nu^s = (\nu^s_x) \) on \( \bar{X} \) defined by
\[
\nu^s_x = \frac{1}{\mathcal{P}_N(s)} \sum_{h \in N} e^{-sd(x,ho)} \text{Dirac}(ho).
\]
The computation shows that
\[
\|\phi_s\|^2 = \sum_{g_1, g_2 \in G \atop \pi(g_1) = \pi(g_2)} e^{-s[d(o,g_1o)+d(o,g_2o)]} = \sum_{g \in G} e^{-sd(o,go)} \left( \sum_{h \in N} e^{-sd(go,ho)} \right)
\]
\[
= \mathcal{P}_N(s) \sum_{g \in G} \left\| \nu^s_g \right\| e^{-sd(o,go)}.
\]

Fix \( a \in \mathbb{R}^*_+ \). According to Corollary 4.10, \( \nu^s \) satisfies the Shadow Principle. By Corollary 4.18 there exists \( C \in \mathbb{R}^*_+ \), such that for every \( \ell \in \mathbb{R}^+_+ \),
\[
\sum_{g \in S(\ell,a)} \left\| \nu^s_g \right\| e^{-sd(o,go)} \leq C \nu^s_a \left( \bar{X} \setminus B(o, \ell) \right).
\]

Unfolding the definition of \( \nu^s \), we get
\[
\mathcal{P}_N(s) \sum_{g \in S(\ell,a)} \left\| \nu^s_g \right\| e^{-sd(o,go)} \leq C \sum_{h \in N} e^{-sd(o,ho)}.
\]
It follows that
\[ \|\phi_s\|^2 \leq C \sum_{k \in \mathbb{N}} \sum_{h \in \mathbb{N}} e^{-sd(o,ho)} \leq C \sum_{k \in \mathbb{N}} \sum_{h \in \mathbb{N}} e^{-sd(o,ho)}. \]

Up to replacing \( C \) by a larger constant, we have
\[ \|\phi_s\|^2 \leq C \sum_{h \in \mathbb{N}} [1 + d(o,ho)] e^{-sd(o,ho)}. \tag{13} \]

Note that the series
\[ -\sum_{h \in \mathbb{N}} d(o,ho) e^{-sd(o,ho)} \]

is the derivative of the Poincaré series of \( N \), hence it converges. Consequently the right-hand side in (13) converges. Thus \( \|\phi_s\| \) is finite, which completes the proof of our claim.

Let \( s,t \in \mathbb{R}_+ \) with \( s > \omega_N \) and \( 2t > \omega_Q \). The scalar product of \( \phi_s \) and \( \psi_t \) can be computed as follows:

\[ (\phi_s,\psi_t) = \sum_{q \in Q} e^{-td(\zeta(o),q\zeta(o))} \left( \sum_{g \in \pi^{-1}(q)} e^{-sd(o,go)} \right) \]
\[ = \sum_{g \in G} e^{-sd(o,go)} e^{-td(\zeta(o),\pi(g)\zeta(o))}. \]

The projection \( \zeta: X \to X/N \) is 1-Lipschitz. Combined with the Cauchy-Schwarz inequality, it gives
\[ \mathcal{P}_G(s + t) \leq (\phi_s,\psi_t) \leq \|\phi_s\| \|\psi_t\| < \infty. \]

Consequently \( s + t \geq \omega_G \). This inequality holds for every \( s,t \in \mathbb{R}_+ \) with \( s > \omega_N \) and \( 2t > \omega_Q \), whence the result. \( \square \)

**Remark.** We keep the notations of the proof. **Corollary 4.29** is sharp.

- Indeed, assume that \( X \) is a Cayley graph of \( G \). If \( Q \) has subexponential growth, then \( \omega_Q = 0 \) and \( Q \) is amenable, so that \( \omega_N = \omega_G \) by **Corollary 4.27**.

- At the other end of the spectrum, assume that \( G = F_r \) is the free group of rank \( r \) acting on its Cayley graph \( X \) with respect to a free basis. Let \( g \in G \setminus \{1\} \). For every \( k \in \mathbb{N} \), denote by \( N_k \) the normal closure of \( g^k \). Then
\[ \lim_{k \to \infty} \omega_{N_k} = \frac{1}{2} \omega_G \quad \text{and} \quad \lim_{k \to \infty} \omega_{G/N_k} = \omega_G. \]
The first limit is due to Grigorchuk [22], see also Champetier [8]. The second limit was proved by Shukhov [43], see also Coulon [15].

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5 Divergent actions

5.1 Contracting tails, continued

We continue here our study of shadows of elements $g \in \mathcal{T}(\alpha, L)$ having a contracting tail. A several places, we will make use of the following variation of [12, Chapitre 3, Lemme 1.11]. Its proof works verbatim as in [12] and is left to the reader.

**Lemma 5.1.** Let $C \in \mathbb{R}_+$. Let $x, y \in X$ and $\gamma: [a, b] \to X$ be a continuous path from $x$ to $y$. Let $\gamma': I' \to X$ a geodesic. Assume that $\gamma$ lies in the $C$-neighborhood of $\gamma'$. Let $[a', b']$ be a subinterval of $I'$. Set $x' = \gamma(a')$ and $y' = \gamma'(b')$. Then $\gamma'$ restricted to $[a', b']$ lies in the $D$-neighborhood of $\gamma$, where

$$D = 2C + d(x, x') + \max\{0, d(x', y') - d(x, y)\}.$$  

**Lemma 5.2.** Let $x, y \in X$ and $\gamma$ be a geodesic of $X$ from $x$ to $y$. Let $\gamma': I \to X$ be a geodesic such that $\gamma$ lies in the $C$-neighborhood of $\gamma'$. Let $s, t \in I$ with $s \leq t$ and set $x' = \gamma(s)$ and $y' = \gamma(t)$. Then $\gamma$ lies in the $D$-neighborhood of $\gamma'$ restricted to $[s, t]$, where $D = 2C + d(x, x'), d(y, y').$

**Proof.** Consider a point $z$ on $\gamma$. By assumption there is a point $z'$ on $\gamma'$ such that $d(z, z') \leq C$. Since $z$ lies between $x$ and $y$ we have $d(x, y) = d(x, z) + d(z, y)$. Using the triangle inequality it yields

$$|d(x', z') + d(z', y') - d(x', y')| \leq 2d(x, x') + 2d(y, y') + 2C.$$  

Hence either $z'$ lies on $\gamma'$ between $x'$ and $y'$ or

$$\min\{d(x', z'), d(y', z')\} \leq d(x, x') + d(y, y') + C.$$  

The conclusion follows from the triangle inequality. 

**Lemma 5.3.** Let $\alpha \in \mathbb{R}_+^*$ and $r, L \in \mathbb{R}_+$ with $L > r + 55\alpha$. Let $x, y_1, y_2 \in X$ with $d(x, y_1) \leq d(x, y_2)$. Assume that the pairs $(x, y_1)$ and $(x, y_2)$ have an $(\alpha, L)$-contracting tail. If $\mathcal{O}_x(y_1, r) \cap \mathcal{O}_x(y_2, r)$ is not empty, then $\mathcal{O}_x(y_2, r)$ is contained in $\mathcal{O}_x(y_1, r + 56\alpha)$.

**Proof.** By assumption there is an $\alpha$-contracting geodesic $\tau_\gamma$ ending at $y_i$, and a projection $p_i$ of $x$ on $\tau_\gamma$ satisfying $d(p_i, y_i) \geq L$. We split the proof in several claims.

**Claim 5.4.** Let $z_2$ be the point on $\tau_2$ such that $d(y_2, z_2) = r + 17\alpha$. Let $c \in \mathcal{O}_x(y_2, r)$ and $\gamma: I \to X$ a complete gradient line for $c$ starting at $x$. Then $z_2$ lies in the $10\alpha$-neighborhood of $\gamma$.

According to Lemma 4.14, $\gamma$ intersects $\mathcal{N}_\alpha(\tau_2)$. Denote by $\gamma(s_2)$ and $\gamma(t_2)$ the entry and exit point of $\gamma$ in $\mathcal{N}_\alpha(\tau_2)$. We know, again from Lemma 4.14, that $d(\gamma(s_2), p_2) \leq 2\alpha$ and $d(\gamma(t_2), y_2) \leq r + 16\alpha$. In particular, the projection $m_2$ of $\gamma(t_2)$ on $\tau_2$ lies between $z_2$ and $y_2$. In other words $p_2, z_2, m_2$ and $y_2$ are aligned in this order along $\tau_2$. The path $\gamma$ restricted to $[s_2, t_2]$ lies in the $3\alpha$-neighborhood of $\tau$ (Lemma 2.2). It follows that $z_2$ is $10\alpha$-close to $\gamma$ (Lemma 5.1).
Claim 5.5. Let \( q \) be a projection of \( z_2 \) onto \( \tau_1 \). Then \( d(q, y_1) \leq r + 40\alpha \).

According to our assumption \( \mathcal{O}_x(y_1, r) \cap \mathcal{O}_x(y_2, r) \) is not empty. Let \( \gamma : I \to X \) be a complete gradient arc for a cocycle \( c \) in this intersection. We apply Lemma 4.14 with \( \tau_1 \). It tells us that \( \gamma \) intersects \( \mathcal{N}_\alpha(\tau_1) \). Moreover if \( \gamma(\alpha_1) \) and \( \gamma(\alpha_1') \) are respectively the entry and exit points of \( \gamma \) in \( \mathcal{N}_\alpha(\tau_1) \), then
\[
d(p_1, \gamma(\alpha_1)) \leq 2\alpha \quad \text{and} \quad d(\gamma(\alpha_1'), y_1) \leq r + 16\alpha.
\]
In particular, if \( q' \) stands for a projection of \( \gamma(\alpha_1') \) on \( \tau_1 \) we have \( d(q', y_1) \leq r + 17\alpha \). According to our previous claim, there is \( u_2 \in I \) such that \( \gamma(u_2) \) is 10\( \alpha \)-close to \( z_2 \).

Assume first that \( u_2 \geq t_1 \). Since \( \tau_1 \) is \( \alpha \)-contracting, the projection of \( \gamma(u_2) \) on \( \tau_1 \) is \( \alpha \)-close to \( q' \). However the projection on \( \tau_1 \) is large-scale 1-Lipschitz, hence \( d(q, q') \leq 15\alpha \). Consequently \( d(q, y_1) \leq r + 32\alpha \).

Suppose now that \( u_2 \leq t_1 \). We first prove that \( u_2 \in [s_1, t_1] \). Combining our assumption with Lemma 2.3 applied to \( \tau_1 \) we have
\[
d(x, z_2) + d(z_2, y_2) \geq d(x, y_2) \geq d(x, y_1) \geq d(x, p_1) + d(p_1, y_1) - 4\alpha.
\]
Hence \( d(x, z_2) - d(x, p_1) \geq L - (r + 21\alpha) \). Since \( d(z_2, \gamma(u_2)) \leq 10\alpha \) and \( d(p_1, \gamma(\alpha_1)) \leq 2\alpha \), we obtain
\[
u_2 - s_1 \geq d(x, \gamma(u_2)) - d(x, \gamma(\alpha_1)) \geq d(x, z_2) - d(x, p_1) - 12\alpha
\]
\[
\geq L - (r + 33\alpha).
\]
Thus \( u_2 \geq s_1 \), as we announced. Since \( \tau_1 \) is \( \alpha \)-contracting, the path \( \gamma \) restricted to \([s_1, t_1]\) lies in the \( 3\alpha \)-neighborhood of \( \tau_1 \). According to Lemma 5.2, there is a point \( q'' \) on \( \tau \) lying between \( p_1 \) and \( y_1 \) such that \( d(\gamma(u_2), q'') \leq 9\alpha \), (hence \( d(z_2, q'') \leq 19\alpha \)) It follows from Lemma 2.3 (applied with \( \tau_1 \)) and the triangle inequality that
\[
d(q'', y_1) \leq d(q''', y_1) - d(q'', y_1) + 4\alpha \leq d(x, y_2) - d(x, z_2) + 23\alpha \leq d(y_2, z_2) + 23\alpha \leq r + 40\alpha
\]
which completes the proof of our claim in the second case.

Claim 5.6. \( \mathcal{O}_x(y_2, r) \) is contained in \( \mathcal{O}_x(y_1, r + 56\alpha) \).

Consider \( c \in \mathcal{O}_x(y_2, r) \). Let \( \gamma : I \to X \) be a complete gradient arc for \( c \) starting at \( x \). According to our first claim, \( z_2 \) is 10\( \alpha \)-closed to a point \( \gamma(u_2) \) for some \( u_2 \in I \). Denote by \( q' \) a projection of \( \gamma(u_2) \) onto \( \tau_1 \). Since the projection onto \( \tau_1 \) is large-scale 1-Lipschitz, we get
\[
d(y_1, q') \leq d(y_1, q) + d(q, q') \leq r + 54\alpha.
\]
Recall \( p_1 \) is a projection of \( x \) on \( \tau_1 \). However
\[
d(p_1, q') \geq d(p_1, y_1) - d(q', y_1) \geq L - (r + 54\alpha) > \alpha.
\]
Since \( \tau_1 \) is \( \alpha \)-contracting, we get that \( \gamma \) intersects \( \mathcal{N}_\alpha(\tau) \) (Lemma 2.3). Moreover \( q' \) is 2\( \alpha \)-closed to a point on \( \gamma \). Hence \( d(y_1, \gamma) \leq r + 56\alpha \). This implies that \( c \in \mathcal{O}_x(y_2, r + 56\alpha) \). \( \square \)
The next statement will be used later to estimate the measures of various saturated sets using a Vitali type argument.

**Lemma 5.7.** Let $\alpha \in \mathbb{R}^*_+$ and $r, L \in \mathbb{R}_+$ with $L > r + 55\alpha$. Let $S$ be a subset of $T(\alpha, L)$. There is a subset $S^* \subset S$ with the following properties

(i) The collection $(\mathcal{O}_g(g_o, r))_{g \in S^*}$ is pairwise disjoint.

(ii) $\bigcup_{g \in S} \mathcal{O}_g(g_o, r) \subset \bigcup_{g \in S^*} \mathcal{O}_g(g_o, r + 56\alpha)$.

**Proof.** Since the action of $G$ on $X$ is proper, we can index the elements $g_0, g_1, g_2 \ldots$ of $S$ such that $d(o, g_i o) \leq d(o, g_{i+1} o)$, for every $i \in \mathbb{N}$. We build by induction a sequence $(i_n)$ as follows. Set $i_0 = 0$. Let $n \in \mathbb{N}$ for which $i_n$ has been defined. We search for the minimal index $j > i_n$ such that

$$\left( \bigcup_{k=0}^n \mathcal{O}_g(g_{i_k} o, r) \right) \cap \mathcal{O}_g(g_j o, r) = \emptyset.$$ 

If such an index exists, we let $i_{n+1} = j$. Otherwise we let $i_{n+1} = i_n$, in other words, the sequence $(i_n)$ eventually stabilizes. Finally we set

$$S^* = \{g_{i_n} : n \in \mathbb{N}\}.$$

Note that (i) directly follows from the construction. Let $g \in S$. If $g$ does not belong to $S^*$, it means that there is $h \in S^*$, with $d(o, ho) \leq d(o, go)$ such that $\mathcal{O}_g(g_o, r) \cap \mathcal{O}_h(h_o, r)$ is non-empty. Hence by Lemma 5.3, the shadow $\mathcal{O}_g(g_o, r)$ is contained in $\mathcal{O}_h(h_o, r + 56\alpha)$. This completes the proof of (ii).

**Lemma 5.8.** Let $\alpha \in \mathbb{R}^*_+$ and $r, L \in \mathbb{R}_+$ with $L > r + 17\alpha$. Let $x, y \in X$ such that $(x, y)$ has an $(\alpha, L)$-contracting tail. Let $K$ be a closed ball of radius $R$ centered at $x$. If $d(x, y) > R + r + 17\alpha$, then for every $c, c' \in \mathcal{O}_x(y, r)$, we have $\|c - c'\|_K \leq 36\alpha$.

**Proof.** Denote by $\tau$ a contracting tail of $(x, y)$ and let $p$ be a projection of $x$ on $\tau$ such that $d(p, y) \geq L$. Fix $x' \in K$ and denote by $p'$ a projection of $x'$ onto $\tau$. Let $q$ be a projection of $c$ on $\tau$, so that $d(y, q) \leq r + 9\alpha$ (Lemma 4.14). We claim that $d(p', q) > 6\alpha$. If $d(p, p') \leq \alpha$, then this is just a consequence of the triangle inequality. Thus we can assume that $d(p, p') > \alpha$. According to Lemma 2.3 we have $d(x, p') \leq d(x, x') + 2\alpha$. Hence

$$d(p', q) \geq d(x, y) - d(x, p') - d(q, y) \geq d(x, y) - d(x, x') - d(q, y) - 2\alpha$$

$$\geq d(x, y) - (R + r + 11\alpha) > 6\alpha.$$ 

It follows from Corollary 3.10 that

$$d(z, q) - 18\alpha \leq d(z, q) \leq d(z, q).$$

This estimate holds for every $x' \in K$. Consider now $x_1, x_2 \in K$. Since $c$ is a cocycle, $c(x_1, x_2) = c(x_1, q) + c(q, x_2)$, thus

$$|c(x_1, x_2) - [d(x_1, q) - d(x_2, q)]| \leq 18\alpha.$$ 

The same argument holds for $c'$, thus $\|c - c'\|_K \leq 36\alpha$. \qed
5.2 The contracting limit set

Contracting limit set. We now introduce a variation of the radial limit set that keeps track of the elements fellow-traveling with some contracting geodesic. Let \( \alpha, r, L \in \mathbb{R}_+ \) and \( x \in X \). The set \( \Lambda_{ctg}(G, x, \alpha, r, L) \) consists of all cocycles \( c \in \partial X \) with the following property: for every \( T \geq 0 \), there exists \( g \in G \) such that

- \( d(x, go) \geq T \),
- \( (x, go) \) has an \( (\alpha, L) \)-contracting tail,
- \( c \in \mathcal{O}_x(go, r) \).

We also let \( \Lambda_{ctg}(G, x, \alpha, r) = \bigcap_{L \in \mathbb{R}_+} \Lambda_{ctg}(G, x, \alpha, r, L) \).

Remark. Observe that the set \( \Lambda_{ctg}(G, x, \alpha, r, L) \) is a non-decreasing (respectively non-increasing) function of \( \alpha \) and \( r \), (respectively \( L \)).

Definition 5.9. The contracting limit set of \( G \) is

\[
\Lambda_{ctg}(G) = \bigcup_{\alpha, r \in \mathbb{R}_+} GA_{ctg}(G, o, \alpha, r).
\]

It follows from the definition that the contracting limit set is \( G \)-invariant and contained in the radial limit set.

Proposition 5.10. Let \( \alpha \in \mathbb{R}_+^* \). For every \( r \in \mathbb{R}_+ \) and \( L > r + 17\alpha \) we have

\[
GA_{ctg}(G, o, \alpha, r, L) \subset \Lambda_{ctg}(G, o, \alpha, r + 18\alpha, L - \alpha).
\]

In particular, \( GA_{ctg}(G, o, \alpha, r) \subset \Lambda_{ctg}(G, o, \alpha, r + 18\alpha) \). Moreover, the contracting limit set can also be described as

\[
\Lambda_{ctg}(G) = \bigcup_{\alpha, r \in \mathbb{R}_+} \Lambda_{ctg}(G, o, \alpha, r).
\]

Proof. Let \( h \in G \) and \( c \) be a cocycle in

\[
h\Lambda_{ctg}(G, o, \alpha, r, L) = \Lambda_{ctg}(G, ho, \alpha, r, L).
\]

For simplicity we let \( b = h^{-1}c \). For every \( n \in \mathbb{N} \), we can find an element \( g_n \in \mathcal{T}(o, L) \) such that \( d(o, g_n o) \geq n \) and \( b \in \mathcal{O}_o(g_n o, r) \). Denote by \( \tau_n \) the contracting tail for the pair \((o, g_n o)\) and \( p_n \) a projection of \( o \) on \( \tau_n \) satisfying \( d(p_n, g_n o) \geq L \). Up to passing to a subsequence, we can assume that \( d(o, g_n o) \geq d(o, h^{-1}o) + L + \alpha \), for every \( n \in \mathbb{N} \).

Let \( n \in \mathbb{N} \). We claim that \( hg_n o \) belongs to \( \mathcal{T}(o, L-\alpha) \). Let \( p'_n \) be a projection of \( h^{-1}o \) on \( \tau_n \). It suffices to prove that \( d(p'_n, g_n o) \geq L \). Indeed, after translating the figure by \( h \), it tells us that \( h\tau_n \) is a contracting tail for the pair \((o, hg_n o)\).
We distinguish two cases. Assume first that \( d(p_n, p'_n) > \alpha \). It follows from Lemma 2.3 that \( d(o, p'_n) \leq d(o, h^{-1} o) + 2\alpha \). The triangle inequality yields

\[
d(p'_n, g_n o) \geq d(o, g_n o) - d(o, p'_n) \geq d(o, g_n o) - d(o, h^{-1} o) - 2\alpha \geq L - \alpha
\]

Assume now that \( d(p_n, p'_n) \leq \alpha \). The triangle inequality yields

\[
d(p'_n, g_n o) \geq d(p_n, g_n o) - d(p_n, p'_n) \geq \alpha
\]

which completes the proof of our claim.

We now prove that \( c \in \mathcal{O}_o(h g_n o, r + 18\alpha) \). Let \( q_n \) be a projection of \( b \) onto \( \tau_n \). According to Lemma 4.14, we have \( d(q_n, g_n o) \leq r + 9\alpha \). Combining the above discussion with the triangle inequality we have

\[
d(p'_n, q_n) \geq d(p'_n, g_n o) - d(q_n, g_n o) \geq L - (r + 10\alpha) > 6\alpha.
\]

On the one hand, by Corollary 3.10, we have

\[
b(h^{-1} o, q_n) \leq d(h^{-1} o, q_n) - 18\alpha,
\]

i.e. \( \langle h^{-1} o, b \rangle_{q_n} \leq 9\alpha \). Consequently

\[
\langle h^{-1} o, b \rangle_{g_n o} \leq \langle h^{-1} o, b \rangle_{q_n} + d(q_n, g_n o) \leq r + 18\alpha.
\]

Recall that \( b = h^{-1} c \). The above inequality implies that \( c \) belongs to the shadow \( \mathcal{O}_o(h g_n o, r + 18\alpha) \), which completes the proof of our claim. Note that \( d(o, h g_n o) \) diverges to infinity, hence \( c \in \Lambda_{ctg}(G, o, \alpha, r + 18\alpha, L - \alpha) \).

The points in the contracting limit set have a specific behavior with respect to the equivalence relation \( \sim \) used to define the reduced horoboundary (see Section 3.2). Proceeding as for the radial limit set (see Lemma 4.22) one checks that the contracting limit set is saturated. The next statement precis this fact.

**Proposition 5.11.** Let \( \alpha \in \mathbb{R}^+ \) and \( r, L \in \mathbb{R}^+ \) with \( L > r + 50\alpha \). Let \( c, c' \in \partial X \) such that \( c \sim c' \). Assume that \( c \) belongs to \( \Lambda_{ctg}(G, o, \alpha, r, L) \). Then \( ||c - c'||_{\infty} \leq 36\alpha \) and \( c' \) belongs to \( \Lambda_{ctg}(G, o, \alpha, r + 33\alpha, L) \). In particular, the saturation of \( \Lambda_{ctg}(G, o, \alpha, r) \) is contained in \( \Lambda_{ctg}(G, o, \alpha, r + 33\alpha) \).

**Proof.** Since \( c \) belongs to \( \Lambda_{ctg}(G, o, \alpha, r, L) \), there exists a sequence of elements \( g_n \in \mathcal{T}(o, L) \) such that \( d(o, g_n o) \) diverges to infinity and \( c \in \mathcal{O}_o(g_n o, r) \) for every \( n \in \mathbb{N} \). For each \( n \in \mathbb{N} \), the pair \( (o, g_n o) \) has an \( (\alpha, L) \)-contracting tail, say \( \tau_n \). Let \( q_n \) (respectively \( q'_n \)) be a projection of \( c \) (respectively \( c' \)) onto \( \tau_n \). We break the proof in several steps.

**Claim 5.12.** For every \( n \in \mathbb{N} \), we have \( d(q_n, q'_n) \leq 15\alpha \).

Let \( \gamma : \mathbb{R}^+ \to X \) be a gradient ray for \( c \) starting at \( o \). Let \( n \in \mathbb{N} \). According to Lemma 4.14, the path \( \gamma \) intersects \( N_\alpha(\tau_n) \). Moreover the exit point \( \gamma(T) \) of \( \gamma \) in \( N_\alpha(\tau_n) \) is \( 7\alpha \)-closed to \( q_n \). Let \( t \geq T \) and \( z_t \) a projection of \( \gamma(t) \) onto \( \tau_n \).
Corollary 3.10. Assume that contrary to our claim $d(q_n', q_n) > 15\alpha$. Hence

$$d(q_n', z_t) \geq d(q_n, q_n') - d(q_n, z_t) > 6\alpha.$$ 

It follows from Corollary 3.10 applied with $c'$ that

$$c'(\gamma(t), q_n') \geq d(q_n', \gamma(t)) - 18\alpha \geq -18\alpha.$$ 

On the other hand, since $\gamma$ is a gradient ray for $c$, we know that

$$c(\gamma(t), q_n') = c(\gamma(t), o) + c(o, q_n') = -t + c(o, q_n').$$ 

These two estimates hold for every $t \geq T$, thus contradicting the fact that $\|c - c'\|_{\infty} < \infty$. It completes the proof of our first claim.

Claim 5.13. For every $n \in \mathbb{N}$, the cocycle $c'$ belongs to $\mathcal{O}_o(g_n o, r + 33\alpha)$. In particular, $c' \in \Lambda_{ctk}(G, o, \alpha, r + 33\alpha, L)$.

Let $n \in \mathbb{N}$. Since $(o, g_n o)$ has a contracting tail, there is a projection $p_n$ of $o$ on $\tau_n$ such that $d(p_n, g_n o) \geq L$. According to Lemma 4.14, $d(q_n, g_n o) \leq r + 9\alpha$. Our previous claim, combined with the triangle inequality, gives

$$d(p_n, q_n) \geq d(p_n, g_n o) - d(q_n', q_n) - d(q_n, g_n o) \geq L - (r + 24\alpha) > 6\alpha.$$ 

Hence by Corollary 3.10, we have $\langle o, c' \rangle_{q_n'} \leq 9\alpha$, which combined with (7) yields

$$\langle o, c' \rangle_{g_n o} \leq \langle o, c' \rangle_{q_n'} + d(q_n, g_n o) + d(q_n, q_n') \leq r + 33\alpha.$$ 

This completes the proof of our second claim.

Claim 5.14. $\|c - c'\|_{\infty} \leq 36\alpha$.

Consider a closed ball $K$ of radius $R$ centered at $o$. If $n \in \mathbb{N}$ is sufficiently large, then $d(o, g_n o) > R + r + 50\alpha$. According to our previous claim $c$ and $c'$ both belong to $\mathcal{O}_o(g_n o, r + 33\alpha)$. It follows from Lemma 5.8 that $\|c - c'\|_K \leq 36\alpha$. This estimate does not depend on $K$, whence the result. 

In a CAT(-1) settings, shadows are known to provide a basis of open neighborhoods for the points in the boundary at infinity. This is no more the case in our context. However we can still approximate saturated subsets of the contracting limit set using shadows. This is the purpose of the next lemma.

Corollary 5.15. Let $\alpha \in \mathbb{R}_+^*$ and $r, L \in \mathbb{R}_+$ with $L > r + 17\alpha$. Let $B$ be a saturated subset of $\Lambda_{ctk}(G, o, \alpha, r, L)$ and $V$ an open subset of $X$ containing $B$. Let $b \in B$. There exists $T \in \mathbb{R}_+$ such that for every $g \in T(\alpha, L)$ with $d(o, go) \geq T$, if $b$ belongs to $\mathcal{O}_o(go, r)$ then $\mathcal{O}_o(go, r) \subset V$. 

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Proof. Assume on the contrary that our statement fails. We can find a sequence of elements $g_n \in T(\alpha, L)$ such that $d(o, g_n o)$ diverges to infinity, $b$ belongs to $\mathcal{O}_o(g_n o, r)$ and $\mathcal{O}_o(g_n o, r) \setminus V$ is non-empty. For every $n \in \mathbb{N}$, we write $c_n$ for a cocycle in $\mathcal{O}_o(g_n o, r) \setminus V$. Up to passing to a subsequence, we can assume that $c_n$ converges to $c \in \bar{X}$. As $V$ is open, $c \not\in V$. We claim that $\|c - b\|_{\infty} < \infty$.

Let $K$ be a closed ball of radius $R$ centered at $o$. As $c_n$ converges to $c$, there is $N \in \mathbb{N}$, such that for every $n \geq N$, we have $\|c_n - c\|_K \leq 1$. By construction $b$ and $c_n$ both belong to $\mathcal{O}_o(g_n o, r)$. According to Lemma 5.8, if $n$ is sufficiently large $\|b - c_n\|_K \leq 36\alpha$ so that $\|b - c\|_K \leq 36\alpha + 1$. This inequality holds for every compact subset $K \subset X$, which completes the proof of our claim. Since $B$ is saturated, $c \in B$. It contradicts the fact that $B \subset V$. \hfill \Box

5.3 Measure of the contracting limit set

From now on, we assume that $G$ is not virtually cyclic and contains a contracting element. According to Corollary 4.10, there are $(\varepsilon, r_0) \in \mathbb{R}_+^* \times \mathbb{R}_+$ such that any $G$-invariant, $\omega_G$-conformal density $\nu = (\nu_x)$ satisfies the Shadow Principle with parameters $(\varepsilon, r_0)$. The goal of this section is to prove that if the action of $G$ on $X$ is divergent, then any such density gives full measure to the contracting limit set. The proof is an application of the Kochen-Stone theorem, which generalizes the second Borel-Cantelli Lemma.

**Proposition 5.16 (Kochen-Stone [29]).** Let $(\Omega, \mu)$ be a probability space. Let $(B_n)$ be a sequence of subsets of $\Omega$ such that

$$\sum_{n \in \mathbb{N}} \mu(B_n) = \infty.$$ 

Assume that there exists $C \in \mathbb{R}_+^*$ such that for every $N \in \mathbb{N},$

$$\sum_{n_1 = 0}^{N} \sum_{n_2 = 0}^{N} \mu(B_{n_1} \cap B_{n_2}) \leq C \left( \sum_{n=0}^{N} \mu(B_n) \right)^2.$$ 

Then

$$\mu \left( \bigcap_{N \in \mathbb{N} \ n \geq N} \bigcup_{n=0}^{N} B_n \right) \geq \frac{1}{C}.$$ 

Recall that for every $\alpha, r, \ell, L \in \mathbb{R}_+$, the set $A_\ell(\alpha, r, L)$ is defined by

$$A_\ell(\alpha, r, L) = \bigcup_{g \in S(\ell, r) \cap T(\alpha, L)} \mathcal{O}_o(g o, r).$$

These are the sets with which we will apply the Kochen-Stone theorem. The aim of the next lemmas is to make sure that the hypotheses of Proposition 5.16 are satisfied.
Lemma 5.17. Let $a \in \mathbb{R}^*_+$. There are $\alpha, r_1, C \in \mathbb{R}^*_+$, such that for every $r \geq r_1$ and $L > r + 17\alpha$, the following holds. Let $\nu = (\nu_x)$ be a $G$-invariant, $\omega_G$-conformal density. For every integer $N \in \mathbb{N}$,
\[
\sum_{g \in G} e^{-\omega_G d(o,go)} \leq C_1 e^{2\omega_G L} \sum_{n=0}^N \nu_o(B_n),
\]
where $B_n = A_{na}(\alpha, r, L)$.

Proof. Recall that $(\varepsilon, r_0)$ are the Shadow Principle parameters which have been fixed once and for all at the beginning of Section 5.3. We denote by $\alpha, r_1, C \in \mathbb{R}^*_+$ the parameters given by Proposition 4.17 applied with $\omega_G$, $a$, and $(\varepsilon, r_0)$. Fix $r \geq r_1$ and $L > r + 17\alpha$. Let $\nu = (\nu_x)$ be a $G$-invariant, $\omega_G$-conformal density. Recall that $(G, \nu)$ satisfies the Shadow Principle with parameters $(\varepsilon, r_0)$. It follows from Proposition 4.17 that for every $\ell \in \mathbb{R}^*_+$,
\[
\sum_{g \in S(\ell,a)} e^{-\omega_G d(o,go)} \leq C e^{2\omega L} \nu_o(A_{\ell+L}(\alpha, r, L)).
\]
Summing this identity we get for every $N \in \mathbb{N}$,
\[
\sum_{n=0}^N \sum_{g \in S(na,a)} e^{-\omega_G d(o,go)} \leq C e^{2\omega L} \sum_{n=0}^N \nu_o(A_{na+L}(\alpha, r, L)).
\]
Note that $A_{na+L}(\alpha, r, L)$ is covered by $A_{(m-1)a}(\alpha, r, L)$ and $A_{ma}(\alpha, r, L)$ where $m = n + \lceil L/a \rceil$. Hence
\[
\sum_{g \in G} e^{-\omega_G d(o,go)} \leq 2C e^{2\omega L} \sum_{n=0}^{N+\lceil L/a \rceil} \nu_o(B_n),
\]
whence the result. \qed

Lemma 5.18. Let $a, \alpha, r \in \mathbb{R}^*_+$. There are $b, C_2 \in \mathbb{R}^*_+$ such that for every $L > r + 17\alpha$, the following holds. Let $\nu = (\nu_x)$ be a $G$-invariant, $\omega_G$-conformal density. For every $N \in \mathbb{N}$, we have
\[
\sum_{n_1=0}^K \sum_{n_2=0}^N \nu_o(B_{n_1} \cap B_{n_2}) \leq C_2 \left( \sum_{g \in G} e^{-\omega_G d(o,go)} \right)^2,
\]
where $B_n = A_{na}(\alpha, r, L)$.
Proof. Let \( L > r + 17\alpha \). Let \( \nu = (\nu_x) \) be a \( G \)-invariant, \( \omega_G \)-conformal density. Let \( N \in \mathbb{N} \). Observe first that

\[
\sum_{n_1=0}^N \sum_{n_2=0}^N \nu_o(B_{n_1} \cap B_{n_2}) \leq 2 \sum_{n_1=0}^N \sum_{n_2=n_1}^N \nu_o(B_{n_1} \cap B_{n_2})
\]

\[
\leq 2 \sum_{n_1=0}^N \sum_{n_2=0}^{N-n_1} \nu_o(B_{n_1} \cap B_{n_1+n_2}).
\]

(14)

Consider now \( n_1, n_2 \in \mathbb{N} \) with \( 0 \leq n_1 \leq n_1 + n_2 \leq N \). By definition, we have

\[
B_{n_1} \cap B_{n_1+n_2} = \bigcup_{(g_1, g_2) \in U} \mathcal{O}_o(g_1 \alpha, r) \cap \mathcal{O}_o(g_1 g_2 o, r),
\]

where \( U \) is the set of pairs \((g_1, g_2) \in G\) with the following properties:

(U1) \( g_1, g_1 g_2 \in T(\alpha, L) \),

(U2) \( g_1 \in S(n_1 a, r) \) and \( g_1 g_2 \in S(n_1 a + n_2 a, r) \).

Let \( g_1, g_2 \in U \) for which \( \mathcal{O}_o(g_1 \alpha, r) \cap \mathcal{O}_o(g_1 g_2 o, r) \) is non-empty. According to Lemma 4.15, we have

\[
d(o, g_2 o) \leq d(g_1 o, g_1 g_2 o) \leq |d(o, g_1 g_2 o) - d(o, g_1 \alpha)| + 4r + 64\alpha.
\]

In particular,

\[
d(o, g_1 g_2 o) \geq d(o, g_1 \alpha) + d(o, g_2 o) - 4r - 64\alpha.
\]

Moreover, combined with (U2), it shows that \( g_2 \in S(n_2 a, 6r + 64\alpha) \). Using Remark 4.11, we estimate the measure of each shadow:

\[
\nu_o\left(\mathcal{O}_o(g_1 \alpha, r) \cap \mathcal{O}_o(g_1 g_2 o, r)\right) \leq \nu_o\left(\mathcal{O}_o(g_1 g_2 o, r)\right)
\]

\[
\leq e^{2\omega_G r} e^{-\omega_G d(o, g_1 g_2 o)}
\]

\[
\leq e^{\omega_G (6r + 64\alpha)} e^{-\omega_G [d(o, g_1 \alpha) + d(o, g_2 o)]}.
\]

Consequently

\[
\nu_o\left(B_{n_1} \cap B_{n_1+n_2}\right) \leq e^{\omega_G (6r + 64\alpha)} \sum_{g_1 \in S(n_1 a, r), g_2 \in S(n_2 a, 6r + 64\alpha)} e^{-\omega_G [d(o, g_1 \alpha) + d(o, g_2 o)]}.
\]

Note that for every \( d \in \mathbb{R}_+ \), an element \( g \in G \) belongs to at most \( \lceil 2d/a \rceil \) spheres of the form \( S(na, d) \) when \( n \) runs over \( \mathbb{N} \). Summing the previous inequality over \( n_1 \) and \( n_2 \), and using (14) we get

\[
\sum_{n_1=0}^N \sum_{n_2=0}^N \nu_o(B_{n_1} \cap B_{n_2}) \leq C_2 \sum_{g_1, g_2 \in G, d(o, g_1 \alpha), d(o, g_2 o) \leq Na+b} e^{-\omega_G [d(o, g_1 \alpha) + d(o, g_2 o)]}
\]

\[
\leq C_2 \left(2 \sum_{g \in G, d(o, g) \leq Na+b} e^{-\omega_G d(o, g)}\right)^2,
\]

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where
\[ b = a + 6r + 64\alpha, \quad \text{and} \quad C_2 = 2e^{\omega_G(6r+64\alpha)} \left[ \frac{12r + 128\alpha}{a} \right]^2 \]

only depends on \( a, \alpha \) and \( r \).

**Proposition 5.19.** Assume that the action of \( G \) on \( X \) is divergent. There exists \( \alpha, r \in \mathbb{R}_+ \) with the following property. Let \( \nu = (\nu_z) \) be a \( G \)-invariant, \( \omega_G \)-conformal density. For every \( L > r + 17\alpha \), we have
\[ \nu_0(\Lambda_{ctg}(G, o, \alpha, r, L)) > 0. \]

**Proof.** Fix \( a \in \mathbb{R}_+^* \). Let \( \alpha, r_1, C_1 \) be the parameters given by Lemma 5.17. Fix \( r \geq r_1 \). Let \( b, C_2 \) be the parameters given by Lemma 5.18 applied with \( a, \alpha \) and \( r \). Choose \( L > r + 17\alpha \) and set \( a' = \max\{a, b, L\} \). We write \( C' \) for the constant given by Corollary 4.18 applied with \( \omega_G, a' \) and \( (\epsilon, r_0) \).

Let \( \nu = (\nu_z) \) be a \( G \)-invariant, \( \omega_G \)-conformal density. For simplicity; for every \( n \in \mathbb{N} \), we let \( B_n = A_{nm}(r, \alpha, L) \). Since the action of \( G \) is divergent, Lemma 5.17 tells us that
\[ \sum_{n \in \mathbb{N}} \nu_0(B_n) = \infty. \]

Recall that \((G, \nu)\) satisfies the Shadow Principle with parameters \((\epsilon, r_0)\). By Corollary 4.18
\[ \sum_{g \in S(\ell, a')} e^{-\omega_G d(o, go)} \leq C, \quad \forall \ell \in \mathbb{R}_+. \]

Since the Poincaré series of \( G \) diverges at \( s = \omega_G \), we deduce from Lemmas 5.17 and 5.18 that there exists \( C' \in \mathbb{R}_+^* \), such that for every sufficiently large \( N \in \mathbb{N} \),
\[ \sum_{n_1=0}^{N} \sum_{n_2=0}^{N} \nu_0(B_{n_1} \cap B_{n_2}) \leq C' \left( \sum_{n=0}^{N} \nu_0(B_n) \right)^2. \]

Applying Proposition 5.16, we observe that
\[ \nu_0(\Lambda_{ctg}(G, o, \alpha, r, L)) = \nu_0 \left( \bigcap_{N \in \mathbb{N}} \bigcup_{n \geq N} B_n \right) \geq \frac{1}{C'}. \]

**Corollary 5.20.** Assume that the action of \( G \) on \( X \) is divergent. There exist \( \alpha, r \in \mathbb{R}_+^* \) with the following property. If \( \nu = (\nu_z) \) is a \( G \)-invariant, \( \omega_G \)-conformal density, then \( \nu_0 \) gives full measure
\[ \Lambda_{ctg}(G, o, \alpha, r) = \bigcap_{L \in \mathbb{R}_+} \Lambda_{ctg}(G, o, \alpha, r, L). \]

In particular, \( \nu_o \) gives full measure to the contracting limit set \( \Lambda_{ctg}(G) \) and thus to the radial limit set \( \Lambda_{rad}(G) \).
Remark 5.21. The corollary is a reminiscence of the fact that contracting elements are “generic” in $G$, see Yang [49].

Proof. Let $\alpha, r$ be the parameters given by Proposition 5.19. Let $\nu = (\nu_x)$ be a $G$-invariant, $\omega_G$-conformal density. Let $L > r + 17\alpha$. For simplicity we let

$$B = GA_{\text{ctg}}(G, o, \alpha, r, L).$$

We claim that $\nu_o$ gives full measure to $B$. Assume on the contrary that the set $A = \bar{X} \setminus B$ has positive measure. We define a new density $\nu^* = (\nu^*_x)$ by

$$\nu_x^* = \frac{1}{\nu_o(A)} \mathbb{1}_A \nu_x.$$

Note that $\nu^*$ is $\omega_G$-conformal. By construction $B$, and thus $A$, is $G$-invariant. Hence $\nu^*$ is also $G$-invariant. It follows from Proposition 5.19, that the $\nu^*_o$ gives positive measure to $B$, a contradiction. According to Proposition 5.10, the set $B$ is contained in $A_{\text{ctg}}(G, o, \alpha, r + 18\alpha, L - \alpha)$. Hence the latter has full measure as well. This fact holds for every $L > r + 17\alpha$. Thus

$$\nu_0 \left( \bigcap_{L \in \mathbb{R}_+} A_{\text{ctg}}(G, o, \alpha, r + 18\alpha, L) \right) = 1,$$

i.e.

$$\nu_0 \left( A_{\text{ctg}}(G, o, \alpha, r + 18\alpha) \right) = 1. \quad \square$$

5.4 Passing to the reduced horoboundary

We now study the restriction to the reduced horoboundary of invariant conformal densities. We still assume that $G$ is not virtually cyclic and contains a contracting element. The Shadow Principle parameters $(\varepsilon, r_0) \in \mathbb{R}_+^* \times \mathbb{R}_+$ are as in the previous section.

Proposition 5.22. Assume that the action of $G$ on $X$ is divergent. There is $C \in \mathbb{R}_+^*$ with the following property. Let $\nu = (\nu_x)$ and $\nu' = (\nu'_x)$ be two $G$-invariant, $\omega_G$-conformal densities. Denote by $\mu = (\mu_x)$ and $\mu' = (\mu'_x)$ their respective restrictions to the reduced horocompactification $(\bar{X}, \mathcal{R})$. Then $\mu_o \leq C \mu'_o$.

Proof. Let $\alpha, r \in \mathbb{R}_+^*$ be the parameters given by Corollary 5.20. Without loss of generality we can assume that $r \geq r_0$. Let $\nu = (\nu_x)$ and $\nu' = (\nu'_x)$ be as in the statement. Let $B \subset \partial X$ be a saturated Borel subset. We want to compare $\mu_o(B) = \nu_o(B)$ and $\mu'_o(B) = \nu'_o(B)$. According to Corollary 5.20, both $\nu_o$ and $\nu'_o$ give full measures to $A_{\text{ctg}}(G, o, \alpha, r)$. Without loss of generality, we can assume that $B$ is contained in $A_{\text{ctg}}(G, o, \alpha, r)$. Let $V$ be an open subset of $\bar{X}$ containing $B$. Choose $L > r + 55\alpha$. Using Corollary 5.15, we build a subset $S \subset T(\alpha, L)$ such that

$$B \subset \bigcup_{g \in S} \mathcal{O}_o(go, r) \subset V.$$

According to Lemma 5.7, there is a subset $S^*$ of $S$ such that
• the collection \( (O_o(g_o, r))_{g \in S^*} \) is pairwise disjoint, and
• \( B \) is covered by \( (O_o(g_o, r + 56\alpha))_{g \in S^*} \).

Since \((G, \nu')\) satisfies the Shadow Principle, we get

\[
\nu_o(B) \leq \sum_{g \in S^*} \nu_o(O_o(g_o, r + 56\alpha)) \leq e^{2\omega G(r + 56\alpha)} \sum_{g \in S^*} e^{-\omega G(d(o, g_o))} \leq C \sum_{g \in S^*} \nu'_o(O_o(g_o, r)),
\]

where \( C = e^{2\omega G(r + 56\alpha)}/\varepsilon \) does not depend on \( \nu \) and \( \nu' \). Hence \( \nu_o(B) \leq C\nu'_o(V) \). This inequality holds for every open subset \( V \) containing \( B \), thus \( \nu_o(B) \leq C\nu'_o(B) \).

**Proposition 5.23.** Let \( \nu = (\nu_x) \) be a \( G \)-invariant, \( \omega_G \)-conformal density and \( \mu = (\mu_x) \) its restriction to the reduced horocompactification \((\bar{X}, \mathcal{R})\). Assume that the action of \( G \) on \( X \) is divergent. Then

(i) \( \mu_o \) is supported on the contracting limit set

(ii) \( \mu_o \) is ergodic.

(iii) \( \mu_o \) is non-atomic, in the sense that no equivalence class for \( \sim \) has positive measure.

(iv) \( \mu \) is a \( G \)-invariant, \( \omega_G \)-quasi-conformal density.

(v) \( \mu \) is almost-unique, that is there is \( C \in \mathbb{R}_+^* \) such that if \( \mu' = (\mu'_x) \) is the restriction to the reduced horocompactification of another \( G \)-invariant, \( \omega_G \)-conformal density, then for every \( x \in X \), we have \( \mu'_x \leq C\mu_x \).

**Proof.** Let \( \alpha, r \in \mathbb{R}_+^* \) be the parameters given by Corollary 5.20 and \( C \in \mathbb{R}_+^* \) the one given by Proposition 5.22. Let \( \nu = (\nu_x) \) be a \( G \)-invariant, \( \omega_G \)-conformal density and \( \mu = (\mu_x) \) its restriction to the reduced horocompactification \((\bar{X}, \mathcal{R})\).

It follows from Corollary 5.20 that \( \mu_o \) gives full measure to the contracting limit set. Let us prove the ergodicity of \( \mu_o \). Let \( B \) be a \( G \)-invariant saturated Borel subset such that \( \mu_o(B) > 0 \). Consider the density \( \nu^* = (\nu^*_x) \) defined by

\[
\nu^*_x = \frac{1}{\nu_o(B)} \mathbb{1}_B \nu_x.
\]

Since \( B \) is \( G \)-invariant, \( \nu^* \) is a \( G \)-invariant, \( \omega_G \)-conformal density. Denote by \( \mu^* = (\mu^*_x) \) its restriction to the reduced horocompactification. It follows from Proposition 5.22 that \( \mu_o \leq C\mu^*_o \). Consequently \( \mu_o(\bar{X} \setminus B) = 0 \), that is \( \mu_o(B) = 1 \).

We now focus on non-atomicity. Let \( c \in \partial X \) and \( F \) its equivalence class. We need to prove that \( \mu_o(F) = 0 \). By Corollary 5.20 \( \mu_o \) only charges the set \( \Lambda_{ctg}(G, o, \alpha, r) \). Without loss of generality, we can assume that \( F \) contains
a cocycle \( c \in \Lambda_{ctg}(G, o, \alpha, r) \). In particular, there exists a sequence \((g_n)\) of elements of \( G \) such that \( d(o, g_n o) \) diverges to infinity and \( c \in O_o(g_n o, r) \). By Proposition 5.11, \( \| c - b \|_\infty \leq 36\alpha \), for every \( b \in F \). Thus \( F \) is contained in \( O_o(g_n o, r + 36\alpha) \), for every \( n \in \mathbb{N} \). It follows from Remark 4.11 that

\[
\mu_o(F) \leq \nu_o(O_o(g_n o, r + 36\alpha)) \leq e^{2\omega_G(r + 36\alpha)} e^{-\omega_G d(o, g_n o)}, \quad \forall n \in \mathbb{N}.
\]

Since \((g_n)\) diverges to infinity, we get \( \mu_o(F) = 0 \).

Let us prove (iv). It follows from the construction that \( \mu \) is \( G \)-invariant, \( \| \mu \| = 1 \), and \( \mu_x \ll \mu_y \), for every \( x, y \in X \). Hence we are only left to prove that \( \mu \) is quasi-conformal. Let \( x, y \in X \). We define two auxiliary maps as follows.

\[
\begin{align*}
\bar{X} & \to \mathbb{R} & & \text{and} & & \bar{X} & \to \mathbb{R} \\
c & \mapsto \inf_{c' \sim c} c'(x, y) & & c & \mapsto \sup_{c' \sim c} c'(x, y)
\end{align*}
\]

We denote them by \( c \mapsto \beta^-_c(x, y) \) and \( c \mapsto \beta^+_c(x, y) \) respectively. As \( \bar{X} \) is separable, one checks that these maps are \( \mathfrak{B} \)-measurable. Let \( B \) be a saturated Borel subset. Using the conformality of \( \nu \) we have

\[
\nu_x(B) \leq \int 1_B(c) e^{-\omega_G c(x, y)} d\nu_y(c) \leq \int 1_B(c) e^{-\omega_G \beta^-_c(x, y)} d\nu_y(c).
\]

Since \( B \) is saturated and \( c \mapsto \beta^-_c(x, y) \) is \( \mathfrak{B} \)-measurable, we get

\[
\mu_x(B) \leq \int 1_B(c) e^{-\omega_G \beta^-_c(x, y)} d\mu_y(c).
\]

This inequality holds for every \( B \in \mathfrak{B} \). Hence

\[
\frac{d\mu_x}{d\mu_y}(c) \leq e^{-\omega_G \beta^-_c(x, y)}, \quad \mu\text{-a.e.}
\]

In the same way, we obtain a lower bound for the Radon-Nikodym derivative with \( \beta^+_c(x, y) \) in place of \( \beta^-_c(x, y) \). By Proposition 5.11, for \( \mu \)-almost every \( c \in \bar{X} \), we have

\[
c(x, y) - 36\alpha \leq \beta^-_c(x, y) \leq \beta^+_c(x, y) \leq c(x, y) + 36\alpha.
\]

Hence \( \mu \) is quasi-conformal. Point (v) now follows from Proposition 5.22 and the quasi-conformality. \( \square \)

### 5.5 More applications

**Proposition 5.24.** Assume that the action of \( G \) on \( X \) is divergent. For every infinite normal subgroup of \( G \) we have

\[
\omega_N > \frac{1}{2} \omega_G.
\]

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Proof. Let \( Q = G/N \) and \( \omega_Q \) be the growth rate of \( Q \) on \( X/N \). According to Corollary 4.29 we have

\[
\omega_N + \frac{1}{2}\omega_Q \geq \omega_G.
\]

Since the map \( X \to X/N \) is 1-Lipschitz, \( \omega_Q \leq \omega_G \). Hence

\[
\omega_N \geq \frac{1}{2}\omega_G.
\]

Suppose now that, contrary to our claim, \( \omega_G = 2\omega_N \). We choose

- a \( G \)-invariant, \( \omega_G \)-conformal density \( \nu = (\nu_x) \) and
- an \( N \)-invariant, \( \omega_N \)-conformal density \( \nu' = (\nu'_x) \) such that the action of \( N \) on \( (\bar{X}, \mathcal{B}, \nu'_{o}) \) is ergodic.

We write \( \mu \) and \( \mu' \) for their respective restrictions to the reduced horocompactification \( (\bar{X}, \mathcal{R}) \). In particular, the action of \( N \) on \( (\bar{X}, \mathcal{R}, \mu'_{o}) \) is ergodic. We claim that \( \mu_{0} \) is absolutely continuous with respect to \( \mu'_{0} \). According to Corollary 4.10, \((G, \nu')\) satisfies the Shadow Lemma for some parameters \((\varepsilon, r_0) \in \mathbb{R}_+ \times \mathbb{R}_+^*\).

By Corollary 5.20, there exists \( \alpha, r \in \mathbb{R}_+ \) such that \( \nu_{o} \) gives full measure to \( \Lambda_{ctg}(G, o, \alpha, r) \). Without loss of generality, we can assume that \( r \geq r_0 \).

For our purpose, it suffices to compare \( \mu \) and \( \mu' \) on \( \Lambda_{ctg}(G, o, \alpha, r) \). Let \( B \) be a saturated subset contained in \( \Lambda_{ctg}(G, o, \alpha, r) \). Let \( V \) be an open set containing \( B \). Fix \( L > r + 55\alpha \). Using Corollary 5.15, we build a subset \( S \subset T(\alpha, L) \) such that

\[
B \subset \bigcup_{g \in S} O_o(go, r) \subset V.
\]

According to Lemma 5.7, there is a subset \( S^* \) of \( S \) such that

- the collection \( (O_o(go, r))_{g \in S^*} \) is pairwise disjoint, and
- \( B \) is covered by \( (O_o(go, r + 56\alpha))_{g \in S^*} \).

Using Remark 4.11 with the density \( \nu \) we get

\[
\nu_o(B) \leq \sum_{g \in S^*} \nu_o(O_o(go, r + 56\alpha)) \leq e^{2\omega_G(r + 56\alpha)} \sum_{g \in S^*} e^{-\omega_Gd(o,go)}.
\]

Recall that for every \( g \in G \), we have \( \|\nu'_g\| \geq e^{-\omega_Nd(o,go)} \). Since \( \omega_G = 2\omega_N \), we obtain

\[
\nu_o(B) \leq e^{2\omega_G(r + 56\alpha)} \sum_{g \in S^*} \|\nu'_g\| e^{-\omega_Nd(o,go)}.
\]

Using now the Shadow Principle with the density \( \nu' \) we obtain

\[
\nu_o(B) \leq C \sum_{g \in S^*} \nu'_o(O_o(go, r)) \leq C \nu'_o(V), \quad \text{where} \quad C = \frac{1}{\varepsilon} e^{2\omega_G(r + 60\alpha)}.
\]
does not depend on $B$. This inequality holds for every open subset $V$ containing $B$, hence $\nu_0(B) \leq C\nu'_0(B)$, i.e. $\mu_0(B) \leq C\mu'_0(B)$. This completes the proof of our claim.

Denote by $f$ the Radon-Nikodym derivative $f = d\mu/d\mu'$. Both $\mu$ and $\mu'$ are $N$-invariant. Hence the set $A = \{ c \in \bar{X} : f(c) > 0 \}$ is $N$-invariant. Note that $\mu'_0(A) > 0$. Indeed otherwise $\mu_0$ would be the zero measure. Since the action of $N$ on $(\bar{X}, \mathcal{R}, \mu)$ is ergodic, we get $\mu'_0(A) = 1$. Hence $\mu_0$ and $\mu'_0$ are in the same class of measures. Since $\mu_0$ is $G$-invariant, $\mu'_0$ is $G$-quasi-invariant. We assumed that $\mu'_0$ is ergodic for the action of $N$. It follows from Lemma 4.5 that $\mu'$ is almost-fixed by $G$. Thus $\omega_N \geq \omega_G$ by Corollary 4.26. This contradicts our assumption and completes the proof.

**Proposition 5.25.** Let $H \subset G$ be a subgroup which is not virtually cyclic and contains a contracting element. Let $\nu = (\nu_x)$ be an $H$-invariant, $\omega_H$-conformal density and $\mu = (\mu_x)$ its restriction to the reduced horocompactification $(\bar{X}, \mathcal{R})$. Assume that the action of $H$ is divergent. If $\mu$ is almost fixed by $G$, then $(G, \nu)$ satisfies the Shadow Principle.

**Proof.** According to Corollary 5.20, there are $\alpha, r_0 \in \mathbb{R}_+^+$ such that $\nu$ gives full measure to $\Lambda_{\text{ctg}}(H, o, \alpha, r_0)$. Proceeding as in the proof of Corollary 4.10, we show that for every $g \in G$ and $r \in \mathbb{R}_+$,

$$\nu_0(\mathcal{O}_o(go, r)) \geq \|\nu_{go}\| e^{-\omega_H d(o, go)} \nu_0(\mathcal{O}_{g^{-1}o}(o, r)).$$

(15)

Choose now $r \geq r_0$ and $g \in G$. We denote by $\mathcal{O}^+_{g^{-1}o}(o, r)$ the saturation of the shadow $\mathcal{O}_{g^{-1}o}(o, r)$. According to Proposition 5.11,

$$\mathcal{O}^+_{g^{-1}o}(o, r) \cap \Lambda_{\text{ctg}}(H, o, \alpha, r) \subset \mathcal{O}_{g^{-1}o}(o, r + 36\alpha).$$

Recall that $\nu$ gives full measure to $\Lambda_{\text{ctg}}(H, o, \alpha, r_0)$, thus to $\Lambda_{\text{ctg}}(H, o, \alpha, r)$ as well. Since $\mu$ is almost fixed by $G$ we have

$$\nu_0(\mathcal{O}_{g^{-1}o}(o, r + 36\alpha)) \geq \mu_0(\mathcal{O}^+_{g^{-1}o}(o, r)) \geq \varepsilon \mu_0(\mathcal{O}^+_{g^{-1}o}(o, r)) \geq \varepsilon \nu_0(\mathcal{O}_{g^{-1}o}(o, r),$$

where $\varepsilon \in \mathbb{R}_+$ does not depend on $g$ and $r$. Combined with (15) it shows that for every $r \geq r_0$, for every $g \in G$, we have

$$\nu_0(\mathcal{O}(go, r + 36\alpha)) \geq \varepsilon \|\nu_{go}\| e^{-\omega_H d(o, go)} \nu_0(\mathcal{O}_{g^{-1}o}(o, r)).$$

According to our assumption, $H$ is not virtually cyclic and contains a contracting element. The conclusion now follows from Proposition 4.9 applied with the group $H$.

**Theorem 5.26.** Let $H$ be a commensurated subgroup of $G$. If the action of $H$ on $X$ is divergent, then the following holds.

(i) Any $H$-invariant, $\omega_H$-conformal density is $G$-almost invariant when restricted to the reduced horocompactification $(\bar{X}, \mathcal{R})$.

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(ii) \(\omega_H = \omega_G\).

(iii) The action of \(G\) on \(X\) is divergent.

Proof. Let \(\nu = (\nu_x)\) be an \(H\)-invariant, \(\omega_H\)-conformal density. We denote by \(\mu = (\mu_x)\) its restriction to the reduced horocompactification \((\bar{X}, \mathcal{R})\). Let \(g \in G\). By definition of commensurability the intersection \(H_0 = H^g \cap H\) has finite index in \(H\). In particular, \(H_0\) is divergent and \(\omega_{H_0} = \omega_H\). Recall that \(\nu^g\) is the image of \(\nu\) under the right action of \(g \in G\). It is an \(H^g\)-invariant, \(\omega_H\)-conformal density, thus an \(H_0\)-invariant, \(\omega_{H_0}\)-conformal density. Similarly \(\nu\) is an \(H_0\)-invariant, \(\omega_{H_0}\)-conformal density. Since \(H_0\) is divergent, Proposition 5.23 tells us that

- there is \(C \in \mathbb{R}_+\) such that \(\mu^g \leq C\mu\),
- the action of \(H^g \cap H\) on \((\bar{X}, \mathcal{R}, \mu_o)\) is ergodic.

Note that \(C\) depends a priori on \(H_o\) and thus on \(g\). Nevertheless, it still proves that \(\mu\) is \(G\)-quasi-invariant. Consequently \(\mu\) is \(C_0\)-almost fixed by \(G\), for some \(C_0 \in \mathbb{R}^*_+\) (Lemma 4.5). We deduce from Proposition 5.25 that \((G, \nu)\) satisfies the Shadow Principle. Point (ii) now follows from Corollary 4.26. Recall that \(\mathcal{P}_H(s) \leq \mathcal{P}_G(s)\), for every \(s \in \mathbb{R}_+\). Since the action of \(H\) on \(X\) is divergent, \(\mathcal{P}_H(s)\) diverges at \(s = \omega_H = \omega_G\). Hence the action of \(G\) on \(X\) is divergent as well, which proves (iii).

We already know that \(\mu\) is almost-fixed by \(G\), so that the map \(\chi: G \to \mathbb{R}\) sending \(g\) to \(\ln \|\mu_g\|\) is a quasi-morphism (Lemma 4.4). We are left to prove that \(\mu\) is actually \(G\)-almost invariant, i.e. \(\chi\) is bounded. Recall that \((G, \nu)\) satisfies the Shadow Principle. It follows from Proposition 4.23 that the critical exponent of the series

\[
\sum_{g \in G} e^{\chi(g)} e^{-sd(\nu_g)} = \sum_{g \in G} \|\nu_g\| e^{-sd(\nu_g)}
\]

is exactly \(\omega_H\). Hence \(\omega_{-\chi} = \omega_{\chi} = \omega_H\). Note also that, since \(\nu\) is \(H\)-invariant, \(\chi(hg) = \chi(g)\) for every \(h \in H\) and \(g \in G\). Using Proposition 4.3, with the quasi-morphism \(-\chi\), we produce an \(H\)-invariant, \(\omega_H\)-conformal density \(\nu^* = (\nu^*_x)\) satisfying the following additional property: there is \(C_1 \in \mathbb{R}^*_+\) such that for every \(g \in G\), for every \(x \in X\), we have

\[
\frac{1}{C_1} \nu_x \leq e^{\chi(g)} g^{-1} \nu_{gx} \leq C_1 \nu_x.
\]

Denote by \(\mu^*\) its restriction to the reduced horocompactification \((\bar{X}, \mathcal{R})\). According to Proposition 5.23(v), there is \(C_2 \in \mathbb{R}^*_+\) such that \(\mu \leq C_2\mu^*\). Recall that \(\mu\) is \(C_0\)-almost fixed by \(G\). Consequently for every \(g \in G\), we have

\[
e^{\chi(g)} \mu_o \leq C_0 \left(g^{-1} \mu_{go}\right) \leq C_0 C_2 \left(g^{-1} \mu^*_o\right) \leq C_0 C_1 C_2 \left(e^{-\chi(g)} \mu^*_o\right).
\]

Since \(\mu_o\) and \(\mu^*_o\) are probability measures, \(\chi\) is bounded, whence the result. \(\square\)
A Strongly positively recurrent actions

A.1 Definition

Let $G$ be a group acting properly, by isometries on a proper, geodesic, metric space $X$. Given a compact subset $K \subset X$, we define a subset $G_K \subset G$ as follows: an element $g \in G$ belongs to $G_K$ if there exist $x, y \in K$ and a geodesic $\gamma$ joining $x$ to $gy$ such that the intersection $\gamma \cap G_K$ is contained in $K \cup gK$. Although $G_K$ is not a subgroup of $G$, its exponential growth rate $\omega(G_K, X)$ is defined in the same way as for the one of $G$.

**Definition A.1.** The *entropy at infinity* of the action of $G$ on $X$ is

$$\omega_\infty(G, X) = \inf_{K} \omega(G_K, X),$$

where $K$ runs over all compact subsets of $X$. The action of $G$ on $X$ is strongly positively recurrent (or statistically convex co-compact) if $\omega_\infty(G, X) < \omega(G, X)$.

We refer the reader to [42, 17] for examples of strongly positively recurrent actions in the context of hyperbolic geometry. Arzhantseva, Cashen and Tao [1, Section 10] also observed that the work of Eskin, Mirzakani, and Rafi [19, Theorem 1.7] implies that the action of the mapping class group on the Teichmüller space endowed with the Teichmüller metric is strongly positively recurrent.

A.2 Divergence

**Proposition A.2.** If the action of $G$ on $X$ is strongly positively recurrent, then it is divergent.

The statement was proved by Yang [48]. We give here an alternative approach in the spirit of Schapira-Tapie [42]. The idea is to build a $G$-invariant, $\omega_G$-conformal density which gives positive measure to the radial limit set. Indeed, according to Proposition 4.23, this will imply that the action of $G$ on $X$ is divergent. As we explained in Remark 4.24 this part of Proposition 4.23 does not require that $G$ contains a contracting element.

First, we give a description of the complement of the radial limit set. To that end we introduce some notations. Given a compact subset $K \subset X$ and $\varepsilon \in \mathbb{R}^+$, we denote by $A_{K,\varepsilon}$ the set of all cocycles $c \in \partial X$ with the following property: there is a point $x \in K$ such that for every $\varepsilon$-quasi-gradient ray $\gamma: \mathbb{R}_+ \to X$ for $c$ starting at $x$, for every $u \in G$, if the intersection $\gamma \cap uK$ is non-empty, then $d(K, uK) \leq 1$.

**Lemma A.3.** The radial set of $G$ satisfies the following inclusion

$$\partial X \setminus \Lambda_{\text{rad}}(G) \subset \bigcap_{K \subset X} G \left( \bigcup_{\varepsilon > 0} A_{K,\varepsilon} \right),$$

where $K$ runs over all compact subsets of $X$. 

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Proof. The proof is by contraposition. Consider a cocycle $c \in \partial X$ that is not in the set
$$\bigcap_{K \subset X} \bigcup_{\varepsilon > 0} G \left( \bigcup_{\varepsilon > 0} A_{K,\varepsilon} \right).$$
There is a compact subset $K \subset X$ such that for every $g \in G$ and $\varepsilon > 0$, the cocycle $c$ does not belong to $gA_{K,\varepsilon}$. Fix $\varepsilon \in (0,1)$ and $x_0 \in K$. In addition we let $g_0 = 1$. We are going to build by induction, a sequence of points $x_1, x_2, \ldots$ in $X$, a sequence of elements $g_1, g_2, \ldots$ in $G$, and a sequence of rays $\gamma_1, \gamma_2, \ldots$, such that for every $i \in \mathbb{N} \setminus \{0\}$ the following holds.

(i) $x_i$ belongs to $g_iK$.

(ii) $c(x_0, x_i) \geq i/2$.

(iii) For every $i \in \mathbb{N} \setminus \{0\}$, the path $\gamma_i$ is a $2^{-i}\varepsilon$-quasi-gradient ray of $c$ starting at $x_{i-1}$ and passing through $x_i$.

Let $i \in \mathbb{N}$. Assume that $x_i \in X$, $g_i \in G$ have been defined. By assumption $c$ does not belong to the set $g_iA_{K,2^{-i+1}\varepsilon}$.

Hence there exists an $2^{-i+1}\varepsilon$-quasi-gradient ray $\gamma_{i+1} : \mathbb{R}_+ \to X$ for $c$ starting at $x_i$ and an element $u_i \in G$ such that $\gamma_{i+1} \cap g_iu_iK$ is non-empty and $d(g_iK, g_iu_iK) > 1$. We let $g_{i+1} = g_iu_i$ and denote by $x_{i+1}$ a point in $\gamma_{i+1} \cap g_{i+1}u_iK$. Since $x_i \in g_iK$ and $x_{i+1} \in g_{i+1}K$, we have $d(x_i, x_{i+1}) > 1$. However $\gamma_{i+1}$ is a quasi-gradient line. Hence
$$c(x_i, x_{i+1}) \geq d(x_i, x_{i+1}) - 2^{-i+1}\varepsilon \geq 1/2.$$ 

Using the induction hypothesis, we get
$$c(x_0, x_{i+1}) \geq c(x_0, x_i) + c(x_i, x_{i+1}) \geq (i + 1)/2.$$ 

Consequently $x_{i+1}$, $g_{i+1}$, and $\gamma_{i+1}$ satisfy the announced properties.

Note that the sequence $(x_i)$ is unbounded. Indeed otherwise $c(x_0, x_i)$ should be bounded as well. Thus we can build an infinite path $\gamma$ by concatenating the restriction of each $\gamma_i$ between $x_{i-1}$ and $x_i$. It follows from the construction that $\gamma$ is an $\varepsilon$-quasi-gradient line for $c$, see Remark 3.3. Moreover $\gamma$ intersects $g_iK$ for every $i \in \mathbb{N}$. One proves using the triangle inequality that $c$ belongs to the radial limit set.

Let $K \subset X$ be a compact subset and $\varepsilon \in \mathbb{R}_+^*$. For every compact subset $F \subset X$, we define $U_{K,\varepsilon}(F)$ to be the set of cocycles $b \in X$ for which there is a cocycle $c \in A_{K,\varepsilon}$ satisfying $\|b - c\|_F < \varepsilon$. Observe that $U_{K,\varepsilon}(F)$ is an open subset of $X$ containing $A_{K,\varepsilon}$. 

\[ \]
Lemma A.4. Let $K \subset X$ be a compact set and $\varepsilon \in \mathbb{R}^*_+$. Fix a base point $o \in K$. There exist $r \in \mathbb{R}_+$ and a finite subset $S \subset G$, such that for every $T \geq \varepsilon$, if $F$ stands for the closed ball of radius $T + r$ centered at $o$, then

$$U_{K,\varepsilon}(F) \cap G o \subset S \left( \bigcup_{k \in G_K \atop d(o, ko) \geq T} O_o(ko, r) \right).$$

Proof. Since the action of $G$ on $X$ is proper, the set

$$S = \{ u \in G : d(K, uK) \leq 1 \}$$

is finite. We fix $r > 2 \operatorname{diam} K + 1$. Let $T \geq \varepsilon$ and $F$ be the closed ball of radius $T + r$ centered at $o$. Let $g \in G$ such that $go$ belongs to $U_{K,\varepsilon}(F)$. We write $b = \iota(go)$ for the corresponding cocycle. By definition, there is $c \in A_{K,\varepsilon}$ such that $\|b - c\|_F < \varepsilon$. Observe first that $d(o, go) > T + r - \varepsilon$. Indeed the map $x \mapsto b(x, go)$ admits a global minimum at $go$, while there exists a $c$-gradient line starting at $go$. We cannot have at the same time $d(o, go) \leq T + r - \varepsilon$ and $\|b - c\|_F < \varepsilon$. In particular, $g \notin S$.

Since $c \in A_{K,\varepsilon}$, there exists $x \in K$, such that for every $\varepsilon$-quasi-gradient ray $\gamma : \mathbb{R}_+ \to X$ for $c$, starting at $x$, if $\gamma$ intersects $uK$ for some $u \in G$, then $u \in S$. Consider now a geodesic $\alpha : [0, \ell] \to X$ from $x$ to $go$. We denote by $s \in [0, \ell]$, the largest time such that $\alpha(s) \in SK$. We now denote by $t \in [s, \ell]$, the smallest time such that the point $z = \alpha(t)$ lies in $hK$ for some $h \in G \setminus S$ (such a time $t$ exists since $\alpha(\ell)$ belongs to $gK$). It follows from the construction that $h$ can be written $h = uk$ with $u \in S$ and $k \in G_K$. Observe that $\langle x, go \rangle_z = 0$. Moreover $d(x, uo) \leq r/2$ and $d(z,uko) \leq r/2$. The triangle inequality yields $\langle uo, go \rangle_{uko} \leq r$, i.e. $go$ belongs to $uO_o(ko, r)$.

We are left to prove that $d(o, ko) \geq T$. It suffices to show that $d(o, z) \geq T + r$. Assume on the contrary that $d(o, z) < T + r$. In particular, both $x$ and $z$ belong to $F$. Since $b$ and $c$ differ by at most $\varepsilon$ on $F$, we get that $c(x, z) > d(x, z) - \varepsilon$. Hence any geodesic from $x$ to $z$ is an $\varepsilon$-quasi-gradient arc for $c$. If we concatenate this path with a gradient ray for $c$ starting at $z$, we obtain an $\varepsilon$-quasi-gradient ray for $c$ starting at $x$ and intersecting $gK$ with $d(K, gK) > 1$. This contradicts the fact that $c$ belongs to $A_{K,\varepsilon}$, and completes the proof.

Proposition A.5. If the action of $G$ on $X$ is strongly positively recurrent, then there is a $G$-invariant, $\omega_G$-conformal density which gives full measure to the radial limit set $\Lambda_{rad}(G)$.

Proof. By definition, there is a compact subset $K \subset X$ such that $\omega_{G,K} < \omega_G$. We fix once and for all a base point $o \in K$. The argument relies on Patterson’s construction recalled in the proof of Proposition 4.3 with $H = G$ and $\chi$ the trivial morphism. For every $s > \omega_G$, we consider the density $\nu^s = (\nu^s_x)$ defined as in (5). As we explained there is a sequence $(s_n)$ converging to $\omega_G$ from above such that $\nu^{s_n}$ converges to a $G$-invariant, $\omega_G$-conformal density $\nu$ supported on $\partial X$. 

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Lemma A.4 applied, we have that there exists $\epsilon > 0$ such that $\omega_{G_K} < \omega_G - \eta$. The weight $\theta$ used to construct $\nu$ is slowly increasing. More precisely, according to (P1) there exists $t_0$ such that for every $t \geq t_0$ and $u \in \mathbb{R}_+$ we have

$$\theta(t + u) \leq e^{\eta u} \theta(t).$$

Let $\epsilon > 0$. Let $r \in \mathbb{R}_+$ and $S \subset G$ be the data provided by Lemma A.4 applied with $K$ and $\epsilon$. For every $T \in \mathbb{R}_+$, we write $F_T$ for the closed ball of radius $T + r$ centered at $o$. Let $s > \omega_G$ and $T \geq \max\{t_0, \epsilon\}$. In view of Lemma A.4, we have

$$\nu_\alpha^*(U_{K,\epsilon}(F_T)) \leq |S| \sum_{d \in G} \nu_\alpha^*(O_\alpha(ko, r)).$$

Let us estimate the measures of the shadows in the sum. Let $k \in G_K$, such that $d(o, ko) \geq T$. Any element $g \in G$ such that $g \in O_\alpha(ko, r)$ can be written $g = ku$ with $u \in G$ and

$$d(o, ko) + d(o, u) - 2r \leq d(o, go) \leq d(o, ko) + d(o, u) - 2r.$$  

Unfolding the definition of $\nu^*$, we get

$$\nu^*_\alpha(O_\alpha(ko, r)) \leq e^{2\eta r} \frac{e^{-sd(o, ko)}}{Q(s)} \sum_{u \in G} \theta(d(o, go)) e^{-sd(o, u)}. \tag{16}$$

Observe that if $d(o, u) \geq t_0$, then it follows from our choice of $t_0$ that

$$\theta(d(o, go)) \leq \theta(d(o, ko) + d(o, u)) \leq e^{\eta d(o, ko)} \theta(d(o, u)).$$

Otherwise, since $d(o, ko) \geq t_0$, we have

$$\theta(d(o, go)) \leq \theta(t_0 + d(o, u) + d(o, ko) - t_0) \leq e^{\eta d(o, ko)} \theta(t_0).$$

We break the sum in (16) according to the length of $u$ and get

$$\nu^*_\alpha(O_\alpha(ko, r)) \leq e^{2\eta r} \frac{e^{-(s-\eta)d(o, ko)}}{Q(s)} \left[\theta(t_0)\Sigma_1(s) + \Sigma_2(s)\right],$$

where

$$\Sigma_1(s) = \sum_{u \in G} e^{-sd(o, u)},$$

$$\Sigma_2(s) = \sum_{u \in G} \theta(d(o, u)) e^{-sd(o, u)}.$$  

Note that $\Sigma_1(s)$ is a finite sum that does not depend on $k$, while $\Sigma_2(s)$ is the remainder of the series $Q(s)$. Hence

$$\nu^*_\alpha(O_\alpha(ko, r)) \leq e^{2\eta r} \left[\frac{\theta(t_0)}{Q(s)} \Sigma_1(s) + 1\right] e^{-(s-\eta)d(o, ko)},$$

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Summing over all long elements \( k \in G_K \), we get

\[
\nu_o^s(U_{K,\varepsilon}(F_T)) \leq |S| e^{2\omega_GR} \sum_{k \in G_K} e^{-(\omega_G-\eta)d(o,ko)}.
\]

Note that \( \Sigma_1(s) \) is bounded, while \( Q(s) \) diverges to infinity. Since \( U_{K,\varepsilon}(F_T) \) is an open subset of \( \bar{X} \), we can pass to the limit and get

\[
\nu_o(U_{K,\varepsilon}(F_T)) \leq |S| e^{2\omega_G R} \sum_{k \in G_K} e^{-(\omega_G-\eta)d(o,ko)}.
\]

The sum corresponds to the remainder of the Poincaré series of \( G_K \) at \( s = \omega_G - \eta \). However, \( \omega_G - \eta > \omega_{G_K} \). Hence this series converges, and its reminder tends to zero when \( T \) approaches infinity. Consequently, for every \( \varepsilon > 0 \),

\[
\nu_o \left( \bigcap_{T \geq 0} U_{K,\varepsilon}(F_T) \right) = 0.
\]

By construction the set \( A_{K,\varepsilon} \) is contained in \( U_{K,\varepsilon}(F_T) \) for every \( T \in \mathbb{R}_+ \). It follows from Lemma A.3 that

\[
\partial X \setminus \Lambda_{rad}(G) \subset G \left( \bigcup_{\varepsilon > 0} \bigcap_{T \geq 0} U_{K,\varepsilon}(F_T) \right).
\]

Since \( G \) is countable we conclude that \( \nu_o(\partial X \setminus \Lambda_{rad}(G)) = 0 \). Recall that \( \nu_o \) is supported on \( \partial X \), thus \( \nu_o(\Lambda_{rad}(G)) = 1 \).

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