A NOTE ON THE SUNDARAM–STANLEY BIJECTION
(OR, VIENNOT FOR UP-DOWN TABLEAUX)

ELIJAH BODISH, BEN ELIAS, DAVID E. V. ROSE, AND LOGAN TATHAM

Abstract. We give a direct proof of a result of Sundaram and Stanley: that the dimension of the space of invariant vectors in a 2k-fold tensor product of the vector representation of sp_{2n} equals the number of (n + 1)-avoiding matchings of 2k points. This can be viewed as an extension of Schensted’s theorem on longest decreasing subsequences. Our main tool is an extension of Viennot’s geometric construction to the setting of up-down tableaux.

1. Introduction

The Robinson–Schensted correspondence is a celebrated combinatorial bijection between permutations and pairs of standard Young tableaux of the same shape:

$$\mathfrak{S}_k \leftrightarrow \{ (P, Q, \lambda) \mid \lambda \vdash k, P, Q \text{ are standard Young tableaux of shape } \lambda \} .$$

Mechanically, one takes a permutation $$w \in \mathfrak{S}_k$$ written in one-line notation $$w(1)w(2) \cdots w(k)$$ and uses Schensted’s bumping algorithm to add one number $$w(t)$$ at a time to the insertion tableaux $$P_t$$, at each step giving tableaux $$P_t$$ (of non-standard content) whose shape is a partition $$\lambda_t \vdash t$$. This sequence of partitions is encoded in the recording tableaux $$Q$$. In other words, the bumping algorithm is a rule to produce from $$w$$ a sequence of steadily growing tableaux $$(P_t, \lambda_t)$$ for $$1 \leq t \leq k$$.

Many useful properties of the permutation $$w \in \mathfrak{S}_k$$ are encoded within the corresponding triple $$\text{RS}(w) = (P, Q, \lambda)$$, but many other features are obfuscated. While one can use Schensted’s reverse bumping algorithm to determine $$w$$ from $$\text{RS}(w)$$, it is not easy to gain intuition for how $$w$$ behaves simply by looking at $$P$$ and $$Q$$.

An example where a feature of a permutation is both well-encoded and obscured by $$\text{RS}$$ is the study of longest increasing (resp. longest decreasing) subsequences. An increasing subsequence is a sequence $$1 \leq t_1 < t_2 < \cdots < t_\ell \leq k$$ such that $$w(t_1) < w(t_2) < \cdots < w(t_\ell)$$; the length of the subsequence is $$\ell$$. Decreasing subsequences are defined analogously. The following is a classic result due to Schensted.

Theorem 1.1 ([8]). Let $$w \in \mathfrak{S}_k$$. The length of the longest increasing (resp. decreasing) subsequence of $$w$$ is equal to the number of columns$$^1$$ (resp. rows) in $$\text{RS}(w)$$.

Despite knowing their length, it is not obvious how to read the longest increasing and decreasing subsequences from the triple $$(P, Q, \lambda)$$.

Example 1.2. Consider the permutation $$w \in \mathfrak{S}_{19}$$ written in one-line notation as

$$w = 2 \ 9 \ 1 \ 15 \ 4 \ 7 \ 13 \ 18 \ 11 \ 19 \ 5 \ 14 \ 3 \ 10 \ 6 \ 17 \ 8 \ 16 \ 12 .$$

\[ ^1 \text{Here we use synecdoche: if } \text{RS}(w) = (P, Q, \lambda) \text{ then by “the number of columns in } \text{RS}(w)\text{” we mean the number of columns in } \lambda. \text{ We continue to use this convention throughout the paper.} \]
We have that
\[
P = \begin{bmatrix}
1 & 3 & 5 & 6 & 8 & 12 \\
2 & 4 & 10 & 14 & 16 \\
7 & 11 & 17 & 19 \\
9 & 13 & 18 \\
15 
\end{bmatrix}, \quad Q = \begin{bmatrix}
1 & 2 & 4 & 7 & 8 & 10 \\
3 & 5 & 6 & 12 & 16 \\
9 & 14 & 17 & 18 \\
11 & 15 & 19 \\
13 
\end{bmatrix}
\]

Correspondingly, the longest increasing subsequence has length 6, and one such subsequence is (indicated in bold):
\[
w = 2 \, 9 \, 1 \, 15 \, 4 \, 7 \, 13 \, 18 \, 11 \, 19 \, 5 \, 14 \, 3 \, 10 \, 6 \, 17 \, 8 \, 16 \, 12
\]
(there are many others). It is not clear how this subsequence is encoded in \(P\) and \(Q\). On the other hand, the longest decreasing subsequence has length 5, and one such subsequence is
\[
w = 2 \, 9 \, 1 \, 15 \, 4 \, 7 \, 13 \, 18 \, 11 \, 19 \, 5 \, 14 \, 3 \, 10 \, 6 \, 17 \, 8 \, 16 \, 12.
\]

In this case, it is tempting to believe that we obtain a longest decreasing subsequence by letting \(w(t_1)\) be the entry in the last row of \(P\) and then successively taking the largest entry that is smaller in the previous row. However, this is false, as our next example shows.

**Example 1.3.** Consider the permutation \(w \in \mathfrak{S}_{19}\) written in one-line notation as
\[
w = 19 \, 10 \, 1 \, 8 \, 18 \, 12 \, 13 \, 11 \, 16 \, 14 \, 3 \, 9 \, 7 \, 4 \, 6 \, 2 \, 15 \, 17 \, 5.
\]

We have that
\[
P = \begin{bmatrix}
1 & 2 & 4 & 5 & 14 & 15 & 17 \\
3 & 6 & 13 \\
7 & 9 & 16 \\
8 & 11 \\
10 \\
12 \\
18 \\
19 
\end{bmatrix}, \quad Q = \begin{bmatrix}
1 & 4 & 5 & 7 & 9 & 17 & 18 \\
2 & 6 & 10 \\
3 & 12 & 15 \\
8 & 19 \\
11 \\
13 \\
14 \\
16 
\end{bmatrix}
\]

The longest increasing subsequence has length 7, and one such subsequence is
\[
w = 19 \, 10 \, 1 \, 8 \, 18 \, 12 \, 13 \, 11 \, 16 \, 14 \, 3 \, 9 \, 7 \, 4 \, 6 \, 2 \, 15 \, 17 \, 5.
\]

The longest decreasing subsequence has length 8, and one such subsequence is
\[
w = 19 \, 10 \, 1 \, 8 \, 18 \, 12 \, 13 \, 11 \, 16 \, 14 \, 3 \, 9 \, 7 \, 4 \, 6 \, 2 \, 15 \, 17 \, 5.
\]

In both cases, it is not clear how to read these subsequences off from \(P\) or \(Q\). In particular, the schema for finding a longest decreasing subsequence in Example 1.2 fails to produce such a sequence here. Indeed, no longest decreasing sequence contains the number 10, the lone entry in its row in \(P\).

Theorem 1.1 links longest decreasing sequences to the representation theory of the Lie algebra \(\mathfrak{gl}_n\). Consider the vector representation \(V = \mathbb{C}^n\) of \(\mathfrak{gl}_n\). Schur–Weyl duality between \(\mathfrak{S}_k\) and \(\mathfrak{gl}_n\) shows that there is a surjective \(\mathbb{C}\)-algebra homomorphism
\[
SW: \mathbb{C}[\mathfrak{S}_k] \to \text{End}_{\mathfrak{gl}_n}(V^\otimes k).
\]

This homomorphism is injective if and only if \(k \leq n\). The group algebra \(\mathbb{C}[\mathfrak{S}_k]\) has a basis \(\{b_w\}\) indexed by permutations \(w \in \mathfrak{S}_k\) (the \(q = 1\) specialization of the Kazhdan–Lusztig basis) such that \(\ker(SW)\) is...
spanned by
\[ \{ b_w \mid \text{RS}(w) \text{ has at least } n + 1 \text{ rows} \} . \]
Consequently, the dimension of the space of \( \mathfrak{gl}_k \)-invariant vectors in \( V^\otimes k \otimes (V^\vee)^\otimes k \) is equal to the number of permutations \( w \in S_k \) such that \( \text{RS}(w) \) has \( n \) or fewer rows. By Theorem 1.1, this is the same as the number of permutations that avoid a length \( n + 1 \) decreasing sequence.

In the present paper, motivated by representation theory, we are particularly interested in the case of longest decreasing subsequences. While Schensted’s proof of Theorem 1.1 in the case of increasing subsequences is straightforward, his arguments for decreasing subsequences are indirect. Work of Viennot [11] remedies this, providing a direct and easy proof of Theorem 1.1 for decreasing subsequences. In fact, our main interest in this paper is not the Robinson–Schensted correspondence, but rather a type \( C \) generalization thereof that is related to the representation theory of the Lie algebra \( \mathfrak{sp}_{2n} \). As we show in [3] Viennot’s proof of Theorem 1.1 extends to type \( C \). However, we first finish discussing the story in type \( A \).

Viennot’s work proceeds via a graphical interpretation of the Robinson–Schensted correspondence in terms of certain colored segments in the first quadrant of \( \mathbb{R}^2 \), that we call Viennot diagrams. We recall this construction and give several examples in [2]. Viennot’s methods are most well-known because they give an easy proof of the following property (which is hard to prove using the bumping algorithm): if \( \text{RS}(w) = (P, Q, \lambda) \), then \( \text{RS}(w^{-1}) = (Q, P, \lambda) \). Viennot’s simple proof of this fact follows by reflecting a Viennot diagram along the diagonal \( y = x \) inside \( \mathbb{R}^2 \), a symmetry which swaps \( w \) and \( w^{-1} \), as well as \( P \) and \( Q \). However, Viennot also provides a straightforward proof of Theorem 1.1. To keep the discussion self-contained (and since the original French text [11] is sufficiently difficult to track down), we recall Viennot’s argument for longest decreasing subsequences in Proposition 2.8 and Corollary 2.11 and for longest increasing subsequences in Proposition 2.12. This provides an intuitive algorithm to find longest decreasing/increasing subsequences. Another useful feature of Viennot’s construction is that it records a “timeline” of the bumping algorithm; see Definition 2.1 and Example 2.2. This will be a crucial feature in our extension of this construction to type \( C \) combinatorics.

1.1. The Sundaram–Stanley bijection. The type \( C \) analogue of the Robinson–Schensted correspondence (dealing with the combinatorics arising in Brauer–Schur–Weyl duality) is due to Sundaram [10], with a variant given by Stanley [9]. In this correspondence, permutations in \( S_k \) are generalized by matchings of \( 2k \) points, and pairs of standard Young tableaux are replaced by so-called up-down tableau.

We now describe this bijection, which is easy to motivate using string diagrams for permutations in \( S_k \).

Suppose we are given a string diagram \( \mathcal{D} \) for a permutation \( w(\mathcal{D}) \in S_k \), e.g.

\[
\mathcal{D} = \begin{array}{ccc}
1 & 2 & 3 \\
\text{\textbullet} & \text{\textbullet} & \text{\textbullet}
\end{array}
\quad \implies \quad w(\mathcal{D}) = 2 4 3 1.
\]

\[\text{We are unaware of the best reference for this well-known fact. As a representation of } C[S_k], V^\otimes k \text{ is a sum of Specht modules for partitions } \lambda \text{ with at most } n \text{ rows. In [2] Theorem 6.5.3, one can find a proof of the theorem (claimed earlier by Kazhdan and Lusztig) that the cell modules for the Hecke algebra agree with the Specht modules with the expected labeling. Thus, Kazhdan-Lusztig basis elements in lower cells than } \lambda \text{ will kill the Specht module associated to } \lambda.\]

\[\text{Indeed, difficult enough that the authors were able to independently prove the results in [2] on decreasing/increasing subsequences (and our generalization to up-down tableaux in [3] before realizing we had been scooped (on the former) by none other than Viennot himself!}\]

\[\text{To be clear, this does not produce a longest decreasing/increasing subsequence directly from the triple } (P, Q, \lambda), \text{ which above we noted is difficult. Instead, such subsequences are recovered from the Viennot diagram, whereas recovering the Viennot diagram from } (P, Q, \lambda) \text{ is still unintuitive. Viennot diagrams form a "bridge" between permutations and standard Young tableaux, making it easier to see information about both sides of the correspondence.}\]
We read string diagrams from bottom-to-top, thus the one-line notation for \( w(D) \) can be obtained by labeling the points at the top of the strands 1, \ldots, \( k \) from left-to-right as in \( \begin{array}{c} 1 \end{array} \), and “sliding” the numbers along the strands to the bottom. By “bending” the top points around to the right, this permutation determines a matching of \( 2k \) fixed points on a line wherein none of first (or last) \( k \) points are matched with each other, e.g.

\[
\begin{array}{cccccc}
2 & 4 & 3 & 1 & 4 & 3 & 2 & 1
\end{array}
\sim
\begin{array}{cccccc}
2 & 4 & 3 & 1 & 4 & 3 & 2 & 1
\end{array}
\]

Here, we label the right endpoints of the strands in the matching by the labels \( \bar{1}, \ldots, \bar{k} \) from right-to-left, and label the left endpoints with the corresponding un-barred symbol. The one-line notation for this matching is therefore

\[
2 4 3 1 \bar{4} \bar{3} \bar{2} \bar{1}.
\]

It is clearly determined by the one-line notation 2431 for \( w \), since the last \( k \) symbols are simply \( \bar{k}, \ldots, \bar{1} \).

Analogously, given an arbitrary matching \( \mathcal{M} \) of \( 2k \) points on a line, we can label the right endpoints of the strands \( \bar{k}, \ldots, \bar{1} \) and label the left endpoints accordingly. This associates to each such matching \( \mathcal{M} \) a word \( w(\mathcal{M}) \) using each of the symbols \{1, \ldots, \( k \), \( \bar{1}, \ldots, \bar{k} \}\) once. In this word, the symbol \( i \) must appear before the symbol \( \bar{i} \), and the symbols \( \bar{k}, \ldots, \bar{1} \) appear in decreasing order. For example:

(1.2)

\[
\mathcal{M} =
\begin{array}{cccccccccccccccccccccccccccccccccc}
7 & 8 & 6 & 5 & 8 & 3 & 7 & 4 & 1 & 6 & 2 & 5 & 4 & 3 & 2 & 1
\end{array}
\implies w(\mathcal{M}) = 7 8 6 5 \bar{8} \bar{3} \bar{7} \bar{4} \bar{1} 6 2 5 \bar{4} \bar{3} \bar{2} \bar{1}.
\]

We will refer to such words \( w(\mathcal{M}) \) as matching words of length \( 2k \). Clearly, there is a bijection between matchings and matching words, since given a matching word we can match the entries \( i \) and \( \bar{i} \).

The Sundaram–Stanley bijection associates an up-down tableau UD(\( \mathcal{M} \)) with \( 2k \) steps to each matching word \( w(\mathcal{M}) \) of length \( 2k \), and thus to each matching \( \mathcal{M} \) of \( 2k \) points. Recall that a standard Young tableau (SYT) \( Q \) with \( k \) boxes can be thought of as a “movie” \( (\lambda_t)_{1 \leq t \leq k} \) of partitions, where one box (the \( t \)-labeled box in \( Q \)) is added in each time interval. In contrast, an up-down tableau UD with \( 2k \) steps is a sequence \((UD_t)_{t=0}^{2k}\) of partitions wherein the Young diagram for UD\(_{t+1}\) is obtained from that of UD\(_t\) by adding or removing one box. In the present paper, all up-down tableaux will be assumed to satisfy UD\(_0\) = \( \emptyset = UD_{2k} \). While the movie \( (\lambda_t) \) corresponding to a SYT can be recorded compactly by decorating the final time slice \( \lambda = \lambda_k \) with numbers (producing \( Q \)), an up-down tableau can not be so compactly encoded (after all, the final partition UD\(_{2k}\) is empty). Thus, the notation for up-down tableaux is more cumbersome (one must remember the whole movie rather than one time slice), even though the concept is not much more difficult.

The up-down tableau UD(\( \mathcal{M} \)) is defined by applying Schensted’s bumping algorithm to the word \( w(\mathcal{M}) \) to add a box for each entry of the form \( i \), and by simply removing the \( i \)-labeled box for each

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\(^5\)We will always assume that the corresponding string diagram is reduced, meaning that pairs of strands cross each other the minimal possible number of times.
entry \( i \). For example, for the word

\[ w(M) = 7 \ 8 \ 6 \ 5 \ 8 \ 3 \ 7 \ 4 \ 1 \ 6 \ 2 \ 5 \ 4 \ 3 \ 2 \ 1 \]

from above, we have

\[
\emptyset, \begin{array}{c|c}
7 & 8 \\
\hline
6 & 5 \\
\end{array}, \begin{array}{c|c|c}
7 & 8 & 5 \\
\hline
6 & 5 & 3 \\
\end{array}, \begin{array}{c|c|c|c}
7 & 8 & 5 & 3 \\
\hline
6 & 5 & 3 & 2 \\
\end{array}, \begin{array}{c|c|c|c|c}
7 & 8 & 5 & 3 & 2 \\
\hline
6 & 5 & 3 & 2 & 1 \\
\end{array}, \begin{array}{c|c|c|c|c|c}
7 & 8 & 5 & 3 & 2 & 1 \\
\hline
6 & 5 & 3 & 2 & 1 & \emptyset \\
\end{array}
\]

We emphasize that the numbers appearing in the Young diagrams in (1.3) are shown here only to exhibit how to obtain UD\((M)\) from \(w(M)\), and \emph{are not} part of the data of an up-down tableau. The up-down tableau simply consists of the sequence of Young diagrams. One can reconstruct the numbering from the up-down tableau, so we often keep the numbers for convenience.

For example, suppose we perform this operation on a matching word coming from a permutation, like \(2431 \emptyset \ 4 \ \emptyset \ 3 \ \emptyset \ 2 \ \emptyset \ 1\). One obtains a special kind of up-down tableau UD, where all boxes are added before any are removed. If the permutation corresponds under Robinson–Schensted to \((P, Q, \lambda)\), then the boxes will be added according to the tableau \(Q\) and then removed according to the tableau \(P\) (as the tableau \(P\) agrees with the numbers labeling UD\(_k\) in this case). That is, \(Q\) corresponds to the movie \((\text{UD}_t)_{t=0}^k\) and \(P\) to the movie \((\text{UD}_{2k-t})_{t=0}^k\). In this way, the up-down tableau records the data of both \(P\) and \(Q\), and the Sundaram–Stanley bijection is compatible with (and thus generalizes) the Robinson–Schensted bijection.

The following theorem of Sundaram–Stanley says that one can reconstruct the matching word \(w(M)\) from the up-down tableau UD\((M)\).

\textbf{Theorem 1.4.} The assignment \(M \mapsto \text{UD}(M)\) determines a bijection between matchings of \(2k\) points and up-down tableaux with \(2k\) steps.

In [10, Section 8], Sundaram establishes a version of this theorem. Specifically, she uses a variant of the assignment \(M \mapsto w(M)\) that instead labels the right endpoints in a matching diagram by the symbols \(\bar{1}, \ldots, \bar{k}\) from \emph{left-to-right}. The resulting bijection between such words and up-down tableaux requires the use of \emph{jeu-de-taquin}. We prefer the bijection defined above (which appears e.g. in Stanley’s work [9, Section 9]) since it avoids the use of jeu-de-taquin and directly generalizes the Robinson–Schensted correspondence.

\textbf{1.2. Decreasing sequences and patterns.} When visualizing a permutation with its string diagram, a decreasing sequence of length \(\ell\) is the same as a choice of \(\ell\) strings for which any pair of strands will cross each other. Said differently, given any subsequence of length \(\ell\), one can view the \(\ell\) corresponding strands as a string diagram for a permutation in \(S_{\ell}\) by ignoring the other \(k-\ell\) strands. The subsequence is then decreasing if and only if the permutation in \(S_{\ell}\) is the longest element, also known as the \emph{half twist}.\footnote{The first box removed is \(k\), the next box removed is \(k-1\), and so on. Reading the movie backwards, one can use the reverse bumping rule to determine how to update the tableau when a box is added.}
Similar ideas apply to matchings of $2k$ points. Let us say that an $\ell$-pattern (or a pattern of length $\ell$) in a matching $M$ is a choice of $\ell$ strings for which any pair of strands will cross one another. This corresponds to the appearance of a half twist inside the matching diagram of the matching. From the perspective of matching words, an $\ell$-pattern corresponds to a decreasing subsequence of un-barred symbols that concludes before any of their barred versions appears. We will refer to such a subsequence as a coexistent decreasing subsequence.

For example, consider the matching word $w(M) = 7\, 8\, 6\, 5\, 8\, 3\, 7\, 4\, 1\, 6\, 2\, 5\, 4\, 3\, 2\, 1$ from (1.2). The bold subsequence $7653$ is a coexistent decreasing subsequence, hence corresponds to a 4-pattern in $M$. On the other hand, the decreasing subsequence $76531$ is not coexistent, since 1 does not appear until after 7. In the matching diagram (1.2), one can see that the 7-labeled string and the 1-labeled string do not cross. In fact, this matching does not contain a 5-pattern.

In this paper we prove the following, which is a generalization of Theorem 1.1.

**Theorem 1.5.** Let $M$ be a matching of $2k$ points. The length of the longest pattern in $M$ is equal to the maximum number of rows appearing in any partition in $UD(M)$.

This result, which in [5, 7] is attributed to Sundaram, is stated in [9, Theorem 16] without proof. A proof can be deduced from the more-general results on set partitions and “vacillating tableau” in [4]; see Section 5 therein.

Our proof of Theorem 1.5 in [3] is based on a simple analogue of Viennot’s geometric construction for up-down tableaux, which we believe is new. By mimicking Viennot’s proof of Theorem 1.1 with one small modification, we prove that a matching has an $\ell$-pattern if and only if some partition in its up-down tableau has at least $\ell$ rows. We also give an easy algorithm to find this $\ell$-pattern.

We conclude the introduction by discussing the connection to representation theory. Consider the vector representation $V = \mathbb{C}^{2n}$ of $\mathfrak{sp}_{2n}$. Brauer–Schur–Weyl duality gives a surjective $\mathbb{C}$-algebra homomorphism

\begin{equation}
\text{BSW}: \text{Brauer}_k \to \text{End}_{\mathfrak{sp}_{2n}}(V^\otimes k).
\end{equation}

The Brauer algebra $\text{Brauer}_k$ (and its quantum analogue the BMW algebra [6, 1]) is typically described diagrammatically using tangle diagrams, and it has a basis in bijection with matchings of $2k$ points. The kernel of the homomorphism BSW is less easy to describe than in type $A$; see [3, Section 2.3] for some discussion and history. Nonetheless, as proved by Sundaram in [10, Theorem 6.15], the dimension of the endomorphism ring on the right-hand side of (1.4) (equivalently, the dimension of the space of $\mathfrak{sp}_{2n}$-invariant vectors in $V^\otimes 2k$) precisely matches the number of up-down tableaux such that all partitions appearing have $n$ or fewer rows. By Theorem 1.5, this agrees with the number of matchings of $2k$ points that avoid $(n + 1)$-patterns.

In [3], we present (the image of) $\text{Brauer}_k$ acting on $V^\otimes k$ (and its quantum analogue) using matchings; see the category $\text{Web}^\times(\mathfrak{sp}_{2n})$ from Section 5.4 therein. In [3, Theorem 5.35], we show that $\text{Web}^\times(\mathfrak{sp}_{2n})$ has a spanning set consisting of matchings that avoid $(n + 1)$-patterns. Using Theorem 1.5 and Sundaram’s result, we are able to prove that the relevant Hom-space in $\text{Web}^\times(\mathfrak{sp}_{2n})$ is isomorphic to $\text{End}_{\mathfrak{sp}_{2n}}(V^\otimes k)$. This was the reason for our interest in Theorem 1.5.

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2. VIENNOT’S GEOMETRIC CONSTRUCTION AND DECREASING SUBSEQUENCES

In [11], Viennot gives a graphical interpretation of the Robinson–Schensted correspondence in terms of certain colored segments in the first quadrant of $\mathbb{R}^2$. This framework has a manifest symmetry, given by reflection across the diagonal, which immediately implies a celebrated and non-obvious property of the Robinson–Schensted correspondence: that interchanging the insertion and recording tableaux corresponds to taking the inverse of the corresponding permutation. However, for our generalization in §3 it will be useful to break this symmetry, and thus present Viennot’s construction as a certain timeline that encodes Schensted’s bumping algorithm.

Definition 2.1 (Viennot’s geometric construction). Let $w \in \mathfrak{S}_k$ be a permutation written in one-line notation as $w = a_1 \cdots a_k$, and let $\mathcal{C}_k = \{c_1 < \cdots < c_k\}$ be a totally ordered set of $k$ colors. The Viennot diagram of $w$ is the collection of points and colored segments in the first quadrant $\{(t, b) | t, b > 0\} \subset \mathbb{R}^2$ defined as follows. For each “time coordinate” $t = 1, 2, \ldots, k$, repeat the following steps:

- Draw the point $(t, a_t)$.
- Draw the vertical segment from $(t, a_t)$ to $(t, k + 1)$ and color it as follows: it begins with color $c_1$ and changes color from $c_i$ to $c_{i+1}$ whenever it passes a point $(t, b)$ where the segment $(t-1, b) \rightarrow (t, b)$ is colored $c_i$.
- Draw horizontal segments $(t, a_i) \rightarrow (t+1, a_i)$ for all $1 \leq i \leq t$ colored as follows: if the segments $(t-1, a_i) \rightarrow (t, a_i)$ and $(t, a_i) \rightarrow (t, a_i-1)$ have the same color $c_j$, then color this segment $c_{j+1}$. Otherwise, color it the same as the segment $(t-1, a_i) \rightarrow (t, a_i)$.

The Viennot diagram is the resulting diagram after step $k$.

Example 2.2. We illustrate the steps in the above algorithm for the permutation $w = 2 4 3 1$ in $\mathfrak{S}_4$. Our colors are red < orange < green (< blue).

Remark 2.3. There are two useful ways to view the resulting Viennot diagram. The first is essentially as described: at the end of the process, each point $(t, a_t)$ determines two multi-colored rays traveling vertically and horizontally to the points $(t, k + 1)$ and $(k + 1, a_t)$, increasing in color as they bump into like-colored rays. We emphasize that nothing is particularly special about the chosen value $k + 1$; we can view these rays as traveling to infinity, and the value $k + 1$ is simply chosen to lie outside the square $[1, k] \times [1, k]$ where the points $(t, a_t)$ may lie.

Another equally useful description is the following: for a given color $c$, the union of the $c$-colored segments determines a collection of non-intersecting lattice paths. Each such path enters the square $[1, k] \times [1, k]$ from above, always travels either downward or rightward within the square, then exits to the right. This point of view will be essential in our proof of Proposition 2.8 below.

Example 2.4. The following is the Viennot diagram for the permutation $w = 2 9 1 15 4 7 13 18 11 19 5 14 3 10 6 17 8 16 12$

\footnote{Our coordinate directions are the “time” direction and the “bumping” direction.}
in $S_{19}$. Our colors are $\text{red} < \text{orange} < \text{green} < \text{blue} < \text{black} (< \cdots)$.

The connection to the Robinson–Schensted correspondence is the following result of Viennot.

**Theorem 2.5** ([11]). Let $w \in S_k$ be a permutation and consider the Viennot diagram assigned to $w$. Then, $RS(w) = (P, Q, \lambda)$, where $P$ is determined by the colors appearing on the right of the Viennot diagram, and $Q$ is determined by the colors appearing on top. More precisely, the $j^{th}$ row of $P$ is filled with the $b$-coordinates of the $c_j$-colored horizontal segments meeting the line $t = k + 1$. Similarly, the $j^{th}$ row of $Q$ is filled with the $t$-coordinates of the $c_j$-colored vertical segments meeting the line $b = k + 1$.

**Example 2.6.** For the permutation in Example 2.4, the first row of $P$ has entries $(1, 3, 5, 6, 8, 12)$, coming from the red segments going to the right. The second row of $P$ has entries $(2, 4, 10, 14, 16)$, coming from the orange segments. Since 5 colors appear in the diagram, $P$ has 5 nonempty rows. Meanwhile, the first row of $Q$ has entries $(1, 2, 4, 7, 8, 10)$, coming from the red segments going to the top.

**Remark 2.7.** As mentioned above, the Viennot diagram of a permutation $w$ gives a visual timeline for Schensted’s bumping algorithm. At time $t$, the value $a_t$ enters the insertion tableaux. If we stop the algorithm after $s$ steps, we obtain a diagram inside a $[1, s] \times [1, k]$ box. Reading tableaux off the right and top walls of the box just as above, we recover the triple $(P_s, Q_s, \lambda_s)$. In other words, $P_s$ is determined by the colors meeting the vertical line at $t = s + \frac{1}{2}$.

The following connects Viennot’s construction to decreasing subsequences. The proof of Lemma 2.9 therein is essentially a retooling of Viennot’s argument for the decreasing subsequence case of Theorem 1.1.

**Proposition 2.8.** Let $w = a_1 \cdots a_k$ be a permutation in $S_k$ written in one-line notation, and let $c_1 < \cdots < c_k$ be the totally ordered set of colors in Viennot’s geometric construction. The color $c_\ell$ appears in the Viennot diagram for $w$ if and only if the list $a_1, \ldots, a_k$ contains a decreasing subsequence of length $\ell$.

**Proof.** As mentioned in Remark 2.3 we can view the union of all segments of color $c_i$ as a collection of non-intersecting lattice paths from points of the form $(t, k + 1)$ to points of the form $(k + 1, b)$ that weakly decrease in the $b$-direction and weakly increase in the $t$-direction. In particular, every corner in
such a path takes the form

and we call these outer corners and inner corners, respectively. We record the following observations:

- Every such \( c_1 \)-colored lattice path contains at least one outer corner.
- For \( i \geq 1 \), each \( c_{i+1} \)-colored outer corner meets exactly one \( c_i \)-colored inner corner, and vice versa:

We call a sequence \( (t_1, b_1), \ldots, (t_r, b_r) \) of points in a Viennot diagram a length \( r \) decreasing sequence if \( t_1 < \cdots < t_r \) and \( b_1 > \cdots > b_r \). By the above observations, our result is a consequence of the following.

**Lemma 2.9.** A Viennot diagram contains a length \( r \) decreasing sequence of \( c \)-colored inner corners if and only if it contains a length \( r+1 \) decreasing sequence of \( c \)-colored outer corners.

Indeed, as observed above, a length \( r \) decreasing sequence of \( c_i \)-colored outer corners is also a length \( r \) decreasing sequence of \( c_{i-1} \)-colored inner corners. Thus, the lemma allows us to construct a length \( r+1 \) decreasing sequence of \( c_{i-1} \)-colored outer corners from a length \( r \) decreasing sequence of \( c_r \)-colored outer corners, and vice-versa. If the color \( c_{\ell} \) appears, we can choose one \( c_{\ell} \)-colored outer corner and iteratively apply the claim to obtain a length \( \ell \) decreasing sequence of \( c_1 \)-colored outer corners. Conversely, given a length \( \ell \) decreasing sequence of \( c_1 \)-colored outer corners, we can run the procedure in reverse to obtain one \( c_{\ell} \)-colored outer corner, implying that the color \( c_{\ell} \) appears in the diagram.

We now prove the lemma. Note that each \( c \)-colored inner corner \( \text{ic} = (t, b) \) determines two associated \( c \)-colored outer corners \( l(\text{ic}) = (l(t), b) \) and \( d(\text{ic}) = (t, d(b)) \) with \( l(t) < t \) and \( d(b) < b \), which are obtained by following the horizontal and vertical segments meeting \( \text{ic} \) leftward and downward until they arrive at outer corners. These outer corners necessarily exist, since the \( c \)-colored lattice path never leaves the first quadrant, thus cannot travel leftward or downward from an inner corner indefinitely.

Now, suppose that a Viennot diagram contains a length \( r \) decreasing sequence \( \text{ic}_1, \ldots, \text{ic}_r \) of \( c \)-colored inner corners, and write \( \text{ic}_j = (t_j, b_j) \). For \( 1 \leq j \leq r+1 \), consider the following subsets:

\[
B_j := \begin{cases} 
(-\infty, t_1) \times [b_1, \infty) & \text{if } j = 1 \\
[t_{j-1}, t_j) \times [b_j, b_{j-1}) & 2 \leq j \leq r \\
[t_r, \infty) \times (-\infty, b_r) & j = r+1.
\end{cases}
\]

Aside from the edge cases \( j = 1 \) and \( j = r+1 \), \( B_j \) is the rectangle between two consecutive corners in the decreasing sequence. In Example 2.10 below, one can see the rectangles \( B_j \) shaded in gray. By construction, the region \( B_{j+1} \) lies completely below and to the right of the region \( B_j \). Thus any collection of \( r+1 \) points wherein each point lies in a distinct region \( B_j \) will yield a decreasing sequence.

It suffices to show that each \( B_j \) contains at least one \( c \)-colored outer corner. For the edge cases, we must have that \( l(\text{ic}_1) \in B_0 \) and \( d(\text{ic}_r) \in B_{r+1} \). For \( 2 \leq j \leq r \), since distinct \( c \)-colored lattice paths in a Viennot diagram never intersect, at least one of \( d(\text{ic}_{j-1}) \) or \( l(\text{ic}_j) \) lies in \( B_j \), because the situation must
fit one of the following schematics\footnote{It is a game of chicken. A vehicle leaves $i_{c_j}$ traveling left, and another leaves $i_{c_{j-1}}$ traveling down, on a deadly collision course! Either they collide (please fetch the outer corner), or (at least) one of the vehicles is a chicken, veering off before the potential collision.}:

\begin{equation}
(2.1)
\end{equation}

This establishes one direction of Lemma 2.9

The other direction is entirely analogous. Let $o_{c1}, \ldots, o_{c_{r+1}}$ be a decreasing sequence of $c$-colored outer corners, and write $o_{c_j} = (s_j, u_j)$. If we consider the subsets

$$B^j := (s_j, s_{j+1}] \times (u_{j+1}, u_j]$$

for $1 \leq j \leq r$, then each such box must contain a $c$-colored inner corner (the schematic is obtained from the one in (2.1) by rotating each picture 180 degrees). Choosing one such inner corner in each $B^j$ gives the desired decreasing sequence. \hfill \Box

\textbf{Example 2.10.} We illustrate the passage from inner to outer corners in Proposition 2.8 using the Viennot diagram from Example 2.8. The decreasing sequence of orange inner corners determines the indicated decreasing sequence of outer corners. (Here, we’ve removed all lattice paths except those that are orange.)

Continuing with this example, the outer orange corners are identified with inner red corners. These inner corners determine the indicated sequence of outer corners (here we display the red and orange
By pairing Theorem 2.5 with Proposition 2.8, we deduce the decreasing sequence case of Theorem 1.1.

**Corollary 2.11.** Let \( w = a_1 \cdots a_k \) be a permutation in \( S_k \) written in one-line notation. The number of rows in \( RS(w) \) equals the length of the longest decreasing subsequence in the list \( a_1, \ldots, a_k \).

**Proof.** By Theorem 2.5, the number of rows in the Young diagram associated to \( w \) equals the largest color \( c_\ell \) appearing in \( w \)'s Viennot diagram. By Proposition 2.8, \( \ell \) is the length of the longest decreasing subsequence in \( a_1, \ldots, a_k \). □

For completeness, we also recall Viennot’s proof of the increasing subsequence case of Theorem 1.1.

**Proposition 2.12.** The number of columns in \( RS(w) \) equals the length of the longest increasing subsequence.

**Proof.** Note that the number of columns is equal to the number of \( c_1 \)-colored lattice paths in the Viennot diagram for \( w \).

We may produce an increasing subsequence with one \( c_1 \)-colored outer corner for each lattice path as follows. Start with any outer corner on the “innermost” \( c_1 \)-colored lattice path (i.e. the lattice path that intersects the line \( b = k + 1 \) at the largest \( t \) value). Travel down from this point until intersecting the next \( c_1 \)-colored lattice path, then to the left until an outer corner is reached. Continuing in this way, we identify an outer corner on each \( c_1 \) colored lattice path, and this sequence of outer corners is increasing by construction.

Conversely, the entries of any increasing subsequence determine outer corners that must lie on distinct \( c_1 \)-colored lattice paths (since the lattice paths in a Viennot diagram always travel downward and rightward), so the length of any increasing subsequence is bounded above by the number of \( c_1 \)-colored lattice paths. □

3. **Up-down Viennot diagrams and patterns**

We now extend Viennot’s geometric construction to up-down tableaux.

**Definition 3.1.** Let \( \mathcal{M} \) be a matching of \( 2k \) points and let \( w(\mathcal{M}) = a_1 \cdots a_{2k} \) be the corresponding matching word. The *up-down Viennot diagram* of \( \mathcal{M} \) is the collection of marked points and colored
segments in the first quadrant \( \{(t, b) \mid t, b > 0\} \subset \mathbb{R}^2 \) defined as follows. For each “time coordinate” \( t = 1, 2, \ldots, 2k \), do the following:

1. If \( a_t \) is un-barred, draw the marking \( \bullet \) at the point \( (t, a_t) \) and draw horizontal and vertical segments following the procedure from Definition 2.1.

2. If \( a_t \) is barred, draw the marking \( \times \) at the point \( (t, a_t) \) and draw horizontal segments as in Definition 2.1 at all \( b \)-values except \( b = a_t \). The horizontal segment at height \( a_t \) ends at the point \( (t, a_t) \), and no vertical segment is drawn starting from this point.

The up-down Viennot diagram, denoted \( V(\mathcal{M}) \), is the resulting diagram after step \( 2k \).

**Example 3.2.** We illustrate the steps in the above algorithm for the matching word 4 2 3 4 3 1 2 1. Our colors are red < orange < green (< blue).

We also give the end result of Definition 3.1 for the matching from (1.2).

**Example 3.3.** The following is the up-down Viennot diagram for the matching word

\[
\begin{array}{cccccccc}
7 & 8 & 6 & 5 & 8 & 3 & 7 & 4 & 1 & 6 & \bar{2} & \bar{5} & \bar{4} & 3 & \bar{2} & \bar{1}.
\end{array}
\]
Proposition 3.4. The assignment $\mathcal{M} \mapsto V(\mathcal{M})$ is a bijection between matchings and up-down Viennot diagrams.

Proof. An up-down Viennot diagram is uniquely determined by the $\bullet$ markings (the locations of the $c_1$-colored outer corners) and the $\times$ markings. These determine, and are determined by, the matching word. \qed

We now arrive at our generalization of Proposition 2.8.

Proposition 3.5. Let $\mathcal{M}$ be a matching of $2k$ points and let $c_1 < \cdots < c_k$ be the totally ordered set of colors in its up-down Viennot diagram $V(\mathcal{M})$. The color $c_\ell$ appears in $V(\mathcal{M})$ if and only if the matching word $w(\mathcal{M})$ contains a coexistent decreasing subsequence of length $\ell$.

Proof. If $(t, b)$ is an (outer/inner) corner in an up-down Viennot diagram, then we claim that the rectangle $[1, t] \times [1, b]$ is identical to a corresponding piece in an ordinary Viennot diagram, i.e. there are no $\times$-marked points inside this rectangle. Indeed, the existence of the corner implies that $\bar{b}$ does not appear until after time $t$, thus neither does $\bar{b}$ for any $b' < b$, since barred indices appear in decreasing order. Thus all $\times$-marked points in the time interval $[1, t]$ have vertical coordinate in $(b, \infty]$.

Choosing a $c_\ell$-colored outer corner $(t, b)$, we can repeat the algorithm of Lemma 2.9 to obtain a decreasing subsequence of length $\ell$ inside the rectangle $[1, t] \times [1, b]$. None of the indices in this decreasing subsequence will appear in barred form before time $t$, so this sequence is coexistent.

Conversely, suppose we have a length $\ell$ coexistent decreasing subsequence $a_{t_1} > a_{t_2} > \cdots > a_{t_\ell}$. Then, $\bar{a}_{t_\ell}$ does not appear until after time $t_\ell$, so neither does $\bar{b}$ for any $b < a_{t_1}$. Thus the rectangle $[1, t_\ell] \times [1, a_{t_1}]$ is identical to a corresponding piece in an ordinary Viennot diagram. The reverse algorithm at the end of Lemma 2.9 will stay within this rectangle, and produces a $c_\ell$-colored outer corner. \qed

Proof of Theorem 1.5. The assignment $V(\mathcal{M}) \mapsto \mathcal{M} \mapsto \text{UD}(\mathcal{M})$ given by composing the bijection from Proposition 3.4 with the Sundaram–Stanley bijection is given as in Remark 2.7, i.e. $\text{UD}(\mathcal{M})$ can be read off from “time slices” in $V(\mathcal{M})$ of the form $t = s + \frac{1}{2}$. It follows that there exists $0 < s < 2k$ such that $\text{UD}(\mathcal{M})_s$ has at least $\ell$ rows if and only if the color $c_\ell$ appears in $V(\mathcal{M})$. By Proposition 3.5, the color $c_\ell$ appears in $V(\mathcal{M})$ if and only if the matching word $w(\mathcal{M})$ has a coexistent decreasing subsequence of length $\ell$, which corresponds to an $\ell$-pattern. \qed

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