HOLOMORPHIC CARTAN GEOMETRY ON MANIFOLDS WITH NUMERICALLY EFFECTIVE TANGENT BUNDLE

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ABSTRACT. Let $X$ be a compact connected Kähler manifold such that the holomorphic tangent bundle $TX$ is numerically effective. A theorem of [11] says that there is a finite unramified Galois covering $M \to X$, a complex torus $T$, and a holomorphic surjective submersion $f : M \to T$, such that the fibers of $f$ are Fano manifolds with numerically effective tangent bundle. A conjecture of Campana and Peternell says that the fibers of $f$ are rational and homogeneous. Assume that $X$ admits a holomorphic Cartan geometry. We prove that the fibers of $f$ are rational homogeneous varieties. We also prove that the holomorphic principal $G$--bundle over $T$ given by $f$, where $G$ is the group of all holomorphic automorphisms of a fiber, admits a flat holomorphic connection.

1. Introduction

Let $X$ be a compact connected Kähler manifold such that the holomorphic tangent bundle $TX$ is numerically effective. (The notions of numerically effective vector bundle and numerically flat vector bundle over a compact Kähler manifold were introduced in [11].) From a theorem of Demailly, Peternell and Schneider we know that there is a finite unramified Galois covering

$$
\gamma : M \to X ,
$$

a complex torus $T$, and a holomorphic surjective submersion

$$
f : M \to T ,
$$

such that the fibers of $f$ are Fano manifolds with numerically effective tangent bundle (see [11] p. 296, Main Theorem]). It is conjectured by Campana and Peternell that the fibers of $f$ are rational homogeneous varieties (i.e., varieties of the form $G/P$, where $P$ is a parabolic subgroup of a complex semisimple group $G$) [10] p. 170, [11] p. 296]. Our aim here is to verify this conjecture under the extra assumption that $X$ admits a holomorphic Cartan geometry.

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Let \((E'_H, \theta')\) be a holomorphic Cartan geometry on \(X\) of type \(G/H\), where \(H\) is a complex Lie subgroup of a complex Lie group \(G\). (The definition of Cartan geometry is recalled in Section 2.) Consider the pullback \(\theta\) of \(\theta'\) to the holomorphic principal \(H\)–bundle \(E_H := \gamma^*E'_H\), where \(\gamma\) is the above covering map. The pair \((E_H, \theta)\) is a holomorphic Cartan geometry on \(M\). Using \((E_H, \theta)\) we prove the following theorem (see Theorem 2.1):

**Theorem 1.1.** There is a semisimple linear algebraic group \(G\) over \(\mathbb{C}\), a parabolic subgroup \(P \subset G\), and a holomorphic principal \(G\)–bundle
\[
\mathcal{E}_G \longrightarrow T ,
\]
such that the fiber bundle \(\mathcal{E}_G/P \longrightarrow T\) is holomorphically isomorphic to the fiber bundle \(f : M \longrightarrow T\).

The group \(G\) in Theorem 1.1 is the group of all holomorphic automorphisms of a fiber of \(f\). Let \(\text{ad}(\mathcal{E}_G) \longrightarrow T\) be the adjoint vector bundle of the principal \(G\)–bundle \(\mathcal{E}_G\) in Theorem 1.1. Let \(K_f^{-1} \longrightarrow M\) be the relative anti–canonical line bundle for the projection \(f\).

We prove the following (see Proposition 3.3 and Proposition 3.4):

**Proposition 1.2.** Let \(X\) be a compact connected Kähler manifold such that \(TX\) is numerically effective, and let \((E'_H, \theta')\) be a holomorphic Cartan geometry on \(X\) of type \(G/H\). Then the following two statements hold:

1. The adjoint vector bundle \(\text{ad}(\mathcal{E}_G)\) is numerically flat.
2. The principal \(G\)–bundle \(\mathcal{E}_G\) admits a flat holomorphic connection.

2. Cartan Geometry and Numerically Effectiveness

Let \(G\) be a connected complex Lie group. Let \(H \subset G\) be a connected complex Lie subgroup. The Lie algebra of \(G\) (respectively, \(H\)) will be denoted by \(\mathfrak{g}\) (respectively, \(\mathfrak{h}\)).

Let \(Y\) be a connected complex manifold. The holomorphic tangent bundle of \(Y\) will be denoted by \(TY\). Let \(E_H \longrightarrow Y\) be a holomorphic principal \(H\)–bundle. For any \(g \in H\), let

\[(2.1)\]
\[
\beta_g : E_H \longrightarrow E_H
\]

be the biholomorphism defined by \(z \mapsto zg\). For any \(v \in \mathfrak{h}\), let

\[(2.2)\]
\[
\zeta_v \in H^0(E_H, TE_H)
\]
be the holomorphic vector field on $E_H$ associated to the one–parameter family of biholomorphisms $t \mapsto \beta_{\exp(tv)}$. Let

$$\text{ad}(E_H) := E_H \times^H \mathfrak{h} \to Y$$

be the adjoint vector bundle associated $E_H$ for the adjoint action of $H$ on $\mathfrak{h}$. The adjoint vector bundle of a principal $G$–bundle is defined similarly.

A holomorphic Cartan geometry of type $G/H$ on $Y$ is a holomorphic principal $H$–bundle

$$p : E_H \to Y$$

(2.3)

together with a $\mathfrak{g}$–valued holomorphic one–form

$$\theta \in H^0(E_H, \Omega^1_{E_H} \otimes_\mathbb{C} \mathfrak{g})$$

(2.4)

satisfying the following three conditions:

1. $\beta_g^* \theta = \text{Ad}(g^{-1}) \circ \theta$ for all $g \in H$, where $\beta_g$ is defined in (2.1),
2. $\theta(z)(\zeta_v(z)) = v$ for all $v \in \mathfrak{h}$ and $z \in E_H$ (see (2.2) for $\zeta_v$), and
3. for each point $z \in E_H$, the homomorphism from the holomorphic tangent space

$$\theta(z) : T_z E_H \to \mathfrak{g}$$

(2.5)

is an isomorphism of vector spaces.

(See [14].)

A holomorphic line bundle $L \to Y$ is called numerically effective if $L$ admits Hermitian structures such that the negative part of the curvatures are arbitrarily small [11, p. 299, Definition 1.2]. If $Y$ is a projective manifold, then $L$ is numerically effective if and only if the restriction of it to every complete curve has nonnegative degree. A holomorphic vector bundle $E \to Y$ is called numerically effective if the tautological line bundle $\mathcal{O}_{\mathbb{P}(E)}(1) \to \mathbb{P}(E)$ is numerically effective.

Let $X$ be a compact connected Kähler manifold such that the holomorphic tangent bundle $TX$ is numerically effective. Then there is a finite étale Galois covering

$$\gamma : M \to X$$

(2.6)

a complex torus $T$ and a holomorphic surjective submersion

$$f : M \to T$$

(2.7)

such that the fibers of $f$ are connected Fano manifolds with numerically effective tangent bundle [11, p. 296, Main Theorem].
Theorem 2.1. Let $(E'_H, \theta')$ be a holomorphic Cartan geometry on $X$ of type $G/H$, where $X$ is a compact connected Kähler manifold such that the holomorphic tangent bundle $TX$ is numerically effective. Then there is

1. a semisimple linear algebraic group $G$ over $\mathbb{C}$,
2. a parabolic subgroup $P \subset G$, and
3. a holomorphic principal $G$–bundle $E_G \to T$,

such that the fiber bundle $E_G/P \to T$ is holomorphically isomorphic to the fiber bundle $f$ in (2.7).

Proof. Let

\begin{equation}
(E_H, \theta)
\end{equation}

be the holomorphic Cartan geometry on $M$ obtained by pulling back the holomorphic Cartan geometry $(E'_H, \theta')$ on $X$ using the projection $\gamma$ in (2.6).

Let

\begin{equation}
E_G := E_H \times^H G \to M
\end{equation}

be the holomorphic principal $G$–bundle obtained by extending the structure group of $E_H$ using the inclusion of $H$ in $G$. So $E_G$ is a quotient of $E_H \times G$, and two points $(z_1, g_1)$ and $(z_2, g_2)$ of $E_H \times G$ are identified in $E_G$ if there is an element $h \in H$ such that $z_2 = z_1 h$ and $g_2 = h^{-1} g_1$. Let

\[ \theta_{MC} : TG \to G \times \mathfrak{g} \]

be the $\mathfrak{g}$–valued Maurer–Cartan one–form on $G$ constructed using the left invariant vector fields. Consider the $\mathfrak{g}$–valued holomorphic one–form

\[ \tilde{\theta} := p_1^* \theta + p_2^* \theta_{MC} \]

on $E_H \times G$, where $p_1$ (respectively, $p_2$) is the projection of $E_H \times G$ to $E_H$ (respectively, $G$), and $\theta$ is the one–form in (2.8). This form $\tilde{\theta}$ descends to a $\mathfrak{g}$–valued holomorphic one–form on the quotient space $E_G$ in (2.9), and the descended form defines a holomorphic connection on $E_G$; see [3] for holomorphic connection. Therefore, the principal $G$–bundle $E_G$ in (2.9) is equipped with a holomorphic connection. This holomorphic connection on $E_G$ will be denoted by $\nabla^G$.

The inclusion map $\mathfrak{h} \hookrightarrow \mathfrak{g}$ produces an inclusion

\[ \text{ad}(E_H) \hookrightarrow \text{ad}(E_G) \]
of holomorphic vector bundles. Using the form \( \theta \), the quotient bundle \( \text{ad}(E_G)/\text{ad}(E_H) \) gets identified with the holomorphic tangent bundle \( TM \). Therefore, we get a short exact sequence of holomorphic vector bundles on \( M \)

\[
(2.10) \quad 0 \rightarrow \text{ad}(E_H) \rightarrow \text{ad}(E_G) \rightarrow TM \rightarrow 0.
\]

The holomorphic connection \( \nabla^G \) on \( E_G \) induces a holomorphic connection on the adjoint vector bundle \( \text{ad}(E_G) \). This induced connection on \( \text{ad}(E_G) \) will be denoted by \( \nabla^{\text{ad}} \). For any point \( x \in T \), consider the holomorphic vector bundle

\[
(2.11) \quad \text{ad}(E_G)^x := \text{ad}(E_G)|_{f^{-1}(x)} \rightarrow f^{-1}(x)
\]

(see (2.7) for \( f \)). Let \( \nabla^x \) be the holomorphic connection on \( \text{ad}(E_G)^x \) obtained by restricting the above connection \( \nabla^{\text{ad}} \).

Any complex Fano manifold is rationally connected [13, p. 766, Theorem 0.1]. In particular, \( f^{-1}(x) \) is a rationally connected smooth complex projective variety. Since \( M \) is rationally connected, the curvature of the connection \( \nabla^x \) vanishes identically (see [4] p. 160, Theorem 3.1). From the fact that \( f^{-1}(x) \) is rationally connected it also follows that \( f^{-1}(x) \) is simply connected [9, p. 545, Theorem 3.5], [12, p. 362, Proposition 2.3]. Since \( \nabla^x \) is flat, and \( f^{-1}(x) \) is simply connected, we conclude that the vector bundle \( \text{ad}(E_G)^x \) in (2.11) is holomorphically trivial.

Let

\[
(2.12) \quad 0 \rightarrow \text{ad}(E_H)|_{f^{-1}(x)} \rightarrow \text{ad}(E_G)^x \xrightarrow{\alpha} (TM)|_{f^{-1}(x)} \rightarrow 0
\]

be the restriction to \( f^{-1}(x) \subset M \) of the short exact sequence in (2.10). Let \( T_xT \) be the tangent space to \( T \) at the point \( x \). The trivial vector bundle over \( f^{-1}(x) \) with fiber \( T_xT \) will be denoted by \( f^{-1}(x) \times T_xT \). Let

\[
(df)|_{f^{-1}(x)} : (TM)|_{f^{-1}(x)} \rightarrow f^{-1}(x) \times T_xT
\]

be the differential of \( f \) restricted to \( f^{-1}(x) \). The kernel of the composition homomorphism

\[
\text{ad}(E_G)^x \xrightarrow{\alpha} (TM)|_{f^{-1}(x)} \xrightarrow{(df)|_{f^{-1}(x)}} f^{-1}(x) \times T_xT
\]

(see (2.12) for \( \alpha \)) will be denoted by \( K^x \). So, from (2.12) we get the short exact sequence of vector bundles

\[
(2.13) \quad 0 \rightarrow K^x \rightarrow \text{ad}(E_G)^x \rightarrow f^{-1}(x) \times T_xT \rightarrow 0
\]

over \( f^{-1}(x) \).
Since both $\text{ad}(E_G)^x$ and $f^{-1}(x) \times T_xT$ are holomorphically trivial, using (2.13) it can be shown that the vector bundle $K^x$ is also holomorphically trivial. To prove that $K^x$ is also holomorphically trivial, fix a point $z_0 \in f^{-1}(x)$, and fix a subspace

(2.14) \[ V_{z_0} \subset \text{ad}(E_G)^{x}_{z_0} \]

that projects isomorphically to the fiber of $f^{-1}(x) \times T_xT$ over the point $z_0$. Since $\text{ad}(E_G)^x$ is holomorphically trivial, there is a unique holomorphically trivial subbundle $V \subset \text{ad}(E_G)^x$ whose fiber over $z_0$ coincides with the subspace $V_{z_0}$ in (2.14). Consider the homomorphism

$V \rightarrow f^{-1}(x) \times T_xT$

obtained by restricting the projection in (2.13). Since this homomorphism is an isomorphism over $z_0$, and both $V$ and $f^{-1}(x) \times T_xT$ are holomorphically trivial, we conclude that this homomorphism is an isomorphism over $f^{-1}(x)$. Therefore, $V$ gives a holomorphic splitting of the short exact sequence in (2.13). Consequently, the vector bundle $\text{ad}(E_G)^x$ decomposes as

(2.15) \[ \text{ad}(E_G)^x = K^x \oplus V. \]

Since $\text{ad}(E_G)^x$ is trivial, from a theorem of Atiyah on uniqueness of decomposition, [2, p. 315, Theorem 2], it follows that the vector bundle $K^x$ is trivial; decompose all the three vector bundles in (2.15) as direct sums of indecomposable vector bundles, and apply Atiyah’s result. From (2.12) we get a short exact sequence of holomorphic vector bundles

(2.16) \[ 0 \rightarrow \text{ad}(E_H)|_{f^{-1}(x)} \rightarrow K^x \rightarrow T(f^{-1}(x)) \rightarrow 0, \]

where $T(f^{-1}(x))$ is the holomorphic tangent bundle of $f^{-1}(x)$. Since $K^x$ is trivial, from (2.11) it follows that the tangent bundle $T(f^{-1}(x))$ is generated by its global sections. This immediately implies that $f^{-1}(x)$ is a homogeneous manifold.

Since $f^{-1}(x)$ is a Fano homogeneous manifold, we conclude that there is a semisimple linear algebraic group $G'$ over $\mathbb{C}$, and a parabolic subgroup $P' \subset G'$, such that $f^{-1}(x) = G'/P'$. Since a quotient space of the type $G'/P'$ is rigid [1, p. 131, Corollary], if follows that any two fibers of $f$ are holomorphically isomorphic.

Let

(2.17) \[ G := \text{Aut}^0(f^{-1}(x)) \]
be the group of all holomorphic automorphisms of $f^{-1}(x)$. It is known that $\mathcal{G}$ is a connected semisimple complex linear algebraic group [II p. 131, Theorem 2]. Since $f^{-1}(x)$ is isomorphic to $\mathcal{G}'/P'$, it follows that $\mathcal{G}$ is a semisimple linear algebraic group over $\mathbb{C}$ of adjoint type (this means that the center of $\mathcal{G}$ is trivial). As before, let

$$z_0 \in f^{-1}(x)$$

be a fixed point. Let

$$\mathcal{P} \subset \mathcal{G}$$

be the subgroup that fixes the point $z_0$. Note that $\mathcal{P}$ is a parabolic subgroup of $\mathcal{G}$, and the quotient $\mathcal{G}/\mathcal{P}$ is identified with $f^{-1}(x)$.

Consider the trivial holomorphic fiber bundle

$$T \times f^{-1}(x) \longrightarrow T$$

with fiber $f^{-1}(x)$. Let $\mathcal{E} \longrightarrow T$ be the holomorphic fiber bundle given by the sheaf of holomorphic isomorphisms from $T \times f^{-1}(x)$ to $M$, where $M$ is the fiber bundle in (2.6); recall that all the fibers of $f$ are holomorphically isomorphic. It is straightforward to check that $\mathcal{E}$ is a holomorphic principal $\mathcal{G}$–bundle, where $\mathcal{G}$ is the group defined in (2.17). Let

$$\varphi : \mathcal{E}_\mathcal{G} := \mathcal{E} \longrightarrow T$$

be this holomorphic principal $\mathcal{G}$–bundle. The fiber of $\mathcal{E}_\mathcal{G}$ over any point $y \in T$ is the space of all holomorphic isomorphisms from $f^{-1}(x)$ to $f^{-1}(y)$.

So there is a natural projection

$$\mathcal{E}_\mathcal{G} \longrightarrow M$$

that sends any $\xi \in \varphi^{-1}(y)$ to the image of the point $z_0$ in (2.18) by the map

$$\xi : f^{-1}(x) \longrightarrow f^{-1}(y).$$

This projection identifies the fiber bundle

$$\mathcal{E}_\mathcal{G}/\mathcal{P} \longrightarrow T$$

with the fiber bundle $M \longrightarrow T$, where $\mathcal{P}$ is the subgroup in (2.19). This completes the proof of the theorem. $\square$
3. Principal bundles over a torus

Let $G_0$ be a reductive linear algebraic group defined over $\mathbb{C}$. Fix a maximal compact subgroup $K_0 \subset G$. Let $Y$ be a complex manifold and $E_{G_0} \longrightarrow Y$ a holomorphic principal $G_0$–bundle over $Y$. A unitary flat connection on $E_{G_0}$ is a flat holomorphic connection $\nabla^0$ on $E_{G_0}$ which has the following property: there is a $C^\infty$ reduction of structure group $E_{K_0} \subset E_{G_0}$ of $E_{G_0}$ to the subgroup $K_0$ such that $\nabla^0$ is induced by a connection on $E_{K_0}$ (equivalently, the connection $\nabla^0$ preserves $E_{K_0}$). Note that $E_{G_0}$ admits a unitary flat connection if and only if $E_{G_0}$ is given by a homomorphism $\pi_1(Y) \longrightarrow K_0$.

Let $P \subset G_0$ be a parabolic subgroup. Let $R_u(P) \subset P$ be the unipotent radical. The quotient group $L(P) := P/R_u(P)$, which is called the Levi quotient of $P$, is reductive (see [8, p. 158, §11.22]). Given a holomorphic principal $P$–bundle $E_P \longrightarrow Y$, let $E_{L(P)} := E_P \times^P L(P) \longrightarrow Y$ be the principal $L(P)$–bundle obtained by extending the structure group of $E_P$ using the quotient map $P \longrightarrow L(P)$. Note that $E_{L(P)}$ is identified with the quotient $E_P/R_u(P)$. By a unitary flat connection on $E_P$ we will mean a unitary flat connection on the principal $L(P)$–bundle $E_{L(P)}$ (recall that $L(P)$ is reductive).

A vector bundle $E \longrightarrow Y$ is called numerically flat if both $E$ and its dual $E^*$ are numerically effective [11, p. 311, Definition 1.17].

**Proposition 3.1.** Let $E_{G_0}$ be a holomorphic principal $G_0$–bundle over a compact connected Kähler manifold $Y$. Then the following four statements are equivalent:

1. There is a parabolic proper subgroup $P \subset G_0$ and a strictly anti–dominant character $\chi$ of $P$ such that the associated line bundle $E_{G_0}(\chi) := E_{G_0} \times^P \mathbb{C} \longrightarrow E_{G_0}/P$

is numerically effective.

2. The adjoint vector bundle $\text{ad}(E_{G_0})$ is numerically flat.

3. The principal $G$–bundle $E_{G_0}$ is pseudostable, and $c_2(\text{ad}(E_{G_0})) = 0$ (see [6] p. 26, Definition 2.3 for the definition of pseudostability).

4. There is a parabolic subgroup $P_0 \subset G_0$ and a holomorphic reduction of structure group $E_{P_0} \subset E_{G_0}$ of $E_{G_0}$ such that $E_{P_0}$ admits a unitary flat connection.

**Proof.** This proposition follows from [7] p. 154, Theorem 1.1 and [3] Theorem 1.2. □
Lemma 3.2. Let $T$ be a complex torus, and let $E_{G_0} \to T_0$ be a holomorphic principal $G_0$–bundle. Let $P \subset G_0$ be a parabolic subgroup. If the four equivalent statements in Proposition 3.1 hold, then the holomorphic tangent bundle of $E_{G_0}/P$ is numerically effective.

Proof. Assume that the four equivalent statements in Proposition 3.1 hold.

Let $\delta : E_{G_0}/P \to T_0$ be the natural projection. Let $T_0 := \ker(d\delta) \subset T(E_{G_0})$ be the relative tangent bundle for the projection $\delta$. The vector bundle $T_0 \to E_{G_0}/P$ is a quotient of the adjoint vector bundle $\text{ad}(E_{G_0})$. Since $\text{ad}(E_{G_0})$ is numerically effective (second statement in Proposition 3.1), it follows that $T_0$ is numerically effective [11, p. 308, Proposition 1.15(i)].

Consider the short exact sequence of vector bundles on $E_{G_0}/P$

\[ 0 \to T_\delta \to T(E_{G_0}/P) \xrightarrow{\delta^*TT_0} 0. \]

Since $\delta^*TT_0$ and $T_\delta$ are numerically effective ($TT_0$ is trivial), it follows that $T(E_{G_0}/P)$ is numerically effective [11, p. 308, Proposition 1.15(ii)]. This completes the proof of the lemma. \qed

As before, $X$ is a compact connected Kähler manifold such that $TX$ is numerically effective, and $(E'_H, \theta')$ be a holomorphic Cartan geometry on $X$ of type $G/H$. Also, $\gamma$ and $f$ are the maps constructed in (2.6) and (2.7) respectively. Let

\[ K_f^{-1} \to M \]

be the relative anti–canonical line bundle for the projection $f$.

Let $G$ be the group in (2.17), and let $E_G \to T$ be the principal $G$–bundle constructed in (2.20). Let $\text{ad}(E_G) \to T$ be the adjoint vector bundle.

Proposition 3.3. Let $X$ is a compact connected Kähler manifold such that $TX$ is numerically effective, and let $(E'_H, \theta')$ be a holomorphic Cartan geometry on $X$ of type $G/H$. Then the relative anti–canonical line bundle $K_f^{-1}$ in (3.1) is numerically effective. Also, the following three statements hold:

1. The adjoint vector bundle $\text{ad}(E_G)$ is numerically flat.
2. The principal $G$–bundle $E_G$ is pseudostable, and $c_2(\text{ad}(E_G)) = 0$.
3. There is a parabolic subgroup $\mathcal{P} \subset G$ and a holomorphic reduction of structure group $\mathcal{E}_\mathcal{P} \subset E_G$ of $E_G$ such that $\mathcal{E}_\mathcal{P}$ admits a unitary flat connection.
Proof. Let $\gamma : M \to X$ be the covering in (2.6), and let $f : M \to T$ be the projection in (2.7). There is a semisimple complex linear algebraic group $G$, a parabolic subgroup $P \subset G$, and a holomorphic principal $G$–bundle $E_G \to T$ such that the fiber bundle $E_G/P \to T$ is holomorphically isomorphic to the one given by $f$ (see Theorem 2.1).

Since the canonical line bundle $K_T \to T$ is trivial, the line bundle $K_f ^{-1}$ is isomorphic to $K_M ^{-1}$. The anti–canonical line bundle $K_M ^{-1}$ is numerically effective because $TM$ is numerically effective. Hence $K_f ^{-1}$ is numerically effective. Recall that $E_G/P = M$ using the projection in (2.21). The line bundle $K_f ^{-1}$ corresponds to a strictly anti–dominant character of $P$ because $K_f ^{-1}$ is relatively ample. Hence the first of the four statements in Proposition 3.1 holds. Now Proposition 3.1 completes the proof of the proposition. □

**Proposition 3.4.** Let $X$ and $(E'_H, \theta')$ be as in Lemma 3.3. The principal $G$–bundle $E_G$ constructed in Theorem 2.1 admits a flat holomorphic connection.

Proof. We know that principal $G$–bundle $E_G$ is pseudostable, and $c_2(\text{ad}(E_G)) = 0$ (see the second statement in Proposition 3.3). Hence the proposition follows from [6, p. 20, Theorem 1.1]. □

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