The quantum bit from relativity of simultaneity on an interferometer

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The patterns of fringes produced by an interferometer have always been important testbeds for our best contemporary theories of physics. Historically, interference has been used to contrast quantum mechanics to classical physics, but recently experiments have been performed that test quantum theory against even more exotic alternatives. A physically motivated family of theories are those where the state space of a two-level system is given by a sphere of arbitrary dimension. This includes classical bits, and real, complex and quaternionic quantum theory. In this sense, special relativity itself can be used to explain why physics should be described by the rules of quantum theory in this setup. Moreover, our result has consequences for actual interference experiments that are currently performed in the lab: they have to be designed carefully to avoid rendering beyond-quantum effects invisible by relativity of simultaneity.

Introduction.— Questions of locality have historically been used to probe the validity of quantum mechanics. A famous example is given by the EPR paradox, the subsequent discussions and experimental tests of which contrast the predictions of quantum theory against classical theory [1, 5]. A relatively new development is to test quantum theory against even more general conceivable non-classical theories (e.g. [6–13]).

Popescu and Rohrlich have asked whether quantum mechanics has the maximal amount of nonlocality, given its peaceful coexistence with special relativity [14] (see also [15, 16]), and showed that, surprisingly, the answer is negative: there are conceivable correlations (now known as PR-boxes) that violate the Bell-CHSH inequality by more than any quantum state, while still not allowing for superluminal information transfer (a property known as no-signalling). This discovery has triggered a whole new area of research, examining the consequences of super-strong nonlocality, and aiming at a simple characterization of quantum correlations (see e.g. [17] and references within).

In this paper, we apply another relativistic consideration to the fundamental question of whether quantum theory could be modified in some regime (either as a generalisation, or replacement in an as-yet-unencountered limit). By noting that the number of clicks in a detector should be agreed on by all observers regardless of their frame of reference, we consider the implications of relativity of simultaneity, where two observers might disagree about the order of events [18]. On interferometers with two spatially separated arms, we show that this principle only allows local transformations and detector click probabilities that are either identical or very close to the predictions of standard complex quantum theory.

Thus, the relativistic structure of spacetime itself enforces in this setting that outcome probabilities of measurements are described by the standard rules of quantum theory. Our results are particularly relevant for a class of experiments that is currently being performed [19–22], which test the validity of quantum mechanics in specific interferometric setups. Furthermore, they give interesting insights on the relation between quantum mechanics and spacetime, which is the major object of study in quantum gravity research [23].

Borrowing some machinery from the framework of general probabilistic theories, we shall begin by introducing a general description for two-level systems, including quantum theory over arbitrary division algebras [24] (e.g. complex numbers, quaternions) and classical bits as special cases. We then explain how to describe an interferometry setup using this framework. Next we explicitly introduce relativistic considerations by examining how aspects of this framework should transform under changes of reference frame. We then consider the restrictions that arise when this is applied to the physical scenario of a two-armed interferometer. Finally, under different physical background assumptions, we analyse which theories are consistent with relativity in this setup.

General two-level systems.— A classical two-level system (bit) is qualitatively different from a quantum one (qubit). In the classical case knowing the probability that the system is in one of the two mutually exclusive alternatives is sufficient to totally predict every aspect of its behaviour, whereas in quantum theory this is not the case. Quantum theory also admits the coherent superposition of possibilities, allowing for interference effects between actions taken on the two possibilities, as opposed to classical theory that only allows for probabilistic mixing.

The additional information in the qubit beyond the classical two possibilities may be represented by statistics associated with additional complementary measurements: alternative measurements that could be made on
the qubit, whose outcomes cannot be simultaneously predicted (due to the uncertainty principle). In complex quantum theory, one can find two further complementary measurements in addition to the classical one. By taking the simultaneously possible expectation values for these three measurements, and representing them as a vector in Euclidean space, one arrives at the three-dimensional Bloch ball.

There has recently been a wave of research results deriving the formalism of quantum theory from simple physical postulates [8–13, 25, 26]. In most of these approaches, the first step is to prove that a two-level system is described by a ball state space; simple assumptions on the information-theoretic behaviour of a generalised bit lead to a natural generalization of the three-dimensional Bloch ball: the \(d\)-dimensional Bloch ball.

Let us now formalise a generalised setup for a system consisting of \(d\) such complementary measurements. The state \(\omega\) of such a two-level system is an element of the \(d\)-dimensional Euclidean unit ball \(B^d \equiv \{ x \in \mathbb{R}^d \mid |x| \leq 1 \} \).

Two-outcome measurements are described by vectors \(e \in \mathbb{R}^d\) with \(|e| = 1\); the probability of the first outcome, if measured on state \(\omega \in B^d\), is \((1 + e \cdot \omega)/2\), and that of the second outcome is \((1 - e \cdot \omega)/2\). Transformations which map states to states are given by \(d \times d\) orthogonal matrices \(R\) acting on \(\omega\). They are reversible because by applying \(R^{-1} = R^T\), the effect of \(R\) can be undone. In general, one has a compact group \(G \subseteq O(d)\) that describes the set of all physically possible reversible transformations on the states.

For \(d = 1\), the unit ball becomes a line segment, and we recover a classical bit. \(\omega = +1\) and \(\omega' = -1\) are the two distinct configurations of a classical spin, and the values in between correspond to probabilistic mixtures. The only non-trivial reversible transformation is the bit flip \(R = -I\), and thus taken together with the trivial transformation \(I\), we see \(G = \mathbb{Z}_2\).

If \(d = 3\), we recover the two-level systems of standard complex quantum theory: Every complex \(2 \times 2\) density matrix \(\rho\) is in one-to-one correspondence with an element \(\omega_\rho = (\omega_\rho^1, \omega_\rho^2, \omega_\rho^3)\) of the Bloch ball \(B^3\) via \(\rho = (I + \sum_{i=1}^3 \omega_\rho^i \sigma_i)/2\), where \(\sigma_i\) are the Pauli matrices. In this representation, the unitary transformations \(\rho \mapsto U\rho U^\dagger\) with \(U \in SU(2)\) are rotations: \(\omega_\rho \mapsto R_U \omega_\rho\), where \(R_U\) is a suitable element of \(SO(3)\). The probability of outcome \(+1\) of a projective measurement with projector \(P = |\psi\rangle \langle \psi|\) can be written \(\text{tr}(P\rho) = (1 + \omega_P \cdot \omega_\rho)/2\), where \(|\omega_P| = 1\). In quantum theory, it is impossible to implement the “universal NOT” map \(R = -I\), even though it is a symmetry of the Bloch ball (this corresponds to the transposition of the density matrix, which violates complete positivity). Thus, the compact group describing the physically possible reversible transformations on a qubit is \(G = SO(3)\).

The special case \(d = 2\) corresponds to quantum theory of the real numbers; \(d = 5\) describes a quaternionic quantum bit [27, 28], while \(d = 9\) describes an octonionic two-level system, which can be seen similarly as in the case of complex quantum theory explained above. The ball state spaces with arbitrary \(d \in \mathbb{N}\) have long been known in mathematical physics as examples of state spaces of Jordan algebras [29], and they have appeared in various places in quantum information theory [30, 31]. All these state spaces have \(N = 2\) perfectly distinguishable states and no more [8, 22], with every pair of antipodal points on the sphere (surface of the ball) describing mutually exclusive alternatives.

**Generalised interferometry.**— States \(\omega = (\omega_1, \ldots, \omega_d)\) of a general Bloch ball state space \(B^d\) can be used to describe the location of a single particle in an interferometer, as in Figure 1. We can parametrise the state space such that one of the components, say \(\omega_1\), determines the probability \(p\) to find the particle in the upper branch, as opposed to the lower branch. According to the general formalism described above, we must have \(p = (1 + \omega_1)/2\). As in quantum theory, there are in general many different states that give the same value \(p\). For example, when \(p = 1/2\) there can be incoherent mixtures of the two classically possible paths (such as the center of the ball), coherent superpositions like states on the ball’s equator, or states at any point in between.
Now consider two agents, say Alice and Bob (A and B in Figure 1), who reside at the spatially separated arms of the interferometer. If Alice performs an operation on the particle that is reversible and local to her arm (such as inserting a phase plate), then this will be described by some element $T_A$ of the group of physically possible reversible transformations $G \subseteq O(d)$. However, since she acts locally, this transformation should not alter the probability $p$ of finding the particle in her branch or the other. Thus, $T_A$ must preserve $\omega_1$ for all states. Hence $T_A \in O(d-1) \cap G$, and the transformation must be an element of the orthogonal group of one dimension less than the ball, describing all maps that preserve the $\omega_1$-axis.

We remark that ball state spaces are a special case in a wider framework known as generalised probabilistic theories (GPTs) [8, 13, 32]. In language adopted from this framework, because the set of $\omega_1$-preserving transformations preserve the statistics associated with the “which path” measurement, this set is referred to as the phase group of the “which path” measurement [33]. This definition arises as a natural generalization of the quantum case, where the transformations preserving the statistics of a measurement in a fixed basis $\{|\xi_j\rangle\}_{j=1...N}$ are the unitaries of the form $U(\phi_1,...,\phi_N) = \sum_{j=1}^{N} e^{i\phi_j}|\xi_j\rangle\langle \xi_j|$, and the $\phi_j$ are the phases associated with the given transformation.

Even if independent agents act locally on the different branches of the interferometer, their interventions will in general affect some global property of the system (e.g. the relative path length), and alter the output statistics of the interferometer (i.e. its interference pattern). However, it remains to be determined if all of the elements in the phase group can be applied by just one agent, or if only a subgroup of these transformations will be available to her.

An operational way to identify the subgroup of the phase transformations associated with an agent acting on a particular branch is to invoke a restriction known as branch locality [34], which says that all states with zero probability of being in a particular branch are left invariant under transformations on that branch. Intuitively, if a particle has no probability whatsoever of travelling down an agent’s branch, there should be no way of telling from the output statistics which phase plate (if any) that agent has inserted. As an example, consider a three-armed interferometer in quantum theory: no transformation applied to the first branch will ever change the relative phase between the second and third branches.

For the spherical state spaces of a two-branch interferometer, branch locality does not introduce any further restrictions: the only state that has zero probability to be found in the upper branch is $\omega = (1,0,...,0) \in \mathbb{R}^d$, and this state is invariant under all transformations in the phase group $O(d-1) \cap G$. The same argument applies to the lower branch, hence every single operation in the phase group can be localised to either of the two branches, whilst remaining consistent with branch locality [34, 35].

Reference frame invariance in the GPT framework.— Local actions of agents in the two interferometer arms are classical space-like separated events. Thus, we will analyse the interferometer within special relativity, and consider the effect of a change of reference frame on the general scenario. The number of clicks in a detector should reflect an objective element of reality. One could envision a setup of a photon detector attached to a bomb, such that the world is blown up if and only if the detector clicks; two observers in different frames of reference should not disagree over the fate of the world. By extension, if different observers do not disagree on whether an event occurred or not (they may still disagree over where, when and in what order relative to other events), they must also agree on the statistics associated with the occurrence of this event. In our formalism, this means that the probabilities $(1 + \omega_p \cdot \omega)/2$ for a positive outcome of measurement $\omega_p$ applied to state $\omega$ should be invariant under a change of reference frame.
Suppose we have two observers, say Rachel and Steven, who both observe the interference experiment, but are moving at relativistic speed relative to each other, as depicted in Figure 2. Then there will be a Lorentz transformation $\Lambda$ relating Rachel’s frame of reference to Steven’s. A priori, the state $\omega'$ that Steven sees might be different from the state $\omega$ experienced by Rachel, so long as they also describe measurements by different vectors $\omega'_P, \omega_P$ such that the outcome probabilities agree (i.e. $\omega_P \cdot T = \omega'_P \cdot T'$. For this to be possible, there would have to be a representation of the Lorentz group, with reversible transformations $T_A \in G$ acting such that $\omega' = T_A \omega$. However, since $G \subseteq O(d)$, this would induce a finite-dimensional unitary representation of the Lorentz group, and it is well-known that the only representation of this kind is the trivial representation $\phi_0$. Moreover, the “which path” degree of freedom is not a geometric degree of freedom\(^1\) relative to some Lorentz covariant property, say, a particle’s momentum direction (like photon polarization, for example), which is why also Wigner’s little group does not apply. Thus $T_A = 1$ for all $A$, and the transformations done by Alice and Bob appear the same in every reference frame.

**Operations on different branches commute.**—

We now analyze in detail the idea \cite{32} that relativity of simultaneity might impose further restrictions on the probabilistic behaviour of spatially extended interferometers. While doing so, we follow the spirit of \cite{37}, but take Peres’ argumentation beyond quantum theory. As before, consider the situation in Figure 2, with two agents: Alice who acts on the upper branch and Bob who acts on the lower branch. As the branches are space-like separated, the transformations applied by Alice and Bob should not cause the particle to jump from one branch to another, and so will belong to the phase group $G_\phi = O(d-1) \cap G$ associated with the “which path” measurement (where $G$ accounts for additional restrictions on the transformations beyond the ball automorphisms). More generally, there will be a subgroup of transformations that can be locally applied by Alice $G_A \subseteq G_\phi$, and a subgroup locally applicable by Bob $G_B \subseteq G_\phi$.

In general, there are situations (in particular on interferometers with more than two arms) where $G_A \neq G_B$, or where $G_A$ and $G_B$ are contained in a proper subgroup of $G_\phi$. However, in Appendix 4, we explain why it is natural in our specific setup to expect that $G_A = G_B = G_\phi$, based on some simple physical arguments about symmetry. Thus, we will now make the explicit assumption that all phase transformations are available locally to both Alice and Bob, and relax this assumption towards the end of this paper. In particular, this assumption is consistent with branch locality, as discussed above.

\footnote{One can imagine that the two material interferometer arms carry classical labels (like “A” and “B”, or “passing Earth” versus “passing Mars”). Then different observers will agree on whether a particle is detected in arm $A$ or arm $B$, and thus on the description of quantum states, regardless of their frames of reference.}

If there is at least one pair of transformations $T_A$ and $T_B$ such that $[T_A, T_B] \neq 0$, then the order in which Alice and Bob choose to apply their transformations will have an observable effect on the output statistics of the interferometer at least for some states. This is particularly problematic for a space-like separation between Alice and Bob: Consider again the two observers Rachel and Steven, moving at relativistic speed relative to each other, as in Figure 3. Although Rachel and Steven must agree what effect either action $T_A$ or $T_B$ would have on the interferometer individually (from the Lorentz invariance of transformations), Steven could in general disagree with Rachel about the order in which the two events occur\(^1\).

In particular, let us say Rachel observes the application of $T_A$ by Alice followed by $T_B$ by Bob; the compound operation is then $T_B T_A$. When changing into Steven’s reference frame, the compound operation should be the same. However, Steven may observe instead that $T_B$ happens before $T_A$, describing the compound operation by $T_A T_B$. This will lead to contradiction unless $T_A T_B = T_B T_A$ in all cases; that is $[T_A, T_B] = 0$ for all $T_A \in G_A$ and $T_B \in G_B$: since $G_A = G_B = G_\phi$, the phase group must be Abelian.

Note the subtle way that relativity impacts this interference setup: on the one hand, the “which path” information is not a geometric degree of freedom that transforms in any non-trivial way under the Lorentz group. In other words, the two material interferometer arms break manifest Lorentz invariance, and all observers see the same transformations $T_A$ and $T_B$. On the other hand, the local choices or applications of the transformations $T_A$ and $T_B$ are classical space-like separated events, which must not admit a unique time-ordering. Thus, $T_A$ and $T_B$ must commute.

**Which theories are consistent with relativity?**—

Before we move on to the general case, let us first assume in analogy to the quantum case that the $d$-dimensional Bloch ball state space carries the group of reversible transformations $G = SO(d)$. Then the phase group becomes $G_\phi = SO(d-1)$. Since all rotations commute only in two dimensions or less, this is only Abelian if $d \leq 3$, showing that the case of standard complex quantum theory is the highest-dimensional possibility allowed by relativity of simultaneity.

**Theorem 1.** Relativity of simultaneity enforces $d \leq 3$ for $d$-dimensional Bloch balls which admit the rotations $SO(d)$ as their reversible transformations. Furthermore, only for $d = 3$, the qubit of standard complex quantum theory, will the allowed group of phase transformations be non-trivial.

For both cases $d = 1$ (classical bit) and $d = 2$ (quantum bit over the real numbers), the only possible phase transformation is the identity, such that there are no actions at all which Alice and Bob could implement on the arms of the interferometer. Phase transformations are only possible if $d = 3$, the standard quantum bit, and in this case we obtain the complex phase $G_\phi = SO(2) = U(1)$. 

\[^1\]
In several recent derivations of quantum theory from simple postulates \[38, 39\], the condition that “\(G_\phi\) is non-trivial and Abelian” appeared as a crucial mathematical property (though in different notation) in the proofs. Here, we obtain an intriguing physical interpretation of this mathematical fact, related to special relativity.

So far we have assumed that \(G = \text{SO}(d)\), but in principle we can allow other groups of reversible transformations \(G \subseteq O(d)\). Let us make the minimal assumption that it is possible to map every pure state, i.e. point on the surface of \(B^d\), to every other by a reversible transformation. This property, namely transitivity on the surface of the Bloch ball \[38\], is true in quantum theory but very well motivated in general, since reversible time evolution should be able to exhaust the set of all pure states. Then we could have \(G = O(d)\) or \(G = \text{SO}(d)\), but there are also proper subgroups \(G\) of \(\text{SO}(d)\) which are still transitive on the surface of the Bloch ball, such as \(SU(d/2)\) for even \(d \geq 4\) \[38\]. It is possible to exhaustively list the groups \[40\] that are transitive on the surface of the sphere \(S^{d-1}\), and the respective phase groups formed by fixing an axis. Using this, and knowledge about the maximality of closed subgroups \[41\] of \(\text{SO}(d)\) and automorphisms \[42\] of \(U(2)\), one finds the following generalization of Theorem \[4\].

**Theorem 2.** If we only demand that reversible time evolution can map every pure state on the \(d\)-dimensional Bloch ball to every other, and that there are non-trivial phase transformations, then relativity of simultaneity allows for the following possibilities and no more:

- \(d = 2\) (the quantum bit over the real numbers), with \(G = \text{O}(2)\) and \(G_\phi = \mathbb{Z}_2\);
- \(d = 3\) (the standard complex quantum bit) with \(G = \text{SO}(3)\) and \(G_\phi = \text{SO}(2) = U(1)\);
- \(d = 4\), with \(G \simeq U(2)\) and \(G_\phi = \text{SO}(2) = U(1)\).

Thus, we get two further solutions beyond Theorem \[4\]. The first one, \(d = 2\), describes a bit in quantum theory over the real numbers. However, this solution seems physically quite unlikely: the phase group is discrete, namely isomorphic to \(\mathbb{Z}_2\), whereas one would expect the group of phase transformations to be continuous, due to continuity of time evolution \[8\].

The \(d = 4\) case is more interesting and quite unexpected. It has recently appeared in totally different context in \[43\]. To understand how \(G\) acts on the Bloch ball, note that every unitary matrix \(U \in U(n)\) can be written \(U = \Re U + i \cdot 3mU\), and then the block matrix

\[
G = \begin{pmatrix}
\Re U & 3mU \\
-3mU & \Re U
\end{pmatrix}
\]  

is a \(2n \times 2n\) real-valued orthogonal matrix. The \(4\)-dimensional \(G\) consists of those maps for \(U \in U(2)\). This is the same group \(U(2)\) that also acts on the pure states in standard complex quantum theory; however, the group actions are different in the way they affect the outcome probabilities of measurements. It is easy to compute the phase group preserving the vector \(\omega_1 = (1, 0, 0, 0)\), resulting in the set of maps of the form

\[
G_\phi(\theta) = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & \cos \theta & 0 & \sin \theta \\
0 & 0 & 1 & 0 \\
0 & -\sin \theta & 0 & \cos \theta
\end{pmatrix}
\].

As one can see, these maps leave the \(\omega_2\)-subspace invariant, and behave in the \((\omega_1, \omega_2, \omega_3)\)-subspace exactly in the same way as phase plates on a standard complex quantum bit. In other words, the group of transformations that can be applied locally on the interferometer arms is isomorphic to the group of standard qubit transformations, which suggests that this theory would behave very similarly to the standard quantum bit for experiments of this kind.

In Theorem \[3\] in the appendix, we relax our assumptions even further, by allowing \(G_A = U(d)\) and \(G_B = U(d)\) to be different (but still demanding that they generate all phase transformations). As we show, this yields exactly one solution in addition to those of Theorem \[2\] the quaternionic quantum quantum bit \((d = 5)\).

**Implications for interference experiments.**— There has been recent experimental activity aimed at testing quantum mechanics against more general theories in interference experiments \[20, 21\]. One proposal \[22\] is due to Peres \[19\], in which an interferometric set-up discriminates between ordinary quantum theory and its quaternionic counterpart, or more generally, whether a quantum two-level system has more than three complementary measurements. Given the results above, it seems particularly tempting to test for a \(d = 4\) Bloch ball, which has one degree of freedom more than the standard complex quantum bit, but still one less than the quaternionic quantum bit \((d = 5)\).

While Peres’ proposal is in principle suitable to test for this, our results show that the actual experimental implementation has to be chosen very carefully: as long as the state space will be probed only by applying different, spatially separated phase plates (which has been proposed for some setups \[44\]), relativity of simultaneity is likely to prevent any detectable difference to the standard complex quantum bit, as we have shown above.

Suppose we generalise even further, enforcing consistency of \(d \geq 5\) Bloch balls with relativity by admitting a proper Abelian subgroup of the phase dynamics (relaxing the condition that the full dynamics satisfy transivity on the sphere, or that all phase transformations can be implemented locally, allowing in particular \(G_A = G_B \subseteq G\)). Elementary group theory \[45\] shows that this Abelian subgroup \(G_A \subseteq G_B\) must be of the form \(\text{SO}(2) \oplus \text{SO}(2) \oplus \ldots \oplus \text{SO}(2)\), with \(n\) addends, some of which may be replaced by a finite subgroup.

For example, quaternionic quantum bits \((d = 5)\) have an Abelian subgroup of phase dynamics isomorphic to the torus \(\text{SO}(2) \oplus \text{SO}(2)\) (but this group cannot be induced
by fixing an axis of a group that is transitive on the surface of the 5-sphere). This shows that local transformations in the interferometer arms will then be described by \( n \) independent complex phases, an observation that brings us suspiciously close to standard complex quantum theory, suggesting that the experimental detection of possible deviations will be very difficult.

**Summary and Outlook.**— In this paper, we have considered how relativity constrains observable interference phenomena, without assuming that the probabilities of detector clicks are necessarily described by quantum mechanics. We have shown that relativity of simultaneity on two-armed interferometers enforces a behaviour that is very close to that of standard complex quantum theory: for two-level systems with state spaces that are \( d \)-dimensional balls, only the standard complex quantum bit with \( d = 3 \) is consistent with this principle.

Relaxing some background assumptions, we obtained an additional exotic four-dimensional case, which behaves identically to the standard qubit for all local transformations in the interferometer arms. Dropping also the symmetry assumption that the local transformations in both arms are identical, we get one final further solution: the quaternionic quantum bit.

Two questions arise: (i) given the findings above, what consequences can we draw for implementations of experiments like the one proposed by Peres [19]? Our results show that symmetries in the experimental setup play a crucial role, but more is to be explored. (ii) How far can one push this type of reasoning: is quantum theory the only theory consistent with relativistic spacetime? Equivalently: is the quantum path integral rule for summing up complex phases a direct consequence of the structure of spacetime? Our results suggest this may well be the case.

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I. APPENDIX: MOTIVATING, INTERPRETING AND RELAXING $\mathcal{G}_A = \mathcal{G}_B = \mathcal{G}_\phi$

We can break down the assumption of the main text, $\mathcal{G}_A = \mathcal{G}_B = \mathcal{G}_\phi$, into two parts:

(i) The assumption that $\mathcal{G}_A = \mathcal{G}_B$;

(ii) the assumption that the smallest subgroup$^2$ of $\mathcal{G}_\phi$ that contains both $\mathcal{G}_A$ and $\mathcal{G}_B$ is the full phase group $\mathcal{G}_\phi$.

Note that in the last section of the paper, we drop Assumption (ii), but keep Assumption (i). This way, we recover the toroidal group of locally applicable operations $\mathcal{G}_A = \mathcal{G}_B = \text{SO}(2) \oplus \cdots \oplus \text{SO}(2)$.

Discussion of Assumption (i).— Let us first discuss a situation where Assumption (i) does not hold, and then argue why it should hold in our two-armed interferometer. Imagine a situation where we have a single particle on four interferometer arms, and two of the arms (say, $|1\rangle$ and $|2\rangle$) are held by Alice, whereas the other two arms ($|3\rangle$ and $|4\rangle$) are held by Bob. We can think of a “which-party”-measurement, that determines whether the particle is at Alice’s or Bob’s location.

Consider a unitary $U$ that acts non-trivially only on the subspace spanned by $|1\rangle$ and $|2\rangle$. This is a phase transformation for the “which-party”-measurement. It can map $|1\rangle$ onto an arbitrary superposition $\alpha|1\rangle + \beta|2\rangle$, and $|2\rangle$ onto an arbitrary orthogonal state in the subspace, while leaving $|3\rangle$ and $|4\rangle$ invariant. This unitary does not alter the probability of finding the particle in Alice’s set of branches.

Clearly, $U$ is a phase transformation that is locally available to Alice, but not to Bob (this can be verified by arguments of branch locality $^{[44]}$, as $U$ will have no observable effect on the output statistics for states that have no support in $|1\rangle$ or $|2\rangle$). Thus, we expect that the map $\rho \mapsto U\rho U^\dagger$ is an element of $\mathcal{G}_A$. There is another operational argument for this: suppose that Alice decides to either apply $U$, or not to apply it. After this, she determines whether the particle is found in her half of the interferometer (by performing a “which-party”-measurement). If she finds the particle in her half, then some properties of the particle will depend on whether she has applied $U$ before or not. In other words, applying $U$ has a locally detectable consequence on outcomes of measurements (conditional on finding

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$^2$ All groups in this paper are assumed to be topologically closed. Since they are subgroups of the Lie group O(d), they must therefore be compact Lie groups themselves (which includes the possibility that they are finite or not connected).
the particle there). These consequences are observable by Alice, but not by Bob. This is the physical origin of the fact that $U \bullet U^\dagger \in G_A$ and $U \bullet U^\dagger \not\in G_B$ – the mechanism that locates this transformation at $A$, but not at $B$.

In our setting, however, no localization mechanism of this kind is possible: whenever Alice or Bob perform a “which-party”-measurement (which is just a “which-path”-measurement), the state will collapse either to $\omega = (1, 0, \ldots, 0)$ or to $\omega = (-1, 0, \ldots, 0)$, independently of any phase transformations that may have been applied before. Thus, there is no physical basis of locating some transformations $T \in G_\phi$ either at $A$ or at $B$. Unless there is an additional mechanism that breaks the symmetry between Alice and Bob, we should expect to have $G_A = G_B$.

This is consistent with the findings of [34, 32], in which branch locality in general provides a mechanism for breaking the symmetry of phase operations between branches by explicitly identifying whether a particular transformation can be done on any particular branch. For two-level state spaces with an uncertainty principle (such as $d$-balls), it was found that branch locality does not break the symmetry between Alice and Bob. The argument above takes this a step further, by noting that no mechanism for breaking the symmetry between branches of an interferometer will have an observable effect on a two-level system subject to an uncertainty principle.

Note that $G_A = G_B$ is a stronger statement than just saying that $G_A$ and $G_B$ are isomorphic: it says that every action that can locally be performed by Alice can in principle also be locally performed by Bob. In a standard quantum Mach-Zehnder interferometer, this is indeed the case: inserting a phase plate of angle $\phi$ into Alice’s arm is completely equivalent to inserting a phase plate of angle $(-\phi)$ into Bob’s arm.

We can also understand this property as saying that these transformations are fully relational: they have no locally detectable consequences whatsoever, but they alter the relation between the two interferometer arms (such as their relative optical path lengths). Thus, one would expect that either both Alice and Bob are able to trigger such a change of relation between the arms (by applying a suitable transformation), or neither of them are. This amounts to $G_A = G_B$.

**Experimental consequences.**— One way to avoid the conclusion of this paper is to disregard the argumentation above, and consider the possibility that physics nevertheless allows for $G_A \neq G_B$, as we will do in Theorem 3 below. Relativity of simultaneity will then only enforce that $[G_A, G_B] = 0$, but not that the groups have to be Abelian individually. In the experiment proposed by Peres [19], which tests the standard complex quantum bit against higher-dimensional ball state spaces, one would then have to make sure that local transformations on slits or interferometer arms are not elements of the intersection $G_A \cap G_B$ – otherwise, one would only test transformations within this Abelian subgroup, which leads to a bunch of standard complex phases $SO(2) \oplus \cdots \oplus SO(2)$ as explained towards the end of the main text.

Experimentally, this would amount to implement transformations that are asymmetrical with respect to the two branches or slits – in other words, transformations on a branch that cannot be undone by transformations on the other branch. This would demand an explicit fundamental asymmetry manifesting itself in the experiment.

**Discussion of Assumption (ii).**— Suppose that the smallest group that contains both $G_A$ and $G_B$, call it $G_{AB}$, is a proper subgroup of all phase transformations $G_\phi$. We can argue that this seems physically implausible as it introduces an additional asymmetry into the physical setup (namely the arbitrary choice of subgroup), and it is not clear what mechanism breaks the symmetry and selects this subgroup.

We will restrict our discussion to the case that $G_{AB}$ is connected (and non-trivial), which is motivated by the continuity of time evolution. In this case, we can also assume that $G_\phi$ is connected, since all transformations in $G_\phi$ which are not connected to the identity will then not have any significance for the interference experiment.

First consider the case that $G_\phi$ is a simple Lie group – that is, it is connected, non-Abelian, and every closed connected normal subgroup is either $\{1\}$ or all of $G_\phi$. It follows that either $G_\phi$ is trivial (in which case there are no local operations on the interferometer arms whatsoever), or that $G_{AB}$ is not a normal subgroup. In the latter case, there are many “copies” of $G_{AB}$ inside of $G_\phi$, namely $XG_{AB}X^{-1}$ for $X \in G_\phi$. But then, one should ask which physical mechanism selects the actual subgroup $G_{AB}$ from this infinitude of possibilities? In other words: the proper selection of subgroup is an additional choice to be made; some element of the physical setup (the interferometer) has to break the symmetry and perform this choice. Clearly, the beamsplitter is the only conceivable part of the setup that could serve this purpose. But then, the beamsplitter would have to do much more than just to prepare a delocalised state, and to select a basis where the particle being in the upper branch corresponds to the state $\omega = (1, 0, \ldots, 0)$ – it would also have to select that subgroup. This would lead to the seemingly paradoxical possibility that there could be beamsplitters which prepare exactly the same states (which behave identically for all measurements), but that make Alice’s and Bob’s local “phase plates” behave as elements of different transformation groups (such that the local phase transformations, followed by a measurement, would yield different statistics).

Now consider the case that $G_\phi$ is not a simple Lie group. Considering all the possible phase groups [40] starting with a group $G$ that is transitive on the sphere, we end up with only one non-simple possibility: that $G_\phi = SO(4)$, which is the phase group for Bloch ball dimension $d = 5$ – that is, the quaternionic quantum bit. In this case, the argument above can be avoided if $G_{AB}$ is a non-trivial normal subgroup of $G_\phi$. There are exactly two groups of this
kind \[46\], both isomorphic to SU(2), which is due to the double cover Spin(4) = SU(2) × SU(2).

Since we should expect \( G_A \) and \( G_B \) to be at least isomorphic as groups, it is easy to see that there is no way to choose them as subgroups of SU(2) such that \([G_A, G_B] = 0\) and such that SU(2) is generated from the union of both groups. Thus, \( G_{AB} \sim SU(2) \) does not work, and we obtain the same conclusion as above also in this special case of the quaternionic qubit: \( G_A \) and \( G_B \) should generate all of \( G_\phi \).

**Dropping (i) and keeping (ii): the quaternionic qubit.**— We can also see what happens if we drop Assumption (i), but stick to Assumption (ii). In this case, \( G_A \) and \( G_B \) may be distinct groups (that have possibly some non-trivial subgroup \( G_A \cap G_B \) in common), but they should still be isomorphic. It is easy to see that

\[
G_A' := \{ G \in G_\phi \mid G = XGX^{-1} \text{ for all } X \in G_A \}
\]

is a normal subgroup of \( G_\phi \), and \( G_B \subseteq G_A' \), thus \( G_A' \) is non-trivial. If \( G_A' \subseteq G_\phi \), then we have found a non-trivial normal subgroup of \( G_\phi \). Consider the other case that \( G_A' = G_\phi \). Since \( G_A \subseteq G_\phi \), this implies that \( G_A \) is Abelian. Repeating the same construction with \( A \leftrightarrow B \) interchanged, we obtain one of the following cases:

- \( G_B' = G_\phi \) as well. But then, \( G_B \) must also be Abelian, and so is the whole phase group \( G_\phi \) — in this case, we are back in the setting of the main text (equivalent to additionally postulating Assumption (i)).

- \( G_B' \subseteq G_\phi \).

Thus, we are either back with both Assumptions (i) and (ii), or we have found that either \( G_A \) or \( G_B \) is contained in a non-trivial proper normal subgroup \( G' \) of \( G_\phi \). Since \( G_A \) and \( G_B \) are assumed to be isomorphic and to generate all of \( G_\phi \), each of them must contain a non-trivial connected subgroup. Thus, the connected component of \( G' \) at the identity, \( G_0' \), is non-trivial. Since it is a characteristic subgroup of \( G' \), and \( G' \) is a normal subgroup of \( G_\phi \), the group \( G_0' \) is a normal (proper non-trivial connected) subgroup of \( G_\phi \). Hence \( G_\phi \) is not a simple Lie group.

As we have seen above, this admits only one possibility: that \( G_\phi = SO(4) \), i.e. we have the quaternionic quantum bit of ball dimension \( d = 5 \) and transformation group \( G = SO(5) \). Thus, dropping Assumption (i), we can summarise our findings in the following theorem:

**Theorem 3.** If we only demand that reversible time evolution can map every pure state on the \( d \)-dimensional Bloch ball to every other, that \( G_A \) and \( G_B \) are isomorphic (but not necessarily identical), and that they generate the full non-trivial group of phase transformations, relativity of simultaneity allows for the following possibilities and no more:

- \( d = 2 \) (the quantum bit over the real numbers), with \( G = O(2) \) and \( G_A = G_B = G_\phi = \mathbb{Z}_2 \);
- \( d = 3 \) (the standard complex quantum bit), with \( G = SO(3) \) and \( G_A = G_B = G_\phi = SO(2) = U(1) \);
- \( d = 4 \), with \( G \simeq U(2) \) and \( G_A = G_B = G_\phi = SO(2) = U(1) \);
- \( d = 5 \) (the quaternionic quantum bit), with \( G = SO(5) \), \( G_\phi = SO(4) \), \( G_A \) the left- and \( G_B \) the right-isoclinic rotations in \( SO(4) \) (or vice versa), such that both are isomorphic to \( SU(2) \) and \( G_A \cap G_B = \{ +I, -I \} \).

Note that \( G_A = G_B \) for \( d = 2, 3, 4 \) means that the same transformations (not just isomorphic transformations) can be applied by Alice and Bob on their interferometer arms, as explained further above. This is how Theorem\[2\] follows as a special case under the additional assumption \( G_A = G_B \).

This theorem also provides some guidance as to what one might realistically hope to find in Peres’ interference experiment \[19\]. The groups \( G_A \) and \( G_B \) for the \( d = 5 \) case can be explicitly constructed \[48\] by considering the quaternionic phase matrices \( U_A = \begin{pmatrix} q_A & 0 \\ 0 & 1 \end{pmatrix} \) and \( U_B = \begin{pmatrix} 1 & 0 \\ 0 & q_B \end{pmatrix} \), with unit quaternions \( q_A, q_B \in \mathbb{H} \), \( |q_A| = |q_B| = 1 \).