Experimental Realization of Nonadiabatic Holonomic Single-qubit Quantum Gates with Optimal Control in a Trapped Ion

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(Dated: June 9, 2020)

Quantum computation with quantum gates induced by geometric phases is regarded as a promising strategy in fault-tolerant quantum computation, due to its robustness against operational noises. However, because of the parametric restriction of previous schemes, the main robust advantage of holonomic quantum gates is smeared. Here, we experimentally demonstrate a solution scheme, demonstrating nonadiabatic holonomic single-qubit quantum gates with optimal control in a trapped $^{171}$Yb$^+$ ion based on three-level systems with resonant drives, which also hold the advantages of fast evolution and convenient implementation. Compared with corresponding previous geometric gates and conventional dynamic gates, the superiority of our scheme is that it is more robust against control amplitude errors, which is confirmed by the measured gate infidelity through both quantum process tomography and random benchmarking methods. In addition, we also outline that nontrivial two-qubit holonomic gates can also be realized within current experimental technologies. Therefore, our experiment validates the feasibility for this robust and fast holonomic quantum computation strategy.

Recently, constructing a powerful practical quantum computer, which is based on the quantum parallelism, has epoch-making significance. However, for a practical large scale quantum system, the control fields and its surrounding environment always generate inevitable influence, which leads to ruin of quantum information. Thus, robust and fast quantum information processing is desired. Interestingly, both Abelian [1] and non-Abelian [2] geometric phases, which just rely on the global properties of the evolution trajectories, have intrinsic features against certain local noises [3–6]. Moreover, considering the decoherence of the quantum systems, nonadiabatic evolution [7] are more preferable than the adiabatic evolution [1, 2] which needs long running-time. Consequently, high-fidelity quantum gates achieved in a nonadiabatic geometric way are currently of high interest [8, 9], and thus considerable attention has been devoted to the nonadiabatic geometric quantum computation [10].

Notably, due to the intrinsic non-commutativity of the non-Abelian geometric phase, it can naturally be used to realize universal quantum gates for the so-called holonomic quantum computation, which is originally proposed based on the adiabatic evolution [11, 12]. But it would suffer from severe environmental induced noise effect, because the coherence time of a quantum system is limited. To overcome this, nonadiabatic holonomic quantum computation (NHQC) schemes [13, 14] have been proposed, which remove the adiabatic condition. After that, the flexibility of holonomic gates were expanded theoretically [15–23] and verified experimentally in superconducting circuits [24, 25], nuclear magnetic resonance (NMR) [26–28], nitrogen-vacancy centers in diamond [29–34], etc. However, because of the parametric restriction, these NHQC schemes become sensitive to systematic errors and this drawback in the worst case could possibly leads to one magnitude higher reduction of fidelity than dynamical gates [35, 36], which thus smears the main advantage of geometric quantum gates.

Recently, theoretical schemes [37, 38] have been proposed to release the parametric constraint of previous NHQC schemes, and an arbitrary holonomic quantum gate there can be achieved in a single-loop evolution. Experimentally, following Ref. [37], shortcut to non-Abelian geometric gates has been demonstrated [39] in a superconducting circuit with off-resonance drives. However, to compatible with pulse shaping, the detuning between the drive and the corresponding qubit should be time-dependent, and what’s more, the detuning and the driving amplitudes should be precisely controlled in a correlated way, which is experimentally difficult. Therefore, implementing a robust NHQC scheme with resonant drives is more preferable in experiments, as it only requires control of the two driving fields.

In this letter, we experimentally demonstrate a nonadiabatic holonomic quantum gate scheme [38] in a trapped $^{171}$Yb$^+$ ion, with a three level configuration, which consists of modulating both time-dependent amplitude and phase of a two-tone resonant microwave drive, to introduce optimal control for the gate evolution. In our realization, characterized by random benchmarking method (RB), the demonstrated average gate fidelity is above 99%, which is mainly restricted by the limited coherent time. Moreover, we demonstrate that our nonadiabatic holonomic gates are more robust against control amplitude errors over both previous NHQC schemes and conventional dynamical gates under a same maximum drive amplitude, which is noteworthy for practical large scale quantum systems. In addition, combining with nontrivial non-
diabatic holonomic two-qubit gates, robust universal NHQC can be achieved in the trapped ions setup within current state-of-art technologies. Therefore, our experiment validates the feasibility towards robust NHQC.

We first address the realization of arbitrary holonomic single-qubit gates in the \{|0\}, \{|1\}\} subspace of a trapped $^{171}$Yb$^+$ ion, with \{|0\} $\equiv |^2S_{1/2}, F = 1, m_F = 0\rangle$, \{|1\} $\equiv |^2S_{1/2}, F = 1, m_F = 1\rangle$ and \{|a\} $\equiv |^2S_{1/2}, F = 0, m_F = 0\rangle$ as an auxiliary state, as shown in Fig. 1(a). Our proposal is realized by driving two microwave fields with time-dependent amplitude and phase resonantly coupled to the transitions \{|0\} $\leftrightarrow$ \{|a\} and \{|1\} $\leftrightarrow$ \{|a\}. This interaction can be described by

$$H_1(t) = \frac{\Omega_0(t)}{2} e^{-i\phi(t)} |0\rangle\langle a| + \frac{\Omega_1(t)}{2} e^{-i\phi_1(t)} |1\rangle\langle a| + \text{H.c.} = \frac{\Omega(t)}{2} e^{-i\phi(t)} \left( \sin \theta |0\rangle - \cos \theta e^{i\varphi} |1\rangle \right) \langle a| + \text{H.c.},$$

(1)

where \(\Omega_j\) and \(\phi_j\) \((j = 0, 1)\) are related to the time-dependent amplitude and phase of the two microwave fields respectively; \(\Omega(t) = \sqrt{\Omega_0^2(t) + \Omega_1^2(t)}\), \(\tan(\theta/2) = \Omega_0(t)/\Omega_1(t)\) with \(\theta\) being time-independently and \(\phi = \phi_0(t) - \phi_1(t) + \pi\) is a constant angle. From the Hamiltonian \(H_1(t)\), the quantum dynamics is induced by a resonant coupling between the bright state \(|b\rangle = \sin(\theta/2)|0\rangle - \cos(\theta/2)e^{i\varphi}|1\rangle\) and the auxiliary state \(|a\rangle\), while leaves the dark state \(|d\rangle = -\cos(\theta/2)e^{-i\varphi}|0\rangle - \sin(\theta/2)|1\rangle\) being decoupled from \(|b\rangle, |a\rangle\) subspace. This means through a cyclic evolution governed by the Hamiltonian \(H_1(t)\), the dark state \(|d\rangle\) is always not changed for certain constant values \(\theta\) and \(\phi\). The quantum dynamics in the \(|b\rangle, |a\rangle\} subspace satisfies the time-dependent Schrödinger equation \(i\frac{\partial}{\partial t} |\psi(t)\rangle = H_1(t) |\psi(t)\rangle\) [40], where the evolution state \(|\psi(t)\rangle\) can generally be parameterized by two time-dependent angles \(\alpha(t), \beta(t)\) and a global time-dependent phase \(f(t)\) as

$$|\psi(t)\rangle = e^{-i f(t)/2} \left( \cos \frac{\alpha(t)}{2} e^{i\beta(t)/2} \sin \frac{\alpha(t)}{2} e^{-i\beta(t)/2} \right).$$

(2)

According to the time-dependent Schrödinger equation, the relationships of the parameters in the evolution state \(|\psi(t)\rangle\) and the Hamiltonian \(H_1(t)\) can be solved as

$$f(t) = \beta_0(t) \cos \alpha(t),$$
$$\dot{\alpha}(t) = \Omega(t) \sin (\beta(t) + \phi_0(t)), $$
$$\dot{\beta}(t) = \Omega(t) \cot \alpha(t) \cos (\beta(t) + \phi_0(t)), $$

(3)

where the dot represents time differential. Especially, under the condition in Eq. (3), we can choose a proper set of variables \(\alpha(t), f(t)\) and \(\beta(t)\) to inversely engineer the Hamiltonian \(H_1(t)\) and dominate the evolution path. Therefore, we can design the evolution path to induce pure non-Abelian geometric phase on the bright state \(|b\rangle\) after a cyclic evolution [37, 38], from which nonadiabatic holonomic quantum gates can be constructed in the \(|0\rangle, |1\rangle\} subspace.

Specifically, during a cyclic evolution with time \(T\), we set \(\alpha(t) = \pi \sin^2(\pi t/T)\) and \(f(t) = \eta [2\alpha - \sin(2\alpha)]\) with \(\eta\) being a constant, which decides a chosen evolution path. For the optimization purpose, the evolution path should be divided into two equal time intervals \([0, T/2]\) and \([T/2, T]\). During the first interval \(t \in [0, T/2]\), the initial value of \(\beta(t)\) is set to \(\beta_0(t) = 0\), and \(\beta_1(T/2) = \int_0^{T/2} f(t) \cos \alpha(t) dt = 0\), the corresponding evolution operator is \(U_1(T/2, 0) = |d\rangle\langle d| + e^{i\gamma_1}|a\rangle\langle b|\), where \(\gamma_1 = -f(T/2)/2\). During the second interval \(t \in [T/2, T]\), we set \(\beta_2(T/2) = \gamma\), and \(\beta_2(T) = \int_{T/2}^T f(t) \cos \alpha(t) dt = \gamma\) with \(\gamma\) being an arbitrary constant angle. Then, the evolution operator is \(U_2(T/2, T/2) = |d\rangle\langle d| + e^{i\gamma_2}|b\rangle\langle a|\), where \(\gamma_2 = f(T/2)/2 + \gamma\). For geometric visualization of the cyclic evolution, the two evolution paths have rotational symmetry on the Bloch sphere, and the cyclic geometric phase is exactly the rotation angle \(\gamma\) corresponding to half of the solid angle of the rotation area as shown in Fig. 1(b). Therefore, even if there are dynamical phase accumulated during the evolution process, the final dynamical phase is zero at the end of the cyclic evolution. Moreover, the dark state is always decoupled. Consequently, after this cyclic evolution, the holonomic matrix is given by \(U(T, 0) = |d\rangle\langle d| + e^{i\gamma} |b\rangle\langle b|\) in the \(|d\rangle, |b\rangle\} subspace, which represents arbitrary holonomic single-qubit gates in the qubit

![FIG. 1. Realization of arbitrary single-qubit nonadiabatic holonomic gates. (a) Hyperfine energy level of $^{171}$Yb$^+$ ion. The qubit states \{|0\} and \{|1\} are encoded in $^2S_{1/2}, F = 1, m_F = 0\rangle$ and $^2S_{1/2}, F = 1, m_F = 1\rangle$, respectively. Two microwaves fields $\omega_{0a}$ and $\omega_{1a}$ are resonantly coupled with transitions \{|0\} $\leftrightarrow$ \{|a\} and \{|1\} $\leftrightarrow$ \{|a\} to generate holonomic gates in the qubit subspace. (b) Illustration of the implemented holonomic gates in Bloch sphere of \{|b\}, \{|a\}\} subspace. The evolution is divided into two steps: first evolving from bright state \{|b\} to auxiliary state \{|a\} and then back with an additional phase. (c) Simplified experimental setup. The ion is trapped in a needle trap which has a pair of radio frequency electrodes and two pair of direct-current electrodes. The Microwaves generated from AWG mix with the microwave signal from signal generator (EB257D) and then this mixed resonant signal would interact with the ion through microwave horn.](image-url)
FIG. 2. QPT results of single-qubit gates. (a) Operation flow of the QPT process. A group of complete basis are used to prepare the initial state and finally every result would be performed quantum state tomography to reconstruct the process matrix. And arbitrary single-qubit holonomic quantum gate that want to be characterized could be implemented in the middle of these sequences. (b) The bar charts of the real and imaginary parts of process matrix of $X$, $H$, $T$ and $S$ gates. The solid black outlines are for the theoretical gates.

basis $\{|0\rangle,|1\rangle\}$ as

$$U(\theta, \phi, \gamma) = e^{i\frac{\gamma}{2}}e^{-i\frac{\theta}{2}\sigma},$$  \hspace{1cm} (4)

where $\sigma = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta)$ and $\sigma$ are Pauli matrices. The above rotation matrix $U$ describes a rotation operation around the axis $\sigma$, by an angle $\gamma$, up to a global phase factor $\exp(i\gamma/2)$.

Our experiment is performed on a trapped $^{171}$Yb$^+$ ion, with the simplified circuits schematically shown in Fig. 1(c). The two energy level difference between $|0\rangle$, $|a\rangle$ and $|0\rangle$, $|1\rangle$ in our qutrit are characterized by $\omega_{0a} = 12.6428$ GHz and $\omega_{01} = 12.5$ MHz, with the corresponding magnetic field is about 8.93 G. This magnetic field is produced by 30 permanent magnets fixed in a circular aluminum holder to avoid the magnetic disturbance [41]. The microwave driving on the qutrit is generated from a 12.4428 GHz signal generator (Agilent E8257D) which is mixed with a 200 MHz microwave signal generated from a arbitrary waveform generator (AWG). After a high pass filter (HPF), this signal will be amplified to about 10 W and then sent to the ion with a microwave horn [44]. Our trap device is shielded with a 1.5 mm thick single layer Mu-metal [42], making the final coherent time about 20 ms for $|1\rangle \leftrightarrow |a\rangle$ transition and about 200 ms for $|0\rangle \leftrightarrow |a\rangle$ transition, characterized by Ramsey experiments. Through optimized time-dependent amplitude and phase modulation, we can couple transitions $|0\rangle \leftrightarrow |a\rangle$ and $|1\rangle \leftrightarrow |a\rangle$ simultaneously, to acquire a target geometric phase gate.

In each cycle, the experiment takes the following procedure: after 1 ms Doppler cooling, the state of the ion is initialized to $|a\rangle$ by 20 $\mu$s optical pumping [43]. Then a resonant microwave between $|a\rangle$ and $|0\rangle$ prepares the state to the $|0\rangle$ state with 99.5% fidelity. The holonomic gates are performed through two microwave fields $\omega_{0a}$ and $\omega_{1a}$, which are modulated according to Eq. (3) [45] in amplitude and phase. Finally, a resonant $\omega_{0a}$ ($\omega_{1a}$) transfer the $|0\rangle$ ($|1\rangle$) to the $|a\rangle$ state for state dependent fluorescence detection through an objective whose numerical aperture (NA) = 0.4.

Here, we first show the experimental scheme by setting the global time-dependent phase $f(t) = \eta[2\alpha - \sin(2\alpha)]$ with $\eta = 1/5$. We characterize the single-qubit holonomic gate through a standard quantum process tomography (QPT) method with the experimental sequences shown in Fig. 2(a). In the QPT process, we first prepare a set of initial states $\{0\rangle,|1\rangle,|0\rangle + |1\rangle)/\sqrt{2},|0\rangle - |1\rangle)/\sqrt{2},|0\rangle + i|1\rangle)/\sqrt{2},|0\rangle - i|1\rangle)/\sqrt{2}\}$ by applying a set of operations $\{I, R_x(\pi), R_y(\pi/2), R_y(-\pi/2), R_x(-\pi/2), R_x(\pi/2)\}$ to state $|0\rangle$ respectively. Then the holonomic gate will be performed immediately. Finally, we measure the output states through quantum state tomography to reconstruct the final states. The process matrix can be estimated with all the results through maximum likelihood estimation method [46]. We use the process fidelity $F_{\text{att}} = |\text{Tr}(\chi_{\text{exp}}\chi_{\text{th}}^\dagger)|$ to evaluate the QPT results, where $\chi_{\text{exp}}$ and $\chi_{\text{th}}$ represent the experimental and theoretical process matrix. The experimental result of four example gates $U(\pi/2,0,\pi),U(\pi/4,0,\pi),U(0,0,\pi/4)$ and $U(0,0,\pi/2)$, which are respectively the $X$, $H$, $T$ and $S$ gates, are shown in Fig. 2(b), and the corresponding gate fidelities are obtained to be $F_X = 97.21 \pm 0.03\%$, $F_H = 97.65 \pm 0.06\%$, $F_T = 97.85 \pm 0.05\%$ and $F_S = 97.43 \pm 0.03\%$, respectively.

Besides, we also use the random benchmarking (RB) method to characterize the performance of single-qubit gates, which is not depend on perfect state preparation and mea-
measurement. A reference RB experiment and an interleaved RB experiment are performed to investigate the fidelity of holonomic geometric phase gates, whose experimental sequences are shown in Fig. 3(a). The results of four holonomic single-qubit gates are shown in Fig. 3(b). The reference RB experiment gives the average fidelity $F_{\text{ave}} = 99\%$ of single-qubit gates in Clifford group. And the interleaved RB experiments give the fidelity of specific holonomic gate, which are $F_X = 99.10\%$, $F_H = 98.90\%$, $F_T = 99.20\%$, $F_S = 99.10\%$ respectively. The remaining infidelity is mainly from decoherence of $|1\rangle$ state, due to the magnetic field disturbance. We further proceed to demonstrate the gate robustness against control amplitude errors, which is one kind of dominant gate error sources for a large scale quantum system. We compare the robust NHQC (characterized as RNHQC) gates ($\eta = 1$) [45] with the conventional NHQC gates ($\eta = 0$) under the same maximum of the driving strength, $(2\pi)10$ KHz in our experiment, which is limited by the power of amplifier. We have experimentally characterized the performance of robust NHQC gates $X$ and $H$ with a single-qubit RB method, as a function of Rabi frequency error $\epsilon$, as well as that for the corresponding conventional NHQC gates, as shown in Figs. 4(a) and 4(b) for robust and conventional NHQC. All experimental results agree very well with the numerical simulations. The comparisons clearly illustrate the distinct advantages of the realized robust NHQC gates against control amplitude errors, especially in large Rabi frequency error for both holonomic single-qubit gates.

In addition, we demonstrated superiority of the quantum gates induced by geometric phases over that of the dynamical phases, against control amplitude errors under the same pulse shape. In our experiment, we compare the holonomic gates ($\eta$) with the dynamical gates ($\eta_D$) [45] under the same driving pulse with the maximum driving strength being $(2\pi)10$ KHz and $\eta = \eta_D$. We have experimentally characterized the performance of $X$ and $H$ holonomic gates with the single-qubit RB method, as a function of Rabi frequency error $\epsilon$, as well as that for the corresponding dynamical gates, as shown in Figs. 4(c) and 4(d). The comparisons clearly prove the superiority of the geometric gates over dynamical gates against control amplitude errors.

In order to achieve a universal quantum computation, two-qubit entangling operations are also necessary. Here, we propose a feasible scheme to implement robust nonadiabatic non-Abelian geometric controlled-phase gate between the internal atomic states ($|0\rangle, |1\rangle$) and motional state of this ion. Only the $\{0,1\}$ subspace of the motional state is considered although there are infinite states in this Hilbert space. In order to perform the geometric controlled-phase gate, a resonant interaction between $|a0\rangle$ and $|11\rangle$ is necessary, and the Hamiltonian of this interaction is

$$H_2(t) = \frac{\tilde{\Omega}(t)}{2} e^{-i\tilde{\phi}(t)} |11\rangle\langle a0| + \frac{\tilde{\Omega}(t)}{2} e^{i\tilde{\phi}(t)} |a0\rangle\langle 11| \quad (5)$$

where $\tilde{\Omega}(t)$ and $\tilde{\phi}(t)$ are the effective coupling strength and phase of the parametric drive respectively. The left part of $|a0\rangle$ and $|11\rangle$ are internal state while the right part are motional state. Similar to the single-qubit case with the Hamiltonian of Eq. (1), we can realize geometric phase gates diag $(e^{i\gamma}, e^{-i\gamma})$ in the subspace $|11\rangle, |a0\rangle$ by modulating the effective coupling strength and phase. When only considering the two-qubit computational subspace $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$, the resulting unitary operation corresponds to a controlled-phase gate with a conditional phase $\gamma$ will be achieved as

$$\tilde{U}(\gamma) = \text{diag} (1, 1, 1, e^{i\gamma}) . \quad (6)$$

In conclusion, we have experimentally demonstrated arbitrary robust nonadiabatic holonomic single-qubit gates with resonant drives. The superiority against control amplitude error of the realized single-qubit geometric gates is verified through comparison with the conventional NHQC gates. Besides, we also demonstrate the robustness against Rabi frequency error of the geometric gates over dynamic gates. The distinct advantage of geometric gates illustrates that they are promising candidates for robust quantum computation. Finally, aiming at a universal robust NHQC, we also propose a scheme for nontrivial two qubit control phase gate, which can be realized with two ion qubits. Therefore, our work validates the feasibility towards robust NHQC in trapped ions.

![Fig. 4. Noise-resilient feature of different single-qubit holonomic quantum gates. The comparison of (a) X and (b) H gate fidelity of RNHQC with $\eta = 1$ and conventional NHQC gate with $\eta = 0$ under the same Rabi frequency error. The results indicates the superiority of robust NHQC method against Rabi frequency error, especially in big error situations. The holonomic gates fidelity of both geometric and dynamic gate of (c) X and (d) H as a function of Rabi frequency error. The results indicates the anti-noise ability of geometric gates is better than that of dynamic gates. The error bars indicate the standard deviation, and each data point is averaged over 2000 realizations.](image-url)
This work was supported by the National Key Research and Development Program of China (Nos. 2017YFA0304100, 2016YFA0302700, and 2016YFA0301803), the National Natural Science Foundation of China (Nos. 11874343, 11774335, 11734015, and 11874156), An-hui Initiative in Quantum Information Technologies (AHY020100, AHY070000), Key Research Program of Frontier Sciences, CAS (No. QYZDY-SSW-SLH003), and the Fundamental Research Funds for the Central Universities (Nos. WK2470000026).

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I. OPTIMAL CONTROL TECHNIQUE

For Hamiltonian $H_1(t)$, there are two adjustable time-dependent parameters, which can incorporate optimal control technique [1, 2] to further enhance the robustness of nonadiabatic geometric quantum gates against control amplitude errors. Here, we consider the static control amplitude error situation, i.e. $\Omega(t) \rightarrow (1 + \epsilon)\Omega(t)$, and the Hamiltonian can be written as

$$H_\epsilon(t) = (1 + \epsilon)\frac{\Omega(t)}{2} e^{-i\phi_\epsilon(t)}|b\rangle\langle a| + \text{H.c.},$$

Under the static control amplitude errors, optimal control technique can be used in two interval for a single-loop evolution. Then, the influence of the control amplitude errors can be evaluated at the end of the first interval $t = T/2$, and the probability amplitude is given as $|\langle\psi(\tau/2)|\psi_\epsilon(\tau/2)|^2 = 1 + \hat{O}_1 + \hat{O}_2 + \ldots$, where $|\psi_\epsilon(\tau/2)|$ is the state with the static control amplitude errors, and $\hat{O}_m$ is the perturbation term of order $m$. For realistic experimental realization, we consider the probability amplitude to the second order as

$$|\langle\psi(\tau/2)|\psi_\epsilon(\tau/2)|^2 \simeq 1 - \epsilon^2 \int_0^{T/2} e^{-i\phi_\epsilon(t)} \sin^2 \alpha dt|^2.$$  

To achieve $|\langle\psi(\tau/2)|\psi_\epsilon(\tau/2)|^2 \simeq 1$, we set $\alpha(t) = \pi \sin^2(\pi t/T)$ and $f(t) = \eta[2\alpha - \sin(2\alpha)]$, $\beta_1(0) = 0$ and $\beta_1(T/2) = \gamma$, which lead to $|\langle\psi(\tau/2)|\psi_\epsilon(\tau/2)|^2 \simeq 1 - \epsilon^2 \sin^2 \eta \pi / (2\eta)^2$. i.e., for non-zero integer $\eta$, $|\langle\psi(\tau/2)|\psi_\epsilon(\tau/2)|^2 \simeq 1$. When $\eta = 0$, the current implementation reduces to the previous NHQC case. For experimental realization, we select $\eta = 1$ to demonstrate the robustness of nonadiabatic geometric gates against control amplitude errors $-0.2 \leq \epsilon \leq 0.2$ compared with the previous NHQC case with $\eta = 0$. For fair compare, both maximum value of $\Omega(t)$ are set to be $\Omega_{\text{max}} = (2\pi)10$ kHz as a restriction. The maximum value of the optimized pulse is bounded by $\Omega_{\text{max}}$, and thus the improvement of the gate performance can only be attributed to the optimal control. The amplitude shapes and phases of experimental microwave fields with $\eta = 0$ and $\eta = 1$ for operation $U(\pi/2, 0, \pi)$ are given in Fig. 1(a) and 1(c).

II. DYNAMICAL GATE

Here, we show the construction of arbitrary quantum gates without canceling dynamical phase in a single-loop evolution. Specifically, a cyclic evolution with time $T$ is divided into two equal time intervals $[0, T/2]$ and $[T/2, T]$. During the first interval $t \in [0, T/2]$, we set $\alpha(t) = \pi \sin^2(\pi t/T)$ and $f(t) = \eta[2\alpha - \sin(2\alpha)]$.
η_D[2\alpha - \sin(2\alpha)], the initial value of \beta(t) is set to \beta_1(0) = 0, and \beta_1(T/2) = \int_0^{T/2} \dot{f}(t) \cos(\alpha(t)) \, dt = 0, the corresponding evolution operator is U_1(T/2, 0) = |d\rangle \langle d| + e^{i\gamma_1} |a\rangle \langle b|, where \gamma_1 = -\dot{f}_1(T/2)/2. During the second interval \( t \in [T/2, T] \), we set \( \alpha(t) = \pi \sin(\pi t/T) \) and \( f(t) = -\eta_D[2\alpha - \sin(2\alpha)] \), \( \beta_2(T/2) = 0 \), and \( \beta_2(T) = \int_{T/2}^T \dot{f}(t) \cos(\alpha(t)) \, dt = 0 \). Then, the evolution operator is \( U_2(T, T/2) = |d\rangle \langle d| + e^{i\gamma_2} |b\rangle \langle a| \), where \( \gamma_2 = -\dot{f}_1(T/2)/2 \). Over all, after this cyclic evolution, the evolution matrix is given by \( U(T, 0) = |d\rangle \langle d| + e^{i\gamma_D} |b\rangle \langle b| \) in the \{ |d\rangle, |b\rangle \} subspace with \( \gamma_D = -\dot{f}_1(T/2) = -2\eta_D \pi \), which represents arbitrary dynamical single-qubit gates in the qubit basis \{ |0\rangle, |1\rangle \} as

\[
U(\theta, \phi, \gamma_D) = e^{i\gamma_D} e^{-i\frac{\gamma_D}{2} \mathbf{n} \cdot \mathbf{\sigma}}
\]

After that, we can experimentally confirm the priority of geometric phase than dynamical phase against control amplitude errors \(-0.2 \leq \epsilon \leq 0.2 \) under the same driving amplitude condition. Due to the dynamical evolution relying on parameter \( \eta_D \), thus, we can construct the geometric gates for same evolution matrix in main text with the conditions \( \eta_D = \eta \) for the same driving amplitude and \( \gamma_D = -2\eta_D \pi = \gamma \) with the same parameters \( \theta, \phi \). The amplitude shapes and phases of experimental microwave fields with \( \eta_D = \eta = -1/2 \) for operation \( U(\pi/4, 0, \pi) \) are given in Fig. 1(b) and 1(d).

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