ON THE DEFINITION OF THE KOBAYASHI-BUSEMAN PSEUDOMETRIC

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Abstract. We prove that the $(2n-1)$-th Kobayashi pseudometric of any domain $D \subset \mathbb{C}^n$ coincides with the Kobayashi–Buseman pseudometric of $D$, and that $2n-1$ is the optimal number, in general.

1. Introduction and results

Let $\mathbb{D} \subset \mathbb{C}$ be the unit disc. Recall first the definitions of the Lempert function $\tilde{k}_D$ and the Kobayashi–Royden pseudometric $k_D$ of a domain $D \subset \mathbb{C}^n$ (cf. [1]):

\[ \tilde{k}_D(z, w) = \inf \{ \tanh^{-1} |\alpha| : \exists \varphi \in \mathcal{O}(\mathbb{D}, D) : \varphi(0) = z, \varphi(\alpha) = w \}, \]

\[ \kappa_D(z; X) = \inf \{ \alpha \geq 0 : \exists \varphi \in \mathcal{O}(\mathbb{D}, D) : \varphi(0) = z, \alpha \varphi'(0) = X \}, \]

where $z, w \in D$, $X \in \mathbb{C}^n$. The Kobayashi pseudodistance $k_D$ can be defined as the largest pseudodistance which does not exceed $\tilde{k}_D$. Note that if $k_D^{(m)}$ denotes the $m$-th Lempert function of $D$, that is,

\[ k_D^{(m)}(z, w) = \inf \{ \sum_{j=1}^{m} \tilde{k}_D(z_j, z_{j+1}) : z_1, \ldots, z_m \in D, z_1 = z, z_m = w \}, \]

then

\[ k_D(z, w) = \inf_m k_D^{(m)}(z, w) = \inf \{ \int_0^1 \kappa_D(\gamma(t); \gamma'(t))dt \}, \]

where the infimum is taken over all piecewise $C^1$-curves $\gamma : [0, 1] \to D$ connecting $z$ and $w$. By a result of M. Y. Pang (see [5]), the Kobayashi–Royden pseudometric is the infinitesimal form of the Lempert function

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for taut domains; more precisely, if $D$ is a taut domain, then

$$\kappa_D(z; X) = \lim_{C \ni t \to 0} \frac{\hat{k}_D(z, z + tX)}{t}. \quad (1)$$

In [3], S. Kobayashi introduces a new invariant pseudometric, called the Kobayashi–Buseman pseudometric in [1]. One of the equivalent ways to define the Kobayashi–Buseman pseudometric $\hat{\kappa}_D$ of $D$ is just to set $\hat{\kappa}_D(z; \cdot)$ to be largest pseudonorm which does not exceed $\kappa_D(z; \cdot)$.

Recall that

$$\hat{\kappa}_D(z; X) = \inf \left\{ \sum_{j=1}^{m} \kappa_D(z; X_j) : m \in \mathbb{N}, \sum_{j=1}^{m} X_j = X \right\}. \quad (2)$$

Thus, it is natural to consider the new function $\kappa_D^{(m)}$, namely,

$$\kappa_D^{(m)}(z; X) = \inf \left\{ \sum_{j=1}^{m} \kappa_D(z; X_j) : \sum_{j=1}^{m} X_j = X \right\}. \quad (3)$$

We call $\kappa_D^{(m)}$ the $m$-th Kobayashi pseudometric of $D$. It is clear that $\kappa_D^{(m)} \geq \kappa_D^{(m+1)}$ and if $\kappa_D^{(m)}(z; \cdot) = \kappa_D^{(m+1)}(z; \cdot)$ for some $m$, then $\kappa_D^{(m)}(z; \cdot) = \kappa_D^{(j)}(z; \cdot)$ for any $j > m$. It is shown in [3] that

$$\kappa_D^{(2n)} = \hat{\kappa}_D. \quad (4)$$

Let now $D \subset \mathbb{C}^n$ be a taut domain. We point out that, using the equalities (1) and (2), M. Kobayashi (see [2]) shows that

$$\hat{\kappa}_D(z; X) = \lim_{C \ni t \to 0} \frac{k_D(z, z + tX)}{t}. \quad (5)$$

Obvious modifications in the proof of this result lead to

$$\lim_{u, v \to z, u \neq v} \frac{k_D^{(m)}(u, v) - k_D^{(m)}(z; u - v)}{||u - v||} = 0. \quad (6)$$

uniformly in $m$ and locally uniformly in $z$; thus,

$$\kappa_D^{(m)}(z; X) = \lim_{C \ni t \to 0} \frac{k_D^{(m)}(z, z + tX)}{t}$$

uniformly in $m$ and locally uniformly in $z$ and $X$.

The aim of this note is the following result which improves (2).

**Theorem 1.** For any domain $D \subset \mathbb{C}^n$ we have that

$$\kappa_D^{(2n-1)} = \hat{\kappa}_D. \quad (7)$$
On the other hand, if \( n \geq 2 \) and
\[
D_n = \{ z \in \mathbb{C}^n : \sum_{j=2}^{n} (2|z_1^3 - z_j^3| + |z_1^3 + z_j^3|) < 2(n - 1) \},
\]
then
\[
\kappa_{D_n}^{(2n-2)}(0; \cdot) \neq \hat{k}_{D_n}(0; \cdot).
\]

Note that the proof below shows that the equality (4) remains true for any \( n \)-dimensional complex manifold.

An immediately consequence of Theorem 1 and the equality (3) is:

**Corollary 2.** For any taut domain \( D \subset \mathbb{C}^n \) one has that
\[
\lim_{w \to z, w \neq z} k_D^{(2n-1)}(z, w) = 1
\]
locally uniformly in \( z \), and \( 2n - 1 \) is the optimal number, in general.

**Remarks.** (i) If \( D \subset \mathbb{C} \), then even \( \tilde{k}_D = k_D \) (cf. [1]).

(ii) Corollary 2 holds for \( n \)-dimensional taut complex manifolds.

(iii) Observe that Corollary 2 may be taken as a very weak version of the following question asked by S. Krantz (see [4]): whether there is a positive integer \( m = m(D) \) such that \( k_D = k_D^{(m)} \).

Let now \( h_S \) be the Minkowski functions of a starlike domain \( S \subset \mathbb{R}^N \), that is, \( h_S(X) = \inf\{ t > 0 : X/t \in S \} \). We may define as above
\[
h_S^{(m)}(X) = \inf\left\{ \sum_{j=1}^{m} h_S(X_j) : \sum_{j=1}^{m} X_j = X \right\}.
\]

Then the Minkowski function \( h_{\hat{S}} \) of the convex hull \( \hat{S} \) of \( S \) is the largest pseudonorm which does not exceed \( h_S \). It follows by a lemma due to C. Carathéodory (cf. [2]) that
\[
h_{\hat{S}} = h_{\hat{S}}^{(N)} = \inf\{ \sum_{j=1}^{M} h_S(X_j) : M \leq N, \sum_{j=1}^{M} X_j = X, X_1, \ldots, X_M \text{ are } \mathbb{R}\text{-linearly independent} \}.
\]

One can easily see that \( N \) is the optimal number for the class of starlike domains in \( \mathbb{R}^N \).

Denote by \( I_{D,z} \) the indicatrix of \( \kappa_D(z; \cdot) \), that is, \( I_{D,z} = \{ X \in \mathbb{C}^n : \kappa_D(z; X) < 1 \} \). Note that \( I_{D,z} \) is a balanced domain (a domain \( B \subset \mathbb{C}^n \) is said to be balanced if \( \lambda X \in B \) for any \( \lambda \in \mathbb{D} \) and any \( X \in B \)). In particular, \( I_{D,z} \) is a starlike domain and hence (2) follows by (6). Similarly, (4) will follow by the following.
Proposition 3. If $B \subset \mathbb{C}^n$ is a balanced domain, then
\begin{equation}
 h_B = h_B^{(2n-1)}.
\end{equation}

Observe that the domain $D_n$ from Theorem 1 is pseudoconvex and balanced, thus $\kappa_{D_n}(0; \cdot) = h_{D_n}$ (cf. \cite{[1]}) and so $\kappa_{D_n}^{(m)}(0; \cdot) = h_{D_n}^{(m)}$. Then inequality (5) is equivalent to
\begin{equation}
 h_{D_n} \neq h_{D_n}^{(2n-2)}.
\end{equation}

2. Proofs

To prove Proposition 3, we shall need the following result.

Lemma 4. Any balanced domain can be exhausted by bounded balanced domains with continuous Minkowski functions.

Proof. Let $B \subset \mathbb{C}^n$ be a balanced domain. Denote by $B_n(z, r) \subset \mathbb{C}^n$ the ball with center $z$ and radius $r$. For $z \in \mathbb{C}^n$ and $j \in \mathbb{N}$, set $F_{n,j,z} := B_n(z, ||z||^2/j)$. We may assume that $B_n(0, 1) \subset B$. Put
\begin{equation}
 B_j := \{z \in B_n(0, j) : F_{n,j,z} \subset B\}, \quad j \in \mathbb{N}.
\end{equation}

Then $(B_j)_{j \in \mathbb{N}}$ is an exhaustion of $B$ by non-empty bounded open sets. We shall show that $B_j$ is a balanced domain with continuous Minkowski functions.

For this, take any $z \in B_j$ and $0 \neq \lambda \in \overline{B}$, and observe that $F_{n,j,\lambda z} \subset \lambda F_{n,j,z} \subset B$. Thus, $B_j$ is a balanced domain.

Since $h_{B_j}$ is an upper semicontinuous function, it remains to prove that it is lower semicontinuous. Assuming the contrary, we may find a sequence of points $(z_k)_{k \in \mathbb{N}}$ converging to some point $z \in \mathbb{C}^n$ and a positive number $c$ such that $h_{B_j}(z_k) < 1/c < h_{B_j}(z)$ for any $k$. Note that $F_{n,j,cz_k} \subset B$, $k \in \mathbb{N}$. Hence $B_n(cz, c^2||z||^2/j) \subset B$. On the other hand, fix $t \in (0, 1)$ such that $h_{B_j}(tcz) > 1$. Then $F_{n,j,tcz} \subset B_n(tcz, c^2||z||^2/j) \subset B$; thus $h_{B_j}(tcz) < 1$, a contradiction. \hfill $\square$

Proof of Proposition 3. First, we shall prove (7) in the case, when $B \subset \mathbb{C}^n$ is a bounded balanced domain with continuous Minkowski function. Fix a vector $X \in \mathbb{C}^n \setminus \{0\}$. Then $h_B(X) \neq 0$ and we may assume that $h_B(X) = 1$. By the continuity of $h_B$ and (6), there exist $\mathbb{R}$-linearly independent vectors $X_1, \ldots, X_m$ ($m \leq 2n$) such that $\sum_{j=1}^m X_j = X$ and
\begin{equation}
 \sum_{j=1}^m h_B(X_j) = 1. \quad \text{Since } h_B \text{ is a norm, the triangle inequality implies that } h_B(X_j) = h_B(X_j), \quad j = 1, \ldots, m. \quad \text{To prove (7), it suffices to show}
\end{equation}
that \( m \neq 2n \). The convexity of \( \hat{B} \) provides a support hyperplane \( H \) for \( \hat{B} \) at \( X \in \partial \hat{B} \), say \( H = \{ z \in \mathbb{C}^n : \text{Re}(z - X, \overline{X_0}) = 0 \} \), \( X_0 \in \mathbb{C}^n \), where \( \langle \cdot , \cdot \rangle \) stands for the Hermitian scalar product in \( \mathbb{C}^n \). Assuming \( m = 2n \) implies that \( H = \{ \sum_{j=1}^{m} \alpha_j \hat{X}_j : \sum_{j=1}^{m} \alpha_j = 1, \alpha_1, \ldots, \alpha_m \in \mathbb{R} \} \), where \( \hat{X}_j := X_j/h_B(X_j) \in \partial \hat{B} \). In particular, \( \partial \hat{B} \) contains a relatively open subset of \( H \). Since \( \hat{B} \) is a balanced domain, it follows that its intersection with the plane, spanned by \( X_0 \), is a disc whose boundary contains a line segment, a contradiction.

Now let \( B \subset \mathbb{C}^n \) be an arbitrary balanced domain. If \( (B_j)_{j=1}^\infty \) is an exhaustion of \( B \) given by Lemma 4, then \( h_{B_j} \searrow h_B \) pointwise and hence \( h_{B_j} \searrow h_B \) by (6). Then (7) follows by the inequalities \( h_B \leq h_{B_j}^{(2n-1)} \leq h_{B_j}^{(2n-1)} \) and the equality \( h_{B_j} = h_{B_j}^{(2n-1)} \) from above. \( \square \)

Proof of the inequality (8). Let \( L_n = \{ z \in \mathbb{C}^n : z_1 = 1 \} \). Then the triangle inequality implies that \( D_n \subset D \times \mathbb{C}^{n-1} \) and

\[
F_n := \partial D_n \cap L_n = \{ z \in \mathbb{C}^n : z_1 = 1, z_j \in \Omega, 2 \leq j \leq n \},
\]

where \( \Omega \) is the set of the third roots of unity. Denoting by \( \Delta \) the convex hull of \( \Omega \), it follows that

\[
\partial \hat{D}_n \cap L_n = \hat{F}_n = \{ 1 \} \times \Delta^{n-1}.
\]

Hence, \( \partial \hat{D}_n \cap L_n \) is a \((2n-2)\)-dimensional convex set. Put \( \tilde{F}_n = \{ Y \in \hat{F}_n : h_{D_n}^{(2n-2)}(Y) = 1 \} \). If \( X \in \tilde{F}_n \), then there exist \( X_1, \ldots, X_m \in \mathbb{C}^n \setminus \{ 0 \} \), \( m \leq 2n - 2 \), such that \( \sum_{j=1}^{m} X_j = X \) and \( \sum_{j=1}^{m} h_{D_n}(X_j) = 1 \) (note that \( D_n \) is taut). Hence, \( X_1/h_{D_n}(X_1), \ldots, X_m/h_{D_n}(X_m) \in F_n \) and \( X \) belongs to their convex hull. Since \( F_n \) is a finite set, it follows that \( \tilde{F}_n \) is contained in a finite union of at most \((2n-3)\)-dimensional convex sets. Thus, \( \tilde{F}_n \neq \hat{F}_n \) which implies that \( h_{D_n} \neq h_{D_n}^{(2n-2)} \). \( \square \)

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