A NEW LAPLACE-TYPE FRACTIONAL DERIVATIVE

A Preprint

Adebowale Sijuwade
Department of Mathematics
Washington State University
Pullman WA, 99163
adebowale.sijuwade@wsu.edu

Mostafa Rezapour
Department of Mathematics
Washington State University
Pullman WA, 99163
mostafa.rezapour@wsu.edu

January 15, 2020

ABSTRACT

In this paper, we present a new derivative via the Laplace transform. The Laplace transform leads to a natural form of the fractional derivative which is equivalent to a Riemann-Liouville derivative with fixed terminal point. We first consider a representation which interacts well with periodic functions, examine some rudimentary properties and propose a generalization. The interest for this new approach arose from recent developments in fractional differential equations involving Caputo-type derivatives and applications in regularization problems.

Keywords Fractional calculus, Laplace Transform

1 Introduction

This section is intended to motivate the usage of fractional differentiation and integration. Let D denote the ordinary derivative $\frac{d}{dt}(\cdot)$. There are many applications of fractional calculus such as: modeling diffusion processes in Atanackovic [1]; heat transfer models in Atangana and Baleanu [3]; continuum mechanics in Mainardi [4]; evolution equations and renewal processes in Kochubei [5]; physics and engineering in Diethelm [8]; robotics, aerospace and biomedicine in Caponetto [13] and financial economics in Kiryakova [16]. Partial fractional differential equations and modelling techniques in hydrodynamics and stochastics can be found in Kilbas et al. [15]. An introductory treatment of fractional differential equations is provided in Miller et al. [18] and Podlubny [20]. One can find several different definitions involving fractional-order integral and derivative operators such as those seen in de Olivera [2] and Samko [19]. In this article, we consider spaces where the functions are continuous or piecewise continuous.

Suppose that $f$ is locally integrable on $\mathbb{R}$. We define distributions by

$$< f(t), \phi(t) > = \int_{\mathbb{R}} f(t)\phi(t)dt,$$

and denote the space of distributions by $\mathcal{D}'$. Consider the delta distribution in the usual sense as a continuous linear functional acting on the space of distributions $\mathcal{D}$ by

$$< \delta(t-c), \phi(t) > = \phi(c) \text{ for } \phi \in \mathcal{D}, c \in \mathbb{R}.$$ 

The derivative of a distribution $\phi$ is given by

$$< \phi'(t), \psi(t) > = - < \phi(t), \psi'(t) >,$$

where $\psi \in \mathcal{D}$. Setting $\phi(t) = \delta(t-c)$, we obtain $\delta'(c)$ on the right hand side. This definition can be easily generalized to higher order derivatives by
\[ <D^{(n)}\delta(t - c), \phi(t)> = (-1)^{(n)}(D^{(n)}\phi)(c). \]

The Fourier transform of a distribution is defined as follows:

\[ <\mathcal{F}(\phi(t))(\omega), \phi(\omega)> = <\phi(t), \mathcal{F}(\phi(\omega))>, \]

where \( \mathcal{F}(\phi(t)) \) is the usual definition of the Fourier transform for real-valued functions.

Now consider the Schwarz class of smooth test functions \( \mathcal{S} \), which decay with their derivatives at infinity. The space of continuous linear functionals on this space is denoted by \( \mathcal{S}' \), which is contained in the set of distributions. Consider the Laplace transform of a distribution \( \phi(t) \) defined by

\[ F(s) = \mathcal{L}(\phi(t)) = \mathcal{F}(\phi(t)e^{-\sigma t})(\mu), \tag{2} \]

where \( s = \sigma + i\mu, \mu < 0 \) and \( \phi(t)e^{-\sigma t} \in \mathcal{S}' \). Let \( f \) be a distribution supported on \((0, \infty)\) such that for \( \sigma > 0 \), \( f(t)e^{-\sigma t} \in \mathcal{S}' \). It follows that the Laplace transform of the Dirac delta function is given by

\[ \mathcal{L}(\delta(t - c))(s) = e^{-cs}, \]

and

\[ \mathcal{L}(\phi'(t))(s) = s\mathcal{L}(\phi(t))(s). \]

A more rigorous treatment of distributional derivatives and generalized functions can be found in Schwarz [6], Kanwal [7], McBride [17] and Zemanian [30]. If \( f : \mathbb{R} \to \mathbb{R} \) is of exponential order, then

\[ F(s) = \mathcal{L}(f(t)) = \int_0^\infty e^{-st}f(t)dt, \]

\[ \mathcal{L}(D^n f) = s^\alpha F(s) - \sum_{k=0}^{n-1} s^k (D^{n-k-1} f(t))|_{t=0}. \tag{3} \]

The left and right Riemann Liouville derivatives of order \( \alpha \in \mathbb{C} \), where \( Re(\alpha) \geq 0 \) on a finite real interval are defined by

\[ \left( \frac{R}{a} D^\alpha \right) g(t) = \frac{1}{\Gamma(n - \alpha)} D^n \int_a^t g(s)(t - s)^{n-\alpha-1}ds, \tag{4} \]

when \( t > a \) and

\[ \left( \frac{L}{b} D^\alpha \right) g(t) = \frac{1}{\Gamma(n - \alpha)} D^n \int_t^b g(s)(t - s)^{n-\alpha-1}ds, \tag{5} \]

when \( t < b \) that resemble the Cauchy formula for integration given by

\[ (D^{-n} f)(t) = \int_a^t \frac{f(s)(t - s)^{\alpha-1}ds}{\Gamma(\alpha)}, \]

where \( t < b, n = \lfloor \alpha \rfloor \). From the identity

\[ D^n f(t) = \lim_{h \to 0} h^{-n} \sum_{k=0}^{n} (-1)^k \binom{n}{k} f(t - kh), \]

we have the Grunwald-Letnikov definition for \( \alpha < 0 \):
and it can be shown that if \( f \in C[a,t] \),

\[
\lim_{h \to 0} \frac{h^{-\alpha} \sum_{k=0}^{n} (-1)^k \binom{\alpha}{k} f(t-kh)}{n} = \frac{1}{\Gamma(\alpha)} \int_{a}^{t} (t-s)^{\alpha-1} f(s) ds,
\]

which connects the Grunwald-Letnikov and Riemann-Liouville approaches. For \( \alpha < 0 \) it is then enough to replace \( \binom{\alpha}{k} \) with \( (-1)^{-k} \binom{\alpha}{k} \) in the classical limit and proceed. The Riemann-Liouville fractional derivative is problematic in that

\[
\lim_{t \to a} aD_t^{-\alpha} f(t) = b_n,
\]

where \( b_n \) for \( n = 0, 1, 2, ... \) are prescribed constants. Consider the \( \alpha \)-th Caputo fractional derivative of \( f(t) \) defined by

\[
C_a D_t^\alpha f(t) = \frac{1}{\Gamma(\alpha-n)} \int_{a}^{t} \frac{f^{(n)}(s)ds}{(t-s)^{\alpha+1-n}} \text{, } (n-1 < \alpha < n).
\]

Integrating by parts, one can see that for \( f \in C^{n+1}([a,T]) \), \( 0 \leq n-1 < \alpha < n \),

\[
C_a D_t^\alpha f(t) \to D^n f(t)
\]
as \( \alpha \to n \). The Caputo derivative unfortunately does not reduce to the classical derivative for \( n-1 < \alpha \leq n \) since its behavior depends on the terminal point, \( a \) as follows

\[
C_a D_t^k f(t) \to \int_{a}^{t} \frac{f^{(n)}(s)ds}{\Gamma(1)} = D^{n-1} f(t) - D^{n-1} f(a).
\]

The Caputo and Riemann-Liouville derivatives do not satisfy all of the expected classical properties. For instance, for \( c \in \mathbb{R} \), \( n-1 < \alpha \leq n \),

\[
C_a D_t^\alpha c = 0 \text{ while } aD_t^\alpha c = \frac{ct^{-\alpha}}{\Gamma(\alpha-n)}.
\]

Fortunately, they do satisfy the semigroup property

\[
C_a D_t^\alpha (C_a D_t^\beta f(t)) = C_a D_t^{\alpha+\beta} f(t), \ \beta = 0, 1, 2, ...
\]

and the expected result

\[
D^{\alpha} (t-c)^{\gamma} = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma-\alpha+1)} (t-c)^{\gamma-\alpha}.
\]

The equivalence of the Riemann-Liouville, Caputo and Grunwald-Letnikov fractional derivatives allows us to verify this as follows:
\[ GL \alpha_t(t) = \frac{1}{\Gamma(-\alpha)} \int_t^0 (t-s)^{-\alpha-1}(s-c)^\gamma ds. \]

Substituting \( u = \frac{t-c}{s-c} \), we have:

\[
\int_0^1 \frac{(t-c)(1-u)^{-\alpha-1}(t-c)u^\gamma(t-c)}{\Gamma(-\alpha)} du = \frac{(t-c)^{\gamma-\alpha}}{\Gamma(-\alpha)} \int_0^1 (1-u)^{-\alpha-1}u^\gamma du = (t-c)^{\gamma-\alpha}\left(\frac{\beta(\gamma+1,-\alpha)}{\Gamma(-\alpha)}\right) = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma-\alpha+1)}(t-c)^{\gamma-\alpha}.
\]

We now shift our attention to fractional derivatives defined via Laplace transforms. Suppose that \( f^{(k)}(0) = 0 \) for \( k = 0, 1, 2, ..., n-1 \). A Laplace-transform based fractional derivative can be defined naturally by

\[ LT \alpha \space f = \mathcal{L}^{-1}(s^\alpha F(s)). \] (10)

In the case that \( f = (t-c)^\gamma \), we have

\[ LT \alpha \space (t-c)^\gamma = \mathcal{L}^{-1}(s^\alpha \mathcal{L}((t-c)^\gamma)) = \Gamma(\gamma+1)\mathcal{L}^{-1}(e^{-cs}s^\alpha-\gamma-1) = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma-\alpha+1)}\mathcal{L}^{-1}(e^{-cs}\mathcal{L}(t^{\gamma-\alpha})) = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma-\alpha+1)}(t-c)^{\gamma-\alpha}H(t-c), \]

where \( H(t) \) is the usual Heaviside function. For simplicity, let \( f(t) = |t-c| \), then

\[ LT \alpha \space f(t) = \mathcal{L}^{-1}(s^\alpha \mathcal{L}((c-t) + 2(t-c)H(t-c))), \]

since

\[ \mathcal{L}^{-1}(s^\alpha \mathcal{L}((c-t) + 2(t-c)H(t-c))) = \mathcal{L}^{-1}(cs^{-\alpha-1} + s^{-\alpha-2}(2e^{-cs} - 1)). \]

When \( \alpha = 1 \), this reduces to \( 2H(t-c) - 1 \), which agrees with the usual distributional derivative \( \frac{d}{dt} \). When \( \alpha = 2 \),

\[ LT \alpha \space |t-c| = (cD^s - 1)\delta(t) + 2H(t-c), \]

otherwise when \( \alpha < 1 \),

\[ LT \alpha \space |t-c| = \frac{c^{\alpha-2}}{\Gamma(\alpha-1)} - \frac{c^{\alpha-3}(2H(t-3) - 1)}{\Gamma(\alpha-2)}. \]
For other values of $\alpha$, the inverse Laplace transform does not exist, since by a standard choice of contour as in Arfken [34], for $p > 0$,

$$\frac{1}{2\pi i} \int_{\gamma-i\sqrt{R^2-\gamma^2}}^{\gamma+i\sqrt{R^2-\gamma^2}} s^p e^{st} ds = \frac{1}{2\pi i} \int_{\frac{\pi}{2}-\sin^{-1}\left(\frac{\gamma}{R}\right)}^{\frac{\pi}{2}+\sin^{-1}\left(\frac{\gamma}{R}\right)} (Re^{i\theta})^p e^{Re^{i\theta}} iRe^{i\theta} d\theta. \tag{11}$$

After integration by parts and a standard application of the Euler’s reflection formula, the above integral results in

$$\int_{-\epsilon}^{0} (-se^{-i\pi})^p e^{st} \frac{ds}{2\pi i} - \frac{1}{2\pi i} \int_{-\epsilon}^{0} (-se^{i\pi})^p e^{st} \frac{ds}{2\pi i} + \int_{-\pi}^{\pi} (Re^{i\theta})^p iRe^{i\theta} \frac{ds}{2\pi i} = \frac{\sin p\pi}{\pi} \int_{t}^{\infty} e^{-st} s^p ds + \frac{e^{p+1} \sin \pi p}{\pi(p+1)}$$

$$= \frac{t^{-p-1}}{\Gamma(-p)},$$

as $R \to \infty$, $\epsilon \to 0$. In the case of the Riemann-Liouville fractional derivative, we obtain

$$\mathcal{L}(0D_0^\alpha f(t)) = s^\alpha \mathcal{L} f(t) - \sum_{k=0}^{n-1} s^k (D^{n-k-1} f(t))_{t=0}$$

$$= s^\alpha F(s) - \sum_{k=0}^{n-1} D^{n-1-k} (g_{n-\alpha} \ast f)_{t=0},$$

where $n = \lceil \alpha \rceil$. Thus, the Laplace transform readily generalizes the left Riemann-Liouville fractional derivative with a lower limit $a = 0$, applying (8). When $\alpha = \pm 1$,

$$\mathcal{L}(D^n f(x)) = \mathcal{L}^{-1}(s^n \mathcal{L}(f)),$$

reduces to

$$s\mathcal{L}\left(\int_0^t f(x)dx\right) = \mathcal{L}(f(t)).$$

Now let

$$g_c(t) = \frac{tc^{-1}H(t)}{\Gamma(c)},$$

then

$$D^\alpha_c f(t) = \mathcal{L}^{-1}(s^\alpha \mathcal{L} f(t)) = \mathcal{L}^{-1}(s^n \mathcal{L}(g_{n-\alpha} \ast f)(t)) = \mathcal{L}^{-1}(\mathcal{L}(g_{n-\alpha})(sF(s))) = g_{n-\alpha}(t) \ast Df(t), \tag{12}$$

which agrees with the left Riemann-Liouville derivative for $a = 0$. We can use the work above to define more general fractional derivatives in terms of a convolution kernel. For a function $\Phi(s, \alpha)$, consider the operator $\phi(s, \alpha)$, such that

$$\mathcal{L}(0RLD_0^\alpha f(t)) = \Phi(s, \alpha)\mathcal{L}(f(t))(s), \tag{13}$$

where $\Phi(s, \pm 1) = s^\pm 1$ and $\Phi(s, 0) = 1$. By writing $\Phi(s, \alpha) = sk(s, \alpha)$ and proceeding as in (12), we are left with

$$\int_0^t K(t - x, \alpha)x^\alpha Df(x)dx, \tag{14}$$
where $K(t, \alpha) = \mathcal{L}^{-1}k(s, \alpha)$. Caputo and Fabrizio [22] propose a Caputo-type fractional derivative with an exponential kernel, considering applications to constitutive equations for dissipation [10]. This derivative is studied in distributional settings in Atanackovic [26], where this operator is shown to obey a viscoelastic consistency result. The Caputo-Fabrizio fractional derivative is criticized in Tarasov [11] and Origuieira et al. [12] for its non-locality. In Ortiguieira et al. [30], criteria for suitable fractional derivative candidates are considered including reduction to a classical derivative, backwards compatibility, a neutral element and the Leibniz rule. The interaction between the Laplace transform and Mittag-Leffler functions can be found in Gorenflo and Mainardi [9]. Other generalizations can be found in Katugampola [27, 28].

In (22), the kernel $K(t, \alpha)$ plays the role of $g_{n-\alpha}$ in (12). A more detailed approach can be found in Oliveira [23] working from the theory of distributions, revealing a deep relationship between the Riemann-Liouville and Caputo fractional derivative definitions. Relaxation processes describe the response of a physical system to external perturbations. Let $f(t)$ denote the response function, which can be assumed to decay monotonically. The Laplace transform allows one to study processes which model an exponential relaxation model. If the relaxation function exponentially decays with the relaxation time, then it will satisfy the differential equation $f'(t) = c_1 f(t)$, where $c_1$ depends on the relaxation time. One can study approximate relaxation models, satisfying the fractional differential equation

$$D_\alpha^\nu f(t) = C(c_1, \alpha) f(t),$$

where $0 < \alpha < 1$, $C(c_1, 1) = c_1$ and $D_\alpha^\nu f(t)$ denotes a general fractional derivative. From (8) and the assumption of monotonicity, Caputo-type fractional derivatives as in (13) are suitable models. Since the Laplace transform of a Caputo-type fractional derivative depends on the associated convolution kernel, (15) can be readily solved. However, these derivatives can be quite sensitive to initial conditions, as seen in the case of the FC and AB generalizations proposed in Kochubei [39], and described in de Oliveira et al [23]. Relaxation processes are important in several areas of physics, including polymer glasses [34], fine-particle systems [35] and magnetic resonance imaging [36].

Fractional derivatives of this type have also been used in backward propagation (BP) algorithms in [37]. BP algorithms require and exhaustive use of the chain rule via the di Bruno formula [20], which can be quite computationally expensive. As in (3), the Laplace transform transforms differentiation in the time domain into multiplication in the frequency domain, which can serve to remedy this situation. In recent literature, fractional derivatives have been used to improve these algorithms by using a fractional gradient descent method over the classical method [24, 25] and shown to to converge more quickly [38]. Integer order methods are also problematic for gradient descent when applied to regularization problems. In particular, for $\ell_1$ regularization, integer order methods do not produce a continuous gradient, worsening the performance. By defining a fractional derivative using the Laplace transform as in (10), the resulting descent method can provide greater compatibility with the $\ell_1$ regularization term. In the next section, we define a new fractional derivative via the Laplace transform in the next section.

2 The $\ell_1$ derivative

Suppose that $\sigma > 0$ and $f$ is a real-valued function such that $\{f(t)e^{-\sigma t}\} \in \mathcal{S}'$.

**Definition 2.1.** Define the $\ell_1$ derivative by

$$\ell_1(Df)(t) = \lim_{\epsilon \to 0} \mathcal{L}^{-1}(s\mathcal{L}'(|f(t - \epsilon)| - |f(\epsilon)|),$$

and we say that $f$ is $\ell_1$-differentiable if the above limit exists.

**Theorem 2.2.**

(a) Let $f_k$ be $\ell_1$-differentiable for $k = 0, 1, 2, ..., n$, then

$$\ell_1 D \left( \sum_{k=0}^{n} c_k f_k \right)(t) = \sum_{k=0}^{n} c_k \left( \ell_1 D f_k \right)(t).$$
(b) Suppose that \( f \) is \( \ell_1 \)-differentiable, then

\[
\ell_1 D^{(\ell_1 Df)}(t) = \lim_{\epsilon \to 0} \mathcal{L}^{-1}(s^2 \mathcal{L}(|f(t) - \epsilon| - |f(\epsilon)|)).
\]  

\textbf{Proof.} (a) is immediate, since

\[
\ell_1 D\left(\sum_{k=0}^{n} c_k f_k\right)(t) = \lim_{\epsilon \to 0} \mathcal{L}^{-1}\left(s \mathcal{L}\left(\sum_{k=0}^{n} |c_k f_k(t) - \epsilon| - |c_k f_k(\epsilon)|\right)\right)
\]

\[
= \lim_{\epsilon \to 0} \mathcal{L}^{-1}\left(s \mathcal{L}\left(|f_k(t) - \epsilon| - |f_k(\epsilon)|\right)\right)
\]

\[
= \sum_{k=0}^{n} c_k \lim_{\epsilon \to 0} \mathcal{L}^{-1}\left(\mathcal{L}(|f_k(t) - \epsilon|) - |f_k(\epsilon)|\right)
\]

\[
= \sum_{k=0}^{n} c_k \left(\ell_1 D(f_k)\right).
\]

To show (b),

\[
\quad = \ell_1 D^{(\ell_1 Df)}(t) = \lim_{\epsilon \to 0} \mathcal{L}^{-1}(s \mathcal{L}(|(\ell_1 Df)(t) - \epsilon|) - |(\ell_1 Df)(\epsilon)|)
\]

\[
= \lim_{\epsilon \to 0} \mathcal{L}^{-1}(s \mathcal{L}(|(\ell_1 Df)(t) - \epsilon|)) - |(\ell_1 Df)(\epsilon)|
\]

\[
\lim_{\epsilon_1 \to 0} \mathcal{L}^{-1}(s \mathcal{L}\left(\lim_{\epsilon_2 \to 0} \mathcal{L}^{-1}(s \mathcal{L}(|f(t) - \epsilon_1 - \epsilon_2|)) - |f(\epsilon_2 - \epsilon_1)|\right) - \lim_{\epsilon_3 \to 0} \mathcal{L}^{-1}(s \mathcal{L}(|f(\epsilon_1 - \epsilon_3)|) - |f(\epsilon_3)|)
\]

\[
= \lim_{\epsilon_1 \to 0} \mathcal{L}^{-1}(s \mathcal{L}\left(\lim_{\epsilon_2 \to 0} \mathcal{L}^{-1}(s \mathcal{L}(|f(t) - \epsilon_1 - \epsilon_2|)) - |f(-\epsilon_1 - \epsilon_2)|\right) - \mathcal{L}^{-1}(s \mathcal{L}(|f(\epsilon_1)|) - |f(0)|)
\]

\[
= \lim_{\epsilon_1 \to 0} \mathcal{L}^{-1}(s \mathcal{L}\left(\lim_{\epsilon_2 \to 0} \mathcal{L}^{-1}(s \mathcal{L}(|f(t) - \epsilon_1|)) - |f(\epsilon_1)|\right) - \mathcal{L}^{-1}(s \mathcal{L}(|f(\epsilon_1)|) - |f(0)|)
\]

\[
= \lim_{\epsilon_1 \to 0} \mathcal{L}^{-1}(s^2 \mathcal{L}(|f(t) - \epsilon_1|)) - |f(\epsilon_1)| - \mathcal{L}^{-1}(s \mathcal{L}(|f(\epsilon_1)|) - |f(0)|)
\]

\[
= \lim_{\epsilon \to 0} \mathcal{L}^{-1}(s^2 \mathcal{L}(|f(t) - \epsilon|) - |f(\epsilon)|).
\]

The above steps steps follow from Dominated Convergence, since

\[
|F(t)e^{-st}| \leq |F(t)|,
\]

and

\[
\int_0^\infty |F(t - \epsilon) - F(t)|dt \to 0,
\]

which yields

\[
\lim_{\epsilon \to 0} \mathcal{L}(|F(t) - \epsilon e^{-st}|) = \mathcal{L}(|F(t)e^{-st}|).
\]
Theorem 2.3. Suppose that $f$ is real-valued, $\Phi : \mathbb{R} \to \mathbb{R}$ is concave and $\ell_1$-differentiable with $\Phi(f(0)) = 0$. Then
\[ \ell_1(D)(\Phi(f)) \geq \mathcal{L}^{-1}|\Phi(\mathcal{L}(f))|. \] (19)

Proof. 
Jensen’s inequality applied to $\Phi$ yields
\[ \Phi(F(s - \epsilon)) \geq e^{-\epsilon} s^{-1} \int_0^\infty se^{-st}\Phi(f(t - \epsilon))dt, \]
since
\[ \int_0^\infty se^{-st}dt = 1. \]
The result follows from the definition of the $\ell_1$ derivative
\[ \ell_1(D)(\Phi(f)) = \lim_{\epsilon \to 0}\mathcal{L}^{-1}\left(s\mathcal{L}(|\Phi(f(t) - \epsilon)|) - |\Phi(f(\epsilon))|\right). \]

Theorem 2.4. Let $m, n \in \mathbb{R}$. We have the following results:

(a) Suppose that $t > 0$. If $P(t)$ is a polynomial of degree $n$, then $\ell_1(D)(P(t))$ is a polynomial of degree $(n - 1)$.
(b) Let $C \in \mathbb{R}$. Then
\[ \ell_1(D)(C) = |C|(\delta(t) - \delta'(t)). \] (20)
(c) Let $m \in \mathbb{R}$. Then
\[ \ell_1(D)(e^{mt}) = \delta(t) + me^{mt} - 1. \] (21)
(d) If $f$ is periodic with period $P > 0$ and $f(t) \neq 0$, then
\[ \ell_1(D)(f(t)) = \frac{f(t)f(t)'}{|f(t)|} - f(0). \] (22)

Proof. Let $k > 0$. Property (a) follows from the computation
\[ \ell_1(D)(t^{2k}) = \lim_{\epsilon \to 0}\mathcal{L}^{-1}(s\mathcal{L}((t - \epsilon)^{2k}) - \epsilon^{2k}) \]
\[ = \lim_{\epsilon \to 0}\mathcal{L}^{-1}(e^{\epsilon s}\Gamma(2k + 1) - \epsilon^{2k}) \]
\[ = \lim_{\epsilon \to 0}\mathcal{L}^{-1}(e^{-\epsilon s}s^{-2k}\Gamma(2k + 1) - \epsilon^{2k}) \]
\[ = t^{2p-1} \]
To show (b),
\[ \ell_1(D)(C) = \lim_{\epsilon \to 0}\mathcal{L}^{-1}(s\mathcal{L}(|C|) - |C|) \]
\[ = \mathcal{L}^{-1}(|C| - |C|s) \]
\[ = |C|(δ(t) - δ'(t)), \]

since \( \mathcal{L}^{-1}(s^n) = D^n \delta(t) \) in the distributional sense.

Property (c) follows from the identity \( \mathcal{L}(f(at)) = \frac{F(s/a)}{a} \).

\[ \ell_1(D)(e^{mt}) = \lim_{\epsilon \to 0} \mathcal{L}^{-1}(s \mathcal{L}(e^{mt}) - e^s) \]

\[ = \lim_{\epsilon \to 0} \mathcal{L}^{-1}(s \mathcal{L}(e^{mt}) - e^s) \]

\[ = \lim_{\epsilon \to 0} \mathcal{L}^{-1}(s \mathcal{L}(e^{mt}) - e^s) \]

\[ = \delta(t) + me^{mt} - 1. \]

\[ \text{Theorem 2.5.} \quad \text{Suppose that } f \text{ is non-vanishing and differentiable.} \]

(a) If \( f \) is periodic with period \( P > 0 \), then

\[ \ell_1(D)(f(t)) = \frac{f(t)f(t)'}{|f(t)|} - f(0). \quad (23) \]

(b) If \( a > 0 \) and \( |f| \) has Laplace transform \( F^*(s) \), then

\[ \ell_1(D)(f(at)) = af'(at)|f(at)|^{-1} - f(0). \]

\[ \textbf{Proof.} \text{ To show (a), applying the identity} \]

\[ \mathcal{L}(|f(t-e)|) = \frac{e^s \mathcal{L}(f)}{1 - e^{-sP}}, \]

we obtain

\[ \ell_1(D)(f(t)) = \lim_{\epsilon \to 0} \mathcal{L}^{-1}\left( s \mathcal{L}(e^{mt}) - e^s \mathcal{L}(f) - |f(\epsilon)| \right). \]

To show (b), we have

\[ \ell_1(D)(f(at)) = \lim_{\epsilon \to 0} \mathcal{L}^{-1}\left( s \mathcal{L}(f(a(t-e))) - |f(\epsilon)| \right) \]

\[ = \lim_{\epsilon \to 0} \mathcal{L}^{-1}\left( sa^{-1}e^{as} \mathcal{L}(|f(at)|) - |f(\epsilon)| \right) \]

\[ = \lim_{\epsilon \to 0} \mathcal{L}^{-1}\left( sa^{-1}e^{as} \mathcal{L}(F^*(s/a)) - |f(\epsilon)| \right) \]

\[ = af'(at)|f(at)|^{-1} - f(0). \]

Now, suppose that \( n \) is a positive integer such that \( 0 < \alpha < n \), then we have

\[ \ell_1(D)(f(at)) = \lim_{\epsilon \to 0} \mathcal{L}^{-1}\left( s \mathcal{L}(|f((t-e))|) - |f(\epsilon)| \right) \]
\[
= \lim_{\epsilon \to 0} \mathcal{L}^{-1}\left(s(\mathcal{L}(|f(t-\epsilon)|) - |f(\epsilon)|s^{-1})\right)
\]
\[
= \lim_{\epsilon \to 0} \mathcal{L}^{-1}\left(s^{\alpha} s^{-\alpha}(\mathcal{L}(|f(t-\epsilon)|) - |f(\epsilon)|s^{-1})\right)
\]
\[
= \lim_{\epsilon \to 0} \mathcal{L}^{-1}\left(\mathcal{L}\left(\frac{t^{n-\alpha-1}}{\Gamma(n-\alpha)}\right)\mathcal{L}(D^n|f(t-\epsilon)| - |f(\epsilon)|)\right).
\]

We now consider generalizations of the work above. One consideration is to adjust the rate of convergence in (16). Let \( f \) be \( \ell_1 \)-differentiable. Define the modified \( \ell_1 \)-fractional derivative of order \( \alpha \in \mathbb{R} \) by

\[
\ell_1 D_1^\alpha f = \mathcal{L}^{-1}(s^\alpha|f(t-\epsilon)| - |t\epsilon|).
\] (24)

In the case that \( f(t) = |t| \), we obtain

\[
\mathcal{L}^{-1}(s^\alpha \cdot (1 - \epsilon + 2e^{-\epsilon s})s^{\alpha-2}).
\]

When \( \alpha = 1 \), this closely behaves like the \( \ell_1 \) derivative. Rewriting the modified fractional derivative, we have

\[
\ell_1 D_1^\alpha f = \lim_{\epsilon \to 0} \mathcal{L}^{-1}s^\alpha \cdot (\mathcal{L}(|f(t-\epsilon)|) - f(\epsilon)) = \ell_1 D_1^\alpha f
\]

\[
= \lim_{\epsilon \to 0^+} \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} e^{st}(s^\alpha \int_0^\infty |f(t-\epsilon)|e^{-st}) - f(\epsilon) dt.
\]

A natural generalization of the derivative above can be seen by replacing the exponential kernel \( e^{-st} \) with the two-parameter Mittag-Leffler function defined by

\[
E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)},
\] (25)

where \( \alpha, \beta \in \mathbb{C} \).

3 Conclusion

In this paper, we have discussed a new derivative defined by the Laplace transform. This representation interacts well with periodic functions and satisfies some of the expected classical properties. This derivative can be readily generalized to one of fractional order. For further research, we seek a fractional operator that is nonlocal yet manages to smoothen the \( \ell_1 \) loss function in gradient descent methods. The search for a fractional gradient with better smoothing properties compatible with \( \ell_1 \) regularization is related to the choice of kernel in (14).

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