On the integrable gravity coupled to fermions

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Abstract

In the present letter we indicate an extension of the pure gravity inverse scattering integration technique to the case when fermions (introduced on the base of supersymmetry) are present. In this way the integrability technique for simple ($N = 1$) supergravity in two space-time dimensions coupled to the matter fields taking values in the Lie algebra of $E_{8(+8)}$ group is developed. This theory contains matter living only in one Weyl representation of $SO(16)$ and represents the reduction to two dimensions of the three-dimensional simple supergravity constructed in [1].

Our spectral linear problem use superspace and covers the complete set of principal bosonic and fermionic equations of motion. This linear system, as in pure gravity, contains only the first order poles with respect to the spectral parameter. The procedure of constructing the exact super-solitonic solutions is outlined.

1 Introduction

The three-dimensional maximal ($N = 16$) supergravity coupled to the matter fields taking values in the Lie algebra of $E_{8(+8)}$ group has been constructed by N. Marcus and J. Schwarz in [2]. The theory has 128 space-time physical scalars parametrizing the coset $E_{8(+8)} / SO(16)$ and their 128 space-time spinorial super-partners. From the point of view of the internal space [where $SO(16)$ symmetry acts] both sets of fields are spinors in such a way that those which are space-time scalars belong to the "positive" chirality representation of $SO(16)$ and their super-partners (which are space-time spinors) are in the inequivalent "negative" chirality representation. This arrangement excludes a possibility to have any truncation to a theory with all fields in one Weyl representation of $SO(16)$. However, H. Nishino and S. Rajpoot [1] have been pointed out that in three-dimensional space-time such common chirality case also exists though only for the simple ($N = 1$) supergravity. This possibility corresponds to quite a new theory which cannot be obtained as particular case from the Marcus-Schwarz construction.
However, it is easy to check that Lagrangian for the $N = 1$ Nishino-Rajpoot theory (without a cosmological constant) can be obtained from Lagrangian of the $N = 16$ Marcus-Schwarz supergravity by the following formal rules:

a) among 16 Marcus-Schwarz gravitinos $\psi^I_\mu$ ($I = 1, 2, ..., 16$) take only one (for example for $I = 1$) to be non-zero,

b) among all internal $\Gamma^I$-matrices of $SO(16)$ (which are symmetric and off-diagonal) chose the first one (again for $I = 1$) to have the upper-right block as unit matrix,

c) in all quantities (apart of $\Gamma^I$-matrices) each dotted index replace by undotted.

This means that also the reduction of the three-dimensional Nishino-Rajpoot supergravity to two dimensions can be derived from the reduction of the Marcus-Schwarz theory to two dimensions using the same three rules. Fortunately, the reduction of the Marcus-Schwarz theory to two dimensions is known and this is two-dimensional integrable $N = 16$ supergravity constructed by H. Nicolai [3]. Hence, the equations of motion of the matter fields in the dimensionally reduced $N = 1$ Nishino-Rajpoot supergravity follow from the Nicolai $N = 16$ two-dimensional supergravity after one apply to the last theory the operations a), b), c). This formal procedure being applied to the equations (8) and (11) of the letter [3] gives the following equations of motion for the matter fields (that is for 128 scalars $\varphi^A$ and 128 spinors $\chi^A$) in the two-dimensional version of the Nishino-Rajpoot theory:

\[
\frac{1}{\alpha} \eta^{\alpha \beta} \left[ (\alpha \varphi^A)_\beta + \frac{1}{4} d^I_{\alpha} \Gamma^{IJ} \alpha B \varphi^B \right] = \frac{i}{8} \Gamma^{IJ} \alpha B \bar{\chi}^C \gamma^\alpha \Gamma^{IJ} \chi^D , \quad (1)
\]

\[
- \frac{1}{\sqrt{\alpha}} \gamma^\alpha \left[ (\sqrt{\alpha} \chi^A)_\alpha + \frac{1}{4} d^I_{\alpha} \Gamma^{IJ} \alpha B \sqrt{\alpha} \chi^B \right] = \frac{1}{16} \gamma^\alpha \Gamma^{IJ} \alpha B \bar{\chi}^C \gamma_\alpha \Gamma^{IJ} \chi^D . \quad (2)
\]

All notations used here are explained in the section 3 of our letter (they are the same as in papers [2,3,1]).

1 In letter [3] there is a misprint with the numerical coefficient in front of the right hand side of equation (8) corresponding to our (1). The correct value for this coefficient (namely $i/8$) have been found by Nicolai and Warner later in [4]. The numerical coefficients in front of the right hand side of our equation (2) and in corresponding equation (11) in letter [3] both are correct. The difference (1/16 and 1/24) is due to the fact that in the r.h.s. of our (2) we used the sum $\gamma_\alpha \ldots \gamma^\alpha$... only over $\alpha = 0, 1$ while in [3] this summation was extended up to the addend $\gamma_3 \ldots \gamma^3$, where $\gamma_3 = \gamma^3 = \gamma_0 \gamma_1$. However, the difference manifests itself only in the aforementioned overall factors.

In our letter we don’t use the special designation $D_\alpha$ for covariant (with respect to the local orthogonal rotations of the frame $V$) derivatives. In all our equations such derivatives are written in the explicit forms.
Remarkably enough the equations (1)-(2) obtained by the above-mentioned formal rules coincide exactly with those derived in exact mathematical way from our superspace linear spectral problem as its self-consistency conditions (see section 3). Consequently, these equations are integrable and exact solitonic type solutions for them can be constructed.

A question might arise why one need to go to superspace to search a new linear spectral problem and cannot extract it from that one already presented in [3] with the aid of the same rules a), b), c)? Such a way, however, will give a Lax pair which will not be able to cover the fermionic equations (2). The reason is that the linear system proposed in [3] is not complete in the sense that fermionic equation (11) of the letter [3] [that one which generates our (2)] do not follow from this linear system and must be added by hands. This circumstance is evident from the fact that the Lax pair of [3] contains only quadratic products of anticommuting fermions and corresponding self-consistency conditions cannot produce any equation of motion containing terms of the odd powers of such fermions. [Besides there is one undesirable point (although not of any principal significance) in relation to the Lax pair constructed in [3] which lead to some technical complications. This is appearance the poles of the second order with respect to the spectral parameter while the corresponding spectral problem in pure gravity has only simple poles]. The approach proposed in the present letter dismiss these nuisances.

2 The multidimensional version of the integrable gravity

It is known [5,6] that in any space-time of dimension $n + 2$ with interval

$$ ds^2 = g_{\alpha\beta} (x^0, x^1) \, dx^\alpha dx^\beta + g_{ik} (x^0, x^1) \, dy^i dy^k $$

(3)

(where $\alpha, \beta, \gamma, ... = 0, 1$ and $i, k, l, ... = 1, 2, ...n$) the Einstein equations are integrable since the equations for the metric coefficients $g_{ik} (x^0, x^1)$ follows from the Lax pair as its self-consistency conditions and the metric components $g_{\alpha\beta} (x^0, x^1)$ can be found by quadratures in terms of the known $g_{ik} (x^0, x^1)$. Without loss of generality the 2-dimensional block $g_{\alpha\beta} (x^0, x^1)$ can be chosen in conformal flat form:

$$ g_{\alpha\beta} = \lambda^2 \eta_{\alpha\beta} , \quad \eta_{\alpha\beta} = \text{diag} (\eta_{00}, \eta_{11}) = \text{diag} (1, -1) . $$

(4)
It is convenient to introduce matrix $G$ (with entries $G_{ik}$) and represent the metric matrix $g$ (with entries $g_{ik}$) as

$$g = \alpha^{2/n} G, \quad \det G = 1, \quad \det g = \alpha^2. \quad (5)$$

Then the components $R_{\alpha i}$ of the Ricci tensor vanish identically and equations $R_{\alpha \beta} = 0$ and $R_{ik} = 0$ can be written in matrix form using the light-like variables $\zeta, \eta$:

$$x^0 = \eta + \zeta, \quad x^1 = \eta - \zeta. \quad (6)$$

Equations $R_{ik} = 0$ are:

$$f,\zeta = 0, \quad (7)$$

$$(\alpha G^{-1} G_{,\zeta})_{,\eta} + (\alpha G^{-1} G_{,\eta})_{,\zeta} = 0. \quad (8)$$

Equations $R_{\alpha \beta} = 0$ are equivalent to the system:

$$\frac{f,\zeta}{f} = \frac{\alpha}{4\alpha,\zeta} \text{Tr} \left[ (G^{-1} G_{,\zeta})^2 \right], \quad (9)$$

$$\frac{f,\eta}{f} = \frac{\alpha}{4\alpha,\eta} \text{Tr} \left[ (G^{-1} G_{,\eta})^2 \right], \quad (10)$$

where

$$f = \frac{\alpha^{(n-1)/n} \lambda^2}{\alpha,\zeta,\alpha,\eta}. \quad (11)$$

If equations (7) and (8) are solved then $f$ can be found by quadratures from equations (9) and (10) [the self-consistency conditions of which are satisfied automatically if $\alpha$ and $G$ are the solutions of the equations (7) and (8)].

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2 All multidimensional matrices (apart from the two-dimensional Dirac gamma-matrices) we designate by the bold letters. Tilde at the top of a matrix means transposition. The functions which depend also on the spectral parameter we designate by the letters with the hat on the top. The simple partial derivatives are denoted by comma.

3 In fact the system $R_{\alpha \beta} = 0$ contains one more equation additional to the (9)-(10) which is of second order for $f$ but it is the direct consequence of (7)-(10) and we can forget about it.

4 The following important point should be stressed here. From the relations (7) follows that solution of equation (8) for the matrix $G$ should satisfy the restriction $\det G = 1$. However, the application of the inverse scattering integration procedure to the equation (8), if taking this restriction from the outset, technically are a little bit awkward. More convenient approach is to solve (8) [under given solution of (7) for $\alpha$] first ignoring any additional restriction for the $\det G$ but at the end of calculation make the simple rescaling of the solution in order to get the necessary condition of the unit determinant. The trick is as follows. If we obtained the solution of the equation (8) with $\det G \neq 1$ we can pass to the new "physical" matrix $G_{(ph)}$ by the transformation $G_{(ph)} = (\det G)^{-1/n} G$. It is simple task to prove that the matrix $G_{(ph)}$ is also a solution of the same equation (8) and simultaneously satisfy the condition $\det G_{(ph)} = 1$. 


The spectral linear problem associated with the main equation (8) we used in [5] contains the differentiation also with respect to the spectral parameter but this is an inessential technical point. Our original Lax representation for the equation (8) can be written also in the following equivalent form:

$$
\hat{G}^{-1}\hat{G}_\zeta = \frac{\alpha}{\alpha - s} G^{-1}G_\zeta, \quad \hat{G}^{-1}\hat{G}_\eta = \frac{\alpha}{\alpha + s} G^{-1}G_\eta,
$$

(12)

where $\hat{G}(\zeta, \eta, s)$ depends on the complex spectral parameter $s$ which depends on the coordinates $\zeta, \eta$ in accordance with differential equations:

$$
\frac{s_\zeta}{s} = \frac{2\alpha_\zeta}{\alpha - s}, \quad \frac{s_\eta}{s} = \frac{2\alpha_\eta}{\alpha + s}.
$$

(13)

The self-consistency requirement for the last equations is satisfied due to the condition (7). The solution of the equations (13) contains one arbitrary complex constant $w$ then parameter $s = s(\zeta, \eta, w)$ has one arbitrary degree of freedom independent of those due to the change of coordinates. This means that in the integrability conditions for the pair (12) all terms containing the different powers of $s$ must vanish separately. The matrix $G$ in the right hand side of (12) is a function on the two coordinates $\zeta$ and $\eta$ only, that is it is treated as unknown ”potential” independent on the parameter $s$. The function $\alpha(\zeta, \eta)$ which is a solution of the wave equation (7) should be considered in (12) as some given external field. The equation of interest (8) should result from the linear spectral system (12) as its self-consistency (integrability) conditions and it is easy to check that this is indeed the case.

For any regular at the point $s = 0$ solution $\hat{G}(\zeta, \eta, s)$ of the Lax pair (12) the solution of the equation (8) for matrix $G(\zeta, \eta)$ follows automatically from the relation:

$$
G(\zeta, \eta) = [\hat{G}(\zeta, \eta, s)]_{s=0}.
$$

(14)

In case of solitonic fields the procedure of integration of the spectral linear problem (12) for matrix $\hat{G}(\zeta, \eta, s)$ consists of the following steps. First we need to have some background solutions $\alpha(\zeta, \eta)$ and $G_0(\zeta, \eta)$ of the gravitational equations (7)-(8) and then to find from equations (12) the corresponding background spectral matrix $\hat{G}_0(\zeta, \eta, s)$. After that we ”dress” $\hat{G}_0(\zeta, \eta, s)$ by the simplest meromorphic (having only isolated first-order poles with respect to the spectral parameter $s$) matrix $\hat{K}(\zeta, \eta, s)$, that is we represent $\hat{G}$ in the form $\hat{G}(\zeta, \eta, s) = \hat{G}_0(\zeta, \eta, s) \hat{K}(\zeta, \eta, s)$. Substituting this form into equations (12) we can find the matrix $\hat{K}(\zeta, \eta, s)$ in terms of the known background solution $\hat{G}_0(\zeta, \eta, s)$ and given wave function $\alpha(\zeta, \eta)$ by pure algebraic procedure (namely this is the principal advantage of the method). The number of poles with respect to the spectral parameter $s$ in matrix $\hat{K}(\zeta, \eta, s)$ is the number of solitons we add to the background. Then the final solution of interest for $G(\zeta, \eta)$ follows from the relation (14).
3 Generalization to the case when fermions are present

Generalization of the scheme described above to the case when a number of fermionic fields are present can be done by introducing the superspace with coordinates $x^0, x^1, \theta^1, \theta^2$ where $\theta^1$ and $\theta^2$ are odd (anticommuting) variables. Now the spectral matrix $\hat{G} (\zeta, \eta, s)$ which appeared in the equations (12) should be replaced by a supermatrix $\hat{\Psi} (\zeta, \eta, \theta^1, \theta^2, s)$ with even (commuting) entries and instead of the simple derivatives $\partial_{\zeta}$ and $\partial_{\eta}$ in (12) we should use the following odd differential operators:

$$D_{\zeta} = \frac{\partial}{\partial \theta^2} - \theta^2 \frac{\partial}{\partial \zeta},$$

$$D_{\eta} = -\frac{\partial}{\partial \theta^1} + \theta^1 \frac{\partial}{\partial \eta}. $$

(15)

(16)

These operators anticommute:

$$D_{\zeta} D_{\eta} + D_{\eta} D_{\zeta} = 0,$$

(17)

and the superspace generalization of the Lax pair (12) becomes:

$$\hat{\Psi}^{-1} D_{\zeta} \hat{\Psi} = \frac{\alpha}{\alpha - s} \hat{\Psi}^{-1} D_{\eta} \hat{\Psi}, \quad \hat{\Psi}^{-1} D_{\eta} \hat{\Psi} = \frac{\alpha}{\alpha + s} \hat{\Psi}^{-1} D_{\zeta} \hat{\Psi},$$

(18)

where $\alpha(x^0, x^1)$ still is an usual even function which does not depends on the $\theta$-coordinates and satisfies the wave equation (11). The spectral parameter $s(x^0, x^1)$ also is even and does not depends on the $\theta$-coordinates and follows from the differential equations (13) after function $\alpha(x^0, x^1)$ is fixed.

For any regular at the point $s = 0$ solution of equations (18) for $\hat{\Psi}$ the matrix of interest $\Psi$ comes from the relation:

$$\Psi (\zeta, \eta, \theta^1, \theta^2) = [\hat{\Psi} (\zeta, \eta, \theta^1, \theta^2, s)]_{s=0}. $$

(19)

The crucial point is that the self-consistency condition $D_{\zeta} (D_{\eta} \hat{\Psi}) + D_{\eta} (D_{\zeta} \hat{\Psi}) = 0$ of the generalized Lax pair (18) reduces only to one supermatrix equation [superspace analog of (13)]:

$$D_{\zeta} (\alpha \Psi^{-1} D_{\eta} \Psi) - D_{\eta} (\alpha \Psi^{-1} D_{\zeta} \Psi) = 0,$$

(20)

which is equivalent to the set of equations of motion for the bosonic and fermionic component fields which are represented by the coefficients in the expansion of $\Psi (\zeta, \eta, \theta^1, \theta^2)$ with respect to the $\theta$-coordinates. The matrix $\Psi (\zeta, \eta, \theta^1, \theta^2)$ has structure:

$$\Psi = J (1 + \theta^1 \Omega_1 + \theta^2 \Omega_2 + \theta^1 \theta^2 H),$$

(21)
\[\Psi^{-1} = \left[ I - \theta^1 \Omega_1 - \theta^2 \Omega_2 - \theta^1 \theta^2 H - \theta^1 \theta^2 (\Omega_1 \Omega_2 - \Omega_2 \Omega_1) \right] J^{-1}, \] (22)

where \( I \) is the unity, \( J(\zeta, \eta) \) and \( H(\zeta, \eta) \) have even entries and both \( \Omega_1(\zeta, \eta) \) and \( \Omega_2(\zeta, \eta) \) consist of the odd entries. If we are able to find the solution of equations (18) for the spectral matrix \( \hat{\Psi}(\zeta, \eta, \theta^1, \theta^2, s) \) (this is the main problem of the method, however, solvable for the solitonic fields) then the solution of the equation (20) for matrix \( \Psi(\zeta, \eta, \theta^1, \theta^2) \) follows automatically from the relation (19) together with component fields of interest \( J, \Omega_1, \Omega_2, H \). This means that these fields will satisfy automatically the differential equations following from (20) after we substitute into it expansions (21)-(22). These differential equations\(^5\) are:

\[\frac{1}{\alpha} \left( \alpha J^{-1} J, \zeta \right)_\eta + \frac{1}{\alpha} \left( \alpha J^{-1} J, \eta \right)_\zeta = \frac{1}{2} \left( J^{-1} J, \zeta \right) \Omega_1^2 - \Omega_2^2 J^{-1} J, \zeta \right), \] (23)

\[\frac{1}{\alpha} \left( \sqrt{\alpha} \Omega_1 \right), \zeta + \frac{1}{2} \left( J^{-1} J, \zeta \right) \Omega_1 - \Omega_1 J^{-1} J, \zeta \right) \Omega_2^2 - \Omega_2^2 J^{-1} J, \eta \right) = 0, \] (24)

\[\frac{1}{\sqrt{\alpha}} \left( \sqrt{\alpha} \Omega_2 \right), \eta + \frac{1}{2} \left( J^{-1} J, \eta \right) \Omega_2 - \Omega_2 J^{-1} J, \eta \right) \Omega_1^2 - \Omega_1^2 J^{-1} J, \zeta \right) = 0, \] (25)

\[H = \frac{1}{2} \left( \Omega_2 \Omega_1 - \Omega_1 \Omega_2 \right). \] (26)

As follows from the foregoing the equations (23)-(25) are integrable because they are the self-consistency conditions of the super Lax pair (18) but what is much more interesting they coincide exactly with equations of motion (1)-(2) for the matter in the Nishino-Rajpoot supergravity if the fields \( J, \Omega_1, \Omega_2 \) take values in the \( E_{8(8)} \) group. To show this coincidence explicitly it is necessary to return in the system (23)-(25) to the Cartesian coordinates \( x^0, x^1 \) defined in (6) and to use instead of the metric \( J \) the orthonormal frame \( V \):

\[J = V V. \] (27)

Instead of the matrices \( \Omega_1, \Omega_2 \) it is convenient to use their similar images \( \Lambda_1, \Lambda_2 \) with respect to the frame \( V \):

\[\Omega_1 = \tilde{V}^{-1} \Lambda_1 \tilde{V} , \Omega_2 = \tilde{V}^{-1} \Lambda_2 \tilde{V} . \] (28)

\(^5\) Direct derivation of the first integrability condition of the Lax pair (18) gives it as \( \left( \alpha J^{-1} J, \zeta + \alpha \Omega_1^2 \right)_\eta + \left( \alpha J^{-1} J, \eta + \alpha \Omega_2^2 \right)_\zeta = 0 \), however using (24) and (25) it can be transformed to the form (23).
The frame current $V^{-1}V_\alpha$ can be decomposed into antisymmetric and symmetric parts:

$$V^{-1}V_\alpha = Q_\alpha + P_\alpha,$$  \hspace{1cm} (29)

where matrices $Q_\alpha$ are antisymmetric and matrices $P_\alpha$ symmetric. From (27) and (29) we obtain the following useful expression for the metric current $J^{-1}J_\alpha$:

$$J^{-1}J_\alpha = 2\tilde{V}^{-1}P_\alpha \tilde{V}.$$  \hspace{1cm} (30)

Now it is easy to see that equations (24) and (25) have a natural Dirac spinorial structure in two-dimensional Minkowski space-time with metric $\eta_{\alpha\beta}$ defined by (4) and with the following gamma-matrices:

$$\gamma^0 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad \gamma_0 = \gamma^0, \quad \gamma_1 = -\gamma^1.$$  \hspace{1cm} (31)

If we introduce the matrix-generalized two-component Dirac-like spinor $\Phi$

$$\Phi = \begin{pmatrix} \Phi_1 \\ \Phi_2 \end{pmatrix} = \begin{pmatrix} \Lambda_1 + \Lambda_2 \\ \Lambda_1 - \Lambda_2 \end{pmatrix},$$  \hspace{1cm} (32)

in which the components $\Phi_1$ and $\Phi_2$ are internal matrices of arbitrary size (their internal matrix structure has no relation to the two-dimensional Dirac algebra in two-dimensional space-time), then in terms of such spinor and frame current components $Q_\alpha, P_\alpha$ equations (23)-(25) take the form:

$$\eta^{\alpha\beta} \left[ \frac{1}{\alpha} (\alpha P_\alpha)_{,\beta} + Q_\alpha P_\beta - P_\beta Q_\alpha \right] = \frac{1}{8} P_\alpha \left( \bar{\Phi} \gamma^\alpha \Phi \right) - \frac{1}{8} \left( \bar{\Phi} \gamma^\alpha \Phi \right) P_\alpha,$$  \hspace{1cm} (33)

$$\frac{1}{\sqrt{\alpha}} \gamma^\alpha (\sqrt{\alpha} \Phi)_{,\alpha} + Q_\alpha (\gamma^\alpha \Phi) - (\gamma^\alpha \Phi) Q_\alpha = \frac{1}{16} (\gamma_\alpha \Phi) \left( \bar{\Phi} \gamma^\alpha \Phi \right) - \frac{1}{16} \left( \bar{\Phi} \gamma^\alpha \Phi \right) \left( \gamma_\alpha \Phi \right),$$  \hspace{1cm} (34)

where

$$\bar{\Phi} = (\Phi_1, \Phi_2) \gamma^0 = i (\Phi_2, -\Phi_1).$$  \hspace{1cm} (35)

Now we can apply these form of the original system (23)-(25) to the case when the fields $J, \Omega_1, \Omega_2$ parametrize the $E_{8(+8)}$ group. This means that the fields $Q_\alpha, P_\alpha, \Phi$ are in the Lie algebra of $E_{8(+8)}$ that is they can be represented as superpositions (with local coefficients) of the generators of this algebra which generators consist of 120 antisymmetric matrices $X^{IJ}$ ($I, J, K, L = 1, 2, ..., 16$) and 128 symmetric matrices $Y^A$ ($A, B, C, D = 1, 2, ..., 128$) each of the size
The defining relations are:

\[ [X^{IJ}, X^{KL}] = \delta^{IL}X^{JK} + \delta^{JK}X^{IL} - \delta^{IK}X^{JL} - \delta^{JL}X^{IK}, \]  
(36)

\[ [X^{IJ}, Y^A] = -\frac{1}{2}\Gamma^{IJ}_{AB}Y^B, \]  
(37)

\[ [Y^A, Y^B] = \frac{1}{4}\Gamma^{IJ}_{AB}X^{IJ}, \]  
(38)

where the constant matrices \( \Gamma^{IJ} \) (with entries \( \Gamma^{IJ}_{AB} \)) are the generators of transformations of the internal Weyl spinors of "positive" chirality under the \( SO(16) \) rotations (see Appendix). The Lie algebra values of the fields of interest are \( [x, y] = \frac{1}{2}\Gamma^{IJ}_{AB}X^{IJ} \).

The physical degrees of freedom in the fields \( q^{IJ}(x^0, x^1) \) and \( p^A(x^0, x^1) \) are those produced by 128 space-time physical scalars \( \varphi^A(x^0, x^1) \) which parametrize the \( E_{8(\pm8)} \) group values of the orthonormal frame \( V = \exp(a^{IJ}X^{IJ} + \varphi^AY^A) \).

Here the 120 components \( a^{IJ}(x^0, x^1) \) are pure gauge objects which can be chosen in any desirable form by an appropriate orthogonal rotation of the frame \( V \) (for example \( a^{IJ} \) can be eliminated, in which case the frame becomes a symmetric matrix). Then all physical degrees of freedom in the frame current (29) come from the scalar fields \( \varphi^A(x^0, x^1) \). The each odd coefficient \( \chi^A(x^0, x^1) \) (for each fixed value of the index \( A \)) in \( \Phi \) represents the two-component Dirac spinor in the two-dimensional space-time \( x^0, x^1 \). In this way the numbers of the physical degrees of freedom (that is the numbers of the arbitrary initial data) for scalars and spinors are the same (256 for scalars since equations of motion for \( \varphi^A \) are of the second order in time and 256 for spinors \( \chi^A \) because they satisfy Dirac-like equations of the first order in time but have twice more components).

Substituting decomposition (39) into equations (33)-(34) and taking into account the relations:

\[ \chi^A = \begin{pmatrix} \chi^A_1 \\ \chi^A_2 \end{pmatrix}, \quad \bar{\chi}^A = i \left( \chi^A_2, -\chi^A_1 \right), \quad \bar{\Phi} = 4e^{-i\pi/4}\chi^A(x^0, x^1)Y^A, \quad \bar{\chi}^A\gamma^\alpha\chi^B = -\bar{\chi}^B\gamma^\alpha\chi^A, \]  
(40)

together with defining commutations (36)-(38), we obtain exactly the equations (1)-(2). These equations are the basic physical equations of motion for the two-dimensional Nishino-Rajpoot theory. All other fields in this theory (conformal

\[ 248 \times 248 \) (the concrete realization of these matrices see, for example, in [7]).
factor $\lambda^2$ and gravitino) can easily be found in terms of the solutions of the system (11)-(2). To find solutions of these principal equations one should return to the system (22)-(26) and construct some (as simple as possible) background solutions for the fields $J^0(0), \Omega^0_1, \Omega^0_2, H^0$ which parametrize the $E_8(8)$ group. Then from (21) follows the background matrix $\Psi^0(0)(\zeta, \eta, \theta_1, \theta_2)$, using which one can integrate the Lax pair (18) to get the corresponding seed solution for the spectral matrix $\hat{\Psi}^0(0)(\zeta, \eta, \theta_1, \theta_2, s)$. All subsequent steps are closely analogous to the procedure described at the end of the previous section. The only difference is that now some part of the calculations should be carried out following the rules of algebra of anticommuting numbers. After adding to the background a number of solitons we will obtain the final solutions for the $n$-solitonic $E_8(8)$ fields $J(\zeta, \eta), \Omega_1(\zeta, \eta), \Omega_2(\zeta, \eta)$ and we can choose any orthonormal frame $V(\zeta, \eta)$ satisfying the relation (27). This frame will give the matrices $Q_\alpha$ and $P_\alpha$ as is prescribed by the relation (29) and from (39) we will obtain the coefficients $q^{IJ}_\alpha(x^0, x^1)$ and $p^A_\alpha(x^0, x^1)$, namely those appearing in the equations (11)-(2). Also with the help of this frame matrix we will extract from $\Omega_1$ and $\Omega_2$ matrices $\Lambda_1$ and $\Lambda_2$ in accordance with formulas (28) and these matrices will give us the matrix spinor $\Phi$ (32). Then the 128 spinors of interest $\chi^A$ will follow from the third relation (39). The quantities $q^{IJ}_\alpha(x^0, x^1)$, $p^A_\alpha(x^0, x^1), \chi^A(x^0, x^1)$ calculated in this way are exactly those which will satisfy automatically equations (11)-(2).

Finally it is necessary to stress that one should not try to find a local supersymmetry directly in the foregoing integrable ansatz for the fields $\varphi^A$ and $\chi^A$ because it corresponds to completely fixed supersymmetry gauges. It follows from the work [3] and rules a), b), c) that after we chose the gauges corresponding to the conformal flat metric $g_{\alpha\beta}(x^0, x^1)$ and to the special form $\psi_\alpha = \gamma_\alpha \varphi$ for the gravitino some residual supersymmetry still remains in the system. This residual freedom can be used to eliminate the superpartner to the function $\alpha(x^0, x^1)$ together with superpartner to the spectral parameter $s(x^0, x^1)$ in the Lax pair. That’s way we can use (without loss of generality) in the super Lax representation (18) the quantities $\alpha$ and $s$ as the usual even functions.

In the two-dimensional supergravities such special gauges give rise to the same miracle as in pure gravity, that is complete separation of the equations of motion for the matter from gravitational potentials and gravitinos. Namely due to this fact the models considered in [3] and here are integrable. To have the general form for these supergravities one should apply the backward supersymmetric transformations which transformations can be extracted from papers [4] and [8].
4 Appendix

The sixteen $256 \times 256$ gamma-matrices $\Gamma^I$ of $SO(16)$ can be chosen symmetric and block off-diagonal:

\[
\Gamma^I = \begin{pmatrix}
0 & \Gamma^I_{\dot{A}A} \\
\Gamma^I_{\dot{A}A} & 0
\end{pmatrix},
\] (41)

where $\Gamma^I_{\dot{A}A}$ is transposed to $\Gamma^I_{\dot{A}A}$. The Clifford relation $\Gamma^I \Gamma^J + \Gamma^J \Gamma^I = 2\delta^{IJ} I$ takes the form:

\[
\sum_{\dot{A}} (\Gamma^I_{\dot{A}A} \Gamma^J_{\dot{B}A} + \Gamma^J_{\dot{A}A} \Gamma^I_{\dot{B}A}) = 2\delta^{IJ} \delta_{AB},
\] (42)
\[
\sum_{\dot{A}} (\Gamma^I_{\dot{A}A} \Gamma^J_{\dot{A}B} + \Gamma^J_{\dot{A}A} \Gamma^I_{\dot{A}B}) = 2\delta^{IJ} \delta_{\dot{A}\dot{B}}.\] (43)

It is known that the real solutions of these equations exist and can be constructed, for example, in the way analogous to what has been done in appendix A of the paper [9] for the group $SO(8)$.

The $SO(16)$ generators of spinorial transformation $\Gamma^{I\dot{J}} = \frac{1}{2} (\Gamma^I \Gamma^J - \Gamma^J \Gamma^I)$ are block diagonal:

\[
\Gamma^{I\dot{J}} = \begin{pmatrix}
\Gamma^{I\dot{J}}_{\dot{A}B} & 0 \\
0 & \Gamma^{I\dot{J}}_{\dot{A}B}
\end{pmatrix},
\] (44)

where

\[
\Gamma^{I\dot{J}}_{\dot{A}B} = \frac{1}{2} \sum_{\dot{A}} \left( \Gamma^I_{\dot{A}\dot{A}} \Gamma^J_{\dot{B}A} - \Gamma^J_{\dot{A}\dot{A}} \Gamma^I_{\dot{B}A} \right),
\] (45)
\[
\Gamma^{I\dot{J}}_{\dot{A}B} = \frac{1}{2} \sum_{\dot{A}} \left( \Gamma^I_{\dot{A}\dot{A}} \Gamma^J_{\dot{A}B} - \Gamma^J_{\dot{A}\dot{A}} \Gamma^I_{\dot{A}B} \right).
\] (46)

The blocks $\Gamma^{I\dot{J}}_{\dot{A}B}$ and $\Gamma^{I\dot{J}}_{\dot{A}B}$ generates the transformations of the internal Weyl spinors of ”positive” and ”negative” chiralities respectively. Both components $\Gamma^{I\dot{J}}_{\dot{A}B}$ and $\Gamma^{I\dot{J}}_{\dot{A}B}$ are antisymmetric under interchange of the upper indices $I, J$ as well as lower indices $A, B$ and $\dot{A}, \dot{B}$.

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