A REGULARITY CRITERION FOR THE 3D FULL COMPRESSIBLE MAGNETOHYDRODYNAMIC EQUATIONS WITH ZERO HEAT CONDUCTIVITY

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Abstract. We establish a regularity criterion for the 3D full compressible magnetohydrodynamic equations with zero heat conductivity and vacuum in a bounded domain.

1. Introduction. In this paper, we consider the 3D full compressible magnetohydrodynamic equations in a bounded domain \( \Omega \subset \mathbb{R}^3 \):

\[
\begin{align*}
\partial_t \rho + \text{div} (\rho u) &= 0, \\
\partial_t (\rho u) + \text{div} (\rho u \otimes u) + \nabla p - \mu \Delta u - (\lambda + \mu) \nabla \text{div} u &= \text{rot} b \times b, \\
C_V [\partial_t (\rho \theta) + \text{div} (\rho u \theta)] - \kappa \Delta \theta + pd\text{div} u &= \frac{\mu}{2} |\nabla u + \nabla u^t|^2 + \lambda (\text{div} u)^2 + \nu |\text{rot} b|^2, \\
\partial_t b + \text{rot} (b \times u) &= \nu \Delta b, \quad \text{div} b = 0,
\end{align*}
\]

with the initial and boundary conditions

\[
\begin{align*}
u &= 0, \quad \frac{\kappa \theta}{\xi_n} = 0, \quad b \cdot n = 0, \quad \text{rot} b \times n = 0 \quad \text{on} \quad \partial \Omega \times (0, \infty), \\
(\rho, u, \theta, b)(\cdot, 0) &= (\rho_0, u_0, \theta_0, b_0) \quad \text{in} \quad \Omega \subset \mathbb{R}^3.
\end{align*}
\]

Here the unknowns \( \rho, u, p, \theta, \) and \( b \) stand for the density, velocity, pressure, temperature, and magnetic field, respectively. The physical constants \( \mu \) and \( \lambda \) are the shear viscosity and bulk viscosity of the fluid, respectively, and satisfy \( \mu > 0 \) and \( \lambda + \frac{3}{2} \mu \geq 0 \). \( C_V > 0 \) is the specific heat at constant volume, \( \kappa > 0 \) is the heat conductivity coefficient, and \( \nu > 0 \) is the magnetic diffusivity coefficient. \( \nabla u^t \) denotes

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the transpose of the matrix $\nabla u$. We assume that $\Omega$ is a bounded and simply connected domain in $\mathbb{R}^3$ with smooth boundary $\partial\Omega$. We use $n$ to denote the outward unit normal vector to $\partial\Omega$.

The full compressible magnetohydrodynamic equations (1)-(4) can be rigorously derived from the compressible Navier-Stokes-Maxwell system [14]. Due to the physical importance of the magnetohydrodynamics, there are a lot of works on the system (1)-(4), among others, we mention [8] on the local strong solutions, [4,9,10] on the global weak solutions, [15,16] on low Mach number limit, and [19] on the time decay of smooth small solutions.

Assume that the pressure take the form $p = R\rho\theta$ with $R > 0$ being the generic gas constant. Below we take $R = 1$ for presentation simplicity.

In [11], Huang and Li proved the following regularity criterion to the system (1)-(4):

$$\rho \in L^\infty(0,T; L^\infty), \quad u \in L^s(0,T; L^r) \quad \text{for} \quad \frac{2}{s} + \frac{3}{r} = 1, \quad 3 < r \leq \infty, \quad (7)$$

with $b$ satisfying the homogeneous Dirichlet boundary condition $b = 0$ on $\partial\Omega \times (0, \infty)$. Later this result was generalized in [7] to the case when $b$ satisfies the boundary condition (5), i.e.,

$$b \cdot n = 0, \quad \text{rot } b \times n = 0 \quad \text{on} \quad \partial\Omega \times (0, \infty). \quad (8)$$

When considering the system (1)-(4) in a two dimensional bounded domain, Lu, Chen and Huang [18] showed the following regularity criterion

$$\text{div } u \in L^1(0,T; L^\infty) \quad (9)$$

with $b$ satisfying the boundary condition $b = 0$ on $\partial\Omega \times (0, \infty)$. Here we remark that the same result can be proved for $b$ satisfying the boundary condition: $b \cdot n = 0$, rot $b = 0$ on $\partial\Omega \times (0, \infty)$. A related weak result was obtained in [6].

Very recently, Huang and Wang [12] establish the following regularity criterion

$$\rho, \theta, b \in L^\infty(0,T; L^\infty) \quad \text{with} \quad 2\mu > \lambda. \quad (10)$$

for the system (1)-(4) in the whole space $\mathbb{R}^3$ with $\kappa = \nu = 0$.

The aim of this paper is to show that the regularity criterion (10) still hold for the system (1)-(4) in a bounded domain with the boundary condition (5) when $\kappa = 0$ and $\nu = 1$. We will prove

**Theorem 1.1.** Let $\kappa = 0$ and $\nu = 1$. For $q \in (3,6]$, assume that the initial data $(\rho_0 \geq 0, u_0, p_0 = R\rho_0\theta_0 \geq 0, b_0)$ satisfy

$$\begin{cases}
\rho_0, p_0 \in W^{1,q}(\Omega), \quad u_0 \in H^1_0(\Omega) \cap H^2(\Omega), \quad b_0 \in H^2 \quad \text{with} \quad \text{div } b_0 = 0 \quad \text{in} \quad \Omega, \\
\rho_0 \cdot n = 0, \quad \text{rot } b_0 \times n = 0 \quad \text{on} \quad \partial\Omega
\end{cases} \quad (11)$$

and the compatibility condition

$$-\mu \Delta u_0 - (\lambda + \mu) \nabla \text{div } u_0 + \nabla p_0 - \text{rot } b_0 \times b_0 = \sqrt{\rho_0}g, \quad (12)$$

with $g \in L^2(\Omega)$. Let $(\rho, u, p, b)$ be a local strong solution to the problem (1)-(6). If (10) holds true with $0 < T < \infty$, then the solution $(\rho, u, p, b)$ can be extended beyond $T > 0$.

**Remark 1.1.** When $\kappa > 0$ and $\nu = 1$, we have a better result (7) which was proved by a very different method, see [7,11].
We mention that when taking \( b = 0 \) in the system (1)-(4), it is reduced to the full compressible Navier-Stokes system and a lot of regularity criteria can be found in [5, 20, 23] and the references cited therein.

The remainder of this paper is devoted to the proof of Theorem 1.1. We give some preliminaries in section 2 and present the proof of Theorem 1.1 in section 3. Below we shall use the letter \( C \) to denote the positive constant which may change from line to line.

2. Preliminaries. First, we consider the boundary value problem for the Lamé operator \( L \)

\[
\begin{aligned}
LU &:= \mu \Delta U + (\mu + \lambda) \nabla \div U = F \quad \text{in} \quad \Omega, \\
U(x) &= 0 \quad \text{on} \quad \partial \Omega.
\end{aligned}
\]

Here \( U := (U_1, U_2, U_3) \), \( F := (F_1, F_2, F_3) \). It is well known that the system (13) is a strongly elliptic system, thus it has a unique weak solution \( U \in H^1_0(\Omega) \) when \( F \in W^{-1,2}(\Omega) \).

**Lemma 2.1.** Suppose \( q \in (1, \infty) \) and \( U \) is a solution of (13). There exists a constant \( C \) depending only on \( \lambda, \mu, q \) and \( \Omega \) such that the following estimates hold:

1. If \( F \in L^q(\Omega) \), then
   \[
   \|U\|_{W^{2,q}(\Omega)} \leq C\|F\|_{L^q(\Omega)};
   \]

2. If \( F \in W^{-1,q}(\Omega) \) (that is, \( F = \div f \) with \( f = (f_{ij})_{3 \times 3} \), \( f_{ij} \in L^q(\Omega) \)), then
   \[
   \|U\|_{W^{1,q}(\Omega)} \leq C\|f\|_{L^q(\Omega)};
   \]

3. If \( F = \div f \) with \( f_{ij} = \hat{\partial}_i h^k_{ij} \) and \( h^k_{ij} \in W^{1,q}_0(\Omega) \) for \( i, j, k = 1, 2, 3 \), then
   \[
   \|U\|_{L^q(\Omega)} \leq C\|h\|_{L^q(\Omega)}.
   \]

**Proof.** The estimates (14) and (15) are well-known for strongly elliptic systems, see [2]. The estimate (16) can be proved by a duality argument with the help of (14).

We state an endpoint estimate for \( L \) in the case \( q = \infty \). Let \( BMO(\Omega) \) stand for the John-Nirenberg space of bounded mean oscillation whose norm is defined by

\[
\|f\|_{BMO(\Omega)} := \|f\|_{L^2(\Omega)} + [f]_{BMO},
\]

with

\[
[f]_{BMO(\Omega)} := \sup_{x \in \Omega, r \in (0, d)} \frac{1}{|\Omega_r(x)|} \int_{\Omega_r(x)} |f(y) - f_{\Omega_r(x)}| dy,
\]

\[
f_{\Omega_r(x)} := \frac{1}{|\Omega_r(x)|} \int_{\Omega_r(x)} f(y) dy.
\]

Here \( \Omega_r(x) := B_r(x) \cap \Omega \), \( B_r(x) \) is a ball with center \( x \) and radius \( r \), \( d \) is the diameter of \( \Omega \) and \( |\Omega_r(x)| \) denotes the Lebesgue measure of \( \Omega_r(x) \).

**Lemma 2.2** ([1]). If \( F = \div f \) with \( f = (f_{ij})_{3 \times 3} \), \( f_{ij} \in L^\infty(\Omega) \cap L^2(\Omega) \), then \( \nabla U \in BMO(\Omega) \) and there exists a constant \( C \) depending only on \( \lambda, \mu \) and \( \Omega \) such that

\[
\|\nabla U\|_{BMO(\Omega)} \leq C(\|f\|_{L^\infty(\Omega)} + \|f\|_{L^2(\Omega)}).
\]

Let us now recall a variant of the Brezis-Wainwright inequality [3].
Lemma 2.3 ([21]). Assume that $\Omega$ is a bounded Lipschitz domain in $\mathbb{R}^3$ and $f \in W^{1,q}$ with $3 < q < \infty$. There exists a constant $C$ depending on $q$ and the Lipschitz property of $\Omega$ such that
\[ \|f\|_{L^\infty(\Omega)} \leq C(1 + \|f\|_{BMO(\Omega)} \ln(e + \|\nabla f\|_{L^q(\Omega)})). \] (18)

Lemma 2.4 ([13]). Let $b$ be a solution to the Poisson equation
\[ -\Delta b = f \quad \text{in} \quad \Omega \]
with the boundary condition
\[ b \cdot n = 0, \text{rot}\ b \times n = 0 \quad \text{on} \quad \partial \Omega. \]
Then it holds
\[ \|\nabla^2 b\|_{L^p} \leq C\|f\|_{L^p} + C\|\nabla b\|_{L^2} \quad \text{with} \quad 1 < p < \infty. \] (19)
In the following proofs, we will use the Poincaré inequality [17]:
\[ \|b\|_{L^2} \leq C(\|\text{div}\ b\|_{L^2} + \|\text{rot}\ b\|_{L^2}) \] (20)
for any $b \in H^1(\Omega)$ with $b \cdot n = 0$ or $b \times n = 0$ on $\partial \Omega$. We will also use the inequality [22]:
\[ \|\nabla b\|_{L^2} \leq C(\|\text{div}\ b\|_{L^2} + \|\text{rot}\ b\|_{L^2}) \] (21)
for any $b \in H^1(\Omega)$ with $b \cdot n = 0$ or $b \times n = 0$ on $\partial \Omega$.

3. Proof of Theorem 1.1. This section is devoted to the proof of Theorem 1.1, we first show a priori estimates. For simplicity, we will take $\nu = C_V = R = 1$.

Testing (2) by $u$, (4) by $b$, summing up the results and using (1) and (10), we obtain that
\[ \frac{1}{2} \frac{d}{dt} \int (\rho|u|^2 + |b|^2)dx + \int (\mu|\nabla u|^2 + (\lambda + \mu)(\text{div} u)^2 + |\text{rot} b|^2)dx = -\int u \nabla p dx = \int \text{div} u dx \leq \frac{\lambda + \mu}{2} \int (\text{div} u)^2 dx + C, \]
which gives
\[ \int (\rho|u|^2 + |b|^2)dx + \int_0^T \int (|\nabla u|^2 + |\text{rot} b|^2)dxdt \leq C. \] (22)
Integrating (3) over $\Omega \times (0,t)$ and using (10) and (22), we find that
\[ \int \rho \theta dx \leq C. \] (23)
By the same calculations as that in [12], we get
\[ \int \rho|u|^4 dx + \int_0^T \int |\nabla u|^2|u|^2 dxdt \leq C, \] (24)
and thus
\[ \int_0^T \|u\|_{L^4}^4 dt \leq C. \] (25)
We define $v \in H^1_0$ satisfying
\[ Lv := \mu \Delta v + (\lambda + \mu)\nabla \text{div} v = \nabla p, \] (26)
and $w := u - v$. Thanks to Lemma 2.1, for any $1 < r < \infty$, there hold
\[ \|\nabla v\|_{L^r(\Omega)} \leq C\|p\|_{L^r(\Omega)}, \|\nabla^2 v\|_{L^r(\Omega)} \leq C\|\nabla p\|_{L^r(\Omega)}. \] (27)
It is easy to see that \( w \) satisfies
\[
\mu \Delta w + (\lambda + \mu) \nabla \div w = \rho \dot{u} - \text{rot} \, b \times b, \tag{28}
\]
where we denote \( \dot{f} := f_t + u \cdot \nabla f \).

Then it follows from Lemma 2.1 that
\[
\| \nabla^2 w \|_{L^2(\Omega)} \leq C \| \dot{\rho} u \|_{L^2(\Omega)} + C \| \text{rot} \, b \times b \|_{L^2(\Omega)}. \tag{29}
\]

Let \( E \) be the specific energy defined by
\[
E := \theta + \frac{|u|^2}{2}.
\]

Then
\[
\partial_t \left( \rho E + \frac{|b|^2}{2} \right) + \text{div} (\rho \dot{u} + pu + |b|^2 u) = \frac{1}{2} \mu \Delta |u|^2 + \mu \text{div} (u \cdot \nabla u) + \lambda \text{div} (u \text{div} u) + \text{div} ((u \cdot b)b) - \text{div} (\text{rot} \, b \times b). \tag{30}
\]

Testing (2) by \( u_t \) and using (1), we deduce that
\[
\frac{1}{2} \frac{d}{dt} \int (\mu |\nabla u|^2 + (\lambda + \mu)(\text{div} u)^2) dx + \int \rho |\dot{u}|^2 dx = \int \rho u_t \cdot (u \cdot \nabla) u dx + \int \left[ \left( p + \frac{1}{2} |b|^2 \right) I_3 - (b \otimes b) \right] : \nabla u_t dx \leq \frac{1}{8} \int \rho |\dot{u}|^2 dx + C \int |u|^2 |\nabla u|^2 dx + \frac{d}{dt} \int \left[ \left( p + \frac{1}{2} |b|^2 \right) I_3 - (b \otimes b) \right] : \nabla u dx - \int p_t \text{div} \, u dx - \int \left[ \frac{1}{2} |b|^2 I_3 - (b \otimes b) \right] : \nabla u dx. \tag{31}
\]

To estimate the third term on the right hand side of (31), we remark that
\[
\left| \int \left[ \left( p + \frac{1}{2} |b|^2 \right) I_3 - (b \otimes b) \right] : \nabla u dx \right| \leq C \int |\nabla u| dx \leq \frac{\mu}{16} \int |\nabla u|^2 dx + C.
\]

Moreover, we have
\[
- \int p_t \text{div} \, v dx = \int \nabla p_t v dx = \int \nu (\mu \Delta v_t + (\lambda + \mu) \nabla \text{div} v_t) dx = - \frac{1}{2} \frac{d}{dt} \int (\mu |\nabla v|^2 + (\lambda + \mu)(\text{div} v)^2) dx. \tag{32}
\]

And according to (30) and (1), it holds
\[
- \int \rho \dot{u} \text{div} \, w dx = - \int \left( \rho E + \frac{|b|^2}{2} \right) \text{div} \, w dx + \int \left( \frac{1}{2} \rho |u|^2 \right) \text{div} \, w dx + \int (b \cdot b_t) \text{div} \, w dx = - \int \left( \rho E u + pu + |b|^2 u - \frac{1}{2} \mu |\nabla u|^2 - \mu (u \cdot \nabla) u - \lambda \text{div} \, u - (u \cdot b)b + \text{rot} \, b \times b \right) \cdot \nabla \text{div} \, w dx - \frac{1}{2} \int \text{div} (\rho u_t) |u|^2 \text{div} \, w dx + \int \rho u_t \text{div} \, w dx + \int b b_t \text{div} \, w dx = - \int \left( 2p \theta u + |b|^2 u - \frac{1}{2} \mu |\nabla u|^2 - \mu (u \cdot \nabla) u - \lambda \text{div} \, u - (u \cdot b)b + \text{rot} \, b \times b \right) \cdot \nabla \text{div} \, w dx + \int \rho u_t \text{div} \, w dx + \int b b_t \text{div} \, w dx
\]
\[ C \int \rho |u|^2 dx + C \|u\|_{L^6}^2 + C \int |u|^2 |\nabla u|^2 dx + C \|\nabla b\|_{L^4}^2 + \delta_1 \|\nabla^2 w\|_{L^2}^2 \\
+ \delta_2 \|\sqrt{\rho} \dot{u}\|_{L^2}^2 + C \|\sqrt{\rho} u\|_{L^4}^2 \|\nabla w\|_{L^2}^2 + C \|\nabla w\|_{L^2}^2 + \delta_3 |b|_{L^2}^2 \leq C + \delta_2 \|\sqrt{\rho} \dot{u}\|_{L^2}^2 + C \|\nabla w\|_{L^2}^2 + \delta_3 |b|_{L^2}^2 \leq C + \delta_2 \|\sqrt{\rho} \dot{u}\|_{L^2}^2 + \delta_3 |b|_{L^2}^2 \quad (33) \]
for any small \(0 < \delta_1, \delta_2, \text{ and } \delta_3\). Here we have used the Gagliardo-Nirenberg inequalities:

\[
\|\nabla w\|_{L^4} \leq C \|\nabla w\|_{L^2}^{\frac{1}{2}} \|\nabla^2 w\|_{L^2}^{\frac{3}{2}} + C \|\nabla w\|_{L^2}, \\
\|\nabla w\|_{L^2} \leq \|\nabla u\|_{L^6} + \|\nabla u\|_{L^2} \leq C + \|\nabla u\|_{L^2}.
\]

Observe that the last term of (31) can be bounded as

\[
- \int \left[ \frac{1}{2} |b|^2 \delta_u + (b \otimes b) \right] : \nabla u dx \leq \delta_3 |b|_{L^2}^2 + C \|\nabla u\|_{L^2}^2. \quad (34)
\]

On the other hand, testing (4) by \(b_t - \Delta b\), we get

\[
\frac{d}{dt} \int \rho \dot{b}^2 dx + \int (|b_t|^2 + |\Delta b|^2) dx \\
= \int \rho (b \times u)^2 dx \leq C \int \|\nabla u\|^2 dx + C \|u\|_{L^6}^2 \|\nabla b\|_{L^2}^2 \leq C \|\nabla u\|_{L^2}^2 + \frac{1}{2} \|\Delta b\|_{L^2}^2 + C \|u\|_{L^6}^2 \|\nabla b\|_{L^2}^2. \quad (35)
\]

Here we have used the inequality

\[
\|\nabla^2 b\|_{L^2} \leq C \|\Delta b\|_{L^2} + C \|\nabla b\|_{L^2} \quad (36)
\]

and the Gagliardo-Nirenberg inequality

\[
\|\nabla b\|_{L^2} \leq C \|\nabla b\|_{L^2} \|\nabla^2 b\|_{L^2} + C \|\nabla b\|_{L^2}^2. \quad (37)
\]

Inserting (32), (33), and (34) into (31), combining them with (35), choosing \(\delta_1, \delta_2, \text{ and } \delta_3\) suitably small, and using the Gronwall inequality, we have

\[
\sup_{0 \leq t \leq T} \int (|\nabla u|^2 + |\nabla b|^2) dx + \int_0^T \int (|\sqrt{\rho} \ddot{u}|^2 + |b_t|^2 + |\nabla^2 b|^2) dx dt \leq C. \quad (38)
\]

Now we are in a position to give a high order regularity estimates of the solutions. The calculations were motivated by [20]. First of all, we rewrite the equation (2) as

\[
\rho \dot{u} + \nabla p - Lu = g := \text{rot } b \times b
\]
to find that

\[
\rho \dot{u} + \rho u \cdot \nabla \dot{u} + \nabla \rho u + \text{div} (\nabla p \otimes u) = \mu [\Delta u_t + \text{div} (\Delta u \otimes u)] \\
+ (\lambda + \mu) [\nabla \text{div} u_t + \text{div} (\nabla \text{div} u \otimes u)] + g_t + \text{div} (g \otimes u).
\]

Testing the above equation by \(u\) and using (1), we have

\[
\frac{1}{2} \frac{d}{dt} \int \rho |\dot{u}|^2 dx - \mu \int \dot{u} |\Delta u_t + \text{div} (\Delta u \otimes u)| dx
\]
\[-(\lambda + \mu) \int \hat{u} [\nabla \text{div} u_t + \text{div} (\nabla \text{div} u \otimes u)] dx \]
\[= \int (p_t \text{div} \hat{u} + (u \cdot \nabla) \hat{u} \cdot \nabla p) dx + \int (g_t + \text{div} (g \otimes u)) \hat{u} dx. \quad (39)\]

As in [20], one can estimate the second and third terms in above equation as follows:

\[-\int \hat{u} [\Delta u_t + \text{div} (\Delta u \otimes u)] dx \geq \int \left(\frac{3}{4} |\nabla \hat{u}|^2 - C |\nabla u|^4\right) dx,\]
and

\[-\int \hat{u} [\nabla \text{div} u_t + \text{div} (\nabla \text{div} u \otimes u)] dx \geq \int \left(\frac{1}{2} (\text{div} \hat{u})^2 - \frac{1}{8} |\nabla \hat{u}|^2 - C |\nabla u|^4\right) dx.\]

Since \(p := \rho \theta\), we rewrite (3) as follows,

\[p_t + \text{div} (pu) + p \text{div} u = \frac{\mu}{2} |\nabla u + \nabla u'|^2 + \lambda (\text{div} u)^2 + |\text{rot} b|^2. \quad (40)\]

Using (40), as in [12, 20], one can estimate the fourth term in (39) as

\[\int (p_t \text{div} \hat{u} + (u \cdot \nabla) \hat{u} \cdot \nabla p) dx \leq C + C \|\nabla u\|_{L^4}^4 + C \|\nabla b\|_{L^4}^4 + \frac{\mu}{8} \|\nabla \hat{u}\|_{L^2}^2. \quad (41)\]

Using the fact that \(b \cdot \nabla b + b \times \text{rot} b = \frac{1}{2} \nabla |b|^2\), and (38), we can bound the last term of (39) as

\[
\int (g_t + \text{div} (g \otimes u)) \hat{u} dx \\
= \int \left[ \text{div} \left( \frac{1}{2} \|b\|_3 \mathbb{I}_{3} - b \otimes b \right) + \text{div} (g \otimes u) \right] \hat{u} dx \\
= - \int \left( \frac{1}{2} \|b\|_3 \mathbb{I}_{3} - b \otimes b + g \otimes u \right) : \nabla \hat{u} dx \\
\leq C \left( \|b\|_{L^4}^4 + \|b\|_{L^\infty} \|\text{rot} b\|_{L^1} \|u\|_{L^3} \right) \|\nabla \hat{u}\|_{L^2} \\
\leq C (1 + \|\nabla b\|_{L^3}) \|\nabla \hat{u}\|_{L^2} \leq \frac{\mu}{8} \|\nabla \hat{u}\|_{L^2}^2 + C \|\nabla b\|_{L^3}^2 + C.
\]

Inserting those estimates into (38) and using the inequalities

\[\|\nabla b\|_{L^4}^4 \leq C \|b\|_{L^6}^2 \|b\|_{H^2}^2,\]
\[\|\nabla u\|_{L^4}^4 \leq \|\nabla u\|_{L^3} \|\nabla u\|_{L^5}^2 \leq C (\|\nabla v\|_{L^6} + \|\nabla w\|_{L^6}^2) \leq C (1 + \|\nabla \hat{u}\|_{L^2}^3),\]

we have

\[
\frac{1}{2} \frac{d}{dt} \int \rho |\dot{u}|^2 dx + \mu \int |\nabla \dot{u}|^2 dx + (\lambda + \mu) \int (\text{div} \dot{u})^2 dx \\
\leq C (1 + \|\nabla u\|_{L^4}^4 + \|b\|_{H^2}^2) \\
\leq C + C \|\nabla \hat{u}\|_{L^2}^2 + C \|b\|_{H^2}^2, \\
\]

which gives

\[\|\sqrt{\rho} \dot{u}\|_{L^2(0,T;L^2)} + \|\dot{u}\|_{L^2(0,T;H^1)} \leq C. \quad (42)\]

By the same calculations as in [12], it is easy to verify that

\[\sup_{0 \leq t \leq T} \|\nabla w\|_{H^1} + \int_0^T (\|\nabla^2 w\|_{L^p}^2 + \|\nabla w\|_{L^\infty}^2) dt \leq C, \quad 2 \leq p \leq 6. \quad (43)\]
Applying \( \partial_t \) to (4), testing the result by \( b_t \), and using (24) and (42), we have
\[
\frac{1}{2} \frac{d}{dt} \int |b_t|^2 \, dx + \int \text{rot} b_t^2 \, dx = - \int \partial_t (b \times u) \text{rot} b_t \, dx
\]
\[
= - \int (b_t \times u + b \times \dot{u} - b \times (u \cdot \nabla) u) \text{rot} b_t \, dx
\]
\[
\leq (\|b_t\|_{L^6} \|u\|_{L^6} + \|b\|_{L^6} \|\dot{u}\|_{L^6} + \|b\|_{L^6} \|u \cdot \nabla\|_{L^2}) \|\text{rot} b_t\|_{L^2}
\]
\[
\leq C(\|b_t\|_{L^6} + \|\dot{u}\|_{L^5} + \|u \cdot \nabla u\|_{L^2}) \|\text{rot} b_t\|_{L^2}
\]
\[
\leq \frac{1}{2} \|\text{rot} b_t\|_{L^2}^2 + C \|b_t\|_{L^2}^2 + C \|\dot{u}\|_{L^6}^2 + C\|u \cdot \nabla u\|_{L^2}^2,
\]
which implies
\[
\|b_t\|_{{L^6}(0, T; L^2)} + \|b_t\|_{{L^2}(0, T; H^1)} \leq C. \tag{44}
\]
This, together with (4) and (38), leads to
\[
\|b\|_{{L^6}(0, T; H^2)} + \|b\|_{{L^2}(0, T; W^{2,0})} \leq C, \tag{45}
\]
where we used
\[
\|u\|_{{L^6}(0, T; W^{1,0})} \leq \|v\|_{{L^6}(0, T; W^{1,0})} + \|w\|_{{L^6}(0, T; W^{1,0})} \leq C.
\]
Direct calculations show that
\[
\frac{d}{dt} \|\nabla\rho\|_{L^q} \leq C(1 + \|\nabla u\|_{L^q}) \|\nabla\rho\|_{L^q} + C\|\nabla^2 u\|_{L^q}, \tag{46}
\]
and
\[
\frac{d}{dt} \|\nabla p\|_{L^q} \leq C(1 + \|\nabla u\|_{L^q}) (\|\nabla p\|_{L^q} + \|\nabla^2 u\|_{L^q}) + C\|\nabla b\|_{L^q} \|\nabla^2 b\|_{L^q}. \tag{47}
\]
We bound the last term of (47) as follow:
\[
\|\nabla b\|_{L^q} \|\nabla^2 b\|_{L^q} \leq C(1 + \|\nabla^2 b\|_{L^q}) \|\nabla^2 b\|_{L^q} \leq C + C\|\nabla^2 b\|_{L^q}^2. \tag{48}
\]
Using Lemmas 2.2 and 2.3, as in [12], it is easy to prove that
\[
\|\nabla\rho\|_{{L^6}(0, T; L^6)} + \|\rho\|_{{L^6}(0, T; L^6)} \leq C, \tag{49}
\]
\[
\|\nabla u\|_{{L^2}(0, T; L^6)} + \|u\|_{{L^6}(0, T; H^2)} \leq C. \tag{50}
\]

Now we can complete our proof of Theorem 1.1 as follows. Suppose that (10) holds, i.e.,
\[
\lim_{T \to T^*} \left( \|\rho\|_{{L^6}(0, T^-; L^6)} + \|\theta\|_{{L^6}(0, T^-; L^6)} + \|b\|_{{L^6}(0, T^-; L^6)} \right) =: M_0 < \infty. \tag{51}
\]

Note that the generic constant \( C \) remains uniformly bounded for all \( T < T^* \), so the functions \( (\rho, u, \theta, b)(\cdot, T^*) := \lim_{T \to T^*} (\rho, u, \theta, b)(\cdot, t) \) satisfy the conditions imposed on the initial data at the time \( t = T^* \). By the above estimates, we have \( \rho \dot{u} \in C([0, T], L^2) \), which gives
\[
\rho \dot{u}(\cdot, T^*) = \lim_{T \to T^*} \rho \dot{u}(\cdot, t) \in L^2.
\]

Hence
\[
\left[ - \mu \Delta u - (\lambda + \mu) \nabla \text{div} u + \nabla p - \text{rot} b \times b \right]_{t = T^*} = \sqrt{\rho(x, T^*)} g_1(x), \tag{52}
\]
with
\[
g_1(x) = \begin{cases} 
\rho^{-\frac{1}{2}}(x, T^*)(\rho \dot{u})(x, T^*), & \rho(x, T^*) > 0, \\
0, & \rho(x, T^*) = 0
\end{cases}
\]
satisfying $g_1 \in L^2$. Thus, $(\rho, u, \theta, b)(x, T^*)$ also satisfies (12). Therefore, one can take $(\rho, u, \theta, b)(x, T^*)$ as the initial data and apply the local well-posed theorem to extend the local strong solution beyond $T^*$. We thus finish the proof of Theorem 1.1.

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