Net-proton number fluctuations in the presence of the QCD critical point

Michał Szymański¹, Marcus Bluhm¹,², Krzysztof Redlich¹,³, Chihiro Sasaki¹

¹ Institute of Theoretical Physics, University of Wroclaw, PL-50204 Wroclaw, Poland
² SUBATECH UMR 6457 (IMT Atlantique, Université de Nantes, IN2P3/CNRS), 4 rue Alfred Kastler, 44307 Nantes, France
³ Extreme Matter Institute EMMI, GSI, Planckstr. 1, D-64291 Darmstadt, Germany

E-mail: michal.szymanski@ift.uni.wroc.pl

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Abstract. Event-by-event fluctuations of the net-proton number studied in heavy-ion collisions provide an important means in the search for the conjectured chiral critical point (CP) in the QCD phase diagram. We propose a phenomenological model in which the fluctuations of the critical chiral mode couple to protons and anti-protons. This allows us to study the behavior of the net-proton number fluctuations in the presence of the CP. Calculating the net-proton number cumulants, \(C_n\) with \(n=1,2,3,4\), along the phenomenological freeze-out line we show that the ratio of variance and mean \(C_2/C_1\), as well as kurtosis \(C_4/C_2\) resemble qualitative properties observed in data in heavy-ion collisions as a function of beam energy obtained by the STAR Collaboration at RHIC. In particular, the non-monotonic structure of the kurtosis and smooth change of the \(C_2/C_1\) ratio with beam energy could be due to the CP located near the freeze-out line. The skewness, however, exhibits properties that are in contrast to the criticality expected due to the CP. The dependence of our results on the model parameters and the proximity of the chemical freeze-out to the critical point are also discussed.

Delineating the phase diagram of quantum chromodynamics (QCD) at finite temperature \(T\) and baryon-chemical potential \(\mu_B\) is a challenging problem in theoretical and experimental studies [1–13]. The Beam Energy Scan (BES) at the Relativistic Heavy Ion Collider (RHIC) [12,14] has been dedicated to the search for the conjectured QCD critical point (CP) through the systematic studies of various observables. In particular, a non-monotonic behavior of the fluctuations of conserved charges with beam energy (\(\sqrt{s}\)) is considered as a conceivable experimental signature of the chiral critical behavior and of the CP in heavy-ion collisions [11,15–20]. Such a behavior is typically associated with the divergence of the correlation length and the fluctuations of the critical mode (\(\sigma\)) at the CP [15,16,21–25]. Another, indirect way of verifying the existence of the CP is given by searching for non-uniform structures in multiplicity distributions due to domain formations in the region of the first-order phase transition adjacent to the critical point [26–28].
Measurements of event-by-event fluctuations in the net-proton number [14, 29, 30, 32], as a proxy for net-baryon number, the net-electric charge [33] and the net-kaon number [34], as a proxy for net-strangeness, have been performed in heavy-ion collisions at RHIC and LHC energies. While non-monotonic structures, in the still preliminary STAR Collaboration data on higher-order net-proton fluctuations were indeed observed [30–32], their unambiguous interpretation as the consequence of the presence of the CP has not been achieved yet [19, 20, 35, 36].

Different QCD-like effective models indicate, that the chiral CP belongs to the Z(2) static universality class of the 3-dimensional Ising model, and that the chiral critical mode can be associated with the order parameter, the chiral condensate [23–25, 37]. For a static and infinite system in thermodynamic equilibrium, the scaling of the fluctuations of $\sigma$ modes with the diverging correlation length at the critical point is governed by the Z(2) critical exponents [38]. However, a medium created in a heavy-ion collision is neither static nor infinite and due to its expansion dynamics, the non-equilibrium effects can play an important role. The temporal growth of the correlation length is already dynamically limited by the phenomenon of critical slowing down [39, 40]. In addition, non-equilibrium effects can lead to the retardation and damping of the critical signals [41–44]. Furthermore, even in thermodynamic equilibrium, the exact charge conservation [45–47], volume fluctuations [46, 48–50], and further sources of non-critical fluctuations in the data [51], as well as, late hadronic stage processes [52–56], can modify signals of the critical fluctuations.

Given the above mentioned challenges it becomes clear that theoretical description and interpretation of data on fluctuation observables require special care and eventually a dynamical framework to match the conditions expected in heavy-ion collisions. Nevertheless, numerical simulations of the critical dynamics require also input from static, equilibrium models to provide an analytic benchmark [44, 50, 57, 58]. One of such phenomenological models, that accounts for the critical fluctuations in heavy-ion collisions has been formulated in Ref. [56]. There, the fluctuations of the critical mode were linked to (anti-)protons and resonances by allowing the particle masses to fluctuate as a consequence of the (partial) mass generation through the coupling to $\sigma$. As was demonstrated in Ref. [56], the non-monotonicity observed in the higher-order fluctuation data of STAR Collaboration [30–32] can be qualitatively described within such formulation. However, in contrast to the experimental data a pronounced peak structure was also found in the lowest-order cumulant ratios of the net-proton fluctuations even for small values of couplings [56].

The objective of this work is to re-examine the model assumptions introduced in Ref. [56] to improve the above discrepancies between the model predictions and STAR data on the variance of net-proton number fluctuations obtained in heavy-ion collisions. The general study is based on the universality for the critical scaling behavior of the net-quark (net-baryon) number susceptibility [59]. We will show that the consistent implementation of critical scaling leads to a much weaker singularity than that seen for the net-proton number variance in Ref. [56]. In fact, under the mean
field approximation of the chiral effective models, the singular part of the net-quark number susceptibility is proportional to the chiral susceptibility multiplied by the order parameter squared [60, 61]. In line with these findings, we propose the modification of the model introduced in Ref. [56] to further consistently identify the influence of criticality due to the existence of the CP on the net-proton number fluctuations. We will quantify different cumulant ratios of these fluctuations along the chemical freeze-out line obtained in heavy-ion collisions. We will also identify systematics of the net-proton number cumulant ratios for different locations of the CP in the \((T,\mu_B)\)-plane, relative to the freeze-out line.

The theoretical tools used in our study are introduced in Section 1. In Section 2, we present results on the properties of the net-proton number fluctuations in the presence of the CP and for different assumptions on its location or the coupling strength between particles and the critical mode. We conclude our findings in Section 3.

1. Modeling critical fluctuations near CP

In the following, we describe the theoretical tools to be used in order to quantify the net-proton number fluctuations in the presence of the QCD critical point. We first define the baseline model which does not contain contributions from chiral critical mode fluctuations. Then we explain how the critical fluctuations can be coupled to particles near the chiral CP. Our approach is motivated by the observation that the critical contribution to the variance of the net-proton number should obey a certain scaling behavior [60, 61]. On the mean-field level, this idea can be extended to higher-order fluctuations. The critical mode fluctuations are obtained by using universality class arguments between QCD and the 3-dimensional Ising spin model. The necessary mapping between the corresponding variables is also discussed.

1.1. Baseline model

As a baseline model for calculating the net-proton number cumulants we employ the hadron resonance gas (HRG). In this model, the pressure of the interacting hadron gas is approximated by the sum of the partial pressures of non-interacting hadrons and their resonances [9, 62]. In the HRG model, the particle density of each particle species is given by the ideal gas expression

\[ n_i(T, \mu_i) = d_i \int \frac{d^3k}{(2\pi)^3} f_i^0(T, \mu_i), \]

where \(d_i\) is the degeneracy factor and

\[ f_i^0 = \frac{1}{(-1)^{B_i} e^{(E_i-\mu_i)/T}}, \]

is the thermal equilibrium distribution function. In Eq. (2), \(E_i = \sqrt{k^2 + m_i^2}\) and \(\mu_i = B_i\mu_B + S_i\mu_S + Q_i\mu_Q\) are the energy and chemical potentials of a particle with mass \(m_i\), baryon number \(B_i\), strangeness \(S_i\) and electric charge \(Q_i\), and \(\mu_X\) is the conjugate to
the conserved charge number \( N_X \). For a grand canonical ensemble, the average number in a constant volume \( V \) is \( \langle N_i \rangle = V n_i \).

In the thermal medium the particle number fluctuates around its mean on an event-by-event basis. These fluctuations can be quantified in terms of cumulants, where the n-th order cumulant is defined as

\[
C_n^i = V T^3 \left. \frac{\partial^{n-1} (n_i/T^3)}{\partial (\mu_i/T)^{n-1}} \right|_T ,
\]

at constant temperature \( T \). In the following, we consider the first four cumulants of the net-proton number \( N_{p-\bar{p}} = N_p - N_{\bar{p}} \), which are given by

\[
C_1 = \langle N_{p-\bar{p}} \rangle = \langle N_p \rangle - \langle N_{\bar{p}} \rangle ,
\]

\[
C_2 = \langle (\Delta N_{p-\bar{p}})^2 \rangle = C_p^2 + C_{\bar{p}}^2 ,
\]

\[
C_3 = \langle (\Delta N_{p-\bar{p}})^3 \rangle = C_p^3 - C_{\bar{p}}^3 ,
\]

\[
C_4 = \langle (\Delta N_{p-\bar{p}})^4 \rangle_c = C_p^4 + C_{\bar{p}}^4 ,
\]

where

\[
\Delta N_{p-\bar{p}} = N_{p-\bar{p}} - \langle N_{p-\bar{p}} \rangle ,
\]

\[
\langle (\Delta N_{p-\bar{p}})^4 \rangle_c = \langle (\Delta N_{p-\bar{p}})^4 \rangle - 3 \langle (\Delta N_{p-\bar{p}})^2 \rangle^2 .
\]

The second equalities in Eqs. (5)-(7) hold only when correlations between different particle species vanish which is the case in the HRG baseline.

In the following, we will ignore the contributions stemming from resonance decays and concentrate solely on the primary proton and anti-proton numbers and their fluctuations. Since the cumulants are volume-dependent, it is useful to consider their ratios in which, ignoring volume fluctuations, the volume dependence cancels out. Consequently,

\[
\frac{C_2}{C_1} = \frac{\sigma^2}{M} , \quad \frac{C_3}{C_2} = S \sigma , \quad \frac{C_4}{C_2} = \kappa \sigma^2 ,
\]

where \( M = C_1 \) is the mean, \( \sigma^2 = C_2 \) the variance, \( \kappa = C_4/C_2^2 \) the kurtosis, and \( S = C_3/C_2^{3/2} \) the skewness. Under the assumption that experimentally measured event-by-event multiplicity fluctuations originate from a thermal source with given \( T \) and \( \mu_X \), one can compare the model results for the cumulant ratios with the experimental data to deduce features of the QCD phase diagram.

1.2. Coupling to critical mode fluctuations

Fluctuations of the critical mode \( \sigma \) near the QCD critical point are expected to affect various experimentally measurable quantities, including the net-proton number cumulants [15, 16, 63]. Currently, there is no general prescription of how to model the effect of critical fluctuations on observables. In [56], inspired by the way in which the particle mass is generated near the chiral transition in the sigma models, the critical
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Mode fluctuations were incorporated into the HRG model by allowing the particle mass to fluctuate on an event-by-event basis around its mean value. Consequently, this leads to fluctuations in the distribution function, \( f_i = f_i^0 + \delta f_i \). Here, the change of the distribution function due to critical mode fluctuations reads

\[
\delta f_i = \frac{\partial f_i}{\partial m_i} \delta m_i = -\frac{g_i}{T} \frac{v_i^2}{\gamma_i} \delta \sigma,
\]

where \( g_i \) is the coupling strength between \( \sigma \) and the particle of type \( i \), which in principle can also depend on \( T \) and \( \mu_X \).

Due to the above modification of the distribution function, proton and anti-proton number fluctuations are no longer independent. Considering only the most singular contributions to the fluctuations (see [64] for a discussion of the impact of less critical contributions), one obtains the following expressions for the net-proton number cumulants influenced by the conjectured critical point,

\[
C_2 = C^p_2 + C^\bar{p}_2 + \langle (V\delta\sigma)^2 \rangle (I_p - I_{\bar{p}})^2,
\]

\[
C_3 = C^p_3 - C^\bar{p}_3 - \langle (V\delta\sigma)^3 \rangle (I_p - I_{\bar{p}})^3
\]

and

\[
C_4 = C^p_4 + C^\bar{p}_4 + \langle (V\delta\sigma)^4 \rangle_c (I_p - I_{\bar{p}})^4,
\]

where

\[
I_i = \frac{g_i d_i}{T} \int \frac{d^3 k}{(2\pi)^3} \frac{v_i^2}{\gamma_i}
\]

and \( \langle (V\delta\sigma)^n \rangle_c \) are the critical mode cumulants.

To calculate the cumulants of the critical mode, we apply universality class arguments which state, that close to the critical point different physical systems belonging to the same universality class, exhibit the same critical behavior characterized by the corresponding critical exponents [65]. Assuming that the QCD critical point belongs to the same universality class as the 3-dimensional Ising model, we identify the order parameter in QCD, \( \sigma \), with the magnetization \( M_I \), i.e. the order parameter of the Ising spin model. This allows us to define the critical mode cumulants as

\[
\langle (V\delta\sigma)^n \rangle_c = \left( \frac{T}{VH_0} \right)^{n-1} \frac{\partial^{n-1} M_I}{\partial h^{n-1}} \bigg|_r,
\]

where \( r = (T - T_c)/T_c \) and \( h = H/H_0 \) are the reduced temperature and magnetic field in the spin model, respectively, and the critical point is located at \( r = h = 0 \).

In our approach, the second cumulant \( C_2 \) of the net-proton number receives critical contributions through the coupling of (anti-)protons to the critical mode via the first derivative of the magnetization \( \partial M_I/\partial h \). In the spin model, this quantity is related to the magnetic susceptibility, and because of universality, to the chiral susceptibility (\( \chi \)) in QCD. The chiral susceptibility however, is known to diverge stronger than the net-baryon number susceptibility \( \chi_B \) [59–61], therefore, the model in its form introduced in Ref. [56] requires some modifications. These can be accomplished by using the following
relation between the net-baryon number and the chiral susceptibilities, which was found in the effective model calculations \[60,61\],

\[ \chi_B \simeq \chi_{B}^{\text{reg}} + \sigma^2 \chi. \] (17)

Here \( \chi_{B}^{\text{reg}} \) is the regular part of the net-baryon number susceptibility, whereas the second term in the above equation constitutes the singular part, \( \chi_{B}^{\text{sing}} \). Although this relation holds on the mean-field level, we will still use it in the present model to capture the correct scaling behavior near the QCD critical point.

Following Eq. (12), the singular part of the second-order cumulant is written, as

\[ C_{2}^{\text{sing}} = \langle (V \delta \sigma)^2 \rangle m_p^2 (J_p - J_{\bar{p}})^2, \] (18)

where

\[ J_i = \frac{g_i d_i}{T} \int \frac{d^3 k}{(2\pi)^3} \frac{v_i^2}{E_i}. \] (19)

In the linear-\( \sigma \) models, the proton mass is related to the chiral condensate. By replacing the factor \( m_p \) in Eq. (18) by \( g_p \sigma \), the critical contribution to the second-order cumulant in Eq. (12) has the same form as the one in Eq. (17). We note that this replacement is only done at leading-order in the derivation, i.e. factors of \( E_i \) remain unaffected. This phenomenological procedure results in the following modification

\[ \gamma_i \rightarrow E_i/(g_i \sigma), \] (20)

which is applied also to the higher-order cumulants of the net-proton number. Consequently, we arrive at

\[ C_2 = C_2^p + C_2^\theta + g_p^2 \sigma^2 \langle (V \delta \sigma)^2 \rangle (J_p - J_{\bar{p}})^2, \] (21)

\[ C_3 = C_3^p - C_3^\theta - g_p^3 \sigma^3 \langle (V \delta \sigma)^3 \rangle (J_p - J_{\bar{p}})^3 \] (22)

and

\[ C_4 = C_4^p + C_4^\theta + g_p^4 \sigma^4 \langle (V \delta \sigma)^4 \rangle (J_p - J_{\bar{p}})^4. \] (23)

As a result, \( C_2 \) is still diverging at the critical point but this divergence is much weaker after the above phenomenological modification, where the critical exponent for \( R \rightarrow 0 \) (\( h \rightarrow 0^+ \)) decreases from \( \beta(1-\delta) \) to \( \beta(3-\delta) \). In the following, we use \( g \) instead of \( g_p \) and apply Eqs. (4) and (21)-(23) together with Eq. (19) to calculate the net-proton number cumulant ratios near CP.

1.3. Magnetic equation of state

For the magnetic equation of state we use a parametric representation, \[66\]

\[ M_I = M_0 R^\beta \theta, \] (24)

which is strictly speaking valid only in the scaling region close to the critical point. Here \( R \) and \( \theta \) are auxiliary variables which depend on \( r \) and \( h \), from which \( R \) measures the distance from the critical point. They are determined by solving the following equations,

\[ r = R(1 - \theta^2), \] (25)

\[ h = R^{\beta \delta} w(\theta), \] (26)
where $\beta$ and $\delta$ are critical exponents and

$$w(\theta) = c\theta(1 + a\theta^2 + b\theta^4)$$

(27)

is an odd polynomial in $\theta$. The parameters entering Eqs. (24)-(27) are determined numerically by the Monte-Carlo simulations or by other theoretical tools such as the $\epsilon$-expansion or functional renormalization group methods. In [66], renormalization group and field theoretical methods were used to determine the critical exponents and coefficients of $w(\theta)$, reading, $\beta = 0.325$ and $\delta = 4.8169$, and $a = -0.76145$, $b = 0.00773$ and $c = 1$. For the normalization constants $H_0$ and $M_0$ in Eqs. (16) and (24) we follow [56] and set exemplarily, $M_0 = 5.52 \times 10^{-2}$ GeV and $H_0 = 3.44 \times 10^{-4}$ GeV$^3$. Differentiation of Eqs. (24)-(27) with respect to $h$ as defined in Eq. (16) allows us to determine the cumulants of the critical mode. These expressions are summarized in the appendix of Ref. [56], and will not be repeated here.

The parametrization of the magnetic equation of state used in this work provides an accurate description of the order parameter close to the critical point. In a simpler, linear parametric representation the coefficients and critical exponents read, $a = -2/3$, $b = 0$ and $c = 3$, $\beta = 1/3$ and $\delta = 5$. In this representation, the cumulants of the critical mode are considerably simpler and can be found in [43, 67]. We note, that the numerical results presented in [56] were obtained using this simpler representation. Qualitatively, features of the results obtained in these parametrizations are very similar, however the non-monotonic structures due to the critical point are quantitatively more pronounced in the parametrization employed here.

1.4. Mapping between spin model and QCD

In order to utilize the universality class argument, one needs a mapping between the reduced temperature $r$ and magnetic field $h$ in the spin model and the QCD temperature $T$ and baryon-chemical potential $\mu_B$. Such a mapping is non-universal. Moreover, it is sensitive to the model assumptions. One of the frequently used models is a linear mapping between the spin model and QCD phase diagrams [43, 68]. There, the main assumptions in the mapping are: (i) The conjectured QCD critical point at $(\mu_{cp}, T_{cp})$ is located at $r = h = 0$ in the Ising model coordinate system, and (ii) the $r$ axis is tangential at the critical point to the first-order phase transition line in QCD, where the positive $r$ direction points towards the QCD crossover region. The orientation of the $h$ axis is not well constrained. In this work we assume that this axis is perpendicular to the $r$ axis and its positive direction points towards the hadronic phase of QCD (see Fig. 1).

To obtain $(r, h)$ corresponding to a given $(\mu_B, T)$ pair it is convenient to introduce an auxiliary coordinate system $(\tilde{r}, \tilde{h})$, originating at the QCD critical point and oriented such that the $\tilde{r}$ axis is parallel to the $\mu_B$ axis. Then the mapping is defined as

$$\tilde{r} = \frac{\mu_B - \mu_{cp}}{\Delta \mu_{cp}},$$

(28)
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Figure 1. (Color online) The model setup for the location of the CP. The filled band between the two dashed curves shows lattice QCD results for the location of the chiral crossover transition obtained by solving Eq. (30) at leading-order for $\kappa_c = 0.007$ (upper curve, $T_{c,0} = 0.163$ GeV) and $\kappa_c = 0.02$ (lower curve, $T_{c,0} = 0.145$ GeV). The green dot shows the critical point with the attached spin model coordinate system (see the main text for details of the mapping) and the first-order phase transition line for larger $\mu_B$. The solid blue line shows the chemical freeze-out curve from [9].

and

$$\tilde{h} = \frac{T - T_{cp}}{\Delta T_{cp}},$$

(29)

where $\Delta T_{cp}$ and $\Delta \mu_{cp}$ are parameters which are connected to the size of the critical region. Following [56] we set $\Delta T_{cp} = 0.02$ GeV and $\Delta \mu_{cp} = 0.42$ GeV. The corresponding point in the spin model coordinate system is obtained by rotation of the auxiliary coordinate system, where the angle is determined by the slope of the first-order phase transition line of QCD at the critical point.

The exact location of the QCD critical point and the slope of the first-order phase transition line are not known. Input provided by lattice QCD calculations may be used to constrain these parameters. The crossover line can be parametrized as

$$T_c(\mu_B) = T_{c,0} \left[ 1 - \kappa_c \left( \frac{\mu_B}{T_c(\mu_B)} \right)^2 + \ldots \right],$$

(30)

where $T_{c,0} = (0.145 \ldots 0.163)$ GeV is the critical temperature at vanishing chemical potential [3,4] and $\kappa_c \simeq 0.007 \ldots 0.059$ is the chiral crossover curvature [69–72]. Then, for given $\mu_{cp}$, $T_{c,0}$ and $\kappa_c$ the temperature $T_{cp}$ and the slope of the first-order phase transition line at $(\mu_{cp}, T_{cp})$ are obtained from Eq. (30).

Finally, to make contact between our model calculations and the experimental data on net-proton fluctuations we calculate the net-proton number cumulants at chemical freeze-out. In this work we use the chemical freeze-out conditions which were determined by analyzing the measured hadron yields [73–78]. The solid blue line in Fig. 1 indicates the recent parametrization from Ref. [9].
2. Numerical results on the net-proton number fluctuations

In the following we discuss the numerical results on different fluctuations of the net-proton number obtained in the above phenomenological approach that correctly embeds the expected scaling behavior of the net-proton variance [60, 61]. To link our results to STAR data we first discuss how the experimentally applied kinematic acceptance cuts can be included into the framework. Then, the impact of our phenomenological modification introduced in Eq. (20) on fluctuation observables is studied. This includes also a discussion of the effects of modifying the coupling strength $g$ and the proximity of the thermal conditions at which the cumulant ratios are evaluated near the QCD critical point.

2.1. Kinematic cuts

In the fluctuation measurements, the investigated phase-space coverage is limited by the detector design and specific demands from the experimental analysis, like e.g. to optimize efficiency. Since the observables depend on the implemented kinematic acceptance, (see e.g. refs. [29] and [30–32, 79]) it is important to incorporate the experimental cuts into our theoretical framework. Following [80], we include restrictions in kinematic rapidity $y$, transverse momentum $k_T$ and azimuthal angle $\phi$ by replacing

$$\int d^3k \rightarrow \int k_T \sqrt{k_T^2 + m_i^2 \cosh y} dk_T dy d\phi,$$

and $E_i \rightarrow \sqrt{k_T^2 + m_i^2 \cosh y}$ in the momentum-integrals. In line with [32], we consider the following phase-space integrations: $-0.5 \leq y \leq 0.5$, $0 \leq \phi \leq 2\pi$ and $0.4 \text{GeV/c} \leq k_T \leq 2 \text{GeV/c}$. We note, that this procedure cannot account for scattering of particles in and out of the acceptance window during the late stage evolution of a medium created in heavy-ion collisions.

2.2. Net-proton number cumulant ratios

The contributions of critical fluctuations to the net-proton number cumulants are sensitive to the value of the coupling $g$ between (anti-)protons and the critical mode. This value may, in principle, depend on $T$ and $\mu_X$, i.e. on the position in the QCD phase diagram where the cumulant ratios are evaluated. In fact, in the quark-meson [81] and NJL [82] models, the meson-nucleon couplings are found to decrease both with increasing $T$ and/or $\mu_B$. In the following, we consider fixed values for $g$ along the chemical freeze-out curve depicted in Fig. 1, i.e. independent of the beam energy $\sqrt{s}$. Typical values for $g$ may be inferred from various effective model calculations. In the linear sigma model this parameter can be related in the ground state to the pion decay constant, $g \approx m_p/f_\pi \approx 10$ [38, 63]. Similar values can be found in non-linear chiral models [83] describing QCD matter in neutron stars. On the other hand, based on different quark-meson models the value of $g \approx 3 - 7$ for the nucleon-meson couplings is
Figure 2. (Color online) Net-proton number cumulant ratios from Eq. (10) calculated following Ref. [56] for $g = 3$ and 5 (red solid and dashed lines, respectively) compared to present model results (blue solid and dashed lines, respectively). For comparison, we also show the preliminary STAR data on the net-proton number fluctuations [32] (squares, where the error bars contain both statistical and systematic errors). Also shown are results for the non-critical baseline (black dotted lines).

well conceivable [84]. To highlight the features of our model results we will use $g$ in the range between 3 and 5.

In the previous work [56], the critical point was exemplarily located at $\mu_{cp} = 0.39$ GeV and $T_{cp} = 0.149$ GeV (see Fig. 1). There, even for rather small values of $g \simeq 3$, the cumulant ratio $C_2/C_1$ exhibited a clear peak structure compared to the non-critical baseline and in contrast to the STAR data [29–32]. This is illustrated in Fig. 2.

The effect of the modified scaling, discussed in Sec. 1.2, is the substantial reduction of the critical contribution to $C_2$ in Eq. (21) implying the disappearance of the maximum in $C_2/C_1$ even for large values of $g \simeq 5$, as seen in Fig. 2. Within error bars, the model results are in agreement with data of the STAR Collaboration [32] on the $C_2/C_1$ ratio. We note, that in order to see a similar maximum in $C_2/C_1$ in this model calculations, as found in [56] for a given location of the critical point, a significantly larger value of $g$, outside the expected range discussed above, would be necessary. Furthermore, differences between $C_2/C_1$ ratios shown in Fig. 2 are independent of a particular choice
Table 1. Considered locations of the QCD critical point in the \((\mu_B, T)\)-plane. The parameters \(T_{c,0}\) and \(\kappa_c\) of the crossover (pseudo-critical) line Eq. (30) needed to determine \(T_{cp}\) and the slope of the first-order phase transition line at the critical point for a given \(\mu_{cp}\) are also listed. The locations of these critical points in the QCD phase diagram are shown in Fig. 3.

| \(CP_i\) | \(\mu_{cp} [\text{GeV}]\) | \(T_{cp} [\text{GeV}]\) | \(T_{c,0} [\text{GeV}]\) | \(\kappa_c\) |
|---|---|---|---|---|
| 1 | 0.390 | 0.149 | 0.156 | 0.007 |
| 2 | 0.420 | 0.141 | 0.155 | 0.010 |
| 3 | 0.450 | 0.134 | 0.155 | 0.012 |

of the phenomenological freeze-out conditions. Indeed, in these studies we have adopted the freeze-out line from Ref. [9], whereas in [56] the \(C_2/C_1\) was calculated along the line from Refs. [85, 86].

The higher-order cumulant ratios of the net-proton number fluctuations are also shown in Fig. 2. The model introduced in [56] exhibits clearly pronounced non-monotonic structures of higher-order cumulant ratios for \(g = 3\) and \(5\). We note, that while the ratios \(C_2/C_1\) and \(C_4/C_2\) are independent of the choice made for the orientation of the \(h\)-axis, the behavior of \(C_3/C_2\) is sensitive to such choice. In [56], the positive direction of the \(h\)-axis was defined to point upward towards larger values of \(T\). Here, we choose the opposite direction as discussed in Sec. 1.4 such that \(g\sigma\) is positive along the chemical freeze-out curve.

As seen in Fig. 2, in the present model calculations, the non-monotonic structures of the higher-order cumulant ratios become strongly suppressed even for \(g = 5\). In fact, for the considered setup, the model results show rather small deviations from the non-critical baseline. Moreover, in contrast to [56], the behavior of \(C_3/C_2\) in the present model does not depend on the orientation of \(h\). This is because the combined \(\theta\)-dependence in \(C_3\), see Eq. (22), of the critical mode fluctuations \(<(V\delta\sigma)^3>\) and the factor \(\sigma^3\) is even. Thus, although \(h\) is an odd function in \(\theta\) as seen from Eqs. (26) and (27), a re-orientation of the \(h\)-axis would have no effect.

The substantial reduction of the critical signal in the net-proton number cumulant ratios seen in our model results is a consequence of (i) the reduced scaling of the critical contributions to \(C_{n=2,3,4}\) in Eqs. (21)-(23) and (ii) the magnitude of the factors \((g\sigma)^n\). As a result of the phenomenological implementation in Eq. (20), the scaling of \(C_n\) at the CP is weakened by an additional \(n\beta\) factor in the critical exponents. Moreover, the factor \(g\sigma\) differs from the vacuum proton mass \(m_p\) in Eq. (18) as employed in [56]. In the present calculations, \(g\sigma\) is of the order of 0.2 – 0.3 GeV for most \(\sqrt{s}\). Its actual values depend on the parameters in the magnetic equation of state, most notably on the value of \(M_0\) (see Sec. 1.3), and the mapping between spin model and QCD (see Sec. 1.4). These values receive support from recent works on the origin of the baryon masses in both lattice QCD [87, 88] and effective models based on parity doubling [89, 90]. There, the baryon masses are found to be given to a large extent by \(\sigma\)-independent contributions.
The properties of different cumulant ratios shown in Fig. 2 can lead to the conclusion, that the monotonic beam energy dependence seen experimentally in $C_2/C_1$ together with the non-monotonicity in the higher-order cumulant ratios cannot be explained simultaneously by a model that (i) includes critical mode fluctuations through the coupling of $\sigma$ with the particles and (ii) obeys the connection between the chiral and net-baryon number susceptibilities observed in effective models [60, 61]. However, one notes that the model results depend not only on the values of the coupling $g$ but also on the non-universal details of the mapping between QCD and the spin model discussed in Sec. 1.4. One of them is the unknown distance of the QCD critical point from the chemical freeze-out conditions at which the fluctuations are determined. To study this effect we keep the chemical freeze-out conditions fixed but vary the location of the critical point in the QCD phase diagram as summarized in Tab. 1 and depicted in Fig. 3.

In Fig. 4 we show the net-proton number fluctuations along the phenomenological freeze-out line at fixed value of the coupling $g = 5$ and assuming different locations of the CP. As evident from this figure, moving the critical point closer to the chemical freeze-out curve leads to an increase of non-monotonic structures in the net-proton number cumulant ratios. While deviations from the non-critical baseline (black dotted lines) remain moderately weak in $C_2/C_1$, they become more pronounced with increasing order of the fluctuations in $C_n/C_2$ ratios.

In Fig. 5 we show the influence of the critical point located at the closest distance $CP_3$ on the energy dependence of the net-proton number fluctuations for different values of the coupling $g$. As can be seen, all cumulant ratios depend strongly on the actual value of $g$. This behavior is expected from Eqs. (21)-(23) and Eq. (19) since cumulants $C_n$ scale as $g^{2n}$. This differs from the model introduced in [56] where, as seen from Eqs. (12)-(15), they scale only as $g^n$. When compared to the STAR data, the results for $C_2/C_1$ and $C_4/C_2$ shown in Fig. 5 are in qualitative agreement with data, given the
uncertainties in the model assumptions. However, the \( C_3/C_2 \) ratio also increases beyond the non-critical baseline towards the lower beam energies in contrast to the STAR data.

From the results shown in Figs. 2, 4 and 5 it is clear that for small couplings \( g \approx 3 \) deviations from the non-critical baseline are negligible in all cumulant ratios irrespective of the studied location of the critical point. By increasing \( g \), non-monotonic structures in the \( \sqrt{s} \)-dependence of the net-proton cumulant ratios develop and are sensitive to the relative distance between the critical point and the chemical freeze-out curve. This behavior is also stronger in the higher-order cumulant ratios.

It is therefore conceivable that, by an appropriate choice of the location of the CP and the model parameters, it is possible to describe the energy dependence of some ratios of the net-proton number fluctuations as seen in the preliminary STAR data. Consequently, a rather strong increase of the \( C_4/C_2 \) ratio beyond the HRG baseline and the smooth dependence of \( C_2/C_1 \) observed in heavy-ion collisions at energies \( \sqrt{s} < 20 \) GeV could be due to the contribution from the CP located near the chemical freeze-out line. However, in such a case the \( C_3/C_2 \) ratio should also exceed the non-critical baseline, which is not seen experimentally.

Thus, based on the proposed model results one could conclude, that the energy dependence of \( C_4/C_2, C_3/C_2 \) and \( C_2/C_1 \) observed in heavy ion collisions at \( \sqrt{s} < 20 \)
Figure 5. (Color online) Net-proton number cumulant ratios from Eq. (10) calculated with the CP$_3$-setup for the location of the CP (see Tab. 1) and for different values of the coupling $g = 3, 4$ and $5$ (orange solid, green long-dashed and red dash-dotted lines, respectively). The preliminary STAR data for the net-proton number fluctuations [32] (squares, where the error bars contain both statistical and systematic errors) are shown for comparison. The non-critical baseline model results are shown by black dotted lines.

GeV by the STAR Collaboration does not follow the systematics expected from the contribution of the CP to the net-proton number fluctuations. This conclusion is consistent with the previous analysis of different fluctuation observables based on lattice QCD and PNJL model results [19, 20]. However, the above requires further theoretical and empirical justifications due to the current uncertainties in model assumptions and experimental data.

3. Conclusions

We have studied the influence of the QCD chiral critical point (CP) on the properties of the n-th order cumulants ($C_n$) of the net-proton number and their ratios. The results were addressed in the context of the recent data from the STAR Collaboration on the energy dependence of the net-proton number fluctuations in Au-Au collisions obtained within the Beam Energy Scan program at the Relativistic Heavy Ion Collider (RHIC).

To calculate the net-proton number cumulants we have proposed a phenomenological model where non-critical fluctuations are obtained from the hadron resonance
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gas (HRG) statistical operator, which is known to describe data on particle yields in heavy-ion collisions and the lowest-order fluctuation observables from lattice QCD. For simplicity, the baryonic sector of the HRG was approximated by contributions from primary protons and anti-protons. To quantify the role of the chiral criticality due to the CP, the phenomenological model was introduced to describe the non-analytic part of the statistical operator in which the fluctuations of the critical chiral mode $\sigma$ are coupled to the (anti-)protons. This was achieved by linking their masses to the $\sigma$ mode, as suggested by different chiral models. Consequently, the (anti-)proton mass and its momentum distribution function fluctuate on an event-by-event basis around its mean or equilibrium value, respectively.

The critical mode fluctuations were determined by applying universality class arguments between QCD and the 3-dimensional Ising spin model. We have extended the phenomenological model introduced in Ref. [56] by accounting for the critical scaling behavior of the net-baryon variance $\chi_B$ suggested by effective chiral models. There $\chi_B$ is linked to the product of the chiral susceptibility and the chiral order parameter squared.

We have found a substantial reduction of the critical mode contributions to the net-proton number fluctuations compared to the results in Ref. [56]. This was mostly due to the reduced critical scaling and the size of the proton mass modification due to the coupling to the $\sigma$ mode. This brings our results for different n-th order cumulants ($C_n$) of the net-proton number, calculated along the phenomenological chemical freeze-out line, closer to the experimental observations made by the STAR Collaboration for the energy dependence of the cumulant ratios in heavy-ion collisions at RHIC. In particular, with an appropriate choice of the model parameters and the location of the CP relative to the chemical freeze-out line, the model can reproduce the smooth energy dependence of $C_2/C_1$ and the increase and non-monotonic variation of $C_4/C_2$ towards lower beam energies, as is observed by the STAR Collaboration. However, a decrease of the $C_3/C_2$ ratio towards lower energies, seen in the STAR data, is inconsistent with the systematics expected from the contribution of the CP which would required an access of this ratio beyond the non-critical baseline. Thus, it is rather unlikely that properties observed in the low energy behavior of different ratios of the net-proton number cumulants in heavy-ion collisions are due to the existence of the critical point near the phenomenological chemical freeze-out line.

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