Sahlqvist-Type Completeness Theory for Hybrid Logic with Binder

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Abstract
In the present paper, we continue the research in [22] to develop the Sahlqvist-type completeness theory for hybrid logic with satisfaction operators and downarrow binders $L(@, \downarrow)$. We define the class of skeletal Sahlqvist formulas for $L(@, \downarrow)$ following the ideas in [10], but we follow a different proof strategy which is purely proof-theoretic, namely showing that for every skeletal Sahlqvist formula $\varphi$ and its hybrid pure correspondence $\pi$, $K_{H(@, \downarrow)} + \varphi$ proves $\pi$, therefore $K_{H(@, \downarrow)} + \varphi$ is complete with respect to the class of frames defined by $\pi$, using a restricted version of the algorithm ALBA+ defined in [22].

Keywords: completeness theory, Hilbert system, hybrid logic with binder, ALBA algorithm

1 Introduction

Hybrid logic Hybrid logics [3] have higher expressivity than modal logics where it is possible to talk about states in the model using nominals that are true at exactly one state. There are also other connectives in hybrid logic which are used to increase the expressive power, e.g. the satisfaction operator $@i \varphi$ which intuitively reads “at the world denoted by $i$, $\varphi$ is true”, and the downarrow binder $\downarrow x. \varphi$ which binds the current world and can refer to the world later in $\varphi$. In the present paper, we use $L$ to denote the language for hybrid logic with nominals, $L(\@)$ with nominals and satisfaction operators, $L(\@, \downarrow)$ with nominals, satisfaction operators and downarrow binders, and $K_{H}$, $K_{H(@)}$, $K_{H(@, \downarrow)}$ to denote their respective basic systems.

Correspondence theory Correspondence theory started as a branch of the model theory of modal logic. We say that a modal formula $\varphi$ corresponds to a first-order formula $\alpha$ if they are valid on exactly the same class of Kripke frames. Sahlqvist [17] and van Benthem [21] gave a syntactic description of certain modal formulas (later called Sahlqvist formulas) which have two nice properties: first of all, they have first-order correspondents, secondly, they axiomatize normal modal logics strongly complete with respect to the class of Kripke frames defined by them.

Correspondence and completeness theory for hybrid logic Existing literature on correspondence and completeness theory for hybrid logic is abundant, see [11, 12, 10, 11, 12, 13, 14, 13, 20, 22]. Gargov and Goranko proved that any extension of $K_{H}$ with pure axioms (formulas that contain nominals only but no propositional variables) is strongly complete. ten Cate and Blackburn [2] showed that any pure extension of $K_{H(@)}$ and $K_{H(@, \downarrow)}$ are strongly complete. ten Cate, Marx and Viana [20] proved that any extension of $K_{H(@)}$ with modal Sahlqvist formulas (with no nominals and with propositional variables only) is strongly complete, and that these two kinds of results cannot be combined in general, since there is a pure formula and a modal Sahlqvist formula which together axiomatize a Kripke-incomplete logic when added to $K_{H(@)}$. Conradi and Robinson [19] studied to what extent can these two results be combined in $L(\@)$, using algorithmic and algebraic method. Zhao [22] studies the correspondence theory for $L(\@, \downarrow)$.
Our contribution The present paper continues the study in [22] on the completeness theory in the spirit of [10], using algorithmic method, which is based on the algorithm ALBA (Ackermann Lemma Based Algorithm) [8, 5], which computes the first-order correspondents of input formulas/inequalities and is guaranteed to succeed on Sahlqvist formulas/inequalities. However, our completeness proof follows a different strategy, which is not algebraic as in [10], but purely proof-theoretic. We define the class of skeletal Sahlqvist formulas (which is a subclass of Sahlqvist formulas defined in [22]) for \( \mathcal{L}(\otimes, \downarrow) \) following the ideas in [10], show that for every skeletal Sahlqvist formula \( \varphi \) and its hybrid pure correspondence \( \pi \), \( K_{\mathcal{H}(\otimes, \downarrow)}^\pi + \varphi \) proves \( \pi \), therefore \( K_{\mathcal{H}(\otimes, \downarrow)}^\pi + \varphi \) is complete with respect to the class of frames defined by \( \pi \), using a restricted version of the algorithm ALBA\(^\dagger\) defined in [22].

Structure of the paper The structure of the paper is as follows: Section 2 presents preliminaries on hybrid logic with satisfaction operators and downarrow binders, including syntax, semantics and basic system \( K_{\mathcal{H}(\otimes, \downarrow)} \). Section 3 provides ingredients on algorithmic correspondence theory. Section 4 defines based Algorithm) [8, 5], which computes the first-order correspondents of input formulas/inequalities \( \phi \) and its hybrid pure correspondence \( \pi \), \( K_{\mathcal{H}(\otimes, \downarrow)}^\pi + \phi \) proves \( \pi \), therefore \( K_{\mathcal{H}(\otimes, \downarrow)}^\pi + \phi \) is complete with respect to the class of frames defined by \( \pi \), using a restricted version of the algorithm ALBA\(\dagger\) defined in [22].

2 Preliminaries on hybrid logic with binder

In the present section we collect the preliminaries on hybrid logic with binder. For more details, see [3, Chapter 14] and [19].

2.1 Language and syntax

Definition 2.1. Given three pairwise disjoint countably infinite sets \( \text{Prop} \) of propositional variables, \( \text{Svar} \) of state variables, \( \text{Nom} \) of nominals, the hybrid language \( \mathcal{L}(\otimes, \downarrow) \) is defined as follows:

\[
\varphi ::= p \mid x \mid i \mid \bot \mid T \mid \neg \varphi \mid \varphi \lor \varphi \mid \varphi \land \varphi \mid \varphi \rightarrow \varphi \mid \Diamond \varphi \mid \Box \varphi \mid @x \varphi \mid @i \varphi \mid \downarrow \varphi,
\]

where \( p \in \text{Prop}, x \in \text{Svar}, i \in \text{Nom} \).

We use \( \bar{p} \) to denote a set of propositional variables and \( \varphi(\bar{p}) \) to indicate that the propositional variables that occur in \( \varphi \) are all in \( \bar{p} \). We use \( \text{Prop}(\varphi) \) to denote the set of all propositional variables occurring in \( \varphi \). We say that a formula is pure if it contains no propositional variables. We define free and bound occurrences of state variables as usual, and say that a hybrid formula is a sentence if it contains no free occurrences of state variables. We define \( \sigma \) to be a substitution that uniformly replaces propositional variables by formulas and terms (nominals or state variables) by terms. We use \( \varphi[\theta/p] \) to denote the substitution replacing \( p \) by \( \theta \) uniformly. We also use \( \varphi[\gamma/\delta] \) to denote the replacement of some occurrences of \( \delta \) in \( \varphi \) by \( \gamma \). In the present paper we will only consider the language with one unary modality.

In the article, we will use inequalities of the form \( \varphi \leq \psi \), where \( \varphi \) and \( \psi \) are formulas, and quasi-inequalities of the form \( \varphi_1 \leq \psi_1 \land \ldots \land \varphi_n \leq \psi_n \Rightarrow \varphi \leq \psi \). We will find it easy to work with inequalities \( \varphi \leq \psi \) in place of implicative formulas \( \varphi \rightarrow \psi \) in Section 4.

2.2 Semantics

Definition 2.2. A Kripke frame is a pair \( \mathcal{F} = (W, R) \) where \( W \) is a non-empty set called the domain of \( \mathcal{F} \), \( R \) is a binary relation on \( W \) called the accessibility relation. A pointed Kripke frame is a pair \( (\mathcal{F}, w) \) where \( w \in W \). A Kripke model is a pair \( \mathcal{M} = (\mathcal{F}, V) \) such that \( V : \text{Prop} \cup \text{Nom} \rightarrow P(W) \) is a valuation on \( \mathcal{F} \) where for all nominals \( i \in \text{Nom} \), \( V(i) \subseteq W \) is a singleton.

An assignment \( g \) on \( \mathcal{M} = (W, R, V) \) is a map \( g : \text{Svar} \rightarrow W \). For any assignment \( g \), any \( x \in \text{Svar} \), any \( w \in W \), we define \( g^x_w \) (the \( x \)-variant of \( g \)) as follows: \( g^x_w(x) = w \) and \( g^x_w(y) = g(y) \) for all \( y \in \text{Svar} \setminus \{x\} \).

Now the satisfaction relation is given as follows: for any Kripke model \( \mathcal{M} = (W, R, V) \), assignment \( g \) on \( \mathcal{M} \), \( w \in W \),
For any formula \( \varphi \), we use \( \llbracket \varphi \rrbracket_{M,g} = \{ w \in W \mid M, g, w \models \varphi \} \) to denote the truth set of \( \varphi \) in \( (M, g) \). \( \varphi \) is globally true on \( (M, g) \) (notation: \( M, g \models \varphi \)) if \( M, g, w \models \varphi \) for every \( w \in W \). \( \varphi \) is valid on a Kripke frame \( F \) (notation: \( F \models \varphi \)) if \( \varphi \) is globally true on \( (F, V, g) \) for each valuation \( V \) and each assignment \( g \).

For the semantics of inequalities and quasi-inequalities, they are given as follows:

- \( M, g, w \models \varphi \leq \psi \) iff \( \text{for all } w \in W, \text{ if } M, g, w \models \varphi, \text{ then } M, g, w \models \psi. \)

- \( M, g, w \models \varphi_1 \leq \varphi_2 \leq \cdots \leq \psi_n \Rightarrow \varphi \leq \psi \) iff \( M, g, w \models \varphi_i \leq \psi_i \) for all \( 1 \leq i \leq n \).

The definitions of validity are similar to formulas. It is easy to see that \( M, g, \models \varphi \leq \psi \) iff \( M, g, \models \varphi \rightarrow \psi \).

### 2.3 Hilbert system

The axioms and inference rules of the Hilbert system \( K_{\mathcal{H}(\alpha, \downarrow)} \) of \( \mathcal{L}(\alpha, \downarrow) \) is given as follows (see [19]):

- (CT) \( \vdash \varphi \) for all classical tautologies \( \varphi \)

- (Dual) \( \vdash \diamond p \leftrightarrow \neg \square \neg p \)

- (K) \( \vdash \square(p \rightarrow q) \rightarrow (\square p \rightarrow \square q) \)

- (K\(\alpha\)) \( \vdash \alpha_1(p \rightarrow q) \rightarrow (\alpha_1 p \rightarrow \alpha_1 q) \)

- (Selfdual) \( \vdash \neg \alpha_1 p \leftrightarrow \alpha_1 \neg p \)

- (Ref) \( \vdash \alpha_1 \top \)

- (Intro) \( \vdash \top \land p \rightarrow \alpha_1 p \)

- (Back) \( \vdash \diamond \alpha_1 p \rightarrow \alpha_1 p \)

- (Agree) \( \vdash \alpha_1 \alpha_1 p \rightarrow \alpha_1 p \)

- (DA) \( \vdash \alpha_1(\downarrow x. \varphi \leftrightarrow \varphi[i/x]) \)

- (Name\(\downarrow\)) \( \vdash \downarrow x. \alpha x. \varphi \rightarrow \varphi, \text{ if } x \text{ does not occur in } \varphi \)

- (BG\(\downarrow\)) \( \vdash \alpha_1 \downarrow x. \alpha_1 \downarrow \alpha_1 x \)
(MP) If \( \vdash \varphi \rightarrow \psi \) and \( \vdash \varphi \) then \( \vdash \psi \)

(SB) If \( \vdash \varphi \) then \( \vdash \varphi^\sigma \), provided that \( \sigma \) is safe for \( \varphi \)

(Nec) If \( \vdash \varphi \) then \( \vdash \square \varphi \)

(Nec\(_\downarrow\)) If \( \vdash \varphi \) then \( \vdash \downarrow_i \varphi \)

(Nec\(_\downarrow\downarrow\)) If \( \vdash \varphi \) then \( \vdash \downarrow \downarrow \varphi \)

We use \( K_{\mathcal{H}(@,\downarrow)} + \Sigma \) to denote the logic system containing all axioms of \( K_{\mathcal{H}(@,\downarrow)} \) and \( \Sigma \), closed under the rules of \( K_{\mathcal{H}(@,\downarrow)} \). We use \( \vdash_{\Sigma} \varphi \) to denote that \( \varphi \) is a theorem of \( K_{\mathcal{H}(@,\downarrow)} + \Sigma \). When \( \Sigma \) is empty, we use the notation \( \vdash \varphi \).

We use \( \Gamma \vdash_{\Sigma} \varphi \) to denote that there are \( \gamma_1, \ldots, \gamma_n \in \Gamma \) such that \( \vdash_{\Sigma} \gamma_1 \land \ldots \land \gamma_n \rightarrow \varphi \). Given a frame class \( \mathcal{F} \), we use \( \Gamma \vDash_{\mathcal{F}} \varphi \) to denote that for any frame \( \mathcal{F} = (W, R) \in \mathcal{F} \), any valuation \( V \) and assignment \( g \) on \( W \), any point \( w \in W \), if \( \mathcal{F}, V, g, w \vDash_{\gamma} \varphi \) for all \( \gamma \in \Gamma \), then \( \mathcal{F}, V, g, w \vDash_{\gamma} \varphi \).

We have the following derived rules and theorems for any \( \Sigma \), which will be useful in Section 7:

(Trans) \( \vdash_{\Sigma} @_j \varphi \land @_i \varphi \rightarrow @_i \varphi \)

(Sym) \( \vdash_{\Sigma} @_i \rightarrow @_j i \)

\( \vdash_{\Sigma} @_i (\beta \land \gamma) \leftrightarrow @_i \beta \land @_i \gamma \)

\( \vdash_{\Sigma} \neg @_i (\alpha \lor \beta) \leftrightarrow \neg @_i \alpha \land \neg @_i \beta \)

\( \vdash_{\Sigma} @_j \alpha \land @_i j \rightarrow @_i \alpha \)

\( \vdash_{\Sigma} @_i p @_j p \leftrightarrow @_j p \)

\( \vdash_{\Sigma} @_i \downarrow x. \varphi \leftrightarrow @_i \varphi[1/x] \)

\( \vdash_{\Sigma} @_i \alpha \land \neg @_k \beta \land \neg @_i (j \rightarrow \neg k) \rightarrow \neg @_i (\alpha \rightarrow \beta) \)

\( \vdash_{\Sigma} @_i (i_1 \lor \ldots \lor i_n) \) for \( 1 \leq i \leq n \)

\( \vdash_{\Sigma} \neg @_i (\neg i_1 \land \ldots \land \neg i_n) \) for \( 1 \leq i \leq n \)

(Res) If \( \vdash_{\Sigma} \gamma \leftrightarrow \delta \), then \( \vdash_{\Sigma} \varphi \leftrightarrow \varphi[\gamma/\delta] \).

**Definition 2.3** (Soundness and Strong Completeness). We say that \( K_{\mathcal{H}(@,\downarrow)} + \Sigma \) is *sound* with respect to \( \mathcal{F} \), if \( \Gamma \vdash_{\Sigma} \varphi \) implies that \( \Gamma \vDash_{\mathcal{F}} \varphi \). We say that \( K_{\mathcal{H}(@,\downarrow)} + \Sigma \) is *strongly complete* with respect to \( \mathcal{F} \), if \( \Gamma \vDash_{\mathcal{F}} \varphi \) implies that \( \Gamma \vdash_{\Sigma} \varphi \).

**Theorem 2.4** (Theorem 9.4.4 in [19]). For any set \( \Sigma \) of pure \( \mathcal{L}(@,\downarrow) \)-sentences, \( K_{\mathcal{H}(@,\downarrow)} + \Sigma \) is sound and strongly complete with respect to the class of frames defined by \( \Sigma \).

### 3 Ingredients of algorithmic correspondence

In this paper, we give a restricted version \( \text{ALBA}^{\text{Restricted}}_{@,\downarrow} \) of the correspondence algorithm \( \text{ALBA}^{\downarrow} \) for hybrid logic with binder defined in [22]. The algorithm \( \text{ALBA}^{\text{Restricted}}_{@,\downarrow} \) transforms the input hybrid formula \( \varphi \rightarrow \psi \) into an equivalent set of pure quasi-inequalities which does not contain occurrences of propositional variables\(^1\).

Since the purpose of the algorithm is to give Hilbert-style proof of the hybrid pure correspondence \( \pi \) of the input skeletal Sahlqvist formula \( \varphi \rightarrow \psi \), the ingredients we will give is different from the one in [22]. They can be listed as follows:

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1. We say that a substitution is *safe* if it does not make free occurrences of state variables \( x \) to be substituted into the scope of \( \downarrow \).
2. Notice that here we do not use the expanded modal language as in [22].
• An algorithm $\text{ALBA}^\downarrow_{\text{Restricted}}$, which transforms a given hybrid formula $\varphi \to \psi$ into equivalent pure quasi-inequalities $\text{Pure}(\varphi \to \psi)$;

• A syntactically identified class of inequalities on which the algorithm is successful;

• A translation of the inequalities and quasi-inequalities involved in the algorithm into hybrid formulas;

• A proof that for a given skeletal Sahlqvist formula $\varphi \to \psi$, for each step of the execution of $\text{ALBA}^\downarrow_{\text{Restricted}}$, the translation of the resulting quasi-inequality is provable in $K_{H(\partial, \mathbf{i})} + (\varphi \to \psi)$, therefore $\pi$ is provable in $K_{H(\partial, \mathbf{i})} + (\varphi \to \psi)$.

In the remainder of the paper, we will give the definition of skeletal Sahlqvist inequalities (Section 4), define a modified version of the algorithm $\text{ALBA}^\downarrow_{\text{Restricted}}$ (Section 5), and success on Sahlqvist inequalities (Section 6). We will give a translation of the inequalities and quasi-inequalities involved in the algorithm into hybrid formulas as well as prove that for a given skeletal Sahlqvist formula $\varphi \to \psi$, for each step of the execution of $\text{ALBA}^\downarrow_{\text{Restricted}}$, the translation of the resulting quasi-inequality is provable in $K_{H(\partial, \mathbf{i})} + (\varphi \to \psi)$ (Section 7).

4 Skeletal Sahlqvist inequalities

In the present section, since we will use the algorithm $\text{ALBA}^\downarrow_{\text{Restricted}}$ which is based on the classification of nodes in the signed generation trees of hybrid modal formulas, we will use the unified correspondence style definition (cf. [7, page 346]) to define skeletal Sahlqvist formulas. We will collect all the necessary preliminaries on skeletal Sahlqvist formulas. For the sake of the algorithm, we will find it convenient to use inequalities $\varphi \leq \psi$ instead of implicative formulas $\varphi \to \psi$.

**Definition 4.1** (Order-type). (cf. [7] page 346) For any $n$-tuple of propositional variables $(p_1, \ldots, p_n)$, an order-type of $(p_1, \ldots, p_n)$ is an element $\varepsilon$ in $\{1, \partial\}^n$. We call $p_i$ has order-type 1 with respect to $\varepsilon$ if $\varepsilon_i = 1$, and write $\varepsilon(i) = 1$ or $\varepsilon(p_i) = 1$; we call $p_i$ has order-type $\partial$ with respect to $\varepsilon$ if $\varepsilon_i = \partial$, and write $\varepsilon(i) = \partial$ or $\varepsilon(p_i) = \partial$. We use $\varepsilon^\partial$ to denote the opposite order-type of $\varepsilon$ where $\varepsilon^\partial(p_i) = 1$ (resp. $\varepsilon^\partial(p_i) = \partial$) if $\varepsilon(p_i) = \partial$ (resp. $\varepsilon(p_i) = 1$).

**Definition 4.2** (Signed generation tree). (cf. [9] Definition 4) The positive (resp. negative) generation tree of any given hybrid formula $\varphi$ is defined as follows:

We first label the root of the generation tree of $\varphi$ with + (resp. −), then label the children nodes as below:

- If a node is labelled with $\lor, \land, \Box, \Diamond, \downarrow, x$, then label the same sign to its children nodes;
- If a node is labelled with $\neg$, then label the opposite sign to its child node;
- If a node is labelled with $\rightarrow$, then label the opposite sign to the first child node and the same sign to the second child node;
- If a node is labelled with $\emptyset$, then label the same sign to the second child node (notice that we do not label the first child node with nominal or state variable).

Nodes in signed generation trees are positive (resp. negative) if they are signed + (resp. −).

**Example 4.3.** The positive generation tree of $+\Diamond(p \lor \neg q) \rightarrow \Diamond q$ is given in Figure 1.

We will use signed generation trees in the inequalities $\varphi \leq \psi$, where we use the positive generation tree $+\varphi$ and the negative generation tree $-\psi$. We call an inequality $\varphi \leq \psi$ uniform in a variable $p_i$ if all occurrences of $p_i$ in $+\varphi$ and $-\psi$ have the same sign, and call $\varphi \leq \psi$ $\varepsilon$-uniform in an array $\bar{p}$ if $\varphi \leq \psi$ is uniform in $p_i$, occurring with the sign indicated by $\varepsilon$ (i.e., $p_i$ has the sign + (resp. −) if $\varepsilon(p_i) = 1$ (resp. $\partial$)), for each $p_i$ in $\bar{p}$. 
For any order-type \( \varepsilon \) over \( n \), any formula \( \varphi(p_1, \ldots, p_n) \), any \( 1 \leq i \leq n \), an \( \varepsilon \)-critical node in a signed generation tree \( \ast \varphi \) (where \( \ast \in \{+, -\} \) is a leaf node \( +p_i \) (when \( \varepsilon_i = 1 \)) or \( -p_i \) (when \( \varepsilon_i = \partial \)). An \( \varepsilon \)-critical branch in a signed generation tree \( \ast \varphi \) is a branch from an \( \varepsilon \)-critical node. The \( \varepsilon \)-critical branches are those which the algorithm \( \text{ALBA}^\downarrow_{\text{Restricted}} \) will solve for. We say that \( \ast \varphi \) agrees with \( \varepsilon \), and write \( \varepsilon(\ast \varphi) \), if every leaf node with a propositional variable \( p \) in the signed generation tree \( \ast \varphi \) is \( \varepsilon \)-critical.

We use \( + \psi \prec \ast \varphi \) (resp. \( - \psi \prec \ast \varphi \)) to denote that an occurrence of a subformula \( \psi \) inherits the positive (resp. negative) sign from the signed generation tree \( \ast \varphi \). We use \( \varepsilon(\gamma) \prec \ast \varphi \) (resp. \( \varepsilon'\theta(\gamma) \prec \ast \varphi \)) to denote that the signed generation subtree \( \gamma \), with the sign inherited from \( \ast \varphi \), agrees with \( \varepsilon \) (resp. \( \varepsilon'\theta \)). A propositional variable \( p \) is positive (resp. negative) in \( \varphi \) if \( +p \prec +\varphi \) (resp. \( -p \prec +\varphi \)) for all occurrences of \( p \) in \( \varphi \).

**Definition 4.4.** (cf. [9, Definition 5]) Nodes in signed generation trees are called skeletal nodes, according to Table 1. For the names of skeletal nodes, see [10, Remark 3.24]. A branch in a signed generation tree is called a skeletal branch if it consists (apart from variable nodes) of skeletal nodes only.

**Definition 4.5** (skeletal Sahlqvist inequalities and formulas). (cf. [10, Definition 2.4]) For any order-type \( \varepsilon \), the signed generation tree \( \ast \varphi(p_1, \ldots, p_n) \) is \( \varepsilon \)-skeletal Sahlqvist if for all \( 1 \leq i \leq n \), every \( \varepsilon \)-critical branch with leaf \( p_i \) is skeletal. An inequality \( \varphi \leq \psi \) is \( \varepsilon \)-skeletal Sahlqvist if the signed generation trees \( +\varphi \) and \( -\psi \) are \( \varepsilon \)-skeletal Sahlqvist. An inequality \( \varphi \leq \psi \) is skeletal Sahlqvist if it is \( \varepsilon \)-skeletal Sahlqvist for some \( \varepsilon \). An implicative formula \( \varphi \rightarrow \psi \) is skeletal Sahlqvist if \( \varphi \leq \psi \) is skeletal Sahlqvist.

**Example 4.6.** Here we give an example of a skeletal Sahlqvist inequality for the order-type \( \varepsilon = (1, 1) \), where the skeletal nodes are marked with \( S \), and the leaf nodes of \( \varepsilon \)-critical branches are marked with \( C \). It is clear that the branch from \( +p_2 \) to \( +\land \) and the branch from \( +p_1 \) to \( +\land \) are both \( \varepsilon \)-critical and skeletal.

### 5 The algorithm \( \text{ALBA}^\downarrow_{\text{Restricted}} \)

In the present section, we define the modified version of the correspondence algorithm \( \text{ALBA}^\downarrow_{\text{Restricted}} \) for hybrid logic with binder, which is a partial version of the algorithm \( \text{ALBA}^\downarrow \) in [22]. First of all, the algorithm receives an input formula \( \varphi \rightarrow \psi \) and transforms it into an inequality \( \varphi \leq \psi \). Then the algorithm goes in three steps.

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3 This name comes from [10].
Figure 2: (1,1)-skeletal Sahlqvist inequality $\Diamond p_1 \land p_2 \leq \Diamond \Box p_1 \lor \Diamond \Box p_2$

1. Preprocessing and first approximation:

In the generation tree of $+\varphi$ and $-\psi$.4

(a) Apply the distribution rules:

i. Push down $+\Diamond, +\downarrow x, +\Box_1, +\Box_x, -\Diamond, -\land, -\rightarrow$ by distributing them over nodes labelled with $+\lor$ which are skeletal nodes (see Figure 3; notice that here we treat $\Box_1$ and $\Box_x$ as unary modality with only the right branch as the input, and $\triangle \in \{\Diamond, \downarrow x, \Box_1, \Box_x\}$, and

ii. Push down $-\Box, -\downarrow x, -\Box_1, -\Box_x, +\land, +\lor, +\rightarrow$ by distributing them over nodes labelled with $-\land$ which are skeletal nodes (see Figure 4 here $\triangle \in \{\Box, \downarrow x, \Box_1, \Box_x\}$).

(b) Apply the splitting rules:

\[
\alpha \lor \beta \leq \gamma \\
\alpha \leq \gamma \beta \leq \gamma \\
\alpha \leq \beta \land \gamma \\
\alpha \leq \beta \land \gamma
\]

4The algorithm relies on signed generation trees in Section 4. We will identify a signed formula with its signed generation tree.
The reduction stage

Here the monotone and antitone variable elimination rules eliminate propositional variables \( p \) where the inequality is semantically monotone or antitone with respect to \( p \).

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5Here the monotone and antitone variable elimination rules eliminate propositional variables \( p \) where the inequality is semantically monotone or antitone with respect to \( p \).

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Figure 4: Distribution rules for \(-\land\)

(c) Apply the monotone and antitone variable-elimination rule\[5\]

\[
\begin{align*}
\frac{\alpha(p) \leq \beta(p)}{\alpha(\bot) \leq \beta(\bot)} & \quad \frac{\beta(p) \leq \alpha(p)}{\beta(\top) \leq \alpha(\top)}
\end{align*}
\]

for \( \beta(p) \) positive in \( p \) and \( \alpha(p) \) negative in \( p \).

We denote by \texttt{Preprocess}(\( \varphi \leq \psi \)) the finite set \( \{ \varphi_i \leq \psi_i \}_{i \in I} \) of inequalities obtained after applying the previous rules exhaustively. Then we apply the following first approximation rule to every inequality in \texttt{Preprocess}(\( \varphi \leq \psi \)):

\[
\varphi_i \leq \psi_i \quad \frac{\varphi_i \leq \psi_i}{i_0 \leq \varphi_i} \quad \frac{\psi_i \leq \neg i_1}{\psi_i \leq \neg i_1}
\]

Here, \( i_0 \) and \( i_1 \) are fresh nominals. Now we get a set of sets of inequalities \( \{ i_0 \leq \varphi_i, \psi_i \leq \neg i_1 \}_{i \in I} \). We call the set \( \{ i_0 \leq \varphi_i, \psi_i \leq \neg i_1 \} \) system.

2. The reduction stage: In this stage, for each \( \{ i_0 \leq \varphi_i, \psi_i \leq \neg i_1 \} \), we apply the following rules to prepare for eliminating all the proposition variables in \( \{ i_0 \leq \varphi_i, \psi_i \leq \neg i_1 \} \):

(a) Substage 1: Decomposing the skeletal branch In the current substage, the following rules are applied to decompose the skeletal branches of the signed skeletal Sahlqvist formula:

i. Splitting rules:

\[
\begin{align*}
i \leq \beta \land \gamma & \quad i \leq \beta \land \gamma & \quad \alpha \lor \beta \leq \neg i
\end{align*}
\]

ii. Approximation rules:

\[
\begin{align*}
i \leq \Diamond \alpha & \quad j \leq \alpha & \quad i \leq \Diamond j & \quad x \leq \Diamond \alpha & \quad \Box \alpha \leq \neg i & \quad \Box \alpha \leq \neg x
\end{align*}
\]

\[
\begin{align*}
i \leq \Diamond j & \quad x \leq \Diamond j & \quad \Box j \alpha \leq \neg i & \quad \Box j \alpha \leq \neg x
\end{align*}
\]

\[
\begin{align*}
i \leq \Diamond j & \quad x \leq \Diamond j & \quad \Box j \alpha \leq \neg i & \quad \Box j \alpha \leq \neg x
\end{align*}
\]
**Substage 2: The Ackermann stage**

In the present substage, we compute the minimal/maximal valuations for propositional variables and use the Ackermann rules to eliminate all the propositional variables. The two rules are the core of ALBA\textsubscript{Restricted}, since their applications eliminate propositional variables. In fact, the previous substage aims at reaching a shape where the Ackermann rules can be applied. Notice that the Ackermann rules are executed on the whole set of inequalities rather than on a single inequality.

The right-handed Ackermann rule:

\[
\begin{align*}
\{ & i_1 \leq p \\
& \vdots \\
& i_n \leq p \\
& j_1 \leq \gamma_1 \\
& j_1 \leq \gamma_1 \\
& j_m \leq \gamma_m \\
& \beta_1 \leq -k_1 \\
& \vdots \}
\end{align*}
\]

is replaced by

\[
\begin{align*}
\{ & j_1 \leq \gamma_1[(i_1 \lor \ldots \lor i_n)/p] \\
& \vdots \\
& j_m \leq \gamma_m[(i_1 \lor \ldots \lor i_n)/p] \\
& \beta_1[(i_1 \lor \ldots \lor i_n)/p] \leq -k_1 \\
& \vdots \\
& \beta_k[(i_1 \lor \ldots \lor i_n)/p] \leq -k_k \\
& \beta_k \leq -k_k
\end{align*}
\]

where each $\beta_i$ is positive, and each $\gamma_j$ negative in $p$.

The left-handed Ackermann rule:

\[
\begin{align*}
& p \leq -i_1 \\
& \vdots \\
& p \leq -i_n \\
& j_1 \leq \gamma_1 \\
\end{align*}
\]

is replaced by

\[
\begin{align*}
& j_1 \leq \gamma_1[(-i_1 \land \ldots \land -i_n)/p] \\
& \vdots \\
& j_m \leq \gamma_m[(-i_1 \land \ldots \land -i_n)/p] \\
& \beta_1[(-i_1 \land \ldots \land -i_n)/p] \leq -k_1 \\
& \vdots \\
& \beta_m[(-i_1 \land \ldots \land -i_n)/p] \leq -k_k \\
& \beta_k \leq -k_k
\end{align*}
\]

where each $\beta_i$ is negative, and each $\gamma_j$ positive in $p$.

\[\text{In the Ackermann stage, for the sake of simplicity, we use } I \text{ to denote both nominals and state variables, since their behaviours at this stage are essentially the same.}\]
3. **Output:** If in the previous stage, for some \( \{i_0 \leq \varphi_i, \psi_i \leq \neg i_1 \} \), the algorithm gets stuck, i.e. some propositional variables cannot be eliminated by the reduction rules, then the algorithm stops and output “failure”. Otherwise, each initial tuple \( \{i_0 \leq \varphi_i, \psi_i \leq \neg i_1 \} \) of inequalities after the first approximation has been reduced to a set of pure inequalities \( \text{Reduce}(\varphi_i \leq \psi_i) \), and then the output is a set of pure quasi-inequalities \( \{\&\text{Reduce}(\varphi_i \leq \psi_i) \Rightarrow i_0 \leq \neg i_1 : \varphi_i \leq \psi_i \in \text{Preprocess}(\varphi \leq \psi)\} \). Finally we uniformly substitute all free occurrences of state variables by fresh nominals, and denote the set of pure quasi-inequalities \( \text{Pure}(\varphi \rightarrow \psi) \).

Since the algorithm \( \text{ALBA}_{\text{Restricted}}^\downarrow \) is a restricted version of the algorithm \( \text{ALBA}^\downarrow \), its soundness follows from the soundness of \( \text{ALBA}^\downarrow \).

**Theorem 5.1** (Soundness of the algorithm). If \( \text{ALBA}_{\text{Restricted}}^\downarrow \) runs successfully on \( \varphi \rightarrow \psi \) and outputs \( \text{Pure}(\varphi \leq \psi) \), then for any Kripke frame \( F = (W, R) \),

\[
F \models \varphi \rightarrow \psi \quad \text{iff} \quad F \models \text{Pure}(\varphi \rightarrow \psi).
\]

**Remark 5.2.** The special feature of the restricted version of the algorithm \( \text{ALBA}_{\text{Restricted}}^\downarrow \) compared with \( \text{ALBA}^\downarrow \) in [22] is that there is no expanded hybrid language needed in \( \text{ALBA}_{\text{Restricted}}^\downarrow \), and there is no tense operators needed due to the fact that we do not need most of the residuation rules in \( \text{ALBA}^\downarrow \) except for \( \neg \). Another feature of \( \text{ALBA}_{\text{Restricted}}^\downarrow \) is that during Stage 2, for each inequality involved, they are of the form \( i \leq \gamma \), \( x \leq \gamma \), \( \gamma \leq \neg i \) or \( \gamma \leq \neg x \), which means that they can be equivalently translated into hybrid formulas of the form \( \Diamond i \gamma \), \( \Diamond x \gamma \), \( \neg \Diamond i \gamma \) or \( \neg \Diamond x \gamma \), as we can see in Section 6 and 7.

### 6 Success of \( \text{ALBA}_{\text{Restricted}}^\downarrow \)

In the present section we show that \( \text{ALBA}_{\text{Restricted}}^\downarrow \) succeeds on all skeletal Sahlqvist inequalities. The proof is similar to [22, Section 7], but we will stress on the special shape of the inequalities involved in the execution of the algorithm.

**Theorem 6.1.** \( \text{ALBA}_{\text{Restricted}}^\downarrow \) succeeds on all skeletal Sahlqvist formulas.

**Definition 6.2** (Definite \( \varepsilon \)-skeletal Sahlqvist inequality). Given an order-type \( \varepsilon \) and \( * \in \{-, +\} \), the signed generation tree \( * \varphi \) is definite \( \varepsilon \)-skeletal Sahlqvist if it is \( \varepsilon \)-skeletal Sahlqvist and there is no \( +\vee, -\wedge \) occurring on an \( \varepsilon \)-critical branch. An inequality \( \varphi \leq \psi \) is definite \( \varepsilon \)-skeletal Sahlqvist if \( +\varphi \) and \( -\psi \) are both definite \( \varepsilon \)-skeletal Sahlqvist.

**Lemma 6.3.** Let \( \{\varphi_i \leq \psi_i\}_{i \in I} = \text{Preprocess}(\varphi \leq \psi) \) obtained by exhaustive application of the rules in Stage 1 on an input \( \varepsilon \)-skeletal Sahlqvist inequality \( \varphi \leq \psi \). Then each \( \varphi_i \leq \psi_i \) is a definite \( \varepsilon \)-skeletal Sahlqvist inequality.

**Proof.** The proof is essentially the same as in [22, Lemma 7.3].

**Lemma 6.4.** Given inequalities \( i_0 \leq \varphi_i \) and \( \psi_i \leq \neg i_1 \) obtained from Stage 1 where \( +\varphi_i \) and \( -\psi_i \) are definite \( \varepsilon \)-skeletal Sahlqvist, by applying the rules in Substage 1 of Stage 2 exhaustively, the inequalities obtained are in one of the following forms:

1. pure inequalities of the form \( i \leq \gamma \), \( x \leq \gamma \), \( \gamma \leq \neg i \) or \( \gamma \leq \neg x \), where \( \gamma \) is pure;
2. inequalities of the form \( i \leq p \) or \( x \leq p \) where \( \varepsilon(p) = 1 \);
3. inequalities of the form \( p \leq \neg i \) or \( p \leq \neg x \) where \( \varepsilon(p) = \partial \);
4. inequalities of the form \( i \leq \delta \) or \( x \leq \delta \) where \( +\delta \) is \( \varepsilon\)-uniform;
5. inequalities of the form \( \delta \leq \neg i \) or \( \delta \leq \neg x \) where \( -\delta \) is \( \varepsilon\)-uniform.
Proof. The proof is similar to [22] Lemma 7.5. The rules in the Substage 1 of Stage 2 treat skeletal nodes in \(+\varphi_i\) and \(¬\psi_i\), except \(+\lor_i\) and \(¬\land_i\). For each rule, without loss of generality, we suppose that we start with an inequality of the form \(i \leq \alpha\). By applying the rules in Substage 1 of Stage 2, the inequalities we obtain are either a pure inequality (i.e. without propositional variables), or an inequality in which the left-hand side (resp. right-hand side) is \(i \lor x\) (resp. \(¬i \lor ¬x\)), and the other side of the inequality is a formula \(\alpha'\) that is a subformula of \(\alpha\), such that \(\alpha'\) has one root connective less than \(\alpha\). In addition, if \(\alpha'\) is on the left-hand side (resp. right-hand side) then \(¬\alpha'\) \((+\alpha')\) is definite \(\varepsilon\)-skeletal Sahlqvist.

By exhaustively applying the rules in the Substage 1 of Stage 2, we eliminate all the skeletal connectives in the \(\varepsilon\)-critical branches, so for non-pure inequalities, they become of form 2, 3, 4 or 5.

In addition, in each inequality, either the left-hand side is \(i\) or \(x\), or the right-hand side is \(¬i\) or \(¬x\), and for each step of Substage 1 of Stage 2, after the applications of the rules, the resulting inequalities still have this property. Therefore, the final pure inequalities are of the form \(i \leq \gamma\), \(x \leq \gamma\), \(\gamma \leq ¬i\) or \(\gamma \leq ¬x\), where \(\gamma\) is pure.

Lemma 6.5. Suppose we have inequalities of the form in Lemma 6.4 then the Ackermann lemmas are applicable and all propositional variables can be eliminated, and for each inequality in the system, either the left-hand side is \(i\) or \(x\), or the right-hand side is \(¬i\) or \(¬x\).

Proof. Easy observation from the syntactic requirements of the Ackermann lemmas.

Proof of Theorem 6.7. Assume we have an \(\varepsilon\)-skeletal Sahlqvist formula \(\varphi \rightarrow \psi\) as input. By Lemma 6.3, we get a set of definite \(\varepsilon\)-skeletal Sahlqvist inequalities. Then by Lemma 6.4, we get inequalities as described in Lemma 6.4. By Lemma 6.5, the inequalities are ready to apply the Ackermann rules, and therefore we can eliminate all the propositional variables and ALBA\textsubscript{restricted} succeeds on the input.

7 Completeness results

In this section, we will prove that given any skeletal Sahlqvist formula \(\varphi \rightarrow \psi\), the logic \(K_{H(\alpha, \downarrow)} + (\varphi \rightarrow \psi)\) is sound and strongly complete with respect to the class of frames defined by \(\varphi \rightarrow \psi\). Our proof strategy is as follows:

- First of all, we give a translation of each quasi-inequality \&Reduce\((\varphi_i \leq \psi_i) \Rightarrow i_0 \leq ¬i_1\) in \Pure\((\varphi \rightarrow \psi)\) into the language \(L(\alpha, \downarrow)\) which results in a set of \(L(\alpha, \downarrow)\)-formulas \(\{\pi_i \mid i \in I\}\), and we will show that \(\varphi \rightarrow \psi\) and the set \(\Pi := \{\pi_i \mid i \in I\}\) define the same class of Kripke frames.

- Secondly, we prove that each \(\pi_i\) is provable in \(K_{H(\alpha, \downarrow)} + (\varphi \rightarrow \psi)\). Therefore, since \(K_{H(\alpha, \downarrow)} + \Pi\) is sound and strongly complete with respect to the class of frames defined by \(\Pi\) (i.e. by \(\varphi \rightarrow \psi\)), we get the soundness and strong completeness of \(K_{H(\alpha, \downarrow)} + (\varphi \rightarrow \psi)\).

7.1 The translation of inequalities and quasi-inequalities into \(L(\alpha, \downarrow)\)-formulas

The key observation in the success proof in Section 6 is that in the systems in Stage 2 and the quasi-inequalities in Stage 3, for each inequality involved, either the left-hand side is \(i\) or \(x\), or the right-hand side is \(¬i\) or \(¬x\). Indeed, the inequality \(i \leq \gamma\) (resp. \(x \leq \gamma\)) is equivalent to the \(L(\alpha, \downarrow)\)-formula \(\alpha_{i\gamma}\) (resp. \(\alpha_{x\gamma}\)), and the inequality \(\gamma \leq ¬i\) (resp. \(\gamma \leq ¬x\)) is equivalent to the \(L(\alpha, \downarrow)\)-formula \(¬\alpha_{i\gamma}\) (resp. \(¬\alpha_{x\gamma}\)). Therefore, the systems obtained in Stage 2 and the quasi-inequalities obtained in Stage 3 are equivalent to a \(L(\alpha, \downarrow)\)-formula.

Definition 7.1 (Translation of inequalities and quasi-inequalities into \(L(\alpha, \downarrow)\)-formulas). We define the translation of the inequalities of the form \(i \leq \gamma\), \(x \leq \gamma\), \(\gamma \leq ¬i\), \(\gamma \leq ¬x\) into \(L(\alpha, \downarrow)\)-formulas as follows:

- \(\text{Tr}(i \leq \gamma) := \alpha_{i\gamma}\);
- \(\text{Tr}(x \leq \gamma) := \alpha_{x\gamma}\);
• $\text{Tr}(\gamma \leq -i) := -\otimes_1 \gamma$;
• $\text{Tr}(\gamma \leq -x) := -\otimes_x \gamma$.

When an inequality is of more than one of the forms above at the same time, we can take any form appearing in the list since they are equivalent.

Given a quasi-inequality $\text{Quasi}$ of the form $\text{Ineq}_i \& \ldots \& \text{Ineq}_n \Rightarrow i \leq -j$ where each of $\text{Ineq}_1, \ldots, \text{Ineq}_n$ is of the form $i \leq \chi$, $x \leq \gamma$, $\gamma \leq -i$ or $\gamma \leq -x$, we define

$$\text{Tr}(\text{Quasi}) := \text{Tr}(\text{Ineq}_1) \land \ldots \land \text{Tr}(\text{Ineq}_n) \Rightarrow -\otimes_1 j$$

Given a set $\text{QuasiSet}$ of quasi-inequalities of the form above, we define

$$\text{Tr}(\text{QuasiSet}) := \bigwedge_{\text{Quasi} \in \text{QuasiSet}} \text{Tr}(\text{Quasi}).$$

**Proposition 7.2.** For each inequality $\text{Ineq}$ of the form $i \leq \gamma$, $x \leq \gamma$, $\gamma \leq -i$ or $\gamma \leq -x$, each quasi-inequality $\text{Quasi}$ of the form described in the definition above, we have that for any Kripke model $M$, any assignment $g$ on $M$,

- $M, g \Vdash \text{Ineq}$ iff $M, g \Vdash \text{Tr}(\text{Ineq})$
- $M, g \Vdash \text{Quasi}$ iff $M, g \Vdash \text{Tr}(\text{Quasi})$
- $M, g \Vdash \text{QuasiSet}$ iff $M, g \Vdash \text{Tr}(\text{QuasiSet})$

### 7.2 Provability of the translations

**Lemma 7.3.** Given an input skeletal Sahlqvist formula $\varphi \rightarrow \psi$, during Stage 1, for each inequality $\theta \leq \chi$ produced by the algorithm, we have $\vdash_{\varphi \rightarrow \psi} \theta \rightarrow \chi$.

**Proof.** We prove by induction on the algorithm steps in Stage 1 that for each inequality $\theta \leq \chi$ produced by the algorithm, we have $\vdash_{\varphi \rightarrow \psi} \theta \rightarrow \chi$.

- For the basic step, obviously $\vdash_{\varphi \rightarrow \psi} \varphi \rightarrow \psi$.
- For the distribution rules, we have that the following equivalences are provable in $\textbf{K}_{H(\alpha, \beta)}$ (therefore in $\textbf{K}_{H(\alpha, \beta)} + (\varphi \rightarrow \psi)$), thus by the (Res) rule, for the inequality $\theta \leq \chi$ obtained by the distribution rule, we have $\vdash_{\varphi \rightarrow \psi} \theta \rightarrow \chi$.

  - $\Diamond (\alpha \lor \beta) \leftrightarrow \Diamond \alpha \lor \Diamond \beta$;
  - $\neg(\alpha \lor \beta) \leftrightarrow -\alpha \land -\beta$;
  - $(\alpha \lor \beta) \land \gamma \leftrightarrow (\alpha \land \gamma) \lor (\beta \land \gamma)$;
  - $\alpha \land (\beta \lor \gamma) \leftrightarrow (\alpha \land \beta) \lor (\alpha \land \gamma)$;
  - $\downarrow x. (\alpha \lor \beta) \leftrightarrow (\downarrow x. \alpha \lor \downarrow x. \beta)$;
  - $\otimes_1 (\alpha \lor \beta) \leftrightarrow (\otimes_1 \alpha \lor \otimes_1 \beta)$;
  - $\otimes_x (\alpha \lor \beta) \leftrightarrow (\otimes_x \alpha \lor \otimes_x \beta)$;
  - $((\alpha \lor \beta) \rightarrow \gamma) \leftrightarrow ((\alpha \rightarrow \gamma) \land (\beta \rightarrow \gamma))$;
  - $\Box (\alpha \land \beta) \leftrightarrow \Box \alpha \land \Box \beta$;
  - $\neg(\alpha \land \beta) \leftrightarrow -\alpha \lor -\beta$;
  - $(\alpha \land \beta) \lor \gamma \leftrightarrow (\alpha \lor \gamma) \land (\beta \lor \gamma)$;
  - $\alpha \lor (\beta \land \gamma) \leftrightarrow (\alpha \lor \beta) \land (\alpha \lor \gamma)$;
  - $\downarrow x. (\alpha \lor \beta) \leftrightarrow (\downarrow x. \alpha \lor \downarrow x. \beta)$;
  - $\otimes_1 (\alpha \land \beta) \leftrightarrow (\otimes_1 \alpha \land \otimes_1 \beta)$;
  - $\otimes_x (\alpha \land \beta) \leftrightarrow (\otimes_x \alpha \land \otimes_x \beta)$;
We prove by induction on the algorithm steps in Stage 2 that for each system translated.

Proof.

Corollary 7.4. Suppose that for the input formula \( \varphi \to \psi \), in Stage 1, before the first-approximation rule, we get a set of inequalities \( \{ \theta_1 \leq \chi_1, \ldots, \theta_n \leq \chi_n \} \), then \( \vdash_{\varphi \to \psi} \theta_i \to \chi_i \) for \( 1 \leq i \leq n \).

Lemma 7.5. For each \( \theta_i \leq \chi_i \) before the first-approximation rule in Stage 1, after the first-approximation rule, we get the system \( \text{Sys} := \{ i_0 \leq \theta_i, \chi_i \leq -i_1 \} \), which corresponds to the quasi-inequality \( \text{Quasi}_i := i_0 \leq \theta_i \land \chi_i \leq -i_1 \to i_0 \leq -i_1 \), then we have that \( \vdash_{\varphi \to \psi} \text{Tr}(\text{Quasi}_i) \).

Proof. By Corollary 7.4, we have that \( \vdash_{\varphi \to \psi} \theta_i \to \chi_i \). Therefore we have the following proof in \( K_{\mathcal{H}(\alpha, \beta)} + (\varphi \to \psi) \).

Now we fix a quasi-inequality \( \text{Quasi}_i := i_0 \leq \theta_i \land \chi_i \leq -i_1 \to i_0 \leq -i_1 \). We will prove that for each system \( \text{Sys} \) obtained during the Stage 2, \( \text{Tr}(\&\text{Sys} \to i_0 \leq -i_1) \) is provable.

Lemma 7.6. Given a quasi-inequality \( \text{Quasi}_i := i_0 \leq \theta_i \land \chi_i \leq -i_1 \Rightarrow i_0 \leq -i_1 \) obtained in Stage 1, for each system \( \text{Sys} \) obtained during the Stage 2, \( \vdash_{\varphi \to \psi} \text{Tr}(\&\text{Sys} \to i_0 \leq -i_1) \).

Proof. First of all, since in each inequality in the system, either the left-hand side is a nominal/state variable, or the right-hand side is the negation of a nominal/state variable, \( \& \text{Sys} \to i_0 \leq -i_1 \) can be translated.

We prove by induction on the algorithm steps in Stage 2 that for each system \( \text{Sys} \) obtained during the Stage 2, \( \vdash_{\varphi \to \psi} \text{Tr}(\&\text{Sys} \to i_0 \leq -i_1) \) is provable.

1. For the splitting rules, obviously \( \vdash_{\varphi \to \psi} \text{Tr}(\text{Quasi}_i) \).

2. For the splitting rules, it suffices to prove that from \( \vdash_{\varphi \to \psi} \Theta_i(\beta \land \gamma) \land \alpha \to \neg \Theta_i i_1 \) one can get \( \vdash_{\varphi \to \psi} \Theta_i \beta \land \Theta_i \gamma \land \alpha \to \neg \Theta_i i_1 \) and from \( \vdash_{\varphi \to \psi} \neg \Theta_i (\alpha \lor \beta) \land \gamma \to \neg \Theta_i i_1 \) one can get \( \vdash_{\varphi \to \psi} \neg \Theta_i \alpha \land \neg \Theta_i \beta \land \gamma \to \neg \Theta_i i_1 \), which follows by the facts that \( \vdash_{\varphi \to \psi} \Theta_i (\beta \land \gamma) \leftrightarrow \Theta_i \beta \and \Theta_i \gamma \) and \( \vdash_{\varphi \to \psi} \neg \Theta_i (\alpha \lor \beta) \leftrightarrow \neg \Theta_i \alpha \and \neg \Theta_i \beta \).

3. For the approximation rule from \( i \leq \diamond \alpha \to j \leq \alpha \) and \( i \leq \diamond j \), it suffices to prove that from \( \vdash_{\varphi \to \psi} \Theta_i \alpha \land \gamma \to \neg \Theta_i i_1 \) one can get \( \vdash_{\varphi \to \psi} \Theta_i \alpha \land \Theta_i \diamond j \land \gamma \to \neg \Theta_i i_1 \), which follows by the fact that \( \vdash_{\varphi \to \psi} \Theta_i \alpha \land \Theta_i \diamond j \to \Theta_i \diamond \alpha \).

4. For the approximation rule from \( x \leq \diamond \alpha \to j \leq \alpha \) and \( x \leq \diamond j \), the proof is similar.

\[ \neg (\alpha \to \beta \land \gamma) \leftrightarrow (\alpha \to \beta) \land (\alpha \to \gamma). \]
• For the approximation rule from $\square \alpha \leq -i$ to $\alpha \leq -j$ and $\square -j \leq -i$, it suffices to prove that from $\vdash_{\varphi \rightarrow \psi} \neg \Box_i \Box \alpha \land \gamma \rightarrow \neg \Box_i \alpha_1$, one can get $\vdash_{\varphi \rightarrow \psi} \neg \Box_j \neg \Box_i \alpha \land \gamma \rightarrow \neg \Box_i \alpha_1$, which follows from the fact that $\vdash_{\varphi \rightarrow \psi} \neg \Box_j \neg \Box_i \alpha \rightarrow \neg \Box_i \alpha$. For the approximation rule from $\square \alpha \leq -x$ to $\alpha \leq -j$ and $\square -j \leq -x$, the proof is similar.

• For the approximation rule from $i \leq \Box_j \alpha$ to $j \leq \alpha$, it suffices to prove that from $\vdash_{\varphi \rightarrow \psi} \Box_i \Box_j \alpha \land \gamma \rightarrow \neg \Box_i \alpha_1$, one can get $\vdash_{\varphi \rightarrow \psi} \Box_j \alpha \land \gamma \rightarrow \neg \Box_i \alpha_1$, which follows from the fact that $\vdash_{\varphi \rightarrow \psi} \Box_j \alpha \implies \Box_i \Box_j \alpha$. For the approximation rule from $x \leq \Box_j \alpha$ to $j \leq \alpha$, from $i \leq \Box_x \alpha$ to $x \leq \alpha$, from $y \leq \Box_x \alpha$ to $x \leq \alpha$, the proof is similar.

• For the approximation rule from $\Box_j \alpha \leq -i$ to $\alpha \leq -j$, it suffices to prove that from $\vdash_{\varphi \rightarrow \psi} \Box_i \Box_j \alpha \land \gamma \rightarrow \neg \Box_i \alpha_1$, one can get $\vdash_{\varphi \rightarrow \psi} \Box_j \alpha \land \gamma \rightarrow \neg \Box_i \alpha_1$, which follows from the fact that $\vdash_{\varphi \rightarrow \psi} \Box_j \alpha \implies \Box_i \Box_j \alpha$. For the approximation rule from $\Box_j \alpha \leq -x$ to $\alpha \leq -j$, from $\Box_x \alpha \leq -i$ to $\alpha \leq -x$, from $\Box_x \alpha \leq -y$ to $\alpha \leq -x$, the proof is similar.

• For the approximation rule from $i \leq j \alpha \leq -i$ to $\alpha \leq -j$, it suffices to prove that from $\vdash_{\varphi \rightarrow \psi} \Box_j \Box_i \alpha \land \gamma \rightarrow \neg \Box_i \alpha_1$, one can get $\vdash_{\varphi \rightarrow \psi} \Box_i \Box_j \alpha \land \gamma \rightarrow \neg \Box_i \alpha_1$, which follows from the fact that $\vdash_{\varphi \rightarrow \psi} \Box_j \Box_i \alpha \implies \Box_i \Box_j \alpha$. For the approximation rule from $x \leq \Box_j \alpha \leq -y$ to $\alpha \leq -y$, the proof is similar.

• For the approximation rule from $\alpha \rightarrow \beta \leq -i$ to $j \leq \alpha$, $\beta \leq -k$ and $j \rightarrow -k \leq -i$, it suffices to prove that from $\vdash_{\varphi \rightarrow \psi} \Box_j (\alpha \rightarrow \beta) \land \gamma \rightarrow \neg \Box_i \alpha_1$, one can get $\vdash_{\varphi \rightarrow \psi} \Box_j \alpha \land \neg \Box_k \beta \land \Box_i (j \rightarrow -k) \land \gamma \rightarrow \neg \Box_i \alpha_1$, which follows from the fact that $\vdash_{\varphi \rightarrow \psi} \Box_j \alpha \land \neg \Box_k \beta \land \Box_i (j \rightarrow -k) \rightarrow \neg \Box_i (\alpha \rightarrow \beta)$. For the approximation rule from $\alpha \rightarrow \beta \leq -x$ to $j \leq \alpha$, $\beta \leq -k$ and $j \rightarrow -k \leq -x$, the proof is similar.

• For the residuation rule from $i \leq -\alpha$ to $\alpha \leq -i$, it suffices to prove that from $\vdash_{\varphi \rightarrow \psi} \Box_i -\alpha \land \gamma \rightarrow \neg \Box_i \alpha_1$, one can get $\vdash_{\varphi \rightarrow \psi} \neg \Box_i \alpha \land \gamma \rightarrow \neg \Box_i \alpha_1$, which follows from the fact that $\vdash_{\varphi \rightarrow \psi} \neg \Box_i \alpha \implies \Box_i -\alpha$.

For the residuation rule from $x \leq -\alpha$ to $\alpha \leq -x$, the proof is similar.

• For the residuation rule from $-\alpha \leq -i$ to $i \leq \alpha$, it suffices to prove that from $\vdash_{\varphi \rightarrow \psi} -\Box_i -\alpha \land \gamma \rightarrow \neg \Box_i \alpha_1$, one can get $\vdash_{\varphi \rightarrow \psi} \Box_i \alpha \land \gamma \rightarrow \neg \Box_i \alpha_1$, which follows from the fact that $\vdash_{\varphi \rightarrow \psi} -\Box_i -\alpha \implies \Box_i \alpha$.

For the residuation rule from $-\alpha \leq -x$ to $x \leq \alpha$, the proof is similar.

\[
\begin{array}{l}
i_1 \leq p \\
\vdots \\
i_n \leq p \\
\beta_1 \leq \gamma_1 \\
\vdots \\
\beta_j \leq \gamma_j \\
\vdots \\
\beta_k \leq -k_k
\end{array}
\]

For the right-handed Ackermann rule from

\[
\begin{array}{l}
i_1 \leq \gamma_1 ([i_1 \lor \ldots \lor i_n]/p) \\
\vdots \\
i_m \leq \gamma_m ([i_1 \lor \ldots \lor i_n]/p) \\
\beta_1 ([i_1 \lor \ldots \lor i_n]/p) \leq -k_1 \\
\vdots \\
\beta_k ([i_1 \lor \ldots \lor i_n]/p) \leq -k_k
\end{array}
\]
without loss of generality we suppose that \( m = k = 1 \), then it suffices to prove that from

\[
\vdash \varphi \rightarrow \psi \text{ if } @i_1 p \land \ldots \land @i_n p \land @j_1 \gamma_1 \land \neg @k_1 \beta_1 \rightarrow \neg @i_1 i_1
\]

one can get

\[
\vdash \varphi \rightarrow \psi @i_1 \gamma_1[(i_1 \lor \ldots \lor i_n)/p] \land \neg @k_1 \beta_1[(i_1 \lor \ldots \lor i_n)/p] \rightarrow \neg @i_1 i_1.
\]

By uniform substitution \((i_1 \lor \ldots \lor i_n)/p\) on

\[
\vdash \varphi \rightarrow \psi @i_1 p \land \ldots \land @i_n p \land @j_1 \gamma_1 \land \neg @k_1 \beta_1 \rightarrow \neg @i_1 i_1,
\]

we can get

\[
\vdash \varphi \rightarrow \psi @i_1 (i_1 \lor \ldots \lor i_n) \land \ldots \land @i_n (i_1 \lor \ldots \lor i_n) \land @j_1 \gamma_1[(i_1 \lor \ldots \lor i_n)/p] \land \neg @k_1 \beta_1[(i_1 \lor \ldots \lor i_n)/p] \rightarrow \neg @i_1 i_1.
\]

Since \( \vdash \varphi \rightarrow \psi @i_1 (i_1 \lor \ldots \lor i_n), \ldots, \vdash \varphi \rightarrow \psi @i_n (i_1 \lor \ldots \lor i_n) \), we have that

\[
\vdash \varphi \rightarrow \psi @i_1 \gamma_1[(i_1 \lor \ldots \lor i_n)/p] \land \neg @k_1 \beta_1[(i_1 \lor \ldots \lor i_n)/p] \rightarrow \neg @i_1 i_1.
\]

- For the left-handed Ackermann rule from

\[
\begin{cases}
  p \leq \neg i_1 \\
  \vdots \\
  p \leq \neg i_m \\
  j_1 \leq \gamma_1 \\
  \vdots \\
  j_m \leq \gamma_m[(\neg i_1 \land \ldots \land i_n)/p] \\
  \beta_1 \leq \neg k_1 \\
  \vdots \\
  \beta_m[(\neg i_1 \land \ldots \land i_n)/p] \leq \neg k_k
\end{cases}
\]

without loss of generality we suppose that \( m = k = 1 \), then it suffices to prove that from

\[
\vdash \varphi \rightarrow \psi \neg @i_1 p \land \ldots \land \neg @i_n p \land @j_1 \gamma_1 \land \neg @k_1 \beta_1 \rightarrow \neg @i_1 i_1
\]

one can get

\[
\vdash \varphi \rightarrow \psi \neg @i_1 \gamma_1[(\neg i_1 \land \ldots \land \neg i_n)/p] \land \neg @k_1 \beta_1[(\neg i_1 \land \ldots \land \neg i_n)/p] \rightarrow \neg @i_1 i_1.
\]

By uniform substitution \((\neg i_1 \land \ldots \land \neg i_n)/p\) on

\[
\vdash \varphi \rightarrow \psi \neg @i_1 p \land \ldots \land \neg @i_n p \land @j_1 \gamma_1 \land \neg @k_1 \beta_1 \rightarrow \neg @i_1 i_1,
\]

we can get

\[
\vdash \varphi \rightarrow \psi \neg @i_1 (\neg i_1 \land \ldots \land \neg i_n) \land \ldots \land @i_n (\neg i_1 \land \ldots \land \neg i_n) \land @j_1 \gamma_1[(\neg i_1 \land \ldots \land \neg i_n)/p] \land \neg @k_1 \beta_1[(\neg i_1 \land \ldots \land \neg i_n)/p] \rightarrow \neg @i_1 i_1.
\]

Since \( \vdash \varphi \rightarrow \psi \neg @i_1 (\neg i_1 \land \ldots \land \neg i_n), \ldots, \vdash \varphi \rightarrow \psi \neg @i_n (\neg i_1 \land \ldots \land \neg i_n), \) we have that

\[
\vdash \varphi \rightarrow \psi @i_1 \gamma_1[(\neg i_1 \land \ldots \land \neg i_n)/p] \land \neg @k_1 \beta_1[(\neg i_1 \land \ldots \land \neg i_n)/p] \rightarrow \neg @i_1 i_1.
\]

\[\square\]

**Corollary 7.7.** Given a skeletal Sahlqvist formula \( \varphi \rightarrow \psi \), for each quasi-inequality \( \text{Quasi in Pure}(\varphi \rightarrow \psi) \), we have that \( \vdash \varphi \rightarrow \psi \text{ Tr}(\text{Quasi}) \), therefore \( \vdash \varphi \rightarrow \psi \text{ Tr(Pure}(\varphi \rightarrow \psi)) \).

**Proof.** It suffices to see that for each pure quasi-inequality produced after Stage 2, by uniformly substitute free occurrences of state variables by fresh nominals, the translation of the resulting pure quasi-inequality is still provable in \( K_{h(a,1)} + (\varphi \rightarrow \psi) \).  

\[\square\]

15
7.3 Main Proof

Now we are ready to prove our main result:

**Theorem 7.8.** For any skeletal Sahlqvist formula \( \varphi \rightarrow \psi \), \( K_{H(\uparrow, \downarrow)} + (\varphi \rightarrow \psi) \) is sound and strongly complete with respect to the class of Kripke frames \( \mathcal{F} \) defined by \( \varphi \rightarrow \psi \).

**Proof.** Our proof strategy is as follows: we prove that for any \( \mathcal{L}(\uparrow, \downarrow) \)-formula set \( \Gamma \) and any \( \mathcal{L}(\uparrow, \downarrow) \)-formula \( \gamma \),

\[
\Gamma \vdash_{\varphi \rightarrow \psi} \gamma \Rightarrow \Gamma \models_{\mathcal{F}} \gamma \Rightarrow \Gamma \vdash_{\text{Tr}(\text{Pure}(\varphi \rightarrow \psi))} \gamma \Rightarrow \Gamma \vdash_{\varphi \rightarrow \psi} \gamma .
\]

• For the first implication, i.e. the soundness part, it is easy.

• For the second implication, from the fact that \( \mathcal{F} \models_{\varphi \rightarrow \psi} \) iff \( \mathcal{F} \models_{\text{Tr}(\text{Pure}(\varphi \rightarrow \psi))} \) (Theorem 5.1) iff \( \mathcal{F} \models_{\text{Tr}(\text{Pure}(\varphi \rightarrow \psi))} \) (corollary of Proposition 7.2) we have that \( \mathcal{F} \) is also defined by \( \text{Tr}(\text{Pure}(\varphi \rightarrow \psi)) \). By Theorem 2.4, we have the completeness of \( K_{H(\uparrow, \downarrow)} + \text{Tr}(\text{Pure}(\varphi \rightarrow \psi)) \) with respect to \( \mathcal{F} \).

• For the third implication, it suffices to show that all theorems of \( K_{H(\uparrow, \downarrow)} + \text{Tr}(\text{Pure}(\varphi \rightarrow \psi)) \) are also theorems of \( K_{H(\uparrow, \downarrow)} + (\varphi \rightarrow \psi) \). To show this, it is enough to prove that \( \vdash_{\varphi \rightarrow \psi} \text{Tr}(\text{Pure}(\varphi \rightarrow \psi)) \), which follows from Corollary 7.7.

By an easy adaptation of the previous results to a set \( \Sigma \) of skeletal Sahlqvist formulas, we have the following corollary:

**Corollary 7.9.** For any set \( \Sigma \) of skeletal Sahlqvist formulas, \( K_{H(\uparrow, \downarrow)} + \Sigma \) is sound and strongly complete with respect to the class of Kripke frames \( \mathcal{F} \) defined by \( \Sigma \).

8 Conclusion

In the present paper, we investigate the completeness theory for hybrid logic with binder \( \mathcal{L}(\uparrow, \downarrow) \). We define the class of skeletal Sahlqvist formulas, and show that for any set \( \Sigma \) of skeletal Sahlqvist formulas, \( K_{H(\uparrow, \downarrow)} + \Sigma \) is sound and strongly complete with respect to the class of Kripke frames \( \mathcal{F} \) defined by \( \Sigma \). Our strategy is to use the algorithm \( \text{ALBA}_{\text{restricted}} \) to transform an input skeletal Sahlqvist formula \( \varphi \rightarrow \psi \) into an equivalent \( \mathcal{L}(\uparrow, \downarrow) \)-formula \( \text{Tr}(\text{Pure}(\varphi \rightarrow \psi)) \), and then show that \( K_{H(\uparrow, \downarrow)} + (\varphi \rightarrow \psi) \) proves \( \text{Tr}(\text{Pure}(\varphi \rightarrow \psi)) \).

Our methodology could also work for \( \mathcal{L}(\uparrow) \), which follows from a restricted version of the algorithm hybrid-ALBA defined in [10]. Indeed, we got inspiration of the definition of skeletal Sahlqvist inequalities from [10]. In [10], Conradie and Robinson gave an algebraic proof of the completeness of \( K_{H(\uparrow)} + \Sigma \) where \( \Sigma \) is a set of skeletal formulas. Our proof can be seen as a proof-theoretic counterpart of their proof.

For future directions, we list the following:

• In [20], ten Cate, Marx and Viana proved that modal Sahlqvist formulas that do not contain occurrences of nominals axiomatize complete logics extending \( K_{H(\uparrow)} \). A future question is whether this result could be extended to the language \( \mathcal{L}(\uparrow, \downarrow) \).

• In [10], Conradie and Robinson proved that for any set \( \Sigma \) of nominally skeletal inductive formulas, the logic \( K_{H(\uparrow)} + \Sigma \) is sound and strongly complete with respect to its class of Kripke frames. A future question is whether this result could be extended to the language \( \mathcal{L}(\uparrow, \downarrow) \).

• In [15], Litak gave an algebraization of hybrid logic with binder \( \mathcal{H}(\uparrow, \downarrow) \). A future question is whether we can use this algebraization to give canonicity proofs of certain formulas to prove completeness results.
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