Qubit Geodesics on the Bloch Sphere from Optimal-Speed Hamiltonian Evolutions

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In the geometry of quantum evolutions, a geodesic path is viewed as a path of minimal statistical length connecting two pure quantum states along which the maximal number of statistically distinguishable states is minimum. In this paper, we present an explicit geodesic analysis of the dynamical trajectories that emerge from the quantum evolution of a single-qubit quantum state. The evolution is governed by an Hermitian Hamiltonian operator that achieves the fastest possible unitary evolution between given initial and final pure states. Furthermore, in addition to viewing geodesics in ray space as paths of minimal length, we also verify the geodesicity of paths in terms of unit geometric efficiency and vanishing geometric phase. Finally, based on our analysis, we briefly address the main hurdles in moving to the geometry of quantum evolutions for open quantum systems in mixed quantum states.

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I. INTRODUCTION

It is well-known that geometry plays a fundamental role in physics. Moreover, it is not unusual to observe that the noncommutative nature of quantum theory has significantly motivated the birth of the newly established field of Quantum Riemannian Geometry (QRG, [1]). This geometry is an extension of classical differential geometry to curved quantum spaces with a noncommutative coordinate algebra. For recent applications of QRG to models of quantum gravity and to formulations of ordinary quantum mechanics in the spirit of classical gravity, we refer to Refs. [2] and [3, 4], respectively.

In this paper, we consider the geometric formulations of quantum mechanics in a more conventional way where the geometry on the space of quantum states [5], either pure [6] or mixed [7], specifies limitations on our capacity of discriminating one state from another by means of measurements. The geometry on the space of quantum states does not express, in general, the actual dynamical evolution of a quantum system [8]. Indeed, not all Hamiltonian evolutions are shortest time Hamiltonian evolutions and, therefore, do not coincide with the geodesic paths on the underlying quantum state space equipped with a suitable metric. However, focusing for simplicity on pure states, there exist optimum Hamiltonians generating optimal-speed evolutions characterized by the shortest duration [9, 10] along with the maximal energy dispersion [11, 12]. For such quantum motions, Hamiltonian curves (i.e., dynamical trajectories) traced by quantum states undergoing actual physical evolutions can be formally shown to be geodesics (i.e., geodesic lines or geodesic paths) on the underlying metricized manifolds.

Following Refs. [6, 10], a geodesic path in the above mentioned geometric formulations of quantum mechanics can be regarded as a path of minimal statistical length connecting two quantum states along which the maximal number of statistically distinguishable states is minimum. In particular, the larger the size of the statistical fluctuations in measurements prepared to distinguish one state from another, the closer points are together. Therefore, optimum Hamiltonian evolutions can be shown to happen along geodesic paths of minimal statistical length. The system evolution occurs while crossing the minimum number of statistically distinguishable quantum states and, in addition, moves quickly through regions in which the energy dispersion is large. The problem of finding a Hamiltonian operator that achieves the minimum travel time (or, alternatively, the highest evolution speed) has been considered for systems in either pure [9, 13, 15] or mixed [14, 17] quantum states. In these works, the emphasis is on the Hamiltonian operator and not on the geodesicity of the dynamical trajectory traced by the actual Hamiltonian evolution. In this regard, it is important to keep in mind that geodesics are curves with a preferred parametrization. Therefore, when characterizing geodesic paths, one needs to specify its parametrization in terms of a coordinate (that is, the affine parameter) along with the path (for instance, the great circle for a two-sphere).

The main goal of our paper is to help create more awareness of the interplay between quantum mechanics and geometry by spelling out how the concept of shortest time Hamiltonian evolution in the quantum information sense coincides with the notion of geodesic path in the geometric sense. To achieve this goal, we consider a shortest time Hermitian Hamiltonian evolution of a two-level quantum system from a pure source state $|A\rangle$ to a pure target state $|B\rangle$. Then, we show that the dynamical trajectory connecting $|A\rangle$ and $|B\rangle$ that emerges from the unitary evolution operator $U(t) = e^{-iHt}$ coincides with the geodesic path between the two single-qubit pure states when the two-dimensional

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Hilbert space $\mathcal{H}_2^1$ is viewed in terms of points on the complex projective Hilbert space $\mathbb{C}P^1$ (or, equivalently, the Bloch sphere $S^2 \cong \mathbb{C}P^1$) equipped with the Fubini-Study metric (or, equivalently, the round metric on the sphere).

For completeness, we point out that the selection of the main goal of our paper is also motivated by our recent geometrically oriented investigations on quantum search algorithms [21, 22], on continuous-time quantum search evolutions [23, 24], on the efficiency of quantum evolutions [26, 27], on the emergence of geodesic paths in different areas of physics (including classical gravity) [28], and, finally, on the intriguing link between propagation of light with maximal degree of coherence and optimal-speed unitary quantum time evolutions [29].

The layout of the rest of the paper is as follows. In Section II, we revisit some preliminary results on the geometry of pure quantum states. Specifically, we discuss the concept of distance between pure states, the concept of quantum line, and the notion of quantum geodesic line with special attention to suitably chosen parametrizations. In Section III, we present a family of optimal-speed Hamiltonian evolutions. We describe properties of a typical Hamiltonian of this family and, in addition, provide an explicit expression of the shortest time quantum dynamical trajectory connecting an arbitrary initial source state to an arbitrary final target state during such unitary evolution. In Section IV, using the geometry of quantum pure states introduced in Section II and focusing on the quantum Hamiltonian motion described in Section III, we verify in an explicit manner the geodesicity of the quantum dynamical trajectory emerging from the chosen optimal-speed Hamiltonian evolution. Once again, we devote special attention to the parametrization of the geodesic paths. In Section V, we verify special attention to the parametrization of the quantum dynamical trajectory introduced in Section III by means of the concepts of geometric efficiency [26, 30] and Berry’s geometric phase [31]. In Section VI, we finally present our final remarks.

II. GEOMETRY OF PURE QUANTUM STATES

In this section, we revisit for completeness some mathematical preliminaries needed to present our main result. After introducing the concept of Fubini-Study metric tensor for pure states, our main goals here can be summarized as follows. First, we discuss general parametrizations of quantum lines in Eqs. (24) and (28). Second, we present quantum geodesic lines as quantum lines satisfying Eq. (43) or, equivalently, as paths of minimal length connecting fixed initial and final states on the Bloch sphere. For further mathematical details on the geometry of pure quantum states, we refer to Refs. [32, 33].

A. Distance between two pure states

In what follows, we introduce the Fubini-Study metric tensor. Recall that the finite distance between two quantum states $|\psi_A\rangle$ and $|\psi_B\rangle$ belonging to a Hilbert space $\mathcal{H}$ can be defined in different ways. For instance, the Fubini-Study distance $d_{FS}(|\psi_A\rangle, |\psi_B\rangle)$ between two quantum states $|\psi_A\rangle$ and $|\psi_B\rangle$ is defined as [33]

$$d_{FS}(|\psi_A\rangle, |\psi_B\rangle) \overset{\text{def}}{=} \lambda \sqrt{1 - |\langle \psi_A | \psi_B \rangle|^2},$$

with $\lambda$ being an arbitrary real constant factor. Alternatively, the Wootters distance $d_W(|\psi_A\rangle, |\psi_B\rangle)$ between two quantum states $|\psi_A\rangle$ and $|\psi_B\rangle$ is given by [33]

$$d_W(|\psi_A\rangle, |\psi_B\rangle) \overset{\text{def}}{=} \lambda \cos^{-1} \left| |\langle \psi_A | \psi_B \rangle| \right|.$$  

Interestingly, given two infinitesimally close neighboring pure quantum states $|\psi(\xi)\rangle$ and $|\psi(\xi + \Delta\xi)\rangle$ that can be distinguished thanks to a real parameter $\xi$, it happens that up to the second order in $\Delta\xi$ with $\Delta\xi \ll 1$, the differential forms of the Wootters and the Fubini-Study distances are equivalent [34, 35].

Following the line of reasoning presented in Ref. [33], let us consider a set of quantum state vectors $\{|\psi(\xi)\rangle\}$ parametrized by the parameters $\xi \overset{\text{def}}{=} (\xi^1, ..., \xi^m)$. The quantity $m$ denotes the number of real parameters assumed to parametrize a quantum state $|\psi(\xi)\rangle$ in $\mathbb{C}P^{n-1}$. For clarity, we assume here that $\mathcal{H}$ is the $n$-dimensional complex Hilbert space $\mathcal{H}_N^2$ of $N$-qubit quantum states with $n = 2^N$ and we focus on the simple case with $n = 2$. Then, regardless of the chosen definition of finite distance, the infinitesimal line element $ds_{FS}^2$ quantifying the distance between two neighboring states $|\psi(\xi)\rangle$ and $|\psi(\xi + d\xi)\rangle$ can be written as

$$ds_{FS}^2 = g_{ab}(\xi) d\xi^a d\xi^b.$$  

The quantity $g_{ab}(\xi)$ in Eq. (3) is defined as [32],

$$g_{ab}(\xi) \overset{\text{def}}{=} \lambda^2 \left[ \gamma_{ab}(\xi) - \beta_a(\xi) \beta_b(\xi) \right],$$

for $a, b = 1, 2$.

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where, given that \( \partial_a \overset{\text{def}}{=} \partial / \partial \xi^a \), we have
\[
\gamma_{ab}(\xi) \overset{\text{def}}{=} \text{Re} \left[ \langle \partial_a \psi(\xi) \mid \partial_b \psi(\xi) \rangle \right], \quad \text{and} \quad \beta_{a}(\xi) \overset{\text{def}}{=} -i \langle \psi(\xi) \mid \partial_a \psi(\xi) \rangle.
\]
Note from Eq. (5) that,
\[
\beta_a(\xi) \beta_b(\xi) = \langle \partial_a \psi(\xi) \mid \psi(\xi) \rangle \langle \psi(\xi) \mid \partial_b \psi(\xi) \rangle,
\]
since \( \partial_a [\langle \psi(\xi) \mid \psi(\xi) \rangle] = 0 \) implies that \( \langle \psi(\xi) \mid \partial_a \psi(\xi) \rangle = -\langle \partial_a \psi(\xi) \mid \psi(\xi) \rangle \). Therefore, using Eqs. (5) and (6), the metric tensor components \( g_{ab}(\xi) \) defined in Eq. (4) become
\[
g_{ab}(\xi) = \lambda^2 \left\{ \text{Re} \left[ \langle \partial_a \psi(\xi) \mid \partial_b \psi(\xi) \rangle \right] - \langle \partial_a \psi(\xi) \mid \psi(\xi) \rangle \langle \psi(\xi) \mid \partial_b \psi(\xi) \rangle \right\}.
\]
For notational simplicity, let us define
\[
A_{ab}(\xi) \overset{\text{def}}{=} \langle \partial_a \psi(\xi) \mid \psi(\xi) \rangle \langle \psi(\xi) \mid \partial_b \psi(\xi) \rangle.
\]
We shall prove that \( A_{ab}(\xi) \, d\xi^a d\xi^b = \text{Re} \, [A_{ab}(\xi)] \, d\xi^a d\xi^b \). Consider,
\[
A_{ab}(\xi) = \text{Re} \, [A_{ab}(\xi)] + i \, \text{Im} \, [A_{ab}(\xi)] = A_{ab}^{(1)}(\xi) + i A_{ab}^{(2)}(\xi).
\]
Then, from Eq. (5) we have
\[
A_{ab}^{(1)}(\xi) = \text{Re} \, [A_{ab}(\xi)]
= \text{Re} \, [\langle \partial_a \psi(\xi) \mid \psi(\xi) \rangle \langle \psi(\xi) \mid \partial_b \psi(\xi) \rangle]
= \text{Re} \, [\langle \partial_a \psi(\xi) \mid \psi(\xi) \rangle^\ast \langle \psi(\xi) \mid \partial_b \psi(\xi) \rangle^\ast]
= \text{Re} \, [\langle \psi(\xi) \mid \partial_a \psi(\xi) \rangle \langle \partial_b \psi(\xi) \mid \psi(\xi) \rangle]
= \text{Re} \, [\langle \partial_b \psi(\xi) \mid \psi(\xi) \rangle \langle \psi(\xi) \mid \partial_a \psi(\xi) \rangle]
= \text{Re} \, [A_{ba}(\xi)]
= A_{ba}^{(1)}(\xi),
\]
and, in addition,
\[
A_{ab}^{(2)}(\xi) = \text{Im} \, [A_{ab}(\xi)]
= \text{Im} \, [\langle \partial_a \psi(\xi) \mid \psi(\xi) \rangle \langle \psi(\xi) \mid \partial_b \psi(\xi) \rangle]
= -\text{Im} \, [\langle \partial_a \psi(\xi) \mid \psi(\xi) \rangle^\ast \langle \psi(\xi) \mid \partial_b \psi(\xi) \rangle^\ast]
= -\text{Im} \, [\langle \partial_b \psi(\xi) \mid \psi(\xi) \rangle \langle \psi(\xi) \mid \partial_a \psi(\xi) \rangle]
= -\text{Im} \, [A_{ba}(\xi)]
= -A_{ba}^{(2)}(\xi).
\]
From Eqs. (10) and (11), we conclude that \( A_{ab}^{(1)}(\xi) \) is symmetric under exchange of indices while \( A_{ab}^{(2)}(\xi) \) is anti-symmetric. Therefore, from the symmetry of \( d\xi^a d\xi^b \), we have \( A_{ab}(\xi) \, d\xi^a d\xi^b = \text{Re} \, [A_{ab}(\xi)] \, d\xi^a d\xi^b \). In conclusion, the metric tensor \( g_{ab}(\xi) \) can be written as
\[
g_{ab}(\xi) = \lambda^2 \text{Re} \, [\langle \partial_a \psi(\xi) \mid \partial_b \psi(\xi) \rangle] - \lambda^2 \text{Re} \, [\langle \partial_a \psi(\xi) \mid \psi(\xi) \rangle \langle \psi(\xi) \mid \partial_b \psi(\xi) \rangle],
\]
that is,
\[
g_{ab}(\xi) = \lambda^2 \text{Re} \, [\langle \partial_a \psi(\xi) \mid \partial_b \psi(\xi) \rangle] - \lambda^2 \text{Re} \, [\langle \partial_a \psi(\xi) \mid \psi(\xi) \rangle \langle \psi(\xi) \mid \partial_b \psi(\xi) \rangle].
\]
Eq. (13) defines the Fubini-Study metric tensor on the manifold of pure quantum states. We point out that it is convenient to set \( \lambda = 2 \) in Eq. (13). This way, limiting our attention to the two-dimensional case, \( g_{ab}(\xi) \) in Eq. (13) becomes a metric tensor on the Bloch sphere with the radius equal to one.

B. Quantum lines

This subsection is divided in two parts. In the first part, we discuss the parametrization of quantum lines. In the second part, we show that geodesic paths are quantum lines of minimal length between two initial and final pure states on the Bloch sphere. Clearly, the notion of distance used here relies on the concept of Fubini-Study metric introduced in the previous subsection.
1. Parametrization of quantum lines

Consider two normalized quantum state vectors $|\psi_A\rangle$ and $|\psi_B\rangle$ belonging to a Hilbert space $\mathcal{H}$ such that,

$$\langle \psi_A | \psi_A \rangle = 1, \text{ and } \langle \psi_B | \psi_B \rangle = 1. \quad (14)$$

Note that we do not require $|\psi_A\rangle$ and $|\psi_B\rangle$ to be orthogonal. Thus, in general, $\langle \psi_A | \psi_B \rangle \neq \delta_{AB}$ with $\delta_{AB}$ denoting the Kronecker delta symbol. From $|\psi_A\rangle$ and $|\psi_B\rangle$, we can consider a one-parameter $\xi \in (0, 1) \subset \mathbb{R}$ that specifies a parametric set of quantum state vectors $|\psi(\xi)\rangle$,

$$|\psi(\xi)\rangle \equiv \mathcal{N}_\xi \left[ (1-\xi) |\psi_A\rangle + e^{i\varphi} \xi |\psi_B\rangle \right]. \quad (15)$$

We point out that $\varphi \in \mathbb{R}$ is a relative phase to be properly selected by imposing that global phase factors are not physically important in quantum mechanics and $\mathcal{N}_\xi$ is a real normalization factor to be chosen in such a manner that $\langle \psi(\xi) | \psi(\xi) \rangle = 1$. Furthermore, we note that $|\psi(\xi)\rangle$ in Eq. (15) is the analogue of a straight line $\vec{r}(\xi)$ in a flat Euclidean space that connects two points $\vec{r}_A$ and $\vec{r}_B$.

$$\vec{r}(\xi) \equiv (1-\xi) \vec{r}_A + \xi \vec{r}_B. \quad (16)$$

Therefore, it appears reasonable to regard the linear combination of the states $|\psi_A\rangle$ and $|\psi_B\rangle$ that defines $|\psi(\xi)\rangle$ in Eq. (15) as a "geodesic" line in the Hilbert space $\mathcal{H}$ that connects these two state vectors. To select the phase $\varphi$, we recall that unlike the classical case in Eq. (16), the quantum case in Eq. (15) requires that global phase factors are physically unimportant. Therefore, state vectors $|\psi_j\rangle$ and $e^{i\varphi_j} |\psi_j\rangle$ with $j \in \{A, B\}$ are physically indistinguishable and represent the same quantum state. For this reason, one needs to impose that the "geodesic" line connecting $|\psi_A\rangle$ and $|\psi_B\rangle$ must coincide with the "geodesic" line connecting $|\tilde{\psi}_A\rangle \equiv e^{i\varphi_A} |\psi_A\rangle$ and $|\tilde{\psi}_B\rangle \equiv e^{i\varphi_B} |\psi_B\rangle$. Specifically, we require

$$\mathcal{N}_\xi \left[ (1-\xi) |\psi_A\rangle + e^{i\varphi} \xi |\psi_B\rangle \right] = \mathcal{N}_\xi \left[ (1-\xi) e^{i\varphi_A} |\psi_A\rangle + e^{i\varphi} \xi e^{i\varphi_B} |\psi_B\rangle \right]$$

$$= \mathcal{N}_\xi \left[ (1-\xi) |\psi_A\rangle + e^{i\varphi} \xi e^{i(\varphi_B-\varphi_A)} |\psi_B\rangle \right]$$

$$\sim \mathcal{N}_\xi \left[ (1-\xi) |\psi_A\rangle + \xi e^{i(\varphi_B-\varphi_A)} e^{i\varphi} |\psi_B\rangle \right], \quad (17)$$

that is,

$$e^{i\varphi} = e^{i(\varphi_B-\varphi_A)} e^{i\varphi}. \quad (18)$$

Observe that in the last line of Eq. (17), the symbol "$\sim$" denotes physical equivalence of quantum states and not mathematical equivalence. Eq. (18) can be satisfied by choosing the phase factor $e^{i\varphi}$ equal to

$$e^{i\varphi} = \frac{\langle \psi_B | \psi_A \rangle}{|\langle \psi_B | \psi_A \rangle|}. \quad (19)$$

Indeed, using Eqs. (18) and (19), we obtain

$$e^{i(\varphi_B-\varphi_A)} e^{i\varphi} = e^{i(\varphi_B-\varphi_A)} \frac{\langle \tilde{\psi}_B | \tilde{\psi}_A \rangle}{|\langle \tilde{\psi}_B | \tilde{\psi}_A \rangle|}$$

$$= e^{i(\varphi_B-\varphi_A)} e^{-i(\varphi_B-\varphi_A)} \frac{\langle \psi_B | \psi_A \rangle}{|\langle \psi_B | \psi_A \rangle|}$$

$$= \frac{\langle \psi_B | \psi_A \rangle}{|\langle \psi_B | \psi_A \rangle|} \quad = e^{i\varphi}. \quad (20)$$

Therefore, employing the expression of the properly identified phase factor $e^{i\varphi}$ in Eq. (19), the quantum line in Eq. (15) can be formally written as

$$|\psi(\xi)\rangle \equiv \mathcal{N}_\xi \left[ (1-\xi) |\psi_A\rangle + \left( \frac{\langle \psi_B | \psi_A \rangle}{|\langle \psi_B | \psi_A \rangle|} \right) \xi |\psi_B\rangle \right]. \quad (21)$$

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The last quantity that we need to specify in Eq. (21) is the real normalization factor $N_\xi$. As mentioned earlier, this can be determined by requiring the normalization condition $\langle \psi (\xi) | \psi (\xi) \rangle = 1$. Specifically, we have

$$1 = \langle \psi (\xi) | \psi (\xi) \rangle = N_\xi^2 [1 - 2\xi (1 - \xi) (1 - \langle |\psi_B|\psi_A\rangle)] ,$$

that is,

$$N_\xi = N_\xi (\xi) \defeq \frac{1}{\sqrt{1 - 2\xi (1 - \xi) (1 - \langle |\psi_B|\psi_A\rangle)}} .$$

(23)

Inserting Eq. (23) into Eq. (21), $|\psi (\xi)\rangle$ becomes

$$|\psi (\xi)\rangle \defeq \frac{[1 - \xi |\psi_A\rangle + \frac{\langle |\psi_B|\psi_A\rangle}{|\psi_B|\psi_A\rangle} \xi |\psi_B\rangle]}{\sqrt{1 - 2\xi (1 - \xi) (1 - \langle |\psi_B|\psi_A\rangle)}} .$$

(24)

At this point, to explicitly show that $|\psi (\xi)\rangle$ in Eq. (24) is indeed a proper geodesic line (that is, a line connecting $|\psi_A\rangle$ and $|\psi_B\rangle$ with shortest length with lengths computed by means of the Fubini-Study metric), it happens to be more convenient employing an alternative parametrization of the state $|\psi (\xi)\rangle$. This particular step is allowed thanks to the parametric-invariance of lengths of curves. A convenient parametrization of $|\psi (\xi)\rangle$ can be given in terms of a new parameter $\theta \in [0, \pi]$,

$$|\psi (\theta)\rangle \defeq N_\theta \left[ \cos \left( \frac{\theta}{2} \right) |\psi_A\rangle + e^{i\phi} \sin \left( \frac{\theta}{2} \right) |\psi_B\rangle \right] ,$$

(25)

where the normalization factor $N_\theta$ can be obtained by imposing the normalization constraint $\langle \psi (\theta) | \psi (\theta) \rangle = 1$. In particular, we have

$$1 = \langle \psi (\theta) | \psi (\theta) \rangle = N_\theta^2 [1 + \sin (\theta) \langle |\psi_B|\psi_A\rangle] ,$$

(26)

that is,

$$N_\theta = N_\theta (\theta) \defeq \frac{1}{\sqrt{1 + \sin (\theta) \langle |\psi_B|\psi_A\rangle}} .$$

(27)

Finally, using Eqs. (25) and (27), $|\psi (\theta)\rangle$ becomes

$$|\psi (\theta)\rangle \defeq \frac{\left[ \cos \left( \frac{\theta}{2} \right) |\psi_A\rangle + e^{i\phi} \sin \left( \frac{\theta}{2} \right) |\psi_B\rangle \right]}{\sqrt{1 + \sin (\theta) \langle |\psi_B|\psi_A\rangle}} ,$$

(28)

where $e^{i\phi}$ in Eq. (28) equals $\langle |\psi_B|\psi_A\rangle / \langle |\psi_B|\psi_A\rangle$. For completeness, we emphasize that $|\psi (\xi)\rangle$ in Eq. (24) and $|\psi (\theta)\rangle$ in Eq. (28) are the same states. In particular, the relation between the two parameters $\xi$ and $\theta$ can be obtained as follows. From the condition,

$$N_\xi \left[ (1 - \xi) |\psi_A\rangle + e^{i\phi} \xi |\psi_B\rangle \right] = N_\theta \left[ \cos \left( \frac{\theta}{2} \right) |\psi_A\rangle + e^{i\phi} \sin \left( \frac{\theta}{2} \right) |\psi_B\rangle \right] ,$$

(29)

we get

$$N_\xi (1 - \xi) = N_\theta \cos \left( \frac{\theta}{2} \right) , \text{ and } N_\xi \xi = N_\theta \sin \left( \frac{\theta}{2} \right) .$$

(30)

Manipulations of Eq. (30) yield,

$$\xi = \xi (\theta) \defeq \frac{\tan \left( \frac{\theta}{2} \right)}{1 + \tan \left( \frac{\theta}{2} \right)} .$$

(31)

Observe that $\xi (0) = 0$, $\xi (\pi) = 1$. Moreover, for $\xi = 0$ and $\theta = 0$, $|\psi\rangle = |\psi_A\rangle$. Finally, for $\xi = 1$ and $\theta = \pi$, $|\psi\rangle = e^{i\phi} |\psi_B\rangle \sim |\psi_B\rangle$. 

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2. Geodesics as quantum lines of minimal length

We want to show here that the quantum line in Eq. (28) is a quantum geodesic line. For simplicity, let us set \( \psi(t) = |\psi(\theta)\rangle = |\psi(\theta)\rangle \) where, in our case, \( \theta_a = \partial_{\theta_a} \) (the path depends on a single parameter). Then, we observe that the single Fubini-Study metric component in Eq. (13) can be written in a number of alternative manners

\[
g_{FS}(\theta) = \lambda^2 \text{Re} \left[ \langle \psi^* | \psi \rangle - \left| \langle \psi^* | \psi \rangle \right|^2 \right] \\
= \lambda^2 \text{Re} \left[ \langle \psi^* | \psi \rangle - \langle \psi^* | \psi \rangle^* \right] \\
= \lambda^2 \text{Re} \left[ \langle \psi^* | \psi \rangle - \langle \psi^* | \psi \rangle \right] \\
= \lambda^2 \text{Re} \left[ \langle \psi^* | \psi \rangle + \langle \psi^* | \psi \rangle \right] \\
= \lambda^2 \text{Re} \left[ \langle \psi^* | \psi \rangle + \langle \psi^* | \psi \rangle \right]. \\
\text{(32)}
\]

In the third to last line of Eq. (32), we have used the fact that \( \langle \psi | \psi \rangle = 1 \) implies that \( \langle \psi | \psi \rangle + \langle \psi | \psi \rangle = 0 \), that is, \( \langle \psi^* | \psi \rangle = - \langle \psi^* | \psi \rangle \). Therefore, \( \text{Re} \left( \langle \psi^* | \psi \rangle \right) = 0 \) and \( \langle \psi^* | \psi \rangle \) is a pure imaginary number \( iA \) with \( A \in \mathbb{R} \). The real quantity \( \mathcal{A} = \mathcal{A}(t) \equiv -i \langle \psi(t) | \psi(t) \rangle \) is a very relevant geometric quantity with a significant physical meaning. Indeed, it is the connection one form that specifies proper covariant differentiation and, in addition, leads to the so-called horizontal lift condition (i.e., \( iA(t) = \langle \psi(t) | \psi(t) \rangle = 0 \) \text{(36)}. Moreover, we see later in Eq. \( \text{(88)} \) that the connection one form is such that its line integral gives the geometric phase. In what follows, we employ the relation

\[
g_{FS}(\theta) = \lambda^2 \text{Re} \left[ \langle \psi^* | \psi \rangle + \langle \psi^* | \psi \rangle \right]. \\
\text{(33)}
\]

We want to use Eq. (28) to evaluate \( g_{FS}(\theta) \) in Eq. (33). We proceed as follows. Note that,

\[
|\psi\rangle = A|\psi_A\rangle + e^{i\varphi}B|\psi_B\rangle, \quad |\psi\rangle = C|\psi_A\rangle + e^{i\varphi}D|\psi_B\rangle, \\
\text{where } A, B, C, \text{ and } D \text{ are given by,}
\]

\[
A \equiv N_\theta \cos \left( \frac{\theta}{2} \right), \quad B \equiv N_\theta \sin \left( \frac{\theta}{2} \right) \\
C \equiv N_\theta \cos \left( \frac{\theta}{2} \right) - \frac{N_\theta}{2} \sin \left( \frac{\theta}{2} \right), \quad D \equiv N_\theta \sin \left( \frac{\theta}{2} \right) + \frac{N_\theta}{2} \cos \left( \frac{\theta}{2} \right) \\
N_\theta \equiv \frac{1}{\sqrt{1 + a \sin(\theta)}}, \quad a \equiv |\langle \psi_B | \psi_A \rangle|, \quad N_\theta \equiv \frac{-(a/2) \cos(\theta)}{|1 + a \sin(\theta)|^2}. \\
\text{(35)}
\]

Inserting Eq. (34) into Eq. (33), we obtain

\[
g_{FS}(\theta) = \lambda^2 \left\{ C^2 + D^2 + 2aCD + |AC + BD + a(AD + BC)|^2 \right\}. \\
\text{(36)}
\]

Then, using Eq. (35) along with performing a number of algebraic manipulations, \( g_{FS}(\theta) \) in Eq. (36) becomes

\[
g_{FS}(\theta) = \frac{\lambda^2 (1 - a^2)}{8a \sin \theta - 2a^2 \cos 2\theta + 2a^2 + 4}, \\
\text{(37)}
\]
III. OPTIMAL-SPEED HAMILTONIAN EVOLUTION

In this section, having introduced the Fubini-Study metric tensor for pure states along with the discussion of parametrizations of quantum geodesic paths as paths of minimal length connecting given initial and final states on the Bloch sphere, our two main tasks can be stated as follows. First, we introduce the Hamiltonian operator $H$ in Eq. (53) that achieves the fastest possible unitary evolution between two given initial and final pure states $|A\rangle$ and $|B\rangle$. Second, we present the shortest time quantum dynamical trajectory in Eq. (54) that emerges from $H$ in Eq. (53) and that connects $|A\rangle$ to $|B\rangle$.

That is,

$$g_{FS}(\theta) = \lambda^2 \frac{1 - a^2}{4|1 + a \sin(\theta)|^2}. \quad (38)$$

As a side remark, we point out that $\langle \psi | \dot{\psi} \rangle = [AC + BD + a(AD + BC)]^2 = 0$. Indeed, this is expected since $iA(t) = \langle \psi(t) | \dot{\psi}(t) \rangle = 0$ is the horizontal lift condition that yields geodesics on the Bloch sphere [32]. Therefore, the length $s_{A \rightarrow B}$ of the line connecting the states $|\psi_A\rangle$ and $|\psi_B\rangle$ defined as,

$$s_{A \rightarrow B} \overset{\text{def}}{=} \int_0^\pi g_{FS}^{1/2}(\theta) d\theta, \quad (39)$$

is given by,

$$s_{A \rightarrow B} = \frac{\lambda}{2} \int_0^\pi \frac{\sqrt{1 - a^2}}{1 + a \sin(\theta)} d\theta. \quad (40)$$

Performing a (Karl Weierstrass) change of variables (that is, $\theta \rightarrow t = t(\theta) \overset{\text{def}}{=} \tan(\theta/2)$), $s_{A \rightarrow B}$ becomes

$$s_{A \rightarrow B} = \frac{\lambda}{2} \frac{\sqrt{1 - a^2}}{1 + a \sin(\theta)} \int_0^\infty \frac{2}{t^2 + 2at + 1} dt. \quad (41)$$

With the help of the Mathematica symbolic software, we get

$$\int_0^\infty \frac{2}{t^2 + 2at + 1} dt = \frac{2}{\sqrt{1 - a^2}} \left[ \tan^{-1}\left( \frac{a + t}{\sqrt{1 - a^2}} \right) \right]_{t=0}^{t=\infty} \quad (42)$$

$$= \frac{2}{\sqrt{1 - a^2}} \left[ \frac{\pi}{2} - \tan^{-1}\left( \frac{a}{\sqrt{1 - a^2}} \right) \right]$$

$$= \frac{2}{\sqrt{1 - a^2}} \cos^{-1}(a).$$

Finally, using Eq. (42) and recalling the definition of $a$ in Eq. (35), the length $s_{A \rightarrow B}$ in Eq. (41) becomes

$$s_{A \rightarrow B} = \lambda \cos^{-1}(|\langle \psi_B | \psi_A \rangle|). \quad (43)$$

Since the length $s_{A \rightarrow B}$ in Eq. (43) of the line connecting the states $|\psi_A\rangle$ and $|\psi_B\rangle$ equals the Wootters distance. For $\lambda = 2$, the Wootters distance equals the angle between the vectors that identify the initial and final states $|\psi_A\rangle$ and $|\psi_B\rangle$ on the Bloch sphere. This angle represents the minimal possible length of the path $\gamma$ on the Bloch sphere connecting $|\psi_A\rangle$ and $|\psi_B\rangle$,

$$\text{Length} \left( \gamma_{A \rightarrow B}^{\text{geodesic}} \right) \leq \text{Length} \left( \gamma_{A \rightarrow B}^{\text{non-geodesic}} \right), \quad (44)$$

for any non-geodesic path $\gamma_{A \rightarrow B}^{\text{non-geodesic}}$. Therefore, we conclude that the quantum line in Eq. (28) is indeed a quantum geodesic line.
A. The Hamiltonian

Following the work presented by Mostafazadeh in Ref. [15], consider a traceless and time-independent Hamiltonian $H$ specified by a spectral decomposition given by $H \equiv E_1 |E_1\rangle \langle E_1| + E_2 |E_2\rangle \langle E_2|$, where $\langle E_2|E_1\rangle = \delta_{21}$ and $E_2 \geq E_1$. Clearly, $\{E_i\}_{i=1,2}$ and $\{|E_i\rangle\}_{i=1,2}$ denote the eigenvalues and the corresponding orthonormal eigenvectors of the Hamiltonian $H$. Moreover, $\delta_{ij}$ with $1 \leq i, j \leq 2$ is the usual Kronecker delta symbol. One is interested in evolving a state $|A\rangle$, not necessarily normalized, into a state $|B\rangle$ in the shortest possible time by maximizing the energy uncertainty $\Delta E$ and obtain $\Delta E = \Delta E_{\text{max}}$, with

$$\Delta E \equiv \left[ \frac{\langle A|H^2|A\rangle}{\langle A|A\rangle} - \left( \frac{\langle A|H|A\rangle}{\langle A|A\rangle} \right)^2 \right]^{1/2}.$$  \hspace{1cm} (45)

We maximize the energy uncertainty $\Delta E$ since we see later in Eq. (81) that the speed of quantum evolution $ds/dt$ along the curve is proportional to the energy uncertainty $\Delta E$, $ds/dt \propto \Delta E$. To get the value of $\Delta E_{\text{max}}$, we observe that an arbitrary unnormalized initial state $|A\rangle$ can be recast as $|A\rangle = \alpha_1 |E_1\rangle + \alpha_2 |E_2\rangle$ where $\alpha_1 \equiv \langle E_1|A\rangle$, $\alpha_2 \equiv \langle E_2|A\rangle \in \mathbb{C}$. Then, after some straightforward algebra, we obtain

$$\Delta E = \frac{E_2 - E_1}{2} \left[ 1 - \left( \frac{\alpha_1^2 - \alpha_2^2}{\alpha_1^2 + \alpha_2^2} \right)^2 \right]^{1/2}. \hspace{1cm} (46)$$

We observe from Eq. (46) that the maximum value of $\Delta E$ is achieved when $|\alpha_1| = |\alpha_2|$ and, in addition, is equal to

$$\Delta E_{\text{max}} \equiv \left( \frac{E_2 - E_1}{2} \right). \hspace{1cm} (47)$$

A main idea underlying Mostafazadeh’s approach in Ref. [15] is expressing $H \equiv E_1 |E_1\rangle \langle E_1| + E_2 |E_2\rangle \langle E_2|$ by means of the initial and final states $|A\rangle$ and $|B\rangle$, respectively, while keeping $\Delta E = \Delta E_{\text{max}}$. To this end, note that $|A\rangle$ and $|B\rangle$ can be decomposed as $|A\rangle = \alpha_1 |E_1\rangle + \alpha_2 |E_2\rangle$ and $|B\rangle = \beta_1 |E_1\rangle + \beta_2 |E_2\rangle$, respectively. Moreover, we must put $|\alpha_1| = |\alpha_2|$ and $|\beta_1| = |\beta_2|$ to satisfy $\Delta E = \Delta E_{\text{max}}$ and, consequently, guarantee minimum travel time $T_{\text{min}}$. Therefore, set $\alpha_2 = e^{i\varphi_\alpha} \alpha_1$ and $\beta_2 = e^{i\varphi_\beta} \beta_1$ with $\varphi_\alpha$ and $\varphi_\beta \in \mathbb{R}$. Then, states $|A\rangle$ and $|B\rangle$ can be recast as

$$|A\rangle = \alpha_1 |E_1\rangle + \alpha_2 |E_2\rangle = \alpha_1 |E_1\rangle + e^{i\varphi_\alpha} \alpha_1 |E_2\rangle, \hspace{1cm} (48)$$

and,

$$|B\rangle = \beta_1 |E_1\rangle + \beta_2 |E_2\rangle = \beta_1 |E_1\rangle + e^{i\varphi_\beta} \beta_1 |E_2\rangle, \hspace{1cm} (49)$$

respectively. Using Eqs. (48) and (49), let us introduce the states $|A\rangle$ and $|B\rangle$ defined by the relations $|E_1\rangle + e^{i\varphi_\alpha} |E_2\rangle = \alpha_1^{-1} |A\rangle \equiv \sqrt{2} |A\rangle$ and $|E_1\rangle + e^{i\varphi_\beta} |E_2\rangle = \beta_1^{-1} |B\rangle \equiv \sqrt{2} e^{-i\varphi_{\alpha - \beta}} |B\rangle$, respectively. The states $|A\rangle$ and $|B\rangle$ are being introduced to express the Fubini-Study and the geodesic distances in terms of the modulus squared of their quantum overlap and, in addition, to recast the optimal-speed quantum Hamiltonian in a convenient form. After some matrix algebra manipulations with Eqs. (48) and (49), we obtain

$$\left( \begin{array}{c} |E_1\rangle \\ |E_2\rangle \end{array} \right) = \frac{\sqrt{2}}{e^{i\varphi_{\alpha - \beta}} - e^{i\varphi_{\alpha + \beta}}} \left( e^{i\frac{\varphi_{\alpha + \beta}}{2}} - e^{-i\varphi_{\alpha}} \right) \left( -e^{i\frac{\varphi_{\alpha - \beta}}{2}} - 1 \right) \left( \beta_1^{-1} e^{i\frac{\varphi_{\alpha - \beta}}{2}} |B\rangle \right). \hspace{1cm} (50)$$

For completeness, we remark that

$$|\langle A|B\rangle|^2 = \frac{\langle A|B\rangle^2}{\langle A|A\rangle \langle B|B\rangle} = \cos^2 \left( \frac{\varphi_{\alpha} - \varphi_{\beta}}{2} \right) = \cos^2 \left( \frac{\theta_{\text{FS}}}{2} \right), \hspace{1cm} (51)$$

where $\theta_{\text{FS}} \equiv \varphi_{\alpha} - \varphi_{\beta} = 2s_{\text{FS}} = s_{\text{geo}}$, with $s_{\text{FS}}$ and $s_{\text{geo}}$ being the Fubini-Study and the geodesic distances, respectively. Finally, observing that $E_2 = -E_3 \equiv E$ since the Hamiltonian $H$ is assumed to be traceless and employing Eq. (50), the spectral decomposition of the Hamiltonian yields

$$H = \frac{iE}{\sin \left( \frac{\varphi_{\alpha} - \varphi_{\beta}}{2} \right)} \left| B \right\rangle \langle A | - \left| A \right\rangle \langle B |. \hspace{1cm} (52)$$
Finally, using Eq. (50) along with recalling the definitions of states $|A\rangle$ and $|B\rangle$, we note that the Hamiltonian in Eq. (52) can be expressed in terms of the initial and final states $|A\rangle$ and $|B\rangle$, respectively, as

$$H = iE \cot \left( \frac{\varphi_\alpha - \varphi_\beta}{2} \right) \left[ |B\rangle \langle A| - |A\rangle \langle B| \right].$$

Eq. (53) describes the correct version of the Hamiltonian specifying the optimal-speed unitary time evolution as originally proposed in Ref. [15]. For completeness, observe that for $H$ in Eq. (53), we correctly get $\langle A|H|A\rangle / \langle A|A\rangle = 0$ and $\Delta E = \left[ \langle A|H^2|A\rangle / \langle A|A\rangle \right]^{1/2} = E = \Delta E_{\text{max}}$.

### B. The quantum dynamical trajectory

Given the Hamiltonian $H$ in Eq. (53), we shall find the quantum dynamical trajectory $t \rightarrow |\psi(t)\rangle$ with $|\psi(t)\rangle \overset{\text{def}}{=} e^{-iHt} |A\rangle$ connecting initial and final states $|A\rangle$ and $|B\rangle$, respectively. We shall find that $|\psi(t)\rangle$ can be written as,

$$|\psi(t)\rangle = \left[ \cos \left( \frac{E}{\hbar} t \right) - \cos \left( \frac{\varphi_\alpha - \varphi_\beta}{2} \right) \sin \left( \frac{E}{\hbar} t \right) \right] |A\rangle + \frac{e^{i\varphi_\alpha} e^{i\varphi_\beta} - e^{i\varphi_\alpha} e^{-i\varphi_\beta}}{2i \sin \left( \frac{\varphi_\alpha - \varphi_\beta}{2} \right)} |B\rangle,$$

where $0 \leq t \leq T_{\text{AB}}$ with $T_{\text{AB}} \overset{\text{def}}{=} h \theta_{\text{FS}} / (2E)$. First, since we restrict our attention to a traceless Hamiltonian with $E_2 = -E_1 = E$, we note that

$$|\psi(t)\rangle = e^{-iHt} |A\rangle = \alpha_1 e^{iEt} |E_1\rangle + \alpha_2 e^{-iEt} |E_2\rangle.$$

Using Eq. (50), $|\psi(t)\rangle$ in Eq. (55) becomes

$$|\psi(t)\rangle = \left[ \cos \left( \frac{E}{\hbar} t \right) - \cos \left( \frac{\varphi_\alpha - \varphi_\beta}{2} \right) \sin \left( \frac{E}{\hbar} t \right) \right] |A\rangle + \frac{e^{i\varphi_\alpha} e^{i\varphi_\beta} - e^{i\varphi_\alpha} e^{-i\varphi_\beta}}{2i \sin \left( \frac{\varphi_\alpha - \varphi_\beta}{2} \right)} |B\rangle.$$

After some tedious but straightforward algebra, we note that

$$\frac{e^{i\varphi_\alpha} e^{i\varphi_\beta} - e^{i\varphi_\alpha} e^{-i\varphi_\beta}}{2i \sin \left( \frac{\varphi_\alpha - \varphi_\beta}{2} \right)} = -\frac{1}{2i} \sin \left( \frac{\varphi_\alpha - \varphi_\beta}{2} \right),$$

and, in addition,

$$\frac{e^{i\varphi_\alpha} e^{i\varphi_\beta} - e^{i\varphi_\alpha} e^{-i\varphi_\beta}}{2i \sin \left( \frac{\varphi_\alpha - \varphi_\beta}{2} \right)} = \frac{1}{2i} \sin \left( \frac{\varphi_\alpha - \varphi_\beta}{2} \right).$$

Therefore, making use of Eqs. (58) and (57), Eq. (56) yields

$$|\psi(t)\rangle = \text{Re} \left[ -i e^{i\varphi_\alpha} e^{i\varphi_\beta} \sin \left( \frac{E}{\hbar} t \right) \sin \left( \frac{\varphi_\alpha - \varphi_\beta}{2} \right) |A\rangle + e^{i\varphi_\alpha} e^{-i\varphi_\beta} \sin \left( \frac{E}{\hbar} t \right) |B\rangle \right].$$

To simplify Eq. (59), we observe that

$$\text{Re} \left[ -i e^{i\varphi_\alpha} e^{i\varphi_\beta} \sin \left( \frac{E}{\hbar} t \right) \sin \left( \frac{\varphi_\alpha - \varphi_\beta}{2} \right) \right] = \cos \left( \frac{E}{\hbar} t \right) \sin \left( \frac{\varphi_\alpha - \varphi_\beta}{2} \right) - \sin \left( \frac{E}{\hbar} t \right) \cos \left( \frac{\varphi_\alpha - \varphi_\beta}{2} \right).$$

Finally, using Eq. (60), $|\psi(t)\rangle$ in Eq. (59) becomes $|\psi(t)\rangle$ in Eq. (54). For completeness, observe that

$$|\psi(0)\rangle = |A\rangle, \quad \text{and} \quad |\psi \left( \frac{\hbar}{E} \frac{\varphi_\alpha - \varphi_\beta}{2} \right)\rangle = e^{i\varphi_\alpha} e^{-i\varphi_\beta} |B\rangle \approx |B\rangle.$$

Moreover, as a consistency check, observe that $\langle \psi(t) | \psi(t) \rangle = 1$ where $\langle A|B\rangle = \exp \left( -i \frac{\varphi_\alpha - \varphi_\beta}{2} \cos (\varphi_\alpha - \varphi_\beta) / 2 \right)$.  

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TABLE I: Schematic description of the steps performed to prove the geodesicity of the path traced by the pure state vector subjected to the chosen quantum evolution. Each step is characterized by a temporal parameter and a corresponding parametrized state vector. The transition from $|\psi(t)\rangle$ to $|\psi_{geo}(\eta)\rangle$ allowed us to explicitly verify the geodesic nature of the path traced by the state vector $|\psi_{geo}(\eta)\rangle$.

### IV. GEODESICITY OF THE QUANTUM DYNAMICAL TRAJECTORY

In this section, exploiting geometric tools to describe pure states as presented in Section II and focusing on the Hamiltonian motion specified in Section III, we explicitly show the geodesicity (as defined in Eq. (43)) of the shortest time quantum dynamical trajectory in Eq. (54) that emerges from the chosen optimal-speed Hamiltonian evolution $H$ in Eq. (53). While doing so, we devote special attention to the parametrization of quantum geodesic paths.

We begin by using the formalism presented in Section II to show that $|\psi(t)\rangle$ in Eq. (54) defines a geodesic path on the Bloch sphere. We start by performing a sequence of two changes of parametrization of the vector state $|\psi(t)\rangle$ to recast $|\psi(t)\rangle$ as $|\psi_{geo}(\xi)\rangle$ given by

$$|\psi(t)\rangle = |\psi_{geo}(\xi(t))\rangle,$$

where,

$$|\psi_{geo}(\xi)\rangle \equiv N_\xi(\xi) \left[ (1 - \xi) |A\rangle + \xi e^{i\phi} |B\rangle \right],$$

with $0 \leq \xi \leq 1$. Recall that the normalization factor $N_\xi(\xi)$ and the phase factor $e^{i\phi}$ in Eq. (63) are given by

$$N_\xi(\xi) \equiv \frac{1}{\sqrt{1 - 2\xi (1 - \xi) (1 - |\langle A|B\rangle|)}},$$

and

$$e^{i\phi} \equiv \frac{\langle B|A\rangle}{|\langle B|A\rangle|} = \frac{\langle A|B\rangle^*}{|\langle B|A\rangle|} = \frac{e^{i\varphi_a - \varphi_B}}{\sqrt{c_{\varphi_a - \varphi_B}}} \cos \left( \frac{\varphi_a - \varphi_B}{2} \right) = e^{i\varphi_a - \varphi_B}.$$

Therefore, to recast $|\psi(t)\rangle$ as $|\psi_{geo}(\xi(t))\rangle$, we need to solve the following algebraic system of equations

$$\begin{cases}
N_\xi(\xi) \xi = \frac{\sin(\xi t)}{\sin(\frac{\varphi_a - \varphi_B}{2})}, \\
N_\xi(\xi) (1 - \xi) = \cos \left( \frac{\xi t}{\hbar} \right) - \frac{\cos \left( \frac{\varphi_a - \varphi_B}{2} \right)}{\sin \left( \frac{\varphi_a - \varphi_B}{2} \right)} \sin \left( \frac{\xi t}{\hbar} \right) .
\end{cases}$$

After some algebra, we get from Eq. (66) that

$$\xi(t) = \frac{\sin \left( \frac{\xi t}{\hbar} \right)}{\cos \left( \frac{\xi t}{\hbar} \right) \sin \left( \frac{\varphi_a - \varphi_B}{2} \right) + \left[ 1 - \cos \left( \frac{\varphi_a - \varphi_B}{2} \right) \right] \sin \left( \frac{\xi t}{\hbar} \right)}.$$

For completeness, we remark that we correctly obtain from the relation in Eq. (67) that

$$\xi(t) = 0, \text{ and } \xi \left( \frac{\hbar}{E} \frac{\varphi_a - \varphi_B}{2} \right) = 1.$$
We also find that the normalization factor $N_{\xi}(\xi)$ in Eq. (64) becomes

$$N_{\xi}(\xi) = \frac{\cos \left(\frac{E t}{\hbar}\right) + 1 - \cos \left(\frac{\xi - \varphi}{2}\right)}{\sin \left(\frac{\xi - \varphi}{2}\right)} \sin \left(\frac{E t}{\hbar}\right).$$

(69)

In our second reparametrization, we recast $|\psi_{\text{geo}}(\xi)\rangle$ in Eq. (63) as

$$|\psi_{\text{geo}}(\xi)\rangle = \left|\tilde{\psi}_{\text{geo}}(\eta(\xi))\right\rangle$$

(70)

where,

$$\left|\tilde{\psi}_{\text{geo}}(\eta)\right\rangle \overset{\text{def}}{=} N_{\eta}(\eta) \left[\cos \left(\frac{\eta}{2}\right)|A\rangle + \sin \left(\frac{\eta}{2}\right)e^{i\phi}|B\rangle\right],$$

(71)

with $0 \leq \eta \leq \pi$. Within this new parametrization, the normalization factor $N_{\eta}(\eta)$ and the phase factor $e^{i\phi}$ in Eq. (71) are given by

$$N_{\eta}(\eta) \overset{\text{def}}{=} \frac{1}{\sqrt{1 + \sin(\eta)|\langle B|A\rangle|}}$$

and $e^{i\phi} \overset{\text{def}}{=} \frac{|\langle B|A\rangle|}{|\langle B|A\rangle|}$,

(72)

respectively. In particular, we find

$$\xi(\eta) = \frac{\tan \left(\frac{\eta}{2}\right)}{1 + \tan \left(\frac{\eta}{2}\right)},$$

(73)

that is,

$$\eta(\xi) = 2 \tan^{-1} \left(\frac{\xi}{1 - \xi}\right),$$

(74)

with $0 \leq \eta(\xi) \leq \pi$. In summary, $|\psi(t)\rangle$ in Eq. (54) can be recast as

$$|\psi(t)\rangle = |\psi_{\text{geo}}(\xi(t))\rangle = \left|\tilde{\psi}_{\text{geo}}(\eta(\xi(t)))\right\rangle,$$

(75)

where,

$$\eta(\xi(t)) \overset{\text{def}}{=} 2 \tan^{-1} \left(\frac{\xi(t)}{1 - \xi(t)}\right),$$

(76)

with $\xi(t)$ given in Eq. (67). Substituting Eq. (67) into Eq. (76), we obtain

$$\eta(t) \overset{\text{def}}{=} 2 \tan^{-1} \left[\frac{\sin \left(\frac{E t}{\hbar}\right)}{\sin \left(\frac{\theta_{\text{FS}}}{2}\right)} \cos \left(\frac{E t}{\hbar}\right) - \cos \left(\frac{\theta_{\text{FS}}}{2}\right) \sin \left(\frac{E t}{\hbar}\right)\right],$$

(77)

with $0 \leq t \leq \frac{\hbar\theta_{\text{FS}}}{2E}$. For consistency check, note that we correctly have $\eta(0) = 0$ and $\eta(t) \rightarrow \pi$ as $t \rightarrow (\hbar\theta_{\text{FS}})/2E$ since

$$\lim_{t \rightarrow \frac{\hbar\theta_{\text{FS}}}{2E}} \frac{\sin \left(\frac{E t}{\hbar}\right)}{\sin \left(\frac{\theta_{\text{FS}}}{2}\right)} \cos \left(\frac{E t}{\hbar}\right) - \cos \left(\frac{\theta_{\text{FS}}}{2}\right) \sin \left(\frac{E t}{\hbar}\right) = +\infty.$$ 

(78)

Having recast $|\psi(t)\rangle$ in Eq. (54) as $|\tilde{\psi}_{\text{geo}}(\eta)\rangle$ in Eq. (71) with $\eta = \eta(t)$ in Eq. (77), we can follow the analysis outlined in the last part of Section II to verify that the distance of the path traced out by $t \mapsto |\psi(t)\rangle$ between $|A\rangle$ and $|B\rangle$ on the Bloch sphere is equal to the Wootters distance. Therefore, we can conclude that the Hamiltonian $H$ in Eq. (53) gives rise to a trajectory $t \mapsto |\psi(t)\rangle$ that represents a geodesic line on the Bloch sphere. This concludes our quantitative discussion. However, before presenting our final remarks, we briefly present two alternative ways to check the geodesicity of a curve in ray space in the next section.
V. ALTERNATIVE CONSISTENCY CHECKS OF THE GEODESICITY OF A CURVE

In the previous section, we have verified in an explicit manner the geodesicity of the curve in projective Hilbert space emerging from the optimal-speed Hamiltonian in Eq. (53) by showing that the curve in Eq. (54) is a minimal length curve (Eq. (43)). In this section, we check the geodesicity property in two additional manners. In the first verification, specified by a necessary and sufficient criterion, we observe that the curve in ray space is a unit geometric efficiency curve \(|\psi(\xi(t))\rangle\) with \(\eta_{\text{geo}}(\eta)\). In the second verification, which provides a necessary but not sufficient criterion, we check that the curve in ray space is a null phase curve (i.e., the geometric phase vanishes \([5, 37, 38]\)).

A. Geodesics as unit geometric efficiency curves

We begin by discussing a geometric measure of efficiency for a quantum evolution \([30]\). Consider an evolution of a state vector \(|\psi(t)\rangle\) specified by the Schrödinger equation, \(i\hbar \partial_t |\psi(t)\rangle = H(t)|\psi(t)\rangle\), with \(t_A \leq t \leq t_B\). Then, following \([30]\), a geometric measure of efficiency \(\eta_{\text{geo}}\) with \(0 \leq \eta_{\text{geo}} \leq 1\) for such a quantum evolution is given by \([20]\)

\[
\eta_{\text{geo}} = 1 - \frac{\Delta s}{s} = \frac{2\cos^{-1} \left( \left|\langle A|B\rangle\right| \right)}{2 \int_{t_A}^{t_B} \Delta E(t) \, dt'},
\]

where \(\Delta s \equiv s - s_0\), \(s_0\) is the distance along the shortest geodesic path joining the distinct initial \(|A\rangle \equiv |\psi(t_A)\rangle\) and final \(|B\rangle \equiv |\psi(t_B)\rangle\) states on the projective Hilbert space and finally, \(s\) denotes the distance along the dynamical trajectory traced by the state vector \(|\psi(t)\rangle\) with \(t_A \leq t \leq t_B\). Clearly, a geodesic quantum evolution is specified by the condition

\[
\eta_{\text{geo}} = 1.
\]

Note that the numerator in Eq. (79) specifies the angle between the state vectors \(|A\rangle\) and \(|B\rangle\) and is equal to the Wootters distance \([6]\). Instead, the denominator in Eq. (79) describes the integral of the infinitesimal distance \(ds\) along the evolution curve in ray space \([30]\),

\[
ds = 2 \sqrt{\Delta E(t)} \sqrt{\hbar} \, dt,
\]

with \(\Delta E \equiv \left( \langle \psi|H^2(t)|\psi\rangle - \langle \psi|H(t)|\psi\rangle^2 \right)^{1/2}\) denoting the square root of the dispersion of the Hamiltonian operator \(H(t)\). Remarkably, Anandan and Aharonov demonstrated that the infinitesimal distance \(ds \equiv 2 \sqrt{\Delta E(t)} \sqrt{\hbar} \, dt\) is linked to the Fubini-Study infinitesimal distance \(d_{\text{FS}}\) \([30]\),

\[
ds_{\text{FS}} \left( |\psi(t)\rangle, |\psi(t + dt)\rangle \right) \equiv 4 \left[ 1 - \left| \langle \psi(t)|\psi(t + dt)\rangle \right|^2 \right] = 4 \frac{\Delta E^2(t)}{\hbar^2} \, dt^2 + \mathcal{O}\left( dt^3 \right),
\]

where \(\mathcal{O}\left( dt^3 \right)\) denotes an infinitesimal quantity equal or higher than \(dt^3\). From the link between \(d_{\text{FS}}\) and \(ds\), one concludes that \(s\) is proportional to the temporal integral of the energy uncertainty \(\Delta E\) of the quantum system and,
in addition, specifies the distance along the quantum evolution of the system in ray space as measured by the Fubini-Study metric. We emphasize that \( \Delta s \) is equal to zero and the efficiency \( \eta \) in Eq. (79) reduces to one when the dynamical curve coincides with the shortest geodesic path joining the initial and final states. Obviously, the shortest possible distance between two orthogonal quantum states in ray space is \( \pi \).

In our problem, \( \langle A | B \rangle \overset{\text{def}}{=} e^{-i\alpha_\alpha - \varphi_\beta} \cos(\varphi_\alpha - \varphi_\beta) / 2 \), \( \Delta E^t \overset{\text{def}}{=} E = \text{const.} \), \( t_A \overset{\text{def}}{=} 0 \), and \( t_B \overset{\text{def}}{=} (\hbar/E) [\varphi_\alpha - \varphi_\beta] / 2 \). Therefore, a simple calculation yields a unit geometric efficiency in Eq. (79), \( \eta = 1 \). Therefore, the geodesicity condition is properly satisfied.

### B. Geodesics as null phase curves

In this second subsection, we check that the curve in ray space is a null phase curve [5, 37, 38]. We shall check this condition by showing that the total phase along the horizontal lift of a geodesic in the projective Hilbert space is zero. This is a necessary (but not sufficient; there are null phase curves that are not necessarily geodesic curves) condition to be satisfied by a geodesic curve in ray space as recently stressed in Refs. [38]. Before presenting the simple check, we provide some basic mathematical background along with some relevant historical remarks on the concept of geometric phase.

**Basic background.** Let \( \mathcal{H} \{ 0 \} \) denote the Hilbert space described by an \( (N+1) \)-dimensional complex vector space of normalized state vectors \( \{ \psi(t) \} \). In quantum mechanics, a physical state is not represented by a normalized state vector \( \psi(t) \). Instead, physical states are represented by a ray. A ray is the one-dimensional subspace \( \{ e^{i\phi(t)} \psi(t) : e^{i\phi(t)} \in U(1) \} \) to which this vector \( \psi(t) \) belongs. Two state vectors \( \psi_1(t) \) and \( \psi_2(t) \) that belong to the same ray are equivalent, \( \psi_1(t) \sim \psi_2(t) \), if \( \psi_1(t) = e^{i\phi_1\psi_2(t)} \psi_2(t) \) for some \( \phi_1(t) \in \mathbb{R} \). The equivalence relation \( \sim \) gives rise to equivalence classes on the \( (2N+1) \)-dimensional sphere \( S^{2N+1} \). The set of all equivalence classes \( S^{2N+1} / U(1) \) determines the space of rays (that is, the space of physical states). In general, \( S^{2N+1} / U(1) \) is called the projective Hilbert space \( \mathcal{P}(\mathcal{H}) \).

This relation between state vectors in Hilbert space and rays in projective Hilbert space mediated by phase factors can be nicely described in terms of the fiber bundle formalism [5, 39]. Roughly speaking, the main ingredients of a fiber bundle are a total space \( E \), a base space \( M \), a fiber space \( F \), a group \( G \) acting on the fibers, and a projection map \( \pi \) that projects the fibers above to points in \( M \). In quantum mechanics, \( \mathcal{H} \{ 0 \} \) plays the role of \( E \), \( \mathcal{P}(\mathcal{H}) \) plays the role of \( M \), \( U(1) \) plays the role of \( G \), fibers in \( F \) are represented by all unit vectors from the same ray and, finally, the projection map \( \pi \) given by

\[
\pi : \mathcal{H} \{ 0 \} \ni \psi(t) \mapsto \pi(\psi(t)) \overset{\text{def}}{=} |\psi(t)\rangle \langle \psi(t)| \in \mathcal{P}(\mathcal{H}),
\]

plays the part of the projection in the fiber bundle construction. For more details on the fiber bundle formalism, we refer to Refs. [39, 41]. A schematic summary of the fiber bundle formalism in quantum mechanics appears in Table II.

Correctly, a path \( t \mapsto |\psi(t)\rangle \) with \( 0 \leq t \leq T \) traced out by a state vector \( |\psi(t)\rangle \) satisfying the evolution equation \( i\hbar h_t |\psi(t)\rangle = H(t) |\psi(t)\rangle \) lies in \( \mathcal{H} \{ 0 \} \). The corresponding path in \( \mathcal{P}(\mathcal{H}) \) can be determined by projecting the path in \( \mathcal{H} \{ 0 \} \) down onto a path in \( \mathcal{P}(\mathcal{H}) \). It is interesting to point out that in a cyclic quantum evolution, the initial and final physical states are the same. Therefore, cyclic evolutions are closed paths in \( \mathcal{P}(\mathcal{H}) \). However, closed paths in \( \mathcal{P}(\mathcal{H}) \) can correspond to open paths in \( \mathcal{H} \{ 0 \} \). Therefore, the initial and final state vectors in a cyclic evolution are on the same fiber, but at different “heights”. Heights are characterized by the emergence of an overall phase factor between the state vectors of interest, \( |\psi(T)\rangle = e^{i\phi_{\text{tot}}(T)} |\psi(0)\rangle \). Part of the total phase \( \phi_{\text{tot}}(T) \) (that is, the geometric phase \( \phi_{\text{geometric}}(T) \)) depends only on the geometry of the fiber bundle.
The geometry of the fiber bundle is characterized by a connection that helps comparing fibers at different points on $\mathcal{P}(\mathcal{H})$. The connection, in turn, can be introduced once one considers the decomposition of the tangent space $T[H\setminus\{0\}]$ to $H\setminus\{0\}$ in terms of an horizontal space $H_{T[H\setminus\{0\}]}$ and a vertical subspace $V_{T[H\setminus\{0\}]}$:

$$T[H\setminus\{0\}] = V_{T[H\setminus\{0\}]} \oplus H_{T[H\setminus\{0\}]}.$$  

(83)

In terms of Eq. (83), a vector $|\psi(t)\rangle \in T[H\setminus\{0\}]$ with $i \psi(t) = d\psi/dt$ can be decomposed as

$$|\psi(t)\rangle = \langle \psi(t)|\psi(t)\rangle + |\hbar\psi(t)\rangle,$$

(84)

where $|\hbar\psi(t)\rangle \equiv \left( |\psi(t)\rangle - \langle \psi(t)|\psi(t)\rangle \right) \perp |\psi(t)\rangle$, $|\hbar\psi(t)\rangle \in H_{T[H\setminus\{0\}]}$, and $\langle \psi(t)|\psi(t)\rangle |\psi(t)\rangle \in V_{T[H\setminus\{0\}]}$.

Observe that the connection one-form $A(t) \equiv -i \langle \psi(t)|\dot{\psi}(t)\rangle$ appears in the definition of $|\hbar\psi(t)\rangle$, the covariant derivative of $|\psi(t)\rangle$. Indeed, consider the projector $P_\parallel \equiv |\psi\rangle \langle \psi|$ onto $|\psi\rangle$ and the projector $P_\perp = I - |\psi\rangle \langle \psi|$ onto states perpendicular to $|\psi\rangle$. Then, $P_\parallel + P_\perp = I$ is a resolution of the identity operator $I$, $P_\parallel^2 = P_\parallel$, $P_\perp^2 = P_\perp$, and $P_\parallel P_\perp = P_\perp P_\parallel = 0$ is the null operator. Using the definitions of $A(t)$, $P_\parallel$, and $P_\perp$, we find $|\hbar\psi(t)\rangle = |\dot{\psi}(t)\rangle - iA(t)|\psi(t)\rangle$.

For further details on this construction, see Ref. [41]. Finally, once a connection is specified, the concept of a horizontal lift can be properly defined. In particular, a horizontal lift is specified by lifting the tangent vectors of a curve in $\mathcal{P}(\mathcal{H})$ to horizontal tangent vectors of a curve in $H\setminus\{0\}$. We are now ready to introduce the concept of Berry’s geometric phase.

Remarks on the geometric phase. In June of 1983, considering an adiabatic (i.e., slow varying parameters) and cyclic quantum evolution of a quantum state $|\psi(t)\rangle$ as an eigenstate of the Hamiltonian $H(t)$ of the system in a time interval $T$, Michael Berry discovered that the quantum state gains a geometrical phase factor in addition to the usual dynamical phase factor.

The original state $|\psi(0)\rangle$ returns to itself up to a phase factor,

$$|\psi(T)\rangle = e^{i\phi_{\text{tot}}(T)}|\psi(0)\rangle.$$

(85)

The total phase $\phi_{\text{tot}}(T)$ in Eq. (85) is the sum of the dynamical phase $\phi_{\text{dynamical}}(T)$ and the geometric phase $\phi_{\text{geometric}}(T)$,

$$\phi_{\text{tot}}(T) = \phi_{\text{dynamical}}(T) + \phi_{\text{geometric}}(T),$$

(86)

with $\phi_{\text{dynamical}}(T)$ defined as

$$\phi_{\text{dynamical}}(T) \equiv -\int_0^T \langle \psi(t)|H(t)|\psi(t)\rangle \frac{d\psi}{d\psi} dt.$$  

(87)

In the fiber bundle description of quantum mechanics, one can regard the space of normalized states as a fiber bundle over the space of rays, with the bundle having a natural connection that allows to compare the phases on two neighboring states. In October of 1983, using this fiber bundle formalism, Barry Simon interpreted in the geometric phase $\phi_{\text{geometric}}(T) \equiv \phi_{\text{tot}}(T) - \phi_{\text{dynamical}}(T)$ as the line integral of the Abelian connection one-form $A$ (for the geometrical phase on $\mathcal{P}(\mathcal{H})$) over a closed path $l$ in the projective Hilbert space $\mathcal{P}(\mathcal{H})$,

$$\phi_{\text{geometric}}(T) = \oint \phi_A = \int_{\Sigma} \mathcal{F}.$$

(88)

The second equality in Eq. (88) is a consequence of Stokes’ theorem (i.e., recasting a line integral as a surface integral), with $\mathcal{F} \equiv DA$ denoting the Abelian curvature two-form and $\Sigma$ being any surface bounded by $l$ in $\mathcal{P}(\mathcal{H})$. The geometrical nature of this phase in Eq. (88) is justified by its dependence solely on the closed path evolution of the ray in the projective Hilbert space and, moreover, by its complete independence on any aspect of the Hamiltonian that governs the dynamical evolution. In 1987, Aharonov and Anandan showed in the adiabaticity requirement is unnecessary for the emergence of geometric phases in cyclic quantum evolutions. In 1988, Samuel and Bhandari introduced in a general setting for Berry’s geometric phase in which neither unitarity nor cyclicity of the quantum evolution are required. In a series of works between 1991 and 1995, Pati devoted a serious effort in describing the relation between phases and distances in quantum evolutions, both cyclic and noncyclic. In 1995, Pati
expressed the geometric phase factor $e^{i\phi_{\text{geometric}}(T)}$ in a noncyclic evolution with $0 \leq t \leq T$ in terms of the horizontal lift $|\tilde{\psi}(t)\rangle$ of a curve in the projective Hilbert space $\mathcal{P}(\mathcal{H})$ as

$$e^{i\phi_{\text{geometric}}(T)} \overset{\text{def}}{=} \frac{\langle \tilde{\psi}(0) | \tilde{\psi}(T) \rangle}{\langle \tilde{\psi}(T) | \tilde{\psi}(T) \rangle}.$$  \hspace{1cm} (89)

The horizontal lift $|\tilde{\psi}(t)\rangle$ is defined in terms of the state $|\psi(t)\rangle$ that satisfies $i\hbar \partial_t |\psi(t)\rangle = \mathcal{H}(t) |\psi(t)\rangle$ as [48],

$$|\tilde{\psi}(t)\rangle \overset{\text{def}}{=} e^{i\int_0^t \mathcal{H}(t') dt'} |\psi(t)\rangle,$$  \hspace{1cm} (90)

with $\langle \tilde{\psi}(t) | \partial_t \tilde{\psi}(t) \rangle = 0$. Interestingly, observe that the phase factor in the horizontal lift $|\tilde{\psi}(t)\rangle$ in Eq. (90) can be expressed in terms of the connection $\mathcal{A}(t)$. Indeed, since $i\hbar \partial_t |\psi(t)\rangle = \mathcal{H}(t) |\psi(t)\rangle$ and $\langle \psi(t) | i\mathcal{A}(t) | \psi(t) \rangle = 0$, we have $i \langle \psi(t) | \mathcal{H} | \psi(t) \rangle = -i\hbar \mathcal{A}(t)$. Therefore, Eq. (90) can be recast as

$$|\tilde{\psi}(t)\rangle \overset{\text{def}}{=} e^{-i\int_0^t \mathcal{A}(t') dt'} |\psi(t)\rangle.$$  \hspace{1cm} (91)

In 1993, Mukunda and Simon provided in [37] a very general setting for the geometric phase for any smooth open curve of unit vectors in Hilbert space by employing the kinematics of the Hilbert space of states of a general quantum system along with a properly defined gauge transformation group. In particular, they showed that the total phase along the horizontal lift of a geodesic curve in $\mathcal{P}(\mathcal{H})$ is zero. Therefore, since the dynamical phase along an horizontal lift is zero, they concluded that geodesics are null (geometric) phase curves. In the geodesic curve scenario, $\langle \tilde{\psi}(0) | \tilde{\psi}(T) \rangle$ is real and positive. Therefore, $\langle \tilde{\psi}(0) | \tilde{\psi}(T) \rangle = |\langle \tilde{\psi}(0) | \tilde{\psi}(T) \rangle|$ in Eq. (89), $e^{i\phi_{\text{geometric}}(T)} = 1$ and, finally, $\phi_{\text{geometric}}(T) = 0$. In conclusion,

$$\phi_{\text{geometric}}(T) = 0$$  \hspace{1cm} (92)

is the necessary but not sufficient condition for geodesic behavior of a curve $\gamma_{\text{geodesic}}$ in $\mathcal{P}(\mathcal{H})$. Before presenting this verification, we present for completeness a quick remark. As pointed out earlier, we stated there exist curves connecting two pure states that are not necessarily the shortest curves for which, however, the gained geometric phase is zero. These curves generalize the concept of geodesic curve and are known in the literature as null phase curves (NPCs, [49][51]). A geodesic is a NPC. The converse, in general, is false. The generalization involved in transitioning from geodesics to NPCs emerges especially when the dimensionality of the complex Hilbert space is greater than or equal to three. Indeed, in a two-dimensional Hilbert space, the ray space is the Poincaré sphere and NPCs are great circles arcs or geodesics on the sphere $S^2$. A NPC connecting any two nonantipodal points on $S^2$ is either the corresponding geodesic, or it may navigate some extended region of the corresponding great circle. When $\dim_{\mathbb{C}} \mathcal{H} > 2$, NPCs are more numerous than geodesics and there are infinitely many NPCs connecting any two nonantipodal points in ray space (against a single geodesic). Examples of NPCs that are not (free) geodesics can be found in the framework of the so-called constrained geodesics [49].

Unlike free geodesics, constrained geodesics are paths of minimum length connecting pairs of points on a smooth submanifold of the complete ray space that specifies the physical system under consideration. Two illustrative physical examples of such curves are constrained geodesics on the submanifolds of single mode coherent states and normalized Gaussian pure states. In both cases, the constrained geodesics differ from the free geodesics. In the Gaussian case, for instance, constrained geodesic paths are assumed to be traversing solely centered normalized Gaussian wave functions and not even superpositions of Gaussians are to be considered. However, in both examples, it happens that constrained geodesics are also null phase curves. For more details, we refer to Ref. [49]. For a formal definition of a NPC in terms of a real and positive Bargmann invariant of third order, we refer to Ref. [50]. We now return to the verification.

The verification. We note that the horizontal lift $|\tilde{\psi}(t)\rangle$ in Eq. (90) equals $|\psi(t)\rangle$ in Eq. (54). Indeed, in our problem, $\langle \psi(t) | \mathcal{H}(t) | \psi(t) \rangle = 0$ with $\mathcal{H}(t)$ in Eq. (53). Alternatively, using Eq. (51), one can use brute force to verify that $\langle \psi(t) | \partial_t \psi(t) \rangle = iA(t) = 0$. We also observe that $T \overset{\text{def}}{=} (\hbar / E)[(\varphi_\alpha - \varphi_\beta) / 2], \ |\psi(0)\rangle \overset{\text{def}}{=} |A\rangle$ in Eq. (48), and $|\psi(T)\rangle \overset{\text{def}}{=} e^{i2\varphi_\alpha - \varphi_\beta / 2} |B\rangle \neq |A\rangle$ with $|B\rangle$ defined in Eq. (49). Therefore, recalling that $\langle A | B \overset{\text{def}}{=} e^{-i(\varphi_\alpha - \varphi_\beta) / 2} |\varphi_\alpha - \varphi_\beta\rangle$, we get $\langle A | \psi(T) \rangle = \cos(\theta_{FS} / 2) \in \mathbb{R}$, with $\theta_{FS} \overset{\text{def}}{=} \varphi_\alpha - \varphi_\beta = 2s_{\text{FS}} = s_{\text{geo}} \leq \pi$. Clearly, $s_{\text{FS}}$ and $s_{\text{geo}}$ denote the Fubini-Study and the geodesic distances, respectively. Finally, employing Eq. (89), we conclude that Eq. (92) is properly fulfilled.

VI. CONCLUDING REMARKS

In this paper, we presented an explicit geodesic analysis of the dynamical trajectories that emerge from the quantum evolution of a single-qubit quantum state. The evolution is governed by an Hermitian Hamiltonian operator that
Geodesics can be viewed as minimal length curves (Eq. (44)),
\[ \gamma_{A\rightarrow B} \leq \gamma_{A\rightarrow B}^{(\text{non-geodesic})}. \] (93)

Geodesics can be regarded as unit geometric efficiency curves (Eq. (80)),
\[ \eta_{\text{geometric}}^{(\text{geodesic})} = 1. \] (94)

Geodesics can be considered as null phase curves (Eq. (92)),
\[ \phi_{\text{geometric}}^{(\text{geodesic})} (T) = 0. \] (95)

As pointed out in the paper, the relations in Eqs. (93) and (94) provide necessary and sufficient geodesicity conditions. The relation in Eq. (95), instead, is only a necessary condition.

From a quantum mechanics standpoint, we focused in this paper on the study of the time-optimal evolution of closed two-level quantum systems specified by pure states driven by the Schrödinger equation. From a geometry viewpoint, we used the Fubini-Study distance measure since it is the only natural choice for a measure that defines “random states”. Specifically, the distance between density matrices for both pure and mixed states must decrease under coarse-graining (i.e., randomization), if the distance expresses statistical distinguishability [52,53]. In this respect, the Fubini-Study metric is the only monotone Riemannian metric on the space of pure quantum states. However, a more realistic scenario is the case of open system dynamics in mixed quantum states [54,55]. In this case, a number of new challenges are expected to emerge. First, one needs to consider general nonunitary quantum evolutions where the dynamics is described by a master equation in the Lindblad form. Finding exact analytical expressions for the actual dynamical trajectories traced by an open quantum system in a mixed quantum state can be rather complicated [17,57]. For an explicit discussion on some conceptual and computational difficulties in finding the time-optimal quantum evolution of mixed states governed by a master equation, we refer to Ref. [17]. Indeed, even limiting the attention to the unitary evolution of closed physical systems, optimal-time evolutions of mixed (pure) states are typically generated by time-varying (constant) Hamiltonians [20]. Second, from a geometric perspective, there are infinitely many monotone Riemannian metrics on the space of mixed quantum states as specified by the Morozova-Cencov-Petz theorem [52,53].

Therefore, there is the freedom to choose a variety of distance measures between mixed states. There is the need to study several measures, each of them with specific physical motivations, convenience, disadvantage. Arguably, the main hurdle when investigating open systems in mixed quantum states is this nonuniqueness of the metric. However, even assuming to have chosen the metric and having ready to use the actual dynamical trajectory of the system, it is generally not straightforward finding closed form expressions of geodesic paths on arbitrary manifolds of mixed quantum states equipped with Riemann metrics of statistical relevance. In a few cases, however, this task can be successfully accomplished [58]. For example, geodesic paths connecting two mixed states are known for some metrics, including the Quantum Fisher information (QFI) metric [59], the Wigner-Yanase metric [60], and the metric based on the trace distance [61]. In particular, in the case of the Bures metric [62,64] (or, alternatively, QFI metric with \( g^{(\text{QFI})}_{\mu\nu} = 4g^{(\text{Bures})}_{\mu\nu} \)), formulas for geodesic paths can be presented in terms of projections of large circles on a sphere in a purifying space [65]. Furthermore, in the Wigner-Yanase metric case [66,67], it is possible to provide explicit expressions for the geodesic path, geodesic distance, and, finally, sectional and scalar curvatures. These relations were
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C. Cafaro and P. M. Alsing, Geometric aspects of analog quantum search evolutions

C. Cafaro, S. Ray, and P. M. Alsing, Minimum time for the evolution to a nonorthogonal quantum state and upper bound of the geometric efficiency of quantum evolutions, Quantum Reports 3, 444 (2021).

C. Cafaro, D. Felice, and P. M. Alsing, Quantum Groverian geodesic paths with gravitational and thermal analogies, Eur. Phys. J. Plus 135, 900 (2020).

C. Cafaro, S. Ray, and P. M. Alsing, Optimal-speed unitary quantum time evolutions and propagation of light with maximal degree of coherence, Phys. Rev. A105, 052425 (2022).

J. Anandan and Y. Aharonov, Geometry of quantum evolution, Phys. Rev. Lett. 65, 1697 (1990).

D. Chruscinski and A. Jamiołkowski, Geometric Phases in Classical and Quantum Mechanics, Birkhäuser (2004).

J. F. Provost and G. Vella, Riemannian structure on manifolds of quantum states, Commun. Math. Phys. 76, 289 (1980).

H. P. Laba and V. M. Tkachuk, Geometric characteristics of quantum evolution: Curvature and torsion, Condensed Matter Physics 20, 130003 (2017).

M. Rovny, C. Cafaro, and A. Plastino, Information and metrics in Hilbert space, Phys. Rev. A55, 1695 (1997).

V. V. Dodonov, O. V. Man’ko, V. I. Man’ko, and A. Wünsche, Energy-sensitive and “classical-like” distances between quantum states, Physica Scripta 59, 81 (1999).

E. J. Birrittella, P. M. Alsing, and C. C. Gerry, The parity operator: Applications in quantum metrology, AVS Quantum Sci. 3, 014701 (2021).

N. Mukunda and R. Simon, Quantum kinematic approach to the geometric phase I. General Formalism, Annals of Physics 228, 205 (1993).

V. Mittal, K. S. Akhilesh, and S. K. Goyal, Geometric decomposition of geodesics and null-phase curves using Majorana star representation, Phys. Rev. A105, 052219 (2022).

M. Nakahara, Geometry, Topology, and Physics, Institute of Physics Publishing Ltd (2003).

T. Eguchi, P. B. Gilkey, and A. J. Hanson, Gravitation, gauge theories, and differential geometry, Phys. Rep. 66, 213 (1980).

A. Bohm, L. J. Boya, and B. Kendrick, Derivation of the geometric phase, Phys. Rev. A43, 1206 (1991).

M. V. Berry, Quantal phase factors accompanying adiabatic changes, Proc. R. Soc. London, Ser. A392, 45 (1984).

B. Simon, Holonomy, the quantum adiabatic theorem, and Berry’s phase, Phys. Rev. Lett. 51, 2167 (1983).

Y. Aharonov and J. Anandan, Phase change during a cyclic quantum evolution, Phys. Rev. Lett. 58, 1593 (1987).

J. Samuel and R. Bhandari, General setting for Berry’s phase, Phys. Rev. Lett. 60, 2339 (1988).

A. K. Pati, Relation between “phases” and “distance” in quantum evolution, Phys. Lett. A159, 105 (1991).

A. K. Pati, On phases and length of curves in a cyclic quantum evolution, Pramana 42, 455 (1994).

A. K. Pati, Geometric aspects of noncyclic quantum evolutions, Phys. Rev. A52, 2576 (1995).

E. M. Rabei, Arvind, N. Mukunda, and R. Simon, Bargmann invariants and geometric phases: A generalized connection, Phys. Rev. A60, 3397 (1999).

N. Mukunda, Arvind, E. Ercolessi, G. Marmo, G. Morandi, and R. Simon, Bargmann invariants, null phase curves, and a theory of the geometric phase, Phys. Rev. A67, 042114 (2003).

S. Chaturvedi, E. Ercolessi, G. Morandi, A. Ibort, G. Marmo, N. Mukunda, and R. Simon, Null phase curves and manifolds in geometric phase theory, J. Math. Phys. 54, 062106 (2013).

D. Petz, Monotone metrics on matrix spaces, Lin. Algebra Appl. 244, 81 (1996).

D. Petz and C. Sudar, Extending the Fisher metric to density matrices, in Geometry in Present Days Science, eds. O. E. Barndorff-Nielsen and E. B. Vendel, World Scientific, pp. 21-34 (1999).

M. M. Taddei, B. M. Escher, L. Davidovich, and R. L. de Matos Filho, Quantum speed limit for physical processes, Phys. Rev. Lett. 110, 050402 (2013).

A. del Campo, I. L. Egusquiza, M. B. Plenio, and S. F. Huelga, Quantum speed limits in open system dynamics, Phys. Rev. Lett. 110, 050403 (2013).

S. Deffner and E. Lutz, Quantum speed limit for non-Markovian dynamics, Phys. Rev. Lett. 111, 010402 (2013).

D. C. Brody and B. Longstaff, Evolution speed of open quantum dynamics, Phys. Rev. Research 1, 033127 (2019).

I. Bengtsson, S. Weis, and K. Zyczkowski, Geometry of the set of mixed quantum states: An apophatic approach, In: P. Kielpinski, S. Ali, A. Odzijewicz, M. Schlichenmaier, T. Voronov, (eds.) Geometric Methods in Physics. Trends in Mathematics. Birkhäuser, Basel (2013).

A. Uhlmann, Geometric phases and related structures, Rep. Math. Phys. 36, 461 (1995).

P. Gibilisco and T. Isola, Wigner-Yanase information on quantum state space: The geometric approach, J. Math. Phys. 44, 3752 (2003).

X. Cai and Y. Zheng, Quantum dynamical speedup in a nonequilibrium environment, Phys. Rev. A95, 052104 (2017).
VI CONCLUDING REMARKS

[62] D. Bures, *An extension of Kakutani’s theorem on infinite product measures to the tensor product of semifinite $\omega^*$-algebras*, Trans. Amer. Math. Soc. **135**, 199 (1969).

[63] A. Uhlmann, *The “transition probability” in the state space of a $*$-algebra*, Rep. Math. Phys. **9**, 273 (1976).

[64] M. Hübner, *Explicit computation of the Bures distance for density matrices*, Phys. Lett. **A163**, 239 (1992).

[65] J. Dittmann, *On the Riemannian metric on the space of density matrices*, Rep. Math. Phys. **36**, 309 (1995).

[66] E. P. Wigner and M. M. Yanase, *Information content of distributions*, Proc. Natl. Acad. Sci. U.S.A. **49**, 910 (1963).

[67] S. Luo, *Wigner-Yanase skew information and uncertainty relations*, Phys. Rev. Lett. **91**, 180403 (2003).

[68] E. Sjöqvist, *Geometry along evolution of mixed quantum states*, Phys. Rev. Research **2**, 013344 (2020).

[69] C. Cafaro and P. M. Alsing, *Complexity of pure and mixed qubit geodesic paths on curved manifolds*, Phys. Rev. D**106**, 096004 (2022).

[70] C. Cafaro and S. A. Ali, *Jacobi fields on statistical manifolds of negative curvature*, Physica D**234**, 70 (2007).

[71] C. Cafaro and S. Mancini, *Quantifying the complexity of geodesic paths on curved statistical manifolds through information geometric entropies and Jacobi fields*, Physica D**240**, 607 (2011).

[72] R. A. Bertlmann and P. Krammer, *Bloch vectors for qudits*, J. Phys. A: Math. Theor. **41**, 235303 (2008).

[73] P. Kurzynski, A. Kolodziejski, W. Laskowski, and M. Markiewicz, *Three-dimensional visualization of a qutrit*, Phys. Rev. A**93**, 062126 (2016).

[74] S. K. Goyal, B. Neethi Simon, R. Singh, and S. Simon, *Geometry of the generalized Bloch ball for qutrits*, J. Phys. A: Math. Theor. **49**, 165203 (2016).