Maxwell-Jüttner distribution for rigidly-rotating flows in spherically symmetric spacetimes using the tetrad formalism

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We consider rigidly rotating states in thermal equilibrium on static spherically symmetric spacetimes. Using the Maxwell-Jüttner equilibrium distribution function, constructed as a solution of the relativistic Boltzmann equation, the equilibrium particle flow four-vector, stress-energy tensor and the transport coefficients in the Marle model are computed. Their properties are discussed in view of the topology of the speed-of-light surface induced by the rotation for two classes of spacetimes: maximally symmetric (Minkowski, de Sitter and anti-de Sitter) and Reissner-Nordström black-hole spacetimes. To facilitate our analysis, we employ a non-holonomic comoving tetrad field, obtained unambiguously by applying a Lorentz boost on a fixed background tetrad.

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I. INTRODUCTION

Due to their simplicity, rigidly rotating systems in thermal equilibrium represent attractive toy-models which can be used to gain insight on the physical features of more complex systems or geometries which exhibit rotation (e.g. rotating Kerr black holes). Such systems can be interesting also from a quantum field theory point of view, where the definition of vacuum states or states at finite temperature is still an open field (for some recent results, see Ref. [1] and references therein).

On Minkowski spacetime, such systems were studied using both kinetic theory and quantum field theory [2–17] and the quantum corrections can be obtained analytically [14, 15, 17]. In this paper, we use the relativistic Boltzmann equation to study the equilibrium states and the transport coefficients of fluids undergoing rigid rotation on static spherically-symmetric background spacetimes, as well as to discuss the topology of the speed of light surface (SOL) which forms due to the rotation.

In order to obtain expressions for the transport coefficients, the Marle model is employed for the Boltzmann collision integral [18]. To facilitate our analysis, we employ non-holonomic tetrad fields [19–21] with respect to which the mass shell condition for the momentum four-vector becomes independent of the background metric, while the calculation of the transport coefficients becomes identical with that on the Minkowski spacetime [6]. The tetrad of the comoving frame is obtained by applying a pure Lorentz boost (i.e. without rotation) on the tetrad of the background metric [22]. The only degrees of freedom available in this procedure correspond to choosing the gauge for the fixed tetrad. Our formulation is sufficiently general to encompass previously studied examples, such as the Minkowski [17] and Schwarzschild [22, 23] spacetimes. We specialize our results to the cases of maximally symmetric spacetimes (Minkowski, de Sitter and anti-de Sitter), as well as for the Reissner-Nordström spacetime.

In Sec. II we discuss the tetrad formalism, which we apply to obtain the transport coefficients in the Marle model. The construction of the comoving frame for rigidly rotating flows on spherically symmetric spaces is presented in Sec. III while Sec. IV is dedicated to the discussion of rigidly rotating thermal states on maximally symmetric and Reissner-Nordström spacetimes. Section V concludes this paper.

II. THE RELATIVISTIC BOLTZMANN EQUATION

We start this section by presenting in Subsec. II A a technique for defining the comoving frame with no unspecified degrees of freedom, which relies on a fixed tetrad field corresponding to the (arbitrary) background spacetime.

Subsection II B reviews the Boltzmann equation written with respect to tetrad fields in conservative form, as described in Ref. [21]. Details regarding the transition from the generally covariant Boltzmann equation to the Boltzmann equation with respect to tetrad fields, as well as from this latter form to the conservative form of the Boltzmann equation, are presented in appendices A and B respectively.

Subsection II C introduces the Maxwell-Jüttner distribution for local thermodynamic equilibrium, as well as the conditions that the macroscopic particle number density, temperature and four-velocity must satisfy in order for the fluid to be in global thermodynamic equilibrium.

Subsection II D ends this section with a computation of the transport coefficients arising when the Marle model is used for the collision integral, which are calculated starting from the Boltzmann equation in conservative form in a manner analogous to that employed on flat space [6].
The expressions for the resulting coefficients, defined by using a covariant generalisation of their flat spacetime definitions, are identical to those obtained in flat spacetime.

**A. Comoving frame**

Considering a fixed spacetime having the line element:

\[ ds^2 = g_{\mu\nu}dx^\mu dx^\nu, \]  

an orthonormal frame \( \{e_\alpha\} \) can be chosen such that the metric is locally flat:

\[ g_{\mu\nu}e_\alpha^\mu e_\beta^\nu = \eta_{\alpha\beta}, \]  

where \( \eta_{\alpha\beta} = \text{diag}(-1,1,1,1) \) is the metric of the Minkowskian model of this spacetime. The tetrad frame vectors \( e_\alpha^\mu \) uniquely determine a set of co-vectors (one-forms) \( \{\omega^\mu\} \) through \( (2.2) \):

\[ \langle e_\beta^\mu, \omega^\alpha \rangle \equiv \omega_\mu^{\alpha} e_\beta^\mu = \delta_\beta^\alpha, \]

The choice of tetrad has 6 degrees of freedom, due to the invariance of Eq. \( (2.2) \) under Lorentz transformation. However, we consider that the tetrad \( \{e_\alpha\} \) is fixed in some predefined gauge, such that it can serve as a reference tetrad for the future development in this chapter.

A comoving frame is defined as an orthonormal frame \( \{e_\alpha\} \) with respect to which the fluid four-velocity is

\[ u^\alpha = u^\mu \omega_\mu^{\alpha} = (1,0,0,0), \]

where \( \{\omega_\mu^{\alpha}\} \) are the co-vectors corresponding to the tetrad vectors \( \{e_\alpha\} \). Eq. \( (2.4) \) implies:

\[ e_0^\mu = u^\mu. \]

Eq. \( (2.5) \) reduces the number of degrees of freedom in Eq. \( (2.2) \) to 3. We eliminate these degrees of freedom by requiring that the comoving frame \( \{e_\alpha\} \) is obtained from the local frame \( \{e_\alpha\} \) by applying a pure Lorentz boost \( L^\alpha_\beta \), such that:

\[ e_\alpha = e_\beta L^\beta_\alpha. \]

The components \( L^\beta_0 \) can be obtained by contracting Eq. \( (2.5) \) with \( \omega_\mu^{\alpha} \):

\[ L^\alpha_0 = u^\alpha \equiv u^\mu \omega_\mu^{\alpha}. \]

The above equation fixes all three degrees of freedom of the genuine Lorentz boost \( L^\beta_\alpha \), which can be written as follows:

\[ L^\alpha_\beta = \begin{pmatrix} u^\alpha & u^j \\ u^i & \delta^i_j + \frac{u^i u_j}{u^0 + 1} \end{pmatrix}. \]

It can be checked that \( L^\beta_\alpha \) is indeed a pseudo-orthogonal matrix:

\[ \eta_{\alpha\beta} L^\beta_\alpha L^\beta_\beta = \eta_{\alpha\beta}, \]

\[ \eta^\beta_\alpha L^\beta_\alpha L^\beta_\beta = \eta^\beta_\beta, \]

satisfying \( L^T L = \) \( 2 \).

**B. Conservative relativistic Boltzmann equation**

The relativistic Boltzmann equation with respect to arbitrary coordinate systems on arbitrary geometries can be written as:

\[ p^\mu \frac{\partial f}{\partial x^\mu} - \Gamma^\nu_{\mu\nu} p^\nu \frac{\partial f}{\partial p^\nu} = C[f], \]

where \( f \equiv f(x^\mu, p^\nu) \) is the Boltzmann distribution function, \( x^\mu \) represent spacetime coordinates and \( p^\mu = (p^0, p^i) \) are the components of the particle four-momentum vector. The time component \( p^0 \) of the momentum 4-vector is fixed by the mass-shell condition:

\[ g_{\mu\nu} p^\mu p^\nu = -m^2, \]

where \( g_{\mu\nu} \) are the components of the spacetime metric. The connection coefficients \( \Gamma^\nu_{\mu\nu} \) appearing in Eq. \( (2.10) \) have the following expression with respect to a coordinate frame:

\[ \Gamma^\lambda_{\mu\nu} = \frac{1}{2} g^{\lambda\sigma} (g_{\sigma\mu,\nu} + g_{\sigma\nu,\mu} - g_{\mu\nu,\sigma}), \]

where a comma denotes differentiation with respect to the coordinates, e.g. \( g_{\sigma\mu,\nu} \equiv \partial_\sigma g_{\mu\nu} \equiv \partial_{\sigma\nu} g_{\mu\nu} \).

The Boltzmann equation \( (2.10) \) can be expressed with respect to the tetrad components of the momentum vector as follows:

\[ p^\alpha e_\alpha^\mu \frac{\partial f}{\partial p^\mu} - \Gamma^i_{\alpha\beta} p^\alpha p^\beta \frac{\partial f}{\partial p^i} = C[f]. \]

For more details on the relation between Eqs. \( (2.10) \) and \( (2.13) \), we refer the reader to Appendix A. The connection coefficients \( \Gamma^i_{\alpha\beta} \) appearing in Eq. \( (2.13) \) can be obtained using:

\[ \Gamma^i_{\alpha\beta} = \eta^{\hat{\gamma}\hat{\delta}} \left( c_{\hat{\gamma}\hat{\delta}\alpha} + c_{\hat{\gamma}\beta\alpha} - c_{\alpha\beta\hat{\delta}} \right), \]

where the Cartan coefficients can be calculated from the commutators of the tetrad vectors \( 26 \):

\[ c_{\hat{\alpha}\hat{\beta}} = \langle \{e_\alpha, e_\beta, \omega_\gamma\} \rangle. \]

In order to derive transport equations for macroscopic quantities, we follow Ref. \( 21 \) and express Eq. \( (2.13) \) in conservative form:

\[ \frac{1}{\sqrt{-g}} \partial_\mu \left( \sqrt{-g} p^\alpha e_\alpha^\mu f \right) - p^0 \partial_0 \left( \Gamma^i_{\alpha\beta} p^\alpha p^\beta f \right) = C[f]. \]

For completeness, we provide the details of the derivation of the transition from Eq. \( (2.13) \) to Eq. \( (2.10) \) in Appendix B. The form \( (2.10) \) of the Boltzmann equation is particularly convenient from a numerical point of view, being directly amenable to finite-element or finite-volume numerical methods. Furthermore, Eq. \( (2.10) \) can
be used to easily derive transport equations for the moments $M^{\alpha_1 \ldots \alpha_n}_{n+1}$ of $f$:

$$\nabla_{\alpha_{n+1}} T^{\alpha_1 \alpha_2 \ldots \alpha_n}_{n+1} = \int \frac{d^3p}{p^0} C[f] p^{\alpha_1} \ldots p^{\alpha_n},$$

(2.17)

where

$$T^{\alpha_1 \alpha_2 \ldots \alpha_n}_{n+1} = \int \frac{d^3p}{p^0} f p^{\alpha_1} \ldots p^{\alpha_n} p^{\alpha_{n+1}}.$$  

(2.18)

In particular, the conservation equation for the particle four-flow $N^\alpha \equiv T^\alpha$ and stress-energy tensor $T^{\alpha \beta}$ can be obtained from Eq. (2.17) for $n = 0$ and $n = 1$:

$$\nabla_\alpha N^\alpha = 0, \quad \nabla_\beta T^{\alpha \beta} = 0.$$  

(2.19)

The right hand sides of the above equations vanish since $1$ and $p^\alpha$ are collision invariants [6], i.e.:

$$\int \frac{d^3p}{p^0} C[f] = \int \frac{d^3p}{p^0} C[f] p^\alpha = 0.$$  

(2.20)

C. Thermodynamic equilibrium

At local equilibrium, the collision integral $C[f]$ vanishes and $f$ is given by [6, 27]:

$$f^{(eq)} = \frac{Z}{(2\pi)^3} \left[ \exp \left( -\beta \mu - \beta p^\alpha u_\alpha - \varepsilon \right) \right]^{-1},$$

(2.21)

where $Z$ represents the number of degrees of freedom, $\beta = 1/T$ is the inverse local temperature, $u_\alpha$ are the covariant components of the macroscopic velocity 4-vector, and $\mu$ is the chemical potential. The constant $\varepsilon$ takes the values $-1$, $0$ and $1$ for the Fermi-Dirac (F-D), Maxwell-Jüttner (M-J) and Bose-Einstein (B-E) distributions, respectively. Since the equilibrium distributions corresponding to the Fermi-Dirac or Bose-Einstein statistics can be inferred from the M-J distribution [17, 28], the focus in this paper will be on the latter distribution, which we give explicitly below:

$$f^{(eq)} = \frac{Z}{(2\pi)^3} \exp \left( \beta \mu + \beta p^\alpha u_\alpha \right).$$

(2.22)

The chemical potential $\mu$ can be eliminated in favor of the particle number density $n$, as follows [6]:

$$f^{(eq)} = \frac{n \beta}{4 \pi m^2 K_2(m \beta)} \exp \left( \beta p^\alpha u_\alpha \right).$$

(2.23)

Direct integration of Eq. (2.23) can be employed to obtain the equilibrium expressions of the particle flow 4-vector $N^\alpha$ and of the stress-energy tensor $T^{\alpha \beta}$:

$$N^\alpha = n u^\alpha,$$

(2.24a)

$$T^{\alpha \beta} = E u^\alpha u^\beta + P \Delta^{\alpha \beta},$$

(2.24b)

where $\Delta^{\alpha \beta}$ is the projector corresponding to the hypersurface orthogonal to $u^\alpha$:

$$\Delta^{\alpha \beta} \equiv n^{\alpha \beta} + u^\alpha u^\beta.$$  

(2.25)

In Eq. (2.24b), the equilibrium energy density $E$ and pressure $P$ have the following expression:

$$E = nmG(\zeta) - P,$$

(2.26a)

$$P = \frac{n}{\beta},$$

(2.26b)

where the relativistic coldness $\zeta$ is defined as [4, 21]:

$$\zeta = m \beta$$

(2.27)

and the function $G(\zeta)$ is defined in terms of modified Bessel functions of the third kind $K_n$ [6]:

$$G(\zeta) = \frac{K_3(\zeta)}{K_2(\zeta)}.$$  

(2.28)

Thus, the inverse temperature $\beta$ uniquely determines the energy density $E$ and hydrostatic pressure $P$ through Eqs. (2.26).

It is worth noting that the trace of $T^{\alpha \beta}$ (2.24b) has the following form:

$$T^{\alpha \beta} = -m^2 \int \frac{d^3p}{p^0} f^{(eq)} = -\varepsilon + 3P = -nmK_1(\zeta)/K_2(\zeta).$$

(2.29)

We end this subsection by noting that when the fluid is in global thermodynamic equilibrium, $f = f^{(eq)}$ everywhere in the spacetime. Substituting Eq. (2.22) into the Boltzmann equation in conservative form (2.10) shows that $\beta \mu$ must be constant, while the vector field $k^\alpha = \beta u^\alpha$ must satisfy the Killing equation [6]:

$$\nabla_\alpha (\beta \mu) = 0, \quad k_{\alpha;\beta} + k_{\beta;\alpha} = 0.$$  

(2.30)

In the above, the semicolon denotes the covariant differentiation. In Section III Eqs. (2.29) will be solved for the case of rigidly rotating thermal distributions on general static spherically symmetric spacetimes.

D. Transport coefficients

In an out-of-equilibrium flow, the distribution function $f$ is generally different from $f^{(eq)}$. In the Eckart decomposition, the particle flow 4-vector $N^\alpha \equiv T^\alpha$ and the stress-energy tensor can be written as:

$$N^\alpha = nu^\alpha,$$

(2.31a)

$$T^{\alpha \beta} = Eu^\alpha u^\beta + (P + \omega)\Delta^{\alpha \beta} + 2q(u^\alpha w^\beta) + u^\alpha \varepsilon^\beta,$$

(2.31b)

where the energy density $E$ and hydrostatic pressure $P$ define the non-equilibrium inverse temperature $\beta$.
The energy density $E$, dynamic pressure $\overline{\rho}$, heat flux $q^\alpha$, and pressure deviator $\pi^{\alpha\beta}$ can be computed from $T^{\alpha\beta}$ using the following expressions:

$$E = u_\alpha u_\beta T^{\alpha\beta},$$

$$P + \overline{\rho} = \frac{1}{3} \Delta^{\alpha\beta} T^{\alpha\beta},$$

$$q^\alpha = -\lambda \Delta^{\alpha\beta} u_\gamma T^{\beta\gamma},$$

$$\pi^{\alpha\beta} = T^{\alpha\beta} - \overline{\rho} T_{\alpha\beta},$$

where the notation $A^{\alpha\beta}$ refers to:

$$A^{\alpha\beta} = \left[ \frac{1}{2} \left( \Delta^{\alpha\gamma} \Delta_{\gamma\beta} + \Delta^{\beta\gamma} \Delta_{\gamma\alpha} - \frac{1}{3} \Delta^{\alpha\beta} \Delta^{\gamma\delta} \right) \right] A^{\gamma\delta}.$$  

In the hydrodynamic limit, the following relations hold for the dynamic pressure, pressure deviator and heat flux:

$$\overline{\rho} = -\eta \nabla \cdot u,$$

$$q^\alpha = -\lambda \Delta^{\alpha\beta} \left( \nabla^{\beta} T - \frac{T}{E + P} \nabla^{\beta} P \right),$$

$$\pi^{\alpha\beta} = -2\mu \nabla^{\alpha} u_{\beta\gamma},$$

where $T = \beta^{-1}$ and the bulk viscosity $\eta$, shear viscosity $\mu$, and thermal conductivity $\lambda$ are the transport coefficients which make the subject of the present subsection.

The values of the transport coefficients depend on the form of the collision operator $C[f]$ in the Boltzmann equation. In general, $C[f]$ is a nonlinear integral operator which drives $f$ towards local thermodynamics equilibrium. The computation of the transport coefficients requires the analysis of the hydrodynamic regime of the Boltzmann equation, for the recovery of which there are various procedures, including: the Chapman-Ensksog procedure, the Grad moments method, and the renormalisation group method. To illustrate the methodology for the computation of the transport coefficients, we employ in this section the single relaxation time models proposed by Marle and Anderson-Witting.

In order for the Marle model to be consistent, the collision invariants $W$ and $\rho^\alpha$ must be preserved. Replacing Eq. in Eq. gives:

$$\nabla_\alpha N^\alpha = -\frac{m}{\tau} \int \frac{d^3p}{p^\beta} (f - f^{\text{eq}}) = \frac{1}{m\tau} \left( T^{\alpha\beta} - T_{E}^{\alpha\beta} \right)$$

$$\nabla_\beta T^{\alpha\beta} = -\frac{m}{\tau} (N^\alpha - N_{E}^\alpha).$$

The above equations can be used to determine the parameters $n_E$, $u^\alpha_E$, and $T_E$ of the Maxwell-Juttner distribution $f^{\text{eq}}$, as well as of the corresponding “equilibrium” stress-energy tensor $T_{E}^{\alpha\beta}$. Since $N^\alpha = nu^\alpha$ and $N_{E}^\alpha = n_{E}u_{E}^\alpha$, the requirement that the right hand side of Eq. vanishes imposes:

$$n_E = n, \quad u^\alpha_E = u^\alpha.$$

Furthermore, using Eq. and by contracting Eq., Eq. reduces to:

$$\frac{nmK_1(\zeta E)}{K_2(\zeta E)} = E_E - 3P_E = E - 3(P + \overline{\rho}).$$

It is important to note that $\beta_E = \zeta E/m$, defined by Eq., does not in general coincide with the inverse temperature $\beta$ of the system, which is defined in terms of the energy density $E$ corresponding to the stress-energy tensor $T_{E}^{\alpha\beta}$ computed from $f$.

The simplified version of the Chapman-Ensksog procedure is performed in three steps: first, $f$ is considered to be close to $f^{\text{eq}}$, in which case it can be written as:

$$f = f^{\text{eq}}(1 + \phi),$$

where $\phi$ is regarded as a small number. Second, the relaxation time $\tau$ is also considered to be small, such that the leading contribution on the left-hand side of Eq. is given by $f^{\text{eq}}$.

$$\nabla_\mu \rho^\alpha \epsilon_{\alpha\beta}^{\mu} f^{\text{eq}} - p^\alpha \frac{\partial}{\partial p^\beta} \left( \Gamma^i_{\alpha\beta} \frac{p^\gamma p^\delta}{p^\mu} f^{\text{eq}} \right) = -\frac{m}{\tau} f^{\text{eq}} \phi.$$  

In the third step, Eq. is used to determine the evolution equations of the equilibrium quantities $n$, $u^\alpha$, and $E_E$:

$$Dn = -n \nabla_\gamma u^\gamma,$$

$$Du^\alpha = -\frac{1}{E_E + P_E} \Delta^{\alpha\gamma} \nabla_\gamma P_E,$$

$$DE_E = -(E_E + P_E) \nabla_\gamma u^\gamma,$$

where

$$D \equiv u^\gamma \nabla_\gamma$$

is the convective derivative. Combining Eqs. and , the convective derivative of the equilibrium temperature $T_E = \beta_E$ can be obtained:

$$DT_E = -\frac{1}{\beta_E c_{\text{v}} E E} \nabla_\gamma u^\gamma.$$
where \(c_{v,E} \equiv \frac{1}{3} (\partial E_E / \partial T_E)\) is the heat capacity, which has the following expression:

\[
c_{v,E} = \zeta_E^2 + 5 \xi_E G_E - \zeta_E^2 G_E^2 - 1, \tag{2.45}
\]

where \(G_E \equiv G(\xi_E)\) is defined in Eq. (2.28).

In the fourth step, the non-equilibrium part \(\delta T^{\alpha \beta} = T^{\alpha \beta} - T^{\alpha \beta}_E\) of the stress-energy tensor is calculated by integrating Eq. (2.41) after a multiplication by \(p^\alpha p^\beta\):

\[
- \frac{m}{\tau} \delta T^{\alpha \beta} = \nabla_\gamma T^{\alpha \beta \gamma}. \tag{2.46}
\]

The third order moment \(T^{\alpha \beta \gamma}_E\) of \(f^{(eq)}\) is known analytically \[3,30\] and has the following expression:

\[
T^{\alpha \beta \gamma}_E = \eta m^2 \left[ \frac{K_1(\xi_E)}{K_2(\xi_E)} u^{\hat{\alpha}} u^{\hat{\beta}} u^{\hat{\gamma}} + \frac{G_E}{\xi_E} (u^{\hat{\alpha}} \eta^{\hat{\beta}} u^{\hat{\gamma}} + u^{\hat{\beta}} \eta^{\hat{\gamma}} u^{\hat{\alpha}} + u^{\hat{\gamma}} \eta^{\hat{\alpha}} u^{\hat{\beta}}) \right]. \tag{2.47}
\]

Performing the contractions in Eqs. (2.32) on Eq. (2.46) gives:

\[
- \frac{1}{\tau} (E - E_E) = n D \left( \frac{3 G_E}{\beta_E} \right) - 2 P E G_E \frac{Dn}{n}, \tag{2.48a}
\]

\[
- \frac{1}{\tau} (P + \bar{\omega} - P_E) = n D \left( \frac{G_E}{\beta_E} \right) + \frac{2}{3} P E G_E \nabla_\gamma u^{\hat{\gamma}}, \tag{2.48b}
\]

\[
- \frac{1}{\tau} \bar{\omega}^\gamma = nm \left( 1 + \frac{5 G_E}{m \beta_E} \right) D u^{\hat{\gamma}} + \Delta \bar{\omega}^{\hat{\beta}} \nabla_\beta \left( \frac{E_E + P_E}{m \beta_E} \right), \tag{2.48c}
\]

\[
- \frac{1}{\tau} \bar{\omega}^{\hat{\alpha}} = 2 P E G_E \nabla^{<\hat{\alpha}} u^\gamma >. \tag{2.48d}
\]

Replacing the convective derivative \(Dn\) from Eq. (2.48a) with the right hand side of Eq. (2.42a) shows that Eq. (2.39) indeed holds, allowing \(\bar{\omega}\) to be cast in the form:

\[
\bar{\omega} = \frac{1}{3} (E - E_E) - (P - P_E). \tag{2.49}
\]

The difference \(P - P_E\) can be expressed in terms of the difference \(E - E_E\) by expanding \(E\) in powers of \(\beta^{-1} - \beta^{-1}_E\), and retaining only the first order term, as follows \[3,30,31\]:

\[
E - E_E = (P - P_E) c_{v,E} + \ldots . \tag{2.50}
\]

Substituting Eq. (2.50) in Eq. (2.49) gives (to first order in \(\beta^{-1} - \beta^{-1}_E\)):

\[
\bar{\omega} = \frac{c_{v,E} - 3}{3 c_{v,E}} (E - E_E). \tag{2.51}
\]

A tedious but straightforward calculation, involving the use of Eqs. (2.42a) and (2.43) to eliminate the convective derivatives in Eq. (2.48a), yields the following expression for the coefficient of bulk viscosity:

\[
\eta = \frac{\tau P_E (3 - c_{v,E})}{3 c_{v,E}^2} \left( 20 G_E + 3 \zeta_E - 3 \zeta_E^2 G_E^2 - 2 \zeta_E^2 G_E - 10 \xi_E G_E^2 + 2 \zeta_E^2 G_E^3 \right). \tag{2.52}
\]

To set Eq. (2.46) in the form in Eq. (2.34b), the convective derivative \(Du^{\hat{\gamma}}\) can be replaced using Eq. (2.42d), while the following identities can be employed in the second term:

\[
\Delta \bar{\omega}^{\hat{\gamma}} \nabla_\beta \left( \frac{E_E + P_E}{m \beta_E} \right) = \Delta \bar{\omega}^{\hat{\beta}} \nabla_\beta \left( P_E G_E \right) = \Delta \bar{\omega}^{\hat{\beta}} \nabla_\beta P_E + \frac{1}{m} (c_{v,E} + 1) \nabla_\beta T_E. \tag{2.53}
\]

The coefficient of thermal conductivity can now be obtained:

\[
\lambda = \frac{\tau P_E}{m} (1 + c_{v,E}). \tag{2.54}
\]

Finally, the coefficient of shear viscosity can be read by comparing Eqs. (2.52d) and (2.48a).}

\[
\mu = \tau P_E G_E. \tag{2.55}
\]

The final ingredient necessary to interpret the transport coefficients is the definition of a relaxation time. According to Ref. [3], the generalisation of the relaxation time to the relativistic case yields the following expression for \(\tau\):

\[
\tau = \frac{1}{n \pi a^2 \nu}. \tag{2.56}
\]

where the mean velocity \(\nu\) can be taken to represent either the average of the Möller velocity \(g_\theta = \sqrt{(v - v_\nu)^2 - (v \times v_\nu)^2}\), or of the modulus of the velocity \(v = cp/p^0\):

\[
\langle g_\theta \rangle = \frac{2}{\xi_E K_1(\zeta_E)^2} \left[ 4 \xi^2 E K_3(2 \zeta_E) + 6 \xi E K_5(2 \zeta_E) \right.
\]

\[
\left. + (3 - 4 \xi^2 E) K_4(2 \zeta_E) - 6 \xi E K_5(2 \zeta_E) - 3 K_6(2 \zeta_E) \right], \tag{2.57a}
\]

\[
\langle \nu \rangle = \frac{\xi_E^2}{K_1(\zeta_E)} \left[ e^{-\xi E} \frac{1 + \xi E}{\xi E} - \Gamma(0, \zeta_E) \right]. \tag{2.57b}
\]

where \(K_i(z)\) is the repeated integral of \(K_0(z)\), defined as \[3,32,36\]:

\[
K_i(z) = \int_0^\infty \frac{e^{-z \cosh t}}{(\cosh t)^i} dt, \tag{2.58}
\]

while \(\Gamma(\nu, z)\) denotes the incomplete Gamma function \[33,36\]:

\[
\Gamma(\nu, z) = \int_z^\infty dt e^{-t} t^{\nu-1}. \tag{2.59}
\]
The tetrad formalism has made possible the analogy between the computation of the transport coefficients on curved spaces with respect to arbitrary coordinate systems and on Minkowski space in Cartesian coordinates. Moreover, the expressions for the coefficients of bulk viscosity, thermal conductivity and shear viscosity are identical to those obtained for Minkowski space, in agreement with Einstein’s equivalence principle. This is not surprising, since the equations defining the transport coefficients, as shown in Eqs. (2.57), are based on the definition of the mean velocity, having the value \( \langle \phi \rangle \), and the average of the Müller velocity \( \langle \eta \rangle \) of the velocity \( \langle v \rangle \) and \( \langle \varepsilon \rangle \) as functions of the relativistic coldness \( \zeta \). The limits (2.60) at small and large \( \zeta \), are confirmed: \( \langle \phi \rangle \) goes to \( \zeta \) at small \( \zeta \) and is well approximated by \( \sqrt{16/\pi \zeta} \) at large \( \zeta \); while \( \langle v \rangle \) goes to 1 (the speed of light) as \( \zeta \rightarrow 0 \), while at large \( \zeta \), it behaves like \( \sqrt{8/\pi \zeta} \).

It is also interesting to note that the maximum value of \( \langle \phi \rangle \) is attained at \( \zeta_{\text{max}} \approx 1.034 \), where \( \langle \phi \rangle \approx 0.876 \).

Using Eq. (2.56) for the definition of \( \tau \), it is convenient to introduce the following notation:

\[
\tilde{\eta} = \frac{a^2 \eta}{m}, \quad \tilde{\lambda} = a^2 \lambda, \quad \tilde{\mu} = \frac{a^2 \mu}{m}, \quad \text{(2.61)}
\]

where the “effective” transport coefficients \( \tilde{\eta}, \tilde{\lambda} \) and \( \tilde{\mu} \) only depend on \( \zeta \) (since \( P_E = \eta m/\zeta \)). Figure 2 shows a comparison of Eqs. (2.57) when the mean velocity is taken to be the average of the Möller velocity or the average of \( v \), in terms of \( \zeta \). It can be seen that, while \( \tilde{\lambda} \) and \( \tilde{\mu} \) decrease monotonically from infinite values at \( \zeta \rightarrow 0 \) to 0 as \( \zeta \rightarrow \infty \), the “effective” bulk viscosity \( \tilde{\eta} \) presents a maximum value at \( \zeta = \zeta_{\text{max}} \), while decreasing to 0 as \( \zeta \rightarrow 0 \) or \( \zeta \rightarrow \infty \). The value of \( \zeta_{\text{max}} \) depends on the definition of the mean velocity, having the value \( \zeta_{\langle \phi \rangle} = 1.342 \) and \( \zeta_{\langle v \rangle} = 1.535 \), when the mean velocity is taken as \( \langle \phi \rangle \) and \( \langle v \rangle \), respectively. A direct comparison of the curves for the transport coefficients corresponding to the relaxation time constructed using \( \langle \phi \rangle \) and \( \langle v \rangle \) reveals that their qualitative behaviour is the same in both cases. Thus, for the remainder of this paper, we will only discuss the case when \( \langle v \rangle \) is employed.
well as Eq. (2.46) describing the non-equilibrium part of the stress-energy tensor, are written in a covariant form, reducing to the Minkowski expressions presented in Ref. [8] in the flat-space limit. The effect of curvature is however felt through the covariant derivatives in Eqs. (2.33), which define the transport coefficients. It is worth writing down the expression for $\nabla \hat{\gamma} u^\hat{\gamma}$ appearing in Eq. (2.34a):

$$\nabla \hat{\gamma} u^\hat{\gamma} = e^\hat{\gamma}_\alpha \partial^\alpha u^\hat{\gamma} + \Gamma^\hat{\gamma}_{\hat{\beta} \hat{\gamma}} u^\hat{\beta} = \Gamma^\hat{\gamma}_{0 \hat{\gamma}},$$

(2.62)

since, in the comoving frame, $u^\hat{\gamma} = (1, 0, 0, 0)$.

We end this section by noting that the expressions we obtained for the connection coefficients depend on the form of the constitutive equations. In this section, we defined the transport coefficients using Eqs. (2.33), which represent the covariant form of the standard definitions on Minkowski space [24]. While other definitions of the transport coefficients are possible [22, 23], in this paper we only consider the covariant formalism presented in this section.

### III. ROTATING FLOWS IN CENTRAL CHARTS

In this section, we consider an application of the formalism presented in Sec. II to the case of flows undergoing rigid rotation on spherically symmetric spacetimes. In Subsec. III A the expression of the inverse temperature $\beta$ and 4-velocity $u^\mu$ are found by solving the Killing equation (2.30). Subsection III B defines the comoving frame using the Lorentz boost (2.8) introduced in Subsec. II A. Subsection III C ends this section with a discussion of the form of the rigidly-rotating equilibrium states on arbitrary static spherically-symmetric spacetimes.

#### A. Four-velocity

Let us consider a central chart (i.e. static and spherically symmetric) whose metric in spherical coordinates $(x^\mu) = (t, r, \theta, \phi)$ may be written in the general form

$$ds^2 = w^2 \left[ -dt^2 + \frac{dr^2}{u^2} + \frac{v^2}{w^2} (d\theta^2 + \sin^2 \theta d\phi^2) \right],$$

(3.1)

where $u$, $v$ and $w$ depend only on the radial coordinate $r$. The non-vanishing Christoffel symbols corresponding to the above metric are given below (the prime denotes differentiation with respect to $r$):

$$\Gamma^t_r u = \frac{u'}{w}, \quad \Gamma^r_r u = \frac{u^2}{w}, \quad \Gamma^r_r r = \frac{w'}{w} - \frac{u'}{u},$$

$$\Gamma^r_{\theta \theta} = \frac{u^2 v^2}{v^2} \left( \frac{w'}{w} - \frac{1}{r} \right), \quad \Gamma^\theta _{\phi \phi} = -\sin \theta \cos \theta,$$

$$\Gamma^\phi _{\phi \phi} = -\frac{u^2 v^2}{v^2} \left( \frac{u'}{w} + \frac{1}{r} - \frac{v'}{v} \right), \quad \Gamma^\theta _{\phi \phi} = \cot \theta,$$

$$\Gamma^\theta _{r \theta} = \Gamma^\phi _{r \phi} = \frac{w'}{w} - \frac{v'}{v}.$$  

(3.2)

For the remainder of this paper, we will consider rigidly rotating flows rotating with constant angular velocity $\Omega$ about the $z$ axis. The only non-vanishing components of the 4-velocity of such flows are $u^t$ and $w^\phi$, which can be found once $k^\mu = (k^t, 0, 0, k^\phi)^T$ is known. Substituting $(\mu, \nu) = (t, r)$ in Eq. (2.30) gives:

$$k_t = C_1 w^2 (r),$$

(3.3)

where $C_1$ is an integration constant. Furthermore, setting $(\mu, \nu) = (r, \phi)$ in Eq. (2.30) gives:

$$k_\phi = \Theta (\theta) \left( \frac{w r}{v} \right)^2.$$  

(3.4)

The function $\Theta (\theta)$ can be determined by setting $(\mu, \nu) = (\theta, \varphi)$:

$$\Theta (\theta) = C_2 \sin^2 \theta,$$

(3.5)

where $C_2$ is an integration constant. Let us consider the norm of $k^\mu$:

$$k^2 = g_{\mu \nu} k^\mu k^\nu = -C_1^2 w^2 + C_2^2 \left( \frac{w r}{v} \right)^2,$$

(3.6)

where $\rho = r \sin \theta$ represents the distance to the $z$ axis. Since $k^2 = -\beta^2$, it is convenient to set $C_1 = -\beta_0$ and $C_2 = \beta_0 \Omega$, such that:

$$k^\mu = \beta_0 (0, 0, 0, \Omega)^T,$$

(3.7a)

$$\beta = \beta(r, \theta) = \beta_0 w \sqrt{1 - \left( \frac{\rho \Omega}{v} \right)^2}.$$  

(3.7b)

The velocity field $u^\mu$ can be obtained by dividing $k^\mu$ (3.7a) by $\beta$ (3.7b):

$$u^\mu = \frac{\gamma}{w(r)} (0, 0, 0, \Omega)^T,$$

(3.8)

where the Lorentz factor $\gamma$ is defined as:

$$\gamma = \frac{1}{\sqrt{1 - \left( \frac{\rho \Omega}{v} \right)^2}}.$$  

(3.9)

#### B. Comoving frame

In this subsection, we follow the steps in section II A in order to define a comoving tetrad for the problem of rigidly rotating flows described in the previous subsection. The first step is to construct a tetrad with respect to which the spacetime metric (3.1) is diagonal. Such a local frame is that of the diagonal gauge, defined as:

$$e_\theta = \frac{1}{w} \partial_t, \quad \omega_\theta = w dt,$$

$$e_\phi = \frac{u}{w} \partial_r, \quad \omega_\phi = \frac{w}{u} dr,$$

$$e_\rho = \frac{w r}{\rho w} \partial_\theta, \quad \omega_\rho = \frac{rw}{\rho w} d\theta,$$

$$e_\varphi = \frac{v}{\rho w} \partial_\phi, \quad \omega_\varphi = \frac{vw}{\rho w} d\phi.$$  

(3.10)
The expression for co-frame one-forms:

\[ \mathbf{d}v \equiv \text{Eq. (2.8)}, \]

while the corresponding co-frame one-forms are given by:

\[ z^\alpha = \text{Eq. (3.8)}. \]

Before ending this section, it is worth giving the metric \( ds^2 = \text{Eq. (3.5)} \) with respect to co-rotating coordinates, defined as \( \theta = t_{\text{static}} \) and \( \varphi = \varphi_{\text{static}} - \Omega t_{\text{static}} \):

\[ ds^2 = u^2 \left[ -\gamma^{-2} dt^2 + \frac{2\rho^2 \Omega}{v^2} dt d\varphi + \frac{dr^2}{u^2} + \frac{r^2}{u^2} d\Omega^2 \right]. \]

Thus, the co-rotating observer sees \( g_{00} \to 0 \) as the Killing horizon (i.e. where \( k^\mu = u_\mu \beta^\nu \) becomes null) is approached:

\[ -g_{00} = w^2 \left( 1 - \frac{2\Omega^2}{v^2} \right) = \frac{\beta^2}{\rho^2} = 0. \]

It can be seen that, on these Killing horizons, the temperature \( \beta^{-1} \) tends to infinity, in agreement with Tolman’s law [37, 38]. In Sec. [IV] we will discuss the structure of these horizons for the particular cases of maximally-symmetric spacetimes and of the Reissner-Nordström black holes.

\section{Equilibrium states}

The distribution function describing equilibrium flows of perfect (i.e. non-viscous) fluids is the equilibrium distribution function, which gives rise to the following particle current 4-vector \( N^\mu \) and stress-energy tensor \( T^{\mu\nu} \):

\[ N^\mu_{\text{eq}} = n u^\mu, \]

\[ T^{\mu\nu}_{\text{eq}} = E u^\mu u^\nu + P \Delta^{\mu\nu}, \]

where the projector \( \Delta^{\mu\nu} \) on the hypersurface orthogonal to \( u^\mu \) is defined as

\[ \Delta^{\mu\nu} = g^{\mu\nu} + u^\mu u^\nu. \]

The energy density \( E \) and hydrostatic pressure \( P \) can be obtained from \( T^{\mu\nu} \) using the following relations:

\[ E = u^\mu u^\nu T^{\mu\nu}, \]

\[ P = \frac{1}{3} \Delta_{\mu\nu} T^{\mu\nu}. \]

Let us now consider the M-J equilibrium distribution function \( f^{\text{eq}} \) written with respect to the tetrad field \( \{ e_\alpha \} \), when \( p_\mu u^\mu = -\rho \):

\[ f^{\text{eq}} \equiv f^{\text{eq}}(Z; \beta) = \frac{Z}{(2\pi)^3} e^{-\beta\rho}, \]

where we have taken a vanishing chemical potential.

With respect to this tetrad, \( N^\alpha \) and \( T^{\alpha\beta} \) in Eq. \( [3.19] \) take the following form:

\[ N^\alpha = \int \frac{d^3p}{p^0} f^{\text{eq}} p^\alpha = (n, 0, 0, 0)^T, \]

\[ T^{\alpha\beta} = \int \frac{d^3p}{p^0} f^{\text{eq}} p^\alpha p^\beta = \text{diag}(E, P, P, P). \]

The integrals in the above equations can be performed analytically in terms of modified Bessel functions [6, 17]:
where \( \beta = \beta_0 \omega \sqrt{1 - \left(\rho \Omega/v\right)^2} \) and

\[
\widetilde{K}_n(m\beta) \equiv \frac{(m\beta)^n}{2^{n-1}(n-1)!} K_n(m\beta)
\] (3.25)

reduces to unity in the massless limit (i.e. \( m \to 0 \)) for all positive integers \( n > 0 \). Eqs. (3.24a) and (3.24c) confirm that the hydrostatic pressure is related to the particle number density \( n \) and local inverse temperature \( \beta \) through (3): \[ P = n \beta^{-1}. \] (3.26)

Setting the mass to 0 in Eqs. (3.24) yields:

\[
n_{M-1} = \frac{Z}{\pi^2 \beta^3}, \quad E_{M-1} = \frac{3Z}{\pi^2 \beta^3}, \quad P_{M-1} = \frac{Z}{\pi^2 \beta^3}
\] (3.27a)

While in this paper we only considered gas particles obeying Maxwell-Jüttner statistics, the above results can readily be extended to Bose-Einstein and Fermi-Dirac statistics, as described in Refs. [17, 28]. Since the modified Bessel functions in the expressions of \( n, E \) and \( P \) in Eqs. (3.24) decrease monotonically as their argument increases, it can be seen that these quantities also decrease monotonically with the increase of \( m \) or \( \beta \). The plots in Fig. 3 show the dependence of the energy density \( E \) (3.25b) and equation of state \( w = P/E \) (bottom) on the inverse temperature \( \beta^{-1} \), for various values of the mass \( m \). The expressions for \( E \) and \( P \) can be found in Eqs. (3.24b) and (3.24c), respectively.

The analysis of \( \beta \) will be focused on the structure of the Killing horizons seen by co-rotating observers, as described by Eq. (3.18). Furthermore, the regimes where \( \bar{\eta} \) is monotonic, or where it exhibits regions of local extremum will be discussed. For simplicity, in this section, we only consider the relaxation time (2.57b) constructed using \( \langle \nu \rangle \) (2.57b), since the results obtained using \( \langle g_\phi \rangle \) are qualitatively similar.

According to Eq. (3.18), the temperature measured by co-rotating observers diverges on the Killing horizons associated with the Killing vector in Eq. (3.7a). In the case when the metric functions \( w \) and \( v \), defined in Eq. (3.11), are non-zero and well defined everywhere in the spacetime, such surfaces represent speed of light surfaces (i.e. co-rotating observers travel at the speed of light):

\[
1 - \left( \frac{\rho \Omega}{v} \right)^2 = 0.
\] (4.1)

The second class refers to horizons which occur in spaces where \( w \) and \( v \) can vanish for some choice of the spacetime coordinates. In the absence of rotation, they coincide with the familiar event or cosmological horizons, for the cases of black holes or of the de Sitter expanding universe, respectively.

Two classes of spherically-symmetric spacetimes will be considered in what follows: the maximally symmetric spacetimes (e.g. the Minkowski, de Sitter and anti-de Sitter spacetimes) will be discussed in Subsec. IV A while Reissner-Nordström spacetimes will be the subject
A. Maximally symmetric spaces

The maximally-symmetric spaces which make the subject of the present subsection represent vacuum solutions of the Einstein equations in the presence of a cosmological constant equal to:

$$\Lambda = 3\epsilon \omega^2,$$

(4.2)

where $\epsilon = 0, 1$ and $-1$ for the Minkowski, de Sitter and anti-de Sitter spacetimes, respectively. The notation $\omega$ refers to the Hubble constant for de Sitter space and to

FIG. 5. The proper radial distance $\tilde{s}_{\text{SOL}}$ \[4.13\] between the origin and the SOL in the equatorial plane ($\sin \theta = 1$) with respect to $\tilde{\Omega} = \Omega / \omega$. The bottom, middle and top lines correspond to the cases $\epsilon = 1$ (dS), $\epsilon = 0$ (Minkowski) and $\epsilon = -1$ (AdS). In the case of Minkowski spacetime, we adopt the convention $\tilde{s} = s = \Omega^{-1}$ and $\tilde{\Omega} = \Omega$ (i.e. the parameter $\omega$ is immaterial in this case).

FIG. 6. Dependence of the effective coefficient of bulk viscosity $\tilde{\eta}$ divided by its maximum value $\tilde{\eta}_{\text{max}}$ on $\tilde{r}$ on (a) dS and (b) AdS spacetimes in the absence of rotation ($\tilde{\Omega} = 0$). Each curve corresponds to a different value of $\zeta_0 = m \beta_0$, where $\beta_0 \equiv \beta (r = 0)$ is the inverse temperature at the coordinate origin. The local maxima exhibited by $\tilde{\eta}$ (highlighted by circular points) appear only when $\zeta_0$ is large (dS) or small (adS), its locations being given by Eqs. \[4.16\] and \[4.17\] for the dS and adS spaces, respectively.
the inverse radius of curvature for anti-de Sitter space. For completeness, we also give the corresponding Ricci scalar:

\[ R = 12\epsilon\omega^2. \tag{4.3} \]

The line element can be written as [33]:

\[ ds^2 = -(1 - \epsilon\omega^2r^2)dt^2 + \frac{dr^2}{1 - \epsilon\omega^2r^2} + r^2d\Omega^2, \tag{4.4} \]

where the radial coordinate \( r \) has the range \([0, \infty)\) on AdS and \( r \in [0, \omega^{-1}]\) on dS. On dS spacetime, the surface \( r = \omega^{-1} \) represents the cosmological horizon. Eq. (4.4) can be put in the form of the generic line element in Eq. (3.1) by making the following identifications:

\[ w = v = \sqrt{1 - \epsilon\omega^2r^2}, \quad u = u^2. \tag{4.5} \]

Using Eq. (3.7b), the following expression can be obtained for the inverse temperature \( \beta \):

\[ \beta = \beta_0\sqrt{1 - (\omega^2 + \Omega^2\sin^2\theta)r^2}, \tag{4.6} \]

where \( \beta_0 \) represents the inverse temperature at the origin \( r = 0 \). Setting \( \Omega = 0 \) in Eq. (4.6) shows that, in the absence of rotations, the local temperature \( \beta^{-1} \) remains constant (Minkowski case), decreases to 0 as \( r \to \infty \) (AdS case) or increases to infinity as the cosmological horizon \( r = \omega^{-1} \) is approached (dS case).

When \( \Omega \neq 0 \), the rotation induces an SOL where \( \beta \) vanishes, such that:

\[ 1 - \epsilon\tilde{r}^2 - \rho^2\tilde{\Omega}^2 = 0, \tag{4.7} \]

where the notation

\[ \tilde{r} = \omega r, \quad \tilde{\rho} = \omega r \sin\theta, \quad \tilde{\Omega} = \frac{\Omega}{\omega}. \tag{4.8} \]

was introduced for convenience. In the case of the Minkowski spacetime (\( \epsilon = 0 \)), the SOL is located where

\[ \rho\Omega = 1. \tag{4.9} \]

Rearranging Eq. (4.7) to

\[ \rho^2\tilde{\Omega}^2 = 1 - \epsilon\tilde{r}^2 \tag{4.10} \]

shows that the repulsive nature of a positive cosmological constant (\( \epsilon = 1 \)), occurring in the case of the dS space, induces a further centrifugal effect, pulling the SOL inwards with increasing \( r \). In the AdS case (\( \epsilon = -1 \),...
the attractive nature of a negative cosmological constant \( \Lambda = -3 \omega^2 \) can play the role of a centripetal force, thus diminishing the effect of rotation.

The position \( \tilde{r}_{\text{SOL}} \) of the SOL can be found from Eq. (4.7):

\[
\tilde{r}_{\text{SOL}} = \frac{1}{\sqrt{\Omega^2 \sin^2 \theta + \epsilon}}.
\]

(4.11)

On dS, the SOL always forms inside the cosmological horizon, being located at

\[
\tilde{r}_{\text{SOL}} = \frac{1}{\sqrt{\Omega^2 \sin^2 \theta + 1}} \quad \text{(de Sitter)}.
\]

(4.12)

It can be seen that the SOL always touches the cosmological horizon on the rotation axis (i.e., \( \theta = 0 \)), where \( \tilde{r}_{\text{SOL}} = 1 \). This behaviour is illustrated in Fig. 4(a).

The situation on AdS is quite different: as shown in Ref. [44], compact manifolds do not exhibit superluminal velocities unless the rotation parameter is sufficiently large. In this case, no SOL forms if \( \tilde{\Omega} < 1 \) and the temperature \( \beta^{-1} \) remains finite throughout the spacetime. For \( \tilde{\Omega} > 1 \), the location of the SOL is given by:

\[
\tilde{r}_{\text{SOL}} = \frac{1}{\sqrt{\Omega^2 \sin^2 \theta - 1}} \quad \text{(anti-de Sitter)},
\]

(4.13)

where \( \theta \) is constrained such that \( \sin \theta \geq \tilde{\Omega}^{-1} \), as shown in Fig. 4(b). Furthermore, Eq. (4.7) implies that for a fixed value of \( \tilde{\Omega} \), the value of \( \beta \) (and indeed of all quantities derived from it, such as \( n, E, P, \tilde{\eta}, \tilde{\mu} \) and \( \tilde{\lambda} \)) is constant on the cone having its apex at the origin, for which

\[
\sin \theta = \tilde{\Omega}^{-1}.
\]

(4.14)

In particular, setting \( \tilde{\Omega} = 1 \) implies that \( \beta \) is constant throughout the equatorial plane.

A more geometric assessment of the location of the SOL is the proper radial distance \( \tilde{s} = \omega s \) from the origin to the SOL, which can be written as follows:

\[
\tilde{s} = \omega \int_0^{\tilde{r}_{\text{SOL}}} d\tilde{r} \sqrt{g_{rr}} = \begin{cases} \arcsin \frac{1}{\sqrt{\Omega^2 \sin^2 \theta + 1}}, & \text{(dS)} \\ \frac{1}{\tilde{\Omega}^{-1}}, & \text{(Minkowski)} \\ \arcsinh \frac{1}{\sqrt{\Omega^2 \sin^2 \theta - 1}}, & \text{(AdS)} \end{cases}
\]

(4.15)

Figure 6 shows that, for fixed \( \tilde{\Omega} \), the distance from the SOL to the rotation axis in the equatorial plane is larger in the AdS and smaller in the dS cases with respect to the same distance in Minkowski space.

The plots in Fig. 6 show the dependence of \( \tilde{\eta} \) on \( \tilde{\tau} \) for various values of the relativistic coldness \( \zeta_0 = m \beta_0 \equiv m \beta(r = 0) \) measured at the origin when \( \tilde{\Omega} = 0 \) for the cases of (a) the dS and (b) the adS spaces. On dS space, \( \beta \) decreases monotonically from the maximum value \( \beta_0 \) at the origin towards 0 on the cosmological horizon (where \( \tilde{r} = 1 \)). For all \( \zeta \leq \zeta_{\text{max}} \approx 1.53508 \), Fig. 6(a) implies that \( \tilde{\eta} \) also decreases monotonically, since in this regime, \( \tilde{\eta} \) shows no local extrema. However, for all \( \zeta_0 > \zeta_{\text{max}} \), \( \tilde{\eta} \) increases up to the maximum value \( \tilde{\eta}_{\text{max}} \approx 3.554 \times 10^{-4} \), attained when:

\[
\tilde{r}_{\text{max}} = \sqrt{1 - \left( \frac{\zeta_{\text{max}}}{\zeta_0} \right)^2},
\]

(4.16)

as can be seen in Fig. 6(a). On dS space, \( \zeta \) increases monotonically from \( \zeta_0 \) at the origin to infinity as \( \tilde{r} \rightarrow \infty \). Figure 6(b) shows that \( \tilde{\eta} \) also decreases monotonically to 0 as \( \tilde{r} \rightarrow \infty \) for all \( \zeta_0 \leq \zeta_{\text{max}} \), while in the case when \( \zeta_0 < \zeta_{\text{max}} \), \( \tilde{\eta} \) attains the maximum value \( \tilde{\eta}_{\text{max}} \) when

\[
\tilde{r}_{\text{max}} = \sqrt{\left( \frac{\zeta_{\text{max}}}{\zeta_0} \right)^2 - 1}.
\]

(4.17)

At non-vanishing values of \( \tilde{\Omega} \), \( \tilde{\eta} \) attains the maximum value \( \tilde{\eta}_{\text{max}} \) at

\[
\tilde{\tau} = \left[ \frac{1 - (\zeta_{\text{max}} / \zeta_0)^2}{\Omega^2 + \epsilon} \right]^{1/2}.
\]

(4.18)

For the dS space, Fig. 7(a) shows that increasing the value of \( \tilde{\Omega} \) decreases the distance to the horizon, while the location of the maximum also decreases according to:

\[
\tilde{r}_{\text{max}} \mid_{\text{dS}} = \frac{1}{\sqrt{1 + \Omega^2}} \sqrt{1 - \left( \frac{\zeta_{\text{max}}}{\zeta_0} \right)^2}.
\]

(4.19)

Figure 7(b) shows that, on Minkowski space, \( \tilde{\eta} \) is constant throughout the spacetime in the absence of rotation, while for non-vanishing values of \( \tilde{\Omega} \), it attains a maximum for all \( \zeta_0 \geq \zeta_{\text{max}} \) located at:

\[
\tilde{r}_{\text{max}} \mid_{\text{Mink}} = \frac{1}{\Omega} \sqrt{1 - \left( \frac{\zeta_{\text{max}}}{\zeta_0} \right)^2}.
\]

(4.20)

On AdS space, three regimes can be distinguished. When \( \tilde{\Omega} > 1 \), an SOL forms and the characteristics of \( \tilde{\eta} \) are similar to the case when dS space is considered. When \( \tilde{\Omega} = 1 \), \( \tilde{\eta} \) is constant throughout the equatorial plane, as implied by Eq. (4.14). These two regimes can be clearly seen in Fig. 7(c). Finally, when \( \tilde{\Omega} < 1 \), no SOL forms and \( \tilde{\eta} \) only attains a maximum when \( \zeta_0 < \zeta_{\text{max}} \), as shown in Fig. 7(d). The position of this maximum increases as \( \tilde{\Omega} \) increases.
FIG. 8. The dependence of $\beta^2/\beta_0^2$ on $\tilde{r} = r/2M$ in the equatorial plane $\sin \theta = 1$ for (a)-(c) $\tilde{Q}$ fixed at (a) 0 (Schwarzschild space), (b) 0.25 and (c) 0.5 (extremal Reissner-Nordström space), for various values of $\tilde{\Omega} = 2M\Omega$. In (d), $\tilde{\Omega}$ is fixed at 0.4 and the charge is varied from $\tilde{Q} = 0$ to $\tilde{Q} = 0.5$. The regions where $\beta^2 > 0$ represent “allowed” regions (where the temperature is finite and well defined), while the points where $\beta = 0$ represent horizons.

### B. Reissner-Nordström metric

The line element of the Reissner-Nordström metric is given by:

$$ds^2 = -\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)dt^2 + \frac{dr^2}{1 - \frac{2M}{r} + \frac{Q^2}{r^2}} + r^2 d\Omega^2,$$

(4.21)
describing the gravitational field of a black hole of mass $M$ and charge $Q$. Comparing the above equation with Eq. (3.1) gives the following expressions for the metric functions:

$$w = v = \sqrt{1 - \frac{2M}{r} + \frac{Q^2}{r^2}}, \quad u = 1 - \frac{2M}{r} + \frac{Q^2}{r^2},$$

(4.22)
such that the inverse temperature $\beta$ (3.15) becomes:

$$\beta = \beta_0 \sqrt{1 - \frac{2M}{r} + \frac{Q^2}{r^2} - \rho^2 \Omega^2},$$

(4.23)

where $\beta_0$ is the inverse temperature at infinity on the rotation axis (i.e. $z = r \cos \theta \to \pm \infty$ and $\rho = 0$). In the absence of rotation, the temperature $\beta^{-1}$ increases from $\beta_0$ at infinity to infinite values as the black hole outer horizon is approached (i.e. $r \to M + \sqrt{M^2 - Q^2}$). To better investigate the topology of the horizon structure when $\Omega > 0$, it is convenient to cast the equation $\beta^2 = 0$ as:

$$1 - \frac{1}{r} + \frac{\tilde{Q}^2}{r^2} - \tilde{\rho}^2 \tilde{\Omega}^2 = 0,$$

(4.24)

where the following notations were introduced:

$$\tilde{r} = \frac{r}{2M}, \quad \tilde{Q} = \frac{Q}{2M}, \quad \tilde{\Omega} = 2M\Omega.$$  

(4.25)

Figure 8 shows the dependence of $\beta^2/\beta_0^2$ on $\tilde{r}$ in the equatorial plane $\sin \theta = 1$ at various values of $\tilde{Q}$ and $\tilde{\Omega}$. In the regions where $\beta^2 > 0$, the local temperature $\beta^{-1}$ is finite and the hydrodynamic moments (3.21) are well defined. At small enough values of $\tilde{\Omega}$, the two intersections of the graph of $\beta^2/\beta_0^2$ with the horizontal axis correspond to the locations of the black hole horizon and of the SOL.
As \( \tilde{\Omega} \) increases, \( \beta^2 \) remains negative for all values of \( \tilde{r} \), showing that the SOL and the black hole event horizon join, forming an exclusion region which incorporates the whole equatorial plane. It is interesting to note that, at fixed \( \tilde{Q} \), the black hole horizon moves outwards as \( \tilde{\Omega} \) is increased, while the SOL moves inwards, as expected.

Figure 9(d) shows that, for fixed \( \tilde{\Omega} \), the black hole horizon moves inwards as \( \tilde{Q} \) is increased, while the SOL is pushed outwards, as expected since the charge \( Q \) has an inverse effect compared to the mass \( M \). It is interesting to note that, in the extremal Reissner-Nordström case, an equilibrium distribution of rotating particles sees an

FIG. 9. The SOL structure of the Reissner-Nordström spacetime for four values of \( \tilde{Q} = Q/2M \). The vertical axis represents the coordinate \( \tilde{z} \equiv z/2M \) along the rotation axis, while the horizontal axis represents the distance \( \tilde{\rho} = r \sin \theta /2M \) from the rotation axis. The contours represent the surfaces where \( \beta = 0 \), i.e. either the black hole horizon (only the outer horizons are shown) or the rotation horizon (i.e. where the SOL induced by the rotation forms). The case (a) shows the horizon structure for the Schwarzschild space (\( Q = 0 \)), while the case (d) represents an extremal Reissner-Nordström black hole (\( Q = M \)).
event horizon near the black hole which dresses the singularity at \( \tilde{r} = 0 \).

The horizon structure of the Reissner-Nordström spacetime is represented in Fig. 10 for various values of \( \tilde{Q} \) and \( \tilde{\Omega} \), with the Schwarzschild case (\( \tilde{Q} = 0 \)) shown in Fig. 10(a) and the extremal Reissner-Nordström case shown in Fig. 10(d). It can be seen that increasing \( \tilde{\Omega} \) at fixed \( \tilde{Q} \) pushes the black hole horizon outwards, while the SOL is pulled inwards. At large enough \( \tilde{\Omega} \), these two horizons merge, thus excluding the entire equatorial plane from the region where \( \beta^2 > 0 \).

In the absence of rotation, \( \zeta \) increases monotonically from 0 on the event horizon up to \( \zeta_0 \) as \( r \to \infty \). In this case, the dependence of \( \tilde{\eta} \) on \( r \) is non-monotonic only when \( \zeta_0 > \zeta_{\text{max}} \), as shown in Fig. 10(a). The points of maxima \( \tilde{r}_{\text{max}} \) which occur outside the outer event horizon are located at:

\[
\tilde{r}_{\text{max}} = \frac{1 + \sqrt{1 - 4\tilde{Q}^2[1 - (\zeta_{\text{max}}/\zeta_0)^2]}}{2[1 - (\zeta_{\text{max}}/\zeta_0)^2]},
\]

valid only for \( \zeta_0 > \zeta_{\text{max}} \), as shown in Figs. 10(a) and 10(b). When \( \zeta_0 < \zeta_{\text{max}} \) and \( \tilde{Q} > 0 \), the points of maxima are located inside the outer horizon.

When the rotation is switched on, the location of the points of maxima is given in the general case when \( \tilde{Q} > 0 \) by a quartic equation. Figs. 10(c) and 10(d) suggest that \( \tilde{\eta} \) can develop two points of maxima with a point of local minimum between the event and rotation horizons. For the cases shown in these plots, it can be seen that the regime where \( \tilde{\eta} \) presents a local minimum can be obtained either by increasing \( \tilde{\Omega} \) at fixed values of \( \zeta_0 \), or by increasing \( \zeta_0 \) at fixed values of \( \tilde{\Omega} \).
V. CONCLUSION

In this paper, we employed the tetrad formalism to study the properties of equilibrium states of gases undergoing rigid-rotation on spherically-symmetric spacetimes. By employing the Boltzmann equation in conservave form [21], we obtained covariant expressions for the transport coefficients when the Marle model for the collision operator is employed. Our results coincide with the expressions on flat spacetime, in agreement with the equivalence principle. In order to study rigidly-rotating thermal states, we employed a comoving tetrad field, which we obtained by performing a Lorentz boost on a fixed tetrad which diagonalizes the background spacetime metric. Using the tetrad formalism, we obtained expressions for the particle flow four-vector and stress-energy tensor corresponding to such states. Furthermore, we discussed the formation of speed of light surfaces and their topology in the cases of maximally symmetric spacetimes (Minkowski, anti-de Sitter, and de Sitter spaces), as well as in the case of the Reissner-Nordström black hole spacetime (including the Schwarzschild and extremal Reissner-Nordström cases).

In constructing the transport coefficients, we considered that the relaxation time is inversely-proportional to the average of the Möller velocity or the modulus of the velocity. Our analysis showed no qualitative differences between the results obtained using the two aforementioned definitions for the mean velocity. We found that the particle number density, energy density, equilibrium pressure, coefficient of thermal conductivity and coefficient of shear viscosity exhibit a monotonic dependence on the inverse temperature $\beta$, such that their properties can be inferred from those of $\beta$. However, since the coefficient of bulk viscosity $\eta$ attains a maximum value at a finite value of $\beta$, while decreasing to 0 as $\beta$ approaches either 0 or infinity, its properties were also studied in detail.

For the case of maximally-symmetric spacetimes, we showed that the speed-of-light surface (SOL) forms closer to the rotation axis on de Sitter (dS) space compared to Minkowski space, while on anti-de Sitter (aDS) space, it forms farther away. Furthermore, no SOL forms on aDS if the rotation parameter $\Omega$ is smaller than the inverse radius of curvature $\omega$. Our analysis also revealed that, on AdS, the inverse temperature $\beta$ and all quantities derived from it (i.e. the stress-energy tensor and the transport coefficients) are constant on cones defined by $\Omega \sin \theta = \omega$. In particular, $\beta$ is constant throughout the equatorial plane when $\Omega = \omega$. We found that the coefficient of bulk viscosity can display a non-monotonic behaviour for certain values of the relativistic coldness $\zeta_0$ measured at the origin of the spacetime.

In the Reissner-Nordström case, the SOL plays the role of a “rotational horizon”, complementing (and indeed enhancing) the event horizon of the black hole. As the rotation parameter $\Omega$ is increased, the distance between the rotational horizon and the rotation axis decreases, while the distance between the event horizon and the rotation axis increases. This is also true for the case of the extremal Reissner-Nordström black hole, where the presence of rotation induces an event horizon which dresses the singularity at the origin. Increasing the black hole charge at fixed rotation parameter has the inverse effect of decreasing the radius of the event horizon, while pushing the rotational horizon away. When the rotation parameter is non-zero, the coefficient of bulk viscosity $\eta$ can exhibit two points of maxima and one local minimum between the event horizon and the speed of light surface.

We would like to highlight the fact that the results presented in this paper represent a solid starting point for a systematic comparison between kinetic theory results and the properties of rigidly-rotating thermal states obtained using quantum field theory on curved spaces. We wish to perform such comparisons [12] for, e.g., rigidly rotating states on the Minkowski spacetime, where analytic results are available from the quantum-field theory approach [14, 15], as well as from the kinetic theory approach [17]. Furthermore, similar comparisons can be performed for the case of the anti-de Sitter space, where analytic results obtained using quantum field theory are already available [14, 11]. Finally, this work can be extended to the case of rigidly-rotating thermal states on axisymmetric space-times, such as the Kerr black hole spacetime.

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Appendix A: Boltzmann equation with respect to non-holonomic tetrad fields

In this section of the appendix, the transition from the Boltzmann equation (2.10) with respect to arbitrary coordinates $\{x^\mu\}$ to Eq. (2.13), where non-holonomic tetrad fields are employed, is presented. Following Ref. [6], it is possible to write the exterior derivative of $f$ as follows:

$$df = \left( \frac{\partial f}{\partial x^\mu} \right) p^\mu \, dx^\nu + \frac{\partial f}{\partial p^i} \, dp^i,$$

$$= \left( \frac{\partial f}{\partial x^\mu} \right) p^\mu \, dx^\nu + \frac{\partial f}{\partial p^i} \, dp^i, \quad (A1)$$

where on the first line, $df/\partial x^\mu$ is taken while considering $p^i$ to be constant. On the second line, the components $p^i = p^\rho \omega^i_\rho$ with respect to the tetrad 1-forms $\{\omega^i\}$ are kept constant. In order to derive the Boltzmann equation when the components of the momentum 4-vector are expressed with respect to non-holonomic tetrad fields, the
derivatives on the first line of Eq. (A.1) must be expressed with respect to derivatives on the second line. Using Eq. (2.11), the following expression can be obtained for the exterior derivative of \( p^0 \):

\[ dp^0 = -\frac{1}{2} \partial_{\mu\nu,\lambda} p^\mu p^\nu p_\lambda dx^\lambda - p_\lambda dp^\lambda, \]

(A2)
such that the exterior derivative of \( p^i \) can be written as:

\[ dp^i = \left( \partial_{\alpha\beta,\mu} p^\mu \frac{p^\alpha p^\beta}{p_0} \right) dx^\mu + \left( \omega^i_\alpha - \omega^i_\beta \frac{p_\beta}{p_0} \right) dp^\beta. \]

(A3)

Substituting the above result in Eq. (A.1) yields the following identifications:

\[ \left( \frac{\partial f}{\partial x^\mu} \right) p^\mu = \left( \frac{\partial f}{\partial x^\mu} \right) p^\mu + \frac{\partial f}{\partial p^\rho} \left( p^\rho \omega^\mu_\alpha - \omega^\mu_\beta g_{\alpha\beta,\mu} \frac{p^\alpha p^\beta}{p_0} \right), \]

\[ \frac{\partial f}{\partial p^\rho} = \left( \omega^\rho_\alpha - \omega^\rho_\beta \frac{p_\beta}{p_0} \right) \frac{\partial f}{\partial p^\beta}. \]

(A4)

The Boltzmann equation can now be written as:

\[ p^\mu \left( \frac{\partial f}{\partial x^\mu} \right) p^\rho - \frac{\partial f}{\partial p^\rho} \left[ -p^\rho p^\nu \omega^\mu_\nu + \Gamma^j_{\mu\nu} p^\nu p^\rho \omega^j_\nu \right. \]

\[ \left. -\omega^\mu_\nu \left( \Gamma^j_{\mu\nu} p^\nu p^\rho p_\nu - g_{\alpha\beta,\mu} \frac{p^\alpha p^\beta}{2p_0} \right) \right] = C[f]. \]

(A5)

The term involving the derivative of the metric \( g_{\alpha\beta,\mu} \) can be written in terms of the Christoffel symbols (2.12):

\[ g_{\alpha\beta,\mu} \frac{p^\alpha p^\beta}{2p_0} = \Gamma^\theta_{\alpha\beta,\mu} \frac{p^\theta}{p_0}, \]

(A6)

while the two terms inside the square bracket on the first line of Eq. (A.5) can be related to the covariant derivative of \( \omega^\mu_\nu \):

\[ -p^\rho p^\nu \partial_{\rho\nu} \omega^\mu_\nu + \Gamma^j_{\mu\nu} p^\nu p^\rho \omega^j_\nu = -p^\rho p^\nu \nabla_\nu \omega^\mu_\nu - \Gamma^0_{\mu\nu} p^\nu p^\rho \omega^\theta_\nu, \]

(A7)

which can be written in terms of the connection coefficients (2.11):

\[ \nabla_\nu \omega^\mu_\nu = \omega^\mu_\nu \nabla_\nu \omega^\nu_\mu = -\Gamma^j_{\alpha\beta,\mu} \omega^\mu_\alpha \omega^\nu_\beta. \]

(A8)

Inserting Eqs. (A6), (A7) and (A8) into Eq. (A.5) gives the final form for the Boltzmann equation:

\[ p^\rho c^\mu_\alpha \left( \frac{\partial f}{\partial x^\mu} \right) p^\rho - \Gamma^j_{\alpha\beta,\mu} \frac{p^\alpha p^\beta}{p_0} \frac{\partial f}{\partial p^\rho} = C[f]. \]

(A9)

Appendix B: Conservative form of the Boltzmann equation written with respect to non-holonomic tetrad fields

In this section of the appendix, we present a derivation of the conservative form (2.10) of the Boltzmann equation (2.13), written with respect to non-holonomic tetrad fields. Even though the relation between these equations was already found in Ref. [21], we present this calculation here for completeness.

The term involving the spatial derivatives of \( f \) in Eq. (2.13) can be put in conservative form as follows:

\[ p^\rho c^\mu_\alpha \frac{\partial f}{\partial x^\mu} = \frac{1}{\sqrt{g}} \partial_\mu \left( \sqrt{g} p^\rho c^\mu_\alpha f \right) - \Gamma^\beta_{\alpha\beta,\mu} p^\rho f, \]

(B1)

where the connection coefficient appears from taking the covariant derivative of \( e^\mu_\alpha \):

\[ \frac{1}{\sqrt{g}} \partial_\mu \left( \sqrt{g} e^\mu_\alpha \right) = \nabla_\mu e^\mu_\alpha = \omega^\mu_\rho \Gamma^\rho_{\alpha\beta} e^\mu_\beta = \Gamma^\beta_{\alpha\beta}. \]

(B2)

The second term in Eq. (2.13) can be written as:

\[ \Gamma^i_{\alpha\beta} p^\alpha p^\beta \frac{\partial f}{\partial p^i} = p^\rho \frac{\partial}{\partial p^i} \left( \Gamma^i_{\alpha\beta} \frac{p^\alpha p^\beta}{p_0} f \right) \]

\[ - f p^\theta \Gamma^i_{\alpha\beta} \frac{\partial}{\partial p^i} \left( \frac{p^\alpha p^\beta}{p_0} \right). \]

(B3)

The term on the second line in Eq. (B.3) can be computed as follows. For the case when the derivative acts on \( p^\alpha \), the following expression is obtained:

\[ \Gamma^i_{\alpha\beta} \frac{\partial p^\alpha}{\partial p^i} = \Gamma^i_{\alpha\beta} \frac{p_\alpha}{p_0} + \Gamma^j_{\alpha\beta} \frac{p_\beta}{p_0} \]

\[ - \Gamma^\rho_{\alpha\beta} \frac{p^\rho}{p_0} \]

\[ = \Gamma^0_{\alpha\beta} \frac{p^\rho}{p_0}, \]

(B4b)

where the term \( \Gamma^j_{\alpha\beta} \) in Eq. (B.4a) vanishes due to the antisymmetry of the connection coefficients in the first two indices. Furthermore, the term \( \Gamma^\rho_{\alpha\beta} p^\rho \) is \( \Gamma^\rho_{\alpha\beta} p^\rho \), since \( \Gamma^\rho_{\alpha\beta} = 0 \). Finally, Eq. (B.4b) by noting that \( \Gamma^\rho_{\alpha\beta} p^\rho = \Gamma^\rho_{\alpha\beta} p^\rho = - \Gamma^\rho_{\alpha\beta} p^\rho = \Gamma^\rho_{\alpha\beta} p^\rho \).

Next, the term involving the derivative \( p^\beta \) can be expressed as:

\[ \Gamma^i_{\alpha\beta} \frac{\partial p^\beta}{\partial p^i} = \Gamma^i_{\alpha\beta} \frac{p_\beta}{p_0} + \Gamma^i_{\alpha\beta} \frac{p_\rho}{p_0} \]

\[ - \Gamma^\rho_{\alpha\beta} \frac{p^\rho}{p_0} + \Gamma^\rho_{\alpha\beta} \frac{p^\rho}{p_0} \]

\[ = \Gamma^\theta_{\alpha\beta} \frac{p^\theta}{p_0} + \Gamma^\theta_{\alpha\beta} \frac{p^\theta}{p_0} \]

(B5b)

where the relation \( \Gamma^i_{\alpha\beta} p^\rho = \Gamma^i_{\alpha\beta} p^\rho + \Gamma^\rho_{\alpha\beta} p^\rho \) was used.

Finally, the term involving the derivative of \( p^\rho \) can be computed as follows:

\[ \Gamma^i_{\alpha\beta} \frac{\partial}{\partial p^i} \left( \frac{1}{p_0} \right) = - \Gamma^i_{\alpha\beta} \frac{p_i}{p_0} \]

\[ - \Gamma^i_{\alpha\beta} \frac{p_i}{p_0} - \Gamma^i_{\alpha\beta} \frac{1}{p_0} \]

(B6b)
Inserting Eqs. (B4c), (B5b) and (B6b) into Eq. (B3) yields:

\[ \Gamma^i_{\hat{\alpha}\hat{\beta}} p^\hat{\alpha} p^\hat{\beta} \frac{\partial f}{\partial p^i} = p^\hat{\alpha} \frac{\partial}{\partial p^i} \left( \Gamma^i_{\hat{\alpha}\hat{\beta}} p^\hat{\alpha} p^\hat{\beta} \right) + \Gamma^\gamma_{\hat{\alpha}\hat{\beta}} p^\hat{\gamma} f. \]  

(B7)

The final result is obtained by combining the above equation with Eq. (B2):

\[ \frac{1}{\sqrt{-g}} \frac{\partial}{\partial \mu} \left( \sqrt{-g} p^\mu e^\alpha f \right) - p^\hat{\alpha} \frac{\partial}{\partial p^\alpha} \left( \Gamma^i_{\hat{\alpha}\hat{\beta}} p^\hat{\gamma} p^\hat{\beta} f \right) = C[f]. \]  

(B8)