Research Article

Atomic Decompositions and John-Nirenberg Theorem of Grand Martingale Hardy Spaces with Variable Exponents

Libo Li and Zhiwei Hao

College of Mathematics and Computing Science, Hunan University of Science and Technology, Xiangtan, Hunan 411201, China

Correspondence should be addressed to Zhiwei Hao; haozhiwei@hnust.edu.cn

Received 19 October 2021; Accepted 27 November 2021; Published 19 January 2022

Academic Editor: Tianqing An

Copyright © 2022 Libo Li and Zhiwei Hao. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

Let $\theta \geq 0$ and $p(\cdot)$ be a variable exponent, and we introduce a new class of function spaces $L_{p(\cdot)} (E)$ in a probabilistic setting which unifies and generalizes the variable Lebesgue spaces with $\theta = 0$ and grand Lebesgue spaces with $p(\cdot) = p$ and $\theta = 1$. Based on the new spaces, we introduce a kind of Hardy-type spaces, grand martingale Hardy spaces with variable exponents, via the martingale operators. The atomic decompositions and John-Nirenberg theorem shall be discussed in these new Hardy spaces.

1. Introduction

The martingale theory is widely studied in the field of mathematical physics, stochastic analysis, and probability. Weisz [1] presented the atomic decomposition theorem for martingale Hardy spaces. Herz [2] established the John-Nirenberg theorem for martingales. Since then, the study of martingale Hardy spaces associated with various functional spaces has attracted a steadily increasing interest. For instance, martingale Orlicz-type Hardy spaces were investigated in [3–6], martingale Lorentz Hardy spaces were studied in [7–9], and variable martingale Hardy spaces were developed in [10–14].

Let $1 < p < \infty$, and the grand Lebesgue space $L_p(\cdot)(E)$ introduced by Iwaniec and Sbordone [15] is defined as the Banach function space of the measurable functions $f$ on finite $E$ such that

$$
\|f\|_{L_p(\cdot)} = \sup_{0 \leq \eta < \xi < 1} \left( \frac{1}{|E|} \int_E |f(x)|^{p(\cdot)\eta} \right)^{1/(p(\cdot) - \eta)} < \infty.
$$

Such spaces can be used to integrate the Jacobian under minimal hypotheses [15]. The grand Lebesgue spaces as a kind of Banach function space were investigated in the papers of Capone et al. [16, 17], Fiorenza et al. [18–21], Kokilashvili et al. [22, 23], and so forth. In particular, grand Lebesgue spaces with variable exponents were studied in [24, 25].

We find that the framework of grand Lebesgue spaces with variable exponents has not yet been studied in martingale theory. This paper is aimed at discussing the variable grand Hardy spaces defined on the probabilistic setting and showing the atomic decompositions and John-Nirenberg theorem in these new Hardy spaces. More precisely, we first present the atomic characterization of grand Hardy martingale spaces with variable exponents. To do so, we introduce the following new notations of atom.

**Definition 1.** Let $p(\cdot)$ be a variable exponent and $\theta \geq 0$. A measurable function $a$ is called a $(1, p(\cdot), \theta)$-atom (resp. $(2, p(\cdot), \theta)$-atom) if there exists a stopping time $\tau$ such that

$$
\begin{align*}
(1) & \quad \mathbb{E}_n a = 0, \forall n < \tau, \\
(2) & \quad \|a \|_{L_p(\cdot)} \leq \|S(a)\|_{L_p(\cdot)}, \|Ma\|_{L_p(\cdot)} \leq \|\chi_{(\tau, \infty)}\|_{L_p(\cdot)}^{-1}.
\end{align*}
$$

See Section 2 for the notation $L_p(\cdot, \theta)$. Denote by $A_p(\cdot, \infty)$ (resp. $A_p(\cdot, \infty)$, $A_M(\cdot, \infty)$) the collection of all sequences of triplet $(a^k, \tau_k, \mu_k)$, where $a^k$ are $(1, p(\cdot), \theta)$-atoms (resp. $(2, p(\cdot), \theta)$-atoms, $(3, p(\cdot), \theta)$-atoms), $\tau_k$ are...
stopping times satisfying (a1) and (a2) in Definition 1, and 
\( \mu_k \) are nonnegative numbers and also
\[
\left\| \sum_{k \in \mathbb{Z}} \mu_k \frac{X_{(\tau_k, \infty)}}{\|X_{(\tau_k, \infty)}\|_{L_{p(|\cdot|)}}} \right\|_{L_{p(|\cdot|)}} < \infty. \tag{2}
\]

Under these definitions, we show the atomic decompositions of the Grand Hardy martingale spaces with variable exponents (see Section 3). To be precise, we prove that for any \( f \in H_{p(|\cdot|)}^q \) (resp. \( Q_{p(|\cdot|), B} \)) if there exists a sequence of triplet \( (A^k, \tau_k, \mu_k) \in \mathcal{A}_p(S, \infty) \) (resp. \( \mathcal{A}_p(S, \infty) \)), so that for each \( n \geq 0 \),
\[
f_n = \sum_{k \in \mathbb{Z}} \mu_k E_n A^k \ a.e.,
\]
\[
\left\| \sum_{k \in \mathbb{Z}} \mu_k \frac{X_{(\tau_k, \infty)}}{\|X_{(\tau_k, \infty)}\|_{L_{p(|\cdot|)}}} \right\|_{L_{p(|\cdot|)}} \approx \left\| f \right\|_{H_{p(|\cdot|)}^q} \left( \text{resp. } \left\| f \right\|_{Q_{p(|\cdot|), B}} \right).
\]

Moreover, we extend the classical John-Nirenberg theorem to the grand variable Hardy martingale spaces. To be precise, under suitable conditions, we present the following one:
\[
\left\| f \right\|_{BMO_{p(|\cdot|)}} \approx \left\| f \right\|_{BMO_{p(\cdot)}}. \tag{4}
\]

See Theorem 11 for the details. This conclusion improves the recent results [12, 26], respectively.

Throughout this paper, \( \mathbb{Z}, \mathbb{N}, \) and \( \mathbb{C} \) denote the integer set, nonnegative integer set, and complex numbers set, respectively. We denote by \( C \) the absolute positive constant, which can vary from line to line. The symbol \( A \leq B \) stands for the inequality \( A \leq CB \). If we write \( A \approx B \), then it stands for \( A \leq B \leq A \).

### 2. Preliminaries

#### 2.1. Grand Lebesgue Spaces with Variable Exponents

Let \( (\Omega, \mathcal{F}, \mathbb{P}) \) be a probability space. An \( \mathcal{F} \)-measurable function \( p(\cdot) : \Omega \to (0, \infty) \) which is called a variable exponent. For convenience, we denote
\[
p_\ast = \text{essinf } \{ p(\omega) : \omega \in \Omega \}, \quad p_\ast = \text{esssup } \{ p(\omega) : \omega \in \Omega \},
\]
\[
p_\ast(B) = \text{essinf } \{ p(\omega) : \omega \in B \} \quad \text{and} \quad p_\ast(B) = \text{esssup } \{ p(\omega) : \omega \in B \}.
\]

Denote by \( \mathcal{P} = \mathcal{P}(\Omega) \) the collection of all variable exponents \( p(\cdot) \) satisfying with \( 1 < p_\ast \leq p_\ast < \infty \). The variable Lebesgue space \( L_{p(\cdot)}(\Omega) \) consists of all \( \mathcal{F} \)-measurable functions \( f \) such that for some \( \lambda > 0 \),
\[
\rho \left( \frac{f}{\lambda} \right) = \int_{\Omega} \left| \frac{f(\omega)}{\lambda} \right|^{p(\omega)} d\mathbb{P} < \infty. \tag{6}
\]

This leads to a Banach function space under the Luxemburg-Nakano norm
\[
\left\| f \right\|_{L_{p(\cdot)}} = \inf \left\{ \lambda > 0 : \rho \left( \frac{f}{\lambda} \right) \leq 1 \right\}. \tag{7}
\]

Based on this, we begin with the definition of the grand Lebesgue space with variable exponent.

**Definition 2.** Suppose that \( p(\cdot) \in \mathcal{P} \) and \( \theta \geq 0 \). We define the grand Lebesgue space with variable exponent \( L_{p(|\cdot|), \theta} = L_{p(\cdot), \theta}(\Omega) \) as the set of all \( \mathcal{F} \)-measurable functions \( f \) satisfying
\[
\left\| f \right\|_{L_{p(|\cdot|), \theta}} = \sup_{0 < \eta < \theta^{-1}} \left\| f \right\|_{L_{p(|\cdot|), \eta}} < \infty. \tag{8}
\]

The Grand Lebesgue space with variable exponent can unify and generalize the various function spaces. To be precise, if \( \theta = 0 \), \( L_{p(|\cdot|), \theta} \) degenerates to the variable Lebesgue space \( L_{p(|\cdot|)} \). If \( \theta = 1 \) and \( p(\cdot) \equiv p \), \( L_{p(|\cdot|), \theta} \) becomes the Grand Lebesgue space \( L_p \).

#### 2.2. Martingale Grand Hardy Spaces via Variable Exponents

Let \( \{ \mathcal{F}_n \}_{n \geq 0} \) be a nondecreasing sequence of sub-\( \sigma \)-algebras of \( \mathcal{F} \) sets with \( \mathcal{F} = \sigma(\bigcup_{n \geq 0} \mathcal{F}_n) \). The expectation operator and the conditional expectation operator relative to \( \mathcal{F}_n \) are denoted by \( \mathbb{E} \) and \( \mathbb{E}_n \), respectively. A sequence \( f = \{ f_n \}_{n \geq 0} \) of random variables is said to be a martingale, if \( f_n \) is \( \mathcal{F}_n \)-measurable, \( \mathbb{E}(\{ f_n \}) < \infty \), and \( \mathbb{E}_n(f_{n+1}) = f_n \) for every \( n \geq 0 \). Denote \( \mathcal{M} \) to be the set of all martingales \( f = \{ f_n \}_{n \geq 0} \) with respect to \( \{ \mathcal{F}_n \}_{n \geq 0} \) such that \( f_0 = 0 \). For \( f \in \mathcal{M} \), write its martingale difference by \( d_nf = f_n - f_{n-1} (n \geq 0, f_{-1} = 0) \). Define the maximal function, the square function, and the conditional square function of \( f \), respectively, as follows:
\[
M_nf = \sup_{n \geq m} |f_n|, \quad Mf = \sup_{n \geq 0} |f_n|,
\]
\[
S_m(f) = \left( \sum_{n=0}^{m} |df_n|^2 \right)^{1/2}, \quad s(f) = \left( \sum_{n=0}^{\infty} |df_n|^2 \right)^{1/2},
\]
\[
s_m(f) = \left( \sum_{n=0}^{m} E_{n-1} |df_n|^2 \right)^{1/2}, \quad s(f) = \left( \sum_{n=0}^{\infty} E_{n-1} |df_n|^2 \right)^{1/2}. \tag{9}
\]

Let \( \Gamma \) be the set of all sequences \( \{ \lambda_n \}_{n \geq 0} \) of nondecreasing, nonnegative, and adapted functions, and \( \lambda_\infty = \lim_{n \to \infty} \lambda_n \). For \( f \in \mathcal{M} \), \( p(\cdot) \in \mathcal{P} \), and \( \theta \geq 0 \), denote
\[
\Gamma[Q_{p(\cdot), \theta}](f) = \left\{ \{ \lambda_n \}_{n \geq 0} \in \Gamma : S_n(f) \leq \lambda_{n-1} \right\}, \quad \Gamma[D_{p(\cdot), \theta}](f) = \left\{ \{ \lambda_n \}_{n \geq 0} \in \Gamma : f_n \leq \lambda_{n-1} \right\}. \tag{10}
\]

Now we introduce the grand martingale Hardy spaces associated with variable exponents as follows:
where the infimum is taken over all the admissible representations of (14).

Proof. Let \( f \in H^s_{p(\cdot),\theta} \). Now consider the stopping time for each \( k \in \mathbb{Z} \):

\[
\tau_k = \inf \left\{ n \in \mathbb{N} : s_{n+1}(f) > 2^k \right\}.
\]

(17)

It is easy to see that the sequence of these stopping times is nondecreasing. For each stopping time \( \tau \), denote \( f^n = f_{n+\tau} \). It is easy to write that

\[
f_n = \sum_{k \in \mathbb{Z}} (f^{n+1}_k - f^n_k).
\]

(18)

For each \( k \in \mathbb{Z} \), let \( \mu_k = 3 \cdot 2^k \|X_{\tau_k<\infty}\|_{L^{p(\cdot),\theta}} \). If \( \mu_k \neq 0 \), we set

\[
a_n^k = \frac{f_n^{k+1} - f_n^k}{\mu_k}, \quad n \in \mathbb{N}.
\]

(19)

If \( \mu_k = 0 \), we set \( a_n^k = 0 \) for each \( n \in \mathbb{N} \). For each fixed \( k \in \mathbb{Z} \), \( (a_n^k)_{n \geq 0} \) is a martingale. Since \( s(f^k) = s_k(f) \leq 2^k \), we get

\[
\frac{1}{\mu_k} s(f^k) \leq \frac{s(f^{k+1}) + s(f^k)}{\mu_k} \leq \left\| X_{\tau_k<\infty} \right\|_{L^{p(\cdot),\theta}}^{-1}.
\]

(20)

We can easily check that \( (a_n^k)_{n \geq 0} \) is a bounded \( L^2 \)-martingale. Hence, there exists an element \( a^k \in L^2 \) such that \( E \tau_k = a^k \). If \( n \leq \tau_k \), then \( a_n^k = 0 \), and \( \beta(a) \leq \left\| X_{\tau_k<\infty} \right\|_{L^{p(\cdot),\theta}}^{-1} \).

Consequently, it implies that \( a^k \) is really a \((1, p(\cdot), \theta)\)-atom.

Denote \( \Lambda_k = \{ \tau_k < \infty \} \). Knowing that \( \{ \tau_k < \infty \} = \{ s(f) > 2^k \} \) and \( \tau_k \) is nondecreasing for each \( k \in \mathbb{Z} \), we obtain \( \Lambda_{k+1} \subseteq \Lambda_k \). Now, we point out that

\[
\sum_{k \in \mathbb{Z}} 3 \cdot 2^k \chi_{\Lambda_k}(x) = 2 \sum_{k \in \mathbb{Z}} 3 \cdot 2^k \chi_{\Lambda_k \backslash \Lambda_{k+1}}(x).
\]

(21)

Indeed, for a fixed \( x_0 \in \Omega \), there is \( k_0 \in \mathbb{Z} \) so that \( x_0 \in \Lambda_{k_0} \) and \( x_0 \in \Lambda_{k_0+1} \), then we have

\[
\sum_{k \in \mathbb{Z}} 3 \cdot 2^k \chi_{\Lambda_k}(x_0) = \sum_{k = -\infty}^{k_0} 3 \cdot 2^k \chi_{\Lambda_k}(x_0) = \sum_{k = -\infty}^{k_0} 3 \cdot 2^k = 3 \cdot 2^{k+1},
\]

(22)
This means

$$\left\| \sum_{k \in \mathbb{Z}} \mu_k \chi_{(t_k,\infty)} \right\|_{L_{p/(\theta)}(\mathbb{P})} = \left\| \sum_{k \in \mathbb{Z}} 2^k \chi_{(t_k,\infty)} \right\|_{L_{p/(\theta)}(\mathbb{P})} = 6 \left\| \sum_{k \in \mathbb{Z}} 2^k \chi_{(t_k,\infty)} \right\|_{L_{p/(\theta)}(\mathbb{P})}$$

$$= 6 \sup_{0 < q < p - 1} \left\{ \sup_{0 < q < p - 1} \left( \frac{2^k \chi_{(t_k,\infty)}(x)}{\lambda} \right)^{p/(\theta) - 1} d\mathbb{P} \leq 1 \right\}$$

$$= 6 \sup_{0 < q < p - 1} \left\{ \sup_{0 < q < p - 1} \left( \frac{2^k \chi_{(t_k,\infty)}(x)}{\lambda} \right)^{p/(\theta) - 1} d\mathbb{P} \leq 1 \right\}$$

$$= 6 \sup_{0 < q < p - 1} \left\{ \sup_{0 < q < p - 1} \left( \frac{2^k \chi_{(t_k,\infty)}(x)}{\lambda} \right)^{p/(\theta) - 1} d\mathbb{P} \leq 1 \right\}$$

$$\leq 6 \sup_{0 < q < p - 1} \left\{ \sup_{0 < q < p - 1} \left( \frac{2^k \chi_{(t_k,\infty)}(x)}{\lambda} \right)^{p/(\theta) - 1} d\mathbb{P} \leq 1 \right\}$$

$$= 6 \sup_{0 < q < p - 1} \left\{ \sup_{0 < q < p - 1} \left( \frac{2^k \chi_{(t_k,\infty)}(x)}{\lambda} \right)^{p/(\theta) - 1} d\mathbb{P} \leq 1 \right\}$$

$$= 6 \sup_{0 < q < p - 1} \left\{ \sup_{0 < q < p - 1} \left( \frac{2^k \chi_{(t_k,\infty)}(x)}{\lambda} \right)^{p/(\theta) - 1} d\mathbb{P} \leq 1 \right\}$$

(23)

For the converse part, according to the definition of (1, p(\cdot), \theta)-atom, we easily conclude

$$s(a^f) = s(a^f) \chi_{(t_k,\infty)} \leq \left\| s(a^f) \right\|_{L_{\infty}} \chi_{(t_k,\infty)} \leq \left\| \chi_{(t_k,\infty)} \right\|_{L_{p/(\theta)}}^{p/(\theta)} \chi_{(t_k,\infty)}$$

(24)

where $a^f$ is the (1, p(\cdot), \theta)-atom and $t_k$ is the stopping time associated with $a^k$ which, when combined with the subadditivity of the operator $s$, yields

$$s(f) \leq \sum_{k \in \mathbb{Z}} \mu_k s(a^k) \leq \sum_{k \in \mathbb{Z}} \mu_k \frac{X_{(t_k,\infty)}}{\left\| \chi_{(t_k,\infty)} \right\|_{L_{p/(\theta)}}}$$

(25)

This implies

$$\left\| f \right\|_{L_{p/(\theta)}} = \left\| s(f) \right\|_{L_{p/(\theta)}} \leq \sum_{k \in \mathbb{Z}} \mu_k \frac{X_{(t_k,\infty)}}{\left\| \chi_{(t_k,\infty)} \right\|_{L_{p/(\theta)}}}$$

(26)

Taking over all the admissible representations of (14) for $f$, we obtain the desired result.

Next, we will characterize $Q_{p(\cdot), \theta}$ and $D_{p(\cdot), \theta}$ by atoms, respectively. The proof is similar to the proof of Theorem 4. For the completeness of this paper, we provide some details.

**Theorem 5.** Suppose $p(\cdot) \in \mathcal{P}$ and $\theta \geq 0$. If the martingale $f = (f_n)_{n \geq 0} \in Q_{p(\cdot), \theta}$ (resp. $D_{p(\cdot), \theta}$), then there exists a sequence of triplet $(a^f, t_k, \mu_k) \in \mathcal{A}_S(p(\cdot), \infty)$ (resp. $H_S(p(\cdot), \infty)$) so that for each $n \in \mathbb{N}$,

$$f_n = \sum_{k \in \mathbb{Z}} \mu_k E_n a^k$$

(27)

$$\sum_{k \in \mathbb{Z}} \mu_k \left\| \chi_{(t_k,\infty)} \right\|_{L_{p/(\theta)}} \leq \left\| f \right\|_{L_{p/(\theta)}}$$

(28)

Conversely, if the martingale $f = (f_n)_{n \geq 0}$ has admissible representation (27), then $f \in Q_{p(\cdot), \theta}$ (resp. $D_{p(\cdot), \theta}$) and

$$\left\| f \right\|_{L_{p/(\theta)}} \leq \inf \left( \sum_{k \in \mathbb{Z}} \mu_k \frac{X_{(t_k,\infty)}}{\left\| \chi_{(t_k,\infty)} \right\|_{L_{p/(\theta)}}} \right)$$

(29)

where the infimum is taken over all the admissible representations of (27).

**Proof.** The proof follows the ideas in Theorem 4, so we omit some details. Suppose $f = (f_n)_{n \geq 0} \in Q_{p(\cdot), \theta}$ (resp. $D_{p(\cdot), \theta}$). We define stopping times as follows:

$$\tau_k = \inf \left\{ n \in \mathbb{N} : \lambda_n > 2^k \right\}, \inf \emptyset = \infty$$

(30)

where $(\lambda_n)_{n \geq 0}$ is an adapted, nondecreasing sequence such that almost everywhere $\left| \sum_{n \geq 1} |S_n(f)| \right| \leq \lambda_{n-1}$ (resp. $\left| f_n \right| \leq \lambda_{n-1}$) and $\lambda_\infty \in L_{p(\cdot), \theta}$.

Let $(a^k)_{k \in \mathbb{Z}}$ and $(\mu_k)_{k \in \mathbb{Z}}$ be defined as in the proof of Theorem 4. And replace $A_k = \{ t_k < \infty \} = \{ \tau_f > 2^k \}$ by the $A_k = \{ t_k < \infty \} = \{ \lambda_\infty > 2^k \}$. Then, we obtain that $f_n = \sum_{k \in \mathbb{Z}} \mu_k E_n a^k$ and (28) still hold.

For the converse part, write

$$\lambda_n = \sum_{k \in \mathbb{Z}} \mu_k \left\| s(a^k) \right\|_{L_{p/(\theta)}} \left( \text{resp.} \lambda_n = \sum_{k \in \mathbb{Z}} \mu_k \left\| M(a^k) \right\|_{L_{p/(\theta)}} \right)$$

(31)

Clearly, $(\lambda_n)_{n \geq 0}$ is a nonnegative, nondecreasing, and adapted sequence with $S_{n+1}(f) \leq \lambda_n$ (resp. $\left| f_n \right| \leq \lambda_n$). Thus, we get

$$\left\| f \right\|_{L_{p/(\theta)}} \leq \inf \left( \sum_{k \in \mathbb{Z}} \mu_k \frac{X_{(t_k,\infty)}}{\left\| \chi_{(t_k,\infty)} \right\|_{L_{p/(\theta)}}} \right)$$

(32)

Taking over all the admissible representations of (27) for $f$, we obtain the desired result.

**Remark 6.** Suppose $p(\cdot) \in \mathcal{P}$ and $\theta \geq 0$. We conclude that the sum $\sum_{k=0}^N \mu_k a^k$ in Theorem 4 converges to $f$ in $H_{p(\cdot), \theta}$ as $M \rightarrow \infty$, $N \rightarrow \infty$. Indeed, it follows by the subadditive property of the operator $s$, we get, for any $M, N \in \mathbb{Z}$ with $M < N$,
Moreover, \( s(f - f^{(N+1)}) \) is decreasing and convergent to 0 (a.e.) as \( N \to \infty \), and \( s(f^{(N+1)}) \) is decreasing and convergent to 0 (a.e.) as \( M \to -\infty \). From this and the dominated convergence theorem in \( L^p_{\alpha,d} \) for \( 0 < \epsilon < p - 1 \) (see [34], Theorem 2.62), it follows that

\[
\|f - \sum_{k=1}^N \|f_{\alpha,k}\|_{L_p} \|f - f^{(N+1)}\|_{L_p} \leq \|s(f - f^{(N+1)}) + s(f^{(N+1)})\|_{L_p} \leq \sup_{0 < \epsilon < p - 1} \|f\|_{L_p} \sup_{0 < \epsilon < p - 1} \|f\|_{L_p} \to 0 \text{ as } M \to -\infty, N \to \infty.
\]

Furthermore, we can also show the norm convergence of the summation \( \sum_{k=1}^N \|f_{\alpha,k}\|_{L_p} \) in Theorems 5 as \( M \to -\infty \), \( N \to \infty \).

\[ 4. \text{ The Generalized John-Nirenberg Theorem} \]

In the sequel of this section, we will often suppose that every \( \mathcal{F}_n \) is generated by countably many atoms. Recall that \( B \in \mathcal{F}_n \) is called an atom, and if for any \( A \subset B \) with \( A \in \mathcal{F}_n \) satisfying \( \mathbb{P}(A) < \mathbb{P}(B) \), we have \( \mathbb{P}(A) = 0 \). We denote by \( A(\mathcal{F}_n) \) the set of all atoms in \( \mathcal{F}_n \). We shall present the generalized John-Nirenberg theorem on grand Lebesgue spaces with variable exponents. For each \( 1 \leq p < \infty \), the Banach space \( BMO_p \) (bounded mean oscillation [35]) is defined as

\[ BMO_p = \left\{ f \in L_p : \mathbb{E}_n ||f - E_n f||_{L_p}^{p} < \infty \right\}. \]

It can be easily shown that the norm of \( BMO_p \) is equivalent to

\[ \|f\|_{BMO_p} = \sup_{\tau \in \mathcal{T}} \mathbb{E}_n ||f - \tau|_{L_p}||_{L_p}, \]

where \( \mathcal{T} \) consists of all stopping times relative to \( \{\mathcal{F}_n\}_{n=0}^{\infty} \). It follows immediately from the John-Nirenberg theorem [2, 30] that \( BMO_p = BMO_1, 1 < p < \infty \).

What is more, in [2], there has

\[ C \cdot p \|f\|_{BMO_p} \geq \|f\|_{BMO_1} \geq \|f\|_{BMO_1}. \]

\[ 7. \text{Definition} \]

For \( p(\cdot) \in \mathcal{P} \) and \( \theta \geq 0 \), the generalized BMO martingale space is defined by

\[ BMO_{p(\cdot),\theta} = \left\{ f \in L_{p(\cdot)} : \|f\|_{BMO_{p(\cdot),\theta}} < \infty \right\}, \]

where

\[ \|f\|_{BMO_{p(\cdot),\theta}} = \sup_{\tau \in \mathcal{T}} \mathbb{E}_n ||f - \tau||_{L_{p(\cdot)}}. \]

\[ 8. \text{Remark} \]

If \( \theta = 0 \), \( BMO_{p(\cdot),\theta} \) degenerates to the variable exponent BMO martingale space \( BMO_{p(\cdot)} \) introduced and studied in [12]. If \( \theta = 1 \) and \( p(\cdot) \equiv p \), \( BMO_{p(\cdot),\theta} \) becomes the grand BMO martingale space \( BMO_{p(\cdot)} \) studied in [26].

In order to establish the generalized John-Nirenberg theorem in the framework of \( BMO_{p(\cdot),\theta} \), we need the following lemmas and notations.

\[ 9. \text{Lemma (Hölder’s inequality, see [34]). Let } p(\cdot), q(\cdot), r(\cdot) \in \mathcal{P} \text{ satisfy} \]

\[ \frac{1}{p(\omega)} + \frac{1}{q(\omega)} = \frac{1}{r(\omega)}, \text{ a.e. } \omega \in \Omega. \]

Then, there exists a constant \( C \) such that for all \( f \in L_{p(\cdot)} \) and \( g \in L_{q(\cdot)} \), we have\( \mathbb{P}(\cdot) \) and

\[ \|fg\|_{L_{r(\cdot)}} \leq C\|f\|_{L_{q(\cdot)}} \|g\|_{L_{r(\cdot)}}. \]

We mention that if the variable exponent \( p(\cdot) \) meets the log-Hölder continuity condition in Euclidean spaces, the inverse Hölder’s inequality holds for characteristic functions in \( L_{p(\cdot)}(\mathbb{R}^n) \) (see [36]). Compared with Euclidean space \( \mathbb{R}^n \), the probability space \( (\Omega, \mathcal{P}) \) has no natural metric structure. Fortunately, Jiao et al. [11, 27] put forward the following condition: there exists an absolute constant \( \kappa \geq 1 \) depending only on \( p(\cdot) \) such that

\[ \mathbb{P}(B)^{p(\cdot)} \leq \kappa \mathbb{P}(\cdot) \in \mathcal{P} \text{ satisfying (43).} \]

\[ 10. \text{Lemma (see [27]). Suppose } p(\cdot) \in \mathcal{P} \text{ satisfying (43).} \]

(1) For each \( B \in \bigcup_{n \geq 0} A(\mathcal{F}_n) \), we get

\[ \mathbb{P}(B) \leq \mathbb{E}_n \mathbb{P}(\cdot) \|f_{\beta,n}\|_{L_{p(\cdot)}}. \]

(2) Let \( q(\cdot) \in \mathcal{P} \text{ satisfy (43). If } r(\cdot) \text{ satisfies} \]

\[ \frac{1}{r(\cdot)} = \frac{1}{p(\cdot)} + \frac{1}{q(\cdot)}, \text{ a.e. } \omega \in \Omega, \]

then \( r(\cdot) \) also satisfies condition (43). Moreover, for each \( B \in \bigcup_{n \geq 0} A(\mathcal{F}_n) \), we deduce

\[ \text{...} \]
If \( p(\cdot) \in \mathcal{P} \) satisfies (43) and \( \theta \geq 0 \). Then, for every \( f \in \text{BMO}_1 \), there has
\[
\| f \|_{\text{BMO}_1} \leq \| f \|_{\text{BMO}_1, \theta} \leq \| f \|_{\text{BMO}_1}.
\]

Proof. If \( p(\cdot) \in \mathcal{P} \) satisfies (43), then we clearly get that \( p(\cdot) - \eta \) also satisfies (43) for \( 0 < \eta < p_1 - 1 \). It follows from Lemmas 9 and 10 that
\[
\left\| \frac{f - f^{r-1}}{L_1} \right\|_{L_1} \leq \left\| \frac{f - f^{r-1}}{L_1} \right\|_{X(\tau^{\infty})} \left\| f - f^{r-1} \right\|_{L_1},
\]
for any \( 0 < \eta < p_1 - 1 \). Here, the variable exponent \((p(\cdot) - \eta)\) is defined by
\[
1 \left( \frac{p(\omega) - \eta}{p(\omega) - \eta} \right) + 1 = 1, \text{ a.e. } \omega \in \Omega.
\]
This is equivalent to the following inequality:
\[
\left\| \frac{f - f^{r-1}}{L_1} \right\|_{X(\tau^{\infty})} \leq \left\| \frac{f - f^{r-1}}{L_1} \right\|_{L_1}.
\]
Hence, we have
\[
\left\| \frac{f - f^{r-1}}{L_1} \right\|_{X(\tau^{\infty})} = \sup_{0 < q < p_1 - 1} q^{\theta(p(\cdot) - \eta)} \left\| \frac{f - f^{r-1}}{L_1} \right\|_{X(\tau^{\infty})} \leq \left\| \frac{f - f^{r-1}}{L_1} \right\|_{X(\tau^{\infty})}.
\]
Taking the supremum over all stopping times, we deduce
\[
\| f \|_{\text{BMO}_1} \leq \| f \|_{\text{BMO}_1, \theta}.
\]
Conversely, from the definition of \( L_{p(\cdot)0} \), we get
\[
\left\| \frac{f - f^{r-1}}{L_1} \right\|_{L_1} = \sup_{0 < q < p_1 - 1} q^{\theta(p(\cdot) - \eta)} \left\| \frac{f - f^{r-1}}{L_1} \right\|_{X(\tau^{\infty})}
\]
\[
\leq \sup_{0 < q < p_1 - 1} q^{\theta(p(\cdot) - \eta)} \left\| \frac{f - f^{r-1}}{L_1} \right\|_{X(\tau^{\infty})}
\]
\[
= \sup_{0 < q < p_1 - 1} q^{\theta(p(\cdot) - \eta)} \left\| \frac{f - f^{r-1}}{L_1} \right\|_{X(\tau^{\infty})}
\]
It follows from Lemma 9 that
\[
\| f \|_{\text{BMO}_1} \leq \| f \|_{\text{BMO}_1, \theta} \leq \| f \|_{\text{BMO}_1}.
\]
where \( q(\cdot) \) satisfies
\[
\frac{1}{p(\omega) - \eta} = \frac{1}{2p_1} + \frac{1}{q(\omega)}, \text{ a.e. } \omega \in \Omega.
\]
Hence, by (38), we deduce that
\[
\| f \|_{\text{BMO}_1} \leq \| f \|_{\text{BMO}_1, \theta} \leq \| f \|_{\text{BMO}_1}.
\]
From what has been discussed above, we draw the conclusion that
\[
\| f \|_{\text{BMO}_1} \leq \| f \|_{\text{BMO}_1, \theta} \leq \| f \|_{\text{BMO}_1}.
\]

Theorem 11 improves the recent results [12, 26], respectively. More precisely, if we consider the case \( \theta = 0 \), then the following result holds:

Corollary 12. If \( p(\cdot) \) satisfies (43) with \( 1 < p_1 \leq p_2 < \infty \), then for \( f \in \text{BMO}_1 \),
\[
\| f \|_{\text{BMO}_1} \leq \| f \|_{\text{BMO}_1}.
\]

And especially for \( \theta = 1 \) and \( p(\cdot) \equiv p \), we get the conclusion as follows.

Corollary 13 (see [26]). Suppose \( 1 < p < \infty \), then for \( f \in \text{BMO}_1 \),
\[
\| f \|_{\text{BMO}_1} \leq \| f \|_{\text{BMO}_1}.
\]

Data Availability

No data is used in the manuscript.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

This project is partially supported by the National Natural Science Foundation of China (Grant Nos. 11801001 and 12101223), Scientific Research Fund of Hunan Provincial Education Department (Grant No. 20C0780), and Scientific
References

[1] F. Weisz, “Martingale Hardy spaces for 0 < p ≤ 1,” Probability Theory and Related Fields, vol. 84, no. 3, pp. 361–376, 1990.

[2] C. Herz, “Bounded mean oscillation and regulated martingales,” Transactions of the American Mathematical Society, vol. 193, no. 0, pp. 199–215, 1974.

[3] Z. Hao and L. Li, “Orlicz-Lorentz Hardy martingale spaces,” Journal of Mathematical Analysis and Applications, vol. 482, no. 1, article 123520, 2020.

[4] Y. Jiao, F. Weisz, G. Xie, and D. Yang, “Martingale Musielak-Orlicz-Lorentz Hardy spaces with applications to dyadic Fourier analysis,” Journal of Geometric Analysis, vol. 31, no. 11, pp. 11002–11050, 2021.

[5] L. Long, F. Weisz, and G. Xie, “Real interpolation of martingale Orlicz Hardy spaces and BMO spaces,” Journal of Mathematical Analysis and Applications, vol. 505, no. 2, article 125565, 2022.

[6] T. Miyamoto, E. Nakai, and G. Sadasue, “Martingale Orlicz-Hardy spaces,” Mathematische Nachrichten, vol. 285, no. 5-6, pp. 670–686, 2012.

[7] K.-P. Ho, “Atomic decompositions, dual spaces and interpolations of martingale Hardy-Lorentz-Karamata spaces,” The Quarterly Journal of Mathematics, vol. 65, no. 3, pp. 985–1009, 2014.

[8] Y. Jiao, L. Peng, and P. Liu, “Atomic decompositions of Lorentz martingale spaces and applications,” Journal of Function Spaces and Application, vol. 7, no. 2, pp. 153–166, 2009.

[9] Y. Jiao, L. Wu, A. Yang, and R. Yi, “The predual and John-Nirenberg inequalities on generalized BMO martingale spaces,” Transactions of the American Mathematical Society, vol. 369, no. 1, pp. 537–553, 2017.

[10] Z. Hao, “Atomic decomposition of predictable martingale Hardy space with variable exponents,” Czechoslovak Mathematical Journal, vol. 65, no. 140, pp. 1033–1045, 2015.

[11] Y. Jiao, F. Weisz, L. Wu, and D. Zhou, “Variable martingale Hardy spaces and their applications in Fourier analysis,” Dissertationes Mathematicae, vol. 550, pp. 1–67, 2020.

[12] Y. Jiao, D. Zhou, Z. Hao, and W. Chen, “Martingale Hardy spaces with variable exponents,” Banach Journal of Mathematical Analysis, vol. 10, no. 4, pp. 750–770, 2016.

[13] F. Weisz, “Doob’s and Burkholder-Davis-Gundy inequalities with variable exponent,” Proceedings of the American Mathematical Society, vol. 149, no. 2, pp. 875–888, 2021.

[14] F. Weisz, “Characterizations of variable martingale Hardy spaces via maximal functions,” Fractional Calculus and Applied Analysis, vol. 24, no. 2, pp. 393–420, 2021.

[15] T. Iwaniec and C. Sbordone, “On the integrability of the Jacobian under minimal hypotheses,” Archive for Rational Mechanics and Analysis, vol. 119, no. 2, pp. 129–143, 1992.

[16] C. Capone and A. Fiorenza, “On small Lebesgue spaces,” Journal of Function Spaces and Applications, vol. 3, no. 1, pp. 73–89, 2005.

[17] C. Capone, M. R. Formica, and R. Giova, “Grand Lebesgue spaces with respect to measurable functions,” Nonlinear Analysis, vol. 85, pp. 125–131, 2013.

[18] G. Di Fratta and A. Fiorenza, “A direct approach to the duality of grand and small Lebesgue spaces,” Nonlinear Analysis, vol. 70, no. 7, pp. 2582–2592, 2009.

[19] A. Fiorenza, “Duality and reflexivity in grand Lebesgue spaces,” Collectanea Mathematica, vol. 51, pp. 131–148, 2000.

[20] A. Fiorenza, B. Gupta, and P. Jain, “The maximal theorem for weighted grand Lebesgue spaces,” Studia Mathematica, vol. 188, no. 2, pp. 123–133, 2008.

[21] A. Fiorenza and C. Sbordone, “Existence and uniqueness results for solutions of nonlinear equations with right hand side in L1,” Studia Mathematica, vol. 127, no. 3, pp. 223–231, 1998.

[22] V. Kokilashvili, “Boundedness criteria for singular integrals in weighted grand Lebesgue spaces,” Journal of Mathematical Sciences, vol. 170, no. 1, pp. 20–33, 2010.

[23] V. Kokilashvili, A. Meskhi, and H. Rafeiro, “Grand Bochner-Lebesgue space and its associate space,” Journal of Functional Analysis, vol. 266, no. 4, pp. 2125–2136, 2014.

[24] V. Kokilashvili and A. Meskhi, “Maximal and Calderón–Zygmund operators in grand variable exponent Lebesgue spaces,” Georgian Mathematical Journal, vol. 21, no. 4, pp. 447–461, 2014.

[25] A. Testici and D. Israfilov, “Approximation by matrix transforms in generalized grand Lebesgue spaces with variable exponent,” Applicable Analysis, vol. 100, no. 4, pp. 819–834, 2021.

[26] Z. Hao and L. Li, “Grand martingale Hardy spaces,” Acta Mathematica Hungarica, vol. 153, no. 2, pp. 417–429, 2017.

[27] Z. Hao and Y. Jiao, “Fractional integral on martingale Hardy spaces with variable exponents,” Fractional Calculus and Applied Analysis, vol. 18, no. 5, pp. 1128–1145, 2015.

[28] F. Weisz, Martingale Hardy Spaces and Their Applications in Fourier Analysis, vol. 1568 of Lecture Notes in Math, Springer-Verlag, Berlin, 1994.

[29] A. M. Garsia, Martingale Inequalities: Seminar Notes on Recent Progress, Math, Lecture Notes Series, Benjamin Inc., New York, 1973.

[30] R. Long, Martingale Spaces and Inequalities, Science Press, Beijing, 1993.

[31] Z. Hao, Y. Jiao, F. Weisz, and D. Zhou, “Atomic subspaces of Lp-martingale spaces,” Acta Mathematica Hungarica, vol. 150, no. 2, pp. 423–440, 2016.

[32] Z. Hao, Y. Jiao, and L. Wu, “John-Nirenberg inequalities with variable exponents on probability spaces,” Tokyo Journal of Mathematics, vol. 38, no. 2, pp. 353–367, 2015.

[33] G. Xie, F. Weisz, D. Yang, and Y. Jiao, “New martingale inequalities and applications to Fourier analysis,” Nonlinear Analysis, vol. 182, pp. 143–192, 2019.

[34] D. Cruz-Uribe and A. Fiorenza, Variable Lebesgue Spaces: Foundations and Harmonic Analysis, Birkhäuser, Heidelberg, 2013.

[35] F. John and L. Nirenberg, “On functions of bounded mean oscillation,” Communications on Pure and Applied Mathematics, vol. 14, no. 3, pp. 415–426, 1961.

[36] E. Nakai and Y. Sawano, “Hardy spaces with variable exponents and generalized Campanato spaces,” Journal of Functional Analysis, vol. 262, no. 9, pp. 3665–3748, 2012.