Linear-Time Approximation Scheme for $k$-Means Clustering of Affine Subspaces

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Abstract
In this paper, we present a linear-time approximation scheme for $k$-means clustering of incomplete data points in $d$-dimensional Euclidean space. An incomplete data point with $\Delta > 0$ unspecified entries is represented as an axis-parallel affine subspace of dimension $\Delta$. The distance between two incomplete data points is defined as the Euclidean distance between two closest points in the axis-parallel affine subspaces corresponding to the data points. We present an algorithm for $k$-means clustering of axis-parallel affine subspaces of dimension $\Delta$ that yields an $(1 + \epsilon)$-approximate solution in $O(nd)$ time. The constants hidden behind $O(\cdot)$ depend only on $\Delta$, $\epsilon$ and $k$. This improves the $O(n^2d)$-time algorithm by Eiben et al. [SODA’21] by a factor of $n$.

2012 ACM Subject Classification I.3.5 Computational Geometry and Object Modeling

Keywords and phrases $k$-means clustering, affine subspaces

Funding This work was supported by the National Research Foundation of Korea (NRF) grant funded by the Korea government (MSIT) (No.2020R1C1C1012742).

1 Introduction
Clustering is a fundamental research topic in computer science, which arises in various applications [11], including pattern recognition and classification, data mining, image analysis, and machine learning. In clustering, the objective is to group a set of data points into clusters so that the points from the same cluster are similar to each other. Usually, input points lie in a high-dimensional space, and the similarity between two points is defined as their distance. Two of the popular clusterings are $k$-median and $k$-means clusterings. In the $k$-means clustering problem, we wish to partition a given point set into $k$ clusters to minimize the sum of squared distances of each point to its cluster center. Similarly, in the $k$-median clustering problem, we wish to partition a given point set into $k$ clusters to minimize the sum of distances of each point to its cluster center.

In this paper, we consider clustering for incomplete data points. The analysis of incomplete data is a long-standing challenge in practical statistics. There are lots of scenarios where entries of points of a given data set are incomplete [2]. For instance, a few questions are left blank on a questionnaire; weather records for a region omit the figures for one weather station for a short period because of a malfunction; stock exchange data is absent for one stock on one day because of a trading suspension. Various heuristic, greedy, convex optimization, statistical, or even ad hoc methods were proposed throughout the years in different practical domains to handle missing data [2].

Gao et al. [8] introduced a geometric approach to deal with incomplete data points for clustering problems. An incomplete point has one or more unspecified entries, which can be
represented as an axis-parallel affine subspace. The distance between two incomplete data points is defined as the Euclidean distance between two closest points in the axis-parallel affine subspaces corresponding to the data points. Since the distance between an axis-parallel affine subspace and a point is well-defined, the classical clustering problems such as $k$-means, $k$-median, and $k$-center can be defined on a set of axis-parallel affine subspaces.

The $k$-center problem in this setting was studied by [8, 9, 13]. Gao et al. [8, 9] focused on the $k$-center clustering for $k \leq 3$, and presented an approximation algorithm for the $k$-center clustering of axis-parallel affine subspaces. Later, Lee and Schulman [13] improved the running time of the algorithm by Gao et al., and then presented an $O(nd)$-time approximation algorithm for the $k$-center clustering problem for a larger $k$. The constant hidden behind $O(\cdot)$ depends on $\Delta, \epsilon$ and $k$. Moreover, they showed that the running time of an approximation algorithm with any approximation ratio cannot be polynomial in even one of $k$ and $\Delta$ unless $P = NP$, and thus the running time of their algorithm is almost tight.

Very recently, Eiben et al. [6] presented an approximation algorithm for the $k$-means clustering of $n$ axis-parallel affine subspaces of dimension $\Delta$. Their algorithm yields an $(1+\epsilon)$-approximate solution in $O(n^2d)$ time with probability $O(1)$. The constant hidden behind $O(\cdot)$ depends on $\Delta, \epsilon$ and $k$. Since the best-known algorithm for the $k$-center clustering in this setting runs in time linear in both $n$ and $d$ (but exponential in both $k$ and $\Delta$), it is a natural question if a similar time bound can be achieved for the $k$-means clustering. In this paper, we resolve this natural question by presenting an $(1+\epsilon)$-approximation algorithm for the $k$-means clustering problem running in time linear in $n$ and $d$.

**Related work.** The $k$-median and $k$-means clustering problems for points in $d$-dimensional Euclidean space have been studied extensively. Since these problems are NP-hard even for $k = 2$ or $d = 2$ [3, 14, 16], the study of $k$-means and $k$-median clusterings have been devoted to obtain $(1+\epsilon)$-approximation algorithms for these problems [1, 5, 7, 10, 12]. These algorithms run in time polynomial time in the input size if one of $k$ and $d$ is constant. Indeed, it is NP-hard to approximate Euclidean $k$-means clustering within a factor better than a certain constant larger than one [3]. That is, the $k$-means clustering problem does not admit a PTAS for arbitrary $k$ and $d$ unless $P=NP$.

Also, the clustering problems for lines (which are not necessarily axis-parallel) also have been studied [15, 17]. More specifically, Ommer and Malik [17] presented a heuristic for $k$-median clustering of lines in three-dimensional space. Later, Marom and Feldman [15] presented an algorithm for computing a corset of size $d k^{O(k)} \log n/\epsilon^2$, which gives a polynomial-time $(1+\epsilon)$-approximation algorithm for the $k$-means clustering of lines in $d$-dimensional Euclidean space.

**Our results.** We present an algorithm for $k$-means clustering of axis-parallel affine subspaces of dimension $\Delta$ that yields an $(1+\epsilon)$-approximate solution in $2^{O(\sqrt{\Delta k} (\log \frac{\Delta}{\epsilon} + k) )} dn$ time with a constant probability. This improves the previously best-known algorithm by Eiben et al [6], which takes $2^{O(\sqrt{\Delta k} (\log \frac{\Delta}{\epsilon}) )} dn^2$ time. Since it is a generalization of the $k$-means clustering problem for points ($\Delta = 0$), it does not admit a PTAS for arbitrary $k$ and $d$ unless $P=NP$. Furthermore, similarly to Lee and Schulman [13], we show in Appendix A that an approximation algorithm with any approximation ratio cannot run in polynomial time in even one of $k$ and $\Delta$ unless $P = NP$. Thus, the running time of our algorithm is almost tight.


2 Preliminaries

We consider points in $\mathbb{R}^d$ with missing entries in some coordinates. Let us denote the missing entry value by $\otimes$, and let $\mathbb{H}^d$ denote the set of elements of $\mathbb{R}^d$ where we allow some coordinates to take the value $\otimes$. Furthermore, we call a point in $\mathbb{H}^d$ a $\Delta$-missing point if at most $\Delta$ of its coordinates have value $\otimes$. We use $[k]$ to denote the set $\{1, \ldots, k\}$ for any integer $k \geq 1$. For any point $u \in \mathbb{H}^d$ and an index $i \in [d]$, we use $(u)_i$ to denote the entry of the $i$-th coordinate of $u$. If it is understood in context, we simply use $u_i$ to denote $(u)_i$.

Throughout this paper, we use $i$ or $j$ to denote an index of the coordinates of a point, and $t$ to denote an index of a sequence (of points or sets). We use $(u_t)_{t \in [k]}$ to denote a $k$-tuple consisting of $u_1, u_2, \ldots, u_k$.

**Distance between two $\Delta$-missing points.** The domain of a point $u$ in $\mathbb{H}^d$, denoted by $\text{dom}(u)$, is defined as the set of coordinate-indices $i \in [d]$ with $(u)_i \neq \otimes$. For a set $I$ of coordinate-indices in $[d]$, we say that $u$ is fully defined on $I$ if $\text{dom}(u) \subseteq I$. Similarly, we say that $u$ is partially defined on $I$ if $\text{dom}(u) \cap I \neq \emptyset$. For a set $P$ of points of $\mathbb{H}^d$ and a set $I$ of coordinate-indices in $[d]$, we use $\text{PD}(P, I)$ to denote the set of points of $P$ fully defined on $I$. Similarly, we use $\text{PD}(S, I)$ to denote the set of points of $P$ partially defined on $I$. The null point is a point $p \in \mathbb{H}^d$ such that $(p)_i = \otimes$ for all indices $i \in [d]$. With a slight abuse of notation, we denote the null point by $\otimes$ if it is understood in context. Also, we sometimes use $I_t$ to denote $\text{dom}(u_t)$ if it is clear in context.

Notice that a $\Delta$-missing point in $\mathbb{H}^d$ can be considered as a $\Delta$-dimensional affine subspace in $\mathbb{R}^d$. The distance between two $\Delta$-missing points in $\mathbb{H}^d$ is defined as the Euclidean distance between their corresponding $\Delta$-dimensional affine subspaces in $\mathbb{R}^d$. More generally, we define the distance between two points $x$ and $y$ in $\mathbb{H}^d$ on a set $I \subseteq [d]$ as

$$d_I(x, y) = \sqrt{\sum_{i \in I} |x_i - y_i|^2},$$

where $|a - b| = 0$ for $a = \otimes$ or $b = \otimes$ by convention.

**The $k$-Means clustering of $\Delta$-missing points.** In this paper, we consider the $k$-means clustering of $\Delta$-missing points of $\mathbb{H}^d$. As in the standard setting (for $\Delta = 0$), we wish to partition a given point set $P$ into $k$ clusters to minimize the sum of squared distances of each point to its cluster center. For any partition $(P_t)_{t \in [k]}$ of $P$ into $k$ clusters such that each cluster $P_t$ is associated with a cluster center $c_t \in \mathbb{R}^d$, the cost of the partition is defined as the sum of squared distances of each point in $P$ to its cluster center.

To be more precise, we define the clustering cost as follows. For a set $P \subseteq \mathbb{H}^d$ and a $\Delta$-missing point $y$, we use $\text{cost}(P, y)$ to denote the sum of squared distances of each point in $P$ to $y$. We also define the cost on a coordinate set $I \subseteq [d]$, denoted by $\text{cost}_I(P, y)$, as the sum of squared distances on $I$ between the points in $P$ and their cluster centers. That is, $\sum_{x \in P} d_I(x, y)^2$. For convention, $\text{cost}_I(P, y) = 0$ for $I = \emptyset$.

The clustering cost $\text{cost}((P_t)_{t \in [k]}, (c_t)_{t \in [k]})$ of clustering $((P_t)_{t \in [k]}, (c_t)_{t \in [k]})$ is defined as $\sum_{t \in [k]} \text{cost}(P_t, c_t)$.

Now we consider two properties of an optimal clustering $((P_t^*)_{t \in [k]}, (c_t^*)_{t \in [k]})$ that minimizes the clustering cost, which will be frequently used throughout this paper. For each cluster $P_t^*$, $\text{cost}(P_t^*, c_t^*)$ is minimized when $c_t^*$ is the centroid of $P_t^*$. That is, $c_t^*$ is the centroid of $P_t^*$. For a set $P$ of points in $\mathbb{H}^d$, the centroid of $P$, denoted by $c(P)$, is defined
23:4 Linear-Time Approximation Scheme for $k$-Means Clustering of Affine Subspaces

as

$$(c(P))_i = \begin{cases} \oplus_i \frac{\sum_{u \in PD(P, i)} u_i}{|PD(P, i)|} & \text{if } PD(P, i) = \emptyset, \\ |PD(P, i)| & \text{otherwise.} \end{cases}$$

Also, the clustering cost is minimized when $(P^*_t)_{t \in [k]}$ forms the Voronoi partition of $P$ induced by $(c^*_t)_{t \in [k]}$. That is, $(P^*_t)_{t \in [k]}$ is the partition of $P$ into $k$ clusters in such a way that $c^*_t$ is the closest cluster point from any point $p$ in $P^*_t$.

### Sampling

Our algorithm uses random sampling to compute an approximate $k$-means clustering. Lemma 1 is a restatement of [1, Lemma 2.1], and Lemma 2 is used in [6] implicitly. Since Lemma 2 is not explicitly stated in [6], we give a sketch of the proof in Appendix ??.

- **Lemma 1 ([1, Lemma 2.1]).** Assume that we are given a set $P$ of points in $\mathbb{H}^d$, an index $i \in [d]$, and an approximation factor $\alpha > 0$. Let $Q$ be a subset of $P$ with $|PD(Q, i)| \geq |c|P|$ for some constant $c$, which is not given explicitly. Then we can compute a point $x$ of $\mathbb{R}$ in $O(|P|d\alpha, \delta)$ time satisfying with probability $\frac{2\alpha}{\delta} 2^{O(-m_{\alpha, \delta} \log(m_{\alpha, \delta}))}$ that

$$\text{cost}_1(Q, x) \leq (1 + \alpha)\text{cost}_1(Q, c(Q)),$$

where $m_{\alpha, \delta} \in O(1/\alpha\delta))$.

- **Lemma 2 ([6]).** Assume that we are given a set $P$ of $\Delta$-missing points in $\mathbb{H}^d$ and an approximation factor $\alpha > 0$. Let $Q$ be a subset of $P$ with $|Q| \geq c|P|$ for some constant $c$ with $0 < c < 1$, which is not given explicitly. Then we can compute a $\Delta$-missing point $u \in \mathbb{H}^d$ in $O(|P|d\lambda)$ time satisfying with probability $\frac{8(\alpha^3 + 1)\lambda^3 + 1}{4(4\Delta)^3 \lambda}$ that

$$\text{cost}_1(Q, u) \leq (1 + \alpha)\text{cost}_1(Q, c(Q)),$$

where $I$ denotes the domain of $u$, and $\lambda = \max\{(\frac{3}{\alpha})^{1/(2\Delta)}, (128\Delta^3)^{1/(2\Delta)}\}$.

**Proof (Sketch).** We randomly select a point $p$ from $P$, and then define $u$ as follows so that $\text{dom}(u) = \text{dom}(p)$. To do this, we choose a random sample $T$ of size $8\lambda$ from $P$. For each coordinate-index $i \in \text{dom}(p) \cap \text{dom}(c(T))$, we set $(u)_i = (c(T))_i$. For each coordinate-index $j \in \text{dom}(p) - \text{dom}(c(T))$, we choose a random sample $T_j$ of size $8\lambda$ from $P$, and set $(u)_j = (c(T_j))_j$.

Eiben et al. showed that $\text{dom}(u) = \text{dom}(p)$ and $\text{cost}_1(Q, u) < (1 + \alpha)\text{cost}_1(Q, c(Q))$ with probability at least $\frac{8(\alpha^3 + 1)\lambda^3 + 1}{4(4\Delta)^3 \lambda}$, where $I$ denotes the domain of $u$. Details of the analysis can be found in the proof of [6, Lemma 17].

### 3 Overview of the Algorithm

To describe our contribution, we first briefly describe a $(1 + \epsilon)$-approximation algorithm for $k$-means clustering for points in $d$-dimensional Euclidean space given by Kumar et al. [12].

Let $P$ be a set of $n$ points in $d$-dimensional Euclidean space, and $((P^*_t)_{t \in [k]}, (c^*_t)_{t \in [k]})$ be an optimal $k$-means clustering for $P$.

### Sketches of [1] and [12].

The algorithm of Kumar et al. [12] consists of several phases of two types: sampling phases and pruning phases. Their idealized strategy as follows. At the beginning of a phase, it decides the type of the phase by computing the index $t$
that maximizes $|P^*_t|$. If the cluster center of $P^*_t$ has not been obtained, the algorithm enters the sampling phase. This algorithm picks a random sample of a constant size from $P_t$ and hopefully this sample would contain enough random samples from $P^*_t$. Then one can compute a good approximation $c_t$ to $c^*_t$ using Lemma 1.

If it is not the case, the algorithm enters a pruning phase, and it assigns each point in $R$ to its closest cluster if their distance is at most $L$, where $L$ denotes the smallest distance between two cluster centers we have been obtained so far. They repeat this until all cluster centers are obtained, and finally obtain a good approximation to $(P^*_t)_{t \in [k]}$.

However, obviously, it is hard to implement this idealized strategy. To handle this, they try all possibilities (for both pruning and sampling phases and for all indices $t \in [k]$) to be updated for sampling phases), and return the best solution found this way. Kumar et al. [12] showed that their algorithm runs in $O(2^{(k/\epsilon)^{O(1)}} n)$ time, and returns an $(1+\epsilon)$-approximate $k$-means clustering with probability $1/2$. Later, Ackermann et al. [1] gave a tighter bound on the running time of this algorithm.

**Sketch of Eiben et al. [6]** To handle $\Delta$-missing points, Eiben et al. generalized the algorithm in [12]. Their idealized strategy (using the counting oracle) can be summarized as follows. It maintains $k$ centers $(u_t)_{t \in [k]}$, which are initially set to the null points. In each sampling phase, it obtains one (or at least $|d - \Delta|$) coordinate of one of the centers.

At the beginning of a phase, it decides the type of the phase by computing the index $t$ that maximizes $|PD(P^*_t, [d - I_t])|$. A sampling phase happens if $|PD(P^*_t, [d - I_t])| > c|R|$, where $R$ denotes the number of points which are not yet assigned to any cluster. In this case, a random sample of constant size from $R$ would contain enough random samples from $|PD(P^*_t, j)|$ with $j \in [d - I_t]$. Thus, using the random sample, one can obtain a good approximation to $(c^*_t)_{t \in [k]}$.

Otherwise, a pruning phase happens. In a pruning phase, the algorithm assigns points which are not yet assigned to any cluster to clusters. Here, a main difficulty is that even though the distance between a point $p$ in $R$ and its closest center $u_t$ is at most $L$, where $L$ denotes the distance between two cluster centers, it is not necessarily that $p \in P^*_t$. They resolved this in a clever way by ignoring $\Delta$ coordinates for comparing the distances from two cluster centers.

**Comparison of our contribution and Eiben et al. [6]**. Our contribution is two-fold: the dependency on $n$ decreases to $O(n)$ from $O(n^2)$, and the dependency of $\Delta$ and $k$ decreases significantly.

First, the improvement on the dependency of $n$ comes from introducing a faster and simpler procedure for a pruning phase. In the previous algorithm, it cannot be guaranteed that a constant fraction of points of $R$ is removed from $R$. This yields the quadratic dependency of $n$ in their running time. We overcome the difficulty they (and we) face in a pruning phase in a different way. For each subset $T$ of $[k]$, we consider the set $S_T$ of points $x \in R$ such that $\text{DOM}(x) \subseteq I_t$ for every $t \in T$ and $\text{DOM}(x) \not\subseteq I_{t'}$ for every $t' \not\in T$. Then $S_T$’s for all subsets $T \in [k]$ form a partition of $R$. In a pruning phase, we choose the set $S_T$ that maximizes $|S_T|$. We show that the size of $S_T$ is at least a constant fraction of $|R|$ (unless we enter the sampling phase). Moreover, in this case, if the distance between a point $p \in S_T$ and its closest center $u_t$ is at most $L$, where $L$ denotes the distance between two cluster centers, it holds that $p \in P^*_t$.

Second, the improvement on the dependency of $\Delta$ and $k$ comes from using the framework of Ackermann et al. [1] to analyze the approximation factor of the algorithm while Eiben et
4 For 2-Means Clustering

In this section, we focus on the case that \( k = 2 \), and in the following section, we show how to generalize this idea to deal with a general constant \( k > 2 \).

4.1 Algorithm Using the Counting Oracle

In this section, we first sketch an algorithm for 2-means clustering assuming that we can access the counting oracle. Let \( (P_1^*, P_2^*) \) be an optimal 2-clustering for \( P \), and \( c_1^* \) and \( c_2^* \) be the centroids of \( P_1^* \) and \( P_2^* \), respectively. The counting oracle takes a subset \( Q \) of \( P \) and a cluster-index \( t = 1, 2 \) as input, and then returns the number of points in \( Q \cap P_t^* \).

In Section 4.3, we will modify this algorithm so that the use of the counting oracle can be avoided without increasing the approximation factor. The assumption made in this subsection makes it easier to analyze the approximation factor of the algorithm.

The algorithm consists of several phases of two types: a sampling phase or a pruning phase. Initially, we set \( u_1, u_2 \) so that \( (u_t)_i = \emptyset \) for all \( t = 1, 2 \) and \( i \in [d] \). In a sampling phase, we obtain values of \( (u_t)_j \) for indices \( t = 1, 2 \) and \( j \in [d] \) which were set to \( \emptyset \). Also, we assign points of \( P \) to one of the two clusters in sampling and pruning phases. The pseudocode of the algorithm is described in Algorithm 1.

At the beginning of each phase, we decide the type of the current phase. Let \( t \) be the cluster-index that maximizes \( |\text{PD}(R \cap P_t^*, [d] - I_t)| \), where \( R \) is the set of points of \( P \) which are not assigned to any cluster. The one of the following cases always happen:

- \( |\text{PD}(R \cap P_t^*, [d] - I_t)| \geq c|R| \) or \( \sum_{t'=1,2} |\text{FD}(R \cap P_{t'}, I_{t'})| \geq c|R| \) for a constant \( c \leq 1/4 \), which will be specified later.

Sampling phase. If the first case happens, we enter a sampling phase. Let \( \alpha \) be a constant, which will be specified later, which is an approximation factor used in Lemmas 1 and 2 for sampling. If \( I_1 \) is empty, we replace \( u_t \) with a \( \Delta \)-missing point in \( \mathbb{R}^d \) obtained from Lemma 2. If \( I_t \) is not empty, then it is guaranteed that \( |I_t| \) is at least \( d - \Delta \). We compute the coordinate-index \( j \) in \([d] - I_t\) that maximizes \( |\text{PD}(R \cap P_t^*, j)| \) using the counting oracle. Clearly, \( (u_t)_j = \emptyset \) and \( |\text{PD}(R \cap P_t^*, j)| \) is at least \( c|R|/\Delta \). Then we replace \( (u_t)_j \) with a value obtained from Lemma 1. At the end of the phase, we check if \( \text{FD}(R, I_1 \cap I_2) \) is not empty. If it is not empty, we assign those points to their closest cluster centers.

Pruning phase. Now consider the second case. In this case, we enter a pruning phase. Instead of obtaining a new coordinate value of \( u_t \), we assign points of \( R \) to one of the clusters as follows. Let \( t' \) be the cluster-index that maximizes \( |\text{FD}(R, I_{t'})| \). Among the points of \( \text{FD}(R, I_{t'}) \), we choose \( |\text{FD}(R, I_{t'})|/2 \) points closest to \( u_{t'} \), and assign them to \( u_{t'} \).

The algorithm terminates when every point of \( P \) is assigned to one of the clusters. Let \( P_t \) be the set of points of \( P \) assigned to \( u_t \) for \( t = 1, 2 \). Also, let \( (c_1, c_2) \) be \( (u_1, u_2) \) when the algorithm terminates. Notice that \( (P_1, P_2) \) is not necessarily the Voronoi partition induced by \( (c_1, c_2) \) because of the pruning phase. Also, by construction, a point \( p \) of \( P \) is assigned to \( u_t \) for a cluster-index \( t \) only when it is fully defined on \( I_t \) at the moment.

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1 We will set \( \alpha = \epsilon/3 \) and \( c = \frac{\alpha}{64(\Delta+1)} \).
Algorithm 1: Idealized 2-Means

\begin{algorithm}
  \caption{Idealized 2-Means}
  \begin{algorithmic}[1]
    \STATE \textbf{input :} A set $P$ of $\Delta$-missing points in the plane
    \STATE \textbf{output :} A $(1 + \epsilon)$-approximate 2-means clustering for $P$
    \STATE $R \leftarrow P$ and $P_1, P_2 \leftarrow \emptyset$
    \STATE Initialize $u_1, u_2$ so that $u_i = \otimes$ for all $i \in [d]$ and all $t \in [2]$;
    \WHILE{$R \neq \emptyset$}
      \STATE Let $t$ be the cluster-index that maximizes $|PD(R \cap P^*_t, [d] - I_t)|$;
      \IF{$|PD(R \cap P^*_t, [d] - I_t)| \geq c|R|$}
        \STATE /* sampling phase */
        \IF{$I_t = \emptyset$}
          \STATE $u_t \leftarrow \Delta$-missing point obtained from Lemma 2;
        \ELSE
          \STATE Let $j$ be the coordinate-index in $[d] - I_t$ that maximizes $PD(R \cap P^*_t, j)$;
          \STATE $(u_t)_j \leftarrow \text{The value obtained from Lemma 2}$;
        \ENDIF
        \STATE Add the points in $FD(R, I_1 \cap I_2)$ closer to $u_1$ than to $u_2$ to $P_1$;
        \STATE Add the points in $FD(R, I_1 \cap I_2)$ closer to $u_2$ than to $u_1$ to $P_2$;
        \STATE $R \leftarrow R - FD(R, I_1 \cap I_2)$;
      \ELSE
        \STATE /* pruning phase */
        \STATE $t \leftarrow$ the cluster-index that maximizes $|FD(R, I_t)|$;
        \STATE $B \leftarrow \text{The first half of } FD(R, I_t) \text{ sorted in ascending order of distance from } u_t$;
        \STATE Assign the points in $B$ to $P_1$;
        \STATE $R \leftarrow R - B$;
      \ENDIF
    \ENDWHILE
    \RETURN $(u_1, u_2)$
  \end{algorithmic}
\end{algorithm}

4.2 Approximation Factor

In this section, we analyze the approximation factor of the algorithm in Section 4.1. To do this, let $S$ be the sequence of sampling phases happened during the execution of the algorithm.

\textbf{Lemma 3.} At any time during the execution, $|I_t| \geq d - \Delta$ or $|I_t| = 0$ for $t = 1, 2$.

\textbf{Proof.} Initially, all coordinates of $u_1$ and $u_2$ are set to $\otimes$, and thus $I_1 = I_2 = \emptyset$. The cluster centers $u_1$ and $u_2$ are updated during the sampling phases only. If $I_t$ is empty for $t = 1, 2$, we use Lemma 2, and thus $u_t$ is updated to a $\Delta$-missing point. If $I_t$ is not empty, we use Lemma 1 and increase $|I_t|$ by one. Therefore, the lemma holds.

\textbf{Corollary 4.} The size of $S$ is at most $2\Delta + 2$.

Now we show that the clustering cost induced by the points assigned to incorrect clusters during the pruning phases is small compared to the clustering cost induced by the points assigned to correct clusters.

Consider two consecutive sampling phases $s$ and $s'$, and consider the pruning phases lying between them. During this period, $u_1$ and $u_2$ remain the same. Let $R$ be the set of
points of $P$ which were not yet assigned to any cluster at the beginning of this period. In each iteration during this period, points of $R$ are assigned to one of the two clusters. Let $R^{(x)}$ be the set of points of $P$ which are not yet assigned to any cluster at the beginning of the $x$th pruning phase (during this period). Let $X_t^{(x)} = \text{FD}(R^{(x)}, I_t)$ for $t = 1, 2$. Also, let $A_t^{(x)}$ denote the set of points of $P$ assigned to $u_t$ at the $x$th iteration. By construction, either $A_1^{(x)} = \emptyset$ or $A_2^{(x)} = \emptyset$, but not both.

Let $X_t$ be the increasing sequence of indices $x$ of $[N]$ with $A_t^{(x)} \neq \emptyset$ for $t = 1, 2$, where $N$ denotes the number of pruning phases lying between the two consecutive sampling phases we are considering. To make the description easier, let $A_t^{(N+1)} = X_t^{(x_t)}$, where $x_t$ is the last index of $X_t$. Then we add $N + 1$ at the end of $X_1$ and $X_2$.

We first analyze the clustering cost induced by the points of $P_1^*$ (and $P_2^*$) assigned to $u_2$ (and $u_1$) during the pruning phases. To do this, we need the following three technical claims, which will be used to prove Lemma 8.

\textbf{Claim 5.} For any index $x \in [N]$, we have the following.

$$|X_1^{(x)} \cap P_2^*| + |X_2^{(x)} \cap P_1^*| < 2c(\Delta + 1)(|X_1^{(x)}| + |X_2^{(x)}|).$$

\textbf{Proof.} Assume to the contrary that $|X_1^{(x)} \cap P_2^*| + |X_2^{(x)} \cap P_1^*| \geq 2c(\Delta + 1)(|X_1^{(x)}| + |X_2^{(x)}|)$. We show that there is a cluster-index $t$ such that $\text{PD}(P_1^* \cap R^{(x)}, [d] - I_t)$ has $c|R^{(x)}|$ points, which contradicts that the $x$th phase is a pruning phase.

To see this, observe the following. Let $R' = R^{(0)} - (X_1^{(0)} \cup X_2^{(0)})$. That is, $R'$ is the set of points which are not yet assigned to any cluster at the beginning of these pruning phases, but whose domains are fully defined on neither $I_1$ nor $I_2$. Therefore, during the pruning phases (in this period), no point of $R'$ is assigned to any cluster.

$$\sum_{t=1,2} |\text{PD}(P_t^* \cap R^{(x)}, [d] - I_t)| = \sum_{t=1,2} |\text{PD}(P_t^* \cap (R' \cup X_1^{(x)} \cup X_2^{(x)}), [d] - I_t)|$$

$$= |P_1^* \cap X_2^{(x)}| + |P_2^* \cap X_1^{(x)}| + \sum_{t=1,2} |R' \cap P_t^*|$$

$$\geq 2c(\Delta + 1)(|X_1^{(x)}| + |X_2^{(x)}| + |R'|)$$

$$= 2c(\Delta + 1)|R^{(x)}|$$

The first and last equalities hold since $R^{(x)}$ is decomposed by $R'$, $X_1^{(x)}$, and $X_2^{(x)}$. The second equality holds because $X_t^{(x)} \subset \text{FD}(R^{(x)}, I_t)$ for $t = 1, 2$ and no point in $R'$ is fully defined on $I_t$ for any cluster-index $t = 1, 2$. The third inequality holds by the fact that $2c(\Delta + 1) < 1$ and the assumption we made at the beginning of this proof.

Let $t$ be the cluster-index that maximizes $|\text{PD}(P_t^* \cap R^{(x)}, [d] - I_t)|$. Then we have $|P_t^* \cap \text{PD}(R^{(x)}, [d] - I_t)| \geq c(\Delta + 1)|R^{(x)}|$. Therefore, the lemma holds.

\textbf{Claim 6.} For any consecutive indices $x'$ and $x$ in $X_1$ with $x' < x$, $|A_1^{(x')} \cap P_2^*| \leq \frac{16c(\Delta + 1)}{1 - 8c(\Delta + 1)} |A_1^{(x)} \cap P_2^*|$.

\textbf{Proof.} We first show that $|A_1^{(x)} \cap P_2^*|$ is at most $8c(\Delta + 1)|A_1^{(x)}|$ as follows.

$$|A_1^{(x)} \cap P_2^*| \leq |X_1^{(x')} \cap P_2^*|$$

$$< 2c(\Delta + 1)(|X_1^{(x')}| + |X_2^{(x')}|)$$

$$\leq 4c(\Delta + 1)|X_2^{(x')}|$$

$$\leq 8c(\Delta + 1)|A_1^{(x)}|.$$
The first inequality holds because $A_1^{(x)}$ is a subset of $X_1^{(x)}$. The second inequality holds by Claim 5 and the third inequality holds since $|X_1^{(x')}| > |X_2^{(x')}|$. If it is not the case, the algorithm would assign a half of $X_2^{(x')}$ to $u_2$, and thus $A_1^{(x')} = \emptyset$, which contradicts that $x \in X_1$. The last equality holds since $|X_1^{(x')}| \leq 2 |A_1^{(x')}$.

Now we give a lower bound of $|A_1^{(x)} \cap P_1^*|$ as follows. The second inequality holds due to the upper bound of $|A_1^{(x)} \cap P_1^*|$ stated above.

$$|A_1^{(x)} \cap P_1^*| = |A_1^{(x)}| - |A_1^{(x)} \cap P_1^*|$$

$$\geq |A_1^{(x)}| - 8c(\Delta + 1)|A_1^{(x)}|$$

$$= (1 - 8c(\Delta + 1))|A_1^{(x)}|$$

By combining the upper and lower bounds, we can obtain the following. The last inequality holds because $|A_1^{(x')}| \leq 2 |A_1^{(x')}$.

$$\frac{|A_1^{(x)} \cap P_1^*|}{|A_1^{(x)} \cap P_1^*|} \leq \frac{8c(\Delta + 1)|A_1^{(x')}|}{1 - 8c(\Delta + 1)|A_1^{(x)}|} \leq \frac{16c(\Delta + 1)}{1 - 8c(\Delta + 1)}.$$  

\[\textbf{Claim 7.} \text{ Let } A_t = \bigcup_{x \in [N]} A_t^{(x)} \text{ for } t = 1, 2.\]

- $\text{cost}(A_1 \cap P_2^*, c_1) \leq 32c(\Delta + 1)\text{cost}(FD(R^{(0)} \cap P_1^*, I_1), c_1)$, and
- $\text{cost}(A_2 \cap P_1^*, c_2) \leq 32c(\Delta + 1)\text{cost}(FD(R^{(0)} \cap P_2^*, I_2), c_2)$.

\[\textbf{Proof.} \text{ We prove the first inequality only. The other is proved analogously. Recall that } X_1 \text{ is the increasing sequence of indices } x \text{ of } [N + 1] \text{ with } A_1^{(x)} \neq \emptyset, \text{ and the last index of } X_1 \text{ is } N + 1. \text{ Consider two consecutive indices } x' \text{ and } x \text{ in } X_1 \text{ with } x' < x. \text{ By definition, for any index } x \in [N], c_1 \text{ is closer to any point of } A_1^{(x')} \text{ than to any point of } A_1^{(x)}. \text{ Therefore, we have}\]

$$\frac{\text{cost}(A_1^{(x')} \cap P_2^*, c_1)}{\text{cost}(A_1^{(x')} \cap P_2^*)} \leq \frac{\text{cost}(A_1^{(x)} \cap P_1^*, c_1)}{\text{cost}(A_1^{(x)} \cap P_1^*)}.$$  

By Claim 5, we have

$$\text{cost}(A_1^{(x)} \cap P_2^*, c_1) \leq \frac{16c(\Delta + 1)}{1 - 8c(\Delta + 1)}\text{cost}(A_1^{(x)} \cap P_1^*, c_1).$$

Note that the points of $P$ assigned to $u_1$ during these consecutive pruning phases are exactly the points in the union of $A_1^{(x)}$s for all indices $x \in [N]$. Moreover, $A_1^{(x)}$s are pairwise disjoint. Also, $\frac{16c(\Delta + 1)}{1 - 8c(\Delta + 1)} \leq 32c(\Delta + 1)$. Therefore,

$$\text{cost}(A_1 \cap P_2^*) \leq 32c(\Delta + 1)\text{cost}(FD(R^{(0)} \cap P_1^*, I_1), c_1).$$

The other inequality can be proved symmetrically.
Lemma 8. Let $\sum_{s \in S} \text{cost}(A^*_t \cap P^*_t, c_1) + \sum_{s \in S} \text{cost}(A^*_t \cap P^*_t, c_2) \leq (64c(\Delta + 1)^2) \sum_{s \in S} \text{cost}_{I^*}(R^*_t, c_{I^*})$.

Proof. Let $S_t$ be the set of sampling phases $s$ in $S$ with $t^s = t$.

$$\sum_{s \in S} \text{cost}(A^*_t \cap P^*_t, c_1) \leq 32c(\Delta + 1) \sum_{s \in S} \text{cost}(FD(R^*_t \cap P^*_t, I^*_t), c_1)$$

$$\leq 32c(\Delta + 1) \sum_{s \in S} \sum_{s' \geq s} \text{cost}_{I^*}(PD(R^*_t \cap P^*_t, I^*'), c_1)$$

$$= 32c(\Delta + 1) \sum_{s' \in S_t} \sum_{s' \geq s} \text{cost}_{I^*}(PD(R^*_t \cap P^*_t, I^*'), c_1)$$

$$\leq 32c(\Delta + 1) \sum_{s' \in S_t} \sum_{s' \geq s} \text{cost}_{I^*}(PD(R^*_t, I^*), c_1)$$

$$\leq 64c(\Delta + 1)^2 \sum_{s \in S_t} \text{cost}_{I^*}(PD(R^*_t, I^*), c_1)$$

The first inequality holds by Claim 4. The second one holds since $s' \geq s$ for two sampling phases $s'$ and $s$ in $S$ with $I^*_s \subset I^*_t$. The third equality holds since it changes only the ordering of summation. The fourth inequality holds since for two sampling phases $s$ and $s'$ in $S$, $R^*_s \subset R^*_t$ if $s' \geq s$. The fifth inequality holds since the number of sampling phases is at most $2(\Delta + 1)$ The last equality holds by definition of $\text{cost}()$.

Analogously, we have

$$\sum_{s \in S} \text{cost}(A^*_t \cap P^*_t, c_2) \leq 64c(\Delta + 1)^2 \sum_{s \in S_2} \text{cost}_{I^*}(R^*_t, c_2).$$

Finally, we have

$$\sum_{s \in S} \text{cost}(A^*_t \cap P^*_t, c_1) + \sum_{s \in S} \text{cost}(A^*_t \cap P^*_t, c_2) \leq 64c(\Delta + 1)^2 \sum_{s \in S} \text{cost}_{I^*}(R^*_t, c_{I^*}).$$

We now analyze the clustering cost induced by $P_t$ excluding the points incorrectly assigned to a cluster during the pruning phases. That is, we give an upper bound on the clustering cost induced by the points assigned during the sampling phases and the points assigned correctly during the pruning phases. Let $T_t$ denote the set of points of $P^*_t$ assigned to $u_t$ during the pruning phases and $S_t$ denote the set of points assigned to $u_t$ during the sampling phases.

Lemma 9. $\text{cost}(S_t \cup T_t, c_1) + \text{cost}(S_t \cup T_t, c_2) \leq \sum_{s \in S} \text{cost}_{I^*}(R^*_t, c_{I^*})$.\hfill $\blacksquare$
Lemma 10. For convenience, let \( p \) with probability \( p \) we have

\[
\text{cost}(S_1, c_1) + \text{cost}(S_2, c_2) \leq \text{cost}(S \cap P_1^*, c_1) + \text{cost}(S \cap P_2^*, c_2).
\]

To see this, consider a point \( p \) in \( \text{PD}((S \cup T_1) \cap P_i^*, I) \). Let \( I_t \) is the domain of the moment when \( p \) is assigned to a cluster. If \( p \) is assigned during a sampling phase, by construction, we have \( \text{dom}(p) \subseteq I_1 \cap I_2 \). If \( p \) is assigned to \( u_t \) during a pruning phase, we have \( \text{dom}(p) \subseteq I_t \).

By combining all properties mentioned above, we have

\[
\text{cost}(S_1 \cup T_1, c_1) + \text{cost}(S_2 \cup T_2, c_2) \leq \sum_{t=1,2} \text{cost}((S \cup T_t) \cap P_t^*, c_t)
= \sum_{s \in S} \text{cost}_{I_t}(\text{PD}((S \cup T_t) \cap P_t^*, I^s), c_t)
\leq \sum_{s \in S} \text{cost}_{I_t}(R_t^s, c_t).
\]

The first inequality comes from \( \| \) and the fact that \( T_t \subseteq P_t^* \), and the second equality holds by the definition of \( \text{cost}(\cdot) \). The last inequality holds since \( \text{PD}((S \cup T_t) \cap P_t^*, I) \) is a subset of \( R_t^s \) for \( s = (t, I) \).

Therefore, it suffices to analyze an upper bound of the sum of \( \text{cost}_{I_t}(R_t^s, c_t) \) for all sampling phases \( s \) in \( S \). We show that it is bounded by \( (1 + \alpha)\text{OPT}_2(P) \) with a constant probability.

**Lemma 10.** \( \sum_{s \in S} \text{cost}_{I_t}(R_t^s, c_t) \leq (1 + \alpha)\text{OPT}_2(P) \), with a probability at least \( p^2 q^{2\Delta} \), where \( q \) and \( p \) are the probabilities in Lemmas 4 and 5.

**Proof.** For convenience, let \( R^s = R_t^s \). The algorithm iteratively obtains the values of \( c_t \) using Lemmas 4 and 5. Let \( s = (t, I) \) be a sampling phase in \( S \). If \( I \) consists of a single coordinate-index, say \( i \), then \( (u_t)_i \) was updated using Lemma 4 during the phase \( s \). Thus we have

\[
\text{cost}_{I_t}(R^s, c_t) \leq (1 + \alpha)\text{cost}_{I_t}(R^t, c(R^t)) \leq (1 + \alpha)\text{cost}_{I_t}(P^*_t, c_t),
\]

with probability \( q \). The first inequality holds by Lemma 4 with probability \( q \). The second inequality holds because \( R^s \) is a subset of \( P_t^* \), and \( c_t \) is the centroid of \( P_t^* \).

Otherwise, that is, if \( I \) consists of more than one coordinate-indices, \( (u_t)_i \)’s were obtained using Lemma 5 for all indices \( i \in I \) during the sampling phase \( s \). Thus we have the following with probability \( p \) by Lemma 5

\[
\text{cost}_{I_t}(R^s, c_t) \leq (1 + \alpha)\text{cost}_{I_t}(R^s, c(R^s)) \leq (1 + \alpha)\text{cost}_{I_t}(P^*_t, c_t).
\]

By Lemma 5, the number of indices obtained by Lemma 4 is at most \( 2\Delta \) in total, and the number of indices obtained by Lemma 5 is exactly two, one for each cluster. Therefore, with probability \( p^2 q^{2\Delta} \), we have

\[
\sum_{s \in S} \text{cost}_{I_t}(R^s, c_t) \leq (1 + \alpha)\text{OPT}_2(P).
\]

\( \blacksquare \)
Algorithm 2: 2-Means((u1, u2), R)

1. $R \leftarrow R - \text{FD}(R, I_1 \cap I_2)$;
2. $E \leftarrow \emptyset, u'_1 \leftarrow u_1$ and $u'_2 \leftarrow u_2$;
3. if $R = \emptyset$ then return $(u_1, u_2)$;
4. for $t=1,2$ do
   5. if $I_t = \emptyset$ then
      6. $u'_t \leftarrow$ the $\Delta$-missing point obtained from Lemma 2
      7. Add the clustering returned by 2-Means((u'_1, u'_2), R)
   8. else
      9. foreach $j \in [d] - I_t$ do
         10. $u'_t \leftarrow u_t$;
         11. $(a'_t)_j \leftarrow$ The value obtained from Lemma 1
         12. Add the clustering returned by 2-Means((u'_1, u'_2), R)
      13. end
   14. end
   15. $t \leftarrow$ the cluster-index that maximizes $|\text{FD}(R, I_t)|$;
16. if $\text{FD}(R, I_t) \geq |R|/3$ then
   17. $B \leftarrow$ The first half of $\text{FD}(R, I_t)$ sorted in ascending order of distance from $u_t$;
   18. Add the clustering returned by 2-Means((u_1, u_2), R - B) to $E$;
19. end
20. return the clustering $(c_1, c_2)$ in $E$ which minimizes $\text{cost}(R, \{c_1, c_2\})$

By combining Lemmas 8, 9 and 10, we have the following lemma.

Lemma 11. For a constant $\alpha > 0$, the algorithm returns an $(1 + 64 c(\Delta + 1)^2)(1 + \alpha)$-approximate 2-means clustering for $P$ with probability at least $p^2 q^{2\Delta}$, where $q$ and $p$ are the probabilities in Lemmas 4 and 5.

4.3 Algorithm without Counting Oracle

The algorithm described in Section 4.3 uses the counting oracle, which seems hard to implement. There are two places where the counting oracle is used: to determine the type of the phase and to determine the coordinate- and cluster-indices to be updated in a sampling phase (at Line 4-5 and at Line 10 of Algorithm 1).

In this section, we show how to avoid using the counting oracle. To do this, we simply try all possibilities: run both sampling and pruning phases, and update each of the indices during a sampling phase. Our main algorithm, 2-Means((u_1, u_2), R), is described in Algorithm 2. Its input consists of cluster centers $(u_1, u_2)$ of a partial clustering of $P$ and a set $R$ of points of $P$ which are not yet assigned to any cluster. Finally, 2-Means((R, P), P) returns an $(1 + 64 c(\Delta + 1)^2)(1 + \alpha)$-approximate 2-means clustering of $P$. By setting $\alpha$ properly, we can obtain an $(1 + \epsilon)$-approximate 2-means clustering of $P$.

Description of the algorithm. In the main algorithm, we run both sampling and pruning phases. In a sampling phase, if $I_t \neq \emptyset$, we simply update $(u_t)_j$ for all coordinate-indices $j \in [d] - I_t$ since we do not know which index maximizes $|PD(R \cap P^*_t, j)|$. At the end of sampling and pruning phases, we call 2-Means recursively. Then we return the clustering
with minimum cost among the clusterings returned the recursive calls. The pseudocode of this algorithm is described in Algorithm \[\text{Algorithm2}\].

**Analysis of the algorithm.** Clearly, the clustering cost returned by 2-Means(\((\otimes, \otimes), P\)) is at most the cost returned by the algorithm described in Section 4.1. In the following, we analyze the running time of 2-Means(\((\otimes, \otimes), P\)). Let \(T(n, \delta)\) be the running time of 2-Means((\(u_1, u_2\)), \(R\)) when \(n = |R|\) and \(\delta = \min\{d - |I_1|, \Delta + 1\} + \min\{d - |I_2|, \Delta + 1\}\). Here, \(\delta\) is an upper bound on the number of updates required to make \(I_1 = [d]\) and \(I_2 = [d]\).

\[\text{Lemma 12. } T(n, \delta) \leq \delta \cdot T(n, \delta - 1) + T\left(\frac{n}{2}, \delta\right) + O\left(\frac{\delta^2 dn}{\alpha}\right).\]

**Proof.** Assume that we call 2-Means((\(u_1, u_2\)), \(R\)) with \(|R| = n\) and \(\delta = \min\{d - |I_1|, \Delta + 1\} + \min\{d - |I_2|, \Delta + 1\}\). The tasks excluding Lines 6–17 and Line 23 take \(O(\delta \Delta^3 dn/\alpha)\) time. In the following, we focus on the running time for completing the tasks described in Lines 6–17 and Line 20.

We can perform Line 9 in \(O(\Delta^3 dn/\alpha)\) time by Lemma 2 and Line 14 in \(O(dn/\alpha)\) time by Lemma 1 for each index \([d] - I_t\). Since \(u_t\) is updated at most \(\delta\) times, this takes \(O(\delta \Delta^3 dn/\alpha)\) in total. Then we recursively call 2-Means((\(u'_1, u'_2\)), \(R\)) at most \(\delta\) times at Lines 6–17. For each call to 2-Means((\(u'_1, u'_2\)), \(R\)), \(\delta\) decreases by one. Therefore, the time for the recursive calls at Lines 6–17 is at most \(\delta \cdot T(n, \delta - 1)\).

Also, 2-Means calls itself recursively at Line 21. In this case, the second parameter, \(R - B\), has size at most \(5n/6\). Therefore, the time for this recursive call is \(T(5n/6, \delta)\). Therefore, the lemma holds.

\[\text{Lemma 13. } T(n, \delta) \leq (6\delta)^{2\delta + 1} \left(\frac{6}{5}\right)^{\delta^2} \Delta^3 dn/\alpha \]

**Proof.** We prove the claim inductively. Basically, \(T(n, 0) \leq O(dn/\alpha)\) and \(T(1, \delta) = O(1)\). For \(\delta \geq 1\) and \(n > 1\), \(T(n, \delta)\) satisfies the following by inductive hypothesis:

\[
T(n, \delta) \leq \delta T(n, \delta - 1) + T\left(\frac{n}{2}, \delta\right) + \delta \cdot \Delta^3 dn/\alpha
\]

\[
\leq (6\delta)^{2\delta + 1} \left(\frac{6}{5}\right)^{\delta^2} \Delta^3 dn/\alpha
\]

By Lemmas 11 and 13, we get the following lemma.

\[\text{Lemma 14. } 2\text{-Means } ((\otimes, \otimes), P) \text{ returns a clustering for } P \text{ of cost at most } (1 + 64c(\Delta + 1)^2)(1 + \alpha)\text{OPT}_2(P) \text{ in } 2^{O(\Delta^{\log \Delta})} d|P| \text{ time with probability } p^2 q^{2\Delta}.\]

We obtain the following theorem by setting \(\alpha = \epsilon/3\) and \(c = \frac{\alpha}{64(\Delta + 1)}\).

\[\text{Theorem 15. } \text{Given a } \Delta\text{-missing } n\text{-point set } P \text{ in } \mathbb{H}^d, \text{ a } (1 + \epsilon)\text{-approximate } 2\text{-means clustering can be found in } 2^{O(\max(\Delta^{\log \Delta}, \Delta^{1/\log \Delta}))} dn \text{ time with probability } 1/2.\]

5. **For k-Means Clustering**

In this section, we describe and analyze for a k-means clustering algorithm. We also describe the algorithm for \(k = 2\) in Appendix 2 to help readers to follow this section. Let \(U = \{u_1, u_2, \ldots, u_k\}\) be a sequence of points in \(\mathbb{H}^d\). For a sequence \(T\) of cluster-indices, we use \(U_T\) to denote a \(|T|\)-tuple consisting of \(u_t\)'s for \(t \in T\). In this section, for any sequence \(\mathcal{A}\) and a set \(T\) of sequential-indices, \(\mathcal{A}_T\) denotes \(|T|\)-subsequence of \(\mathcal{A}\) consisting of the \(t\)-th entries of \(\mathcal{A}\) for \(t \in T\).
We first sketch an algorithm for \(k\)-clustering assuming that we can access the counting oracle. Let \(P^* = \langle P_1^*, \ldots, P_k^* \rangle\) be an optimal \(k\)-clustering for \(P\) induced by the centroids \(C^* = \langle c_1^*, \ldots, c_k^* \rangle\). The counting oracle takes a subset of \(P\) and a cluster-index \(t \in [k]\), and it returns the number of points in the subset which are contained in \(P^*_t\). Then in Section 5.3, we show how to do this without using the counting oracle.

The algorithm consists of several phases of two types: a sampling phase or a pruning phase. We initialize \(U\) so that \((u_t)_i = \emptyset\) for all \(t \in [k]\) and \(i \in [d]\). In a sampling phase, we obtain values of \((u_t)_j\) for indices \(t \in [k]\) and \(j \in [d]\) which were set to \(\emptyset\). Also, we assign points of \(P\) to one of the \(k\) clusters in sampling and pruning phases. Finally, \(U\) is updated to \((1 + \epsilon)\)-approximate clustering as we will see later. In the following, we use \(C\) to denote the output of the algorithm. The pseudocode of the algorithm is described in Algorithm 3.

At any moment during the execution of the algorithm, we maintain the set \(R\) of remaining points and a \(k\)-tuple \(U = \langle u_1, \ldots, u_k \rangle\) of (partial) centers in \(\mathbb{H}^d\). Also, we let \(I_t\) be \(\text{dom}(u_t)\). Initially, \(R\) is set to \(P\), and \(U\) is set to the \(k\)-tuple of null points. The algorithm terminates if \(R = \emptyset\), and finally \(U\) becomes a set of points in \(\mathbb{R}^d\). At the beginning of each phase, we decide the type of the current phase. For this purpose, we consider the partition \(\mathcal{F}\) of \(R\).
Lemma 16. For any point \( x \in R \), there exists a unique set in \( \mathcal{F} \) containing \( x \).

Proof. We first show that the sets in \( \mathcal{F} \) are pairwise disjoint. Consider two proper subsets \( T \) and \( T' \) of \([k]\). Then we have \( S_T \cap S_{T'} = \emptyset \). To see this, let \( t \) be a cluster-index in \( T - T' \). By definition, for a point \( p \in S_T \), we have \( \text{DOM}(p) \subseteq I_t \), and for a point \( p \in S_{T'} \), we have \( \text{DOM}(p) \not\subseteq I_t \).

Thus, it suffices to show that the union of the sets in \( \mathcal{F} \) is \( R \). For any point \( p \in R \), let \( T \) be the set of cluster-indices \( t \) in \([k]\) with \( \text{DOM}(p) \subseteq I_t \). By definition, \( p \) is contained in \( S_T \), which is a set of \( \mathcal{F} \). Therefore, the lemma holds.

At the beginning of each phase, we decide which type of the current phase. Let \( t \) be the cluster-index of \([k]\) that maximizes \( |\text{PD}(R \cap P^*_t, |d| - I_t)| \), where \( R \) is the set of points of \( P \) which are not assinged to any cluster. The one of the following cases always happen: \( |\text{PD}(R \cap P^*_t, |d| - I_t)| \geq c|R| \), or there exists a set \( T \) of cluster-indices in \([k]\) such that \( |S_T \cap (\cup_{t \in T} P^*_t)| \geq c|R| \) for any constant \( c < 1/(2^k + k) \), which will be specified later.

Lemma 17. One of the following always holds for any constant \( c < 1/(2^k + k) \).

1. \( |\text{PD}(R \cap P^*_t, |d| - I_t)| \geq c|R| \) for a cluster-index \( t \in [k] \), or
2. \( |S_T \cap (\cup_{t \in T} P^*_t)| \geq c|R| \) for a proper subset \( T \) of \([k]\).

Proof. By the definition of \( \mathcal{F} \), we have

\[
\text{PD}(R \cap P^*_t, |d| - I_t) = \bigcup_{t \in T \subseteq [k]} S_T \cap P^*_t.
\]

Thus, the union of \( \{\text{PD}(R \cap P^*_t, |d| - I_t)\}_{t \in [k]} \{S_T \cap (\cup_{t \in T} P^*_t)\}_{T \subseteq [k]} \) decompose \( R \). Therefore, the lemma holds.

**Sampling phase.** If the first case happens, we enter a sampling phase. Let \( \alpha \) be a constant, which will be specified later. We use it as an approximation factor used in Lemmas 1 and 2 for sampling. If \( I_t \) is empty, we replace \( u_t \) with a \( \Delta \)-missing point in \( \mathbb{R}^d \) obtained from Lemma 2. If \( I_t \) is not empty, then it is guaranteed that \( |I_t| \) is at least \( d - \Delta \). We compute the coordinate-index \( j \in |d| - I_t \) that maximizes \( |\text{PD}(R \cap P^*_t, j)| \) using the counting oracle. Clearly, \( (u_t)_j = \emptyset \) and \( |\text{PD}(R \cap P^*_t, j)| \) is at least \( c|R|/\Delta \). Then we replace \( (u_t)_j \) with a value obtained from Lemma 4. At the end of the phase, we check if \( \text{FD}(R, \cap_{t \in [k]} I_t) \) is not empty. If it is not empty, we assign those points to their closest cluster centers.

**Pruning phase.** Otherwise, we enter a pruning phase. Instead of obtaining a new coordinate value of \( u_t \), we assign points of \( R \) to cluster centers in a pruning phase. To do this, we find a proper subset \( T \) of \([k]\) which maximizes \( |S_T| \). Then among the points of \( S_T \), we choose the \( \lfloor |S_T|/2 \rfloor \) points closest to their closest centers in \( u_T \). Then we assign each of them to its closest center in \( u_T \). In this way, points in \( \cup_{t \in T} P^*_t \) might be assigned (incorrectly) to \( u_t \) for a cluster-index \( t \in T \). We call such a point a *stray point*.

---

2 Notice that \( T \) might be empty, but it is also a proper subset of \([k]\).
3 We set \( \alpha = \epsilon/3 \), and \( c = \frac{\alpha}{2^{2^d}(\Delta + 1)} \).
5.2 Analysis of the Approximation Factor

In this section, we analyze the approximation factor of the algorithm. We let $S$ be the sequence of sampling phases happened during the execution of the algorithm.

**Lemma 18.** At any time during the execution, $|I_t| \geq d - \Delta$ or $|I_t| = 0$ for $t \in [k]$.

**Proof.** Initially, all coordinates of $u_t$ are set to $\emptyset$, and thus $I_t = \emptyset$. The cluster centers $u_t$ are updated during the sampling phases only. If $I_t$ is empty for $t \in [k]$, we use Lemma 2 and thus $u_t$ is updated to a $\Delta$-missing point. If $I_t$ is not empty, we use Lemma 1 and increase $|I_t|$ by one. Therefore, the lemma holds.

**Corollary 19.** The size of $S$ is at most $k(\Delta + 1)$.

5.2.1 Clustering Cost Induced by Stray Points

We first analyze the clustering cost induced by the stray points assigned during the pruning phases. Consider the consecutive pruning phases lying between two adjacent sampling phases in $S$. Let $N$ denote the number of such pruning phases. During this period, $u_t$ remains the same for each cluster-index $t \in [k]$. For a proper subset $T$ of $[k]$, let $A_T^{(x)}$ denote the set of points of $S_T$ assigned to $u_t$ in the $x$th iteration (during this period). Let $R_T^{(x)}$ be the set of points of $P$ which are not yet assigned to any cluster at the beginning of the $x$th iteration (during this period). Let $X_T^{(x)} = R_T^{(x)} \cap S_T$. By construction, in the $x$th iteration, there exists a unique index-set $T$ with $A_T^{(x)} \neq \emptyset$. Furthermore, let $X_T$ be the increasing sequence of indices $x$ of $[N]$ with $A_T^{(x)} \neq \emptyset$. For convenience, let $A_T^{(N+1)} = X_T^{(N)}$. We denote $P_T^* = \cup_{t \in T} P_T^*$ and $P_T^\prime = \cup_{t \in T} P_T^\prime$. We need the following three technical claims, which will be used to prove Lemma 24.

**Claim 20.** For any index $x \in [N]$, we have the following.

$$\sum_{T \subseteq [k]} |X_T^{(x)} \cap P_T^*| < c k(\Delta + 1) \sum_{T \subseteq [k]} |X_T^{(x)}|.$$

**Proof.** Assume to the contrary that $\sum_{T \subseteq [k]} |X_T^{(x)} \cap P_T^*| \geq c k(\Delta + 1) \sum_{T \subseteq [k]} |X_T^{(x)}|$. In the following, we show that there is a cluster-index $t$ such that $|PD(P_t^* \cap R_t^{(x)}[d - I_t])| \geq c |R_t^{(x)}|$. This contradicts that $x$th phase is a pruning phase. Let $R' = R^{(0)} \cup X_T^{(0)}$. That is, $R'$ is the set of points which are not yet assigned to any cluster at the beginning of these pruning phases, but whose domains are not contained in $S_T$ for any proper subset of $[k]$. Therefore, during the pruning phases (in this period), no point of $R'$ is assigned to any cluster.

$$\sum_{t \in [k]} |PD(P_t^* \cap R_t^{(x)}, [d] - I_t)| = \sum_{t \in [k]} |PD(P_t^* \cap (R' \cup (\cup_{T \subseteq [k]} X_T^{(x)})), [d] - I_t)|$$

$$= \sum_{t \in [k]} \sum_{T \subseteq [k]} |P_t^* \cap X_T^{(x)}| + \sum_{t \in [k]} |P_t^* \cap R'|$$

$$= \sum_{T \subseteq [k]} \sum_{t \in T} |P_t^* \cap X_T^{(x)}| + \sum_{t \in [k]} |P_t^* \cap R'|$$

$$\geq c k(\Delta + 1) \sum_{T \subseteq [k]} |X_T^{(x)}| + |R'|$$

$$= c k(\Delta + 1) |R_t^{(x)}|.$$
The first and the last inequality hold since $R'$ and $X_T^{(x)}$ partition $R^{(x)}$. The second equality holds since $X_T^{(x)} \subseteq S_T$. Also, $\text{dom}(p) \subseteq I_t$ for any point $p \in S_T$ if and only if $t \in T$. The equalities hold since they only change the ordering of summation and the inequality holds by assumption and the fact $ck(\Delta + 1) \leq 1$. Therefore, the claim holds. ▶

Claim 21. For any consecutive indices $x$ and $x'$ in $X_T$ with $x' < x$, the following holds for any proper subset $T$ of $[k]$:

$$\frac{|A_T^{(x')} \cap P_T^*|}{|A_T^{(x)} \cap P_T^*|} \leq \frac{4 \cdot 2^k ck(\Delta + 1)}{1 - 2 \cdot 2^k ck(\Delta + 1)}.$$  

Proof. We first show that $|A_T^{(x')} \cap P_T^*|$ is at most $4c(2\Delta + 2)|X_T^{(x)}|$ as follows.

$$|A_T^{(x')} \cap P_T^*| \leq |X_T^{(x')} \cap P_T^*|$$

$$< ck(\Delta + 1) \left( \sum_{T' \subseteq [k]} |X_T^{(x')}| \right)$$

$$\leq 2^k ck(\Delta + 1) |X_T^{(x')}|$$

$$\leq 2 \cdot 2^k ck(\Delta + 1) |A_T^{(x')}|.$$  

The first inequality holds because $A_T^{(x')}$ is a subset of $X_T^{(x')}$. The second inequality holds by Claim 20 and the third inequality holds since $T$ maximizes $|X_T^{(x')}|$. If it is not the case, then $A_T^{(x')} = \emptyset$ since the algorithm would assign the first half of $X_T^{(x')}$ for a set $T' \subseteq [k]$ other than $T$. This contradicts for $x \in X_T$. The last inequality holds since $|X_T^{(x')}| \leq 2|A_T^{(x')}|$.

Now we give a lower bound of $|A_T^{(x')} \cap P_T^*|$. The second inequality holds due to the upper bound of $|A_T^{(x')} \cap P_T^*|$ stated above.

$$|A_T^{(x')} \cap P_T^*| = |A_T^{(x')}| - |A_T^{(x')} \cap P_T^*|$$

$$\geq |A_T^{(x')}| - 2 \cdot 2^k ck(\Delta + 1) |A_T^{(x')}|$$

$$= (1 - 2 \cdot 2^k ck(\Delta + 1)) |A_T^{(x')}|.$$  

By combining the bounds, we can obtain $|A_T^{(x')} \cap P_T^*| \leq 2 \cdot 2^k ck(\Delta + 1) |A_T^{(x')}|$. The last inequality holds by $|A_T^{(x')}| \leq 2|A_T^{(x')}|$. Therefore, the claim holds. ◆

Claim 22. Let $A_T = \bigcup_{x \in [N]} A_T^{(x)}$. For any proper subset $T$ of $[k]$, the following holds:

$$\text{cost}(A_T \cap P_T^*, C_T) \leq 8 \cdot 2^k ck(\Delta + 1) \text{cost}(X_T^{(0)} \cap P_T^*, C_T).$$  

Proof. Note that each $c_t$ of $C$ is extension of $u_t$. Recall that $X_T$ is the increasing sequence of indices $x$ of $[N+1]$ with $A_T^{(x)} \neq \emptyset$ and the last index is $N+1$. By definition, for any index $x \in [N]$, $C_T$ is closer to any point of $A_T^{(x)}$ than to any point of $X_T^{(x)}$. Consider two consecutive indices $x$ and $x'$ in $X_T$ with $x' < x$.

$$\frac{\text{cost}(A_T^{(x')} \cap P_T^*, C_T)}{|A_T^{(x')} \cap P_T^*|} \leq \frac{\text{cost}(A_T^{(x')} \cap P_T^*, C_T)}{|A_T^{(x')} \cap P_T^*|}.$$  

Therefore, the claim holds. ◆
Claim 23. For a sampling phase $s'$ in $S$ and a proper subset $T$ of $[k]$, let $X$ be a point subset of $S_T$ which are not assigned at the end of a sampling phase $s'$. Then we have,

\[
\text{cost}(X \cap P^*_T, c_T) \leq \sum_{s \in S} \text{cost}_{I'}(PD(X \cap P^*_s, I^s), c_{t^s}),
\]

where the summation is taken over all sampling phases $s$ in $S$ which $s \preceq s'$ and $t^s \in T$.

Proof. Note that all points of $X \subset S_T$ are fully defined on $I_t$ for every cluster-indices $t \in T$ during the pruning phases between $s'$ and its consecutive sampling phase. Also, $I^s_t \subset I^s_{t'}$ for sampling phases $s$ with $s \preceq s'$. Then, by definition of cost($\cdot$), we have the following,

\[
\text{cost}(X \cap P^*_T, c_T) \leq \sum_{t \in T} \text{cost}(X \cap P^*_t, c_t)
\]

\[
= \sum_{t \in T} \text{cost}_{I'}(FD(X \cap P^*_t, I^s_{t'}), c_t)
\]

\[
\leq \sum_{x \in X} \sum_{s \preceq s'} \text{cost}_{I'}(PD(X \cap P^*_s, I^s), c_{t^s}).
\]
phase $s$. Then, we have the following:

$$\sum_{s \in S} \sum_{T \subseteq [k]} \text{cost}(A_T^s \cap P_T^s, C_T) \leq 8 \cdot 2^k c_k (\Delta + 1) \sum_{s \in S} \sum_{T \subseteq [k]} \text{cost}(X_T^s \cap P_T^s, C_T)$$

$$\leq 8 \cdot 2^k c_k (\Delta + 1) \sum_{s \in S} \sum_{T \subseteq [k]} \sum_{s' \leq s'} \sum_{t' \in T} \text{cost}_{I'}(\text{PD}(X_{T}^{s'} \cap P_{T}^{s'}, I'), c_{t'})$$

$$= 8 \cdot 2^k c_k (\Delta + 1) \sum_{s \in S} \sum_{s' \leq s'} \sum_{t' \in T} \text{cost}_{I'}(\text{PD}(X_{T}^{s'} \cap P_{T}^{s'}, I'), c_{t'})$$

$$\leq 8 \cdot 2^k c_k (\Delta + 1) \sum_{s \in S} \sum_{s' \leq s'} \text{cost}_{I'}(\text{PD}(R^s, I^s), c_{t'})$$

$$\leq 8 \cdot 2^k c_k (\Delta + 1) \sum_{s \in S} \sum_{s' \leq s'} \text{cost}_{I'}(R^s, c_{t'})$$

The first and second inequalities hold by Claims 22 and 28. The third one holds since it changes only the ordering of summation. The fourth one holds since for a fixed sampling phase $s'$ in $S$, $X_{T}^{s'}$ are disjoint for all proper subsets $T$ of $[k]$. Also notice that, for two sampling phases $s, s'$ in $S$ and a proper subset $T$ of $[k]$, $R^s$ contains $X_{T}^{s'} \cap P^s_{T}$ if $s \preceq s'$. The fifth one holds since the size of $S$ is at most $k(\Delta + 1)$. The last holds by the definition of $\text{cost}(\cdot)$.

### 5.2.2 Clustering Cost Induced by Non-Stray Points

We then give an upper bound on the clustering cost induced by non-stray points. The first term in the following lemma is the clustering cost induced by points assigned during the sampling phases, and the second term is the clustering cost induced by non-stray points assigned during the pruning phases.

**Lemma 25.** Let $S$ be the set of points assigned during the sampling phases.

$$\text{cost}(S, C) + \sum_{s \in S} \sum_{T \subseteq [k]} \text{cost}(A_T^s \cap P_T^s, C_T) \leq \sum_{s \in S} \text{cost}_{I'}(R^s, c_{t'})$$

**Proof.** During the sampling phases, we assign points in $S$ according to the Voronoi partition of $u_{[k]}$. Thus, we have the following. The last inequality holds since we assign a point during a sampling phase if it is fully defined on $\text{dom}(u_t)$ for all $t \in [k]$.

$$\text{cost}(S, C) \leq \sum_{t \in [k]} \text{cost}(S \cap P_t^s, c_t) = \sum_{s \in S} \text{cost}_{I'}(\text{PD}(S \cap R^s, I), c_{t'})$$

For the second term of this claim, we have the following:

$$\sum_{s' \in S} \sum_{T \subseteq [k]} \text{cost}(A_T^{s'} \cap P_T^{s'}, C_T) \leq \sum_{s' \in S} \sum_{T \subseteq [k]} \sum_{s \preceq s'} \sum_{t' \in T} \text{cost}_{I'}(\text{PD}(A_T^{s'} \cap P_T^{s'}, I'), c_{t'})$$

$$= \sum_{s \in S} \sum_{s' \preceq s} \sum_{t' \in T} \text{cost}_{I'}(\text{PD}(A_T^{s'} \cap P_T^{s'}, I'), c_{t'})$$

$$\leq \sum_{s \in S} \sum_{s' \preceq s} \sum_{t' \in T} \text{cost}_{I'}(\text{PD}(A_T^{s'} \cap R^s, I^s), c_{t'})$$

$$\leq \sum_{s \in S} \text{cost}_{I'}(\text{PD}(A \cap R^s, I^s), c_{t'})$$
where \( A \) denotes the set of points of \( P \) assigned during the pruning phases. The first equality holds by Claim \[\text{Claim 23}\] The second and the last inequalities hold since they change only the ordering of summation. The third equality holds since \( A_T^* \cap P_{t'}^* \subset R^k \) if \( s \leq s' \).

By combining previous properties, we have:

\[
\text{cost}(S,C) + \sum_{s \in S} \sum_{T \subseteq [k]} \text{cost}(A_T^* \cap P_{t'}^*, C_T) \leq \sum_{s \in S} \text{cost}_I(D(P \cap R^k, I^*), c_{t'}) \\
+ \sum_{s \in S} \text{cost}_I(D(A \cap R^k, I^*), c_{t'}) \\
\leq \sum_{s \in S} \text{cost}_I(R^k, c_{t'}). \quad \blacktriangledown
\]

Since the total clustering cost is bounded by the sum of \( \text{cost}_I(R^k, c_t) \) for all sampling phases \( s = (t, I) \) within a factor of \((1 + 8 \cdot 2^k c^2 k^2 (\Delta + 1)^2)\) by Lemma \[\text{Lemma 24}\] and \[\text{Lemma 25}\], it suffices to bound the sum of \( \text{cost}_I(R^k, c_t) \) for all sampling phases \( s = (t, I) \). The following lemma can be proved using Lemmas \[\text{Lemma 1}\] and \[\text{Lemma 2}\]. Its proof can be found in Appendix ??.

\[\blacktriangledown \text{Lemma 26.} \quad \text{For a constant } \alpha > 0, \sum_{s \in S} \text{cost}_I(R^k, c_{t'}) \leq (1 + \alpha)\text{opt}_k(P) \text{ with a probability at least } p^k q^k \Delta, \text{ where } q \text{ and } p \text{ are the probabilities in Lemmas } \text{Lemma 1} \text{ and } \text{Lemma 2}. \quad \blacktriangledown\]

\[\text{Proof.}\] The algorithm iteratively obtains the values of \( c_t \) using Lemmas \[\text{Lemma 1}\] and \[\text{Lemma 2}\]. For a sampling phase \( s = (t, I) \) in \( S \), if \( I \) consists of a single coordinate-index, say \( i \), then \( (u_t)_i \) was updated using Lemma \[\text{Lemma 1}\]. Thus we have

\[
\text{cost}_I(R^k, c_t) \leq (1 + \alpha)\text{cost}_I(R^k, c(R^k)) \leq (1 + \alpha)\text{cost}_I(P_{t'}^*, c'_{t'}). \]

The first inequality holds by Lemma \[\text{Lemma 1}\] with probability \( q \). The second inequality holds because \( R^k \) is a subset of \( P_{t'}^* \), and \( c_t \) is the centroid of \( P_{t'}^* \).

Otherwise, that is, if \( I \) consists of more than one coordinate-indices, \( (u_t)_i \)'s were obtained using Lemma \[\text{Lemma 2}\] for all indices \( i \in I \). Thus we have the following with a probability at least \( p \) by Lemma \[\text{Lemma 2}\]

\[
\text{cost}_I(R^k, c_t) \leq (1 + \alpha)\text{cost}_I(R^k, c(R^k)) \leq (1 + \alpha)\text{cost}_I(P_{t'}^*, c'_{t'}). \]

By Lemma \[\text{Lemma 3}\] the number of indices obtained by Lemma \[\text{Lemma 1}\] is at most \( k \Delta \) in total, and the number of indices obtained by Lemma \[\text{Lemma 2}\] is exactly \( k \), one for each cluster. Therefore, with a probability at least \( p^k q^k \Delta \), we have

\[
\sum_{s \in S} \text{cost}_I(R^k, c_{t'}) \leq (1 + \alpha)\text{opt}_k(P). \quad \blacktriangledown
\]

We can obtain the following lemma by combining previous properties.

\[\blacktriangledown \text{Lemma 27.} \quad \text{For a constant } \alpha > 0, \text{ the algorithm returns an } (1 + 8 \cdot 2^k c k^2 (\Delta + 1)^2)(1 + \alpha)-\text{approximate } k\text{-means clustering for } P \text{ with probability at least } p^k q^k \Delta, \text{ where } q \text{ and } p \text{ are the probabilities in Lemmas } \text{Lemma 1} \text{ and } \text{Lemma 2}. \quad \blacktriangledown\]

### 5.3 Algorithm without Counting Oracle

The algorithm we have described uses the counting oracle in two places: to determine the type of the phase and selecting a pair of the cluster-index and coordinate-index to be updated in a sampling phase. In this section, we explain how to avoid using the counting oracle. To do this, we simply try all possible cases: run both phases and update each possible cluster
for all indices during a sampling phase. The main algorithm, \textit{k-Means}(\mathcal{U}, R), is described in Algorithm 4. Its input consists of cluster centers \mathcal{U} of a partial clustering of \mathcal{P} and a set \mathcal{R} of points of \mathcal{P} which are not yet assigned. Finally, \textit{k-Means} (\otimes[k], P) returns an \((1 + 8 \cdot 2^k c k^2 (\Delta + 1)^2)(1 + \alpha)\)-approximate \(k\)-means clustering of \mathcal{P}, where \otimes[k] denotes the \(k\)-tuple of \(\mathbb{H}^d\) which all elements has \(\otimes\) only for its all \(d\)-coordinates. By setting \(\alpha\), we can obtain an \((1 + \epsilon)\)-approximate.

The clustering cost returned by \textit{k-Means} (\otimes[k], P) is at most the cost returned by the algorithm which uses the counting oracle in Section 5.1. In the following, we analyze the running time of \textit{k-Means} (\otimes[k], P). Let \(T(n, \delta)\) be the running of \textit{k-Means} (\mathcal{U}, R) when \(n = |R|\) and \(\delta = \sum_{t \in [k]} \min\{d − |I_t|, \Delta + 1\}\). Here, \(\delta\) is an upper bound on the number of updates required to make \(I_t = [d]\) for every cluster-index \(t\) in \([k]\).

**Claim 28.** \(T(n, \delta) \leq \delta \cdot T(n, \delta - 1) + T\left(\left(1 - \frac{1}{2^\delta + 1 - 2}\right) n, \delta\right) + O\left(\frac{4k^2 \Delta^3 dn}{\alpha}\right)\)

**Proof.** In a sampling phase, \textit{k-Means} calls itself at most \(\delta\) times recursively with different parameters. Each recursive call takes \(T(n, \delta - 1)\) time. Also, the time for updating cluster centers takes \(O(\delta \Delta^3 dn/\alpha)\) in total by Lemma 1 and 2. For a pruning phase, we compute \(|R \cap S_T|\) for each \(T \subset [k]\) in total \(O(dn)\) time, and then choose the first half of \(S_T\) in increasing order of the distances from \(u_T\) in total \(O(kdn)\) time. The recursive call invoked in the pruning phase takes \(T\left(\left(1 - \frac{1}{2^\delta + 1 - 2}\right) n, \delta\right)\) time. We have \(\delta + 1\) clusterings returned by recursive calls in total, and we can choose \(c_{[k]}\) in \(O(\delta kdn)\) time. Thus, the claim holds.

**Claim 29.** \(T(n, \delta) \leq (2\delta(2^k - 1))^{\delta+1}(1 + \frac{1}{2^\delta + 1 - 2})^2 \Delta^3 kdn/\alpha\)

**Proof.** We prove the claim inductively. Basically, \(T(n, 0)\) and \(T(1, \delta) = O(kd)\). For \(\delta \geq 1\) and \(n > 1\), \(T(n, \delta)\) satisfies the following by inductive hypothesis:
Theorem 30. Linear-Time Approximation Scheme for $k$-Means Clustering of Affine Subspaces

$$T(n, \delta) \leq \delta \cdot T(n, \delta - 1) + T\left(\left(1 - \frac{1}{2k+1} - \frac{1}{2}k^{2k+1} - 2\right)n, \delta\right) + \delta \cdot \Delta^3 kdn/\alpha$$

$$\leq (2\delta(2^k - 1))^{2^{k+1}} \left(1 + \frac{1}{2k+1} - 3\right)^{\delta^2} \Delta^3 kdn/\alpha$$

$$\cdot \left(\frac{2^{k+1} - 3}{2k+1 - 2}\right)^{2k-1} \left(\frac{2^{k+1} - 3}{2k+1 - 2}\right)^{\delta^2} \frac{1}{(2k+1 - 2)^{2k}}$$

$$\leq (2\delta(2^k - 1))^{2^{k+1}} \left(1 + \frac{1}{2k+1} - 3\right)^{\delta^2} \Delta^3 kdn/\alpha. \quad \blacktriangleleft$$

In summary, $k$-Means $(\otimes[k], P)$ returns a clustering cost at most $(1 + 8 \cdot 2^k ck^2(\Delta + 1)^2)(1 + \alpha)\text{OPT}_k(P)$ in $2O(k^2 \Delta + k \Delta \log \Delta + \Delta^2 - \log \alpha) d[P]$ time with probability at least $p^k q^k \Delta$. We obtain the following theorem by setting $\alpha = \epsilon/3$, and $c = \frac{a}{8 \cdot 2^k k^2(\Delta + 1)^2}$.

**Theorem 30.** Given a $\Delta$-missing $n$-point set $P$ in $\mathbb{R}^d$, a $(1 + \epsilon)$-approximate solution to the $k$-means clustering problem can be found in $2O(\max\{\Delta^k(\log \Delta + k), \frac{\epsilon}{\alpha}(\log \frac{1}{\alpha})\}) d[P]$ time with a constant probability $1/2$.

**Proof.** We prove that for a constant $\epsilon > 0$ and $\Delta$-missing points $P$ of $\mathbb{R}^d$, we can find a $(1 + \epsilon)$-approximation of $k$-means clustering of $P$ in $2O(\max\{\Delta^k(\log \Delta + k), \frac{\epsilon}{\alpha}(\log \frac{1}{\alpha})\}) d[P]$ time with a constant probability. We denote the time complexity as $T \cdot dn$, where $n = |P|$. As mentioned in Section 2, $k$-Means $(\otimes[k], P)$ returns a clustering cost at most $(1 + 8 \cdot 2^k ck^2(\Delta + 1)^2)(1 + \alpha)\text{OPT}_k(P)$ in $2O(T) d[P]$ time with probability at least $p^k q^k \Delta$, where $T = k^2 \Delta + k \Delta \log \Delta + \Delta^2 - \log \epsilon - k \log p - k \Delta \log q$.

We set $\alpha = \epsilon/3$, and $c = \frac{a}{8 \cdot 2^k k^2(\Delta + 1)^2}$. Therefore, we have the following for $\lambda$ which defined in Lemma 2.

- $- \log c = O(k + \log \Delta - \log \epsilon)$, and
- $\lambda \in O(\max\{\epsilon^{-1/2\Delta}, \Delta^{3/2\Delta}\})$

We consider two cases: Case (i) $\Delta^{3/2\Delta} \in O(\epsilon^{-1/2\Delta})$ and Case (ii) $\epsilon^{-1/2\Delta} \in O(\Delta^{3/2\Delta})$.

**Case (i) $\Delta^{3/2\Delta} \in O(\epsilon^{-1/2\Delta})$ :** In this case, $\lambda \in O(\epsilon^{-1/2\Delta}) \subset O(1/\epsilon)$ for $\Delta \geq 1$. Thus, we deduce the following for the probabilities $q$ and $p$ defined in Lemmas 1 and 2.

- $- \log p \in O(\Delta \lambda(\log \Delta - \log \epsilon)) \subset O(\frac{\Delta}{\alpha}(k - \log \epsilon))$
- $- \log q \in O(\frac{\Delta}{\alpha}(\log \Delta - \log \alpha)) \subset O(\frac{\Delta}{\alpha}(k - \log \epsilon))$

Thus,

$$T \in O\left(\frac{\Delta k}{\epsilon}(k - \log \epsilon)\right).$$

**Case (ii) $\epsilon^{-1/2\Delta} \in O(\Delta^{3/2\Delta})$ :** In such case, we have $\lambda \in O(\Delta^{3/2\Delta}) \subset O(\Delta^{1.5})$;

- $- \log p \in O(\Delta \lambda(\log \Delta - \log \epsilon)) \subset O(\Delta^{2.5}(k + \log \Delta))$
- $- \log q \in O(\frac{\Delta}{\alpha}(\log \Delta - \log \alpha)) \subset O(\Delta^{3}(k + \log \Delta))$

Therefore, we have;

$$T \in O(\Delta^{3}k(k + \log \Delta)).$$

Finally, we have

$$2O(T) \cdot dn \in 2O(\max\{\Delta^k(\log \Delta + k), \frac{\epsilon}{\alpha}(\log \frac{1}{\alpha})\}) d[P]. \quad \blacktriangleleft$$
References

1. Marcel R. Ackermann, Johannes Blömer, and Christian Sohler. Clustering for metric and nonmetric distance measures. *ACM Transactions on Algorithms*, 6(4), September 2010.

2. P.D. Allison. *Missing Data*. Number no. 136 in Missing Data. SAGE Publications, 2001. URL: https://books.google.co.kr/books?id=ZtYArHXjpB8C

3. Daniel Aloise, Amit Deshpande, Pierre Hansen, and Preyas Popat. NP-hardness of Euclidean sum-of-squares clustering. *Machine learning*, 75(2):245–248, 2009.

4. Pranjal Awasthi, Moses Charikar, Ravishankar Krishnaswamy, and Ali Kemal Sinop. The hardness of approximation of Euclidean $k$-means. In *Proceedings of the 31st International Symposium on Computational Geometry (SoCG 2015)*, 2015.

5. Ke Chen. On coresets for $k$-median and $k$-means clustering in metric and euclidean spaces and their applications. *SIAM Journal on Computing*, 39(3):923–947, 2009.

6. Eduard Eiben, Fedor V Fomin, Petr A Golovach, Willian Lochet, Fahad Panolan, and Kirill Simonov. EPTAS for $k$-means clustering of affine subspaces. In *Proceedings of the Thirty-Second ACM-SIAM Symposium on Discrete Algorithms (SODA 2021)*, pages 2649–2659, 2021.

7. Dan Feldman and Michael Langberg. A unified framework for approximating and clustering data. In *Proceedings of the 43th Annual ACM Symposium on Theory of Computing (STOC 2011)*, pages 569–578, 2011.

8. Jie Gao, Michael Langberg, and Leonard J Schulman. Analysis of incomplete data and an intrinsic-dimension helly theorem. *Discrete & Computational Geometry*, 40(4):537–560, 2008.

9. Jie Gao, Michael Langberg, and Leonard J. Schulman. Clustering lines in high-dimensional space: Classification of incomplete data. 7(1), 2010.

10. Sariel Har-Peled and Akash Kushal. Smaller coresets for $k$-median and $k$-means clustering. *Discrete & Computational Geometry*, 37(1):3–19, Jan 2007.

11. Anil Kumar Jain, M. Narasimha Murty, and Patrick J. Flynn. Data clustering: A review. *ACM Computing Surveys*, 31(3):264–323, 1999.

12. Amit Kumar, Yogish Sabharwal, and Sandeep Sen. Linear-time approximation schemes for clustering problems in any dimensions. *Journal of the ACM*, 57(2):1–32, 2010.

13. Euiwoong Lee and Leonard J Schulman. Clustering affine subspaces: hardness and algorithms. In *Proceedings of the twenty-fourth annual ACM-SIAM symposium on Discrete algorithms (SODA 2013)*, pages 810–827, 2013.

14. Meena Mahajan, Prajakta Nimbhorkar, and Kasturi Varadarajan. The planar $k$-means problem is NP-hard. *Theoretical Computer Science*, 442:13–21, 2012.

15. Yair Marom and Dan Feldman. $k$-means clustering of lines for big data. In *Advances in Neural Information Processing Systems*, volume 32, 2019.

16. Nimrod Megiddo. On the complexity of some geometric problems in unbounded dimension. *Journal of Symbolic Computation*, 10(3):327–334, 1990.

17. Björn Ommer and Jitendra Malik. Multi-scale object detection by clustering lines. In *Proceedings of the IEEE 12th International Conference on Computer Vision (ICCV 2009)*, pages 484–491, 2009.
Our algorithm is almost tight in the sense that it is exponential in both $k$ and $\Delta$ but linear in both $n$ and $d$.

**Theorem 31** ([13]). *For fixed $k \geq 3$ and $\epsilon > 0$, there is no algorithm that computes an $(1 + \alpha)$-approximate $k$-means clustering in time polynomial of $n, d$ and $\Delta$ unless P=NP.*

**Proof.** We reduce graph $k$-coloring to the $k$-means clustering problem. Given $G = (V, E)$ with $V = \{v_1, \ldots, v_n\}$ and $E = \{(v_1, 1, v_1, 2), \ldots, (v_m, 1, v_m, 2)\}$, we construct a set $S$ of $n$ points of $\mathbb{R}^m$ as follows. Each point $p_t$ in $S$ corresponds to each vertex $v_t$ for each $t \in [n]$, and the $i$th coordinate corresponds to $(v_i, v_{i,2})$ for each $i \in [m]$. For each $t \in [n]$ and $i \in [m]$, let

$$(p_t)_i = \begin{cases} 
-1 & \text{if } v_{t} = v_{i,1}, \\
+1 & \text{if } v_{t} = v_{i,2}, \\
\otimes & \text{otherwise.}
\end{cases}$$

Assume that $G$ is $k$-colorable if and only if $S$ can be partitioned into $k$ subsets such that no two points in the same subset have fixed values -1 and +1 in the same coordinates. The $k$-means clustering cost of such a partition is 0 by definition. The number of points in $S$ is $n$, the number of coordinates is $m$, and the number of missing entries for each point is at most $m$. If there is an algorithm that computes an $(1 + \epsilon)$-approximate $k$-means clustering in time polynomial in $n, d$ and $\Delta$, we can solve graph $k$-coloring in time polynomial in $n$ and $m$, implying $P = NP$.

**Theorem 32** ([4]). *For fixed $\Delta \geq 0$ and $\epsilon > 0$, there is no algorithm that computes an $(1 + \epsilon)$-approximate $k$-means clustering in time polynomial of $n, d$ and $k$ unless P=NP.*