Generalized concurrence and limits of separability for two qutrits

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Abstract
We present an extension of the Wootters concurrence [1] for the case of two qutrits in mixed states. The reduction of our extension to the case of two levels shows complete agreement with Wootters concurrence for two qubits. As an explicit example, we compute the concurrence for a family of symmetric states and we obtain the bounds on the limit for separability. Our results are compared with those of the negativity.

1 Introduction
Quantum entanglement is of great current interest because of its key role in the modern theory of quantum information [2, 3]. Quantum entanglement of bipartite pure states is well understood, including the general case of arbitrary n-dimensional bipartite systems. Several interesting measures of entanglement have emerged and are in agreement with the definition of reduced entropy given by the von Neumann entropy of the reduced density matrix [4]. For the case of mixed states some measures have also emerged but most of them are not easily computable for a general density matrix. The exceptions are the concurrence of Wootters (and closed expression for the entanglement of formation, EOF) [1] for two qubits and the generalized negativity of Vidal [5] for an arbitrary pair of qubits. However, Vidal’s negativity is defined on the basis of the Peres-Horodecki criteria which is found to be a necessary but not sufficient condition for separability in higher dimensional systems than 2 × 3 [7]. Finally, in Sec. 4 we present our conclusions.

2 Concurrence transformation
Concurrence for two qubits systems is obtained by means of a composition of the complex conjugate and the transformation performed by \(\sigma_y\). The action of \(\sigma\) on a single qubit \(|\psi\rangle = \sum_{i=0,1} c_i |i\rangle\) can be understood in a general form as a transformation which kills the original state and generates a new state \(|\psi'\rangle\). The new state is orthogonal to \(|\psi\rangle\) if and only if \(|\psi\rangle\) is maximally nonuniform in the \(c_i\), i.e., if, \(|\psi_2\rangle\) is one of the elements of the basis. Otherwise, the generated state \(|\psi'_2\rangle\) is not orthogonal to \(|\psi_2\rangle\), and, in a special case, is collinear to \(|\psi_2\rangle\), if and only if, it is made of states with equal weights of the elements in the basis. Since \(\sigma_y\) has other special properties like hermiticity, unitarity, a square root of identity, and to has diagonal elements equal to zero, one is motivated to search for an extended version of \(\sigma_y\) for qutrits which fulfils all of these properties. However, it can be shown that all these properties can not be fulfilled simultaneously by one operator in higher dimensional systems. This arises from the fact that the symmetry of a two-level systems is, in many
ways, exceptional. We focus our search for a similar transformation fulfilling not all but the fundamental properties of \( \sigma_y \) in order to measure the concurrence of a pair of qutrits. We look for a hermitian operator as a basis of our generalized concurrence so that the transformation might be associated to a physical quantity, to have all diagonal elements equal to zero so that the transformation kills the original state, and instead of a flip or inverter operator we will say, to be a split-level operator which is a more general class of transformation. We define split-level operator \( S_{n,j} \) as a transformations that splits the \( j \)-th component of the \( n \)-dimensional ket-vector state \( |\psi\rangle \) into a superposition of \( n-1 \) orthogonal components. The generalized version of \( \sigma_y \) can be obtained by considering the well known connection between \( \sigma_y \) and the ladder operators \( J_{\pm} \):

\[
\sigma_y = -\frac{i}{\hbar} (J_+ - J_-)
\]  

for two-level systems, the ladder operator can be written in the matrix representation as

\[
J_+ = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad J_- = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}
\]

Then Eq. (1) can be written as \( \sigma_y = -\frac{1}{\hbar} O_2 \), where

\[
O_2 = e^{i\pi/2} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + e^{3i\pi/2} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}
\]  

This picture of ladder operators can be thought as a reduction of a more general split-level operators. Furthermore, as can be seen clearly from this picture, \( J_+ \) and \( J_- \) define the set of all possible split-level operators for two-level systems, and \( \sigma_y \) is the sum of all possible operators for two level systems with suitable relative phases. It is then straightforward that for three-level systems the set of split-level operators \( S_{3,i} \) are given, in the matrix representation, by

\[
S_{3,1} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad S_{3,2} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad S_{3,3} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}
\]

The corresponding operator for two qutrits can be initially defined as

\[
O'_3 = e^{i\pi/3} S_{3,1} + e^{3i\pi/3} S_{3,2} + e^{5i\pi/3} S_{3,3}
\]  

Although the relative phases have been introduced following the same sense as that of the case of qubits (equally spaced along the whole phase space), the operator \( O'_3 \) is not hermitian as it should be. These equally spaced phases in general do not lead to hermitian operators even in the case of even-dimensional systems, different from two-level systems. This is again a feature that only occurs in the exceptional dimension of two-level systems.

Then, we will say, instead of equally spaced relative phases between split-level operators, that the relative phases are to be uniformly and maximally spaced in one semi-plane so that the hermiticity of \( O \) can be granted. These relative phases can be obtained for three level systems by introducing the set of diagonal matrices \{\( H_{3,i} \)\}

\[
H_{3,1} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & -i \end{pmatrix} \quad H_{3,2} = \begin{pmatrix} -i & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & i \end{pmatrix} \quad H_{3,3} = \begin{pmatrix} i & 0 & 0 \\ 0 & -i & 0 \\ 0 & 0 & -1 \end{pmatrix}
\]

which can be generated by permuting the diagonal entries obtained from rotation of \( \pi/2 \) in the phase space starting from \( \pi/2 \), i.e., \( e^{i\pi/2} \), \( e^{2i\pi/2} \), and \( e^{3i\pi/2} \). Now if we define the vector \( \mathbf{H} = H_{3,1} \hat{u}_1 + H_{3,2} \hat{u}_2 + H_{3,3} \hat{u}_3 \), where the \( \hat{u}_i \) are orthogonal unitary vectors, and the vector \( \mathbf{S}_3 = S_{3,1} \hat{u}_1 + S_{3,2} \hat{u}_2 + S_{3,3} \hat{u}_3 \), then the transformation for qutrits can be redefined as

\[
O_3 = H_3 \cdot S_3
\]

In the matrix representation it reads

\[
O_3 = \begin{pmatrix} 0 & -i & i \\ i & 0 & -i \\ -i & i & 0 \end{pmatrix}
\]

In terms of an alternate convenient decomposition, \( O_3 \) can be written as

\[
O_3 = i (X - X^\dagger)
\]

where \( X \) is the cyclic (but not hermitian) Pauli matrix for qutrits. A complete characterization of the Pauli matrices for qutrits can be found in Ref. 3.

This decomposition suggest a simpler interpretation of \( O_3 \). Since \( X \) and \( X^\dagger \) are the cyclic ladder operators for two qutrits (\( X \) increases and \( X^\dagger \) decreases, cyclicly the level of a single qutrit), we can extend the definition of \( O_3 \) to a general case of qutrits as the composition of all possible cyclic operator with suitable relative phases.
Summarizing the characteristics of $O_3$, we have that it is hermitian, non-unitary, and its action on a single qutrit can be described as follows

$$O|j\rangle = i(|j + 1\rangle - |j - 1\rangle)$$  \hspace{1cm} (9)

where $j \pm 1$ is modulo 3.

### 3 Concurrence for two qutrits in a pure state

Now we extend the concurrence of Wootters by defining the concurrence for two qutrits in pure state $|\psi_{3 \times 3}\rangle = |\psi\rangle$ as

$$C_3(\psi) = |\langle \psi | \tilde{\psi} \rangle|$$  \hspace{1cm} (10)

where

$$|\tilde{\psi}\rangle = (O_3 \otimes O_3)|\psi^*\rangle$$  \hspace{1cm} (11)

with $O_3$ as defined in [3], and $|\psi^*\rangle$ being the complex conjugate of $|\psi\rangle$. For a vector state of two qutrits, in the standard basis \{ $|i, j\rangle$ \},

$$|\psi\rangle = \sum_{i,j=0}^{2} \alpha_{ij}|i, j\rangle$$  \hspace{1cm} (12)

with, $\sum_{i,j=0}^{2} \alpha_{ij}^2 = 1$, concurrence (10) can be written as

$$C_3(\psi) =$$  \hspace{1cm} (13)

$$\left| \alpha_{00} + \alpha_{11} + \alpha_{22} \right|^2 + \left( \alpha_{01} + \alpha_{12} + \alpha_{20} \right)^2$$

$$+ \left( \alpha_{02} + \alpha_{21} + \alpha_{10} \right)^2 - \left( \alpha_{00} + \alpha_{11} + \alpha_{22} \right)^2$$

$$- \left( \alpha_{01} + \alpha_{12} + \alpha_{20} \right)^2 - \left( \alpha_{02} + \alpha_{21} + \alpha_{10} \right)^2$$

In the Schmidt basis \{ $|i, i\rangle$ \}, state (12) can be decomposed as

$$|\psi\rangle = \sum_{i=0}^{2} \beta_i|i, i\rangle$$  \hspace{1cm} (14)

and then (10) reduces to

$$C_3(\psi) = \left| \sum_{i=0}^{2} \beta_i \right|^2 - 1$$  \hspace{1cm} (15)

where the $\beta_i$ are the Schmidt coefficients.

It is remarkable that results (13) and (15) include the lower dimensional case of two qubits. The corresponding results for two qubits can be recovered by setting to zero all the coefficients within the level to be eliminated. For instance, if we set to zero all the $\alpha_{ij}$ and $\alpha_{ji}$ in (13), and consequently, the Schmidt coefficient $\beta_2$ in (15), then, the new expressions coincide with those for two qubits, namely,

$$C_2(\psi) = 2|\alpha_{00}\alpha_{11} - \alpha_{01}\alpha_{10}|$$  \hspace{1cm} (16)

and, respectively,

$$C_2(\psi) = 2\beta_0\beta_1$$  \hspace{1cm} (17)

$C_2$ predicts an amount of entanglement of 2 for maximally entangled states against the value 1 for the case of two qubits. This is an expected result, since the amount of entanglement should increase with the dimensionality of the system. This might be associated with the possibility of two different flip action in three level systems against only one flip in two levels, and suggests than in the general case of two qunits (n-level particles) the concurrence of maximally entangled states is given by

$$C_{n,\text{max}} = n - 1$$  \hspace{1cm} (18)

By comparing our results with those of G. Vidal we find exact agreement between one of his negativities, the so-called robustness ($N_{SS}$). The negativity is defined as the sum of the negative eigenvalues of the partial transpose of the density matrix, so that it measures the degree to which the partial trace of the density matrix $(\rho^{T_A})$ fails to be positive. It is given by (Eq. (1) in Ref. [8])

$$N = \frac{\|\rho^{T_A}\| - 1}{2}$$  \hspace{1cm} (19)

Where $\|\|$ is the trace norm. For the case of pure states, the negativity $N$ (Eq. (47) in Ref. [5]) reduces exactly to half of Eq. (19) and then, robustness ($N_{SS} = 2N$) matches exactly our result. This result gives a prescription to generate negativity by a procedure similar to that of the concurrence given by Pauli spin operator.

On the other hand, J. Cereceda [10] has obtained an extension of concurrence for two qutrits systems by means of decomposing the standard density matrix into the SU(3) generators. His concurrence ($C_c$) for the case of two qutrits in pure state (14) is given by (Eq. (21) in Ref. [11])

$$C_c(\psi) = \sqrt{3 \left( (\beta_0\beta_1)^2 + (\beta_1\beta_2)^2 + (\beta_2\beta_0)^2 \right)}$$  \hspace{1cm} (20)

This result does not match ours. Besides, $C_c$ is not a general result extendable for mixed states, and apart from its normalization drawback, $C_c$ does not yield the expected values for the case of two qubits. For example, it predicts a value of concurrence $\frac{\sqrt{3}}{2}$ for partially entangled qutrits states against 1, as it should be since the amount of entanglement of these two partially entangled qutrits is the same as that of two maximally entangled qubits.
Concurrence of two qutrits in a mixed state

For the case of a mixed state \( \rho = \sum_i p_i |\psi_i\rangle \langle \psi_i | \) of two qutrits, concurrence is defined as the average concurrence of the pure states making up \( \rho \) minimized over all possible decompositions of \( \rho \). That is

\[
C_3(\rho) = \min_\|\| \sum_i p_i C_3(\psi_i) \tag{21}
\]

where, \( \min_\|\| \) means the minimization over all possible decompositions of \( \rho \) and \( C_3(\psi_i) \) is the concurrence of the \( i \)-th pure state in the ensemble that defines the state \( \rho \).

For two qutrits in a mixed state the concurrence is then given by

\[
C_3(\rho) = \max\{0, 2\lambda_1 - \sum_{i=1}^{9} \lambda_i \} \tag{22}
\]

where the \( \lambda_i \) (with \( i = 1, 2, ..., 9 \)) are the square roots of the eigenvalues of the non-Hermitian matrix \( \tilde{\rho} \tilde{\rho}^* \) in decreasing order, where

\[
\tilde{\rho} = (O_3 \otimes O_3) \rho^* (O_3 \otimes O_3) \tag{23}
\]

with \( \rho^* \) being the complex conjugate of \( \rho \). The result \( \eqref{22} \) is a conjecture at this stage. The motivation for this is the equivalence obtained and proved for the case of two qubit systems by Wootters \[1\]. In the following section we explore this result for the case of a standard Werner state, which allowed us to gain some confidence in order to presume the validity of this result.

4.1 An Explicit Example: Standard Werner states

In two level systems a Werner state is defined as a composition of one fraction \( |x\rangle \) of a singlet \( (\Psi_+, \Phi_+) \) and one fraction \( (1-x) \) of the maximally mixed density matrix \( (\frac{1}{4}\delta_{i\mu,j\nu}) \). For two qutrits, we consider standard Werner kind of states as a single fraction of maximally entangled qutrits and a random fraction of impurity (coherence), as follows

\[
\rho_w = \frac{x}{3} \sum_{i,j=0}^{2} |i,i\rangle\langle j,j| + \frac{1-x}{9} I \tag{24}
\]

where \( I \) is the identity matrix in the Schmidt basis. For this case, the nine eigenvalues of \( \rho \tilde{\rho} \) are

\[
\begin{align*}
0 & \text{ of multiplicity 5} \\
\frac{1}{9} (1-x)^2 & \text{ of multiplicity 3} \\
\frac{1}{9} (1+5x)^2 & \text{ of multiplicity 1}
\end{align*}
\]

among which the last one is the greatest for \( 0 \leq x \leq 1 \), and then, concurrence \( C_3(\rho_w) \) is given by

\[
C_3(\rho_w) = \max\{0, \frac{8}{3} x - \frac{2}{3} \} \tag{26}
\]

which yields that our measure is zero for \( x \leq \frac{1}{3} \). For \( x > \frac{1}{3} \) concurrence \( \eqref{26} \) increases linearly with \( x \), up to its maximum value 2, for \( x = 1 \), as expected for maximally entangled qutrits.

From this result we can recover the results for two-level systems by setting some of the entries in the density matrix to be zero. Let us drop the third level \( |2\rangle \) by setting the Schmidt coefficients \( \beta_0 = \beta_1 = \frac{1}{2} \), and \( \beta_2 = 0 \) in the definition of the maximally entangled state (now, actually partial entangled qutrit) for the standard Werner state \( \eqref{24} \), so that \( |\psi'_\text{max}\rangle = \sum_{i=0}^{2} (1 - \delta_{1,2}) (1 - \delta_{2,j}) \beta_i |i,i\rangle = \sum_{i=0}^{1} \beta_i |i,i\rangle \). Also we have to drop the elements \( I_{2\mu,j\nu} \), \( I_{12,j\nu} \), \( I_{1\mu,2\nu} \) and \( I_{\mu,j,2} \) in the maximally mixed density matrix, so that the elements in maximally mixed component of the standard Werner state would read \( \frac{1}{16} (1- \delta_{2\mu,j\nu}) (1- \delta_{1\mu,j\nu}) (1- \delta_{\mu,j,2}) \delta_{\mu,j,2} = \frac{1}{4} \delta_{\mu,j,2} (1- \delta_{\mu,j,2}) \). Meanwhile computing the concurrence for this particular case we obtain four eigenvalues of \( \rho' \tilde{\rho}' \) different to zero, namely, \( \frac{1}{16} (1-x)^2 \) (of multiplicity 2), and

\[
\frac{1}{16} \left( 1 + 12x - 5x^2 \pm 4x \sqrt{1 + 12x - 9x^2} \right)
\]

Then, the concurrence is given by

\[
C_{3\rightarrow 2}(\rho_w) = \max\{0, c\} \tag{27}
\]
with
\[ c = \frac{1}{3} \sqrt{1 + 12x - 5x^2 + 4x \sqrt{1 + 12x - 9x^2}} - \frac{1}{4} \sqrt{1 + 12x - 5x^2 - 4x \sqrt{1 + 12x - 9x^2}} - \frac{1}{2} (1 - x) \]

This implies that this density matrix becomes separable for all \( c \leq 0 \), i.e., \( x \leq \frac{1}{3} \) which is exactly the prescribed value by Peres \cite{Peres} for two qubit systems. On the other hand, the maximum concurrence for this case is 1, which also coincides with the results for maximally entangled qubits, as expected.

For standard Werner states \cite{Vidal}, Vidal’s negativity and robustness (expressions (51) and (52) in Ref. \cite{Vidal}, with \( d = 3 \), and \( g = 0 \) which are the dimension of the two qudits, and a parameter used to consider a more general symmetric state, respectively) read
\[
\mathcal{N}(\rho_w) = \frac{1}{4} (|1 - 3x| + |1 + 3x|) - \frac{1}{2} \tag{28}
\]
and
\[
\mathcal{N}_{SS}(\rho_w) = \frac{1}{2} ||6x - 1|| - 1 = 2\mathcal{N}(\rho) \tag{29}
\]
respectively. These two measure have maximum values of 1 and 2, respectively, but according to them, for \( x \leq \frac{1}{3} \), \( \rho \) becomes separable while our measure indicates that it occurs for values of \( x \leq \frac{1}{4} \). This implies that our measure does detect some entangled states that neither negativity nor robustness do. In fact, negativities do not make difference between separability in two qubits and two qudits. This can be explained because negativities are defined on the basis of the Peres-Horodocki criteria and, as mentioned in the introduction, the PPT is a necessary but not sufficient condition for higher dimension than \( 2 \times 3 \) which implies that the negativities do not detect some PPT entangled states in these cases which include the case of the present work.

Other explicit examples, as well as some guide lines to the extension of this work to the case of arbitrary dimensional systems will be presented in further papers \cite{Wootters,Peres}.

5 Conclusions

We have generalized the Wootters’s definition of concurrence for two-level bipartite systems. We identify a composite transformation to the original state which includes a complex conjugation of the state and a flip action performed by the Pauli matrix \( \sigma_y \). Starting from this analysis we generalize the characteristics for a good “flip” operator in three-level bipartite systems which leads us to a unique operator \( O_3 \) which does not have all the properties of \( \sigma_y \) but the three fundamental one: to be a composition of split-level operators, to have null diagonal elements, and hermiticity. Then we define concurrence \( C_3 \) for two qudits systems on the basis of the transformation given by the operator \( O_3 \) and the complex conjugate. This definition lead us to a very well behaved measure of entanglement for pure states. For mixed states we presume that the average concurrence of the pure states in the decomposition of the density matrix \( \rho \), minimized over all possible decompositions, reduces in the same way that Wootters concurrence reduces for two qubits systems in terms of the eigenvalues of \( \rho \). We explored this measure on explicit examples and some of the remarkable results are: 1. Concurrence in two qudits is stronger than in two qubits, 2. The extended concurrence can be reduced to the case of two qubits recovering exactly the same well known results for this case, 3. Separability for a density matrix with single fraction of maximally entangled qudits and a random fraction of impurity (decoherence) occurs for a higher random fraction of impurity that for the case of Werner states of two qubits, namely, our results shows that the needed random fraction of impurity in the density matrix to be separable is \( \frac{1}{3} \), against \( \frac{2}{3} \) in the case of two qubits, which is consistent with the previous results, 4. Compared with other measures, ours shows some differences that enhance its convenience, namely, regarding the close measures of G. Vidal, the results show that those measures do not

![Figure 2: MQE of two qudits in standard Werner state](image-url)
detect some entangled states of two qutrits that our concurrence does. This result finds foundations in the fact that those measures have been built on basis of the Peres criteria which is not a sufficient condition for separability in higher systems than \(2 \times 3\).

These results suggest that our presumption about the analytic reduction of concurrence for mixed states is valid and that our operator \(O_3\) gives a prescription to generate negativity by a procedure similar to that of the concurrence given by Pauli spin operator. It is also remarkable that our concurrence is an effectively computable measure and seems to be extendable to the case of \(n \times n\) quantum systems as it will be presented in a further paper [12].

The main contribution of this work is to generate an effectively computable measure of quantum entanglement for the case of qutrit bipartite systems which consists on the first step toward the definition of a general measure of quantum entanglement for two qunits (\(n\)-level particles) systems. This work is also, in general, a contribution toward the characterization and conceptualization of the quantum entanglement.

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