WEIGHT POSETS ASSOCIATED WITH GRADINGS OF SIMPLE LIE ALGEBRAS, WEYL GROUPS, AND ARRANGEMENTS OF HYPERPLANES

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1. INTRODUCTION

The set of weights of a finite-dimensional representation of a reductive Lie algebra has a natural poset structure ("weight poset"). Studying certain combinatorial problems related to antichains in weight posets, we realised that the best setting is provided by the representations associated with \( \mathbb{Z} \)-gradings of simple Lie algebras [13]. This article, which can be regarded as a sequel to [13], is devoted to a general theory of ideals (antichains) in the corresponding weight posets. Although the subject has interesting representation-theoretic aspects, we work here almost exclusively in the combinatorial setup. Specifically, our main object is going to be a \( \mathbb{Z} \)-graded root system.

Let \( V \) be an \( n \)-dimensional Euclidean space, with inner product \( ( , ) \), and let \( \Delta \) be an irreducible, crystallographic root system spanning \( V \). We refer to [2, 7] for basic definitions and properties of root systems. Let \( \Delta^+ \) be a set of positive roots and \( \Pi = \{ \alpha_1, \ldots, \alpha_n \} \) the set of simple roots in \( \Delta^+ \). The usual partial order "\( \preceq \)" in \( \Delta^+ \) is defined by the requirement that \( \gamma \) covers \( \mu \) if and only if \( \gamma - \mu \in \Pi \). A \( \mathbb{Z} \)-grading of \( \Delta \) is a disjoint union \( \Delta = \bigsqcup_{i \in \mathbb{Z}} \Delta(i) \) such that if \( \gamma_1 \in \Delta(i_1), \gamma_2 \in \Delta(i_2), \) and \( \gamma_1 + \gamma_2 \) is a root, then \( \gamma_1 + \gamma_2 \in \Delta(i_1 + i_2) \). Then \( \Delta(0) \) is a root system in its own sense. We always assume that \( \Delta^+ \) is compatible with \( \mathbb{Z} \)-grading, which means that

\[
\Delta^+ = \Delta(0)^+ \sqcup \Delta(1) \sqcup \Delta(2) \sqcup \ldots,
\]

where \( \Delta(0)^+ \) is a set of positive roots in \( \Delta(0) \). Then \( \Pi = \sqcup_{i \geq 0} \Pi(i) \), where \( \Pi(i) = \Pi \cap \Delta(i) \), and \( \Pi(0) \) is a set of simple roots for \( \Delta(0) \). Each \( \Delta(i), i \geq 1 \), can be regarded as a sub-poset of \( \Delta^+ \), and we are primarily interested in the poset \( \Delta(1) \).

Let \( J_-(\Delta(1)) \) be the set of lower (= order) ideals in \( \Delta(1) \). We relate \( J_-(\Delta(1)) \) to certain elements in the Weyl group \( W \) of \( \Delta \) and certain hyperplane arrangements inside the Coxeter arrangement of \( \Delta \). The Weyl group of \( \Delta(0), W(0) \), is a parabolic subgroup of \( W \). Let \( W^0 \) be the set of minimal length coset representatives for \( W/W(0) \). It is known that

\[
W^0 = \{ w \in W \mid w(\alpha) \in \Delta^+ \quad \forall \alpha \in \Delta(0)^+ \},
\]

see [7, 1.10]. Let \( N(w) = \{ \gamma \in \Delta^+ \mid -w(\gamma) \in \Delta^+ \} \) be the inversion set of \( w \in W \) and \( w \mapsto \ell(w) = \# N(w) \) the length function on \( W \). By a classical result of Kostant [8, Prop. 5.10],

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$M \subset \Delta^+$ is the inversion set of some $w$ if and only if both $M$ and $\Delta^+ \setminus M$ are closed under addition. Such an $M$ is said to be bi-convex.

Our basic results on the lower ideals in $\Delta(1)$ and related elements of $W^0$ are presented in Section 3. It is readily seen that if $w \in W^0$, then $I_w := N(w) \cap \Delta(1)$ is a lower ideal of $\Delta(1)$, which yields the map

$$\tau : W^0 \to \mathcal{I}_-(\Delta(1)), \quad w \mapsto \tau(w) := I_w.$$ 

For any $I \in \mathcal{I}_-(\Delta(1))$, we construct two extreme bi-convex subsets of $\Delta^+$ that belong to $\bigcup_{k \geq 1} \Delta(k)$ and whose 1-component is $I$, see Theorems 3.3 and 3.4. This implies that $\tau$ is onto and $\tau^{-1}(I)$ contains a unique element of minimal and of maximal length. These two elements of $W$ are said to be the minimal and the maximal elements of $I$, denoted $w_{I,\text{min}}$ and $w_{I,\text{max}}$ respectively. Furthermore, we observe that $\tau^{-1}(I)$ is an interval w.r.t. the weak Bruhat order "\leq" on $W^0$; that is, $\tau^{-1}(I) = \{ w \in W^0 \mid w_{I,\text{min}} \leq w \leq w_{I,\text{max}} \}$, see Theorem 3.6.

Let $W^0_{\text{min}}$ (resp. $W^0_{\text{max}}$) be the subset of $W^0$ that consists of the minimal (resp. maximal) elements of all lower ideals. We provide a characterisation of each subset that does not refer to lower ideals. Set $\Delta(\geq k) = \bigcup_{j \geq k} \Delta(j)$ and $\Delta(\leq k) = \bigcup_{j \leq k} \Delta(j)$. Then

$$W^0_{\text{min}} = \{ w \in W^0 \mid w^{-1}(\alpha) \in \Delta(\geq -1) \text{ for all } \alpha \in \Pi \},$$

(Theorem 3.7);

$$W^0_{\text{max}} = \{ w \in W^0 \mid w^{-1}(\alpha) \in \Delta(\leq 1) \text{ for all } \alpha \in \Pi \}.$$ (Theorem 3.8)

We also point out a connection between an involution on $\mathcal{I}_-(\Delta(1))$, involution on $W^0$, and the subset $W^0_{\text{min}}$ and $W^0_{\text{max}}$ (Proposition 3.9).

As an application of our minimal/maximal elements, we describe the antichains related to the lower ideals. Let $\min(M)$ and $\max(M)$ denote the minimal and maximal elements of a subset $M$ w.r.t. the poset structure of $\Delta(1)$. For $I \in \mathcal{I}_-(\Delta(1))$, one may consider two antichains: $\max(I)$ and $\min(\Delta(1) \setminus I)$. Given $\gamma \in \Delta(1)$, our result is that

- $\gamma \in \max(I)$ if and only if $w_{I,\text{min}}(\gamma) \in -\Pi$, see Theorem 4.1;
- $\gamma \in \min(\Delta(1) \setminus I)$ if and only if $w_{I,\text{max}}(\gamma) \in \Pi$, see Theorem 4.2.

Associated with $\Delta^+$ and $\Delta(0)^+$, there are two open dominant chambers, $\mathcal{C}^o = \{ v \in V \mid (v, \alpha) > 0 \ \forall \alpha \in \Pi \}$ and $\mathcal{C}(0)^o = \{ v \in V \mid (v, \alpha) > 0 \ \forall \alpha \in \Pi(0) \}$. The chambers $w(\mathcal{C}^o)$, $w \in W$, are said to be small. Let $\mathcal{H}_\gamma$ denote the hyperplane in $V$ orthogonal to $\gamma \in \Delta^+$. The hyperplanes $\mathcal{H}_\gamma$ with $\gamma \in \Delta(1)$ dissect $\mathcal{C}(0)^o$ into certain regions, and we prove that there is a natural bijection between $\mathcal{I}_-(\Delta(1))$ and the set of these regions. Moreover, if $R^+_I \subset \mathcal{C}(0)^o$ is the open region corresponding to $I$, then $w^1_{I,\text{min}}(\mathcal{C}^o)$ is the unique small chamber in $R^+_I$ closest to $\mathcal{C}^o$ and $w^1_{I,\text{max}}(\mathcal{C}^o)$ is the unique small chamber in $R^+_I$ farthest from $\mathcal{C}^o$ (Theorem 5.1). This result prompts considering the hyperplane arrangement $\mathcal{A}_{\Delta(0,1)} = \{ \mathcal{H}_\gamma \mid \gamma \in \Delta(0)^+ \cup \Delta(1) \}$ in $V$. 


It is well known that the whole Coxeter arrangement $A_\Delta = \{H_\gamma \mid \gamma \in \Delta^+\}$ is free and its exponents are just the usual exponents of $W$ [10, Ch. 6]. We conjecture that the arrangement $A_\Delta(0, 1)$ is also free and its exponents are determined by certain partition associated with $\Delta(0)^+ \cup \Delta(1)$ (Conjecture 5.3). Actually, this is a special case of a more general conjecture that is discussed in the Introduction of [16]. Moreover, by [16, Theorem 11.1], that general conjecture and hence our Conjecture 5.3 are true if $\Delta$ is classical or of type $G_2$. For $\gamma \in \Delta$, let $[\gamma : \alpha_i]$ be the coefficient of $\alpha_i$ in the expression of $\gamma$ via the simple roots. The height of $\gamma$ is $ht(\gamma) = \sum_{i=1}^n [\gamma : \alpha_i]$. We deduce from Conjecture 5.3 that

$$\#J_-(\Delta(1)) = \prod_{\gamma \in \Delta(1)} \frac{ht(\gamma) + 1}{ht(\gamma)}.$$  

This equality has also been proved in [13], by ad hoc methods, for the abelian and extra-special gradings of $\Delta$ (see Section 2.3 for their definitions).

An inspiring observation is that, to a great extent, the theory of lower ideals in $\Delta(1)$ is parallel (similar) to the theory of upper (= ad-nilpotent) ideals in the poset $\Delta^+$. The latter will be referred to as the affine theory, because it requires the use of the affine Weyl group $\tilde{W}$ and the affine root system $\tilde{\Delta}$. We discuss this parallelism in Section 6.

In Appendix A, we give a case-free proof of an observation in [16, Prop. 3.1] to the effect that certain sequence associated with an upper ideal of $\Delta^+$ is, actually, a partition. This fact is also needed for Conjecture 5.3.

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2. Weight posets and gradings of simple Lie algebras

Let $(\mathcal{P}, \preceq)$ be a finite poset. A lower (resp. upper) ideal $I$ is a subset of $\mathcal{P}$ such that if $\mu \in I$ and $\nu \preceq \mu$ (resp. $\nu \succeq \mu$), then $\nu \in I$. Let $\mathcal{J}_-(\mathcal{P})$ be the set of lower ideals, $\mathcal{J}_+(\mathcal{P})$ the set of upper ideals, and $\mathcal{A}\mathcal{n}(\mathcal{P})$ the set of antichains in $\mathcal{P}$. For any $M \subset \mathcal{P}$, let $\min(M)$ (resp. $\max(M)$) denote the set of minimal (resp. maximal) elements of $M$ with respect to ‘$\preceq$’. The following three maps set up bijections between the respective pairs of sets:

$$I \in \mathcal{J}_-(\mathcal{P}) \mapsto \max(I) \in \mathcal{A}\mathcal{n}(\mathcal{P}), \quad I \in \mathcal{J}_+(\mathcal{P}) \mapsto \min(I) \in \mathcal{A}\mathcal{n}(\mathcal{P}),$$

$$I \in \mathcal{J}_-(\mathcal{P}) \mapsto I^c := \mathcal{P} \setminus I \in \mathcal{J}_+(\mathcal{P}).$$

Both $\mathcal{J}_-(\mathcal{P})$ and $\mathcal{J}_+(\mathcal{P})$ are graded posets under inclusion, with the rank function $I \mapsto \#I$. The rank-generating function of either of them is

$$M_\mathcal{P}(t) = \sum_{I \in \mathcal{J}_-(\mathcal{P})} t^{\#I}.$$  

It is also called the $M$-polynomial of $\mathcal{P}$ in [13]. Clearly, $M_\mathcal{P}(1) = \#\mathcal{J}_-(\mathcal{P}) = \#\mathcal{A}\mathcal{n}(\mathcal{P})$.  


2.1. Gradings of simple Lie algebras and root systems. Although we are primarily interested in combinatorics of posets related to \( \mathbb{Z} \)-gradings of root systems, it is instructive and helpful to keep in mind that a \( \mathbb{Z} \)-grading of \( \Delta \) is an offspring of a \( \mathbb{Z} \)-grading of the corresponding simple Lie algebra \( g \). This provides a broader perspective and adds some geometric flavour and intuition to one’s considerations. (We refer to [19, Ch. 3, §3] for generalities on gradings of semisimple Lie algebras.)

Let \( g = u^- \oplus t \oplus u \) be a fixed triangular decomposition, where \( t \) is a Cartan subalgebra of \( g \). The associated root system \( \Delta(g, t) \) is \( \Delta \), and \( V = t^*_g \) is the \( \mathbb{R} \)-span of \( \Delta \) in \( t^* \). If \( g_\gamma \) is the root space for \( \gamma \in \Delta \), then \( u = \bigoplus_{\gamma \in \Delta^+} g_\gamma \). Write \( s_\gamma \) for the reflection in \( W \) with respect to \( \gamma \in \Delta \). Let \( \theta \) be the highest root in \( \Delta^+ \). Recall that \( h(t) = h - 1 \), where \( h \) is the Coxeter number of \( \Delta \).

Let \( g = \bigoplus_{i \in \mathbb{Z}} g(i) \) be a \( \mathbb{Z} \)-grading. Since any derivation of \( g \) is inner, we have \( g(i) = \{ x \in g \mid [\tilde{h}, x] = ix \} \) for a unique (semisimple) element \( \tilde{h} \in g(0) \). The element \( \tilde{h} \) is said to be defining for the grading in question. Here \( g(0) \) is the centraliser of \( \tilde{h} \), hence a reductive Lie algebra. Without loss of generality, one may assume that \( \tilde{h} \in t \) and \( \alpha(\tilde{h}) \geq 0 \) for all \( \alpha \in \Pi \). Then \( t \subset g(0) \), \( g(0) = (g(0) \cap u^-) \oplus t \oplus (g(0) \cap u) \) is a triangular decomposition of \( g(0) \), and

\[
\begin{align*}
  u = (g(0) \cap u) \oplus g(1) \oplus g(2) \oplus \ldots.
\end{align*}
\]

If \( \Delta(i) = \{ \gamma \in \Delta \mid \gamma(\tilde{h}) = i \} \), then \( \Delta(i) \) is the set of roots of \( g(i) \), and \( \Delta = \bigsqcup_{i \in \mathbb{Z}} \Delta(i) \) is a compatible \( \mathbb{Z} \)-grading of \( \Delta \) in the sense of Introduction, i.e., Eq. (1.1) holds. We also have \( \Pi = \bigsqcup_{i \geq 0} \Pi(i) \), where \( \Pi(i) = \{ \alpha \in \Pi \mid \alpha(\tilde{h}) = i \} \), and \( \Pi(0) \) is the set of simple roots in \( \Delta(0)^+ = \Delta(0) \cap \Delta^+ \).

Each \( g(i) \) is a \( g(0) \)-module, and therefore \( \Delta(i) \) has a natural poset structure as the set of weights of a \( g(0) \)-module. In case of compatible gradings, this weight poset structure on \( \Delta(i) \) coincides with the restriction of ‘\( \preceq \)’ to \( \Delta(i) \), see [13, Remark 2.9]. More precisely, if \( \gamma, \gamma' \in \Delta(i) \), then \( \gamma \) covers \( \gamma' \) if and only if \( \gamma - \gamma' \in \Pi(0) \). Therefore, \( \gamma' \preceq \gamma \) if and only if \( \gamma - \gamma' \) is a nonnegative integer linear combination of \( \Pi(0) \).

Set \( b(0)^- = (g(0) \cap u^-) \oplus t \) and \( b(0) = t \oplus (g(0) \cap u) \). These are two opposite Borel subalgebras of \( g(0) \). A link between combinatorics and geometry is provided by the following simple observation, which we do not pursue in this article.

**Proposition 2.1.** There is a bijection between the lower (resp. upper) ideals of \( (\Delta(1), \preceq) \) and the \( b^-(0) \)-stable (resp. \( b(0) \)-stable) subspaces of \( g(1) \).

**Proof.** If \( I \in \mathcal{J}_-(\Delta(1)) \) or \( I \in \mathcal{J}_+(\Delta(1)) \), then \( c_I = \bigoplus_{\gamma \in I} g_\gamma \) is the corresponding \( b^-(0) \)-stable or \( b(0) \)-stable subspace of \( g(1) \). The details are left to the reader. \( \square \)

2.2. Standard \( \mathbb{Z} \)-gradings. By [18, §1.2, §2.1], if one is interested in possible \( g(0) \)-modules \( g(i) \), and hence in possible posets \( \Delta(i) \), then it suffices to consider the \( g(0) \)-modules \( g(1) \).
for all semisimple $\mathfrak{g}$. (For $i > 1$, the problem is reduced to considering the induced $\mathbb{Z}$-grading of a certain semisimple subalgebra of $\mathfrak{g}$.) For this reason, it suffices to consider defining elements $\tilde{h} \in \mathfrak{t}$ such that $\alpha(\tilde{h}) \in \{0, 1\}$, i.e., $\Pi = \Pi(0) \sqcup \Pi(1)$. The corresponding $\mathbb{Z}$-gradings (of both $\mathfrak{g}$ and $\Delta$) are said to be standard. More precisely, if $\#\Pi(1) = k$, then we call it a $k$-standard grading. A standard $\mathbb{Z}$-grading can be represented by the Dynkin diagram of $\mathfrak{g}$, where the vertices in $\Pi(1)$ are coloured. If $\Pi(1) = \{\alpha_{i_1}, \ldots, \alpha_{i_k}\}$, then the $\alpha_{i_j}$’s are precisely the lowest weights of the simple $\mathfrak{g}(0)$-modules in $\mathfrak{g}(1)$, the centre of $\mathfrak{g}(0)$ is $k$-dimensional, and $\mathfrak{g}(1)$ is a direct sum of $k$ simple $\mathfrak{g}(0)$-modules. In this case, the poset $\Delta(1)$ is the disjoint union of $k$ subposets corresponding to the simple summands of $\mathfrak{g}(1)$. Therefore, all enumerative problems for $\Delta(1)$ reduce to $1$-standard gradings.

The weight posets $\Delta(i)$, $i > 0$, can be visualised as follows. Let $\mathcal{H}(\Delta^+)$ be the Hasse diagram of $(\Delta^+, \preceq)$. If $\gamma' - \gamma = \alpha \in \Pi$, then the edge connecting $\gamma$ and $\gamma'$ in $\mathcal{H}(\Delta^+)$ is said to be of type $\alpha$. Given a standard $\mathbb{Z}$-grading of $\mathfrak{g}$, let us remove from $\mathcal{H}(\Delta^+)$ all the edges of types from $\Pi(1)$. This yields a disconnected graph. Each connected component of it is the Hasse diagram of either the set of positive roots of a simple factor of $\mathfrak{g}(0)$ (if it contains roots from $\Pi(0)$) or the weight poset of a simple $\mathfrak{g}(0)$-module in some $\mathfrak{g}(i)$, $i > 0$. The set of weights of a simple $\mathfrak{g}(0)$-module in some $\mathfrak{g}(i)$, $i \geq 1$, is precisely the set of roots $\gamma$ with fixed values $[\gamma : \alpha]$ for all $\alpha \in \Pi(1)$, see e.g. [19, 3.5].

2.3. Special classes of $\mathbb{Z}$-gradings. In [13, Sect. 3, 4], we considered in details the following two classes of $\mathbb{Z}$-gradings of $\mathfrak{g}$ and hence of $\Delta$:

The abelian case: $\mathfrak{g} = \mathfrak{g}(-1) \oplus \mathfrak{g}(0) \oplus \mathfrak{g}(1)$.

Here $\mathfrak{g}(0) \oplus \mathfrak{g}(1)$ is a parabolic subalgebra and $\mathfrak{g}(1)$ is its abelian nilradical. In this case $\mathfrak{g}(1)$ is a simple $\mathfrak{g}(0)$-module and therefore such a grading is $1$-standard. If $\Pi(1) = \{\tilde{\alpha}\}$, then upon the identification of $\mathfrak{t}_R$ and $\mathfrak{t}_{\tilde{\alpha}}$, the defining element $\tilde{h}$ appears to be the minuscule fundamental weight $\varphi_{\tilde{\alpha}}$ of the dual root system $\Delta^\vee$. As is well known, the admissible simple roots $\tilde{\alpha}$ are characterised by the property that $[\theta : \tilde{\alpha}] = 1$ [3, Ch. VIII, §7, n°3].

The extra-special case: $\mathfrak{g} = \mathfrak{g}(-2) \oplus \mathfrak{g}(-1) \oplus \mathfrak{g}(0) \oplus \mathfrak{g}(1) \oplus \mathfrak{g}(2)$ and $\dim \mathfrak{g}(2) = 1$.

Any simple Lie algebra has a unique, up to conjugation, $\mathbb{Z}$-grading of this form, and without loss of generality, we may assume that $\Delta(2) = \{\theta\}$. Upon the identification of $\mathfrak{t}_R$ and $\mathfrak{t}_{\theta^\vee}$, the defining element $\tilde{h}$ is recognised as the coroot $\theta^\vee$. That is, $\Delta(i) = \{\gamma \in \Delta \mid (\gamma, \theta^\vee) = i\}$ and $W(0)$ is the stabiliser of $\theta$ (or $\theta^\vee$) in $W$. Since here $\Pi(1) = \{\alpha \in \Pi \mid (\gamma, \theta^\vee) \neq 0\}$, we see that $\mathfrak{g}(1)$ is a simple $\mathfrak{g}(0)$-module if and only if $\theta$ is a multiple of a fundamental weight, i.e., $\mathfrak{g}$ is not of type $\mathfrak{A}_n$.

The following simple lemma is one of our main tools for inductive arguments in subsequent sections.

Lemma 2.2. Suppose that the roots $\mu, \nu_1, \nu_2$ have the property that $\nu_1 + \nu_2 \in \Delta$ and $\mu + \nu_1 + \nu_2 \in \Delta$. Then $\mu + \nu_1$ or $\mu + \nu_2$ is also a root.
Proof. 1) If \((\mu + \nu_1 + \nu_2, \nu_1 + \nu_2) > 0\), then \((\mu + \nu_1 + \nu_2, \nu_1) > 0\) or \((\mu + \nu_1 + \nu_2, \nu_2) > 0\). Hence \(\mu + \nu_2\) or \(\mu + \nu_1\) is a root.

2) If \((\mu + \nu_1 + \nu_2, \nu_1 + \nu_2) \leq 0\), then \((\mu, \nu_1 + \nu_2) < 0\). Hence \((\mu, \nu_1) < 0\) or \((\mu, \nu_2) < 0\), i.e., again \(\mu + \nu_1\) or \(\mu + \nu_2\) is a root. \(\square\)

3. Elements of \(W^0\) Associated with the Lower Ideals in \(\Delta(1)\)

In this section, \(\Delta = \bigcup_{i \in \mathbb{Z}} \Delta(i)\) is a \(\mathbb{Z}\)-grading and \(\Delta^+ = \Delta(0)^+ \cup \Delta(\geq 1)\). Recall that \(I \in \mathcal{J}_-(\Delta(1))\) if and only if whenever \(\gamma \in I, \mu \in \Delta(0)^+,\) and \(\gamma - \mu \in \Delta,\) then \(\gamma - \mu \in I\). Then \(I^c := \Delta(1) \setminus I \in \mathcal{J}_+(\Delta(1))\). That is, if \(\gamma \in \Delta^c, \mu \in \Delta(0)^+,\) and \(\gamma + \mu \in \Delta,\) then \(\gamma + \mu \in I^c\).

For any \(I \subset \Delta^+\), we set \(I^1 = I\) and if \(I^{k-1} \neq \emptyset\), then \(I^k = (I + I^{k-1}) \cap \Delta\) for \(k \geq 2\). Then \(\langle I \rangle := \bigcup_{k \geq 1} I^k \subset \Delta^+\).

Lemma 3.1. \(\langle I \rangle\) is a closed subset of \(\Delta^+\).

Proof. Suppose that \(\gamma_i \in I^{k_i}, i = 1, 2\) and \(\gamma_1 + \gamma_2\) is a root. Our goal is to prove that \(\gamma_1 + \gamma_2 \in I^{k_1 + k_2}\). Without loss of generality, we may assume that \(k_1 \leq k_2\). Arguing by induction, we assume that the required property holds for all \((k_1', k_2')\) such that either \(k_1' + k_2' < k_1 + k_2\) or \(k_1' + k_2' = k_1 + k_2\) and \(k_1' < k_1\).

- If \(k_1 = 1\), then \(\gamma_1 + \gamma_2 \in I^{k_1+k_2}\) by the very definition of \(I^k\).
- If \(k_1 > 1\), then \(\gamma_1 = \gamma_1' + \gamma_2''\) with \(\gamma_1' \in I\) and \(\gamma_2'' \in I^{k_1-1}\). By Lemma 2.2, we then have \(\gamma_1' + \gamma_2' \in \Delta\) or \(\gamma_1'' + \gamma_2' \in \Delta\). Hence, by the induction assumption, \(\gamma_1' + \gamma_2' \in I^{k_2+1}\) or \(\gamma_1'' + \gamma_2' \in I^{k_1+k_2-1}\); and in either case we also conclude that \(\gamma_1' + \gamma_2' \in I^{k_1+k_2}\). \(\square\)

Now, let us turn to the case in which \(I \subset \Delta(1)\). Then \(I^k \subset \Delta(k)\) for all \(k \geq 1\). Consequently, \(\langle I \rangle \subset \Delta(\geq 1)\).

Proposition 3.2. If \(I \in \mathcal{J}_-(\Delta(1))\), then \(I^k \in \mathcal{J}_-(\Delta(k))\) for any \(k \geq 1\). Likewise, if \(I \in \mathcal{J}_+(\Delta(1))\), then \(I^k \in \mathcal{J}_+(\Delta(k))\) for any \(k \geq 1\).

Proof. Argue by induction on \(k\) and use Lemma 2.2. \(\square\)

Theorem 3.3. If \(I \in \mathcal{J}_-(\Delta(1))\), then \(\langle I \rangle\) is a bi-convex subset of \(\Delta^+\).

Proof. By Lemma 3.1, \(\langle I \rangle\) is closed. Set \((I^k)^c = \Delta(k) \setminus I^k\) for \(k \geq 1\). Then
\[
\overline{\langle I \rangle} := \Delta^+ \setminus \langle I \rangle = \Delta(0)^+ \cup I^c \cup (I^2)^c \cup \ldots,
\]
and our goal is to prove that \(\overline{\langle I \rangle}\) is closed, too. Assuming that this is not the case, one can find \(\mu', \mu'' \in \overline{\langle I \rangle}\) such that \(\mu' + \mu'' \in \langle I \rangle\). Since \(\Delta(0)^+\) is closed and each \((I^k)^c\) is an upper ideal (use Proposition 3.2!), one has to only consider the case in which neither \(\mu'\) nor \(\mu''\) belong to \(\Delta(0)^+\). Specifically, assume that \(\mu' \in (I^i)^c\) and \(\mu'' \in (I^j)^c\) with \(i, j \geq 1\), but \(\mu' + \mu'' \in I^{i+j}\). Arguing by induction, we may assume that \(i + j\) is the smallest integer with such property. By the recursive definition of \(\langle I \rangle\), one has
\[
\mu' + \mu'' = \gamma_1 + \gamma_{i+j-1} \in I^{i+j},
\]
where \( \gamma_k \in I^k \). Since \((\mu' + \mu'', \gamma_1 + \gamma_{i+j-1}) > 0\), we may assume that, say, \((\mu', \gamma_1) > 0\) and hence \( \nu := \mu' - \gamma_1 = \gamma_{i+j-1} - \mu'' \in \Delta(i-1) \). Now, there are two possibilities for \( i \).

(a) \( i = 1 \). Then \( \nu \in \Delta(0) \). If \( \nu \in \Delta(0)^+ \), then \( \mu' = \gamma_1 + \nu \in I \). If \( \nu \in \Delta(0)^- \), then \( \mu'' = \gamma_j - \nu \in \hat{I} \). In either case, this contradicts the assumption on \( \mu', \mu'' \).

(b) \( i > 1 \). Then \( \nu = \mu' - \gamma_1 \in \Delta(i-1) \subseteq \Delta^+ \). If \( \nu \in I^{-1} \), then \( \nu' \in I^i \), a contradiction. If \( \nu \in (I^{-1})^c \), then

\[
(\mu' - \gamma_1) + \mu'' = \gamma_{i+j-1} \in I^{i+j-1},
\]

which contradicts the minimality of \( i + j \).

Thus, \( \langle I \rangle \) is closed, and we are done. \( \square \)

**Theorem 3.4.** If \( I \in \mathcal{J}_-(\Delta(1)) \), then \( \Delta(\geq 1) \setminus \langle I^c \rangle \) is a bi-convex subset of \( \Delta^+ \).

**Proof.** All the necessary ideas are already contained in the previous proof.

1. The complement in \( \Delta^+ \) of the indicated subset is \( \langle I^c \rangle \cap \Delta(0)^+ \). Here \( \langle I^c \rangle = \bigcup_{k \geq 1} (I^c)^k \) is closed by Lemma 3.1, and, by Proposition 3.2, each \( (I^c)^k \) is an upper ideal of \( \Delta(k) \). Therefore, \( \langle I^c \rangle \cap \Delta(0)^+ \) is closed, too.

2. To prove that \( \Delta(\geq 1) \setminus \langle I^c \rangle = \bigcup_{k \geq 1} (\Delta(k) \setminus (I^c)^k) \) is closed, one uses the fact that each \( \Delta(k) \setminus (I^c)^k \) is a lower ideal and repeats *mutatis mutandis* the inductive argument of the previous proof. \( \square \)

By Theorem 3.3, there is a unique \( w \in W \) such that \( N(w) = \langle I \rangle \). In particular,

\[
N(w) \cap \Delta(1) = I.
\]

Since \( N(w) \subseteq \Delta(\geq 1) \), we also have \( w \in W^0 \). Furthermore, if \( w' \in W^0 \) also satisfies (\( \clubsuit \)), then \( N(w') \supset \langle I \rangle = N(w) \). Thus, \( w \) is the unique element of minimal length in \( W^0 \) such that the 1-component of \( N(w) \) is \( I \). We shall say that \( w \) is the minimal element of \( I \) and denote it by \( w_{I, \text{min}} \). Likewise, by Theorem 3.4, there is a unique \( \tilde{w} \in W^0 \) such that \( N(\tilde{w}) = \Delta(\geq 1) \setminus \langle I^c \rangle \). Clearly, the 1-component of \( N(\tilde{w}) \) is \( I \). Furthermore, if \( w' \in W^0 \) also satisfies (\( \clubsuit \)), then \( \Delta^+ \setminus N(w') \supset \langle I^c \rangle \cup \Delta(0)^+ = \Delta^+ \setminus N(\tilde{w}) \). Thus, \( \tilde{w} \) is the unique element of maximal length in \( W^0 \) such that the 1-component of \( N(\tilde{w}) \) is \( I \). For this reason, we say that \( \tilde{w} \) is the maximal element of \( I \) and denote it by \( w_{I, \text{max}} \).

**Remark 3.5.** It is readily seen that if \( w \in W^0 \), then \( I_w := N(w) \cap \Delta(1) \) is a lower ideal in \( \Delta(1) \). This provides the natural map \( \tau : W^0 \to \mathcal{J}_-(\Delta(1)) \), \( w \mapsto I_w \). An offspring of Theorems 3.3 and 3.4 is that \( \tau \) is onto and we have two sections \( s_{\text{min}}, s_{\text{max}} : \mathcal{J}_-(\Delta(1)) \to W^0 \) for \( \tau \), where \( s_{\text{min}}(I) = w_{I, \text{min}} \) and \( s_{\text{max}}(I) = w_{I, \text{max}} \).

Recall that the *weak Bruhat order* \( \leq \) on (any subset of) \( W \) is defined by the condition that \( w \leq w' \) if and only if \( N(w) \subseteq N(w') \). As a consequence of preceding results, we obtain the following interesting fact.

**Theorem 3.6.** For any \( I \in \mathcal{J}_-(\Delta(1)) \), \( \tau^{-1}(I) \) is an interval with respect to the weak Bruhat order in \( W^0 \). Namely, \( \tau^{-1}(I) = \{ w \in W^0 \mid w_{I, \text{min}} \leq w \leq w_{I, \text{max}} \} \).
Proposition 3.9. Proof. If \( w \in \tau^{-1}(I) \), then \( N(w) \cap \Delta(1) = I \) and hence
\[
N(w_{I,\text{min}}) = \langle I \rangle \subset N(w) \subset \Delta(\geq 1) \setminus \langle I^c \rangle = N(w_{I,\text{max}}),
\]
in view of the definitions of \( w_{I,\text{min}} \) and \( w_{I,\text{max}} \). That is, \( w_{I,\text{min}} \leq w \leq w_{I,\text{max}} \).

The other implication is obvious. \( \square \)

Definition 1. The set of minimal elements of \( W^0 \) is \( W^0_{\text{min}} = \{ w_{I,\text{max}} \mid I \in \mathcal{J}_-(\Delta(1)) \} \);

The set of maximal elements of \( W^0 \) is \( W^0_{\text{max}} = \{ w_{I,\text{max}} \mid I \in \mathcal{J}_-(\Delta(1)) \} \).

Our next aim is to provide alternative descriptions of the sets \( W^0_{\text{min}} = s_{\text{min}}(\mathcal{J}_-(\Delta(1))) \) and \( W^0_{\text{max}} = s_{\text{max}}(\mathcal{J}_-(\Delta(1))) \).

Theorem 3.7. \( W^0_{\text{min}} = \{ w \in W^0 \mid w^{-1}(\alpha) \in \Delta(\geq -1) \text{ for all } \alpha \in \Pi \} \).

Proof. (i) Suppose that \( w = w_{I,\text{min}} \) and \( w^{-1}(\alpha) \in \Delta(-k) \) for some \( \alpha \in \Pi \) and \( k \geq 1 \). More precisely, if \( w^{-1}(\alpha) = -\gamma \), then \( w(\gamma) = -\alpha \). Hence \( \gamma \in I^k \). Assume that \( k \geq 2 \). Then \( \gamma = \gamma' + \gamma'' \) with \( \gamma' \in I \) and \( \gamma'' \in I^{k-1} \). Here we would obtain that \( -\alpha = w(\gamma') + w(\gamma'') \) is a sum of two negative roots, which is absurd. Thus, \( k \leq 1 \).

(ii) Conversely, suppose that \( w \in W^0 \) has the property that \( w^{-1}(\alpha) \in \Delta(\geq -1) \) for all \( \alpha \in \Pi \). Set \( I = N(w) \cap \Delta(1) \). Then \( I \in \mathcal{J}_-(\Delta(1)) \), because \( w \in W^0 \). Therefore \( \langle I \rangle = N(w_{I,\text{min}}) \) and \( N(w_I) \subset N(w) \). The last inclusion implies that \( w = uw_{I,\text{min}} \) for some \( u \in W \) such that \( \ell(u) = \ell(u') + \ell(w_{I,\text{min}}) \), see e.g. [16, Lemma 5.1]. Assume that \( u \neq 1_W \). Then \( w = s_{\alpha}u'w_{I,\text{min}} \) for some \( \alpha \in \Pi \) such that \( \ell(u) = 1 + \ell(u') \) and therefore
\[
N(w) = N(u'w_{I,\text{min}}) \cup (u'w_{I,\text{min}})^{-1}(\alpha).
\]
Since \( \ell(u'w_{I,\text{min}}) = \ell(u') + \ell(w_{I,\text{min}}) \), we have \( N(u'w_{I,\text{min}}) \supset N(w_{I,\text{min}}) \supset I \). Therefore \( (u'w_{I,\text{min}})^{-1}(\alpha) \in \Delta(k) \) and hence \( k \geq 2 \). Then \( w^{-1}(\alpha) = -(u'w_{I,\text{min}})^{-1}(\alpha) \in \Delta(-k) \), which contradicts the assumption on \( w \). Thus, \( w = w_{I,\text{min}} \), and we are done. \( \square \)

Theorem 3.8. \( W^0_{\text{max}} = \{ w \in W^0 \mid w^{-1}(\alpha) \in \Delta(\leq 1) \text{ for all } \alpha \in \Pi \} \).

Proof. The proof is similar to the previous one and left to the reader. \( \square \)

Below we point out a relationship between an involution on \( \mathcal{J}_-(\Delta(1)) \), involution on \( W^0 \), and the subsets \( W^0_{\text{min}} \) and \( W^0_{\text{max}} \). Let \( w_0 \in W \) and \( \tilde{w}_0 \in W(0) \) be the respective longest elements. It is easily seen that if \( w \in W^0 \), then \( w_0w\tilde{w}_0 \in W^0 \). Therefore, the mapping
\[
w \in W^0 \mapsto i(w) := w_0w\tilde{w}_0 \in W^0
\]
is a well-defined involution on \( W^0 \), see [6]. For any \( I \in \mathcal{J}_-(\Delta(1)) \), we have defined the dual lower ideal \( I^* \) by \( I^* = \tilde{w}_0(\Delta(1) \setminus I) \), see [13, Sect. 2]. Note that \( \#I + \#I^* = \#\Delta(1) \).

Proposition 3.9. For any \( w \in W^0 \), we have
(i) \( (I_w)^* = I_{i(w)} \).
(ii) \( w \in W^0_{\text{min}} \) if and only if \( i(w) \in W^0_{\text{max}} \). More precisely, \( i(w_{I,\text{min}}) = w_{I^*,\text{max}} \).
Proof. (i) We have \( N(w_0w\tilde{w}_0) = \Delta^+ \setminus N(w\tilde{w}_0) \). Since \( \ell(w\tilde{w}_0) = \ell(w) + \ell(\tilde{w}_0) \), one also has \( N(w_0w\tilde{w}_0) = N(\tilde{w}_0) \cup (\tilde{w}_0)^{-1}N(w) = \Delta(0)^+ \cup \tilde{w}_0(N(w)) \) [16, Lemma 5.1]. Therefore, \( N(w_0w\tilde{w}_0) \cap \Delta(1) = \Delta(1) \setminus \tilde{w}_0(N(w) \cap \Delta(1)) \). That is, \( I_w(\tilde{w}_0) = \Delta(1) \setminus \tilde{w}_0(I_w) = (I_w)^* \).

(ii) Combine part (i), characterisations of \( W_0^{\min} \) and \( W_0^{\max} \) in Theorems 3.7, 3.8, and the following properties of the longest elements: \( w_b \) takes \( \Pi \) to \( -\Pi \); whereas \( \tilde{w}_0 \) takes each \( \Delta(i) \) to itself and also \( \Delta(0)^+ \) to \( -\Delta(0)^+ \).

For any subset \( S \subset W \), define its Poincaré polynomial by \( S(t) = \sum\limits_{w \in S} t^{\ell(w)} = \sum\limits_{w \in S} t^{#N(w)} \).

The celebrated Kostant-Macdonald identity [9] says that

\[
W(t) = \prod_{\gamma \in \Delta^+} \left( \frac{1 - t^{ht(\gamma)+1}}{1 - t^{ht(\gamma)}} \right).
\]

In particular, \( #W = \prod_{\gamma \in \Delta^+} \frac{ht(\gamma)+1}{ht(\gamma)} \).

Example 3.10. Let \( \Delta = \bigsqcup_{i=-1}^1 \Delta(i) \) be an abelian grading. Then Theorems 3.7 and 3.8 immediately imply that \( W_0^{\min} = W_0^{\max} = W^0 \). Therefore \( \#J_-(\Delta(1)) = #W^0 \). Furthermore, \( i(w) = w \) if and only if \( (I_w)^* = I_w \). For any parabolic subgroup \( W(0) \subset W \), we have

\[
\#\{w \in W^0 \mid i(w) = w\} = W^0(-1),
\]

see [6, 14]. Therefore, in the abelian case, \( W^0(-1) \) equals the number of self-dual lower ideals in \( \Delta(1) \). This has already been proved in [17]. In the abelian case, \( W^0(t) \) coincides with the rank-generating function for the poset of lower ideals, see e.g. [13, Sect. 3], i.e., \( W^0(t) = M_{\Delta(1)}(t) \) and thereby \( M_{\Delta(1)}(-1) \) is the number of self-dual lower ideals.

Remark 3.11. For the non-abelian \( \mathbb{Z} \)-gradings (i.e., if \( \Delta(2) \neq \emptyset \)), \( W_0^{\min} \) and \( W_0^{\max} \) are different proper subsets \( W^0 \). Moreover, the polynomials \( W_0^{\min}(t), W_0^{\max}(t), \) and \( M_{\Delta(1)}(t) \), which have the same value at \( t = 1 \), are different. For the reader convenience, we compare explicit formulae for all these polynomials:

\[
M_{\Delta(1)}(t) = \sum_{w \in W_0^{\min}} t^{#(N(w) \cap \Delta(1))} = \sum_{w \in W_0^{\max}} t^{#(N(w) \cap \Delta(1))},
\]

\[
W_0^{\min} = \sum_{w \in W_0^{\min}} t^{#N(w)}, \quad W_0^{\max} = \sum_{w \in W_0^{\max}} t^{#N(w)}.
\]

We have conjectured in [13, Conjecture 5.2] (and verified in many cases) that \( M_{\Delta(1)}(-1) \) yields the number of self-dual lower ideals in \( \Delta(1) \) for any \( \mathbb{Z} \)-grading. That is, in a sense, \( M_{\Delta(1)}(t) \) is the most appropriate \( t \)-analogue of \( #J_-(\Delta(1)) \).

Example 3.12. Let \( \Delta = \bigsqcup_{i=-2}^2 \Delta(i) \) be an extra-special grading. As \( \Delta(2) = \{\theta\} \), it follows from Theorem 3.7 that, for \( w \in W^0 \), we have \( w \in W_0^{\min} \) if and only if \( w^{-1}(\alpha) \neq -\theta \) for all \( \alpha \in \Pi \), i.e., \( -w(\theta) \notin \Pi \). Likewise, by Theorem 3.8, \( w \in W_0^{\max} \) if and only if \( w(\theta) \notin \Pi \). Hence here \( W^0 = W_0^{\min} \cup W_0^{\max} \). As \( W(0) \) is the stabiliser of \( \theta \) in \( W \), we have \( W^0\theta = W\theta \),
Remark. If a \( \mathbb{Z} \)-grading is neither abelian nor extra-special, then \( W^0 \neq W^0_{\min} \cup W^0_{\max} \).

Suppose now that the \( \mathbb{Z} \)-grading in question is 1-standard. More precisely, \( \Pi = \Pi(0) \cup \Pi(1) \) and \( \Pi(1) = \{ \tilde{\alpha} \} \). For any \( w \in W^0 \), we look at the coefficient of \( \tilde{\alpha} \) for the roots \( w^{-1}(\alpha) \), \( \alpha \in \Pi \). Namely, write

\[
w^{-1}(\alpha) = k_{\alpha}(w)\tilde{\alpha} + \sum_{\alpha_i \in \Pi(0)} l_i(w)\alpha_i
\]

and consider the mapping \( \eta : W^0 \to \mathbb{Z}^n, \eta(w) = (k_{\alpha}(w))_{\alpha \in \Pi} \).

**Theorem 3.13.** (i) The mapping \( \eta \) is injective;
(ii) \( \eta(W^0_{\min}) = \{ (k_{\alpha}(w))_{\alpha \in \Pi} \mid k_{\alpha}(w) \geq -1 \text{ for all } \alpha \in \Pi \} \);
(iii) \( \eta(W^0_{\max}) = \{ (k_{\alpha}(w))_{\alpha \in \Pi} \mid k_{\alpha}(w) \leq 1 \text{ for all } \alpha \in \Pi \} \).

**Proof.** (i) Let \( \{ \varpi_{\alpha}^{\vee} \}_{\alpha \in \Pi} \) be the fundamental weights of the dual Lie algebra \( g^{\vee} \) corresponding to \( \Pi \). In other words, \( (\alpha, \varpi_{\beta}^{\vee}) = \delta_{\alpha\beta} \) for all \( \alpha, \beta \in \Pi \), i.e., \( \{ \varpi_{\alpha}^{\vee} \}_{\alpha \in \Pi} \) is the dual basis to \( \Pi \). Then \( W(0) \) is the stabiliser of \( \varpi_{\alpha}^{\vee} \) in \( W \) and all weights \( w(\varpi_{\alpha}^{\vee}), w \in W^0 \), are different. We have \( (w(\varpi_{\alpha}^{\vee}), \alpha) = (\varpi_{\alpha}^{\vee}, w^{-1}(\alpha)) = k_{\alpha}(w) \). Whence \( w(\varpi_{\alpha}^{\vee}) = \sum_{\alpha \in \Pi} k_{\alpha}(w)\varpi_{\alpha}^{\vee} \).

(ii), (iii). This readily follows from Theorems 3.7 and 3.8, because \( w^{-1}(\alpha) \in \Delta(1) \) if and only if \( k_{\alpha}(w) = i \). \( \square \)

**Remark.** The above proof suggests to regard \( \eta \) as a mapping from \( W^0 \) to the lattice \( \mathcal{L} = \{ \sum_{\alpha \in \Pi} k_{\alpha}\varpi_{\alpha}^{\vee} \mid k_{\alpha} \in \mathbb{Z} \} \simeq \mathbb{Z}^n \) in \( V \). Set also \( \mathcal{C}_{\geq -1} = \{ \sum_{\alpha \in \Pi} k_{\alpha}\varpi_{\alpha}^{\vee} \mid k_{\alpha} \geq -1 \forall \alpha \in \Pi \} \) and \( \mathcal{C}_{\leq 1} = \{ \sum_{\alpha \in \Pi} k_{\alpha}\varpi_{\alpha}^{\vee} \mid k_{\alpha} \leq 1 \forall \alpha \in \Pi \} \). Then Theorem 3.13 asserts that

\[
\eta(W^0_{\min}) = W \cdot \varpi_{\alpha}^{\vee} \cap \mathcal{C}_{\geq -1} \text{ and } \eta(W^0_{\max}) = W \cdot \varpi_{\alpha}^{\vee} \cap \mathcal{C}_{\leq 1}.
\]

Thus, the minimal or maximal elements of \( W^0 \) are in a natural one-to-one correspondence with certain subsets of the \( W \)-orbit of \( \varpi_{\alpha}^{\vee} \).

**Example 3.14.** The abelian gradings are 1-standard and then \( \varpi_{\alpha}^{\vee} \) is a minuscule fundamental weight of \( g^{\vee} \). Then \( (\varpi_{\alpha}^{\vee}, \gamma) \in \{-1, 0, 1\} \) for all \( \gamma \in \Delta \) [3, Ch. VIII, §7, n⁰3]. Consequently, the whole orbit \( W \cdot \varpi_{\alpha}^{\vee} \) belongs to \( \mathcal{C}_{\geq -1} \cap \mathcal{C}_{\leq 1} \). Here we again obtain that all elements of \( W^0 \) are both maximal and minimal, and therefore \( \#2\Delta(1) = \#W^0 \).

4. Extreme roots associated with the lower ideals in \( \Delta(1) \)

Recall that any lower (resp. upper) ideal of a poset \( \mathcal{P} \) is determined by its maximal (resp. minimal) elements. Below, we describe these extreme elements (roots) for the ideals in \( \mathcal{P} = \Delta(1) \), using the corresponding minimal and maximal elements of \( W^0 \).
Theorem 4.1. If $I \in \mathcal{I}(-\Delta(1))$ and $\gamma \in \Delta(1)$, then $\gamma \in \max(I)$ if and only if $w_{I,\min}(\gamma) \in -\Pi$.

Proof. Write $w$ for $w_{I,\min}$ in this proof. Recall that $\gamma \in I$ if and only if $w(\gamma) \in -\Delta^+$. 

(i) If $\gamma \in I$ and $\gamma \not\in \max(I)$, then $\gamma = \gamma' - \delta$ for some $\gamma' \in I$ and $\delta \in \Delta(0)^+$. Then $w(\gamma) = w(\gamma') - w(\delta)$ is a sum of negative roots.

(ii) Conversely, if $\gamma \in I$ and $w(\gamma) \not\in -\Pi$, then $w(\gamma) = -\delta_1 - \delta_2$, where $\delta_i \in \Delta^+$. Hence $-w^{-1}(\delta_1) - w^{-1}(\delta_2) = \gamma \in \Delta(1)$. Set $\mu_i = -w^{-1}(\delta_i)$, so that $\gamma = \mu_1 + \mu_2$. Without loss of generality, we may assume that $\mu_2$ is positive. Let us consider possible levels of $\mu_2$ and consequences of that for $\gamma$.

1. The case in which $\mu_2 \in \Delta(0)^+$ is impossible, since $w(\mu_2) = -\delta_2$ and $w \in W^0$.

2. Suppose that $\mu_2 \in \Delta(1)$. Since $w(\mu_2)$ is negative, we have $\mu_2 \in I$. Furthermore, here $\mu_1 \in \Delta(0)$. As in (1), the case $\mu_1 \in \Delta(0)^+$ is impossible. Hence $-\mu_1 \in \Delta(0)^+$ and then $\gamma = \mu_1 + \mu_2 < \mu_2$, i.e., $\gamma \not\in \max(I)$.

3. Suppose that $\mu_2 \in \Delta(k)$, $k \geq 2$. Let us show that there is another decomposition $\gamma = \tilde{\mu}_1 + \tilde{\mu}_2$ such that $\tilde{\mu}_2 \in \Delta(\tilde{k})$ with $0 < \tilde{k} < k$.

Since $w(\mu_2)$ is negative, we have $\mu_2 \in I^k$ by the very definition of $w = w_{I,\min}$. Hence, $\mu_2 = \mu' + \mu''$, where $\mu' \in I^{k'}, \mu'' \in I^{k''}$, and $k' + k'' = k$. As $\gamma = \mu_1 + \mu' + \mu''$, we have $\mu_1 + \mu' \in \Delta$ or $\mu_1 + \mu'' \in \Delta$, see Lemma 2.2. By symmetry, it suffices to consider the first possibility. Then we set $\tilde{\mu}_1 = \mu_1 + \mu'$, $\tilde{\mu}_2 = \mu''$, and $\tilde{k} = k''$.

Thus, one can gradually descend to the case $\tilde{k} = 1$ and conclude using (2) that $\gamma \not\max(I)$.

Theorem 4.2. For $I \in \mathcal{I}(-\Delta(1))$ and $\gamma \in \Delta(1)$, we have $\gamma \in \min(I^c)$ if and only if $w_{I,\max}(\gamma) \in \Pi$.

Proof. This proof is similar (and “dual”) to the proof of Theorem 4.1. Write $w$ for $w_{I,\max}$ in this proof. Recall that $\gamma \in I^c$ if and only if $w(\gamma) \in \Delta^+$.

(i) If $\gamma \in I^c \setminus \min(I^c)$, then $\gamma = \gamma' + \delta$ for some $\gamma' \in I^c$ and $\delta \in \Delta(0)^+$. Then $w(\gamma) = w(\gamma') + w(\delta)$ is a sum of positive roots.

(ii) Conversely, if $\gamma \in I^c$ and $w(\gamma) \not\in \Pi$, then $w(\gamma) = \delta_1 + \delta_2$, where $\delta_i \in \Delta^+$. Hence $w^{-1}(\delta_1) + w^{-1}(\delta_2) = \gamma \in \Delta(1)$. Set $\mu_i = w^{-1}(\delta_i)$. Without loss of generality, we may assume that $\mu_2$ is positive. Let us consider possible levels of $\mu_2$ and consequences of that for $\gamma$.

1. Suppose that $\mu_2 \in \Delta(0)^+$. Then $\mu_1 \in \Delta(1)$ and $w(\mu_1) \in \Delta^+$. Hence $\mu_1 \in I^c$ and $\gamma = \mu_1 + \mu_2 \not\in \min(I^c)$.

2. Suppose that $\mu_2 \in \Delta(1)$. Then $\mu_2 \in I^c$ and $\mu_1 \in \Delta(0)$.

   - If $\mu_1$ is positive, then again $\gamma = \mu_1 + \mu_2 \not\in \min(I^c)$.

   - The case in which $\mu_1 \in -\Delta(0)^+$ is impossible, since $w(\mu_1) = \delta_1$ and $w \in W^0$.

3. Suppose that $\mu_2 \in \Delta(k)$, $k \geq 2$. Let us show that there is another decomposition $\gamma = \tilde{\mu}_1 + \tilde{\mu}_2$ such that $\tilde{\mu}_2 \in \Delta(\tilde{k})$ with $0 < \tilde{k} < k$. 

Since $w(\mu_2) \in \Delta^+$, we have $\mu_2 \in (I^c)^k$ by the very definition of $w = w_{I,\max}$. Hence, $\mu_2 = \mu' + \mu''$, where $\mu' \in (I^c)^k$, $\mu'' \in (I^c)^{k''}$, and $k + k'' = k$. As $\gamma = \mu_1 + \mu' + \mu''$, we have $\mu_1 + \mu' \in \Delta$ or $\mu_1 + \mu'' \in \Delta$, see Lemma 2.2. By symmetry, it suffices to handle the first possibility. Then we set $\tilde{\mu}_1 = \mu_1 + \mu'$, $\tilde{\mu}_2 = \mu''$, and $\tilde{k} = k''$.

Thus, one can gradually descend to the case $\tilde{k} = 1$ and conclude using (2) that $\gamma \notin \max(I^c)$.

$$\square$$

5. Dominant chambers and arrangements of hyperplanes

For $\gamma \in \Delta$, let $H_\gamma$ be the hyperplane in $V$ orthogonal to $\gamma$. Then $A = \{H_\gamma \mid \gamma \in \Delta^+\}$ is the Coxeter arrangement associated with $\Delta$. The connected components of $V \setminus (\bigcup_{\gamma \in \Delta^+} H_\gamma)$ are called (open) chambers. Each chamber is an open simplicial cone in $V$, and $W$ acts simply transitively on the set of chambers. The dominant open chamber is $C^0 = \{v \in V \mid (v, \alpha) > 0 \ \forall \alpha \in \Pi\}$. The closure of $C^0$ is denoted by $C$. If $K', K''$ are two chambers, then the distance between them, $d(K', K'')$, is the number of hyperplanes in $A$ that separate them. As is well known, $d(C, w(C)) = \ell(w)$. More precisely, the hyperplane $H_\gamma$ separates $C$ and $w(C)$ if and only if $\gamma \in N(w^{-1})$, see [2, Chap. VI, §1, Prop. 17].

In this section, we will consider certain sub-arrangements of $A_\Delta$ and their relationship to ideals/antichains in the poset $\Delta(1)$. The first of them is $A_\Delta(0) = \{H_\gamma \mid \gamma \in \Delta(0)^+\}$, the Coxeter arrangement associated with $\Delta(0)$. The corresponding big dominant chamber is $C(0)^0 = \{v \in V \mid (v, \alpha) > 0 \ \forall \alpha \in \Pi(0)\}$ and its closure is denoted by $C(0)$. It follows readily from the definition of $W^0$ (see Eq. (1.2)), that $w \in W^0$ if and only if $w^{-1}(C) \subset C(0)$. In particular, the big dominant chamber $C(0)$ is the union of $\#W^0$ “small” chambers.

Theorem 5.1.

(i) The hyperplanes $H_\gamma, \gamma \in \Delta(1)$, dissect the cone $C(0)$ into certain regions (cones) that are in a natural one-to-one correspondence with the ideals of $\Delta(1)$ (and we write $R_i^0$ for the open region corresponding to $I \in \mathcal{J}_-(\Delta(1))$);

(ii) if $w \in W^0_{\min}$, then $w^{-1}(C^0)$ is the unique small chamber in $R^0_{I_w}$ that is closest to $C^0$;

(iii) if $w \in W^0_{\max}$, then $w^{-1}(C^0)$ is the unique small chamber in $R^0_{I_w}$ that is farthest from $C^0$;

Proof. (i) Given $I \in \mathcal{J}_-(\Delta(1))$, define the open region (cone), $R^0_I$, corresponding to $I$ as follows:

$$R^0_I = \{x \in C^0 \mid (x, \gamma) > 0 \text{ if } \gamma \notin I \& (x, \gamma) < 0 \text{ if } \gamma \in I\}.$$  

Using the fact that $\tau : W^0 \to \mathcal{J}_-(\Delta(1))$ is onto, one immediately obtains that $R^0_I \neq \emptyset$ for any $I$. Indeed, if $\tau(w) = I$, then $H_\gamma (\gamma \in \Delta(1))$ separates $C^0$ and $w^{-1}(C^0)$ if and only if $\gamma \in N(w) \cap \Delta(1) = I$. Therefore, $w^{-1}(C^0) \subset R^0_I$. Furthermore, any chamber $w^{-1}(C^0)$, $w \in W^0$, belongs to some region $R^0_I$, which means that the closed regions $R_I (I \in \mathcal{J}_-(\Delta(1)))$ exhaust the big dominant chamber $C(0)$. 


(ii),(iii) This follows from (i) and the fact that \( w_{I, \min} \) (resp. \( w_{I, \max} \)) is the unique element of minimal (resp. maximal) length in \( \tau^{-1}(I) \).

These properties suggest to consider the sub-arrangement \( A_\Delta(0,1) \) of \( A_\Delta \) that contains only the hyperplanes \( H_\gamma \) corresponding to \( \gamma \in \Delta(0)^+ \cup \Delta(1) \). Set \( \eta_i = \#\{ \gamma \in \Delta(0)^+ \cup \Delta(1) \mid \text{ht}(\gamma) = i \} \) and consider the associated sequence \( \mathcal{P}(0,1) = (\eta_1, \eta_2, \ldots) \).

**Lemma 5.2.** The sequence \( \mathcal{P}(0,1) \) is a partition, i.e., \( \eta_1 \geq \eta_2 \geq \ldots \). In addition, \( \eta_1 > \eta_2 \).

**Proof.** This is a particular case of a more general observation, see [16, Prop. 3.1]. However, that proof consists of a reference to case-by-case and computer computations. For this reason, we provide a general case-free proof in the Appendix, see Proposition A.1.

Note also that, for the standard gradings, the inequality \( \eta_i > \eta_2 \) readily stems from the fact that \( \Delta(0)^+ \cup \Delta(1) \) contains all simple roots, i.e., \( \eta_i = \text{rk} \Delta \).

**Conjecture 5.3.** The arrangement \( A_\Delta(0,1) \) is free and its exponents are given by the dual partition \( \mathcal{P}(0,1)^t \) to \( \mathcal{P}(0,1) \).

This is a special case of a general conjecture discussed in [16]. Namely, let \( \mathcal{I} \subset \Delta^+ \) be an arbitrary upper ideal and \( A_\Delta(\mathcal{I}^c) = \{ H_\gamma \mid \gamma \notin \mathcal{I} \} \subset A_\Delta \). Sommers and Tymoczko conjecture that the arrangement \( A_\Delta(\mathcal{I}^c) \) is free and its exponents are given by the dual partition to \( (\lambda_1, \lambda_2, \ldots) \), where \( \lambda_i = \#\{ \gamma \in \Delta^+ \setminus \mathcal{I} \mid \text{ht}(\gamma) = i \} \). (The string \( (\lambda_1, \lambda_2, \ldots) \) is really a partition, see Proposition A.1.) By [16, Theorem 11.1], this general conjecture, and thereby Conjecture 5.3, are true if \( \Delta \) is of type \( A_n, B_n, C_n, D_n, \) and \( G_2 \). Using this conjecture, one derives a closed formula for the number of lower ideals (antichains) in \( \Delta(1) \).

**Theorem 5.4.** It follows from Conjecture 5.3 that

\[
\#(\mathcal{B}_-(\Delta(1))) = \#\text{Ann}(\Delta(1)) = \prod_{\gamma \in \Delta(1)} \frac{\text{ht}(\gamma) + 1}{\text{ht}(\gamma)}.
\]

**Proof.** Let \( b_1, \ldots, b_n \) be the exponents of the free arrangement \( A_\Delta(0,1) \). By the factorisation result of Terao (see [10, Theorem 4.137], the characteristic polynomial of \( A_\Delta(0,1) \) is \( \chi_{(0,1)}(t) = \prod_{i=1}^n (t - b_i) \). By a theorem of Zaslavsky [20], the total number of regions of \( A_\Delta(0,1) \) equals \((-1)^n \chi_{(0,1)}(-1) = \prod_{i=1}^n (b_i + 1) \). By definition of the dual partition, if \( \eta_i = \#\{ \gamma \in \Delta(0)^+ \cup \Delta(1) \mid \text{ht}(\gamma) = i \} \), then \( \eta_i - \eta_{i+1} \) is the number of exponents that are equal to \( i \). Therefore,

\[
\prod_{i=1}^n (b_i + 1) = \prod_{\gamma \in \Delta(0)^+ \cup \Delta(1)} \frac{\text{ht}(\gamma) + 1}{\text{ht}(\gamma)}.
\]

Since the arrangement \( A_\Delta(0,1) \) is \( W(0) \)-invariant and \( \mathcal{C}(0) \) is a fundamental domain for the \( W(0) \)-action, the number of regions inside \( \mathcal{C}(0) \) equals \( \prod_{i=1}^n (b_i + 1)/\#W(0) \). On the other hand, the Kostant-Macdonald identity (3.1) implies that \( \#W(0) = \prod_{\gamma \in \Delta(0)^+} \frac{\text{ht}(\gamma) + 1}{\text{ht}(\gamma)} \).
Combining all these formulae, we conclude that the number of regions of \( A_\Delta(0, 1) \) inside \( \mathcal{C}(0) \) equals \( \prod_{\gamma \in \Delta(1)} \frac{ht(\gamma) + 1}{ht(\gamma)} \). Finally, by Theorem 5.1(i), the last number also gives the number of antichains (ideals) in \( \Delta(1) \). \hfill \Box

**Remark 5.5.** Formula (5.1) for \( \#A_\Delta(\Delta(1)) \) appears already in [13] as a consequence of a general conjectural formula for \( M_{\Delta(1)}(t) \) [13, Conj. 5.1]. Now, our theory of minimal/maximal elements in \( W^0 \), a relationship to arrangements, and partial results of [16] allow us to conclude that (5.1) holds for all classical cases and \( G_2 \). However, the present approach does not provide new information on \( M_{\Delta(1)}(t) \), because there seems to be no relationship between the arrangement \( A_\Delta(0, 1) \) and the rank-generating function \( M_{\Delta(1)}(t) \).

**Example 5.6.** In the abelian case, we have \( A_\Delta(0, 1) = A_\Delta \) and the exponents of the Coxeter arrangement \( A_\Delta \) are the usual exponents of the Weyl group \( W \) [10, Theorem 6.60]. Hence \( \chi_{A_\Delta}(t) = \prod_{i=1}^n (t - m_i) \) and \( (-1)^n \chi_{A_\Delta}(-1) = \prod_{i=1}^n (m_i + 1) = \#W \), as required.

As usual, we arrange the exponents in the non-decreasing order: \( 1 = m_1 \leq m_2 \leq \ldots \leq m_n = h - 1 \). If \( n \geq 2 \), then \( m_1 < m_2 \) and \( m_{n-1} < m_n \).

**Example 5.7.** In the extra-special case, \( W(0) \) is the stabiliser of \( \theta \) and \( A_\Delta(0, 1) \) is just the deleted arrangement \( A' = A_\Delta \setminus \mathcal{H}_\theta \). It is known that \( A' \) is free and the exponents of \( A' \) are \( m_1, \ldots, m_{n-1}, m_n - 1 \) (combine Theorems 4.51 and 6.104 in [10]). Therefore \( (-1)^n \chi_{A'}(-1) = (m_1 + 1) \ldots (m_{n-1} + 1)m_n = \#W_\Delta \). Since \( \#W/\#W(0) \) is the number of long roots in \( \Delta \), the number of the \( W(0) \)-dominant regions of \( A' \) is

\[
\frac{\#W}{\#W(0)} \cdot \frac{h - 1}{h} = \Pi_{I'} h \cdot \frac{h - 1}{h} = \Pi_{I'} (h - 1),
\]

which is the number of antichains in \( \Delta(1) \). This was computed earlier in [13, Section 4], see also Example 3.12.

**Example 5.8.** For the 1-standard \( \mathbb{Z} \)-grading of \( g = E_7 \) with \( \Pi(1) = \{ \alpha_7 \} \), we have \( g(0) \simeq gl(7) \) and \( g(1) = \wedge^3(\mathbb{C}^7) \) is the third fundamental representation. Here the numbering of \( \Pi \) follows [19, Tables]. Then

\[
P(0, 1) = (7, 6^4, 5^2, 4^2, 3, 2, 1, 1) \quad \text{and} \quad P(0, 1)^f = (13, 11, 10, 9, 7, 5, 1).
\]

Therefore, the conjectural exponents of \( A_\Delta(0, 1) \) are 1, 5, 7, 9, 10, 11, 13 and then the number of lower ideals in \( \Delta(1) \) is 252.

6. **AFFINE VERSUS FINITE THEORY**

In this section, we compare the theory of upper (or ad-nilpotent) ideals of \( \Delta^+ \) (the *affine theory*) and our theory of lower ideals in \( \Delta(1) \) related to a \( \mathbb{Z} \)-grading of \( \Delta \) (the *finite theory*).
We begin with the necessary notation. Recall that $V = \bigoplus_{i=1}^{n} \mathbb{R} \alpha_{i}$ and $(,)$ is a $W$-invariant inner product on $V$. As usual, $\mu^V = 2\mu/(\mu, \mu)$ is the coroot for $\mu \in \Delta$ and $Q^V = \bigoplus_{i=1}^{n} \mathbb{Z} \alpha_{i}^V$ is the coroot lattice in $V$. Letting $\hat{V} = V \oplus \mathbb{R} \delta \oplus \mathbb{R} \lambda$, we extend the inner product $(,)$ on $\hat{V}$ so that $(\delta, V) = (\lambda, V) = (\delta, \delta) = (\lambda, \lambda) = 0$ and $(\delta, \lambda) = 1$. Set $\alpha_{0} = \delta - \theta$.

Then
\[
\hat{\Delta} = \{ \Delta + k\delta \mid k \in \mathbb{Z} \} \text{ is the set of affine (real) roots;}
\hat{\Delta}^+ = \Delta^+ \cup \{ \Delta + k\delta \mid k \geq 1 \} \text{ is the set of positive affine roots;}
\hat{\Pi} = \Pi \cup \{ \alpha_{0} \} \text{ is the corresponding set of affine simple roots.}
\]

For any $\gamma \in \hat{\Delta}$, the reflection $s_{\gamma} \in GL(\hat{V})$ is defined in the usual way, via the extended inner product, and the affine Weyl group, $\hat{W}$, is the subgroup of $GL(\hat{V})$ generated by the reflections $s_{\alpha}$, $\alpha \in \hat{\Pi}$. As is well known, $\hat{W}$ is also a semi-direct product of $W$ and $Q^\vee$ [2, 7]. It follows that $\hat{W}$ has two natural actions:

(a) the linear action on $\hat{V}$;
(b) the affine-linear action on $V$.

Using the linear action, one defines the inversion set $\hat{N}(w) = \{ \gamma \in \hat{\Delta}^+ \mid w(\gamma) \in -\hat{\Delta}^+ \}$ and the length $\hat{l}(w) = \#\hat{N}(w)$ for any $w \in \hat{W}$.

The affine theory is well-developed, and we present below notable correlations with results of this article. An overview of the “affine” results discussed below can also be found in [12, Section 2].

1) By the very definition, $\hat{\Delta}$ is $\mathbb{Z}$-graded, with $\hat{\Delta}(k) = \Delta + k\delta$, $k \in \mathbb{Z}$. Extending our previous terminology to the affine case, one can say that this $\mathbb{Z}$-grading is 1-standard. The unique affine simple root in $\hat{\Delta}(1)$ is $\alpha_{0}$ and the parabolic subgroup $\hat{W}(0)$ is just $W$.

Accordingly, the set of minimal length coset representatives is
\[
\hat{W}^0 = \{ w \in \hat{W} \mid w(\alpha) \in \hat{\Delta}^+ \text{ for all } \alpha \in \Pi \}
\]
(such elements of $\hat{W}$ are called dominant in [12].) Let $\mathcal{I}$ be an upper ideal of the poset $(\hat{\Delta}^+, \preceq)$, i.e., $\mathcal{I} \in \mathcal{J}(\hat{\Delta}^+)$. The affine theory gets off the ground when one replaces $\mathcal{I}$ with $\delta - \mathcal{I} = \{ \delta - \gamma \mid \gamma \in \mathcal{I} \} \subset \hat{\Delta}(1)$ and seeks for a characterisation of $\delta - \mathcal{I}$ is terms of $\hat{W}$, or rather, in terms of $\hat{W}^0$. Note that $\delta - \mathcal{I}$ becomes a lower ideal in the negative part of $\hat{\Delta}(1) \simeq \Delta$.

2) Given $\mathcal{I} \in \mathcal{J}_{+}(\hat{\Delta}^+)$, the first basic result is that there is a unique element $w_{\mathcal{I}, \min} \in \hat{W}^0$ of minimal length such that $\hat{N}(w_{\mathcal{I}, \min}) \cap \hat{\Delta}(1) = \delta - \mathcal{I}$. Namely,
\[
(6.1) \quad \hat{N}(w_{\mathcal{I}, \min}) = \bigcup_{k \geq 1} (k\delta - \mathcal{I}^k) = \bigcup_{k \geq 1} (\delta - \mathcal{I})^k.
\]

The key point is to prove that the RHS is a bi-convex subset of $\hat{\Delta}^+$, see [4, Sect. 2]. Hence our Theorem 3.3 is a “finite” analogue of that result. Then the set of minimal elements of $\mathcal{I}$ (called generators of $\mathcal{I}$ in [11, 12]), i.e., maximal elements of $\delta - \mathcal{I}$ can be characterised via $w_{\mathcal{I}, \min}$, see [11, Theorem 2.2]. The corresponding “finite” assertion is our Theorem 4.1.
3) Since $\hat{W}$ and $\hat{\Delta}$ are infinite, one cannot always provide an element $w_{I,\text{max}} \in \hat{W}^0$ of maximal length such that $\hat{N}(w_{I,\text{max}}) \cap \hat{\Delta}(1) = \delta - I$. Sommers proves [15] that such a maximal element exists if and only if $I \subset \Delta^+ \setminus \Pi$. In that case, $w_{I,\text{max}}$ can be used for describing the maximal elements of $\Delta^+ \setminus I$, i.e., the minimal elements of $\hat{\Delta}(1) \setminus (\delta - I)$, see [15, Cor. 6.3]. Our Theorems 3.3 and 4.2 provide finite analogues of this for all lower ideals in $\Delta(1)$.

4) In the finite case, $\Delta(1)$ is the weight poset of a weight multiplicity free representation of $g(0)$, and the maximal and minimal elements in $W_0$ exist for all lower ideals. But the adjoint representation of $g$ is not weight multiplicity free (unless $g = sl_2$). Therefore, in the affine case, one considers only the weight multiplicity free part of $g$ corresponding to $\Delta^+$. A related disadvantage is that $\Delta^+ \setminus I$ shouldn’t be called a “lower ideal” and that $w_{I,\text{max}}$ does not always exists, see 3) above.

5) Among the advantages of the affine case are the following:

- $\hat{W} = W \rtimes Q^\vee$ is a semi-direct product having two related actions (on $V$ and $\hat{V}$);
- $\delta$ is a $\hat{W}$-invariant element of $\hat{V}$ and all the pieces $\hat{\Delta}(k)$ are isomorphic;

These properties often help in computations and allow to achieve more complete results. On the other hand, an advantage of the finite theory is that both $W$ and $W(0)$ contain the elements of maximal length, which yields a natural involution on $W^0$ and provides a relationship between $W^0_\text{min}$ and $W^0_\text{max}$ in Proposition 3.9.

6) There are at least two approaches to computing the total number of upper ideals (antichains) in $\Delta^+$, which are discussed below.

(6a) There is a natural bijection between $J_+(\Delta^+)$ and the $W$-dominant regions of the Catalan arrangement

$$\text{Cat}(\Delta) = \{ \mathcal{H}_{\gamma,k} \mid \gamma \in \Delta^+, \ k = -1, 0, 1 \},$$

where $\mathcal{H}_{\gamma,k} = \{ v \in V \mid (\gamma, v) = k \}$. Then an explicit formula for the characteristic polynomial of $\text{Cat}(\Delta)$ yields a formula for $\#J_+(\Delta^+)$, see [1]. A finite counterpart of this approach is implemented in Section 5, $\text{Cat}(\Delta)$ being replaced with $A_\Delta(0, 1)$. In particular, the “finite” analogue of the above bijection is our Theorem 5.1.

(6b) There is a natural bijection between the set of minimal elements in $\hat{W}^0$, denoted $\hat{W}^0_\text{min}$, and the points of certain convex polytope $\mathcal{D}_\text{min} \subset V$ lying in $Q^\vee$ [5, Prop. 3]. This polytope is $\hat{W}$-conjugate to a dilated fundamental alcove of $\hat{W}$, and the number

$$\#(\mathcal{D}_\text{min} \cap Q^\vee) = \#\hat{W}^0_\text{min} = \#J_+(\Delta^+)$$

can be computed via a result of Haiman, see [5, Section 3] for details. To construct a bijection $\hat{W}^0_\text{min} \leftrightarrow \mathcal{D}_\text{min} \cap Q^\vee$, Cellini and Papi use the semi-direct product structure of $\hat{W}$. However, one can notice that the following synthetic procedure works. If $w_{I,\text{min}}$ is defined by (6.1) and ‘*’ denotes the affine-linear action of $\hat{W}$, then the point of $Q^\vee$ corresponding to $I$ is merely $w_{I,\text{min}} * 0$. 


**Warning.** Cellini and Papi [4, 5] give the definition of the inversion set \( \hat{N}(w) \) with the inverse of \( w \in \hat{W} \). Therefore, their minimal element corresponding to \( I \) is the inverse of ours, and hence the points of \( Q^\vee \) corresponding to \( \hat{W}_0 \) are also different.

Since \( W = \hat{W}(0) \) is the stabiliser of \( 0 \in V \) w.r.t. the affine-linear action, a finite analogue of the Cellini-Papi bijection is the following. Suppose that a \( \mathbb{Z} \)-grading of \( \Delta \) is 1-standard and \( \Pi(1) = \{ \tilde{\alpha} \} \). Then \( W(0) \) is the stabiliser of the fundamental weight \( \varpi_{\tilde{\alpha}} \) (Section 3) and we need the cardinality of \( W_0 \varpi_{\tilde{\alpha}} \). This subset of the orbit \( W \cdot \varpi_{\tilde{\alpha}} = W_0 \varpi_{\tilde{\alpha}} \) is explicitly described, see Theorem 3.13 and Remark afterwards, but we are unable (yet) to infer from this a way to compute the cardinality.

**Remark 6.1.** There are many other aspects of the affine theory that are not mentioned above. Developing their “finite” counterparts can (and will) be the subject of forthcoming publications.

**Appendix A. A Partition Associated With an Upper Ideal of \( \Delta^+ \)**

Let \( I \) be an upper ideal of the poset \( (\Delta^+, \preceq) \) and \( I^c = \Delta^+ \setminus I \). Define

\[
\lambda_i = \# \{ \gamma \in I^c \mid \text{ht}(\gamma) = i \}.
\]

Our goal is to give a case-free proof of the following observation, see [16, Prop. 3.1].

**Proposition A.1.** The sequence \( (\lambda_1, \lambda_2, \ldots) \) is a partition of the number of roots of \( I^c \). That is, \( \lambda_1 \geq \lambda_2 \geq \ldots \). Moreover, if \( I \neq \Delta^+ \), then \( \lambda_1 > \lambda_2 \).

**Proof.** We use some properties of a principal nilpotent element in the corresponding simple Lie algebra \( g \). Recall that \( g = u \oplus t \oplus u^- \) is a fixed triangular decomposition, \( \Delta^+ \) is the set of \( t \)-roots in \( u \), and \( g_\gamma \) is the roots space corresponding to \( \gamma \in \Delta \). Take \( e = \sum_{\alpha \in \Pi} e_\alpha \), where \( e_\alpha \) is a nonzero element of \( g_\alpha \). After work of Dynkin and Kostant in 1950’s, it is known that \( e \) is a principal nilpotent element of \( g \). Specifically, we need the following properties of the centraliser \( z_g(e) \) of \( e \):

\[
z_g(e) \subseteq u \quad \text{and} \quad \dim z_g(e) = n = \text{rk} g.
\]

The \( t \)-roots in the derived subalgebra \( u' = [u, u] \) are exactly the non-simple positive roots, hence \( u' \) is of codimension \( n \) in \( u \). Combining the above properties, we see that the mapping \( \text{ad}(e) : u \to u' \) is onto. Moreover, both vector spaces are graded:

\[
u = \bigoplus_{i=1}^{h-1} u(i) \quad \text{and} \quad u' = \bigoplus_{i=2}^{h-1} u(i),
\]

where \( u(i) = \bigoplus_{\gamma : \text{ht}(\gamma) = i} g_\gamma \), and \( \text{ad}(e) \) is a homomorphism of degree 1. Let \( c_I = \bigoplus_{\gamma \in I} g_\gamma \) be the \( b \)-stable subspace of \( u \) corresponding to \( I \). The quotient spaces \( u/c_I \) and \( u'/(c_I \cap u') \)
inherit the above grading and the commutative diagram

\[
\begin{array}{ccc}
  u & \xrightarrow{\text{ad}(e)} & u' \\
  \downarrow & & \downarrow \\
  u/\mathfrak{c}_I & \xrightarrow{\text{ad}(e)} & u'/ (\mathfrak{c}_I \cap u')
\end{array}
\]

shows that the map in the bottom row is also graded surjective, of degree 1. Furthermore, let \( \tilde{u}(i) \) be the component of grade \( i \) in \( u/\mathfrak{c}_I \). Then \( u/\mathfrak{c}_I = \bigoplus_{i \geq 1} \tilde{u}(i) \), \( u'/ (\mathfrak{c}_I \cap u') = \bigoplus_{i \geq 2} \tilde{u}(i) \), and \( \dim \tilde{u}(i) = \lambda_i \). Consequently, the graded surjectivity implies that \( \lambda_i \geq \lambda_{i+1} \) for all \( i \).

Finally, if \( \mathfrak{c}_I \neq u \), then \( \tilde{u}(1) \neq 0 \), and the image of \( e \in u(1) \subset u \) in \( \tilde{u}(1) \subset u/\mathfrak{c}_I \) is a \textit{nonzero} element in the kernel of \( \text{ad}(e) \). Hence \( \lambda_1 > \lambda_2 \). \( \square \)

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