Nature of quasiparticle excitations in the fractional quantum Hall effect

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(Dated: November 1, 2018)

We investigate the 1/3 fractional quantum Hall state with one and two quasiparticle excitations. It is shown that the quasiparticle excitations are best described as excited composite fermions occupying higher composite-fermion quasi-Landau levels. In particular, the composite-fermion wave function for a single quasiparticle has 15% lower energy than the trial wave function suggested by Laughlin, and for two quasiparticles, the composite fermion theory also gives new qualitative structures.

PACS numbers: 71.10.Pm, 73.43.-f

I. INTRODUCTION

Interacting electrons in two dimensions form a collective quantum fluid under the application of an intense magnetic field that exhibits the remarkable phenomenon of the fractional quantum Hall effect (FQHE). An early trial wave function proposed by Laughlin for the ground state at the filling factor \( \nu = 1/m \), \( m \) odd, turned out to work well. The composite-fermion (CF) theory applies to a broader range of phenomena, while also providing a new interpretation for the physics of the \( \nu = 1/m \) state, as a state of composite fermions at an effective filling of \( \nu^* = 1 \). While the wave function for the \( \nu = 1/m \) ground state from the CF theory is the same as that in Ref. 3, the wave functions for the excitations are different, which gives an opportunity to test the validity of the CF theory at \( \nu = 1/m \) itself.

We note that when speaking of “quasiparticles” in this paper, we really will mean “quasiparticle excitations” of an incompressible FQHE state. In a broader sense, the quasiparticles, namely the objects that are weakly interacting, are known to be composite fermions; many properties of the FQHE state and various related phenomena can be understood in terms of almost free composite fermions. We will see that the quasiparticle excitations are also nothing but the familiar composite fermions.

Different candidate theories for the charged quasiparticle excitations of the incompressible quantum Hall states have been studied in the past.\(^{3,4,5,6,7,8}\) Two competing views are as follows. In Ref. 3, the quasiparticle was thought of as a “vortex,” based on which a wave function was proposed. In Ref. 4, the quasiparticle was viewed as an excited composite fermion, which produces a different wave function. Many studies\(^{5,6,7,8}\) have compared the two wave functions, and, irrespective of the geometry and the type of interaction employed in these studies, the CF wave function has been found to be superior.

Our objective in this work is to compare the two theories for systems containing more than one quasiparticle. A qualitative difference between the two approaches has been noted in recent microscopic studies of multi-quasiparticle states\(^{9,10}\) in the context of fractional statistics; the statistics parameter, which is a statement regarding the winding properties of quasiparticles around one another, was found to possess a definite value only for the CF wave functions. Our present study finds qualitative differences as well as substantial quantitative deviations between the two approaches.

II. SINGLE QUASIPARTICLE

We will concentrate below on \( \nu = 1/3 \), and begin, with one quasiparticle in the disc geometry. We will assume Coulomb interaction and investigate systems with up to 160 electrons.

The trick of piercing a flux quantum adiabatically through the system motivates the following wave functions for the quasiparticle:

\[
\Psi_{L}^{qp} = e^{-\sum_{j}|z_{j}|^{2}/4} \prod_{l} \left( 2 \frac{\partial}{\partial z_{l}} \right) \prod_{j<k} (z_{j} - z_{k})^{3}, \tag{1}
\]

where the position of the \( j \)th electron is denoted by \( z_{j} \equiv x_{j} - iy_{j} \), and the length is measured in units of the magnetic length \( \ell \equiv \sqrt{\hbar c/eB} \). This wave function describes a quasiparticle located at the origin.

In the CF theory, the wave functions for the ground as well as excited states are constructed by analogy to the corresponding wave functions at an effective filling factor, which is \( \nu^* = 1 \) in the case of \( \nu = 1/3 \). These are shown schematically in Fig. 1. The ground state for \( \nu = 1/3 \) takes the form:

\[
\Psi_{GS}^{\nu} = \prod_{j<k} (z_{j} - z_{k})^{2} \Phi_{1}, \tag{2}
\]

where \( \Phi_{1} \) represents the incompressible integral quantum Hall ground state at filling factor unity, and \( \mathcal{P} \) is an operator that projects the state onto the lowest Landau level (LL). This wave function is identical to the ground state trial wave function in Ref. 3:

\[
\prod_{j<k} (z_{j} - z_{k})^{3} e^{-\sum_{j}|z_{j}|^{2}/4}. \tag{3}
\]

Further extending the analogy, the quasiparticle is analogous to the integral quantum Hall state in which the
ψ_{1}^{L} is different from \( \psi_{1}^{L} \). The lowest LL projected form of this wave function is fairly complicated and would have been difficult to guess without guidance from the CF theory. The presence of factors of the type \((z_j - z_i)\) in the denominator is not a problem because they are canceled by similar factors in the numerator.

In order to investigate which wave function gives a better description of the quasiparticle, we compute the expectation values of the Coulomb energy for each state:

\[
E_{1}^{L} = \frac{\langle \Psi_{1}^{L} | V | \Psi_{1}^{L} \rangle}{\langle \Psi_{1}^{L} | \Psi_{1}^{L} \rangle},
\]

(6)

\[
E_{1}^{CF} = \frac{\langle \Psi_{1}^{L} | V | \Psi_{1}^{L} \rangle}{\langle \Psi_{1}^{L} | \Psi_{1}^{L} \rangle},
\]

(7)

with the total Coulomb energy

\[
V = \sum_{j<k} \frac{1}{|z_j - z_k|}.
\]

(Energies will be measured in units of \( e^2/\ell \).) Since both trial wave functions are in the lowest LL, and the ground state is the same, the difference

\[
\Delta E_{1}^{L} = E_{1}^{L} - E_{1}^{CF}
\]

(9)

gives the energy difference between the quasiparticle energies predicted by the two theories.

\( \psi_{1}^{L} \) has many derivatives with respect to the coordinates, the number of which increases with the number of electrons; this does not allow a direct evaluation of \( E_{1}^{L} \) via Monte Carlo sampling according to \( |\psi_{1}^{L}|^2 \). However, an evaluation of \( E_{1}^{L} \) in Eq. (6) is possible by a method similar to that used in Ref. 2. The partial integration of all derivatives reduces the denominator and the numerator, respectively, to the forms

\[
\langle \Psi_{1}^{L} | \Psi_{1}^{L} \rangle = \int \prod_{k} d z_k |\psi^{GS}|^2 \prod_{k} (|z_k|^2 - 2),
\]

(10)

\[
\langle \Psi_{1}^{L} | V | \Psi_{1}^{L} \rangle = \int \prod_{k} d z_k d z_k \tilde{V} |\psi^{GS}|^2 \prod_{k} (|z_k|^2 - 2),
\]

(11)

with

\[
\tilde{V} = \sum_{j<k} \left[ \frac{1}{|z_j - z_k|} + \frac{|z_j - z_k|^5(|z_j|^2 - 2)(|z_k|^2 - 2)}{3} \right] \times \left( 9 + 2|z_j - z_k|^2 - 3|z_j - z_k|^4 
- 2(|z_j - z_k|^2 + 3)\text{Re}[z_j z_k (z_j^* z_k^*)] \right).
\]

(12)

Defining \( S \equiv \text{sgn}[\prod_k (|z_k|^2 - 2)] \), we can evaluate \( E_{1}^{L} \) by the expression

\[
E_{1}^{L} = \frac{\langle S \tilde{V} \rangle}{\langle S \rangle},
\]

(13)

where \( \langle \ldots \rangle \) denotes the Monte Carlo average with the weighting function \( |\psi^{GS}|^2 \prod_k (|z_k|^2 - 2) \). As a test, for

FIG. 1: Schematic depiction of (a) the ground state, and (b) the single quasiparticle excitation at \( \nu = 1/3 \) in terms of composite fermions. The composite fermions are shown as electrons with two arrows, where the arrows represent the vortices bound to composite fermions. The labels \( n \) and \( l \) denote the CF quasi-Landau level index and the angular momentum of the composite fermion, respectively.
Landau levels, as shown schematically in Fig. 3(a). We work employed a model with short range interaction potential.

The reason for the discrepancy is unclear. We cannot reach the conclusion that the CF wave function has lower energy. It is of interest to note that the energy of \( \Psi_{\text{QP}} \) is higher than that of \( \Psi_{\text{CF}} \) by approximately 15% of the quasiparticle energy (\( \approx 0.07 \)). (We note that an earlier calculation on the sphere obtained a difference of \( \approx 0.005 \) between the two quasiparticle wave functions; the reason for the discrepancy is unclear. We cannot compare our results directly with those in Ref. 6 as that work employed a model with short range interaction potential.)

### III. TWO QUASIPARTICLES

Next we consider states with two quasiparticles. The CF theory provides the following wave function for two quasiparticles at the origin:

\[
\Psi_{\text{CF}}^{[N-2,2]} = \mathcal{P} \prod_{j<k} (z_j - z_k)^2 
\begin{array}{cccc}
  z_1^* & z_2^* & \cdots \\
  z_1^* z_1 & z_2^* z_2 & \cdots \\
  z_1 & z_2 & \cdots \\
  \vdots & \vdots & \cdots \\
  z_1^{N-3} & z_2^{N-3} & \cdots 
\end{array} 
\times \exp \left[ -\frac{1}{4} \sum_j |z_j|^2 \right].
\]

Here the superscript \( [N-2,2] \) represents the occupation number in each CF quasi-Landau level, namely, \( N-2 \) electrons for \( n = 0 \) and 2 electrons for \( n = 1 \) CF quasi-Landau levels, as shown schematically in Fig. 3(a). We will not show the explicit lowest LL projected form here, which is significantly more complicated than that for a single CF quasiparticle.

A generalization of Eq. (11) to two quasiparticles is also straightforward:

\[
\Psi_{\text{QP}}^{[2]} = e^{-\sum_j |z_j|^2/4} \prod_l \left( \frac{\partial}{\partial z_l} \left( 2 \frac{\partial}{\partial z_l} \right) \prod_{j<k} (z_j - z_k)^3 \right) \]

which is identical to the wave function considered in Ref. 6 for two quasiparticles at the origin.

As for one quasiparticle, we have computed the difference

\[
\Delta E^{2\text{QP}} = E^{2\text{QP}} - E^{[N-2,2]},
\]

where

\[
E^{2\text{QP}} = \left\langle \Psi_{\text{QP}}^{2\text{QP}} \right| V \left| \Psi_{\text{QP}}^{2\text{QP}} \right\rangle / \left\langle \Psi_{\text{QP}}^{2\text{QP}} \right| \Psi_{\text{QP}}^{2\text{QP}} \rangle,
\]

\[
E^{[N-2,2]} = \left\langle \Psi_{\text{CF}}^{[N-2,2]} \right| V \left| \Psi_{\text{CF}}^{[N-2,2]} \right\rangle / \left\langle \Psi_{\text{CF}}^{[N-2,2]} \right| \Psi_{\text{CF}}^{[N-2,2]} \rangle.
\]

For composite fermions, the energy can be calculated using the methods described previously for fairly large
systems. The evaluation of the Coulomb energy for \( \Psi_{L}^{2qp} \) involves similar difficulty as the energy of a single quasiparticle, caused by the presence of many derivatives. What makes matters worse here is that \( \Psi_{L}^{2qp} \) has twice as many derivatives as \( \Psi_{L}^{1qp} \), producing more singular terms in \( \tilde{V} \). Consequently, the convergence of the integration becomes poor and the method employed in the preceding section turns out not to be a clever way to treat \( \Psi_{L}^{2qp} \).

As an alternative method, we have expanded \( \Psi_{L}^{2qp} \) in the series

\[
\Psi_{L}^{2qp} = \sum_{(l_{j}),\{m_{j}\},\{n_{j}\}} C(\{l_{j}\},\{m_{j}\},\{n_{j}\})D(l_{1},\ldots,l_{N})
\times D(m_{1},\ldots,m_{N})D(n_{1},\ldots,n_{N})
\times \exp \left( -\frac{1}{4} \sum_{j} |z_{j}|^2 \right),
\]

where

\[
D(n_{1},\ldots,n_{N}) = \left| \begin{array}{cccc}
\left( \frac{\partial}{\partial z_{1}} \right)^{n_{1}} & 1 & \left( \frac{\partial}{\partial z_{2}} \right)^{n_{2}} & 1 \\
\left( \frac{\partial}{\partial z_{1}} \right)^{n_{1}} & z_{1} & \left( \frac{\partial}{\partial z_{2}} \right)^{n_{2}} & z_{2} \\
\vdots & \vdots & \vdots & \vdots \\
\left( \frac{\partial}{\partial z_{1}} \right)^{n_{1}} & z_{1}^{N-1} & \left( \frac{\partial}{\partial z_{2}} \right)^{n_{2}} & z_{2}^{N-1} \\
\end{array} \right|
\]

and the primed summation runs under the restriction: \( l_{j},m_{j},n_{j} \geq 0 \) and \( l_{j} + m_{j} + n_{j} = 2 \) for all \( j \). The expansion in Eq. (19) is possible because the polynomial part in \( \Psi_{GS}^{L} \) is the cube of a determinant. It has an advantage over the previous method; we can obtain the analytic form of \( \Psi_{L}^{2qp} \) without the explicit derivatives, which allows a direct evaluation of the wave function itself. It is also seen that the determinant in Eq. (20) vanishes when all of \( n_{i} \)'s are nonzero. For \( N = 3 \) the numerical calculation using the wave function in Eq. (19) produced \( E_{L}^{2qp} = 1.2047 \pm 0.0001 \), which compares well with the exact value (87/128)\( \sqrt{\pi} \approx 1.204715 \).

Since the number of non-vanishing terms in Eq. (19) increases rapidly, the integer coefficients \( C(\{l_{j}\},\{m_{j}\},\{n_{j}\}) \) were obtained numerically up to \( N = 9 \). We plot \( \Delta E^{2qp} \) as a function of \( N \) in Fig. (a). It is observed that \( \Psi_{L}^{2qp} \) has a much higher energy than \( \Psi_{CF}^{[N-2,2]} \) even for a small number of electrons, unlike for a single quasiparticle. This discrepancy increases as the system grows. It can be observed that the thermodynamic limit for the energy difference is \( \approx 0.16 \), which is surprisingly big, even larger than the gap to creating a quasiparticle quasihole pair out of the ground state (which is roughly 0.1). It far exceeds the naive estimate \( 2\Delta E^{1qp} \approx 0.022 \).

A qualitative difference between the two approaches appears at the level of two quasiparticles. The two wave functions considered above have different total angular momenta:

\[
M_{CF}^{[N-2,2]} = \frac{1}{2}(3N^2 - 7N + 4),
\]
\[
M_{L}^{2qp} = \frac{1}{2}(3N^2 - 7N).
\]

(The largest occupied single electron orbital has the same angular momentum for the two states, though.) There is no obvious way of constructing a generalization of \( \Psi_{L}^{1qp} \) with an angular momentum \( \Psi_{CF}^{[N-2,2]} \). However, it turns out that the CF theory contains an excited state for two quasiparticles with angular momentum \( M_{L}^{2qp} \):

\[
\Psi_{CF}^{[N-2,1,1]} = \mathcal{P} \prod_{j<k} (z_{j} - z_{k})^2 \left( \begin{array}{c}
(z_{1}^*)^2 (z_{2}^*)^2 \\
z_{1}^* z_{2}^* \\
1 1 \\
z_{1} z_{2} \\
\vdots \vdots \\
z_{1}^{N-3} z_{2}^{N-3} \\
\end{array} \right) \times \exp \left( -\frac{1}{4} \sum_{j} |z_{j}|^2 \right). \tag{23}
\]

This wave function contains one excited composite fermion in the second CF quasi-Landau level but the other in the third CF quasi-Landau level, as indicated by the notation \( [N-2,1,1] \). See Fig. (b).
FIG. 5: Density profiles of $\Psi_{2q}^{L}$, $\Psi_{CF}^{[N-2,2]}$, and $\Psi_{CF}^{[N-2,1,1]}$ for (a) $N = 6$, and (b) $N = 7$.

The energy of $\Psi_{CF}^{[N-2,1,1]}$ is expected to be higher than that of $\Psi_{CF}^{[N-2,2]}$, since the former has larger "kinetic energy" for the composite fermions, which is of the order of the $1/3$ gap. We have also computed $\Delta E^{2q} = E_{2q}^{L} - E_{CF}^{[N-2,1,1]}$ with $E_{CF}^{[N-2,1,1]} \equiv \langle \Psi_{CF}^{[N-2,1,1]} | V | \Psi_{CF}^{[N-2,1,1]} \rangle / \langle \Psi_{CF}^{[N-2,1,1]} | \Psi_{CF}^{[N-2,1,1]} \rangle$ as a function of the system size. As shown in Fig. (b), the energy difference is reduced compared with $\Delta E^{2q}$. This suggests that $\Psi_{2q}^{L}$ actually resembles the CF $[N-2,1,1]$ state. That is also verified in the density plot of three states, which is shown in Fig. 5.

It is obvious that as more and more quasiparticles are created by repeated application of $\prod_l (2 \partial / \partial z_l)$, the resulting state will contain, in the CF interpretation, composite fermions in higher and higher CF quasi-Landau levels; this state will have much higher energy than the state with all CF quasiparticles in the second CF quasi-Landau level.

IV. CONCLUSION

Our study confirms the description of quasiparticles as composite fermions in an excited composite-fermion quasi-Landau level. This physics not only gives the best available microscopic wave functions for the quasiparticles, but also brings out new qualitative structures for multi-quasiparticle states. While we have concentrated on the vicinity of $\nu = 1/3$ above, it should be noted that the CF theory explains the general phenomenon of the FQHE, giving accurate wave functions for the ground states, quasiparticles, quasiholes, and states containing many quasiparticles and quasiholes at arbitrary filling factors.

Acknowledgments

Partial support by the National Science Foundation under grant no. DMR-0240458 is gratefully acknowledged.

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