QMA-hardness of Consistency of Local Density Matrices with Applications to Quantum Zero-Knowledge
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Abstract

We provide several advances to the understanding of the class of Quantum Merlin-Arthur proof systems (QMA), the quantum analogue of NP. Our central contribution is proving a longstanding conjecture that the Consistency of Local Density Matrices (CLDM) problem is QMA-hard under Karp reductions. The input of CLDM consists of local reduced density matrices on sets of at most \(k\) qubits, and the problem asks if there is an \(n\)-qubit global quantum state that is locally consistent with all of the \(k\)-qubit local density matrices. The containment of this problem in QMA and the QMA-hardness under Turing reductions were proved by Liu [APPROX-RANDOM 2006]. Liu also conjectured that CLDM is QMA-hard under Karp reductions, which is desirable for applications, and we finally prove this conjecture. We establish this result using the techniques of simulatable codes of Grilo, Slofstra, and Yuen [FOCS 2019], simplifying their proofs and tailoring them to the context of QMA.

In order to develop applications of CLDM, we propose a framework that we call locally simulatable proofs for QMA: this provides QMA proofs that can be efficiently verified by probing only \(k\) qubits and, furthermore, the reduced density matrix of any \(k\)-qubit subsystem of an accepting witness can be computed in polynomial time, independently of the witness. Within this framework, we show several advances in zero-knowledge in the quantum setting. We show for the first time a commit-and-open computational zero-knowledge proof system for all of QMA, as a quantum analogue of a “sigma” protocol. We then define a Proof of Quantum Knowledge, which guarantees that a prover is effectively in possession of a quantum witness in an interactive proof, and show that our zero-knowledge proof system satisfies this definition. Finally, we show that our proof system can be used to establish that QMA has a quantum non-interactive zero-knowledge proof system in the secret parameter setting.

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1 Introduction

The complexity class QMA is the quantum analogue of NP, the class of problems whose solutions can be verified in deterministic polynomial time. More precisely, in QMA, an all-powerful prover produces a quantum proof that is verified by a quantum polynomially-bounded verifier. Given the probabilistic nature of quantum computation, we require that for true statements, there exists a quantum proof that makes the verifier accept with high probability (this is called completeness), whereas all “proofs” for false statements are rejected with high probability (which is called soundness).

The class QMA was first defined by Kitaev [KSV02], who also showed that deciding if a $k$-local Hamiltonian problem has low-energy states is QMA-complete. The importance of this result is two-fold: first, from a theoretical computer science perspective, it is the quantum analogue of the Cook-Levin theorem, since it establishes the first non-trivial QMA-complete problem. Second, it shows deep links between physics and complexity theory, since the $k$-local Hamiltonian problem is an important problem in many-body physics. Thus, a better understanding of QMA would lead to a better understanding of the power of quantum resources in proof verification, as well as the role of quantum entanglement in low-energy states.

Follow-up work strengthened our understanding of this important complexity class, e.g., by showing that QMA is contained in the complexity class PP [KW00]; that it is possible to reduce completeness and soundness errors without increasing the length of the witness [MW05]; understanding the difference between quantum and classical proofs [AK07, GKS16, FK18]; the possibility of perfect completeness [Aar09]; and, more recently, the relation of QMA with non-local games [NV17, NV18, CGJV19].

Also, much follow-up work focused on understanding the complete problems for QMA, mostly by improving the parameters of the QMA-hard Local Hamiltonian problem, or making it closer to models more physically relevant [KR03, Liu06, KKR06, OT08, CM14, HNN13, BC18]. In 2014, a survey of QMA-complete languages [ Boo14] contained a list of 21 general problems that are known to be QMA-complete$^2$, and since then, the situation has not drastically changed. This contrasts with the development of NP, where only a few years after the developments surrounding 3–SAT, Karp published a theory of reducibility, including a list of 21 NP-complete problems [Kar72]; while 7 years later, a celebrated book by Garey and Johnson surveyed over 300 NP-complete problems [GJ90].$^3$

Recently, the role of QMA in quantum cryptography has also been explored. For instance, several results used ideas of the QMA-completeness of the Local Hamiltonian problem in order to perform verifiable delegation of quantum computation [FHM18, Mah18, Gri19]. Furthermore, another line of work studies zero-knowledge protocols for QMA [BJSW16, BJSW20, VZ20]; which is extremely relevant, given the fundamental importance of zero-knowledge protocols for NP in cryptography.

Despite the multiple advances in our understanding of QMA and related techniques, a number of fundamental open questions remain. In this work, we solve some of these open problems by showing: (i) QMA-hardness of the Consistency of Local Density Matrix (CLDM) problem under Karp reductions; (ii) “commit-and-open” Zero-Knowledge (ZK) proof of quantum knowledge (PoQ) protocols for QMA; and (iii) a non-interactive zero-knowledge (NIZK) protocol in the secret parameter scenario.

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$^1$PP is the complexity class of decision problems that can be solved by probabilistic polynomial-time algorithms with error strictly smaller than $\frac{1}{2}$.

$^2$We remark that these problems can be clustered as variations of a handful of base problems.

$^3$The first edition of Garey and Johnson [GJ90] was published in 1979.
algorithm whose history state\(^4\) is \textit{locally simulatable}, meaning that the reduced density matrices on any small set of qubits is efficiently computable (without knowledge of the quantum witness). In order to be able to explain our results in more details and appreciate their contribution to a better understanding of QMA, we first give an overview of these areas and how they relate to these particular problems.

1.1 Background

In this section, we discuss the background on the topics that are relevant to this work, summarizing their current state-of-the-art.

\textbf{Consistency of Local Density Matrices (CLDM).} The Consistency of Local Density matrices problem (CLDM) is as follows: given the classical description of local density matrices \(\rho_1, \ldots, \rho_m\), each on a set of at most \(k\) qubits and for a global system of \(n\) qubits, is there a state \(\tau\) that is consistent with such reduced states? Liu [Liu06] showed that this problem is in QMA and that it is QMA-hard under Turing reductions, i.e., a deterministic polynomial time algorithm with access to an oracle that solves CLDM in unit time can solve any problem in QMA.

We remark that this type of reduction is rather troublesome for QMA, since the class is not known (nor expected) to be closed under complement, i.e., it is widely believed that QMA \(\neq\) coQMA. If this is indeed the case, then Turing reductions do not allow a black-box generalization of results regarding the CLDM problem to all problems in QMA. This highlights the open problem of establishing the QMA-hardness of the CLDM problem under Karp reductions, i.e., to show an efficient mapping between yes- and no-instances of any QMA problem to yes- and no-instances of CLDM, respectively.

\textbf{Zero-Knowledge (ZK) Proofs for QMA.} In an interactive proof, a limited party, the verifier, receives the help of some untrusted powerful party, the prover, in order to decide if some statement is true. This is a generalization of a proof, where we allow multiple rounds of interaction. As usual, we require that the completeness and soundness properties hold. For cryptographic applications, the zero-knowledge (ZK) property is often desirable: here, we require that the verifier learn nothing from the interaction with the prover. This property is formalized by showing the existence of an efficient simulator, which is able to reproduce (i.e., simulate) the output of any given verifier on a yes instance (without having direct access to the actual prover or witness)\(^5\).

As paradoxical as it sounds, statistical zero-knowledge interactive proofs are known to be possible for a host of languages, including the Quadratic Non-Residuosity, Graph Isomorphism, and Graph Non-Isomorphism problems [GMW91, GMR89]; furthermore, all languages that can be proven by multiple provers (MIP) admits perfect zero-knowledge MIPs [BGKW88]. What is more, by introducing computational assumptions, it was shown that all languages that admit an interactive proof system also admit a zero-knowledge interactive proof system [BOGG+88]. Zero-knowledge interactive proof systems have had a profound impact in multiple areas, including cryptography [GMW87] and complexity theory [Vad07].

We now briefly review the zero-knowledge interactive proof system for the NP-complete problem of Graph 3-colouring (3-COL). This is a 3-message proof system, and has the additional property that,

\footnote{\textsuperscript{4}See Equation (1).}

\footnote{\textsuperscript{5}Different definitions of “reproduce” result in different definitions of zero-knowledge protocols. A protocol is \textit{perfect zero-knowledge} if the distribution of the output of the simulator is exactly the same as the distribution of output of transcripts of the protocol. A protocol is \textit{statistical zero-knowledge} if such distributions are statistically close. Finally, a protocol is \textit{computational zero-knowledge} if no efficient algorithm can distinguish both distributions. The convention is that in the absence of such specification, we are considering the case of computational zero-knowledge.}
given a witness, the prover is efficient. As a first message, the prover commits to a permutation of the given 3-colouring (meaning that the prover randomly permutes the colours to obtain colouring \( c \), and produces a list \( \langle v_i, \text{commit}(c(v_i)) \rangle \), using a cryptographic primitive commit which is a commitment scheme). In the second message, the verifier chooses uniformly at random an edge \( \{v_i, v_j\} \) of the graph. The prover responds with the information that allows the verifier to open the commitments to the colouring of the vertices of this edge (and nothing more). The verifier accepts if and only if the revealed colours are different. It is easy to see that the protocol is complete and sound. For the zero-knowledge property, the simulator consists in a process that guesses which edge will be requested by the verifier and commits to a colouring that satisfies the prover in case this guess is correct. If the guess is incorrect, the technique of rewinding allows the simulator to re-initialize the interaction until it is eventually successful. Protocols that follow the commit-challenge-response structure of this proof system are called \( \Sigma \)-protocols\(^6\) and, due to their simplicity, they play a very important role, for instance in the celebrated Fiat-Shamir transformation [FS87].

The foundations of zero-knowledge in the quantum world were established by Watrous, who showed a technique called quantum rewinding [Wat09] which is used to show the security of some classical zero-knowledge proofs (including the protocol for 3-COL described above), even against quantum adversaries. The importance of this technique is that quantum measurements typically disturb the measured state. When we consider quantum adversaries, such difficulties concern even classical proof systems, due to the rewinding technique that is ubiquitous (see example in the case of 3-COL above). Indeed, in the quantum setting, intermediate measurements (such as checking if the guess is correct) may compromise the success of future executions, since it is not possible a priori to “rewind” to a previous point in the execution in a black-box way.

Another dimension where quantum information poses new challenges is in the study of interactive proof systems for quantum languages. We point out that Liu [Liu06] observed very early on that the CLDM problem should admit a simple zero-knowledge proof system following the “commit-and-open” approach, as in the 3-COL protocol. Inspired by this observation, recent progress has established the existence of zero-knowledge protocols for all of QMA [BJSW16, BJSW20]. We note that although the proof system used there is reminiscent of a \( \Sigma \)-protocol, there are a number of reasons why it is not a “natural” quantum analogue of a \( \Sigma \) protocol. These include: (i) the use of a coin-flipping protocol, which makes the communication cost higher than 3 messages; (ii) the fact that the verifier’s message is not a random challenge; and (iii) the final answer from the prover is not only the opening of some committed values.

Recently, Vidick and Zhang [VZ20] showed how to make classical all of the interaction between the verifier and the prover in [BJSW16, BJSW20], by considering argument systems\(^7\) instead of proof systems. In their protocol, they compose the result of Mahadev [Mah18] for verifiable delegation of quantum computation by classical clients with the zero-knowledge protocol of [BJSW16, BJSW20].

**Zero-Knowledge Proofs of Knowledge (PoK)**. In a zero-knowledge proof, the verifier becomes convinced of the existence of a witness, but this a priori has no bearing on the prover actually having in her possession such a witness. In some circumstances, it is important to guarantee that the prover actually has a witness. This is the realm of a zero-knowledge proof of knowledge (PoK) [GMR89, BG93].

\(^6\)The Greek letter \( \Sigma \) visualizes the flow of the protocol.

\(^7\)Argument systems are a relaxation of proof systems where the prover is also bounded to polynomial-time computation, and, for positive instances, the prover is provided a witness to the NP instance. This model allows much more efficient protocols which enables it to be used in practice [BCSH14, PHGR16, BSCR19].
We give an example to depict this subtlety. Let us consider the task of anonymous credentials [Cha83]. In this setting, Alice wants to authenticate into some online service using her private credentials. In order to protect her credentials, she could engage in a zero-knowledge proof; this, however would be unsatisfactory, since the verifier in this scenario would become convinced of the existence of accepting credentials, which does not necessarily translate to Alice actually being in the possession of these credentials. To remedy this situation, the PoK property establishes an “if-and-only-if” situation: if the verifier accepts, then we can guarantee that the prover actually knows a witness. This notion is formally defined by requiring the existence of an extractor, which is polynomial-time process that outputs a valid witness when given oracle access to some prover \( P^* \) that makes the verifier accept with high enough probability.

In the quantum case, there has been some positive results in terms of the security of classical proofs of knowledge for NP against quantum adversaries [Unr12]. However, in the fully quantum case (that is, proofs of quantum knowledge for QMA), no scheme has been proposed. One of the possible reasons why no such proof of quantum knowledge protocols was proposed is the lack of a simple zero-knowledge proof for QMA.

Non-Interactive Zero-Knowledge Proofs (NIZK). The interactive nature of zero knowledge proof systems (for instance, in \( \Sigma \)-protocols) means that in some situations they are not applicable since they require the parties to be simultaneously online. Therefore, another desired property of such proof systems is that they are non-interactive, which means the whole protocol consists in a single message from the prover to the verifier. Non-interactive zero-knowledge proofs (NIZK) is a fundamental construction in modern cryptography and has far-reaching applications, for instance to cryptocurrencies [BSCG+14].

We note that NIZK is known to be impossible in the standard model [GO94], i.e., without extra assumptions, and therefore NIZK has been considered in different models. In one of the models most relevant in cryptography, we assume a common reference string (CRS) [BFM88], which can be seen as a trusted party sending a random string to both the prover and the verifier. In another model, the trusted party is allowed to send different (but correlated) messages to the prover and the verifier; this is called the secret parameter setup [PS05]. Classically, this model has been shown to be very powerful, since even its statistical zero-knowledge version is equivalent to all of the problems in the complexity class AM (this is the class that contains problem that can be verified by public-coin polynomial-time verifiers). As mentioned in [PS05], this model encompasses another model for NIZK where the prover and the verifier perform an offline pre-processing phase (which is independent of the input) and then the prover provides the ZK proof [KMO89]. This inclusion holds since the parties could perform secure multi-party computation to compute the trusted party’s operations.

In the quantum case, very little is known on non-interactive zero-knowledge. Chailloux, Ciocan, Kerenidis and Vadhan studied this problem in a setup where the message provided by the trusted party can depend on the instance of the problem [CCKV08]. Recently, some results also showed that the Fiat-Shamir transformation for classical protocols is still safe in the quantum setting, in the quantum random oracle model [LZ19, DFMS19, Cha19]. One particular and intriguing open question is the possibility of NIZKs for QMA.
1.2 Results

As we have shown so far, the state-of-the-art in the study of QMA is that the body of knowledge is still developing, and that there are some specific goals that, if achieved, would help us better understand QMA and devise new protocols for quantum cryptography. Given this context, we present now our results in more detail.

Our first result (Section 3) is to show that the CLDM problem is QMA-hard under Karp reductions, solving the 14-year-old problem proposed by Liu [Liu06].

**Main Result.** The CLDM problem is QMA-complete under Karp reductions.

We capture the techniques used in establishing the above into a new characterization of QMA that provides the best-of-both worlds in terms of two proof systems for QMA in an abstract way: we define SimQMA as the complexity class with proof systems that are (i) locally verifiable (as in the Local Hamiltonian problem), and (ii) every reduced density matrix of the witness can be efficiently computed (as in the CLDM problem). This results is the basis for our applications to quantum cryptography:

**Application 1.** SimQMA = QMA.

Next, we define a quantum notion of a classical Σ-protocol, which we call a Ξ-protocol (please note, both a Σ and Ξ protocol is also referred to throughout as “commit-and-open” protocols.) Using our characterization given in Application 1, we show a QMA-complete language that admits a Ξ-protocol. Taking into account the importance of Σ protocols for zero-knowledge proofs, we are able to show (Section 5) a quantum analogue of the celebrated [GMW91] paper:

**Application 2.** All problems in QMA admit a computational zero-knowledge Ξ-proof system.

Then we provide the definition of Proof of Quantum Knowledge (PoQ). In short, we say that a proof system is a PoQ if there exists a quantum polynomial-time extractor \( K \) that has oracle access to a quantum prover which makes the verifier accept with high enough probability, and the extractor is able to output a sufficiently good witness for a “QMA-relation”. We note that this definition for a PoQ is not a straightforward adaptation of the classical definition; this is because NP has many properties such as perfect completeness, perfect soundness and even that proofs can be copied, that are not expected to hold in the QMA case. More details are given in Section 6. We are then able to show that our Ξ protocol for QMA described in Result 2 is PoQ. This is the first proof of knowledge for QMA.

**Application 3.** All problems in QMA admit a zero-knowledge proof of quantum knowledge proof system and a statistical zero-knowledge proof of quantum knowledge argument system.

We remark that using techniques for post-hoc delegation of quantum computation [FHM18], our PoQ for QMA may be understood as a proof-of-work for quantum computations, since it could be used to convince a verifier that the prover has indeed created the history state of some pre-defined computation. This is very relevant in the scenario of testing small-scale quantum computers in

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8Besides being an excellent symbolic reminder of the interaction in a 3-message proof system, Ξ is chosen as a shorthand for what we might otherwise call a \( q \Sigma \) protocol, due to the resemblance with the pronunciation as “\( c \)sigma”.

9This definition is joint work with Coladangelo, Vidick and Zhang [CVZ20].

10See also independent and concurrent work by Coladangelo, Vidick and Zhang [CVZ20].
the most adversarial model possible: the zero-knowledge property ensures that the verifier learns nothing but the truth of the statement, while the PoQ property means that the prover has indeed prepared a ground state with the given properties. Comparatively, all currently known protocols either make assumptions on the devices, or certify only the answer of the computation, but not the knowledge of the prover.

Finally, using the techniques of Application 2, we show that every problem in QMA has a non-interactive statistical zero-knowledge proof in the secret parameter model. We are even able to strengthen our result to the complexity class QAM (recall that in a QAM proof system, the verifier first sends a random string to the prover, who answers with a quantum proof). Note that QAM trivially contains QMA.

Application 4. All problems in QAM have a non-interactive statistical zero-knowledge protocol in the secret parameter model.

Note that, as in the classical case [PS05], our result also implies a QNIZK protocol where the prover and the verifier run an offline (classical) pre-processing phase (independent of the witness) and then the prover sends the quantum ZK proof to the verifier. We note also that even though these models are less relevant to the cryptographic applications of NIZK, we think that our result moves us towards a QNIZK protocol for QMA in a more standard model.

1.3 Techniques

The starting point for our results are locally simulatable codes, as defined in [GSY19]. We give now a rough intuition on the properties of such codes and leave the details to Section 4.

First, a quantum error correcting code is \( s \)-simulatable if there exists an efficient classical algorithm that outputs the reduced density matrices of codewords on every subset of at most \( s \) qubits. Importantly, this algorithm is oblivious of the logical state that is encoded. We note that it was already known that the reduced density matrices of codewords hide the encoded information, since quantum error correcting codes can be used in secret sharing protocols [CGL99], and in [GSY19] they show that there exist codes such that the classical description of the reduced density matrices of the codewords can be efficiently computed. Next, [GSY19] extends the notion of simulatability of logical operations on encoded data as follows. Recalling the theory of fault-tolerant quantum computation, according to which some quantum error-correcting codes allow computations over encoded data by using “transversal” gates and encoded magic states. The definition of \( s \)-simulatability is extended to require that the simulator also efficiently computes the reduced density matrix on at most \( s \) qubits of intermediate steps of the physical operations that implement a logical gate on the encoded data (again, by transversal gates and magic states).

Example 1. Let us suppose that the encoding map \( \text{Enc} \) admits transversal application of the one-qubit gate \( G \), i.e., \( G^{\otimes N}\text{Enc}(|\psi\rangle) = \text{Enc}(G|\psi\rangle) \). The simulatability property requires that the density matrices on at most \( s \) qubits of \( (G^{\otimes t} \otimes I^{\otimes (N-t)})\text{Enc}(|\psi\rangle) \) should be efficiently computed, for every \( 0 \leq t \leq N \).

In [GSY19], the authors show that the concatenated Steane code is a locally simulatable code. With this tool, in [GSY19], it is shown that every MIP\(^*\) protocol\(^{11}\) can be made zero-knowledge, thus quantizing the celebrated result of [BGKW88]. Here, we provide an alternative proof for the

\(^{11}\text{MIP}^*\) is the set of languages that admit a classical multi-prover interactive proof, where, in addition, the provers share entanglement
simulatability of concatenated Steane codes. Our new proof is much simpler than the proof provided in [GSY19], but it holds for a slightly weaker statement (but which is already sufficient to derive the results in [GSY19]). Then, for the first time, we apply the techniques of simulatable codes from [GSY19] to QMA, which enables us to solve many open problems as previously described.

In order to explain our approach to achieving our main result, we first recall the quantum Cook-Levin theorem proved by Kitaev [KSV02]. In his proof, Kitaev uses the circuit-to-Hamiltonian construction [Fey82], mapping an arbitrary QMA verification circuit $V = U_T \ldots U_1$ to a local Hamiltonian $H_V$ that enforces that low energy states are history states of the computation, i.e., a superposition of the snapshots of $V$ for every timestep $0 \leq t \leq T$:

$$|\Phi\rangle = \frac{1}{\sqrt{T+1}} \sum_{t=0}^{T+1} |t\rangle \otimes U_t \ldots U_1 |\psi_{\text{init}}\rangle.$$  \(1\)

In the above, the first register is called the clock register, and it encodes the timestep of the computation, while the second register contains the snapshot of the computation at time $t$, i.e., the quantum gates $U_1, \ldots, U_t$ applied to the initial state $|\psi_{\text{init}}\rangle = |\phi\rangle|0\rangle^A$, that consists of the quantum witness and auxiliary qubits. The Hamiltonian $H_V$ also guarantees that $|\psi_{\text{init}}\rangle$ has the correct form at $t = 0$, and that the final step accepts, i.e., the output qubit is close to $|1\rangle$.

In [GSY19], they note that an important obstacle to making a state similar to $|\Phi\rangle^{12}$ locally simulatable is its dependence on the witness state $|\phi\rangle$. The solution is to consider a different verification algorithm $V'$ that implements $V$ on encoded data, much like in the theory of fault-tolerant quantum computing. In more details, for a fixed locally simulatable code, $V'$ expects the encoding of the original witness $\text{Enc}(|\phi\rangle)$ and then, with her raw auxiliary states, she creates encodings of auxiliary states $\text{Enc}(|0\rangle)$ and magic states $\text{Enc}(|\text{MS}\rangle)$, and then performs the computation $V$ through transversal gates and magic state gadgets, and finally decodes the output qubit. This gives rise to a new history state:

$$|\Phi'\rangle = \frac{1}{\sqrt{T'+1}} \sum_{t=0}^{T'+1} |t\rangle \otimes U'_t \ldots U'_1 |\psi'_{\text{init}}\rangle,$$  \(2\)

where $|\psi'_{\text{init}}\rangle = \text{Enc}(|\phi\rangle)|0\rangle^A'$ and $U'_1, \ldots, U'_{T'}$ are the gates of $V'$ described above. Using the techniques from [GSY19],\(^{13}\) we can show that from the properties of the locally simulatable codes, the reduced density matrix on every set of $5$ qubits of $|\Phi'\rangle$ can be efficiently computed. In this work, we prove that these reduced density matrices are in fact QMA-hard instances of CLDM. More concretely, we show that these reduced density matrices of a hypothetical history state of an accepting QMA-verification can always be computed, and there exists a global state (namely the history state) consistent with these reduced density matrices if and only if the original QMA verification accepts with overwhelming probability (and therefore we are in the case of a yes-instance).

Our main result opens up a number of possible applications to cryptographic settings. However, as we discussed in Section 1.2 we face a tradeoff. In CLDM, we have the description of the local density matrices, which yields a zero-knowledge $\Xi$ protocol. On the other hand, the QMA verification

\(^{12}\)In [GSY19], they are simulating history states for MIP$^*$ computation and therefore they need to deal also with arbitrary Provers' operations.

\(^{13}\)We remark that we also need to fix a small bug in their proof. The bug fix deals with technicalities regarding $V'$ and $|\psi'\rangle$ that are beyond the scope of this overview. See Section 4.2.1 and Remark 4.7 for more details.
for CLDM is non-local: we need multiple copies of the global state to perform tomography on the reduced states,\textsuperscript{14} instead of a single copy that is needed in the Local Hamiltonian problem.

In order to combine these two desired properties in a single object, we describe a powerful technique that we call \textit{locally simulatable proofs}. In a locally simulatable proof system for some problem $A = (A_{\text{yes}}, A_{\text{no}})$, we require that: (i) the verification test performed by the verifier acts on at most $k$ out of the $n$ qubits of the proof, and (ii) for every $x \in A_{\text{yes}}$, there exists a locally simulatable witness $|\psi\rangle$, i.e., a state $|\psi\rangle$ that passes all the local tests and such that for every $S \subseteq [n]$ with $|S| \leq k$, it is possible to compute the reduced state of the $|\psi\rangle$ on $S$ efficiently (without the help of the prover). Notice that we have no extra restrictions on $x \in A_{\text{no}}$, since any quantum witness should make this verifier reject with high probability.

We then show that all problems in QMA admit a locally simulatable proof system. In order to achieve this, we use the local tests on the encoded version of the QMA verification algorithm that come from the Local Hamiltonian problem, together with the fact that the history state of such computation is a low-energy state and is simulatable (which is used to establish the QMA-hardness of CLDM).

We remark that a direct classical version of locally simulatable proofs as we define them is impossible. This is because, given the local values of a classical proof, it is always possible to reconstruct the full proof by gluing these pieces together. The fact that this operation is hard to perform quantumly is intrinsically related to entanglement: given the local density matrices, it is not a priori possible to know which parts are entangled in order to glue them together. As discussed in the next section, this allows us to achieve a type of simple zero-knowledge protocol that defies all classical intuition.

### 1.3.1 Locally Simulatable Proofs in Action

We now sketch how each of Application 2–Application 3 is obtained via the lens of locally simulatable proofs.

**Zero Knowledge.** We use the characterization QMA = SimQMA to give a new zero-knowledge proof system for QMA. Our protocol is much simpler than previous results \cite{BJSW16, BJSW20}, and it follows the “commit-challenge-response” structure of a $\Sigma$-protocol. Since our commitment is a quantum state (the challenge and response are classical), we call this type of protocol a “$\Xi$-protocol” (see Section 1.2).

The main idea is to use the quantum one-time pad to split the first message in the protocol into a quantum and a classical part. More concretely, the prover sends $X^a Z^b |\psi\rangle$ and commitments to each bit of $a$ and $b$ to the verifier, where $|\psi\rangle$ is a locally simulatable quantum witness for some instance $x$ and $a$ and $b$ are uniformly random strings. The verifier picks some $c \in [m]$, which corresponds to one of the tests of the simulatable proof system, and asks the prover to open the commitment of the encryption keys to the corresponding qubits. The honest prover opens the commitment corresponding to the one-time pad keys of the qubits involved in test $c$. The verifier then checks if: (i) the openings are correct and, (ii) the decrypted reduced state passes test $c$.

Assuming the existence of unconditionally binding and computationally hiding commitment schemes, we show that our protocol is a computational zero-knowledge proof system for QMA. Completeness and soundness follow trivially, whereas the zero-knowledge property is established by

\textsuperscript{14}See Lemma 3.3.
constructing a simulator that exploits the properties of the locally simulatable proof system and the rewinding technique of Watrous [Wat09].

To the best of our knowledge, this is the first time that quantum techniques are used in zero-knowledge to achieve a commit-and-open protocol that requires no randomization of the witness. Indeed, for reasons already discussed, all classical zero-knowledge $\Sigma$ protocols require a mapping or randomization of the witness (e.g. in the 3-COL protocol, this is the permutation that is applied to the coloring before the commitment is made). We thus conclude that quantum information enables a new level of encryption that is not possible classically: the “juicy” information is present in the global state, whose local parts are fully known [GSY19].

**Proof of Quantum Knowledge for QMA**.

As discussed in Section 1.2, our first challenge here is to define a Proof of Quantum Knowledge (PoQ). We recall that in the classical setting, we require an extractor that outputs some witness that passes the NP verification with probability $1$, whenever the verifier accepts with probability greater than some parameter $\kappa$, known as the knowledge error.

In the quantum case, given: (i) that we are not able to clone quantum states and (ii) QMA is not known to be closed under perfect completeness, the best that we can hope for is to extract some quantum state that would pass the QMA verification with some probability to be related to the acceptance probability in the interactive protocol, whenever this latter value is above some threshold $\kappa$.

To define a PoQ, we first fix the verification algorithm $V_x$ for some instance of a problem in QMA. We also assume $P^*$ to be a prover that makes the verifier accept with probability at least $\varepsilon > \kappa$ in the $\Xi$ protocol.\(^\text{15}\) We assume that $P^*$ only performs unitary operations on a private and message registers. We then define a quantum polynomial-time algorithm $K$ that has oracle access to $P^*$, meaning that $K$ can execute the unitary operations of $P^*$, their inverse operations and has access to the message register of $P^*$.\(^\text{16}\) The protocol is said to be a Proof of Quantum Knowledge if $K$ outputs, with non-negligible probability, some quantum state $\rho$ that would make $V_x$ accept with probability at least $q(\varepsilon, n)$, where $q$ is known as the quality function, or aborts otherwise.

The difficulty in showing that our $\Xi$ protocols are PoQs lies in the fact that any measurement performed by the extractor disturbs the state held by $P^*$, and therefore when we rewind $P^*$ by applying the inverse of his operation, we do not come back to the original state. We overcome this difficulty in the following way. We set $\kappa$ to be some value very close to 1, namely $\kappa = 1 - \frac{1}{p(n)}$ for some large enough polynomial $p$. Our extractor starts by simulating $P^*$ on the first message of the $\Xi$ protocol, and then holds the (supposed) one-time-padded state and the commitments to the one-time-pad keys. $K$ follows by iterating over all possible challenges of the $\Xi$ protocol, runs $P^*$ on this challenge, perform the verifier’s check and then rewinds $P^*$. By the assumption that $P^*$ has a very high acceptance probability, the measurements performed by $K$ do not disturb the state too much, and in this case, $K$ can retrieve the correct one-time pads for every qubit of the witness. If $K$ is successful (meaning that $k$ is able to open every committed bit), then $K$ can decode the original one-time-padded state and it is a good witness for $V_x$ with high probability.

We then analyse the sequential repetition of the protocol, that allows us to have a PoQ with exponentially small knowledge error $\kappa$, and extracts one good witness from $P^*$ (out of the polynomially many copies that $P^*$ should have in order to cause the verifier to accepted in the multiple runs of the protocol).

\(^{15}\)Note that we reserve the word “verifier” here for the $\Xi$ protocol and refer to $V_x$ as the QMA verification algorithm.

\(^{16}\)This model is already considered by [Unr12] in his work of quantum proofs of knowledge for NP.
Non-Interactive zero knowledge proof for QMA in the secret parameter model.

Finally, in Section 7, we achieve our non-interactive statistical zero-knowledge protocol for QMA in the secret parameter setting using techniques similar to our Ξ protocol: the trusted party chooses the one-time pad key and a random (and small) subset of these values that are reported to the verifier. Since the prover does not know which are the values that were given to the verifier, he should act as in the Ξ-protocol, but now the verifier does not actually need to ask for the openings, since the trusted dealer has already sent them. Although this is a less natural model, we hope that this result will shed some light in developing QNIZK proofs for QMA in more commonly-used models.

1.4 Open problems

Further QMA-complete languages. We note that a number of problems are currently known to be QMA-complete under Turing reductions, including the $N$-representability \cite{LCV07} and bosonic $N$-representability problems \cite{WMN10} as well as the universal functional of density function theory (DFT) \cite{SV09}. It is an open question if these problems can be shown to be QMA-complete under Karp reductions using the techniques presented in our work.

Complexity of $k$ CLDM for $k < 5$. We prove in this work that 5-CLDM is QMA-hard under Karp reductions. We leave as an open problem proving if the problem is still QMA-complete for $k < 5$.

Marginal reconstruction problem. We remark that the classical version of CLDM is defined as follows: given the description of $m$ marginal distributions on sets of bits $C_1, \ldots, C_m$, such that $|C_i| \leq k$, decide if there is a probability distribution that is close to those marginals, or such a distribution does not exist. This problem was proven NP-complete by Pitowsky \cite{Pit91}, and its containment in NP is proved by using the fact that such distribution can be seen as a point $p$ in the correlation polytope in a polynomial-size Hilbert space. In this case, by Caratheodory’s theorem, $p$ is a convex combination of polynomially many vertices of such polytope, and therefore these vertices serve as the NP-proof and a linear program verifies if there is a convex combination of them that is consistent with the marginals of the problem’s instance.

The difference here is that the proof and the marginals are different (but connected) objects. We leave as an open problem if we can extract a notion of a locally simulatable classical proof from this (or any other) problem, and its applications to cryptography and complexity theory. In particular, we wonder if there is a natural zero-knowledge protocol for this problem.

Applications of quantum ZK protocols. In classical cryptography, ZK and PoK protocols are a fundamental primitive since they are crucial ingredients in a plethora of applications. We discussed in Section 1.2 that our quantum ZK PoQ for QMA could be used as a proof-of-work for quantum computations. An interesting open problem is finding other settings in which the benefits of our simple ZK protocols for QMA can be applied. We list now some possibilities that could be explored in future work: authentication with uncloneable credentials \cite{CDS94}; proof of quantum ownership \cite{BJM19}; or ZK PoQ verification for quantum money \cite{AC12}.

Practical ZK protocols for QMA. Even if we reach a conceptually much simpler ZK protocol for QMA, the resources needed for it are still very far from practical. We leave as an open problem if

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17In \cite{LCV07}, the authors reduce the Local Hamiltonian problem on qubits into the Local Hamiltonian problem on fermions, and then they propose a Turing reduction from LH on fermions to the $N$-representability problem. The missing step is reducing CLDM directly to the $N$-representability problem, which might be straightforward, but needs a formal proof.
one could devise other protocols that are more feasible from a physical implementation viewpoint, which could include classical communication protocols based on the protocols proposed by Vidick and Zhang [VZ20], or device-independent ones based on the ideas of Grilo [Gri19].

Non-interactive Zero Knowledge protocols for QMA in the CRS model. In this work, we propose a QNIZK protocol where the information provided by the trusted dealer is asymmetric. We leave as an open problem if one could devise a protocol where the dealer distributes a common reference string (CRS) (or shared EPR pairs) to the prover and the verifier.

A possible way of achieving such non-interactive protocol would be to explore the properties of $\Xi$-protocols, as done classically with $\Sigma$-protocols. For instance, the well-known Fiat-Shamir transformation [FS87] allows us to make $\Sigma$-protocols non-interactive (in the Random Oracle model). We wonder if there is a version of this theorem when the first message can be quantum.

Witness indistinguishable/hiding protocols for QMA. Classically, there are two weaker notions that can substitute for ZK in different applications. In Witness Indistinguishable (WI) proofs, we require that the verifier cannot distinguish if she is interacting with a prover holding a witness $w_1$ or $w_2$, for any $w_1 \neq w_2$. In Witness Hiding (WH), we require that the verifier is not able to cook-up a witness for the input herself. We note that zero-knowledge implies both such definitions, and we leave as an open problem finding WI/WH protocols for QMA with more desirable properties than the known ZK protocols.

Computational Zero-Knowledge proofs vs. Statistical Zero-Knowledge arguments. Classically, it is known that the class of problems with computational ZK proofs is closely related to the class of problems with statistical ZK arguments [OV07]. We wonder if this relation is also true in the quantum setting.

1.5 Concurrent and subsequent works

Concurrently to this work, Bitansky and Shmueli [BS20] proposed the first quantum zero-knowledge argument system for QMA with constant rounds and negligible soundness. Their main building block is a non black-box quantum extractor for a post-quantum commitment scheme.

Also concurrently to this work, Coladangelo, Vidick and Zhang [CVZ20] proposed a non-interactive argument of quantum knowledge for QMA with a quantum setup phase (that is independent of the witness) and a classical online phase. Subsequently to our work, Alagic, Childs, Grilo and Hung [ACGH20] proposed the first non-interactive zero-knowledge argument system for QMA where the communication is purely classical. Their protocol works in the random oracle model with setup.

All of these protocols achieve only computational soundness and their security relies on stronger cryptographic assumptions (namely the Learning with Errors assumption, as well as post-quantum fully homomorphic encryption). Their proof structure follows by combining powerful cryptographic constructions based on these primitives to achieve their results. On the other hand, the key idea in our protocols is to use structural properties of QMA and with that, we can achieve a zero-knowledge protocol with statistical soundness under the very weak assumption that post-quantum one-way functions exist.
1.6 Differences with previous version

In a previous version of this work\textsuperscript{18}, we used the results of [GSY19] almost in a black-box way. In contrast, in the current version, we provide a new proof for the technical results that we need from [GSY19]; this not only makes this work self-contained, but it also provides a conceptually much simpler proof. More concretely, the sketch that was presented in Appendix A of the previous version has now become a full proof in Section 4. We note that this also allowed us to find a small bug in the proof of [GSY19], and provide a relatively easy fix (see Remark 4.7).

Fermi Ma pointed out a bug in the proof sketch of a proposal of statistical zero-knowledge argument for QMA that we have in previous versions of this paper.

1.7 Structure

The remainder of this document is structured as follows: Section 2 presents Preliminaries and Notation. In Section 3, we prove our results on the CLDM problem and we present our framework of simulatable proofs, with the technical portion of this contribution appearing in Section 4. Section 5 establishes the zero-knowledge $\Xi$ protocol for QMA, while in Section 6, we define a proof of quantum knowledge and show that the interactive proof system satisfies the definition. Finally, in Section 7, we show a non-interactive zero-knowledge proof for QMA in the secret parameter model.

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2 Preliminaries

2.1 Notation

For $n \in \mathbb{N}$, we define $[n] := \{0, \ldots, n - 1\}$. For some finite set $S$, we denote $s \in_S S$ as an element $s$ picked uniformly at random from $S$. We say that a function $f$ is negligible ($f(n) = \operatorname{negl}(n)$), if for every constant $c$, we have $f(n) = o\left(\frac{1}{n^c}\right)$. Given two discrete probability distributions $P$ and $Q$ over the domain $\mathcal{X}$, we define its statistical distance as $d(P, Q) = \sum_{x \in \mathcal{X}} |P(x) - Q(x)|$.

2.2 Quantum computation

We assume familiarity with quantum computation, and refer to [NC00] for the definition of basic concepts such as qubits, quantum states (pure and mixed), unitary operators, quantum circuits and quantum channels.

\textsuperscript{18}available at https://arxiv.org/abs/1911.07782v1
For an $n$-qubit state $\rho$ and an $m$-qubit state $\sigma$, we define $\rho^S \otimes \sigma^\overline{S}$ as the $m + n$-qubit quantum state $\tau$ that consists of the tensor product of $\rho$ and $\sigma$ where the qubits of $\rho$ are in the positions indicated by $S \subseteq [m + n]$, $|S| = n$, and the qubits of $\sigma$ are in the positions indicated by $\overline{S}$, with the ordering of the qubits consistent with the ordering in $\rho$ and $\sigma$, as well as the integer ordering in $S$ and $\overline{S}$. We extend this notation and write $A^S \otimes B^\overline{S}$ for operators $A$ and $B$ acting on $|S|$ and $|\overline{S}|$ qubits, respectively.

We define quantum gates with sans-serif font ($X, Z, \ldots$), and we define $I, X, Y$ and $Z$ to be the Pauli matrices, $P_k = \{|I, X, Y, Z\}^\otimes k$, $H$ to be the Hadamard gate, CNOT to be the controlled-Not gate, $P = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$ and $T = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{pmatrix}$. A Clifford circuit is a quantum circuit composed of Clifford gates: $I, X, Y, H, \text{CNOT}$ and $P$. It is well-known that universal quantum computation can be achieved with Clifford and $T$ gates.

For an operator $A$, the trace norm is $\|A\|_{\text{tr}} := \text{Tr} \left( \sqrt{A^\dagger A} \right)$, which is the sum of the singular values of $A$. For two quantum states $\rho$ and $\sigma$, the trace distance between them is

$$D(\rho, \sigma) = \frac{1}{2} \|\rho - \sigma\|_{\text{tr}} = \max_P \text{Tr} \left( P(\rho - \sigma) \right),$$

where the maximization is taken over all possible projectors $P$.

If $D(\rho, \sigma) \leq \varepsilon$, we say that $\rho$ and $\sigma$ are $\varepsilon$-close. If $\varepsilon = \text{negl}(n)$, then we say that $\rho$ and $\sigma$ are statistically indistinguishable, and we write $\rho \approx_s \sigma$.

If for every polynomial-time algorithm $A$, we have that

$$|\text{Pr}[A(\rho) = 1] - \text{Pr}[A(\sigma) = 1]| \leq \text{negl}(n),$$

then we say that $\rho$ and $\sigma$ are computationally indistinguishable, and we write $\rho \approx_c \sigma$.

For some $S \subseteq \{0, 1\}^*$, let $\{\Psi_x\}_{x \in S}$ and $\{\Phi_x\}_{x \in S}$ be two families of quantum channels from $q(|x|)$ qubits to $r(|x|)$ qubits, for some polynomials $q$ and $r$. We say that these two families are \textit{computationally indistinguishable}, and denote it by $\Psi_x \approx_c \Phi_x$, if for every $x \in S$ and polynomial $s$ and $k$ and every state $\sigma$ on $q(|x|) + k(|x|)$ and every polynomial-size circuit acting on $r(|x|) + k(|x|)$ qubits, it follows that

$$|\text{Pr}[Q((\Psi_x \otimes I)(\sigma)) = 1] - \text{Pr}[Q((\Phi_x \otimes I)(\sigma)) = 1]| \leq \text{negl}(n).$$

Finally, we state a result on rewinding by [Wat09].

\textbf{Lemma 2.1 (Lemma 9 of [Wat09])}. Let $Q$ be an quantum circuit that acts on an $n$-qubit state $|\psi\rangle$ and $m$ auxiliary systems $|0\rangle$. Let

$$p(\psi) = \|(0| \otimes I)Q(|\psi\rangle \otimes |0\rangle^\otimes m)\|^2 \text{ and } |\phi(\psi)\rangle = \frac{1}{\sqrt{p(\psi)}}((0| \otimes I)Q(|\psi\rangle \otimes |0\rangle^\otimes m).$$

Let $p_0, q \in (0, 1)$ and $\varepsilon \in (0, \frac{1}{2})$ such that $\varepsilon |p(\psi) - q| < \varepsilon$, $p_0(1 - p_0) \leq 1(1 - q)$, and $p_0 \leq p(\psi)$. Then there is a quantum circuit $R$ of size at most

$$O \left( \frac{\log(1/\varepsilon) \text{size}(Q)}{p_0(1 - p_0)} \right),$$

such that on input $|\psi\rangle$, $R$ computes the quantum state $\rho(\psi)$ that satisfies

$$\langle \phi(\psi)|\rho(\psi)|\phi(\psi)\rangle \geq 1 - 16\varepsilon \frac{\log^2 \frac{1}{\varepsilon}}{p_0^2(1 - p_0)^2}.$$
2.3 Complexity classes

In this section, we define several complexity classes that are considered in this work.

**Definition 2.2 (QMA).** A promise problem $A = (A_{yes}, A_{no})$ is in QMA if there exist polynomials $p, q$ and a polynomial-time uniform family of quantum circuits $\{Q_n\}$, where $Q_n$ takes as input a string $x \in \Sigma^*$ with $|x| = n$, a $p(n)$-qubit quantum state $|\psi\rangle$, and $q(n)$ auxiliary qubits in state $|0\rangle^{\otimes q(n)}$, such that:

**Completeness:** If $x \in A_{yes}$, there exists some $|\psi\rangle$ such that $Q_n$ accepts $(x, |\psi\rangle)$ with probability at least $1 - \text{negl}(n)$.

**Soundness:** If $x \in A_{no}$ for any state $|\psi\rangle$, $Q_n$ accepts $(x, |\psi\rangle)$ with probability at most $\text{negl}(n)$.

We say that a witness for $x$ is good if it makes the verification algorithm accept with probability $1 - \text{negl}(|x|)$.

We define now a quantum interactive protocol between two parties.

**Definition 2.3 (Quantum interactive protocol between $A$ and $B$ ($A \leftrightarrow B$)).** Let $\mathcal{A}$ and $\mathcal{B}$ be the private registers of parties $A$ and $B$, respectively, and $\mathcal{M}$ be the message register. A quantum interactive protocol between $A$ and $B$ is a sequence of unitaries $U_0, \ldots, U_m$ where $U_i$ acts on registers $\mathcal{A}$ and $\mathcal{M}$ for even $i$, and on registers $\mathcal{B}$ and $\mathcal{M}$ for odd $i$. The size of the register $\mathcal{A}, \mathcal{B}$ and $\mathcal{M}$, the number $m$ of messages and the complexity of the allowed $U_i$ is defined by each instance of the protocol. We can also consider interactive protocols where $A$ outputs some value after interacting with $B$, and we also denote such output as $(A \leftrightarrow B)$.

**Definition 2.4.** A promise problem $A = (A_{yes}, A_{no})$ is in QZK if there is an interactive protocol $(V \vdash P)$, where $V$ is polynomial-time and is given some input $x \in A$ and outputs a classical bit indicating acceptance or rejection of $x$, $P$ is unbounded, and the following holds

**Completeness:** If $x \in A_{yes}$, $\Pr[(V \vdash P) = 1] \geq 1 - \text{negl}(n)$.

**Soundness:** If $x \in A_{no}$ for all $P^*$, we have that $\Pr[(V \vdash P^*) = 1] \leq \frac{1}{\text{poly}(n)}$.

**Computational zero-knowledge:** For any $x \in A_{yes}$ and any polynomial-time $V'$ that receives the inputs $x$ and some state $\zeta$, there exists a polynomial-time quantum channel $\mathcal{S}_{V'}$ that also receives $x$ and $\zeta$ as input such that $(V' \vdash P) \approx_c \mathcal{S}_{V'}$.

Following the result by Pass and Shelat [PS05], we define now the notion of non-interactive zero-knowledge proofs in the secret parameter model.

**Definition 2.5 (Non-interactive statistical zero-knowledge proofs in the secret parameter model).** A triple of algorithms $(D, P, V)$ is a non-interactive statistical zero-knowledge proof in the secret parameter model for a promise problem $A = (A_{yes}, A_{no})$ where $D$ is a probabilistic polynomial time algorithm, $V$ is a probabilistic polynomial time algorithm and $P$ is an unbounded algorithm such that there exists a negligible function $\varepsilon$ such that the following conditions follow:

**Completeness:** for every $x \in A_{yes}$, there exists some $P$

$$\Pr[(r_P, r_V) \leftarrow D(1|x|); \pi \leftarrow P(x, r_P); V(x, r_V, \pi) = 1] \geq 1 - \varepsilon(n).$$
**Soundness:** for every \( x \in A_{\text{no}} \) and every \( P \)

\[
Pr[(r_P, r_V) \leftarrow D(1^{|x|}); \pi \leftarrow P(x, r_P); V(x, r_V, \pi) = 1] \leq \varepsilon(n).
\]

**Statistical zero-knowledge:** there is a probabilistic polynomial time algorithm \( S \) such that for every \( x \in A_{\text{yes}} \), the statistical distance of the distribution of the output of \( S(x) \) and the distribution of \( (r_V, \pi) \) for \( (r_P, r_V) \leftarrow D(1^{|x|}) \) and \( \pi \leftarrow P(x, r_P) \) is \( \text{negl}(n) \).

### 2.4 Local Hamiltonian problem

We discuss now the Local Hamiltonian problem, the quantum analog of MAX-SAT problem.

**Definition 2.6.** For \( k \in \mathbb{N}, a, b \in \mathbb{R} \) with \( a < b \), the \( k \)-Local Hamiltonian problem with parameters \( a \) and \( b \) is the following promise problem. Let \( n \) be the number of qubits of a quantum system. The input is a set of \( m(n) \) Hamiltonians \( H_0, \ldots, H_{m(n)−1} \) where \( m \) is a polynomial in \( n \), for all \( i \) we have that \( \|H_i\| \leq 1 \) and each \( H_i \) acts on \( k \) qubits out of the \( n \) qubit system. For \( H = \sum_{j=1}^{m(n)} H_j \) the following two conditions hold.

**Yes.** There exists a state \( n \)-qubit state \( |\varphi\rangle \) such that \( \langle \varphi | H | \varphi \rangle \leq a \cdot m(n) \).

**Yes.** For every \( n \)-qubit state \( |\varphi\rangle \) it holds that \( \langle \varphi | H | \varphi \rangle \geq b \cdot m(n) \).

In the proof of containment in QMA for the \( k \)-Local Hamiltonian problem for \( b - a > \frac{1}{\text{poly}(n)} \), Kitaev showed how to estimate the energy of a local term. The verification procedure consists of picking one term uniformly at random, and then measuring the energy of the that term. Notice that this verification procedure can be seen as \( m \) POVMs, each one acting on at most a \( k \)-qubit system. We record this in the following lemma.

**Lemma 2.7.** There exists a verification algorithm for the \( k \)-Local Hamiltonian problem with parameters \( a, b \in \mathbb{R}, b - a \geq \frac{1}{\text{poly}(n)} \), consisting of picking one of \( m = \text{poly}(n) \) \( k \)-qubit POVMs \( \{\Pi_1, I - \Pi_1\}, \ldots, \{\Pi_{m(|x|)}, I - \Pi_{m(|x|)}\} \) and accepting if and only if the witness projects onto \( \Pi_i \).

In Kitaev's proof of QMA-hardness of the Local Hamiltonian problem, he uses the circuit-to-Hamiltonian construction proposed by Feynman [Fey82] in order to reduce arbitrary QMA verification procedures to time-independent Hamiltonians in a way that the Local Hamiltonian has low-energy states if and only if the QMA verification accepts with high probability.

More concretely, Kitaev shows a reduction from a quantum circuit \( V \) consisting of \( T \) gates \( U_1, \ldots, U_T \) acting on a \( p \)-qubit state \( |\psi\rangle \) provided by the prover and an auxiliary register \( |0\rangle^\otimes q \) to some Hamiltonian \( H_V = \sum_{i \in [m]} H_i \), where the terms \( H_1, \ldots, H_m \) act on \( T + p + q \) qubits, and they range between the following types:

- **clock consistency** \( H_{t,\text{clock}} = |01\rangle\langle 01|_{t,t+1}, \) for \( 0 \leq t \leq T - 1 \)
- **initialization** For \( j \in [q], H_{j,\text{init}}^\text{init} = |0\rangle_{0} \otimes |1\rangle_{T+p+j}, \)
- **propagation** Let \( J \) be the set of qubits on which \( U_t \) acts non-trivially,
  \[
  H_0^{\text{prop}} = \frac{1}{2} (|0\rangle_{0} \langle 0|_{0} + |10\rangle_{0,1} \langle 0|_{0} - |10\rangle_{0,1} \langle 0|_{0} - |01\rangle_{1} \langle 1|_{c} \langle 0|_{c} - (U_t)^{J_0} \langle 1|_{c} \langle 0|_{c} - (U_t^{J_0})^T)_{J_0}.
  \]
A commitment scheme is a two-phase protocol between two parties, the Sender and Receiver. In the first phase, the Sender, who holds some message $m$, unknown by the Receiver, sends a commitment $c$ to the Receiver. In the second phase, the Receiver will reveal the committed value $m$. We require two properties of the protocol: hiding, meaning that the Receiver cannot guess $m$ from $c$, and binding, which stays that the Receiver cannot decide to open value $m' \neq m$ when $c$ was committed for $m$. 

$$H_T^{prop} = \frac{1}{2} \left( |10\rangle\langle10|_{T-1,T} + |1\rangle\langle1|_T - |1\rangle\langle0|_T \otimes (U_0)_{J_T} - |0\rangle\langle1|_c(T) \otimes (U^\dagger_T)_{J_T} \right)$$

For $1 \leq t \leq T-1$, $H_t^{prop} = \frac{1}{2} \left( |100\rangle\langle100|_{t-1,t,t+1} + |110\rangle\langle110|_{c(t-1,t,t+1)} - |110\rangle\langle100|_{t-1,t,t+1} \otimes (U_t)_{J_t} - |100\rangle\langle110|_{t-1,t,t+1} \otimes (U^\dagger_t)_{J_t} \right)$

output $H^{out} = |1\rangle\langle1|_T \otimes |0\rangle\langle0|_{T+1}$

The facts that we need from this reduction are summarized in the following lemma.

**Lemma 2.8 ([KSV02]).** If there exists some state $|\psi\rangle$ that makes $V$ accept with probability $1 - \text{negl}(n)$, then the history state

$$\frac{1}{\sqrt{T+1}} \sum_{t \in [T+1]} |\text{unary}(t)\rangle \otimes U_t \ldots U_1(|\psi\rangle \otimes 0)^\otimes q$$

has energy $\text{negl}(n)$ according to $H_V$. If every quantum state $|\psi\rangle$ makes $V$ reject with probability at least $\varepsilon$, then the groundenergy of $H_V$ is at least $\Omega\left(\frac{1}{T^{0.5}}\right)$.

### 2.5 Quantum error correcting codes and fault-tolerant quantum computing

For $n > k$, an $[[N, K]]$ quantum error correcting code (QECC) is a mapping from a $K$-qubit state $|\psi\rangle$ into an $N$-qubit state $\text{Enc}(|\psi\rangle)$. The distance of an $[[N, K]]$ QECC is $D$ if for an arbitrary quantum operation $E$ acting on $(D - 1)/2$ qubits, the original state $|\psi\rangle$ can be recovered from $E(\text{Enc}(|\psi\rangle))$, and in this case we call it an $[[N, K, D]]$ QECC. For a $[[N, 1, D]]$-QECC and its encoding $\text{Enc}$, we overload notation and for an $k$-qubit system $\phi$ we write $\text{Enc}(\phi) := \text{Enc}^{\otimes k}(\phi)$.

Given a fixed QECC, we say that some $k$-qubit gate $U$ can be applied transversally, if we apply $U^{\otimes n} \text{Enc}(|\psi\rangle) = \text{Enc}(U|\psi\rangle)$ where $|\psi\rangle$ is a $k$-qubit state and the $i$-th tensor of $U$ acts on the $i$-th qubit of the encodings of each qubit of $|\psi\rangle$. It is known that no code admits a universal set of transversal gates [EK09]. In order to overcome this difficulty to achieve fault-tolerant computation, one can use tools from computation by teleportation. In this case, provided a resource called magic states, one can simulate the non-transversal gate by applying some procedure on the target qubit and the magic state and such procedure contains only (classically controlled) transversal gates.

In this work, we consider the $k$-fold concatenation of the Steane code. We simply state the properties we need from it in this work and we refer to [GSY19] for further details. The $k$-fold concatenation of the Steane code is $[[7^k, 1, 3^k]]$ QECC such that all Clifford operations can be performed transversally and the $T$ gates can be applied using the magic state $T|+\rangle$.

### 2.6 Commitment schemes

A commitment scheme is a two-phase protocol between two parties, the Sender and Receiver. In the first phase, the Sender, who holds some message $m$, unknown by the Receiver, sends a commitment $c$ to the Receiver. In the second phase, the Receiver will reveal the committed value $m$. We require two properties of the protocol: hiding, meaning that the Receiver cannot guess $m$ from $c$, and binding, which stays that the Receiver cannot decide to open value $m' \neq m$ when $c$ was committed for $m$. 

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It is well-known that there are no commitment schemes with unconditionally hiding and unconditionally binding properties, but such schemes can be achieved if either of the properties holds only computationally.\textsuperscript{19}

In this work, we consider commitment schemes which are unconditionally binding but computationally hiding. There are recent quantum-secure instantiations of these schemes, assuming the hardness of Learning Parity with Noise [JKPT12, XXW13, BKL15].

We present now the formal definition of such schemes.

**Definition 2.9** (Computationally hiding and unconditionally binding commitment schemes). Let \( \eta \) be some security parameter, \( p \) be some polynomial, and \( \mathcal{M}, \mathcal{C}, \mathcal{D} \subseteq \{0,1\}^{p(\eta)} \) be the message space, commitment space and opening space, respectively. A computationally hiding and unconditional binding commitment scheme consists of a pair of algorithms \((\text{commit}, \text{verify})\), where

- \( \text{commit} \) takes as input a value from \( \mathcal{M} \) and some value in \( \mathcal{C} \times \mathcal{D} \),
- \( \text{verify} \) takes as input a value from \( \mathcal{M} \times \mathcal{C} \times \mathcal{D} \) and outputs some value in \( \{0,1\} \)

with the following properties:

**Correctness:** If \((c, d) \leftarrow \text{commit}(m)\), then \(\text{verify}(m, c, d) = 1\).

**Computationally hiding:** For any polynomial-time quantum adversary \( A \) and \((c, d) \leftarrow \text{commit}(m)\)

\[
\Pr[A(c) = m] \leq \frac{1}{|\mathcal{M}|} + \text{negl}(\eta).
\]

**Unconditionally binding:** For any \((m, d)\) and \((m', d')\), it follows that

\[
\text{verify}(m, c, d) = \text{verify}(m', c, d') = 1 \implies m = m'.
\]

3 Consistency of local density matrices and locally simulatable proofs

In this section, prove that the CLDM problem is in QMA (Section 3.1), and we present our Theorem on the QMA-hardness of the CLDM problem (Section 3.2).\textsuperscript{20}

One drawback of the containment of CLDM in QMA is that the verification procedure must check a super-constant number of qubits of the witness. Notice that for the local-Hamiltonian problem, checking a constant number of qubits is sufficient for an inverse polynomial completeness/soundness gap, but then we do not have the full knowledge of reduced density matrices of the witness as in CLDM. Our techniques allow us to define a new object called Locally Simulatable proofs, where we have the best of both worlds: full knowledge of the reduced density matrices of the witness and local verification. We present our framework of Locally Simulatable proofs in Section 3.3 (see Section 1.3.1 for an overview on how these locally simulatable proofs are used and Sections 5 and 7 for details).

Let us start by formally defining the CLDM problem.

\textsuperscript{19}Where unconditionally means that the property is guaranteed even against unbounded adversaries, in contrast to computational case, when the property is only guaranteed against quantum polynomial-time adversaries.

\textsuperscript{20}Note that, together with the technical details in Section 4, our proof is self-contained.
Definition 3.1 (Consistency of local density matrices problem (CLDM) [Liu06]). Let \( n \in \mathbb{N} \). The input to the consistency of local density matrices problem consists of \((C_1, \rho_1), \ldots, (C_m, \rho_m)\) where \( C_i \subseteq [n] \) and \(|C_i| \leq k\); and \( \rho_i \) is a density matrix on \(|C_i|\) qubits and each matrix entry of \( \rho_i \) has precision \( \text{poly}(n) \). Given two parameters \( \alpha \) and \( \beta \), assuming that one of the following conditions is true, we have to decide which of them holds.

- **Yes.** There exists some \( n\)-qubit quantum state \( \tau \) such that for every \( i \in [m] \), \( \left\| \text{Tr}_{C_i}(\tau) - \rho_i \right\|_{\text{tr}} \leq \alpha \).
- **No.** For every \( n\)-qubit quantum state \( \tau \), there exists some \( i \in [m] \) such that \( \left\| \text{Tr}_{C_i}(\tau) - \rho_i \right\|_{\text{tr}} \geq \beta \).

Remark 3.2. Note that in [Liu06], the definition of the problem sets \( \alpha = 0 \). In our case, we define the problem more generally, otherwise we would only achieve QMA_1 hardness (the version of QMA with perfect completeness) rather than QMA-hardness.

### 3.1 Consistency of local density matrices is in QMA.

In the proof of containment of CLDM in QMA, Liu uses a characterization of QMA called QMA+ [AR03]. For completeness, we start by showing the containment of CLDM in QMA, by presenting a standard verifier for the problem, which is a straightforward composition of the results from [AR03] and [Liu06].

**Lemma 3.3.** The consistency of local density matrices problem is in QMA for any \( k = O(\log n) \), and \( \alpha, \beta \) such that \( \varepsilon := \frac{\beta}{2^{k}} - \alpha \geq \frac{1}{\text{poly}(n)} \).

**Proof.** Let \((C_1, \rho_1), \ldots, (C_m, \rho_m)\) be an instance for CLDM. Let \( p \) be a polynomial such that \( p(n)\varepsilon^2 = \Omega(n) \), the verification system expects some state \( \psi \) consisting of \( p(n) \) copies of the state \( \tau \) that is supposed to be consistent with the local density matrices. The verifier then picks \( i \in [m] \) and \( P \in P_{|C_i|} \) uniformly at random. The verifier then measures each (supposed) one of the \( p(n) \) copies according to the observable \( P^{C_i} \otimes I^{C_i^c} \), and let \( \hat{p} \) be the average of its outcomes. The verifier accepts if and only if \( |\hat{p} - \text{Tr}(P\rho_i)| \leq \alpha + \frac{\varepsilon}{2} \).

In the completeness case, we have that \( \psi = \tau^{\otimes \varepsilon} \) and in this case, each of the \( p(n) \) measurements is 1 with probability \( \text{Tr}\left(\left( P^{C_i} \otimes I^{C_i^c}\right)\tau \right) \) for every \( i \). By Hoeffding’s inequality, we have that with probability at least \( 1 - 2\exp( p(n)\varepsilon^2 / 8 ) = 1 - \text{negl}(n) \),

\[
\left| \text{Tr}\left(\left( P^{C_i} \otimes I^{C_i^c}\right)\tau \right) - \hat{p} \right| \leq \frac{\varepsilon}{2}.
\]

Using the fact that \( \tau \) is consistent with \( \rho_i \), we also have that

\[
\left| \text{Tr}\left(\left( P^{C_i} \otimes I^{C_i^c}\right)\tau \right) - \text{Tr}(P\rho_i) \right| \leq \alpha,
\]

and therefore by the triangle inequality, the verifier accepts with probability at least \( 1 - \text{negl}(n) \).

For soundness, let \( \psi_j \) be the reduced density state considering the register of the \( j \)th copy of the state. Let \( \tau = \frac{1}{p(n)} \sum_j \psi_j \). Since we have a no-instance, there exists some \( i \in [m] \) such that \( \left\| \text{Tr}_{C_i}(\tau) - \rho_i \right\|_{\text{tr}} \geq \beta \). Let us write

\[
\text{Tr}_{C_i}(\tau) - \rho_i = \sum_{P \in P_{|C_i|}} \gamma_P P,
\]

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for $\gamma_P = \text{Tr}( (P_{C_i} \otimes I_{C_i}^c) \tau ) - \text{Tr}(P \rho_i)$. There must be some choice of $P \in \mathcal{P}_{|C_i|}$ such that $|\gamma_P| \geq \frac{\beta}{4^{m^2}}$. Notice that by the definition of $\tau$,

$$\text{Tr}( (P_{C_i} \otimes I_{C_i}^c) \tau ) = \frac{1}{p(n)} \sum_j \text{Tr}( (P_{C_i} \otimes I_{C_i}^c) \psi_j ).$$

In this case, if we measure each register corresponding to the $j$th copy of the state, then the expected value of its average is $\text{Tr}( (P_{C_i} \otimes I_{C_i}^c) \tau )$. Let $\hat{p}$ be the average of the outcomes of the performed measurements. Again using Hoeffding’s inequality, we have that with probability at least $1 - 2 \exp(p(n)\varepsilon^2/8) = 1 - \text{negl}(n)$,

$$\left| \text{Tr}( (P_{C_i} \otimes I_{C_i}^c) \tau ) - \hat{p} \right| \leq \frac{\varepsilon}{2},$$

and therefore, we have that for this fixed $i$ and $P$, the prover accepts with probability $\text{negl}(n)$. Since such $i$ is picked with probability $\frac{1}{m}$ and such a $P$ is picked with probability at least $\frac{1}{4^m}$, the overall acceptance is at most $1 - O\left(\frac{1}{m^4\varepsilon}\right)$ (where we account for the negligible factors inside the $O$-notation).

3.2 Consistency of local density matrices is QMA-hard

We show now that CLDM is QMA-hard under standard Karp reductions.

**Theorem 3.4.** The consistency of local density matrices problem is QMA-hard under Karp reductions.

At a high level, the proof consists in showing a verification algorithm for every problem in QMA such that the reduced density matrices of the history state of the verification procedure for yes-instances is simulatable. More precisely, we show that for any fixed $s$, there exists a verification algorithm $V_{x}^{(s)} = U_T \cdots U_1$ such that there is a classical algorithm that outputs the classical description of the reduced density matrix of

$$\Phi = \frac{1}{T + 1} \sum_{t,t' \in [T+1]} |\text{unary}(t)\rangle\langle\text{unary}(t')| \otimes U_t \cdots U_t' \left( \psi^{(s)} \otimes |0\rangle\langle 0| \otimes A \right) U_t^\dagger \cdots U_{t'}^\dagger,$$

on any subset of $s$ qubits in time $\text{poly}(|x|, 2^s)$. Here, $\psi^{(s)}$ is some witness that makes $V_{x}^{(s)}$ accept with probability $1 - \text{negl}(|x|)$. This is formalized in the following Lemma.

**Lemma 3.5 (Simulation of history states).** For any problem $A = (A_{\text{yes}}, A_{\text{no}})$ in QMA and $s \in \mathbb{N}$, there is a uniform family of verification algorithms $V_x^{(s)} = U_T \cdots U_1$ for $A$ that acts on a witness of size $p(|x|)$ and $q(|x|)$ auxiliary qubits such that there exists a polynomial-time deterministic algorithm $\text{Sim}_{V^{(s)}}$ that on input $x \in A$ and $S \subseteq [T + p + q]$ with $|S| \leq 3s + 2$, $\text{Sim}_{V^{(s)}}(x, S)$ outputs the classical description of an $|S|$-qubit density matrix $\rho(x, S)$ with the following properties

1. If $x$ is a yes-instance, then there exists a witness $\psi^{(s)}$ that makes $V_{x}^{(s)}$ accept with probability at least $1 - \text{negl}(n)$ such that $\| \rho(x, S) - \text{Tr}_S(\Phi) \|_{\text{tr}} \leq \text{negl}(n)$, where

   $$\Phi = \frac{1}{T + 1} \sum_{t,t' \in [T+1]} |\text{unary}(t)\rangle\langle\text{unary}(t')| \otimes U_t \cdots U_1 \left( \psi^{(s)} \otimes |0\rangle\langle 0| \otimes A \right) U_t^\dagger \cdots U_{t'}^\dagger,$$

   is the history state of $V_{x}^{(s)}$ on the witness $\psi^{(s)}$.

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21In particular, notice that this can only be true if $x$ is a yes-instance.
2. Let $H_i$ be one term from the circuit-to-local Hamiltonian construction from $V_x^{(s)}$ and $S_i$ be the set of qubits on which $H_i$ acts non-trivially. Then for every $x \in A$, $\text{Tr}(H_i \rho(x, S_i)) = 0$.

Section 4 is devoted to a self-contained proof of Lemma 3.5. This proof is inspired by Lemmas 15, 16 and 17 of [GSY19]. However, here we provide a simpler proof for a simpler statement, and we also fix a small bug in their proof.

Assuming the above Lemma, we are now ready to prove Theorem 3.4.

Proof of Theorem 3.4. Let $A = (A_{yes}, A_{no})$ be a promise problem in QMA and let $V_x^{(s)}$ be the QMA verification circuit for $A$ stated in Lemma 3.5.

Given input $x \in A$, our reduction consists of using Sim$_{V_x^{(s)}}$ to compute the CLDM instance

$$\{(S, \rho(x, S)) : S \text{ is a subset of the qubits in the history state with } |S| \leq 5\}. \quad (4)$$

We show now that if $x \in A_{yes}$, there exists a state that is consistent with all $\rho(x, S)$. Let $\psi^{(s)}$ and $V_x^{(s)}$ be given by Lemma 3.5 for $s = 5$ (and thus $V_x^{(s)}$ accepts $\psi^{(s)}$ with probability $1 - \text{negl}(n)$).

Then, also by Lemma 3.5, we have that the history state $\Phi$ of the computation of $V_x^{(s)}$ on $\psi^{(s)}$ is consistent with the reduced density matrices defined in Equation (4).

We show now that if $x \in A_{no}$, for every state $\tau$, there exists some $S \subseteq [T + p + q]$ with $|S| \leq 5$, such that $\|\text{Tr}_{S}(\tau) - \rho(x, S)\|_{tr} > \frac{1}{T^5}$. Let us assume, by way of contradiction, that there exists a $\tau$ such that for every $S \subseteq [T + p + q]$ with $|S| \leq 5$, $\|\text{Tr}_{S}(\tau) - \rho(x, S)\|_{tr} \leq \frac{1}{T^5}$. Let $H_{V_x^{(s)}}$ be the 5-local Hamiltonian resulting from the circuit-to-Hamiltonian construction on circuit $V_x^{(s)}$. We show that in this case, $\tau$ has energy $O\left(\frac{1}{T^5}\right)$ with respect to $H_{V_x^{(s)}}$, which is a contradiction, since $x \in A_{no}$ and therefore $H_{V_x^{(s)}}$ has ground energy $\Omega\left(\frac{1}{T\tau}\right)$ by Lemma 2.8. This finishes the proof.

Let $S_i$ be the set of at most 5-qubits on which the $i$-th term of $H_{V_x^{(s)}}$ acts and $\rho_i = \rho(x, S_i)$. We have that the energy of such $\tau$ is at most

$$\text{Tr}(H\tau) = \sum_i \text{Tr}(H_i \text{Tr}_{S_i}(\tau)) \leq \sum_i \left(\text{Tr}(H_i \rho_i) + \frac{1}{T^5}\right) \leq O\left(\frac{1}{T^4}\right),$$

where the first inequality comes from the assumption that $\|\text{Tr}_{S_i}(\tau) - \rho_i\|_{tr} < \frac{1}{T^5}$ for all $i$, and the second inequality follows since there are $O(T)$ terms and from Lemma 3.5 we have that $\text{Tr}(H_i \rho_i) = 0$. \hfill \Box

3.3 Locally simulatable proofs

As previously mentioned, we present here the framework of locally simulatable proofs, which combines in an abstract way the strong points of the local-Hamiltonian problem and CLDM. More concretely, locally simulatable proofs allow us to perform QMA verification by only checking a constant number of qubits (as in the local-Hamiltonian problem), while having full knowledge of the reduced density matrices of small subsets of the qubits of a good witness (as in CLDM).

\footnote{In [GSY19], they are simulating history states for MIP* computation and therefore they need to deal also with arbitrary provers’ operations}
**Definition 3.6 (k-SimQMA).** A promise problem \( A = (A_{\text{yes}}, A_{\text{no}}) \) is in the complexity class \( k\text{-SimQMA} \) with soundness \( \beta(n) \leq 1 - \frac{1}{\text{poly}(n)} \) if there exist polynomials \( m, p \) such that given \( x \in A \), there is an efficient deterministic algorithm that computes \( m(|x|) \)-qubit POVMs \( \{\Pi_1, I - \Pi_1\}, \ldots, \{\Pi_m(|x|), I - \Pi_m(|x|)\} \), that act on some quantum state of size \( p(|x|) \), such that:

**Simulatable completeness:** If \( x \in A_{\text{yes}} \), there exist a \( p(|x|) \)-qubit state \( \tau \), that we call a simulatable witness, and a set of \( k \)-qubit density matrices \( \{\rho(x, S)\}_{S \subseteq [p(n)]} \) that can be computed in polynomial time from \( x \), such that for all \( c \in [m] \)

\[
\text{Tr}(\Pi_c \tau) \geq 1 - \text{negl}(|x|),
\]

and for every \( S \subseteq [p(n)] \) of size \( k \)

\[
\|\text{Tr}_S(\tau) - \rho(x, S)\|_\text{tr} \leq \text{negl}(|x|).
\]

**Soundness:** If \( x \in A_{\text{no}} \), for any \( p(|x|) \)-qubit state \( \tau \) we have that

\[
\frac{1}{m} \sum_{c \in [m]} \text{Tr}(\Pi_c \tau) \leq \beta(|x|).
\]

**Lemma 3.7.** Every problem in QMA is in \( 5\text{-SimQMA} \).

**Proof (Sketch).** We can consider the POVMs that arise from the verification of the Local Hamiltonian problem and the density matrices that are the simulation of the history state of the computation. From Lemma 2.7 and Lemma 3.5, the result follows. \( \square \)

4 Simulation of history states

In this section, we prove Lemma 3.5. Technically, we achieve this result using a recent notion defined by Grilo, Slofstra and Yuen [GSY19] called simulatable codes. In a simulatable code, any \( s \)-qubit reduced density matrix of a codeword can be efficiently computed independently of the encoded state. In order to stress this independence of the encoded state, we say that the reduced density matrix is simulated. Furthermore, the reduced density states of the intermediate steps of the physical computation corresponding to a logical gate on encoded data (either through transversal gates or computation by teleportation with magic states) can also be efficiently computed. We refer to Example 1 and Section 1.3 for a more detailed overview of such a simulation.

**Definition 4.1.** Let \( C \) be a \([N, 1, D]\)–QECC that allows universal quantum computation on the encoded data by applying logical gates from a universal gateset \( \mathcal{G} \) with transversal gates (and possibly with the help of magic states). Let \( G \in \mathcal{G} \) be a logical gate acting on \( k_G \) qubits, \( U_1^{(G)}, \ldots, U_\ell^{(G)} \) be the transversal circuit that is applied to the physical qubits of the encoding of a \( k_G \)-qubit state and logically applies \( G \) on the data through \( \ell = \text{poly}(N) \) physical gates, with the help of an \( m_G \)-qubit magic state \( \tau_G \). We say that \( C \) is \( s \)-simulatable if there is a deterministic algorithm \( \text{Sim}_C \) that receives as input \( G \in \mathcal{G} \), a value \( 0 \leq t \leq \ell \) and a subset \( S \subseteq [N(m_G + k_G)] \) with \( |S| \leq s \) and \( \text{Sim}_C(G, t, S) \) runs in time \( \text{poly}(2^N) \) and outputs the classical description \(^{23}\) of an \( |S| \)-qubit density matrix \( \rho(G, t, S) \) such that for every \( k_G \)-qubit state \( \sigma \)

\[
\rho(G, t, S) = \text{Tr}_S \left( (U_\ell^{(G)} \cdots U_1^{(G)}) \text{Enc}(\sigma \otimes \tau_G) (U_\ell^{(G)} \cdots U_1^{(G)})^\dagger \right).
\]

\(^{23}\)The classical description of a density matrix consists of each entry of the matrix with \( \text{poly}(n) \) bits of precision.
Remark 4.2. In this work, we consider mostly QECCs that admit transversal Clifford gates and then use magic state $|T\rangle$ to compute $T$-gates. Thus, for concreteness, we take that $\mathcal{G} = \{\text{CNOT}, P, H, T\}$. In this case $k_G = 2$ for $G = \text{CNOT}$ gate and $k_G = 1$ for the other gates, and $\tau_T = |T\rangle\langle T|$ and no magic state is needed for the other gates. Notice that the gates $U_1^{(G)}, \ldots, U_\ell^{(G)}$ are publicly known, and therefore they do not need to be a parameter for $\text{Sim}_C$.

Remark 4.3. Definition 4.1 is slightly weaker from the one defined in [GSY19]. There, the runtime of the simulator is $\text{poly}(2^s)$ (whereas ours is $\text{poly}(2^N)$). That is, the runtime of their simulator depends on the number of qubits whose density matrix is simulated whereas in our case, our simulator depends on the size of the codeword. However, we notice that in the applications of simulatable codes in [GSY19] as well as here, QECCs of constant size are considered, and therefore our weaker definition suffices.

We call these codes locally simulatable codes in order to emphasize that only small parts of codewords can be simulated. In particular, in [GSY19], it was shown that the concatenated Steane code is locally simulatable for some $s$.

Lemma 4.4 ([GSY19]). For every $k > \log(s + 3)$, the $k$-fold concatenated Steane code is $s$-simulatable.

The proof of Lemma 4.4 in [GSY19] is somewhat involved, going through extensive calculations using the stabilizer formalism for QECCs. Here, as a side contribution of this work (see Section 4.3), we present a new, simpler, proof for Lemma 4.4. This not only makes our contribution self-contained, but we believe it might facilitate the use of such notions in other contexts. Our proof relies on the fact that quantum error correcting codes are good for “hiding secrets”: if we consider a subset of qubits of a quantum error correcting code smaller than the number or errors that can be corrected, then this reduced density matrix is independent of the encoded state. We push this observation further and show that this independence of the encoded state also works for intermediate steps of the computation on encoded data. We describe the detailed connection between QECCs and secret sharing in Section 4.1 and provide our new proof of Lemma 4.4 in Section 4.3 (and for an intuitive description, recall the discussion in Section 1.3).

In Section 4.2, we prove Lemma 3.5 assuming Lemma 4.4 (for pedagogical reasons, our proof of Lemma 4.4 is deferred to a later section, as described above). Very roughly, the proof proceeds by lifting the simulation of QECC to the simulation of the history states of QMA computation. More concretely, we start with a verification algorithm $V_x$ for some problem $A$ in QMA and we transform it into another QMA verification algorithm $V_x^{(s)}$ for $A$ with the same completeness and soundness parameters, but which allows us to perform the simulation. The verification circuit $V_x^{(s)}$ expects an encoded version of the QMA witness $|\psi\rangle$ of $V_x$, where each qubit of $|\psi\rangle$ is encoded using a simulatable code, and then performs $V_x$ on the encoded data using techniques from fault-tolerant quantum computation. With this new encoded verification circuit for QMA, along with the techniques from [GSY19], we are able to show how to compute reduced density matrices of the history state of $V_x^{(s)}$ when a good witness is provided. The details of how to construct $V_x^{(s)}$ are presented in Section 4.2.1 and we show in Section 4.2.2 that the history state of $V_x^{(s)}$ is efficiently simulatable.\footnote{We remark, however, that our proof does not work for the stronger notion of simulatability that would come with different parameters of Definition 4.1 as discussed in Remark 4.3.}
Figure 1: Circuit that encodes subsystem $A$ of some input state, SWAPs a subregister $S$, $|S| \leq (D - 1)/2$, of the encoding, and then applies the error correction procedure on the remaining state.

4.1 QECCs and secret sharing

Here, we provide a proof of what can be seen as a “composable”\(^{25}\) statement to the fact that quantum error correcting codes can be used for secret sharing [CGL99]. This is used later in Sections 4.2 and 4.3.

Lemma 4.5. Let $C$ be an $[[N, 1, D]]$-QECC, $A$ be a quantum register (possibly entangled with some environment register $E$), and $A'$ be the register that is the output of encoding a quantum system in register $A$ under $C$. Let also $S$ be a subset of size at most $(D - 1)/2$ of the qubits in $A'$ and $\mathcal{S} = A' \setminus S$. Then we have that there exists some state $\tau_S$ such that for all $|\psi\rangle_{AE}$

\[
\text{Tr}_S((\text{Enc} \otimes I)(|\psi\rangle<\psi|_{AE})) = \tau_S \otimes \text{Tr}_A(|\psi\rangle<\psi|_{AE}),
\]

where the encoding on the LHS only acts on register $A$.

**Proof.** Let us consider the circuit given in Figure 1 that receives $|\psi\rangle_{AE}$, encodes register $A$, swaps the qubits of $S$ with $|0\rangle^{\otimes|S|}$ and finally applies the correction procedure of the QECC with the qubits in $\mathcal{S}$ and the fresh auxiliary qubits. Notice that since $|S| \leq (D - 1)/2$, for any qubit $\rho$, we have that the error correction procedure maps $|0\rangle(|0\rangle^{\otimes|S|} \otimes \text{Tr}_S(\text{Enc}(\rho)))$ to $\text{Enc}(\rho)$, and therefore we have that $\phi = \tau_S|\psi\rangle_{AE} \otimes (\text{Enc} \otimes I)(|\psi\rangle<\psi|_{AE})$, where $\tau_S|\psi\rangle_{AE} = \text{Tr}_S((\text{Enc} \otimes I)(|\psi\rangle<\psi|_{AE})$ and we have a tensor product structure, since we have that $|\psi\rangle$ is a pure state. We show now that $\tau_S|\psi\rangle_{AE}$ has to be independent of $|\psi\rangle_{AE}$, i.e., there exists some $\tau_S$ such that $\tau_S|\sigma\rangle_{AE} = \tau_S$ for all $|\sigma\rangle_{AE}$. Notice that this finishes the proof since it implies

\[
\text{Tr}_S((\text{Enc} \otimes I)(|\psi\rangle<\psi|_{AE})) = \tau_S \otimes \text{Tr}_A(|\psi\rangle<\psi|_{AE}).
\]

Let us then prove the existence of $\tau_S$. Let us assume towards a contradiction that there exist some state $|\rho\rangle$, orthogonal to $|\psi\rangle$ such that $\tau_S|\psi\rangle \neq \tau_S|\rho\rangle$. By linearity, we have that $\tau_S|\psi\rangle \neq \tau_S|\psi\rangle + \tau_S|\rho\rangle$. Notice that we can repeat the above circuit as many times as we want and get $\tau_S^{\otimes k}|\psi\rangle$ (resp. $\tau_S^{\otimes k}|\psi\rangle + |\rho\rangle$) given a single copy of $|\psi\rangle$ (resp. $\frac{1}{\sqrt{2}}(|\psi\rangle + |\rho\rangle)$). In particular if we pick sufficiently large $k$, we have a way to distinguish $|\psi\rangle$ from $\frac{1}{\sqrt{2}}(|\psi\rangle + |\rho\rangle)$ with probability strictly larger than $\frac{1}{2}$ [HW12], but this is a contradiction since the best success probability is $\left|\langle\psi|\left(\frac{1}{\sqrt{2}}(|\psi\rangle + |\rho\rangle)\right)\right|^2 = \frac{1}{2}$. \hfill $\square$

The result in [CGL99] (summarized in Corollary 4.6) follows directly from Lemma 4.5 by considering a trivial system $E$.\(^{25}\)

---

\(^{25}\)By composable, we mean that it also considers the purification of the encoded state and we make no claim regarding the notion of Universal Composability in cryptography.
Corollary 4.6. Let $C$ be an $[[N, 1, D]]$-QECC. For any $S$ such that $|S| \leq (D - 1)/2$, there exists some density matrix $\tau_S$ such that for all $\psi$

$$
\tau_S = \text{Tr}_S(\text{Enc}(\psi)).
$$

4.2 Proof of simulation of history states

Here, we present the full proof of Lemma 3.5 (as discussed, this assumes Lemma 4.4, which is proved in a self-contained way later in Section 4.3). We start by defining the verification algorithm $V_x^{(s)}$ stated in Lemma 3.5 (Section 4.2.1), and then in Section 4.2.2, we show that the history state of the computation of $V_x^{(s)}$ on some good witness $\psi^{(s)}$ is simulatable.

4.2.1 The circuit $V_x^{(s)}$

First, we present the verification algorithm $V_x^{(s)}$ stated in Lemma 3.5. For that, we start with an arbitrary uniform family of circuits $\{V_x\}$ that we have from the definition of QMA. By definition, $V_x$ receives as input a $p'(|x|)$-qubit quantum state, for some polynomial $p'$, and auxiliary qubits such that if $x \in A_{\text{yes}}$ there exists an input state $|\psi\rangle$ such that $V_x$ outputs 1 with probability $1 - \text{negl}(|x|)$, whereas if $x \in A_{\text{no}}$, for all inputs $|\psi\rangle$, $V_x$ outputs 1 with probability $\text{negl}(|x|)$. We depict such a circuit in Figure 2.

![Figure 2: Verification circuit for some problem $A = (A_{\text{yes}}, A_{\text{no}})$ in QMA](image)

We have that, in order to efficiently compute reduced density matrices of the history state of $V_x$ on a good witness $|\psi\rangle$, it is necessary to efficiently compute reduced density matrices of $|\psi\rangle$, and in general, this is not known to be possible. Therefore, our approach here is to modify the circuit $V_x$ into $V_x^{(s)}$ such that completeness and soundness do not change, but such that we are able to efficiently compute the reduced density matrices of a good witness $\psi^{(s)}$, of the snapshots of the $V_x^{(s)}$ computation on $\psi^{(s)}$, and of the history state of such a computation. We do this modification in two steps. In the first one, we consider $V_x^{\text{otp}}$ which is expected to receive a one-time pad of the witness, along with the one-time pad keys. $V_x^{\text{otp}}$ then uncomputes the one-time pad encryption and performs the original computation. The verification circuit $V_x^{\text{otp}}$ is depicted in Figure 3.

It is easy to see that the completeness and soundness of $V_x^{\text{otp}}$ are unchanged compared to $V_x$. The reason why we perform this trivial modification is that, in the honest encrypted version, when we trace out all qubits of the witness but one, the remaining qubit is in the totally mixed state. This helps fix a small bug in [GSY19], as explained in Remark 4.7. For some witness $|\psi\rangle$ of $V_x$ of size $p'$, we define the associated witness for $V_x^{\text{otp}}$ as $\psi^{\text{otp}} = \frac{1}{2^p'} \sum_{a, b} |a, b\rangle\langle a, b| \otimes X^a Z^b |\psi\rangle Z^b X^a$.

In the second step, we encode the witness for $V_x^{(s)}$ with an $[[N, 1, D]]$ quantum error correcting code $C$ that is $(3s + 2)$-simulatable, and that the encoding, decoding and error detection procedures of $C$ have complexity $\text{poly}(N)$. In this work, we set $C$ to be the log$(3s + 5)$-fold concatenated Steane code which yields a $(3s + 2)$-simulatable code from Lemma 4.4.
Figure 3: Verification circuit $V_{otp}^x$ for a QMA problem. Here, the honest $\psi_{otp} = \frac{1}{2^{2p'}} \sum_{a,b} |a,b\rangle \langle a,b| \otimes X^a Z^b |\psi\rangle \langle \psi| Z^b X^a$, where $|\psi\rangle$ is a $p'$-qubit state.

1. Receive as witness the state $\phi(s) = \text{Enc}(\psi_{otp}) = \frac{1}{2^{2p'}} \sum_{a,b} \text{Enc}(|a,b\rangle \langle a,b| \otimes X^a Z^b |\psi\rangle \langle \psi| Z^b X^a)$

2. Run ChkEnc:
   2.1. Check if each logical qubit of the witness is encoded under $C$, and reject if this is not the case

3. Run ResGen:
   3.1. For each auxiliary qubit, encode $|0\rangle$ under $C$
   3.2. For every $T$-gate of $V_x$, create $|T\rangle$ and encode it under $C$

4. Run $\text{Enc}(V_{otp}^x)$:
   4.1. Undo the (encoded) one-time pad by transversally applying CNOT and C-Z gates.
   4.2. Simulate each gate of $V_x$, either transversally, or using unitary $T$-gadgets.

5. Run Dec:
   5.1. Decode the output bit, and accept or reject depending on its value.

Figure 4: Detailed description of $V_x^{(s)}$

We define $V_x^{(s)}$ as follows. It is supposed to receive the witness of $V_{otp}^x$ encoded under $C$ and then $i)$ verifies if each qubit of the witness is correctly encoded under $C$, $ii)$ creates encodings of auxiliary $|0\rangle$ and $|T\rangle$ under $C$, $iii)$ performs an encoded version of $V_{otp}^x$, either using transversal Clifford gates or performing the $T$-gadget described on Figure 9 (Section 4.3), and $iv)$ decodes the output of the computation. We describe $V_x^{(s)}$ more formally in Figure 4 and depict it in Figure 5.

The completeness and soundness of $V_x^{(s)}$ are straightforward: the acceptance probability of $V_x^{(s)}$ on

$$\text{Enc}(\psi_{otp}) = \frac{1}{2^{2p'}} \sum_{a,b} |a,b\rangle \langle a,b| \otimes \text{Enc}(X^a Z^b |\psi\rangle \langle \psi| Z^b X^a)$$

(5)

is exactly the same as $V_{otp}^x$ on $\psi_{otp}$; and witnesses that are orthogonal to such states are rejected with probability 1, since $V_x^{(s)}$ first checks if the witness is correctly encoded.
Figure 5: Verification circuit $V_x(s)$ for the QMA problem, where $\psi(s) = \text{Enc}(\psi_{\text{otp}}) = \frac{1}{2^{p'}} \sum_{a,b} \text{Enc}(|a, b\rangle\langle a, b| \otimes X^a Z^b |\psi\rangle\langle \psi| Z^b X^a)$ and $|\psi\rangle$ is a $p'$-qubit state.

Notice that the size of the circuit $V_x(s)$ is at most $\text{poly}(|x|, N)$-times bigger than $V_x$. Notice also that if we assume that $V_x$ contains only $\{H, \text{CNOT}, P, T\}$ gates, then all gates in $V_x(s)$ belong to the gateset $\{X, Z, P, H, \text{CNOT}, c(P), T\}$. This implies that each gate of $V_x(s)$ acts on at most 2 qubits.

Remark 4.7. We note that in the proof of $\text{MIP}^* = \text{ZK-MIP}^*$ in [GSY19], when the encoded version of the protocol is defined (similar to Figure 5), the verifier does not check if the provers’ answers lie in the codespace, which could potentially hurt soundness. We can easily address this issue by adding the procedure ChkEnc to check if the witness lies in the codespace. However, such a modification could harm the simulatability of the history state. In order to allow such a simulation (which will be proven in the next subsection), we added the intermediate circuit $V_x^{\text{otp}}$ where we consider the quantum one-time padded version of the witness (along with the one-time pad keys). Such modifications can be easily incorporated in the context of [GSY19].

4.2.2 Simulation of $V_x(s)$

The goal of this section is prove Lemma 3.5, where the circuit $V_x(s)$ was presented in Section 4.2.1. Our final goal is to show that the reduced density matrices of the history state

$$\frac{1}{T+1} \sum_{t, t' \in [T+1]} |\text{unary}(t)\rangle\langle \text{unary}(t')| \otimes U_t \cdots U_1(\psi(s) \otimes |0\rangle\langle 0| \otimes q) U_1^\dagger \cdots U_t^\dagger$$

of the computation of $V_x^s$ on a good witness $\psi(s)$ can be simulated. We also show that the simulations have low-energy according to the local terms of the circuit-to-Hamiltonian construction.

In order to prove it, we first show similar properties (i.e. simulatability and low-energy) for every snapshot

$$U_t \cdots U_1(\psi(s) \otimes |0\rangle\langle 0| \otimes q) U_1^\dagger \cdots U_t^\dagger$$

of the computation of $V_x(s)$ on a good witness $\psi(s)$ (Lemma 4.8), and also for small intervals of the history state (Lemma 4.9), i.e.,

$$\frac{1}{|I|} \sum_{t, t' \in I} |\text{unary}(t)\rangle\langle \text{unary}(t')| \otimes U_t \cdots U_1(\psi(s) \otimes |0\rangle\langle 0| \otimes q) U_1^\dagger \cdots U_t^\dagger,$$

for $I = \{t_1, t_1 + 1, \ldots, t_2\}$, $|I| \leq s + 1$.

26Here, we denote $c(P)$ as the controlled-P gate.
Lemma 4.8. Let \( A = (A_{\text{yes}}, A_{\text{no}}) \) be a problem in QMA, and \( V_{x}^{(s)} = U_{T} \cdots U_{1} \) be the verification algorithm described in Section 4.2.1 for some input \( x \in A \), where \( V_{x}^{(s)} \) acts on a \( p(|x|) \)-qubit witness and \( q(|x|) \) auxiliary qubits. Then there exists a polynomial-time deterministic algorithm \( \text{Sim}_{\text{snap}}^{V_{x}^{(s)}} \) that on input \( x \in A \), \( t \in [T + 1] \) and \( Y \subseteq [T + p + q] \) with \( |Y| \leq 3s + 2 \), \( \text{Sim}_{\text{snap}}^{V_{x}^{(s)}}(x, t, Y) \) outputs the classical description of an \(|Y|\)-qubit density matrix \( \rho(x, t, Y) \) with the following properties:

1. If \( x \) is a yes-instance, for any witness \( \psi^{(s)} \) in the form of Equation (5) that makes \( V_{x}^{(s)} \) accept with probability \( 1 - \negl(n) \), we have that
   \[
   \|\rho(x, t, Y) - \text{Tr}_{Y}(U_{t} \cdots U_{1}(\psi^{(s)} \otimes |0\rangle\langle 0|)U_{1}^{†} \cdots U_{t}^{†})\|_{\text{tr}} \leq \negl(n).
   \]

2. For any auxiliary qubit \( j \in Y \), we have that \( \text{Tr}_{j}(\rho(x, 0, Y)) = |0\rangle\langle 0| \).

3. Let \( t_{d} \) be the step just before decoding, \( t \geq t_{d} \), and \( E \subseteq Y \) be the set of qubits of the encoding of the output qubit in \( Y \). We have that
   \[
   \text{Tr}_{E}(\rho(x, t, Y)) = \text{Tr}_{E}(U_{t} \cdots U_{t_{d}+1} \text{Enc}(|1\rangle|1\rangle)U_{t_{d}+1}^{†} \cdots U_{t}^{†}).
   \]

Proof. We prove the result by showing how to compute the reduced state of
   \[
   \Delta_{x} = U_{t} \cdots U_{1}(\psi^{(s)} \otimes |0\rangle\langle 0|)U_{1}^{†} \cdots U_{t}^{†},
   \]
where \( \psi^{(s)} \) is a hypothetical good witness for \( V_{x}^{(s)} \).

Let us denote by \( E_{i} \) the set of qubits of the encoding of the \( i \)-th qubit (even if at step \( t \), \( i \)-th logical qubit is unencoded but later on the computation it will be so). We depict such regiset of qubits in Figure 6. We will split our analysis in three phases: before all the qubits are encoded, the logical computation and the decoding. For each of the phases, we have the following definitions:

- \( t_{0} \) and \( t_{1} \): For a fixed \( t \), we define \( t_{0} \leq t \leq t_{1} \) such that at \( t_{0} \) and \( t_{1} \), all qubits are all fully encoded or fully unencoded (i.e., at step \( t_{0} \) and \( t_{1} \) there are no operations such as performing a logical gate, encoding, decoding, etc.);

- \( Q \): for fixed \( t_{0}, t_{1} \), we let \( Q \) be the set of qubits on which operations \( U_{t_{0}+1}, \ldots, U_{t_{1}} \) act; and

- \( \mathcal{U} \): we define \( \mathcal{U} \) as the set logical qubits that are still unencoded by step \( t_{1} \).
Let us consider the case before all qubits are encoded (i.e. until the last step of ResGen). Let $t_0 \leq t \leq t_1$, where $t_0$ is the timestamp just before starting the operation of timestamp $t$ (which consists on either checking the encoding of a qubit of the witness or creating a resource state) and $t_1$ be the timestamp after this operation is performed. Let us assume, for simplicity of notation, that the qubits in $Q$ are the first $|Q|$ qubits of the state (and for the other cases follow analogously, but the states are permuted). Using Lemma 4.5, we have that

$$
\xi_{Y,t_0} := \text{Tr}_{Y \cup Q}(\Delta_{t_0}) = \text{Tr}_{Q \setminus Y}(\Delta_{t_0}) \otimes \left( \bigotimes_{i \notin U} \tau_{(Y \setminus Q) \cap E_i} \right) \otimes \left( \bigotimes_{i \in U} |0\rangle\langle0| \otimes |Y \cap E_i| \right),
$$

(6)

where we use the facts that every qubit in $Q$ is encoded at step $t_0$ (so by Lemma 4.5 we have the tensor product structure) and that for every $i \in U$ we have that $E_i \cap Q = \emptyset$, since we know that the $i$-th qubit is still unencoded by step $t_1$.

By Lemma 4.4, $\text{Sim}_c$ can compute $\tau_{Y \cap E_i}$ in time $\text{poly}(2^N)$ without knowing $\Delta_{t_0}$. Therefore, if we can compute the $N$-qubit state $\text{Tr}_{\tau}(\Delta_{t_0})$ in time $\text{poly}(2^N)$, $\text{Sim}_V^{\text{snap}}$ can compute $\rho(x, t, Y)$ by computing $\text{Tr}_{\tau}(\Delta_{t_0})$, classically simulating the unitaries $U_{t_0+1}, \ldots, U_t$, tracing out the qubits in $Q \setminus Y$ and finally appending $\tau_{(Y \setminus Q) \cap E_i}$ and the auxiliary qubits. The runtime is $\text{poly}(2^N, |x|)$, and Equation (6) implies that the first property of $\text{Sim}_V^{\text{snap}}$ holds for $t$ on the first phase. Notice also that since the state $\xi_{Y,0}$ has $|0\rangle|0\rangle$ on all auxiliary qubits, the second property of $\text{Sim}_V^{\text{snap}}$ holds.

We describe now how to compute $\text{Tr}_{\tau}(\Delta_{t_0})$ in time $\text{poly}(2^N)$. When $t$ lies in some intermediate step of ChkEnc, in other words, when $V_{x}^{(s)}$ checks the encoding of qubit $q$, $\text{Tr}_{\tau}(\Delta_{t_0}) = \text{Enc}(\text{Tr}_{\tau}(\psi_{\text{otp}})) = \text{Enc}(\text{Tr}_{\tau}(I/2))$, since we defined the honest witness for $\psi_{\text{otp}}$, the $q$-th qubit is either a one-time padded state (without its one-time pad key) or it is the key for the one-time pad (and therefore it is also totally mixed). The result follows for the encode checking phase since the encoding of the totally mixed state can be trivially computed in time $\text{poly}(2^N)$. We remark that this is the exact (and only) part of the proof where we need the properties of the intermediate verifier $V_{x}^{\text{otp}}$. If $t$ lies in some intermediate step of ResGen, i.e., when $V_{x}^{(s)}$ creates the encoding of the auxiliary $|0\rangle$ or $|T\rangle$ qubits, we have that $\text{Tr}_{\tau}(\Delta_{t_0}) = |0\rangle|0\rangle|Q\rangle$, whose description can be trivially computed.

We now consider the second phase, where the logical computation is performed. Here, let $t_0$ be the timestamp at the beginning of the logical computation performed at time $t$. In this case, we have that

$$
\Delta_{t} = U_{t} \cdots U_{t_0+1} \Delta_{t_0} U_{t_0+1}^\dagger \cdots U_{t}^\dagger,
$$

and all qubits of $\Delta_{t_0}$ are fully encoded. This corresponds to the simulation in the middle of the application of a logical gate, and by Lemma 4.4, $\text{Tr}_{\tau}(\Delta_{t})$ can be efficiently computed.\textsuperscript{28}

Finally, we reach the third phase and let $t_d$ be the timestep where the decoding starts. Since every qubits at timestep $t_d$ is fully encoded, we can define a quantum state $\gamma$ such that $\Delta_{t_d} = \text{Enc}(\gamma)$ and then we define $\Delta_{t_d} = \frac{\text{Enc}(|1\rangle|1\rangle \otimes I \rangle\langle \gamma|}{\text{Tr}(|1\rangle|1\rangle \otimes I \langle \gamma|}$. As in Equation (6), $\text{Sim}_V^{\text{snap}}$ can compute the reduced state on the encoding of qubits of $\Delta_{t_d}$ that are not the output by Lemma 4.4, and for the output qubit, $\text{Sim}_V^{\text{snap}}$ can classically simulate $U_{t_d+1}, \ldots, U_t$ on $\text{Enc}(|1\rangle|1\rangle)$ and then trace out the qubits not in $Y$.

\textsuperscript{27}In our self-contained proof, this appears in Lemma 4.10.

\textsuperscript{28}In our self-contained proof, this appears in Lemma 4.11 for Clifford computations and in Lemma 4.12 for T computation with magic states.

28
These operations can be performed in time \( \text{poly}(2^N, |x|) \). By construction we have the third property of \( \text{Sim}_{\text{V}(s)}^{\text{snap}} \).

Notice that if \( \Delta_{t_d} \) is indeed the \( t_d \)-th step of the computation \( V_x^{(s)} \) on a good witness \( \psi^{(s)} \), we have that \( \text{Tr}(\Delta_{t_0} - \Delta_{t_0}) \leq \text{negl}(|x|) \), and the first property of \( \text{Sim}_{\text{V}(s)}^{\text{snap}} \) follows for \( t \geq t_d \). □

We now show simulatability of intervals of the history state.

**Lemma 4.9.** Let \( A = (A_{\text{yes}}, A_{\text{no}}) \) be a problem in QMA, and \( V_x^{(s)} = U_T \cdots U_1 \) be the verification algorithm described in Section 4.2.1 for some input \( x \in A \) and \( s \geq 5 \), where \( V_x^{(s)} \) acts on a \( p(|x|) \) qubit witness and \( q(|x|) \) auxiliary qubits. Then there exists a polynomial-time deterministic algorithm \( \text{Sim}_{\text{V}(s)}^{\text{Int}} \) that on input \( x \in A, I = \{t_1, t_1 + 1, \ldots, t_2\} \subseteq [T+1], t_2 - t_1 \leq s + 1 \) and \( S \subseteq [T+p+q], |S| \leq s \), and \( \text{Sim}_{\text{V}(s)}^{\text{Int}}(x, I, S) \) runs in time \( \text{poly}(|x|, N) \) and outputs the classical description of an \(|S|\)-qubit density matrix \( \rho(x, I, S) \) with the following properties

1. If \( x \) is a yes-instance, then there exists a witness \( \psi^{(s)} \) that makes \( V_x^{(s)} \) accept with probability at least \( 1 - \text{negl}(n) \) such that \( \|\rho(x, I, S) - \text{Tr}_I(\Phi_I)\|_1 \leq \text{negl}(n) \), where

\[
\Phi_I = \frac{1}{|I|} \sum_{t,t' \in I} |\text{unary}(t)\rangle\langle\text{unary}(t')| \otimes U_t \cdots U_1 \left( \psi^{(s)} \otimes |0\rangle\langle0|^{\otimes q} \right) U_1^\dagger \cdots U_{t'}^\dagger,
\]

is an interval of the history state of \( V_x^{(s)} \) on the witness \( \psi^{(s)} \).

2. Let \( H_i \) be a clock term \( (H_{i}^{\text{clock}}) \), initialization term \( (H_{i}^{\text{init}}) \) or the output term \( (H_{i}^{\text{out}}) \) from the circuit-to-Hamiltonian construction of \( V_x^{(s)} \)\(^{29} \) and \( S_i \) be the set of qubits on which \( H_i \) acts non-trivially. Then for every \( x \in A \), we have \( \text{Tr}(H_i \rho(x, I, S_i)) = 0 \).

3. For any propagation term \( H_i^{\text{prop}} \) from the circuit-to-Hamiltonian construction of \( V_x^{(s)} \) and the corresponding set of qubits \( S_i \), and for every \( x \in A \) and \( I \) such that \( |t, t+1 \in I \) or \( t, t+1 \not\in I \), we have \( \text{Tr}(H_i^{\text{prop}} \rho(x, I, S_i)) = 0 \).

**Proof.** For simplicity, let \( \Delta_{t,t'} = U_t \cdots U_1 \left( \psi^{(s)} \otimes |0\rangle\langle0|^{\otimes q} \right) U_1^\dagger \cdots U_{t'}^\dagger \).

Let \( C \) be the set of clock qubits and \( W \) be the set of working qubits. We have that

\[
\text{Tr}_S(\Phi_I) = \frac{1}{|I|} \sum_{t,t' \in I} \text{Tr}_{ST\not\subset C}( |\text{unary}(t)\rangle\langle\text{unary}(t')|) \otimes \text{Tr}_{ST\not\subset W}(\Delta_{t,t'})
\]

\[
= \frac{1}{|I|} \sum_{t,t' \in I} \text{Tr}_{ST\not\subset C}( |\text{unary}(t)\rangle\langle\text{unary}(t')|) \otimes \text{Tr}_{ST\not\subset W}(U_t \cdots U_{t+1} \Delta_{t_1,t_1} U_{t+1}^\dagger \cdots U_{t'}^\dagger)
\]

\[
= \frac{1}{|I|} \sum_{t,t' \in I} \text{Tr}_{ST\not\subset C}( |\text{unary}(t)\rangle\langle\text{unary}(t')|) \otimes \text{Tr}_{G \setminus S}(U_t \cdots U_{t+1} \text{Tr}_{ST\not\subset W}(\Delta_{t_1,t_1}) U_{t+1}^\dagger \cdots U_{t'}^\dagger),
\]

where \( G \) is the set of qubits on which \( U_{t_1}, \ldots, U_{t_2} \) act. Notice that since \( t_2 - t_1 \leq |S| + 1 \), and each \( U_j \) acts on at most 2 qubits, so we have \( |G| \leq 2(|S| + 1) \) and \( |S \cup G| \leq 3s + 2 \).

\(^{29}\)See Section 2.4.
Let \( Y = (S \cap W) \cup G \). \( \text{Sim}^\text{Int}_{V^{(s)}}(x, I, S) \) starts by running \( \text{Sim}^\text{snap}_{V^{(s)}}(x, t_1, Y) \) from Lemma 4.8 to compute the state \( \tilde{\rho}(x, t_1, Y) \) such that \( \| \hat{\rho}(x, t_1, Y) - \text{Tr}_Y(\Delta_{t_1, t_1}) \| \leq \text{negl}(|x|) \) for some yes-instance \( x \). Then \( \text{Sim}^\text{Int}_{V^{(s)}}(x, t_1, Y) \) computes

\[
\rho(x, I, S) = \text{Tr}_{S \cup C}(\text{unary}(t))|\text{unary}(t')\rangle \otimes \text{Tr}_{G \setminus S} \left( U_t \ldots U_{t+1} \text{Tr}_{S \cup C}(\tilde{\rho}(x, t_1, Y)) U_{t+1}^\dagger \ldots U_t^\dagger \right)
\]

(7)

from \( \tilde{\rho}(x, t_1, Y) \) in time \( \text{poly}(2^N, |x|) \) and it follows that for a yes-instance \( x \) and a good witness \( \psi^{(s)} \),

\[
\| \text{Tr}_Y(\Phi_I) - \rho(x, I, S) \|_\text{tr} \leq \text{negl}(n),
\]

which proves the runtime and first property of \( \text{Sim}^\text{Int}_{V^{(s)}} \).

We show now that the output of \( \text{Sim}^\text{Int}_{V^{(s)}} \) has energy 0 with respect to the clock, initialization and output terms of \( H_{V^{(s)}} \), proving the second property of \( \text{Sim}^\text{Int}_{V^{(s)}} \). For that, we consider a 5-local term \( H_i \) and the set of qubits \( S_i \) on which \( H_i \) acts non-trivially. We prove the property for each type of local terms.

- If \( H_i \) is a clock constraint, since \( \rho(x, I, S_i) \) always output reduced density matrices on clock registers that are consistent with valid unary encodings, \( \text{Tr}(H_i \rho(x, I, S_i)) = 0 \) for every \( I \).

- If \( H_i \) is a initialization constraint \( H_i^{\text{init}} = |01\rangle \langle 01| \) and \( S_i \) consists of the first clock qubit and the \( j \)-th auxiliary qubit. We consider two subcases: if \( 0 \notin I \), \( \rho(x, I, S_i) \) has energy 0 because the content in the clock qubit is \( |1\rangle \langle 1| \) and \( \text{Tr}(|01\rangle \langle 01| \otimes |\gamma\rangle \langle \gamma|) = 0 \); if \( 0 \in I \), then \( t_1 = 0 \) and by the second property of \( \text{Sim}^\text{snap}_{V^{(s)}} \), it follows that \( \tilde{\rho}(x, 0, \{j\}) = |0\rangle \langle 0| \), and therefore \( \text{Tr}(|01\rangle \langle 01| \rho(x, I, S_i)) = \text{Tr}(|1\rangle \langle 1| \tilde{\rho}(x, 0, \{j\})) = 0 \).

- If \( H_i \) is the constraint \( H_i^{\text{out}} = |10\rangle \langle 10| \) and \( S_i \) consists of the last clock qubit and the output qubit. We again have two cases: if \( T \notin I \), \( \rho(x, I, S_i) \) has energy 0 because of the clock qubit (as in the previous cases); if \( T \in I \), then since \( |S_i| \leq s \), by the third property of \( \text{Sim}^\text{snap}_{V^{(s)}} \), it follows that \( \rho(x, t_0, S_i) = \text{Tr}_C(U_t \ldots U_{t+1} \text{Enc}(|1\rangle \langle 1| U_{t+1}^\dagger \ldots U_t^\dagger) \), where \( t_d \) is the step just before the decoding. This implies that

\[
\text{Tr}(|10\rangle \langle 10| \rho(x, I, S_i)) = \text{Tr}(|10\rangle \langle 10| \otimes U_T \tilde{\rho}(x, T - 1, S_i) U_T^\dagger) = \text{Tr}(|10\rangle \langle 10| |11\rangle \langle 11|) = 0.
\]

For the propagation term \( H_i^{\text{prop}} \), we have again two subcases: if \( t, t + 1 \notin I \), \( \rho(x, I, S_i) \) has energy 0 because of the clock qubits (as in the previous cases); if \( t, t + 1 \in I \), notice that any state

\[
\sum_{t_1 + 1 \leq t \leq t_2} |\text{unary}(t - t_1)\rangle |\text{unary}(t' - t_1)\rangle \otimes U_{t_2} \ldots U_{t+1} \sigma U_{t+1}^\dagger \ldots U_{t_2}^\dagger
\]

has energy 0 \(^{30}\) since it has the correct propagation of the unitary \( U_t \) at step \( t \). This proves the third property of \( \text{Sim}^\text{Int}_{V^{(s)}} \).

\(^{30}\)Here we use again the notation \( t_i = \min(I) \) and \( t_2 = \max(I) \).
We depict such intervals in Figure 7. Notice that since $|x| < |t_0| - |S|$, we have the timestamps which are strictly smaller than the $i$-th traced out qubit. The filled boxes in blue represent the traced out qubits and the ones in red represent the qubits that are not traced out. Finally, the interval $I_j$ contains the timestamps which are strictly smaller than the $i$-th traced out qubit and at least the $(i-1)$-st traced out qubit (or 0 when $i = 1$).

**Lemma 3.5 (restated).** For any problem $A = (A_{\text{yes}}, A_{\text{no}})$ in QMA and $s \in \mathbb{N}$, there is a uniform family of verification algorithms $V_{x}^{(s)} = U_T \cdots U_1$ for $A$ that acts on a witness of size $p(|x|)$ and $q(|x|)$ auxiliary qubits such that there exists a polynomial-time deterministic algorithm $\text{Sim}_{V(x)}$ that on input $x \in A$ and $S \subseteq [T + p + q]$ with $|S| \leq 3s + 2$, $\text{Sim}_{V(x)}(x, S)$ outputs the classical description of an $|S|$-qubit density matrix $\rho(x, S)$ with the following properties

1. If $x$ is a yes-instance, then there exists a witness $\psi^{(s)}$ that makes $V_{x}^{(s)}$ accept with probability at least $1 - \text{negl}(n)$ such that $\|\rho(x, S) - \text{Tr}_{\mathbb{N}}(\Phi)\|_{tr} \leq \text{negl}(n)$, where

   $$\Phi = \frac{1}{2^{T+1}} \sum_{t, t' \in [T+1]} \text{unary}(t) \langle \text{unary}(t') | \otimes U_T^\dagger \cdots U_1^\dagger \left( \psi^{(s)} \otimes |0\rangle^\otimes q \right) U_1 \cdots U_T,$$

   is the history state of $V_{x}^{(s)}$ on the witness $\psi^{(s)}$.

2. Let $H_t$ be one term from the circuit-to-local Hamiltonian construction from $V_{x}^{(s)}$ and $S_t$ be the set of qubits on which $H_t$ acts non-trivially. Then for every $x \in A$, $\text{Tr}(H_t \rho(x, S_t)) = 0$.

**Proof.** We let $V_{x}^{(s)}$ be the circuit for $A$ as defined in Section 4.2.1 and we follow the notation of Lemma 4.9 with $\Delta_{t, t'} = U_T \cdots U_1 (\psi^{(s)} \otimes |0\rangle^\otimes q) U_T^\dagger \cdots U_1^\dagger$.

Let $C_{tr} = \{i_1, \ldots, i_{|C_{tr}|} \} \subseteq \{1, \ldots, T\}$ denote the set of clock qubits that are not in $S$, and let $i_1 < i_2 < \cdots < i_{|C_{tr}|}$. Let us partition $[T + 1]$ into the intervals $I_1, \ldots, I_{|C_{tr}|+1}$ such that $I_j$ contains the timestamps which are strictly smaller than the $i_j$ and at least the $i_{j-1}$ (or 0 when $i = 1$). More formally, we have

- $I_1 = \{t : t \in [T+1] \text{ and } t < i_1\}$;
- For $2 \leq j < |C_{tr}|$, $I_j = \{t : t \in [T+1] \text{ and } i_{j-1} \leq t < i_j\}$;
- $I_{|C_{tr}|} = \{t : t \in [T+1] \text{ and } t \geq i_{|C_{tr}|}\}$.

We depict such intervals in Figure 7. Notice that since $|C_{tr}| \geq n - |S|$, it follows that $|I_j| \leq |S| + 1$. We have that

$$\text{Tr}_{t}(\text{unary}(t) | \text{unary}(t') \rangle) = \text{Tr}_{t} \left( |1\rangle^\otimes t \otimes |1\rangle^\otimes (t' - t) \otimes |0\rangle^\otimes (T - t') \right)$$

$$= \text{Tr} \left( \prod_{j \in \{1, \ldots, T\}} |t_j \rangle \langle t_j|_{i_j} \right) \prod_{j \neq i \text{ even}} |t_j \rangle \langle t_j|_{i_j} \prod_{j \neq i \text{ odd}} |t_j \rangle \langle t_j|_{i_j}$$
where in the second equality we remove the crossterms that vanish and in the third equality we have that

\[ \text{Tr}_{C_{tr}}(\Phi) = \frac{1}{T+1} \sum_{t,t' \in [T+1]} \text{Tr}_{C_{tr}}(\text{unary}(t)\langle\text{unary}(t') | \otimes \Delta_{t,t'}) \]

\[ = \frac{1}{T+1} \sum_{t,t' \in I_j} \text{Tr}_{C_{tr}}(\text{unary}(t)\langle\text{unary}(t') | \otimes \Delta_{t,t'}) \]

\[ = \sum_{j=1}^{\ell} \frac{|I_j|}{T+1} \text{Tr}_{C_{tr}}(\Phi_{I_j}), \tag{8} \]

where in the second equality we remove the crossterms that vanish and in the third equality we regroup the terms in states with the form \( \Phi_I = \frac{1}{|I|} \sum_{t,t' \in [T]} \text{unary}(t)\langle\text{unary}(t') | \otimes \Delta_{t,t'} \).

We can then define \( \text{Sim}_{V(s)}(x,S) \) as follows:

1. Compute the set \( C_{tr} \);
2. Compute the intervals \( I_1, \ldots, I_{|C_{tr}|+1} \);
3. Run \( \text{Sim}_{V(s)}^{\text{int}} \) from Lemma 4.9 to compute \( \rho(x,I_j,S) = \text{Sim}_{V(s)}^{\text{int}}(x,I_j,S) \);  
4. Output \( \rho(x,S) = \sum_{j=1}^{[C_{tr}]} \frac{|I_j|}{T+1} \rho(x,I_j,S) \).

Since \( \text{Sim}_{V(s)}^{\text{int}} \) runs in time \( poly(2^N,|x|) \), so does \( \text{Sim}_{V(s)} \). From Equation (8) and the fact that \( \|\rho(x,I_j,S) - \text{Tr}_{\mathcal{T}}(\Phi_{I_j})\|_{tr} \leq \text{negl}(n) \), by Lemma 4.9, it follows that \( \|\rho(x,S) - \text{Tr}_{\mathcal{T}}(\Phi)\|_{tr} \leq \text{negl}(n) \).

Notice that a propagation term \( H_t^{prop} \) acts on the clock qubits \( t \) and \( t + 1 \) and therefore, when we simulate its corresponding set of qubits, \( t, t + 1 \not\in C_{tr} \). It follows by the definition of the intervals \( I_j \) that in this case either \( t, t + 1 \not\in I_j \) or \( t + 1 \not\in I_j \). Thus, we have by the second and third properties of \( \text{Sim}_{V(s)}^{\text{int}} \) of Lemma 4.9 that for all intervals \( I_j \) and local terms \( H_t \) of the circuit-to-Hamiltonian construction of \( V_x(s) \) with corresponding set of qubits \( S_{x, \text{int}} \), \( \text{Tr}(H_t \rho(x,I_j,S)) = 0 \), and the same holds for \( \rho(x,S) \) by convexity.

4.3 New proof for locally simulatable codes

Finally, we now provide our new (simpler) proof for Lemma 4.4. We split our proof in three parts: in Lemma 4.10, we show that codewords are simulatable; then in Lemma 4.11 we show that intermediate steps of transversal Clifford gates are simulatable; finally in Lemma 4.12 we prove that intermediate steps of T-gadgets are simulatable. These three parts together prove Lemma 4.4 in a straightforward way. Note that our proofs are applicable for any QECC that admits transversal Clifford gates and Clifford gadgets for non-Clifford gates (such as T) with magic states.

Let us start by showing how to compute reduced density matrices on a small set of qubits of a codeword of a QECC.

**Lemma 4.10.** Let \( C \) be an \([N,1,D]\)-QECC whose encoding procedure \( \text{Enc} \) has complexity \( poly(N) \) and let \( \rho \) be a qubit. Then there exists a classical algorithm \( \text{Sim}_{C}^{\text{int}} \) that on input \( S \subseteq [N], |S| \leq (D - 1)/2 \), runs in time \( poly(2^N) \) and outputs the classical description of \( \rho(S) \) such that for every qubit \( \sigma \), we have

\[ \rho(S) = \text{Tr}_{\mathcal{T}}(\text{Enc}(\sigma)). \]
Proof. From Corollary 4.6, we have that \(|S| \leq (D - 1)/2\) implies that there exists some state \(\tau_S\) such that for all \(\psi\)

\[
\tau_S = \text{Tr}_\Sigma(\text{Enc}(\psi)).
\]

In this case, the algorithm \(\text{Sim}_C^{\text{CW}}\) can then output Equation (9) by classically computing the classical description of the trace, and then computing \(\rho(S) = \text{Tr}_\Sigma(\text{Enc}(|0\rangle\langle 0|)) = \text{Tr}_\Sigma(\text{Enc}(\sigma))\) in time \(\text{poly}(2^N)\).

We now apply the Lemma 4.10 to show how to compute the reduced density matrix on a state in the intermediate steps of transversal Clifford computation on encoded data.

**Lemma 4.11.** Let \(\mathcal{C}\) be the \([[N, 1, D]]-\text{QECC}\) obtained by the \(k\)-fold concatenation of the Steane code. Let \(G \in \{\text{H, CNOT, P}\}, m_G \in \{1, 2\}\) be the number of qubits on which \(G\) act and \(U_1, \ldots, U_N\) be the physical gates that implement \(G\) transversally on the encoding of \(m_G\) qubits under \(\mathcal{C}\), i.e., \(U_i\) applies \(G\) on the \(i\)-th qubit of the encoding of each qubit of an \(m_G\)-qubit system. Then there exists a classical algorithm \(\text{Sim}_C^{\text{Cliff}}\) that on input \(G, 0 \leq t \leq N\) and subset \(S, |S| \leq (D - 1)/4\), we have that \(\text{Sim}_C^{\text{Cliff}}(G, t, S)\) runs in time \(\text{poly}(2^N)\) and outputs the classical description of a state \(\rho(G, t, S)\) such that for every \(m_G\)-qubit state \(\sigma\)

\[
\rho(G, t, S) = \text{Tr}_\Sigma\left((U_t \cdots U_1)\text{Enc}(\sigma)(U_t \cdots U_1)^\dagger\right).
\]

Moreover, \(\text{Sim}_C^{\text{Cliff}}\) can be modified to receive a classical bit \(b\) and \(\text{Sim}_C^{\text{Cliff}}(G, t, S, b)\) outputs the classical description of a state \(\rho(G, t, S, b)\) such that

\[
\rho(G, t, S, b) = \text{Tr}_\Sigma\left((U_t^b \cdots U_1^b)\text{Enc}(\sigma)(U_t^b \cdots U_1^b)^\dagger\right).
\]

**Proof.** Since tensor products of one-qubit Pauli matrices form a basis for the space of all matrices, we have that

\[
\text{Tr}_\Sigma\left((U_t \cdots U_1)\text{Enc}(\sigma)(U_t \cdots U_1)^\dagger\right) = \frac{1}{2^{|S|}} \sum_{P \in \mathcal{P}_{|S|}} \text{Tr}\left((U_t \cdots U_1)\text{Enc}(\sigma)(U_t \cdots U_1)^\dagger P\right) \cdot P,
\]

where we abuse notation and inside the trace, we extend \(P\) to act on \(mN\) qubits, acting non-trivially on the qubits in \(S\) and trivially (i.e., identity) on \(\overline{S}\). Considering a single term in the above sum, we have that

\[
\text{Tr}\left((U_t \cdots U_1)\text{Enc}(\sigma)(U_t \cdots U_1)^\dagger P\right) = \text{Tr}\left((U_t \cdots U_1)\text{Enc}(\sigma)P'(U_t \cdots U_1)^\dagger\right) = \text{Tr}(\text{Enc}(\sigma)P') = \text{Tr}_R\left(\text{Tr}_\Sigma(\text{Enc}(\sigma)) P'\right),
\]

where \(P'\) is the unique tensor product of Pauli matrices defined by \(PU_1 \cdots U_1 = U_t \cdots U_1 P'\) (where we consider again \(P\) acting on \(mN\) qubits as discussed above), the second equality holds from the cyclic property of trace, and \(R\) is the set of qubits on which \(P'\) acts non-trivially (i.e., not identity). Note that since each \(U_i\) acts on a distinct set of 2 physical qubits and \(P\) acts non-trivially on at

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31In other words, for every \(2^i \times 2^i\) complex matrix \(M\), we have \(M = \frac{1}{2^i} \sum_{P \in \mathcal{P}_i} \text{Tr}(MP) \cdot P\).
most $|S|$ qubits, it follows that $P'$ acts non-trivially on at most $2|S|$ qubits and therefore $|R| \leq 2|S|$. In this case, $\text{Sim}_{\text{Cliff}}$ can compute the classical description of $\rho(R) = \text{Tr}_R(\text{Enc}(\sigma))$ using $\text{Sim}^{\text{CW}}_C$, since $|R| \leq 2|S| \leq (D - 1)/2$. With the classical description of $\rho(R)$ computed by $\text{Sim}^{\text{CW}}_{\nu(r)}$ from Lemma 4.10, $\text{Sim}_{\text{Cliff}}$ can easily compute

$$\rho(G, t, S) = \frac{1}{2|S|} \sum_{P \in \mathcal{P}_{|S|}} \text{Tr}(\rho(R)P') \cdot P = \text{Tr}_S \left( \left( U_t \cdots U_1 \right) \text{Enc}(\sigma) \left( U_t \cdots U_1 \right) \right),$$

by iterating over all $P \in \mathcal{P}_{|S|}$, computing $P'$ and $\text{Tr}(\sigma P')$ and then summing up $\text{Tr}(\sigma P') \cdot P$.

Moreover, we can easily adapt the above method to the situation where $G$ is controlled by a bit $b$: if $b = 0$, run $\text{Sim}^{\text{CW}}_C(S)$ on the qubits, since no computation is performed on the qubits; whereas if $b = 1$, run $\text{Sim}_{\text{Cliff}}(G, t, S)$, since $G$ is being performed on the qubits.

Figure 8: Gadget for performing $T$-gate. In Figure 8a, we consider the gadget on unencoded qubits, whereas in Figure 8b, it is performed on encoded data. Note that in Figure 8b, the control qubit needs to be decoded in order to perform transversal $X^c$ and $P^c$ operations and we highlight the operations done at the decoded level in red.

Figure 9: Unitary version of the $T$-gadget described in Figure 8b. We split the gadget into two phases (which is denoted by a dashed line). In the first phase, all computation happens at the logical level. In the second phase, one qubit is decoded and some operations now happen at the physical level, which is highlighted in red.

We now address the remaining part: computing using magic state gadgets. Unfortunately, we cannot consider the encoded version of the well-known gadget to compute the $T$-gate using $|T\rangle$ magic states (see Figures 8a and 8b), since for our applications, we require that our circuit be unitary. For that, we use the unitary version of Figure 8b, described in Figure 9.

The following lemma shows how to simulate the reduced density matrices on the computation of $T$-gadgets.

**Lemma 4.12.** Let $C$ be a $[[N, 1, D]]$-QECC be the $k$-fold concatenated Steane code. Let $U_1, \ldots, U_t$ be the unitary circuit for the $T$-gadget described in Figure 9. Then there exists a classical algorithm $\text{Sim}^T_C$
that on input $0 \leq t \leq \ell$ and subset $S, |S| \leq (D - 1)/4$, we have that $\text{Sim}_C^T(t, S)$ runs in time $\text{poly}(2^N)$ and outputs the classical description of an $|S|$-qubit state $\rho(t, S)$ such that for every qubit $\sigma$

$$\rho(t, S) = \text{Tr}_S((U_t \cdots U_1)\text{Enc}(|0\rangle\langle 0| \otimes \sigma \otimes |T\rangle\langle T|)(U_t \cdots U_1)^{\dagger}).$$  \hspace{1cm} (11)$$

Proof. Notice that if $t$ lies in the first phase of Figure 9, i.e., the transversal application of any of the first two CNOT gadgets, then the simulation is already covered by $\text{Sim}_C^{\text{Cliff}}$ of Lemma 4.11. The challenging part here is when $t$ lies in the second phase of the gadget since $i$) one qubit is completely decoded in this computation and $ii$) we are now applying controlled Cliffords whose control-qubit is not classical. For simplicity, we assume $\sigma = |\psi\rangle\langle \psi|$ and the extension to mixed states follows by convexity.

For $i \in \{1, 2, 3\}$, let $E_i$ be the set of qubits of the encoding of the $i$-th logical qubit, ordered from top to bottom. Let $|\phi\rangle = \text{CNOT}_{2,1}\text{CNOT}_{3,2}|\psi\rangle|T\rangle$. Notice that

$$\text{Tr}_1(|\phi\rangle\langle \phi|) = \frac{1}{2}(|0\rangle\langle 0| \otimes |\psi\rangle\langle \psi| + |1\rangle\langle 1| \otimes XP^\dagger|\psi\rangle\langle \psi|PX),$$

and by Lemma 4.5, we have that

$$\text{Tr}_{\cup E_2 \cup E_3}(\text{Enc}(|\phi\rangle\langle \phi|)) = \frac{1}{2} \sum_{b \in \{0,1\}} \tau_{S \cap E_1} \otimes \text{Enc}(|b\rangle\langle b| \otimes (XP^\dagger)^b|\psi\rangle\langle \psi|(PX)^b),$$

for some $\tau_{S \cap E_1}$ independent of the encoded state.

In order to prove the simulatability of the second phase of Figure 9, we consider two subcases: when $t$ lies in the encoding/decoding of the second qubit; and during the transversal application of the controlled $X$ and $P$ gates.

When $t$ lies within the decoding/encoding of the second qubit, notice that the third qubit is not touched by any operation and therefore the third qubit is a codeword (for some unknown logical qubit). Using Lemma 4.5 again, we have that

$$\text{Tr}_{E_2 \setminus S}(\sigma) = \frac{1}{2} \sum_{b \in \{0,1\}} \tau_{S \cap E_1} \otimes \text{Enc}(|b\rangle\langle b|) \otimes \tau_{S \cap E_3},$$

Therefore, in order to compute Equation (11), $\text{Sim}_C^T$ can compute the classical description of the state $\frac{1}{2} \sum_{b \in \{0,1\}} \text{Enc}(|b\rangle\langle b|)$ at the decoding stage corresponding to $t$ and then compute its reduced density matrix on the qubits $S \cap E_2$ and output it (along with $\tau_{S \cap E_2}$ and $\tau_{S \cap E_3}$). An analogous argument holds if $t$ lies within the re-encoding of the second qubit.

Finally, when $t$ lies in the transversal application of $X^b$ or $P^h$, notice that right after the decoding in the circuit described in Figure 9, we have the state

$$\text{Tr}_{\cup E_2 \cup E_3}(\text{Enc}(|\phi\rangle\langle \phi|)) = \frac{1}{2} \sum_{b \in \{0,1\}} \tau_{S \cap E_1} \otimes |b\rangle\langle b| \otimes |0\rangle\langle 0|^{N-1} \otimes \text{Enc}(X^b(P^\dagger)^b|\psi\rangle\langle \psi|P^hX^b),$$

and we want to apply a classically-controlled transversal Clifford gate with a known control qubit which is chosen uniformly at random. This case is covered by the second part of Lemma 4.11, finishing the proof. \hfill \square
5 Zero-knowledge $\Xi$-protocol for QMA

In this section, we show that simulatable proof systems lead to a zero-knowledge protocols with a very simple proof structure, which can be classified in the “commit-challenge-response” framework. As mentioned in Section 1.1, when all the messages are classical, such type of protocols are called $\Sigma$-protocols. We extend this definition to the quantum setting by allowing the first message to be a quantum state and in this case we call it a $\Xi$-protocol.

Definition 5.1 ($\Xi$-protocol). An $\Xi$-protocol consists of a three-round protocol between a prover and a verifier and it takes the following form:

**Commitment:** In the first round, the prover sends some initial quantum state.

**Challenge:** In the second round, the verifier sends a uniformly random challenge $c \in [m]$.

**Open:** The prover answers the challenge $c$ with some classical value.

For simplicity we denote $\Xi$-QZK as the class of problems that have a $\Xi$ computational quantum zero-knowledge proof system.

5.1 Protocol

| Notation | Meaning |
|----------|---------|
| $n$      | Number of the qubits in the SimQMA proof |
| $k$      | Locality parameter |
| $\Pi_c$  | POVM corresponding to a check of SimQMA proof system |
| $S_c$    | Set of qubits on which $\Pi_c$ acts non-trivially |
| $m$      | Number of different SimQMA checks |
| $\rho_S$ | Reduced density matrix of the proof on set $S$ of qubits for $|S| = k$ |
| $\tau$   | Quantum state that is supposed to pass the checks and be consistent with all local density matrices up to negligible error |
| $\sigma(c)$ | $\rho_{S_c} \otimes |0\rangle\langle 0|^{S_c}$ |
| $\zeta$  | Side-information of a malicious verifier |
| $\tilde{\phi}_{a,b}$ | $X^a Z^b \phi X^a Z^b$, for a $q$-qubit quantum state $\phi$ and $a, b \in \{0, 1\}^q$ |

Figure 10: Notation reference

We describe in Figure 11 the zero knowledge $\Xi$ protocol for QMA, whose informal description was given in Section 1.3.
Let $A = (A_{\text{yes}}, A_{\text{no}})$ be a problem in $k$-SimQMA with soundness $\delta$, $x \in A_{\text{yes}}$, $\Pi_c$ be the set of POVMs for $x$, and $\tau$ be a (supposed) simulatable witness for $x$.

1. Prover picks $a, b \in \{0, 1\}^n$ and $r \in \mathbb{R}$, $\mathbb{R}$ is all of the possible randomness needed to commit to $2n$ bits.

2. Prover sends $\tilde{\tau}_{a, b} \otimes |\text{comm}_{a, b}^r \rangle \langle \text{comm}_{a, b}^r|$, where $\text{comm}_{a, b}^r$ is the commitment to each bit of $a$ and $b$.

3. The verifier sends $c \in \mathbb{R} \times [m]$.

4. The prover opens the commitment for $a|_{S_c}$ and $b|_{S_c}$, where $S_c$ is the set of qubits on which $\Pi_c$ acts non-trivially.

5. If the commitments do not open, the verifier rejects.

6. The verifier measures $X^a|_{S_c} Z^a|_{S_c} \tilde{\tau}_{a, b} Z^a|_{S_c} X^a|_{S_c}$ with POVMs $\{\Pi_c, I - \Pi_c\}$, and accepts if and only if the outcome is $\Pi_c$.

Figure 11: Zero-knowledge $\Xi$-protocol for SimQMA.

5.2 Computational zero-knowledge proof for QMA

The goal of the section is to prove that every language in QMA has a $\Xi$-protocol that is a quantum computational zero-knowledge proof system if we assume that the commitment used in Figure 11 is computationally hiding and unconditionally binding.

We first state two lemmas that will be proved in Sections 5.2.1 and 5.2.2, respectively.

Lemma 5.2. The protocol in Figure 11 has completeness $1 - \text{negl}(n)$ and soundness $\delta$.

Lemma 5.3. The protocol in Figure 11 is computational zero-knowledge.

We can then state the main theorem of this section.

Theorem 5.4. $\text{QMA} \subseteq \Xi$-$\text{QZK}$.

Proof. Direct from Lemmas 3.7, 5.2 and 5.3.

5.2.1 Proof of Lemma 5.2

Lemma 5.2 (revisited). The protocol in Figure 11 has completeness $1 - \text{negl}(n)$ and soundness $\delta$.

Proof. By Definition 3.6, if $x \in A_{\text{yes}}$, the prover can follow the protocol honestly with some $\tau$ that is consistent with all the POVMs and the local density matrices. In this case, the acceptance probability is exponentially close to 1.

Let us now analyze the case for $x \in A_{\text{no}}$. Let $\psi \otimes |z\rangle \langle z|$ be the state sent by the prover in the first message, where $\psi$ is supposed to be the copies of the one-time padded state that is consistent with the POVMs and the reduced density matrices, and $z$ is the commitment to the one-time pad keys. We assume, without loss of generality, that $|z\rangle$ is a classical value, since the verifier can measure it as soon as she receives it, and the prover can send the $z$ that maximizes the acceptance probability.
For challenge $c$, the prover answers with $|w_c⟩$, where again we assume to be a classical value for the same reasons as above. Since the commitment scheme is unconditionally binding, we can define the strings $a, b ∈ \{0, 1\}^n$ to be the string containing the unique bits that could be open for the corrected committed bits, or 0 if the commitment is defective.

Let $S_c ⊆ [n]$ be defined as in Figure 11. Notice that $|w_c⟩$ is supposed to be the opening of bits of $a$ and $b$ in the subset $S_c$. Let $D_c$ be the event that $w_c$ is the correct opening for all of such bits, and $1_{D_c}$ be the indicator variable for such event.

We have then that the acceptance probability is

\[
\frac{1}{m} \sum_{c ∈ [m]} 1_{D_c} \text{Tr} \left( \Pi_c X^a |s_c⟩⟨s_c| Z^b |s_c⟩⟨s_c| X^a |s_c⟩ \right)
\]

\[
\leq \frac{1}{m} \sum_{c ∈ [m]} \text{Tr} \left( \Pi_c X^a |s_c⟩⟨s_c| Z^b |s_c⟩⟨s_c| X^a |s_c⟩ \right)
\]

\[
= \frac{1}{m} \sum_{c ∈ [m]} \text{Tr} \left( \Pi_c X^a Z^b |s_c⟩⟨s_c| X^a |s_c⟩ \right)
\]

\[
\leq \max_\phi \frac{1}{m} \sum_c \text{Tr} (\Pi_c \phi)
\]

\[
\leq \delta.
\]

where in the equality we use the fact that $\Pi_c$ only acts on the qubits in $S_c$, and the last inequality follows since $x ∈ A_{no}$ and from Definition 3.6 the SimQMA protocol has soundness $\delta$. □

5.2.2 Proof of Lemma 5.3

We prove now the zero knowledge property of the protocol.

Before presenting the simulator, let us analyze how the verification algorithm behaves. We can assume, without loss of generality that the verifier is composed of two verification algorithms $\hat{V}_1$ and $\hat{V}_2$.

For $\hat{V}_1$, since the classical part of the message can be copied and the challenge sent by the verifier is measured by the prover, we can assume $\hat{V}_1$ acts like the following

\[
\sum_{a,b,r} \hat{V}_1 (\tilde{\rho}_{a,b} \otimes |\text{comm}_{a,b}^r⟩⟨\text{comm}_{a,b}^r| \otimes ζ) \hat{V}_1^†
\]

\[
= \sum_{a,b,r,c} \rho_{a,b,c,r} φ_{a,b,c,r} \otimes |\text{comm}_{a,b}^r⟩⟨\text{comm}_{a,b}^r| \otimes |c⟩⟨c|,
\]

where $\sum_{a,b,c,r} \rho_{a,b,c,r} = 1$ and we have traced-out the copy of $|c⟩⟨c|$ that was sent to the prover (and measured).

The message $|c⟩$ is sent to the prover, who answers then with some value $|o_c⟩$, i.e., the opening of the commitments corresponding to the challenge $c$.

The verifer then outputs

\[
\sum_{a,b,r,c} p_{ρ,a,b,c,r} \hat{V}_2 (φ_{ρ,a,b,c,r} \otimes |\text{comm}_{a,b}^r⟩⟨\text{comm}_{a,b}^r| \otimes |c⟩⟨c| \otimes |o_c⟩⟨o_c|) \hat{V}_2^†.
\]
Let $A = (A_{yes}, A_{no})$ be a problem in $k$-SimQMA, $x \in A_{yes}$, and $\rho_c = \rho(x, S_c)$ be the local density matrix of a simulatable witness $\tau$ for $x$ on the qubits corresponding to $\Pi_c$.

1. Pick $c \in \mathbb{S}[m]$, $a, b \in \{0, 1\}^n$, $r \in \mathbb{S}$

2. Create the state $\tilde{\sigma}(c)_{a,b} \otimes |\text{comm}_r\rangle \langle \text{comm}_r| \otimes \zeta$, where $\sigma(c) = \rho_c^{S_c} \otimes |0\rangle \langle 0|^S_c$

3. Run $\hat{V}_1$ on $\tilde{\sigma}(c)_{a,b} \otimes |\text{comm}_r\rangle \langle \text{comm}_r| \otimes \zeta$

4. Measure the last register in the computational basis and abort if it is not $|c\rangle$

5. Otherwise, append the register $|o_c\rangle$, apply $\hat{V}_2$ and output the result.

Figure 12: Simulator Sim for ZK $\Xi$-protocol for QMA

In order to show zero-knowledge, we start by showing that for a fixed $c$, $\sum_{a,b} p_{\sigma,a,b,c,r}$ is independent of $\sigma$, up to negligible factors, if the commitment scheme is hiding. We denote $R = |\mathbb{S}|$.

**Lemma 5.5.** Let $p_c = \frac{1}{R} \sum_{r} p_{I,0,0,c,r}$. Then for any $\sigma$, we have that

$$\left| \frac{1}{2^n R} \sum_{a,b,r} p_{\sigma,a,b,c,r} - p_c \right| \leq \text{negl}(n),$$

where the probabilities are defined as Equation (15) for the polynomial-time adversary $\hat{V}_1$.

**Proof.** Let us suppose that there exist some state $\rho$, a challenge $c$ and a polynomial $q$ such that

$$\left| p_c - \left( \frac{1}{2^n R} \sum_{a,b,r} p_{\sigma,a,b,c,r} \right) \right| \geq q(n).$$

(17)

Then it is possible to distinguish the states

$$\frac{1}{2^n R} \sum_{a,b,r} \tilde{\sigma}_{a,b} \otimes |\text{comm}_r\rangle \langle \text{comm}_r|$$

and

$$\frac{1}{R} \sum_{r} |\text{comm}_{0,0}\rangle \langle \text{comm}_{0,0}|$$

by appending $\zeta$, applying $\hat{V}_1$ and measuring the challenge register in the computational basis.

However, since the commitment scheme is computationally hiding, we have that

$$\frac{1}{2^n R} \sum_{a,b,r} \tilde{\sigma}_{a,b} \otimes |\text{comm}_r\rangle \langle \text{comm}_r| \approx_c \frac{1}{2^n R} \sum_{a,b,r} |\text{comm}_{0,0}\rangle \langle \text{comm}_{0,0}|$$

(18)

and therefore these states are indistinguishable. We conclude that the assumption in Equation (17) is false. 

\[ \square \]
Lemma 5.6. The Simulator described in Figure 12 does not abort with probability at least \( \frac{1}{m} - \text{negl}(n) \). In this case, its output is \( \text{negl}(n) \)-close to

\[
\frac{1}{2^{2n}R} \sum_{a,b,c,r} \tilde{V}_2 \left( (I \otimes |c\rangle\langle c|) \tilde{V}_1 \left( \sigma(c)_{a,b} \otimes |\text{comm}^r_{a,b}\rangle\langle \text{comm}^r_{a,b}| \otimes \zeta \right) \right) \tilde{V}_1^\dagger (I \otimes |c\rangle\langle c|) \otimes |o_c\rangle\langle o_c|) \tilde{V}_2^\dagger.
\]

Proof. The state of \( \text{Sim} \) after step 2 is

\[
\frac{1}{2^{2n}Rm} \sum_{a,b,c,r} \sigma(c)_{a,b} \otimes |\text{comm}^r_{a,b}\rangle\langle \text{comm}^r_{a,b}| \otimes \zeta \otimes |c\rangle\langle c|
\]

(20)

Sim runs \( \tilde{V}_1 \) on the first three registers of the state in Equation (20), resulting in

\[
\frac{1}{2^{2n}Rm} \sum_{a,b,c,r} \tilde{V}_1 \left( \sigma(c)_{a,b} \otimes |\text{comm}^r_{a,b}\rangle\langle \text{comm}^r_{a,b}| \otimes \zeta \right) \tilde{V}_1^\dagger \otimes |c\rangle\langle c|
\]

\[
= \frac{1}{2^{2n}Rm} \sum_{a,b,c,r} p_{\sigma(c),a,b,c,r} \sigma(c)_{a,b,c,r} \otimes |\text{comm}^r_{a,b}\rangle\langle \text{comm}^r_{a,b}| \otimes |c\rangle\langle c| \otimes |c\rangle\langle c|
\]

(21)

Notice that \( \text{Sim} \) does not abort when \( c = c' \), and this event happens with probability

\[
\frac{1}{m} \sum_c \frac{1}{2^{2n}R} \sum_{a,b,c,r} p_{\sigma(c),a,b,c,r} \geq \frac{1}{m} \sum_c (p_c - \text{negl}(n)) = \frac{1}{m} - \text{negl}(n),
\]

(22)

where the inequality follows from Lemma 5.5 and the equality from the fact that \( \sum_c p_c = 1 \).

In order to provide the output of the simulator, conditioned that it did not abort, we post-select in Equation (21) the event that \( c = c' \), which gives us

\[
\frac{1}{2^{2n}Rp_{\text{succ}}} \sum_{a,b,c,r} (1 \otimes |c\rangle\langle c|) \tilde{V}_1 \left( \sigma(c)_{a,b} \otimes |\text{comm}^r_{a,b}\rangle\langle \text{comm}^r_{a,b}| \otimes \zeta \right) \tilde{V}_1^\dagger (1 \otimes |c\rangle\langle c|) \otimes |c\rangle\langle c|
\]

\[
\approx \frac{1}{2^{2n}R} \sum_{a,b,c,r} (1 \otimes |c\rangle\langle c|) \tilde{V}_1 \left( \sigma(c)_{a,b} \otimes |\text{comm}^r_{a,b}\rangle\langle \text{comm}^r_{a,b}| \otimes \zeta \right) \tilde{V}_1^\dagger (1 \otimes |c\rangle\langle c|) \otimes |c\rangle\langle c|,
\]

(23)

where we set \( p_{\text{succ}} = \frac{1}{2^{2n}R} \sum_{a,b,c,r} p_{\sigma(c),a,b,c,r} \) to be the probability that \( \text{Sim} \) does not abort and the approximation holds by Equation (22).

Finally, \( \text{Sim} \) only needs to append the last register with \( |o_c\rangle \), which can be performed efficiently given \( a, b \) and \( c \), and apply \( \tilde{V}_2 \).

\[\square\]

Lemma 5.7. The output of a simulator that does not abort is computationally indistinguishable from the output of the malicious verifier in the protocol in Figure 11.

Proof. In the real protocol, the first message sent by the prover is

\[
\frac{1}{2^{2n}R} \sum_{a,b,r} \tau_{a,b} \otimes |\text{comm}^r_{a,b}\rangle\langle \text{comm}^r_{a,b}|,
\]

and the verifier applies \( \tilde{V}_1 \) on the message sent by the prover and the side-information \( \zeta \), resulting in the state

\[
\frac{1}{2^{2n}R} \sum_{a,b,c,r} (1 \otimes |c\rangle\langle c|) \tilde{V}_1 \left( \tau_{a,b} \otimes |\text{comm}^r_{a,b}\rangle\langle \text{comm}^r_{a,b}| \otimes \zeta \right) \tilde{V}_1^\dagger (1 \otimes |c\rangle\langle c|).
\]
On challenge $|c\rangle$, the prover then answers with $|o_c\rangle$, the openings of the corresponding commitments. The verifier then applies $\hat{V}_2$, and outputs

$$\frac{1}{2^{2nR}} \sum_{a,b,c,r} \hat{V}_2 \left( (1 \otimes |c\rangle\langle c|) \hat{V}_1 (\overline{\tau}_{a,b} \otimes \text{comm}_{a,b}^r \otimes \zeta) \hat{V}_1^\dagger (1 \otimes |c\rangle\langle c|) \otimes |o_c\rangle\langle o_c|) \right) \hat{V}_2.$$ 

We show now that this state is indistinguishable from the state that is output by the simulator, proved in Lemma 5.6. For simplicity, let $\xi_{a,b,r} = \text{comm}_{a,b}^r \otimes \zeta$. Up to normalization factors, we have that

$$\sum_{a,b,c,r} \hat{V}_2 \left( (1 \otimes |c\rangle\langle c|) \hat{V}_1 (\overline{\tau}_{a,b} \otimes \xi_{a,b,r}) \hat{V}_1^\dagger (1 \otimes |c\rangle\langle c|) \otimes |o_c\rangle\langle o_c|) \right) \hat{V}_2^\dagger \approx c \sum_{a,b,c,r} \hat{V}_2 \left( (1 \otimes |c\rangle\langle c|) \hat{V}_1 \left( \text{Tr}_{S_c} (\sigma_{a,b} \otimes \xi_{a,b,r}) \otimes \zeta \right) \hat{V}_1^\dagger (1 \otimes |c\rangle\langle c|) \otimes |o_c\rangle\langle o_c|) \right) \hat{V}_2^\dagger \quad (24)$$

$$\approx_s \sum_{a,b,c,r} \hat{V}_2 \left( (1 \otimes |c\rangle\langle c|) \hat{V}_1 \left( \text{Tr}_{S_c} (\sigma_{a,b} \otimes \xi_{a,b,r}) \otimes \zeta \right) \hat{V}_1^\dagger (1 \otimes |c\rangle\langle c|) \otimes |o_c\rangle\langle o_c|) \right) \hat{V}_2^\dagger \quad (25)$$

where in the first approximation we use the fact that the commitments of $a|S_c$ and $b|S_c$ are never revealed and that the commitment is computationally hiding, and the second approximation holds since we assume that $\|\text{Tr}_{S_c} (\sigma) - \rho_c\|_F \leq \text{negl}(n)$ by Definition 3.6.

By the definition of $\xi_{a,b,r}$ and Lemma 5.6, the state of Equation (25) is $\text{negl}(n)$-close to the output of the simulator.

We are finally ready to prove Lemma 5.3.

**Lemma 5.3 (revised).** The protocol in Figure 11 is computational zero-knowledge.

**Proof.** Notice that from Lemmas 2.1 and 5.5, there exists a quantum algorithm $\text{Sim}'$ that runs in time

$$O \left( m \text{poly}(n) \left( \text{time} \left( \hat{V}_1 \right) + \text{time} \left( \hat{V}_2 \right) \right) \right)$$

whose output is $\text{negl}(n)$-close to the output of $\text{Sim}$, conditioned on not aborting. From Lemma 5.7, the output of $\text{Sim}'$ is computationally indistinguishable from the run of the real protocol, and therefore it can be used as the simulator, finishing our proof.

**5.3 Decreasing the soundness error**

We remark that unlike the protocol in [GSY19], we do not know how to show parallel repetition for our protocol. The problem here is that for an $\ell$-fold version of our protocol, the simulator, as in Figure 12, would need to correctly answer the question for each one of these $\ell$ copies of the game, what would happen with probability $\frac{1}{m^\ell}$, and therefore the rewinding technique would have an exponential cost in $\ell$.

However, as in the classical case, we can show that our protocol accepts sequential repetition, since the guess for each of the iterations is performed independently.
Lemma 5.8. Consider the $\ell$-fold sequential repetition of the QZK $\Xi$-protocol, where $1 \leq \ell = \text{poly}(n)$ and the verifier accepts if and only if each sequential run accepts. Then this is a quantum zero-knowledge protocol for SimQMA with completeness $1 - \text{negl}(|x|)$ and soundness $O(\delta(|x|)^\ell)$.

Proof. The completeness and soundness properties hold trivially.

Let us now argue about the zero-knowledge property. Notice that an honest prover $P$ has an $\ell$-fold tensor product of the honest witness, and uses a single copy per iteration. In this case, we can run the simulator $\ell$ times sequentially, using the output of the $i$th run as the side-information of the $(i + 1)$st run. \qed

6 Proofs of quantum knowledge

In this section, we define a Proof of Quantum Knowledge (Section 6.1) and then prove that the Zero-knowledge protocols presented in the previous sections (Section 6.2) satisfy this new definition.

6.1 Definition

The content of this subsection was written in collaboration with Andrea Coladangelo, Thomas Vidick and Tina Zhang and a similar version of it also appears in their concurrent and independent work [CVZ20].

A Proof of Knowledge (PoK) is an interactive proof system for some relation $R$ such that if the verifier accepts some input $x$ with high enough probability, then she is convinced that the prover knows some witness $w$ such that $(x, w) \in R$. This notion is formalized by requiring the existence of an efficient extractor $K$ that is able to output a witness for $x$ when $K$ is given oracle access to the prover (and is able to rewind his actions).

Definition 6.1 (Classical Proof of Knowledge [BG93]). Let $R \subseteq X \times Y$ be a relation. A proof system $(P, V)$ for $R$ is a Proof of Knowledge for $R$ with knowledge error $\kappa$ if there exists a polynomial $p > 0$ and a polynomial-time machine $K$ such that for any classical interactive machine $P^*$ that makes $V$ accept some instance $x$ of size $n$ with probability at least $\varepsilon > \kappa(n)$, we have

$$\Pr \left[ (x, K^{P^*(x,y)}(x)) \in R \right] \geq p \left( (\varepsilon - \kappa(n)), \frac{1}{n} \right).$$

In the definition, $y$ corresponds to the side-information that $P^*$ has, possibly including some $w$ such that $(x, w) \in R$.

PoKs were originally defined only considering classical adversaries, and this notion was first studied in the quantum setting by Unruh [Unr12]. The first issue that arises in the quantum setting is which type of query $K$ could be able to perform. To solve this, we assume that $P^*$ always performs some unitary operation $U$. Notice that this can be done without loss of generality since (i) we can consider the purification of the prover, (ii) all the measurements can be performed coherently, and (iii) $P^*$ can keep track of the round of communication in some internal register and $U$ implicitly controls on this value. Then, the quantum extractor $K$ has oracle access to $P^*$ by performing $U$ and $U^\dagger$ on the message register and private register of $P^*$, but $K$ has no direct access to the latter. We denote the extractor $K$ with such an oracle access to $P^*$ as $K^{[P^*(x,\rho)]}$, where here $\rho$ is the (quantum) side-information held by $P^*$. 42
**Definition 6.2 (Quantum Proof of Knowledge [Unr12]).** Let \( R \subseteq X \times Y \) be a relation. A proof system \((P, V)\) for \( R\) is a Quantum Proof of Knowledge for \( R\) with knowledge error \( \kappa \) if there exists a polynomial \( p > 0 \) and a quantum polynomial-time machine \( K \) such that for any quantum interactive machine \( P^* \) that makes \( V \) accept some instance \( x \) of size \( n \) with probability at least \( \varepsilon > \kappa(n) \), then

\[
\Pr \left[ \left( x, K^{P^*(x, \rho)}(x) \right) \in R \right] \geq p \left( (\varepsilon - \kappa(n)), \frac{1}{n} \right).
\]

**Remark 6.3.** In the fully classical case of Definition 6.1, the extractor could repeat the procedure \( \text{poly}( (\varepsilon - \kappa(n)) ) \) times in order to increase the success probability. We notice that this is not known to be possible for a general quantum \( P^* \), since the final measurement to extract the witness would possibly disturb the internal state of \( P^* \), making it impossible to simulate the side-information that \( P^* \) had originally in the subsequent simulations.

We finally move on to the full quantum setting, where we want a Proof of Quantum Knowledge (PoQ). Here, at the end of the protocol, we want the verifier to be convinced that the prover has a quantum witness for the input \( x \).

The first challenge is defining the notion a “relation” between the input \( x \) and some quantum state \(| \psi \rangle\). Classically, we implicitly consider relations for NP languages by fixing some verification algorithm \( V \) for it, and defining \((x, w) \in R\) if and only if \( V \) accepts the input \( x \) with the witness \( w \).

Quantumly, the situation is a bit more delicate, since a witness \(| \psi \rangle\) leads to acceptance probability \( \Pr[Q(x, |\psi\rangle) = 1] \). This issue also appears with probabilistic complexity classes such as MA, and we can solve it by fixing some parameter \( \gamma \) and defining the relation to contain \((x, |\psi\rangle)\) for all quantum states \(| \psi \rangle\) that lead to acceptance probability at least \( \gamma \). The difference here is that we need also to consider the mixture of such quantum states, since they are also valid witnesses in a QMA protocol. Therefore, fixing some quantum verifier \( Q \) and \( \alpha \), we define a quantum relation as follows

\[ R_{Q, \gamma} = \{ (x, \sigma) : Q \text{ accepts } (x, \sigma) \text{ with probability at least } \gamma \} . \]

Notice that with \( R_{Q, \gamma} \), we implicitly define subspaces \( \{S_x\}_x \) such that \((x, \sigma) \in R_{Q, \gamma} \) if and only if \( \sigma \in S_x \).

With this in hand, we can define a QMA-relation.

**Definition 6.4 (QMA-relation).** Let \( A = (A_{\text{yes}}, A_{\text{no}}) \) be a problem in QMA, and let \( Q \) be an associated quantum polynomial-time verification algorithm (which takes as input an instance and a witness), with completeness \( \alpha \) and soundness \( \beta \). Then, we say that \( R_{Q, \gamma} \) is a QMA-relation with completeness \( \alpha \) and soundness \( \beta \) for the problem \( A \). In particular, we have that for \( x \in A_{\text{yes}} \), there exists some \(| \psi \rangle\) such that \((x, |\psi\rangle) \in R_{Q, \alpha} \) and for \( x \in A_{\text{no}} \), for every \( \rho \) and \( \varepsilon > 0 \) it holds that \((x, \rho) \notin R_{Q, \beta + \varepsilon} \).

We can finally define a Proof of Quantum Knowledge.

**Definition 6.5 (Proof of Quantum Knowledge).** Let \( R_{Q, \gamma} \) be a QMA relation. A proof system \((P, V)\) is a Proof of Quantum Knowledge for \( R_{Q, \gamma} \) with knowledge error \( \kappa(n) > 0 \) and quality \( q \), if there exists a polynomial \( p > 0 \) and a polynomial-time machine \( K \) such that for any quantum interactive machine \( P^* \) that makes \( V \) accept some instance \( x \) of size \( n \) with probability at least \( \varepsilon > \kappa(n) \), we have

1. \( K^{P^*(x, \rho)}(x) \) aborts with probability at most \( p \left( (\varepsilon - \kappa(n)), \frac{1}{n} \right) \), and

2. if \( K^{P^*(x, \rho)}(x) \) does not abort, it outputs the quantum state \( \phi \) such that \((x, \phi) \in R_{Q,q(\varepsilon, \frac{1}{n})} \).
6.2 Proof of quantum-knowledge for our $\Xi$-protocol

We show now that the $\Xi$-protocol of Figure 11 is a Proof of Quantum Knowledge with knowledge error inverse polynomially close to 1.

Let $A = (A_{yes}, A_{no})$ to be a problem in $k$-SimQMA, $x \in A$, $\{\Pi_c\}$ be the set of POVMs for $x$ and $P^*$ be a prover that makes the verifier accept with probability at least $\kappa(n) \geq 1 - \frac{1}{2m^2}$ in the $\Xi$-protocol of Figure 11.

1. Run $P^*$ and store the first message $\psi \otimes |z\rangle\langle z|$.

2. For every challenge $c$
   2.1. Simulate $P^*$ on challenge $c$
   2.2. Check (coherently) if the answer correctly opens the committed value, if not abort
   2.3. Copy the opening of the committed values
   2.4. Run $P^*$ backwards on challenge $c$

3. Let $a, b \in \{0, 1\}^n$ be the opened strings

4. Output $X^aZ^b\psi Z^bX^a$

Figure 13: Single-shot Knowledge extractor $K$

Lemma 6.6. Let $A$ and $\{\Pi_c\}$ be as defined in Figure 13 and $Q$ be the verification algorithm for $A$ that consists of picking $c \in [m]$ uniformly at random and measuring the provided witness with $\Pi_c$. Let $\kappa(n) = 1 - \frac{1}{2m^2}$ and $K$ be the $\text{poly}(n)$-time extractor defined in Figure 13. If a quantum interactive machine $P^*$ makes $V$ accept the instance $x$ of size $n$ with probability at least $\varepsilon := 1 - \delta > \kappa(n)$, then we have that

1. $K^{P^*(x,\rho)}(x)$ aborts with probability at most $1 - m^2\delta$, and

2. if $K^{P^*(x,\rho)}(x)$ does not abort, it outputs the quantum state $\phi$ such that $(x, \phi) \in R_{Q,1-\delta - m^2\delta}$.

Proof. Let $a, b \in \{0, 1\}^{2n}$ be the unique values that can be opened by the classical value $|z\rangle$ sent by the prover (or zeroes if the commitments are mal-formed). Doing the same calculations of Equations (12) to (13), and assuming that the acceptance probability of the original protocol is at least $1 - \delta$, it follows that

$$\left( x, X^aZ^b\psi X^aZ^b \right) \in A_{Q,1-\delta}. \tag{26}$$

Our goal now is show how to retrieve the values $a$ and $b$, without damaging the quantum state $\psi$ too much.

Let $\mu_{VM}^P$ be the state shared by the verifier and prover after the commitment phase. Since the message (for honest verifiers) is always a classical value, we can model $P^*$’s behaviour with the unitary $U_c$ performed by him on challenge $c$. Let also $\Pi$ be the projection of $V$ onto the acceptance subspace, $\Pi_c = U_c^\dagger \Pi U_c$ be the operation that performs the prover’s unitary for challenge $c$, performs the measurement of the verifier, and then undoes the prover’s unitary.
Given that $P^*$ makes $V$ accept with probability at least $1 - \delta$ and each challenge is picked with probability $\frac{1}{m}$, it follows that for any challenge $c$, $V$ accepts with probability at least $1 - m\delta$, otherwise the acceptance probability would be strictly less than $1 - \delta$. In other words, it follows that for every $c$, we have that

$$\text{Tr} \left( \hat{\Pi}_c \mu \hat{\Pi}_c \right) \geq 1 - m\delta,$$

which implies that

$$\text{Tr} \left( \hat{\mu} \right) \geq 1 - m^2\delta, \quad \text{for} \quad \hat{\mu} = \hat{\Pi}_m \ldots \hat{\Pi}_1 \mu \hat{\Pi}_1 \ldots \hat{\Pi}_m \quad (27)$$

Let $O$ be the register where the verifier holds the original state $\psi$ and $\phi = \text{Tr}_O(\hat{\mu})$. We have that

$$D(\psi, \phi) = D(\text{Tr}_O(\mu), \text{Tr}_O(\hat{\mu})) \leq D(\mu, \hat{\mu}) \leq m^2\delta. \quad (28)$$

where in the equality we use the definition of $\psi$, $\phi$ and the register $O$, the first inequality follows since trace distance is contractive under CPTP maps and the last inequality holds by Equation (27).

Notice that the decision of an abort by $K$ is strictly less restrictive than the rejections of an honest verifier, since the verifier also tests that the commitment correctly opens. In this case, we can conclude that $K$ does not abort with probability at least $1 - m^2\delta$, proving item 1 of the statement.

Then, if we condition on the event that $K$ does not abort, all the committed information is opened and since the commitment is biding, $K$ holds the unique values $a, b \in \{0, 1\}^n$ that can be opened from $z$. $K$ finishes by outputting $X^aZ^b\phi Z^bX^a$. It follows from Equation (28) and the fact that trace distance is preserved under unitary operations, we have

$$D \left( X^aZ^b\psi X^aZ^b, X^aZ^b\phi X^aZ^b \right) \leq m^2\delta,$$

and therefore

$$\left( x, X^aZ^b\phi X^aZ^b \right) \in R_{Q,1-\delta-m^2\delta},$$

which finishes the proof of item 2 of the statement.

### 6.2.1 Sequential repetition

In the previous section, we have the quantum extractor that works if the knowledge error is very high, namely inverse polynomially close to 1. We show here how to decrease the knowledge leakage, by considering the sequential repetition of the $\Xi$-protocol.
Let $A = (A_{yes}, A_{no})$ to be a problem in $k$-SimQMA, $x \in A$, $\{\Pi_c\}$ be the set of POVMs for $x$ and $P^*$ be a prover that makes the verifier accept with probability at least $\kappa(n) \geq (1 - \frac{1}{2m^2})^\ell$ in the $\ell$-fold sequential repetition of $\Xi$-protocol of Figure 11.

1. $K$ chooses $i \in [\ell]$
2. For $0 < j < i$
   2.1. Pick $c_j$ uniformly at random and put it in the message register
   2.2. Run $P^*$ with $c_j$
   2.3. If $V$ would reject, abort
3. Run Single-shot extractor for the $i$-th game

Figure 14: Knowledge extractor $K$

**Lemma 6.7.** Fix $\ell \geq 1$. If some quantum interactive machine $P^*$ that makes $V$ accept some instance $x$ of size $n$ in $\ell$ sequential repetitions of the $\Xi$-protocol with probability at least $\varepsilon$, then there exists $i \in [\ell]$, such that the probability that $P^*$ passes game $i$, conditioned on the event that $P^*$ passed the games $1, \ldots, i - 1$, is at least $\varepsilon - \ell$.

**Proof.** Let us prove this by contradiction. Let $E_j$ be the event that $P^*$ passes the game $j$. So let us assume that for all $j$, $\Pr[E_j|E_1 \wedge \ldots \wedge E_{j-1}] < \varepsilon - \ell$.

Notice that we can bound the overall acceptance probability as

$$\varepsilon = \Pr[E_1 \wedge \ldots \wedge E_\ell] = \Pr[E_1] \Pr[E_2|E_1] \ldots \Pr[E_\ell|E_1 \wedge \ldots \wedge E_{\ell-1}] < (\varepsilon - \ell)^\ell = \varepsilon,$$

which is a contradiction. $\square$

**Lemma 6.8.** Let $A$ and $\{\Pi_c\}$ be defined as in Figure 14 and $Q$ be the verification algorithm for $A$ that consists of picking $c \in [m]$ uniformly at random and measuring the provided witness with $\Pi_c$. Let $\kappa(n) = (1 - \frac{1}{2m^2})^\ell$ and $K$ be the $\text{poly}(n)$-time extractor defined in Figure 14. If a quantum interactive machine $P^*$ makes $V$ accept the instance $x$ of size $n$ with probability at least $\varepsilon := (1 - \delta)^\ell > \kappa(n)$, then we have that

$$\Pr\left[\left(x, K^{P^*(x,\rho)}(x)\right) \in R_{Q,1-\delta-m^2\delta}\right] \geq \frac{\varepsilon}{\ell}(1 - m^2\delta).$$

and $K$ runs in time $\text{poly}(n)$.

**Proof.** From Lemma 6.7, we know that there exists at least one value of $i^* \in [k]$ such that the success of probability in the $i^*$-th round, conditioned on the event of success on rounds $1, \ldots, i^*$ is $1 - \delta$, and we recall that by definition $\delta < \frac{1}{2m^2}$. We have then that the value $i$ guessed by $K$ is equal to $i^*$ with probability at least $\frac{1}{2}$. 46
Let us assume now that \( i = i^* \). Using a slightly different notation from Lemma 6.6, let \( \mu \) be the initial state of \( P^* \) and \( \hat{\Pi}_j = \prod_{j \neq i} U_j V_j \) be the verifier operation in round \( j \ V_j \), followed by the provers’ operation operation on round \( j \), and finally the projection onto the acceptance subspace on the \( j \)-th round.

The probability that \( K \) does not abort in the first \( i - 1 \) steps is

\[
\text{Tr} \left( \prod_{i=1}^{i-1} \hat{\Pi}_i \mu \hat{\Pi}_1 \ldots \hat{\Pi}_{i-1} \right) \geq \text{Tr} \left( \prod_{j=i}^{\ell} \hat{\Pi}_j \ldots \hat{\Pi}_1 \mu \hat{\Pi}_1 \ldots \hat{\Pi}_j \right) = \varepsilon,
\]

where the equality holds since we assume that \( P^* \) makes the verifier accept with probability \( \varepsilon \).

Remark 6.9. We notice that that \( P^* \) would need multiple copies of the witness in order to pass the sequential repetitions of the \( \Xi \)-protocol. However, the extractor of Figure 14 can only extract one such copy. It could be easily extended to output a constant number of copies and we leave as an open problem achieving better PoQ extractors for our protocol.

7 Non-interactive zero-knowledge protocol for QMA in the secret parameter model

In this section, using similar techniques of Section 5, we show that all problems in SimQMA have a QNISZK protocol in the secret parameter model, quantizing the result by Pass and Shelat [PS05].

We start by defining the model.

Definition 7.1 (Quantum non-interactive proofs in the classical secret parameter model). A triple of algorithms \((D, P, V)\) is called a quantum non-interactive proof in the secret parameter model for a promise problem \( A = (A_{yes}, A_{no}) \) where \( D \) is a probabilistic polynomial time algorithm, \( V \) is a quantum polynomial time algorithm and \( P \) is an unbounded quantum algorithm such that there exists a negligible function \( \varepsilon \) such that the following conditions follow:

Completeness: for every \( x \in A_{yes} \), there exists some \( P \)

\[
\Pr[(r_P, r_V) \leftarrow D(1^{\|x\|}); \pi \leftarrow P(x, r_P); V(x, r_V, \pi) = 1] \geq 1 - \varepsilon(n).
\]

Soundness: for every \( x \in A_{no} \) and every \( P \)

\[
\Pr[(r_P, r_V) \leftarrow D(1^{\|x\|}); \pi \leftarrow P(x, r_P); V(x, r_V, \pi) = 1] \leq \varepsilon(n).
\]

Statistical zero-knowledge: for every \( x \in A_{yes} \), there is a polynomial time algorithm \( S \) such that for the state \( \sigma = S(x) \) and \( \rho = \sum_{(r_V, s_P) \leftarrow D(1^{\ell})} P_{r_V, s_P} |r_V\rangle \otimes P(x, r_P) \) we have that \( \sigma \approx_s \rho \).

The non-interactive protocol is very similar to the \( \Xi \) protocol, with a small (but crucial) change: instead of using commitments for the one-time pad key, the trusted party picks these values and reveals just a constant number of these values to the verifier (and all of them to the prover). Let us be a bit more precise. The trusted party picks uniformly at random the one-time pad keys \( a \) and \( b \)
and sends them to the prover. For the verifier, the trusted party sends \( S, a|_S \) and \( b|_S \), where \( S \) is a random subset of \( k \) indices of \( a \) and \( b \).

The prover uses \( a \) and \( b \) to one-time pad the simulatable proof and sends this one-time padded state to the verifier.

The verifier picks one of the checking terms uniformly at random. If the qubits corresponding to the chosen term are not in \( S \), the verifier accepts. Otherwise, the verifier uses \( a|_S \) and \( b|_S \) to decrypt the qubits corresponding to such term, and finally performs the check. The completeness of the protocol is straightforward. For soundness, we have that with inverse polynomial probability the revealed bits will allow the verifier to check the desired term of the encoded history state. Finally, the zero-knowledge property holds since the quantum proof is simulatable.

We describe now the protocol more formally.

Let \( A = (A_{\text{yes}}, A_{\text{no}}) \) be a problem in \( k\text{-SimQMA} \) with soundness \( \delta \), \( x \in A \), \( \{\Pi_c\} \) be the set of POVMs for \( x \), and \( \tau \) be a (supposed) simulatable witness for \( x \).

1. \( D \) picks \( a, b \in \{0, 1\}^n \) and \( S \subseteq [n] \), with \( |S| = k \), uniformly at random.
2. \( D \) sends \((a, b)\) to the prover and \((S, a|_S, b|_S)\) and to the verifier.
3. The prover sends \( \tilde{\tau}_{a,b} \) to the verifier
4. The verifier picks \( c \in \{0, 1\}^m \)
5. If the qubits corresponding to \( \Pi_c \) are not in \( S \), the verifier accepts
6. Otherwise, the verifier applies \( Xa|_S Zb|_S \) to the qubits in \( \Pi_c \) and accepts according to its output.

Figure 15: QNIZK protocol in the secret parameter model for SimQMA

Remark 7.2. We make the verifier pick one of the terms and then check with the set \( S \) in order to simplify the proof of soundness. The verifier could instead check some \( \Pi_c \) that matches with \( S \) or accept when this is not possible and this protocol would still be sound.

Lemma 7.3. The protocol in Figure 15 has completeness \( 1 - \text{negl}(|x|) \) and soundness \( 1 - \frac{1 - \delta}{n^k} \).

Proof. If \( x \in A_{\text{yes}} \), then the prover sends the honest one-time pad of the witness \( \tau \) that makes the verifier of the QMA protocol accept with probability exponentially close to 1.

In this case, if the qubits of the randomly chosen \( \Pi_c \) are not included in \( S \), the verifier always accepts, otherwise \( V \) accepts with probability exponentially close to 1, by the properties of \( \tau \).

Let \( x \in A_{\text{no}} \), \((a, b)\) and \((S, a|_S, b|_S)\) be the values sent by the trusted party to the prover and verifier, respectively. Let also \( \rho \) be the quantum state sent by the prover and \( \sigma = Xa|_S Zb|_S \rho Xa|_S Zb|_S \).

By definition, the acceptance probability of the protocol is then

\[
\frac{1}{m} \sum_{S, c} Pr[S_c \not\subseteq S] + Pr[S_c \subseteq S] \text{Tr} \left( \Pi_c Xa|_{S_c} Zb|_{S_c} \rho Zb|_{S_c} Xa|_{S_c} \right)
\]

\[
= \left(1 - \frac{1}{n^k}\right) + \frac{1}{n^k} \text{Tr} \left( \frac{1}{m} \sum_{c} \Pi_c Xa|_{S_c} Zb|_{S_c} \rho Zb|_{S_c} Xa|_{S_c} \right).
\]
Notice that
\[
\text{Tr} \left( \frac{1}{m} \sum_c \Pi_c X^{a|s_c} Z^{b|s_c} \rho Z^{b|s_c} X^{a|s_c} \right) = \text{Tr} \left( \frac{1}{m} \sum_c \Pi_c \sigma \right) \leq \max_{\tau} \text{Tr} \left( \frac{1}{m} \sum_c \Pi_c \tau \right) \leq \delta
\]

where the first equality holds since \( \Pi_c \) is acting only on the decoded values and the last inequality holds since \( x \in A_{\text{no}} \).

Therefore, the overall acceptance probability is at most \( 1 - \frac{1 - \delta}{n^k} \).

Let \( A = (A_{\text{yes}}, A_{\text{no}}) \) to be a problem in \( k\)-SimQMA, \( x \in A_{\text{yes}} \), and \( \rho_c = \rho(x, S_c) \) be the local density matrix of a simulatable witness \( \tau \) for \( x \) on the qubits corresponding to \( \Pi_c \).

1. The simulator picks random values \( a, b \in \{0, 1\}^\ell \) and \( S \subseteq [\ell] \)
2. Simulator computes the (constant-size) reduced density matrix \( \sigma \) of the qubits in positions \( S \) and let \( \sigma(S) = \rho^S \otimes I^S \)
3. Output \( |S\rangle\langle S| \otimes |a\rangle\langle a| \otimes |b\rangle\langle b| \otimes \sigma \).

**Figure 16:** Simulator for the QNIZK protocol

**Lemma 7.4.** The protocol is statistical zero-knowledge.

**Proof.** We show that the protocol is statistical zero-knowledge by showing the density matrices of the output of the simulator and the real protocol are close.

In the real protocol, let \( \tau \) be the simulatable proof in the QMA protocol for a yes-instance. In the honest run of the protocol, we have that the view of the verifier after the prover sends the message is
\[
\frac{1}{2^n(n^k)} \sum_{a,b,S} |S\rangle\langle S| \otimes |a\rangle\langle a| \otimes |b\rangle\langle b| \otimes \tau_{a,b}. \tag{31}
\]

Notice that since we are averaging over all possible values of \( a_S \) and \( b_S \), Equation (31) is equal to
\[
\rho_p = \frac{1}{2^{|S|}(|S|)} \sum_{a, b, S} |S\rangle\langle S| \otimes |a\rangle\langle a| \otimes |b\rangle\langle b| \otimes \left( \tilde{\tau}_{a_S, b_S} \otimes I^S \right),
\]
where \( \tau_S = Tr_{\overline{S}}(\tau) \).

By definition, the output of the simulator is
\[
\rho_s = \frac{1}{2^{|S|}(|S|)} \sum_{S, a, b, S} |S\rangle\langle S| \otimes |a\rangle\langle a| \otimes |b\rangle\langle b| \otimes \left( \tilde{\rho}_{a_S, b_S} \otimes I^S \right).
\]

To conclude the proof, we have that
\[
D(\rho_p, \rho_s) \leq D(\tau_S, \sigma_S) \leq \text{negl}(n),
\]
where the first inequality holds since the trace distance is subadditive under tensor product and preserved under unitary operations. The second inequality follows from Definition 3.6. \qed
Theorem 7.5. Every problem in QMA has a QNISZK in the secret parameter model.

Proof. Direct from Lemmas 3.7, 7.3 and 7.4.

Remark 7.6. In the cryptography literature, there is a notion called adaptive soundness and zero-knowledge where the witness is chosen after the trusted party provides the secret parameter (or CRS). We notice that our protocols can also handle these stronger notions.

7.1 Extension to QAM

In [KLGN19], the authors generalize both the complexity classes QMA to allow interaction between the verifier and prover to allow public randomness, both classical (i.e., classical coins) and quantum (i.e., sharing EPR pairs). In this framework, we consider the class QAM, where the verifier sends random coins to the prover, who then answers with a quantum state. We notice that in [KLGN19], this complexity class is called $cq$QAM.

Definition 7.7 (QAM). A promise problem $A = (A_{\text{yes}}, A_{\text{no}})$ is in QAM if and only if there exist polynomials $r$, $p$, $q$ and a polynomial-time uniform family of quantum circuits $\{Q_{r,n}\}$, where $Q_{r,n}$ takes as input a string $x \in \Sigma^*$ with $|x| = n$, a $p(n)$-qubit quantum state $|\psi\rangle$, and $q(n)$ auxiliary qubits in state $|0\rangle^{\otimes q(n)}$, such that:

Completeness: If $x \in A_{\text{yes}}$, $\Pr_r[\exists |\psi_r\rangle \text{ s.t. } Q_n \text{ accepts } (x, |\psi\rangle)] \geq 2/3$.

Soundness: If $x \in A_{\text{no}}$, $\Pr_r[\forall |\psi_r\rangle Q_n \text{ accepts } (x, |\psi\rangle)] \leq 1/3$.

It is straightforward to generalize Definition 3.6 and define SimQAM where the POVMs and the reduced density matrices depend also in the public random string $r$. It is not hard to see that we can also generalize Lemma 3.7 and show that QAM = SimQAM.

In this case, our QNIZK protocol can be adapted to this complexity class, by just making the trusted party pick also the $r$ uniformly at random and sending it to both the prover and the verifier. Given a fixed $r$, the same arguments as shown for SimQMA hold.

Theorem 7.8. Every problem in QAM has a QNISZK in the secret parameter model.

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