The Torus-Equivariant Cohomology of Nilpotent Orbits

Peter Crooks

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Abstract

We consider aspects of the geometry and topology of nilpotent orbits in finite-dimensional complex simple Lie algebras. In particular, we give the equivariant cohomologies of the regular and minimal nilpotent orbits with respect to the action of a maximal compact torus of the overall group in question.

1 Introduction

1.1 Generalities

Throughout, we let \( G \) be a connected, simply-connected complex simple linear algebraic group. Let \( K \subseteq G \) be a maximal compact subgroup, and fix a maximal torus \( T \subseteq K \). Set \( H := T_C \), a maximal torus of \( G \). Denote by \( g, k, t, \) and \( h \) the Lie algebras of \( G, K, T, \) and \( H \), respectively. Let \( \text{Ad} : G \to \text{GL}(g) \) and \( \text{ad} : g \to \text{gl}(g) \) denote the adjoint representations of \( G \) and \( g \), respectively. Let \( \Delta \subseteq \text{Hom}(T, U(1)) = \text{Hom}(H, \mathbb{C}^*) \) denote the resulting collection of roots of \( g \) with respect to the adjoint representation of \( T \). By fixing a Borel subgroup \( B \subseteq G \) containing \( H \), we specify collections \( \Delta^+, \Delta^- \subseteq \Delta \) of positive and negative roots, respectively. Let \( \Pi \subseteq \Delta^+ \) denote the resulting collection of simple roots.

Recall that a point \( \xi \in g \) is called nilpotent if the vector space endomorphism \( \text{ad}_\xi : g \to g \) is nilpotent. Recall also that the nilpotent cone is the closed subvariety \( N \) of \( g \) consisting of the nilpotent elements. We call an adjoint \( G \)-orbit a nilpotent orbit if it is contained in \( N \). As an orbit of an algebraic \( G \)-action, any nilpotent orbit is a smooth locally closed subvariety of \( g \).

It is well-known that there exist only finitely many nilpotent orbits of \( G \). Indeed, if \( G = \text{SL}_n(\mathbb{C}) \), then one can use Jordan canonical forms to give an explicit indexing of the nilpotent orbits by the partitions of \( n \).

Furthermore, the nilpotent orbits constitute an algebraic stratification of \( N \) (see [6]). In other words, we have the partial order on the set of nilpotent orbits given by \( \Theta_1 \leq \Theta_2 \) if and only if \( \Theta_1 \subseteq \overline{\Theta_2} \) (the Zariski-closure of \( \Theta_2 \) in \( N \)). Hence,

\[
\overline{\Theta} = \bigcup_{\Omega \leq \Theta} \Omega
\]

*Department of Mathematics. University of Toronto. Toronto, ON, Canada.
peter.crooks@utoronto.ca
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for all nilpotent orbits Θ.

It turns out that the set of nilpotent orbits has a unique maximal element, Θ_{reg}, and a unique minimal non-zero element, Θ_{min}. These distinguished orbits are called the regular and minimal nilpotent orbits, respectively. The former consists precisely of the regular nilpotent elements of g, while the latter is the orbit of a root vector for a long root.

1.2 Context

The study of nilpotent orbits lies at the interface of algebraic geometry, representation theory, and symplectic geometry. Indeed, one has the famous Springer resolution

$$\mu : T^*(G/B) \to N$$

of the singular nilpotent cone (see [6]). The fibres of μ over a given nilpotent orbit Θ are isomorphic as complex varieties, and this isomorphism class is called the Springer fibre of Θ. The Springer correspondence then gives a realization of the irreducible complex W-representations on the Borel-Moore homology groups of the Springer fibres (see [6]).

From the symplectic standpoint, we note that coadjoint G-orbits are canonically complex symplectic manifolds. Since the Killing form on g provides an isomorphism between the adjoint and coadjoint representations of G, it follows that adjoint G-orbits (and in particular, nilpotent G-orbits) are naturally complex symplectic manifolds.

Some attention has also been given to the matter of computing topological invariants of nilpotent orbits. In [7], Collingwood and McGovern compute the fundamental group of each nilpotent orbit in the classical Lie algebras. Also, Juteau’s paper [11] gives the integral cohomology groups of the minimal nilpotent orbit in each of the finite-dimensional complex simple Lie algebras. Additionally, Biswas and Chatterjee compute \(H^2(\Theta; \mathbb{R})\) for Θ any nilpotent orbit in a finite-dimensional complex simple Lie algebra (see their paper [2]).

Our contribution is a computation of the \(T\)-equivariant cohomology algebras of the G-orbits Θ_{reg} and Θ_{min}. (To this end, \(H^*_T(X)\) shall always denote the \(T\)-equivariant cohomology over \(\mathbb{Q}\) of a \(T\)-manifold X.) We state our result below.

**Theorem 1.**

(i) \(H^*_T(\Theta_{\text{reg}}) \cong H^*(G/B; \mathbb{Q})\)

(ii) Let \(\alpha \in \Delta_+\) be the highest root, and let \(\Xi := \{\beta \in \Pi : (\alpha, \beta) = 0\}\). Let \(W_\Xi\) be the subgroup of \(W\) generated by the reflections \(s_\beta, \beta \in \Xi\). Then, \(H^*_T(\Theta_{\text{min}})\) is isomorphic to the quotient of

\[
\{f \in \text{Map}(W/W_\Xi, H^*_T(pt)) : (w \cdot \beta)((f([w]) - f([ws_\beta])) \\
\forall w \in W, \beta \in \Delta_-, (\alpha, \beta) \neq 0\}
\]

by the ideal generated by the map \(W/W_\Xi \to H^*_T(pt), [w] \mapsto w \cdot \alpha\).

1.3 Structure of the Article

Section 2 is devoted to an examination of the regular nilpotent orbit. Specifically, we establish a few facts concerning the structure of the \(G\)-stabilizer \(C_G(\eta)\) of a point
$\eta \in \Theta_{\text{reg}}$. We then give a new description of $\Theta_{\text{reg}} \cong G/C_G(\eta)$ as a $T$-manifold (see Theorem 3). This description is suitable for purposes of computing $H^*_T(\Theta_{\text{reg}})$.

Section 3 treats the case of the minimal nilpotent orbit, but the approach differs considerably from that adopted when studying $\Theta_{\text{reg}}$. We begin by introducing a natural $\mathbb{C}^*$-action on nilpotent orbits. Via this action, we define $\mathbb{P}(\Theta_{\text{min}})$, a smooth closed subvariety of $\mathbb{P}(\mathfrak{g})$. This variety has interesting properties beyond those materially relevant to computing $H^*_T(\Theta_{\text{min}})$. In particular, $\mathbb{P}(\Theta_{\text{min}})$ is naturally a symplectic manifold, and the $T$-action on $\Theta_{\text{min}}$ descends to a Hamiltonian action on $\mathbb{P}(\Theta_{\text{min}})$. Accordingly, we give an explicit description of $\mathbb{P}(\Theta_{\text{min}})$ (see 3.2) and use it to find the moment polytope of $\mathbb{P}(\Theta_{\text{min}})$ (see 3.3).

In 3.4, we use GKM Theory to provide a description of $H^*_T(G/P)$, where $P \subseteq G$ is a parabolic subgroup containing $T$. This is done in recognition of the fact (which we prove in 3.5) that $\mathbb{P}(\Theta_{\text{min}})$ is $G$-equivariantly isomorphic to $G/P_\Xi$, where $P_\Xi$ is the parabolic determined by $\Xi$.

Then remains to relate the graded algebras $H^*_T(G/P_\Xi)$ and $H^*_T(\Theta_{\text{min}})$. This is achieved via the Thom-Gysin sequence in $T$-equivariant cohomology, which allows us to exhibit $H^*_T(\Theta_{\text{min}})$ as a quotient of $H^*_T(G/P_\Xi)$. Indeed, we take the quotient of $H^*_T(G/P_\Xi)$ by the ideal generated by the $T$-equivariant Euler class of the associated line bundle $G \times_{P_\Xi} \mathfrak{g}_\alpha \to G/P_\Xi$.

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2 The Regular Nilpotent Orbit

Throughout this section, we may actually take $G$ to be semisimple. Now, recall that an element $\xi \in \mathfrak{g}$ is called regular if the dimension of the Lie algebra centralizer $C_\mathfrak{g}(\xi) = \{X \in \mathfrak{g} : [X, \xi] = 0\}$ coincides with the rank of $\mathfrak{g}$. The regular nilpotent elements of $\mathfrak{g}$ actually constitute $\Theta_{\text{reg}}$.

Let us construct a reasonably standard representative of $\Theta_{\text{reg}}$. Indeed, for each $\beta \in \Pi$, choose a root vector $e_\beta \in \mathfrak{g}_\beta \setminus \{0\}$. Consider the nilpotent element

$$\eta := \sum_{\beta \in \Pi} e_\beta.$$ 

In [12], Kostant proved that $\eta \in \Theta_{\text{reg}}$. Furthermore, one can easily prove that $C_\mathfrak{g}(\eta)$ belongs to the positive nilpotent subalgebra $\mathfrak{n}_+ := \bigoplus_{\beta \in \Delta^+} \mathfrak{g}_\beta$.

Let $C_\mathfrak{g}(\eta) = \{g \in G : \text{Ad}_g(\eta) = \eta\}$ be the $G$-stabilizer of $\eta$. This gives an isomorphism $\Theta_{\text{reg}} \cong G/C_\mathfrak{g}(\eta)$ of complex $G$-varieties, where the action of $C_\mathfrak{g}(\eta)$ on $G$ is given by $x : g \mapsto gx^{-1}$, $x \in C_\mathfrak{g}(\eta)$, $g \in G$. 

3
Having realized $\Theta_{\text{reg}}$ in this way, we turn our attention to $C_G(\eta)$. To this end, we recall that the inner automorphism group (or adjoint group) of $\mathfrak{g}$ is the subgroup $\text{Int}(\mathfrak{g})$ of $\text{GL}(\mathfrak{g})$ generated by all automorphisms of the form $e^{ad \xi}$, $\xi \in \mathfrak{g}$. Since $\text{Ad}_{\exp(\xi)} = e^{ad \xi}$ for all $\xi \in \mathfrak{g}$, it follows that $\text{Int}(\mathfrak{g})$ is precisely the image of the adjoint representation $\text{Ad}: G \rightarrow \text{GL}(\mathfrak{g})$. Hence, $\text{Int}(\mathfrak{g})$ is a connected Zariski-closed subgroup of $\text{GL}(\mathfrak{g})$.

We shall require the below theorem concerning the structure of the $\text{Int}(\mathfrak{g})$-stabilizer $C_{\text{Int}(\mathfrak{g})}(\eta)$ of $\eta$.

**Theorem 2.** The centralizer $C_{\text{Int}(\mathfrak{g})}(\eta)$ is a connected abelian unipotent subgroup of $\text{Int}(\mathfrak{g})$.

This is Theorem 2.6 in [13].

We note that a connected unipotent complex linear algebraic group is isomorphic to affine space as a variety. For our purposes, the relevant observation is that $C_{\text{Int}(\mathfrak{g})}(\eta)$ is isomorphic as a complex manifold to $\mathbb{C}^n$ for some $n$.

**Proposition 1.** The inclusion of the centre $Z(G) \hookrightarrow C_G(\eta)$ is a homotopy-equivalence.

**Proof.** Note that $\phi: C_G(\eta) \rightarrow C_{\text{Int}(\mathfrak{g})}(\eta)$, $g \mapsto \text{Ad}_g$, is a surjective Lie group morphism. Since $G$ is connected, $Z(G)$ is the kernel of the adjoint representation, and hence is also the kernel of $\phi$. This yields a fibre bundle

$$Z(G) \rightarrow C_G(\eta) \xrightarrow{\phi} C_{\text{Int}(\mathfrak{g})}(\eta).$$

Since the base space $C_{\text{Int}(\mathfrak{g})}(\eta)$ is contractible, our bundle is trivial. Noting that the inclusion of a fibre in a trivial bundle over $C_{\text{Int}(\mathfrak{g})}(\eta)$ is a homotopy-equivalence, the inclusion $Z(G) \hookrightarrow C_G(\eta)$ is also a homotopy-equivalence. \qed

An immediate corollary of Proposition 1 is the existence of an isomorphism of graded $\mathbb{Z}$-algebras between the $G$-equivariant cohomology $H^*_G(\Theta_{\text{reg}}; \mathbb{Z})$ of the regular nilpotent orbit and the group cohomology $H^*_{\text{gp}}(Z(G); \mathbb{Z})$ of the finite group $Z(G)$.

**Corollary 1.** $H^*_G(\Theta_{\text{reg}}; \mathbb{Z}) \cong H^*_{\text{gp}}(Z(G); \mathbb{Z})$

**Proof.** Recall that

$$H^*_G(\Theta_{\text{reg}}; \mathbb{Z}) \cong H^*_G(G/C_G(\eta); \mathbb{Z}) = H^*((G/C_G(\eta))/G; \mathbb{Z}),$$

where $(G/C_G(\eta))/G$ is the quotient of $EG \times (G/C_G(\eta))$ by the diagonal action of $G$. This quotient is homeomorphic to $EG/C_G(\eta)$, and hence

$$H^*_G(\Theta_{\text{reg}}; \mathbb{Z}) \cong H^*(EG/C_G(\eta); \mathbb{Z}).$$

By Proposition 1, $EG/Z(G) \rightarrow EG/C_G(\eta)$ is a fibre bundle with contractible fibre $C_G(\eta)/Z(G)$. Hence,

$$H^*(EG/C_G(\eta); \mathbb{Z}) \cong H^*(EG/Z(G); \mathbb{Z}).$$

However, we may take $EG/Z(G)$ to be the classifying space $BZ(G)$, whose singular cohomology coincides with the group cohomology $H^*_{\text{gp}}(Z(G); \mathbb{Z})$. This completes the proof. \qed
Corollary 2. (i) There is a natural complex Lie group isomorphism $C_G(\eta)_0 \cong C_{\text{Int}(\mathfrak{g})}(\eta)$, where $C_G(\eta)_0$ is the identity component of $C_G(\eta)$.

(ii) There is a natural central extension

$$1 \to Z(G) \to C_G(\eta) \to C_G(\eta)_0 \to 1,$$

and the inclusion $C_G(\eta)_0 \to C_G(\eta)$ is a splitting. In particular, $C_G(\eta)$ is the internal direct product $Z(G) \times C_G(\eta)_0$.

Proof. Since $Z(G) \to C_G(\eta)$ is a homotopy equivalence, it induces a group isomorphism

$$\pi_0(Z(G)) \xrightarrow{\cong} \pi_0(C_G(\eta)) \cong C_G(\eta)/C_G(\eta)_0.$$

Hence, if $\omega \in C_G(\eta)/Z(G)$ is a coset, then there exists a unique $g \in C_G(\eta)_0$ for which $[g] = \omega$. Now, recall that $C_G(\eta)/Z(G) \to C_{\text{Int}(\mathfrak{g})}(\eta), [g] \mapsto \text{Ad}_g$, is an isomorphism. Hence, if $f \in C_{\text{Int}(\mathfrak{g})}(\eta)$, then there exists a unique $g \in C_G(\eta)_0$ for which $\text{Ad}_g = f$. Accordingly, $\varphi : C_{\text{Int}(\mathfrak{g})}(\eta) \to C_G(\eta)_0$, $\text{Ad}_g \mapsto g, g \in C_G(\eta)_0$, is a well-defined complex Lie group isomorphism.

For the second part, note that one always has the central extension

$$1 \to Z(G) \to C_G(\eta) \xrightarrow{\pi} C_{\text{Int}(\mathfrak{g})}(\eta) \to 1,$$

where $\pi : C_G(\eta) \to C_{\text{Int}(\mathfrak{g})}(\eta)$ is the projection map. By replacing $C_{\text{Int}(\mathfrak{g})}(\eta)$ with the isomorphic copy $C_G(\eta)_0$ and setting $\psi := \varphi \circ \pi : C_G(\eta) \to C_G(\eta)_0$, we obtain the central extension

$$1 \to Z(G) \to C_G(\eta) \xrightarrow{\psi} C_G(\eta)_0 \to 1.$$

Note that the inclusion $C_G(\eta)_0 \to C_G(\eta)$ splits this sequence. \hfill $\square$

Theorem 3. The regular nilpotent orbit $\Theta_{\text{reg}}$ is $T$-equivariantly diffeomorphic to a product $K/Z(G) \times V$, where $V$ is a finite-dimensional real vector space on which $T$ acts trivially.

Proof. Earlier, we noted that the centralizer $C_G(\eta)$ belonged to the nilpotent subalgebra $\mathfrak{n}_+$. Letting $N$ denote the connected closed subgroup of $G$ with Lie algebra $\mathfrak{n}_+$, this fact implies the inclusion $C_G(\eta)_0 \subseteq N$.

Letting $A$ denote the connected closed subgroup of $G$ with (real) Lie algebra $\mathfrak{a} \subseteq \mathfrak{g}$, the Iwasawa decomposition gives a diffeomorphism $\Phi : K \times A \times N \xrightarrow{\cong} G,$

$$(k, a, n) \mapsto \text{kan}.$$ 

Now, let $Z(G) \times C_G(\eta)_0$ act on $G$ via the $C_G(\eta)$-action on $G$ and the isomorphism $Z(G) \times C_G(\eta)_0 \to C_G(\eta)$. Explicitly, this action is given by

$$(z, h) : g \mapsto g(z h)^{-1},$$

$$(z, h) \in Z(G) \times C_G(\eta)_0, g \in G.$$ We enlarge this to a $T \times (Z(G) \times C_G(\eta)_0)$-action with $T$ acting on $G$ by left-multiplication. Note that $\Phi$ is then a $T \times (Z(G) \times C_G(\eta)_0)$-manifold isomorphism for the action of $T \times (Z(G) \times C_G(\eta)_0)$ on $K \times A \times N$ defined by

$$(t, z, h) : (k, a, n) \mapsto (tkz^{-1}, a, nh^{-1}),$$

$$(t, z, h) \in T \times (Z(G) \times C_G(\eta)_0), (k, a, n) \in K \times A \times N.$$
\[(t, z, h) \in T \times (Z(G) \times C_G(\eta)_{0}), (k, a, n) \in K \times A \times N.\] It follows that
\[\Theta_{\text{reg}} \cong G/C_G(\eta) = G/((Z(G) \times C_G(\eta)_{0})\]
is \(T\)-equivariantly diffeomorphic to the quotient
\[(K \times A \times N)/(Z(G) \times C_G(\eta)_{0}),\]
endowed with its residual \(T\)-action. The latter is clearly \(T\)-equivariantly diffeomorphic to
\[K/Z(G) \times A \times N/C_G(\eta)_{0},\]
where \(T\) acts by left-multiplication on the factor \(K/Z(G)\) and trivially on the factors \(A\) and \(N/C_G(\eta)_{0}\). Since \(A\) is diffeomorphic to its Lie algebra, it remains only to establish that \(N/C_G(\eta)_{0}\) is diffeomorphic to a real vector space. However, this follows from the fact that a quotient of a nilpotent connected simply-connected Lie group by a connected closed subgroup is diffeomorphic to a real vector space (see [13]).

**Corollary 3.** There is an isomorphism \(H^*_T(\Theta_{\text{reg}}) \cong H^*(G/B; \mathbb{Q})\).

**Proof.** By Theorem 3 \(H^*_T(\Theta_{\text{reg}}) \cong H^*_T(K/Z(G))\). Since \(Z(G)\) is a finite group, the action of \(T\) on \(K/Z(G)\) is locally free, and
\[H^*_T(K/Z(G)) \cong H^*(T\backslash K/Z(G); \mathbb{Q}) \cong H^*(T\backslash K; \mathbb{Q}) \cong H^*(G/B; \mathbb{Q}).\]

\[3\] The Minimal Nilpotent Orbit

**3.1 A \(C^*\)-Action on Nilpotent Orbits**

Fix a non-zero nilpotent orbit \(\Theta \subseteq \mathfrak{g}\) and a point \(\xi \in \Theta\). By the Jacobson-Morozov Theorem, there exist a semisimple element \(h \in \mathfrak{g}\) and a nilpotent element \(f \in \mathfrak{g}\) for which \((\xi, h, f)\) is an \(\mathfrak{sl}_2(\mathbb{C})\)-triple with nil-positive element \(\xi\). We note that for all \(\lambda \in \mathbb{C}\),
\[
\text{Ad}_{\exp(\lambda h)}(\xi) = e^{\text{ad}_{\lambda h}}(\xi) = e^{2\lambda} \xi.
\]
From this calculation, it follows that \(\Theta\) is invariant under the scaling action of \(C^*\) on \(\mathfrak{g}\). Accordingly, we introduce
\[\mathbb{P}(\Theta) := \Theta/C^*,\]
a smooth quasi-projective subvariety of \(\mathbb{P}(\mathfrak{g})\). Since the actions of \(G\) and \(C^*\) on \(\mathfrak{g}\) commute, the \(G\)-action descends to the quotients \(\mathbb{P}(\Theta)\) and \(\mathbb{P}(\mathfrak{g})\).

We remark that \(\mathbb{P}(\Theta)\) has a rich geometric structure. To see this, choose a \(K\)-invariant Hermitian inner product \(\langle , \rangle : \mathfrak{g} \otimes_{\mathbb{R}} \mathfrak{g} \to \mathbb{C}\). This yields a \(K\)-invariant Kähler structure on \(\mathbb{P}(\mathfrak{g})\). Since the usual action of \(U(n + 1)\) on \(\mathbb{P}^n\) is Hamiltonian, so too is the action of \(K\) on \(\mathbb{P}(\mathfrak{g})\). Furthermore, one has the moment map \(\Phi : \mathbb{P}(\mathfrak{g}) \to \mathfrak{t}^*\) defined by
\[
\Phi([\xi])(X) = \frac{\text{Im}(\langle [X, \xi], \xi \rangle)}{\langle \xi, \xi \rangle},
\]
where \( X \in \mathfrak{g} \setminus \{0\} \) and \( \eta \in \mathfrak{k} \) (see [8] for a derivation of \( \Phi \)). Note that the Kähler structure on \( \mathbb{P}(\mathfrak{g}) \) restricts to a \( K \)-invariant Kähler structure on the smooth subvariety \( \mathbb{P}(\Theta) \), and the action of \( K \) on \( \mathbb{P}(\Theta) \) is Hamiltonian.

It should be noted that \( \mathbb{P}(\Theta) \) is generally not projective. However, \( \mathbb{P}(\Theta_{\min}) \) is the \( G \)-orbit in \( \mathbb{P}(\mathcal{N}) \) of minimal dimension, meaning that it is a closed (hence projective) subvariety of \( \mathbb{P}(\mathfrak{g}) \). This will be crucial to our study of \( \mathbb{P}(\Theta_{\min}) \), and subsequently to our description of \( \Theta_{\min} \) itself.

### 3.2 Description of the \( T \)-Fixed Points

Let us take a moment to examine the Hamiltonian action of \( T \) on \( \mathbb{P}(\Theta) \), where \( \Theta \subseteq \mathfrak{g} \) is a non-zero nilpotent orbit. We have

\[
\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\beta \in \Delta} \mathfrak{g}_\beta,
\]

the weight space decomposition of the representation \( \text{Ad}|_T \). Note that a point in \( \mathbb{P}(\mathfrak{g}) \) is fixed by \( T \) if and only if it is a class of vectors in \( \mathfrak{g} \setminus \{0\} \) with the property that \( T \) acts by scaling each vector. In other words,

\[
\mathbb{P}(\mathfrak{g})^T = \mathbb{P}(\mathfrak{h}) \cup \{ \mathfrak{g}_\beta : \beta \in \Delta \}.
\]

With this description, we may determine \( \mathbb{P}(\Theta)^T \). Indeed, since \( \mathfrak{h} \) consists of semisimple elements of \( \mathfrak{g} \) while \( \Theta \) consists of non-zero nilpotent elements, we find that \( \mathfrak{h} \cap \Theta = \emptyset \). Hence,

\[
\mathbb{P}(\Theta)^T = \{ \mathfrak{g}_\beta : \beta \in \Delta, \mathfrak{g}_\beta \cap \Theta \neq \emptyset \},
\]

a finite set. In particular, \( \mathbb{P}(\Theta)^T \) is non-empty if and only if \( \Theta \) is the orbit of a root vector.

Let us take a moment to provide a more refined description of \( \mathbb{P}(\Theta)^T \). To this end, we will require the below lemma.

**Lemma 1.** Let \( \beta, \gamma \in \Delta \) be roots. The root spaces \( \mathfrak{g}_\beta \) and \( \mathfrak{g}_\gamma \) are \( G \)-conjugate if and only if \( \beta \) and \( \gamma \) are conjugate under \( W \).

**Proof.** Suppose that \( w \in W \) and that \( \beta = w \cdot \gamma \). Choosing a representative \( g \in N_G(H) \) of \( w \), this means precisely that \( \beta = \gamma \circ \varphi_{g^{-1}}|_H \), where \( \varphi_{g^{-1}} : G \to G \) is conjugation by \( g^{-1} \). Given \( h \in H \) and \( \xi \in \mathfrak{g}_\beta \), note that

\[
\text{Ad}_h(\text{Ad}_g(\xi)) = \text{Ad}_g(\text{Ad}_{g^{-1}hg}(\xi))
\]

\[
= \text{Ad}_g(\beta(g^{-1}hg)\xi)
\]

\[
= \text{Ad}_g(\gamma(h)\xi)
\]

\[
= \gamma(h)(\text{Ad}_g(\xi)).
\]

It follows that \( \mathfrak{g}_\gamma = \text{Ad}_g(\mathfrak{g}_\beta) \).

Conversely, suppose that \( g \in G \) and that \( \mathfrak{g}_\gamma = \text{Ad}_g(\mathfrak{g}_\beta) \). Consider the Zariski-closed subgroup

\[
L := \{ x \in G : \text{Ad}_x(\mathfrak{g}_\gamma) = \mathfrak{g}_\gamma \},
\]
noting that $H, gHg^{-1} \subseteq L$. Since $H$ and $gHg^{-1}$ are maximal tori of $L$, there exists $x \in L$ for which $xHx^{-1} = gHg^{-1}$. Hence, $x^{-1}g \in N_G(H)$ and $\text{Ad}_{x^{-1}g}(g_{\beta}) = g_{\gamma}$. We may therefore assume that $g \in N_G(H)$. Now, let $w \in W$ denote the class of $g$. Given $h \in H$ and $\xi \in g_{\beta}$, we find that

$$\langle w \cdot \beta(h) \xi, \xi \rangle = \beta(g^{-1}hg)\xi$$

$$= \text{Ad}_{g^{-1}hg}(\xi)$$

$$= \text{Ad}_{g^{-1}}(\gamma(h) \text{Ad}_g(\xi))$$

$$= \gamma(h)\xi.$$ 

It follows that $\gamma = w \cdot \beta$. 

Since $g$ is a simple Lie algebra, the root system associated with the pair $(g, h)$ is irreducible. Hence, there are at most two distinct root lengths (namely, those of the long and short roots), and the roots of a given length constitute an orbit of $W$ in $\Delta$. By Lemma 1, there are at most two nilpotent $G$-orbits $\Theta$ for which $\mathbb{P}(\Theta)^T$ is non-empty, the orbits of root vectors for the short and long roots. Furthermore, if $\Theta$ is the orbit of a root vector $e_{\beta} \in g_{\beta} \setminus \{0\}$, $\beta \in \Delta$, then $\mathbb{P}(\Theta)^T$ is the union of the points $g_{\gamma}$ for all $\gamma \in \Delta$ with length equal to that of $\beta$. Since $\Theta_{\min}$ is the orbit of a long root vector, $\mathbb{P}(\Theta_{\min})^T = \{g_{\gamma} : \gamma \in \Delta_{\text{long}}\}$, where $\Delta_{\text{long}} \subseteq \Delta$ is the set of long roots.

### 3.3 The Moment Polytope of $\mathbb{P}(\Theta_{\min})$

Note that the moment map $\Phi : \mathbb{P}(g) \to \mathfrak{k}^*$ considered in 3.1 can be modified to obtain a moment map for the Hamiltonian action of $T$ on $\mathbb{P}(\Theta_{\min})$. Indeed, we denote by $\mu : \mathbb{P}(\Theta_{\min}) \to \mathfrak{t}^*$ the moment map given by the composition

$$\mathbb{P}(\Theta_{\min}) \hookrightarrow \mathbb{P}(g) \overset{\Phi}{\to} \mathfrak{k}^* \to \mathfrak{t}^*.$$

Recall that

$$\mathbb{P}(\Theta_{\min})^T = \{g_{\beta} : \beta \in \Delta_{\text{long}}\}.$$ 

Given $\beta \in \Delta_{\text{long}}$, choose a point $e_{\beta} \in g_{\beta} \setminus \{0\}$. Note that for $X \in \mathfrak{t}$,

$$\mu(g_{\beta})(X) = \frac{\text{Im}((\langle X, e_{\beta} \rangle, e_{\beta}))}{\langle e_{\beta}, e_{\beta} \rangle} = \frac{\text{Im}(d_{e_{\beta}}(X)(e_{\beta}, e_{\beta}))}{\langle e_{\beta}, e_{\beta} \rangle} = \text{Im}(d_{e_{\beta}}(X)),$$

where $d_{e_{\beta}} : \mathfrak{t} \to i\mathbb{R}$ is the morphism of real Lie algebras induced by $\beta : T \to U(1)$. If one regards the weight lattice $\text{Hom}(T, U(1))$ as included into $\mathfrak{t}^*$ in the usual way, then our above calculation takes the form

$$\mu(g_{\beta}) = \beta.$$ 

The moment polytope $\mu(\mathbb{P}(\Theta_{\min}))$ is then the convex hull of $\Delta_{\text{long}}$ in $\mathfrak{t}^*$. 

8
3.4 Partial Flag Varieties as GKM Manifolds

Let us consider the matter of computing $H^*_T(P(\Theta_{\text{min}}))$. To this end, choose a long root $\alpha \in \Delta_{\text{long}}$, so that $g_\alpha \in P(\Theta_{\text{min}})^T$. Let $Q$ denote the $G$-stabilizer of $g_\alpha$. Since $G/Q \cong P(\Theta_{\text{min}})$ is projective, $Q$ is a parabolic subgroup of $G$. Accordingly, we will address the more general issue of computing the $T$-equivariant cohomology of the partial flag variety $G/P$, where $P \subseteq G$ is a parabolic subgroup containing $T$. Indeed, we will establish that $G/P$ is a GKM (Goresky-Kottwitz-MacPherson) manifold, allowing us to subsequently deploy some well-known machinery to compute its $T$-equivariant cohomology (see [5] and [9]).

Let us recall the definition of a GKM manifold.

**Definition 1.** A compact $T$-manifold $X$ is called a GKM manifold if

(i) $X^T$ is finite, and

(ii) for every codimension-one subtorus $S \subseteq T$, $\dim(X^S) \leq 2$.

Let us briefly address the significance of this notion in the context of computing $T$-equivariant cohomology. Suppose that $X$ is a GKM manifold as in Definition 1. If $S \subseteq T$ is a subtorus of codimension one and $Y$ is a connected component of $X^S$, then $Y \cap X^T \neq \emptyset$. In particular, $Y$ is $T$-invariant. Furthermore, $Y = \{e\}$ or $Y$ is isomorphic as a $T$-manifold to $S^2$ on which $T$ acts via some non-trivial character $\alpha_Y \in \text{Hom}(T,U(1))$. In the latter case, $Y^T$ consists of two points, $x_Y^+$ and $x_Y^-$. Let $\{Y_j\}_{j=1}^n$ be the collection of those two-spheres in $X$ arising as connected components of fixed point submanifolds of codimension-one subtori (henceforth called distinguished two-spheres). The inclusion $X^T \hookrightarrow X$ induces an injective graded algebra morphism $H^*_T(X) \hookrightarrow H^*_T(X^T) = \text{Map}(X^T, H^*_T(\text{pt}))$ with image

$$\{ f \in \text{Map}(X^T, H^*_T(\text{pt})) : \forall j \in \{1, \ldots, n\}, \alpha_{Y_j}|(f(x_Y^+)-f(x_Y^-)) \} \cong H^*_T(X).$$

Note that Definition 1 is precisely the definition of GKM manifold given in [10], where the authors exhibited certain homogeneous spaces of a compact connected simply-connected semisimple Lie group as GKM manifolds. Below is a statement of their result.

**Theorem 4.** Let $M$ be a compact connected simply-connected semisimple Lie group. Let $R \subseteq M$ be a maximal torus, and let $U$ be a closed subgroup of $M$ containing $R$. Assume that $M/U$ is oriented. Then, the left-multiplicative action of $R$ renders $M/U$ a GKM manifold.

For the duration of this section, let us fix a parabolic subgroup $P \subseteq G$ satisfying $B \subseteq P$. Note that $P$ is then the standard parabolic subgroup $P_{\Lambda}$ generated by $B$ and the root subgroups $\{U^-_{-\beta} := \exp(\mathfrak{g}_{-\beta}) : \beta \in \Lambda\}$ for some unique subset $\Lambda$ of $\Pi$.

**Corollary 4.** The partial flag variety $G/P$ is a GKM manifold for the left-multiplicative action of $T$.

**Proof.** The Iwasawa decomposition of $G$ tells us that $G = KB$. In particular, $K$ acts transitively on $G/P$. Since the $K$-stabilizer of the identity coset $[e] \in G/P$ is $K \cap P$, we have a $K$-manifold isomorphism $K/(K \cap P) \cong G/P$. It will therefore suffice to establish
that $K/(K \cap P)$ is a GKM manifold for the left-multiplicative action of $T$. For this, we will invoke Theorem 4. We need only note that $K$ is connected, simply-connected, and semisimple (since $G$ is), that $T \subseteq K \cap P$, and that $K/(K \cap P)$ is oriented (as $G/P$ is).

It thus remains to determine the fixed points $(G/P)^T$ and the distinguished two-spheres. Accordingly, we will require the below analogue of Theorem 2.2 of [10].

**Lemma 2.** Let $S \subseteq T$ be a subtorus. The image of $(G/B)^S$ under the fibration $G/B \xrightarrow{\varphi} G/P$ is $(G/P)^S$.

**Proof.** Consider the fibration $\psi : K/T \to K/(K \cap P)$. By Theorem 2.2 of [10], $\psi((K/T)^S) = (K/(K \cap P))^S$. Since each of the maps in the commutative diagram

$$
\begin{array}{ccc}
K/T & \xrightarrow{\psi} & K/(K \cap P) \\
\approx & & \approx \\
G/B & \xrightarrow{\varphi} & G/P
\end{array}
$$

is $T$-equivariant, the desired result follows. \qed

We immediately obtain a description of $(G/P)^T$. Indeed, note that $(G/B)^T = \{[k] : k \in N_K(T)\}$. Hence, $(G/P)^T$ is identified with $N_K(T)/(N_K(T) \cap P) \cong W/W_P$, where $W_P$ is the subgroup of $W$ generated by the simple reflections $\{s_\beta : \beta \in \Delta\}$ (see [2]). Let us now determine the distinguished two-spheres in $G/P$.

**Lemma 3.** A submanifold $X \subseteq G/P$ is a distinguished two-sphere if and only if it is related by the action of $N_K(T)$ to a distinguished two-sphere containing the identity coset $[e]$.

**Proof.** Suppose that $X$ is a two-sphere arising as a component of $(G/P)^S$ for some codimension-one subtorus $S \subseteq T$. Note that $X^T = \{[k_1], [k_2]\}$ for some $k_1, k_2 \in N_K(T)$. Furthermore, $R := k_1^{-1}S k_1$ is a codimension-one subtorus of $T$ and $k_1^{-1}X \cong S^2$ is a component of $(G/P)^R$ containing $[e]$. The proof of the converse is then a simple reversal of this argument. \qed

Accordingly, we will temporarily restrict our attention to the distinguished two-spheres in $G/P$ containing $[e]$. Let $X \subseteq G/P$ denote one such two-sphere. Note that

$$T_{[e]}(G/P) \cong g/p \cong \bigoplus_{\beta \in \Delta \setminus \Delta_P} g_\beta$$

as complex $T$-modules, where $\Delta_P$ is the set of roots whose root spaces belong to $p$. Since $T_{[e]}X$ is a complex one-dimensional $T$-invariant subspace of $T_{[e]}(G/P)$, $T_{[e]}X \cong g_\beta$ for some $\beta \in \Delta \setminus \Delta_P$. In [10], it is then concluded that $X^T = \{[e], [s_\beta]\} \subseteq W/W_P$.\[10\]
Lemma 4. Let $X \subseteq G/P$ be a distinguished two-sphere containing $[e]$, so that $Y := kX$ is a distinguished two-sphere containing $[w]$. If $\beta \in \Delta$ is the weight with which $T$ acts on $T[e]X$, then $w \cdot \beta$ is the weight with which $T$ acts on $T[w]Y$.

Proof. Consider the automorphism $\phi : G/P \to G/P$, $[g] \mapsto [kg]$, noting that $(d[e]\phi)[T[e]X : T[e]X \to T[w]Y$ is a complex vector space isomorphism. Furthermore, $\phi(t[g]) = (ktk^{-1})\phi([g])$ for all $t \in T$ and $g \in G$, so that $d[e]\phi((k^{-1}tk)v) = td[e]\phi(v)$ for all $v \in T[e](G/P)$. Hence, if $u \in T[w]Y$, then $u = d[e]\phi(v)$ for some $v \in T[e]X$ and

$$tu = td[e]\phi(v) = d[e]\phi((k^{-1}tk)v) = d[e]\phi(\beta(k^{-1}tk)v) = \beta(k^{-1}tk)d[e]\phi(v) = (w \cdot \beta)(t)u$$

for all $t \in T$. \hfill \Box

Let us summarize our findings.

Theorem 5. (i) There is a natural bijection $W/W_P \cong (G/P)^T$.

(ii) Fix $[w] \in W/W_P \cong (G/P)^T$. Given $\beta \in \Delta \setminus \Delta_P$, there exists a unique distinguished two-sphere $X \subseteq G/P$ with $X^T = \{[w],[w\beta]\}$, and with the property that $w \cdot \beta$ is the weight of $T[w]X$. Every distinguished two-sphere containing $[w]$ arises in this way.

(iii) We have a graded algebra isomorphism

$$H^*_T(G/P) \cong \{f \in \text{Map}(W/W_P \to H^*_T(pt)) : (w \cdot \beta)(f([w]) - f([w\beta])) \quad \forall w \in W, \beta \in \Delta \setminus \Delta_P\}$$

3.5 A Description of $\Theta_{\text{min}}$ and $\mathbb{P}(\Theta_{\text{min}})$

We devote this section to explicit descriptions of $\Theta_{\text{min}}$ and $\mathbb{P}(\Theta_{\text{min}})$ as homogeneous $G$-varieties. As noted earlier, the latter space is $G$-equivariantly isomorphic to a partial flag variety $G/P$. Accordingly, we shall begin by finding a parabolic subgroup $P \subseteq G$ with this property. In order to proceed, however, we will require the below result.

Theorem 6. Let $\Phi$ be an irreducible root system with collection of simple roots $\Sigma \subseteq \Phi$. 


(i) There exists a unique maximal root \( \beta \in \Phi \) (called the highest root).

(ii) This root is long.

(iii) We have \( \langle \beta, \gamma \rangle \geq 0 \) for all \( \gamma \in \Sigma \).

For a proof, the reader might refer to Propositions 19 and 23 in [14].

Denote by \( \alpha \in \Delta_+ \) the highest root, and choose a root vector \( e_\alpha \in g_\alpha \setminus \{0\} \). Note that \( [e_\alpha] = g_\alpha \in \mathbb{P}(\Theta_{\text{min}})^T \). Let \( C_g(e_\alpha) \) denote the centralizer of \( e_\alpha \) with respect to the adjoint representation of \( g \).

**Lemma 5.** \( C_g(e_\alpha) \) is a \( t \)-submodule of \( g \).

**Proof.** This is a straightforward application of the Jacobi identity. Indeed, suppose that \( X \in t \) and \( Y \in C_g(e_\alpha) \). Note that

\[
[[X, Y], e_\alpha] = [X, [Y, e_\alpha]] - [Y, [X, e_\alpha]] = -d e\langle X \rangle [Y, e_\alpha] = 0.
\]

In other words, \( C_g(e_\alpha) \) is a sum of \( t \)-submodules of the \( t \)-weight spaces occurring in the adjoint representation of \( t \) on \( g \). The summand coming from the trivial weight space \( \mathfrak{h} \) is just \( \ker(d e\langle \alpha \rangle) \), where we regard \( d e\langle \alpha \rangle \) as belonging to \( \mathfrak{h}^* \) instead of \( \mathfrak{t}^* \). Furthermore, if \( \beta \in \Delta \), then \( g_\beta \subseteq C_g(e_\alpha) \) if and only if \( [g_\alpha, g_\beta] = \{0\} \). Hence, we have established that

\[
C_g(e_\alpha) = \ker(d e\langle \alpha \rangle) \oplus \bigoplus_{\{\beta \in \Delta : [g_\alpha, g_\beta] = \{0\}\}} g_\beta.
\]

Now, let \( C_G(e_\alpha) \) and \( Q := C_G([e_\alpha]) \) be the \( G \)-stabilizers of \( e_\alpha \in \Theta_{\text{min}} \) and \( [e_\alpha] \in \mathbb{P}(\Theta_{\text{min}}) \), respectively. The inclusion \( C_G(e_\alpha) \subseteq Q \) yields an inclusion of Lie algebras \( C_g(e_\alpha) \subseteq q := \text{Lie}(Q) \). Since \( \dim \mathbb{C} q = \dim \mathbb{C} C_q(e_\alpha) + 1 \) (a consequence of comparing the dimensions of \( \Theta_{\text{min}} \) and \( \mathbb{P}(\Theta_{\text{min}}) \)), and since \( \mathfrak{h} \subseteq q \) (as \( H \) stabilizes \( [e_\alpha] \)), we must have

\[
q = \mathfrak{h} \oplus \bigoplus_{\{\beta \in \Delta : [g_\alpha, g_\beta] = \{0\}\}} g_\beta.
\]

In light of our having chosen \( \alpha \) to be the highest root, \( [g_\alpha, g_\beta] = \{0\} \) for all \( \beta \in \Delta_+ \). It thus remains to determine those negative roots whose root spaces appear as summands of \( q \).

**Lemma 6.** If \( \beta \in \Delta_- \), then \( [g_\alpha, g_\beta] = \{0\} \) if and only if \( \langle \alpha, \beta \rangle = 0 \).

**Proof.** Suppose that \( [g_\alpha, g_\beta] = \{0\} \). Choose \( h_\beta \in [g_\beta, g_-] \) such that \( d e\langle h_\beta \rangle = \langle \alpha, \beta \rangle \). Also, select \( e_\beta \in g_\beta \) and \( f_\beta \in g_- \) such that \( h_\beta = [e_\beta, f_\beta] \). By assumption, \( [e_\beta, e_\alpha] = 0 \). Since \( \alpha \) is the highest root, we also have \( [f_\beta, e_\alpha] = 0 \). Hence,

\[
0 = [e_\beta, [f_\beta, e_\alpha]] - [f_\beta, [e_\beta, e_\alpha]] = ([e_\beta, f_\beta], e_\alpha)
\]
\[ \langle \alpha, \beta \rangle = 0. \]

Conversely, suppose that \( \langle \alpha, \beta \rangle = 0 \). It will suffice to prove that \( \alpha + \beta \) is not a weight of the adjoint representation. Since these weights are \( W \)-invariant, it will actually suffice to prove that \( s_\beta(\alpha + \beta) \) is not a weight of \( g \). However, the orthogonality assumption implies that \( s_\beta(\alpha + \beta) = \alpha - \beta \). Also, \( \alpha - \beta > \alpha \), meaning that \( \alpha - \beta \) cannot be a weight of \( g \).

Now, suppose that

\[ \beta = \sum_{\gamma \in \Pi} a_\gamma \gamma, \]

\( a_\gamma \in \mathbb{Z}_{\leq 0} \), is the expression of \( \beta \) as a linear combination of simple roots. Since \( \langle \alpha, \gamma \rangle \geq 0 \) for all \( \gamma \in \Pi \), we see that \( \langle \alpha, \beta \rangle = 0 \) if and only if \( \langle \alpha, \gamma \rangle = 0 \) whenever \( a_\gamma \neq 0 \). In other words, \( \langle \alpha, \beta \rangle = 0 \) if and only if \( \beta \) is a linear combination of those simple roots orthogonal to \( \alpha \).

Accordingly, let us set

\[ \Xi := \{ \beta \in \Pi : \langle \alpha, \beta \rangle = 0 \}. \]

We have shown that \( Q = P_\Xi \), the parabolic subgroup of \( G \) determined by the simple roots in \( \Xi \).

We thus have the below result concerning the \( G \)-variety structure of \( \mathbb{P}(\Theta_{\min}) \).

**Theorem 7.** There is a \( G \)-variety isomorphism \( \mathbb{P}(\Theta_{\min}) \cong G/P_\Xi \).

Let us now address the \( G \)-variety structure of \( \Theta_{\min} \). To this end, we denote by \( L \overset{\pi}{\to} \mathbb{P}(g) \) the tautological line bundle over \( \mathbb{P}(g) \). Recall that for \( \xi \in g \setminus \{0\} \), we have \( \pi^{-1}(\{\xi\}) = \text{span}_\mathbb{C}\{\xi\} \). Furthermore, the tautological bundle is \( G \)-equivariant, with the \( G \)-action on the total space \( L \) given by

\[ g : ([\xi], v) \mapsto ([\text{Ad}_g(\xi)], \text{Ad}_g(v)), \]

\( g \in G, \xi \in g \setminus \{0\}, v \in \text{span}_\mathbb{C}\{\xi\} \).

Let \( E \overset{\varphi}{\to} \mathbb{P}(\Theta_{\min}) \) denote the pullback of \( L \) along the inclusion \( \mathbb{P}(\Theta_{\min}) \hookrightarrow \mathbb{P}(g) \). Note that \( E \) inherits from \( L \) the structure of a \( G \)-equivariant line bundle over \( \mathbb{P}(\Theta_{\min}) \). Furthermore, \( \Theta_{\min} \) \( G \)-equivariantly (and also \( \mathbb{C}^* \)-equivariantly) includes into \( E \) as a smooth open subvariety, namely the complement \( E^* \) of the zero-section. Accordingly, we will describe \( \Theta_{\min} \) by more closely examining \( E \).

Since \( \mathbb{P}(\Theta_{\min}) \) is the homogeneous \( G \)-variety \( G/P_\Xi \), we may exhibit \( E \) as an associated bundle for the one-dimensional \( P_\Xi \)-representation \( \varphi^{-1}([e_\alpha]) = g_\alpha \). More precisely, let \( G \times_{P_\Xi} g_\alpha \) denote the quotient of \( G \times g_\alpha \) by the equivalence relation

\[ (gp, v) \sim (g, \text{Ad}_p(v)), \]

\( p \in P_\Xi, g \in G, v \in g_\alpha \). Consider the map \( G \times_{P_\Xi} g_\alpha \to G/P_\Xi \) given by projection from the first component, whose fibres are then naturally complex vector spaces. The bundle
$G \times_{P_\mathbb{Z}} \mathfrak{g}_\alpha \to G/P_\mathbb{Z}$ is $G$-equivariant by virtue of the left-multiplicative $G$-action on the first component of $G \times_{P_\mathbb{Z}} \mathfrak{g}_\alpha$.

We have an isomorphism $\mathcal{E} \cong G \times_{P_\mathbb{Z}} \mathfrak{g}_\alpha$ of $G$-equivariant holomorphic line bundles over $G/P_\mathbb{Z}$, where we are regarding $\mathcal{E}$ as a line bundle over $G/P_\mathbb{Z}$. We therefore have the below description of $\Theta_{\text{min}}$.

**Theorem 8.** There is an isomorphism of $G$-equivariant holomorphic principal $\mathbb{C}^*$-bundles over $G/P_\mathbb{Z}$ between $\Theta_{\text{min}}$ and $(G \times_{P_\mathbb{Z}} \mathfrak{g}_\alpha)^*$.

### 3.6 The $T$-Equivariant Cohomology of $\Theta_{\text{min}}$

Let us use the description of $\Theta_{\text{min}}$ provided in §3.5 to compute $H_T^*(\Theta_{\text{min}})$. To this end, we have the equivariant Thom-Gysin sequence

$$\cdots \to H^2_T(G/P_\mathbb{Z}) \to H^1_T(G \times_{P_\mathbb{Z}} \mathfrak{g}_\alpha) \to H^0_T((G \times_{P_\mathbb{Z}} \mathfrak{g}_\alpha)^*) \to \cdots$$

associated with the zero-section $G/P_\mathbb{Z}$ in $G \times_{P_\mathbb{Z}} \mathfrak{g}_\alpha$ and its complement $(G \times_{P_\mathbb{Z}} \mathfrak{g}_\alpha)^*$. We can say considerably more about this sequence in our context, but it will require a brief computation of the $T$-equivariant Euler class $\text{Eul}_T(N) \in H^2_T(G/P_\mathbb{Z})$ of the normal bundle $N \cong G \times_{P_\mathbb{Z}} \mathfrak{g}_\alpha$ of the zero-section in $G \times_{P_\mathbb{Z}} \mathfrak{g}_\alpha$. Indeed, we will give the restriction $\text{Eul}_T(G \times_{P_\mathbb{Z}} \mathfrak{g}_\alpha)[w] \in H^2_T(\text{pt}) \cong \text{Sym}^1(\text{Hom}(T, U(1)) \otimes_{\mathbb{Z}} \mathbb{Q})$ to each fixed point $[w] \in W/W_{P_\mathbb{Z}} \cong (G/P_\mathbb{Z})^T$.

**Lemma 7.** If $w \in W$, then $\text{Eul}_T(G \times_{P_\mathbb{Z}} \mathfrak{g}_\alpha)[w] = w \cdot \alpha$.

**Proof.** Let $i_{[w]} : \{[w]\} \hookrightarrow G/P_\mathbb{Z}$ be the inclusion, and let $i_{[w]}^* : H^*_T(G/P_\mathbb{Z}) \to H^*_T(\text{pt})$ be the associated map on equivariant cohomology. Note that

$$\text{Eul}_T(G \times_{P_\mathbb{Z}} \mathfrak{g}_\alpha)[w] = i_{[w]}^*(\text{Eul}_T(G \times_{P_\mathbb{Z}} \mathfrak{g}_\alpha))$$

$$= \text{Eul}_T((i_{[w]})^*(G \times_{P_\mathbb{Z}} \mathfrak{g}_\alpha))$$

$$= \text{Eul}_T((G \times_{P_\mathbb{Z}} \mathfrak{g}_\alpha)[w]),$$

where $(G \times_{P_\mathbb{Z}} \mathfrak{g}_\alpha)[w]$ is the fibre over $[w]$. Now, choose a representative $k \in N_K(T)$ of $w$, noting that any element of this fibre is of the form $[(k, \xi)]$, $\xi \in \mathfrak{g}_\alpha$. Note that for $t \in T$,

$$t \cdot [(k, \xi)] = [(tk, \xi)] = [(k(k^{-1}tk), \xi)]$$

$$= [(k, (k^{-1}tk) \cdot \xi)]$$

$$= [(k, \alpha(k^{-1}tk)\xi)]$$

$$= (w \cdot \alpha)(t)[(k, \xi)].$$

Hence, $w \cdot \alpha = \text{Eul}_T((G \times_{P_\mathbb{Z}} \times \mathfrak{g}_\alpha)[w]) = \text{Eul}_T(G \times_{P_\mathbb{Z}} \mathfrak{g}_\alpha)[w]$. \qed

In particular, the image of $\text{Eul}_T(G \times_{P_\mathbb{Z}} \mathfrak{g}_\alpha)$ in $H^*_T((G/P_\mathbb{Z})^T)$ is non-zero. Since restriction gives an inclusion of $H^*_T(G/P_\mathbb{Z})$ into $H^*_T((G/P_\mathbb{Z})^T)$ as a subalgebra, and since $H^*_T((G/P_\mathbb{Z})^T)$ (a direct sum of polynomial rings) has no zero-divisors, we conclude that
Eul$_T(G \times_{P_\mathbb{Z}} g_\alpha)$ is not a zero-divisor in $H^*_T(G/P_\mathbb{Z})$. It follows that our Thom-Gysin sequence splits into the short-exact sequences
\[
0 \to H^{i-2}_T(G/P_\mathbb{Z}) \to H^i_T(G \times_{P_\mathbb{Z}} g_\alpha) \to H^i_T((G \times_{P_\mathbb{Z}} g_\alpha)^*) \to 0.
\]
(For a proof, see [1].)

For a second useful refinement of our Thom-Gysin sequence, we note that restriction to the zero-section gives a $T$-equivariant homotopy equivalence between $G \times_{P_\mathbb{Z}} g_\alpha$ and $G/P_\mathbb{Z}$. It follows that the associated restriction map $H^*_T(G \times_{P_\mathbb{Z}} g_\alpha) \to H^*_T(G/P_\mathbb{Z})$ is an isomorphism. Using this isomorphism, we shall replace $H^*_T(G \times_{P_\mathbb{Z}} g_\alpha)$ in our short-exact sequences to obtain
\[
0 \to H^{i-2}_T(G/P_\mathbb{Z}) \to H^i_T(G \times_{P_\mathbb{Z}} g_\alpha) \to H^i_T((G \times_{P_\mathbb{Z}} g_\alpha)^*) \to 0.
\]

The map $H^{i-2}_T(G/P_\mathbb{Z}) \to H^i_T(G/P_\mathbb{Z})$ is multiplication by $\text{Eul}_T(G \times_{P_\mathbb{Z}} g_\alpha)$ (see [4], for instance). Furthermore, the map $H^*_T(G/P_\mathbb{Z}) \to H^*_T((G \times_{P_\mathbb{Z}} g_\alpha)^*)$ is the map $\psi^*$ on equivariant cohomology induced by the projection $\psi : (G \times_{P_\mathbb{Z}} g_\alpha)^* \to G/P_\mathbb{Z}$. (This follows from the fact that the bundle projection $G \times_{P_\mathbb{Z}} g_\alpha \to G/P_\mathbb{Z}$ and zero-section $G/P_\mathbb{Z} \to G \times_{P_\mathbb{Z}} g_\alpha$ give inverse maps on equivariant cohomology.)

The above analysis yields two immediate corollaries. Firstly, the $T$-equivariant Betti numbers $b^i_T(\Theta_{\min})$ of $\Theta_{\min}$ are given by
\[
b^i_T(\Theta_{\min}) = b^i_T(G/P_\mathbb{Z}) - b^{i-2}_T(G/P_\mathbb{Z}).
\]

Secondly, $\psi^* : H^*_T(G/P_\mathbb{Z}) \to H^*_T(\Theta_{\min})$ is a surjective graded algebra morphism. Its kernel is $\langle \text{Eul}_T(G \times_{P_\mathbb{Z}} g_\alpha) \rangle$, the ideal of $H^*_T(G/P_\mathbb{Z})$ generated by the equivariant Euler class $\text{Eul}_T(G \times_{P_\mathbb{Z}} g_\alpha) \in H^2_T(G/P_\mathbb{Z})$. In particular, there is a graded algebra isomorphism
\[
H^*_T(\Theta_{\min}) \cong H^*_T(G/P_\mathbb{Z})/\langle \text{Eul}_T(G \times_{P_\mathbb{Z}} g_\alpha) \rangle.
\]

Using Lemma [7] and Theorem [5] and noting that $W_{P_\mathbb{Z}} = W_\mathbb{Z}$ is the subgroup of $W$ generated by the reflections $\{s_\beta\}_{\beta \in \mathbb{Z}}$, we obtain the below more explicit description of $H^*_T(\Theta_{\min})$.

**Theorem 9.** $H^*_T(\Theta_{\min})$ is isomorphic to the quotient of
\[
\{ f \in \text{Map}(W/W_\mathbb{Z}, H^*_T(pt)) : (w \cdot \beta)(f([w]) - f([ws_\beta])) = 0 \}
\]
by the ideal generated by the map $W/W_\mathbb{Z} \to H^*_T(pt)$, $[w] \mapsto w \cdot \alpha$.

### 3.7 An Example

Let us compute the equivariant cohomology of the minimal nilpotent orbit of $G = \text{SL}_2(\mathbb{C})$. To this end, let $T \subseteq G$ be the compact real form of the standard maximal torus of $G$. Note that $\Delta = \{-2, 2\} \subseteq \mathbb{Z} \cong \text{Hom}(T, U(1))$ is the resulting collection of roots. Letting $B \subseteq G$ be the Borel subgroup of upper-triangular matrices, we find that $\alpha = 2$ is the highest root. It is not orthogonal to any of the simple roots, so that $\Xi = \emptyset$. Hence, $P_\mathbb{Z} = B$ and $\Delta_{P_\mathbb{Z}} = \{2\}$. The Weyl group $W$ is $\mathbb{Z}/2\mathbb{Z}$, and the generator acts
by negation on the weight lattice. The subgroup $W_\Xi$ is trivial. In particular, $G/P_\Xi$ has two $T$-fixed points.

Since $\alpha$ is identified with $2x \in \mathbb{Q}[x] \cong H^*_T(\text{pt})$, Theorem 5 implies that $H^*_T(G/P_\Xi)$ includes into $H^*_T(\text{pt}) \oplus \mathbb{Q}[x]^{\oplus 2}$ as the subalgebra

$$H^*_T(G/P_\Xi) \cong \{(f_1(x), f_2(x)) \in \mathbb{Q}[x]^{\oplus 2} : x|(f_1(x) - f_2(x))\}$$

$$= \{(f_1(x), f_2(x)) \in \mathbb{Q}[x]^{\oplus 2} : f_1(0) = f_2(0)\}.$$

Indeed, we have recovered the $U(1)$-equivariant cohomology of the two-sphere with the rotation action of $U(1)$.

Lemma 7 tells us that $\text{Eul}_T(N) = (2x, -2x)$ when included into $\mathbb{Q}[x]^{\oplus 2}$. Hence,

$$H^*_T(\Theta_{\min}) \cong \{(f_1(x), f_2(x)) \in \mathbb{Q}[x]^{\oplus 2} : f_1(0) = f_2(0)\}/\langle(x, -x)\rangle.$$

Note that this is generated as a $\mathbb{Q}$-algebra by $y := [(x, 0)]$. The relation is $y^2 = 0$, so that

$$H^*_T(\Theta_{\min}) \cong \mathbb{Q}[y]/\langle y^2 \rangle,$$

with $y$ an element of grading degree two.

We remark that this is consistent with Corollary 8. Indeed, if $G = \text{SL}_2(\mathbb{C})$, then $\Theta_{\min} = \Theta_{\text{reg}}$. Hence, $H^*_T(\Theta_{\min}) = H^*_T(\Theta_{\text{reg}})$. Corollary 8 tells us that the latter is isomorphic to the ordinary cohomology of $G/B \cong \mathbb{P}^1$.

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