Tight Lower Bounds for the RMR Complexity of Recoverable Mutual Exclusion

DAVID YU CHENG CHAN, University of Calgary, Canada
PHILIPP WOELFEL, University of Calgary, Canada

We present a tight RMR complexity lower bound for the recoverable mutual exclusion (RME) problem, defined by Golab and Ramaraju [8]. In particular, we show that any $\Omega$-process RME algorithm using only atomic read, write, fetch-and-store, fetch-and-increment, and compare-and-swap operations, has an RMR complexity of $\Omega(\log n/\log \log n)$ on the CC and DSM model. This lower bound covers all realistic synchronization primitives that have been used in RME algorithms and matches the best upper bounds of algorithms employing swap objects (e.g., [5, 6, 10]).

Algorithms with better RMR complexity than that have only been obtained by either (i) assuming that all failures are system-wide [7], (ii) employing fetch-and-add objects of size $(\log n)^{o(1)}$ [12], or (iii) using artificially defined synchronization primitives that are not available in actual systems [6, 9].

CCS Concepts: • Theory of computation → Shared memory algorithms; • Software and its engineering → Mutual exclusion; • Computer systems organization → Reliability.

Additional Key Words and Phrases: recoverable mutual exclusion, asynchronous system, shared memory, fetch and increment

1 INTRODUCTION

Recent research on the mutual exclusion problem has focused on recoverable algorithms [3, 5–12, 15]. Here, a process may crash at any point during the execution, upon which its entire local state is being reset, including all its local program variables. Shared memory, however, is not affected by process crashes. When a process crashes, it starts a recovery routine that allows it to resume participation in the mutual exclusion protocol. This variant of mutual exclusion has been motivated by recent advances in non-volatile memory architectures [8].

The standard complexity measure for mutual exclusion algorithms is RMR complexity. The RMR complexity of conventional, non-recoverable $n$-process mutual exclusion is well understood: If only read-write registers and compare-and-swap objects are available, then a worst-case RMR complexity of $\Theta(\log n)$ is optimal [2, 16]. Using other standard synchronization primitives, such as fetch-and-store (swap) or fetch-and-increment, the RMR complexity can be reduced to $O(1)$ [4, 13, 14].

Recoverable mutual exclusion is seemingly harder: Many algorithms use fetch-and-store objects, and the best ones achieve an RMR complexity of $\Theta(\log n/\log \log n)$ [6, 10]. To study what it takes to achieve better RMR complexity than that, artificially defined synchronization primitives have been used that do not exist in actual hardware [6, 9]. Katzan and Morrison [12] observed that one can obtain an RMR complexity of $O(\log w, n)$ using $w$-bit fetch-and-add objects. In particular, if $w = n^e$ for some $e > 0$, then constant RMR complexity is possible. But it is a standard (and reasonable) assumption that $w = O(\log n)$. Even for poly-logarithmic values of $w$ their algorithm does not beat the best known upper bounds of $O(\log n/\log \log n)$.

It is therefore not surprising that it has been stated as an open problem (see e.g., [3, 12]), whether there are algorithms with better RMR complexity. In this paper we provide an answer, for almost all standard synchronization primitives that have been used to solve RME, with the exception of fetch-and-add. (For fetch-and-add objects that can store $w = (\log n)^{o(1)}$ bits, the lower bound does not apply due to [12], but for the more realistic assumption $w = (\log n)^{O(1)}$ it remains open if there exist algorithms with $o(\log n/\log \log n)$ RMR complexity.)
Theorem 1. Any deadlock-free \( n \)-process RME algorithm, where all objects support only read, write, fetch-and-store, fetch-and-increment, and compare-and-swap operations, has RMR complexity \( \Omega(\log n / \log \log n) \) in the CC and the DSM model.

This lower bound is tight: It shows that the algorithms by Golab and Hendler [6] (for the CC model) and Jayanti, Jayanti, and Joshi [10] (for the CC and DSM model) are optimal. Both algorithms use registers and fetch-and-store objects, and [6] also uses compare-and-swap. The lower bound demonstrates for the first time, that RME is strictly harder than non-recoverable mutual exclusion, because, as mentioned above, the latter has constant RMR complexity if fetch-and-store or fetch-and-increment objects and registers are available [4, 13, 14]. Chan and Woelfel [3] gave an RME algorithm with constant \( \text{amortized} \) RMR complexity using fetch-and-increment and compare-and-swap objects in addition to registers. Thus, our result separates worst-case from amortized RMR complexity for RME. Interestingly, for non-recoverable mutual exclusion, worst-case and amortized RMR complexity is the same for any subset of primitives (that includes at least read-write registers) to which our lower bound applies. It is also worth pointing out that our lower bound applies to all deadlock-free RME algorithms, and does not rely on the critical-section re-entry property, which is usually required for RME.

In addition, Golab and Hendler [7] showed that with the stricter system-wide failure model (all processes crash simultaneously), there exist RME algorithms with \( O(1) \) RMR complexity. Thus our lower bound also demonstrates that RME is strictly harder in the more general failure model that allows processes to crash independently than in the system-wide failure model.

Recently, Dhoked and Mittal [5] gave an algorithm that adapts to the number of process crashes (in the “recent” past), \( F \). In particular, they achieve an RMR complexity of \( O(\min(\sqrt{F}, \log n / \log \log n)) \). In our proof we construct an execution, in which one process incurs \( \Omega(\log n / \log \log n) \) RMRs, even though each process attempts to enter the critical section at most once and crashes at most once. This shows that the RMR complexity of mutual exclusion algorithms, such as the one by Dhoked and Mittal, can only adapt to the total number of crashes—bounding the number of crashes per process does not suffice to improve RMR complexity.

2 PRELIMINARIES

We consider the standard asynchronous shared memory model, where \( n \) processes with unique IDs communicate by executing atomic operations (called steps) on shared base objects.

A mutual exclusion algorithm is a shared (implemented) object that supports methods \( \text{Enter()} \) and \( \text{Exit()} \), and calls to these methods must alternate, starting with \( \text{Enter()} \). A process is in the critical section when it has finished an \( \text{Enter()} \) call but not yet called \( \text{Exit()} \). It is in the remainder section, if it is not in the critical section and has no pending \( \text{Enter()} \) or \( \text{Exit()} \) call. Such algorithms satisfy at least two conditions: Mutual exclusion requires that no two processes are in the critical section at the same time, and deadlock-freedom requires that some process with a pending \( \text{Enter()} \) or \( \text{Exit()} \) call will eventually finish this call, provided that all processes that are not in the remainder section keep taking steps.

A recoverable mutual exclusion (RME) algorithm provides an additional method, \( \text{Recover()} \). It is assumed that a process may crash at any point. (Formally, a process performs a crash step, which does not alter any shared memory objects.) If a process that is not in the remainder section crashes, all its local variables are reset to their initial values, and the process immediately begins executing method \( \text{Recover()} \). Deadlock-freedom is only required if the number of crashes is finite. (Otherwise a process might repeatedly crash in its critical section, preventing other processes from
making progress.) In addition to mutual exclusion and deadlock-freedom, RME algorithms must satisfy critical section re-entry, which means that if a process crashes in the critical section, then it will reenter the critical section before any other process. Our lower bound is independent of that property (deadlock-freedom and mutual exclusion are sufficient).

There are two common models used for the complexity analysis of mutual exclusion algorithms. In the cache-coherent (CC) model, processes are equipped with caches, and whenever a process performs a read operation it obtains a cache-copy of the corresponding memory location. Any non-read operation of that memory location invalidates all cache copies. A process’s operation incurs a remote memory reference (RMR), if it is a read operation and the process has no valid cache copy, or if it is not a read operation. In the distributed shared memory (DSM) model, the shared memory is partitioned into segments, one for each process. An operation on a shared memory location incurs an RMR if and only if that memory location is not in the calling process’s memory segment. Almost all work on the efficiency of (recoverable and non-recoverable) mutual exclusion algorithms has considered RMR complexity in one of those two models.

A passage of a process begins when it calls Enter() and ends when the process crashes or when it finishes its following Exitt() call. A super-passage of a process begins when the process calls Enter() and when it completes its following Exitt() call. (Note that in the absence of process crashes, all passages and super-passages coincide.) The RMR complexity of a mutual exclusion algorithm is the maximum number of RMRs a process may incur in any passage.

For our lower bound proof, base objects can store values from arbitrary (even uncountable) domains. We assume that processes can perform atomically any of the following operations on a shared object with value $x$:

- **read():** returns $x$;
- **FAS($x'$):** writes $x'$ and returns $x$;
- **CAS($y$, $x'$):** writes $x'$ and returns true, provided that $x = y$; otherwise it leaves the value unchanged and returns false.
- **FAI():** writes $x + 1$ and returns $x$, provided that $x$ is an integer. Otherwise it does not change the value of the object and returns $x$.

Note that a FAS operation is strictly stronger than a write operation (which does not return anything), so our lower bound proof does not consider write operations separately.

### 3 THE RME LOWER BOUND PROOF

We consider an arbitrary algorithm that solves the RME problem with $o(\log n)$ RMR complexity.

**Assumptions:** We make the following assumptions w.l.o.g.:

(A1) In the critical section, each process performs operation(s) that incur at least one RMR.
(A2) Since the algorithm incurs $o(\log n)$ RMRs in every passage of every execution, we assume $n$ is sufficiently large such that every passage of every execution incurs no more than $\log n$ RMRs.
(A3) Each process begins at most one super-passage, i.e., it leaves the remainder section at most once. (Note that this assumption makes our proof stronger, since a solution for the RME problem that allows multiple super-passages per process clearly also solves the RME problem in the scenario where each process can begin at most one super-passage.)

**Definitions:** We define the following:
• Let $\mathcal{P} = \{1, \ldots, n\}$ be the set of processes and $\mathcal{R}$ be the set of objects.
• Given any array $A[0..2^n - 1]$ and any set $S \subseteq \mathcal{P}$, we use $A[S]$ to denote $A[\sum_{p \in S} 2^p - 1]$.
• A schedule is a sequence over $\{p, \hat{p} : p \in \mathcal{P}\}$, where $p$ denotes a non-crash step by process $p$, and $\hat{p}$ denotes a crash-step by $p$.
• Let $C_0$ denote the initial configuration.
• For each schedule $\sigma$, a configuration $C$, and a register $R$, we define the following:
  - $P(\sigma)$: the set of all processes that have steps in $\sigma$.
  - $E(C, \sigma)$: the execution determined by $\sigma$ starting in configuration $C$. Note: An execution is a sequence of events, where each event corresponds to a step by some process and contains the following information: The process that is executing the step, the shared memory operation that process is executing, the object (register) on which the shared memory operation is executed, and whether the shared memory operation incurs an RMR.
  - $\text{val}_p(C, \sigma)$: the value of $R$ at the end of $E(C, \sigma)$.
  - $\text{state}_p(C, \sigma)$: the state of $p$ at the end of $E(C, \sigma)$.
  - $\text{last}_p(C, \sigma)$: the process that last performed an operation on $R$ at the end of $E(C, \sigma)$; or $\perp$ if no process has ever performed an operation on $R$.
  - $F(C, \sigma)$: the set of processes that have finished their super-passage at the end of $E(C, \sigma)$.

We also define $E(\sigma) = E(C_0, \sigma)$, $\text{val}_p(\sigma) = \text{val}_p(C_0, \sigma)$, $\text{state}_p(\sigma) = \text{state}_p(C_0, \sigma)$, $\text{last}_p(\sigma) = \text{last}_p(C_0, \sigma)$, and $F(\sigma) = F(C_0, \sigma)$.

### 3.1 Overview of the Proof

Our goal is to show that the algorithm has $\Omega(\log n / \log \log n)$ RMR complexity even when each process begins at most one super-passage and crashes at most once. Towards that end, we will construct a schedule $\sigma_\text{goal}$ such that during $E(\sigma_\text{goal})$:

- Each process begins at most one super-passage, and crashes at most once.
- Some process never crashes and never enters the critical section, yet incurs $\Omega(\log n / \log \log n)$ RMRs.

On a very high level the construction follows the outline of Anderson and Kim’s lower bound for non-recoverable mutual exclusion algorithms [1]. Their proof applies only to read-write registers. In order to deal with stronger primitives, we have to crash processes at opportune points in time, so that they "forget" information they may have observed (e.g., as a result of FAI or FAS operations).

We begin with a simple observation: if multiple processes are 'actively' attempting to enter the critical section, then they cannot safely enter the critical section before discovering one another, lest they violate mutual exclusion. Thus throughout the proof, we will construct several closely related schedules in which we attempt to maximize both the number of these active processes and the number of RMRs they incur without discovering one another.

More formally, let $\sigma_{\text{round}}[0..\infty][0..2^n - 1]$ be an initially empty table of schedules with an unbounded number of rows and $2^n$ columns. Roughly speaking, for every non-negative integer $i$, the $i$-th row of the table will contain only schedules in which the active processes have incurred at least $i$ RMRs. For every integer $s \in \{0, 1, \ldots, 2^n - 1\}$, we associate the $s$-th column with the unique set $S \subseteq \mathcal{P}$ of processes such that $s = \sum_{p \in S} 2^p - 1$. Then the $s$-th column will contain only schedules in which only the processes in $S$ can begin super-passages.
Filling the first row of the table is simple: in the empty schedule, every active process has incurred 0 RMRs, and the set of processes that have begun super-passages is $\emptyset$, a subset of every possible set of processes. Thus we set every cell of $\sigma_{\text{round}[0]}[0..2^n-1]$ to contain the empty schedule.

The proof then proceeds in rounds, where in each round $i \geq 1$, we fill in some cells of the $i$-th row with schedules derived by appending more steps to the schedules in the $(i-1)$-th row. Since we only want schedules in which the active processes do not discover one another, many of the cells in each row will be left with the value $\perp$, indicating that we did not find a schedule matching the required criteria. Thus as we go down through the rows of the table, the number of cells in each row that we fill with schedules decreases.

As such, the goal of each round is to limit this decrease, such that $\Omega(\log n/\log \log n)$ rounds complete before the number of schedules becomes too few to continue. After which, every schedule in the final round would have active processes that incur $\Omega(\log n/\log \log n)$ RMRs without entering the critical section (or crashing).

To facilitate this, we maintain a number of invariants on every row of schedules that we construct. Roughly speaking, these invariants are:

1. There is a maximal schedule which has the maximal number of active processes, and all other schedules are 'sub'-schedules that correspond to every possible subset of the active processes in the maximal schedule. This invariant ensures that if the maximal schedule cannot be extended without allowing some active processes to discover one another, then a sub-schedule can be extended and made into the new maximal schedule for the next round.
2. The state of every process is the same in every schedule it is part of. This invariant ensures that the active processes have not discovered one another, since they have the same state in a schedule where there are no other active processes.
3. For each register, its value in each schedule depends only on whether the schedule contains the process that last accessed it in the maximal schedule. This invariant ensures that register values are sufficiently similar across different schedules that it becomes difficult for the active processes to later distinguish between different schedules.
4. In every schedule, each process crashes at most once, and every process that is within a super-passage has not yet entered the critical section. The invariant makes the proof significantly simpler, since it prevents interactions between the active processes and the inactive processes that have already entered the critical section but not yet completed their super-passage.
5. In the DSM model, the registers that are owned by active processes have not been accessed by any other active process. This invariant also simplifies the proof, since it prevents non-RMR-incurring steps from allowing an active process to discover another active process, and thus allows the proof to focus on the RMR-incurring steps.
6. In the CC model, for each process $p$, the set of registers that $p$ has valid cache copies of is identical over all schedules that contain $p$. This invariant ensures that in the CC model, for each process $p$, the number of RMRs incurred by $p$ is the same in every schedule it is part of.
7. In the $i$-th row, every active process in every schedule has incurred at least $i$ RMRs.

It is easy to see that these invariants all hold for row 0. Furthermore, for every non-negative integer $i$, let $n_i$ be the number of active processes in the maximal schedule of row $i$. Then the first invariant asserts that row $i$ has $2^n_i$ schedules. Moreover, to show that $\Omega(\log n/\log \log n)$ rounds can be completed, it suffices to show that for every integer $i \geq 1$, $n_i > n_{i-1}/\Theta(\log^{O(1)} n)$. 


Each round of the proof is divided into two phases: a setup phase in which non-RMR-incurring steps are appended to the schedules until every active process in every schedule is poised to incur an RMR, and a contention phase, in which RMR-incurring steps are appended in specific orders that limit the fraction of active processes discovered.

In the setup phase, multiple non-RMR-incurring step(s) are appended for each active process until they are poised to incur an RMR. By the above invariants, the non-RMR-incurring steps appended for each process are the same in every schedule that contains the process. This is because each process begins with the same state in every schedule that contains the process, and then:

- In the DSM model, its non-RMR-incurring steps only access its own registers, which have never been accessed by any other active process, and thus these steps intuitively provide no new information that would cause the process to change its next steps.
- In the CC model, its non-RMR-incurring steps would be reads on registers that it has valid cache copies of in every schedule that contains it. Then, since the process already has valid cache copies of these registers, they intuitively provide no new information that would cause the process to change its next steps.

In the contention phase, our construction method differs depending on the relative number of registers that the active processes are poised to access (in the maximal schedule).

In a low contention scenario, the active processes are poised to access a relatively large number of registers, and so on average, each register has relatively few processes poised to access it. In this case, we construct a graph with nodes representing the active processes, and edges that intuitively indicate processes that could discover one another: either because they are poised to access the same register, or they are poised to access a register that is owned or previously accessed by another active process. Since the contention is relatively low, the resulting graph is relatively sparse, and thus contains a relatively large independent set. We now discard any schedule that contains any process outside of this independent set, so that the remaining schedules only contain active processes that would not discover one another with their next step. The remaining schedules then have a single step appended for each active process, and then are used to fill the next row of \( \sigma_{\text{round}}[0, \infty][0, 2^n - 1] \). It is straightforward to show that the above invariants still hold for this new row of schedules. Furthermore, due to the relative largeness of the independent set, it is also straightforward to show that \( n_i > n_{i-1}/O(\log^{O(1)} n) \) for every row \( i \geq 1 \) constructed in a low contention scenario.

In a high contention scenario, the active processes are poised to access a relatively small number of registers, and so on average, each register has relatively many processes poised to access it. In this scenario, it is often inevitable that some active processes are discovered by the others, and these active processes must then be inactivated by allowing them to enter the critical section, then complete their super-passage. To further complicate matters, each such process could discover \( o(\log n) \) other active processes before completing its super-passage, and these discovered processes must then be removed (schedules that contain such processes are discarded). Nevertheless, we can limit the number of discovered processes as follows.

First, we determine the plurality type of operation that the plurality of active processes are poised to perform. Every active process that is not poised to perform this plurality type of operation is then removed (schedules that contain such processes are discarded). Note that since there are only a constant number of operation types, a constant fraction of the active processes must remain.

We then divide these remaining active processes into groups of \( O(\log^{O(1)} n) \) processes, such that within each group, all processes are poised to access the same register (we remove any active processes that cannot be placed into such groups, the number of which is at most a constant fraction of the remaining active processes). Then within each group,
we select two active processes (preferentially those applying operations that would change the value of the register) that we call the alpha processes. Every schedule that does not contain all of the alpha processes is then discarded.

Intuitively, these alpha processes are the processes that will be discovered: they will be crashed, and then allowed to run until they complete their super-passage. Any other active processes that they discover along the way will be removed (schedules that contain such processes are discarded).

Now recall that each process incurs at most \( o(\log n) \) RMRs during its super-passage, whereas each group contains \( O(\log^{O(1)} n) \) processes. Thus we can ensure that a constant fraction of the groups still contain active processes that have not been discovered. Then one undiscovered active process in each such group, called the beta process, is allowed to take an RMR-incurred step that is intuitively hidden by the steps of the alpha processes in its group as follows:

- If the plurality type of operation is read, then since reads do not change the value of a register, the beta process can safely perform its read between the reads of the alpha processes without affecting the value of the register.
- If the plurality type of operation is fetch-and-store, then since fetch-and-stores completely overwrite the value of a register, the beta process can safely perform its fetch-and-store between the fetch-and-stores of the alpha processes without affecting the final value of the register.
- If the plurality type of operation is fetch-and-increment, we can replace the fetch-and-increment of the first alpha process with a fetch-and-increment by the beta process, and the final value of the register will remain the same.
- If the plurality type of operation is compare-and-swap, then there must be an ordering of the alpha and beta processes such that the beta process fails its compare-and-swap operation, and so has no effect on the final value of the register.

Roughly speaking, this allows the beta processes to take their RMR-incurred steps without changing the value of any register, and although the alpha processes can immediately discover the beta processes, they will immediately crash and forget the beta processes, and will never discover the beta processes again. Thus we can construct schedules that allow the beta processes to remain active without being discovered; all other remaining active processes are removed. Since a constant fraction of the groups of \( O(\log^{O(1)} n) \) processes yield an undiscovered beta process for the new maximal schedule, we can also prove that \( n_i > n_{i-1}/O(\log^{O(1)} n) \) for every row \( i \geq 1 \) constructed in a high contention scenario.

Thus, regardless of whether each round \( i \geq 1 \) has a low contention phase or a high contention phase, \( n_i > n_{i-1}/O(\log^{O(1)} n) \). By the first invariant, the number of schedules in each row \( i \) is \( 2^n_i \). So \( \Omega(\log n/\log \log n) \) rounds complete before the number of schedules becomes too few to continue. After which, the schedules in the final round would have active processes that incur \( \Omega(\log n/\log \log n) \) RMRs without entering the critical section (or crashing). Consequently, the algorithm has \( \Omega(\log n/\log \log n) \) RMR complexity.

**3.2 Proof Details**

**Invariants:** To prove the main theorem, we will iteratively construct arrays of schedules.

Let \( i \) be a non-negative integer, and \( A[0..2^n - 1] \) be an array such that each array entry contains either a schedule or \( \perp \). Then we say that \( A[0..2^n - 1] \) is **i-compliant** if it satisfies the following invariants:

(I1) For every set \( S \subseteq \mathcal{P} \), if \( A[S] \neq \perp \), then \( P(A[S]) \subseteq S \). (Note that this implies \( F(A[S]) \subseteq S \).

(I2) There is a unique set \( S_{\text{max}} \subseteq \mathcal{P} \) such that for every set \( S \subseteq \mathcal{P} \), \( A[S] \neq \perp \) if and only if \( F(A[S_{\text{max}}]) \subseteq S \subseteq S_{\text{max}} \).
(I3) For every process \( p \in S_{\text{max}} \) and every set \( S \subseteq P \) that contains \( p \), if \( A[S] \neq \bot \), then \( \text{state}_p(A[S]) = \text{state}_p(A[S_{\text{max}}]) \).

(II) \( F(A[S]) = F(A[S_{\text{max}}]) \) for every set \( S \subseteq P \) with \( A[S] \neq \bot \). (Note that this invariant immediately follows from Invariants (I1), (I2), and (I3).)

(III) For every register \( R \in \mathcal{R} \), there is a value \( y_R \) such that for every set \( S \subseteq P \), if \( A[S] \neq \bot \), then:

\[
\text{val}_R(A[S]) = \begin{cases} 
\text{val}_R(A[S_{\text{max}}]) & \text{if } \text{last}_R(A[S_{\text{max}}]) \in S \\
y_R & \text{otherwise}
\end{cases}
\]

Note that it is possible that \( y_R = \text{val}_R(A[S_{\text{max}}]) \). Furthermore, if \( \text{last}_R(A[S_{\text{max}}]) = \bot \notin S \), then \( \text{val}_R(A[S]) = y_R \) for every set \( S \subseteq P \) with \( A[S] \neq \bot \).

(IV) For every set \( S \subseteq P \) with \( A[S] \neq \bot \), during \( E(A[S]) \), each process crashes at most once, and each process that is not in \( F(A[S]) \) never crashes.

(V) For every set \( S \subseteq P \) with \( A[S] \neq \bot \), each process that is not in \( F(A[S]) \) does not enter the critical section during \( E(A[S]) \).

(VI) In the DSM model, for every process \( p \in S_{\text{max}} \setminus F(A[S_{\text{max}}]) \), every register \( R \in \mathcal{R} \) owned by \( p \), and every set \( S \subseteq P \) with \( A[S] \neq \bot \), \( R \) can only be accessed by \( p \) during \( E(A[S]) \). (Or equivalently, In the DSM model, for every set \( S \subseteq P \) such that \( A[S] \neq \bot \), during \( E(A[S]) \), each register \( R \in \mathcal{R} \) can only be accessed by its owner if the owner of \( R \) is in \( S_{\text{max}} \setminus F(A[S_{\text{max}}]) \).

(VII) In the CC model, for every process \( p \in S_{\text{max}} \setminus F(A[S_{\text{max}}]) \), there is a set \( \mathcal{R}_p \) of registers such that for every set \( S \subseteq P \) that contains \( p \), if \( A[S] \neq \bot \), then the set of registers that \( p \) has valid cache copies of at the end of \( E(A[S]) \) is exactly \( \mathcal{R}_p \). (Or equivalently, for every set \( S \subseteq P \) such that \( A[S] \neq \bot \), and every process \( p \in S \setminus (S_{\text{max}} \setminus F(A[S_{\text{max}}])) \), the set of registers that \( p \) has valid cache copies of at the end of \( E(A[S]) \) is exactly the same as at the end of \( E(A[S_{\text{max}}]) \)).

(VIII) For every set \( S \subseteq P \) and every process \( p \in S \setminus F(A[S]) \), if \( A[S] \neq \bot \), then \( p \) incurs at least \( i \) RMRs during \( E(A[S]) \).

Let \( i \) be a non-negative integer, and \( A[0..2^n - 1] \) be an array that is \( i \)-compliant. Then we denote by \( S_{\text{max}}(A[0..2^n - 1]) \) the unique set of invariant (I2).

Let \( \sigma_{\text{round}}[0..2^n - 1] \) be a table with all entries initially containing \( \bot \). Our goal is to fill in the table such that for every non-negative integer \( i \), either \( \sigma_{\text{round}}[i, 0..2^n - 1] \) is \( i \)-compliant, or \( i \in \Omega(\log n/\log \log n) \).

Let \( d \) be a sufficiently large constant and \( k = \log^d n \).

**Base Case:** For every set \( S \subseteq P \), let \( \sigma_{\text{round}}[0, S] \) be set to the empty schedule (so every entry of \( \sigma_{\text{round}}[0, 0..2^n - 1] \) is the empty schedule). Clearly, the array \( \sigma_{\text{round}}[0, 0..2^n - 1] \) is 0-compliant with \( S_{\text{max}}(\sigma_{\text{round}}[0, 0..2^n - 1]) = P \) and has \( 2^n \) non-\( \bot \) entries.

We now iterate through \( i = 1, 2, \ldots \) as follows:

**i-th Iteration (Termination Phase):** If \( \sigma_{\text{round}}[i-1, 0..2^n - 1] \) is not \((i-1)\)-compliant or has less than \( 2^i(k^3) \) non-\( \bot \) entries, terminate.

**i-th Iteration (Setup Phase):** For every set \( S \subseteq P \), let \( \sigma_{\text{old}}[S] = \sigma_{\text{round}}[i-1][S] \). So the array \( \sigma_{\text{old}}[0..2^n - 1] \) is \((i-1)\)-compliant.
Tight Lower Bounds for the RMR Complexity of Recoverable Mutual Exclusion

Thus by Invariant (I2), there is a unique set \( S_{\text{max}}^\text{old} \subseteq P \) such that \( \sigma_{\text{old}}(S_{\text{max}}^\text{old}) \neq \perp \) and for every set \( S \subseteq P \), \( \sigma_{\text{old}}[S] \neq \perp \) if and only if \( F(\sigma_{\text{old}}(S_{\text{max}}^\text{old})) \subseteq S \subseteq S_{\text{max}}^\text{old} \). So by definition, \( S_{\text{max}}(\sigma_{\text{old}}(0...2^n - 1)) = S_{\text{max}}^\text{old} \). Then for every process \( p \in S_{\text{max}}^\text{old} \setminus F(\sigma_{\text{old}}(S_{\text{max}}^\text{old})) \), let \( S_p = \{ p \} \cup F(\sigma_{\text{old}}(S_{\text{max}}^\text{old})) \). Note that \( F(\sigma_{\text{old}}(S_{\text{max}}^\text{old})) \subseteq S_p \subseteq S_{\text{max}}^\text{old} \), so \( \sigma_{\text{old}}[S_p] \neq \perp \).

Now for every process \( p \in S_{\text{max}}^\text{old} \setminus F(\sigma_{\text{old}}(S_{\text{max}}^\text{old})) \), let \( \sigma_p \) be a schedule consisting only of the maximum non-negative number of non-crash steps of \( p \) such that any RMRs incurred by \( p \) in \( E(\sigma_{\text{old}}[S_p] \circ \sigma_p) \) were also incurred in \( E(\sigma_{\text{old}}[S_p]) \). Then let \( C_p \) be the configuration at the end of \( E(\sigma_{\text{old}}[S_p]) \). So by definition, \( p \) does not incur any RMRs in \( E(C_p, \sigma_p) \).

Furthermore, if \( \sigma_p \) is finite, then an RMR would be incurred by \( p \) at the end of \( E(\sigma_{\text{old}}[S_p] \circ \sigma_p \circ p) \).

**Lemma 2.** For every process \( p \in S_{\text{max}}^\text{old} \setminus F(\sigma_{\text{old}}(S_{\text{max}}^\text{old})) \), \( \sigma_p \) is finite.

**Proof.** Let \( p \) be any process in \( S_{\text{max}}^\text{old} \setminus F(\sigma_{\text{old}}(S_{\text{max}}^\text{old})) \). Suppose, for contradiction, that \( \sigma_p \) is infinite.

Recall that \( S_p = \{ p \} \cup F(\sigma_{\text{old}}(S_{\text{max}}^\text{old})) \), so \( F(\sigma_{\text{old}}(S_{\text{max}}^\text{old})) \subseteq S_p \subseteq S_{\text{max}}^\text{old} \). Since \( \sigma_{\text{old}}(0...2^n - 1) \) is \( i - 1 \)-compliant with \( S_{\text{max}}(\sigma_{\text{old}}(0...2^n - 1)) = S_{\text{max}}^\text{old} \), by Invariants (I2) and (I4), \( \sigma_{\text{old}}[S_p] \neq \perp \) and \( F(\sigma_{\text{old}}[S_p]) = F(\sigma_{\text{old}}(S_{\text{max}}^\text{old})) \). So \( p \in S_p \setminus F(\sigma_{\text{old}}(S_{\text{max}}^\text{old})) \). Thus by Invariant (I7), \( p \) does not enter the critical section during \( E(\sigma_{\text{old}}[S_p]) \).

By definition, \( C_p \) is the configuration at the end of \( E(\sigma_{\text{old}}[S_p]) \) and \( p \) does not incur any RMRs in \( E(C_p, \sigma_p) \). So by Assumption (A1), \( p \) does not enter the critical section during the infinite execution \( E(\sigma_{\text{old}}[S_p] \circ \sigma_p) \).

Now recall that \( S_p = \{ p \} \cup F(\sigma_{\text{old}}(S_{\text{max}}^\text{old})) \), so every process \( q \neq p \) is either in \( F(\sigma_{\text{old}}[S_p]) \) or not in \( S_p \). By Invariant (I1), every process \( q \neq p \) takes no steps in \( E(\sigma_{\text{old}}[S_p]) \). Since \( \sigma_p \) contains only steps of \( p \) and \( p \neq S_p \), every process \( q \neq S_p \) also takes no steps in \( E(\sigma_{\text{old}}[S_p] \circ \sigma_p) \). So in \( E(\sigma_{\text{old}}[S_p] \circ \sigma_p) \), \( p \) takes infinitely many steps without entering the critical section while every process \( q \neq p \) is in the remainder section — contradicting that \( E(\sigma_{\text{old}}[S_p] \circ \sigma_p) \) is an execution of an algorithm that solves the RME problem.

Since \( \sigma_{\text{old}}(0...2^n - 1) \) is \( (i - 1) \)-compliant with \( S_{\text{max}}(\sigma_{\text{old}}(0...2^n - 1)) = S_{\text{max}}^\text{old} \), by Invariant (I2), for every set \( S \subseteq P \), \( \sigma_{\text{old}}[S] \neq \perp \) if and only if \( F(\sigma_{\text{old}}(S_{\text{max}}^\text{old})) \subseteq S \subseteq S_{\text{max}}^\text{old} \). So for each set \( S \subseteq P \) such that \( \sigma_{\text{old}}[S] \neq \perp \), let \( p_{1,S} \) be the process with the smallest ID in \( S \setminus F(\sigma_{\text{old}}(S_{\text{max}}^\text{old})) \), \( p_{2,S} \) be the process with the second smallest ID, and so on. Then let \( C_S \) be the configuration at the end of \( E(\sigma_{\text{old}}[S]) \), and let \( \sigma_S = \sigma_{p_{1,S}} \circ \sigma_{p_{2,S}} \circ \ldots \). Note that by Lemma 2 and the fact that the system has only \( n \) processes, \( \sigma_S \) is a finite schedule.

**Lemma 3.** For every set \( S \subseteq P \) such that \( \sigma_{\text{old}}[S] \neq \perp \):

(S1) No RMRs are incurred during \( E(C_S, \sigma_S) \).

(S2) For each process \( p \in S \), \( state_p(\sigma_{\text{old}}(C_S, \sigma_S)) = state_p(C_S, \sigma_S) \).

(S3) For each process \( p \in S \setminus F(\sigma_{\text{old}}(S_{\text{max}}^\text{old})) \), \( p \) incurs an RMR at the end of \( E(C_S, \sigma_S \circ p) \).

(S4) For each process \( p \in S \setminus F(\sigma_{\text{old}}(S_{\text{max}}^\text{old})) \), \( p \) has not left the critical section at the end of \( E(C_S, \sigma_S) \).

(S5) \( F(\sigma_{\text{old}}[S]) = F(\sigma_{\text{old}}(S) \circ \sigma_S) \).

(S6) In the DSM model, each register \( R \in \mathcal{R} \) can only be accessed by its owner during \( E(C_S, \sigma_S) \).

(S7) In the DSM model, for each process \( p \in S \setminus F(\sigma_{\text{old}}(S_{\text{max}}^\text{old})) \) and each register \( R \in \mathcal{R} \) owned by \( p \), \( val_R(\sigma_{\text{old}}(C_S, \sigma_S) = val_R(C_S, \sigma_S) \).

(S8) In the CC model, each register \( R \in \mathcal{R} \) can only be read during \( E(C_S, \sigma_S) \).

(S9) In the CC model, during \( E(C_S, \sigma_S) \), each process \( p \) can only read registers that it already has valid cache copies of.

**Proof.** Let \( S \subseteq P \) be any set of processes such that \( \sigma_{\text{old}}[S] \neq \perp \). Since \( \sigma_{\text{old}}(0...2^n - 1) \) is \( (i - 1) \)-compliant with \( S_{\text{max}}(\sigma_{\text{old}}(0...2^n - 1)) = S_{\text{max}}^\text{old} \), by Invariant (I2), for every set \( S' \subseteq P \), \( \sigma_{\text{old}}[S'] \neq \perp \) if and only if \( F(\sigma_{\text{old}}(S_{\text{max}}^\text{old})) \subseteq S' \subseteq S_{\text{max}}^\text{old} \). Thus \( F(\sigma_{\text{old}}(S_{\text{max}}^\text{old})) \subseteq S \subseteq S_{\text{max}}^\text{old} \). Then since \( S \subseteq S_{\text{max}}^\text{old} \), by definition we have that for every process \( p \in S \setminus F(\sigma_{\text{old}}(S_{\text{max}}^\text{old})) \), \( S_p = \{ p \} \cup F(\sigma_{\text{old}}(S_{\text{max}}^\text{old})) \).

9
Since \( \sigma_{\text{old}}[0..2^n - 1] \) is \((i - 1)\)-compliant, by Invariant (I3), for every process \( p \in S \setminus F(\sigma_{\text{old}}[S_{\text{max}}]) \) and every process \( q \in S_p, \text{state}_{p}(\sigma_{\text{old}}[S]) = \text{state}_{q}(\sigma_{\text{old}}[S_p]). \) Furthermore, by Invariant (I4), \( F(\sigma_{\text{old}}[S_{\text{max}}]) = F(\sigma_{\text{old}}[S]). \) So by Invariant (I7), every process \( p \in S \setminus F(\sigma_{\text{old}}[S_{\text{max}}]) \) has not entered the critical section during \( E(\sigma_{\text{old}}[S]). \)

The proof now differs depending on the model:

**CC Model:** In the CC model, any step that does not incur an RMR must be a read operation on a register that the invoking process already has a valid cache copy of. Thus, by definition, for every process \( p \in S \setminus F(\sigma_{\text{old}}[S_{\text{max}}]), p \) only performs read operations on registers that it already has valid cache copies of during \( E(C_p, \sigma_p). \)

Since \( \sigma_{\text{old}}[0..2^n - 1] \) is \((i - 1)\)-compliant, by Invariant (I9), for every process \( p \in S \setminus F(\sigma_{\text{old}}[S_{\text{max}}]), \) the set of registers that \( p \) has valid cache copies of is in same in \( C_p \) as in \( C_S. \) Furthermore, read operations clearly cannot invalidate any valid cache copies. Thus, by the definition of \( \sigma_S, \) observe that for every process \( p \in S \setminus F(\sigma_{\text{old}}[S_{\text{max}}]), \) the operations performed by \( p \) during \( E(C_S, \sigma_S) \) are the same as in during \( E(C_p, \sigma_p), \) i.e., \( p \) only performs read operations on registers that it already has valid cache copies of during \( E(C_S, \sigma_S) \) ((S9)).

This implies the following:

- Since RMRs are not incurred by any read operation on a register that the invoking process already has a valid cache copy of, no RMRs are incurred during \( E(C_S, \sigma_S) \) ((S1)).
- Since only read operations are performed during \( E(C_S, \sigma_S), \) each register can only be read during \( E(C_S, \sigma_S) \) ((S8)).
- By definition, \( \sigma_S \) contains only steps of processes in \( S \setminus F(\sigma_{\text{old}}[S_{\text{max}}]). \) Thus for every process \( p \in S, \) observe that \( \text{state}_{p}(C_S, \sigma_S) = \text{state}_{p}(C_p, \sigma_p) \) ((S2)).

Now recall that for every process \( p \in S \setminus F(\sigma_{\text{old}}[S_{\text{max}}]), \) the set of registers that \( p \) has valid cache copies of is in same in \( C_p \) as in \( C_S. \) Then, since (i) \( \text{state}_{p}(C_S, \sigma_S) = \text{state}_{p}(C_p, \sigma_p), \) (ii) the valid cache copies of \( p \) are the same in \( C_p \) as in \( C_S, \) (iii) new cache copies cannot be created by reading registers that valid cache copies already exist for, and (iv) \( p \) incurs an RMR in \( E(C_p \circ \sigma_p, p) \) by the definition of \( \sigma_p, \) observe that \( p \) also incurs an RMR at the end of \( E(C_S, \sigma_S \circ p) \) ((S3)).

- Since every process \( p \in S \setminus F(\sigma_{\text{old}}[S_{\text{max}}]) \) has not entered the critical section during \( E(\sigma_{\text{old}}[S]) \) and no RMRs are incurred during \( E(C_S, \sigma_S), \) by Assumption (A1), every process \( p \in S \setminus F(\sigma_{\text{old}}[S_{\text{max}}]) \) has not left the critical section at the end of \( E(C_S, \sigma_S) \) ((S4)).

Then, since \( \sigma_S \) contains only non-crash steps, no process completes during \( E(C_S, \sigma_S). \) Thus \( F(\sigma_{\text{old}}[S]) = F(\sigma_{\text{old}}[S] \circ \sigma_S) \) ((S5)).

**DSM Model:** In the DSM model, any step that does not incur an RMR must be an operation on a register owned by the invoking process. Thus, by definition, for every process \( p \in S \setminus F(\sigma_{\text{old}}[S_{\text{max}}]), p \) only performs operations on its own registers during \( E(C_p, \sigma_p). \)

Since \( \sigma_{\text{old}}[0..2^n - 1] \) is \((i - 1)\)-compliant with \( S_{\text{max}} = S_{\text{max}}, \) by Invariant (I8), for every process \( p \in S_{\text{max}} \setminus F(\sigma_{\text{old}}[S_{\text{max}}]), \) every register \( R \in R \) owned by \( p, \) and every set \( S' \subseteq P \) with \( \sigma_{\text{old}}[S'] \neq \bot, R \) can only be accessed by \( p \) during \( E(\sigma_{\text{old}}[S']), \) so \( \text{last}_R(\sigma_{\text{old}}[S']) \) is either \( p \) or \( \bot. \) Thus by Invariant (I5), if \( \text{init}_R \) is the initial value of \( R, \) then:

\[
\text{val}_R(\sigma_{\text{old}}[S']) = \begin{cases} \text{val}_R(\sigma_{\text{old}}[S_{\text{max}}]) & \text{if } p \in S' \\ \text{init}_R & \text{otherwise} \end{cases}
\]
So for every process \( p \in S \setminus F(\sigma_{\text{old}}[S_{\text{max}}]) \) and every register \( R \in \mathcal{R} \) owned by \( p \), \( \text{val}_R(\sigma_{\text{old}}[S]) = \text{val}_R(\sigma_{\text{old}}[S_p]) \). Furthermore, operations on registers not owned by \( p \) clearly cannot change the value of registers owned by \( p \). Consequently, by the definition of \( \sigma_S \), observe that for every process \( p \in S \setminus F(\sigma_{\text{old}}[S_{\text{max}}]) \), the operations performed by \( p \) during \( E(C_S, \sigma) \) are the same as in during \( E(C_p, \sigma_p) \), i.e., \( p \) only performs operations on its own registers during \( E(C_S, \sigma_S) \).

This implies the following:

- Since RMRs are not incurred by any operation on a register owned by the invoking process, no RMRs are incurred during \( E(C_S, \sigma_S) \) ((S1)).
- Since each process only accesses its own registers during \( E(C_S, \sigma_S) \), for each register \( R \in \mathcal{R} \), \( R \) can only be accessed by its owner during \( E(C_S, \sigma_S) \) ((S6)).
- Furthermore, since the operations performed by each process \( p \in S \setminus F(\sigma_{\text{old}}[S_{\text{max}}]) \) during \( E(C_S, \sigma_S) \) are the same as in during \( E(C_p, \sigma_p) \), \( \text{val}_p(C_p, \sigma_p) = \text{val}_p(C_S, \sigma_S) \) ((S7)).
- By definition, \( \sigma_S \) contains only steps of processes in \( S \setminus F(\sigma_{\text{old}}[S_{\text{max}}]) \). Thus for every process \( p \in S \setminus F(\sigma_{\text{old}}[S_{\text{max}}]) \), observe that \( \text{状态}_p(C_S, \sigma_S) = \text{状态}_p(C_p, \sigma_p) \) ((S2)).
- Furthermore, by the definition of \( \sigma_p \), \( p \) incurs an RMR at the end of \( E(C_p, \sigma_p \circ p) \), i.e., \( p \) is poised to access a register that it does not own at the end of \( E(C_p, \sigma_p) \). Thus, since \( \text{状态}_p(C_S, \sigma_S) = \text{状态}_p(C_p, \sigma_p) \), \( p \) is also poised to access a register that it does not own at the end of \( E(C_S, \sigma_S) \), and so \( p \) also incurs an RMR at the end of \( E(C_S, \sigma \circ p) \) ((S3)).
- Since every process \( p \in S \setminus F(\sigma_{\text{old}}[S_{\text{max}}]) \) has not entered the critical section during \( E(\sigma_{\text{old}}[S]) \) and no RMRs are incurred during \( E(C_S, \sigma_S) \), by Assumption (A1), every process \( p \in S \setminus F(\sigma_{\text{old}}[S_{\text{max}}]) \) has not left the critical section at the end of \( E(C_S, \sigma_S) \) ((S4)).

Then, since every process \( p \in S \setminus F(\sigma_{\text{old}}[S_{\text{max}}]) \) has not left the critical section at the end of \( E(C_S, \sigma_S) \), every process \( p \in S \setminus F(\sigma_{\text{old}}[S_{\text{max}}]) \) has not completed its super-passage at the end of \( E(C_S, \sigma_S) \). Thus \( F(\sigma_{\text{old}}[S]) = F(\sigma_{\text{old}}[S] \circ \sigma) \) ((S5)).

We now construct a new array \( \sigma_{\text{setupA}}[0..2^n - 1] \) such that for every set \( S \subseteq \mathcal{P} \), \( \sigma_{\text{setupA}}[S] = \perp \) if \( \sigma_{\text{old}}[S] = \perp \); otherwise \( \sigma_{\text{setupA}}[S] = \sigma_{\text{old}}[S] \circ \sigma_S \).

**Lemma 4.** Except for Invariant (I7), \( \sigma_{\text{setupA}}[0..2^n - 1] \) is \((i-1)\)-compliant with \( S_{\text{max}}(\sigma_{\text{setupA}}[0..2^n - 1]) = S_{\text{old}} \).

**Proof.** For every set \( S \subseteq \mathcal{P} \), if \( \sigma_{\text{setupA}}[S] \neq \perp \), then by construction, \( \sigma_{\text{setupA}}[S] = \sigma_{\text{old}}[S] \circ \sigma_S \). Since \( \sigma_{\text{old}}[0..2^n - 1] \) is \((i-1)\)-compliant, by Invariant (I1), \( P(\sigma_{\text{old}}[S]) \subseteq S \). By the definition of \( \sigma_S \), \( \sigma_S \) contains only steps of processes in \( S \). Thus \( P(\sigma_{\text{setupA}}[S]) \subseteq S \) (Invariant (I1)).

Since \( \sigma_{\text{old}}[0..2^n - 1] \) is \((i-1)\)-compliant with \( S_{\text{max}}(\sigma_{\text{old}}[0..2^n - 1]) = S_{\text{old}} \), by Invariant (I2), for every set \( S \subseteq \mathcal{P} \), \( \sigma_{\text{old}}[S] \neq \perp \) if and only if \( F(\sigma_{\text{old}}[S_{\text{max}}]) \subseteq S \subseteq S_{\text{old}} \). By construction, for every set \( S \subseteq \mathcal{P} \), \( \sigma_{\text{setupA}}[S] = \perp \) if and only if \( \sigma_{\text{old}}[S] = \perp \). Furthermore, by Lemma 3 ((S5)), \( F(\sigma_{\text{old}}[S_{\text{max}}]) = F(\sigma_{\text{setupA}}[S_{\text{max}}]) \). Thus for every set \( S \subseteq \mathcal{P} \), \( \sigma_{\text{setupA}}[S] \neq \perp \) if and only if \( F(\sigma_{\text{setupA}}[S_{\text{max}}]) \subseteq S \subseteq S_{\text{old}} \) (Invariant (I2)).

By Lemma 3 ((S2)), for every set \( S \subseteq \mathcal{P} \) such that \( \sigma_{\text{old}}[S] \neq \perp \), and every process \( p \in S \), \( \text{状态}_p(\sigma_{\text{setupA}}[S_p]) = \text{状态}_p(C_p, \sigma_p) \circ \text{状态}_p(C_S, \sigma_S) = \text{状态}_p(\sigma_{\text{setupA}}[S]) \). By construction, for every set \( S \subseteq \mathcal{P} \), \( \sigma_{\text{setupA}}[S] = \perp \) if and only if \( \sigma_{\text{old}}[S] = \perp \). Furthermore, we have already proven that for every set \( S \subseteq \mathcal{P} \), \( \sigma_{\text{setupA}}[S] \neq \perp \) if and only if
Thus as we wanted, for every set $S \subseteq \mathcal{P}$ that contains $p$, if $\sigma_{\text{setupA}}[S] \neq \bot$, then $\text{state}_p(\sigma_{\text{setupA}}[S]) = \text{state}_p(\sigma_{\text{setupA}}[S_{\text{max}}])$ (Invariant (I3)).

Since we have already proven that Invariants (I1), (I2), and (I3) hold for $\sigma_{\text{setupA}}[0..2^n - 1]$, it immediately follows that Invariant (I4) also holds.

Since $\sigma_{\text{old}}[0..2^n - 1]$ is $(i-1)$-compliant, Invariant (I6) holds for $\sigma_{\text{old}}[0..2^n - 1]$. Thus no new cache copies can be created during $\sigma_{\text{old}}[0..2^n - 1]$.

Since $\sigma_{\text{old}}[0..2^n - 1]$ is $(i-1)$-compliant, Invariant (I8) holds for $\sigma_{\text{old}}[0..2^n - 1]$. By Lemma 3 ((S6)), for every set $S \subseteq \mathcal{P}$ such that $\sigma_{\text{old}}[S] \neq \bot$, each register $R \in \mathcal{R}$ can only be accessed by its owner during $E(C$, $\sigma_{\text{old}})$. Thus Invariant (I8) also holds for $\sigma_{\text{setupA}}[0..2^n - 1]$.

Since $\sigma_{\text{old}}[0..2^n - 1]$ is $(i-1)$-compliant, Invariant (I9) holds for $\sigma_{\text{setupA}}[0..2^n - 1]$. In the CC model, by Lemma 3 ((S8)), for every set $S \subseteq \mathcal{P}$ such that $\sigma_{\text{old}}[S] \neq \bot$, each register $R \in \mathcal{R}$ can only be read during $E(C$, $\sigma_{\text{old}})$. Thus no valid cache copy can be invalidated during $E(C$, $\sigma_{\text{old}})$. Furthermore, by Lemma 3 ((S8)), during $E(C$, $\sigma_{\text{old}})$, each process $p$ can only read registers that it already has valid cache copies of. Thus no new cache copies can be created during $E(C$, $\sigma_{\text{old}})$.

Consequently, Invariant (I9) also holds for $\sigma_{\text{setupA}}[0..2^n - 1]$.

Since $\sigma_{\text{old}}[0..2^n - 1]$ is $(i-1)$-compliant, Invariant (I10) holds for $\sigma_{\text{old}}[0..2^n - 1]$. Then, since no steps are removed in the construction of the schedules for $\sigma_{\text{setupA}}[0..2^n - 1]$, clearly Invariant (I10) also holds for $\sigma_{\text{setupA}}[0..2^n - 1]$.

We will now prove that Invariant (I5) holds for $\sigma_{\text{setupA}}[0..2^n - 1]$ as follows. Let $R \in \mathcal{R}$ be any register. Our goal is to show that there exists a value $y_R$ such that for every set $S \subseteq \mathcal{P}$, if $\sigma_{\text{setupA}}[S] \neq \bot$, then:

$$\text{val}_R(\sigma_{\text{setupA}}[S]) = \begin{cases} \text{val}_R(\sigma_{\text{setupA}}[S_{\text{max}}]) & \text{if } \text{last}_R(\sigma_{\text{setupA}}[S_{\text{max}}]) \in S \\ y_R & \text{otherwise} \end{cases}$$

Note that since $\sigma_{\text{old}}[0..2^n - 1]$ is $(i-1)$-compliant with $S_{\text{max}}(\sigma_{\text{old}}[0..2^n - 1]) = S_{\text{max}}^{\text{old}}$ by Invariant (I5), there is a value $y_R$ such that for every set $S \subseteq \mathcal{P}$, if $\sigma_{\text{old}}[S] \neq \bot$, then:

$$\text{val}_R(\sigma_{\text{old}}[S]) = \begin{cases} \text{val}_R(\sigma_{\text{old}}[S_{\text{max}}]) & \text{if } \text{last}_R(\sigma_{\text{old}}[S_{\text{max}}]) \in S \\ y_R & \text{otherwise} \end{cases}$$

First, suppose that $R$ is not accessed during $E(C$, $\sigma_{\text{old}})$ for every set $S \subseteq \mathcal{P}$ such that $\sigma_{\text{old}}[S] \neq \bot$. Then since $\sigma_{\text{old}}[S] = \bot$ if and only if $\sigma_{\text{setupA}}[S] = \bot$, and $R$ is not accessed during $E(C$, $\sigma_{\text{old}})$, $\text{last}_R(\sigma_{\text{old}}[S_{\text{max}}]) = \text{last}_R(\sigma_{\text{setupA}}[S_{\text{max}}])$. Thus as we wanted, for every set $S \subseteq \mathcal{P}$, if $\sigma_{\text{setupA}}[S] \neq \bot$, then:

$$\text{val}_R(\sigma_{\text{setupA}}[S]) = \begin{cases} \text{val}_R(\sigma_{\text{setupA}}[S_{\text{max}}]) & \text{if } \text{last}_R(\sigma_{\text{setupA}}[S_{\text{max}}]) \in S \\ y_R & \text{otherwise} \end{cases}$$

So suppose instead that there exists a set $S' \subseteq \mathcal{P}$ such that $\sigma_{\text{old}}[S'] \neq \bot$, and a process $p \in S' \setminus F(\sigma_{\text{old}}[S_{\text{max}}])$ that accesses $R$ during $E(C$, $\sigma_{\text{old}})$. The proof now differs depending on the model.

In the DSM model, by Lemma 3 ((S8)), $p$ must be the owner of $R$. Then since $\sigma_{\text{old}}[0..2^n - 1]$ is $(i-1)$-compliant, by Invariant (I8), either $\text{last}_R(\sigma_{\text{old}}[S_{\text{max}}]) = p$ or $\text{val}_R(\sigma_{\text{old}}[S_{\text{max}}]) = y_R$. Furthermore, by Lemma 3 ((S6) and (S7)), for every set $S \subseteq \mathcal{P}$ such that $\sigma_{\text{old}}[S] \neq \bot$, if $p \in S$, then $\text{val}_R(C$, $\sigma_{\text{old}}) = \text{val}_R(C_p$, $\sigma_{\text{old}})$; otherwise $\text{val}_R(C$, $\sigma_{\text{old}}) = y_R$. Thus
as we wanted, for every set \( S \subseteq \mathcal{P} \), if \( \sigma_{\text{setupA}}[S] \neq \bot \), then:

\[
\text{val}_{\mathcal{R}}(\sigma_{\text{setupA}}[S]) = \begin{cases} 
\text{val}_{\mathcal{R}}(\sigma_{\text{setupA}}[S_{\text{max}}]) & \text{if } \text{last}_{\mathcal{R}}(\sigma_{\text{setupA}}[S_{\text{max}}]) \in S \\
y_{\mathcal{R}} & \text{otherwise}
\end{cases}
\]

Finally, in the CC model, by Lemma 3 ((S9)), \( p \) already has a valid cache copy of \( R \) in \( C_{S'} \). So since \( \sigma_{\text{old}}[0..2^n-1] \) is \((i-1)\)-compliant, by Invariant (I9), for every set \( S \subseteq \mathcal{P} \) such that \( \sigma_{\text{old}}[S] \neq \bot \), and \( p \in S \), \( \text{val}_{\mathcal{R}}(\sigma_{\text{old}}[S]) = \text{val}_{\mathcal{R}}(\sigma_{\text{old}}[S']) \). Thus either \( \text{last}_{\mathcal{R}}(\sigma_{\text{old}}[S_{\text{max}}]) = p \) or \( \text{val}_{\mathcal{R}}(\sigma_{\text{old}}[S_{\text{max}}]) = y_{\mathcal{R}} \). Note that if any process other than \( p \) also accesses \( R \) during \( E(C_{S'}, \mathcal{S}) \), then \( \text{val}_{\mathcal{R}}(\sigma_{\text{old}}[S_{\text{max}}]) = y_{\mathcal{R}} \). Furthermore, by Lemma 3 ((S8)), for every set \( S \subseteq \mathcal{P} \) such that \( \sigma_{\text{old}}[S] \neq \bot \), \( \text{val}_{\mathcal{R}}(\sigma_{\text{old}}[S]) = \text{val}_{\mathcal{R}}(\sigma_{\text{setupA}}[S]) \). Thus observe that as we wanted, for every set \( S \subseteq \mathcal{P} \), if \( \sigma_{\text{setupA}}[S] \neq \bot \), then:

\[
\text{val}_{\mathcal{R}}(\sigma_{\text{setupA}}[S]) = \begin{cases} 
\text{val}_{\mathcal{R}}(\sigma_{\text{setupA}}[S_{\text{max}}]) & \text{if } \text{last}_{\mathcal{R}}(\sigma_{\text{setupA}}[S_{\text{max}}]) \in S \\
y_{\mathcal{R}} & \text{otherwise}
\end{cases}
\]

Consequently we have proven that Invariant (I5) holds for \( \sigma_{\text{setupA}}[0..2^n-1] \).

\[\square\]

We now construct another array \( \sigma_{\text{setupB}}[0..2^n-1] \) with the goal of satisfying Invariant (I7) as follows. If no process is within the critical section at the end of the \( E(\sigma_{\text{setupA}}[S_{\text{max}}]) \), we simply construct \( \sigma_{\text{setupB}}[0..2^n-1] \) such that \( \sigma_{\text{setupB}}[0..2^n-1] = \sigma_{\text{setupA}}[0..2^n-1] \). Furthermore, we define \( S_{\text{max}} = S_{\text{old}} \).

Otherwise, to avoid violating mutual exclusion, there must be exactly one process \( p \in S_{\text{max}} \setminus F(\sigma_{\text{setupA}}[S_{\text{max}}]) \) such that at the end of the \( E(\sigma_{\text{setupA}}[S_{\text{max}}]) \), \( p \) is within the critical section. Then note that by Lemma 4 and Invariant (I3), for every set \( S \subseteq \mathcal{P} \) such that \( \sigma_{\text{setupA}}[S] \neq \bot \), if \( p \in S \) then \( p \) is also within the critical section at the end of \( E(\sigma_{\text{setupA}}[S]) \); otherwise no process is within the critical section at the end of \( E(\sigma_{\text{setupA}}[S]) \). Thus we construct \( \sigma_{\text{setupB}}[0..2^n-1] \) such that for every set \( S \subseteq \mathcal{P} \), if \( p \in S \), then \( \sigma_{\text{setupB}}[S] = \bot \); otherwise \( \sigma_{\text{setupB}}[S] = \sigma_{\text{setupA}}[S] \). Furthermore, we define \( S_{\text{max}} = S_{\text{old}} \setminus \{p\} \).

**Lemma 5.** This new array \( \sigma_{\text{setupB}}[0..2^n-1] \) is \((i-1)\)-compliant with \( S_{\text{max}}(\sigma_{\text{setupB}}[0..2^n-1]) = S_{\text{max}} \).

**Proof.** By Lemma 4, except for Invariant (I7), \( \sigma_{\text{setupA}}[0..2^n-1] \) is \((i-1)\)-compliant with \( S_{\text{max}}(\sigma_{\text{setupA}}[0..2^n-1]) = S_{\text{old}} \).

If no process is within the critical section at the end of the \( E(\sigma_{\text{setupA}}[S_{\text{max}}]) \), then \( \sigma_{\text{setupB}}[0..2^n-1] = \sigma_{\text{setupA}}[0..2^n-1] \). Thus by Lemma 4 and Lemma 3 ((S4)), Invariant (I7) also holds for \( \sigma_{\text{setupB}}[0..2^n-1] \), and so it follows that \( \sigma_{\text{setupB}}[0..2^n-1] = \sigma_{\text{setupA}}[0..2^n-1] \) is \((i-1)\)-compliant with \( S_{\text{max}}(\sigma_{\text{setupB}}[0..2^n-1]) = S_{\text{max}} = S_{\text{old}} \).

Otherwise, there is exactly one process \( p \in S_{\text{max}} \setminus F(\sigma_{\text{setupA}}[S_{\text{max}}]) \) such that \( p \) is within the critical section at the end of the \( E(\sigma_{\text{setupA}}[S_{\text{max}}]) \), and for every set \( S \subseteq \mathcal{P} \), if \( p \in S \), then \( \sigma_{\text{setupB}}[S] = \bot \); otherwise \( \sigma_{\text{setupB}}[S] = \sigma_{\text{setupA}}[S] \). By Lemma 4, except for Invariant (I7), \( \sigma_{\text{setupB}}[0..2^n-1] \) is \((i-1)\)-compliant with \( S_{\text{max}}(\sigma_{\text{setupB}}[0..2^n-1]) = S_{\text{old}} \). Thus observe that by the construction of \( \sigma_{\text{setupB}}[0..2^n-1] \), Invariants (I1), (I2), (I3), (I4), (I5), (I6), (I8), (I9), (I10) must also hold for \( \sigma_{\text{setupB}}[0..2^n-1] \) with \( S_{\text{max}}(\sigma_{\text{setupB}}[0..2^n-1]) = S_{\text{max}} = S_{\text{old}} \setminus \{p\} \).

By the construction of \( \sigma_{\text{setupB}}[0..2^n-1] \), for every set \( S \subseteq \mathcal{P} \), if \( p \in S \), then \( \sigma_{\text{setupB}}[S] = \bot \). Thus for every set \( S \subseteq \mathcal{P} \), if \( \sigma_{\text{setupB}}[S] \neq \bot \), then \( p \notin S \). Consequently, for every set \( S \subseteq \mathcal{P} \) such that \( \sigma_{\text{setupB}}[S] \neq \bot \), no process is within the critical section at the end of \( E(\sigma_{\text{setupB}}[S]) \). Therefore by Lemma 3 ((S4)), Invariant (I7) holds for \( \sigma_{\text{setupB}}[0..2^n-1] \), and thus \( \sigma_{\text{setupB}}[0..2^n-1] \) is \((i-1)\)-compliant with \( S_{\text{max}}(\sigma_{\text{setupB}}[0..2^n-1]) = S_{\text{max}} \).

\[\square\]
**i-th Iteration (Decision Phase):** Then for each register \( R \in \mathcal{R} \), let \( B_R \) be the set of processes poised to access \( R \) at the end of \( E(\sigma_{\text{setup}}[S_{\text{max}}]) \). (So \( B_R \cap F(\sigma_{\text{setup}}[S_{\text{max}}]) = \emptyset \).) Further, let
\[
H = \bigcup_{R \in \mathcal{R}} B_R,
\]
and
\[
L = S_{\text{max}} \setminus (H \cup F(\sigma_{\text{setup}}[S_{\text{max}}][])).
\]

**i-th Iteration (Low Contention Phase if \(|L| \geq |H|\):** We begin by constructing an undirected graph where the processes in \( L \) are the nodes, and for every pair of nodes \( p \) and \( q \), we connect an edge between \( p \) and \( q \) if and only if at least one of the following is true at the end of \( E(\sigma_{\text{setup}}[S_{\text{max}}]) \):

- \( p \) and \( q \) are poised to access the same register.
- \( p \) is poised to access a register owned by \( q \), or vice versa.
- \( p \) is poised to access a register that \( q \) has previously performed an operation on, or vice versa.

Thus in this graph:

- Since \(|B_R| < k\) for every register \( R \in \mathcal{R} \) that processes in \( L \) are poised to access, there are at most \( k|L| \) edges representing processes that are poised to access the same register.
- Since every register \( R \in \mathcal{R} \) is owned by at most one process, there are at most \( |L| \) edges representing processes that are poised to access a register owned by some process in \( L \) (one edge for each process, connecting it to the owner of the register it is poised to access).
- By Assumption (A2), every passage incurs at most \( \log n \) RMs. By Lemma 5, \( \sigma_{\text{setup}}[0..2^n - 1] \) is \((i-1)\)-compliant. Recall that \( L \cap F(\sigma_{\text{setup}}[S_{\text{max}}]) = \emptyset \). So by Invariant (I6), each process in \( L \) has never crashed, and so has started at most one passage. Thus each process in \( L \) has performed operations on at most \( \log n \) registers that are not owned by itself. (Note that registers owned by itself are excluded because their associated edges would have already been added in the previous step.) Then since \(|B_R| < k\) for every register \( R \in \mathcal{R} \) that processes in \( L \) are poised to access, there are at most \( k|L| \log n \) edges representing processes that are poised to access a register that some process in \( L \) has previously performed an operation on (\( k \) edges from each of the \( \log n \) registers previously accessed by each process in \( L \)).

Hence, the total number of edges is at most \( k|L| + |L| + k|L| \log n < 3k|L| \log n \).

Let \( I \) be the maximum independent set of the graph.

**Lemma 6.** \(|I| \geq |L|/(7k \log n)|\).

**Proof.** Since there are at most \( 3k|L| \log n \) edges, the average degree of the graph is at most \( 6k \log n \). The lemma immediately follows by Turan’s Theorem. \( \square \)

Let \( S_I = F(\sigma_{\text{setup}}[S_{\text{max}}]) \cup I \). Note that since \( I \not\subseteq L \), \( F(\sigma_{\text{setup}}[S_{\text{max}}]) \subseteq S_I \subseteq S_{\text{max}} \). We now construct a new array \( \sigma_{\text{low}A}[0..2^n - 1] \) such that for every set \( S \subseteq \mathcal{P} \), if \( S \not\subseteq S_I \), then \( \sigma_{\text{low}A}[S] = 1 \); otherwise \( \sigma_{\text{low}A}[S] = \sigma_{\text{setup}}[S] \).

**Lemma 7.** This new array \( \sigma_{\text{low}A}[0..2^n - 1] \) is \((i-1)\)-compliant with \( S_{\text{max}} \)(\( \sigma_{\text{low}A}[0..2^n - 1] \)) = \( S_I \).

**Proof.** By Lemma 5, \( \sigma_{\text{setup}}[0..2^n - 1] \) is \((i-1)\)-compliant with \( S_{\text{max}} \)(\( \sigma_{\text{setup}}[0..2^n - 1] \)) = \( S_{\text{max}} \). By construction, \( \sigma_{\text{low}A}[0..2^n - 1] \) is simply a modification of \( \sigma_{\text{setup}}[0..2^n - 1] \) where every set \( S \subseteq \mathcal{P} \) that contains any process in \( S_{\text{max}} \\setminus S_I \)
has had $\sigma_{\text{low}(S)}$ set to $\bot$, where $F(\sigma_{\text{setup}(S_{\text{max}})}) \subseteq S_f \subseteq S_{\text{setup}}$. It suffices to observe that every invariant still holds with $S_{\text{max}}(\sigma_{\text{low}(0.2^n - 1)}) = S_f$, and thus $\sigma_{\text{low}(0.2^n - 1)}$ is $(i - 1)$-compliant with $S_{\text{max}}(\sigma_{\text{low}(0.2^n - 1)}) = S_f$. □

Now for each set $S \subseteq P$ such that $\sigma_{\text{low}(S)} \neq \bot$, let $C'_S$ be the configuration at the end of $E(\sigma_{\text{low}(S)})$, and let $\sigma'_S$ be the schedule consisting of exactly one non-crash step by each process in $S \setminus F(\sigma_{\text{low}(S)})$ in order from the process with the smallest ID to the process with the largest ID. Note that $\sigma'_S$ is a finite schedule since there are only $n$ processes in the system. Also note that by Lemma 7, $\sigma_{\text{low}(0.2^n - 1)}$ is $(i - 1)$-compliant with $S_{\text{max}}(\sigma_{\text{low}(0.2^n - 1)}) = S_f$, so $\sigma_{\text{low}(S)} \neq \bot$ if and only if $F(\sigma_{\text{low}(S)}) \subseteq S \subseteq S_f$. Thus $\sigma'_S$ contains exactly one non-crash step of each process in $I \cap S$ and no other steps.

**Lemma 8.** For every set $S \subseteq P$ such that $\sigma_{\text{low}(S)} \neq \bot$:

(L1) For each register $R \in R$, $R$ is accessed by at most one process $p \in I \cap S$ during $E(C'_S, \sigma'_S)$ and no other processes.

(L2) For each process $p \in I$, if $p$ accesses a register $R$ during $E(C'_S, \sigma'_S)$, then the owner of $R$ is not in $I \setminus \{p\}$.

(L3) For each process $p \in I$, if $p$ accesses a register $R$ during $E(C'_S, \sigma'_S)$, then $R$ has never been accessed by any process in $\{p\} \setminus (\sigma_{\text{low}(S)})$.

(L4) For each process $p \in I$, during $E(C'_S, \sigma'_S)$, $p$ cannot invalidate any cache copy of any process in $I \setminus \{p\}$.

(L5) For each process $p \in I$, if $p$ accesses a register $R$ during $E(C'_S, \sigma'_S)$, then there is a value $y_R$ such that for every set $S' \subseteq P$, if $\sigma_{\text{low}(S')} \neq \bot$, then:

$$\text{val}_{R}(\sigma_{\text{low}(S')}) = \left\{ \begin{array}{ll} \text{val}_{R}(\sigma_{\text{low}(S)}) & \text{if } p \in S' \\ y_R & \text{otherwise} \end{array} \right.$$  

Note that this implies that for each register $R \in R$, if $R$ is accessed during $E(C'_S, \sigma'_S)$ then $\text{val}_{R}(\sigma_{\text{low}(S)}) = \text{val}_{R}(\sigma_{\text{low}(S')})$.

(L6) For each register $R \in R$, if $R$ is accessed during $E(C'_S, \sigma'_S)$ then $\text{val}_{R}(C'_S, \sigma'_S) = \text{val}_{R}(C'_S, \sigma'_S)$.

(L7) For each process $p \in S$, $\text{state}_{p}(C'_S, \sigma'_S) = \text{state}_{p}(C'_S, \sigma'_S)$.

(L8) Each process in $I \cap S$ incurs exactly one RMR during $E(C'_S, \sigma'_S)$.

(L9) For each process $p \in S \setminus F(\sigma_{\text{low}(S)})$, $p$ has not left the critical section during $E(\sigma_{\text{low}(S)} \circ \sigma'_S)$.

(L10) $F(\sigma_{\text{low}(S)}) = F(\sigma_{\text{low}(S)} \circ \sigma'_S)$.

**Proof.** Let $S \subseteq P$ be any set of processes such that $\sigma_{\text{low}(S)} \neq \bot$. By Lemma 7, $\sigma_{\text{low}(0.2^n - 1)}$ is $(i - 1)$-compliant with $S_{\text{max}}(\sigma_{\text{low}(0.2^n - 1)}) = S_f$. So by Invariant (I2), for every set $S' \subseteq P$, $\sigma_{\text{low}(S')} \neq \bot$ if and only if $F(\sigma_{\text{low}(S)}) \subseteq S' \subseteq S_f$. Thus $F(\sigma_{\text{low}(S)}) \subseteq S \subseteq S_f$.

Furthermore, by Invariants (I3) and (I4), for every process $p \in S$, $\text{state}_{p}(\sigma_{\text{low}(S)}) = \text{state}_{p}(\sigma_{\text{low}(S)})$ and $F(\sigma_{\text{low}(S)}) = F(\sigma_{\text{low}(S)})$. So by Invariant (I7), every process $p \in S \setminus F(\sigma_{\text{low}(S)})$ has not entered the critical section during $E(\sigma_{\text{low}(S)})$.

Now recall that $\sigma'_S$ contains exactly one non-crash step of each process in $I \cap S$ and no other steps. Also recall that by construction, for every process $p \in S$, $\text{state}_{p}(\sigma_{\text{low}(S)}) = \text{state}_{p}(\sigma_{\text{low}(S)})$ and $\text{state}_{p}(\sigma_{\text{setup}(S)}) = \text{state}_{p}(\sigma_{\text{setup}(S)})$. Thus since $I$ is an independent set of the graph we constructed earlier:

- No pair of processes in $I$ are poised to access the same register at the end of $E(\sigma_{\text{setup}(S_{\text{max}})})$. Since $\text{state}_{p}(\sigma_{\text{low}(S)}) = \text{state}_{p}(\sigma_{\text{setup}(S_{\text{max}})})$ for every process $p \in S$, no pair of processes in $I$ are poised to access the same register at the end of $E(\sigma_{\text{low}(S)})$. Thus every process in $I \cap S$ accesses a different register during $E(C'_S, \sigma'_S)$ (L10).
Finally, recall that \( \text{setup}\{S\} \) incurs an RMR at the end of \( \text{setup}\{S\} \). By Lemma 7, \( \text{setup}\{S\} \) and any process \( p \) in \( S \), no process \( p \) in \( S \) is poised to access a register that has previously been accessed by a different process \( q \) in \( S \) at the end of \( \text{setup}\{S\} \). For each process \( p \) in \( I \), if \( p \) accesses a register \( R \) during \( E(C', \sigma') \), then the owner of \( R \) is not in \( I \setminus \{ p \} \). (L2).

By Lemma 7, \( \text{setup}\{S\} \), no process \( p \) in \( S \) is poised to access a register that has previously been accessed by a different process \( q \) in \( S \) at the end of \( \text{setup}\{S\} \). For each process \( p \) in \( I \), if \( p \) accesses a register \( R \) during \( E(C', \sigma') \), then \( R \) has never been accessed by any process in \( I \setminus \{ p \} \) during \( E(\text{setup}\{S\}) \). Consequently, for each process \( p \) in \( I \), if \( p \) accesses a register \( R \) during \( E(C', \sigma') \), no process in \( I \setminus \{ p \} \) makes a cache copy of \( R \) during \( E(\text{setup}\{S\}) \). (L3).

Furthermore, since \( \text{setup}\{S\} \) and any process \( p \) in \( S \), no process \( p \) in \( S \) is poised to access a register that has previously been accessed by a different process \( q \) in \( S \) at the end of \( \text{setup}\{S\} \). Thus for each process \( p \) in \( I \), if \( p \) accesses a register \( R \) during \( E(C', \sigma') \), then \( R \) has never been accessed by any process in \( I \setminus \{ p \} \) during \( E(\text{setup}\{S\}) \). Consequently, for each process \( p \) in \( I \), if \( p \) accesses a register \( R \) during \( E(C', \sigma') \), no process in \( I \setminus \{ p \} \) makes a cache copy of \( R \) during \( E(\text{setup}\{S\}) \). (L4).

By Lemma 7, \( \text{setup}\{S\} \), no process \( p \) in \( S \) is poised to access a register that has previously been accessed by a different process \( q \) in \( S \) at the end of \( \text{setup}\{S\} \). Thus for each process \( p \) in \( I \), if \( p \) accesses a register \( R \) during \( E(C', \sigma') \), then \( R \) has never been accessed by any process in \( I \setminus \{ p \} \) during \( E(\text{setup}\{S\}) \). Consequently, for each process \( p \) in \( I \), if \( p \) accesses a register \( R \) during \( E(C', \sigma') \), no process in \( I \setminus \{ p \} \) makes a cache copy of \( R \) during \( E(\text{setup}\{S\}) \). (L5).

By Lemma 7, \( \text{setup}\{S\} \), no process \( p \) in \( S \) is poised to access a register that has previously been accessed by a different process \( q \) in \( S \) at the end of \( \text{setup}\{S\} \). Thus for each process \( p \) in \( I \), if \( p \) accesses a register \( R \) during \( E(C', \sigma') \), then \( R \) has never been accessed by any process in \( I \setminus \{ p \} \) during \( E(\text{setup}\{S\}) \). Consequently, for each process \( p \) in \( I \), if \( p \) accesses a register \( R \) during \( E(C', \sigma') \), no process in \( I \setminus \{ p \} \) makes a cache copy of \( R \) during \( E(\text{setup}\{S\}) \). (L6).

By Lemma 7, \( \text{setup}\{S\} \), no process \( p \) in \( S \) is poised to access a register that has previously been accessed by a different process \( q \) in \( S \) at the end of \( \text{setup}\{S\} \). Thus for each process \( p \) in \( I \), if \( p \) accesses a register \( R \) during \( E(C', \sigma') \), then \( R \) has never been accessed by any process in \( I \setminus \{ p \} \) during \( E(\text{setup}\{S\}) \). Consequently, for each process \( p \) in \( I \), if \( p \) accesses a register \( R \) during \( E(C', \sigma') \), no process in \( I \setminus \{ p \} \) makes a cache copy of \( R \) during \( E(\text{setup}\{S\}) \). (L7). (Recall that \( \sigma' \) consists of exactly one non-crash step of each process in \( I \cap S \) and no other steps, so the states of other processes do not change.)

Since \( \sigma(S) \neq \bot \), by construction, \( \sigma(S) \neq \bot \). By Lemma 3 ((S3)), for each process \( p \) in \( S \setminus F(\sigma(S)) \), \( p \) incurs an RMR at the end of \( E(\text{setup}\{S\} \circ p) \). By construction and Lemma 4 (Invariant (I4)), \( \text{setup}\{S\} = \text{setup}\{S\} = \text{setup}\{S\} \), and \( F(\sigma(S)) = F(\text{setup}\{S\} \circ p) = F(\text{setup}\{S\} \circ p) = F(\text{setup}\{S\} \circ p) \). Thus for each process \( p \) in \( S \setminus F(\sigma(S)) \), \( p \) incurs an RMR at the end of \( E(\text{setup}\{S\} \circ p) \). By definition, \( S = S \cup F(\sigma(S)) \), so since \( S \subseteq S \setminus F(\sigma(S)) \), \( S = \cup S \). Finally, recall that \( \sigma(S) \) contains exactly one non-crash step of each process in \( I \cap S \) and no other steps. Consequently,
since we have already proven that each process \( I \cap S \) accesses a different register during \( E(C'_S, \sigma'_S) \), observe that each process in \( I \cap S \) must incur exactly one RMR during \( E(C'_S, \sigma'_S) \) ((L8)).

By Lemma 7, \( \sigma_{\text{lowA}}[0.2^n - 1] \) is \((i - 1)\)-compliant, so by Invariant (I7), each process that is not in \( F(\sigma_{\text{lowA}}[S]) \) does not enter the critical section during \( E(\sigma_{\text{lowA}}[S]). \) Recall that \( \sigma'_S \) consists of exactly one non-crash step of each process in \( I \cap S = S \setminus F(\sigma_{\text{lowA}}[S]) \) and no other steps. Thus, with only one step, although a process could enter the critical section during \( E(C'_S, \sigma'_S) \), it cannot have taken any steps within the critical section. Thus by Assumption (A1), no process can leave the critical section during \( E(C'_S, \sigma'_S) \). So for each process \( p \in S \setminus F(\sigma_{\text{lowA}}[S]), \) \( p \) has not left the critical section during \( E(\sigma_{\text{lowA}}[S] \circ \sigma'_S) \) (L9). Therefore, since no process in \( S \setminus F(\sigma_{\text{lowA}}[S]) \) has left the critical section during \( E(\sigma_{\text{lowA}}[S] \circ \sigma'_S) \), no process in \( S \setminus F(\sigma_{\text{lowA}}[S]) \) has completed its super-passage during \( E(\sigma_{\text{lowA}}[S] \circ \sigma'_S). \) Thus \( F(\sigma_{\text{lowA}}[S]) = F(\sigma_{\text{lowA}}[S] \circ \sigma'_S) \) (L10)).

We now construct a new array \( \sigma_{\text{lowB}}[0.2^n - 1] \) such that for every set \( S \subseteq \mathcal{P}, \sigma_{\text{lowB}}[S] = \bot \) if \( \sigma_{\text{lowA}}[S] = \bot \); otherwise \( \sigma_{\text{lowB}}[S] = \sigma_{\text{lowA}}[S] \circ \sigma'_S \).

**Lemma 9.** Except for Invariant (I7), \( \sigma_{\text{lowB}}[0.2^n - 1] \) is \( i \)-compliant with \( S_{\text{max}}(\sigma_{\text{lowB}}[0.2^n - 1]) = S_I \).

**Proof.** For every set \( S \subseteq \mathcal{P}, \) if \( \sigma_{\text{lowB}}[S] \neq \bot \), then by construction, \( \sigma_{\text{lowB}}[S] = \sigma_{\text{lowA}}[S] \circ \sigma'_S \). By Lemma 7, \( \sigma_{\text{lowA}}[0.2^n - 1] \) is \((i - 1)\)-compliant, so by Invariant (I1), \( P(\sigma_{\text{lowA}}[S]) \subseteq S. \) By the definition of \( \sigma'_S \), \( \sigma'_S \) contains only steps of processes in \( S \cap I \). Thus \( P(\sigma_{\text{lowB}}[S]) \subseteq S \) (Invariant (I1)).

By Lemma 7, \( \sigma_{\text{lowA}}[0.2^n - 1] \) is \((i - 1)\)-compliant with \( S_{\text{max}}(\sigma_{\text{lowA}}[0.2^n - 1]) = S_I \). So by Invariant (I2), for every set \( S \subseteq \mathcal{P}, \sigma_{\text{lowA}}[S] = \bot \) if and only if \( F(\sigma_{\text{lowA}}[S]) \subseteq S \subseteq S_I. \) By construction, for every set \( S \subseteq \mathcal{P}, \sigma_{\text{lowB}}[S] = \bot \) if and only if \( \sigma_{\text{lowA}}[S] = \bot \). Furthermore, by Lemma 8 (L10), \( F(\sigma_{\text{lowA}}[S]) = F(\sigma_{\text{lowB}}[S]). \) Thus for every set \( S \subseteq \mathcal{P}, \sigma_{\text{lowB}}[S] = \bot \) if and only if \( F(\sigma_{\text{lowB}}[S]) \subseteq S \subseteq S_I \) (Invariant (I2)).

By Lemma 8 (L7), for every set \( S \subseteq \mathcal{P} \) such that \( \sigma_{\text{lowA}}[S] \neq \bot \), and every process \( p \in S, \) \( \text{state}_p(\sigma_{\text{lowA}}[S]) = \text{state}_p(C'_S, \sigma_S) \) (Invariant (I3)). Therefore, since we have already proven that for every set \( S \subseteq \mathcal{P}, \sigma_{\text{lowB}}[S] = \bot \), then \( \text{state}_p(\sigma_{\text{lowB}}[S]) = \text{state}_p(C'_S, \sigma_S) \) (Invariant (I3)).

By Lemma 8 (L5), for every process \( p \in I, \) if \( p \) accesses a register \( R \) during \( E(C'_S, \sigma'_S) \), then there is a value \( y_R \) such that for every set \( S' \subseteq \mathcal{P}, \) if \( \sigma_{\text{lowA}}[S'] \neq \bot \), then:

\[
\text{val}_R(\sigma_{\text{lowA}}[S']) = \begin{cases} 
\text{val}_R(\sigma_{\text{lowA}}[S]) & \text{if } p \in S' \\
y_R & \text{otherwise}
\end{cases}
\]

Note that since \( p \) accesses \( R \) during \( E(C'_S, \sigma'_S) \), by Lemma 8 (L11), \( \text{last}_R(\sigma_{\text{lowA}}[S]) = p. \) By Lemma 7, \( \sigma_{\text{lowA}}[0.2^n - 1] \) is \((i - 1)\)-compliant, so if \( p \in S', \) then \( p \) also accesses \( R \) during \( E(C'_S, \sigma'_S) \). Thus by Lemma 8 (L6), if \( p \in S', \) then \( \text{val}_R(\sigma_{\text{lowB}}[S']) = \text{val}_R(\sigma_{\text{lowB}}[S]). \) Furthermore, by Lemma 8 (L1), for every set \( S \subseteq \mathcal{P} \) such that \( \sigma_{\text{lowA}}[S] \neq \bot, \) \( R \) cannot be accessed by any process other than \( p \) during \( E(C'_S, \sigma'_S) \). Therefore if \( p \notin S', \) then \( R \) cannot be accessed during \( E(C'_S, \sigma'_S), \) and so \( \text{val}_R(\sigma_{\text{lowB}}[S']) = y_R. \) Thus we have that for every register \( R \in \mathcal{R} \) such that some process \( p \) accesses
Then, since any register that is not accessed by any process during \(E(C'_S, \sigma'_S)\) clearly does not change its state, observe that Invariant (I5) holds for \(\sigma_{lowB}[0..2^n - 1]\).

By Lemma 7, \(\sigma_{lowA}[0..2^n - 1]\) is \((i - 1)\)-compliant, so Invariant (I6) holds for \(\sigma_{lowA}[0..2^n - 1]\). For every set \(S \subseteq \mathcal{P}\) such that \(\sigma_{lowA}[S] \neq \bot\), \(\sigma'_S\) contains no crash steps. Thus Invariant (I6) also holds for \(\sigma_{lowB}[0..2^n - 1]\).

By Lemma 7, \(\sigma_{lowA}[0..2^n - 1]\) is \((i - 1)\)-compliant with \(S_{max}(\sigma_{lowA}[0..2^n - 1]) = S_I\). By Invariant (I8), for every process \(p \in S_I \setminus F(\sigma_{lowA}[S_I])\), every register \(R \in \mathcal{R}\) owned by \(p\), and every set \(S \subseteq \mathcal{P}\) with \(\sigma_{lowA}[S] \neq \bot\), \(R\) can only be accessed by \(p\) during \(E(\sigma_{lowA}[S])\). By Lemma 8 ((L2)), for every set \(S \subseteq \mathcal{P}\) such that \(\sigma_{lowA}[S] \neq \bot\), for each process \(p \in I\), if \(p\) accesses a register \(R\) during \(E(C'_S, \sigma'_S)\), then the owner of \(R\) is not in \(I \setminus \{p\} = (S_I \setminus F(\sigma_{lowA}[S_I])) \setminus \{p\}\). In other words, for every set \(S \subseteq \mathcal{P}\) such that \(\sigma_{lowA}[S] \neq \bot\), for each process \(p \in I\), if \(p\) owns a register \(R\), then no other process in \(I = S_I \setminus F(\sigma_{lowA}[S_I])\) accesses \(R\) during \(E(C'_S, \sigma'_S)\). By Lemma 8 ((L10)), \(F(\sigma_{lowA}[S_I]) = F(\sigma_{lowB}[S_I])\). Consequently, for every process \(p \in S_I \setminus F(\sigma_{lowA}[S_I])\), every register \(R \in \mathcal{R}\) owned by \(p\), and every set \(S \subseteq \mathcal{P}\) with \(\sigma_{lowB}[S] \neq \bot\), \(R\) can only be accessed by \(p\) during \(E(\sigma_{lowB}[S])\) (Invariant (I8)).

By Lemma 7, \(\sigma_{lowA}[0..2^n - 1]\) is \((i - 1)\)-compliant with \(S_{max}(\sigma_{lowA}[0..2^n - 1]) = S_I\). By Invariant (I3), for every process \(p \in S_I\) and every set \(S \subseteq \mathcal{P}\) that contains \(p\), if \(\sigma_{lowA}[S] \neq \bot\), then \(stat_{\mathcal{R}}(\sigma_{lowA}[S]) = stat_{\mathcal{R}}(\sigma_{lowA}[S_I])\). Furthermore, by Invariant (I9), in the CC model, for every process \(p \in S_I \setminus F(\sigma_{lowA}[S_I])\), there is a set \(\mathcal{R}_p\) of registers such that for every set \(S \subseteq \mathcal{P}\) that contains \(p\), if \(\sigma_{lowA}[S] \neq \bot\), then the set of registers that \(p\) has valid cache copies of at the end of \(E(\sigma_{lowA}[S])\) is exactly \(\mathcal{R}_p\). Recall that for every set \(S \subseteq \mathcal{P}\) such that \(\sigma_{lowA}[S] \neq \bot\), \(\sigma'_S\) contains exactly one non-crash step of each process in \(S_I\) and no other steps. By Lemma 8 ((L1)), for every set \(S \subseteq \mathcal{P}\) such that \(\sigma_{lowA}[S] \neq \bot\), each register is accessed by at most one process \(p \in I \cap S\) during \(E(C'_S, \sigma'_S)\) and no other processes. Furthermore, by Lemma 8 ((L5)), for every set \(S \subseteq \mathcal{P}\) such that \(\sigma_{lowA}[S] \neq \bot\), for every register \(R \in \mathcal{R}\), if \(R\) is accessed during \(E(C'_S, \sigma'_S)\), then \(val_{\mathcal{R}}(\sigma_{lowA}[S_I]) = val_{\mathcal{R}}(\sigma_{lowA}[S])\). Thus for every process \(p \in S_I\) and every set \(S \subseteq \mathcal{P}\) such that \(\sigma_{lowA}[S] \neq \bot\), during both \(E(C'_S, \sigma'_S)\) and \(E(C'_S, \sigma'_S)\), \(p\) performs the same operation on the same register which begins with the same value, causing the resulting state and response. There are two cases: either this operation that \(p\) performs on \(R\) is a read operation, or it is not.

If it is a read operation, then no cache copies are invalidated by the read, and \(p\) creates a new valid cache copy of \(R\) during both \(E(C'_S, \sigma'_S)\) and \(E(C'_S, \sigma'_S)\), thus observe that for every set \(S' \subseteq \mathcal{P}\) that contains \(p\), if \(\sigma_{lowB}[S'] \neq \bot\), then the set of registers that \(p\) has valid cache copies of is exactly \(\mathcal{R}_p \cup \{R\}\). If it is not a read operation, then cache copies can be invalidated, but by Lemma 8 ((L4)), the invalidated cache copies cannot belong to any process in \(I \setminus \{p\} = (S_I \setminus F(\sigma_{lowA}[S_I])) \setminus \{p\}\). Thus the cache copies of every process in \(S_I \setminus F(\sigma_{lowA}[S_I]) \setminus \{p\}\) are unaffected, whereas observe that for every set \(S' \subseteq \mathcal{P}\) that contains \(p\), if \(\sigma_{lowB}[S'] \neq \bot\), then the set of registers that \(p\) has valid cache copies of is exactly \(\mathcal{R}_p \cup \{R\}\). By Lemma 8 ((L10)), \(F(\sigma_{lowA}[S_I]) = F(\sigma_{lowB}[S_I])\). Consequently, in both cases, for every process \(p \in S_I \setminus F(\sigma_{lowA}[S_I])\), there is a set \(\mathcal{R}'_p\) of registers (namely either \(\mathcal{R}_p \cup \{R\}\) or \(\mathcal{R}_p \setminus \{R\}\) where \(R\) is the one register that \(p\) is poised to access at the end of \(E(\sigma_{lowA}[S_I])\) such that for every set \(S \subseteq \mathcal{P}\) that contains \(p\), if \(\sigma_{lowA}[S] \neq \bot\), then the set of registers that \(p\) has valid cache copies of is exactly \(\mathcal{R}'_p\) (Invariant (I9)).

By Lemma 7, \(\sigma_{lowA}[0..2^n - 1]\) is \((i - 1)\)-compliant, so by Invariant (I10), for every set \(S \subseteq \mathcal{P}\) and every process \(p \in S \setminus F(\sigma_{lowA}[S])\), if \(\sigma_{lowB}[S] \neq \bot\), then \(p\) incurs at least \(i - 1\) RMRs during \(E(\sigma_{lowA}[S])\). By Lemma 8 ((L8)), for every
set $S \subseteq \mathcal{P}$ such that $s_{\text{low}B}[S] \neq \bot$, each process in $I \cap S$ incurs exactly one RMR during $E(C')$. By construction, for every set $S \subseteq \mathcal{P}$, if $s_{\text{low}B}[S] \neq \bot$, then $s_{\text{low}B}[S] = s_{\text{low}B}[\sigma_{\text{low}B}] \circ \sigma'_{\mathcal{S}'},$ i.e., every process in $I \cap S = S \setminus F(s_{\text{low}B}[S])$ incurs exactly one more RMR during $E(s_{\text{low}B}[S])$ than during $E(s_{\text{low}B}[S])$. By Lemma 8 ((L10)), $F(s_{\text{low}B}[S]) = F(s_{\text{low}B}[S])$ for every set $S \subseteq \mathcal{P}$ such that $s_{\text{low}B}[S] \neq \bot$. Thus for every set $S \subseteq \mathcal{P}$ and every process $p \in S \setminus F(s_{\text{low}B}[S]),$ if $s_{\text{low}B}[S] \neq \bot$, then $p$ incurs at least i RMRs during $E(s_{\text{low}B}[S])$ (Invariant (I10)).

We now construct another array $s_{\text{low}B}[0..2^n - 1]$ with the goal of satisfying Invariant (I7) as follows. If no process is within the critical section at the end of $E(s_{\text{low}B}[S])$, we simply construct $s_{\text{low}B}[0..2^n - 1]$ such that $s_{\text{low}B}[0..2^n - 1] = \bot$. Furthermore, we define $s_{\text{low}B} = S_t$.

Otherwise, to avoid violating mutual exclusion, there must be exactly one process $p \in S_t \setminus F(s_{\text{low}B}[S])$ such that at the end of $E(s_{\text{low}B}[S])$, $p$ is within the critical section. Then note that by Lemma 9 and Invariant (I3), for every set $S \subseteq \mathcal{P}$ such that $s_{\text{low}B}[S] \neq \bot$, if $p \in S$ then $p$ is also within the critical section at the end of $E(s_{\text{low}B}[S])$; otherwise no process is within the critical section at the end of $E(s_{\text{low}B}[S])$. Thus we construct $s_{\text{low}B}[0..2^n - 1]$ such that for every set $S \subseteq \mathcal{P}$, if $p \in S$, then $s_{\text{low}B}[S] = \bot$; otherwise $s_{\text{low}B}[S] = s_{\text{low}B}[S]$. Furthermore, we define $s_{\text{low}B} = S_t \setminus \{p\}$.

**Lemma 10.** This new array $s_{\text{low}B}[0..2^n - 1]$ is i-compliant with $s_{\text{low}B}[0..2^n - 1] = s_{\text{low}B}$.  

Proof. By Lemma 9, except for Invariant (I7), $s_{\text{low}B}[0..2^n - 1]$ is i-compliant with $s_{\text{low}B}[0..2^n - 1] = S_t$.  

If no process is within the critical section at the end of $E(s_{\text{low}B}[S])$, then $s_{\text{low}B}[0..2^n - 1] = s_{\text{low}B}[0..2^n - 1]$. Thus by Lemma 9 and Lemma 8 ((L9)), Invariant (I7) also holds for $s_{\text{low}B}[0..2^n - 1]$, and so it follows that $s_{\text{low}B}[0..2^n - 1] = s_{\text{low}B}[0..2^n - 1]$ is i-compliant with $s_{\text{low}B}[0..2^n - 1] = s_{\text{low}B}$.  

Otherwise, there is exactly one process $p \in S_t \setminus F(s_{\text{low}B}[S])$ such that $p$ is within the critical section at the end of $E(s_{\text{low}B}[S])$, and for every set $S \subseteq \mathcal{P}$, if $p \in S$, then $s_{\text{low}B}[S] = \bot$; otherwise $s_{\text{low}B}[S] = s_{\text{low}B}[S]$. By Lemma 9, except for Invariant (I7), $s_{\text{low}B}[0..2^n - 1]$ is i-compliant with $s_{\text{low}B}[0..2^n - 1] = S_t$. Thus observe that by the construction of $s_{\text{low}B}[0..2^n - 1]$, Invariants (I1), (I2), (I3), (I4), (I5), (I6), (I8), (I9), (I10) must all also hold for $s_{\text{low}B}[0..2^n - 1]$ with $s_{\text{low}B} = s_{\text{low}B} = S_t \setminus \{p\}$.  

By the construction of $s_{\text{low}B}[0..2^n - 1]$, for every set $S \subseteq \mathcal{P}$, if $p \in S$, then $s_{\text{low}B}[S] = \bot$; otherwise $s_{\text{low}B}[S] = s_{\text{low}B}[S]$. By Lemma 9, except for Invariant (I7), $s_{\text{low}B}[0..2^n - 1]$ is i-compliant with $s_{\text{low}B}[0..2^n - 1] = S_t$. Thus observe that by the construction of $s_{\text{low}B}[0..2^n - 1]$, Invariants (I1), (I2), (I3), (I4), (I5), (I6), (I8), (I9), (I10) must all also hold for $s_{\text{low}B}[0..2^n - 1]$ with $s_{\text{low}B} = s_{\text{low}B} = S_t \setminus \{p\}$. Finally, we terminate this i-th iteration by setting $s_{\text{setup}} = s_{\text{setup}}[i..2^n - 1] = s_{\text{low}B}[0..2^n - 1]$.

**i-th Iteration (High Contention Phase if $|L| < |H|$):** By Lemma 5, $s_{\text{setup}}[0..2^n - 1]$ is (i-1)-compliant with $s_{\text{low}B}[0..2^n - 1] = s_{\text{max}}$. Recall that $H = \bigcup_{R \in \mathcal{R}} B_R$, where for every register $R \in \mathcal{R}$, $B_R$ is the set of processes poised to access $R$ at the end of $E(s_{\text{setup}}[S_{\text{setup}}])$.

We start divide the processes in $H$ into groups of exactly $k$ processes such that within each group, all processes are poised to access the same register at the end of $E(s_{\text{setup}}[S_{\text{setup}}])$. We make as many such groups as possible. Then let $H_1$ be the set of processes in the resulting groups, i.e., $H_1$ is a modification of $H$ where all processes that are not in any group are removed. Note that by this construction, $H = \bigcup_{R \in \mathcal{R}} B_R$, $|H_1| > |H|/2$.

Next, let $H_2$ be a modification of $H_1$ such that for each process $p \in H_1$, $p$ is in $H_2$ if and only if both of the following are true:


• No process in \(H_1\) is poised to access a register owned by \(p\) at the end of \(E(\sigma_{\text{setup}}(S_{\text{setup}}^\max))\).

• No process in \(H_1\) is poised to access a register \(R\) at the end of \(E(\sigma_{\text{setup}}(S_{\text{setup}}^\max))\) such that \(\text{last}_R(\sigma_{\text{setup}}(S_{\text{max}}^\max)) = p\).

Note that since \(H_1\) is composed of groups of exactly \(k\) processes that are all poised to access the same register at the end of \(E(\sigma_{\text{setup}}(S_{\text{setup}}^\max))\), there are at most \(|H_1|/k\) registers that are poised to be accessed by processes in \(H_1\), and thus there are at most \(2|H_1|/k\) processes removed in the construction of \(H_2\) from \(H_1\).

Then let \(H_3\) be a modification of \(H_2\) such that each remaining group of \(H_2\) with at least \(k/4\) processes is shrunk to contain only \(k/4\) processes, and all other groups are removed. Since at most \(2|H_1|/k\) processes were removed in the construction of \(H_2\) from \(H_1\), and \(k > \log n\), at least half of the processes remain, and so it is easy to see that at least a quarter of the groups in \(H_2\) remain with at least \(k/4\) processes. So \(|H_3| \geq |H_1|/16 > |H|/32\).

Next, recall that all registers support only read, fetch-and-store (FAS), fetch-and-increment (FAI), and compare-and-swap (CAS) operations. Let \(\text{opt}_H\) denote one of these 4 operation types such that the plurality of processes in \(H\) are poised to perform an operation of type \(\text{opt}_H\) at the end of \(E(\sigma_{\text{setup}}(S_{\text{max}}^\max))\). Then let \(H_4\) be a modification of \(H_3\) such that for each process \(p \in H_3\), \(p\) is in \(H_4\) if and only if \(p\) is poised to perform an operation of type \(\text{opt}_H\) at the end of \(E(\sigma_{\text{setup}}(S_{\text{setup}}^\max))\). Since there are only 4 operation types, \(|H_4| \geq |H_3|/4\).

Now let \(H_5\) be a modification of \(H_4\) such that each group of \(H_4\) with at least \(k/32\) processes is shrunk to contain only \(k/32\) processes, and all other groups are removed. Since each group of \(H_3\) originally had exactly \(k/4\) processes and \(|H_4| \geq |H_3|/4\) where \(k > \log n\), it is easy to see that at least an eighth of the groups in \(H_4\) remain with at least \(k/32\) of the processes. So \(|H_5| \geq |H_3|/64 \geq |H_1|/1024 > |H|/2048\).

Let \(h\) be the number of remaining groups in \(H_5\). Then since \(H_5\) only contains groups with exactly \(k/32\) processes, \(h = 32|H_5|/k\). We arbitrarily order these groups, and construct an array \(G[0..h - 1]\) such that for every integer \(j \in \{0, 1, \ldots, h - 1\}\), \(G[j]\) is the \(j\)-th group in the ordering. Then for every integer \(j \in \{0, 1, \ldots, h - 1\}\), let \(R[j]\) be the register that every process in \(G[j]\) is poised to access at the end of \(E(\sigma_{\text{setup}}(S_{\text{setup}}^\max))\).

Finally, let \(S_H = F(\sigma_{\text{setup}}(S_{\text{setup}}^\max)) \cup H_5\). We now construct a new array \(\sigma_{\text{high}}[0..2^n - 1]\) such that for every set \(S \subseteq \mathcal{P}\), if \(S \nsubseteq S_H\), then \(\sigma_{\text{high}}[S] = \bot\); otherwise \(\sigma_{\text{high}}[S] = \sigma_{\text{setup}}[S]\).

**Lemma 11.** This new array \(\sigma_{\text{high}}[0..2^n - 1]\) is \((i - 1)\)-compliant with \(S_{\text{max}}(\sigma_{\text{high}}[0..2^n - 1]) = S_H\). Furthermore, for every integer \(j \in \{0, 1, \ldots, h - 1\}\):

- For every set \(S \subseteq \mathcal{P}\) such that \(\sigma_{\text{high}}[S] \neq \bot\), every process in \(G[j] \cap S\) is poised to access \(R[j]\) at the end of \(E(\sigma_{\text{high}}[S])\).
- The owner of \(R[j]\) is not in \(S_H \setminus F(\sigma_{\text{high}}(S_H))\).
- For every set \(S \subseteq \mathcal{P}\) such that \(\sigma_{\text{high}}[S] \neq \bot\), \(\text{val}_{R[j]}(\sigma_{\text{high}}[S]) = \text{val}_{R[j]}(\sigma_{\text{high}}(S_H))\).

**Proof.** By Lemma 5, \(\sigma_{\text{setup}}[0..2^n - 1]\) is \((i - 1)\)-compliant with \(S_{\text{max}}(\sigma_{\text{setup}}[0..2^n - 1]) = S_{\text{setup}}\). By construction, \(\sigma_{\text{high}}[0..2^n - 1]\) is simply a modification of \(\sigma_{\text{setup}}[0..2^n - 1]\) where every set \(S \subseteq \mathcal{P}\) that contains any process in \(\sigma_{\text{setup}}(S_{\text{setup}}^\max) \setminus S_H\) has had \(\sigma_{\text{high}}[S]\) set to \(\bot\), where \(F(\sigma_{\text{setup}}(S_{\text{setup}}^\max)) \subseteq S_H \subseteq S_{\text{setup}}\). It suffices to observe that since every invariant still holds with \(S_{\text{max}}(\sigma_{\text{high}}[0..2^n - 1]) = S_H, \sigma_{\text{high}}[0..2^n - 1]\) is \((i - 1)\)-compliant with \(S_{\text{max}}(\sigma_{\text{high}}[0..2^n - 1]) = S_{\text{setup}}\). So by Invariant (I3), for every process \(p \in S_{\text{setup}}\) and every set \(S \subseteq \mathcal{P}\) that contains \(p, \text{if } \sigma_{\text{setup}}[S] \neq \bot\), then...
Tight Lower Bounds for the RMR Complexity of Recoverable Mutual Exclusion

\[ \text{state}_p(\sigma_{\text{setup}}[S]) = \text{state}_p(\sigma_{\text{setup}}[S_{\text{max}}]) \]

Thus for every process \( p \in G[j] \) and every set \( S \subseteq P \) that contains \( p \), if \( \sigma_{\text{setup}}[S] \neq \perp \), then \( p \) is also poised to access \( R[j] \) at the end of \( E(\sigma_{\text{setup}}[S]) \).

By the construction of \( \sigma_{\text{highA}}[0..2^n - 1] \), for every set \( S \subseteq P \) such that \( \sigma_{\text{highA}}[S] \neq \perp \), \( \sigma_{\text{highA}}[S] = \sigma_{\text{setup}}[S] \). So for every process \( p \in G[j] \) and every set \( S \subseteq P \) that contains \( p \), if \( \sigma_{\text{highA}}[S] \neq \perp \), then \( p \) is also poised to access \( R[j] \) at the end of \( E(\sigma_{\text{highA}}[S]) \). Thus for every set \( S \subseteq P \) such that \( \sigma_{\text{highA}}[S] \neq \perp \), every process in \( G[j] \cap S \) is poised to access \( R[j] \) at the end of \( E(\sigma_{\text{highA}}[S]) \).

Now recall that \( S_H = F(\sigma_{\text{setup}}[S_{\text{max}}]) \cup H_5 \), where \( H_5 \subseteq H_2 \). By construction, for each process \( p \in H_1 \), \( p \) is in \( H_2 \) if and only if both of the following are true:

- No process in \( H_1 \) is poised to access a register owned by \( p \) at the end of \( E(\sigma_{\text{setup}}[S_{\text{max}}]) \).
- No process in \( H_1 \) is poised to access a register \( R \) at the end of \( E(\sigma_{\text{setup}}[S_{\text{max}}]) \) such that \( \text{last}_R(\sigma_{\text{setup}}[S_{\text{max}}]) = p \).

We have already shown that \( \sigma_{\text{highA}}[0..2^n - 1] \) is \((i-1)\)-compliant with \( S_{\text{max}}(\sigma_{\text{setup}}[0..2^n - 1]) = S_{\text{setup}} \). So by Invariant (I2), for every process \( R \in \mathcal{R} \), there is a value \( y_R \) such that for every set \( S \subseteq P \), if \( \sigma_{\text{setup}}[S] \neq \perp \), then:

\[ \text{val}_R(\sigma_{\text{setup}}[S]) = \begin{cases} \text{val}_R(\sigma_{\text{setup}}[S_{\text{max}}]) & \text{if } \text{last}_R(\sigma_{\text{setup}}[S_{\text{max}}]) \in S \\ y_R & \text{otherwise} \end{cases} \]

We have just shown that \( \text{last}_R[j](\sigma_{\text{setup}}[S_{\text{max}}]) \) is not in \( S_H \setminus F(\sigma_{\text{highA}}[S_H]) \). So either \( \text{last}_R[j](\sigma_{\text{setup}}[S_{\text{max}}]) \in F(\sigma_{\text{highA}}[S_H]) \) or \( \text{last}_R(\sigma_{\text{setup}}[S_{\text{max}}]) \notin S_H \).

By the construction of \( \sigma_{\text{highA}}[0..2^n - 1] \), for every set \( S \subseteq P \), if \( \sigma_{\text{highA}}[S] \neq \perp \), then \( \sigma_{\text{highA}}[S] = \sigma_{\text{setup}}[S] \). Thus if \( \text{last}_R[j](\sigma_{\text{setup}}[S_{\text{max}}]) \) is in \( F(\sigma_{\text{highA}}[S_H]) \), then for every set \( S \subseteq P \) such that \( \sigma_{\text{highA}}[S] \neq \perp \), \( \text{val}_R[j](\sigma_{\text{highA}}[S]) = \text{val}_R[j](\sigma_{\text{setup}}[S_{\text{max}}]) \).

Otherwise \( \text{last}_R(\sigma_{\text{setup}}[S_{\text{max}}]) \) is not in \( S_H \). Since we have already proven that \( \sigma_{\text{highA}}[0..2^n - 1] \) is \((i-1)\)-compliant with \( S_{\text{max}}(\sigma_{\text{highA}}[0..2^n - 1]) = S_H \), by Invariant (I2), for every set \( S \subseteq P \) such that \( \sigma_{\text{highA}}[S] \neq \perp \), \( S \subseteq S_H \) and so \( \text{last}_R(\sigma_{\text{setup}}[S_{\text{max}}]) \) is not in \( S \). Thus for every set \( S \subseteq P \) such that \( \sigma_{\text{highA}}[S] \neq \perp \), \( \text{val}_R(\sigma_{\text{highA}}[S]) = y_R \).

So in both cases, for every set \( S \subseteq P \) such that \( \sigma_{\text{highA}}[S] \neq \perp \), \( \text{val}_R[j](\sigma_{\text{highA}}[S]) = \text{val}_R[j](\sigma_{\text{highA}}[S_H]) \).

We now iterate over \( j \in \{0, 1, \ldots, h - 1\} \) to construct two arrays \( a_1[0..h-1] \) and \( a_2[0..h-1] \) of processes and an array \( a[0..h-1] \) of processes as follows. If \( \text{opt}_H \) \( \neq \text{CAS} \), then let \( a_1[j] \) and \( a_2[j] \) be two arbitrary but distinct processes in \( G[j] \), and let \( a[j] = a_1[j] \circ a_2[j] \). Otherwise, consider the register \( R[j] \) that every process in \( G[j] \) is poised to access at the end of \( E(\sigma_{\text{highA}}[S_H]) \). Let \( u_j = \text{val}_R[j](\sigma_{\text{highA}}[S_H] \circ a[0] \circ a[1] \circ \ldots \circ a[j - 1]) \). Then let \( a[j] \) be any process in \( G[j] \) such that \( a[j] \) is about to perform a \( \text{CAS}(u, u') \) operation where \( u' \neq u_j \); if no such process exists, then let \( a[j] \) be any process in \( G[j] \). Next, let \( a[j] \) be any process in \( G[j] \setminus \{a_1[j]\} \). Finally, let \( a[j] = a_1[j] \circ a_2[j] \).

For every integer \( j \in \{0, 1, \ldots, h - 1\} \), let \( \sigma_{\alpha}^j \) be the concatenation of all schedules in \( a[0..j] \), i.e., \( \sigma_{\alpha}^j = a[0] \circ a[1] \circ \ldots \circ a[j] \). Then let \( \sigma_{\alpha} = \sigma_{\alpha}^{h-1} \). Furthermore, let \( S_{\alpha} \) be the set of all processes with steps in \( \sigma_{\alpha} \). Note that since
Thus observe that to avoid violating deadlock freedom, there must exist a schedule \( \sigma_F \) such that:

- \( \sigma_F \) begins with exactly one crash step of every process in \( S_a \), and contains no other crash steps.
- \( \sigma_F \) contains only steps of processes in \( S_a = S_F \setminus F(\sigma_{\text{highA}}[S_H]) \), i.e., \( P(\sigma_F) = S_a \).
- During \( E(\sigma_{\text{highA}}[S_F] \circ \sigma_a \circ \sigma_F) \), every process in \( S_F \) begins and then completes a super-passage, i.e., \( F(\sigma_{\text{highA}}[S_F] \circ \sigma_a \circ \sigma_F) = S_F \).

Let \( C_F \) be the configuration at the end of \( E(\sigma_{\text{highA}}[S_F] \circ \sigma_a) \). Then let \( \mathcal{R}_F \) be the set of every register that is accessed during \( E(\mathcal{C}_F, \sigma_F) \) (after the crash steps of every process in \( S_a \) at the beginning of \( \sigma_F \)). Next, let \( \mathcal{D} \subseteq \mathcal{P} \) be the set of every process \( p \in H_5 \setminus S_a \) such that there exists a register \( R \in \mathcal{R}_F \) such that either \( p \) owns \( R \), or \( \text{last_R}(\sigma_{\text{highA}}[S_H]) = p \).

**Lemma 12.** \(|\mathcal{D}| \leq 2|S_a| \log n = \frac{2^k}{k} |H_5| \log n\).

**Proof.** First, consider each register \( R \in \mathcal{R}_F \) such that \( R \) is owned by a process in \( S_a \). Since \( \mathcal{D} \subseteq H_5 \setminus S_a \), the owner of \( R \) is not in \( \mathcal{D} \). Furthermore, by Lemma 11, \( \sigma_{\text{highA}}[0..2^n - 1] \) is \((i - 1)\)-compliant with \( S_{\text{max}}(\sigma_{\text{highA}}[0..2^n - 1]) = \mathcal{H}_5 \), so by Invariant (I8), since the owner of \( R \) is in \( S_a \subseteq H_5 \subseteq S_H \setminus F(\sigma_{\text{highA}}[S_H]) \), \( R \) cannot be accessed by any process in \( H_5 \setminus S_a \) during \( E(\sigma_{\text{highA}}[S_H]) \). Thus intuitively, each register \( R \in \mathcal{R}_F \) that is owned by a process in \( S_a \) does not contribute any processes to \( \mathcal{D} \).

So it suffices to consider the registers in \( \mathcal{R}_F \) that are not owned by any process in \( S_a \). By Assumption (A2), each process accesses at most \( \log n \) registers that it does not own during a passage. Thus there are at most \(|S_a| \log n \) registers in \( \mathcal{R}_F \) that are not owned by any process in \( S_a \). Consequently, \(|\mathcal{D}| \leq 2|S_a| \log n = \frac{2^k}{k} |H_5| \log n \). \(\square\)

Now let \( H_6 \) be a modification of \( H_5 \) where every process \( p \in H_5 \) is in \( H_6 \) if and only if \( p \not\in \mathcal{D} \). Note that since \( \mathcal{D} \subseteq H_5 \setminus S_a \), \( S_a \subseteq H_5 \subseteq H_6 \). By Lemma 12, \(|\mathcal{D}| \leq \frac{2^k}{k} |H_5| \log n \), so \(|H_6| \geq |H_5| - \frac{2^k}{k} |H_5| \log n \). For sufficiently large \( k \) (\( k \geq 256 \log n \)), \( \frac{2^k}{k} |H_5| \log n \leq 0.5 |H_5| \). Thus, at least half of the processes in \( H_5 \) remain in \( H_6 \), and so it is easy to see that at least a quarter of the groups in \( H_6 \) remain with at least \( k/128 \) processes (out of the \( k/32 \) originally in \( H_6 \)). Furthermore, since \( S_a \subseteq H_6 \), for \( 0 \leq j \leq h - 1 \), since \( S_a \) contains \( \{a_1[j], a_2[j] \} \subseteq G[j] \cap H_6 \), and so \( \{a_1[j], a_2[j] \} \subseteq G'[j] \).

Now for every integer \( j \in \{0, 1, \ldots, h - 1\} \), let \( \beta(j) = 0 \) if \( |G'[j]| < k/160 \); otherwise, let \( \beta(j) \) be an arbitrary process in \( G'[j] \setminus \{a_1[j], a_2[j] \} \). Then let:

\[
S_{\beta} = \bigcup_{j=0}^{h-1} \{a_1[j], a_2[j], \beta(j)\}
\]
So:

\[ S_\beta = S_\alpha \cup \bigcup_{j=0}^{h-1} \{ \beta_1[j] \} \]

Note that by construction, \( S_\alpha \subseteq S_\beta \subseteq H_0 \subseteq H_5 \).

**Lemma 13.** \(|S_\beta \setminus S_\alpha| > \frac{|H|}{204.8k} \).

**Proof.** Since at least a quarter of the \( h \) groups in \( H_0 \) have at least \( k/128 \) processes, \( \beta_1[j] \neq \emptyset \) for at least a quarter of the integers \( j \in \{0, 1, \ldots, h-1\} \). Thus \(|S_\beta \setminus S_\alpha| \geq 0.25h \cdot |H_5|/k \).

Now recall that \(|H_5| \geq |H_3|/64 \geq |H_1|/1024 > |H_1|/2048 \). So \(|S_\beta \setminus S_\alpha| \geq 10|H_5|/k > \frac{|H|}{204.8k} \).  

Next, let \( S_B = S_\beta \cup F(\sigma_{\text{highA}}[S_H]) \). We now construct a new array \( \sigma_{\text{highB}}[0..2^n-1] \) such that for every set \( S \subseteq P \), if \( S \notin S_B \), then \( \sigma_{\text{highB}}[S] = \perp \); otherwise \( \sigma_{\text{highB}}[S] = \sigma_{\text{highA}}[S] \). Note that since \( S_\beta \subseteq H_0 \subseteq H_5 \) and \( S_H = F(\sigma_{\text{highA}}[S_H]) \cup H_5 \), \( F(\sigma_{\text{highA}}[S_H]) \subseteq S_B \subseteq S_H \).

**Lemma 14.** This new array \( \sigma_{\text{highB}}[0..2^n-1] \) is \((i-1)\)-compliant with \( S_{\text{max}}(\sigma_{\text{highB}}[0..2^n-1]) = S_B \). Furthermore, for every integer \( j \in \{0, 1, \ldots, h-1\} \):

- For every set \( S \subseteq P \) such that \( \sigma_{\text{highB}}[S] \neq \perp \), every process in \( G[j] \cap S \) is poised to access \( R[j] \) at the end of \( E(\sigma_{\text{highB}}[S]) \).
- The owner of \( R[j] \) is not in \( S_B \setminus F(\sigma_{\text{highB}}[S]) \).
- For every set \( S \subseteq P \) such that \( \sigma_{\text{highB}}[S] \neq \perp \), \( \text{val}_{R[j]}(\sigma_{\text{highB}}[S]) = \text{val}_{R[j]}(\sigma_{\text{highB}}[S_B]) \).

In addition, for every register \( R \in \mathcal{R}_F \),

- The owner of \( R \) is not in \( S_B \setminus F(\sigma_{\text{highB}}[S_B]) \).
- For every set \( S \subseteq P \) such that \( \sigma_{\text{highB}}[S] \neq \perp \), \( \text{val}_{R}(\sigma_{\text{highB}}[S]) = \text{val}_{R}(\sigma_{\text{highB}}[S_B]) \).

**Proof.** By Lemma 11, \( \sigma_{\text{highA}}[0..2^n-1] \) is \((i-1)\)-compliant with \( S_{\text{max}}(\sigma_{\text{highA}}[0..2^n-1]) = S_H \). By construction, \( \sigma_{\text{highB}}[0..2^n-1] \) is simply a modification of \( \sigma_{\text{highA}}[0..2^n-1] \) where every set \( S \subseteq P \) that contains any process in \( S_H \setminus S_B \) has had \( \sigma_{\text{highB}}[S] \) set to \( \perp \), where \( F(\sigma_{\text{highA}}[S_H]) \subseteq S_B \subseteq S_H \). It suffices to observe that since every invariant still holds with \( S_{\text{max}}(\sigma_{\text{highB}}[0..2^n-1]) = S_B \), \( \sigma_{\text{highB}}[0..2^n-1] \) is \((i-1)\)-compliant with \( S_{\text{max}}(\sigma_{\text{highB}}[0..2^n-1]) = S_B \).

Furthermore, by Lemma 11, for every integer \( j \in \{0, 1, \ldots, h-1\} \):

- For every set \( S \subseteq P \) such that \( \sigma_{\text{highA}}[S] \neq \perp \), every process in \( G[j] \cap S \) is poised to access \( R[j] \) at the end of \( E(\sigma_{\text{highA}}[S]) \).
- The owner of \( R[j] \) is not in \( S_H \setminus F(\sigma_{\text{highA}}[S_H]) \).
- For every set \( S \subseteq P \) such that \( \sigma_{\text{highA}}[S] \neq \perp \), \( \text{val}_{R[j]}(\sigma_{\text{highA}}[S]) = \text{val}_{R[j]}(\sigma_{\text{highA}}[S_H]) \).

By the construction of \( \sigma_{\text{highB}}[0..2^n-1] \), for every set \( S \subseteq P \) such that \( \sigma_{\text{highB}}[S] \neq \perp \), \( \sigma_{\text{highB}}[S] = \sigma_{\text{highA}}[S] \). Thus for every integer \( j \in \{0, 1, \ldots, h-1\} \) and every set \( S \subseteq P \) such that \( \sigma_{\text{highB}}[S] \neq \perp \), every process in \( G[j] \cap S \) is poised to access \( R[j] \) at the end of \( E(\sigma_{\text{highA}}[S]) \).

By Lemma 11, \( \sigma_{\text{highA}}[0..2^n-1] \) is \((i-1)\)-compliant. So by Invariant (I4), \( F(\sigma_{\text{highA}}[S_H]) = F(\sigma_{\text{highA}}[S_B]) = F(\sigma_{\text{highB}}[S_B]) \). Then, since \( S_B \subseteq S_H \), for every integer \( j \in \{0, 1, \ldots, h-1\} \), the owner of \( R[j] \) is not in \( S_B \setminus F(\sigma_{\text{highB}}[S_B]) \).
Next, for every integer $j \in \{0, 1, \ldots, h - 1\}$ and every set $S \subseteq P$ such that $\sigma_{\text{highB}}[S] \neq \bot$,
\[
val_R[j](\sigma_{\text{highB}}[S]) = val_R[j](\sigma_{\text{highA}}[S]) \\
= val_R(j)(\sigma_{\text{highA}}[S_H]) \\
= val_R[j](\sigma_{\text{highA}}[S_B]) \\
= val_R[j](\sigma_{\text{highB}}[S_B])
\]
Thus we have proven that for every integer $j \in \{0, 1, \ldots, h - 1\}$:
- For every set $S \subseteq P$ such that $\sigma_{\text{highB}}[S] \neq \bot$, every process in $G[j] \cap S$ is poised to access $R[j]$ at the end of $E(\sigma_{\text{highB}}[S])$.
- The owner of $R[j]$ is not in $S_B \cap F(\sigma_{\text{highB}}[S_B])$.
- For every set $S \subseteq P$ such that $\sigma_{\text{highB}}[S] \neq \bot$, $val_R[j](\sigma_{\text{highB}}[S]) = val_R[j](\sigma_{\text{highB}}[S_B])$.

By definition, $D \subseteq P$ is the set of every process $p \in H_0 \setminus S_a$ such that there exists a register $R \in R_F$ such that either $p$ owns $R$, or $last_R(\sigma_{\text{highA}}[S_H]) = p$. By construction, $H_0 \cap D = \emptyset$, so $S_B = S_B \cup F(\sigma_{\text{highA}}[S_H])$. Furthermore, recall that $F(\sigma_{\text{highA}}[S_H]) = F(\sigma_{\text{highA}}[S_B])$. Thus for every register $R \in R_F$, the owner of $R$ is not in $S_B \setminus F(\sigma_{\text{highB}}[S_B])$.

By Lemma 11, $\sigma_{\text{highA}}[0..2^n - 1]$ is $(i-1)$-compliant with $S_{\text{max}}(\sigma_{\text{highB}}[0..2^n - 1]) = S_H$. So by Invariant (I5), for every register $R \in R_F$, there is a value $y_R$ such that for every set $S \subseteq P$, if $\sigma_{\text{highA}}[S] \neq \bot$, then:
\[
val_R(\sigma_{\text{highA}}[S]) = \begin{cases} 
val_R(\sigma_{\text{highA}}[S_H]) & \text{if } last_R(\sigma_{\text{highA}}[S_H]) \in S \\
y_R & \text{otherwise}
\end{cases}
\]
Since $last_R(\sigma_{\text{highA}}[S_H]) \in D$ and $D \cap S_B = \emptyset$, for every set $S \subseteq P$ such that $F(\sigma_{\text{highA}}[S_H]) = F(\sigma_{\text{highB}}[S_B]) \subseteq S \subseteq S_B \subseteq S_H$, $val_R(\sigma_{\text{highB}}[S]) = y_R$. Furthermore, since we have already proven that $\sigma_{\text{highB}}[0..2^n - 1]$ is $(i-1)$-compliant with $S_{\text{max}}(\sigma_{\text{highB}}[0..2^n - 1]) = S_B$, by Invariant (I2), for every set $S \subseteq P$ such that $\sigma_{\text{highB}}[S] \neq \bot$, $F(\sigma_{\text{highB}}[S_B]) \subseteq S \subseteq S_B$. Thus for every register $R \in R_F$, and every set $S \subseteq P$ such that $\sigma_{\text{highB}}[S] \neq \bot$, $val_R(\sigma_{\text{highB}}[S]) = val_R(\sigma_{\text{highB}}[S_B])$.

We now iterate over $j \in \{0, 1, \ldots, h - 1\}$ to construct an array $\beta[0..h - 1]$ of schedules as follows. Recall that by definition,
- $\beta_1[j]$ is $\emptyset$ if $|G'[j]| < k/160$; otherwise, $\beta_1[j]$ is an arbitrary process in $G'[j] \setminus \{a_1[j], a_2[j]\}$.
- $\nu_j = val_R[j](\sigma_{\text{highA}}[S_H] \circ a_1^{j-1})$.

If opt$_H$ is CAS and $\beta_1[j] \neq \emptyset$, then let $\nu_j$ and $\nu_j'$ be such that $\beta_1[j]$ is poised to perform a CAS($\nu_j$, $\nu_j'$) operation on $R[j]$ at the end of $E(\sigma_{\text{highB}}[S_B])$. Note that by Lemma 14 and Invariant (I3), $\beta_1[j]$ would also be poised to perform a CAS($\nu_j$, $\nu_j'$) operation on $R[j]$ at the end of $E(\sigma_{\text{highB}}[S])$ for every set $S \subseteq P$ such that $\sigma_{\text{highB}}[S] \neq \bot$ and $\beta_1[j] \in S$.

We then define:
\[
\beta[j] = \begin{cases} 
\beta_1[j] \circ a_2[j] & \text{if } |G'[j]| < k/160 \\
\beta_1[j] \circ a_2[j] & \text{if opt$_H$ is FAI} \\
\beta_1[j] \circ a_1[j] \circ a_2[j] & \text{if opt$_H$ is CAS and } \nu_j \neq \nu_j \\
a_1[j] \circ \beta_1[j] \circ a_2[j] & \text{otherwise}
\end{cases}
\]
By Lemma 14, $\sigma_{highB}[0..2^n - 1]$ is $(i - 1)$-compliant with $S_{max}(\sigma_{highB}[0..2^n - 1]) = S_B$. Furthermore, $S_a \subseteq S_B \subseteq S_B$, so $F(\sigma_{highB}[S_B]) \cup S_a \subseteq S_B$. Thus by Invariant (I2), for every set $S \subseteq \mathcal{P}$ such that $F(\sigma_{highB}[S_B]) \cup S_a \subseteq S \subseteq S_B$, $\sigma_{highB}[S] \neq \bot$.

So for every set $S \subseteq \mathcal{P}$ such that $F(\sigma_{highB}[S_B]) \cup S_a \subseteq S \subseteq S_B$, let $C'_S$ be the configuration at the end of $E(\sigma_{highB}[S])$, and let $\sigma'_S$ be a modification of $\sigma_a$ such that for every integer $j \in \{0, 1, \ldots, h - 1\}$, if $\beta_1[j] \notin \emptyset$ and $\beta_1[j] \subseteq S$, then $\alpha[j]$ is replaced by $\beta[j]$ in $\sigma'_S$. Note that by this construction, $\sigma'_S$ contains exactly one non-crash step of each process in $S_B \cap S$ and no other steps.

**Lemma 15.** For every set $S \subseteq \mathcal{P}$ such that $F(\sigma_{highB}[S_B]) \cup S_a \subseteq S \subseteq S_B$, the set of registers accessed during $E(C'_S, \sigma'_S)$ is exactly the set of registers accessed during $E(C'_S, \sigma_a)$ and exactly the set of registers in $R[0..h - 1]$.

**Proof.** First, recall that for every set $S \subseteq \mathcal{P}$ such that $F(\sigma_{highB}[S_B]) \cup S_a \subseteq S \subseteq S_B$, $\sigma_{highB}[S] \neq \bot$. So let $S \subseteq \mathcal{P}$ be a set of processes such that $F(\sigma_{highB}[S_B]) \cup S_a \subseteq S \subseteq S_B$.

By Lemma 14, for every integer $j \in \{0, 1, \ldots, h - 1\}$, every process in $G[j] \cap S$ is poised to access $R[j]$ at the end of $E(\sigma_{highB}[S])$. By construction, both $\sigma'_S$ and $\sigma_a$ contain at most one non-crash step of each process in $S_B \cap S$ and no other steps. Since $S_B \subseteq H_B$, where $G[0..h - 1]$ are the groups of processes that constitute $H_B$, every process with a step in $\sigma'_S$ or $\sigma_a$ is poised to access a register in $R[0..h - 1]$ at the end of $E(\sigma_{highB}[S])$. Therefore every register accessed during either $E(C'_S, \sigma'_S)$ or $E(C'_S, \sigma_a)$ is in $R[0..h - 1]$.

Next, by construction, for every integer $j \in \{0, 1, \ldots, h - 1\}$, both $\sigma'_S$ and $\sigma_a$ contain a non-crash step of $\alpha_2[j]$. By Lemma 14, for every integer $j \in \{0, 1, \ldots, h - 1\}$, $\alpha_2[j]$ is poised to access $R[j]$ at the end of $E(\sigma_{highB}[S])$. So every register in $R[0..h - 1]$ is accessed during both $E(C'_S, \sigma'_S)$ and $E(C'_S, \sigma_a)$.

Thus we have shown that the set of registers accessed during $E(C'_S, \sigma'_S)$ is exactly the set of registers accessed during $E(C'_S, \sigma_a)$ and exactly the set of registers in $R[0..h - 1]$.

Now for every integer $j \in \{0, 1, \ldots, h - 1\}$, let $\sigma'_S[j]$ and $\sigma_a[j]$ be the suffixes of $\sigma'_S$ and $\sigma_a$ that contain only the steps of processes in $G[0..j]$. (So for every integer $j \in \{0, 1, \ldots, h - 1\}$, $\sigma_a[j] = \sigma_a[j]$.) We also define $\sigma'_S[1..] = \sigma_a[1..] = \emptyset$.

**Lemma 16.** For every register $R \in \mathcal{R}$ every integer $j \in \{0, 1, \ldots, h - 1\}$, and every set $S \subseteq \mathcal{P}$ such that $F(\sigma_{highB}[S_B]) \cup S_a \subseteq S \subseteq S_B$, $val_R(C'_S, \sigma'_S[j]) = val_R(C'_S, \sigma_a[j])$.

**Proof.** First, recall that for every set $S \subseteq \mathcal{P}$ such that $F(\sigma_{highB}[S_B]) \cup S_a \subseteq S \subseteq S_B$, $\sigma_{highB}[S] \neq \bot$. So let $S \subseteq \mathcal{P}$ be a set of processes such that $F(\sigma_{highB}[S_B]) \cup S_a \subseteq S \subseteq S_B$.

By definition, $\sigma'_S[1..] = \sigma_a[1..] = \emptyset$. Thus for every register $R \in \mathcal{R}$, $val_R(C'_S, \sigma'_S[1..]) = val_R(C'_S, \sigma_a[1..])$. So it suffices to show that for every register $R \in \mathcal{R}$ and every integer $j \in \{0, 1, \ldots, h - 1\}$, if $val_R(C'_S, \sigma'_S[j - 1..]) = val_R(C'_S, \sigma_a[j - 1..])$, then $val_R(C'_S, \sigma'_S[j..]) = val_R(C'_S, \sigma_a[j..])$.

Thus let $j$ be an integer in $\{0, 1, \ldots, h - 1\}$, and suppose that for every register $R \in \mathcal{R}$, $val_R(C'_S, \sigma'_S[j - 1..]) = val_R(C'_S, \sigma_a[j - 1..])$. By the construction of $\sigma'_S$, $\alpha[j]$ is replaced by $\beta[j]$ if and only if $\beta_1[j] \notin \emptyset$ and $\beta_1[j] \subseteq S$. Thus if either $\beta_1[j] \notin \emptyset$ or $\beta_1[j] \notin S$, then for every register $R \in \mathcal{R}$, $val_R(C'_S, \sigma'_S[j..]) = val_R(C'_S, \sigma_a[j..])$ as wanted.

Otherwise, $\beta_1[j] \notin \emptyset$ and $\beta_1[j] \subseteq S$. Then by the definition of $\beta[j]$: $\beta[j] = \begin{cases} \beta_1[j] \circ \alpha_2[j] & \text{if opt}_H \text{ is FAI} \\ \beta_1[j] \circ \alpha_1[j] \circ \alpha_2[j] & \text{if opt}_H \text{ is CAS and } \alpha_2[j] \notin \alpha[j] \\ \alpha_1[j] \circ \beta_1[j] \circ \alpha_2[j] & \text{otherwise} \end{cases}$
Since $\alpha_1[j]$ and $\alpha_2[j]$ are in $S_{\alpha} \subseteq S$, all of $\alpha_1[j]$, $\alpha_2[j]$, and $\beta_1[j]$ are in $G[j] \cap S$. So by Lemma 14, all of $\alpha_1[j]$, $\alpha_2[j]$, and $\beta_1[j]$ are poised to access $R[j]$ at the end of $E(\sigma_{\text{highB}}[S])$. By construction, both $\sigma'_B$ and $\sigma_\alpha$ contain at most one non-crash step of each process and no other steps. Thus for each process $p \in \{\alpha_1[j], \alpha_2[j], \beta_1[j]\}$, $\text{state}_p(C'_S, \sigma'_B[j] = j - 1)) =$ 
\text{state}_p(C'_S, \sigma_\alpha[j] = j - 1))$.

Therefore all of $\alpha_1[j]$, $\alpha_2[j]$, and $\beta_1[j]$ are poised to access $R[j]$ at the end of both $E(C'_S, \sigma'_B[j] = j - 1))$ and $E(C'_S, \sigma_\alpha[j] = j - 1))$. Thus for every register $R \in R \setminus R[j]$, $\text{val}_R(C'_S, \sigma'_B[j]) = \text{val}_R(C'_S, \sigma_\alpha[j])$. It now suffices to show that $\text{val}_R[j](C'_S, \sigma'_B[j]) = \text{val}_R[j](C'_S, \sigma_\alpha[j])$.

If $\text{opt}_H$ is read, then since reads do not change the value of a register, $\text{val}_R[j](C'_S, \sigma'_B[j]) = \text{val}_R[j](C'_S, \sigma_\alpha[j])$ as wanted.

If $\text{opt}_H$ is FAS, then $\alpha[j] = \alpha_1[j] \circ \alpha_2[j]$ and $\beta[j] = \alpha_1[j] \circ \beta_1[j] \circ \alpha_2[j]$. Let $v_2$ be the value such that $\alpha_2[j]$ is poised to perform FAS($v_2$) at the end of both $E(C'_S, \sigma'_B[j] = j - 1))$ and $E(C'_S, \sigma_\alpha[j] = j - 1))$. Then $\text{val}_R[j](C'_S, \sigma'_B[j]) = v_2$ and $\text{val}_R[j](C'_S, \sigma_\alpha[j]) = v_2$. Thus $\text{val}_R[j](C'_S, \sigma'_B[j]) = \text{val}_R[j](C'_S, \sigma_\alpha[j])$ as wanted.

If $\text{opt}_H$ is FAI, then $\alpha[j] = \alpha_1[j] \circ \alpha_2[j]$ and $\beta[j] = \beta_1[j] \circ \alpha_2[j]$. So $\text{val}_R[j](C'_S, \sigma_\alpha[j]) = \text{val}_R[j](C'_S, \sigma_\alpha[j] = j - 1)) + 2$ and $\text{val}_R[j](C'_S, \sigma'_B[j]) = \text{val}_R[j](C'_S, \sigma'_B[j] = j - 1)) + 2$. Thus $\text{val}_R[j](C'_S, \sigma'_B[j]) = \text{val}_R[j](C'_S, \sigma_\alpha[j])$ as wanted.

Finally, consider the case where $\text{opt}_H$ is CAS. Recall that $v_j = \text{val}_R[j](\sigma_{\text{highA}}[S_H] \circ \sigma_\alpha[j - 1])$. By Lemma 11, $\text{val}_R[j](\sigma_{\text{highA}}[S_H]) = \text{val}_R[j](\sigma_{\text{highA}}[S])$. Then, since $\sigma_\alpha[j - 1]$ contains at most one step by each process, observe that $\text{val}_R[j](\sigma_{\text{highA}}[S_H] \circ \sigma_\alpha[j - 1]) = \text{val}_R[j](\sigma_{\text{highA}}[S] \circ \sigma_\alpha[j - 1])$. By definition, since $\sigma_{\text{highA}}[S] \neq \perp$, $\sigma_{\text{highA}}[S] = \sigma_{\text{highA}}[S]$. Thus $v_j = \text{val}_R[j](\sigma_{\text{highA}}[S] \circ \sigma_\alpha[j - 1])$. Then, since for every register $R \in R$, $\text{val}_R[j](C'_S, \sigma'_B[j] = j - 1)) = \text{val}_R[j](C'_S, \sigma_\alpha[j - 1])$ too.

By definition, either $\alpha_1[j]$ is poised to perform a CAS($v_j, \psi'$) operation where $\psi' \neq v_j$ at the end of $E(\sigma_{\text{highA}}[S_H])$, or no process in $G[j]$ is poised to perform a CAS($v_j, \psi'$) operation where $\psi' \neq v_j$ at the end of $E(\sigma_{\text{highA}}[S_H])$. By Lemma 11, $\sigma_{\text{highA}}[0.2^n - 1]$ is $(i - 1)$-compliant, with $S_{\text{max}}(\sigma_{\text{highA}}[0.2^n - 1]) = S_H$. So by Invariant (I3), since $\sigma_{\text{highA}}[S] = \sigma_{\text{highA}}[S] \neq \perp$ and $\{\alpha_1[j], \alpha_2[j], \beta_1[j]\} \subseteq S$, for each process $p \in \{\alpha_1[j], \alpha_2[j], \beta_1[j]\}$, $\text{state}_p(\sigma_{\text{highA}}[S_H]) = \text{state}_p(\sigma_{\text{highA}}[S])$.

Now recall that for each process $p \in \{\alpha_1[j], \alpha_2[j], \beta_1[j]\}$, $\text{state}_p(C'_S, \sigma'_B[j] = j - 1)) = \text{state}_p(\sigma_{\text{highB}}[S]) = \text{state}_p(C'_S, \sigma_\alpha[j - 1])$. So either $\alpha_1[j]$ is poised to perform a CAS($v_j, \psi'$) operation where $\psi' \neq v_j$ at the end of both $E(C'_S, \sigma'_B[j] = j - 1))$ and $E(C'_S, \sigma_\alpha[j - 1])$, or no process in $\{\alpha_1[j], \alpha_2[j], \beta_1[j]\}$ is poised to perform a CAS($v_j, \psi'$) operation where $\psi' \neq v_j$ at the end of both $E(C'_S, \sigma'_B[j] = j - 1))$ and $E(C'_S, \sigma_\alpha[j - 1])$.

In the latter case, $\text{val}_R[j](C'_S, \sigma'_B[j]) = \text{val}_R[j](C'_S, \sigma'_B[j] = j - 1)) = v_j$ and $\text{val}_R[j](C'_S, \sigma_\alpha[j]) = \text{val}_R[j](C'_S, \sigma'_B[j] = j - 1)) = v_j$. Thus $\text{val}_R[j](C'_S, \sigma'_B[j]) = \text{val}_R[j](C'_S, \sigma_\alpha[j])$ as wanted.

In the former case, since $\beta_1[j] \in S$, $\beta_1[j]$ is poised to perform a CAS($v_j, \psi'$) operation on $R[j]$ at the end of $E(\sigma_{\text{highA}}[S])$. Recall that for each process $p \in \{\alpha_1[j], \alpha_2[j], \beta_1[j]\}$, $\text{state}_p(C'_S, \sigma'_B[j] = j - 1)) = \text{state}_p(\sigma_{\text{highB}}[S]) = \text{state}_p(C'_S, \sigma_\alpha[j - 1])$. So $\beta_1[j]$ is also poised to perform a CAS($v_j, \psi'$) operation on $R[j]$ at the end of both $E(C'_S, \sigma'_B[j] = j - 1))$ and $E(C'_S, \sigma_\alpha[j - 1])$.

By definition, since $\text{opt}_H$ is CAS, if $v_j \neq v_\beta$, then $\beta[j] = \beta_1[j] \circ \alpha_1[j] \circ \alpha_2[j]$ or $\beta[j] = \alpha_1[j] \circ \beta_1[j] \circ \alpha_2[j]$. So if $v_j \neq v_\beta$, then in $E(C'_S, \sigma'_B[j])$, $\beta_1[j]$ performs an unsuccessful CAS($v_\beta, \psi'_B$) on $R[j]$ when $R[j]$ contains $v_j \neq v_\beta$. Otherwise $v_j = v_\beta$, so in $E(C'_S, \sigma'_B[j])$, $\beta_1[j]$ performs an unsuccessful CAS($v_j, \psi'_B$) on $R[j]$ immediately after $\alpha_1[j]$ successfully changes the value of $R[j]$ from $v_j$ to $\psi' \neq v_j$. Thus regardless of whether $v_j = v_\beta$, the CAS($v_\beta, \psi'_B$) operation by $\beta_1[j]$ does not change the value of $R[j]$ during $E(C'_S, \sigma'_B[j])$. Consequently, $\text{val}_R[j](C'_S, \sigma'_B[j]) = \text{val}_R[j](C'_S, \sigma_\alpha[j])$, as wanted.
Lemma 17. For every set \( S \subseteq \mathcal{P} \) such that \( F(\sigma_{\text{highB}}[S_B]) \cup S_a \subseteq S \subseteq S_B \):

(H1) For every process \( p \in \mathcal{P} \), if \( p \) accesses a register \( R \) during \( E(C'_S, \sigma'_S) \), then the owner of \( R \) is not in \( S_B \setminus F(\sigma_{\text{highB}}[S_B]) \).

(H2) For every integer \( j \in \{0, 1, \ldots, h-1\} \), \( \text{val}_{R[j]}(\sigma_{\text{highB}}[S]) = \text{val}_{R[j]}(\sigma_{\text{highB}}[S_B]) \).

(H3) In the CC model, for every process \( p \in \mathcal{P} \), \( \sigma_{\text{highB}}[S] \) is exactly the same as \( \sigma_{\text{highB}}[S_B] \) at the end of \( E(\sigma_{\text{highB}}[S] \circ \sigma'_S) \) is exactly the same as \( \sigma_{\text{highB}}[S_B] \).

(H4) For every integer \( j \in \{0, 1, \ldots, h-1\} \), \( \text{val}_{R[j]}(C'_S, \sigma'_S) = \text{val}_{R[j]}(C'_{SB}, \sigma'_{SB}) \).

(H5) For each process \( p \in \mathcal{P} \), \( \text{state}_p(C'_S, \sigma'_S) = \text{state}_p(C'_{SB}, \sigma'_{SB}) \).

(H6) Each process in \( S_B \setminus S_a \) incurs exactly one RMR during \( E(C'_S, \sigma'_S) \).

(H7) For each process \( p \in \mathcal{P} \), \( \sigma_{\text{highB}}[S] \) has not left the critical section during \( E(\sigma_{\text{highB}}[S] \circ \sigma'_S) \).

(H8) \( F(\sigma_{\text{highB}}[S]) = F(\sigma_{\text{highB}}[S_B]) \).

Proof. First, recall that for every set \( S \subseteq \mathcal{P} \) such that \( F(\sigma_{\text{highB}}[S_B]) \cup S_a \subseteq S \subseteq S_B \), \( \sigma_{\text{highB}}[S] \neq \perp \). So let \( S \subseteq \mathcal{P} \) be a set of processes such that \( F(\sigma_{\text{highB}}[S_B]) \cup S_a \subseteq S \subseteq S_B \).

By Lemma 15, the set of registers accessed during \( E(C'_S, \sigma'_S) \) is exactly the set of registers in \( R[0..h-1] \). By Lemma 14, for every integer \( j \in \{0, 1, \ldots, h-1\} \):

- \( \text{val}_{R[j]}(\sigma_{\text{highB}}[S]) = \text{val}_{R[j]}(\sigma_{\text{highB}}[S_B]) \) \( (\text{H2}) \).
- The owner of \( R[j] \) is in \( S_B \setminus F(\sigma_{\text{highB}}[S_B]) \) \( (\text{H1}) \).

Thus for every process \( p \in \mathcal{P} \), if \( p \) accesses a register \( R \) during \( E(C'_S, \sigma'_S) \), then the owner of \( R \) is not in \( S_B \setminus F(\sigma_{\text{highB}}[S_B]) \) \( (\text{H1}) \).

By Lemma 14, \( \sigma_{\text{highB}}[0..2^n-1] \) is \( (i-1) \)-compliant with \( S_{\text{max}}(\sigma_{\text{highB}}[0..2^n-1]) = S_B \). So by Invariant (I3), for every process \( p \in \mathcal{P} \), \( \sigma_{\text{highB}}[S_B] \) is the set of registers that \( p \) has valid cache copies of at the end of \( E(\sigma_{\text{highB}}[S]) \) at the same as at the end of \( E(\sigma_{\text{highB}}[S_B]) \).

Suppose \( \text{opt}_{\text{HT}} \) is read. Then by Lemma 14 and Invariant (I3), for every process \( p \in \mathcal{P} \), \( \text{state}_p(\sigma_{\text{highB}}[S]) = \text{state}_p(\sigma_{\text{highB}}[S_B]) \). So at the end of both \( E(\sigma_{\text{highB}}[S]) \) and \( E(\sigma_{\text{highB}}[S_B]) \), \( p \) is poised to perform a read operation on the same register \( R \). Then, since \( \sigma'_S \) contains exactly one non-crash step of each process in \( S \setminus S_B \), the set of registers that \( p \) has valid cache copies of at the end of \( E(\sigma_{\text{highB}}[S] \circ \sigma'_S) \) is the union of \( R \) and the set of registers that \( p \) has valid cache copies of at the end of \( E(\sigma_{\text{highB}}[S_B] \circ \sigma'_{SB}) \). Furthermore, since \( \sigma'_{SB} \) contains exactly one non-crash step of each process in \( S_B \setminus S_a \), the set of registers that \( p \) has valid cache copies of at the end of \( E(\sigma_{\text{highB}}[S] \circ \sigma'_{SB}) \) is exactly the same as at the end of \( E(\sigma_{\text{highB}}[S_B] \circ \sigma'_{SB}) \).

Now suppose \( \text{opt}_{\text{HT}} \) is not read. Then by Lemma 15, the set of registers accessed during \( E(C'_S, \sigma'_S) \) is exactly the set of registers in \( R[0..h-1] \) and exactly the set of registers accessed during \( E(C'_{SB}, \sigma'_{SB}) \). Thus, since every non-read operation invalidates all cache copies on a register, all cache copies of registers in \( R[0..h-1] \) are invalidated during both \( E(C'_S, \sigma'_S) \) and \( E(C'_{SB}, \sigma'_{SB}) \). So for every process \( p \in \mathcal{P} \), \( \sigma_{\text{highB}}[S_B] \) is the set of registers that \( p \) has valid cache copies of at the end of \( E(\sigma_{\text{highB}}[S] \circ \sigma'_S) \) is exactly the same as at the end of \( E(\sigma_{\text{highB}}[S_B] \circ \sigma'_{SB}) \).

Consequently, regardless of \( \text{opt}_{\text{HT}} \), in the CC model, for every process \( p \in \mathcal{P} \), \( \sigma_{\text{highB}}[S] \) is exactly the same as at the end of \( E(\sigma_{\text{highB}}[S_B] \circ \sigma'_{SB}) \) \( (\text{H8}) \).

27
Now consider \( E(C'_S, \sigma_a) \) and \( E(C'_S, \sigma_a) \). By construction, \( \sigma_a \) contains exactly one non-crash step of every process in \( S_t \) and no other steps. By Lemma 14, \( \sigma_{\text{highB}}[0..2^n-1] \) is \((i-1)\)-compliant with \( S_{\text{max}}(\sigma_{\text{highB}}[0..2^n-1]) = S_B \). So by Invariants (I2) and (I3), since \( F(\sigma_{\text{highB}}[S_B]) \cup S_t \subseteq S \subseteq S_B \), for every process \( p \in S \supseteq S_t \), \( \text{state}_p(\sigma_{\text{highB}}[S]) = \text{state}_p(\sigma_{\text{highB}}[S_B]) \). By Lemma 15, the set of registers accessed during \( E(C'_S, \sigma_a) \) is exactly the set of registers in \( R[0..h-1] \) and exactly the set of registers accessed during \( E(C'_S, \sigma_a) \). Since we have already proven (H2), for every integer \( j \in \{0, 1, \ldots, h-1\}, \text{val}_R[j](\sigma_{\text{highB}}[S_B]) = \text{val}_R[j](\sigma_{\text{highB}}[S_B]) \). Therefore during both \( E(C'_S, \sigma_a) \) and \( E(C'_S, \sigma_a) \), the same set \( S_t \) of processes begin in the same states, and perform the same operations in the same order on the same registers which also begin with the same values. So for every integer \( j \in \{0, 1, \ldots, h-1\}, \text{val}_R[j](C'_S, \sigma_a) = \text{val}_R[j](C'_S, \sigma_a) \). Thus by Lemma 16, for every integer \( j \in \{0, 1, \ldots, h-1\}:

\[
\text{val}_R[j](C'_S, \sigma'_S) = \text{val}_R[j](C'_S, \sigma_a)
= \text{val}_R[j](C'_S, \sigma_a)
= \text{val}_R[j](C'_S, \sigma'_S)
\]

So for every integer \( j \in \{0, 1, \ldots, h-1\}, \text{val}_R[j](C'_S, \sigma'_S) = \text{val}_R[j](C'_S, \sigma'_S) \) ((H4)).

Now let \( j' \) be an integer in \( \{0, 1, \ldots, h-1\} \). Then by Lemma 16:

\[
\text{val}_R[j'](C'_S, \sigma'_S[j'-1]) = \text{val}_R[j'](C'_S, \sigma_a[j'-1])
= \text{val}_R[j'](C'_S, \sigma_a[j'-1])
= \text{val}_R[j'](C'_S, \sigma'_S[j'-1])
\]

Let \( C'_S \) and \( C'_S \) be the configurations at the end of \( E(C'_S, \sigma'_S[j'-1]) \) and \( E(C'_S, \sigma'_S[j'-1]) \) respectively. So \( R[j'] \) has the same value in both \( C'_S \) and \( C'_S \). In addition, recall that by the definition of \( \sigma'_S[j'-1] \) and \( \sigma'_S[j'-1] \), processes in \( G[j'] \cap S \) do not take any steps during \( E(C'_S, \sigma'_S[j'-1]) \) and \( E(C'_S, \sigma'_S[j'-1]) \). Thus, for every process \( p \in G[j'] \cap S \), since \( \text{state}_p(\sigma_{\text{highB}}[S]) = \text{state}_p(\sigma_{\text{highB}}[S_B]) \), the state of \( p \) is still the same in both \( C'_S \) and \( C'_S \), i.e., \( p \) is poised to perform the same operation on \( R[j'] \) in both \( C'_S \) and \( C'_S \).

Now suppose that \( \beta_1[j'] \) is in \( S \subseteq S_B \). Then \( \sigma'_S[j', j'] = \sigma'_S[j'-1] \circ \beta_1[j'] \) and \( \sigma'_S[j', j'] = \sigma'_S[j'-1] \circ \beta_1[j'] \). Furthermore, in addition to \( \sigma'_S[j', j'] = \sigma'_S[j'-1] \circ \beta_1[j'] \) and \( \sigma'_S[j', j'] = \sigma'_S[j'-1] \circ \beta_1[j'] \), healthier \( \text{state}_p(\sigma'[S]) = \text{state}_p(\sigma'[S]) \) ((H5)).

Thus we have shown that for every integer \( j \in \{0, 1, \ldots, h-1\}, \text{state}_p(C'_S, \sigma'_S[j]) = \text{state}_p(C'_S, \sigma'_S[j]) \). Now note that for every process \( p \in S \setminus S_t \), since \( S \subseteq S_B \), there exists an integer \( j \in \{0, 1, \ldots, h-1\} \) such that \( p = \beta_1[j] \). So for each process \( p \in S \setminus S_t \), \( \text{state}_p(C'_S, \sigma'_S[j]) = \text{state}_p(C'_S, \sigma'_S[j]) \)
every process in $S_r \cap S$ incurs exactly one RMR during $E(C_S', \sigma'_S)$. In the CC model, every process in $S_r \cap S$ is poised to perform a non-read operation or read a register that it does not have a valid cache copy of in $C_S'$, and so every process in $S_r \cap S$ incurs exactly one RMR during $E(C_S', \sigma'_S)$. Thus, in both the DSM and CC models, each process in $S_r \cap S$ incurs exactly one RMR during $E(C_S', \sigma'_S)$ ((H6)).

By Lemma 14, $\sigma_{\text{high}}[0..2^a - 1]$ is $(i-1)$-compliant, so by Invariant (I7), each process that is not in $F(\sigma_{\text{high}}[S])$ does not enter the critical section during $E(\sigma_{\text{high}}[S])$. Recall that $\sigma'_S$ consists of exactly one non-crash step of each process in $S_r \cap S = S \setminus F(\sigma_{\text{high}}[S])$ and no other steps. Thus, with only one step, although a process could enter the critical section during $E(C_S', \sigma'_S)$, it cannot have taken any steps within the critical section. Thus by Assumption (A1), no process can leave the critical section during $E(C_S', \sigma'_S)$. So for each process $p \in S \setminus F(\sigma_{\text{high}}[S])$, $p$ has not left the critical section during $E(\sigma_{\text{high}}[S] \circ \sigma'_S)$ ((H7)). Therefore, since no process in $S \setminus F(\sigma_{\text{high}}[S])$ has left the critical section during $E(\sigma_{\text{high}}[S] \circ \sigma'_S)$, no process in $S \setminus F(\sigma_{\text{high}}[S])$ has completed its super-passage during $E(\sigma_{\text{high}}[S] \circ \sigma'_S)$. Thus $F(\sigma_{\text{high}}[S]) = F(\sigma_{\text{high}}[S] \circ \sigma'_S)$ ((H8)).

Now recall that $S_F = S_\alpha \cup F(\sigma_{\text{high}}[S_F])$ and that by Lemma 11 and Invariant (I4), $F(\sigma_{\text{high}}[S_F]) = F(\sigma_{\text{high}}[S_B]) = F(\sigma_{\text{high}}[S_B])$. Thus $S_F = S_\alpha \cup F(\sigma_{\text{high}}[S_B])$, and so $F(\sigma_{\text{high}}[S_B]) \cup S_\alpha \subseteq S_F \subseteq S_B$. Furthermore, recall that by definition, since $S_F = \sigma_\alpha \cup F(\sigma_{\text{high}}[S_F])$, $\sigma_\alpha = \sigma_{S_F}$. Thus $E(C_{S_F}', \sigma_{S_F}') = E(C_{S_F}', \sigma_{S_F})$.

**Lemma 18.** For every register $R \in \mathcal{R}_F$, and every set $S \subseteq \mathcal{P}$ such that $F(\sigma_{\text{high}}[S_B]) \cup S_\alpha \subseteq S \subseteq S_B$, $\text{val}_{\mathcal{R}}(C_{S_F}', \sigma_{S_F}') = \text{val}_{\mathcal{R}}(C_{S_F}', \sigma_\alpha)$.

**Proof.** Let $S \subseteq \mathcal{P}$ be a set of processes such that $F(\sigma_{\text{high}}[S]) \cup S_\alpha \subseteq S \subseteq S_B$. By Lemma 14 and Invariant (I2), since $F(\sigma_{\text{high}}[S]) \cup S_\alpha \subseteq S_F \subseteq S_B$ and $F(\sigma_{\text{high}}[S]) \cup S_\alpha \subseteq S \subseteq S_B$, both $\sigma_{\text{high}}[S_F] \neq S_F$ and $\sigma_{\text{high}}[S] \neq S_F$. So by Lemma 14, for every register $R \in \mathcal{R}_F$, $\text{val}_{\mathcal{R}}(\sigma_{\text{high}}[S_F]) = \text{val}_{\mathcal{R}}(\sigma_{\text{high}}[S_B]) = \text{val}_{\mathcal{R}}(\sigma_{\text{high}}[S])$.

Now consider $E(C_{S_F}', \sigma_{S_F}')$ and $E(C_{S_F}', \sigma_{S_F})$. By Lemma 15, only registers in $R[0..h - 1]$ are accessed during both executions. So for every register $R \in \mathcal{R}_F$, if $R$ is not one of $R[0..h - 1]$, then $\text{val}_{\mathcal{R}}(C_{S_F}', \sigma_{S_F}') = \text{val}_{\mathcal{R}}(C_{S_F}', \sigma_{S_F})$.

Next, by Lemma 17 ((H4)), for every integer $j \in \{0, 1, \ldots, h-1\}$, $\text{val}_{\mathcal{R}[j]}(C_{S_F}', \sigma_{S_F}') = \text{val}_{\mathcal{R}[j]}(C_{S_F}', \sigma_{S_F}) = \text{val}_{\mathcal{R}[j]}(C_{S_F}', \sigma_{S_F}') = \text{val}_{\mathcal{R}[j]}(C_{S_F}', \sigma_{S_F})$. Therefore, for every register $R \in \mathcal{R}_F$, regardless of whether $R$ is one of $R[0..h - 1]$, $\text{val}_{\mathcal{R}}(C_{S_F}', \sigma_{S_F}') = \text{val}_{\mathcal{R}}(C_{S_F}', \sigma_{S_F}) = \text{val}_{\mathcal{R}}(C_{S_F}', \sigma_\alpha)$.}

Next, recall that there exists a schedule $\sigma_F$ such that:

- $\sigma_F$ begins with exactly one crash step of every process in $S_\alpha$, and contains no other crash steps.
- $\sigma_F$ contains only steps of processes in $S_\alpha = S_F \setminus F(\sigma_{\text{high}}[S_F])$.
- During $E(\sigma_{\text{high}}[S_F] \circ \sigma_\alpha \circ \sigma_F)$, every process in $S_F$ begins and then completes a super-passage, i.e., $F(\sigma_{\text{high}}[S_F] \circ \sigma_\alpha \circ \sigma_F) = S_F$.

By Lemma 11 and Invariant (I4), $F(\sigma_{\text{high}}[S_F]) = F(\sigma_{\text{high}}[S_B]) = F(\sigma_{\text{high}}[S_B])$. By construction, $\sigma_{\text{high}}[S_F] = \sigma_{\text{high}}[S_F]$. Thus:

- $\sigma_F$ begins with exactly one crash step of every process in $S_\alpha$, and contains no other crash steps.
- $\sigma_F$ contains only steps of processes in $S_\alpha = S_F \setminus F(\sigma_{\text{high}}[S_F])$.
- During $E(\sigma_{\text{high}}[S_F] \circ \sigma_\alpha \circ \sigma_F)$, every process in $S_F$ begins and then completes a super-passage, i.e., $F(\sigma_{\text{high}}[S_F] \circ \sigma_\alpha \circ \sigma_F) = S_F$.

Further recall that by definition, $C_F$ is the configuration at the end of $E(\sigma_{\text{high}}[S_F] \circ \sigma_\alpha)$, and $\mathcal{R}_F$ is the set of every register that is accessed during $E(C_F, \sigma_F)$ (after the crash steps of every process in $S_\alpha$ at the beginning of $\sigma_F$).
Now for every set $S \subseteq \mathcal{P}$ such that $F(\sigma_{\text{highC}}[S]) \cup S_\alpha \subseteq S \subseteq S_B$, let $C'_S$ be the configuration at the end of $E(\sigma_{\text{highB}}[S]) \circ \sigma_\alpha$. Then $E(\sigma_{\text{highA}}[S]) \circ \sigma_\alpha = E(\sigma_{\text{highB}}[S]) \circ \sigma'_\alpha$, so $C_F = C'_S$. 

**Lemma 19.** For every set $S \subseteq \mathcal{P}$ such that $F(\sigma_{\text{highB}}[S]) \cup S_\alpha \subseteq S \subseteq S_B$, during both $E(C'_S, \sigma_F)$ and $E(C'_S, \sigma_F) = E(C_F, \sigma_F)$, the same set of processes (namely $S_\alpha$) crash, then perform the same operations in the same order on the same set of registers (namely $R_F$) and so must reach the same resulting states.

**Proof.** By Lemma 18, for every register $R \in R_F$ and every set $S \subseteq \mathcal{P}$ such that $F(\sigma_{\text{highB}}[S]) \cup S_\alpha \subseteq S \subseteq S_B$, $R$ has the same value in $C_F = C'_S$ as in $C'_S$. Furthermore, by the definition of $\sigma_F$, $\sigma_F$ begins with a crash step of every process in $P(\sigma_F) = S_\alpha$. The lemma immediately follows. ☐

We now construct a new array $\sigma_{\text{highC}}[0..2^n - 1]$ such that for every set $S \subseteq \mathcal{P}$, if $F(\sigma_{\text{highB}}[S]) \cup S_\alpha \subseteq S \subseteq S_B$, then $\sigma_{\text{highC}}[S] = \sigma_{\text{highB}}[S] \circ \sigma'_S \circ \sigma_F$; otherwise $\sigma_{\text{highC}}[S] = \bot$.

**Lemma 20.** This new array $\sigma_{\text{highC}}[0..2^n - 1]$ is $i$-compatible with $S_{\text{max}}(\sigma_{\text{highC}}[0..2^n - 1]) = S_B$. Furthermore, $F(\sigma_{\text{highC}}[S]) = F(\sigma_{\text{highB}}[S]) \cup S_\alpha$.

**Proof.** For every set $S \subseteq \mathcal{P}$, if $\sigma_{\text{highC}}[S] \neq \bot$, then by construction, $\sigma_{\text{highC}}[S] = \sigma_{\text{highB}}[S] \circ \sigma'_S \circ \sigma_F$. By Lemma 14, $\sigma_{\text{highB}}[0..2^n - 1]$ is $(i - 1)$-compatible, so by Invariant (II), $F(\sigma_{\text{highB}}[S]) \subseteq S$. By the definition of $\sigma'_S$, $\sigma'_S$ contains only steps of processes in $S \cap S_\beta$. By the definition of $\sigma_F$, $\sigma_F$ contains only steps of processes in $S_\alpha \subseteq S$. Thus $F(\sigma_{\text{highC}}[S]) \subseteq S$ (Invariant (II)).

Now for every set $S \subseteq \mathcal{P}$ such that $F(\sigma_{\text{highB}}[S]) \cup S_\alpha \subseteq S \subseteq S_B$, consider $F(\sigma_{\text{highC}}[S]) = F(\sigma_{\text{highB}}[S]) \circ \sigma'_S \circ \sigma_F$. By Lemma 17 ((H6)), $F(\sigma_{\text{highB}}[S]) = F(\sigma_{\text{highB}}[S]) \circ \sigma'_S = F(C'_S, \sigma'_S)$. By definition, every process in $S_\alpha$ completes its super-passage during $E(C_F, \sigma_F)$. So by Lemma 19, every process in $S_\alpha$ also completes its super-passage during $E(C'_S, \sigma_F)$. Thus $F(\sigma_{\text{highC}}[S]) = F(\sigma_{\text{highB}}[S]) \cup S_\alpha$. Therefore, $F(\sigma_{\text{highC}}[S]) = F(\sigma_{\text{highB}}[S]) \cup S_\alpha$.

Furthermore, by Lemma 14 and Invariants (II) and (I4), for every set $S \subseteq \mathcal{P}$ such that $F(\sigma_{\text{highB}}[S]) \cup S_\alpha \subseteq S \subseteq S_B$, $F(\sigma_{\text{highB}}[S]) = F(\sigma_{\text{highB}}[S])$, and so $F(\sigma_{\text{highC}}[S]) = F(\sigma_{\text{highC}}[S]) \cup S_\alpha = F(\sigma_{\text{highB}}[S]) \cup S_\alpha = F(\sigma_{\text{highC}}[S])$.

By construction, for every set $S \subseteq \mathcal{P}$, if $F(\sigma_{\text{highB}}[S]) \cup S_\alpha \subseteq S \subseteq S_B$, then $\sigma_{\text{highC}}[S] = \sigma_{\text{highB}}[S] \circ \sigma'_S \circ \sigma_F$; otherwise $\sigma_{\text{highC}}[S] = \bot$. By Lemma 14, $\sigma_{\text{highB}}[0..2^n - 1]$ is $(i - 1)$-compatible with $S_{\text{max}}(\sigma_{\text{highB}}[0..2^n - 1]) = S_B$. So by Invariant (II), for every set $S \subseteq \mathcal{P}$, $\sigma_{\text{highB}}[S] \neq \bot$ if and only if $F(\sigma_{\text{highB}}[S]) \subseteq S \subseteq S_B$. Thus for every set $S \subseteq \mathcal{P}$, if $F(\sigma_{\text{highB}}[S]) \cup S_\alpha \subseteq S \subseteq S_B$, then $\sigma_{\text{highC}}[S] = \bot$ and $\sigma_{\text{highC}}[S] = \sigma_{\text{highB}}[S] \circ \sigma'_S \circ \sigma_F \neq \bot$. Then, since we have already proven that $F(\sigma_{\text{highC}}[S]) = F(\sigma_{\text{highB}}[S]) \cup S_\alpha$, for every set $S \subseteq \mathcal{P}$, $\sigma_{\text{highC}}[S] = \bot$ if and only if $F(\sigma_{\text{highB}}[S]) \subseteq S \subseteq S_B$ (Invariant (II)).

Furthermore, we have already shown that for every set $S \subseteq \mathcal{P}$ such that $F(\sigma_{\text{highB}}[S]) \cup S_\alpha \subseteq S \subseteq S_B$, $F(\sigma_{\text{highC}}[S]) = F(\sigma_{\text{highB}}[S])$. Thus, since we just proved that Invariant (II) holds for $\sigma_{\text{highC}}[0..2^n - 1]$ with $S_{\text{max}}(\sigma_{\text{highB}}[0..2^n - 1]) = S_B$ and $F(\sigma_{\text{highB}}[S]) = F(\sigma_{\text{highB}}[S]) \cup S_\alpha$, it follows that for every set $S \subseteq \mathcal{P}$ such that $\sigma_{\text{highC}}[S] = \bot$, $F(\sigma_{\text{highC}}[S]) = F(\sigma_{\text{highB}}[S])$ (Invariant (I4)).

By Lemma 17 ((H5)), for every set $S \subseteq \mathcal{P}$ such that $F(\sigma_{\text{highB}}[S]) \cup S_\alpha = F(\sigma_{\text{highB}}[S]) \subseteq S \subseteq S_B$, for every process $p \in S \setminus S_\alpha$, state(p$(C'_S, \sigma'_S)$) = state(p$(C'_S, \sigma'_S)$). Since $\sigma_F$ only contains steps of processes in $S_\alpha$, for every set $S \subseteq \mathcal{P}$ such that $F(\sigma_{\text{highB}}[S]) \cup S_\alpha = F(\sigma_{\text{highB}}[S]) \subseteq S \subseteq S_B$, for every process $p \in S \setminus S_\alpha$, state(p$(C'_S, \sigma'_S)$) = state(p$(C'_S, \sigma'_S)$). Thus for every set $S \subseteq \mathcal{P}$ such that $F(\sigma_{\text{highB}}[S]) \subseteq S \subseteq S_B$, for every process $p \in S \setminus S_\alpha$, state(p$(\sigma_{\text{highC}}[S])) = state(p$(\sigma_{\text{highB}}[S]))$. Furthermore, we have already proven that $F(\sigma_{\text{highC}}[S]) = F(\sigma_{\text{highB}}[S]) \cup S_\alpha$, and that Invariants (II) and (I4) hold for $\sigma_{\text{highC}}[0..2^n - 1]$ with $S_{\text{max}}(\sigma_{\text{highC}}[0..2^n - 1]) = S_B$, for every set $S \subseteq$
Tight Lower Bounds for the RMR Complexity of Recoverable Mutual Exclusion

\( \mathcal{P} \) such that \( F(σ_{highC}[S_B]) ⊆ S ⊆ S_B \), \( S_a \) is in both \( F(σ_{highC}[S]) \) and \( F(σ_{highC}[S_B]) \), so for every process \( p ∈ S_a \), \( state_p(σ_{highC}[S_B]) = state_p(σ_{highC}[S]) \). Consequently, for every set \( S ⊆ \mathcal{P} \) such that \( σ_{highC}[S] ≠ ⊥ \), for every process \( p ∈ S \), regardless of whether \( p \) is in \( S_a \), \( state_p(σ_{highC}[S_B]) = state_p(σ_{highC}[S]) \) (Invariant (I3)).

Next, by Lemma 19, for every set \( S ⊆ \mathcal{P} \) such that \( F(σ_{highC}[S_B]) ⊆ S ⊆ S_B \), and every register \( R ∈ \mathcal{R}_F \), \( val_R(σ_{highC}[S]) = val_R(σ_{highC}[S_F]) \). Furthermore, only registers in \( \mathcal{R}_F \) are accessed during \( E(C'_S, σ_F) \). So for every integer \( j ∈ \{0, 1, \ldots, h-1\} \), if \( R[j] \notin \mathcal{R}_F \), then by Lemma 17 ((H14)), \( val_{R[j]}(C'_S, σ_F) = val_{R[j]}(C'_S, σ'_F) \), i.e., \( val_{R[j]}(σ_{highC}[S]) = val_{R[j]}(σ_{highC}[S_B]) \). Thus we have shown that for every register \( R ∈ \mathcal{R} \) such that either \( R ∈ \mathcal{R}_F \) or \( R \) is one of \( R[0..h-1] \), and every set \( S ⊆ \mathcal{P} \) such that \( F(σ_{highC}[S]) ⊆ S ⊆ S_B \), \( val_R(σ_{highC}[S]) = val_R(σ_{highC}[S_B]) \). Then since we have already proven that Invariant (I2) holds for \( σ_{highC}[0..2^n - 1] \) with \( S_{max}(σ_{highC}[0..2^n - 1]) = S_B \), for every register \( R ∈ \mathcal{R} \) such that either \( R ∈ \mathcal{R}_F \) or \( R \) is one of \( R[0..h-1] \), and every set \( S ⊆ \mathcal{P} \) such that \( σ_{highC}[S] ≠ ⊥ \), \( val_R(σ_{highC}[S]) = val_R(σ_{highC}[S_B]) \). So for every register \( R ∈ \mathcal{R} \) such that either \( R ∈ \mathcal{R}_F \) or \( R \) is one of \( R[0..h-1] \), regardless of \( last_R(σ_{highC}[S]) \), if \( y_R = val_R(σ_{highC}[S]) \), then we have that for every set \( S ⊆ \mathcal{P} \), if \( σ_{highC}[S] ≠ ⊥ \), then:

\[
val_R(σ_{highC}[S]) = \begin{cases} val_R(σ_{highC}[S_B]) & \text{if } last_R(σ_{highC}[S_B]) ∈ S \\ y_R & \text{otherwise} \end{cases}
\]

Now consider the registers that are not in \( \mathcal{R}_F \) and not one of \( R[0..h-1] \), i.e., the registers that are not accessed during \( E(C'_S, σ'_F \circ σ_F) \) for every set \( S \subset \mathcal{P} \) such that \( F(σ_{highC}[S_B]) \subset S \subset S_B \). By Lemma 14, \( σ_{highB}[0..2^n - 1] \) is \((i-1)\)-compliant with \( S_{max}(σ_{highC}[0..2^n - 1]) = S_B \). So by Invariant (I5), for each such register \( R \), there exists a value \( y_R \) such that for every set \( S \subset \mathcal{P} \), if \( σ_{highB}[S] ≠ ⊥ \), then:

\[
val_R(σ_{highB}[S]) = \begin{cases} val_R(σ_{highB}[S_B]) & \text{if } last_R(σ_{highB}[S_B]) ∈ S \\ y_R & \text{otherwise} \end{cases}
\]

Then, since each such register \( R \) is not accessed during \( E(C'_S, σ'_F \circ σ_F) \) for every set \( S \subset \mathcal{P} \) such that \( F(σ_{highC}[S_B]) \subset S \subset S_B \), and we have already proven that Invariant (I2) holds for \( σ_{highC}[0..2^n - 1] \) with \( S_{max}(σ_{highC}[0..2^n - 1]) = S_B \):

\[
val_R(σ_{highC}[S]) = \begin{cases} val_R(σ_{highC}[S_B]) & \text{if } last_R(σ_{highC}[S_B]) ∈ S \\ y_R & \text{otherwise} \end{cases}
\]

Consequently, for every register \( R ∈ \mathcal{R} \), regardless of whether \( R ∈ \mathcal{R}_F \) and whether \( R \) is one of \( R[0..h-1] \), there is a value \( y_R \) such that for every set \( S \subset \mathcal{P} \), if \( σ_{highC}[S] ≠ ⊥ \), then:

\[
val_R(σ_{highC}[S]) = \begin{cases} val_R(σ_{highC}[S_B]) & \text{if } last_R(σ_{highC}[S_B]) ∈ S \\ y_R & \text{otherwise} \end{cases}
\]

So Invariant (I5) holds for \( σ_{highC}[0..2^n - 1] \).

By Lemma 14, \( σ_{highB}[0..2^n - 1] \) is \((i-1)\)-compliant. So by Invariant (I6), for every set \( S \subset \mathcal{P} \) with \( σ_{highB}[S] ≠ ⊥ \), during \( E(σ_{highB}[S]) \), each process crashes at most once, and each process that is not in \( F(σ_{highB}[S]) \) never crashes.

By construction, for every set \( S \subset \mathcal{P} \) with \( σ_{highC}[S] ≠ ⊥ \), \( σ'_C \) does not contain any crash steps. Furthermore, \( σ_F \) contains exactly one crash step for each process in \( S_a \) and no other crash steps.

We have already proven that \( F(σ_{highC}[S_B]) = F(σ_{highB}[S_B]) \cup S_a \) and that Invariant (I4) holds for \( σ_{highC}[0..2^n - 1] \). By Lemma 14, Invariant (I4) also holds for \( σ_{highC}[0..2^n - 1] \). So for every set \( S \subset \mathcal{P} \) with \( σ_{highC}[S] ≠ ⊥ \), \( F(σ_{highC}[S]) = F(σ_{highB}[S]) \cup S_a \).
So for every process $p \not\in F(\sigma_{\text{highC}}[S_B])$, $p$ never crashes during $E(\sigma_{\text{highC}}[S])$. Furthermore, since $S_a \not\in F(\sigma_{\text{highB}}[S])$, processes in $S_a$ never crash during $E(\sigma_{\text{highB}}[S])$, and thus crash at most once during $E(\sigma_{\text{highB}}[S]) = E(\sigma_{\text{highB}}[S] \circ \sigma'_S \circ \sigma_F)$. Finally, processes in $F(\sigma_{\text{highB}}[S])$ do not have any (crash) steps in either $\sigma'_S$ or $\sigma_F$, so they also still crash at most once during $E(\sigma_{\text{highB}}[S]) = E(\sigma_{\text{highB}}[S] \circ \sigma'_S \circ \sigma_F)$. Consequently, for every set $S \subseteq \mathcal{P}$ with $\sigma_{\text{highB}}[S] \neq \bot$, during $E(\sigma_{\text{highB}}[S])$, each process crashes at most once, and each process that is not in $F(\sigma_{\text{highB}}[S])$ never crashes (Invariant (I6)).

Now suppose, for contradiction, that for some set $S \subseteq \mathcal{P}$ with $\sigma_{\text{highB}}[S] \neq \bot$, some process $p$ that is not in $F(\sigma_{\text{highB}}[S])$ enters the critical section during $E(\sigma_{\text{highB}}[S]) = E(\sigma_{\text{highB}}[S] \circ \sigma'_S \circ \sigma_F)$. Since we have already proven that Invariant (I1) holds for $\sigma_{\text{highB}}[0..2^n - 1]$, $p \in S$. Furthermore, we have also already shown that $F(\sigma_{\text{highB}}[S]) = F(\sigma_{\text{highB}}[S]) \cup S_a$, so $p \not\in F(\sigma_{\text{highB}}[S]) \cup S_a$.

By Lemma 14, $\sigma_{\text{highB}}[0..2^n - 1]$ is $(i-1)$-compliant. So by Invariant (I7), since $p \not\in F(\sigma_{\text{highB}}[S])$, $p$ does not enter the critical section during $E(\sigma_{\text{highB}}[S])$. Furthermore, by the definition of $\sigma_F$, since $p \not\in S_a$, $\sigma_F$ contains no steps of $p$. Thus $p$ must be one of the processes that take a step during $E(C'_S, \sigma'_S)$, and must enter the critical section with this one step. By Assumption (A1), a process that enters the critical section cannot leave the critical section before incurring an RMR within the critical section. Thus, since $\sigma_F$ contains no steps of $p \not\in S_a$, $p$ remains in the critical section throughout $E(C'_S, \sigma_F)$.

Now consider the processes in $S_a$. Since $\sigma_{\text{highB}}[0..2^n - 1]$ is $(i-1)$-compliant, and $S_a \cap F(\sigma_{\text{highB}}[S]) = \emptyset$, the processes in $S_a$ do not enter the critical section during $E(\sigma_{\text{highB}}[S])$. Since $p$ is in the critical section in $C'_S$, and $\sigma'_S$ contains at most one step of each process, to avoid violating mutual exclusion, each process in $S_a$ cannot enter the critical section with its at most one step taken during $E(C'_S, \sigma'_S)$ by Assumption (A1). Thus during $E(\sigma_{\text{highB}}[S] \circ \sigma'_S)$, the processes in $S_a$ do not enter the critical section. Furthermore, since $p$ remains in the critical section throughout $E(C'_S, \sigma_F)$, to avoid violating mutual exclusion, the processes in $S_a$ also do not enter the critical section during $E(\sigma_{\text{highB}}[S] \circ \sigma'_S \circ \sigma_F)$.

However, we have already shown that $F(\sigma_{\text{highB}}[S]) = F(\sigma_{\text{highB}}[S]) \cup S_a$. Thus we have that during $E(\sigma_{\text{highB}}[S]) = E(\sigma_{\text{highB}}[S] \circ \sigma'_S \circ \sigma_F)$, every process in $S_a$ completes its super-passage without entering the critical section — a contradiction. Consequently, for every set $S \subseteq \mathcal{P}$ with $\sigma_{\text{highB}}[S] \neq \bot$, each process that is not in $F(\sigma_{\text{highB}}[S])$ does not enter the critical section during $E(\sigma_{\text{highB}}[S])$ (Invariant (I7)).

Next, by Lemma 14, for every register $R \in \mathcal{R}$ such that either $R$ is one of $R[0..h - 1]$ or $R \in \mathcal{R}_F$, the owner of $R$ is not in $S_B \setminus F(\sigma_{\text{highB}}[S_B])$. By Lemma 15, the definition of $\mathcal{R}_F$, and Lemma 19, for every set $S \subseteq \mathcal{P}$ such that $\sigma_{\text{highC}}[S] \neq \bot$, each register $R \in \mathcal{R}$ is only accessed during $E(\sigma_{\text{highB}}[S]) = E(\sigma_{\text{highB}}[S] \circ \sigma'_S \circ \sigma_F)$ if either $R$ is one of $R[0..h - 1]$ or $R \in \mathcal{R}_F$. Therefore, for every register $R \in \mathcal{R}$, if $R$ is accessed during $E(C'_S, \sigma'_S \circ \sigma_F)$, then the owner of $R$ is not in $S_B \setminus F(\sigma_{\text{highB}}[S_B])$. Consequently, for every set $S \subseteq \mathcal{P}$ such that $\sigma_{\text{highC}}[S] \neq \bot$, during $E(C'_S, \sigma'_S \circ \sigma_F)$, each register $R \in \mathcal{R}$ cannot be accessed if the owner of $R$ is in $S_B \setminus F(\sigma_{\text{highB}}[S_B])$.

By Lemma 14, Invariant (I8) holds for $\sigma_{\text{highB}}[0..2^n - 1]$ and $S_{\text{max}}(\sigma_{\text{highB}}[0..2^n - 1]) = S_B$. So in the DSM model, for every set $S \subseteq \mathcal{P}$ such that $\sigma_{\text{highB}}[S] \neq \bot$, during $E(\sigma_{\text{highB}}[S])$, each register $R \in \mathcal{R}$ can only be accessed by its owner if the owner of $R$ is in $S_B \setminus F(\sigma_{\text{highB}}[S_B])$. Thus for every set $S \subseteq \mathcal{P}$ such that $\sigma_{\text{highC}}[S] \neq \bot$, during $E(\sigma_{\text{highC}}[S]) = E(\sigma_{\text{highB}}[S] \circ \sigma'_S \circ \sigma_F)$, each register $R \in \mathcal{R}$ can only be accessed by its owner if the owner of $R$ is in $S_B \setminus F(\sigma_{\text{highB}}[S_B]) \cup S_a$. Consequently, in the DSM model, for every set $S \subseteq \mathcal{P}$ such that $\sigma_{\text{highC}}[S] \neq \bot$, during $E(\sigma_{\text{highC}}[S]) = E(\sigma_{\text{highB}}[S] \circ \sigma'_S \circ \sigma_F)$, each register $R \in \mathcal{R}$ can only be accessed by its owner if the owner of $R$ is in $S_B \setminus F(\sigma_{\text{highB}}[S_B])$ (Invariant (I8)).

By Lemma 17 (H3), in the CC model, for every set $S \subseteq \mathcal{P}$ such that $F(\sigma_{\text{highB}}[S_B]) \cup S_a \subseteq S \subseteq S_B$, and every process $p \in S \setminus F(\sigma_{\text{highB}}[S_B])$, the set of registers that $p$ has valid cache copies of at the end of $E(\sigma_{\text{highB}}[S] \circ \sigma'_S)$ is exactly the
same as at the end of \( E(\sigma_{highB}[S]) \circ \sigma_{highB} \). By Lemma 19, for every register \( R \in \mathcal{R} \), if a non-read operation is performed on \( R \) during \( E(C'_{\sigma}, \sigma_T) \), it is also performed on \( R \) during \( E(C''_{\sigma}, \sigma_T) \). By definition, \( \sigma_T \) contains only steps of processes in \( S_{\sigma} \). So for every set \( S \subseteq \mathcal{P} \) such that \( F(\sigma_{highB}[S]) \cup S_{\sigma} \subseteq S \subseteq S_{B} \), and every process \( p \in S \setminus (F(\sigma_{highB}[S]) \cup S_{\sigma}) \), the set of registers that \( p \) has valid cache copies of at the end of \( E(\sigma_{highC}[S]) = E(\sigma_{highB}[S] \circ \sigma_{highC} \circ \sigma_T) \) is exactly the same as at the end of \( E(\sigma_{highB}[S]) = E(\sigma_{highB}[S] \circ \sigma_{highC} \circ \sigma_T) \).

Now recall that we have already proven that \( F(\sigma_{highC}[S]) = F(\sigma_{highB}[S]) \cup S_{\sigma} \), and that Invariant (2) holds for \( \sigma_{highC}[0..2^n - 1] \) with \( S_{\max}(\sigma_{highC}[0..2^n - 1]) = S_{B} \). Thus for every set \( S \subseteq \mathcal{P} \) such that \( \sigma_{highC}[S] \neq \bot \), and every process \( p \in S \setminus (S_{B} \setminus F(\sigma_{highC}[S])) \), the set of registers that \( p \) has valid cache copies of at the end of \( E(\sigma_{highC}[S]) \) is exactly the same as at the end of \( E(\sigma_{highB}[S]) \) (Invariant (9)).

Finally, by Lemma 14, \( \sigma_{highB}[0..2^n - 1] \) is \((i-1)\)-compliant, so by Invariant (10), for every set \( S \subseteq \mathcal{P} \) and every process \( p \in S \setminus F(\sigma_{highB}[S]) \), if \( \sigma_{highB}[S] \neq \bot \), then \( p \) incurs at least \( i - 1 \) RMRs during \( E(\sigma_{highB}[S]) \). By Lemma 17 ((H6)), for every set \( S \subseteq \mathcal{P} \) such that \( F(\sigma_{highB}[S]) \cup S_{\sigma} \subseteq S \subseteq S_{B} \), each process in \( S_{B} \setminus S \) incurs exactly one RMR during \( E(C'_{\sigma}, \sigma_{T}) \). By construction, for every set \( S \subseteq \mathcal{P} \), if \( \sigma_{highC}[S] \neq \bot \), then \( \sigma_{highC}[S] = \sigma_{highB}[S] \circ \sigma_{highC} \circ \sigma_T \), i.e., every process in \( S_{B} \setminus S = S \setminus F(\sigma_{highB}[S]) \) incurs at least one more RMR during \( E(\sigma_{highC}[S]) \) than during \( E(\sigma_{highB}[S]) \). Thus for every set \( S \subseteq \mathcal{P} \) and every process \( p \in S \setminus F(\sigma_{highB}[S]) \) if \( \sigma_{highC}[S] \neq \bot \), then \( p \) incurs at least \( i \) RMRs during \( E(\sigma_{highC}[S]) \).\( \square \)

Now recall that we have already proven that \( F(\sigma_{highB}[S]) = F(\sigma_{highB}[S]) \cup S_{\sigma} \), and that Invariant (4) holds for \( \sigma_{highC}[0..2^n - 1] \). So for every set \( S \subseteq \mathcal{P} \), if \( \sigma_{highB}[S] \neq \bot \), then \( F(\sigma_{highC}[S]) = F(\sigma_{highB}[S]) \cup S_{\sigma} \). Thus \( (S \setminus F(\sigma_{highB}[S])) \subseteq (S \setminus F(\sigma_{highC}[S])) \). Therefore for every set \( S \subseteq \mathcal{P} \) and every process \( p \in S \setminus F(\sigma_{highB}[S]) \) if \( \sigma_{highC}[S] \neq \bot \), then \( p \) incurs at least \( i \) RMRs during \( E(\sigma_{highC}[S]) \) (Invariant (10)).

Finally, we terminate this \( i \)-th iteration by setting \( \sigma_{round}[i, 0..2^n - 1] = \sigma_{highC}[0..2^n - 1] \).

**Analysis**: For every non-negative integer \( i \), if \( \sigma_{round}[i, 0..2^n - 1] \) is \( i \)-compliant, then let \( S'_{max} = S_{max}(\sigma_{round}[i, 0..2^n - 1]) \), and let \( n_i = |S'_{max} \setminus F(\sigma_{round}(i, S'_{max}))| \).

**Lemma 21.** For every non-negative integer \( i \), if \( \sigma_{round}[i, 0..2^n - 1] \) has non-\( \perp \) entries, then \( \sigma_{round}[i, 0..2^n - 1] \) is \( i \)-compliant.

**Proof.** If \( i = 0 \), then every entry of \( \sigma_{round}[0, 0..2^n - 1] \) is the empty schedule. Clearly, the array \( \sigma_{round}[0, 0..2^n - 1] \) is \( 0 \)-compliant.

So suppose \( i > 0 \). Thus if \( \sigma_{round}[i, 0..2^n - 1] \) has non-\( \perp \) entries, then either \( \sigma_{round}[i, 0..2^n - 1] = \sigma_{lowB}[0..2^n - 1] \) or \( \sigma_{round}[i, 0..2^n - 1] = \sigma_{highC}[0..2^n - 1] \). The lemma immediately follows from Lemma 10 and Lemma 20. \( \square \)

**Lemma 22.** For every positive integer \( i \), if \( \sigma_{round}[i, 0..2^n - 1] \) is \( i \)-compliant, then \( n_i \geq n_{i-1}/(640 \log^{d+1} n) - 2 \)

**Proof.** Since \( \sigma_{round}[i, 0..2^n - 1] \) is \( i \)-compliant, either \( \sigma_{round}[i, 0..2^n - 1] = \sigma_{lowB}[0..2^n - 1] \) or \( \sigma_{round}[i, 0..2^n - 1] = \sigma_{highC}[0..2^n - 1] \).

**Case 1.** \( \sigma_{round}[i, 0..2^n - 1] = \sigma_{lowB}[0..2^n - 1] \).

Then by Lemma 10, \( S_{max}(\sigma_{lowB}[0..2^n - 1]) = S_{lowB}[0..2^n - 1] \). Thus \( n_i = |S_{lowB}[0..2^n - 1] \setminus F(\sigma_{lowB}(S_{lowB}[0..2^n - 1]))| \).

Now recall that in the construction of \( \sigma_{lowB}[0..2^n - 1] \), we checked whether there exists a process \( p \in S_{1} \setminus F(\sigma_{lowB}(S_{1})) \) such that \( p \) is within the critical section at the end of \( E(\sigma_{lowB}[S_{1}]) \). If such a process \( p \) exists, then we set \( \sigma_{lowB}[0..2^n - 1] \) to be a simple modification of \( \sigma_{lowB}[0..2^n - 1] \) where every set \( S \subseteq \mathcal{P} \) that contains \( p \) has had \( \sigma_{lowB}[S] \) set to \( \perp \); otherwise we set \( \sigma_{lowB}[0..2^n - 1] = \sigma_{lowB}[0..2^n - 1] \). By Lemma 9, \( S_{max}(\sigma_{lowB}[0..2^n - 1]) = S_{max}(\sigma_{lowB}[0..2^n - 1]) \)
Lemma 4. Invariant (I2) holds for $|S_{\text{max}} \setminus F(\sigma_{\text{old}}[S_{\text{max}}])| \geq |S_I \setminus F(\sigma_{\text{old}}[S_I])| - 1$. Thus $n_I \geq |S_I \setminus F(\sigma_{\text{old}}[S_I])| - 1$.

Then recall that by the construction of $\sigma_{\text{old}}[0.2^n - 1]$, for every set $S \subseteq P$, $\sigma_{\text{old}}[S] = \bot$ if and only if $\sigma_{\text{old}}[S] = \bot$. By Lemma 9 and Lemma 7, Invariant (I2) holds for both $\sigma_{\text{old}}[0.2^n - 1]$ and $\sigma_{\text{old}}[0.2^n - 1]$.

Now recall that in the construction of $\sigma_{\text{old}}[S_{\text{max}}]$.

Also recall that by the construction of $\sigma_{\text{old}}[S_{\text{max}}]$.

By construction, $I \cap F(\sigma_{\text{setup}}[S_{\text{setup}}]) = \emptyset$. Thus $n_I \geq |S_I \setminus F(\sigma_{\text{old}}[S_I])| - 1$.

By Lemma 6, $|I| \geq |L|/(7k \log n)$. By construction, $|L| \geq 0.5|S_{\text{setup}}[S_{\text{setup}}]|$. Thus $n_I \geq |L|/(7k \log n) - 1 \geq |S_{\text{setup}}[S_{\text{setup}}]|/(409.6k) - 1$.

Case 2. $\sigma_{\text{setup}}[0.2^n - 1] = \sigma_{\text{high}}[0.2^n - 1]$. Then by Lemma 20, $|S_{\text{high}}[0.2^n - 1]| = S_{\text{high}}[0.2^n - 1]$. Thus $n_I \geq |S_{\text{high}}[0.2^n - 1]| - 1$.

Recall that by definition, $S_{\text{high}} = S_{\beta} \cup F(\sigma_{\text{high}}[S_{\text{high}}])$. Furthermore, by Lemma 20, $F(\sigma_{\text{high}}[S_{\text{high}}]) = F(\sigma_{\text{high}}[S_{\text{high}}]) \cup S_{\alpha}$.

Also recall that by the construction of $\sigma_{\text{high}}[0.2^n - 1]$, for every set $S \subseteq P$, $\sigma_{\text{high}}[S] = \bot$; otherwise $\sigma_{\text{high}}[S] = \sigma_{\text{high}}[S]$. Thus by Lemma 11 and Invariant (I4), $F(\sigma_{\text{high}}[S_{\text{high}}]) = F(\sigma_{\text{high}}[S_{\text{high}}])$. Finally, recall that by construction, $S_{\alpha} \cap F(\sigma_{\text{high}}[S_{\text{high}}]) = \emptyset$ and $S_{\alpha} \cap F(\sigma_{\text{high}}[S_{\text{high}}]) = \emptyset$. Therefore:

\[
S_{\text{high}} \setminus F(\sigma_{\text{high}}[S_{\text{high}}]) = S_{\text{high}} \setminus (F(\sigma_{\text{high}}[S_{\text{high}}]) \cup S_{\alpha}) = S_{\text{high}} \setminus (F(\sigma_{\text{high}}[S_{\text{high}}]) \cup S_{\alpha}) = S_{\text{high}} \setminus S_{\alpha}
\]

By Lemma 13, $|S_{\text{high}} \setminus S_{\alpha} > |L|/204.6k$. Thus $n_I \geq |S_{\text{high}} \setminus F(\sigma_{\text{high}}[S_{\text{high}}])| > |L|/204.6k$.

By construction, $|L| \geq 0.5|S_{\text{setup}}[S_{\text{setup}}]|$. Thus $n_I \geq |S_{\text{setup}}[S_{\text{setup}}]|/(409.6k) - 1$.

So in both cases, $n_I \geq |S_{\text{setup}}[S_{\text{setup}}]|/(409.6k \log n) - 1$.

Now recall that in the construction of $\sigma_{\text{setup}}[0.2^n - 1]$, we checked whether there exists a process $p \in S_{\text{old}} \setminus F(\sigma_{\text{setup}}[S_{\text{old}}])$ such that $p$ is within the critical section at the end of $E(\sigma_{\text{setup}}[S_{\text{old}}])$. If such a process $p$ exists, then we set $\sigma_{\text{setup}}[0.2^n - 1]$ to be a simple modification of $\sigma_{\text{setup}}[0.2^n - 1]$ where every set $S \subseteq P$ that contains $p$ has had $\sigma_{\text{setup}}[S]$ set to $\bot$; otherwise we set $\sigma_{\text{setup}}[0.2^n - 1] = \sigma_{\text{old}}[S_{\text{setup}}]$ by the construction of $\sigma_{\text{setup}}[0.2^n - 1]$. By Lemma 4, $S_{\text{max}}(\sigma_{\text{setup}}[0.2^n - 1]) = S_{\text{max}}(\sigma_{\text{old}}[S_{\text{max}}])$. Thus $n_I \geq |S_{\text{max}} \setminus F(\sigma_{\text{setup}}[S_{\text{setup}}])|/(409.6k \log n) - 1$.

Recall that by the construction of $\sigma_{\text{setup}}[0.2^n - 1]$, for every set $S \subseteq P$, $\sigma_{\text{setup}}[S] = \bot$ if and only if $\sigma_{\text{old}}[S] = \bot$. By Lemma 4, Invariant (I2) holds for $\sigma_{\text{setup}}[0.2^n - 1]$ and $S_{\text{max}}(\sigma_{\text{setup}}[0.2^n - 1]) = S_{\text{max}}(\sigma_{\text{old}}[S_{\text{max}}])$. Thus $n_I \geq |S_{\text{max}} \setminus F(\sigma_{\text{setup}}[S_{\text{max}}])|/(409.6k \log n) - 2 \geq |S_{\text{max}} \setminus F(\sigma_{\text{old}}[S_{\text{max}}])|/(409.6k \log n) - 2$.
Finally, recall that $\sigma_{old}[0.2^n - 1]$ is simply $\sigma_{round}[i - 1, 0..2^n - 1]$. So $|S_{max}^{old} \setminus F(\sigma_{old}[S_{max}^{old}])| = n_{i-1}$. Consequently, $n_i \geq n_{i-1}/(409.6k \log n) - 2$. Then, since $k = \log^d n$, $n_i \geq n_{i-1}/(409.6 \log^{d+1} n) - 2$. □

Since $S_{max}(\sigma_{round}[0, 0..2^n - 1]) = \mathcal{P}$ and $\sigma_{round}[0, \mathcal{P}]$ is the empty schedule, $F(\sigma_{round}[0, \mathcal{P}]) = \emptyset$ and $n_0 = n$. By Lemma 22, for every positive integer $i$, if $\sigma_{round}[i, 0..2^n - 1]$ is $I$-compliant, then $n_i \geq n_{i-1}/O(\log^{d+1} n)$.

Consequently, if $I$ is the largest positive integer such that $\sigma_{round}[I, 0..2^n - 1]$ is $I$-compliant, then $I$ is $\Omega(\log n / \log \log n)$.

So $\sigma_{round}[I, S_{max}^I]$ contains a schedule such that:

- Since we reach the $I$-th iteration, $\sigma_{round}[I - 1, 0..2^n - 1]$ has at least $2^{(k^i)}$ non-empty entries, i.e., $n_{I-1} \geq k^3 = \log^3 n$. So $n_I \geq \log^3 n/(640 \log^{d+1} n) - 2$, which for a sufficiently large constant $d$, $n_I \geq \log^d n$. Thus $|S_{max}^I \setminus F(\sigma_{round}[I, S_{max}^I])| \geq \log^d n$.
- For every process $p \in S_{max}^I \setminus F(\sigma_{round}[I, S_{max}^I]), p$ incurs at least $I$ RMRs during $E(\sigma_{round}[I, S_{max}^I])$ (Invariant (I10)).
- For every process $p \in S_{max}^I \setminus F(\sigma_{round}[I, S_{max}^I]), p$ never crashes during $E(\sigma_{round}[I, S_{max}^I])$ (Invariant (I6)).
- For every process $p \in S_{max}^I \setminus F(\sigma_{round}[I, S_{max}^I]), p$ never enters the critical section during $E(\sigma_{round}[I, S_{max}^I])$ (Invariant (I7)).

Thus we have proven Theorem 1.

4 CONCLUSION

We proved a tight RMR lower bound for RME, which applies to almost all standard shared memory primitives that have been used to solve the problem. The lower bound separates the RMR complexity of mutual exclusion in the traditional, non-recoverable model from the recoverable model, for systems that provide fetch-and-store and fetch-and-increment objects in addition to registers and compare-and-swap objects. It applies to objects of arbitrary (even unbounded) size.

RME can be solved in constant RMRs with fetch-and-add primitives of size $n^{Ω(1)}$ bits [5], so obviously our lower bound cannot be extended to cover such primitives. But it remains an open problem, whether fetch-and-add operations can help, under the standard assumption that objects can store only $O(\log n)$-bits. We believe that this is not the case. In fact, we conjecture that in general objects that can only store $O(\log n)$ bits of information are not sufficient to break through the $\Omega(\log n / \log \log n)$ RMR complexity barrier.

ACKNOWLEDGMENTS

Support is gratefully acknowledged from the Natural Science and Engineering Research Council of Canada (NSERC) under Discovery Grant RGPIN/2019-04852, and the Canada Research Chairs program.

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