Matrix Models of AdS Gravity

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We explore the connection between anti-deSitter supergravity and gauge theory, in the context of bound states of many D1 and D5 branes. The near-horizon $AdS_3 \times S^3$ supergravity describes the identity sector of the conformal field theory produced by the brane dynamics. A variant of anomaly inflow (for the 2d conformal anomaly) is involved. Dynamical matter fields on $AdS_3$ couple to the chiral ring and its descendant fields on the branes. We propose a map between boundary conformal field theory and bulk supergravity/matter dynamics, which is strongly reminiscent of matrix models of 2d gravity.
1. Introduction

The duality between open and closed string processes is an old story [1]; the string diagrammatic expansion involving Riemann surfaces with boundary contains boundaries of the moduli space involving factorization on open as well as closed string states. Thus there has been persistent speculation (see [2] for but one example) that closed string dynamics (gravity) is somehow induced from open string dynamics (gauge theory). The most recent examples are matrix theory [3], where the leading long-range gravitational interaction arises as an effect of open string quantum fluctuations, and Maldacena’s proposal [4] that anti-deSitter supergravity is in some sense ‘dual’ to supersymmetric gauge theory. In the latter case, the near-horizon geometry of M2, M5, D3, and D1+D5 branes (in the large $N$ limit at large, fixed $g_{str}, N$) is $AdS_p \times S^d$ for appropriate values of $p$ and $d$. This limit holds fixed the radii of the anti-deSitter space and the sphere while the Planck length $\ell_{pl} \to 0$, so that gravity is classical. The corresponding brane theories are conformally invariant in the limit; their superconformal symmetries match the isometries of $AdS_p \times S^d$. Thus far, Maldacena’s conjecture has been used to make a series of predictions about gauge theory in the limit $N \to \infty$, $g_{str}, N$ large, which are difficult to test via a corresponding calculation in the gauge theory.

The issue at hand is whether gravity encodes gauge theory correlation functions, or is merely coupled to gauge dynamics. We are by now accustomed to the idea that short-distance dynamics in one is related to long-distance dynamics in the other (c.f. [5]). In the D3 brane case, it is expected [4,6,7] that operators coupling to non-BPS excitations of the gauge theory have scaling dimensions that are of order $\ell/\ell_{str}$, where $\ell$ is the radius of the anti-deSitter space; then to discern them would involve the stringy resonances in closed string theory. This is what one would expect on the basis of worldsheet (channel) duality – open string dynamics is only obtained upon summing over an infinite tower of resonances in the closed string sector. Truncation of the dynamics to the lowest states takes an average over microstates of the gauge theory. Thus, the information encoded by classical supergravity should be a subset of the information in the gauge dynamics. The massless closed strings couple to the energy-momentum tensor, etc., of open string theory. One therefore expects that the classical spacetime action of the massless level of closed string theory is the generating function of the insertion of currents and other ‘geometrical’ operators in the planar limit of open string field theory.

The present work analyzes a situation where both sides of the brane/supergravity
dynamics are under some level of control, in order to study the conjectured correspondence of [4,6,7]. We will work with the D1-D5 system in IIB string theory, since on the one hand, the low-energy supergravity is topological in the infrared, and coupled to an assortment of matter fields arising from Kaluza-Klein reduction; and on the other hand, the IR dynamics on the branes is a 1+1 dimensional $\mathcal{N} = (4,4)$ sigma model whose properties have been extensively investigated. For studies of this system, see for example [8,9] (and [10] for general reviews on black holes in string theory). We will discuss some aspects of the IR conformal field theory on the branes in section 2, and of $AdS_3 \times S^3$ supergravity in section 3.

In this system, there is a precise correspondence between the identity sector of the IR conformal field theory on the branes and the low-energy limit of supergravity on $AdS_3 \times S^3$, which we develop in section 4. The core ideas are the relation of $AdS_3$ supergravity as a Chern-Simons theory [11,12] on the group $SU(1,1|2) \times SU(1,1|2)$ to the corresponding WZW theory on the boundary [13]; and the connection between $SL(2,R) \simeq SU(1,1)$ Chern-Simons/WZW theory and gravitational Ward identities in 1+1 dimensions [14,15,16].

Because the system has less than maximal supersymmetry, the supergravity theory has matter fields that are not in the same supermultiplet as the graviton. We will find in section 5 that the chiral ring of the brane theory couples to such matter fields, although a complete catalogue is not generated. As in [6,7], there appears to be a close connection between this geometrical sector of the brane theory (the chiral ring and its descendant fields) and the dimensionally reduced, low-energy supergravity/matter theory. The many results on absorption into branes [17-22] support the idea of a duality between the low-energy supergravity/matter theory and the chiral correlators of the brane theory, since emission and absorption probabilities can be calculated either as a chiral correlators on the brane, or as transmission and reflection amplitudes in the supergravity/matter theory; the results are the same.

After assembling the key ingredients of the relation between CFT data on the brane and supergravity/matter perturbations in the bulk, in section 6 we propose a map between the supergravity theory and the brane conformal field theory; roughly, the partition function of the brane conformal field theory in an arbitrary background worldsheet metric and gauge field, with couplings to chiral operators turned on, is expected to be the wavefunctional of the corresponding bulk supergravity/matter system, with the extra dimension of
AdS$_3$ spacetime appearing from the dependence of the CFT partition function on metric data (the $S^3$ arises from the background gauge field). This is strikingly similar to the way the matrix model of noncritical string theory ‘grows’ an extra dimension (for an extensive review and further references, see [23]); indeed, we will find a plethora of parallels between the two.

2. The D1-D5 system

The low-energy dynamics of the bound states of $Q_1$ D-strings and $Q_5$ D5-branes in type IIB string theory is described (taking the four spatial directions along the D5-branes that are transverse to the D1-branes to be compactified on a small torus) by the Higgs branch of 1+1 dimensional $SU(Q_1) \times SU(Q_5)$ gauge theory coupled to hypermultiplets in the $(Q_1, Q_5)$ and its conjugate, as well as adjoint hypermultiplets in each group. The infrared limit of this system is an $\mathcal{N} = (4, 4)$ sigma model on a target space $\mathcal{M}$ which is a blowup of the orbifold $S^{Q_1 Q_5} T^4$. The sigma model fields will be denoted $Y_{(A)}^i$, $\psi_+^{(A)}_{A\alpha}$, $\psi_-^{(A)}_{A\dot{\beta}}$, $A = 1, ..., Q_1 Q_5$. Here $a, \dot{b}$ denote spinor indices, and $i$ a vector index, in the tangent group of the $T^4$; and $\alpha, \dot{\beta}$ are spinor indices of the space transverse to both the worldsheet and the $T^4$. The space $\mathcal{M}$ is hyperKähler, hence the IR sigma model is conformally invariant. Further compactification of the spatial coordinate of this gauge theory on a circle of radius much larger than those of the internal $T^4$ puts the conformal field theory in finite volume; then the asymptotic level density for $L_0 \gg \bar{L}_0$ computes the black hole entropy [24]. Excitations of the fermions in the sigma model account for the entropy of spinning black holes [25].

An alternative presentation of this 1+1 field theory arises after T-duality along two circles of the internal $T^4$ (coordinates $x^6, ..., x^9$), which turns the D1-D5 system into a set of intersecting 3-branes, $Q_1$ of which are wrapped around (say) the 67 direction and $Q_5$ of which wrap the 89 direction. The Higgs branch is now realized as the geometrical phase where the intersection loci are blown up, making a single large Riemann surface whose genus is $Q_1 Q_5$ [26]. The $Q_1 Q_5$ hypermultiplets are the blowup modes of the degenerate Riemann surface given by the brane construction (very similar configurations have been used to study $\mathcal{N} = 2$ gauge theories in four dimensions [27], however we are interested in the field theory limit as opposed to the ‘MQCD’ limit studied there).

The orbifold $S^{Q_1 Q_5}(T^4)$ has a large variety of twisted sectors. The marginal perturbation of the sigma model on $\mathcal{M}$ by the $\mathbb{Z}_2$ twist field that interchanges two copies of $T^4$
deforms this manifold to the desired target space, the moduli space of instantons on $T^4$. While the states of the theory after this perturbation are no longer eigenstates of the orbifold holonomy, the predominant configurations intertwine all the copies of $T^4$. We can thus treat the theory as if it were in the maximally twisted sector of the unperturbed orbifold. In this sector, the fields expand in oscillators having mode numbers $n + \frac{m}{Q_1 Q_5}$, $n, m \in \mathbb{Z}$.

The chiral ring of this sigma model contains operators built out of superfields whose lowest components are the fermions $\psi^{a\alpha}_{+(A)}$, $\psi^{a\dot{\alpha}}_{-(A)}$; these must be contracted with tensor fields $T^{A_1 \ldots A_l}_{a_1 \alpha_1 \ldots b_1 \beta_1 \ldots}$, and combined into invariants under the symmetric group acting on the $A$ index. The superfield whose lowest component is

$$O^{a\alpha}_{b\dot{\beta}} = \sum_{A=1}^{Q_1 Q_5} \psi^{a\alpha}_{+(A)} \psi^{(A)}_{- b\dot{\beta}}$$

(2.1)

generates the chiral ring and its descendants in the untwisted sector through the operator product expansion. It also has the nice feature that, when acting on the low-lying states in the maximally twisted sector, its lowest creation operators increase the level number by $1/Q_1 Q_5$; in contrast, operators such as the $SU(2)$ current $J^\alpha_+ = \psi^{a\alpha}_{+(A)} \psi^{a\alpha}_{+(\dot{A})}$ increase the level number by one. Thus the ‘soft’ perturbations of the black hole states are by operators invariant under the orbifold procedure and of the type (2.1), where the $A$ index is contracted between left- and right-movers.

Highest weight operators preserving an $\mathcal{N} = 2$ subalgebra of the $\mathcal{N} = 4$ may be constructed as follows. The $\mathcal{N} = 4$ supercharges are

$$G^{\dot{a}\dot{\alpha}}_+ = \sum_{A=1}^{Q_1 Q_5} \oint \partial_+ Y_{b}^{(A)\dot{a}} \psi^{b\alpha}_{+(\dot{A})}$$

$$G^{b\dot{\beta}}_- = \sum_{A=1}^{Q_1 Q_5} \oint \partial_- Y_{a}^{(A)b} \psi^{a\dot{\beta}}_{-(\dot{A})}.$$  

(2.2)

Denote the components of the $SU(2)$ indices $\alpha, \dot{\beta}$ by the labels $(1, 2)$. The $l+1$-fold product

$$O^{a_1 \ldots a_{l+1}}_{b_1 \ldots b_{l+1}} = \prod_{i=1}^{l+1} O^{a_1}_{b_1 i}$$

(2.3)

is annihilated by the supercharges $G^{\dot{a}1}_{\pm}$. Acting by $G^{\dot{a}2}_{+}$ and $G^{\dot{b}2}_{-}$ for fixed $\dot{a}, \dot{b}$ gives the highest component of a superfield under a particular $\mathcal{N} = (2, 2)$ subalgebra, which is also
highest weight of spin \( l \) under the global \( SO(4) = SU(2)_L \times SU(2)_R \). Acting with the global \( SO(4) \) lowering operators (\( \oint J^\alpha_+ \sigma^{-\alpha} \), \( \oint J^\alpha_- \sigma^{-\alpha} \)) fills out the spin \( l \) spherical harmonic.

The simplest chiral operator, \( \Phi_{(0)}^{ij} = \partial_+ Y_{(\lambda)}^i \partial_+ Y_{(\lambda)}^j \), has \( l = 0 \) and was the one used in the first calculations of absorption by D-brane black holes \([17,18] \). The operators

\[
\Phi_{(l)}^{ij} = \gamma^a_{\alpha\dot{a}} \gamma^b_{\beta\dot{b}} G^a_{\alpha\dot{a}} G^b_{\beta\dot{b}} \mathcal{O}_{ab1...al}^{a1...al},
\]

are similar to those used in the coupling of the brane theory to higher angular momentum perturbations in the bulk supergravity \([19,21] \). There exist many other operators in the chiral ring. There are operators with \( h \neq \bar{h} \), corresponding to perturbations with nonzero angular momentum in \( AdS_3 \); and with different \( SU(2)_L \) and \( SU(2)_R \) quantum numbers, needed to couple to higher-spin fields on \( S^3 \). There are also many operators with higher spin in the tangent group of \( T^4 \), obtained by symmetrizing on the \( a, b \) indices of the \( \mathcal{O}_{ab}^{a\dot{a}} \).

Finally, there is orbifold cohomology from the twist fields of \( S^{Q_1:Q_5}(T^4) \). The simplest such fields come from the \( \mathbb{Z}_2 \) twist that interchanges two copies of \( T^4 \). Taking linear combinations \( Y_1 \pm Y_2 \) of the coordinates on the two \( T^4 \)'s, the odd coordinate lives on \( T^4/\mathbb{Z}_2 \). In this way one finds 16 basic twist fields which have dimension \( h, \bar{h} = (\frac{1}{2}, \frac{1}{2}) \) and \( SU(2)_L, R \) spins \( j, \bar{j} = (\frac{1}{2}, \frac{1}{2}) \). Because the orbifold is nonabelian, the set of these \( \mathbb{Z}_2 \) twists closes on all the higher twist fields (any permutation can be obtained by a product of transpositions). Thus, all of the untwisted operators appear in the operator product of the basic superfield \((2.4)\), and all of the twisted operators appear in the operator product of the basic \( \mathbb{Z}_2 \) twist superfields; this fact will be important when we discuss their role in the AdS/CFT correspondence below.

### 3. 3d supergravity

The classical geometry of the extremal D1-D5 system compactified on a four-torus of volume \( v \) is

\[
ds^2 = (H_1 H_5)^{-1/2} dx^+ dx^- + (H_1 H_5)^{1/2} (dr^2 + r^2 d\Omega_3^2) + \left( \frac{H_1}{H_5} \right)^{1/2} dx_{\text{int}}^2
\]

\[
H_1 = 1 + \frac{g_{\text{str}}^2 Q_1}{v r^2}
\]

\[
H_5 = 1 + \frac{g_{\text{str}}^2 Q_5}{r^2}.
\]

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1. Similar operators appear in the orbifold construction of \([22] \); for instance the operators that split the degeneracy of the gauge couplings of the different \( SU(N) \) factors in their construction.
The near-horizon geometry is $AdS_3 \times S^3 \times T^4$, with metric (after a rescaling $r = r_0\ell_\text{str}g_6\sqrt{Q_1Q_5}^{-1}$)

$$ds^2 = \ell_\text{str}^2g_6\sqrt{Q_1Q_5}\left[r^2dx^+dx^- + \left(\frac{dr}{r}\right)^2 + d\Omega_3^2\right] + \left(\frac{Q_1}{vQ_5}\right)^{1/2}dx^2_{\text{int}}. \quad (3.2)$$

The radius $\ell$ of the anti-deSitter space and the three-sphere is $\ell^2 = g_6\ell_\text{str}^2(Q_1Q_5)^{1/2}$, whereas the volume of the $T^4$ is $\frac{Q_1}{vQ_5}; g_6 = g_\text{str}\ell_\text{str}v^{-1/2}$ is the effective fundamental string coupling. This geometry is semiclassical in the limit $g_6 \to 0$, with $g_6(Q_1Q_5)^{1/2}$ held fixed and large. At energies where the Kaluza-Klein modes on $S^3$ are not appreciably excited, the bulk supergravity theory is three-dimensional with a negative cosmological constant. This theory has 16 supercharges and its infrared dynamics is conveniently realized as 2+1 Chern-Simons gauge theory with gauge group $SU(1,1|2) \times SU(1,1|2)$; the level is $k = Q_1Q_5$. Of the bosonic sector, $SU(1,1) \times SU(1,1) \simeq SL(2,R) \times SL(2,R)$ describes 2+1 anti-deSitter gravity with dreibein $E$ and spin connection $\Omega$ in terms of ‘gauge’ fields $A = \Omega + \mathcal{E}/\ell$ and $\tilde{A} = \Omega - \mathcal{E}/\ell$

$$\frac{1}{16\pi G} \int \mathcal{E}(R - 2\Lambda) = \frac{k}{4\pi} \left(\int (AdA + \frac{2}{3}A^3) - \int (\tilde{A}d\tilde{A} + \frac{2}{3}\tilde{A}^3)\right) \quad (3.3)$$

top surface terms. Here $\Lambda = -\ell^{-2}$, and $k = \ell/(4G)$. The $SU(2) \times SU(2)$ part of the Chern-Simons gauge group is the gauged R-symmetry, for which the Chern-Simons term is a superpartner of the cosmological constant \[11\] (in particular, supersymmetry quantizes the cosmological constant). There is, of course, the standard kinetic term for these gauge fields, which is irrelevant in the infrared. The classical values of the connections $A, \tilde{A}$ on $AdS_3$ are (in the isotropic coordinates (3.2), in which displacements are referred to the $AdS_3$ scale $\ell$)

$$A = \begin{bmatrix} \frac{1}{2}dr/r & rdx^+ \\ 0 & -\frac{1}{2}dr/r \end{bmatrix}, \quad \tilde{A} = \begin{bmatrix} -\frac{1}{2}dr/r & 0 \\ rdx^- & \frac{1}{2}dr/r \end{bmatrix}. \quad (3.4)$$

\[12\] It is sometimes said that this geometry yields an exact string sigma model, since $AdS_3 \times S^3 \times T^4$ is a group manifold. However, that sigma model has an NS $B$-field turned on, whereas the geometry we are interested in has an RR $B$-field excited. Thus the conformal sigma model describes the IR limit of D-string dynamics near a D1-D5 black string, and does not describe massless closed strings in the limit of interest $g_\text{str} \to 0$ (the dual theory with an NS $B$-field would be at very large coupling). Hence the applicability of the group-manifold sigma model remains to be seen.
A model of a black hole in $AdS_3$ results from identification of $AdS_3$ under an isometry \[30\]. One can represent the $AdS_3$ space as a hypersurface in $\mathbb{R}^{2,2}$ via

\[
X = \frac{1}{\ell} \begin{pmatrix}
T_1 + X_1 & T_2 + X_2 \\
-T_2 + X_2 & T_1 - X_1
\end{pmatrix}, \quad \det|X| = 1.
\]

(3.5)

Then the identification $X \sim \rho_L X \rho_R$, with

\[
\rho_L = \begin{pmatrix}
e^{\pi(r_+-r_-)} & 0 \\
0 & e^{-\pi(r_+-r_-)}
\end{pmatrix}, \quad \rho_R = \begin{pmatrix}
e^{\pi(r_+ + r_-)} & 0 \\
0 & e^{-\pi(r_+ + r_-)}
\end{pmatrix},
\]

(3.6)

has the characteristics of a black hole in $AdS_3$, with (dimensionless) mass and ‘angular momentum’

\[
M = r_+^2 + r_-^2, \quad J = \frac{2\ell r_+ r_-}{8G}.
\]

(3.7)

This geometry appears as the near-horizon geometry of five-dimensional near-extremal black holes obtained by compactification of the spatial coordinate $\phi (x^\pm = t \pm \phi)$, which have been the subject of extensive study \[8,9\]. A set of gauge potentials for this solution is

\[
A = \begin{pmatrix}
\frac{1}{2}d\rho & z_+ e^\rho dx^+ \\
z_+ e^\rho dx^- & -\frac{1}{2}d\rho
\end{pmatrix}, \quad \tilde{A} = \begin{pmatrix}
-\frac{1}{2}d\rho & z_- e^{-\rho} dx^- \\
z_- e^{-\rho} dx^+ & \frac{1}{2}d\rho
\end{pmatrix},
\]

(3.8)

where $z_\pm = r_\pm \pm r_\mp$. The coordinate $\rho$ covers the exterior of the horizon, $r^2 = r_+^2 \text{ch}^2 \rho - r_-^2 \text{sh}^2 \rho \geq r_+^2$. The $\rho$ coordinate is singular in the extremal limit $r_\pm \to 0$, in which case it is better to use the coordinate $r$. Then the extremal $M = J = 0$ background is just (3.4), with $\phi$ periodically identified. Similarly, the extremal $\ell M = 8GJ \neq 0$ limit is described by the simple modification

\[
A = \begin{pmatrix}
\frac{1}{2}(dr/r) & rdx^+ \\
\frac{8GJ}{\ell} dx^+ & -\frac{1}{2}(dr/r)
\end{pmatrix}, \quad \tilde{A} = \begin{pmatrix}
-\frac{1}{2}(dr/r) & 0 \\
r dx^- & \frac{1}{2}(dr/r)
\end{pmatrix}.
\]

(3.9)

The substitution $J \to p(x^+)$ describes a family of travelling wave solutions \[31\].

The group of diffeomorphisms that preserves the asymptotic form of the $AdS_3$ metric (3.4) was shown by Brown and Henneaux to constitute the Virasoro algebra of conformal transformations of the boundary at infinity \[32,33\], with central charge $c = 6k$. A simple way to see this \[36\] is to note that the 2+1 Chern-Simons theory on a manifold with boundary is essentially a 1+1 chiral WZW model for the corresponding group \[13\]; furthermore, the asymptotic boundary conditions – that $A, \tilde{A}$ approach (3.4) as $r \to \infty$ – are equivalent \[30\] to the restriction on the WZW currents that yields the Hamiltonian reduction of the $SL(2, \mathbb{R})$ WZW model to Liouville theory \[37\]. The Liouville action is the
classical generating function for Virasoro Ward identities. Thus one understands the
presence of the asymptotic symmetry group as a consequence of the $SL(2,R)$ WZW/Liouville
theory carrying classical central charge $c = 6k$ on the boundary at infinity. The canonical
generators $T_{\pm\pm}$ of the Virasoro symmetry are

$$T_{++} = -\frac{k}{2} \text{Tr}(\alpha^2 + 2\alpha \partial_+ A_+ + A_+ A_+)$$
$$T_{--} = -\frac{k}{2} \text{Tr}(\alpha^2 + 2\alpha \partial_- \bar{A}_- + \bar{A}_- \bar{A}_-) ,$$

where $\alpha$ is a constant in the Lie algebra such that $\text{Tr}\alpha^2 = \frac{1}{2}$ (e.g. $\alpha = \frac{1}{2}\sigma^3$). The
supersymmetrization of these results is straightforward; the group $SU(1,1|2) \times SU(1,1|2)$
is the global part of the $\mathcal{N} = (4,4)$ superconformal group in 1+1 dimensions, so the
Liouville theory generalizes to the $\mathcal{N} = (4,4)$ Liouville theory [38].

The conformal algebra also makes its appearance in the 2+1 black hole [33-35]. Loosely
speaking, the main effect of the identification (3.6) is to map the conformal algebra from
the conformal plane at the boundary of $AdS_3$ to the conformal cylinder at the boundary
of the black hole spacetime (3.3),(3.4). This results in a shift of $L_0$ by $h_{\text{min}} = c/24$ due to
the Schwarzian of the conformal transformation involved. One has

$$M = \frac{8G}{\ell} (L_0 + \bar{L}_0) \ , \quad J = L_0 - \bar{L}_0$$

(and thus the ‘angular momentum’ $J$ is the third (‘momentum’) BPS charge carried by
five dimensional black holes); equivalently,

$$r_{\pm} = \frac{4G}{\ell} \left[ \left( \frac{cL_0}{6} \right)^{1/2} \pm \left( \frac{c\bar{L}_0}{6} \right)^{1/2} \right].$$

Because of the shift in $L_0, \bar{L}_0$, anti-deSitter space has ‘negative mass’ $L_0 = -c/24$, or
$M = -1$. The global supersymmetries preserved by (3.11) are periodic in $\phi$ [39]; the black
hole has the ‘Ramond’ supersymmetry appropriate to a cylindrical geometry.

On the basis of the asymptotic symmetry group with its classical central charge $c = 6k$,
it has been argued [34,35,40,41] that one can compute the black hole entropy

$$S = \frac{2\pi r_+ \ell}{4G}$$

purely from the algebra of diffeomorphisms, independent of considerations of supersym-
metry or string theory. However, having a Virasoro algebra with central charge $c$ implies
the asymptotic level density

$$S \sim 2\pi \left( \frac{cL_0}{6} \right)^{1/2} + 2\pi \left( \frac{c\bar{L}_0}{6} \right)^{1/2}$$
only if the underlying conformal field theory is unitary. In fact, the prime counterexample is the Liouville model \cite{42}, whose density of states grows as $3.14$, but with $c_{\text{eff}} = c - h_{\text{min}} = 1$ appearing in place of $c$. Therefore, one cannot conclude from the Virasoro algebra alone that the microstates are accounted for. As we will see below, it is the D1-D5 field theory that provides the underlying unitary dynamics.

Another calculation of the entropy is that of Carlip \cite{43}, who analyzes the $SL(2, R) \times SL(2, R)$ WZW model that results when a boundary is placed at the black hole horizon. In a sense, one is attempting to compute the black hole entropy as a kind of partition function of the black hole ‘horizon degrees of freedom’. This calculation is puzzling for a number of reasons. It proceeds by imposing $L_0 + \bar{L}_0 = 0$ on the effective WZW model on the horizon. Carlip argues that the zero modes of the current algebra contribute a large negative amount in the black hole geometry, $L_0 = j(1 - j)/(k + 2)$ with $j \sim k^2 r_+$, allowing for a large oscillator excitation at $L_0 + \bar{L}_0 = 0$. The states with $L_0 + \bar{L}_0 = 0$ have the same asymptotic level density as the Bekenstein-Hawking entropy \cite{43}. But also because of this negative shift (as well as the indefinite metric on $SL(2, R)$), many of the states being counted do not have positive norm; the relation to an honest count of microstates in a unitary theory such as \cite{24} is unclear. Another difference with the latter computation is that the predominant states in the effective sigma model of \cite{24} are spread almost uniformly over the spatial coordinate $\phi$, since the effective radius is $k = Q_1 Q_5$ times larger than that seen by bulk supergravity fields \cite{28}. In contrast, the states counted in \cite{43} have a high oscillation number in this spatial direction. Finally, in the present context we are instructed to extend this computation to supergravity, where the boundary WZW model has 8 more fermionic and 3 more bosonic fields for both left- and right-movers; a count of the $L_0 + \bar{L}_0 = 0$ states in the horizon $SU(1,1|2)$ WZW model would no longer find the correct entropy.

4. Conformal anomaly inflow

We have a puzzle: how does the supergravity theory in the asymptotic region of anti-deSitter space “know” about the density of states of the brane dynamics? One possibility is that the branes are indeed located there. In this section, we suggest a rather different possibility – the fact that the gravitational action (3.3) is a Chern-Simons term means that it could transport the conformal anomaly of the branes to distant parts of spacetime by an anomaly inflow mechanism.
Chern-Simons gauge theory on a manifold $M$ with boundary takes the form (4.1)

$$S_{CS}[A] = \frac{k}{4\pi} \int_M (AdA + \frac{2}{3} A^3) - \frac{k}{4\pi} \int_{\partial M} A_u A_v.$$  

Here we suppose that the boundary of the 3-manifold $M$ is parametrized by coordinates $u, v$. The last term is not gauge invariant; rather, if $A = g^{-1} \dd A g + g^{-1} dg$, then one of the components of $\dd A$ is fixed at the boundary (e.g. $\dd A_u$), and in addition one obtains the gauged chiral WZW model

$$S[A] = S[\dd A] - kS_{WZW}[g, \dd A]$$

In the situation at hand, the chiral $SL(2,\mathbb{R})$ WZW model is known to be equivalent to the generating functional for Ward identities in 2d gravity [13,16].

Let us pause to review these results. Ordinarily, in quantizing Chern-Simons theory one chooses the holomorphic polarization where $A_u$ are coordinates of the wavefunction; instead, one may identify the boundary values of the connection $A$ with the 2d spin connection $\omega$ and zweibein $e$ via $A^{0,\pm} = (\omega, e^\pm)$, and work in the nonstandard polarization where $(e^+_u, e^+_v, \omega_u)$ are coordinates. Then the Gauss Law constraints $G^a = 0$ on the wave functional are precisely the gravitational Ward identities for two-dimensional diffeomorphisms and local Lorentz transformations, with Virasoro central charge $c = -6k$ at the classical level [13,16,14]. The wave functional $\Psi[e^+_u, e^+_v, \omega_u]$ of $SL(2,\mathbb{R})$ Chern-Simons theory in this unorthodox polarization satisfies [15]

$$G^a \Psi[e^+_u, e^+_v, \omega_u] = 0,$$

(4.3)

These constraints enforce the $SL(2,\mathbb{R})$ invariance of the boundary wave functional. Parametrizing the zweibein as

$$e^+ = e^\varphi (du + \mu dv),$$

$$e^- = e^\bar{\varphi} (dv + \bar{\mu} du),$$

(4.4)

Here and below we sometimes use coordinates $u, v$ in place of $x^\pm$ to avoid confusion between worldvolume and tangent space $\pm$ indices.
the constraints (4.3) are solved by the effective gravitational action

\[ \Psi[\varphi, \mu, \omega] = \exp\left[ \frac{ic}{24\pi} (S_O[\varphi, \mu, \omega] + S_L[\varphi, \mu] + S_V[\mu]) \right], \tag{4.5} \]

where \( c = -6k \), and

\[ S_O[\varphi, \mu, \omega] = \int dudv \left[ \frac{1}{2} \mu \omega^2 - \omega (\partial_v \varphi - \mu \partial_u \varphi - \partial_u \mu) \right] \]

\[ S_L[\varphi, \mu] = \int dudv \left[ \frac{1}{2} \partial_u \varphi \partial_v \varphi + \mu \left( \frac{1}{2} (\partial_u \varphi)^2 + \partial_u^2 \varphi \right) \right] \tag{4.6} \]

\[ S_V[\mu] = \int dudv \frac{\partial_v F}{\partial_u F} \left( \frac{\partial^3 F}{\partial_u^3 F} - 2 \left( \frac{\partial^2 F}{\partial_u^2 F} \right)^2 \right). \]

In the last equation, \( \mu \) is related to \( F \) implicitly by the relations \( \partial_v f = \mu \partial_u f \), and \( F = f^{-1} \); equivalently, \( S_V \) is defined by the Virasoro Ward identity

\[ \mathcal{V} \exp\left[ \frac{ic}{24\pi} S_V[\mu] \right] = 0, \]

\[ \mathcal{V} = \left( \partial_v - \mu \partial_u - 2(\partial_u \mu) \right) \frac{\delta}{\delta \mu} + \frac{ic}{24\pi} \partial_u^3 \mu. \tag{4.7} \]

At the special point \( \omega = \varphi = 0 \), the Beltrami parameter \( \mu \) is given in terms of \( SL(2, R) \) currents as \(^{14}\)

\[ \mu(u, v) = J^{-}(v) - 2uJ^{0}(v) + u^2J^{+}(v), \tag{4.8} \]

and the stress tensor \( T = \frac{1}{k+2} J^a J^a + \partial J^0 \) gives straightforwardly \( c \sim -6k \) in the semiclassical limit. This analysis extends to \( \mathcal{N} = (4, 4) \) supergeometry, with \( SL(2, R) \) replaced by \( SU(1, 1|2) \) (for the \( \mathcal{N} = 1 \) case, see \(^{14}\)). The bulk supergravity effective action is thus the generating functional of current insertions in the conformal field theory, and therefore universal.

It is important that the theory induced by Chern-Simons dynamics on the boundary is the chiral WZW model, describing the gravitational anomalies of only the left- or right-moving degrees of freedom; otherwise, we would have two sets of gravitational fields on the boundary. It has been argued \(^{36}\) that the two chiralities formally combine to make the nonchiral gravitational effective action. There may be subtleties with zero modes common to both left and right sectors, however. We will leave such questions to future work.

The proposal of \(^{6,7}\) couples bulk supergravity to gauge theory on the boundary via couplings of the form \( \int H \Phi \), where \( H \) is a bulk field, and \( \Phi \) is a chiral ring operator in the gauge theory. Witten proposed that it is natural to associate the gauge theory dynamics with the boundary at infinity in anti-deSitter space. We shall take a somewhat
different path; in the end, the picture we will develop bears a striking similarity with the matrix model of noncritical strings [23]. Consider Chern-Simons supergravity on a manifold which is locally $AdS_3$, with an outer boundary at infinity and an inner boundary along some (timelike) cylinder, so that the spacetime is topologically an annulus times $\mathbb{R}$. The IR supergravity dynamics is described by the action (4.1). Place on this inner boundary the conformal sigma model that represents the IR dynamics of the D1-D5 system (the precise sense in which this is meant will be described below). This sigma model has a conformal anomaly $\hat{c} = \frac{2}{3}c = 4k$. A conformal reparametrization on the boundary conformal field theory generates a gravitational anomaly with coefficient $\hat{c}$; this anomaly is cancelled by ‘conformal anomaly inflow’ from a combination diffeomorphism and local Lorentz transformation in the bulk (matching the conformal transformation on the boundary), which generates the gravitational WZW action with coefficient $-\hat{c}$. This is a standard anomaly inflow story [15]: the novelty here is that 2+1 gravity is itself a Chern-Simons theory to which the anomaly inflow mechanism applies with respect to diffeomorphisms and local Lorentz transformations. For the annulus spatial geometry, the Chern-Simons theory is

\[ S[A] = S[\bar{A}] - kS_{\text{inner WZW}}[g, \bar{A}] + kS_{\text{outer WZW}}[g, \bar{A}] ; \quad (4.9) \]

one can think of the inner WZW term cancelling the conformal anomaly of the matter on the branes, while the latter reconstitutes it on the outer boundary – in effect, the conformal anomaly is transported to the outer boundary even if the degrees of freedom are elsewhere (or as we shall suggest in a bit, nowhere in particular). This fact explains the results of [34].

The boundary values of the 2+1 connection should be matched to the background geometry in which the boundary conformal field theory is written. Specifically, on the boundary we have

\[ (A^0_u, A^+_u, A^+_v) = (\omega_u, e^+_u, e^+_v) , \quad (\bar{A}^0_v, \bar{A}^-_v, \bar{A}^-_u) = (\omega_v, e^-_v, e^-_u) . \quad (4.10) \]

This has the intriguing effect that, when we evaluate the gauge potentials $A, \bar{A}$ on their classical values (3.4) for anti-deSitter space, the location of the inner boundary in $AdS_3$ translates into the background worldsheet geometry on the static gauge D1-D5 sigma model. The restriction of the inner boundary to surfaces of constant $\varphi$ amounts to a kind of minisuperspace approximation. The sigma model is in the NS sector in order to match the global (super)isometries. A similar story applies for the 2+1 black hole geometry. Again,
to match bulk and boundary supersymmetries, the boundary conformal field theory is in the Ramond sector (and dominated by the sector of maximal twist \(28\), as discussed in section 2). In particular, the \(M = 0\) extremal black hole state is well-approximated in the CFT by the Ramond sector ground state of the CFT obtained by acting on the NS vacuum with the Ramond twist operator of maximal order \(k = Q_1 Q_5\).

The matching of the bulk supergravity onto the boundary determines the bulk \(SU(1,1|2)\) gauge fields in terms of the expectation values of the corresponding currents in the conformal field theory. Consider for example the extremal black hole \((3.9)\), or rather its extension to travelling wave solutions. The classical stress tensor \((3.10)\) is

\[
k A_u^+ A_u^- = p(u) ;
\]

this matches that of the conformal field theory (using the identifications \((4.10),(4.4)\))

\[
4\pi A_u^+ \frac{\delta S_\sigma}{\delta A_v^+} = \frac{\delta S_\sigma}{\delta \mu} = T_{uu}(x^+) \tag{4.12}
\]

provided we identify \(p(x^+) = T_{++}(x^+)\). There is a similar story for the general black hole \((3.8)\). Note that a Legendre transformation \(A_u^- = \frac{4\pi i}{k} \frac{\delta}{\delta A_v^+}\) relates the gravity and brane calculations. It is important in order for the matching to work, that the coordinates in the bulk and boundary theories must be compatible; for instance, it was shown in \([46]\) that one can find a coordinate transformation of the travelling wave metric that locally puts it into the vacuum \(AdS_3\) form \((3.2)\). Similarly, making a conformal transformation \(u \to f(u)\) on the boundary CFT produces a term \(\frac{\delta}{\delta u}\{f, u\}\) in the stress tensor \(T_{++}\), where \(\{f, u\}\) is the Schwarzian derivative. Unless the coordinates match properly, the bulk geometry will not correctly reflect the stress-energy of the sources on the boundary (pieces involving the Schwarzian of the coordinate transformation from one to the other will be missed).

### 5. Coupling to matter

The dimensionally reduced type IIB supergravity theory of course consists of more than \(SU(1,1|2)\) Chern-Simons supergravity. Matter fields in the bulk couple to the chiral ring and its descendant fields on the boundary. The bosonic sector of the D=6 supergravity

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4 Afficionados of Liouville theory will recognize that the quantities \(p(u) = \sigma^2 + \partial_u \sigma; \sigma = \partial_u \log(G); G = (\partial_u F)^{-1/2}\) employed in \([46]\) are the essential characteristics of the classical solution to Liouville theory, related to the coordinate transformation \(u \to F(u)\).
theory with 32 supercharges contains the graviton $g_{\mu\nu}$, 5 tensor fields $B_{\mu\nu}$, 16 vectors $A_\mu$, and 25 scalars parametrizing the coset manifold $SO(5,5)/SO(5)\times SO(5)$. The graviton modes $h_{ij}$ on the internal $T^4$ are minimally coupled scalars, and couple to the $4k$ scalars $Y_{(A)}^{(i)}$ ($i = 1, \ldots, 4; A = 1, \ldots, k$) on the boundary conformal field theory via the interaction term (for modes that are S-waves on the $S^3$)

$$V_{l=0} = \frac{1}{4\pi\alpha'_\text{eff}} \int dx^+ dx^- h_{ij}(x^+, x^-) \partial_+ Y_{(A)}^{i} \partial_- Y_{(A)}^{j}, \quad (5.1)$$

with the effective inverse string tension $\alpha'_\text{eff} = \ell_\text{str}^4 g_6(k)^{1/2} \equiv \ell^2 [21,20]$; $r_0$ is the location of the boundary. (The direct coupling between the internal metric and these fields is perhaps easiest to see in the geometrical representation of the Higgs branch in terms of D3 branes discussed in section 2.) Absorption coefficients have been calculated in the $AdS_3$ black hole geometry and compared with this coupling in [17]. Replacing $h_{ij} \rightarrow h_{ij} + B_{ij}^{(RR)}$ incorporates the RR $B$-field polarized along $T^4$.

For higher angular momenta, an interaction [19-21] has been suggested for angular momenta $(\frac{l}{2}, \frac{l}{2}) \in SU(2) \times SU(2)$ of the form

$$\frac{1}{4\pi\alpha'_\text{eff}} \int dx^+ dx^- e^{-l\varphi} h_{ij}^{(l)}(x^+, x^-) \Phi_{(l)}^{ij}; \quad (5.2)$$

The factor of $e^{l\varphi}$ is the appropriate gravitational dressing in conformal coordinates.\footnote{In arbitrary coordinates, one must replace $\nabla = \partial \rightarrow (1 - \mu\bar{\mu})^{-1}e^{-\varphi}(\partial + \mu\bar{\mu}) + \omega + a$, and $\det e = e^{2\varphi} \rightarrow (1 - \mu\bar{\mu})e^{2\varphi}$. The background gauge field $a$ covariantizes the transformation properties under $SU(2)_{L,R}$; we will suppress it for now to simplify the discussion.}

The worldsheet operators coupled to $h_{ij}^{(l)}$ are elements of the chiral ring given in section 2. As an aside, recall that it was also proposed in [21] that, in the effective string picture, there is an upper bound $l_{\text{max}} = k$ on the $SO(4)$ angular momenta that can be absorbed by the black string without further suppression by powers of the frequency $\omega$; however it was unclear what property of general relativity was responsible for this suppression. The point is that the higher partial waves are spherical harmonics on the $S^3$, which couple to the $SU(2) \times SU(2)$ Chern-Simons gauge fields of the effective supergravity theory (these gauge fields are modes of the graviton in the original six-dimensional supergravity before Kaluza-Klein reduction to the IR theory on $AdS_3$); the $SU(2)$ representations are only integrable (unitary) for $l \leq k/2$.

There are of course many other modes of the reduced supergravity theory, some of which have been discussed in the literature [22]. The couplings found to date fit the
pattern of [6,7] – they couple bulk operators to the chiral ring of the boundary conformal field theory and its descendant fields. For example, the $s$-wave mode of the fixed scalar describing the volume fluctuations of the internal torus couples to the sum (over $A$) of the product of energy-momentum tensors $T_{++}^{(A)} T_{--}^{(A)}$; and the $s$-wave mode of the vector field $V_{\mu i}$ (called an ‘intermediate scalar’ in [22]), arising from the components of the metric and $B^{(RR)}$-field with one index in the internal torus and one index in $AdS_3$, couples to $\partial_+ Y_{(\Lambda \alpha)}^{(A)} T_{++}^{(A)}$. The 16 $\mathbb{Z}_2$ twist fields discussed in section 2 have the appropriate quantum numbers to be identified with the 16 vector fields $A_{\mu}$. Much of the structure expected from the supergravity side has been reproduced by coupling the bulk modes to particular scaling operators on the boundary conformal field theory [17-22]. It is important to realize that there are vastly many more operators in the chiral ring than there are in the supergravity multiplet. The latter contains only spins up to two on the internal $T^4$, whereas very high spin operators are obtained in the products of the $O_{a \alpha}^{a \alpha}$ of section 2. The fact that these operators are formed in the product of the $\Phi_{ij}^{(l)}$ suggests that they should be considered as coupling to composite operators in the supergravity theory. Similarly, the couplings to higher-order twist fields should be viewed as composites of those for the basic $\mathbb{Z}_2$ twist operators.

6. A matrix model analogy

The form (5.1),(5.2) of the interaction of bulk scalars with boundary chiral fields is not quite what one expects from supergravity. Rather, the quantity $e^{-l \varphi} h_{ij}^{(l)}$ is the asymptotic form of the solution of the scalar wave equation for the corresponding bulk field $H_{ij}^{(l)}(r, x^+, x^-)$, provided we identify $r$ with $e^\varphi$ as suggested by the preceding analysis. The bulk perturbations $H_{ij}^{(l)}$ are scalar fields on $AdS_3$ obeying the wave equation ($z = 1/r$)

$$\left( z \frac{\partial}{\partial z} z^{-1} \frac{\partial}{\partial z} + \frac{\partial^2}{\partial x^+ \partial x^-} - \frac{l(l+2)}{z^2} \right) H_{ij}^{(l)} = 0 ,$$

whose solutions are

$$H_{ij}^{(l)} = \varepsilon_{ij} \exp[i(p_+ x^+ + p_- x^-)] \left( \frac{p}{r} \right) K_{l+1}(p/r) ;$$

here $p^2 = p_+ p_-$, and $\varepsilon_{ij}$ is a polarization tensor. Naively, one would have expected the worldsheet operator $\Phi_{ij}^{(l)}$ to be gravitationally dressed by the metric $e^{-l \varphi} = r^{-l}$; this only agrees with (5.2) at asymptotically small values of $p/r$.\footnote{It is interesting to note that the scale at which these two start to differ is $r \sim p$, the distance scale that [4] associates with a given energy scale in the brane theory.}
This relation – between the naive gravitational dressing of the chiral operators, and the solutions to the scalar field wave equation – is strongly reminiscent of the relation between scaling operators and macroscopic loops in noncritical string theory. There are indeed many qualitative similarities between the setup we are proposing – the partition function on a 3d annular spacetime – and the macroscopic loop amplitudes of two-dimensional gravity. The annulus geometry of noncritical string theory has the property that, when the proper length of one boundary shrinks to zero size, the loop operator may be expanded in a series of local operators (which is equivalent to setting boundary conditions on the microscopic hole). For a given scaling operator, the partition function – considered as a function of the length of the other boundary – is the wave function of the local operator.

Now think of the outer boundary of $AdS_3$ as a ‘macroscopic loop’ shrunk to zero size (the analogue of the boundary length in 2d gravity is the $AdS_3$ radial coordinate $1/r$). The boundary conditions set there define a set of ‘microscopic’ operators. On the other hand, think of the inner boundary as another macroscopic loop (of finite size $1/r$). Consider the two-dimensional sigma model action in a background $N = (4, 4)$ geometry

\[ S_{2d} = S_{\sigma}[Y, \psi; \omega, \varphi, \mu, a] + \sum_{l=0}^{k} V_l[Y, \psi; \varphi, \omega, \mu, a; h^{(l)}]. \] (6.3)

Here $a$ is a background gauge field that covariantizes the $SU(2)_{L,R}$ dependence (see the footnote after eq. (5.2)). Also, we have included only the minimal scalars on the internal torus; however, the extension to other chiral fields coupling to other bulk matter fields is obvious. From this action we may define the wavefunctional of a three-dimensional effective theory via

\[ \Psi_{\text{eff}, 3d}[\mathcal{E}, \Omega, A_{L,R}; H^{(l)}_{ij}] = \int D\mathcal{Y} D\psi e^{-S_{2d}}. \] (6.4)

In this map, the metric is determined by the map (4.10) between two-dimensional and three-dimensional geometrical data; the $SU(2)$ connection $a$ determines $A_{L,u} = a_u$ and $A_{R,v} = a_v$; and the matter field $H$ is defined in three dimensions on the hypersurface determined by the inner boundary:

\[ H^{(l)} = H^{(l)}(r = e^{\varphi(x^+, x^-)}, x^+, x^-) \xrightarrow{r \to \infty} r^{-l} h^{(l)}_{ij}(x^+, x^-). \] (6.5)

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7 The similarities between the CFT/AdS correspondence and 2d gravity have been noticed by many people, including the authors of [2] and N. Seiberg, and probably many others.
The results of section 4 show that this map is manifestly correct when all the chiral couplings $H$ vanish; then (6.4) is a wavefunctional for Chern-Simons supergravity. We see yet another similarity to noncritical string theory in the generation of another dimension of spacetime – the radial coordinate of $AdS_3$ – through the scale of the background geometry (this is rather different from the matrix model of M-theory [3]). In addition, the background gauge field $a$ generates the three-sphere of $AdS_3 \times S^3$ via gauge fields $a_u = g^{-1} \partial u g$, $a_v = \partial_v h \cdot h^{-1}$; the fields $g$, $h$ compensate the transformation properties of $\Phi_{ij}^{(l)}$ under $SU(2)_{L,R}$ transformations. In conformal gauge, the field space $\varphi$, $g$ of $\mathcal{N} = (4, 4)$ Liouville theory, together with the base space coordinates $x^\pm$, parametrize $AdS_3 \times S^3$; the Liouville fields determine an embedding of the branes’ worldvolume into spacetime. Curiously, the analogue of KPZ scaling [14] is just the opposite here – whereas the boundary conditions on the macroscopic loop of noncritical strings matched onto relevant scaling operators in the UV (large negative $\varphi$), in 2+1 gravity the analogous scaling operators are irrelevant, and the boundary conditions are set in the IR (large positive $\varphi$).

The claim of [4,6,7] is that the effective action of bulk supergravity/matter at $k = Q_1 Q_5 = \infty$ is classical supergravity coupled to the $SO(5)_{SO(5) \times SO(5)}$ Kaluza-Klein scalars $H_{ij}^{(l)}$. Consequently, in this limit the log of (6.4) should be the Hamilton-Jacobi functional of this system. We regard the classical 2+1 gravity/matter theory as the analogue of the tachyon effective field theory of noncritical strings at cosmological constant $\mu = 0$. The quantity playing the role of $\mu$ in the 2+1 theory is the coupling $k = \ell/4G$, which is being sent to infinity to make 2+1 gravity classical. Pursuing this analogy further, we would say that 2+1 gravity/matter generates the ‘bulk S-matrix’ through the prescription of Witten [7], whereas the full brane CFT at finite $k$ constructs the ‘wall S-matrix’ (a kind of realization of ‘t Hooft’s ‘brick wall for black holes’ [48]). In other words, at $\mu = 0$ in the matrix model, a perturbation created at large negative $\varphi$ in the Liouville coordinate (the UV region of proper size) never returns as it travels toward larger values of $\varphi$ (the IR region). The scattering of noncritical strings off the Liouville potential – the ‘wall S-matrix’ – is nonperturbative in $\mu$.8 Similarly, at infinite $k$, an infalling matter perturbation in $AdS_3$ will not return from beyond the event horizon. The unitary S-matrix for black holes cannot be detected in perturbative supergravity (the expansion around $k = \infty$).9

8 Or more accurately, the expansion of the S-matrix in powers of $\mu$ about $\mu = 0$ is ill-defined.
9 Also in gravity, the vacuum state (both in $AdS_3$ and the black hole spacetime) seems to have a ‘double-sided’ nature – two asymptotic regions can occur. It would be fascinating if the effect
A number of subtleties plagued the development of noncritical string theory, that seem unavoidable here as well. One problem was operator mixing due to contact terms in situations where linear combinations of allowed scaling dimensions could sum to zero \cite{49}. Precisely that situation occurs here as well, and we may expect it to complicate the identification of bulk supergravity/matter and boundary CFT perturbations. All of the operators considered in section 2 have integer scaling dimensions, and we concluded that most of them were composites of the basic supergravity fields $H_{ij}, B^{(RR)}_{ij}$, etc. – just as one would expect of couplings generated by contact interactions.

To summarize, we propose that (6.3) describes bulk supergravity interacting with a set of internal (black hole) degrees of freedom at low energies, via a kind of functional integral transform (6.4). Correlation functions of the two dimensional conformal field theory determine those of gravity, but not typically vice-versa, except (according to \cite{6,7}) at infinite $k = Q_1 Q_5$. The absorption calculations of \cite{17-22,47} confirm that the correspondence is working, at least at the level of two-point correlations. Specifically, the greybody factors for emission and absorption, are obtained on the one hand in gravity by solving the wave equation of an incoming or outgoing scalar field in the background 3d geometry; and on the other hand, in the 2d conformal field theory they are generated by the matrix elements of chiral perturbations like (5.1), (5.2) in the ensemble of microstates that are the brane description of a black hole.

Our analysis so far has been closer in spirit to the Lorentzian signature investigations of \cite{6}. How does it compare to the Euclidean analysis of \cite{7} for the computation of correlation functions? In the present context, one should continue to hyperbolic three-space $\mathbb{H}_3$, which is formally the vacuum solution to $SL(2,\mathbb{C})$ Chern-Simons gravity; the inner boundary becomes the two-sphere boundary surrounding a horospherical ball, tangent to infinity, that has been removed. The bulk fields that couple to the boundary operators are excited on this sphere, leading to a perturbation of the CFT. Similarly, the Euclidean $AdS_3$ black hole \cite{50} has the topology of a solid torus – a disk times $S^1$. The connections $A, \tilde{A}$ are the analytic continuations of (3.8), with $t_{\text{Eucl}} = it, r_{\text{Eucl}} = -i|r|$. Our prescription involves the wavefunction on a toroidal boundary surrounding the horizon, obtained by cutting out a hole containing the origin of the disk. Holonomies of the Euclidean connection (3.8) determine the complex modulus of this torus \cite{50}; as we have seen, these holonomies of turning on finite $k$ is to throw away the other region, just as the ‘Seiberg bound’ seems to do in noncritical string theory.
are directly related to the $L_0, \bar{L}_0$ eigenvalues in the sum over states of the CFT. The pure supergravity theory codes the current sector correlators of the conformal field theory, which are guaranteed to match properly between CFT and supergravity. One is thus interested in the torus partition function of the CFT with appropriate operators turned on. The prescription of [7] is conjectured to compute the integrated chiral field correlators of the Euclidean CFT on the torus.[4]

7. Final remarks

It is interesting to contrast the model of low-energy black hole dynamics presented here with the CGHS model [51]. This 1+1 dimensional model of black hole formation and evaporation can be cast as the interaction of bulk matter fields with a dynamical boundary [52]. Conservation of stress-energy forces the mirror to accelerate away from incoming radiation, and above a certain threshold the mirror accelerates away forever, modelling a black hole since the incident radiation never returns. The present model differs from the CGHS model in that the reflecting wall has many internal degrees of freedom, which absorb and thermalize the incoming radiation rather than trying to immediately reflect it back to infinity. These internal degrees of freedom are lacking in the CGHS model (and in general relativity in higher dimensions). These degrees of freedom are also what one needs to account for the entropy. Because we have all the relevant low-energy degrees of freedom, it is hoped that an analysis of the dynamics of our model will help explain what fails in the standard field-theoretic analyses of the information problem, and how string theory describes physics beyond the horizon. This region would appear in an analytic continuation of the background geometric data of the 2d conformal field theory, thus realizing a speculation of [53] as to how matrix theory encodes this part of spacetime.

The analysis of near-extremal five dimensional black strings completely parallels that given above, so we will not repeat it. The brane theory is an $\mathcal{N} = (4,0)$ superconformal field theory in the infrared, with $\hat{c} = 4k = 4Q_1Q_2Q_3$ in terms of a triplet of brane numbers $Q_i$; correspondingly, the supergravity theory in the bulk involves the Chern-Simons action for the group $SU(1,1|2) \times SL(2,R)$.

10 An obvious generalization of these two Euclidean situations is to quotient $\mathbb{H}_3$ by a finitely generated discrete group that the turns the boundary into the covering space of a Riemann surface $\Sigma$; it would be interesting to work out the physical significance of this construction. A natural candidate is multiple $AdS_3$ Euclidean black holes, one for each handle.
Although our considerations have made heavy use of rather special properties of two dimensional conformal field theory and three dimensional gravity, one is led to suppose that the same thing is happening in the D3 brane system with $SU(2, 2|4)$ supersymmetry as well. There appears to be a similar connection between the scaling properties of the brane theory and the Schwinger terms in the algebra of stress tensors [54] (see also [55]). The analogue of $e^\varphi$ in our considerations should be played by the conformal scale factor $f$ of [7].

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