Remark on a conjecture of Mukai
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Introduction

The conjecture mentioned in the title appears actually as a question in [M] (Problem 4.11):

**Conjecture.** Let $C$ be a general curve, and $E$ a stable rank 2 vector bundle on $C$ with $\text{det} \ E = K_C$. The multiplication map $\mu_E : S^2 H^0(C, E) \to H^0(C, S^2 E)$ is injective.

Let $\mathcal{M}_K$ be the moduli space of stable rank 2 vector bundles $E$ on $C$ with $\text{det} \ E = K_C$, and let $\mathcal{M}_K^n$ be the subvariety of $\mathcal{M}_K$ parametrizing bundles with $h^0(E) = n$. As explained in *loc. cit.*, the conjecture implies that $\mathcal{M}_K^n$ is smooth, of codimension $\frac{1}{2} n(n + 1)$; this would give an analogue of the Brill-Noether theory for rank 2 vector bundles with canonical determinant.

In this note we prove the conjecture in a very particular case:

**Proposition 1.** The conjecture holds if $h^0(E) \leq 6$.

As a corollary we obtain that $\mathcal{M}_K^n$ is smooth of codimension $\frac{1}{2} n(n + 1)$ in $\mathcal{M}$ for $n \leq 6$. Another consequence is that Mukai’s conjecture holds for $g \leq 9$ (see §2).

The idea of the proof is the following. Put $n = h^0(E)$; assume for simplicity that $E$ is generated by its global sections, and that the map $\mu_E$ annihilates a non-degenerate (symmetric) tensor. This means that the image of the map $\mathbb{P}_C(E) \to \mathbb{P}^{n-1}$ defined by the tautological line bundle is contained in a smooth quadric; equivalently, the map from $C$ into the Grassmann variety $G(2, n)$ associated to $E$ factors through the orthogonal Grassmannian $GO(2, n)$. Now for $n \leq 6$, the restriction of the Plücker line bundle $\mathcal{O}_G(1)$ to $GO(2, n)$ is the sum of two (possibly equal) globally generated line bundles; pulling back to $C$ we obtain a decomposition of $\text{det} \ E = K_C$ which turns out to contradict Brill-Noether theory for a general curve.

The last part of the argument breaks down for $n \geq 7$, since then the restriction map $\text{Pic}(G(2, n)) \to \text{Pic}(GO(2, n))$ is an isomorphism. In fact we will show that the result cannot be improved without strengthening the hypotheses that we use on $C$ and $E$.

1. Proof of the main result

We will say that a curve $C$ is *Brill-Noether general* if for any line bundle $L$...
on \( C \), the multiplication map

\[
H^0(C, L) \otimes H^0(C, K_C \otimes L^{-1}) \longrightarrow H^0(C, K_C)
\]

is injective. A general curve of given genus is Brill-Noether general; if \( S \) is a K3 surface with \( \text{Pic}(S) = \mathbb{Z}[H] \), a general element of the linear system \( |H| \) is Brill-Noether general [L].

We will use this property in the following way:

**Lemma 1.** Let \( C \) be a Brill-Noether general curve and \( L, L' \) two line bundles on \( C \) such that \( h^0(K_C \otimes (L \otimes L')^{-1}) \geq 1 \). Then:

a) The multiplication map \( H^0(C, L) \otimes H^0(C, L') \rightarrow H^0(C, L \otimes L') \) is injective;

b) If \( L' = L \), we have \( h^0(L) \leq 1 \).

**Proof:** Choose a non-zero section \( s \in H^0(C, (K_C \otimes (L \otimes L')^{-1})) \), and consider the commutative diagram

\[
\begin{array}{ccc}
H^0(L) \otimes H^0(L') & \longrightarrow & H^0(L \otimes L') \\
1 \otimes s \downarrow & & \downarrow s \\
H^0(L) \otimes H^0(K_C \otimes L^{-1}) & \longrightarrow & H^0(K_C).
\end{array}
\]

By our hypothesis on \( C \) the bottom horizontal map is injective; it follows that the top one is injective.

If \( L' = L \), we get that the map \( H^0(L)^{\otimes 2} \rightarrow H^0(L^{\otimes 2}) \) is injective; if \( H^0(L) \) contains two linearly independent elements \( s, t \), the tensor \( s \otimes t - t \otimes s \) is non-zero and belongs to the kernel of that map, a contradiction.

Let \( E \) be a vector bundle on a curve \( C \). If the multiplication map \( \mu_E \) is injective, the same property holds for all subbundles of \( E \); thus \( E \) satisfies

\((\star)\) For every sub-line bundle \( L \subset E \), the map \( \mu_L : S^2H^0(C, L) \rightarrow H^0(C, L^2) \) is injective.

By [T], this condition is automatically satisfied if \( C \) is general and any sub-line bundle of \( E \) has degree \( \leq g + 1 \). Thus Proposition 1 is a consequence of the following more precise result:

**Proposition 2.** Let \( C \) be a Brill-Noether general curve, and \( E \) a rank 2 vector bundle on \( C \) with \( \det E = K_C \), satisfying condition \((\star)\). Then any non-zero tensor \( \tau \in S^2H^0(C, E) \) such that \( \mu_E(\tau) = 0 \) has rank \( > 6 \).

**Proof:** a) The general set-up

Let \( C \) be a curve, \( E \) a rank 2 vector bundle on \( C \) with \( \det E = K_C \), \( \tau \) an element of \( \text{Ker} \mu_E \) of rank \( n \). This means that we can find linearly independent
elements \( s_1, \ldots, s_n \) of \( H^0(C, E) \) such that \( \tau = s_1^2 + \cdots + s_n^2 \). Let \( F \subset E \) be the image of the map \( O^n_C \to E \) defined by \( s_1, \ldots, s_n \). Then \( s_1, \ldots, s_n \) are sections of \( F \) which satisfy \( s_1^2 + \cdots + s_n^2 = 0 \) in \( H^0(C, S^2 F) \). By property (⋆) \( F \) is a rank 2 subsheaf of \( E \), with determinant \( K_C(-A) \) for some effective divisor \( A \).

Let \( P := P_C(F) \), and let \( O_P(1) \) be the tautological line bundle on \( P \). Through the canonical isomorphisms

\[ H^0(P, O_P(1)) \isom H^0(C, F) \quad H^0(P, O_P(2)) \isom H^0(C, S^2 F) \]

the multiplication map \( \mu_F \) is identified with \( \mu_{O_P(1)} \); thus we can view \( s_1, \ldots, s_n \) as global sections of \( O_P(1) \), which generate \( O_P(1) \) and satisfy \( s_1^2 + \cdots + s_n^2 = 0 \).

In other words, the image of the morphism \( \varphi : P \to \mathbb{P}^{n-1} \) defined by \( (s_1, \ldots, s_n) \) is contained in the smooth quadric \( Q \) defined by \( X_1^2 + \cdots + X_n^2 = 0 \).

Let \( \gamma \) be the map of \( C \) into the Grassmann variety \( G(2, n) \) associated to the surjective homomorphism \( O^n_C \to F \); by definition \( F \) is the pull back of the universal quotient bundle on \( G(2, n) \). The determinant of that bundle is the Plücker line bundle \( O_G(1) \) on \( G(2, n) \), so we get \( \gamma^*O_G(1) \isom K_C(-A) \).

For each \( x \in C \), the point \( \gamma(x) \in G(2, n) \) corresponds to the line \( \varphi(P(F_x)) \) in \( \mathbb{P}^{n-1} \). So the fact that the image of \( \varphi \) is contained in \( Q \) means that \( \gamma \) factors through the orthogonal grassmannian \( GO(2, n) \) of lines contained in \( Q \):

\[ \gamma : C \xrightarrow{\gamma_0} GO(2, n) \hookrightarrow G(2, n) . \]

b) The cases \( n = 4 \) and \( n = 5 \)

Now we will assume \( n \leq 6 \) and that \( C \) is Brill-Noether general, and derive a contradiction. The case \( n \leq 3 \) is trivial (\( GO(2, n) \) is empty). Consider the case \( n = 4 \). Then \( GO(2, 4) \) parametrizes the lines in a smooth quadric in \( \mathbb{P}^5 \); it has two components, both isomorphic to \( \mathbb{P}^1 \). Inside \( G(2, 4) \), which is a quadric in \( \mathbb{P}^5 \), each of these components is a conic. Thus \( \gamma \) factors as

\[ \gamma : C \xrightarrow{\gamma_0} \mathbb{P}^1 \hookrightarrow G(2, 4) , \]

with \( O_G(1)|_{\mathbb{P}^1} = O_{\mathbb{P}^1}(2) \). Thus \( L = \gamma_0^*O_{\mathbb{P}^1}(1) \) satisfies \( L^2 = K_C(-A) \) and \( h^0(L) \geq 2 \), contradicting Lemma 1 b).

Suppose \( n = 5 \). Let \( V \) be a 4-dimensional vector space, with a non-degenerate alternate form \( \omega \) and an orientation \( \Lambda^4 V \isom C \). The orthogonal \( W \) of \( \omega \) in \( \Lambda^2 V \) is a 5-dimensional vector space with a non-degenerate quadratic form given by the wedge product. The isotropic 2-planes in \( W \) are of the form \( \ell \wedge \ell^\perp \), for \( \ell \in \mathbb{P}(V) \). Thus \( GO(2, 5) \subset G(2, 5) \) is identified with \( \mathbb{P}(V) \) embedded in \( G(2, W) \) by \( \ell \mapsto \ell \wedge \ell^\perp \). The corresponding map from \( \mathbb{P}(V) \) to \( \mathbb{P}(\Lambda^2 W) \) is quadratic, so again \( O_G(1)|_{\mathbb{P}(V)} = O_{\mathbb{P}(V)}(2) \), and we conclude exactly as above.
c) The case $n = 6$

Let again $V$ be a 4-dimensional vector space with an orientation. Then $\Lambda^2 V$ is a 6-dimensional quadratic vector space; the corresponding quadric in $P(V)$ is the Grassmannian $G(2, V)$. The lines contained in $G(2, V) \subset P(\Lambda^2 V)$ correspond to the 2-planes $\ell \cap H \subset \Lambda^2 V$ where $\ell \subset H \subset V$, $\dim \ell = 1$, $\dim H = 3$; thus $GO(2, 6) \subset G(2, 6)$ is identified with the incidence variety $Z \subset P(V) \times P(V^*)$, embedded in $G(2, V)$ by $(\ell, H) \mapsto \ell \cap H$. Fixing $\ell$ or $H$ one sees that the restriction of $O_G(1)$ to $Z$ is the pull back of $O_P(1) \boxtimes O_P(1)$.

The map $\gamma_O : C \to Z$ gives by projection two maps $u : C \to P(V)$ and $u' : C \to P(V^*)$. Put $L = u^* O_{P(V)}(1)$, $L' = u'^* O_{P(V^*)}(1)$. Then $L \otimes L' \cong K_C(\lambda)$, and we deduce from $u$ and $u'$ two homomorphisms

$$v : V^* \to H^0(C, L) \quad v' : V \to H^0(C, L') .$$

Consider the map

$$V \otimes V^* \xrightarrow{v \otimes v'} H^0(L) \otimes H^0(L') \to H^0(L \otimes L') ;$$

the fact that $(u, u') : C \to P(V) \times P(V^*)$ factors through $Z$ means that the identity element of $V \otimes V^* \cong \text{End}(V)$ goes to zero in $H^0(L \otimes L')$, and therefore already in $H^0(L) \otimes H^0(L')$ by Lemma 1 a).

Put $K = \ker u$, $K' = \ker u'$. The kernel of $v \otimes v'$ is $K \otimes V^* + V \otimes K'$; any element of this kernel has rank $\leq \dim K + \dim K'$. Since the identity tensor has rank 4, we get $\dim K + \dim K' \geq 4$.

Suppose $\dim K = \dim K' = 2$. Identifying again $V \otimes V^*$ to $\text{End}(V)$, we can write $\text{Id}_V = p + q$, with $\text{Im} p \subset K$ and $\text{Ker} q \supset K'^\perp$. This implies that $p$ and $q$ are orthogonal projectors, and therefore that $K' = K^\perp$. Then $P(K) \times P(K^\perp)$ is contained in $Z$, and $\gamma_O$ factors as

$$C \xrightarrow{(u, u')} P(K) \times P(K^\perp) \hookrightarrow GO(2, 6) .$$

Let $(\ell, H) \in P(K) \times P(K^\perp)$; then $\ell \subset K$ and $H \supset K$, so the line $P(\ell \cap H)$ in $P(\Lambda^2 V) = P^5$ contains the point $p := P(\Lambda^2 K)$. In other words, the lines in $Q$ parametrized by $C$ all pass through $p$, hence are contained in the tangent hyperplane $T_p(Q)$. Therefore the scroll $\varphi(P)$ in $P^5$ is contained in $T_p(Q)$; this is impossible because $\varphi$ is defined by 6 linearly independent sections of $O_P(1)$.

Suppose now $\dim K = 3$, so that the image of $C$ in $P(V^*)$ is the point $P(K^\perp)$. Then $\gamma_O$ factors as

$$C \xrightarrow{u} P(K) \hookrightarrow GO(2, 6) ;$$

the image of $P(K)$ in $GO(2, 6)$ is the family of lines contained in $P(\Lambda^2 K) \subset P(\Lambda^2 V) = P^5$. Thus $\varphi$ maps $P$ to the projective plane $P(\Lambda^2 K)$, which is again impossible. The same argument applies when $\dim K' = 3$. $\blacksquare$
2. Consequences and comments

The following lemma must be well-known:

**Lemma 2.** Let $C$ be a Brill-Noether general curve, and $E$ a semi-stable rank $2$ vector bundle on $C$ with $\det E = K_C$. Then $h^0(E) \leq \frac{g}{2} + 2$.

**Proof:** Let $L$ be a sub-line bundle of $E$ of maximal degree. The exact sequence

$$0 \to L \to E \to K_C \otimes L^{-1} \to 0$$

gives $h^0(E) \leq h^0(L) + h^0(K_C \otimes L^{-1})$. Since $h^0(L) h^0(K_C \otimes L^{-1}) \leq g$ and $h^0(L) \leq h^0(K_C \otimes L^{-1})$ by semi-stability, the required inequality holds unless $h^0(L)$ is 0 or 1. In that case we have

$$h^0(L) + h^0(K_C \otimes L^{-1}) = 2h^0(L) + g - 1 - \deg L \leq g + 1 - \deg L$$

and $\deg L \geq \frac{g}{2} - 1$ by [N], hence $h^0(E) \leq \frac{g}{2} + 2$. ■

In particular $g \leq 9$ guarantees $h^0(E) \leq 6$, hence:

**Corollary.** Mukai’s conjecture holds for $g \leq 9$.

We will show that Proposition 2 cannot be improved without further hypotheses:

**Proposition 3.** For each integer $s \geq 5$, there exists a Brill-Noether general curve $C$ of genus $g = 2s$ and a stable rank $2$ vector bundle $E$ on $C$ with $\det E = K_C$, satisfying $\star$, such that $\text{Ker} \, \mu_E$ contains a tensor of rank $7$.

**Proof:** We choose a K3 surface $S$ with $\text{Pic}(S) = \mathbb{Z}[H]$, $(H^2) = 4s - 2$, and take for $C$ a general element of the linear system $|H|$. Then $C$ has genus $2s$ and is Brill-Noether general; in particular it admits a finite set $\mathcal{P}$ of line bundles $L$ with $h^0(L) = 2$, $\deg L = s + 1$. Choose one of these line bundles, say $L$; the Lazarsfeld bundle $E_L$ is defined by the exact sequence

$$0 \to E_L^* \to H^0(C, L) \otimes \mathcal{O}_S \to L \to 0.$$  

It turns out that the restriction $E$ of $E_L$ to $C$ does not depend on the choice of $L \in \mathcal{P}$; we have $\det E = K_C$, $h^0(E) = s + 2$ (see [V]), and for each $L \in \mathcal{P}$ an exact sequence

$$0 \to L \to E \to K_C \otimes L^{-1} \to 0.$$  

**Lemma 3.** Let $M$ be a sub-line bundle of $E$. Then either $M \in \mathcal{P}$, or $h^0(M) \leq 1$ and $\deg M \leq s$.

**Proof:** Put $M' := K_C \otimes M^{-1}$; we may assume that the $E/M$ is isomorphic to $M'$, so that

$$s + 2 = h^0(E) \leq h^0(M) + h^0(M') = 2h^0(M) + g - 1 - \deg M = 2h^0(M') + g - 1 - \deg M'.$$
As in the proof of Lemma 2 this implies that either $h^0(M)$ or $h^0(M')$ is $\leq 2$.

If $h^0(M) \leq 2$, we get $\deg M \leq s + 1$, hence either $M \in \mathcal{P}$, or $\deg M \leq s$ and $h^0(M) \leq 1$.

Assume $h^0(M') \leq 2$, so that $h^0(M) \geq s$. We get again $\deg M' \leq s + 1$, hence $\deg M \geq 3s - 3$. On the other hand the exact sequence preceding the Lemma shows that $M$ injects into $K_C \otimes L^{-1}$ for each $L \in \mathcal{P}$; since Card($\mathcal{P}$) $\geq 2$ the inclusion must be strict, so $\deg M < \deg K_C \otimes L^{-1} = 3s - 3$, a contradiction. 

It follows that $E$ is stable and satisfies condition $(\ast)$. We observe that $S^2E$ is isomorphic to $K_C \otimes \mathcal{E}nd_0(E)$, where $\mathcal{E}nd_0(E)$ is the sheaf of trace 0 endomorphisms of $E$; thus Serre duality and the stability of $E$ imply $h^1(S^2E) = h^0(\mathcal{E}nd_0(E)) = 0$, hence $h^1(S^2E) = 3g - 3$ by Riemann-Roch.

In $S^2H^0(E)$ the locus of tensors of rank $\leq 7$ has dimension $7(s - 1)$, while the kernel of $\mu_E$ has codimension $\leq 6s - 3$. Therefore the intersection of these subvarieties is not reduced to 0 – in fact it has dimension $\geq s - 4$. By Proposition 2 all the tensors in this intersection have rank 7. 

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