On a class of canonical systems corresponding to matrix string equations: general-type and explicit fundamental solutions and Weyl–Titchmarsh theory

Alexander Sakhnovich

Abstract

An important representation of the general-type fundamental solutions of the canonical systems corresponding to matrix string equations is established using linear similarity of a certain class of Volterra operators to the squared integration. Explicit fundamental solutions of these canonical systems are also constructed via the GBDT version of Darboux transformation. Examples and applications to dynamical canonical systems are given. Explicit solutions of the dynamical canonical systems are constructed as well. Three appendices are dedicated to the Weyl–Titchmarsh theory for canonical systems, transformation of a subclass of canonical systems into matrix string equations (and of a smaller subclass of canonical systems into matrix Schrödinger equations), and a linear similarity problem for Volterra operators.

MSC(2020): 34A05, 34B20, 37J06, 45D05, 46N20

Keywords: canonical system, matrix string equation, dynamical canonical system, fundamental solution, Volterra operator, Darboux matrix, explicit generalized eigenfunction.
1 Introduction

Canonical (spectral canonical) systems have the form

\[ w'(x, \lambda) = i\lambda JH(x)w(x, \lambda), \quad J := \begin{bmatrix} 0 & I_p \\ I_p & 0 \end{bmatrix} \quad \left( w' := \frac{d}{dx}w \right), \]  

(1.1)

where \( i \) is the imaginary unit \((i^2 = -1)\), \( \lambda \) is the so-called spectral parameter, \( I_p \) is the \( p \times p \) \((p \in \mathbb{N})\) identity matrix, \( \mathbb{N} \) stands for the set of positive integer numbers, \( H(x) \) is a \( 2p \times 2p \) matrix valued function (matrix function), and \( H(x) \geq 0 \) (that is, the matrices \( H(x) \) are self-adjoint and the eigenvalues of \( H(x) \) are nonnegative). Canonical systems are important objects of analysis, being perhaps the most important class of the one-dimensional Hamiltonian systems and including (as subclasses) several classical equations. They have been actively studied in many already classical as well as in various recent works (see, e.g., \([1, 8, 13, 15, 21, 33–37, 43, 45, 50, 52]\) and numerous references therein). We will also consider (and construct explicit solutions) for a more general class

\[ w'(x, \lambda) = i\lambda jH(x)w(x, \lambda), \quad H(x) = H(x)^*, \quad j := \begin{bmatrix} I_{m_1} & 0 \\ 0 & -I_{m_2} \end{bmatrix}, \]  

(1.2)

where \( m_1, m_2 \in \mathbb{N} \). Here, we set \( m_1 + m_2 =: m \), \( H \) is an \( m \times m \) locally integrable matrix function, and \( H(x)^* \) means the complex conjugate transpose of the matrix \( H(x) \). System (1.2) will be called a generalized canonical system and the corresponding matrix function \( H \) will be called a generalized Hamiltonian.

In the case \( m_1 = m_2 =: p \), it is easily checked that \( j \) and \( J \) are unitarily similar:

\[ J = \Theta j \Theta^*, \quad \Theta := \frac{1}{\sqrt{2}} \begin{bmatrix} I_p & -I_p \\ I_p & I_p \end{bmatrix}, \]  

(1.3)

that is (assuming \( H(x) \geq 0 \)), system (1.2) is equivalent to (1.1) (see Appendix B for details). We call the system

\[ w'(x, \lambda) = i\lambda jH(x)w(x, \lambda), \quad H(x) \geq 0 \quad (m_1 = m_2 = p) \]  

(1.4)
canonical (as well as the equivalent system (1.1)). The matrix function $H(x)$ is called the Hamiltonian of this system.

In most works on canonical systems the less complicated $2 \times 2$ Hamiltonian case (i.e., the case $p = 1$) is dealt with although the cases with other values of $p$ ($p > 1$) are equally important. Interesting recent works [11,30,53] on $2 \times 2$ canonical systems and string equations also contain some useful references. Here, we deal with the case of $2p \times 2p$ Hamiltonians ($p \geq 1$).

Well-known Dirac (or Dirac-type) systems are equivalent to a special subclass of canonical systems (see [15, 17, 43, 45, 50] and references therein). The Hamiltonians corresponding to Dirac systems (after we switch from the representation (1.1) to the representation (1.4)), have the form

$$H(x) = \gamma(x)^* \gamma(x), \quad \gamma(x)j\gamma(x)^* = -I_p,$$

(1.5)

where $\gamma$ are $p \times 2p$ matrix functions. For instance, formulas (1.7), (1.11), (1.12), and (1.26) in [45] lead to the representation (1.5).

The Hamiltonians, which we consider in this paper, have the form

$$H(x) = \beta(x)^* \beta(x), \quad \beta(x)j\beta(x)^* = 0 \quad (p \geq 1),$$

(1.6)

where $\beta$ are again $p \times 2p$ matrix functions. Thus, canonical systems (1.4) with Hamiltonians of the form (1.6) are dual in a certain way to the class of canonical systems corresponding to Dirac systems. Under some natural conditions, systems (1.4), (1.6) are also equivalent to the matrix string equations (see [50, Chapter 11] and Appendix B in our paper). Canonical systems (1.4) with some special Hamiltonians of the form (1.6) appear, for instance, as the linear systems auxiliary to nonlinear second harmonic generation equations [25, 26].

We note that first order symplectic systems $y'(x) = F(x)y(x)$ are actively studied (see [4,9,10,29] and references therein). In the reformulation for our case, symplecticity means the equality

$$F(x)^*j + jF(x) + \mu(x)F(x)^*jF(x) = 0.$$  

Thus, generalized canonical systems (1.2) (where $m_1 = m_2$) are symplectic, with $\mu \equiv 0$. Canonical systems (1.4), (1.6), which are our main topic in this paper, remain symplectic for any choice of $\mu(x)$. 

3
The normalization condition

\[ \beta'(x) j \beta(x)^* = i I_p \]  

(1.7)

for Hamiltonians of the form (1.6) is essential for the construction of fundamental solutions and solving inverse problems. In Appendix B, we show that matrix Schrödinger equations may be transformed into canonical systems (1.4), (1.6), (1.7) satisfying certain additional condition. There is considerable interest in generalized Schrödinger equations (e.g., in Schrödinger equations with distributional potentials, see some references in [12]). One can say that systems (1.4), (1.6), (1.7) present an important generalization of the of the matrix Schrödinger equations. Canonical systems with Hamiltonians satisfying (1.6), (1.7) were briefly considered in [49, 50]. However, local boundedness of \( \beta'' \) was required there instead of the local square-integrability of \( \beta'' \), which we require in the next section. In Section 2, we represent the fundamental solutions for this case as the transfer matrix function from [47, 49, 50]. For this purpose, we use the linear similarity of the operator \( K = i \beta(x) j \int_0^x \beta(t)^* \cdot dt \) to the operator (2.2) of squared integration as well as the form of the corresponding similarity transformation operator \( V \) (see Theorem C.1 and its proof in Appendix C).

The representation of the fundamental solutions in Section 2 is important in itself and (in view of the interconnections between fundamental solutions and Weyl–Titchmarsh functions) it also presents a crucial step in solving the inverse problem to recover canonical system from the Weyl–Titchmarsh function.

Some basic results and notions on the Weyl-Titchmarsh theory of the general-type canonical systems (1.4) are described in Appendix A. The results are conveniently reformulated in terms of system (1.4) instead of system (1.1), and, what is essentially more important, certain redundant conditions contained in [45, Appendix A] are removed.

In other sections of the paper we study explicit solutions of systems (1.2) with generalized Hamiltonians \( dj + \beta(x)^* \beta(x) \) as well as explicit solutions and corresponding Weyl–Titchmarsh (Weyl) functions of the canonical systems (1.4), (1.6). We note that explicit solutions of Dirac systems and the corresponding Weyl–Titchmarsh theory have been studied sufficiently well
(see, e.g., [20, 38, 45]) but the situation with the systems (1.4), (1.6) is quite different.

Explicit solutions of canonical systems and their properties are of essential theoretical and applied interest. Various versions of Bäcklund-Darboux transformations and related dressing and commutation methods [6, 7, 16, 18, 22, 27, 31, 32, 54] are fruitful tools in the construction of explicit solutions of linear and integrable nonlinear equations. Bäcklund-Darboux transformations for canonical and dynamical canonical systems, respectively, were constructed in [39] and [42]. More precisely, GBDT (generalized Bäcklund-Darboux transformation) was constructed for these systems. It is important that GBDT (see, e.g., [20, 27, 38, 44, 45] and references therein) is characterized by the generalized matrix eigenvalues (not necessarily diagonal) and the corresponding generalized eigenfunctions. In Section 3, generalized matrix eigenvalues and generalized eigenfunctions are denoted by $\mathcal{A}$ and $\Lambda(x)$, respectively.

Although GBDT for canonical systems was obtained in [39], a crucial step of constructing the generalized eigenfunctions $\Lambda(x)$ (which is necessary for constructing explicitly Hamiltonians and fundamental solutions) is done in the present paper. More precisely, the procedure works in the following way. We start with some initial systems (1.2), where initial Hamiltonians $H(x)$ are comparatively simple, and construct explicitly the fundamental solutions and generalized eigenfunctions for these systems. (In particular, some considerations from [40, 44] were helpful for this purpose.) Using generalized eigenfunctions, the transformed generalized Hamiltonians and so called Darboux matrices are constructed as well. Recall that Darboux matrix for generalized canonical systems is the matrix function $\Upsilon(x, \lambda)$ satisfying the equation

$$\Upsilon'(x, \lambda) = i\lambda(j\tilde{H}(x)\Upsilon(x, \lambda) - \Upsilon(x, \lambda)jH(x)),$$

where $H$ is the initial generalized Hamiltonian and $\tilde{H}$ the transformed one. In this way, we obtain fundamental solutions $\tilde{W}$ for a wide class of the transformed systems (i.e., systems with the transformed generalized Hamiltonians $\tilde{H}(x)$). Indeed, it is easy to see that $\tilde{W}$ is expressed via the fundamental solution $W$ of the initial system and the Darboux matrix, namely, $\tilde{W}(x, \lambda) = \Upsilon(x, \lambda)W(x, \lambda)$. 

5
Some preliminaries on GBDT for the generalized canonical systems are
given, and transformed generalized Hamiltonians and Darboux matrices
are constructed in Section 3. Generalized eigenfunctions are constructed explicitly in Section 4. Explicit formulas for fundamental solutions of the initial systems and for the Weyl functions of the transformed canonical systems on the semi-axis \([0, \infty)\) are established in Section 5. It is shown in Section 6 that the second equality in (1.6) (i.e., the equality \(\beta(x) j \tilde{\beta}(x)^* = 0\)) for the initial matrix function \(\beta(x)\) yields the equalities \(\tilde{\beta}(x) j \tilde{\beta}(x)^* = 0\) and \(\tilde{\beta}(x)' j \tilde{\beta}(x)^* = \beta(x)' j \beta(x)^*\) for the transformed matrix function \(\tilde{\beta}(x)\). Some interesting examples are treated in Section 7.

There are important connections between spectral and dynamical characteristics as well as between spectral and dynamical systems (see, e.g., [2, 5, 24, 41, 42, 51] and references therein). In particular, GBDT for the spectral canonical systems (1.4) is closely related to the GBDT for the dynamical canonical system

\[
H(x) \frac{\partial}{\partial t} Y(x, t) = j \frac{\partial}{\partial x} Y(x, t) \quad (m_1 = m_2 = p), \quad H(x) \geq 0, \quad x \geq 0. \tag{1.8}
\]

We note that the invertibility of \(H(x)\) was assumed for the dynamical canonical system considered in [42], and system (1.8) slightly differs from the one in [42]. Dynamical canonical systems are of interest in mechanics and control theory (see, e.g., [23]). The GBDT formula for \(Y\) and some explicit examples of \(H\) and \(Y\) are also discussed in Section 7.

As usual, \(\mathbb{R}\) stands for the real axis, \(\mathbb{R}_+ = \{r : r \in \mathbb{R}, \ r \geq 0\}\), \(\mathbb{C}\) stands for the complex plane, the open upper half-plane is denoted by \(\mathbb{C}_+\), and \(\overline{a}\) means the complex conjugate of \(a\). The notation \(\Re(a)\) stands for the real part of \(a\), and \(\Im(a)\) denotes the imaginary part of \(a\). The notation \(\text{diag}\{d_1, \ldots\}\) stands for the diagonal (or block diagonal) matrix with the entries (or blocks) \(d_1, \ldots\) on the main diagonal. The space of square-integrable functions on \((0, b)\) \((0 < b \leq \infty)\) is denoted by \(L_2(0, b)\) and the corresponding space of \(p\)-dimensional column vector functions is denoted by \(L_2^p(0, b)\). By \(L_2^{\alpha \times q}(0, b)\) we denote the class of \(p \times q\) matrix functions with the entries belonging to \(L_2(0, b)\). The notation \(I\) stands for the identity operator. The norm \(\|A\|\) of the \(n \times n\) matrix \(A\) means the norm of \(A\) acting in the space \(\ell_2^n\) of the sequences of length \(n\). The class of bounded operators acting from the
Hilbert space $H_1$ into Hilbert space $H_2$ is denoted by $B(H_1, H_2)$, and we set $B(H) := B(H, H)$.

2 General-type fundamental solutions

In this section, we study canonical system (1.4) satisfying conditions (1.6) and (1.7):

\[ H(x) = \beta(x)^*\beta(x), \quad \beta(x)j\beta(x)^* = 0, \quad \beta'(x)j\beta(x)^* = iI_p. \]  

(2.1)

Let us consider the system (1.4), (2.1) on some finite interval $[0, T]$ ($T > 0$). The linear similarity of the operators $K \in B(L^2_2(0, T))$ and $A \in B(L^2_2(0, T))$, where

\[ Kf = i\beta(x)j\int_0^x \beta(t)^*f(t)dt, \quad Af = \int_0^x (t - x)f(t)dt, \]  

(2.2)

is essential for us. Here, the operator $A$ is introduced as the squared integration multiplied by $-1$. (Recall that in the case of Dirac systems the analog of $A$ is the integration multiplied by $i$.) It is easy to see that

\[ K - K^* = i\beta(x)j\int_0^T \beta(t)^* \cdot dt. \]  

(2.3)

If $\beta''(x) \in L^p_{2 \times 2p}(0, T)$, we have (according to Theorem C.1) $K = VAV^{-1}$, which we substitute into (2.3). Multiplying both parts of the derived equality by $V^{-1}$ from the left and by $(V^*)^{-1}$ from the right, we obtain the operator identity

\[ AS - SA^* = i\Pi j\Pi^*, \]  

(2.4)

where

\[ S = V^{-1}(V^*)^{-1} > 0, \quad \Pi h = \Pi(x)h, \quad \Pi(x) := (V^{-1}\beta)(x), \]  

(2.5)

$\Pi \in B(C^{2p}, L^2_2(0, T))$, \quad $\Pi(x) \in L^p_{2 \times 2p}(0, T)$, \quad $h \in C^{2p}$. \]  

(2.6)

Note that $\Pi$ above is the operator of multiplication by the matrix function $\Pi(x)$ and the operator $V^{-1}$ is applied to $\beta$ (in the expression $V^{-1}\beta$) columnwise. The transfer matrix function corresponding to the so called $S$-node
\( w_A(\lambda) = w_A(T, \lambda) = I_{2p} - ij\Pi^* S^{-1}(A - \lambda I)^{-1}\Pi, \quad (2.7) \)

and was first introduced and studied in [47]. We introduce the projectors
\( P_\ell \in B(L^2_p(0, T), L^2_p(0, \ell)): \)
\[
(P_\ell f)(x) = f(x) \quad (0 < x < \ell, \quad \ell \leq T). \quad (2.8)
\]

Now, we set
\[
S_\ell = P_\ell SP_\ell^*, \quad V_\ell = P_\ell V P_\ell^*, \quad A_\ell = P_\ell AP_\ell^*, \quad \Pi_\ell = P_\ell \Pi, \quad (2.9)
\]
\[
w_A(\ell, \lambda) = I_{2p} - ij\Pi^*_\ell S_{-1}^\ell(A_\ell - \lambda I)^{-1}\Pi_\ell, \quad (2.10)
\]

Since \( V \) is a triangular operator, \( V^{-1} \) is triangular as well, and we have
\( P_\ell V^{-1} = P_\ell V^{-1} P_\ell^* P_\ell. \) Hence, taking into account \((2.5)\) and \((2.9)\) we derive
\[
P_\ell V^{-1} P_\ell^* V_\ell = P_\ell V^{-1} P_\ell^* P_\ell V P_\ell^* = P_\ell V^{-1} V P_\ell^* = I, \quad (2.11)
\]
\[
S_\ell = P_\ell V^{-1}(V^*)^{-1} P_\ell^* = P_\ell V^{-1} P_\ell^* P_\ell (V^*)^{-1} P_\ell^*. \quad (2.12)
\]

It follows that
\[
V^{-1}_\ell = P_\ell V^{-1} P_\ell^*, \quad S_\ell = V^{-1}_\ell (V^*_\ell)^{-1}. \quad (2.13)
\]

We also have \( P_\ell A = P_\ell AP_\ell^* P_\ell. \) Thus, multiplying both parts of \((2.4)\) by \( P_\ell \)
from the left and by \( P_\ell^* \) from the right (and using \((2.9), (2.13), \) and the last
equality in \((2.5)) we obtain
\[
A_\ell S_\ell - S_\ell A_\ell^* = i\Pi_\ell j\Pi_\ell^*, \quad \Pi_\ell(x) = (V^{-1}_\ell \beta)(x) \quad (0 < x < \ell). \quad (2.14)
\]

Clearly \( w_A(\ell, \lambda) \) coincides with \( w_A(T, \lambda) \) when \( \ell = T. \)

**Remark 2.1** Relations \((2.10), (2.13)\) and \((2.14)\) show that \( S_\ell \) and \( w_A(\ell, \lambda) \)
may be defined via \( V_\ell \) (and \( \beta(x) \) given on \([0, \ell] \)) precisely in the same way
as \( w_A(T, \lambda) \) is constructed via \( V \) (and \( \beta(x) \) given on \([0, T] \)). Moreover, according
to Remark C.2, \( V_\ell \) may be constructed in the same way as \( V \), and
so \( w_A(\ell, \lambda) \) does not depend on the choice of \( \beta(x) \) for \( \ell < x < T \) and the
choice of \( T \geq \ell. \) In particular, \( w_A(\ell, \lambda) \) is uniquely defined on the semi-axis
\( 0 < \ell < \infty \) for \( \beta(x) \) considered on the semi-axis \( 0 \leq x < \infty. \)
The fundamental solution of the canonical system (1.4), where Hamiltonian has the form (2.1) may be expressed via the transfer functions \( w_A(\ell, \lambda) \) using continuous factorization theorem [50, p. 40] (see also [45, Theorem 1.20] as a more convenient for our purposes presentation).

**Theorem 2.2** Let the Hamiltonian of the canonical system (1.4) have the form (2.1), where \( \beta(x) \) is a \( p \times 2p \) matrix function two times differentiable and such that \( \beta''(x) \in L^p_2(0, T) \), if the canonical system is considered on the finite interval \([0, \beta]\), or the entries of \( \beta''(x) \) are locally square integrable, if the canonical system is considered on \([0, \infty)\).

Then, the fundamental solution \( W(x, \lambda) \) of the canonical system normalized by \( W(0, \lambda) = I_{2p} \) admits representation

\[
W(\ell, \lambda) = w_A(\ell, \frac{1}{\lambda}).
\]

**Proof.** First, we fix some \( 0 < T < \infty \) and consider \( \beta(x) \) on \([0, T]\). It is easy to see that the projectors \( P_\ell \) and the triple \( \{A, S, \Pi\} \) satisfy conditions of [45, Theorem 1.20]. Hence, according [45, Theorem 1.20] the matrix function \( w_A(\ell, \frac{1}{\lambda}) \) is the normalized fundamental solution of the canonical system (1.4) with Hamiltonian

\[
H(\ell) = \frac{d}{d\ell} \int_0^\ell \Pi_\ell(x)^*S^{-1}_\ell \Pi_\ell(x)dx,
\]

where \( S^{-1}_\ell \) is applied to \( \Pi_\ell(x) \) columnwise. Using the second equalities in (2.13) and (2.14), we rewrite (2.16) in the form

\[
H(\ell) = \frac{d}{d\ell} \int_0^\ell \beta(x)^*\beta(x)dx = \beta(\ell)^*\beta(\ell),
\]

and the statement of the theorem is proved on \([0, T]\). Taking into account Remark 2.1, we see that the statement of the theorem is valid on \([0, \infty)\) as well. ■

**Remark 2.3** The operators \( S_\ell \) satisfying (2.14) are so called structured operators. The study of the structured operators in inverse problems takes roots in the seminal note [28] by M.G. Krein and was developed by L.A. Sakhnovich in [48–50].
3 GBDT: Darboux matrices for generalized canonical systems

Let us consider systems (1.2) on finite or semi-infinite intervals. Without loss of generality, we choose either the intervals \( \mathbb{I}_T = [0, T] \) \((T < \infty)\) or the semi-axis \( \mathbb{R}_+ = [0, \infty) \). We also fix an initial generalized Hamiltonian \( H(x) = H(x)^* \). Given an initial \( m \times m \) generalized Hamiltonian \( H(x) \), each GBDT is (as usual) determined by some \( n \times n \) matrices \( A \) and \( S(0) = S(0)^* \) \((n \in \mathbb{N})\) and by an \( n \times m \) matrix \( \Lambda(0) \) which satisfy the matrix identity

\[
AS(0) - S(0)A^* = i\Lambda(0)j\Lambda(0)^*.
\]  

(3.1)

Taking into account the initial values \( \Lambda(0) \) and \( S(0) \) (and using the matrix \( A \) and the matrix function \( H(x) \)) we introduce matrix functions \( \Lambda(x) \) and \( S(x) = S(x)^* \) via the equations:

\[
\Lambda'(x) = -iAA\Lambda(x)jH(x), \quad S'(x) = \Lambda(x)jH(x)j\Lambda(x)^*.
\]  

(3.2)

It is easy to see that (3.1) and (3.2) yield [39] the identity

\[
AS(x) - S(x)A^* \equiv i\Lambda(x)j\Lambda(x)^*.
\]  

(3.3)

Remark 3.1 We note that, similar to the case of the general-type fundamental solutions in Section 2, we also use operator identities and transfer matrix function in Lev Sakhnovich form in our GBDT constructions. However, instead of the infinite-dimensional operators in Section 2, identities (3.1) and (3.3) are written for matrices. Here, we use calligraphic letter \( A \) and \( S \) instead \( A \) and \( S \) in Section 2 (and the notation \( \Lambda \) instead of \( \Pi \)) for the elements of the \( S \)-node (of the triple \( \{A, S, \Lambda\} \)). The so called Darboux matrix from Darboux transformations is represented in GBDT (for each \( x \)) as the transfer matrix function. More precisely, we will show that in the points of invertibility of \( S(x) \) (for the case of the generalized canonical system) the Darboux matrix is expressed via

\[
w_A(x, \lambda) = I_m - ij\Lambda(x)^*S(x)^{-1}(A - \lambda I_n)^{-1}\Lambda(x)
\]  

(3.4)

(see (3.11)). The dependence of \( S, \Lambda \) and \( w_A \) on \( x \) is of basic importance and greatly differs from the dependence of \( S_\ell, \Pi_\ell \) and \( w_A(\ell, \lambda) \) on \( \ell \) in Section 2.
According to [39], \( w_A(x, \lambda) \) satisfies the equation
\[
\begin{align*}
w'_A(x, \lambda) &= (i\lambda jH(x) - \tilde{q}_0(x))w_A(x, \lambda) - i\lambda w_A(x, \lambda)jH(x); \\
\tilde{q}_0(x) &= j\Lambda(x)^*S(x)^{-1}\Lambda(x)jH(x) - jH(x)j\Lambda(x)^*S(x)^{-1}\Lambda(x).
\end{align*}
\] (3.5)

Note that (3.5) follows directly from (3.2)–(3.4). Moreover, (3.3) yields (see [39] or [45, (1.88)]):
\[
w_A(x, \mu)^*jw_A(x, \lambda) = j + i(\mu - \lambda) \times \Lambda(x)^*(\mathcal{A}^* - \mu I_n)^{-1}\mathcal{S}(x)^{-1}(\mathcal{A} - \lambda I_n)^{-1}\Lambda(x).
\] (3.7)

Relation (3.8) shows that, under conditions \( \det((\mathcal{A} - \lambda I_n) \neq 0 \) and \( \det(\mathcal{A}^* - \lambda I_n) \neq 0 \), \( w_A \) is invertible and
\[
w_A(x, \lambda)^{-1} = jw_A(x, \overline{\lambda})^*j.
\] (3.8)

Further we assume that
\[
\det \mathcal{A} \neq 0,
\] (3.9)
and so \( w_A(x, 0) \) is well defined (in the points of invertibility of \( S(x) \)). We note that (3.5) yields
\[
w'_A(x, 0) = -\tilde{q}_0(x)w_A(x, 0),
\] (3.10)
and we set
\[
v(x, \lambda) := w_A(x, 0)^{-1}w_A(x, \lambda).
\] (3.11)

Formulas (3.5), (3.10) and (3.11) imply that
\[
\begin{align*}
v'(x, \lambda) &= i\lambda j\tilde{H}(x)v(x, \lambda) - i\lambda v(x, \lambda)jH(x), \\
j\tilde{H}(x) &= w_A(x, 0)^{-1}jH(x)w_A(x, 0).
\end{align*}
\] (3.12)
(3.13)

Thus, one can see that \( j\tilde{H} \) is linear similar to \( jH \). Moreover, in view of (3.8) we can rewrite (3.13) in the form
\[
\tilde{H}(x) = w_A(x, 0)^*H(x)w_A(x, 0).
\] (3.14)
According to (3.14), the equality $\tilde{H}(x) = \tilde{H}(x)^*$ is valid. Hence, $\tilde{H}(x)$ is the transformed generalized Hamiltonian of the transformed generalized canonical system

$$\bar{w}'(x, \lambda) = i\lambda j\tilde{H}(x)w(x, \lambda), \quad \tilde{H}(x) = \tilde{H}(x)^* \quad (x \geq 0). \quad (3.15)$$

Clearly, $\tilde{H} \geq 0$ if $H \geq 0$, and $\tilde{H} > 0$ if $H > 0$. Therefore, in the case of an initial canonical system, the transformed system is also canonical. By virtue of (3.12), a fundamental solution $\tilde{W}$ of the transformed system is given by the formula

$$\tilde{W}(x, \lambda) = v(x, \lambda)W(x, \lambda), \quad (3.16)$$

where $W$ is a fundamental solution of the initial system.

**Remark 3.2** If $H(x) \geq 0$ and $S(0) > 0$, the second equation in (3.2) implies that $S(x) > 0$ for $x \geq 0$. In particular, $S(x)$ is invertible.

**Remark 3.3** In view of (3.12) (or (3.16)) the matrix function $v(x, \lambda)$ is the so called Darboux matrix of the generalized canonical system.

According to (3.7), (3.8) and (3.11), the representation of $v(x, \lambda)$ in terms of $\Lambda(x)$ and $S(x)$ may be simplified. Namely, we have

$$v(x, \lambda) = jw_A(x, 0)^*jw_A(x, \lambda)$$

$$= I_m - i\lambda j\Lambda(x)^*(A^*)^{-1}S(x)^{-1}(A - \lambda I_n)^{-1}\Lambda(x). \quad (3.17)$$

### 4 Explicit solutions of the transformed generalized canonical systems

Consider the case, where the initial generalized Hamiltonian $H(x)$ has the form

$$H(x) = dj + \beta(x)^*\beta(x), \quad \beta(x) := \begin{bmatrix} e^{icx} I_{m_1} & e^{-icx} \alpha \end{bmatrix}. \quad (4.1)$$

Here, $\beta(x)$ is an $m_1 \times m$ matrix function, $\alpha$ is an $m_1 \times m_2$ matrix function and

$$c, d \in \mathbb{R}; \quad \alpha \alpha^* = I_{m_1} \quad (m_2 \geq m_1). \quad (4.2)$$
In view of (4.1) and (4.2), we have
\[ \beta(x)j\beta(x)^* \equiv 0. \] (4.3)

Recall that the matrix function \( \Lambda(x) \) is determined by \( \Lambda(0) \) and by the system
\[ \Lambda'(x) = -i\Lambda(x)jH(x). \] (4.4)

We construct generalized eigenfunction \( \Lambda(x) \) in the case (4.1) explicitly.

**Proposition 4.1** Let (4.1) and (4.2) hold. Then, the matrix function
\[ \Lambda(x) = \begin{bmatrix} \Phi_1(x) & \Phi_2(x) \end{bmatrix} \]

such that
\[ \Phi_1(x) = \exp\{ix(cI_n &dash- d\mathcal{A})\}(e^{ixQf_1} + e^{-ixQf_2}), \] (4.5)
\[ \Phi_2(x) = \exp\{-ix(cI_n + d\mathcal{A})\}(e^{ixQ(\mathcal{A} + cI_n + Q)A^{-1}f_1} + e^{-ixQ(\mathcal{A} + cI_n - Q)A^{-1}f_2})\alpha, \] (4.6)

where \( f_k \) are \( n \times m_1 \) matrices, \( Q \) is an \( n \times n \) matrix and
\[ \mathcal{A}Q = Q\mathcal{A}, \quad Q^2 = c(2\mathcal{A} + cI_n), \] (4.7)
satisfies (4.4).

**Proof.** Using (4.1)–(4.6) we derive
\[ \Lambda(x)j\beta(x)^* = -e^{-idx}\mathcal{A}(e^{ixQ(cI_n + Q)A^{-1}f_1} + e^{-ixQ(cI_n - Q)A^{-1}f_2}). \] (4.8)

It follows from (4.5) that
\[ \frac{d}{dx}\Phi_1(x) = -id\mathcal{A}\Phi_1(x) + i\exp\{ix(cI_n &dash- d\mathcal{A})\} \]
\[ \quad \times (e^{ixQ(cI_n + Q)f_1} + e^{-ixQ(cI_n - Q)f_2}). \] (4.9)

According to (4.6), we also have
\[ \frac{d}{dx}\Phi_2(x) = -id\mathcal{A}\Phi_2(x) + i\exp\{-ix(cI_n + d\mathcal{A})\} \]
\[ \quad \times (e^{ixQ(cI_n - Q)f_1} + e^{-ixQ(cI_n + Q)A^{-1}f_2}). \] (4.10)
Since $\Lambda(x) = [\Phi_1(x) \quad \Phi_2(x)]$ and $H(x)$ has the form (4.1) (where (4.2) holds) relations (4.7)--(4.10) imply (4.4). ■

It is easy to see that one can set $Q = 0$ (in the Proposition 4.1) in the case $c = 0$. A more interesting case, where $c = 0$ and (4.7) holds, is generated by the matrices $A$ and $Q$ of the form

\[
A = \xi I_{2r} + \begin{bmatrix} 0 & A_{12} \\ 0 & 0 \end{bmatrix} \quad (\xi \in \mathbb{C}), \quad Q = \begin{bmatrix} 0 & Q_{12} \\ 0 & 0 \end{bmatrix}
\]  

(4.11)

(where $A$ and $Q$ are $2r \times 2r$ matrices, $A_{12}$ and $Q_{12}$ are some $r \times r$ matrices) or by the block diagonal matrices with the blocks of the same form as the matrices on the right-hand sides of the equalities in (4.11).

The next immediate corollary of [44, Proposition B.1] (and its proof) deals with the case $c \neq 0$.

**Corollary 4.2** Let $c \neq 0$, let $\det(2A + cI_n) \neq 0$, and let $E$ be the similarity transformation matrix and $J$ Jordan normal form in the representation

\[
c(2A + cI_n) = EJE^{-1}.
\]  

(4.12)

Then, $Q$ satisfying (4.7) may be constructed explicitly and has the form

\[
Q = EDE^{-1},
\]  

(4.13)

where $D$ is a block diagonal matrix with the blocks of the same orders as the corresponding Jordan blocks of $J$. Moreover, the blocks of $D$ are upper triangular Toeplitz matrices (or scalars if the corresponding blocks of $J$ are scalars). If $z$ is the eigenvalue of some block of $J$, then the entries on the main diagonal of the corresponding block of $D$ equal $\sqrt{z}$ (and one can fix any of the two possible values of $\sqrt{z}$ for this main diagonal).

Given generalized eigenfunction $\Lambda(x)$, one can construct (explicitly) the fundamental solution $W(x, \lambda)$ of the transformed generalized canonical system using relations (3.4), (3.11), (3.16) and the second equality in (3.2). We note that an explicit expression for $W(x, \lambda)$, which we need for this purpose, is constructed similar to the way it is done in Proposition 5.1.
5 The case of the spectral canonical systems

1. It follows from (3.8), (3.14), and (4.3) that the transformed generalized Hamiltonians constructed in Section 4 have the form

\[ \tilde{H}(x) = dw_A(x,0)^* jw_A(x,0) + \tilde{\beta}(x)^* \tilde{\beta}(x) = dj + \tilde{\beta}(x)^* \tilde{\beta}(x), \]

\[ \tilde{\beta}(x) := \beta(x)w_A(x,0), \quad \tilde{\beta}(x)j\tilde{\beta}(x)^* = \beta(x)j\beta(x)^* = 0. \]

Here, \( \tilde{\beta} \) is the corresponding transformation of \( \beta \). Setting

\[ d = 0, \quad m_1 = m_2 =: p, \]

we obtain a class of canonical systems

\[ \tilde{\omega}'(x,\lambda) = i\lambda j\tilde{H}(x)\tilde{\omega}(x,\lambda), \quad \tilde{H}(x) = \tilde{\beta}(x)^* \tilde{\beta}(x) \geq 0, \quad \tilde{\beta}(x)j\tilde{\beta}(x)^* = 0. \]

Further in the text we normalize the fundamental solutions \( W \) and \( \tilde{W} \) of the systems (1.2) and (3.15), respectively, setting

\[ W(0,\lambda) = \tilde{W}(0,\lambda) = I_{2p}. \]

We write down the Hamiltonian \( H(x) \) given by (4.1), (4.2), and (5.3) in the form

\[ H(x) = e^{-icxj} \mathcal{K} e^{icxj}, \quad \mathcal{K} := \begin{bmatrix} I_p & \alpha \\ \alpha^* & I_p \end{bmatrix} (\alpha\alpha^* = I_p). \]

**Proposition 5.1** The fundamental solution of the canonical system (1.4), where \( \Im(\lambda) \neq 0 \), the Hamiltonian \( H \) is given by (5.6) and \( c \neq 0 \), has the form

\[ W(x,\lambda) = e^{-icxj} E(\lambda) \begin{bmatrix} e^{iz_1(\lambda)c} I_p & 0 \\ 0 & e^{iz_2(\lambda)c} I_p \end{bmatrix} E(\lambda)^{-1}, \quad E = \begin{bmatrix} E_1 & E_2 \end{bmatrix}, \]

\[ E_i := \begin{bmatrix} -\alpha \\ \frac{1}{\lambda}(\lambda + c - z_i) I_p \end{bmatrix}, \quad z_i^2 = c(2\lambda + c) \quad (i = 1, 2), \quad \Im(z_1) > 0. \]
Proof. It is easy to see that $E$ is invertible (one may consider, for instance, the linear span of the rows of $E$, which coincides with $\mathbb{C}^{2p}$). Moreover, using the equality
\[ \lambda - \frac{1}{\lambda} (\lambda + c)(\lambda + c - z_i) = \frac{z_i}{\lambda} (\lambda + c - z_i), \]
we have
\[ EZE^{-1}E = EZ = (\lambda jK + cj)E \quad \text{for} \quad Z = \text{diag}\{z_1I_p, z_2I_p\}. \quad (5.9) \]
It follows that
\[ EZE^{-1} = \lambda jK + cj. \quad (5.10) \]
Relations (5.7) and (5.10) yield
\[ W(x, \lambda) = e^{-ixj}e^{ix(\lambda jK + cj)}, \quad (5.11) \]
and for $W(x, \lambda)$ of the form (5.11) we immediately obtain
\[ W'(x, \lambda) = i\lambdajej^{-ixj}Ke^{ixj}W(x, \lambda), \quad W(0, \lambda) = I_{2p}. \quad (5.12) \]
Taking into account (5.6) and (5.12), we see that $W$ given by (5.7), (5.8) is, indeed, the normalized fundamental solution of the canonical system described in the proposition. ■

2. Further in this section, we assume that
\[ \mathcal{S}(0) > 0, \quad c \neq 0, \quad (5.13) \]
so that the statements of Remark 3.2 and Proposition 5.1 may be used. After normalization (5.5) formula (3.16) takes the form
\[ \widetilde{W}(x, \lambda) = v(x, \lambda)W(x, \lambda)v(0, \lambda)^{-1}. \quad (5.14) \]
For system (5.1)–(5.3), in view of (3.11), (5.7) and (5.14) we obtain
\[ \tilde{\beta}(x)\widetilde{W}(x, \lambda)v(0, \lambda)E_1(\lambda) = e^{iz_1(\lambda)}\tilde{\beta}(x)w_A(x, \lambda)e^{-ixj}E_1(\lambda). \quad (5.15) \]
Taking into account (5.15) (and some definitions and considerations on Weyl–Titchmarsh theory in Appendix A), we derive the following theorem.
Theorem 5.2  Canonical system with Hamiltonian of the form (5.1)–(5.3) on \([0, \infty)\), where (5.13) holds, has a unique Weyl function (Weyl’s limit point case). This Weyl function is given explicitly by the formula

\[
\varphi(\lambda) = \begin{bmatrix} 0 & I_p \end{bmatrix} v(0, \lambda) E_1(\lambda) \left( \begin{bmatrix} I_p & 0 \end{bmatrix} v(0, \lambda) E_1(\lambda) \right)^{-1},
\]

where \(E_1\) has the form (5.8).

Proof. First, we note that formulas (3.7)–(3.11) (and [45, Corollary E.3]) yield

\[
v(0, \lambda)^*jv(0, \lambda) \geq j, \quad v(0, \lambda)jv(0, \lambda)^* \geq j \quad (\lambda \in \mathbb{C}_+).
\]

(5.17)

It easily follows from (5.8) and (5.17) that

\[
E_1 \left( -\frac{c}{2} + \varepsilon i \right)^* j E_1 \left( -\frac{c}{2} + \varepsilon i \right) > 0,
\]

\[
\left[ I_p \quad 0 \right] v(0, \lambda) j v(0, \lambda)^* \begin{bmatrix} I_p \\ 0 \end{bmatrix} > 0,
\]

(5.18)

where the first inequality holds (at least) for small \(\varepsilon > 0\) and the second inequality holds for all \(\lambda \in \mathbb{C}_+\) (excluding the part of spectrum of \(A\) situated in \(\mathbb{C}_+\)). Hence (see, e.g., [45, Proposition 1.43]), \(\det \left( \begin{bmatrix} I_p & 0 \end{bmatrix} v(0, \lambda) E_1(\lambda) \right) \neq 0\), and so

\[
\det \left( \begin{bmatrix} I_p & 0 \end{bmatrix} v(0, \lambda) E_1(\lambda) \right) \neq 0 \quad \text{for} \quad \lambda \in \mathbb{C}_+,
\]

(5.19)

excluding, possibly, some isolated points. In other words, \(\varphi(\lambda)\) in (5.16) is well defined.

Now, we will show that for such \(\lambda\) that \(\Im(z_1(\lambda))\) is sufficiently large (excluding, possibly, isolated points) the relation

\[
\tilde{\beta}(x) \tilde{W}(x, \lambda) \begin{bmatrix} I_p \\ \varphi(\lambda) \end{bmatrix} \in L_2^{p \times p}(0, \infty)
\]

(5.20)

is valid. Indeed, the matrix functions \(\beta(x)\) and \(e^{-icxj}\) on the right-hand side of (5.15) are bounded. In view of (3.2), we have

\[
\int_0^r S(x)^{-1} \Lambda(x) j \beta(x)^* \beta(x) j \Lambda(x)^* S(x)^{-1} = S(0)^{-1} - S(r)^{-1} \leq S(0)^{-1}.
\]
Therefore, we obtain
\[ \beta(x) j \Lambda(x)^* S(x)^{-1} \in L_2^{p \times n}(0, \infty). \] (5.21)

Finally, Proposition 4.1 and Corollary 4.2 show that the matrix function
\[ e^{i z_1(\lambda)x} \Lambda(x) \] is bounded for sufficiently large values of \( \Im(z_1(\lambda)) \). Taking into account the definition (3.4) of \( w_A \) and considerations above, we see that the right-hand side of (5.15) belongs \( L_2^{p \times p}(0, \infty) \) (for sufficiently large values of \( \Im(z_1(\lambda)) \)). Thus, the left-hand side of (5.15) belongs \( L_2^{p \times p}(0, \infty) \) as well, and so (5.20) holds for \( \varphi(\lambda) \) given by (5.16).

Assume that for some \( \lambda = \lambda_0 \in \mathbb{C}_+ \) we have (5.20) and also have
\[ \tilde{\beta}(x) \tilde{W}(x, \lambda_0) = I \]
\[ \tilde{\beta}(x) \tilde{W}(x, \lambda_0) E_1(\lambda_0) = E_2(\lambda_0), \] (5.22)

We will show (by contradiction) that this is impossible for sufficiently large values of \( \Im(z_1(\lambda_0)) \). Indeed, since (5.20) implies
\[ \beta(x) \tilde{W}(x, \lambda_0) v(0, \lambda_0) E_1(\lambda_0) = \tilde{E}(\lambda_0) \]
additional relations (5.22) yield the existence of \( f \in \mathbb{C}^p \) such that
\[ \tilde{\beta}(x) \tilde{W}(x, \lambda_0) = v(0, \lambda_0) E_1(\lambda_0) \] (5.23)

On the other hand, taking into account that \( z_2(\lambda) = -z_1(\lambda) \) we similar to (5.15) derive
\[ \tilde{\beta}(x) \tilde{W}(x, \lambda) = e^{-i z_1(\lambda)x} \beta(x) w_A(x, \lambda) e^{-icxj} E_2(\lambda). \] (5.24)

Next, we should consider \( g(x, \lambda) = \beta(x) w_A(x, \lambda) e^{-icxj} \) in a more detailed way, and we note that according to (3.2), (3.4), (4.1), (4.2), (4.5), and (4.6) the entries \( g_{ik} \) of \( g \) admit representation
\[ g_{ik}(x, \lambda) = \sum_{s=1}^{N_1} P_s(\lambda) x^{\ell_s} e^{h_s x} / \left( P(\lambda) \sum_{s=1}^{N_2} x^{n_s} e^{\zeta_s x} \right). \] (5.25)

where \( P \) and \( P_s \) are polynomials, and \( N_1, P_s, \ell_s \) and \( h_s \) depend on \( i, k \). Moreover, similar to (5.19) one can show that (excluding isolated points \( \lambda \)) we have
\[ \det \left( g(x, \lambda) \tilde{E}(\lambda) \right) \neq 0, \quad \tilde{E}(\lambda) := \left[ \begin{array}{c} -\alpha \\ (\lambda + c)/\lambda \end{array} \right]. \]
where \( \hat{E} \) is the “rational part” of \( E_2 \). It follows that

\[
g(x, \lambda)E_2(\lambda)f \neq 0. \tag{5.26}
\]

Taking into account (5.24)–(5.26), we see that (5.23) (and so (5.22)) does not hold for sufficiently large values of \( \Im(z(\lambda_0)) \).

Since (5.22) does not hold for sufficiently large values of \( \Im(z(\lambda_0)) \) (excluding, may be, isolated points), there is an open domain in \( \mathbb{C}_+ \), where \( \varphi(\lambda) \) (given by (5.16)) is uniquely defined via (5.20). Thus, each Weyl function of our system coincides with \( \varphi(\lambda) \) in this domain (see the Definition A.2 of the Weyl functions). Recall that Weyl functions are holomorphic in \( \mathbb{C}_+ \). Hence, the Weyl function of our system is unique (and its existence follows from Proposition A.1). We see that the Weyl function exists, is unique and coincides with \( \varphi(\lambda) \) in some domain. Therefore, \( \varphi(\lambda) \) given by (5.16) is the Weyl function and admits holomorphic continuation in all \( \mathbb{C}_+ \). ■

\textbf{Remark 5.3} In Proposition 5.1 and Theorem 5.2, we assume that \( c \neq 0 \). The constructions are much simpler when \( c = 0 \). In particular, we recall that \( \beta_j\beta^* \equiv 0 \) (see (4.3)). Moreover, \( \mathcal{K} \) given in (5.6) equals \( \beta^*\beta \) as \( c = 0 \). Hence, \( \mathcal{K}_j\mathcal{K} = 0 \). Therefore, using (5.6) we have:

\[
H(x) \equiv \mathcal{K}, \quad W(x, \lambda) = e^{i\lambda xj\mathcal{K}} = I_{2p} + i\lambda xj\mathcal{K}
\]

for the case \( c = 0 \).

\section{Matrix string equation}

Consider again the case of the initial canonical systems

\[
w'(x, \lambda) = i\lambda jH(x)w(x, \lambda),
\]

where \( H(x) = \beta(x)^*\beta(x) \) and \( \beta(x) \) are \( p \times 2p \) matrix functions. According to (3.14), the transformed Hamiltonians (of the GBDT-transformed canonical systems (3.15)) have the form

\[
\tilde{H}(x) = \tilde{\beta}(x)^*\tilde{\beta}(x), \quad \tilde{\beta}(x) = \beta(x)w_A(x, 0). \tag{6.1}
\]
When the matrix functions $\beta(x)$ have the form presented in (4.1) (and (4.2), (5.3) hold), our assertions below show (in view of Appendix B) that the considered transformed canonical systems correspond to a special subclass of string equations. Thus, our explicit formulas may be transferred for the case of string equations as explained in Remark B.3.

**Proposition 6.1** Let $\beta(x)$ satisfy the equality

$$\beta(x)j\beta(x)^* = 0. \quad (6.2)$$

Then, $\tilde{\beta}(x)$ satisfies the relations

$$\tilde{\beta}(x)j\tilde{\beta}(x)^* = 0, \quad \tilde{\beta}(x)'j\tilde{\beta}(x)^* = \beta(x)'j\beta(x)^*. \quad (6.3)$$

**Proof.** Recall (see, e.g., (3.8)) that

$$w_A(x, 0)jw_A(x, 0)^* = j. \quad (6.4)$$

The first equality in (6.3) easily follows from (6.4) (and was already stated in (5.2)). Formulas (3.10), (6.4) and the second equality in (6.1) imply that

$$\tilde{\beta}(x)'j\tilde{\beta}(x)^* = \beta(x)'w_A(x, 0)jw_A(x, 0)^*\beta(x)^* + \beta(x)w_A'(x, 0)jw_A(x, 0)^*\beta(x)^*$$

$$= \beta(x)'j\beta(x)^* - \beta(x)\tilde{q}_0(x)j\beta(x)^*. \quad (6.5)$$

The definition (3.6) of $\tilde{q}_0$ and the equality (6.2) yield

$$\beta(x)\tilde{q}_0(x)j\beta(x)^* = 0. \quad (6.6)$$

The second equality in (6.3) is immediate from (6.5) and (6.6). $\blacksquare$

**Corollary 6.2** Let $\beta(x)$ be given by (4.1), where $c = \frac{1}{2}$ and $\alpha\alpha^* = I_p$. Then, (6.2) holds and

$$\tilde{\beta}(x)'j\tilde{\beta}(x)^* = iI_p. \quad (6.7)$$
7 Examples and applications

1. Let us consider explicit examples of the Hamiltonians \( \tilde{H}(x) = \tilde{\beta}(x)\ast \tilde{\beta}(x) \), corresponding Darboux matrices \( u(x, \lambda) \), fundamental solutions \( \tilde{W}(x, \lambda) \), and Weyl functions \( \varphi(\lambda) \).

Example 7.1 In our first example, we assume that

\[
p = n = 1, \quad A = a \neq \pi \quad (a \in \mathbb{C}), \quad c \neq 0, \quad d = 0, \quad (7.1)
\]

where the condition \( a \neq \pi \) provides an easy recovery of \( S(x) \) from (3.3).

Recall that according to the second equalities in (4.1), (4.2), and (4.7), we have

\[
\beta(x) := [e^{i\alpha} e^{-i\alpha}] \quad |\alpha| = 1, \quad Q = \sqrt{2ac + c^2}. \quad (7.2)
\]

In order to define the sign of the square root above, we assume that \( \Im(Q) > 0 \).

By virtue of (4.5), (4.6)) and (7.1), we obtain

\[
a\Lambda(x) = [a(f_1e^{ixQ} + f_2e^{-ixQ}) \quad a((a + c + Q)f_1 e^{ixQ} + (a + c - Q)f_2 e^{-ixQ})] \\
\times e^{i\alpha}, \quad (7.3)
\]

where \( f_1 \) and \( f_2 \) are scalars and \( a\Lambda \) is written down more conveniently than \( \Lambda \). It follows from (3.3) and (7.3) that

\[
S(x) = \frac{i}{a - \pi} \left( |f_1 e^{ixQ} + f_2 e^{-ixQ}|^2 \\
- \frac{1}{|a|^2} \left| (a + c + Q)f_1 e^{ixQ} + (a + c - Q)f_2 e^{-ixQ} \right|^2 \right), \quad (7.4)
\]

and the requirement \( S(0) > 0 \) takes the form

\[
i(\bar{a} - a)(|a(f_1 + f_2)|^2 - |(a + c + Q)f_1 + (a + c - Q)f_2|^2) > 0. \quad (7.5)
\]

Relations (3.4), (5.2) and (7.1)–(7.3) yield

\[
\tilde{\beta}(x) = \beta(x) - \frac{i}{a|a|^2S(x)} \left( \bar{a}(f_1 e^{-ix\tilde{Q}} + f_2 e^{ix\tilde{Q}}) \\
- \bar{\pi}(a + c + Q)f_1 e^{-ix\tilde{Q}} + (a + c - Q)f_2 e^{ix\tilde{Q})} \right) (a\Lambda(x)). \quad (7.6)
\]
According to (3.17) and (7.1), the corresponding Darboux matrix is given by the formula
\[
v(x, \lambda) = I_2 - \frac{i\lambda}{a[a^2(a - \lambda)S(x)]} j(a\Lambda(x)) (a\Lambda(x))^*.
\] (7.7)

Formulas (5.7), (5.14) and (7.3), (7.7) give explicitly fundamental solutions of the canonical systems with \(\tilde{\beta}\) of the form (7.6). In view of (5.16) and (7.7), the Weyl functions \(\varphi\) of such canonical systems on \([0, \infty)\) have the form:
\[
\varphi(\lambda) = \psi_1(\lambda)/\psi_2(\lambda),
\] (7.8)
\[
\psi_1(\lambda) = a[a^2S(0)(a - \lambda)(\lambda + c - z_1(\lambda))]
+ i\alpha((\overline{\alpha} + c + Q)f_1 + (\overline{\alpha} + c - Q)f_2)\vartheta(\lambda),
\] (7.9)
\[
\psi_2(\lambda) = a[a^2S(0)(\lambda - a)\lambda - i\alpha(f_1 + f_2)\vartheta(\lambda),
\] (7.10)

where \(z_1(\lambda) = \sqrt{c(2\lambda + c)}\) \((\Im(z_1) > 0)\),
\[
h(\lambda) := a\Lambda(0)E_1(\lambda) = a((a + c + Q)f_1 + (a + c - Q)f_2)(\lambda + c - z_1(\lambda))
- a\alpha(f_1 + f_2)\lambda.
\] (7.11)

Example 7.2

Now, assume that
\[
p = 1, \quad n = 2, \quad c = 0, \quad d = 0, \quad A = \begin{bmatrix} \xi & a \\ 0 & \xi \end{bmatrix} \quad (\xi \in \mathbb{R}, \ \xi \neq 0),
\] (7.12)
\[
Q = \begin{bmatrix} 0 & q \\ 0 & 0 \end{bmatrix}, \quad f_1 = \begin{bmatrix} f \\ 0 \end{bmatrix}, \quad f_2 = \begin{bmatrix} 0 \\ g \end{bmatrix}; \quad q, f, g \in \mathbb{C}, \quad f \neq 0, \quad g \neq 0.
\] (7.13)

In this case, we have
\[
\beta = \begin{bmatrix} 1 & \alpha \end{bmatrix}, \quad e^{\pm ixQ} = I_2 \pm ixQ,
\] (7.14)
\[
(A - \lambda I_2)^{-1} = (\xi - \lambda)^{-1}I_2 - (\xi - \lambda)^{-2}\begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix}.
\] (7.15)

Hence, formulas (4.5), (4.6) and simple calculations yield
\[
\Lambda(x) = \begin{bmatrix} f - igg & -gg(ix + \xi^{-1}) \\ g & g \end{bmatrix}
\] (7.16)
In view of (7.16), the required matrix identity (3.1) may be written in the form

\[
\begin{bmatrix}
a S_{21}(0) - \pi S_{12}(0) & a S_{22}(0) \\
-\overline{a} S_{22}(0) & 0
\end{bmatrix}
= i \begin{bmatrix}
|f|^2 - |g q \xi^{-1}|^2 & \overline{f}(f + g q \xi^{-1}) \\
g(f + g q \xi^{-1}) & 0
\end{bmatrix},
\]
where \(S_{ik}\) are the entries of \(S\). Hence, we cannot choose an arbitrary entry \(a\) in \(A\) but demand \(f + g q \xi^{-1} \neq 0\) and choose \(a\) and \(S_{22}(0)\) satisfying the following conditions (which is always possible):

\[
a S_{22}(0) = \overline{f}(f + g q \xi^{-1}), \quad a \neq 0, \quad S_{22}(0) > 0.
\]

Next, we choose \(S_{12}(0)\) (and so \(S_{21}(0) = \overline{S_{12}(0)}\)) such that (7.17) holds, and we choose such \(S_{11}(0) > 0\) that \(S(0) > 0\).

Since \(\xi \in \mathbb{R}\), we cannot use (3.3) in order to recover \(S(x)\) from \(\Lambda(x)\) and construct \(S(x)\) in a different way. It follows from (4.1), (7.12) and (7.16) that

\[
\Lambda(x) j \beta^* = \begin{bmatrix} C_1 x + C_2 \\ C_3 \end{bmatrix}, \quad C_1 = i(\pi - 1)q g, \quad C_2 = f + \pi q g \xi^{-1}, \quad C_3 = g(1 - \alpha).
\]

Therefore, the second equality in (3.2) yields

\[
\begin{aligned}
S(x) &= S(0) + \int_0^x \Lambda(t) j \beta^* (\Lambda(t) j \beta^*)^* dt \\
&= S(0) + \left[ \frac{1}{3} |C_1|^2 x^3 + \Re(C_1 C_2^*) x^2 + |C_2|^2 x - \frac{1}{2} C_1 C_3^* x^2 + C_2 C_3 x \right].
\end{aligned}
\]

Using (7.19), we rewrite the equality (5.2) for \(\tilde{\beta}\) (transformed \(\beta\)) in the form

\[
\tilde{\beta}(x) = \begin{bmatrix} 1 & \alpha \end{bmatrix} - i \left[ C_1 x + C_2 \overline{C_3} \right] S(x)^{-1} A^{-1} \Lambda(x),
\]
where \(S(x), A\) and \(\Lambda(x)\) are given in (7.21), (7.12) and (7.16), respectively. Finally, the Darboux matrix \(v(x, \lambda)\) is expressed via \(\Lambda(x)\) and \(S(x)\) in (3.17), and the expression for the corresponding fundamental solution \(\tilde{W}\) follows from (5.14) and Remark 5.3.
2. Relations (3.2) and (3.3) imply an important equality (see [42, (2.13)]):

\[(\Lambda^* S^{-1})' = i H j \Lambda^* S^{-1} A + \tilde{q}_0^* \Lambda^* S^{-1}.\] (7.23)

We assume that \(S(0) > 0\) and \(H(x) \geq 0\), that is, \(S(x) > 0\) for \(x \geq 0\), and so \(S(x)^{-1}\) is well defined (see Remark 3.2). In view of (3.8), (3.10) and (7.23), for \(\tilde{H}\) of the form (3.14) and \(Y\) given by

\[Y(x, t) = j w_A(x, 0)^* \Lambda(x)^* S(x)^{-1} e^{i t A},\] (7.24)

we have

\[\tilde{H}(x) \frac{\partial}{\partial t} Y(x, t) = j \frac{\partial}{\partial x} Y(x, t) \ (m_1 = m_2 = p), \ x \geq 0.\] (7.25)

In other words, the \(2p \times n\) matrix function \(Y\) (or, equivalently, the columns of \(Y\)) satisfies the dynamical canonical system (7.25).

Taking into account (3.3) and (3.4), we rewrite \(w_A(x, 0)^* \Lambda(x)^* S(x)^{-1}\) in a simpler form (in terms of \(\Lambda(x)\) and \(S(x)\)):

\[w_A(x, 0)^* \Lambda(x)^* S(x)^{-1} = \Lambda(x)^* (A^*)^{-1} S(x)^{-1} A^{-1}.\]

Hence,

\[Y(x, t) = j \Lambda(x)^* (A^*)^{-1} S(x)^{-1} e^{i t A} A^{-1}.\] (7.26)

**Proposition 7.3** Let the initial Hamiltonian \(H(x) \geq 0\) be given, and let the relations (3.1), \(S(0) > 0\), and \(\det A \neq 0\) hold. Then, \(Y\) of the form (7.26) satisfies dynamical canonical system (7.25), where the transformed Hamiltonian \(\tilde{H}\) is given by (3.14).

In this way, explicit expressions for \(\Lambda\) and \(S\) in Examples 7.1 and 7.2 give us explicit expressions for \(Y(x, t)\). Moreover, it is immediate from (7.12) that \(e^{i t A}\) in (7.26) takes under assumptions of Example 7.2 a simple form

\[e^{i t A} = e^{i t \xi} \left( I_2 + i t a \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \right).\] (7.27)
A Canonical systems: Weyl–Titchmarsh theory

Consider generalized canonical system (1.2). It is immediate that the fundamental solution $W$ of (1.2) satisfies the equality

$$\frac{d}{dx}(W(x, \mu)^*jW(x, \lambda)) = i(\lambda - \mu)W(x, \lambda)^*H(x)W(x, \lambda).$$  \hfill (A.1)

In view of (A.1) (for the case $\mu = \lambda$) and of the normalization $W(0, \lambda) = I_m$ always assumed in this appendix, we have

$$\int_0^r W(x, \lambda)^*H(x)W(x, \lambda)dx = \frac{i}{\lambda - \bar{\lambda}}(j - W(r, \lambda)^*jW(r, \lambda)), \quad \hfill (A.2)$$

for $\lambda \not\in \mathbb{R}$ and $r \geq 0$. Moreover, (A.1) for the case $\mu = \lambda$ implies that

$$W(r, \lambda)^*jW(r, \lambda) \equiv j \equiv W(r, \lambda)jW(r, \lambda)^*. \hfill (A.3)$$

Further in the appendix, we will deal with the general-type (i.e., not necessarily related to explicit solutions) canonical system (1.4) on $[0, \infty)$. Since $H \geq 0$, formula (A.2) yields

$$W(r_2, \lambda)^*jW(r_2, \lambda) \leq W(r_1, \lambda)^*jW(r_1, \lambda) \leq j, \quad \hfill (A.4)$$

$$j \leq W(r_1, \lambda)^*jW(r_1, \lambda) \leq W(r_2, \lambda)^*jW(r_2, \lambda) \quad (r_1 \leq r_2, \quad \lambda \in \mathbb{C}_+). \quad \hfill (A.5)$$

Next, introduce the families $\mathcal{N}(r)$ of linear-fractional (Möbius) transformations

$$\phi(r, \lambda) = (\mathcal{W}_{21}(r, \lambda)\mathcal{P}_1(\lambda) + \mathcal{W}_{22}(r, \lambda)\mathcal{P}_2(\lambda))$$

$$\times (\mathcal{W}_{11}(r, \lambda)\mathcal{P}_1(\lambda) + \mathcal{W}_{12}(r, \lambda)\mathcal{P}_2(\lambda))^{-1}, \hfill (A.6)$$

where $\mathcal{W}_{ik}$ and $\mathcal{P}_k$ are $p \times p$ matrix functions,

$$\mathcal{W}(r, \lambda) = \{\mathcal{W}_{ik}(r, \lambda)\}_{i,k=1}^2 := jW(r, \lambda)^*j, \hfill (A.7)$$

and $\mathcal{P}_1(\lambda), \mathcal{P}_2(\lambda)$ are pairs of meromorphic in $\mathbb{C}_+$ matrix functions (so called nonsingular pairs with property-$j$) such that

$$\mathcal{P}_1(\lambda)^*\mathcal{P}_1(\lambda) + \mathcal{P}_1(\lambda)^*\mathcal{P}_1(\lambda) > 0, \quad \begin{bmatrix} \mathcal{P}_1(\lambda)^* & \mathcal{P}_2(\lambda)^* \end{bmatrix} \begin{bmatrix} \mathcal{P}_1(\lambda) \\ \mathcal{P}_2(\lambda) \end{bmatrix} \geq 0, \hfill (A.8)$$
where the first inequality holds in one point (at least) of $\mathbb{C}_+$, and the second inequality holds in all the points of analyticity of $\mathcal{P}_k$ ($k = 1, 2$). It follows from (A.8) by contradiction that

$$\det \left( \mathcal{W}_{11}(r, \lambda) \mathcal{P}_1(\lambda) + \mathcal{W}_{12}(r, \lambda) \mathcal{P}_2(\lambda) \right) \neq 0. \quad \text{(A.9)}$$

Indeed, formulas (A.3), (A.4) and (A.7) imply that $\mathcal{W}(r, \lambda)^* j \mathcal{W}(r, \lambda) \geq j$, which yields

$$\left[ \mathcal{P}_1(\lambda)^* \mathcal{P}_2(\lambda)^* \right] \mathcal{W}(r, \lambda)^* j \mathcal{W}(r, \lambda) \left[ \begin{array}{c} \mathcal{P}_1(\lambda) \\ \mathcal{P}_2(\lambda) \end{array} \right] \geq 0 \quad \text{(A.10)}$$

for such $\lambda$ in $\mathbb{C}_+$ that (A.8) holds. On the other hand, if we have

$$\det \left( \mathcal{W}_{11}(r, \lambda) \mathcal{P}_1(\lambda) + \mathcal{W}_{12}(r, \lambda) \mathcal{P}_2(\lambda) \right) = 0,$$

then (for some $g \neq 0$)

$$(\mathcal{W}_{11}(r, \lambda) \mathcal{P}_1(\lambda) + \mathcal{W}_{12}(r, \lambda) \mathcal{P}_2(\lambda)) g = 0 \quad (g \in \mathbb{C}^p), \quad \text{(A.11)}$$

and so we obtain

$$g^* \left[ \mathcal{P}_1(\lambda)^* \mathcal{P}_2(\lambda)^* \right] \mathcal{W}(r, \lambda)^* j \mathcal{W}(r, \lambda) \left[ \begin{array}{c} \mathcal{P}_1(\lambda) \\ \mathcal{P}_2(\lambda) \end{array} \right] g < 0. \quad \text{(A.12)}$$

Clearly, (A.12) contradicts (A.10).

Let us rewrite (A.6) in the form

$$\left[ \begin{array}{c} I_p \\ \phi(\lambda) \end{array} \right] = j \mathcal{W}(r, \lambda)^* j \mathcal{W}(r, \lambda) \left[ \begin{array}{c} \mathcal{P}_1(\lambda) \\ \mathcal{P}_2(\lambda) \end{array} \right] (\mathcal{W}_{11}(r, \lambda) \mathcal{P}_1(\lambda) + \mathcal{W}_{12}(r, \lambda) \mathcal{P}_2(\lambda))^{-1}. \quad \text{(A.13)}$$

Now, setting

$$\mathfrak{A}(r, \lambda) := \mathcal{W}(r, \lambda)^* j \mathcal{W}(r, \lambda),$$

and using (A.3), we see that formulas (A.6)–(A.8) (i.e., the relation $\phi \in \mathcal{N}(r)$) yield

$$\left[ I_p \phi(\lambda)^* \right] \mathfrak{A}(r, \lambda) \left[ I_p \phi(\lambda) \right] \geq 0. \quad \text{(A.14)}$$
Moreover, according to (A.4), (A.13) and (A.14), \( \phi(\lambda) \) is holomorphic and contractive in \( \mathbb{C}_+ \). On the other hand, if \( \phi \) satisfies (A.14), we set

\[
\begin{bmatrix}
\mathcal{P}_1(\lambda) \\
\mathcal{P}_2(\lambda)
\end{bmatrix} = W(r, \lambda) \begin{bmatrix} I_p \\ \phi(\lambda) \end{bmatrix},
\]  
(A.15)

and see that the relations (A.6)–(A.8) are valid. Thus,

\[
\phi(\lambda) \in \mathcal{N}(r)
\]  
(A.16)

is equivalent to (A.14). Therefore, according to (A.4), \( \mathcal{N}(r_2) \) is embedded in \( \mathcal{N}(r_1) \):

\[
\mathcal{N}(r_2) \subseteq \mathcal{N}(r_1) \quad (r_1 < r_2).
\]  
(A.17)

By virtue of Montel’s theorem, there is a sequence \( \{\phi_k(\lambda)\} \) such that

\[
\phi_k \in \mathcal{N}(r_k), \quad r_k \to \infty \quad (\text{for } k \to \infty),
\]  
(A.18)

and \( \phi_k(\lambda) \) tend uniformly (on any compact in \( \mathbb{C}_+ \)) to some matrix function \( \varphi(\lambda) \). Thus, \( \varphi(\lambda) \) is holomorphic and satisfies (A.14) for any \( r > 0 \). In other words,

\[
\varphi(\lambda) \in \bigcap_{r>0} \mathcal{N}(r).
\]  
(A.19)

Let us write down \( \mathcal{N}(r) \) in the Weyl matrix disk form. Taking into account (A.4) and (A.13), we obtain

\[
-A_{22}(r_2, \lambda) \geq -A_{22}(r_1, \lambda) \geq I_p \quad (r_2 > r_1);
\]  
(A.20)

\[
A(r_2, \lambda)^{-1} \geq A(r_1, \lambda)^{-1} \geq j,
\]  
(A.21)

\[
(A(r, \lambda)^{-1})_{11} = (A_{11}(r, \lambda) - A_{12}(r, \lambda)A_{22}(r, \lambda)^{-1}A_{21}(r, \lambda))^{-1} \geq I_p,
\]  
(A.22)

where \( A_{ik}(r, \lambda) \) and \( (A(r, \lambda)^{-1})_{ik} \), respectively, are \( p \times p \) blocks of \( A(r, \lambda) \) and \( A(r, \lambda)^{-1} \). The invertibility of \( A_{11} - A_{12}A_{22}^{-1}A_{21} \) in (A.22) follows from the invertibility of \( A \) and \( A_{22} \) (for this and for the equality in (A.22) see [50, p. 21]). In particular, we derive from (A.20) and (A.22) that the following positive definite square roots are uniquely defined:

\[
\rho_L(r, \lambda) = (-A_{22}(r_2, \lambda)^{-1})^{1/2},
\]  
(A.23)

\[
\rho_R(r, \lambda) = (A_{11}(r, \lambda) - A_{12}(r, \lambda)A_{22}(r, \lambda)^{-1}A_{21}(r, \lambda))^{1/2}.
\]  
(A.24)
Here, $\rho_L$ and $\rho_R$ are the so called left and right semi-radii of the Weyl disk. The inequality (A.14) may be rewritten in the form of the Weyl disk parametrization of the values $\phi(r, \lambda)$ (similar, e.g., to the parametrization [14, (2.19)] for Dirac systems):

$$\phi(r, \lambda) = \rho_L(r, \lambda)\omega(r, \lambda)\rho_R(r, \lambda) - \mathfrak{A}_{22}(r, \lambda)^{-1}\mathfrak{A}_{21}(r, \lambda) \quad (\omega^* \omega \leq I_p), \quad (A.25)$$

where $\omega(r, \lambda)$ are $p \times p$ matrices and $\phi \in \mathcal{N}_p$. Recall that the matrix inequality $B_2 \geq B_1 \geq 0$ yields $B_2^{1/2} \geq B_1^{1/2}$ (see, e.g., [3]). Hence, in view of (A.20)–(A.24) the left and right semi-radii are non-increasing.

By $L^2(H)$ we denote the space of vector functions on $\mathbb{R}_+$ with the scalar product

$$(f_1, f_2)_H = \int_0^\infty f_2(x)^*H(x)f_1(x)dx.$$ 

**Proposition A.1** Let $H(x)$ ($x \geq 0$) be the Hamiltonian of a canonical system. Then, there is $\varphi(\lambda)$, which satisfies (A.19) for this system. If (A.19) holds, the columns of $W(x, \lambda) \begin{bmatrix} I_p \\ \varphi(\lambda) \end{bmatrix}$ belong $L^2(H)$, that is,

$$\int_0^\infty \begin{bmatrix} I_p & \varphi(\lambda)^* \end{bmatrix} W(x, \lambda)^*H(x)W(x, \lambda) \begin{bmatrix} I_p \\ \varphi(\lambda) \end{bmatrix} dx < \infty \quad (\lambda \in \mathbb{C}_+). \quad (A.26)$$

**Proof.** We already proved that $\bigcap_{r>0} N(r)$ is non-empty. Moreover, in view of (A.14), for any $\varphi$ satisfying (A.19) and any $r > 0$ we have

$$\begin{bmatrix} I_p & \varphi(\lambda)^* \end{bmatrix} \mathfrak{A}(r, \lambda) \begin{bmatrix} I_p \\ \varphi(\lambda) \end{bmatrix} \geq 0. \quad (A.27)$$

Taking into account (A.2) and (A.27), we derive

$$\int_0^r \begin{bmatrix} I_p & \varphi(\lambda)^* \end{bmatrix} W(x, \lambda)^*H(x)W(x, \lambda) \begin{bmatrix} I_p \\ \varphi(\lambda) \end{bmatrix} dx$$

$$\leq \frac{i}{\lambda - \bar{\lambda}} \begin{bmatrix} I_p & \varphi(\lambda)^* \end{bmatrix} j \begin{bmatrix} I_p \\ \varphi(\lambda) \end{bmatrix} \leq \frac{i}{\lambda - \bar{\lambda}} I_p, \quad (A.28)$$

and (A.26) follows. $\blacksquare$
Definition A.2 Holomorphic (in $\mathbb{C}_+$) $p \times p$ matrix functions $\varphi(\lambda)$, such that the inequality (A.26) holds, are called Weyl–Titchmarsh (Weyl) functions of the canonical system (1.4) on $[0, \infty)$.

Proposition A.1 implies that Weyl function always exists.

B Canonical systems and matrix string and Schrödinger equations: interconnections

1. In view of (1.3), canonical systems (1.4) with Hamiltonians $H(x)$ of the form (1.6) may be transformed into systems (1.1) with Hamiltonians $\mathcal{H}$:

$$W'(x, \lambda) = i\lambda JH(x)W(x, \lambda), \quad \mathcal{H} = \vartheta(x)^*\vartheta(x), \quad \vartheta(x)J\vartheta(x)^* = 0, \quad (B.1)$$

using the transformation

$$W = \Theta w, \quad \mathcal{H} = \Theta H\Theta^*, \quad \vartheta(x) = \beta^* \Theta^*.$$

Clearly, the inverse transformation works as well, that is, systems (1.4), (1.6) and systems (B.1) are equivalent.

It will be convenient to repeat here the transformation (from [48, Ch. 4] or [50, Section 11.1]) of the system (B.1) into the matrix string equation. We partition $p \times 2p$ matrix function $\vartheta(x)$ into $p \times p$ blocks $\vartheta(x) = [\vartheta_1(x) \quad \vartheta_2(x)]$.

We assume that $\det(\vartheta_1(x)) \neq 0$, and we also require that $\vartheta_1(x)^{-1}\vartheta_2(x)$ is absolutely continuous and its derivative is invertible. We set

$$\mathcal{Y}(x, \lambda) = \vartheta(x)W(x, \lambda), \quad \mathcal{Z}(x, \lambda) = \vartheta_1(x)^{-1}\vartheta(x)W(x, \lambda), \quad (B.3)$$

$$\mathcal{X}(x) := \left(i(\vartheta_1(x)^{-1}\vartheta_2(x))^\prime\right)^{-1} = \mathcal{X}(x)^*.$$  

(B.4)

The self-adjointness of $\mathcal{X}(x)$ above follows from the last equality in (B.1).

According to (B.1) and (B.3), we have

$$W'(x, \lambda) = i\lambda J\vartheta(x)^*\mathcal{Y}(x, \lambda), \quad (B.5)$$

$$\mathcal{Z}'(x, \lambda) = (\vartheta_1(x)^{-1}\vartheta(x))^\prime W(x, \lambda) = \left[0 \quad (\vartheta_1(x)^{-1}\vartheta_2(x))^\prime\right] W(x, \lambda). \quad (B.6)$$
Finally, taking into account (B.4)–(B.6), we see that $\mathcal{Z}(x, \lambda)$ satisfies the matrix string equation

$$
\frac{d}{dx} \left( \kappa(x) \frac{d}{dx} \mathcal{Z}(x, \lambda) \right) = \lambda \vartheta_1(x)^* \mathcal{Y}(x, \lambda) = \lambda \omega(x) \mathcal{Z}(x, \lambda),
$$

(B.7)

$$
\omega(x) := \vartheta_1(x)^* \vartheta_1(x) > 0.
$$

(B.8)

2. Now, consider the matrix Schrödinger equation

$$
- \mathcal{Z}''(x, \lambda) + u(x) \mathcal{Z}(x, \lambda) = \lambda \mathcal{Z}(x, \lambda) \quad (u(x) = u(x)^*),
$$

(B.9)

where $u$ is a $p \times p$ matrix function. The transformation of (B.9) into the canonical system of the form (B.1), such that

$$
\vartheta''(x) = u(x) \vartheta(x),
$$

(B.10)

and $\vartheta(x)$ is normalized at $x = 0$ by

$$
B(0) = \Theta := \frac{1}{\sqrt{2}} \begin{bmatrix} I_p & I_p \\ -I_p & I_p \end{bmatrix},
$$

(B.11)

where

$$
B(x) := \begin{bmatrix} \vartheta(x) \\ \vartheta'(x) \end{bmatrix},
$$

(B.12)

is described in [50, Section 11.2]. The interconnections between the spectral theories of systems (B.1), (B.10) and equations (B.9) are also studied there. It is easily checked (see also [34] for the case $p = 1$) that the above-mentioned transformation in [50, Section 11.2] works in the opposite direction as well.

Namely, starting from the canonical system (B.1), (B.10), (B.11) one comes to the Schrödinger equation (B.9). Indeed, according to [50, (2.10)]

$$
B(0)^* J_1 B(0) = \Theta_1^* J_1 \Theta_1 = J,
$$

(B.13)

$$
J_1 := i \begin{bmatrix} 0 & -I_p \\ I_p & 0 \end{bmatrix},
$$

where $J$ is given in (1.1). Moreover, the equalities (B.10) and (B.12) yield

$$
B'(x) = \begin{bmatrix} 0 & I_p \\ u(x) & 0 \end{bmatrix} B(x).
$$

(B.14)
The relations (B.13) and (B.14) imply that
\[ B(x)^*J_1B(x) = B(0)^*J_1B(0) = J, \]  
(B.15)
and so
\[ B(x)JB(x)^* = J. \]  
(B.16)

**Remark B.1** Formula (B.16) shows that the equalities
\[ \vartheta(x)J\vartheta(x)^* = 0, \quad \vartheta'(x)J\vartheta(x)^* = iI_p \]  
(B.17)
follow from (B.10)–(B.12).

Finally, setting
\[ Z(x,\lambda) = \vartheta(x)W(x,\lambda) \]  
(B.18)
and taking into account (B.1), (B.10) and (B.17), we derive
\[ Z'' = uZ - 2\lambda Z + \lambda Z, \]
that is, \( Z(x,\lambda) \) satisfies matrix Schrödinger equation (B.9). Formula (B.18) describes the connection between the solutions of the canonical system (B.1), (B.10), (B.11) and of the corresponding Schrödinger equation (B.9).

3. Since matrix Schrödinger equations may be transformed (see [50]) into canonical systems satisfying (B.10)–(B.12) (and by virtue of Remark B.1), they are also equivalent to a subclass of canonical systems (1.4) with Hamiltonians of the form (2.1).

**Remark B.2** It is easy to see that in the case of our explicit formulas (4.1), (4.5), (4.6) the matrix function \( \vartheta(x) = \beta(x)\Theta^* \) satisfies (B.10), where \( u = -c^2I_p \). However, \( \tilde{\vartheta}(x) = \tilde{\beta}(x)\Theta^* \) does not satisfy (B.10) (excluding, possibly, some special cases).

**Remark B.3** Formulas (B.3), (B.4), (B.7), and (B.8) show that explicit expressions for the Hamiltonians and fundamental solutions constructed in this paper generate explicit expressions for the matrix string equations and their solutions as well.
C On linear similarity to squared integration

We will consider similarity transformations of linear integral operators $K$ in $L^2_2(0, T)$ ($0 < T < \infty$):

$$K = i\beta(x)j \int_0^x \beta(t)^* \cdot dt, \quad \beta(x)j\beta(x)^* \equiv 0, \quad \beta'(x)j\beta(x)^* \equiv iI_p,$$

(C.1)

where $\beta(x)$ is a $p \times 2p$ matrix function and

$$j = \begin{bmatrix} I_p & 0 \\ 0 & -I_p \end{bmatrix}.$$  

(C.2)

Recall that the operator $A$ is introduced in (2.2). The class of operators $K = \int_0^x K(x, t) \cdot dt$, which are linear similar to $A$ above, was studied (for the case of the scalar kernel function $K(x, t)$) in the essential for our considerations paper [46]. Here, we study an important special subclass (C.1) of such operators under reduced smoothness conditions on $K(x, t)$. We include the matrix case (i.e., the case $p > 1$) and present a complete proof of the similarity result.

**Theorem C.1** Let operator $K$ be given by the first equality in (C.1), and let $\beta(x)$ satisfy the second and third equalities in (C.1). Assume that $\beta(x)$ is two times differentiable and the entries of $\beta''(x)$ are square-integrable, that is, $\beta''(x) \in L^{p \times 2p}_2(0, T)$. Then, $K$ is linear similar to $A$:

$$K = VAV^{-1}, \quad V = u(x)(I + \int_0^x V(x, t) \cdot dt),$$

(C.3)

where $u(x)$ is a $p \times p$ matrix function, which is unitary (i.e., $u^* = u^{-1}$) and absolutely continuous on $[0, T]$, and

$$\sup \|V(x, t)\| < \infty \quad (0 \leq t \leq x \leq T).$$

(C.4)

**Proof.** In the proof, we construct an operator $V$, which satisfies theorem’s conditions. This $V$ is closely related to *transformation operators in inverse spectral and scattering theories.*

Step 1. Together with $K$, we consider the operators:

$$\tilde{K} := i\beta''(x)j \int_0^x \beta(t)^* \cdot dt, \quad (I - \tilde{K})^{-1} = \int_0^x R(x, t) \cdot dt.$$  

(C.5)
The operator $\tilde{K}$ has a semi-separable kernel, and so (see, e.g., [19, Section IX.2]) the matrix function $R$ in (C.5) has the form

$$R(x,t) = i\beta''(x)u_1(x)u_1(t)^{-1}j\beta(t)^* \quad (0 \leq t \leq x),$$

where the $2p \times 2p$ matrix function $u_1$ is the normalized fundamental solution of the system

$$u_1'(x) = ij\beta(x)^*\beta''(x)u_1(x), \quad u_1(0) = I_{2p}. \quad \text{(C.7)}$$

Introduce the $p \times p$ matrix function $g(x)$ by the equalities

$$g(0) = I_p, \quad g'(0) = \frac{i}{2}\beta'(0)j\beta'(0)^*, \quad \text{(C.8)}$$

$$g''(x) = i(I - \tilde{K})\left(\beta''(x)u_1(x)(j\beta'(0)^* + \frac{i}{2}j\beta(0)^*\beta'(0)j\beta'(0)^*)\right),$$

where the operator $(I - \tilde{K})$ on the right-hand side of (C.9) is applied columnwise to the $p \times p$ matrix function above. Further in the proof, we study the matrix function

$$y(x,z) := (I - z^2K)^{-1}g(x). \quad \text{(C.10)}$$

(Since $K$ has a semi-separable kernel, one can write down a more explicit expression for $y$ as well.) Differentiating two times both parts of the equality $(I - z^2K)y(x,z) = g(x)$ and taking into account (C.1), (C.5), we derive

$$y''(x,z) = g''(x) - z^2y(x,z) + i z^2\beta''(x)j \int_0^x \beta(t)^*y(t,z)dt$$

$$= g''(x) - z^2(I - \tilde{K})y(x,z). \quad \text{(C.11)}$$

It also easily follows from (C.1), (C.8), and (C.10) that

$$y(0,z) = I_p, \quad y'(0,z) = \frac{i}{2}\beta'(0)j\beta'(0)^*. \quad \text{(C.12)}$$

Applying (columnwise) $(I - \tilde{K})^{-1}$ to both parts of (C.11), using (C.5)
and (C.9), and integrating by parts, we obtain

\[ y''(x, z) = - \int_0^x R(x, t)y''(t, z)dt \]

\[ + i\beta''(x)u_1(x)(j\beta'(0)^* + \frac{i}{2}j\beta(0)^*\beta'(0)^*) - z^2 y(x, z) \]

\[ = - R(x, t)y'(t, z)\bigg|_0^x + \left( \frac{\partial}{\partial t}R(x, t)y(t, z) \right)\bigg|_0^x \]

\[ + i\beta''(x)u_1(x)(j\beta'(0)^* + \frac{i}{2}j\beta(0)^*\beta'(0)^*) - z^2 y(x, z) \]

\[ - \int_0^x \left( \frac{\partial^2}{\partial t^2}R(x, t) \right)y(t, z)dt. \] \hspace{1cm} (C.13)

In view of the last equality in (C.1), we have \((\beta'(x)j\beta(x)^*)' = 0\), which yields

\[ u_2(x) := i\beta''(x)j\beta(x)^* = -i\beta'(x)j\beta'(x)^*. \] \hspace{1cm} (C.14)

Here, \(u_2\) is a \(p \times p\) matrix function. Relations (C.6), (C.7), and (C.14) imply:

\[ R(x, x) = u_2(x); \quad \frac{\partial}{\partial t}R(x, t)\bigg|_{t=x} = -u_2(x)^2 + u_3(x), \] \hspace{1cm} (C.15)

\[ u_2(x)' = u_3(x)^* - u_3(x), \quad u_3(x) := i\beta''(x)j\beta'(x)^*; \] \hspace{1cm} (C.16)

\[ \frac{\partial}{\partial t}R(x, t)\bigg|_{t=0} = i\beta''(x)u_1(x)(j\beta'(0)^* - j\beta(0)^*u_2(0)); \quad (C.17) \]

\[ R(x, 0) = i\beta''(x)u_1(x)j\beta(0)^*. \] \hspace{1cm} (C.18)

Taking into account (C.12) and (C.14)–(C.17), we rewrite (C.13) in the form:

\[ y''(x, z) + z^2 y(x, z) = - u_2(x)y'(x, z) - (u_2(x)^2 - u_3(x))y(x, z) \]

\[ - \int_0^x \left( \frac{\partial^2}{\partial t^2}R(x, t) \right)y(t, z)dt, \] \hspace{1cm} (C.19)

where

\[ \frac{\partial^2}{\partial t^2}R(x, t) = h_1(x)h_2(t), \quad h_1(x) = i\beta''(x)u_1(x) \in L_2^{p \times 2p}(0, T), \] \hspace{1cm} (C.20)

\[ h_2(t) = u_1(t)^{-1}(j\beta(t)^*u_2(t)^2 - j\beta(t)^*u_3(t)^* - j\beta'(t)^*u_2(t) + j\beta''(t)^*), \]

\[ h_1 \in L_2^{p \times 2p}(0, T), \quad h_2 \in L_2^{2p \times p}(0, T). \] \hspace{1cm} (C.21)
We introduce the $p \times p$ matrix functions $y_1(x, z)$ and $u(x)$ by the equalities
\[ y_1(x, z) = u(x)^{-1}y(x, z); \quad u'(x) = -\frac{1}{2}u_2(x)u(x), \quad u(0) = I_p. \]  
(C.22)

Since $u_2^* = -u_2$ and $u(0) = I_p$, we obtain $u^* = u^{-1}$. Thus, using formulas (C.16), (C.20), and (C.22), we rewrite (C.19) in the form
\[ y''_1(x, z) + z^2y_1(x, z) = u_4(x)y_1(x, z) - u(x)^*h_1(x)\int_0^x h_2(t)u(t)y_1(t, z)dt, \]  
(C.23)
\[ u_4(x) := u(x)^*\left(\frac{1}{2}(u_3(x) + u_3(x)^*) - \frac{3}{4}u_2(x)^2\right)u(x). \]  
(C.24)

In view of (C.12) and (C.22), the initial conditions for $y_1$ take the form
\[ y_1(0, z) = I_p, \quad y'_1(0, z) = y'(0, z) - u'(0) = 0. \]  
(C.25)

Step 2. Let us construct the solution of system (C.23) with the initial conditions (C.25) as a series
\[ y_1(x, z) = \sum_{k=0}^{\infty} \psi_k(x, z), \quad \psi_0(x, z) = \cos(zx)I_p, \]  
(C.26)
\[ \psi_k(x, z) := \int_0^x \cos(z(x-t))\int_0^t \left(u_4(s)\psi_{k-1}(s, z) - \int_0^s F(s, \eta)\psi_{k-1}(\eta, z)d\eta\right)dsdt \quad (k \geq 1), \]  
(C.27)
\[ F(x, t) := u(x)^*h_1(x)h_2(t)u(t). \]  
(C.28)

Clearly, for $k \geq 1$ we have
\[ \psi''_k(x, z) = -z^2\psi_k(x, z) + u_4(x, z)\psi_{k-1}(x, z) - \int_0^x F(x, \eta)\psi_{k-1}(\eta, z)d\eta. \]  
(C.29)

Thus, if the corresponding series converge, the matrix function $y_1$ given by (C.26)–(C.28) satisfies (C.23). Convergences follow from the representation
\[ \psi_k(x, z) = \int_0^x \cos(z\zeta)\mathcal{V}_k(x, \zeta)d\zeta \quad (k \geq 1), \]  
(C.30)

35
which is proved by induction. Indeed, setting $k = 1$ in (C.27), taking into account that $\psi_0(s, z) = \cos(zs)$ and
\[
\cos(z(x - t)) \cos(zs) = \frac{1}{2} \left( \cos \left( z(x - t - s) \right) + \cos \left( z(x - t + s) \right) \right),
\]
\[
(C.31)
\]
\[
\cos(z(x - t)) \cos(z\eta) = \frac{1}{2} \left( \cos \left( z(x - t - \eta) \right) + \cos \left( z(x - t + \eta) \right) \right),
\]
\[
(C.32)
\]
and changing variables and order of integration, we derive:
\[
\psi_1(x, z) = \int_0^x \cos(z\zeta) \mathcal{V}_1(x, \zeta) d\zeta,
\]
\[
(C.33)
\]
\[
\mathcal{V}_1(x, \zeta) = \frac{1}{2} \left( \int_0^{(x + \zeta)/2} u_4(t) dt + \int_0^{(x - \zeta)/2} u_4(t) dt - \int_{(x + \zeta)/2}^x \ddot{F}(t, x - t + \zeta) dt 
\right.
\]
\[
- \int_{(x - \zeta)/2}^{x - \zeta} \ddot{F}(t, x - t - \zeta) dt - \int_{(x - \zeta)}^{x - \zeta} \ddot{F}(t, \zeta + t - x) dt \right),
\]
\[
(C.34)
\]
\[
\ddot{F}(t, \eta) := \int_{\eta}^t \mathcal{F}(s, \eta) ds.
\]
\[
(C.35)
\]
Assuming that (C.30) holds for $k - 1$ and $\psi_k$ is given by (C.27), we use a similar procedure (i.e., formulas of the (C.31) type, change of variables and change of order of integration) and obtain (C.30) for $k$, where
\[
2\mathcal{V}_k(x, \zeta) = \int_{x - \zeta}^{x} \int_{\zeta + t - x}^{t} u_4(s) \mathcal{V}_{k-1}(s, \zeta + t - x) ds dt 
\]
\[
+ \int_{(x + \zeta)/2}^{x} \int_{\zeta + x - t}^{t} u_4(s) \mathcal{V}_{k-1}(s, \zeta + x - t) ds dt 
\]
\[
+ \int_{(x - \zeta)/2}^{x - \zeta} \int_{x - t - \zeta}^{t} u_4(s) \mathcal{V}_{k-1}(s, x - t - \zeta) ds dt 
\]
\[
- \int_{x - \zeta}^{x} \int_{\zeta + t - x}^{t} \mathcal{F}(s, \eta) \mathcal{V}_{k-1}(\eta, \zeta + t - x) d\eta ds dt 
\]
\[
- \int_{(x + \zeta)/2}^{x} \int_{\zeta + x - t}^{t} \mathcal{F}(s, \eta) \mathcal{V}_{k-1}(\eta, \zeta + x - t) d\eta ds dt 
\]
\[
- \int_{(x - \zeta)/2}^{x - \zeta} \int_{x - t - \zeta}^{t} \mathcal{F}(s, \eta) \mathcal{V}_{k-1}(\eta, x - t - \zeta) d\eta ds dt.
\]
\[
(C.36)
\]
Thus, the representation (C.30) is proved. Moreover, one can choose such $C(T) = C > 0$ that
\begin{align*}
\int_0^T \|h_k(t)\| dt &\leq C \quad (k = 1, 2), \quad \int_0^T \|u_4(t)\| dt \leq C^2, \quad (C.37) \\
\sup_{0 \leq \zeta \leq x \leq T} \|V_1(x, \zeta)\| &\leq C. \quad (C.38)
\end{align*}

We assume that the inequalities (C.37) and (C.38) hold. In particular, the inequality
\begin{equation}
\|V_k(x, \zeta)\| \leq \frac{(3C^2)^{k-1}}{(k-1)!} Cx^{k-1} \quad (k \geq 1) \quad (C.39)
\end{equation}
is fulfilled for $k = 1$. If (C.39) is valid for $V_{k-1}$, relations (C.36)–(C.38) imply that (C.39) is valid for $V_k$. Hence, (C.39) is proved. Therefore, the series $\sum_{k=1}^{\infty} \|V_k(x, \zeta)\|$ is convergent. Thus, the series in (C.26) converges as well, and (in view of (C.26), (C.30), (C.39)) we have
\begin{align*}
y_1(x, z) &= \cos(zx)I_p + \int_0^x \cos(z\zeta) V(x, \zeta) d\zeta, \quad (C.40) \\
V(x, \zeta) &:= \sum_{k=1}^{\infty} V_k(x, \zeta), \quad \sup_{0 \leq \zeta \leq x \leq T} \|V(x, \zeta)\| < \infty. \quad (C.41)
\end{align*}

It is immediate from (C.40) and (C.41) that $y_1(0, z) = I_p$. In order to calculate the initial value $y'_1(0, z)$, we differentiate both sides of (C.27) and obtain
\begin{align*}
\psi'_k(x, z) &= -z \int_0^x \sin(z(x-t)) \int_0^t \left( u_4(s) \psi_{k-1}(s, z) \\
&\quad - \int_0^s F(s, \eta) \psi_{k-1}(\eta, z) d\eta \right) ds dt (k \geq 1). \quad (C.42)
\end{align*}

Now, the equality $y'_1(0, z) = 0$ easily follows from (C.26) and (C.42). Summing up, we have shown (in this step of the proof) that $y_1$ of the form (C.40) satisfies (C.23) and (C.25).
Step 3. Next, we show that the solution $y_1$ of (C.23), (C.25) is unique. Multiplying both parts of (C.23) by the operator $A$ (given in (2.2)) and using (C.25), we derive

$$By_1 = I_p + z^2 Ay_1, \quad B = I - \int_0^x B(x,t) \cdot dt := I + Au_4 - A \int_0^x \mathcal{F}(x,t) \cdot dt.$$

(C.43)

From (2.2), (C.28), and (C.43), after simple calculations we obtain

$$\mathcal{B}(x,t) = \left[ I_p \quad xI_p \quad u_7(x) \right] \begin{bmatrix} u_5(t) - tu_4(t) \\ u_4(t) + u_6(t) \\ h_2(t)u(t) \end{bmatrix}, \quad \begin{bmatrix} u_5(t) := -\int_0^t s u(s) h_1(s) ds \cdot h_2(t)u(t), \\ u_6(t) := \int_0^t u(s) h_1(s) ds \cdot h_2(t)u(t), \\ u_7(x) := \int_0^x (s - x)u(s) h_1(s) ds. \end{bmatrix}$$

(C.44)

Here, $u_4$ is given in (C.24) and the following transformation is used:

$$\int_0^x (t - x) \int_0^t \mathcal{F}(t,s) \cdot ds dt = \int_0^x (s - x) \int_0^x \mathcal{F}(s,t) ds \cdot dt \quad \text{(C.46)}$$

$$= \int_0^x \int_0^x (s - x)u(s) h_1(s) ds \cdot h_2(t)u(t) \cdot dt$$

$$- \int_0^x \int_0^x (s - x)u(s) h_1(s) ds \cdot h_2(t)u(t) \cdot dt.$$

Since $B$ is a triangular operator and the integral part of $B$ has a semi-separable kernel, it easily follows (see, e.g., [19, Section IX.2]) that $B$ is invertible and $B^{-1}$ is a bounded operator. (In fact, the integral part of $B$ is a Volterra operator from Hilbert-Schmidt class and $B^{-1} - I$ is again a triangular Volterra operator with a semi-separable kernel.) Thus, we rewrite (C.43) as

$$y_1 = (I - z^2 B^{-1}A)^{-1} B^{-1} I_p.$$

(C.47)

Now, it is easy to see that $y_1$ is unique. Recall that this unique solution admits representation (C.3), and so, taking into account (C.22), we obtain

$$y(x, z) = V \cos(zx) I_p,$$

(C.48)
where $V$ is given by the second equality in (C.3). One easily checks that

$$(I - z^2 A)^{-1} I_p = \cos(zx) I_p.$$  

(C.49)

In view of (C.10), (C.48), and (C.49), we have

$$(I - z^2 K)^{-1} g = V(I - z^2 A)^{-1} I_p.$$  

(C.50)

Presenting the resolvents in both parts of (C.50) as series, we rewrite (C.50) in the form $K^ng = VA^n I_p$. In particular, setting $n = 0$, we derive $g = VI_p$. The substitution $g = VI_p$ into $K^ng = VA^n I_p$ yields

$$(n \geq 0).$$

(C.51)

It follows that

$$(KV) A^n I_p = K(VA^n I_p) = K^{n+1} VI_p = VA^{n+1} I_p = (VA) A^n I_p.$$  

(C.52)

One can easily see (using, e.g., Weierstrass approximation theorem) that the closed linear span of the columns of the matrix functions $A^n I_p$ $(n \geq 0)$ coincides with $L^p_2(0, T)$. Therefore, (C.52) implies $KV = VA$, and (C.3) follows. The required properties of $u$ and $V$ have already been proved. ■

Remark C.2 It is important for the study of the canonical systems on the semi-axis $[0, \infty)$ that, according to (C.26)–(C.28), (C.30), and (C.41), the matrix function $V(x, \zeta)$ in the domain $0 \leq \zeta \leq x \leq \ell$ is uniquely determined by $\beta(x)$ on $[0, \ell]$ (and does not depend on the choice of $\beta(x)$ for $\ell < x < T$ and the choice of $T \geq \ell$).

Acknowledgments This research was supported by the Austrian Science Fund (FWF) under Grant No. P29177.

References

[1] D.Z. Arov and H. Dym, Bitangential direct and inverse problems for systems of integral and differential equations, Cambridge University Press, Cambridge, 2012.
[2] M.I. Belishev and V.S. Mikhailov, *Inverse problem for a one-dimensional dynamical Dirac system (BC-method)*, Inverse Problems **30** (2014), Art. 125013.

[3] R. Bellman, *Some inequalities for the square root of a positive definite matrix*, Linear Algebra Appl. **1** (1968), 321–324.

[4] M. Bohner and O. Došlý, *Oscillation of symplectic dynamic systems*, ANZIAM J. **46** (2004), 17–32.

[5] J. Breuer, E. Ryckman, and B. Simon, *Equality of the spectral and dynamical definitions of reflection*, Comm. Math. Phys. **295** (2010), 531–550.

[6] J.L. Cieslinski, *Algebraic construction of the Darboux matrix revisited*, J. Phys. A **42** (2009), Art. 404003.

[7] A. Constantin and R. Ivanov, *Dressing method for the Degasperis-Procesi equation*, Stud. Appl. Math. **138** (2017), 205–226.

[8] L. de Branges, *Hilbert spaces of entire functions*, Prentice-Hall, Englewood Cliffs, N.J., 1968.

[9] O. Došlý, J. Elyseeva, and R. Šimon Hilscher, *Symplectic difference systems: oscillation and spectral theory*, Birkhäuser/Springer, Cham, 2019.

[10] O. Došlý and R. Šimon Hilscher, *Disconjugacy, transformations and quadratic functionals for symplectic dynamic systems on time scales*, J. Differ. Equations Appl. **7** (2001), no. 2, 265–295.

[11] J. Eckhardt and A. Kostenko, *The inverse spectral problem for indefinite strings*, Invent. Math. **204** (2016), 939–977.

[12] J. Eckhardt, F. Gesztesy, R. Nichols, A. Sakhnovich, and G. Teschl, *Inverse spectral problems for Schrödinger-type operators with distributional matrix-valued potentials*, Differential Integral Equations **28** (2015), 505–522.
[13] J. Eckhardt, A. Kostenko, and G. Teschl, *Spectral asymptotics for canonical systems*, J. Reine Angew. Math. 736 (2018), 285–315.

[14] B. Fritzsche, B. Kirstein, I.Ya. Roitberg, and A.L. Sakhnovich, *Weyl theory and explicit solutions of direct and inverse problems for Dirac system with a rectangular matrix potential*, Oper. Matrices 7 (2013), 183–196.

[15] B. Fritzsche, B. Kirstein, and A.L. Sakhnovich, *Weyl functions of generalized Dirac systems: integral representation, the inverse problem and discrete interpolation*, J. Anal. Math. 116 (2012), 17–51.

[16] F. Gesztesy, *A complete spectral characterization of the double commutation method*, J. Funct. Anal., 117 (1993), 401–446.

[17] F. Gesztesy and A. Sakhnovich, *The inverse approach to Dirac-type systems based on the A-function concept*, J. Funct. Anal. 279 (2020), Art. 108609.

[18] F. Gesztesy and G. Teschl, *On the double commutation method*, Proc. Amer. Math. Soc., 124 (1996), 1831–1840.

[19] I. Gohberg, S. Goldberg, and M.A. Kaashoek, *Classes of linear operators*, Vol. I, Birkhäuser, Basel, 1990.

[20] I. Gohberg, M.A. Kaashoek, and A.L. Sakhnovich, *Pseudocanonical systems with rational Weyl functions: explicit formulas and applications*, J. Differential Equations 146 (1998), 375–398.

[21] I. Gohberg and M.G. Krein, *Theory and applications of Volterra operators in Hilbert space*, Transl. math. monographs. 24, Amer. Math. Soc., Providence, RI, 1970.

[22] C.H. Gu, H. Hu and Z. Zhou. *Darboux transformations in integrable systems. Theory and their applications to geometry*, Springer, Dordrecht, 2005.
[23] B. Jacob, K. Morris, and H. Zwart, *C₀-semigroups for hyperbolic partial differential equations on a one-dimensional spatial domain*, J. Evol. Equ. **15** (2015), 493–502.

[24] V. Jakšić, B. Landon, and A. Panati, *A note on reflectionless Jacobi matrices*, Comm. Math. Phys. **332** (2014), 827–838.

[25] D.J. Kaup, *Simple harmonic generation: an exact method of solution*, Stud. Appl. Math. **59** (1978), 25–35.

[26] D.J. Kaup and H. Steudel, *Recent results on second harmonic generation*, Contemporary Math. **326** (2003), 33–48.

[27] A. Kostenko, A. Sakhnovich, and G. Teschl, *Commutation Methods for Schrödinger Operators with Strongly Singular Potentials*, Math. Nachr. **285** (2012), no. 4, 392–410.

[28] M.G. Krein, *Continuous analogues of propositions on polynomials orthogonal on the unit circle* (Russian), Dokl. Akad. Nauk SSSR (N.S.) **105** (1955), 637–640.

[29] W. Kratz and R. Šimon Hilscher, *Rayleigh principle for linear Hamiltonian systems without controllability*, ESAIM Control Optim. Calc. Var. **18** (2012), 501–519.

[30] H. Langer, *Transfer functions and local spectral uniqueness for Sturm-Liouville operators, canonical systems and strings*, Integral Equations Operator Theory **85** (2016), 1–23.

[31] V.A. Marchenko, *Nonlinear equations and operator algebras*, D. Reidel, Dordrecht, 1988.

[32] V.B. Matveev and M.A. Salle, *Darboux transformations and solitons*, Springer, Berlin, 1991.

[33] V. Mogilevskii, *Spectral and pseudospectral functions of Hamiltonian systems: development of the results by Arov–Dym and Sakhnovich*, Methods Funct. Anal. Topology **21** (2015), no. 4, 370–402.
[34] C. Remling, *Spectral theory of canonical systems*, De Gruyter, Berlin, 2018.

[35] R. Romanov, *Order problem for canonical systems and a conjecture of Valent*, Trans. Amer. Math. Soc. **369** (2017), 1061–1078.

[36] R. Romanov and H. Woracek, *Canonical systems with discrete spectrum*, J. Funct. Anal. **278** (2020), Art. 108318.

[37] J. Rovnyak and L.A. Sakhnovich, *Pseudospectral functions for canonical differential systems. II*, in Oper. Theory Adv. Appl. **218**, Birkhäuser/Springer, Basel, 2012, pp. 583–612.

[38] A.L. Sakhnovich, *Dressing procedure for solutions of nonlinear equations and the method of operator identities*, Inverse problems **10** (1994), 699–710.

[39] A.L. Sakhnovich, *Iterated Bäcklund–Darboux transform for canonical systems*, J. Funct. Anal. **144** (1997), 359–370.

[40] A.L. Sakhnovich, *Second harmonic generation: Goursat problem on the semi-strip, Weyl functions and explicit solutions*, Inverse Problems **21** (2005), 703–716.

[41] A.L. Sakhnovich, *Dynamical and spectral Dirac systems: response function and inverse problems*, J. Math. Phys. **56** (2015), Art. 112702.

[42] A.L. Sakhnovich, *Dynamical canonical systems and their explicit solutions*, Discrete Contin. Dyn. Syst. **37** (2017), 1679–1689.

[43] A.L. Sakhnovich, *New “Verblunsky-type” coefficients of block Toeplitz and Hankel matrices and of corresponding Dirac and canonical systems*, J. Approx. Theory **237** (2019), 186–209.

[44] A.L. Sakhnovich, *Einstein, σ-model and Ernst-type equations and non-isospectral GBDT version of Darboux transformation*, arXiv:2003.13024.
[45] A.L. Sakhnovich, L.A. Sakhnovich and I.Ya. Roitberg, *Inverse Problems and Nonlinear Evolution Equations. Solutions, Darboux Matrices and Weyl–Titchmarsh Functions*, De Gruyter, Berlin, 2013.

[46] L.A. Sakhnovich, *The spectral analysis of Volterra operators and some inverse problems* (Russian), Dokl. Akad. Nauk SSSR (N.S.) 115 (1957), 666–669.

[47] L.A. Sakhnovich, *On the factorization of the transfer matrix function*, Sov. Math. Dokl. 17 (1976), 203–207.

[48] L.A. Sakhnovich. *Factorization problems and operator identities*, Russian Math. Surveys 41 (1986), 1–64.

[49] L.A. Sakhnovich, *The method of operator identities and problems in analysis*, St. Petersburg Math. J. 5 (1994), 1–69.

[50] L.A. Sakhnovich, *Spectral theory of canonical differential systems, method of operator identities*, Birkhäuser, Basel, 1999.

[51] L.A. Sakhnovich, *Dirac equation: the stationary and dynamical scattering problems*, in Oper. Theory Adv. Appl. 263, pp. 407–424, Birkhäuser/Springer, Cham, 2018.

[52] M. Suzuki, *An inverse problem for a class of canonical systems and its applications to self-reciprocal polynomials*, J. Anal. Math. 136 (2018), 273–340.

[53] H. Woracek, *Asymptotics of eigenvalues for a class of singular Krein strings*, Collect. Math. 66 (2015), 469–479.

[54] V.E. Zakharov and A.V. Mikhailov, *On the integrability of classical spinor models in two-dimensional space-time*, Comm. Math. Phys. 74 (1980), 21–40.

Alexander Sakhnovich
Faculty of Mathematics, University of Vienna,
Oskar-Morgenstern-Platz 1, A-1090 Vienna, Austria,
e-mail: oleksandr.sakhnovych@univie.ac.at