NEW SHARP CUSA–HUYGENS TYPE INEQUALITIES FOR TRIGONOMETRIC AND HYPERBOLIC FUNCTIONS

Abstract. We prove that for $p \in (0, 1]$, the double inequality

$$
\frac{1}{3}p \cos px + 1 - \frac{1}{3}p < \frac{\sin x}{x} < \frac{1}{3}p \cos qx + 1 - \frac{1}{3}q
$$

holds for $x \in (0, \pi/2)$ if and only if $0 < p \leq p_0 \approx 0.77086$ and $\sqrt{15}/5 = q_1 \leq q \leq 1$. While its hyperbolic version holds for $x > 0$ if and only if $0 < p \leq p_1 = \sqrt{15}/5$ and $q \geq 1$. As applications, some more accurate estimates for certain mathematical constants are derived, and some new and sharp inequalities for Schwab-Borchardt mean and logarithmic means are established.

1. Introduction

The Cusa and Huygens (see, e.g., [1]) states that for $x \in (0, \pi/2)$, the inequality

$$
\frac{\sin x}{x} < \frac{2 + \cos x}{3}
$$

holds true. Its version of hyperbolic functions refers to (see [2]) the inequality

$$
\frac{\sinh x}{x} < \frac{2 + \cosh x}{3}
$$

holds for $x > 0$, and it is known as hyperbolic Cusa–Huygens inequality (see [2]).

There are many improvements, refinements and generalizations of (1.1) and (1.2), see [3], [4], [5], [6], [7], [8], [9]; [11], [12], [13], [14], [15], [16], [17], [18], [19], [20], [21].

Now we focus on the bounds for $(\sin x)/x$ in terms of $\cos px$, where $x \in (0, \pi/2)$, $p \in (0, 1]$. In 1945, Iyengar [22] (also see [23, subsection 3.4.6]) proved that for $x \in (0, \pi/2)$,

$$
\cos px \leq \frac{\sin x}{x} \leq \cos qx
$$

holds with the best possible constants

$$p = \frac{1}{\sqrt{3}} \quad \text{and} \quad q = \frac{2}{\pi} \arccos \frac{2}{\pi}.
$$

Moreover, the following chain of inequalities hold:

$$
\cos x \leq \frac{\cos x}{1 - x^2/3} \leq (\cos x)^{1/3} \leq \cos \frac{x}{\sqrt{3}} \leq \frac{\sin x}{x} \leq \cos qx \leq \cos \frac{x}{2} \leq 1.
$$

Qi et al. [24] showed that

$$
\frac{\cos^2 x}{2} < \frac{\sin x}{x}
$$

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holds for \( x \in (0, \pi/2) \). Klén et al. [25, Theorem 2.4] pointed out that the function \( p \mapsto (\cos px)^{1/p} \) is decreasing on \((0, 1)\) and for \( x \in \left(-\sqrt{27/5}, \sqrt{27/5}\right) \)

\[
\cos^2 \frac{x}{2} \leq \frac{\sin x}{x} \leq \cos^3 \frac{x}{3} \leq \frac{2 + \cos x}{3}
\]

are valid. Subsequently, Yang [8] (also see [26]) gave a refinement of (1.6), which states that for \( p, q \in (0, 1) \) the double inequality

\[
(\cos px)^{1/p} < \frac{\sin x}{x} < (\cos qx)^{1/q}
\]

holds for \( x \in (0, \pi/2) \) if and only if \( p \in \left[p_0^*, 1\right) \) and \( q \in (0, 1/3) \), where \( p_0^* \approx 0.3473 \). Moreover, the double inequality

\[
\left(\cos \frac{x}{3}\right)^{\alpha} < \frac{\sin x}{x} < \left(\cos \frac{x}{3}\right)^{3}
\]

with the best exponents \( \alpha = 2 \frac{\ln \pi - \ln 2}{\ln 4 - \ln 3} \approx 3.1395 \) and 3. Also, he pointed out that the value range of variable \( x \) such that (1.6) holds can be extended to \((0, \pi)\). Very recently, Yang [21] gave another improvement of (1.6), that is, for \( x \in (0, \pi/2) \) the inequalities

\[
\sin \frac{x}{x} < \left(\frac{3}{2} \cos \frac{x}{2} + \frac{1}{3}\right)^2 < \cos^3 \frac{x}{3} < \frac{2 + \cos x}{3}
\]

are true.

An important improvement for the inequality in (1.5) is due to Neuman [2]:

\[
\cos^{4/3} \frac{x}{2} = \left(\frac{1 + \cos x}{2}\right)^{2/3} < \frac{\sin x}{x}, \; x \in (0, \frac{\pi}{2}).
\]

Lv et al. [27] showed that for \( x \in (0, \pi/2) \) inequalities

\[
\left(\cos \frac{x}{2}\right)^{4/3} < \frac{\sin x}{x} < \left(\cos \frac{x}{2}\right)^{\theta}
\]

hold, where \( \theta = 2 \frac{\ln \pi - \ln 2}{\ln 2} = 1.3030... \) and 4/3 are the best possible constants. By constructing a decreasing function \( p \mapsto (\cos px)^{1/(3p^2)} \) \( (p \in (0, 1]) \), Yang [9] showed that the double inequality

\[
(\cos p^*_1 x)^{1/(3p^*_1)} < \frac{\sin x}{x} < \cos^{5/3} \frac{x}{\sqrt{5}}
\]

is true for \( x \in (0, \pi/2) \) with the best constants \( p^*_1 \approx 0.45346 \) and \( 1/\sqrt{5} \approx 0.44721 \).

It follows that

\[
(\cos x)^{1/3} < \cos^{1/2} \frac{\sqrt{5}x}{3} < \cos^{2/3} \frac{x}{\sqrt{2}} < \cos \frac{x}{\sqrt{3}} < \cos^{4/3} \frac{x}{2} < (\cos p^*_1 x)^{1/(3p^*_1)}
\]

\[
\sin \frac{x}{x} < \cos^{5/3} \frac{x}{\sqrt{5}} < \cos \frac{x}{\sqrt{6}} < \cos^3 \frac{x}{3} < \cos^{16/3} \frac{x}{4} < e^{-x^2/6} < \frac{2 + \cos x}{3}
\]

are valid for \( x \in (0, \pi/2) \).

For the bounds for \( (\sinh x) / x \) in terms of \( \cosh px \), it is known that the inequalities

\[
\frac{\sinh x}{x} < \cosh^3 \frac{x}{3} < \frac{2 + \cosh x}{3}
\]

holds true for \( x > 0 \) (see [18]), which is exactly derived by the inequalities for means

\[
L < A_{1/3} < \frac{2G + A}{3}
\]
(see e.g. [28], [29], [30]), where $L$, $A_p$, $G$ and $A$ stand for the logarithmic mean, power mean of order $p$, geometric mean and arithmetic mean or positive numbers $a$ and $b$ defined by

$$
L(a, b) = \frac{a - b}{\ln a - \ln b} \quad \text{if } a \neq b \text{ and } L(a, a) = a,
$$

$$
A_p(a, b) = \left(\frac{a^p + b^p}{2}\right)^{1/p} \quad \text{if } p \neq 0 \text{ and } A = A_0(a, b) = \sqrt{ab},
$$

$G = A_0$ and $A = A_1$, respectively. Zhu in [31] proved that for $p > 1$ or $p \leq 8/15$, and $x \in (0, \infty)$, the inequality

$$
\left(\sinh \frac{x}{x}\right)^q > p + (1 - p) \cosh x
$$

is true if and only if $q \geq 3(1 - p)$. It follows by letting $p = 1/2$ and $q = 3/2$ that

$$
\frac{\sinh x}{x} > \cosh^{4/3} \frac{x}{2}
$$

holds for $x > 0$ (also see [2, (2.8)]). Yang [19] showed that the inequality

$$
\sinh \frac{x}{x} > \left(\cosh px\right)^{1/(3p^2)}
$$

holds for all $x > 0$ if and only if $p \geq 1/\sqrt{5}$ and its reverse holds if and only if $0 < p \leq 1/3$. And, the function $p \mapsto \left(\cosh px\right)^{1/(3p^2)}$ is decreasing on $(0, \infty)$.

The aim of this paper is to determine the best $p$ such that the inequalities

$$
\frac{\sin x}{x} < (>) \frac{1}{3p^2} \cos px + 1 - \frac{1}{3p^2}, \quad p \in (0, 1), \quad x \in (0, \pi/2),
$$

$$
\frac{\sinh x}{x} < (>) \frac{1}{3q^2} \cosh px + 1 - \frac{1}{3q^2}, \quad p, x \in (0, \infty)
$$

hold true.

Our main results are contained in the following theorems.

**Theorem 1.** For $p \in (0, 1]$ and $x \in (0, \pi/2)$, the double inequality

$$
\frac{1}{3p^2} \cos px + 1 - \frac{1}{3p^2} < \frac{\sin x}{x} < \frac{1}{3q^2} \cos qx + 1 - \frac{1}{3q^2}
$$

holds if and only if $0 < p \leq p_0 \approx 0.77086$ and $0.77460 \approx \sqrt{15}/5 = p_1 \leq q \leq 1$, where $p_0$ is the unique root of the equation

$$
F_p \left(\frac{\pi}{2}\right) = \frac{2}{\pi} - \left(\frac{1}{3p^2} \cos \frac{\pi}{2} + 1 - \frac{1}{3p^2}\right) = 0
$$

on $(0, 1)$. And, the bound for $(\sin x)/x$ given in (1.14) is increasing with respect to parameters $p$ or $q$.

**Theorem 2.** For $p, x > 0$, the double inequality

$$
\frac{1}{3p^2} \cosh px + 1 - \frac{1}{3p^2} < \frac{\sinh x}{x} < \frac{1}{3q^2} \cosh qx + 1 - \frac{1}{3q^2}
$$

holds if and only if $0 < p \leq p_1 = \sqrt{15}/5$ and $q \geq 1$. And, the bound for $(\sinh x)/x$ given in (1.16) is increasing with respect to parameters $p$ or $q$. 

Remark 1. The weighted basic inequality of two positive numbers of \(a\) and \(b\) tell us that for \(\alpha \in [0,1]\), the inequality \(a^\alpha (1-\alpha) b \geq a^\alpha b^{1-\alpha}\). It is reversed if and only if \(\alpha \geq 1\) or \(\alpha \leq 0\) (see [22]). Hence, taking into account (1.11) and (1.14) we see that

(i) if \(p \in [p^*_1, 1/\sqrt{3}]\), where \(p^*_1 \approx 0.45346\), then
\[
\sin{x}{x} > (\cos{px})^{1/(3p^2)} > \frac{1}{3p^2} \cos{px} + 1 - \frac{1}{3p^2};
\]

(ii) if \(p \in (1/\sqrt{3}, p_0)\), where \(p_0 \approx 0.77086\), then
\[
\sin{x}{x} > \frac{1}{3p^2} \cos{px} + 1 - \frac{1}{3p^2} > (\cos{px})^{1/(3p^2)}.
\]

In the same way, (1.15) together with (1.10) leads us to

(iii) if \(p \in [1/\sqrt{5}, 1/\sqrt{3}]\), then
\[
\sinh{x}{x} > (\cosh{px})^{1/(3p^2)} > \frac{1}{3p^2} \cosh{px} + 1 - \frac{1}{3p^2};
\]

(iv) if \(p \in (1/\sqrt{3}, \sqrt{15}/5)\), then
\[
\sinh{x}{x} > \frac{1}{3p^2} \cosh{px} + 1 - \frac{1}{3p^2} > (\cosh{px})^{1/(3p^2)}.
\]

Taking \(p = 3/4, 1/\sqrt{7}, 2/3, 1/\sqrt{3}\) and \(q = \sqrt{3}/5, \sqrt{2}/3, \sqrt{3}/2, 1\) in Theorem [1] we have

Corollary 1. For \(x \in (0, \pi/2)\), the inequalities
\[
(1.17) \cos{\frac{x}{\sqrt{3}}} < \frac{3}{4} \cos{\frac{2x}{\sqrt{3}}} + \frac{1}{4} < \frac{4}{5} \cos{\frac{3x}{\sqrt{2}}} + \frac{1}{5} < \cos{\frac{4x}{\sqrt{2}}} + \frac{1}{11} < \frac{\sin{x}}{x}
\]
\[
< \frac{5}{9} \cos{\frac{5x}{\sqrt{2}}} + \frac{4}{9} < \cos^2{\frac{x}{\sqrt{6}}} + \frac{1}{9} \cos{\frac{6x}{\sqrt{2}}} + \frac{5}{9} < \frac{1}{3} \cos{x} + \frac{2}{3}.
\]

Putting \(p = \sqrt{3}/5, 3/4, 1/\sqrt{2}, 2/3, 1/\sqrt{3}\) and \(q = 1, 2/\sqrt{3}\) in Theorem [2] we have

Corollary 2. For \(x > 0\), the inequalities
\[
(1.18) \cosh{\frac{x}{\sqrt{3}}} < \frac{3}{4} \cosh{\frac{2x}{\sqrt{3}}} + \frac{1}{4} < \frac{4}{5} \cosh{\frac{3x}{\sqrt{2}}} + \frac{1}{5} < \cosh{\frac{4x}{\sqrt{2}}} + \frac{1}{11} \cosh{\frac{5x}{\sqrt{2}}} + \frac{4}{9} < \frac{\sinh{x}}{x} < \frac{1}{3} \cosh{x} + \frac{2}{3} < \frac{1}{2} \cosh^2{\frac{x}{\sqrt{3}}} + \frac{1}{2}.
\]

2. Proof of Theorem 1

In order to prove Theorem [1] we need some lemmas.

Lemma 1. For \(x \in (0, \pi/2)\), the function \(p \mapsto U_p(x)\) defined on \([0,1]\) by
\[
U_p(x) = \frac{1}{3p^2} \cos{px} + 1 - \frac{1}{3p^2} \quad \text{if } p \in (0,1) \quad \text{and } U_0(x) = 1 - \frac{x^2}{6}
\]
is increasing.

Proof. Differentiation yields
\[
\frac{\partial U_p}{\partial p} = \frac{1}{3p^3} (2 - 2 \cos{px} - px \sin{px}) = \frac{12px}{3p^3} \left( \frac{\sin{\frac{px}{2}}}{\frac{px}{2}} - \cos{\frac{px}{2}} \right) \sin{\frac{px}{2}} > 0,
\]
which completes the proof. \(\square\)
Lemma 3. Let the function $F_p$ be defined on $(0,\pi/2)$ by

\begin{equation}
F_p(x) = \frac{\sin x}{x} - \left(\frac{1}{3p^2} \cos px + 1 - \frac{1}{3p^2}\right) \text{ if } p \in (0,1] \text{ and } F_0(x) = \frac{\sin x}{x} - 1 + \frac{x^2}{2}.
\end{equation}

(i) If $F_p(x) < 0$ for all $x \in (0,\pi/2)$, then $p \in [p_1, 1]$, where $p_1 = 15/2 = 0.77460$.

(ii) If $F_p(x) > 0$ for all $x \in (0,\pi/2)$, then $p \in [0,p_0]$, where $p_0 \approx 0.77086$.

Proof. (i) If $F_p(x) < 0$ for all $x \in (0,\pi/2)$, then we have

\begin{equation}
\lim_{x \to 0} \frac{\sin x}{x} - \left(\frac{1}{3p^2} \cos px + 1 - \frac{1}{3p^2}\right) = -\frac{1}{360} (5p^2 - 3) \leq 0,
\end{equation}

which leads to $p \in (\sqrt{15}/5, 1]$.

(ii) If $F_p(x) > 0$ for all $x \in (0,\pi/2)$, then we have

$F_p\left(\frac{\pi}{2}\right) = \frac{2}{3} - \left(\frac{1}{3p^2} \cos \frac{p\pi}{2} + 1 - \frac{1}{3p^2}\right) > 0.$

From Lemma we see that the function $p \mapsto F_p(\pi/2^-)$ is decreasing on $[0,1]$, which together with the facts

$F_{1/2}\left(\frac{\pi}{2}\right) = \frac{2}{\pi} - \frac{2}{3}\sqrt{2} + \frac{1}{3} > 0 \text{ and } F_1\left(\frac{\pi}{2}\right) = \frac{2}{\pi} - \frac{2}{3} < 0$

gives that there is a unique number $p_0 \in (1/2, 1)$ such that $F_p(\pi/2^-) > 0$ for $p \in (0,p_0)$ and $F_p(\pi/2^-) < 0$ for $p \in (p_0,1)$. Solving the equation $F_p(\pi/2^-) = 0$ for $p$ by mathematical computer software we find that $p_0 \approx 0.77086$.

This completes the proof.

□

Lemma 3. Let $c \in (0,3/5]$ and let the sequence $(a_n(c))$ be defined by

\begin{equation}
a_n(c) = 3 - (2n+1)e^{n-1}.
\end{equation}

Then (i) $a_n(c) \geq 0$ for $n \in \mathbb{N}$; (ii) for $n \geq 3$, we have

\[1 < \frac{a_{n+1}(c)}{a_n(c)} \leq \frac{a_{n+1}(3/5)}{a_n(3/5)} \leq \frac{11}{5}.\]

Proof. (i) We first show that $a_n(c) \geq 0$ for $n \in \mathbb{N}$.

A simple computation leads to

\[a_{n+1}(c) - a_n(c) = (2n+1)e^{n-1} - (2n+3)e^n = e^{n-1}(2n+1) - (2n+3)c \geq 0,
\]

which implies that $a_{n+1}(c) \geq a_n(c) \geq a_1(c) = 0$.

(ii) Since $a_1(c) = 0$, $a_2(c) = 3 - 5c \geq 0$, $a_n(c) > 0$ for $n \geq 3$, if we can show that the function

\[(c,n) \mapsto \frac{a_{n+1}(c)}{a_n(c)} = \frac{3 - (2n+3)c^n}{3 - (2n+1)c^{n-1}}\]

is increasing in $c$ on $(0,3/5]$ and decreasing in $n \geq 3$, then we have

\[1 = \frac{a_{n+1}(0)}{a_n(0)} < \frac{a_{n+1}(c)}{a_n(c)} < \frac{a_{n+1}(3/5)}{a_n(3/5)} \leq \frac{a_{n+1}(3/5)}{a_n(3/5)} = \frac{11}{5},\]
Proof of Theorem 1. Expanding in power series gives
\[
2^n a_n^2 (c) \left( \frac{a_{n+1} (c)}{a_n (c)} \right)' = (4n^2 + 8n + 3) c^n - 3n (2n + 3) c + (6n^2 - 3n - 3) := h_n (c),
\]
where
\[
h_n' (c) = -n (2n + 3) (3 - (2n + 1) c^{n-1}) = -n (2n + 3) a_n (c) < 0,
\]
which is clearly positive due to that $h_3 (3/5) = 876/125 > 0$ and $(4n^2 - 14n - 5) = n(4n - 14) - 5 \geq 3$ for $n \geq 4$. This reveals that $h_n' (c) > 0$, that is, $(c, n) \mapsto a_{n+1} (c) / a_n (c)$ is increasing in $c$ on $(0, 3/5)$.

On the other hand, we have
\[
\frac{a_{n+1} (c)}{a_n (c)} - \frac{a_{n+2} (c)}{a_{n+1} (c)} = c^n - 4c^{n+1} + 6 (c - 1)^2 n + 15c^2 - 18c + 3 \geq \frac{c^n}{a_n (c) a_{n+1} (c)} \left( 4c^{n+1} + 6 (c - 1)^2 \times 3 + 15c^2 - 18c + 3 \right) \geq \frac{c^n}{a_n (c) a_{n+1} (c)} (4c^{n+1} + 33 (\frac{7}{11} - c) (1 - c)) > 0,
\]
where the first inequality holds due to $a_n (c) > 0$ for $n \geq 3$, while the last one holds since $c \in (0, 3/5]$. This means that $(c, n) \mapsto a_{n+1} (c) / a_n (c)$ is decreasing with $n \geq 3$.

Thus we complete the proof of this assertion. \qed

Now we are in a position to prove Theorem 1.

Proof of Theorem 1. Expanding in power series gives
\[
F_p (x) = \frac{\sin x}{x} - \left( \frac{1}{3p^2} \cos px + 1 - \frac{1}{3p^2} \right) = \sum_{n=0}^\infty (-1)^n \frac{x^{2n}}{(2n+1)!} \left( \frac{1}{3p^2} \sum_{n=0}^\infty (-1)^n \frac{(px)^{2n}}{(2n)!} + 1 - \frac{1}{3p^2} \right)
\]
\[
= \sum_{n=2}^\infty (-1)^n \frac{3 - (2n + 1) p^{2n-2} x^{2n}}{3 (2n + 1)!},
\]
where $a_n (c)$ is defined by (2.3). Considering the function $f_p (x) = x^{-4} F_p (x)$, we have
\[
(2.4) \quad f_p (x) = x^{-4} F_p (x) = \frac{3 - 5p^2}{360} + \sum_{n=3}^\infty (-1)^n \frac{a_n (p^2)}{3 (2n + 1)!} x^{2n-4},
\]
and differentiation yields
\[ f_p'(x) = \sum_{n=3}^{\infty} (-1)^n \frac{(2n-4) a_n (p^2)}{3(2n+1)!} x^{2n-5} \]
\[ = \sum_{n=3}^{\infty} (-1)^n u_n(x), \]
where
\[ u_n(x) = \frac{(2n-4) a_n (p^2)}{3(2n+1)!} x^{2n-5}. \]

Utilizing Lemma 3 we get that for \( p^2 \in (0, 3/5] \) and \( n \geq 3, \)
\[ \frac{u_{n+1}(x)}{u_n(x)} = \frac{(2n-2) a_{n+1} (p^2)}{3(2n+3)!} x^{2n-3} \times \frac{(2n-4) a_n (p^2)}{3(2n+1)!} x^{2n-5} \]
\[ < \frac{1}{(2 \times 3 - 4) (2 \times 3 + 3)} \times 1 \times \frac{11}{5} \times \frac{\pi^2}{4} = \frac{11\pi^2}{360} < 1, \]
which implies that the power series \( \sum_{n=3}^{\infty} (-1)^n u_n(x) \) is a Leibniz type alternating one, and so \( f_p'(x) < 0 \) for \( p^2 \in (0, 3/5]. \)

(i) We first prove the second inequality in (1.14) holds, where \( p_1 = \sqrt{15}/5 \) is the best. As shown previously, we see that \( f_{p_1} \) is decreasing on \( (0, \pi/2) \), and therefore,
\[ f_{p_1}(x) < f_{p_1}(0^+) = \lim_{x \to 0^+} (x^{-4} F_{p_1}(x)) = \frac{1}{360} (3 - 5p_1^2) = 0, \]
which together with (2.21) yields \( F_{p_1}(x) < 0 \) for \( x \in (0, \pi/2) \).

Next we prove \( p_1 = \sqrt{15}/5 \) is the best. If there is another \( p_1' < p_1 \) such that the second inequality in (1.14) holds for \( x \in (0, \pi/2) \), then by Lemma 2 there must be \( p_1' \in [p_1, 1] \), which yields a contradiction. Therefore, \( p_1 = \sqrt{15}/5 \) can not be replaced with other smaller ones.

(ii) Now we prove the first inequality in (1.14) holds with the best constant \( p_0 \approx 0.77088 \). Since \( p_0^2 \in (0, 3/5] \), \( f_{p_0} \) is also decreasing on \( (0, \pi/2) \), and so
\[ f_{p_0}(x) > f_{p_0}\left(\frac{\pi}{2}\right) = \lim_{x \to \pi/2^-} (x^{-4} F_{p_0}(x)) = \left(\frac{\pi}{2}\right)^{-4} F_{p_0}\left(\frac{\pi}{2}\right) = 0, \]
where the last equality is true due to \( p_0 \) is the unique root of the equation (1.15) on \( (0, 1) \). It together with (2.24) gives \( F_{p_0}(x) > 0 \) for \( x \in (0, \pi/2) \).

Lastly, we show that \( p_0 \) is the best. Assume that there is another \( p_0' > p_0 \) such that \( F_{p_0'}(x) > 0 \) for \( x \in (0, \pi/2) \). Then by Lemma 2 there must be \( p_0' \in [0, p_0] \), which is clear a contradiction. Consequently, \( p_0 \) can not be replaced by other larger numbers.

Thus the proof is complete. \( \square \)

Remark 2. Application of the conclusion that \( f_p'(x) < 0 \) for \( x \in (0, \pi/2) \) if \( p^2 \in (0, 3/5] \) gives \( f_p(X) = f_p(x) < f_p(0^+) \), that is,
\[ \left(\frac{\pi}{2}\right)^{-4} F_p\left(\frac{\pi}{2}\right) < x^{-4} F_p(x) < \lim_{x \to 0^+} x^{-4} F_p(x) = \frac{3 - 5p^2}{360}. \]
which can be changed into

\[(2.5) \left( \frac{1}{3p^2} \cos px + 1 - \frac{1}{3p^2} \right) + c_0(p) x^4 < \frac{\sin x}{x} < \left( \frac{1}{3p^2} \cos px + 1 - \frac{1}{3p^2} \right) + c_1(p) x^4,\]

where \(c_0(p) = (\pi/2)^{-4} F_p(\pi/2^-)\) and \(c_1(p) = (3 - 5p^2) / 360\) are the best constants. Then

(i) when \(p = p_1 = \sqrt{15}/5\), we have

\[c_0(p_1)x^4 + \left( \frac{1}{3p_1^2} \cos p_1x + 1 - \frac{1}{3p_1^2} \right) < \frac{\sin x}{x} < \left( \frac{1}{3p_1^2} \cos p_1x + 1 - \frac{1}{3p_1^2} \right),\]

where \(c_0(p_1) = (\pi/2)^{-4} F_{p_1}(\pi/2^-) \approx -7.2618 \times 10^{-5}\) and \(c_1(p_1) = 0\) are the best possible constants;

(ii) when \(p = p_0 \approx 0.77086\), we get

\[\left( \frac{1}{3p_0} \cos p_0x + 1 - \frac{1}{3p_0^2} \right) < \frac{\sin x}{x} < \left( \frac{1}{3p_0} \cos p_0x + 1 - \frac{1}{3p_0^2} \right) + c_1(p_0) x^4,\]

where \(c_0(p_0) = 0\) and \(c_1(p_0) = (3 - 5p_0^2) / 360 \approx 8.0206 \times 10^{-5}\) are the best constants.

3. Proof of Theorem 2

For proving Theorem 2 we first give the following lemmas.

**Lemma 4.** For \(x \in (0, \infty)\), the function \(p \mapsto V_p(x)\) defined on \([0, \infty)\) by

\[V_p(x) = \frac{1}{3p^2} \cosh px + 1 - \frac{1}{3p^2}\]

if \(p \neq 0\) and \(V_0(x) = 1 + \frac{x^2}{6}\)

is increasing.

**Proof.** Differentiation yields

\[
\frac{\partial V_p}{\partial p} = \frac{1}{3p^4} \left( px \sinh px - 2 \cosh px + 2 \right)
\]

\[= \frac{2x}{3p^2} \left( \cosh \frac{px}{2} - \frac{\sinh \frac{px}{2}}{\frac{px}{2}} \right) \sinh \frac{px}{2} > 0,
\]

which completes the proof. \(\square\)

**Lemma 5.** Let the function \(G_p\) be defined on \((0, \infty)\) by

\[G_p(x) = \frac{\sinh x}{x} - \left( \frac{1}{3p^2} \cos px + 1 - \frac{1}{3p^2} \right)\]

if \(p \neq 0\) and \(G_0(x) = \frac{\sinh x}{x} - 1 - \frac{x^2}{3p^2}\).

(i) If \(G_p(x) < 0\) for all \(x \in (0, \infty)\), then \(p \geq 1\).

(ii) If \(G_p(x) > 0\) for all \(x \in (0, \pi/2)\), then \(p \leq p_1 = \sqrt{15}/5 \approx 0.77460\).

**Proof.** In order to prove the desired results, we need the following two relations:

\[(3.1) \quad \lim_{x \to 0} \frac{G_p(x)}{x^4} = -\frac{1}{360} (5p^2 - 3),\]

\[(3.2) \quad \lim_{x \to \infty} \frac{G_p(x)}{e^{px}} = \begin{cases} -\frac{1}{6p^2} & \text{if } p > 1, \\
-\frac{1}{6} & \text{if } p = 1, \\
\infty & \text{if } 0 < p < 1, \\
\infty & \text{if } p = 0.\end{cases}\]

The first one follows by expanding in power series:

\[G_p(x) = -\frac{1}{360} (5p^2 - 3)x^4 + o(x^6).\]
To obtain the second one, it needs to note that
\[ e^{-px}G_p(x) = e^{(1-p)x} \frac{1-e^{-2px}}{2x} - \frac{1-e^{-2px}}{3p} \left(1 - \frac{1}{3p}\right) e^{-px}, \]
which gives (3.2).

(i) If \( G_p(x) < 0 \) for all \( x \in (0, \infty) \), then we have \( \lim_{x \to 0} x^{-4}G_p(x) \leq 0 \) and \( \lim_{x \to \infty} e^{-px}G_p(x) \leq 0 \). These together with (3.1) and (3.2) give \( p \geq 1 \).

(ii) If \( G_p(x) > 0 \) for all \( x \in (0, \infty) \), then we have \( \lim_{x \to 0} x^{-4}G_p(x) \geq 0 \) and \( \lim_{x \to \infty} e^{-px}G_p(x) \geq 0 \). These together with (3.1) and (3.2) indicate \( p \leq 1 \). \( \square \)

We now can prove Theorem 2.

Proof of Theorem 2. The necessity follows by Lemma 5. To prove the sufficiency, we expanding \( G_p(x) \) in power series to get
\[ G_p(x) = \sinh x - (\frac{1}{3p} \cosh px + 1 - \frac{1}{3p}) = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n+1)!} - \left(\frac{1}{3p^2} \sum_{n=0}^{\infty} \frac{(px)^{2n}}{(2n)!} + 1 - \frac{1}{3p^2}\right) \]
\[ = \sum_{n=2}^{\infty} \frac{3 - (2n + 1)p^{2n-2}}{3(2n + 1)!} x^{2n} = \sum_{n=2}^{\infty} \frac{a_n(p^2)}{3(2n + 1)!} x^{2n}. \]
It is derived from Lemma 3 that \( a_n(p^2) \geq 0 \) if \( 0 < p \leq \sqrt{15}/5 \), and clearly, \( a_n(p^2) < 0 \) if \( p \geq 1 \). \( \square \)

4. Applications

As simple applications of main results, we present some precise estimations for certain special functions and constants in this section.

The sine integral is defined by
\[ \text{Si}(t) = \int_0^t \frac{\sin x}{x} dx. \]
Some estimates for sine integral can be seen \([33, 34, 35, 26, 9]\). Now we give a new result.

Proposition 1. For \( t \in (0, \pi/2) \) and \( p \in (0, \sqrt{15}/5) \), we have
\[ \sin pt + \left(1 - \frac{1}{3p^2}\right) t + \frac{c_0(p) t^5}{5} < \text{Si}(t) < \sin pt + \left(1 - \frac{1}{3p^2}\right) t + \frac{c_1(p) t^5}{5}, \]
where \( c_0(p) = (\pi/2)^{-4} F_p(\pi/2) \) and \( c_1(p) = (3 - 5p^2)/360 \), here \( F_p(\pi/2) \) is defined by (1.15). Particularly, putting \( p = 0^+, 2/3 \), we have
\[ \frac{t}{18} - \frac{1}{18} t^3 + \frac{2\pi^3 - 48\pi + 96}{15\pi^3} t^5 < \text{Si}(t) < \frac{t}{18} - \frac{1}{18} t^3 + \frac{1}{600} t^5, \]
\[ \frac{9}{8} \sin t - \frac{3}{4} t + \frac{2(16 - 5\pi)}{5\pi^5} t^5 < \text{Si}(t) < \frac{9}{8} \sin t + \frac{3}{4} t + \frac{7}{16200} t^5, \]
and then,
\[ 1.3705 \approx \frac{2}{5} \pi - \frac{1}{300} \pi^3 + \frac{1}{5} < \text{Si} \left(\frac{\pi}{2}\right) < \frac{2}{5} \pi - \frac{1}{144} \pi^3 + \frac{1}{19 \times 200} \pi^5 \approx 1.3714, \]
\[ 1.3706 \approx \frac{1}{16} \pi + \frac{9\sqrt{3}}{16} + \frac{1}{5} < \text{Si} \left(\frac{\pi}{2}\right) < \frac{1}{8} \pi + \frac{7}{518400} \pi^5 + \frac{9\sqrt{3}}{16} \approx 1.3711. \]
Proof. Integrating each sides in (2.5) over $[0, t]$ yields (4.1). Taking the limits of the left and right hand sides in (4.1) as $p \to 0^+$ gives (4.2), and putting $p = 2/3$ in (4.1) leads to (4.3). Substituting $t = \pi/2$ into (4.2) and (4.3) we get the last two approximations of $\text{Si} \left( \frac{\pi}{2} \right)$.

It is known that
\[
\int_0^\infty x \sinh x \, dx = \frac{1}{2} \psi'(\frac{1}{2}) = \frac{\pi^2}{4},
\]
where $\psi'$ is the tri-gamma function defined by
\[
\psi'(t) = \int_0^\infty \frac{xe^{-tx}}{1-e^{-x}} \, dx.
\]
We define
\[
Sh(t) = \int_0^t \frac{x}{\sinh x} \, dx.
\]
Then by (1.16) we have
\[
3 \cosh x + 2 < \frac{x}{\sinh x} < \frac{9}{5 \cosh(\sqrt{15}x/5) + 4}.
\]
Integrating over $[0, t]$ and calculating lead to

**Proposition 2.** For $t > 0$, we have
\[
\sqrt{3} \ln \left( \frac{\sqrt{3}-\sqrt{15}+2}{\sqrt{3}+\sqrt{15}+2} \right) - \sqrt{3} \ln \left( 2 - \sqrt{3} \right) < Sh(t) < 2\sqrt{15} \arctan \left( \frac{5e^{\sqrt{15}/5} + 4}{3} \right) - 2\sqrt{15} \arctan 3.
\]
In particular, we have
\[
4.5621 \approx 2\sqrt{3} \ln \left( 2 + \sqrt{3} \right) < \psi'(\frac{1}{2}) < 2\sqrt{15} \pi - 4\sqrt{15} \arctan 3 \approx 4.9845.
\]

The Catalan constant [36]
\[
G = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2} = 0.9159655941772190...
\]
is a famous mysterious constant appearing in many places in mathematics and physics. Its integral representations contain the following [37]
\[
G = \frac{1}{\pi} \int_0^\infty \frac{\arctan x}{x} \, dx = \frac{1}{2} \int_0^{\pi/2} \frac{x}{\sin x} \, dx = \frac{\pi^2}{16} - \frac{\pi}{4} \ln 2 + \int_0^{\pi/4} \frac{x^2}{\sin^2 x} \, dx.
\]
We present an estimation for $G$ below.

**Proposition 3.** We have
\[
0.91586 \approx \frac{\sqrt{15}}{\pi} \ln \left( \frac{4 \cos \frac{\sqrt{15}}{5} + 3 \sin \frac{\sqrt{15}}{5} + 5}{4 \cos \frac{\sqrt{15}}{5} - 3 \sin \frac{\sqrt{15}}{5} + 5} \right) < G < \frac{\sqrt{15}}{\pi} \ln \left( \frac{11\sqrt{2-2\sqrt{2-3\sqrt{2+3\sqrt{2+3+32}}}}}{11\sqrt{2-2\sqrt{2-3\sqrt{2+3\sqrt{2+3+32}}}}} \right) \approx 0.91675,
\]
Proof. From the fourth and fifth inequalities in (1.17) we obtain that for $x \in (0, \pi/2)$, the two-side inequality
\[
\frac{1}{9} \cos \frac{\sqrt{15}}{5} + \frac{4}{9} < \frac{x}{\sin x} < \frac{1}{12} \cos \frac{4\pi}{5} + \frac{1}{17},
\]
holds true. Integrating both sides over \([0, \pi/2]\) yields
\[
\int_0^{\pi/2} \frac{dx}{\sin x} < \int_0^{\pi/2} \frac{x}{\cos \frac{x}{2} + \frac{4}{3}} \, dx < \int_0^{\pi/2} \frac{dx}{\cos \frac{x}{2} + \frac{4}{3}}.
\]
Direct computations give
\[
\int_0^{\pi/2} \frac{dx}{\cos \frac{15x}{5} + \frac{4}{3}} = \frac{\sqrt{15}}{2} \ln \frac{4 \cos \frac{3x}{20} - 3 \sin \frac{3x}{20} + 5}{4 \cos \frac{3x}{20} + 3 \sin \frac{3x}{20} + 5} \approx 1.8317,
\]
\[
\int_0^{\pi/2} \frac{dx}{\cos \frac{3x}{4} + \frac{11}{27}} = \frac{2\sqrt{15}}{5} \ln \frac{11\sqrt{2-\sqrt{15}}\sqrt{2+3\sqrt{2}+32}}{11\sqrt{2-\sqrt{15}}\sqrt{2+3\sqrt{2}+32}} \approx 1.8335.
\]
Utilizing the the second formula in (4.5) (4.4) follows. □

We close this paper by giving some inequalities for bivariate means.

The Schwab-Borchardt mean of two numbers \(a \geq 0\) and \(b > 0\), denoted by \(SB(a, b)\), is defined as [38, Theorem 8.4], [39, (2.3)]

\[
SB(a, b) = \begin{cases} 
\sqrt{\frac{b}{a}} - a^2 \arccos(a/b) & \text{if } a < b, \\
\frac{a}{\sqrt{b^2 - a^2}} & \text{if } a = b, \\
\sqrt{\frac{b}{a}} - b^2 \arccosh(a/b) & \text{if } a > b.
\end{cases}
\]

The properties and certain inequalities involving Schwab-Borchardt mean can be found in [40], [41]. We now establish a new inequality for this mean.

For \(a < b\), letting \(x = \arccos(a/b)\) in the fourth inequality of (1.17) and using half-angle and triple-angle formulas for cosine function, and multiplying two sides by \(b\), we get

\[
SB(a, b) \geq \frac{16b}{27} \left( \sqrt{1 + \frac{1+4(a/b)^3-3a/b}{2}} + \frac{11b}{27} \right)
\]

\[
= \frac{16b}{27} \left( \sqrt{2(b-2a)^2(a+b) + 2b^{3/2}} \right)^{1/2} b^{1/4} + \frac{11b}{27}.
\]

For \(a > b\), letting \(x = \arccosh(a/b)\) in the inequality connecting the fourth and sixth members of (1.17) and using half-angle and triple-angle formulas for hyperbolic cosine function, and multiplying two sides by \(b\), we get the same inequality as above.

**Proposition 4.** For \(a, b > 0\), we have

\[
(4.7) \quad SB(a, b) \geq \frac{8\sqrt{b}}{27} \left( |b-2a| \sqrt{\frac{a+b}{2} + b^{3/2}} \right)^{1/2} b^{1/4} + \frac{11b}{27}.
\]

**Remark 3.** From the inequality (4.7), it is easy to get

\[
SB(a, b) \geq \frac{11 + 8\sqrt{2}}{27} b \approx 0.82643 \times b
\]
due to \(|b-2a| \geq 0\). It seems to new and interesting.

For \(a, b > 0\), with \(x = (1/2) \ln (a/b)\), we have

\[
\frac{\sinh x}{x} = \frac{L(a, b)}{G(a, b)} \quad \cos px = \frac{(a^p + b^p) / 2}{\left( \sqrt{ab} \right)^p} = \frac{A_p(a, b)}{G_p(a, b)},
\]
and by Theorem 2 we immediately get the following

**Proposition 5.** For $a, b > 0$ with $a \neq b$, the double inequality

$$(4.8) \quad \frac{1}{5} A^{\sqrt{5}/5} G^{1-\sqrt{5}/5} + \frac{4}{5} G < L < \frac{1}{3} A + \frac{2}{3} G$$

holds with the best constants $p_1 = \sqrt{15}/5$ and 1. And, the function

$$p \mapsto \frac{1}{3p^2} A^p G^{1-p} + \left(1 - \frac{1}{3p^2}\right) G$$

if $p \neq 0$ and $G + \frac{(\ln b - \ln a)^2}{24}$ if $p = 0$

is increasing on $\mathbb{R}$.

**Remark 4.** Accordingly, Corollary 3 can be changed into the chain of inequalities for means:

$$A^{1/\sqrt{3}} G^{1-1/\sqrt{3}} < \frac{2}{5} A^{2/3} G^{1/3} + \frac{1}{5} G < \frac{2}{3} A^{1/\sqrt{3}} G^{1-1/\sqrt{3}} + \frac{1}{3} G$$

$$< \frac{16}{5} A^{3/4} G^{1/4} + \frac{11}{20} G < \frac{2}{5} A^{\sqrt{15}/5} G^{1-\sqrt{15}/5} + \frac{4}{5} G < L$$

$$< \frac{1}{3} A + \frac{2}{3} G < \frac{1}{2} A^{2/\sqrt{3}} G^{1-2/\sqrt{3}} + \frac{1}{2} G.$$

**Remark 5.** In [19], Yang obtained a sharp lower bound $A_{q_0}^{1/(3q_0)} G^{1-1/(3q_0)}$ for the logarithmic mean $L$, where $q_0 = 1/\sqrt{5}$, and pointed out that this one seems to be superior to most of known ones. Now, we derive a new sharp lower bound $\frac{5}{9} A^{p_1} G^{1-p_1} + \frac{2}{9} G$ for $L$, where $p_1 = \sqrt{15}/5$. We claim that the latter is better than the former. In fact, we have

$$(4.9) \quad L > \frac{5}{9} A^{p_1} G^{1-p_1} + \frac{2}{9} G > A_{q_0}^{1/(3q_0)} G^{1-1/(3q_0)}$$

if and only if $q \geq q_0 = 1/\sqrt{5}$. In order for the second inequality in (4.9) to hold, it suffices that for $x > 0$

$$D(x) = \frac{1}{5p^2} \cosh p_1 x + 1 - \frac{1}{3p^2} - (\cosh qx)^{1/(3q^2)} > 0$$

if and only if $q \geq q_0 = 1/\sqrt{5}$.

The necessity can be obtained by $\lim_{x \to 0} x^{-4} D(x) \geq 0$, which follows by expanding in power series

$$D(x) = \frac{1}{72} x^4 \left(p_1^2 + 2q^2 - 1\right) + o(x^6).$$

Then, $q \geq \sqrt{(1 - p_1^2)/2} = 1/\sqrt{5}$.

Since $q \mapsto (\cosh qx)^{1/(3q^2)}$ is decreasing on $(0, \infty)$ proved in [19] Lemma 2], to prove $D(x) \geq 0$ if $q \geq q_0$, it suffices to show that $D(x) \geq 0$ when $q = q_0$.

Differentiation yields

$$D'(x) = \frac{1}{3p_1} \sinh p_1 x - \frac{1}{3q_0} \cosh^{1/(3q_0^2)} - q_0 x \sinh q_0 x$$

$$= \frac{\sinh q_0 x}{3q_0} \left( \frac{q_0 \sinh p_1 x}{p_1 \sinh q_0 x} \cos^{1/(3q_0^2)} - q_0 x \right) - \frac{\sinh q_0 x}{3q_0} \times L \left( \frac{q_0 \sinh p_1 x}{p_1 \sinh q_0 x} \cos^{1/(3q_0^2)} - q_0 x \right) \times D_1(x),$$

where

$$D_1(x) = \ln \left( \frac{q_0 \sinh p_1 x}{p_1 \sinh q_0 x} \right) - \left( \frac{1}{3q_0^2} - 1 \right) \ln \cosh q_0 x.$$
Differentiating $D_1(x)$ gives

$$D'_1(x) = \frac{D_2(x)}{6q_0 \sinh 2q_0x \sinh p_1x},$$

where

$$D_2(x) = 4(-\sinh p_1x \sinh^2 q_0x - 3q_0^2 \cosh^2 q_0x \sinh p_1x$$

$$+ 3q_0 \sinh p_1x \sinh^2 q_0x + 3p_1q_0 \cosh p_1x \cosh q_0x \sinh q_0x).$$

Utilizing "product into sum" formulas and expanding in power series lead to

$$D_2(x) = -2\left(6q_0^2 - 1\right) \sinh p_1x + (3p_1q_0 - 1) \sinh (p_1x + 2q_0x)$$

$$+ (3p_1q_0 + 1) \sinh (2q_0x - p_1x)$$

$$= \sum_{n=1}^{\infty} d_n \frac{p_1^{2n-1}x^{2n-1}}{(2n-1)!},$$

where

$$d_n = (3p_1q_0 - 1) \left(1 + \frac{2q_0}{p_1}\right)^{2n-1} + (3p_1q_0 + 1) \left(\frac{2q_0}{p_1} - 1\right)^{2n-1} - 2 \left(6q_0^2 - 1\right)$$

Now we show that $d_n \geq 0$ for $n \geq 1$. A simple verification yields $d_1 = d_2 = 0$, $d_3 = 64/45 > 0$. Suppose that $d_n > 0$ for $n > 3$, that is,

$$(3p_1q_0 + 1) \left(\frac{2q_0}{p_1} - 1\right)^{2n-1} > 2 \left(6q_0^2 - 1\right) - (3p_1q_0 - 1) \left(1 + \frac{2q_0}{p_1}\right)^{2n-1}.$$ Then,

$$d_{n+1} = (3p_1q_0 - 1) \left(1 + \frac{2q_0}{p_1}\right)^{2n+1} + (3p_1q_0 + 1) \left(\frac{2q_0}{p_1} - 1\right)^{2n+1} - 2 \left(6q_0^2 - 1\right)$$

$$\times \left(\frac{2q_0}{p_1} - 1\right)^{2} - 2 \left(6q_0^2 - 1\right)$$

$$= 8q_0 \frac{p_1}{p_1^2} \left(p_1 (3p_1q_0 - 1) \left(1 + \frac{2q_0}{p_1}\right)^{2n-1} - (6q_0^2 - 1) (p_1 - q_0)\right).$$

Since $(3p_1q_0 - 1) = (3\sqrt{3} - 5) / 5 > 0$, using binomial expansion we get

$$\frac{p_1^2}{8q_0} d_{n+1} > p_1 (3p_1q_0 - 1) \left(1 + (2n - 1) \frac{2q_0}{p_1}\right) - (6q_0^2 - 1) (p_1 - q_0)$$

$$= 2q_0 (3p_1q_0 - 1) (2n - 1) + q_0 (3p_1^2 + 6q_0^2 - 6p_1q_0 - 1)$$

$$= 2q_0 (3p_1q_0 - 1) (2n - 1) - 2q_0 (3p_1q_0 - 1)$$

$$= 4q_0 (3p_1q_0 - 1) (n - 1) > 0,$$

where the third equality holds due to $3p_1^2 + 6q_0^2 = 3$. By mathematical induction, we have proven $D_2(x) \geq 0$ for $n \geq 1$. It follows that $D'_1(x) > 0$, which means that $D_1$ is increasing on $(0, \infty)$, and then $D_1(x) > \lim_{x \to 0^+} D(x) = 0$. This in turn implies that $D$ is increasing on $(0, \infty)$, and therefore, $D(x) \geq D(0^+) = 0$, which proves the sufficiency.
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