The global properties of the finiteness and continuity of the Lorentzian distance

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Abstract

It is well-known that global hyperbolicity implies that the Lorentzian distance is finite and continuous. By carefully analysing the causes of discontinuity of the Lorentzian distance, we show that in most other respects the finiteness and continuity of the Lorentzian distance is independent of the causal structure. The proof of these results relies on the properties of a class of generalised time functions introduced by the authors in [16].

Keywords: Lorentzian geometry, Lorentzian distance, causal structure, time function

MSC2010: 53C50, 83C65

1 Introduction

It is well-known that the Lorentzian distance in a globally hyperbolic manifold is finite and continuous, [1, Lemma 4.5]. There are a handful of other results that describe the properties of manifolds with continuous Lorentzian distances, e.g. [1 Theorem 4.24] and [10 Theorems 2.2, 2.4 and 3.6]. These results suggest that the Lorentzian distance should be related, at least in a conformal sense, to other conditions in the causal hierarchy.

We show that this is not so, apart from the few results mentioned above. Finiteness and continuity, both jointly and separately, are almost entirely independent of the causal hierarchy, Theorem 4.1. To prove this result we use new class of generalised time functions introduced by the authors.

In [16] the authors’ gave the following characterisation of the finiteness of the Lorentzian distance and proved a version of the Lorentzian distance formula, particular versions of which were proved in [4, 8, 12]: see [5] for a review and [2] for a recent reformulation.

Theorem 1.1 (Finiteness of the Lorentzian distance [16]). Let $(M, g)$ be a Lorentzian manifold. The Lorentzian distance is finite if and only if there exists a function $f : M \rightarrow \mathbb{R}$ so that $\text{esssup } g(\nabla f, \nabla f) \leq -1$.

Such a function is necessarily monotonic on timelike curves.

Theorem 1.2 (The Lorentzian distance formula). If $(M, g)$ has finite Lorentzian distance then for all $p, q \in M$

$$d(p, q) = \inf \{|f(q) - f(p)| : f : M \rightarrow \mathbb{R}, \text{esssup } g(\nabla f, \nabla f) \leq -1\}.$$

Theorems 1.1 and 1.2 were proven by showing how to construct a sufficiently large number of functions, that we call surface functions, with the appropriate properties, see Definition
1.4 It is natural to wonder if these functions also characterise continuity and could provide a converse to the globally hyperbolic result [1, Lemma 4.5] mentioned in the first paragraph. The versions of the Lorentzian distance formula reviewed in [5], all require the assumption of a condition in the causal hierarchy, e.g., global hyperbolicity or stable causality. This reduces their applicability, but allows for stronger regularity conditions on the functions involved. In particular, Minguzzi [8, Theorem 97] has shown that in stably causal manifolds the finiteness and continuity of the Lorentzian distance is equivalent to the Lorentzian distance formula holding.

We have already shown that if the Lorentzian distance is continuous then all surface functions are continuous [16, Corollary 3.15]. With the assumption that the Lorentzian distance is finite we prove the converse to [16, Corollary 3.15] in Theorem 2.6.

Theorem 2.6. Let $M$ have finite Lorentzian distance. The Lorentzian distance is continuous if and only if every function in $\mathcal{S}(M)$ is continuous.

Thus Theorems 2.6 and 1.2 provide a generalisation of Minguzzi’s result that drops the assumption of stably causality.

Corollary 1.3. Let $(M, g)$ be a Lorentzian distance. The Lorentzian distance is finite and continuous if and only if $\mathcal{S}(M) \neq \emptyset$ and every element of $\mathcal{S}(M)$ is continuous.

In Theorem 2.7 we rephrase the necessary and sufficient conditions for continuity of the Lorentzian distance in terms of the limiting behaviour of lengths of curves. In the process we remove the requirement for the Lorentzian distance to be finite, and this relies heavily on the characterisation of continuity by the functions $\mathcal{S}(M)$.

Theorem 2.7. Let $(M, g)$ be a Lorentzian manifold. The Lorentzian distance is discontinuous if and only if there exists $x, y \in M$, with $d(x, y) < \infty$, and $(x_i)_{i \in \mathbb{N}} \subset M$ a future directed sequence converging to $x$, $(y_i)_{i \in \mathbb{N}} \subset M$ a past directed sequence converging to $y$ and a sequence of curves $(\gamma_i)_{i \in \mathbb{N}}$ so that for all $i \in \mathbb{N}$, $\gamma_i \in \Omega_{x_i, y_i}$, and at least one of the following is true:

- $\lim_{i \to \infty} L(\gamma_i \setminus I^+(x)) > 0$,
- $\lim_{i \to \infty} L(\gamma_i \setminus I^-(y)) > 0$.

Section 3 studies the relationship between the conformal structure and the finiteness and continuity of the Lorentzian distance, as in [10]. We show that the causes of discontinuity of the Lorentzian distance are of two kinds: those invariant under conformal transformations, and those which can be introduced or removed by conformal transformations. We follow up in subsection 3.1 by presenting a few results, both new and old, that follow easily from the preceding material. These results relate the Lorentzian distance to causal structure. The results of Sections 2 and 3 taken together indicate that there is a weak relationship between finiteness and continuity of the Lorentzian distance and the causal hierarchy. This confirms the impression left by [10].

In Section 4 we show that this weak relationship is very weak. We give a series of examples showing that finiteness and continuity of the Lorentzian distance, both jointly and separately, are almost entirely independent of all the “standard” causality conditions weaker than global hyperbolicity, see Theorem 4.1. Causal structure, the Lorentzian distance function and the surface functions that we employ are all global objects, of which the causal structure is always conformally invariant. Our results indicate that while the causal structure and good properties of the Lorentzian distance are largely independent, properties of the Lorentzian distance and the surface functions are tightly intertwined, c.f. Theorems 1.1, 1.2 and 2.6.

1.1 Notation

A Lorentzian manifold, $(M, g)$, is a smooth, Hausdorff, paracompact manifold, $M$, equipped with a Lorentzian metric, $g$. We will not always mention the metric explicitly. Two manifolds, $(M, g), (N, h)$ are conformally related if $M = N$ and there exists $\Omega : M \to \mathbb{R}$ so that $h = \Omega^2 g$. 


When necessary we will explicitly mention the metric, for example $L(\gamma; g)$ is the arc length of $\gamma$ with respect to $g$, $\Gamma^+(x; g)$ is the future of $x$ with respect to $g$, $d(x, y; g)$ is the Lorentzian distance between $x$ and $y$ with respect to $g$, and so on. Unless otherwise mentioned, we assume that curves are piecewise $C^1$, with everywhere non-zero tangent vector, and we treat them as both sets and functions. For example, $\gamma : [0, 1] \to M$ is a curve and if $x \in M$ then by $\gamma \cap \Gamma^+(x)$ we mean the subcurve of $\gamma$ whose image is $\gamma([0, 1]) \cap \Gamma^+(x)$. We make use of several of limit curve results for continuous causal curves. These results are scattered in a variety of sources, we have collected those that we need in Appendix A.

A subset, $B$, of $M$ is an achronal boundary if there exists $F \subset M$ so that $F = \Gamma^+(F)$ and $B = \partial F$. A set $U \subset M$ is convex is any two points in $U$ can be joined by a unique geodesic curve contained in $U$.

Given $x, y \in M$ let $\Omega_{x,y}$ be the, possibly empty, set of future directed piecewise smooth timelike curves from $x$ to $y$.

**Definition 1.4.** Let $S \subset M$ be an achronal boundary such that $M = \Gamma^+(S) \cup S \cup \Gamma^-(S)$ and for all $x \in M$, $d(x, S)$ and $d(S, x)$ are finite. The function $\tau_S : M \to \mathbb{R}$ defined by

$$
\tau_S(x) := \begin{cases} 
  d(S, x) & x \in \Gamma^+(S) \\
  0 & x \in S \\
  -d(x, S) & x \in \Gamma^-(S)
\end{cases}
$$

will be called the surface function of $S$. The set of all surface functions induced by an achronal surface as above will be denoted $S(M)$.

The surface functions are differentiable a.e., monotonically increasing on all timelike curves, and satisfy $\text{ess sup} g(\nabla \tau_S, \nabla \tau_S) \leq -1$; see [1]. If the Lorentzian distance is finite then at least one of these functions exists, [16]. Otherwise our notation and definitions follow [1].

### 2 Characterising continuity

We already know from the proof of Theorem [1] see [16], that finiteness of the Lorentzian distance is equivalent to the existence of a surface function. In this section we give two characterisations of continuity of the Lorentzian distance: one in terms of surface functions, and one in terms of the behaviour of lengths of curves. Our first lemma is “half” of our final result on the behaviour of lengths of curves.

**Lemma 2.1.** Let $(M, g)$ be a Lorentzian manifold, and $(x, y) \in M \times M$. If the Lorentzian distance is discontinuous at $(x, y)$ then for all future directed sequences $(x_i)_{i \in \mathbb{N}}$ converging to $x$, and all past directed sequences $(y_i)_{i \in \mathbb{N}}$ converging to $y$, there exists a sequence of future directed curves $(\gamma_i)_{i \in \mathbb{N}}$ so that $\gamma_i \in \Omega_{x_i, y_i}$, $\lim_{i \to \infty} L(\gamma_i) = \lim_{i \to \infty} d(x_i, y_i) > d(x, y)$ and at least one of the following is true:

- $\lim_{i \to \infty} L(\gamma_i \setminus \Gamma^+(x)) > 0$,
- $\lim_{i \to \infty} L(\gamma_i \setminus \Gamma^-(y)) > 0$.

**Proof.** Since the Lorentzian distance is discontinuous at $(x, y)$ there exists a sequence, $((u_i, v_i))_{i \in \mathbb{N}} \subset M \times M$, so that $(u_i, v_i) \to (x, y)$ and $\lim_{i \to \infty} d(u_i, v_i) \neq d(x, y)$. Since the Lorentzian distance is lower semi-continuous, [1] Lemma 4.4, $\lim_{i \to \infty} d(u_i, v_i) > d(x, y)$. Let $(x_i)_{i \in \mathbb{N}}$ be a future directed sequence converging to $x$ and $(y_i)_{i \in \mathbb{N}}$ a past directed sequence converging to $y$. For all $i \in \mathbb{N}$ there exists $N \in \mathbb{N}$ so that for all $j \geq N$, $u_j, v_j \in \Gamma^+(y_i) \cap \Gamma^-(x_i)$, hence $d(x_i, y_i) \geq \lim_{j \to \infty} d(u_j, v_j)$. Thus $\lim_{i \to \infty} d(x_i, y_i) \geq \lim_{i \to \infty} d(u_j, v_j) > d(x, y)$. By construction $\Omega_{x_i, y_i} \neq \emptyset$. The existence of a sequence of curves $(\gamma_i)_{i \in \mathbb{N}}$ so that $\gamma_i \in \Omega_{x_i, y_i}$, $\lim_{i \to \infty} L(\gamma_i) = \lim_{i \to \infty} d(x_i, y_i) > d(x, y)$ now follows directly from the definition of the Lorentzian distance.
Lemma 2.3. The proof of the Lorentzian distance formula given in [16]. The following lemma is a technical result which summarises a technique first used in the proof of the Lorentzian distance formula given in [16].

It remains to show that at least one of

\[
\lim_{i \to \infty} L\left(\gamma_i \setminus I^+(x)\right) > 0, \quad \text{or} \quad \lim_{i \to \infty} L\left(\gamma_i \setminus I^-(y)\right) > 0,
\]

is true. For each \(i \in \mathbb{N}\), let \(\gamma_i^- = \gamma_i \setminus I^+(x)\) and \(\gamma_i^+ = \gamma_i \setminus I^-(y)\). These sub-curves are not necessarily disjoint. We have that \(L(\gamma_i) = L(\gamma_i^- \cup \gamma_i^+) + L(\gamma_i \cap I^+(x) \cap I^-(y))\). Since \(\lim_{i \to \infty} L(\gamma_i \cap I^+(x) \cap I^-(y)) \leq d(x, y)\) and \(\lim_{i \to \infty} L(\gamma_i) > d(x, y)\) it is the case that \(\lim_{i \to \infty} L(\gamma_i^-) > 0\). This implies that at least one of \(\lim_{i \to \infty} L(\gamma_i^-) > 0\) or \(\lim_{i \to \infty} L(\gamma_i^+) > 0\) is true as required.

The “obvious” converse of Lemma 2.1 is not true. Namely we can have a sequence of curves with the strange ‘limiting length’ behaviour, but still have continuity of the Lorentzian distance at the point in question. The following example illustrates this behaviour.

Example 2.2. Let \(M = \mathbb{R}^2 \setminus \{(1, t) \in \mathbb{R}^2 : t \in [1, 2]\}\). Let \(V = \{(1 + s, t + s) \in \mathbb{R}^2 : t \in (1, 2), s \in (0, \infty)\}\). Let \(x = (0, 0), y = (0, 4), (\gamma_i)_{i \in \mathbb{N}}\) be a future directed sequence converging to \(x\) and \((\gamma_i)_{i \in \mathbb{N}}\) be a past directed sequence converging to \(y\). We now show that \(d\) is continuous at \((x, y)\) and that there exists a sequence of curves \((\gamma_i)_{i \in \mathbb{N}}\) so that for all \(i \in \mathbb{N}\), \(\gamma_i \in \Omega_{x, y}\), and \(\lim_{i \to \infty} L(\gamma_i \setminus I^+(x)) > 0\). Hence we give a counter example to the converse of Lemma 2.1. The situation is depicted in Figure 1.

We show that any curve from \(x_i\) to \(y_i\) through \(V\) must have length less than 4. As the metric is flat, for any sequence of curves \((\gamma_i)_{i \in \mathbb{N}}\) so that for each \(i\), \(\gamma_i \in \Omega_{x, y}\), we have that \(\lim_{i \to \infty} L(\gamma_i \setminus I^+(y)) = 0\) and \(\lim_{i \to \infty} L(\gamma_i \setminus (I^+(x) \cup V)) = 0\). Hence to get an upper bound for \(\lim_{i \to \infty} L(\gamma_i)\) we can calculate the limit of the lengths of the longest timelike geodesic from \(y_i\) to the boundary of \(V\) and the length of the longest timelike geodesic in \(V\). The limit of the lengths of the timelike geodesic from \(y_i\) to the boundary of \(V\) will be equal to the length of longest timelike geodesic from \(y\) to \(V\).

The portion of \(V\) in the past of \(y\) is the quadrilateral whose four vertices are \((1, 1), (1, 2), (\frac{3}{2}, \frac{3}{2}), (2, 2)\). The reverse triangle inequality implies that the longest timelike geodesic from \(y\) to the line segment from \((1, 2)\) to \((\frac{3}{2}, \frac{3}{2})\) is the straight line from \(y\) to \((1, 2)\). Similarly the reverse triangle inequality implies that the longest timelike geodesic in \(V\) is the straight line from \((1, 1)\) to \((\frac{3}{2}, \frac{3}{2})\). A little care is needed here since the points \((1, 2)\) and \((1, 1)\) are not in the manifold. The lengths of the two geodesics can be calculated as \(\sqrt{3}\) and \(\sqrt{2}\). Since \(\sqrt{3} + \sqrt{2} < 4\) we have demonstrated the claim.

Since the metric is flat, geodesics are straight lines, thus a simple calculation shows that \(d(x, y) = 4\). This implies that \(d\) is continuous at \((x, y)\). It remains to show that \(\lim_{i \to \infty} L(\gamma_i \setminus I^+(x)) > 0\). This follows immediately as \(V \subset M \setminus I^+(x)\) and \(V\) is open. Note that as \(V\) is open and as \(V \subset I^+(x_i)\) for all \(i \in \mathbb{N}\) then if \(v \in V\) then \(d\) is discontinuous at \((x, v)\) \(\triangle\).

It is the case, however, that the presence of a sequence of curves \((\gamma_i)_{i \in \mathbb{N}}\) with the properties described in the conclusion of Lemma 2.1 does imply discontinuity of the Lorentzian distance. We just don’t know where the discontinuity occurs. Thus if insistence that the discontinuity occurs at \((x, y)\) is dropped then this form of the converse of Lemma 2.1 does hold. The statement is in Theorem 2.7. The rest of this section sets out to prove this.

The proof of the converse relies on surface functions to detect when discontinuity occurs, without needing to know precisely where the discontinuity is located. Hence, along the way to our ultimate goal, we prove a simple characterisation of continuity in Lorentzian manifolds with finite Lorentzian distance.

The following lemma is a technical result which summarises a technique first used in the proof of the Lorentzian distance formula given in [16].

Lemma 2.3. Let \(M\) have finite Lorentzian distance. If \(x \in M\) and \(y \in I^+(x)\) then there exists an achronal boundary \(S\) so that

1. \(M = I^-(S) \cup S \cup I^+(S)\),
2. \(x \in S\),
Figure 1: An illustration of the proof that the Lorentzian distance is continuous at \((x, y) = ((0, 0), (0, 4))\) given in Example 2.2. In each diagram the boundary of the future \(x\) is given by the upward sloped dashed lines and the boundary of the past of \(y\) is given by the downward sloped dashed lines. The hashed area is \(V\) and the black stripe is the line which has been removed from the manifold. In the left hand diagram \(x_i\) is a representative element of the future directed sequence converging to \(x\), \(y_i\) is a representative of the past directed sequence converging to \(y\) and \(\gamma_i\) is a representation of the sequence of timelike curves given in the example. In the right hand diagram the two thinner lines from \(y\) to \(V\) and inside \(V\) are the two maximal timelike geodesics that are used to show that the maximum length of a curve from \((0, 0)\) to \((0, 4)\) is less that \(\sqrt{2} + \sqrt{3}\).
3. if $\gamma$ is a timelike curve from $S$ to $y$ then $\gamma \cap S \subset \partial I^+(x) \cap S$, and,
4. for all $z \in M$, $d(z, S) < \infty$ and $d(S, z) < \infty$.

**Proof.** Since the Lorentzian distance is finite [16, Lemma 3.7] and [16, Lemma 3.12] imply that there exists $S_1 \subset M$ an achronal boundary so that $M = I^+(S_1) \cup S_1 \cup \Gamma^-(S_1)$ and that, for all $z \in M$ $d(z, S_1) < \infty$ and $d(S_1, z) < \infty$. We now modify $S_1$ by adding/removing bits to its past/future to define a surface $S$ with the required property. The situation is depicted in Figure 2.

Let $S_2 = \partial (\Gamma^- (y) \cup \Gamma^- (S_1))$. Since $\Gamma^- (y) \cup \Gamma^- (S_1)$ is a past set $S_2$ is an achronal boundary. If $y \in \Gamma^- (S_1) \cup S_1$ then $S_1 = S_2$ so that $M = I^+(S_2) \cup S_2 \cup \Gamma^-(S_2)$. If $y \in I^+(S_1)$ then $y \in S_2$ and hence $\Gamma^- (S_2) = \Gamma^- (y) \cup I^-(S_1)$. Some definition chasing shows that $I^+(S_2) = I^+(S_1) \setminus \Gamma^- (y)$. Since $S_1 = S_1 \setminus \Gamma^- (y) \cup (S_1 \cap I^- (y))$ we have that $M = I^+(S_2) \cup S_2 \cup I^-(S_2)$ as required.

By construction for all $z \in M$, $d(z, S_2) \leq \max \{d(z, S_1), d(z, y)\}$. Since the Lorentzian distance is finite $d(z, S_2) < \infty$. Similarly since $I^+(S_2) \subset I^+(S_1)$, for all $z \in M$, $d(S_2, z) \leq d(S_1, z) < \infty$. Let $S = \partial (I^+(x) \cup I^+(S_2))$. The time dual of the argument about $S_2$ shows that $S$ is achronal, $M = I^+(S) \cup S \cup \Gamma^-(S)$ and that for all $z \in M$, $d(z, S)$ and $d(S, z)$ are finite valued.

We now show that $x \in S$. If $y \in I^+(S_1) \cup S_1$ then $y \in S_2$ so that $x \in \Gamma^- (S_2)$. If $y \in I^+(S_1)$ then $y \in \Gamma^- (S_2)$ so that $x \in \Gamma^- (S_2)$. Thus, in either case $x \in \Gamma^- (S_2)$. By construction this implies that $x \in S$ as required.

Let $\gamma : [0, 1] \to M$ be a future directed curve from $S$ to $y$. By construction $S = S_2 \setminus I^+(x) \cap \partial (I^+(x) \cap \Gamma^- (S_2))$. Since $y \in I^+(x)$, the point $\gamma(0)$ is contained in $\partial (I^+(x) \cap \Gamma^- (S_2)) = \partial (I^+(x) \cap S)$. As $\gamma \cap S = \{\gamma(0)\}$ we have the result. \[\square\]

**Corollary 2.4.** Let $M$ have finite Lorentzian distance, $(x, y) \in M \times M$, $(x_i)_i \in \mathbb{N}$ be a future directed sequence converging to $x$, $(y_i)_i \in \mathbb{N}$ a past directed sequence converging to $y$ and $(\gamma_i)_i \in \mathbb{N}$ a sequence of curves so that for all $i \in \mathbb{N}$, $\gamma_i \in \Omega_{x_i, y_i}$. Then there exists an achronal boundary $S \subset M$ so that $M = I^+(S) \cup S \cup \Gamma^-(S)$, $x \in S$ and there exists $N \in \mathbb{N}$ so that for all $i \geq N$, $\gamma_i \setminus I^+(x) \subset \Gamma^-(S)$.

**Proof.** Choose $n \in \mathbb{N}$ and apply Lemma 2.3 using $x \in M$ and $y_n \in I^+(x)$. The result is an achronal boundary $S$ so that

1. $M = I^-(S) \cup S \cup I^+(S)$,
2. $x \in S$.
3. if $\gamma$ is a timelike curve from $S$ to $y_n$ then $\gamma \cap S \subset S \cap \partial I^+(x)$, and,
4. for all $z \in M$, $d(z, S) < \infty$ and $d(S, z) < \infty$.

The situation is depicted in Figure 3. We now show that for all $i \geq n$, $\gamma_i \setminus I^+(x) \subset \Gamma^-(S)$. Let $i \geq n$. Since $x_i \in I^-(x) \subset \Gamma^-(S)$ and $y_i \in I^+(x) \subset I^+(S)$, [14, Proposition 3.15] implies that there exists a unique point $z \in \gamma_i \cap S$. As $\gamma_i \cap \Gamma^+(S)$ is a timelike curve from $S$ to $y$ we know that $z \in S \cap \partial I^+(x)$. Hence $\gamma_i \setminus I^+(x) \subset \Gamma^-(S)$ as required. \[\square\]

**Corollary 2.5.** If in addition to the assumptions of Corollary 2.4, $\lim_{i \to \infty} L(\gamma_i \setminus I^+(x)) > 0$ then the surface function of $S$ is discontinuous at $x$.

**Proof.** By assumption the surface function of $S$ satisfies $\tau_S(x_i) = d(x_i, S) > \lim_{i \to \infty} L(\gamma_i \setminus I^+(x)) > 0 = \tau_S(x)$. Since $(x_i)_i$ converges to $x$, $\tau_S$ is discontinuous at $x$. \[\square\]

Observe that Corollary 2.5 does not say that the Lorentzian distance is discontinuous at $x$.

**Theorem 2.6.** Let $M$ have finite Lorentzian distance. The Lorentzian distance is continuous if and only if every function in $S(M)$ is continuous.
Figure 2: The three figures above illustrate the construction of $S$ given in the proof of Lemma 2.3 when $y \in I^+(S_1)$.

Figure 3: The figure presents the essence of the geometric argument used in the proof of Corollary 2.4 to show that $\gamma_i \setminus I^+(x) \subset I^-(S)$. The curve from $x_i$ to $y_i$ represents $\gamma_i$. The solid part of this curve is $\gamma_i \setminus I^+(x)$ the dashed portion is $\gamma_i \cap I^+(x)$. Note that $d(x_i, S) > L(\gamma_i \setminus I^+(x))$. Hence if $\lim_{i \to \infty} L(\gamma_i \setminus I^+(x)) > 0$ then $\lim_{i \to \infty} d(x_i, S) > 0$. 
Proof. It is already known that if the Lorentzian distance is continuous then every element of \( S(M) \) is continuous, [10, Corollary 3.15].

Suppose that the Lorentzian distance is discontinuous. Lemma 2.1 implies that there exists \( x, y \in M, (x_i)_{i \in \mathbb{N}} \subset M \) a future directed sequence converging to \( x \), \( (y_i)_{i \in \mathbb{N}} \subset M \) a past directed sequence converging to \( y \) and a sequence of curves \( (\gamma_i)_{i \in \mathbb{N}} \) so that \( \gamma_i \in \Omega_{x, y} \), 
\[
\lim_{i \to \infty} d(x_i, y_i) = \lim_{i \to \infty} L(\gamma_i) > d(x, y) \quad \text{and at least one of the following is true:}
\]
\[\begin{align*}
\bullet \quad & \lim_{i \to \infty} L(\gamma_i \setminus I_+(x)) > 0, \\
\bullet \quad & \lim_{i \to \infty} L(\gamma_i \setminus I_-(y)) > 0.
\end{align*}\]

Hence Corollary 2.5 or its time dual, gives the result. \( \square \)

It is now possible to remove the assumption of finiteness from Theorem 2.6 and thus prove a converse to Lemma 2.1

**Theorem 2.7.** Let \((M, g)\) be a Lorentzian manifold. The Lorentzian distance is discontinuous if and only if there exists \( x, y \in M, \) with \( d(x, y) < \infty, \) and \((x_i)_{i \in \mathbb{N}} \subset M \) a future directed sequence converging to \( x, (y_i)_{i \in \mathbb{N}} \subset M \) a past directed sequence converging to \( y \) and a sequence of curves \( (\gamma_i)_{i \in \mathbb{N}} \) so that for all \( i \in \mathbb{N}, \gamma_i \in \Omega_{x, y}, \) and at least one of the following is true:
\[\begin{align*}
\bullet \quad & \lim_{i \to \infty} L(\gamma_i \setminus I_+(x)) > 0, \\
\bullet \quad & \lim_{i \to \infty} L(\gamma_i \setminus I_-(y)) > 0.
\end{align*}\]

This theorem expresses the idea that the Lorentzian distance is discontinuous if and only if there is a sequence of curves whose lengths limit to a non-zero value when the limit “should” be zero (with some regularity assumptions included). To put that into context, the curves \( \gamma_i \setminus I_+(x) \) limit to a causal curve in a null surface and so lengths of the curves “should” also limit to zero. This is what happens in globally hyperbolic manifolds (just as the theorem implies) and what happens when the limit curve between \( x \) and \( y \) exists. One way to interpret the theorem is that if the limiting null surface isn’t really null or when the limit isn’t sufficiently uniform then discontinuities of the Lorentzian distance result, see Examples 2.2, 3.1, and 3.2.

The theorem does not claim that the Lorentzian distance is discontinuous at \((x, y)\). Example 2.2 shows that the conditions of the theorem can be satisfied but the Lorentzian distance is continuous at \((x, y)\). What is important is that the non-zero limit of the lengths of the given sub-curves implies that there exists some pair \((u, v) \in M \times M\) at which the Lorentzian distance is discontinuous. The proof relies, in a non-trivial way, on Theorem 2.6 to avoid direct specification of \((u, v)\). That is, by appealing to the continuity of the globally defined surface functions the need to explicitly identify \(u \) and \( v \) can be avoided.

Proof. Lemma 2.1 proves the “if” portion of the result. So suppose that there exists \( x, y \in M, \) with \( d(x, y) < \infty, \) and \((x_i)_{i \in \mathbb{N}} \subset M \) a future directed sequence converging to \( x, (y_i)_{i \in \mathbb{N}} \subset M \) a past directed sequence converging to \( y \) and a sequence of curves \( (\gamma_i)_{i \in \mathbb{N}} \) so that \( \gamma_i \in \Omega_{x, y}, \) and such that at least one of the following is true;
\[\begin{align*}
\bullet \quad & \lim_{i \to \infty} L(\gamma_i \setminus I_+(x)) > 0, \\
\bullet \quad & \lim_{i \to \infty} L(\gamma_i \setminus I_-(y)) > 0.
\end{align*}\]

Since these two conditions are time duals of each other we can, without loss of generality, assume that for all \( i \in \mathbb{N}, \lim_{i \to \infty} L(\gamma_i \setminus I_+(x)) > 0. \)

If for all \( i, d(x_i, y_i) = \infty \) then as \( d(x, y) < \infty \) the Lorentzian distance is discontinuous at \((x, y)\). So, suppose that there exists \( j \in \mathbb{N} \) so that \( d(x_j, y_j) < \infty. \)

Let \( N = I^+(x_j) \cap I^-(y_j). \) We consider \( N \) as a submanifold with the induced metric. By assumption \( N \) has a finite Lorentzian distance. Let \( d_N \) denote the Lorentzian distance on \( N \) induced by the ambient metric. By definition for any \( u, v \in N, d_N(u, v) \leq d(u, v). \) Since \( N \)
is a causal diamond for any \( u, v \in N \) if \( \gamma \) is a timelike curve from \( u \) to \( v \) in \( M \) then \( \gamma \subset N \). This implies that \( d_N(u,v) = d(u,v) \).

For all \( i \in \mathbb{N} \), let \( \tilde{x}_i = x_{i+j}, \tilde{y}_i = y_{i+j} \) and \( \tilde{\gamma}_i \) is \( \gamma_{i+j} \). Then \( (\tilde{x}_i)_{i \in \mathbb{N}} \subset N \) is a future directed sequence converging to \( x \) in \( N \), \( (\tilde{y}_i)_{i \in \mathbb{N}} \subset N \) is a past directed sequence converging to \( y \) in \( N \) and \( (\tilde{\gamma}_i)_{i \in \mathbb{N}} \) is a sequence of curves in \( N \) so that \( \tilde{\gamma}_i \in \Omega_{\tilde{x}_i,\tilde{y}_i} \). By assumption, \( \lim_{i \to \infty} L(\gamma_i \setminus \Gamma^+(x)) > 0 \). Corollary 2.4 implies that there exists a discontinuous eikonal function on \( N \) induced by some achronal boundary. Theorem 2.6 implies that there exists \( u, v \in N \) so that \( d_N \) is discontinuous at \( (u,v) \). Since \( N \) is open and \( d_N = d|_{N \times N} \), \( d \) is discontinuous at \( (u,v) \).

3 The Lorentzian distance and conformal transformations

Both finiteness and continuity of the Lorentzian distance can be altered by conformal transformations, e.g. [10, Theorems 2.4 and 3.6]. Theorem 2.7 implies that this relationship can be studied by understanding how conformal transformations change the lengths of curves.

The set \( V \) of Example 2.2 is conformally invariant, hence the “causes of discontinuity” of the Lorentzian distance in this case can not be removed by a conformal transformation. To support the idea that discontinuities of the Lorentzian distance can either be conformally invariant or removable (or introducable!) by conformal transformation, we begin our study with two examples which, in contrast to Example 2.2, illustrate discontinuities which can be removed by conformal transformations.

Example 3.1. Let \( M = \mathbb{R}^2 \setminus \{(0,0)\} \). Let \( g \) be the metric given by

\[
g = \frac{1}{(x^2 + y^2)^2} (-dy^2 + dx^2).\]

Let \( x = (0,-1) \). Let \( F = \Gamma^+((0,0)) \) considered as a subset of \( M \). A simple calculation shows that for all \( y \in F \), \( d(x,y) = \infty \). The Lorentzian distance is continuous on \((M,g)\). The manifold \((M,g)\) is causally continuous.
Figure 5: Schematic representation of the construction used in the second part of Example 3.1.

Let \( \tilde{F} = \{ (x,y) \in M : y > 0, |2x| < y \} \). One can think of \( \tilde{F} \) as the future of \( (0,0) \) with respect to the metric \(-2dy^2 + dx^2\). Let \( \rho : M \to [0,1] \) be a bump function so that \( \rho|_{\tilde{F}} = 1 \) and \( \rho|_{M \setminus \tilde{F}} = 0 \). Let \( h = \left( \frac{\rho(x,y)}{\sqrt{x^2+y^2}} + 1 - \rho(x,y) \right)(-dy^2 + dx^2) \).

Let \( y \in F \). Then there exists \( \tilde{y} \in \tilde{F} \cap \Gamma^-(y) \). Again a simple computation shows that \( d(x,y;h) = \infty \). Let \( y \in \partial F \). Then, by definition of \( \rho \), \( d(x,y;h) < \infty \). Thus the Lorentzian distance of \( h \) is discontinuous at \((x,y)\). Since \( M \) is distinguishing, Theorem 3.20 implies that \((M,h)\) is causally continuous.

In Propositions 3.3 and 3.6 we show how this form of discontinuity is related to conformal transformations. \( \triangle \)

**Example 3.2.** Let \( N = \mathbb{R}^2 \setminus \{ (0,y) \in \mathbb{R}^2 : y \in [-1,1] \} \) equipped with the metric, \( g \), induced by 2-dimensional Minkowski space. Let \( M = \mathbb{R}^2 \setminus \{ (0,0) \} \). We claim that there is a diffeomorphism \( \phi : N \to M \), so that \( \phi(\partial \Gamma^+((0,1))) = \partial \Gamma^+((0,0)) \) and \( \phi(\partial \Gamma^-((0,-1))) = \partial \Gamma^-((0,0)) \). Equip \( M \) with the push forward, \( \phi_*g \).

Let \( x = (-1,-1) \) and \( y = (1,1) \). By construction \( d_{\phi_*g}(x,y) = 0 \) as there are no timelike curves in \( M \) starting at \( x \) and ending at \( y \). Let \( (x_i)_{i \in \mathbb{N}} \) be a future directed sequence converging to \( x \) and for each \( i \in \mathbb{N} \) let \( \gamma_i \in \Omega_{x_i,y} \). By construction the length of each \( \gamma_i \) in \( M \) is the same as the length of \( \phi^{-1}(\gamma_i) \) and therefore is larger than 1. Thus \( \lim_{i \to \infty} d_{\phi_*g}(x_i,y) = d(x,y) \) and \( d \) is discontinuous at \((x,y)\).

It remains to show that the claimed diffeomorphism \( \phi \) exists. Let \[ h(t) = \begin{cases} \exp\left(\frac{-1}{t}\right) & \text{if } t > 0 \\ 0 & \text{if } t \leq 0 \end{cases} \]

and \[ k(t) = 2 - \frac{h\left(\frac{1}{2} - t\right)}{h\left(\frac{1}{2} - t\right) + h\left(\frac{1}{2} + t\right)} \] Then \[ \phi(x,y) = \begin{cases} (x,y-1) & \text{if } (x,y) \in \Gamma^+((0,1)) \\ (x,y+k) & \text{if } (x,y) \in N \setminus (\Gamma^+((0,1)) \cup \Gamma^-((0,-1))) \\ (x,y+1) & \text{if } (x,y) \in \Gamma^-((0,0)) \] is the required diffeomorphism. Since \( M \) is distinguishing, Theorem 3.20 implies that \((M,\phi_*g)\) is causally continuous. \( \triangle \)

**Proposition 3.3.** Let \((M,g)\) be a Lorentzian manifold. If the Lorentzian distance is discontinuous at \((x,y) \in M \times M\) then there exists a past inextendible timelike curve \( \lambda \) so that for all \( u \in \Gamma^-(x), \lambda \subset \Gamma^+(u) \) and \( \Gamma^+(y) \subset \Gamma^+(\lambda) \).
From above we know that $i > j$. By construction for all $b$, the past is complete. This would imply that $I \cap \mathbb{R}$ is past inextendible $I$. By construction $I$ is past inextendible $I$. 

Choose $h$ a complete Riemannian metric. We are free to parametrise each $\gamma_i$ with respect to the arc length induced by $h$. Hence for each $i$ there exists $b_i > 0$ so that $\gamma_i \cap [0, b_i] \rightarrow M$. From above we know that $b_i \rightarrow \infty$. Let $\gamma$ be the past directed timelike limit curve from $y$ given by iterated application of Lemma [A.10] starting with a compact neighbourhood of $y$. We will, in an abuse of notation, denote the subsequence of $\gamma_i$ that is uniformly convergent to $\gamma$ on compact subsets by $(\gamma_i)$. Note that, by construction, $\gamma$ is causal.

Since $b_i \rightarrow \infty$ we know that $\gamma$ has infinite $h$ arc length. This implies that $\gamma$ is inextendible to the past. As $\gamma$ is past inextendible $I^+(\gamma)$ is a terminal indecomposable future set of [6, Theorem 2.3]. Theorem 2.1 of [6] implies that there exists $\lambda$ a past inextendible timelike curve so that $I^+(\lambda) = I^+(\gamma)$. If $\lambda$ had finite $h$ arc length $\lambda$ would be extendible to the past as $(M, h)$ is complete. This would imply that $I^+(\lambda)$ was not a terminal indecomposable future set. Since this is a contradiction $\lambda$ must have infinite $h$ arc length.

By construction $I^+(y) \subset I^+(\lambda)$. Let $u \in I^-(x)$ then there exists $i \in \mathbb{N}$ so that $u \in I^-(x_i)$. By construction for all $j > i$, $\gamma_j \subset I^+(u)$ this implies that $\gamma \subset I^+(u)$ so that $\lambda \subset I^+(\lambda) = I^+(\gamma) \subset I^+(\lambda)$ as required.

**Proposition 3.4.** Let $(M, g)$ be a chronological Lorentzian manifold. If the Lorentzian distance is infinite at $(x, y) \in M \times M$ then there exists a past inextendible timelike curve $\lambda$ so that for all $u \in I^-(x)$, $\lambda \subset I^+(u)$ and $I^+(y) \subset I^+(\lambda)$.

**Proof.** Let $(\gamma_i)_{i \in \mathbb{N}} \subset \Omega_{x, y}$ be such that $L(\gamma_i) \rightarrow \infty$. Let $\gamma$ be the past directed continuous limit curve through $y$.

Suppose that $\gamma$ is contained in an open pre-compact subset $U$. By Lemma [A.8] $L(\gamma) = \infty$. In particular, as $L(\gamma) > 0$ we know that $\gamma$ contains a timelike segment. If $\gamma$ contains curve segments which are null then via application of [14] Propositions 2.18 and 2.19, in the submanifold $U$, we can find a timelike curve with length greater than or equal to $\gamma$ contained in $U$. In an abuse of notation denote this timelike curve by $\gamma$. Note that by construction this new $\gamma$ is also such that $L(\gamma) = \infty$.

Since $\gamma \subset U$, $\gamma$ is compact. Let $\epsilon > 0$ and choose an auxiliary Riemannian metric $h$ on $M$. Since $\gamma$ is compact it is covered by a finite number of convex normal neighbourhoods of $h$-radius less than $\epsilon$. This implies that $\gamma$ must return to at least one of the covering
neighbourhoods after leaving it. Since this is true for all $\epsilon$ we can construct a sequence $(p_i)_{i \in \mathbb{N}} \subset \gamma$ that has accumulation points in $\gamma$. Since $\gamma$ is closed, $U$ is open and $\overline{U}$ is compact there exists some subsequence of $(p_i)$ that is convergent. In an abuse of notation denote this subsequence by $(p_j)$ and let $p \in \gamma$ be the limit point.

Let $q \in \Gamma^+(p) \cap \gamma$. Since $p \in \Gamma^-(q)$ and as $U$ is open there exists $N \in \mathbb{N}$ so that for all $j > N$, $p_j \in \Gamma^-(q)$. Since the finite $\epsilon$-$h$-radius open covers of $\overline{U}$ were made up of convex normal neighbourhoods we know that there exists some $j > N$ so that $q \in \Gamma^-(p_j)$. This implies that there exists a closed timelike curve. The result now follows from the same arguments as for the proof of Proposition 3.3. 

A “conformal converse” to Proposition 3.3 is possible, see Proposition 3.6. Minkowski space with a point removed demonstrates that the conformal factor is necessary.

**Lemma 3.5.** Let $(M,g)$ be a Lorentzian manifold. If there exists $x,y \in M$ and a past inextendible timelike curve $\lambda$ so that for all $u \in \Gamma^-(x)$, $\lambda \subset \Gamma^+(u)$ and $\Gamma^+(y) \subset \Gamma^+(\lambda)$, then there exists a conformal factor $\Omega$ so that for all $z \in \Gamma^+(y)$, $d(x,z;\Omega^2g) = \infty$.

**Proof.** Without loss of generality we assume that $\lambda$ is parametrised so that for some $b \in \mathbb{R}$, $\lambda : [0,b) \to M$. Choose $f : M \to \mathbb{R}$ a smooth function so that for all $x \in \Gamma^+(\lambda)$, $f(x) \neq 0$ and such that for all $t \in [0,b)$

$$\exp\left(\frac{1}{f(\lambda(t))}\right) \sqrt{-g(\lambda'(t),\lambda'(t))} \geq \frac{1}{b-t}.$$ 

Choose $U \subset M$ an open subset so that $\lambda \subset U$ and $\overline{U} \subset \Gamma^+(\lambda)$. Let $\rho : M \to \mathbb{R}$ be such that $\rho|_{\overline{U}} = 1$ and $\rho|_{\Gamma^-(\lambda)} = 0$. Define $\Omega : M \to \mathbb{R}$ by $\Omega(x) = \sqrt{1 - \rho(x) + \rho(x) \exp(f(x)^{-1})}$. Let $h = \Omega^2g$ be the Lorentzian metric conformally related to $g$ by $\Omega$.

By construction, for all $a \in \mathbb{R}$,

$$L(\lambda;h) = \int_a^b \exp(f(\lambda(t))^{-1}) \sqrt{-g(\lambda'(t),\lambda'(t))} dt = \infty,$$

where the last equality follows by definition of $f$.

If $z \in \Gamma^+(\lambda;g) = \Gamma^+(\lambda;h)$ then there exists $a \in [0,b)$ so that $\lambda([a,b)) \subset \Gamma^-(z;h)$. By assumption this implies that for all $u \in \Gamma^-(x;h) = \Gamma^-(x;g)$, $d(u,z;h) = \infty$. 

**Proposition 3.6.** Let $(M,g)$ be a Lorentzian manifold with finite Lorentzian distance. If there exists $x,y \in M$ and a past inextendible timelike curve $\lambda$ so that for all $u \in \Gamma^-(x)$, $\lambda \subset \Gamma^+(u)$ and $\Gamma^+(y) \subset \Gamma^+(\lambda)$, then there exists a conformal factor $\Omega$ so that for all $p \in \Gamma^+(x)$ and all $q \in \partial \Gamma^+(y)$ the Lorentzian distance of $(M,\Omega^2g)$ is discontinuous at $(p,q)$.

**Proof.** Let $\Omega : M \to \mathbb{R}$ be the conformal factor constructed in Lemma 3.5, let $h = \Omega^2g$ and let $d(u,z;h) = d(u,z;g)$. By assumption $d(u,z;g) < \infty$. Thus for all $u \in \Gamma^-(x;h)$ and $z \in \partial \Gamma^+(\lambda;h)$ the Lorentzian distance of $(M,h)$ is discontinuous at $(u,z)$, whence the result.

We also have the following lemma, which follows the same proof, that can occasionally be useful when the Lorentzian distance is not known to be finite. The result is an alternate form of [18] Lemma 2.1 with a different construction of the required conformal factor.

**Lemma 3.7.** Let $(M,g)$ be a Lorentzian manifold. If there exists $x,y \in M$ so that $d(x,y) = 0$ and a past inextendible timelike curve $\lambda$ so that for all $u \in \Gamma^-(x)$, $\lambda \subset \Gamma^+(u)$ and $\Gamma^+(y) \subset \Gamma^+(\lambda)$, then there exists a conformal factor $\Omega$ so that $(M,\Omega^2g)$ has discontinuous Lorentzian distance at $(x,y)$.

**Proof.** Using the function $\Omega$ of Lemma 3.5 shows that for all $z \in \Gamma^+(y)$, $d(x,z;\Omega^2g) = \infty$. The Lorentzian distance is therefore discontinuous at $(x,y)$ since $d(x,y;\Omega^2g) = 0$. 

12
“Removing” a discontinuity is harder than introducing one as more control is needed over the lengths of curves. We begin our study of finding conformally related continuous Lorentzian distances by characterising the conformally invariant causal structure that introduces discontinuities of the Lorentzian distance.

**Lemma 3.8.** Let \((M, g)\) be a Lorentzian manifold and \(x \in M\). If \((x_i)_{i \in \mathbb{N}}\) and \((u_i)_{i \in \mathbb{N}}\) are future directed sequences converging to \(x\) then \(\bigcap_{i \in \mathbb{N}} I^+(x_i) = \bigcap_{i \in \mathbb{N}} I^+(u_i)\).

**Proof.** For all \(i \in \mathbb{N}, x \in I^+(u_i)\). Since \(I^+(u_i)\) is open there exists \(j \in \mathbb{N}\) so that \(x_j \in I^+(u_i)\). This implies that \(\bigcap_{i \in \mathbb{N}} I^+(x_i) \subset \bigcap_{i \in \mathbb{N}} I^+(u_i)\). As the argument above is symmetric in the two sequences we have that \(\bigcap_{i \in \mathbb{N}} I^+(x_i) = \bigcap_{i \in \mathbb{N}} I^+(u_i)\). \(\square\)

**Definition 3.9.** Let \((M, g)\) be a Lorentzian manifold. For each \(x \in M\), let \((x_i)_{i \in \mathbb{N}}\) be a future directed sequence converging to \(x\) and define

\[
\text{Miss}^+(x) := \text{Int} \left( \left( \bigcap_{i \in \mathbb{N}} I^+(x_i) \right) \setminus I^+(x) \right).
\]

Likewise \(\text{Miss}^-(x)\) is the interior of \(\left( \bigcap_{i \in \mathbb{N}} I^-(y_i) \right) \setminus I^-(x)\) for any past directed sequence \((y_i)_{i \in \mathbb{N}}\) converging to \(x\).

Lemma 3.8 shows that Definition 3.9 is independent of the choice of future (or past) directed sequence.

We think of the elements of \(\text{Miss}^+(x)\) as points which “should” be in \(I^+(x)\) but due to some global feature of causality are missing. The sets \(\text{Miss}^\pm(x)\) are related to Sorkin and Woolgar’s \(K^+\) relation [17].

**Proposition 3.10.** If \((M, g)\) is a Lorentzian manifold then for all \(x \in M\) and all conformal factors \(\Omega\), \(\text{Miss}^+(x) = \text{Miss}^+(x; \Omega^2 g)\).

**Proof.** This follows directly from the identity: for all \(y \in M\), \(I^+(y) = I^+(y; \Omega^2 g)\). \(\square\)

**Example 3.11.** The set \(V\) of Example 2.2 is \(\text{Miss}^+((0, 0))\). \(\triangle\)

**Proposition 3.12.** Let \((M, g)\) be a Lorentzian manifold. If there exists \(x \in M\) so that \(\text{Miss}^+(x) \neq \emptyset\) then every conformally related Lorentzian distance, including the trivially conformally related distance, is discontinuous.

**Proof.** Let \(y \in \text{Miss}^+(x)\) and let \((y_i = y)_{i \in \mathbb{N}}\) be the trivial past directed sequence converging to \(y\) and \((x_i)\) any future directed sequence converging to \(x\). Since \(\text{Miss}^+(x)\) is open there exists \(z \in \text{Miss}^+(x) \cap I^+(y)\). Hence there exists a timelike curve \(\gamma \in \Omega \times y \cap \text{Miss}^+(x)\). Since \(z \in \text{Miss}^+(x)\), for each \(i \in \mathbb{N}\) there exists \(\lambda_i \in \Omega_{x_i, z}\). For each \(i \in \mathbb{N}\), let \(\gamma_i\) be the concatenation of \(\lambda_i\) and \(\gamma\). By construction \(y \not\in I^+(x)\) hence \(d(x, y) = 0\). But \(d(x_i, y) \geq L(\gamma_i) \geq L(\gamma) > 0\). Thus the Lorentzian distance is discontinuous. The result now follows from Proposition 3.10. \(\square\)

Proposition 3.12 has a conformal inverse. The conformal factor must satisfy a certain property that places restrictions on the global structure of the manifold. The following definition describes that property. The subsequent lemma describes the implied global structure. Proposition 3.10 gives the “conformal” converse.

**Definition 3.13.** Let \((M, g)\) be a Lorentzian manifold. A function \(\Omega : M \to \mathbb{R}\) is called a length suppressing conformal factor if for all \(\epsilon > 0\) there exists \(K \subset M\) a compact set so that if \(\gamma\) is a causal curve in \(M\) then \(L(\gamma \setminus K; \Omega^2 g) < \epsilon\).

The following Lemma describes when such a conformal factor exists, subtly generalising a well-known result in the literature. If the manifold is strongly causal then there exists a length suppressing conformal factor so that the conformally related metric has finite diameter, [10] Lemma 2.3. The earliest proof of this result, known to the authors, is [3] Theorem 1. Our result applies to non-strongly causal manifolds, e.g. [13] Figure 23 or [10] Figure 7, and in cases where acausal behaviour is restricted to a compact set, e.g. Example 1.2.
Lemma 3.14. Let \((M, g)\) be a Lorentzian manifold. There exists a length suppressing conformal transformation if and only if for all compact exhaustions, \((K_i)_{i \in \mathbb{N}}\), of \(M\) there exists \(N \in \mathbb{N}\) so that \(i \geq N\) implies that there exists \(k_i \in \mathbb{R}^+\) so that for all causal curves \(\gamma\), 
\[L((\gamma \cap K_i) \setminus K_N) \leq k_i.\]

Proof. Suppose that there exists a compact exhaustion, \((K_i)_{i \in \mathbb{N}}\), and \(N \in \mathbb{N}\) so that \(i \geq N\) implies that there exists \(k_i \in \mathbb{R}^+\) so that for all causal curves \(\gamma\), 
\[L((\gamma \cap K_i) \setminus K_N) \leq k_i.\]

Choose \(\epsilon > 0\). For any \(x \in M\), \(N \in \mathbb{N}\) so that \(x \in K_i \setminus \text{int}(K_{i-1})\). Theorem 2.7 would give the result. We now show that the limits of the particular \(L(\gamma \cap K_i) \setminus K_N) \leq k_i\).

Thus, 
\[L((\gamma \cap K_i) \setminus K_N) \leq 1 + \Omega((K_{i+1}) \setminus K_N) \leq k_i\]

as required.

Suppose that there exists a length suppressing conformal factor, \(\Omega\). Let \((K_i)_{i \in \mathbb{N}}\) be a compact exhaustion of \(M\). Choose \(\epsilon > 0\). Then there exists \(K \subset M\) compact so that 
\[L(\gamma \setminus K; \Omega^2 g) < \epsilon.\]

Choose \(N \in \mathbb{N}\) so that \(K \subset K_N\) and let \(i \geq N\). Since \(K_i\) is compact there exists \(h_i \in \mathbb{R}^+\) so that \(0 < h_i \leq \min\{\Omega(x) : x \in K_i\}\). Let \(\gamma\) be a causal curve then
\[\epsilon > L(\gamma \setminus K_N; \Omega^2 g) \geq L((\gamma \cap K_i) \setminus K_N; \Omega^2 g) = h_i L((\gamma \cap K_i) \setminus K_N; g).\]

Hence 
\[L((\gamma \cap K_i) \setminus K_N; g) < \frac{\epsilon}{h_i},\]

and we may take 
\[k_i = \frac{\epsilon}{h_i}\]
to get the result.

Corollary 3.15. Let \((M, g)\) be a Lorentzian manifold, let \((K_i)_{i \in \mathbb{N}}\) be a compact exhaustion of \(M\) so that there exists \(N \in \mathbb{N}\) so that \(i \geq N\) implies that there exists \(k_i \in \mathbb{R}^+\) so that for all causal curves \(\gamma\), 
\[L((\gamma \cap K_i) \setminus K_N) \leq k_i\]

and let \(\Omega\) be a length suppressing conformal factor. If \(\max\{d(x, y) : x, y \in K_N\} < \infty\) then \((M, \Omega^2 g)\) has finite diameter.

Proof. Let \(k = \max\{d(x, y) : x, y \in K_N\}\). Choose \(\epsilon > 0\). By construction there exists a compact set \(K\) so that for all timelike curves \(\gamma\), 
\[L(\gamma \setminus K; \Omega^2 g) < \epsilon.\]

There exists \(i \geq N\) so that \(K \subset K_i\). In this case 
\[L((\gamma \cap K_i) \setminus K_N) \leq k_i\]

since \(\Omega\) is continuous there exists \(w > 0\) so that for all \(x \in K_i\) \(|\Omega(x)| < w\). Thus
\[L(\gamma; \Omega^2 g) \leq L(\gamma \setminus K; \Omega^2 g) + L((\gamma \cap K_i) \setminus K_N; \Omega^2 g) + L(\gamma \cap K_N; \Omega^2 g) \leq \epsilon + w k_i + w k.
\]

As this holds for any timelike curve the diameter of \((M, \Omega^2 g)\) is less than or equal to \(\epsilon + w k_i + w k\).

We can now present the “conformal” converse of Proposition 3.12. This is a generalisation of [10, Lemma 2.3] and [3, Theorem 1].

Proposition 3.16. Let \((M, g)\) be a Lorentzian metric. If \(\Omega : M \to \mathbb{R}\) is a length suppressing conformal factor and for all \(x \in M\), \(\text{Miss}^+(x) = \emptyset = \text{Miss}^-(x)\) then \((M, \Omega^2 g)\) has continuous Lorentzian distance.

Proof. Let \(x, y \in M\) and suppose that there exists \((x_i)_{i \in \mathbb{N}} \subset M\) a future directed sequence converging to \(x\), \((y_i)_{i \in \mathbb{N}} \subset M\) a past directed sequence converging to \(y\), and suppose that there exists a sequence of curves \((\gamma_i)_{i \in \mathbb{N}}\) so that for all \(i \in \mathbb{N}\) \(\gamma_i \in \Omega_{x_i, y_i}\). If it were the case that \(\lim_{i \to \infty} L(\gamma_i \setminus \text{int}(I^+(x))) = 0\) and \(\lim_{i \to \infty} L(\gamma_i \setminus \text{int}(I^-(x))) = 0\) then, as \(x, y \in M\) are arbitrary, Theorem 2.7 would give the result. We now show that the limits of the particular curve lengths are in fact equal to 0.

Let \(h = \Omega^2 g\). Choose \(\epsilon > 0\) and \(K \subset M\) a compact set so that if \(\lambda\) is a timelike curve in \(M\) then 
\[L(\lambda \setminus K; h) < \epsilon.\]

Such a compact set exists since we assume that a length suppressing conformal factor exists. We know that for each \(i \in \mathbb{N}\),
\[L(\gamma_i \setminus I^+(x); h) = L((\gamma_i \setminus I^+(x)) \cap K; h) + L((\gamma_i \setminus I^+(x)) \setminus K; h) < L((\gamma_i \setminus I^+(x)) \cap K; h) + \epsilon.\]
Suppose, for a contradiction, that there exists a subsequence, \((\gamma_k)\), of \((\gamma_i)\) such that
\[
\lim_{i \to \infty} L \left( (\gamma_k_i \setminus I^+(x)) \cap K; h \right) \neq 0.
\]

Lemma A.10 implies the existence of a limit curve, \(\gamma\), in \(K\) and Lemma A.8 implies that
\[
0 \neq \lim_{i \to \infty} L \left( (\gamma_k_i \setminus I^+(x)) \cap K; h \right) \leq L(\gamma).
\]
This implies that \(\gamma\) has some timelike subcurve \(\mu\). By definition for all \(i \in \mathbb{N}\), \(\gamma_i \subset I^+(x_i)\). Thus \(\gamma \subset \bigcap_i I^+(x_i)\). By construction \(\gamma \not\subset I^+(x)\), hence \(\gamma \subset (\bigcap_i I^+(x_i)) \setminus I^+(x)\). Let \(p, q \in \mu \subset \gamma\) so that \(q \in \Gamma^+(p)\). Then as \(p, q \in (\bigcap_i I^+(x_i)) \setminus I^+(x)\) We know that \(\emptyset \neq I^+(p) \cap I^-(q) \subset (\bigcap_i I^+(x_i)) \setminus I^+(x)\). Therefore Miss\(^+\)(\(x\)) \(\neq \emptyset\). This is a contradiction and hence \(\lim_{i \to \infty} L \left( (\gamma_k_i \setminus I^+(x)) \cap K; h \right) = 0\).

We now know that \(\lim_{i \to \infty} L(\gamma_i \setminus I^+(x); h) < \epsilon\). Since \(\epsilon\) was arbitrary, we see that in fact \(\lim_{i \to \infty} L(\gamma_i \setminus I^+(x); h) = 0\). The time reverse of the above arguments shows that \(\lim_{i \to \infty} L(\gamma_i \setminus I^-(y); h) = 0\) also, as required.

We summarise the results of this section.

**Theorem 3.17.** Let \((M, g)\) be a Lorentzian manifold. If the Lorentzian distance is either

1. infinite and \(M\) is chronological, or
2. discontinuous,

then there exists \(x \in M\), \(y \in I^+(x)\) and an inextendible incomplete past directed timelike curve so that for all \(u \in I^-(x)\), \(\lambda \subset I^+(u)\) and \(I^+(y) \subset I^+(\lambda)\).

**Proof.** This follows from Propositions 3.3 and 3.4. \(\square\)

**Theorem 3.18.** If there exists \(x \in M\), \(y \in I^+(x)\) and an inextendible incomplete past directed timelike curve so that for all \(u \in I^-(x)\), \(\lambda \subset I^+(u)\) and \(I^+(y) \subset I^+(\lambda)\), then

1. the Lorentzian distance is conformally infinite, and
2. if either there exists \(u \in I^-(x)\) and \(v \in I^+(y)\) so that \(d(u, v) < \infty\) or if \(d(x, y) = 0\) then the Lorentzian distance is conformally discontinuous.

**Proof.** Lemma 3.3 proves the “not finite” case. Proposition 3.6 and Lemma 3.7 prove the “not continuous” case. \(\square\)

The need for \(M\) to be chronological in Theorem 3.11 is necessary. The needed example is given by the Misner spacetime, \([11, \text{Page } 171]\). The additional conditions needed in the “discontinuous” case of Theorem 3.12 are also necessary. The counter example is provided by a totally vicious manifold with a point removed.

### 3.1 Immediate applications

The results of Section 3 provide a collection of tools that can be put to use to prove interesting results. For example,

**Theorem 3.19.** If \((M, g)\) is a strongly causal manifold then there exists a conformal transformation \(\Omega\) so that \((M, \Omega^2 g)\) has finite Lorentzian distance. Either

1. there exists \(x\) so that Miss\(^+\)(\(x\)) \(\neq \emptyset\) or Miss\(^-\)(\(x\)) \(\neq \emptyset\), or
2. \(\Omega\) can be chosen such that \((M, \Omega^2 g)\) also has continuous Lorentzian distance.

**Proof.** By definition every strongly causal manifold has a compact exhaustion that satisfies the conditions of Lemma 3.14, see the proof of \([10, \text{Lemma } 2.3]\). Hence the manifold carries a length suppressing conformal factor. Corollary 2.3 and Proposition 3.16 prove the result. \(\square\)
Theorem 3.19 is a generalisation of [10, Theorem 2.4]. Next we relate the sets $\text{Miss}^\pm(x)$ outer continuity of $I^\pm$.

Proposition 3.20. Let $M$ be a Lorentzian manifold. The set-valued function $I^+$ is outer continuous if and only if for all $x \in M$, $\text{Miss}^+(x) = \emptyset$.

Proof. Suppose that there exists $x \in M$ so that $\text{Miss}^+(x) \neq \emptyset$. Since $\text{Miss}^+(x)$ is open it contains a compact set. The definition of $\text{Miss}^+(x)$ now implies that the manifold is not causally continuous.

Suppose that $M$ is not causally continuous. Then there exists $x \in M$ and a compact set $K \subset M \setminus I^+(x)$ so that for all neighbourhoods $U$ of $x$ there exists $y \in U$ so that $K \cap I^+(y) \neq \emptyset$. Let $(x_i)_{i \in \mathbb{N}}$ be a future directed sequence converging to $x$. By assumption for each $i$ there exists $k_i \in K \cap I^+(x_i)$. As the sequence $(k_i)_{i \in \mathbb{N}}$ lies in the compact set $K$ some subsequence has a limit point $k \in K$. Since $K \subset M \setminus I^+(x)$ there exists an open neighbourhood $U$ of $k$ so that $U \subset M \setminus I^+(x)$. Let $V \subset U \cap I^+(k)$. Since $(x_i)$ is future directed for all $j \geq i$, $k_j \in I^+(x_i)$. Hence for all $i \in \mathbb{N}$, $k \in I^+(x_i)$, so for all $i \in \mathbb{N}$, $V \subset I^+(x_i)$. By construction $\emptyset \neq V \subset \text{Miss}^+(x)$, as required.

Proposition 3.20 has the implication that if the manifold is mildly well-behaved causally then the sets, $\text{Miss}^\pm(x)$, are related to the very strong causality condition, causal continuity.

Theorem 3.21. Let $(M, g)$ be a distinguishing Lorentzian manifold. The manifold $M$ is causally continuous if and only if for all $x \in M$, $\text{Miss}^+(x) = \emptyset = \text{Miss}^-(x)$.

Proof. This follows from Proposition 3.20 and the definition of causal continuity.

We also obtain a new proof of a known relation between continuity of the Lorentzian distance and causality in the distinguishing case.

Theorem 3.22 (Theorem 4.24 of [1]). Let $(M, g)$ be a distinguishing Lorentzian manifold. If the Lorentzian distance is continuous then the manifold is causally continuous.

Proof. This follows from Proposition 3.12 and Theorem 3.21.

It is possible to generalise [10, Theorem 2.4] further.

Theorem 3.23. Let $(M, g)$ be a distinguishing Lorentzian manifold and let $\Omega$ be a length suppressing conformal factor. The manifold is causally continuous if and only if $(M, \Omega^2 g)$ has continuous Lorentzian distance.

Proof. Theorem 3.21 and Proposition 3.16 show that causal continuity implies that $(M, \Omega^2 g)$ has continuous Lorentzian distance. Proposition 3.12 and Theorem 3.21 show that if $M$ is distinguishing and $(M, \Omega^2 g)$ has a continuous Lorentzian distance then $M$ is causally continuous.

Our techniques also give a short proof of the following well-known implication.

Theorem 3.24 (Lemma 4.5 of [1]). If $(M, g)$ is globally hyperbolic then the Lorentzian distance is finite and continuous.

Proof. By definition, for all $u, v \in M$, $J^+(u) \cap J^-(v)$ is compact. Proposition 3.3 implies that the Lorentzian distance is continuous. Lemmas A.8 and A.10 imply that the Lorentzian distance is finite.
4 Finiteness, continuity and the causal hierarchy

The previous sections of this paper should give the reader the impression that the finiteness and continuity of the Lorentzian distance do not connect well with rungs in causal hierarchy. In this section we exactly describe the relationship between finiteness, continuity and the causal hierarchy. We consider here a collection of standard causality conditions, [10], whose relations are described in Figure 7.

Figure 7: The causality conditions appearing in Section 4. An arrow, \( \rightarrow \), from one condition to another indicates that the first condition implies the second. For example, global hyperbolicity implies causally simple.

Theorem 4.1. Let \((M, g)\) be a Lorentzian manifold, and \(d\) the Lorentzian distance function.

1. The condition “\(d\) is finite and continuous” is independent of each of the following causality conditions on \((M, g)\): causally simple, causally continuous, stably causal, strongly causal, distinguishing, causal and chronological.

2. The condition “\(d\) is finite” is independent of each of the following causality conditions on \((M, g)\): causally simple, causally continuous, stably causal, strongly causal, distinguishing, causal. The condition “\(d\) is finite” implies the condition “chronological”, while totally vicious implies “\(d\) is not finite”.

3. The condition “\(d\) is continuous” is independent of each of the following causality conditions on \((M, g)\): causally simple, causally continuous, stably causal, strongly causal, distinguishing, causal and chronological. Totally vicious implies “\(d\) is continuous”.

4. More precisely, for each of the pairs of causal conditions (not \(A, B\)),

- not causally simple, causally continuous
- not stably causal, strongly causal
- not strongly causal, distinguishing
- not distinguishing, causal
- not causal, chronological

there exists four Lorentzian manifolds satisfying \(B\) but not \(A\) and such that the Lorentzian distance is respectively

- finite and continuous,
- finite and discontinuous,
- infinite and continuous,
- infinite and discontinuous.

The condition (not causally continuous, stably causal) implies that every conformally related manifold is discontinuous and there exists two Lorentzian manifolds, satisfying the condition, so that the Lorentzian distance is finite and infinite respectively.

The proof of this theorem is a collection of examples and counterexamples divided into nine cases based on the causality conditions: causally simple, causally continuous, stably causal, strongly causal, distinguishing, causal, chronological, and totally vicious. We include globally hyperbolic in the discussion below for completeness.
4.1 Totally vicious

In a totally vicious manifold the Lorentzian distance, considered as an extended function \(d : M \times M \to [0, \infty]\), is constant (infinite) and therefore continuous. Finiteness of the Lorentzian distance clearly does not hold. Thus any totally vicious manifold is an example of a continuous but not finite manifold. This proves the last statements of items 2 and 3 in Proposition 3.12.

4.2 Chronological

Example 4.1 below, presents a chronological non-causal manifold with finite and continuous Lorentzian distance.

By removing a vertical (parallel to \(\mathbb{R}\)) line segment from the manifold in Example 4.1, points \(x, y\) so that Miss^{-}(x) \neq \emptyset and Miss^{-}(y) \neq \emptyset are introduced. Thus the resulting manifold will be chronological non-causal with finite but discontinuous Lorentzian distance, by Proposition 3.12.

By removing any point not on the closed null curve (in order to preserve non-causality) from the manifold given in Example 4.2 and applying Lemma 3.15, we obtain a conformal transformation which produces a chronological non-causal manifold with infinite and discontinuous Lorentzian distance. Proposition 3.16 gives the proof of discontinuity.

Producing a chronological non-causal manifold with infinite and continuous distance is slightly more awkward since we need to avoid, for example, the situations described by Proposition 3.10 and Lemma 3.7. Example 4.3 provides the details.

This proves all statements about chronology in Theorem 4.1.

Note also that Examples 2.2 and 3.2 present chronological manifolds with discontinuous but finite distance, where the discontinuities arise via different mechanisms. Example 3.1 gives a chronological manifold with continuous but not finite distance.

Example 4.2. Let \(M = S^{1} \times \mathbb{R}\) equipped with the metric

\[
g = -s(t)dt^{2} + 2\sqrt{1 - s(y)^{2}}dt\,d\theta + s(t)d\theta^{2}.
\]

Let \(s(t) = \arctan^{2}(t)\) so that for large positive and negative \(t\) the metric is approximately \(-dt^{2} + d\theta^{2}\), whereas for \(t\) close to 0 the metric is approximately \(2dt\,d\theta\). Thus light cones “tip over” close to the “waist” \(\{(x, y) : y = 0\}\), which is a closed null curve.

Let \(\gamma(\tau) = (t(\tau), \theta(\tau)) \in M\) be a geodesic with affine parameter \(\tau\) so that if \(\epsilon \in \{-1, 0, 1\}\) then

\[
g(\gamma', \gamma') = -s(t(\tau))(t'(\tau))^{2} - 2\sqrt{1 - s(t(\tau))^{2}}t'(\tau)\theta'(\tau) + s(t(\tau))(\theta'(\tau))^{2} = \epsilon.
\]

Since the coefficients of the metric do not depend on \(\theta\) there is a second constant of integration, \(q \in \mathbb{R}\), given by

\[
q = -\sqrt{1 - s(t(\tau))^{2}}t'(\tau) + s(t(\tau))\theta'(\tau).
\]

By substituting the equation involving \(\epsilon\) into the square of the equation involving \(q\), the following equation for \(t'(\tau)\) can be derived

\[
(t'(\tau))^{2} + \epsilon s(t(\tau)) = q^{2}.
\]

This equation has the implicit solution, when at least one of \(q\) and \(\epsilon\) is non-zero,

\[
\int_{t(K)}^{t(\tau)} \frac{1}{\sqrt{q^{2} - \epsilon s(\xi)}} \, d\xi = K \pm \tau, \quad (1)
\]

where \(K\) is a constant of integration that can be taken to be 0 since \(\tau\) is affine.

By a standard cutting the corner argument, [14] Page 7.6 and Definition 2.13, for any \((t, \theta) \in M\), \(t < 0\), the longest timelike curve from \((t, \theta)\) to any point \((0, \phi)\) will be a curve
which approaches but never reaches the $t = 0$ surface. Since we can arrange for a sequence of length maximising curves that approach but never reach the waist to be contained in a compact subset of $M$ the limit geodesic will exist, Lemma 4.10. This limit curve will be a timelike geodesic, Lemma 4.8. Hence there exists a timelike geodesic from $(t, \theta)$ that approaches but does not reach the waist.

Let $\gamma(t) = (t(\tau), \theta(\tau))$ be a unit length, future directed, timelike geodesic from $(a, b)$, $a < 0$, so that for all $\tau \in \text{dom}(\gamma)$, $t(\tau) < 0$ and $\gamma$ winds around the waist at $t = 0$. This implies that $\gamma$ is inextendible, $t'(\tau) > 0$ and that $t'(\tau) \to 0$ as $t(\tau)$ approaches $0$. Equation (1) implies that $\tau$ has a maximum, which we denote by $\tau_\gamma$, and so $\gamma$ is incomplete and inextendible. For $\epsilon = -1$, Equation (1) implies that the maximum of $\tau$ will occur when $q = 0$. This implies that $\theta'(K)$ is given by solving $\sqrt{1 - s(t(K)^2)t'(K) + s(t(K))\theta'(K)} = 0$. Hence the Lorentzian distance from a level $t$ surface to the waist is independent of $\theta$. By definition of an affine parameter $L(\gamma) = \tau_\gamma - K$. Since equation (1) has smooth dependence on the domain of integration and as the metric is symmetric about the $t = 0$ surface we know that the Lorentzian distance is continuous in a neighbourhood of the $t = 0$ surface. It is clear that the Lorentzian distance is continuous everywhere else in $M$ as both the future and past of the $t = 0$ surface are globally hyperbolic submanifolds. Hence the Lorentzian distance is continuous on all of $M$. \triangle

Example 4.3. Let $(M, g)$ be the manifold of Example 4.2. Choose $(\tau, s) \in M$, $\tau > 0$, and let $U, V \subset M$ be open neighbourhoods, with compact closure, of $(\tau, s)$ so that $U \subset \Omega^2(\{(t, \theta) \in M : t = 0\})$, $V \subset U$ and $U$ is homeomorphic to the 2-dimensional ball. Choose $\rho : M \to [0, 1]$ with support in $U$ so that $\rho|_V = 1$. Define $\Omega : M \setminus \{(\tau, s)\} \to \mathbb{R}$ by $\Omega(\tau, \theta) = (\tau - \epsilon)^2 + (\theta - \epsilon)^2$, where $\epsilon \in \mathbb{R}$ is chosen so that $\min\{\Omega(p) : p \in U\} > 2$. Let $O = M \setminus \{(\tau, s)\}$ and equip $O$ with the metric $h = (1 + \rho(\Omega - 1))^2g$.

The manifold $(N, h)$ is chronological and not causal since there are no closed timelike curves but there is a closed null curve, i.e. the surface $\{(t, \theta) \in N : t = 0\}$. The Lorentzian distance induced by $h$ is infinite and continuous for the same reasons that the Lorentzian distance of Example 4.1 is infinite and continuous. \triangle

4.3 Causal

By construction there is only one closed null curve in the manifold of Example 4.2. In addition there are no closed timelike curves. Hence if a point is removed from the closed null curve the manifold will be causal. Let $(0, \theta_1)$ and $(0, \theta_2)$, $\theta_1 \neq \theta_2$, be in the manifold. Then $\Omega((0, \theta_1)) = \Omega((0, \theta_2))$. Removing a point from the closed null curve does not effect this set equivalence. Thus the manifold of Example 4.2 with a point removed from the closed null curve is causal but not distinguishing. In particular this new manifold has finite and continuous Lorentzian distance.

To build a causal, not distinguishing, Lorentzian manifold with infinite and continuous Lorentzian distance we can use the technique given in Example 4.3. Start with the manifold of Example 4.2 remove a point on the closed null curve and apply a conformal transformation built as in Example 4.3.

Removing a vertical line from Example 4.2 as in Subsection 4.2 and removing a point from the closed null curve, produces a causal but not distinguishing manifold with finite and discontinuous.

Removing a vertical line from Example 4.3 as in Subsection 4.2 and removing a point from the closed null surface, produces a causal but not distinguishing manifold with infinite and discontinuous.

This proves all statements about causality in Theorem 4.1.

Examples 2.2, 3.2, and 3.1 apply in this case too.
4.4 Distinguishing

Example 4.4 below presents a distinguishing non-strongly causal manifold with finite but discontinuous distance.

Because only horizontal “half”-infinite lines have been removed from $[-4, 4] \times \mathbb{R}$ to obtain the manifold $N$ of Example 4.4 for all $x \in N$ we have $\text{Miss}^-(x) = \text{Miss}^+(x) = \emptyset$. Hence we can apply Proposition 3.16 to Example 4.4 to get a distinguishing non-strongly causal manifold with finite and continuous distance.

Applying Lemma 3.5 produces a distinguishing non-strongly causal manifold with infinite but discontinuous distance.

Production of a distinguishing non-strongly causal manifold with infinite and continuous distance is more complicated, but can be achieved by following the method of Example 4.3, see Example 4.5.

This proves all of the statements about the distinguishing case in Theorem 4.1. Examples 2.2, 3.2, and 3.1 apply in this case too.

Example 4.4. Let $N = ([0, 1] \times \mathbb{R}) \setminus \{ (0, x) : x \leq 1 \} \cup \{ (-1, x) : x \geq -1 \}$. Define two points $(t, x), (s, y) \in N$ to be equivalent, $(t, x) \sim (s, y)$, if and only if $x = y$ and $t = \pm s$. Let $M = N/\sim$ be the quotient manifold: see Figure 8. Equip $N$ with the metric $-dt^2 + dx^2$.

This induces a metric on $M$. This is a standard example of a distinguishing non-stably causal manifold, e.g. Figure 38 of [7, Page 193].

Let $\epsilon_1, \epsilon_2 > 0$ and let $\gamma : [0, 3] \rightarrow M$ be the curve given by

$$
\gamma(t) = \begin{cases}
(0, \epsilon_1) + t(1, 1) & t \in [0, 1), \\
(1, 1 + \epsilon_1) + (t - 1)(-2, -2 - \epsilon_1 - \epsilon_2) & t \in [1, 2], \\
(-1, -1 - \epsilon_2) + t(1, 1) & t \in [2, 3].
\end{cases}
$$

For $\epsilon_1$ and $\epsilon_2$ small enough, the curve $\gamma$ will be timelike.

Choose $U$ an open neighbourhood about $(0, 0)$. By taking $\epsilon_1$ and $\epsilon_2$ arbitrarily small we can see that $\gamma$ starts in $U$ and returns to it. Thus $M$ is not strongly causal. The manifold is distinguishing, this can be checked directly. 

Example 4.5. Let $N$ and $M$ be as in Example 4.4. Let $U, V \subset \mathbb{R}^2$ be pre-compact open neighbourhoods of $(-1, -1) \in \mathbb{R}^2$ so that $\overline{V} \subset U$ and $U \setminus \{ (1, x) : x \leq 1 \} \cup \{ (-1, x) : x \geq -1 \} \subset N$.

Let $\rho : \mathbb{R}^2 \rightarrow [0, 1]$ be a bump function with support in $U$ and such that $\rho|_V = 1$. Let $\Omega : \mathbb{R}^2 \rightarrow \mathbb{R}$ be defined by $\Omega(t, x) = \frac{(t+1)^2 + (x+1)^2}{(t+1)^2 + (x+1)^2}$, where $c$ is chosen so that $\min\{\Omega(p) : 

Figure 8: An illustration of Example 4.4. The thick lines without two cross hatches have been removed from the manifold. The thick lines with cross hatches have been identified. The dashed line is a null surface of geometric importance. The black dot marks the $(-1, -1)$ point.
$p \in U \} \geq 2$. Define the metric $h$ on $M$ by restriction of the metric $(1 + \rho(\Omega - 1))^2(-dt^2 + dx^2)$ on $N$. The manifold $(M, h)$ is distinguishing but not strongly causal, as causal structure is conformally invariant. The induced Lorentzian distance is infinite and continuous by the same arguments used in Example 3.1.

4.5 Strongly causal

Example 4.6 gives a strongly causal but not stably causal manifold with finite and discontinuous Lorentzian distance.

Since the lines that have been removed to produce the manifold $N$ in Example 4.6 are spacelike and "half"-infinite we know that for all $x \in M$, where $M$ is defined as in Example 4.6, $\text{Miss}^{-1}(x) = \text{Miss}^{+1}(x) = \emptyset$. Hence we can applying Proposition 3.16 to Example 4.6 to get a strongly causal non-stably causal manifold with finite and continuous distance.

Applying Lemma 3.5 to Example 4.6 produces a strongly causal non-stably causal manifold with infinite but discontinuous distance.

Applying Proposition 3.16 to Example 4.6 gives a strongly causal non-stably causal manifold with finite and continuous distance. The same construction as used in Examples 4.3 and 4.5 can now be applied to the point $(0, -1) \in M$ of Example 4.6 to produce an example of a strongly causal non-stably causal manifold with infinite and continuous distance.

This proves all the statements about strongly causal in Theorem 4.1.

Examples 2.2, 3.2, and 3.1 apply in this case too.

Example 4.6 ([11, Figure 9]). Let $\tilde{N} = [-2, 2] \times \mathbb{R}$. Let $L_1 = \{(0, x) \in N : x \geq -1\}$, $L_2 = \{(1, x) \in N : x \leq 1\}$ and $L_3 = \{(-1, x) \in N : x \leq 1\}$, $N = N \setminus (L_1 \cup L_2 \cup L_3)$. Let $M$ be the manifold given by identifying on $N$ the lines $\{2\} \times \mathbb{R}$ and $\{-2\} \times \mathbb{R}$: see Figure 9. That is, if $\sim$ is the equivalence relation given by identifying $(2, x)$ with $(-2, x)$ for all $x \in \mathbb{R}$ then $M = N / \sim$. The manifold $N$ carries the metric induced by inclusion into Minkowski space $\mathbb{R}^{1,1}$, and this metric induces a metric on $M$, which we denote by $g$. The Lorentzian metric $(M, g)$ is strongly causal but not stably causal. Any small widening of the light cones will allow a closed time-like curve to be created. The Lorentzian distance is finite and discontinuous on $M$. △

4.6 Stably causal

Example 2.2 is a stably causal and not causally continuous manifold with finite and discontinuous Lorentzian distance.
Applying a conformal transformation as in Lemma 3.5 to Example 2.2 will produce a manifold with infinite and discontinuous Lorentzian distance.

Next recall that a stably causal manifold is distinguishing. Thus a stably causal manifold is causally continuous if and only if for all points \( x \) in the manifold we have Miss\(^-\)(\( x \)) = Miss\(^+\)(\( x \)) = \emptyset. Thus Proposition 3.12 implies that every stably causal and not causally continuous manifold has discontinuous Lorentzian distance, proving part of the final statement of Theorem 4.1.

Minkowski space with a point removed has finite and continuous Lorentzian distance and is stably causal, causally continuous but not causally simple.

The conformal transformations used in Examples 4.3 and 4.5 can be applied to Minkowski space with a point removed to produce a stably causal, causally continuous not causally simple, manifold with infinite and continuous Lorentzian distance.

This proves all the statements about stable causal in Theorem 4.1.

### 4.7 Causally continuous

Example 3.2 gives a causally continuous not causally simple manifold with finite and discontinuous Lorentzian distance.

Example 3.1 presents a causally continuous not causally simple manifold with infinite and discontinuous Lorentzian distance.

Minkowski space with a point removed is causally continuous, not causally simple, with finite and continuous Lorentzian distance.

Applying a conformal transformation as in Example 4.3 or Example 4.5 to Minkowski space with a point removed produces a causally continuous, not causally simple, with infinite and continuous Lorentzian distance.

This proves all the statements about causally continuous in Theorem 4.1.

### 4.8 Causally simple

Example 4.7, below, gives a causally simple non-globally hyperbolic manifold with finite and continuous Lorentzian distance.

Construction of a causally simple non-globally hyperbolic manifold with finite and discontinuous distance is complicated by the need to maintain causal simplicity. Note that the manifold \( \tilde{M} \) in Example 4.7 is a submanifold of the manifold \( M \) in Example 3.2. Restricting the metric \( \phi^*g \) on \( M \) to \( \tilde{M} \) produces a causally simple non-globally hyperbolic manifold with finite and discontinuous distance.

Applying a conformal transformation as in Lemma 3.5 to the manifold constructed in Example 4.7 produces a causally simple non-globally hyperbolic manifold with infinite and discontinuous distance.

Using the technique illustrated in Examples 4.3 or Example 4.5 on the manifold of Example 4.7 gives a causally simple non-globally hyperbolic manifold with infinite and continuous Lorentzian distance.

**Example 4.7 (Figure 10).** Let

\[
\tilde{M} = \mathbb{R}^2 \setminus \{(x, y) \in \mathbb{R}^2 : x \leq -1\} \cup \{(x, y) \in \mathbb{R}^2 : x \geq 0, |y| \leq 2x\}
\]

with the metric \(-dy^2 + dy^2\) induced by the inclusion of \( \tilde{M} \) into 2-dimensional Minkowski space: see Figure 10. This is a causally simple non-globally hyperbolic manifold with finite and continuous Lorentzian distance. \(\triangle\)
4.9 Globally hyperbolic

The Lorentzian distance is necessarily finite and continuous in a globally hyperbolic manifold, [1, Lemma 4.5].

A Appendix - a limit curve theorem

This appendix collects a few important details about limit curves together. In contrast to existing results, we emphasise the case of continuous causal curves. In the differentiable case the results below are classical, see [1, 9, 14]. Where possible we cite the related results.

Definition A.1 ([1, Page 54]). A continuous curve \( \gamma : (a, b) \to M, a, b \in \mathbb{R}, a < b \), is a future directed causal curve if for all \( t \in (a, b) \) there exists \( \epsilon > 0 \) and a convex normal neighbourhood, \( U \), of \( \gamma(t) \) with \( \gamma(t-\epsilon, t+\epsilon) \subset U \) so that for any \( t_1, t_2 \in (t-\epsilon, t+\epsilon), t_1 < t_2 \), there is a smooth future directed causal curve lying in \( U \) from \( \gamma(t_1) \) to \( \gamma(t_2) \).

Continuous causal curves are discussed in some detail in [18, Definition 2.3 ff].

Lemma A.2. Every continuous causal curve has a parameterisation with respect to which it is locally Lipschitz and thus differentiable almost everywhere.

Proof. The proof can be found in [1, Pages 75 and 76]. It uses the Lorentzian metric to show that every continuous causal curve is locally Lipschitz.

Lemma A.3 ([13, Lemma 5.9]). If \( U \subset M \) is a open convex set, then the function \( \Delta : U \times U \to TM \) defined by \( \Delta(x, y) := \exp^{-1}_x(y) \in T_xM \) is smooth.

Definition A.4 ([14, Definition 2.13]). Let \( U \subset M \) be an open convex set, define \( \Phi(x, y) = g(\Delta(x, y), \Delta(x, y)) \).

Lemma A.5. Let \( U \subset M \) be an open convex normal neighbourhood and for all \( u, v \in U \) let \( \gamma_{uv} \) be the unique geodesic from \( u \) to \( v \) in \( U \). If \( (x_i)_{i\in\mathbb{N}} \) is a sequence in \( U \) converging to \( x \in U \) and \( (y_i)_{i\in\mathbb{N}} \) is a sequence in \( U \) converging to \( y \in U \) and for all \( i \in \mathbb{N}, y_i \in I^+(x_i) \) then \( \lim_{i\to\infty} L(\gamma_{x_iy_i}) = L(\gamma_{xy}) \).

Proof. We can always parametrise \( \gamma_{x_iy_i} \) so that \( \gamma_{x_iy_i} : [0, 1] \to M \). In this case, by definition, \( \gamma_{x_iy_i}'(1) = \Delta(x_i, y_i) \) and we can see that \( L(\gamma_{x_iy_i}) = \sqrt{-\Phi(x_i, y_i)} \). Hence \( L(\gamma_{xy}) \) depends continuously on \( x_i, y_i \) and so \( \lim_{i\to\infty} L(\gamma_{x_iy_i}) = L(\gamma_{xy}) \).

A slightly more nuanced approach, handling the cases \( \Phi(x_i, y_i) = 0 \) and \( \Phi(x_i, y_i) \neq 0 \) for all \( i \in \mathbb{N} \) would give a little additional insight into the differential dependence of \( L(\gamma_{x_iy_i}) \) on \( x_i, y_i \). We shall not need this, however.
**Definition A.6 (see [14] Definitions 7.1 and 7.4).** Let $\gamma: [a, b] \to M$, $a, b \in \mathbb{R}$, $a < b$, be a continuous causal curve. A partition of $[a, b]$ is a finite subset $\{t_i \in [a, b] : i = 1, \ldots, m\}$ so that $t_1 = a < t_2 < \cdots < t_{m-1} < t_m = b$ and for all $i = 1, \ldots, m - 1$ there exists an open convex normal neighbourhood containing $\gamma(t_i)$ and $\gamma(t_{i+1})$. Let $\Xi(\gamma)$ denote the set of all such partitions of $I$. Note that $\Xi(\gamma)$ depends not only on the domain of $\gamma$ but also on $\gamma$ due to the requirement that for all $i$ in a partition $\gamma(t_i)$ and $\gamma(t_{i+1})$ are in a common open convex normal neighbourhood.

Define the length of $\xi \in \Xi(\gamma)$ as

$$L(\xi, \gamma) = \sum_{i=1}^{n-1} L(\gamma_{\gamma(t_i), \gamma(t_{i+1})}),$$

where $\gamma_{\gamma(t_i), \gamma(t_{i+1})}$ is the, now awkward expression for the, unique geodesic from $\gamma(t_i)$ to $\gamma(t_{i+1})$ lying in the assumed convex normal neighbourhood. Define the length of $\gamma$ as

$$L(\gamma) = \inf \{L(\xi, \gamma) : \xi \in \Xi(\gamma)\},$$

where, in an abuse of notation, we overloaded the symbol $L$. If the domain of $\gamma: I \to \mathbb{R}$ is not compact then we define

$$L(\gamma) = \sup \{L(\gamma|_K) : K \subset I \text{ is compact}\},$$

once again abusing the symbol $L$.

Since geodesics are length maximising if $\xi' \subset \xi \in \Xi(\gamma)$ then $L(\xi) < L(\xi')$ and if $\xi'' = \xi \cup \xi'$, $\xi, \xi', \xi'' \in \Xi(\gamma)$ then $L(\xi'') \leq \min\{L(\xi), L(\xi')\}$. As $L(\xi) \geq 0$ for all $\xi \in \Xi(\gamma)$ the length of $\gamma$ is well defined and finite for curves with compact domain.

**Lemma A.7 (see [14] Definition 7.4).** Let $\gamma: I \to \mathbb{R}$ be a future directed locally Lipschitz causal curve then

$$\int_I \sqrt{-g(\gamma', \gamma')} \, dt = \inf \{L(\xi, \gamma) : \xi \in \Xi(\gamma)\} = \sup \{L(\gamma|_K) : K \subset I \text{ is compact}\}.$$

*Proof.* This follows from standard results regarding the relationship of rectifiable curves, local Lipschitz continuity and path integrals. \[\Box\]

**Lemma A.8 (see [9] Theorem 2.4) or [11] Proposition 8.2).** Let $(M, h)$ be a Riemannian manifold and let $b \in \mathbb{R} \cup \{\infty\}$. For each $i \in \mathbb{N}$, let $\gamma_i: [0, b] \to M$ be a future directed causal curve. If there exists a continuous causal curve $\gamma: [0, b] \to M$ so that the sequence $(\gamma_i)_{i \in \mathbb{N}}$ converges to $\gamma$ uniformly on compact subsets of $[0, b)$, with respect to the distance induced by $h$, then $\lim_{i \to \infty} L(\gamma_i) \leq L(\gamma)$.

*Proof.* Let $K \subset I$ be compact. We will show that for all $\xi \in \Xi(\gamma|_K)$, with $m$ elements, there exists $N \in \mathbb{N}$ so that $i \geq N$ implies that there exists $\xi_i \in \Xi(\gamma|_K)$ so that $\xi = \xi_i$ and for all $i = 1, \ldots, m - 1$ the geodesics $\gamma_{\gamma_i(t_i), \gamma_i(t_{i+1})}$ and $\gamma_{\gamma_i(t_i), \gamma_i(t_{i+1})}$ are in the same convex normal neighbourhood.

Choose $\xi \in \Xi(\gamma)$ and assume that $\xi$ has $m$ elements. For each $j = 1, \ldots, m - 1$ choose $U_j$ a convex normal neighbourhood so that $\gamma(t_j), \gamma(t_{j+1}) \in U_j$. Since there are only a finite number of $U_j$ there exists $\epsilon > 0$ so that for all $j = 1, \ldots, m - 1$ the ball based at $t_j$ of radius $\epsilon$ is contained in $U_j$, and $U_j$ is compact. Since $K$ is compact, by assumption there exists $N \in \mathbb{N}$ so that $i \geq N$ implies that for all $t \in K$, $d(\gamma_i(t), \gamma(t)) < \epsilon$, where $d$ is the Riemannian distance induced by $h$. In particular for each $j = 1, \ldots, m - 1$, $i > N$ implies that $\gamma_i(t_j) \in U_j \cap U_{j+1}$. This implies that for all $j = 1, \ldots, m - 1$, $i \geq N$ $\gamma_{\gamma_i(t_j), \gamma_i(t_{j+1})} \subset U_j$. Thus $\xi \in \Xi(\gamma_i)$ as required.

Since uniform convergence on compact subsets implies pointwise convergence, Lemma A.5 implies that for all $\epsilon > 0$ and all $j = 1, \ldots, m - 1$ there exists $N(\epsilon, j) \in \mathbb{N}$ so that $i \geq N(\epsilon, j)$ implies that $L(\gamma_{\gamma_i(t_j), \gamma_i(t_{j+1})}) < L(\gamma_{\gamma_i(t_j), \gamma_i(t_{j+1})}) + \epsilon$. Since $j = 1, \ldots, m - 1$ this implies that for all $\epsilon > 0$ there exists $N(\epsilon) \in \mathbb{N}$ so that $i \geq N$ implies that $L(\xi_i, \gamma|_K) < L(\xi_i, \gamma|_K) + \epsilon$. 24
This implies that \( L(\gamma|_K) \leq L(\gamma|_K) + \epsilon. \) Taking supremum over \( K \) we get that for all \( \epsilon > 0 \) there exists \( N(\epsilon) \in \mathbb{N} \) so that \( i \geq N(\epsilon) \) implies that \( L(\gamma_i) \leq L(\gamma) + \epsilon. \) This implies that \( \lim_{i \to \infty} L(\gamma_i) \leq L(\gamma) \) as required. \( \square \)

**Lemma A.9** (see \[3\] Lemma 2.7 or \[1\] Proposition 3.31). Let \( b \in \mathbb{R} \cup \{\infty\}. \) For each \( i \in \mathbb{N}, \) let \( \gamma_i : (a, b) \to M \) be a future directed causal curve. If there exists a continuous curve \( \gamma : (a, b) \to M \) so that the sequence \( (\gamma_i)_{i \in \mathbb{N}} \) converges pointwise to \( \gamma, \) then \( \gamma \) is a future directed causal curve.

**Proof.** Let \( t \in (a, b) \) and choose \( U \) an open convex normal neighbourhood containing \( \gamma(t). \) Since \( U \) is open there exists \( \epsilon > 0 \) so that \( \gamma(t - \epsilon, t + \epsilon) \subset U. \) Let \( t_1, t_2 \in (t - \epsilon, t + \epsilon), \) \( t_1 < t_2. \) By assumption there exists \( N \in \mathbb{N} \) so that \( i \geq N \) implies that \( \gamma_i(t_1), \gamma_i(t_2) \in U. \) Without loss of generality we can assume that \( N = 0. \)

For each \( i \in \mathbb{N} \) as each \( \gamma_i \) is future directed causal and as \( U \) is convex normal there exists a future directed causal geodesic in \( U \) from \( \gamma_i(t_1) \) to \( \gamma_i(t_2). \) Let \( v_i = \Delta(\gamma_i(t_1), \gamma_i(t_2)) \) then \( (v_i)_{i \in \mathbb{N}} \) converges to \( v = \Delta(\gamma(t_1), \gamma(t_2)), \) by Lemma A.3. Since each \( \gamma_i \) is future directed and timelike \( g(T, v_i) \leq 0 \) and \( g(v_i, v_i) \leq 0. \) Taking the limit with respect to \( i \) shows that \( g(T, v) \leq 0 \) and \( g(v, v) \leq 0. \) Hence \( v \in T_{\gamma(t_1)}M \) is future directed and causal. By construction \( \exp_{\gamma(t_1)}(v) = \gamma(t_2). \) Thus as \( U \) is convex normal the unique geodesic between \( \gamma(t_1) \) and \( \gamma(t_2) \) is the curve \( t \mapsto \exp_{\gamma(t_1)}(tv) \) which is future directed and causal as required. \( \square \)

Like all limit curve results the lemma below is based on Arzelà's Theorem, \[1\] Theorem 3.30. Ours is, essentially, a more precise form of \[1\] Proposition 3.31 and \[7\] Lemma 6.2.1. For a detailed study of limit curve theorems refer to \[9\].

**Lemma A.10.** Let \( M \) be a manifold and let \( d : M \times M \to \mathbb{R} \) be the distance induced by a complete Riemannian metric. Let \( B \) be a bounded subset of \( M. \) Let \( \gamma_i : I_i \to M \) be a sequence of \( C^0 \) curves in \( M, \) so that for some \( N \in \mathbb{N}, n > N \) implies that \( \gamma_n \subset \overline{B}, \) where \( I_i = [0, b_i], \) \( b_i \in \mathbb{R} \) or \( [0, \infty) \) in which case we set \( b_i = \infty. \) Furthermore, we assume that each \( \gamma_i \) is parametrised so that for all \( i \in \mathbb{N} \) and for all \( t_1, t_2 \in I_i, \) \( d(\gamma_i(t_1), \gamma_i(t_2)) \leq |t_1 - t_2|. \)

Let \( b = \sup b_i; \) if \( b = \infty \) let \( Y_1 = [0, b_i) \) and \( X = [0, \infty), \) otherwise let \( Y_1 = [0, b_i] \) and \( X = [0, b]. \) Then there exists a sequence of strictly monotonic increasing, bijective, smooth changes of parameter \( f_i : X \to Y_1, \) so that there is a subsequence of \( (\gamma_i \circ f_i)_{i \in \mathbb{N}} \) that converges uniformly, on compact subsets of \( X, \) to a \( C^0 \) curve \( \gamma : X \to M, \) which lies in \( \overline{B}. \)

**Proof.** We have three cases to consider. If \( b = \infty \) and \( b_i = \infty, \) let \( f_i(x) = x. \) If \( b = \infty \) and \( b_i \neq \infty, \) let \( f_i(x) = \frac{2b_i}{\pi} \arctan \left( \frac{x}{2b_i} \right). \) Otherwise \( b < \infty \) and we let \( f_i(x) = \frac{b_i}{b} x. \) In any case, we know that \( f_i : X \to Y_1 \) is a strictly monotonic, increasing, bijective smooth function.

We now show that, in any case, we have the relation \( f_i(u) - f_i(v) \leq u - v \) and therefore that \( \{f_i : i \in \mathbb{N}\} \) is a uniformly equicontinuous family. When \( b < \infty \) we know that \( f_i(u) - f_i(v) = \frac{b_i}{b} u - \frac{b_i}{b} v = \frac{b_i}{b} (u - v) \leq u - v. \)

When \( b = \infty \) and \( b_i = \infty \) we know that \( f_i(u) - f_i(v) = u - v \leq u - v. \) When \( b = \infty \) and \( b_i < \infty \) we note that \( \frac{d}{dx} f_i = \frac{1}{1 + \left( \frac{x}{2b_i} \right)^2} \leq 1. \)

Let \( g(x) = x, \) then as \( f_i(0) = 0, g(0) = 0 \) and \( 0 < \frac{d}{dx} f_i < \frac{d}{dx} g \) for all \( x > 0, \) we know that \( f_i(x) \leq g(x) \) for all \( x > 0. \) Therefore we have that \( f_i(u) - f_i(v) \leq g(u) - g(v) = u - v, \) as required.
We now show that, in any case, \( \{ \tilde{\gamma}_i = \gamma_i \circ f_i : i \in \mathbb{N} \} \) is a uniformly equicontinuous family. By assumption for all \( i \in \mathbb{N} \) and all \( t_1, t_2 \in I \), we know that \( d(\tilde{\gamma}_i(t_1), \tilde{\gamma}_i(t_2)) \leq |t_1 - t_2| \). Thus for all \( t_1, t_2 \in X \) we know that
\[
\begin{align*}
    d(\tilde{\gamma}_i(t_1), \tilde{\gamma}_i(t_2)) &= d(\gamma_i \circ f_i(t_1), \gamma_i \circ f_i(t_2)) \\
    &\leq |f_i(t_1) - f_i(t_2)| \\
    &\leq |t_1 - t_2|.
\end{align*}
\]
Since \( |t_1 - t_2| \) does not depend on \( i \), the collection of functions \( \gamma_i \circ f_i = \tilde{\gamma}_i : X \to M \) is uniformly equicontinuous.

We now show that \( \{ \tilde{\gamma}_i(t) : i \in \mathbb{N} \} \) is bounded for each \( t \in X \). Let \( t \in X \) and let \( x_i = \tilde{\gamma}_i(t) \). Since \( \tilde{\gamma}_i(X) = \gamma_i(Y_i) \), we can see that for all \( n > N, x_n \in \mathcal{B} \), by assumption. The set \( X_N = \{ x_i : i \leq N \} \) is finite and because \( \mathcal{B} \) is bounded there must exist \( B \in \mathbb{R}^+ \) so that \( d(x_i, x_j) < B \) for all \( i, j \). Hence \( \{ \tilde{\gamma}_i(t) : i \in \mathbb{N} \} \) is bounded for each \( t \in X \). So, by Arzelà’s theorem \([1] \) Theorem 3.30], there exists some \( C^0 \) curve \( \gamma : X \to M \) such that there is a subsequence of \( \{ \tilde{\gamma}_i \}_{i \in \mathbb{N}} \) which converges uniformly to \( \gamma \) on compact subsets of \( X \).

To show that \( \gamma \in \mathcal{B} \) we must show that for all \( t \in X \), \( \gamma(t) \in \mathcal{B} \). As there is a subsequence \( \{ \tilde{\gamma}_{k_i} \}_{i \in \mathbb{N}} \) of \( \{ \tilde{\gamma}_i \} \) that converges to \( \gamma \) uniformly on compact subsets of \( X \) and as \( [t, t + \varepsilon] \) is a compact subset of \( X \), for some \( \varepsilon > 0 \), we can conclude that \( \tilde{\gamma}_{k_i}(t) \to \gamma(t) \). We know, however, that for all \( n > N \), \( \tilde{\gamma}_n(t) \in \mathcal{B} \), thus there exists some \( m_0 \in \mathbb{N} \) so that for all \( i > m_0 \), \( k_i > N \) and therefore \( \tilde{\gamma}_{k_i}(t) \in \mathcal{B} \). Hence \( \gamma(t) \in \mathcal{B} \) as required. \( \square \)

References

[1] J. K. Beem, P. Ehrlich, and K. Easley. *Global Lorentzian Geometry (Pure and Applied Mathematics)*. CRC Press, 2 edition, 1996.
[2] D. Canarutto and E. Minguzzi. The distance formula in algebraic spacetime theories. [arXiv:1902.00591](https://arxiv.org/abs/1902.00591).
[3] C. J. S. Clarke. On the geodesic completeness of causal space-times. *Mathematical Proceedings of the Cambridge Philosophical Society*, 69 (2) (1971), 319–323.
[4] N. Franco. Global eikonal condition for Lorentzian distance function in noncommutative geometry. *SIGMA. Symmetry, Integrability and Geometry: Methods and Applications*, 6 (2010), 064. [arXiv:1003.5651](https://arxiv.org/abs/1003.5651).
[5] N. Franco. The Lorentzian distance formula in noncommutative geometry. *Journal of Physics: Conference Series*, 968 (1) (2018), 012005.
[6] R. Geroch, E. H. Kronheimer, and R. Penrose. Ideal points in space-time. *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences*, 327 (1571) (1972), 545–567.
[7] S. W. Hawking and G. F. R. Ellis. *The Large Scale Structure of Space-Time (Cambridge Monographs on Mathematical Physics)*. Cambridge University Press, 1975.
[8] E. Minguzzi. Causality theory for closed cone structures with applications. *to appear in Reviews in Mathematical Physics*, doi 10.1142/S0129055X19300012.
[9] E. Minguzzi. Limit curve theorems in Lorentzian geometry. *Journal of Mathematical Physics*, 49 (9) (2008), 092501.
[10] E. Minguzzi. Characterization of some causality conditions through the continuity of the Lorentzian distance. *Journal of Geometry and Physics*, 59 (7) (2009), 827–833.
[11] E. Minguzzi and M. Sánchez. The causal hierarchy of spacetimes. *Recent developments in pseudo-Riemannian geometry*, ESI Lect. Math. Phys., 299–358, 2008.
[12] V. Moretti. Aspects of noncommutative Lorentzian geometry for globally hyperbolic spacetimes. *Reviews in Mathematical Physics*, 15 (10) (2003), 1171–1217.
[13] B. O’Neill. *Semi-Riemannian Geometry With Applications to Relativity*, 103 of Pure and applied mathematics. Academic Press, 1983.
[14] R. Penrose. *Techniques of Differential Topology in Relativity*. Society for Industrial Mathematics, 1972. CBMS-NSF Regional Conference Series in Applied Mathematics.

[15] RectifiableCurve. Rectifiable curve. *Encyclopedia of Mathematics*. [http://www.encyclopediaofmath.org/index.php?title=Rectifiable_curve&oldid=29205](http://www.encyclopediaofmath.org/index.php?title=Rectifiable_curve&oldid=29205). Accessed: 2018-11-06.

[16] A. Rennie and B. E. Whale. Generalised time functions and finiteness of the Lorentzian distance. *Journal of Geometry and Physics*, **106** (2016), 108–121.

[17] R D Sorkin and E Woolgar. A causal order for spacetimes with Lorentzian metrics: proof of compactness of the space of causal curves. *Classical and Quantum Gravity*, **13** (7) (1996), 1971–1993.

[18] B. E. Whale, M. J. S. L. Ashley, and S. M. Scott. Generalizations of the abstract boundary singularity theorem. *Classical and Quantum Gravity*, **32** (13) (2015), 135001.