IN Variant Algebraic Surfaces and Constrained Systems

PAULO R. DA SILVA AND OTÁVIO H. PEREZ

Abstract. We study flows of smooth vector fields $X$ over invariant surfaces $M$ which are levels of rational first integrals. It leads us to study constrained systems, that is, systems with impasses. We identify a subset $I \subset M$ which we call “pseudo-impasse” set and analyze the flow of $X$ by points of $I$. Systems well known in the literature exemplify our results: Lorenz, Chen, Falkner-Skan and Fisher-Kolmogorov. We also study 1-parameter families of integrable systems and unfolding of minimal sets. Our main tool is the geometric singular perturbation theory.

1. Introduction

Let $X: \mathbb{R}^3 \to \mathbb{R}^3$ be a smooth vector field. A trajectory of $X$ is a smooth curve $\varphi(t)$ satisfying that

$$\dot{\varphi}(t) = X(\varphi(t)), \quad t \in I \subseteq \mathbb{R}.$$ 

We say that a smooth function $H: \mathbb{R}^3 \to \mathbb{R}$ is a first integral of $X$ if it satisfies $\langle \nabla H, X \rangle = 0$. It means that $H(\varphi(t)) = H(\varphi(0))$, for any $t \in I$.

Our main goal is to describe the flow of $X$ on invariant algebraic surfaces, that is on $M = H^{-1}(0)$ with $H$ being a polynomial function.

Here we focus our attention on surfaces $M = H^{-1}(0)$ with $H(x, y, z) = f(x, y)z - g(x, y)$, where $f, g$ are polynomials. Considering these surfaces is not very restrictive. In fact, as we will see in the examples below, many surfaces, including singular parts or being disconnected, can be represented in this way.

The surface $M$ can be written as the disjoint union of two subsets: $G_M$ and $I_M$. The first one is the graphic of $z = (g/f)(x, y)$, and the second one represents the subset of $M$ that cannot be written as a graphic. The set $I_M$ is called pseudo-impasse set. Geometrically, $I_M \subset M$ is a set of lines which are parallel to the $z$-axis. See Proposition 4, section 3.

Before we present our results, let’s start with an example to indicate what kind of problem we are interested in.

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**Example.** Consider the vector field $X(x, y, z) = (2y, 2xz, (x^2 - 1)z - (y^2 - 1))$ where $M = H^{-1}(0)$ defines a smooth algebraic invariant surface, with $H(x, y, z) = (x^2 - 1)z - (y^2 - 1)$. We can write $M = \mathcal{G}_M \cup \mathcal{I}_M$, where $\mathcal{G}_M = \{(x, y, z) \in M | z = (y^2 - 1)/(x^2 - 1), x \neq \pm 1\}$ and $\mathcal{I}_M = \{(x, y, z) \in M | y^2 - 1 = 0 = x^2 - 1, z \in \mathbb{R}\}$. $\mathcal{G}_M$ is a graphic and the pseudo impasse $\mathcal{I}_M$ is a set of four lines that are orthogonally projected on the $xy$-plane. See figure 1. The flow of $X$ on $\mathcal{G}_M$ is described by

\begin{equation}
\dot{x} = 2y, \ (x^2 - 1)\dot{y} = 2x(y^2 - 1).
\end{equation}

Additional effort is needed to describe the flow in $\mathcal{I}_M$. The projections of $\mathcal{I}_M$ on $xy$-plane are hyperbolic equilibrium points of the system $\dot{x} = (x^2 - 1)2y, \ \dot{y} = 2x(y^2 - 1)$. It is easy to see that $\mathcal{I}_M$ does not contain any equilibrium point and the four lines are not invariant.

![Figure 1. Phase portrait of $X(x, y, z) = (2y, 2xz, (x^2 - 1)z - (y^2 - 1))$ (left) and surface $M$ (right). The pseudo impasse set $\mathcal{I}_M$ is denoted in blue.](image)

Systems written as (1) are known as constrained systems (or impasse systems). Constrained systems have been widely studied in the literature. In [21] the author classified normal forms in $\mathbb{R}^2$ and in [19] the authors gave normal forms defined in $\mathbb{R}^n$, $n \geq 3$. Both references assume that the impasse manifold is smooth. Applications in electrical circuits can be found in [18].

Below we briefly list some results that we have proved about flows and impasses.

- The flow of $X$ on the algebraic invariant manifold $M$ is determined by a constrained system $A(x, y)(\dot{x}, \dot{y}) = F(x, y), \ (x, y) \in \mathbb{R}^2$ (see section 2 for a precise definition). The projection of $\mathcal{I}_M$ on $\mathbb{R}^2$ lies on the impasse set $\mathcal{I} = \{(x, y) : \det A(x, y) = 0\}$ and it is a set of equilibrium points of the adjoint vector field $A^*F$. See Theorem 7 section 3.
Let \((x, y) \in \mathbb{R}^2\) be the projection of a line \(r \subset \mathcal{I}_M\). Then \((x, y) \in \mathcal{I}\) is a hyperbolic equilibrium point for the adjoint vector field \(A^* F\) if, and only if, \((x, y)\) is a hyperbolic node. Moreover, \(r\) does not contain equilibrium points of \(X\), \(r\) is transversal to the flow and \(r\) does not contain any singular points of the surface \(M\). See Proposition 11 and Theorem 15, section 3. If \((x, y) \in \mathcal{I}\) is a non-hyperbolic equilibrium point then, under some conditions, \(r\) intersects the singular part of \(M\) or \(r\) is invariant by the flow. See Theorem 16, section 3.

In section 4 we exemplify our results with well-known systems in the literature, for example, Falkner-Skan Equation, Lorenz System and Chen System. We discuss how the study of flows on invariant surfaces by means of a constrained system can be extended in higher dimensions.

In the second part of this paper we consider 1-parameter families of first integrals and unfoldings of minimal sets. In our approach, singular perturbation theory [9] is the main tool. More precisely, we prove that equilibrium points and periodic orbits of smooth system which are contained in algebraic invariant surfaces persist under small perturbations.

Example. Let \(H_\varepsilon : \mathbb{R}^3 \to \mathbb{R}\) be a family of smooth functions given by \(H_\varepsilon(x, y, z) = x - y^2 - \varepsilon z\). For each \(\varepsilon > 0\), \(H_\varepsilon\) is a first integral of \(X_\varepsilon(x, y, z) = (2y(y + z) - \varepsilon(y - z), y + z, y - z)\). If \(\varphi_0\) is a trajectory of \(X_0\) in the level \(H_0(x, y, z) = 0\), then \(\varphi_0\) is a solution of
\[
0 = x - y^2, \quad \dot{y} = y + z, \quad \dot{z} = y - z.
\]

System (2) is an algebraic differential equation and it describes the slow flow on the slow manifold of a singular perturbation problem. Furthermore, Proposition 19, section 5, says that \(\varphi_0\) is a trajectory of \(X_0\) if, and only if, \(\varphi_0\) is a solution of (2). This allows us to study the flow of \(X_\varepsilon\) for \(\varepsilon > 0\). More precisely, we study the flow of \(X_\varepsilon\) in the levels \(H_\varepsilon(x, y, z) = 0\) for \(\varepsilon > 0\) using singular perturbation problems. For this purpose, Fenichel Theorem is our main tool. Concerning to 1-parameter families of smooth vector fields we prove:

- Let \(H_\varepsilon\) be a family of first integrals of \(n\)-dimensional smooth vector fields \(X_\varepsilon\). The equilibria of \(X_\varepsilon\) on \(\{H_\varepsilon = 0\}\) are equilibria of a singular perturbation problem. If \(p_0 \in \{H_0 = 0\}\) is an equilibrium point of \(X_0\), then there exists a sequence of equilibrium points \(p_\varepsilon\) of \(X_\varepsilon\), satisfying that \(p_\varepsilon \to p_0\) and \(p_\varepsilon \in \{H_\varepsilon = 0\}\). See Proposition 20, section 5.
- Under some conditions, periodic orbits of \(X_0\) contained in \(\{H_0 = 0\}\) persist under small perturbations. See Proposition 21, section 5.

The paper is organized as follows. In section 2 we present basic concepts and definitions concerning constrained systems and singular perturbation
theory. We start section 3 studying some geometric properties of the algebraic invariant surface, and then we present results that relate the flows on $M$ to constrained systems. We exemplify these results in section 4 with well-known systems in the literature, such as Falkner-Skan, Lorenz and Chen systems. We also discuss how the problem of describe flows on invariant surfaces by means of constrained systems extends in higher dimensions. Finally, in section 5 we deal with families of first integrals and slow-fast systems.

2. Preliminaries on the Geometric Singular Perturbation Theory and Impasses

In this section we present basic concepts and definitions concerning constrained systems and singular perturbation theory. We also refer [19, 21] and [9] for an introduction of constrained systems and singular perturbation theory, respectively.

2.1. Constrained systems. A constrained system (or impasse system) is given by

$$A(x)\dot{x} = F(x), \quad x \in \mathbb{R}^n$$

where $F : \mathbb{R}^n \to \mathbb{R}^n$ is a smooth vector field and $A(x)$ is a square matrix of order $n$ whose entries $a_{ij}(x)$ smoothly depend on $x$. This kind of system generalizes vector fields because at the points where $\det A(x) \neq 0$ we can rewrite (3) as $\dot{x} = A^{-1}(x)F(x)$. On the other hand, at the points where $\det A(x) = 0$ (called impasse points) we cannot assure the existence and/or uniqueness of the solutions. Another particularity of (3) is the existence of the impasse manifold, defined by

$$\mathcal{I} = \{x \in \mathbb{R}^n | \det A(x) = 0\}.$$ 

We can draw the phase portrait of (3) as follows. We relate the system (3) to the vector field $A^*F$, where $A^*$ is the adjoint matrix of $A$ characterized by $AA^* = A^*A = \det(A)I$. Thus, the phase portrait of (3) can be seen as the phase portrait of the vector field $A^*F$ by removing the impasse points from its orbits and inverting its orientation where $\det A < 0$. As in [4], the vector field $A^*F$ will be called adjoint vector field.

It can be found normal forms for constrained systems in $\mathbb{R}^2$ and $\mathbb{R}^n$ with $n \geq 3$ in [21] and [19], respectively. In both references, the authors studied the dynamics of (3) in the neighborhood of a regular point $x_0 \in \mathcal{I}$, that is, $d(\det A(x))(x_0) \neq 0$. Moreover, they adopted the hypothesis that the trajectories of the system (3) either do not intercept $\mathcal{I}$, or intercept $\mathcal{I}$ in a finite number of isolated points.

Let $x_0 \in \mathcal{I}$ be a regular impasse point and consider the following conditions.

(A) The vector space $\ker A(x_0)$ is transversal to $\mathcal{I}$.

(B) The vector $F(x_0)$ does not belong to the range of $A(x_0)$. 
By linear algebra, we know that \( \text{Im} A = \ker A^* \) and \( \ker A = \text{Im} A^* \). Therefore, condition (A) means that the vector field \( A^*F \) is transversal to \( \mathcal{I} \) at \( x_0 \) and condition (B) means that \( x_0 \) is not an equilibrium point of \( A^*F \).

**Definition 1.** Let \( x_0 \in \mathcal{I} \) be a regular impasse point of (3).

1. \( x_0 \) is non singular if \( x_0 \) satisfies conditions (A) and (B).
2. \( x_0 \) is a K-singularity (kernel singularity) if \( x_0 \) satisfies (B) and does not satisfy (A).
3. \( x_0 \) is an R-singularity (range singularity) if \( x_0 \) satisfies (A) and does not satisfy (B).
4. \( x_0 \) is an RK-singularity (range-kernel singularity) if \( x_0 \) does not satisfy conditions (A) and (B).

2.2. Slow–fast systems and Fenichel Theory. A *singularly perturbed system* is a system of the form

\[
\varepsilon \dot{x} = f(x, y, \varepsilon), \quad \dot{y} = g(x, y, \varepsilon),
\]

where \( f, g \) are smooth, \( x \in \mathbb{R}^n \), \( y \in \mathbb{R}^m \) and \( \varepsilon > 0 \) is small. The dot \( \cdot \) denotes the derivative with respect to \( t \).

The parameter \( \varepsilon \) measures the variation rate of \( x \) and \( y \). When \( \varepsilon = 0 \), the system (5) reduces to the differential-algebraic system

\[
0 = f(x, y, 0), \quad \dot{y} = g(x, y, 0).
\]

System (6) is called a *reduced problem* or *slow equation*. By taking \( t = \varepsilon \tau \) in (5), we obtain the system

\[
x' = f(x, y, \varepsilon), \quad y' = \varepsilon g(x, y, \varepsilon),
\]

where \( ' \) denotes the derivative with respect to \( \tau \). In the limit \( \varepsilon = 0 \), we obtain the fast equation (or layer problem) given by

\[
x' = f(x, y, 0), \quad y' = 0.
\]

System (8) is reduced to \( n \) differential equation with respect to the fast variable \( x \), which depends on the slow variable \( y \) as a parameter. For \( \varepsilon > 0 \), systems (5) and (7) are equivalents.

The set

\[
S_0 = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^m | f(x, y, 0) = 0\}
\]

is the phase space of (6) and it is known in the literature as a *critical manifold* or *slow manifold*. Notice that \( S_0 \) is the set of equilibrium points of (8).

We can interpret the phase portrait (5) and (7) when \( \varepsilon \) is close to 0 as follows. A point outside \( S_0 \) moves from a stable fast fiber according to the dynamics of (8), until it reaches a stable branch of \( S_0 \). Then, the dynamics change to (6). If the corresponding solution reaches a singularity or a bifurcation point (where \( S_0 \) loses stability), thus the dynamics changes to (8).
A point \( p = (x, y) \in S_0 \) is *normally hyperbolic* if \( f_x(p, 0) \) is a matrix whose all eigenvalues have nonzero real parts being \( k^s \) eigenvalues with negative real parts and \( k^u \) eigenvalues with positive real parts. The set of all normally hyperbolic points of \( S_0 \) is denoted by \( \mathcal{NH}(S_0) \).

The following theorem is one of the most important results of singular perturbation theory and it is due to Fenichel. Such result describes how is the flow of system (5) for \( \varepsilon \) is sufficiently small. See [9] for details.

**Theorem 2.** Let \( N_0 \subset \mathcal{NH}(S_0) \) be a \( j \)-dimensional normally hyperbolic compact submanifold of \( S_0 \) for [6], with a \((j + j^s)\)-dimensional local stable manifold \( W^s \) and a \((j + j^u)\)-dimensional local unstable manifold \( W^u \). Suppose that system (5) satisfies \( f, g \in C^r \). Then for \( \varepsilon > 0 \) sufficiently small the following statements are true.

F1. There exists a \( C^{r-1} \) family of compact locally invariant manifolds \( \{N_\varepsilon\}_\varepsilon \) of (5) converging to \( N_0 \), according Hausdorff distance. Moreover, \( N_\varepsilon \) is diffeomorphic to \( N_0 \).

F2. There exist \( C^{r-1} \) families of \((j + j^s + k^s)\)-dimensional and \((j + j^u + k^u)\)-dimensional manifolds \( N^s_\varepsilon \) and \( N^u_\varepsilon \), respectively, such that \( N^s_\varepsilon \) and \( N^u_\varepsilon \) are the local stable and unstable manifolds of \( N_\varepsilon \).

### 2.3. Catastrophes as invariant surfaces.

The slow flow (6) of a singular perturbation problem, with \((x, y, z) \in \mathbb{R}^3\), is given by a constrained system

\[
0 = V_x(x, y, z), \quad \dot{y} = \beta(x, y, z), \quad \dot{z} = \gamma(x, y, z),
\]

where \( V \) is the *potential function*. According Takens [20], under topological equivalence, there are 12 normal forms of generic constrained differential equations.

| \( V(x, y, z) \) | \( X(x, y, z) \) | Type |
|------------------|-----------------|------|
| \( \frac{\partial V}{\partial x} \) | (0, 1, 0) | Flow-box |
| | (0, 0, y) | Source |
| | (0, y, 0) | Saddle |
| | (0, -y, -z) | Sink |
| \( \frac{\partial V}{\partial x} + yx \) | (0, 1, 0) | Flow-box 1 |
| | (0, -1, 0) | Flow-box 2 |
| | (0, 3x + z, 1) | Source |
| | (0, -3x + z, 1) | Sink |
| | (0, -z, 1) | Saddle |
| | (0, -z, 1) | Focus |
| \( \frac{\partial V}{\partial x} + \frac{\partial V}{\partial z} + yx \) | (0, 1, 0) | Flow-box 1 |
| | (0, -1, 0) | Flow-box 2 |

Following Proposition, whose proof can be found in [11], says the slow flow on the slow manifold is given by a constrained system.

**Proposition 3.** Consider system (10) and a point \( p \) satisfying

(a) \( V_x(p) = 0, V_{xx}(p) = 0; \)
(b) \( V_{xy}(p) \neq 0 \) or \( V_{xz}(p) \neq 0. \)

Then system (10) can be written as

\[
- V_{xx} \dot{x} = V_{xy} \beta + V_{xz} \gamma, \quad \dot{z} = \gamma,
\]

where \( y = \xi(x, z) \) is the solution of \( V_x(p) = 0. \)
If $V(x, y, z) = \frac{x^2}{2}$, then equation (11) defines a smooth system.

If $V(x, y, z) = \frac{x^3}{3} + yx$, then equation (11) defines a constrained system where $I = \{(x, z) \in \mathbb{R}^2 | x = 0\}$. For the Flow-Box normal forms, system (11) is given by $-2x\dot{x} = \mp 1$, $\dot{z} = 0$, where the adjoint vector field is $\dot{x} = \mp 1$, $\dot{z} = 0$. It follows that the origin is non singular. The source and sink normal forms are $-2x\dot{x} = \pm 3x + z$, $\dot{z} = 1$, where the adjoint vector field is $\dot{x} = \pm 3x + z$, $\dot{z} = -2x$. Since the origin is a node for the adjoint vector field, it is a R-singularity. The same occurs for the saddle and focus normal forms. For the saddle normal form, system (11) takes form $-2x\dot{x} = -z$, $\dot{z} = 1$, where the adjoint vector field is $\dot{x} = -z$, $\dot{z} = -2x$. For the focus normal form, system (11) takes form $-2x\dot{x} = x + z$, $\dot{z} = 1$, where the adjoint vector field is $\dot{x} = x + z$, $\dot{z} = -2x$.

If $V(x, y, z) = \frac{x^4}{4} + \frac{yz^2}{2} + yx$, then the impasse set of (11) is given by $I = \{(x, z) \in \mathbb{R}^2 | 3x^2 + z = 0\}$ and the flow-box normal forms are given by $-(3x^2 + z)\dot{x} = 1$, $\dot{z} = 0$, where the adjoint vector field is $\dot{x} = 1$, $\dot{z} = 0$. In this case, the origin is a K-singularity.

**Figure 2.** Phase portraits of (11) for the source (left) and sink (right) cases.

**Figure 3.** Phase portraits of (11) for the saddle (left) and focus (right) cases.
3. Phase portrait on invariant algebraic surfaces

Let $X$ be a smooth vector field in $\mathbb{R}^3$ with polynomial first integral $H(x, y, z) = f(x, y)z - g(x, y)$. Denote $M = H^{-1}(0) = G_M \cup I_M$ where $G_M$ is the graphic of $z = (g/f)(x, y)$ and $I_M \subset M$ is the pseudo-impasse subset of $M$. The next proposition summarizes some properties of $M$.

**Proposition 4.** Let $Z_f$ and $Z_g$ be the zero sets of $f$ and $g$, respectively.

(a) If $Z_f \cap Z_g = \emptyset$, then $I_M = \emptyset$ and $M = G_M$.

(b) If $Z_f \cap Z_g$ is a set containing isolated points, then $I_M$ is a set formed by lines which are parallel to the $z$-axis. See figure 5.

(c) If $Z_f$ and $Z_g$ coincide in an open set of $\mathbb{R}^2$, then $I_M$ is a surface that is orthogonal to the $xy$-plane.

(d) If there exist $p \neq q \in \mathbb{R}^2$ such that $f(p)f(q) < 0$ and $I_M = \emptyset$ then $M$ is disconnected.

(e) If $p$ is a singular point of $M$, then $p \in I_M$.

**Proof.** We can see the sets $Z_f$ and $Z_g$ as curves in $\mathbb{R}^2$. This implies that $I_M$ is a set of lines such that they are parallel to the $z$-axis and they intercept the $xy$-plane at the points of intersection of the curves $Z_f$ and $Z_g$. Then it follows that the items (a), (b) and (c) are true. For item (d), suppose that $I_M = \emptyset$, that is, $M = G_M$. By hypothesis, $f$ assumes positives and negatives values, therefore we can write $M = G^+_M \cup G^-_M$, where $G^+_M = \{f(x, y) > 0, z = (g/f)(x, y)\}$ and $G^-_M = \{f(x, y) < 0, z = (g/f)(x, y)\}$. The sets $G^+_M$ and $G^-_M$ are open sets contained in $M$ such that $G^+_M \cap G^-_M = \emptyset$, thus the surface is disconnected. Now note that a point $p = (x_0, y_0, z_0) \in M$ is singular if, and only if, $\nabla H(p) = 0$. In other words, $p$ will be a singular point of $M$ if

$$f_x(x_0, y_0)z_0 = g_x(x_0, y_0), \quad f_y(x_0, y_0)z_0 = g_y(x_0, y_0), \quad f(x_0, y_0) = 0.$$  

The last equation assures that $p \in I_M$ and thus item (e) holds.

The set $I_M$ is not necessarily the set of all singular points of $M$. As we shall see, there are examples of regular surfaces such that $I_M \neq \emptyset$. Moreover, the hypothesis that $f$ assumes positive and negative values is crucial...
in Proposition 4.

In the particular case where $X$ is a polynomial vector field we can $\dot{x} = X(x)$ to an analytic system on a closed ball of radius one, whose interior is diffeomorphic to $\mathbb{R}^3$ and its boundary, the 2-dimensional sphere $S^2$, plays the role of the infinity. This closed ball is denoted by $\mathbb{D}^3$ and called the *Poincaré ball*, because the technique for doing such an extension is precisely the *Poincaré compactification* for a polynomial differential system in $\mathbb{R}^3$, which is described in details in [6]. Besides, we also can extend $M$ to the infinity. In [11] the authors proved the following Lemma in order to obtain the expression of a surface at infinity.

**Lemma 5.** Let $H : \mathbb{R}^3 \to \mathbb{R}$ be a polynomial of degree $m$ and $H(x, y, z) = 0$ be an algebraic surface. The extension of this surface to the boundary of the Poincaré ball $x^2 + y^2 + z^2 = 1$ is obtained solving the system $w^m H(x/w, y/w, z/w) = 0$, $w = 0$.

If $P$ is a polynomial, denote the degree of $P$ by $d(P)$. We can write $P$ as $P(x, y) = \sum_{j=0}^{m} P_j(x, y)$, where $P_j$ is a homogeneous polynomial of degree $j$.

**Proposition 6.** The extension of the surface $M$ to infinity contains the poles $(0, 0, \pm 1)$. In particular, if $d(f) \geq d(g)$ then such extensions contain the big circle $\{z = 0\}$.

**Proof.** If $H$ is a polynomial of degree $m$ we have three cases to consider.

- $d(g) = m$ and $d(f) < m - 1$: Lemma 5 provides $g_m(x, y) = 0$ and therefore the extension of $M$ is the same as the extension of the surface $\{g_m(x, y) = 0, z \in \mathbb{R}\}$.
- $d(g) < m$ and $d(f) = m - 1$: Lemma 5 provides $f_{m-1}(x, y)z = 0$, and this implies that the extension of $M$ is the union of the great circle $\{z = 0\}$ with the extensions of the surface $f_{m-1}(x, y) = 0$.
- $d(g) = m$ and $d(f) = m - 1$: Lemma 5 provides $f_{m-1}(x, y)z - g_m(x, y) = 0$.
Since \( g_m(x, y) \) and \( f_{m-1}(x, y) \) are homogeneous polynomials of degree \( m \) and \( m - 1 \) respectively, the poles \((0, 0, \pm 1)\) belongs to the projection at the infinity. \( \square \)

Let \( X(x, y, z) = (\alpha(x, y, z), \beta(x, y, z), \gamma(x, y, z)) \), be a smooth vector field with \( \alpha, \beta \) given by \( \alpha(x, y, z) = \sum_{i=0}^{m} \alpha_i(x, y)z^i, \beta(x, y, z) = \sum_{j=0}^{n} \beta_j(x, y)z^j \).

**Theorem 7.** If \( M = \{ f(x, y)z - g(x, y) = 0 \} \) is an invariant algebraic surface of \( X = (\alpha, \beta, \gamma) \), then there exists a constrained system defined in \( \mathbb{R}^2 \) such that:

1. The impasse curve of such system is given by \( I = \{(x, y)|f(x, y) = 0\} \).
2. On \( \mathbb{R}^2 \setminus I \), the orbits of the constrained system are the projections of the ones on \( \mathcal{G}_M \).
3. The projection of \( \mathcal{I}_M \) on \( \mathbb{R}^2 \) is contained in \( I \) and it is a set of equilibrium points of the adjoint system.

**Proof.** If \((x, y, z) \in \mathcal{G}_M\) then \( z = \xi(x, y) = (g/f)(x, y) \). Thus the orbits of \( X \) on \( \mathcal{G}_M \) are obtained from the solutions of

\[
\dot{x} = \sum_{i=0}^{m} \alpha_i(x, y) \left[(g/f)(x, y)\right]^i, \quad \dot{y} = \sum_{j=0}^{n} \beta_j(x, y) \left[(g/f)(x, y)\right]^j, \tag{12}
\]

Rewriting (12) we get the constrained system

\[
f^m \dot{x} = \sum_{i=0}^{m} f^{m-i} g^i \alpha_i, \quad f^n \dot{y} = \sum_{j=0}^{n} f^{n-j} g^j \beta_j, \tag{13}
\]

where \( f^k(x, y) = \left(f(x, y)\right)^k \) and \( g^l(x, y) = \left(g(x, y)\right)^l \).

The impasse curve of (13) is \( I = \{(x, y)|f(x, y) = 0\} \) and system (13) has the phase portrait of system (12) on the region \( f(x, y) \neq 0 \). Therefore, the orbits of \( X \) on \( \mathcal{G}_M \) are the image by \( \xi \) of the orbits of (13) on \( \mathbb{R}^2 \setminus I \).

The adjoint system of (13) is given by

\[
\dot{x} = f^n \left( \sum_{i=0}^{m} f^{m-i} g^i \alpha_i \right), \quad \dot{y} = f^m \left( \sum_{j=0}^{n} f^{n-j} g^j \beta_j \right). \tag{14}
\]

Moreover, the projection of \( \mathcal{I}_M \) on \( \mathbb{R}^2 \) is given by \( \{(x, y)|f(x, y) = 0 = g(x, y)\} \), which is a set containing equilibrium points of the adjoint vector field (14). \( \square \)

Theorem 7 shows us that outside the impasse curve \( I \) the orbits of system (13) are the orbits of \( X \) on \( \mathcal{G}_M \). Thus, our objective is to describe the orbits of \( X \) by points on \( \mathcal{I}_M \).

In the proof of Theorem 7 observe that \( I \) is a curve of equilibrium points for the adjoint system (14). This leads us to start our studies with less degenerate cases. For this, we will consider a system in \( \mathbb{R}^3 \) given by
\( \dot{x} = \alpha(x, y), \quad \dot{y} = \sum_{j=0}^{n} \beta_j(x, y)z^j, \quad \dot{z} = \gamma(x, y, z), \)

which has invariant algebraic surface \( M = H^{-1}(0), \) \( H(x, y, x) = f(x, y)z - g(x, y). \) As in the proof of Theorem 7 we obtain the constrained system

\[
\dot{x} = \alpha, \quad f^n \dot{y} = g^n \beta_n + \sum_{j=0}^{n-1} f^{n-j} g^j \beta_j,
\]

whose adjoint system is given by

\[
\dot{x} = f^n \alpha, \quad \dot{y} = g^n \beta_n + \sum_{j=0}^{n-1} f^{n-j} g^j \beta_j.
\]

We already know by Theorem 7 that the projection \( \Pi(I_M) \subset I \) is a set containing only equilibrium points of the adjoint system. These are the points that we must study to understand the flows of (15) on \( I_M \subset M. \)

The next result classifies the impasse points by means of the functions \( f \) and \( g. \)

**Proposition 8.** Let \( p \in \mathbb{R}^2 \) be an impasse point of system (16).

1. \( p \) is a R-singularity if and only if \( g(p) = 0 \) or \( \beta_n(p) = 0. \)
2. \( p \) is a K-singularity if and only if \( f_y(p) = 0. \)
3. \( p \) is a RK-singularity if and only if \( p \) is R-singularity and K-singularity simultaneously.
4. \( p \) is non singular if and only if \( p \) does not satisfy any of the previous conditions.

**Proof.** It follows directly from Definition 1 and the expression of the adjoint system (17).

We say that an impasse point of system (16) \( p \) is **R-singularity of first kind** if \( g(p) = 0 \) and **R-singularity of second kind** if \( \beta_n(p) = 0. \) The set of R-singularities of first kind is exactly the projection of \( I_M \) on the \( xy \)-plane. Moreover, it may appear R-singularities of first and second kind simultaneously.

**Corollary 9.** Consider the adjoint system (17) and the surface \( M = H^{-1}(0), \) \( H(x, y, x) = f(x, y)z - g(x, y). \)

1. Let \( p \) be a singular point of \( M. \) Thus the projection of \( p \) on \( xy \)-plane is a R-singularity of first kind for the adjoint system.
2. If there exists \( p \neq q \in \mathbb{R}^2 \) such that \( f(p)f(q) < 0 \) and the impasse curve \( I \) does not have R-singularities of first kind then \( M \) is disconnected.
Proof. For item 1, recall that Proposition 6 assures that \( p \in \mathcal{I}_M \). It follows from the last remark that the projection of \( p \) is a R-singularity of first kind.

If \( \mathcal{I} \) does not have R-singularities of first kind, then \( \mathcal{I}_M \) is empty. By Proposition 4, the item 2 is true. \( \square \)

Let \( r \) be a parallel line with respect to the \( z \)-axis and \( q \in \mathcal{I} \) the point in which \( r \) intercepts the \( xz \)-plane. From Corollary 9 we know that if \( q \) is not a R or RK-singularity of first kind, then \( r \) does not intersect the surface \( M \). On the other hand, as a consequence of Proposition 6 we have that \( r \) intersects \( M \) at infinity.

**Corollary 10.** If \( q \in \mathcal{I} \) is an isolated K-singularity, R-singularity of second kind or RK-singularity of second kind, then the line \( r = \{ (p, z) | z \in \mathbb{R} \} \) intersects \( M \) at the infinity.

Proof. We already know by Proposition 4 that the extension of \( M \) at the infinity contains the poles \((0, 0 \pm 1)\). The corollary follows observing that the line reaches such poles at the infinity. \( \square \)

**Remark.** We say that \( H(x, y, z) \) is a Darboux polynomial of \( X \) if it satisfies \((\nabla H) \cdot X = \mu H\), where \( \mu = \mu(x, y, z) \) is a real polynomial called the cofactor of \( H(x, y, z) \). The surface \( H(x, y, z) = 0 \) is an invariant algebraic surface.

**Example.** The polynomial \( H(x, y, z) = yz - 1 \) is a Darboux polynomial of \( X(x, yz) = (0, z, -z^3) \) with cofactor \( \mu(x, y, z) = -z^2 \). The flows on \( M = H^{-1}(0) \) are described by the constrained system \( \dot{x} = 0, y \dot{y} = 1 \). All points on \( \mathcal{I} = \{ y = 0 \} \) are non singular. It follows from Corollary 9 that \( M \) is disconnected. See Figure 6.

**Example.** The polynomial \( H(x, y, z) = (x + y^2)z - 1 \) is Darboux polynomial for \( X(x, yz) = (0, z, -2y^3z^3) \) with cofactor \( \mu(x, y, z) = -2yz^2 \). The flows on \( M = H^{-1}(0) \) are described by the constrained system \( \dot{x} = 0, (x + y^2) \dot{y} = 1 \). The origin is a K-singularity. Moreover, it follows from Corollary 9 that \( M \) is disconnected. See Figure 7.

**Example.** Let \( \lambda \in \mathbb{R} \) be a parameter satisfying \(|\lambda| > 1\). The polynomial \( H(x, y, z) = (y - x)z^2 - \lambda yz^3 \) is Darboux polynomial for \( X(x, yz) = (1, \lambda yz, z^2) \). The flows on \( M = H^{-1}(0) \) are described by the constrained system \( \dot{x} = 1, (y - x) \dot{y} = \lambda y \). The origin is a hyperbolic equilibrium point for the adjoint vector field, whose eigenvalues are \(-1\) and \( \lambda \) and its eigenvectors are \( v_+ = (1, 0) \) and \( v_\lambda = (1, 1 + \lambda) \). If \( \lambda > 1 \) the origin will be a saddle point and if \( \lambda < 1 \) the origin will be a node. In both cases, the origin is a R-singularity of second kind. See Figure 8.

**Example.** Let \( \lambda \in \mathbb{R} \) be a parameter satisfying \( \lambda > 0 \). The polynomial \( H(x, y, z) = (y - x)z - 1 \) is Darboux polynomial for \( X(x, yz) = (1 + \lambda^2, -(\lambda x + y)z, (\lambda x + y)z^3) \) with cofactor \( \mu(x, y, z) = (\lambda x + y)z^2 \). The flows
on $M = H^{-1}(0)$ are described by the constrained system $\dot{x} = \frac{1 + \lambda^2}{\lambda}$, \( \dot{y} = -(\lambda x + y) \). The origin is a R-singularity of second kind. Moreover, it is a hyperbolic focus for the adjoint vector field whose eigenvalues are $-\frac{1 \pm \sqrt{-3-4\lambda^2}}{2}$. See figure [9]

**Example.** The polynomial $H(x, y, z) = y^2 z - (x + y)$ is a first integral for $X(x, y, z) = (\frac{1}{4}, -\frac{1}{2}(yz + \frac{1}{2}), z^2)$ and the flows on $M = H^{-1}(0)$ are described by $\dot{x} = \frac{1}{4}, \dot{y} = -\frac{1}{2}(\frac{3y}{2} + x)$. The origin is a R-singularity of first kind for the system whose eigenvalues are $-\frac{1}{2}$ and $-\frac{1}{4}$. See figure [10]

**Figure 6.** Phase portrait of $X(x, yz) = (0, z, -z^3)$ (left) and surface $M$ (right).

**Figure 7.** Phase portrait of $X(x, yz) = (0, z, -2yz^3)$ (left) and surface $M$ (right).
Figure 8. Phase portrait of $X(x, yz) = (1, \lambda yz, z^2 - \lambda yz^3)$ for $\lambda > 1$, for $\lambda < 1$ and surface $M$ (right).

Figure 9. Phase portrait of $X(x, y, z) = \left(\frac{1+\lambda^2}{\lambda}, -(\lambda x+y)z, (\lambda x+y)z^3\right)$ (left) and surface $M$ (right).

Figure 10. Phase portrait of $X(x, y, z) = \left(\frac{1}{2}, -\frac{1}{2}(yz + \frac{1}{2}), z^2\right)$ (left) and surface $M$ (right). The pseudo impasse set $\mathcal{I}_M$ is denoted in blue.

If $n \geq 2$ the linearization of the adjoint vector field $J\tilde{X}(x, y)$ is given by

\begin{equation}
\begin{pmatrix}
  f^{n-1}(nf_{x}\alpha + f_{xx}) \\
  g^{n-1}(ng_{x}\beta_n + g_{nx}) + \sum_{i=0}^{n-1} S_{ix} \\
  g^{n-1}(ng_{y}\beta_n + g_{ny}) + \sum_{i=0}^{n-1} S_{iy}
\end{pmatrix}
\end{equation}
where
\[ S_{tx} = f^{n-i-1} g^{i-1} \left( f g \beta_{tx} + ((n-i)f_x g + if_y ) \beta_i \right), \]
and
\[ S_{ty} = f^{n-i-1} g^{i-1} \left( f g \beta_{ty} + ((n-i)f_y g + if_x ) \beta_i \right). \]

If \( p \in \mathbb{R}^2 \) is a R-singularity (of first or second kind), then \( p \) will be non-hyperbolic. In particular, if \( p \) is R-singularity of first kind then \( J \tilde{X}(p) \) is identically zero.

3.1. **Particular case: \( \beta(x,y) \) with degree 1 in the variable \( z \).** In what follows, we suppose that \( X \) is
\[ \dot{x} = \alpha(x,y), \quad \dot{y} = \beta_1(x,y)z + \beta_0(x,y), \quad \dot{z} = \gamma(x,y,z), \]
whose flows are described by the constrained system
\[ \dot{x} = \alpha(x,y), \quad f(x,y)\dot{y} = g(x,y)\beta_1(x,y) + f(x,y)\beta_0(x,y), \]
and its adjoint system is
\[ \dot{x} = f(x,y)\alpha(x,y), \quad \dot{y} = g(x,y)\beta_1(x,y) + f(x,y)\beta_0(x,y). \]

The linearization \( J \tilde{X}(x,y) \) of (21) is
\[ \begin{pmatrix} f_x \alpha + f \alpha_x & f_y \alpha + f \alpha_y \\ (f \beta_{0x} + g \beta_{1x}) + (f_x \beta_0 + g_x \beta_1) & (f \beta_{0y} + g \beta_{1y}) + (f_y \beta_0 + g_y \beta_1) \end{pmatrix}. \]

Another justify for our choice is that all the examples in the next section (Falkner-Skan Equation, Lorenz System and Chen System) are in the form (19). Moreover, linear vector fields can be written as (19).

It is important to observe that it may exist a function \( \tilde{f} : \mathbb{R}^2 \rightarrow \mathbb{R} \) such that \( f(x,y) = \tilde{f}(x,y)\beta_1(x,y) \). In fact, this is the case for Lorenz System and Chen System. This implies that System (19) gives rise to the constrained system
\[ \dot{x} = \alpha(x,y), \quad \tilde{f}(x,y)\dot{y} = g(x,y) + \tilde{f}(x,y)\beta_0(x,y), \]
whose adjoint vector field is
\[ \dot{x} = \tilde{f}(x,y)\alpha(x,y), \quad \dot{y} = g(x,y) + \tilde{f}(x,y)\beta_0(x,y). \]

System (23) does not have R-singularities of second kind and \( I_M \) is the union of two sets: lines that intersect the \( xy \)-plane at the points of \( \{ \tilde{f}(x,y) = 0 = g(x,y) \} \) and the set of lines that intersect the \( xy \)-plane at the points of \( \{ \beta_1(x,y) = 0 = g(x,y) \} \). System (23) describes the flow on the first set only.

In Lorenz and Chen Systems, the sets \( \{ \tilde{f}(x,y) = 0 \} \) and \( \{ \beta_1(x,y) = 0 \} \) are the same. Therefore, we will focus in this case.

**Proposition 11.** Consider system (20) and let \( r = \{(x_0,y_0,z)\} \subset I_M \) be a line. Then
The point \(q = (x_0, y_0)\) is a RK-singularity of first kind or a R-singularity of first and second kind.

(2) If \(q\) is a RK-singularity of first kind, then \(q\) is a hyperbolic equilibrium point if, and only if, \(q\) is a node with one eigenvalue.

(3) If \(q\) is a R-singularity of first and second kind, then \(q\) is non hyperbolic.

Proof. Since \(H(x, y, z) = f(x, y)z - g(x, y)\) defines an invariant surface for (19), then the equation \(z^2 f_y \beta_1 + z \left( \alpha f_x + \beta_0 f_y - \beta_1 g_y \right) - \left( \alpha g_x + \beta_0 g_y \right) + \gamma f = \mu H\) holds for \((x, y, z) \in \mathbb{R}^3\). In particular, this still true for points in the pseudo-impasse set \(\mathcal{I}_M\), that is, points such that \(f(x, y) = g(x, y) = 0\). Then we have \(z^2 f_y \beta_1 + z \left( \alpha f_x + \beta_0 f_y - \beta_1 g_y \right) - \left( \alpha g_x + \beta_0 g_y \right) = 0\). In particular, we have a polynomial in the \(z\) variable which is identically zero. Therefore its coefficients are identically zero for all \((x, y) \in \mathbb{R}^2\), and it implies that

\[
(25) \quad f_y \beta_1 = 0, \quad \alpha f_x + \beta_0 f_y - \beta_1 g_y = 0, \quad \alpha g_x + \beta_0 g_y = 0.
\]

The first equation in (25) tells us that a R-singularity of first kind \(q = (x_0, y_0)\) satisfies \(f_y(q) = 0\) or \(\beta_1(q) = 0\), and therefore \(q\) is a RK-singularity of first kind or a R-singularity of first and second kind. For the second statement, if \(q\) is RK-singularity of first kind then the linearization of (21) is \(J\tilde{X}(x, y) = \begin{pmatrix} f_x \alpha & f_y \alpha \\ f_x \beta_0 & f_y \beta_0 \end{pmatrix}\), and it implies that \(q\) is hyperbolic if, and only if, \(f_x \alpha\) and \(g_y \beta_1\) are non-zero. Moreover, the second equation in (25) assures that \(f_x \alpha = g_y \beta_1\), and then \(q\) is a hyperbolic node for the adjoint vector field (21). Finally, for the third statement, if \(q\) is a R-singularity of first and second kind then the linearization of (21) is \(J\tilde{X}(x, y) = \begin{pmatrix} f_x \alpha & f_y \alpha \\ f_x \beta_0 & f_y \beta_0 \end{pmatrix}\), and its eigenvalues are 0 and \(f_x \alpha + f_y \beta_0\). Therefore \(q\) is non hyperbolic.

\[\square\]

Remark 12. Proposition 11 concerns systems of the form (20), and in general the statements 2 and 3 are not true for systems of the form (23). In fact, according our examples, systems like (23) can have a hyperbolic node of the adjoint system with two eigenvectors.

Theorem 13. Suppose that \(H : \mathbb{R}^3 \to \mathbb{R}\) given by \(H(x, y, z) = f(x, y)z - g(x, y)\) defines an algebraic invariant surface for (19). Assume that \(\mathcal{Z}_f\) intercepts \(\mathcal{Z}_g\) at isolated points.

(1) Let \(p = (x_0, y_0, z_0) \in \mathcal{I}_M\) be a point and let \(r \subset \mathcal{I}_M\) be the line through \(p\). If \(X(p) \in r\) then \(q = (x_0, y_0)\) is a non hyperbolic equilibrium point of the adjoint vector field (21).

(2) If \(p = (x_0, y_0, z_0)\) is a singular point for \(M\) then \(q = (x_0, y_0)\) is a non hyperbolic equilibrium point for the adjoint vector field (21).

(3) If \(q = (x_0, y_0)\) is a hyperbolic RK-singularity of first kind for the adjoint system (21), then the line \(r = \{(q, z), z \in \mathbb{R}\} \subset \mathcal{I}_M\) does not contain any equilibrium point of (19).
Proof. Since $X(p) \in r$, then $X(p) = (0,0,\gamma(p))$. In particular, we have $\alpha(x_0,y_0) = 0$ and therefore the linearization $J\tilde{X}(x_0, y_0)$ of the adjoint system (21) at $(x_0, y_0)$ is \[
abla\alpha
\begin{pmatrix} f_x\beta_0 + g_x\beta_1 \\ f_y\beta_0 + g_y\beta_1 \end{pmatrix}.
\]
The eigenvalues are 0 and $f_y\beta_0 + g_y\beta_1$, thus $q$ is not hyperbolic. For the second statement, Corollary 9 assures that $q$ is R-singularity of first kind and therefore $q$ is an equilibrium point for the adjoint system (21). Since $p$ is singular, we have the equations $f_x(x_0, y_0)z_0 = g_x(x_0, y_0)$, $f_y(x_0, y_0)z_0 = g_y(x_0, y_0)$, $f(x_0, y_0) = 0$, and therefore the linearization $J\tilde{X}(q)$ of (21) computed in $q$ is \[
abla\alpha
\begin{pmatrix} f_x\alpha \\ f_y\alpha \end{pmatrix}.
\]
The eigenvalues are 0 and $f_x\alpha + f_y(\beta_0 + 1)$. It follows that $q$ is non hyperbolic. Finally, for the third statement, since $q = (x_0, y_0)$ is a hyperbolic R-singularity of first kind then $f(q) = g(q) = f_y(q) = 0$. The linearization $J\tilde{X}(x, y)$ of (21) computed at $q$ is \[
abla\alpha
\begin{pmatrix} f_x\alpha \\ f_y\alpha \end{pmatrix}.
\]
In particular, $\alpha(q) \neq 0$ and therefore the line $r = \{ (x_0, y_0, z), z \in \mathbb{R} \} \subset \mathcal{I}_M$ does not contain any equilibrium point of system (19).

Corollary 14. Let $r \subset \mathcal{I}_M$ be an invariant line of (19). Then the projection of $r$ on the xy-plane is a non hyperbolic equilibrium point for (21).

Proof. Note that for all $p \in r$, $X(p) \in r$. By the first item of the Theorem (13), the Corollary is true.

Corollary 15. If $q = (x_0, y_0)$ is a hyperbolic RK-singularity of first kind of the adjoint system (21) then the line $r = \{ (x_0, y_0, z), z \in \mathbb{R} \} \subset \mathcal{I}_M$ satisfies:

1. The vector field (19) is transversal to $r$.
2. There are no singular point of $M$ in $r$.
3. The line $r$ does not contain any equilibrium points of system (19).

The next results concern to the flow in a line $r \subset \mathcal{I}_M$ which its projection is a non hyperbolic point for the adjoint vector field (21).

Theorem 16. Let $q = (x_0, y_0) \in \mathcal{I}$ be an isolated R-singularity of (21) and $r$ be the line through $q$ such that $r \subset \mathcal{I}_M$.

1. If $\nabla f(q)$ and $\nabla g(q)$ are linearly independent and $q$ is of first and second kind, then $r$ is invariant by the flows of (19). Moreover, the linearization of (21) computed at $q$ is zero.
2. If $\nabla f(q)$ and $\nabla g(q)$ are linearly independent and $q$ is a RK-singularity, then $r$ is invariant by the flows of (19) if, and only if, $\alpha(q) = 0$.
3. If $\nabla f(q)$ and $\nabla g(q)$ are linearly dependent, then there is a singular point of $M$ in $r$.

Proof. Proposition 11 implies that $q$ is a non hyperbolic point of the adjoint system (21). Since $q$ is a R-singularity of first and second kind, then equation (25) becomes $\alpha f_x + \beta_0 f_y = 0$, $\alpha g_x + \beta_0 g_y = 0$. We can rewrite the previous
Therefore we have \( r = \frac{1}{\alpha} \), \( \beta = 0 \), \( \gamma = 0 \) at \( q = (0, 0) \). Since \( \nabla f(q) \) and \( \nabla g(q) \) are linearly independent, it follows that \((\alpha(q), \beta_0(q)) = (0, 0)\). Therefore we have \( \alpha(q) = \beta_0(q) = \beta_1(q) = 0 \) and \( r \) is an invariant set for (19). Moreover, the linearization of (21) computed at \( q \) is zero. For the second statement, since \( q \) is a RK-singularity then \( g_y(q) \neq 0 \) and equation (25) becomes \( \beta_0 = -\frac{\alpha y}{g_y}, \beta_1 = \frac{f_y}{g_y} \), and then \( \alpha(q) = 0 \) is a necessary and sufficient condition to assure that \( M \) is invariant. Finally, if \( \nabla f(q) \) and \( \nabla g(q) \) are linearly dependent then there is \( c \neq 0 \) such that \( \nabla g = c \nabla f \). Therefore we have \( \nabla H(q, z) = \left((z - c)f_x, (z - c)f_y, 0\right) \), and then \((x_0, y_0, c)\) is a singular point of \( M \).

**Proposition 17.** Let \( p = (x_0, y_0, z_0) \in \mathcal{I}_M \) be a singular point of \( M \) and let \( r \) be the line through \( p \) such that \( r \subset \mathcal{I}_M \). Then \( X(p) \in r \) if, and only if, the linearization of the adjoint system (21) at \( q = (x_0, y_0) \) is identically zero.

**Proof.** Since \( p \) is singular, we have the equations \( f_x(x_0, y_0)z_0 = g_x(x_0, y_0), f_y(x_0, y_0)z_0 = g_y(x_0, y_0), f(x_0, y_0) = 0 \), and therefore the linearization of (21) at \( q \) is \( \nabla \widetilde{X}(q) = \begin{pmatrix} f_x(\beta_0 + \alpha z_0) & f_y(\beta_0 + \alpha z_0) \\ f_x(\beta_0 + z_0\beta_1) & f_y(\beta_0 + z_0\beta_1) \end{pmatrix} \).

Remember that we are supposing that \( \mathcal{I} \) is regular, thus \( f_x(q) \neq 0 \) or \( f_y(q) \neq 0 \). Then \( \nabla \widetilde{X}(q) \) is identically null if, and only if, \( \alpha(q) = 0 \) and \( \beta_0(q) + z_0\beta_1(q) = 0 \).

**Corollary 18.** Let \( r = \{(x_0, y_0, z)|z \in \mathbb{R}\} \subset \mathcal{I}_M \) be a line parallel to the \( z \)-axis. If \( r \) is a singular subset of \( M \), then \( r \) is invariant by (19) if, and only if, the linearization of (21) at \( q = (x_0, y_0) \) is identically null.

**Proof.** Observe that for all \( p \in r \) we have \( X(p) \in r \).

4. **Examples**

In this section we apply our results to four well-known equations: Falkner-Skan equation (derived from fluid dynamics); Lorenz equation (meteorological studies); Chen equation (shows chaotic behavior) and Fisher-Kolmogorov equation (related to population dynamics).

4.1. **Falkner-Skan equation.**

The Falkner-Skan equation was studied in [3] and it is given by \( f'''' + f f' + \lambda(1 - (f')^2) = 0 \), where \( \lambda \) is a parameter. This equation describes a model in fluid dynamics and it describes a model of the steady two-dimensional flow of a slightly viscous incompressible fluid past a wedge. We can express this equation as a system of differential equations

\[
\begin{align*}
\dot{x} &= y, \\
\dot{y} &= z, \\
\dot{z} &= -xz - \lambda(1 - y^2).
\end{align*}
\]

In [14] the authors proved that \( H(x, y, z) = 2xz + (1 - y^2) \) is the only Darboux polynomial of (26) when \( \lambda = \frac{1}{2} \). Observe that \( \langle H_x, H_y, H_z \rangle, X \rangle = -xH(x, y, z) \). Denote \( M = \{H(x, y, z) = 0\} \). The pseudo impasse set \( \mathcal{I}_M \subset \)
$M$ is given by the lines $r_{1,2} = \{(0, \pm 1, z), z \in \mathbb{R}\}$. Moreover, $M$ is a regular surface.

Let $f(x, y) = 2x$ and $g(x, y) = y^2 - 1$. The dynamics of (26) on $M$ are described by

$$\dot{x} = y, \ 2xy = y^2 - 1,$$

whose impasse curve is given by $I = \{x = 0\}$. Its adjoint vector field is

$$\dot{x} = 2xy, \ \dot{y} = y^2 - 1,$$

and the equilibrium points $p_{1,2} = (0, \pm 1)$ are RK-singularities of first kind on $I$. Note that the points $p_{1,2}$ are the projections of the lines $r_{1,2}$. Observe that there are no equilibrium points of (28) outside $I$, therefore there are no equilibrium points of (26) on $G_M$ when $\lambda = \frac{1}{2}$.

The linearization of (28) is given by $J\tilde{X}(x, y) = \begin{pmatrix} 2y & 2x \\ 0 & 2y \end{pmatrix}$, and thus $p_1$ is an unstable hyperbolic node and $p_2$ is a stable hyperbolic node for the adjoint vector field (28). It follows from Theorem 15 that the flows of (26) with $\lambda = \frac{1}{2}$ is transversal to $\mathcal{I}_M$ and $\mathcal{I}_M$ does not contain any equilibrium point of (26). See figure 11.

Moreover, note that when we project the flows of (26) with $\lambda = \frac{1}{2}$ on the $xz$-plane we obtain the systems $\dot{x} = \pm \sqrt{xz + 1}, \ \dot{z} = 0$, where $\dot{z} = 0$ means that the trajectories of (26) on $M$ are contained in planes parallel to the $xy$-plane. Another way to verify this fact is observing that at the points of $M$ the system (26) takes form $\dot{x} = y, \ \dot{y} = z, \ \dot{z} = 0$.

\[\text{Figure 11. Phase portrait of (27) (left) and surface } M \text{ (right).}\]

The pseudo impasse set $\mathcal{I}_M$ is denoted in blue.

4.2. Lorenz System.

The Lorenz System is given by

$$\dot{x} = s(-x + y), \ \dot{y} = rx - y - xz, \ \dot{z} = xy - bz,$$

where $(x, y, z) \in \mathbb{R}^3$ are variables and $(s, r, b) \in \mathbb{R}^3$ are parameters. This model was proposed by Lorenz in 1963 (see [16]) in order to study meteorological phenomena. When $(s, r, b) = (10, 28, \frac{8}{3})$, this system presents chaotic
behavior. In [15] the authors gave all the six invariant algebraic surfaces for the Lorenz System (29). In [3] the flows on such invariant surfaces were considered and in [11] the authors studied the global dynamics of (29).

Three out of six invariant algebraic surfaces can be written in the form

\[ M = \{ H_i(x,y,z) = 0 \} \]

with

\[ H_i(x,y,z) = f_i(x,y)z - g_i(x,y), \quad i = 1, 2, 3. \]

They are:

- **Case (a)**: \((r, s, b) = (r, 0, \frac{1}{3})\),
  
  \[ f_1(x,y) = -\frac{4}{3}x^2 \quad g_1(x,y) = 4x^2 - 16(1 - r) \]

- **Case (b)**: \((r, s, b) = (r, 1, 4)\),
  
  \[ f_2(x,y) = -\frac{4}{3}x^2 \quad g_2(x,y) = 4x^2 - 8xy + 4y^2 - x^4 \]

- **Case (c)**: \((2s - 1, s, 6s - 2)\),
  
  \[ f_3(x,y) = -\frac{4}{3}x^2 \quad g_3(x,y) = 4s^2y^2 - 4s(4s - 2)xy + (4s - 2)x^2 - x^4 \]

In what follows, we analyze the flows of (29) on such surfaces.

**Case (a).** Since \((r, s, b) = (r, 0, \frac{1}{3})\), system (29) takes form

\[ \dot{x} = \frac{1}{3}(-x + y), \quad \dot{y} = rx - y - xz, \quad \dot{z} = xy, \]

and its flows on \(M_1 = \{ H_1(x,y,z) = 0 \} \) is described by the constrained system

\[ \dot{x} = \frac{1}{3}(-x + y), \quad \frac{4}{3}x\dot{y} = \frac{4}{3}x(rx - y) + \frac{4}{9}y^2 + \frac{8}{9}xy - \frac{4}{3}x^2 - x^4, \]

whose adjoint vector field is given by

\[ \dot{x} = \frac{4}{9}x(-x + y), \quad \dot{y} = \frac{4}{3}x(rx - y) + \frac{4}{9}y^2 + \frac{8}{9}xy - \frac{4}{3}x^2 - x^4. \]

The origin \((0,0) \in I_1 = \{ x = 0 \} \) is the only equilibrium point for (32). Since \(I_{M_1} = \{ (0,0,z) | z \in \mathbb{R} \} \) is a singular subset of \(M_1\) and the linearization of (32) computed at \((0,0)\) is zero, it follows from Proposition 17 that \(I_{M_1}\) is an invariant set of (30). This fact also can be verified observing that all points on \(I_{M_1}\) are equilibrium points. Notice that outside \(I_1\) there are no equilibrium points for (32), therefore there are no equilibrium points of (30) on \(G_{M_1}\). See figure 12.

**Case (b).** Suppose \((r, s, b) = (r, 1, 4)\). For \(r < 1\) the function \(f_2\) is always nonzero, for \(r = 1\) it has only one root and for \(r > 1\) it has two roots. Since
we are interested in the cases where \( f_2 \) has real roots (because it is in this case that constrained systems rise), we will study the cases where \( r = 1 \) and \( r > 1 \).

For \( r = 1 \), system (29) is
\[
\dot{x} = -x + y, \quad \dot{y} = x - y - xz, \quad \dot{z} = -4z + xy,
\]
and its flows on \( M_2 = \{-4x^2z - (4x^2 - 8xy + 4y^2 - x^4) = 0\} \) are described by
\[
\dot{x} = -x + y, \quad 4x\dot{y} = 4x(x - y) + (4x^2 - 8xy + 4y^2 - x^4),
\]
whose adjoint system is
\[
\dot{x} = 4x(-x + y), \quad \dot{y} = 4x(x - y) + (4x^2 - 8xy + 4y^2 - x^4).
\]
The origin \((0,0) \in \mathcal{I}_2 = \{x = 0\}\) is the only equilibrium point of (35). Since \( \mathcal{I}_{M_2} = \{(0,0,z) | z \in \mathbb{R}\} \) is a singular set of \( M_2 \) and the linearization of (35) computed at \((0,0)\) is zero, it follows from Proposition 17 that \( \mathcal{I}_{M_2} \) is invariant for (33). This fact also can be checked observing that at the points on \( \mathcal{I}_{M_2} \) the vector field (33) is \((0,0,-4z)\). See figure 13.

For \( r > 1 \), system (29) is
\[
\dot{x} = -x + y, \quad \dot{y} = rx - y - xz, \quad \dot{z} = -4z + xy.
\]
Denote \( A(x,r) = -4x^2 - 16(1 - r), \quad M_2 = \{A(x,r)z - (4r^2x^2 - 8xy + 4y^2 - x^4) = 0\} \). The flow of (36) on \( M_2 \) is described by the constrained system
\[
\dot{x} = -x + y, \quad A(x,r)\dot{y} = A(x,r)(rx - y) - x(4r^2x^2 - 8xy + 4y^2 - x^4),
\]
whose adjoint system is
\[
\dot{x} = A(x,r)(-x + y), \quad \dot{y} = A(x,r)(rx - y) - x(4r^2x^2 - 8xy + 4y^2 - x^4).
\]
In this case, \( \mathcal{I}_2^\pm = \{\pm2\sqrt{r-1}, y \in \mathbb{R}\} \) are impasse curves for system (37). The pseudo impasse set in \( M_2 \) are the lines
\[
\mathcal{I}_{M_2}^\pm = \{\pm2\sqrt{r-1}, \pm2\sqrt{r-1}, z \in \mathbb{R}\}.
\]
Observe that \((\pm2\sqrt{r-1}, \pm2\sqrt{r-1}, r - 1) \in \mathcal{I}_{M_2}^\pm\) are singular points of \( M_2 \) and equilibrium points for (36). However, all the other points on \( \mathcal{I}_{M_2}^\pm \) are regular points. System (38) has three equilibrium points, and two of them are on the impasse curve. More precisely, \((\pm2\sqrt{r-1}, \pm2\sqrt{r-1}) \in \mathcal{I}_2^\pm\) computed in such points is identically zero. The third equilibrium point of (38) is the origin and it is a hyperbolic saddle. Therefore, there is a saddle point in \( \mathcal{G}_{M_2} \). See figure 14.

Case (c). Since \((r,s,b) = (2s - 1, s, 6s - 2)\), system (29) takes form
\[
\dot{x} = s(-x + y), \quad \dot{y} = (2s - 1)x - y - xz, \quad \dot{z} = -(6s - 2)z + xy,
\]
Denote \( g_3(x,y) = 4s^2y^2 + (4s - 2)x^2 - 4s(4s - 2)xy - x^4 \). The flow of (39) on \( M_3 = \{H_3(x,y,z) = 0\} \) is given by
\[
\dot{x} = s(-x + y), \quad 4s\dot{y} = 4s((2s - 1)x - y) + g_3(x, y),
\]
whose adjoint system is

\begin{equation}
\dot{x} = 4s^2x(-x + y), \quad \dot{y} = 4sx((2s - 1)x - y) + g_3(x, y).
\end{equation}

The impasse curve is given by \( I_3 = \{ x = 0 \} \) and the origin \((0, 0)\) is the only equilibrium point for (41). Since \( I_{M_3} = \{ (0, 0, z) | z \in \mathbb{R} \} \) is a singular set of \( M_3 \) and the linearization of (41) computed at \((0, 0)\) is zero, it follows from Proposition 17 that \( I_{M_3} \) is invariant by the flows of (39). This fact also can be checked observing that on \( I_{M_3} \) the system (39) is \( X = (0, 0, -(6s - 2)z) \).

Observe that outside \( I_3 \) system (41) has two equilibrium points given by \((\pm 2\sqrt{1 + 3s^2 - 4s}, \pm 2\sqrt{1 + 3s^2 - 4s})\). Such points are different when \( s > 1 \) or \( s < \frac{1}{3} \) and they collide at the origin when \( s = 1 \) or \( s = \frac{1}{3} \). Therefore there are two equilibrium points of (39) on \( G_{M_3} \) when \( s > 1 \) or \( s < \frac{1}{3} \). See figure 15.

4.3. Chen System.

Chen System is given by

\begin{equation}
\dot{x} = a(-x + y), \quad \dot{y} = (c - a)x + cy - xz, \quad \dot{z} = xy - bz,
\end{equation}
where \((x, y, z) \in \mathbb{R}^3\) are variables and \((a, b, c) \in \mathbb{R}^3\) are parameters. This model was proposed for the first time in [5] and it shows a chaotic behavior for a convenient choice of \(a, b\) and \(c\). In [17] was given six invariant algebraic surfaces for Chen System (42). In [2] the authors studied the dynamics of (42) on such surfaces and in [13] the global dynamics of (42) was considered.

Two out of six invariant algebraic surfaces studied in such references can be written in the form
\[
M_i = \{ H_i^{-1}(0) \} \quad \text{com} \quad H_i(x, y, z) = f_i(x, y) z - g_i(x, y),
\]
i = 4, 5. They are:

| Case | \((a, b, c)\) | \(f_i(x, y)\) | \(g_i(x, y)\) |
|------|----------------|----------------|----------------|
| (d)  | \((-\frac{c}{3}, 0, c)\) | \(-\frac{2}{3}cx^2\) | \(4c^2x - 8c^2xy - 8c^2x^2 - 8c^2x^2 - 8c^2x^2 - 8c^2x^2 - x^2\) |
| (e)  | \((-c, -4c, c)\) | \(4c^2x - 8c^2xy - 8c^2x^2 - 8c^2x^2 - 8c^2x^2 - 8c^2x^2 - x^2\) |

Observe that for all \(c \in \mathbb{R}\) the function \(f_5(x, y) = 4cx^2 + 48c^3\) is always non-zero. This implies that the pseudo impasse set \(\mathcal{I}_M\) is empty and the flows on \(M_5\) are described by a smooth system.

When \((a, b, c) = (-\frac{c}{3}, 0, c)\), system (42) is
\[
\dot{x} = -\frac{c}{3}(-x + y), \quad \dot{y} = c\left(\frac{4}{3}x + y\right) - xz, \quad \dot{z} = xy,
\]
and its flows on \(M_4 = \{ H_4(x, y, z) = 0 \}\) are described by
\[
\dot{x} = -\frac{c}{3}(-x + y), \quad \frac{4}{3}cx\dot{y} = \frac{4}{3}c^2x\left(\frac{4}{3}x + y\right) - g_5(x, y),
\]
whose adjoint system is
\[
\dot{x} = \frac{4}{9}c^2x(x - y), \quad \dot{y} = \frac{4}{3}c^2x\left(\frac{4}{3}x + y\right) - g_5(x, y).
\]

The impasse curve is given by \(\mathcal{I}_4 = \{ x = 0 \}\) and the origin \((0, 0)\) is the only equilibrium point for (45). Since \(\mathcal{I}_{M_4} = \{(0, 0, \alpha) | \alpha \in \mathbb{R}\}\) is a singular set of \(M_4\) and the linearization of (45) computed at \((0, 0)\) is zero, it follows from Proposition 17 that \(\mathcal{I}_{M_4}\) is an invariant set for the flows of (43).
that outside $\mathcal{I}_4$ there are no equilibrium points for (45), therefore there are no equilibrium points for (43) on $G_{M_4}$. See figure 16.

Figure 16. Phase portrait of system (44) (left) and surface $M_4$ (right). The pseudo impasse set $\mathcal{I}_M$ is denoted in blue.

4.4. **Constrained systems in higher dimensions and flows on invariant hypersurfaces.** In this section we discuss how the previous problems can be extended to higher dimensions. In order to do this, we illustrate our discussion by means of an example.

The Fisher-Kolmogorov Equation was introduced in [10] and it is a model for populational dynamics. Such equation is given by $u_t = u_{xx} + u - u^3$. In [7] the authors proposed the Extended Fisher-Kolmogorov Equation (EFK-equation for short) given by $u_t = -\gamma u_{xxxx} + u_{xx} + u - u^3$, $\gamma > 0$.

Observe that for $\gamma = 0$, EFK-equation becomes the regular Fisher-Kolmogorov equation. For stationary solutions (solutions in which do not depend on time $t$), the EFK-equation reduces to

$$-\gamma u_{xxxx} + u_{xx} + u - u^3 = 0, \gamma > 0.$$

Applying some transformations and changes of variables, we can express the stationary solutions of the EFK-equation by means of the polynomial system

$$\dot{x} = y, \dot{y} = z, \dot{z} = w, \dot{w} = x - qz - x^3,$$ where $(x, y, z, w) \in \mathbb{R}^4$ and $q \in \mathbb{R}$ is negative.

In [12] the authors proved that $H(x, y, z, w) = \frac{qy^2-z^2-z^4}{2} + \frac{x^4}{4} + wy$ is a first integral for (46) and therefore $H^{-1}(k)$ defines an invariant algebraic hypersurface. For $k \neq 0$ and $k \neq -\frac{1}{4}$, $H^{-1}(k)$ is a smooth 3-dimensional hypersurface. Note that $H$ is written in the form $H(x, y, z, w) = f(x, y, z)w - g(x, y, z)$, and therefore we can define the sets $G_M$ and $\mathcal{I}_M$, where $M = H^{-1}(k)$. Moreover, Proposition 4 is still true for $n$-dimensional hypersurfaces.
The authors also proved in [12] that the flows on $H^{-1}(k)$ can be described by the constrained system
\begin{equation}
\dot{x} = y, \quad \dot{y} = z, \quad 4y\dot{z} = 4k + 2(x^2 + z^2 - qy^2) - x^4,
\end{equation}
whose adjoint vector field is
\begin{equation}
\dot{x} = 4y^2, \quad \dot{y} = 4yz, \quad \dot{z} = 4k + 2(x^2 + z^2 - qy^2) - x^4.
\end{equation}

The ideas used to prove this fact are the same used in the proof of Theorem 7. Note that the impasse surface of (47) is the $xz$-plane on $\mathbb{R}^3$, and the projection of the pseudo impasse set $I_M = \{(x, y, z, w) | y = 0 = \frac{y^2 - x^2 - z^2}{2} + \frac{x^4}{4} + k, w \in \mathbb{R}\}$ is a one-dimensional curve of equilibrium points for the adjoint vector field (48).

Although Proposition 4 and Theorem 7 can be easily extended in higher dimensions, it is harder to generalize Theorem 15 and Proposition 11. In the 2-dimensional case, the projection of the pseudo impasse set is a set of equilibrium points contained in the impasse curve. In higher dimensions, if $M$ is a $n$-dimensional hypersurface embedded in $\mathbb{R}^{n+1}$, then $I_M$ and $I$ will be a $(n-1)$-dimensional submanifold of $M$ and $\mathbb{R}^n$, respectively. Moreover, the projection of $I_M$ is a $(n-2)$-dimensional submanifold of $I$ and all their points are equilibrium points for the $n$-dimensional constrained system. This case require a detailed analysis.

5. Unfolding minimal sets in 1-parameter families of invariant algebraic surfaces

In this section we consider 1-parameter families of smooth vector fields
\begin{equation}
X_\varepsilon(x, y) = \left(\alpha_\varepsilon(x, y), \beta_\varepsilon(x, y)\right), \quad \varepsilon \downarrow 0,
\end{equation}
where $x \in \mathbb{R}$ and $y \in \mathbb{R}^m$. Assume that $H_\varepsilon(x, y)$ is a smooth first integral and denote
\begin{equation}
M_\varepsilon = H_\varepsilon^{-1}(0),
\end{equation}
the corresponding invariant surface. We also assume that the vector field and the first integral vary smoothly with respect to the parameter $\varepsilon$.

The trajectories of (49) are the solutions of
\begin{equation}
\dot{x} = \alpha_\varepsilon(x, y), \quad \dot{y} = \beta_\varepsilon(x, y).
\end{equation}

Using singular perturbation theory and Fenichel Theorem 2 as main tools, we study the persistence of equilibrium points and periodic orbits using normal hyperbolicity.

Since $H_\varepsilon$ is a first integral of (51), for each $\varepsilon$ sufficiently small we have $\alpha_\varepsilon(H_\varepsilon)_x + \beta_\varepsilon(H_\varepsilon)_y = 0$. Thus we can rewrite system (51) as
\begin{equation}
\dot{x} = -\left(\frac{\beta_\varepsilon(H_\varepsilon)_y}{(H_\varepsilon)_x}\right), \quad \dot{y} = \beta_\varepsilon(x, y).
\end{equation}
The smooth dependence on $\varepsilon$ implies that $\alpha_\varepsilon \to \alpha_0$, $\beta_\varepsilon \to \beta_0$ and $H_\varepsilon \to H_0$ in the $C^1$-topology.

If $\varphi_0(t) = (x_0(t), y_0(t))$ is a solution of (51) satisfying that $H_0(\varphi_0(0)) = 0$, then $\varphi_0(t)$ is a solution for the differential-algebraic equation

$$0 = H_0(x, y), \quad \dot{y} = \beta_0(x, y).$$

Conversely, consider the differential-algebraic equation (53) and let $p_0 \in \mathcal{N}\mathcal{H}(M_0)$. There is a neighborhood $V \subset \mathcal{N}\mathcal{H}(M_0)$ of $p_0$ such that if $\varphi_0(t) = (x_0(t), y_0(t))$, $\varphi_0(0) = p_0$ is a solution of (53) in $V$, then $\varphi_0(t)$ is a solution of

$$\dot{x} = -\left(\frac{\beta_0(H_0)y}{(H_0)_x}\right), \quad \dot{y} = \beta_0(x, y).$$

The neighborhood $V$ is the neighborhood in which $(H_0)_x \neq 0$. Differentiating $H_0(x, y) = 0$ with respect to $t$, we get $0 = \dot{x}(H_0)_x + \dot{y}(H_0)_y$, $\dot{y} = \beta_0(x, y)$, as desired. We summarize these previous facts in the following Proposition.

**Proposition 19.** Let $H_0 : \mathbb{R}^{m+1} \to \mathbb{R}$ be a smooth first integral of system (51) with $\varepsilon = 0$ and $p_0 \in \mathcal{N}\mathcal{H}(M_0)$ be a normally hyperbolic point. Then there exists a neighborhood $V \subset \mathcal{N}\mathcal{H}(M_0)$ of $p_0$ such that if $\varphi_0(t) = (x_0(t), y_0(t)), \varphi_0(0) = p_0$, is a solution of the slow system (53) in $V$ if, and only if, $\varphi_0(t)$ is a solution of (54).

In other words, Proposition 19 says that $\varphi_0$ is an orbit on the normally hyperbolic part of the slow manifold if, and only if, $\varphi_0$ is an orbit of (54) on the level $H_0 = 0$.

Now consider the singularly perturbed system

$$\varepsilon \dot{x} = H_\varepsilon(x, y), \quad \dot{y} = \beta_\varepsilon(x, y).$$

We remark that $M_0$ is the slow manifold of (55). Take $p_0 = (x_0, y_0) \in \mathcal{N}\mathcal{H}(M_0)$. Since $H_\varepsilon \to H_0$ in the $C^1$-topology we have

$$H_\varepsilon(p_0) \to H_0(p_0), \quad \frac{\partial^i H_\varepsilon}{\partial x^i}(p_0) \to \frac{\partial^i H_0}{\partial x^i}(p_0), \quad \frac{\partial^i H_\varepsilon}{\partial y^i}(p_0) \to \frac{\partial^i H_0}{\partial y^i}(p_0).$$

**Remark.** Let $N_\varepsilon$ be the family of locally invariant surface of (55) given by Fenichel’s Theorem 2 converging to a compact subset $N_0 \subset \mathcal{N}\mathcal{H}(M_0)$. Then there is a compact subset $\overline{M}_\varepsilon \subset \mathcal{N}\mathcal{H}(M_\varepsilon)$ diffeomorphic to $N_\varepsilon$, for $\varepsilon$ sufficiently small.

From now on we denote $\mathcal{N}\mathcal{H}(M_\varepsilon) = \{(x, y) \in M_\varepsilon : (H_\varepsilon)_x \neq 0\}$. The next statements aim to relate the dynamics of (52) on $\mathcal{N}\mathcal{H}(M_\varepsilon)$ with the dynamics of the singular perturbation problem (55). We will always suppose that $\mathcal{N}\mathcal{H}(M_0)$ and $V$ is the neighborhood given by Proposition 19. The idea is to use the normal hyperbolicity and Fenichel Theorem 2 to study the persistence of equilibrium points and periodic orbits.
Proposition 20. Let \( p_0 \in N_0 \) be an equilibrium point of (52) (for \( \varepsilon = 0 \)). Then

(a): For \( \varepsilon > 0 \) sufficiently small, there is a sequence of points \( p_\varepsilon \in Nh(M_\varepsilon) \) which converges to \( p_0 \) and, for each \( \varepsilon, p_\varepsilon \) is an equilibrium point of (52).

(b): \( p_\varepsilon \in Nh(M_\varepsilon) \) is an equilibrium point of (52) (for \( \varepsilon > 0 \)) if, and only if, \( p_\varepsilon \) is an equilibrium point for (55).

Proof. Since \( p_0 \) is an equilibrium point, it follows from Fenichel’s Theorem that for \( \varepsilon > 0 \) sufficiently small, there is a sequence of points \( p_\varepsilon \in N_\varepsilon \) which converges to \( p_0 \) such that \( p_\varepsilon \) is an equilibrium point of the singularly perturbed problem (55). The manifold \( N_\varepsilon \) is locally invariant and normally hyperbolic for (55).

In particular, for each \( \varepsilon \) we have \( \beta_\varepsilon(p_\varepsilon) = \gamma_\varepsilon(p_\varepsilon) = 0 \) and thus \( p_\varepsilon \) is an equilibrium point for (52). Since \( H_\varepsilon(p_\varepsilon) = 0 \) and \( p_\varepsilon \in N_\varepsilon \) we have \( (H_\varepsilon)_x(p_\varepsilon) \neq 0 \) for \( \varepsilon \) sufficiently small. Thus \( p_\varepsilon \in Nh(M_\varepsilon) \) and statement (a) is true. Statement (b) follows directly.

For the next result, we will suppose that (55) is a singularly perturbed system of the form

\[
\varepsilon \dot{x} = H_\varepsilon(x,y), \quad \dot{y} = \beta_\varepsilon(y),
\]

that is, we require that \( \beta_\varepsilon \) does not depend on \( x \). This assumption is not a simple convenience. In fact, since (55) is well defined in the normally hyperbolic part of the slow manifold, by the Implicit Function Theorem there is a function \( \Phi_\varepsilon \) such that \( \Phi_\varepsilon(y) = x \).

We also suppose that (52) is a system of the form

\[
\dot{x} = -\left( \frac{\beta_\varepsilon(H_\varepsilon)_y}{(H_\varepsilon)_x} \right), \quad \dot{y} = \beta_\varepsilon(y).
\]

Proposition 21. Let \( \varphi_0(t) \subset V \) be a periodic orbit of (57) (for \( \varepsilon = 0 \)). Then

(a): For \( \varepsilon > 0 \) sufficiently small, there is a sequence of orbits \( \{\varphi_\varepsilon(t)\} \subset Nh(M_\varepsilon) \) which converges to \( \{\varphi_0(t)\} \), according Hausdorff distance, such that for each \( \varepsilon \), \( \{\varphi_\varepsilon(t)\} \) is a periodic orbit of (57).

(b): If \( \{\psi_\varepsilon(t)\} \subset Nh(M_\varepsilon) \) is a periodic orbit of (57) for \( \varepsilon > 0 \), then \( \{\psi_\varepsilon(t)\} \) is a periodic orbit of (56).

Proof. Let \( \varphi_0(t) = (x_0(t),y_0(t)) \) be a periodic orbit of (57) (for \( \varepsilon = 0 \)) in \( V \). Thus \( \varphi_0(t) \) is a periodic orbit for (56), for \( \varepsilon = 0 \), by Proposition 19. Note that \( \{\varphi_0(t)\} \) is compact, then it follows from Fenichel’s Theorem that for \( \varepsilon \) sufficiently small, there is a sequence of periodic orbits \( \varphi_\varepsilon(t) = (x_\varepsilon(t),y_\varepsilon(t)) \) for (56) which converges to \( \varphi_0 \) and such that \( \{\varphi_\varepsilon\} \subset N_\varepsilon \).

On \( Nh(M_\varepsilon) \), we have \( (H_\varepsilon)_x \neq 0 \). By the Implicit Function Theorem, there is a function \( \Phi_\varepsilon \) such that we can write \( Nh(M_\varepsilon) \) as the graphic of \( \Phi_\varepsilon \).
Define \( \overline{\varphi}_\varepsilon(t) = \left( \Phi_\varepsilon \circ y_\varepsilon(t), y_\varepsilon(t) \right) \). Since \( \varphi_\varepsilon \) is periodic, \( y_\varepsilon(t) \) is periodic and then \( \overline{\varphi}_\varepsilon(t) \) is periodic. Moreover, \( \overline{\varphi}_\varepsilon \subset \mathcal{N}\mathcal{H}(M_\varepsilon) \) because \( \overline{\varphi}_\varepsilon \) is contained in the graphic of \( \Phi_\varepsilon \). We also have that \( \overline{\varphi}_\varepsilon(t) \) is solution of (57) because \( \dot{y} = \beta_\varepsilon(\overline{\varphi}_\varepsilon(t)) \) and \( H_\varepsilon \circ \overline{\varphi}_\varepsilon(t) = 0 \) implies \( \dot{x} = -\frac{\beta_\varepsilon(H_\varepsilon) + \gamma_\varepsilon(H_\varepsilon)}{(H_\varepsilon)_x} \).

Finally, we have that \( \{ \overline{\varphi}_\varepsilon \} \) converges to \( \{ \varphi_0 \} \) because \( \Phi_\varepsilon \) converges to \( \Phi_0 \) and we can write \( \varphi_0(t) = \left( \Phi_0(y_0(t)), y_0(t) \right) \). For item (b), observe that \( \dot{y} = \beta_\varepsilon(\psi_\varepsilon(t)) \) and \( H_\varepsilon \circ \psi_\varepsilon(t) = 0 \), and then \( \{ \psi_\varepsilon(t) \} \) is periodic of (56).

This completes the proof. \( \square \)

**Example.** Consider \( H(x, y, z, \varepsilon) = x - \left( \frac{x^2 + z^2}{2} \right) - \varepsilon h(x, y) \), \( \beta(y, z, \varepsilon) = -z + y(1 - y^2 - z^2) + \varepsilon, \gamma(y, z, \varepsilon) = y + z(1 - y^2 - z^2) + \varepsilon \), where \( h : \mathbb{R}^2 \to \mathbb{R} \) is smooth. Then \( H_\varepsilon \) is a first integral of

\begin{equation}
\dot{x} = \beta_\varepsilon(y + \varepsilon h_y) + \gamma_\varepsilon(z + \varepsilon h_z), \quad \dot{y} = \beta, \quad \dot{z} = \gamma.
\end{equation}

For \( \varepsilon = 0 \), by Proposition 19 the flow on \( M_0 \) is given by the algebraic differential equation \( 0 = x - \left( \frac{x^2 + z^2}{2} \right), \dot{y} = -z + y(1 - y^2 - z^2), \dot{z} = y + z(1 - y^2 - z^2) \).

Note that there is a stable limit cycle on \( M_0 \) and the origin is an equilibrium point. It follows from Propositions 20 and 21 that for \( \varepsilon > 0 \) sufficiently small such compact orbits persist for system (58). This fact also can be checked noticing that the system \( \dot{y} = -z + y(1 - y^2 - z^2), \dot{z} = y + z(1 - y^2 - z^2) \) is structurally stable.

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São Paulo State University (UNESP), Institute of Biosciences, Humanities and Exact Sciences, Rua C. Colombo, 2265, CEP 15054–000. S. J. Rio Preto, São Paulo, Brazil.

E-mail address: paulo.r.silva@unesp.br
E-mail address: otavioperez@hotmail.com