Gauge theory
of second class constraints
without extra variables.

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Abstract

We show that any theory with second class constraints may be cast into a gauge theory if one makes use of solutions of the constraints expressed in terms of the coordinates of the original phase space. We perform a Lagrangian path integral quantization of the resulting gauge theory and show that the natural measure follows from a superfield formulation.
1 Introduction.

A covariant quantization of theories with second class constraints is in general a difficult task. A general method is the so called conversion method \[1\] in which additional variables are added in such a way that the second class constraints are converted into first class ones. This allows then for the use of conventional covariant quantization methods for general gauge theories. In this paper we show that another way to introduce additional variables, which also enables one to cast the original theory into a gauge theory, is to make use of coordinates on the constraint surface parametrized in terms of a redundant number of variables. The natural way to do this, which exactly yields the needed number of variables, is to choose the redundant variables to be the coordinates of the enveloping, original phase space. No new additional variables are then added. This is the procedure we shall follow in this paper. If \( x^i \) denotes the coordinates on the original phase space and \( \theta^\alpha(x) = 0 \) the constraints, then we shall make use of functions \( \bar{x}^i(x) \) satisfying the conditions

\[
\theta^\alpha(\bar{x}^i(x)) = 0, \tag{1.1}
\]

\[
\bar{x}^i(\tilde{x}) = \tilde{x}^i \tag{1.2}
\]

for whatever choice of solution \( \tilde{x}^i \) of \( \theta^\alpha(\tilde{x}) = 0 \). (The last property is a normalization of \( \bar{x}^i(x) \).) We shall show that \( \bar{x}^i(x) \) is gauge invariant and that the original theory may be cast into a gauge theory simply by replacing \( x^i \) by \( \bar{x}^i(x) \) in the (first order) action. Conventional gauge theoretical quantization methods are then applicable.

The two conditions above were previously considered in [2]. However, there \( \bar{x}^i(x) \) were also required to satisfy a closed Poisson algebra, which when combined with (1.1), yields the condition

\[
\{\bar{x}^i(x), \bar{x}^j(x)\} = \{x^i, x^j\}_D|_{x\rightarrow \bar{x}(x)}, \tag{1.3}
\]

where the bracket on the left-hand side is the original Poisson bracket on the original phase space and where the bracket on the right-hand side is the Dirac bracket. Solutions to these three conditions were then investigated by means of the general ansatz

\[
\bar{x}^i(x) = x^i + \sum_{k=1}^\infty X^i_{\alpha_1\cdots\alpha_k}(x)\theta^{\alpha_1}(x)\cdots\theta^{\alpha_k}(x). \tag{1.4}
\]

This ansatz automatically satisfies (1.2) and it was shown that the coefficient functions \( X^i_{\alpha_1\cdots\alpha_k}(x) \) are possible to choose in many different ways to solve the condition (1.1). However, although the condition (1.3) is more restrictive a general form for the solutions of all three conditions were derived and a mapping procedure for a covariant quantization by means of these solutions was proposed.

In this paper the philosophy is different. Here we derive some general properties that only follow from the existence of functions \( \bar{x}^i(x) \) satisfying the first two conditions, (1.1) and (1.2). We show that the fact that these solutions of the constraints
are expressed in terms of a redundant number of variables always provide for the possibility of a gauge theoretical treatment of theories with second class constraints.

In section 2 we give the precise setting for our considerations. In section 3 we present an auxiliary gauge theory and define some properties needed for our constructions. In section 4 we demonstrate the general existence of $\bar{x}^i(x)$ by means of an explicit integral equation. In section 5 we prove that $\bar{x}^i(x)$ is gauge invariant and in section 6 we show how the gauge invariant action is constructed by means of $\bar{x}^i(x)$ and how it may be quantized. In section 7 we introduce a superfield formulation which is needed in the path integral quantization to derive the appropriate measure in a natural way.

2 The setting.

Let $x^i$, $i = 1, \ldots, 2n$, be coordinates on a symplectic supermanifold $\mathcal{M}$, $\dim \mathcal{M} = 2n$, where $\varepsilon_i \equiv \varepsilon(x^i)$ are the Grassmann parities of $x^i$. Let, furthermore, there be a nondegenerate two-form $\omega$ on $\mathcal{M}$:

$$\omega = \frac{1}{2} \omega_{ij}(x) dx^i \wedge dx^j (-1)^{\varepsilon_i}, \quad \text{sdet } \omega_{ij} \neq 0,$$

which is required to be closed ($\partial_i = \partial/\partial x^i$):

$$d\omega = 0 \iff \partial_i \omega_{jk}(x)(-1)^{\varepsilon_i+1}\varepsilon_k + \text{cycle}(i, j, k) = 0.\quad (2.2)$$

Since $\omega$ is nondegenerate there exists an inverse $\omega^{ij}$ in terms of which the super Poisson bracket is defined by

$$\{A(x), B(x)\} = A(x) \partial_i \omega^{ij}(x) \partial_j B(x),$$

$$\omega^{ij}(x) \omega_{jk}(x) = \delta_k^i.$$  \quad (2.3)

$\omega_{ij}(x)$ and $\omega^{ij}(x)$ have the symmetry properties ($\varepsilon(\omega^{ij}) = \varepsilon(\omega_{ij}) = \varepsilon_i + \varepsilon_j$)

$$\omega_{ij}(x) = \omega_{ji}(x)(-1)^{(\varepsilon_i+1)\varepsilon_j}, \quad \omega^{ij}(x) = -\omega^{ji}(x)(-1)^{\varepsilon_i\varepsilon_j}.\quad (2.4)$$

The Poisson bracket (2.3) satisfies the Jacobi identities since (2.2) implies

$$\omega^{ij} \partial_i \omega^{jk}(x)(-1)^{\varepsilon_i\varepsilon_k} + \text{cycle}(ijk) = 0.\quad (2.5)$$

On $\mathcal{M}$ we consider a Hamiltonian theory with the Hamiltonian $H(x)$. Furthermore, we let the theory be constrained by the conditions $\theta^\alpha(x) = 0$, where the Grassmann parity of $\theta^\alpha$ is arbitrary and denoted by $\varepsilon_{\alpha} \equiv \varepsilon(\theta^\alpha)$. $\theta^\alpha$ are linearly independent implying that the rank of $\partial_i \theta^\alpha$ is equal to the number of constraints. (The rank consists of two blocks, one for the even part and one for the odd part. By the rank we mean in the following the sum of the two.) We are particularly interested in the case when the constraints are of second class. In this case we require the number of constraints to be $2m < 2n$ and that $\theta^\alpha$ satisfy

$$\text{Rank } \left\{ \theta^\alpha(x), \theta^\beta(x) \right\}_{\theta = 0} = 2m,\quad (2.6)$$
\[
\text{Rank } \frac{\partial \theta^\alpha(x)}{\partial x^i} \bigg|_{\theta=0} = 2m. \tag{2.7}
\]

\(\theta^\alpha(x) = 0\) determines a constraint surface \(\Gamma\) which in the case of second class constraints is a symplectic supermanifold of dimension \(2(n - m)\).

## 3 An auxiliary gauge theory from general projection matrices

Let us introduce the functions \(Z_i^\alpha(x)\) satisfying the property

\[
\theta^\beta \overset{\leftarrow}{\partial}_i Z_i^\alpha = \delta^\beta_\alpha, \quad \varepsilon(Z_i^\alpha) = \varepsilon_i + \varepsilon_\alpha. \tag{3.1}
\]

By means of \(Z_i^\alpha\) we define the general projection matrices \(P^i_j\) by

\[
P^i_j(x) \equiv \delta^i_j - Z_i^\alpha(x) \left( \theta^\alpha(x) \overset{\leftarrow}{\partial}_j \right). \tag{3.2}
\]

\(P^i_j\) satisfy then the following properties

\[
\theta^\alpha(x) \overset{\leftarrow}{\partial}_i P^i_j(x) = 0, \tag{3.3}
\]

\[
P^i_j(x) Z_i^\alpha(x) = 0, \tag{3.4}
\]

\[
P^i_k(x) P^k_j(x) = P^i_j(x). \tag{3.5}
\]

From (3.1) it follows that the rank of \(Z_i^\alpha\) is the same as the rank of \(\partial_\alpha \theta^\alpha\). For second class constraints we have therefore

\[
\text{rank } Z_i^\alpha(x) = 2m, \quad \text{rank } P^i_j(x) = 2(n - m), \tag{3.6}
\]

from (2.7).

By means of \(P^i_j(x)\) we may define the differential operator

\[
\overset{\leftarrow}{\nabla}_i \equiv \overset{\leftarrow}{\partial}_k P^k_i(x), \tag{3.7}
\]

which due to (3.3) and (3.4) satisfies the properties

\[
\theta^\alpha(x) \overset{\leftarrow}{\nabla}_i = 0, \tag{3.8}
\]

\[
\overset{\leftarrow}{\nabla}_i Z_i^\alpha(x) = 0. \tag{3.9}
\]
As an additional condition on \( Z^i_\alpha(x) \) we require that the differential operators (3.7) satisfy the closed algebra

\[
\left[ \vec{\nabla}_i, \vec{\nabla}_j \right] = \vec{\nabla}_k U^k_{ij}(x),
\]

which partly is a consistency condition for (3.8). Notice that \( U^k_{ij}(x) \) is not uniquely defined by this algebra since (3.10) is unaffected by the replacement \( U^k_{ij}(x) \rightarrow U^k_{ij}(x) + Z^k_\alpha(x) f^\alpha_{ij}(x) \) due to (3.9).

The two conditions (3.1) and (3.10) on \( Z^i_\alpha \) allow us to view the differential operators \( \vec{\nabla}_i \) as gauge generators in a theory in which the constraint variables \( \theta^\alpha(x) \) are the physical gauge invariant variables. Any action which only depends on \( \theta^\alpha(x) \) is then gauge invariant and the resulting gauge theory is reducible due to (3.9). From the prescription in [3] the master action for this reducible gauge theory is (using the short-handed DeWitt notation)

\[
S = S(\theta) + x^*_i P^i_k C^k + \frac{1}{2} C^*_k U^k_{ij} C^j (-1)^{\epsilon_i} + C^*_i Z^i_\alpha C^\alpha - C^*_1 U^1_{\alpha\beta} C^\beta (-1)^{\epsilon_1} + \frac{1}{6} C^*_1 U^1_{ijk} C^j C^i (-1)^{\epsilon_i + \epsilon_j + \epsilon_k},
\]

where \( S(\theta) \) is an arbitrary action only depending on \( \theta^\alpha(x) \), \( C^i (\varepsilon(C^i) = \varepsilon_i + 1) \) are ghosts, \( C^*_1 \) (\( \varepsilon(C^*_1) = \varepsilon_1 \)) ghosts for ghosts, and where \( x^*_i, C^*_i \), and \( C^*_1 \) are antifields to \( x^i, C^i \), and \( C^*_1 \) with opposite Grassmann parities to the latter. The master equation \( (S, S) = 0 \) contains all the previous conditions. In fact, if we start from the master action (3.11) and require \( (S, S) = 0 \) for arbitrary actions \( S(\theta) \) all conditions are generated.

4 An integral equation for the solution \( \bar{x}^i(x) \)

Let us introduce the functions \( \bar{x}^i = \bar{x}^i(\lambda, x), \; i = 1, \ldots, 2n \), where \( \lambda \) is a bosonic parameter. Let furthermore this function satisfy the equation

\[
\frac{d\bar{x}^i}{d\lambda} = Z^i_\alpha(\bar{x}) \theta^\alpha(\bar{x}), \; \bar{x}^i(0, x) = x^i,
\]

where \( Z^i_\alpha(x) \) is defined in (3.1). This equation implies

\[
\frac{d\theta^\alpha(\bar{x})}{d\lambda} = \theta^\alpha(\bar{x}) \frac{\partial}{\partial \bar{x}^i} \frac{d\bar{x}^i}{d\lambda} = \theta^\alpha(\bar{x})
\]

due to (3.1). Hence we have

\[
\theta^\alpha(\bar{x}) = e^{\lambda} \theta^\alpha(x).
\]

This implies in turn that

\[
\theta^\alpha(\bar{x}) = 0 \; \text{for} \; \bar{x}^i(x) \equiv \lim_{\lambda \to -\infty} \bar{x}^i(\lambda, x).
\]

4
Since the equation (4.1) may always be solved, \( \bar{x}^i(x) \) always exists. The equation (4.1) may be integrated to the following nonlinear Volterra integral equation

\[
\bar{x}^i(\lambda, x) = x^i + \int_0^\lambda d\sigma e^\alpha Z^i_\alpha(\bar{x}(\sigma, x))\theta^\alpha(x),
\] (4.5)

where we have made use of (4.3). By means of iterations one may then obtain an expression of the form (1.4) for \( \bar{x}^i(x) \) used in [2] generalized to coordinates with arbitrary Grassmann parities. To the lowest orders in \( \theta^\alpha \) we get explicitly

\[
\bar{x}^i(x) \equiv \lim_{\lambda \to -\infty} \bar{x}^i(\lambda, x) = x^i - Z^i_\alpha(x)\theta^\alpha(x) + \frac{1}{2} Z^i_\alpha(x) \partial_k Z^k_\beta(x) \theta^\beta(x) \theta^\alpha(x) + \cdots
\] (4.6)

The solution for \( \bar{x}^i(x) \) above imply that

\[
\bar{x}^i \partial_k = P^i_m(\bar{x})\sigma^m_k(x),
\] (4.7)

where \( \sigma^m_k(x) \) is an invertible matrix function normalized such that \( \sigma^m_k(x)|_{\theta(x)=0} = \delta^m_k \). Notice, however, that we may always replace \( \sigma^m_k(x) \) by

\[
\sigma^m_k(x) \longrightarrow \sigma^m_k(x) + Z^m_\alpha(\bar{x}(x))M^\alpha_k(x)
\] (4.8)

without affecting (4.7) due to (3.4). The expression (4.7) satisfies the consistency condition

\[
0 = \theta^\alpha(\bar{x}) \partial_k = \theta^\alpha(\bar{x}) \frac{\partial}{\partial \bar{x}^l} \left( \bar{x}^i \partial_k \right)
\] (4.9)

due to (3.3). The integrability conditions \( (\bar{x}^i \partial_k \partial_l = 0) \) of (4.7) may be written as

\[
P^i_j(\bar{x}) \left( \sigma^j_{[k}(x) \partial_l] - U^j_{nm}(\bar{x}(x))\sigma^m_k(x)\sigma^n_l(x)(-1)^{\varepsilon_k\varepsilon_l} \right) = 0,
\] (4.10)

where we have required \( Z^i_\alpha(x) \) also to satisfy (3.10). Due to (3.4) eq. (4.10) implies

\[
\sigma^j_{[k}(x) \partial_l] = U^j_{nm}(\bar{x}(x))\sigma^m_k(x)\sigma^n_l(x)(-1)^{\varepsilon_k\varepsilon_l} + Z^j_\alpha \sigma^\alpha_{kl},
\] (4.11)

where \( \sigma^\alpha_{kl} \) is antisymmetric in the lower indices. Notice that antisymmetry here is meant in a super sense: \( [ij] = ij - ji(-1)^{\varepsilon_i\varepsilon_j} \). The relations (4.11) may equivalently be written as

\[
(\sigma^{-1})^r_{[p} \partial_k (\sigma^{-1})^k_{q]} = (\sigma^{-1})^r_j U^j_{pq}(\bar{x}) + (\sigma^{-1})^r_j Z^j_\alpha(\bar{x})\sigma^\alpha_{kl}(\sigma^{-1})^l_p (\sigma^{-1})^k_q (-1)^{\varepsilon_p\varepsilon_k}.
\] (4.12)

Solutions of these conditions always exist since \( \bar{x}^i(x) \) exists.
5 Gauge invariance of $\bar{x}^i(x)$

The expression (4.7) implies now

$$\bar{x}^i(x) \tilde{G}_\alpha(x) = 0, \quad \varepsilon(\tilde{G}_\alpha) = \varepsilon_\alpha, \quad (5.1)$$

where

$$\tilde{G}_\alpha(x) = \tilde{\partial}_k G^k_\alpha(x), \quad G^k_\alpha(x) \equiv (\sigma^{-1})^k_m(x) Z^m_\alpha(\bar{x}(x)). \quad (5.2)$$

The integrability conditions (4.12) imply furthermore that the algebra of $G_\alpha$ is closed:

$$[\tilde{G}_\alpha(x), \tilde{G}_\beta(x)] = \tilde{G}_\gamma(x) U^\gamma_{\alpha\beta}(x), \quad (5.3)$$

where

$$U^\gamma_{\alpha\beta}(x) = V^\gamma_{\alpha\beta}(\bar{x}) - \sigma^m_{\alpha\gamma}(x)(\sigma^{-1})^n_m(x)(\sigma^{-1})^k_n(x) Z^k_\alpha(\bar{x}) Z^l_\beta(\bar{x})(-1)^{\varepsilon_l(\varepsilon_m + \varepsilon_\alpha)}, \quad (5.4)$$

where in turn $V^\gamma_{\alpha\beta}(x)$ is defined in (5.8) below and $\sigma^m_{ij}(x)$ in (4.11).

**Proof:** We have

$$[\tilde{G}_\alpha(x), \tilde{G}_\beta(x)] = \tilde{\partial}_i G^i_{[\alpha} \tilde{\partial}_j G^j_{\beta]},$$

$$G^i_{[\alpha} \tilde{\partial}_j G^j_{\beta]} = (\sigma^{-1})^i_m(x)(Z^m_{\alpha\gamma}(\bar{x}) \nabla_n Z^n_{\beta\gamma}(\bar{x})) +$$

$$+ (\sigma^{-1})^i_{[m}(x) \tilde{\partial}_j (\sigma^{-1})^j_{n]}(x) Z^m_{\beta\gamma}(\bar{x}) Z^m_{\alpha\gamma}(\bar{x})(-1)^{\varepsilon_l(\varepsilon_m + \varepsilon_\alpha)}, \quad (5.5)$$

where we have made use of the expression (4.7) in the first term. The first term is now zero due to (3.9). By means of the integrability conditions (4.12) we get then

$$G^i_{[\alpha} \tilde{\partial}_j G^j_{\beta]} = -(\sigma^{-1})^i_j U^l_{jk} Z^k_\alpha Z^j_\beta(-1)^{\varepsilon_\alpha \varepsilon_j} -$$

$$-(\sigma^{-1})^i_j Z^l_\gamma(\bar{x}) \sigma^m_{mn}(x)(\sigma^{-1})^n_i(x)(\sigma^{-1})^m_k(x) Z^k_\alpha(\bar{x}) Z^l_\beta(\bar{x})(-1)^{\varepsilon_l(\varepsilon_m + \varepsilon_\alpha)}. \quad (5.6)$$

Now

$$P^i_{[j} U^l_{jk} Z^k_\alpha Z^j_\beta(-1)^{\varepsilon_\alpha \varepsilon_j} = P^i_{[j} \tilde{\partial}_l P^l_{kj} Z^k_\alpha Z^j_\beta(-1)^{\varepsilon_\alpha \varepsilon_j} = 0 \quad (5.7)$$

due to (3.4). Hence, we must have

$$U^l_{jk} Z^k_\alpha Z^j_\beta(-1)^{\varepsilon_\alpha \varepsilon_j} = -Z^l_\gamma V^\gamma_{\alpha\beta}. \quad (5.8)$$

This inserted into (5.6) yields (5.3) with (5.4).
6 Construction of gauge invariant actions and their quantization

The action (\(\lambda_\alpha\) are Lagrange multipliers)
\[
S[x] = \int dt \left( V_i(x) \dot{x}^i - H(x) - \lambda_\alpha \theta^\alpha(x) \right)
\]  
(6.1)
describes dynamics of the type we consider although not in its most general form since the equations of motion imply
\[
\omega_{ij} = \partial_i V_j + \partial_j V_i(-1)^{(\epsilon_i+1)(\epsilon_j+1)},
\]  
(6.2)
which means that the two-form (2.1) here is exact. (A general action may be written as in [4].) The action (6.1) is in a first order form which according to a basic theorem allows us to construct an equivalent action by replacing \(x_i\) in (6.1) by the solutions \(\bar{x}^i(x)\) of the constraints, i.e. \(\theta^\alpha(\bar{x}) = 0\). The equivalent action is then
\[
S[\bar{x}] = \int dt \left( V_i(\bar{x}) \dot{\bar{x}}^i - H(\bar{x}) \right).
\]  
(6.3)
This action is now gauge invariant. We have
\[
S[\bar{x}] \frac{\delta}{\delta \bar{x}^i(x)} G^i_\alpha(x) = 0,
\]  
(6.4)
where \(G^i_\alpha\) is given in (5.2). \(S[\bar{x}]\) may be quantized by a Lagrangian path integral method (see [3]). The master action is
\[
S = S[\bar{x}] + \int dt \left( x_i^* G^i_\alpha(x) C^\alpha + \frac{1}{2} C^* U_{\alpha\beta}^\gamma(x) C^\beta C^\alpha(-1)^{\varepsilon^\alpha} \right),
\]  
(6.5)
where \(C^\alpha\) are ghosts with Grassmann parity \(\varepsilon(C^\alpha) = \varepsilon^\alpha + 1\), and \(x_i^*\) and \(C^*\) are antifields to \(x^i\) and \(C^\alpha\) with opposite Grassmann parities to the latter. \(U_{\alpha\beta}^\gamma\) is given by (5.4). The master equation \((S, S) = 0\) is satisfied by the properties of \(G^i_\alpha(x)\) in (5.2).

7 A superspace formulation

The path integral quantization of the gauge invariant action (6.3) using the master action (6.5) does not determine the natural measure. Here we demonstrate that this measure is directly obtained if we make use of a superfield formulation. We follow then the particular formulation given in [5, 6]. The coordinates \(x^i\) on the supersymplectic manifold are then turned into superfields according to the following rule:
\[
x^i \rightarrow x^i(\tau) \equiv x^i_0 + \tau x^i_1, \quad \varepsilon(x^i(\tau)) = \varepsilon_i,
\]  
(7.1)
where \( \tau \) is an odd Grassmann parameter. \( x_0^i \) represents the original coordinates \( x^i \) with Grassmann parity \( \varepsilon_i \) while \( x_1^i \) is the superpartner to \( x_0^i \) with Grassmann parity \( \varepsilon(x_1^i) = \varepsilon_i + 1 \). We may then define superfunctions \( Z^i_\alpha(x(\tau)) \) and \( P_{ij}^i(x(\tau)) \) which are equal to the previously considered functions with \( x^i \) replaced by the superfields \( 7.1 \). We may therefore also determine superfield solutions, \( \bar{x}^i(x(\tau)) \), to the constraints satisfying

\[
\theta^\alpha(\bar{x}^i(x(\tau))) = 0
\]

along the lines of section 4. In fact, \( \bar{x}^i(x(\tau)) \) are equal to the solutions \( 4.5, 4.6 \) with \( x^i \) replaced by \( x_i(\tau) \) which is easily seen by expanding \( 7.2 \) in \( \tau \). These super solutions will then be gauge invariant in the super sense

\[
\bar{x}^i(x(\tau)) \leftarrow G^i_\alpha(\tau) = 0,
\]

\[
\bar{x}^i(x(\tau)) \equiv \partial_i G^i_\alpha(x(\tau)),
\]

\( 7.3 \)

where \( G^i_\alpha(x(\tau)) \) is given by \( 5.2 \) with the replacement \( 7.1 \).

Instead of the original action \( 6.1 \) we consider here the superfield action \( 5, 6 \)

\[
S'[x(\cdot)] \equiv \int dt d\tau \left( V_i(x(\tau))Dx^i(\tau)(-1)^{\varepsilon_i} - Q(x(\tau), \tau) - \lambda_\alpha(\tau)\theta^\alpha(x(\tau)) \right),
\]

\( 7.4 \)

where \( Q \) is an odd function of the superfields \( 7.1 \) and \( \tau \), and where \( V_i \) is the superpotential in \( 6.1 \) here expressed in terms of the superfield \( 7.1 \). \( \lambda_\alpha(\tau) \) is an independent superfield (Lagrange multiplier) with Grassmann parity \( \varepsilon(\lambda_\alpha) = \varepsilon_\alpha + 1 \). \( D \) is the odd differential operator

\[
D \equiv \frac{d}{d\tau} + \tau \frac{d}{dt} \quad \Rightarrow \quad D^2 = \frac{d}{dt}.
\]

\( 7.5 \)

A variation of the action \( 7.4 \) yields the equations

\[
Dx^i(\tau) = -\{Q(\tau), x^i(\tau)\} - \lambda_\alpha(\tau)\{\theta^\alpha(x(\tau)), x^i(\tau)\}, \quad \theta^\alpha(x(\tau)) = 0.
\]

\( 7.6 \)

The consistency conditions

\[
0 = D\theta^\beta(x(\tau)) = -\{Q(\tau), \theta^\beta(x(\tau))\} - \lambda_\alpha(\tau)\{\theta^\alpha(x(\tau)), \theta^\beta(x(\tau))\}
\]

\( 7.7 \)

determine \( \lambda_\alpha \) in the case of second class constraints. The resulting expression for \( \lambda_\alpha \) inserted back into \( 7.6 \) yields then the equation

\[
Dx^i(\tau) = -\{Q(\tau), x^i(\tau)\}D,
\]

\( 7.8 \)

where we make use of the Dirac bracket. This in turn implies by means of \( 7.3 \)

\[
\dot{x}^i(\tau) = \{x^i(\tau), H(x(\tau))\}_D, \quad H(x(\tau)) \equiv \partial_\tau Q(\tau) - \frac{1}{2}\{Q(\tau), Q(\tau)\}_D.
\]

\( 7.9 \)

This demonstrates how the equations from the superaction \( 7.4 \) reduces to the equations from \( 6.1 \) together with the equations for the superpartners. If the theory
allows for a supersymmetric formulation, then \( Q \) may be chosen to have no explicit \( \tau \)-dependence. In this case (7.4) is manifestly supersymmetric.

A gauge invariant superfield action is now obtained if one replaces \( x^i(\tau) \) in (7.4) by the solutions of (7.2), \( \text{i.e.} \overline{\dot{x}}^i(x(\tau)) \). This action, \( S'[\overline{x}] \), may then be quantized using the superfield formulation in [5, 6]. Antibrackets and \( \Delta \)-operators are defined by

\[
(F, G) \equiv F \int \frac{\delta}{\delta \Phi^A(\tau, t)} d\tau dt \frac{\delta}{\delta \Phi^A(\tau, t)} G - (F \leftrightarrow G)(-1)^{(\varepsilon_F+1)(\varepsilon_G+1)},
\]

\[
\Delta \equiv -\int \frac{\delta}{\delta \Phi^A(\tau, t)} d\tau dt \frac{\delta}{\delta \Phi^A(\tau, t)}, \tag{7.10}
\]

where \( \Phi^A \) are super antifields with Grassmann parity \( \varepsilon(\Phi^A) = \varepsilon(\Phi^A) = \varepsilon_A \). The functional derivatives satisfy

\[
\frac{\delta}{\delta \Phi^A(\tau, t)} \Phi^B(\tau', t') = \delta^B_A \delta(\tau - \tau') \delta(t - t') = \Phi^B(\tau, t) \frac{\delta}{\delta \Phi^A(\tau', t')},
\]

\[
\varepsilon \left( \frac{\delta}{\delta \Phi^A} \right) = \varepsilon \left( \frac{\delta}{\delta \Phi^A} \right) = \varepsilon_A + 1. \tag{7.11}
\]

We use the conventions

\[
\int d\tau \tau = 1, \quad \delta(\tau - \tau') = \tau - \tau',
\]

\[
\Rightarrow \quad \int f(\tau') \delta(\tau - \tau') d\tau' = f(\tau) = \int d\tau' \delta(\tau' - \tau) f(\tau'). \tag{7.12}
\]

If the super field-antifields pairs, \( \Phi^A \) and \( \Phi_A \), in (7.10) are defined as follows

\[
\Phi^A(\tau) = \Phi^A_0 + \tau \Phi^A_1,
\]

\[
\Phi_A(\tau) \equiv \Phi^*_A - \Phi^*_0 A, \quad \varepsilon(\Phi^A) = \varepsilon(\Phi^A) \equiv \varepsilon_A, \tag{7.13}
\]

then the expressions (7.10) reduce to the conventional expressions in terms of the fields \( \Phi^A \) and the corresponding antifields \( \Phi^*_A \), \( \varepsilon(\Phi^*_A) = \varepsilon_A + 1, (a = 0, 1) \). Notice that

\[
(\Phi^A(\tau, t), \Phi_B(\tau', t')) = -(\Phi^A_0(t), \Phi^*_B(t')) \tau' + \tau(\Phi^A_1(t), \Phi^*_B(t')) = \\
\delta^A_B \delta(\tau - \tau') \delta(t - t'). \tag{7.14}
\]

The functional derivatives satisfying (7.11) are in terms of the component fields (7.13) given by

\[
\frac{\delta}{\delta \Phi^A(\tau, t)} = \tau \frac{\delta}{\delta \Phi^A_0(t)} + (-1)^{\varepsilon_A} \frac{\delta}{\delta \Phi^A_1(t)} \tau', \quad \frac{\delta}{\delta \Phi^A(\tau, t)} = -\frac{\delta}{\delta \Phi^A_0(t)} \tau + \frac{\delta}{\delta \Phi^A_1(t)},
\]

\[
\frac{\delta}{\delta \Phi_A(\tau, t)} = \tau \frac{\delta}{\delta \Phi^*_A (t)} + \frac{\delta}{\delta \Phi^*_0 A(t)}, \quad \frac{\delta}{\delta \Phi^*_A (\tau, t)} = -\frac{\delta}{\delta \Phi^*_A (t)} \tau + (-1)^{\varepsilon_A} \frac{\delta}{\delta \Phi^*_0 A(t)}. \tag{7.15}
\]
With these tools at hand we get the master action

\[ S = S'[\bar{x}(\cdot)] + \int \bar{x}_i(\tau, t) d\tau dt G^i_\alpha(x(\tau, t)) C^\alpha(\tau, t) + \frac{1}{2} \int \tilde{C}_\alpha(\tau, t) d\tau dt U^\alpha_{\alpha\beta}(x(\tau, t)) C^\beta(\tau, t) C^\alpha(\tau, t)(-1)^{\varepsilon_\alpha} - \int \tilde{C}^\alpha(\tau, t) d\tau dt \lambda_\alpha(\tau, t)(-1)^{\varepsilon_\alpha}, \]

(7.16)

where \( S'[\bar{x}(\cdot)] \) is the action (7.4) with \( x^i(\tau) \) replaced by \( \bar{x}^i(x(\tau)) \) in (7.2) and where the last term is a standard nonminimal term to allow for a gauge fixing delta function. Notice that \( S'[\bar{x}(\cdot)] \) does not contain \( \lambda_\alpha \) in (7.4) and that \( \lambda_\alpha \) in the last term is a new variable. The master action (7.16) satisfies

\[ (S, S) = 0, \quad \Delta S = 0, \]

(7.17)

where the antibracket and the \( \Delta \)-operator are given by (7.10) for the set of superfields \( \Phi^A = \{ x^i, C^\alpha, \bar{C}^\alpha, \lambda_\alpha \} \) and their super antifields. Note that \( \varepsilon(C^\alpha) = \varepsilon_\alpha + 1 \) and \( \varepsilon(\bar{C}^\alpha) = \varepsilon_\alpha \). The second equality in (7.17) follows from the locality in \( \tau \) of \( S \) which yields a factor zero and implies that \( S \) also satisfies the quantum master equation which in turn implies that no quantum corrections of the natural measure in the path integral is required. In order to gauge fix the master action (7.16) we need a gauge fixing fermion, \( \Psi \), expressed in terms of the superfields \( \Phi^A \) such that the super antifields are determined through the equations

\[ \tilde{\Phi}_A(\tau, t) = \frac{\delta}{\delta \Phi_A(\tau, t)} = (-1)^{\varepsilon_A} \frac{\delta}{\delta \Phi_A(\tau, t)} \Psi. \]

(7.18)

A possible choice is

\[ \Psi = \int \theta^\alpha(x(\tau, t)) d\tau dt \tilde{C}_\alpha(\tau, t) = \int d\tau dt \tilde{C}_\alpha(\tau, t) \theta^\alpha(x(\tau, t)). \]

(7.19)

Eq.(7.18) yields then

\[ \tilde{x}_i(\tau, t) = \tilde{C}_\alpha(\tau, t) \theta^\alpha(x(\tau, t)) \tilde{\partial}_i(-1)^{\varepsilon_i}, \quad \tilde{C}_\alpha(\tau, t) = 0, \]

\[ \tilde{\lambda}_\alpha(\tau, t) = 0, \quad \tilde{C}^\alpha(\tau, t) = \theta^\alpha(x(\tau, t)). \]

(7.20)

With these expressions inserted into (7.16) we obtain the gauged fixed action

\[ S_\Psi = S'[\bar{x}(\cdot)] + \int d\tau dt \tilde{C}_\alpha(\tau, t) \theta^\alpha(x(\tau, t)) \tilde{\partial}_i G^i_\alpha(x(\tau, t)) C^\beta(\tau, t) - \int d\tau dt \lambda_\alpha(\tau, t) \theta^\alpha(x(\tau, t)). \]

(7.21)

In the path integral the last term yields the delta-function \( \delta(\theta^\alpha) \) after integration over \( \lambda_\alpha \). In the presence of this delta-function \( \bar{x}^i(x) = x^i \), and the middle term yields
unity after integration over $C_\alpha$ and $\bar{C}_\alpha$ since $G^i_\beta\big|_{\theta=0} = Z^i_\beta$ implies $\theta^\alpha G^i_\beta\big|_{\theta=0} = \delta^\alpha_\beta$ due to (3.1). Thus, the path integral over $S_\Psi$ reduces to the path integral over the first and last term in (7.21) which is equivalent to a path integral over the original action (7.1). Integration over the superpartner $x^i_1$ in (7.1) yields then the expected measure as was shown in (3.1).

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