CREMONA TRANSFORMATIONS OF WEIGHTED PROJECTIVE PLANES, ZARISKI PAIRS, AND RATIONAL CUSPIDAL CURVES

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Abstract. In this work, we study a family of Cremona transformations of weighted projective planes which generalize the standard Cremona transformation of the projective plane. Starting from special plane projective curves we construct families of curves in weighted projective planes with special properties. We explain how to compute the fundamental groups of their complements, using the blow-up-down decompositions of the Cremona transformations, we find examples of Zariski pairs in weighted projective planes (distinguished by the Alexander polynomial) and construct a family of quasi-homogeneous rational cuspidal curves which can be used to construct normal surface singularities having rational homology sphere link. Finally, normal surface singularities having integral homology sphere link are constructed using similar techniques.

Introduction

This paper deals with curves in surfaces with normal singularities and the interplay between their topological and algebraic properties.

In this direction we provide a family of examples of curves in weighted projective planes using a generalization of the classical Cremona transformations. This allows us to construct infinitely many pairs of curves in weighted projective planes defining linearly equivalent divisors and the same local type of singularities, whose embeddings are not homeomorphic. Moreover, whose complements have non-isomorphic fundamental groups. This is known in the literature as Zariski pairs when referred to plane projective curves [2] since Zariski provided the first example of such phenomenon in [34]. The curves are obtained from a smooth cubic and three tangent lines via a weighted Cremona transformation 1.4. These groups are distinguished using two different techniques. In section 2.1 a topological approach is given by obtaining presentations of the groups. These presentations, which in general are complicated to calculate, can be managed from the original curve via Cremona transformation in a very explicit geometric way. To complete this example, we also present the more algebraic approach via cyclic coverings as was originally used by...
Zariski and later developed by Steenbrink [32, Lemma 3.14], Libgober [22], Esnault-Viehweg [14], Vaquié [33], and the first author [2]. Our method uses a generalization of [14] given in [4], see section 2.3.

Section 3 is devoted to developing some methods to construct rational cuspidal curves which will be useful in the later sections.

The second part of the paper focuses on local properties of surface singularities. Our main goal is to provide examples of surface germs whose link is a rational (or even integral) homology sphere. A source of examples is given by superisolated singularities. In section 4 we introduce the determinant of a surface singularity as the absolute value of the determinant of the intersection matrix of a resolution. In §4.1 a result is given to calculate this determinant using partial resolutions. A surface singularity has a rational homology sphere link if and only if the dual graph of a (partial) resolution is a tree whose vertices are rational curves. Moreover, it will be integral if its determinant is one. We use this criterion to study weighted Lé-Yomdin singularities and to describe infinite families with rational links.

In particular, following the ideas in [1, 27], one can use the Zariski pairs obtained in section 2 to construct weighted Lé-Yomdin singularities having the same Alexander polynomials, the same abstract topology, but different embedded topology. It would be hopeless to compute the Jordan form of the complex monodromy (the actual invariant that distinguishes the embedded topology) without the use of the techniques in this paper.

The last part is devoted to solving two problems on surface singularities with a rational sphere link. Namely, in §5.1 we study Brieskorn-Pham surface singularities \{x^a + y^b + z^c = 0\} \subset \mathbb{C}^3 as a special case of weighted Lé-Yomdin. We illustrate how to recover classical results in a simple way, namely to characterize which ones have a rational sphere link and show that the only integral homology spheres occur in the classical case, that is, whenever \((a, b, c)\) are pairwise coprime. Besides Brieskorn-Pham singularities, more examples are provided in §5.2 using weighted Cremona transformations and Kummer covers.

András Némethi asked us if it was possible to find singularities with integral homology sphere link in the realm of weighted Lé-Yomdin singularities. The only ones we found are the already known Brieskorn-Pham singularities. As an alternative, in §5.3, a new family of surface singularities is presented following [28]. We give conditions for this family to have a rational homology sphere link. Moreover, this family provides infinitely many new examples of integral homology sphere links which may answer the question by András Némethi in the affirmative.

1. Quotient singularities and weighted Cremona transformations

The main objects of this work will be weighted projective planes (and lines) and quotient singularities. A quotient singularity is a normal space which is locally isomorphic to \((X, 0)\) where \(X\) is the quotient of \(\mathbb{C}^n\) by the action of a cyclic group \(\mu_m \subset \mathbb{C}^*\) given by

\[
\zeta \cdot (x_1, \ldots, x_n) = (\zeta^{a_1} x_1, \ldots, \zeta^{a_n} x_n), \quad \zeta^m = 1, (x_1, \ldots, x_n) \in \mathbb{C}^n.
\]

If \(\gcd(m, a_1, \ldots, a_n) = 1\), the action is faithful. We denote this singularity by \(\frac{1}{m}(a_1, \ldots, a_n)\). There are trivial some equivalences of quotient singularities such as
\[
\frac{1}{m}(a_1, \ldots, a_n) = \frac{1}{m}(da_1, \ldots, da_n) \text{ if } \gcd(m, d) = 1. \text{ A less obvious one is given by }
\frac{1}{m}(a_1, \ldots, a_n) \cong \frac{d}{m} \left( \frac{a_1}{d}, \frac{a_2}{d}, \ldots, \frac{a_n}{d} \right) \text{ if } d = \gcd(m, a_2, \ldots, a_n).
\]

1.1. Curves in quotient surface singularities. We introduce some notation for germs of curves in a quotient surface singularity \( S := \frac{1}{d}(a, b) \) (with \( a, b, d \) pairwise coprime and \( d > 1 \)). Let \( \pi : \mathbb{C}^2 \to S \) be the quotient map. Any germ of curve \( C \subset S \) is defined as the zero locus of a non-constant equivariant germ \( f \in \mathbb{C}\{x, y\} \).

The following notions appeared already in Dolgachev’s work [12].

**Definition 1.1.** A germ of curve \( C \) is said to be quasi-smooth if \( C \) is smooth as an abstract curve; and it is said to be extremely quasi-smooth if \( f \) has multiplicity one.

**Remark 1.2.** There are simple characterizations of the above concepts in terms of a minimal resolution \( \hat{S} \to S \); recall that its dual graph is a bamboo whose vertices represent smooth rational divisors. A curve is quasi-smooth if its strict transform in \( \hat{S} \) is a curvette of an exceptional divisor, that is, smooth and transversal to it. Moreover, it is extremely quasi-smooth if this divisor is an end of the bamboo. In the particular case \( \frac{1}{d}(1, 1) \), any quasi-smooth curve is extremely quasi-smooth, and any linear form can be the multiplicity-1 component of \( f \). Otherwise, in \( \frac{1}{d}(a, b) \) \((a, b) \neq (1, 1)\) the equivariant part of multiplicity 1 of a extremely quasi-smooth \( f \) can only be given by the eigenspaces of the cyclic action, in our notation, either \( x \) or \( y \).

1.2. Weighted projective planes.

In this section we briefly describe weighted projective planes in order to fix some notation. A weight is a triple \( \omega := (e_1, e_2, e_3) \in \mathbb{Z}_{>0}^3 \) such that \( \gcd \omega = 1 \). The weighted projective plane \( \mathbb{P}_\omega^2 \) is a normal surface obtained as the quotient of \( \mathbb{C}^3 \setminus \{0\} \) by the action of \( \mathbb{C}^* \) given by
\[
t \cdot (x, y, z) = (t^{e_1} x, t^{e_2} y, t^{e_3} z), \quad t \in \mathbb{C}^*, (x, y, z) \in \mathbb{C}^3 \setminus \{0\}.
\]

Weighted projective lines are defined in a similar way. The symbol \([x : y : z]_\omega\) stands for points in \( \mathbb{P}_\omega^2 \), for orbits in \( \mathbb{C}^3 \setminus \{0\} \) or their closure in \( \mathbb{C}^3 \). This variety is covered by three charts. The first one is
\[
(x, y) \mapsto [x : y : 1]_\omega,
\]
which is only injective if the source of this map is \( \frac{1}{e_3}(e_1, e_2) \). The other charts are defined accordingly.

Define \( d_k := \gcd(e_i, e_j) \) and \( \alpha_k := \frac{e_k}{d_k} \), \( \{i, j, k\} = \{1, 2, 3\} \). Note that \( \eta := (\alpha_1, \alpha_2, \alpha_3) \) are pairwise coprime. According to the properties described above, the map
\[
\mathbb{P}_\omega^2 \xrightarrow{\text{t}} \mathbb{P}_\eta^2
\]
(1.1)
\[
[x : y : z]_\omega \xrightarrow{\text{t}} [x^{d_1} : y^{d_2} : z^{d_3}]_\omega
\]
is well defined since
\[
t \cdot [x : y : z]_\omega = [t^{e_1} x : t^{e_2} y : t^{e_3} z]_\omega \mapsto [t^{d_1 e_1} x^{d_1} : t^{d_2 e_2} y^{d_2} : t^{d_3 e_3} z^{d_3}]_\omega = t^{d_1 d_2 d_3} [x^{d_1} : y^{d_2} : z^{d_3}]_\omega
\]
since \( d_i e_i = \alpha_i d_1 d_2 d_3 \). Moreover, one can easily check that it is an isomorphism.
One may consider \( \mathbb{P}^2_\omega \) and \( \mathbb{P}^2_\eta \) in a slightly different way, see also [12]. The plane \( \mathbb{P}^2_\eta \) has at most 3 singular points at \( P_\omega := [1 : 0 : 0]_\eta \) (if \( \alpha_1 > e_2 e_3 \)), \( P_\eta := [0 : 1 : 0]_\eta \) (if \( \alpha_2 > e_1 e_3 \)), and \( P_\omega := [0 : 0 : 1]_\eta \) (if \( \alpha_3 > e_1 e_2 \)). The plane \( \mathbb{P}^2_\omega \) can be considered as an orbifold if the quotient charts are not normalized. We will see the precise meaning of this difference later.

### 1.3. Weighted blow-ups.

Let us consider now \( \omega := (p, q) \in \mathbb{Z}^2_{> 0} \), \( \gcd \omega = 1 \). The \( \omega \)-weighted blow-up of \( \mathbb{C}^2 \) at the origin is the map \( \pi_\omega : \mathbb{C}^2_\omega \to \mathbb{C}^2 \) where

\[
\mathbb{C}^2_\omega := \{(x, u) \in \mathbb{C}^2 \times \mathbb{P}^1_\omega | x \in u\}.
\]

This normal variety is represented with two charts. The first one is

\[
\sigma : \frac{1}{p}(-1, q) \to \mathbb{C}^2_\omega, \quad (x, y) \mapsto ((x^p, x^q y), [1 : y]_\omega);
\]

and the second one is analogous and modeled on \( \frac{1}{q}(p, -1) \). The exceptional divisor of \( \pi_\omega \) is a projective line which contains the singular points \((0, [1 : 0]_\omega)\) (if \( p > 1 \)) and \((0, [0 : 1]_\omega)\) (if \( q > 1 \)) of the surface \( \mathbb{C}^2_\omega \). Note that the curvettes of this divisor are extremely quasi-smooth if either \( p \) or \( q \) equal 1.

Let us study now 3-dimensional weighted blow-ups. We recover the previous notations for \( \omega, \eta \in \mathbb{Z}^3_{> 0} \). We consider \( \Pi_\omega : \mathbb{C}^3_\omega \to \mathbb{C}^3 \) where

\[
\mathbb{C}^3_\omega := \{(x, u) \in \mathbb{C}^3 \times \mathbb{P}^2_\omega | x \in u\}.
\]

The normal variety is now represented with three charts. The first one is

\[
\sigma : \frac{1}{p}(-1, q, r) \to \mathbb{C}^3_\omega, \quad (x, y, z) \mapsto ((x^p, x^q y, x^r z), [1 : y : z]_\omega);
\]

and the other two charts can be analogously defined and have \( \frac{1}{q}(p, -1, r) \) and \( \frac{1}{r}(p, q, -1) \) as quotient charts. Let us study the local structure of \( \mathbb{C}^3_\omega \) at \( E_\omega := \Pi_\omega^{-1}(0) \); since \( \Pi_\omega \) is an isomorphism outside this exceptional divisor the points not in \( E_\omega \) are smooth. Note that \( E_\omega \cong \mathbb{P}^2_\omega \cong \mathbb{P}^2_\eta \), see (1.1); the quasi-homogeneous coordinates of \( \mathbb{P}^2_\omega \) will be denoted as \([x_\eta : y_\eta : z_\eta]\). For the sake of simplicity we will denote the elements of \( E_\omega \) only by its \( \omega \)-quasi-homogeneous coordinates.

\[
P_x = [1 : 0 : 0]_\omega, \quad P_y = [0 : 1 : 0]_\omega, \quad P_z = [0 : 0 : 1]_\omega,
\]

\[
X = \{[0 : y : z]_\omega | yz \neq 0\}, \quad Y = \{[x : 0 : z]_\omega | xz \neq 0\}, \quad Z = \{[x : y : 0]_\omega | xy \neq 0\}.
\]

The stratification of \( E_\omega \) in terms of the singular points of the ambient space is as follows:

- **0-dimensional strata:**

\[
\mathcal{P}_x = \begin{cases} 0 & \text{if } p = 1 \\ \{P_x\} & \text{otherwise,} \end{cases}
\]

where \( P_x \) has type \( \frac{1}{p}(-1, q, r) \) and is singular if \( p > 1 \). The remaining strata \( \mathcal{P}_y \) and \( \mathcal{P}_z \) are defined accordingly.
• 1-dimensional strata:

\[ \mathcal{L}_x = \begin{cases} 
\emptyset & \text{if } a = 1 \\
X \cup \{P_y\} & \text{if } 1 < a = q \neq r \\
X \cup \{P_z\} & \text{if } 1 < a = r \neq q \\
X & \text{if } 1 < a = q = r \\
\hat{X} & \text{otherwise.} 
\end{cases} \]

The transversal singularities in the middle cases are \( \frac{1}{a}(-1,p) \). The remaining strata \( \mathcal{L}_y \) and \( \mathcal{L}_z \) are defined accordingly.

• 2-dimensional stratum: \( T \) is the smooth stratum, which contains \( \{[x : y : z]_\omega \mid xyz \neq 0\} \). It contains also \( X \) (resp. \( \hat{Y} \), resp. \( \hat{Z} \)) if \( a = 1 \) (resp. \( b = 1 \), resp. \( c = 1 \)) and \( P_x \) (resp. \( P_y \), resp. \( P_z \)) if \( p = 1 \) (resp. \( q = 1 \), resp. \( r = 1 \)).

1.4. Weighted Cremona transformations.

The most well-known Cremona transformation of \( \mathbb{P}^2 \) corresponds to the birational map \([x : y : z] \mapsto [yz : xz : xy]\); geometrically, this map is the composition of the blow-ups at \([1 : 0 : 0],[0 : 1 : 0],[0 : 0 : 1]\) and the contractions of the strict transforms of the lines \( x = 0, y = 0, z = 0 \) which become pairwise disjoint \((-1)\)-lines in the blown-up plane.

In this section we generalize this transformation to a birational map from a weighted projective plane to \( \mathbb{P}^2 \). Let us fix \( \mathbb{P}^2_\omega, \omega := (p,q,r) \), where \( p, q, r \) are pairwise coprime. Consider two positive integers \( \tilde{p}, \tilde{q} \) such that \( pp + q\tilde{q} = r + pq \) (they exist from standard semigroup properties). These arithmetic data provide the following rational map

\[ \mathbb{P}^2_{\omega, \tilde{p}, \tilde{q}} \longrightarrow \mathbb{P}^2 \]

\([x : y : z]_{\omega} \Longleftrightarrow [y^p z : x^q z : x^r y^{\tilde{q}}]_{\omega} \).

It is a well-defined rational map (not a morphism) since the three coordinates have \( \omega \)-degree equal to \( pq + r \). It is in fact a birational map whose inverse is given by

\[ \mathbb{P}^2 \longrightarrow \mathbb{P}^2_{\omega, \tilde{p}, \tilde{q}} \]

\([x : y : z] \Longleftrightarrow [y^{\frac{1}{p}} z^{\frac{1}{q}} : x^{\frac{1}{q}} z^{\frac{1}{r}} : x^{\frac{1}{r}} y^{\frac{1}{\tilde{q}}}]_{\omega} \).

Note that this map is well defined. Assume \( x_0 \) (resp. \( y_0, z_0 \)) is such that \( x_0^p = x \) (resp. \( y_0^q = y, z_0^r = z \)) and choose for instance \( x_1 = \zeta_p x_0 \). Let \( \tilde{q} \in \mathbb{Z} \) such that \( q\tilde{q} \equiv 1 \mod p \). As a consequence, \( r\tilde{q} \equiv (r + pq)\tilde{q} \equiv (pp + q\tilde{q})\tilde{q} \equiv \tilde{q} \mod p \). Then

\[ [y_0 z_0^p : x_1 z_0^{\frac{q}{r}} z_0^\tilde{q} : x_1^{\frac{1}{r}} z_0^{\frac{1}{p}}]_{\omega} = [\zeta_p y_0 z_0 : \zeta_p^q x_0 z_0^{\frac{q}{r}} : \zeta_p^{\frac{1}{r}} x_0^{\frac{1}{q}} y_0^{\frac{1}{p}}]_{\omega} = [y_0 z_0^p : x_0 z_0^{\frac{q}{r}} : x_0^{\frac{1}{r}} y_0^{\frac{1}{p}}]_{\omega}. \]

A similar argument applies to other choices of roots of \( y^\frac{1}{p} \) and \( z^\frac{1}{r} \). These equations completely determine the birational map, but a more geometric description will be useful.

**Proposition 1.3.** The map \( \Phi_{\omega, \tilde{p}, \tilde{q}} \) is the composition of the following blow-ups and downs:

1. Three simultaneous blow-ups:
   (a) Type \((p,q)\) at \([0 : 0 : 1]_\omega \equiv \frac{1}{q}(p,q)\).
   (b) Type \((1,\tilde{p})\) at \([0 : 1 : 0]_\omega \equiv \frac{1}{p}(p,\tilde{p}) = \frac{1}{q}(p,p\tilde{q}+q\tilde{q}) = \frac{1}{q}(1,\tilde{p})\).
(c) Type \((1, \tilde{q})\) at \([1 : 0 : 0]\).

(2) Three simultaneous blow-downs:
(a) Type \((1, 1)\) at \([0 : 0 : 1]\).
(b) Type \((q, \tilde{p})\) at \([1 : 0 : 0]\).
(c) Type \((p, \tilde{q})\) at \([0 : 1 : 0]\).

Proof. Let us start with the three blow-ups in \(\mathbb{P}_\omega^2\). We obtain a normal rational surface \(S\). The preimage of three axes appear in Figure 1, containing the strict transforms \(L_x, L_y, L_z\) of the lines and the exceptional components \(E_x, E_y, E_z\). The self-intersections and the type of the singular points are computed using [8, Theorem 4.3].

\[
\frac{1}{p}(\tilde{q}, -1) \quad \frac{-\bar{r}}{pq} \quad \frac{1}{q}(\tilde{p}, -1) \\
L_y \quad \frac{1}{p\bar{q}} \quad E_z \\
\frac{1}{q}(p, -1) \quad \frac{-\bar{r}}{q\bar{p}} \quad \frac{1}{q}(q, -1) \\
E_x \quad L_x \quad \frac{-\bar{r}}{pq} \\
-1 \quad \frac{-\bar{r}}{p} \quad E_y \\
L_z
\]

Figure 1. Weighted blow-ups of \(\mathbb{P}^2\) in \(S\)

The strict transforms of the lines coincide with the exceptional components of a \((p, \tilde{q})\)-blowing-up \((L_y)\), a \((q, \tilde{p})\)-blowing-up \((L_x)\) and a standard blowing-up \((L_z)\). The result of the triple blowing-down is \(\mathbb{P}^2\).

This geometric expression will be useful for the study of curves in \(\mathbb{P}_\omega^2\) via their transforms in \(\mathbb{P}^2\).

2. Zariski pairs on weighted projective planes

In this section, we are going to use the Cremona transformations in §1.4 to produce Zariski pairs in weighted projective planes. By a Zariski pair we mean two curves embedded in the same surface whose combinatorics are the same, but whose embeddings are non-homeomorphic. As in the classical case of curves in the projective plane, the combinatorics of curves in a weighted projective plane is encoded by the degrees of its irreducible components and the dual graph of a minimal resolution of the curve (where the strict transforms of the irreducible components of the curve are marked).

In this section we will produce families of Zariski pairs of irreducible curves. Let us start with the combinatorics defined by a smooth projective cubic and three tangent lines at inflection points. Note that such lines are necessarily non-concurrent and hence the remaining singular points are three nodes. This combinatorics admits a Zariski pair of sextics, see [2], and their embeddings are distinguished by the algebraic property of whether or not the inflection points of the cubic (that is, the three singular points of the sextic) are aligned. The image by a standard Cremona transformation of the smooth cubics (using the three tangent lines at the
axes) produces a Zariski pair of irreducible sextics with three $E_6$-points. In this case, the embeddings can be proved to be different showing that the fundamental group of their complements are not isomorphic.

Our strategy is to replace this Cremona transformation by the inverse of those described in §1.4.

2.1. Fundamental groups of complements.

We start by recalling the two possible fundamental groups of the complements of the sextic curves given as the union of a smooth cubic and three tangent lines at inflection points.

Proposition 2.1 ([3]). Let $C$ be a smooth cubic with three tangent lines $X, Y, Z$ at inflections which are not aligned. Then, $\pi_1(\mathbb{P}^2 \setminus (C \cup X \cup Y \cup Z))$ is abelian.

In [3], the fundamental group of the other member of the Zariski pair is also computed; since it is non-abelian, this invariant distinguishes the two members. For our purpose, we need a more geometrical presentation of the group involving meridians for all the irreducible components and such that the meridians close to the nodes are made explicit.

Proposition 2.2 ([5]). Let $C$ be a smooth cubic with three tangent lines $X, Y, Z$ at inflections which are aligned. Then, $\pi_1(\mathbb{P}^2 \setminus (C \cup X \cup Y \cup Z))$ is

\begin{equation}
\langle c, \ell_x, \ell_y, \ell_z \mid [\ell_x, \ell_x] = [\ell_y, \ell_y] = [\ell_z, \ell_z] = [c, \ell_x^{-1} \ell_x] = [c, \ell_y^{-1} \ell_y] = c\ell_x c\ell_y c\ell_z = 1 \rangle
\end{equation}

where $c$ is a meridian of $C$, and $\ell_x, \ell_y, \ell_z$ are meridians of $X, Y, Z$, respectively; moreover the meridians of the lines correspond to meridians close to the double points.

Let us fix $\Phi := \Phi_{\omega, \rho, \delta}$ as in §1.4, and let us denote by $\tilde{C} \subset \mathbb{P}^2$ the strict transform of the smooth cubic $C$ by $\Phi$, where the lines $X, Y, Z$ have equations $x = 0, y = 0, z = 0$, respectively. Consider the homogeneous polynomials of degree 3

$H_\lambda(x, y, z) := x^3 + y^3 + z^3 + 3xy(\lambda^{-1}x + \lambda y) + 3xz(x + z) + 3yz(\lambda^{-1} y + \lambda z)$,

where $\lambda^3 = 1$. The curve $C_\lambda = \{H_\lambda = 0\}$ is a smooth cubic which is tangent to the line $L_x$ at the inflection point $[0 : 1 : -\lambda]$ and analogously for $Y$ at $[-1 : 0 : 1]$, and $Z$ at $[1 : -\lambda : 0]$. Note that for the cubic $C_1$ the three inflection points are contained in the line $x + y + z = 0$. However, for the smooth cubic $C_{\exp \frac{\pi i}{3}}$ the three inflection points are not aligned.

Corollary 2.3. In the non-aligned case, $\pi_1(\mathbb{P}^2 \setminus \tilde{C})$ is isomorphic to $\mathbb{Z}/3(pq + r)$.

Proof. The space $\mathbb{P}^2 \setminus \tilde{C}$ is homeomorphic to $S \setminus (\tilde{C} \cup L_x \cup L_y \cup L_z)$ (see Figure 1) and the space $\mathbb{P}^2 \setminus (\tilde{C} \cup X \cup Y \cup Z)$ is homeomorphic to $S \setminus (\tilde{C} \cup L_x \cup L_y \cup L_z \cup E_x \cup E_y \cup E_z)$, where $\tilde{C}$ denotes the strict transform of $C$ in $S$. As a consequence of [18, Lemma 4.18] the kernel of the epimorphism

\begin{equation}
\pi_1(\mathbb{P}^2 \setminus (C \cup X \cup Y \cup Z)) \rightarrow \pi_1(\mathbb{P}^2 \setminus \tilde{C})
\end{equation}

is the normal subgroup generated by the meridians of $E_x, E_y, E_z$ in $S$. Since the source is an abelian group by Proposition 2.1, the group $\pi_1(\mathbb{P}^2 \setminus \tilde{C})$ is abelian as well. Hence it coincides with $H_1(\mathbb{P}^2 \setminus \tilde{C}; \mathbb{Z}) \cong \mathbb{Z}/\deg(\tilde{C})$, since $\tilde{C}$ contains the vertices of $\mathbb{P}^2_\omega$.

In order to compute the other fundamental group we need a technical result.
Lemma 2.4. Let \( \pi : \mathcal{C}_{(p,q)} \to \mathbb{C}^2 \) be the \((p,q)\)-blow-up of the origin in \( \mathbb{C}^2 \) and let \( E \) denote its exceptional component. Let \( X, Y \subset \mathbb{C}^2 \) be the axes (curves of equations \( x = 0, \ y = 0, \) respectively), and let us keep this notation for their strict transforms. Let \( U := \mathbb{C}^2 \setminus (X \cup Y) \equiv \mathcal{C}_{(p,q)} \setminus (E \cup X \cup Y) \).

If \( \mu_X, \mu_Y, \mu_X \) denote meridians of the respective curves in \( \pi_1(U) \equiv \mathbb{Z}\mu_X \oplus \mathbb{Z}\mu_Y \), then (multiplicative notation) \( \mu_E = \mu_X^p \mu_Y^q \).

Proof. Consider \((1,1)\) as the base point, then \( \mu_X \) is the loop \( t \mapsto (e^{2i\pi t}, 1) \), while \( \mu_Y \) is the loop \( t \mapsto (1, e^{2i\pi t}) \). Let us pick a chart of \( \mathcal{C}_{(p,q)} \), say

\[
\frac{1}{p}(-1, q) \to \mathbb{C}^2 \\
[(x, y)] \longmapsto (x^p, x^q y).
\]

The base point in the chart is the class of \((1, 1)\); the equation of \( E \) is \( x = 0 \) and hence \( \mu_E \) is represented by \( t \mapsto [(e^{2i\pi t}, 1)] \). Hence, in \( \mathbb{C}^2 \) is represented by \( t \mapsto (e^{2ip\pi t}, e^{2iq\pi t}) \) and the result follows. \( \square \)

Proposition 2.5. In the aligned case, \( \pi_1(\mathbb{P}^2_\omega \setminus \mathcal{C}) \) is isomorphic to \( \mathbb{Z}/3(pq + r) \) if \( 2 \) divides \( pq\tilde{q} \) and to

\[(\ell, u \mid \ell^{pq+r} = 1, u^3 = \ell^2)\]

otherwise. This group is a central extension of \( \mathbb{Z}/2 \ast \mathbb{Z}/3 \) by a cyclic group of order \( \frac{pq+r}{2} \).

Proof. Following the proof of Corollary 2.3, the epimorphism described in (2.2) also holds in this case. Hence a presentation of \( \pi_1(\mathbb{P}^2_\omega \setminus \mathcal{C}) \) can be given once meridians of \( E_x, E_y, E_z \) are written in terms of the generators given in (2.1). Since the meridians \( \ell_x, \ell_y, \ell_z \) of the lines in the presentation (2.1) are homotopic to meridians close to the double points, by Lemma 2.4 we have that \( \ell_x, \ell_y, \ell_z \) are meridians of \( E_x, \ell_y^p \ell_z^q \) is a meridian of \( E_x, \) and \( \ell_y^p \ell_z^q \) is a meridian of \( E_y \). Hence a presentation of \( \pi_1(\mathbb{P}^2_\omega \setminus \mathcal{C}) \) can be obtained by adding the relations

\[(\ell, u \mid \ell^{pq+r} = 1, \ell^3 = 1)\]

to the presentation given in (2.1).

Finally, let us simplify this presentation. As a first step one can eliminate \( \ell_x \), since \( \ell_x = \ell_{y^{-1}} \). Also, choose \( \tilde{p}, \tilde{q} \in \mathbb{Z} \) such that \( \tilde{q} \tilde{q} - \tilde{p} \tilde{p} = 1 \). Note that \( \ell_y, \ell_z \) commute; then the remaining two relations in (2.4) become

\[
\ell_y^{-p} \ell_z^q = \ell_y^q \ell_z^p = 1 \implies \begin{cases} \ell_y^{pq+r} = \ell_z^{pq+r}, \\
\ell_y \ell_z = \ell_{y^{pq+r}+\tilde{p}^{pq+r}}. \end{cases}
\]

In fact, this is an equivalence. Let us denote \( \ell := \ell_z \) and \( u := \ell \). Since \([c, \ell_y \ell] = [c, \ell_y^{-1} \ell] = 1 \), one has

\[
1 = c \ell_y^{-1} c \ell_y \ell = c \ell_y^{-1} c \ell_y \ell \ell^{-1} c \ell \ell^{-1} c \ell \longleftarrow c \ell_y^{-1} (\ell_y \ell) \ell \ell^{-1} c \ell \iff c \ell = c \ell \ell^{-1} c \ell = c \ell \ell^{-1} = (c \ell)^3 \ell^{-2} \iff \ell^2 = u^3.
\]

Hence \( \pi_1(\mathbb{P}^2_\omega \setminus \mathcal{C}) \) admits a presentation

\[(\ell, u \mid \ell^{pq+r} = 1, [u, \ell^{pq+r+\tilde{q}^{pq+r}}] = 1, u^3 = \ell^2).\]
Note that, using $\ell^2 = u^3$, the relation $[u, \ell^{\hat{p} \hat{q} + \hat{q} \hat{p}}] = 1$ can be either eliminated or replaced by $[u, \ell] = 1$ depending on the parity of $\hat{p} \hat{q} + \hat{q} \hat{p}$. In addition, $\ell$ can also be eliminated using $\ell^{pq+r} = 1$ and $u^3 = \ell^2$ in case $pq + r$ is odd. In particular, if $\hat{p} \hat{q} + \hat{q} \hat{p}$ is even or $pq + r$ is odd, then (2.5) becomes an abelian group. Otherwise, one obtains the presentation (2.3).

It is immediate to verify that $\hat{p} \hat{q} + \hat{q} \hat{p}$ is odd and $pq + r$ even if and only if $pqr \hat{p} \hat{q}$ is odd, which ends the proof. □

**Corollary 2.6.** The derived subgroup $F$ of $\pi_1(\mathbb{P}^2_\omega \setminus \tilde{C})$ (in the non-abelian case) is the direct product of $\mathbb{Z}/(\mathbb{Z}/2)$ and a free group of rank 2. The characteristic polynomial of the action of the monodromy on $F \otimes \mathbb{C}$ is $t^2 - t + 1$.

### 2.2. A family of Zariski pairs of irreducible weighted projective curves.

Summarizing, let $\omega = (p, q, r)$ be pairwise coprime positive integers, and $\hat{p}, \hat{q}$ such that $\hat{p} \hat{q} + \hat{q} \hat{p} = pq + r$. Consider $C$ a smooth projective cubic and $\Phi_1$ (resp. $\Phi_2$) weighted Cremona transformation from $\mathbb{P}^2_\omega$ to $\mathbb{P}^2$ with respect to three tangent lines to $C$ at aligned (resp. non-aligned) inflection points.

**Theorem 2.7.** Under the conditions above, if $pqr \hat{p} \hat{q}$ is odd then $(\Phi_i^*(C), \Phi_i^*(C))$ is a Zariski pair of irreducible weighted projective curves of degree $3(pq + r)$ in $\mathbb{P}^2_\omega$.

**Proof.** Since both $\Phi_i$, $i = 1, 2$ are birational and $C$ is irreducible, then $\Phi_i^*(C)$, $i = 1, 2$ are both irreducible as well. Also, the singularities of $\Phi_i^*(C)$ are determined locally by the singularities of the union of $C$ and the lines used for the Cremona transformation $\Phi_i$. Hence, $\Phi_1^*(C)$ and $\Phi_2^*(C)$ have the same combinatorics. Finally, if $pqr \hat{p} \hat{q}$ is odd, then by Proposition 2.5 and Corollary 2.3 the fundamental groups of their complements are not isomorphic. This ends the proof. □

### 2.3. Cyclic covers and their irregularity à la Esnault-Viehweg.

The purpose of this section is to prove Theorem 2.7 via a generalization of the Alexander polynomial method, that is, the calculation of invariants associated with cyclic covers of the weighted projective plane ramified along the curves. In particular, we will calculate the dimension of the equivariant part of the cover with respect to the action of the deck transformation. This approach was originally used by Zariski [34] for sextics with six cusps in the projective plane. Later on, Libgober [22] and Esnault [13] made significant progress in this direction for cyclic covers and projective plane. Also Esnault-Viehweg [14] gave the tools that allowed the first author in [2], Sabbah [31], and Loeser-Vaquie [24] to find descriptions of the irregularity of cyclic covers. This approach was extended by Libgober [23] for abelian covers. The approach presented here is a generalization of Esnault-Viehweg’s and was developed by the authors for cyclic covers of surfaces with abelian quotient singularities and $\mathbb{Q}$-resolutions (or partial resolutions) in [4].

Let $\rho : \hat{X} \to \mathbb{P}^2_\omega$ be the cyclic cover of $\mathbb{P}^2_\omega$ ramified along a reduced curve $C$ of degree $d$. Consider $\hat{X} = \rho^{-1}(\mathbb{P}^2_\omega \setminus (C \cup \text{Sing} \mathbb{P}^2_\omega))$ the unramified part of the cover and let $\sigma : \hat{X} \to \hat{X}$ be a generator of the monodromy of the unramified cover.

Let $\pi : Y \to \mathbb{P}^2_\omega$ be a $\mathbb{Q}$-embedded resolution of $C$. For $P \in \text{Sing} \mathbb{C}$, let $\Gamma_P$ be the dual graph of the exceptional divisor of $\pi$ over $P$. For any $v$ vertex of $\Gamma_P$ we will denote by $E_v$ the associated exceptional divisor over $P$ and by $m_v$ (resp. $\nu_v - 1$) the coefficient of $E_v$ in the divisor $\pi^*C$ (in $K_\pi$, the relative canonical divisor).
The following result describes a method to recover the dimension of the different eigenspaces of $H^1(X;\mathbb{C})$ with respect to the monodromy action (or deck transformation) of the cover. A more general result can be found in [4, Theorem 4.4] for non-reduced divisors, but we state it here for covers associated with reduced divisors.

**Theorem 2.8** ([4, Theorem 4.4]). The dimension of the eigenspace of $\sigma^*$ acting on $H^1(X;\mathbb{C})$ for $k = \exp \frac{2\pi i k}{d}$, $0 < k < d$, equals $\dim \ker\pi^{(k)}$ where

$$
\pi^{(k)} : H^0\left(\mathbb{P}^2_\omega, \mathcal{O}_{\mathbb{P}^2_\omega}(kH + K_{\mathbb{P}^2_\omega})\right) \longrightarrow \bigoplus_{P \in \text{Sing} \mathcal{C}} \frac{\mathcal{O}_{\mathbb{P}^2_\omega,P}(kH + K_{\mathbb{P}^2_\omega})}{\mathcal{M}^{(k)}_{\mathcal{C},P}},
$$

is naturally defined given $H$ a divisor of degree 1, $K_{\mathbb{P}^2_\omega}$ the canonical divisor, and $\mathcal{M}^{(k)}_{\mathcal{C},P}$ the following $\mathcal{O}_{\mathbb{P}^2_\omega,P}$-module of quasi-adjunction

$$
\mathcal{M}^{(k)}_{\mathcal{C},P} := \left\{ g \in \mathcal{O}_{\mathbb{P}^2_\omega,P}(kH + K_{\mathbb{P}^2_\omega}) \mid \text{mult}_{E_w} \pi^* g > \frac{\nu_v}{d} \right\}.
$$

Our purpose will be to calculate $\dim \ker\pi^{(k)} + \dim \ker\pi^{(d-k)}$ for certain $k$ and $d = 3(pq + r)$ for the $d$-cyclic cover of the curves in the family presented in section 2.1.

Under the conditions of Theorem 2.7, that is, $pqr\bar{p}\bar{q}$ odd, let us consider the curve $\tilde{\mathcal{C}}_\lambda := \Phi_{\lambda,\bar{p},\bar{q}} \mathcal{C}_\lambda$ as defined in section 2.1. This curve has, in general, three singular points at the vertices $P_r, P_g, P_z$. Recall that for $\zeta := \exp \frac{2\pi i}{d}$ an easy computation given in Corollary 2.3 shows that the fundamental group of $\mathbb{P}^2_\omega \setminus \tilde{\mathcal{C}}_\lambda$ is abelian and hence the first cohomology group of any cyclic cover ramified along $\tilde{\mathcal{C}}_\lambda$ vanishes.

In order to understand the maps $\pi^{(k)}$ and the corresponding modules of quasi-adjunction $\mathcal{M}^{(k)}_{\mathcal{C},P}$ described in Theorem 2.8 one needs to study the singular points of $\tilde{\mathcal{C}} := \tilde{\mathcal{C}}_1$ in $\mathbb{P}^2_\omega$. Recall that $\text{Sing} \tilde{\mathcal{C}} \supseteq \{P_r, P_g, P_z\}$. More precisely, we will restrict our attention to the case $\frac{k}{d} = \frac{5}{6}$. Since $d = 3(pq + r)$, the degrees of the curves involved in $\pi^{(k)}$ is

$$
d_k = \frac{5d}{6} - (p + q + r) = \frac{5pq + 3r}{2} - (p + q).
$$

**Proposition 2.9.** A $\mathbb{Q}$-resolution of $(\tilde{\mathcal{C}}, P_z)$ has a dual graph with two vertices and its exceptional set is shown in Figure 2. Then $\mathcal{M}^{(k)}_{\mathcal{C},P_z}$, $k = \frac{5d}{6}$ is defined by the following conditions on germs $g \in \mathcal{O}_{\mathbb{P}^2_\omega,P_z}(d_k)$:

$$
\text{mult}_{E_{rz}} \pi^* g \geq \frac{5pq - 2(p + q)}{2r} + \frac{1}{2}, \quad \text{mult}_{E_w} \pi^* g \geq \frac{15pq - 6(p + q)}{2r} + 2.
$$

![Figure 2. A $\mathbb{Q}$-resolution of $(\tilde{\mathcal{C}}, P_z)$](image-url)
Hence the Newton polygon of this equation is a segment of slope $-i$, i.e., the total transform is $x^g$ given by the germs (2.6)

\[ ε \text{ component } E \text{ transform (through a smooth ambient point) has equation } x - y^3 = 0. \]

One can check that the multiplicity of the relative canonical divisor is $\nu_v = \frac{p+q}{r}$. To complete the resolution, we perform a (3,1)-blow-up, producing a new component $E_w$ for which $m_w = 3 \left(\frac{3pq}{r} + 1\right)$ and $\nu_w = \frac{3pq+2}{r}$.

By definition, the module of quasi-adjunction $M_{C,P_x}^{(k)}$ is a submodule of $O_x(d_k)$ satisfying

\[ \text{ mult } E_{v_z} \pi^* g > \frac{km_v}{d} - \frac{5pq - 2(p + q)}{2r}, \]

\[ \text{ mult } E_{v_z} \pi^* g > \frac{km_w}{d} - \frac{15pq + 3r - 6(p + q)}{2r}. \]

Finally, note that the class of $g$ imposes extra conditions, namely, if $H = V(h)$, $h \in O_x(1)$, then $\text{ mult } E_{v_z} \pi^*(g/h_{d_k})$ must be an integer for $v \in \{v_z, w\}$. Using (2.6) we can write $\text{ mult } E_{v_z} \pi^* g = \frac{5pq - 2(p + q)}{2r} + \varepsilon_{v_z}$, for some $\varepsilon_{v_z} \in \mathbb{Q}_{>0}$. Hence,

\[ \text{ mult } E_{v_z} \pi^* (g/h_{d_k}) = \frac{5pq - 2(p + q)}{2r} + \varepsilon_{v_z} - \frac{d_k}{r} = \varepsilon_{v_z} - \frac{3r}{2} \in \mathbb{Z}. \]

This implies $\varepsilon_{v_z} = \varepsilon_w = \frac{1}{2} + n_z, n_z \in \mathbb{Z}_{\geq 0}$. Analogously for $v = w$ one obtains

\[ \text{ mult } E_{v} \pi^* (g/h_{d_k}) = \frac{15pq + 3r - 6(p + q)}{2r} + \varepsilon_{w} - \frac{3d_k}{r} = \varepsilon_{w} + \frac{3}{2} \in \mathbb{Z}, \]

which implies $\varepsilon_w = \frac{1}{2} + n_w, n_w \in \mathbb{Z}_{\geq 0}$ and ends the proof. \( \square \)

**Proposition 2.10.** A $\mathbb{Q}$-resolution of $(\tilde{C}, P_x)$ is obtained with one weighted blow-up. Then $M_{C,P_x}^{(k)}$ is defined by the following condition on germs $g \in O_{P_x} (d_k)$:

\[ \text{ mult } E_{v_z} \pi^* g \geq \frac{3}{\gcd(3, p)} \cdot \frac{p + 3\tilde{q} - 2}{2p} + 1 \]

**Proof.** We follow the same ideas as in the proof of Proposition 2.9. Locally we work with the cubic $xy^2 - (y + z)^3 = 0$ (this cubic has a flex at $(0:1:-1)$). Then, the local equation of $\tilde{C}$ at $[1 : 0 : 0]_\omega$, regarded as $[(0, 0)] \in \frac{1}{p}(q, r) = \frac{1}{p}(1, \tilde{q})$, is $y^2z^3 - (z + y^3)^3 = 0$. We can change the coordinates (not affecting the action) where the equation becomes $y^3(z - y^3)^3 - z^3 = 0$. In these new coordinates the Newton polygon is non-degenerated and the singularity is resolved with a blowing-up with exceptional component $E_{v_z}$. Its weight is $(3, p + 3\tilde{q})$ if $\gcd(3, p) = 1$ and $(1, \frac{p}{3} + \tilde{q})$ otherwise.
The invariants are

\[ m_{v_x} = \frac{3p + 3\tilde{q}}{p}, \quad \nu_{v_x} = \frac{p + 3\tilde{q} + 3}{p} \]

Let us compute the quasi-adjunction module \( M_{^{(k)}D,P_x} \), as a submodule of \( \mathcal{O}_{x}(\tilde{d}_k) := \mathcal{O}_{\mathbb{P}^2,P_x}(\tilde{d}_k) \), where \( \tilde{d}_k \) is such that \( q\tilde{d}_k \equiv d_k \mod p \), which implies that \( \tilde{d}_k \equiv \frac{p + 3\tilde{q} - 2}{2} \). The condition for a germ \( g \in \mathcal{O}_{x}(\tilde{d}_k) \) to be in \( M_{^{(k)}D,P_x} \) is:

\[ \text{mult}_{E_{v_x}} \pi^{*}g > \frac{3p + 3\tilde{q} - 2}{2p}. \]

As above, the restriction given by \( g \in \mathcal{O}_{x}(\tilde{d}_k) \) leads to

\[ \text{mult}_{E_{v_x}} (\pi^{*}g)_{\tilde{d}_k} = \frac{3p + 3\tilde{q} - 2}{2p} + \varepsilon_{v_x} - \frac{3\tilde{d}_k}{p} = \varepsilon_{v_x} \in \mathbb{Z}. \]

Hence, \( \varepsilon_{v_x} \in \mathbb{Z}_{>0} \). \( \square \)

**Lemma 2.11.** Let \( g \) be a weighted homogeneous polynomial satisfying \( \deg_{\omega} g = \frac{5(pq + r) - 2(p + q + r)}{2} \) and \( g \in \ker \pi^{(k)} \). Then \( g(x, y, z) = x^{\frac{5}{2}(q + \tilde{p} - 2)}y^{\frac{3}{2}(p + \tilde{q} - 2)}f(x, y, z) \) where \( \deg_{\omega} f = pq + r \) and

\[ \text{mult}_{E_{v_x}} \pi^{*}f(x, y, 1) \geq \frac{pq}{r}, \quad \text{mult}_{E_{v_x}} \pi^{*}f(x, y, 1) \geq \frac{3pq}{r} + \frac{1}{2}, \]

\[ \text{mult}_{E_{v_x}} \pi^{*}f(1, y, z) \geq \frac{3\tilde{q}}{\gcd(3, p)p} + 1, \quad \text{mult}_{E_{v_x}} \pi^{*}f(x, 1, z) \geq \frac{3\tilde{p}}{\gcd(3, q)q} + 1. \]

**Proof.** The exponent of \( x^{\alpha} \) as a factor of \( g \) is given by the maximal value \( \alpha \in \mathbb{Z}_{>0} \), such that the divisor \( V(g) - \alpha Y \) is effective. Using the generalization of Noether’s multiplicity Theorem in this context (see [8, Thm. 4.3.(4)]) and Proposition 2.9 one obtains

\[ (V(g) \cdot Y)_{P_x} \geq \frac{(\text{mult}_{E_{v_x}} \pi^{*}g) \cdot (\text{mult}_{E_{v_x}} \pi^{*}y)r}{pq} \]

\[ \geq \frac{(5pq - 2(p + q))}{2pr} + \frac{1}{2p} = \frac{1}{2p} (5pq - 2p - 2q + r). \]

Hence,

\[ ((V(g) - \alpha Y) \cdot Y)_{P_x} \geq \frac{1}{2pr} (5pq - 2p - 2q(1 + \alpha) + r). \]

Analogously, at \( P_z \) one can use Proposition 2.10 to obtain

\[ (V(g) \cdot Y)_{P_z} \geq \left( 3\frac{p + 3\tilde{q} - 2}{2p} + 1 \right) \frac{1}{p + 3\tilde{q}} = \frac{5p + 9\tilde{q} - 6}{2p(p + 3\tilde{q})}, \]

regardless of the value of \( \gcd(3, p) \). Hence,

\[ ((V(g) - \alpha Y) \cdot Y)_{P_z} \geq \frac{5p + 9\tilde{q} - 6(1 + \alpha)}{2p(p + 3\tilde{q})}. \]
Then a global computation of the intersection multiplicity can be bounded by
\[
((V(g) - \alpha Y) \cdot Y)_{P_x} \geq ((V(g) - \alpha Y) \cdot Y)_{P_x} + ((V(g) - \alpha Y) \cdot Y)_{P_x}
\]
\[
\geq \frac{1}{2pr} (5pq - 2p - 2q(1 + \alpha) + r) + \frac{5p + 9q - 6 - 6\alpha}{2p(p + 3\bar{q})}
\]
\[
= \frac{1}{2pr} (5pq + r - 2(p + q + r) - 2\alpha q) - \frac{1}{p} + \frac{5p + 9q - 6 - 6\alpha}{2p(p + 3\bar{q})}
\]
\[
= \deg_x(V(g) - \alpha Y) \cdot \deg_y(Y) \frac{pq}{pq} + \frac{3(p + \bar{q} - 2 - 2\alpha)}{2p(p + 3\bar{q})}.
\]

By Bézout’s Theorem for weighted projective planes, \(\alpha = \frac{1}{2}(p + \bar{q} - 2)\). The same calculation applies to the divisor \(X\). This shows that \(g = x^\beta y^\alpha f(x, y, z)\), \(\beta = \frac{1}{2}(q + \bar{p} - 2)\) where
\[
\deg(f) = \frac{1}{2} (5(pq + r) - 2(p + q + r) - 2\alpha q - 2\beta p)
\]
\[
= \frac{1}{2} (5(pq + r) - 2(p + q + r) - (p + \bar{q} - 2)q - (q + \bar{p} - 2)p) = pq + r.
\]
The last equality follows from \(p\bar{p} + q\bar{q} = pq + r\).

The last part follows immediately from Propositions 2.9 and 2.10 and the additivity properties of the multiplicity. □

The local algebraic information obtained in this section will help us effectively study the morphism \(\pi^{(k)}\) described in Theorem 2.8. Let us use the notation introduced before Theorem 2.7 and at the beginning of this section, let us also denote by \(X_1\) (resp. \(X_2\)) the cyclic cover of \(\mathbb{P}^2\) of order \(d = 3(pq + r)\) ramified along \(\Phi^*(\mathcal{C})\) (resp. \(\Phi^*_2(\mathcal{C})\)). Finally, denote by \(L^{(k)}_i\) the invariant part of \(H^1(X_i; \mathcal{O}_{X_i})\) with respect to the action of the monodromy by multiplication by \(\exp \frac{2\pi i k}{d}\). Likewise, we denote by \(\pi^{(k)}\) the map described in Theorem 2.8 for the curve \(\Phi^*_i(\mathcal{C})\). The discussion above shows the following.

Proposition 2.12. If the product \(pqr\bar{p}\bar{q}\) is odd and \(\frac{k}{d} = \frac{\beta}{6}\), then \(\dim \ker \pi^{(k)}_1 = 0\) and \(\dim \ker \pi^{(k)}_2 = 1\).

Proof. By Lemma 2.11, the image by \(\Phi_i\) of \(V(f)\) is a line passing through the three flexes. The existence of this line for \(\Phi^*_2(\mathcal{C})\) but not for \(\Phi^*_1(\mathcal{C})\) ends the proof. □

Proof. Since the curves \(\Phi^*_1(\mathcal{C})\) and \(\Phi^*_2(\mathcal{C})\) have the same combinatorics and the same local type of singularities, the target space for \(\pi^{(k)}_1\) and \(\pi^{(k)}_2\) are the same. Therefore Proposition 2.12 implies \(\dim \ker \pi^{(k)}_1 = 1 + \dim \ker \pi^{(k)}_2\) for \(\frac{k}{d} = \frac{\beta}{6}\).

By Theorem 2.8, \(\dim \ker \pi^{(k)}_i = \dim L^{(k)}_i\) is a birational invariant of \(X_i\) and thus \(X_1 \nRightarrow X_2\), which implies that the fundamental groups of \(\mathbb{P}^2 \setminus \Phi^*_1(\mathcal{C})\) and \(\mathbb{P}^2 \setminus \Phi^*_2(\mathcal{C})\) are not isomorphic and thus \((\Phi^*_1(\mathcal{C}), \Phi^*_2(\mathcal{C}))\) forms a Zariski pair. □

3. SOME RATIONAL CUSPIDAL CURVES ON WEIGHTED PROJECTIVE PLANES

The study of rational cuspidal curves in \(\mathbb{P}^2\) is a classical subject. There is an extensive literature about them, and we recommend the beautiful paper [16] reviewing this topic, the most relevant conjectures, and bibliography. Two outstanding conjectures have been solved recently by Koras and Palka: the Nagata-Coolidge conjecture [20], that is, any rational cuspidal curve can be transported to a line
via a Cremona transformation and such curves can have at most four singular points [21]. There is a strong knowledge of such curves in \( \mathbb{P}^2 \) which have helped for the solution of these conjectures and other important problems, like the semigroup conjecture in [16], which was proven in [10].

Only one rational cuspidal curve in \( \mathbb{P}^2 \) possesses four cusps: a quintic curve with singular locus \( 3A_6 + 3A_2 \). There are many of them with three singular points, see [17] for an infinite family. The simplest one is the cuspidal quartic with three cusps. The standard Cremona transformation is a way to produce this curve, namely, the standard Cremona transformation of a smooth conic with respect to three of its tangent lines produces a tricuspidal quartic. Note that the blowing-up of the vertices does not affect the curve, and the blowing-downs produce the three cusps.

3.1. Rational cuspidal curves via weighted Cremona transformations.

In this section, we will study the strict transforms of this conic using the inverse of the weighted Cremona transformations introduced in §1.4. As a first stage, let us compute their fundamental groups. As in §2, let us start with the arrangement of a smooth conic and three lines, giving a presentation which contains suitable meridians for all the components.

Let \( C \) be a smooth conic and let \( X, Y, Z \) be three distinct tangent lines to \( C \). If the equations of the lines are \( x = 0, y = 0, z = 0 \), respectively, then the equation of \( C \) (up to a suitable change of coordinates) is

\[
x^2 + y^2 + z^2 - 2(yz + xz + xy) = 0.
\]

The fundamental group of the complement of the smooth conic and three tangent lines is the Artin group of the triangle \( T(4, 4, 2) \) (i.e. [9]). However, for our purposes, it is more suitable to use a presentation with a more geometrical interpretation. We present it here for completeness, but its proof is immediate using the classical Zariski-van Kampen method (as in [11]).

**Proposition 3.1.** The fundamental group of \( \mathbb{P}^2 \setminus (C \cup X \cup Y \cup Z) \) is isomorphic to

\[
\langle c, \ell_x, \ell_y, \ell_z \mid [\ell_x, \ell_y] = [\ell_x, \ell_z] = [\ell_y, \ell_z] = \ell_y c \ell_x c \ell_z = 1 \rangle.
\]

The element \( c \) is a meridian of \( C \), and \( \ell_x, \ell_y, \ell_z \) are meridians of \( X, Y, Z \), respectively. Moreover, \( (\ell_x, \ell_y) \) are meridians close to \([0 : 0 : 1]\), \( (\ell_x, \ell_z) \) are meridians close to \([0 : 1 : 0]\), and \( (\ell^c_y = c^{-1} \ell_y c, \ell_z) \) are meridians close to \([0 : 0 : 1]\).

Considering \( u := c \ell_z \), the above presentation of \( \pi_1(\mathbb{P}^2 \setminus (C \cup X \cup Y \cup Z)) \) can be alternatively written as:

\[
\langle u, \ell_x, \ell_y, \ell_z \mid [\ell_x, \ell_y] = [\ell_x, \ell_z] = [\ell^u_y, \ell_z] = u \ell_y u \ell_x \ell_z^{-1} = 1 \rangle.
\]

As in the section §2, fixing \( \omega, \tilde{p}, \tilde{q} \), we consider the birational map \( \Phi \) and we denote by \( \tilde{C} \) the strict transform of \( C \) by \( \Phi \).

**Proposition 3.2.** Let \( d := \gcd(p + 2\tilde{q}, q + 2\tilde{p}) \). Then \( \pi_1(\mathbb{P}^2_{\omega} \setminus \tilde{C}) \) is the semidirect group \( (\mathbb{Z}/d) \mathbb{A} \rtimes (\mathbb{Z}/2(pq + r))B \) where the action is given by \( B \cdot A = A^{-1} \). Hence the group has size \( 2d(pq + r) \) and is abelian if and only if \( d = 1 \).

**Remark 3.3.** Note that \( d \) is odd, since \( p, q \) cannot be simultaneously even. Moreover,

\[
\gcd(d, p) = \gcd(p, \tilde{q}, q + 2\tilde{p}) = \gcd(p, q\tilde{q}, q + 2\tilde{p}) = \gcd(p, r, q + 2\tilde{p}) = 1.
\]
and analogously \( \gcd(d, q) = 1 \). Note also
\[
\gcd(d, \hat{p}) = \gcd(d, \hat{p}, p + 2\hat{q}, q) = 1,
\]
and hence \( \gcd(d, \hat{q}) = 1 \). The following congruences can easily be checked:
\[
p\hat{p} \equiv -2\hat{p}\hat{q} \equiv q\hat{q} \mod d.
\]
Moreover,
\[
\gcd(d, pq + r) = \gcd(d, pq + r, q(p + 2\hat{q})) = \gcd(d, pq + r, p(q - 2\hat{p})) = \gcd(d, pq + r, 2p\hat{p}) = 1.
\]

**Proof of Proposition 3.2.** The presentation of \( \pi_1(\mathbb{P}^2_\omega \setminus \hat{C}) \) is obtained from (3.2) by adding the relations which kill the meridians of the exceptional divisors \( E_x, E_y, E_z \):
\[
(3.3) \quad \ell_x\ell_y = \ell_x^p\ell_y^q = u^{-1}\ell_y^p u\ell_x^q = 1.
\]

Let us first check that the abelianization of this quotient is \( \mathbb{Z}/2(pq + r) \). We will denote by \( \bullet \) the class of \( \bullet \) in the abelianization. Note that \( [\ell_z] = [u]^2 \); moreover, using Bézout’s identity and the equations in (3.3), both \( [\ell_x] \) and \( [\ell_y] \) can be expressed in terms of \( [\ell_z] \). Hence, the abelianization is cyclic. A presentation matrix in terms of the generators \([\ell_y], [u] \) is given by
\[
\begin{pmatrix}
q & 2\hat{p} \\
-p & 2\hat{q}
\end{pmatrix}
\]
whose determinant, \( 2(pq + r) \), is the size of the abelianization.

Let us study now the group itself. Note first that \( \ell_y \) can be eliminated from (3.3) as \( \ell_y = \ell_x^{-1} \). Let us check that \( u^2 \) is central. The last relation in (3.2) can be written as
\[
u^2 = \ell_x\ell_x^{-1}(u^{-1}\ell_x u).
\]

We deduce that \( u^2 \) commutes with \( \ell_x \), since it commutes with each factor; hence \( u^2 \) also commutes with \( u\ell_x u^{-1} \). Also note that
\[
\ell_x^p = \ell_x^{-\hat{q}}, \quad \ell_x^q = u\ell_x^p u^{-1} \implies \ell_x = \ell_x^{-\hat{p}} u\ell_x^{-\hat{q}} u^{-1}.
\]

Then \( u^2 \) commutes with \( \ell_x \) and it is central.

Using the last relation in (3.2) \( \ell_x \) can also be eliminated as \( \ell_x = u\ell_x^{-1} u\ell_x \). The presentation of the group becomes:
\[
\langle u, \ell_x | [\ell_x, u\ell_x^{-1} u] = [\ell_x^u, u\ell_x^{-1} u\ell_x] = \ell_x^p(u\ell_x^{-1} u\ell_x)^q = u^{-1}\ell_x^{-q} u(u\ell_x^{-1} u\ell_x)^\hat{p} = 1 \rangle
\]
which can be further simplified using the centrality of \( u^2 \):
\[
\langle u, \ell_x | [\ell_x, u^2] = [\ell_x, u\ell_x u^{-1}] = u^{2\hat{q}}\ell_x^{p+\hat{q}}(u\ell_x u^{-1})^{-\hat{q}} = u^{2\hat{p}}(u\ell_x u^{-1})^{-(q+\hat{p})}\ell_x^p = 1 \rangle
\]

The map \( \pi_1(\mathbb{P}^2_\omega \setminus \hat{C}) \to \mathbb{Z}/2 \) given by \( u \mapsto 1 \) and \( \ell_x, \ell_z \mapsto 0 \) is well defined. A presentation of its kernel \( K \) is obtained using Reidemeister-Schreier method. The generators are \( X_0 := \ell_x, X_1 := u\ell_x u^{-1}, \) and \( U := u^2 \). The first two relations imply that the group is abelian; the other relations yield:
\[
u^{2\hat{q}}\ell_x^{p+\hat{q}}(u\ell_x u^{-1})^{-\hat{q}} = 1 \implies U^{\hat{q}} X_1^{p+\hat{q}}X_0^{-\hat{q}} = U^{\hat{q}} X_0^{p+\hat{q}} X_1^{-\hat{q}} = 1,
\]
\[
u^{2\hat{p}}(u\ell_x u^{-1})^{-(q+\hat{p})}\ell_x^p = 1 \implies U^{\hat{p}} X_1^{-(q+\hat{p})}X_0^\hat{p} = U^{\hat{p}} X_0^{-(q+\hat{p})} X_1^\hat{p} = 1.
\]
Let us express these relations in a matrix (the rows represent the relations and
the columns stand for the generators). These relations become (recall
\(d = \gcd(p + 2\tilde{q}, q + 2\tilde{p})\)):

\[
\begin{pmatrix}
  d & -d & 0 \\
  -\tilde{q} & p + \tilde{q} & \tilde{q} \\
  \tilde{p} & -(q + \tilde{p}) & \tilde{p}
\end{pmatrix}
\begin{pmatrix}
  X_0 \\
  X_1
\end{pmatrix}
\sim
\begin{pmatrix}
  d & 0 & 0 \\
  -\tilde{q} & p & \tilde{q} \\
  \tilde{p} & -q & \tilde{p}
\end{pmatrix}
\begin{pmatrix}
  X_0X_1^{-1} \\
  X_1
\end{pmatrix}.
\]

The determinant of this matrix is \(d(pq + r)\) which is the size of \(K\); the greatest
common divisor of the 2-minors divides \(d\) and \(2p\tilde{p}\), i.e., it is 1. Hence, the group \(K\) is
cyclic of order \(d(pq + r)\). One also has that \(D := X_0X_1^{-1}\) is of order \(d\). Considering
the product of the second relation to the power \(q\) and the third relation to the
power \(p\) one obtains

\[1 = Dp\tilde{p} - q\tilde{q}U = pq + r\]

From the order in the abelianization, we deduce that \(U\) is of order \(pq + r\). Hence \(K\)
is the direct product of the cyclic group of order \(d\) generated by \(D\) and the cyclic
group of order \(pq + r\) generated by \(U\). The conjugation by \(u\) satisfies \(uUu^{-1} = U\)
and \(uDu^{-1} = D^{-1}\). The result follows. □

The singularities of these curves can be computed as in Propositions 2.9 and 2.10,
see Figure 3.

\[
\begin{align*}
E_{v_z}^2 &= -\frac{2pq + r}{pq} \\
E_{w}^2 &= -\frac{1}{2} \\
E_{v_x}^2 &= -\frac{p\gcd(2,q)}{2(p+2q)}(-p,2) \\
E_{v_y}^2 &= -\frac{p\gcd(2,q)}{2(q+2p)}(-q,2)
\end{align*}
\]

\textbf{Figure 3.} Singularities of the rational cuspidal curves

3.2. Rational cuspidal curves via weighted Kummer covers.

There is another simple way to produce rational cuspidal curves in weighted
projective planes from this arrangement of curves. It is quite simple but it will be
shown to be useful in the upcoming sections. Let \(a, b, c\) be pairwise coprime and
let \(\omega = (e_1, e_2, e_3), e_1 = bc, e_2 = ac, e_3 = ab\). Following §1.2 note that \(\eta = (1,1,1)\)
and thus there is an isomorphism \(\mathbb{P}_\omega^2 \to \mathbb{P}^2\) given by \([x : y : z]_\omega \mapsto [x^a : y^b : z^c]\). This
map gives a geometrical interpretation to the group

\[G_{a,b,c} := \pi_1(\mathbb{P}^2 \setminus (\mathcal{C} \cup X \cup Y \cup Z))/(\ell_x^a = \ell_y^b = \ell_z^c = 1)\].
as the orbifold group of $\mathbb{P}^2_{\omega}$ with respect to the curve $C \cup X \cup Y \cup Z$ and index $e(C) = 0, n(X) = a, n(Y) = b, n(Z) = c$ as defined in [6]. Briefly, if $X$ is a smooth projective surface, $D = D_1 \cup \ldots \cup D_s$ is a normal crossing union of smooth hypersurfaces, and $n_i := n(D_i) \in \mathbb{Z}_{\geq 0}$, then one can define the orbifold fundamental group $\pi_{1}^{\text{orb}}(X)$ of $X$ with respect to $D$ with indices $n_i$ the quotient of the group $\pi_1(X \setminus D)$ by the normal subgroup generated by $\gamma_i^{n_i}$, where $\gamma_i$ is a meridian of $D_i$. If $D$ is not normal crossing, then one resolves to a normal crossing divisor by blowing up points and defines the index at an exceptional divisor as the least common multiple of the indices of the components passing through the point (in this context we set $\text{lcm}(0, n) = 0$).

**Proposition 3.4.** The abelianization of the group $G_{a,b,c}$ is $\mathbb{Z}/2abc$. The group is abelian if and only if $1 \in \{a, b, c\}$.

**Proof.** The computation for the abelianization is straightforward. Assume $c = 1$, then

$$G_{a,b,1} = \langle \ell_x, u \mid [\ell_x, u\ell_x u] = 1, \ell_x^a = (u\ell_x u)^b = 1 \rangle = \langle \ell_x, v \mid [\ell_x, v^2] = 1, \ell_x^a = \ell_x^b v^b = 1 \rangle.$$  

Using Bézout’s identity we can express $\ell_x$ in terms of $v$ and the result follows.

For the case $a, b, c \neq 1$, let us consider the double cover of $\mathbb{P}^2$ ramified along $\mathcal{C}$. It is $\mathbb{P}^1 \times \mathbb{P}^1$ and the preimage $\tilde{\mathcal{C}}$ of $\mathcal{C}$ is the diagonal; the three lines are transformed in pairs of vertical-horizontal lines $X_\pm, Y_\pm, Z_\pm$ intersecting $\tilde{\mathcal{C}}$, see Figure 4.

**Figure 4.** Double cover ramified along $\mathcal{C}$

Note that the fundamental group of $\mathbb{P}^1 \times \mathbb{P}^1 \setminus (\tilde{\mathcal{C}} \cup X_+ \cup X_- \cup Y_+ \cup Y_- \cup Z_+ \cup Z_-)$ is isomorphic to the fundamental group of the projective complement of Ceva’s arrangement. The kernel $K_{a,b,c}$ of the map $G_{a,b,c} \to \mathbb{Z}/2, u \mapsto 1$ and the images of the other generators vanish, is an index 2 subgroup of $G_{a,b,c}$ and it is an orbifold fundamental group for the above configuration in $\mathbb{P}^1 \times \mathbb{P}^1$; the projection on each component induces orbifold morphisms onto $\mathbb{P}^1_{a,b,c}$ where $\mathbb{P}^1_{a,b,c}$ is an orbifold modelled on $\mathbb{P}^1$, with three quotient points of order $a, b, c$, respectively, and its orbifold fundamental group is isomorphic to $\langle \mu_1, \mu_2, \mu_3 \mid \mu_1^a = \mu_2^b = \mu_3^c = \mu_3 \mu_2 \mu_1 = 1 \rangle$, a triangle group. The combination of the two projections induces an epimorphism of $K_{a,b,c}$ onto $\pi_{1}^{\text{orb}}(\mathbb{P}^1_{a,b,c})$. The triangle is hyperbolic if $\{a, b, c\} \neq \{2, 3, 5\}$ and hence its group is infinite. If $(a, b, c) = (2, 3, 5)$, the triangle group is the alternating
3.3. A rational cuspidal curve with four cusps.

We end this section with a nice example in \( \mathbb{P}^2_{(1,1,2)} \), which is a curve of degree 6 with 4 ordinary cusps (as the maximum in \( \mathbb{P}^2 \)). Let us explain how to construct it via Cremona transformations. Let us start with a tricuspidal quartic \( \mathcal{C}_0 \); this curve, dual of the nodal cubic, has a bitangent line \( L \). Let \( P_0 \in \mathcal{C}_0 \cap L \); its blown-up produces a ruled surface \( \Sigma_1 \), where the negative section \( E \) is the exceptional component and \( \mathcal{C}_1 \), the strict transform of \( \mathcal{C}_0 \) has three cusps and one tangent fiber. Let us consider the Nagata transformation at \( \mathcal{C}_1 \cap E \); the result is \( \Sigma_2 \) and the blow-down of the negative section produces \( \mathbb{P}^2_{(1,1,2)} \). The strict transform \( \mathcal{C} \) of \( \mathcal{C}_1 \) is the desired curve.

**Proposition 3.5.** The fundamental group \( \pi_1(\mathbb{P}^2_{(1,1,2)} \backslash (\mathcal{C} \cup \{P_2\})) \) has a presentation

\[
\langle s, t, u | sts = tst, sus = usu, tut = utu, (stu)^2 = 1 \rangle.
\]

**Proof.** Following the construction it is the fundamental group of \( \Sigma_2 \setminus (\mathcal{C}_2 \cup E_2) \) where \( \mathcal{C}_2 \) is the strict transform of \( \mathcal{C} \) and \( E_2 \) is the negative section. The Zariski-Van Kampen method applied to the ruling yields the result. \( \square \)

3.4. Milnor fibers.

We have presented in §2 and in §3 several examples of irreducible quasi-projective curves such that their (maybe orbifold) fundamental groups are non-abelian. As a consequence their cones are quasi-homogeneous non-isolated surface singularities in \( \mathbb{C}^3 \) with non simply-connected Milnor fibers.

If \( F \in \mathbb{C}[x, y, z] \) is a homogeneous polynomial of degree \( d \), an important topological invariant is its Milnor fiber which is \( d\)-cyclic cover of the complement of the tangent cone \( \mathcal{C}_d \) in \( \mathbb{P}^2 \), defined by an epimorphism \( \pi_1(\mathbb{P}^2 \backslash \mathcal{C}_d) \to \mathbb{Z}/d \). If \( \omega \) is a weight and \( F \in \mathbb{C}[x, y, z] \) is an \( \omega\)-quasi-homogeneous polynomial of \( \omega\)-degree \( d \), its Milnor fiber \( F = 1 \) can also be recovered as a \( d\)-cyclic orbifold cover of \( \mathbb{P}^2 \setminus \mathcal{C}_d \) (the complement of the tangent \( \omega\)-quasi-cone) defined by an epimorphism \( \pi_1^{orb}(\mathbb{P}^2 \setminus \mathcal{C}_d) \to \mathbb{Z}/d \). If the elements of \( \omega \) are pairwise coprime and the vertices are in \( \mathcal{C}_d \), then the notions of \( \pi_1 \) and \( \pi_1^{orb} \) coincide. If it is not the case, the notion of orbifold fundamental groups apply.

The curves obtained via the Cremona transformation provide homogeneous singularities with simple topology and such that the Milnor fiber has non-trivial fundamental group. The following result is a direct consequence of Proposition 3.2.

**Proposition 3.6.** Let us assume take \( \omega = (p, q, r) \) (pairwise coprime) and \( \bar{p}, \bar{q} \) such that \( p\bar{q} = 1 \). Let

\[F_{\omega, p, q}(x, y, z) = y^{2p}z^2 + x^{2q}z^2 + x^{2\bar{p}}y^{2\bar{q}} - 2z(x^qy^pz + x^{q+\bar{p}}y^r + x^q(y^{p+\bar{q}})) = 0\]

be a homogeneous singularity. Then, the fundamental group of its Milnor fiber is cyclic of order \( \gcd(p + 2\bar{q}, q + 2\bar{p}) \).
More complicated fundamental groups can be obtained if we choose the orbifold variant. Let \( a, b, c > 1 \) pairwise coprime. Let \( \omega = (bc, ac, ab) \) be a weight, and let

\[
F_{a,b,c}(x,y,z) = x^{2a} + y^{2b} + z^{2c} - 2(x^ay^b + x^az^c + y^bz^c).
\]

As a direct consequence of Proposition 3.4 we obtain the following result.

**Proposition 3.7.** The fundamental group of the Milnor fiber of \( F_{a,b,c} = 0 \) is infinite and non-abelian.

### 4. Weighted Lé-Yomdin surface singularities

In this section we study the relationship between (weighted) projective plane curves and normal surface singularities whose link is a rational (or integral) homology sphere.

#### 4.1. The determinant of a normal surface singularity.

Let \((S,0)\) be a germ of normal surface singularity and let \( K \) be its link. It is well known that \( K \) is a graph manifold whose plumbing decorated graph is the dual graph \( \Gamma \) of a simple normal crossing resolution. Each vertex \( v \) of \( \Gamma \) is decorated with two numbers \((g_v, e_v)\), where \( g_v \) is the genus of the corresponding irreducible component \( E_v \) and \( e_v \) is its self-intersection. Let \( A \) be the intersection matrix of the graph; recall that \( A \) is negative definite. The following result is classical.

**Proposition 4.1.** The free part of \( H_1(K;\mathbb{Z}) \) has rank \( 2 \sum_v g_v + \text{Rank} H_1(\Gamma;\mathbb{Z}) \) and its torsion part is isomorphic to \( \text{coker} A \). In particular, its cardinality is \( \det(-A) \).

As a consequence (which can easily be proven), the determinant \( \det(-A) \) does not depend on the resolution. This justifies the definition of the determinant of a normal surface singularity.

**Definition 4.2.** The **determinant** \( \det S \) of a normal surface singularity \( S \) is defined as \( \det(-A) \), where \( A \) is the intersection matrix of any resolution of \( S \).

As a consequence, one has the following combinatorial criteria to detect rational (resp. integral) homology sphere singularities, that is, surface singularities whose link is a rational (resp. integral) homology sphere.

**Corollary 4.3.** The surface singularity \( S \) is a rational (resp. integral) homology sphere if and only if all \( g_v \)'s vanish and \( \Gamma \) is a tree (resp. and \( \det S = 1 \)).

#### 4.2. Superisolated and Lé-Yomdin singularities.

In [16], the authors relate hypersurface singularities whose link is a rational homology sphere with rational cuspidal curves using superisolated singularities. In our search for more examples of rational (or integral) surface singularities, a generalization of this method will be discussed here. For the sake of completeness we present the classical result.

**Definition 4.4.** Let \((S,0) \subset (\mathbb{C}^3,0)\) be the germ of a hypersurface singularity with equation \( F = f_d + f_{d+k} + \ldots \), where the previous decomposition is the decomposition in homogeneous parts. Assume \( f_d \neq 0, k > 0 \). Let \( C_m := V(f_m) \) denote the projective zero locus in \( \mathbb{P}^2 \) of the homogeneous polynomial \( f_m \). We say that \( S \) is a **Lé-Yomdin singularity** if \( \text{Sing}(C_d) \cap C_{d+k} = \emptyset \). If \( k = 1 \), \( S \) is called a **superisolated** singularity.
Superisolated singularities were introduced by Luengo [25]: they can be solved by one blow-up. In [26], the authors show that the link of a superisolated singularity is a rational homology sphere if and only if all the irreducible components of $C_d$ are cuspidal rational and if the curve is reducible they only intersect at one point. Besides the smooth case, no other one provides an integral homology sphere as can be deduced from the following result in [25]. We reproduce the proof since it will be generalized for other classes of singularities.

**Proposition 4.5** ([25]). Let $S$ be a superisolated singularity with tangent cone $C_d$ of degree $d$. Let $\Pi: \hat{C}^3 \to C^3$ be the blow-up of $0 \in S \subset C^3$ and $\pi: \hat{S} \to S$ the restriction of $\Pi$ to the strict transform of $S$. If $E \cong \mathbb{P}^2$ is the exceptional divisor of $\Pi$, then the exceptional divisor of $\pi$ is $C_d = E \cap \hat{S}$.

Moreover, if $C_{d,1}, \ldots, C_{d,s}$ denote the irreducible components of $C_d$, $d_i := \deg C_{d,i}$. Then,

$$(C_{d,i} \cdot C_{d,i})_S = -d_i(d - d_i + 1), \quad (C_{d,i} \cdot C_{d,j})_S \cdot P = (C_{d,i} \cdot C_{d,j})_{\mathbb{P}^2} \cdot P, \ i \neq j, \ P \in \operatorname{Sing} C_d.$$  

**Proof.** Let us assume that $[0:0:1] \in \operatorname{Sing} C_d$. We can fix the usual chart of the blowing-up. Assume that $S = \{F = 0\}$, where $F = f_d + f_{d+1} + \ldots$; in the chart $(x, y, z) \mapsto (xz, yz, z)$ and $E = \{z = 0\}$, $\hat{S} = \{f_d(x, y, 1) + z(f_{d+1}(x, y, 1) + \ldots) = 0\}$, i.e., $C_d = \{z = f_d(x, y, 1) = 0\}$. In the neighborhood of $P$, $(E, C_d)$ and $(\hat{S}, C_d)$ are isomorphic. We deduce that for $i \neq j$, $(C_{d,i} \cdot C_{d,j})_S \cdot P = (C_{d,i} \cdot C_{d,j})_{\mathbb{P}^2} \cdot P.$

The surfaces $E$ and $\hat{S}$ are generically transversal, namely outside $\operatorname{Sing} C_d$. The Euler class $e(E) = -L$, where $L$ is a line in $E$. Then,

$$(C_{d,i} \cdot C_{d,i})_S = (e(E) \cdot C_{d,i})_{\mathbb{P}^2} = -d_i.$$  

Also

$$(C_{d} \cdot C_{d,i})_S = (C_{d,i} \cdot C_{d,i})_S + \sum_{j \neq i} (C_{d,j} \cdot C_{d,i})_S = (C_{d,i} \cdot C_{d,i})_S + \sum_{j \neq i} (C_{d,j} \cdot C_{d,i})_{\mathbb{P}^2} = (C_{d,i} \cdot C_{d,i})_S + d_i(d - d_i),$$  

and the result follows. 

Although $\pi$ is not necessarily a resolution with normal crossings, $\det S$ can be recovered from it using its intersection matrix; it is a classical result which will follow from a later proposition.

**Corollary 4.6.** If $S$ is as above, then

$$\det S = (d + 1)^{s-1} \cdot d_1 \cdots d_s.$$  

In particular, if $C_d$ is irreducible, then $\det S = d$.

**Proof.** By Proposition 4.5, the diagonal terms of the intersection matrix for $\pi$ equal $-d_i(d - d_i + 1)$ and the non-diagonal terms are $d_i \cdot d_j$. Replacing the first row by the sum of all rows, one obtains $-(d_1, \ldots, d_s)$. If we add the new first row multiplied by $d_i$ times the $i^{th}$-row ($i > 1$), all the non-diagonal terms vanish and the diagonal term becomes $-d_i(d - d_i + 1).$

For Lê-Yomdin singularities we follow the same strategy. If $S = \{F = 0\}$ with $F = f_d + f_{d+k} + \ldots$, and we keep the notation above, the main difference is that $\hat{S}$ is no longer smooth, in general. If $P \in \operatorname{Sing} C_d$, then the local equation of $\hat{S}$ at $P$ is $z^k - f(x, y) = 0$ where $f(x, y) = 0$ is the local equation of $C_d$ at $P$. Intersection
theory can be used also in normal surfaces, see [29] for definition and [8] for useful tips. As the following result shows the intersection form of a partial resolution is also useful.

**Lemma 4.7.** Let \((S, 0)\) be a normal surface singularity and let \(\pi : (X, D) \to (S, 0)\) be a proper birational morphism which is an isomorphism outside \(D = \pi^{-1}(0)\) on the normal surface \(X\). Let \(A\) be the intersection matrix for \(D\). Then,

\[
\det S = \det(-A) \prod_{P \in D} \det(X, P).
\]

**Proof.** Note first that the product in the formula is finite since only a finite number of singular points may arise. Let \(\sigma : (Y, E) \to (X, D)\) be a resolution of the singularities of \(X\). Let \(B\) the intersection matrix of \(E\). Instead of expressing this matrix in terms of the irreducible components of \(E\), we replace the strict transforms of the components of \(D\) by their total transforms.

Then, \(B\) is replaced by a matrix \(\tilde{B}\), with the same determinant, which is a diagonal sum of \(A\) and the intersection matrices of the singular points. Then,

\[
\det S = \det(-B) = \det(-\tilde{B}) = \det(-A) \prod_{P \in \Sing X} \det(X, P). \qed
\]

**Proposition 4.8.** Let \(S\) be a \(k\)-Lé-Yomdin singularity with tangent cone \(C_d\) of degree \(d\). With the notations of Proposition 4.5,

\[
(C_{d,i} \cdot C_{d,j})_S = \frac{d_i(d - d_i + k)}{k}, \quad (C_{d,i} \cdot C_{d,j})_{S,P} = \frac{(C_{d,i} \cdot C_{d,j})_{P,P}}{k}, \quad i \neq j, P \in \Sing C_d.
\]

**Proof.** We follow the guidelines of the proof of Proposition 4.5. Note that it is not true any more that in the neighborhood of \(P \in \Sing C_d\) the germs \((E, C_d)\) and \((\hat{S}, C_d)\) are isomorphic. However, the projection \(\rho(x, y, z) := (x, y)\) restricts to a \(k:1\) proper map \((\hat{S}, C_d) \to (E, C_d)\). Since \(\pi^*(\pi_* C_{d,i}) = kC_{d,i}\) we have that for \(i \neq j\)

\[
(C_{d,i} \cdot C_{d,j})_{S,P} = \frac{1}{k} (\pi^* \pi_* C_{d,i} \cdot \pi^* \pi_* C_{d,j})_{S,P} = \frac{1}{k} (C_{d,i} \cdot C_{d,j})_{P,P}.
\]

For the self-intersections we apply the same ideas:

\[
(C_d \cdot C_{d,i})_S = (\rho(E) \cdot C_{d,i})_{P,P} = -d_i
\]

\[
(C_d \cdot C_{d,i})_S = (C_{d,i} \cdot C_d)_{S} + \sum_{j \neq i} (C_{d,j} \cdot C_{d,i})_{S} = (C_{d,i} \cdot C_{d,i})_{S} + \frac{d_i(d - d_i)}{k},
\]

and the result follows. \(\qed\)

A similar proof to the one of Corollary 4.6 provides the following result.

**Corollary 4.9.** If \(S\) is a \(k\)-Lé-Yomdin as above then

\[
\det S = d_1 \cdot \ldots \cdot d_s \cdot \left(\frac{d + k}{k}\right)^{s-1} \prod_{P \in \Sing C_d} \det S_{P,k},
\]

where

\[
S_{P,k} = \{z^k = f_P(x, y) \mid P \in \Sing C_d\},
\]

and \(f_P(x, y) = 0\) is a local equation of \(C_d\) at \(P\).

In particular, if \(C_d\) is smooth, then \(\det S = d\).
Example 4.10. Let $S_k$ be the singularity $\{z^k = x^2 + y^2\}$, then $\det S_k = k$. Denote by $T_k$ the singularity $\{z^k = x^2 + y^3\}$, then we have

$$\det T_k = \begin{cases} 1 & \text{if } \gcd(k, 6) = 1, 6 \\ 3 & \text{if } \gcd(k, 6) = 2 \\ 2 & \text{if } \gcd(k, 6) = 3. \end{cases}$$

Note that for $\gcd(k, 6) = 6$, the graph has one vertex with genus 1.

Conjecture 1. Let $C : f(x, y) = 0$ be a germ of a reduced plane curve singularity, and let $S_k : z^k = f(x, y)$ be a cyclic germ of surface. Let $N$ be the order of the semisimple factor of the monodromy of $C$. Then $\det S_k$ is a quasi-polynomial in $k$ of period $N$.

Proposition 4.11. A $k$-Lê-Yomdin singularity with tangent cone $C_d$ has as link a rational homology sphere if and only if $C_d$ is a union of rational cuspidal curves with only one intersection point and the links of the $k$-cyclic singularities associated to the singular points of $C_d$ have also a rational homology sphere as a link.

The proof of this proposition is a direct consequence of the previous result. The fact that we do not have a closed formula for the determinant of a cyclic germ makes difficult to ensure that Lê-Yomdin singularities do not provide integral homology sphere links, but our experimentation leads to this conjecture.

Conjecture 2. No $k$-Lê-Yomdin singularity $k > 1$ has an integral homology sphere link.

4.3. Weighted Lê-Yomdin singularities.

We are going to generalize these families of singularities using weighted homogeneous curves. We use the notation $\omega, \eta, \varepsilon, \varepsilon_1, \varepsilon_2, \varepsilon_3$ etc. introduced in §1. The following notion of weighted Lê-Yomdin singularity was introduced in [7].

Definition 4.12. A hypersurface $(S, 0) := \{F = 0\}$ is an $(\omega, k)$-weighted Lê-Yomdin singularity if the following holds. Let $F := f_d + f_{d+k} + \ldots$ be the decomposition in $\omega$-weighted homogeneous forms, then $\operatorname{Jac}(f_d) \cap V(f_{d+k}) = \emptyset$.

In order to relate geometrically this definition with the definition of superisolated and Lê-Yomdin singularities, let us consider the weighted blow-up $\Pi_\omega : \mathbb{C}^3_\omega \to \mathbb{C}^3$. We have stratified in §1 the exceptional divisor $E_\omega \cong \mathbb{P}^2_\omega \cong \mathbb{P}^2_\eta$ following the singularities of $\mathbb{C}^3_\omega$. We deal with two curves $C_d, C_{d+k} \subset E_\omega$. In general $f_d(x, y, z) = x^{\varepsilon_x}y^{\varepsilon_y}z^{\varepsilon_z}g(x^a, y^b, z^c)$, where $\varepsilon_x, \varepsilon_y, \varepsilon_z \in \{0, 1\}$ and $g$ is $\eta$-weighted homogeneous of degree $\frac{d-\varepsilon_x-\varepsilon_y-\varepsilon_z}{abc}$. If we see this curve in $\mathbb{P}^2_\eta$ its equation is $x^{\varepsilon_x}y^{\varepsilon_y}z^{\varepsilon_z}g(x, y, z, \eta) = 0$.

Proposition 4.13. Let $S = \{F = 0\}$ be an $(\omega, k)$-weighted Lê-Yomdin singularity with $\omega$-quasi-tangent cone $C_d = \{f_d = 0\}$. Let $\Pi_\omega$ be the $\omega$-blow-up, $E \cong \mathbb{P}^2_\omega \cong \mathbb{P}^2_\eta$ is the exceptional divisor and $\hat{S}$ is the strict transform (and a partial resolution) of $S$. Recall the stratification of $E_\omega = \mathcal{P} \cup \mathcal{L} \cup \mathcal{T}$ as given in section 1.3. The structure of $\hat{S}$ along $P \in C_d = E_\omega \cap \hat{S}$ is as follows:

1. $P \in \mathcal{T}$.
2. If $P \notin \operatorname{Sing}C_d$ then $\hat{S}$ is smooth at $P$ and $E \cap P \hat{S}$.
If \( P \in \text{Sing} \mathcal{C}_d \) then \( P \notin \mathcal{C}_{d+k} \). There are local coordinates \( U, V, W \) centered at \( P \) such that \( E_\omega = \{ W = 0 \} \), \( \mathcal{C}_d = \{ W = g(U, V) = 0 \} \) and \( \tilde{S} = \{ W^k = g(U, V) \} \); in particular \( \tilde{S} \) is smooth at \( P \) if and only if \( k = 1 \) (but it is not transversal to \( E_\omega \)).

(2) \( P \in \mathcal{L}_y \) (a similar statement holds for \( \mathcal{L}_x, \mathcal{L}_z \)).

(2.a) If \( \mathcal{C}_d \) is transversal to \( C \) at \( P \) then \( (\tilde{S}, P) \cong \frac{1}{r}(-1, q) \). In the quotient ambient space the situation is similar to (1.a).

(2.b) If \( \mathcal{C}_d \) coincides with \( \tilde{Y} \) near \( P \) then \( \tilde{S} \) is smooth at \( P \). In the quotient ambient space the situation is similar to (1.a).

(2.c) If \( \mathcal{C}_d \nmid \tilde{Y} \), i.e., \( f_d(x + t, y, 1) \) is of order \( > 1 \) (\( P = [t : 0 : 1] \)) then \( P \notin \mathcal{C}_{d+k} \). The germ \((\tilde{S}, P)\) is isomorphic to \( z^k = f_d(x + t, y, 1) \) in \( \frac{1}{r}(0, q, -1) \), where \( z = 0 \) is the equation of \( E_\omega \).

(3) \( P = P_\ast \) (a similar statement holds for \( P_x, P_y \)).

(3.a) If \( \mathcal{C}_d \) is extremely quasi-smooth at \( P \) (i.e. \( f_d(x, y, 1) \) is of order 1) the situation is as in (1.a) replacing the ambient smooth space by \( \frac{1}{r}(y, q, -1) \). Let \( h_1(y, x) \) be the linear part of \( f_d(x, y, 1) \).

(3.a.i) If \( h_1(y, x) \) is proportional to \( x \), then \((\tilde{S}, P) \cong \frac{1}{r}(y, -1) \).

(3.a.ii) If \( h_1(y, x) \) is proportional to \( y \), then \((\tilde{S}, P) \cong \frac{1}{r}(q, -1) \).

(3.a.iii) For the other cases, \( p \equiv q \mod r \) and the above cases coincide.

(3.b) If \( \mathcal{C}_d \) is not extremely quasi-smooth at \( P \) (i.e., \( f_d(x, y, 1) \) is of order > 1), then \( P \notin \mathcal{C}_{d+k} \) and \( d+k \equiv 0 \mod r \). The germ \((\tilde{S}, P)\) is isomorphic to \( z^k = f_d(x, y, 1) \) in \( \frac{1}{r}(y, q, -1) \), where \( z = 0 \) is the equation of \( E_\omega \).

**Proof.** In order to simplify the computations, a suitable choice of \( \omega \) allows to fix \( P = [0 : 0 : 1] \). The general situation is as follows. The total transform of \( S \) is equal to \( \tilde{S} + dE_\omega \). The equations in the chart are:

\[
z^d(f_d(x, y, 1) + z^k \underbrace{f_{d+k}(x, y, 1 + \ldots)}_{q(x, y)}) = 0.
\]

Let us denote by (*) the condition \( f_d(x, y, 1) \) has multiplicity 1 at \((0,0)\).

\[
\begin{align*}
P \notin \mathcal{C}_{d+k} & \iff q(x, y) \text{ is a unit,} \\
P \in \mathcal{C}_{d+k} & \implies (*)
\end{align*}
\]

We change the coordinates \( x, y, z \) by \( x_1, z_1 \) where \( y_1 = y \),

\[
(x_1, z_1) = \begin{cases} (x, zp(x, y)^{\frac{1}{r}}) & \text{if not (*)} \\
(f_d(x, y, 1) + z^kq(x, y), z) & \text{if (*)}
\end{cases}
\]

Up to a permutation of coordinates, the action of \( \mu_r \) on \((x_1, y_1, z_1)\) reads as in \((x, y, z)\). If \( P \notin \mathcal{C}_{d+k} \) then \( d+k \equiv 0 \mod r \).

In these coordinates \( E_\omega : z_1 = 0 \) and \( \mathcal{C}_d : W = g(x_1, y_1) = 0 \), where

\[
g(x_1, y_1) = \begin{cases} f_d(x_1, y_1, 1) & \text{if not (*)} \\
x_1 & \text{if (*)}
\end{cases}
\]

The local equations for \( dE_\omega + \tilde{S} \) are \( z_1^d(z_1^k + g) = 0 \). If (*) holds, both look like two surfaces in a quotient ambient space whose preimages in \( \mathbb{C}^3 \) are smooth and transversal.
When \( P \in T \) one can choose \( \omega = (1,1,1) \) and \( (1) \) follows. For the case \( P \in L_y \) one can choose \( \omega = (b,q,b) \) and show \( (2) \). If \( P = P_z \), then the choice \( \omega = (p,q,r) \) applies and the case \( (3) \) also follows. \( \square \)

The divisor \( C_d \) has an irreducible decomposition in \( s + \varepsilon_x + \varepsilon_y + \varepsilon_z \) components \( \varepsilon_x X + \varepsilon_y Y + \varepsilon_z Z + \hat{C}_d \), where \( \hat{C}_d = \sum_{i=1}^s C_{d,i} \) and \( \varepsilon_x, \varepsilon_y, \varepsilon_z \in \{0,1\} \). Recall that \( abc \) divides \( \deg \).

Consider the stratification of a weighted projective plane as above. We call a curve in a weighted projective plane _stratified smooth_ if it is smooth, intersects the axes transversally and does not contain the vertices.

**Proposition 4.14.** Let \( S \) be an \((\omega,k)\)-weighted Lé-Yomdin singularity with quasi-tangent cone \( C_d \subset \mathbb{P}^2_\omega \) of degree \( d \). With the notation of Proposition 4.8:

1. \( (C_{d,i} \cdot C_{d,j})_S = -\frac{d(d-d_i+k)}{kpqr} \).
2. If \( C_i = X \), then \( (X \cdot X)_S = -\frac{a^2d(d-p+k)}{kpqr} \). A similar formula holds for \( Y, Z \).
3. If \( i \neq j \), then \( (C_i \cdot C_j)_S = -\frac{(C_i)_{\omega \cdot \omega \cdot \omega \cdot \omega}}{kpqr} \).
4. If \( C_{d,j} \) is replaced by \( X \), then \( (3) \) is multiplied by \( a \). A similar formula holds for \( Y, Z \).
5. \( (X \cdot Y)^{S, P} = \frac{ab}{k^2} \). A similar formula holds for the other pairs involving \( X, Y, Z \).

**Proof.** We follow the ideas in the proofs of Propositions 4.5 and 4.8. Note that \( e(E_\omega) \) is a divisor of \( \mathbb{P}^2_\omega \) of degree \( 1 \) and that Bézout’s Theorem for the \( \omega \)-projective plane states that the sum of the intersection numbers of two divisors equals the product of the degrees divided by \( pqr \).

When \( a, b, c = 1 \) the proof follows exactly the same ideas. When at least one of them is greater than \( 1 \), both \( E_\omega \) and \( \hat{S} \) intersect at components which may be singular in the ambient space. For \( (2), (4) \) and \( (5) \) we have to make the distinction between \( E_\omega \cong \mathbb{P}^2_\omega, \hat{S} \subset \mathbb{C}^2_\omega \) as orbifolds and \( E_\omega \cong \mathbb{P}^2_\omega, \hat{S} \) as abstract surfaces. Recall that if \( \sigma : \mathbb{P}^2_\omega \to \mathbb{P}^2_\eta \) is the standard isomorphism, then \( \sigma^*(X) = aX \). \( \square \)

**Corollary 4.15.** If \( S \) is an \((\omega,k)\)-weighted Lé-Yomdin as above and \( A \) is the intersection matrix of the blowing-up, then

\[
\det S = a^{2s} \cdot b^{2s} \cdot c^{2s} \cdot d_1 \cdot \ldots \cdot d_h \cdot \left( \frac{d+k}{kpqr} \right)^{s-1} \prod_{p \in S} \det(\hat{S}_{k,p}),
\]

where \( \hat{S}_{k,p} \) is the surface singularity at \( P \) as described in Proposition 4.13.

In particular, if \( C_d \) is stratified smooth, then \( \det S = \frac{d^h}{pq} \).

5. Normal surface singularities with rational homology sphere links

In this section we will use the results and strategies presented in section 4 in order to exhibit examples of weighted Lé-Yomdin singularities whose links are rational homology spheres, generalizing the strategy in [16]. We will be using Proposition 4.11 in the context of weighted Lé-Yomdin singularities.
5.1. Brieskorn-Pham singularities.

We will interpret these singularities as Lé-Yomdin singularities and study their \( \mathbb{Q} \)-resolution graph. Consider \( \omega_0 = (a, b, c) \) and the Brieskorn-Pham singularity
\[
S = \{ F_{\omega_0} = x^a + y^b + z^c = 0 \} \subset (\mathbb{C}^3, 0),
\]
where \( a, b, c \) are not assumed to be pairwise coprime.

Denote by \( e := \gcd(a, b, c) \), \( (e_1, e_2, e_3) = (\frac{a}{e}, \frac{b}{e}, \frac{c}{e}) \) and \( d_k = \gcd(e_i, e_j) \), where \( \{i, j, k\} = \{1, 2, 3\} \). Note that \( \alpha_i := \frac{e_i}{d_k} \in \mathbb{Z}_{>0} \) are pairwise coprime. If
\[
\omega = \left( \frac{e_1 e_2 e_3}{d_1 d_2 d_3}, \frac{e_1 e_3}{d_1 d_2 d_3}, \frac{e_2 e_3}{d_1 d_2 d_3} \right) = (d_1 \alpha_2 \alpha_3, d_2 \alpha_1 \alpha_3, d_3 \alpha_1 \alpha_2),
\]
then \( F_{\omega_0}(x, y, z) \) is an \( \omega \)-weighted homogeneous polynomial of degree \( d := \frac{abc}{e^2 d_1 d_2 d_3} \) and hence \( S \) can be viewed as an \((\omega, k)\)-weighted Lé-Yomdin singularity for any \( k \geq 1 \). Following the general construction,
\( f_{\tilde{d}} = F_{\omega_0}(x, y, z) = g(x^{\alpha_1}, y^{\alpha_2}, z^{\alpha_3}) = 0 \)
can be considered a curve in \( \mathbb{P}^2_{\omega} \cong \mathbb{P}^2_{\omega_0} \) for \( \eta = (d_1, d_2, d_3) \) of \( \eta \)-degree \( d_\eta = ed_1 d_2 d_3 \) given by the equation \( g(x, y, z) = x^{ed_2 d_3} + x^{ed_1 d_3} + z^{ed_1 d_2} = 0 \). Its genus is
\[
\frac{d_\eta(d_\eta - |\eta|)}{2|\eta|} + 1 = \frac{e^2 d_1 d_2 d_3 - e(d_1 + d_2 + d_3) + 2}{2}.
\]
Since the curve \( C_{\tilde{d}} \) is transversal to the axes we obtain that the exceptional locus of \( \tilde{S} \) has (in the intersection with the axes) \( ed_\eta \) cyclic points of order \( \alpha_i \). The determinant of the singularity is
\[
d = \frac{d_1 d_2 d_3 (\alpha_1 \alpha_2 \alpha_3)^2}{(\alpha_1 d_1, d_2, d_3)}^e = \alpha_1^{ed_1 - 1} \alpha_2^{ed_2 - 1} \alpha_3^{ed_3 - 1}.
\]
As a consequence of this discussion one obtains the following.

**Proposition 5.1.** The Brieskorn-Pham singularity \( S = \{ F_{\omega_0} = x^a + y^b + z^c = 0 \} \subset (\mathbb{C}^3, 0) \) is a rational homology sphere singularity if and only if either \( d_1 = d_2 = d_3 = 1, e = 2 \) or \( d_i = d_j = e = 1 \) for some \( i \neq j \).

Moreover, it is an integral homology sphere if and only if the exponents are pairwise coprime.

5.2. Examples coming from Cremona transformations and Kummer covers.

The purpose of this section is provide more candidates to surface singularities with rational homology sphere links by applying the techniques used in §3. In particular, we will start with the strict transforms of the conic by the Cremona transformations.

In order to do so one needs \( \omega := (p, q, r) \) pairwise coprime, and \( p, q, r \in \mathbb{Z}_{>0} \) such that \( pq + r = pp + qq \), the weighted homogeneous polynomial
\[
f_{\omega}(x, y, z) = f(y^p z, x^q z, x^r y^q)
\]
has \( \omega \)-degree \( 2(pq + r) \) and defines a rational curve in \( \mathbb{P}^2_{\omega} \) which is smooth outside the vertices. Assume for simplicity that \( pq + r < pqr \). Hence for any generic quasi-homogeneous polynomial \( g_\omega(x, y, z) \) of degree \( 2pq r \), \( F := f_{\omega} + g_\omega \) defines an \((\omega, k)\)-weighted Lé-Yomdin singularity, for \( k = 2(pqr - pq - r) \). A partial resolution of this singularity has an exceptional locus which is a rational curve with three singular points. In most cases the link of this singularity is a rational homology sphere. For simplicity, we will prove it in a special case.
Proposition 5.2. With the previous notation, take \( p = 1, \tilde{p} = r, \) and \( \tilde{q} = 1. \) Then, for any \( q, r > 1 \) satisfying \( \gcd(3, k) = \gcd(3, qr - q - r) = 1 \) and a generic \( g_\omega \), the equation \( \{ F = f_\omega + g_\omega = 0 \} \subset \mathbb{C}^3 \) defines a surface singularity with a rational homology sphere link.

Proof. We study the strict transform of this singularity at \( P_z, P_y, P_z \) after an \( \omega \)-weighted blow-up. At \( P_z \), the ambient space is smooth and the strict transform has equation
\[
0 = x^k + y^2z^2 + z^2 + y^2 - 2yz(z + y + 1) = x^k + (y_1 + z)^2z^2 + y_1^2 - 2(y_1 + z)z(2z + y_1)
\]
if \( y_1 = y - z \). This is topologically equivalent to \( 0 = x^k + y^2 + z^3 \). Since \( \gcd(3, qr - q - r) = 1 \), by Proposition 5.1 the link of this singularity is a rational homology sphere.

At \( P_y \), the ambient space is \( \frac{1}{q}(1, -1, r) \) and the strict transform has equation
\[
f_\omega(x, y, z) = y^k + z^2 + x^{2q}z^2 + x^{2r} - 2z(x^qz + x^r + x^{q+r}) = y^k + z^2 - 2x^{q+2r} + \ldots
\]
if \( z_1 = z - x^r. \) This change of variable is compatible with the action; this equation defines a singularity in \( \mathbb{C}^3 \) whose link is a rational homology sphere, and so is the case in the quotient manifold.

By symmetry arguments, the same happens for \( P_z \). Hence, \( F \) defines a singularity whose link is a rational homology sphere. \( \Box \)

Let us use the orbifold approach. Given \( (a, b, c) \) pairwise coprime consider \( \omega := (bc, ac, ab) \); the normalized \( \eta \) is \( (1, 1, 1) \) and the isomorphism \( \mathbb{P}^2_\omega \rightarrow \mathbb{P}^2 \) is given by \( [x : y : z]_\omega \mapsto [x^a : y^b : z^c] \), see (1.1). This isomorphic can be seen as a weighted Kummer cover and the homogeneous polynomial
\[
f_\omega(x, y, z) = f(x^a, y^b, z^c) = x^{2a} + y^{2b} + z^{2c} - 2(y^b z^c + x^a z^c + x^a y^b)
\]
of \( \omega \)-degree \( 2abc \), which defines a rational curve in \( \mathbb{P}^2_\omega \cong \mathbb{P}^2 \) and it is tangent to the axes. In most cases the link of this singularity is a rational homology sphere. Let us study a special case.

Proposition 5.3. For any generic quasi-homogeneous polynomial \( g_\omega(x, y, z) \) of degree \( 3abc \), and \( a, b, c \) odd numbers \( \{ F := f_\omega + g_\omega = 0 \} \subset \mathbb{C}^3 \) defines a surface singularity with a rational homology sphere link.

Proof. Note that \( \{ F = 0 \} \) defines an \( (\omega,k) \)-weighted Lê-Yomdin singularity, for \( k = abc \). A partial resolution of this singularity has an exceptional locus which is a rational curve with three singular points (corresponding to the tangencies). In most cases the link of this singularity is a rational homology sphere.

By symmetry reasons we study only the strict transform of this singularity at the tangency point with \( Y \) after an \( \omega \)-weighted blow-up. After a change of coordinates the local equation of \( F \) is
\[
0 = z^{abc} + (x + 1)^2a + y^{2b} + 1 - 2(y^b + (x + 1)^a + (x + 1)^a y^b)
\]
\[
= z^{abc} + a^2x^2 - 2y^b + \ldots
\]
and the ambient space is \( \frac{1}{b}(0, ac, -1) \). Since \( a, b, c \) are odd numbers, by Proposition 5.1 this equation defines a singularity in \( \mathbb{C}^3 \) whose link is a rational homology sphere, and so is the case in the quotient manifold. \( \Box \)
5.3. New examples of integral homology sphere surface singularities.

Note that the only integral homology spheres we have found are well known in the literature, which justifies this subsection.

Following ideas of the third named author, Vey, and Vos, we present an infinite family of normal surface singularities which are complete intersection in \(\mathbb{C}^4\) and whose links are integral homology spheres. The examples given here can be generalized to any dimension.

Let \(n_0, n_1, n_2, n_3 \in \mathbb{Z}_{>0}\) and \(b_{20}, b_{21}, b_{30}, b_{31}, b_{32} \in \mathbb{Z}_{\geq 0}\). Consider \(S\) the surface singularity in \((\mathbb{C}^4, 0)\) defined by

\[
S = \{f_1 + f_2 = f_3 = 0\} \subset (\mathbb{C}^4, 0), \quad \begin{cases} f_1 = x_1^{n_1} - x_0^{n_0}, \\
 f_2 = x_2^{n_2} - x_0^{b_{20}}x_1^{b_{21}}, \\
 f_3 = x_3^{n_3} - x_0^{b_{30}}x_1^{b_{31}}x_2^{b_{32}}. \end{cases}
\]

Note that the family of surfaces \(S\) contains the Brieskorn-Pham surface singularities, for instance when \(n_0 = b_{20} = 1, b_{21} = b_{31} = b_{32} = 0\).

The purpose of this section is to show when the link of \(S\) is a rational homology sphere as well as to characterize when it is integral. The idea is to resolve \(S\) with \(\mathbb{Q}\)-normal crossings and apply Lemma 4.7 to compute \(\det S\). In order to do so we consider the Cartier divisors of \(S\) defined by \(Y = \{f_1 = 0\}\) and \(H_i = \{x_i = 0\}, \ i = 0, 1, 2\). This family was recently studied in Vos’ PhD thesis in a more general context and we just briefly discuss here the construction of the partial resolution obtained in [28, section 5].

**Theorem 5.4.** Let \(S \subset (\mathbb{C}^4, 0)\) be the surface singularity defined above. Assume \(n_0, n_1, n_2, n_3 \in \mathbb{Z}_{>0}\) are pairwise coprime, then \(S\) is a rational homology sphere. Moreover, in that case \(S\) is an integral homology sphere singularity if and only if \(m := \gcd(n_3, b_{20}n_1 + b_{21}n_0) = 1\).

**Proof.** Let \(\pi_1 : \mathring{\mathbb{C}}^4 \rightarrow \mathbb{C}^4\) be the weighted blow-up at the origin of \(\mathbb{C}^4\) with weights \(w_1 = (\frac{n_0}{m}, \frac{n_1}{m}, \frac{n_2}{m}, \frac{n_3}{m})\) where \(n = n_0n_1n_2n_3\). The exceptional divisor of \(\pi_1\) is the weighted projective variety \(E_1 = \mathbb{P}^3_{w_1}\). The assumption on the integers \(n_i, i = 0, .., 3\) being pairwise coprime implies that the exceptional divisor \(E_1\) of the restriction \(\varphi_1 = \pi_1|_{\mathring{S}} : \mathring{S} \rightarrow S\) is a rational irreducible curve which contains three singular points of \(\mathring{S}\), namely \(Q_0 = H_0 \cap E_1, Q_1 = H_1 \cap E_1,\) and \(P_1 = H_2 \cap E_1 = Y \cap E_1,\) see figure 5. The local type of the singularities at \(Q_0\) and \(Q_1\) are given by

\[
Q_0 : \begin{cases} \mathring{S} = \frac{1}{n_0}(n_1n_2n_3, -1) \\
 E_1 : x_1 = 0, \quad \hat{H}_0^{\text{red}} : x_0 = 0, \end{cases} \quad Q_1 : \begin{cases} \mathring{S} = \frac{1}{n_1}(-1, n_0n_2n_3) \\
 E_1 : x_0 = 0, \quad \hat{H}_1^{\text{red}} : x_1 = 0. \end{cases}
\]

Around \(P_1\) the surface \(\mathring{S}\) can be described inside \(\frac{1}{n_2n_3}(-1, n_0n_1n_3, n_0n_1n_2)\) as the set of zeros of \(x_2^{n_2} - x_0^{b_2} + x_1^{n_1} - x_0^{b_1}x_2^{b_2} + (x_2^{n_2} - x_0^{b_2})R_2(x_0, x_2)\) where \(b_i = b_{i0} \frac{n_i}{m} + \cdots + b_{i,n_i-1} \frac{n_i}{m} - n, i = 2, 3,\) and \(R_i(0, x_2) = 0\). Since the monomial with higher order will not play any role in the resolution of \(S\), roughly speaking the situation at \(P_1\) with variables \([x_0, x_2, x_3]\) can be thought of as

\[
P_1 : \begin{cases} \mathring{S} = \{x_0^{b_2} + x_2^{n_2} + x_3^{n_3} = 0\} \subset \frac{1}{n_2n_3}(-1, n_0n_1n_3, n_0n_1n_2) \\
 E_1 : x_0 = 0, \quad \hat{H}_2^{\text{red}} : x_2 = 0, \quad \hat{Y} : x_0^{b_2} + x_2^{n_2} = 0. \end{cases}
\]
The points $Q_0$ and $Q_1$ already have $\mathbb{Q}$-normal crossings, so one does not need to blow them up anymore. Consider the previous coordinates around $P_1$ and let $\pi_2$ be the blow-up at $P_1$ with weights $w_2 = (1, \frac{k_0}{n_2}, \frac{k_1}{n_2})$. The exceptional divisor of $\pi_2$ is $E_2 = \mathbb{P}^2_{w_2}/G$ where $G$ is a cyclic group of order $n_2 n_3$ acting diagonally as in (5.2). The exceptional divisor $E_2$ of the restriction $\varphi_2 |_{\hat{S}} : \hat{S} \to \hat{S}$ is again a rational irreducible curve containing $2 + m$ cyclic quotient singular points of $\hat{S}$, namely $Q_{12} = E_1 \cap E_2$, $P_2 = \hat{Y} \cap E_2$, and $m$ points $Q_{2j} \in H_2 \cap E_2$.

The composition $\varphi = \varphi_1 \circ \varphi_2 : \hat{S} \to S$ is a $\mathbb{Q}$-resolution of $S$ and the order of the groups at $Q_{12}$, $Q_{2j}$, and $P_2$ are $d$, $n_2$, and $\frac{m n_3}{n_1}$ respectively. Since the $\mathbb{Q}$-resolution graph is a tree and the exceptional divisors are isomorphic to $\mathbb{P}^1$ the link of $S$ is a rational homology sphere. In order to compute $\det S$ one needs to calculate the self-intersection numbers $E_i^2 = -a_i$, $i = 1, 2$, which can be done by exploiting our information on the curve $\hat{Y}$ in the partial resolution of $S$. First, note that the intersection of $E_2$ with $\hat{Y}$ at $P_2$ is $m$. Second,

$$\varphi^* Y = \hat{Y} + N_1 E_1 + N_2 E_2$$

where $N_1 = n_0 n_1 n_2 n_3 = n$ and $N_2 = \frac{k_0 + n_0}{m}$. Since $E_i \cdot \varphi^* Y = 0$, $i = 1, 2$, one obtains that $a_1 = \frac{N_1}{N_1 d}$ and $a_2 = \frac{m + N_1}{N_2}$. Therefore the determinant of the intersection matrix is given by

$$\det(A) = \det \left( \begin{array}{cc} -a_1 & \frac{1}{d} \\ \frac{1}{d} & -a_2 \end{array} \right) = \frac{m}{N_1 d}.$$ 

By Lemma 4.7 one has

$$\det S = \det(-A)n_0 n_1 d n_2 n_3 \frac{n_3}{m} = n_2^{n-1}.$$ 

Therefore by Corollary 4.3 the link of $S$ is a integral homology sphere if and only if $\det S = 1$, or equivalently, $m = 1$ as claimed.
Remark 5.5. If the exponents $n_i$’s are not pairwise coprime, then $E_1 = \bigsqcup_j E_{1j}$ has $n_{23} = \gcd(n_2, n_3)$ irreducible components and $E_2$ is irreducible. They have genus
\[ g(E_{1j}) = \frac{1}{2} \left( \frac{n_{123}}{n_{23}} - 1 \right) \left( \frac{n_{023}}{n_{23}} - 1 \right) \quad \text{and} \quad g(E_2) = \frac{1}{2} (n_{23} - 1) (m - 1), \]
where $m = \gcd(n_3, b)$ with $b = b_{20}n_1 + b_{21}n_0$. The determinant of $S$ can be rewritten as
\[ \det S = \left( \frac{b}{m} \right)^{n_{23} - 1} \left( \frac{N_1}{\alpha} \right)^{n_{123} - n_{23}} \left( \frac{N_1}{\beta} \right)^{n_{023} - n_{23}} \left( \frac{n_2}{n_{23}} \right)^{m - 1} \]
where $N_1 = \lcm(n_0, n_1, n_2, n_3)$, $\alpha = \lcm(n_1, n_2, n_3)$, $\beta = \lcm(n_0, n_2, n_3)$, and $n_{ijk} = \frac{n_i n_j n_k}{\lcm(n_i, n_j, n_k)}$. From here it can easily be shown that the link of $S$ is an integral homology sphere if and only if $\gcd(n_i, n_j) = 1$, $i \neq j$, and $m = 1$. The details are left to the reader.

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