KOSZUL DUALITY FOR OPERADS

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Table of Contents

0. Introduction.
1. Operads in algebra and geometry.
2. Quadratic operads.
3. Duality for \(dg\) - operads.
4. Koszul operads.

0. Introduction.

(0.1) The purpose of this paper is to relate two seemingly disparate developments. One is the theory of graph cohomology of Kontsevich [Kon 2 - 3] which arose out of earlier works of Penner [Pe] and Kontsevich [Kon 1] on the cell decomposition and intersection theory on the moduli spaces of curves. The other is the theory of Koszul duality for quadratic associative algebras which was introduced by Priddy [Pr] and has found many applications in homological algebra, algebraic geometry and representation theory (see e.g., [Be] [BGG] [BGS] [Ka 1] [Man]). The unifying concept here is that of an operad.

This paper can be divided into two parts consisting of chapters 1, 3 and 2, 4, respectively. The purpose of the first part is to establish a relationship between operads, moduli spaces of stable curves and graph complexes. To each operad we associate a collection of sheaves on moduli spaces. We introduce, in a natural way, the cobar complex of an operad and show that it is nothing but a (special case of the) graph complex, and that both constructions can be interpreted as the Verdier duality functor on sheaves.

In the second part we introduce a class of operads, called quadratic, and introduce a distinguished subclass of Koszul operads. The main reason for introducing Koszul operads (and in fact for writing this paper) is that most of the operads "arising from nature" are Koszul, cf. (0.8) below. We define a natural duality on quadratic operads (which is
analogous to the duality of Priddy [Pr] for quadratic associative algebras) and show that it is intimately related to the cobar - construction, i.e., to graph complexes.

(0.2) Before going further into discussion of the results of the paper, let us make some comments for the reader not familiar with the notion of an operad. Operads were introduced by J.P. May [May] in 1972 for the needs of homotopy theory. Since then it was gradually realized that this concept has in fact a fundamental significance for mathematics in general. From an algebraic point of view, an operad is a system of data that formalizes properties of a collection of maps $X^n \to X$, a certain set for each $n = 1, 2, ...$, which are closed under permutations of arguments of the maps and under all possible superpositions. Such a collection may be generated by iterated compositions of some primary maps, called generators of the operad, and the whole structure of the operad may be determined, in principle, by giving the the list of relations among the generators. This is very similar to defining a group by generators and relations. As for groups, the consideration of the whole operad and not only generators and relations presents several obvious advantages. For instance, we can develop “homological algebra” for operations, i.e., study higher syzygies. Note, in particular, that all the types of algebras encountered in practice (associative, Lie, Poisson, Jordan etc.) are governed by suitable operads (we take the binary operations involved as generators and the identities which they satisfy as relations).

The place of operads among other structures may be illustrated (very roughly) by the following table.

| Spin 1 | Modules | Vector bundles | Maxwell equations | Yang – Mills theory |
|--------|---------|----------------|-------------------|---------------------|
| Spin 2 | Algebras | Manifolds | Linear gravity equations | Einstein gravity |
| Spin 3 | Operads | ?(Moduli spaces) | Rarita – Schwinger equations | ? (Conformal field theory) |

Question marks indicate that the name put at the corresponding square of the table is just the first approximation to an unknown ultimate name. The relation of operads to algebras in the first column is similar to the relation of algebras to modules. Given an algebra $A$, one has a notion of an $A$-module. Similarly, given an operad $\mathcal{P}$, there is a notion of a $\mathcal{P}$-algebra. Further, for any $\mathcal{P}$-algebra $A$, there is a well defined abelian category of $A$-modules, see §1.6 below. This explains the hierarchy of the first column of the above table. In the geometric column, vector bundles on a fixed manifold correspond to modules over the (commutative) algebra of functions on the manifold. Similarly, giving an operad
is analogous to fixing the class of geometric objects we want to consider. This is equivalent to studying the moduli space of these geometric objects. The relevance of the notion of an operad to conformal field theory can be justified by the following fact which plays a crucial role in the theory of operads and in the present paper in particular:

\[ \mathcal{M} = \{ \overline{M}_{0,n+1}, n = 2, 3, \ldots \} \] (Grothendieck-Knudsen moduli spaces of stable genus 0 curves with \( n + 1 \) punctures) has a natural structure of an operad of smooth manifolds.

(0.3) The paper is organized as follows. Chapter 1 begins with introducing a category of trees that plays the key role (cf. [BV]) throughout the paper. We then recall the definition of an operad and produce a few elementary examples of operads. Next, the above mentioned operad \( \mathcal{M} \) formed by Grothendieck-Knudsen moduli spaces is described in some detail. The significance of this operad for the whole theory of operads is explained: any operad can be described as a collection of sheaves on \( \mathcal{M} \).

(0.4) Chapter 2 is devoted to quadratic operads, the ones generated by binary operations subject to relations involving three arguments only. Most of the structures that one encounters in algebra, e.g., associative, commutative, Lie, Poisson, Jordan, etc. algebras correspond to quadratic operads.

Given a quadratic operad \( \mathcal{P} \), we define the quadratic dual operad \( \mathcal{P}^! \) in the way analogous to the definition of quadratic duality (Priddy) of quadratic associative algebras. In particular, the operads \( \text{Com} \) governing commutative and \( \text{Lie} \) (governing Lie algebras) are quadratic dual to each other. In some informal sense, as a correspondence between the categories of (differential graded) commutative and Lie algebras, this relation goes back at least to the work of Quillen [Q 1-2] and Moore [Mo]. Our theory exhibits a very simple and precise algebraic fact which is the reason for this relation. The operad \( \text{As} \) describing associative (not necessary commutative) algebras is self-dual in our sense.

There is a natural concept of a quadratic algebra over a quadratic operad and for the dual quadratic operads \( \mathcal{P} \) and \( \mathcal{Q} \) we construct a duality between quadratic (super-) algebras over \( \mathcal{P} \) and \( \mathcal{Q} \). For the case of associative algebras (whose operad is self-dual) we recover the construction of Priddy.

Quadratic operads have several nice features. In §2.2 we introduce on the category of such operads the internal \( \text{hom} \) in the spirit of Manin [Man]. We show that the quadratic duality can be interpreted as \( \text{hom}(-, \text{Lie}) \) where \( \text{Lie} \) is the Lie operad which therefore plays the role of a dualizing object in our theory.

(0.5) In chapter 3 we introduce a contravariant duality functor \( \mathbf{D} \) on the category of differential graded (\( dg \)-) operads (as opposed to the quadratic duality functor \( \mathcal{P} \mapsto \mathcal{P}^! \) studied in the previous chapter). We present various approaches to duality. From the
algebraic point of view, the duality $\mathbf{D}$ is an analogue of the cobar construction and a generalization of the tree part of graph complexes. From the geometric point of view, the duality is an analogue of the Verdier duality for sheaves. In more detail, let $\mathcal{W}_{n+1}$ be the moduli space of $n$-labelled trees with metric, cf. [Pe] [Kon 2]. This is a contractible topological space with a natural cell decomposition. This cell decomposition is dual, in a sense, to the canonical stratification of $\overline{\mathcal{M}}_{0,n+1}$. Moreover, the collection $\mathcal{W} = \{\mathcal{W}_{n+1}\}$ forms a co-operad which plays the role dual to that of the operad $\mathcal{M}$. We show that any $dg$-operad gives rise to a compatible collection of constructible complexes on the spaces $\mathcal{W}_{n+1}$, one for each $n$. Furthermore the collection arising from the $\mathbf{D}$-dual $dg$-operad turns out to be formed by the Verdier duals of the complexes corresponding to the original operad. The spaces $\mathcal{W}_{n+1}$ are similar in nature to Bruhat-Tits buildings, and there is yet another description of duality in terms of sheaf cohomology, which is reminiscent of the Deligne-Lusztig duality [DL 1-3] for representations of finite Chevalley groups.

Next, to any $dg$-operad we associate its generating function which is in fact (see Definition 3.1.8) a certain formal map $\mathbf{C}^r \to \mathbf{C}^r$. It turns out (Theorem 3.3.2) that for $dg$-operads dual in our sense their generating maps are, up to signs, composition inverses to each other. Recall (see, e.g., [BGS] [Lö]) that for associative algebra $A$ over a field $k$ its cobar-construction (i.e., a suitable $dg$-model for the Yoneda algebra $\text{Ext}^\bullet_{A}(k,k)$) has generating function which is multiplicative inverse to the generating function of $A$. The role of composition (change of coordinates on a manifold) versus multiplication (change of coordinates in a vector bundle) for generating functions of operads also fits nicely into the table above.

(0.6) Chapter 4 is devoted to Koszul operads, the quadratic operads whose quadratic dual is quasi-isomorphic (canonically) to the $\mathbf{D}$-dual. We prove that the operads $\mathcal{A}s, \mathcal{C}om$ and $\mathcal{L}ie$ are Koszul. Associated to any quadratic operad is its Koszul complex. We show that a quadratic operad $\mathcal{P}$ is Koszul if and only if its Koszul complex is exact, which is equivalent also to vanishing of higher homology for free $\mathcal{P}$-algebras. Given a Koszul operad $\mathcal{P}$, we introduce the notion of a Homotopy $\mathcal{P}$-algebra which reduces in the special cases of Lie and Commutative algebras to that introduced earlier by Schlessinger and Stasheff [SS] and exploited in an essential way by Kontsevich [Kon 2]. In fact, Koszul operads provide the most natural framework for the "formal non-commutative geometry".

(0.7) The concept of an operad in its present form had several important precursors. One should mention the formalism of "theories" of Lawvere (see [BV]) and the pioneering work of Stasheff [St] on homotopy associative $H$-spaces. A little earlier, in the 1955 paper [L], Lazard considered what we would now call formal groups in an operad. He used the notion of "analyseur" which is essentially equivalent (though formally different) to the modern notion of an operad. In (2.2.14) we give a natural interpretation of Lazard’s "Lie
theory” for formal groups in analyseurs in terms of Koszul duality. We are grateful to Y.I. Manin for pointing out to us the reference [L].

In late 1950’s Kolmogoroff and Arnold [Kol] [Ar 1] have studied, in connection with Hilbert’s 13-th problem, what in our present language is the operad of continuous operations $\mathbb{R}^n \to \mathbb{R}$. It was proved in these papers that any continuous function in $n \geq 3$ variables can be represented as a superposition of continuous functions in only one and two variables, i.e., the above operad is generated by unary and binary operations. From the point of view of the present paper (0.2), this result raises an interesting question whether all the relations among the binary generators follow from those provided by functions of three variables. It seems that this question has never been addressed. We would like to thank I.M. Gelfand for drawing our attention to Kolmogoroff’s work.

(0.8) Our interest in this subject originated from an attempt to explain a striking similarity between combinatorics involved in the Graph complex, introduced by Kontsevich in his work on Chern-Simons theory, and combinatorics of the Grothendieck-Knudsen moduli spaces used by A. Beilinson and the first author in their work on local geometry of moduli spaces of $G$-bundles [BG 1]. In particular, the main motivation for the study of Koszul operads begun in this paper is the investigation of the operads formed by Clebsch-Gordan spaces, see (1.3.12) below, for representations of quantum groups and affine Lie algebras. In a future publication we plan to show that these operads are Koszul. In short, Koszul algebras should be replaced by Koszul operads whenever the category under consideration has a tensor-type (e.g., fusion) structure.

(0.9) We are very much indebted to Maxim Kontsevich whose ideas stimulated most of our constructions. We are also very grateful to Ezra Getzler who informed us about his work in progress with J.D.S. Jones [Ge J] and was the first to suggest that the constructions we were working with were related to operads, the notion unknown to the first author at the time (June 1992). His remarks helped to clarify several important points. We much benefited from conversations with Sasha Beilinson, whose current joint work with the first author (see [BG 1], [BG 2]) is closely related to the subject. We would like to thank I.M. Gelfand, Y.I. Manin, J.P. May, V.V. Schechtman and J.D. Stasheff for the discussions of the results of this paper. Several people kindly responded to the call for comments to the preliminary version. In particular, we are indebted to D. Wright for the references [MTWW] and [W] enabling us to give a proper attribution of Theorem 3.3.9 and to A.A. Voronov for correcting our use of det-spaces. Special thanks are due to J.D. Stasheff for pointing out numerous inaccuracies in the earlier version.
1. OPERADS IN ALGEBRA AND GEOMETRY

1.1. Preliminaries on trees.

(1.1.1) By a tree we mean in this paper a non-empty connected oriented graph $T$ without loops (oriented or not) with the following property: there is at least one incoming edge and exactly one outgoing edge at each vertex of $T$, see Fig. 1a.

Trees are viewed as abstract graphs (1-dimensional topological spaces), a plane picture being irrelevant. We allow some edges of a tree to be bounded by a vertex at one end only. Such edges will be called external. All other edges (those bounded by vertices at both ends) will be called internal. Any tree has a unique outgoing external edge, called the output or the root of the tree, and several ingoing external edges, called inputs or leaves of the tree. Similarly, the edges going in and out of a vertex $v$ of a tree will be referred to as inputs and outputs at $v$. A tree with possibly several inputs and a single vertex is called a star. There is also a tree (see Fig. 1b) with a single input and without vertices called the degenerate tree.

We use the notation $\text{In}(T)$ for the set of input edges of a tree $T$; for any vertex $v \in T$ we denote by $\text{In}(v)$ the set of input edges at $v$. Similarly, we denote by $\text{Out}(T)$ the unique output edge (root) of $T$ and for every vertex $v \in T$ we denote by $\text{Out}(v)$ the output edge at $v$.

Let $I$ be a finite set. A tree $T$ equipped with a bijection between $I$ and the set $\text{In}(T)$ will be referred to as $I$-labelled tree or an $I$-tree for short. Two $I$-trees $T, T'$ are called isomorphic if there exists an isomorphism of trees $T \to T'$ preserving orientations and the labellings of the inputs.

We denote by $[n]$ the finite set $\{1, 2, \ldots, n\}$ and call $[n]$-trees simply $n$-trees.
(1.1.2) **Composition.** Let \( I_1 \) be a set and let \( I_2 \) be another set with a marked element \( i \in I_2 \). Define the composition of \( I_1 \) and \( I_2 \) along \( i \) as \( I = I_1 \circ_i I_2 = I_1 \cup (I_2 \setminus \{i\}) \). Given an \( I_1 \)-tree \( T_1 \) and an \( I_2 \)-tree \( T_2 \), let \( T = T_1 \circ_i T_2 \) be the \( I \)-tree obtained by identifying the output of \( T_1 \) with the \( i \)-th input of \( T_2 \) as depicted in Fig. 2.

The tree \( T \) is called the *composition* of \( T_1 \) and \( T_2 \) (along \( i \)). Note that any tree can be obtained as an iterated composition of stars.

Let \( T \) be a tree and \( v \xrightarrow{e} w \) an internal edge of \( T \). Then we can form a new tree \( T/e \) by contracting \( e \) into a point. This new point is a vertex of \( T/e \) denoted by \( <e> \). Clearly, we have

\[
\text{In}(<e>) = \text{In}(w) \circ_e \text{In}(v).
\]

If \( T \) is \( I \)-labelled then so is \( T/e \). If \( T, T' \) are two \( I \)-labelled trees then we write \( T \xrightarrow{e} T' \) if the tree \( T' \) is isomorphic (as a labelled tree) to \( T/e \).

Write \( T \geq T' \) if there is a sequence of edge contractions \( T \rightarrow T_1 \rightarrow \ldots \rightarrow T_k \rightarrow T' \). Thus, \( \geq \) is a partial order on the set of all \( I \)-trees. An \( I \)-star is the unique minimal element with respect to that order.

A tree \( T \) (with at least one vertex) is called a *binary tree* if there are exactly 2 inputs at each vertex of \( T \). Binary trees are the maximal elements with respect to the partial order \( \leq \). It can be shown that the number of non-isomorphic binary \( n \)-trees is equal to 
\[
(2n - 3)!! = 1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2n - 3).
\]
A simple proof of this fact using generating functions will be given in Chapter 3.

(1.1.4) **The category of trees.** Let \( T, T' \) be trees (viewed as 1-dimensional topological spaces). By a *morphism* from \( T \) to \( T' \) we understand a continuous surjective map \( f : T \rightarrow T' \) with the following properties:

(i) \( f \) takes each vertex to a vertex and each edge into an edge or a vertex.
(ii) \( f \) is monotone, i.e., preserves the orientation.
(iii) The inverse image of any point of \( T' \) under \( f \) is a connected subtree in \( T \).
Thus any morphism is a composition of an isomorphism and several edge contractions. In this way we get a category which we denote $Trees$.

(1.1.5) Let $\Sigma_n$ denote the symmetric group of order $n$. For any two sets $I, J$ of the same cardinality we denote by $\text{Iso}(I, J)$ the set of all bijections $I \to J$. We write $\Sigma_I$ for $\text{Iso}(I, I)$, so that $\Sigma_n = \text{Iso}([n], [n])$. Clearly, $\text{Iso}(I, J)$ is a principal homogeneous left $\Sigma_I$-space and a principal homogeneous right $\Sigma_J$-space.

Let $W$ be a vector space (over some field $k$) with an action of $\Sigma_n$. There is a canonical way to construct (out of $W$) a functor $I \mapsto W(I)$ from the category of $n$-element sets and bijections to the category of $k$ vector spaces. Namely, put

\[ W(I) = \left( \bigoplus_{f \in \text{Iso}([n], I)} W \right)_{\Sigma_n}, \]

the coinvariants with respect to the simultaneous action of $\Sigma_n$ on $\text{Iso}([n], I)$ and $W$. The original space $W$ is recovered as the value of this functor on the set $[n]$. Similarly, if $W$ is a set, or topological space with $\Sigma_n$-action, we can construct a functor $I \mapsto W(I)$ from the category of $n$-element sets and their bijections to the category of sets, or topological spaces as above, by the following analog of (1.1.6):

\[ W(I) = \text{Iso}([n], I) \times_{\Sigma_n} W. \]
1.2. $k$ - linear operads.

(1.2.1) Let $k$ be a field of characteristic 0. A $k$ - linear operad $\mathcal{P}$ is a collection $\{\mathcal{P}(n), n \geq 1\}$ of $k$ - vector spaces equipped with the following set of data:

(i) An action of the symmetric group $\Sigma_n$ on $\mathcal{P}(n)$ for each $n \geq 1$.

(ii) Linear maps (called compositions)

$$\gamma_{m_1, \ldots, m_l} : \mathcal{P}(l) \otimes \mathcal{P}(m_1) \otimes \ldots \otimes \mathcal{P}(m_l) \longrightarrow \mathcal{P}(m_1 + \ldots + m_l)$$

for all $m_1, \ldots, m_l \geq 1$. We write $\mu(\nu_1, \ldots, \nu_l)$ instead of $\gamma_{m_1, \ldots, m_l}(\mu \otimes \nu_1 \otimes \ldots \otimes \nu_l)$.

(iii) An element $1 \in \mathcal{P}(1)$, called the unit, such that $\mu(1, \ldots, 1) = \mu$ for any $l$ and any $\mu \in \mathcal{P}(l)$.

It is required that these data satisfy the conditions (associativity and equivariance with respect to symmetric group actions) specified by May ([May], §1). These conditions are best expressed in terms of trees. Observe first that the datum (i) allows to assign to any finite set $I$ a vector space $\mathcal{P}(I)$ as in (1.1.5). Next, we associate to any tree $T$ the vector space

(1.2.2) \[ \tilde{\mathcal{P}}(T) = \bigotimes_{v \in T} \mathcal{P}(\text{In}(v)). \]

To the degenerate tree without vertices we associate, by definition, the field $k$.

Note in particular that to the trees $T(n)$ and $T(m_1, \ldots, m_l)$ depicted in Fig.3 we associate the spaces $\mathcal{P}(n)$ and $\mathcal{P}(l) \otimes \mathcal{P}(m_1) \otimes \ldots \otimes \mathcal{P}(m_l)$.

Thus the datum (ii) gives a map

(1.2.3) \[ \tilde{\mathcal{P}}(T(m_1, \ldots, m_l)) \longrightarrow \tilde{\mathcal{P}}(T(m_1 + \ldots + m_l)) = \mathcal{P}(m_1 + \ldots + m_l). \]

For any $n$ - tree $T$ there exists a sequence of trees

(1.2.4) \[ T = T_0 \rightarrow T_1 \rightarrow \ldots \rightarrow T_r = T(n) \]
where each $T_i$ is obtained from $T_{i-1}$ by replacing a fragment of type $T(m_1,\ldots, m_l)$ by $T(m_1 + \ldots + m_l)$. So maps (1.2.3) give rise to a sequence of maps

$$\tilde{P}(T) = \tilde{P}(T_0) \to \tilde{P}(T_1) \to \ldots \to \tilde{P}(T_r) = P(n).$$

The associativity condition is equivalent to the requirement that the composite map $\tilde{P}(T) \to P(n)$ does not depend on the choice of a sequence (1.2.4).

(1.2.5) Observe that the tree $T(m_1 + \ldots + m_l)$ is obtained from $T(m_1,\ldots, m_l)$ by contracting all the $l$ internal edges. The existence of unit $1 \in P(1)$ makes it possible to decompose the map (1.2.3) corresponding to this contraction, into more elementary ones, each consisting of contracting a single edge.

Namely, let $T$ be a tree, $v \xrightarrow{e} w$ be an internal edge of $T$ and $T/e$ be the tree obtained by contracting $e$. Let $\langle e \rangle$ be the vertex of $T/e$ obtained from the contracted edge. We define a map

$$\mathcal{P}(\text{In}(v)) \otimes \mathcal{P}(\text{In}(w)) \longrightarrow \mathcal{P}(\text{In}(v) \circ_e \text{In}(w)) \overset{(1.1.3)}{=} \mathcal{P}(\text{In}(\langle e \rangle))$$

by the formula $\mu \otimes \nu \mapsto \mu(1,\ldots, \nu,\ldots,1)$ where $\nu$ is placed at the entry corresponding to the edge $e$. By tensoring this map with the identity elsewhere on $T$, we obtain a map

$$\tilde{\gamma}_{T,e} : \tilde{P}(T) \rightarrow \tilde{P}(T/e).$$

(1.2.6)

More generally, if $I$ is a finite set and $T, T'$ are two $I$ - trees such that $T \geq T'$ then by composing maps of the type $\tilde{\gamma}_{T,e}$ we get a map

$$\tilde{\gamma}_{T,T'} : \tilde{P}(T) \to \tilde{P}(T')$$

(1.2.7)

which is well - defined due to the associativity condition.

Thus a $k$ - linear operad $\mathcal{P}$ gives rise to a functor

$$\tilde{P} : \text{Trees} \longrightarrow \text{Vect}, \ T \mapsto \tilde{P}(T)$$

equipped with the following additional structures:

(i) For any trees $T_1$ and $T_2$ and any $j \in \text{In}(T_2)$ one has a functorial isomorphism

$$\Phi_{T_1,T_2} : \tilde{P}(T_1) \otimes \tilde{P}(T_2) \rightarrow \tilde{P}(T_1 \circ_j T_2).$$

(1.2.8)

(ii) The isomorphisms in (i) satisfy the associativity constraint saying that for any trees $T_1, T_2, T_3$ and any $i \in \text{In}(T_2), j \in \text{In}(T_3)$ the following diagram commutes:
(1.2.9) Let $V$ be a $k$-vector space. Its operad of endomorphisms, $\mathcal{E}_V$, consists of vector spaces

$$\mathcal{E}_V(n) = \text{Hom}(V^\otimes n, V).$$

with compositions and the $\Sigma_n$-action on $\mathcal{E}_V(n)$ being defined in an obvious way. We have $\mathcal{E}_V(1) = \text{End}(V)$.

(1.2.10) Observe that for any $k$-linear operad $\mathcal{P}$ the space $K = \mathcal{P}(1)$ has a natural structure of an associative $k$-algebra with unit. Conversely, if $K$ is a $k$-algebra then the collection $\{\mathcal{P}(1) = K, \mathcal{P}(n) = \{0\}, n > 1\}$ forms an operad. Furthermore, for an arbitrary $k$-linear operad $\mathcal{P}$ the space $\mathcal{P}(n)$ has several $\mathcal{P}(1)$-module structures. These structures are summarized in the following definition.

(1.2.11) Definition. Let $K$ be an associative $k$-algebra with unit. A $K$-collection is a $K$-collection $E = \{E(n), n \geq 2\}$ of $k$-vector spaces equipped with the following structures:

(i) A left $\Sigma_n$-action on $E(n)$, for each $n \geq 2$.
(ii) A structure of a left $K$-module and a right $K^\otimes n$-module on $E(n)$, $n \geq 2$.

These structures are required to satisfy the following compatibility condition: for any $s \in \Sigma_n$ and any $\lambda_1, ..., \lambda_n \in K$, $a \in E(n)$ we have

$$s (a \cdot (\lambda_1 \otimes ... \otimes \lambda_n)) = a \cdot (\lambda_{s(1)} \otimes ... \otimes \lambda_{s(n)}).$$

If $\mathcal{P}$ is a $k$-linear operad and $K = \mathcal{P}(1)$ then $\{\mathcal{P}(n), n \geq 2\}$ is, clearly, a $K$-collection.

(1.2.12) It will be convenient for the future purposes to give a reduced, in a certain sense, version of the tree formalism above.

We call a tree $T$ reduced if there are at least two inputs at each vertex $v \in T$ (the degenerate tree without vertices is also assumed to be reduced).

Let $K$ be an associative $k$-algebra and $E$ a $K$-collection. As in (1.1.5) we extend $E$ to a functor on finite sets and bijections. To each reduced tree $T$ we associate a vector space $E(T)$ as follows. For the degenerate tree $\to$ we set $E(\to) = K$ and for a tree $T$ with non-empty set of vertices we set

(1.2.13) $$E(T) = \bigotimes_{v \in T} K E(\text{In}(v)).$$
This tensor product is taken over the ring $K$ by using the $(K, K^\otimes \text{In}(v))$-bimodule structure on each $E(\text{In}(v))$, see Fig.4.

Explicitly, this means that $E(T)$ is the quotient of the $k$-tensor product $\bigotimes_v E(\text{In}(v))$ by associativity conditions described as follows. Suppose that $e \mapsto \lambda_e$ is any $K$-valued function on the set of all edges such that $\lambda_e = 1$ for all external edges. Then for any collection of $a_v \in E(\text{In}(v))$ we impose the relation

$$\bigotimes_{v \in T} a_v \cdot \left( \bigotimes_{e \in \text{In}(v)} \lambda_e \right) = \bigotimes_{v \in T} \lambda_{\text{Out}(v)} \cdot a_v.$$ 

(1.2.14) Now let $\mathcal{P} = \{\mathcal{P}(n)\}$ be a $k$-linear operad and $K = \mathcal{P}(1)$. Since $\{\mathcal{P}(n), n \geq 2\}$ form a $K$-collection, we can assign to any reduced tree $T$ a vector space $\mathcal{P}(T)$ by formula (1.2.13). This assignment has the following two fundamental structures:

(i) A linear map

$$\gamma_{T,T'} : \mathcal{P}(T) \to \mathcal{P}(T')$$

defined whenever $T \geq T'$.

(ii) An isomorphism

$$\mathcal{P}(T_1 \circ_i T_2) \to \mathcal{P}(T_1) \otimes_K \mathcal{P}(T_2)$$

given for any $i \in \text{In}(T_2)$.

The maps in (i) and (ii) are induced by (1.2.7) and (1.2.8), respectively.

Observe further that the assignment $T \to \mathcal{P}(T)$ extends to a functor on the (sub)category of reduced trees, see (1.1.4).

(1.2.16) Convention. In the rest of this paper we shall consider only reduced trees, unless specified otherwise.
1.3. Algebraic operads.

(1.3.1) Let \( \mathcal{P}, \mathcal{Q} \) be two \( k \)-linear operads. A morphism of operads \( f : \mathcal{P} \to \mathcal{Q} \) is a collection of linear maps which are equivariant with respect to \( \Sigma_n \)-action, commute with compositions \( \gamma_{m_1, \ldots, m_l} \) in \( \mathcal{P} \) and \( \mathcal{Q} \) and take the unit of \( \mathcal{P} \) to the unit of \( \mathcal{Q} \), see [May].

(1.3.2) Definition. Let \( \mathcal{P} \) be a \( k \)-linear operad. A \( \mathcal{P} \)-algebra is a \( k \)-vector space \( A \) equipped with a morphism of operads \( f : \mathcal{P} \to \mathcal{E}_A \) where \( \mathcal{E}_A \) is the operad of endomorphisms of \( A \), see (1.2.9).

Clearly, giving a structure of a \( \mathcal{P} \)-algebra on \( A \) is the same as giving a collection of linear maps

\[
f_n : \mathcal{P}(n) \otimes A^\otimes n \to A
\]

satisfying natural associativity, equivariance and unit conditions. We write \( \mu(a_1, \ldots, a_n) \) for \( f_n(\mu \otimes (a_1 \otimes \ldots \otimes a_n)), \mu \in \mathcal{P}(n), a_i \in A. \)

(1.3.4) Free algebras. Let \( \mathcal{P} \) be a \( k \)-linear operad and \( K = \mathcal{P}(1) \). As noted in (1.2.10), \( K \) is an associative \( k \)-algebra and every \( \mathcal{P}(n) \) is a left \( K \)-module and a right \( K^\otimes n \)-module.

Let \( V \) be any left \( K \)-module. Form the following graded vector space

\[
F_\mathcal{P}(V) = \bigoplus_{n \geq 1} \left( \mathcal{P}(n) \otimes_{K^\otimes n} V^\otimes n \right)_{\Sigma_n}
\]

where \( V^\otimes n \) is the \( n \)-th tensor power of \( V \) over \( k \).

(1.3.6) Lemma. Compositions in \( \mathcal{P} \) induce natural maps \( \mathcal{P}(n) \otimes F_\mathcal{P}(V)^\otimes n \to F_\mathcal{P}(V) \). There maps make \( F_\mathcal{P}(V) \) into a \( \mathcal{P} \)-algebra.

We call \( F_\mathcal{P}(V) \) the free \( \mathcal{P} \)-algebra generated by \( V \). Note that \( F_\mathcal{P}(V) \) has a natural grading given by the decomposition (1.3.5).

Here are some other examples of operads and algebras:

(1.3.7) Associative algebras. For any \( n \) let \( \mathcal{A}s(x_1, \ldots, x_n) \) denote the free associative \( k \)-algebra on generators \( x_1, \ldots, x_n \) (i.e., the algebra of non-commutative polynomials). Let \( \mathcal{A}s(n) \subset \mathcal{A}s(x_1, \ldots, x_n) \) be the subspace spanned by monomials containing each \( x_i \) exactly once. There are exactly \( n! \) such monomials namely \( x_{s(1)} \cdot \ldots \cdot x_{s(n)}, s \in \Sigma_n \). Clearly, \( \mathcal{A}s(n) \) has a natural \( \Sigma_n \)-action and as a \( \Sigma_n \)-module it is isomorphic to the regular
representation of $\Sigma_n$. The collection $\mathcal{A}s= \{\mathcal{A}s(n)\}$ forms an operad called the associative operad. The composition maps (see (1.2.1))

$$\mathcal{A}s(l) \otimes \mathcal{A}s(m_1) \otimes ... \otimes \mathcal{A}s(m_l) \longrightarrow \mathcal{A}s(m_1 + ... + m_l)$$

are given by substituting the monomials $\phi_1 \in \mathcal{A}s(m_1), ..., \phi_l \in \mathcal{A}s(m_l)$ in place of genera-
tors $x_1, ..., x_l$ into a monomial $\psi \in \mathcal{A}s(l)$.

We leave to the reader to verify that an $\mathcal{A}s$ - algebra is nothing but an associative algebra in the usual sense (possibly without unit).

(1.3.8) Commutative (associative) algebras. For any $n$ let $\text{Com}(x_1, ..., x_n) = k[x_1, ..., x_n]$ be the free commutative algebra on generators $x_1, ..., x_n$ (i.e., the algebra of polynomials). There exists precisely one monomial containing each $x_i$ exactly once, namely $x_1 \cdot ... \cdot x_n$. Let $\text{Com}(n) \subset \text{Com}(x_1, ..., x_n)$ be the 1-dimensional subspace spanned by this monomial. The collection $\text{Com} = \{\text{Com}(n)\}$ forms an operad with respect to the trivial actions of $\Sigma_n$ on $\text{Com}(n)$ and compositions defined similarly to (1.3.7). We call $\text{Com}$ the commutative operad.

Again, it is straightforward to see that a $\text{Com}$ - algebra is nothing but a commutative associative algebra in the usual sense (possibly without unit).

(1.3.9) Lie algebras. For any $n$ let $\text{Lie}(x_1, ..., x_n)$ be the free Lie algebra over $k$ generated by $x_1, ..., x_n$. Let $\text{Lie}(n) \subset \text{Lie}(x_1, ..., x_n)$ be the subspaces spanned by all bracket monomials containing each $x_i$ exactly once. Note that such monomials are not all linearly independent due to the Jacobi identity. The subspace $\text{Lie}(n)$ is invariant under the action of $\Sigma_n$ on $\text{Lie}(x_1, ..., x_n)$ by permutations of $x_i$. It is known that

(1.3.10) \[ \dim \text{Lie}(n) = (n - 1)! \]

Moreover, if $k$ is algebraically closed, then A. Klyachko [Kl] an isomorphism of $\Sigma_n$ - modules

(1.3.11) \[ \text{Lie}(n) \cong \text{Ind}_{\mathbb{Z}/n}^{\Sigma_n} (\chi) \]

where $\chi$ is the 1-dimensional representation of the cyclic group $\mathbb{Z}/n$ sending the generator into a primitive $n$ - th root of 1 (the induced module does not depend on the choice of such a $\chi$).

The collection $\text{Lie} = \{\text{Lie}(n)\}$ forms an operad with respect to the composition operations defined similarly to the ones described in (1.3.4). An algebra over the operad $\text{Lie}$ is the same as a Lie algebra in the usual sense.

It should be clear to the reader at this point how to construct operads governing other types of algebras encountered in practice: Poisson algebras, Jordan algebras etc.
(1.3.12) A collection of finite-dimensional $k$-vector spaces

$$\left\{ E^j(a_1, \ldots, a_r), \ a_1, \ldots, a_r \in \mathbb{Z}_+, \ \sum a_i \geq 1, \ j = 1, \ldots, r \right\}$$

is called an $r$-fold collection if the following holds:

(i) For any $a_1, \ldots, a_r \in \mathbb{Z}_+$ the space $E^j(a_1, \ldots, a_r)$ is equipped with an action of the group $\Sigma_{a_1} \times \cdots \times \Sigma_{a_r}$.

(ii) $E^j(0, \ldots, 0, 1, 0, \ldots, 0)$ (1 on the $j$-th place) = \begin{cases} 0 & \text{if } i \neq j \\ k & \text{if } i = j \end{cases}$.

Now let $K$ be a semisimple $k$-algebra and $\{M_1, \ldots, M_r\}$ be a complete collection of (the isomorphism classes of) simple left $K$-modules. Given a $K$-collection $E = \{E(n), n \geq 2\}$ (1.2.11), we regard $E(n)$ as a left module over the algebra $\Pi_n = K^\otimes n \otimes K^{op}$. For any $a_1, \ldots, a_r \in \mathbb{Z}_+$ such that $\sum a_i = n$, we define

(1.3.13) $E^i(a_1, \ldots, a_r) = \text{Hom}_{\Pi_n}(M_1^\otimes a_1 \otimes \cdots \otimes M_r^\otimes a_r \otimes M_i^*, E(n))$.

The $k$-vector spaces $E^j(a_1, \ldots, a_r)$ thus defined clearly form an $r$-fold collection. It will be called the $r$-fold collection associated to the $K$-collection $E$.

(1.3.14) **Clebsch-Gordan spaces and operads.** Let $\mathcal{A}$ be a semisimple Abelian $k$-linear category equipped with a symmetric monoidal structure $\otimes$ (for example, the tensor category of finite-dimensional representations of a finite, or an algebraic group).

Fix any integer $r \geq 1$ and any set $X_1, \ldots, X_r$ of pairwise non-isomorphic simple objects of $\mathcal{A}$. Let $X = \bigoplus_{i=1}^r X_i$. Associated to $X$ is the operad $\mathcal{P}_X$ defined by collection of vector spaces

(1.3.15) $\mathcal{P}_X(N) = \text{Hom}_{\mathcal{A}}(X^\otimes n, X)$.

Observe that $\mathcal{P}_X(1) = \bigoplus \text{End}(X_i)$ is a semisimple $k$-algebra. We put $K = \mathcal{P}_X(1)$ and view the collection (1.3.15) as a $K$-collection. Clearly, the $r$-fold collection associated to that $K$-collection via (1.3.13) is formed by the Clebsch-Gordan spaces

(1.3.16) $\mathcal{P}^i(a_1, \ldots, a_r) = \text{Hom}_{\mathcal{A}}(X_1^\otimes a_1 \otimes \cdots \otimes X_r^\otimes a_r, X_i)$.

(1.3.17) **Operads in monoidal categories.** Let $\text{Vect}$ be the category of $k$-vector spaces. Note that the concept of a $k$-linear operad appeals only to the category $\text{Vect}$ with its symmetric monoidal structure given by the tensor product. Clearly, one can define operads in any symmetric monoidal category, i.e., a category $\mathcal{A}$ (not necessarily additive
or abelian) equipped with a bifunctor $\otimes : A \times A \to A$ and natural associativity and commutativity constraints for this functor [Mac].

All the preceding constructions (e.g., the notion of a $\mathcal{P}$-algebra) can be carried over to the setup of operads in any symmetric monoidal category $(A, \otimes)$. Given such an operad, one defines, as in (1.1.5), for any finite set $I$, an object $\mathcal{P}(I) \in A$ and for tree $T$ an object $\tilde{\mathcal{P}}(T) \in A$.

In this section we concentrate on algebraic examples of categories $A$.

**1.3.18 Two categories of graded vector spaces.** Let $gVect^+$ be the category whose objects are graded vector spaces $V^\bullet = \bigoplus_{i \in \mathbb{Z}} V^i$ and morphisms are linear maps preserving the grading. For $v \in V^i$ we write $\deg(v) = i$. We introduce on $gVect^+$ a symmetric monoidal structure defining with the tensor product

$$ (V^\bullet \otimes W^\bullet)^m = \bigoplus_{i+j=m} V^i \otimes W^j. $$

The associativity morphism $V^\bullet \otimes (W^\bullet \otimes X^\bullet) \to (V^\bullet \otimes W^\bullet) \otimes X^\bullet$ takes $v \otimes (w \otimes x) \mapsto (v \otimes w) \otimes x$. The commutativity isomorphism $V^\bullet \otimes W^\bullet \to W^\bullet \otimes V^\bullet$ takes $v \otimes w \mapsto w \otimes v$.

The symmetric monoidal category $gVect^-$ has the same objects, morphisms, tensor product and associativity isomorphism as $gVect^+$ but the commutativity isomorphism is changed to be

$$ v \otimes w \mapsto (-1)^{\deg(v) \cdot \deg(w)} w \otimes v. $$

Any $k$-linear operad $\mathcal{P}$ can be regarded as an operad in either of the categories $gVect^\pm$ and so we can speak about $\mathcal{P}$-algebras in these categories. For example, if $\mathcal{P} = \mathcal{Com}$ is the commutative operad, then a $\mathcal{Com}$-algebra in $gVect^+$ is a commutative associative algebra equipped with a grading compatible with the algebra structure. A $\mathcal{Com}$-algebra in $gVect^-$ is an associative graded algebra which is graded (or super-) commutative.

**1.3.21 The determinant operad $\Lambda$.** In the operad $\mathcal{Com}$ each space $\mathcal{Com}(n)$ is 1-dimensional and equipped with the trivial action of $\Sigma_n$. We now introduce an operad $\Lambda$ in the category $gVect^-$ which is an “odd” analog of $\mathcal{Com}$.

We define $\Lambda(n)$ to be the 1-dimensional vector space $\Lambda^n(k^n)$ (the sign representation of $\Sigma_n$) placed in degree $(1 - n)$. In order to describe compositions in $\Lambda$ we first describe what is to be a $\Lambda$-algebra in $gVect^-$. We consider graded vector spaces $A = \bigoplus A_i$ with one binary operation $(a, b) \mapsto ab$ satisfying the following identities:

1. $\deg(ab) = \deg(a) + \deg(b) - 1$,
2. $ab = (-1)^{\deg(a) + \deg(b) + 1} ba$,.
(iii) \( a(bc) = (-1)^{\deg(a)+1} (ab)c \).

The operation \( ab \) corresponds to the generator \( \mu \in \Lambda(2) \). The condition (i) means that \( \deg(\mu) = -1 \) and the condition (ii) means that \( \Sigma_2 \) acts on \( \mu \) by the sign representation.

We fix integers \( d_1, \ldots, d_n \) and let \( \Lambda(x_1, \ldots, x_n) \) be the free algebra with the above identities generated by symbols \( x_i \), of degree \( d_i \).

\[ \text{(1.3.22) Lemma.} \quad \text{The subspace } E_{d_1, \ldots, d_n} \text{ in } \Lambda(x_1, \ldots, x_n) \text{ spanned by non-associative monomials containing each } x_i \text{ exactly once, has dimension 1. Moreover, any two such monomials are proportional with coefficients } \pm 1. \]

The meaning of this lemma is that the identities (i) - (iii) above are consistent, i.e., do not lead to the contradiction \( 1=0 \). In other words, any two ways of comparing the signs of the monomials lead to the same result.

\[ \text{Proof of the lemma:} \quad \text{As in Mac Lane’s axiomatics of symmetric monoidal categories [Mac], it is enough to check the three elementary ambiguities, i.e., the two ways of comparing } a(bc) \text{ and } (bc)a, \text{ of } (ab)c \text{ and } c(ab) \text{ and of } a(b(cd)) \text{ and } ((ab)c)d. \text{ We leave this to the reader.} \]

Another way to see Lemma 1.3.22 can be based on the following remark (made to us by E. Getzler). If \( A \) is a graded commutative algebra (i.e., a \( Com \) - algebra in the category \( gVect^- \)) then the same algebra with the grading shifted by one and new multiplication \( a \ast b = (-1)^{\deg(a)}ab \) will satisfy the identities (i) - (iii) above.

\[ \text{(1.3.23) To finish the construction of the operad } \Lambda, \text{ we take, in the situation of the Lemma 1.3.22, } n \text{ generators } x_1, \ldots, x_n \text{ of degree 0. We denote } \Lambda'(n) = E_{0, \ldots, 0} \text{ the 1-dimensional subspace from this lemma. The space } \Lambda'(n) \text{ is } \Sigma_n \text{- invariant and the action of } \Sigma_n \text{ on } \Lambda'(n) \text{ is given by the sign representation. To see the last assertion it is enough to show that any transposition } (ij) \text{ acts by } (-1). \text{ But we can take the basis vector of } \Lambda'(n) \text{ given by any product } ...x_ix_j... \text{ in which } x_i \text{ and } x_j \text{ are bracketed together. By (ii) we have } x_ix_j = -x_jx_i \text{ whence the assertion.} \]

So we can identify \( \Lambda'(n) \) with \( \Lambda(n) = \bigwedge^n k^n \) and the substitution of monomials in place of generators, as in (1.3.7), defines the operad structure on \( \Lambda \).
1.4. Geometric operads.

(1.4.1) The category of topological space has an obvious symmetric monoidal structure given by the Cartesian product. Operads in this category will be called topological operads. (This was the original context of [May].)

Given a topological operad $P$, the total homology spaces $H_\bullet(P(n), k)$ form an operad in the category $gVect^-$ (by Künneth formula). Furthermore, for any $q \geq 0$, the subspaces $H_{q(n-1)}(P(n), k)$ form a sub-operad.

(1.4.2) An important example of a topological operad is given by the little $m$-cubes operad $C_m$ of Boardman - Vogt - May [BV] [May]. By definition, $C_m(n)$ is the space of numbered $n$-tuples of non-intersecting $m$-dimensional cubes inside the standard cube $I^n$, with faces parallel to those of $I^m$. The operad $C_2$ of little squares has an algebro-geometric analog which will be particularly interesting for us and which we proceed to describe.

(1.4.3) The moduli space $M(n)$. Let $M_{0,n+1}$ be the moduli space of $(n + 1)$-tuples $(x_0, ..., x_n)$ of distinct points on the complex projective line $\mathbb{C}P^1$ modulo projective automorphisms. Choose a point $\infty \in \mathbb{C}P^1$ so $\mathbb{C}P^1 = \mathbb{C} \cup \{\infty\}$. Letting $x_0 = \infty$, one gets an isomorphism of $M_{0,n+1}$ with the moduli space of $n$-tuples of distinct points on $\mathbb{C}$ modulo affine automorphisms.

The space $M_{0,n+1}$ has a canonical compactification $M(n) \supset M_{0,n+1}$ introduced by Grothendieck and Knudsen [De 2] [Kn]. The space $M(n)$ is the moduli space of stable $(n + 1)$-pointed curves of genus 0, i.e., systems $(C, x_0, ..., x_n)$ where $C$ is a possibly reducible curve (with at most nodal singularities) and $x_0, ..., x_n \in C$ are distinct smooth points such that:

(i) Each component of $C$ is isomorphic to $\mathbb{C}P^1$.

(ii) The graph of intersections of components of $C$ is a tree.

(iii) Each component of $C$ has at least 3 special points where by a special point we mean either one of the $x_i$ or a singular point of $C$.

The space $M_{0,n+1}$ forms an open dense part of $M(n)$ consisting of $(C, x_0, ..., x_n)$ such that $C$ is isomorphic to $\mathbb{C}P^1$. The space $M(n)$ is a smooth complex projective variety of dimension $n - 2$. It has the following elementary construction, see [BG 1] [FM] [Ka 3].

Let $\text{Aff} = \{x \mapsto ax + b\}$ be the group of affine transformations of $\mathbb{C}$. The group $\text{Aff}$ acts diagonally on $\mathbb{C}^n$ preserving the open part $\mathbb{C}^n_*$ formed by points with pairwise distinct coordinates. As we noted before, we have an isomorphism $M_{0,n+1} \cong \mathbb{C}^n_*/\text{Aff}$. Denote by $\Delta \subset \mathbb{C}^n$ the principal diagonal, i.e., the space of points $(x, x, ..., x)$. Then we have an
embedding
\[ M_{0,n+1} = C^n/\text{Aff} \subset (C^n - \Delta)/\text{Aff} = \mathbb{C}P^{n-2}. \]

The coordinate axes in \( C^n \) give \( n \) distinguished points \( p_1, ..., p_n \in \mathbb{C}P^{n-2} \). Let us blow up all the points \( p_i \), then blow up the proper transforms (= closures of the preimages of some open parts) of the lines \( <p_i, p_j> \), then blow up the proper transforms of the planes \( <p_i, p_j, p_k> \), and so on. It can be shown that the resulting space is isomorphic to \( M(n) \).

1.4.4 The configuration operad \( M \). The family of spaces \( M = \{ M(n), n \geq 1 \} \), forms a topological operad. The symmetric group action on \( M(n) \) is given by

\[ (C, x_0, ..., x_n) \mapsto (C, x_0, x_{s(1)}, ..., x_{s(n)}), \quad s \in \Sigma_n. \]

The composition map

\[ M(l) \times M(m_1) \times ... \times M(m_l) \longrightarrow M(m_1 + ... + m_l) \]

is defined by

\[ (C, y_0, ..., y_l), (C^{(1)}, x_0^{(1)}, ..., x_{m_1}^{(1)}), ..., (C^{(l)}, x_0^{(l)}, ..., x_{m_l}^{(l)}) \mapsto \]

\[ (C', y_0, x^{(1)}_1, ..., x_{m_1}^{(1)}, ..., x^{(l)}_1, ..., x_{m_l}^{(l)}) \]

where \( C' \) is the curve obtained from the disjoint union \( C \sqcup C^{(1)} \sqcup ... \sqcup C^{(l)} \) by identifying \( x_0^{(i)} \) with \( y_i, i = 1, ..., l \) (Fig.5)

We call \( M \) the configuration operad, since \( M(n) \) can be regarded as (compactified) configuration spaces of points on \( \mathbb{C}P^1 \).

1.4.5 The stratification of the space \( M(n) \). Given a point \( (C, x_0, ..., x_n) \in M(n) \), we associate to it an \( n \) - tree \( T = T(C, x_0, ..., x_n) \) as follows. The vertices of \( T \) correspond to the irreducible components of \( C \). The vertices corresponding to two components \( C_1, C_2 \)
are joined by an (internal) edge if \( C_1 \cap C_2 \neq \emptyset \). An external edge is assigned to each of
the marked points \( x_0, ..., x_n \). The input edge labelled by \( i \neq 0 \) is attached to the vertex
_corresponding to the component containing \( x_i \). The output edge is attached to the vertex
corresponding to the component containing \( x_0 \). The property (iii) of stable curves ensures
that \( T(C, x_0, ..., x_n) \) is a reduced tree.

For any reduced \( n \)-tree \( T \) let \( \mathcal{M}(T) \subset \mathcal{M}(n) \) be the subset consisting of points
\((C, x_0, ..., x_n)\) such that \( T(C, x_0, ..., x_n) = T \). In this way we obtain an algebraic stratification
\( \mathcal{M}(n) = \bigcup \mathcal{M}(T) \). This stratification has the following properties (cf. [BG 1]):

\[(1.4.6) \quad \text{codim } \mathcal{M}(T) = \# \text{ internal edges of } T.\]

\[(1.4.7) \quad \mathcal{M}(T) \subset \overline{\mathcal{M}(T')} \iff T \geq T'.\]

In particular, 0-dimensional strata are labelled by binary trees. Codimension 1 strata correspond to trees with two vertices. Their closures are precisely the irreducible components of \( \mathcal{M}(n) - M_{0,n+1} \) which is a normal crossing divisor. Moreover, the entire
stratification above can be recovered by intersecting these components in all possible ways.
In addition, we have the following result.

\[(1.4.8) \textbf{Proposition.} \text{ There are canonical direct product decompositions}\]

\[
\mathcal{M}(T) = \prod_{v \in T} M_{0,|\text{In}(v)|+1},
\]

\[
\overline{\mathcal{M}(T)} = \prod_{v \in T} \mathcal{M}(\text{In}(v)).
\]

In particular, the closure of each stratum is smooth.

Let \( T(m_1, ..., m_l) \) be the tree in Fig.3.

\[(1.4.9) \textbf{Corollary.} \text{ The structure maps (1.2.1)(ii) of the operad } \mathcal{M} \text{ can be identified with}
the embedding of the stratum}\]

\[
\mathcal{M}(l) \times \mathcal{M}(m_1) \times ... \times \mathcal{M}(m_l) = \mathcal{M}(T(m_1, ..., m_l) \hookrightarrow \mathcal{M}(n).
\]
1.5. Operads and sheaves.

(1.5.1) Let \( P \) be a \( k \)-linear operad. The compositions in \( P \) make it possible to construct, for any \( n \), a sheaf \( \mathcal{F}_P(n) \) on the moduli space \( \mathcal{M}(n) \), as we now proceed to explain.

Let \( X \) be any CW-complex and \( S = \{X_\alpha\} \) be a Whitney stratification [GM] of \( X \) into connected strata \( X_\alpha \). We say that a sheaf \( \mathcal{F} \) of \( k \)-vector spaces on \( X \) is \( S \)-combinatorial if the restriction of \( \mathcal{F} \) to each stratum \( X_\alpha \) is a constant sheaf. Thus “\( S \)-combinatorial” is more restrictive than “\( S \)-constructible” [KS] [GM] which means that \( \mathcal{F}|_{X_\alpha} \) are only locally constant.

It is well known that giving an \( S \)-combinatorial sheaf \( \mathcal{F} \) is equivalent to giving the following linear algebra data:

(i) Vector spaces \( F_\alpha = H^0(X_\alpha, \mathcal{F}) \), one for each stratum \( X_\alpha \);
(ii) Generalization maps \( g_{\alpha\beta} : F_\alpha \rightarrow F_\beta \) defined whenever \( X_\alpha \subset X_\beta \) and satisfying the transitivity condition:

\[
g_{\alpha\gamma} = g_{\beta\gamma} \circ g_{\alpha\beta} \quad \text{if} \quad X_\alpha \subset X_\beta \subset X_\gamma.
\]

(1.5.2) We now consider the moduli space \( \mathcal{M}(n) \) defined in §1.4 together with the stratification \( \mathcal{M}(n) = \bigcup_T \mathcal{M}(T) \) labelled by the set of \( n \)-trees. A sheaf on \( \mathcal{M}(n) \) combinatorial with respect to this stratification will be referred to as a “combinatorial sheaf of \( \mathcal{M}(n) \)”.

Let \( P \) be a \( k \)-linear operad. To any \( n \)-tree \( T \) we have associated in (1.2.13) a vector space \( P(T) \) and to any pair of \( n \)-trees \( T' \leq T \) we have associated a linear map \( \gamma_{T,T'} : P(T) \rightarrow P(T') \). Note that we have an inclusion \( \mathcal{M}(T) \subset \mathcal{M}(T') \) precisely when \( T' \leq T \). Thus associating to a stratum \( \mathcal{M}(T) \) the space \( \mathcal{M}(T) \) as generalization maps, we get a combinatorial sheaf on \( \mathcal{M}(n) \) which we denote by \( \mathcal{F}_P(n) \).

The sheaves \( \mathcal{F}_P(n) \) for different \( n \) enjoy certain compatibility with the operad structure on \( \{\mathcal{M}(n)\} \). Those compatibility properties are formalized in the next subsection. Before proceeding to do this, let us agree on the following shorthand notation. If \( X,Y \) are topological spaces, \( \mathcal{F} \) is a sheaf on \( X \) and \( \mathcal{G} \) is a sheaf on \( Y \), then the sheaf \( p_X^* \mathcal{F} \otimes p_Y^* \mathcal{G} \) on \( X \times Y \), where \( p_X : X \times Y \rightarrow X, p_Y : X \times Y \rightarrow Y \) are the natural projections, will be shortly denoted by \( \mathcal{F} \otimes \mathcal{G} \). Similarly for more spaces.

(1.5.3) Sheaves on an operad. Let \( Q \) be a topological operad and

\[
\gamma_{m_1,\ldots,m_l} : Q(l) \times Q(m_1) \times \ldots \times Q(m_l) \rightarrow Q(m_1 + \ldots + m_l)
\]

be its structure maps. A sheaf on \( Q \) is, by definition, a collection \( \mathcal{F} = \{\mathcal{F}(n)\} \) where \( \mathcal{F}(n) \) is a sheaf of \( k \)-vector spaces on \( Q(n) \) together with the following data:
(i) A structure of $\Sigma_m$ - equivariant sheaf on $F(n)$.

(ii) For each $m_1, ..., m_l$, a homomorphism of sheaves on $Q(l) \times Q(m_1) \times ... \times Q(m_l)$:

$$r_{m_1,...,m_l} : \gamma^{*}_{m_1,...,m_l} F(m_1 + ... + m_l) \to F(l) \otimes F(m_1) \otimes ... \otimes F(m_l).$$

(iii) A homomorphism $\epsilon : F(1) \to k$ where $F(1)$ is the stalk of the sheaf $F(1)$ at the point $1 \in Q(1)$.

These data should satisfy three conditions of associativity, equivariance and counit which we now explain.

(1.5.4) The coassociativity condition. Observe that the associativity condition for the topological operad $Q$ amounts to the commutativity, for all $l, m_1, ..., m_l$, $m_{ij}, i = 1, ..., l, j = 1, ..., \nu_i$, of the diagrams

$$Q(l) \times \prod_i Q(m_i) \times \prod_{i,j} Q(m_{ij}) \xrightarrow{1 \times \gamma} Q(l) \times \prod_i Q\left(\sum_j m_{ij}\right)$$

Correspondingly, the structure data (ii) for a sheaf $F$ give rise to the following diagram of sheaves on $Q(l) \times \prod_i Q(m_i) \times \prod_{i,j} Q(m_{ij})$:

$$(\gamma \circ (1 \times \gamma))^* F\left(\sum_{i,j} m_{ij}\right) \xrightarrow{r} (1 \times \gamma)^* \left(F(l) \otimes \bigotimes_i F\left(\sum_j m_{ij}\right)\right)$$

It is required that all such diagrams commute.

(1.5.5) The equivariance condition. It is required, first of all, that the map $r_{m_1,...,m_l}$ commutes with the natural action of $\Sigma_m \times ... \times \Sigma_m$ on $F(m_1 + ... + m_l)$ and $F(m_1) \otimes ... \otimes F(m_l)$.

Secondly, for any permutation $s \in \Sigma_l$ we denote by $\tilde{s} \in \Sigma_{m_1+...+m_l}$ the block permutation which moves segments of lengths $m_1, ..., m_l$ according to $s$. Note that the equivariance condition for the topological operad $Q$ implies the commutativity of the diagram

$$\text{Id} \times s \quad Q(l) \times Q(m_1) \times ... \times Q(m_l) \xrightarrow{\gamma} Q(m_1 + ... + m_l)$$

It is now required that the following diagram of sheaves on $Q(l) \times Q(m_1) \times ... \times Q(m_l)$ commutes:
\[
\left( (\gamma \circ (\tilde{s} \times \text{Id}))^* \mathcal{F}(m_{s(1)} + \ldots + m_{s(l)}) \right) \overset{\text{r}}{\rightarrow} (s \times \text{Id})^* \left[ \mathcal{F}(l) \otimes \mathcal{F}(m_{s(1)}) \otimes \ldots \otimes \mathcal{F}(m_{s(l)}) \right]
\]
\[
\gamma^* \mathcal{F}(m_1 + \ldots + m_l) \overset{\text{r}}{\rightarrow} \mathcal{F}(l) \otimes \mathcal{F}(m_1) \otimes \ldots \otimes \mathcal{F}(m_l)
\]
where “eq.” denotes morphisms of equivariance.

(1.5.6) The counit condition. Recall that the element \(1 \in Q(1)\) is such that the composition
\[
Q(l) \overset{j}{\rightarrow} Q(l) \times Q(1) \times \ldots \times Q(1) \overset{\gamma}{\rightarrow} Q(l)
\]
where \(j(q) = (q, 1, \ldots, 1)\), is the identity map. It is required, in addition, that the composition
\[
\mathcal{F}(l) = (\gamma \circ j)^* \mathcal{F}(l) \overset{\text{r}}{\rightarrow} j^* \left( \mathcal{F}(l) \otimes \mathcal{F}(1) \otimes \ldots \otimes \mathcal{F}(1) \right) \overset{\text{1d} \otimes \epsilon \otimes \ldots \otimes \epsilon}{\rightarrow} \mathcal{F}(l)
\]
is the identity homomorphism.

This completes the definition of a sheaf on a topological operad.

(1.5.7) Example. Let \(Pt\) be the topological operad having \(Pt(n) = \{pt\}\) (one-point space) for any \(n\) and all the structure maps being the identities. We call \(Pt\) the trivial operad (it defines associative and commutative \(H\)-spaces, see [May]). Let \(\mathcal{F}\) be a sheaf on \(Pt\). For any \(n\) the sheaf \(\mathcal{F}\) on the space \(Pt(n)\) is just a vector space \(F(n)\). The structure data (i) - (iii) of (1.5.3) amount to \(\Sigma_n\) - action on the space \(F(n)\), linear maps
\[
F(m_1 + \ldots + m_l) \rightarrow F(l) \otimes F(m_1) \otimes \ldots \otimes F(m_l)
\]
and a linear functional \(\epsilon : F(1) \rightarrow k\).

It is immediate to see that these data make the collection of the dual vector spaces \(F(n)^*\) into a \(k\)-linear operad. Conversely, every \(k\)-linear operad \(P\) with finite-dimensional spaces \(P(n)\) defines a sheaf on the operad \(Pt\).

(1.5.8) Remark. One might say that the \(F(n)\) in the above example form a cooperad, a structure dual to that of an operad. We prefer to postpone the discussion of this structure until Chapter 3.

The considerations of (1.5.7) can be generalized as follows.

(1.5.9) Proposition. Let \(Q\) be a topological operad and \(\mathcal{F}\) be a sheaf on \(Q\). Then the graded spaces \(H^\bullet(Q(n), \mathcal{F}(n))^*\) (the duals to the total cohomology spaces) form an operad in the category \(g\text{Vect}^-\) of graded vector spaces. Moreover, for any \(q \geq 0\) the subspaces \(H^q(n-1)(Q(n), \mathcal{F}(n))^*\) form a sub-operad.

Proof: Obvious from the axioms. Left to the reader.
The fact (1.4.1) that the homology spaces $H_\bullet(Q(n), k)$ form an operad is a particular case of this proposition. Indeed, for any topological operad $Q$ the constant sheaves $k_{Q(n)}$ form a sheaf on $Q$.

(1.5.10) Let $Q$ be a topological operad and $\mathcal{F}$ be a sheaf on $Q$. We say that $\mathcal{F}$ is an *iso* - sheaf if all the maps $r_{m_1,...,m_l}$ (data (ii) of a sheaf) are isomorphisms. Now we can formulate the precise relation between $k$ - linear operads and sheaves on the configuration operad $\mathcal{M}$.

(1.5.11) **Theorem.** a) If $\mathcal{P}$ be a $k$ - linear operad, then the sheaves $\mathcal{F}_{\mathcal{P}}(n)$ on the spaces $\mathcal{M}(n)$ introduced in (1.5.2) form a sheaf $\mathcal{F}_{\mathcal{P}}$ on the operad $\mathcal{M}$.

b) If $\mathcal{P}(1) = k$ then $\mathcal{F}_{\mathcal{P}}$ is an iso - sheaf.

c) Any combinatorial iso - sheaf $\mathcal{F}$ on $\mathcal{M}$ has the form $\mathcal{F}_{\mathcal{P}}$ for some $k$ - linear operad $\mathcal{P}$ with $\mathcal{P}(1) = k$.

The proof is straightforward and left to the reader.

(1.5.12) **Remark.** If one replaces in (1.5.11)(c) the adjective “combinatorial” by “constructible” then one obtains a notion of a *braided operad* introduced by Fiedorowicz [F].
1.6. Modules over an algebra over an operad.

(1.6.1) Let $\mathcal{P}$ be a $k$ - linear operad and $A$ be a $\mathcal{P}$ - algebra. An $A$ - module (or a $(\mathcal{P}, A)$ - module, if $\mathcal{P}$ is to be specified explicitly) is a $k$ - vector space $M$ together with a collection of linear maps

$$q_n : \mathcal{P}(n) \otimes A^{\otimes (n-1)} \otimes M \rightarrow M, \quad n \geq 1$$

satisfying the following conditions:

(i) (Associativity) For any natural numbers $n, r_1, ..., r_n$, any elements $\lambda \in \mathcal{P}(n)$, $\mu_i \in \mathcal{P}(r_i)$, $a_{ij} \in A$, $i = 1, ..., n - 1$ and also $a_{n1}, ..., a_{nr_{n-1}} \in A$, $m \in M$ we have the equality

$$q_{m_1 + ... + m_n}(\lambda(\mu_1, ..., \mu_n) \otimes a_{11} \otimes ... \otimes a_{1r_1} \otimes a_{21} \otimes ... \otimes a_{nr_{n-1}} \otimes m) =$$

$$= q_n\left(\lambda \otimes \mu_1(a_{11}, ..., a_{1r_1}) \otimes ... \otimes \mu_{n-1}(a_{n-1,1}, ..., a_{n-1,r_{n-1}}) \otimes \right.$$}

$$\left.\otimes q_{r_n}(\mu_n \otimes a_{n1} \otimes ... \otimes a_{nr_{n-1}} \otimes m)\right).$$

(ii) (Equivariance) The map $q_n$ is equivariant with respect to the action of $\Sigma_{n-1} \subset \Sigma_n$ on $\mathcal{P}(n) \otimes A^{\otimes (n-1)} \otimes M$ given by

$$s(\lambda \otimes (a_1 \otimes ... \otimes a_{n-1}) \otimes m) = s^*(\lambda) \otimes (a_{s(1)} \otimes ... \otimes a_{s(n-1)}) \otimes m.$$

(iii) (Unit) We have $q_1(1 \otimes 1 \otimes m) = m$ for any $m \in M$ where $1 \in \mathcal{P}(1)$ is the unit of $\mathcal{P}$ and $1$ is the unit element of $k = A^{\otimes 0}$.

We often write $\lambda(a_1, ..., a_{n-1}, m)$ for $q_n(\lambda \otimes (a_1 \otimes ... \otimes a_{n-1}) \otimes m)$.

(1.6.2) Examples. a) Every $\mathcal{P}$ - algebra is a module over itself.

b) Let $\mathcal{P} = \mathcal{As}$ and $A$ be a $\mathcal{P}$ - algebra, i.e., an associative algebra. An $(\mathcal{As}, A)$ - module is the same as an $A$ - bimodule in the usual sense. Indeed, let us realize $\mathcal{As}(n)$ as the space of non-commutative polynomials in $x_1, ..., x_n$ spanned by the monomials $x_{s(1)} \cdot ... \cdot x_{s(n)}$, $s \in \Sigma_n$, see (1.3.6). Given an $A$ - bimodule $M$, we define the map $q_n : \mathcal{As}(n) \otimes A^{\otimes (n-1)} \otimes M \rightarrow M$ by the rule

$$q_n(x_{s(1)} \cdot ... \cdot x_{s(n)} \otimes (a_1 \otimes ... \otimes a_{n-1}) \otimes m) = a_{s(1)} \cdot ... \cdot a_{s(n)}$$

where we set $a_n = m$.

c) Let $\mathcal{P} = \mathcal{Com}$ and $A$ be a $\mathcal{P}$ - algebra, i.e., a commutative algebra. A $(\mathcal{Com}, A)$ - module is the same as an $A$ - module in the usual sense.
d) Let $\mathcal{P} = \text{Lie}$ and $A$ be a Lie algebra. A $(\text{Lie}, A)$-module is the same as an $A$-module (representation of the Lie algebra $A$) in the usual sense.

(1.6.3) Let $M, M'$ be two $(\mathcal{P}, A)$-modules. A morphism of modules $f : M \to M'$ is a $k$-linear map such that $f(\lambda(a_1, ..., a_{n-1}, m)) = \lambda(a_1, ..., a_{n-1}, f(m))$ for any $n$, any $\lambda \in \mathcal{P}(n), a_i \in A, m \in M$. Clearly, all $(\mathcal{P}, A)$-modules form an Abelian category.

(1.6.4) The universal enveloping algebra. As for every Abelian category, it is natural to expect that the category of $(\mathcal{P}, A)$-modules can be described in terms of left modules over a certain associative algebra. Such an algebra indeed exists and is called the universal enveloping algebra of $A$, to be denoted $U_A$ or $U_{\mathcal{P}}(A)$. Here is the construction.

By definition, $U_A$ is generated by symbols $X(\lambda; a_1, ..., a_{n-1})$, for every $n$, every $\lambda \in \mathcal{P}(n)$ and every $a_1, ..., a_{n-1} \in A$. (In particular, when $n = 1$ then no $a_i$ should be specified and we have generators $X(\lambda), \lambda \in \mathcal{P}(1)$). These symbols are subject to conditions of polylinearity with respect to each argument and to the following identifications:

\begin{equation}
X(\lambda; \mu_1(a_{11}, ..., a_{1r_1}), ..., \mu_{n-1}(a_{n-1,1}, ..., a_{n-1,r_{n-1}})) = \\
= X(\lambda(\mu_1, ..., \mu_{n-1}, \mathbf{1}); a_{11}, ..., a_{n-1,r_{n-1}})
\end{equation}

for any $\lambda \in \mathcal{P}(n), \mu_i \in \mathcal{P}(m_i), a_{ij} \in A$.

The multiplication in the algebra $U_A$ is given by the formula:

\[X(\lambda; a_1, ..., a_{n-1})X(\mu, b_1, ..., b_{l-1}) = X(\lambda(\mathbf{1}, ..., \mathbf{1}, \mu); a_1, ..., a_{n-1}, b_1, ..., b_{l-1}).\]

It is immediate to verify that we get in this way an associative algebra $U_A$ and the image of $X(\mathbf{1})$ is the unit of this algebra.

Heuristically, $X(\lambda; a_1, ..., a_{n-1})$ corresponds to the operator

\[m \mapsto \lambda(a_1, ..., a_{n-1}, m)\]

which acts on every $(\mathcal{P}, A)$-module $M$.

The following fact is obvious and its proof is left to the reader.

(1.6.6) Proposition. Let $A$ be a $\mathcal{P}$-algebra. The category of $A$-modules is equivalent to the category of left modules over the associative algebra $U_{\mathcal{P}}(A)$.

(1.6.7) Examples. a) If $\mathcal{G}$ is a Lie algebra then $U_{\text{Lie}}(\mathcal{G})$ defined above is the ordinary universal enveloping algebra of $\mathcal{G}$.

b) If $A$ is an associative algebra and we regard it as an $As$-algebra then $U_{As}(A) = A \otimes A^{\text{op}}$. 

26
c) If $A$ is an accosiative commutative algebra and we regard it as $\text{Com}$-algebra then $U_{\text{Com}}(A) = A$.

(1.6.8) By construction, the universal enveloping algebra $U_P(A)$ has the form

$$U_P(A) = \bigoplus P(n) \otimes A^{\otimes(n-1)} / \equiv$$

where $\equiv$ is the equivalence relation described in (1.6.5). The $n$-th summand above is just the space of generators $X(\lambda; a_1, \ldots, a_{n-1})$.

Observe that $U(A) = U_P(A)$ has a natural (multiplicative) filtration $F_\bullet$ where $F_nU(A)$ is the subspace consisting of images of generators $X(\lambda; a_1, \ldots, a_{r-1})$ for $r \leq n$. 
2. QUADRATIC OPERADS.

2.1. Description of operads by generators and relations.

(2.1.1) Let $K$ be an associative $k$-algebra. Given an arbitrary $K$-collection $E = \{E(n), n \geq 2\}$ (1.2.11) we define an operad $F(E)$ called the free operad generated by $E$. By definition

$$F(E)(n) = \bigoplus_{n \text{-trees } T} E(T)$$

where $T$ runs over isomorphism classes of $n$-trees and $E(T)$ was defined in (1.2.13). The composition maps (1.2.1)(ii) for $F(E)$

$$F(E)(l) \otimes F(E)(m_1) \otimes ... \otimes F(E)(m_l) \to F(E)(m_1 + ... + m_l)$$

are defined by means of maps

(2.1.2) $E(T) \otimes E(T_1) \otimes ... \otimes E(T_l) \to E(T(T_1, ..., T_l))$

where $T$ is an $l$-tree, $T_i, i = 1, ..., l$, is a $m_i$-tree and $T(T_1, ..., T_l)$ is their composition, see Fig.6.

The definition of the map (2.1.2) is obvious, keeping in mind that both the LHS and RHS are tensor products of the same spaces but over different rings (some over $k$ and some over $K$).

(2.1.3) Ideals. Let $\mathcal{P} = \{\mathcal{P}(n)\}$ be a $k$-linear operad. A (two-sided) ideal in $\mathcal{P}$ is a collection $\mathcal{I}$ of vector subspaces $\mathcal{I}(n) \subset \mathcal{P}(n)$ satisfying the following three conditions:
(i) For each $n$ the space $\mathcal{I}(n)$ is preserved by the action of $\Sigma_n$ on $\mathcal{P}(n)$.

(ii) If $\lambda \in \mathcal{P}(n), \mu_1 \in \mathcal{P}(m_1), \ldots, \mu_n \in \mathcal{P}(m_n)$, and for at least one $j$ we have $\mu_j \in \mathcal{I}(m_j)$, then the composition $\lambda(\mu_1, \ldots, \mu_n)$ belongs to $\mathcal{I}(m_1 + \ldots + m_n)$.

(iii) If $\lambda \in \mathcal{I}(n)$ and $\mu_i \in \mathcal{P}(m_i), i = 1, \ldots, n$, then $\lambda(\mu_1, \ldots, \mu_n) \in \mathcal{I}(m_1 + \ldots + m_n)$.

If $\mathcal{I}$ is an ideal in an operad $\mathcal{P}$ then we can construct the quotient operad $\mathcal{P}/\mathcal{I}$ with components $(\mathcal{P}/\mathcal{I})(n) = \mathcal{P}(n)/\mathcal{I}(n)$ (quotient linear spaces). The conditions (ii) and (iii) above imply that compositions in $\mathcal{P}$ induce well defined compositions in $\mathcal{P}/\mathcal{I}$.

It is straightforward to see that the kernel of a morphism of $k$-linear operads $f : \mathcal{P} \rightarrow \mathcal{Q}$ is an ideal in $\mathcal{P}$.

(2.1.4) Let $V$ be a finite-dimensional $k$-vector space and let $\mathcal{L}(V), \mathcal{A}(V), \mathcal{C}(V)$ be respectively the free Lie algebra, free associative (tensor) algebra and free commutative (polynomial) algebra generated by $V$. There are canonical linear maps

$$\mathcal{L}(V) \xhookrightarrow{\epsilon} \mathcal{A}(V) \xrightarrow{\pi} \mathcal{C}(V).$$

Both maps are the identity on $V$ and are uniquely determined by the requirement that $\pi$ is a homomorphism of associative algebras and $\epsilon$ is a homomorphism of Lie algebras (with the structure of a Lie algebra on $\mathcal{A}(V)$ given by $[a,b] = ab - ba$). The maps (2.1.5) give rise to a collection of linear maps

$$\mathcal{L}(n) \xhookrightarrow{\epsilon_n} \mathcal{A}(n) \xrightarrow{\pi_n} \mathcal{C}(n), \quad n = 1, 2, \ldots$$

such that $\pi_n \circ \epsilon_n = 0$ for $n \geq 2$.

These maps give morphisms of operads $\mathcal{L} \xhookrightarrow{\epsilon} \mathcal{A} \xrightarrow{\pi} \mathcal{C}$. Let $\mathcal{L}^+$ be the collection of spaces $\{\mathcal{L}(n), n \geq 2\}$ and $(\mathcal{L}^+)$ be the minimal ideal in $\mathcal{A}$ containing $\mathcal{L}^+$. The proof of the following result is left to the reader.

(2.1.6) Proposition. $(\mathcal{L}^+) = \text{Ker}(\pi)$ that is, $\mathcal{C} \cong \mathcal{A}/(\mathcal{L}^+)$. 

(2.1.7) Quadratic operads. Let $K$ be a semisimple $k$-algebra. Let $E$ be a $(K, K^\otimes 2)$-bimodule with an involution $\sigma : E \rightarrow E$ such that

$$\sigma(\lambda e) = \lambda \sigma(e), \quad \sigma(e \cdot (\lambda_1 \otimes \lambda_2)) = \sigma(e) \cdot (\lambda_2 \otimes \lambda_1), \quad \forall \lambda, \lambda_1, \lambda_2 \in K, e \in E.$$

We form the space $E \otimes_K E$, the tensor product with respect to the right $K$-module structure on the first factor given by $e \cdot \lambda = e \cdot (\lambda \otimes 1)$. This space has two structures:

(i) A $\Sigma_2$-action given by the action of $\sigma$ on the second factor $E$.

(iii) A structure of $(K, K^\otimes 3)$-bimodule.
Therefore the induced $\Sigma_2$ - module $\text{Ind}_{\Sigma_2}^{\Sigma_3}(E \otimes_K E)$ inherits the $(K, K^{\otimes 3})$ - bimodule structure. Let $R \subset \text{Ind}_{\Sigma_2}^{\Sigma_3}(E \otimes_K E)$ be a $\Sigma_3$ - stable $(K, K^{\otimes 3})$ - sub - bimodule. To any such data $(E, R)$ we associate an operad $\mathcal{P}(K, E, R)$ in the following way.

We form the $K$ - collection $\{E(2) = E, E(n) = 0, n > 2\}$ (denoted also by $E$) and the corresponding free operad $F = F(E)$. Observe that $F(E)(3) = \text{Ind}_{\Sigma_2}^{\Sigma_3}(E \otimes_K E)$, see Fig.7.

More generally,

\begin{equation}
F(E)(n) = \bigoplus_{\text{binary trees } T} E(T).
\end{equation}

Let $(R)$ be the ideal in $F(E)$ generated by the subspace $R \subset F(E)(3)$. We put $\mathcal{P} = \mathcal{P}(K, E, R) = F(E)/(R)$. An operad of type $\mathcal{P}(K, E, R)$ is called a quadratic operad with the space of generators $E$ and the space of relations $R$. Note that $K, E, R$ can be recovered from $\mathcal{P}$ as $K = \mathcal{P}(1), E = \mathcal{P}(2)$, and $R = \text{Ker}\{F(E)(3) \to \mathcal{P}(3)\}$.

(2.1.9) The quadratic duality. Given a semisimple $k$ - algebra $K$ and a finite - dimensional left $K$ - module $V$ with a $\Sigma_n$ - action, we define $V^\vee = \text{Hom}_K(V, K)$. This is a right $K$ - module, i.e., a left module over the opposite algebra $K^{op}$. We always equip $V^\vee$ with the transposed action of $\Sigma_n$ twisted by the sign representation.

Let $\mathcal{P} = \mathcal{P}(K, E, R)$ be a quadratic operad. The space $E^\vee$ has a natural structure of $(K^{op}, K^{op} \otimes K^{op})$ - bimodule. Observe that $F(E^\vee)(3) = (F(E))(3)^\vee$. Let $R^\perp \subset F(E^\vee)(3)$ be the orthogonal complement of $R$. It is stable under the $\Sigma_3$ - action and the three $K^{op}$ - actions on $F(E^\vee)(3)$. We define the dual quadratic operad $\mathcal{P}^!$ to be

\[ \mathcal{P}^! = \mathcal{P}(K^{op}, E^\vee, R^\perp). \]

(2.1.10) Examples. Suppose that $K = k$ and $\mathcal{P} = \mathcal{P}(k, E, R)$ is a quadratic operad. A $\mathcal{P}$ - algebra is a vector space $A$ with several binary operations (parametrized by $E$) which are
subject to certain identities (parametrized by $R$) each involving three arguments. This is precisely the way of defining the types of algebras most commonly encountered in practice. In particular, the operads $As, Com, Lie$ governing, respectively, associative, commutative and Lie algebras, are quadratic.

Note that the structure constants of an algebra $A$ over a quadratic operad satisfy quadratic relations. For example, if $\{e_i\}$ is a basis of $A$, define $\mu : A \times A \to A$ by $\mu(e_i \otimes e_j) = \sum c^k_{ij}e_k$. The condition that $\mu$ is associative means that

$$\sum_k c^k_{ij}c^m_{kl} = \sum_k c^k_{jk}c^m_{ik} \quad \forall i, j, m.$$ 

Similarly in other examples. This explains the name “quadratic operad.”

(2.1.11) Theorem. We have isomorphisms of operads

$$Com^! = Lie, \quad Lie^! = Com, \quad As^! = As.$$ 

The idea that the “worlds” of commutative, Lie and associative algebras are, in some sense, dual to each other as described above, was promoted by Drinfeld [Dr] and Kontsevich [Kon 2] (the $Com − Lie$ duality was implicit already in Quillen’s paper [Q 1] and in Moore’s Nice talk [Mo]). The concept of quadratic operads and their Koszul duality allows us to make this idea into a theorem. Observe also that the isomorphism $Com \cong As/(Lie^+)$ of Proposition 2.1.6 is, in a sense, “self - dual”.

Proof of the theorem: The group $\Sigma_3$ has three irreducible representations which we denote 1 (the identity representation), $Sgn$ (the sign representation) and $V_2$ (the 2-dimensional representation in the hyperplane $\sum x_i = 0$ in the 3-dimensional space of $(x_1, x_2, x_3)$). Both $Com$ and $Lie$ have 1-dimensional spaces of generators with $\Sigma_2$ acting trivially on $Com(2)$ and by sign on $Lie(2)$. An elementary calculation of group characters shows that we have isomorphisms of $\Sigma_3$ - modules

$$F(Com(2))(3) = 1 \oplus V_2, \quad F(Lie(2))(3) = Sgn \oplus V_2.$$ 

This implies that we have

$$Com(3) = 1, \quad R_{Com} = V_2, \quad Lie(3) = V_2, \quad R_{Lie} = Sgn.$$ 

The duality between $Com$ and $Lie$ follows.

To prove that $As^! = As$, we consider the space $F(As(2))(3)$. This space has dimension 12 and is spanned by the 12 expressions of the form $x_{s(1)}(x_{s(2)}x_{s(3)}), x_{s(1)}(x_{s(2)}x_{s(3)}), s \in \Sigma_3$. 

31
We introduce on this space a scalar product $<,>$ by setting all these products orthogonal to each other and putting

$$<x_i(x_jx_k),x_i(x_jx_k)> = \text{sgn} \left( \begin{array}{ccc} 1 & 2 & 3 \\ i & j & k \end{array} \right), \quad <(x_ix_j)x_k,(x_ix_j)x_k> = -\text{sgn} \left( \begin{array}{ccc} 1 & 2 & 3 \\ i & j & k \end{array} \right).$$

This product is sign-invariant with respect to the $\Sigma_3$-action, i.e., $<s(\mu),s(\nu)> = \text{sgn}(s) <\mu,\nu>$. The (6-dimensional) space of relations $R_{\mathcal{A}s}$ is spanned by all the associators $(x_i(x_jx_k) - (x_ix_j)x_k$. This space coincides with its own annihilator with respect to the described scalar product. This shows that $\mathcal{A}s^1 = \mathcal{A}s$. 
2.2. The analogs of Manin’s tensor products. The Lie operad as a dualizing object.

(2.2.1) In this section we consider only quadratic operads $\mathcal{P}$ with $\mathcal{P}(1) = k$. Such an operad is defined by a vector space of generators $E = \mathcal{P}(2)$ with an involution $\sigma$ (action of $\Sigma_2$) and an $\Sigma_3$-invariant subspace $R$ of relations inside $F(E)(3)$ where $F(E)$ is the free operad generated by $E$. Observe that $F(E)(3)$ is the direct sum of three copies of $E \otimes E$, see Fig.6. So we refer to this space as $3(E \otimes E)$.

Our aim is to present an operad-theoretic version of [Man].

(2.2.2) Let $\mathcal{A}$ be any symmetric monoidal category and $\mathcal{P}, \mathcal{Q}$ be two operads in $\mathcal{A}$. The collection of objects $\mathcal{P}(n) \otimes \mathcal{Q}(n)$ forms a new operad in $\mathcal{A}$ denoted by $\mathcal{P} \otimes \mathcal{Q}$. The main property of this operad is that if $A$ is a $\mathcal{P}$-algebra and $B$ is a $\mathcal{Q}$-algebra (in $\mathcal{A}$) then $A \otimes B$ is a $\mathcal{P} \otimes \mathcal{Q}$-algebra.

We are interested in the case of $k$-linear operads, i.e., $\mathcal{A} = Vect$.

(2.2.3) Suppose that $\mathcal{P}, \mathcal{Q}$ are quadratic operads with $\mathcal{P}(1) = \mathcal{Q}(1) = k$ so $\mathcal{P}(2)$ and $\mathcal{Q}(2)$ are their spaces of generators. We denote by $\mathcal{P} \circ \mathcal{Q}$ the sub-operad in $\mathcal{P} \otimes \mathcal{Q}$ generated by $\mathcal{P}(2) \otimes \mathcal{Q}(2)$. Let $R_P \subset 3(\mathcal{P}(2) \otimes \mathcal{P}(2))$ and $R_Q \subset 3(\mathcal{Q}(2) \otimes \mathcal{Q}(2))$ be the spaces of relations of $\mathcal{P}$ and $\mathcal{Q}$. Then the operad $\mathcal{P} \circ \mathcal{Q}$ is described as follows.

The space of generators of $\mathcal{P} \circ \mathcal{Q}$ is, by definition, $\mathcal{P}(2) \otimes \mathcal{Q}(2)$. The third component of the free operad generated by $\mathcal{P}(2) \otimes \mathcal{Q}(2)$ is $3(\mathcal{P}(2) \otimes \mathcal{Q}(2))$ which can be regarded in two ways as:

1. $(\mathcal{P}(2) \otimes \mathcal{P}(2)) \otimes 3(\mathcal{Q}(2) \otimes \mathcal{Q}(2))$ and regarded as such, it contains the subspace $(\mathcal{P}(2) \otimes \mathcal{P}(2)) \otimes R_Q$.
2. $3(\mathcal{P}(2) \otimes \mathcal{P}(2)) \otimes (\mathcal{Q}(2) \otimes \mathcal{Q}(2))$ and regarded as such, it contains the subspace $R_P \otimes (\mathcal{Q}(2) \otimes \mathcal{Q}(2))$.

(2.2.4) Proposition. If $\mathcal{P}, \mathcal{Q}$ are quadratic then $\mathcal{P} \circ \mathcal{Q}$ is also quadratic with the space of generators $\mathcal{P}(2) \otimes \mathcal{Q}(2)$ and the space of relations

$$(\mathcal{P}(2) \otimes \mathcal{P}(2)) \otimes R_Q + (R_P \otimes (\mathcal{Q}(2) \otimes \mathcal{Q}(2))).$$

(The sum is not necessarily direct.)

The proof is straightforward.

Thus $\mathcal{P} \circ \mathcal{Q}$ is the analog of the white circle product $A \circ B$ for associative algebras considered by Manin [Man].

(2.2.5) Given two quadratic operads $\mathcal{P}, \mathcal{Q}$ as before we define the quadratic operad $\mathcal{P} \bullet \mathcal{Q}$ (the black circle product, cf. [Man]) to have the same space of generators $\mathcal{P}(2) \circ \mathcal{Q}(2)$ as
\( \mathcal{P} \otimes \mathcal{Q} \) but the space of relations
\[
( (\mathcal{P}(2) \otimes \mathcal{P}(2)) \otimes R_{\mathcal{Q}} ) \cap ( R_{\mathcal{P}} \otimes (\mathcal{Q}(2) \otimes \mathcal{Q}(2)) ).
\]

(2.2.6) Theorem. Each of the products \( \circ, \bullet \) defines on the category of quadratic operads (and morphisms defined in (1.3.1)) a symmetric monoidal structure. Moreover, we have:

(a) \( (\mathcal{P} \circ \mathcal{Q})^! = \mathcal{P}^! \cdot \mathcal{Q}^! \).

(b) \( \text{Hom}(\mathcal{P} \cdot \mathcal{Q}, \mathcal{R}) = \text{Hom}(\mathcal{P}, \mathcal{Q}^! \circ \mathcal{R}) \). In particular, the commutative operad is a unit object with respect to \( \circ \) and the Lie operad is a unit object with respect to \( \bullet \).

Proof: a) It is similar to the argument of Manin [Man] for quadratic algebras by using the obvious relations between sums, intersections and orthogonal complements of subspaces in a vector space.

b) The fact that \( \text{Com} \) is a unit object with respect to \( \circ \) follows because \( \text{Com}(n) = k \) for any \( n \). The fact that \( \text{Lie} \) is a unit object with respect to \( \bullet \), follows from part a) and the fact that \( \text{Lie}^! = \text{Com} \).

(2.2.7) Given quadratic operads \( \mathcal{P} \) and \( \mathcal{Q} \), we define the quadratic operad \( \text{hom}(\mathcal{P}, \mathcal{Q}) \) as follows. Its space of generators is set to be
\[
\text{hom}(\mathcal{P}, \mathcal{Q})(2) = \text{Hom}_k(\mathcal{P}(2), \mathcal{Q}(2)).
\]

The space of relations \( R_{\text{hom}(\mathcal{P}, \mathcal{Q})} \) is defined to be the minimal subspace in \( F(\text{Hom}_k(\mathcal{P}(2), \mathcal{Q}(2))))(3) \) such that the canonical map
\[
\text{can} : \mathcal{Q}(2) \to \text{Hom}_k(\mathcal{P}(2), \mathcal{Q}(2)) \otimes \mathcal{P}(2) = \mathcal{P}(2)^* \otimes \mathcal{Q}(2) \otimes \mathcal{P}(2)
\]
extends to a morphism of operads \( \mathcal{Q} \hookrightarrow \text{hom}(\mathcal{P}, \mathcal{Q}) \circ \mathcal{P} \). More precisely, we define \( R_{\text{hom}(\mathcal{P}, \mathcal{Q})} \) to be minimal among subspaces \( J \) with the property that
\[
J \otimes (\mathcal{P}(2) \otimes \mathcal{P}(2)) \subset F(\text{Hom}_k(\mathcal{P}(2), \mathcal{Q}(2))) \otimes \mathcal{P}(2))(3)
\]
contains the image of the embedding
\[
\text{can}_* : F(\mathcal{Q}(2))(3) \hookrightarrow F(\text{Hom}_k(\mathcal{P}(2), \mathcal{Q}(2)) \otimes \mathcal{P}(2))(3)
\]
induced by \( \text{can} \).

If \( \mu_1, \ldots, \mu_m \) is a basis of \( \mathcal{P}(2) \), \( \nu_1, \ldots, \nu_n \) is a basis of \( \mathcal{Q}(2) \) and \( a_{ij} \), \( i = 1, \ldots, m \), \( j = 1, \ldots, n \) is the corresponding basis of \( \text{Hom}(\mathcal{P}(2), \mathcal{Q}(2)) \) then the elements
\[
\tilde{\nu}_i = \sum_j a_{ij} \otimes \mu_j \in (\text{hom}(\mathcal{P}, \mathcal{Q}) \circ \mathcal{P})(2)
\]
satisfy all the quadratic relations holding for actual \( \nu_i \).

34
(2.2.8) **Theorem.** We have a natural isomorphism of operads

\[ \text{hom}(\mathcal{P}, \mathcal{Q}) \cong \mathcal{P} \bullet \mathcal{Q}. \]

**Proof:** Simple linear algebra.

(2.2.9) **Corollary.** a) For any quadratic operad \( \mathcal{P} \) we have

\[ \mathcal{P} \bullet = \text{hom}(\mathcal{P}, \text{Lie}). \]

b) There exists a morphism of operads \( \text{Lie} \to \mathcal{P} \otimes \mathcal{P} \) which takes the generator of the 1-dimensional space \( \text{Lie}(2) \) into the identity operator in \( \mathcal{P}(2) \otimes \mathcal{P}(2) = \mathcal{P}(2) \otimes \mathcal{P}(2)^* \). In particular, for any \( \mathcal{P} \)-algebra \( A \) and \( \mathcal{P} \)-algebra \( B \) the vector space \( A \otimes_k B \) has a natural Lie algebra structure.

(2.2.10) Let us give another interpretation of the operad \( \text{hom}(\mathcal{P}, \mathcal{Q}) \) in terms of algebras.

Suppose that \( V_1, ..., V_m, W \) are \( k \)-vector spaces. By a \( \mathcal{P} \)-multilinear map \( V_1 \times \cdots \times V_m \to W \) we understand an element

\[ \Phi \in F_{\mathcal{P}}(V_1^* \oplus \cdots \oplus V_m^*) \otimes W \]

which is homogeneous of degree 1 with respect to dilatations of any of \( V_i \). (Recall (1.3.6) that \( F_{\mathcal{P}} \) means the free \( \mathcal{P} \)-algebra generated by a vector space). Such maps form a vector space which we denote by \( \text{Mult}_{\mathcal{P}}(V_1, ..., V_m|W) \). For example \( \mathcal{P}(n) = \text{Mult}_{\mathcal{P}}(k, ..., k|k) \) (\( m \) copies of \( k \) before the bar), cf. (1.3.7) - (1.3.9). The space of ordinary multilinear maps, i.e., \( \text{Hom}_k(V_1 \otimes \cdots \otimes V_m, W) \) is nothing but \( \text{Mult}_{\text{Com}}(V_1, ..., V_m|W) \) where \( \text{Com} \) is the commutative operad.

Similarly to usual multilinear maps, \( \mathcal{P} \)-multilinear maps can be composed. In particular, for any vector space \( V \) the spaces

\[ \mathcal{E}_{\mathcal{P}, V}(n) = \text{Mult}_{\mathcal{P}}(V, ..., V|V) \quad (n \text{ copies of } V) \]

form a \( k \)-linear operad which we call the **operad of endomorphisms of \( V \) in \( \mathcal{P} \)**.

(2.2.12) Let \( \mathcal{Q} \) be another \( k \)-linear operad. By a \( \mathcal{Q} \)-algebra in \( \mathcal{P} \) we mean a \( k \)-vector space \( A \) together with a morphism of operads \( f : \mathcal{Q} \to \mathcal{E}_{\mathcal{P}, A} \). When \( \mathcal{P} = \text{Com} \) is the commutative operad, we get the usual notion of a \( \mathcal{Q} \)-algebra.

(2.2.13) **Theorem.** Let \( \mathcal{P} \) and \( \mathcal{Q} \) be quadratic operads with \( \mathcal{P}(1) = \mathcal{Q}(1) = k \). Then \( \mathcal{Q} \)-algebras in \( \mathcal{P} \) are the same as \( \text{hom}(\mathcal{P}, \mathcal{Q}) \)-algebras (in the usual sense).

**Proof:** Let \( A \) be a \( \mathcal{Q} \)-algebra in \( \mathcal{P} \). The corresponding morphism \( f : \mathcal{Q} \to \mathcal{E}_{\mathcal{P}, A} \) is defined by its second component

\[ f_2 : \mathcal{Q}(2) \to \mathcal{E}_{\mathcal{P}, A}(2) = (\mathcal{P}(2) \otimes (A^*)^\otimes \Sigma_2) \otimes A. \]
By taking a partial transpose of $f_2$ we get a $\Sigma_2$ - equivariant map

$$f_2^\dagger : \text{Hom}_k(\mathcal{P}(2), \mathcal{Q}(2)) \rightarrow \text{Hom}_k(A \otimes A, A) = \mathcal{E}_A(2).$$

In order that a given linear map $f_2$ come from a morphism of operads $f : Q \rightarrow \mathcal{E}_{\mathcal{P}, A}$, the quadratic relations among the generators of $Q$ should be satisfied. By definition of relations in $\text{hom}(\mathcal{P}, \mathcal{Q})$ this is equivalent to the condition that $f_2^\dagger$ extends to a morphism $\text{hom}(\mathcal{P}, \mathcal{Q}) \rightarrow \mathcal{E}_A$, i.e., that we have on $A$ a structure of a $\text{hom}(\mathcal{P}, \mathcal{Q})$ - algebra.

(2.2.14) Lazard - Lie theory for formal groups in operads and Koszul duality.

The idea of considering formal groups in operads goes back to an important paper of Lazard [L]. He used the concept of “analyseur” which is essentially equivalent to the modern notion of operad. The main result of Lazard is a version of Lie theory (= correspondence between (formal) Lie groups and Lie algebras) for his generalized formal groups. It turns out that Lazard - Lie theory has a very transparent interpretation in terms of Koszul duality for operads.

We start with formulating basic definitions in the modern language. Let $\mathcal{P}$ be a $k$ - linear operad. We assume $\mathcal{P}(1) = k$. Let $W$ be a $k$ - vector space and

$$\hat{F}_\mathcal{P}(W) = \prod_{n \geq 1} (\mathcal{P}(n) \otimes W^\otimes n)_{\Sigma_n},$$

the completed free $\mathcal{P}$ - algebra on $W$. If $z_1, ..., z_r$ form a basis of $W$ then elements of $\hat{F}_\mathcal{P}(W)$ can be regarded as formal series in $z_1, ..., z_r$ whose terms are products of $z_i$ with respect to operations in $\mathcal{P}$. Thus, for example, if $\mathcal{P} = \text{Com}$, we get the usual power series algebra, if $\mathcal{P} = \mathcal{A}s$, we get the algebra of non - commutative power series etc. Note that our series do not have constant terms since all the free algebras are without unit.

There is a natural projection (on the first factor)

$$\hat{F}_\mathcal{P}(W) \rightarrow W.$$

It can be thought of as the differential at zero.

Let $V, W$ be two $k$ - vector spaces. We define the space of formal $\mathcal{P}$ - maps from $V$ to $W$ as

$$\text{FHom}_\mathcal{P}(V, W) = V \otimes \hat{F}_\mathcal{P}(W^*).$$

For $\Phi \in \hat{F}_\mathcal{P}(W)$ we denote by $d_0 \Phi \in \text{Hom}_k(V, W)$ its differential at 0, i.e., the image of $\Phi$ under the natural projection to $V \otimes W^*$ induced by (2.2.16). There are obvious composition maps

$$\text{FHom}_\mathcal{P}(V, W) \times \text{FHom}_\mathcal{P}(W, X) \rightarrow \text{FHom}_\mathcal{P}(V, X)$$

36
(given by inserting of power series into arguments of other power series) which make the collection of vector spaces and formal $\mathcal{P}$-maps into a category.

A simple generalization of the classical inverse function theorem shows that $\Phi \in \text{FHom}_\mathcal{P}(V,W)$ is invertible if and only if $d_0\Phi \in \text{Hom}_k(V,W)$ is invertible.

\textbf{(2.2.17) Definition.} Let $\mathcal{P}$ be a $k$-linear operad. A $\mathcal{P}$-formal group is a pair $(V, \Phi)$ where $V$ is a $k$-vector space and $\Phi \in \text{FHom}_\mathcal{P}(V \oplus V, V)$ is a formal $\mathcal{P}$-map satisfying the two conditions:

(i) $d_0\Phi : V \otimes V \to V$ is the addition map $(v, v') \mapsto v + v'$.

(ii) $\Phi$ is associative i.e., we have the equality $\Phi(x, \Phi(y, z)) = \Phi(\Phi(x, y), z)$ of formal maps $V \oplus V \oplus V \to V$.

A formal homomorphism of formal $\mathcal{P}$-groups $(V, \Phi)$ and $(W, \Psi)$ is a formal $\mathcal{P}$-map $f \in \text{FHom}_\mathcal{P}(V, W)$ such that $f(\Phi(x, y)) = \Psi(f(x), f(y))$.

\textbf{(2.2.18)} If $(V, \Phi)$ is a $\mathcal{P}$-formal group, then we define, following Lazard, its Lie bracket by

$$[x, y] = \Phi(x, y) - \Phi(y, x) \quad \text{mod. cubic terms}.$$ 

This construction makes $V$ into a Lie algebra in $\mathcal{P}$ in the sense of (2.2.12). Furthermore, we obtain, from Theorem 2.2.13 and the results of Lazard [L], the following theorem.

\textbf{(2.2.19) Theorem.} Let $\mathcal{P}$ be a quadratic operad and $\mathcal{P}^!$ be its quadratic dual. The category of finite-dimensional $\mathcal{P}$-formal groups (and their formal homomorphisms) is equivalent to the category of finite-dimensional $\mathcal{P}^!$-algebras.

A special case of this theorem corresponding to $\mathcal{P} = \mathcal{A}s$, the associative operad, was pointed out to us earlier by M. Kontsevich. Since $\mathcal{A}s^! = \mathcal{A}s$, the theorem in this case says that for formal groups defined by means of power series with non-commuting variables the role of Lie algebras is played by associative algebras (possibly without unit).
2.3. Quadratic algebras over a quadratic operad.

(2.3.1) Let \( P \) be a \( k \) - linear operad and \( A \) a \( P \) - algebra. An *ideal* in \( A \) is a linear subspace \( I \subset A \) such that for any \( n \), any \( \mu \in P(n) \) and any \( a_1, ..., a_{n-1} \in A, i \in I \), we have \( \mu(a_1, ..., a_{n-1}, i) \in I \). Given any ideal \( I \subset A \) the quotient vector space \( A/I \) has a natural structure of a \( P \) - algebra.

(2.3.2) Quadratic algebras. Let \( P = P(K, E, R) \) be a quadratic operad (2.1.7) so that \( K = P(1) \) is a semisimple \( k \) - algebra and \( E = P(2) \) is a \((K, K^{\otimes 2})\) - bimodule. Let also \( V \) be a \( K \) - bimodule. The tensor product \( E \otimes_K V^{\otimes 2} \) has a natural structure of a \((K, K^{\otimes 2})\) - bimodule (the left \( K \) - action comes from that on \( E \) and the right \( K^{\otimes 2} \) - action comes from that on \( V^{\otimes 2} \)). Moreover, there is a \( \Sigma_2 \) - action on \( E \otimes_K V^{\otimes 2} \) given by

\[
\sigma(e \otimes (v_1 \otimes v_2)) = \sigma(e) \otimes (v_2 \otimes v_1).
\]

Observe that the space of coinvariants \( \big( E \otimes_K V^{\otimes 2} \big)_{\Sigma_2} \) inherits a \( K \) - bimodule structure.

Let \( S \subset \big( E \otimes_K V^{\otimes 2} \big)_{\Sigma_2} \) be a \( K \) - sub - bimodule. Given \( V \) and \( S \) we construct a \( P \) - algebra \( A = A(V, S) \) as follows.

Let \( F_P(V) \) be the free \( P \) - algebra generated by \( V \), see (1.3.5). Observe that \( F_P(V)_2 \), the degree 2 graded component of \( F_P(V) \), is \( \big( E \otimes_K V^{\otimes 2} \big)_{\Sigma_2} \). Let \( (S) \subset F_P(V) \) be the ideal generated by \( S \) (= the minimal ideal containing \( S \)). Put

\[
A(V, S) = F_P(V)/(S).
\]

An algebra of this type will be called a *quadratic \( P \) - algebra* (with the space of generators \( V \) and the space of relations \( S \)).

Observe that \( A(V, S) \) has a natural grading \( A(V, S) = \bigoplus_{i \geq 1} A_i(V, S) \). Furthermore, \( P \), as any operad in the category \( Vect \), can be regarded as an operad in the category \( gVect^+ \) of graded vector spaces (1.4.2)(c) and \( A(V, S) \) is a \( P \) - algebra in this category.

(2.3.6) Quadratic superalgebras. Let \( P = P(K, E, R) \) be a quadratic operad as before. We now view \( P \) as an operad in the other category of graded vector spaces (1.4.2) namely \( gVect^- \). A \( P \) - algebra in this category will be referred to as a (graded) \( P \) - *superalgebra*.

Let \( V \) be a \( K \) - bimodule. We replace the \( \Sigma_2 \) - action on \( E \otimes_K V^{\otimes 2} \) given by (2.3.3) by the following one:

\[
\sigma(e \otimes (v_1 \otimes v_2)) = -\sigma(e) \otimes (v_2 \otimes v_1).
\]
Let $S \subset \left( E \otimes_{K \otimes^2} V \otimes^2 \right)_{\Sigma_2}$ be a $K$ - sub - bimodule where the coinvariants are taken with respect to the new action (2.3.7). Given such data $(V, S)$, we define a $\mathcal{P}$ - superalgebra $A(V, S)^-$ as the quotient as the free $\mathcal{P}$ - superalgebra generated by $V$ (placed in degree 1) by the ideal generated by $S$.

(2.3.8) Quadratic duality. Let $\mathcal{P} = \mathcal{P}(K, E, R)$ be a quadratic operad and $\mathcal{P}^! = \mathcal{P}(K^{op}, E^\vee, R^\perp)$ the dual operad (2.1.9). Given a quadratic $\mathcal{P}$ - algebra $A = A(V, S)$ we define the quadratic $\mathcal{P}^!$ - superalgebra $A^! = A(V^\vee, S^\perp)^-$. Here $V^\vee = \text{Hom}_K(V, K)$ and

$$S^\perp \subset \left( \left( E \otimes_{K \otimes^2} V \otimes^2 \right)_{\Sigma_2} \right)^\vee = \left( \left( V^\vee \otimes^2 \otimes_{K \otimes^2} E^\vee \right) \otimes_{(K^{op}) \otimes^2} \left( V^\vee \right) \otimes^2 \right)_{\Sigma_2}$$

is the annihilator of $S$.

In a similar way, given a quadratic $\mathcal{P}$ - superalgebra $B$, we define the dual $\mathcal{P}^!$ - algebra (in the category $gVect^+$) $B^!$. The assignment $A \mapsto A^!$ gives a 1-1 correspondence between quadratic $\mathcal{P}$ - algebras (resp. superalgebras) and quadratic $\mathcal{P}^!$ - superalgebras (resp. algebras).

(2.3.9) Examples. a) Let $\mathcal{P} = A$s be the associative operad. Then $A^s^! = A$s. Note that an $A$s - super algebra (i.e., an $A$s - algebra in $gVect^-$) is the same as an $A$s - algebra (in $gVect^+$). Both concepts give the usual notion of a graded associative algebra. In this case the quadratic duality (2.3.8) reduces to the well known Koszul duality for quadratic associative algebras introduced by Priddy [Pr].

b) Let $\mathcal{P} = \text{Com}$ be the commutative algebra and $A$ be a quadratic $\text{Com}$ - algebra i.e., a quadratic commutative (associative) algebra. Let $A^\text{as}_s^!$ be the Koszul dual of $A$ as of an associative algebra. One can show, see [Q 2] that $A^\text{as}_s^!$ has a structure of a graded - commutative Hopf algebra. Hence $A^\text{as}_s^!$ is the enveloping algebra of a certain graded Lie superalgebra (i.e., a $\text{Lie}$ - algebra in the category $gVect^-$) which we denote $\mathcal{G} = \text{Prim}(A^\text{as}_s^!)$. We have $A^! = \mathcal{G}$.

(2.3.10) Quadratic duality and enveloping algebras. Let $\mathcal{P} = \mathcal{P}(K, E, R)$ be a quadratic operad. Let $A$ be a quadratic $\mathcal{P}$ - algebra. Then we have (1.6.4) the universal enveloping algebra $U_{\mathcal{P}}(A)$, which is an associative algebra in ordinary sende. If $A$ is a quadratic $\mathcal{P}$ - superalgebra, the same construction as in (1.6.4) define its universal enveloping superalgebra $U_{\mathcal{P}}^-(A)$ which is an $\mathcal{A}s$ - algebra in the category $gVect^-$. As noted in (2.3.9)(a), we can regard $U_{\mathcal{P}}^-(A)$ as an ordinary graded associative algebra.

(2.3.11) Theorem. If $A$ is a quadratic $\mathcal{P}$ - algebra and $A^!$ the dual quadratic $\mathcal{P}^!$ - superalgebra then the universal enveloping algebras $U_{\mathcal{P}}(A), U_{\mathcal{P}^!}(A^!)$ are quadratic associative algebras in the ordinary sense and

$$(U_{\mathcal{P}}(A))^! = U_{\mathcal{P}^!}(A^!)$$
Proof: Let $P = P(K, E, R)$ and $A = A(V, S)$. The algebra $U_P(A)$ has an obvious grading in which the generator $X(\lambda; a_1, ..., a_{n-1})$, $\lambda \in P(n)$, $a_i \in A_{n_i}$, has degree $i_1 + ... + i_{n-1}$. The degree 1 component of $U_P(A)$ is linearly spanned by $X(\lambda, a)$, $\lambda \in P(2) = E$, $a \in A_1 = V$ and is isomorphic to $E \otimes_K V$. Obviously this component generates $U_P(A)$ as an algebra.

To describe the relations among these generators, consider the space $Y = F(E)(3) \otimes V \otimes V$ (here and in the remainder of this section all tensor products are taken over $K$). This space splits into the direct sum of three components $Y = L_1 \oplus L_2 \oplus L_3$ depicted in Fig. 8.

The letters “$K$” on some of the edges mean that the tensor product over $K$ is taken with respect to the structures of left/right $K$-module represented by the ends of this edge. Note that every $L_i$ and hence $X$ is a $K$-bimodule with respect to the actions corresponding to the edges not marked “$K$”. Elements of $L_1$ can be viewed as formal expressions $\sum X(\lambda_i; a_i)X(\mu_i; b_i)$, as can elements of $L_3$. The group $\Sigma_2$ permuting the left two inputs of the trees in Fig. 8, maps $L_1$ isomorphically to $L_3$ and preserves $L_2$. Thus, denoting $X = Y_{\Sigma_2}$ the space of coinvariants, we have

$$X = E \otimes V \otimes E \otimes V \oplus E \otimes (E \otimes V^\otimes_2)_{\Sigma_2}.$$ 

Let $W \subset X$ be the image, under the canonical projection $Y \rightarrow X$, of the subspace $R \otimes V^\otimes_2$, where $R \subset F(E)(3)$ is the space of relations of $P$.

(2.3.12) Proposition. The algebra $U_P(A)$ is defined by the space of generators $E \otimes V$ and the space of quadratic relations

(2.3.13) $E \otimes V \otimes E \otimes V \cap (W + E \otimes S)$

where $S \subset (E \otimes V^\otimes_2)_{\Sigma_2}$ is the space of relations in $A$ and the intersection is taken inside $X$.

Proof: Straightforward. Left to the reader.

Similarly, the algebra $U_{P'}(A')$ has the space of generators $E^\vee \otimes V^\vee$ and the space of relations

(2.3.14) $E^\vee \otimes V^\vee \otimes E^\vee \otimes V^\vee \cap (W^\perp + E^\vee \otimes S^\perp)$. 

40
We want to show that (2.3.13) and (2.3.14) are the orthogonal complement to each other. This is a particular case of the following general lemma.

**Lemma.** Let \( X \) be any \( K \)-bimodule decomposed into a direct sum of bimodules \( X = M \oplus N \). Let \( X^\vee = M^\vee \oplus N^\vee \) be the corresponding decomposition of \( X^\vee = \text{Hom}_K(X,K) \). Let \( W \subset X \) and \( L \subset N \) be any sub-bimodules. Then the orthogonal complement (in \( M^\vee \)) of \( M \cap (W + L) \) coincides with \( M^\vee \cap (W_X^\perp + L_N^\perp) \), where \( W_X^\perp \) and \( L_N^\perp \) are the orthogonal complements in \( X \) and \( N \), respectively.

**Proof of the lemma:** Since \( K \) is semisimple, we can write \( N = L \oplus P \) where \( P \) is another sub-bimodule, so \( X = M \oplus L \oplus P \) and \( X^\vee = M^\vee \oplus L^\vee \oplus P^\vee \). If \( Z \subset X \) is any sub-bimodule then

\[
(2.3.16) \quad (M \cap Z)_{M}^1 = \text{Im} \left\{ (M \cap Z)_{X}^1 \xrightarrow{\pi} M^\vee \right\} = M^\vee \cap ((M \cap Z)_{X}^1 + L^\vee + P^\vee),
\]

where \( \pi : X^\vee \to M^\vee \) is the projection dual to the embedding \( M \hookrightarrow X \), i.e., the projection along \( L^\vee \oplus P^\vee \). Let us apply this to \( Z = W + L \) and note that

\[
(M \cap (W + L))_{X}^1 = M_Z^1 + (W_X^1 \cap L_X^1) = L^\vee + P^\vee + (W_X^1 \cap (M^\vee + P^\vee)).
\]

We get that the RHS of (2.3.16) is equal to

\[
(2.3.17) \quad M^\vee \cap (W_X^1 \cap (M^\vee + P^\vee) + L^\vee + P^\vee).
\]

To finish the proof of the lemma, it remains to show that (2.3.17) coincides with \( M^\vee \cap (W_X^1 + P^\vee) \). (Note that \( P^\vee \) is the same as \( L_N^\perp \).)

To show this, suppose that \( m = w + p \in M^\vee \cap (W_X^1 + P^\vee) \), so that \( w \in W_X^1, p \in P^\vee \). Then \( w = m - p \in M^\vee + P^\vee \), so writing \( m = w + 0 + p \), we get that \( m \) belongs to (2.3.17). Conversely, let \( m \) belong to (2.3.17), so \( m = w + l + p \) with \( l \in L^\vee, p \in P^\vee \) and \( w \in W_X^1 \cap (M^\vee + P^\vee) \), so \( w = m' + p', m' \in M^\vee, p' \in P^\vee \). Then we have \( m = m' + p' + l + p \) and, since \( X^\vee = M^\vee \oplus L^\vee \oplus P^\vee \), we get \( m = m', p + p' = 0, l = 0 \). This \( m = w + p \), wo \( m \in M^\vee \cap (W_X^1 + P^\vee) \). Lemma 2.3.15 and hence Theorem 2.3.11 are proven.
3. DUALITY FOR $dg$ - OPERADS.

3.1. $dg$ - operads. Generating maps.

(3.1.1) Let $dg Vect$ be the symmetric monoidal category of differential graded ($dg$ - ) vector spaces, i.e., of complexes over the base field $k$. By definition, an object of $dg Vect$ is a graded vector space $V^\bullet$ together with a linear map (differential) $d : V^\bullet \to V^\bullet$ of degree 1 such that $d^2 = 0$. Morphisms are linear maps preserving gradings and differentials. The tensor product of two complexes $V^\bullet$ and $W^\bullet$ is defined by (1.4.3) with the differential given by the Leibnitz rule

$$d(v \otimes w) = d(v) \otimes w + (-1)^i v \otimes d(w), \quad v \in V^i, w \in W^j.$$

The symmetry isomorphism $V^\bullet \otimes W^\bullet \to W^\bullet \otimes V^\bullet$ is the same as in the category $gVect^-$, see (1.4.2 (c)). As usual, for a complex $V^\bullet \in dg Vect$ we define the dual complex $V^*\bullet$ by

$$(V^*)^i = (V^{-i})^*, \quad d_{V^*} = (d_V)^*$$

and the shifted complex $V^\bullet[i], \subset \mathbb{Z},$ by

$$(V^\bullet[i])^j = V^{i+j}, \quad d_{V^\bullet[i]} = (-1)^i d_V.$$

An operad in the category $dg Vect$ is called a $dg$ - operad (over $k$). An algebra over a $dg$ - operad in the category $dg Vect$ will be called a $dg$ - algebra. Note that any $k$ - linear operad $P$ can be regarded as a $dg$ - operad (each $P(n)$ is placed in degree 0).

(3.1.4) For any $dg$ - operad $P$ the collection of cohomology vector spaces $H^\bullet P(n)$ form an operad $H^\bullet(P)$ in the category $gVect^-$. A morphism of $dg$ - operads $f : P \to Q$ is called a quasi - isomorphism if the induced morphism $H^\bullet(f) : H^\bullet(P) \to H^\bullet(Q)$ is an isomorphism. Similarly for $dg$ - algebras over $dg$ - operads.

(3.1.5) A $dg$ - operad $P$ will be called admissible if the following conditions hold:

(i) Each $P(n)$ is a finite - dimensional $dg$ - vector space.

(ii) The space $P(1)$ is concentrated in degree 0, and is a semisimple $k$ - algebra.

Given a semisimple $k$ - algebra $K$, we denote by $dg OP(K)$ the category of admissible $dg$ - operads $P$ with $P(1) = K$ and with morphisms identical on first components.
Our next aim is to define a kind of “generating function” for an admissible dg-operad $\mathcal{P}$. It will be convenient for the future to work in a slightly greater generality.

Let $K$ be a semisimple $k$-algebra. By a $K$-dg-collection we mean a collection $E = \{E(n), n \geq 2\}$ of finite-dimensional dg-vector spaces $E(n)$ together with a left $\Sigma_n$-action and a structure of $(K, K^{\otimes n})$-bimodule on each $E(n)$ which satisfy the compatibility condition identical to the one given in (1.2.11).

Clearly if $\mathcal{P}$ is an admissible dg-operad and $K = \mathcal{P}(1)$ then $\mathcal{P}(n), n \geq 2$, form a $K$-dg-collection. Given any $K$-dg-collection $E$, one defines the free dg-operad $F(E)$ as in (2.1.1).

By an $r$-fold dg-collection we mean, similarly to (1.3.15), a collection of finite-dimensional complexes $E^i(a_1, \ldots, a_r)$, $a_i \in \mathbb{Z}_+, i = 1, \ldots, r$, $\sum a_i \geq 1$ of $\Sigma_{a_1} \times \cdots \times \Sigma_{a_r}$-modules such that

$$E^i(0, \ldots, 0, \underbrace{1}_{j}, 0, \ldots, 0) = \begin{cases} k & \text{for } i = j; \\ 0 & \text{for } i \neq j \end{cases}$$

Given a semisimple $k$-algebra $K$ with $r$ simple summands and a $K$-dg-collection $E$, we define the $r$-fold dg-collection associated to $E$ by the formula identical to (1.3.16).

### (3.1.8) Definition.

Let $K$ be a semisimple $k$-algebra, $r$ be the number of simple summands in $K$ and $E$ be a $K$-dg-collection. The generating map of $E$ is the $r$-tuple of formal power series

$$g_E^{(i)}(x_1, \ldots, x_r) = \sum_{a_1, \ldots, a_r=0}^{\infty} \chi[E^i(a_1, \ldots, a_r)] \frac{x_1^{a_1}}{a_1!} \cdots \frac{x_r^{a_r}}{a_r!}, \quad i = 1, \ldots, r$$

where $\{E^i(a_1, \ldots, a_r)\}$ is the $r$-fold dg-collection associated to $E$ and $\chi$ stands for the Euler characteristic.

The $r$-tuple $g_E(x) = (g_E^{(1)}(x), \ldots, g_E^{(r)}(x))$ will be regarded as a formal map

$$g_E : \mathbb{C}^r \to \mathbb{C}^r, \quad x = (x_1, \ldots, x_r) \mapsto (g_E^{(1)}(x), \ldots, g_E^{(r)}(x)).$$

Note that by (3.1.7) we have

$$g_E^{(i)}(x) = x_i + (\text{higher order terms}).$$

In particular, for any admissible dg-operad $\mathcal{P}$ we have its generating map $g_\mathcal{P}$.
Special case: If $\mathcal{P}$ is a $k$-linear operad with $\mathcal{P}(1) = k$ then its generating map is a single power series

$$g_\mathcal{P}(x) = \sum_{n=1}^{\infty} \dim \mathcal{P}(n) \frac{x^n}{n!}.$$  

(3.1.12) Examples. a) The operads $\mathcal{A}s, \mathcal{C}om$ and $\mathcal{L}ie$, see (1.3.7) - (1.3.9), are admissible operads with $K = k$ and trivial dg-structure. Therefore $r = 1$ and the generating maps of these operads are the following power series in one variable:

$$g_{\mathcal{A}s}(x) = \frac{1}{1-x} - 1, \quad g_{\mathcal{C}om}(x) = e^x - 1, \quad g_{\mathcal{L}ie}(x) = -\log(1-x).$$

b) Let $\mathcal{A}$ be the symmetric monoidal category of representations of the group $\mathbb{Z}/2$. It has two (1-dimensional) irreducible objects: the trivial representation $I$ and the sign representation $J$. Taking $X = I \oplus J$, we define an admissible $k$-linear operad $\mathcal{P}$ with $\mathcal{P}(n) = \text{Hom}_{\mathcal{A}}(X \otimes^n, X)$, see (1.3.12). We have $\mathcal{P}(1) = k \oplus k$. As explained in (1.3.16), the 2-fold collection associated to $\mathcal{P}$ consists of Clebsch-Gordan spaces

$$\text{Hom}_{\mathcal{A}}(I \otimes^a \otimes J \otimes^b, I) = \begin{cases} 0 & \text{for } b \text{ odd} \\ k & \text{for } b \text{ even} \end{cases}$$

$$\text{Hom}_{\mathcal{A}}(I \otimes^a \otimes J \otimes^b, J) = \begin{cases} 0 & \text{for } b \text{ even} \\ k & \text{for } b \text{ odd} \end{cases}.$$

Therefore the generating map of $\mathcal{P}$ is given by the formulas

$$g^{(1)}_\mathcal{P}(x_1, x_2) = \sum_{a, b} \dim \text{Hom}_{\mathcal{A}}(I \otimes^a \otimes J \otimes^b, I) \frac{x_1^a x_2^b}{a! b!} = e^{x_1} \cosh(x_2) - 1,$$

$$g^{(2)}_\mathcal{P}(x_1, x_2) = \sum_{a, b} \dim \text{Hom}_{\mathcal{A}}(I \otimes^a \otimes J \otimes^b, J) \frac{x_1^a x_2^b}{a! b!} = e^{x_1} \sinh(x_2).$$

(3.1.13) It is possible to refine the generating map of a $K$-dg-collection so as to take into account not only the dimensions of the graded components of the complexes $E^i(a_1, \ldots, a_r)$ but also the symmetric groups actions.

Let $\mathcal{R}_n$ be the Grothendieck group of the category of finite-dimensional representations of the symmetric group $\Sigma_n$ over $k$. For any such representation $V$ we denote by $[V]$ its class in $\mathcal{R}_n$. The maps

$$\mathcal{R}_m \otimes \mathcal{R}_n \to \mathcal{R}_{m+n}, \quad [V] \otimes [W] \mapsto \text{Ind}_{\Sigma_m \times \Sigma_n}^{\Sigma_{m+n}}(V \otimes W)$$
make $\mathcal{R} = \bigoplus_{n \geq 0} \mathcal{R}_n$ into a commutative graded ring. It is well known [Macd] that $\mathcal{R}$ is isomorphic to the ring $\mathbb{Z}[e_1,e_2,...]$ of symmetric functions in infinitely many variables. There is a natural ring homomorphism $h : \mathcal{R} \to \mathbb{Z}$ defined by

\begin{equation}
(3.1.15)
    h([V]) = \frac{\dim V}{n!}, \quad [V] \in \mathcal{R}_n.
\end{equation}

The tensor product $\mathcal{R}_{a_1} \otimes \ldots \otimes \mathcal{R}_{a_r}$ is naturally identified with the Grothendieck group of representations of the Cartesian product $\Sigma_{a_1} \times \ldots \times \Sigma_{a_r}$. It is a direct summand of the ring $\mathcal{R}^{\otimes r} = \bigoplus \mathcal{R}_{a_1} \otimes \ldots \otimes \mathcal{R}_{a_r}$. If $V^*$ is any finite-dimensional complex of $\Sigma_{a_1} \times \ldots \times \Sigma_{a_r}$-modules then by $[V^*] \in \mathcal{R}^{\otimes r}$ we denote the alternating sum of classes of $V^i$ in $\mathcal{R}_{a_1} \otimes \ldots \otimes \mathcal{R}_{a_r} \subset \mathcal{R}^{\otimes r}$.

\begin{equation}
(3.1.16)
    \text{Let now } K \text{ be a semisimple } k-\text{algebra with } r \text{ simple summands and } E \text{ be a } K-\text{dg-}\text{collection. We define the \textit{refined generating map} of } E \text{ to be the } r-\text{tuple of formal power series}
\end{equation}

\begin{equation}
(3.1.17)
    G^i_E(x_1,...,x_r) = \sum [E^i(a_1,...,a_r)] x_1^{a_1} \ldots x_r^{a_r} \in \mathcal{R}^{\otimes r}[[x_1,...,x_r]]
\end{equation}

where, as in (3.1.8), $\{E^i(a_1,...,a_r)\}$ is the $r$-fold $dg$-collection associated to $E$ (note that there are no denominators).

Let $h^{\otimes r} : \mathcal{R}^{\otimes r} \to \mathbb{Z}$ be the tensor power of the homomorphism (3.1.15)

\begin{equation}
(3.1.18)
    h_r : \mathcal{R}^{\otimes r}[[x_1,...,x_r]] \to \mathbb{Z}[[x_1,...,x_r]]
\end{equation}

be the associated homomorphism of power series rings. Then the numerical generating map of $E$ can be recovered as

\begin{equation}
(3.1.19)
    g^{(i)}_E(x_1,...,x_r) = h_r(G^i_E(x_1,...,x_r)).
\end{equation}
3.2. The cobar-duality.

(3.2.0) For a finite-dimensional $k$-vector space $V$ we denote by $\text{Det}(V)$ the top exterior power of $V$.

Let $T$ be a tree (1.1.1). We denote by $\text{Ed}(T)$ the set of all edges of $T$ except the output edge $\text{Out}(T)$. We denote by $\text{Det}(T)$ the 1-dimensional vector space $\text{Det}(k^{\text{Ed}(T)})$. Similarly, let $\text{ed}(T)$ be the set of internal edges of $T$ and let $\text{det}(T) = \text{Det}(k^{\text{ed}(T)})$. The number of internal edges of $T$ will be denoted $|T|$.

(3.2.1) Let $\mathcal{P}$ be an admissible $dg$-operad (3.1.5), so that $K = \mathcal{P}(1)$ is a semisimple $k$-algebra. For any $n \geq 2$ we construct a complex

$$\mathcal{P}(n)^* \otimes \text{det}(k^n) \xrightarrow{\delta} \bigoplus_{\text{n-trees } T \atop |T| = 1} \mathcal{P}(T)^* \otimes \text{det}(T) \xrightarrow{\delta} \cdots \xrightarrow{\delta} \bigoplus_{\text{n-trees } T \atop |T| = n - 2} \mathcal{P}(T)^* \otimes \text{det}(T)$$

where the sums are over isomorphism classes of (reduced) $n$-trees and $\mathcal{P}(T)$ was defined in (1.2.13). The differential $\delta$ is defined by its matrix elements

$$\delta_{T',T} : \mathcal{P}(T')^* \otimes \text{det}(T') \longrightarrow \mathcal{P}(T)^* \otimes \text{det}(T)$$

where $T, T'$ are $n$-trees, $|T| = i, |T'| = i - 1$. By definition, $\delta_{T',T} = 0$ unless $T' = T/e$ is obtained from $T$ by contracting an internal edge $e$. If this is the case, we set

$$\delta_{T',T} = (\gamma_{T,T'})^* \otimes l_e$$

where $\gamma_{T,T'}$ is the composition map from (1.2.15) and the map $l_e : \text{det}(T') \rightarrow \text{det}(T)$ is defined by the formula

$$l_e(f_1 \wedge \ldots \wedge f_m) = e \wedge f_1 \wedge \ldots \wedge f_m.$$ 

In this formula we use the natural identification $\text{ed}(T) = \text{ed}(T') \cup \{e\}$ and regard $e$ as a basis vector of $k^{\text{ed}(T)}$.

(3.2.5) It is straightforward to verify that $\delta^2 = 0$, i.e., (3.2.2) is a complex. We normalize the grading of this complex by placing the sum over $T$ with $|T| = i$ in degree $i + 1$.

Observe that, for a $dg$-operad $\mathcal{P}$, each term of (3.2.2) is a $dg$-vector space whose differential we denote by $d$. Clearly $d$ commutes with $\delta$ so (3.2.2) is a complex of $dg$-vector spaces i.e., a double complex.

(3.2.6) We now define a collection of $dg$-vector spaces $C(\mathcal{P})(n), n \geq 1$. For $n = 1$ we put $C(\mathcal{P})(1) = K^{\text{op}}$, placed in degree 0 (with trivial differential). For $n > 1$ we define $C(\mathcal{P})(n)$ to be the total complex (= $dg$-vector space) associated to the double complex (3.2.2).
(3.2.7) **Theorem.**  

a) The collection $C(P) = \{C(P)(n), n \geq 1\}$ has a natural structure of an admissible $dg$-operad. 

b) The correspondence $P \mapsto C(P)$ extends to contravariant functor $C : dgOP(K) \to dgOP(K^{op})$ (notation of (3.1.5)). This functor takes quasi-isomorphisms to quasi-isomorphisms.

**Proof:** a) The complexes $P(n), n \geq 2$, obviously form a $K-dg$-collection (3.1.6). Therefore the shifted dual complexes $P(n)^*[-1]$ form a $K^{op}-dg$-collection (shift and duality are defined by (3.1.2-3)). We denote this collection by $P^*[-1]$. Thus we can form the free $dg$-operad $F(P^*[-1])$.

(3.2.8) **Lemma.** For any $n$ the complex $F(P^*[-1])(n)$ is isomorphic to the total complex of the double complex (3.2.2), the latter taken with the internal differential $d$ (induced by that on $P$) only.

The lemma is proved by immediate inspection. The only point that needs explanation is the appearance of vector spaces $\text{det}(T)$ in (3.2.2). The reason for this is part b) of the following lemma.

(3.2.9) **Lemma.**

a) Let $I$ be a finite set of $m$ elements and $W_i^*, i \in I$ be $dg$-vector spaces. Then there is a canonical isomorphism

$$\phi : \bigotimes_{i \in I} (W_i^*[-1]) \approx \left( \bigotimes_{i \in I} W_i^* \right)[-m] \otimes \text{det}(k^I).$$

b) Let $E = \{E(n), n \geq 2\}$, be a $K-dg$-collection, $E[-1]$ be the collection of shifted $dg$-vector spaces and $T$ be a tree with $m$ vertices. Then there is a canonical isomorphism

$$E[-1|(T) \cong E(T)[-m] \otimes \text{det}(T).$$

**Proof:** a) To define $\bigotimes_{i \in I} W_i^*$, we should choose some ordering $(i_1, ..., i_m)$ on $I$ and consider the product $W_{i_1}^* \otimes ... \otimes W_{i_m}^*$. Any other ordering will give the product related to this one by a uniquely defined isomorphism. The same for $\bigotimes W_i^*[-1]$. Let now $w_i \in W_i^*[-1]$ be some elements and let $w'_i$ denote the same elements but regarded as elements of $W_i^*$. We define

$$\phi(w_{i_1} \otimes ... \otimes w_{i_m}) = w'_{i_1} \otimes ... \otimes w'_{i_m} \otimes (i_1 \wedge ... \wedge i_m).$$

The proof that $\phi$ is independent on the choice of ordering $(i_1, ..., i_m)$ follows from the definition of the symmetric monoidal structure in $dgVect$ and is left to the reader.

b) Let $\text{Vert}(T)$ be the set of vertices of $T$. Let $\Gamma \subset T$ be the subtree consisting of all vertices and all internal edges. This is a contractible 1-dimensional simplicial complex so
taking its chain complex we get the exact sequence

\[ 0 \rightarrow k^{\text{ed}(T)} \rightarrow k^{\text{Vert}(T)} \rightarrow k \rightarrow 0 \]

whence the space \( \text{Det}(k^{\text{Vert}(T)}) \) is canonically identified with \( \text{det}(T) = \text{Det}(k^{\text{ed}(T)}) \). Now part (b) of the lemma follows from part (a), since \( E(T) = \bigotimes_{v \in \text{Vert}(T)} E(\text{In}(v)) \).

Lemmas 3.2.9 and 3.2.8 are proven.

(3.2.10) By Lemma 3.2.8, the compositions in the free operad \( F(P^*[-1]) \) give rise to maps of graded vector spaces

\[ (3.2.11) \quad C(P)(l) \otimes C(P)(m_1) \otimes \ldots \otimes C(P)(m_l) \rightarrow C(P)(m_1 + \ldots + m_l) \]

which satisfy the Leibnitz rule with respect to internal differentials \( d \) in \( C(P)(n) \). To complete the proof of Theorem 3.2.7 (a), it remains to check that the maps (3.2.11) satisfy also the Leibnitz rule with respect to the differentials \( \delta \) in \( C(P)(n) \) defined in (3.2.3). We leave this straightforward checking to the reader. This completes the proof of part (a) of Theorem 3.2.7. Part (b) is straightforward.

(3.2.12) We call the \( dg \)-operad \( C(P) \) the \textit{cobar} - construction of \( P \). We define the dual \( dg \)-operad \( D(P) \) by

\[ (3.2.13) \quad D(P) = C(P) \otimes \Lambda = \{ C(P)(n) \otimes \Lambda(n) \} , \]

where \( \Lambda \) is the determinant operad (1.3.21) and the product \( \otimes \) is defined in (2.2.2). Recall that \( \Lambda(n) \) is a 1-dimensional vector space in degree \((1 - n)\) with the sign action of \( \Sigma_n \). Hence \( D(P)(n) \) is the \( dg \)-vector space associated to the following complex which differs from (3.2.2) by shifting the grading by \((1 - n)\) and by replacing \( \text{det} \) by \( \text{Det} \):

\[ (3.2.14) \quad \ldots \rightarrow \bigoplus_{|T| = n - 3} P(T)^* \otimes \text{Det}(T) \rightarrow \bigoplus_{|T| = n - 2} P(T)^* \otimes \text{Det}(T) . \]

The grading in (3.2.14) is arranged so that the rightmost term is placed in degree 0.

It follows from the definitions that the correspondence \( P \mapsto D(P) \) extends to a contravariant functor \( D : dg\text{OP}(K) \rightarrow dg\text{OP}(K^\text{op}) \) taking quasi - isomorphisms to quasi - isomorphisms.

(3.2.15) Example. Let \( P = \text{Com} \) be the commutative operad so that \( \text{Com}(n) = k \) for every \( n \), see (1.3.8). The \( n \)-th component of \( D(P) \) is the \textit{tree complex}

\[ \bigoplus_{|T|=0} \text{Det}(T) \rightarrow \bigoplus_{|T|=1} \text{Det}(T) \rightarrow \ldots \rightarrow \bigoplus_{|T|=n-2} \text{Det}(T) \]

48
This is a special case of more general graph complexes considered by Kontsevich [Kon 2]. In the paper [BG 1] it was proven by using Hodge theory that for \( k = C \) this complex is exact everywhere except the rightmost term and the cokernel of the rightmost differential is naturally isomorphic to \( \mathcal{L}ie(n) \), the \( n \)-th space of the Lie operad. Thus we have an isomorphism \( D(\text{Com}) \cong \mathcal{L}ie \). We give a purely algebraic proof of this later in Chapter 4.

\textbf{(3.2.16) Theorem.} For any admissible \( dg \)-operad \( \mathcal{P} \) there is a canonical quasi-isomorphism \( D(D(\mathcal{P})) \rightarrow \mathcal{P} \).

\textit{Proof:} Let us write down \( D(D(\mathcal{P}))(n) \) explicitly. By definition (3.2.14),

\[
D(D(\mathcal{P}))(n) = \bigoplus_{\text{n-trees}} \left[ \otimes_{v \in S} D(\mathcal{P})(\text{In}(v))^* \otimes \det(S) \right].
\]

Substituting here the definitions (1.1.6) of the value on a set of the functor corresponding to \( D(\mathcal{P}) \), and keeping track of cancellation of some \( \det \)-factors, we get

\textbf{(3.2.17)}

\[
D(D(\mathcal{P}))(n) = \bigoplus_{T \cong T'} \otimes_{v \in T} \mathcal{P}(\text{In}(v)) \otimes_{w \in T'} \det(T_w).
\]

Here the summation is over the isomorphism classes of pairs \( T, T' \) of \( n \)-trees such that \( T \geq T' \) i.e., \( T' \) can be obtained from \( T \) by contracting some (possibly empty) set of edges. For \( w \in T' \) we denote by \( T_w \) the subtree of \( T \) contracted into \( w \). Division by a 1-dimensional vector space is understood as tensoring with the dual space.

The construction may be understood better using Fig. 9a where vertices \( w \) of the tree \( T' \) are indicated as big circles ("regions") containing inside them the corresponding trees \( T_w \). Let \( T_n \) denote the unique \( n \)-tree with a single vertex. The summand in (3.2.17) corresponding to the pair \( T_n \geq T_n \) is nothing but \( \mathcal{P}(n) \) (Fig. 9b):
It is straightforward to verify that the summand $P(n)$ is in fact a quotient complex of the whole complex $D(D(P))(n)$, i.e., the projection to this summand along all other summands is a (surjective) morphism of complexes which we denote by $f_n$. It is also easily verified that we get in this way a morphism of $dg$ - operads $f : D(D(P)) \to P$. Let us show that $f$ is a quasi-isomorphism i.e., that each subcomplex $\ker (f_n) \subseteq D(D(P))(n)$, is acyclic. Note that $\ker (f_n)$, as part of $D(D(P))(n)$, is actually a triple complex so its differential is a sum of three partial differentials $d_1 + d_2 + d_3$. The differential $d_1$ is induced by the differential in $P$; the differential $d_2$ is induced by the composition in $P$ (which induces the second differential in $D(P)$). Finally, $d_3$ is induced by the composition in $D(P)$, i.e., by the grafting of trees.

It is enough to show that $\ker f_n$ is acyclic with respect to $d_3$. If $T$ is an $n$ - tree with more than one vertex then the summands in (3.2.17) with all $T' \leq T$ form a subcomplex $K_T^* \subseteq (\ker (f_n), d_3)$ and $\ker (f_n)$ is the direct sum of such $K_T^*$. We shall prove that each $K_T^*$ is acyclic.

The differential $d_3$ in $K_T^*$ consists purely in redrawing the boundaries among regions in Fig. 8 (a). More precisely, $K_T^*$ is the tensor product of the vector space $P(T)$ and a purely combinatorial complex $C_T^*$ where

$$C_T^i = \bigoplus_{T' \leq T} \bigotimes_{w \in T'} \det(T_w)^*.$$

Observe that specification of a tree $T' \leq T$ is equivalent to a specification of a subset of internal edges of $T$ which are contracted in $T'$. We see that $C_T^*$ is isomorphic to the augmented chain complex of a simplex whose vertices correspond to internal edges of $T$. Thus, $C_T^*$ and $K_T^*$ are acyclic. Theorem 3.2.16 follows.

3.2.18 Proposition. For any admissible $dg$ - operad $P$ with $P(1) = k$ there exists a natural morphism of $dg$ - operads $\lambda : D(Com) \to P \otimes D(P)$. In particular, for any $dg$ - algebra $A$ over $P$ and any $dg$ - algebra $B$ over $D(P)$ the tensor product $A \otimes_k B$ has a natural structure of a $D(Com)$ - algebra.

This result is analogous to Corollary 2.2.9 (b). The precise meaning of the analogy will be explained in Chapter 4.

Proof: Since $Com(n) = k$ for every $n$, we find that $D(Com)(n)$ is the $dg$ - vector space whose $(-i)$ - th graded piece is the direct sum of spaces $\det(T)$ corresponding to $n$ - trees $T$ with $n - 2 - i$ interior edges. We define the required maps $\lambda_n : D(Com)(n) \to P(n) \otimes D(P)(n)$ by prescribing their restrictions on the summands $\det(T)$. We define

$$\lambda_n|_{\det(T)} : \det(T) \to P(n) \otimes P^*(T) \otimes \det(T) \subseteq D(P)(n)$$
to be $\text{Id}_{\text{Det}(T)} \otimes \gamma_T^\dagger$, where the element $\gamma_T^\dagger \in \mathcal{P}(n) \otimes \mathcal{P}^*(T)$ is the transpose of the map $\gamma_T : \mathcal{P}(T) \to \mathcal{P}(n)$ given by the composition in $\mathcal{P}$, see (1.2.4).

This defines the maps $\lambda_n$. The proof that these maps form a morphism of operads, is straightforward and left to the reader.
3.3. The generating map of the dual dg - operad.

(3.3.1) Let \( \mathcal{P} \) be an admissible dg - operad and \( Q = D(\mathcal{P}) \) be its dual (3.2.13). Let \( r \) be the number of simple summands of the semisimple algebra \( \mathcal{P}(1) = Q(1)^{op} \) and let \( g_{\mathcal{P}}, g_{\mathcal{Q}} : C^r \rightarrow C^r \) be the generating maps of \( \mathcal{P}, \mathcal{Q} \), see (3.1.8). Let also \( G_{\mathcal{P}}, G_{\mathcal{Q}} \) be the refined generating maps, see (3.1.16).

The following is the main result of this section.

(3.3.2) Theorem. a) We have the following identity of formal maps \( C^r \rightarrow C^r \):

\[ g_{\mathcal{Q}}(-g_{\mathcal{P}}(-x)) = x, \quad x = (x_1, ..., x_r). \]

b) The same identity holds for the refined generating maps \( G_{\mathcal{P}} \) and \( G_{\mathcal{Q}} \).

(3.3.3) Example. Let \( \mathcal{P} = \text{Com} \) be the commutative operad. As we saw in (3.1.12), \( g_{\mathcal{P}}(x) = e^x - 1 \). By (3.2.15), \( Q = D(\mathcal{P}) \) is quasi - isomorphic to the operad \( \mathcal{L}ie \) so \( g_{\mathcal{Q}} = g_{\mathcal{L}ie}(x) = -\log(1 - x) \). One sees that these two series satisfy Theorem 3.3.2.

(3.3.4) Let \( K = \mathcal{P}(1) \). As a first step towards the proof of our theorem we describe the \( r \)-fold dg - collection associated to the \( \mathcal{K}^{op} - dg \) - collection \( D(\mathcal{P}) = \{ D(\mathcal{P})(n) \} \).

By an \( r \)-colored tree we understand a tree \( T \) together with a function \( c \) ("coloring") from the set of all edges of \( T \) to \( \{1, ..., r\} \). For such a tree \( T \) and a vertex \( v \in T \) let \( \text{In}_i(v) \) denote the set of input edges at \( v \) of color \( i \). By an \( (a_1, ..., a_r) \)-tree we mean an \( r \)-colored tree \( T \) with \( a_i \) inputs of color \( i \), which are labelled by the numbers \( 1, 2, ..., a_i \).

(3.3.5) Let \( \{ P^i(a_1, ..., a_r) \} \) be the \( r \)-fold dg - collection associated to \( \mathcal{P} \). For any \( i \) we use the \( \Sigma_{a_1} \times ... \times \Sigma_{a_r} \) - action on each \( P^i(a_1, ..., a_r) \) to define, similarly to (1.1.6), a functor

\[ (I_1, ..., I_r) \mapsto P^i(I_1, ..., I_r) \]

on the category of \( r \)-tuples of finite sets and bijections. Let also \( \{ Q^i(a_1, ..., a_r) \} \) be the \( r \)-fold dg - collection associated to the dual dg - operad \( Q = D(\mathcal{P}) \). Recall that \( |T| \) denotes the number of internal edges of a tree \( T \).

(3.3.6) Proposition. Each \( Q^i(a_1, ..., a_r) \) is isomorphic, as a graded vector space with \( \Sigma_{a_1} \times ... \times \Sigma_{a_r} \) - action, to

\[ \bigoplus_{(a_1, ..., a_r)-\text{trees}} \bigotimes_{v \in T} P^{c(\text{Out}(v))}(\text{In}_1(v), ..., \text{In}_r(v))^* \left[ |T| - \sum a_i - 2 \right] \otimes \det(T). \]

The proof is straightforward.
To prove Theorem 3.3.2, we invoke a purely algebraic formula describing the inversion of a formal (or analytic) map \( g : \mathbb{C}^r \to \mathbb{C}^r \). This formula is due to J. Towber, see [W], Th. 3.10 or [MTWW], Th. 2.13. (We are grateful to D. Wright for pointing out the references to this formula, which was rediscovered by us.)

Suppose we have \( r \) formal power series

\[
g^{(i)}(x_1, \ldots, x_r) = x_i + \sum_{a_1, \ldots, a_r \geq 0} p^{(i)}(a_1, \ldots, a_r) \frac{x_1^{a_1}}{a_1!} \cdots \frac{x_r^{a_r}}{a_r!}
\]

where \( p^{(i)}(a_1, \ldots, a_r) \) are some complex coefficients and let

\[
y_i = g^{(i)}(x_1, \ldots, x_r)
\]

be a formal change of variables given by these series. Since \( g^{(i)}(x) = x_i + \text{terms of order } \geq 2 \), this change of variables is formally invertible, i.e., we can express \( x_i \) as power series in \( y_1, \ldots, y_r \). The question is to find the coefficients of these power series provided the coefficients \( p^{(i)}(a_1, \ldots, a_r) \) are known.

**Theorem 3.3.9.** The inverse of the formal map (3.3.8) is given by

\[
x_i = h_i(y_1, \ldots, y_r) = y_i + \sum_{a_1, \ldots, a_r \geq 0} q^{(i)}(a_1, \ldots, a_r) \frac{y_1^{a_1}}{a_1!} \cdots \frac{y_r^{a_r}}{a_r!}
\]

where

\[
q^{(i)}(a_1, \ldots, a_r) = \sum_{m=0}^{a_1+\ldots+a_r-1} (-1)^{a_1+\ldots+a_r-m} \left\{ \sum_{(a_1, \ldots, a_r) - \text{trees } T \mid |T| = m, c(\text{Out}(T)) = i} \prod_{v \in T} p^c(\text{Out}(v)) \right. \left( |\text{In}_1(v)|, \ldots, |\text{In}_r(v)| \right) \}
\]

Several other inversion formulas for analytic maps can be found in [Sy] [Gd] [Jo] [BCW] [MTWW].

**Theorem 3.3.12.** We prefer to give here a simple direct proof of Theorem 3.3.9, partly for the convenience of the reader, partly because an intermediate lemma in the proof will be used later. The proof proceeds by direct substitution of the proposed answer into the claimed
equation. Suppose first that the series $h_i$ (i.e., the coefficients $q^{(i)}(a_1,\ldots,a_r)$) in (3.3.10) are arbitrary. Let us form the composition

$$f^{(i)}(y) = g^{(i)}(h^{(1)}(y),\ldots,h^{(r)}(y)), \quad y = (y_1,\ldots,y_r)$$

and write

$$f^{(i)}(y) = y_i + \sum_{a_1,\ldots,a_r \geq 0} u^{(i)}(a_1,\ldots,a_r) \frac{x_1^{a_1}}{a_1!} \cdots \frac{x_r^{a_r}}{a_r!}.$$ 

It is immediate to get a general formula for the coefficients $u^{(i)}(a_1,\ldots,a_r)$. Call a rooted tree $T$ short if $T$ has no consecutive internal edges (see Fig. 10)

The lowest vertex of a tree $T$ (the one which is adjacent to the output edge Out($T$)) will be denoted by Lw($T$).

**Lemma.** We have

$$u^{(i)}(a_1,\ldots,a_r) =$$

$$\sum_{\text{short } (a_1,\ldots,a_r) - \text{trees } T \atop c(\text{Out}(T)) = i} \left\{ p^{c(\text{Out}(T))}(|\text{In}_1(\text{Lw}(T))|, \ldots, |\text{In}_r(\text{Lw}(T))|) \times \prod_{v \in T, v \neq \text{Lw}(T)} q^{c(\text{Out}(v))}(|\text{In}_1(v)|, \ldots, |\text{In}_r(v)|) \right\}$$

where $T$ runs over short $(a_1,\ldots,a_r)$ - trees such that the vertex Lw($T$) may have just one input edge but all other vertices have $\geq 2$ input edges.

The lemma is proved by explicit substitution of power series.

Let us now substitute into Lemma 3.3.13 the particular values of the coefficients $q^{(i)}(a_1,\ldots,a_r)$ from (3.3.11). We get a formula for $u^{(i)}(a_1,\ldots,a_r)$ which involves a summation over all
(\(a_1, \ldots, a_r\)) - labelled trees \(T\), not necessarily short, such that the lowest vertex \(Lw(T)\) is allowed to have one input (whose color coincides with the color of \(\text{Out}(T)\)), but no other vertex is allowed to have one input. That means that any summand in the arising formula for \(u^{(i)}(a_1, \ldots, a_r)\) will enter twice: it will correspond once to a tree whose lowest vertex has \(\geq 2\) inputs and it will correspond for the second time to the same tree but with a one-input vertex appended at the bottom. These two summands will enter with opposite signs, hence they will cancel each other. Therefore all the coefficients \(u^{(i)}(a_1, \ldots, a_r)\) are equal to zero. So \(f_i(y) = y_i\) and Theorem 3.3.9 is proven.

This completes the proof of part (a) of Theorem 3.3.2. The proof of part (b) is entirely similar and consists of repeating the argument in the new context when the coefficients of power series are not just numbers but elements of the ring \(\mathbb{R} \otimes r\). We omit the details.

**Example.** Let \(b_n\) be the number of non-isomorphic binary \(n\)-trees. Let \(\mathcal{P}\) be the \(k\)-linear operad with \(\mathcal{P}(1) = \mathcal{P}(2) = k\), \(\mathcal{P}(n) = 0, n \geq 3\). Applying the definition, we see that the complex \(D(\mathcal{P})(n)\) consists of one vector space \(V(n)\), of dimension \(b_n\), placed in degree 0. The generating maps of \(\mathcal{P}\) and \(D(\mathcal{P})\) are therefore \(g(x) = x + x^2/2\) and \(h(x) = x + \sum_{n=2}^{\infty} b_n x^n / n!\), respectively. The identity \(g(-h(-x)) = x\) yields \(h(x) = 1 - \sqrt{1 - 2x}\) whence \(b_n = (2n - 3)!! = 1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2n - 3)\).
3.4. The configuration operad and duality.

(3.4.1) Let $X$ be a topological space. A complex of sheaves of $k$-vector spaces on $X$ can be regarded as a sheaf with values in the Abelian category $dgVect$ of $dg$-vector spaces. Such objects will be referred to as $dg$-sheaves.

For a $dg$-sheaf $\mathcal{F}^\bullet$ on $X$ we denote by $R\Gamma(X, \mathcal{F}^\bullet)$ the $dg$-vector space of global sections of the canonical Godement resolution of $\mathcal{F}^\bullet$, see [Go] [KS]. The cohomology spaces of $R\Gamma(X, \mathcal{F}^\bullet)$ are $H^i(X, \mathcal{F}^\bullet)$, the usual topological hypercohomology with coefficients in $\mathcal{F}^\bullet$.

Given two topological spaces $X_1, X_2$ and $dg$-sheaves $\mathcal{F}_i^\bullet$ on $X_i$, there is a natural morphism

$$R\Gamma(X_1, \mathcal{F}_1^\bullet) \otimes R\Gamma(X_2, \mathcal{F}_2^\bullet) \to R\Gamma(X_1 \times X_2, \mathcal{F}_1^\bullet \otimes \mathcal{F}_2^\bullet)$$

which is a quasi-isomorphism.

(3.4.3) Let $Q$ be a topological operad. The notion of a $dg$-sheaf on $Q$ is completely analogous to the notion of a sheaf on $Q$ in (1.5.3). Similarly to (1.5.9) we have:

(3.4.4) Proposition. If $Q$ is a topological operad and $\mathcal{F}^\bullet$ is a $dg$-sheaf on $Q$ then the collection

$$R\Gamma(Q, \mathcal{F}^\bullet) = \{R\Gamma(Q(n), \mathcal{F}^\bullet(n))^*, \ n \geq 1\}$$

forms a $dg$-operad, so that the cohomology spaces $H^\bullet(Q(n), \mathcal{F}^\bullet(n))^*$ form an operad in the category $gVect^-$. 

(3.4.5) Example: logarithmic forms of the configuration operad. Let $k = \mathbb{C}$ be the field of complex numbers and let $\mathcal{M}$ be the configuration operad (1.4.4). For any $n \geq 1$ let $j_n : M_{0,n+1} \hookrightarrow \mathcal{M}(n)$ be the embedding of the open stratum and $\underline{\mathbf{C}}_{M_{0,n+1}}$ be the constant sheaf on $M_{0,n+1}$. Let $D(n) = \mathcal{M}(n) - M_{0,n+1}$. This is a divisor with normal crossings in $\mathcal{M}(n)$, consisting of $2^{n-1} - 1$ smooth components $\overline{\mathcal{M}(T)}$, where $T$ is an $n$-tree with exactly one internal edge. Let $\Omega^\bullet_{\mathcal{M}(n)}(\log D(n))$ be the corresponding logarithmic de Rham complex [D 1]. Consider the collection of shifted and twisted complexes

$$\Omega^\bullet_{\mathcal{M}}(\log D) = \left\{ \Omega^\bullet_{\mathcal{M}(n)}(\log D(n))[n-2] \otimes \text{Det}(\mathbb{C}^n), \ n \geq 1 \right\}.$$

This collection has a natural structure of a $dg$-sheaf on $\mathcal{M}$, whose structure maps (1.5.3)(ii) are given by the Poincaré residue maps ([D 1], n. 3.1.5.2). Let us briefly recall this notion in the generality we need. Let $X$ be a smooth variety and $Y \subset X$ be a divisor with normal crossings which we assume to consist of smooth components $Y_1, ..., Y_M$. Let
$Z \subset Y$ a codimension $m$ subvariety given by intersection of some $m$ of these components: $Z = \bigcap_{i \in I} Y_i$, $|I| = m$, and let $\gamma : Z \hookrightarrow X$ be the embedding. The Poincaré residue is the map

\[(3.4.7) \quad \text{res}: \gamma^* \Omega^*_X(\log Y)[m] \otimes \text{Det}(C^I) \longrightarrow \Omega^*_Z(\log(Z \cap Y)).\]

The appearance of $\text{Det}(C^I)$ stems from the fact that the operations of taking the residues along individual hypersurfaces $Y_i, i \in I$ anticommute with each other due to the residue theorem.

Returning to our situation, let $m_1, ..., m_l \geq 1$ be given. The structure map of the operad $M$

$$\gamma_{m_1, ..., m_l} : M(l) \times M(m_1) \times ... \times M(m_l) \to M(m_1 + ... + m_l),$$

described in (1.4.4), is a closed embedding whose image is the intersection of several components of the divisor $D(m_1 + ... + m_l)$. If we denote the set of these components by $I$ then we have a canonical identification

$$\text{Det}(C^l) \cong \left(\text{Det}(C^l) \otimes \bigotimes \text{Det}(C^{m_i})\right)^* \otimes \text{Det}(C^{m_1 + ... + m_l}).$$

Thus the maps (3.4.6) indeed supply the necessary structure.

(3.4.8) **The gravity operad $\mathcal{G}$.** We denote the $dg$-operad $R\Gamma(M, \Omega^*_M(\log D))^*$ simply by $\tilde{\mathcal{G}}$ and by $\mathcal{G}$ we denote the cohomology operad of $\tilde{\mathcal{G}}$ i.e., the operad in the category of graded vector spaces $gVect^-$ given by $\mathcal{G}(n) = H^\bullet(\tilde{\mathcal{G}}(n))$. Following a suggestion of E. Getzler, we call $\mathcal{G}$ the gravity operad. Since $\Omega^*_M(n)(\log D(n))$ is quasi-isomorphic to the direct image $Rf_{n,*}C_{M_0,n+1}$, we have

$$H^i(M(n), \Omega^*(\log D(n))) = H^i(M_{0,n+1}, C).$$

Thus the $i$-th component of the graded vector space $\mathcal{G}(n)$ is $H_{n-2-i}(M_{0,n+1}, C) \otimes \text{Det}(C^n)$.

(3.4.9) **Proposition.** There is a quasi-isomorphism of $dg$-operads $\tilde{\mathcal{G}} \to \mathcal{G}$ (where $\mathcal{G}$ is considered with zero differential).

**Proof:** Let $L^i(n)$ be the space of global logarithmic $i$-forms on $M(n)$. It is known [ESV] that $L^i(n)$ consists of closed forms and is naturally identified with $H^i(M_{0,n+1}, C)$. Thus the embedding of the graded vector space $L^\bullet(n) = \bigoplus L^i(n)$ with zero differential into $R\Gamma(M(n), \Omega^\bullet_M(n)(\log D(n)))$ is a quasi-isomorphism. Denote this embedding by $\phi_n$. Clearly the Poincaré residue homomorphisms preserve global logarithmic forms so they make the collection of graded vector spaces $L^\bullet(n)^*[-n+2] \otimes \text{Det}(C^n)$ into an operad.
which is isomorphic to \( G \). Taking duals of the embeddings \( \phi_n \) above we get the required quasi-isomorphism \( \tilde{G} \to G \).

**3.4.10** Let \( \mathcal{P} \) be an admissible \( dg \)-operad. As in (1.5.2) we associate to \( \mathcal{P} \) a \( dg \)-sheaf \( \mathcal{F}_\mathcal{P} \) on the configuration operad \( \mathcal{M} \). By (3.4.4) the complexes \( \mathcal{R}\Gamma(\mathcal{M}(n), \mathcal{F}_\mathcal{P}(n))^* \) form another \( dg \)-operad. It is described as follows.

**3.4.11** Theorem. Suppose that \( k = \mathbb{C} \). Then the operad \( \mathcal{R}\Gamma(\mathcal{M}, \mathcal{F}_\mathcal{P})^* \) is quasi-isomorphic to \( D(\mathcal{P} \otimes \mathcal{G}) \) where \( D \) is the duality for \( dg \)-operads introduced in (3.2.13).

*Proof:* Let \( X \) be a finite (compact) CW-complex and \( S = \{X_\alpha\} \) be its Whitney stratification by locally closed CW-subcomplexes. We denote by \( j_\alpha : X_\alpha \hookrightarrow X \) the embeddings of the strata. Let also \( X_{\geq m} \) be the union of strata of dimensions \( \geq m \). This is an open subset in \( X \) and we denote by \( j_{\geq m} : X_{\geq m} \hookrightarrow X \) the embedding. Let \( \mathcal{F} \) be any (\( dg \))-sheaf on \( X \). Let \( \mathcal{F}_\alpha = j_\alpha^* \mathcal{F} \) be the restrictions of \( \mathcal{F} \) to the strata. Note that \( \mathcal{F} \) has a decreasing filtration

\[
\mathcal{F} = (j_{\geq 0})! j_{\geq 0}^* \mathcal{F} \supset (j_{\geq 1})! j_{\geq 1}^* \mathcal{F} \supset \ldots
\]

with quotients

\[
\bigoplus_{\dim X_\alpha = m} j_\alpha! j_\alpha^* \mathcal{F}.
\]

Here \( j_\alpha! \) are the direct images with proper support (extensions by zero). We can regard this filtration as a Postnikov system realizing \( \mathcal{F} \) as a convolution [Ka 4] of the following complex of objects of \( D^b(\text{Sh}_X) \), the derived category of sheaves on \( X \):

\[
\bigoplus_{\dim X_\alpha = 0} j_\alpha! j_\alpha^* \mathcal{F} \to \bigoplus_{\dim X_\alpha = 1} j_\alpha! j_\alpha^* \mathcal{F}[1] \to \bigoplus_{\dim X_\alpha = 2} j_\alpha! j_\alpha^* \mathcal{F}[2] \to \ldots
\]

By replacing \( j_\alpha! j_\alpha^* \mathcal{F} \) with appropriate injective resolutions, we can realize (3.4.13) as an actual double complex of sheaves on \( X \). By applying the functor \( \mathcal{R}\Gamma(X, -) \) to (3.4.13) we get that \( \mathcal{R}\Gamma(X, \mathcal{F}) \) is the total complex of the double complex

\[
\bigoplus_{\dim X_\alpha = 0} \mathcal{R}\Gamma_c(X_\alpha, j_\alpha^* \mathcal{F}) \to \bigoplus_{\dim X_\alpha = 1} \mathcal{R}\Gamma_c(X_\alpha, j_\alpha^* \mathcal{F})[1] \to \ldots
\]

where \( \mathcal{R}\Gamma_c \) is the derived functor of global sections with compact support. The horizontal grading in this complex is so normalized that the sum with \( \dim X_\alpha = 0 \) has horizontal degree 0.

Assume now that \( \mathcal{F} \) is \( S \)-combinatorial (1.5.1) and given by (\( dg \)-)vector spaces \( F_\alpha \). Then we have the equalities

\[
\text{3.4.15} \quad \mathcal{R}\Gamma_c(X_\alpha, j_\alpha^* \mathcal{F}) = \mathcal{R}\Gamma_c(X_\alpha, \mathbb{C}) \otimes F_\alpha.
\]
We specialize now to the case when \( X = \mathcal{M}(n) \), the stratification \( S \) consists of \( \mathcal{M}(T) \) and \( \mathcal{F} = \mathcal{F}_\mathcal{P}(n) \) is the sheaf corresponding to an admissible \( dg \)-operad \( \mathcal{P} \). For an \( n \)-tree \( T \) let \( D(T) \subset \overline{\mathcal{M}(T)} \) be the complement to \( \mathcal{M}(T) \). This is a divisor with normal crossing. As usual, we denote by \( |T| \) the number of internal edges in \( T \) so \( \dim_{\mathbb{C}} \mathcal{M}(T) = n - 2 - |T| \).

Note that we have the following quasi-isomorphisms

\[
R\Gamma_c(\mathcal{M}(T), \mathcal{C}) \cong R\Gamma(\mathcal{M}(T), \mathcal{C})^*[-2(n - 2 - |T|)] \cong R\Gamma(\overline{\mathcal{M}(T)}, \Omega^*_{\overline{\mathcal{M}(T)}}(\log D(T))^*[-2(n - 2 - |T|)]).
\]

(3.4.16)

Here the first quasi-isomorphism is the Poincaré duality. The second quasi-isomorphism is the consequence of the fact [D 1] that the logarithmic de Rham complex is quasi-isomorphic to the full direct image of the constant sheaf. Note that in virtue of the product decompositions (1.4.8) and the definition of the \( dg \)-operad \( \tilde{\mathcal{G}} \) we have a quasi-isomorphism

\[
R\Gamma \left( \overline{\mathcal{M}(T)}, \Omega^*_{\overline{\mathcal{M}(T)}}(\log D(T)) \right)^* \cong \tilde{\mathcal{G}}(T)[n - 2 - |T|] \otimes \text{Det}(T),
\]

where \( \text{Det}(T) \) was introduced in (3.2.0). Taking into account Proposition 3.4.9, we see that the double complex (3.4.14), i.e., \( R\Gamma(\mathcal{M}(n), \mathcal{F}_\mathcal{P}(n)) \), can be replaced by the double complex

\[
\bigoplus_{|T|=0} \mathcal{P}(T) \otimes \mathcal{G}(T) \otimes \text{Det}(T) \rightarrow \bigoplus_{|T|=1} \mathcal{P}(T) \otimes \mathcal{G}(T) \otimes \text{Det}(T) \rightarrow ...
\]

So the dual complex will be \( D(\mathcal{P} \otimes \mathcal{G})(n) \) as claimed.

(3.4.17) Example. Taking \( \mathcal{P} = \text{Com} \), the commutative operad, we get that every \( \mathcal{F}_\mathcal{P}(n) \) is the constant sheaf \( \mathcal{C} \) on \( \mathcal{M}(n) \). Thus the operad formed by the total homology spaces \( H_\bullet(\mathcal{M}(n), \mathcal{C}) \) is quasi-isomorphic to \( D(\mathcal{G}) \).
3.5. The building co-operad and duality.

(3.5.1) By a *cooperad* in a symmetric monoidal category \((\mathcal{A}, \otimes)\) we mean an operad in the opposite category \(\mathcal{A}^{op}\). Explicitly, a cooperad \(\mathcal{B}\) is a collection of objects \(\mathcal{B}(n) \in \mathcal{A}, n \geq 1\) with \(\Sigma_n\) - action on each \(\mathcal{B}_n\) and morphisms

\[
\rho_{m_1, \ldots, m_l} : \mathcal{B}(m_1 + \ldots + m_l) \to \mathcal{B}(l) \otimes \mathcal{B}(m_1) \otimes \ldots \otimes \mathcal{B}(m_l)
\]

satisfying the conditions dual to the corresponding conditions for an operad.

If a collection \(\mathcal{C} = \{\mathcal{C}(n)\}\) forms a cooperad in the category of \((dg, -)\) vector spaces then the dual \(dg\) - vector spaces \(\mathcal{C}(n)^*\) form a \(dg\) - operad \(\mathcal{C}^*\). A \(dg\) - cooperad \(\mathcal{C}\) is called *admissible* if \(\mathcal{C}^*\) is an admissible operad in the sense of (3.1.5).

Cooperads in the category of topological spaces will be called *topological cooperads*.

(3.5.2) The building \(\mathcal{W}(n)\). Let \(T\) be an \(n\) - tree. By a *metric* on \(T\) we mean an assignment of a positive number (length) to each internal edge of \(T\). The external edges can be thought of as having length 1. Let \(\mathcal{W}(n)\) be the set of isometry classes of all \(n\) - trees with metrics. This set has a natural topology. In plain words, when the length of some edge is going to 0, we say that the limit tree is obtained by contracting this edge into a point. For any \(n\) - tree \(T\) (without metric) let \(\mathcal{W}^0(T)\) denote the subset in \(\mathcal{W}(T)\) consisting of all trees with metric isomorphic to \(T\). This subset is clearly a cell (a product of several copies of \(\mathbb{R}_+\), corresponding to internal edges of \(T\)). Thus \(\mathcal{W}(n)\) is a union of these non - compact cells. For example, the space \(\mathcal{W}(3)\) is the union of three half - lines glued along a common end (Fig. 11):

The minimal cell in \(\mathcal{W}(n)\) has dimension 0 and corresponds to the \(n\) - tree \(T_n\) with a single vertex (and no internal edges). Maximal cells of \(\mathcal{W}(n)\) correspond to binary trees. Thus, the cells in \(\mathcal{W}(n)\) are parametrized by the same set as the strata in the moduli space \(\mathcal{M}(n)\), see (1.4.5). But the closure relations among the cells are *dual* to those among the strata in \(\mathcal{M}(n)\), where the maximal stratum corresponds to the tree with one vertex and 0 -dimensional strata correspond to binary trees.
We call $\mathcal{W}(n)$ the building of $n$-trees since it is in many respects similar to the Bruhat-Tits building.

**3.53 The building cooperad.** Suppose given natural numbers $m_1, \ldots, m_l$. We define a map

$$\rho_{m_1, \ldots, m_l} : \mathcal{W}(m_1 + \ldots + m_l) \to \mathcal{W}(l) \times \mathcal{W}(m_1) \times \ldots \times \mathcal{W}(m_l)$$

as follows. Let $T(m_1, \ldots, m_l)$ be the $(m_1 + \ldots + m_l)$-tree drawn in Fig. 3 above. Let $(T', \mu')$ be any other metrized $(m_1 + \ldots + m_l)$-tree. Let $T''$ be the maximal tree which is obtained from both $T'$ and $T(m_1, \ldots, m_l)$ by contracting edges. The tree $T''$ is naturally divided into blocks $T''_1, \ldots, T''_l$, $T''_\infty$ where $T''_i$ has $m_i$ inputs and $T''_\infty$ has $l$ inputs, see Fig. 12.

Each block $T''_\nu$, $\nu = 1, \ldots, \infty$, is naturally equipped with a metric $\mu''_\nu$ on its internal edges. We redefine the lengths of edges which become loose edges in $T''_\nu$ by letting their lengths be equal to 1. The map $\rho_{m_1, \ldots, m_l}$ takes

$$(T', \mu') \mapsto ((T''_\infty, \mu''_\infty), (T''_1, \mu''_1), \ldots, (T''_\nu, \mu''_\nu)).$$

**3.54 Proposition.** The collection of maps $\rho_{m_1, \ldots, m_l}$ and the natural actions of $\Sigma_n$ on $\mathcal{W}(n)$, $n \geq 1$, define on the collection of $\mathcal{W}(n)$ the structure of a topological cooperad which will be denoted by $\mathcal{W}$.

We call $\mathcal{W}$ the building cooperad. Note that the structure maps $\rho_{m_1, \ldots, m_l}$ for $\mathcal{W}$ are surjective, whereas the structure maps for the operad $\mathcal{M}$ are, dually, injective.

Just as any operad gives rise to a sheaf on the topological operad $\mathcal{M}$ (Theorem 1.5.11), any cooperad gives a sheaf on the cooperad $\mathcal{W}$. Let us give the corresponding definitions.

**3.55 Definition.** Let $\mathcal{B}$ is a topological cooperad and let

$$\rho_{m_1, \ldots, m_l} : \mathcal{B}(m_1 + \ldots + m_l) \to \mathcal{B}(l) \times \mathcal{B}(m_1) \times \ldots \times \mathcal{B}(m_l)$$

be its structure maps (3.5.2). A sheaf (resp. a dg - sheaf) $\mathcal{F}$ on $\mathcal{B}$ consists of the following data:
(i) A collection of $\Sigma_n$-equivariant sheaves (resp. $dg$-sheaves) $F(n)$ on $B(n)$, one for each $n \geq 1$;
(ii) Homomorphisms of sheaves on $B(m_1 + ... + m_l)$

$$\mu_{m_1,...,m_l} : \rho_{m_1,...,m_l}^*(F(l) \otimes F(m_1) \otimes ... \otimes F(m_l)) \rightarrow F(m_1 + ... + m_l).$$

These data should satisfy the compatibility conditions dual to those given in (1.5.4) - (1.5.6).

If, for example, all the spaces $B(n)$ consist of a single point then a $dg$-sheaf on $B$ is the same as a $dg$-operad. More generally, we have the following.

(3.5.6) Proposition. If $B$ is a topological cooperad and $F$ is a $dg$-sheaf on $B$ then the collection of complexes $R\Gamma(B(n), F(n))$ forms a $dg$-operad. Similarly, the complexes $R\Gamma_c(B(n), F(n))$ (the derived functors of sections with compact support) form a $dg$-operad.

Let $R\Gamma(B, F)$ and $R\Gamma_c(B, F)$ denote the $dg$-operads thus obtained.

(3.5.7) Definition. A sheaf $F$ on a topological cooperad $B$ is called an iso-sheaf if the morphisms

$$F(l) \otimes F(m_1) \otimes ... \otimes F(m_l) \rightarrow (\rho_{m_1,...,m_l})^*F(m_1 + ... + m_l)$$

obtained from $\mu_{m_1,...,m_l}$ by adjunction, are isomorphisms and higher direct images $R^j(\rho_{m_1,...,m_l})^*F(m_1 + ... + m_l)$ vanish for $j \geq 1$.

(3.5.8) Let $Q$ be any $k$-linear cooperad. Then $Q(1)$ is a $k$-coalgebra and each $Q(n)$ is a $(Q(1)^{\otimes n}, Q(1))$-bi-comodule. As in (1.1.5), we extend the collection of $\Sigma_n$-modules $Q(n)$ to a functor $I \mapsto Q(I)$ on finite sets and bijections. For any tree $T$ we define, similarly to (1.2.13), the space

$$Q(T) = \bigodot_{v \in T} Q(\text{In}(v))$$

where $\bigodot$ is the cotensor product of comodules over a coalgebra. If $Q(1) = k$ then $Q(T) = \bigotimes_{v \in T} Q(\text{In}(v))$ is the usual tensor product over $k$. If all $Q(n)$ are finite-dimensional over $k$ then we can use the operad $Q^*$ to define $Q(T) = (Q^*(T))^*$ where $Q^*(T)$ was defined by (1.2.13).

If $T \leq T'$ then the cooperad structure on $Q$ defines a linear map $\rho_{T,T'} : Q(T) \rightarrow Q(T')$. The condition $T \leq T'$ means that the cell $W^0(T)$ is contained in the closure of $W^0(T')$. Therefore the maps $\rho_{T,T'}$ give rise to a combinatorial (1.5.1) sheaf $G_Q(n)$ on $W(n)$ with fiber $Q(T)$ on $W^0(T)$.

Here is the “dual” of Theorem 1.5.11.
(3.5.9) Theorem. Let \( Q \) be a \( k \)-linear operad. Then:

a) The sheaves \( G_Q(n) \) on the spaces \( W(n) \) form a sheaf \( G_Q \) on the cooperad \( W \).
b) If \( Q(1) = k \) then \( G_Q \) is an iso-sheaf.
c) Any iso-sheaf \( G \) on \( W \) such that each \( G(n) \) is constant on each cell \( W^0(T) \), has the form \( G_Q \) for some \( k \)-linear cooperad \( Q \) with \( Q(1) = k \).

The proof is straightforward and left to the reader.

In a similar way, for any \( dg \)-operad \( Q \) we construct a \( dg \)-sheaf \( G^\bullet_Q \) on \( W \).

(3.5.10) Let \( X \) be a CW-complex and \( S \) be its stratification into strata which are topological manifolds. Recall [KS] that the Verdier duality gives a contravariant functor \( F^\bullet \mapsto V(F^\bullet) \) from the derived category of \( dg \)-sheaves on \( X \) with \( S \)-constructible cohomology into itself.

The following result gives two different sheaf-theoretic interpretations of the duality \( D \) on \( dg \)-operads introduced in §3.2.

(3.5.11) Theorem. Let \( Q \) be an admissible \( dg \)-cooperad (3.5.1) so that \( Q^\ast \) is an admissible \( dg \)-operad. Then:

a) There is a natural quasi-isomorphism of \( dg \)-operads

\[
D(Q^\ast) \cong R\Gamma_c(W, G^\bullet_Q) \otimes \Lambda
\]

where \( \Lambda \) is the determinant \( dg \)-operad (1.3.21).

b) For any \( n \), the \( dg \)-sheaves \( G^\bullet_Q(n) \) and \( G_{D(Q^\ast)} \) on \( W(n) \) are Verdier dual to each other.

Proof: Let \((X, S)\) be any space stratified into cells. Giving an \( S \)-combinatorial \( (dg-) \) sheaf \( F \) on \( X \) is the same (1.5.11) as giving \( (dg-) \) vector spaces \( F_\sigma = R\Gamma(\sigma, F) \) together with generalization maps \( g_{\sigma \tau} : F_\sigma \to F_\tau \) for \( \sigma \subset \tau \) satisfying transitivity conditions. For any cell \( \sigma \) let \( OR(\sigma) = H_c^{\dim(\sigma)}(\sigma, k) \) be its (1-dimensional) orientation space. Note that for the cell \( W^0(T) \) in the building \( W(n) \) corresponding to an \( n \)-tree \( T \), the space \( OR(W^0(T)) \) can be naturally identified with the space \( \det(T) \), see (3.2.0). Theorem 3.5.11 is a consequence of the following well known combinatorial construction of the Verdier duality and hypercohomology functors.

(3.5.12) Proposition. a) Let \((X, S)\) be as above and \( F \) be an \( S \)-combinatorial \( dg \)-sheaf on \( X \) given by \( dg \)-vector spaces \( F_\sigma \) and maps \( g_{\sigma \tau} \). Then:

a) The \( dg \)-vector space \( R\Gamma_c(X, F) \) is quasi-isomorphic naturally to (the total \( dg \)-vector space arising from) the complex

\[
\bigoplus_{\dim(\sigma) = 0} F_\sigma \otimes OR(\sigma) \longrightarrow \bigoplus_{\dim(\sigma) = 1} F_\sigma \otimes OR(\sigma) \longrightarrow ...
\]
where the sum over $\sigma$ with $\dim(\sigma) = m$ is placed in degree $m$.

b) The Verdier dual $dg$-sheaf $V(\mathcal{F})$ is represented by the collection of by $dg$-vector spaces associated to complexes

$$V(\mathcal{F})_\sigma = \left\{ ... \rightarrow \bigoplus_{\tau \supset \sigma} F^*_\tau \otimes \text{OR}(\tau) \rightarrow \bigoplus_{\tau \supset \sigma} F^*_\tau \otimes \text{OR}(\tau) \rightarrow F^*_\sigma \otimes \text{OR}(\sigma) \right\},$$

where the sum over $\tau$ with $\dim(\tau) = \dim(\sigma) + m$ is placed in degree $(-m)$.

This completes the proof of Theorem 3.5.11.

The isomorphism $D(\text{Com}) \cong \text{Lie}$, see (3.2.15), yields the following.

(3.5.13) Corollary. The cohomology of $\mathcal{W}(n)$ with compact support (and constant coefficients) are as follows:

$$H^i_c(\mathcal{W}(n), k) = \begin{cases} 0, & i \neq n-2 \\ \text{Lie}(n), & i = n-2. \end{cases}$$

This shows that the space $\text{Lie}(n)$ is analogous to the Steinberg representation of the group $GL_n(\mathbb{F}_q)$. Note that the standard Steinberg representation of this group, see [Lu] has dimension $q^{n(n-1)/2} = q \cdot q^2 \cdot ... \cdot q^{n-1}$. The modified Steinberg (discrete series) representation introduced by Lusztig [Lu] has dimension $(q-1)(q^2-1)...(q^{n-1}-1)$. Both numbers can be seen as $q$-analog of $(n-1)! = \dim \text{Lie}(n)$. The role of the Lie operad as the dualizing module in our theory (2.2.9) is analogous to the role of the Steinberg representation in the Deligne - Lusztig theory [DL 1-3].
4. KOSZUL OPERADS.

4.1. Koszul operads and Koszul complexes.

(4.1.1) Let $K$ be a semisimple $k$-algebra and $P = \mathcal{P}(K,E,R)$ be a quadratic operad. Let $P^! = \mathcal{P}(K^{op}, E^\vee, R^\perp)$ be the dual quadratic operad and $D(P)$ be the dual $dg$-operad. Observe that for every $n$ the degree 0 part, $D(P)(n)^0$, of the $dg$-vector space $D(P)(n)$ is equal to $F(E^\vee)(n)$, the $n$-th space of the free operad generated by the single space $E$, i.e.,

$$D(P)(n)^0 = \bigoplus_{\text{binary } n-\text{trees } T} E^\vee(T) \otimes \text{Det}(T) = F(E^\vee)(n).$$

We define a morphism of $dg$-operads $\gamma_P : D(P) \to P^!$ (here $P^!$ is equipped with the trivial $dg$-structure) to be given by compositions

$$D(P)(n) \to D(P)(n) = F(E^\vee)(n) \to F(E^\vee)(n)/(R^\perp) = P^!(n).$$

(4.1.2) Lemma. For every $n$ the morphism $\gamma_P$ induces an isomorphism $H^0(D(P)(n)) \to P^!(n)$.

Proof: Observe that the penultimate term

$$D(P)(n)^{-1} = \bigoplus_T \mathcal{P}^*(T) \otimes \text{Det}(T)$$

is the sum over the $n$-trees $T$ such that all but one vertices of $T$ are binary and just one vertex is ternary. Note also that $P(3)^\vee$ is dual to the space of relations of the quadratic operad $P^!$. Thus, the image of the last differential in $D(P)(n)$ is precisely the space of consequences of the relations in $P^!$, whence the statement of the lemma.

(4.1.3) Definition. A quadratic operad $P$ is called Koszul if the morphism $\gamma_P : D(P) \to P^!$ is a quasi-isomorphism, i.e., each complex $D(P)(n)$ is exact everywhere but the right end.

(4.1.4) Proposition. a) A quadratic operad $P$ is Koszul if and only if so is $P^!$.

b) Let $r$ be the number of simple summands of the algebra $P(1) = P^!(1)^{op}$ and let $g_P, g_P^! : \mathbb{C}^r \to \mathbb{C}^r$ be the generating maps of $P$ and $P^!$ respectively. If $P$ is Koszul, we have the formal power series identity

$$g_P^!( -g_P( -x) ) = x, \quad x = (x_1, \ldots, x_r).$$
Proof: a) Form the composition

\[ D(\mathcal{P}^f) \xrightarrow{D(\gamma_{\mathcal{P}})} D(D(\mathcal{P})) \xrightarrow{f_{\mathcal{P}}} \mathcal{P}, \]

where \(f_{\mathcal{P}}\) is the quasi-isomorphism constructed in Theorem 3.2.16. It is immediate that this composition coincides with \(\gamma_{\mathcal{P}}\). If \(\mathcal{P}\) is Koszul then \(\gamma_{\mathcal{P}}\) and hence \(D(\gamma_{\mathcal{P}})\) are quasi-isomorphisms, whence the statement.

b) Follows from Theorem 3.3.2 and the observation that quasi-isomorphic dg-operads have the same generating maps.

(4.1.5) Let \(K\) be a semisimple \(k\)-algebra and \(P = \{P(n)\}\) be a \(K\)-collection (1.2.11). We extend the collection \(P\) to a functor \(I \mapsto P(I)\) on the category of finite sets and their bijections as in (1.1.5). For any surjection \(f : I \rightarrow J\) of finite sets we put

\[ P(f) = \bigotimes_{j \in J} P(f^{-1}(j)). \]

If, in addition, \(P\) is an operad then for any composable pair of surjections \(I \xrightarrow{f} J \xrightarrow{g} H\) the compositions in \(P\) give rise to a map

\[ \mu_{g,f} : P(g) \otimes P(f) \rightarrow P(gf). \]

Let \(P, Q\) be two \(K\)-collections. We define a new \(K\)-collection \(P(Q)\), called the composition of \(P\) and \(Q\), as follows:

\[ (4.1.6) \quad P(Q)(n) = \bigoplus_{m=1}^{n} P(Q)(n)^m, \quad \text{where} \quad P(Q)(n)^m = \bigoplus_{f : [n] \rightarrow [m]} [P(m) \otimes A Q(f)]_{\Sigma_m} \]

and the last sum is taken over all surjections \([n] \rightarrow [m]\).

(4.1.7) Proposition. The generating map (3.1.8) of the collection \(P(Q)\) is given by the composition

\[ g_{P(Q)}(x) = g_P(g_Q(x)), \quad x = (x_1, \ldots, x_r). \]

Proof: This follows from Lemma 3.3.13.

(4.1.8) Koszul complexes. Recall [Pr] that the Koszul complex of a quadratic \(K\)-algebra \(A\) is the vector space \(A \otimes_K (A^!)^\vee\) (where \((A^!)^\vee = \text{Hom}_{K^{op}}(A^!, K^{op})\)) equipped with the differential defined in a certain natural way.

Let now \(\mathcal{P} = \mathcal{P}(K, E, R)\) be a quadratic operad and \(\mathcal{P}^f\) be the dual operad. We define the \(n\)-th Koszul complex of \(\mathcal{P}\) to be the \(n\)-th space of the composition \(\mathcal{P}(((\mathcal{P}^f)^!)^!(n))\).
Here \((\mathcal{P}^1)^\vee\) is the \(K\) - collection consisting of \(\mathcal{P}^1(n)^\vee = \text{Hom}_{K^{op}}(\mathcal{P}^1(n), K^{op})\) with the \(\Sigma_n\) - action being the transposed one twisted by sign. We put a grading on the space \(\mathcal{P}((\mathcal{P}^1)^\vee)(n)\) by (4.1.6) and define the differential \(d\) in the following way.

For \(X \in \mathcal{P}(2)\) and a surjection \(g : [m + 1] \to [m]\) let

\[\mu_{g,X} : \mathcal{P}(m) \to \mathcal{P}(m + 1)\]

be the operator of composition with \(X\) along \(g\).

For \(\Xi \in \mathcal{P}^1(2) = \mathcal{P}(2)^\vee\) and surjections \(g : [m + 1] \to [m], h : [n] \to [m + 1]\) let

\[\mu_{g,h,\Xi} : \mathcal{P}^1(h) \to \mathcal{P}^1(hg)\]

be the operator of composition with \(\Xi\) induced by \(\mu_{g,h}\).

Let us write the identity element

\[\text{Id}_{\mathcal{P}(2)} \in \text{Hom}_K(\mathcal{P}(2), \mathcal{P}(2)) = \mathcal{P}(2) \otimes_K \mathcal{P}^1(2)\]

in the form \(\sum X_i \otimes \Xi_i\) where \(X_1, \ldots, X_d \in \mathcal{P}(2)\) and \(\Xi_1, \ldots, \Xi_d \in \mathcal{P}^1(2)\). For a surjection \(f : [n] \to [m]\) we define a map

\[d_f : \mathcal{P}(m) \otimes (\mathcal{P}^1)^\vee(f) \to \bigoplus_{h : [n] \to [m+1]} \mathcal{P}(m + 1) \otimes (\mathcal{P}^1)^\vee(h)\]

by

\[d_f = \sum_{g : [m + 1] \to [m]} \sum_{i=1}^d \mu_{h,X_i} \otimes \mu_{g,h,\Xi_i}^*, \quad gh = f\]

Clearly \(d_f\) is \(\Sigma_n\) - equivariant, hence factors through the spaces of \(\Sigma_n\) - coinvariants. Therefore the formula \(d = \sum_{f : n \to m} d_f\) defines a linear map

\[d : \mathcal{P}((\mathcal{P}^1)^\vee)(n)^m \to \mathcal{P}((\mathcal{P}^1)^\vee)(n)^{m+1}\]

(4.1.10) Proposition. The morphisms \(d\) satisfy, for various \(m\), the condition \(d^2 = 0\) thus making each \(\mathcal{P}((\mathcal{P}^1)^\vee)(n)\) into a complex.

(4.1.11) To prove Proposition 4.1.10, we introduce the following notation. Let \(E = \mathcal{P}(2)\) and \(F(E)\) be the corresponding free operad. By definition (2.1.1), \(F(E)(n) = \bigoplus E(T)\) (sum over binary \(n\) - trees). Let \(R \subset F(E)(3)\) be the space of relations of \(\mathcal{P}\). Call an \(n\) - tree \(T\) a \(1\)-ternary tree if \(T\) contains exactly one ternary vertex \(v = v(T)\), all other

\[67\]
vertices being binary. For such a tree $T$ there are exactly three binary trees $T', T'', T'''$ such that $T$ can be obtained from each of them by contracting an edge. We denote by

$$R_T \subset E(T') \oplus E(T'') \oplus E(T''') \subset F(E)(n)$$

the result of substituting $R$ at the place $v$. Thus,

$$P(n) = \frac{F(E)(n)}{\sum_{1 - \text{ternary trees } T} R_T}, \quad P^!(n)^\vee = \bigcap_{1 - \text{ternary trees } T} R_T.$$

Looking at any summand of $P((P^!)(n)^m$ we find that this is a subquotient of $F(E)(n)$. More precisely, for a surjection $f : [n] \to [m]$ let $T(F)$ be the reduced tree obtained by depicting the map $f$ (and ignoring vertices with one input (Fig.13).

This tree has one vertex (root) at the bottom and several vertices at the top. We have

$$P^!(f) \otimes (P^!(f)) = \frac{\bigcap_{1 - \text{ternary trees } T: T \geq T(f), v(T) \to \text{top}} R_T}{\left( \sum_{1 - \text{ternary trees } T: T \geq T(f), v(T) \to \text{bottom}} R_T \right) \cap \left( \bigcap_{1 - \text{ternary trees } T: T \geq T(f), v(T) \to \text{top}} R_T \right)}.$$

Applying $d$ amounts to deleting some $R_T$ from the intersection in the numerator of (4.1.12) and adding some other $R_T$ to the sum in the denominator, according to surjections $g : [m + 1] \to [m]$. Hence applying $d^2$ to (4.1.12) moves some term $R_T$ from the numerator to the denominator. Therefore $d^2 = 0$.

Now we state the main result of this section.

**4.1.13 Theorem.** Let $\mathcal{P}$ be a quadratic operad. The following conditions are equivalent:

(i) $\mathcal{P}$ is Koszul.

(ii) The Koszul complexes $\mathcal{P}((\mathcal{P}^!)(n)^\vee)$ are exact for all $n \geq 2$.

**4.1.14** We prove Theorem 4.1.13 by reducing it to a similar result about quadratic associative algebras or, rather, quadratic categories, a result which was proven in the required generality in [BGS].
For any operad $\mathcal{P}$ we construct, following Mac Lane, a certain PROP $[\text{Ad}]$. By definition, this is a category $\text{Cat}(\mathcal{P})$ whose objects are symbols $[n], n = 0, 1, 2, \ldots$. The morphisms are defined by

$$\text{Hom}_{\text{Cat}(\mathcal{P})}([n], [m]) = \bigoplus_{f: [n] \to [m]} \mathcal{P}(f), \quad \text{for} \quad n \geq m,$$

where $f$ runs over all surjections $[n] \to [m]$. For $n < m$ we set $\text{Hom}_{\text{Cat}(\mathcal{P})}([n], [m]) = 0$. The composition in the category $\text{Cat}(\mathcal{P})$ is induced by maps $\mu_{g,f}$, see (4.1.5). It is well known $[\text{Ad}]$ that $\text{Cat}(\mathcal{P})$ has a natural structure of a symmetric monoidal category given on objects by $[m] \otimes [n] = [m + n]$. We write $\mathcal{P}(n, m) = \text{Hom}_{\text{Cat}(\mathcal{P})}([n], [m])$. Thus the space $\mathcal{P}(n)$ of our operad can be written as $\mathcal{P}(n, 1)$.

If $\mathcal{P}$ is a quadratic operad then $\text{Cat}(\mathcal{P})$ is a quadratic category ($\mathbb{Z}_+\text{-}algebra$) in the sense of [BGS, §3] and $\text{Cat}(\mathcal{P}^!)$ is the dual quadratic category. Let us compare the cobar - duals and the Koszul duals for $\mathcal{P}$ and $\text{Cat}(\mathcal{P})$. The cobar - dual category of $\text{Cat}(\mathcal{P})$ is the category $\mathbf{D}((\text{Cat}(\mathcal{P}))$ with the same objects $[m]$ as $\text{Cat}(\mathcal{P})$. The complex $\text{Hom}_{\mathbf{D}(\text{Cat}(\mathcal{P}))}([n], [m])$ is given by

$$\mathcal{P}(n, m)^* \to \bigoplus_{n > r > m} \mathcal{P}(n, r)^* \otimes_{\mathcal{P}(r, r)} \mathcal{P}(r, m)^* \to \cdots$$

$$\to \bigoplus_{n > r_1 > r_2 > m} \mathcal{P}(n, r_1)^* \otimes_{\mathcal{P}(r_1, r_1)} \mathcal{P}(r_1, r_2)^* \otimes_{\mathcal{P}(r_2, r_2)} \mathcal{P}(r_2, m)^* \to \cdots$$

The category $\text{Cat}(\mathcal{P})$ is Koszul if and only if each of these complexes is exact off the rightmost term. Observe that

$$\text{Hom}_{\mathbf{D}(\text{Cat}(\mathcal{P}))}([n], [m]) = \left(\text{Hom}_{\mathbf{D}(\text{Cat}(\mathcal{P}))}([n], [1])\right)^{\otimes m}$$

and

$$\text{Hom}_{\mathbf{D}(\text{Cat}(\mathcal{P}))}([n], [1]) = \mathbf{D}(\mathcal{P})(n)$$

is the $n$ -th complex of the cobar - operad $\mathbf{D}(\mathcal{P})$. This gives the following.

**Lemma.** A quadratic operad $\mathcal{P}$ is Koszul if and only if $\text{Cat}(\mathcal{P})$ is a Koszul quadratic category.

Let us now recall the interpretation of Koszulness for categories in terms of Koszul complexes. Let $\mathcal{Q} = \mathcal{P}^!$. For the category $\text{Cat}(\mathcal{P})$ we have, according to [BGS, n.4.4], the complexes in the form

$$\left\{ \mathcal{Q}(n, m)^* \to \mathcal{Q}(n, m+1)^* \otimes_{\mathcal{K}[\Sigma_{m+1}]} \mathcal{P}(m+1, m) \to \mathcal{Q}(n, m+2)^* \otimes_{\mathcal{K}[\Sigma_{m+2}]} \mathcal{P}(m+2, m) \to \cdots \right\}$$
\[ \to Q(n, n - 1)^* \otimes_{K[\Sigma_{n-1}]} P(n - 1, m) \to P(n, m) \].

Let us denote the above complex by \( K^\bullet(n, m) \). Again, \( K^\bullet(n, m) \) is the \( m \) - th tensor power of the complex \( K^\bullet(n, 1) \) and \( K^\bullet(n, 1) \) is the \( n \) - th Koszul complex of the quadratic operad \( P \). Thus Koszulness of \( P \) is equivalent to the exactness of all the Koszul complexes for \( \text{Cat}(P) \) hence is equivalent to exactness of the complexes \( K^\bullet(n, 1) \) only, i.e., of Koszul complexes of \( P \). This concludes the proof.
4.2. Homology of algebras over a quadratic operad and Koszulness.

(4.2.1) Given a quadratic operad \( \mathcal{P} = \mathcal{P}(K, E, R) \) and a \( \mathcal{P} \) - algebra \( A \), we define a chain complex \( (C_\bullet^\mathcal{P}(A), d) \) as follows. We put

\[
C_n(A) = C_n^\mathcal{P}(A) = (A^\otimes n \otimes K^\otimes n \mathcal{P}^1(n)^\vee)_{\Sigma_n}.
\]

To define \( d_n : C_n(A) \to C_{n-1}(A) \), we first define a map

\[
\bar{d}_n : \left( A^\otimes n \otimes \left( \bigoplus_{\text{binary} \ T} E(T) \right) \otimes \det(T) \right)_{\Sigma_n} \rightarrow \left( A^\otimes (n-1) \otimes \left( \bigoplus_{(n-1)-\text{trees} \ S} E(S) \otimes \det(S) \right) \right)_{\Sigma_{n-1}}.
\]

If \( T \) is an \( n \) - tree, we call a vertex \( v \in T \) extremal if all inputs of \( v \) are inputs of \( T \). For such a \( v \) we define the set \([n]/v\) obtained by replacing the subset \( \text{In}(v) \subset [n] \) by a single element. Let \( T/v \) be the \([n]/v\) - tree obtained by erasing \( v \) and replacing it by an external edge.

Let \( T \) be a binary \( n \) - tree and \( v \in T \) an extremal vertex. The \( \mathcal{P} \) - action on \( A \) defines a map

\[
d_{T,v} : A^\otimes n \otimes E(T) \otimes \det(T) \rightarrow A^\otimes [n]/v \otimes E(T/v) \otimes \det(T/v).
\]

We define \( \bar{d}_n \) to be given by the matrix with entries \( \tilde{d}_{T,v} \). (Note that the ambiguity in numbering the set \([n]/v\) by 1, 2, ..., \( n-1 \) will disappear after we take coinvariants of \( \Sigma_{n-1} \).) Observe that \( C_n(A) \subset (A^\otimes n \otimes \bigoplus E(T) \otimes \det(T))_{\Sigma_n} \).

(4.2.2) Proposition. For any \( n \) the map \( \bar{d}_n \) takes the subspace \( C_n(A) \) into \( C_{n-1}(A) \). The maps \( d_n \) defined as restrictions of \( \bar{d}_n \) to \( C_n(A) \), satisfy \( d_{n-1} \circ d_n = 0 \).

Proof: Let \( X \) be an element of \( A^\otimes n \otimes R_T \) for some 1-ternary tree \( T \) (here \( R_T \) stands for the space of relations at the ternary vertex of \( T \)). Let \( v \in T \) be the ternary vertex and \( T_1, T_2, T_3 \) be the binary trees obtained by splitting this vertex. Suppose first that the vertex \( v \in T \) is extremal. Let \( e_i \) be the edge of \( T_i \) which is contracted into \( v \). Let also \( v_i \in T_i \) be the source of the edge \( e_i \). Then \( v_i \) is an extremal vertex. Therefore after erasing \( v_i \) the edge \( e_i \) will become external and cannot be contracted.

If \( v \) is not extremal then all the extremal vertices of \( T_i \) come from extremal binary vertices of \( T \). If \( w \in T \) is such a vertex then \( T/w \) is 1-ternary and \( \sum_{i=1}^3 \bar{d}_{T_i,w}(X) \) belongs to \( R_{T/w} \). Clearly all 1-ternary \((n-1)\) - trees can be represented as \( T/w \). So if \( X \in A^\otimes n \otimes R_T \) for all 1-ternary \( n \) - trees \( T \) then \( \bar{d}_{n}(X) \in A^\otimes (n-1) \otimes R_S \) for all 1-ternary \((n-1)\) - trees \( S \). This shows that \( \bar{d}_n(C_n(A)) \subset C_{n-1}(A) \).
Let us prove that $\bar{d}_{n-1} \circ \bar{d}_n = 0$ on $C_n(A)$. Let $T$ be a 1-ternary tree $T$ whose 1-ternary vertex $v$ is extremal. Suppose that $X \in A^{\otimes n} \otimes R_T \subset \bigoplus_{i=1}^3 A^{\otimes n} \otimes E(T_i)$ where $T_i$, are as before. Denote by $w_i$ the end of the edge $e_i$. Then $w_i$ will become extremal after erasing $v_i$. Thus

$$\sum_{i=1}^3 \bar{d}_{T_i/e_i, w_i} (\bar{d}_{T_i, v_i}(X))$$

is a sum of quantities like $r(a, b, c)$ with $r \in R \subset F(E)(3)$ and $a, b, c \in A$. Since $A$ is a $P$-algebra and $R$ is the space of relations for $P$, every such quantity is equal to 0. This implies that $\bar{d}_{n-1}(\bar{d}_n(X)) = 0$ if $X \in \bigcap_T A^{\otimes n} \otimes R_T$. The proposition follows.

(4.2.3) Definition. The complex $C_\bullet(A) = \bigoplus C_n(A)$ with the differential $d$ defined above is called the chain complex of the $P$-algebra $A$. Its homology will be denoted $H_n(A)$ or $H_P^n(A)$.

(4.2.4) Examples. a) Let $P = As$ be the associative operad. The dual operad $P!$ is isomorphic to $P$ itself. Let $A$ be an associative algebra. Since $P!(n)$ is the regular representation of $\Sigma_n$, we find that the complex $C_\bullet(A)$ has the form

$$\ldots \to A \otimes A \otimes A \to A \otimes A \to A.$$

A straightforward calculation of the differential shows that it is the standard Hochschild complex of $A$. Thus, $H_P^n(A)$ is the Hochschild homology of $A$.

b) Let $P = Lie$ be the Lie operad. Then $P! = Com$ is the commutative operad; in particular, $P!(n)$ is the trivial 1-dimensional $\Sigma_n$-module and $P!(n)^\vee$ is the sign representation. Let $G$ be a Lie algebra. By the above, its chain complex, as a $P$-algebra, has the form

$$\ldots \to \Lambda^3 G \to \Lambda^2 G \to G.$$

Again, a straightforward calculation of the differential shows that this is the standard Chevalley - Eilenberg chain complex of $G$, so that $H_P^n(G) = H_i(G, k)$ is the Lie algebra homology of $G$ with constant coefficients.

c) Similarly, if $P = Com$ is the commutative operad, and $A$ is a commutative algebra, one finds that $H_P^n(A)$ is the Harrison homology of $A$, see e.g., [Lo].

(4.2.5) Theorem. Let $P$ be a quadratic operad. Then $P$ is Koszul if and only if for any free $P$-algebra $F = F_P(V)$ we have $H_P^i(F) = 0$ for $i > 0$.

Proof: Let $F_n$ denote the free $P$-algebra on generators $x_1, \ldots, x_n$. This algebra has an obvious $(\mathbb{Z}_+)^n$-grading such that $\deg(x_i) = (0, \ldots, 1, \ldots, 0)$ (the unit on the $i$-th place) and $\deg(\mu(a, b)) = \deg(a) + \deg(b)$ for any $a, b \in F_n, \mu \in P(2)$.

The chain complex $C_\bullet^P(F_n)$ also inherits this grading. The following result which is immediate from the definitions implies the "if" part of Theorem 4.2.5.

72
(4.2.6) Proposition. The \( n \)-th Koszul complex \( K^n_P \) of \( P \) is isomorphic to the multihomogeneous part of \( C^P(F_n) \) of multi-degree \((1, \ldots, 1)\).

Before proving the “only if” part of (4.2.5), let us mention the following corollary from what we have already done.

(4.2.7) Corollary. The operads \( A_s, Com, Lie \) are Koszul.

Proof: It is well known that a free associative algebra without unit has higher Hochschild homology trivial and a free Lie algebra has higher homology with constant coefficients trivial. This proves that \( A_s \) and \( Lie \) are Koszul. The case of \( Com \) follows by duality (4.1.4).

Another situation of applicability of Theorem 4.2.5 is provided by the work of E. Getzler and J.D.S. Jones [Ge J]. They proved, in the context of their work on iterated integrals, the vanishing of the homology of free algebras over the operads associated to homology of configuration space.

(4.2.8) End of the proof of Theorem 4.2.5. Let \( A \) be any \( P \)-algebra. Along with the chain complex \( C_n^P(A) \) we introduce the “big” chain complex \( BC_n^P(A) \). By definition,

\[
(4.2.9) \quad BC_n^P(A) = \bigoplus_{i+j=n \atop j \leq 0} \left( \bigoplus \left( A^i \otimes P(T) \otimes \text{Det}(T) \right) \right) .
\]

The differential \( d : BC_n^P(A) \to BC_{n-1}^P(A) \) is given by the following two types of matrix elements:

\[
d_{T,e} : A^i \otimes P(T) \otimes \text{Det}(T) \to A^i \otimes P(T/e) \otimes \text{Det}(T/e)
\]
defined for any \( i \)-tree \( T \) and any internal edge \( e \in Ed(T) \), and

\[
d_{T,v} : A^i \otimes P(T) \otimes \text{Det}(T) \to A^i[v] \otimes P(T/v) \otimes \text{Det}(T/v)
\]
defined for any \( i \)-tree \( T \) and any extremal vertex \( v \in T \).

The operator \( d_{T,e} \) is \( 1 \otimes \mu_{T,e} \otimes l^*_e \) where \( \mu_{T,e} : P(T) \to P(T/e) \) is induced by the composition in \( P \) and the map \( l^*_e : \text{Det}(T) \to \text{Det}(T/e) \) is dual to the exterior multiplication by \( e \). The operator \( d_{T,v} \) is induced, in a similar way, by the \( P \)-action on \( A \).

It is immediate to verify that \( d^2 = 0 \). The embedding \( P^1(n)^{\vee} \subset \bigoplus_{\text{binary trees } T} R(T) \) gives rise to an embedding of complexes

\[
(4.2.10) \quad j : C^P(A) \to BC^P(A).
\]
The complex $BC_P^P(A)$ has an increasing filtration $F$ with the $m$-th term $F_mBC_P^P(A)$ being the sum of the summands in (4.2.9) with $i \leq m$. The associated graded complex has the form

$$\text{gr}_m^F BC_P^P(A) = A^\otimes m \otimes D(P)(m)^\vee$$

where $D$ is the duality for $dg$-operads (3.2.13). This implies the following result.

(4.2.11) Proposition. If the operad $P$ is Koszul then the embedding (4.2.11) is a quasi-isomorphism for any $P$-algebra $A$.

To complete the proof of Theorem 4.2.5, it is enough, in view of Proposition 4.2.11, to establish the following.

(4.2.11) Proposition. Let $P$ be any $k$-linear operad, $K = P(1)$ and and $A = F_P(V)$ be the free $P$-algebra generated by a finite-dimensional $K$-module $V$. Then

$$H_i(BC_P^P(A)) = \begin{cases} 0, & i > 0 \\ V, & i = 0. \end{cases}$$

Proof: To each tree $T$ and a function $\nu : \text{In}(T) \to \{1, 2, 3, \ldots\}$ we associate the vector space

$$C(T, \nu) = \text{Det}(T) \otimes \bigotimes_{v \in T} P(\text{In}(v)) \otimes \bigotimes_{i \in \text{In}(T)} A_{\nu(i)}$$

where $A_n, n = 1, 2, 3, \ldots$, denotes the degree $n$ component of $A$. By construction the graded vector space $BC_P^P(A)$ is the direct sum of spaces $C(T, \nu)$ over all (isomorphism classes of) pairs $(T, \nu)$.

Given a pair $(T, \nu)$ we call an input edge $e \in \text{In}(T)$ non-degenerate if $\nu(e) > 1$. Given a non-degenerate edge $e \in \text{In}(T)$ we define the degeneration of $(T, \nu)$ along $e$ as the pair $(\hat{T}, \hat{\nu})$ where $\hat{T}$ is obtained by attaching at $e$ a star (1.1.1) with $\nu(e)$ inputs. The function $\hat{\nu}$ is set equal 1 on the new inputs and remain unchanged on other inputs. A pair $(T, \nu)$ is called non-degenerate if for any extremal vertex $v \in T$ there is an edge $e \in \text{In}(v)$ such that $\nu(e) > 1$. A non-degenerate pair cannot be obtained by degeneration.

Let $\text{Nd}(T, \nu)$ denote the number of non-degenerate input edges of $T$. We introduce an increasing filtration $G$ on $BC_P^P(A)$ by putting

$$G_i BC_P^P(A) = \bigoplus_{\#(\text{vertices of } T) + \text{Nd}(T, \nu) \leq i} C(T, \nu).$$

The differential preserves the filtration. Furthermore, the complex $\text{gr}_i^G BC_P^P(A)$ splits into a direct sum of complexes $E_\bullet(T, \nu)$ labelled by non-degenerate pairs $(T, \nu)$. These complexes have the form

$$E_\bullet(T, \nu) = \bigoplus_{(\hat{T}, \hat{\nu})} C(\hat{T}, \hat{\nu})$$

74
where the sum runs over all possible pairs \((\hat{T}, \hat{\nu})\) obtained from \((T, \nu)\) by (iterated) degeneration. Such pairs \((\hat{T}, \hat{\nu})\) are parametrized by all possible subsets of the set \(S\) of non-degenerate inputs of \(T\). It is clear that the corresponding spaces \(C(\hat{T}, \hat{\nu})\) are all the same. Observe that subsets of \(S\) correspond to faces of a simplex with \(|S|\) vertices (denote this simplex by \(\Delta\)). Moreover, we find that \(E_\bullet(T, \nu)\) is the tensor product of the fixed vector space \(C(T, \nu)\) and the augmented chain complex of \(\Delta\). This complex is acyclic unless \(S = \emptyset\). The only non-degenerate pair \((T, \nu)\) with \(S = \emptyset\) consists of \(T = \{\to\}\) (the tree with no vertices) and \(\nu = 1\). This gives \(H_0(BC_\bullet(A)) = V\), and the theorem follows.

(4.2.12) Homotopy \(\mathcal{P}\) - algebras. Let \(\mathcal{P} = \mathcal{P}(K, E, R)\) be a Koszul quadratic operad and \(\mathcal{P}_!\) its quadratic dual. Then the canonical morphism of operads (4.1.1) \(\gamma_\mathcal{P} : \mathbf{D}(\mathcal{P}_!) \to \mathcal{P}\) is a quasi-isomorphism. Hence, any \(\mathcal{P}\) - algebra can be viewed as a \(\mathbf{D}(\mathcal{P}_!)\) - algebra. This motivates the following

(4.2.13) Definition. A \(dg\) - algebra over \(\mathbf{D}(\mathcal{P}_!\)) is called a homotopy \(\mathcal{P}\) - algebra.

(4.2.14) Proposition. Let \(A = \bigoplus A_n\) be a graded \(K\) - bimodule with \(\dim(A_n) < \infty\) for all \(n\). A giving a structure of a homotopy \(\mathcal{P}\) - algebra on \(A\) is the same as giving a non-homogeneous differential \(d\) on the free algebra \(\mathbf{F}_{\mathcal{P}_!}(A^*[1])\) satisfying the conditions:

(i) \(d^2 = 0\);

(ii) \(d\) is a derivation with respect to any binary operation in \(\mathcal{P}_!\), i.e., we have

\[
d(\mu(a, b)) = \mu(d(a), b) + (-1)^{\deg(a)}\mu(a, d(b)), \quad \mu \in \mathcal{P}(2), a, b \in \mathbf{F}_{\mathcal{P}_!}(A^*[1]).
\]

Proof: Let \(d\) be a differential in the free algebra \(F = \mathbf{F}_{\mathcal{P}_!}(A^*[1])\) satisfying (i) and (ii). Recall that the free algebra has a natural grading \(F = \bigoplus_{n \geq 1} F_n\) where (1.3.5)

\[
F_n = (\mathcal{P}_!(n) \otimes_K (A^*)^\otimes n)_{\text{sgn}}
\]

(the subscript “sgn” stands for the “anti-invariants” of the symmetric group action). We decompose the differential \(d\) into homogeneous components \(d = d_1 + d_2 + ...\) where the component \(d_i\) shifts degree by \(i - 1\), i.e., \(d_i : F_j \to F_{j+i-1}, \forall j\). The equality \(d^2 = 0\) yields, in particular \(d_2^2 = 0\).

Observe that the operad \(\mathcal{P}_!\) being quadratic, the algebra \(F\) is generated by its degree 1 component \(F_1 = \mathcal{P}_!(1) \otimes_K A^* = A^*\). Hence, the differential \(d\) is completely determined, due to the property (ii), by its restriction to \(F_1\). Separating the degrees, we see that this restriction is given by a collection of maps \(d_m : A^* = F_1 \to F_m, m = 1, 2, ...\). In view of (4.2.15), we may view \(d_m\) as a \(\Sigma_m\) - invariant element

\[
d_m \in \text{Det}(k^m) \otimes \mathcal{P}_!(m) \otimes_K (A^*)^\otimes m \otimes_K A.
\]
Next, we replace, using (1.1.6), the integer \( m \) (4.2.16) by any \( m \)-element set. Further, for any \( n \)-tree \( T \), we form the tensor product of the elements \( d_m \) over all vertices of \( T \) to get an element \( \bigotimes_{v \in T} d_{\text{In}(v)} \). Rearranging the factors in the tensor product, we get

\[
(4.2.17) \quad \bigotimes_{v \in T} d_{\text{In}(v)} \in \text{Det}(T) \otimes \bigotimes_{v \in T} \left( \mathcal{P}^1(\text{In}(v)) \otimes_K \left( \bigotimes_{e \in \text{In}(v)} A^* \right) \otimes_K A \right)
\]

where the factor \( A \) on the right corresponds to the output edge of \( v \) and \( \text{Det}(T) \) was introduced in (3.2.0). Each internal edge of \( T \) occurs in the above tensor product twice, once as an input at some vertex, contributing to the factor \( A^* \) and once as an output at some vertex, contributing to the factor \( A \). Contracting the pairs of factors \( A^* \) and \( A \) corresponding to each internal edge and using the notation (1.2.13), we obtain from (4.2.17) a well defined element

\[
d(T) \in \text{Det}(k^n) \otimes \mathcal{P}^1(T) \otimes_K (A^*)^\otimes n \otimes A.
\]

This element can be regarded a morphism

\[
(4.2.18) \quad d(T) : \mathcal{P}^1(T) \to \mathcal{P}^1(T)^* \otimes \text{Det}(k^n) \to \text{Hom}(A^\otimes n, A).
\]

The morphisms (4.2.18), assembled for various \( n \)-trees together, define, for each \( n \), a morphism

\[
D(\mathcal{P}^1)(n) \to \text{Hom}(A^\otimes n, A) = \mathcal{E}_A(n).
\]

One can check by a direct calculation that these morphisms give rise to a morphism of operads \( D(\mathcal{P}^1) \to \mathcal{E}_A \) if and only if the original differential \( d \) on the free algebra \( F \) satisfies the property (i) of Proposition 4.2.13. That makes \( A \) a \( D(\mathcal{P}^1) \)-algebra. The opposite implication is proved by reversing the above argument.

(4.2.19) Examples. a) Let \( \mathcal{P} = \text{Lie} \) be the Lie operad so \( \mathcal{P}^1 = \text{Com} \). Since \( \text{Com}(n) = k \) for any \( n \), a structure of a \( D(\text{Com}) \)-algebra on a \( dg \)-vector space \( A \) is determined by specification of \( n \)-ary antisymmetric operations \([x_1, \ldots, x_n]\) for any \( n \geq 2 \) (these operations correspond to the basis vectors of \( \text{Com}(n) \)). Proposition 4.2.13 in this case gives the equivalence of two definitions [SS] of a homotopy Lie algebra: first as a vector space with brackets \([x_1, \ldots, x_n]\) satisfying the generalized Jacobi identity and, the second, as a vector space \( A \) with a differential on the exterior algebra \( \Lambda^\otimes(A^*) \), satisfying the Leibnitz rule.

b) If we take \( \mathcal{P} = \text{Com} \), we get a notion of a homotopy \( \text{Com} \)-algebra which is a \( dg \)-algebra over \( D(\text{Lie}) \). Such algebras are particular cases of algebras “associative and commutative up to all higher homotopies” (May algebras) [HS 1] [KM 1, 2]. More precisely, a homotopy \( \text{Com} \)-algebra is strictly commutative and equipped with a system of natural
homotopies which ensure, in particular, the associativity of the cohomology algebra. This structure is not the same as a homotopy between $ab$ and $ba$ in an associative $dg$-algebra, the data often referred to as "homotopy commutativity".
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