Analysis of the 3D acoustic cloaking problems using optimization method

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Abstract. Control problems for the 3D model of acoustic scattering which describes scattering acoustic waves by a permeable obstacle with the form of a spherical layer are considered. These problems arise while developing the design technologies of acoustic cloaking devices using the wave flow method. The solvability of direct and control problems for the acoustic scattering model under study is proved. The sufficient conditions which provide local uniqueness and stability of optimal solutions are established.

1. Introduction
Recent significant research has focused on design of invisibility cloaking devices for material bodies. Beginning with pioneering papers [1, 2, 3], a large number of publications was devoted to developing different methods of solving the cloaking problems. The transformation optics (TO) method proposed in [2] is the most popular cloaking method lying on the base of the so-called passive cloaking strategy. The first applications of the TO method were connected with solving electromagnetic cloaking problems. Then after proof of the equivalence of the 2D Maxwell equations to the 2D equations of acoustics of anisotropic medium, this theory was expanded to acoustic cloaking at first in 2D case [4] and then in general 3D case [5, 6]. We also mention the papers [7, 8, 9] devoted to the proof of existence of acoustic cloaking shells and to the studies of properties of solutions of the acoustic cloaking problem. In [10], the invisibility problem in X-ray tomography was studied.

It should be emphasized that the technical realization of solutions obtained in cited references is connected with substantial difficulties. One of approaches of overcoming these difficulties consists of replacing the exact cloaking problem by the approximate cloaking problem for which solutions admit relatively simple technical realization (see, e.g., [11, 12, 13, 14] and references therein). Another approach is based on using the optimization method of solving inverse problems. This method was applied in papers [15, 16] devoted to numerical analysis of 2D cloaking problems and in [17, 18, 19] when studying impedance cloaking problems.

Optimization method is also applied in this paper while theoretical analysis of acoustic cloaking problems in which the cloaking effect is achieved due to the choice of variable parameters of inhomogeneous isotropic liquid medium filling the acoustic cloaking shell. We begin with formulation of the direct scattering problem. Let $\Omega$ be the bounded domain in $\mathbb{R}^3$ which is topologically equivalent to the spherical layer with the interior domain $\Omega_i$, the external domain $\Omega_e^\infty$ and, in the general case, with the curvilinear boundary $\Gamma$ consisting of two components:
internal $\Gamma_i$ and external $\Gamma_e$ (see Figure 1). We assume that the domains $\Omega_i$, $\Omega$ and $\Omega_e^\infty$ are filled with liquid medium, and let $\rho$ and $c$ be functions given in $\Omega$ which have the sense of the density of the medium and the speed of sound in the domain $\Omega$, $\rho_0=\text{const}$ and $c_0=\text{const}$ are the constant density of the medium and the speed of sound in the domains $\Omega$, $\Omega_i$ and $\Omega_e^\infty$ respectively, $\omega$ is the angular frequency. We introduce a constant wave number $k=\frac{\omega}{c_0}$. For brevity, we refer to triple $(\Omega, \rho, c)$ as an acoustic shell.

Let us assume that the field $p^{\text{inc}}$ arises in the external part $\Omega_e^\infty$ of the region $\Omega$. The incidence of this field to the shell results in the appearance of the refracted field $p$ in the domain $\Omega$, the internal field $p_i$ in $\Omega_i$, and the field $p_s$ scattered by this shell in the domain $\Omega_e^\infty$. The problem of determining the fields $p_i$ in $\Omega_i$, $p$ in $\Omega$ and $p_s$ in $\Omega_e^\infty$ is reduced to finding the fields $p_i$ in $\Omega_i$, $p$ in $\Omega$ and $p_e=p^{\text{inc}}+p_s$ in $\Omega_e^\infty$ from the following relations:

$$\Delta p_i + k^2 p_i = 0 \text{ in } \Omega_i, \quad \Delta p_e + k^2 p_e = 0 \text{ in } \Omega_e^\infty, \quad \rho \text{div}(\frac{1}{\rho} \nabla p) + k^2 \eta p = 0 \text{ in } \Omega,$$

$$p = p_i, \quad \frac{1}{\rho} \frac{\partial p}{\partial n} = \frac{1}{\rho_0} \frac{\partial p_i}{\partial n} \text{ on } \Gamma_i, \quad p = p_e, \quad \frac{1}{\rho} \frac{\partial p}{\partial n} = \frac{1}{\rho_0} \frac{\partial p_e}{\partial n} \text{ on } \Gamma_e,$$

$$\frac{\partial p^s(x)}{\partial r} - ikp^s(x) = o(r^{-1}) \text{ as } r = |x| \to \infty.$$  

Here, in particular, $\eta = c^2_0/c^2$ is a refraction index and (3) has the meaning of the Sommerfeld radiation condition in $R^3$.

In this paper the inverse problem for the model (1)–(3) arising while developing the design technologies of cloaking devices using the wave flow method is considered. This problem in exact formulation consists of finding variable parameters $\rho$ and $\eta$ of the medium filling the domain $\Omega$ from the condition of perfect cloaking any object placed in the internal domain $\Omega_i$. By optimization method, this problem is reduced to the problem of minimization of the following cost functional:

$$I_0(P) = ||P - p^d||_Q^2 \equiv \int_Q ||P - p^d||^2 dx.$$  

Here, $P$ is the function equalled to $p_i$ in $\Omega_i$, $p$ in $\Omega$ and $p^{\text{inc}}+p_s$ in $\Omega_e^\infty$, function $p^d \in L^2(Q)$ describes the field measured in some subset $Q \subset \Omega_i \cup \Omega_e^\infty$. As controls, we take the pair $\lambda = 1/\rho$, $\mu = \lambda \eta$ uniquely connected with the pair $(\rho, \eta)$ of physical parameters at $\rho > 0, \eta > 0$. 

Figure 1. Geometry of cloaking shell.

Figure 2. Geometry of bounded problem.
The main goal of this paper is to perform theoretical analysis of control problems with tracking-type functionals of type (4) for the model (1)–(3). Keeping this goal in mind, we first reduce unbounded problem (1)–(3) to an equivalent problem considered in bounded domain, and it is shown that under appropriate assumptions the weak solution of the latter problem exists, it is unique and depends continuously on the datum (incident field $p^{inc}$). Then, we prove the existence of a solution of our control problem and deduce the optimality system. Based on its’ analysis, we prove the uniqueness results for concrete control problems and derive the stability estimates of optimal solutions. It is to note that this paper contains detailed proofs of some theorems formulated in [20].

2. Study of the direct problem
Let us introduce certain functional spaces which will be used while studying the direct and control problems for model (1)–(3). Let $B_R$ be the ball of radius $R$ with the boundary $\Gamma_R$ containing $\Omega, \Omega_e \equiv \Omega^\infty \cap B_R$ (see Figure 2). We use the spaces $H^1(\Omega), H^1(\Omega_e), H^1(B_R), L^2(\Omega), L^2(\Gamma_r), H^{1/2}(\Gamma_R), L^\infty(\Omega)$ and $H^s(\Omega)$ with the norms $\| \cdot \|_{1,\Omega}, \| \cdot \|_{1,\Omega_e}$, $\| \cdot \|_{1,B_R}, \| \cdot \|_{\Omega}, \| \cdot \|_{Q}, \| \cdot \|_{\Gamma_r}, \| \cdot \|_{1/2,\Gamma_R^\prime}, \| \cdot \|_{-1/2,\Gamma_R^\prime}, \| \cdot \|_{L^\infty(\Omega)}$ and $\| \cdot \|_{s,\Omega}$ respectively. We put $L^\infty_0(\Omega) = \{ \eta \in L^\infty(\Omega) : \eta(x) \geq \eta_0 \text{ in } \Omega \}, \eta_0 = \text{const} > 0$. We also need the space $H^1(\Delta,\Omega_e) = \{ v \in H^1(\Omega_e) : \Delta v \in L^2(\Omega_e) \}$ equipped with the norm $\| v \|_{H^1(\Delta,\Omega_e)} = \| v \|_{1,\Omega_e}^2 + \| \Delta v \|_{1,\Omega_e}^2$ and the space $\mathcal{P}^{inc} = \{ v \in H^1(\Omega_e) : \Delta v + k^2v = 0 \text{ in } \Omega_e \} \subset H^1(\Delta,\Omega_e)$ equipped with the norm $\| \cdot \|_{1,\Omega_e}$. The latter will serve for describing the restrictions of incident waves to the domain $\Omega_e$. The space $X = H^1(B_R)$ equipped with the Hilbert norm $\| \cdot \|_X = \| \cdot \|_{B_R}$ is defined and denoted by $X^*$ the space antidual of $X$.

As in [17, 18], we introduce the Dirichlet-to-Neumann operator $T \in \mathcal{L}(H^{1/2}(\Gamma_R), \ H^{-1/2}(\Gamma_R))$, which maps an arbitrary function $h$ in $H^{1/2}(\Gamma_R)$ to the function $\partial \hat{p}/\partial n \in H^{-1/2}(\Gamma_R)$. Here, $\hat{p}$ is the solution of the Dirichtlet external problem $\Delta \hat{p} + k^2\hat{p} = 0$ in $\Omega^\infty \setminus B_R$, $\hat{p}|_{\Gamma_R} = g$ satisfying the radiation condition (3). It is known that problem (1)–(3) considered in unbounded domain $R^3$ is equivalent in a certain sense to boundary value problem (1), (2), considered in the ball $B_R$ under the following additional condition for $p_s$ on $\Gamma_R$:

$$\partial p_s/\partial n = Tp_s \text{ on } \Gamma_R. \quad (5)$$

We begin with derivation of the weak formulation of problem (1)–(3). To this end, we multiply the first two equations in (1) considered in $\Omega_i \cup \Omega_e$ by the function $\bar{\Phi}/\rho_0$, where $\Phi \in X$ is the test function, and $\bar{\Phi}$ denotes the complex conjugate function to $\Phi$, then we multiply the last equation in (1) considered in $\Omega$ by $\bar{\Phi}/\rho$, integrate over $\Omega_i \cup \Omega_e$, or $\Omega$, apply Green’s formulas in $\Omega_i, \Omega_e$ and $\Omega$, and combine the obtained identities. Using the relation $p = p^{inc} + p_s$ in $\Omega_e$ and the boundary conditions in (2), (5) we arrive to the following identity for triple $P = (p_i, p, p_e) \in X$:

$$a_u(P, \Phi) \equiv a_0(P, \Phi) + a^u(P, \Phi) = \langle f^{inc}, \Phi \rangle \quad \forall \Phi \in X. \quad (6)$$

Here and below, $u$ denotes the pair $(\lambda, \mu)$ where $\lambda = \rho^{-1}, \mu = \eta \eta, a_u : X \times X \rightarrow C$ and $f^{inc} : X \rightarrow C$ are the sesquilinear and antilinear forms defined by

$$a^u(P, \Phi) = a(\lambda, \mu; P, \Phi) = (\lambda \nabla P, \nabla \Phi) - k^2(\mu P, \Phi) \equiv \int_\Omega (\lambda \nabla P \cdot \nabla \bar{\Phi} - k^2 \mu P \bar{\Phi}) dx, \quad (7)$$

$$a_0(P, \Phi) = \rho_0^{-1} \int_{\Omega_i \cup \Omega_e} (\nabla P \cdot \nabla \Phi - k^2 P \bar{\Phi}) dx - \rho_0^{-1} \int_{\Gamma_R} (TP) \bar{\Phi} d\sigma, \quad (8)$$

$$\langle f^{inc}, \Phi \rangle = -\rho_0^{-1} \int_{\Gamma_R} (Tp^{inc}) \bar{\Phi} d\sigma + \rho_0^{-1} \int_{\Gamma_R} \frac{\partial p^{inc}}{\partial n} \bar{\Phi} d\sigma, \quad \Phi \in X. \quad (9)$$
The integral $\int_{\Gamma_R}^{}$ in (8) and (9) denotes the duality pairing between the spaces $H^{-1/2}(\Gamma_R)$ and \( H^{1/2}(\Gamma_R) \). Identity (6) represents the weak formulation of problem (1), (2), (5) and its solution \( P \in X \) is called as a weak solution of problem (1)–(3).

We remind that by trace theorem for any function \( \Phi \in X \) there exists the trace \( \Phi|_{\Gamma_R} \in H^{1/2}(\Gamma_R) \) while for any function \( p^{inc} \in \mathcal{P}^{inc} \) there exists the normal trace \( \partial p^{inc}/\partial n|_{\Gamma_R} \in H^{-1/2}(\Gamma_R) \) and the following estimates hold:

\[
\|\Phi\|_{1/2,\Gamma_R} \leq C_R\|\Phi\|_X \quad \forall \Phi \in X, \quad \|\partial p^{inc}/\partial n\|_{-1/2,\Gamma_R} \leq C'_R\|p^{inc}\|_{1,\Omega_e} \quad \forall p^{inc} \in \mathcal{P}^{inc}.
\]  

(10)

Here, \( C_R \) and \( C'_R \) are constants which depend on \( \Omega, R \) and may be on \( k \) but are independent of \( \Phi \in X \) and \( p^{inc} \in \mathcal{P}^{inc} \). Using (10) and estimating

\[
\|\lambda\|_{L^\infty(\Omega)} \leq C_s\|\lambda\|_{s,\Omega} \quad \forall \lambda \in H^s(\Omega), \; s > 3/2
\]  

(11)

with the constant \( C_s \) depending on \( \Omega \) which follows from the continuous and compact embedding \( H^s(\Omega) \subset L^\infty(\Omega) \) at \( s > 3/2 \), one can derive the following estimates:

\[
|\langle \lambda \nabla P, \nabla \Phi \rangle| \leq \|\lambda\|_{L^\infty(\Omega)} \|P\|_X \|\Phi\|_X \leq C_s\|\lambda\|_{s,\Omega} \|P\|_X \|\Phi\|_X,
\]

(12)

\[
|\langle \mu P, \Phi \rangle| \leq \|\mu\|_{L^\infty(\Omega)} \|P\|_X \|\Phi\|_X \leq C_r\|\mu\|_{r,\Omega} \|P\|_X \|\Phi\|_X,
\]

(13)

\[
|a_0(P, \Phi)| \leq C_1\|P\|_X \|\Phi\|_X \quad \|\langle f^{inc}, \Phi \rangle\| \leq C_1\|p^{inc}\|_{1,\Omega_e} \|\Phi\|_X.
\]

(14)

Here, \( C_1 \) is a constant which depends on \( \Omega, R, \rho_0 \) and \( k \).

We note that the sesquilinear form \( a_u \) introduced in (6), (7), (8) defines the pair of mutually adjoint operators \( A_u : X \to X^* \) and \( A^* : X \to X^* \) by

\[
\langle A_u P, \Phi \rangle = a_u(P, \Phi) = \langle A_u^* \Phi, P \rangle \quad \forall P, \Phi \in X
\]

(15)

and variational problem (6) is equivalent to the operator equation \( A_u P = f^{inc} \). Using (10)–(14) and properties of the operator \( T \) (see, e.g., [17]), one can prove that \( A_u \) is a Fredholm operator, and besides it is injective if \( \lambda \in H^s_{\lambda_0}(\Omega), \; \mu \in H^r_{\mu_0}(\Omega) \) where \( s > 5/2, r > 3/2, \lambda_0 = \text{const} > 0, \mu_0 = \text{const} > 0 \). We denote by \( A^{-1}_u : X^* \to X \) the inverse operator to \( A_u \) and sets \( C_u = |A^{-1}_u| \).

It is clear that for any \( f^{inc} \in X^* \) problem (6) has a unique solution \( P_u = A^{-1}_u(f^{inc}) \) which satisfies the estimate \( \|P_u\|_X \leq C_u\|f^{inc}\|_{X^*} \), where constant \( C_u \) depends on the pair \( u = (\lambda, \mu) \). Furthermore, if \( \lambda \) and \( \mu \) belong to bounded sets \( K_1 \subset H^s_{\lambda_0}(\Omega) \) and \( K_2 \subset H^r_{\mu_0}(\Omega) \) respectively then one can prove arguing as in [17] that the following estimate holds for the solution \( P_u \) of problem (6):

\[
\|P_u\|_X \leq C_2\|p^{inc}\|_{1,\Omega_e} \quad \forall u = (\lambda, \mu) \in K_1 \times K_2.
\]

(16)

Here, the constant \( C_2 \) is independent of \( u \). Let us formulate this result as a theorem.

**Theorem 1.** Let \( \Gamma \in C^{k,1} \), \( K_1 \subset H^s_{\lambda_0}(\Omega) \) and \( K_2 \subset H^r_{\mu_0}(\Omega) \) be the nonempty bounded sets where \( s > 5/2, r > 3/2, \lambda_0 = \text{const} > 0, \mu_0 = \text{const} > 0 \) and let \( u = (\lambda, \mu) \in K_1 \times K_2 \). Then for an arbitrary incident field \( p^{inc} \in \mathcal{P}^{inc} \) problem (6) has a unique solution \( P_u \in X \) which satisfies the estimate (16) where constant \( C_2 \) depends on \( \Omega, R, \rho_0, k \) and sets \( K_1 \) and \( K_2 \) but is independent of \( u = (\lambda, \mu) \).

3. **Study of control problem. The optimality system**

In this section, a precise formulation of our control problem for acoustic scattering model under study is given, the existence of optimal solutions is proved and the optimality system which describes necessary conditions of extremum for the control problem is derived. We remind that the control problem for scattering model (1)–(3) consists of minimization of
certain cost functional depending on the state (acoustic field \( P = \langle p, p, p \rangle \in X \)) and controls \( \lambda = \rho^{-1} \in H^s_{\ast} (\Omega) \) and \( \mu = \lambda_0 \in H^r_{\ast\ast} (\Omega) \) satisfying the state equation. The latter has the form of the weak formulation (6) of direct scattering problem (1)–(3). We shall assume that controls \( \lambda \) and \( \mu \) are changed over certain sets \( K_1 \) and \( K_2 \). More precisely, the following conditions are assumed to hold:

(j) \( K_1 \subset H^s_{\ast 0} (\Omega) \) and \( K_2 \subset H^r_{\ast\ast 0} (\Omega) \) are nonempty convex closed sets where \( s > 5/2, r > 3/2, \lambda_0 = \text{const} > 0, \mu_0 = \text{const} > 0; \alpha_0 = \text{const} > 0. \)

Let \( K = K_1 \times K_2, u = (\lambda, \mu) \). Define operator \( G : X \times K \times P^{inc} \rightarrow X^* \) by \( \langle G(P, u, p^{inc}), \Phi \rangle = a_u(P, \Phi) - \langle f^{inc}, \Phi \rangle \) for any \( \Phi \) and rewrite weak formulation (6) of problem (1)–(3) as operator equation \( G(P, u, p^{inc}) = 0. \) The following constrained minimization problem is considered:

\[
J(P, u) = \frac{a_0}{2} I(P) + \frac{\alpha_1}{2} \| \lambda \|^2_{\ast, \Omega} + \frac{\alpha_2}{2} \| \mu \|^2_{\ast\ast, \Omega} \rightarrow \inf, \quad (P, u) \in X \times K, \quad G(P, u, p^{inc}) = 0. \tag{17}
\]

Here, \( I : X \rightarrow \mathbb{R} \) is a cost functional. Denote by \( Z_{ad} = Z_{ad}(p^{inc}) = \{(P, u) \in X \times K : G(P, u, p^{inc}) = 0, (P, u) < \infty\} \) the set of admissible pairs for problem (17).

**Theorem 2.** Let \( I : X \rightarrow R \) be a weakly lower semicontinuous functional under assumptions (i) and (j), \( p^{inc} \in P^{inc} \) and \( Z_{ad} \) be nonempty set. Let \( \alpha_1 \geq 0, \alpha_2 \geq 0 \) and \( K_1, K_2 \) be bounded sets or \( \alpha_1 > 0, \alpha_2 > 0 \) and functional \( I(P) \) be bounded below. Then control problem (17) has at least one solution \((P, u) \in X \times K \).

Proof. Let \( \{(P_m, u_m)\} \) where \( u_m = (\lambda_m, \mu_m), m = 1, 2, \ldots \) be a minimizing sequence for which

\[
a_0(P_m, \Phi) + a(\lambda_m, \mu_m; P_m, \Phi) = (f^{inc}, \Phi) \forall \Phi \in X, \tag{18}
\]

\[
\lim_{m \rightarrow \infty} J(P_m, u_m) = \inf_{(P, u) \in Z_{ad}} J(P, u) = J^*. \tag{19}
\]

From (19), condition (j) and Theorem 1, the following estimates follow: \( \| \lambda_m \|_{s, \Omega} \leq c_1, \| \mu_m \|_{r, \Omega} \leq c_2, \| P_m \|_X \leq c_3 \). Here, \( c_1, c_2, c_3 \) are constants which are independent of \( m \). From these estimates, it follows that there exist weak limits \( \lambda \in K_1 \subset H^s_{\ast 0} (\Omega), \mu \in K_2 \subset H^r_{\ast\ast 0} (\Omega), \hat{P} \in X \) of some subsequences of sequences \{\( \lambda_m \), \{\( \mu_m \), \{\( P_m \).\} Using these facts, compactness of embedding \( H^s (\Omega) \subset L^\infty (\Omega) \) at \( t > 3/2 \) we conclude (passing if necessary to subsequences) that \( P_m \rightarrow \hat{P} \in X \) weakly in \( X, \lambda_m \rightarrow \hat{\lambda} \in K_1, \mu_m \rightarrow \hat{\mu} \in K_2 \) strongly in \( L^\infty (\Omega) \). Let us show that triple \((\hat{P}, \hat{\lambda}, \hat{\mu})\) satisfies the following identity:

\[
a_0(\hat{P}, \Phi) + a(\hat{\lambda}, \hat{\mu}; \hat{P}, \Phi) = a_0(\hat{P}, \Phi) + \int (\nabla \hat{\Phi} \cdot \nabla \Phi - k^2 \hat{P} \Phi) dx = 0 \quad \forall \Phi \in X. \tag{20}
\]

To this end, we pass to limit in \( (18) \) as \( m \rightarrow \infty \). It is clear that linear term \( a_0(P_m, \Phi) \) passes to the term \( a_0(\hat{P}, \Phi) \) as \( m \rightarrow \infty \) while for the difference \( a(\lambda_m, \mu_m; P_m, \Phi) - a(\hat{\lambda}, \hat{\mu}; \hat{P}, \Phi) \) there is

\[
a(\lambda_m, \mu_m; P_m, \Phi) - a(\hat{\lambda}, \hat{\mu}; \hat{P}, \Phi) =
\]

\[
= ((\lambda_m - \hat{\lambda}) \nabla P_m, \nabla \Phi) + \hat{\lambda}(\nabla P_m - \nabla \hat{P}, \nabla \Phi) - k^2[(\mu_m - \hat{\mu}) P_m, \Phi) + (\hat{\mu}(P_m - \hat{P}, \Phi)]. \tag{21}
\]

Using (12), (13), we derive that as \( m \rightarrow \infty \)

\[
|((\lambda_m - \hat{\lambda}) \nabla P_m, \nabla \Phi)| \leq \| \lambda_m - \hat{\lambda} \|_{L^\infty (\Omega)} \| P_m \|_X \| \Phi \|_X \rightarrow 0, \quad |(\hat{\lambda}(\nabla P_m - \nabla \hat{P}), \nabla \Phi)| \rightarrow 0 \quad \forall \Phi \in X,
\]

\[
|((\mu_m - \hat{\mu}) P_m, \Phi)| \leq \| \mu_m - \hat{\mu} \|_{L^\infty (\Omega)} \| P_m \|_X \| \Phi \|_X \rightarrow 0, \quad |(\hat{\mu}(P_m - \hat{P}, \Phi)| \rightarrow 0 \quad \forall \Phi \in X. \tag{22}
\]

Thus, passing to the limit in \( (18) \) as \( m \rightarrow \infty \) and using (22), we obtain (20). This means that \( G(\hat{P}, \hat{u}, p^{inc}) = 0 \) where \( \hat{u} = (\hat{\lambda}, \hat{\mu}) \). Since \( J \) is weakly lower semicontinuous on \( X \times K \), we obtain that \( J(\hat{P}, \hat{u}) = J^* \) which proves the theorem. \( \blacksquare \)
It should be noted that the assertion of Theorem 2 is valid for functional $I_0$ defined by (4) since it is weakly lower semicontinuous and nonnegative.

The next step in the study of control problem (17) is to establish sufficient conditions on the input data in problem (17) under which its solution is unique and stable. For this purpose, make use the approach developed in [17, 18] for the 2D impedance cloaking problems is used.

It is based on the derivation and analysis of the optimality system describing the first-order necessary conditions of an extremum in the problem (17). However, since problem (17) is stated for complex-valued functions, it has to be preliminarily decomplexified. As a result, it is reduced to equivalent control problem considered in the class of real valued functions. Then based on [21], the optimality system can be derived for the latter “real” control problem. Finally, this “real” optimality system is transformed to “complex” optimality system corresponding to the initial problem (17). Using this scheme and arguing as in [17, 18] the following theorem can be proved.

**Theorem 3.** Let the pair $(\hat{P}, \hat{u}) \equiv (\hat{P}, \hat{\lambda}, \hat{\mu}) \in X \times K$ be the solution of problem (17) under conditions (i), (j) and let the functional $I(P)$ be continuously differentiable with respect to state in the point $\hat{P}$. Then there is a unique Lagrange multiplier $R \in X$ which is the solution of the Euler-Lagrange equation

$$a_0(\Psi, R) + a(\hat{\lambda}, \hat{\mu}; \Psi, R) = -(\alpha_0/2)\langle I'_P(\hat{P}), \Psi \rangle \forall \Psi \in X$$

(23)

and the following variational inequalities are fulfilled:

$$\alpha_1(\hat{\lambda} - \lambda), \Omega + \Re((\lambda - \hat{\lambda}) \nabla \hat{\nabla}, \nabla R) \geq 0 \quad \forall \lambda \in K_1,$$

(24)

$$\alpha_2(\hat{\mu} - \mu), \Omega - \Re k^2((\mu - \hat{\mu}) \hat{P}, R) \geq 0 \quad \forall \mu \in K_2.$$  

(25)

Simple analysis using the definition (15) of operator $A^*_u$ adjoint of $A_u$ shows that the identity (23) is equivalent to the operator equation

$$A^*_u R = (-\alpha_0/2)I'_P(\hat{P}).$$

(26)

Taking into account (26), we shall refer below to (23) as an adjoint problem. We also note that in the particular case when $I(P) = I_0(P)$ (23) takes the form

$$a_0(\Psi, R) + a(\hat{\lambda}, \hat{\mu}; \Psi, R) = -a_0(\Psi, \hat{P} - p^d)_Q \forall \Psi \in X.$$  

Adjoint problem (23) together with variational inequalities (24), (25) and direct problem (6) form the optimality system for problem (17). The optimality system plays an important role in studying the properties of solutions of the control problem. On its basis, efficient numerical algorithms of solving problem (17) can be developed (please refer to [20] about some of them for the case of layered cloaking shell). In addition, on the base of analysis of the optimality system one can establish the sufficient conditions on the initial data providing the uniqueness and stability of solutions of particular extremal problems.

We restrict ourselves by the formulation of respective result for the control problem

$$J(P, \lambda, \mu) = \frac{\alpha_0}{2} \|P - p^d\|^2_Q + \frac{\alpha_1}{2} \|\lambda\|^2_{\Omega} + \frac{\alpha_2}{2} \|\mu\|^2_{\Omega} \to \inf,$$

$$G(P, \lambda, \mu, p^{inc}) = 0, \quad (P, \lambda, \mu) \in X \times K_1 \times K_2$$

(27)

corresponding to the functional $I_0(P)$ in (4). Denote by $(P_1, \lambda_1, \mu_1)$ an arbitrary solution of problem (27) corresponding to given functions $p^d = p^d_1$ and $p^{inc} = p^{inc}_1$. By $(P_2, \lambda_2, \mu_2)$,
solution of the same problem corresponding to the perturbed functions \( \tilde{p}^d = p_2^d \) and \( \tilde{p}^{inc} = p_2^{inc} \) is denoted. We put
\[
M_P = C_2 \sup_{p^{inc} \in K^{inc}} \|p^{inc}\|_{1,\Omega_e}, \quad M_P^0 = M_P + \max(\|p_1^d\|_Q, \|p_2^d\|_Q), \quad a = 2C_2M_P^0, \quad b = 2C_2M_P^0M_{P^{-1}},
\]
and assume that the following conditions are satisfied:
\[
\alpha_1(1 - \varepsilon) > 5\alpha_0VC_2^2C_s^2M_PM_P^0, \quad \alpha_2(1 - \varepsilon) > 5\alpha_0C_2^2C_r^2M_PM_P^0,
\]
where \( \varepsilon \in (0, 1) \) is an arbitrary number. The following theorem can be proved.

**Theorem 4.** Let, in addition to conditions (i), (j), \( K_1, K_2 \) and \( K^{inc} \subset T^{inc} \) be bounded sets and let the triple \( (P_1, \lambda_1, \mu_1) \) be the solution of problem (27) corresponding to given functions \( p_1^d \in L^2(Q) \) and \( p_1^{inc} \in K^{inc}, \quad l = 1, 2 \). Assume that conditions (28) are satisfied. Then the following estimates hold true:
\[
\|P_1 - P_2\|_Q \leq \|p_1^d - p_2^d\|_Q + \varphi(\|p_1^{inc} - p_2^{inc}\|_{1,\Omega_e}),
\]
\[
\|\lambda_1 - \lambda_2\|_{s,\Omega} \leq \sqrt{\alpha_0/\varepsilon_0} \Delta, \quad \|\mu_1 - \mu_2\|_{r,\Omega} \leq \sqrt{\alpha_0/\varepsilon_2} \Delta,
\]
\[
\|P_1 - P_2\|_X \leq C_2[MP(C_s\sqrt{\alpha_0/\varepsilon_0}) + k^2C_r\sqrt{\alpha_0/\varepsilon_2} \Delta] + C_2\|p_1^{inc} - p_2^{inc}\|_{1,\Omega_e}.
\]

Here \( \Delta \) is defined by
\[
\Delta = (1/2)\|p_1^d - p_2^d\|_Q + \varphi(\|p_1^{inc} - p_2^{inc}\|_{1,\Omega_e}).
\]

We note that the estimates (30), (31) have the meaning of stability estimates of optimal solutions for control problem (27). These estimates are valid only if the parameters \( \alpha_1 \) and \( \alpha_2 \) entering into (27) are positive and satisfy conditions (28). This means that terms \( (\alpha_1/2)\|\lambda_1\|_{s,\Omega}^2 \) and \( (\alpha_2/2)\|\lambda_1\|_{s,\Omega}^2 \) in the expression for the minimized functional in (27) have a regularizing effect on control problem under study.

4. Conclusion
In this paper, the control problems for the 3D model of acoustic scattering by permeable inhomogeneous isotropic obstacle having the form of the spherical layer are studied. These problems arise while developing the design technologies of acoustic cloaking devices using the optimization method. The solvability of the general control problem has been proved, the optimality system which describes the necessary conditions of extremum derived and sufficient conditions to the data to provide uniqueness and stability of optimal solutions established. The separate paper of the authors will be devoted to develop numerical algorithms of solving cloaking problems on the base of the optimality system and to analyse results of numerical experiments.

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