Rectangles, integer vectors and hyperplanes of the hypercube

E. Gioan , I. P. Silva

Abstract. We introduce a family of nonnegative integer vectors - primitive vectors - defining hyperplanes of the real affine cube over $\mathbb{C}^n := \{-1, 1\}^n$ and study their properties with respect to the rectangles of the cube. As a consequence we give a short proof that, for small dimensions ($n \leq 7$), the real affine cube can be recovered from its signed rectangles and its signed cocircuits complementary of its facets and skew-facets.

1 Introduction

We consider $\mathbb{C}^n := \{-1, 1\}^n$. $\text{Aff}(\mathbb{C}^n)$ denotes the oriented matroid of the affine dependencies of $\mathbb{C}^n$ over $\mathbb{R}$, the real affine cube.

The oriented matroid $\text{Aff}(\mathbb{C}^n)$ can be defined in several equivalent ways. We recall its definition in terms of the family of signed cocircuits. A signed cocircuit of $\text{Aff}(\mathbb{C}^n)$ is an ordered pair $X = (X^+, X^-)$ satisfying the following two conditions:

1) $H := \mathbb{C}^n \setminus (X^+ \cup X^-)$ is the set of all elements of $\mathbb{C}^n$ spanning the same affine hyperplane $H$ of $\mathbb{R}^n$.

2) If the hyperplane $H := \mathbb{C}^n \setminus (X^+ \cup X^-)$ is defined by a linear equation $H : x \cdot h = b$ ($h, b \in \mathbb{Z}^{n+1}$) then either $X^+ := \{v \in \mathbb{C}^n : v \cdot h > b\}$ and $X^- := \{v \in \mathbb{C}^n : v \cdot h < b\}$ or $X^+ := \{v \in \mathbb{C}^n : v \cdot h < b\}$ and $X^- := \{v \in \mathbb{C}^n : v \cdot h > b\}$.

To every hyperplane $H : x \cdot h = b$ of the matroid $\text{Aff}(\mathbb{C}^n)$ is associated the pair of opposite signed cocircuits, $X = (X^+, X^-)$ and $\neg X = (X^-, X^+)$. This pair encodes the 2-partition of $\mathbb{C}^n \setminus H = X^+ \uplus X^-$ into the points lying on each one of the open half-spaces of $\mathbb{R}^n$ defined by the affine hyperplane spanned by $H$.

The oriented matroid $\text{Aff}(\mathbb{C}^n)$ is defined by the collection of all its signed cocircuits.

We suggest that the reader consults [2] as general reference on oriented matroids and [5] as general reference on matroids.

The (signed) rectangles of $\mathbb{C}^n$ are the shortest (signed) circuits of
$\text{Aff}(C^n)$. They are the signed sets of the form $\pm R$ with $R = (\{u, v\}, \{u', v'\}) = u^i v^i + u'^i v'^i$, where $(u, u', v, v')$ are the vertices of a rectangle of $\mathbb{R}^n$ with diagonals $uv, u'v'$. We denote by $\mathcal{R}$ the family of the signed rectangles the real affine cube.

The Facets of $C^n$ are the subsets: $H_{i^+} := \{v \in C^n : v_i = 1\}$ and $H_{i^-} := \{v \in C^n : v_i = -1\}$, $i = 1, \ldots, n$. The skew-facets of $C^n$ are the subsets: $H_{ij^+} := \{v \in C^n : v_i = v_j\}$ and $H_{ij^-} := \{v \in C^n : v_i = -v_j\}$, $1 \leq i < j \leq n$. They are hyperplanes of the real affine cube as well as its shortest cocircuits.

The signed cocircuits of $\text{Aff}(C^n)$ complementary of the facets and of the skew facets are denoted respectively: $X_{i^+} = (H_{i^-}, \emptyset)$, $X_{i^-} = (H_{i^+}, \emptyset)$ and $X_{ij^+} = (H_{i^+} \cap H_{j^-}, H_{i^-} \cap H_{j^+})$, $X_{ij^-} = (H_{i^+} \cap H_{j^-}, H_{i^-} \cap H_{j^-})$.

We denote $\mathcal{F}$ the family of the signed cocircuits complementary of the facets of the skew facets of the real cube $\text{Aff}(C^n)$.

**Definition 1.1** An oriented cube (canonically oriented cube of [7]) is an oriented matroid over $C^n$ containing as signed circuits the signed rectangles of $\mathcal{R}$ and, as signed cocircuits, the signed cocircuits of $\mathcal{F}$.

In this note we give a short proof of the following theorem (Theorem 3.1) whose proof is mentioned in [7].

**Theorem** For $n \leq 7$ the real affine cube $\text{Aff}(C^n)$ is the unique oriented cube.

This is a small step towards an answer to the next Question 1. A positive answer to this question with the results of [7] would imply the existence of a purely combinatorial characterization of affine/linear dependencies of $\pm 1$ vectors over the reals:

**Question.** [7] Is the real cube the unique oriented cube?

Our proof of the theorem uses the encoding of $\text{Aff}(C^n)$ in terms of the family $\mathcal{H}_n$ of non-negative integer vectors of $\mathbb{N}^n_{0+1}$ that defines its hyperplanes and signed cocircuits up to symmetries of the non-oriented matroid of the real affine cube (see [6]). This is briefly recalled in section 3, where the complete proof is presented.

In the next section 2 we define primitive vectors of $C^n$ and from there a recursive family of hyperplanes - primitive hyperplanes of $\text{Aff}(C^n)$ whose behaviour with respect to the net of signed rectangles implies that the corresponding signed cocircuits of the real cube must be signed cocircuits of every oriented cube.
We point out that as a direct consequence of the main theorem we obtain (for \( n \leq 7 \)) a new proof of the following conjecture of M. Las Vergnas also open for \( n > 7 \):

**Las Vergnas cube conjecture (4):** \( Aff(C^n) \) is the unique orientation of the (non-oriented) real affine cube.

This conjecture was verified computationally for \( n \leq 7 \), by J. Bokowski et al [3].

## 2 Nonnegative integer vectors, rectangles and hyperplanes of oriented cubes

### 1.1. Parallel Strata and rectangles

We are interested in nonnegative integer vectors \( h \in \mathbb{N}^n_0 \) and how they stratify the vertices of the cube \( C^n \in \mathbb{R}^n \) into parallel levels (strata) orthogonal to \( h \).

**Definition 2.1 (h-levels of \( C^n \))**

Given \( h \in \mathbb{N}^n_0 \) we consider \( |h| := \sum_{i=1}^{n} h_i \).

For every \( a \in \{0, \ldots, |h|\} \) the \( a \)-level of \( h \), denoted \( S_a(h) \) or simply \( S_a \), is the set of vertices of the cube \( C^n \) defined as:

\[
S_a = S_a(h) := \{ v \in C^n : h \cdot v = |h| - 2a \}
\]

Clearly \( S_{|h|−a} = −S_a \), in particular \( S_0 = \{1\} \) and \( S_{|h|} = \{-1\} \).

Eventually \( S_a = \emptyset \).

**Definition 2.2 (Realizable h - rectangles)**

An \( h \)-rectangle is a sequence of four natural numbers \( r = (a \leq b \leq c \leq d) \) such that \( d = b + c − a \) and \( 0 \leq a \) and \( d \leq |h| \). We distinguish between 1, 2, 3 and 4- rectangles according to the number of different \( h \)-levels they cross.

A 1-rectangle is of the form \( r_1 = (a \leq b \leq c \leq d) \), a 2-rectangle is of the form \( r_2 = (a \leq b < b = b) \), 3-rectangle is of the form \( r_3 = (a < b = b < c) \) and a 4-rectangle is of the form \( r_4 = (a < b < c < d) \).

An \( h \)-rectangle \( r = (a \leq b \leq c \leq d) \) is realizable if there is a signed geometric rectangle, \( R = v_a^+ v_b^- v_c^- v_d^+ \), of \( Aff(C^n) \) such that \( v_a \in S_a(h) \), \( v_b \in S_b(h) \), \( v_c \in S_c(h) \) and \( v_d \in S_d(h) \).
Definition 2.3 (Embedding of levels) Given two $h$-levels, $S_a, S_b$ we say that $S_a$ is embedded in $S_b$, written $S_a \hookrightarrow S_b$, if there is a spanning tree $\Gamma_a$ of the complete graph $K(S_a)$ such that for every edge $\{u, u'\}$ of $\Gamma_a \subseteq S_a$ there is a pair of vertices $v, v' \in S_b$ such that $R = u + u' - v - v' +$ is a signed rectangle of the cube.

Remark. Notice that if $S_a$ is embedded in $S_b$, $S_a \hookrightarrow S_b$, then the affine span of $S_a$ must be parallel to the affine span of $S_b$. The notion of embedding of levels is from this point view a combinatorial version of parallelism in $\mathbb{R}^n$.

Notation. We usually identify an element $v \in C^n$ with the subset $\alpha \subseteq [n]$ of its negative entries, more precisely with the sequence of the elements of $\alpha$ written by increasing order. Example: $(1, 1, 1, 1) \in C^4 \equiv \emptyset$, $(1, -1, -1, 1) \in C^4 \equiv 23$.

Let $h \in \mathbb{N}_0^{n-1}$ and $g = (h, g) \in \mathbb{N}_0^n$. The $g$-levels and $h$-levels are related in the following way, for every $a \in \mathbb{N}_0$, $0 \leq a \leq |g|$:

$$S_a(g) = S_a(h)n^+ \cup S_{a-g}(h)n^-$$

where $S_a(h)n^+ = \{(v, 1) \in C^n : v \in S_a(h)\}$ and $S_{a-g}(h)n^- = \{(v, -1) \in C^n : v \in S_{a-g}(h)\}$.

Examples
1) The vector $h = (1, 1, 1, 2)$ has 6 levels: $S_0 = \emptyset$, $S_1 = \{1, 2, 3\}$, $S_2 = \{12, 13, 23, 4\}$, $S_3 = \{14, 24, 34, 123\}$, $S_4 = \{124, 134, 234\}$, and $S_5 = \{1234\}$. Observe that for every $a \neq 2$ one has $S_a \hookrightarrow S_2$, as represented in Figure 2. Note also that $S_2 \hookrightarrow S_3$ but $S_2 \not\hookrightarrow S_4$.

2) The vector $h = (1, 2, 2, 3)$ has 9 levels. No level $S_b = S_b(h)$ satisfies the property $\forall a \neq 2$, $S_a \hookrightarrow S_b$. 

Figure 1:
Figure 2: The (1,1,1,2)-levels and embeddings of the levels $S_a$, $a \neq 2$ in $S_2$.

**Definition 2.4** (primitive vectors)

A vector $h \in \mathbb{N}_0^n$ is called a primitive vector if all its 3 and 4- numerical rectangles are realizable.

**Examples.** The vectors $(1,1,1)$, $(1,1,2)$, $(0,1,1)$, are primitive vectors of $C^3$.

The next Proposition gives, in particular, a recursive construction of primitive vectors of $\mathbb{R}^n$.

**Proposition 2.1** (Properties of primitive vectors)

Let $h = (h_1, \ldots, h_n)$ be a primitive vector of $C^n$. Then:

(i) For every $0 \leq a \leq |h|$, the $a$-level $S_a$ of $h$ is nonempty, moreover for $a \neq 0, |h|$, every level $S_a$ contains at least two elements.

(ii) For every $g \in \mathbb{N}$, $0 \leq g \leq \frac{|h|}{4} + 1$, $g = (h, g)$ is a primitive vector of $C^{n+1}$.

**Proof.** (i) Immediate from the definition, since every 3-rectangle must be realizable.

(ii) We have $S_a(g) = S_a(h)(n+1)^+ \cup S_{a-g}(h)(n+1)^-$.

Let $r = a < b \leq c < d$ be a 3 or 4- rectangle of $g = (h, g)$. If $d < |h|$ or $a > g$ the rectangle $r$ is certainly $h$- realizable. In the first case with elements of $S_a(h)(n+1)^+$, in the second case with elements of $S_{a-g}(h)(n+1)^-$. 

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If \( a < g \) and \( d > |h| \), then, since \( a + d = b + c > |h| \) we can guarantee that \( c \geq g \). Consider the rectangle \( r' = (a < b; c - g < d - g) \). If \( r' \) is a 2-rectangle then \( a = c - g \) and \( b = d - g \). Take \( v_a \in S_a(h(n + 1)^+ \text{ and } v_b \in S_b(h)(n + 1)^+ \). Then \( v_c = (v_a, -1) \in S_c(g) \) and \( v_d = (v_b, -1) \in S_d(g) \) and \( R' = ((v_a, 1), (v_b, 1); (v_c, 1), (v_d, 1)) \). Similarly, if \( r' \) is a 3 or 4 \( h \)-rectangle then it is realizable in \( S_a(h)(n + 1)^+ \). Let \( R' = ((v_a, 1), (v_b, 1); (v_{c-g}, 1), (v_{d-g}, 1)) \) be a realization of \( r' \) then \( R = ((v_a, 1), (v_b, 1); (v_{c-g}, -1), (v_{d-g}, -1)) \) must be a realization of \( r \) in \( C^n \).

**Theorem 2.1** (primitive vectors and oriented cubes)

Let \( \mathcal{M} = \mathcal{M}(C^n) \) be an oriented cube and \( cl : 2^{C^n} \to 2^{C^n} \) its closure operator.

Assume that there is a primitive vector \( h \) of \( C^n \) such that some \( h \)-level \( S_b \) has the following property:

\[
(E) \quad \forall a \neq b \quad S_a \hookrightarrow S_b
\]

Then either

(i) \( cl(S_b) = C^n \), or

(ii) \( cl(S_b) = S_b \) is a hyperplane of \( \mathcal{M} \) whose complement is the support of the pair \( \pm X_b \) of signed cocircuits of \( \mathcal{M} \) defined by \( X_b = (\cup_{a < b} S_a, \cup_{a > b} S_a) \).

**Proof.** First notice that, by definition of embedding, the hypothesis that \( S_b \) satisfies condition \((E)\) implies that if one element \( v \in S_a \), \( a \neq b \) belongs to \( cl(S_b) \) then \( S_a \subseteq cl(S_b) \). On the other hand if some element \( v \in S_a \), \( a \neq b \) is not \( cl(S_b) \) then, by orthogonality with the signed rectangles of the embedding, \( R = (u_i^+, u_i^-, v_i^+, v_i^-) \), all the elements of \( S_a \) must have the same sign in any signed cocircuit \( V = (V^+, V^-) \), complementary of \( cl(S_b) \), i.e. one must have either \( S_a \subseteq V^+ \) or \( S_a \subseteq V^- \). With these remarks in mind we now prove the theorem.

**Claim 1.** if there is \( v \in cl(S_b) \) such that \( v \in S_{b-1} \cup S_{b+1} \) then \( cl(S_b) = C^n \).

We may assume \( n > 1 \) and therefore \( b \neq 0, |h| \).

Assume that there is \( v \in S_{b-1} \cap cl(S_b) \). In this case, by the above remarks \( S_{b-1} \subseteq cl(S_b) \). And because the 3-rectangle \( r = (b - 1 < b < b + 1) \) is realizable, orthogonality of the signed covectors complementary of \( cl(S_b) \) with any geometric realization \( R = u_{b-1}^+ u_b^- v_b^- v_{b+1}^+ \) of \( r \), implies that \( v_{b+1} \) must be in \( cl(S_b) \) and therefore \( S_{b+1} \subseteq cl(S_b) \).

Similarly if \( v \in S_{b+1} \cap cl(S_b) \) orthogonality with the same geometric realization \( R \) of the 3-rectangle \( r = (b - 1 < b = b + 1) \) implies that \( S_{b+1} \cup S_{b-1} \subseteq cl(S_b) \). So, if some \( v \in S_{b-1} \cup S_{b+1} \) in the closure of \( S_b \) then both levels \( S_{b-1} \) and \( S_{b+1} \) are contained in \( cl(S_b) \).
Now, once $S_{b+1} \cup S_{b-1} \subseteq \text{cl}(S_b)$, considering the sequence of geometric realizations of the 3-rectangles $r_b = (b < b+1 = b+1 < b+2)$, $r_{b+1} = (b+1 < b+2 = b+2 < b+3)$, \ldots, $r_{n-2} = (n-2 < n-1 = n-1 < n)$ we conclude, successively, that $S_{b+2} \subseteq \text{cl}(S_b) \ldots S_n \subseteq \text{cl}(S_b)$. Similarly using a sequence of 3-rectangles $r_{b-2} = (b-2 < b-1 = b-1 < b)$, \ldots, $r_2 = (2 < 1 = 1 < 0)$ we also deduce that $S_{b-2}, \ldots, S_0 \subseteq \text{cl}(S_b)$ and therefore that $\text{cl}(S_b) = C^n$.

**Claim 2.** If $(S_{b-1} \cup S_{b+1}) \cap \text{cl}(S_b) = \emptyset$ then $\text{cl}(S_b) = S_b$.

In this case we use the fact that $M$ is a cube canonically oriented and prove that the only way of signing the complement $C^n \setminus S_b$, of $S_b$, orthogonally to the rectangles of $M$ is $\pm X_b$ where $X_b = (\cup_{a<b} S_a, \cup_{a>b} S_a)$. This proves simultaneously that $\text{cl}(S_b) = S_b$ and that $X_b$ must be a cocircuit of $M$.

If $(S_{b-1} \cup S_{b+1}) \cap \text{cl}(S_b) = \emptyset$ then $\text{cl}(S_b)$ is a flat of $M$ whose complement is the support of (signed) covectors of $M$. Using a realization $R = (u^+_b, u_b^-, v_b^-, v_{b+1}^+)$ of the 3-rectangle $r = (b-1 < b = b < b+1)$ we conclude that $S_{b-1}$ and $S_{b+1}$ must have different signs in any covector $V = (V^+, V^-)$ complementary of $\text{cl}(S_b)$. We may assume w.l.o.g. that $S_{b-1} \subseteq V^+$ and that $S_{b+1} \subseteq V^-$.

Consider the sequence of 3-rectangles with the "upper" vertex in $S_b$ or $S_{b-1}$ defined by: $r_{b+2} = (b < b+1 = b+1 < b+2), s_{b+3} = (b-1 < b+1 = b+1 < b+3), r_{b+4} = (b < b+2 = b+2 < b+4), s_{b+5} = (b-1 < b+2 = b+2 < b+5)$, \ldots. Orthogonality with geometric realizations of these rectangles implies successively that $S_{b+2}, S_{b+3}, \ldots, S_{|b|} \subseteq V^-$.

Similarly orthogonality with realizations of the sequence of 3-rectangles with "lower" vertex in $S_b$ or $S_{b+1}$ defined by: $r_{b-2} = (b-2 < b-1 = b-1 < b), s_{b-3} = (b-3 < b-1 = b-1 < b+1), r_{b-4} = (b-4 < b-2 = b-2 < b), s_{b-5} = (b-5 < b-2 = b-2 < b+1), \ldots$. leads us to conclude successively that $S_{b-2}, S_{b-3}, \ldots, S_0 \subseteq V^+$. That means there is a unique signature of the complement of $S_b$ orthogonal to the signed rectangles of $M$ that signature is defined by $X_b$. The unicity of the signature and the fact that its support is the complement of $S_b$ imply that $X_b$ is a signed cocircuit of $M$ and also that $\text{cl}(S_b) = S_b$ is a hyperplane of $M$. □

The next Proposition gives a recursive way of constructing hyperplanes and the corresponding signed cocircuits of any oriented cube.
Theorem 2.2 (recursive definition of primitive hyperplanes and cocircuits of oriented cubes)
Let \( h \) be a primitive vector of \( C^{n-1} \) for which there exists \( b \in \mathbb{N}_0 \) satisfying the following two conditions:

\[ (h-i) \quad \forall a \in \mathbb{N}_0, \ 0 < a < |h|, \ a \neq b \quad S_a(h) \hookrightarrow S_b(h) \]
\[ (h-ii) \quad \text{The signed set } X_b := (\cup_{a < b} S_a(h), \cup_{a > b} S_a(h)) \text{ is a signed cocircuit of every oriented cube } \mathcal{M}(C^{n-1}) \]

Then, every vector \( g = (h, g) \in \mathbb{N}_0^n \), with \( 0 \leq g \leq b \), satisfies the following two properties:

\[ (g-i) \quad \forall a \in \mathbb{N}_0, \ 0 < a < |g|, \ a \neq b \quad S_a(g) \hookrightarrow S_b(g) \]
\[ (g-ii) \quad \text{The signed set } \tilde{X}_b := (\cup_{a < b} S_a(g), \cup_{a > b} S_a(g)) \text{ is a signed cocircuit of every oriented cube } \mathcal{M}(C^n) \]

Proof. Let \( \tilde{M} = \mathcal{M}(C^n) \) be an oriented cube. Consider vectors \( h, g \in \mathbb{N}_0^{n-1} \) and \( g = (h, g) \in \mathbb{N}_0^n \) in the conditions of the Theorem. Consider \( \tilde{H} := \text{cl}(S_b(g)) \).
We prove the theorem in three steps.

(I) \( \tilde{H} \) is a hyperplane of \( \tilde{M} \), whose pair of signed cocircuits restricts in the facet \( H_{n^+} \) of \( \tilde{M} \) to the pair of signed cocircuit \( \pm X_b \) of \( S_b(h) \).

(II) \( g \) satisfies \((g-i)\).

(III) The unique extension of the cocircuit \( X_b \) of the facet \( H_{n^+} \) to a signed covector of \( \tilde{M} \) complementary of \( S_b(g) \) and orthogonal to the rectangles of \( C^n \) is the signed set \( \tilde{X}_b \) defined \((g-ii)\).

(I) Since \( h \) satisfies \((h-i)\) we know that \( S_{b-g}(h) \hookrightarrow S_b(h) \) implying also that \( S_{b-g}(h) n^- \hookrightarrow S_b(h) n^+ \). Consequently the ranks of \( S_b(g) \) and \( S_b(h) n^+ \) are related by:

\[ rk(S_b(g)) = rk(S_b(h) n^+) + 1. \]

The restriction of the canonically oriented cube \( \tilde{M} \) to a facet is an oriented cube of rank \( rk(\tilde{M}) - 1 \). The set \( H := S_b(h) n^+ \) is, by condition \((h-i)\) a hyperplane of the facet \( H_{n^+} \) of \( \tilde{M} \) and therefore a hyperline of \( \tilde{M} \) implying that \( rk(H) = rk(\tilde{M}) - 2 \). Replacing in \((1)\) we have:

\[ rk(S_b(g)) = rk(H) + 1 = rk(\tilde{M}) - 1 \]

Therefore \( \tilde{H} := \text{cl}(\tilde{M}(S_b(g))) \) must be a hyperplane of \( \tilde{M} \) and its complement the support of a pair of signed cocircuits of \( \tilde{M} \). Moreover, one of these signed cocircuits of \( \tilde{M} \) must restrict in the facet \( H_{n^+} \) to the signed cocircuit \( X_b := (\cup_{a < b} S_a(h) n^+, \cup_{a > b} S_a(h) n^+) \) completing the proof of Step (I).
We consider separately two cases: $h-i$

For $a$, $g \leq a \leq |h|$, $S_a(g) = S_a(h)n^+ \cup S_{a-g}(h)n^-$ intersects both facets $H_{n^+}$ and $H_{n^-}$. The hypothesis that $h$ satisfies $(h-i)$ guarantees that $S_a(g)n^+ \hookrightarrow S_b(h)n^+$ and also that $S_{a-g}(h)n^- \hookrightarrow S_b(h)n^+$ in order to conclude that $S_a(g) \hookrightarrow S_b(h)$ it is enough to prove that there is a geometric realization of the 2-rectangle $r_2 = (a = a; b = b)$ of the form $R = ((u_a, 1)^+, (u_{a-g}, -1)^-, (v_{b-g}, -1)^-, (v_b, 1)^+)$ with $(u_a, 1) \in S_a(h)n^+$, $(u_{a-g}, -1) \in S_{a-g}(h)n^-$, $(v_{b-g}, -1) \in S_b(h)n^-$ and $(v_b, 1) \in S_b(h)n^+$.

Consider the 3 or 4 rectangle of $r' = (a - g < a; b - g < b)$ of $h$. The vector $h$ is a primitive vector of $C^{n-1}$ so there is a geometric realization $R' = (u_{a-g}, u_a^-, v_{b-g}, v_b^+)$ of this rectangle in $C^{n-1}$ leading the desired realization of the 2-rectangle $r_2$ in $C^n$ and concluding the proof that $g$ satisfies $(g-i)$.

(III) $g$ satisfies $(g-ii)$

We know from (I) that there is a signed cocircuit $\tilde{X}$ of $\tilde{\mathcal{M}}$ complementary of the hyperplane $\tilde{H}$ whose restriction to the facet $H_{n^+}$ is $X_b$. The fact that $g$ satisfies $(g-ii)$, proved in (II), implies directly that the unique extension of $X_b$ to $H_{n^-} \setminus S_{b-g}(h)n^-$ orthogonal to the rectangles of $\tilde{\mathcal{M}}$ must satisfy the conditions: $\cup_{a < b} S_a(g) \subseteq \tilde{X}^+$ and $\cup_{b < a \leq |h|} S_a(g) \subseteq \tilde{X}^-$. Concerning the levels $S_a(g)$ with $a > |h|$ the fact that they are embedded in $S_b(g)$ implies that all the elements in each level will have the same sign however some more arguing is needed before concluding that they must all be negative in $\tilde{X}$.

We consider separately two cases: Case 1) $a \leq 2b$ and case 2) $a > 2b$.

Case 1) In this case consider the 3-rectangle of $g$: $r = (2b - a < b = b < a)$. If the rectangle $r' = (b - g < a - g; 2b - a < b)$ of $h$ is a 3 or 4 rectangle of $h$ then, since $h$ is primitive, $r'$ is realizable. Consider a realization $R' = (u_{b-g}, v_{a-g}, v'_{2b-a}, u'_b)$ of $r'$ in $C^{n-1}$. Then clearly $R = (v'_{2b-a}, 1)^+ (u'_b, 1)^- (u_{a-g}, -1)^- (v_{a-g}, -1)^+$ is a geometric realization of $r$ in $C^n$ and by orthogonality with this circuit in any extension of $X_b$ to $H_{n^-} \setminus S_{b-g}(h)n^-$ the sign of $(v_{a-g}, -1)$ must be negative, implying that $S_a(g) \subseteq \tilde{X}^-$. In the case $r'$ is a 2-rectangle of $h$, which occurs when $g = a - b$, take $u_{b-g} \in S_{b-g}(h)$ and $v_b \in S_b(h)$. Clearly $R = (u_{a-g}, 1)^+ (v_b, 1)^- (u_{b-g}, -1)^- (v_b, -1)^+$ is a geometric realization of $r$ in $C^n$ and we also conclude that $S_a(g) \subseteq \tilde{X}^-$. 

(II) $g$ satisfies $(g-i)$.

Notice that for every $a$, $0 \leq a < g$, $S_a(g) = S_a(h)n^+$ and since $h$ satisfies $(h-i)$ it is clear that in this case we have: $S_a(g)n^+ \hookrightarrow S_b(h)n^+$ and therefore $S_a(g)n^+ \hookrightarrow S_b(g)$. For $a > |h|$, since $S_a(g) = S_{a-g}(h)n^- \hookrightarrow S_b(h)n^+$ and the result in this case is also a direct consequence of the fact that $h$ satisfies $(h-i)$.
In this case \( r = (|h| - (a - b) < b < |h| < a) \) is a 4-rectangle of \( g \) (notice that \( b < |h| < a \)) and \( r' = (b - g < a - g; |h| - (a - b) < |h|) \) is a rectangle of \( h \). If \( r' \) is a 3 or 4-rectangle of \( h \) then, since \( h \) is primitive, there is a realization \((u_{b-g}, v_{a-g}; v'_{|h| - (a - b)}, u'_{|h|})\) of \( r' \) in \( C^{n-1} \) and \( R = v'_{|h| - (a - b), 1}^+ (u_{b-g}, -1)^- (u'_{|h|}, 1)^- (v_{a-g} - 1)^+ \) is geometric realization of \( r \). By orthogonality with this circuit we conclude as before that \( S_b(g) \subseteq \tilde{X}^- \). The case \( r' \) is a 2-rectangle the argument is similar as in the previous case, leading to the conclusion that the unique extension of \( X_b \) to \( H_n \setminus S_{b-g}(h)n^- \) orthogonal to the rectangles of \( \tilde{M} \) is the signed vector \( \tilde{X}_b \) of the theorem, thus proving that \( g \) satisfies \((g - ii) \). □

The next Corollary whose proof is left to the reader restates Theorem 3.2 as a recursive procedure for constructing from signed cocircuits of every canonically oriented cube over \( C^n \) signed cocircuits of every canonically oriented cube over \( C^{n+1} \).

**Corollary 2.1** Let \( h \in \mathbb{N}^n \) be a primitive vector such that the signed cocircuit of \( \text{Aff}(C^n) \) defined by \( S_b(h) : x.h = |h| - 2b \) is a signed cocircuit of every oriented cube \( M(C^n) \). For every \( c \leq b \) let \( g := (h, b - c) \), then the signed cocircuit of \( \text{Aff}(C^{n+1}) \) defined by \( S_b(g) : g.x = |g| - 2b \) is a signed cocircuit of every oriented cube \( M(C^{n+1}) \).

We are now ready to prove the main theorem.

### 3 The main Theorem

**Theorem 3.1** For \( n \leq 7 \) the oriented matroid \( \text{Aff}(C^n) \) is the unique oriented cube.

We recall from [7] that every oriented cube over \( C^n \) must have rank \( n+1 \). The next Proposition about matroids then guarantees that in order to prove theorem 3.1 we “only” have to prove that every signed cocircuit of \( \text{Aff}(C^n) \) must be a signed cocircuit of every oriented cube \( M(C^n) \).

**Proposition 3.1** Consider two matroids \( M, M' \), without loops, over the same set \( E \) and with the same rank \( r \). Let \( \mathcal{H}, \mathcal{H}' \) denote the families of hyperplanes respectively of \( M \) and \( M' \) and assume that \( \mathcal{H} \subseteq \mathcal{H}' \). Then \( \mathcal{H} = \mathcal{H}' \) and \( M = M' \).
Proof. Suppose that there is a hyperplane $H' \in \mathcal{H}' \setminus \mathcal{H}$. Let $B = \{h_1, \ldots, h_{r-1}\}$ be a basis of $H'$ in $M'$. Since $|B| < r$, $B$ is contained in some hyperplane $H \in \mathcal{H}$ of $M$. The assumption that $\mathcal{H} \subset \mathcal{H}'$ implies that $H \in \mathcal{H}'$ and in $\mathcal{M}'$ we have $B \subset H \cap H'$. Now $H \cap H'$ is a flat of $M'$ whose rank is less than the rank of $H'$ contradicting the assumption that $B$ is a basis of $H'$.

The explicit definition of $\text{Aff}(C^n)$ for $n \leq 7$ that we use is in terms of the family of signed cocircuits, defined by the following family $\mathcal{H}_n$ of non-negative integer vectors, up to automorphisms of the class of orientations (i.e. of the unsigned arrangement of hyperplanes representing the oriented matroid)

$$\mathcal{H}_n := \{(h_1 \leq h_2 \leq \ldots \leq h_n \leq h_{n+1}) \in \mathbb{N}_0^{n+1} : \gcd(h_1, \ldots, h_{n+1}) = 1 \text{ and } x(h_1, \ldots, h_n) = h_{n+1} \text{ defines a hyperplane of } \text{Aff}(C^n)\}.$$ 

Each vector $(h_1 \leq h_2 \leq \ldots \leq h_n \leq h_{n+1}) \in \mathcal{H}_n$ of the form $(0^{\alpha_0}, h_1^{\alpha_1}, \ldots, h_k^{\alpha_k})$ with $\alpha_0 \geq 0 \alpha_1, \ldots, \alpha_k \geq 1$ and $\alpha_0 + \alpha_1 + \ldots + \alpha_k = n + 1$ determines exactly $\binom{n+1}{\alpha_0, \ldots, \alpha_k} 2^{\alpha_0-\alpha_k}$ distinct hyperplanes (and pairs of signed cocircuits) of $\text{Aff}(C^n)$.

We recall that there is a natural procedure to generate vectors of $\mathcal{H}_n$ from vectors of $\mathcal{H}_{n-1}$ and a recursive family $\mathcal{G}_n \subseteq \mathcal{H}_n$ that we briefly recall from [6].

**Definition 3.1 (the family $\mathcal{G}_n$) [6]**

Given $h = (h_1 \leq h_2 \leq \ldots \leq h_n) \in \mathcal{H}_{n-1}$. For $i = 1, \ldots, n$ let $h_i := (h_1, \ldots, h_{i-1}, h_{i+1}, \ldots, h_n)$, then the level $S_b(h_i)$ with $b = \frac{|h_i|-h_i}{2}$ is a hyperplane of $\text{Aff}(C^{n-1})$ and for every $c \leq b$, such that $S_c(h_i) \neq \emptyset$ the vector $g := (h_i, b-c, |h_i|-b-c)$ is a vector of $\mathcal{H}_n$.

We denote by $\mathcal{G}_n$ the family of vectors of $\mathcal{H}_n$ which is obtained (up to reordering of entries) in this way.

The next Theorem gives the explicit definition of $\mathcal{H}_n$ as well as the relation between $\mathcal{H}_n$ and $\mathcal{G}_n$, for $n \leq 7$.

**Theorem A [6]**

Consider $\mathcal{H}_n := \{(h_1 \leq \ldots \leq h_n \leq h_{n+1}) \in \mathbb{N}_0^{n+1} : \gcd(h_1, \ldots, h_{n+1}) = 1 \text{ and } (h_1, \ldots, h_n) \cdot x = h_{n+1} \text{ defines a hyperplane of } \text{Aff}(C^n)\}$. The list $\mathcal{H}_n$ as well as its sublist $\mathcal{G}_n$ of definition 3.1 is the following for $n \leq 7$:

- $\mathcal{H}_1 = \mathcal{G}_1 = \{(1, 1)\}$;
- $\mathcal{H}_2 = \mathcal{G}_2 = \{(0, 1, 1)\}$;
- $\mathcal{H}_3 = \mathcal{G}_3 = \{(0, 0, 1, 1), (1, 1, 1)\}$;
- $\mathcal{H}_4 = \mathcal{G}_4 = \{(0, 0, 0, 1, 1), (0, 1, 1, 1), (1, 1, 1, 1)\}$;
- $\mathcal{H}_5 = \mathcal{G}_5 = \{(0, 0, 0, 0, 1, 1), (0, 0, 1, 1, 1), (0, 1, 1, 1, 1), (1, 1, 1, 1, 1)\}$;
- $\mathcal{H}_6 = \mathcal{G}_6 = \{(0, 0, 0, 0, 0, 1, 1), (0, 0, 0, 1, 1, 1), (0, 0, 1, 1, 1, 1), (0, 1, 1, 1, 1, 1), (1, 1, 1, 1, 1, 1)\}$;
- $\mathcal{H}_7 = \mathcal{G}_7 = \{(0, 0, 0, 0, 0, 0, 1, 1), (0, 0, 0, 0, 1, 1, 1), (0, 0, 0, 1, 1, 1, 1), (0, 0, 1, 1, 1, 1, 1), (0, 1, 1, 1, 1, 1, 1), (1, 1, 1, 1, 1, 1, 1)\}$. 

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\[H_4 = \mathcal{G}_4 = \{(0, 0, 0, 1, 1), (0, 1, 1, 1, 1), (1, 1, 1, 1, 2)\};\]
\[H_5 = \mathcal{G}_5 = \{(0, 0, 0, 0, 1, 1), (0, 0, 1, 1, 1, 1), (0, 1, 1, 1, 1, 2), (1, 1, 1, 1, 1, 1),\]
\[\quad (1, 1, 1, 1, 2), (1, 1, 1, 2, 2), (1, 1, 1, 2, 3)\};\]
\[H_6 = \mathcal{G}_6 = \{(0, 0, 0, 0, 0, 1, 1, 1, 1, 1, 1), (0, 1, 1, 1, 1, 2), (1, 1, 1, 1, 1, 2),\]
\[\quad (1, 1, 1, 1, 1, 4), (1, 1, 1, 1, 2, 2), (1, 1, 1, 2, 2, 2), (1, 1, 1, 2, 2, 3),\]
\[\quad (1, 1, 1, 2, 3, 3), (1, 1, 1, 3, 3, 3), (1, 1, 2, 2, 2, 3), (1, 1, 2, 2, 2, 3),\]
\[\quad (1, 1, 2, 2, 3, 4), (1, 1, 2, 3, 3, 5), (1, 1, 2, 3, 3, 4)\};\]
\[H_7 = \mathcal{G}_7 \cup \{(1, 1, 2, 4, 4, 5, 6, 7), (1, 1, 2, 4, 4, 6, 7, 9), (1, 2, 3, 3, 4, 4, 5, 6), (1, 2, 3, 4, 4, 5, 6, 7),\]
\[\quad (1, 2, 3, 4, 5, 6, 7, 8), (1, 3, 3, 3, 4, 4, 5), (1, 3, 3, 4, 5, 6, 7), (2, 2, 2, 2, 3, 3, 3),\]
\[\quad (2, 2, 2, 3, 3, 3, 4, 7), (2, 2, 3, 3, 3, 4, 4, 5), (2, 2, 3, 3, 4, 4, 5, 7), (2, 3, 3, 4, 4, 5, 6)\}.\]

We are now ready to prove the main Theorem.

**Proof of Theorem 3.1.**

Consider an oriented cube \(M(C^n)\).

In order to prove that every signed cocircuit defined by a vector \(h \in H_n\) is a signed cocircuit of \(M(C^n)\), it is enough to prove that one of the signed cocircuits defined by an equation \(h_i x = h_i\) is a signed cocircuit of \(M(C^n)\). This is a consequence of the behaviour of the families of signed sets, \(R\) and \(F\), under the symmetries of the real cube (\( \mathbb{R} \)).

1) **Cases** \(n \leq 6\).

For \(n \leq 5\) and every \(h \in H_n\), \(h_i\) is a primitive vector for every \(i = 1, \ldots, n+1\).

Theorem 3.2. then guarantees that all the hyperplanes and corresponding signed cocircuits of \(\mathcal{A}ff(C^n)\) defined by vectors of \(\mathcal{G}_{n+1} = H_{n+1}\) must be signed cocircuits of \(M(C^n)\) and therefore, by Proposition 3.2 \(M(C^n) = \mathcal{A}ff(C^n)\).

2) **Case** \(n = 7\).

**Claim 2.A** The signed cocircuits of \(\mathcal{A}ff(C^7)\) defined by the vectors of \(\mathcal{G}_7\) must be signed cocircuits of \(M(C^7)\).

The arguments applied in the previous cases to vectors of \(H_6\) lead to the conclusion that the hyperplanes and signed cocircuits of \(\mathcal{A}ff(C^7)\) defined by 109 of the 131 vectors of \(\mathcal{G}_7\) must be hyperplanes and signed cocircuits of \(M(C^7)\). The remaining 22 vectors of \(\mathcal{G}_7\) arise from the following 5 vectors of \(H_6\): (1, 1, 1, 1, 3, 3, 4), (1, 1, 2, 2, 2, 3, 3), (1, 1, 2, 2, 3, 3, 4), (1, 1, 2, 3, 3, 4, 5) and (1, 1, 1, 2, 3, 3, 5).
Each one of these 5 vectors has one restriction $h_1$ that is not primitive. In the case of the first 4 vectors, the restriction $h_1$, in the case of the fifth the restriction $h_4$, so we can not apply Theorem 3.2 to conclude that the hyperplanes of $G_n$ obtained extending the hyperplanes $h_1.x = h_i = S_b(h_1)$, with $b = \left\lfloor \frac{|h_i| - b_c}{2} \right\rfloor$ in the facet $H_{7+}$ with an $h_1$-level in the facet $H_{7-}$ must be signed cocircuits of $M(C^7)$ needs some further verification.

Actually in all the cases the proof goes in the same way, generalizing the proof of theorem 3.2:

First we verify that in all the 5 cases the $h_i$-levels embed in the level $S_b(h_i): h_1.x = b (b = \left\lfloor \frac{|h_i| - b_c}{2} \right\rfloor)$.

Secondly, for all $c$, $0 \leq c \leq b$ for which $g := (h_1, b - c, |h_i| - b - c)$ is one of the 109 vectors of $G_n$ arising from primitive restrictions of vectors of $H_6$ there is nothing to prove. We retain those $c’s$ (e.g. $c \geq h_7$) for which $g$ is one of the remaining 22 vectors of $G_n$. For those we verify that $S_a(g) \hookrightarrow S_b(g)$.

Note that for that it is enough to prove that there is a 2-rectangle whose edges in each level connect the facets $H_{7+}$ and $H_{7-}$.

The embeddings $S_a(g) \hookrightarrow S_b(g)$ imply: first, that for every $v \in S_c(h_1)7^-$ the hyperplane $cl(S_b(h_1)7^+ \cup v)$ of $M(C^7)$ must contain all the elements of $S_c(h_1)7^-$. Next, that the only way of extending the signed cocircuit of $S_b(h_i)$ in the facet $H_{7+}$ to the complementary of $S_c(h_1)7^-$ in the facet $H_{7-}$, orthogonally to the signed rectangles, must be the signed cocircuit of $Aff(C^7)$ complementary of $S_b(g)$.

Once Claim 2A. is proved we start proving:

**Claim 2.B** All the signed cocircuits of $Aff(C^7)$ determined by the 12 vectors of $H_7 \setminus G_7$ must be signed cocircuits of $M(C^7)$.

Of these twelve vectors eight differ in exactly one entry from a vector of $G_7$, meaning that they span hyperplanes of $\mathbb{R}^n$ parallel to the affine span of some hyperplane of $Aff(C^7)$ that we already know is a hyperplane of $M(C^7)$.

These eight vectors, together with the vector of $G_7$ differing in one entry are:

$(1, 1, 2, 4, 4, 5, 6, 7)$, $(1, 2, 3, 3, 4, 4, 5, 6)$ and $(1, 2, 3, 4, 4, 5, 6, 7)$ all differing in one entry from $(1, 1, 2, 3, 4, 4, 5, 6) \in G_7$,

$(1, 2, 3, 4, 5, 6, 7, 8)$ and $(1, 1, 2, 3, 4, 5, 6, 8) \in G_7$,

$(2, 2, 2, 2, 3, 3, 3, 5)$ and $(1, 2, 2, 2, 2, 3, 3, 3) \in G_7$,

$(2, 2, 2, 3, 3, 3, 4, 7)$ and $(1, 2, 2, 2, 3, 3, 3, 4) \in G_7$.

In all the cases, let $h$ be a restriction of the vector of $G_7$ defining a hyperplane $S_c(h)$ that we already know must be in $M(C^7)$ and let $S_b(h)$ be the level corresponding to the hyperplane of $Aff(C^7)$ whose pair of signed
cocircuits we want to conclude must be in $\mathcal{M}(C^7)$. In order to do so we proceed in all the cases in the same way:

(i) verify that in $\mathcal{M}$, $rk(S_b(h)) \leq 7$. For that, if necessary using the rectangles, we reduce to 7 the number of elements needed to span all the elements of that $h$-level.

(ii) verify that for all $a \neq b$, $S_a(h) \rightarrow S_b(h)$. This implies, in particular, that the hyperplane $S_a(h)$ is embedded in $S_b(h)$ and consequently, for an element $v_a \in S_a$ we have $rk(cl(S_b \cup v_a)) \leq rk(S_b) + 1$. On the other hand the embedding $S_a \rightarrow S_b$ implies that $rk(cl(S_b \cup v_a)) = rk(cl(S_b \cup S_a)) = rk(\mathcal{M}) = 8$ and we conclude that $rk(S_b) = 7$ and $cl(S_b)$ must be a hyperplane of $\mathcal{M}$.

(iii) the fact that for all $a \neq b$, $S_a \rightarrow S_b$ implies that in the signed cocircuit complementary of $cl(S_b)$ all the elements of $S_a$ must all have the same sign, or be all contained in the hyperplane $cl(S_b)$. In the eight cases a ”ladder” of signed rectangles leads the conclusion that the pair of signed cocircuits of $\text{Aff}(C^7)$ complementary of $S_b$ is the unique signature of the complement of $S_b$ orthogonal to the rectangles of $C^7$ and therefore, these cocircuits of $\text{Aff}(C^7)$ must be cocircuits of $\mathcal{M}$.

Of the four remaining vectors of $\mathcal{H}_7 \setminus \mathcal{G}_7$ the following two:

$(1, 1, 2, 4, 4, 6, 7, 9)$ and $(1, 2, 3, 3, 4, 5, 5, 6)$ differ in exactly one entry from the vectors, respectively, $(1, 1, 2, 4, 4, 5, 6, 7)$ and $(1, 2, 3, 3, 4, 5, 6, 7)$ of $\mathcal{H}_7$ whose signed cocircuits we already know must be in $\mathcal{M}$ and we proceed exactly as before in order to conclude that the corresponding signed cocircuits of $\text{Aff}(C^7)$ must be signed cocircuits of $\mathcal{M}$ too.

Finally we are left with the two vectors $(1, 3, 3, 4, 4, 5, 5)$ and $(1, 3, 3, 4, 5, 5, 6, 7)$. In both cases we prove that the pair of signed cocircuits of the hyperplane defined by $S_b(h_8) := h_8 \cdot x = h_8 (b = \frac{|h_8| - h_8}{2})$ must be cocircuits of $\mathcal{M}$. In both cases the hyperplane contains exactly seven elements of $C^7$, therefore we know that $rk(cl(S_b)) \leq 7$.

To conclude that $cl(S_b) = S_b$ is a hyperplane whose complement, signed as in $\text{Aff}(C^7)$ must be signed cocircuits of $\mathcal{M}$ we proceed as before: first, we verify that $\forall a \neq b, S_a \leftrightarrow S_b$. Next use the net of signed rectangles to conclude that if one element of a different level $S_a$ belongs to $cl(S_b)$ then $cl(S_b) = C^7$, implying that $cl(S_b) = S_b$. Then, essentially the same ladder of rectangles proves the unicity of the signature, concluding the Proof of Theorem 3.1.
4 Final Remarks

In this note we defined recursively a family of nonnegative integer vectors orthogonal to hyperplanes of the real cube - primitive vectors. We proved that the corresponding primitive hyperplanes and signed cocircuits of the real cube $\text{Aff}(C^n)$ must be hyperplanes and signed cocircuits of every oriented cube. The proof relied upon a direct characterization of the behaviour of these nonnegative integer vectors with respect to the net of rectangles of $\text{Aff}(C^n)$.

Although explicit descriptions of the affine cube for $n = 8$ have been computed [1], and our methods apply to this case, an approach of this next case without computer aid would still be too lengthy. A better understanding of the real affine cube and its symmetries is needed.

For instance, note that the hyperplanes defined by primitive vectors are strictly contained in the family $G_n$ (definition 3.1) which for $n \geq 7$ is strictly contained in the family $H_n$. Although we think it must be true, we were not able to prove that the hyperplanes and signed cocircuits determined by the vectors of $G_n$ must be hyperplanes and signed cocircuits of every oriented cube. This would, in particular, simplify further our proof of the case $n = 7$.

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E. Gioan
e-mail adress: emeric.gioan@lirmm.fr
LIRMM (UMR 5506-CC477), 161r. Ada 34095 Montpellier Cedex 5 - FRANCE

I. P. Silva
e-mail adress: ipsilva@fc.ul.pt
Dep. de Matemática, Faculdade de Ciencias da Universidade de Lisboa
Edício C6, Campo Grande, 1749-016 Lisboa, PORTUGAL