Volume Expansion Rate and The Age of The Universe

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Abstract

Under four assumptions such that 1) Einstein’s theory of gravity is correct, 2) existence of foliation by geodesic slicing, 3) the trace of the extrinsic curvature, \( K \), is negative at the present time in observed region, that is, the observed universe is now expanding, 4) the strong energy condition is satisfied, we show that \( H_0 t_0 \leq 1 \), where \( H_0 \equiv -K/3 \) agrees with the Hubble parameter in the case of a homogeneous and isotropic universe, and \( t_0 \) is the age of the Universe. If \( H_0 t_0 > 1 \) is confirmed observationally, at least one of the four assumptions is incorrect.

**Key word**: Hubble parameter
Recent observations of Cepheid variables in NGC4571 (Pierce et al. 1994) and M100 (Freedman et al. 1994) suggest that the Hubble parameter \( H_0 \) is \( \sim 80 \text{km/s/Mpc} \) (Fukugita et al. 1993; Jacoby et al. 1992; van den Bergh 1992). Such a high value of \( H_0 \) may contradict the age estimate of our universe using, for example, globular clusters (Demanqul et al. 1991; Renzini 1991) so that \( H_0 t_0 > 1 \). However one may consider an inhomogeneous universe such that \( H_0 \) in our neighborhood is high (Turner et al. 1992) but global \( H_0 \) is low enough to agree with \( t_0 \). In the previous paper (Nakao et al. 1995) we considered a simple inhomogeneous model in which we are in a void expressed by an open Friedmann universe and showed that \( H_0 t_0 \leq 1 \) even in such an inhomogeneous model as long as the peculiar velocity correction is perfect. In this paper, we extend our argument to a more general inhomogeneous universe.

We shall put four assumptions on the evolution of the universe.

1) First we assume that Einstein’s theory of gravity is the correct theory to describe the evolution of the universe after the Planck time. Then the basic equations in (3+1)-formalism become: the constraint equations are given by

\[
^{(3)}R + K^2 = K_{ij} K^{ij} + 16\pi \rho_H, \tag{1}
\]
\[
K_{ij}^j - K_{ii} = 8\pi J_i, \tag{2}
\]

the evolution equations

\[
\frac{\partial K_{ij}^j}{\partial t} = \alpha (^{(3)}R_i^j + KK_{ij}^j) - \alpha_{[ij]} - 8\pi \left( S_i^j + \frac{1}{2} \delta_i^j (\rho_H - S_i^j) \right) - K_{i[m}^m \beta_{j]i} + K_{i[m}^j \beta_{m}^j + K_{i[m}^j \beta_{m}^j, \tag{3}
\]
\[
\frac{\partial \gamma_{ij}}{\partial t} = -2\alpha K_{ij} + \beta_{[ij]} + \beta_{j]i}, \tag{4}
\]

where \( \alpha \) and \( \beta_i \) are the lapse function and the shift vector, respectively, and the vertical bar means the covariant derivative with respect to the 3-metric \( \gamma_{ij} \). \( \rho_H, J_i \) and \( S_{ij} \) are defined by

\[
\rho_H = T^{\mu\nu} n_\mu n_\nu, \tag{5}
\]
\[ J_i = -T^{\mu\nu}n_\mu h_{i\nu}, \]  
\[ S_{ij} = T^{\mu\nu}h_{i\mu}h_{j\nu}, \]

and
\[ h_{\mu\nu} = g_{\mu\nu} + n_\mu n_\nu, \]

where \( n^\mu \) and \( T^{\mu\nu} \) are the normal vector to \( t = \text{constant} \) hypersurface and the energy momentum tensor of the matter, respectively.

2) We assume the existence of a foliation by geodesic slicing, (i.e, \( \alpha = 1 \)) at least in the region over which we can, in principle, observe.\(^*\) Further for simplicity we choose \( \beta_i = 0 \) so that the line element can be expressed as
\[ ds^2 = -dt^2 + \gamma_{ij} dx^i dx^j. \]

As for the existence of the foliation by the geodesic slices, see Appendix A.

3) The observed universe is expanding, i.e., \( K < 0.\)\(^†\)

4) The strong energy condition is satisfied, i.e., \( S^l_l + \rho_H \geq 0.\)

Now from the trace of Eq.(4), we have
\[ K = -\frac{1}{\sqrt{\gamma}} \frac{\partial \sqrt{\gamma}}{\partial t}, \]

where \( \gamma \) is the determinant of \( \gamma_{ij}. \) We shall call \(-K\) the volume expansion rate and define \( H_v \) as
\[ H_v(t, x) = -\frac{K(t, x)}{3}. \]

\(^*\) The region over which we can, in principle, observe is described mathematically as the causal past \( J^-(p) \) of our present world point denoted by \( p. \)

\(^†\)In mathematical language, the signature of \( K \) is negative in the intersection of each geodesic slice with our observed region, \( O_p \cap J^-(p), \) where \( O_p \) is the neighborhood of \( p. \)
For a homogeneous and isotropic universe, \( H_v \) agrees with the Hubble parameter. The trace of Eq.(3) with the aid of the Hamiltonian constraint equation, Eq.(1), yields

\[
\frac{\partial K}{\partial t} = \frac{1}{3} K^2 + (K_{ij}^T)^2 + 4\pi(S_i^t + \rho_H),
\]

where \( K_{ij}^T \) is the traceless part of \( K_{ij} \). From the assumption 4), Eq.(12) implies that \( K \) is a monotonically increasing function with respect to \( t \) and hence from the assumptions 2) and 3), \( K \) is always negative in “our past”\(^†\)

Dividing Eq.(12) by \( K^2 \), we have

\[
\frac{1}{K^2} \frac{\partial K}{\partial t} = \frac{1}{3} + f,
\]

where

\[
f = \frac{1}{K^2} ((K_{ij}^T)^2 + 4\pi(S_i^t + \rho_H)) \geq 0,
\]

from the assumptions 4).

Integrating Eq.(13) from \( t = t_i \) to \( t_0 \), we have

\[
-\frac{1}{K(0)} + \frac{1}{K(i)} = \frac{1}{3}(t_0 - t_i) + \int_{t_i}^{t_0} f \, dt \\
\geq \frac{1}{3}(t_0 - t_i),
\]

where \( K(0) = K(t_0, x) \) and \( K(i) = K(t_i, x) \). Since \( K \) is always negative, \( H_v = -K(i)/3 > 0 \), and hence using expression (11), we can rewrite Eq.(15) as

\[
H_{v0}(t_0 - t_i) \leq 1 - \frac{H_{v0}}{H_v} \leq 1,
\]

where \( H_{v0} = -K(0)/3 \). Choosing \( t_i \) to be at the initial singularity of the universe \( t = 0 \), we obtain \( H_{v0}t_0 \leq 1 \).

However, it should be noted that \( t_i \) can be of course freely chosen. We can choose \( t_i \) to be the equal time \( t_{eq} \) or to be the decoupling time \( t_{dec} \) of our universe. The important fact

\(^†\)Here “our past” means the subset of \( J^{-}(p) \) which is connected to our observed region by the timelike geodesic \( n^\mu \) perpendicular to the geodesic slices.
shown here is just that the period from arbitrary \( t_i \) to the present time \( t_0 \) is bounded by \( H_{t_0}^{-1} \) (see Appendix B for the more detailed argument).

Equation (16) is a local equation. If \( H_{t_0}t_0 > 1 \) is confirmed observationally in our neighborhood, at least one of the four assumptions which lead to Eq.(16) is incorrect. Four possibilities are:

1. Einstein’s gravity is not correct.

2. The foliation by geodesic slices does not exist (on any averaging scale smaller than our observed scale; see Appendix A).

3. The observed universe is now not expanding.

4. The strong energy condition is not satisfied.

To include the cosmological constant belongs to the fourth possibility. However we note that this is not the only one. We will discuss each possibility elsewhere (Nakamura et al. 1995).

Since \( H_{t_0} \) is defined by the volume expansion rate (Eq.(11)), we should consider the relation between the observed Hubble parameter \( H_0 \) and \( H_{t_0} \). In order to do so, we need to investigate the null geodesic \( k^\mu \) and the distance-redshift relation. By virtue of the comoving coordinate Eq.(9), the angular frequency of the light ray for the comoving observer and comoving source (perhaps the cluster of galaxy) is given simply by \( \omega = k^t \) and hence \( k^\mu = (\omega, k^i) \).

It is sufficient for our purpose to see the time-component of the geodesic equation,

\[
\frac{d\omega}{d\lambda} = K_{ij}k^ik^j,
\]

and the null condition which is given by

\[
\omega = \frac{d\ell}{d\lambda} \equiv \sqrt{\gamma_{ij}k^ik^j}.
\]

Integrating Eq.(17), we obtain
\[
\omega_0 = \omega_e + \int_{\lambda_e}^{\lambda_0} K_{ij} k^i k^j d\lambda,
\]

(19)

where \(\omega_0\) and \(\omega_e\) are respectively the observed angular frequency and emitted one. Here we assume that the proper distance \(\Delta \ell\) between the observer and the source is so small that the integrand in Eq.(19) does not change rapidly. This condition may correspond to \(|K_{ij}|\Delta \ell \ll 1\) (or \(|K_{ij}|\omega_e \Delta \lambda \ll 1\), where \(\Delta \lambda = \lambda_0 - \lambda_e\), because \(\Delta \ell \sim \omega_e \Delta \lambda\) by Eq.(18)). Then from the above equation, the redshift \(z\) is approximately written as

\[
z = \frac{\omega_e}{\omega_0} - 1 \sim -\frac{1}{\omega_o} \left( K_{ij}^T + \frac{1}{3} \gamma_{ij} K \right) k^i k^j \Delta \lambda.
\]

(20)

Using Eq.(18), we get the Hubble law in an inhomogeneous universe as

\[
z \sim H_0 \Delta \ell - K_{ij}^T \frac{k^i k^j}{\omega_o^2} \Delta \ell.
\]

(21)

In general, \(K_{ij}^T\) does not vanish. However, if the observer stays in an almost isotropic region with the linear Hubble law \(z \sim H_0 \Delta \ell\), the second term of the R.H.S. in the above equation is much smaller than the first term near the observer: If the observer stays in the almost isotropic region, such a situation can be approximated by a Tolman-Bondi space-time in which the observer is at the symmetric center. In this space-time, denoting the comoving radial coordinate by \(r\), \(K_{ij}^T k^i k^j \propto r^2 \propto \Delta \ell^2\) near the observer. This means that if the observer can find the effect of the second term in Eq.(21), the Hubble law is not linear. Conversely, we can ignore the second term in Eq.(21) in the situation in which the Hubble law is linear.

Lauer and Postman (1994) suggested from the observations of brightest cluster galaxies over \(0.01 \leq z \leq 0.05\) that the Hubble flow is essentially uniform and isotropic. In this case we may regard \(H_{i0} \sim H_0\). So \(H_{i0} > 1\) requires the four possibilities stated above even if our universe is so inhomogeneous globally that \(H_v\) in other places with distance greater than 100Mpc is much smaller than 80km/s/Mpc.

We would like to thank M. Fukugita for useful lectures on the Hubble parameter and T. Tanaka and M. Siino for their useful discussion. We are grateful to S.A. Hayward for his
careful reading of the English. This work was supported by Grant-in-Aid Scientific Research of the Ministry of Education 04234104 and Grant-in-Aid for Scientific Research Fellowship 2925.

**APPENDIX: A**

The geodesic slicing condition imposes the hypersurface unit normal $n^\mu$ to be tangent to timelike geodesics. Here it should be noted that the nearby timelike geodesics have a tendency to cross with each other in region curved by gravity, e.g. due to the concentration of matter. If the crossing of the timelike geodesics with tangent $n^\mu$ occurs, $n^\mu$ becomes multi-valued at this crossing point. This means that the hypersurface becomes singular at this point since the normal vector and the normal direction to the hypersurface can not be defined uniquely there. Hence a foliation by regular geodesic slices through this crossing point does not exist. If we consider the galaxy, star or much smaller object, e.g., a stone, the timelike geodesics through those objects may cross on the free fall time determined by the energy density of the object (Smarr & York 1978). Hence, rigorously speaking, the foliation by geodesic slicing beyond the shortest free fall time scale of the system may not exist.

However here it should be noted that the existence of the foliation by geodesic slices in an approximate sense depends on what scale we consider. When we investigate, for example, the formation of a star, we consider the matter averaged over an appropriate scale so that the matter can be treated as a continuous quantity (the metric tensor correspondingly becomes an averaged one). In such a treatment the geodesic slicing may be applicable during the free fall time determined by the averaged density of the star in the above sense, not the free fall time determined by the density of the nuclei of the atoms themselves and we know that such an averaged treatment well describes the dynamics of the star.

Here we are considering the averaged matter and metric tensor in a cosmological sense: the averaging scale should be determined so that the free fall time scale agrees with the age of the universe. It is usually considered that under such an averaging treatment a bound
object such as a cluster of galaxies can be regarded as a particle which follows the geodesic $n^\mu$, assuming that the rotation of the velocity field associated with the cluster of galaxies is negligible.

**APPENDIX: B**

The result Eqs.(15) and (16) is essentially the same as the well known fact that within the proper time $\tau \leq H^{-1}_{v_0}$ measured toward the past there exists a conjugate point to a hypersurface $\Sigma$ with $K_{(0)} = -3H_{v_0}$ (Hawking & Ellis 1973; Wald 1984). However it is worthy to note that under the assumptions 1)~4) the occurrence of the conjugate point to $\Sigma$ means the existence of singularities by almost the same argument as Wald’s one (Wald 1984).

Before we proceed to our discussion, we shall see briefly the Wald’s singularity theorem (Wald 1984). Assuming the Einstein equations and

A) the space-time is globally hyperbolic,

B) the strong energy condition holds,

C) there exists a Cauchy surface $\Sigma$ for which the trace of the extrinsic curvature satisfies

$$K \leq -3H_{v_0} < 0 \text{ everywhere},$$

then $H_{v_0}t_0 \leq 1$, where $t_0$ is the proper time of all past directed timelike curves from $\Sigma$.

The proof is very simple. *If there exists a timelike curve which is longer than $H_{v_0}^{-1}$ from $\Sigma$, there exists a maximal length curve which is longer than $H_{v_0}^{-1}$ and has no conjugate points from the global hyperbolicity. This implies a contradiction with the conditions A) and C) because these conditions imply that all timelike geodesics have conjugate points.*

From the above theorem, we can read that the age of the universe is bounded by $H_{v_0}^{-1}$. However it should be noted that the above conditions are global statements. Even if we observe the universe perfectly, we can not, in principle, confirm whether the above conditions are satisfied. Thus global assumptions in the above theorem are not suitable for the present
problem. In this paper, our discussion is restricted only in the observed and observable region and we consider what we can say from the observation on $H_0$.

Here, we shall show the existence of the singularity in the past under our assumptions 1)~4) although it seems to be trivial. Suppose that the universe $J^-(p)$ is foliated past-completely by geodesic slicing beyond the time $t - t_i > H_{eo}^{-1}$, that is, the age of the universe is longer than $H_{eo}^{-1}$. On the other hand, the foliation by geodesic slicing must break down within $t - t_i > H_{eo}^{-1}$ because we see from Eq. (15) that $K_{(i)} < 3[t_0 - t_i - H_{eo}^{-1}]^{-1} \to -\infty$ for $t_0 - t_i \to H_{eo}^{-1}$. This contradicts the past-complete foliation by geodesic slices. Hence there exists a past incomplete timelike geodesic $n^\mu$ perpendicular to the geodesic slices. Especially, the proper time of the timelike geodesic $n^\mu$ in our past (see footnote) is bounded by $H_{eo}^{-1}$. Hence, the assumptions 1)~4) represents a kind of Big Bang cosmology.

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