N = 2 Supermultiplet of Currents 
and Anomalous Transformations 
in Supersymmetric Gauge Theory

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Abstract

We examine some properties of supermultiplet consisting of the $U(1)_{J}$ current, extended supercurrents, energy-momentum tensor and the central charge in $N = 2$ supersymmetric Yang-Mills theory. The superconformal improvement requires adding another supermultiplet beginning with the $U(1)_{R}$ current. We determine the anomalous (quantum mechanical) supersymmetry transformation associated with the central charge and the energy-momentum tensor to one-loop order.

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1This work is supported in part by Grant-in-Aid for Scientific Research (07640403) from the Ministry of Education, Science and Culture, Japan.
1 Introduction

It is well-known that the $U(1)_R$ current, the supercurrent $S^a_\alpha$ and the energy-momentum tensor $T^{ab}$ form a supermultiplet of currents in theories with $N = 1$ supersymmetry\[1\]. (For a review, see\[2\] for example). In the pure super Yang-Mills case, this supermultiplet is

$$J^{(V)}_{\alpha\dot{\alpha}} \equiv W_\alpha \bar{W}_{\dot{\alpha}},$$

whereas in the scalar Wess-Zumino model it is

$$J^{(S)}_{\alpha\dot{\alpha}} \equiv -\frac{1}{3} \bar{D}_{\dot{\alpha}} \Phi^\dagger D_\alpha \Phi - \frac{2}{3} \Phi^\dagger i \left(\overset{\leftrightarrow}{\partial}\right)_{\alpha\dot{\alpha}} \Phi .$$

The second term in eq. (2) is necessary for the superconformal improvement $\sigma \cdot S^a = 0, T^a_\alpha = 0$. The corresponding situation in theories with extended supersymmetry, namely, the $N = 2$ counterpart of supermultiplet of currents does not seem to have been discussed actively before except through coupling to $N = 2$ supergravity \[3\]. In this note, we wish to add a few things on this point.

We first note that it is the $U(1)_J$ current which is related to the energy momentum tensor and the density ( albeit the total derivative \[4\]) of the central charge via the $N = 2$ supersymmetry transformations. This current is known to be anomaly free. The $U(1)_R$ current, which is anomalous, comes in when one improves the supercurrent and the energy-momentum tensor. This is presented in the next section. In section three, we examine anomalous (quantum mechanical) properties of the supersymmetry transformation \[5\] acting on the $U(1)_J$ current. Operating the transformation twice, we are able to derive a quantum mechanical property associated with the central charge of the extended supersymmetry algebra and that of the energy-momentum tensor. The details of our computation as well as its implications will be discussed elsewhere. We follow the notation of \[6\].

2 Classical Transformation Law of Currents

It is most convenient to begin our discussion with the abelian case; we can then organize the entire $N = 2$ supermultiplet of currents in terms of the
\[ N = 2 \text{ chiral and antichiral superfields;} \]
\[
\mathcal{A} = \Phi + i\sqrt{2}\eta W - \frac{1}{4}\eta\eta\bar{D}^2\Phi^\dagger \]
\[
\mathcal{A}^\dagger = \Phi^\dagger - i\sqrt{2}\bar{\eta} \bar{W} - \frac{1}{4}\bar{\eta}\bar{\eta}\bar{D}^2\Phi .
\]

Define
\[
V^a_j \equiv -\frac{1}{2}(D_\theta A)\sigma^a\bar{D}_\theta A^\dagger + \frac{1}{2}(D_\eta A)\sigma^a\bar{D}_\eta A^\dagger ,
\]
where \(D_{\xi(i)}\) is the ordinary superderivative with respect to the Grassmann number \(\xi^{(i)}\) with \(\xi^{(1)} = \theta, \xi^{(2)} = \eta\). We find that eq. (4) contains \(U(1)_J\) current
\[
J^a_j \equiv -\psi\sigma^a\bar{\psi} + \lambda\sigma^a\bar{\lambda} .
\]
as the lowest component. This current is anomaly free. Higher components are found to be
\[
-\sqrt{2}i\psi\sigma^a\sigma^b\theta\partial_b A^* - v_{bc}\theta\sigma^a\bar{\sigma}^b\bar{\lambda} , \theta \text{ term ,}
\]
\[
+\sqrt{2}i\lambda\sigma^a\sigma^b\eta\partial_b A^* - v_{bc}\eta\sigma^a\bar{\sigma}^b\bar{\psi} , \eta \text{ term ,}
\]
\[
-2\sqrt{2}i\partial_b(A^*\psi^b) - \sqrt{2}\epsilon^{abcd}\theta_b(A^*v_{cd}) , \eta\theta \text{ terms ,}
\]
\[
-i\psi\sigma^a\partial^b\bar{\psi} + i\partial^b\psi\sigma^a\bar{\psi} - i\lambda\sigma^a\partial^b\bar{\lambda} + i\partial^b\lambda\sigma^a\bar{\lambda}
\]
\[
-2\partial^a A\partial^b A^* - 2\partial^b A\partial^a A^* - 2\psi^b v^{ac} + 2i\epsilon^{abcd}\partial_c A\partial_d A^* + \tilde{v}^b v^{ac}
\]
\[
-\tilde{v}^a v^{bc} + g^{ab}(2\partial_c A\partial^c A^* + \frac{1}{2}\psi^c v_{cd}) , \theta\sigma^a\bar{\theta}, \eta\sigma^a\bar{\eta} \text{ terms ,}
\]
which are respectively the supercurrent \(S^a_{(1)}\), the current for extended supersymmetry \(S^a_{(2)}\), the density of the central charge \(C^a\) and the energy-momentum tensor \(T^{ab}\). (We have used equations of motion. The remaining components are either zero or a total derivative.)

Eqs. (3), (4) do not obey \(\sigma \cdot S_{(i)} = 0\), \((i = 1, 2)\) condition. Neither does eq. (4) obey the tracelessness condition. To make them obey these conditions, i.e. to carry out the superconformal improvement, we introduce another \(N = 2\) supermultiplet of currents:
\[
V^a_R \equiv -\frac{1}{2}D_\theta A\sigma^a\bar{D}_\theta A^\dagger - \frac{1}{2}D_\eta A\sigma^a\bar{D}_\eta A^\dagger - 2iA \partial^a A^\dagger .
\]
The lowest component is the $R$-current

$$J_R^a \equiv 2i(A^\dagger \partial^a A - (\partial^a A^\dagger) A) - \psi^a \bar{\psi} - \lambda^a \bar{\lambda} \ .$$  

(11)

The higher components do not seem to have straightforward interpretation. These can be used, however, for the superconformal improvement. Note that the $\theta$ and $\eta$ components of $\mathcal{V}_J^a$ and those of $\mathcal{V}_R^a$ satisfy

$$\mathcal{V}_J^a|_{\theta = \bar{\xi} \theta} = - \mathcal{V}_J^a|_{\eta = \bar{\xi} \eta} = 2\sqrt{2} i \psi^b \sigma^b \bar{\xi} \partial_b A^*$$

(12)

$$\mathcal{V}_R^a|_{\theta = \bar{\xi} \theta} = \mathcal{V}_R^a|_{\eta = \bar{\xi} \eta} = 4\sqrt{2} i \psi^b \sigma^b \bar{\xi} \partial_b A^* \ .$$

(13)

The trace of the $\theta \sigma^b \bar{\theta}$ term and that of the $\eta \sigma^b \bar{\eta}$ term in $\mathcal{V}_J^a$ are respectively $4 \partial_a A \partial_a A^*$ and $-4 \partial_a A \partial_a A^*$ whereas the trace of the $\theta \sigma^b \bar{\theta}$ term and that of the $\eta \sigma^b \bar{\eta}$ term in $\mathcal{V}_R^a$ are $8 \partial_a A \partial_a A^*$. Knowing these, let us define

$$\mathcal{V}_{imp}^{a(1)} \equiv \mathcal{V}_J^a - \frac{1}{2} \mathcal{V}_R^a \ , \ \mathcal{V}_{imp}^{a(2)} \equiv \mathcal{V}_R^a + \frac{1}{2} \mathcal{V}_R^a \ .$$

(14)

The first one $\mathcal{V}_{imp}^{a(1)}$ accomplishes the superconformal improvement for $S_{(1)}^a$ and $T_{ab}$. The second one does for $S_{(2)}^a$ and $T_{ab}$.

To conclude, it is $U(1)_J$ which is connected to the center of the supersymmetry algebra via the supersymmetry transformations. The superconformal improvement is done by either one of the two in eq. (14).

Having known the basic algebraic structure, let us turn to the nonabelian case. No $N = 2$ superfield formulation useful for our purpose seems to be known [7, 8, 2]. We carry out straightforward computation by components. It is straightforward to derive $S_{(i)}^a$ as an extended supertransformation on $J_J^a$.

$$\delta_{(i)} J_J^a = \xi^{(i)} S_{(i)}^a + \bar{\xi}^{(i)} \bar{S}_{(i)}^a ,$$

$$S_{(1)\alpha}^a = i tr \left\{ \sqrt{2} \left( \sigma^b \bar{\sigma}^a \psi \right)^\alpha \partial_b A^* - i \left( \sigma^b c \sigma^a \bar{\lambda} \right)_\alpha \left[ A^*, A \right] \right\} ,$$

$$\bar{S}_{(1)\dot{\alpha}}^a = - i tr \left\{ \sqrt{2} \left( \bar{\sigma}^b \sigma^a \bar{\psi} \right)^\dot{\alpha} \partial_b A - i \left( \bar{\sigma}^b c \sigma^a \lambda \right)^\dot{\alpha} \left[ A^*, A \right] \right\} ,$$

$$S_{(2)\alpha}^a = - i tr \left\{ + \sqrt{2} \left( \sigma^b \bar{\psi} \right)^\alpha \partial_b A^* + i \left( \sigma^b c \sigma^a \bar{\psi} \right)^\alpha \left[ A^*, A \right] \right\} .$$
\[ S^{\alpha} \frac{\delta}{\delta} = i tr \left\{ \frac{1}{2} \left( \sigma^a \bar{\psi} \right)_\alpha [A^*, A] \right\}, \]

\[ \mathcal{S}^{a}_{(2)} = \frac{1}{2} \left( \sigma^a \bar{\psi} \right)_\alpha [A^*, A], \]

Transforming once more, we obtain the energy-momentum tensor \( T^{ab} \) and the density \( C^a \) of the central charge.

\[-2 \xi^1 \sigma_b \bar{\xi}^1 T^{ab} \equiv \delta_1 \delta_2 J^a_j = \xi^1 \sigma_b \bar{\xi}^1 (2) \left[ tr \left\{ -D^a A^* D^b A - D^b A^* D^a A \right. \right. \]

\[-i \bar{\psi} \sigma^a D^b \psi - i \lambda \sigma^a D^b \bar{\lambda} - v^{ac} v_{bc} \big\} - g^{ab} \mathcal{L} \bigg] + \xi \sigma_b \bar{\xi} tr \left\{ -2 i \epsilon^{abcd} D_c A D_d A^* \right. \]

\[ + \frac{1}{2} \epsilon^{abcd} v_{cd} [A^*, A] \bigg\} + \xi \sigma_b \bar{\xi} tr \left\{ v^a_{b} \bar{v}^{ac} - \bar{v}^b_{c} v^{ac} \right\}. \]

\[ \bar{v}^{ab} \equiv \frac{i}{2} \epsilon^{abcd} v_{cd}. \]

\[ C^a \equiv \delta_1 \delta_2 J^a_j = tr \left\{ 2 \sqrt{2} i D_b (A^* v^{ba}) + \sqrt{2} \epsilon^{abcd} D_d (A^* v_{bc}) \right\}. \]

3 Quantum Mechanical Transformations

In the remainder of this note, we examine how some of these currents written in the interaction picture receive quantum corrections by studying supersymmetry transformations discussed above. This will be done to one-loop. As currents are in general composite operators and the ones considered here are not gauged, it is perfectly consistent that the quantum mechanical counterpart of the classical transformations contains new operators. For that reason, this term may be referred to as anomalous transformations. The corresponding calculation in the \( N = 1 \) case has been given in [5]. Let us write generically

\[ \delta_\theta J^a_j (x | \epsilon) = \theta S_1^a (x | \epsilon) + \theta \Delta_1^a (x | \epsilon) \]

\[ \delta_\eta J^a_j (x | \epsilon) = \eta S_2^a (x | \epsilon) + \eta \Delta_2^a (x | \epsilon) \]

\[ \bar{\delta}_\theta \delta_\theta J^a_j (x | \epsilon) = \bar{\theta} \theta T^a (x | \epsilon) + \Delta_1^a (x | \epsilon; \theta, \bar{\theta}) \]

\[ \bar{\delta}_\eta \delta_\eta J^a_j (x | \epsilon) = \eta \theta C^a (x | \epsilon) + \Delta_2^a (x | \epsilon; \eta, \theta) \].

(17)
Here the $U(1)_J$ current in the Heisenberg picture regularized by gauge invariant point splitting is

$$J_a^J(x) = tr \left[ \bar{\Psi}(x + \frac{\epsilon}{2}) \sigma^a U_\epsilon \Psi(x - \frac{\epsilon}{2}) U_{-\epsilon} - \bar{\Lambda}(x + \frac{\epsilon}{2}) \sigma^a U_\epsilon \Lambda(x - \frac{\epsilon}{2}) U_{-\epsilon} \right]$$

$$U_\epsilon \equiv P \exp \left[ -i \frac{1}{2} \int_{x - \frac{\epsilon}{2}}^{x + \frac{\epsilon}{2}} dx^a v_a(x) \right].$$  \hspace{1cm} (18)

We will use bold face letters for the operators in the Heisenberg picture.

It is easy to see that $\Delta^a_i(x \mid \epsilon) = 0 \ i = 1, 2$ by antisymmetry under $\Psi \leftrightarrow \Lambda$. We will now compute $\Delta^a_{\text{center}}(x \mid \epsilon; \eta, \theta)$ and $\Delta^a_{\text{EM}}(x \mid \epsilon; \theta, \bar{\theta})$ to one-loop order.

We find

$$\Delta^a_{\text{center}}(x \mid \epsilon; \eta, \theta) = -\frac{i}{2} e^{htr} \left\{ -\sqrt{2} D_c A^* \left( x + \frac{\epsilon}{2} \right) \left[ \right. \eta \sigma_b \bar{\Psi}(x), \theta \sigma^c \sigma^a \Psi \left( x - \frac{\epsilon}{2} \right) \right] \right.$$ 

$$-i \bar{\Psi} \left( x + \frac{\epsilon}{2} \right) \sigma^a \sigma^d \eta \left[ \theta \sigma_b \bar{\Lambda}(x), v_{cd} \left( x - \frac{\epsilon}{2} \right) \right] \right.$$ 

$$+ \bar{\Psi} \left( x + \frac{\epsilon}{2} \right) \sigma^a \eta \left[ \theta \sigma_b \bar{\Lambda}(x), D \left( x - \frac{\epsilon}{2} \right) \right] \right.$$ 

$$-\sqrt{2} \theta \sigma_b \delta^c \eta \bar{\Psi} \left( x + \frac{\epsilon}{2} \right) \left[ D_c A^* \left( x \right), \sigma^a \Psi \left( x - \frac{\epsilon}{2} \right) \right] \right.$$ 

$$-\sqrt{2} D_c A^* \left( x + \frac{\epsilon}{2} \right) \left[ \theta \sigma_b \bar{\Lambda}(x), \eta \sigma^c \sigma^a \Lambda \left( x - \frac{\epsilon}{2} \right) \right] \right.$$ 

$$+ i \bar{\Lambda} \left( x + \frac{\epsilon}{2} \right) \sigma^a \sigma^d \eta \left[ \eta \sigma_b \bar{\Psi}(x), v_{cd} \left( x - \frac{\epsilon}{2} \right) \right] \right.$$ 

$$- \bar{\Lambda} \left( x + \frac{\epsilon}{2} \right) \sigma^a \theta \left[ \eta \sigma_b \bar{\Psi}(x), D \left( x - \frac{\epsilon}{2} \right) \right] \right.$$ 

$$+ i \bar{\Lambda} \left( x + \frac{\epsilon}{2} \right) \sigma^a \sigma^d \theta \left[ \eta \sigma_b \bar{\Psi}(x), v_{cd} \left( x - \frac{\epsilon}{2} \right) \right] \right.$$ 

$$- \bar{\Lambda} \left( x + \frac{\epsilon}{2} \right) \sigma^a \theta \left[ \eta \sigma_b \bar{\Psi}(x), D \left( x - \frac{\epsilon}{2} \right) \right] \right.$$ 

$$+ \sqrt{2} \delta \sigma_b \delta^c \eta \bar{\Lambda} \left( x + \frac{\epsilon}{2} \right) \left[ D_c A^* \left( x \right), \sigma^a \Lambda \left( x - \frac{\epsilon}{2} \right) \right] \left\} \right..$$  \hspace{1cm} (19)

Let

$$\lim_{\epsilon \to 0} \Delta^a_{\text{center}}(x \mid \epsilon; \eta, \theta) = \sum_{a=1}^{4} I_a + \sum_{a=5}^{9} I'_a,$$  \hspace{1cm} (20)

where $I_a$ $a = 1 \sim 4$ and $I'_a$ $a = 5 \sim 9$ represent the respective contributions from the $a$-th line in eq. (19) evaluated to one-loop in the interaction picture.

Space does not permit us to present one-loop computation of all $I_a$’s and $I'_a$’s we have carried out. We just demonstrate our calculation in a sample
We thank Alyosha Morozov for discussions on a related topic.

4 Acknowledgements

We have used

\[ I_1 = \lim_{\epsilon \to 0} \frac{i}{2}(-\sqrt{2}) \text{tr} \left\{ T^r \left[ \eta \sigma_b \bar{\psi}(x), T^s \right] \right\} e^b \]

\times \int d^4 y \frac{1}{g^2} \frac{1}{\sqrt{2}} \text{tr} \left\{ T^u \left[ T^v, T^w \right] \right\} < \partial_c A^{(r)*}(x + \frac{\epsilon}{2}) A^{(t)}(y) >

\times \theta \sigma^c \bar{\sigma}^a < \psi^{(s)}(x - \frac{\epsilon}{2}) \bar{\psi}^{(u)}(y) > \lambda^{(v)}(y)

\begin{align*}
= \lim_{\epsilon \to 0} \frac{1}{2} g^2 & t_{ad}(\eta \sigma_b \bar{\psi}(x) T_{(ad)}^v) \right) \int d^4 y \int \frac{d^4 q}{(2\pi)^4 i} \frac{d^4 p}{(2\pi)^4 i} \theta \sigma^c \bar{\sigma}^a \sigma^d \lambda^{(v)}(y) \\
\times \epsilon^b & P_{c d} \epsilon i(p-q)(x-y) \epsilon i(p+q) \frac{i}{2} \\
= \left( \frac{1}{2} \right)^2 & \frac{g_2}{32\pi^2} t_{r(ad)} \eta \sigma_b \bar{\psi}(x) \left( \theta \sigma^b \bar{\sigma}^a \sigma^d \partial_b \lambda(x) - \theta \sigma^c \bar{\sigma}^a \sigma^b \partial_c \lambda(x) \right) . \quad (21)
\end{align*}

We have used

\[ \lim_{\epsilon \to 0} \int \frac{d^4 p}{(2\pi)^4 i} \frac{d^4 q}{(2\pi)^4 i} \epsilon^b P_{c d} \epsilon i(p-q)(x-y) \epsilon i(p+q) \frac{i}{2} = \frac{i}{64\pi^2} \left( \delta_{c d} \frac{\partial}{\partial x^d} - \delta_{d c} \frac{\partial}{\partial x^d} \right) \delta^{(4)}(x-y) . \quad (22) \]

Collecting all contributions, we find the final answer:

\[ \lim_{\epsilon \to 0} \Delta_{\text{center}}^a(x \mid \epsilon) = i \left( \frac{1}{2} \right)^2 \frac{g_2}{32\pi^2} \left[ \partial_b \left( -\eta \bar{\psi} \bar{\sigma}^{ab} \lambda - 2\theta \sigma^{ab} \eta \bar{\psi} \lambda \right) \\
+ \frac{3}{2} \eta \theta \bar{\psi} \partial^a \lambda - \lambda \partial^a \bar{\psi} \right] - \eta \sigma^{ab} \theta \left( \bar{\psi} \sigma_b \sigma_c \partial^c \lambda + \bar{\lambda} \sigma_b \sigma_c \partial^c \bar{\psi} \right) . \quad (23) \]

As for the energy-momentum tensor, a similar calculation yields

\[ \Delta_{EM}^a (x \mid \epsilon, \theta, \bar{\theta}) = \theta \sigma_b \bar{\sigma} \Delta_{EM}^{ab} (x \mid \epsilon) \]

\[ \lim_{\epsilon \to 0} \Delta_{EM}^{ab}(x \mid \epsilon) = -i \left( \frac{1}{2} \right)^2 \frac{g_2}{64\pi^2} \text{tr} \left[ 3\eta^{ab} \left( (\partial^c \lambda) \sigma_c \bar{\lambda} - \lambda \sigma_c \partial^c \bar{\lambda} \right) \\
+ \left( (\partial^a \lambda) \sigma^b \bar{\lambda} - \lambda \sigma^b \partial^a \bar{\lambda} \right) - \left( (\partial^b \lambda) \sigma^a \bar{\lambda} - \lambda \sigma^a \partial^b \bar{\lambda} \right) + i \epsilon^{abcd} \partial_c (\lambda \sigma_d \lambda) \right] . \quad (24) \]

4 Acknowledgements

We thank Alyosha Morozov for discussions on a related topic.
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