Infinitesimally tight Lagrangian orbits

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Abstract
We describe isotropic orbits for the restricted action of a subgroup of a Lie group acting on a symplectic manifold by Hamiltonian symplectomorphisms and admitting an $\text{Ad}^*$-equivariant moment map. We obtain examples of Lagrangian orbits of complex flag manifolds, of cotangent bundles of orthogonal Lie groups, and of products of flags. We introduce the notion of infinitesimally tight and study the intersection theory of such Lagrangian orbits, giving many examples.

Keywords
Lagrangian orbits · Product of flags · Infinitesimally tight · Orthogonal Lie group

Mathematics Subject Classification 53D12 · 14M15

1 Introduction
Let $(M, ω)$ be a connected symplectic manifold, $G$ a Lie group with Lie algebra $g$, and $L$ a Lie subgroup of $G$. Assume that there exists a Hamiltonian action of $G$ on $M$ which admits an $\text{Ad}^*$-equivariant moment map $μ : M → g^*$. The purpose of this paper is to study those orbits $Lx$ with $x ∈ M$ that are Lagrangian submanifolds of $(M, ω)$, or more generally, isotropic submanifolds. We also discuss some essential features of the intersection theory of such Lagrangian orbits, namely the concepts of locally tight and infinitesimally tight Lagrangians. The famous Arnold–Givental conjecture, proved in many cases, predicts that the number of intersection points of a Lagrangian $L$ and its image $ϕ(L)$ by the flow of a Hamiltonian vector field can be estimated from below by the sum of its $\mathbb{Z}_2$ Betti numbers:
The concepts of tightness address those Lagrangians which attain the lower bound, and are therefore of general interest in symplectic geometry.

For us additional motivation to study Lagrangians and their intersection theory comes from questions related to the Homological Mirror Symmetry conjecture and in particular from concepts of objects and morphisms in the so-called Fukaya–Seidel categories, which are generated by Lagrangian vanishing cycles (and their thimbles) with prescribed behavior inside of symplectic fibrations. In [7, Thm. 2.2] it was shown that the usual height function from Lie theory gives adjoint orbits of semisimple Lie groups the structure of symplectic Lefschetz fibrations. These give rise to what is known as Landau–Ginzburg (LG) models. We wish to study the Fukaya–Seidel category of these LG models. Finding Lagrangian submanifolds and understanding their intersection theory inside a compactification is an initial tool to investigate possible thimbles. The Fukaya–Seidel category of the LG model for the adjoint orbit of \( \mathfrak{sl}(2, \mathbb{C}) \) was calculated in [3] and such LG models was shown to have no projective mirrors (Theorems 4.1 and 7.6).

Products of flag manifolds occur as compactifications of adjoint orbits of semisimple noncompact Lie groups, see [8, Sec. 3], and this originated our particular interest in finding Lagrangian submanifolds inside products of flags. Moreover, minimal noncompact semisimple orbits were shown in [4] to satisfy the KKP conjecture, but a verification of the KKP conjecture for general semisimple orbits, which remains to be done, would also require better understanding of the Lagrangians inside their compactification. Hence we have various motivations to search for Lagrangians inside noncompact adjoint orbits, the compact ones, that is, the flag manifolds, and products.

This paper is divided as follows. In Sect. 2, we give a simple characterization of isotropic orbits. If \( L \) denotes the Lie algebra of \( L \) and \( L' \) its derived algebra, we have

\[
|L \cap \varphi(L)| \geq \sum b_L(L; \mathbb{Z}_2).
\]

Proposition 1 An orbit \( Lx \) is isotropic if and only if \( \mu(x) \) belongs to the annihilator \( (L')^\circ \) of \( L' \).

In Sect. 3, we use Proposition 1 to characterize isotropic orbits in the cotangent bundle of an orthogonal Lie group. In particular, if \( G \) is a semisimple Lie group and \( T^*(G) \approx G \times g^* \) is its cotangent bundle we prove:

Corollary 1 The only isotropic orbits by the natural left and right actions of \( G \) on \( T^*(G) \) are of the form \( G(g, 0) \) for all \( g \in G \). Such orbits are Lagrangian and Hamiltonian isotopic to \( G \).

In Sect. 4, we consider compact semisimple Lie groups. Endowing adjoint orbits \( \text{Ad}(U)(iH_0) \) with the Kirillov–Kostant–Souriau symplectic form, we prove that the orbit of a proper subgroup \( L \subset U \) through the origin \( iH_0 \) is isotropic if and only if \( L' \subset (iH_0)^\perp \). For example, the orbit of \( \text{SO}(n) \) through the origin of any flag of \( SU(n) \) is Lagrangian.

An interesting example happens when \( U = \text{SU}(3) \) and \( L = U_H \) is the isotropy group in \( U \) of the element \( H = i\text{diag}[2, -1, -1] \). We have that \( (u_H)^\perp \) intersects the three types of adjoint orbits of \( SU(3) \), namely the flags \( \mathbb{C}P^2 \), \( \text{Gr}_2(3, \mathbb{C}) \) and \( \mathbb{F}(1, 2) \). We prove that the only possible isotropic orbits of \( L \) passing through \( H \) are:

- the trivial one, that is, a single point in \( \mathbb{C}P^2 \),
- a (2 dimensional) Lagrangian in \( \text{Gr}_2(3, \mathbb{C}) \), and
- a (3 dimensional) Lagrangian in the flag \( \mathbb{F}(1, 2) \).
In Sect. 5, we study Lagrangian orbits in products of flag manifolds with respect to the diagonal and shifted diagonal actions, showing:

**Theorem 1** A product of flags \( F_{\Theta} \times F_{\Theta^*} \) admits an isotropic orbit by the diagonal action if and only if \( F_{\Theta^*} \) is the dual flag \( F_{\Theta}^* \) of \( F_{\Theta^*} \).

Acting by subgroups of the type \( \Delta^m = \{(u, mum^{-1}) \in U \times U : u \in U\} \) we obtain:

**Proposition 4** Inside the product \( F_{\Theta} \times F_{\Theta^*} \), for each \( m \in U \), there exists a unique isotropic orbit of the diagonal action by the subgroup \( \Delta^m \). Such an orbit is Lagrangian and it is given by the graph of the map \(-\text{Ad} (m) : \text{Ad} (U) (iH) \to \text{Ad} (U) (i\sigma (H))\).

In particular, when \( m = e \) is the identity in \( U \), we prove that there exists a unique Lagrangian orbit of the diagonal action of \( U \) on \( F_{\Theta} \times F_{\Theta^*} \) given as the graph of \(-\text{id} : \text{Ad} (U) (iH) \to \text{Ad} (U) (i\sigma (H))\). Furthermore, as an important feature of the orbits by shifted diagonals is stated as:

**Theorem 2** All Lagrangian orbits in \( F_{\Theta} \times F_{\Theta^*} \) of Proposition 4 belong the same Hamiltonian isotopy class.

Section 6 is dedicated to the study of tight immersions. We explore a new concept which we call infinitesimally tight (Definition 2). This notion is equivalent to the concept of locally tight given by [15] (Definition 1). In other words,

**Theorem 3** Let \( G \) be a Lie group and \( M \) a homogeneous space together with a \( G \)-invariant symplectic form \( \omega \). Then a Lagrangian submanifold \( \mathcal{L} \subset M \) is infinitesimally tight if and only if \( \mathcal{L} \) is locally tight.

As an example we show that the Lagrangian orbit \( S^3 \) of \( U(2) \) in the flag \( F(1,2) \) is infinitesimally tight. In further generality, we obtain:

**Corollary 3** The Lagrangian orbits of type

\[
\Gamma \{ -\text{Ad} (m) : \text{Ad} (U) (iH) \to \text{Ad} (U) (i\sigma (H))\}
\]

corresponding to the shifted diagonals \( \Delta^m \) are infinitesimally tight in \( F_{\Theta} \times F_{\Theta^*} \).

In Appendix A) we describe the KKS symplectic form on adjoint orbits of orthogonal Lie groups. Finally, in Appendix B we give a list of open problems about Lagrangian orbits.

### 2 Isotropic orbits

Let \((M, \omega)\) be a connected symplectic manifold and \( \cdot : G \times M \to M \) a Hamiltonian action of a Lie group \( G \) on \( M \). If \( g \) is the Lie algebra of \( G \) and \( g^* \) its dual vector space, this means that the action is symplectic and that there exists a smooth map \( \mu : M \to g^* \), called moment map, such that for all \( X \in g \)

\[
d\hat{\mu}(X) = i\tilde{X}\omega
\]

where \( \hat{\mu}(X) : M \to \mathbb{R} \) is the smooth map defined by \( \hat{\mu}(X)(x) = \mu(x)(X) \) and

\[
\tilde{X}(x) = \frac{d}{dt} e^{tX} \cdot x \bigg|_{t=0}, \quad x \in M,
\]
is the fundamental vector field associated to \( X \). Identity (1) implies that \( \widetilde{X} \) is the Hamiltonian vector field of \( \tilde{\mu}(X) \). If \( \text{Ad}^* : G \to \text{GL}(\mathfrak{g}^*) \) denotes the coadjoint representation of \( G \), a moment map \( \mu : M \to \mathfrak{g}^* \) is called \( \text{Ad}^*\text{-equivariant} \) if
\[
\mu (g \cdot x) = \text{Ad}^* (g) \mu (x) \quad g \in G, \ x \in M.
\]

**Remark 1** If \( \phi_g : M \to M \) is defined by \( \phi_g (x) = g \cdot x \) for all \( x \in M \), since \( \cdot : G \times M \to M \) is a symplectic action, then \( \phi_g^* \omega = \omega \), or equivalently, \( \mathcal{L}_X \omega = 0 \) for all \( X \in \mathfrak{g} \). Therefore, \( \widetilde{X} \) is locally Hamiltonian but not necessarily globally Hamiltonian. This is the reason why not every symplectic action is a Hamiltonian action. The latter happens if for instance \( H_1 \) is connected and semisimple (only if \( H_1 (\mathfrak{g}, \mathbb{R}) = H^2 (\mathfrak{g}, \mathbb{R}) = 0 \)), then the symplectic action has an \( \text{Ad}^*\text{-equivariant} \) moment map, see [17] or [14].

Recall that a submanifold \( \iota : \mathcal{L} \hookrightarrow M \) of a symplectic manifold \((M, \omega)\) is called **isotropic** if \( \iota^* \omega = 0 \). If moreover \( \dim (\mathcal{L}) = 1/2 \dim (M) \) then \( \mathcal{L} \) is called **Lagrangian**. From now on, we assume that \( \cdot : G \times M \to M \) is a Hamiltonian action for a connected symplectic manifold \((M, \omega)\) which admits an \( \text{Ad}^*\text{-equivariant} \) moment map \( \mu \). Let \( L \) be a Lie subgroup of \( G \) with Lie algebra \( \mathfrak{l} \). The problem considered here is to describe those orbits \( Lx (x \in M) \) of \( L \) that are Lagrangian submanifolds of \( M \), or more generally isotropic. The following arguments use the moment map \( \mu \) to give necessary and sufficient conditions for the orbit \( Lx \) to be isotropic.

If \( X, Y \in \mathfrak{g} \), it is well known that the Poisson bracket of \( \tilde{\mu}(X) \) and \( \tilde{\mu}(Y) \) is given by
\[
\{ \tilde{\mu}(X), \tilde{\mu}(Y) \} = \omega (\widetilde{X}, \widetilde{Y}) = -\widetilde{X} \cdot \tilde{\mu}(Y) = \widetilde{Y} \cdot \tilde{\mu}(X).
\]

Therefore, for all \( x \in M \)
\[
\widetilde{X} \cdot \tilde{\mu}(Y) (x) = \frac{d}{dt} \tilde{\mu}(Y) (e^{tX} x) |_{t=0} = \frac{d}{dt} \mu (e^{tX} x) |_{t=0} (Y)
= \frac{d}{dt} \text{Ad}^* (e^{tX} \mu (x) |_{t=0} (Y) = (\text{ad}^* (X) \mu (x)) (Y)
= -\mu (x) ([X, Y]).
\]

The above computation implies two things. The first one is that \( \tilde{\mu} \) defines a Lie algebra homomorphism between \( \mathfrak{g} \) and \( C^\infty (M) \) seen as Lie algebra with the Poisson bracket. The second one is that
\[
\omega_x (\widetilde{X} (x), \widetilde{Y} (x)) = 0 \quad \text{if and only if} \quad \mu (x) [Y, X] = 0. \tag{2}
\]
Recall also that the tangent space to the orbit \( Lx \) at the point \( x \) is given by
\[
T_x (Lx) = \{ \widetilde{X} (x) : X \in \mathfrak{l} \}.
\]

Therefore, \( \omega_x \) vanishes identically on \( T_x (Lx) \) if and only if \( \mu (x) [X, Y] = 0 \) for all \( X, Y \in \mathfrak{l} \). Hence, we obtain the following characterization of those orbits of \( L \) that are isotropic.

**Proposition 1** An orbit \( Lx \) is isotropic if and only if \( \mu (x) \) belongs to the annihilator \( (\mathfrak{l}^\circ) \) of the derived algebra \( \mathfrak{l}' \) of \( \mathfrak{l} \).

**Proof** Choose \( y \in Lx \). By observation (2), the tangent space \( T_y (Lx) = T_y (Ly) \) is isotropic if and only if \( \mu (y) [X, Y] = 0 \) for all \( X, Y \in \mathfrak{l} \), that is, if and only if \( \mu (y) \) belongs to the
annihilator of \( l' \). But, \( \mu (y) \in (l')^o \) if and only if \( \mu (x) \in (l')^o \) since if \( y = gx \ (g \in L) \) then 
\[
\mu (y) = \text{Ad} (g) \mu (x)
\]
and therefore \( \mu (y) \) annihilates \( l' \) if and only if \( \mu (x) (\text{Ad} (g) l') = \mu (x) (l') = 0 \) given that \( l' \) is invariant by every automorphism of \( l \).

\[\square\]

\textbf{Remark 2}  
It is worth noticing that the criterion given in Proposition 1 needs to be verified only at a single point of the orbit \( Lx \) given that the annihilator \((l')^o\) is invariant by the coadjoint action.

Proposition 1 can also be applied to the case \( L = G \), although, the reasoning can be carried out for a pair of groups \( L \subset G \). If the action of \( G \) is Hamiltonian the same is true for the action of \( L \) and the moment map \( \mu_L \), as well as for the moment map \( \mu \) for the restriction of the action of \( G \), that is, \( \mu_L (x) = \mu (x)|_L \) directly by definition. (The examples we will consider suggest to use a pair of groups \( L \subset G \) and to take \( M \) as a homogeneous space of \( G \).)

In the particular case when \( (G, \langle \cdot, \cdot \rangle) \) is an orthogonal Lie group (see Definition 3) the moment map can be interpreted as a map with values in \( g \). In such case \((l')^o\) becomes the orthogonal complement of \( l' \) with respect to the invariant scalar product \( \langle \cdot, \cdot \rangle_e \) induced over \( g \). Accordingly, an orbit \( Lx \) is isotropic if and only if \( \mu (x) \) belongs to the orthogonal complement of \( l' \). In particular, this is the case when \( G \) is a compact Lie group where there exists an invariant inner product on \( g \) and when \( G \) is a semisimple Lie group replacing the inner product by the Cartan–Killing form.

\section{Orthogonal Lie groups}

Let \( G \) be a connected Lie group with Lie algebra \( g \) and \( g^* \) its dual vector space. The cotangent bundle of \( G \) is isomorphic to the trivial vector bundle \( G \times g^* \) through the isomorphism of vector bundles \( \lambda : T^* (G) \to G \times g^* \) defined by

\[
\lambda (g, \alpha_g) = (g, \alpha_g \circ (dL_g)_e) \quad (g, \alpha_g) \in T^* (G).
\]

By means of left and right multiplications on \( G \) we can define two natural left actions of \( G \) on itself, called left and right action, which are given by \( L : G \times G \to G \) and \( R : G \times G \to G \) and are defined by \( L(g, h) = L_e(h) = gh \) and \( R(g, h) = R_{g^{-1}}(h) = hg^{-1} \), respectively. These actions can be lifted to \( T^* (G) \) allowing us to define two left actions of \( G \) on \( T^* (G) \) as follows.

\[
\tilde{L} : G \times T^* (G) \to T^* (G), \quad \tilde{L}(g, (h, \alpha_h)) = (gh, \alpha_h \circ (dL_{g^{-1}})_h)
\]

and

\[
\tilde{R} : G \times T^* (G) \to T^* (G), \quad \tilde{R}(g, (h, \alpha_h)) = (hg^{-1}, \alpha_h \circ (dR_g)_{hg^{-1}}).
\]

On the coordinates defined by the formula (3) these left actions are expressed as

\begin{align}
\tilde{L}_\lambda (g, (h, \alpha)) &= (\lambda \circ \tilde{L}_\lambda \circ \lambda^{-1})(h, \alpha) = (gh, \alpha) \quad \text{and} \\
\tilde{R}_\lambda (g, (h, \alpha)) &= (\lambda \circ \tilde{R}_\lambda \circ \lambda^{-1})(h, \alpha) = (hg^{-1}, \text{Ad}^*(g) (\alpha)).
\end{align}

If \( \theta_0 \) denotes the Liouville 1-form of \( T^* (G) \), then on the coordinates (3) it is given by \( \theta = (\lambda^{-1})^* \theta_0 \) as

\[
\theta_{g, \alpha}(v(g), \beta) = \alpha((dL_{g^{-1}})_g(v(g))),
\]
and the canonical symplectic form $\omega_0$ of $T^*(G)$ is $\omega = (\lambda^{-1})^*\omega_0 = -d\theta$ which is given explicitly as
\[
\omega_{(g,\alpha)}((v(g), \beta), (u(g), \gamma)) = \gamma((dL_{g^{-1}})_g(v(g))) - \beta((dL_{g^{-1}})_g(u(g))) + \alpha(((dL_{g^{-1}})_g(v(g))), (dL_{g^{-1}})_g(u(g))),
\]
where $(g, \alpha) \in G \times g^*$ and $(v(g), \beta), (u(g), \gamma) \in T_{(g,\alpha)}(G \times g^*) \approx T_gG \times g^*$. In these terms it is simple to check that (4) and (5) are symplectic actions of $G$ over $T^*(G) \approx G \times g^*$. Therefore, as $\omega_0$ (and $\omega$) is an exact symplectic form we have the following $\text{Ad}^*$-equivariant moment maps, see [1]. For the left action
\[
\mu_L : T^*(G) \to g^*, \quad \mu_L(g, \alpha_g) = \alpha_g \circ (dR_g)e,
\]
which on the coordinates of $G \times g^*$ is given by
\[
\mu_L^\lambda : G \times g^* \to g^*, \quad \mu_L^\lambda(g, \alpha) = (\mu_L \circ \lambda^\lambda)(\alpha) = \text{Ad}^*(g)(\alpha).
\]
Analogously, for the right action
\[
\mu_R : T^*(G) \to g^*, \quad \mu_R(g, \alpha_g) = -\alpha_g \circ (dL_g)e,
\]
and also
\[
\mu_R^\lambda : G \times g^* \to g^*, \quad \mu_R^\lambda(g, \alpha) = (\mu_L \circ \lambda^\lambda)(\alpha) = -\alpha.
\]
Let $(g, \langle \cdot, \cdot \rangle)$ be an orthogonal Lie group and $(g, \langle \cdot, \cdot \rangle_e)$ is its respective orthogonal Lie algebra. If $\mathfrak{z}(g)$ denotes the center of $\mathfrak{g}$ then we obtain

**Proposition 2** Let $(g, \alpha)$ be an element of $G \times g^*$. Then the orbit $G(g, \alpha)$ by the left action of $G$ on $G \times g^*$ is isotropic if and only if $\text{Ad}(g)(X\alpha) \in \mathfrak{z}(g)$. On the other hand, the orbit $G(g, \alpha)$ by the right action of $G$ on $G \times g^*$ is isotropic if and only if $X\alpha \in \mathfrak{z}(g)$.

**Proof** Let $(g, \alpha)$ be an element of $G \times g^*$ and $X\alpha$ the unique element of $g$ such that $\alpha(\cdot) = \langle X\alpha, \cdot \rangle_e$. As $\langle \cdot, \cdot \rangle$ is a bi-invariant pseudo-metric over $G$, then $\langle \text{Ad}(g)(X), \text{Ad}(g)(Y) \rangle_e = \langle X, Y \rangle_e$ for all $g \in G$ and $X, Y \in g$. As $\langle \cdot, \cdot \rangle_e$ is an invariant scalar product on $g$, by Proposition 1 the orbit $G(g, \alpha)$ by the left action of $G$ on $G \times g^*$ is isotropic if and only if
\[
\mu_L^\lambda(g, \alpha)(\langle X, Y \rangle) = 0 \text{ for all } X, Y \in g,
\]
but
\[
\mu_L^\lambda(g, \alpha)(\langle X, Y \rangle) = \text{Ad}^*(g)(\alpha)(\langle X, Y \rangle) = \alpha(\text{Ad}_{g^{-1}}(\langle X, Y \rangle)) = \langle X\alpha, \text{Ad}(g^{-1})(\langle X, Y \rangle) \rangle_e = \langle \text{Ad}(g)(X\alpha), [X, Y] \rangle_e.
\]
Therefore, the orbit $G(g, \alpha)$ is isotropic if and only if $\langle \text{Ad}(g)(X\alpha), [X, Y] \rangle_e = 0$ for all $X, Y \in g$. As $\langle \cdot, \cdot \rangle_e$ is nondegenerate the above happens if and only if $[\text{Ad}(g)(X\alpha), X] = 0$ for all $X \in g$, that is, $\text{Ad}(g)(X\alpha) \in \mathfrak{z}(g)$.

By a similar way, for the right action we have that
\[
\mu_R^\lambda(g, \alpha)(\langle X, Y \rangle) = -\alpha(\langle X, Y \rangle) = -\langle X\alpha, [X, Y] \rangle_e = -\langle X\alpha, X, Y \rangle_e.
\]
Thus, the orbit $G(g, \alpha)$ by the right action of $G$ on $G \times g^*$ is isotropic if and only if
\[
\langle X\alpha, X, Y \rangle_e = 0 \text{ for all } X, Y \in g,
\]
that is, $X\alpha \in \mathfrak{z}(g)$.

An immediate consequence of the previous Proposition is the following.

**Corollary 1** Let $G$ be a semisimple Lie group. The only orbits by the left and right actions of $G$ on $G \times g^*$ that are isotropic are of the form $G(g, 0)$ for all $g \in G$. Such orbits are Lagrangian and Hamiltonian isotopic to $G$. 

\[\square\]
Proof As $G$ is a semisimple Lie group, then $\mathfrak{g}$ is a semisimple Lie algebra. Therefore $\mathfrak{z}(\mathfrak{g}) = 0$. Thus, as $\text{Ad}(g) : \mathfrak{g} \to \mathfrak{g}$ is a linear isomorphism, the result follows immediately. \qed

4 Complex flag manifolds

Let $U$ be a compact semisimple Lie group with Lie algebra $\mathfrak{u}$. The adjoint orbits of $U$ in $\mathfrak{u}$ are the flags of the complex group $G$ which has the Lie algebra $\mathfrak{g} = \mathfrak{u}_\mathbb{C}$. If $\langle \cdot, \cdot \rangle$ denotes the Cartan–Killing form on $\mathfrak{u}$, the Kirillov–Kostant–Souriau (KKS) symplectic form is given by

$$\omega_x \left( \widetilde{X} (x), \widetilde{Y} (x) \right) = \langle x, [X, Y] \rangle$$

and the Hamiltonian vector field $\widetilde{X} = \text{ad} (X)$ corresponds to the Hamiltonian function $\mu(x) = \langle x, X \rangle$. Therefore, the moment map $\mu$ is the identity map, which is (evidently) equivariant (for more details see Appendix A).

Remark 3 Adjoint orbits are the only homogeneous spaces of a compact semi-simple group $U$ for which the action of $U$ is Hamiltonian. It is so because if the action of $U$ on a manifold $M$ is transitive and Hamiltonian, then the moment map $\mu$ is a covering space over the adjoint orbit (the complex flag). Since complex flags are simply connected, we conclude that $M$ is itself the adjoint orbit, see [14, p. 341].

Let $t = i\mathfrak{h}_\mathbb{R}$ be the Lie algebra of a maximal torus $T = \text{exp} t = \text{exp} U$. Here $\mathfrak{h}_\mathbb{R}$ is defined as follows. As the restriction of the Cartan–Killing form to $t$ is nondegenerate, if $\alpha$ is a root of $t$, there exists a unique $H_\alpha \in t$ such that $\alpha(\cdot) = \langle H_\alpha, \cdot \rangle$. The real subspace generated by $H_\alpha$, with $\alpha$ root of $t$, is denoted by $\mathfrak{h}_\mathbb{R}$. Then every adjoint orbit has the form $\text{Ad} (U) (iH_0)$ with $H_0 \in \mathfrak{h}_\mathbb{R}$, which may be chosen in the closure of the positive Weyl chamber. According to Sect. 2, a subgroup $L \subset U$ with Lie algebra $\mathfrak{l}$ admits an isotropic orbit in $\text{Ad} (U) (iH_0)$ if and only if $\text{Ad} (U) (iH_0) \cap (t')^\perp \neq \emptyset$, that is, if there exists $u \in U$ such that $\text{Ad} (u) (iH_0) \in (t')^\perp$. In such a case, the isotropic $L$-orbits are the orbits of $X \in \text{Ad} (U) (iH_0) \cap (t')^\perp$. Focusing on the subgroup instead, these observations can be reinterpreted as follows. If $L \subset U$ is a proper subgroup, then $L$ has an isotropic orbit in some flag $\text{Ad} (U) (iH_0)$ with $H_0 \neq 0$. The reason is that if $L$ is proper then $(t')^\perp \neq \{0\}$ and if $0 \neq X \in (t')^\perp$ then $X$ belongs to some orbit $\text{Ad} (U) (iH_0)$ with $H_0 \neq 0$. A particular case is the orbit of $L$ through the origin $x_0 = iH_0$. It is isotropic if and only if $t' \subset (iH_0)^\perp$. A sufficient condition is that $t' \subset t^\perp$.

Example 1 Choose $U = \text{SU} \,(n)$ and $L = \text{SO} \,(n)$. Take $t \subset \mathfrak{u} \,(n)$ as the subalgebra of diagonal matrices (tr = 0). A matrix in $\mathfrak{s} \mathfrak{o} \,(n)$ (real anti-symmetric matrix) has zeros in the diagonal which implies that $\mathfrak{s} \mathfrak{o} \,(n) \subset t^\perp$. Thus, the orbit of $\text{SO} \,(n)$ through the origin of any flag is isotropic. Since these orbits have half the dimension of the respective flags they are in fact Lagrangian orbits. This example gives an instance of the well-known construction of the immersion of real flags into complex flags.

Example 2 Example 1 may be generalized as follows: choose a Weyl basis of $\mathfrak{g} = \mathfrak{u}_\mathbb{C}$ containing $X_\alpha \in \mathfrak{g}_\alpha$ with $[X_\alpha, X_\beta] = m_{\alpha, \beta}X_{\alpha + \beta}$. For each root $\alpha$ define $A_\alpha = X_\alpha - X_{-\alpha}$. Then, $A_\alpha \in \mathfrak{u}$ (by the canonical construction of $\mathfrak{u}$) and

$$[A_\alpha, A_\beta] = m_{\alpha, \beta}A_{\alpha + \beta} + m_{-\alpha, \beta}A_{\alpha - \beta}.$$ 

Consequently, the subspace $\mathfrak{l}$ generated by the $A_\alpha$’s is a subalgebra perpendicular to the Cartan subalgebra. Therefore, for any Lie group $L$ with Lie algebra $\mathfrak{l}$, the orbits $Lx_0$ through
Example 3 Choose $H \in t$ and let $U_H = \{ u \in U : \text{Ad}(u)H = H \}$ be the centralizer of $H$ in $U$. Its Lie algebra is

$$u_H = \{ X \in u : [H, X] = 0 \} = t \oplus \sum_{\alpha(H) = 0} u_{\alpha},$$

where $u_{\alpha} = u \cap (g_{\alpha} \oplus g_{-\alpha})$. For example, if $H$ is regular, then $u_H = t$ and since $t$ is abelian, $t' = \{0\}$, and accordingly every flag (adjoint orbit) intersects $(t')^\perp = u$ and therefore $U_H$ has isotropic orbits in all flags. In fact, all these orbits of $T$ are isotropic since $t$ is Abelian.

Example 4 To obtain a more interesting example, take $u = su(3)$ and

$$H = i \text{diag}\{2, -1, -1\}.$$  

Then $u_H$ is given by matrices of the form

$$\begin{pmatrix} it & 0 \\ 0 & A \end{pmatrix} \quad (6)$$

with $A \in u(2)$ and $it + \text{tr}A = 0$. The derived algebra $u'_H$ is given by the matrices in $u_H$ such that $t = 0$, $A \in su(2)$ and $\dim \mathbb{R} u'_H = 3$. The orthogonal complement $(u'_H)^\perp$ is the 5-dimensional space of matrices

$$X = \begin{pmatrix} 2it & z & w \\ -\bar{z} & -it & 0 \\ -\bar{w} & 0 & -it \end{pmatrix} \quad t \in \mathbb{R}, z, w \in \mathbb{C}. \quad (7)$$

This orthogonal complement intersects the 3 types of adjoint orbits which give the flags $\mathbb{C}\mathbb{P}^2$, $\text{Gr}_2(3, \mathbb{C})$ and $\mathbb{F}(1, 2)$. In the case of $\mathbb{F}(1, 2)$ it is easy to find a matrix in $(u'_H)^\perp$ which is regular. For example, the matrix

$$\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in (u'_H)^\perp$$

which has eigenvalues $i, -i, 0$. This matrix is regular and its adjoint orbit is the maximal flag. For the cases of partial flags the calculations require a further details.

(1) The case of $\mathbb{C}\mathbb{P}^2$. Here we need to find a matrix in $(u'_H)^\perp$ which is conjugate to a matrix of the type $D = \text{diag}\{it2a, -ia, -ia\}$ with $a > 0$. Any matrix $X$ in (7) has trace 0 and its characteristic polynomial has the form $\lambda^3 + F\lambda + G$ with $F = 3t^2 + |z|^2 + |w|^2$ and $G = -\det X = it (2t^2 + |w|^2 + |z|^2)$. The characteristic polynomial of $D$ is $\lambda^3 + 3a^2\lambda + 2iAa^3$. The two matrices are conjugate if and only if their characteristic polynomials coincide since both are anti-Hermitian. This happens if and only if

$$2a^3 = t \left(2t^2 + |z|^2 + |w|^2\right)$$

$$3a^2 = 3t^2 + |z|^2 + |w|^2.$$

If $t = 0$, then $\det X = 0$ and the eigenvalues of $X$ are 0 and $\pm i \sqrt{A}$ (since $A > 0$). In such case, $X$ is regular and not conjugate to $D$. On the other hand, we can multiply both matrices
by a positive constant without affecting conjugation. Thus, it is enough to verify the existence of solutions for \( t = \pm 1 \). Setting \( t = 1 \), the previous equations become

\[
2a^3 = 2 + |z|^2 + |w|^2
\]
\[
3a^2 = 3 + |z|^2 + |w|^2
\]

which is equivalent to \( 2a^3 - 2 = 3a^2 - 3 = |w|^2 + |z|^2 \). The first equality gives the polynomial \( 2a^3 - 3a^2 + 1 = 0 \) which has roots 1 with multiplicity 2 and \(-\frac{1}{2}\). Thus, we must have \( a = 1 \). However, if \( a = 1 \) then \( 2a^3 - 2 = 0 \) which implies that \( z = w = 0 \). Hence the only possible solution is obtained by taking \( X = \text{diag}(2i, -i, -i) \). On the other hand, if \( t = -1 \) then the equations become

\[
2a^3 = -2 - |z|^2 - |w|^2
\]
\[
3a^2 = 3 + |z|^2 + |w|^2
\]

that is, \(-2a^3 - 2 = 3a^2 - 3 = |w|^2 + |z|^2\). The first equality gives the polynomial \( 2a^3 + 3a^2 - 1 = 0 \) which has roots \(-1\) with multiplicity 2 and \(\frac{1}{2}\). The only possible solution here is obtained by taking \( a = 1/2 \) which would give the contradiction \( |w|^2 + |z|^2 = -9/4 \). Therefore a conjugation does not exist. Summing up, the only possible isotropic orbit in the projective plane is the trivial one (reduced to a point) passing through \( H = \text{diag}(2i, -i, -i) \).

(ii) The case of \( \text{Gr}_2(3, \mathbb{C}) \). We need to find a matrix in (7) conjugate to \( \text{diag}(ia, ia, -2ia) \) with \( a > 0 \). The equations are similar to the ones just considered. The characteristic polynomial now becomes \( \lambda^3 + 3a^2\lambda - 2ia^3 \) and the equations required for existence of a conjugation are

\[
-2a^3 = t(2t^2 + |z|^2 + |w|^2)
\]
\[
3a^2 = 3t^2 + |z|^2 + |w|^2.
\]

Once again, it is enough to verify the cases when \( t = \pm 1 \). For \( t = 1 \) we have \(-2a^3 - 2 = 3a^2 - 3 = |z|^2 + |w|^2\). The first equality is the same one found in the second case of the previous example, and gives no solution. On the other hand, for \( t = -1 \) we have \(-2a^3 + 2 = 3a^2 - 3 = |z|^2 + |w|^2\). The polynomial is \( 2a^3 + 3a^2 - 5 = 0 \) whose roots are 1 and \( -\frac{5}{4} \pm \frac{1}{4}i \sqrt{15} \). Thus, we must have \( a = 1 \) and consequently \(|z|^2 + |w|^2 = 0\), that is, \( z = w = 0 \). Therefore the only solution is the orbit of \( H_1 = \text{diag}(i, i, -2i) \).

Opposite to the case of \( \mathbb{C}P^2 \), this isotropic orbit in \( \text{Gr}_2(3, \mathbb{C}) \) is not trivial. In fact, a generic matrix in \( u_H \) has the form

\[
Y = \begin{pmatrix} 2it & 0 & 0 \\ 0 & -it & z \\ 0 & \bar{z} & -it \end{pmatrix} \quad t \in \mathbb{R}, \ z \in \mathbb{C}.
\]

If \( H \) is the diagonal matrix in (6), then

\[
\text{ad} \ (Y) \ H = - [H, Y] = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -3iz \\ 0 & 3iz \bar{z} & 0 \end{pmatrix}.
\]

The tangent space to the orbit is generated by the latter matrices with \( z \) varying in \( \mathbb{C} \). Hence, the orbit has real dimension 2. Since \( \dim_{\mathbb{R}} \text{Gr}_2(3, \mathbb{C}) = 4 \), this isotropic orbit is in fact Lagrangian.

(iii) The case of \( \mathbb{F}(1, 2) \). To conclude this example we need to analyze the isotropic orbits of \( U_H \) in the maximal flag \( \mathbb{F} = \mathbb{F}(1, 2) \). To do so we need to choose a realization of \( \mathbb{F} \) as
adjoint orbit $\text{Ad} \ (U) \ (i \ H_0)$ and to analyze the orbits of the action of $U_H$ in the intersection of this orbit with the orthogonal complement $(u'_H)^\perp$, which is the 5 dimensional space formed by the matrices

$$X = \begin{pmatrix} 2it & z & w \\ -\bar{z} & -it & 0 \\ -\bar{w} & 0 & -it \end{pmatrix} \quad t \in \mathbb{R}, \ z, w \in \mathbb{C}. \quad (8)$$

Choose $iH_0 = \text{diag}(i, 0, -i)$, which is a regular element. Then, the intersection $(u'_H)^\perp \cap \text{Ad} \ (U) \ (i \ H_0)$ is formed by matrices of the form (8), whose eigenvalues are 0 and $\pm i$, since $\text{Ad} \ (U) \ (i \ H_0)$ is the set of matrices in $su \ (3)$ which have the same eigenvalues as $i \ H_0$. Since 0 is an eigenvalue of $X \in (u'_H)^\perp \cap \text{Ad} \ (U) \ (i \ H_0)$, we must have $X = -it \ (2t^2 + |w|^2 + |z|^2)$ which happens if and only if $t = 0$, given that $t \in \mathbb{R}$. Thus, the characteristic polynomial of $X$ becomes $\lambda^3 + (|z|^2 + |w|^2) \lambda$ implying that $|z|^2 + |w|^2 = 1$. This shows that $(u'_H)^\perp \cap \text{Ad} \ (U) \ (i \ H_0)$ is formed by the matrices satisfying

$$|z|^2 + |w|^2 = 1$$

describing the sphere $S^3$ in $\mathbb{C}^2$. On the other hand, the group $U_H$ is isomorphic to $U \ (2)$ and its action in $(u'_H)^\perp \cap \text{Ad} \ (U) \ (i \ H_0)$ is the same action of $U \ (2)$ in $S^3$. Therefore $U_H$ acts transitively in $(u'_H)^\perp \cap \text{Ad} \ (U) \ (i \ H_0)$ which is the unique isotropic orbit. In fact, Lagrangian since its dimension is $3 = \frac{1}{2} \dim_\mathbb{R} \mathbb{F}$.

Still in this case, it is interesting to regard the Lagrangian orbit ($\approx S^3$) more intrinsically, in terms of flags of subspaces $(V_1 \subset V_2)$ with $\dim V_i = i$. This is done observing that

$$k = \frac{\sqrt{2}}{2} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \quad (9)$$

and

$$\text{Ad} \ (k) \begin{pmatrix} i & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -i \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} i & 0 & 0 \\ 0 & 0 & -i \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} \in (u'_H)^\perp \cap \text{Ad} \ (U) \ (i \ H_0).$$

Since $x_0 = iH_0$ is the origin of the maximal flag, this conjugation means that the Lagrangian orbit of $U_H$ is the orbit through $kx_0$. In the other hand, looking at $\mathbb{F} \ (1, 2)$ as the set of flags $(V_1 \subset V_2)$ with $\dim V_i = i$ the origin is $f_0 = \langle \langle e_1 \rangle \subset \langle e_1, e_2 \rangle \rangle$ where $\{e_1, e_2, e_3\}$ is the canonical basis of $\mathbb{C}^3$. In this representation, the Lagrangian orbit of $U_H$ is the orbit through

$$kf_0 = \langle \langle ke_1 \rangle \subset \langle ke_1, ke_2 \rangle \rangle.$$

By the expression (9) of $k$ it follows that $ke_1 = \frac{\sqrt{2}}{2} \ (e_1 + e_3)$ and $ke_2 = e_2$ implying that

$$kf_0 = \langle \langle e_1 + e_3 \rangle \subset \langle e_1 + e_3, e_2 \rangle \rangle.$$  

Here, $U_H$ is the embedding of $U \ (2)$ in $SU \ (3)$ given by the matrices

$$\begin{pmatrix} z & 0 \\ 0 & g \end{pmatrix} \quad |z| = 1, \quad g \in U \ (2).$$
The orbit of this group through $kf_0$ is given by 

$$U_H k f_0 = \{(e_1 + u e_3) \subset (e_1 + u e_3, u e_2) : u \in SU (2)\}$$

where in this expression $SU (2)$ is seen as the unitary group of the subspace generated by $e_2, e_3$.

**Example 5** Let $Z_H$ be the complexification of $U_H$ and let $\mathfrak{z}_H$ be the complexification of $\mathfrak{u}_H$. Then $Z_H$ is the centralizer of $H$ in $G$, and $\mathfrak{z}_H$ is the centralizer of $H$ in $\mathfrak{g}$. As observed in [5], if $U_H x$ is a Lagrangian orbit (in a Kähler manifold $M$), then the orbit $Z_H x$ of the complexification is open. The reason is that the tangent space $T_x U_H x$ is Lagrangian and therefore the subspace $J (T_x U_H x)$ is the complement of $T_x U_H x$ in $T_x M$ where $J$ is a complex structure. The tangent space $T_x Z_H x$ contains $J (T_x U_H x)$ since, if $X \in \mathfrak{u}_H$ then $J (\tilde{X} (x)) = i \tilde{X} (x)$ and $i X \in \mathfrak{z}_H$.

In the previous examples, $U_H$ has a Lagrangian orbit in the maximal flag, and therefore $Z_H$ has an open orbit.

**Remark 4** – The examples using $SU (3)$ above show that in general the projection between 2 flags does not take isotropic orbits to isotropic orbits. In fact, the projection of the Lagrangian orbit in the maximal flag $F (1, 2)$ to the projective plane $\mathbb{C} \mathbb{P}^2$ is not isotropic, give that it does not project on the origin, which is the only isotropic orbit in $\mathbb{C} \mathbb{P}^2$.

– If a subgroup $U_H$ has a Lagrangian orbit in a flag $F_{\theta}$ then its complexification $Z_H$ has an open orbit in $F_{\theta}$. The converse is not true, since in the previous example $U_H$ only has a Lagrangian orbit in the maximal flag. Therefore $Z_H$ has an open orbit in the maximal flag, and so it follows that $Z_H$ has an open orbits in every flag.

– **ISOTROPY REPRESENTATION:** Let $U$ and $L \subset U$ be compact groups. The isotropic orbits of $L$ inside the adjoint orbits of $U$ are essentially given by the orbits of the isotropy representation of $L$ in $U / L$. For example, is $l' = l$ then $(l')^\perp = l^\perp$, which is identified to the tangent space at the origin of $U / L$. The adjoint representation of $L$ in $(l')^\perp = l^\perp$ is the isotropy representation. So that the orbits of $L$ in $\text{Ad} (l')^\perp = l^\perp$ which are the isotropic orbits in the adjoint orbits of $U$ are the same as the orbits by the isotropy representation. When $l' \neq l$, the situation does not change much. In fact, because of compactness $l$ is reductive and $l = \mathfrak{l} \oplus l'$ where $\mathfrak{l}$ is the centre of $l$. Moreover, $\mathfrak{l}$ is orthogonal to $l'$ (with respect to the Cartan–Killing form of $u$). Thus, $(l')^\perp = \mathfrak{l} \oplus l^\perp$. The adjoint orbits in $l^\perp$ are the orbits of the isotropy representation of $L$ on $U / L$. Now if $X = Z + Y \in \mathfrak{z}_H \oplus l^\perp$ then 

$$\text{Ad} (L) (Z + Y) = Z + \text{Ad} (L) Y$$

which means that the $L$-orbit of $X$ is the translation by $Y$ of the orbit of $Y \in l^\perp$, which is an orbit of the isotropy representation.

Summing up, the orbits of the isotropy representation of $L$ on $U / L$ are the same orbits that occur as isotropic orbits in the adjoint orbits of $U$ (the flags of $G = U_C$).

– Suppose that $G$ is a complex semisimple Lie group, not necessarily compact, with $U$ the real compact form of $G$. As the moment map $\mu$ is the identity of $\text{Ad} (G) (H)$, an orbit $L x$ is isotropic if and only if $x \in (l')^\perp$ where the orthogonal is taken with respect to the Cartan–Killing form $(\cdot , \cdot )$. Some examples of this case are:

1. If $L = U$ then $(u')^\perp = u^\perp = i u = s$ (the symmetric part of the Cartan decomposition $g = u \oplus i u$). Thus $U$ has isotropic orbits in $\text{Ad} (G) (H)$ if and only if $H \in s$. In such case there exists a unique isotropic orbit, which is the flag $\text{Ad} (G) (H) \cap s$. 

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2. If \( I \) is a Borel subalgebra (minimal parabolic) \( p = \mathfrak{h} \oplus n^+ \) then \( p' = n^+ \) and \((p')^\perp\) is the opposite Borel subalgebra \( p^- = \mathfrak{h} \oplus n^- \).

5 Products of flags

Assume here that \( U \) is a compact and connected semisimple Lie group with Lie algebra \( \mathfrak{u} \). Let \( G \) be a complex Lie group with Lie algebra \( \mathfrak{g} := \mathfrak{u}_C \). The goal here is to describe isotropic or Lagrangian orbits in a product of flags \( F_{\Theta_1} \times F_{\Theta_2} \) of \( G \). A cartesian product \( F_{\Theta_1} \times F_{\Theta_2} \) of 2 flags of \( G \) may be regarded as a flag of \( G \times G \) and therefore it may be seen as an adjoint orbit of the compact part \( U \times U \) as studied in Sect. 4. The group \( U \) itself (as does \( G \)) acts in the product \( F_{\Theta_1} \times F_{\Theta_2} \) by the diagonal action \( k (x, y) = (kx, ky), k \in U, (x, y) \in F_{\Theta_1} \times F_{\Theta_2} \), that is, \( U \) is seen as the subgroup \( U \times U \) given by the diagonal \( \Delta_U = \{(u, u) : u \in U\} \) and the diagonal action is the restriction of the action of \( U \times U \) to \( \Delta_U \).

Let us now describe the isotropic orbits by the diagonal action. The Lie algebra of \( U \times U \) is \( u \times u \) and the Lie algebra of the diagonal \( \Delta_U \) is the diagonal \( \Delta_u = \{(X, X) : X \in u\} \). Everything is constructed using Cartesian products: Cartan subalgebras \( t \times t \) and \( \mathfrak{h} \times \mathfrak{h} \), maximal torus \( T \times T \), Weyl chamber \( a^+ \times a^+ \) \((a = \mathfrak{h}_R)\) and the flag manifolds that are orbits \( \text{Ad} (U \times U)(iH_1, iH_2) \) with \((H_1, H_2) \in a^+ \times a^+ \) where

\[
\text{Ad} (U \times U)(iH_1, iH_2) = \text{Ad} (U) iH_1 \times \text{Ad} (U) iH_2.
\]

An adjoint orbit (product flag) \( \text{Ad} (U \times U)(iH_1, iH_2) \) admits an isotropic orbits by the diagonal action if and only if it intercepts the orthogonal \( \Delta_u^\perp \) of the derived algebra \( \Delta'_u \) of \( \Delta_u \approx u \) with \( \Delta'_u = \Delta_u \) since \( u \) is semisimple. The orthogonal subspace is given by

\[
\Delta_u^\perp = \{(X, -X) : X \in u\}
\]

since the Cartan–Killing form of \( u \times u \) is the sum of the forms in each coordinate. Thus, the adjoint orbit \( \text{Ad} (U \times U)(iH_1, iH_2) \) has an isotropic orbit by the diagonal action if and only if there exist \( u_1, u_2 \in U \) such that

\[
\text{Ad} (u_1)(iH_1) = -\text{Ad} (u_2)(iH_2). \quad (10)
\]

**Remark 5** Denote by \( U^p := U \times \cdots \times U \) the product of \( U \ p \)-times. If we consider an arbitrary product of flags \( F_{\Theta_1} \times \cdots \times F_{\Theta_p} \) which can be identified with an orbit \( \text{Ad} (U^p)(iH_1, \ldots, iH_p) \), then it is easy to show that this has an isotropic orbit by the diagonal action if and only if there exist \( u_1, \ldots, u_p \in U \) such that \( \sum_{j=1}^p \text{Ad}(u_j)(iH_j) = 0 \).

For the case of two flags, \( (10) \) implies that \( iH_2 = \text{Ad} \left( u_2^{-1} u_1 \right) (-iH_1) \) which means that \( iH_2 \) belongs to the adjoint orbit of \( -iH_1 \). This in turn is equivalent to the statement that the flags \( \mathcal{F}_{\Theta_1} = \text{Ad} (U)(iH_1) \) and \( \mathcal{F}_{\Theta_2} = \text{Ad} (U)(iH_2) \) are dual in the sense that \( \Theta_2 = \sigma \Theta_1 \) where \( \sigma \) is the symmetry of the Dynkin diagram given by \( \sigma = -w_0 \) and \( w_0 \) is the main involution (element of greatest length) of the Weyl group \( \mathbb{W} \). In fact, \( H_2 = \iota (H_1) = -w_0 (H_1) \) if and only if \( -H_1 \) belongs to the adjoint orbit of \( H_2 \). Summing up,

**Theorem 1** A product of flags \( \mathcal{F}_{\Theta_1} \times \mathcal{F}_{\Theta_2} \) admits an isotropic orbit by the diagonal action if and only if \( \mathcal{F}_{\Theta_2} \) is the dual flag \( \mathcal{F}_{\Theta_1^*} \) of \( \mathcal{F}_{\Theta_1} \).

Assuming that the flags are dual, that is, \( -iH_1 \in \text{Ad} (U)(iH_2) \), then the isotropic orbits by the diagonal action on \( \text{Ad} (U \times U)(iH_1, iH_2) \) are those that pass through elements of the type
$(X, -X)$ inside the adjoint orbit. Given an element $(X, -X) \in \text{Ad}(U)(iH_1) \times \text{Ad}(U)(iH_2)$ set $X = \text{Ad}(u)(iH_1)$ with $u \in U$. Then, $-X = \text{Ad}(u)(-iH_1)$, that is,

$$(X, -X) = (\text{Ad}(u)(iH_1), \text{Ad}(u)(-iH_1))$$

which means that $(X, -X)$ belongs to the diagonal orbit of $(iH_1, -iH_1)$ and reciprocally, the elements of the diagonal orbit of $(iH_1, -iH_1)$ have the form $(X, -X)$. In this case, there exists a unique isotropic orbit by the diagonal action. This isotropic orbit has the following geometric interpretation: the map $-\text{id}$ of $u$ takes the orbit $\text{Ad}(U)(iH)$ to the orbit $\text{Ad}(U)(-iH) = \text{Ad}(U)(i\sigma(H))$ defining a diffeomorphism between the flag $\mathbb{F}_\theta$ and its dual flag $\mathbb{F}_{\theta^*}$. Since the isotropic orbit of the diagonal action is given by

$$\{(X, -X) \in \text{Ad}(U)(iH) \times \text{Ad}(U)(-iH) : X \in \text{Ad}(U)(iH)\},$$

we conclude that this isotropic orbit is the graph of the diffeomorphism defined by the antipodal map $-\text{id}$. Such graph has dimension $\dim \mathbb{F}_\theta = \dim \mathbb{F}_{\theta^*}$. Therefore, the isotropic orbit is in fact Lagrangian.

In conclusion, we have obtained the following description of isotropic orbits.

**Proposition 3** There exists a unique isotropic orbit of the diagonal action of $U$ on $\mathbb{F}_\theta \times \mathbb{F}_{\theta^*}$. Such orbit is Lagrangian and is given as the graph of $-\text{id} : \text{Ad}(U)(iH) \rightarrow \text{Ad}(U)(i\sigma(H))$.

### 5.1 Shifted diagonals as Lagrangians

Variations of the diagonal action may be obtained by the action on a product of flags by subgroups of the type

$$\Delta^m = \{(u, mum^{-1}) \in U \times U : u \in U\}$$

for any given $m \in U$. The Lie algebra of $\Delta^m$ is

$$\Delta^m_u = \{(X, \text{Ad}(m)X) \in u \times u : X \in u\}.$$

This is isomorphic to $u$ and its orthogonal complement is given by

$$\left(\Delta^m_u\right)^\perp = \{(X, -\text{Ad}(m)X) \in u \times u : X \in u\}.$$

Thus, the diagonal action by the subgroup $\Delta^m$ has an isotropic orbit in the flag $\text{Ad}(U \times U)(iH_1, iH_2)$ if and only if such orbit contains the elements of the form $(X, -\text{Ad}(m)X)$. This happens if and only if there exist elements $u_1, u_2 \in U$ such that $\text{Ad}(u_1)(iH_1) = -\text{Ad}(m^{-1}u_2)(iH_2)$, that is, $-iH_1 = \text{Ad}(v)(iH_2)$ where $v = u_1^{-1}m^{-1}u_2$. Therefore, similarly to what happen for the diagonal action, such isotropic orbits only exist in the products $\mathbb{F}_\theta \times \mathbb{F}_{\theta^*}$ of dual flags, namely when $-iH_1$ belongs to the adjacent orbit of $iH_2$. Now, it is simple to see that the elements of the orbit of $(iH_1, -\text{Ad}(m)(iH_1))$ by the diagonal action $\Delta^m$ have the form $(X, -\text{Ad}(m)(X))$. Reciprocally, given an element $(X, -\text{Ad}(m)(X))$ in $\text{Ad}(U)(iH_1) \times \text{Ad}(U)(iH_2)$, if $X = \text{Ad}(u)(iH_1)$ with $u \in U$ then

$$(X, -\text{Ad}(m)(X)) = (\text{Ad}(u)(iH_1), -\text{Ad}(m)(\text{Ad}(u)(iH_1)))$$

$$= (\text{Ad}(u)(iH_1), \text{Ad}(mum^{-1})(-\text{Ad}(m)(iH_1))).$$

This means that $(X, -\text{Ad}(m)(X))$ belongs to the orbit of $(iH_1, -\text{Ad}(m)(iH_1))$ by the diagonal action of $\Delta^m$. Therefore, here there also exists a unique isotropic orbit by the diagonal action of the subgroup $\Delta^m$ and this is given by the graph of $-\text{Ad}(m)$.
Remark 6 A direct computation allows us to show that an arbitrary product of flags $\mathbb{F}_{\Theta_1} \times \cdots \times \mathbb{F}_{\Theta_{p+1}} = \Ad(U^{p+1}) (iH_1, \ldots, iH_{p+1})$ has an isotropic orbit by the diagonal action of the subgroup

$$\{(u, m_1 um_1^{-1}, \ldots, m_p um_p^{-1}) \in U^{p+1} : u \in U\},$$

if and only if there exist $u_1, \ldots, u_{p+1} \in U$ such that

$$\Ad(u_1)(iH_1) + \sum_{j=1}^{p} \Ad(m_j^{-1} u_{j+1})(iH_{j+1}) = 0.$$

Summing up,

Proposition 4 Inside the product $\mathbb{F}_{\Theta} \times \mathbb{F}_{\Theta^*}$, for each $m \in U$, there exists a unique isotropic orbit of the diagonal action by the subgroup $\Delta^m = \{(u, mum^{-1}) : u \in U\}$. Such an orbit is Lagrangian and it is given by the graph of the map $-\Ad(m) : \Ad(U)(iH) \to \Ad(U)(i\sigma(H))$.

Next we prove that each pair of Lagrangian orbits in $\mathbb{F}_{\Theta} \times \mathbb{F}_{\Theta^*}$ of the previous proposition are Hamiltonian isotopic.

Lemma 1 Let $m_1, m_2 \in U$ with $m_1 \neq m_2$. Denote by

$$L_1 = \Gamma \{-\Ad(m_1) : \Ad(U)(iH) \to \Ad(U)(i\sigma(H))\}$$

and

$$L_2 = \Gamma \{-\Ad(m_2) : \Ad(U)(iH) \to \Ad(U)(i\sigma(H))\},$$

the Lagrangian orbits in $M = \mathbb{F}_{\Theta} \times \mathbb{F}_{\Theta^*}$ by the diagonal action of the subgroups $\Delta^{m_1}$ and $\Delta^{m_2}$, respectively. Then $L_1$ is Hamiltonian isotopic to $L_2$.

Proof Since $U$ is compact and connected, the exponential map $e : u \to U$ is surjective (see [14, p. 243]). Therefore, for $m_2 m_1^{-1}$, there exists $X \in u$ such that $e^X = m_2 m_1^{-1}$. We define $\varphi : [0, 1] \times M \to M$ as

$$\varphi(t, \Ad(u_1)(iH), \Ad(u_2)(-iH)) = \Ad(u_1)(iH), \Ad(e^{tX} u_2)(-iH),$$

for all $t \in [0, 1]$. As the KKS symplectic form is $\Ad$-invariant and we consider in $M$ the product symplectic form ($\KKS$ symplectic form in each coordinate $\omega := p_1^* \omega_1 + p_2^* \omega_2$) we have that $\varphi_t$ is a symplectomorphism for all $t \in [0, 1]$. Moreover

$$\varphi(0, \Ad(u_1)(iH), \Ad(u_2)(-iH)) = (\Ad(u_1)(iH), \Ad(u_2)(-iH)),$$

that is, $\varphi_0 = \text{id}_M$, and for $(\Ad(u)(iH), -\Ad(m_1 u)(iH)) \in L_1$ we get

$$\varphi(1, \Ad(u)(iH), \Ad(m_1 u)(-iH))) = (\Ad(u)(iH), \Ad(e^X m_1 u)(-iH)) = (\Ad(u)(iH), -\Ad(m_2 u)(iH)) \in L_2.$$

That is, $\varphi(1, L_1) = L_2$. Thus, $\varphi$ is a symplectic isotopy which deforms the Lagrangian orbit $L_1$ to the Lagrangian orbit $L_2$.

Now, let us see that $\varphi$ is actually a Hamiltonian isotopy. Recall that each of our flag manifolds here is endowed with the $\KKS$ symplectic form and for each $X \in u$ the Hamiltonian
Infinitesimally tight Lagrangian orbits

vector field $\tilde{X} = \text{ad} \,(X)$ has Hamiltonian function $\tilde{\mu}(X) (x) = \langle x, X \rangle$. As $\text{Ad}(e^{tX}) = e^{t\text{ad}(X)}$, we have that

$$\frac{d}{dt} \psi_t (\text{Ad}(u_1) (iH), \text{Ad}(u_2) (-iH)) = \frac{d}{dt} (\text{Ad}(u_1) (iH), \text{Ad}(e^{tX} u_2) (-iH))$$

$$= \left(0, \frac{d}{dt} e^{t\text{ad}(X)} (\text{Ad}(u_2) (-iH)) \right)$$

$$= (0, \text{Ad}(X) \circ e^{t\text{ad}(X)} (\text{Ad}(u_2) (-iH)))$$

$$= (\tilde{0}, \tilde{X}) \left( \text{Ad}(u_1) (iH), \text{Ad}(e^{tX} u_2) (-iH) \right)$$

$$= (\tilde{0}, \tilde{X}) (\psi_t (\text{Ad}(u_1) (iH), \text{Ad}(u_2) (-iH))).$$

That is, $\frac{d}{dt} \psi_t = (\tilde{0}, \tilde{X}) \circ \psi_t$ for all $t \in [0, 1]$. Hence, as $(\tilde{0}, \tilde{X})$ is a Hamiltonian vector field on $M$ it follows that $\psi_t$ is a Hamiltonian isotopy. Thus, $L_1$ and $L_2$ are Hamiltonian isotopic.

**Theorem 2** All Lagrangian orbits in $\mathbb{F}_\Theta \times \mathbb{F}_{\Theta^*}$ of Proposition 4 belong the same Hamiltonian isotopy class.

6 **Infinitesimally tight immersions**

Oh in [15] studied tight Lagrangian submanifolds of $\mathbb{C}P^n$ and posed the question of classifying all possible tight Lagrangian submanifolds in Hermitian symmetric spaces. In particular, he asked whether the real forms are the only possible tight Lagrangian submanifolds. Later, C. Gorodski and F. Podestà classified those compact tight Lagrangian submanifolds which have the $\mathbb{Z}_2$-homology of a sphere in the case of irreducible compact homogeneous Kähler manifolds [9]. The concept of tightness has applications to the problem of Hamiltonian volume minimization. For instance, Kleiner and Oh showed that the standard $\mathbb{R}P^n$ inside $\mathbb{C}P^n$ is tight and has the least volume among its Hamiltonian deformations. Some features of intersection theory of Lagrangian submanifolds including some Lagrangian orbits in complex hyperquadrics can be found in [6,11].

Here we explore the concept of *infinitesimally tight* which we show to be equivalent to the notion of locally tight. We give examples of infinitesimally tight Lagrangians, and we prove that the Lagrangians orbits by the diagonal and shifted diagonal actions in the product of two flags, found in the previous sections, are infinitesimally tight. Let $G$ be a Lie group and $M$ a homogeneous space together with a $G$-invariant symplectic form $\omega$, that is, the action of $G$ on $(M, \omega)$ is symplectic.

**Definition 1** [15] A Lagrangian submanifold $L$ in $M$ is called *globally tight* (respectively *locally tight*) if for all $g \in G$ (respectively $g$ near the identity) such that $L$ intersects $g \,(L)$ transversally, we have

$$\# (L \cap g \,(L)) = \text{SB} (L, \mathbb{Z}_2)$$

where $\# (\cdot)$ is the number of intersection points, and $\text{SB} (L, \mathbb{Z}_2)$ is the sum of the $\mathbb{Z}_2$ Betti numbers, that is, the sums of the dimensions of the homologies of $L$ with $\mathbb{Z}_2$ coefficients.

**Remark 7** In [10] a Lagrangian submanifold $L$ of a Kähler manifold $M$ is called *globally tight* (or *locally tight*) if the conditions of Definition 1 are satisfied for isometries of $M$. The definition of [10] is directed to Hermitian symmetric spaces, this is why it refers to
isometries of $M$. Definition 1 adapts the concept of [10] and considers more general symmetric homogeneous spaces.

**Remark 8** The equality appearing in Definition 1 is the lower bound of the inequality of the Arnold–Givental conjecture, namely $(\#(\mathcal{L} \cap g(\mathcal{L})) \geq \text{SB}(\mathcal{L}, \mathbb{Z}_2))$. The conjecture has been proven in many cases, see for instance [16] and the survey [13].

Denote by $\tilde{X}$ the fundamental vector field associated to an element $X \in \mathfrak{g}$.

**Definition 2** Let $\mathcal{L}$ be a submanifold of $M$. An element $X \in \mathfrak{g}$ is called transversal to $\mathcal{L}$ if it satisfies the following 2 conditions

1. for any $x \in L$, if $\tilde{X}(x) \in T_x \mathcal{L}$, then $\tilde{X}(x) = 0$, and
2. the set

$$f_{\mathcal{L}}(X) = \{x \in L : 0 = \tilde{X}(x) \in T_x \mathcal{L}\}$$

is finite.

In other words, $\tilde{X}$ is only tangent to $\mathcal{L}$ at most at finitely many points where it vanishes. A Lagrangian submanifold $\mathcal{L}$ in a homogeneous space $M$ is called infinitesimally tight if the equality

$$\#(f_{\mathcal{L}}(X)) = \text{SB}(\mathcal{L}, \mathbb{Z}_2),$$

is satisfied for any $X \in \mathfrak{g}$ such that $\tilde{X}$ is transversal to $\mathcal{L}$.

**Example 6** Let $\mathcal{L}$ be a maximal circle in the sphere $S^2$ considered as a homogeneous manifold, then $\mathcal{L}$ is locally tight, globally tight and infinitesimally tight. This happens because the Hamiltonian vector fields on $S^2$ are generated by the moments of rotation around the $x, y$ and $z$ axis.

**Theorem 3** Let $G$ be a Lie group and $M$ a homogeneous space together with a $G$-invariant symplectic form $\omega$. Then a Lagrangian submanifold $\mathcal{L} \subset M$ is infinitesimally tight if and only if $\mathcal{L}$ is locally tight.

**Proof** Let $\iota : \mathcal{L} \to M$ be a Lagrangian submanifold of $M$. By Weinstein’s neighborhood Theorem [18], to decide whether $L$ is locally tight or infinitesimally tight, we may assume that $M = T^* L$. Let us denote by $V_x := \pi^{-1}(x)$ the (vertical) fibre of $\pi : T^* L \to L$ at $x$. Let $X \in \mathfrak{g}$ and $\tilde{X}$ the corresponding fundamental vector field. At each point $x \in \mathcal{L}$ we may write

$$\tilde{X}(x) = i(x) \oplus v(x),$$

where $i(x) = (i^* \tilde{X})(x) \in T_x \mathcal{L}$, $v(x) \in T V_x$, and $\oplus$ denotes metric orthogonal.

Assume $\tilde{X}$ is transversal to $\mathcal{L}$, then by definition, if $v(x) = 0$, (that is, if $\tilde{X}(x)$ is tangent to $\mathcal{L}$), then we also have that $i(x) = 0$. So that the zeros of $\tilde{X}|_\mathcal{L}$ and the zeros of $v$ coincide. Let $\epsilon << 0$. Since $g = \exp(\varepsilon Z)$ is close to the identity, then the submanifold $g(\mathcal{L})$ intersects $\mathcal{L}$ at the points where $v$ vanishes. But by the assumption of transversality these are precisely the points where $\tilde{X}$ vanishes. It is important to note that in this case the flow is determined by means of the exponential map of $G$. Now, assuming that $\mathcal{L}$ is infinitesimally tight, it then follows that $\#(\mathcal{L} \cap g(\mathcal{L})) = \#f_{\mathcal{L}}(X) = \text{SB}(\mathcal{L}, \mathbb{Z}_2)$ so that $\mathcal{L}$ is locally tight.

Conversely, assume that $\mathcal{L}$ is locally tight. For each $x \in \mathcal{L}$ we follow the integral curve of $\tilde{X}$ until time $\epsilon << 0$ and call the new point $\mathcal{L}'(x)$. Then for small $\epsilon$, the set of all such points $\mathcal{L}'(x)$ with $x \in \mathcal{L}$ forms a new Lagrangian $\mathcal{L}'$ (here, the flow of every fundamental vector field determines a symplectomorphism. In fact, when we assume the existence of a
moment map, as consequence of this symplectomorphism is actually Hamiltonian), then \( L' \) is in fact a section of \( T^* L \). Therefore, we have that \( L' \) intersects the zero section \( L \) precisely at the points \( x \in L \) where \( \tilde{X} \) vanishes, so that, assuming \( L \) is locally tight, we get 

\[
\# f_L (X) = \# (L \cap g(L)) = SB (L, \mathbb{Z}_2).
\]

Thus, \( L \) is infinitesimally tight. \( \Box \)

**Remark 9** It is simple see that the conclusion of Theorem 3 also holds true if \( M \) is a symplectic manifold and \( G \) acts on \( M \) by symplectomorphisms.

**Example 7** We consider the case of \( u(2) \subset su(3) \). Example 4 of Sect. 4 considers the subgroup \( U_H \approx U (2) \) inside \( U = SU (3) \) which has Lie algebra given by the matrices

\[
u_H = \left\{ \begin{pmatrix} it & 0 \\ 0 & A \end{pmatrix} : A \in u (2), it + trA = 0 \right\}.
\]

A Lagrangian orbit \( L \) of this group occurs only inside the maximal flag \( \mathbb{F} (1,2) \), in which case \( L = S^3 \) in the space of matrices \( (\approx \mathbb{C}^2) \), and we have

\[
(u_H)^\perp = \left\{ X_\beta = \begin{pmatrix} 0 & \beta \\ -\beta^T & 0 \end{pmatrix} : \beta = (z_1, z_2) \in \mathbb{C}^2 \right\}.
\]

More precisely, \( L = S^3 = \text{Ad} (U) (iH_0) \cap (u_H)^\perp \) where \( iH_0 = \text{diag}(i, 0, -i) \). If \( X_\beta, X_\gamma \in (u_H)^\perp \), then

\[
[X_\beta, X_\gamma] = \begin{pmatrix} -\beta \gamma^T + \gamma \beta^T & 0 \\ 0 & -\beta^T \gamma + \gamma^T \beta \end{pmatrix} = \begin{pmatrix} i \text{Im} \gamma \beta^T & 0 \\ 0 & -\beta^T \gamma + \gamma^T \beta \end{pmatrix} \in u_H.
\]

In particular, if \( x = X_\gamma \in L \) and \( X_\beta \in (u_H)^\perp \) then \( \tilde{X}_\beta (x) = \text{ad} (X_\beta) (x) \in u_H \) and \( \tilde{X}_\beta (x) = 0 \) if and only if \( \gamma \beta^T \) is real. This happens if and only if \( \gamma \) is a real multiple of \( \pm \beta \). Therefore, any \( 0 \neq X_\beta \in (u_H)^\perp \) is transversal to \( L \) (in the sense of Definition 2) and has singularities at the antipodal points \( \mathbb{R}X_\beta \cap S^3 \).

On the other hand, if \( Y \in u (2) = u_H \) then \( \tilde{Y} \) is tangent to \( L = S^3 \) and consequently is not transversal. Finally, if \( Z = X_\beta + Y \) with \( X_\beta \neq 0 \neq Y \in u (2) \) then \( \tilde{Z} (X_\beta) \neq T_x \mathbb{L} \) if \( x \) is not a singularity of \( \tilde{X}_\beta \) since in such a case \( \tilde{X}_\beta (x) \neq T_x \mathbb{L} \) is \( \tilde{Y} (x) \in T_x \mathbb{L} \). Thus, \( Z = X_\beta + Y \) is transversal to \( L \) if and only if the singularities of \( \tilde{X}_\beta \) are also singularities of \( \tilde{Y} \), which in turn occurs if and only if \( [Y, X_\beta] = 0 \), given that the singularities of \( X_\beta \) belong to \( \mathbb{R}X_\beta \). The condition \( [Y, X_\beta] = 0 \) still holds true when \( Y = 0 \), that is, \( Z = X_\beta \). Summing up, \( Z = X_\beta + Y \) is transversal if and only if \( X_\beta \neq 0 \) and \( [Y, X_\beta] = 0 \). Therefore, we conclude that transversal elements have precisely 2 singularities, thus in agreement with the sum of Betti numbers of \( L = S^3 \).

Hence, we have obtained

**Proposition 5** The Lagrangian orbit \( L = S^3 \) of \( U (2) \) in the flag \( \mathbb{F} (1,2) \) is infinitesimally tight.

**Example 8** (Diagonal action) In Sect. 5 we established that the set

\[
\mathcal{L} = \{ (X, -X) \in \text{Ad} (U) (iH) \times \text{Ad} (U) (-iH) : X \in \text{Ad} (U) (iH) \}
\]
inside the product of a flag $F_{\Theta} = \text{Ad} (U) (iH)$ by its dual $F_{\Theta^*} = \text{Ad} (U) (-iH)$ is the unique Lagrangian orbit of the diagonal action of $U$ in $F_{\Theta} \times F_{\Theta^*}$. This orbit is infinitesimally tight when $F_{\Theta} \times F_{\Theta^*} = \text{Ad} (U) (iH) \times \text{Ad} (U) (-iH)$ is regarded as an adjoint orbit of $U \times U$. To verify it, the first step is to find elements $(Y, Z) \in u \times u$ which are transversal to $L$ in the sense of Definition 2. The tangent space to $L$ at $(x, y) \in L$ is given by

$$T_{(x,y)}L = \left\{ (\tilde{A}(x), \tilde{A}(y)) : A \in u \right\}.$$ 

So that the tangent space $T_{(X,-X)}L$ of the orbit in the product is

$$T_{(X,-X)}L = \{ ([A, X], -[A, X]) : A \in u \}.$$ 

Accordingly, $T_{(X,-X)}L \in F_{\Theta} \times F_{\Theta^*}$ if and only if there exists $A \in u$ such that $[Y, X] = [A, X]$ and $[Z, X] = -[A, X]$, that is, $[Y, X] = -[Z, X]$, or alternatively, precisely when $X$ is a singularity of $Y + Z$ in the flag $F_{\Theta} = \text{Ad} (U) (iH)$. Therefore, the first condition for transversality says that $Y + Z$ has a finite number of singularities over the flag $F_{\Theta}$. The second condition requires $(\tilde{Y}(x), \tilde{Z}(y)) = 0$ when $(\tilde{Y}(x), \tilde{Z}(y)) \in T_{(X,-X)}L$, which means that $[Y, X] = -[Z, X] = 0$. Consequently, a pair $(Y, Z) \in u \times u$ is transversal to the Lagrangian orbit $L$ of the diagonal action on $F_{\Theta} \times F_{\Theta^*}$ if and only if $Y + Z$ has a finite number of singularities on $F_{\Theta^*}$, which are singularities of $Y$ as well as singularities of $Z$. Now, if $A \in u$ then $\tilde{A} = \text{ad} (A)$ on an adjoint orbit $\text{Ad} (U) (iH)$ is the Hamiltonian vector field of the height function $\mu(A)(x) = (A, x)$. Therefore, the singularities of $\tilde{A}$ coincide with the singularities of $\mu(A)$. There are finitely many singularities if and only if $A$ is a regular element. In such case, the singularities are parametrized by the Weyl group $W$. The number of singularities equals the cardinality of $W/W_{\Theta}$ (compare [7], Proposition 2.4).

Thus, we have obtained the following final characterization of the elements of $u \times u$ that are transversal to $L$.

**Proposition 6** Let $F_{\Theta} = \text{Ad} (U) (iH)$. A pair $Y, Z \in u \times u$ is transversal to a Lagrangian orbit $L$ of the diagonal action on $F_{\Theta} \times F_{\Theta^*}$ if and only if $Y + Z$ is regular in $u$ and both $Y$ and $Z$ belong to the intersection of the centralizers $w (iH)$, for all $w$ in the Weyl group $W$.

**Corollary 2** The Lagrangian orbit $L$ of the diagonal action is infinitesimally tight in $F_{\Theta} \times F_{\Theta^*}$.

**Proof** The Lagrangian orbit $L$ is diffeomorphic to $F_{\Theta}$ and the sum of the Betti numbers of $F_{\Theta}$ (also with $Z$ coefficients) is the number of fixed points (singularities) of a regular element, which is the cardinality of the quotient $W/W_{\Theta}$. $\square$

A similar argument gives us a more general collection of infinitesimally tight Lagrangians:

**Corollary 3** The Lagrangian orbits of type

$$\Gamma \{ -\text{Ad} (m) : \text{Ad} (U) (iH) \to \text{Ad} (U) (i\sigma (H)) \}$$

corresponding to the shifted diagonals $\Delta^m$ are infinitesimally tight in $F_{\Theta} \times F_{\Theta^*}$.

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Appendix A: KKS symplectic form on orthogonal groups

Transferring the KKS symplectic form to an adjoint orbit of a semisimple Lie group it is a well known construction, see for instance [2]. We describe the more general case of the KKS symplectic form on the adjoint orbit of an orthogonal Lie group. Let $G$ be a real connected Lie group of dimension $n$ and $\mathfrak{g}$ its Lie algebra. For every $g \in G$, we denote by $L_g : G \to G$ (resp. $R_g : G \to G$) the left (resp. right) multiplication of $g$ on $G$. Recall that a pseudo-metric on a smooth manifold is a symmetric and nondegenerate $(0, 2)$ tensor field with constant index. A pseudo-metric $\langle \cdot, \cdot \rangle$ over $G$ is called bi-invariant if $L_g$ and $R_g$ are isometries of $(G, \langle \cdot, \cdot \rangle)$ for all $g \in G$. If $\text{Ad} : G \to \text{GL}(\mathfrak{g})$ and $\text{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ denote the adjoint representations of $G$ and $\mathfrak{g}$, respectively, then to have a bi-invariant pseudo-metric $\langle \cdot, \cdot \rangle$ on $G$ is equivalent to having a scalar product $\langle \cdot, \cdot \rangle_0 : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ such that any of the following statements are satisfied

1. $\text{Ad}_g : \mathfrak{g} \to \mathfrak{g}$ is a linear isometry of $(\mathfrak{g}, \langle \cdot, \cdot \rangle_0)$ for all $g \in G$, that is
   \[ \langle \text{Ad}(g)X, \text{Ad}(g)Y \rangle_0 = \langle X, Y \rangle_0, \quad g \in G, \quad X, Y \in \mathfrak{g}. \] (11)

2. $\text{ad}(X) : \mathfrak{g} \to \mathfrak{g}$ is an infinitesimal isometry of $(\mathfrak{g}, \langle \cdot, \cdot \rangle_0)$ for all $X \in \mathfrak{g}$, that is
   \[ \langle \text{ad}(X)(Y), Z \rangle_0 + \langle Y, \text{ad}(X)Z \rangle_0 = 0, \quad X, Y, Z \in \mathfrak{g}. \] (12)

A scalar product which satisfies the identity (12) is called invariant.

**Definition 3** 1. The pair $(G, \langle \cdot, \cdot \rangle)$ is called an orthogonal Lie group if $\langle \cdot, \cdot \rangle$ is a bi-invariant pseudo-metric on $G$.

2. If $\mathfrak{g}$ is a finite-dimensional Lie algebra and $\langle \cdot, \cdot \rangle_0$ is an invariant scalar product on $\mathfrak{g}$ the pair $(\mathfrak{g}, \langle \cdot, \cdot \rangle_0)$ is called an orthogonal Lie algebra.

It is simple to see that there exists a bijective correspondence between simply connected orthogonal Lie groups and orthogonal Lie algebras. If $\langle \cdot, \cdot \rangle_0 : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ is an invariant scalar product, then

\[ \langle u(g), v(g) \rangle_0 := \langle (dL_{g^{-1}})_g(u(g)), (dL_{g^{-1}})_g(v(g)) \rangle_0 \quad g \in G, \]

is a bi-invariant pseudo-metric on $G$.

**Example 9** Examples of orthogonal Lie groups are the compact Lie groups, semisimple Lie groups, the cotangent bundle of a Lie group, and the $\lambda$-oscillator groups. For the last example see [12].

Recall that the tangent space to the adjoint orbit $\text{Ad}(G)(H)$ of an element $H \in \mathfrak{g}$ is given by $T_{X_0}(\text{Ad}(G)(H)) = \{ [X, X_0] : X \in \mathfrak{g} \}$ where $\tilde{X} = \text{ad}(X)$ is the fundamental vector field by the adjoint action associated to $X \in \mathfrak{g}$. On the other hand, if $\mathfrak{g}^*$ denotes the dual vector space of $\mathfrak{g}$ and $\text{Ad}^* : G \to \text{GL}(\mathfrak{g}^*)$ the coadjoint representation of $G$, the tangent space to the coadjoint orbit $\text{Ad}^*(G)(\alpha)$ of an element $\alpha \in \mathfrak{g}^*$ is $T_{\beta}(\text{Ad}^*(G)(\alpha)) = \{-\beta \circ \text{ad}(X) : X \in \mathfrak{g}\}$. Here $\tilde{\alpha}^* = \text{ad}^*(X)$ is the fundamental vector field by the coadjoint action of $X \in \mathfrak{g}$. Let $(G, \langle \cdot, \cdot \rangle)$ be an orthogonal Lie group and $(\mathfrak{g}, \langle \cdot, \cdot \rangle_0)$ its respective orthogonal Lie algebra. As $\langle \cdot, \cdot \rangle_0 : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ is nondegenerate, the map $\varphi : \mathfrak{g} \to \mathfrak{g}^*$ defined by $\varphi(X) = \langle X, \cdot \rangle_0$ is a linear isomorphism. If $\alpha \in \mathfrak{g}^*$, we denote by $X_\alpha$ the only $X_\alpha \in \mathfrak{g}$ such that $\alpha(\cdot) = \varphi(X_\alpha)$.

---

1 If $V$ is a finite-dimensional space a scalar product on $V$ is an application $\mu_0 : V \times V \to \mathbb{R}$ which is bilinear symmetric and nondegenerate.
Lemma 2 If \((G, \langle \cdot, \cdot \rangle)\) is an orthogonal Lie group, then \(\varphi : g \to g^*\) is equivariant with respect the adjoint and coadjoint actions. Consequently, the adjoint and coadjoint representations of \(g\) are isomorphic.

Proof The first claim is an immediate consequence of identity (11). If we put \(g = e^{iX}\) in the formula \(\text{Ad}^*(g) \circ \varphi = \varphi \circ \text{Ad}(g)\) and apply derivative at \(t = 0\) to both sides of the last formula we get that \(\text{ad}^*(X) \circ \varphi = \varphi \circ \text{ad}(X)\) for all \(X \in g\).

Assume \((G, \langle \cdot, \cdot \rangle)\) is an orthogonal Lie group. It is simple to see that Lemma 2 implies that \(\varphi^{-1}(\text{Ad}^*(G)(\alpha)) = \text{Ad}(G)(X_0)\) for all element \(\alpha\) of \(g^*\). Therefore, we have that

Proposition 7 If \((G, \langle \cdot, \cdot \rangle)\) is an orthogonal Lie group, \(\varphi : g \to g^*\) maps adjoint orbits diffeomorphically and \(G\)-equivariantly onto coadjoint orbits. Moreover, there exists a symplectic form \(\Omega\) on the adjoint orbit \(\text{Ad}(G)(X)\) such that the adjoint action restricted to \(\text{Ad}(G)(X)\) determines a symplectic action of \(G\).

Proof If we consider the restriction of the adjoint action and the coadjoint action of \(G\) on \(\text{Ad}(G)(X)\) and \(\text{Ad}^*(G)(\varphi(X))\), respectively, the first claim is clear. On the other hand, it is well known that for all \(\alpha \in g^*\), the coadjoint orbit \(\text{Ad}^*(G)(\alpha)\) is a symplectic manifold with the symplectic form given by

\[
\omega_\beta(\text{ad}^*(Y)(\beta), \text{ad}^*(Y)(\beta)) = \beta([X, Y]).
\]

Moreover, it holds that for all \(g \in G\), the map \(\text{Ad}^*(g)|_{\text{Ad}^*(G)(\alpha)}\) is a symplectomorphism of \((\text{Ad}^*(G)(\alpha), \omega)\). Therefore, the pullback of \(\omega\) by \(\varphi\) induces a symplectic form \(\Omega\) on the adjoint orbit \(\text{Ad}(G)(X)\) which satisfies that \(\text{Ad}(g)|_{\text{Ad}(G)(X)}\) is a symplectomorphism of \((\text{Ad}(G)(X_0), \Omega)\) for all \(g \in G\) since by Lemma 2 we have that

\[
(\text{Ad}(g))^* \Omega = (\text{Ad}(g))^* (\varphi^* \omega) = (\varphi \circ \text{Ad}(g))^* \omega = (\text{Ad}^*(g) \circ \varphi)^* \omega = \varphi^* \omega = \Omega.
\]

Explicitly, the symplectic form on \(\text{Ad}(G)(X)\) is given by

\[
\omega_{X_0}(\text{ad}(Y)(X_0), \text{ad}(Z)(X_0)) = \varphi(X_0)([Y, Z]) = [X_0, [X, Y]]_e.
\]

An immediate consequence is

Corollary 4 The map \(\mu : \text{Ad}(G)(X) \to g^*\) defined by \(\mu(X_0) = \varphi(X_0)\) defines an \(\text{Ad}^*\)-equivariant moment map for \((\text{Ad}(G)(X_0), \Omega)\).

Proof For all \(Y \in g\) and \(X_0 \in \text{Ad}(G)(X)\) we get

\[
\text{d}\hat{\mu}(Y)_{X_0}(\text{ad}(Z)(X_0)) = \left. \frac{d}{dt} \text{Ad} \left( e^{tZ} \right) (X_0) \right|_{t=0} = [Y, [Z, X_0]]_e = \Omega_{X_0}(\text{ad}(Y)(X_0), \text{ad}(Z)(X_0)).
\]

That is, \(\text{d}\hat{\mu}(Y) = \iota_Y \Omega\). Lemma 2 implies that this moment map is \(\text{Ad}^*\)-equivariant since

\[
\mu(\text{Ad}(g)(X_0)) = \varphi(\text{Ad}(g)(X_0)) = \text{Ad}^*(g)(\varphi(X_0)) = \text{Ad}^*(g)(\mu(X_0)).
\]
Appendix B: Open questions

In this appendix we establish some problems whose interest arises from the results proved in this paper and the motivation was expressed previously.

**Problem** Find a characterization of Lagrangian orbits in terms of the moment map $\mu$ (analogous to Proposition 1) in the case when $\mu$ is not equivariant for the adjoint representation, but only with respect to an affine representation.

**Problem** Find the flags where $L$ admits an isotropic orbit, or more specifically, a Lagrangian one. Equivalently, find the “types” of elements of $u$ in the orthogonal complement of $(t)^\perp$.

**Problem** Determine the pairs $H_1, H_2 \in a^+$ such that $U_{H_i}$ has isotropic or Lagrangian orbit in $F_{\mathfrak{so}_2}$. This problem was partially solved in [5], who classified the linear compact groups that have Lagrangian orbits for projective spaces (therefore not just subgroups that are centralizers of the tori). Beware of not jumping to the false conclusion of thinking that “because the flag is a projective submanifold then the classification given by [5], solves also the case of flags”. This does not solve the problem, since a Lagrangian submanifold of a projective submanifold is only isotropic in projective space, given that it has less than half the dimension.

**Problem** Problem B is likely to be related to the following question about semisimple algebras: let $\mathfrak{p}$ be a parabolic subalgebra. Determine the types of nilpotent orbits that intersect the nilradical $n$ of $\mathfrak{p}$.

**Problem** When an orbit $U_{H}x$ is Lagrangian its dimension is half of the dimension of the orbit $Z_{H}x$ which is open. A natural question is whether $Z_{H}x$ is the cotangent bundle of $U_{H}x$, that is, whether there exists a symplectic diffeomorphism between $Z_{H}x$ and $T^* (U_{H}x)$ where the cotangent bundle is considered with the canonical symplectic form. This problem is inspired in the general theorem of Weinstein [18] that identifies the cotangent bundle $T^* \mathcal{L}$ of a Lagrangian submanifold $\mathcal{L}$ with a tubular neighborhood of $\mathcal{L}$.

**Problem** An extension of Problem B is to ask when does it happen that an orbit $U_{H}x$ is isotropic but not Lagrangian. We may expect the following situation to hold in general: a) $Z_{H}x$ is a symplectic submanifold; b) $U_{H}x$ is a Lagrangian submanifold of $Z_{H}x$ and c) $Z_{H}x \approx T^* (U_{H}x)$.

**Problem** Generalize all results proved in the case of product of two flags for arbitrary products of flags.

**Problem** Generalize Example 7 for orbits of of the Grassmanians

$$SU(n)/S(U(k) \times U(n-k))$$

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