Mass formulae of 4-dimensional dilaton black holes

Tadashi Okai

Department of Physics, University of Tokyo
Bunkyo-ku, Tokyo 113, Japan

Abstract

Integral and differential mass formulae of 4-dimensional stationary and axisymmetric Einstein-Maxwell-dilaton systems are derived. The total mass (energy) of these systems are expressed in terms of other physical quantities such as electric charge of the black hole suitably modified due to the existence of the dilaton field. It is shown that when we vary slightly the fields (metric of the spacetime $g_{\mu \nu}$, $U(1)$–gauge potential $A_\mu$, and dilaton $\phi$) in such a way as they obey classical equations of motion, the variation of the dilaton does not contribute explicitly to the variation of the total mass, but contributes only through the variation of the electric charge of the black hole.
1 Introduction

One of the great achievements of black hole physics in late 1960’s and 1970’s, which have formed the standard picture of black holes\cite{1}, was the area theorem proved by Hawking. It has consequently stimulated the investigations on the close analogy between black hole mechanics and thermodynamics\cite{2} \cite{3}. They revealed the parallelism between the area of the horizon and the entropy, the surface gravity and the temperature. Furthermore, Hawking made it clear that the surface gravity of the horizon is proportional to the temperature of blackbody radiation of black holes, induced by their strong gravitational force\cite{4}.

The study of string theory, on the other hand, offered a new insight into gravity: the Einstein equation is a low-energy approximation in string theory\cite{5}. New types of actions including gravity were then considered in the study of low energy phenomenology of string theory and in the study of string compactification (including Kaluza-Klein’s)\cite{6}. In recent years, several stringy black hole solutions were discovered and their physical properties were examined\cite{7} \cite{8} \cite{9}.

In this article, we examine the relation between total mass and other physical quantities in the Einstein-Maxwell-dilaton black holes. The similarity of black hole mechanics and thermodynamics established in early 1970’s is re-investigated in the context of stringy black holes. The differential mass formula for Einstein-Maxwell theory was given by

$$\delta M = \Phi_H \delta Q_H + \Omega_H \delta J + T \delta A,$$

where $\Phi_H$, $Q_H$, $\Omega_H$, $J$, $T$, $A$ denote electric potential of electromagnetic gauge field measured at the horizon, electric charge of the black hole, angular velocity of the horizon, total angular momentum of the system, Hawking temperature, area of the horizon, respectively. (They will be precisely defined in subsequent sections.) When we look at the action of Einstein-Maxwell-dilaton theory

$$S = \int d^4x \sqrt{-g} (R - 2(\partial\phi)^2 - e^{-2\phi} F^2),$$

the existence of the dilaton field seems, at first sight, to add extra degrees of freedom to the variation of the total mass. We will show, however, that this does not happen and the effect of the dilation is absorbed into a suitable redefinition of $Q_H$. This result is consistent with the “no-hair conjecture”\cite{10} in stringy black holes.

The organisation of this article is the following. In section 2, we recall general setups for the description of stationary and axisymmetric spacetimes and derive the integral mass formula of Einstein-Maxwell-dilaton theory. In section 3, we obtain the differential mass formula.
The basic strategy is to use special properties of Killing vectors. Section 4 is devoted to conclusions and discussions.

2 Integral mass formula of dilaton black holes

We begin by the action (1). The equations of motion are

$$G_{\mu\nu} - T^{(F)}_{\mu\nu} - T^{(dil)}_{\mu\nu} = 0, \quad \nabla_\mu(e^{-2\phi}F^{\mu\nu}) = 0, \quad \Box\phi + \frac{1}{2}e^{-2\phi}F^2 = 0,$$

where we put

$$T^{(F)}_{\mu\nu} := e^{-2\phi}(2F_{\mu\alpha}F_\nu^{\alpha} - \frac{1}{2}F^2g_{\mu\nu}), \quad T^{(dil)}_{\mu\nu} := (2\partial_\mu\phi\partial_\nu\phi - (\partial\phi)^2g_{\mu\nu}).$$

We consider stationary and axisymmetric systems throughout this article. Furthermore, we restrict ourselves to the systems of “rotating bodies.” That is to say, the spacetime is invariant under the simultaneous inversion of time $t$ and azimuthal angle $\phi$. By the existence of a regular horizon and the gauge freedom of the metric, we may suppose, without loss of generality, that the metric be

$$ds^2 = -\sqrt{\Delta(r)f(r, \theta)} \left( \chi dt^2 - \frac{1}{\chi}(d\phi - \omega dt)^2 \right) + g(r, \theta) \left( \frac{dr^2}{\Delta(r)} + d\theta^2 \right),$$

where $f$, $g$, $\chi$, $\omega$ are some functions of $(r, \theta)$ and $\Delta$ of $r$ only (See, e.g., [11][12]). We do not need detailed information of $f$, $g$, $\chi$, $\omega$. Important is that the horizon is characterised by $\Delta(r) = 0$ and the determinant of the metric of the 2-dimensional space spanned by the vectors $\partial_t$ and $\partial_\phi$ is $g_{tt}g_{\phi\phi} - g_{t\phi}^2 = -f^2\Delta$. Therefore

$$g_{tt}g_{\phi\phi} - g_{t\phi}^2 \rightarrow 0$$
$$g_{rr} \rightarrow \infty$$

as one approaches the horizon, whereas $(g_{tt}g_{\phi\phi} - g_{t\phi}^2)g_{rr}$ remains finite. This property is basic in proving the lemma 3 in section 3.

Let $\Sigma$ be a 3-dimensional submanifold of the spacetime defined by a time-slice of $\{t = \text{constant}\}$. $\Sigma$ has its spatial infinity, which we denote symbolically by $\infty$, and is bounded by the outer horizon $H$:

$$\partial\Sigma = \{\infty\} \cup \{H\}.$$
Let us recall now several physical quantities needed to develop integral and differential mass formulae\cite{3}\cite{11}. Let
\[ k^\cdot := \left( \frac{\partial}{\partial t} \right), \quad m^\cdot := \left( \frac{\partial}{\partial \phi} \right) \]
be the time translational killing vector and axisymmetric Killing vector, respectively. The angular velocity of the black hole and the electric potential measured at the horizon are defined by
\[ \Omega_H := -g^\phi_{\phi} \quad \text{and} \quad \Phi_H := [l^\mu A_\mu]_H, \]
respectively.

Since we are now assuming the spacetime to be axisymmetric, so are the spatial infinity $\infty$ and the outer horizon $H$. This means that the axial Killing vector $m^\cdot$ is tangent to $\infty$, $H$, and $\Sigma$. Thus $m^\mu d\sigma_\mu = 0$, $m^\mu dS_{\mu\nu} = 0$ (both on $\infty$ and $H$), where $d\sigma_\mu$ and $dS_{\mu\nu}$ are the dual of volume elements of $\Sigma$ and $\partial \Sigma$, respectively.

Another Killing vector, which is null on $H$ and is normal to $H$, is defined as follows:
\[ l^\cdot := \left( \frac{\partial}{\partial t} \right) + \Omega_H \left( \frac{\partial}{\partial \phi} \right). \]
(Note that $l^\cdot$ is tangent to the 3-dimensional space that the 2-dimensional space $H$ sweeps with time evolution.) The other null vector orthogonal to $H$ is denoted by $n^\cdot$, which is normalised such that $n_\mu l^\mu = -1$. Then the dual of the surface element of $\partial \Sigma$ is expressed as
\[ dS_{\mu\nu} = \frac{1}{2} (l_\mu n_\nu - n_\mu l_\nu) dA = l_\mu n_\nu dA, \]
where $dA$ is the surface area element of $\partial \Sigma$. We use semicolon “;” in the meaning of the covariant differentiation $\nabla_\mu$ throughout this paper.

The total mass $M$ of the system, the total angular momentum $J$, and the total “modified” electric charge $\tilde{Q}$ are defined by the surface integral at spatial infinity:
\[ M := -\frac{1}{4\pi} \int_{\infty}^H k^\mu_{\alpha} dS_{\mu\nu}, \quad J := -\frac{1}{8\pi} \int_{\infty}^H m^\mu_{\nu} dS_{\mu\nu}, \quad \tilde{Q} := -\frac{1}{4\pi} \int_{\infty} e^{-2\phi} F^{\mu\nu} dS_{\mu\nu}. \]
These quantities are expressed in other ways:
\[ M = \frac{1}{4\pi} \int_{H} k^{\alpha\beta} dS_{\alpha\beta} - \frac{1}{4\pi} \int (T^{\alpha\beta} - \frac{1}{2} T g^{\alpha\beta}) k_\beta d\sigma_\alpha. \quad (2) \]
Here we have used Stokes’ theorem
\[ \int V^\mu_{\nu\cdot} d\sigma_\mu = \left( \int_{\infty} + \int_{H} \right) V^\mu_{\nu\cdot} dS_{\mu\nu} \quad \text{for all} \ V^\mu_{\nu\cdot} = V^{[\mu\nu]} \]
and the equation of motion $R_{\mu\nu} = T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T$. Note that any Killing vector $\xi^\mu$ satisfies the equality $\xi^\mu \gamma_\alpha = -R^\mu \gamma_\nu \xi^\nu$. Similarly, total angular momentum and modified electric charge are rewritten as

$$J = \frac{1}{8\pi} \int_H m^{\mu\nu} dS_{\mu\nu} + \frac{1}{8\pi} \int T^{(F)}_{\mu\nu} m^\mu d\sigma^\nu =: J_H + J_F,$$

$$\tilde{Q} = \frac{1}{4\pi} \int_H e^{-2\phi} F^{\mu\nu} dS_{\mu\nu} =: \tilde{Q}_H.$$

We naturally assume the stationarity and axisymmetry of dilaton and electromagnetic field as well as the metric. Thus

$$(\mathcal{L}_k g)_{\mu\nu} = (\mathcal{L}_k \phi) = (\mathcal{L}_m g)_{\mu\nu} = (\mathcal{L}_m \phi) = (\mathcal{L}_m A)_{\mu} = 0,$$

where $\mathcal{L}$ denotes the Lie derivative. Therefore

$$T^{(dil)}_{\mu\nu} m^\mu d\sigma^\nu = \left(2\phi_{;\mu} \phi_{;\nu} - (\partial \phi)^2 g_{\mu\nu}\right) m^\mu d\sigma^\nu$$

vanishes. Thus there is no contribution of dilaton to the total angular momentum. And we note here that surface gravity $\kappa$ and electric potential $\Phi_H$ are constants on the horizon if the dominant energy condition of the energy momentum tensor is satisfied. It is naturally satisfied in our present case. We do no explain details of these quantities. The reader is referred to [3][11] for the discussion of their physical meaning and property.

First, the following lemma is a starting point of the integral mass formula.

**Lemma 1**

Let $A$ be the area of the event horizon of the black hole and $T := \kappa/(8\pi)$ be the Hawking temperature of the horizon. Then the total mass (energy) of the system is given by

$$\frac{M}{2} = \int T_{\alpha\beta} k^\alpha d\sigma^\beta + \frac{1}{16\pi} \int R(k^\mu d\sigma_{\mu}) + \Omega_H J_H + TA.$$

**Proof**

Using Stokes’ theorem,

$$\frac{M}{2} = -\frac{1}{8\pi} \int_\infty k^{\alpha\beta} dS_{\alpha\beta} = -\frac{1}{8\pi} \int k^{\alpha\beta ;\beta} d\sigma_{\alpha} + \frac{1}{8\pi} \int k^{\alpha\beta} dS_{\alpha\beta}.$$

Next, using the property of Killing vectors $k^{\alpha ;\beta} = -R^{\alpha \beta}$,

$$\frac{M}{2} = \frac{1}{8\pi} \int R^{\alpha \beta} k^{\alpha} d\sigma_{\beta} + \frac{1}{8\pi} \int (l^{\alpha\beta} - \Omega_H m^{\alpha\beta}) dS_{\alpha\beta}$$

$$= \frac{1}{8\pi} \int T_{\alpha\beta} k^\alpha d\sigma^\beta + \frac{1}{16\pi} \int R(k^\alpha d\sigma_{\alpha}) + TA + \Omega_H J_H. \quad \text{Proof ends.}$$
This lemma is followed by the proposition 1, which states the integral mass formula of Einstein-Maxwell-dilaton theory.

**proposition 1**

Under these assumptions, total mass of the system $M$ is rephrased as

$$
\frac{M}{2} = TA + \Omega_H (J_H + J_F) + \bar{Q}_H \Phi_H + \frac{1}{16\pi} \int (R - 2(\partial \phi)^2 - e^{-2\phi} F^2)(k^\mu d\sigma_\mu).
$$

**proof**

Our claim is

$$
\frac{1}{8\pi} \int T_{\alpha \beta} k^\alpha d\sigma^\beta = \Phi_H \bar{Q}_H + \Omega_H J_F - \frac{1}{16\pi} \int (2(\partial \phi)^2 + e^{-2\phi} F^2)(kd\sigma).
$$

First,

$$
\frac{1}{8\pi} \int T_{\alpha \beta} k^\alpha d\sigma^\beta = \frac{1}{8\pi} \int e^{-2\phi} (2F_{\alpha \mu} F_\beta^\mu - \frac{1}{2} F^2 g_{\mu \nu}) (l^\alpha - \Omega_H m^\alpha) d\sigma^\beta \\
+ \frac{1}{8\pi} \int (2\phi_{\alpha \beta} - (\partial \phi)^2 g_{\mu \nu}) k^\alpha d\sigma^\beta \\
= \frac{1}{4\pi} \int e^{-2\phi} F_{\alpha \mu} F_\beta^\mu l^\alpha d\sigma^\beta + \Omega_H J_F \\
- \frac{1}{16\pi} \int (2(\partial \phi)^2 + e^{-2\phi} F^2)(kd\sigma).
$$

Thus, if we prove

$$
\frac{1}{4\pi} \int e^{-2\phi} F_{\alpha \mu} F_\beta^\mu l^\alpha d\sigma^\beta = \Phi_H \bar{Q}_H,
$$

then the proof of the proposition will be completed. From the Killing symmetry of $A_\mu$, we have $F_{\alpha \mu} F_\beta^\mu l^\alpha = l^\alpha (A_{\mu,\alpha} - A_{\alpha,\mu}) F_\beta^\mu = -(l^\alpha A_\alpha)_{,\mu} F_\beta^\mu$. Therefore, using the equation of motion of $A_\mu$,

$$
\frac{1}{4\pi} \int e^{-2\phi} F_{\alpha \mu} F_\beta^\mu l^\alpha d\sigma^\beta = \frac{1}{4\pi} \int e^{-2\phi} (l^\alpha A_\alpha)_{,\mu} F^\beta \mu d\sigma^\beta \\
= \frac{1}{4\pi} \int (e^{-2\phi} (l^\alpha A_\alpha)) F^\beta \mu d\sigma^\beta \\
= \frac{1}{4\pi} \int_\Sigma (l^\alpha A_\alpha) F^\beta \mu dS_{\beta \mu} \\
= \Phi_H \bar{Q}_H. \quad \text{Proof ends.}
$$

Let us check that the electric potential is constant on $H$ in the Einstein-Maxwell-dilaton theory as well as in the Einstein-Maxwell. Actually, $R_{\mu \nu} l^\mu l^\nu = 0$ holds on $H$. Using the equation of motion, we get $(T^{(F)}_{\mu \nu} + T^{(\text{dil})}_{\mu \nu}) l^\mu l^\nu = 0$. Now

$$
T^{(\text{dil})}_{\mu \nu} l^\mu l^\nu = (2\partial_\mu \phi \partial_\nu \phi - g_{\mu \nu} (\partial \phi)^2) l^\mu l^\nu = 0
$$
is easily seen from the Killing symmetry. Thus $T^{(F)}_{\mu\nu}l^\mu l^\nu = 0$. The rest of the proof is exactly the same as that of the Einstein-Maxwell theory. Therefore $\Phi_H$ is constant even when dilaton is coupled.

We note here that $J_F$ is rewritten, by using $F_{\alpha\mu}F_{\beta}^{\mu}m^\alpha = m^\alpha(A_{\mu;\alpha} - A_{\alpha;\mu})F_{\beta}^{\mu} = -(m^\alpha A_{\alpha})_{;\mu}F_{\beta}^{\mu}$, in the following way:

$$J_F = \frac{1}{4\pi} \int_H e^{-2\phi}(m^\mu A_\mu)F^{\alpha\beta} dS_{\alpha\beta}.$$  

### 3 Differential mass formula

In this section, we carry out the variational calculation and show the differential mass formula, which is stated in the proposition 2. When we vary black hole solutions, we preserve the horizon, stationarity and axisymmetry of the black hole:

1) Since we assume the spacetime be stationary and axisymmetric, we should preserve these symmetries before and after the variation.

2) There is a freedom of the general coordinate transformation. Using this freedom, we can require that the horizon $H$ itself be invariant before and after the variation and that the null vector $l$ remain normal to $H$.

Thus, $\delta k^\mu = \delta m^\mu = 0$, $\delta l^\mu = \delta \Omega_H m^\mu$, and $(\delta l)^{[\mu} l^{\nu]} = 0$ holds. These conditions are important in order to prove the lemma 2.

**proposition 2**

$$\delta M = T \delta A + \Omega_H(\delta J_H + \delta J_F) + \Phi_H \delta \tilde{Q}_H. \quad (3)$$

We provide some lemmas below in order to prove the proposition 2.

**lemma 2**

$$\delta \left( \frac{1}{16\pi} \int R(k^\mu d\sigma_\mu) \right) = -\frac{1}{16\pi} \int G^{\mu\nu} \delta g_{\mu\nu}(k^\alpha d\sigma_\alpha) - \frac{1}{4\pi} \left( \int_{\infty} + \int_H \right) \delta g_{\mu[;\beta]} k^\alpha dS_{\alpha\beta}$$

$$= -\frac{1}{16\pi} \int G^{\mu\nu} \delta g_{\mu\nu}(k^\alpha d\sigma_\alpha) - \frac{1}{2} \delta T - A\delta T - J_H \delta \Omega_H. \quad (4)$$

The proof of the lemma 2 is given in [3].
lemma 3
When the Einstein-Maxwell-dilaton system is stationary and axisymmetric, and the spacetime has a regular horizon, the following equality holds:
\[ \int_H l^\mu \delta A_\mu F^{\alpha\beta} dS_{\alpha\beta} = -2 \int_H \delta A_\mu F^{\mu\alpha} l^\beta dS_{\alpha\beta}. \]

proof of lemma 3
According to the assumption of the symmetry, we can assume without loss of generality that
\[ A = A_\mu dx^\mu = A_t dt + A_\phi d\phi \]
and that the dual of the surface element \( dS_{\mu\nu} \) has the only component \( dS_{tr} \). Then,
\[ \delta A_\mu F^{\mu\alpha} l^\beta dS_{\alpha\beta} = (\delta A_t F^{\phi t} l^r - \delta A_t F^{\phi r} l^t) dS_{tr} \]
and
\[ \delta A_\mu l^\mu F^{\alpha\beta} dS_{\alpha\beta} = 2(\delta A_t + \Omega H \delta A_\phi) F^{tr} dS_{tr}. \]
Since we have assumed the spacetime to have regular horizons, \(-g_{tt} g_{\phi\phi} + (g_{t\phi})^2\) goes to zero and \(g_{rr}\) goes to infinity with \([-g_{tt} g_{\phi\phi} + (g_{t\phi})^2]/g_{rr}\) remaining finite as we approach the horizon. Therefore we can calculate on the horizon
\[ g^{rr} \Omega H g^{tt} = -g^{rr} g_{\phi\phi} \cdot \frac{-g_{\phi\phi}}{g_{tt} g_{\phi\phi} - (g_{t\phi})^2} = \frac{-g^{rr} g_{t\phi}}{g_{tt} g_{\phi\phi} - (g_{t\phi})^2} = g^{rr} g^{t\phi}, \]
\[ g^{rr} \Omega H g^{\phi t} = -g^{rr} g_{\phi\phi} \cdot \frac{-g_{\phi\phi}}{g_{tt} g_{\phi\phi} - (g_{t\phi})^2} = \frac{-g^{rr} g_{tt}}{g_{tt} g_{\phi\phi} - (g_{t\phi})^2} = g^{rr} g^{\phi\phi}. \]
Thus we get
\[ \left[ \Omega H F^{tr} \right]_H = \Omega_H \left[ g^{rr} (g^{tt} F_{tr} + g^{\phi t} F_{\phi r}) \right]_H \]
\[ = \left[ g^{rr} (g^{\phi t} F_{tr} - g^{\phi\phi} F_{\phi r}) \right]_H = \left[ F^{\phi r} \right]_H. \]
Therefore
\[ \delta A_\mu l^\mu F^{\alpha\beta} dS_{\alpha\beta} = 2(\delta A_t + \Omega H \delta A_\phi) F^{tr} dS_{tr} = 2(\delta A_t F^{tr} + \delta A_\phi F^{\phi r}) dS_{tr} \]
\[ = -2 \delta A_\mu F^{\mu\alpha} l^\beta dS_{\alpha\beta} \]
holds on the horizon. This proves the lemma 3.

lemma 4
\[ \delta \left( -\frac{1}{16\pi} \int e^{-2\phi} F^2 (k^\mu d\sigma_\mu) \right) \]
\[ = -\frac{1}{16\pi} \int \left( e^{-2\phi} \left( 2 F^{\alpha\mu} F_{\alpha}{}^\nu - \frac{1}{2} F^2 g^{\mu\nu} \right) \right) \delta g_{\mu\nu} (k^\alpha d\sigma_\alpha) \]
\[ + \frac{1}{8\pi} \int \delta \phi e^{-2\phi} F^2 (k^\mu d\sigma_\mu) - \tilde{Q}_H \delta \Phi_H - J_F \delta \Omega_H. \]
proof of lemma 4
Let $\delta_g := g_{\mu \nu}$-variation, $\delta_A := A_{\mu}$-variation, and $\delta_\phi := \phi$-variation. Then
\[
\delta_g \left( -\frac{1}{16\pi} \int e^{-2\phi} F^2(k^\mu d\sigma_\mu) \right) = -\frac{1}{16\pi} \int e^{-2\phi} \left( 2 F^{\mu \alpha} F_{\alpha} - \frac{1}{2} F^2 g_{\mu \nu} \right) \delta g_{\mu \nu} (k^\beta d\sigma_\beta),
\]
\[
\delta_\phi \left( -\frac{1}{16\pi} \int e^{-2\phi} F^2(k^\mu d\sigma_\mu) \right) = -\frac{1}{8\pi} \int \delta \phi e^{-2\phi} F^2(k^\mu d\sigma_\mu)
\]
is easily obtained. The proof of the lemma is completed if we prove
\[
(\#) := \delta_A \left( -\frac{1}{16\pi} \int e^{-2\phi} F^2(k^\mu d\sigma_\mu) \right) = -\bar{Q}_H \delta \Phi_H - J_F \delta \Omega_H.
\]
From the equation of motion of $A_{\mu}$ and the Killing symmetry of the system, we have
\[
e^{-2\phi} F^{\alpha \beta \mu} = (e^{-2\phi} F^{\alpha \beta \delta} A_{\beta} l^\mu)_{,\alpha} - e^{-2\phi} F^{\alpha \beta \delta} A_{\beta} l^\mu_{,\alpha}
\]
\[
= (e^{-2\phi} F^{\alpha \beta \delta} A_{\beta} l^\mu)_{,\alpha} - l^\alpha (e^{-2\phi} F^{\mu \beta \delta} A_{\beta})_{,\alpha}
\]
\[
= (e^{-2\phi} F^{\alpha \beta \delta} A_{\beta} l^\mu)_{,\alpha} - (e^{-2\phi} F^{\mu \beta \delta} A_{\beta} l^\alpha)_{,\alpha}.
\]
Thus, using this formula,
\[
(\#) = -\frac{1}{4\pi} \int e^{-2\phi} F^{\alpha \beta \delta} A_{\beta} l^\mu dS_{\mu \alpha}
\]
\[
= \frac{1}{4\pi} \int e^{-2\phi} F^{\alpha \beta \delta} A_{\mu} l^\mu dS_{\alpha \beta},
\]
here we have used the lemma 3. Applying
\[
[l^\mu \delta A_\mu]_H = \delta \Phi_H - (m^\mu A_\mu) \delta \Omega_H \quad \text{and} \quad J_F = \frac{1}{4\pi} \int_H e^{-2\phi} (m^\mu A_\mu) F^{\alpha \beta} dS_{\alpha \beta}
\]
to the above, we get
\[
(\#) = -\bar{Q}_H \delta \Phi_H - J_F \delta \Omega_H.
\]
We have completed the proof of lemma 4.

lemma 5
\[
\delta \left( -\frac{1}{16\pi} \int 2(\partial \phi)^2 (k d\sigma) \right) = -\frac{1}{16\pi} \int (-2 \phi^{,\mu} \phi^{,\nu} + (\partial \phi)^2 g^{\mu \nu}) \delta g_{\mu \nu} (k d\sigma) + \frac{1}{4\pi} \int \delta \phi (\Box \phi)(k d\sigma).
\]

proof of lemma 5
\[
\delta \left( -\frac{1}{16\pi} \int 2(\partial \phi)^2 (k d\sigma) \right) = (\delta_g + \delta_\phi) \left( -\frac{1}{16\pi} \int 2(\partial \phi)^2 (k d\sigma) \right)
\]
\[
= -\frac{1}{16\pi} \int (\delta_g + \delta_\phi)(\phi^{,\mu} \phi^{,\nu} g_{\mu \nu} (k d\sigma) + \frac{1}{4\pi} \int (\delta_\phi)(\phi^{,\mu} \phi^{,\nu} (k d\sigma).
\]
Now we can use the Killing symmetry and we get

\[(\delta \phi)_{,\mu} \phi^{\mu} k^\alpha = (\delta \phi \phi^{\mu} k^\alpha)_{,\mu} - \delta \phi (\square \phi) k^\alpha - \delta \phi \phi^{\mu} k^\alpha \; ; \mu \]

\[= (\delta \phi \phi^{\mu} k^\alpha)_{,\mu} - \delta \phi (\square \phi) k^\alpha - k^\mu (\delta \phi \phi^{\alpha})_{, \alpha} \]

\[= (\delta \phi \phi^{\mu} k^\alpha)_{,\mu} - (\delta \phi \phi^{\alpha} k^\mu)_{,\mu} - \delta \phi (\square \phi) k^\alpha.\]

Plugging this equality into eq.\((\ref{eq:7})\), we obtain

\[-\frac{1}{4 \pi} \int (\delta \phi)_{,\mu} \phi^{\mu} (k d\sigma) = -\frac{1}{2 \pi} \int (\delta \phi) \phi^{\mu} k^\alpha dS_{\alpha \mu} + \frac{1}{4 \pi} \int \delta \phi (\square \phi) (k d\sigma) \]

\[= -\frac{1}{4 \pi} \int (\delta \phi) \phi^{\mu} k^\alpha (l_{\mu} n_{\nu} - n_{\mu} l_{\nu}) dA + \frac{1}{4 \pi} \int \delta \phi (\square \phi) (k d\sigma) \]

\[= \frac{1}{4 \pi} \int \delta \phi (\square \phi) (k d\sigma).\]

This proves the lemma 5.

Putting the eq.s\((\ref{eq:4}), (\ref{eq:5}),\) and \((\ref{eq:6})\) together, we obtain the differential mass formula \((\ref{eq:DMMF})\).

4 Conclusion and discussion

We presented the integral and differential mass formulae of Einstein-Maxwell-dilaton system. Electric charge usually defined in Einstein-Maxwell system \(Q_H := \frac{1}{4 \pi} \int_H F_{\mu \nu} dS_{\mu \nu}\) is now replaced by \(\tilde{Q}_H\). This is quite reasonable when we look at the equation of motion of the electromagnetic field. Interesting is that the only term, in the differential mass formula, explicitly related to the variation of dilaton is \(\delta \tilde{Q}_H\). Dilaton in the Einstein-Maxwell-dilaton theory is an extra degree of freedom in the action, compared with the Einstein-Maxwell theory. It does not give, however, any further contribution to the globally defined quantities such as mass, charge, or angular momentum.

We conventionally interpret the 1st term of the right hand side of the eq.\((\ref{eq:2})\) as the mass of the black hole and the 2nd term as the contribution to the total mass of the matter (electromagnetic field and dilaton field in the present case) outside the horizon\([3]\). Now let us consider an extremally charged spherical black hole configuration\([3][8]\):

\[ds^2 = -(1 - 2M/r) dt^2 + \frac{dr^2}{(1 - 2M/r)} + (1 - 2M/r) d\Omega,\]

\[A_{\mu} dx^\mu = \sqrt{2M} dt / r, \quad e^{2\phi} = (1 - 2M/r).\]

Clearly, \(\int_H k^{\alpha \beta} dS_{\alpha \beta} = 0\) holds when the black hole is extremal. If we take it literally, the mass of the black hole is zero. But, we should note that this zero mainly comes from the fact
that the area of the horizon vanishes as the black hole approaches its extremality. The total mass of the system is defined, in one way, by using a virtual 2-dimensional surface and test body of unit mass placed on it (See, e.g., [11]). This method clearly breaks down when the area of the 2-dimensional surface under consideration shrinks to zero. On the other hand, the modified electric charge $\tilde{Q}_H$ is nonzero and finite even when the area shrinks to zero. This suggests us that the modification of the definition of mass might be needed as well as that of electric charge. We leave this problem for future investigation.

Acknowledgments. It is of great pleasure to express my gratitude to Prof. T. Eguchi for sincere advice, guidance, and valuable comments. I would like to acknowledge Prof. J. Arafune for fruitful discussion and encouragement. I am indebted to Prof. A. Kato and all the members of the Elementary particle theory group in the University of Tokyo for support and encouragement.

References

[1] S.W. Hawking and G.F.R. Ellis, “The large scale structure of space-time,” Cambridge University Press, 1973

[2] J. Bekenstein, Phys. Rev. D7 (1973) 2333;
   J. Bekenstein, Phys. Rev. D12 (1975) 3077

[3] B. Carter, “Black Hole Equilibrium States II,” in Black Holes, ed. C. DeWitt and B. S. DeWitt (New York: Gordon & Breach, Les Houches lecture notes);
   J. M. Bardeen, B. Carter, and S. W. Hawking, Comm. Math. Phys. 31 (1973) 161

[4] S. W. Hawking, Phys. Rev. 13 (1976) 191;
   J. B. Hartle and S. W. Hawking, Phys. Rev. D13 (1976) 2188

[5] M. B. Green, J. H. Schwarz, and E. Witten, “Superstring Theory,” Cambridge University Press, 1987

[6] C. G. Callan, J. A. Harvey, and A. Strominger, Nucl. Phys. B359 (1991) 611;
   C. G. Callan, D. Friedan, E. J. Martinec, and M. J. Perry, Nucl. Phys. B262 (1985) 593;
   C. G. Callan, R. C. Myers, and M. J. Perry, Nucl. Phys. B311 (1988/1989) 673;
   D. J. Gross and J. H. Sloan, Nucl. Phys. B291 (1987) 41
[7] G.W.Gibbons and K.Maeda, Nucl. Phys. B298 (1988) 741

[8] D.Garfinkle, G.T.Horowitz, and A.Strominger, Phys. Rev. D43 (1991) 3140;
   G.T.Horowitz, “The Dark Side of String Theory: Black Holes and Black Strings,” Santa
   Barbara preprint, UCSBTH-92-32, hep-th/9210119, and the references therein

[9] J.Preskill, A.Shapere, S.Trivedi, and F.Wilczek, Mod. Phys. Lett. A6 (1991) 2353;
   C.F.E.Holzhey and F.Wilczek, Nucl. Phys. B380 (1992) 447

[10] W.Israel, Phys. Rev. 25 (1967) 1776;
    D.C.Robinson, Phys. Rev. 10 (1974) 458;
    D.C.Robinson, Phys. Rev. Lett. 34 (1975) 905;
    B.Carter, Phys. Rev. Lett. 26 (1971) 331

[11] R.M.Wald, “General relativity,” the University of Chicago press, 1984

[12] S.Chandrasekhar, “Mathematical theory of black holes,” Oxford University press, 1983