POSITIVE MAPS WHICH MAP THE SET OF RANK K PROJECTIONS
ONTO ITSELF

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Abstract. Extending Wigner’s theorem we give a characterization of positive maps of $B(H)$ into
itself which map the set of rank k projections onto itself.

One form of the celebrated Wigner’s theorem [5] is that if $\phi$ is a linear map of the bounded
operators $B(H)$ on a Hilbert space $H$ into itself with the property that it maps the set of rank 1
projections bijectively onto itself, then $\phi$ is of the form

(*)

$$\phi(a) = UaU^*$$
or

$$\phi(a) = Ua^t U^*,$$

where $a^t$ is the transpose of $a$ with respect to a fixed orthonormal basis for $H$, and $U$ is a unitary
operator. In the paper [2] Sarbicki, Chruscinski and Mozrzymas generalized this to the case when
$H$ is of finite dimension $n$ with $n$ a prime number, and the set of rank 1 projections is replaced
by rank $k$ projections, where $k$ is a natural number strictly smaller than $n$. They gave a counter
example to the conclusion (*) when $n$ is not a prime. In that case $\phi$ is no longer a positive map.

In the present note we make the extra assumption that $\phi$ is a positive unital map. Then for any
Hilbert space we obtain the conclusion (*). Closely related results have been obtained by Molnar
[1].

Recall that an atomic masa in $B(H)$ is a maximal abelian subalgebra $A$ generated by the rank
1 projections corresponding to the vectors in an orthonormal basis for $H$. Thus if $H$ is finite
dimensional each maximal abelian subalgebra is atomic. We start with a lemma. See also [1],
Lemma 2.1.5.

Lemma 1. Let $p \in B(H)$ be a rank 1 projection and $A$ an atomic masa in $B(H)$ containing $p$. Let
$k$ be a natural number, $k < \text{dim}H$. Then there exist $k + 1$ projections $P_1, \ldots, P_{k+1}$ in $A$ such that

$$p = \frac{1}{k} \sum_{j=2}^{k+1} P_j - \frac{k-1}{k} P_1.$$

Proof. Let $p_1 = p, p_2, \ldots, p_{k+1}$ be mutually orthogonal rank 1 projections in $A$. Let

$$P_j = \sum_{i=1, i \neq j}^{k+1} p_i, j = 1, \ldots, k + 1.$$

Then $P_j$ is a projection of rank $k$, and $p_i \leq P_j$ for all $j \neq i$, so $p_i \leq P_j$ for $k$ of the projections $P_j$.
It is therefore an easy computation to show the above formula. The proof is complete.

Theorem 2. Let $\phi$ be a positive unital map of $B(H)$ into itself such that $\phi$ maps the set of projections
of rank $k$ in $B(H), k < \text{dim}H$, onto itself. Then $\phi$ is of the form (*).

Proof. Since each projection of rank $k$ in $B(H)$ is in the image under $\phi$ of a rank $k$ projection, it
follows from Lemma 1 that the rank 1 projections are in image of $\phi$, hence each finite rank operator
is in the image of the finite rank operators. By continuity of $\phi$ it follows that $\phi$ when restricted to the compact operators $C(H)$, maps $C(H)$ onto a norm dense subset of itself.

The definite set $D$ of $\phi$ is the set of self-adjoint operators $a$ such that $\phi(a^2) = \phi(a)^2$. Let $Q$ be a projection of rank $k$; then $P = \phi(Q)$ is a projection of rank $k$, hence

$$\phi(Q^2) = \phi(Q) = P = P^2 = \phi(Q)^2,$$

so that $Q \in D$. By [4], Proposition 2.1.7, $D$ is a norm closed Jordan subalgebra of $B(H)$, so by the same argument as above $D \cap C(H) = C(H)_{sa}$, the self-adjoint operators in $C(H)$. Furthermore, the restriction of $\phi$ to $D$ is a Jordan homomorphism. Since $C(H)$ is irreducible, by [3], Corollary 3.4, $\phi$ is either a homomorphism or an anti-homomorphism on $C(H)$. But $C(H)$ is a simple $C^*$-algebra, so $\phi$ is either an automorphism or an anti-automorphism of $C(H)$.

Let now $\omega_x$ be a vector state on $B(H)$. Then if $p$ is the rank 1 projection onto the 1-dimensional subspace of $H$ generated by $x$, then for $a \in B(H)$,

$$\omega_x(a) = (ax, x) = Tr(pap).$$

Since $\phi$ is a Jordan automorphism of $C(H)$ there is a unit vector $y$ such that if $q$ is the rank 1 projection onto the subspace spanned by $y$, then $\phi(q) = p$. Thus for $a \in B(H)$, since $q \in D$, we have, using [4], Proposition 2.1.7,

$$\omega_x(\phi(a)) = Tr(p\phi(a)p) = Tr(\phi(q)\phi(a)\phi(q)) = Tr(\phi(qa)q) = Tr(\phi(\omega_y(a))q) = \omega_y(a)Tr(p) = \omega_y(a).$$

We have thus shown that each vector state composed with $\phi$ is a vector state. Hence by [4], Theorem 3.3.2 $\phi$ is of the desired form (*). The proof is complete.

References

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