Universal Baxter TQ-relations for open boundary quantum integrable systems

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Abstract

Based on properties of the universal R-matrix, we derive universal Baxter TQ-relations for quantum integrable systems with (diagonal) open boundaries associated with \( U_q(\hat{sl}_2) \). The Baxter TQ-relations for the open XXZ-spin chain are images of these universal Baxter TQ-relations.

Keywords: Baxter Q-operator, Baxter TQ-relation, K-operator, L-operator, reflection equation, universal R-matrix

1 Introduction

Baxter Q-operators \[1\] are fundamental objects in quantum integrable systems. They give information about the eigenfunctions and Bethe roots. In particular, Bazhanov, Lukyanov and Zamolodchikov \[2\] defined Baxter Q-operators as traces of the universal R-matrix over q-oscillator representations of one of the Borel subalgebras of the quantum affine algebra \( U_q(\hat{sl}_2) \). The universal R-matrix is an element of the tensor product of two Borel subalgebras. Hence, the Baxter Q-operators are universal in the sense that they are elements of a Borel subalgebra and thus do not depend on the concrete quantum space of states on which the operators act. Baxter Q-operators for concrete physical models can be obtained by specifying representations of the Borel subalgebra. Much work has been done related to this 'q-oscillator construction' of Baxter Q-operators (see, for example, the following papers and references therein: \[3\] \[4\] \[5\] \[6\] \[7\] \[8\] \[9\] \[10\] \[11\] \[12\] \[13\] \[14\] \[15\] \[16\] \[17\] for the trigonometric case; \[18\] \[19\] \[20\] \[21\] for the rational case; \[22\] \[23\] \[24\] \[25\] \[26\] for some other methods). Moreover, systematic studies related to this from the point of view of the asymptotic representation theory of quantum affine algebras were done in \[27\] \[28\] \[29\].

All these works are for quantum integrable systems with periodic boundary condition. In contrast with the case for models with periodic boundary condition, there are only a few works \[30\] \[31\] \[32\] on the q-oscillator construction \[1\] of Baxter Q-operators for models with open boundary conditions. The first breakthrough on this topic was brought by

\[1\] See also \[33\] \[34\] \[35\] \[36\] for different approach.
Frassek and Szecsenyi [30] who proposed Baxter Q-operators for the (diagonal) open XXX-spin chain ($q = 1$ case). In [31], we proposed universal Baxter Q-operators for (diagonal) open boundary quantum integrable systems associated with $U_q(\widehat{sl}_2)$ and gave Baxter Q-operators for open XXZ-spin chains as holomorphic images (for a tensor product of the fundamental representation) of them. One of the fundamental equations for Baxter Q-operators are the so-called Baxter TQ-relations [1]. Vlaar and Weston [32] proved an operator Baxter TQ-relation for the (diagonal) open XXZ-spin chain based on a representation theoretical method. Thus a Baxter Q-operator for the open XXZ-spin chain proposed in [31] indeed satisfies the TQ-relation. The purpose of this paper is to supplement [31] and give universal analogues of Baxter TQ-relations for (diagonal) open boundary quantum integrable systems. Our universal TQ-relations are equations in $U_q(\widehat{sl}_2)$. The Baxter TQ-relation in [32] follows from our universal Baxter TQ-relation as a holomorphic image.

The layout of this paper is as follows. In Section 2, we summarize the definitions of quantum algebras. In Section 3, we review L-operators, which are building blocks of T- and Q-operators. In particular, the L-operators are defined as holomorphic images of the universal R-matrix. In Section 4, we recall solutions (K-operators) [31] of the reflection equation, which are asymptotic limits of solutions of the intertwining relations for the augmented q-Onsager algebra [37, 38]. The solutions are expressed in terms of Cartan elements of q-oscillator algebras. In Section 5, we present the universal Baxter TQ-relations, which are our main results. In Appendix A, we review Khoroshkin and Tolstoy’s explicit formula [41, 42] of the universal R-matrix. In Appendix B, we explain the derivation of the universal TQ-relations. In Appendix C, we reconsider the dressed reflection equation in relation to the universal R-matrix. In Appendix D, we discuss unitarity relations of R-operators in general situation. Throughout the paper we assume that the deformation parameter $q$ is not a root of unity and use the following notation.

Notation:

- For any elements $X, Y$ of the quantum algebras, the q-commutator is defined by $[X, Y]_q = XY - qYX$. In particular, we set $[X, Y]_1 = [X, Y]$.

- The q-Pochhammer symbol is defined by $(x; q)_k = \prod_{j=0}^{k-1} (1 - qx^j)$. In particular, we will use $(x; q)_\infty = \lim_{k \to \infty} (x; q)_k = \prod_{j=0}^{\infty} (1 - qx^j)$ for $|q| < 1$. For more detail, see for example, page 38 in [44].

- A q-analogue of the exponential function is defined by

$$\exp_q(x) = 1 + \sum_{k=1}^{\infty} \frac{x^k}{(k)_q!},$$

(1.1)

where $(k)_q = (1)_q(2)_q \cdots (k)_q, (0)_q = 1, (k)_q = (1 - q^k)/(1 - q)$. This has infinite

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2On the level of the eigenvalues, the Baxter TQ-relation for the open XXZ-spin chain was derived in [39] (see also, [40] and references therein).
product expressions.

\[ \exp_q(x) = \begin{cases} 
(1 - q)x; q)_{\infty}^{-1} & \text{for } |q| < 1, \\
((q^{-1} - 1)x; q^{-1})_{\infty} & \text{for } |q| > 1.
\] (1.2)

Thus the inverse of (1.1) can be obtained via \( \exp_q(x)^{-1} = \exp_{q^{-1}}(-x) \). For more detail, see for example, page 47 in [44].

• We introduce free parameters \( s_0, s_1 \in \mathbb{Z} \). In particular, we set \( s = s_0 + s_1 \).

• \( \lambda = q - q^{-1}, [x]_q = (q^x - q^{-x})\lambda^{-1} \) for \( x \in \mathbb{C} \).

2 Quantum algebras

In this section, we review quantum algebras. There is overlap among this section and the corresponding sections in [31, 43]. We also refer to [45, 44, 46] for review on this subject.

2.1 The quantum affine algebra \( U_q(\hat{sl}_2) \)

The quantum affine algebra \( U_q(\hat{sl}_2) \) (at level 0, i.e. the quantum loop algebra) is a Hopf algebra generated by the elements \( e_i, f_i, q^{\xi h_i} \) for \( i \in \{0, 1\} \) and \( \xi \in \mathbb{C} \) obeying the following relations:

\[ q^{0h_i} = q^0 = 1, \quad q^{\xi h_i} q^{\eta h_i} = q^{(\xi + \eta)h_i}, \quad q^{\xi h_0} q^{\xi h_1} = 1, \] (2.1)

\[ [e_i, f_j] = \delta_{ij} \frac{q^{h_i} - q^{-h_i}}{q - q^{-1}}, \quad q^{\xi h_i} e_j q^{-\xi h_i} = q^{\xi a_{ij}} e_j, \quad q^{\xi h_i} f_j q^{-\xi h_i} = q^{-\xi a_{ij}} f_j, \] (2.2)

\[ [e_i, [e_i, e_j, e_j] q^{-2}] = [f_i, [f_i, f_j, f_j] q^{-2}] q^2 = 0 \quad i \neq j, \quad \xi, \eta \in \mathbb{C}, \] (2.3)

where \((a_{ij})_{0 \leq i, j \leq 1}\) is the Cartan matrix

\[ (a_{ij})_{0 \leq i, j \leq 1} = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}. \]

The algebra has an automorphism \( \sigma \) defined by

\[ \sigma(e_0) = e_1, \quad \sigma(f_0) = f_1, \quad \sigma(q^{\xi h_0}) = q^{\xi h_1}, \] \[ \sigma(e_1) = e_0, \quad \sigma(f_1) = f_0, \quad \sigma(q^{\xi h_1}) = q^{\xi h_0}, \quad \sigma(q) = q. \] (2.4)

The algebra also has an anti-automorphism \( ^t \) defined by

\[ e_i^t = q^{-1-h_i} f_i, \quad f_i^t = e_i q^{1+h_i}, \quad (q^{\xi h_i})^t = q^{\xi h_i}, \quad q^t = q, \quad i = 0, 1. \] (2.5)

\[ ^3 \text{For any } a \in \mathbb{C} \text{ and a Cartan element } \mathcal{H}, \text{ we denote } q^a q^\mathcal{H} \text{ as } q^{a+\mathcal{H}}. \]
Note that this means $\sigma(ab) = \sigma(a)\sigma(b)$ and $(ab)^t = b^ta^t$ for $a, b \in U_q(\widehat{sl}_2)$. We use the following co-multiplication $\Delta : U_q(\widehat{sl}_2) \to U_q(\widehat{sl}_2) \otimes U_q(\widehat{sl}_2)$:

$$\Delta(e_i) = e_i \otimes 1 + q^{-h_i} \otimes e_i,$$
$$\Delta(f_i) = f_i \otimes q^{h_i} + 1 \otimes f_i,$$
$$\Delta(q^{\epsilon h_i}) = q^{\epsilon h_i} \otimes q^{\epsilon h_i}. \tag{2.6}$$

We will also use the opposite co-multiplication defined by

$$\Delta' = p \circ \Delta,$$
$$p(X \otimes Y) = Y \otimes X, \quad X, Y \in U_q(\widehat{sl}_2). \tag{2.7}$$

Co-unit, anti-pode and grading element $d$ are not used in the present paper.

The Borel subalgebras $B_+$ and $B_-$ are generated by the elements $e_i, q^{\epsilon h_i}$ and $f_i, q^{\epsilon h_i}$, respectively, where $i \in \{0, 1\}, \xi \in \mathbb{C}$. There exists a unique element $\mathcal{R}$ in a completion of $B_+ \otimes B_-$ called the universal R-matrix which satisfies the following relations

$$\Delta'(a) \mathcal{R} = \mathcal{R} \Delta(a) \quad \text{for} \quad \forall \ a \in U_q(\widehat{sl}_2), \tag{2.8}$$

$$\begin{align*}
(\Delta \otimes 1) \mathcal{R} &= \mathcal{R}_{13} \mathcal{R}_{23}, \\
(1 \otimes \Delta) \mathcal{R} &= \mathcal{R}_{13} \mathcal{R}_{12},
\end{align*}$$

where $\mathcal{R}_{12} = \mathcal{R} \otimes 1, \mathcal{R}_{23} = 1 \otimes \mathcal{R}, \mathcal{R}_{13} = (p \otimes 1) \mathcal{R}_{23}$. The Cartan part of the universal R-matrix can be isolated as $\mathcal{R} = \tilde{\mathcal{R}}q^{\frac{h_1 h_2}{2}}$, where $\tilde{\mathcal{R}}$ is called the reduced universal R-matrix, which is a series on $\{e_0 \otimes 1, e_1 \otimes 1, 1 \otimes f_0, 1 \otimes f_1\}$. We symbolically write this as $\tilde{\mathcal{R}} = \tilde{\mathcal{R}}(\{e_0 \otimes 1, e_1 \otimes 1, 1 \otimes f_0, 1 \otimes f_1\})$. The Yang-Baxter equation

$$\mathcal{R}_{12} \mathcal{R}_{13} \mathcal{R}_{23} = \mathcal{R}_{23} \mathcal{R}_{13} \mathcal{R}_{12}, \tag{2.9}$$

follows from the relations (2.8). We will also use

$$\overline{\mathcal{R}} = p\mathcal{R} = \mathcal{R}_{21}, \tag{2.10}$$

which is an element of a completion of $B_- \otimes B_+$. Taking note on the first relation in (2.8), one can show that the universal R-matrices commute with the co-multiplication of any Cartan elements

$$[\mathcal{R}, q^{\epsilon h_i} \otimes q^{\epsilon h_i}] = [\overline{\mathcal{R}}, q^{\epsilon h_i} \otimes q^{\epsilon h_i}] = 0 \quad \text{for} \quad \xi \in \mathbb{C}, \quad i = 1, 2. \tag{2.11}$$

The following relations follow from the uniqueness of the universal R-matrix [18] and the fact that (2.4) is an automorphism of each Borel subalgebra.

$$(\sigma \otimes \sigma)\mathcal{R} = \mathcal{R}, \quad (\sigma \otimes \sigma)\overline{\mathcal{R}} = \overline{\mathcal{R}}. \tag{2.12}$$

$^4$We will use similar notation for the L-operators to indicate the space on which they non-trivially act.
2.2 The quantum algebra $U_q(sl_2)$

The quantum algebra $U_q(sl_2)$ is generated by the elements $E, F, q^\xi H$ for $\xi \in \mathbb{C}$ obeying the following relations:

$$q^0 H = q^0 = 1, \quad q^\xi H q^{\eta H} = q^{(\xi + \eta) H}, \quad q^{2\xi} E, \quad q^{2\xi} F q^{-2\xi} = q^{-2\xi} F,$$

$$[E, F] = \frac{q^H - q^{-H}}{q - q^{-1}}, \quad \xi, \eta \in \mathbb{C}. \quad (2.13)$$

The upper (resp. lower) Borel subalgebra is generated by the elements $E, q^{\xi H}$ (resp. $F, q^{\xi H}$). The Casimir element

$$C = F E + \frac{q^{H+1} + q^{-H-1}}{(q - q^{-1})^2} = EF + \frac{q^{H-1} + q^{-H+1}}{(q - q^{-1})^2} \quad (2.14)$$

is central in $U_q(sl_2)$. We have an automorphism

$$\sigma(E) = F, \quad \sigma(F) = E, \quad \sigma(q^{\xi H}) = q^{-\xi H} \quad \sigma(q) = q, \quad (2.15)$$

and an anti-automorphism

$$E^t = q^{-H-1} F, \quad F^t = E q^{H+1}, \quad (q^{\xi H})^t = q^{\xi H}, \quad q^t = q \quad (2.16)$$

of the algebra. These are $U_q(sl_2)$ analogues of (2.4) and (2.5), respectively. There is an algebra homomorphism called evaluation map $\text{ev}_x: U_q(\hat{sl}_2) \rightarrow U_q(sl_2)$,

$$e_0 \mapsto x^{s_0} F, \quad f_0 \mapsto x^{-s_0} E, \quad q^{\xi h_0} \mapsto q^{-\xi H}, \quad e_1 \mapsto x^{s_1} E, \quad f_1 \mapsto x^{-s_1} F, \quad q^{\xi h_1} \mapsto q^{\xi H}, \quad (2.17)$$

where $x \in \mathbb{C}^\times$ is the spectral parameter. We introduce an operation to permute the parameters $s_0$ and $s_1$:

$$\zeta(s_0) = s_1, \quad \zeta(s_1) = s_0, \quad (2.18)$$

One can verify consistency of these:

$$\sigma \circ \text{ev}_x = (\zeta \cdot \text{ev}_x) \circ \sigma, \quad (2.19)$$

$$(\text{ev}_x(a))^t = \text{ev}_{x^{-1}}(a^t) \quad \text{for} \quad a \in U_q(sl_2), \quad (2.20)$$

where $\circ$ is composition of maps and $\zeta \cdot \text{ev}_x$ is the map (2.17) with the replacement of the parameters $(s_0, s_1) \rightarrow (s_1, s_0)$. In case the objects on which (2.19) is acting do not depend on the parameters $s_0$ and $s_1$, (2.19) reduces to $\sigma \circ \text{ev}_x = \zeta \circ \text{ev}_x \circ \sigma$. The fundamental representation $\pi$ of $U_q(sl_2)$ is given by $\pi(E) = E_{12}, \pi(F) = E_{21}$ and $\pi(q^{\xi H}) = q^{\xi} E_{11} + q^{-\xi} E_{22}$, where $E_{ij}$ is the $2 \times 2$ matrix unit whose $(k, l)$-element is $\delta_{i,k}\delta_{j,l}$. The composition $\pi_x = \pi \circ \text{ev}_x$ gives the (fundamental) evaluation representation of $U_q(sl_2)$.

\footnote{We follow [10] and consider the general gradation of the algebra.}
For the fundamental representation, we define an algebra automorphism $\sigma$ and an algebra anti-automorphism $^t\sigma$ of the algebra of $2 \times 2$ matrices over $\mathbb{C}$ by

$$\sigma(E_{ij}) = E_{i-1,j+1}, \quad (2.21)$$

$$E_{ij}^t = E_{ji}, \quad i, j = 1, 2. \quad (2.22)$$

In this case, the anti-homomorphism $^t\sigma$ coincides with transposition of $2 \times 2$ matrices. We have an identity of algebra homomorphisms

$$\pi \circ \sigma = \sigma \circ \pi \quad (2.23)$$

and an identity of algebra anti-homomorphisms

$$\pi(a^t) = (\pi(a))^t \quad \text{for} \quad a \in U_q(sl_2), \quad (2.24)$$

which justify our use of the same symbol for different maps.

### 2.3 q-oscillator algebras

We introduce two kinds of oscillator algebras $\text{Osc}_i$ ($i = 1, 2$). They are generated by the elements $e_i, f_i, q^{\pm h_j}$ (for $\xi \in \mathbb{C}$) obeying the following relations:

$$q^{0h_1} = q^0 = 1, \quad q^{\pm h_1}q^{\mp h_1} = q^{(\xi \mp \eta)h_1}, \quad q^{\pm h_1}e_1q^{-\pm h_1} = q^{2\xi}e_1, \quad q^{\pm h_1}f_1q^{-\pm h_1} = q^{-2\xi}f_1, \quad (2.27)$$

$$\begin{align*}
q^{0h_2} &= q^0 = 1, \\
n^{\pm h_2}q^{\pm h_2} &= q^{(\xi \pm \eta)h_2}, \quad q^{\pm h_2}f_2q^{-\pm h_2} &= q^{2\xi}f_2, \quad q^{\pm h_2}f_2q^{-\pm h_2} &= q^{-2\xi}f_2,
\end{align*} \quad (2.28)$$

$^6$Osc$^1$ is the same as the one defined by eq.(2.25) in [31]; while Osc$^2$ is slightly different from the one defined by eq.(2.26) in [31]. Let $h_2, e_2, f_2$ be $2 \times 2$ matrices. We define an algebra automorphism $^t\mu : U_q(sl_2) \to \text{Osc}_1$ (see for example [41]) by the relations

$$\rho_\mu(E) = e_1(q^\mu - q^{-\mu-h_1-2}), \quad \rho_\mu(F) = f_1, \quad \rho_\mu(q^{E_H}) = q^{E_H}, \quad \mu, \xi \in \mathbb{C}. \quad (2.25)$$

This realizes the Verma module of $U_q(sl_2)$ with the highest weight $\mu$ on the Fock space. The generators of the q-oscillator algebra $\text{Osc}_1$ can be obtained by contraction of $U_q(sl_2)$ (cf. [40]), and eq. (2.30) in [31]. (The similarity transformation by the factor $q^{-\mu_h}$ is related to the change of basis mentioned in footnote 7 in [31].)

$$\begin{align*}
e_1 &= \lim_{q^{-\mu} \to 0} q^{-\mu_{h_1}}\rho_\mu(E)q^{\mu_{h_1}}q^{-\mu_{h_1}}, \quad f_1 = \lim_{q^{-\mu} \to 0} q^{-\mu_{h_1}}\rho_\mu(F)q^{\mu_{h_1}}q^{-\mu_{h_1}}, \\
q^{\pm h_1} &= \lim_{q^{-\mu} \to 0} q^{-\mu_{h_1}}\rho_\mu(q^{E_H})q^{-\mu_{h_1}}, \quad (2.26)
\end{align*}$$

where the limit is taken with respect to $\mu$ ($q$ is constant).
Note that \( \text{Osc}_2 \) can be realized in terms of \( \text{Osc}_1 \):

\[
e_2 = f_1, \quad f_2 = e_1, \quad q^{\hbar_2} = q^{-\hbar_1}.
\] (2.29)

The following relations follow from (2.27) and (2.28):

\[
[e_1, f_1] = \frac{q^{\hbar_1}}{q - q^{-1}}, \quad [e_2, f_2] = -\frac{q^{-\hbar_2}}{q - q^{-1}},
\] (2.30)

\[
[e_1, f_1]_{q^{-2}} = \frac{1}{q - q^{-1}}, \quad [e_2, f_2]_{q^2} = -\frac{q^2}{q - q^{-1}}.
\] (2.31)

We will use anti-involutions of \( \text{Osc}_i \) (analogues of (2.16)) defined by

\[
e_i^t = q^{-\hbar_i} f_i, \quad f_i^t = e_i q^{\hbar_i + 1}, \quad (q^{\xi\hbar_i})^t = q^{\xi\hbar_i},
\] (2.32)

where \((ab)^t = b^t a^t\) holds for any \(a, b \in \text{Osc}_i, i = 1, 2\). We define the homomorphism

\[
\rho_x^{(i)} : B_+ \rightarrow \text{Osc}_i, i = 1, 2
\] by the relations

\[
\rho_x^{(i)}(e_0) = x^s_0 f_i, \quad \rho_x^{(i)}(e_1) = x^{s_1} e_i, \quad \rho_x^{(i)}(q^{\xi\hbar_0}) = q^{-\xi \hbar_i}, \quad \rho_x^{(i)}(q^{\xi\hbar_1}) = q^{\xi \hbar_i},
\] (2.33)

or the homomorphism \(\rho_x^{(i)} : B_- \rightarrow \text{Osc}_i, i = 1, 2\) by the relations

\[
\rho_x^{(i)}(f_0) = x^{-s_0} e_i, \quad \rho_x^{(i)}(f_1) = x^{-s_1} f_i, \quad \rho_x^{(i)}(q^{\xi\hbar_0}) = q^{\xi \hbar_i}, \quad \rho_x^{(i)}(q^{\xi\hbar_1}) = q^{\xi \hbar_i}.
\] (2.34)

These maps are related each other as (cf. (2.29))

\[
\rho_x^{(i)} = (\zeta \cdot \rho_x^{(1)}) \circ \sigma.
\] (2.35)

Here \(\zeta \cdot \rho_x^{(1)}\) is the map (2.33) or (2.34) with the replacement of the parameters \((s_0, s_1) \rightarrow (s_1, s_0)\). In case the objects on which (2.35) is acting do not depend on the parameters \(s_0\) and \(s_1\), (2.35) reduces to \(\rho_x^{(2)} = \zeta \circ \rho_x^{(1)} \circ \sigma\).

## 3 L-operators

In this section, we review various L-operators, which are building blocks of T-and Q-operators. They are holomorphic images of the universal R-matrix in various representations of Borel subalgebras of \(U_q(\hat{sl}_2)\). We will make use of the product expression of the universal R-matrix given by Khoroshkin and Tolstoy \([41, 42]\), which is reviewed in Appendix A. Their universal R-matrix was already reviewed by several authors (see for example, \([50, 51, 10, 12, 15, 31, 26]\)). In particular, a pedagogical account on how to evaluate it in the context of Baxter Q-operators can be found in \([10, 12]\).

### 3.1 L-operators for T-operators

We define the universal L-operators by

\[
\mathcal{L}(x) = (\pi_x \otimes 1)\mathcal{R}, \quad \overline{\mathcal{L}}(x) = (\pi_x \otimes 1)\overline{\mathcal{R}}, \quad x \in \mathbb{C}.
\] (3.1)
We define the universal L-operators for Q-operators by

\[ \mathcal{L}(x) = \left( 1 + \lambda \mathcal{E}_{12} \otimes \sum_{k=0}^{\infty} \left( -q^{-1} \right)^k x^{ks+s_k} f_{\alpha+k\delta} \right) \times \left( \mathcal{E}_{11} \otimes \exp \left( -\lambda \sum_{k=1}^{\infty} \frac{(-x_s)^k}{q^k + q^{-k}} f_{k\delta} \right) \right) \times \left( 1 + \lambda \mathcal{E}_{21} \otimes \sum_{k=0}^{\infty} (-q^{-1})^k x^{ks+s_0} f_{\delta-a+k\delta} \right) \left( \mathcal{E}_{11} \otimes q^{k} + \mathcal{E}_{22} \otimes q^{-k} \right), \quad (3.2) \]

\[ \overline{\mathcal{L}}(x) = \left( 1 + \lambda \mathcal{E}_{12} \otimes \sum_{k=0}^{\infty} (-q)^k x^{-ks-s_k} e_{\alpha+k\delta} \right) \times \left( \mathcal{E}_{11} \otimes \exp \left( -\lambda \sum_{k=1}^{\infty} \frac{(-x^{-s})^k}{q^k + q^{-k}} e_{k\delta} \right) \right) \times \left( 1 + \lambda \mathcal{E}_{21} \otimes \sum_{k=0}^{\infty} (-q)^k x^{-ks-s_0} e_{\delta-a+k\delta} \right) \left( \mathcal{E}_{11} \otimes q^{k} + \mathcal{E}_{22} \otimes q^{-k} \right). \quad (3.3) \]

Here the root vectors \( \{ e_{\alpha+k\delta}, e_{k\delta}, e_{\delta-a+k\delta}, f_{\alpha+k\delta}, f_{k\delta}, f_{\delta-a+k\delta} \} \) can be expressed in terms of the basic generators \( e_{\alpha} = e_1, e_{\delta-a} = e_0, f_{\alpha} = f_1, f_{\delta-a} = f_0 \) via (A3) and (A4).

Evaluating the second components of these universal L-operators in the fundamental evaluation representation \( \pi_1 = \pi_x|_{x=1} \), we obtain the R-matrices of the 6-vertex model.

\[ R(x) = q^{x} \phi(x)(1 \otimes \pi_1)\mathcal{L}(x) = \left( \begin{array}{cccc} q - q^{-1} x_s & 0 & 0 & 0 \\ 0 & 1 - x_s & \lambda x^{s_1} & 0 \\ 0 & \lambda x^{s_0} & 1 - x_s & 0 \\ 0 & 0 & 0 & q - q^{-1} x_s \end{array} \right), \quad (3.4) \]

\[ \overline{R}(x) = q^{x} \phi(x^{-1})(1 \otimes \pi_1)\overline{\mathcal{L}}(x) = \left( \begin{array}{cccc} q - q^{-1} x^{-s} & 0 & 0 & 0 \\ 0 & 1 - x^{-s} & \lambda x^{-s_1} & 0 \\ 0 & \lambda x^{-s_0} & 1 - x^{-s} & 0 \\ 0 & 0 & 0 & q - q^{-1} x^{-s} \end{array} \right), \quad (3.5) \]

where the overall factor is defined by \( \phi(x) = e^{-\Lambda(x^{-q^{-1}})}, \Lambda(x) = \sum_{k=1}^{\infty} \frac{q^{2k} - q^{-2k}}{k(q^x + q^{-x})} x^k \).

### 3.2 L-operators for Q-operators

We define the universal L-operators for Q-operators by

\[ \mathcal{L}^{(a)}(x) = (\rho_x^{(a)} \otimes 1)\mathcal{R}, \quad \overline{\mathcal{L}}^{(a)}(x) = (\rho_x^{(a)} \otimes 1)\overline{R}, \quad x \in \mathbb{C}, \quad a = 1, 2. \quad (3.6) \]
One can calculate these based on the explicit expression of the universal R-matrix in Appendix A. In particular, \( L^{(1)}(x) \) and \( \overline{L}^{(2)}(x) \) have simple expressions:

\[
L^{(1)}(x) = \exp_{q^{-2}}(\lambda x^{s_1} e_1 \otimes f_\alpha) \exp \left( \sum_{k=1}^{\infty} \frac{(-1)^{k-1} x^{sk}}{|2k|_q} \otimes f_\delta \right) \exp_{q^{-2}}(\lambda x^{-s_0} f_1 \otimes f_{\delta-\alpha}) q^{\frac{h_1}{2} \otimes h_1},
\]

\[
\overline{L}^{(2)}(x) = \\
= \exp_{q^{-2}}(\lambda x^{-s_1} f_2 \otimes e_\alpha) \exp \left( \sum_{k=1}^{\infty} \frac{(-q^2 x^{-s})^k}{|2k|_q} \otimes e_\delta \right) \exp_{q^{-2}}(\lambda x^{-s_0} e_2 \otimes e_{\delta-\alpha}) q^{\frac{h_2}{2} \otimes h_1}.
\]

Direct evaluations of \( L^{(2)}(x) \) and \( \overline{L}^{(1)}(x) \) contain infinite products of q-exponential functions (at least in the root ordering which we have adapted). In order to avoid complicated expressions, we use the relations (2.12), (2.35), (2.29) and \( \sigma = \sigma^{-1} \) for (3.7) and (3.8), to get

\[
L^{(2)}(x) = \zeta \circ (1 \otimes \sigma) L^{(1)}(x) = \\
= \exp_{q^{-2}}(\lambda x^{s_0} f_2 \otimes f_{\delta-\alpha}) \exp \left( \sum_{k=1}^{\infty} \frac{(-q^2 x^{s})^k}{|2k|_q} \otimes \overline{f}_\delta \right) \exp_{q^{-2}}(\lambda x^{-s_0} e_2 \otimes f_\alpha) q^{\frac{h_2}{2} \otimes h_1},
\]

\[
\overline{L}^{(1)}(x) = \zeta \circ (1 \otimes \sigma) \overline{L}^{(2)}(x) = \\
= \exp_{q^{-2}}(\lambda x^{-s_0} e_1 \otimes e_{\delta-\alpha}) \exp \left( \sum_{k=1}^{\infty} \frac{(-1)^{k-1} x^{-sk}}{|2k|_q} \otimes \overline{e}_\delta \right) \exp_{q^{-2}}(\lambda x^{-s_1} f_1 \otimes e_\alpha) q^{\frac{h_1}{2} \otimes h_1},
\]

where the root vectors \( \overline{e}_\delta \) and \( \overline{f}_\delta \), which can be expressed in terms of the basic generators \( e_1, e_0, f_1, f_0 \), come from (A1). Now we evaluate the second component of the universal L-operators in the fundamental evaluation representation \( \pi_1 = \pi_x | x = 1 \). We normalize the L-operators as

\[
L^{(i)}(x) = \phi^{(i)}(x)(1 \otimes \pi_1) L^{(i)}(x), \quad \overline{L}^{(i)}(x) = \phi^{(i)}(x^{-1})(1 \otimes \pi_1) \overline{L}^{(i)}(x),
\]

where \( \phi^{(i)}(x) = e^{-\Phi(x^i)}, \Phi(x) = \sum_{k=1}^{\infty} \frac{1}{k(q^{x+q^{-1}} - 1)} x^k, \ i = 1, 2 \). These are L-operators for Q-operators for XXZ-spin chain. Explicitly, one obtains

\[
L^{(1)}(x) = \left( \frac{q^{\frac{b_1}{2}}}{\lambda x^{s_1} e_1 q^{\frac{b_1}{2}}} - \frac{\lambda x^{-s_0} f_1 q^{-\frac{b_1}{2}}}{\lambda x^{s_1} e_1 q^{\frac{b_1}{2}}} q^{-\frac{b_1}{2}} - q^{-1} x^s q^{\frac{b_1}{2}} \right).
\]

\footnote{See eq. (B.21) in [31].}

\footnote{The L-operators with the superscript \(^{(1)}\) are in the same convention as the ones in [31], but the ones with \(^{(2)}\) are superficially different since we have slightly changed the definition of the q-oscillator algebra \( \text{Osc}_2 \).}
\[ L^{(2)}(x) = \left( \begin{array}{cc} q^{\frac{h_2}{2}} - q^{-1}x^s q^{-\frac{h_2}{2}} & \lambda x^{s_0} f_2 q^{-\frac{h_2}{2}} \\ \lambda x^{s_1} e_2 q^{\frac{h_2}{2}} & q^{-\frac{h_2}{2}} \end{array} \right), \quad (3.13) \]

\[ \bar{L}^{(1)}(x) = \left( \begin{array}{cc} q^{\frac{h_1}{2}} & \lambda x^{s_1} f_1 q^{-\frac{h_1}{2}} \\ \lambda x^{s_0} e_1 q^{\frac{h_1}{2}} & q^{-\frac{h_1}{2}} - q^{-1}x^{-s} q^{\frac{h_1}{2}} \end{array} \right), \quad (3.14) \]

\[ \bar{L}^{(2)}(x) = \left( \begin{array}{cc} q^{\frac{h_2}{2}} & \lambda x^{s_1} f_2 q^{-\frac{h_2}{2}} \\ \lambda x^{s_0} e_2 q^{\frac{h_2}{2}} & q^{-\frac{h_2}{2}} - q^{-1}x^{-s} q^{\frac{h_2}{2}} \end{array} \right). \quad (3.15) \]

In addition to the above L-operators, we need L-operators proportional to the inverse of them.  

\[ \bar{L}^{(1)}(x) = \left( \begin{array}{cc} q^{\frac{h_1}{2}} - q^{-1}x^s q^{-\frac{h_1}{2}} & \lambda x^{s_0} f_1 q^{-\frac{h_1}{2}} \\ \lambda x^{s_1} e_1 q^{\frac{h_1}{2}} & -q^{-1}x^s q^{\frac{h_1}{2}} \end{array} \right), \quad (3.18) \]

\[ \bar{L}^{(1)}(x) = \left( \begin{array}{cc} q^{\frac{h_1}{2}} - q^{-1}x^{-s} q^{-\frac{h_1}{2}} & \lambda x^{s_1} f_1 q^{-\frac{h_1}{2}} \\ \lambda x^{s_0} e_1 q^{\frac{h_1}{2}} & -q^{-1}x^{-s} q^{\frac{h_1}{2}} \end{array} \right), \quad (3.19) \]

\[ \bar{L}^{(2)}(x) = \left( \begin{array}{cc} -q^{-1}x^s q^{-\frac{h_2}{2}} & \lambda x^{s_0} f_2 q^{-\frac{h_2}{2}} \\ \lambda x^{s_1} e_2 q^{\frac{h_2}{2}} & q^{-\frac{h_2}{2}} - q^{-1}x^s q^{\frac{h_2}{2}} \end{array} \right), \quad (3.20) \]

\[ \bar{L}^{(2)}(x) = \left( \begin{array}{cc} -q^{-1}x^{-s} q^{-\frac{h_2}{2}} & \lambda x^{s_1} f_2 q^{-\frac{h_2}{2}} \\ \lambda x^{s_0} e_2 q^{\frac{h_2}{2}} & q^{-\frac{h_2}{2}} - q^{-1}x^{-s} q^{\frac{h_2}{2}} \end{array} \right). \quad (3.21) \]

The L-operators with the superscript \(^{(2)}\) can be obtained from the ones with \(^{(1)}\): 

\[ L^{(2)}(x) = \zeta \circ (1 \otimes \sigma) L^{(1)}(x), \quad (3.22) \]

\[ \bar{L}^{(2)}(x) = \zeta \circ (1 \otimes \sigma) \bar{L}^{(1)}(x), \quad (3.23) \]

\[ \bar{L}^{(2)}(x) = \zeta \circ (1 \otimes \sigma) \bar{L}^{(1)}(x), \quad (3.24) \]

\[ \bar{L}^{(2)}(x) = \zeta \circ (1 \otimes \sigma) \bar{L}^{(1)}(x). \quad (3.25) \]

\(^9\)We could interpret these as follows (cf. \[31\]). Consider universal L-operators of the form:

\[ \tilde{L}^{(a)}(x) = (\rho^{(a)}_x \otimes 1) R^{-1} = \bar{L}^{(a)}(x)^{-1}, \quad \bar{L}^{(a)}(x) = (\rho^{(a)}_x \otimes 1) R^{-1} = \tilde{L}^{(a)}(x)^{-1}, \quad a = 1, 2. \quad (3.16) \]

The L-operators \[5.18, 9.20\] are images of these:

\[ \tilde{L}^{(i)}(x) = \phi^{(i)}(x^{-1})(1 \otimes \pi_1) \tilde{L}^{(i)}(x), \quad \bar{L}^{(i)}(x) = \phi^{(i)}(x)(1 \otimes \pi_1) \bar{L}^{(i)}(x), \quad (3.17) \]

where \( \phi^{(i)}(x) = (-x^{-s} q^{-1}) e^{-\Phi(x^s q^s)} \). There is a useful identity \( \phi^{(1)}(x) \bar{\phi}^{(1)}(x) = 1 - q^{-1}x^{-s} \).
One can check that these L-operators satisfy
\[ L^{(a)}(x) \overline{L}^{(a)}(x) = \overline{L}^{(a)}(x) L^{(a)}(x) = 1 - q^{-1} x^{-s}, \]
(3.26)
\[ \tilde{L}^{(a)}(x) \overline{L}^{(a)}(x) = \overline{L}^{(a)}(x) \tilde{L}^{(a)}(x) = 1 - q^{-1} x^{-s}, \]
(3.27)
\[ g_2 L^{(a)}(xq^2 t_2) g_2^{-1} \overline{L}^{(a)}(x) t_2 = \overline{L}^{(a)}(x) t_2 g_2 L^{(a)}(xq^2 t_2) g_2^{-1} = q^2 - q^{-1} x^{-s}, \]
(3.28)
\[ g_2 \tilde{L}^{(a)}(xq^2 t_2) g_2^{-1} \overline{L}^{(a)}(x) t_2 = \overline{L}^{(a)}(x) t_2 g_2 \tilde{L}^{(a)}(xq^2 t_2) g_2^{-1} = q^2 - q^3 x^s, \]
(3.29)
where \( a = 1, 2 \), \( g = \text{diag}(q^{s_0 - s_1}, q^{-s_0 - s_1}) \), \( g_2 = 1 \otimes g \), and \( t_2 \) is the transposition in the second component of the tensor product. Note that the matrix \( g \) is invariant under the map \( \zeta \circ \sigma \):
\[ \zeta \circ \sigma(g) = g. \]
(3.30)

One can also check the following relations for the L-operators:
\[ L^{(a)}(x)^{t_1 t_2} = \overline{L}^{(a)}(x^{-1}), \]
\[ \tilde{L}^{(a)}(x)^{t_1 t_2} = L^{(a)}(x^{-1}), \]
\[ \tilde{L}^{(a)}(x)^{t_1 t_2} = \overline{L}^{(a)}(x^{-1}), \]
\[ \tilde{L}^{(a)}(x)^{t_1 t_2} = \overline{L}^{(a)}(x^{-1}), \]
(3.31)
where \( t_1 \) is the anti-involution \( (2.32) \) in the first component of the tensor product. The relations (3.26) and (3.27), and (3.28) and (3.29) can be interchanged by (3.31). Moreover (3.26)-(3.29) and (3.31) for \( a = 2 \) follow from the ones for \( a = 1 \) via (3.22)-(3.25) and (3.30).

## 4 Reflection equation and K-operators

A systematic approach for construction of quantum integrable systems with open boundaries was developed by Sklyanin \[39\]. The key equation for this is the reflection equation (boundary Yang–Baxter equation) \[52\]. We start from the following form of the reflection equation and the dual reflection equation for the R-matrices \[ (3.4) \] and \[ (3.5) \]:
\[ R_{12} \left( \frac{y}{x} \right) K_1(x) \overline{R}_{12} (xy) K_2(y) = K_2(y) R_{12} \left( \frac{1}{xy} \right) K_1(x) \overline{R}_{12} \left( \frac{x}{y} \right), \]
(4.1)
\[ R_{12} \left( \frac{y}{x} \right) K_1(x)^{t_1} g_2 \overline{R}_{12} \left( x y^{-2} \right) g_2^{-1} K_2(y)^{t_2} = K_2(y)^{t_2} g_2^{-1} R_{12} \left( \frac{q^2}{x y} \right) g_2 K_1(x)^{t_1} \overline{R}_{12} \left( \frac{x}{y} \right), \]
(4.2)
where \( x, y \in \mathbb{C}^\times \), \( K_1(x) = K(x) \otimes 1 \), \( K_2(y) = 1 \otimes K(y) \). The most general non-diagonal \( 2 \times 2 \) matrix solutions of the reflection equations are known in \[53, 54, 55\]. The diagonal solutions of (4.1) and (4.2) are specialization of them:
\[ K(x) = \begin{pmatrix} x^{s_0} \epsilon_+ + x^{-s_1} \epsilon_- & 0 \\ 0 & x^{-s_0} \epsilon_+ + x^s \epsilon_- \end{pmatrix}, \]
(4.3)
\[ \overline{K}(x) = \begin{pmatrix} q^{-1} x^{s_0} \tau_+ + q x^{-s_1} \tau_- & 0 \\ 0 & q x^{-s_0} \tau_+ + q^{-1} x^s \tau_- \end{pmatrix}, \]
(4.4)
where $\epsilon_+ \text{ and } \tau_\pm$ are scalar parameters\textsuperscript{10}. We assume $\epsilon_+ \epsilon_- \tau_+ \tau_- \neq 0$ since we will deal with solutions which contain $\epsilon_+^{-1}, \epsilon_-^{-1}, \tau_+^{-1}$ or $\tau_-^{-1}$. We remark that these solutions (4.3) and (4.4) are related each other by the following transformation\textsuperscript{55}:

$$
\overline{K}(x) = K^t \left(xq^{-\frac{1}{2}}\right) g|_{\epsilon_+ = \tau_+}.
$$

(4.5)

In addition to (2.18), we assume

$$
\zeta(\epsilon_+) = \epsilon_- , \quad \zeta(\epsilon_-) = \epsilon_+ , \quad \zeta(\tau_+) = \tau_- , \quad \zeta(\tau_-) = \tau_+.
$$

(4.6)

The K-matrices (4.3) and (4.4) are invariant under the operation $\zeta \circ \sigma$:

$$
\zeta \circ \sigma(K(x)) = K(x) , \quad \zeta \circ \sigma(\overline{K}(x)) = \overline{K}(x).
$$

(4.7)

Next, we consider the reflection equations and the dual reflection equations for the L-operators for Q-operators (3.12), (3.14), (3.18) and (3.19):

$$
L_{12}^{(a)} \left(\frac{y}{x}\right) K_1^{(a)}(x) \overline{L}_{12}^{(a)}(xy) K_2(y) = K_{2}(y)L_{12}^{(a)} \left(\frac{1}{xy}\right) K_1^{(a)}(x) \overline{L}_{12}^{(a)} \left(\frac{x}{y}\right),
$$

(4.8)

$$
\overline{L}_{12}^{(a)} \left(\frac{y}{x}\right) \overline{K}_1^{(a)}(x)^t g_2 \overline{L}_{12}^{(a)} \left(xyq^{-\frac{1}{2}}\right) g_2^{-1} \overline{K}_2(y)^t = \overline{K}_2(y)^t g_2^{-1} \overline{K}_1^{(a)}(x)^t g_2 \overline{L}_{12}^{(a)} \left(\frac{x}{y}\right), \quad a = 1, 2.
$$

(4.9)

We have solutions (K-operators)\textsuperscript{31} \textsuperscript{11} of these equations for $a = 1$.

$$
K^{(1)}(x) = \kappa^{(1)}(x)^{-1} x^{s_0 h_1} \exp^{-1} \left(-\frac{\epsilon_- x^s q^{-h_1}}{\lambda \epsilon_+}\right),
$$

(4.10)

$$
\overline{K}^{(1)}(x) = \overline{\kappa}^{(1)}(x)^{-1} x^{s_0 h_1} q^{-h_1} \exp^{-1} \left(-\frac{\tau_- x^s q^{2-h_1}}{\lambda \tau_+}\right),
$$

(4.11)

where the normalization functions are defined by

$$
\kappa^{(1)}(x) = \exp^{-1} \left(-\frac{\epsilon_- x^s}{\lambda \epsilon_+}\right),
$$

(4.12)

$$
\overline{\kappa}^{(1)}(x) = \exp_q^{-1} \left(-\frac{\tau_- x^s}{\lambda \tau_+}\right).
$$

(4.13)

We remark that the same type of normalization is used in\textsuperscript{32}. The solutions (4.10) and (4.11) are related each other by the following transformation:

$$
\overline{K}^{(1)}(x) = \left(1 + \frac{\tau_- x^s q^{-2}}{\tau_+}\right)^{-1} K^{(1)}(x)^{-1} x^{-1} q^{-1} q^{20-5 h_1} |_{\epsilon_+ = \tau_+}.
$$

(4.14)

\textsuperscript{10}Up to an overall factor, $\overline{K}(xq^{\frac{1}{2}}) \overline{g}^{-2}$ coincides with $\overline{K}(x)$ in eq. (4.30) in\textsuperscript{31}.

\textsuperscript{11}$\overline{K}^{(1)}(x) \overline{K}^{(1)}(x)$ corresponds to $\overline{\kappa}^{(1)}(x)$ in page 434 in\textsuperscript{31}.
Solutions for $a = 2$ follow from the first ones (4.10) and (4.11):

$$K^{(2)}(x) = \zeta(K^{(1)}(x)), \quad \bar{K}^{(2)}(x) = \zeta(\bar{K}^{(1)}(x)).$$

(4.15)

One can check this by applying $\zeta \circ (1 \otimes \sigma)$ to (4.8) and (4.9) for $a = 1$ (with the help of (3.22), (3.25), (3.30) and (4.7)).

5 Universal Baxter TQ-relation

In this section, we apply a universal version of Sklyanin’s dressing method \cite{39} to the K-operators in the previous section and obtain more general solutions of the reflection equation. Then we define universal T- and Q-operators for open boundaries integrable systems \cite{31}, which are elements in $U_q(sl_2)$. We will present the universal TQ-relations among them.

We define the universal dressed K-operator for a T-operator by

$$K(x) = \mathcal{L}(x^{-1})(K(x) \otimes 1)\mathcal{L}(x), \quad x \in \mathbb{C}^\times.$$  

(5.1)

One can show that (5.1) satisfies the following universal dressed reflection equation for a T-operator (see Appendix C).

$$R_{12}(x^{-1}y)K_{13}(x)R_{12}(xy)K_{23}(y) = K_{23}(y)R_{12}(x^{-1}y^{-1})K_{13}(x)R_{12}(xy), \quad x, y \in \mathbb{C}^\times.$$  

(5.2)

We define the universal T-operator by

$$T(x) = (\text{tr} \otimes 1)\left((K(x^{-1}) \otimes 1)K(x)\right), \quad x \in \mathbb{C}^\times,$$  

(5.3)

where the trace is taken over the space $\text{End}(\mathbb{C}^2)$. This is invariant under the operation $\zeta \circ \sigma$:

$$\zeta \circ \sigma(T(x)) = T(x).$$  

(5.4)

One can show this by using the relations: $(1 \otimes \sigma)\mathcal{L}(x) = \zeta \circ (\sigma \otimes 1)\mathcal{L}(x)$, $(1 \otimes \sigma)\bar{\mathcal{L}}(x) = \zeta \circ (\sigma \otimes 1)\bar{\mathcal{L}}(x)$ (these follow from (2.12), (2.19), (2.23), (4.7) and $\text{tr} \sigma(A) = \text{tr} A$ for any $2 \times 2$ matrix $A$). We define the universal dressed K-operators for Q-operators by

$$K^{(a)}(x) = \mathcal{L}^{(a)}(x^{-1})(K^{(a)}(x) \otimes 1)\mathcal{L}^{(a)}(x), \quad x \in \mathbb{C}^\times, \quad a = 1, 2.$$  

(5.5)

One can prove that (5.5) satisfy the following universal dressed reflection equations for Q-operators (see Appendix C).

$$L_{12}^{(a)}(x^{-1}y)K_{13}^{(a)}(x)L_{12}^{(a)}(xy)K_{23}(y) = K_{23}(y)L_{12}^{(a)}(x^{-1}y^{-1})K_{13}^{(a)}(x)L_{12}^{(a)}(xy^{-1}), \quad x, y \in \mathbb{C}^\times, \quad a = 1, 2.$$  

(5.6)
We define the universal Q-operators [31] by

$$Q^{(a)}(x) = (\text{tr}_{W_a} \otimes 1) \left( \tilde{K}^{(a)}(x^{-1})K^{(a)}(x) \right), \quad x \in \mathbb{C}^\times, \quad a = 1, 2, \quad (5.7)$$

where $g^{(a)} = q^{(\omega^{-1})h_a}$ and $W_a$ are Fock spaces generated by $\text{Osc}_a$. Note that the second Q-operator follows from the first one

$$Q^{(2)}(x) = \zeta \circ \sigma(Q^{(1)}(x)). \quad (5.8)$$

One can show this by using the relations [33], [34] and [41]. We find that the universal T-and Q-operators satisfy the following universal TQ-relations (see Appendix B for derivation).

$$(q^2 - q^4x^{2a})Q^{(a)}(q^{\frac{1}{2}}x)T(x) = \omega_1^{(a)}(x)\omega_1^{(a)}(x)Q^{(a)}((q^{\frac{1}{2}}x)q^{h_2-a} \quad + \omega_2^{(a)}(x)\omega_2^{(a)}(x)Q^{(a)}(q^2x)q^{-h_2-a}, \quad a = 1, 2, \quad (5.9)$$

where $\omega_1^{(a)}(x), \omega_2^{(a)}(x), \omega_1^{(a)}(x), \omega_2^{(a)}(x)$ are defined by

$$\omega_1^{(1)}(x) = (\epsilon_+x^{s_0} + \epsilon_-x^{-s_1}), \quad \omega_2^{(1)}(x) = (1 - x^{-2a})(\epsilon_+x^{-s_0} + \epsilon_-q^{-2s_1})q^{-2}, \quad (5.10)$$

and

$$\omega_2^{(2)}(x) = \zeta(\omega_1^{(1)}(x)), \quad \omega_2^{(2)}(x) = \zeta(\omega_1^{(1)}(x)), \quad j = 1, 2. \quad (5.11)$$

The universal T-and Q-operators commute with any Cartan elements of $U_q(\hat{sl}_2)$. One can show this by the relation $(1 \otimes q^{h_i})R = (q^{-\xi h_i} \otimes 1)R(q^{\xi h_i} \otimes q^{h_i})$ from [21], the fact that Cartan elements commute with the K-operators, and cyclicity of the trace. Thus one can remove the factors $q^{h_2-a}$ and $q^{-h_2-a}$ in (5.9) by setting $Q^{(a)}(x) = x^{\frac{h_2-a}{2}}Q^{(a)'}(x)$. The universal T-operator [5.3] and Q-operators (5.7) belong to $U_q(\hat{sl}_2)$. Thus the universal TQ-relations (5.9) are equations in $U_q(\hat{sl}_2)$. Evaluating these for various representations of $U_q(\hat{sl}_2)$, we obtain a wide class of T-and Q-operators. For example, the T-and Q-operators acting on $(\mathbb{C}^2)^\otimes L$ are given by (see eqs. (G.22) and (G.24) in [31].

$$T(x) = \Psi(x, \{\xi_i\}) (\pi_{\xi_1} \otimes \cdots \otimes \pi_{\xi_L}) \Delta^{\otimes(L-1)}T(x),$$

$$= (\text{tr} \otimes 1^{\otimes L}) \left( K_0(x^{-1})R_{0L}(x^{-1}\xi_{L-1}) \cdots R_{01}(x^{-1}\xi_1) \times K_0(x)\bar{R}_{01}(x\xi_1) \cdots \bar{R}_{0L}(x\xi_{L-1}) \right). \quad (5.12)$$

\[1^{\otimes L} \text{Note that the following relations follow from (2.8): } (1 \otimes \Delta^{\otimes(L-1)})R = R_{0L} \cdots R_{02}R_{01}, \quad (1 \otimes \Delta^{\otimes(L-1)})\bar{R} = \bar{R}_{01}\bar{R}_{02} \cdots \bar{R}_{0L}. \quad \text{Here we label the space over which the trace is to be taken (auxiliary space) as 0.}

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\[ Q^{(a)}(x) = \Psi^{(1)}(x, \{ \xi_i \}) (\pi_{\xi_1} \otimes \cdots \otimes \pi_{\xi_L}) \Delta^{\otimes(L-1)} Q^{(a)}(x), \]
\[ = (\text{tr}_{W_a} \otimes 1^{\otimes L}) \left( K_0^{(a)}(x^{-1}) L_{01}^{(a)}(x^{-1} \xi_1^{-1}) \cdots L_{01}^{(a)}(x^{-1} \xi_1^{-1}) \right) \times \left( K_0^{(a)}(x) L_{01}^{(a)}(x \xi_1^{-1}) \cdots L_{01}^{(a)}(x \xi_1^{-1}) \right), \quad a = 1, 2, \quad (5.13) \]

where \( \xi_1, \ldots, \xi_L \in \mathbb{C}^\times \) are inhomogeneities on the spectral parameter in the quantum space. The overall factors are given by
\[ \Psi(x, \{ \xi_i \}) = q^L \prod_{k=1}^L \phi(x^{-1} \xi_k^{-1}) \phi(x \xi_k), \quad (5.14) \]
\[ \Psi^{(1)}(x, \{ \xi_i \}) = \prod_{k=1}^L \phi^{(1)}(x^{-1} \xi_k^{-1}) \phi^{(1)}(x \xi_k), \quad a = 1, 2. \quad (5.15) \]

We remark that (5.13) give Q-operators for the open XXZ-spin chain. Moreover, our Q-operators (5.13) reduce to Q-operators for the open XXX-spin chain similar to the ones in [30] in the rational limit \( q \to 1 \). Applying \((\pi_{\xi_1} \otimes \cdots \otimes \pi_{\xi_L}) \Delta^{\otimes(L-1)}\) to (5.13), we obtain the Baxter TQ-relations for the open XXZ-spin chain.

\[ (q^2 - q^4 x^{2s}) Q^{(a)}(q^{\frac{1}{2}} x) T(x) = \omega^{(a)}_1(x) \omega^{(a)}_1(x) \chi_1(x) Q^{(a)}(q^{\frac{1}{2}} x) \eta^{3-2a} \]
\[ + \omega^{(a)}_2(x) \omega^{(a)}_2(x) \chi_2(x) Q^{(a)}(q^{\frac{3}{2}} x) \eta^{3+2a}, \quad a = 1, 2, \quad (5.16) \]

where \( \omega^{(a)}_1(x), \omega^{(a)}_2(x), \omega^{(a)}_1(x) \) and \( \omega^{(a)}_2(x) \) are defined by (5.10), and \( \chi_1(x) \) and \( \chi_2(x) \) are calculated as
\[ \chi_1(x) = \frac{\Psi^{(1)}(q^{\frac{1}{2}} x, \{ \xi_i \}) \Psi(x, \{ \xi_i \})}{\Psi^{(1)}(q^{\frac{1}{2}} x, \{ \xi_i \})} \]
\[ = q^L \prod_{k=1}^L \frac{\phi^{(1)}(q^{-\frac{1}{2}} x^{-1} \xi_k^{-1}) \phi^{(1)}(q^{-\frac{1}{2}} x^{-1} \xi_k) \phi(x^{-1} \xi_k)}{\phi^{(1)}(q^{\frac{1}{2}} x^{-1} \xi_k^{-1}) \phi^{(1)}(q^{\frac{1}{2}} x^{-1} \xi_k)} \]
\[ = q^L \prod_{k=1}^L \left( 1 - q^{-2} (x \xi_k)^{-s} \right) \left( 1 - q^{-2} (x \xi_k^{-1})^{-s} \right), \quad (5.17) \]
\[ \chi_2(x) = \frac{\Psi^{(1)}(q^{\frac{1}{2}} x, \{ \xi_i \}) \Psi(x, \{ \xi_i \})}{\Psi^{(1)}(q^{\frac{1}{2}} x, \{ \xi_i \})} \]
\[ = q^L \prod_{k=1}^L \frac{\phi^{(1)}(q^{-\frac{1}{2}} x^{-1} \xi_k^{-1}) \phi^{(1)}(q^{-\frac{1}{2}} x^{-1} \xi_k) \phi(x^{-1} \xi_k)}{\phi^{(1)}(q^{\frac{1}{2}} x^{-1} \xi_k^{-1}) \phi^{(1)}(q^{\frac{1}{2}} x^{-1} \xi_k)} \]
\[ = q^L \prod_{k=1}^L \left( 1 - (x \xi_k)^{-s} \right) \left( 1 - (x \xi_k^{-1})^{-s} \right). \quad (5.18) \]

\(^{13}\)\(\Psi^{(1)}(x, \{ \xi_i \}) \) and \( \Psi(x, \{ \xi_i \}) \) in [31] correspond to \( \Psi^{(1)}(x, \{ \xi_i \})^{-1} \) and \( q^k \Psi(x, \{ \xi_i \})^{-1} \), respectively.
The factors $\eta$ and $\eta^{-1}$ are diagonal matrices.

$$\eta = \begin{pmatrix} q & 0 \\ 0 & q^{-1} \end{pmatrix} \otimes L, \quad \eta^{-1} = \begin{pmatrix} q^{-1} & 0 \\ 0 & q \end{pmatrix} \otimes L.$$  \hspace{1cm} (5.19)

Up to convention, \eqref{5.16} for $a = 1$ coincides \cite{4} with the TQ-relation in \cite{32}.

Commutativity of the universal T-and Q-operators can be shown based on a refinement of the Sklyanin’s method \cite{39} (with the help of \eqref{3.26}-\eqref{3.29}, \eqref{3.31}, \eqref{1.9} and \eqref{5.6}) as explained in Appendix G in \cite{31}:

$$Q^{(a)}(x) T(y) = T(y) Q^{(a)}(x), \quad a = 1, 2, \quad x, y \in \mathbb{C}. \hspace{1cm} (5.20)$$

Evaluating \eqref{5.20} for $(\pi_{\xi_1} \otimes \cdots \otimes \pi_{\xi_L}) \Delta^{\otimes(L-1)}$, we also obtain

$$Q^{(a)}(x) T(y) = T(y) Q^{(a)}(x), \quad a = 1, 2, \quad x, y \in \mathbb{C}. \hspace{1cm} (5.21)$$

6 Concluding remarks

In this paper, we gave universal Baxter TQ-relations for diagonal open boundary integrable systems associated with $U_q(\hat{sl}_2)$. This supplements and expands the discussions in \cite{31}. By fixing the representation on the quantum space, we recovered the Baxter TQ-relation for the open XXZ spin chain \cite{32}.

One of the unsolved problems related to this paper is generalization to quantum integrable systems with non-diagonal open boundaries. The key objects for construction of the Baxter Q-operators for open boundary integrable systems are K-operators. The K-operators for Q-operators can be obtained as asymptotic limits (or contraction) of generic K-operators which are expressed in terms of generators of symmetry algebras. In the case of the Yangian $Y(sl_2)$, generic K-operators for general non-diagonal boundaries were constructed in \cite{56}, and in the case of $U_q(\hat{sl}_2)$, generic K-operators for general triangular boundaries were constructed in \cite{43}.

The generalization to the higher rank case is also not fully understood yet. In \cite{57}, diagonal K-operators for $U_q(\hat{g}_n)$ were expressed in terms of Cartan elements of a quotient of $U_q(g_n)$. Non diagonal K-matrices for the symmetric tensor representations of $U_q(\hat{sl}_n)$ were constructed in \cite{58} (see also \cite{59} for some aspect on $n = 2$ case). By taking limits of these, one will be able to obtain a subset of the K-operators for Q-operators for the higher rank case.

\footnotesize

- Set $s_0 = s_1 = 1, s = 2$. In this case, $R(x) = \overline{R}(x^{-1})$ and $L^{(1)}(x) = \overline{L}^{(1)}(x^{-1})$ hold. Then we make identification: $T(x) = q^{2L+1} \epsilon_{+} x^2 T^V(x^{-1}), \quad Q^{(a)}(x) = (\epsilon_{+} q^{-1} x^{-2}/\epsilon_{-}) T^W(q^{2} x^{-1}) = (\epsilon_{+} q^{-1} x^{-2}/\epsilon_{-}) \left( q^{-1} x^{-1} \right)^{\otimes L} \otimes L(x)$, $\xi_j = t_j^{-1}, -\epsilon_{-}/\epsilon_{+} = \xi, -\epsilon_{+}/\epsilon_{-} = \xi, \quad L = N, \quad K(x) = -\epsilon_{+} x K^V(x^{-1}), \quad \overline{K}(x) = -q\epsilon_{-} x^{-1} \overline{K}^V(x), \quad R(x) = q(R(x) \text{ in eq. (2.15)} \text{ in } [32], \quad L^{(1)}(x) = L(q^2 x, 1), \quad K^{(1)}(x) = K^W(q^2 x, 1), \quad \overline{K}^{(1)}(x) = (\epsilon_{+} q^{-1} x^{-2}/\epsilon_{-}) \overline{K}^V(x, 1), \quad e_1 = -q^{-2} \lambda^{-1} a, \quad f_1 = -q^2 \lambda^{-1} a^\dagger, \quad h_1 = -2 D$, where the quantities in the left hand sides are in the notation of the present paper, and those in the right hand sides are mainly expressed in the notation of [32]. One have to replace $q$ with $q^{-1}$ and $x$ with $x^{-1}$, and reverse the ordering of the lattice sites to make comparison.
Another way to construct Baxter TQ-relations for open boundaries would be to use a generating function of the T-operators for the anti-symmetric representations. In the case of the periodic boundary condition, it is a column-ordered determinant over a function of the monodromy matrix for a transfer matrix \[60\]. For models with open boundaries, one may have to use a dressed K-matrix (in our case, the universal dressed K-operator \((5.1)\)) instead of the usual monodromy matrix.

It is known that T-operators can be expressed as concise Wronskian-like determinants (Casoratian) in terms of Q-operators (in addition to the references for Baxter Q-operators referred in Introduction, see also \[61, 62, 63, 64, 65\] and references therein). In contrast with integrable systems with periodic boundary, not much is known about this for integrable systems with open boundaries (cf. \[66\]).

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**Appendix A: The universal R-matrix**

In this section, we briefly review the product expression of the universal R-matrix given by Khoroshkin and Tolstoy in \[41, 42\]. Their universal R-matrix was already reviewed by several authors (see for example, \[50, 51, 10, 12, 15, 31, 26\]). Here we basically follow these in the convention in Appendix A in \[31\].

Let \( \{ \alpha + k\delta \}_{k=0}^{\infty} \cup \{ k\delta \}_{k=1}^{\infty} \cup \{ \delta - \alpha + k\delta \}_{k=0}^{\infty} \) be a positive root system of \( \hat{sl}_2 \) in the notation of \[41\]. We fix the root ordering as \( \alpha + (k - 1)\delta \prec \alpha + k\delta \prec l\delta \prec (l + 1)\delta \prec \delta - \alpha + m\delta \prec \delta - \alpha + (m - 1)\delta \) for any \( k, l, m \in \mathbb{Z}_{\geq 1} \). In this case, the universal R-matrix has the following explicit expression:

\[
\mathcal{R} = \mathcal{K}^+ \mathcal{K}' \mathcal{K}^- q^\frac{1}{2} h_1 \otimes h_1,
\]

where each element is defined by

\[
\mathcal{K}^+ = \prod_{k=0}^{\infty} \operatorname{exp}_{q^{-2}} \left( \lambda e_{\alpha + k\delta} \otimes f_{\alpha + k\delta} \right),
\]

\[
\mathcal{K}' = \exp \left( \lambda \sum_{k=1}^{\infty} \frac{k}{[2k]_q} e_{k\delta} \otimes f_{k\delta} \right),
\]

\[
\mathcal{K}^- = \prod_{k=0}^{\infty} \operatorname{exp}_{q^{-2}} \left( \lambda e_{\delta - \alpha + k\delta} \otimes f_{\delta - \alpha + k\delta} \right).
\]
Let $e_\alpha = e_1$, $e_{\delta - \alpha} = e_0$, $f_\alpha = f_1$, $f_{\delta - \alpha} = f_0$. Then the other root vectors are defined by the following recursion relations:

$$e_{\alpha + k\delta} = [2]_q^{-1} [e_{\alpha + (k-1)\delta}, e']_{k\delta},$$

$$e'_{k\delta} = [e_{\alpha + (k-1)\delta}, e_{\delta - \alpha}]_{q^{-2}},$$

$$e_{\delta - \alpha + k\delta} = [2]_q^{-1} [e', e_{\delta - \alpha + (k-1)\delta}],$$

$$f_{\alpha + k\delta} = [2]_q^{-1} [f_{\delta}, f_{\alpha + (k-1)\delta}],$$

$$f'_{k\delta} = [f_{\delta - \alpha}, f_{\alpha + (k-1)\delta}]_q^2,$$

$$f_{\delta - \alpha + k\delta} = [2]_q^{-1} [f_{\delta - \alpha + (k-1)\delta}, f'_{k\delta}], \quad k \in \mathbb{Z}_{\geq 1},$$

and the following generating functions:

$$\lambda \sum_{k=1}^{\infty} e_{k\delta} z^{-k} = \log \left(1 + \lambda \sum_{k=1}^{\infty} e'_{k\delta} z^{-k}\right),$$

$$-\lambda \sum_{k=1}^{\infty} f_{k\delta} z^{-k} = \log \left(1 - \lambda \sum_{k=1}^{\infty} f'_{k\delta} z^{-k}\right), \quad z \in \mathbb{C}.$$

In general, root vectors contain many ($q$-deformed) commutators. However, simplification occurs under the evaluation map. For $k \in \mathbb{Z}_{\geq 0}$, we have

$$\text{ev}_x(e_{\alpha + k\delta}) = (-1)^k x^{ks+s_1} q^{-kH} E,$$

$$\text{ev}_x(e_{\delta - \alpha + k\delta}) = (-1)^k x^{ks+s_0} F q^{-kH},$$

$$\text{ev}_x(f_{\alpha + k\delta}) = (-1)^k x^{-ks-s_1} F q^{kH},$$

$$\text{ev}_x(f_{\delta - \alpha + k\delta}) = (-1)^k x^{-ks-s_0} q^{kH} E,$$

and for $k \in \mathbb{Z}_{\geq 1},$

$$\text{ev}_x(e'_{k\delta}) = (-1)^{k-1} x^{ks} q^{-(k-1)H-k} \left(\lambda [k]_q C - \frac{[k-1]_q q^H + [k+1]_q q^{-H}}{\lambda}\right),$$

$$\text{ev}_x(e_{k\delta}) = \frac{(-1)^{k-1} q^{-k} x^{ks}}{(q - q^{-1})k} \left(C_k - (q^k + q^{-k}) q^{-kH}\right),$$

$$\text{ev}_x(f'_{k\delta}) = (-1)^{k-1} x^{-ks} q^{(k-1)H+k} \left(-\lambda [k]_q C + \frac{[k+1]_q q^H + [k-1]_q q^{-H}}{\lambda}\right),$$

$$\text{ev}_x(f_{k\delta}) = -\frac{(-1)^{k-1} q^{k} x^{-ks}}{(q - q^{-1})k} \left(C_k - (q^k + q^{-k}) q^{kH}\right),$$

18
where the central elements $C_k$ are defined by

$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1} C_k}{k} z^{-k} = \log(1 + \lambda^2 C z^{-1} + z^{-2}), \quad z \in \mathbb{C}. \quad (A13)$$

In addition to the root vectors defined above, we will also use another set of root vectors. Let $\overline{e}_\alpha = e_\alpha$, $\overline{e}_{\delta-a} = e_{\delta-a}$, $\overline{f}_\alpha = f_\alpha$, $\overline{f}_{\delta-a} = f_{\delta-a}$. Then the other root vectors are defined by the following recursion relations:

$$\overline{e}_{\alpha+k\delta} = [2]_q^{-1} [\overline{e}_{\alpha+(k-1)\delta}, \overline{e}_\delta],$$

$$\overline{e}_{\delta-a+k\delta} = [2]_q^{-1} [\overline{e}_\delta, \overline{e}_{\delta-a+(k-1)\delta}],$$

$$\overline{f}_{\alpha+k\delta} = [2]_q^{-1} [\overline{f}_{\delta-a+(k-1)\delta}, \overline{f}_{\alpha}],$$

$$\overline{f}_{\delta-a+k\delta} = [2]_q^{-1} [\overline{f}_{\delta-a+(k-1)\delta}, \overline{f}_{\delta}], \quad k \in \mathbb{Z}_{\geq 1}, \quad (A14)$$

and the following generating functions:

$$-\lambda \sum_{k=1}^{\infty} q^{-2k} \overline{e}_{k\delta} z^{-k} = \log \left( 1 - \lambda \sum_{k=1}^{\infty} q^{-2k} \overline{e}_{k\delta} z^{-k} \right),$$

$$\lambda \sum_{k=1}^{\infty} q^{2k} \overline{f}_{k\delta} z^{-k} = \log \left( 1 + \lambda \sum_{k=1}^{\infty} q^{2k} \overline{f}_{k\delta} z^{-k} \right), \quad z \in \mathbb{C}. \quad (A15)$$

One can prove the following relations by induction.

$$\sigma(e_{\alpha+k\delta}) = q^{-2k} \overline{e}_{\alpha+k\delta}, \quad \sigma(e_{\delta-a+k\delta}) = q^{-2k} \overline{e}_{\delta-a+k\delta},$$

$$\sigma(f_{\alpha+k\delta}) = q^{2k} \overline{f}_{\alpha+k\delta}, \quad \sigma(f_{\delta-a+k\delta}) = q^{2k} \overline{f}_{\delta-a+k\delta} \text{ for } k \in \mathbb{Z}_{\geq 0},$$

$$\sigma(e'_{k\delta}) = -q^{-2k} \overline{e}_{k\delta}, \quad \sigma(f'_{k\delta}) = -q^{2k} \overline{f}_{k\delta},$$

$$\sigma(e_{k\delta}) = -q^{-2k} \overline{e}_{k\delta}, \quad \sigma(f_{k\delta}) = -q^{2k} \overline{f}_{k\delta} \text{ for } k \in \mathbb{Z}_{\geq 1}. \quad (A16)$$

**Appendix B: Derivation of the universal Baxter TQ-relations**

In this section, we derive the universal Baxter TQ-relations (5.9) under the assumption that convergence and cyclicity of the traces in the T-and Q-operators hold. Baxter TQ-relations with fixed quantum spaces were already derived for the open XXX-spin chains.

---

15 This assumption was already verified for concrete models in [30, 32], but remains an open problem on the level of the universal Baxter Q-operators.
and for the open XXZ-spin chains \[32\]. Here we consider the problem on the level of the universal T-and Q-operators.

We introduce two kinds of elements \( G, \overline{G} \in \text{Osc}_1 \otimes \text{End}(\mathbb{C}^2) \) and their inverse:

\[
G = q^{\frac{h_1}{2}} \otimes E_{11} + q^{\frac{h_1}{2}} \otimes E_{22} - \lambda q^{\frac{-h_1}{2} - \frac{\sigma_0}{2}} f_1 \otimes E_{12}, \quad (B1)
\]

\[
G^{-1} = q^{\frac{h_1}{2}} \otimes E_{11} + q^{-\frac{h_1}{2}} \otimes E_{22} + \lambda q^{\frac{-h_1}{2} - \frac{\sigma_0}{2}} f_1 \otimes E_{12}, \quad (B2)
\]

\[
\overline{G} = q^{-\frac{h_1}{2}} \otimes E_{11} + q^{\frac{h_1}{2}} \otimes E_{22} - \lambda q^{-\frac{-h_1}{2} + \frac{\sigma_1}{2}} f_1 \otimes E_{12}, \quad (B3)
\]

\[
\overline{G}^{-1} = q^{\frac{h_1}{2}} \otimes E_{11} + q^{-\frac{h_1}{2}} \otimes E_{22} + \lambda q^{-\frac{-h_1}{2} + \frac{\sigma_1}{2}} f_1 \otimes E_{12}. \quad (B4)
\]

One can check the following relations by direct calculations.

\[
G^{-1} (\rho^{(1)}_{xq^{-\frac{1}{2}}} \otimes \pi_1) \Delta(q^{\xi_{h0}}) G = q^{-\xi_{(h_1+1)}} \otimes E_{11} + q^{-\xi_{(h_1-1)}} \otimes E_{22}
\]

\[
= (q^{\frac{-2h_1}{2x}} h_1)^{\rho^{(1)}_{xq^{\frac{1}{2}}}} (q^{\xi_{(h_0-1)}} q^{\frac{-2h_1}{2x}} h_1) \otimes E_{11} + (q^{\frac{-2h_1}{2x}} h_1)^{\rho^{(1)}_{xq^{-\frac{1}{2}}}} (q^{\xi_{(h_0+1)}} q^{\frac{-2h_1}{2x}} h_1) \otimes E_{22}, \quad (B5)
\]

\[
G^{-1} (\rho^{(1)}_{xq^{\frac{1}{2}}} \otimes \pi_1) \Delta(q^{\xi_{h1}}) G = q^{\xi_{(h_1+1)}} \otimes E_{11} + q^{\xi_{(h_1-1)}} \otimes E_{22}
\]

\[
= (q^{\frac{-2h_1}{2x}} h_1)^{\rho^{(1)}_{xq^{\frac{1}{2}}}} (q^{\xi_{(h_1+1)}} q^{\frac{-2h_1}{2x}} h_1) \otimes E_{11} + (q^{\frac{-2h_1}{2x}} h_1)^{\rho^{(1)}_{xq^{-\frac{1}{2}}}} (q^{\xi_{(h_1-1)}} q^{\frac{-2h_1}{2x}} h_1) \otimes E_{22}, \quad (B6)
\]

\[
G^{-1} (\rho^{(1)}_{xq^{\frac{1}{2}}} \otimes \pi_1) \Delta(e_0) G = x^{s_0} q^{1-\frac{2h_1}{2x}} f_1 \otimes E_{11} + x^{s_0} q^{-1-\frac{2h_1}{2x}} f_1 \otimes E_{22} + x^{s_0} \otimes E_{21}
\]

\[
= (q^{\frac{-2h_1}{2x}} h_1)^{\rho^{(1)}_{xq^{\frac{1}{2}}}} (e_0) q^{\frac{-2h_1}{2x}} h_1 \otimes E_{11} + (q^{\frac{-2h_1}{2x}} h_1)^{\rho^{(1)}_{xq^{-\frac{1}{2}}}} (e_0) q^{\frac{-2h_1}{2x}} h_1 \otimes E_{22} + x^{s_0} \otimes E_{21}, \quad (B7)
\]

\[
G^{-1} (\rho^{(1)}_{xq^{\frac{1}{2}}} \otimes \pi_1) \Delta(e_1) G = x^{s_1} q^{1-\frac{2h_1}{2x}} e_1 \otimes E_{11} + x^{s_1} q^{-1-\frac{2h_1}{2x}} e_1 \otimes E_{22}
\]

\[
= (q^{\frac{-2h_1}{2x}} h_1)^{\rho^{(1)}_{xq^{\frac{1}{2}}}} (e_1) q^{\frac{-2h_1}{2x}} h_1 \otimes E_{11} + (q^{\frac{-2h_1}{2x}} h_1)^{\rho^{(1)}_{xq^{-\frac{1}{2}}}} (e_1) q^{\frac{-2h_1}{2x}} h_1 \otimes E_{22}, \quad (B8)
\]

\[
\overline{G}^{-1} (\rho^{(1)}_{xq^{\frac{1}{2}}} \otimes \pi_1) \Delta'(q^{\xi_{h0}}) \overline{G} = q^{\xi_{(h_1+1)}} \otimes E_{11} + q^{-\xi_{(h_1-1)}} \otimes E_{22}
\]

\[
= (q^{\frac{-2h_1}{2x}} h_1)^{\rho^{(1)}_{xq^{\frac{1}{2}}}} (q^{\xi_{(h_0-1)}} q^{\frac{-2h_1}{2x}} h_1) \otimes E_{11} + (q^{\frac{-2h_1}{2x}} h_1)^{\rho^{(1)}_{xq^{-\frac{1}{2}}}} (q^{\xi_{(h_0+1)}} q^{\frac{-2h_1}{2x}} h_1) \otimes E_{22}, \quad (B9)
\]

\[
\overline{G}^{-1} (\rho^{(1)}_{xq^{\frac{1}{2}}} \otimes \pi_1) \Delta'(q^{\xi_{h1}}) \overline{G} = q^{\xi_{(h_1+1)}} \otimes E_{11} + q^{\xi_{(h_1-1)}} \otimes E_{22}
\]

\[
= (q^{\frac{-2h_1}{2x}} h_1)^{\rho^{(1)}_{xq^{\frac{1}{2}}}} (q^{\xi_{(h_1+1)}} q^{\frac{-2h_1}{2x}} h_1) \otimes E_{11} + (q^{\frac{-2h_1}{2x}} h_1)^{\rho^{(1)}_{xq^{-\frac{1}{2}}}} (q^{\xi_{(h_1-1)}} q^{\frac{-2h_1}{2x}} h_1) \otimes E_{22}, \quad (B10)
\]

\[
\overline{G}^{-1} (\rho^{(1)}_{xq^{\frac{1}{2}}} \otimes \pi_1) \Delta'(f_0) \overline{G} = x^{-s_1} q^{1-\frac{2h_1}{2x}} e_1 \otimes E_{11} + x^{-s_0} q^{-1-\frac{2h_1}{2x}} e_1 \otimes E_{22}
\]
\[
= (q^{\frac{z_1 \cdots z_n h_{1}}{2}} \rho_{xy}^{(1)} (f_0 q^{\frac{\Delta h_{1}}{2}}) \otimes E_{11} + (q^{\frac{z_1 \cdots z_n h_{1}}{2}} \rho_{xy}^{(1)} (f_0 q^{\frac{\Delta h_{1}}{2}}) \otimes E_{22}, \quad (B11)
\]
\[
\bar{G}^{-1} \rho_{xy}^{(1)} \otimes \pi_1) \Delta' (f_1) \bar{G} = x^{-s_1} q^{1- \frac{\Delta}{2}} f_1 \otimes E_{11} + x^{s_1} q^{1- \frac{\Delta}{2}} f_1 \otimes E_{22} + x^{-s_1} \otimes E_{21}
\]
\[
= (q^{\frac{z_1 \cdots z_n h_{1}}{2}} \rho_{xy}^{(1)} (f_1 q^{\frac{\Delta h_{1}}{2}}) \otimes E_{11} + (q^{\frac{z_1 \cdots z_n h_{1}}{2}} \rho_{xy}^{(1)} (f_1 q^{\frac{\Delta h_{1}}{2}}) \otimes E_{22} + x^{-s_1} \otimes E_{21}, \quad (B12)
\]
where \( x \in \mathbb{C} \). Let us apply \([B5]-[B12]\) to the second equation in \([2.8]\). Taking note on the fact that the co-multiplication of the universal R-matrix has the form \([16]\) \((\Delta \otimes 1) \mathcal{R} = \tilde{\mathcal{R}} (\{ \Delta(e_0) \otimes 1, \Delta(e_1) \otimes 1, 1 \otimes 1 \otimes f_0, 1 \otimes 1 \otimes f_1 \}) q^{\Delta h_{11} / 2}\), and the relations \( E_{ij} E_{kl} = \delta_{ik} \delta_{jl} \), we obtain
\[
\mathcal{G}_{12}^{-1} \mathcal{L}_{13}^{(1)} (x q^{-\frac{\Delta}{2}}) \mathcal{L}_{23} (x) \mathcal{G}_{12} =
\]
\[
= (q^{\frac{z_1 \cdots z_n h_{1}}{2}} \otimes 1 \otimes 1) \mathcal{L}_{13}^{(1)} (x q^{\frac{\Delta}{2}}) (q^{\frac{z_1 \cdots z_n h_{1}}{2}} \otimes E_{11} \otimes q^{\frac{\Delta}{2}} h_{1})
\]
\[
+ (q^{\frac{z_1 \cdots z_n h_{1}}{2}} \otimes 1 \otimes 1) \mathcal{L}_{13}^{(1)} (x q^{\frac{\Delta}{2}}) (q^{\frac{z_1 \cdots z_n h_{1}}{2}} \otimes E_{22} \otimes q^{\frac{\Delta}{2}} h_{1})
\]
\[
+ \lambda x^{s_0} \mathcal{F}_{13} (x) (1 \otimes E_{21} \otimes q^{\frac{\Delta}{2}} h_{1}), \quad (B13)
\]
\[
\bar{G}_{12}^{-1} \bar{L}_{13}^{(1)} (x q^{\frac{\Delta}{2}}) \bar{L}_{23} (x) \bar{G}_{12} =
\]
\[
= (q^{\frac{z_1 \cdots z_n h_{1}}{2}} \otimes 1 \otimes 1) \bar{L}_{13}^{(1)} (x q^{-\frac{\Delta}{2}}) (q^{\frac{z_1 \cdots z_n h_{1}}{2}} \otimes E_{11} \otimes q^{\frac{\Delta}{2}} h_{1})
\]
\[
+ (q^{\frac{z_1 \cdots z_n h_{1}}{2}} \otimes 1 \otimes 1) \bar{L}_{13}^{(1)} (x q^{\frac{\Delta}{2}}) (q^{\frac{z_1 \cdots z_n h_{1}}{2}} \otimes E_{22} \otimes q^{\frac{\Delta}{2}} h_{1})
\]
\[
+ x^{-s_1} \lambda \bar{F}_{13} (x) (1 \otimes E_{21} \otimes q^{\frac{\Delta}{2}} h_{1}), \quad (B14)
\]
where \( \mathcal{F}_{13} (x) \) and \( \bar{F}_{13} (x) \) are elements in \( \text{Osc}_1 \otimes \text{End} (\mathbb{C}^2) \otimes \mathcal{B}_- \) and \( \text{Osc}_1 \otimes \text{End} (\mathbb{C}^2) \otimes \mathcal{B}_+ \), respectively. \([17]\) One can also show the following relations by direct calculations.

\[\text{See also similar discussions in section 4 in [13].}\]
\[\text{Although explicit expressions of them are not necessary for the proof of the universal Baxter TQ-relation, one can calculate them based on the explicit expression of the universal R-matrix. For example, we obtain}\]
\[
\mathcal{F}_{13} (x) = (q^{\frac{z_1 \cdots z_n h_{1}}{2}} \otimes 1 \otimes 1) \mathcal{L}_{13}^{(1)} (x q^{-\frac{\Delta}{2}}) (q^{\frac{z_1 \cdots z_n h_{1}}{2}} \otimes 1 \otimes 1)
\]
\[
\times q^{-\frac{\Delta}{2}} h_{11} \otimes 1 \otimes 1 \exp_{q^{-\frac{\Delta}{2}}}^{-1} (\lambda x^{s_0} q^{-1- \frac{\Delta}{2}} f_1 \otimes 1 \otimes f_0) \left( 1 \otimes 1 \otimes \sum_{k=0}^{\infty} (-q^{-1} x^k) k f_{n-a+k} \right)
\]
\[
\times \exp_{q^{-\frac{\Delta}{2}}} (\lambda x^{s_0} q^{-1- \frac{\Delta}{2}} f_1 \otimes 1 \otimes f_0) q^{\frac{\Delta}{2}} h_{11} \otimes 1 \otimes 1. \quad (B15)
\]
Evaluating \([B13]\) and \([B14]\) in the fundamental evaluation representation in the third component of the tensor product, we obtain
\[
\mathcal{G}_{12}^{-1} \mathcal{L}_{13}^{(1)} (x q^{-\frac{\Delta}{2}}) R_{23} (x) \mathcal{G}_{12} =
\]
\[
\]
This type of relations (B16) and (B17) are known in [7]. Baxter TQ-relations, appeared in [32]. Define permutation operators by [30]. Moreover, the diagonal parts of (B18) and (B19), which are essential in the proof of We remark that rational analogues of (B18) and (B19) were previously considered in [30]. Moreover, the diagonal parts of (B18) and (B19), which are essential in the proof of Baxter TQ-relations, appeared in [32]. Define permutation operators by

\[
p_{12}(X \otimes Y \otimes Z) = Y \otimes X \otimes Z, \quad p_{23}(X \otimes Y \otimes Z) = X \otimes Z \otimes Y;
\]

\[
p_{13}(X \otimes Y \otimes Z) = Z \otimes Y \otimes X \quad \text{for} \quad X, Y, Z \in U_q(s\mathfrak{l}_2). \quad (B21)
\]

Applying \(p_{13} \circ p_{12}\) to (2.9), we obtain

\[
R_{23}\overline{R}_{12}\overline{R}_{13} = \overline{R}_{13}\overline{R}_{12}R_{23}. \quad (B22)
\]

Then we evaluate (B22) under \(\rho_x^{(1)} \otimes \pi_y \otimes 1 \quad (x, y \in \mathbb{C}^\times)\), to get

\[
\mathcal{L}_{23}(y)\overline{\mathcal{L}}_{12}^{(y)} (xy^{-1})\mathcal{L}_{13}(x) = \mathcal{L}_{13}(x)\overline{\mathcal{L}}_{12}^{(y)} (xy^{-1})\mathcal{L}_{23}(y). \quad (B23)
\]

\[
= (q - q^{-1}x^s)(q^{\frac{s-1}{2}})^{h_1} \otimes 1 \otimes 1)\mathcal{L}_{13}^{(y)}(xy^{-1})(q^{\frac{s-1}{2}})^{h_1} \otimes E_{11} \otimes \pi(q^{\frac{s-1}{2}})) \quad + (1 - x^s)(q^{\frac{s-1}{2}})^{h_1} \otimes 1 \otimes 1)\mathcal{L}_{13}^{(y)}(xy^{-1})(q^{\frac{s-1}{2}})^{h_1} \otimes E_{22} \otimes \pi(q^{\frac{s-1}{2}})) \quad + \lambda x^{-s}(q^{\frac{s-1}{2}})^{h_1} \otimes 1 \otimes 1)\mathcal{L}_{13}^{(y)}(xy^{-1})(q^{\frac{s-1}{2}})^{h_1} \otimes E_{21} \otimes E_{12}), \quad (B16)
\]

\[
(1 - x^s)(q^{\frac{s-1}{2}})^{h_1} \otimes 1 \otimes 1)\mathcal{L}_{13}^{(y)}(xy^{-1})(q^{\frac{s-1}{2}})^{h_1} \otimes E_{22} \otimes \pi(q^{\frac{s-1}{2}})) \quad + \lambda x^{-s}(q^{\frac{s-1}{2}})^{h_1} \otimes 1 \otimes 1)\mathcal{L}_{13}^{(y)}(xy^{-1})(q^{\frac{s-1}{2}})^{h_1} \otimes E_{21} \otimes E_{12}), \quad (B16)
\]

\[
\mathcal{G}_{12}\mathcal{L}_{13}^{(y)}(xy^{-1})R_{23}(x)\mathcal{G}_{12} =
\]

\[
= (q - q^{-1}x^s)(q^{\frac{s-1}{2}})^{h_1} \otimes 1 \otimes 1)\mathcal{L}_{13}^{(y)}(xy^{-1})(q^{\frac{s-1}{2}})^{h_1} \otimes E_{11} \otimes \pi(q^{\frac{s-1}{2}})) \quad + (1 - x^s)(q^{\frac{s-1}{2}})^{h_1} \otimes 1 \otimes 1)\mathcal{L}_{13}^{(y)}(xy^{-1})(q^{\frac{s-1}{2}})^{h_1} \otimes E_{22} \otimes \pi(q^{\frac{s-1}{2}})) \quad + \lambda x^{-s}(q^{\frac{s-1}{2}})^{h_1} \otimes 1 \otimes 1)\mathcal{L}_{13}^{(y)}(xy^{-1})(q^{\frac{s-1}{2}})^{h_1} \otimes E_{21} \otimes E_{12}), \quad (B17)
\]

This type of relations (B16) and (B17) are known in [7].
Now we can show the relation \((5.9)\) for \(a = 1\) step by step as follows:\(^{18}\)

\[
(q^2 - q^4 x^{2s}) Q^{(a)}(q^\frac{1}{2} x) T(x)
\]
\[
= (q^2 - q^4 x^{2s}) \text{tr}_1 \left( \overline{K}_1^{(1)} (x^{-1} q^{-\frac{1}{2}}) K_{13}^{(1)} (x q^\frac{1}{2}) \right) \text{tr}_2 \left( K_2(x^{-1}) K_{23}(x) \right)
\]
\[
= (q^2 - q^4 x^{2s}) \text{tr}_{12} \left( \overline{K}_1^{(1)} (x^{-1} q^{-\frac{1}{2}}) t_1 K_{13}^{(1)} (x q^\frac{1}{2}) t_1 K_2(x^{-1}) K_{23}(x) \right)
\]
\[
= (q^2 - q^4 x^{2s}) \text{tr}_{12} \left( \overline{K}_1^{(1)} (x^{-1} q^{-\frac{1}{2}}) K_2(x^{-1}) K_{13}^{(1)} (x q^\frac{1}{2}) t_1 K_{23}(x) \right)
\]
\[
= \text{tr}_{12} \left( \overline{K}_1^{(1)} (x^{-1} q^{-\frac{1}{2}}) K_2(x^{-1}) g_2 L_{12}^{(1)} (x^{-2}q^{-\frac{5}{2}}) t_1 g_2 L_{12}^{(1)} (x^{-2}q^{-\frac{5}{2}}) t_1 K_{13}^{(1)} (x q^\frac{1}{2}) t_1 K_{23}(x) \right)
\]
\[
= \text{tr}_{12} \left( \overline{K}_1^{(1)} (x^{-1} q^{-\frac{1}{2}}) K_2(x^{-1}) g_2 L_{12}^{(1)} (x^{-2}q^{-\frac{5}{2}}) t_2 \cdot \left( K_{13}^{(1)} (x q^\frac{1}{2}) L_{12}^{(1)} (x^{-2}q^{-\frac{5}{2}}) t_1 t_2 K_{23}(x) \right) \right)
\]
\[
= \text{tr}_{12} \left( \overline{K}_1^{(1)} (x^{-1} q^{-\frac{1}{2}}) g_2 L_{12}^{(1)} (x^{-2}q^{-\frac{5}{2}}) g_2 K_2(x^{-1}) t_2 \right.
\]
\[
\cdot \left( L_{13}^{(1)} (x^{-1} q^{-\frac{1}{2}}) K_1^{(1)} (x q^\frac{1}{2}) L_{12}^{(1)} (x^{-1}) L_{12}^{(1)} (x^{-2}q^{-\frac{5}{2}}) L_{23}(x^{-1}) K_2(x) \overline{L}_{23}(x) \right)^{t_1} \right)
\]
\[
= \text{tr}_{12} \left( \overline{K}_1^{(1)} (x^{-1} q^{-\frac{1}{2}}) g_2 L_{12}^{(1)} (x^{-2}q^{-\frac{5}{2}}) g_2 K_2(x^{-1}) t_2 \right.
\]
\[
\cdot \left( L_{13}^{(1)} (x^{-1} q^{-\frac{1}{2}}) K_1^{(1)} (x q^\frac{1}{2}) L_{23}(x^{-1}) L_{12}^{(1)} (x^{-2}q^{-\frac{5}{2}}) L_{13}^{(1)} (x q^\frac{1}{2}) K_2(x) \overline{L}_{23}(x) \right)^{t_1} \right)
\]
\[
= \text{tr}_{12} \left( \overline{K}_1^{(1)} (x^{-1} q^{-\frac{1}{2}}) g_2 L_{12}^{(1)} (x^{-2}q^{-\frac{5}{2}}) g_2 K_2(x^{-1}) t_1 t_2 \right.
\]
\[
\cdot \left( L_{13}^{(1)} (x^{-1} q^{-\frac{1}{2}}) L_{23}(x^{-1}) K_1^{(1)} (x q^\frac{1}{2}) L_{12}^{(1)} (x^{-2}q^{-\frac{5}{2}}) K_2(x) \overline{L}_{13}^{(1)} (x q^\frac{1}{2}) \overline{L}_{23}(x) \right) \right)
\]
\[
= \text{tr}_{12} \left( \overline{K}_1^{(1)} (x^{-1} q^{-\frac{1}{2}}) g_2 L_{12}^{(1)} (x^{-2}q^{-\frac{5}{2}}) g_2 \overline{K}_2(x^{-1}) \overline{G}_{12}^{(1)} t_1 t_2 \right.
\]
\[
\cdot \left( G_{12}^{(1)} L_{13}^{(1)} (x^{-1} q^{-\frac{1}{2}}) L_{23}(x^{-1}) G_{12} \cdot G_{12}^{(1)} K_1^{(1)} (x q^\frac{1}{2}) L_{12}^{(1)} (x^{-2}q^{-\frac{5}{2}}) K_2(x) \overline{G}_{12} \right) \right)
\]

\(^{18}\)Here \(\text{tr}_1 = \text{tr}_{V_1} \otimes 1 \otimes 1\), \(\text{tr}_2 = 1 \otimes \text{tr} \otimes 1\), and the third component of the tensor product is in \(U_q(sl_2)\). The parts which contribute to the trace \(\text{tr}_1\) are linear combinations of the terms of the from \(e_1^n f_1^n q^{2b_1}\), \(n \in \mathbb{Z}_{\geq 0}, \xi \in \mathbb{C}\). They are invariant under the anti-involution \(\xi \leftrightarrow q^{-\xi}\); \((e_1^n f_1^n q^{2b_1})^* = e_1^n f_1^n q^{2b_1}\). We will also use the invariance \(g^* = g\), \(\overline{K}^{(1)}(x)^* = \overline{K}^{(1)}(x)\) and \(\overline{K}(x)^* = \overline{K}(x)\).
Applying in this section, we prove (5.6) based on a universal version of Sklyanin’s method [39].

The parts which do not contribute to the trace are

\[
\omega_1^{(1)}(x)q^{-(\frac{2s_0^{-1}-1}{2s_0})h_1}K^{(1)}(x^{-\frac{1}{2}})\otimes E_{11} \otimes 1
\]

Applying the map \(\omega_2^{(1)}(x)q^{\frac{2s_0}{3s_0}-\frac{1}{2}}h_1K^{\prime (1)}\) to (2.12), we obtain

\[
= \text{tr}_{12}\left(\left(\omega_1^{(1)}(x)q^{-(\frac{2s_0^{-1}-1}{2s_0})h_1}K^{(1)}(x^{-\frac{1}{2}})\otimes E_{11} \otimes 1\right)\right)
\]

where \(\ldots\) are the parts which do not contribute to the trace. Applying the map \(\zeta \circ \sigma\) to (5.9) for \(a = 1\), one can show (5.9) for \(a = 2\).

**Appendix C: Proof of the universal dressed reflection equation**

In this section, we prove [5.6] based on a universal version of Sklyanin’s method [39]. Applying \(p_{23}\) and \(p_{23} \circ p_{13} \circ p_{12}\), respectively, to (2.13), we obtain

\[
\mathcal{R}_{13}\mathcal{R}_{12}\mathcal{R}_{23} = \mathcal{R}_{23}\mathcal{R}_{12}\mathcal{R}_{13}, \quad (C1)
\]

\[
\mathcal{R}_{12}\mathcal{R}_{13}\mathcal{R}_{23} = \mathcal{R}_{23}\mathcal{R}_{13}\mathcal{R}_{12}. \quad (C2)
\]
Then we evaluate (2.9) and (C1)–(C2) under $\rho_{x}^{(a)} \otimes \pi_{y} \otimes 1$ ($a = 1, 2; x, y \in \mathbb{C}^\times$), to get

$$L_{12}^{(a)}(xy^{-1})L_{13}^{(a)}(x)L_{23}(y) = L_{23}(y)L_{13}^{(a)}(x)L_{12}^{(a)}(xy^{-1}), \tag{C3}$$

$$L_{13}^{(a)}(x)L_{12}^{(a)}(xy^{-1})L_{23}(y) = L_{23}(y)L_{12}^{(a)}(xy^{-1})L_{13}^{(a)}(x), \tag{C4}$$

$$L_{23}(y)L_{13}^{(a)}(x)L_{12}(xy^{-1}) = L_{12}(xy^{-1})L_{13}^{(a)}(x)L_{23}(y). \tag{C5}$$

One can prove (5.6) step by step as follows:

$$L_{12}^{(a)}(x^{-1}y)K_{13}^{(a)}(x)K_{12}^{(a)}(xy)K_{23}(y) = \text{apply (5.6)}$$

$$= L_{12}^{(a)}(x^{-1}y)L_{13}^{(a)}(x^{-1})K_{1}^{(a)}(x)L_{12}^{(a)}(xy)L_{23}(y^{-1})K_{2}(y)L_{23}(y) \tag{C6}$$

One can also prove \( (5.2) \) similarly (evaluate (2.9), (B22), (C1) and (C2) under $\pi_{x} \otimes \pi_{y} \otimes 1$ $x, y \in \mathbb{C}^\times$).

\(^{19}\)We also remark that a universal dressed reflection equation in [31] (eq. (G.7) in [31]) can be proven similarly (evaluate (2.9) and (B22), (C1) and (C2) under $ev_{x} \otimes \pi_{y} \otimes 1$ $x, y \in \mathbb{C}^\times$). In fact, (5.4) (for $a = 1$) is a limit of eq. (G.7) in [31] ($q^{-\mu} \to 0$, in the notation of [31]), and (5.2) is the image of eq. (G.7) in [31] under $\pi \otimes 1 \otimes 1$. 


Appendix D: Generic unitarity relations of R-operators

It is known that the R-matrices ((3.4) and (3.5)) of the 6-vertex model satisfy the unitarity relation

\[ R(x)R(x) = R(x)R(x) = (q^2 + q^{-2} - x^s - x^{-s})I \otimes I, \]  

(D1)

where \( I \) is the \( 2 \times 2 \) unit matrix. Here we reconsider this type of relations in general situation. We define the generic R-operators by

\[ R(xy^{-1}) = R(x, y) = (ev_x \otimes ev_y)R, \quad R(xy^{-1}) = R(x, y) = (ev_x \otimes ev_y)\overline{R}, \]  

(D2)

where \( x, y \in \mathbb{C} \times \mathbb{C} \). For any finite dimensional irreducible representations \( \chi_1, \chi_2 \) of \( U_q(sl_2) \), we set

\[ R_{\chi_1,\chi_2}(x) = (\chi_1 \otimes \chi_2)R(x), \quad \overline{R}_{\chi_1,\chi_2}(x) = (\chi_1 \otimes \chi_2)\overline{R}(x). \]  

(D3)

Then the following relation holds \[67\]

\[ R_{\chi_1,\chi_2}(x)R_{\chi_1,\chi_2}(x) = R_{\chi_1,\chi_2}(x)R_{\chi_1,\chi_2}(x) = S(x)(I_1 \otimes I_2), \]  

(D4)

where \( S(x) \) is a scalar function \(^{20}\) on \( x \), and \( I_1 \) and \( I_2 \) are unit matrices. This implies the following generic unitarity relation:

\[ R(x)\overline{R}(x) = R(x)R(x) = C(x), \]  

(D5)

where \( C(x) \) is central on \( U_q(sl_2) \otimes U_q(sl_2) \). Derivation of (D5) from (D4) is given as follows \[68\]. Let \( \mathfrak{g} \) be a finite dimensional Lie algebra. The following proposition is well known.

**Proposition** (Page 71, Proposition 5.11 in \[46\]) Let \( u \in U_q(\mathfrak{g}) \). If \( u \) annihilates all finite dimensional irreducible \(^2 U_q(\mathfrak{g}) \)-modules, then \( u = 0 \).

Then one can show the following.

**Corollary** If \( u \in U_q(\mathfrak{g}) \) is scalar on all finite dimensional irreducible \( U_q(\mathfrak{g}) \)-modules, then \( u \) belongs to the center of \( U_q(\mathfrak{g}) \).

**Proof.** Take any element \( a \in U_q(\mathfrak{g}) \) and apply Proposition to \([u,a] = 0 \). This means that \( u \) is central since \( a \) is arbitrarily. \( \square \)

Let us regard (D4) as a matrix with respect to the second component of the tensor product. The matrix elements of this matrix are scalar (or 0) for any finite dimensional irreducible representation \( \chi_1 \) of \( U_q(sl_2) \). Then Corollary suggests (D5).

---

\(^{20}\) \( S(x) \equiv 1 \) in the normalization of the R-matrices in \[67\]. But this is not the case with our R-matrices. 
\(^2\) The word ‘irreducible’ is not explicitly written in the corresponding proposition in \[46\]. However this is not matter since any finite dimensional representation of \( U_q(\mathfrak{g}) \) is completely reducible.

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