LIPSCHITZ METRIC FOR THE TWO-COMPONENT CAMASSA– HOLM SYSTEM

KATRIN GRUNERT
Department of Mathematical Sciences
Norwegian University of Science and Technology
NO-7491 Trondheim, Norway

HELGE HOLDEN
Department of Mathematical Sciences
Norwegian University of Science and Technology
NO-7491 Trondheim, Norway
and
Centre of Mathematics for Applications
University of Oslo
NO-0316 Oslo, Norway

XAVIER RAYNAUD
Centre of Mathematics for Applications
University of Oslo
NO-0316 Oslo, Norway

Abstract. We construct a Lipschitz metric for conservative solutions of the Cauchy problem on the line for the two-component Camassa–Holm system

\begin{align*}
  u_t - u_{txx} + 3uu_x - 2u_xu_{xx} - uu_{xxx} + \rho \rho_x = 0, \\
  \rho_t + (uu) x = 0
\end{align*}

with given initial data \((u_0, \rho_0)\). The Lipschitz metric \(d_{DM}\) has the property that for two solutions \(z(t) = (u(t), \rho(t), \mu_t)\) and \(\tilde{z}(t) = (\tilde{u}(t), \tilde{\rho}(t), \tilde{\mu}_t)\) of the system we have

\[ d_{DM}(z(t), \tilde{z}(t)) \leq C_{M,T} d_{DM}(z_0, \tilde{z}_0) \quad \text{for} \quad t \in [0,T]. \]

Here the measure \(\mu_t\) is such that its absolutely continuous part equals the energy \((u^2 + u^2 x + \rho^2) (t) dx\), and the solutions are restricted to a ball of radius \(M\).

1. Introduction

The two-component Camassa–Holm (2CH) system, which was first derived in [22, Eq. (43)], is given by

\begin{align*}
  u_t - u_{txx} + 3uu_x - 2u_xu_{xx} - uu_{xxx} + \rho \rho_x &= 0, \quad (1.1a) \\
  \rho_t + (uu) x &= 0, \quad (1.1b)
\end{align*}

or equivalently

\begin{align*}
  u_t + uu_x + P_x &= 0, \quad (1.2a) \\
  \rho_t + (uu) x &= 0, \quad (1.2b)
\end{align*}

where \(P\) is implicitly defined by

\[ P - P_{xx} = u^2 + \frac{1}{2} u_x^2 + \frac{1}{2} \rho^2. \quad (1.3) \]

The Camassa–Holm equation [6, 7] is obtained by considering the case when \(\rho\) vanishes identically. The aim of this article is to present the construction of a Lipschitz metric for this system on the real line with vanishing asymptotics, that is, \(u \in H^1\).

1991 Mathematics Subject Classification. Primary: 35Q53, 35B35; Secondary: 35Q20.

Key words and phrases. Two-component Camassa–Holm system, Lipschitz metric, conservative solutions.
and \( \rho \in L^2 \). The conservative solutions to (1.2) are constructed in [15] for non-vanishing asymptotics. A Lipschitz metric for the system with periodic boundary conditions is given in [17]. We here combine the two approaches by constructing a Lipschitz metric for conservative, decaying solutions. The preservation of the energy is needed in the proofs so that the construction of the metric only applies to vanishing asymptotics. Here we rather describe and motivate the general ideas behind the construction, which we hope can be of interest in the study of other related equations. For more background on the two-component Camassa–Holm system, we refer to [15] and the references therein. For related papers, see [4, 5, 20, 19].

2. Relaxation of the equations by the introduction of Lagrangian coordinates

The change of coordinates from Eulerian to Lagrangian coordinates has relaxation properties which are well-known for the Burgers equation, viz.

\[
\begin{align*}
    u_t + uu_x &= 0. 
\end{align*}
\]  

(2.1)

Lagrangian coordinates are defined by characteristics

\[
    y_t(t, \xi) = u(t, y(t, \xi)),
\]

which give the position of a particle which moves in the velocity field \( u \) and its velocity, known as the Lagrangian velocity, is given by

\[
    U(t, \xi) = u(t, x), \quad x = y(t, \xi).
\]

The method of characteristics consists of rewriting (2.1) in terms of the Lagrangian variables and yields

\[
    \begin{align*}
        y_t &= U, \\
        U_t &= 0. 
    \end{align*}
\]  

(2.2)

Comparing (2.1) to (2.2), we observe that we start with a nonlinear and partial (derivatives with respect to \( t \) and \( x \)) differential equation and end up with a linear and ordinary (derivative only with respect to \( t \)) differential equation. We get rid of the nonlinear convection term, and (2.2) is nothing but Newton’s law, which states that the acceleration is constant in the absence of forces. A well-known drawback of the change of coordinates from Eulerian to Lagrangian coordinates is that it doubles the dimension of the problem: We start with a scalar equation and end up with a system of dimension two. This is an important issue and we will deal with it in Section 4. However, in return, we gain the possibility to represent a larger class of objects or, more precisely in our case, to increase the regularity of the unknown functions. Let us make this imprecise statement clearer by an example and, to do so, we drop the dependence in \( t \) in the notation, as we look at singularities in the space variable. The function \( u(x) \) can be represented by its graph \( (x, u(x)) \) but this graph can itself be represented as a parametric curve, namely, \( (y(\xi), U(\xi)) \) and, as we know, the set of graphs is smaller than the set of parametric curves. As far as regularity is concerned, the Heaviside function

\[
    h(x) = \begin{cases} 
        0 & \text{if } x < 0, \\
        1 & \text{if } x \geq 0, 
    \end{cases}
\]

is only of bounded variation but it can be represented in Lagrangian coordinates by the following pair of more regular (in this case Lipschitz) functions

\[
    \begin{align*}
        y(\xi) &= \begin{cases} 
            \xi & \text{if } \xi < 0, \\
            0 & \text{if } \xi \in [0, 1), \\
            \xi - 1 & \text{if } \xi \geq 1, 
        \end{cases} \\
        H(\xi) &= \begin{cases} 
            0 & \text{if } \xi < 0, \\
            \xi & \text{if } \xi \in [0, 1), \\
            1 & \text{if } \xi \geq 1 
        \end{cases}
    \end{align*}
\]  

(2.3)
Indeed, \((x, h(x))\) and \((y(\xi), H(\xi))\) represent one and the same curve, except for the vertical line joining the origin to the point \((0, 1)\). We will return to this example later. The solution of the Camassa–Holm equation (i.e., where \(\rho\) vanishes identically) experiences in general wave breaking (i.e., loss of of regularity in the sense that the spatial derivative becomes unbounded while keeping the \(H^1\) norm finite) in finite time ([9, 10, 11]) and the antisymmetric peakon-antipeakon solution, which is described in [19] and depicted in Figure 1, helps us to understand how the solutions can be prolonged in a way which preserves the energy.

At collision time \(t_c\), we have
\[
\lim_{t \to t_c} u(t, x) = 0 \text{ in } L^\infty, \\
\lim_{t \to t_c} u_x(t, 0) = -\infty,
\]
while the \(H^1\) norm is constant so that \(\lim_{t \to t_c} \|u(t, \cdot)\|_{H^1} = \|u(0, \cdot)\|_{H^1}\). To obtain the conservative solution, we need to track the amount and the location of the concentrated energy. The function \(u\) alone cannot provide this information as \(u(t_c, \cdot)\) is identically zero. Thus, we have to introduce an extra variable to describe the solutions. In Lagrangian variables, it takes the form of the cumulative energy \(H(t, \xi)\), which is given by
\[
H(t, \xi) = \int_{-\infty}^{y(t, \xi)} (u^2 + u_x^2 + \rho^2)(x)dx. \tag{2.4}
\]
We will introduce later its counter-part in Eulerian variables. Equation (1.1b) transports the density \(\rho\). Formally, after changing variables, we have \(\rho(x)\,dx = \rho(y)\,dy = \rho(y)\,y_\xi\,d\xi\), so that the Lagrangian variable corresponding to \(\rho\) is given by
\[
r(t, \xi) = \rho(t, y(t, \xi))y_\xi(t, \xi). \tag{2.5}
\]

Next, we rewrite (1.2) in the Lagrangian variables \((y, U, H, r)\). We obtain the following system
\[
\zeta_t = U, \\
U_t = -Q, \\
H_t = U^3 - 2PU, \\
r_t = 0, \tag{2.6}
\]
where \(\zeta(t, \xi) = y(t, \xi) - \xi\),
\[
P(t, \xi) = \frac{1}{4} \int_{\mathbb{R}} \exp(-|y(t, \xi) - y(t, \eta)|)(U^2 y_\xi + H_\xi)(t, \eta)d\eta. \tag{2.7}
\]
and
\[ Q(t, \xi) = -\frac{1}{4} \int_{\mathbb{R}} \text{sign}(y(t, \xi) - y(t, \eta)) \exp(-|y(t, \xi) - y(t, \eta)|)(U^2 y_\xi + H_\xi) (t, \eta) d\eta. \]

See [15] for more details on this derivation. After differentiation, we obtain
\[ y_\xi t = U_\xi, \quad \text{(2.9a)} \]
\[ U_\xi t = \frac{1}{2} H_\xi + (\frac{1}{2} U^2 - P)y_\xi, \quad \text{(2.9b)} \]
\[ H_\xi t = (3U^2 - 2P)U_\xi - 2QUy_\xi, \quad \text{(2.9c)} \]
\[ r_t = 0. \quad \text{(2.9d)} \]

This system is semilinear and we recognize some features observed earlier for the Burgers equation: We start from a nonlinear partial differential equation and we end up with a system of ordinary differential equations which is semilinear. We consider the system as an ordinary differential equation because the order of the spatial derivative is the same on both sides of the equation, so that the existence and uniqueness of solutions can be established by a contraction argument. Finally, it is important to recall in this section the geometric nature of the Camassa–Holm equation. The equation is a geodesic in the group of diffeomorphism for the $H^1$ norm, see, e.g., [12], as the Burgers equation for the $L^2$ norm. Using the connection between geometry and fluid mechanics, as presented in [1], the function $t \mapsto y(t, \xi)$ can then be understood as a path in the group of diffeomorphisms. Thus besides the relaxation properties we have just described, this interpretation adds a direct geometrical relevance to use of Lagrangian coordinates, see also [13] for the system.

3. Semigroup in Lagrangian coordinates

In [15, Theorem 3.2], we prove by a contraction argument that short-time solutions to (2.6) exist in a Banach space, which we will here denote $E$ and define as follows. Let $V$ be the Banach space defined by
\[ V = \{ f \in L^\infty \mid f_\xi \in L^2 \} \]
and the norm of $V$ is given by $\|f\|_V = \|f\|_{L^\infty} + \|f_\xi\|_{L^2}$. We set $E$
\[ E = V \times H^1 \times V \times L^2 \]
with the following norm $\|X\| = \|\zeta\|_V + \|U\|_{H^1} + \|H\|_V + \|r\|_{L^2}$ for any $X = (\zeta, U, H, r) \in E$. Given a constant $M > 0$, we denote by $B_M$ the ball
\[ B_M = \{ X \in E \mid \|X\| \leq M \}. \]

Short-time solutions of (2.9) cannot in general be extended to global solutions. The challenge is to identify an appropriate set of initial data for which one can construct global solutions that at the same time preserve the structure of the equations, allowing us to return to the Eulerian variables. There are intrinsic relations between the variables in (2.9) that need to be conserved by the solution. This is handled by the set $G$ defined below. In particular, the set $G$ is preserved by the flow.

**Definition 3.1.** The set $G$ is composed of all $(\zeta, U, H, r) \in E$ such that
\[ (\zeta, U, H, r) \in [W^{1,\infty}]^3 \times L^\infty, \quad \text{(3.2a)} \]
\[ y_\xi \geq 0, H_\xi \geq 0, y_\xi + H_\xi > 0 \text{ almost everywhere, and } \lim_{\xi \to -\infty} H(\xi) = 0, \quad \text{(3.2b)} \]
\[ y_\xi^2 H_\xi = y^2_\xi U^2 + U^2_\xi + r^2 \text{ almost everywhere,} \quad \text{(3.2c)} \]

where we denote $y(\xi) = \zeta(\xi) + \xi$. 


The condition \( \eta \geq 0 \) implies that the mapping \( \xi \mapsto y(\xi) \) is almost a diffeomorphism. The solution develop singularities exactly when this mapping ceases to be a diffeomorphism, that is, when \( \eta = 0 \) in some regions. The condition \((3.2c)\) shows that the variables \((y, U, H, r)\) are strongly coupled. In fact, when \( \eta \neq 0 \), we can recover \( H \) from \((3.2c)\). It reflects the fact that \( H \) represents, in Lagrangian coordinates, the energy density of \( u \) and \( \rho \) (that is, \((u^2 + u_x^2 + \rho^2)dx\) in Eulerian coordinates) and therefore, when the solution is smooth, it can be computed from the variables \( y, U, \) and \( r \). Note that the coupling between \( H \) and \((y, U, r)\) disappears when \( \eta = 0 \), which is precisely the moment when collisions occur and when we need the information \( H \) provides on the energy to prolong the solution. The identity makes also clear the smoothing property of the Camassa–Holm system.

As in [15, Theorem 3.6], we obtain the Lipschitz continuity of the semigroup

**Theorem 3.2.** For any \( \bar{X} = (\bar{y}, \bar{U}, \bar{H}, \bar{r}) \in \mathcal{G} \), the system \((2.6)\) admits a unique global solution \( X(t) = (y(t), U(t), H(t), r(t)) \in C^1(\mathbb{R}^+, E) \) with initial data \( \bar{X} = (\bar{y}, \bar{U}, \bar{H}, \bar{r}) \). We have \( X(t) \in \mathcal{G} \) for all times. If we equip \( \mathcal{G} \) with the topology induced by the \( E \)-norm, then the mapping \( S: \mathcal{G} \times \mathbb{R}^+ \rightarrow \mathcal{G} \) defined by

\[
S_t(\bar{X}) = X(t)
\]

is a Lipschitz continuous semigroup. More precisely, given \( M > 0 \) and \( T > 0 \), there exists a constant \( C_M \) which depends only on \( M \) and \( T \) such that, for any two elements \( X_\alpha, X_\beta \in \mathcal{G} \cap B_M \), we have

\[
\|S_t X_\alpha - S_t X_\beta\| \leq C_M \|X_\alpha - X_\beta\| \tag{3.3}
\]

for any \( t \in [0, T] \).

### 4. Relabeling symmetry

The equations are well-posed in Lagrangian coordinates. We want to transport this result back to Eulerian coordinates. If the two sets of coordinates were in bijection, then it would be straightforward but, as mentioned earlier, Lagrangian coordinates increase the number of unknowns from two \((u, \rho)\) to four \((\text{the components of } X)\), which indicates that such a bijection does not exist. There exists a redundancy in Lagrangian coordinates and the goal of this section is precisely to identify this redundancy, in order to be able to define the correct equivalence classes. This redundancy is also present in the case of the Burgers equation when we define the Cauchy problem for both \((2.1)\) and \((2.2)\). To the initial condition \( u(0, x) = u_0(x) \) for \((2.1)\), there corresponds infinitely many parametrizations of the initial conditions for \((2.2)\) given by

\[
y(0, \xi) = f(\xi), \quad U(0, \xi) = u_0(f(\xi)),
\]

for an arbitrary diffeomorphism \( f \). As also mentioned earlier, the representation of a graph is uniquely defined by a single function while there are infinitely many different parametrizations of any given curve. We will use the term relabeling for this lack of uniqueness in the characterization of one and the same curve.

We now define the relabeling functions as follows.

**Definition 4.1.** We denote by \( G \) the subgroup of the group of homeomorphisms from \( \mathbb{R} \) to \( \mathbb{R} \) such that

\[
f - \text{Id} \quad \text{and} \quad f^{-1} - \text{Id} \quad \text{both belong to } W^{1, \infty}, \tag{4.1a}
\]

\[
f_\xi - 1 \quad \text{belongs to } L^2, \tag{4.1b}
\]
where $\text{Id}$ denotes the identity function. Given $\kappa > 0$, we denote by $G_\kappa$ the subset $G_\kappa$ of $G$ defined by

$$G_\kappa = \{ f \in G \mid \|f - \text{Id}\|_{W^{1,\infty}} + \|f^{-1} - \text{Id}\|_{W^{1,\infty}} \leq \kappa \}.$$

We refine the definition of $\mathcal{G}$ in Definition 3.1 by introducing the subsets $\mathcal{F}_\kappa$ and $\mathcal{F}$ as

$$\mathcal{F}_\kappa = \{ X = (y,U,H,r) \in \mathcal{G} \mid y + H \in G_\kappa \},$$

and

$$\mathcal{F} = \{ X = (y,U,H,r) \in \mathcal{G} \mid y + H \in \mathcal{G} \}.$$

The regularity requirement on the relabeling functions given in Definition 4.1 and the definition of $\mathcal{F}$ are introduced in order to be able to define the action of $G$ on $\mathcal{F}$, that is, for any $X = (y,U,H,r) \in \mathcal{F}$ and any function $f \in \mathcal{G}$, the function $(y \circ f, U \circ f, H \circ f, r \circ f \xi)$ belongs to $\mathcal{F}$ and we will denote it by $X \circ f$. This corresponds to the relabeling action. Note that relabeling acts differently on primary functions, as $y$, $U$ and $H$ (in this case, we have $(U,f) \mapsto U \circ f$) and on derivatives or densities, as $y_x$, $U_x$, $H_x$ and $r$ (in that case we have $(r,f) \mapsto r \circ f \xi$). The space $\mathcal{F}$ is preserved by the governing equation (2.6) and, as expected, the semigroup of solutions in Lagrangian coordinates preserves relabeling, i.e., we have the following result.

**Lemma 4.2** ([15, Theorem 4.8]). The mapping $S_t$ is equivariant, that is,

$$S_t(X \circ f) = S_t(X) \circ f$$

for any $X \in \mathcal{F}$ and $f \in \mathcal{G}$.

Now that we have identified the redundancy of Lagrangian coordinates as the action of relabeling, we want to handle it by considering equivalence classes. However, equivalence classes are rather abstract objects which will be hard to work with from an analytical point of view. We consider instead the section defined by $\mathcal{F}_0$, which contains one and only one representative for each equivalence class, so that the quotient $\mathcal{F}/\mathcal{G}$ is in bijection with $\mathcal{F}_0$. Let us denote by $\Pi$ the projection of $\mathcal{F}$ into $\mathcal{F}_0$ defined as

$$\Pi(X) = X \circ (y + H)^{-1}$$

for any $X = (y,U,H,r) \in \mathcal{F}$. By definition, we have that $X$ and $\Pi(X)$ belong to the same equivalence class. We can check that the mapping $\Pi$ is a projection, i.e., $\Pi \circ \Pi = \Pi$, and that it is also invariant, i.e., $\Pi(X \circ f) = \Pi(X)$. It follows that the mapping $[X] \mapsto \Pi(X)$ is a bijection from $\mathcal{F}/\mathcal{G}$ to $\mathcal{F}_0$.

5. **Eulerian coordinates**

In the method of characteristics, once the equation is solved in Lagrangian coordinates, we recover the solution in Eulerian coordinates by setting $u(t,x) = U(t,y^{-1}(t,x))$, where $y^{-1}(t,x)$ denotes—assuming it exists—the inverse of $\xi \mapsto y(t,\xi)$. The Burgers equation and the Camassa–Holm equation develop singularity because $y$ does not remain invertible. In the case of the Burgers equation, $u$ becomes discontinuous but the Camassa–Holm equation enjoys more regularity and $u$ remains continuous. This is a consequence of the preservation of the $H^1$ norm, but it can also be seen from the Lagrangian point of view. Indeed, even if $y$ is not invertible, we can define $u(t,x)$ as

$$u(t,x) = U(t,\xi) \text{ for any } \xi \text{ such that } x = y(t,\xi).$$

This is well-defined because if there exist $\xi_1$ and $\xi_2$ such that $x = y(t,\xi_1) = y(t,\xi_2)$, then $y_x(t,\xi) = 0$ for all $\xi \in [\xi_1,\xi_2]$ because $y$ is non-decreasing, see (3.2b). Then, by (3.2c), we get $U_x(t,\xi) = 0$ so that $U(t,\xi_1) = U(t,\xi_2)$. Furthermore, as we explained earlier in the case of a peakon-antipeakon collision, some information is
needed about the energy to prolong the solution after collision. If \( y \) is invertible, we recover the energy density in Eulerian coordinates as

\[
(u^2 + u_x^2 + \rho^2) \, dx = \frac{H}{y} \circ y^{-1} \, d\xi,
\]

which corresponds to the push-forward of the measure \( H \, d\xi \) with respect to \( y \), i.e.,

\[
(u^2 + u_x^2 + \rho^2) \, dx = y_#(H \, d\xi).
\]

However, when \( y \) is not invertible (5.1) cannot be used and \( y_#(H \, d\xi) \) may not be absolutely continuous so that (5.2) will not hold either. It motivates the introduction of the energy \( \mu \) defined here as \( y_#(H \, d\xi) \), which represents the energy of the system.

**Definition 5.1.** The set \( \mathcal{D} \) consists of all triples \((u, \rho, \mu)\) such that

1. \( u \in H^1 \), \( \rho \in L^2 \), and
2. \( \mu \) is a positive Radon measure whose absolutely continuous part, \( \mu_{ac} \), satisfies

\[
\mu_{ac} = (u^2 + u_x^2 + \rho^2) \, dx.
\]

It can be shown (see [15, Section 4]) that the identity (3.2c) is somehow equivalent to (5.3) but it is clear that, from an analytical point of view, it easier to deal with an algebraic identity like (3.2c) than with a property like (5.3) which immediately requires tools from measure theory. We can show that \( \mathcal{D} \) and \( \mathcal{F}_0 \) are in bijection, and the mappings between the two are given in the following definition. The first one has been already explained.

**Definition 5.2.** Given any element \( X \) in \( \mathcal{F}_0 \), then \((u, \rho, \mu)\) defined as follows

\[
u(x) = U(\xi) \text{ for any } \xi \text{ such that } x = y(\xi),
\]

\[
\rho(x) = y_#(r \, d\xi), \quad \mu = y_#(H \, d\xi),
\]

belongs to \( \mathcal{D} \). We denote by \( M : \mathcal{F}_0 \to \mathcal{D} \) the map which to any \( X \) in \( \mathcal{F}_0 \) associates \((u, \rho, \mu)\).

The mapping, which we denoted by \( L \), from \( \mathcal{D} \) to \( \mathcal{F}_0 \) is defined as follows.

**Definition 5.3.** For any \((u, \rho, \mu)\) in \( \mathcal{D} \) let

\[
\begin{align*}
y(\xi) &= \sup\{y \mid \mu((\infty, y)) + y < \xi\}, \\
H(\xi) &= \xi - y(\xi), \\
U(\xi) &= u \circ y(\xi), \\
r(\xi) &= \rho \circ y(\xi) y(\xi).
\end{align*}
\]

We can see that the lack of regularity of \( u \), which will occur when \( \mu \) is singular or very large, is transformed into regions where the function \( y \) is constant or almost constant. Using the relabeling degree of freedom, we manage to rewrite functions in \( L^2 \) and measures as bounded functions (in \( L^\infty \)). For example, for the peakon-antipeakon collision depicted in Figure 1, the initial data given by \( u_0(x) = \rho_0(x) = 0 \) and \( \mu = \delta(x) \, dx \), which corresponds to the collision time, \( t_c \), when the total energy is equal to one, yields \( r(\xi) = U(\xi) = 0 \) with \( y(\xi) \) and \( H(\xi) \) as defined in (2.3). We can check that, in this case \( \delta(x) \, dx = y_#(H \, d\xi) \). Finally, we define the semigroup \( T_t \) of conservative solutions in the original Eulerian variables \( \mathcal{D} \) as

\[
T_t := M11S_t L.
\]
6. Lipschitz metric for the semigroup

We apply the construction of the semigroup $T_t$ in Section 5, and we can check, as done in [15, Theorem 5.2], that, for given initial data $(u_0, \rho_0, \mu_0)$, if we denote 
\[
(u(t), \rho(t), \mu_t) = T_t(u_0, \rho_0, \mu_0),
\]
then $(u, \rho)$ are weak solutions to (1.2). Moreover, 
\[
\mu_t(\mathbb{R}) = \mu_0(\mathbb{R})
\]
so that the solutions are conservative. Our goal is to define a metric on $D$ which makes the semigroup Lipschitz continuous. The Lipschitz continuity is a property of a semigroup which can be used to establish its uniqueness, see [3] and [2, Theorem 2.9]. By our construction, a metric for the semigroup $T_t$ is readily available. We can simply transport the topology of the Banach space $E$ from $F_0$ to $D$ and obtain, for two elements $(u, \rho, \mu)$ and $(\tilde{u}, \tilde{\rho}, \tilde{\mu})$, 
\[
d_D((u, \rho, \mu), (\tilde{u}, \tilde{\rho}, \tilde{\mu})) = \|L(u, \rho, \mu) - L(\tilde{u}, \tilde{\rho}, \tilde{\mu})\|_E. \tag{6.1}
\]
We have 
\[
d_D(T_t(u, \rho, \mu), T_t(\tilde{u}, \tilde{\rho}, \tilde{\mu})) = \|\Pi_S t L(u, \rho, \mu) - \Pi_S t L(\tilde{u}, \tilde{\rho}, \tilde{\mu})\|_E.
\]
It can be proven that the projection $\Pi$ is continuous (see [15, Lemma 4.6]), but it is not Lipschitz (at least, we have been unable to prove it). Thus, even if $S_t$ is Lipschitz continuous, the semigroup $T_t$ is only continuous with respect to the metric $d_D$ defined by (6.1). In the definition (6.1) of the metric, we let the section $F_0$ play a special role, but this section is arbitrarily chosen. The set $F_0$ is by construction nonlinear (because of (3.2c)) and to use a linear norm to measure distances does not respect that. In fact, we want to measure the distance between equivalence classes. A natural starting point is to define, for $X_\alpha, X_\beta \in F$, $\bar{J}(X_\alpha, X_\beta)$ as 
\[
\bar{J}(X_\alpha, X_\beta) = \inf_{f, g \in G} \|X_\alpha \circ f - X_\beta \circ g\|. \tag{6.2}
\]
The function $\bar{J}$ is relabeling invariant, that is, $\bar{J}(X_\alpha \circ f, X_\beta \circ g) = \bar{J}(X_\alpha, X_\beta)$ and measures precisely the distance between two equivalence classes. However, we have to deal with the fact that the linear norm of $E$ does not play well with relabeling: It is not invariant with respect to relabeling, i.e., we do not have 
\[
\|X \circ f\| = \|X\|. \tag{6.3}
\]
However, such a norm exists. Let 
\[
B = \{X \in L^\infty \mid X_\xi \in L^1\}.
\]
Then, 
\[
\|X \circ f\|_B = \|X \circ f\|_{L^\infty} + \|X_\xi \circ f f_\xi\|_{L^1} = \|X\|_{L^\infty} + \|X_\xi\|_{L^1} = \|X\|_B.
\]
To cope with the lack of relabeling invariance of $\bar{J}$, we introduce $J$ defined as follows.

**Definition 6.1.** Let $X_\alpha, X_\beta \in F$, we define $J(X_\alpha, X_\beta)$ as 
\[
J(X_\alpha, X_\beta) = \inf_{f_1, f_2 \in G} \left(\|X_\alpha \circ f_1 - X_\beta\| + \|X_\alpha - X_\beta \circ f_2\|\right). \tag{6.4}
\]
The function $J$ is not relabeling invariant, but we have $J(X_\alpha, X_\beta) = 0$ if $X_\alpha$ and $X_\beta$ both belong to the same equivalence class. Moreover, the relabeling invariance is not strictly needed for our purpose and the following weaker property is enough. Given $X_\alpha, X_\beta \in F$ and $f \in G_\alpha$, we have 
\[
J(X_\alpha \circ f, X_\beta \circ f) \leq CJ(X_\alpha, X_\beta) \tag{6.5}
\]
for some constant $C$ which depends only on $\kappa$, see [16]. Note that, if the norm $E$ were invariant, that is, (6.3) were fulfilled, then the function $J$ and $\bar{J}$ would be equivalent, because we would have $\bar{J} \leq J \leq 2J$. 
Remark 6.2. We will make use of the following notation. The variable $X$ is used as a standard notation for $(y, U, H, r)$. By the $L^\infty$ norm of $X$, we mean
\[ \|X\|_{L^\infty} = \|y - 1\|_{L^\infty} + \|U\|_{L^\infty} + \|H\|_{L^\infty}, \]
and, by the $L^2$ norm of the derivative $X_\xi$, we mean
\[ \|X_\xi\|_{L^2} = \|y_\xi - 1\|_{L^2} + \|U_\xi\|_{L^2} + \|H_\xi\|_{L^2} + \|r\|_{L^2}, \]
and, similarly,
\[ \|X_\xi\|_{L^\infty} = \|y_\xi - 1\|_{L^\infty} + \|U_\xi\|_{L^\infty} + \|H_\xi\|_{L^\infty} + \|r\|_{L^\infty}. \]

Proof. The first part of the proof is identical to [16] and we reproduce it here for convenience. For any $X_\alpha, X_\beta \in \mathcal{F}_0$, we have
\[ \|X_\alpha - X_\beta\|_{L^\infty} \leq 2J(X_\alpha, X_\beta). \]

We have
\[ \|X_\alpha - X_\beta\|_{L^\infty} \leq \|X_\alpha - X_\alpha \circ f\|_{L^\infty} + \|X_\alpha \circ f - X_\beta\|_{L^\infty}. \]

It follows from the definition of $\mathcal{F}_0$ that $0 \leq y_\xi \leq 1$, $0 \leq H_\xi \leq 1$ and $|U_\xi| \leq 1$ so that $\|X_{\alpha,\xi}\|_{L^\infty} \leq 3$. We also have
\[ \|f - 1\|_{L^\infty} = \|(y_\alpha + H_\alpha) \circ f - (y_\beta + H_\beta)\|_{L^\infty} \leq \|X_\alpha \circ f - X_\beta\|_{L^\infty}. \]

Hence, from (6.12), we get
\[ \|X_\alpha - X_\beta\|_{L^\infty} \leq 4\|X_\alpha \circ f - X_\beta\|_{L^\infty}. \]

In the same way, we obtain $\|X_\alpha - X_\beta\|_{L^\infty} \leq 4\|X_\alpha - X_\beta \circ f\|_{L^\infty}$ for any $f \in G$. After adding these two last inequalities and taking the infimum, we get (6.11). For any $\varepsilon > 0$, we consider a finite sequence $\{X_n\}_{n=0}^N \subset \mathcal{F}_0$ such that $X_0 = X_\alpha$ and $X_N = X_\beta$ and $\sum_{i=1}^N J(X_{n-1}, X_n) \leq d(X_\alpha, X_\beta) + \varepsilon$. We have
\[ \|X_\alpha - X_\beta\|_{L^\infty} \leq \sum_{n=1}^N \|X_{n-1} - X_n\|_{L^\infty} \leq 2 \sum_{n=1}^N J(X_{n-1}, X_n) \leq 2(d(X_\alpha, X_\beta) + \varepsilon). \]

After letting $\varepsilon$ tend to zero, we get
\[ \|X_\alpha - X_\beta\|_{L^\infty} \leq 2d(X_\alpha, X_\beta). \]
The second inequality in (6.10) follows from the definitions of \( J \) and \( d \). Indeed, we have
\[
d(X_\alpha, X_\beta) \leq J(X_\alpha, X_\beta) \leq 2 \| X_\alpha - X_\beta \|.
\]
It is left to prove that \( d \) defines a metric. The symmetry is intrinsic in the definition of \( J \) while the construction of \( d \) from \( J \) takes care of the triangle inequality. From (6.10), we get that \( d(X_\alpha, X_\beta) = 0 \) implies \( (y_\alpha, U_\alpha, H_\alpha) = (y_\beta, U_\beta, H_\beta) \). By (3.2c), we get that \( r_\alpha^2 = r_\beta^2 \), but we cannot yet conclude that \( r_\alpha = r_\beta \). Let us define
\[
R_\alpha(\xi) = \int_{-\infty}^{\xi} r_\alpha(\eta)e^{-|\eta|} \, d\eta \quad \text{and} \quad R_\beta(\xi) = \int_{-\infty}^{\xi} r_\beta(\eta)e^{-|\eta|} \, d\eta.
\]
Then, we have, for any \( f \in G \),
\[
R_\alpha(\xi) - R_\beta(\xi) = -\int_{-\infty}^{\xi} r_\alpha(\eta)e^{-|\eta|} \, d\eta + \int_{-\infty}^{\xi} r_\alpha \circ f f_\xi(e^{-|f(\eta)|}) - e^{-|\eta|} \, d\eta
\]
\[
+ \int_{-\infty}^{\xi} (r_\alpha \circ f f_\xi - r_\beta)e^{-|\eta|} \, d\eta,
\]
which implies
\[
\| R_\alpha - R_\beta \|_{L^\infty} \leq \| f - \text{Id} \|_{L^\infty} + \left\| \int_{-\infty}^{\xi} r_\alpha \circ f f_\xi(e^{-|f(\eta)|}) - e^{-|\eta|} \, d\eta \right\|_{L^\infty}
\]
\[
+ \left\| r_\alpha \circ f f_\xi - r_\beta \right\|_{L^2}.
\]
We have that
\[
\int_{-\infty}^{\xi} r_\alpha \circ f f_\xi(e^{-|f(\eta)|}) - e^{-|\eta|} \, d\eta = \int_{-\infty}^{\xi} r_\alpha \circ f f_\xi(e^{-|f(\eta)|}) (1 - e^{-|f(\eta)|}) \, d\eta
\]
implies
\[
\left\| \int_{-\infty}^{\xi} r_\alpha \circ f f_\xi(e^{-|f(\eta)|}) - e^{-|\eta|} \, d\eta \right\|_{L^\infty} \leq \left\| e^{|f(\xi)|} - 1 \right\|_{L^\infty} \| r_\alpha \|_{L^2} \left\| e^{-|\xi|} \right\|_{L^2}
\]
\[
\leq C \| r_\alpha \|_{L^2} \| f - \text{Id} \|_{L^\infty},
\]
for \( C = e \) if we assume that \( \| f - \text{Id} \|_{L^\infty} \leq 1 \). Since \( X_\alpha \in F_0 \) so that \( y_\xi \leq 1 \), we get from (3.2c) that \( \| r_\alpha \|_{L^2} \leq \| H_\alpha \|_{L^\infty}^{1/2} \). Collecting the results obtained so far, we find that
\[
\| R_\alpha - R_\beta \|_{L^\infty} \leq (2 + C \| H_\alpha \|_{L^\infty}^{1/2}) \| X_\alpha \circ f - X_\beta \|_{L^\infty} \leq 1.
\]
for any \( \| f - \text{Id} \|_{L^\infty} \leq 1 \). Let us now assume that \( d(X_\alpha, X_\beta) = 0 \). For any \( \varepsilon > 0 \), we can find a sequence such that
\[
\sum_{n=1}^{N} \| X_n \circ f_n - X_{n-1} \| \leq \varepsilon.
\]
Using (6.13) and (6.14), we get \( \| f_n - \text{Id} \|_{L^\infty} \leq \varepsilon \) and prove by induction that
\[
\| H_n \|_{L^\infty} \leq \sum_{i=1}^{n} \| X_i \circ f_i - X_{i-1} \|_{L^\infty} + \| H_\alpha \|_{L^\infty},
\]
for all \( n \leq N \). Indeed, we have
\[
\| H_{n+1} \|_{L^\infty} = \| H_{n+1} \circ f_{n+1} \|_{L^\infty}
\]
\[
\leq \| H_{n+1} \circ f_{n+1} - H_n \|_{L^\infty} + \| H_n \|_{L^\infty}
\]
\[
\leq \sum_{i=1}^{n+1} \| X_i \circ f_i - X_{i-1} \|_{L^\infty} + \| H_\alpha \|_{L^\infty},
\]
after using the induction hypothesis. From (6.18), we get
\[
\| H_n \|_{L^\infty} \leq \varepsilon + \| H_\alpha \|.
\]
Hence, by choosing $\varepsilon \leq 1$, and using repeatedly (6.17), we obtain
\[
\|R_\alpha - R_\beta\|_{L^\infty} \leq \sum_{n=1}^{N} \|R_n - R_{n-1}\|_{L^\infty}
\leq (2 + C(\varepsilon + \|H_\alpha\|_{L^\infty})^{1/2}) \sum_{n=1}^{N} \|X_\alpha \circ f - X_\beta\|
\leq (2 + C(\varepsilon + \|H_\alpha\|_{L^\infty})^{1/2}) \varepsilon.
\]
After letting $\varepsilon$ tend to zero, this last inequality implies that $R_\alpha = R_\beta$ so that $r_\alpha = r_\beta$, which concludes the proof that $d$ is a metric.

The Lipschitz estimate for the semigroup $S_t$ given in (3.3) is valid for initial data in $B_M$. Hence, as we want to use the same Lipschitz estimate for any of the $X_n$ in the sequence defining the metric in (6.9), we have to redefine this metric and require that all $X_n$ belong to $F_0 \cap B_M$. The problem is that $B_M$ is not preserved by the semigroup $S_t$, and we will not be able to use the same distance at later times. This is why we introduce the set
\[
F^M = \{X = (x, U, H, r) \in F \mid \|H\|_{L^\infty} \leq M\},
\]
which is preserved by both relabeling and the semigroup. Note that $F^M$ has a simple physical interpretation as it corresponds to the set of all solutions which have total energy bounded by $M$. Moreover, following closely the proof of [16, Lemma 3.4], we obtain that for $X \in F_0$, the sets $B_M$ and $F^M$ are in fact equivalent, i.e., there exists $\bar{M}$ depending only on $M$ such that
\[
F_0 \cap F^M \subset B_{\bar{M}}.
\]
We set $F_0^M = F_0 \cap F^M$ and define the metric $d^M$ as follows.

**Definition 6.5.** Let $d^M$ be the distance on $F_0^M$ which is defined, for any $X_\alpha, X_\beta \in F_0^M$, as
\[
d^M(X_\alpha, X_\beta) = \inf \sum_{n=1}^{N} J(X_{n-1}, X_n)
\]
where the infimum is taken over all finite sequences $\{X_n\}_{n=0}^{N} \subset F_0^M$ such that $X_0 = X_\alpha$ and $X_N = X_\beta$.

We can now state our main stability theorem

**Theorem 6.6.** Given $T > 0$ and $M > 0$, there exists a constant $C_{M,T}$ which depends only on $M$ and $T$ such that, for any $X_\alpha, X_\beta \in F_0^M$ and $t \in [0, T]$, we have
\[
d^M(\Pi S_t X_\alpha, \Pi S_t X_\beta) \leq C_{M,T} d^M(X_\alpha, X_\beta).
\]
In fact due to the use of equivalent notations, the proof of the theorem is identical to [16, Theorem 3.6]. Here, we propose to present a simplified proof where we assume that the norm of $E$ is invariant with respect to relabeling, that is, (6.3) holds. By doing so, we hope that some general ideas behind the construction of the metric becomes clearer. Much of the construction can be understood from the illustration in Figure 2. In this figure, we denote $X_\alpha^t = \Pi S_t (X_\alpha \circ f_0)$, $X_\beta^t = \Pi S_t (X_\beta \circ g_1)$ and $X_\beta^t = \Pi S_t (X_\beta \circ f_1)$. Let us imagine the (very improbable) case where the infimum in (6.20) and the infimum in (6.4) both are reached, so that $d^M(X_\alpha, X_\beta) = \|X_\alpha \circ f_0 - X_\beta \circ g_1\| + \|X_1 \circ f_1 - X_\beta \circ g_1\|$. Then, we have
\[
d^M(X_\alpha^t, X_\beta^t) \leq J(X_\alpha^t, X_\beta^t) \leq J(S_t(X_\alpha \circ f_0), S_t(X_\beta \circ g_0)) + J(S_t(X_1 \circ f_1), S_t(X_\beta \circ g_1))
\leq \|S_t(X_\alpha \circ f_0) - S_t(X_\beta \circ g_0)\| + \|S_t(X_1 \circ f_1) - S_t(X_\beta \circ g_1)\|.
\]
The horizontal curves represent points which belong to the same equivalence class.

\[
\leq C_{M,T} \left( \|X_\alpha \circ f_0 - X_1 \circ g_0\| + \|X_1 \circ f_1 - X_\beta \circ g_1\| \right)
= C_{M,T} d_M(X_\alpha, X_\beta),
\]

which corresponds to the Lipschitz estimate of Theorem 6.6.

**Simplified proof of Theorem 6.6.** As we mentioned earlier, when the norm is invariant, then \(J\) and \(\bar{J}\) are equivalent. Here, it is simpler to consider \(\bar{J}\). For any \(\varepsilon > 0\), there exist a finite sequence \(\{X_n\}_{n=0}^N\) in \(F_M^0\) and functions \(\{f_n\}_{n=0}^{N-1}\), \(\{g_n\}_{n=0}^{N-1}\) in \(G\) such that \(X_0 = X_\alpha, X_N = X_\beta\) and

\[
\sum_{i=1}^N \|X_{n-1} \circ f_{n-1} - X_n \circ g_{n-1}\| \leq d_M(X_\alpha, X_\beta) + \varepsilon. \tag{6.22}
\]

Since \(B_M\), where \(M\) is defined so that (6.19) holds, is preserved by relabeling, we have that \(X_n \circ f_n\) and \(X_n \circ g_{n-1}\) belong to \(B_M\). From the Lipschitz stability result given in (3.3), we obtain that

\[
\|S_t(X_{n-1} \circ f_{n-1}) - S_t(X_n \circ g_{n-1})\| \leq C_{M,T} \|X_{n-1} \circ f_{n-1} - X_n \circ g_{n-1}\|, \tag{6.23}
\]

where the constant \(C_{M,T}\) depends only on \(M\) and \(T\). Introduce

\[
\tilde{X}_n = X_n \circ f_n, \tilde{X}_1^t = S_t(\tilde{X}_n), \text{ for } n = 0, \ldots, N-1,
\]

and

\[
\tilde{X}_n = X_n \circ g_{n-1}, \tilde{X}_1^t = S_t(\tilde{X}_n), \text{ for } n = 1, \ldots, N.
\]

Then (6.22) rewrites as

\[
\sum_{i=1}^N \|\tilde{X}_{n-1} - \tilde{X}_n\| \leq d_M(X_\alpha, X_\beta) + \varepsilon \tag{6.24}
\]

while (6.23) rewrites as

\[
\|\tilde{X}_{n-1}^t - \tilde{X}_n^t\| \leq C_{M,T} \|\tilde{X}_{n-1} - \tilde{X}_n\|. \tag{6.25}
\]

We have

\[
\Pi(\tilde{X}_0^t) = \Pi \circ S_t(X_0 \circ f_0) = \Pi \circ (S_t(X_0) \circ f_0) = \Pi \circ S_t(X_0) = \bar{S}_t(X_\alpha)
\]
and similarly \( \Pi(\tilde{X}_h^n) = \Pi S_t(X_\beta) \). We consider the sequence which consists of \((\Pi X^n_t)^{N=0}_{n=1}\) and \(S_t(X_\beta)\). Using the property that \(\mathcal{F}^M\) is preserved both by relabeling and by the semigroup, we obtain that \((\Pi X^n_t)^{N=0}_{n=1}\) and \(S_t(X_\beta)\) belong to \(\mathcal{F}^M\) and therefore also to \(\mathcal{F}^M_0\). The endpoints are \(\Pi S_t(X_\alpha)\) and \(\Pi S_t(X_\beta)\). From the definition of the metric \(d_M\), we get

\[
d_M(\tilde{S}_t(X_\alpha), \tilde{S}_t(X_\beta)) \leq \sum_{n=1}^{N-1} \tilde{J}(\Pi X_{n-1}^t, \Pi X_n^t) + \tilde{J}(\Pi X_{N-1}^t, S_t(X_\beta))
= \sum_{n=1}^{N-1} \tilde{J}(X_{n-1}^t, \tilde{X}_n^t) + \tilde{J}(X_{N-1}^t, \tilde{X}_N^t), \tag{6.26}
\]
due to the invariance of \(\tilde{J}\) with respect to relabeling. By using the equivariance of \(S_t\), we obtain that

\[
\tilde{X}_n^t = S_t(\tilde{X}_n) = S_t((\tilde{X}_n \circ f_n^{-1}) \circ g_{n-1}) = S_t(\tilde{X}_n) \circ (f_n^{-1} \circ g_{n-1}) = \tilde{X}_n^t \circ (f_n^{-1} \circ g_{n-1}). \tag{6.27}
\]
Hence we get from (6.26) that

\[
d_M(\tilde{S}_t(X_\alpha), \tilde{S}_t(X_\beta)) \leq \sum_{n=1}^{N-1} \tilde{J}(\tilde{X}_{n-1}^t, \tilde{X}_n^t) + \tilde{J}(\tilde{X}_{N-1}^t, \tilde{X}_N^t)
\leq \sum_{n=1}^N \| \tilde{X}_{n-1}^t - \tilde{X}_n^t \| \tag{6.10}
\leq C_{M,T} \sum_{n=1}^N \| \tilde{X}_{n-1} - \tilde{X}_n \| \tag{6.25}
\leq C_{M,T} (d_M(X_\alpha, X_\beta) + \varepsilon).
\]
After letting \(\varepsilon\) tend to zero, we obtain (6.21). \(\square\)

The Lipschitz stability of the semigroup \(T_t\) follows then naturally from Theorem 6.6. It holds on sets of bounded energy. Let \(\mathcal{D}^M\) be the subsets of \(\mathcal{D}\) defined as

\[
\mathcal{D}^M = \{(u, \rho, \mu) \in \mathcal{D} | \mu(\mathbb{R}) \leq M\}. \tag{6.28}
\]
On the set \(\mathcal{D}^M\) we define the metric \(d_{\mathcal{D}^M}\) as

\[
d_{\mathcal{D}^M}((u, \rho, \mu), (\tilde{u}, \tilde{\rho}, \tilde{\mu})) = d_M(L(u, \rho, \mu), L(\tilde{u}, \tilde{\rho}, \tilde{\mu})), \tag{6.29}
\]
where the metric \(d_M\) is defined as in Definition 6.5. This definition is well-posed as, from the definition of \(L\), we have that if \((u, \rho, \mu) \in \mathcal{D}^M\), then \(L(u, \rho, \mu) \in \mathcal{F}^M_0\).

**Theorem 6.7.** The semigroup \((T_t, d_{\mathcal{D}})\) is a continuous semigroup on \(\mathcal{D}\) with respect to the metric \(d_{\mathcal{D}}\). The semigroup is Lipschitz continuous on sets of bounded energy, that is: Given \(M > 0\) and a time interval \([0, T]\), there exists a constant \(C_{M,T}\), which only depends on \(M\) and \(T\) such that for any \((u, \rho, \mu)\) and \((\tilde{u}, \tilde{\rho}, \tilde{\mu})\) in \(\mathcal{D}^M\), we have

\[
d_{\mathcal{D}^M}(T_t(u, \rho, \mu), T_t(\tilde{u}, \tilde{\rho}, \tilde{\mu})) \leq C_{M,T} d_{\mathcal{D}^M}((u, \rho, \mu), (\tilde{u}, \tilde{\rho}, \tilde{\mu})) \tag{6.30}
\]
for all \(t \in [0, T]\). Let \((u, \rho, \mu)(t) = T_t((u_0, \rho_0, \mu_0))\), then \((u(t, x), \rho(t, x))\) is weak solution of the Camassa–Holm equation (1.2).

We conclude the section about this metric by mentioning that, even if the construction of the metric is abstract, it can be compared with standard norms, cf. [16, Section 5], so that it can be used in practice, for example in the study of numerical schemes \([8, 21]\).
REFERENCES

[1] V. Arnold and B. Khesin. *Topological Methods in Hydrodynamics.* Springer-Verlag, New York, 1998.

[2] A. Bressan. *Hyperbolic Systems of Conservation Laws. The One-Dimensional Cauchy Problem.* Oxford University Press, Oxford, 2000.

[3] A. Bressan. Contractive metrics for nonsmooth evolutions. In *Nonlinear Partial Differential Equations,* (H. Holden, K.H. Karlsen, eds.), Abel Symposia, Vol. 7, Springer, Berlin-Heidelberg, pp. 13–25, 2012.

[4] A. Bressan and A. Constantin. Global conservative solutions of the Camassa–Holm equation. *Arch. Ration. Mech. Anal.* 183:215–239, 2007.

[5] A. Bressan, H. Holden, and X. Raynaud. Lipschitz metric for the Hunter–Saxton equation. *J. Math. Pures Appl.* 94:68–92, 2010.

[6] R. Camassa and D. D. Holm. An integrable shallow water equation with peaked solutions. *Phys. Rev. Lett.* 71(11):1661–1664, 1993.

[7] R. Camassa, D. D. Holm, and J. Hyman. A new integrable shallow water equation. *Adv. Appl. Mech.* 31:1–33, 1994.

[8] D. Cohen and X. Raynaud. Convergent numerical schemes for the compressible hyperelastic rod wave equation. *Numerische Mathematik* 122(1):1–59, 2012.

[9] A. Constantin and J. Escher. Global existence and blow-up for a shallow water equation. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)* 26:303–328, 1998.

[10] A. Constantin and J. Escher. Wave breaking for nonlinear nonlocal shallow water equations. *Acta Math.* 181:229–243, 1998.

[11] A. Constantin and J. Escher. On the blow-up rate and the blow-up set of breaking waves for a shallow water equation. Math. Z. 233:75–91, 2000.

[12] A. Constantin and B. Kolev. Least action principle for an integrable shallow water equation. *J. Nonlinear Math. Phys.* 4:471–474, 2001.

[13] J. Escher, M. Kohlmann and J. Lenells. The geometry of the two-component Camassa–Holm and Degasperis–Procesi equations. *J. Geom. Phys.* 61(2): 436–452, 2011.

[14] K. Grunert, H. Holden, and X. Raynaud. Lipschitz metric for the periodic Camassa–Holm equation. *J. Differential Equations* 250: 1460–1492, 2011.

[15] K. Grunert, H. Holden, and X. Raynaud. Global solutions for the two-component Camassa–Holm system. *Communications in Partial Differential Equations* 37(12): 2245–2271, 2012.

[16] K. Grunert, H. Holden, and X. Raynaud. Lipschitz metric for the Camassa–Holm equation on the line. *Discrete Contin. Dyn. Syst.* 33:2809–2827, 2013.

[17] K. Grunert, H. Holden, and X. Raynaud. Periodic conservative solutions for the two-component Camassa–Holm system. In *Spectral Analysis, Differential Equations and Mathematical Physics. A Festschrift in Honor of Fritz Gesztesy's 60th Birthday,* (H. Holden, B. Simon, and G. Teschl, eds.), Proc. Symp. Pure Math., Vol. 87, Amer. Math. Soc., 2013, to appear.

[18] H. Holden and X. Raynaud. Global conservative solutions of the Camassa–Holm equation—a Lagrangian point of view. *Comm. Partial Differential Equations* 32:1511–1549, 2007.

[19] H. Holden and X. Raynaud. Global conservative multipeakon solutions of the Camassa–Holm equation. *J. Hyperbolic Differ. Equ.* 4:39–64, 2007.

[20] H. Holden and X. Raynaud. Global conservative solutions of the generalized hyperelastic-rod wave equation. *J. Differential Equations* 233:448–484, 2007.

[21] H. Holden and X. Raynaud. A numerical scheme based on multipeakons for conservative solutions of the Camassa–Holm equation. In *Hyperbolic Problems: Theory, Numerics, Applications,* (S. Benzoni-Gavage, D. Serre, eds.) Springer, Heidelberg, 873–881, 2008.

[22] P. J. Olver and P. Rosenau. Tri-hamiltonian duality between solitons and solitary-wave solutions having compact support. *Phys. Rev. E,* 53(2):1900–1906, 1996.

E-mail address: katrin.grunert@univie.ac.at
E-mail address: holden@math.ntnu.no
E-mail address: xavierra@cma.uio.no