Entanglement of superposition and pair coherent states

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Quantum entanglement is regarded to be a physical resource in quantum information processing. Though entangled states are routinely being formed in laboratories, there remains many open problems in classifying and understanding them. Discrete dimensional states are well studied and we have necessary and sufficient conditions for separability for 2 $\otimes$ 2 and 2 $\otimes$ 3 dimensions. Continuous variable states are often encountered in experimental situations. However there remains many open problems in understanding entanglement of continuous variable or infinite dimensional states. It ought to be mentioned that criteria for separability for such states have been obtained either using Peres-Horodecki conditions or using EPR-like operators. But fortunately the measure of entanglement of a bipartite pure system, whether continuous or discrete, is well understood. The entanglement of a bipartite pure state is completely defined by von Neumann entropy of the reduced density matrix of either of the parties. Thus if $|\Psi\rangle$ is a bipartite state shared by two parties $A$ and $B$, its entanglement measure is defined as

$$E(\Psi) = S(Tr_A |\Psi\rangle\langle\Psi|) = S(Tr_B |\Psi\rangle\langle\Psi|).$$  \hspace{1cm} (1)

It is well known that entanglement arises due to superposition of states i.e., it is not possible, in general, to describe a composite state by assigning only a single state vector to any of the subsystems. Given this basic relationship between superposition and entanglement, it is natural to link entanglement of some state $|\Gamma\rangle$ to the entanglement of the states appearing in its superposition expansion. It is surprising that this link was investigated in Ref. only recently. One of the possible reasons could be that general characterization and measures of entanglement are still not on the firm footing and most investigations are focused only in that direction. Since for a bipartite system these issues have been resolved, it is possible to start investigating the link between entanglement and superposition. In Ref. it was demonstrated that have if a bipartite state $|\Gamma\rangle$ expanded as a superposition of states $|\Psi\rangle$ and $|\Phi\rangle$ i.e.,

$$|\Gamma\rangle = \alpha|\Psi\rangle + \beta|\Phi\rangle$$  \hspace{1cm} (2)

then how $E(\Gamma)$ is related to $E(\Psi)$ and $E(\Phi)$. This relationship is a complex one in general and depends on orthogonality relation between $|\Psi\rangle$ and $|\Phi\rangle$. It becomes particularly simple if $|\Psi\rangle$ and $|\Phi\rangle$ are biorthogonal

$$E(\Gamma) = |\alpha|^2 E(\Phi) + |\beta|^2 E(\Psi) + h(|\alpha|^2),$$  \hspace{1cm} (3)

where, $h(x) = -x \log_2 x - (1 - x) \log_2 (1 - x)$ is the binary entropy function, and $|\alpha|^2 + |\beta|^2 = 1$. Increase in the entanglement in the superposition (entanglement gain) should satisfy an upper bound

$$E(\Gamma) - \left(|\alpha|^2 E(\Phi) + |\beta|^2 E(\Psi)\right) \leq 1,$$  \hspace{1cm} (4)

But for the cases when the superposed states are not biorthogonal eq.(3) does not satisfy. In this case one can obtain an upper bound on $E(\Gamma)$ while the entanglement gain can become greater than one ebit. Given a its entanglement can be described by eq.(1). The procedure given in Ref. provides valuable insights into how different kind of superpositions could contribute in constructing the entanglement of a state like $|\Gamma\rangle$. Afterwards several authors have further investigated and generalized this result. Among the new results include different measures of entanglement, multipartite entanglement, including more than two states in superposition and finding a new and tighter bounds.

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In this work we consider an example of an entangled non-Gaussian radiation field to analyze the role of entanglement of superposition. The basic aim of this work is to analyze the role of entanglement of superposition in a continuous variable or infinite dimensional system. For this purpose we choose an example of a pair coherent state \[|\zeta, q\rangle\]. These states exhibit nonclassical properties\cite{12} and they have been extensively studied for violation of Bell inequalities\cite{13,14}. Nonseparability for these states has been established using Peres-Horodecki criterion in Ref.\cite{4}.

A pair coherent state \[|\zeta, q\rangle\] is a state of a two-mode radiation field\cite{11} which can be regarded as a specific kind of superposition of biorthogonal states. In what follows we show that one can construct the entanglement of a pair coherent state using the relationship between the superposition and entanglement provided by eq.(3). One can also directly calculate the entanglement of the state in a straight forward fashion (see eq.(11) below) which could be similar to the method given in Ref.\cite{9}.

But the usage of eq.(3) for the calculation requires an iterative procedure to superpose the infinite states in eq.(7). However such iterative procedure, we believe, may be useful in providing insight into quantifying the relative significance of the terms in the superposition expansion of state. This in turn can be helpful in constructing approximate states from the exact ones.

Partial transpose of eq.(8) was shown to have negative eigenvalues and therefore the nonseparability\cite{4}:

\[\lambda_{nm} = \frac{1}{I_0(2|\zeta|)} \frac{|\zeta|^n}{(n!)}^2, \quad \forall n\]

\[\lambda_{nm}^\pm = \pm \frac{1}{I_0(2|\zeta|)} \frac{|\zeta|^{n+m}}{n!m!}, \quad \forall n \neq m.\]

One can in fact construct negativity \[\mathcal{N}(\rho)\] by finding absolute sum of negative eigenvalues in lieu of a computable measure\cite{12} of entanglement as,

\[\mathcal{N}(\rho) = \left| \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \lambda_{nm} \right| = 1 - \frac{2|\zeta|}{I_0(2|\zeta|)}\]

In the limit \[|\zeta| \to 0, \mathcal{N}(\rho) \to 0\] which is indicating that there is no entanglement. Since in this limit only \[|0, 0\rangle\] state will survive in eq.(7) and all such states like \[|n, n\rangle\] are separable. But for higher values of \[|\zeta|\] the negativity increase monotonically as depicted in Fig.(1).

Entanglement \[E(|\zeta, 0\rangle)\] can be written as:

\[E(|\zeta, 0\rangle) = -\sum_{n=0}^{\infty} \frac{|\zeta|^{2n}}{I_0(2|\zeta|)n!^2} \log_2 \left( \frac{I_0(2|\zeta|)n!^2}{|\zeta|^{2n}} \right) .\]

In order to compute the entanglement of superposition let us first consider a set of orthogonal bipartite state \[\{|i, i\rangle\}_{i=0,1,\ldots,\infty}\]. Next consider a superposition of the first two states i.e., \[N_0\{|0, 0\rangle + \zeta |1, 1\rangle\}\] where, \[\zeta\] is a complex number. From this, one can define a normalized state \[|S1\rangle\]

\[|S1\rangle = N_0\gamma_{S1}(|0, 0\rangle + \zeta |1, 1\rangle)\]
where, $\gamma_{S_1} = \frac{1}{N_0\sqrt{1+|\zeta|^2}}$. Since $|S_1\rangle$ is expanded using two biorthogonal states, the entanglement of superposition $E(S_1)$ is written as,

$$E(S_1) = N_0^2 \gamma_{S_1}^2 \left(E(|0,0\rangle) + |\zeta|^2 E(|1,1\rangle)\right) + h_1 \left(N_0^2 \gamma_{S_1}^2\right).$$  \hspace{1cm} (13)

As all the states in the orthonormal set that we have considered are separable, entanglement entropies of each individual state is zero and consequently we have;

$$E(S_1) = h_1 \left(N_0^2 \gamma_{S_1}^2\right),$$  \hspace{1cm} (14)

which also describes the gain in the entanglement by the superposition and it also satisfies the condition (3). The gain always remain less than one ebit and only in the limit $|\zeta| \to 1$, it becomes one ebit. Now consider yet another state $|S_2\rangle$ defined as a superposition of states $|S_1\rangle$ and $|2,2\rangle$ as follows:

$$|S_2\rangle = N_0 \gamma_{S_2} \left(\gamma_{S_1}|S_1\rangle + |2,2\rangle\right)$$  \hspace{1cm} (15)

where the normalization constant is $\gamma_{S_2} = \left(N_0 \sqrt{\sum_{k=0}^{2} \frac{|\zeta|^2}{k!^2}}\right)^{-1}$. The entanglement $E(S_2)$ for the state defined by eq.(15) can be written as:

$$E(S_2) = \left(\frac{\gamma_{S_2}}{\gamma_{S_1}}\right)^2 E(S_1) + h_2 \left(\frac{\gamma_{S_2}}{\gamma_{S_1}}\right)^2$$  \hspace{1cm} (16)

where, $h_2$ is the gain for the second superposition carried out in defining $|S_1\rangle$ and it can be shown to satisfy condition (3). Similarly one can construct a superposition of the type

$$|S_n\rangle = N_0 \gamma_{S_n} \left(\sum_{k=0}^{n} \frac{\zeta^k}{k!} |k,k\rangle\right)$$  \hspace{1cm} (17)

where, the normalization constant $\gamma_{S_n} = \left(N_0 \sqrt{\sum_{k=0}^{n} \frac{|\zeta|^2}{k!^2}}\right)^{-1}$. Entanglement $E(S_n)$ arising by superposing $|S_{n-1}\rangle$ and $|n,n\rangle$ state can be written as;

$$E(S_n) = \left(\frac{\gamma_{S_n}}{\gamma_{S_{n-1}}}\right)^2 E(S_{n-1}) + h_n \left(\frac{\gamma_{S_n}}{\gamma_{S_{n-1}}}\right)^2$$  \hspace{1cm} (18)

In the limit $n \to \infty$ the gain in the entanglement $h_n$ becomes zero because $\left(\frac{\gamma_{S_n}}{\gamma_{S_{n-1}}}\right) \to 1$. Thus in the large $n$ limit $E(S_n) \sim E(S_{n-1})$ and the further superpositions may not contribute significantly to the entanglement characteristics of the state. $E(S_n)$ can be expressed in terms of $h_n$ as follows:

$$E(S_n) = \sum_{k=1}^{n} \left(\frac{\gamma_{S_n}}{\gamma_{S_k}}\right)^2 h_k \left(\frac{\gamma_{S_k}}{\gamma_{S_{k-1}}}\right)^2.$$  \hspace{1cm} (19)
FIG. 2: Variation of the von Neumann entropy (eq. (11)) of the pair coherent state with $|\zeta|$

FIG. 3: Variation of the entanglement $E(S_3)$ and $E(S_1)$ of the pair coherent state with $|\zeta|$

It should be noted that $E(S_n)$ cannot be written as a simple sum of $n$ gains. Each time when one superposes a new state with $|S_n\rangle$, a redefinition of all the previously superposed states becomes necessary. Alternatively eq. (19), after some algebraic manipulations, can be rewritten as

$$E(S_n) = -\sum_{k=0}^{n} N_0^2 \gamma_{Sn}^2 \frac{|\zeta|^{2n}}{k!^2} \log_2 \left( N_0^2 \gamma_{Sn}^2 \frac{|\zeta|^{2n}}{k!^2} \right)$$

(20)

For the limiting case when $n \to \infty$, the factor $N_0^2 \gamma_{Sn}^2 \to \frac{1}{I_0(2|\zeta|)}$ and consequently $E(S_n) \to E(|\zeta, 0\rangle)$. i.e., one obtains the entanglement of a pair coherent state using the entanglement of superposition. In Fig. (2), we plot $E(|\zeta, 0\rangle) = S_a$ as a function of $|\zeta|$, which increases with $E(|\zeta, 0\rangle) = S_a$. This is fine because each mode contains infinite number of states. This plot is essentially the same as in Ref. [4]. Figure (3) shows two curves describing entanglement of states $|S_1\rangle$ and $|S_3\rangle$. The lower curve corresponds to $E(S_1)$, while the upper curve corresponds to $E(S_3)$. The upper curve provides a good approximation to the entanglement of a pair coherent state $E(|\zeta, 0\rangle)$ shown in Fig. (1). Superposing more and more states, with the given range of $|\zeta|$, do not alter the entanglement in a significant way.

Next we consider a state $|\Gamma_s\rangle$ which is given as a superposition of a pair coherent state and a number state i.e.,

$$|\Gamma_s\rangle = \alpha|\zeta, 0\rangle + \beta \frac{\zeta^n}{\sqrt{I_0(m)}} |m, m\rangle$$

(21)

States $|\Gamma_s\rangle$ and $|m, m\rangle$ are nonorthogonal. Density matrix for eq. (21) can be written as $\rho = \frac{1}{||\Gamma_s||^2} |\Gamma_s\rangle \langle \Gamma_s|$, where $||\Gamma_s||^2 = |\alpha|^2 + g^2 f_m$ and $g^2 = |\beta|^2 + \alpha \beta^* + \beta \alpha^*$. The eigenvalues $\lambda^n$ of the reduced density matrix $\rho^A = Tr_B(\rho)$ can be found to be

$$\lambda^n = \frac{|\alpha|^2 + g^2 \delta_{mn}}{||\Gamma_s||^2}$$

(22)

where, $f_n = \frac{|\zeta|^{2n}}{I_0(m)^n}$. The integer $m$ is fixed by the number state chosen for the superposition in eq. (21) and $n = 0, 1, ... \infty$. It is easy to check that $\sum_{n=0}^{\infty} \lambda^n = 1$ and $0 \leq \lambda \leq 1$. From the above one can write entanglement $E(\Gamma_s)$ as

$$E(\Gamma_s) = -\sum_{n=0}^{\infty} \lambda^n \log_2 \lambda^n$$

(23)
It is rather easy to sum the above series but before that we make following parameterization. Let $|\alpha|^2 + |\beta|^2 f_m = 1$, $\alpha = \cos \theta$ and $\beta = \frac{\sin \theta}{\sqrt{f_m}}$. One can now express $||\Gamma_s||^2 = 1 + 2 \cos^2 \theta \sin \theta \sqrt{f_m}$ and $g^2 f_m = \sin^2 \theta + 2 \cos \theta \sin \theta \sqrt{f_m}$.

Finally one can write entanglement $E(\Gamma_s)$ for state $|\Gamma_s\rangle$ as

$$||\Gamma_s||^2 E(\Gamma_s) = -||\Gamma_s||^2 \log_2 \frac{1}{||\Gamma_s||^2} - |\alpha|^2 \log_2 |\alpha|^2 (1 - f_m) + |\alpha|^2 E(|\zeta, 0\rangle) + |\alpha|^2 F_1 + F_2$$

(24)

Factors $F_1$ and $F_2$ that we have introduced are given by,

$$F_1 = f_m \log_2 f_m \quad F_2 = - \left[ (|\alpha|^2 + g^2) f_m \right] \log_2 \left[ (|\alpha|^2 + g^2) f_m \right]$$

(25)

Gain in the entanglement $\Delta E$ for the superposition of the kind given in eq.(21) can be defined as

$$\Delta E = E(\Gamma_s) - |\alpha|^2 E(|\zeta, 0\rangle),$$

as states like $|m, m\rangle$ are separable, $E(|m, m\rangle)$ is not required in the above definition.

Since $\sum_{n=0}^{\infty} \lambda^n = 1$ and $0 \leq \lambda \leq 1$ for entire parameter space, $E(\Gamma_s)$ remains positive. This is consistent with the definition of von Neumann entropy. In fig.(4) we have plotted $E(\Gamma_s)$ as a function of $|\zeta|$ for $m = 0$ and $\theta = \pi/4$. If one compares this plot with the fig.(2) one finds that values of $E(\Gamma_s)$ are not always larger than that of $E(|\zeta, 0\rangle)$. A more quantitative comparison can come from studying the behavior of the gain. Fig.(5) shows plot of $\Delta E$ as a function of $|\zeta|$ and the values of other parameters same as fig.(4). It clearly shows that gain in the entanglement become negative for certain values of $|\zeta|$. For higher values of $|\zeta|$, $\Delta E$ first becomes zero and increases afterwards. It must be noted that for non zero value of $m$ also the gain can become negative. In fig.(6) we plot $\Delta E$ for $m = 1$ and $\theta = \pi/4$. Again the gain becomes negative for the higher values of $|\zeta|$ than $m = 0$ case. But further increasing the $m$ values the entanglement gain remains positive. It must be noted here that superposition can reduce the entanglement or make it to zero [3]. But the kind of superposition we are describing is different from the illustration given in Ref. [3].

For the present choice of parameter we have $\theta = \pi/4$. There is no phase difference between the $m$-th state in the expansion of $|\zeta, 0\rangle$ in harmonic oscillator bases and with the number state given in eq.(21). Thus superposition of a pair coherent state and a number state can alter entanglement characteristic in a complex manner. The gain in entanglement can be negative, zero or positive depending on the values of parameter $\zeta$ takes.

In conclusion, we have constructed the entanglement of pair coherent state by considering successive gain arising due to superposition of the number states. We have shown that only a few states in the superposition expansion can contribute significantly, this can be qualitatively expected from the expansion in eq.(7). However, the method used in analysing the entanglement of superposition can provide a better way of approximating the terms. We have shown that for a certain parameter space (as shown in fig.(5)) the gain in entanglement can become negative when superposing a pair coherent state with a number space.
The figure shows entanglement gain due to the superposition a pair coherent state and a number state as a function of $|\zeta|$ for $m = 0$.

The figure shows entanglement gain due to the superposition a pair coherent state and a number state as a function of $|\zeta|$ for $m = 1$.

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