Effects of interference in the dynamics of a spin-1/2 transverse XY chain driven periodically through quantum critical points

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Received 27 February 2009
Accepted 6 April 2009
Published 6 May 2009

Abstract. We study the effects of interference on the quenching dynamics of a one-dimensional spin 1/2 XY model in the presence of a transverse field \( h(t) \) which varies sinusoidally with time as \( h = h_0 \cos \omega t \), with \(|t| \leq t_f = \pi/\omega\). We have explicitly shown that the finite values of \( t_f \) make the dynamics inherently dependent on the phases of probability amplitudes, which had been hitherto unseen in all cases of linear quenching with large initial and final times. In contrast, we also consider the situation where the magnetic field consists of an oscillatory as well as a linearly varying component, i.e., \( h(t) = h_0 \cos \omega t + t/\tau \), where the interference effects lose importance in the limit of large \( \tau \). Our purpose is to estimate the defect density and the local entropy density in the final state if the system is initially prepared in its ground state. For a single crossing through the quantum critical point with \( h = h_0 \cos \omega t \), the density of defects in the final state is calculated by mapping the dynamics to an equivalent Landau–Zener problem by linearizing near the crossing point, and is found to vary as \( \sqrt{\omega} \) in the limit of small \( \omega \). On the other hand, the local entropy density is found to attain a maximum as a function of \( \omega \) near a characteristic scale \( \omega_0 \). Extending to the situation of multiple crossings, we show that the role of finite initial and final times of quenching are manifested non-trivially in the interference effects of certain resonance modes which solely contribute to the production of defects. Kink density as well as the diagonal entropy density show oscillatory dependence on the number of full cycles of oscillation. Finally, the inclusion of a linear term in the transverse field on top of the oscillatory component results in a kink density which decreases continuously with \( \tau \) while it increases monotonically.
The phenomena of quantum phase transitions occurring at absolute zero temperature have attracted serious attention of scientists in recent years [1, 2]. A quantum critical point is associated with a diverging length scale ($\xi$) and a diverging timescale ($\xi\tau$) which satisfy the scaling forms $\xi \sim |d|^{-\nu}$ and $\xi\tau \sim \xi^z$ in the vicinity of the quantum critical point. Here $d$ denotes the deviation from the critical point, and $\nu$, $z$ are the corresponding critical exponents. Following the possibility of experimental studies of quantum systems trapped in optical lattices [3], there is recent interest in theoretical studies of the related models [4, 5]. When a parameter of the Hamiltonian of the system is swept across a quantum critical point, the diverging relaxation time near the quantum critical point forces the system to undergo non-adiabatic evolution irrespective of the rate of change of parameters. If the system is in its ground state at the initial time, non-adiabatic transitions are manifested in the occurrence of non-zero ‘defects’ (called kinks) and non-zero local entropy in the system.

According to the Kibble–Zurek argument, if a parameter of the Hamiltonian is varied as $t/\tau$, the density of defects ($n$) in the final state is expected to scale as $n \sim \tau^{-\nu d/(\nu z+1)}$, where $d$ is the spatial dimensionality of the system [6]–[9]. The above scaling form has been verified for quantum spin systems quenched through critical points [10]–[13] and also generalized to the cases of nonlinear quenching [14] when a parameter is quenched as $h(t) \sim |t/\tau|^{a}\text{sgn}(t)$, for gapless systems [15] and also for quantum systems with disorder [16] and systems coupled to the external environment [17]. Recently, a generalized
form of the Kibble–Zurek scaling has been introduced which includes a situation where the system is quenched through the multicritical point [18] which shows that the general expression for kink density can be given as \( n \sim \tau^{-d/2z_2} \), where \( z_2 \) determines the scaling of the off-diagonal term of the equivalent Landau–Zener problem close to the critical point.

In parallel to the studies on spin chains in condensed matter physics, great progress has also been made in the realm of quantum optics in exploring the dynamics of two-level systems undergoing Landau–Zener tunnelling due to oscillatory temporal variation of the parameters [19]–[22]. The Landau–Zener transition probabilities have been calculated for single crossing as well as for multiple crossings. Superposition of a linear field along with the sinusoidally varying energy levels gives rise to an altogether different situation, which has also been studied thoroughly in recent years [23].

In our present work we exploit the techniques used in the above papers to explore the dynamics of a one-dimensional spin 1/2 chain undergoing quantum phase transitions due to the application of an oscillatory or an oscillatory as well as a linearly varying magnetic field and compare the results with the earlier findings. Investigation of the dynamics of the one-dimensional transverse XY model under repeated quenching of a linearly varying transverse magnetic field has been carried out in a recent work [24] and it was shown that the defect density decreases in the reverse path though the entropy density increases monotonically with the number of quenches. However, the scaling of the defect density and the local entropy density when the quantum critical point is crossed due to a sinusoidal variation of the magnetic field has not been attempted before. Sinusoidal quenching puts an upper bound on the initial and final times, which makes the process of coarse graining invalid in the present scenario. The resultant dynamics of the system becomes dependent on the phases of the probability amplitudes, leading to the occurrence of constructive or destructive interferences. The spin-1/2 transverse XY chain [25] discussed in this paper is described by the Hamiltonian

\[
H = -\frac{1}{2} \sum_n (J_x \sigma_n^x \sigma_{n+1}^x + J_y \sigma_n^y \sigma_{n+1}^y + h \sigma_n^z),
\]

(1)

where the \( \sigma \)s are Pauli spin matrices satisfying the usual commutation relations and the interactions and the transverse field are denoted by \( J_x, J_y \) and \( h \), respectively, with \( J_x, J_y \) and \( h \) all non-random. The interaction strengths \( J_x \) and \( J_y \) are always time independent whereas we shall use a time-dependent transverse field in the subsequent sections.

The Hamiltonian in equation (1) can be exactly diagonalized using the Jordan–Wigner (JW) transformation which maps a spin-1/2 system to a system of spinless fermions [25, 26] given by

\[
c_n = \exp \left( \frac{i \pi}{2} \sum_{j=1}^{n-1} \sigma_j^z \right) \sigma_n^-,
\]

(2)

where \( \sigma_n^\pm = (\sigma_n^x \pm i \sigma_n^y)/2 \) and the operator \( \sigma_n^z = 2c_n^\dagger c_n - 1 \) and the operators \( c_n \) satisfy the fermionic anti-commutation relations. In terms of JW fermions, the above Hamiltonian can be rewritten in Fourier space with a periodic boundary condition as

\[
H = -\sum_{k>0} \{ [(J_x + J_y) \cos k + h](c_k^\dagger c_k + c_{-k}^\dagger c_{-k}) + i(J_x - J_y) \sin k \ (c_k^\dagger c_{-k} + c_k c_{-k}) \}.
\]

(3)

doi:10.1088/1742-5468/2009/05/P05005
Diagonalizing the Hamiltonian in terms of the Bogoliubov fermions, we arrive at an expression for the gap in the excitation spectrum given by [25, 26]

\[ \epsilon_k = |h^2 + J_x^2 + J_y^2 + 2h(J_x + J_y)\cos k + 2J_x J_y \cos 2k|^{1/2}. \] (4)

The gap given in equation (4) vanishes at \( h = \mp(J_x + J_y) \) for wavevectors \( k = 0 \) and \( \pi \) respectively, signalling a quantum phase transition from a ferromagnetically ordered phase to a quantum paramagnetic phase known as the ‘Ising’ transition [26]. On the other hand, the vanishing of the gap at \( J_x = J_y \) for \( |h| < (J_x + J_y) \) at an ordering wavevector \( k_0 = \cos^{-1}(-h/2J_x) \) signifies a quantum phase transition, belonging to a different universality class from the Ising transitions, between two ferromagnetically ordered phases.

The advantage of employing the JW transformation is that when projected to the two-dimensional subspace spanned by \( |0\rangle \) and \( |k, -k\rangle \), the Hamiltonian takes the form \( H = \sum_k H_k \) where the reduced Hamiltonian is written in the form

\[ H_k(t) = -[h + (J_x + J_y) \cos k]I_2 + \begin{bmatrix} h + (J_x + J_y) \cos k & i(J_x - J_y) \sin k \\ -i(J_x - J_y) \sin k & -h - (J_x + J_y) \cos k \end{bmatrix}, \]

where \( I_2 \) denotes the \( 2 \times 2 \) identity matrix. Therefore, the many-body problem is effectively reduced to the problem of a two-level system with two levels denoted by the states \( |0\rangle \) and \( |k, -k\rangle \) which we refer to as diabatic basis vectors. We shall denote the basis vectors \( |0\rangle \) and \( |k, -k\rangle \) as \( |1\rangle \) and \( |2\rangle \), respectively, in this work for notational convenience and refer to the states as diabatic energy levels.

The paper is organized in the following way. We have already discussed the model that we are going to study. In section 2, we discuss the case of oscillatory magnetic field but restrict our attention to the situation when the quantum critical points are crossed only once, while in section 3 the possibility of multiple crossing is included. Section 4 is used to discuss the dynamics with a magnetic field which has both linearly varying and oscillatory components. In every situation results obtained through approximate analytical methods are contrasted with the numerical ones obtained by direct integration. Conclusions and a summary of the results are presented in section 5.

2. Oscillatory quenching through a quantum critical point: single crossing

In this section, we shall study the spin chain driven by an oscillatory transverse field given by \( h(t) = h_0 \cos \omega t \) from an initial time \( -\pi/\omega \) to a final time 0 so that it crosses the gapless point only once during the course of evolution. The system is initially prepared in the ground state \( |1\rangle \) whereas the final ground state is \( |2\rangle \). We shall evaluate the probability of the state \( |1\rangle \) in the final state due to non-adiabatic evolution through the gapless point. As discussed above, the transverse XY chain Hamiltonian can be written as a direct sum of \( 2 \times 2 \) reduced Hamiltonian matrices \( H_k \). For an oscillatory transverse field, the reduced Hamiltonian gets modified to

\[ H_k(t) = \begin{bmatrix} h_0 \cos \omega t + (J_x + J_y) \cos k & i(J_x - J_y) \sin k \\ -i(J_x - J_y) \sin k & -h_0 \cos \omega t - (J_x + J_y) \cos k \end{bmatrix}. \]

For any given mode \( k \), the instantaneous energy gap of the Hamiltonian is minimum at a time \( t_{0,k} \) such that \( h_0 \cos \omega t_{0,k} + (J_x + J_y) \cos k = 0 \) where the diabatic levels cross each
other. On the other hand, the energy gap vanishes for the wavevectors \( k = 0 \) and \( \pi \) at times \( \cos \omega t = \mp (J_x + J_y) \cos k/h_0 \), respectively, signalling the quantum phase transition of the Ising class.

Denoting the probability amplitudes for the states \( |1\rangle \) and \( |2\rangle \) as \( C_{1,k}(t) \), \( C_{2,k}(t) \), respectively, a general state vector in the reduced Hilbert space is written as

\[
\psi_k(t) = C_{1,k}(t)|1\rangle + C_{2,k}(t)|2\rangle.
\]  

Henceforth, we set \((J_x + J_y) = J\), and the modulus of the off-diagonal terms \(= |J_x - J_y|\) \(\sin k\) is denoted by \(\Delta_k\). Also the modulus of the rate of change of the diagonal terms at time \( t = t_{0,k} \) (given by \( h_0 \omega \sin \omega t_{0,k} \)) is denoted by \(\alpha_k\). Using the transformations

\[
C_{1,k}(t) = C_{1,k}(t)e^{-i\int_0^t dt'[h_0 \cos \omega t + J \cos k]},
\]

\[
C_{2,k}(t) = C_{2,k}(t)e^{-i\int_0^t dt'[-h_0 \cos \omega t - J \cos k]},
\]

Equation (6) we can rewrite the Schrödinger equation describing the time evolution of the probability amplitudes in the form

\[
i \frac{dC_{1,k}(t)}{dt} = \Delta_k C_{2,k}(t)e^{2i\int_0^t dt'[h_0 \cos \omega t + J \cos k]},
\]

\[
i \frac{dC_{2,k}(t)}{dt} = \Delta_k C_{1,k}(t)e^{-2i\int_0^t dt'[h_0 \cos \omega t + \cos k]}.
\]  

It should be noted that for large values of \([h_0 \cos \omega t + J \cos k]\), the phase factors on the rhs of equation (7) oscillate rapidly over time. As a result the amplitudes \(C_{1,k}(t), C_{2,k}(t)\), averaged over small intervals of time, remain basically constant over time. On the other hand, close to \( t = t_{0,k} \), the phase factors assume stationary values, thus leading to non-adiabatic transition between the energy levels [22, 27, 28].

In this section, we prepare the system in the ground state with initial conditions at \( \omega t = -\pi \), i.e., \( C_{1,k}(-\pi) = 1 \) and \( C_{2,k}(-\pi) = 0 \), and the state evolves to \( t = 0 \) so the spin chain crosses the gapless quantum critical point only once. Using equations (7), one can arrive at the differential equation describing the amplitude \(C_{1,k}(t)\) given by

\[
\frac{d^2C_{1,k}}{dt^2} - 2i(h_0 \cos \omega t + J \cos k) \frac{dC_{1,k}}{dt} + \Delta_k^2 C_{1,k} = 0
\]

with the probability of a defect in the final state at \( t = 0 \) given by \( p_k = |C_{1,k}(0)|^2 \). The maximum contribution to the non-adiabatic transition probability comes from near the points where the energy gap between the instantaneous levels is minimum. We can therefore linearize the term \( h_0 \cos \omega t \) in the neighbourhood of \( t_{0,k} \) to get

\[
\frac{d^2C_{1,k}}{dt^2} - 2i(-h_0 \omega(t - t_{0,k}) \sin \omega t_{0,k}) \frac{dC_{1,k}}{dt} + \Delta_k^2 C_{1,k} = 0.
\]

Equation (9) resembles the standard Landau–Zener transition problem [28] for the linear quenching of the magnetic field with a variation of the field \( h = t/\tau_{\text{eff}} \) where \( \tau_{\text{eff}} \) is given by the rate of change of the diagonal terms of the Hamiltonian (1). Therefore, let us define \( \alpha_k = 1/\tau_{\text{eff}} = (d/dt)(\epsilon_1 - \epsilon_2)|_{t_{0,k}} = 2h_0 \omega \sin \omega t_{0,k} = 2\omega \sqrt{h_0^2 - J^2 \cos^2 k} \). The non-adiabatic excitation probability [28] is given as \( p_k = |C_{1,k}(0)|^2 = e^{-2\pi \tau_{\text{eff}}} \), where
\[ \gamma_k = \Delta_k^2 / \|(d/dt)(\epsilon_1 - \epsilon_2)\| \big|_{\epsilon_{0,k}}, \] leading to

\[ p_k = e^{-\pi \Delta_k^2 / \omega \sqrt{h_0^2 - J^2 \cos^2 k}}. \] (10)

At this point, the natural question to ask is that of for what values of the parameters \( h_0 \) and \( \omega \) the above relation of \( p_k \) is applicable. Of course, we have used the non-adiabatic transition probability of the standard linear Landau–Zener problem where time \( t \) evolves from \(-\infty \) to \(+\infty \). The linearization near the crossing point employed above holds good only for small \( \omega \). More precisely, as discussed below, the linearization approximation is applicable when the time period of one cycle of the magnetic field (2\( \pi / \omega \)) is much greater than the Landau–Zener transition time \( (T_{LZ,k}) \) for a single crossing. The dimensionless Landau–Zener transition time \([29,30]\) is defined as \( \kappa_{LZ} = \sqrt{\alpha_k} T_{LZ,k} = |C_{2,k}(\pi/\omega)|^2 / |(d/d\omega)|C_{2,k}(0)|^2 \approx |C_{2,k}(+\infty)|^2 / |(d/d\omega)|C_{2,k}(0)|^2 \), where \( \kappa = \sqrt{\alpha_k} \). Using the above definition, we find that \( T_{LZ,k} \sim \Delta_k / \alpha_k \) in the adiabatic limit \( (\Delta_k^2 / \alpha_k \gg 1) \) whereas in the non-adiabatic limit \( (\Delta_k^2 / \alpha_k \ll 1) \), \( T_{LZ,k} \) is given as \( T_{LZ,k} \sim 1 / \sqrt{\alpha_k} \). It should be noted that for the linear quenching of the magnetic field \( h(t) = t / \tau, \alpha_k = 1 / \tau \).

Generalizing to the case of periodic quenching, \( T_{LZ,k} \sim \Delta_k / (2\omega \sqrt{h_0^2 - J^2 \cos^2 k}) \) in the adiabatic limit, while in the non-adiabatic limit, \( T_{LZ,k} \sim 1 / (2\sqrt{2\omega h_0^2 - J^2 \cos^2 k}) \). The transitions are localized around \( \epsilon_{0,k} \) as compared to the time period for one cycle of the magnetic field if \( T_{LZ,k} \) is less than the time period of one oscillation, i.e., \( T_{LZ,k} \ll \pi / \omega \). This means that for \( h_0^2 \gg J^2 \), in the adiabatic limit, \( \Delta_k \ll h_0 \), and in the non-adiabatic limit, \( \omega \ll h_0 \). This leads to the conclusion that the equation (10) describing the non-adiabatic transition probability is valid only for large \( h_0 \) and small \( \omega \). In the defect density for small \( \omega \) (i.e., \( \omega < \pi (J_x - J_y)^2 / \sqrt{h_0^2 - J^2} \)) only the modes close \( k \sim 0, \pi \) contribute, resulting in a kink density at \( t = 0 \) given by

\[ n \approx \frac{\omega \sqrt{h_0^2 - J^2}}{\pi |J_x - J_y|} \left[ \frac{\omega}{\pi} - \frac{(J_x - J_y)^2}{\sqrt{h_0^2 - J^2}} \right]. \] (11)

The analytical results for \( p_k \) and hence the density of defects (obtained from equation (11)) match exactly with the transition probabilities obtained by numerical integration of the Schrödinger equations for \( h_0 \gg \Delta_k, \omega \) as shown in figures 1 and 2. From equation (11), we find that in the limit \( h_0 \gg J \), the defect density shows a scaling form \( n \sim (h_0 \omega)^{1/2} \) which can be generalized using the Kibble–Zurek argument that assumes that the non-adiabatic transition is only dominant at a time when the characteristic timescale of the system is of the order of the rate of change of the Hamiltonian \([6,7,9]\). In the limit of large \( h_0 \) and small \( \omega \), we can generalize the above prescription to derive a scaling form for the defect density for a single crossing of the quantum critical point due to the periodic driving of the transverse field given by \( n \sim (h_0 \omega)^{d/(\nu z + 1)} \) where \( \nu \) and \( z \) are the exponents associated with the quantum critical point and \( d \) is the spatial dimension of the system.
Figure 1. \( p_k \) versus \( k \) as obtained numerically (solid line) and analytically (dashed line) for \( h_0 = 20, \omega = 0.0003, |J_x - J_y| = 0.05, \) and \( J = 1 \).

Figure 2. \( n \) versus \( \omega \) for \( h_0 = 20, |J_x - J_y| = 0.05 \) and \( J = 1 \). The solid line is found by numerically integrating \( p_k \) over \( k \), and the dashed line is the plot of equation (11). Analytical and numerical results match exactly for lower values of \( \omega \).

2.1. Entropy

Following a recent paper by Barankov and Polkovnikov [31], we define the diagonal entropy \( s_d(k) \) for each \( k \) mode by

\[
s_d(k) = -\sum_l \rho_{ll,k} \ln \rho_{ll,k},
\]

where \( \rho_{ll,k} = \langle l | \rho_k | l \rangle \), \( \rho_k \) being the instantaneous density matrix of the system for the mode \( k \). One advantage of using the diagonal entropy is that it follows the thermodynamical relations which are expected to be followed by entropy defined at higher temperatures. The diagonal entropy becomes identical to the previously defined Von Neumann entropy

doi:10.1088/1742-5468/2009/05/P05005
Figure 3. Variation of the diagonal entropy density with $\omega$ for a half-cycle with $h_0 = 20, |J_x - J_y| = 0.05$ and $J = 1$, as obtained by numerical integration of $s_d(k)$ using the analytical expressions for $p_k$ (equations (10)). The entropy for a half-cycle attains maxima near $\omega \sim \omega_0 = \pi (J_x - J_y)^2 / h_0 \ln 2$.

$(s_{VN})$, given by $s_{VN} = -\int_0^\pi \text{tr}(\rho_k \ln \rho_k) \, dk / \pi$, when the off-diagonal terms in the density matrix, evaluated at the final time, go to zero upon coarse graining over $k$ space [11, 12, 24]. We have checked the variation of the diagonal entropy density, evaluated at the final time, with $\omega$ by numerically integrating $s_d(k) = p_k \ln p_k + (1 - p_k) \ln (1 - p_k)$ over all $k$, with $p_k$ obtained from equation (10). It is seen that the entropy attains a maximum near $\omega = \omega_0 = \pi (J_x - J_y)^2 / h_0 \ln 2$ where the non-adiabatic transition probability (see equation (10)) for the mode $k = \pi / 2$ is half, and falls off on either side of $\omega_0$ (see figure 3). It should be noted that $\omega_0$ closely resembles the characteristic timescale $\tau_0$ appearing in the case of linear quenching [11].

3. Oscillatory quenching through a quantum critical point: multiple crossings

Let us now focus on the case of repeated quenching when the spin chain is periodically driven through the quantum critical point. In the present section, it will be shown that interference plays a major role in deciding the behaviour of the system, and for some choices of parameters, the phases will add up destructively to make the tunnelling probability almost zero. In order to be able to treat the successive Landau–Zener transitions as independent events, the time between two successive crossings has to be greater than the Landau–Zener transition time for a single crossing as mentioned in the previous section. To attain this limit we shall once again restrict our study to large values of $h_0$ and small $\omega$. The system is prepared in the state $|1\rangle$ at time $t = 0$. The diagonal terms of the Hamiltonian for each of the modes $k$ vanish, and consequently the gap becomes minimum when the magnetic field crosses the $-J \cos k$ line, as shown in figure 4.

When the system approaches the crossing points of the diabatic levels, the energy gap is minimum, leading to large relaxation time and the system fails to evolve adiabatically and the non-adiabatic transitions take place. On the other hand, away from the
crossing points, the system follows adiabatic dynamics. Consequently, the evolution matrices associated with the system are different for close to and away from the crossing points [21, 22]. Between the crossings, the system evolves following the matrix

$$ G_j = \begin{bmatrix} e^{-i\theta_j} & 0 \\ 0 & e^{i\theta_j} \end{bmatrix}, $$

where $j$ denotes the direction in which the system goes through the crossing points. The LZ crossing in the non-adiabatic region can be approximately described by the evolution matrix

$$ G_{LZ,j} = \begin{bmatrix} \cos \frac{\chi}{2} & \sin \frac{\chi}{2} e^{i\theta_{LZ,j}} \\ -\sin \frac{\chi}{2} e^{-i\theta_{LZ,j}} & \cos \frac{\chi}{2} \end{bmatrix}, $$

where the angle $\chi$ is given by

$$ \sin^2 \frac{\chi}{2} = 1 - \exp(-2\pi\gamma), $$

where $\gamma$ is defined in the previous section and $j$ once again defines the direction of quenching (with respect to the crossing point). We have suppressed the notation $k$ denoting the wavevector for the time being. Also

$$ \theta_{LZ,1} \approx \frac{\pi}{2} + \theta_{Stokes}, $$

$$ \theta_{LZ,2} \approx \frac{\pi}{2} - \theta_{Stokes}, $$

$$ \theta_{Stokes} = \frac{\pi}{4} + \arg \Gamma(1-i\gamma) + \gamma(\ln \gamma - 1), $$

where $\Gamma(x)$ is the gamma function and $\theta_{Stokes} \to \pi/4$ or $\theta_{Stokes} \to 0$, as $\gamma \to 0$ or $\gamma \to \infty$ respectively. If the system is repeatedly quenched with the sinusoidal field a series of
Landau–Zener crossings take place with the evolution of the system described by the successive application of the matrices defined above. More specifically for one full cycle, i.e., $\omega t$ going from 0 to $2\pi$, one can write the complete evolution matrix as a product of $G_j$ and $G_{LZ,j}$, given by

$$G = G_{LZ,2}G_2G_{LZ,1}G_1,$$

(17)

which can be generalized for $N$ complete cycles to the form

$$G_N = (G_{LZ,2}G_2G_{LZ,1}G_1)^N.$$

(18)

The probability amplitude of the states $|i\rangle C_{i,N}$ ($i = 1, 2$) at the final time $\omega t = 2N\pi$ therefore obeys the relation

$$[C_{1,N}, C_{2,N}] = (G_{LZ,2}G_2G_{LZ,1}G_1)[C_{1,0}, C_{2,0}],$$

(19)

where $C_{1,0} = 1$ and $C_{2,0} = 0$ at initial time $t = 0$. A little bit of algebra yields

$$G_{LZ,2}G_2G_{LZ,1}G_1 = \begin{bmatrix} g_{11} & g_{21} \\ -g_{21}^* & g_{11}^* \end{bmatrix},$$

(20)

where

$$g_{11} = \cos^2 \frac{\chi}{2} e^{-i(\theta_1 + \theta_2)} - \sin^2 \frac{\chi}{2} e^{i(-\theta_{LZ,1} + \theta_{LZ,2} - \theta_1 + \theta_2)},$$

$$g_{21} = \sin \frac{\chi}{2} \cos \frac{\chi}{2} (e^{i(\theta_{LZ,1} + \theta_1 - \theta_2)} + e^{i(\theta_{LZ,2} + \theta_1 + \theta_2)}).$$

(21)

Denoting the probability that for mode $k$ the system is in state $|2\rangle$ after the $n$th crossing by $Q_{n,k}$, we get

$$Q_{1,k} = (1 - p_k)$$

(22)

as seen in the previous section and for one complete full period of oscillation

$$Q_{2,k} = 4p_k(1 - p_k)\sin^2(\theta_{\text{Stokes}} + \theta_2).$$

(23)

For small anisotropy, i.e., $(J_x - J_y)^2 \ll \omega\sqrt{h_0^2 - J^2 \cos^2 k}$ we have $\theta_{\text{Stokes}} \rightarrow \pi/4$, and $\theta_1$ and $\theta_2$ are given by

$$\theta_2 = 2 \int_0^{\pi/2} \sqrt{(h_0 \cos \omega t + J \cos k)^2 + \Delta_k^2} \approx \left( \frac{2h_0 + J\pi \cos k}{\omega} \right),$$

(24)

$$\theta_1 = 2 \int_{\pi/2}^{\pi} \sqrt{(h_0 \cos \omega t + J \cos k)^2 + \Delta_k^2} \approx \left( \frac{-2h_0 + J\pi \cos k}{\omega} \right).$$

(25)

Substituting equations (24) and (25) in equation (23), we get [19, 20]

$$Q_{2,k} = 4p_k(1 - p_k)\sin^2 \left( \frac{2h_0 + J\pi \cos k}{\omega} + \frac{\pi}{4} \right).$$

(26)

The $Q_{2,k}$s obtained numerically and analytically as given in equation (26) are plotted as a function of $k$ in figures 5 and 6. The numerical plot is in fairly good agreement with the analytical results. It is seen that $Q_{2,k}$ oscillates, with the tunnelling probability going to zero for many ks, showing the signatures of constructive and destructive interferences.

doi:10.1088/1742-5468/2009/05/P05005
Figure 5. Variation of $Q_{2,k}$ with $k$ for $h_0 = 20, |J_x - J_y| = 0.005$, $\omega = 0.01$ and $J = 1$. The widely spaced dashed line is analytical, and the numerical data points are shown on the closely spaced dashed line.

Figure 6. Variation of $Q_{2,k}$ with $k$ for $h_0 = 20, |J_x - J_y| = 0.005$, $\omega = 0.01$ and $J = 1$ obtained by averaging out the oscillations of the data shown in figure 5. The dashed line is analytical, and the solid line is numerical. The smooth dotted line is the plot of coarse grained excitation probability $\bar{Q}_{2,k}$ as obtained from equation (27).

It is clear that for very small $\omega$, $Q_{2,k}$ oscillates rapidly with $k$ due to the presence of the sinusoidal term in equation (26). As a result the coarse grained or average $Q_{2,k}$ (denoted by $\bar{Q}_{2,k}$), obtained by integrating each $Q_{2,k}$ over a small range around that $k$ followed by normalization, gives

$$\bar{Q}_{2,k} = 2p_k(1 - p_k),$$

as obtained previously for repetition under linear quenching [24]. It has been shown in recent works [11, 12, 24] that, for linear quenching, we can evaluate the transition probabilities at the end of each cycle by using the coarse grained density matrix only,
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Figure 7. Variation of actual and coarse grained kink density and entropy density with $\omega$ for $h_0 = 20$, $|J_x - J_y| = 0.005$ and $J = 1$ for one complete cycle, obtained by numerically integrating $Q_{2,k}$ and $Q_{2k}$ in equation (26) and equation (27) respectively. The actual and coarse grained plots for kink density match reasonably well, whereas we see a significant difference in the case of entropy density, even though their qualitative behaviours show similarity for a wide range of $\omega$. The peaks of the plots occur near $\omega_0$.

thereby simplifying the problem greatly by neglecting the off-diagonal terms in the matrix. Analogously, in the present case also, the characteristic timescale associated with the rate of change of the off-diagonal terms in the density matrix for a mode $k$ sets the critical value of $\omega$ below which we can safely describe the tunnelling probability by the coarse grained expression of $Q_{2k}$ only, thereby yielding equation (27) \[19\]. One concludes that in the limit of very small $\omega$, the time interval between two successive crossings is large enough to destroy the phase information in the coarse grained probabilities. The variation of actual as well as coarse grained kink density and entropy density for one complete cycle has been shown in figure 7.

To generalize to the case of many complete periods, it is useful to recast equation (20) in the form \[21\]

$$G_{LZ,2}G_{2}G_{LZ,1}G_{1} = \begin{pmatrix} \cos \frac{\zeta}{2} & \sin \frac{\zeta}{2} e^{i\phi} \\ -\sin \frac{\zeta}{2} e^{-i\phi} & \cos \frac{\zeta}{2} \end{pmatrix} \begin{pmatrix} e^{-i\theta/2} & 0 \\ 0 & e^{i\theta/2} \end{pmatrix}. \quad (28)$$

We shall call the diagonal matrix $U_1$ and the other $U_2$, which successively operate on the column matrix $(C_{1,k}, C_{2,k})$. The angles $\zeta$ and $\theta$ are given as

$$\sin^2 \frac{\zeta}{2} \approx 4 \sin^2 \chi \cos^2 \left( \frac{\theta_{LZ,1} - \theta_{LZ,2}}{2} - \theta_2 \right), \quad (29)$$

$$\theta = \tan^{-1} \frac{A}{B}, \quad (30)$$

where

$$A = \cos^2 \frac{\chi}{2} \sin (\theta_1 + \theta_2) + \sin^2 \frac{\chi}{2} \sin (\theta_{LZ,1} - \theta_{LZ,2} + \theta_1 - \theta_2),$$

$$B = \sin (\theta_1 - \theta_2) \sin (\theta_{LZ,1} - \theta_{LZ,2}) + \cos (\theta_1 - \theta_2) \cos (\theta_{LZ,1} - \theta_{LZ,2}).$$

doi:10.1088/1742-5468/2009/05/P05005
Figure 8. Graph showing the behaviour of the excitation probability as a function of the number of full cycles for $h_0 = 20, (J_x - J_y) = 0.005, J = 1$ and $\omega = 0.01$. The dashed line is obtained analytically for $k = 81.9521^\circ$ with integral $2J \cos k/\omega = 28$, while the solid line is the analytical graph for $k = 25.8419^\circ$ with integral $2J \cos k/\omega = 180$. Numerical data points shown on the dashed line corresponding to $k = 81.9521^\circ$ match exactly with the analytical results.

and

$$B = \cos^2 \chi \cos (\theta_1 + \theta_2) - \sin^2 \chi \cos (\theta_{LZ,1} - \theta_{LZ,2} + \theta_1 - \theta_2),$$

$$\phi \approx \frac{\theta_{LZ,1} + \theta_{LZ,2}}{2} - \theta_2.$$  \hspace{1cm} (31)

The dynamics described by equation (28) can be understood by exploring the properties of the rotation matrices $U_1$ and $U_2$. The role of $U_2$ is to rotate a vector about an axis in the $x$-$y$ plane by an angle of $\zeta$, whereas the matrix $U_1$ brings about a rotation of the vector by an angle $\theta$ around the $z$ axis only [32]. If $\theta$ is a multiple of $2\pi$, which we call the resonance condition, the $z$-axis rotation does not affect the dynamics and the small oscillations of $\zeta$ add up constructively to produce full oscillations between the diabatic states $|1\rangle$ and $|2\rangle$ (see figure 8). On the other hand, if $\theta$ differs from a multiple of $2\pi$ by more than $\zeta$, then the rotations about angle $\zeta$ do not add up constructively, and the oscillations will be suppressed, thus resulting in an effective rotation about an axis almost parallel to the $z$ axis only [21] (see figure 9). In the present context, the resonance condition is given by

$$\theta \approx 2(\theta_1 + \theta_2) = \frac{4\pi J \cos k}{\omega} = 2n\pi,$$  \hspace{1cm} (32)

i.e.,

$$\frac{2J \cos k}{\omega} = n.$$  \hspace{1cm} (33)
Figure 9. Graph showing the behaviour of the excitation probability as a function of the number of full cycles, obtained analytically for $h_0 = 20$, $|J_x - J_y| = 0.005$, $J = 1$, $\omega = 0.01$, $k = 80^\circ$ and non-integral $2J \cos k/\omega$. As expected, the excitation probability varies randomly and does not differ appreciably from its initial value.

Therefore for the resonance conditions, we can write

$$
\begin{bmatrix}
    C_{1,N} \\
    C_{2,N}
\end{bmatrix}
= \pm
\begin{pmatrix}
    \cos \frac{\zeta}{2} & \sin \frac{\zeta}{2} e^{i\phi} \\
    -\sin \frac{\zeta}{2} e^{-i\phi} & \cos \frac{\zeta}{2}
\end{pmatrix}
^N
\begin{bmatrix}
    C_{1,0} \\
    C_{2,0}
\end{bmatrix}.
$$

From this formalism it is clearly seen that for the resonance conditions, after $N$ complete cycles, we get $N$ successive rotations by the small angle $\zeta$. This causes oscillations in the probabilities of the two states with frequency given by

$$
\Omega = \frac{\zeta}{2\pi/\omega} = \frac{\omega \zeta}{2\pi} = \frac{\omega}{\pi} \sin^{-1} \left[ 2\sqrt{1 - p_k^2} \cos \left( \theta_{\text{Stokes}} - \frac{2h_0 + J\pi \cos k}{\omega} \right) \right].
$$

Since $\zeta$ depends on the wavevector $k$, the probability for each resonant mode oscillates with its own characteristic frequency. As a result the kink density and the entropy density obtained by integrating over all modes show an oscillatory behaviour (see figure 10). The oscillatory behaviour of the entropy density observed here is an artefact of retaining the phase information of the off-diagonal terms of the density matrix. It can be shown that in the absence of phase information, $|C_{1,k}(\bar{t})|^2, |C_{2,k}(\bar{t})|^2 \to 1/2$ after each crossing, and as a result the entropy density of the system increases monotonically [24] with the number of crossings.

4. Quenching by a magnetic field varying both linearly and periodically

In this section, we study the defect generation in a transverse $XY$ spin chain driven by a time-dependent magnetic field $h(t)$ which consists of a linear part as well as an oscillatory part given by $h(t) = t/\tau + h_0 \cos \omega t$ where $\tau$ denotes the rate of the linear part of the quenching. In the limit $h_0 \to 0$, the dynamics problem reduces to the well studied Kibble–Zurek problem of linear quenching, while in the other limit of $\tau \to \infty$, one should recover
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Figure 10. Graph showing the behaviour of the kink density (solid line) and entropy density (dashed line) as a function of the number of full cycles for $h_0 = 20, |J_x - J_y| = 0.005, J = 1$, and $\omega = 0.01$, as obtained by numerically integrating the excitation probabilities obtained by using equation (19).

the results presented in earlier sections. The presence of both the linear and periodic terms non-trivially modifies the density of defects in the final state as shown below. The reduced Hamiltonian in the present situation is given by

$$H_k(t) = \begin{bmatrix} \epsilon & i(J_x - J_y) \sin k \\ -i(J_x - J_y) \sin k & -\epsilon \end{bmatrix},$$

with $\epsilon = t/\tau + h_0 \cos \omega t + J \cos k$ and we shall once again recall the parameters $\alpha_k$ and $\Delta_k$ as defined before.

For a given wavevector $k$, the instantaneous energy gap is minimum at times $t_{0,k}$ such that

$$\frac{t_{0,k}}{\tau} + h_0 \cos \omega t_{0,k} + J \cos k = 0. \quad (35)$$

Since $\cos \omega t_{0,k} \leq 1$ always, from the above equation we can conclude that all the $t_{0,k}$s occur in the time interval $-\tau(J + h_0) < t_{0,k} < \tau(J + h_0)$. Also, since the time between two successive $t_{0,k}$s is of the order of $\pi/\omega$, we can estimate the number of times that the gap goes to minimum for any $k$ as $\sim (2h_0 \tau / (\pi/\omega))$, with the minimum number of times being 1. The situation is depicted in figure 11.

In the adiabatic limit ($\Delta_k^2/\alpha_k \gg 1$) the Landau–Zener transition time [29, 33] ($\tau_{LZ}$) is given as $\tau_{LZ} \sim \Delta_k^2/\alpha_k$, and in the non-adiabatic case, we have $\tau_{LZ} \sim 1/\sqrt{\alpha_k}$. It should be noted in the present case that the rate of change of diagonal terms is $\alpha_k = |(d/dt)(t/\tau) + h_0 \cos \omega t + J \cos k)|_{t_{0,k}} = 2|1/\tau - h_0 \omega \sin \omega t_{0,k}|$. Hence for our theory to be valid, i.e., to get widely separated non-overlapping LZ transitions, we need $\tau \ll (\pi/\omega(J_x - J_y))$ for the adiabatic and $\sqrt{\tau} \ll (\pi/\omega)$ for the non-adiabatic situations.

We prepare the system in its ground state at $t \rightarrow -\infty$ with $|C_{1,k}(-\infty)|^2 = 1$ and the probability of defects for the mode $k$ in the final state at $t \rightarrow +\infty$ is given by the probability $|C_{1,k}(+\infty)|^2$. For the linear as well as periodic driving, equation (8) when
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Figure 11. Graph showing the behaviour of the diabatic energy levels with time when a linearly varying magnetic field is applied in addition to the oscillatory term. The inclined dotted line is the plot of \( t/10 \), the solid line is \((t/10) + \cos t\), and the dashed line parallel to the \( x \) axis is the constant \(-J \cos k = 1\).

Figure 12. \( p_k \) versus \( k \) (in degrees) for the case when gap becomes minimum only once, with \( \tau = 2, h_0 = 1, \omega = 0.1, |J_x - J_y| = 0.05 \) and \( J = 10 \). The dashed line is analytical and the numerical results shown as data points coincide exactly with the analytical values.

linearized around the crossing point \( t = t_{0,k} \) gets modified to

\[
\frac{d^2C_{1,k}}{dt^2} + 2i \left( \left( \frac{1}{\tau} - h_0 \omega \sin \omega t_{0,k} \right) (t - t_{0,k}) \right) \frac{dC_{1,k}}{dt} + \Delta_{k}^2 C_{1,k} = 0, \tag{36}
\]

which leads to the non-adiabatic transition probability

\[
p_k = e^{-(\pi \Delta_k^2)/(1/\tau) - h_0 \omega \sin \omega t_{0,k})}. \tag{37}
\]

A plot of \( p_k \) as a function of \( k \) for the above mentioned case is shown in figure 12.

doi:10.1088/1742-5468/2009/05/P05005
Figure 13. Kink density \( n \) versus \( \tau \) for \( h_0 = 1, \omega = 0.1, |J_x - J_y| = 0.05 \) and \( J = 10 \). The solid line is the plot of equation (38), and the dashed line is obtained by numerical integration of the analytical expression of \( p_k \) as given in equation (37). We get exact matching between the two results for low \( \tau \) only, as expected from the theory. Only single crossing occurs for the range of \( \tau \) shown in the figure.

In the limit of small \( \tau \) and large \( \omega \), we can expand the excitation probability as

\[
p_k \approx 1 - \left( \frac{\pi (J_x - J_y)^2 \sin^2 k}{1/\tau - h_0 \omega \sin \omega t_{0,k}} \right).
\]

Further, for \( 1/\tau \gg h_0 \omega \sin \omega t_{0,k} \), we can write the expression for the kink density as

\[
n \approx \frac{1}{\pi} \int_0^{\pi} \left[ 1 - \frac{\pi (J_x - J_y)^2 \sin^2 k}{1/\tau - h_0 \omega \sin \omega t_{0,k}} \right] dk
\approx 1 - \frac{\pi (J_x - J_y)^2}{2\tau}.
\]

The approximate equation given in equation (38) matches perfectly with the numerical integration results (see figure 13). On the other hand, in the limit of large \( \tau \) and small \( \omega \), only the modes close to \( k = 0 \) or \( \pi \) contribute to the defect density, and by considering only the 0 and \( k \) modes in \( t_{0,k} \), we can arrive at an approximate analytical expression given by

\[
n = \frac{1}{\pi} \int_0^{\pi} p_k \, dk = \frac{\pi \sqrt{(1/\tau) - h_0 \omega \sin \omega t_{0,0}}}{2|J_x - J_y|} + \frac{\pi \sqrt{(1/\tau) - h_0 \omega \sin \omega t_{0,\pi}}}{2|J_x - J_y|}.
\]

The kink densities as a function of \( \tau \) for the non-adiabatic and adiabatic cases, as obtained from equations (38) and (39) respectively, are plotted in figures 13 and 14 together with the corresponding numerically obtained values. As expected, in the case of non-adiabatic evolution, we get exact matching between analytical and numerical results only for low values of \( \tau \) for which \( (1/\tau) \gg h_0 \omega \sin t_{0,k} \), whereas in the case of adiabatic evolution, we see good agreement between the analytical and numerical results only when \( (1/\tau) \) is not close to \( h_0 \sin \omega t_{0,0} \) or \( h_0 \sin \omega t_{0,\pi} \), since around these values of \( \tau \), the effects of \( h_0 \sin \omega t_{0,k} \) for \( k \neq 0, \pi \) become important.

\[\text{doi:10.1088/1742-5468/2009/05/P05005}\]
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Figure 14. Kink density $n$ versus $\tau$ for $h_0 = 1, \omega = 0.000001, \Delta = 0.05$ and $J = 10$. Only single crossing occurs for the range of $\tau$ shown in the figure. The dashed line is found by numerical integration of equation (37), and the solid line is obtained by using equation (39). We get exact matching between the two results except when $1/\tau$ is very close to $h_0 \omega \sin \omega t_0,0$ or $h_0 \omega \sin \omega t_{0,\pi}$.

It should be noted that for the single-crossing case, using a line of argument similar to that in section 2, one can propose a generalized Kibble–Zurek scaling form of the defect density in the final state given as

$$ n \sim a_0 \left[ \left| \frac{1}{\tau} - h_0 \omega \sin \omega t_{0,0} \right|^{\nu d / (\nu z + 1)} \right] + a_\pi \left[ \left| \frac{1}{\tau} - h_0 \omega \sin \omega t_{0,\pi} \right|^{\nu d / (\nu z + 1)} \right], $$

(40)

where $a_0$ and $a_\pi$ are two constants.

Now we concentrate on the situation of multiple crossings of the energy minima. For multiple crossings to occur for a given wavevector $k$, figure 11 implies that there should exist a $\bar{t}$ such that $(1/\tau) - h_0 \omega \sin \omega \bar{t} = 0$, i.e., $\sin \omega \bar{t} = (1/\tau h_0 \omega)$. This is possible only if $|(1/\tau h_0 \omega)| \leq 1$. The Schrödinger equations for the probability amplitudes can be put in the form

$$ \frac{dC_{1,k}}{dt} = \Delta_k e^{i((t^2/\tau) + (2h_0 \sin \omega t/\omega) + 2Jt \cos k)} C_{2,k}, $$

(41)

$$ \frac{dC_{2,k}}{dt} = \Delta_k e^{i((t^2/\tau) + (2h_0 \sin \omega t/\omega) + 2Jt \cos k)} C_{1,k}. $$

(42)

Using the relation $e^{\pm iz \sin \omega t} = \sum_{r=-\infty}^{\infty} J_r(z) e^{\pm i r \omega t}$, where $J_r(z)$ is the Bessel function of first kind of order $r$, given by

$$ J_r(\eta) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m! \Gamma(m + r + 1)} \left( \frac{\eta}{2} \right)^{2m+1}, $$

(43)

we recast the equation to the form

$$ \frac{dC_{1,k}}{dt} = \Delta_k \sum_{r=-\infty}^{\infty} J_r \left( \frac{2h_0}{\omega} \right) e^{i((t^2/\tau) + 2Jt \cos k + r \omega t)} C_{2,k}. $$

(44)

doi:10.1088/1742-5468/2009/05/P05005
The terms on the RHS of equation (44) being rapidly varying in time, \( \frac{dC_{1,k}}{dt} \) attains a non-negligible value only when the phase is stationary. Using
\[
\frac{t^2}{\tau} + 2Jt \cos k + r \omega t = \frac{1}{\tau} \left( t + \frac{2Jt \cos k + r \omega \tau}{2} \right)^2 - \frac{(2Jt \cos k + r \omega \tau)^2}{4\tau}
\]
we find that \((dC_{1,k}/dt)\) is non-negligible only close to \( t = -(2Jt \cos k + r \omega \tau)/2 \), with \( r = 0, \pm 1, \pm 2, \ldots \). The above relation implies the existence of an effective phase transition for each value of \( r \). Choosing \( r = l \) (say) and denoting \( C_{i,k} \) by \( C_{i,k,i} \) \((i = 1, 2)\), we get
\[
\frac{dC_{1,l,k}}{dt} = \Delta_k J_t e^{-i(2Jt \cos k + l\omega \tau)^2/4\tau} e^{i(1/\tau)(t+(2Jt \cos k + l\omega \tau)/2)^2} C_{2,l,k}.
\]
Invoking the transformation to a new variable \( s = t + l\omega \tau/2 \), we get
\[
\frac{dC_{1,l,k}}{ds} = \Delta_k J_t e^{-i(\ell \omega^2 r^2 + 4Jr^2 l\omega \cos k)/4\tau} e^{-iJ^2 r \cos^2 k e^{i(l/\tau)(s + J^2 \cos k)^2} C_{2,l,k}.
\]
Let us compare with the purely linearly quenching case \((h_0 = 0)\), when the above equation gets modified to
\[
\frac{dC_{1,l,k}}{ds} = \Delta_k e^{-iJ^2 r \cos^2 k e^{i(l/\tau)(s + J^2 \cos k)^2} C_{2,l,k}.
\]
The role of periodic modulation on top of the linear quenching is to renormalize \( \Delta_k \) to \( \Delta_k^{(s)} \) with \( \Delta_k^{(s)} = \Delta_k J_t e^{-i(\ell \omega^2 r^2 + 4Jr^2 l\omega \cos k)/4\tau} \). Note that \( \Delta_k^{(s)} \) also vanishes at the quantum critical point for the modes \( k = 0 \) and \( \pi \).

The dividing of the probability amplitudes for a given wavevector into different \( l \) values by using Bessel’s function can be visualized in the following way: the two energy levels for the wavevector \( k \) are assumed to consist of a number of sublevels [23], with probability amplitudes of the \( l \)th level being denoted by \( C_{1,l,k} \) and \( C_{2,l,k} \). Each sublevel undergoes a level crossing only once through the course of the dynamics, and for the \( l \)th transition for the mode \( k \), the incoming state (given by \( C_{l-1,k} \)) and the outgoing state (given by \( C_{l,k} \)) are connected by the transfer matrix [23]
\[
M_{l,k} = \begin{bmatrix}
D_{l,k} & \beta_{l,k} e^{-i(\ell \omega^2 r^2 + 4Jr^2 l\omega \cos k)/4\tau} \\
-\beta_{l,k} e^{-i(\ell \omega^2 r^2 + 4Jr^2 l\omega \cos k)/4\tau} & D_{l,k}
\end{bmatrix},
\]
where \( D_{l,k} = \sqrt{p_{l,k}} \) and \( \beta_{l,k} = \text{sgn} J_t(\eta) \sqrt{1 - p_{l,k} e^{-i\phi_{l,k}}} \), in which \( p_{l,k} = e^{-\pi \tau \Delta^2 J_t(\eta)^2} \), \( \eta = 2h_0/\omega \) and \( \phi_{l,k} \) is the Stokes phase given by
\[
\phi_{l,k} = \pi/4 + \arg \Gamma(1 - i\delta_{l,k}) + \delta_{l,k} (\ln \delta_{l,k} - 1),
\]
where \( \delta_{l,k} = \tau \Delta^2 J_t(\eta)^2/2 \), in terms of the gamma function \( \Gamma(z) \).

It should be noted that \( J_t(\eta) \to 0 \) for large \( l \), and \( \sum_{l=-\infty}^{\infty} J_t^2(\eta) = 1 \). So the transition is confined to a finite region, and the infinite series of recursive relations for \( l \to \infty \) converges to a finite value. In the case of \( l \) for which \( J_t(\eta) \approx 0 \), \( M_{l,k} \) gets reduced to an identity matrix. Hence taking \( J_t(\eta) = 0 \) \forall l > l_1 \) we can write the state vector \( \vec{C}_{l,k} = (C_{1,l,k}, C_{2,l,k}) \) as
\[
\vec{C}_{l,k} = M_{l,k} \vec{C}_{l-1,k} = M_{l,k} M_{l-1,k} \cdots M_{0,k} \cdots M_{-l+1,k} M_{-l,k} \vec{C}_{in,k}.
\]
Results obtained by numerically solving the Schrödinger equation (dotted line) match reasonably well with the analytical ones (dashed line).

where $\tilde{C}_{in,k}$ denotes the initial condition. The probability of excitation at infinite time is given by $p_k(\infty) = |C_{1,k}(\infty)|^2$:

$$p_k(\infty) = |C_{1,k}(\infty)|^2 = |[1 \ 0]M_{l,k}M_{l-1,k} \cdots M_{0,k} \cdots M_{-l+1,k}M_{-l,k}\tilde{C}_{in,k}|^2.$$  (51)

We are interested in evaluating the defect density in the final state in the limit of large $\tau$ and hence the off-diagonal terms in the matrix $M_{l,k}$ vanish upon coarse graining, i.e., upon integration over all $k$. This approximation leads to the final result for the coarse grained non-adiabatic transition probability given by

$$|C_{1,l,k}|^2 = p_{l,k}|C_{1,l-1,k}|^2 + (1 - p_{l,k})|C_{2,l-1,k}|^2$$  (52)

after neglecting the cross-terms. The effects of the critical points are manifested by $\Delta k$ vanishing at $k = 0, \pi$ and those modes do not evolve from their initial state.

The comparison between kink densities obtained by numerical integration of the Schrödinger equation and by using the approximate analytical equations (51) and (52) for different $\tau$ has been shown in figure 15. We also plot the diagonal entropy density in figures 16 and 17. Although, the defect density in the final state decays with increasing $\tau$, the entropy density is found to increase and ultimately saturates as $\tau$ increases. This is in sharp contrast to the case for linear quenching [12, 24] where it attains a maximum at a characteristic scale $\tau = \tau_0$, and falls off on either side of $\tau_0$. This behaviour can be explained in the following way: for very small $\tau$ the system fails to evolve appreciably and therefore remains very close to its initial ordered state at the final time. For larger values of $\tau$, the probabilities change which introduces disorder in the final state, leading to higher values of the entropy density. For very large $\tau$, the linear term becomes insignificant compared to the oscillatory term in the Hamiltonian. Consequently, only the oscillatory quenching term contributes to the dynamics of the system, keeping the entropy of the system constant for a fixed value of $\omega$. It is to be noted that the interaction between the levels depends on the value of $J_l(2h_0/\omega)$ which saturates in the limit of large $\omega$ if $h_0$ is held fixed irrespective of the values of $l$ and $k$. As a result, the kink density and also entropy density saturate in the limit of large $\omega$, as shown in figure 17.

Figure 15. Graph showing the variation of the kink density with $\tau$ for multiple crossings occurring with $h_0 = 1$, $(J_x - J_y) = 0.05$, $(J_x + J_y) = 10$, and $\omega = 0.1$. Results obtained by numerically solving the Schrödinger equation (dotted line) match reasonably well with the analytical ones (dashed line).
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Figure 16. Graph showing the variation of the kink density (solid line) and entropy density (dashed line) with $\tau$ as obtained analytically for $h_0 = 1$, $(J_x - J_y) = 0.05$, $(J_x + J_y) = 10$, and $\omega = 0.1$.

Figure 17. Graph showing the variation of the kink density (solid line) and entropy density (dashed line) with $\omega$ as obtained analytically for $h_0 = 1$, $(J_x - J_y) = 0.05$, $(J_x + J_y) = 10$, and $\tau = 50$.

5. Conclusion

In conclusion, we have studied the effects of interference in the quenching dynamics of a one-dimensional transverse $XY$ spin chain in the presence of a time-dependent magnetic field $h(t) = h_0 \cos \omega t$ or $h(t) = t/\tau + h_0 \cos \omega t$. The system is initially prepared in its ground state and we estimate the defect density and entropy density in the final state following the quench using both approximate analytical and direct numerical integration techniques. In all the situations, the analytical and numerical results are found to be in good agreement. Our observations presented in the paper are summarized as follows.

Firstly, we have studied the defect density in the final state following a single crossing by linearizing the oscillatory magnetic field round the time at which the instantaneous
energy gap is minimum. We show that in the limit of large $h_0$ and small $\omega$, the defect density scales as $\sqrt{\omega}$. The observation is supported by numerical solution of the Schrödinger equation in the limit of small $\omega$. On the other hand, the diagonal entropy density shows a maximum at a characteristic frequency scale $\omega_0$ as defined in the text. Effects of interference are invisible in the case of a single crossing only. We also suggest an equivalent Kibble–Zurek scaling relation for the defect density in the above situation.

In the next section we generalized to the multiple-crossing situation where the interference of the probability densities plays a dominant role. We use two different transfer matrices valid close to and away from the crossing points. We show that for a full cycle of oscillation the results obtained for repeated linear quenching [24] when the off-diagonal terms of the density matrix are coarse grained, leading to loss of phase information which gives rise to constructive and destructive interferences, are a valid approximation in the limit of small $\omega$. For multiple crossings, we show that there exist resonance wavevectors for which the non-adiabatic transition probability oscillates between zero and one with the number of crossings following a characteristic $k$-dependent frequency. As a result the defect density also exhibits an oscillatory behaviour. The entropy density also shows a similar dependence on the number of crossings, which is in stark contrast with the linear quenching case, in which exclusion of the interference effects in the excitation probabilities causes the entropy density to increase monotonically with the number of crossings. It may be noted that a similar oscillatory behaviour is observed for the central spin of the quantum Heisenberg chain [34].

Lastly, we study the quenching of the spin chain in the presence of a magnetic field which is varying linearly with time as $t/\tau$ and also modulated by a periodically varying part $h_0 \cos \omega t$. For the single-crossing case, we once again use the linearization method which predicts a defect density that is in fair agreement with the numerically obtained result. For multiple crossings, we again invoke the transfer matrix approach to evaluate the cross grained defect density. In this case it has been shown that for large values $\tau$ we can safely neglect the phase information, and hence the effects of interference, by coarse graining the density matrix. Our analytical and numerical results show that the defect density decreases with increasing $\tau$ for a given $\omega$ whereas when $\omega$ is varied with $\tau$ fixed, the defect density saturates for higher values of $\omega$. The entropy density also exhibits a monotonic increase as a function of $\tau$ with fixed $\omega$, an observation that is in sharp contrast with the linear quenching case where the entropy density attains a maximum at a characteristic timescale $\tau_0$ [12]. This may be an artefact of the integrability of the model which gets decoupled into independent two-level systems [35].

Acknowledgments

AD acknowledges R Moessener and the hospitality of MPIPKS, Dresden, where a major part of this work was done. We acknowledge Anatoli Polkovnikov, Diptiman Sen and Uma Divakaran for critical comments and helpful discussions.

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