In this work we present a novel approach to the ray optics limit: we rewrite the dynamical Maxwell equations in Schrödinger form and prove Egorov-type theorems, a robust semiclassical technique. We implement this scheme for periodic light conductors, photonic crystals, thereby making the quantum-light analogy between semiclassics for the Bloch electron and ray optics in photonic crystals rigorous. Our main results, Theorems 3.3 and 4.1, give a ray optics limit for quadratic observables and, among others, apply to local averages of energy density, the Poynting vector and the Maxwell stress tensor. Ours is the first rigorous derivation of ray optics equations which include all sub-leading order terms, some of which are also new to the physics literature. While the ray optics limit we prove initially (Theorem 3.3) applies to photonic crystals of any topological class, we also consider the ray optics limit for real electromagnetic fields propagating in non-gyrotropic photonic crystals. Such an extension is non-trivial, because the ray optic limit for real fields is necessarily a multiband problem.

Key words: Maxwell equations, ray optics, semiclassical limit, Egorov theorem

MSC 2010: 81Q20, 35Q60, 35Q61, 35S05
1 Introduction

The main idea of ray optics is to approximate full electrodynamics as given by the source-
free Maxwell equations in a medium

\[ \mathbf{d} \frac{d}{dt} \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} = \begin{pmatrix} +\nabla \times \mathbf{H} \\ -\nabla \times \mathbf{E} \end{pmatrix}, \]  
\[ \mathbf{d} \cdot \left( \begin{pmatrix} \varepsilon \mathbf{E} + \chi \mathbf{H} \\ \chi^* \mathbf{E} + \mu \mathbf{H} \end{pmatrix} \right) = 0, \]  
(dynamical eqns.) \hspace{1cm} (1.1a)

\[ \mathbf{d} \cdot \left( \begin{pmatrix} \varepsilon \mathbf{E} + \chi \mathbf{H} \\ \chi^* \mathbf{E} + \mu \mathbf{H} \end{pmatrix} \right) = 0, \]  
(no sources eqns.) \hspace{1cm} (1.1b)
by simpler Hamiltonian equations of motion of the form
\[ \dot{r} = +\nabla r \varOmega + \mathcal{O}(\lambda), \quad (1.2a) \]
\[ \dot{k} = -\nabla k \varOmega + \mathcal{O}(\lambda). \quad (1.2b) \]

Here, the material weights electric permittivity \( \varepsilon = \varepsilon(\lambda) \), magnetic permeability \( \mu = \mu(\lambda) \) and bi-anisotropic tensor \( \chi = \chi(\lambda) \) are \( 3 \times 3 \)-matrix-valued functions which describe the response of the medium to the impinging electromagnetic waves; the presence of the perturbation parameter \( \lambda \ll 1 \) indicates that the material weights are modulated compared to their unperturbed counterparts (see Assumption 2.2 for the case considered in this paper). The material weights enter (1.2) implicitly via the dispersion relation \( \varOmega(r, k) \), and indeed, one of the tasks in justifying a ray optics limit is to determine \( \varOmega \) from the weights and the initial state.

The advantage of ray optics equations (1.2) is that they provide a simpler, effective description of the propagation of light in a medium, i.e., we can study solutions of an ODE to understand the behavior of a PDE. Ray optics are used in a wide variety of circumstances, and newfound applications to fields such as computer vision and image processing (see e.g. [STZ99; RG09]) mean it still is an area of active research. One may also think of more sophisticated ray optics equations which include polarization as a classical spin degree of freedom. Instead of having to solve (1.1) for \( (E(t), H(t)) \), ray optics equations describe a light wave by its position \( r \) and its wave vector \( k \), and the wave front propagates with group velocity \( \dot{r} \) along the trajectory \( (r(t), k(t)) \). However, a priori it is not at all clear in what sense (1.2) approximates (1.1), and how to quantify the error.

The purpose of this paper is to derive the ray optics limit in a novel way by rewriting the dynamical Maxwell equations (1.1a) in Schrödinger form and proving an Egorov theorem, a well-known and robust semiclassical technique. While most derivations of ray optics (see e.g. [Som98, Chapter 5.4], [Per00, Chapter 2] and [OMN06]) employ what would be called “semiclassical wavepacket methods” in the context of quantum mechanics, our technique does not rely on the localization of \( (E, H) \) around some \( (r_0, k_0) \) in phase space. More specifically, we identify a class of electromagnetic observables \( F: L^2(\mathbb{R}^3, \mathbb{C}^n) \rightarrow \mathbb{R} \) for which there exists a ray optics observable \( f \) (i.e. a function of \( (r, k) \)) so that in the simplest case we can express
\[ F(E(t), H(t)) = \int_{\mathbb{R}^3} dr \int_{\mathbb{R}^3} dk (f \circ \Phi^t)(r, k) \omega_{E,H}(r, k) + \mathcal{O}(\lambda^2) \quad (1.3) \]
as a phase space average (cf. Theorems 3.3 and 4.1 for details). The ray optics observable \( f \) is transported along the flow \( \Phi^t \) associated to (1.2), and integrated against the Wigner transform \( \omega_{E,H} \) (cf. Corollary 3.5). Note that in order to reduce the error from \( \mathcal{O}(\lambda) \) to \( \mathcal{O}(\lambda^2) \) in equation (1.3) we need to include \( \mathcal{O}(\lambda) \) terms to the ray optics equations (1.2). Apart from the local energy density, our results also cover local averages of
1 Introduction

the Poynting vector, the field amplitudes and the components of the Maxwell stress tensor (see Section 3.3). Note that the error term in (1.3) can be estimated uniformly in \((E, H)\) as long as we keep the field energy fixed. Our approach overcomes two major limitation of "wavepacket techniques": Mathematically, these are notoriously hard to justify. And physically, given that they depend on a judicious choice of initial state, it is hard to go beyond leading order and compute the \(O(\lambda)\) corrections which often contain novel physical effects.

We will illustrate how to implement a ray optics limit via semiclassical techniques for photonic crystals, periodically patterned light conductors. Just as in case of the Bloch electron the periodic structure modifies the dispersion relation: whereas in quantum mechanics \(\frac{1}{2m}k^2 + V(r)\) has to be replaced by an energy band function \(E_n(k) + V(r)\), the so-called semiclassical limit of the Bloch electron (see e. g. [PST03a; DL11a] and references therein), also in case of photonic crystals \(c|k|\) has to be substituted by \(\Omega(r, k) = \tau^2(r)\omega_n(k) + O(\lambda)\) where \(\omega_n(k)\) is a frequency band function and the modulation \(\tau(r)^2\) is due to the external perturbation. And just like in the case of the Bloch electron, we rely on the presence of a spectral gap, i.e. \(\omega_n(k)\) is a non-degenerate frequency band which does not intersect or merge with other bands. The choice of band not only enters the dispersion relation, but also determines the subspace \(\Pi_{\lambda} \ni (E, H)\) on which (1.3) holds. Moreover, finding the form of the \(O(\lambda)\) terms in (1.2) is crucial, because these first-order corrections are believed to explain topological effects [RH08; OMN06].

Our first main result, Theorem 3.3, rigorously establishes the ray optics limit for two classes of observables, scalar and non-scalar quadratic observables (cf. Definition 3.2). Apart from generic conditions on the material weights, no restrictions such as topological triviality of the frequency band \(\omega_n\) or the presence of symmetries needs to be imposed, in the parlance of [DL14b] Theorem 3.3 applies to photonic crystals of any topological class. We follow the ideas of Stiepan and Teufel, but it is necessary to generalize their procedure to include non-scalar observables to cover prominent examples such as the Poynting vector and the Maxwell stress tensor.

The second main result, Theorem 4.1, concludes the discussion of [DL14a, Section 5]: it explains how ray optics also applies to real electromagnetic fields propagating in non-gyrotropic media where the material weights are real. As elaborated in the reference and Section 4, this is non-trivial because real states are necessarily a linear combination of an even number of bands – single bands do not support real states. The crucial ingredient here is Proposition 4.2 which reduces the twin-band case to the single-band case covered by Theorem 3.3. And given that the reality of the material weights does not imply that all Chern numbers vanish, we close by showing that real states in non-gyrotropic PhCs exhibit no topological effects due to Chern numbers (Proposition 5.3).

Up until this work the exact form of the ray optics equations had been an open problem, even on the level of physics the exact form of the ray optics equations had not yet been established: Raghu and Haldane proposed their ray optics equations by analogy to
the corresponding quantum system, the Bloch electron. Subsequently, only three works attempted to derive ray optics equations systematically: Onoda et al [OMN06] used variational techniques developed by Sundaram and Niu [SN99], and their ray optics equation differs to sub-leading order (where all topological contributions enter) from those of Raghu and Haldane. The second work is by Esposito and Gerace [EG13] who derive only the equation for $\dot{r}$ via standard perturbation theory. None of these equations coincide with the equations we have found, though (cf. Proposition 3.9). The only rigorous work we are aware of is [APR13], and they justify the eikonal approximation via a multiscale WKB ansatz. However, Allaire et al crucially assume in [APR13, Hypothesis 1.1] that the perturbation of the material weights is a second-order effect, e. g. $\varepsilon(\lambda) = \varepsilon(0) + O(\lambda^2)$, meaning the perturbation is of the same order of magnitude as the error in (1.3).

The equations we have derived are one-band equations, and in principle, one may wonder whether degenerate bands are a non-scalar feature of a certain class of PhCs? Fortunately, for most the answer is no: in materials where $\varepsilon = c \mu$ and $\chi = 0$ (e. g. vacuum or certain YIG 2d PhCs [Poz98; WCJ+08]) each band is two-fold polarization-degenerate. But typically, $\varepsilon$ and $\mu$ vary independently, thereby breaking this symmetry. Nevertheless, in periodic waveguide arrays where the contrast is very low (of the order of $10^{-4} \sim 10^{-3}$ [Lon09; RZP+13]), the degeneracy of the two polarization states is broken only at the subleading order. Here, we reckon one needs to include a classical spin degree of freedom in the ray optics equations.

**Outline** The essential ingredient for the derivation of ray optics is to bring the Maxwell equations (1.1) in Schrödinger form, something which we explain in Section 2. There we also introduce other necessary objects and notation, and state all assumptions. Because the adiabatically perturbed Maxwell operator (which takes the place of the hamilton operator) is a pseudodifferential operator [DL14c, Theorem 1.3], standard semiclassical techniques can be applied to yield ray optics equations. Those approximate full electrodynamics in the sense of an Egorov theorem (Section 3). In Section 4 we then treat the ray optics limit for real fields propagating in non-gyrotropic PhCs by reducing it to the one-band case. Our work closes with a discussion of our results in Section 6. Some auxiliary results are put into an appendix.

**Acknowledgements** We would like to take the opportunity to thank Stefan Teufel for useful feedback and friendly discussions.
2 Schrödinger formalism of the Maxwell equations

Let us proceed to clearly define the mathematical problem. For the purpose of this paper we restrict ourselves to linear, lossless media meaning that the material weights

\[ W^{-1}(x) := \begin{pmatrix} \varepsilon(x) & \chi(x) \\ \chi^*(x) & \mu(x) \end{pmatrix} \]

which quantify the response of the medium are frequency-independent and take values in the hermitian $6 \times 6$-matrices. We will always make the following assumptions:

**Assumption 2.1 (Material weights)** Assume that $W^{-1} \in L^\infty(\mathbb{R}^3, \text{Mat}_\mathbb{C}(6))$ is positive, selfadjoint, bounded and has a bounded inverse $W$. We say that the weights are real if and only if $[C, W] = 0$ where $C$ denotes complex conjugation.

Throughout the main body of the paper, we will make a conscious attempt to cut down on technical details which are not necessary to understand the strategy of the proofs.

2.1 First-order Schrödinger framework of electromagnetism

As our starting point we recast the Maxwell equations as a Schrödinger equation

\[ i\partial_t \Psi = M_w \Psi \]  

by multiplying both sides of (1.1a) by $iW$ and restricting oneself to electromagnetic fields $\Psi = (E, H) \in L^2(\mathbb{R}^3, \mathbb{C}^6)$ which satisfy (1.1b) in the distributional sense. Based on this precise formulation of the “quantum-light analogy” we can systematically adapt techniques from applied mathematics and quantum physics to classical electromagnetism. Here, the electromagnetic field $\Psi = (E, H)$ plays the role of the wave function and the Maxwell operator

\[ M_w := W \text{ Rot} = W \begin{pmatrix} 0 & +i\nabla \times \\ -i\nabla \times & 0 \end{pmatrix} \]

takes the place of the Schrödinger operator. $\nabla \times E = \nabla \times E$ is the curl for vector fields on $\mathbb{R}^3$, and we will frequently make use of this notation to connect the matrix

\[ v^x \psi := \begin{pmatrix} 0 & -v_3 & +v_2 \\ +v_3 & 0 & -v_1 \\ -v_2 & +v_1 & 0 \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix} = v \times \psi \]

to any vectorial quantity $v$ such as the canonical basis vectors $e_j$, $j = 1, 2, 3$, of $\mathbb{C}^3$. Moreover, the the Maxwell operator is selfadjoint [DL14c, Theorem 2.1] on the Hilbert space.
2.2 Adiabatically perturbed photonic crystals

We are interested in the propagation of light in adiabatically perturbed PhCs where the periodic material weights are perturbed in a specific manner:

**Assumption 2.2 (Slowly modulated weights)** Suppose the material weights are of the form \( W_\lambda(x) = S^{-2}(\lambda x) W(x) \) where

(i) the periodic contribution \( W \) satisfies Assumption 2.1 and is periodic with respect to some lattice \( \Gamma \cong \mathbb{Z}^3 \), and

(ii) the slow modulation \( S \) is either of the form \( S(\lambda x) := \tau^{-1}(\lambda x) \) when \( \chi \neq 0 \) or

\[
S(\lambda x) := \begin{pmatrix}
\tau^{-1}(\lambda x) & 0 \\
0 & \tau^{-1}(\lambda x)
\end{pmatrix}
\] (2.4)

in case \( \chi = 0 \).

The functions \( \tau, \tau_\epsilon, \tau_\mu \in C^\infty(\mathbb{R}^3) \) are always assumed to be positive, \( \tau(0) = \tau_\epsilon(0) = \tau_\mu(0) = 1 \) and bounded away from 0 and \( +\infty \). In case of modulation (2.4), one defines \( \tau := \sqrt{\tau_\epsilon \tau_\mu} \).

We will use the index \( \lambda \) systematically, e. g. \( \delta_\lambda := \delta_{W_\lambda} \). Objects with the index 0 denote the periodic case, and we can write \( M_\lambda = S(\lambda x)^{-2} M_0 \) where \( S(\lambda x) \) denotes the operator of multiplication by \( S(\lambda x) \). We will use this notation for multiplication operators also for other variables.

The periodic Maxwell operator \( M_0 \cong \int_{\mathbb{B}} dk \ M_0(k) \),

\[
M_0(k) := W(\hat{y}) \begin{pmatrix}
0 & -(-i\nabla_y + k)^x \\
(+i\nabla_y + k)^x & 0
\end{pmatrix},
\]

fibers in crystal momentum \( k \in \mathbb{B} \ (\mathbb{B} \cong \mathbb{T}^3 \) being the Pontryagin dual of the lattice \( \Gamma \), usually referred to as Brillouin zone) via the Zak transform

\[
(Z\Psi)(k,y) := \sum_{\gamma \in \Gamma} e^{-ik\cdot(y+\gamma)} \Psi(y+\gamma),
\] (2.5)
2 Schrödinger formalism of the Maxwell equations

and apart from essential spectrum at $\omega = 0$ due to unphysical gradient fields, $\sigma(M_0(k)) = \bigcup_{n \in \mathbb{Z}} \{\omega_n(k)\}$ is purely discrete and consists of frequency bands [DL14c, Theorem 1.4]. With the exception of the ground state bands (which have a linear dispersion around $k = 0$ and $\omega = 0$), all Bloch functions $\varphi_n$ are locally analytic away from frequency band crossings. Note that unlike periodic Schrödinger operators, the Maxwell operator is not bounded from below. In fact, symmetries such as complex conjugation induce relations between bands of different signs [DL14b]: if complex conjugation $C$ commutes with the material weights (i.e. $W$ is real), then the periodic Maxwell operator satisfies $CM_0(k)C = -M_0(-k)$. Consequently, if $\varphi_n(k)$ is an eigenfunction of $M_0(k)$ to $\omega_n(k)$, then $C\varphi_n(-k)$ is an eigenfunction of $M_0(k)$ to $-\omega_n(k)$. Such pairings of twin bands will become crucial to understanding the ray optics limit of real states, because $C\varphi_n(-k) \not\propto \varphi_n(k)$ implies these are eigenfunctions to distinct eigenvalues of $M_0(k)$. Put another way, single bands cannot support real states (cf. discussion in [DL14a, Section 4.1]), a fact which will be crucial in Section 4.

2.3 Auxiliary representations

Our choice of representation exploits (i) the periodicity and (ii) gets rid of the $\lambda$-dependence of the Hilbert spaces. Just like in quantum mechanics, a change of representation is mitigated by a unitary map. The Zak transform $Z : \mathcal{H}_\lambda \to \mathcal{H}_0$ defined in (2.5) above makes use of the periodicity.

In a second step, we use the unitary $S(i\lambda\nabla_k) : \mathcal{H}_\lambda \to \mathcal{H}_0$ to map the problem onto the (fibered) Hilbert space of the unperturbed, periodic system (cf. also [DL14c, Section 2.2]). And because the unperturbed weights $W$ are periodic, $Z\mathcal{H}_0 \simeq L^2(\mathbb{B}) \otimes \mathcal{H}_0$ decomposes into the “slow” space $L^2(\mathbb{B})$ and the “fast” space $\mathcal{H}_0$ which is defined as $L^2(\mathbb{T}^3, \mathbb{C}^6)$ endowed with the scalar product

$$\langle \varphi, \psi \rangle_{\mathcal{H}_0} := \langle \varphi, W^{-1}\psi \rangle_{L^2(\mathbb{T}^3, \mathbb{C}^6)}.$$

Alternatively, we could have reversed the order of the transformations because $ZS(\lambda\hat{k}) = S(i\lambda\nabla_k)Z$.

2.4 The Maxwell operator as a $\Psi$DO

The last ingredient we need is that the Maxwell operator

$$M_\lambda = \mathcal{D}p_\lambda^S(M_\lambda) := Z^{-1}S(i\lambda\nabla_k)^{-1}\mathcal{D}p_\lambda(M_\lambda)S(i\lambda\nabla_k)Z$$

$$= S(\lambda\hat{k})^{-1}Z^{-1}\mathcal{D}p_\lambda(M_\lambda)ZS(\lambda\hat{k})$$

8
can also be seen as a pseudodifferential operator (cf. [DL14c, Theorem 1.3]) associated to
\[ M_{\lambda}(r, k) = M_0(r, k) + \lambda M_1(r, k) \]
\[ := \tau^2(r) M_0(k) - \lambda \tau^2(r) \frac{i}{2} (\nabla, \ln \frac{r}{\tau} \mu)(r) \cdot \Sigma \] (2.6)

where \( \Sigma := (\Sigma_1, \Sigma_2, \Sigma_3) \) is an operator-valued vector with components
\[ \Sigma_j := W \left( \begin{array}{c} 0 \\ e_j^x \\ 0 \end{array} \right). \]

The equivariance of the operator-valued function
\[ M_{\lambda}(r, k - \gamma^*) = e^{+i\gamma^* \cdot \hat{k}} M_{\lambda}(r, k) e^{-i\gamma^* \cdot \hat{k}}, \quad \forall r, k \in \mathbb{R}^3, \gamma^* \in \Gamma^* \] (2.7)

with respect to translations in the dual lattice \( \Gamma^* \) ensures that its Weyl quantization
\[ \mathcal{D}_p(M_{\lambda}) := \frac{1}{(2\pi)^6} \int_{\mathbb{R}^3} dr \int_{\mathbb{R}^3} dk \int_{\mathbb{R}^3} dr' \int_{\mathbb{R}^3} dk' e^{+i(k' \cdot r - r' \cdot k)} M_{\lambda}(r, k) e^{-i(k' \cdot (i\lambda \nabla) - r' \cdot \hat{k})} \] (2.8)

associated to the slow variables \( r \mapsto i\lambda \nabla \hat{k} \) and \( k \mapsto \hat{k} \) (multiplication by \( k \)) defines an equivariant operator on \( H_{\lambda} \) (cf. Appendix A and references therein). Note that while the bi-anisotropic tensor \( \chi \) is absent in [DL14c], the results there immediately generalize: in case \( \chi \neq 0 \), the modulation \( S(r) = \tau^{-1}(r) \) is scalar and a quick computation yields \( M_{\lambda} = S^{-1} M_0(\cdot) S^{-1} = \tau^2 M_0(\cdot) \) agrees with (2.6) after setting \( \tau^e = \tau^\mu \).

**Remark 2.3 (Notation used here compared to [DL14c; DL14a])** In an attempt to unburden the notation, we deviate from our earlier works. For instance, \( M_{\lambda} \) as given by equation (2.3) is a selfadjoint operator on \( Z_\delta_0 \) (cf. Appendix A and references therein). Note that while the bi-anisotropic tensor \( \chi \) is absent in [DL14c], the results there immediately generalize: in case \( \chi \neq 0 \), the modulation \( S(r) = \tau^{-1}(r) \) is scalar and a quick computation yields \( M_{\lambda} = S^{-1} M_0(\cdot) S^{-1} = \tau^2 M_0(\cdot) \) agrees with (2.6) after setting \( \tau^e = \tau^\mu \).

**3 The ray optics limit**

Now we come to the main course of the paper, a rigorous justification of ray optics via a semiclassical limit. Roughly speaking, if the initial state is associated to a single, non-degenerate frequency band \( \omega(k) \in \sigma(M_0(k)) \) which does not intersect or merge with other bands, the dispersion relation which enters in the ray optics equations is no longer \( c |k| \) but proportional to the frequency band function \( \omega(k) \) to leading order. Let us be more precise and enumerate the conditions on the frequency band:

---

9
3 The ray optics limit

Assumption 3.1 Suppose $\omega(k)$ is a non-degenerate frequency band of $M_0(k)$ with Bloch function $\varphi(k)$ that is isolated in the sense that

$$\inf_{k \in \mathbb{B}} \text{dist}(\omega(k), \sigma(M_0(k)) \setminus \omega(k)) > 0. \quad (3.1)$$

Next, we need to clarify what we mean when we say “states associated to the frequency band $\omega$” in case the PhC is perturbed. The perturbation deforms the subspace $Z^{-1}\text{ran} \pi_0(\hat{k})$ where $\pi_0(k) := |\varphi(k)\rangle \langle \varphi(k)|$, and the range of the superadiabatic projection

$$\Pi_\lambda = \mathcal{O}_{||\lambda||}(\lambda) = \mathcal{O}_{||\lambda||}(\lambda)$$

takes its places. Apart from being an orthogonal projection, its other defining property is

$$[M_\lambda, \Pi_\lambda] = \mathcal{O}_{||\lambda||}(\lambda). \quad (3.2)$$

The existence and explicit construction of $\Pi_\lambda$ relies on the gap condition (3.1) and pseudo-differential techniques (cf. [DL14a, Proposition 1]). Equation (3.2) also implies that $\text{ran} \Pi_\lambda$ is left invariant by the dynamics up to errors of arbitrarily small order in $\lambda$.

For electromagnetic waves from the almost invariant subspace $\text{ran} \Pi_\lambda$, we are going to rigorously justify the analog of the semiclassical limit for the Bloch electron where the periodic structure of the ambient medium modifies the dispersion relation to

$$\Omega = \Omega_0 + \lambda \Omega_1 := \tau^2 \omega - \lambda \tau^2 P \cdot \nabla \ln \tau \epsilon \tau \mu. \quad (3.3)$$

To leading order, the frequency band function $\omega(k)$ is modulated by the perturbation $\tau(r)^2 = \tau_\epsilon(r) \tau_\mu(r)$, i.e. the frequency depends on the change in the speed of light. The $\mathcal{O}(\lambda)$ term is sensitive to the details of the modulation, and only appears if $\epsilon$ and $\mu$ are modulated differently; moreover, it includes the imaginary part of the complex Poynting vector,

$$P(k) := \text{Im} \int_{\mathbb{S}^2} \text{d}y \bar{\varphi}(k, y) \times \varphi^H(k, y).$$

Note that this expression works for both types of perturbations, in case $\chi \neq 0$ we take $\tau_\epsilon = \tau_\mu$ and the last term vanishes. In addition to $\Omega_1$, there are also other $\mathcal{O}(\lambda)$ contributions as we will see below.

3.1 The notion of observables in electromagnetism

Our goal is to derive a ray optics limit for observables. Here, the similarity between the ray optics and the semiclassical limit is misleading, because conceptually, the notion of “physical observable” in quantum mechanics and electromagnetism differ from one
3.1 The notion of observables in electromagnetism

another. While quantum observables are usually represented by selfadjoint operators, in electromagnetism they are suitable functions of the fields

\[ F : L^2(\mathbb{R}^3, \mathbb{C}^6) \longrightarrow \mathbb{R}. \]

Examples of observables in electromagnetism include the energy density \( e(x, \Psi) := \frac{1}{2} \Psi(x) \cdot W^{-1}(x) \Psi(x) \), the Poynting vector

\[ S(x, \Psi) := \text{Re} \left( \overline{\psi^E(x) \times \psi^H(x)} \right), \]

even the fields themselves, e.g. \( \delta^E(\Psi) := \psi^E(x) \), as well as their local averages. This is in stark contrast to quantum mechanics where the wave function itself cannot be observed. At the end of the day, electromagnetism, even if written in the language of quantum mechanics, is still a classical theory.

Secondly, just as not every quantum observable has a semiclassical limit, not every observable in electromagnetism has a ray optics limit – at least not via Theorem 3.3. More specifically, we will consider only observables here which are quadratic in the fields:

**Definition 3.2 (Quadratic observables)** Suppose the electromagnetic observable

\[ F(\Psi) = \langle \Psi, \Omega p^Z_{\lambda}(f)\Psi \rangle_{\lambda} \quad (3.4) \]

is defined in terms of a PDO associated to a function \( f \).

(i) We call \( F \) scalar if \( f \equiv f \otimes \text{id}_{h_0} \) and \( f \in C^\infty_b(\mathbb{R}^6, \mathbb{C}) \) are periodic in \( k \).

(ii) We call \( F \) non-scalar if \( f \in C^\infty_b(\mathbb{R}^6, \mathcal{B}(h_0)) \) is an operator-valued function satisfying the equivariance condition (2.7).

Just to be clear: we do not claim that these are the only observables with a mathematically sensible ray optics limit, far from it. Nevertheless, Definition 3.2 covers most physically relevant observables such as local averages of the field amplitudes, the energy density, the Poynting vector, the Minkowski momentum, and the Maxwell stress tensor. We will explore these concrete examples in more detail in Section 3.3. Moreover, while we may certainly view any scalar observable as a non-scalar observable, we see that the ray optics limit for scalar observables is simpler and more elegant.

**Theorem 3.3 (The ray optics limit)** Suppose Assumptions 2.2 and 3.1 hold true, and \( F \) is a quadratic observable associated to \( f \) as in Definition 3.2. Then we have a ray optics limit in the following sense:

(i) For scalar observables where \( f \in C^\infty_b(\mathbb{R}^6, \mathbb{C}) \), the ray optics flow \( \Phi^\lambda \) associated to the hamiltonian equations

\[ \begin{pmatrix} \dot{r} \\ \dot{k} \end{pmatrix} = \begin{pmatrix} -\lambda \Xi & +\text{id} \\ -\text{id} & 0 \end{pmatrix} \begin{pmatrix} \nabla_r \Omega \\ \nabla_k \Omega \end{pmatrix}, \quad (3.5) \]
which include the Berry curvature tensor \( \Xi := (\nabla_k \times i(\varphi, \nabla_k \varphi)_{\hbar_0})^\perp \) as part of the sympletic form, approximates the full light dynamics for \( \Psi \in \text{ran} \Pi_\lambda \) and bounded times in the sense

\[
\mathcal{F}(\Psi(t)) = \mathcal{F}(e^{-i\mathcal{M}_\lambda \cdot t} \cdot \Psi) = \left( \Psi, \mathcal{O} \mathcal{P} \mathcal{S} \mathcal{Z}_\lambda \left( f \circ \Phi^\lambda \right) \Psi \right)_\lambda + O(\lambda^2).
\]  

(ii) For non-scalar observables where \( f \in C^\infty_b(\mathbb{R}^6, B(\hbar_0)) \), the ray optics flow \( \Phi^\lambda \) associated to the Hamiltonian equations

\[
\begin{pmatrix}
\dot{r} \\
\dot{k}
\end{pmatrix} = \begin{pmatrix}
0 & +\text{id} \\
-\text{id} & 0
\end{pmatrix} \begin{pmatrix}
\nabla_r \varOmega \\
\nabla_k \varOmega
\end{pmatrix}
\]  

approximates the full light dynamics for \( \Psi \in \text{ran} \Pi_\lambda \) and bounded times in the sense

\[
\mathcal{F}(\Psi(t)) = \mathcal{F}(e^{-i\mathcal{M}_\lambda \cdot t} \cdot \Psi) = \left( \Psi, \mathcal{O} \mathcal{P} \mathcal{S} \mathcal{Z}_\lambda \left( f_\circ \Phi^\lambda \right) \Psi \right)_\lambda + O(\lambda^2)
\]  

where we have transported the modified non-scalar observable \( f_\circ := \pi_\lambda f \pi_\lambda \circ \pi_\lambda + O(\lambda^2) \) along the flow \( \Phi^\lambda \). Put another way, for non-scalar observables the effect of the projection does not modify the sympletic form of the ray optics equations but changes the function \( f \) which defines the quadratic observable \( \mathcal{F} \).

Remark 3.4 We can immediately extend Theorem 3.3 to quadratic observables of the type

\[
\mathcal{F}(\rho) = \text{Tr}_{\mathcal{H}_1} \left( \rho \mathcal{O} \mathcal{P} \mathcal{S} \mathcal{Z}_\lambda (f) \right)
\]  

where \( \rho = \rho^* \geq 0 \) is a suitable trace-class operator that describes a mixture of different electromagnetic states. Although this generalization is physically relevant and meaningful, from a mathematical point of view the passage from (3.4) to (3.5) is totally trivial.

For scalar quadratic observables we can express (3.6) as a phase space average of \( f \) with respect to the Wigner transform.

Corollary 3.5 Suppose we are in the setting of Theorem 3.3 (i) where \( f \) is scalar. Then equation (3.6) can be recast as

\[
\mathcal{F}(\Psi(t)) = \int_{\mathbb{R}^3} dr \int_{\mathbb{R}^6} dk f \circ \Phi^\lambda(r, k) w_\Phi(r, k) + O(\lambda^2 \|\Psi\|_{\mathcal{H}_1}^2)
\]  

where \( w_\Phi(r, k) \) denotes the reduced Wigner transform of \( \Psi \in L^2(\mathbb{R}^3, \mathbb{C}^6) \) given by

\[
w_\Phi(r, k) := \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} dy e^{-ik \cdot y} \Psi(r + \frac{1}{2} y) \cdot \Psi(r - \frac{1}{2} y).
\]  

12
3.1 The notion of observables in electromagnetism

We postpone a discussion of two observables with a ray optics limit, the local averages of the energy density and the Poynting vector, to Section 3.3 below.

Remark 3.6 Scalar observables have a somewhat simpler ray optics limit, because here a geometric correction, the Berry curvature, enters in the symplectic form. For non-scalar observables though, the Weyl commutator \([f, \pi_\lambda] = O(1)\) is not small as

\[ f \pi_0 \neq \pi_0 f \] (3.11)

holds. Consequently, instead of getting an \(O(\lambda)\) correction in the symplectic form, we need to replace the function \(f\) by

\[
f_{\rho_0} = \pi_\lambda f \frac{\pi_\lambda}{\pi_\lambda} + O(\lambda^2)
= \langle \varphi, f \varphi \rangle_{h_0} \pi_0 + \lambda \left( \pi_1 f \pi_0 + \pi_0 f \pi_1 - \frac{1}{2} \{ \pi_0, f \} \right) \pi_0 - \frac{1}{2} \pi_0 \{ f, \pi_0 \}. \] (3.12)

Note that the term proportional to \(\pi_0\) vanishes identically as \(\pi_0\) is a function of \(k\) only. The crucial idea of Stiepan and Teufel [ST13] was to avoid including this \(O(\lambda)\) term by modifying the symplectic form. However, their derivation relies on \([f, \pi_0] = 0\) and

\[ \{ \pi_0, f \} = -\{ f, \pi_0 \}. \]

Corollary 3.7 The explicit expression for \(\pi_\lambda\) in Theorem 3.3 is \(\pi_\lambda = |\varphi\rangle \langle \varphi | + \lambda \pi_1 + O(\lambda^2)\) with

\[
\pi_1 = \left( -\frac{i}{2} \nabla \cdot \ln \frac{\Sigma \varphi}{\tau} + i \nabla \cdot \ln \tau \cdot |\varphi\rangle \langle \varphi | (M_0(\cdot) + \omega) \right) R_\lambda^\dagger + \text{adjoint}
\]

where \(R_\lambda^\dagger(k) := \pi_0^\dagger (M_0(k) - \omega(k))^{-1} \pi_0^\dagger\) is the reduced resolvent of the periodic Maxwell operator. The modified ray optics observable \(f_{\rho_0}\) computes to

\[
f_{\rho_0} = \langle \varphi, f \varphi \rangle_{h_0} \pi_0 + \lambda \left( \langle \varphi, [f, \pi_1]_{+} \varphi \rangle_{h_0} - \frac{1}{2} \langle \varphi, [\nabla_k \pi_0, \nabla_r f] \varphi \rangle_{h_0} \right) \pi_0
+ \lambda \left( \langle \varphi, f \varphi \rangle_{h_0} \pi_1 - \frac{1}{2} \langle \varphi, \nabla_r f \varphi \rangle_{h_0} \right) \right) \]

where \([f, \pi_1]_{+} := f \pi_1 + \pi_1 f\) and \([\nabla_k \pi_0, \nabla_r f] := \nabla_k \pi_0 \cdot \nabla_r f - \nabla_r f \cdot \nabla_k \pi_0.\) We point out that unlike the first two terms in the above for \(f_{\rho_0}\) which are proportional to \(\pi_0,\) the third term is completely offdiagonal with respect to \(\pi_0.\)
The interested reader may find the computation in Appendix B.

**Remark 3.8** In case \( f \) takes values in the selfadjoint operators, the above expression for \( f_{ro} \) simplifies to

\[
f_{ro} = \langle \varphi, f \varphi \rangle_{\mathcal{H}_0} \pi_0^++\lambda \left( 2 \Re \langle \varphi, \pi_1 \varphi \rangle_{\mathcal{H}_0} - \langle \varphi, \nabla_r f \varphi \rangle_{\mathcal{H}_0} \cdot A - \Im \langle \varphi, \nabla_r f \cdot \nabla_k \varphi \rangle_{\mathcal{H}_0} \right) \pi_0^+ + \lambda \left( \langle \varphi, f \varphi \rangle_{\mathcal{H}_0} \pi_1 + \langle \varphi, \nabla_r f \varphi \rangle_{\mathcal{H}_0} \cdot [\nabla_k \pi_0, \pi_0] \right)
\]

where \( A := i \langle \varphi, \nabla_k \varphi \rangle_{\mathcal{H}_0} \) is the vector associated with the Berry connection. Typically, the selfadjoint observables of interest are of the form \( f = \rho WI \) where \( \rho \) is a scalar function which localizes on a domain \( \Lambda \subset \mathbb{R}^3 \) and \( I \) is a suitable \( 6 \times 6 \) hermitian matrix. If we think of \( \rho \) as a smoothened version of the characteristic function \( 1_{\Lambda} \), then \( \nabla_r f(r) \approx n(r) WI \) where \( n(r) \in \mathbb{R}^3 \) is the external normal to \( r \in \partial \Lambda \), and \( n(r) = 0 \) whenever \( r \not\in \partial \Lambda \). With this in mind, we can distinguish “bulk”-type contributions to \( f_{ro} \) that are proportional to \( \rho \),

\[
\langle \varphi, I \varphi \rangle_{L^2(\mathbb{R}^3, C^5)} \pi_0 + 2 \Re \langle \varphi, I \pi_1 \varphi \rangle_{L^2(\mathbb{R}^3, C^5)} \pi_0 + \langle \varphi, I \varphi \rangle_{L^2(\mathbb{R}^3, C^5)} \pi_1,
\]

and an \( \mathcal{O}(\lambda) \) part of “boundary” type which is localized around \( \partial \Lambda \),

\[
-\langle \varphi, I \varphi \rangle_{L^2(\mathbb{R}^3, C^5)} (n \cdot A) \pi_0 - (n \cdot A') \pi_0 + \langle \varphi, I \varphi \rangle_{L^2(\mathbb{R}^3, C^5)} (n \cdot [\nabla_k \pi_0, \pi_0])
\]

where \( A' := \Im \langle \varphi, I \nabla_k \varphi \rangle_{L^2(\mathbb{R}^3, C^5)} \).

### 3.2 Egorov theorems and proofs

The main ingredients are two Egorov theorems, one for scalar and one for non-scalar observables. We first treat the scalar case in detail, and then proceed to the non-scalar case where we only discuss the necessary modifications.

#### 3.2.1 An Egorov theorem for scalar observables

For this simpler class of scalar observables, we can directly apply the results of Stiepan and Teufel [ST13]. The main technical advantage of their technique compared to earlier works such as [PST03a; DL11a] is that they do not need to assume the triviality of the Bloch bundle.

**Proposition 3.9 (Egorov theorem for scalar observables)** Suppose we are in the setting of Theorem 3.3 (i). Then for any scalar observable associated to \( f \in C^\infty_b (\mathbb{R}^3, \mathbb{C}) \) which is periodic in \( k \), the full light dynamics can be approximated by ray optics for bounded times,

\[
\left\| \Pi_\lambda \left( e^{i M^{-1} \lambda} D_p^{Z} (f) e^{-i M^{-1} \lambda} - D_{\lambda}^{Z} (f \circ \Phi^\lambda_t) \right) \Pi_\lambda \right\|_{\mathcal{B}(H_{\lambda, 0})} = \mathcal{O}(\lambda^2 |t|).
\]
To help separate computations from technical arguments, we start with the following

**Lemma 3.10** Suppose we are in the setting of Proposition 3.9. Then in both cases ($\chi = 0$ or $\chi \neq 0$ and $\tau_e = \tau_\mu$) the dispersion relation $\Omega$ characterized by

$$\pi_\lambda^2 (\Omega - M_\lambda) \pi_\lambda = O(\lambda^2)$$

computes to be (3.3).

**Proof** While the final result holds true for both cases, $\chi = 0$ and $\chi \neq 0$, we detail the computations for $\chi = 0$ where electric permittivity and magnetic permeability may be scaled separately. In case $\chi \neq 0$ we set $\tau_\varepsilon = \tau_\mu$, and all terms which contain gradients of the ratio $\tau_\varepsilon/\tau_\mu = 1$ vanish. The explicit expression for the dispersion relation $\Omega(r, k) = \text{Tr}_{h_0} \left( M_\lambda(r, k) \pi_0(k) \right) + M_{TS}$

is determined by equations (17) and (18) in [ST13], and consists of two parts. The first contribution is the expectation value of the symbol. The second, the so-called Teufel-Stiepan term $M_{TS} := \text{Tr}_{h_0} \left( \{ \pi_0| M_0| \pi_0 \} \right) = 0$, vanishes in our case for the same reason as in equation (3.13) – $\pi_0(k)$ depends only on crystal momentum.

The trace terms are merely a fancy way to write the expectation value with respect to $\varphi$. Clearly, the leading-order term

$$\Omega_0(r, k) = \text{Tr}_{h_0} \left( M_0(r, k) \pi_0(k) \right)$$

is just the band function scaled by $\tau$. For the sub-leading term we first compute

$$\left\langle \begin{pmatrix} \varphi^E(k) \\ \varphi^H(k) \end{pmatrix}, \mathcal{W} \begin{pmatrix} 0 & e_j^x \\ e_j^y & 0 \end{pmatrix} \begin{pmatrix} \varphi^E(k) \\ \varphi^H(k) \end{pmatrix} \right\rangle_{h_0} =$$

$$= \left\langle \begin{pmatrix} \varphi^E(k) \\ \varphi^H(k) \end{pmatrix}, \begin{pmatrix} e_j \times \varphi^H(k) \\ e_j \times \varphi^E(k) \end{pmatrix} \right\rangle_{L^2(\mathbb{T}^3, \mathbb{C}^2)} =$$

$$= - \int_{\mathbb{T}^3} dy \ e_j \cdot \left( \varphi^E(k, y) \times \varphi^H(k, y) - \varphi^E(k, y) \times \varphi^H(k, y) \right)$$

$$= -i 2 P_j(k).$$

This now yields

$$\Omega_1 = \text{Tr} \left( M_1 \pi_0 \right) = -\tau^2 \frac{i}{2} \sum_{j=1}^3 \partial_{r_j} \ln \frac{\tau_e}{\tau_\mu} \left\langle \begin{pmatrix} \varphi^E \\ \varphi^H \end{pmatrix}, \mathcal{W} \begin{pmatrix} 0 & e_j^x \\ e_j^y & 0 \end{pmatrix} \begin{pmatrix} \varphi^E \\ \varphi^H \end{pmatrix} \right\rangle_{h_0}$$

$$= -\tau^2 \mathcal{P} \cdot \nabla \ln \frac{\tau_e}{\tau_\mu}.$$
3 The ray optics limit

When \( \chi \neq 0 \) the perturbation \( S(r) = \tau^{-1}(r) \) is scalar. That means the last term in \( \mathcal{M}_2(r, k) = \tau^2(r) M_0(k) + 0 \) vanishes. Seeing as \( \mathcal{M}_1 \) does not enter in the computation of the Teufel-Stiepan term \([ST13, \text{equation (18)}]\), we immediately deduce \( \Omega_1 = 0 \). □

Proof (Proposition 3.9) The modifications to the proofs in \([ST13]\) are of purely technical nature. Nevertheless, for the benefit of the reader we will sketch the general strategy of Stiepan and Teufel's work, and explain the necessary modifications.

Notation Given the quantum mechanical context their notation is different and clashes with ours: Stiepan and Teufel consider a hamiltonian (operator) \( \hat{H} = H_0 + \epsilon H_1 \) which corresponds to the Maxwell operator \( M_\lambda \) and its symbol \( M_\lambda = M_0 + \lambda M_1 \). The relevant symbol classes such as \( \text{AS}^m_{0,eq}(B(h_1, h_2)) \) are defined in Definition A.1. The analog of the semiclassical hamiltonian \( h = h_0 + \epsilon h_1 \) is the dispersion relation \( \Omega \), and to avoid a notational clash we have renamed the components of the extended Berry curvature as given by \([ST13, \text{equation (23)}]\) to \( \Xi_{kk}, \Xi_{kr}, \Xi_{rk} \) and \( \Xi_{rr} \). At this point we have already obtained the explicit expressions of the dispersion relation in Lemma 3.10.

We need to verify that Proposition 2, Proposition 3 and Theorem 2 in \([ST13]\) can be extended to the case of the slowly modulated periodic Maxwell operator.

Facts on the Maxwell operator and the superadiabatic projection First, the Maxwell operator is unbounded and defined in terms of an equivariant symbol

\[
\mathcal{M}_\lambda \in \text{AS}^1_{1,eq}(B(\mathbb{S}^1, L^2(T^3)'))
\]

where \( \mathcal{D} \) defined in \([DL14c, \text{equation (32)}]\) is the domain of the periodic Maxwell operator \( M_0(k) \) and \( \mathcal{M}_\lambda \) is given by equation (2.6) (cf. \([DL14c, \text{Corollary 4.3}]\)). Moreover, from \([DL14a, \text{Proposition 1}]\) we know the superadiabatic projection \( \Pi_\lambda = \mathcal{P} M_\lambda + \mathcal{O}(\lambda^\infty) \) associated to an isolated band exists and is \( \mathcal{O}(\lambda^\infty) \)-close in norm to a \( \Psi DO \) with symbol

\[
\pi_\lambda \in \text{AS}^0_{0,eq}(B(L^2(T^3), \mathbb{C}^6)) \cap \text{AS}^1_{0,eq}(B(L^2(T^3), \mathbb{C}^6), 0)).
\]

As explained in Appendix A, equivariance is preserved by the Weyl product. Moreover, all of the error terms below are in \( \text{AS}^0_{0,eq}(B(L^2(T^3), \mathbb{C}^6)) \).

Step 1: Pull the projection into the commutator A simple computation yields

\[
\pi_\lambda [\mathcal{M}_\lambda, f] = [\pi_\lambda \mathcal{M}_\lambda f, \pi_\lambda] + \mathcal{O}(\lambda^\infty),
\]

and all we need to check is that all the terms are in \( \text{AS}^0_{0,eq}(B(L^2(T^3), \mathbb{C}^6)) \) which then quantize to bounded operators by a variant of the Caldéron-Vaillancourt theorem (cf. the
3.2 Egorov theorems and proofs

discussion in [DL14c, Section 4.1] and [Teu03, Proposition B.5]): a priori the left-hand side is an element of \( AS_0^{m,\text{eq}}(B(L^2(\mathbb{T}^3, \mathbb{C}^6))) \) by the composition properties of symbols, but the equivariance condition implies that for any \( m > 0 \) we have in fact
\[
AS_0^{m,\text{eq}}(B(L^2(\mathbb{T}^3, \mathbb{C}^6))) = AS_0^{0,\text{eq}}(B(L^2(\mathbb{T}^3, \mathbb{C}^6))).
\]

**Step 2: Replace \( M_\lambda \) by \( \Omega \)** Adapting the arguments from [ST13, Proposition 2] readily yields
\[
\pi_\lambda \# (M_\lambda - \Omega) \# \pi_\lambda = \pi_0 \left( \pi_\lambda \# (M_\lambda - \Omega) \# \pi_\lambda \right) + \mathcal{O}(\lambda^3) = \mathcal{O}(\lambda^2). \tag{3.17}
\]
As argued in Step 1 above, left- and right-hand side are in \( AS_0^{0,\text{eq}}(B(L^2(\mathbb{T}^3, \mathbb{C}^6))) \). The computation (after making the necessary changes in notation) is identical, and one gets (3.17). Consequently, we obtain
\[
\begin{align*}
\left[ \pi_\lambda \# M_\lambda \# \pi_\lambda, \pi_\lambda \# f \# \pi_\lambda \right]_f &= \left[ \pi_\lambda \# (M_\lambda - \Omega) \# \pi_\lambda, \pi_\lambda \# f \# \pi_\lambda \right]_f + \\
&\quad + \left[ \pi_\lambda \# (M_\lambda - \Omega) \# \pi_\lambda, \pi_\lambda \# f \# \pi_\lambda \right]_f \\
&= \left[ \pi_\lambda \# (M_\lambda - \Omega) \# \pi_\lambda, \pi_\lambda \# f \# \pi_\lambda \right]_f + \mathcal{O}(\lambda^3).
\end{align*}
\]

**Step 3: Pull the projection out of the commutator** Then after replacing \( M_\lambda \) with the dispersion relation \( \Omega \) we pull the projection back out of the commutator,
\[
\left[ \pi_\lambda \# (M_\lambda - \Omega) \# \pi_\lambda, \pi_\lambda \# f \# \pi_\lambda \right]_f = \pi_\lambda \# \left[ \Omega, f \right]_f \# \pi_\lambda + \mathcal{O}(\lambda^3) \tag{3.18}
\]
albeit at the expense of an extra \( \mathcal{O}(\lambda^2) \) term. Note that the equality is exact.

**Step 4: Approximate commutator with \( \lambda \)-corrected Poisson bracket** Now we develop all Moyal commutators in \( \lambda \), keeping only terms up to \( \mathcal{O}(\lambda^2) \): since \( \Omega \) and \( f \) are scalar, the even powers in the Moyal commutator
\[
\left[ \Omega, f \right]_f = -\lambda i \{ \Omega, f \} + \mathcal{O}(\lambda^3)
\]
vanish. For the other two commutators, it suffices to keep only the leading-order term. Thus, after replacing \( M_\lambda \) by \( \Omega \) in \( \pi_\lambda \# [M_\lambda, f]_\# \pi_\lambda \), and replacing the Moyal commutators with Poisson brackets at the expense of an \( \mathcal{O}(\lambda^2) \) error, we can write
\[
\begin{align*}
\frac{1}{\lambda} \pi_\lambda \# [M_\lambda, f]_\# \pi_\lambda &= \pi_\lambda \# \left\{ \Omega, f \right\} - \lambda i \left[ \{ \Omega, \pi_0 \}, \{ f, \pi_0 \} \right]_\# \pi_\lambda + \mathcal{O}(\lambda^2) \\
&= \pi_\lambda \# \left\{ \Omega, f \right\} \# \pi_\lambda + \mathcal{O}(\lambda^2) \tag{3.19}
\end{align*}
\]
3 The ray optics limit

in terms of a \( \lambda \)-corrected Poisson bracket

\[
\{ \Omega, f \}_\lambda := X_\Omega \cdot \nabla f := \begin{pmatrix} -\lambda \Xi^{kk} + \text{id} & \lambda \Xi^{kr} \\ -\lambda \Xi^{rk} & -\lambda \Xi^{rr} \end{pmatrix} \begin{pmatrix} \nabla_r \Omega \\ \nabla_k \Omega \end{pmatrix} \cdot \begin{pmatrix} \nabla_r f \\ \nabla_k f \end{pmatrix}.
\]  (3.20)

The explicit formula for the modified symplectic form (whose \( O(\lambda) \) contribution is also called extended Berry curvature), \([ST13, \text{equation (23)}]\), simplifies tremendously since \( \pi_0 \) depends only on \( k \): the terms which involve derivatives of \( \pi_0 \) with respect to \( r \) vanish, i.e. \( \Xi^{rr} = 0 \) and \( \Xi^{rk} = 0 = \Xi^{kr} \). Thus, only the ordinary Berry curvature survives and we obtain the usual Berry curvature for the remaining contribution, \( \Xi^{kk} = \Xi \).

Step 5: A Duhamel argument The ray optics equations (3.5) which define the flow \( \Phi^\lambda \) can alternatively be written as

\[
\begin{align*}
\dot{r}_j &= \{ \Omega, r_j \}_\lambda \\
\dot{k}_j &= \{ \Omega, k_j \}_\lambda
\end{align*}
\]

where \( \{ \cdot, \cdot \}_\lambda \) is the Poisson bracket defined in equation (3.20) above. Thus, observables evolve according to

\[
\begin{align*}
\frac{df}{dt} &= \{ \Omega, f \}_\lambda \\
\frac{df}{dt} &= \{ \Omega, f \}_\lambda \\
\frac{df}{dt} &= \{ \Omega, f \}_\lambda
\end{align*}
\]

This concludes the proof. \( \square \)

3.2.2 The case of non-scalar observables

Even if observables are not scalar, one can still derive an Egorov theorem by slightly modifying the proof of Proposition 3.9. Here, the main idea is to evolve \( f_\text{ro} \) which is obtained by truncating the expansion of \( \pi_\lambda \Pi f_\| \pi_\lambda \) after the first order.
Proposition 3.11 (Egorov theorem for non-scalar observables) Suppose we are in the setting of Theorem 3.3 (ii). Then for all \( f \in C_0^\infty(\mathbb{R}^6, E(h_0)) \) satisfying the equivariance condition (2.7) the full light dynamics can be approximated by ray optics for bounded times,

\[
\left\| \Pi_\lambda \left( e^{i T M_\lambda} D p_\lambda^{S} (f) e^{-i T M_\lambda} - D p_\lambda^{S} (f_0 \circ \Phi_\lambda^1) \right) \Pi_\lambda \right\|_{L^2(h_0)} = O(\lambda^2 |t|). \tag{3.21}
\]

**Proof** Up until Step 3 the proof can be taken verbatim from that of Proposition 3.9. Instead of proceeding as in equation (3.18) in Step 4, we replace \( \pi_\lambda \# f \# \pi_\lambda \) with \( \pi_\lambda \# f_0 \# \pi_\lambda \). While the two agree up to \( O(\lambda^2) \), just like in equation (3.17) the \( O(\lambda^2) \) term commutes with \( \pi_0 \), and thus, the error we introduce in

\[
\left[ \pi_\lambda \# f \# \pi_\lambda, \pi_\lambda \right] = \pi_\lambda \# \left[ \Omega, \pi_\lambda \# f \# \pi_\lambda \right] + O(\lambda^\infty)
\]

is in fact \( O(\lambda^3) \). The double commutator term in (3.18) is zero as

\[
\left[ \pi_\lambda \# f \# \pi_\lambda, \pi_\lambda \right] = O(\lambda^\infty)
\]

vanishes to any order. That means there are no \( O(\lambda) \) which modify the symplectic form either, and we have to replace \( \{\cdot, \cdot\}_\lambda \) with the usual Poisson bracket in equation (3.19) and Step 5 of the proof. Consequently, the resulting ray optics equations are (3.7) which compared to (3.5) are missing the Berry curvature in the symplectic form. This finishes the proof. \( \square \)

### 3.2.3 Proof of Theorem 3.3

With these intermediate results in hand, the proof of the ray optics limit is straightforward.

**Proof (Theorem 3.3)** Given that \( \Psi \in \text{ran} \Pi_\lambda \), we can insert the projection free of charge,

\[
\mathcal{F}(\Psi(t)) = \left\langle \Psi, e^{i T M_\lambda} D p_\lambda^{S} (f) e^{-i T M_\lambda} \Psi \right\rangle = \left\langle \Psi, \Pi_\lambda e^{i T M_\lambda} D p_\lambda^{S} (f) e^{-i T M_\lambda} \Pi_\lambda \Psi \right\rangle. \tag{3.22}
\]

Suppose \( f \) is scalar, then the claim follows from Proposition 3.9. Similarly, Proposition 3.11 implies part (ii) for non-scalar \( f \). \( \square \)

### 3.3 The ray optics limit for certain observables

Our main result, Theorem 3.3, applies directly to a number of physical observables, and we will discuss the local field energy as well as the local average of the Poynting vector in...
detail. Other examples include local averages of the quadratic components of the fields, the components of the Maxwell-Minkowski stress tensor and Minkowski’s electromagnetic momentum.

### 3.3.1 The local field energy

The local field energy is an example of a scalar quadratic observable: while Egorov-type theorems do not allow one to infer information on the pointwise behavior of the local energy density

$$e(x, \Psi) := \frac{1}{2} \Psi(x) \cdot W_\lambda^{-1}(x) \Psi(x),$$

it does apply to local averages. Pick any closed set $\Lambda \subset \mathbb{R}^3$ of positive Lebesgue measure. Next, we choose a smoothened characteristic function $\rho \in C^\infty_b(\mathbb{R}^3, \mathbb{R})$, meaning $\rho|_{\Lambda} = 1$ and $\rho$ vanishes on $\mathbb{R}^3 \setminus \Lambda^\delta$ for some $\delta > 0$ where

$$\Lambda^\delta := \left\{ r \in \mathbb{R}^3 \mid \text{dist}(r, \Lambda) < \delta \right\}$$

is a “thickened” version of the set $\Lambda$. Then

$$\mathcal{E}_\rho(\Psi) := \frac{1}{2} \int_{\mathbb{R}^3} dx \rho(\lambda x) e(x, \Psi)$$

$$= \frac{1}{2} \langle \Psi, (\lambda x) \Psi \rangle_{\lambda} = \frac{1}{2} \langle \Psi, Dp^s_{\lambda}(\rho) \Psi \rangle_{\lambda}$$

is in good approximation the field energy contained inside of the stretched domain

$$\Lambda_\lambda := \{ x \in \mathbb{R}^3 \mid \lambda x \in \Lambda \}$$

provided the “thickness” $\delta$ of the transition layer where $\rho \to 0$ is small. With this proviso, we will call $\mathcal{E}_\rho$ the field energy localized in the volume $\Lambda_\lambda$.

Clearly, $\rho$ defines the scalar, quadratic observable $\mathcal{E}_\rho$, and thus, Theorem 3.3 and Corollary 3.5 apply: for $\Psi \in \text{ran} \Pi_\lambda$ we can approximate

$$\mathcal{E}_\rho(\Psi(t)) = \frac{1}{2} \langle \Psi, (Dp^s_{\lambda}(\rho \circ \Phi^t_\lambda)) \Psi \rangle_{\lambda} + O(\lambda^2 |t|)$$

$$= \frac{1}{2} \int_{\mathbb{R}^3} dr \int_{\mathbb{R}^3} dk \rho \circ \Phi^t_\lambda(r, k) W_{\rho}(r, k) + O(\lambda^2 |t|).$$

with the help of the ray optics flow associated to (3.5).
3.3 The ray optics limit for certain observables

3.3.2 The Poynting vector

In the theory of electromagnetism the Poynting vector

$$S(x, \Psi) := \text{Re} \left( \psi^\star(x) \times \psi^{ii}(x) \right)$$

is proportional (up to a factor $c^2$) to the Abraham momentum density. Indeed, it appears in the local energy conservation law (cf. [Ber82, equation (38)])

$$\partial_t e(x, \Psi(t)) + \nabla_x \cdot S(x, \Psi(t)) = 0 \quad (3.24)$$

as the balancing term to the energy flux. Surprisingly, the three components of $S$ are linked to what would be called the “current operator” in quantum mechanics,

$$j_n := \frac{1}{\lambda}[M_\lambda, \lambda \hat{x}_n] = S^{-2}(\lambda \hat{x}) \mathcal{W} \begin{pmatrix} 0 & -e_n^x \\ +e_n^x & 0 \end{pmatrix} =: \mathcal{D}p^{\nabla Z}_\lambda(s_n), \quad (3.25)$$

with

$$s_n := S^{-1} \mathcal{W} \begin{pmatrix} 0 & -e_n^x \\ +e_n^x & 0 \end{pmatrix} S^{-1},$$

and a quick computation reveals

$$\int_{\mathbb{R}^3} dx \, 2S_n(x, \Psi) = \langle \Psi, j_n \Psi \rangle_\lambda = \langle \Psi, \mathcal{D}p^{\nabla Z}_\lambda(s_n) \Psi \rangle_\lambda.$$

As one can see right away, even when the perturbation is scalar $s_n$ defines a non-scalar observable.

There are in fact two interesting quantities connected to the Poynting vector, the net flux across $\partial \Lambda_\lambda$ as well as the local average of $S$ across $\Lambda_\lambda$. The first can be accessed via $E_\rho$ with the help of local energy conservation (3.24): Taking the time-derivative of $E_\rho(\Psi(t))$ approximately yields the net momentum flux over the “surface” $\partial \Lambda_\lambda$,

$$\frac{d}{dt} E_\rho(\Psi(t)) = \frac{1}{2} \left\langle \Psi, \frac{d}{dt} \left( e^{+i \hat{x}_\lambda} \rho(\lambda \hat{x}) e^{-i \hat{x}_\lambda} \right) \Psi \right\rangle_\lambda = \frac{1}{2} \left\langle \Psi, \nabla \rho(\lambda \hat{x}(t)) \cdot j(t) \Psi \right\rangle_\lambda,$$

because the support of the derivative $\text{supp} \nabla \rho \subseteq \Lambda_\delta \setminus \Lambda_\lambda$ is contained in the “boundary layer” of $\Lambda_\delta$ for $\delta$ sufficiently small which is a thickened version of the boundary $\partial \Lambda_\lambda$. 

21
Now the ray optics limit for scalar observables applies to the energy contained in $\Lambda_\lambda$. Note that the error term in equation (3.23) grows linearly in time, and consequently, the error of the time-derivative is still of $O(\lambda^2)$,

$$
\frac{d}{dt} E_{\rho} (\Psi(t)) = \frac{1}{2} \left\langle \Psi, \nabla \rho (\lambda \hat{x}(t)) \cdot j(t) \Psi \right\rangle_{\lambda} = \frac{1}{2} \left\langle \Psi, D_{\rho_\lambda} \left( \frac{d}{dt} \rho \circ \Phi^\lambda \right) \Psi \right\rangle_{\lambda} + O(\lambda^2) \\
= \frac{1}{2} \left\langle \Psi, D_{\rho_\lambda} \left( \nabla \rho (r(t)) \cdot j(t) \right) \Psi \right\rangle_{\lambda} + O(\lambda^2). 
$$

(3.26)

Using formal arguments, we see that these results are consistent with the local energy conservation law (3.24):

$$
\frac{d}{dt} E_{\rho} (\Psi(t)) \approx \int_{\Lambda} dx \, \partial_t \varepsilon(x, \Psi(t)) = - \int_{\partial \Lambda} d\eta(x) \cdot S(x, \Psi(t)) \\
\approx \int_{\mathbb{R}^3} dx \, \nabla_x \rho(\lambda x) \cdot S(x, \Psi(t)) = \frac{1}{2} \left\langle \Psi, \nabla \rho (\lambda \hat{x}(t)) \cdot j(t) \Psi \right\rangle_{\lambda}
$$

where $d\eta(x)$ is the measure on $\partial \Lambda$ with surface normal pointing outwards.

The field momentum inside of $\Lambda$ is accessible via Theorem 3.3 (ii),

$$
S_{\rho,n} (\Psi(t)) := \frac{1}{2} \left\langle \Psi(t), D_{\rho_\lambda} (\rho s_n) \Psi(t) \right\rangle_{\lambda} = \frac{1}{2} \left\langle \Psi, D_{\rho_\lambda} \left( f_{\rho_\lambda} \circ \Phi^\lambda \right) \Psi \right\rangle_{\lambda} + O(\lambda^2 |t|),
$$

although we need to replace $f := \rho s_n$ with $f_{\rho_\lambda} = \pi_{\rho_\lambda}^\# f \pi_{\rho_\lambda} + O(\lambda^2)$ and the flow $\Phi^\lambda$ of (3.5) by that associated to the ray optics equations (3.7) which omit the Berry curvature in the symplectic form. Instead, several terms that are linked to the geometry of the Bloch bundle appear at $O(\lambda)$ in $f_{\rho_\lambda}$.

### 3.3.3 Other quadratic observables relevant in electrodynamics

At least four more observables, all of them non-scalar, fit into the category of quadratic observables once they are localized by a smoothened characteristic function $\rho$. We leave the details such as finding the appropriate operator-valued function to the reader.

The averaged quadratic component of the electric field

$$
|E_{\rho,n}|^2 := \int_{\mathbb{R}^3} dx \, \rho(\lambda x) |E_n(x)|^2
$$

22
and a similar expression for the magnetic field falls into the category set forth by Definition 3.2.

Apart from the local averages of the Poynting vector \( S(x, \Psi) \) and of the related Abraham momentum density \( G^A(x, \Psi) := c^{-2} S(x, \Psi) \), for the case \( \chi = 0 \) there is a second momentum observable in electromagnetism, the Minkowski momentum density

\[
G^M(x, \Psi) := \text{Re} \left( \tau^{-4}(\lambda x) \left( \bar{e}(x) \bar{\psi}(x) \right) \times \left( \mu(x) \bar{\psi}^H(x) \right) \right).
\]

The relation between the Abraham and the Minkowski momentum densities as well as their physical interpretation are delicate topics in classical electrodynamics known as the Abraham-Minkowski controversy (see e. g. [PNH+07]).

Similarly, local averages of the components of angular momentum (defined with respect to either Abraham or Minkowski momentum density)

\[
L^{A/M}_\rho(\Psi) := \int_{\mathbb{R}^3} dx \rho(\lambda x) \left( x \times G^{A/M}(x, \Psi) \right)
\]
as well as the components of the components of the Maxwell stress tensor (for \( \chi = 0 \))

\[
\tau^{j, n}_\rho := \int_{\mathbb{R}^3} dx \rho(\lambda x) \left( \tau^{-2}_e(\lambda x) \bar{\psi}_j(x) \left( e(x) \bar{\psi}^H(x) \right)_n + \right.
\]

\[
\left. + \tau^{-2}_\mu(\lambda x) \bar{\psi}_j^H(x) \left( \mu(x) \bar{\psi}^H(x) \right)_n - \delta_{j,n} e(x, \Psi) \right)
\]

are other examples of quadratic observables covered by Theorem 3.3. To each one of those quadratic observables one can associate a symbol similar to the form considered in Remark 3.8 as the reader can easily verify.

### 4 Ray optics equations for real electromagnetic fields

One of the insights of our earlier works [DL14a; DL14b] is that single frequency bands cannot support real electromagnetic fields. And certainly, from a physical perspective understanding the ray optics limit for real fields is crucial if our results are to have physical significance.

Real states are only supported by non-gyrotropic materials where the weights are real, i. e. if \( C \) denotes complex conjugation they satisfy \( CW = W = WC \). Then the Maxwell operator \( M_\lambda \) anticommutes with \( C \) (cf. also [DL14b, Section 2.2]), and frequency bands come in pairs: if \( \varphi_n(k) \) is an eigenfunction of \( M_\lambda(k) \) to \( \omega_n(k) \), then \( \varphi_n(-k) \) is an eigenfunction of \( M_\lambda(k) \) to \( -\omega_n(-k) \).

\[
M_\lambda(k) \varphi_n(k) = \omega_n(k) \varphi_n(k) \iff M_\lambda(k) C \varphi_n(-k) = \omega_n(-k) C \varphi_n(-k).
\]
4 Ray optics equations for real electromagnetic fields

So let us focus on a non-degenerate, isolated band \( \omega_+(k) > 0 \) in the sense of Assumption 3.1. Then also its symmetric twin \( \omega_-(k) = -\omega_+(k) \) with Bloch function \( \varphi_-(k) = \varphi_+(k) \) is non-degenerate and isolated. And since real and imaginary parts of \( \varphi_+ \),

\[
\psi_{\text{Re}}(k) := \frac{1}{\sqrt{2}} (\varphi_+(k) + \varphi_-(k)) = \frac{1}{\sqrt{2}} (\varphi_+(k) + \varphi_-(k)), \quad (4.1a)
\]

\[
\psi_{\text{Im}}(k) := \frac{1}{i \sqrt{2}} (\varphi_+(k) - \varphi_-(k)) = \frac{1}{i \sqrt{2}} (\varphi_+(k) - \varphi_-(k)), \quad (4.1b)
\]

are linear combinations of distinct bands, we see that single Bloch bands cannot support real states. Consequently, the relevant almost invariant subspace associated to the pair \( \{ \omega_+, \omega_- \} \) is the range of \( \Pi_{\lambda} = \Pi_{\lambda+} + \Pi_{\lambda-} + \mathcal{O}_{\|\|}(\lambda^\infty) \).

The existence of these three projections is covered by [DL14a, Proposition 1]. To understand why \( \Pi_{\lambda} \) factors into the sum of the superadiabatic projections \( \Pi_{\pm, \lambda} \) associated to the positive/negative frequency bands \( \omega_{\pm} \), one has to look at how they are constructed (see also the proof of Lemma 7 in [DL14a]): \( \Pi_{\lambda} \) is defined as a spectral projection of \( \Omega_{\lambda}^{\pm}(\pi_{\lambda}) \), and the symbol of this PDO in turn is defined as the sum of two contour integrals with respect to the local Moyal resolvents – one for \( \omega_+ \) and one for \( \omega_- \). Each of these contour integrals yields \( \pi_{\lambda+} \) and \( \pi_{\lambda-} \), respectively, and thus, \( \pi_{\lambda} = \pi_{\lambda+} + \pi_{\lambda-} \) holds in the sense of semiclassical symbols.

These arguments also yield that \( \Pi_{\lambda+} \) and \( \Pi_{\lambda-} \) are related by the symmetry \( C \),

\[
C \Pi_{\lambda+} C = \Pi_{\lambda-} + \mathcal{O}_{\|\|}(\lambda^\infty). \quad (4.3)
\]

To make explicit that we restrict ourselves to real electromagnetic fields, let us introduce the \( \mathbb{R} \)-linear real part projection

\[
\text{Re} := \frac{1}{2} (\text{id} + C).
\]

With these definitions in hand, we are able to formulate our second main result:

**Theorem 4.1** Suppose we are in the setting of Theorem 3.3, and assume in addition that the material weights are real, \( CW = W \) and the functions which define the quadratic observables satisfy

\[
(\epsilon f)(r, k) := \overline{f(r, -k)} = f(r, k). \quad (4.4)
\]

Then we have a ray optics limit for real electromagnetic fields in the following sense:
(i) For scalar observables where \( f \in C^\infty_b(\mathbb{R}^6, \mathbb{C}) \), the ray optics flow \( \Phi^\lambda \) associated to the hamiltonian equations (3.5) approximates the full light dynamics for \( \Psi \in \text{Re} (\text{ran} \Pi_\lambda) \) and bounded times in the sense
\[
\mathcal{F}(\Psi(t)) = 4 \left\{ \Psi, \text{Re} \Pi_{+\lambda} \mathcal{D}_{\lambda}^Z (f \circ \Phi^\lambda_t) \Pi_{+\lambda} \text{Re} \Psi \right\}_\lambda + \mathcal{O}(\lambda^2). \tag{4.5}
\]

(ii) For non-scalar observables where \( f \in C^\infty_b(\mathbb{R}^6, \mathcal{B}(\mathfrak{H}_0)) \), the ray optics flow \( \Phi^\lambda \) associated to the hamiltonian equations (3.7) approximates the full light dynamics for \( \Psi \in \text{Re} (\text{ran} \Pi_\lambda) \) and bounded times in the sense
\[
\mathcal{F}(\Psi(t)) = 4 \left\{ \Psi, \text{Re} \Pi_{+\lambda} \mathcal{D}_{\lambda}^Z (f_{\Pi_0} \circ \Phi^\lambda_t) \Pi_{+\lambda} \text{Re} \Psi \right\}_\lambda + \mathcal{O}(\lambda^2) \tag{4.6}
\]
where as before \( f_{\Pi_0} = \pi_\lambda \text{Re} f \pi_\lambda + \mathcal{O}(\lambda^2) \).

To understand why it suffices to look at the dynamics in the positive frequency subspace and where the factor of 4 stems from, it is instructive to consider the perfectly periodic case (\( \lambda = 0 \)). For each \( k \in \mathbb{B} \) the subspace of real initial states \( (\text{ran} \text{Re} \Pi_0)(k) \) has real dimension 2, i.e., complex dimension 1. In fact, we can write \( \Psi \in \text{ran} \text{Re} \Pi_0 \) as a real linear combination of \( \psi_{\text{Re}} \) and \( \psi_{\text{Im}} \) from above,
\[
\Psi = Z^{-1} (\alpha_{\text{Re}} \psi_{\text{Re}} + \alpha_{\text{Im}} \psi_{\text{Im}}),
\]
where the coefficients \( \alpha_{\text{Re}}(k) = \alpha_{\text{Re}}(-k) \) and \( \alpha_{\text{Im}}(k) = \alpha_{\text{Im}}(-k) \) are real and depend on \( k \in \mathbb{B} \). Put another way, the positive frequency contribution and its phase completely determine the real state,
\[
\Psi = Z^{-1} (\alpha_{\text{Re}} \psi_{\text{Re}} + \alpha_{\text{Im}} \psi_{\text{Im}})
\]
\[
= \frac{1}{2} Z^{-1} \left( \left( a_+ / \sqrt{\tau} - i a_- / \sqrt{\tau} \right) \varphi_+ + \left( a_+ / \sqrt{\tau} + i a_- / \sqrt{\tau} \right) \varphi_- \right)
\]
\[
= \text{Re} \ Z^{-1} \left( \left( a_+ / \sqrt{\tau} - i a_- / \sqrt{\tau} \right) \varphi_+ \right),
\]
and the positive and negative frequency contributions are equal in magnitude (norm). This “phase locking” explains why all information is contained in \( \Pi_+ \Psi \). The appearance of the factor of 4 also has a simple explanation: plugging (4.2) into \( D_{\lambda}^Z f \Pi_\lambda \text{Re} \) gives four terms where we have abbreviated \( F(t) = e^{i t M_\lambda} \mathcal{D}_{\lambda}^Z (f) e^{-i t M_\lambda} \), and each of them by itself contains enough information to reconstruct the original expression.

There is a second point which deserves more explanation, namely the restriction to observables which are defined through the function \( f = cf \), because the associated pseudodifferential operators commute with complex conjugation, meaning that they share the same symmetry as the dynamics \( e^{-i t M_\lambda} \). Otherwise \( \mathcal{D}_{\lambda}^Z (f) \) mixes the positive and negative frequency contributions, and it is no longer possible to look at, say, the \( \Pi_{+\lambda} F_\lambda(t) \Pi_{+\lambda} \) component to obtain \( \Pi_{\lambda} F_\lambda(t) \Pi_{\lambda} \text{Re} \).
4.1 Reduction to single-band ray optics

The main ingredient in the proof is a way to reduce the two-band case to the single-band case discussed in Section 3. This is achieved through the following Proposition which merely relies on equations (4.2) and (4.3).

Proposition 4.2 Suppose an orthogonal projection \( \Pi_{\lambda} \) satisfies equations (4.2) and (4.3), and \( F \in \mathcal{B}(L^2(\mathbb{R}^3, \mathbb{C}^3)) \) commutes with \( C \). Then we have

\[
\Pi_{\lambda} F \Pi_{\lambda} \text{Re} = 4 \text{Re} \Pi_{+\lambda} F \Pi_{+\lambda} \text{Re} + O_{\|/\|}(\lambda^\infty).
\]

Remark 4.3 Note that for this Proposition, we do not need to assume that the \( \Pi_{\pm\lambda} \) are associated to single, non-degenerate bands. Projections associated to symmetrically chosen, finite families of bands where

\[
\sigma_{\text{rel}}(-k) = \sigma_{\text{rel}}(k) := \bigcup_{n \in I} \{\omega_n(k)\}, \quad I \subset \mathbb{Z},
\]

also satisfy equations (4.2) and (4.3).

The main ingredient in the proof is the following

Lemma 4.4 \( 2 \text{Re} \Pi_{\pm\lambda} \text{Re} = \Pi_{\lambda} \text{Re} + O_{\|/\|}(\lambda^\infty) \)

Proof Writing out \( 2 \text{Re} = \text{id} + C \) and using equation (4.3), we obtain

\[
4 \text{Re} \Pi_{+\lambda} \text{Re} = \Pi_{+\lambda} + C \Pi_{+\lambda} + \Pi_{+\lambda} C + C \Pi_{+\lambda} = (\Pi_{+\lambda} + \Pi_{-\lambda}) (\text{id} + C) + O_{\|/\|}(\lambda^\infty) = 2 \Pi_{\lambda} \text{Re} + O_{\|/\|}(\lambda^\infty).
\]

Proof (Proposition 4.2) We use \([\Pi_{\lambda}, \text{Re}] = O_{\|/\|}(\lambda^\infty)\) and \(\text{Re}^2 = \text{Re}\) to commute \(\text{Re}\) with \(\Pi_{\lambda}\) and \(F\), incurring an \(O_{\|/\|}(\lambda^\infty)\) error in the process, and then employ Lemma 4.4.

\[
\Pi_{\lambda} F \Pi_{\lambda} \text{Re} = \text{Re} \Pi_{\lambda} F \Pi_{\lambda} \text{Re} + O_{\|/\|}(\lambda^\infty) = 2 \text{Re} \Pi_{\pm\lambda} \text{Re} F \Pi_{\lambda} \text{Re} + O_{\|/\|}(\lambda^\infty).
\]

Next, we write out \(\Pi_{\pm\lambda} = \Pi_{+\lambda} + \Pi_{-\lambda} + O_{\|/\|}(\lambda^\infty)\) and collect terms,

\[
\ldots = 2 \text{Re} \Pi_{+\lambda} F \Pi_{\lambda} \text{Re} + O_{\|/\|}(\lambda^\infty) = 2 \text{Re} \Pi_{\pm\lambda} F (\Pi_{+\lambda} + \Pi_{-\lambda}) \text{Re} + O_{\|/\|}(\lambda^\infty) = 2 \text{Re} (\Pi_{+\lambda} F \Pi_{+\lambda} + \Pi_{-\lambda} F \Pi_{-\lambda}) \text{Re} + O_{\|/\|}(\lambda^\infty).
\]

To show that the sign in the term involving \(\Pi_{-\lambda}\) can be flipped, we insert complex conjugations via \(\text{Re} = \text{Re} C = C \text{Re}\) and use \(\Pi_{-\lambda} = C \Pi_{+\lambda} C + O_{\|/\|}(\lambda^\infty)\) to relate the projections for positive and negative frequency bands:

\[
\text{Re} \Pi_{-\lambda} F \Pi_{-\lambda} \text{Re} = \text{Re} C \Pi_{+\lambda} C F C \Pi_{+\lambda} \text{Re} + O_{\|/\|}(\lambda^\infty) = \text{Re} \Pi_{+\lambda} F \Pi_{+\lambda} \text{Re} + O_{\|/\|}(\lambda^\infty).
\]

That shows the claim.
The last result we need shows that \( \text{Re } \mathcal{O}_\lambda \mathfrak{p}_Z(f) \text{Re} \) is the pseudodifferential operator associated to \( \text{Re } f := \frac{1}{2} (f + \epsilon f) \) (4.7)

where \((\epsilon f)(r, k) = \overline{f(r, -k)}\) is the function associated to \( C \mathcal{O}_\lambda \mathfrak{p}_Z(f) C = \mathcal{O}_\lambda \mathfrak{p}_Z(\epsilon f) \).

**Lemma 4.5** Let \( f \in C^\infty_b(\mathbb{R}^6, B(L^2(\mathbb{T}^3, \mathbb{C}^6))) \) be an operator-valued function satisfying the equivariance condition (2.7). Then the following holds:

(i) \( \text{Re } \mathcal{O}_\lambda \mathfrak{p}_Z(f) \text{Re} = \mathcal{O}_\lambda \mathfrak{p}_Z(f) \text{Re} = \text{Re } \mathcal{O}_\lambda \mathfrak{p}_Z(\epsilon f) \)

(ii) If \( \epsilon f = f \), then \([ \mathcal{O}_\lambda \mathfrak{p}_Z(f), \text{Re} ] = 0\).

**Proof** (i) The crucial ingredient is \( C \mathcal{O}_\lambda \mathfrak{p}_Z(f) C = \mathcal{O}_\lambda \mathfrak{p}_Z(\epsilon f) \) which has been proven as Lemma 5 in [DL14a]. Consequently, we compute

\[
4 \text{Re } \mathcal{O}_\lambda \mathfrak{p}_Z(f) \text{Re} = \mathcal{O}_\lambda \mathfrak{p}_Z(f) + \mathcal{O}_\lambda \mathfrak{p}_Z(f) C + C \mathcal{O}_\lambda \mathfrak{p}_Z(f) + C \mathcal{O}_\lambda \mathfrak{p}_Z(f) C
\]

\[
= \mathcal{O}_\lambda \mathfrak{p}_Z(f) (id + C) + C \mathcal{O}_\lambda \mathfrak{p}_Z(f) C (id + C)
\]

\[
= \mathcal{O}_\lambda \mathfrak{p}_Z(f + \epsilon f) (id + C) = 4 \mathcal{O}_\lambda \mathfrak{p}_Z(\epsilon f) \text{Re}.
\]

Factoring \( id + C \) to the left yields the other equality.

(ii) This follows immediately from (i).

4.2 Proof of ray optics for real fields

**Proof (Theorem 4.1)** We merely have to check whether \( F := e^{+i\lambda M_3} \mathcal{O}_\lambda \mathfrak{p}_Z(f) e^{-i\lambda M_3} \) defines a bounded operator that commutes with \( C \). The boundedness follows from the Caldéron-Vaillancourt theorem (cf. [Teu03, Theorem B.5]), and \([ F, \text{Re} ] = 0\) follows from \([ e^{-i\lambda M_3}, \text{Re} ] = 0 \), \( \epsilon f = f \) and Lemma 4.5. Thus, Proposition 4.2 applies,

\[
\Pi_\lambda e^{+i\lambda M_3} \mathcal{O}_\lambda \mathfrak{p}_Z(f) e^{-i\lambda M_3} \Pi_\lambda \text{Re} = 4 \text{Re } \Pi_\lambda e^{+i\lambda M_3} \mathcal{O}_\lambda \mathfrak{p}_Z(f) e^{-i\lambda M_3} \Pi_\lambda \text{Re} + \mathcal{O}(\lambda^\infty),
\]

and we have reduced the problem of twin band ray optics to the single band case. Now for scalar \( f \) we can relate the left-hand side of (4.8) to the ray optics equations (3.5) for the single band case via Proposition 3.9. The case of non-scalar \( f \) is analogous.
5 Absence of topological effects due to Chern classes in non-gyrotropic materials

Lastly, let us take the opportunity to explain why the $C$ symmetry of the Maxwell operator for non-gyrotropic materials has leads to the absence of topological effects due to Chern classes. In the language of the Cartan-Altland-Zirnbauer classification scheme for topological insulators \cite{AZ97, SRF08}, $C$ acts as a “particle-hole symmetry” because it anti-commutes with the Maxwell operator (which is why some authors refer to it as a pseudo-symmetry). Hence, photonic crystals with $C$-symmetry are either of class BDI or D (cf. \cite{DL14b, Table 4.3}), depending on whether the bi-anisotropic tensor $\chi$ vanishes or not. While these topological phases are not trivial, nevertheless we can show the absence of topological effects due to Chern classes. This is not completely obvious since at least for class D, the frequency bands $\omega_{\pm}$ by themselves need not have zero Chern charge, the $C$-symmetry just implies that the Bloch bundle associated to the pair of bands $\{\omega_+, \omega_-\}$ is trivial (cf. Proposition 5 and Remark 4 in \cite{DL14a}).

For the purpose of this section, let us use “Chern trivial” as a short-hand for “the Chern classes vanish” (i.e. triviality in the sense of class A). We caution the reader that Chern trivial is not one and the same thing as trivial in the sense of class BDI and D, these classes support topological invariants other than Chern classes (see e.g. \cite{SP14} for class BDI). Nevertheless, to avoid tongue twisters we will use this simplification. With that in mind, let us first give the following

**Definition 5.1 (States in trivial Chern phases)**

(i) We say that $\Psi \in \text{ran} \Pi_0$, $\Psi \neq 0$, is in a trivial Chern phase if and only if all three associated Chern numbers vanish, i.e. $c_1(\hat{\Psi} \langle \hat{\Psi} |) = 0 \in \mathbb{Z}^3$ where $\hat{\Psi} = \|\Psi\|^{-1} \Pi_0 \Psi$.

(ii) For $\lambda$ small enough, we say $\Psi \in \text{ran} \Pi_\lambda$ is in a trivial Chern phase if and only if $\Pi_0 \Psi$ is Chern trivial in the sense of (i).

The idea of this definition is to consider $\Pi_\lambda$ as a deformation of $\Pi_0$. In fact, the projection associated to any finite resummation $\Pi_\lambda^{(N)} := I_{\{1/\lambda, \lambda\}} \left( \sum_{m=0}^{N} \lambda^m \mathcal{O} \mathcal{D} \mathcal{P}_S^z (\pi_n) \right)$ can be seen as a homotopy $\lambda \mapsto \Pi_\lambda^{(N)}$ in the space of projections which deforms $\Pi_0$ into $\Pi_\lambda$. And since for weak enough perturbations, spectral gaps do not close, the Chern numbers of the “deformed” $\Pi_\lambda$ has to agree with the Chern numbers of $\Pi_0$.

The crucial observation is that all real states are necessarily topologically trivial.

**Lemma 5.2** States $\Upsilon \in \text{Re} \left( \text{ran} \Pi_\lambda \right)$, $\Upsilon \neq 0$, are in a trivial Chern phase.

**Proof** It suffices to consider the case $\lambda = 0$. So let $\Upsilon \in \text{Re} \text{ran} \Pi_0 \setminus \{0\}$. Without loss of generality, we may assume that $\Upsilon$ is normalized so that $|\Upsilon\rangle \langle \Upsilon|$ is an orthogonal projection. By definition, we know that $(Z \Upsilon)(k) = a_{Re}(k) \psi_{Re}(k) + a_{Im}(k) \psi_{Im}(k)$ where
\[ \alpha_{\text{Re}}(k), \alpha_{\text{Im}}(k) \in \mathbb{R} \] are real coefficients and the \( \psi_{\text{Re,Im}}(k) \) are defined as in equation (4.1). \( C \mathcal{Y} = \mathcal{Y} \) and \( C(\mathcal{Z} \mathcal{Y})(k) = (\mathcal{Z} \mathcal{Y})(-k) \) for almost all \( k \in \mathbb{B} \) are equivalent, and consequently the associated projection satisfies

\[ C \langle (\mathcal{Z} \mathcal{Y})(k) \rangle \langle (\mathcal{Z} \mathcal{Y})(-k) \rangle C = \langle (\mathcal{Z} \mathcal{Y})(-k) \rangle \langle (\mathcal{Z} \mathcal{Y})(-k) \rangle. \]

Hence, the associated bundle is topologically trivial (cf. e. g. [DL11b, Theorem 4.6]).

What makes photonic crystals with \( C \)-symmetry special is that real states are preserved under time-evolution:

**Proposition 5.3** Suppose Assumption 2.2 holds and the material weights are real. Moreover, let \( \Pi_{\lambda} \) be the superadiabatic projection satisfying equations (4.2) and (4.3). Then we have

\[ e^{+i t M_{\lambda}} \left( \text{Re} \Pi_{\lambda} \text{Re} \right) e^{-i t M_{\lambda}} = \text{Re} \Pi_{\lambda} \text{Re} + O(\lambda^\infty). \tag{5.1} \]

In particular, this means that states in a trivial Chern phase \( \Psi \in \text{Re ran} \Pi_{\lambda} \) remain in a trivial Chern phase under the time evolution up to \( O(\lambda^\infty) \).

**Proof** Equation (5.1) follows from

\[ C e^{-i t M_{\lambda}} C = e^{+i t CM_{\lambda} C} = e^{-i t M_{\lambda}} \]

and \( [M_{\lambda}, \Pi_{\lambda}] = O(\lambda^\infty) \) which is one of the defining relations of the superadiabatic projection (cf. [DL14a, Proposition 1]).

The Chern triviality of the time-evolved state \( \Psi(t) = e^{-i t M_{\lambda}} \Psi \) is an immediate consequence of (5.1), because up to an error \( O(\lambda^\infty) \) the time-evolved fields \( \Psi(t) \) remain real and an element of ran \( \Pi_{\lambda} \).

Note that these arguments do not invoke ray optics at all. However, as ray optics for real states approximates the full light dynamics, also in this approximation we are unable to see topological effects due to Chern classes.

**Corollary 5.4** Assume we are in the context of Theorem 4.1, and assume \( \Psi \in \text{Re ran} \Pi_{\lambda} \). Then in the ray optics limit, there are no topological effects attributable to non-trivial Chern numbers.

These last two statements furnish the physical intuition exhibited in previous works on ray optics in PhCs (e. g. [RH08; OMN06]) with an explanation: even though the Berry curvature appears in the ray optics equations (3.5) for scalar quadratic observables, there are no topological effects due to Chern classes.

Lastly, let us mention that these arguments do not preclude the possibility of topological effects in photonic crystals of class D and BDI, effects which are linked to other topological invariants.
6 Quantum-light analogies and differences: comparison of semiclassics and ray optics

The premise of this article was to rigorously establish the quantum-light analogy between semiclassics for the Bloch electron and ray optics in photonic crystals. However, we need to clearly distinguish between analogies in the mathematical structures and similarities in the physics of crystalline solids and photonic crystals. From the perspective of mathematics it is not at all surprising that the semiclassical equations

\begin{align}
\dot{r} &= +\nabla_k \hbar - \lambda \Xi \dot{k} \\
\dot{k} &= -\nabla_r \hbar + \dot{r} \times B
\end{align}

(6.1a)

(6.1b)

for a Bloch electron subjected to an external electromagnetic field \((-\nabla_r \phi, B)\) indeed resemble equation (3.5) where the semiclassical hamiltonian

\[ h(r,k) = \left( E_n(k) + \phi(r) \right) + \mathcal{O}(\lambda) \]

takes the place of the dispersion relation (3.3) (see \[\text{PST03a}\] and references therein for details). The presence of the anomalous velocity term \(\Xi \dot{k}\) in the ray optics equations was key in the early works \[\text{OMN04; RH08}\] to anticipate topologically protected edge modes in photonic crystals. In fact, \[\text{OMN06; RH08; EG13}\] all contain the same semiclassical argument showing the quantization of the transverse conductivity for the quantum system: in case the Bloch electron is subjected to a constant electromagnetic field and the magnetic flux through the unit cell is rational, the effect of \(B\) can be subsumed by using magnetic Bloch bands, and the average current carried by a filled band

\[ j = \int_B \dot{r} = \int_B \dot{k} \left( \nabla_k E_n(k) - \varepsilon \Xi(k) E \right) = \varepsilon c \times E \]

(6.2)

is proportional to the antisymmetric matrix \(c = \frac{1}{2\pi} \int_B \nabla_k \times A(k)\) made up of the first Chern numbers and the electric field \(E\). While suggestive the argument does not work for photonic crystals for reasons that are important and independent of finding a photonic analog of the transverse conductivity.

**Typical states** The leading-order term in (6.2) vanishes because the band is completely filled. Such states are typical for semiconductors and isolators where the Fermi energy \(E_F\) lies in a gap. Even when one includes finite-temperature effects, these are typically seen as perturbations of the (zero temperature) Fermi projection

\[ p_F = \mathbb{1}_{(-\infty,E_F)}(H). \]
However, the Maxwell equations describe classical waves, and there is no exclusion principle which forbids us to populate the same frequency band more than once.

Experiments usually rely on a laser to selectively populate a frequency band. Thus, states are typically peaked around some $k_0 \in \mathbb{B}$ and a frequency $\omega_0$, one may think of a laser beam which impinges on the surface of a photonic crystal: the frequency of the laser light fixes the spectral region, and the angle with respect to the surface normal determines $k_0$. A fully filled band would correspond to a carefully concocted cocktail of light moving in all different directions at specific frequencies, something that seems to be much harder to achieve if at all possible. Hence, we have to take the Brillouin zone average with respect to the reduced Wigner transform

$$w^\text{red}_\Psi(r, k) := \sum_{\gamma \in \Gamma} w\Psi(r, k + \gamma^*)$$

obtained by zone folding the usual Wigner transform $w\Psi$, and $w^\text{red}_\Psi(r, k)$ is now peaked around $k_0$ rather than constant in $k$.

**Observables** We have consciously avoided to call $\mathcal{O}\mathcal{P}_\lambda(f)$ the (Weyl) quantization of the classical observable $f$, as the operator $\mathcal{O}\mathcal{P}_\lambda(f)$ is not an observable in classical electromagnetism – those are functionals of the fields. While this distinction may seem pedantic and unnecessary, it is crucial if one wants to imbue expressions such as

$$\int \mathbb{B}^3 \, dr \int \mathbb{B} \, dk \, f(t, r, k) \, w^\text{red}_\Psi(r, k)$$

with physical meaning. In fact, depending on the type of observable, scalar or non-scalar, we have two different ray optics equations to choose from. For instance, our discussion in Section 3.3.2 explains that only the net energy flux across a surface uses $\dot{r} = +\nabla_s \Omega - \lambda \Xi \nabla_s \Omega$, local averages of the Poynting vector require one to use simpler ray optics equations which omit the anomalous velocity term at the expense of having to insert a more complicated function $f_{\lambda} = \langle \varphi, f \psi \rangle_{\lambda}^\text{h} + \mathcal{O}(\lambda)$ into the integral over phase space.

All in all, while the hamiltonian equations (6.1) and (3.5) look very similar on the surface, the physics they describe is very different. The presence of the anomalous velocity term incorrectly suggests one is able to repeat the arguments of (6.2): ignoring that completely filled frequency bands are hard to come by and that it is unclear what physical quantity the Brillouin zone average of

$$\dot{r} = +\nabla_s \Omega + \lambda \omega \Xi \nabla_s (\tau^2) + \mathcal{O}(\lambda^3)$$

corresponds to, it still would not lead to an expression proportional to $c$. 

31
Designing an experiment to probe the $O(\lambda)$ effects Nevertheless, the $O(\lambda)$ contributions to the ray optics equations contain interesting physics, and the question comes to mind whether it is possible to engineer an experiment where these effects are particularly strong. The reason the leading-order term in (6.2) is identically 0 is the complete filling of the energy band. In photonics, we can turn this premise on its head, instead of indiscriminately exciting a whole band, we can populate states with pin point accuracy. We propose to use states in the slow or frozen mode regime (see e. g. [FV06]): Here, we are interested in critical points of the frequency band function where in addition to $\nabla_k \omega_n(k_0) = 0$ at least also the second-order derivatives vanish, $\text{Hess} \omega_n(k_0) = 0$. To see why, one needs to consider the density of states (DOS) $D(\omega)$ – a quantity which is well-defined because away from 0, the spectrum of periodic Maxwell operators is believed to be absolutely continuous (proven under additional regularity assumptions on the material weights in [Mor00; Sus00; KL01]). For simplicity, let us assume that in the vicinity of $k_0$, the frequency band behaves as

$$\omega_n(k) = \omega_0 + a(k - k_0)^{p} + O((k - k_0)^{p+1})$$

for some integer $p \geq 2$. Then a simple scaling argument yields that the contribution of the band $\omega_n$ near $\omega_0$ to the DOS is

$$D(\omega) \approx b(\omega - \omega_0)^{\frac{1}{3} - 1}$$

where the factor 3 stems from the dimension of the ambient space $\mathbb{R}^3$. For generic critical points $p = 2$ and $D(\omega)$ vanishes at $\omega_0$ – there are no states to populate. The additional condition $\text{Hess} \omega(k_0) = 0$ implies $p \geq 3$, and the DOS either remains non-zero and finite at $\omega_0$ ($p = 3$) or diverges ($p \geq 4$). These heuristic considerations allow us to conclude that for $p = 3$ the leading-order term

$$\int_{\mathcal{B}} dk \nabla_k \omega(k) w^{\text{red}}(r, k) \approx 0,$$

vanishes, but there are sufficiently many states to excite because $D(\omega_0) \neq 0$.

## A Pseudodifferential calculus for equivariant operator-valued symbols

The main point of [DL14c] was to explain how $M_{\lambda} = \mathcal{D}p^Z_{\lambda}(\mathcal{M}_{\lambda})$ can be understood as the pseudodifferential operator associated to the semiclassical symbol (2.6). We content ourselves giving only the necessary definitions and refer the interested reader to [DL14c, Section 4] and references therein. Simply put, $\mathcal{D}p_{\lambda}$ maps $r$ onto $i\lambda \nabla_k$ and $k$ onto the
The formal expression (2.8) for $\mathcal{D}_p(f)$ needs to be interpreted properly: Assume $h_1$ and $h_2$ are Banach or Hilbert spaces; in our applications, they stand for $L^2(\mathbb{T}^3, C^6)$, $b_0$ and $\sigma$. We recall that $h_0$ is $L^2(\mathbb{T}^3, C^6)$ with scalar product weighted by $W^{-1}$ and $\sigma \subseteq h_0$ is the domain of the unperturbed fibered Maxwell operator endowed with the graph norm. On all these spaces the action of the multiplication operators $e^{i\gamma^* \cdot \cdot}$ are well defined. A function $f \in C^\infty(\mathbb{R}^6, B(h_1, h_2))$ is called equivariant if and only if

$$f(r, k - \gamma^*) = e^{i\gamma^* \cdot \cdot} f(r, k) e^{-i\gamma^* \cdot \cdot}$$  \ \ (A.1) holds for all $(r, k) \in \mathbb{R}^6$ and $\gamma^* \in \Gamma^*$. Operator-valued Hörmander symbols

$$S^m_\rho(B(h_1, h_2)) := \{ f \in C^\infty(\mathbb{R}^6, B(h_1, h_2)) \mid \forall \alpha, \beta \in \mathbb{N}_0^2 : \|f\|_{m, \alpha, \beta} < \infty\}$$

of order $m \in \mathbb{R}$ and type $\rho \in [0, 1]$ are defined through the usual seminorms

$$\|f\|_{m, \alpha, \beta} := \sup_{(r, k) \in \mathbb{R}^6} \left(\sqrt{1 + k^2}^{-m+|\rho|} \|\partial^\alpha_k \hat{f}(r, k)\|_{B(h_1, h_2)}\right), \quad \alpha, \beta \in \mathbb{N}_0^2,$$

where $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. The class of symbols $S^m_\rho(B(h_1, h_2))$ which satisfy the equivariance condition (2.7) are denoted with $S^m_{\rho, eq}(B(h_1, h_2))$; similarly, $S^m_{\rho, per}(B(h_1, h_2))$ is the class of $\Gamma^*$-periodic symbols, $f(r, k - \gamma^*) = f(r, k)$. Lastly, we introduce the notion of

**Definition A.1 (Semiclassical symbols)** Assume $h_j, j = 1, 2$, are Banach spaces as above. A map $f : [0, \lambda_0) \rightarrow S^m_{\rho, eq}(B(h_1, h_2))$, $\lambda \mapsto f$, is called an equivariant semiclassical symbol of order $m \in \mathbb{R}$ and weight $\rho \in [0, 1]$, that is $f \in AS^m_{\rho, eq}(B(h_1, h_2))$, if there exists a sequence $\{f_n\}_{n \in \mathbb{N}_0^3} \in S^m_{\rho, eq}(B(h_1, h_2))$, such that for all $N \in \mathbb{N}_0$, one has

$$\lambda^{-N} \left( f - \sum_{n=0}^{N-1} \lambda^n f_n \right) \in S^{m-N\rho}_{\rho, eq}(B(h_1, h_2))$$

uniformly in $\lambda$ in the sense that for any $a, b \in \mathbb{N}_0^3$, there exist constants $C_{a, b} > 0$ so that

$$\left\| f - \sum_{n=0}^{N-1} \lambda^n f_n \right\|_{m, a, b} \leq C_{a, b} \lambda^N$$

holds for all $\lambda \in [0, \lambda_0)$.

The Fréchet space of periodic semiclassical symbols $AS^m_{\rho, per}(B(h_1, h_2))$ is defined analogously.

For such symbols $\mathcal{D}_p(f) : S^m_{eq}(\mathbb{R}^3, h_1) \rightarrow S^m_{eq}(\mathbb{R}^3, h_2)$ makes sense as a linear, continuous map between equivariant distributions (cf. [DL14c, p. 90]), and under certain conditions on $f$ the restriction of $\mathcal{D}_p(f)$ to $L^2_{eq}(\mathbb{R}^3, h_1) \subset S^m_{eq}(\mathbb{R}^3, h_1)$ maps into

$$L^2_{eq}(\mathbb{R}^3, h_2) := \{ \Psi \in L^2_{loc}(\mathbb{R}^3, h_2) \mid \Psi(k - \gamma^*) = e^{i\gamma^* \cdot \cdot} \Psi(k) \ \text{a.e.} \ \forall \gamma^* \in \Gamma^* \}.$$
Another building block of pseudodifferential calculus is the Moyal product \( \sharp \) implicitly defined through \( \mathcal{O}_\sharp(f \sharp g) := \mathcal{O}_\sharp(f) \mathcal{O}_\sharp(g) \). It defines a bilinear continuous map

\[
\sharp : S^m_{p,\text{eq}}(B(h_1, h_2)) \times S^{m_2}_{p,\text{eq}}(B(h_2, h_3)) \longrightarrow S^{m_1+m_2}_{p,\text{eq}}(B(h_1, h_3))
\]

which has an asymptotic expansion

\[
f \sharp g = \sum_{n=0}^\infty \lambda^n (f \sharp g)_{(n)} = f g - \frac{\lambda}{2} \{f, g\} + \mathcal{O}(\lambda^2)
\]

where \( \{f, g\} := \sum_{i=1}^3 (\partial_{\tau_i} f \partial_{\tau_i} g - \partial_{\tau_i} f \partial_{\tau_i} g) \) is the usual Poisson bracket. Each term \((f \sharp g)_{(n)}(r, k)\) is a sum of products of derivatives of \( f \) and \( g \) evaluated at \((r, k)\).

For technical reasons, we need to distinguish between the oscillatory integral \( f \sharp g \) and the formal sum \( \sum_{n=0}^\infty \lambda^n (f \sharp g)_{(n)} \). As \( f \) and \( g \) are defined only on some common, open subset of \( \mathbb{R}^6 \), this formal sum also makes sense if \( f \) and \( g \) are defined only on some common, open subset of \( \mathbb{R}^6 \).

**B Computation of \( \pi_1 \) and \( f_{ro} \)**

**Proof (Lemma 3.7) Moyal projection** The terms of the projection are computed order-by-order from the projection and commutation defects which are responsible for the block-diagonal and block-off-diagonal contributions, respectively (cf. [PST03b, equations (4)–(8)]). As \( \pi_0 \) is a function of \( k \) only, the projection defect

\[
\lambda G_1 + \mathcal{O}(\lambda^2) = \pi_0 \pi_0 - \pi_0 = 0
\]

vanishes, and thus, also \( \pi_0^1 = 0 \).

The off-diagonal term is derived from the commutation defect

\[
\lambda F_1 + \mathcal{O}(\lambda^2) = \frac{1}{2} M \pi_0 + \mathcal{O}(\lambda^2)
\]

\[
= \lambda \left( \frac{1}{2} M + \frac{1}{2} \nabla \cdot \nabla M \right) \pi_0 + \mathcal{O}(\lambda^2)
\]

\[
= \lambda \left( \frac{1}{2} M + \frac{1}{2} \nabla \cdot \nabla M \right) \pi_0 + \mathcal{O}(\lambda^2)
\]

where we have used the abbreviation

\[
\Sigma_j := W \begin{pmatrix} 0 & e_j^x \\ e_j^y & 0 \end{pmatrix}.
\]
The offdiagonal part is now the sum of two terms,

\[
\pi_{10} = \pi_0 F_1 \pi_0^\dagger \left( \mathcal{M}_0 - \tau^2 \omega \right)^{-1} \pi_0^\dagger \pi_0 \left( \mathcal{M}_0 - \tau^2 \omega \right)^{-1} \pi_0 F_1 \pi_0 \\
= \tau^{-2} \pi_0 F_1 \pi_0^\dagger \left( \mathcal{M}_0(\cdot) - \omega \right)^{-1} \pi_0^\dagger + \tau^{-2} \pi_0^\dagger \left( \mathcal{M}_0(\cdot) - \omega \right)^{-1} \pi_0 F_1 \pi_0,
\]

and because the second term is the adjoint of the first, it suffices to look at only one of them. We first need to figure out the offdiagonal parts of \( F_1 \), and because it is purely offdiagonal, \( F_1 = \pi_0 F_1 \pi_0^\dagger + \pi_0^\dagger F_1 \pi_0 \), we can leave out one of the projections:

\[
\pi_0 F_1 = \tau^2 \sum_{j=1}^3 \left( -\partial_j \ln \frac{\tau}{\tau_0} \pi_0 \Sigma_j + 2 \partial_j \ln \tau \left[ \mathcal{M}_0(\cdot), \pi_0 \partial_k \pi_0 \right]_\pi \right)
\]

Let us compute each bit in turn: the \( \Sigma_j \) define selfadjoint operators on \( \mathfrak{h}_0 \), and hence,

\[
\pi_0 \Sigma_j = |\varphi\rangle \langle \Sigma_j \varphi|,
\]

while the term involving the anticommutator

\[
\left[ \mathcal{M}_0(\cdot), \pi_0 \partial_k \pi_0 \right]_\pi = \pi_0 \partial_k \pi_0 (\mathcal{M}_0(\cdot) + \omega)
\]

\[
= |\varphi\rangle \langle \varphi| \left( |\partial_k \varphi\rangle \langle \varphi| + |\varphi\rangle \langle \varphi| \partial_k \varphi \right) (\mathcal{M}_0(\cdot) + \omega)
\]

\[
= (|\varphi\rangle \langle \partial_k \varphi| + \langle \varphi| \partial_k \varphi \rangle \mathfrak{h}_0 \pi_0) (\mathcal{M}_0(\cdot) + \omega)
\]

Putting everything together, we obtain

\[
\pi_0 \pi_1 \pi_0^\dagger = \sum_{j=1}^3 \left( -\frac{1}{2} \partial_j \ln \frac{\tau}{\tau_0} |\varphi\rangle \langle \Sigma_j \varphi| + i \partial_j \ln \tau (|\varphi\rangle \langle \partial_k \varphi| + \\
+ \langle \varphi| \partial_k \varphi \rangle \mathfrak{h}_0 \pi_0) (\mathcal{M}_0(\cdot) + \omega) \right) \pi_0^\dagger (\mathcal{M}_0(\cdot) - \omega)^{-1} \pi_0^\dagger
\]

\[
= \sum_{j=1}^3 \left( -\frac{1}{2} \partial_j \ln \frac{\tau}{\tau_0} |\varphi\rangle \langle \Sigma_j \varphi| + i \partial_j \ln \tau |\varphi\rangle \langle \partial_k \varphi| \left( \mathcal{M}_0(\cdot) + \omega \right) \right)
\]

\[
\cdot \pi_0^\dagger (\mathcal{M}_0(\cdot) - \omega)^{-1} \pi_0^\dagger
\]

for one of the two contributions to \( \pi_1 = \pi_0 \pi_1 \pi_0^\dagger + (\pi_0 \pi_1 \pi_0^\dagger)^\dagger \).

**Ray optics observable** There are two types of terms in (3.12), two terms involving \( \pi_1 \) and two with Poisson brackets. Let us start with the former: since \( \pi_1 \) is completely offdiagonal,
we can compute the sum of the first two terms as
\[ \pi_1 f \pi_0 + \pi_0 f \pi_1 = \pi_0 \pi_1 \pi_0 f \pi_0 + \pi_0 f \pi_0 \pi_1 \pi_0 + \pi_0 \pi_1 \pi_0 f \pi_0 + \pi_0 f \pi_0 \pi_1 \pi_0 \]
\[ = \left( \langle \varphi, \pi_1 \pi_0 f \varphi \rangle_{h_0} + \langle \varphi, f \pi_0 \pi_1 \varphi \rangle_{h_0} \right) \pi_0 + \langle \varphi, f \varphi \rangle_{h_0} \pi_0 + \left( \langle \varphi, \pi_0 \pi_1 \pi_0 f \pi_0 + \pi_0 f \pi_0 \pi_1 \pi_0 \right) \]
\[ = \left( \langle \pi_0 \pi_1 \varphi, f \varphi \rangle_{h_0} + \langle \varphi, f \pi_0 \pi_1 f \pi_0 \rangle_{h_0} \right) \pi_0 + \langle \varphi, f \varphi \rangle_{h_0} \pi_1 \]
\[ = \langle \varphi, [f, \pi_1] \varphi \rangle_{h_0} \pi_0 + \langle \varphi, f \varphi \rangle_{h_0} \pi_1. \]

The only terms that remain are the two Poisson brackets,
\[ \{ \pi_0, f \} \pi_0 + \pi_0 \{ f, \pi_0 \} = \sum_{j=1}^{3} \left( \partial_{\psi_j} \pi_0 \partial_j f \pi_0 - \pi_0 \partial_j f \partial_j \pi_0 \right). \]

We insert
\[ \partial_{\psi_j} \pi_0 = \pi_0 \partial_{\psi_j} \pi_0 \pi_0 + \pi_0 \pi_0 \partial_{\psi_j} \pi_0 = \pi_0 \partial_{\psi_j} \pi_0 + \partial_{\psi_j} \pi_0 \pi_0 \]
into the above and compute
\[ \cdots = \pi_0 \partial_{\psi_1} \pi_0 \partial_j f \pi_0 + \partial_{\psi_2} \pi_0 \pi_0 \partial_j f \pi_0 - \pi_0 \partial_j f \pi_0 \partial_j \pi_0 - \pi_0 \partial_j f \partial_j \pi_0 \]
\[ = \left( \langle \varphi, \partial_{\psi_1} \pi_0 \partial_j f \varphi \rangle_{h_0} - \langle \varphi, \partial_j f \partial_{\psi_1} \pi_0 \rangle_{h_0} \right) \pi_0 + \langle \varphi, \partial_j f \pi_0 \partial_j \varphi \rangle_{h_0} \partial_{\psi_1} \pi_0 - \langle \varphi, \partial_j f \varphi \rangle_{h_0} \partial_{\psi_1} \pi_0 \]
\[ = \langle \varphi, [\partial_{\psi_1}, \pi_0, \partial_j f] \varphi \rangle_{h_0} \pi_0 + \langle \varphi, \partial_j f \pi_0 \partial_j \varphi \rangle_{h_0} \partial_{\psi_1} \pi_0 \]

where we have omitted the sum for brevity. Thus, the ray optics observable computes to
\[ f_{\text{ro}} = \langle \varphi, f \varphi \rangle_{h_0} \pi_0 + \lambda \left( \langle \varphi, [f, \pi_1] \varphi \rangle_{h_0} - \frac{1}{2} \langle \varphi, [\nabla_k \pi_0, \nabla f] \varphi \rangle_{h_0} \right) \pi_0 + \lambda \left( \langle \varphi, f \varphi \rangle_{h_0} \pi_1 - \frac{1}{2} \langle \varphi, \nabla f \varphi \rangle_{h_0} \cdot [\nabla_k \pi_0, \pi_0] \right) \]

where by definition \([\nabla_k \pi_0, \nabla f] := \nabla_k \pi_0 \cdot \nabla f - \nabla f \cdot \nabla_k \pi_0\). To obtain a simplified expression in case \(f = f^*\) takes values in the selfadjoint operators, we note that \(f = f^*\) implies \((\partial_j f)^* = \partial_j f\), and consequently, we obtain
\[ \langle \varphi, [\partial_{\psi_j}, \pi_0, \partial_j f] \varphi \rangle_{h_0} = \langle \varphi, [\partial_{\psi_j}, \varphi \partial_j f + \varphi \partial_j f \partial_{\psi_j}] \varphi \rangle_{h_0} + \langle \varphi, [\varphi \partial_j f \partial_{\psi_j}] \varphi \rangle_{h_0} \]
\[ = \langle \varphi, \partial_{\psi_j} \varphi \rangle_{h_0} \langle \varphi, \partial_j f \varphi \rangle_{h_0} - \langle \varphi, \partial_j f \partial_{\psi_j} \varphi \rangle_{h_0} + \langle \partial_{\psi_j} \varphi, \partial_j f \varphi \rangle_{h_0} \varphi \rangle_{h_0} \varphi \rangle_{h_0} + \langle \partial_{\psi_j} \varphi, \partial_j f \varphi \rangle_{h_0} \varphi \rangle_{h_0} \varphi \rangle_{h_0} \]
\[ = 2 \langle \varphi, \partial_{\psi_j} \varphi \rangle_{h_0} \langle \varphi, \partial_j f \varphi \rangle_{h_0} - 2 \Im \langle \varphi, \partial_j f \partial_{\psi_j} \varphi \rangle_{h_0}. \]
Thus, the commutator terms sum up to

\[
\{ \pi_0, f \} \pi_0 + \pi_0 \{ f, \pi_0 \} = \left( 2 \langle \varphi, \partial_k \varphi \rangle_{h_0} \langle \varphi, \partial_r f \varphi \rangle_{h_0} - i2 \text{Im} \langle \varphi, \partial_r f \varphi \rangle_{h_0} \right) \pi_0 + \\
+ \langle \varphi, \partial_r f \varphi \rangle_{h_0} \left[ \partial_k, \pi_0, \pi_0 \right],
\]

and overall, we yield the desired expression for

\[
f_{ro} = \langle \varphi, f \varphi \rangle_{h_0} \pi_0 + \lambda \left( \langle \varphi, [f, \pi_1] + \varphi \rangle_{h_0} \pi_0 + \langle \varphi, f \varphi \rangle_{h_0} \pi_1 + \\
- \frac{1}{2} \left( 2 \langle \varphi, \nabla_k \varphi \rangle_{h_0} \cdot \langle \varphi, \nabla_r f \varphi \rangle_{h_0} - i2 \text{Im} \langle \varphi, \nabla_r f \cdot \nabla_k \varphi \rangle_{h_0} \right) \pi_0 + \\
+ \langle \varphi, \nabla_r f \varphi \rangle_{h_0} \cdot [\nabla_k \pi_0, \pi_0] \right) \right) + \\
\lambda \left( \langle \varphi, f \varphi \rangle_{h_0} \pi_1 + \langle \varphi, \nabla_r f \varphi \rangle_{h_0} \cdot [\nabla_k \pi_0, \pi_0] \right).
\]

□

References

[APR13] G. Allaire, M. Palombaro, and J. Rauch. Diffraction of Bloch Wave Packets for Maxwell’s Equations. Commun. Contemp. Math. 15, 1–36, 2013.

[AZ97] A. Altland and M. R. Zirnbauer. Non-standard symmetry classes in mesoscopic normal-superconducting hybrid structures. Phys. Rev. B 55, 1142–1161, 1997.

[Ber82] E. E. Bergmann. Electromagnetic Propagation in Homogeneous Media with Hermitian Permeability and Permittivity. The Bell System Technical Journal 61, 935–948, 1982.

[DL11a] G. De Nittis and M. Lein. Applications of Magnetic ΨDO Techniques to SAPT – Beyond a simple review. Rev. Math. Phys. 23, 233–260, 2011.

[DL11b] G. De Nittis and M. Lein. Exponentially Localized Wannier Functions in Periodic Zero Flux Magnetic Fields. J. Math. Phys. 52, 112103, 2011.

[DL14a] G. De Nittis and M. Lein. Effective Light Dynamics in Perturbed Photonic Crystals. Commun. Math. Phys. 332, 221–260, 2014.

[DL14b] G. De Nittis and M. Lein. On the Role of Symmetries in Photonic Crystals. Annals of Physics 350, 568–587, 2014.

[DL14c] G. De Nittis and M. Lein. The Perturbed Maxwell Operator as Pseudodifferential Operator. Documenta Mathematica 19, 63–101, 2014.
References

[EG13] L. Esposito and D. Gerace. Topological aspects in the photonic crystal analog of single-particle transport in quantum Hall systems. Phys. Rev. A 88, 013853, 2013.

[FV06] A. Figotin and I. Vitebskiy. Frozen light in photonic crystals with degenerate band edge. Phys. Rev. E 74, 066613, 2006.

[KL01] P. Kuchment and S. Levendorskiï. On the Structure of Spectra of Periodic Elliptic Operators. Transactions of the American Mathematical Society 354, 537–569, 2001.

[Lon09] S. Longhi. Quantum-optical analogies using photonic structures. Laser & Photonics Reviews 3, 243–261, 2009.

[Mor00] A. Morame. The absolute continuity of the spectrum of Maxwell operator in periodic media. J. Math. Phys. 41, 7099–7108, 2000.

[OMN04] M. Onoda, S. Murakami, and N. Nagaosa. Hall Effect of Light. Phys. Rev. Lett. 93, 083901, 2004.

[OMN06] M. Onoda, S. Murakami, and N. Nagaosa. Geometrical aspects in optical wave-packet dynamics. Phys. Rev. E 74, 066610, 2006.

[PST03a] G. Panati, H. Spohn, and S. Teufel. Effective dynamics for Bloch electrons: Peierls substitution. Commun. Math. Phys. 242, 547–578, 2003.

[PST03b] G. Panati, H. Spohn, and S. Teufel. Space-Adiabatic Perturbation Theory. Adv. Theor. Math. Phys. 7, 145–204, 2003.

[Per00] V. Perlick. Ray Optics, Fermat’s Principle, and Applications to General Relativity. Vol. 61. Lecture Notes in Physics. Springer-Verlag, 2000.

[PNH+07] R. N. C. Pfeifer, T. A. Nieminen, N. R. Heckenberg, and H. Rubinsztein-Dunlop. Colloquium: Momentum of an electromagnetic wave in dielectric media. Rev. Mod. Phys. 79, 1197–1216, 2007.

[Poz98] D. M. Pozar. Microwave Engineering. Wiley, 1998.

[RH08] S. Raghu and F. D. M. Haldane. Analogs of quantum-Hall-effect edge states in photonic crystals. Phys. Rev. A 78, 033834, 2008.

[RG09] A. Rangarajan and K. S. Gurmooorthy. A Schrödinger Wave Equation Approach to the Eikonal Equation: Application to Image Analysis. In: Energy Minimization Methods in Computer Vision and Pattern Recognition. Ed. by D. Cremers, Y. Boykov, A. Blake, and F. Schmidt. Vol. 5681. Lecture Notes in Computer Science. Springer Berlin Heidelberg, 2009, 140–153.

[RZP+13] M. C. Rechtsman, J. M. Zeuner, Y. Plotnik, Y. Lumer, D. Podolsky, F. Dreisow, S. Nolte, M. Segev, and A. Szameit. Photonic Floquet topological insulators. Nature 496, 196–200, 2013.
References

[Rob87] D. Robert. Autour de l’Approximation Semi-Classique. Birkhäuser, 1987.

[SRF+08] A. P. Schnyder, S. Ryu, A. Furusaki, and A. W. W. Ludwig. Classification of topological insulators and superconductors in three spatial dimensions. Phys. Rev. B 78, 195125, 2008.

[STZ99] K. Siddiqi, A. Tannenbaum, and S. W. Zucker. A Hamiltonian Approach to the Eikonal Equation. In: Energy Minimization Methods in Computer Vision and Pattern Recognition. Ed. by E. Hancock and M. Pelillo. Vol. 1654. Lecture Notes in Computer Science. Springer-Verlag, 1999, 1–13.

[Som98] C. G. Someda. Electromagnetic Waves. CRC Press Inc, 1998.

[SP14] J. Song and E. Prodan. $\text{AIII and BDI topological systems at strong disorder}$. Phys. Rev. B 89, 224203, 2014.

[ST13] H.-M. Stiepan and S. Teufel. Semiclassical approximations for Hamiltonians with operator-valued symbols. Commun. Math. Phys. 320, 821–849, 2013.

[SN99] G. Sundaram and Q. Niu. Wave-packet dynamics in slowly perturbed crystals: Gradient corrections and Berry-phase effects. Phys. Rev. B 59, 14915–14925, 1999.

[Sus00] T. Suslina. Absolute continuity of the spectrum of periodic operators of mathematical physics. Journées Équations aux dérivées partielles 2000, 1–13, 2000.

[Teu03] S. Teufel. Adiabatic Perturbation Theory in Quantum Dynamics. Vol. 1821. Lecture Notes in Mathematics. Springer-Verlag, 2003.

[WCJ+08] Z. Wang, Y. D. Chong, J. D. Joannopoulos, and M. Soljačić. Reflection-Free One-Way Edge Modes in a Gyromagnetic Photonic Crystal. Phys. Rev. Lett. 100, 013905, 2008.