SECONDARY HOMOTOPY GROUPS

HANS-JOACHIM BAUES AND FERNANDO MURO

Abstract. Secondary homotopy groups supplement the structure of classical homotopy groups. They yield a 2-functor on the groupoid-enriched category of pointed spaces compatible with fiber sequences, suspensions and loop spaces. They also yield algebraic models of \((n-1)\)-connected \((n+1)\)-types for \(n \geq 0\).

Introduction

The computation of homotopy groups of spheres in low degrees in [Tod62] uses heavily secondary operations termed Toda brackets. Such bracket operations are defined by pasting tracks where a track is a homotopy class of homotopies. Since Toda brackets play a crucial role in homotopy theory it seems feasible to investigate the algebraic nature of tracks. Therefore we shift focus from homotopy groups \(\pi_n X\) to secondary homotopy groups

\[ \pi_{n,*} X = (\pi_{n,1} X \xrightarrow{\partial} \pi_{n,0} X) \]

defined in this paper. Here \(\partial\) is a homomorphism of groups with \(\text{Coker} \ \partial = \pi_n X\) and \(\text{Ker} \ \partial = \pi_{n+1} X\), \(n \geq 1\).

The adjective “secondary” in the title complements the word “group”, so secondary homotopy groups are secondary groups appearing in homotopy theory. The words “secondary groups” stand for a variety of dimension 2 generalizations of the notion of group, like Whitehead’s crossed modules [Whi49], or reduced or stable quadratic modules in the sense of [Bau91]. There is not a well-established terminology in the literature to designate all these 2-dimensional “groups” and we believe that “secondary groups” has the advantage of being new (so no confusion with older concepts is created) and short.

The groups \(\pi_{n,0} X\) and \(\pi_{n,1} X\) are defined directly by use of continuous maps \(f: S^n \to X\) and tracks of such maps to the trivial map, so that \(\pi_{n,*} X\) is actually a functor in \(X\). For \(n \geq 2\) the definition involves the new concept of Hopf invariant for tracks.

We show that the homomorphism \(\partial\) has additional algebraic structure, namely \(\pi_{1,*} X\) is a crossed module, \(\pi_{2,*} X\) is a reduced quadratic module and \(\pi_{n,*} X\), \(n \geq 3\), is a stable quadratic module.

Crossed modules were introduced by J. H. C. Whitehead in [Whi49] and, in fact, for a reduced \(CW\)-complex \(X\) with 1-skeleton \(X^1\) our secondary homotopy group

1991 Mathematics Subject Classification. 18D05, 55Q25, 55S45.
Key words and phrases. secondary homotopy groups, groupoid-enriched category, crossed module, reduced (stable) quadratic module, Hopf invariant of tracks.

The second author was partially supported by the project MTM2004-01865 and the MEC postdoctoral fellowship EX2004-0616.

1
\( \pi_{1,*}X \) is weakly equivalent to the crossed module

\[ \pi_2(X, X^1) \longrightarrow \pi_1 X^1 \]

studied by [Whi49]. Similarly if \( X \) is an \((n - 1)\)-reduced \( CW \)-complex then \( \pi_{n, *} X \) for \( n \geq 2 \) is weakly equivalent to the quadratic modules obtained in [Bau91] in terms of the cell structure of \( X \).

The topological and functorial definition of secondary homotopy groups \( \pi_{n,*} X \) in this paper is crucial to understand new properties of the corresponding concepts in the literature. For example, we are able to determine the algebraic properties of the loop and suspension operators on secondary homotopy groups. As main new results, we describe the fiber sequence for secondary homotopy groups, and we show that secondary homotopy groups form a 2-functor on the groupoid-enriched category of pointed spaces. We also prove that secondary homotopy groups are algebraic models for two-stage spaces, i.e. spaces with only two non-trivial homotopy groups in consecutive dimensions. Such two-stage spaces generalize Eilenberg-MacLane spaces, which are spaces with a single non-trivial homotopy group.

The crucial topological tool that we introduce in this paper for the construction of the secondary homotopy groups is the Hopf invariant for tracks. This very basic piece of homotopy theory does not seem to have been considered before. It generalizes the classical Hopf invariant for maps. In this paper we only need to compute Hopf invariants for tracks between one-point unions of spheres. However we believe that a notion of Hopf invariant for more general tracks would deserve to be studied elsewhere.

Hopf invariant computations for tracks are the basis of the applications obtained in the sequels of this paper, see [BM05b, BM05a, BM06]. These computations are achieved by using geometric tools such as orthogonal group actions and Clifford algebras. This stresses the relevance of the definition of secondary homotopy groups by topological means.

In [BM05a] we determine the algebraic nature of smash product operations on secondary homotopy groups. For the (stable) secondary homotopy groups of spectra this leads in [BM06] to an algebraic “tensor product” approximating the smash product of spectra on a secondary level.

The computation of the algebra of secondary cohomology operations in [Bau06] shows examples where secondary homotopy groups can be algebraically determined successfully. It is the aim of the authors to generalize the theory of [Bau06], concerning the Eilenberg-MacLane spectrum, for general spectra.

Moreover, we will discuss in a sequel of this paper generalized Whitehead products for secondary homotopy groups. In fact, J. H. C. Whitehead introduced in [Whi41] Whitehead products as an additional algebraic structure of homotopy groups. We may consider the secondary homotopy groups together with their algebraic properties also as such an enriching structure.

**Further connections with the literature.** The secondary groups considered in this paper (i.e. crossed modules and reduced or stable quadratic modules) are equivalent, at least up to homotopy, to some other algebraic structures in the literature, such as categorical groups, which can also be braided or symmetric. There are strict and non-strict categorical groups. Strict algebraic objects, like the secondary groups we use, are often more convenient to work with. Garzón, Miranda and del Río [GMdR02] endow the fundamental groupoid of an iterated loop space with
the structure of a categorical group, which can be braided or symmetric according to the number of loopings. Our secondary homotopy groups could be regarded as strictifications of these. Indeed the objects of the categorical groups in [CMR12] are the same as the generators of our secondary homotopy groups. The connection between the secondary groups that we use and categorical groups can be established through Conduché’s 2-crossed modules [Con84], see [BCC93], [BC91] and [BC97].

More precisely, the equivalence between categorical groups and 2-crossed modules can be found in [BCC93], [BC91] where the authors define strict “homotopy categorical groups” for simplicial groups. Given a pointed space $X$ one can consider the Kan loop group $GS\bullet X$ on the reduced singular simplicial set of the pointed space $X$. The categorical groups defined in [BCC93], [BC91] applied to $GS\bullet X$ are models for two-stage spaces. Since the same result holds for our secondary homotopy groups there must be a “weak equivalence” between both constructions. The definition of an explicit equivalence is however out of the scope of this paper.

The connection between 2-crossed modules and secondary groups is established in [BC97], where the authors show that quadratic modules are highly connected 2-crossed modules. Indeed they have nilpotency degree 2, so they are just one step away from being abelian. This yields an advantage of secondary homotopy groups over the categorical groups constructed in [BCC93] and [BC91]. In addition our construction does not rely on the Kan loop group $GS\bullet X$. On the contrary it is purely topological since it uses continuous maps and tracks.

**Acknowledgement.** The authors are very grateful to the referee for his careful reading of the paper and for many valuable suggestions.

1. **Tracks between maps**

We consider the category $\textbf{Top}^\ast$ of compactly generated pointed spaces $X = (X, \ast)$ and pointed maps $f: X \to Y$. For any (unpointed) space $X$ we define $X_+ = X \cup \{\ast\}$ as the same space with an outer base-point $\ast$. The smash product of two pointed spaces is defined by

$$X \wedge Y = (X \times Y)/(X \times \ast \cup \ast \times Y).$$

It is associative and commutative, since it defines a symmetric monoidal structure on the category $\textbf{Top}^\ast$.

Homotopies $IX \to Y$ are defined by using the reduced cylinder $IX = I_+ \wedge X$, where $I = [0, 1]$ is the unit interval, with structure maps

$$X \vee X \xrightarrow{i} IX \xrightarrow{p} X.$$

Here $\vee$ is the symbol for the coproduct, $i$ is the inclusion of the boundary and $p$ is the projection. Given two maps $f, g: X \to Y$ a track $H: f \Rightarrow g$ is a homotopy class of homotopies $IX \to Y$, from $f$ to $g$, relative to the boundary. By abuse of language we denote a homotopy and the represented track by the same symbol. In diagrams tracks will be denoted as follows.

$$X \xrightarrow{f} Y \quad \text{with} \quad H$$
The trivial track $0_f^\square \colon f \Rightarrow f$ is represented by $fp \colon IX \to Y$ and the inverse of a track $H \colon f \Rightarrow g$ is $H^\square \colon g \Rightarrow f$. The vertical composition of tracks

\[
\begin{array}{c}
\xymatrix{X \ar[r]^f \ar[d]_h \ar@/^1pc/[rr]^{H} & Y \\
& Y \ar[u]_h &}
\end{array}
\]

is defined by pasting homotopies representing $H$ and $K$ and is denoted by

\[
\begin{array}{c}
\xymatrix{X \ar@/_1pc/[rr]_H \ar[r]^f \ar[d]_h & Y \ar[u]_h & \\
& K \ar[u]_H &}
\end{array}
\]

One can also compose horizontally a track as in diagram (1.2) with maps $k \colon W \to X$ and $l \colon Y \to Z$ to obtain tracks $Hk \colon fk \Rightarrow gk$ and $lH \colon lf \Rightarrow lg$ in the obvious way. If we have a diagram like

\[
\begin{array}{c}
\xymatrix{X \ar@/_1pc/[rr]^f \ar[r]^{f'} \ar[d]_g & Y \ar[r]^{g'} & Z \\
& Y \ar[u]_g &}
\end{array}
\]

the equality

\[(g'H)(Hf) = (H'g)(f'H)\]

holds and this element is the horizontal composition of $H$ and $H'$ denoted by juxtaposition

\[
\begin{array}{c}
\xymatrix{X \ar@/^1pc/[rr]^{f'H} \ar[r]^{f'} \ar[d]_g & Z \ar[u]_g & \\
& Y \ar[u]_g &}
\end{array}
\]

Tracks endow $\text{Top}^*$ with the structure of a groupoid-enriched category.

Recall that a groupoid-enriched category $\mathbf{C}$ is the same as a 2-category where all 2-morphisms are vertically invertible. In this paper we will work with some other groupoid-enriched categories, apart from $\text{Top}^*$. We will always use the same notation as above for 2-morphisms, identity 2-morphisms (also called trivial 2-morphisms), vertical and horizontal composition, and vertical inverses. A morphism $f \colon X \to Y$ in $\mathbf{C}$ has an automorphism group $\text{Aut}_\mathbf{C}(f)$ given by the set of 2-morphisms $f \Rightarrow f$ and the vertical composition. The homotopy category of $\mathbf{C}$ is the ordinary category $\mathbf{C}/\simeq$ obtained by identifying two morphisms $f, g \colon X \to Y$ in $\mathbf{C}$ provided there exists a 2-morphism between them $H \colon f \Rightarrow g$. For instance if $\mathbf{C} = \text{Top}^*$ then the homotopy category is the usual one. A 2-functor $\varphi \colon \mathbf{C} \to \mathbf{D}$ between groupoid-enriched categories with the same objects which is the identity on objects is said to be a weak equivalence if $\varphi$ induces an isomorphism between the automorphism groups $\text{Aut}_\mathbf{C}(f) \cong \text{Aut}_\mathbf{D}(\varphi(f))$ for any morphism $f$ in $\mathbf{C}$ and an isomorphism between the homotopy categories $\mathbf{C}/\simeq \cong \mathbf{D}/\simeq$. 
Remark 1.3. The groupoid-enriched category $\textbf{Top}^*$ has a strict zero object, the one-point space $\ast$. In particular zero maps are defined. Such a groupoid-enriched category has the crucial property that any 2-morphism composed with a zero map becomes automatically an identity 2-morphism.

Maps from a coproduct $X \vee Y$ in $\textbf{Top}^*$ are given by pairs of maps $(f_1, f_2): X \vee Y \to Z$. Similarly a track $H: (f_1, f_2) \Rightarrow (g_1, g_2)$ between maps $(f_1, f_2), (g_1, g_2): X \vee Y \to Z$ is given by a pair of tracks $H = (H_1, H_2)$ with $H_i: f_i \Rightarrow g_i$ ($i = 1, 2$). This means that the one-point union of pointed spaces is an enriched coproduct in $\textbf{Top}^*$.

The suspension $\Sigma X$ is the quotient space $IX/(X \vee X) = S^1 \wedge X$. It defines a 2-functor

$$\Sigma: \textbf{Top}^* \to \textbf{Top}^*.$$ 

The suspension of a track $H: f \Rightarrow g$ between maps $f, g: X \to Y$ represented by a homotopy $H: IX \to Y$ is the track $\Sigma H: \Sigma f \Rightarrow \Sigma g$ represented by the homotopy

$$I\Sigma X = I_+ \wedge S^1 \wedge X \cong S^1 \wedge I_+ \wedge X \xrightarrow{S^1 \wedge H} S^1 \wedge Y = \Sigma Y.$$ 

We will use the identifications

$$(1.4) \quad \Sigma(X \vee Y) = \Sigma X \vee \Sigma Y, \quad \Sigma^n S^0 = S^n, \quad n \geq 0.$$ 

For the definition of homotopy groups we choose a particular co-H-group structure on $S^1$ given by maps $\mu: S^1 \to S^1 \vee S^1$ and $\nu: S^1 \to S^1$ satisfying the usual properties. We use explicitly these maps in many constructions throughout this paper, however these constructions do not depend on this choice since the maps $\mu$ and $\nu$ are unique up to a canonical track.

The loop space functor $\Omega$ is the right-adjoint of the suspension $\Sigma$. The adjoint of a map $f: \Sigma X \to Y$ is denoted by $ad(f): X \to \Omega Y$. The adjoint of the identity map $1: \Sigma X \to \Sigma X$ is a natural inclusion

$$(1.5) \quad ad(1): X \to \Omega \Sigma X.$$ 

As a pointed set the $n$-fold loop space $\Omega^n X$ is the set of pointed maps $S^n \to X$ and the base-point corresponds to the trivial map. By using the interchange homeomorphism of the smash product we see that suspensions and cylinders commute up to natural isomorphism in $\textbf{Top}^*$, $I\Sigma X \cong IIX$. However one has to be careful with signs because the interchange of factors in $S^1 \wedge S^1 = S^2$ induces $-1$ on the homotopy group $\pi_2$.

2. Groups of nilpotency degree 2

Consider the forgetful functor from groups to pointed sets $\textbf{Gr} \to \textbf{Set}^*$. This functor has a left adjoint

$$\langle \cdot \rangle: \textbf{Set}^* \to \textbf{Gr}: A \mapsto \langle A \rangle.$$ 

Here $\langle A \rangle$ is the quotient of the free group with basis $A$ by the normal subgroup generated by the base-point $* \in A$. This group is isomorphic to the free group with basis $A - \{*\}$. We denote $\vee_A S^1 = \Sigma A$. As usual we identify the fundamental group of $\vee_A S^1$ with a free group, i.e. $\pi_1(\vee_A S^1) = \langle A \rangle$. The free group of nilpotency class 2 (free nil-group for short), generated by the pointed set $A$, is the quotient

$$\langle A \rangle_{nil} = \frac{\langle A \rangle}{\Gamma_3 \langle A \rangle}.$$
where $\Gamma_3(A)$ is the $3^{rd}$ term of the lower central series of $\langle A \rangle$, i.e. the subgroup generated by triple commutators $[x, [y, z]]$ ($x, y, z \in \langle A \rangle$). In this paper we always write group laws additively, even for non-abelian groups, so that the commutator is $[x, y] = -x - y + x + y$. The free abelian group $\mathbb{Z}[A]$ on a pointed set $A$ is the abelianization of $\langle A \rangle$ and of $\langle A \rangle_{\text{nil}}$.

If $\text{gr}$, $\text{nil}$ and $\text{ab}$ are the categories of free groups, free nil-groups and free abelian groups, respectively, then there are obvious nilization and abelianization functors

$$\begin{array}{ccc}
\text{gr} & \xrightarrow{\text{ab}} & \text{ab} \\
\downarrow \text{nil} & & \downarrow \text{ab} \\
\text{nil} & & \langle A \rangle_{\text{nil}}
\end{array}$$

Let $\langle A \rangle_{\text{nil}} \to \mathbb{Z}[A]$ be the natural projection carrying $x$ to $\{x\}$. Since the commutator bracket in $\langle A \rangle_{\text{nil}}$ is bilinear the homomorphism

$$\partial : \otimes^2 \mathbb{Z}[A] \to \langle A \rangle_{\text{nil}}, \quad \partial(\{x\} \otimes \{y\}) = [x, y],$$

is well defined. Here the tensor square of an abelian group $A$ is denoted by $\otimes^2 A = A \otimes A$. This fact is relevant since it allows us to define a groupoid enrichment of $\text{nil}$ which is connected to the tracks between one-point unions of spheres through the Hopf invariant, see the next section. This connection and its consequences are crucial for the definition of secondary homotopy groups.

Let $T : A \otimes B \to B \otimes A$ be the interchange isomorphism $T(a \otimes b) = b \otimes a$. The reduced tensor square is the following cokernel

$$\otimes^2 A \xrightarrow{1 + T} \otimes^2 A \xrightarrow{\sigma} \otimes^2 A.$$

We denote $\sigma(a \otimes b) = a \hat{\otimes} b$. We define the functor $\otimes_n^2$ as

$$\otimes_n^2 = \begin{cases} 
\otimes^2, & \text{if } n = 2; \\
\otimes^3, & \text{if } n \geq 3.
\end{cases}$$

Here we write $a \otimes b \in \otimes_n^2 A$ with $a \otimes b = a \hat{\otimes} b$ for $n \geq 3$. Moreover, $\Gamma_n$ is the functor

$$\Gamma_n = \begin{cases} 
\Gamma, & \text{if } n = 2; \\
- \otimes \mathbb{Z}/2, & \text{if } n \geq 3;
\end{cases}$$

where $- \otimes \mathbb{Z}/2$ is the ordinary tensor product of abelian groups and $\Gamma$ is Whitehead’s universal quadratic functor, see [W50]. There is a natural exact sequence

$$\begin{array}{cccc}
\Gamma_n \mathbb{Z}[A] & \to & \otimes^2 \mathbb{Z}[A] & \xrightarrow{\partial} & \langle A \rangle_{\text{nil}} \to & \mathbb{Z}[A]
\end{array}$$

Here the first arrow is induced by the function sending $x \in \mathbb{Z}[A]$ to $x \otimes x \in \otimes^2 \mathbb{Z}[A]$, see for example [Bau91]. Moreover, these exact sequences fit into a natural commutative diagram

$$\begin{array}{cccc}
\Gamma \mathbb{Z}[A] & \xrightarrow{\sigma} & \otimes^2 \mathbb{Z}[A] & \xrightarrow{\partial} & \langle A \rangle_{\text{nil}} \to & \mathbb{Z}[A] \\
\downarrow & & \downarrow & & \downarrow & \\
\mathbb{Z}[A] \otimes \mathbb{Z}/2 & \xrightarrow{\sigma} & \otimes^2 \mathbb{Z}[A] & \xrightarrow{\partial} & \langle A \rangle_{\text{nil}} \to & \mathbb{Z}[A]
\end{array}$$
where the upper $\partial$ is the homomorphism in $B$ above, and the lower $\partial$ is a factorization of $A$ through the reduced tensor square.

3. Nil-tracks and Hopf invariants of tracks

In this section we introduce a new homotopy invariant which is crucial for the definition of secondary homotopy groups and for the further development of this theory. It is a Hopf invariant for tracks which generalizes the most classical Hopf invariant for maps, see below. In this paper we only consider the Hopf invariant of tracks between one-point unions of spheres. However it would be interesting to develop a more general notion.

Definition 3.1. Let $f, g$ be maps $f, g: S^1 \to \vee^A S^1$ where $A$ is a discrete pointed set, and let

$$\Sigma^{n-1}f, \Sigma^{n-1}g: S^n \to \vee^A S^n$$

be their $(n-1)$-fold suspensions, $n \geq 1$. A track $H: \Sigma^{n-1}f \Rightarrow \Sigma^{n-1}g$, represented by a homotopy $H: IS^n \to \vee^A S^n$, is said to be a nil-track if the adjoint

$$ad(H): IS^1 \longrightarrow \Omega^{n-1}(\vee^A S^n)$$

of the map

$$S^{n-1} \wedge I_+ \wedge S^1 \cong I_+ \wedge S^{n-1} \wedge S^1 = IS^n \xrightarrow{H} \vee^A S^n.$$ 

induces a trivial homomorphism

$$0 = H_2 ad(H): H_2(IS^1, S^1 \vee S^1) \longrightarrow H_2(\Omega^{n-1}(\vee^A S^n), \vee^A S^1).$$

The adjoint of $H$ sends the boundary of the cylinder $IS^1$ into $\vee^A S^1$ since $H$ restricted to the boundary is an $(n-1)$-fold suspension. Of course for $n = 1$ all tracks $H$ above are nil-tracks since $H_2 ad(H)$ maps to the trivial group.

Let $f, g$ be now maps between wedges of 1-spheres $f, g: \vee^B S^1 \to \vee^A S^1$, and let $\Sigma^{n-1}f, \Sigma^{n-1}g$ be their $(n-1)$-fold suspensions. A track $H: \Sigma^{n-1}f \Rightarrow \Sigma^{n-1}g$ is a nil-track if all restricted tracks $H_{ib}$ are nil-tracks where $i_b: S^n \to \vee^B S^n$ is the inclusion given by $b \in B - \{\ast\}$.

The homology groups involved in the definition of nil-tracks are computable. Indeed,

$$H_2(IS^1, S^1 \vee S^1) \cong H_2(\Sigma S^1) = H_2 S^2 \cong \mathbb{Z}.$$ 

Moreover,

$$H_2(\Omega^{n-1}(\vee^A S^n)) \cong H_2(\Omega^{n-1}(\vee^A S^n), \vee^A S^1)$$

is an isomorphism, and the Pontrjagin product

$$\otimes^2 \mathbb{Z}[A] = H_1(\Omega^{n-1}(\vee^A S^n)) \otimes H_1(\Omega^{n-1}(\vee^A S^n)) \longrightarrow H_2(\Omega^{n-1}(\vee^A S^n))$$

is an isomorphism for $n = 2$ and induces an isomorphism for $n \geq 2$

$$\otimes^2 \mathbb{Z}[A] \cong H_2(\Omega^{n-1}(\vee^A S^n)), \tag{3.2}$$

compare notation in [2,3].

Definition 3.3. Let $n \geq 2$. Given a track $H: \Sigma^{n-1}f \Rightarrow \Sigma^{n-1}g$ for maps $f, g: S^1 \to \vee^A S^1$ the Hopf invariant of $H$ is defined as

$$\text{Hopf}(H) = (H_2 ad(H))(1) \in \otimes^2 \mathbb{Z}[A],$$
where we apply the homology functor $H_2$ as in Definition 3.1. In particular, $H$ is a nil-track if and only if \( \text{Hopf}(H) = 0 \). More generally, if $H: \Sigma^{n-1} f \to \Sigma^{n-1} g$ is a track for maps $f, g: \vee_B S^1 \to \vee_A S^1$, the Hopf invariant of $H$ is the homomorphism

\[
\text{Hopf}(H): \mathbb{Z}[B] \to \otimes^n_2 \mathbb{Z}[A]
\]

defined by $\text{Hopf}(H)(b) = \text{Hopf}(H_ib)$, where $i_b: S^1 \subset \vee_B S^1$ is the inclusion of the factor corresponding to $b \in B - \{\ast\}$. Such a track $H$ is a nil-track if and only if $\text{Hopf}(H) = 0$.

In case $n = 1$ we have $\text{Hopf}(H) = 0$ for any track $H$ as above.

**Remark 3.4.** Any element $x \in \pi_3 \vee_A S^2$ determines a track $x: 0 \to 0$ for the trivial map $0: S^2 \to \vee_A S^2$. This track is given by the homotopy $I S^2 \to \Sigma S^2 = S^3 \to \vee_A S^2$, where the first map is the obvious projection. The reader can check that $-\text{Hopf}(x)$ is the classical Hopf invariant of $x$. The sign is due to the fact that in order to define the Hopf invariant of $x$ as a track we need to consider the map $I_+ \wedge S^1 \wedge S^1 \cong S^1 \wedge I_+ \wedge S^1$ interchanging the first two factors of the smash product and this map induces $-1: S^3 \to S^3$ up to homotopy.

The next results are crucial for this paper.

**Theorem 3.5.** Let $f, g: \vee_A S^1 \to \vee_B S^1$ be two maps and $n \geq 1$. If a nil-track

\[
N_{f,g}: \Sigma^{n-1} f \Rightarrow \Sigma^{n-1} g
\]

exists then it is unique. Moreover, $N_{f,g}$ exists if and only if

- $\pi_1 f = \pi_1 g: \langle A \rangle \to \langle B \rangle$, if $n = 1$;
- or $(\pi_1 f)_{\text{nil}} = (\pi_1 g)_{\text{nil}}: \langle A \rangle_{\text{nil}} \to \langle B \rangle_{\text{nil}}$, if $n \geq 2$.

Furthermore, trivial tracks are nil-tracks and the vertical and horizontal composition of nil-tracks are also nil-tracks.

The case $n = 1$ is obvious since a one-point union of circles is aspherical, so a track between maps $f, g: \vee_A S^1 \to \vee_B S^1$ is necessarily unique provided it exists, it is always a nil-track as we have noticed above, and it is well-known that $H$ exists if and only if $f$ and $g$ induce the same homomorphism on fundamental groups. For $n \geq 2$ this theorem is an immediate consequence of the following one.

**Theorem 3.6.** Let $n \geq 2$ and let $f, g: \vee_A S^1 \to \vee_B S^1$ be maps such that for any $x \in \langle A \rangle_{\text{nil}}$ we have $(\pi_1 g)_{\text{nil}}(x) = (\pi_1 f)_{\text{nil}}(x) + \partial \alpha(x)$ for some homomorphism $\alpha: \mathbb{Z}[A] \to \otimes^n_2 \mathbb{Z}[B]$. Then there exists a unique track $H: \Sigma^{n-1} f \Rightarrow \Sigma^{n-1} g$ with Hopf invariant $\text{Hopf}(H) = \alpha$ and conversely. Moreover, the Hopf invariant of tracks satisfies the following formulas. Given a diagram

\[
\begin{array}{ccc}
\Sigma^{n-1} f & \Rightarrow & \Sigma^{n-1} g \\
\vee_A S^n & \Rightarrow & \vee_B S^n \\
\Sigma^{n-1} h & \Rightarrow & \Sigma^{n-1} h
\end{array}
\]

the equation

1. $\text{Hopf}(K \square H) = \text{Hopf}(K) + \text{Hopf}(H)$ holds.
Furthermore, if we consider the diagram

\[ \begin{array}{ccc}
\vee_A S^n & \xrightarrow{\Sigma^n-1 f} & \vee_B S^n \\
\downarrow H & & \downarrow H \\
\vee_C S^n & \xrightarrow{\Sigma^n-1 g} & \vee_D S^n
\end{array} \]

then

1. \( \text{Hopf}(H(\Sigma^{n-1}k)) = \text{Hopf}(H)(\pi_1 k)_{ab}, \)
2. \( \text{Hopf}(H(\Sigma^{n-1}h)) = (\otimes^2 H)_{ab} \text{Hopf}(H). \)

In addition given a track \( H : \Sigma^{n-1} f \Rightarrow \Sigma^{n-1} g \) between maps \( f, g : \vee_A S^1 \rightarrow \vee_B S^1 \) one gets the following equations.

3. \( \text{Hopf}(\Sigma H) = 0 \) if \( n = 1 \),
4. \( \text{Hopf}(\Sigma H) = \sigma \text{Hopf}(H) \) if \( n = 2 \),
5. \( \text{Hopf}(\Sigma H) = \text{Hopf}(H) \) if \( n \geq 3 \).

This theorem is a simple consequence of the theory developed in [Bau91] VI.3.13 and the fact that one-point unions of 1-spheres do not have higher-dimensional homotopy groups.

Let \( \mathbf{S}(n) \subset \mathbf{Top}^* \) be the full groupoid-enriched subcategory of one-point unions of \( n \)-spheres and let \( \mathbf{gr} \) be the category of free groups regarded as a groupoid-enriched category with only the trivial 2-morphisms. Then there is a 2-functor

\[ \pi_1 : \mathbf{S}(1) \rightarrow \mathbf{gr} \]

given by the fundamental group, \( \pi_1(\vee_A S^1) = \langle A \rangle \). This 2-functor is a weak equivalence. This follows easily from [Bau91] VI.3.13 and the fact that one-point unions of 1-spheres do not have higher-dimensional homotopy groups.

For \( n \geq 2 \) we consider the groupoid-enriched subcategory \( \bar{\mathbf{S}}(n) \subset \mathbf{S}(n) \) of suspended maps. Here objects of \( \mathbf{S}(n) \) are one-point unions of 1-spheres \( \vee_A S^1 \), maps \( f, g : \vee_A S^1 \rightarrow \vee_B S^1 \) in \( \mathbf{S}(n) \) are maps in \( \mathbf{Top}^* \) and 2-morphisms \( H : f \Rightarrow g \) in \( \bar{\mathbf{S}}(n) \) are tracks \( H : \Sigma^{n-1} f \Rightarrow \Sigma^{n-1} g \) in \( \mathbf{Top}^* \). The inclusion

\[ \Sigma^{n-1} : \bar{\mathbf{S}}(n) \subset \mathbf{S}(n) \]

is given by the \((n-1)\)-fold suspension on objects and morphisms and it is the identity on 2-morphisms. This is actually a weak equivalence of groupoid-enriched categories. See [Bau91] VI.4.7.

We now consider the algebraic groupoid-enriched category \( \mathbf{nil}(n) \) defined as follows. Objects and morphisms are the same as in \( \mathbf{nil} \). A 2-morphism \( \alpha : \varphi \Rightarrow \psi \) between homomorphisms \( \varphi, \psi : \langle A \rangle_{ab} \rightarrow \langle B \rangle_{ab} \) is a homomorphism \( \alpha : \mathbb{Z}[A] \rightarrow \otimes^n \mathbb{Z}[B] \) such that \( \varphi(x) + \partial \{x\} = \psi(x) \) for any \( x \in \langle A \rangle_{ab} \), i.e. a 2-morphism \( \alpha : \varphi \Rightarrow \psi \) can only exist if the abelianizations coincide \( \varphi_{ab} = \psi_{ab} \), and in this case \( \alpha \) is a lift of the pointwise difference \( -\varphi(x) + \psi(x) \), which lies in the commutator subgroup of \( \langle B \rangle_{nil} \), to the tensor square \( \otimes^2 \mathbb{Z}[B] \), compare the exact sequence \((2.3)\).

The vertical composition is given by addition of abelian group homomorphisms, and for the horizontal composition one uses the abelianization functor \( ab : \mathbf{nil} \rightarrow \mathbf{ab} \) and the bifunctor

\[ \text{Hom}_{ab}(-, \otimes^n_{ab}) : \mathbf{ab}^{op} \times \mathbf{ab} \rightarrow \mathbf{Ab}. \]

For any \( n \geq 2 \) there is a weak equivalence of groupoid-enriched categories

\[ \text{Hopf} : \bar{\mathbf{S}}(n) \rightarrow \mathbf{nil}(n) \]
defined by $\vee_A S^1 \Rightarrow \langle A \rangle_{nil}, \ f \mapsto (\pi_1 f)_{nil}$ and $H \mapsto Hopf(H)$, where we use the Hopf invariant for tracks. Compare [Bau91] VI.4.7. This weak equivalence is compatible on the left hand side with the suspension functor

$$\Sigma: \overline{S}(n) \rightarrow \overline{S}(n + 1)$$

which is the identity on objects and morphisms and on tracks it is given by the suspension of tracks in $Top^*$, and on the right hand side with the 2-functors

$$gr \rightarrow nil(2) \rightarrow nil(n), \ n \geq 3,$$

given by the nilization and the natural projection $\sigma: \otimes^2 \rightarrow \otimes^2$ respectively. Here we set $\overline{S}(1) = S(1)$ for $n = 1$. Theorem 3.6, and therefore Theorem 3.5, follows readily from this.

4. SECONDARY HOMOTOPIES GROUPS OF A POINTED SPACE

We now introduce secondary homotopy groups which enrich the structure of the classical homotopy groups $\pi_n X$ of a pointed space.

**Definition 4.1.** Let $X$ be a pointed space and $n \geq 1$. The secondary homotopy group $\pi_{n,*}X$ is the map

$$\partial: \pi_{n,1}X \rightarrow \pi_{n,0}X$$

defined as follows. Let

$$\pi_{n,0}X = \begin{cases} \langle \Omega X \rangle, & n = 1; \\ \langle \Omega^n X \rangle_{nil}, & n \geq 2. \end{cases}$$

Here the $n$-fold loop space, regarded as a discrete pointed set, generates a free (nil-) group. Moreover, $\pi_{n,1}X$ is the set of equivalence classes $[f, F]$ represented by a map $f: S^1 \rightarrow \vee_{\Omega^n X} S^1$ and a track

$$\begin{array}{c}
S^n \\
\downarrow F \\
\Sigma_{n-1} S^n \\
\downarrow ev \\
X
\end{array}$$

Here the pointed space

$$S^\natural X = \vee_{\Omega^n X} S^n = \Sigma^n \Omega^n X$$

is the $n$-fold suspension of the $n$-fold loop space $\Omega^n X$, where $\Omega^n X$ is regarded as a pointed set with the discrete topology. Hence $S^\natural X$ is the coproduct of $n$-spheres indexed by the set of non-trivial maps $S^n \rightarrow X$, and $ev: S^\natural X \rightarrow X$ is the obvious evaluation map. Moreover, for the sake of simplicity given a map $f: S^1 \rightarrow \vee_{\Omega^n X} S^1$ we will denote $f_{ev} = ev(\Sigma^{n-1} f)$, so that $F$ in the previous diagram is a track $F: f_{ev} \Rightarrow 0$. The equivalence relation $[f, F] = [g, G]$ holds provided the nil-track $N_{f,g}: \Sigma^{n-1} f \Rightarrow \Sigma^{n-1} g$ exists, see Theorem 3.5, and the composite track in the
following diagram is the trivial track.

\[
\begin{array}{c}
\begin{tikzpicture}
    \node (A) at (0,0) {$S^n$};
    \node (B) at (2,0) {$X$};
    \node (C) at (-2,0) {$S^n_0$};
    \node (D) at (0,-3) {$0$};
    \node (E) at (0,1) {$0$};
    \node (F) at (0,-1) {$0$};
    \draw[->] (A) to node[above]{$\Sigma^{n-1} f$} (B);
    \draw[->] (B) to node[above]{$\Sigma^{n-1} g$} (C);
    \draw[->] (A) to node[left]{$F$} (B);
    \draw[->] (C) to node[below]{$G$} (A);
    \draw[->] (A) to node[right]{$\Omega^n g$} (D);
    \draw[->] (B) to node[right]{$\Omega^n f$} (E);
    \draw[->] (C) to node[right]{$\Omega^n X$} (F);
    \draw[->] (B) to node[above]{$ev$} (B);
    \end{tikzpicture}
\end{array}
\]

That is \( F = G \Box (ev N_{f,g}) \). The map \( \partial \) is defined by the formula

\[
\partial[f,F] = \begin{cases} 
(\pi_1 f)(1), & n = 1, \\
(\pi_1 f)_{nil}(1), & n \geq 2,
\end{cases}
\]

where \( 1 \in \pi_1 S^1 = \mathbb{Z} \).

A map \( g : X \to Y \) in \( \text{Top}^* \) induces a map \( \pi_{n,*} g : \pi_{n,*} X \to \pi_{n,*} Y \) given by the following commutative diagram.

\[
\begin{array}{c}
\begin{tikzpicture}
    \node (A) at (0,0) {$\pi_{n,1} X$};
    \node (B) at (1,0) {$\pi_{n,1} Y$};
    \node (C) at (0,-1) {$\pi_{n,0} X$};
    \node (D) at (1,-1) {$\pi_{n,0} Y$};
    \draw[->] (A) to node[above]{$\pi_{n,1} g$} (B);
    \draw[->] (C) to node[below]{$\pi_{n,0} g$} (D);
    \draw[->] (A) to node[left]{$\partial$} (C);
    \draw[->] (B) to node[right]{$\partial$} (D);
\end{tikzpicture}
\end{array}
\]

Here the lower homomorphism \( \pi_{n,0} g = \langle \Omega^n g \rangle \) is induced by the map of pointed sets \( \Omega^n g : \Omega^n X \to \Omega^n Y \). Moreover, an element \( [f,F] \in \pi_{n,1} X \) is sent by the upper map \( \pi_{n,1} g \) to \( [(\Omega^n g) f, gF] \).

\[
\begin{array}{c}
\begin{tikzpicture}
    \node (A) at (0,0) {$S^n$};
    \node (B) at (2,0) {$X$};
    \node (C) at (-2,0) {$S^n_0$};
    \node (D) at (0,-3) {$0$};
    \node (E) at (0,1) {$0$};
    \node (F) at (0,-1) {$0$};
    \draw[->] (A) to node[above]{$\Sigma^{n-1} f$} (B);
    \draw[->] (B) to node[above]{$\Sigma^{n-1} g$} (C);
    \draw[->] (A) to node[left]{$F$} (B);
    \draw[->] (C) to node[below]{$G$} (A);
    \draw[->] (A) to node[right]{$\Omega^n g$} (D);
    \draw[->] (B) to node[right]{$\Omega^n f$} (E);
    \draw[->] (C) to node[right]{$\Omega^n X$} (F);
    \draw[->] (B) to node[above]{$ev$} (B);
    \end{tikzpicture}
\end{array}
\]

We also define \( \pi_{n,*} X \) for \( n = 0 \) as follows.

**Definition 4.3.** For \( n = 0 \) let \( \pi_{0,*} X \) be the **fundamental pointed groupoid** of the pointed space \( X \) for which \( \pi_{0,0} X \) is \( X \) regarded as a discrete pointed set and \( \pi_{0,1} X \) is the set of tracks between points in \( X \). For this we recall that a **pointed groupoid** is a small category \( G \) with a distinguished object \( * \in \text{Ob} G \) such that all morphisms are isomorphisms. A morphism of pointed groupoids \( F : G \to H \) is a functor preserving the distinguished object \( F(*) = * \), and the category of pointed groupoids is denoted by \( \text{grd}^* \). The morphism \( F \) is a **weak equivalence** if it induces a bijection between the pointed sets of isomorphism classes of objects \( \text{Iso}(F) : \text{Iso}(G) \cong \text{Iso}(H) \) and if \( F : \text{Aut}_G(x) \cong \text{Aut}_H(F(x)) \) is an isomorphism for any object \( x \) in \( G \). The fundamental groupoid is a functor \( \pi_{0,*} : \text{Top}^* \to \text{grd}^* \) in the obvious way.

We now study the algebraic structure of secondary homotopy groups \( \pi_{n,*} X \) with \( n \geq 1 \).
Proposition 4.4. For $n \geq 1$ there is a group structure on $\pi_{n,1}X$ such that the map $\partial : \pi_{n,1}X \to \pi_{n,0}X$ is a homomorphism. Moreover, the rows in (4.2) are also group homomorphisms.

Proof. We define the sum of two elements $[f,F], [g,G] \in \pi_{n,1}X$ by the following diagram

$$
\begin{array}{c}
S^n \xrightarrow{\Sigma_n^{-1} \mu} S^n \lor S^n \xrightarrow{\Sigma_n^{-1} \phi} S^n \xrightarrow{ev} X \\
\end{array}
$$

i.e.

$$
[f,F] + [g,G] = [(f,g)\mu, (F,G)(\Sigma_n^{-1} \mu)].
$$

One can readily check by using Theorem 3.5 that this operation is associative and $[0,0] \square$ is a unit element. The inverse of an element $[f,F]$ is represented by

$$
\begin{array}{c}
S^n \xrightarrow{\Sigma_n^{-1} \nu} S^n \lor S^n \xrightarrow{\Sigma_n^{-1} \phi} S^n \xrightarrow{ev} X \\
\end{array}
$$

i.e.

$$
-[f,F] = [f\nu,F(\Sigma_n^{-1} \nu)].
$$

To see this, and in order to introduce the reader to “track arguments”, we now prove by using diagrams that $[f\nu,F(\Sigma_n^{-1} \nu)]$ is indeed the inverse of $[f,F]$.

By definition the sum $[f,F] + [f\nu,F(\Sigma_n^{-1} \nu)]$ is represented by diagram

$$
\begin{array}{c}
S^n \xrightarrow{\Sigma_n^{-1} \mu} S^n \lor S^n \xrightarrow{\Sigma_n^{-1}(1 \lor \nu)} S^n \xrightarrow{\Sigma_n^{-1}(f,f)} S^n \xrightarrow{ev} X \\
\end{array}
$$

This kind of argument will be very common throughout the whole paper. Indeed we use it here again in order to remove the track $F$ from (c) in such a way that the
pasting of (c) is the same as the pasting of (d).

\[ S^n \xrightarrow{\Sigma^{n-1} \mu} S^n \vee S^n \xrightarrow{\Sigma^{n-1}(1,\nu)} S^n \xrightarrow{f} S^X \xrightarrow{ev} X \]

The composite map in the bottom remains constant along diagrams (a)–(d). Moreover, we have also shown that the pasting of these diagrams is also always the same. Therefore (a) and (d) represent the same element in \( \pi_{n,1}X \). We began with (a) which is \([f,F]+[f\nu,F(\Sigma^{n-1}\nu)]\) and we finish with (d) which represents the unit element \([0,0]\). Here we use the equivalence relation defining \( \pi_{n,1}X \).

The reader can now easily check that \( \partial \) is indeed a homomorphism.

**Definition 4.5.** A crossed module \( \partial: M \to N \) is a group homomorphism such that \( N \) acts on the right of \( M \) (the action will be denoted exponentially) and the homomorphism \( \partial \) satisfies the following two properties (\( m, m' \in M, n \in N \)):

1. \( \partial(mn) = -n + \partial(m) + n \),
2. \( m\partial(m') = -m' + m + m' \).

A morphism \( (f_0, f_1): \partial \to \partial' \) between crossed modules \( \partial: M \to N \) and \( \partial': M' \to N' \) is a commutative square in the category of groups

\[
\begin{array}{ccc}
M & \xrightarrow{f_1} & M' \\
\partial & & \partial' \\
N & \xrightarrow{f_0} & N'
\end{array}
\]

such that for any \( m \in M \) and \( n \in N \) the formula \( f_1(mn) = f_1(m)f_0(n) \) holds. Such a morphism is a weak equivalence if it induces isomorphisms \( \text{Ker} \partial \cong \text{Ker} \partial' \) and \( \text{Coker} \partial \cong \text{Coker} \partial' \). The category of crossed modules will be denoted by \( \text{cross} \). A crossed module \( \partial: M \to N \) is 0-free if \( N = \langle E \rangle \) is a free group.

**Proposition 4.6.** The group \( \pi_{1,0}X \) acts on \( \pi_{1,1}X \) in such a way that \( \partial: \pi_{1,1}X \to \pi_{1,0}X \) is a crossed module. Moreover, the induced map \( \pi_{1,*}g \) in (4.2) is a crossed module morphism.

**Proof.** Let \( \alpha: S^1 \to S^1 \vee S^1 \) be any map inducing \( \pi_1 \alpha: \mathbb{Z} \to \langle a, b \rangle: 1 \mapsto -a + b + a \). Any \( x \in \pi_{1,0}X \) can be identified with the homotopy class of a map \( \tilde{x}: S^1 \to S^1_X \). The automorphism

\[
(-)^x: \pi_{1,1}X \to \pi_{1,1}X: [f,F] \mapsto [f,F]^x
\]

is defined as follows: Let \([f,F]^x\) be given by the map

\[ S^1 \xrightarrow{\alpha} S^1 \vee S^1 \xrightarrow{(\tilde{x},f)} S^1_X \]

and the track

\[
\begin{array}{ccc}
S^1 & \xrightarrow{x} & S^1_X \\
\alpha & \xleftarrow{(1,0)} & S^1 \vee S^1 \xrightarrow{(\tilde{x},f)} S^1_X \\
N & \xleftarrow{(0^2,F)} & ev
\end{array}
\]

such that for any \( m \in M \) and \( n \in N \) the formula \( f_1(mn) = f_1(m)f_0(n) \) holds. Such a morphism is a weak equivalence if it induces isomorphisms \( \text{Ker} \partial \cong \text{Ker} \partial' \) and \( \text{Coker} \partial \cong \text{Coker} \partial' \). The category of crossed modules will be denoted by \( \text{cross} \). A crossed module \( \partial: M \to N \) is 0-free if \( N = \langle E \rangle \) is a free group.

**Proposition 4.6.** The group \( \pi_{1,0}X \) acts on \( \pi_{1,1}X \) in such a way that \( \partial: \pi_{1,1}X \to \pi_{1,0}X \) is a crossed module. Moreover, the induced map \( \pi_{1,*}g \) in (4.2) is a crossed module morphism.

**Proof.** Let \( \alpha: S^1 \to S^1 \vee S^1 \) be any map inducing \( \pi_1 \alpha: \mathbb{Z} \to \langle a, b \rangle: 1 \mapsto -a + b + a \). Any \( x \in \pi_{1,0}X \) can be identified with the homotopy class of a map \( \tilde{x}: S^1 \to S^1_X \). The automorphism

\[
(-)^x: \pi_{1,1}X \to \pi_{1,1}X: [f,F] \mapsto [f,F]^x
\]

is defined as follows: Let \([f,F]^x\) be given by the map

\[ S^1 \xrightarrow{\alpha} S^1 \vee S^1 \xrightarrow{(\tilde{x},f)} S^1_X \]

and the track

\[
\begin{array}{ccc}
S^1 & \xrightarrow{x} & S^1_X \\
\alpha & \xleftarrow{(1,0)} & S^1 \vee S^1 \xrightarrow{(\tilde{x},f)} S^1_X \\
N & \xleftarrow{(0^2,F)} & ev
\end{array}
\]
Here $N$ is a nil-track. By using the elementary properties of nil-tracks in Theorem 3.5 the reader can check that this is a well-defined action, independent of the choice of $\alpha$. Equation (1) in Definition 4.5 is immediate. Let us now check that (2) holds. Consider $[f, F], [g, G] \in \pi_{1, X}$. On one hand $[f, F] + [g, G]$ is represented by

(a)

On the other hand $- [g, G] + [f, F] + [g, G]$ is represented by

(b)

Factoring $(G, F)$ we can insert a 2-cell to obtain

(c)

so the pasting of (b) is the same as the pasting of (c). Now we observe that the pasting of (a) and (c) coincide, hence we are done. \hfill \Box

**Definition 4.7.** A reduced quadratic module $(\omega, \partial)$ is a sequence of group homomorphisms

$$N_{ab} \otimes N_{ab} \xrightarrow{\omega} M \xrightarrow{\partial} N$$

such that, if $N \rightarrow N_{ab}: x \mapsto \{x\}$ is the projection onto the abelianization, then the following equations hold for any $x, y \in N$ and $a, b \in M$,

\begin{enumerate}
    \item $\partial \omega(\{x\} \otimes \{y\}) = -x - y + x + y$,
    \item $\omega(\{\partial a\} \otimes \{\partial b\}) = -a - b + a + b$,
    \item $\omega(\{\partial a\} \otimes \{x\} + \{x\} \otimes \{\partial a\}) = 0$.
\end{enumerate}

Moreover, it is a stable quadratic module if the following condition, stronger than (3), is satisfied,

\begin{enumerate}
    \item[(4)] $\omega(\{x\} \otimes \{y\} + \{y\} \otimes \{x\}) = 0$.
\end{enumerate}

Condition (4) says that $\omega: \otimes^2 N_{ab} \rightarrow M$ in a stable quadratic module factors through the natural projection $\tilde{\sigma}: \otimes^2 N_{ab} \rightarrow \otimes^2 \tilde{N}_{ab}$. The factorization will also be denoted by $\omega: \otimes^2 N_{ab} \rightarrow M$. We call $(\omega, \partial: M \rightarrow N)$ 0-free if $N = \langle E \rangle_{\text{nil}}$ is a free nil-group.
A morphism of reduced or stable quadratic modules \((f_1, f_0): (\omega, \partial) \rightarrow (\omega', \partial')\) is just a commutative diagram of the form

\[
\begin{array}{c}
N_{ab} \otimes N_{ab} \xrightarrow{\omega} M \xrightarrow{\partial} N \\
\otimes^2(f_0)_{ab} \downarrow \quad \downarrow f_1 \quad \downarrow f_0 \quad \downarrow (N')_{ab} \otimes (N')_{ab} \xrightarrow{\omega'} M' \xrightarrow{\partial'} N'
\end{array}
\]

It is a \textit{weak equivalence} if it induces isomorphisms \(\text{Ker} \partial \cong \text{Ker} \partial'\) and \(\text{Coker} \partial \cong \text{Coker} \partial'\). We will write \textbf{quad} and \textbf{squad} for the categories of reduced and symmetric quadratic modules, respectively. See [Bau91] IV.C.1.

We now define a map \(\omega\) as in Definition 4.7 for secondary homotopy groups \(\pi_{n,*}X\) with \(n \geq 2\). Consider the diagram

\[
\begin{array}{c}
S^n \\
\Sigma^{n-1} \beta \\
S^n \
\Sigma^n \vee S^n \xrightarrow{\Sigma^{n-1}(\tilde{x}, \tilde{y})} S^n_X \xrightarrow{ev} X
\end{array}
\]

where \(\beta: S^1 \rightarrow S^1 \vee S^1\) is given such that \((\pi_1 \beta)_{nil}(1) = -a - b + a + b \in (a, b)_{nil}\) is the commutator. The track \(B\) is determined by \(\text{Hopf}(B) = -a \otimes b \in \otimes^2 \mathbb{Z}[a, b]\).

Given \(x \otimes y \in \otimes^2_n(\pi_{n,0}X)_{ab}\) let \(\tilde{x}, \tilde{y}: S^1 \rightarrow \vee_{\Omega^n \times S^1} S^1\) be maps with \((\pi_1 \tilde{x})_{ab}(1) = x\) and \((\pi_1 \tilde{y})_{ab}(1) = y\). Then the diagram

\[
\begin{array}{c}
S^n \\
\Sigma^{n-1} \beta \\
S^n \vee S^n \
\Sigma^{n-1}(\tilde{x}, \tilde{y}) \xrightarrow{S^n_X} S^n_X \xrightarrow{ev} X
\end{array}
\]

represents an element

\[
\omega(x \otimes y) = [(\tilde{x}, \tilde{y})\beta, ev(\Sigma^{n-1}(\tilde{x}, \tilde{y}))B] \in \pi_{n,1}X.
\]

**Proposition 4.9.** For \(n \geq 2\) the homomorphism of groups

\[
\omega: (\pi_{n,0}X)_{ab} \otimes (\pi_{n,0}X)_{ab} \longrightarrow \pi_{n,1}X
\]

given by (4.8) is well defined. Moreover, \((\omega, \partial)\) is a reduced quadratic module for \(n = 2\) and a stable quadratic module for \(n \geq 3\). Furthermore, (4.2) is a reduced quadratic module homomorphism for all \(n \geq 2\).

**Proof.** Given \(x \in \otimes^2_n(\pi_{n,0}X)_{ab}\) we alternatively define

\[
\omega(x) = [\omega(x)_1, ev\omega(x)_2] \in \pi_{n,1}X
\]

by choosing \(\omega(x)_1\) and \(\omega(x)_2: (\omega(x)_1)_{ev} \Rightarrow 0\) as in diagram

\[
\begin{array}{c}
S^n \\
\Sigma^{n-1}(\omega(x)_1) \xrightarrow{\omega(x)_1} \Sigma^n(\omega(x)_1) \xrightarrow{ev} X
\end{array}
\]

Here \(\omega(x)_1: S^1 \rightarrow \vee_{\Omega^n \times S^1} S^1\) is a map with \((\pi_1 \omega(x)_1)_{nil}(1) = \partial(x)\) and \n
\[
\omega(x)_2: \Sigma^{n-1}\omega(x)_1 \Rightarrow 0
\]
is the unique track with $\text{Hopf}(\omega(x)_2) = -x$. Such a track exists and is unique by Theorem 5.6. The elementary properties of nil-tracks and more generally of the Hopf invariant for tracks, see Theorem 3.6, show that the element $\partial$ since, provided we remove the nil-track $X$.

Moreover, it is not difficult to see by using Theorem 3.6 that can be freely removed from this diagram without altering the result of the pasting.

Indeed $\bar{\omega}(\text{nil-tracks})$, and by Remark 1.3, we see that the commutator therefore by standard “universal example” arguments, by the elementary properties of nil-tracks, and by Remark 1.3 we see that the commutator $-\partial f, F) + g, G \in \pi_{n,1}X$ is represented by the following diagram

$$
\begin{array}{cccc}
S^n & \xrightarrow{\Sigma^{-1}_n} & S^n & \xrightarrow{(0^2, \omega(x)_2)} S^n \\
N & \xrightarrow{(1,0)} & (0^2, \omega(x)_2) & \xrightarrow{\text{ev}} X
\end{array}
$$

since, provided we remove the nil-track $N$ from the diagram, the pasting of the two lower tracks is the inverse of the upper track, and by Remark 1.3 the nil-track $N$ can be freely removed from this diagram without altering the result of the pasting. Moreover, it is not difficult to see by using Theorem 5.6 that

$$
\text{Hopf}(((\Sigma^{-1}_n f) N^\square) \square ((0^2, \omega(x)_2)(\Sigma^{-1}_n \mu))) = -x.
$$

therefore $\bar{G} = ((\Sigma^{-1}_n f) N^\square) \square ((0^2, \omega(x)_2)(\Sigma^{-1}_n \mu))$ and the claim follows.

The map $\beta : S^1 \to S^1 \lor S^1$ induces the commutator of the generators on $\pi_1$, therefore by standard “universal example” arguments, by the elementary properties of nil-tracks, and by Remark 1.3 we see that the commutator $-\partial f, F) + g, G \in \pi_{n,1}X$ is represented by the following diagram

$$
\begin{array}{cccc}
S^n & \xrightarrow{\Sigma^{-1}_n \beta} & S^n & \xrightarrow{(F,G)} S^n \\
\xrightarrow{(F,G)} & \xrightarrow{(0^2, \omega(x)_2)} & \xrightarrow{\text{ev}} & X
\end{array}
$$

By using Remark 1.3 twice, as in the proof of Proposition 4.4 we deduce that the pasting of the previous diagram coincides with the pasting of

$$
\begin{array}{cccc}
S^n & \xrightarrow{\Sigma^{-1}_n \beta} & S^n & \xrightarrow{(F,G)} S^n \\
\xrightarrow{(F,G)} & \xrightarrow{(0^2, \omega(x)_2)} & \xrightarrow{\text{ev}} & X
\end{array}
$$

Therefore (2) in Definition 4.7 is satisfied.

If $n \geq 3$ equation (3) in Definition 4.7 is an immediate consequence of the fact that $\partial((x) \otimes \{y\} + \{y\} \otimes \{x\}) = 0$. Let us now check (3) in Definition 4.7 in
case \( n = 2 \). Suppose that we have \([f, F] \in \pi_{2,1}X\) and \( x \in \pi_{2,0}X \). We choose \( \tilde{x} \colon S^1 \to \vee_{\Omega^2 X} S^1 \) such that \((\pi_1 \tilde{x})_{\mathrm{nil}}(1) = x\). Then by claim (*) and (4.8) we have that \( \omega(\{\partial[f,F]\} \otimes \{x\} + \{x\} \otimes \{\partial[f,F]\}) \) is represented by the diagram

Here \( N \) is a nil-track. By Remark 1.3 the pasting of this diagram coincides with the pasting of

and again by Remark 1.3 the pasting of this diagram is the same as the pasting of

By using Theorem 3.6 one can readily check that \((0,1)(B \square N \square ((i_2, i_1)B \square (\Sigma \nu)))\) is a nil-track, and therefore this diagram represents the trivial element in \( \pi_{2,1}X \). \( \square \)
For $n \geq 0$ we define the category $\text{cross}(n)$ as follows.

\[
\text{cross}(n) = \begin{cases} 
\text{grd}^*, & \text{pointed groupoids if } n = 0; \\
\text{cross}, & \text{crossed modules if } n = 1; \\
\text{rquad}, & \text{reduced quadratic modules if } n = 2; \\
\text{squad}, & \text{stable quadratic modules if } n \geq 3.
\end{cases}
\]

(4.10)

Theorem 4.11. Secondary homotopy groups are well-defined functors

\[ \pi_{n,*}: \text{Top}^\ast \longrightarrow \text{cross}(n), \quad n \geq 0. \]

This result generalizes the well-known fact on classical homotopy groups which are functors

\[ \pi_n: \text{Top}^\ast \longrightarrow \text{group}(n), \quad n \geq 0, \]

where

\[
\text{group}(n) = \begin{cases} 
\text{Set}^*, & \text{pointed sets if } n = 0; \\
\text{Gr}, & \text{groups if } n = 1; \\
\text{Ab}, & \text{abelian groups if } n \geq 2.
\end{cases}
\]

(4.12)

Moreover, we have functors $(n \geq 0)$

\[
h_0: \text{cross}(n) \longrightarrow \text{group}(n), \\
h_1: \text{cross}(n) \longrightarrow \text{group}(n + 1).
\]

(4.13)

The functor $h_0$ is defined as the cokernel of the group homomorphism $\partial: M \rightarrow N$ for a crossed module $\partial$ or a reduced or stable quadratic module $(\omega, \partial)$, and for $G$ a pointed groupoid $h_0 G = \text{Iso}(G)$ is the pointed set of isomorphism classes of objects. Similarly $h_1$ is the kernel of $\partial: M \rightarrow N$ for crossed modules and reduced and stable quadratic modules, and $h_1 G = \text{Aut}_G(*)$ is the automorphism group of the distinguished object. In particular a morphism $f$ in $\text{cross}(n)$ is a weak equivalence for $n \geq 1$ if and only if $h_0 f$ and $h_1 f$ are isomorphisms.

In Proposition 5.1 below we show that there are natural isomorphisms $(n \geq 0)$

\[
h_0 \pi_{n,*} X \cong \pi_n X \quad \text{and} \quad h_1 \pi_{n,*} X \cong \pi_{n+1} X.
\]

(4.14)

Our definition of $\pi_{n,*} X$ above is a “singular” and hence functorial version of secondary homotopy groups. For many purposes it suffices to consider smaller models of $\pi_{n,*} X$ by choosing a subset of $\Omega^n X$ which generates $\pi_n X$. Let us make precise this observation.

Proposition 4.15. Let $X$ be a pointed space. If $E_0 \rightarrow X$ is a pointed map between pointed sets then there is a unique pointed groupoid $\pi_{0,*}(X, E_0)$ with object set $E_0$ endowed with a full and faithful functor

\[ \pi_{0,*}(X, E_0) \longrightarrow \pi_{0,*} X \]

given by $E_0 \rightarrow X$ on object sets. This morphism of pointed groupoids is a weak equivalence provided any component of $X$ has points in the image of $E_0$. Moreover,
a map of pointed sets \( E_1 \to \Omega X \) induces a crossed module morphism by the pull-back

\[
\begin{array}{ccc}
\pi_{1,1}(X, E_1) & \longrightarrow & \pi_{1,1}X \\
\partial & \downarrow \text{pull} & \partial \\
\pi_{1,0}(X, E_1) = (E_1) & \longrightarrow & (\Omega X) = \pi_{1,0}X
\end{array}
\]

which is a weak equivalence

\[
\pi_{1,*}(X, E_1) \sim \pi_{1,*}X
\]

provided the loops in the image of \( E_1 \) generate the group \( \pi_1 X \). Furthermore, for \( n \geq 2 \) the a map of pointed sets \( E_n \to \Omega^n X \) induces a reduced (stable if \( n \geq 3 \)) quadratic module morphism by the pull-back

\[
\begin{array}{ccc}
\otimes^2(\pi_{n,0}(X, E_n))_{ab} & \longrightarrow & \otimes^2\mathbb{Z}[\Omega^n X] = \otimes^2(\pi_{n,0}X)_{ab} \\
\omega & \downarrow \text{pull} & \omega \\
\partial & \downarrow \partial & \downarrow \\
\pi_{n,1}(X, E_n) & \longrightarrow & \pi_{n,1}X \\
\pi_{n,0}(X, E_n) = (E_n)_{nil} & \longrightarrow & (\Omega^n X)_{nil} = \pi_{n,0}X
\end{array}
\]

which is a weak equivalence

\[
\pi_{n,*}(X, E_n) \sim \pi_{n,*}X, \quad n \geq 2
\]

provided the \( n \)-loops in the image of \( E_n \) generate the abelian group \( \pi_n X \).

This proposition can be used to reduce the number of generators of a secondary homotopy group, as one can check in the following example.

**Remark 4.16.** So far we have not computed any secondary homotopy group. Now, with the help of Proposition 4.15, we give a small model for the secondary homotopy group \( \pi_{n,*} (\vee E S^n) \) of a wedge of spheres indexed by the pointed set \( E \). For this we notice that there is a pointed inclusion \( E \subset \Omega^n (\vee E S^n) \) sending \( e \in E - \{ \ast \} \) to the inclusion of the corresponding factor of the wedge \( S^n \subset \vee E S^n \). Then we have a weak equivalence

\[
\pi_{n,*} (\vee E S^n, E) \sim \pi_{n,*} (\vee E S^n), \quad n \geq 1.
\]

For \( n = 1 \) one easily checks that \( \pi_{1,*} (\vee E S^1, E) \) is

\[
\pi_{1,1} (\vee E S^1, E) = 0 \quad \partial \quad \pi_{1,0} (\vee E S^1, E) = \langle E \rangle.
\]

For \( n = 2 \) the reduced quadratic module \( \pi_{2,*} (\vee E S^2, E) \) is given by the following diagram, see (2.3)

\[
\begin{array}{ccc}
\otimes^2(\pi_{n,0}(\vee E S^n, E))_{ab} & \longrightarrow & \pi_{n,1}(\vee E S^n, E) \\
\omega & \longrightarrow & \pi_{n,0}(\vee E S^n, E) \\
\otimes^2\mathbb{Z}[E] & \longrightarrow & \otimes^2\mathbb{Z}[E] \\
\partial & \longrightarrow & \langle E \rangle_{nil}
\end{array}
\]
This follows from the fact that the next diagram is a pull-back

\[
\begin{array}{ccc}
\otimes^2\mathbb{Z}[E] & \rightarrow & \otimes^2\mathbb{Z}[\Omega^2 \vee E S^2] = \otimes^2(\pi_{2,0} \vee E S^2)_{ab} \\
\downarrow & & \downarrow \omega \\
\otimes^2\mathbb{Z}[E] & \rightarrow & \pi_{2,1} \vee E S^2 \\
\phi & & \phi \\
\downarrow & & \downarrow \partial \\
\langle E \rangle_{nil} & \rightarrow & \langle \Omega^2 \vee E S^2 \rangle_{nil} = \pi_{2,0} \vee E S^2 \\
\end{array}
\]

Here the homomorphism \(\phi\) is defined as follows. Given \(x \in \otimes^2\mathbb{Z}[E]\) the element \(\phi(x) = [\phi_1(x), \phi_2(x)]\) is given by a map

\[
\phi_1(x): S^1 \xrightarrow{\bar{\phi}(x)} \vee E S^1 \subset \vee \Omega^2 \vee E S^1
\]

with \((\pi_1 \bar{\phi}(x))_{nil}(1) = \partial(x)\) and the unique track

\[
\begin{array}{ccc}
S^2 & \rightarrow & \vee E S^1 \\
\Sigma_{\phi_1(x)} \vee E S^1 & \rightarrow & \vee E S^2 \\
\end{array}
\]

with Hopf invariant \(Hopf(\phi_2(x)) = -x\). Here we use the fact that the composite \(ev(\Sigma_{\phi_1(x)}) = \Sigma \bar{\phi}(x)\) is a suspension. For \(n \geq 3\) the stable quadratic module \(\pi_{n,0}(\vee E S^n, E)\) is given by the following diagram, see (2.3).

\[
\begin{array}{ccc}
\otimes^2(\pi_{n,0}(\vee E S^n, E))_{ab} & \rightarrow & \pi_{n,1}(\vee E S^n, E) \\
\downarrow & & \downarrow \partial \\
\otimes^2\mathbb{Z}[E] & \rightarrow & \otimes^2\mathbb{Z}[E] \\
\sigma & & \sigma \\
\downarrow & & \downarrow \partial \\
\langle E \rangle_{nil} & \rightarrow & \langle \Omega^2 \vee E S^2 \rangle_{nil}
\end{array}
\]

This can be easily checked as in the case \(n = 2\) by using the Hopf invariant for tracks.

5. Homotopy groups of fibers

We first obtain by secondary homotopy groups the classical homotopy groups \(\pi_n X\) as in the next result.

**Proposition 5.1.** For all \(n \geq 1\) there is a natural exact sequence of groups

\[
\pi_{n+1}X \xleftarrow{\iota} \pi_{n,1}X \xrightarrow{\partial} \pi_{n,0}X \xrightarrow{q} \pi_n X,
\]

where \(q\) sends a basis element of \(\pi_{n,0}X\), which is a map \(f: S^n \rightarrow X\), to its homotopy class in \(\pi_n X\); and \(\iota\) carries the homotopy class of \(f: S^{n+1} \rightarrow X\) to the element \([0, pf] \in \pi_{n,1}X\), where \(p: IS^n \rightarrow \Sigma S^n = S^{n+1}\) is the obvious projection.

**Proof.** Obviously \(q\) is surjective. Any element \(x \in \pi_{n,0}X\) is represented by a map \(\tilde{x}: S^1 \rightarrow \vee \Omega^\infty X S^1\), i.e. \((\pi_1 \tilde{x})_{nil}(1) = x\). It is immediate to notice that \(q(x)\) is the homotopy class of \(\tilde{x} ev: S^n \rightarrow X\). If \(q(x) = 0\) then there exists a track \(H: \tilde{x} ev \rightarrow 0\), and the pair \([\tilde{x}, H] \in \pi_{n,1}X\) satisfies \(\partial(\tilde{x}, H) - q(x)\). It is immediate to notice that \(q \partial = 0\) and \(\partial q = 0\). The injectivity of \(\iota\) is also easy to check, actually \(\pi_{n+1}X\) is isomorphic to the subgroup of \(\pi_{n,1}X\) given by the elements which can be represented with a 0 in the first coordinate. Finally suppose that for some \([f, F] \in \pi_{n,1}X\) we
have \( \partial [f, F] = 0 \), then the nil-track \( N : 0 \to \Sigma^{n-1} f \) is defined and \([f, F] = [0, F \Box N] \), hence we are done. \( \square \)

We now introduce the (algebraic) fiber of a map in \( \text{cross}(n) \) for \( n \geq 1 \).

**Definition 5.2.** Let \( f : \partial \to \partial' \) be a crossed module morphism

\[
\begin{array}{c}
M \\
\downarrow \partial \\
N
\end{array}
\quad 
\begin{array}{c}
M' \\
\downarrow \partial' \\
N'
\end{array}
\]

We define the fiber \( \text{Fib}(f) \) as the crossed module \( \text{Fib}(f) : \text{Fib}_1(f) \to \text{Fib}_0(f) \) where \( \text{Fib}_0(f) \) is the following pull-back

\[
\begin{array}{c}
\text{Fib}_0(f) \\
\downarrow \partial \\
N
\end{array}
\quad 
\begin{array}{c}
M' \\
\downarrow \partial' \\
N'
\end{array}
\]

\( \text{Fib}_1(f) = M \) and the homomorphism \( \text{Fib}(f) : \text{Fib}_1(f) \to \text{Fib}_0(f) \) is induced by \( (\partial, f^1) : M \to N \times M' \). The action of \( \text{Fib}_0(f) \) on \( \text{Fib}_1(f) \) is the pull-back along \( \partial' \) of the action of \( N \) on \( M \). The axioms of a crossed module are easily verified. There is also a natural crossed module morphism \( j : \text{Fib}(f) \to \partial \) given by the square

\[
\begin{array}{c}
\text{Fib}_1(f) \\
\downarrow \text{Fib}(f) \\
\text{Fib}_0(f)
\end{array}
\quad 
\begin{array}{c}
M \\
\downarrow \partial \\
N
\end{array}
\]

Let \( f : (\omega, \partial) \to (\omega', \partial') \) be now a reduced/stable quadratic module morphism

\[
\begin{array}{c}
\otimes^2 N_{ab} \quad \otimes^2 f_{ab}^0 \\
\downarrow \omega \\
M
\end{array}
\quad 
\begin{array}{c}
\otimes^2 (N')_{ab} \\
\downarrow \omega' \\
M'
\end{array}
\quad 
\begin{array}{c}
\partial \\
\downarrow f^1 \\
N
\end{array}
\quad 
\begin{array}{c}
\partial' \\
\downarrow f'^0 \\
N'
\end{array}
\]

The fiber \( \text{Fib}(f) \) is a reduced/stable quadratic module

\[
\otimes^2 (\text{Fib}_0(f))_{ab} \to \text{Fib}_1(f) \xrightarrow{\text{Fib}(f)} \text{Fib}_0(f)
\]

where \( \text{Fib}(f) : \text{Fib}_1(f) \to \text{Fib}_0(f) \) is defined as in the crossed module case and the first homomorphism is the composite

\[
\otimes^2 (\text{Fib}_0(f))_{ab} \xrightarrow{\otimes^2 \partial'_{ab}} \otimes^2 N_{ab} \xrightarrow{\omega} M.
\]

The natural reduced/stable quadratic module morphism \( j : \text{Fib}(f) \to (\omega, \partial) \) is also defined as above.
Lemma 5.3. Let $f: \partial \to \partial'$ be a morphism of crossed modules, then there is an exact sequence

$$h_1 \text{Fib}(f) \overset{h_1 \partial}{\longrightarrow} h_1 \partial' \overset{h_1 \delta}{\longrightarrow} h_0 \text{Fib}(f) \overset{h_0 \partial}{\longrightarrow} h_0 \partial' \overset{h_0 \delta}{\longrightarrow} 0.$$  

This exact sequence is natural in $f$. Moreover, it is also available for reduced or stable quadratic module morphisms $f: (\omega; \partial) \to (\omega'; \partial')$.

Proof. The homomorphism $\delta$ is determined by the inclusion $M' \hookrightarrow N \times M': m' \mapsto (0, m')$. The proof of the exactness is a simple exercise. \hfill \Box

Theorem 5.4. Let $f: X \to Y$ be a map between pointed spaces and let $F_f$ be the homotopy fiber of $f$. Then for all $n \geq 1$ there is a natural morphism in $\text{cross}(n)$

$$\xi: \pi_n \ast F_f \longrightarrow \text{Fib}(\pi_n \ast f)$$

which induces an isomorphism

$$\pi_n F_f \cong h_0 \text{Fib}(\pi_n \ast f)$$

and an exact sequence

$$\pi_{n+2} Y \longrightarrow \pi_{n+1} F_f \longrightarrow h_1 \text{Fib}(\pi_n \ast f),$$

where the first arrow is the boundary homomorphism in the long exact sequence in homotopy. By using the isomorphism (1) above and Proposition 5.1 we can naturally identify the exact sequence in Lemma 5.3 extended on the left by the exact sequence (2) with the following piece of the long exact sequence of homotopy groups

$$\pi_{n+2} Y \to \pi_{n+1} F_f \to \pi_{n+1} X \to \pi_n Y \to \pi_n F_f \to \pi_n X \to \pi_n Y.$$  

Proof. Recall that $F_f$ is a pull-back

$$\begin{array}{ccc}
F_f & \longrightarrow & Y^I \\
\xi & \downarrow \text{pull} & \downarrow \text{ev}_0 \\
X & \longrightarrow & Y
\end{array}$$

where $Y^I$ is the space of based maps $([0, 1], 1) \to (Y, \ast)$ and $\text{ev}_0$ is the evaluation at $0 \in [0, 1]$.

The morphism $\xi$ consists of two morphisms, the upper one is

$$\xi^1 = \pi_{n,1} \xi: \pi_{n,1} F_f \to \pi_{n,1} X = \text{Fib}_1(\pi_n \ast f).$$

We now construct the map $\xi_0: \pi_{n,0} F_f \to \text{Fib}_0(\pi_n \ast f)$. For this we consider on the one hand the morphism $\pi_{n,0} \xi: \pi_{n,0} F_f \to \pi_{n,0} X$ induced by $f$. On the other hand we define a homomorphism

$$\bar{\xi}: \pi_{n,0} F_f \to \pi_{n,1} Y$$

as follows: an element $z \in \pi_{n,0} F_f$ is represented by a map $\bar{z}: S^1 \to \vee_{\Omega^1 F_f} S^1$ with $(\pi_1 \bar{z})(1) = z$ if $n = 1$ or $(\pi_1 \bar{z})_{|\text{m}'}(1) = z$ if $n \geq 2$. The map

$$S^n \overset{\pi^n \Omega^n \bar{f}}{\longrightarrow} S^n \overset{\pi^n \Omega^n \bar{f}}{\longrightarrow} Y^I$$

has an adjoint

$$ad(\text{ev}(\Sigma^n \Omega^n \bar{f})(\Sigma^{n-1} \bar{z})): IS^n \longrightarrow Y,$$
this adjoint represents a track \( \text{ad}(e\nu(\Sigma^n\Omega^n f)(\Sigma^n-1\hat{\varepsilon})) = (\Sigma^n\Omega^n (f\hat{\varepsilon})) \varepsilon) = 0 \), and
\[ \xi^0 = (\pi_n\hat{e}, \xi): \pi_{n,0} F \rightarrow \text{Fib}(\pi_{n,0} F) \].

It is immediate to check that \( \pi_n \hat{e} \) and \( \xi \) define a homomorphism to the pull-back
\[ \xi^0 = (\pi_{n,0} \hat{e}, \xi): \pi_{n,0} F \rightarrow \text{Fib}(\pi_{n,0} F) \].

Now it is easy to check that \( \xi \) is indeed a morphism in \( \text{cross}(n) \).

By Proposition 6.1 and Lemma 5.3 we obtain from \( \xi \) a diagram with exact rows
\[ \pi_{n+2} F \rightarrow \pi_{n+1} F \rightarrow \pi_{n+1} X \rightarrow \pi_n Y \rightarrow \pi_n F \rightarrow \pi_n X \rightarrow \pi_n Y \]
\[ \xymatrix{ & \pi_{n+2} F \ar[d]^{e} \ar[r] & \pi_{n+1} F \ar[d]^{e} \ar[r] & \pi_{n+1} X \ar[r] \ar[d]^{e} & \pi_n F \ar[d]^{e} \ar[r] & \pi_n X \ar[r] \ar[d]^{e} & \pi_n Y \ar[d]^{e} }\]

It is easy to see that this diagram commutes, and hence the theorem follows from the five lemma.

**Corollary 5.5.** Let \( f: X \rightarrow Y \) be a map between pointed spaces and let \( F_f \) be the homotopy fiber of \( f \). If \( \pi_{n+2} f: \pi_{n+2} X \rightarrow \pi_{n+2} Y \) is surjective then there is a weak equivalence in \( \text{cross}(n) \), \( n \geq 1 \),
\[ \xi: \pi_{n,0} F_f \sim \text{Fib}(\pi_{n,0} F). \]

### 6. Suspension and Loop Functors

Homotopy groups \( \pi_n X \) are objects in the category \( \text{group}(n) \), \( n \geq 0 \), see \( \text{4.12} \).

There are forgetful functors
\[ \phi_n: \text{group}(n) \rightarrow \text{group}(n-1) \]
given by \( \phi_n = 1_{\text{Ab}} \) for \( n \geq 3 \) and by the obvious forgetful functors \( \phi_2: \text{Ab} \rightarrow \text{Gr} \), \( \phi_1: \text{Gr} \rightarrow \text{Set}^* \).

It is a classical result that for any pointed space \( X \) there are natural isomorphisms \( n \geq 0 \)
\[ \Omega: \pi_n \Omega X \cong \phi_{n+1} \pi_{n+1} X \text{ in } \text{group}(n). \]

The analogue of this isomorphism for secondary homotopy groups is as follows.

There are forgetful functors
\[ \phi_n: \text{cross}(n) \rightarrow \text{cross}(n-1) \]
see \( \text{4.10} \), given by \( \phi_n = 1_{\text{squad}} \) for \( n \geq 4 \) and by the functors
\[ \phi_3: \text{squad} \rightarrow \text{rquad}, \]
\[ \phi_2: \text{rquad} \rightarrow \text{cross}, \]
\[ \phi_1: \text{cross} \rightarrow \text{grd}^*. \]

The functor \( \phi_3 \) in \( \text{4.10} \) is obvious, since stable quadratic modules are special reduced quadratic modules. Given a reduced quadratic module \( (\omega, \partial) \) we have \( \phi_2(\omega, \partial) = \partial: M \rightarrow N \) in \( \text{cross} \), with the action of \( N \) on \( M \) defined by
\[ m^n = m + \omega[\partial m] \otimes \{n\}. \]

Finally if \( \partial: M \rightarrow N \) is a crossed module then the pointed groupoid \( \phi_1 \partial \) in \( \text{grd}^* \)
has \( N \) as a set of objects. Moreover the set of all morphisms in \( \phi_1 \partial \) is the semidirect product \( N \ltimes M \), which is the group structure on the set \( N \times M \) defined by the formula
\[ (n, m) + (n', m') = (n + n', m^n + m'), \]
and the structure maps of the groupoid (identities, source and target)

\[ N \xrightarrow{i} N \ltimes M \xrightarrow{s,t} N \]

are \( i(n) = (n, 0) \), \( s(n, m) = n \) and \( t(n, m) = n + \partial m \). The composition law \( \circ \) is determined by the formula

\[ (n + \partial m, m') \circ (n, m) = (n, m + m') \]

The forgetful functors \( \phi_n \) in (6.2) clearly commute with \( h_0 \) and \( h_1 \) in (4.13), that is,

\[ h_i \phi_n = \phi_n h_i, \quad n \geq 1, i = 0, 1. \]

**Theorem 6.5.** There is a natural weak equivalence in \( \mathsf{cross}(n) \)

\[ \Omega : \pi_n \Omega X \longrightarrow \phi_{n+1} \pi_{n+1} X, \quad n \geq 0, \]

which induces the isomorphism \( \Omega : \pi_n \Omega X \cong \phi_{n+1} \pi_{n+1} X \) in \( h_0 \) and \( -\Omega : \pi_{n+1} \Omega X \cong \phi_{n+2} \pi_{n+2} X \) in \( h_1 \). This weak equivalence is an isomorphism for \( n \geq 2 \).

**Proof.** Let us first consider the case \( n \geq 3 \).

We have \( \pi_{n,0} \Omega X = (\Omega^{n+1} X)_{\text{nil}} = \pi_{n+1,0} X \). We define a group homomorphism \( \pi_{n,1} \Omega X \rightarrow \pi_{n+1,1} X \) sending \([f,F]\) with \( f : S^1 \rightarrow \Omega^{n+1} X \) and

\[ S^n \xrightarrow{\Sigma^1 f} S^n_{\Omega X} \xrightarrow{ev} \Omega X \]

to \([f, \text{ad}(F)]\) where \( \text{ad}(F) \) is the adjoint track

\[ S^{n+1} \xrightarrow{\text{ad}(F)} S^{n+1}_{X} \xrightarrow{ev} X \]

Here we use that \( \Sigma S^n_{\Omega X} = S^n_{\Omega X} \) and \( \text{ad}(ev(\Sigma^{n-1} f)) = ev(\Sigma^n f) \).

The reader can check that the diagram

\[ (a) \quad (\pi_{n,0} \Omega X)_{ab} \otimes (\pi_{n,0} \Omega X)_{ab} \xrightarrow{\omega} \pi_{n,1} \Omega X \xrightarrow{\partial} \pi_{n,0} \Omega X \]

\[ (\pi_{n+1,0} X)_{ab} \otimes (\pi_{n+1,0} X)_{ab} \xrightarrow{\omega} \pi_{n+1,1} X \xrightarrow{\partial} \pi_{n+1,0} X \]

commutes, so it is a morphism of stable quadratic modules. Moreover, the following diagram commutes

\[ (b) \quad \pi_{n+1} \Omega X \xrightarrow{i} \pi_{n,1} \Omega X \xrightarrow{\partial} \pi_{n,0} \Omega X \xrightarrow{q} \pi_n \Omega X \]

\[ \pi_{n+2} \Omega X \xrightarrow{i} \pi_{n+1,1} X \xrightarrow{\partial} \pi_{n+1,0} X \xrightarrow{q} \pi_{n+1} X \]

Here the exact rows are given by Proposition 5.1 and the arrows with \( \cong \) are (up to sign) the usual isomorphisms of homotopy groups, therefore the central vertical arrow in \( (a) \) is an isomorphism by the five lemma.
For \( n = 2 \) we have \( \pi_{2,0} \Omega X = \langle \Omega^3 X \rangle_{nil} = \pi_{3,0} X \) and there is a homomorphism \( \pi_{2,1} \Omega X \to \pi_{3,1} X \) defined as above. This homomorphism makes commutative diagrams (a) and (b), therefore it defines an isomorphism of reduced quadratic modules.

For \( n = 1 \) there is an obvious epimorphism \( \pi_{1,0} \Omega X = \langle \Omega^2 X \rangle_{nil} = \pi_{2,0} X \). One can also define a homomorphism \( \pi_{1,1} \Omega X \to \pi_{2,1} X \) as above. It is easy to check that the following square defines the desired crossed module morphism

\[
\begin{array}{ccc}
\pi_{1,1} \Omega X & \overset{\partial}{\longrightarrow} & \pi_{1,0} \Omega X \\
\downarrow & & \downarrow \\
\pi_{2,1} X & \overset{\partial}{\longrightarrow} & \pi_{2,0} X
\end{array}
\]

Moreover, the following diagram commutes

\[
\begin{array}{ccc}
\pi_{2} \Omega X & \overset{i}{\longrightarrow} & \pi_{1,1} \Omega X & \overset{\partial}{\longrightarrow} & \pi_{1,0} \Omega X & \overset{q}{\longrightarrow} & \pi_{1} \Omega X \\
\downarrow & \cong & \downarrow & \cong & \downarrow & \cong & \downarrow \\
\pi_{3} X & \overset{i}{\longrightarrow} & \pi_{2,1} X & \overset{\partial}{\longrightarrow} & \pi_{2,0} X & \overset{q}{\longrightarrow} & \pi_{2} X
\end{array}
\]

This diagram is the analogue to (b) and shows that (c) is a weak equivalence.

Now for \( n = 0 \) we define the functor \( \pi_{0,*} \Omega X \to \phi_{1} \pi_{1,*} X \). On objects it is given by the inclusion \( \text{Ob} \pi_{0,*} \Omega X = \Omega X \subset \langle \Omega X \rangle = \text{Ob} \phi_{1} \pi_{1,*} X \). Given any object \( f \in \Omega X \) in \( \pi_{0,*} \Omega X \) we consider the inclusion \( \bar{f} : S^1 \to S^1_X \) of the factor of the coproduct \( S^1_X \) corresponding to \( f \). Clearly the adjoint \( \text{ad}(\bar{f}) : S^0 \to \Omega X \) is the inclusion of the point \( f \in \Omega X \). If \( g \in \Omega X \) is another object then a morphism \( H : f \to g \) in \( \pi_{0,*} \Omega X \) is just a track \( H : \text{ad}(\bar{f}) \Rightarrow \text{ad}(\bar{g}) \) in \( \text{Top}^* \). The functor sends the morphism \( H \) to the element in \( \pi_{1,0} X \times \pi_{1,1} X \), which is \( \pi_{1,0} X \times \pi_{1,1} X \) as a set, with \( (\pi_1 \bar{f})(1) \) in the left coordinate and right coordinate given by the map

\[
S^1 \overset{\mu}{\longrightarrow} S^1 \vee S^1 \overset{\nu \vee 1}{\longrightarrow} S^1 \vee S^1 \overset{(\bar{g}, \bar{f})}{\longrightarrow} S^1_X
\]

and the track

\[
\begin{array}{ccc}
S^1 & \overset{\mu}{\longrightarrow} & S^1 \vee S^1 & \overset{\nu \vee 1}{\longrightarrow} & S^1 \vee S^1 & \overset{(f, g)}{\longrightarrow} & S^1_X & \overset{ev}{\longrightarrow} & X \\
\downarrow & \uparrow N & \downarrow & \uparrow (\bar{g}, \bar{f}) = (ad(H), 0) & \downarrow & \uparrow (ad(H), 0) \circ \mu & \downarrow & \uparrow ev & \downarrow
\end{array}
\]

Here \( N \) is a nil-track and \( \text{ad}(H) : \bar{f} \Rightarrow \bar{g} \) is the adjoint of the track \( H \). We leave it to the reader to check that \( \pi_{0,*} \Omega X \to \phi_{1} \pi_{1,*} X \) is a well-defined functor. One can use again Proposition 5.1 to check that this functor is an equivalence.

The functors \( \phi_n \) in (6.6) have left adjoints

\[
(6.6) \quad \text{Ad}_n : \text{group}(n - 1) \longrightarrow \text{group}(n)
\]

given by \( \text{Ad}_n = 1_{\text{Ab}} \) for \( n \geq 3 \),

\[
\text{Ad}_2 : \text{Gr} \longrightarrow \text{Ab}, \text{ the abelianization;}
\]

\[
\text{Ad}_1 : \text{Set}^* \longrightarrow \text{Gr}, \text{ taking free group.}
\]
These adjoints can be used to define for a pointed space $X$ the natural suspension morphisms

$$\Sigma: \text{Ad}_{n+1}\pi_nX \to \pi_{n+1}\Sigma X$$

as the adjoint of

$$\pi_nX \overset{\pi_n\text{ad}(1)}{\longrightarrow} \pi_n\Omega\Sigma X \cong \phi_{n+1}\pi_{n+1}\Sigma X.$$ 

Here we use the map $\text{ad}(1): X \to \Omega\Sigma X$ which is adjoint to the identity in $\Sigma X$ and the natural isomorphism $\Omega$. Now we generalize the situation for secondary homotopy groups.

The functors $\phi_n$ in (6.2) have left adjoints,

$$(6.7) \quad \text{Ad}_n: \text{cross}(n-1) \to \text{cross}(n)$$
given by $\text{Ad}_n = 1_{\text{squad}}$ if $n \geq 4$,

$$(6.8) \quad \text{Ad}_3: \text{rquad} \to \text{squad}, \quad \text{Ad}_2: \text{cross} \to \text{rquad}, \quad \text{Ad}_1: \text{grd}^* \to \text{cross}.$$ 

**Lemma 6.9.** The functors in (6.8) preserve 0-free objects and weak equivalences between them.

Here by convention we set all pointed groupoids to be 0-free.

**Proof of Lemma 6.9.** For $\text{Ad}_1$ the lemma follows from Lemma 6.12 below.

For $\text{Ad}_2$ and $\text{Ad}_3$ the lemma follows from the technical fact that the suspension functors between crossed and quadratic complexes described in [Bau91] and [Mur05], which are extensions of $\text{Ad}_2$ and $\text{Ad}_3$, are compatible with the homotopy relation in the category of totally free (i.e. cofibrant) crossed or quadratic complexes. In addition we use that 0-free crossed or quadratic modules are exactly the truncations of totally free crossed or quadratic complexes. \[\square\]

The functor $\text{Ad}_3$ is the stabilization in [Bau91] IV.C.3. It is defined as follows. Given a reduced quadratic module

$$(\omega, \partial) = (\otimes^2 N_{ab} \overset{\omega}{\longrightarrow} M \overset{\partial}{\longrightarrow} N)$$

the stabilized stable quadratic module

$$\text{Ad}_3(\omega, \partial) = (\otimes^2 N_{ab} \overset{\omega}{\longrightarrow} M_\Sigma \overset{\partial_\Sigma}{\longrightarrow} N)$$

is given by the group $M_\Sigma$ obtained by quotienting out in $M$ the relations

$$\omega(a \otimes b + b \otimes a), \quad a, b \in N_{ab},$$

and the homomorphisms $\omega_\Sigma$ and $\partial_\Sigma$ are induced by $\omega$ and $\partial$, respectively, in the obvious way.

The functor $\text{Ad}_2$ in (6.8) is the suspension functor in [Mur05] 3.3. Given a crossed module $\partial: M \to N$ the reduced quadratic module

$$\text{Ad}_2 \partial = (\otimes^2 N_{ab} \overset{\omega}{\longrightarrow} M_\Sigma \overset{\partial}{\longrightarrow} N_{nil})$$

is given by the group $M_\Sigma$ which is a quotient of $M \times (\otimes^2 N_{ab})$ by the relations

$$(-m + m^n, 0) = (0, \{\partial(m)\} \otimes \{n\}) = (0, -\{n\} \otimes \{\partial(m)\}),$$
for any \( m \in M \) and \( n \in N \); and the homomorphisms \( \delta \) and \( \omega \) are defined by the following formulas, \( m \in M, \, n, n' \in N, \)
\[
\begin{align*}
\delta(m, \{n\} \otimes \{n'\}) &= \partial(m) + [n, n'], \\
\omega(\{n\} \otimes \{n'\}) &= \{0\} \otimes \{n'\}.
\end{align*}
\]

Finally we describe the functor \( \text{Ad}_1 \). Let \( G \) be a groupoid with object pointed set \( \text{Ob}G \) and morphism set \( \text{Mor}G \). The crossed module \( \text{Ad}_1G \) is the quotient of the free crossed module, see [Bau91], generated by the function

\[
\text{Mor}G \rightarrow \langle \text{Ob}G \rangle
\]

by the relations \( u = g + f \) for \( u, f, g \in \text{Mor}G \) with \( u = fg \), the composition of \( f \) and \( g \) in \( G \). One readily checks that \( \text{Ad}_1 \) is the adjoint of \( \phi_1 \).

The functor \( h_0 \) commutes with \( \text{Ad}_n \)
\[
(6.10) \quad h_0 \text{Ad}_n = \text{Ad}_n h_0, \quad n \geq 1.
\]
This follows from the definition of \( \text{Ad}_n \) above for \( n \geq 2 \) and from Lemma 6.12 below in case \( n = 1 \). For \( h_1 \) the corresponding commutativity law is not true in general, compare Lemma 6.12 below.

**Theorem 6.11.** There are natural morphisms in \( \text{cross}(n + 1) \)

\[
\Sigma: \text{Ad}_{n+1}\pi_{n+1}X \rightarrow \pi_{n+1} \Sigma X, \quad n \geq 0,
\]

which induce the classical suspension homomorphism \( \Sigma: \text{Ad}_{n+1}\pi_{n+1}X \rightarrow \pi_{n+1} \Sigma X \) in \( h_0 \), and for \( n \geq 3 \) the homomorphism \(-\Sigma: \text{Ad}_{n+2}\pi_{n+1}X \rightarrow \pi_{n+2} \Sigma X \) in \( h_1 \). Moreover, for \( n \geq 3 \) the morphism \( \Sigma \) is a weak equivalence provided \( X \) is \( m \)-connected and \( n \leq 2m - 1 \). It is also a weak equivalence for \( n = 2 \) provided \( X \) is simply connected, and for \( n = 1 \) if \( X \) is connected. Furthermore, \( \Sigma \) is always a weak equivalence for \( n = 0 \).

In the proof of this theorem we will use the following lemma.

**Lemma 6.12.** For any pointed groupoid \( G \) there are natural isomorphisms

1. \( h_0 \text{Ad}_1G = \langle \text{Iso}G \rangle \),
2. \( h_1 \text{Ad}_1G = \bigoplus_{x \in \text{Iso}(G)} (\text{Aut}_G(x))_{ab} \otimes R. \)

Here \( R \) is the group ring of \( \langle \text{Iso}G \rangle \).

**Proof.** The crossed module \( \text{Ad}_1G \) defined above is the truncation

\[
N_1\text{FBG}/d(N_2\text{FBG}) \rightarrow N_0\text{FBG}
\]

of the Moore complex \( N_1\text{FBG} \) of the Milnor construction \( \text{FBG} \) on the classifying space \( BG \) of the pointed groupoid \( G \), see [Kan58] and [GJ99] I.1.4 and V.6. To see this we have on the 0-level

\[
N_0\text{FBG} = \langle \text{Ob}G \rangle,
\]

and on the 1-level the set \( \text{Mor}G \) in \( \text{Ad}_1G \) is mapped to \( N_1\text{FBG}/d(N_2\text{FBG}) \) by sending \( h: U \rightarrow V \) to the coset modulo \( d(N_3\text{FBG}) \) of the element

\[
-1_U + h \in N_1\text{FBG} \subset F_1BG = \langle \text{Mor}G \rangle.
\]

This can be checked by computing \( N_1\text{FBG}/d(N_2\text{FBG}) \) in terms of generators and relations. In order to carry out this computation one uses the Reidemeister-Schreier method, see [MKS66], which simplifies in this particular case since the simplicial
identities hold in $FBG$ and the boundaries and degeneracies in this simplicial group are homomorphisms between free groups on pointed sets induced by maps between the generating pointed sets.

By the previous observation the kernel and cokernel of $Ad_1 G$ are the $\pi_1$ and $\pi_2$ of the suspension $\Sigma |BG|$ of the geometric realization $|BG|$ of the classifying space of $G$, so the lemma follows from elementary facts from homotopy theory.

We also remark that given $x \in \text{Iso}(G)$ and a representative $\tilde{x} \in \text{Ob} G$ of $x$ the group $(\text{Aut}_G(\tilde{x}))_{ab}$ does not depend on the choice of $\tilde{x}$, up to natural isomorphism, therefore we can denote it by $(\text{Aut}_G(x))_{ab}$.

□

Proof of Theorem 6.11. Consider the morphism

$$\pi_{n,*}X \xrightarrow{\pi_{n,*} \text{ad}(1)} \pi_{n,*} \Omega \Sigma X \xrightarrow{\Omega} \phi_{n+1} \pi_{n+1,*} \Sigma X,$$

where $\text{ad}(1): X \to \Omega \Sigma X$ is the adjoint of the identity in $\Sigma X$ and $\Omega$ is given by Proposition 6.5. The morphism in the statement is the adjoint of this one. For $n \geq 3$ the range where this morphism is a weak equivalence follows from Proposition 5.1 and the classical suspension theorem for ordinary homotopy groups.

For $n = 1$ the theorem follows from Proposition 8.5 below and [Mur05] 4.8. For $n = 2$ we use Proposition 8.5 and [Bau91] IV.C. For this we use that we are dealing with 0-free objects and that $Ad_n$ preserves weak equivalences between them, see Lemma 6.9.

If $n = 0$ we have $\text{Iso}(\pi_{0,*}X) = \pi_0 X$ and for any $x \in \text{Ob} \pi_{0,*}X$, $\text{Aut}_{\pi_{0,*}}(x) = \pi_1(X,x)$. By using elementary homotopy theory one can check that

$$\pi_1 \Sigma X \cong \langle \pi_0 X \rangle$$

and

$$\pi_2 \Sigma X \cong \bigoplus_{x \in \pi_0 X} (\pi_1(X,x))_{ab} \otimes \mathbb{Z}(\pi_0 X).$$

Now it is enough to notice that isomorphisms in Lemma 6.12 are compatible with the two isomorphisms above and Proposition 5.1 (in this last case up to sign $-1$ in kernel).

□

7. SECONDARY HOMOTOPY GROUPS AS 2-FUNCTORS

In Section 4.1 we have defined the secondary homotopy group functors $(n \geq 0)$

$$\pi_{n,*}: \text{Top}^* \to \text{cross}(n).$$

As we recalled in Section 11 the category of pointed spaces is a groupoid-enriched category, therefore it is reasonable to wonder whether $\pi_{n,*}$ is a 2-functor. This is known to be true if $n = 0$. In this section we prove that it is actually true for any $n \geq 0$.

We recall from [Bau91] the definition of homotopies or 2-morphisms in the categories of crossed modules and reduced quadratic modules.
Definition 7.1. Suppose that we have two crossed modules \( \partial : M \to N, \partial' : M' \to N' \) and two morphisms \( f, g : \partial \to \partial' \) given by

\[
\begin{array}{ccc}
M & \xrightarrow{f_1, g_1} & M' \\
\partial & \downarrow & \partial' \\
N & \xrightarrow{f_0, g_0} & N'
\end{array}
\]

A 2-morphism \( \alpha : f \Rightarrow g \) is a function \( \alpha : N \to M' \) such that for any \( x, y \in N \) and any \( m \in M \) the following equalities hold:

1. \( \alpha(x + y) = \alpha(x)^{f_0(y)} + \alpha(y), \)
2. \( g_0(x) = f_0(x) + \partial'(\alpha(x)), \)
3. \( g_1(m) = f_1(m) + \alpha(\partial(m)). \)

If we now have two reduced quadratic modules

\[
\otimes^2 N_{ab} \xrightarrow{\omega} M \xrightarrow{\partial} N,
\]

\[
\otimes^2 (N')_{ab} \xrightarrow{\omega'} M' \xrightarrow{\partial'} N',
\]

and two morphisms \( f, g : (\omega, \partial) \to (\omega', \partial') \)

\[
\begin{array}{ccc}
\otimes^2 N_{ab} & \xrightarrow{\omega} & M \xrightarrow{\partial} N \\
\otimes^2 (N')_{ab} & \xrightarrow{\omega'} & M' \xrightarrow{\partial'} N'
\end{array}
\]

a 2-morphism \( \alpha : f \Rightarrow g \) is just a 2-morphism \( \alpha : \phi_2 f \Rightarrow \phi_2 g \) in the category of crossed modules. Here we use the forgetful functor \( \phi_2 \) in (6.3). More precisely, \( \alpha : N \to M' \) is a function such that for any \( x, y \in N \) and \( m \in M \) the following equations hold:

1. \( \alpha(x + y) = \alpha(x) + \alpha(y) + \omega'\{ -f_0(x) + g_0(x) \} \otimes \{ f_0(y) \}), \)
2. \( g_0(x) = f_0(x) + \partial'(\alpha(x)), \)
3. \( g_1(m) = f_1(m) + \alpha(\partial(m)). \)

The 2-morphisms for stable quadratic module morphisms are the same as 2-morphisms for the corresponding reduced quadratic module morphisms. In particular the forgetful functors \( \phi_n \) in (6.2) become automatically 2-functors which are full and faithful at the level of 2-morphisms for \( n \geq 2 \).

The 2-morphisms in the category \( \text{grd}^* \) of pointed groupoids are just natural transformations between functors. For \( n = 0 \) the functor \( \phi_1 \) in (6.2) is also a 2-functor. More precisely, if \( \partial : M \to N \) and \( \partial' : M' \to N' \) are crossed modules and \( \alpha : N \to M' \) is a 2-morphism \( \alpha : f \Rightarrow g \) between two morphisms \( f, g : \partial \to \partial' \) then the natural transformation \( \phi_1 \alpha : \phi_1 f \Rightarrow \phi_1 g \) between the pointed groupoid morphisms \( \phi_1 f, \phi_1 g : \phi_1 \partial \to \phi_1 \partial' \) is given by the morphisms \( (f_0(n), \alpha(n)) : f_0(n) \to g_0(n) \) in \( \phi_1 \partial' \) which are natural in \( n \in N \).

Proposition 7.2. The category \( \text{cross}(n) \) with 2-morphisms as in Definition 7.1 is a groupoid-enriched category. Moreover, for all \( n \geq 1 \) the functor \( \Lambda d_n \) in (6.3) is a 2-functor which is adjoint to \( \phi_n \) in (6.3) as a groupoid-enriched functor.
Proof. For $n = 0$ it is well-known that $\text{cross}(n)$ is a groupoid-enriched category. We only need to carry out the proof of the first part of the statement for crossed modules since the groupoid-enriched structure in $\text{rquad}$ and $\text{squad}$ is pulled back through the forgetful functors in $\Delta_3$.

In this proof $\partial_i : M_i \to N_i = \langle E_i \rangle$ will denote a crossed module for $i = 0, 1, 2, 3$.

Let $f, g, h : \partial_1 \to \partial_2$ be crossed module morphisms and let $\alpha : f \Rightarrow g, \beta : g \Rightarrow h$ be vertically composable 2-morphisms. The vertical composition is defined by $(\beta \square \alpha)(x) = \alpha(x) + \beta(x)$ for any $x \in M_1$. The inverse 2-morphism $\alpha^{-1} : g \Rightarrow f$ is defined by $\alpha^{-1}(x) = -\alpha(x)$ and the trivial 2-morphism $0_f : f \Rightarrow f$ is $0_f(x) = 0$.

Suppose that we have a diagram

$$
\begin{array}{c}
\partial_0 \xrightarrow{k} \partial_1 \xrightarrow{\alpha} \partial_2 \xrightarrow{h} \partial_3
\end{array}
$$

Then the two possible horizontal compositions $\alpha k : f k \Rightarrow g k, h \alpha : h f \Rightarrow g f$ are defined as $\alpha k = \alpha k_0 : N_0 \to M_3$ and $h \alpha = h k_1 : N_1 \to M_4$.

Suppose now that we have a diagram

$$
\begin{array}{c}
\partial_0 \xrightarrow{f} \partial_1 \xrightarrow{g} \partial_2 \xrightarrow{f'} \partial_2
\end{array}
$$

Let us check the equality

$$(g' \alpha) \square (\alpha' f) = (\alpha' g) \square (f' \alpha).$$

Given $x \in M_0$ by using the equations defining crossed modules and 2-morphisms we get

$$
((g' \alpha) \square (\alpha' f))(x) = \alpha' f_0(x) + g'_1 \alpha(x)
$$

$$
= \alpha' f_0(x) + f'_1 \alpha(x) + \alpha' \partial_1 \alpha(x)
$$

$$
= \alpha' f_0(x) + f'_1 \alpha(x) + \alpha'(-f_0(x) + g_0(x))
$$

$$
= \alpha' f_0(x) + f'_1 \alpha(x) + \alpha'(-f_0(x))f_0^\alpha g_0(x) + \alpha'(g_0(x))
$$

$$
= \alpha' f_0(x) + f'_1 \alpha(x) - \alpha'(f_0(x))f_0^\alpha(-f_0(x) + g_0(x)) + \alpha'(g_0(x))
$$

$$
= \alpha' f_0(x) + f'_1 \alpha(x) - \alpha'(f_0(x))f_0^\alpha \partial_1 \alpha(x) + \alpha'(g_0(x))
$$

$$
= \alpha' f_0(x) + f'_1 \alpha(x) - \alpha'(f_0(x))f_0^\alpha \alpha(x) + \alpha'(g_0(x))
$$

$$
= \alpha' f_0(x) - \alpha'(f_0(x)) + f'_1 \alpha(x) + f'_1 \alpha(x) + \alpha'(g_0(x))
$$

$$
= \alpha' f_0(x) + \alpha'(g_0(x))
$$

$$
= ((\alpha' g) \square (f' \alpha))(x).
$$

Hence (a) holds and $\text{cross}(1)$ is indeed a groupoid-enriched category.

Now we define the 2-functors $\text{Ad}_n$ at the level of 2-morphisms. For $n \geq 4$ they are identity 2-morphism functors. For $n = 3$, given a 2-morphism $\alpha : N \to M'$ between two reduced quadratic module morphisms from $(\omega, \partial)$ to $(\omega', \partial')$ the 2-morphism $\text{Ad}_3 \alpha : N \to M'_{\omega}$ is the composition of $\alpha$ with the natural projection $M' \to M'_{\omega}$. For $n = 2$, if $\alpha : N \to M'$ is a 2-morphism between two crossed module morphisms from $\partial$ to $\partial'$ then $\text{Ad}_2 \alpha : N_{nil} \to (M')^{\Sigma_1}$ is defined as $\langle \text{Ad}_2 \alpha \rangle(n) = (\alpha(n), 0)$ for $n \in N_{nil}$. Finally for $n = 1$, if $\alpha : f \Rightarrow g$ is a natural transformation between two pointed
groupoid morphisms \( f, g: G \to G' \) which is given by a collection of morphisms \( \alpha(X): f(X) \to g(X) \) in \( G' \) for \( X \in \text{Ob} G \) then \( \text{Ad}_1 \alpha: \langle \text{Ob} G \rangle \to \langle \text{Ad}_1 G' \rangle \) is the unique 2-morphism between crossed module morphisms satisfying \( (\text{Ad}_1 \alpha)(X) = \alpha(X) \).

**Theorem 7.3.** The secondary homotopy groups are 2-functors \((n \geq 0)\)

\[
\pi_{n,*}: \text{Top}^* \to \text{cross}(n).
\]

In the proof of Theorem 7.3 we will use the following general construction.

**Definition 7.4.** Let \( X \) be a pointed space. Given two maps \( f, g: S^1 \to \Omega^n X S^1 \) and a track \( H: f_{ev} \Rightarrow g_{ev} \) we define \( r(H) \in \pi_{n,1} X \) as follows. Let \( \varepsilon: S^1 \to \Sigma S^1 \) be a map with \( \pi_1 \varepsilon: \mathbb{Z} \to (a, b) \) satisfying \( (\pi_1 \varepsilon)(1) = -a + b \), or just \( (\pi_1 \varepsilon)_{nil}(1) = -a + b \) if \( n \geq 2 \), then \( r(H) \) is represented by the map

\[
S^1 \xrightarrow{\varepsilon} S^1 \vee S^1 \xrightarrow{[f,g]} \vee_{\Omega^n X} S^1
\]

and the track

\[
\begin{array}{c}
S^n \\
\downarrow \varepsilon \\
S^n \\
\downarrow \varepsilon \\
S^n \\
\downarrow \varepsilon
\end{array} \xrightarrow{0} \xrightarrow{g_{ev}} \xrightarrow{(1,1)} \xrightarrow{\mu_{ev}} X
\]

The \( r \)-construction may be regarded as a machine to generate 2-morphisms in \( \text{cross}(n) \) between secondary homotopy groups. Some of the axioms of a 2-morphism in \( \text{cross}(n) \) are checked in the following lemma for the \( r \)-construction.

**Lemma 7.5.** Let \( X \) be a pointed space. Given \( f, g, h: S^1 \to \vee_{\Omega^n X} S^1 \), \( H: f_{ev} \Rightarrow g_{ev} \) and \( K: g_{ev} \Rightarrow h_{ev} \) the following formulas hold in \( \pi_{n,1} X \),

1. \( \partial r(H) = -(\pi_1 f)(1) + (\pi_1 g)(1) \) if \( n = 1 \),
2. \( \partial r(H) = -(\pi_1 f)_{nil}(1) + (\pi_1 g)_{nil}(1) \) if \( n \geq 2 \),
3. \( r(K \square H) = r(H) + r(K) \).

**Proof.** Equations (1) and (2) are clear. The element \( r(H) + r(K) \) is represented by the following diagram

\[
\begin{array}{c}
S^n \\
\downarrow \varepsilon \\
S^n \\
\downarrow \varepsilon \\
S^n \\
\downarrow \varepsilon
\end{array} \xrightarrow{0} \xrightarrow{S^n \vee S^n} \xrightarrow{[f,g]} \vee_{\Omega^n X} S^n
\]

By Remark 7.3 we can insert the track \( (K, 0^\square) \) in (a) so that the pasting of (a) and (b) below remains the same

\[
\begin{array}{c}
S^n \\
\downarrow \varepsilon \\
S^n \\
\downarrow \varepsilon \\
S^n \\
\downarrow \varepsilon
\end{array} \xrightarrow{0} \xrightarrow{S^n \vee S^n} \xrightarrow{[f,g]} \vee_{\Omega^n X} S^n
\]
Pasting some tracks in (b) we obtain

\[(c)\]

\[
\begin{array}{c}
S^n \\
\Sigma^{n-1} \mu \rightarrow S^n \\
\Sigma^{n-1} (ev) \rightarrow S^n \\
\Sigma^{n-1} (f,g,h) \rightarrow S^n_X \rightarrow ev \rightarrow X
\end{array}
\]

One can factor (c) as

\[(d)\]

\[
\begin{array}{c}
S^n \\
\Sigma^{n-1} \mu \rightarrow S^n \\
\Sigma^{n-1} (ev) \rightarrow S^n \\
\Sigma^{n-1} (f,g,h) \rightarrow S^n_X \rightarrow ev \rightarrow X
\end{array}
\]

Diagram (d) represents the same element in \(\pi_{n,1}X\) as

\[(e)\]

\[
\begin{array}{c}
S^n \\
\Sigma^{n-1} \mu \rightarrow S^n \\
\Sigma^{n-1} (ev) \rightarrow S^n \\
\Sigma^{n-1} (f,g,h) \rightarrow S^n_X \rightarrow ev \rightarrow X
\end{array}
\]

where the two \(N\) denote different nil-tracks. The pasting of (e) coincides with the pasting of

\[(f)\]

\[
\begin{array}{c}
S^n \\
\Sigma^{n-1} \mu \rightarrow S^n \\
\Sigma^{n-1} (ev) \rightarrow S^n \\
\Sigma^{n-1} (f,g,h) \rightarrow S^n_X \rightarrow ev \rightarrow X
\end{array}
\]

and (f) represents \(r(K \Box H)\), hence (3) follows.

In the next lemma we check the derivation property for the \(r\)-construction.

**Lemma 7.6.** Let \(X, Y\) be pointed spaces. Given \(x, y: S^1 \rightarrow \vee \Omega^n XS^1, f, g: \vee \Omega^n XS^1 \rightarrow \vee \Omega^n YS^1,\) and \(H: ev(\Sigma^{n-1} f) \Rightarrow ev(\Sigma^{n-1} g)\), we have the following equalities.
in $\pi_{n,1}Y$:

\[
r(H(\Sigma^{n-1}(x,y))((\Sigma^{n-1})\mu)) = r(H(\Sigma^{n-1}x))^{\pi_1(fy)(1)} + r(H(\Sigma^{n-1}y)) \text{ if } n = 1,
\]

and if $n \geq 2$

\[
r(H(\Sigma^{n-1}x)) + r(H(\Sigma^{n-1}y)) + \omega([-\pi_1(fx)(1) + \pi_1(gx)(1)] ; \{\pi_1(fy)(1)\}).
\]

**Proof.** Suppose that $n = 1$. Let $\varpi : S^1 \to S^1 \vee \cdots \vee S^1$ be a map with

$$\pi_1 \varpi : \mathbb{Z} \to \langle a_1, a_2, a_3, a_4, a_5 \rangle : 1 \mapsto -a_1 - a_2 + a_3 + a_1 - a_4 + a_5.$$ 

The element $r(H(\Sigma^{n-1}x))^{\pi_1(fy)(1)} + r(H(\Sigma^{n-1}y))$ is represented by the diagram

(a)

\[
S^n_{\Sigma \rightarrow n} \vee S^n_{\Sigma \rightarrow n} \xrightarrow{(i_1,i_2,i_2)} S^n_{\Sigma \rightarrow n} \vee S^n_{\Sigma \rightarrow n} \xrightarrow{(0^2,H,0^2)} S^n_{\Sigma \rightarrow n} \vee S^n_{\Sigma \rightarrow n} \xrightarrow{\Sigma^{-1}(f,g)} Y
\]

where $z = \Sigma^{-1}(i_1y,i_2x,i_3x,i_2y,i_3y)$ and $N$ is a nil-track. By Remark 1.3 we can add the track $(H,0^2)$ to (a) so that the pasting of (a) and (b) below remains the same

(b)

\[
S^n_{\Sigma \rightarrow n} \vee S^n_{\Sigma \rightarrow n} \xrightarrow{(i_1,i_2,i_2)} S^n_{\Sigma \rightarrow n} \vee S^n_{\Sigma \rightarrow n} \xrightarrow{(0^2,H,0^2)} S^n_{\Sigma \rightarrow n} \vee S^n_{\Sigma \rightarrow n} \xrightarrow{\Sigma^{-1}(f,g)} Y
\]

Pasting some tracks in (b) we obtain

(c)

\[
S^n_{\Sigma \rightarrow n} \vee S^n_{\Sigma \rightarrow n} \xrightarrow{(i_1,i_2,i_2)} S^n_{\Sigma \rightarrow n} \vee S^n_{\Sigma \rightarrow n} \xrightarrow{(0^2,H,0^2)} S^n_{\Sigma \rightarrow n} \vee S^n_{\Sigma \rightarrow n} \xrightarrow{\Sigma^{-1}(f,g)} Y
\]

The pasting of (c) is the same as the pasting of

(d)

\[
S^n_{\Sigma \rightarrow n} \vee S^n_{\Sigma \rightarrow n} \xrightarrow{(i_1,i_2,i_2)} S^n_{\Sigma \rightarrow n} \vee S^n_{\Sigma \rightarrow n} \xrightarrow{(0^2,H,0^2)} S^n_{\Sigma \rightarrow n} \vee S^n_{\Sigma \rightarrow n} \xrightarrow{\Sigma^{-1}(f,g)} Y
\]
Diagram (d) represents the same element in $\pi_{1,1}Y$ as

\[
\begin{array}{ccccccccc}
S^n & \xrightarrow{\Sigma^{-1}e} & S^n \vee S^n & \xrightarrow{(1,1)} & S^n_X \vee S^n_X & \xrightarrow{(H,\theta^2)} & S^n_Y & \xrightarrow{ev} & Y \\
\downarrow \Sigma^{-1}\varepsilon & & \downarrow N & & \downarrow (H^{-1}(f,g)) & & \downarrow \Sigma^{-1}(f,g) & & \downarrow ev \\
S^n & \xrightarrow{\Sigma^{-1}\varepsilon} & S^n \vee S^n & \xrightarrow{\Sigma^{-1}(x,y)\vee(x,y)} & S^n \vee S^n & \xrightarrow{\Sigma^{-1}(x,y)\vee(x,y)} & S^n & \xrightarrow{ev} & Y
\end{array}
\]

Here the two $N$ denote different nil-tracks. Diagram (e) represents $r(H(\Sigma^{-1}(x,y))(\Sigma^{-1}\mu))$, hence the case $n=1$ follows.

Suppose now that $n \geq 2$. By using Theorem 3.6 and the claim (*) in the proof of Proposition 4.9 the element

\[
\pi_1(fx) + \pi_1(gx)
\]

is represented by the diagram (f)

\[
\begin{array}{ccccccccc}
S^n & \xrightarrow{\Sigma^{-1}\varepsilon} & S^n \vee S^n & \xrightarrow{\Sigma^{-1}(x,y)\vee(x,y)} & S^n \vee S^n & \xrightarrow{(H,\theta^2)} & S^n_Y & \xrightarrow{ev} & Y \\
\downarrow \Sigma^{-1}\varepsilon & & \downarrow Q & & \downarrow (H^{-1}(f,g)) & & \downarrow \Sigma^{-1}(f,g) & & \downarrow ev \\
S^n \vee S^n & \xrightarrow{\Sigma^{-1}(\mu\vee\mu)} & S^n \vee S^n & \xrightarrow{\Sigma^{-1}(x,y)\vee(x,y)} & S^n & \xrightarrow{ev} & Y
\end{array}
\]

where $Q$ is the unique nil-track with Hopf invariant

\[-(i_1(\pi_1(x))_{ab}(1) + i_2(\pi_1(x))_{ab}(1)) \otimes (i_1(\pi_1(y))_{ab}(1)) \in \Sigma^2(\mathbb{Z}[\Omega^nX] \oplus \mathbb{Z}[\Omega^nX]),\]

for $i_1, i_2: \mathbb{Z}[\Omega^nX] \to \mathbb{Z}[\Omega^nX] \oplus \mathbb{Z}[\Omega^nX]$ the inclusion of the factors of the direct sum.

By Theorem 3.6 the track $(1,1)Q$ is a nil-track, hence diagram (f) also represents the element $r(H(\Sigma^{-1}(x,y))(\Sigma^{-1}\mu))$, and we are done. \hfill \square

Now we are ready to prove the main result of this section.

**Proof of Theorem 7.3.** This is known for $n = 0$. Suppose now that $n \geq 1$. Let $f, g: X \to Y$ be two maps and $f_*: \pi_{n,*}X \to \pi_{n,*}Y$ the induced morphisms in $\text{cross}(n)$. Moreover, let $H: f \Rightarrow g$ be a track. We define a 2-morphism $\pi_{n,*}H = H_*: f_* \Rightarrow g_*$ in the following way. The function $H_*: \pi_{n,0}X \to \pi_{n,1}Y$ sends an element $x \in \pi_{n,0}X$ represented by a map $\tilde{x}: S^1 \to \vee_{\Omega^nX} S^1$ with $(\pi_1\tilde{x})(1) = x$ if $n = 1$ and $(\pi_1\tilde{x})(n)(1) = x$ if $n \geq 2$ to the element in $\pi_{n,1}Y$ represented by the map (a) and the track (b) below.

Let $\varepsilon: S^1 \to S^1 \vee S^1$ be a map with $(\pi_1\varepsilon)(1) = -a + b \in \langle a, b \rangle$. The map (a) is defined by

\[
\begin{align*}
(a) \quad S^1 & \xrightarrow{\varepsilon} S^1 \vee S^1 \xrightarrow{\varepsilon \vee x} (\vee_{\Omega^nX} S^1) \vee (\vee_{\Omega^nX} S^1) \\
& \xrightarrow{\Sigma^{-1}(f,g)} \vee_{\Omega^nY} S^1.
\end{align*}
\]
The track (b) is given by

\[
\begin{array}{c}
\Sigma^n_0 \xrightarrow{0} S^n \\
\Sigma^n_1 \xrightarrow{(1,1)} S^n \\
\Sigma^n_{(k \vee k)} \xrightarrow{(1,1)} S^n \vee S^n \\
\end{array}
\]

Here \(N\) is a nil-track. With the terminology introduced in Definition 7.4 we have \(H_*(x) = r(H ev(\Sigma^{-1} \tilde{x}))\) for the track \(H ev(\Sigma^{-1} \tilde{x}): ev(\Sigma^n \Omega^n f)(\Sigma^{-1} \tilde{x}) = f \circ ev(\Sigma^{-1} \tilde{x}) = g \circ ev(\Sigma^n \Omega^n g)(\Sigma^{-1} \tilde{x})\).

The proof of equation (1) in the definition of 2-morphisms in \(\text{cross}(n)\) follows from Lemma 7.6. Equation (2) follows from Lemma 7.5 (1) or (2). Equation (3) follows from the fact that given \([k, K] \in \pi_{n+1} X\) the pasting of the following tracks coincide.

The track (c) below represents \(H_* \partial[k, K]\)

\[
\begin{array}{c}
\Sigma^n_0 \xrightarrow{0} S^n \\
\Sigma^n_1 \xrightarrow{(1,1)} S^n \\
\Sigma^n_{(k \vee k)} \xrightarrow{(1,1)} S^n \vee S^n \\
\end{array}
\]

Using Remark 1.3 we can introduce the track \(K\) on the top of (c) without altering the result of the pasting.
Again by Remark 1.3, we can remove the nil-track \( N \) from (d) and the pasting will still be the same.

\[
\begin{array}{c}
\xymatrix{
S^n \ar[r]^{\Sigma^{n-1} \varepsilon} & S^n \vee S^n \ar[r]^{\Sigma^{n-1}(k \vee k)} & S^n_X \vee S^n_X \ar[r]^{\Sigma^n \Omega^n(f,g)} & S^n_Y \ar[r]^{ev} & Y
}
\end{array}
\]

Pasting some maps and tracks in (e) we obtain

\[
\begin{array}{c}
\xymatrix{
S^n \ar[r]^{\Sigma^{n-1} \varepsilon} & S^n \vee S^n \ar[r]^{\Sigma^{n-1}(k \vee k)} & S^n_X \vee S^n_X \ar[r]^{\Sigma^n \Omega^n(f,g)} & S^n_Y \ar[r]^{ev} & Y
}
\end{array}
\]

Again by Remark 1.3, we can remove \((H, 0^\Box)\) from (f) and the pasting of (f) and (g) below coincide.

\[
\begin{array}{c}
\xymatrix{
S^n \ar[r]^{\Sigma^{n-1} \varepsilon} & S^n \vee S^n \ar[r]^{\Sigma^{n-1}(k \vee k)} & S^n_X \vee S^n_X \ar[r]^{\Sigma^n \Omega^n(f,g)} & S^n_Y \ar[r]^{ev} & Y
}
\end{array}
\]

Diagram (g) represents \(-f_*[k, K] + g_*[k, K]\), hence equation (3) holds.

The vertical composition of 2-morphisms \( f \Rightarrow g \Rightarrow h \) is preserved by Lemma 1.6. The proof of the fact that \( \pi_{n,*} \) preserves horizontal composition is straightforward and it is left to the reader.

\begin{proposition}

The inclusion \( \text{cross}_f(n) \subset \text{cross}(n) \) of the full subcategory of 0-free objects induces an equivalence of categories \((n \geq 0)\)

\[
\text{cross}_f(n)/\simeq \xrightarrow{\sim} \text{Hom} \text{cross}(n),
\]

where the homotopy category \( \text{Ho} \) is obtained by inverting weak equivalences.

\end{proposition}

\begin{proof}

For \( n = 0 \) this result is well-known. For \( n = 1 \) this is a consequence of the fact that \( \text{cross} \) has a model category structure where 0-free objects are the cofibrant objects, see [CM07], and the homotopy relation derived from the cylinders on cofibrant objects is given by the tracks defined above. In a similar way one obtains the result for \( n \geq 2 \).

\end{proof}
8. $k$-Invariants

Let $K(G, n)$ be the Eilenberg-MacLane space with $\pi_n K(G, n) = G$. Following Eilenberg-MacLane's notation we write $H^m(G, n, A)$ for the $m$-dimensional cohomology of the space $K(G, n)$ with coefficients in the abelian group $A$. Here we allow $A$ to be a $G$-module in case $n = 1$. In this case $H^m(G, 1, A) = H^m(G, A)$ is the ordinary cohomology (with local coefficients) of the group $G$.

For any connected CW-complex $X$ we write

$$k_n(X) \in H^{n+2}(\pi_n X, n, \pi_{n+1}X)$$

for the first $k$-invariant of the $(n-1)$-connected cover $X(n)$. Recall that $X(n)$ is the homotopy fiber of the canonical map from $X$ to its $(n-1)$-type, $X \to P_{n-1}X$, where $P_{n-1}X$ is a Postnikov section of $X$.

If $n = 1$ then $k_1(X)$ is the usual first $k$-invariant of a connected CW-complex $X$, represented by the crossed module

$$\partial: \pi_2(X, X^1) \to \pi_1 X^1,$$

determined by the skeletal filtration of $X$, see [MW50]. Otherwise, if $n \geq 2$

$$H^{n+2}(\pi_n X, n, \pi_{n+1} X) = \text{Hom}(\Gamma_n \pi_n X, \pi_{n+1} X)$$

and $k_n(X): \Gamma_n \pi_n X \to \pi_{n+1} X$ is induced by the function $\eta^*: \pi_n X \to \pi_{n+1} X$ which sends the homotopy class of $\alpha$: $S^n \to X$ to the homotopy class of $\alpha(\Sigma^{n-2} \eta^*): S^{n+1} \to X$, where $\eta^*: S^3 \to S^2$ is the Hopf map. Compare notation in [23].

The first secondary homotopy group $\pi_1, X$ is a crossed module, see Proposition 4.6. By Proposition 5.1 and [MW50] this crossed module represents an element

$$k(\pi_1, X) \in H^3(\pi_1 X, \pi_2 X).$$

In general, any crossed module $\partial$ defines a cohomology class $k(\partial) \in H^3(h_0 \partial, h_1 \partial)$, see [MW50].

For $n \geq 2$ the $n$-dimensional secondary homotopy group of $X$ defines a homomorphism

$$k(\pi_{n, X}): \Gamma_n \pi_n X \to \pi_{n+1} X,$$

as follows. Let $k(\pi_{n, X})$ be the unique homomorphism fitting into the following commutative diagram

\[
\begin{array}{ccc}
\Gamma_n(\pi_{n, X}) & \oplus \pi_n^*(\pi_{n, X}) & \\
| & | & \\
\Gamma_n \pi_n X & | & \pi_{n+1} X \to \pi_{n+1} X \\
| & q & |
\end{array}
\]

Here the upper horizontal arrow is the injection in (8.1), and $q$ and $\omega$ appear in Proposition 5.1.

In general any 0-free reduced quadratic module $(\omega, \partial)$ defines a homomorphism

$$k(\omega, \partial): \Gamma h_0(\partial, \omega) \to h_1(\omega, \partial),$$

as in [23] and any 0-free stable quadratic module $(\omega, \partial)$ defines accordingly a homomorphism

$$k(\omega, \partial): h_0(\omega, \partial) \otimes \mathbb{Z}/2 \to h_1(\omega, \partial).$$
Theorem 8.2. For any connected CW-complex $X$ and any $n \geq 1$ the equality $k_n(X) = k(\pi_{n,*}X)$ holds.

Proof. We can suppose without loss of generality that the 1-skeleton $X^1 = \vee E S^1$ is just a one-point union of 1-spheres. One can easily check that $\pi_1(X,E)$ in Proposition 8.1 coincides with $\partial : \pi_2(X, X^1) \to \pi_1 X^1$, hence the theorem follows for $n = 1$.

We now prove the theorem for $n \geq 2$. Suppose that we have $x \in \pi_{n,0}X$ and we choose $\tilde{x} : S^1 \to \vee \Omega X S^1$ with $(\pi_1 \tilde{x})_{nil}(1) = x$. Then $\omega(\{x\} \otimes \{x\}) \in \pi_{n,1}X$ is represented by

\[
\begin{array}{cccc}
S^n & \to & S^n & \to \vee & S^n \ast S^n & \to \vee \Omega X S^1 & \xrightarrow{ev} & X \\
\xrightarrow{\Sigma_{n-1} \beta} & & \xrightarrow{\Sigma_{n-1}(\tilde{x}, \tilde{x})} & & \xrightarrow{S^1 X ev} & \\
0 & & 0 & & & & & \\
\end{array}
\]

The pasting of this is the same as the pasting of

\[
\begin{array}{cccc}
S^n & \to & S^n & \to \vee & S^n & \to \vee \Omega X S^1 & \xrightarrow{ev} & X \\
\xrightarrow{\Sigma_{n-1} \beta} & & \xrightarrow{(1,1)} & & \xrightarrow{\Sigma_{n-1} \beta} & \\
0 & & 0 & & & & & \\
\end{array}
\]

By Theorem 3.6

\[
\text{Hopf}((1,1)B) = -1 \in \otimes_2 \mathbb{Z} = \begin{cases} 
\mathbb{Z}, & \text{if } n = 2; \\
\mathbb{Z}/2, & \text{if } n \geq 3.
\end{cases}
\]

Moreover, $(\pi_1(1,1)\beta)_{nil} = 0$, therefore by using the definition of $\iota$ in Proposition 5.1, Theorem 3.6, Remark 3.4 and the characterization of $\eta : S^3 \to S^2$ up to homotopy as the unique map with Hopf invariant 1 we get that $\omega(\{x\} \otimes \{x\}) = \iota(q(x)(\Sigma_{n-2} \eta))$, hence we are done. \qed

Let $\text{types}_n^1$ be the category of pointed $(n-1)$-connected CW-complexes $X$ with $\pi_m(X,x_0) = 0$ for all $m \geq n + 2$ and all $x_0 \in X$.

Proposition 8.3. The functor $\pi_{n,*} : \text{types}_n^1 \to \text{cross}(n)$ induces an equivalence of categories $(n \geq 0)$

$$
\pi_{n,*} : \text{Ho types}_n^1 \sim \to \text{Ho cross}(n),
$$

where the homotopy category $\text{Ho}$ is obtained by localizing with respect to weak equivalences.

For the proof of Proposition 8.3 we recall the following functors.

Let $\text{CW}_n$ be the category of CW-complexes $X$ with trivial $(n-1)$-skeleton $X^{n-1} = \ast$ and cellular maps. There is a “cellular” functor

\[
P_{n+1} \sigma : \text{CW}_n / \sim \longrightarrow \text{cross}_f(n) / \sim .
\]

If $n = 1$ this functor sends a CW-complex $X$ to the crossed module

$$
\partial : \pi_2(X, X^1) \to \pi_1 X^1
$$

given by the boundary operator in the long exact sequence of homotopy groups, see [Mac49] and [MW50]. If $n \geq 2$ the the reduced (stable if $n \geq 3$) quadratic module
$P_n\sigma(X)$ is the truncation of the totally free quadratic complex $\sigma(X)$ defined in [Bau91] IV.C,

$$\otimes^2 C_n(X) \xrightarrow{\omega} \sigma_{n+1}(X)/d(\sigma_{n+2}(X)) \xrightarrow{\delta} \sigma_n(X),$$

compare [Bau91] IV.10.4 and [Mur05] 4.

**Proposition 8.5.** The functor $P_{n+1}\sigma$ in (8.4) is naturally isomorphic to $\pi_{n,*} : \text{CW}_n/\simeq \rightarrow \text{cross}_f(n)/\simeq$ for all $n \geq 1$.

**Proof.** If $X$ is $(n-1)$-reduced then $X^n = \vee_E S^n$ for some pointed set $E$. The inclusion of spheres in the wedge $X^n \subset X$ determines a pointed inclusion $E \subset \Omega^n X$. One can easily check that $P_{n+1}\sigma(X)$ is isomorphic to $\pi_{n,*}(X, E)$ in Proposition 4.15. Now the natural isomorphism in the statement is given by the weak equivalence $P_{n+1}\sigma(X) \simeq \pi_{n,*}(X, E) \rightarrow \pi_{n,*}X$ in Proposition 4.15. Compare Proposition 7.7. □

**Proof of Proposition 8.3.** For $n = 0$ this is a well-known result. For $n \geq 1$ this follows from Propositions 4.14 and 7.7 and the fact that $P_{n+1}\sigma$ in 8.4 does induce an equivalence of categories $P_{n+1}\sigma : Ho\text{types}_n \rightarrow Ho\text{cross}(n)$. This is shown in [Bau91] III.8.2 for $n = 1$. For $n = 2$ the proof follows as in the case $n = 1$, this case is considered even in the non-simply connected case in [Bau91] IV.10.1. The case $n \geq 3$ can be easily proved along the lines of the $n = 1$ and $n = 2$ cases, i.e. by using [Bau91] III.8.5, III.8.8 and IV.C.14. □

**REFERENCES**

[Bau91] H.-J. Baues, *Combinatorial Homotopy and 4-Dimensional Complexes*, Walter de Gruyter, Berlin, 1991.

[Bau06] ———, *The algebra of secondary cohomology operations*, Progress in Math. 247, Birkhäuser, 2006.

[BC91] M. Bullejos and A. M. Cegarra, *A 3-dimensional nonabelian cohomology of groups with applications to homotopy classification of continuous maps*, Canad. J. Math. 43 (1991), no. 2, 265–296.

[BC97] H.-J. Baues and D. Conduché, *On the 2-type of an iterated loop space*, Forum Math. 9 (1997), no. 6, 721–738.

[BCC93] M. Bullejos, P. Carrasco, and A. M. Cegarra, *Cohomology with coefficients in symmetric cat-groups. An extension of Eilenberg-Mac Lane’s classification theorem*, Math. Proc. Cambridge Philos. Soc. 114 (1993), no. 1, 163–189.

[BM05a] H.-J. Baues and F. Muro, *Smash products for secondary homotopy groups*, Preprint of the Max-Planck-Institut für Mathematik MPIM2006-38, http://arxiv.org/abs/math.AT/0604031, 2005.

[BM05b] ———, *The symmetric action on secondary homotopy groups*, Preprint of the Max-Planck-Institut für Mathematik MPIM2006-37, http://arxiv.org/abs/math.AT/0604030, 2005.

[BM06] ———, *Secondary algebras associated to ring spectra*, In preparation, 2006.

[Con84] D. Conduché, *Modules croisés généralisés de longueur 2*, J. Pure Appl. Algebra 34 (1984), no. 2-3, 155–178.

[GJ99] P. J. Goerss and J. F. Jardine, *Simplicial Homotopy Theory*, Progress in Mathematics, no. 174, Birkhäuser Verlag, Basel, 1999.

[GM97] A. R. Garzón and J. G. Miranda, *Homotopy theory for (braided) CAT-groups*, Cahiers Topologie et Géom. Différentielle Catégoriques 38 (1997), no. 2, 99–139.

[GMdR02] A. R. Garzón, J. G. Miranda, and A. del Río, *Tensor structures on homotopy groupoids of topological spaces*, Int. Math. J. 2 (2002), no. 5, 407–431.
[Kan58] D. M. Kan, *A combinatorial definition of homotopy groups*, The Annals of Mathematics 67 (1958), no. 2, 282–312.

[Mac49] S. MacLane, *Cohomology theory in abstract groups III. operator homomorphisms of kernels*, Ann. of Math. (2) 50 (1949), 736–761.

[MKS66] W. Magnus, A. Karras, and D. Solitar, *Combinatorial Group Theory*, Interscience, New York, 1966.

[Mur05] F. Muro, *Suspensions of crossed and quadratic complexes, co-H-structures and applications*, Trans. Amer. Math. Soc. 357 (2005), no. 9, 3623–3653.

[MW50] S. MacLane and J. H. C. Whitehead, *On the 3-type of a complex*, Proc. Nat. Acad. Sci. 36 (1950), 41–48.

[Tod62] H. Toda, *Composition methods in homotopy groups of spheres*, Annals of Mathematics Studies, No. 49, Princeton University Press, Princeton, N.J., 1962.

[Whi41] J. H. C. Whitehead, *On adding relations to homotopy groups*, Ann. of Math. (2) 42 (1941), 409–428.

[Whi49] ———, *Combinatorial homotopy II*, Bull. Amer. Math. Soc. 55 (1949), 453–496.

[Whi50] ———, *A certain exact sequence*, Ann. Math. 52 (1950), 51–110.