Abstract. In this paper, using a geometric method introduced in [12] and initiated by [4], we derive an asymptotic swaption implied volatility at the first-order for a general stochastic volatility Libor Market Model. This formula is useful to quickly calibrate a model to a full swaption matrix. We apply this formula to a specific model where the forward rates are assumed to follow a multi-dimensional CEV process correlated to a SABR process. For a caplet, this model degenerates to the classical SABR model and our asymptotic swaption implied volatility reduces naturally to the Hagan-al formula [11]. The geometry underlying this model is the hyperbolic manifold \( \mathbb{H}^{n+1} \) with \( n \) the number of Libor forward rates.

1. Introduction

The BGM model [6],[15] has recently been the focus of much attention as it gives a theoretical justification for pricing caps-floors using the classical Black-Scholes formula. The basic (physical) random variables are given by the Libor forward rates which are assumed to follow a correlated log-normal process. As the forward swap rate model implied by the BGM model is quite complicated (the swap forward rate is not log-normally distributed), the calibration to a swaption matrix is difficult. An asymptotic swaption implied volatility (at the zero-order in the swaption maturity) was initially derived by Rebonato [18] and Hull-White [9] for the (log-normal) BGM model. Such formula has been obtained by assuming that the ratio of a forward Libor rate over the swap rate and the derivative of the swap rate according to a forward Libor rate are almost constant (and therefore equal to their values at the spot).

Despite its great success, the BGM model presents the same drawbacks as the classical Black-Scholes theory: as the forward rates follow a correlated log-normal process, the model is not able to calibrate the full swaption matrix in/out-the money (in particular the caplets smile) and give a good dynamics to the Libor rates. The incorporation of a swaption smile can be obtained by introducing more elaborated models which should be flexible enough to calibrate caplets and a grid of swaption volatilities (not necessary at the money) across all swaption expiries and underlying swap maturities. One property that these models must still share is their ability to quickly calibrate the swaption matrix without using complicated numerical routines such as Monte-Carlo simulation which are usually noisy and time-consuming. In this context, Andersen-Andreasen introduced the CEV Libor Market Model (LMM) [1] which assumes that each forward rate follows a CEV process, and showed how to obtain asymptotic swaption smile. Their method is still based on the Rebonato “freezing” argument which is not completely mathematically justified. Recently, for this specific model, Kawai found a better asymptotic formula using the Wiener chaos expansion [16]. Although

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giving more flexibility than the BGM model, the CEV LMM model is still not able to calibrate the swaption matrix for in/out strikes and in this context, we are naturally led to use stochastic volatility LMM. The literature on this subject is not particularly large. Andersen-al introduced a LMM where the Libors follow a multi-dimensional correlated CEV process coupled (but uncorrelated) to a Heston model \[2\,3\] and recently Piterbarg modifies this model to incorporate a term structure \[17\]. Using an averaging principle, Piterbarg derives an asymptotic volatility. Note that as these models are uncorrelated to the stochastic volatility, the swaption fair value is simply given by the fair price in the case of a local volatility model conditional to the stochastic volatility process as explained by the Hull-White decomposition \[10\]. An asymptotic expression can then be generated by approximating the moments of the volatility process \[2\].

For pricing exotic options (such as bermudan swaptions for example), it is simpler or more natural to model directly the forward swap rate with a stochastic volatility process. For example, the SABR model \[11\] was introduced to fulfill this goal. An asymptotic swaption smile formula (at the first-order) was derived for this specific model and helps to calibrate quickly the model to liquid market data. In this context, it is natural to try to reconcile/unify both benchmark models, the BGM and SABR models. We therefore introduce a LMM where the forward rates follow a multi-dimensional CEV process (with one beta for each forward) correlated to a SABR model. As it is the case for the SABR model, we impose that the libors are correlated to an unique volatility and it is therefore not possible to follow the Andersen-al \[3\] method (i.e. the Hull-White decomposition) to derive an asymptotic swaption smile.

In this paper, we pursue our previous work on the application of the heat kernel expansion on a Riemannian manifold endowed with an Abelian connection \[12\] to derive an asymptotic smile formula for a swaption. The plan of this paper is as follows: in the first part, we will recall some definitions and present a list of recent Libor Market Models. In the second part, we apply the heat kernel expansion to derive an asymptotic swaption smile formula at the first-order valid for any LMM. In the third part, we present our stochastic LMM and apply this general formula. We will prove that the geometry underlying this model is the hyperbolic manifold \(\mathbb{H}^{n+1}\) with \(n\) the number of forward rates. Furthermore, we show that the "freezing" argument is no longer valid when we try to price a swaption in/out the money: the libors should in fact be frozen to the saddle-point (constrained on a particular hyperplane) which minimizes the geodesic distance on \(\mathbb{H}^{n+1}\).

2. Libor Market Models

We denote by \(F_k(t) \equiv F(t, T_{k-1}, T_k)\) with the forward rate resetting at \(T_{k-1}\) with \(\tau = T_{k-1} - T_k\) the tenor. As the product of the bond \(P(t, T_k)\) with the forward rates \(F_k(t)\) is a difference of two bonds with maturity \(T_{k-1}\) and \(T_k\), \(\frac{1}{2}(P(t, T_{k-1}) - P(t, T_k))\), and therefore a traded asset, \(F_k\) is a (local) martingale under \(Q^k\), the (forward) measure associated with the numéraire \(P(t, T_k)\). Therefore, we assume the following driftless dynamics

\[
\begin{align*}
  dF_k(t) &= \sigma_k(t)\Phi_k(a, F_k)dW_k , \forall t \leq T_{k-1} , \; k = 1, \ldots, n \\
  dW_k dW_l &= \rho_{kl}(t)dt
\end{align*}
\]

with the initial conditions \(a(t = 0) = a\) and \(F_k(t = 0) = F_k^0\).

In order to achieve some flexibility, we assume that the (normal) local volatility \(\Phi_k(a, F_k)\) depends on a hidden Markov process \(a\) (to be specified later) representing a stochastic volatility. We therefore assume that all the forward rates are coupled with the same stochastic volatility \(a\). (Table 1) presents a list of the different functional forms for \(\Phi_k\) used in the literature. The BGM, (limited) CEV and shifted log-normal models correspond to local volatility models \((a = 1)\) and the others to stochastic volatility models with a unique stochastic volatility \(a\) driven by a Heston

\[
\begin{align*}
  d\Phi_k &= \alpha_k(t)\Phi_k dW_k + \beta_k(t)\Phi_k dt, \\
  dW_k &= \sigma_k(t)\Phi_k dt + \rho_{kl}(t)\Phi_k dW_l
\end{align*}
\]
As the conditional probability at the zero-order is proportional to 
\[ \frac{dF}{dF} \]
with \( \epsilon \) a small positive number.

The local volatility associated to the forward swap rate (\( s_{\alpha\beta} \)) can be found in two steps: First by finding an asymptotic expansion
\[ ds_{\alpha\beta} = \sum_{k=\alpha+1}^{\beta} \frac{\partial s_{\alpha\beta}}{\partial F_k} \sigma_k(t)\Phi_k(a, F_k)dZ_k \]
and then by doing the integration over \( \mathbb{B} \). The first step is achieved via the heat kernel expansion technique summarized in appendix A. and the second step using the Laplace saddle-point method explained briefly in appendix B.

3.0.1. Saddle-point. As the conditional probability at the zero-order is proportional to \( e^{-\frac{d(x, x^0)^2}{4}} \) (see appendix A.) with \( d(x, x^0) \) the geodesic distance between the points \( x = (a, \{F_i\}_{i=1,\ldots,n}) \) and
\( x^0 = (\alpha, \{ F^0_i \}_{i=1, \ldots, n}) \), the saddle-point corresponds to the point \( x \) on the submanifold \( s_{\alpha\beta} = s \) which minimizes the geodesic distance \( d(x, x^0) \) \[ (3.2) \]

\[ (a^*, \{ F^*_i \}) \equiv (a, \{ F_i \}) \text{ such as } \min_{a,F_i} [d(x, x^0)^2] \]

Introducing a Lagrange multiplier, \( \lambda \), this is equivalent to \[ (3.3) \]

\[ (a^*, \{ F^*_i \}) \equiv (a, \{ F_i \}) \text{ such as } \min_{a,F_i,\lambda} [d(x, x^0)^2 + \lambda(s_{\alpha\beta}(F) - s)] \]

### 3.1. Asymptotic local volatility

Plugging our asymptotic expression for the conditional probability \( \psi(\cdot, t \mid s) \) into \( (3.1) \) and doing the integration over \( \mathbb{B} \) using the Laplace method (see appendix B.), we finally obtain the local volatility at the first-order

\[ (\sigma_{loc}^{\alpha\beta})^2(t, s) = \sum_{i,j=1}^n \rho_{ij}(t)\sigma_i(t)\sigma_j(t)f_{ij}(F^*, a^*) \]

\[ 1 + 2t \sum_{\mu,\nu=1}^{n+1} A^{\mu\nu}(\partial_{F_i}f_{ij}(F^*, a^*) \partial_{F_j}f_{ij}(F^*, a^*)) - \sum_{\gamma,\delta=1} A^{\gamma\delta}(\partial_{\gamma\delta}d^2) + \partial_{F_i}f_{ij}(F^*, a^*) \]

with \( f_{ij}(F, a) = a^2C_i(F_j)C_j(F)\partial_{F_i}\sigma_{\alpha\beta}\partial_{F_j}\sigma_{\alpha\beta}, \psi(F, a) = \sqrt{\gamma}P \) and \( A^{\mu\nu} = [\partial_{\mu\nu}d^2]^{-1} \). Note that as opposed to other asymptotic methods presented in the literature, this formula is exact at \( t \to 0 \). A similar zero-order formula (independent of the time \( t \) for \( \sigma_i(t), \rho_{ij}(t) \) constant) was derived for a general multi-dimensional local volatility model by [4]. Moreover, in the expansion, we assumed that the time \( t \) is small but we have made no assumption that \( F_k \) is close to the spot libor or that the volatility of volatility is small.

### 3.2. Asymptotic Smile

The asymptotic smile can be derived in two steps from the asymptotic local volatility: first, we have

\[ ds = \frac{\sigma_{loc}^{\alpha\beta}(t, s)}{\sigma_{loc}^{\alpha\beta}(t, s_0)}\sigma_{loc}^{\alpha\beta}(t, s_0)dW_t \]

and doing a change of local time \( t' = \int_0^t \sigma_{loc}^{\alpha\beta}(u, s,u^2)du \), we now obtain the associated local volatility model for the swap rate

\[ ds_{\alpha\beta} = \tilde{\sigma}_{loc}^{\alpha\beta}(t, s)dW'_t \]

with \( \tilde{\sigma}_{loc}^{\alpha\beta}(t, s) = \frac{\sigma_{loc}^{\alpha\beta}(t, s)}{\sigma_{loc}^{\alpha\beta}(t, s_0)} \). Secondly, we know that there is a one-to-one correspondence between this local volatility and the smile [12] given at the first-order by

\[ (3.4) \]

\[ \sigma_{BS}^{\alpha\beta}(K, T_\alpha) = \sqrt{\int_0^{T_\alpha} (\sigma_{loc}^{\alpha\beta})^2(u, 0)du} \sigma_{BS}^{\alpha\beta}(K) \left( 1 + \frac{1}{2} \int_0^{T_\alpha} (\sigma_{loc}^{\alpha\beta})^2(u, 0)du \sigma_{BS}^{\alpha\beta}(K) \right) \]

\[ \sigma_{BS}^{0\beta}(K_0) = \frac{\ln(K)}{\int_0^{K_0} \frac{dz}{C(z)}} \]

\[ \sigma_{BS}^{0\beta}(K_1) = \frac{1}{\left( \int_0^{K_0} \frac{dz}{C(z)} \right)^2} \ln(K) + \frac{1}{\left( \frac{\sigma_{BS}^{0\beta}(K_0)^2 K_0 \sigma_{BS}^{0\beta}(K_0)}{C(K_0)} + \frac{\sigma_{BS}^{0\beta}(0, 0)\partial_{F_i}f_{ij}(0, 0)}{C(f_{av})} \right)} \]

with \( C(f) \equiv \frac{\sigma_{loc}^{\alpha\beta}(0, K)}{\sigma_{loc}^{\alpha\beta}(0, s_0)}f_{av} = \frac{\sigma_{loc}^{\alpha\beta}(0, K)}{2} \).
4. SABR-LMM Model

We have seen that the asymptotic local and implied volatilities can be computed if we know the geodesic distance and a parametrization of geodesic curves on $\mathcal{M}^{n+1}$. This is the case for the hyperbolic space $\mathbb{H}^{n}$ for all $n$. This manifold has a lot of important properties. As such, it appears to be the perfect toy model (usually its Lorentzian version $\text{AdS/dS}$) in a number of domain: chaos, cosmology, string theory, .... In the first part, we present our BGM-LMM-SABR model and show that the underlying geometry is $\mathbb{H}^{n+1}$ (with $n$ the number of forward Libor rates). Using this connection, we will find an asymptotic local volatility and an asymptotic swaption implied volatility.

4.1. Dynamics. We introduce the SABR-LMM model, given by the following SDE under the spot Libor measure $\mathbb{Q}$ (associated to the numéraire $B_d(t) = \prod_{j=1}^{\beta(t)-1} (1 + \tau_j F_j(T_j-1)) P(t,T_{\beta(t)-1})$ where $\beta(t) = m$ if $T_{m-2} < t < T_{m-1}$)

\[
\begin{align*}
  dF_k &= a^2 B^k(t,F) dt + \sigma_k(t)aC_k(F_k) dZ_k \\
  da &= -\nu^2 a^2 b^k(t,F) + \nu a dZ_{n+1}; \ dZ_i dZ_j = \rho_{ij}(t) dt & i, j = 1, \cdots, n + 1
\end{align*}
\]

with

\[
\begin{align*}
  C_k(F_k) &= \phi_k F_k^\beta_k \\
  B^k(t,F) &= \sum_{j=\beta(t)}^k \tau_j \rho_{jk} \sigma_k(t) \sigma_j(t) C_k(F_k) C_j(F_j) \quad \frac{1}{1 + \tau_j F_j}
\end{align*}
\]

The functions $C_k(F_k)$ have been scaled by $\phi_k$ and therefore we can impose that $\sigma_k(0) = 0$. The stochastic equation for $a$ was written in the spot Libor measure in order to get a SDE independent of a specific underlying swap $s_{\alpha,\beta}$ or a forward bond. Under the forward swap measure $\mathbb{Q}^{\alpha,\beta}$, we have

\[
\begin{align*}
  dF_k &= a^2 b^k(t,F) dt + \sigma_k(t)aC_k(F_k) dZ_k \\
  da &= -\nu^2 a^2 b^k(t,F) + \nu a dZ_{n+1}; \ dZ_i dZ_j = \rho_{ij}(t) dt & i, j = 1, \cdots, n + 1
\end{align*}
\]

with

\[
\begin{align*}
  b^k(t,F) &= \sum_{j=\alpha+1}^{\beta} (2j-k-1) \tau_j P(t,T_j) \sum_{i=\min(k+1,j+1)}^{\max(k,j)} \tau_i \rho_{ka} \sigma_i(t) \sigma_k(t) C_i(F_i) C_k(F_k) \quad \frac{1}{1 + \tau_i F_i} \\
  C_{\alpha,\beta}(t) &= \sum_{i=\alpha+1}^{\beta} \tau_i P(t,T_i) \\
  b^k(t,F) &= \sum_{i=\alpha+1}^{\beta} \tau_i \omega_i(t) \sum_{k=\beta(t)}^i \frac{\tau_k C_k(F_k) \rho_{ka} \sigma_k(t)}{1 + \tau_k F_k(t)}
\end{align*}
\]

and with $\omega_i(t) = \frac{\prod_{k=\beta(t)}^{\beta(t)} \frac{1}{1 + \tau_k F_k(t)}}{\sum_{j=\alpha+1}^{\beta} \tau_j P(t,T_j) \sum_{i=\min(k+1,j+1)}^{\max(k,j)} \tau_i \rho_{ka} \sigma_i(t) \sigma_k(t) C_i(F_i) C_k(F_k) \quad \frac{1}{1 + \tau_i F_i}}$. Note that the forward-rate dynamics under the forward measure $\mathbb{Q}^k$ is much simpler and given by the following stochastic differential equations (SDE)

\[dF_k(t) = \sigma_k(t)aC_k(F_k) dW_k, \ dW_k dW_p = \rho_{kp}(t) dt\]
As it is the case for the BGM model, we can use a piecewise parametric form or a functional form for the serial volatilities $\sigma_i(t)$ and the correlation $\rho_{ij}(t)$ (here full rank) as
\[
\begin{align*}
\sigma_i(t) &= N_i[(\alpha(T_i - t) + d)e^{-b(T_i - t)} + c] \forall t \leq T_i - 1 \\
\rho_{ij}(t = 0) &= \rho_L + (1 - \rho_L)e^{-(\delta_A - \delta_B \min(T_i - T_j - 1))}
\end{align*}
\]

The constants $N_i$ are fixed such as $\sigma_i(0) = 1$. The model depends on $9 + 3n$ parameters (see Tab. 2) which are calibrated on the swaption matrix. In the next subsection, we derive the metric, the geodesic distance and the Abelian connection underlying this model.

### 4.2. Hyperbolic geometry.

By definition, the infinitesimal distance (at $t = 0$) between the point $x^\alpha$ and $x^\alpha + dx^\alpha$ (5.1)\footnote{Electronic copy available at: https://ssrn.com/abstract=877762} is given by $\rho^{ij} \equiv [\rho^{-1}]_{ij}$, $(i, j) = (1, \cdots, n)$ and $\rho^{\alpha} \equiv [\rho^{-1}]_{\alpha \alpha}$ are the components of the inverse of the correlation matrix $\rho$.

\[
ds^2 = \frac{2}{\nu^2}a^2 \left( \sum_{i,j=1}^{n} \rho^{ij} \frac{\nu dF_i}{C_i(F_i)} \frac{\nu dF_j}{C_j(F_j)} + 2 \sum_{i=1}^{n} \rho^{\alpha} \nu dF_i d\alpha + \rho^{\alpha \alpha} da^2 \right)
\]

After some algebraic manipulations, we show that in the new coordinates $[x_k]_{k=1\cdots n+1}$ ($L$ is the Cholesky decomposition of the (reduced) correlation matrix: $[\hat{L}]_{i,j=1\cdots n}$),

\[
\begin{align*}
x_k &= \sum_{i=1}^{n} \nu \hat{k}_i \int_{F_i}^{F_i^0} \frac{dF_i}{C(F_i)} + \sum_{i=1}^{n} \rho^{\alpha} \hat{L}_{ik} a, \quad k = 1, \cdots, n \\
x_{n+1} &= (\rho^{\alpha \alpha} - \sum_{i,j}^{n} \rho^{\alpha} \rho^{\alpha \alpha} \rho_{ij})^{1/2} a
\end{align*}
\]

the metric becomes
\[
ds^2 = \frac{2(\rho^{\alpha \alpha} - \sum_{i,j}^{n} \rho^{\alpha} \rho^{\alpha \alpha} \rho_{ij}) \sum_{i=1}^{n} dx_i^2 + dx_{n+1}^2}{x_{n+1}^2}
\]

Written in the coordinates $[x_i]$, the metric is therefore the standard hyperbolic metric on $H^{n+1}$ modulo a constant factor $\mu = 2(\rho^{\alpha \alpha} - \sum_{i,j}^{n} \rho^{\alpha} \rho^{\alpha \alpha} \rho_{ij})$. This factor can be eliminated by doing a change of time $t' = \mu^{-1}t$. In order to compute our saddle-point (3.3), we need the geodesic distance which is given by [19]

**Proposition 4.2.1.** The geodesic distance $d(x, x')$ on $H^{n+1}$ is given by
\[
d(x, x') = \cosh^{-1}[1 + \frac{\sum_{i=1}^{n+1} (x_i - x_i')^2}{2x_{n+1}x_{n+1}'}]
\]

Using the geodesic distance on $H^{n+1}$ between the points $x = ([F]_k, a)$ and the initial point $x' = ([0]_k, \alpha)$, $q_i = [F_i, \frac{dF_i}{C_i(F_i)}]$ given by
\[
d(x, x') = \cosh^{-1}[1 + \frac{\nu^2 \sum_{i,j=1}^{n} \rho^{\alpha \alpha} q_i q_j + 2\nu(\alpha - \alpha) \sum_{i=1}^{n} \rho^{\alpha \alpha} q_i + (\alpha - \alpha)^2 \rho^{\alpha \alpha}}{2(\rho^{\alpha \alpha} - \sum_{i,j=1}^{n} \rho^{\alpha} \rho^{\alpha \alpha} \rho_{ij}) \alpha}]
\]

### Table 2. SABR-LMM: $9 + 3n$ parameters

| BGM parameters | a, b, c, d, $\phi_i, \rho L, \delta_A, \delta_B$ |
|----------------|-----------------------------------------------|
| CEV parameters | $\beta_i, i = 1, \cdots, n$                   |
| SABR parameters| $\alpha, \nu, \rho_{\alpha \alpha} i = 1, \cdots, n$ |
we derive the following non-linear equations \( 3.3 \) satisfied by the saddle-point \( a^*(s), q_i^* \) which implicitly depends on \( s \), the swaption strike:

\[
\begin{align*}
(4.4) & & a^*(s)^2 \rho^{aa} = \alpha^2 \rho^{aa} - 2\nu \sum_{i=1}^{n} \rho^{ia} q_i^* + \nu^2 \sum_{i,j=1}^{n} \rho^{ij} q_i^* q_j^* \\
(4.5) & & \frac{(\rho_{ij} (a^*(s) - \alpha))^n}{a^*(s) (\cosh(d(a^*, \{ q_i^* \})))^2 - 1} = \frac{\lambda \partial s_{\alpha \beta}}{\partial q_i} \\
\end{align*}
\]

with

\[
q_i^* = \phi_i^{-1} \int_{F_i^0}^{F_i^*} x^{-\beta_i} \, dx
\]

The saddle-point is determined by solving these non-linear equations \( 4.4, 4.5 \) and an approximation (which could be used as a guess solution in a numerical optimization routine) is found by linearizing these equations around the spot Libor rates (i.e. \( q_i = 0 \))

\[
\begin{align*}
(4.6) & & \lambda^*(s) = \sum_{\rho, q=1} \omega_{ij} \rho_{ij} q_{ij} \\
(4.7) & & \frac{F_i^*}{F_i^0} = 1 + \frac{\sum_{j=1}^{n} \tilde{\rho}_{ij} (s - s_0)}{\sum_{\rho, q=1} \omega_{ij} \rho_{ij} q_{ij}} + o((s - s_0)^2)
\end{align*}
\]

with \( \omega_i \equiv \partial s_{\alpha \beta} (q_i = 0) \) and \( \tilde{\rho}_{ij} = \rho_{ij} - \rho_{ia} \rho_{ja} / \rho^{aa} \). Note that when the strike is close to at-the-money, the saddle-points are close to the spot Libors and \( a^* = \alpha \). Moreover, by using the explicit expression for the hyperbolic distance, the Van-Blake-Morette determinant is

\[
\Delta(F, a, \alpha) = \frac{d(a, F|\alpha)}{\cosh^2(d(a, F|\alpha)) - 1}
\]

4.3. Connection. The Abelian connection is given by \( 5.1b \)

\[
\begin{align*}
\mathcal{A}_i & = \frac{1}{C_i(F_i)} \sum_{j=1}^{n} \rho_{ij} \left( \frac{b^i(t, F)}{C_j(F_j)} - \frac{\partial_t C_j(F_j)}{2} \right) - \nu \rho^{ia} b^a(F, t) \\
\mathcal{A}_a & = \frac{1}{\nu} \sum_{j=1}^{n} \rho_{ij} \left( \frac{b^i(t, F)}{C_j(F_j)} - \frac{\partial_t C_j(F_j)}{2} \right) - \nu \rho^{aa} b^a(F, t)
\end{align*}
\]

where we have used that

\[
\sqrt{g} = \frac{2^{n+1}}{\nu a^{n+1} \prod_{i=1}^{n} C_i(F_i)}
\]

Finally, the Abelian 1-form connection is

\[
\mathcal{A} = \frac{1}{\nu} \sum_{j=1}^{n} \left( \frac{b^i(t, F)}{C_j(F_j)} - \frac{\partial_t C_j(F_j)}{2} \right) \left( \nu \sum_{i=1}^{n} \rho_{ij} dq_i + \rho^{aa} da \right) - b^a(t, F)(\nu \sum_{i=1}^{n} \rho_{ij} dq_i + \rho^{aa} da)
\]

In order to compute the log of the parallel gauge transport \( ln(\mathcal{P})(a, q|\alpha) = \int_0^t \mathcal{A} \), we need to know a parametrization of the geodesic curve on \( \mathbb{H}^{n+1} \). However, we can directly find \( ln(\mathcal{P})(a, q|\alpha) \) if we approximate the drifts \( b^F(t, F) \) by their values at the Libor spots (and \( t = 0 \)). A similar approximation was done in the Hagan-al formula \( 11 \) as was shown in \( 12 \). Modulo this approximation,

\[
ln(\mathcal{P})(a, q|\alpha) \approx \frac{1}{\nu} \sum_{j=1}^{n} \left( \frac{b^i(0, F^0)}{C_j(F_j^0)} - \frac{\partial_t C_j(F_j^0)}{2} \right) \left( \nu \sum_{i=1}^{n} \rho_{ij} q_i + \rho^{aa} (a - \alpha) \right) - b^a(0, F^0)(\nu \sum_{i=1}^{n} \rho_{ij} q_i + \rho^{aa} (a - \alpha))
\]
4.4. Asymptotic Smile-Summary. The asymptotic local volatility is given by (3.3)

\[
(c_{loc}^{αβ})^2(t, s) = \sum_{i,j=1}^{n} \rho_{ij} \sigma_i(t) \sigma_j(t) f_{ij}(a, F)^2 (1 + 2') \sum_{i,j=1}^{n} A^{\nu \nu} \left( \frac{\partial f_{ij}(a^*, F^*)}{f_{ij}(a^*, F^*)} \right) + 2 \frac{\partial f_{ij}(a^*, F^*)}{f_{ij}(a^*, F^*)} \frac{\partial \nu \psi(a^*, F^*) - \nu \sigma(a^*, F^*)}{\nu \sigma(a^*, F^*)} - \sum_{i,j=1}^{n} A^{\nu \nu} \frac{\partial f_{ij}(a^*, F^*)}{f_{ij}(a^*, F^*)} \frac{\partial \nu \psi(a^*, F^*) - \nu \sigma(a^*, F^*)}{\nu \sigma(a^*, F^*)})
\]

with \((a^* ≡ a^*(s), F^* ≡ \{F_t^*(s)\}_t)\) the saddle-point satisfying the equations (4.4) approximated by (4.6) and

\[
f_{ij}(a, F) = a^2 C_i(F_i) C_j(F_j) \frac{\partial s_{αβ}}{\partial F_i} \frac{\partial s_{αβ}}{\partial F_j}, \quad \psi(a, F) = \sqrt{g} \Delta F, \quad A^{αβ} = [\partial_{αβ} \sigma^2]^{-1}
\]

\[
d(a, F) = \cosh^{-1} \left[ 1 + \frac{\nu^2 \sum_{i,j=1}^{n} \rho_{ij} q_i q_j + 2 \nu (a - α) \sum_{i=1}^{n} \rho_{ii} q_i + (a - α)^2)}{2(\rho_{aα} - \sum_{i,j=1}^{n} \rho_{ij} q_i q_j \alpha \alpha)} \right]
\]

\[
ln(P)(a, q(α)) \sim \frac{1}{\nu} \sum_{i=1}^{n} \left( \frac{b'(0, F^0)}{C_i(F_i)} - \frac{\partial C_j(F_j)}{2} \right) \left( \nu \sum_{i=1}^{n} \rho_{ii} q_i + \rho_{ii} (a - α) \right) - b'(0, F^0) \left( \nu \sum_{i=1}^{n} \rho_{ii} q_i + \rho_{ii} (a - α) \right)
\]

\[
\Delta(F, a, α) = \frac{d(a, F|α)}{\sqrt{\cosh^2(d(a, F|α)) - 1}}
\]

\[
\sqrt{g} = \frac{2 \nu + \text{det}[ρ]^{-\frac{1}{2}}}{\nu a^{1+n} \prod_{i=1}^{n} C_i(F_i)}
\]

Note that this expression is exact when \(t\) goes to zero. The smile at the first-order is then obtained by plugging the above expression into (3.4) with \(t' = \frac{\nu^2}{2(\rho_{aα} - \sum_{i,j=1}^{n} \rho_{ij} q_i q_j)} t\).

Remark 4.4.1 (Libor CEV model). Note that our model reduces for \(ν\) goes to zero (and \(α ≡ 1\)) to the Andersen-Andreasen CEV libor model (with different CEV parameters for each libors) and the above expressions degenerates into

\[
f_{ij}(F) = C_i(F_i) C_j(F_j) \frac{\partial s_{αβ}}{\partial F_i} \frac{\partial s_{αβ}}{\partial F_j}
\]

\[
d(F) = \sqrt{\sum_{i,j=1}^{n} \rho_{ij} q_i q_j}
\]

\[
ln(P)(q) = \sum_{i=1}^{n} \left( \frac{b'(0, F^0)}{C_i(F_i)} - \frac{\partial C_j(F_j)}{2} \right) \sum_{i=1}^{n} \rho_{ii} q_i
\]

\[
\Delta(F, F^0) = 1
\]

\[
\sqrt{g} = \frac{2 \nu + \text{det}[ρ]^{-\frac{1}{2}}}{\nu a^{1+n} \prod_{i=1}^{n} C_i(F_i)}
\]

with the saddle-points (4.4) satisfying the non-linear equations (modulo the constraint \(s_{αβ} = s\))

\[
\rho_{ij} q_i^* = λ \frac{\partial s_{αβ}}{\partial q_i}
\]
4.5. Comments and Numerical Tests. It is interesting to note that for \( n = 1 \), i.e. for a caplet, the caplet asymptotic smile reduces to the classical SABR formula by construction. Moreover, the asymptotic local volatility is given at the zero-order by

\[
(\sigma_{loc}^{n\beta})^2(s,t) = \sum_{i,j=1}^{n} \rho_{ij}(t)\sigma_i(t)\sigma_j(t)a_s^2(F^*)C_i(F^*)C_j(F^*)\frac{\partial s_{\alpha\beta}}{\partial F_i}(F^*)\frac{\partial s_{\alpha\beta}}{\partial F_j}(F^*)
\]

with \( F^* \) depending implicitly on \( s \) via (4.4), (4.5). At this stage, it is useful to recall how a similar asymptotic local volatility is derived using the "freezing" argument. The forward swap rate satisfies the following SDE in the forward swap numéraire \( Q^{\alpha\beta} \)

\[
ds_{\alpha\beta} = \sum_{k=1}^{n} \frac{\partial s_{\alpha\beta}}{\partial F_k} \sigma_k(t) a C_k(F_k) dZ_k
\]

The "freezing" argument consists in assuming that the terms \( \frac{\partial s_{\alpha\beta}}{\partial F_k} \) and \( C(s,F) \) are almost constant. Therefore, the SDE (4.8) can be approximated by

\[
ds_{\alpha\beta} = \sum_{k=1}^{n} \frac{\partial s_{\alpha\beta}}{\partial F_k} (F_0) \sigma_k(t) \frac{C_k(F_0)}{C_k(s)} C_k(s) dZ_k
\]

and the local volatility is

\[
(\sigma_{loc}^{n\beta})^2(s,t) = \sum_{i,j=1}^{n} \rho_{ij}(t)\sigma_i(t)\sigma_j(t)a_s^2(s)C_i(F_0)C_j(F_0)\frac{\partial s_{\alpha\beta}}{\partial F_i}(F_0)\frac{\partial s_{\alpha\beta}}{\partial F_j}(F_0)C_i(s)C_j(s)
\]

We can reproduce this formula for the swaption smile at-the-money \( ^3 \) as the saddle-point Libor rates coincides with the spot rates. This is not the case for in/out-the-money swaption. Therefore our expression (exact at the zero-order) shows that the freezing argument is no longer correct when we try to fit a swaption implied smile in/out-the-money. In the following, we have tested our expression (exact at the zero-order) against the Andersen-Andreasen asymptotic formula (Formula F2) \( ^1 \) in the case \( \nu = 0 \). The accuracy of these approximations are examined using Monte-Carlo (MC) prices as a benchmark. Following \( ^6 \), we consider five scearni (see Tables 3-4-5-6-7). In the following tables, the implied volatility is reported and the numbers in brackets are the errors (in basis points i.e. true volatility times 10\(^4\)) corresponding to the implied volatility computed using the F1 or F2 formula minus the MC implied volatility. An \( x \times y \) swaption has an option maturity of \( x \) years, a swap length of \( y \) years and a tenor of one year. We set a time-step for Monte-Carlo \( \delta = 0.125 \) and \( 2^{16} \) paths\( ^2 \). Our formula F1 is more accurate than F2.

| Scenario A: Libor volatility \( \lambda_i(t) = 5\% \). Libor \( L_i(0) = 6\% \). \( \beta = 0.5 \) |
|---|---|---|---|
| Swaption | strike | MC | F1 | F2 |
| 4\%(ITM) | 22.42% | 22.41% (-1) | 22.61% (19) |
| 5 \times 15 | 6\%(ATM) | 20.33% | 20.41% (8) | 20.46% (13) |
| 8\%(OTM) | 18.92% | 18.93% (1) | 19.01% (10) |
| 4\%(ITM) | 22.41% | 22.51% (11) | 22.67% (26) |
| 10 \times 10 | 6\%(ATM) | 20.38% | 20.41% (3) | 20.50% (12) |
| 8\%(OTM) | 18.93% | 18.93% (-1) | 19.05% (12) |

\( ^3 \)An at-the-money swaption (ATM) has a strike \( K \) equal to the spot rate \( s_{\alpha\beta}(0) \) and an out-of-the money (OTM) (resp. in-the-money (ITM)) swaption has \( K < s_{\alpha\beta}(0) \) (resp. \( K > s_{\alpha\beta}(0) \)).

\( ^2 \)We have used a predictor-corrector scheme with a Brownian bridge.

Electronic copy available at: https://ssrn.com/abstract=877762
Swaption & Strike & MC & F1 & F2 \\
\hline
5.08\% (ITM) & 18.12\% & 18.20\% (8) & 18.17\% (5) \\
5 \times 15 & 7.26\% (ATM) & 16.51\% & 16.61\% (10) & 16.63\% (12) \\
& 9.44\% (OTM) & 15.38\% & 15.38\% (0) & 15.56\% (18) \\
& 5.55\% (ITM) & 17.80\% & 17.81\% (1) & 17.89\% (9) \\
& 10 \times 10 & 7.93\% (ATM) & 16.26\% & 16.33\% (7) & 16.38\% (11) \\
& 10.31\% (OTM) & 15.17\% & 15.19\% (2) & 15.32\% (15) \\
\hline

Table 4. Scenario B: Libor volatility $dL_t = 0.25(0.17 + 0.002(T_{i-1} - t))L_t^3dW_t$. Libor $L_i(0) = \log(a + bi)$. $L_0(0) = 5\%$, $L_19(0) = 9\%$. $\beta = 0.5$

Swaption & Strike & MC & F1 & F2 \\
\hline
5.08\% (ITM) & 14.89\% & 14.97\% (8) & 15.08\% (19) \\
5 \times 15 & 7.26\% (ATM) & 13.73\% & 13.79\% (4) & 13.81\% (8) \\
& 9.44\% (OTM) & 12.92\% & 12.91\% (-1) & 12.92\% (0) \\
& 5.55\% (ITM) & 14.52\% & 14.53\% (1) & 14.64\% (12) \\
& 10 \times 10 & 7.93\% (ATM) & 13.33\% & 13.38\% (5) & 13.40\% (7) \\
& 10.31\% (OTM) & 12.51\% & 12.51\% (0) & 12.54\% (3) \\
\hline

Table 5. Scenario C: Libor volatility $dL_t = 0.25(0.17 - 0.002(T_{i-1} - t))L_t^3dW_t$. Libor $L_i(0) = \log(a + bi)$. $L_0(0) = 5\%$, $L_19(0) = 9\%$. $\beta = 0.5$

Swaption & Strike & MC & F1 & F2 \\
\hline
5.08\% (ITM) & 19.19\% & 19.33\% (14) & 19.38\% (19) \\
5 \times 15 & 7.26\% (ATM) & 17.59\% & 17.72\% (13) & 17.75\% (16) \\
& 9.44\% (OTM) & 16.46\% & 16.49\% (3) & 16.61\% (15) \\
& 5.55\% (ITM) & 18.92\% & 18.94\% (2) & 19.06\% (14) \\
& 10 \times 10 & 7.93\% (ATM) & 17.31\% & 17.39\% (8) & 17.45\% (14) \\
& 10.31\% (OTM) & 16.18\% & 16.21\% (3) & 16.32\% (14) \\
\hline

Table 6. Scenario D: $dL_t = 0.05L_t^3\left(\frac{b_1(t)}{\sqrt{b_1(t)^2 + b_2(t)^2}}dW_1 + \frac{b_2(t)}{\sqrt{b_1(t)^2 + b_2(t)^2}}dW_2\right)$. $b_1(t) = \rho e^{-k_1(T_{i-1} - t)} + \theta e^{-k_2(T_{i-1} - t)}$, $b_2(t) = \sqrt{1 - \rho^2}e^{-k_1(T_{i-1} - t)}$. $\rho = 0.99$, $\theta = -0.99$, $k_1 = k_2 = 0.54$. Libor $L_i(0) = \log(a + bi)$. $L_0(0) = 5\%$, $L_19(0) = 9\%$. $\beta = 0.5$

5. Conclusion

In this short note, we have introduced a LMM model coupled to a SABR stochastic volatility process. By using the heat kernel expansion technique in the short time limit, we have obtained an asymptotic swaption implied volatility at the first-order, compatible with the Hagan-al classical formula for caplets. Moreover, we have seen that this exact expression (when the expiry is very short) is incompatible with the analog expression obtained using the freezing argument.
Table 7. Scenario E: Scenario D with $\beta = 0.3$

| Swaption | Strike MC | F1 | F2 |
|----------|-----------|----|----|
| 5 x 15   | 7.26% (ATM) | 29.47% | 28.47% |
| 9.44% (OTM) | 26.92% | 27.14% |
| 10 x 10  | 7.93% (ITM) | 33.65% | 28.88% |
| 5.55% (ITM) | 31.76% | 27.49% |
| 10.31% (OTM) | 26.01% | 29.96% |
| 26.92% | 7.26% (ATM) | 26.18% | 26.01% |
| 26.68% (67) | 26.88% (41) | 28.18% (17) | 26.18% |

### Appendix A: Heat Kernel Expansion

An short-time expansion of the conditional probability for a multi-dimensional Ito diffusion process can be achieved using the heat kernel expansion. In that purpose, the Kolmogorov equation is rewritten as the heat kernel equation on a $(n)$-dimensional Riemannian manifold $M^n$ endowed with an Abelian connection as explained in [12] [13] [14]. Let’s assume that our multi-dimensional stochastic equations (in $Q^n$) are written as

$$dx^\mu = b^\mu(x,t)dt + \sigma^\mu(x,t)dW^\mu$$

with $dW^\mu dW^\nu = \rho_{\mu\nu}(t)dt$. Then, the metric $g_{\mu\nu}$ depends only on the diffusion terms $\sigma_\mu$ and the connection $A_\mu$ on the drift terms $b^\mu$ as well

\begin{align}
(5.1a) & \quad g_{\mu\nu}(t, x) = \frac{2\rho^{\nu\mu}(t)}{\sigma_\mu(t, x)\sigma_\nu(t, x)}, \quad \mu, \nu = 1 \ldots n, \quad \rho^{\mu\nu} \equiv [\rho^{-1}]_{\mu\nu} \\
(5.1b) & \quad A^\mu(t, x) = \frac{1}{2}(b^\mu(t, x) - \sum_{\nu=1}^{n+1} g^{-\frac{1}{2}} \partial_\nu(g^{\nu\nu}(g^{\mu\nu}(t, x)))) , \quad \mu = 1 \ldots n + 1
\end{align}

with $g(t, x) \equiv det[g_{\mu\nu}(t, x)]$. In terms of these functions, the asymptotic solution to the Kolmogorov equation in the short-time limit is given by

$$p(x, t|x^0) = \frac{\sqrt{g(x)}}{(4\pi t)^{\frac{n}{2}}} \sqrt{\Delta(x, x^0)} \mathcal{P}(x, x^0)e^{-\frac{2t}{\rho(x^0)}} \sum_{n=1}^{\infty} a_n(x, x^0)t^n , \quad t \to 0$$

- Here, $\sigma(x, x^0)$ is the Synge world function equal to one half of the square of geodesic distance $d(x, x^0)$ between $x$ and $x^0$ for the metric $g_{\mu\nu}(x, t = 0)$. This distance is defined as the minimizer of

$$d(x, x^0)^2 = \min_{C} \int_0^T g_{\mu\nu}(x, t = 0) \frac{dx^\mu}{dt} \frac{dx^\nu}{dt} dt$$

and $t$ parameterizes the curve $C(x, x^0)$ joining $x(t = 0) \equiv x^0$ and $x(T) \equiv x$.

- $\Delta(x, x^0)$ is the so-called Van Vleck-Morette determinant

$$\Delta(x, x^0) = g(x, 0)^{-\frac{1}{2}} det(-\frac{\partial^2 \sigma(x, x^0)}{\partial x \partial x^0})g(x^0, 0)^{-\frac{1}{2}}$$

and $g(x, 0) = det[g_{\mu\nu}(x, 0)]$

- $\mathcal{P}(x, x^0)$ is the parallel transport of the Abelian connection along the geodesic $C(x^0, x)$ from the point $x$ to $x^0$

$$\mathcal{P}(x^0, x) = e^{-\int_{C(x^0, x)} A_\mu(t, x^0, x)dx^\mu}$$
The \( a_i(x, x^0) (a_0(x, x^0) = 1) \) are smooth functions on \( M \) and depend on geometric invariants such as the scalar curvature \( R \). More details can be found in [12].

**Appendix B: Saddle-point method**

The integration over \( B \) is obtained by using a saddle-point method which consists in approximating at the first order the integral \( \int f(x) e^{\phi(x)} dx \) in the limit \( \epsilon \) large by [8]

\[
\int f(x) e^{\phi(x)} dx \sim_{\epsilon > 1} f(x^\ast) e^{\phi(x^\ast)} (1 + \frac{1}{\epsilon} \left( - \frac{\partial_{\alpha\beta}}{2f} A_{\alpha\beta} + \frac{\partial_{\alpha} f}{2f} \partial_{\beta\gamma} \phi A_{\alpha\beta} A_{\gamma\delta} \right) + \frac{1}{8} \frac{\partial_{\alpha\beta\gamma\delta}}{72} \phi A_{\alpha\beta} A_{\gamma\delta} A_{\mu\nu})
\]

with \( A^{\alpha\beta} = [\partial_{\alpha\beta} \phi]^{-1} \), \( dx \equiv \prod_{i=1}^n dx_i \) and \( x^\ast \) the saddle-point (which minimizes \( \phi(x) \)). This expression can be obtained by developing \( \phi(x) \) and \( f(x) \) in series around \( x^\ast \). The quadratic part in \( \phi(x) \) leads to a Gaussian integration over \( x \) which can be performed.

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