Nonexistence of quasi-harmonic sphere with large energy

Jiayu Li\textsuperscript{a}, Yunyan Yang\textsuperscript{b}

\textsuperscript{a}Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100190, P. R. China
\textsuperscript{b}Department of Mathematics, Information School, Renmin University of China, Beijing 100872, P. R. China

Abstract

Nonexistence of quasi-harmonic spheres is necessary for long time existence and convergence of harmonic map heat flows. Let \( (N, h) \) be a complete noncompact Riemannian manifolds. Assume the universal covering of \((N, h)\) admits a nonnegative strictly convex function with polynomial growth. Then there is no quasi-harmonic spheres \( u : \mathbb{R}^n \to N \) such that

\[
\lim_{r \to \infty} r^n e^{-\frac{|x|^2}{4}} \int_{|x| \leq r} e^{-\frac{|u|^2}{4}} |\nabla u|^2 dx = 0.
\]

This generalizes a result of the first named author and X. Zhu (Calc. Var., 2009). Our method is essentially the Moser iteration and thus very simple.

Key words: Quasi-harmonic sphere; Harmonic map heat flow

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1. Introduction

Let \((N, h)\) be a complete noncompact Riemannian manifolds. By the Nash embedding theorem, there exists sufficiently large \( K \) such that \((N, h)\) is isometrically embedded in \( \mathbb{R}^K \). We say that a map \( w : \mathbb{R}^n \to N \hookrightarrow \mathbb{R}^K \) is a quasi-harmonic sphere if it satisfies

\[
\Delta u = \frac{1}{2} x \cdot \nabla u + A(u)(\nabla u, \nabla u),
\]

where \( A(\cdot, \cdot) \) is the second fundamental form of \((N, h)\) in \( \mathbb{R}^K \). The Quasi-harmonic sphere arose from the study of singularities of harmonic map heat flows \[3, 7\]. It is closely related to the global smoothness and convergence of the harmonic map heat flow \[1, 2, 3\].

In \[6\], we have proved that if the universal covering of \((N, h)\) admits a nonnegative strictly convex function with polynomial growth, then there is no quasi-harmonic spheres of finite energy, namely

\[
\int_{\mathbb{R}^n} e^{-\frac{|u|^2}{4}} |\nabla u|^2 dx < \infty.
\]

The proof is based on the monotonicity inequality for \( u \) and John-Nirenberg inequality. In this note, we will use the Moser iteration to prove a stronger result. Precisely we have the following:
Theorem 1.1 Let $(N, h)$ be a complete Riemannian manifold. Assume that $u$ is a quasi-harmonic sphere from $\mathbb{R}^n$ ($n \geq 3$) to $(N, h)$. Let $(\tilde{N}, \tilde{h})$ be the universal covering of $(N, h)$. Suppose $(\tilde{N}, \tilde{h})$ admits a nonnegative strictly convex function $\tilde{f} \in C^2(\tilde{N})$ with polynomial growth, i.e. $\nabla^2 \tilde{f}(y)$ is positive definite for every $y \in \tilde{N}$ and

$$\tilde{f}(y) \leq C(1 + \tilde{d}(y, y_0))^{2m}$$

for some $y_0 \in \tilde{N}$ and positive integer $m$, where $\tilde{d}(y, y_0)$ is the distance between $y$ and $y_0$. If

$$\lim_{r \to \infty} r^n e^{-\frac{r^2}{4}} \int_{|x| \leq r} e^{-\frac{|u|^2}{r^2}} |\nabla u|^2 dx = 0,$$  \hspace{1cm} (1.3)

then $u$ is a constant map.

We remark that if $u$ satisfies (1.3), then its energy $\int_{\mathbb{R}^n} e^{-\frac{|u|^2}{r^2}} |\nabla u|^2 dx$ may be infinite. In this sense, the conclusion of Theorem 1.1 is stronger than that of [6]. When $(N, h)$ is the standard real line $\mathbb{R}$, the quasi-harmonic sphere becomes a quasi-harmonic function, which is a solution to the equation

$$\Delta u - \frac{1}{2} x \cdot \nabla u = 0 \quad \text{in} \quad \mathbb{R}^n.$$ 

To prove Theorem 1.1, here we will use the Moser iteration instead of using the monotonicity inequality for quasi-harmonic sphere and the John-Nirenberg inequality for BMO space in [6]. Avoiding hard work from harmonic analysis, our method looks very simple.

A special case of Theorem 1.1 is the following:

Corollary 1.2 Let $u$ be a quasi-harmonic function. If (1.3) is satisfied, then $u$ is a constant.

In view of Theorem 4.2 in [5], any positive quasi-harmonic function $u : \mathbb{R}^n \to \mathbb{R}$ with polynomial growth must be a constant. This is based on the gradient estimate. Its assumption can be interpreted by

$$\int_{|x| \leq r} e^{-\frac{|u|^2}{r^2}} |\nabla u|^2 dx \leq C(n)P(r),$$  \hspace{1cm} (1.4)

where $C(n)$ is a universal constant and $P(r)$ is a polynomial with respect to $r$. Obviously the hypothesis (1.3) is much weaker than (1.4). Hence the conclusion of Corollary 1.2 is better than that of Theorem 4.2 in [5].

In the remaining part of this note, we will prove Theorem 1.1.

2. Proof of Theorem 1.1

Let $u : \mathbb{R}^n \to (N, h) \to \mathbb{R}^K$ be a quasi-harmonic sphere satisfying (1.1). Denote

$$w(r) = \int_{S^{n-1}} (2|u|^2 - |\nabla u|^2) d\theta = \int_{S^{n-1}} (|u|^2 - \frac{1}{r^2} |u|^2) d\theta.$$  \hspace{1cm} (2.1)
It follows from (1.1) that \( \langle \Delta u, u_r \rangle = \frac{\partial}{\partial r} |u_r|^2 \), and thus \( \int_{S^{n-1}} \langle \Delta u, u_r \rangle d\theta = \frac{\partial}{\partial r} \int_{S^{n-1}} |u_r|^2 d\theta \). Integration by parts implies

\[
\frac{d}{dr} w(r) = \int_{S^{n-1}} \left( \frac{2}{r} |u_r|^2 + \left( r - \frac{2n - 2}{r} \right) |u_r|^2 \right) d\theta. \tag{2.2}
\]

For details of deriving (2.2), we refer the reader to [6].

**Lemma 2.1** Let \( w(r) \) be defined by (2.1), \( w^+(r) \) be the positive part of \( w(r) \), and \( u \) be a quasi-harmonic sphere from \( \mathbb{R}^n \) to \( (N, h) \). Suppose

\[
\int_0^\infty e^{-\frac{r^2}{4}} w^+(r)r^{n-1} dr \leq o(r^n e^{\frac{r^2}{4}}) \quad \text{as} \quad r \to \infty. \tag{2.3}
\]

Then there exists a constant \( C \) depending only on \( n \) and \( w(2n) \) such that

\[
\int_{\mathbb{R}^n} (d_N(u(x), u(0)))^2 \ dx \leq Cr^{n+1},
\]

where \( d_N(\cdot, \cdot) \) denotes the distance function on \( (N, h) \).

**Proof.** We can see from (2.2) that \( w'(r) \geq 0 \) for \( r \geq \sqrt{2n - 2} \). We claim that \( w(r) \leq 0 \) for every \( r \geq \sqrt{2n - 2} \). Suppose not, there exists some \( r_0 \geq \sqrt{2n - 2} \) such that \( w(r_0) > 0 \). Then \( w(r) \geq w(r_0) > 0 \) for every \( r > r_0 \) and

\[
w'(r) \geq \left( r - \frac{2n - 2}{r} \right) w(r). \tag{2.4}
\]

We have by integrating \( w'(r)/w(r) \) from \( r_0 \) to \( r \)

\[
w(r) \geq w(r_0) e^{r_0/r} r^{2n-2} e^{-\frac{2n-2}{r} r^n}.
\]

Hence

\[
\int_{r_0}^\infty e^{-\frac{r^2}{4}} w(r)r^{n-1} dr \geq w(r_0) e^{r_0/r} \int_{r_0}^\infty r^{1-n} e^{-\frac{2n-2}{r} r^n} dr
\]

\[
\geq w(r_0) e^{r_0/r} \int_{r_0}^\infty r e^{\frac{r^2}{4}} dr
\]

\[
= 2w(r_0) e^{r_0/r} \int_{r_0}^\infty r^{n-2} e^{-\frac{2n-2}{r} r^n} dr.
\]

This contradicts the assumption (2.3) and thus confirms our claim.

Now we estimate the growth order of the integral \( \int_{\mathbb{R}^n} (d_N(u(x), u(0)))^2 \ dx \). For simplicity, we denote \( d_N(u(x), u(0)) \) by \( d_N(x) \). In the polar coordinates in \( \mathbb{R}^n \), we always identify \( (r, \theta) \) with \( x \). Notice that \( d_N(r, \theta) \leq \int_0^r |u_r| ds \) and one needs the following estimates, which can be obtained by
using the Hölder inequality, the above claim, (2.1) and (2.4).

\[
\int_{S^{n-1}} \left( \int_0^r |u_r| ds \right)^2 d\theta \leq \int_{S^{n-1}} r \left( \int_0^r |u_r|^2 ds \right) d\theta \leq r \int_0^{2n} \int_{S^{n-1}} |u_r|^2 d\theta ds + r \int_{2n}^{\infty} \int_{S^{n-1}} |u_r|^2 d\theta ds \leq Cr + r \int_{2n}^{\infty} \frac{w'(s)}{s^{n-2}} ds \leq Crw(s)ds \leq Crw(2n) \leq Cr,
\]

where \( C \) is a constant depending only on \( n \) and \( w(2n) \). Hence we have

\[
\int_{B_r} d_n^2(x)dx \leq \int_0^{2n} \int_{S^{n-1}} \left( \int_0^r |u_r| ds \right)^2 d\theta \leq C \int_0^{r^2} t^{n-1} dt \leq C r^{n+1}.
\]

This concludes the lemma. \( \square \)

The following Lemma is elementary:

**Lemma 2.2** For every function \( f \) defined on \( \mathbb{R}^n \), if there exists \( k \in \mathbb{N} \) such that

\[
\int_{B_r} |f(x)| dx \leq C_1 r^k + C_2
\]

for some constants \( C_1 \) and \( C_2 \), then we have

\[
\int_{\mathbb{R}^n} e^{-\frac{|x|^2}{4}} |f(x)| dx < \infty.
\]

**Proof.** For sufficiently large \( r \), it is easy to see that

\[
\int_{\mathbb{R}^n \setminus B_r} e^{-\frac{|x|^2}{4}} |f(x)| dx = \sum_{j=1}^{\infty} \int_{B_{2^j} \setminus B_{2^{j-1}}} e^{-\frac{|x|^2}{4}} |f(x)| dx \leq \sum_{j=1}^{\infty} e^{-4^{-2j^2}} \int_{B_{2^j}} |f(x)| dx \leq \sum_{j=1}^{\infty} e^{-4^{-2j^2}} (2^k) C_1 r^k + C_2) \leq Cr^k e^{-\frac{|x|^2}{4}}.
\]
for some constant $C$ depending only on $C_1$ and $C_2$. This immediately implies

$$
\lim_{r \to \infty} \int_{\mathbb{R}^n \cap B_r} e^{-\frac{|x|^2}{4}} |f(x)| \, dx = 0,
$$

and thus gives the desired result. \qed

We will use the Moser iteration of the following simple version (see for example Chapter 8 in [8]):

**Theorem A** Let $u \geq 0$ be a weak solution of $\text{div}(a\nabla u) \geq 0$ in $B_{2\delta}(x_0)$, where $\delta > 0$ is a constant, $x_0 \in \mathbb{R}^n$, $a = a(x)$ satisfies $0 < \lambda \leq a(x) \leq \Lambda$ in $B_{2\delta}(x_0)$. Then for any $p > 0$, there exists a constant $C$ depending only on $\Lambda/\lambda$, $n$ and $p$ such that

$$
\sup_{B_{\delta}(x_0)} u \leq C \left( \frac{1}{|B_{2\delta}(x_0)|} \int_{B_{2\delta}(x_0)} u^p \, dx \right)^{1/p}.
$$

For application of Theorem A, the following observation is crucial:

**Lemma 2.3** Let $\rho(x) = e^{-\frac{|x|^2}{4}}$ on $\mathbb{R}^n$. Then for all $r > 1$ and $x^* \in \mathbb{B}_r = \{x \in \mathbb{R}^n : |x| \leq r\}$, there holds

$$
\sup_{x, y \in \mathbb{B}_r(x^*)} \frac{\rho(x)}{\rho(y)} \leq e^2.
$$

**Proof.** Assume $x \in \mathbb{B}_{\frac{3}{4}}(x^*)$. It is easy to see that

$$
\left( |x^-|^2 - \frac{2}{r} \right)^2 \leq |x|^2 \leq \left( |x^-|^2 + \frac{2}{r} \right)^2.
$$

Hence for $x, y \in \mathbb{B}_{\frac{3}{4}}(x^*)$,

$$
\frac{\rho(x)}{\rho(y)} \leq \exp \left( \frac{1}{4} \left( |x^-|^2 + \frac{2}{r} \right)^2 - \frac{1}{4} \left( |x^-|^2 - \frac{2}{r} \right)^2 \right) \leq \exp \left( \frac{2|x^-|^2}{r} \right).
$$

Note that $x^* \in \mathbb{B}_{\frac{3}{4}}$, we get the desired result. \qed

Now we are ready to prove Theorem 1.1 by using Theorem A.

**Proof of Theorem 1.1.** Let $\tilde{f} \in C^2(\tilde{N})$ be a nonnegative strictly convex function with polynomial growth, $u : \mathbb{R}^n \to (N, h) \leftrightarrow \mathbb{R}^K$ be a quasi-harmonic sphere, and $\tilde{u} \in C^2(\tilde{N})$ be a lift of $u$. Define a function $\phi = \tilde{f} \circ \tilde{u}$. Let $\rho(x) = e^{-\frac{|x|^2}{4}}$. Then we have by a straightforward calculation

$$
\text{div}(\rho \nabla \phi) = \rho \nabla^2 \tilde{f}(\tilde{u}(x))(\nabla \tilde{u}, \nabla \tilde{u}) \geq 0.
$$

(2.5)
Assume $x^* \in \mathbb{B}_r$, such that $\phi(x^*) = \sup_{\mathbb{B}_r} \phi$. It follows from the weak maximum principle for (2.5) that $x^* \in \partial \mathbb{B}_r$. By Lemma 2.3, we can apply Theorem A to the equation (2.5) in the ball $\mathbb{B}_{\frac{1}{2}}(x^*)$. This together with the hypothesis on $\tilde{f}$ implies that for any $p > 0$ and $r > 1$

$$\left( \frac{1}{|\mathbb{B}_r|} \int_{\mathbb{B}_r} \phi^2 \, dx \right)^{1/2} \leq \sup_{\mathbb{B}_r} \phi \leq \sup_{\mathbb{B}_{\frac{1}{2}}(x^*)} \phi \leq C \left( \frac{1}{|\mathbb{B}_{\frac{1}{2}}(x^*)|} \int_{\mathbb{B}_{\frac{1}{2}}(x^*)} \phi^p \, dx \right)^{1/p} \leq C r^{n/p} \left( \int_{\mathbb{B}_{\frac{1}{2}}(x^*)} \phi^p \, dx \right)^{1/p} \leq C r^{n/p} \left( \int_{\mathbb{B}_{\frac{1}{2}}(x^*)} (1 + \tilde{d}^{2mp}(x)) \, dx \right)^{1/p},$$

where $\tilde{d}(x) = \tilde{d}_{\tilde{N}}(\tilde{u}(x), \tilde{u}(0))$ denotes the distance between $\tilde{u}(x)$ and $\tilde{u}(0)$ on the universal covering space $\tilde{N}$ of $N$. $C$ is some constant depending only on $n$ and $p$. Clearly the assumption (1.3) implies (2.3). Notice that Lemma 2.1 still holds when $u$ is replaced by $\tilde{u}$, we have by choosing $p = 1/m$ in the above inequality,

$$\int_{\mathbb{R}^n} \phi^2 \, dx \leq C r^{n+2(n+1)2m},$$

where $C$ is a constant depending only on $n$, $m$ and $\tilde{u}$. From Lemma 2.2, we can see that

$$\int_{\mathbb{R}^n} \rho \phi^2 \, dx < \infty. \quad (2.6)$$

Take a cut-off function $\eta \in C_0^\infty(\mathbb{B}_{2r})$, $\eta \geq 0$ on $\mathbb{B}_{2r}$, $\eta \equiv 1$ on $\mathbb{B}_r$, and $|\nabla \eta| \leq \frac{2}{r}$. Testing the equation (2.5) by $\eta^2 \phi$, we obtain

$$\int_{\mathbb{R}^n} \eta^2 \rho |\nabla \phi|^2 \, dx \leq - \int_{\mathbb{R}^n} 2 \eta \rho \eta \nabla \eta \nabla \phi \, dx \leq 2 \left( \int_{\mathbb{R}^n} \eta^2 \rho |\nabla \phi|^2 \, dx \right)^{1/2} \left( \int_{\mathbb{R}^n} \phi^2 \rho |\nabla \eta|^2 \, dx \right)^{1/2}.$$

This together with (2.6) leads to

$$\int_{\mathbb{B}_r} \rho |\nabla \phi|^2 \, dx \leq \frac{C}{r^2}$$

for some constant $C$ depending only on the integral in (2.6). Passing to the limit $r \to \infty$, we have $|\nabla \phi| \equiv 0$, which together with (2.5) and that $\nabla^2 \tilde{f}$ is positive definite implies that $|\nabla \tilde{u}| \equiv 0$. Hence $\tilde{u}$ is a constant map and thus $u$ is also a constant map. \hfill \square

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References

[1] K. Chang, W. Ding and R. Ye: Finite time blow-up of the heat flow of harmonic maps, *J. Diff. Geom.* **36**: 507-515, 1992.

[2] W. Ding and F. Lin: A generalization of Eells-Sampson’s theorem, *J. Partial Diff. Eq.* **5**: 13-22, 1992.

[3] J. Li and G. Tian: A blow up formula for stationary harmonic maps, *Inter. Math. Res. Notices* **14**: 735-755, 1998.

[4] M. Struwe: On the evolution of harmonic maps in higher dimensions, *J. Diff. Geom.* **28**: 485-502, 1988.

[5] J. Li and M. Wang: Liouville theorems for self-similar solutions of heat flows, *J. Eur. Math. Soc.* **11**: 207-221, 2009.

[6] J. Li and X. Zhu: Non existence of quasi-harmonic spheres, *Cal. Var.*, 2009, DOI 10.1007/s00526-009-0271-0

[7] F. Lin and C. Wang: Harmonic and quasi-harmonic spheres, *Commun. Anal. Geom.* **7**: 397-429, 1999.

[8] D. Gilbarg and N. Trudinger: Elliptic partial differential equations of second order. Springer-Verlag Berlin Heidelberg, 2001.