Reflections in Conics, Quadrics and Hyperquadrics via Clifford Algebra

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Abstract.
In this paper we present a new and not fully employed geometric algebra model. With this model a generalization of the conformal model is achieved. We discuss the geometric objects that can be represented. Furthermore, we show that the Pin group of this geometric algebra corresponds to inversions with respect to axis aligned quadrics. We discuss the construction for the two- and three-dimensional case in detail and give the construction for arbitrary dimension.

Key Words: Clifford algebra, geometric algebra, generalized inversion, conic, quadric, hyperquadric.
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Introduction

1. Algebraic Background
Before we introduce the conic geometric algebra we review some algebraic background.

1.1. Geometric Algebra

Definition 1: Let $V$ be a real valued vector space of dimension $n$. Furthermore, let $b : V \rightarrow \mathbb{R}$ be a quadratic form on $V$. The pair $(V, b)$ is called quadratic space.

We denote the Matrix belonging to $b$ by $B_{ij}$ with $i \leq j, j \leq n$. Therefore $b(x_i, x_j) = B_{ij}$ for some basis vectors $x_i$ and $x_j$.

Definition 2: The Clifford algebra is defined by the relation

$$x_i x_j + x_j x_i = 2B_{ij}, \quad 1 \leq i, j \leq n. \quad (1)$$

Usually the algebra is denoted by $\mathcal{C}l(V, b)$. By Silvester’s law of inertia we can always find a basis $\{e_1, \ldots, e_n\}$ of $V$ such that $e_i^2$ is either 1, -1 or 0.
**Definition 3:** The number of basis vectors that square to \((1, -1, 0)\) is called *signature* \((p, q, r)\). If \(r \neq 0\) we call the geometric algebra *degenerated*. We will denote a Clifford algebra by \(\mathcal{C} \ell_{(p,q,r)}\).

**Remark 1:** A quadratic real space with signature \((p, q, 0)\) is abbreviated by \(\mathbb{R}^{p,q}\).

With the new basis \(\{e_1, \ldots, e_n\}\) the relations \((1)\) become

\[ e_i e_j + e_j e_i = 0, \quad i \neq j. \]

In the remainder of this paper we shall abbreviate the product of basis elements with lists

\[ e_{12\ldots k} := e_1 e_2 \ldots e_k, \text{ with } 0 \leq k \leq n. \]

The \(2^n\) monomials

\[ e_{i_1} e_{i_2} \ldots e_{i_k}, \quad 0 \leq k \leq n \]

form the standard basis of the Clifford algebra. Furthermore, a Clifford algebra is the direct sum \(\bigoplus_{i=0}^n \bigwedge^i V\) of all exterior products \(\bigwedge^i V\) of any grade \(0 \leq i \leq n\) where \(e_{k_1} \ldots e_{k_i}\), \(k_1 < \cdots < k_i\) form a basis of \(\bigwedge^i V\). Therefore, a Clifford algebra is a graded algebra and its dimension is calculated by

\[ \dim \mathcal{C} \ell_{(p,q,r)} = \sum_{i=0}^n \dim \bigwedge^i V = \sum_{i=0}^n \binom{n}{i} = 2^n. \]

Moreover, the Clifford algebra \(\mathcal{C} \ell_{(p,q,r)}\) can be decomposed in an even and an odd part

\[ \mathcal{C} \ell_{(p,q,r)} = \mathcal{C} \ell^+_{(p,q,r)} \oplus \mathcal{C} \ell^-_{(p,q,r)} = \bigoplus_{i \text{ even}} \bigwedge^i V \oplus \bigoplus_{i \text{ odd}} \bigwedge^i V. \]

The even part \(\mathcal{C} \ell^+_{(p,q,r)}\) is a subalgebra, because the product of two even graded monomials must be even graded since the generators cancel only in pairs. Elements contained in \(\bigwedge^i V\) are called *i-blades* and the \(\mathbb{R}\)-linear combination of i-blades is called a *multi-vector*. The product of invertible 1-blades is called a *versor*.

### 1.2. Clifford Algebra Automorphisms

For our purposes two automorphisms that exist on each Clifford algebra are interesting. The *conjugation* is an *anti-involution* denoted by an asterisk, see [5]. Its effect on generators is given by \(e_i^* = -e_i\). There is no effect on scalars. Extending the conjugation by using linearity yields

\[ (e_{i_1} e_{i_2} \ldots e_{i_k})^* = (-1)^k e_{i_k} \ldots e_{i_2} e_{i_1}, \quad 0 \leq i_1 < i_2 < \cdots < i_k \leq n. \]  \(\text{(2)}\)

The geometric product of a 1-blade \(v = \sum_{i=1}^n x_i e_i \in \bigwedge^1 V\) with its conjugate results in

\[ vv^* = -x_1^2 - x_2^2 - \cdots - x_p^2 + x_{p+1}^2 + \cdots x_{p+q}^2 = -b(v,v), \]

where \(v = (x_1, \ldots x_n)^T \in \mathbb{R}^n\).
Definition 4: The inverse element for a versor \( v \in \mathcal{C}\ell(p,q,r) \) is defined by

\[
v^{-1} := \frac{v^*}{N(v)},
\]

with \( N(v) := vv^* \).

The map \( N : \mathcal{C}\ell(p,q,r) \mapsto \mathcal{C}\ell(p,q,r) \) is called the norm of the Clifford algebra. For general multivectors \( M \in \mathcal{C}\ell(p,q,r) \) inverse elements exist and are defined through the relation \( MM^{-1} = M^{-1}M = 1 \), but the determination is more difficult and can be found in [3]. Note that in general not every element is invertible. The other automorphism we are dealing with is the main involution. It is denoted by \( \alpha \) and defined by

\[
\alpha(e_{i_1}e_{i_2}\ldots e_{i_k}) = (-1)^k e_{i_1}e_{i_2}\ldots e_{i_k}, \quad 0 \leq i_1 < i_2 < \cdots < i_k \leq n.
\]

The main involution has no effect on the even subalgebra and it commutes with the conjugation, i.e., \( \alpha(M^*) = \alpha(M)^* \) for arbitrary \( M \in \mathcal{C}\ell(p,q,r) \).

1.3. Clifford Algebra Products

On 1-blades, i.e., vectors \( a, b \in \bigwedge^1 V \) we can write the inner product in terms of the geometric product

\[
a \cdot b := \frac{1}{2}(ab + ba).
\]

A generalization of the inner product to blades can be found in [5]. For \( \mathcal{A} \in \bigwedge^k V, \mathcal{B} \in \bigwedge^l V \) the generalized inner product is defined by

\[
\mathcal{A} \cdot \mathcal{B} := [\mathcal{A}\mathcal{B}]_{|k-l|},
\]

where \([\cdot]_m \in \mathbb{N} \) denotes the grade-\( m \) part. There is another product on 1-blades, i.e., the outer (or exterior) product

\[
a \wedge b := \frac{1}{2}(ab - ba).
\]

This product can also be generalized to blades, see again [5]. For \( \mathcal{A} \in \bigwedge^k V, \mathcal{B} \in \bigwedge^l V \) the generalized outer product is defined by

\[
\mathcal{A} \wedge \mathcal{B} := [\mathcal{A}\mathcal{B}]_{|k+l|}.
\]

From equation (3) and (4) it follows that for 1-blades the geometric product can be written as the sum of the inner and the outer product

\[
ab = a \cdot b + a \wedge b.
\]

More general this can be defined for multivectors with the commutator and the anti-commutator product, see [6]. For treating geometric entities within this algebra context the definition of the inner product null space and its dual the outer product null space is needed.
Definition 5: The inner product null space (IPNS) of a blade \( \mathfrak{A} \in \bigwedge^k V \), cf. [6], is defined by
\[
\text{NI}(\mathfrak{A}) := \left\{ v \in \bigwedge^1 V : v \cdot \mathfrak{A} = 0 \right\}.
\]
Moreover, the outer product null space (OPNS) of a blade \( \mathfrak{A} \in \bigwedge^k V \) is defined by
\[
\text{NO}(\mathfrak{A}) := \left\{ v \in \bigwedge^1 V : v \wedge \mathfrak{A} = 0 \right\}.
\]

1.4. Pin and Spin groups

With respect to the geometric product the units of a Clifford algebra denoted by \( \mathcal{C}_{\ell}^{\times}(p,q,r) \) form a group.

Definition 6: The Clifford group is defined by
\[
\Gamma(\mathcal{C}_{\ell}(p,q,r)) := \left\{ g \in \mathcal{C}_{\ell}(p,q,r) \mid \alpha(g)v^{-1}g^{-1} \in \bigwedge^1 V \text{ for all } v \in \bigwedge^1 V \right\}.
\]
A proof that \( \Gamma(\mathcal{C}_{\ell}(p,q,r)) \) is indeed a group with respect to the geometric product can be found in [4]. We define two important subgroups of the Clifford group.

Definition 7: The Pin group is the subgroup of the Clifford group with \( N(g) = 1 \).
\[
\text{Pin}(p,q,r) := \left\{ g \in \mathcal{C}_{\ell}(p,q,r) \mid gg^* = 1 \text{ and } \alpha(g)v^{-1}g^{-1} \in \bigwedge^1 V \text{ for all } v \in \bigwedge^1 V \right\}.
\]
Furthermore, we define the Spin group by \( \text{Pin}(p,q,r) \cap \mathcal{C}_{\ell}^+(p,q,r) \)
\[
\text{Spin}(p,q,r) := \left\{ g \in \mathcal{C}_{\ell}^+(p,q,r) \mid gg^* = 1 \text{ and } \alpha(g)v^{-1}g^{-1} \in \bigwedge^1 V \text{ for all } v \in \bigwedge^1 V \right\}.
\]

Remark 2: For non-degenerated Clifford algebras the Pin group is a double cover of the orthogonal group of the quadratic space \((V,b)\). Moreover, the Spin group is a double cover of the special orthogonal group of \((V,b)\).

2. Quadric Geometric Algebra

The geometric algebra we will study was first introduced by Zamora [9]. We discuss the planar, i.e., two-dimensional case in any detail before we move on to higher dimensions.

2.1. The Embedding

The conic geometric algebra for the two-dimensional case is constructed with a Clifford algebra over a six-dimensional vector space \( V = \mathbb{R}^6 \). The quadratic form we are using is derived by the quadratic form of the conformal geometric algebra used in [2]:
\[
B = \begin{pmatrix}
0 & 0 & -1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0
\end{pmatrix}.
\]
The signature of the resulting algebra is \((p, q, r) = (4, 2, 0)\). For every axis a conformal embedding is performed. Therefore, we have the embedding \(\epsilon : \mathbb{R}^2 \to \mathcal{C}_\ell(4,2,0)\).

\[
(x, y)^T \mapsto e_1 + x e_2 + \frac{1}{2} x^2 e_3 + e_4 + ye_5 + \frac{1}{2} y^2 e_6. \quad (5)
\]

Affine points \((x, y)^T \in \mathbb{R}^2\) are embedded as null vectors. This means

\[
\epsilon(p)^2 = 0 \text{ for } p \in \mathbb{R}^2. \quad (6)
\]

Furthermore, every axis is embedded in a conformal way, so we have a homogeneous factor for every axis. To keep things simple we assume that all homogeneous factors are equal. The projection on the generator subspace spanned by \(e_1, e_2\) and \(e_3\) is denoted by subscript \(x\) and the projection on \(e_4, e_5, e_6\) by subscript \(y\). Due to the fact, that the embedding is conformal for every axis we get the additional conditions:

\[
\epsilon(p)_x^2 = 0, \quad \epsilon(p)_y^2 = 0. \quad (7)
\]

In the following we call 1-blades satisfying \((6)\) and \((7)\) embedded points even though we know that they represent circles with radius \(r = 0\). The inner product of two embedded points \(p_1 = (x_1, y_1)\) and \(p_2 = (x_2, y_2)\) results

\[
\epsilon(p_1) \cdot \epsilon(p_2) = -\frac{1}{2} x_1^2 + x_1 x_2 - \frac{1}{2} x_1^2 - \frac{1}{2} x_2^2 - \frac{1}{2} y_1 y_2 - \frac{1}{2} y_1 y_2 - \frac{1}{2} y_2^2
\]

\[
= -\frac{1}{2} \left( (x_2 - x_1)^2 + (y_2 - y_1)^2 \right)
\]

\[
= -\frac{1}{2} d_E(p_1, p_2),
\]

where \(d_E(p_1, p_2)\) denotes the Euclidean distance between the points \(p_1\) and \(p_2\). Note that this formula only is true for normalized nullvectors. This means that the homogeneous factor has to be one. We call these points normalized. They are characterized by:

\[
-e_3 \cdot P = 1, \quad -e_6 \cdot P = 1. \quad (8)
\]

The distance between an arbitrary embedded point and \(e_3\) respectively \(e_6\) is constant. Therefore, we can interpret these elements as points at infinity. Furthermore, the combination of the conditions \((8)\) results in

\[-(e_3 + e_6) \cdot P = 2.\]

Thus, the element \(e_3 + e_6\) can also be interpreted as point at infinity. The elements \(e_3\) and \(e_6\) represent the ideal points belonging to each axis and \(e_3 + e_6\) represents a point at infinity belonging to both axes. There are 1-blades that fulfil the conditions \((6)\) and \((7)\) without having a preimage in \(\mathbb{R}^2\). For example \(e_3, e_6, e_3 + e_6\) and algebra elements of the form:

\[
U_1 = e_1 + x_0 e_2 + \frac{1}{2} x^2 e_3 + e_6, \quad U_2 = e_3 + e_4 + y_0 e_5 + \frac{1}{2} y^2 e_6.
\]

If we determine the Euclidean distance of an embedded point to \(U_1\) or \(U_2\) the result is complex and depends on \(x_0\) respectively \(y_0\). Hence, these elements do not represent points.
2.2. Geometric entities

To calculate the preimage $\epsilon^{-1}$ of $p \in \bigwedge^1 V$ representing an embedded point, i.e., an algebra element fulfilling (6) and (7), we determine its IPNS ($\text{NI}(p)$) with respect to the embedding. This is called geometric inner product null space and dual geometric outer product null space, see [6].

**Definition 8:** The geometric inner product null space (GIPNS) and dual the geometric outer product null space (GOPNS) of a $k$-blade $\mathfrak{A} \in \bigwedge^k V$ is defined by
\[
\text{NI}_G(\mathfrak{A}) := \{ (x,y)^T \in \mathbb{R}^2 : \epsilon(x,y) \cdot \mathfrak{A} = 0 \}, \\
\text{NO}_G(\mathfrak{A}) := \{ (x,y)^T \in \mathbb{R}^2 : \epsilon(x,y) \land \mathfrak{A} = 0 \}.
\]

**Remark 3:** When dealing with an algebra element and the belonging geometric entity we will explicitly mention what null space is meant. For example we will talk about inner product conics.

Before we start the examination of geometric objects occurring in this model we define the pseudo-scalars that are necessary to change from inner product to outer product null spaces.

**Definition 9:** On the one hand the pseudo scalar
\[
J = e_2 \land e_5 \land e_1 \land e_4 \land (e_3 + e_6)
\]
maps outer product null spaces to inner product null spaces
\[
J : \bigwedge^i V \rightarrow \bigwedge^{k-i} V, \quad \bigwedge^i V \ni \mathbf{v} \mapsto \mathbf{v} \cdot J \in \bigwedge^{k-i} V,
\]
with $i \in \{1, \ldots, 4\}$ and $k = 5$ for the planar quadric geometric algebra. On the other hand
\[
J^* := e_2 \land e_5 \land e_3 \land e_6 \land (e_1 + e_4)
\]
maps dual elements to normal elements. There is no difference in left- or right multiplication with the pseudo-scalars, since the result describes the same geometric entity except for a homogeneous factor.

With this definition we get
\[
\text{NI}_G(\mathfrak{A}) = \text{NO}_G(\mathfrak{A} \cdot J), \quad \text{NO}_G(\mathfrak{A}) = \text{NI}_G(\mathfrak{A} \cdot J^*).
\]

Note that dualization is done with the inner product. Now we take a look at the inner product null space of 1-blades that are not embedded points. Therefore, at least one of the conditions (6) or (7) is not satisfied. Let $\mathfrak{A} = -2a_1e_1 + 2a_2e_2 - a_3e_3 - 2a_4e_4 + 2a_5e_5 - a_6e_6$ be a general 1-blade. The GIPNS results
\[
\text{NI}_G(\mathfrak{A}) = \{ (x,y)^T \in \mathbb{R}^2 : \epsilon(x,y) \cdot \mathfrak{A} = 0 \}
\]
\[
eq \{ (x,y)^T \in \mathbb{R}^2 : a_1x^2 + 2a_2x + a_3 + a_4y^2 + 2a_5y + a_6 = 0 \}.
\]

The solution set of the GIPNS is an axis aligned conic, because there is no term containing $xy$. A conic can be represented by a symmetric matrix:
\[
\begin{pmatrix}
1 & x & y \\
\end{pmatrix}
\begin{pmatrix}
a_{11} & a_{12} & a_{13} \\
a_{12} & a_{22} & a_{23} \\
a_{13} & a_{23} & a_{33}
\end{pmatrix}
\begin{pmatrix}
1 \\
x \\
y
\end{pmatrix} = 0.
\]
Therefore, we can define a bijection between conics (or symmetric matrices) and 1-blades by

\[ \kappa : V^2 \ni c \rightarrow c \in \bigwedge^1 \mathcal{C}_{(4,2,0)}, \]  

(9)

\[
\begin{pmatrix}
a_0 & a_2 & a_5 \\
a_2 & a_1 & 0 \\
a_5 & 0 & a_4
\end{pmatrix}
\mapsto 2a_1e_1 - 2a_2e_2 + a_3e_3 + 2a_4e_4 - 2a_5e_5 + a_6e_6.
\]

For the bijection (9) we assume that \( a_3 := \frac{1}{2}a_0 \) and \( a_6 := \frac{1}{2}a_0 \). It would be sufficient to demand that \( a_3 + a_6 = a_0 \) to result in the same conic, because the constant value is equal to \( a_3 + a_6 \). This does not change the GIPNS of the conic.

After dualization an inner product conic becomes an outer product conic that is a 4-blade and can be generated by the outer product of 4 embedded points. Hence, these 4 points lay on the conic, because

\[ p_i \in \text{INOG}(p_1 \wedge p_2 \wedge p_3 \wedge p_4), \text{ for } i = 1, \ldots, 4. \]

The natural question that arises is: Is there a way to classify conics in this model? For this purpose we study the incidence of the conics with the three additional ideal points. As first we look at the entities \( a \in \bigwedge^1 \mathcal{C}_{(4,2,0)} \) that contain all three ideal points \( e_3, e_6 \) and \( e_3 + e_6 \). Thus, we get the conditions

\[ a \cdot e_3 = 2a_1 = 0, \quad a \cdot e_6 = 2a_4 = 0, \quad a \cdot (e_3 + e_6) = 2a_1 + 2a_4 = 0. \]

(10)

Thus, \( a_1 \) and \( a_4 \) have to vanish. The corresponding algebra element hast the form

\[ l = 2a_2e_2 - a_3e_3 + 2a_5e_5 - a_6e_6. \]

Its GIPNS is calculated by

\[ \text{NIL}_G(l) = \left\{ (x,y)^T \in \mathbb{R}^2 \mid 2a_2x + 2a_5y + a_3 + a_6 = 0 \right\}. \]

Hence, every line passes through \( e_3, e_6 \) and \( e_3 + e_6 \). An algebra element that contains just \( e_3 \) (\( e_6 \)) is a parabola that axis is the x-axis (y-axis). If an element contains both ideal elements \( e_3 \) and \( e_6 \) it automatically contains also \( e_3 + e_6 \) and therefore represents a line. An element that contains \( e_3 + e_6 \), but neither \( e_3 \) nor \( e_6 \), is given by the condition \( a \cdot (e_3 + e_6) = 2a_1 + 2a_4 = 0. \) This means \( a_1 = -a_4 \) and the belonging conic is an equilateral hyperbola, i.e., the asymptotes include an angle of 90\(^\circ\). The other types, i.e., circle, ellipse a pair of parallel lines, a pair of intersecting lines and a double line can be generated by either choosing four points and calculating the wedge product of the belonging algebra elements or simply by using the bijection between conics and the algebra elements.

**Remark 4:** Note, that this description of conics also contains complex conics.

In the most general case the 2-blades belong to inner product point quadruples. This can be seen from

\[ \text{NIL}_G(a \wedge b) = \text{NIL}_G(a) \cap \text{NIL}_G(b), \]

(11)

see [9]. Therefore, 2-blades represent all points belonging to both conics that are represented by the 1-blades. If two non-degenerated conics do not intersect, the corresponding 2-blade
represents a complex inner product point quadruple. Furthermore, we can see by dualization, that 3-blades belong to outer product point quadruples. For two inner product lines \( l_1, l_2 \) the belonging 2-blade \( l_1 \wedge l_2 \) represents an inner product point pair, where one of the points is their affine intersection point and the other the point at infinity.

**Example 1:** Let us generate a conic through four points. Therefore, we choose four points and embed them via (5).

\[
\begin{align*}
p_1 &= (-1, 0), \rightarrow p_1 = e_1 - e_2 + \frac{1}{2}e_3 + e_4, \quad p_2 = (1, 0), \rightarrow p_2 = e_1 + e_2 + \frac{1}{2}e_3 + e_4, \\
p_3 &= (0, -1), \rightarrow p_3 = e_1 + e_4 - e_5 + \frac{1}{2}e_6, \quad p_4 = (-1, 0), \rightarrow p_4 = e_1 + e_4 + e_5 + \frac{1}{2}e_6.
\end{align*}
\]

The corresponding inner product representation is calculated by

\[
c = \mathbf{j} \cdot (p_1 \wedge p_2 \wedge p_3 \wedge p_4) = 4e_1 - e_3 + 4e_4 - e_6.
\]

The GIPNS is given by

\[
\text{NI}_G(c) = \left\{ (x, y)^T \in \mathbb{R}^2 \mid -x^2 + 1 - y^2 = 0 \right\}.
\]

With the bijection (9) we see easily that \( c \) is the image of the conic given by the diagonal matrix determined with diagonal \((1, -1, -1)\), i.e., a unit circle.

### 2.3. Transformations

In this section we discuss transformations in this algebra. The Clifford algebra \( \mathcal{C}_{(4,2,0)} \) corresponds to the quadratic space \( \mathbb{R}^{(4,2)} \) and therefore 1-blades represent reflections in hyperplanes in this space. It is clear that these transformations will induce transformation that are not linear in the base space \( \mathbb{R}^2 \) because the embedding \( \epsilon \) is quadratic. From the last section we know that 1-blade correspond to conics. Furthermore, a transformation acts via the sandwich operator and results again a \( k \)-blade when applied to a \( k \)-blade \( k = 1, \ldots, 4 \). We begin with an example

**Example 2:** Let \( c \) be the circle from example 1

\[
c = 4e_1 - e_3 + 4e_4 - e_6
\]

and let \( p = e_1 + e_2 + \frac{1}{2}e_3 + e_4 + 2e_5 + 2e_6 \) be \( \epsilon(1, 2) \). Applying the sandwich operator to \( p \) results in

\[
c' = \alpha(c)p\epsilon^{-1} = 5e_1 + e_2 - \frac{1}{2}e_3 + 5e_4 + 2e_5 + e_6.
\]

Now we check if this entity is still an embedded point. Therefore, the conditions (6) and (7) have to be checked. Condition (6) is satisfied, but condition (7) not

\[
c'^2_x = 6, \quad c'^2_y = -6.
\]
Hence, \( c' \) can not be interpreted as embedded point respectively as circle with radius zero. The GIPNS of \( c' \) is given by

\[
\text{NI}_G(c') = \left\{(x, y)^T \in \mathbb{R}^2 \mid -\frac{5}{2}x^2 + x - \frac{5}{2}y^2 + 2y - \frac{1}{2} = 0\right\}.
\]

This represents a pair of complex lines intersecting in the real point \((\frac{1}{5}, \frac{2}{5})\).

Example 2 shows that in general a conic is mapped to another conic. We can not map a circle of radius zero to a circle of radius zero and define a mapping for points on this way. Thus, we interpret 1-blades as conics and study the effect of the transformations applied to conics.

**Theorem 1:** A conic represented by the 1-blade \( a \) is point wise fixed under the transformation induced by itself. Furthermore, these transformation are involutions.

**Proof.** First we show that the conic belonging to the transformation is fixed point-wise. Therefore, we look at the effect of a general 1-blade

\[
a = -2a_1e_1 + 2a_2e_2 - a_3e_3 - 2a_4e_4 + 2a_5e_5 - a_6e_6
\]

to itself:

\[
\alpha(a)aa^{-1} = \alpha(a) = -a.
\]

Multiplication with a homogeneous factor does not change the GIPNS. Thus, the result is the conic represented by \( a \) again. To show that the points of the conic \( a \) are fix under the transformation induced by \( a \) we examine the effect of \( a \) on the intersection points of the conic \( a \) with all lines containing the point \((0, 0)\). These lines are given by

\[
l(x, y) = \mathbb{J} \cdot (e(0,0) \wedge e(x, y) \wedge e_3 \wedge e_6)
\]

\[
= \mathbb{J} \cdot \left((e_1 + e_4) \wedge (e_1 + xe_2 + \frac{1}{2}x^2e_3 + e_4 + ye_5 + \frac{1}{2}y^2e_6) \wedge e_3 \wedge e_6\right)
\]

\[
= 2ye_2 - 2xe_5.
\]

The intersection of all these lines with the conic \( a \) is represented by

\[
l(x, y) \wedge a = -2a_3ye_23 + 4a_1ye_12 - 4a_4ye_24 - 4a_1xe_15 + 4(a_5y + a_2x)e_{25}
\]

\[
-2a_3xe_35 - 2a_6ye_{26} - 4a_4xe_{45} + 2a_6xe_{56}.
\]

This 2-blade represents the intersection point pair of the conic with \( l(x, y) \). The application of the transformation induced by \( a \) to \( l(x, y) \wedge a \) results

\[
\alpha(a)(l(x, y) \wedge a)a^{-1} = -2a_3ye_23 + 4a_1ye_12 - 4a_4ye_24 - 4a_1xe_15 + 4(a_5y + a_2x)e_{25}
\]

\[
-2a_3xe_35 - 2a_6ye_{26} - 4a_4xe_{45} + 2a_6xe_{56}.
\]

This shows, that all intersection point pairs of the line bundle with the conic are fix and therefore the whole conic is fixed point-wise. To see that the transformation is an involution we have to apply it twice to an arbitrary \( k \)-blade \( \mathbb{B} \)

\[
\alpha(a)\alpha(a)\mathbb{B}a^{-1}a^{-1} = \alpha(a^2)\mathbb{B}(a^2)^{-1} = \mathbb{B}.
\]

The last equality follows because \( a^2 \) is a real number. \( \square \)
Due to the fact, that these transformations are represented as reflections with respect to hyperplanes in $\mathbb{R}^{(4,2)}$, they are involutions and fix the corresponding hyperplane point-wise. This is the reason why we interpret these transformations as reflections or inversions with respect conics. Furthermore, the whole group of transformations is generated by the 1-blades. Note that the image of an axis aligned conic is always an axis aligned conic in this model and that intersection point quadruples of a conic with the reflection conic stay fixed, no matter if the intersection points are real or complex valued.

**Remark 5:** The group of conformal transformations of a quadratic space $\mathbb{R}^{(p,q)}$ can be described as the Pin group of a Clifford algebra $C\ell_{(p+1,q+1,0)}$, see [7]. Therefore, the group of conformal transformations of the Minkowski space $\mathbb{R}^{(3,1)}$ is isomorphic to the group of inversions in axis aligned conics.

### 2.4. Effect on lines and points

We have seen in example 2 that a circle with radius zero is mapped to a pair of complex lines intersecting in a real point. Therefore, we have to search for a better description of points in this model. One way to describe points as 2-blades is to examine the intersection of two intersecting in a real point. Therefore, we have to search for a better description of points.

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**Theorem 2:** The image of a line under an inversion with respect to a conic represented by the 1-blade $a \in \wedge^1 C\ell_{(4,2,0)}$ is a conic. Moreover, for non-degenerated conics this conic is the image of $a$ under an affine transformation, i.e., translation and scalar multiplication.

**Proof.** To show this we concentrate on conics with no terms in $x$ respectively $y$. We can do this because we are just interested in the type of the image conic. Furthermore, we can perform translations by two reflections in parallel lines and thus we can carry over the results in main position to arbitrary position. Furthermore, we just show this theorem for non degenerate conics. A conic with no terms in $x$ respectively $y$ is given by

$$a = 2a_1e_1 + \frac{1}{2}a_0e_3 + 2a_4e_4 + \frac{1}{2}a_0e_6.$$
Since we are interested in real conics the variables \( a_0, a_1 \) and \( a_4 \) are not allowed to have the same sign. In the later we assume the conic is real. Now we can look at the matrix of the conic

\[
M = \begin{pmatrix}
1 & 0 & 0 \\
0 & \frac{a_1}{a_0} & 0 \\
0 & 0 & \frac{a_4}{a_0}
\end{pmatrix}.
\]

The image of the line set (13) is calculated by

\[
\alpha(a) l(p_1, p_2) a^{-1} = -\frac{4a_1(x_1 y_2 - y_1 x_2)}{a_0} e_1 - 2(y_1 - y_2)e_2 - \frac{4a_4(x_1 y_2 - y_1 x_2)}{a_0} e_4 + 2(x_1 - x_2)e_5.
\]

The matrix of the corresponding conic is given by

\[
N(p_1, p_2) = \frac{1}{a_0} \begin{pmatrix}
0 & a_0(y_2 - y_1) & a_0(x_1 - x_2) \\
0 & 2c_1(x_1 y_2 - y_1 x_2) & 0 \\
0 & 2a_4(x_1 y_2 - y_1 x_2)
\end{pmatrix}.
\]

From this representation we see immediately that lines through the center of the conic are fixed, but not point wise. To transform this matrix to diagonal form we apply the transformation

\[
p \mapsto \begin{pmatrix} 1 & 0 & 0 \\ \alpha & 1 & 0 \\ \beta & 0 & 1 \end{pmatrix} p, \text{ with } \alpha = -\frac{a_0(y_1 - y_2)}{2a_1(x_1 y_2 - y_1 x_2)} \text{ and } \beta = \frac{a_0(x_1 - x_2)}{2a_4(x_1 y_2 - y_1 x_2)}.
\]

Here \( p = (1, x, y) \in \mathbb{P}^2(\mathbb{R}) \) is a point of the projective plane. The effect of this coordinate transformation on the conic is calculated by

\[
N'(p_1, p_2) = T^{-T} N(p_1, p_2) T^{-1} = \begin{pmatrix}
-\frac{a_0^2(a_4(y_1 - y_2)^2 + a_1(x_1 - x_2)^2)}{2a_4 a_1(x_1 y_2 - y_1 x_2)} & 0 & 0 \\
0 & \frac{2a_1(x_1 y_2 - y_1 x_2)}{a_0} & 0 \\
0 & 0 & \frac{2a_4(x_1 y_2 - y_1 x_2)}{a_0}
\end{pmatrix}.
\]

If we look at the affine part of the conic, we see that this results in

\[
N'(p_1, p_2) = \begin{pmatrix}
1 & 0 & 0 \\
0 & ka_1 & 0 \\
0 & 0 & ka_4
\end{pmatrix}, \text{ with } k = -\frac{4a_0^2(x_1 y_2 - y_1 x_2)^2 a_1}{a_0^2(a_4(y_1 - y_2)^2 + a_1(x_1 - x_2)^2)}.
\]

Therefore, the image is identical to the conic corresponding to \( a \) except for a translation \( T \) and a scaling \( k \). Furthermore, lines are mapped to real conics.

To illustrate this Figure 1 shows the reflections of three intersecting lines with respect to a circle, an ellipse, a parabola and a hyperbola. The inversion conic \( a_i, i = 1, \ldots, 4 \) is coloured red while the pairs (line, image of the line) are presented in the same colour. The inversion conics respectively are from left to right and from up to down given by

\[
a_1 : x^2 + y^2 = 1, \quad a_2 : \frac{25}{16}x^2 + \frac{16}{25}y^2 = 1, \quad a_3 : x^2 - y = 1, \quad a_4 : x^2 - \frac{3}{4}y^2 = 1.
\]
3. Subgroups

In this section we examine some subgroups that are embedded naturally in the Pin respectively Spin group of the conic geometric algebra for the plane.

3.1. Rotations in the plane

First, we concentrate on the group that is generated by inversions with respect to lines passing through the origin. Note that we will study the effect of these mappings applied to points embedded via (12). We take two lines through the origin

\[ l_1 = \mathbf{J} \cdot (e(0,0) \wedge e(\cos \varphi, \sin \varphi) \wedge e_3 \wedge e_6) , \]
\[ l_2 = \mathbf{J} \cdot (e(0,0) \wedge e(\cos \psi, \sin \psi) \wedge e_3 \wedge e_6). \]

Furthermore we are interested in orientation preserving transformation, i.e., elements from the Spin group. The composition of two reflections in \( l_1 \) and \( l_2 \) is given by their geometric product

\[ l_1 l_2 = (\sin \psi \sin \varphi + \cos \psi \cos \varphi) + (\cos \psi \sin \varphi - \sin \psi \cos \varphi)e_{25} \]

with the addition theorems it follows

\[ = \cos(\varphi - \psi) + \sin(\varphi - \psi)e_{25}. \]
The square of \( e_{25} \) is \(-1\). This means that the consecutive reflection in two lines through the origin results in an algebra element that can be interpreted as complex number with norm 1. It is a well known result that rotations in the plane can be described with normed complex numbers. Let us look at the effect of such an element applied to a point that is described by \( \mathbf{R} \). Let \( \mathbf{R} = \cos \varphi + \sin \varphi e_{25} \) be an element in the form of Eq. (14) and \( \mathbf{p} = 2e_{25} + x_0(e_{35} - e_{56}) + y_0(e_{23} + e_{26}) \) a point of the form (12). We calculate

\[
\mathbf{p}' = \alpha(\mathbf{R})\mathbf{p}\mathbf{R}^{-1} = \mathbf{R}\mathbf{p}\mathbf{R}^{-1}
\]

\[
= 2e_{25} + (-2\sin \varphi \cos \varphi x_0 + 2\cos(\varphi)^2 y_0 - y_0)(e_{23} + e_{26}) + (2\sin \varphi \cos \varphi y_0 + 2\cos(\varphi)^2 x_0 - x_0)(e_{35} - e_{56}).
\]

The GIPNS of this entity can be computed or we can simply read the coordinates of the image point.

\[
x = 2\cos(\varphi)^2 x_0 + 2\cos \varphi \sin \varphi y_0 - x_0 = \cos(2\varphi)x_0 + \sin(2\varphi)y_0,
\]

\[
y = 2\cos(\varphi)^2 y_0 - 2\cos \varphi \sin \varphi x_0 - y_0 = \cos(2\varphi)y_0 - \sin(2\varphi)x_0.
\]

Therefore, we can see that this transformation indeed is a rotation about the origin with the angle \(2\varphi\). Due to the fact that these elements are isomorphic to the complex numbers, they form a double cover of the group \( SO(2) \).

**Remark 6:** From the fact that 1-blades are mapped to 1-blades by a reflection in a line it follows that axis aligned conics have to be mapped to axis aligned conics. Therefore, these mappings can not be interpreted point wise for conics.

### 3.2. Translations

Now we aim at the group of Euclidean displacements \( SE(2) \). Therefore, we will show that two consecutive reflections in parallel lines results in a translation. The group of Euclidean displacements can be generated as semidirect product of \( SO(2) \) and \( T(2) \), which describes the abelian translation group. Let \( l_1 \) and \( l_2 \) be two parallel lines and let \( t_1, t_2 \) be their distances from the origin. The lines are given by

\[
l_1(\varphi, t_1) = 2\sin \varphi e_2 - t_1 e_3 - 2\cos \varphi e_5 - t_1 e_6,
\]

\[
l_2(\varphi, t_2) = 2\sin \varphi e_2 - t_2 e_3 - 2\cos \varphi e_5 - t_2 e_6.
\]

The composition can be expressed with the geometric product.

\[
\mathfrak{T}(\varphi, t_1, t_2) = l_1(\varphi, t_1)l_2(\varphi, t_2)
\]

\[
= 2 + (t_1 - t_2)\sin \varphi(e_{23} + e_{26}) + (t_1 - t_2)\cos \varphi(e_{35} - e_{56}). \tag{15}
\]

Applying the sandwich operator to a point \( \mathbf{p} \) results in

\[
\alpha(\mathfrak{T})\mathbf{p}\mathfrak{T}^{-1} = \mathfrak{T}\mathbf{p}\mathfrak{T}^{-1}
\]

\[
= 2e_{25} + (y_0 - 2t_2\cos \varphi + 2t_1\cos \varphi)(e_{23} + e_{26}) + (x_0 - 2t_1\sin \varphi + 2t_2\sin \varphi)(e_{35} - e_{56}).
\]
The image is determined by

\[ x = x_0 - \sin \varphi(2(t_1 - t_2)), \quad y = y_0 + \cos \varphi(2(t_1 - t_2)). \]

Therefore, the transformation is a translation in the normal direction of the given lines \( l_1 \) and \( l_2 \).

### 3.3. The group of planar Euclidean displacements

The composition of translations and rotations generates the whole group of planar Euclidean displacements \( SE(2) \). Furthermore, we can now examine the group that is generated by rotations and translations as subgroup of the Spin group. These algebra elements have the form

\[ a_0 + a_1 e_{25} + a_2(e_{23} + e_{26}) + a_3(e_{35} - e_{56}). \]

The multiplication table of the geometric product for the generators is in Table 1. Hence, this is indeed a subgroup of the Spin group. Furthermore, it is isomorphic to the subgroup of dual quaternions, called planar dual quaternions. We can define a bijection by sending \( e_{25} \) to \( i \), \( (e_{23} + e_{26}) \) to \( \epsilon j \) and \( (e_{35} - e_{56}) \) to \( \epsilon k \).

**Remark 7:** If we restrict ourself to reflections in lines and in circles, we are able to describe the group of conformal transformations of the plane.

### 3.4. Inversions applied to points

In this subsection we study the effect of reflections with respect to a conic in principal position, \( \text{i.e.} \), it is centered at the origin, on points. The points are embedded as intersection points of two lines, as discussed in the previous section. Furthermore, we have to note that the transformations map point pairs to point pairs. Hence, the pair of intersection points of two lines (the affine and the ideal point) are mapped to a point pair. All lines pass through \( \infty \) and so the image of every line must pass through the image of this point. The generalization to conics without principal position is reached with the application of a coordinate transformation. The inversion conic is given by

\[ a = \frac{1}{2} c_0(e_3 + e_6) + 2c_1e_1 + 2c_2e_4. \]
A point is represented as intersection set of two lines, see \((12)\)
\[
p = 2e_{25} + y_0(e_{23} + e_{26}) + x_0(e_{35} - e_{56}).
\]

Applying the sandwich operator to the point results in
\[
p' = \alpha(a)p a^{-1} = -\frac{2y_0c_1}{c_0}e_{12} + \frac{2x_0c_1}{c_0}e_{15} - 2e_{25} + \frac{2y_0c_2}{c_0}e_{24} + \frac{2x_0c_2}{c_0}e_{45}.
\]
This is not of the form \((12)\) and therefore it is not the representation of the intersection of two lines. The GIPNS is calculated
\[
\mathbf{NI}_G(p') = \{ (x, y)^T \in \mathbb{R}^2 \mid -2c_1(-x_0y + x y_0)e_1 - (y_0c_1x^2 + 2y_0c_2y^2)e_2 \\
-2c_2(-x_0y + x y_0)e_4 + (x_0c_1x^2 + x_0c_2y^2 + 2x_0c_5)e_5 = 0 \}
\]
The solution set of this GIPNS can be written as
\[
x = -\frac{2c_0x_0}{c_1x_0^2 + c_2y_0^2}, \quad y = -\frac{2c_0y_0}{c_1x_0^2 + c_2y_0^2}.
\]
Note that we excluded the solution \(x = 0, y = 0\).

**Remark 8:** Inversion with respect to conics that are not axis aligned can be performed by the composition of an inversion and a rotation.

### 4. Generalization to higher dimensions

The main advantage of this geometric algebra model is its flexibility. It is no problem to change the dimension. We will discuss the model for the \(n\)-dimensional case and we will show some examples for the three-dimensional case. The construction is done in the same way as in section 2. We start with a real vector space of dimension \(n\). For each axis we use a conformal embedding. Therefore the dimension of the geometric algebra is \(2^{3n}\) and its quadratic form is given by

\[
B = \begin{pmatrix} D & & & \\ & D & & \\ & & \ddots & \\ & & & D \end{pmatrix}\text{, } D = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}.
\]

The embedding \(\epsilon\) is realized by
\[
\epsilon : \mathbb{R}^n \to \mathcal{C}_{(2n, n, 0)},
\]
\[
(x_1, \ldots, x_n) \mapsto e_1 + x_1e_2 + \frac{1}{2}x_1^2e_3 + \cdots + e_{3n-2} + x_ne_{3n-1} + \frac{1}{2}x_n^2e_{3n}.
\]

The conditions for an embedded point \((6)\) and \((7)\) generalize to
\[
\epsilon(p)^2 = 0, \quad \epsilon(p)^2_{x_1} = 0, \quad \epsilon(p)^2_{x_2} = 0, \ldots \quad \epsilon(p)^2_{x_n} = 0.
\]
We define the pseudo-scalar analogue to definition 9.
Definition 10: The pseudo scalar $\mathcal{J}$ that maps outer product null spaces to inner product null spaces is defined by

$$\mathcal{J} = \bigwedge_{i \mod 3=2}^n e_i \wedge \bigwedge_{j \mod 3=1}^n e_j \wedge \sum_{k \mod 3=0}^n e_k.$$ 

Inner product null spaces can be mapped to outer product null spaces with the pseudo scalar

$$\mathcal{J}^* := \bigwedge_{i \mod 3=2}^n e_i \wedge \bigwedge_{j \mod 3=0}^n e_j \wedge \sum_{k \mod 3=1}^n e_k.$$ 

Blades of grade 1 correspond to inner product axis aligned hyperquadrics. With the dimension, the number of objects that can be represented, grows. Blades of grade $k, k \leq n$ correspond to the intersection of $k$ hyperquadrics.

4.1. Quadrics in three Dimensions

To construct the conic geometric algebra for the three-dimensional space we use the quadratic space $\mathbb{R}^{(6,3)}$ given by the nine-dimensional real vector space $\mathbb{R}^9$ together with the quadratic form

$$B = \begin{pmatrix} D & D \\ D & D \end{pmatrix}, \quad D = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}.$$ 

For three dimensions the embedding $\epsilon$, see Eq (16), has the following form

$$\epsilon : \mathbb{R}^3 \to \mathcal{C}(6,3,0),$$

$$(x, y, z) \mapsto e_1 + xe_2 + \frac{1}{2}x^2 e_3 + e_4 + ye_5 + \frac{1}{2}y^2 e_6 + e_7 + ze_8 + \frac{1}{2}z^2 e_9.$$ 

The conditions for an embedded point (6) and (7) generalize to

$$\epsilon(p)^2 = 0, \quad \epsilon(p)_x^2 = 0, \quad \epsilon(p)_y^2 = 0, \quad \epsilon(p)_z^2 = 0.$$ 

Moreover, the pseudo scalars are given by

$$\mathcal{J} = (e_2 \wedge e_5 \wedge e_8) \wedge (e_1 \wedge e_4 \wedge e_7) \wedge (e_3 + e_6 + e_9),$$

$$\mathcal{J}^* = (e_2 \wedge e_5 \wedge e_8) \wedge (e_3 \wedge e_6 \wedge e_9) \wedge (e_1 + e_4 + e_7).$$ 

The belonging geometric algebra has dimension $2^9 = 512$. An axis aligned quadric in $\mathbb{R}^3$ has six degrees of freedom. This can be seen from the symmetric matrix belonging to a quadric, that has in general ten free entries. The fact that we are treating axis aligned quadrics reduces the number of free entries to seven. Furthermore, this matrix representation is homogeneous.
and therefore we have six degrees of freedom. The bijection between axis aligned quadrics and 1-blades can be defined as

\[
\kappa : V^2 \ni q \rightarrow q \in \bigwedge^1 \mathcal{C}(6,3,0),
\]

\[
\begin{pmatrix}
  c_0 & c_2 & c_5 & c_8 \\
  c_2 & c_1 & 0 & 0 \\
  c_5 & 0 & c_4 & 0 \\
  c_8 & 0 & 0 & c_7
\end{pmatrix} \mapsto 2c_1e_1 - 2c_2e_2 + \frac{1}{3}c_0e_3 + 2c_4e_4 - 2c_5e_5 + \frac{1}{3}c_9e_6
\]

\[
+ 2c_7e_7 - 2c_8e_8 + \frac{1}{3}c_9e_9.
\]

Now we define an inner product inversion quadric.

\[
a = (\epsilon p_1 \land \epsilon p_2 \land \epsilon p_3 \land \epsilon p_4 \land \epsilon p_5 \land \epsilon p_6) \cdot \mathbf{3}
\]

\[
= \frac{25}{9}e_1 - \frac{3}{8}e_3 + 4e_4 - \frac{3}{8}e_6 + \frac{36}{25}e_7 - \frac{3}{8}e_9,
\]

where \(p_1 = (\frac{9}{10}, 0, 0)^T, p_2 = (-\frac{9}{10}, 0, 0)^T, p_3 = (0, \frac{3}{4}, 0)^T, p_4 = (0, -\frac{3}{4}, 0)^T, p_5 = (0, 0, \frac{5}{4})^T, p_6 = (0, 0, -\frac{5}{4})^T\). The GIPNS of \(a\) is given by

\[
\text{NI}_G(a) = \left\{(x, y, z)^T \in \mathbb{R}^3 \mid \frac{100}{81}x^2 + \frac{16}{9}y^2 + \frac{16}{25}z^2 - 1 = 0\right\}.
\]

In Figure 2 you can see the inversion of the unit cube \([-1, 1]^3\] with respect to the ellipsoid defined by \(a\). The image of every side of the cube, \(i.e.,\) of every plane is an ellipsoid passing through the origin. As we did for the planar case we can now direct our attention on the intersection of three planes to get a point pair containing one affine point and one ideal. We choose these planes to be parallel to the coordinate planes and passing through a given point \(p = (x_0, y_0, z_0)\). Expressed in the quadric geometric algebra \(\mathcal{C}(6,3,0)\) we get

\[
p = (\epsilon(x_0, y_0, z_0) \land \epsilon(0, 0, z_0) \land \epsilon(x_0, 0, z_0) \land \epsilon_3 \land \epsilon_6 \land \epsilon_9) \cdot \mathbf{3}
\]

\[
\land (\epsilon(x_0, y_0, z_0) \land \epsilon(0, y_0, 0) \land \epsilon(x_0, y_0, 0) \land \epsilon_3 \land \epsilon_6 \land \epsilon_9) \cdot \mathbf{3}
\]

\[
\land (\epsilon(x_0, y_0, z_0) \land \epsilon(x_0, 0, 0) \land \epsilon(x_0, 0, y_0) \land \epsilon_3 \land \epsilon_6 \land \epsilon_9) \cdot \mathbf{3}
\]

\[
= 3\epsilon_{258} + (\epsilon_{358} - \epsilon_{568} + \epsilon_{589})x_0 + (\epsilon_{238} + \epsilon_{268} - \epsilon_{289})y_0 + (-\epsilon_{235} + \epsilon_{256} + \epsilon_{259})z_0.
\]

**Remark 9:** A representation of the group of Euclidean displacements \(SE(3)\) can be obtained by studying the composition of reflections in planes. Planes are 1-blades that are generated by \((\epsilon p_1 \land \epsilon p_2 \land \epsilon p_3 \land \epsilon_3 \land \epsilon_6 \land \epsilon_9) \cdot \mathbf{3}\).

The effect of the inversion given by \(a\) on point pairs can be written as

\[
f(x, y, z) = \frac{45^2}{2^2(30^2y^2 + 25^2x^2 + 18^2z^2)}(x, y, z)^T.
\]

Note that the point \(\infty\) at infinity gets mapped to the origin and that the origin is mapped to \(\infty\). Therefore, we have a map from \(\mathbb{R}^3 \setminus \{0\} \rightarrow \mathbb{R}^3\). One main advantage of this method is that we can calculate the image ellipsoid of one cube side, \(i.e.,\) one plane directly by applying
the sandwich operator to the plane that is expressed as quadric. For example the plane passing through \( p_1 = [1, 1, 1], p_2 = [1, 1, -1], p_3 = [1, -1, 1] \) can be expressed as 1-blade by

\[
\mathfrak{P} = (\epsilon p_1 \wedge \epsilon p_2 \wedge \epsilon p_3 \wedge e_3 \wedge e_6 \wedge e_9) \cdot \mathfrak{J} = 3e_2 + e_3 + e_6 + e_9.
\]

To check that this indeed the representation of the plane we compute the GIPNS

\[
\text{NI}_G(\mathfrak{P}) = \left\{ (x, y, z)^T \in \mathbb{R}^3 \mid 1 - x = 0 \right\}.
\]

Applying the sandwich operator to \( \mathfrak{P} \) results in

\[
\mathfrak{E} = \alpha(a) \mathfrak{P} a^{-1} = -\frac{200}{27} e_1 - 3e_2 - \frac{32}{3} e_4 - \frac{96}{25} e_7
\]

with GIPNS

\[
\text{NI}_G(\mathfrak{E}) = \left\{ (x, y, z)^T \in \mathbb{R}^3 \mid \frac{10^2}{9^2} x^2 - x + \frac{4^2}{3^2} y^2 + \frac{4^2}{5^2} z^2 = 0 \right\}.
\]

This is one ellipsoid you can see in Figure 2. Furthermore, we can intersect two planes \( \mathfrak{P}_1, \mathfrak{P}_2 \) to get a inner product line that is a edge of the cube. After that we can apply the sandwich operator to the line and get the intersection curve (an ellipse) of the two ellipsoids that are the images of \( \mathfrak{P}_1, \mathfrak{P}_2 \).

**Remark 10:** It is preferable to compute the sandwich operator with conjugation instead inversion. This means we use \( \alpha(a) \mathfrak{X} a^* \) with \( a \in \mathcal{Cl}_{(p,q,r)}^{\mathbb{C}} \) and \( \mathfrak{X} \in \wedge V \). For this representation the element \( a \) has to be normed \( a a^* = 1 \). In general the modified sandwich operator is faster, because calculating the inverse of a Clifford algebra element is extremely expensive.

The second example for three dimensions is an axis aligned hyperboloid of two sheet that is generated by

\[
a = (\epsilon p_1 \wedge \epsilon p_2 \wedge \epsilon p_3 \wedge \epsilon p_4 \wedge \epsilon p_5 \wedge \epsilon p_6) \cdot \mathfrak{J} = -6e_1 + e_3 + \frac{9}{8} e_4 + e_6 + \frac{9}{8} e_7 + e_9.
\]
Figure 4: Inversion with respect to a cylinder

with \( p_1 = (-1, 0, 0), p_2 = (1, 0, 0), p_3 = (2, 0, 4), p_4 = (2, 0, -4), p_5 = (2, 4, 0), p_6 = (2, -4, 0). \)

We calculate the GIPNS

\[
\text{NI}_G(a) = \left\{ (x, y, z)^T \in \mathbb{R}^3 \mid x^2 - 1 - \frac{3}{16}y^2 - \frac{3}{16}z^2 = 0 \right\}.
\]

The mapping applied to point pairs results in

\[
f(x, y, z) = \frac{16}{16x^2 - 3y^2 - 3z^2}(x, y, z)^T.
\]

The effect of this mapping on the cube \([1, 3] \times [-1, 1] \times [-1, 1]\) is presented in Fig. \[3\]. Figure \[4\] shows the effect of an inversion with respect to a cylinder given by \( a = 3e_1 - 2e_3 + 3e_4 - 2e_6 - 2e_9 \) applied to a cube. The equation of the cylinder is derived as \( x^2 + y^2 = 4 \).

Planes that are not parallel to the axis of the cylinder are mapped to paraboloids. Another example is presented in Figure \[5\]. The inversion quadric is an elliptic paraboloid given by \( a = 3e_1 - 2e_3 + 12e_4 - 2e_6 + 6e_8 - 2e_9 \) respectively by \( x^2 + y^2 - 4z + 4 = 0 \).

5. Conclusion

The geometric algebra presented in this article serves for a lot of applications. A generalization of inversions with respect to conics, quadrics and even hyperquadrics in any dimension is possible with the use of the sandwich operator. Axis aligned hyperquadrics are simply represented as 1-blades. Furthermore, this model serves as a generalization of the conformal geometric algebra, see \[1\]. Classical representations of groups are embedded in this algebra naturally.
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