Automatic sequences defined by Theta functions and some infinite products

Shuo LI

1 Introduction

Let $p(x) \in C(x)$ be a rational function satisfying the condition $p(0) = 1$ and $q$ an integer larger than 1, in this article we will consider the expansion in power series of the infinite product

$$f(x) = \prod_{s=0}^{\infty} p(x^{q^s}) = \sum_{i=0}^{\infty} c_i x^i,$$

and study when the sequence $(c_i)_{i \in \mathbb{N}}$ is $q$-automatic. This topic has been studied by many authors, such as [Dum93], [DN15] and [CR18], using analytical approach, here we want to review this topic by a basic algebraic approach.

The main result is that for given integers $q \geq 2$ and $d \geq 0$, there exist finitely many polynomials of degree $d$ defined over the field of rational numbers $\mathbb{Q}$, such that $f(x) = \prod_{s=0}^{\infty} p(x^{q^s}) = \sum_{i=0}^{\infty} c_i x^i$ is a $q$-automatic power series.

2 Definitions and generality

Definition Let $(a_n)_{n \in \mathbb{N}}$ be a sequence, we say it is $q$-automatic if the set

$$Ker((a_i)_{i \in \mathbb{N}}) = \{(a_{ql+b})_{n \in \mathbb{N}}| l \in \mathbb{N}, 0 \leq b < q^l\}$$

is finite. This set will be called the $q$-kernel of $(a_n)_{n \in \mathbb{N}}$.

For every couple of integers $(l, b)$ satisfying $l \in \mathbb{N}, 0 \leq b < q^l$, let us define a relation $R_{l,b}$ over the sequence space: we say $R_{l,b}((a_n)_{n \in \mathbb{N}}, (b_n)_{n \in \mathbb{N}})$ if and only if

$$\forall n \in \mathbb{Z}, b_n = a_{ql+n+b}.$$

Definition Let $\sum_{i=0}^{\infty} a_i x^i$ be a power series, we say it is $q$-automatic if the sequence of coefficients $(a_n)_{n \in \mathbb{N}}$ is $q$-automatic.

Similarly we define operators $O_{l,b}$ over the space of power series:

$$O_{l,b}(\sum_{n=0}^{\infty} a_n x^n) = \sum_{n=0}^{\infty} a_{ql+n+b} x^n.$$
Now let us consider a detailed version of a well-known theorem, see, for example, [AS92].

**Proposition 1** Let \( f \in F((x)) \) be a \( k \)-automatic power series, then there exist polynomials \( a_0(x), a_1(x), \ldots, a_m(x) \in F[x] \) with \( a_0(x)a_m(x) \neq 0 \) such that

\[
\sum_{i=0}^{m} a_i(x) f(x^k^i) = 0.
\]

Furthermore, the coefficients of \( a_0(x), a_1(x), \ldots, a_m(x) \) depend only on \( R_{l,b} \) relations over the \( q \)-kernel of the sequence of the coefficients of \( f \).

**Proof** Let \( B \) denote the \( k \)-kernel of the sequence of coefficients of \( f \), and \( N \) denote the cardinal of \( B \). We can then associate each element in \( B \) with a power series by

\[
(a_n)_{n \in \mathbb{N}} \rightarrow \sum_{n=0}^{\infty} a_n x^n.
\]

Let \( B' \) denote the image of \( B \) by the previous map. For each power series in \( B' \), we have

\[
\sum_{i=0}^{\infty} a_i x^i = \sum_{i=0}^{k-1} x^i \left( \sum_{j=0}^{\infty} a_{kj+i} x^{kj} \right).
\]

Remark that if the sequence \((a_n)_{n \in \mathbb{N}}\) is in \( B \), then \((a_{kn+j})_{n \in \mathbb{N}}\) is also in \( B \), for \( j = 0, 1, \ldots, k-1 \). If we write

\[
\sum_{i=0}^{\infty} a_i x^i = \sum_{(b_n)_{n \in \mathbb{N}} \in B'} \sum_{i=0}^{\infty} c_b b_i x^{ki},
\]

Then

\[
c_b = \begin{cases} 
 x^i, & \text{if } R_{1,i}((a_n)_{n \in \mathbb{N}},(b_n)_{n \in \mathbb{N}}) \\
 0, & \text{otherwise}.
\end{cases}
\]

Particularly, we can do the same thing for \( f(x), f(x^k), \ldots, f(x^{kn}) \):

\[
\begin{align*}
 f(x) &= \sum_{(b_n)_{n \in \mathbb{N}} \in B} c^1_b \sum_{i=0}^{\infty} b_i x^{k_{N+i}}, \\
 f(x^k) &= \sum_{(b_n)_{n \in \mathbb{N}} \in B} c^2_b \sum_{i=0}^{\infty} b_i x^{k_{N+i}}, \\
 \vdots \\
 f(x^{kn}) &= \sum_{(b_n)_{n \in \mathbb{N}} \in B} c^N_b \sum_{i=0}^{\infty} b_i x^{k_{N+i}},
\end{align*}
\]

with \( c^j_b \) defined only by \( R_{l,b} \) relations. But as the cardinal of \( B' \) is \( N \), the linear forms at the right-hand side of above equalities are linearly dependent. As a result, if we neglect the linear dependence between elements in \( B' \), we can have a linear dependence between \( f(x), f(x^k), \ldots, f(x^{kn}) \) such that the coefficients depend only on \( c^j_b \). So these coefficients depend only on \( R_{l,b} \) relations.

Here we make this proposition precise by some examples:
Example Let us consider a periodic sequence
\[ a, b, a, b, a, b, a, b, \ldots \]
which is 2-automatic.

Now let us write down the associated power sequence \( F(x) = a + bx + ax^2 + bx^3 + \ldots \) and two other sequences \( A(x) = a + ax + ax^2 + ax^3 + \ldots \), \( B(x) = b + bx + bx^2 + bx^3 + \ldots \) with constant coefficients.

So
\[
\begin{align*}
F(x) &= A(x^2) + xB(x^2) \\
A(x) &= (1 + x)A(x^2) \\
B(x) &= (1 + x)B(x^2)
\end{align*}
\]

so we have the following dependence:
\[
\begin{align*}
F(x) &= (1 + x^2)(1 + x^4)A(x^8) + x(1 + x^2)(1 + x^4)B(x^8) \\
F(x^2) &= (1 + x^4)A(x^8) + x^2(1 + x^4)B(x^8) \\
F(x^4) &= A(x^8) + x^4B(x^8)
\end{align*}
\]

\( F(x) \) satisfies the functional equation
\[
(x^8 - x^6 + x^4 - x^2)((1 + x^2)F(x^2) - F(x)) = (x^4 - x^3 + x^2 - x)(1 + x^4)((1 + x^4)F(x^4) - F(x^2))
\]

This functional equation does not depend on the values of \( a \) and \( b \).

Example Let us consider the Thue-Morse sequence
\[ a, b, b, a, b, a, a, b, b, a, a, b, a, b, b, a, \ldots \]
which is 2-automatic.

Now let us write down the associated power sequence \( F(x) = a + bx + bx^2 + ax^3 + \ldots \) and another sequence \( G(x) = b + ax + ax^2 + bx^3 + \ldots \), by changing \( a \) to \( b \) and \( b \) to \( a \):

So
\[
\begin{align*}
F(x) &= F(x^2) + xG(x^2) \\
G(x) &= G(x^2) + xF(x^2)
\end{align*}
\]

so we have the following dependence:
\[
\begin{align*}
G(x^2) &= G(x^4) + x^2F(x^4) \\
x^2G(x^4) &= F(x^2) - F(x^4) \\
x^2G(x^2) &= x^2G(x^4) + x^4F(x^4)
\end{align*}
\]

\( F(x) \) satisfies the functional equation
\[
(x^4 - 1)F(x^4) + (1 + x)F(x^2) - xF(x) = 0
\]

This functional equation does not depend on the values of \( a \) and \( b \).
Proposition 2 For a given functional equation \( F : \sum_{s=0}^{m} a_s(t) f(t^{k^s}) = 0 \), there exist finitely many polynomials \( p_1, p_2, \ldots, p_r \) with \( p_i(0) = 1, \forall i \in [0, r] \), such that the associated theta functions \( G_r(x) = \prod_{s=0}^{\infty} p_r(x^{q^s}) \) satisfying equation \( F \).

Proof If \( p(x) \) is a such polynomial satisfying \( p(0) = 1 \). Let us denote by \( G(x) \) the associated power series. By hypothesis, it satisfies the functional equation \( F \):

\[
\sum_{s=0}^{m} a_s(x) G(x^{q^s}) = 0.
\]

On the other hand, the power series \( G \) satisfies another functional equation:

\[
G(x) = p(x) G(x^q).
\]

Plugging the second equation into the first one, we get

\[
\sum_{s=0}^{m} a_s(x) \prod_{r=s}^{m} p(x^{q^{m-r}}) = 0.
\]

An observation is that all terms in the sum contain a factor \( p(x^{q^{m-1}}) \) except the last one. So we have

\[
p(x^{q^{m-1}}) | a_m(x)
\]

with \( p(0) = 1 \), so there are finitely many choices for \( p(x) \).

Proposition 3 For a fixed number \( k \), there are finitely many polynomials \( p_1, p_2, \ldots, p_r \) such that the theta functions \( G_j(x) = \prod_{s=0}^{\infty} p_j(x^{q^s}) \) are \( q \)-automatic and the sizes of their \( q \)-kernels are bounded by \( k \).

Proof Fixing the size of the \( q \)-kernel, we fix the number of possibilities of \( R_{i,b} \) relations, so the possible functional equations, and we conclude by Proposition 4.2.

3 Infinite product of polynomials

Let \( p = \sum_{i=1}^{n} a_i x^i \) be a polynomial with coefficients in \( \mathbb{C} \) and \( q \) be an integer larger than 1. It is known that the coefficients of the power series

\[
f(x) = \prod_{s=0}^{\infty} p(x^{q^s})
\]

form a \( q \)-regular sequence \cite{Dum93}, here we want to study when this sequence is \( q \)-automatic.

Firstly, let us suppose that the degree of \( p \), noted \( \deg(p) \), satisfies \( q^{k-1} < \deg(p) \leq q^k \) for some \( k \in \mathbb{N} \) and write

\[
f(x) = \prod_{s=0}^{\infty} p(x^{q^s}) = \sum_{i=1}^{\infty} c_i x^i.
\]

Then the coefficients \( c_i \) satisfy a recurrence relation:
\[ c_{nq+r} = \sum_{0 \leq j \leq q^k} a_j c_{n+ \frac{r-j}{q}} \]  

for all \( r \) such that \( 0 \leq r \leq q-1 \) and \( c_n = 0 \) for all negative indices.

**Lemma 1** The sequences \((c_{qn+i-j})_{n \in \mathbb{N}}\), for all \( i \) and \( j \) such that \( 0 \leq i \leq q-1 \) and \( 0 \leq j \leq 2q^k \), can be represented as linear combinations of sequences \( \{ (c_{n-i})_{n \in \mathbb{N}} | 0 \leq i < 2q^k \} \).

**Proof** Because of the previous equality, we have
\[ c_{nq+i-j} = \sum_{0 \leq s \leq q^k} a_s c_{n+ \frac{i-s}{q}} \]

for all \( n, i, j \) defined as above. Now let us check that all sequences appearing on the right-hand side of these equalities are in the set defined in the statement. It is enough to calculate the shifting indices and we have the bounds as follows,
\[-2q^k < -3q^{k-1} \leq \frac{i-j-s}{q} \leq 0\]

which proves the statement.

**Example** Let us consider the case where \( p(x) = 1 + x + x^2 + x^3 + x^4 \) and \( q = 2 \), the sequence of coefficients of the power series \( F(x) = \prod_{s=0}^{\infty} p(x^{q^s}) \) is denoted by \((c_n)_{n \in \mathbb{N}}\), so we have
\[ p(x) = (1 + x + x^2 + x^3 + x^4)F(x^2) \]

from which we can deduce
\[ c_{2n} = c_n + c_{n-1} + c_{n-2}, \]
\[ c_{2n+1} = c_n + c_{n-1}. \]

Using the above lemma, we get
\[
\begin{pmatrix}
  c_{2n} \\
  c_{2n-1} \\
  c_{2n-2} \\
  c_{2n-3} \\
  c_{2n-4} \\
  c_{2n-5} \\
  c_{2n-6} \\
  c_{2n-7} \\
  c_{2n-8}
\end{pmatrix} = 
\begin{pmatrix}
  1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
  0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
  0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
  0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
  0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\
  0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0
\end{pmatrix} 
\begin{pmatrix}
  c_n \\
  c_{n-1} \\
  c_{n-2} \\
  c_{n-3} \\
  c_{n-4} \\
  c_{n-5} \\
  c_{n-6} \\
  c_{n-7} \\
  c_{n-8}
\end{pmatrix}
\]

and

5
Because of the previous fact, we can introduce some transition matrices: for all integers \( r \) such that \( 0 \leq r \leq q - 1 \) let us define \( \Gamma_r \) as a square matrix of size \( 2q^k + 1 \) satisfying

\[
\Gamma_r \begin{pmatrix} c_n \\ c_{n-1} \\ \vdots \\ c_{n-2q^k} \end{pmatrix} = \begin{pmatrix} c_{qn+r} \\ c_{qn+r-1} \\ \vdots \\ c_{qn+r-2q^k} \end{pmatrix}
\]

for all \( n \in \mathbb{N} \).

Let us denote by \( G \) the semi-group generated by all \( \Gamma_r \) and multiplication.

**Proposition 4**  \( a \in \{ c_n \mid n \in \mathbb{N} \} \) if and only if there exists a matrix \( g \in G \) such that \( a \) is the first element in the first row of the matrix \( g \), in other words, \( a = g(1,1) \). Furthermore, \( (c_n)_{n \in \mathbb{N}} \) is automatic if and only if \( G \) is a finite semi-group.

**Proof**  The first part of this proposition is trivial, for any \( r \in \mathbb{N} \), let us consider its \( q \)-ary expansion \( r = s_{k_1}s_{k_1-1}...s_0 \). Using Lemma 4.1, we have

\[
\begin{pmatrix} c_r \\ c_{r-1} \\ \vdots \\ c_{r-2q^k} \end{pmatrix} = \Gamma_{s_{k_1}} \Gamma_{s_{k_1-1}} \cdots \Gamma_{s_0} \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix},
\]

which proves the first part of the statement.

For the second part, let us define maps \( \gamma_r \) for all integers \( r \) by \( \gamma_r(n) = q(q(...q(q(n) + s_0)... + s_{k_1-1}) + s_{k_1}) \) for all \( n \in \mathbb{N} \) if \( r = s_{k_1}s_{k_1-1}...s_0 \). Then there is an equality for all \( r \):

\[
\begin{pmatrix} c_{\gamma_r(0)} & c_{\gamma_r(1)} & \cdots & c_{\gamma_r(2q^k)} \\ c_{\gamma_r(0)-1} & c_{\gamma_r(1)-1} & \cdots & c_{\gamma_r(2q^k)-2q^k} \\ \vdots & \vdots & & \vdots \\ c_{\gamma_r(0)-2q^k} & c_{\gamma_r(1)-2q^k} & \cdots & c_{\gamma_r(2q^k)-2q^k} \end{pmatrix} = \Gamma_{s_{k_1}} \Gamma_{s_{k_1-1}} \cdots \Gamma_{s_0} \begin{pmatrix} a_0 & a_1 & \cdots & a_{2q^k} \\ 0 & a_0 & \cdots & a_{2q^k-1} \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & a_0 \end{pmatrix}.
\]

But the last matrix in the above equality is constant and invertible, so each element of a matrix \( g \in G \) is a finite linear composition of elements in the sequence \( (c_n)_{n \in \mathbb{N}} \), so the finiteness of elements in \( (c_n)_{n \in \mathbb{N}} \) is equivalent to the finiteness of elements in \( G \). And using the fact that \( (c_n)_{n \in \mathbb{N}} \) is an automatic sequence, we conclude the statement.
Proposition 5  For given integers \( q \geq 2 \) and \( d \geq 0 \), there exist finitely many polynomials of degree \( d \) defined over the field of rational numbers \( \mathbb{Q} \), such that \( \prod_{x=0}^{\infty} p(x^d) = \sum_{i=1}^{\infty} c_i x^i \) is a \( q \)-automatic power series.

Proof  Suppose that the sequence \((c_n)_{n \in \mathbb{N}}\) generated by \( \prod_{x=0}^{\infty} p(x^d) = \sum_{i=1}^{\infty} c_i x^i \) is automatic. Let us consider a sequence of matrices \((\Gamma_n)_{n \in \mathbb{N}}\), such that \( \Gamma_i \) are defined as above for \( i = 0, 1, \ldots, q-1 \) and \( \Gamma_{qi+j} = \Gamma_i \Gamma_j \) for all \( i \geq 1 \) and \( j = 0, 1, \ldots, q-1 \).

It is easy to see that this matrix sequence is automatic because \( G \) is finite. And also the automata of this matrix sequence is the same as the one of \((c_n)_{n \in \mathbb{N}}\), because \( c_n \) is exactly the element at the position \((1, 1)\) of the matrix \( \Gamma_n \). To conclude the statement, we have to prove two things: firstly the number of automata generating the sequences \((\Gamma_n)_{n \in \mathbb{N}}\) is finite, secondly, the output functions for each automaton are also finite.

For the first point, it is enough to show that \(|G|\) is bounded by a function depending only on \( d \) and \( q \), which is proved by Theorem 1.3 of [MS77]. It says that given naturals \( n \) and \( k \), there exist, up to semi-group isomorphism, only a finite number of finite sub-semi-groups of \( M_n(F) \) generated by at most \( k \) elements.

For the second point, it is a consequence of Proposition 4.5.

Proposition 6  Let \( f \) be a polynomial satisfying the hypothesis in Proposition 4.5, then all its coefficients belong to \( \mathbb{Z} \).

Proof  Let us denote by \( d \) the degree of \( f \) and write down all coefficients of \( f \) in the form \( a_i = \frac{b_i}{q_i} \) such that \((p_i, q_i) = 1\), and similarly for all coefficient of \( F \), let us write down \( c_i = \frac{r_i}{t_i} \) with \((r_i, t_i) = 1\). If there are some coefficients of \( f \) which are rational numbers but not integers, then there exist a prime \( p \) and two integers \( d_1 \) and \( d_2 \) satisfying:

\[
d_1 = \max \{ t \mid t \in \mathbb{N}, \exists q_i, p^i | q_i \} \quad \text{and} \quad d_2 = \max \{ t \mid t \in \mathbb{N}, \exists t_i, p^i | t_i \}
\]

with \( d_1 > 0, d_2 > 0 \). In fact, because of the hypothesis, there exists \( a_i = \frac{b_i}{q_i} \) with \( q_i \neq 1 \). So there exists a prime \( p \) such that \( p | q_i \), thus \( d_1 \neq 0 \). Let us suppose \( a_j = \frac{p_j}{q_j} \) with the smallest index such that \( p^{d_1} | q_j \). Now let us check

\[
c_j = a_j + \sum_{q_k+s=j, k>0} a_k c_s.
\]

If \( c_j = \frac{r_j}{t_j} \) with \( p | t_j \) then \( d_2 \geq 1 \); otherwise, there are some \( a_k, c_j \) such that \( p^{d_1} | q_k t_j \), but with the assumption of smallest index, \( p^{d_1} \nmid q_k \), so \( p | t_j \) thus \( d_2 \geq 1 \).

Let \( l_1 \) be the smallest index such that \( p^{d_1} | q_{l_1} \) and similarly let \( l_2 \) be the smallest index such that \( p^{d_2} | s_{l_2} \). Now let us consider the coefficient \( c_{l_2 q+l_1} \), which can be calculated as

\[
c_{l_2 q+l_1} = \sum_{0 \leq i \leq d_2, j+i=l_2 q+l_1} a_i c_j.
\]

Let us consider the sum at the right-hand side, for any couple of \((a_i, c_j)\), if \( i < t_1 \), then \( p^{d_1} \nmid q_i \), the maximality of \( d_2 \) leads to \( p^{d_1+d_2} \nmid q_i t_j \); similarly, if \( i > t_1 \), then \( j < t_2 \) thus \( p^{d_2} \nmid t_j \), so that \( p^{d_1+d_2} \nmid q_i t_j \); but if \( i = t_1 \), then \( j = t_2 \), so \( p^{d_1} | q_i \) and \( p^{d_2} | t_i \). As a result, \( p^{d_1+d_2} | c_{l_2 q+l_1} \), contradicts the maximality of \( d_2 \).
4 Rational functions generated by infinite products

Here we consider the following question: for a given polynomial \( p \) and an integer \( q \), when does \( F(x) = \prod_{s=0}^{\infty} p(x^{q^s}) \) equal a rational function. This question has already been studied in [DN15] when restricting the polynomial to the cyclotomic case, this section can be considered as a generalization of the previous work.

**Proposition 7** Let \( p \) be a polynomial taking coefficients over \( \mathbb{C} \) and \( q \) be an integer larger than 1, then there is an equivalence between:

1. \( \prod_{s=0}^{\infty} p(x^{q^s}) \) is a rational function.
2. there exists a polynomial \( Q(x) \) such that \( p(x) = \frac{Q(x)}{Q(x)} \) and all roots of \( Q(x) \) are roots of unity, if \( \delta \) is a root of \( Q(x) \) then \( \delta^{q^t} \) is a root of \( Q \) for all \( t \in \mathbb{N} \).

**Proof** (2) implies (1) is straightforward, let us check (1) implies (2).

Let \( F(x) = \prod_{s=0}^{\infty} p(x^{q^s}) \) be a rational function, say \( F(x) = \frac{P(x)}{Q(x)} \), where \( P(x) \) and \( Q(x) \) are coprime, using the functional equation \( F(x) = p(x)F(x^q) \), we get

\[
\frac{P(x)Q(x^q)}{P(x^q)Q(x)} = p(x).
\]

As \( \deg(p(x)) > 0 \), so that \( \deg(Q(x)) > \deg(P(x)) \), and \( P(x^q)|P(x)Q(x^q) \) if \( \deg(P(x)) > 0 \), then \( P(x^q) \) and \( Q(x^q) \) should have at least one common root, which contradicts that \( P(x) \) and \( Q(x) \) are coprime, so we have

\[
F(x) = \frac{1}{Q(x)}
\]

and

\[
p(x) = \frac{Q(x^q)}{Q(x)}
\]

Now let us study the roots of \( Q(x) \), let us suppose \( 0 \leq |r_1| \leq |r_2| \leq ... \leq |r_m| \) where \( r_i \) are the roots of \( Q(x) \) and \( |r_i| \) is the modulus of \( r_i \). Firstly \( |r_m| \) can not be too large, if \( |r_m| > 1 \) then each root of \( Q(x^q) \) should have a modulus strictly smaller than \( |r_m| \), on the other hand \( Q(x)|Q(x^q) \), which is impossible. For the same reason, \( |r_1| \) can not be a real number between 0 and 1. So \( |r_1| \) are either 0 or 1, but if \( x|Q(x) \), the infinite product of \( p(x) \) will not converge, so \( |r_1| = 1 \) for all roots of \( Q(x) \). Using once more \( Q(x)|Q(x^q) \), if \( \delta \) is a root of \( Q(x) \) then it is a root of \( Q(x^q) \) which implies \( \delta^{q^t} \) is a root of \( Q(x) \), we can do it recursively and we obtain \( \delta^{q^t} \) is a root of \( Q \) for all \( t \in \mathbb{N} \), as a corollary, \( \delta \) can only be a root of unity. So we prove (2) using (1).

5 Infinite product of inverse of polynomials

In this section, we consider the power sequence defined as follows:

\[
F(x) = \prod_{s=0}^{\infty} \frac{1}{p(x^{q^s})} = \sum_{i=0}^{\infty} c_i x^i,
\]
where $q$ is an integer larger than 1 and $p = \sum_{i=0}^{n} b_i x^i$ is a polynomial such that $p(0) = 1$ defined as before.

Such a sequence satisfies the functional equation

$$F(x) = \frac{1}{p(x)} F(x^q).$$

If we write $\frac{1}{p(x)} = \sum_{i=0}^{\infty} a_i x^i$, then

$$c_{qn+i} = \sum_{j=0}^{n} a_{qj+i}c_{n-j},$$

for all $n \in \mathbb{N}$ and $i$ such that $0 \leq i \leq q - 1$.

**Proposition 8** If the coefficients of the power series $F(x) = \prod_{s=0}^{\infty} \frac{1}{p(x^q)} = \sum_{i=0}^{\infty} c_i x^i$ take finitely many values in $\mathbb{C}$, then the roots of $f$ are all of modulus 1.

**Proof** Firstly, let us prove that the moduli of all roots of $p$ are not smaller than 1. Otherwise, let us chose one of those which have smallest modulus, say $\alpha$, because of the above definition, we can conclude that $p(\alpha^k) \neq 0$ for all $k$ larger than 1.

Let us consider the equality,

$$\prod_{s=0}^{\infty} \frac{1}{p(x^q)} = \sum_{i=0}^{\infty} c_i x^i,$$

the right-hand side converges when $x$ tends to $\alpha$ while the left-hand side diverges, in fact $\prod_{s=1}^{\infty} \frac{1}{p(\alpha^q)}$ converges to a non-zero value because

$$\log(\prod_{s=1}^{\infty} \frac{1}{p(\alpha^q)}) = -\sum_{s=1}^{\infty} \log(p(\alpha^q))$$

which converges, however, $\frac{1}{p(x^q)}$ has a pole at $x = \alpha$.

Secondly, let us prove that the moduli of all roots of $p$ are not larger than 1. Otherwise, let us chose one of them, say $\beta$, and an integer $t$ such that $|\beta|^q > |a|/|b| + 1$, where $|a|$ is the largest modulus of the sequence $(c_i)_{i \in \mathbb{N}}$ and $|b|$ is the smallest non-zero modulus of this sequence. Now consider the following series

$$\frac{1}{1 - \beta} \prod_{s=1}^{\infty} \frac{1}{p(x^q)} = \sum_{i=0}^{\infty} d_i x^i.$$

It is easy to see that $\{d_i | i \in \mathbb{N}\}$ is finite, because such a series can be obtained by multiplying a polynomial to $F(x)$, but on the other hand, we have the inequality,
However, the sequence $s$ can easily check for all $i$ also $q$-regular there exists a $t$ such that $0 \leq t, 0 \leq t - 1, \forall i$ and $ord(A_{t,i}(\frac{s_i}{p(x)})) \geq ord(\frac{s_i}{p(x)})$ and we define $s_{i+1} = A_{t,i}(\frac{s_i}{p(x)})$, so we can easily check

$$A_{t,i}(s_{i}F(x)) = A_{t,i}(\frac{s_i}{p(x)})F(x) = s_{i+1}F(x),$$

and by induction

$$A_{t,1}A_{t-1}...A_{t_0}(F(x)) = s_{i+1}F(x).$$

However,

$$ord(s_i) < ord(s_{i+1}),$$

the sequence $s_i$ are linearly independent, so $F(x)$ can not be a regular sequence.

$$|d_{q^i}| = |\sum_{j=0}^{i} \beta^{q_j} c_{q^j(i-j)}| \geq -|a| \sum_{j=0}^{i-1} |\beta^{q_j}| + |b||\beta^{q_i}| > 0$$

which diverges. This contradicts the fact that $\{d_i | i \in N\}$ is finite. In conclusion, the roots of $f$ are all of modulus 1.

**Proposition 9** If the power series $F(x) = \prod_{i=0}^{\infty} \frac{1}{p(x^i)} = \sum_{i=0}^{\infty} c_i x^i$ is a $q$-regular sequence, then the roots of $p$ are all roots of unity, furthermore, the order of each root is multiple of $q$.

**Proof** If $F(x) = \prod_{i=0}^{\infty} \frac{1}{p(x^i)} = \sum_{i=0}^{\infty} c_i x^i$ is a $q$-regular sequence, then $F'(x) = \sum_{i=1}^{\infty} c_i x^{i-1}$ is also $q$-regular. On the other hand, we know $\frac{F'(x)}{F(x)} = \prod_{i=0}^{\infty} p(x^i)$ is $q$-regular, so

$$ord(\frac{F'}{F}) = (log F)'$$

is $q$-regular. In the same way we have $(log F(x'))'$ is $q$-regular so that

$$(log F(x))' = (log F(x'))' =\frac{p'(x)}{p(x)}$$

is $q$-regular, then we conclude by Theorem 3.3 [AS92] that all roots are roots of unity.

To prove the second part, we use a method introduced in [Bec94]. We firstly define some notation. Let us denote by $A_{t,i}$ the operator of power series:

$$A_{t,i}(\sum_{j=0}^{\infty} a_j x^j) = \sum_{j=0}^{\infty} a_{q^j x_{j+i}} x^{q^j x_{j+i}}$$

for all $i$ such that $0 \leq i \leq q^i - 1$.

If there exists a root of $p$ which’s order is not a multiple of $q$, say $\alpha$, then for all formal power series $f$, let us define $ord(f(x))$ to be the order of pole of $f$ at point $\alpha$. It is easy to check that there exists a $t \in \mathbb{N}$ such that for all $f \in F[[x]]$, $ord(f(x)) = ord(f(x^t))$ so there are some $i$ such that $ord(f(x)) \leq ord(A_{t,i}(f(x)))$.

Now let us define a sequence of power series $(s_i)_{i \in \mathbb{N}}$ and a sequence of integer $(I_i)_{i \in \mathbb{N}}$ such that

$s_0 = 1$, $0 \leq I_i \leq q^i - 1, \forall i$ and $ord(A_{t,i}(\frac{s_i}{p(x)})) \geq ord(\frac{s_i}{p(x)})$ and we define $s_{i+1} = A_{t,i}(\frac{s_i}{p(x)})$, so we can easily check

$$A_{t,i}(s_{i}F(x)) = A_{t,i}(\frac{s_i}{p(x)})F(x) = s_{i+1}F(x),$$

and by induction

$$A_{t,1}A_{t-1}...A_{t_0}(F(x)) = s_{i+1}F(x).$$

However,

$$ord(s_i) < ord(s_{i+1}),$$

the sequence $s_i$ are linearly independent, so $F(x)$ can not be a regular sequence.
Theorem 1 If the power series $F(x) = \prod_{s=0}^{\infty} \frac{1}{p(x^s)} = \sum_{i=0}^{\infty} c_i x^i$ is a $q$-regular sequence, then there exists a polynomial $Q(x)$ such that $p(x) = \frac{Q(x^d)}{Q(x)}$, so $F(x)$ can be written as

$$F(x) = Q(x) \prod_{i=1}^{\infty} R(x^q),$$

where $R(x) = \frac{Q(x^q)}{Q(x)F(x)}$, which is a polynomial.

6 Applications

In this section we will consider some examples of automatic power series of type

$$F(x) = \prod_{s=0}^{\infty} p(x^{i_s}) = \sum_{i=1}^{\infty} c_i x^i,$$

where $p$ is a polynomial of degree $d$ with coefficients in $Q$ and $l \geq 2$. It has been proved by Proposition 4.5 that the number of such polynomials $p$ is fixed once given the degree $d$ of the polynomial and $l$. But when $l$ and $d$ are both large, it will be difficult to compute the semi-group of matrix discussed in Section 4.2. Here we show a method applied on a particular example to generate the couples $(p, l)$ such that $\prod_{s=0}^{\infty} p(x^{i_s}) = \sum_{i=1}^{\infty} c_i x^i$ is an automatic power sequence.

Let us consider firstly the power series $F_1(x)$ defined by $p_1(x) = 1 + x - x^3 - x^4$ and $l = 2$, it is easy to check that

$$F_1(x) = \prod_{s=0}^{\infty} p_1(x^{2^s}) = \prod_{s=0}^{\infty} (1 + x^{2^s}) \prod_{s=0}^{\infty} (1 - (x^3)^{2^s}).$$

And it is well known that $\prod_{s=0}^{\infty} (1 + x^{2^s}) = \frac{1}{1-x} = \sum_{i=1}^{\infty} x^i$ and $\prod_{s=0}^{\infty} (1 - x^{2^s}) = \sum_{i=1}^{\infty} b_n x^i$, where $(b_n)_{n \in \mathbb{N}}$ is the Thue-Morse sequence beginning with 1, −1. So the coefficient of term $x^n$ in $F_1(x)$, say $f_1(n)$, can be calculated by

$$f_1(n) = \sum_{3i \leq n} b_i.$$
matrices are $x A^2 A x F \alpha$ let us consider the the power series $\sum_{i=0}^{\infty} x^i$ of size 4 $(x^2 + x + 1)(x^6 - 1)(x^4 - 1)$ and $l = 4$, the transition matrices of this polynomial are

$$\alpha_0 = \begin{pmatrix} 1 & -1 & -1 & 1 \\ 0 & 0 & -1 & 1 \\ 0 & 1 & -2 & 1 \\ 0 & 1 & -1 & -1 \end{pmatrix}, \quad \alpha_1 = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix},$$

$$\alpha_2 = \begin{pmatrix} 1 & -2 & 1 & 0 \\ 1 & -1 & 0 & 0 \\ 1 & -1 & 1 & 0 \\ 0 & 1 & -2 & 1 \end{pmatrix}, \quad \alpha_3 = \begin{pmatrix} 0 & -1 & 1 & 0 \\ 1 & -2 & 1 & 0 \\ 1 & -1 & 1 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix},$$

If we define a sequence of matrices $(\alpha_n)_{n \in \mathbb{N}}$ by $\alpha_{n+1} = \alpha_n \alpha_1$, $0 \leq i \leq 3$, then the $n$-th coefficient of $F_2(x)$ is $f_2(n) = \alpha_n(1, 1)$. However the matrices $\alpha_i$ for $i = 0, 1, 2, 3$ are all of form $\begin{pmatrix} A_i & 0 \\ B_i & C_i \end{pmatrix}$ with $A_i$ of size $4 \times 4$, $B_i$ of size $4 \times 1$, $C_i$ of size $1 \times 1$ and 0 the 0-matrix of size $1 \times 4$, so $\alpha_n(1, 1)$ can be calculated only by the multiplications between $A_i$. Remarking that this four matrices are nothing else then $\Gamma_0^2, \Gamma_1 \Gamma_0, \Gamma_0 \Gamma_1, \Gamma_0^2$, we conclude that the sequence $(f_2(n))_{n \in \mathbb{N}}$ is bounded so 4-automatic.

By the same method, the power series $F_3(x)$ defined by $p_3(x) = 1 + x + x^2 - x^4 - x^5 + x^7 + x^8 - x^{10} - x^{11} - x^{12} = (x^2 + x + 1)(x^6 + 1)(1 - x^4)$ and $l = 4$ is also 4-automatic. In fact, its transition matrices are

$$\beta_0 = \begin{pmatrix} 1 & -1 & 1 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 1 & -1 & 1 \end{pmatrix}, \quad \beta_1 = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 1 & -1 & 0 \end{pmatrix},$$

$$\beta_2 = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & -1 \end{pmatrix}, \quad \beta_3 = \begin{pmatrix} 0 & 1 & -1 & 0 \\ 1 & 0 & -1 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \end{pmatrix},$$
and once more they are of form \( \begin{pmatrix} A_i & 0 \\ B_i & C_i \end{pmatrix} \) with \( A_0 = -\Gamma_0 \Gamma_0 \Gamma_1, A_1 = -\Gamma_1 \Gamma_0 \Gamma_0, A_2 = \Gamma_0 \Gamma_1 \Gamma_0, A_3 = \Gamma_1 \Gamma_1 \Gamma_0 \).

Furthermore, as

\[
\prod_{s=0}^{\infty} ((x^2)^{4^s} + 1)(x^{4^s} + 1) = \prod_{s=0}^{\infty} \frac{(x^4)^{4^s} - 1}{x^{4^s} + 1} = \frac{1}{1-x}
\]

we have

\[
(1-x)F_2(x) = \prod_{s=0}^{\infty} ((x^2)^{4^s} + x^{4^s} + 1)((x^6)^{4^s} - 1) \frac{(x^4)^{4^s} - 1}{((x^2)^{4^s} + 1)(x^{4^s} + 1)} = \prod_{s=0}^{\infty} (x^{9})^{4^s} - (x^{6})^{4^s} + (x^{3})^{4^s} + 1,
\]

\[
(1-x)F_3(x) = \prod_{s=0}^{\infty} ((x^2)^{4^s} + x^{4^s} + 1)((x^6)^{4^s} + 1) \frac{(x^4)^{4^s} - 1}{((x^2)^{4^s} + 1)(x^{4^s} + 1)} = \prod_{s=0}^{\infty} -(x^{9})^{4^s} + (x^{6})^{4^s} - (x^{3})^{4^s} + 1.
\]

**Proposition 10** The power series

\[
F_2(x) = \prod_{s=0}^{\infty} ((x^2)^{4^s} + x^{4^s} + 1)((x^6)^{4^s} - 1)(x^{4^s} - 1)
\]

and

\[
F_3(x) = \prod_{s=0}^{\infty} ((x^2)^{4^s} + x^{4^s} + 1)((x^6)^{4^s} + 1)(-(x^{4^s} + 1))
\]

are 4-automatic.

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