Lower Bounds on the Bayes Risk of the Bayesian BTL Model with Applications to Random Graphs

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Abstract—We consider the problem of aggregating pairwise comparisons to obtain a consensus ranking order over a collection of objects. We employ the popular Bradley-Terry-Luce (BTL) model in which each object is associated with a skill parameter which allows us to probabilistically describe the pairwise comparisons between objects. In particular, we employ the Bayesian BTL model which allows for meaningful prior assumptions and to cope with situations where the number of objects is large and the number of comparisons between some players is small or even zero. For the conventional Bayesian BTL model and the same model with home advantage, we derive information-theoretic lower bounds on the Bayes risk of estimators for norm-based distortion functions. We compare the information-theoretic lower bounds with the Bayesian Cramér-Rao lower bounds we derive for the case when the Bayes risk is the mean square error. We draw parallels between pairwise comparisons in the BTL model and inter-player games represented as edges in an Erdős-Rényi graph and analyze the effect of connectedness and various tree structures on the lower bounds. We illustrate the utility of the bounds through simulations by comparing them with the error performance of an expectation maximization-based inference algorithm for the Bayesian BTL model.

Index Terms—Information-theoretic lower bounds, Ranking, BTL model, Random graphs

I. INTRODUCTION

Ranking systems are ubiquitous in daily life as they form essential ingredients in several applications, including electoral preference learning, personalized ad targeting, recommender systems, etc. A ranking system collates the opinions of its survey participants and obtains the true underlying ranking order that best agrees with the majority opinion. The ranking order corresponding to the majority opinion is often referred to as the consensus ranking.

When queried about the ranking order of $q$ items, the survey participants will usually share a list of $\ell \leq q$ items in the order of preference. A large body of works consider permutations of the set $\{1, \ldots, q\}$ as observed ranking orders, i.e., $\ell = q$, and define a parameterized probability density function over the $q!$ permutations [1]–[3]. Several other works assume observations consisting of the top-$\ell$ rated items, where $\ell < q$, and derive inference algorithms for such parametric and non-parametric ranking models [4]–[6]. Often the survey participants prefer providing quick responses in the form of pairwise preferences, especially if $q$ is large. Typically, such pairwise preferences are in response to queries of the form, “Is item $i$ better than item $j$?” (thus, $q = 2$). These observations naturally arise in applications such as sports where two teams play against each other, elections where two candidates face-off, or social choice [7] etc.

Among the ranking models for pairwise preferences [3], the Bradley-Terry-Luce (BTL) model is a popular, simple yet powerful model [8]. The BTL model associates a skill parameter to each item that is being compared. Several authors have addressed the problem of rank aggregation in the BTL model. In [9], the authors use the majorization-minimization approach to infer the skill parameters of the BTL model. The rank centrality algorithm proposed in [10] is another popular approach, where the authors derive, using the theory of Markov chains and random walks, finite sample error rates between the skill parameters of the BTL model and those estimated by the algorithm. Counting algorithms such as Copeland counting [11] and the weighted counting algorithm [12] have been also proposed for rank aggregation in the BTL model. In [13], the authors consider ranking under the BTL model along with several other models and obtain upper bounds on the sample complexity. The conditions for recovering the entries of the pairwise comparison matrix of a more general class of models, which is based on a strong stochastic transitivity property and includes the BTL model as a particular case, have also been derived in [14].

On the other hand, one could also use Bayesian methods for estimating the parameters of the BTL model by incorporating prior information into the pairwise comparison ranking models. In fact, this approach has a long history in modeling animal behavior using the theory of dominance hierarchies [15]. In the case of animal behavior, maximum likelihood estimates of the skill parameters under the BTL model often do not converge to finite values (i.e., they are ill-conditioned), and the Bayesian methods are used as regularization techniques resulting in convergent (and well-conditioned) inference algorithms [16]–[18]. More recent works have also investigated Bayesian preference learning in the setting where the pairwise comparisons are assumed to follow the probit model. This is a model in which each item is associated with a parameterized utility model (instead of a skill parameter) based on a Gaussian process. For inference, gradient descent algorithms [19], [20] and expectation propagation algorithms are proposed [21].

A tractable generalized Bayesian BTL model was proposed in [22]. Here, the authors associate the Gamma distribution as a prior density for the skill parameters. They show that using an appropriate latent variable, it is possible to derive an expectation minimization (EM) algorithm to infer the skill parameters. They propose such algorithms for several extensions of the BTL model such as the BTL model with home advantage, National University of Singapore (NUS) (emails: elemine@nus.edu.sg, ranjitha.p@gmail.com, vtan@nus.edu.sg). The third author is also with the Department of Mathematics, NUS.

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advantage and with ties. We focus on this line of Bayesian BTL models, and derive lower bounds on minimax risks.

A. Contributions

In this work, we consider the Bayesian BTL model proposed in [22]. Under this modeling assumption, we derive lower bounds on the Bayes risks of estimators for norm-based distortion functions, which also serve as lower bounds on their minimax risks. More specifically, by using two separate lines of analyses, we obtain the following results:

- In Section III-A, Theorem 2 states a family of information-theoretic lower bounds on the Bayes risks of estimators for norm-based distortion functions. These bounds reveal that the Bayes risks of estimators for the BTL model dominates the function $n^{-r/2}$ asymptotically in the case of $r$-norm to power $r$ distortion functions. The bounds in (20) are obtained via the evaluation of a family of information-theoretic lower bounds proposed by Xu and Raginsky [23] and which we state in Theorem 1.

B. Paper Outline

In Section II, we provide basic notations, describe the Bayesian BTL model, and state preliminaries for the derivations of the lower bounds on the Bayes risk. In Section III, we state our main analytical results in Theorems 2 and 3. We also discuss the nature of the obtained lower bounds for the Bayes risk and provide some numerical experiments. In Section IV, we describe the effect of the graph structure on the derived lower bounds by answering the three questions we posed and numerically verifying our observations. In Section V, we state Theorems 4 and 5, and we analyze the performance of the obtained lower bounds for the Bayesian BTL model with home-field advantage. We defer most proofs to the Appendices or the supplementary material [25].
for \( x, \alpha, \beta \in \mathbb{R}_{++} \). The parameters \( \alpha \) and \( \beta \) are, respectively, the shape and rate parameters and \( \Gamma(\cdot) \) is the Gamma function. We denote the diagamma function by \( \psi(\cdot) = \Gamma'(\cdot)/\Gamma(\cdot) \), and the Beta function by \( B(x, y) \), for \( x, y \in \mathbb{R}_{++} \). We use \( O(\cdot) \) denote the Big-O notation. We also use the notation \( \lesssim \) to say that a function is asymptotically less than or equal to another, i.e., \( f(x) \lesssim g(x) \) holds if and only if \( \limsup_{x \to \infty} f(x)/g(x) \leq 1 \). Similarly, \( \gtrsim \) is used to denote the asymptotic inequality in the opposite direction.

### A. Preliminaries for the Bayesian BTL model

We now introduce background needed for the later sections.

1) Ranking from pairwise comparisons: Consider a collection of \( k \geq 2 \) items indexed by \([k]\). The outcomes of \( n \in \mathbb{N} \) pairwise comparisons between the items of this collection consists of a record of the form:

\[
\{(i_1, j_1, \ell_1), \ldots, (i_n, j_n, \ell_n)\} \in (\mathcal{I}_n[k] \times \{0, 1\})^n, \tag{3}
\]

where \((i_m, j_m) \in \mathcal{I}_n[k], \) for each \( m \in [n] \), indicates the indices of the item pairs being compared at the \( m \)-th comparison, and \( \ell_m := I \{i_m \text{ beats } j_m\} \) is the corresponding label indicating the preferred item. For each pair of items \((i, j) \in \mathcal{I}_n[k], \) the problem of ranking from pairwise comparisons postulates the existence of underlying pairwise preference probabilities such that item \( i \) is preferred over item \( j \) with probability \( P_{ij} \in [0, 1] \) and the opposite is true with probability \( P_{ji} = 1 - P_{ij} \). Moreover, the pairwise comparisons between item pairs are independent. The pairwise preference probabilities collectively form an underlying pairwise preference matrix \( P \in [0, 1]^{n \times n} \), and the class of all such matrices is given by:

\[
\mathcal{P} := \left\{ P \in [0, 1]^{n \times n} : \begin{array}{c}
P_{ji} = 1 - P_{ij}, \forall (i, j) \in \mathcal{I}_n[k], \\
P_{ii} = 0, \forall i \in [k]
\end{array} \right\}. \tag{4}
\]

The goal of ranking is to recover an accurate estimate of \( P \in \mathcal{P} \), where the accuracy is measured with respect to a desired norm. In this work, we will mainly be interested with the \( L^1 \)-norm and the squared \( L^2 \)-norm, although the information-theoretic lower bounds will be valid for arbitrary norms.

2) Definition of the BTL model: Multiple of statistical models for ranking have been proposed in the literature by imposing additional conditions on the structure of the permissible pairwise preference matrices [26]. The BTL model associates to each item \( i \in [k] \) a skill/rate parameter \( \lambda_i \in \mathbb{R}_{++} \) such that

\[
P_{ij} := \frac{\lambda_i}{\lambda_i + \lambda_j}, \tag{5}
\]

for all \( i, j \in [k] \). In other words, the model is governed by the following subclass of distributions:

\[
\mathcal{P}_{BTL} := \left\{ P \in \mathcal{P} : \exists \lambda \in \mathbb{R}^k_{++} \text{ s.t. } P_{ij} = \frac{\lambda_i}{\lambda_i + \lambda_j}, \forall (i, j) \in \mathcal{I}_n[k] \right\}. \tag{6}
\]

where \( \lambda := (\lambda_1, \ldots, \lambda_k) \). The task of a ranking algorithm for the BTL model is thus to recover an accurate estimate of the skill/rate parameters \( \lambda \).

From the definition of the class \( \mathcal{P}_{BTL} \), it can be seen that the parameter vector \( \lambda \) induces a family of conditional densities \( \{p(\cdot|\lambda) : \lambda \in \mathbb{R}^k_{++}\} \) on the observation space \( \{\mathcal{I}_n[k] \times \{0, 1\}\}^n \). In describing the induced densities, as the order of the pairs in the comparison does not matter by the assumption of independence, it will be sufficient and convenient to extract from the record in (3) two quantities for any pair of items \((i, j) \in [k] \): The first is the number of comparisons in which element \( i \) is preferred over element \( j \), which is denoted by \( w_{ij} \), and the second is the total number of comparisons between elements \( i \) and \( j \), which is denoted by \( n_{ij} \). Observe that we must necessarily have

\[
n = \sum_{(i, j) \in \mathcal{I}_n[k]} n_{ij} = \frac{1}{2} \sum_{(i, j) \in [k]} n_{ij}, \tag{7}
\]

and \( n_{ij} = w_{ij} + w_{ji}, \) for any \((i, j) \in [k] \). In the scope of this work, we will further assume that \( \mathbf{N} := (n_{ij}) \in \mathbb{N}^{k \times k} \) is a matrix that is fixed \textit{a priori}, and the comparisons are performed accordingly.\(^1\) Now, a data sample can be described by the matrix \( \mathbf{W} := (w_{ij}) \in \mathbb{N}^{k \times k} \). Correspondingly, we let \( \Omega := (\Omega_{ij}) \in \mathbb{R}^{k \times k} \) denote the random data matrix, i.e., \( w_{ij} \) is assumed to be the realization of a random variable \( \Omega_{ij} \), for all \((i, j) \in [k] \). Then, one can show with few simple manipulations that, for each \( \lambda \in \mathbb{R}^k_{++} \), the basic BTL model assumption results in the following conditional distributions for the data

\[
\Omega | \lambda \sim p(\mathbf{W} | \lambda) = \prod_{(i, j) \in \mathcal{I}_n[k]} p(w_{ij} | \lambda_i, \lambda_j), \tag{8}
\]

\[
= \prod_{(i, j) \in \mathcal{I}_n[k]} B(w_{ij}; n_{ij}, P_{ij}), \tag{9}
\]

where \( \Omega_{ij} | \lambda_i, \lambda_j \sim p(w_{ij} | \lambda_i, \lambda_j) = B(w_{ij}; n_{ij}, P_{ij}) \). See Lemma 1 in supplementary material [25] for a proof.

3) Bayesian estimation framework: Now, let us establish the Bayesian estimation framework we are interested in for the purpose of deriving lower bounds on the performance of any ranking procedure for the rate parameters \( \lambda \). In the standard Bayesian estimation framework, the unknown parameter vector is treated as a random vector \( \Lambda := (\lambda_1, \ldots, \lambda_k) \in \mathbb{R}^k_{++} \) and the parameter space is endowed with a prior distribution \( p(\lambda) \) on \( \Lambda \). Then, it is assumed that, for a given realization \( \Lambda = \lambda \) and for a fixed \( \mathbf{N} \), a data sample \( \mathbf{W} \) is generated according to the density \( p(\mathbf{W} | \lambda) \). The joint distribution of the pair \( (\Omega, \Lambda) \) for fixed \( \mathbf{N} \) is now uniquely determined by \( p(\Lambda, \mathbf{W}) = p(\lambda)p(\mathbf{W} | \lambda) \). In this case, the \textit{Bayes risk} for estimating \( \Lambda \) from \( \Omega \) with respect to a given distortion function \( d : \mathbb{R}^k_{++} \times \mathbb{R}^k_{++} \to \mathbb{R}^+ \) is defined as

\[
R_B := \inf_{\varphi} \mathbb{E}[d(\Lambda, \varphi(\Omega))], \tag{10}
\]

where \( \varphi(\cdot) : \mathbb{N}^k_{++} \times \mathbb{N}^k_{++} \to \mathbb{R}^k_{++} \) is any estimator of \( \Lambda \).

4) Choice of prior distributions: Several works [22], [27], [28], which perform Bayesian estimation for the basic BTL model and its generalizations, assign a Gamma distributed prior \( \lambda_i \sim G(\lambda_i; a_i, b_i) \) to each skill/rate parameter

\(^1\)The question of how to “optimally” choose \( \mathbf{N} \) for a fixed budget \( n \) will be addressed later in Section IV in the context of random graphs.
i ∈ [k], where \(a := (a_i), b := (b_i) \in \mathbb{R}^k_{i=1}^2.\) With this assumption, we have

\[
A \sim p(\lambda) = \prod_{i \in [k]} p(\lambda_i) = \prod_{i \in [k]} G(\lambda_i; a_i, b_i),
\]

(11)

and by (8) and (11), we get the following overall expression:

\[
p(\lambda, W) = \prod_{(i, j) \in I_0[k]} B(w_{ij}; n_{ij}, P_{ij}) \prod_{i \in [k]} G(\lambda_i; a_i, b_i).
\]

(12)

The prior in (11), which we will be assuming throughout this paper, turns out to be a convenient choice, justified by what is called in the literature “the Thurstonian interpretation” of the BTL model [29]. In fact, the probability that an item is preferred over another in a pairwise comparison in the BTL model can be naturally seen as begin determined by the shortest of two exponentially distributed random arrival times with rate parameters given by the respective skill/rate parameters of the items. Namely, the correspondence \(P_{ij} = \mathbb{P}[\tau_{si} < \tau_{sj}]\) can be established, for each pair \((i, j) \in I_0[k]\) and for all \(s = 1, \ldots, n_{ij},\) by defining the latent random variables \(\tau_{si} \sim \mathcal{E}(\lambda_i)\) and \(\tau_{sj} \sim \mathcal{E}(\lambda_j),\) where \(\mathcal{E}(\lambda)\) is the exponential distribution with rate \(\lambda.\)

In their study [22], Caron and Doucet introduced the following set of latent random variables

\[
Z_{ij} = Z_{ji} := \min_{k=1}^{n_{ij}} \{\tau_{ki}, \tau_{kj}\},
\]

(13)

for \((i, j) \in I_0[k],\) and remark that the reduced amount of missing information in (13) with respect to the exponentially distributed random arrival times leads to faster rates of convergence for the EM and the data augmentation algorithms they propose for performing Bayesian inference in BTL models. These new latent variables will also be useful in our information-theoretic lower bound derivations of Sections III-A and V-A. We denote the random matrix of the sum of shortest arrival times associated to the comparisons of item pairs by \(Z := (Z_{ij}) \in \mathbb{R}^{k \times k}\) and its realization by \(\zeta := (\zeta_{ij}) \in \mathbb{R}^{k \times k}.\) It is observed in [22, Eq. (2.1)] that

\[
Z_{ij} | \lambda_i, \lambda_j \sim p(\zeta_{ij} | \lambda_i, \lambda_j) = G(\zeta_{ij}; n_{ij}, \lambda_i + \lambda_j)
\]

(14)

for all \((i, j) \in I[k].\)

B. Preliminaries for the Lower Bounds on the Minimax Risk

The definition of the Bayes risk introduced in (10) relies on the choice of a specific prior distribution on the skill parameters of the model \(\lambda.\) It is clear that different priors will, in general, result in different values of the Bayes risk. A more general notion of risk associated to estimation problems is the minimax risk, which can be defined in our context as follows:

\[
R_M := \inf_{\mathcal{W}} \sup_{\mathcal{A} \sim p(\lambda)} \mathbb{E}[d(\mathcal{A}, \varphi(\Omega))].
\]

(15)

Note that the minimax risk is computed by choosing an estimator that minimizes the maximum Bayes risk, and thus, \(R_M \geq R_B.\)

1) Information-theoretic lower bounds on the Bayes risk:

Although several techniques exist to compute lower bounds on the minimax risk of estimation and optimization problems (see for instance [30]) our focus will be on computing asymptotic lower bounds on the Bayes risk, which will automatically serve as asymptotic lower bounds on the minimax risk. In particular, the lower bounds we derive in Sections III-A and V-A will make use of the following result from [23] presented among several other general lower bounds on the Bayes risk developed via information-theoretic quantities.

**Theorem 1:** [23, Theorem 3] Let \(\|\cdot\|\) be an arbitrary norm in \(\mathbb{R}^d\) and let \(r \geq 1.\) The Bayes risk for estimating the parameter \(\mathbf{X} \in \mathbb{R}^d\) based on the sample \(\mathbf{Y}\) with respect to the distortion function \(d(x, \hat{x}) = \|x - \hat{x}\| r\) satisfies

\[
R_B \geq \sup_{\mathcal{P}(\mathbf{M}) \times \mathbb{Y}} \frac{d}{r} (V_d \Gamma \left(1 + \frac{d}{r}\right) - r/d)
\]

\[
\times e^{-\left(\frac{d}{r} - b(\mathbf{M})\right) r/d},
\]

(16)

where \(V_d\) denotes the volume of the unit ball in \((\mathbb{R}^d, \|\cdot\|).\)

2) BCRB and HCRB on the Bayes risk: As a precursor to the sections that follow, we define the Mean-Square Error (MSE) matrix and the Fisher Information Matrix (FIM) [31], and state the assumptions under which we derive the BCRB and Hybrid Cramér-Rao bound (HCRB) in this paper. Consider a general estimation problem where the unknown vector \(\mathbf{X} \in \mathbb{R}^k\) can be split into sub-vectors \(X = [X_r^T, X_d^T]^T,\) where \(X_r \in \mathbb{R}^m\) consists of random parameters distributed according to a known distribution, and \(X_d \in \mathbb{R}^{k-m}\) consists of deterministic parameters. Let \(\varphi(\mathbf{Y})\) denote an estimator of \(\mathbf{X}\) as a function of the observations \(\mathbf{Y} .\) The MSE matrix \(\mathbb{E} \mathbf{X}\) is defined as

\[
\mathbb{E} \mathbf{X} := \mathbb{E} \left[ (\mathbf{X} - \varphi(\mathbf{Y}))(\mathbf{X} - \varphi(\mathbf{Y}))^T \right],
\]

(17)

where the expectation is taken over both \(\mathbf{X}\) and \(\mathbf{Y}\). The first step in obtaining Cramér-Rao-type lower bounds is to derive the FIM [31]. In this paper, we use the notation \(\mathbb{I}^{X}\) to represent the FIM under the different modeling assumptions. Typically, \(\mathbb{I}^{X}\) is expressed in terms of the individual blocks of submatrices, where the \((i, j)^{th}\) block is given by

\[
\left[\mathbb{I}^{X}\right]_{ij} := -\mathbb{E} \left[ (\nabla_i \varphi)(\nabla_j \varphi)^T \log p(\mathbf{Y}, \mathbf{X}_r | \mathbf{X}_d) \right],
\]

(18)

where \(\nabla\) denotes the gradient with respect to the vector \(\mathbf{X}\). Then, assuming that the MSE matrix \(\mathbb{E} \mathbf{X}\) exists and the FIM is non-singular, a lower bound on the MSE matrix \(\mathbb{E} \mathbf{X}\) is given by the inverse of the FIM:

\[
\mathbb{E} \mathbf{X} \succeq (\mathbb{I}^{X})^{-1}.
\]

(19)

For example, when \(X_r \neq \emptyset\) and \(X_d = \emptyset, \mathbb{I}^{X}\) represents a Bayesian Information Matrix (BIM) and the corresponding lower bound on the MSE matrix is called the BCRB. When \(X_r \neq \emptyset\) and \(X_d \neq \emptyset, \mathbb{I}^{X}\) represents a Hybrid Information Matrix (HIM), and the corresponding lower bound on the MSE matrix is called as the HCRB. Finally, the Bayes risk for the squared \(L^2\) norm can be lower bounded by the trace of the inverse of the FIM.
III. MAIN ANALYTICAL RESULTS

A. Information-Theoretic Lower Bounds

The next theorem states the main result of this subsection. Its proof will be given at the end.

Theorem 2: Consider the Bayesian BTL model introduced in Section II-A. Let \( \| \cdot \| \) denote an arbitrary norm in \( \mathbb{R}^k \). For any \( r \geq 1 \), let \( d(\lambda, \hat{\lambda}) = \| \lambda - \hat{\lambda} \|_r \) be the distortion function, where \( \lambda := \varphi(W) \) is an estimator of \( \lambda \) based on data sample \( W \) for a fixed \( N \). The Bayes risk \( R_B \) and the minimax risk \( R_M \) defined respectively in (10) and (15) for estimating \( \Lambda \in \mathbb{R}^{k \times +} \) are asymptotically lower bounded by:

\[
R_M \geq R_B \geq_n \frac{k}{r^c} \left( V_k \left( 1 + \frac{k}{r} \right) \right)^{-r/k} e^{-r E_{BTL}(N, a, b)}
\]

where \( V_k \) denotes the volume of the unit ball in \( \mathbb{R}^k \) and  

\[
n_i := \frac{1}{2} \sum_{j \in [k] \setminus \{i\}} n_{ij},
\]

for all \( i \in [k] \), and

\[
E_{BTL}(N, a, b) := \frac{1}{k} \sum_{i \in [k]} \left( -\frac{1}{2} \log 2\pi + \log b_i - \psi(a_i) + \frac{1}{2} \log \left( a_i + \frac{1}{2} \sum_{j \in [k] \setminus \{i\}} n_{ij} \right) \right).
\]

Corollary 1: If \( a_i = a \) and \( b_i = b \), for each \( i \in [k] \), one can further lower bound the expression in (20) via Jensen’s inequality as follows:

\[
R_B \geq_n \frac{k}{r^c} \left( V_k \left( 1 + \frac{k}{r} \right) \right)^{-r/k} \times \exp \left\{ -r \left( -\frac{1}{2} \log 2\pi + \log b - \psi(a) \right) \right\}.
\]

Consequently, for the \( L^1 \) norm (\( r = 1 \)), we get:

\[
R_B \geq_n \sqrt{\frac{\pi}{2}} e^{-(\log b - \psi(a)+1)} \frac{k}{\sqrt{a/k + n}}.
\]

and for the squared \( L^2 \) norm (\( r = 2 \)), we get

\[
R_B \geq_n e^{-2(\log b - \psi(a)+1)} \frac{k}{a/k + n}.
\]

In proving Theorem 2, we will use the lower bound given in Theorem 1 and the relation we introduce next.

Proposition 1: For the Bayesian BTL model introduced in Section II-A, we have

\[
\frac{1}{k} (I(\Lambda; \Omega Z) - h(\Lambda)) \leq_n E_{BTL}(N, a, b),
\]

where \( n_i \) is defined in (21) and \( E_{BTL}(N, a, b) \) in (22).

The proof of Proposition 1 is given in Appendix A.

Proof of Theorem 2: We first observe that

\[
R_B = \inf_{\varphi} \mathbb{E}[\ell(\lambda, \varphi(W))] \geq \inf_{\varphi'} \mathbb{E}[\ell(\lambda, \varphi'(W, Z))] =: R_{B'}.
\]

Now, taking \( d \leftarrow k, X \leftarrow \Lambda, \) and \( T \leftarrow (W, Z) \) in Theorem 1, the proof of the claimed asymptotic lower bound in Theorem 2 follows by lower bounding \( R_{B'} \) via the unconditional version of the lower bound in (16) and then using the relation (26) derived in Proposition 1.

B. Bayesian Cramér-Rao Lower Bounds (BCRBs)

In this part, we derive the BCRB, which is a well-known lower bound on the MSE of an estimator, i.e., when the squared \( L^2 \) norm is used as the distortion measure in the definition of the Bayes risk. In contrast to the information-theoretic bounds derived in the previous section, the BCRB does not require the auxiliary variable \( Z \).

Theorem 3: For the Bayesian BTL model introduced in Section II-A, the BCRB on the MSE matrix \( E^A \) of the unknown random skill/rate parameter vector \( \Lambda \) with the parameterized distribution given by (11) is given by \( E^A \succeq (I^A)^{-1} \), where the entries of the BIM are given by

\[
[I^A]_{i,j} = (a_i - 1)T_1(a_i, b) + \sum_{j \in [k] \setminus \{i\}} n_{ij} T_2(a_i, a_j, b),
\]

for \( i \in [k] \), and

\[
[I^A]_{i,j} = -n_{ij} T_3(a_i, a_j, b)
\]

for \((i, j) \in I[k] \). The functions \( T_1, T_2, T_3 \) are defined as:

\[
T_1(a_i, b) := \mathbb{E} \left[ \Lambda_i^{-2} \right] = \frac{b^2 \Gamma(a_i - 2)}{\Gamma(a_i)},
\]

\[
T_2(a_i, a_j, b) := \frac{b^2(a_i - 2) \Gamma(a_i - 2)}{\Gamma(a_i)} \times \left[ \frac{a_j}{a_i + a_j - 2} - \frac{\Gamma(a_j + 1)}{(a_i + a_j - 1) \Gamma(a_j)} \right],
\]

\[
T_3(a_i, a_j, b) := \frac{b^2(a_i - 1) \Gamma(a_i - 1)}{\Gamma(a_i)} \times \left[ \frac{(a_j - 1) \Gamma(a_j - 1)}{(a_i + a_j - 1) \Gamma(a_i)} - \frac{1}{a_i + a_j - 2} \right].
\]

The Bayes risk for the squared \( L^2 \) norm is lower bounded as

\[
R_B \geq \text{Tr}((I^A)^{-1}).
\]

The proof of Theorem 3 is given in Appendix B. Note that the BCRB given in Theorem 3 can be expressed as a function of \( a/b^2 \), i.e., the variance of the prior distribution in (11).

C. Discussions

Let us comment on the derived lower bounds.

1) Effect of priors: To simplify the discussion, we let \( a_i = a \) and \( b_i = b \) for each \( i \in [k] \). In [22], the prior (11) is chosen such that \( b = ak - 1 \), for \( a \in \mathbb{R}_{++} \) and \( k \in \mathbb{N} \). This choice ensures that \( \sum_{i \in [k]} \lambda_i = 1 \), and it is justified by the fact that \( b \) acts as a scaling parameter with no influence on inference [22, Section 5]. This latter observation is reflected as well in the information-theoretic lower bound of Theorem 2 and the BCRB of Theorem 3 which depend on \( b \) only as a multiplicative scaling factor given by \( 1/b^2 \). Let us next examine the behavior of the derived lower bounds in two...
distribution in (11) are chosen as BCRB since the former has been derived by including the proposed by [22] for norm alongside the MSE performance of the EM algorithm. We now present some simulation results. Fig. 1 displays plots of the information-theoretic and BCRB lower bounds of Theorem 2, respectively. Figure is generated for the information-theoretic and BCRB lower bounds derived in the previous section and we answer the questions (q.1)–(q.3) we posed in the Introduction. Among all tree graphs with k nodes and k − 1 edges, we focus on two extremal tree structures, the first one being the star graph which has one central node with edges to every other node, and the second one being the chain graph which consists of an arbitrary ordering of the k nodes with edges between pairs j and j + 1 for every j ∈ [k].

IV. EFFECT OF GRAPH STRUCTURE ON BOUNDS

In any ranking procedure, the subset of pairs of items being compared induces a comparison graph. Let G := ([n], E) be a comparison graph such that if the item pair (i, j) ∈ I0[k] belongs to the edge set E with edge weights eij ∈ N, then the items i and j are being compared nij = eij times. In this section, we investigate the effect of the graph structure on the information-theoretic and BCRB lower bounds derived in the previous section and we answer the questions (q.1)–(q.3) we posed in the Introduction. Among all tree graphs with a fixed budget for n, the maximum and minimum values of the lower bound on the Bayes risk in (20) is achieved by the following water-filling solution for n_i defined in (21):

\[ n_i = (\mu - a_i)^+, \quad \text{for } i \in [k], \]

where \( \mu \) is chosen such that

\[ \sum_{i \in [k]} (\mu - a_i)^+ = n. \]

Proof: It is easy to see that the allocation of \( n_i \)'s, for all \( i \in [k] \), which maximizes \( E_{\text{ITL}}(N, a, b) \) defined in (22), and thus minimizes the lower bound in (20), is given by the water-filling solution, since this optimization corresponds to the problem of maximizing

\[ \sum_{i \in [k]} \frac{1}{2} \log (a_i + n_i), \]

subject to the constraints \( \sum_{i \in [k]} n_i = n \) and \( n_i \in N \), see for instance the discussion in [32, Chapter 9.4].

Corollary 3: Suppose that \( a_i = a, \) for all \( i \in [k] \). Among all tree graphs with a fixed budget for n, as defined in (7), the maximum and minimum values of the lower bound on the Bayes risk in (20) are achieved by the extremal star and chain graphs, respectively.

The proof is provided in Appendix C. We are now ready to answer the first two questions we posed in the Introduction.

5 Any additional information regarding data can only decrease the lower bounds on the Bayes and minimax risks.
Given a fixed budget $n$, as defined in (7), and $a_i = a$, for all $i \in [k]$, Corollary 2 implies that any connected regular graph results in an optimal allocation minimizing the lower bounds on the Bayes risk in (20) among all connected graphs. One such graph is the fully connected graph with an equal number of pairwise comparisons with $n_i = n/k$ per node, for all $i \in [k]$, and $n_{ij} = 2n/(k(k - 1))$ per edge, for all $(i, j) \in \mathcal{I}[k]$. Another one is the cycle graph with an equal number of pairwise comparisons $n_{ii+1} = n_{1k} = n/k$ per edge, for all $i \in [k - 1]$.

Among all tree graphs, the chain and star graphs minimizes and maximizes, respectively, the information-theoretic lower bounds on the Bayes risk in (20) for a given fixed budget $n$, as defined in (7).

Fig. 3 illustrates the information-theoretic lower bounds as a function of the sample size in the discussed graph topologies.

B. Optimal Allocations in Connected Graphs and Trees for the Bayesian Cramér-Rao Lower Bounds

Here, we analyze the dependence of BCRB on graph topologies. Let $\mathbf{I}_{st}^A$, $\mathbf{I}_{ch}^A$, $\mathbf{I}_{ra}^A$, and $\mathbf{I}_{fc}^A$ denote the FIMs for a star graph, a chain graph, a random tree graph, and a fully connected graph, respectively. In Fig. 4, we provide numerical evidence that for a given large budget $n$, as defined in (7), the Bayes risks of various graph topologies satisfy

$$\text{Tr}((\mathbf{I}_{st}^A)^{-1}) \leq \text{Tr}((\mathbf{I}_{ch}^A)^{-1}) \leq \text{Tr}((\mathbf{I}_{ra}^A)^{-1}) \leq \text{Tr}((\mathbf{I}_{fc}^A)^{-1}).$$

Thus, we conjecture that the answers (a.1) and (a.2) stated in the previous subsection are also valid for the BCRB when the budget $n$ is large.

C. Phase Transitions

To answer (q.3), we plot in Fig. 5 the information-theoretic lower bound and the BCRB as functions of the normalized
edge probability of the random ER graph for various values of \( k \) with \( n \) fixed. Here is the conclusion we draw from Fig. 5.

(a.3) The information-theoretic lower bounds derived in Theorems 2 do not demonstrate a phase transition, albeit a decrease is observed with increasing normalized edge probability. Thus, the bounds do not provide much information in terms of graph connectedness. On the other hand, the BCRB derived in Theorem 3 demonstrates a phase transition when the graph is almost connected corresponding to normalized probability 1. Note that the phase transition occurs even when the graph may not be connected due to the inherent regularization present in the Bayesian nature of the problem. In particular, the priors allow for pairs of vertices to have zero counts.

V. EXTENSIONS TO THE BTL MODEL WITH HOME-FIELD ADVANTAGE

It is reasonable to expect that in some applications, such as sport competitions, teams will have a better chance of winning when they play at home (compared to when they play in their opponent’s home ground). The BTL model with home-field advantage [22] takes into account this asymmetry by associating to each item \( i \in [k] \), a skill/rate parameter \( \lambda_i \in \mathbb{R}_+ \) as before, but such that

\[
P_{ij} = \begin{cases} 
Q_{ij} := \frac{\theta_i}{\theta_i + \theta_j}, & \text{if } i \text{ is home}, \\
Q_{ji} := \frac{\lambda_j}{\lambda_i + \theta_j}, & \text{if } j \text{ is home},
\end{cases}
\]

(38)

where a new variable \( \theta \in \mathbb{R}_+ \) is introduced to model the strength of the home-field advantage (\( \theta > 1 \)) or disadvantage (\( \theta < 1 \)). Let \( w_{ij}^h \) denote the number of comparisons in which \( i \) is home and wins, \( ji \) plays when \( i \) is at home, so that \( n_{ij} = n_{ij}^h + n_{ji}^h \).

Note that the matrix \( N^h := (n_{ij}^h) \in \mathbb{N}^{k \times k} \) is not necessarily symmetric. As before, we assume that the total budget matrix \( \mathbf{N} := (n_{ij}) \in \mathbb{N}^{k \times k} \) is fixed a priori. In this new model, the data can be described by \( \mathbf{W}^h := (w_{ij}^h) \in \mathbb{N}^{k \times k} \), and one can write

\[
p(\mathbf{W}^h | \lambda, \theta) = \prod_{(i,j) \in \mathcal{I}[k]} B(w_{ij}^h; n_{ij}^h, Q_{ij}) B(n_{ji}^h - w_{ji}^h; n_{ji}^h, Q_{ij}',)
\]

(39)

by observing that \( \Omega_{ij}^h \sim B(n_{ij}^h; n_{ij}^h, Q_{ij}) \) holds for home-field wins, and \( n_{ji}^h - \Omega_{ji}^h \sim B(n_{ji}^h - w_{ji}^h; n_{ji}^h, Q_{ij}') \) for foreign- or away-field wins. Using the prior in (11), Caron and Doucet introduces the following latent variables [22, Eq. (11)]:

\[
Z_{ij}^h | \lambda_i, \lambda_j, \theta \sim \mathcal{P}(\zeta_{ij}^h | \lambda_i, \lambda_j, \theta) = \mathcal{G}(\zeta_{ij}^h; n_{ij}^h, \theta \lambda_i + \lambda_j),
\]

(40)

for all \( (i, j) \in \mathcal{I}[k] \), and they also showed that [22, eq. (17)]

\[
\Lambda_i | \mathbf{W}^h, \zeta^h, \theta \sim p(\lambda_i | \mathbf{W}^h, \zeta^h, \theta)
\]

\[
= \mathcal{G} \left( \lambda_i; a_i + \sum_{j \in [k] \setminus \{i\}} w_{ij}^h + \sum_{j \in [k] \setminus \{i\}} (n_{ji}^h - w_{ji}^h), b_i + \theta \sum_{j \in [k] \setminus \{i\}} \zeta_{ij}^h \right),
\]

(41)

for \( i \in [k] \), where \( \zeta_{ij}^h = (\zeta_{ij}^h) \in \mathbb{R}^{k \times k} \). As before, we use the symbols \( \Omega^h := (\Omega_{ij}^h) \in \mathbb{N}^{k \times k}, \mathbf{Z}^h := (\mathbf{Z}_{ij}^h) \in \mathbb{R}^{k \times k}, \) and \( \Lambda := (\Lambda_i) \) to denote the corresponding random matrices in the home-field advantage model, e.g., \( \Omega^h \) refers to the data random variable with realizations given by \( \mathbf{W}^h \). Without loss of generality, we allow a prior distribution on the home-field advantage parameter such that \( \theta \sim p_\theta(\theta) \), where \( p_\theta \) is a distribution with support \( (1, \infty) \).

A. INFORMATION-THEORETIC LOWER BOUNDS WITH HOME-FIELD ADVANTAGE

The next theorem provides a family of lower bounds obtained for the new model via Theorem 1. Its proof is provided at the end of this section.

Theorem 4: Let \( d(\lambda, \hat{\lambda}) := \| \lambda - \hat{\lambda} \|^r \) denote the distortion function for an arbitrary norm \( \| \cdot \| \) in \( \mathbb{R}^k \) with \( r \geq 1 \). Let \( \hat{\lambda} := \phi(\mathbf{W}) \) be an estimator of \( \lambda \). The Bayes risk \( R_B \) for estimating the parameter \( \lambda \in \mathbb{R}^k \) based on a sample \( \mathbf{W}^h \) in the Bayesian BTL model with home-field advantage is asymptotically lower bounded by the following expression:

\[
R_B = \inf_{\phi} \mathbb{E}[d(\lambda, \phi(\Omega))]
\]

\[
\geq n_i \frac{k}{r^k} \left( V_k \lambda \frac{1}{r} \right)^{-r/k} e^{-r E_{HA}(N^h, a, b, p_\theta)}
\]

(42)

where \( V_k \) denotes the volume of the unit ball in \( \mathbb{R}^k, \| \cdot \| \), \( n_i \) is defined in (21), and

\[
E_{HA}(N^h, a, b, p_\theta) = \frac{1}{k} \sum_{i \in [k]} \left( -\frac{1}{2} \log (2\pi) + \log b_i \right)
\]

\[
- \psi(a_i) + \frac{1}{2} \log \left( a_i + \sum_{j \in [k] \setminus \{i\}} F_{ij}(n_{ij}^h, n_{ji}^h, a_i, b_i, p_\theta) \right),
\]

(43)
with
\[ F_{ij}(n_{ij}^h, n_{ij}^b, a_i, b_i, p_{\Theta}) = \mathbb{E} \left[ \Theta \Lambda_i \over \Theta \Lambda_i + \Theta \Lambda_j \right] n_{ij}^h + \mathbb{E} \left[ \Lambda_j \over \Lambda_i + \Theta \Lambda_j \right] n_{ij}^b, \quad (44) \]
for any \((i, j) \in \mathcal{I}[k]\).

**Corollary 4:** The lower bound in (43) justifies our basic intuition that one must choose \(n_{ij}^h = n_{ij}^b\) to cancel the effect of any home-field advantage or disadvantage, since
\[ \mathbb{E} \left[ \Lambda_i \over \Lambda_i + \Theta \Lambda_j \right] = \mathbb{E} \left[ \Lambda_j \over \Lambda_j + \Theta \Lambda_i \right] = 1 - \mathbb{E} \left[ \Theta \Lambda_i \over \Theta \Lambda_i + \Theta \Lambda_j \right]. \quad (45) \]

Thus, symmetric matrices \(N^h\) lead to \(E_{HA}(N^h, a, b, p_{\Theta}) = E_{BTL}(N, a, b)\) given by (22).

Suppose that the symmetry condition is not satisfied, i.e., \(n_{ij}^h \neq n_{ij}^b\) holds for some pairs of items \((i, j) \in \mathcal{I}_o[k]\). In this case, we want to analyze how the home-field advantage parameter affects the family of information-theoretic lower bounds. For this purpose, we now discuss a special case of Theorem 4, where we evaluate (44) by symbolic computing software for deterministic \(\Theta = \theta > 1\) and constants \(a_i = a\) and \(b_i = b\), for all \(i \in [k]\). In this case, we get
\[ \mathbb{E} \left[ \theta \Lambda_i \over \theta \Lambda_i + \theta \Lambda_j \right] = f(a, \theta) \]
\[ := a \left( -1 + \frac{1}{\theta} \right)^{-2a} \theta^{-a} B[1 - \theta, 2a, 1 - a], \quad (46) \]
where \(B[z, x, y]\) is the incomplete beta function [33]. Therefore, we see that (44) does not actually depend on the scale parameter \(b\) of the Gamma prior in (11) (and this is true for both random and deterministic \(\Theta\)). Moreover, it can be verified that \(\lim_{\theta \to 1} f(a, \theta) = 1/2\) holds as expected, and \(\lim_{\theta \to \infty} f(a, \theta) = 1\). In particular, for \(a = 2\), (46) reduces to the following simpler form
\[ f(2, \theta) = \frac{\theta(2 + 3\theta - 6\theta^2 + 3\theta^3 + 6\theta \log \theta)}{(-1 + \theta)^4}. \quad (47) \]

It can be verified that function \(f(2, \theta)\) is increasing concave in \(\theta > 1\), for any \(a \in \mathbb{R}_{++}\). Moreover, \(f(10) \approx 0.87\) and \(f(100) \approx 0.98\). Fig. 6 illustrates the impact of the parameter \(\theta > 1\) to the information theoretic lower bounds on the Bayes risk of the BTL model given by (42) for two choices of the matrix \(N^h\) for \(k = 10\) items. In fact, letting \(n_{ij}^o = \alpha n_{ij}\), for \((i, j) \in \mathcal{I}_o[k]\) and \(\alpha \in (0.5, 1)\), (44) equals \((2\alpha - 1)f(2, \theta) + (1 - \alpha)n_{ij}\), for \((i, j) \in \mathcal{I}_o[k]\), and \(\alpha n_{ij} - (2\alpha - 1)f(2, \theta)n_{ij}\), for \((i, j) \in \mathcal{I}_o[k] \setminus \mathcal{I}_o[k]\). The observed behavior in Fig. 6 can be better understood by inspecting the latter relations.

The key ingredients of the proof of Theorem 4 are Theorem 1 and the following proposition.

**Proposition 2:** We have
\[ \frac{1}{k} \left( I(\Lambda; \Omega Z) - h(\Lambda) \right) \leq_{n_i} E_{HA}(N^h, a, b, p_{\Theta}), \quad (48) \]
where \(n_i\) is defined in (21) and \(E_{HA}(N^h, a, b, p_{\Theta})\) in (43).

The proof of Proposition 2 is provided in the supplementary material [25]. We omit the proof of the Theorem 4 since it is proved using similar steps to the proof of Theorem 2.

**B. Hybrid Cramér-Rao Lower Bounds with Home-Field Advantage**

We derive the HCRB for the BTL model with home-field advantage described in (38). The HCRB is obtained using the HIM computed over the random vector \(\Lambda\), and the deterministic parameter \(\theta > 1\), and hence the bound is hybrid. The likelihood is given by [22]
\[ p(W^h, \lambda|\theta) = \prod_{i \in [k]} b_{n_i} \Gamma(a_i) \lambda_i^{a_i} e^{-\lambda_i} \times \prod_{(i, j) \in \mathcal{I}[k]} \left( n_{ij}^h \right)^{\theta \lambda_i} \left( \theta \lambda_i + \lambda_j \right)^{n_{ij}^b} \left( \theta \lambda_i + \lambda_j \right)^{n_{ij}^b} \left( \theta \lambda_i + \lambda_j \right)^{n_{ij}^b}, \quad (49) \]
where we recall that \(w_{ij}^h\) denote the number of comparisons in which \(i\) is at home and beats \(j\), and \(n_{ij}^b\) denote the total number of times \(i\) and \(j\) plays when \(i\) is at home, for all \((i, j) \in \mathcal{I}[k]\). In the following, we state the HCRB.

**Theorem 5:** Define the expectations of \(\frac{\Lambda_i^j \Lambda_j^i}{(\theta \Lambda_i + \Lambda_j)^2}\) and \(\frac{1}{(\theta \Lambda_i + \Lambda_j)^2}\) for \(t_i, t_j \in (-\infty, \infty)\) respectively as
\[ \mu_{\Lambda_i, \Lambda_j}(t_i, t_j, \theta) := \mathbb{E} \left[ \frac{\Lambda_i^j \Lambda_j^i}{\theta \Lambda_i + \Lambda_j} \right]. \quad (50) \]
\[
\nu_{\Lambda_1, \Lambda_2} (t_i, t_j, \theta) := \mathbb{E} \left[ \frac{1}{(\theta \Lambda_1 + \Lambda_2)^2} \right]. 
\]

Given the joint probability distribution in (49), the HCRB on the MSE matrix \(E^{\lambda, \theta} \) of the unknown hybrid vector \(\lambda, \theta\), where the parameterized distribution of the skill parameter \(\lambda\) is given by (11) and home-advantage parameter \(\theta\) is deterministic, is given by \(E^{\lambda, \theta} \geq (I_{\Lambda, \theta})^{-1}\), where

\[
I_{\Lambda, \theta} := \begin{bmatrix}
H^\lambda & H^{\lambda, \theta} \\
(H^{\lambda, \theta})^T & \mathbf{H}^\theta
\end{bmatrix}
\]

and where

\[
[H^\lambda]_{i,i} := \frac{(a_i - 1)b^2(A_i - 2)}{\Gamma(a_i)}
\]

\[
+ \sum_{j \in [k] \setminus \{i\}} n_{ij}^h \theta \nu_{\Lambda_i, \Lambda_j} (-1, 1, \theta)
\]

\[
+ \sum_{(i,j) \in [k]} n_{ij}^h \theta \nu_{\Lambda_i, \Lambda_j} (1, -1, \theta), \forall i \in [k]
\]

\[
[H^\theta]_{i,j} := - \left[ n_{ij}^h \theta \nu_{\Lambda_i, \Lambda_j} (-1, 1, \theta)
\right. \\
\left. + n_{ij}^h \theta \nu_{\Lambda_i, \Lambda_j} (1, -1, \theta) \right], \forall (i,j) \in \mathcal{I}[k],
\]

\[
[H^{\lambda, \theta}]_{i,j} := \sum_{(i,j) \in [k]} \left[ n_{ij}^h \mu_{\Lambda_i, \Lambda_j} (0, 0, \theta)
\right. \\
\left. - \sum_{(i,j) \in [k]} n_{ij}^h \nu_{\Lambda_i, \Lambda_j} (2, 0, \theta),
\right.
\]

\[
- \sum_{(i,j) \in [k]} \left[n_{ij}^h \theta \nu_{\Lambda_i, \Lambda_j} (1, 0, 0) - n_{ij}^h \theta \nu_{\Lambda_i, \Lambda_j} (1, 0, 0) \right]
\]

\[
\forall i \in [k],
\]

where expressions for the quantities \(\mu_{\Lambda_i, \Lambda_j} (t_i, t_j, \theta)\) and \(\nu_{\Lambda_i, \Lambda_j} (t_i, t_j, \theta)\) are provided in Lemmas 5 and 6 in the supplementary material [25].

In Section III-B, we saw that BCRB computation involves obtaining the mean of \(\lambda_{\Lambda_i, \Lambda_j}\) w.r.t. \(\Lambda_i\) and \(\Lambda_j\), which is straightforward as shown in the proof of Theorem 3. However, due to the presence of the home-field advantage parameter \(\theta\), which appears in the above model through an equivalent ratio given by \(\frac{\theta \Lambda_i}{\theta \Lambda_i + \Lambda_j}\), deriving the expressions of the mean of \(\frac{\theta \Lambda_i}{\theta \Lambda_i + \Lambda_j}\) is not straightforward. We derive the mean and generalize it to obtain the expressions for \(\mu_{\Lambda_i, \Lambda_j} (t_i, t_j, \theta)\) and \(\nu_{\Lambda_i, \Lambda_j} (t_i, t_j, \theta)\) in the supplementary material.

VI. CONCLUSIONS

We presented two asymptotic lower bounds on the Bayes risk for learning the skill or rate parameters of the Bayesian BTL model \(\lambda\). From these bounds, we made progress in understanding the effect of the various graph structures (indicating the pairs of players who play against one another) on the Bayes risk of the Bayesian BTL model.

There are three directions for future research. First, we would like to assess the tightness of the derived lower bounds. From Figs. 1 and 2, it appears that the bounds are increasingly tight as the sample size \(n \to \infty\). Showing that this is true analytically would be of tremendous theoretical interest and would confirm that the answers to the three questions are based not only on lower but also upper bounds. Second, we would like to show that (37) is true, which would imply that the BCRB allows us to make the same conclusions on graph structures as the information-theoretic lower bound. Finally, we would like to use the bounds to gain further intuition of how the structure of the comparison graph affects the minimax risk. Some questions of interest include: Does the fully-connected graph outperform a simple cycle (this was left unexplored in answer (a1))? For a fixed number of edges, do planar graphs generally outperform non-planar ones?

APPENDIX A

PROOF OF PROPOSITION 1

Proof: We first note that

\[
I(\lambda; \Omega, Z) = \mathbb{E} \left[ \log \frac{p(\lambda|\Omega, Z)}{p(\lambda)} \right] = \mathbb{E} \left[ \log \frac{p(\lambda|\Omega, Z)}{p(\lambda)} \right].
\]

From the last expression, it is easy to see that we have

\[
I(\lambda; \Omega, Z) - h(\lambda) = \mathbb{E} \left[ \log p(\lambda|\Omega, Z) \right].
\]

On the other hand, by Lemma 3 given in the supplementary material [25], we know that the skill parameters of the Bayesian BTL model satisfies the following conditional density

\[
p(\lambda|\mathbf{w}, \zeta) = \prod_{i \in [k]} \mathcal{G}(\Lambda_i; a_i + w_i, b_i + \zeta_i),
\]

where

\[
w_i := \sum_{j \in [k] \setminus \{i\}} w_{ij}
\]

denote the total number of wins of an item \(i\) and we define

\[
\zeta_i := \sum_{j \in [k] \setminus \{i\}} \zeta_{ij}
\]

for all \(i \in [k]\). The random variables corresponding to these realizations are denoted as \(\Omega_i\) and \(Z_i\), respectively.

Hence, we need to compute

\[
I(\lambda; \Omega, Z) - h(\lambda)
\]

\[
= \mathbb{E} \left[ \log \left( \prod_{i \in [k]} \mathcal{G}(\Lambda_i; a_i + \Omega_i, b_i + Z_i) \right) \right]
\]

\[
= \sum_{i \in [k]} \mathbb{E} \left[ \log \mathcal{G}(\Lambda_i; a_i + \Omega_i, b_i + Z_i) \right]
\]

For that purpose, we first claim that

\[
\lim_{n_i \to \infty} \log \left( 1 + O \left( \mathbb{E} \left[ \frac{1}{a_i + \Omega_i} \right] \right) \right) = 0,
\]

where \(n_i\) is defined in (21). For the proof, see Lemma 4 in the supplementary material [25]. Now, using the identifications \(M \leftarrow \Lambda_i, A \leftarrow a_i + \Omega_i,\) and \(B \leftarrow b_i + Z_i\), we thus get, by
For the final term, we note that
\[ I(\Lambda; \Omega, Z) - h(\Lambda) \leq \sum_{i \in [k]} \left( -\frac{1}{2} \log(2\pi) - \mathbb{E} \log \Lambda_i + \frac{1}{2} \log (a_i + \mathbb{E} [\Omega_i]) + \mathbb{E} [(a_i + \Omega_i) - (b_i + Z_i)\Lambda_i] + \log \left( 1 + O \left( \frac{1}{a_i + \Omega_i} \right) \right) \right), \tag{65} \]
as \( n_i \to \infty \). We are only left to compute the others terms in (65). We start by computing
\[
\mathbb{E} [a_i + \Omega_i] = \mathbb{E} \left[ a_i + \sum_{j \in [k] \setminus \{i\}} \Omega_{ij} \right] \tag{66}
= a_i + \sum_{j \in [k] \setminus \{i\}} \mathbb{E} [\Omega_{ij} | \Lambda_i, \Lambda_j] \tag{67}
= a_i + \sum_{j \in [k] \setminus \{i\}} \mathbb{E} [\frac{n_{ij}}{\Lambda_i + \Lambda_j}] \tag{68}
\]
where (68) follows from \( \Omega_{ij} | \lambda_i, \lambda_j \sim \mathcal{B}(w_{ij}; n_{ij}, P_{ij}) \). Now, we compute
\[
\mathbb{E} [(b_i + Z_i)\Lambda_i] = \mathbb{E} \left[ \left( b_i + \sum_{j \in [k] \setminus \{i\}} Z_{ji} \right) \Lambda_i \right] \tag{69}
= b_i \mathbb{E} [\Lambda_i] + \sum_{j \in [k] \setminus \{i\}} \mathbb{E} [Z_{ji} | \Lambda_i] \tag{70}
= b_i \mathbb{E} [\Lambda_i] + \sum_{j \in [k] \setminus \{i\}} \mathbb{E} [\Lambda_i \mathbb{E} [Z_{ji} | \Lambda_i, \Lambda_j]] \tag{71}
\]
\[
= \frac{b_i a_i}{b_i} + \sum_{j \in [k] \setminus \{i\}} \mathbb{E} \left[ \frac{\Lambda_i n_{ij}}{\Lambda_i + \Lambda_j} \right] \tag{72}
= a_i + \sum_{j \in [k] \setminus \{i\}} \mathbb{E} \left[ \frac{n_{ij} \Lambda_i}{\Lambda_i + \Lambda_j} \right] \tag{73}
\]
where (72) follows from the fact that \( Z_{ji} | \lambda_i, \lambda_j \sim \mathcal{G}(\zeta_{ij}; n_{ij}, \lambda_i + \lambda_j) \) and \( \Lambda_i \sim \mathcal{G}(\lambda_i, a_i, b_i) \). Thus, we conclude from (68) and (73) that the two terms cancel each other: \( \mathbb{E} [(a_i + \Omega_i) - (b_i + Z_i)\Lambda_i] = 0 \). Next, we note that for \( \Lambda_i \sim \mathcal{G}(\lambda_i, a_i, b_i) \), we have \( \mathbb{E} \log \Lambda_i = \psi(a_i) - \log b_i + 1 - \log \Gamma(a_i) \). For the final term, we note that
\[
\frac{1}{2} \log (a_i + \mathbb{E} [\Omega_i]) = \frac{1}{2} \log \left( a_i + \frac{1}{2} \sum_{j \in [k] \setminus \{i\}} n_{ij} \right). \tag{74}
\]
Therefore, we obtain (26). This concludes the proof of the proposition. \( \blacksquare \)

**Proposition 3:** Let \( M, A \) and \( B \) be three non-negative random variables for which we define the random variable \( \mathcal{G}(M; A, B) \), where \( A \) and \( B \) determine respectively the shape and rate parameters of a random Gamma distribution of \( M \). Then, as \( \mathbb{E} [1/A] \to 0 \),
\[
\mathbb{E} \log \mathcal{G}(M; A, B) \leq -\frac{1}{2} \log(2\pi) - \mathbb{E} \log M + \frac{1}{2} \log \mathbb{E} [A] + \mathbb{E} [A - BM] + \log \left( 1 + O \left( \mathbb{E} \left[ \frac{1}{A} \right] \right) \right). \tag{75}
\]
\[
\text{Proof: We start by writing}
\]
\[
\log \mathcal{G}(M; A, B) = \log \left( \frac{BA}{\Gamma(A)} M^{A-1} e^{-BM} \right)
= A \log B - \log \Gamma(A) + (A - 1) \log M - BM. \tag{76}
\]
Now, since the Gamma function can be approximated using Stirling’s formula [35], we obtain
\[
\log \Gamma(x) = \frac{1}{2} \log(2\pi) + x \log x - \frac{1}{2} \log x - x + \log \left( 1 + O \left( \frac{1}{x} \right) \right). \tag{77}
\]
Hence, we obtain the following asymptotic expression for (75):
\[
\log \mathcal{G}(M; A, B) = A \log B + (A - 1) \log M - BM - \left( \frac{1}{2} \log(2\pi) + A \log A - \frac{1}{2} \log A - A + \log \left( 1 + O \left( \frac{1}{A} \right) \right) \right). \tag{78}
\]
Thus, to prove the claim in (75), we need to compute an upper bound on \( \mathbb{E} \log \mathcal{G}(M; A, B) \). First, we show that
\[
\mathbb{E} [A \log (BM) - A \log A] \leq 0. \tag{79}
\]
To prove this claim, we write
\[
\mathbb{E} [A \log (BM) - A \log A] = \mathbb{E} [A \mathbb{E} \log (BM) \mid B] - A \log A \tag{80}
\]
\[
\leq \mathbb{E} [A \log (\mathbb{E} [BM] \mid B)] - A \log A \tag{81}
\]
\[
\leq \mathbb{E} [A \log (B \mathbb{E} [M] \mid AB)] - A \log A \tag{82}
\]
\[
\leq \mathbb{E} [A \log \left( \frac{BA}{B} \right) - A \log A] = 0, \tag{83}
\]
where (a) follows by Jensen’s inequality for concave functions, and (b) follows by the fact that \( \mathbb{E} [M \mid AB] = A/B \) holds for the Gamma distribution [34]. Next, by Jensen’s inequality,
\[
\mathbb{E} \left[ \frac{1}{2} \log A \right] \leq \frac{1}{2} \log \mathbb{E} [A], \tag{84}
\]
and
\[
\mathbb{E} \left[ \log \left( 1 + O \left( \frac{1}{A} \right) \right) \right] \leq \log \left( 1 + O \left( \mathbb{E} \left[ \frac{1}{A} \right] \right) \right). \tag{85}
\]
By upper bounding \( \mathbb{E} \log \mathcal{G}(M; A, B) \) via (79), (85), and (86), we obtain the claim of the lemma. \( \blacksquare \)
The above equation can be re-written as

\[
\log p(W, \lambda) = \sum_{i=1}^{k} \sum_{j=1}^{k} w_{ij} \log \lambda_i
\]

\[
- \sum_{i=1}^{k} \sum_{j=1+1}^{k} n_{ij} \log (\lambda_i + \lambda_j)
\]

\[
+ [a_i \log b - \log \Gamma(a_i) + (a_i - 1) \log \lambda_i - b \lambda_i].
\]  

(87)

Differentiating (87) w.r.t. \( \lambda_i \), we obtain

\[
\frac{\partial \log p(W, \lambda)}{\partial \lambda_i} = \frac{a_i - 1 + \sum_{j=1}^{k} w_{ij}}{\lambda_i}
\]

\[
- \sum_{j \in [k] \setminus \{i\}} \left( \frac{w_{ij}}{\lambda_i + \lambda_j} + \frac{w_{ji}}{\lambda_i + \lambda_j} \right)
\]

\[
= \frac{a_i - 1 + \sum_{j=1}^{k} w_{ij}}{\lambda_i} - \sum_{j \in [k] \setminus \{i\}} \frac{n_{ij}}{\lambda_i + \lambda_j}.
\]  

(88)

for \( i \in [k] \), where we used the fact that \( n_{ij} = w_{ij} + w_{ji} \) holds for all \( (i, j) \in \mathcal{I}[k] \). Differentiating (89) again w.r.t. \( \lambda_i \), we obtain

\[
\frac{\partial^2 \log p(W, \lambda)}{\partial \lambda_i^2} = -\frac{a_i - 1 + \sum_{j=1}^{k} w_{ij}}{\lambda_i^2}
\]

\[
+ \sum_{j \in [k] \setminus \{i\}} \frac{n_{ij}}{(\lambda_i + \lambda_j)^2}.
\]  

(90)

for \( i \in [k] \). Similarly, differentiating (87) w.r.t. \( \lambda_i \) and \( \lambda_j \) we obtain

\[
\frac{\partial^2 \log p(W, \lambda)}{\partial \lambda_i \partial \lambda_j} = \frac{n_{ij}}{(\lambda_i + \lambda_j)^2}.
\]  

(91)

for \( (i, j) \in \mathcal{I}[k] \). In order to obtain the BCRB, we take the expectation of (90) and (91) w.r.t. the joint density function. Since \( \mathbb{E}[\Omega_{ij}|\lambda_i, \lambda_j] = \frac{n_{ij}}{\lambda_i(\lambda_i + \lambda_j)^2} \), we have

\[
[\Gamma^A]_{i,i} = \mathbb{E} \left[ \frac{1 - a_i}{\lambda_i^2} + \sum_{j \in [k] \setminus \{i\}} \frac{n_{ij} \lambda_j}{\Lambda_i(\lambda_i + \lambda_j)^2} \right].
\]  

(92)

Evaluating the above expression we get (28). Furthermore, we compute the off-diagonal terms as

\[
[\Gamma^A]_{i,j} = -n_{ij} T_3(a_i, a_j, b)
\]  

(93)

for \( (i, j) \in \mathcal{I}[k] \).
weight assignments for star graphs with central node \( i^* = 1 \), the edge weight assignment of \( G_S \) minimizes the sum in (98) by the concavity of the logarithm function. Now, suppose that we shift part of the weight of an edge \( n_{ij} > 0 \), for \( j \in [k] \setminus \{1\} \), to create a new edge with weight \( n_{ji} \) such that \( i \in [k] \setminus \{1\} \). Since we have

\[
\frac{\partial S}{\partial n_i} = \frac{2n_i}{a + n_i} - \frac{2n_i}{a + 2n_i} \sum_{i' \in [k] \setminus \{i^*\}} n_{i'} > 0, \quad (99)
\]

for all \( i \in [k] \setminus \{i^*\} \), we conclude that the sum in (98) will be increased by the new configuration. Suppose instead that from the star graph configuration we shift part of the weight from the most heavy edge \( n_{12} > 0 \) to create a new edge with weight \( n_{ji} \) such that \( i \in [k] \setminus \{1\} \) and \( j \in [k] \setminus \{2\} \). We can actually think of this transition as if it was done in two stages: At the first stage, we shift the weight from the edge \( n_{12} \) to the edge \( n_{1i} \), and at the second stage we shift the weight from the edge \( n_{1i} \) to \( n_{ji} \). But we know from the previous arguments that both stages of this transition will necessarily increase the sum in (98). Finally, we note that the types of deviations we considered are exhaustive, since for the graph to be connected, we must have \( n_i > 0 \), for each \( i \in [k] \), i.e., in any deviation we consider at least one element of each row of the adjacency matrix \( N \) of the graph must be non-zero. So, the proof of the claim for the star graph is complete.

Next we proceed with the proof of the claim concerning the chain graph. Note that any tree has exactly \( k - 1 \) non-zero edges \( n_{ji} \), for \( (i, j) \in I_0 \), and \( n_i > 0 \), for all \( i \in [k] \). To prove the extremality of the chain graph among trees, one can easily show that starting from the chain graph configuration, shifting any weight from any of the upper diagonal edges (in the adjacency matrix) into any position on its right (and similarly shifting the weights in the symmetrical positions of the matrix to preserve the overall symmetry) will result an increase in the sum in (98), and hence decrease in the lower bound on the Bayes risk. Similarly, removing any such weight entirely from the elements in the upper diagonal edges will decrease in the lower bound on the Bayes risk. This proves that the chain graph is the optimal configuration minimizing the lower bound on the Bayes risk in (42) among all trees.

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