Geodesic structure of the 2 + 1 black hole

Norman Cruz\textsuperscript{1,2†}, Cristián Martínez\textsuperscript{1‡} and Leda Peña\textsuperscript{1}

\textsuperscript{1} Departamento de Física, Facultad de Ciencias, Universidad de Chile,
Casilla 653, Santiago, Chile.

\textsuperscript{2} Departamento de Física, Facultad de Ciencia, Universidad de Santiago de Chile,
Casilla 307, Santiago, Chile.

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Abstract

Null and timelike geodesics around a 2+1 black hole are determined. Complete geodesics of both types exist in the rotating black-hole background, but not in the spinless case. Upper and lower bounds for the radial size of the orbits are given in all cases and the possibility of passing from one black hole exterior spacetime to another is discussed using the Penrose diagrams. An analysis of particle motions by means of effective potentials and orbit graphs are also included.

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\textsuperscript{†}Electronic address: ncruz@usachvm1.usach.cl

\textsuperscript{‡}Electronic address: jzanelli@abello.seci.uchile.cl
I. INTRODUCTION

The study of timelike and null geodesics is an adequate way to visualize the main features of a spacetime. The paths of freely moving particles and photons in the four-dimensional Schwarzschild metric is the key to understand various important physical phenomena: planetary motions, gravitational lensing, radar delay, etc. The three-dimensional black hole solution was reported in [1] (BTZ solution hereafter). This solution is a spacetime of constant negative curvature, but it differs from anti-de Sitter (adS) space in its global properties; it is obtained from adS space through identifications by means of a discrete subgroup of its isometry group SO(2, 2) [2]. Here we attempt a complete review of the motion of massive and massless test particles in the BTZ black hole.

The first study of geodesics around this black hole was done in [3]. In order to obtain the equations of motion for massive particle, these authors use the Hamilton-Jacobi formalism. However, the simplicity of this lower dimensional model makes the use of that method unnecessary; the symmetries of the BTZ solution enables us to derive directly the equations for timelike and also null geodesics. Exact solutions can be found for the timelike geodesic equations, not only for extreme values of angular momentum of the black hole as in [3], but also for all cases. Moreover, a different interpretation about the oscillatory character of radial timelike solution is given using the Penrose diagram.

This article is organized as follows. The section II is devoted to derive the geodesic equations. These are easy obtained using the constants of motion associated to the two Killing vectors and the normalization condition for the tangent to geodesic curves.

In the section III we first solve the radial equation and bounds for radial motion are determined. From this, we conclude that the rotating black hole is a geodesically complete spacetime. Moreover, the complete geodesics allow the passage from one exterior black hole spacetime to another. An explanation based on Penrose diagram is given.

We also note that while massive particles always fall into the event horizon and no stable orbits are possible [3], massless particles can escape. This important issue makes the
analogy between this three-dimensional model and its four-dimensional counterpart more
tight: thermodynamic phenomena (Hawking radiation) as well as energy extraction from a
rotating black hole (Penrose process) are possible. At last, in this section, the remaining
geodesic equations are integrated and their solutions are tabulated together with the radial
bounds.

It is possible to analyze, in a direct way, the main geodesic features by defining an
effective potential [4]. We present this analysis in the section [4]. In [3], only one of the two
possible solutions for the effective potential was discussed. The study of both solutions is
necessary, for instance, to understand the Penrose process and to determine all the accesible
regions for test particles. An analysis of dragging of inertial frames is also included.

At the last section, we summarize ours results and discuss briefly the charged black hole.

II. GEODESIC EQUATIONS

The action considered in [1] is

\[ I = \frac{1}{2\pi} \int \sqrt{-g} (R + 2\ell^{-2}) d^2x dt + B, \]  

(1)

where \( B \) is a surface term, and the radius of curvature \( \ell = (-\Lambda)^{-1/2} \) provides the length
scale necessary in order to have a horizon (in \( 2 + 1 \) dimensions the mass is dimensionless,
and \( \Lambda \) is the cosmological constant). The Einstein equations are solved by the black hole
field

\[ ds^2 = -N^2 dt^2 + N^{-2} dr^2 + r^2 (N^\phi dt + d\phi)^2, \]

(2)

where the squared lapse \( N^2 \) and the angular shift \( N^\phi \) are given by

\[ N^2 = -M + \frac{r^2}{\ell^2} + \frac{J^2}{4r^2} \quad \text{and} \quad N^\phi = -\frac{J}{2r^2}, \]

(3)

with \(-\infty < t < \infty\), \( 0 < r < \infty \), and \( 0 \leq \phi < 2\pi \).

The constant of integration \( M \) is the conserved charge associated with asymptotic invari-
ance under time displacements (mass) and \( J \) is that associated with rotational invariance
(angular momentum). The lapse function vanishes for two values of \( r \) given by
\[ r_{\pm} = \ell M^{\frac{1}{2}} \left[ 1 \pm \sqrt{1 - \left( \frac{J M}{\ell} \right)^2} \right]^{\frac{1}{2}}, \tag{4} \]

where \( r_+ \) is the black-hole horizon (\( M > 0 \) and \( |J| \leq M \ell \)).

The surface of infinite redshift is given by the relation \( g_{tt}(r = r_{\text{erg}}) = 0 \), where \( r_{\text{erg}} = \ell M^{\frac{1}{2}} \). These three values obey the inequality \( r_- \leq r_+ \leq r_{\text{erg}} \).

The Killing vectors associated to the BTZ metric are two \[ \frac{\partial}{\partial t}, \frac{\partial}{\partial \phi} \] and \( \frac{\partial}{\partial \phi} \). Thus, the constants of motion along the geodesic are

\[ E = -g_{ab}\xi^a u^b = \left[ -M + \frac{r^2}{\ell^2} \right] \left( \frac{dt}{d\lambda} \right) + \frac{1}{2} \left( \frac{d\phi}{d\lambda} \right), \tag{5} \]

where \( \xi^a = (\partial/\partial t)^a \) denotes the static Killing vector, and

\[ L = g_{ab}\Phi^a u^b = r^2 \left( \frac{d\phi}{d\lambda} \right) - \frac{1}{2} \left( \frac{dt}{d\lambda} \right), \tag{6} \]

where \( \Phi^a = (\partial/\partial \phi)^a \) is the rotational Killing vector and \( u^a = dx^a/d\lambda \) is the tangent to a curve parameterized by \( \lambda \), which is normalized by the condition

\[ u^a u_a = -m^2, \tag{7} \]

where \( m \) is 1 for timelike geodesic and 0 for null geodesics.

The constant \( E \) cannot be interpreted as the local energy of the particle at infinity since the black-hole field is not asymptotically flat.

Making the following re-scalings,

\[ \hat{r} = \frac{r}{\ell \sqrt{M}}, \quad \hat{\phi} = \frac{\phi}{\sqrt{M}}, \quad \hat{t} = \frac{\sqrt{M}}{\ell} t, \]

\[ \hat{\lambda} = \frac{\lambda}{\ell}, \quad \hat{E} = \frac{E}{\sqrt{M}}, \quad \hat{L} = \frac{L}{\ell \sqrt{M}}, \quad \hat{J} = \frac{J}{\ell M}, \]

and using (5), (6) and (7), we obtain the geodesic equations (in what follows we will omit the caret on the variables):

\[ r^2 \dot{r}^2 = -m^2 \left( r^4 - r^2 + \frac{J^2}{4} \right) + (E^2 - L^2) r^2 + L^2 - JEL, \tag{8} \]

\[ \dot{\phi} = \frac{(r^2 - 1)L + \frac{1}{2}J E}{(r^2 - r_+^2)(r^2 - r_-^2)}, \tag{9} \]

\[ \dot{\lambda} = \frac{E r^2 - \frac{1}{2} J L}{(r^2 - r_+^2)(r^2 - r_-^2)}, \tag{10} \]
where dot means $\frac{d}{d\lambda}$.

Equations (8) to (10) describe the motion of test particles in 2+1 black-hole background. From these equations we will obtain the orbits for massless and massive particles and the effective radial potential.

### III. SOLUTIONS

The radial equation (8) can be integrated directly. The solution for timelike geodesics ($m = 1$) is

$$r^2(\lambda) = \frac{1}{2} \left[ \alpha + \gamma \sin 2(\lambda - \lambda_0) \right],$$

(11)

and for null geodesics ($m = 0$),

$$r^2(\lambda) = \begin{cases} 
\alpha(\lambda - \lambda_0)^2 - \frac{\beta}{\alpha} & \text{if } \alpha \neq 0 \\
2\sqrt{\beta}(\lambda - \lambda_0) & \text{if } \alpha = 0 \text{ and } \beta \neq 0 \\
\text{constant} & \text{if } \alpha = 0 \text{ and } \beta = 0,
\end{cases}$$

(12)

with $\alpha = E^2 - L^2 + m^2$, $\beta = L^2 - JE L - \frac{m^2 J^2}{4}$ and $\gamma = \sqrt{\alpha^2 + 4\beta}$.

It is clear from (11) that there always exist a finite upper bound $r_{\text{max}}$ for the radial coordinate, which verifies $r_{\text{max}} > r_+$. Thus, massive particles can not escape from the black hole. This inequality can be seen as follows: replacing $r^2$ by $u + r_+^2$, (8) reads

$$\frac{\dot{u}^2}{4} = -u^2 + B_+ u + C_+^2,$$

(13)

where

$$B_\pm = E^2 - L^2 + m^2(1 - 2r_\pm^2) \quad \text{and} \quad C_\pm = r_\pm \left( E - \frac{J L}{2r_\pm^2} \right).$$

It is easy to see that the right hand side of (13) has two real roots for $u$, one positive, $u_1$, and the other negative, $-u_2$. Thus, $\dot{u}$ is real only for $-u_2 \leq u \leq u_1$, this implies that $r_{\text{max}}^2 = r_+^2 + u_1 > r_+^2$ since $u_1$ is positive. We also note from (8) that there is a lower bound ($r_{\text{min}}$) different from zero if
\[ \alpha > 0 \quad \text{and} \quad \beta < 0. \]  

Similarly, replacing \( r^2 \) by \( v + r_0^2 \) in Eq. (8), we conclude that the lower bound verifies \( r_{\text{min}} < r_- \).

If the conditions (14) are not satisfied, timelike geodesics terminate at the singularity \( r = 0 \). Hence, this spacetime is timelike geodesically complete, i.e., all timelike geodesics, with the exception of those which terminate at the singularity, have infinite affine lengths both in the past and future directions. A necessary condition for \( \beta < 0 \) is \( J \neq 0 \), hence all the massive particles hit the singularity in the spinless black hole.

In the same way, we observe from (12) that this spacetime is also null geodesically complete provided that \( J \) doesn’t vanish. Hence the rotating solution is geodesically complete.

Unlike the case \( m = 1 \), for null geodesics \( r_{\text{max}} \) can be infinite. This occurs if \( E^2 \geq L^2 \) (\( \alpha \geq 0 \)). Thus, massless particles can escape from the black hole.

Another remarkable feature of the massless case is the existence of circular orbits of any radius. This occurs if \( \alpha \) and \( \beta \) vanish simultaneously which corresponds to set \( E = JL \) with \( |J| = 1 \) (extreme black hole). The same condition allows a circular timelike geodesic \[3\], but its radius is just the horizon.

In order to solve the geodesic equations (8) and (10), we consider three cases, \( J = 0 \), \( 0 < |J| < 1 \), and \( |J| = 1 \). The solutions for \( \phi \) and \( t \) are shown in Table I and the radial bounds in Table II. The trajectories of massless and massive particles are drawn in the Figs. 4 and 5.

The radial coordinate for timelike geodesic is a periodic function of the affine parameter \( \lambda \). The bounds for this motion are higher and lower than the outer and inner horizons respectively. One can understand this behavior using the Penrose diagram for the rotating black hole \[2\], that we include in the Fig. 2. The diagram is formed by an infinite sequence of regions I \( (r_+ \leq r < \infty) \), II \( (r_- < r < r_+) \) and III \( (0 < r \leq r_-) \). We see that

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1 The nature of this singularity is discussed in Ref. [2]
the timelike geodesic $A$ that begin at a point in region I can cross $r_+$ and $r_-$ and hits the singularity. However, the geodesic $B$ skips the singularity, crossing $r_+$ and $r_-$ infinitely many times. It is essentially a consequence of extending maximally the 2+1 spacetime and of the timelike character of the singularity, as shown in the Fig. [1]. Notice from the expression for the $t$ coordinate given in Table I, any radiation emitted by a particle crossing $r_+$, will be infinitely red-shifted to an external observer at I, and infinitely blue-shifted when crossing $r_-$. 

IV. EFFECTIVE POTENTIAL

The radial equation (8) is a quadratic function of $E$, and can be written as

$$\dot{r}^2 = (E - V^+_\text{eff})(E - V^-\text{eff}),$$

where the roots $V^\pm\text{eff}$ are

$$V^\pm\text{eff}(r) = \frac{J}{2r^2} \pm \frac{1}{r^2} \sqrt{(r^2 - r^2_+)(r^2 - r^2_-)(L^2 + m^2r^2)}.$$ 

Both roots coincide at the event horizon.

It is straightforward, but long, to show that for timelike geodesics $V^+_\text{eff}$ ($V^-\text{eff}$) is a monotonically increasing (decreasing) function in region I; this implies that there are no stable orbits in that region. We also see that it is not possible to define the effective potential in region II. For $m = 0$, $V^\pm\text{eff}$ tends asymptotically to $\pm L$, except for the extreme black hole. In that case, $r^2_+ = r^2_- = 1/2$, and the effective potential has one constant branch allowing a circular orbit, as mentioned in Sec. [11].

A. Non-rotating case

In Fig. [4] the effective potentials for $M > 0$, $M = 0$, $-1$ are shown. The case $M = -1$ corresponds to an adS space; bound orbits are allowed. The vacuum state of the black-hole configuration is obtained for $M = 0$, ($r_+ = 0$). In this case all particles fall to the origin.
Fig. 5 shows the effective potentials for null geodesics. We note that in the cases $M > 0$ massless particles with $E \geq L$ can escape from the black hole. Moreover, these potentials have a similar form to those for massive particles with zero angular momentum in the four-dimensional Schwarzschild black hole.

B. Rotating case

The effective potentials for massive and massless particles are shown in figures 6 and 7. As in the case with $J = 0$, only massless particles can escape from the black hole. When $JL > 0$ only test particles with $E > 0$ have positive local energy measured in a locally nonrotating frame (LNRF). Nevertheless, when $JL < 0$ particles with positive energy in the LNRF can have $E < 0$. These negative energy state exist in the ergoregion and enables a Penrose process as in the four-dimensional case [3]. Nevertheless, as massive particles can not escape from the black hole, energy extraction is only possible with massless particles.

When $J$ is different from zero, the inertial frames are dragged with angular velocity $w$ given by

$$w = \frac{J}{2r^2}.$$ (17)

In the four dimensional Kerr black hole the angular velocity of the frame dragging depend on mass, has the same sign as $J$ and it falls off for large $r$ as $r^{-3}$.

V. SUMMARY AND DISCUSSION

In this article we have solved exactly the timelike and null geodesic equations for the 2+1 black hole. The solutions reflect the singularities of coordinates at the horizons as can be seen from Table 1 and Fig. 3. We show that only the rotating black hole is geodesically complete. The spacelike character of the singularity at $r = 0$, however, makes the spinless black hole geodesically incomplete. Since massive particles cannot escape from black hole, the Penrose process can take place with massless particles only.
The metric of the charged 2+1 black hole \[1\] is obtained by performing the change
\[N^2 \rightarrow N^2 + \frac{1}{2}Q^2 \log \left( \frac{r}{r_0} \right),\]
where \(Q\) is the electric charge of black hole and \(r_0\) is a constant. In this case we can have two, one or no horizons \[3\] and then the effective potential could have local minima and produce a very different behavior for test particles. Therefore, the charged case requires further study in order to reveal its geodesic structure.

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FIGURES

FIG. 1. Penrose diagram for the maximally extended nonextremal rotating black hole \(0 < J^2 < 1\).

FIG. 2. Orbits around the spinless black hole a) \(m = 1\) with \(E = 2\) and \(L = 2\), b) \(m = 0\) with \(E > L\) and \(E < L\).

FIG. 3. Orbits around the rotating black hole a) \(m = 1\) with \(E = 2\), \(L = 2\), \(J = 0.5\), b) \(m = 1\) with \(E = 2\), \(L = 2\), \(J = -0.5\). As it can be seen from Table I, the expression that relates \(\phi\) and \(r\) diverges at the horizons. Thus, the trajectories approaching the horizons will spiral round them an infinite number of times.

FIG. 4. Examples of effective potential for massive particles for different values of \(M\).

FIG. 5. Examples of effective potential for massless particles for different values of \(M\).

FIG. 6. Effective potential for massive particles with \(JL < 0\) and \(JL > 0\). The regions between \(V_{\text{eff}}^+\) and \(V_{\text{eff}}^-\) are forbidden.

FIG. 7. Effective potential for massless particles with \(JL < 0\) and \(JL > 0\). The regions between \(V_{\text{eff}}^+\) and \(V_{\text{eff}}^-\) are forbidden.
TABLE I. Timelike ($m = 1$) and null ($m = 0$) geodesics for different values of $J$. We are defined
\[
f(x; B; C) = \log \left| \frac{x}{2C\sqrt{-m^2x^2 + Bx + C^2}} \right|^{\frac{1}{2}}
\]
a and $R(x; B; C) = \sqrt{-m^2x^2 + Bx + C^2}$.

| case $J$ | orbit equation |
|-----------|----------------|
| $J = 0$   | \[
\begin{align*}
\phi &= \pm \frac{1}{2} f(r^2; E^2 - L^2 + m^2; L) + \phi_0 \\
t &= \pm \frac{1}{2} f(r^2 - 1; E^2 - L^2 - m^2; E) + t_0
\end{align*}
\] |
| $0 < J^2 < 1$ $^b$ | \[
\begin{align*}
\phi &= \pm \frac{1}{4\sqrt{1 - J^2}} J \left[ \frac{1}{r_+} f(r^2 - r^2_+; B_+; C_+) - \frac{1}{r_-} f(r^2 - r^2_-; B_-; C_-) \right] + \phi_0 \quad ^c \\
t &= \pm \frac{1}{\sqrt{1 - J^2}} \left[ r_+ f(r^2 - r^2_+; B_+; C_+) - r_- f(r^2 - r^2_-; B_-; C_-) \right] + t_0
\end{align*}
\] |
| $J^2 = 1$  | \[
\begin{align*}
\phi &= \pm \sqrt{\frac{1}{8}} J \left[ f(x; B; C) + \frac{R(x; B; C)}{Cx} \right] + \phi_0 \quad ^d \\
t &= \pm \sqrt{\frac{1}{8}} \left[ f(x; B; C) - \frac{R(x; B; C)}{Cx} \right] + t_0
\end{align*}
\] |

$^a$ Notice that $f(x; B; 0)$ is $x$ independent.

$^b$ The case $J = 0$ is obtained directly using the prescription $\lim_{J \to 0} \frac{J}{r_-} = \pm 2$.

$^c$ $B_\pm = E^2 - L^2 + m^2(1 - 2r^2_\pm)$ and $C_\pm = r_\pm \left( E - \frac{JL}{2r^2_\pm} \right)$.

$^d$ Here $x = r^2 - \frac{1}{2}$, $B = E^2 - L^2$ and $C = \sqrt{\frac{1}{2}}(E - L \text{sgn} J)$. 

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TABLE II. Radial bounds for timelike ($m = 1$) and null ($m = 0$) geodesics for different values of $J$.

| case | radial bounds |
|------|----------------|
| $m = 1$ | $0 < r^2 \leq \frac{1}{2} \left[ \sqrt{((E + L)^2 + 1)((E - L)^2 + 1)} + E^2 - L^2 + 1 \right]$ |
| $J = 0$ | $m = 0$ | $0 < r^2 \leq \begin{cases} \infty & \text{if } E^2 \geq L^2 \\ \frac{L^2}{L^2 - E^2} & \text{if } E^2 < L^2 \end{cases}$ |
| $m = 1$ | $\frac{1}{2} [\alpha + \gamma]^a \geq r^2 \geq \begin{cases} \frac{1}{2} [\alpha - \gamma] & \text{if } \alpha > 0 \text{ and } \beta < 0 \\ 0 & \text{otherwise} \end{cases}$ |
| $0 < J^2 \leq 1$ | $m = 0$ | $\frac{JEL - L^2}{E^2 - L^2} \leq r^2 < \infty$ if $E^2 \geq L^2$ and $L^2 \geq JEL$ |
| | | $0 < r^2 \leq \frac{L^2 - JEL}{L^2 - E^2}$ if $E^2 < L^2$ and $L^2 > JEL$ |

$^a \alpha = E^2 - L^2 + 1, \beta = L^2 - JEL - J^2/4, \gamma = \sqrt{((E + L)^2 + 1 + J)((E - L)^2 + 1 - J)}.$
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