Exact renormalization group equation and decoupling in quantum field theory

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Abstract

New proves of decoupling of massive fields in several quantum field theories are derived in the effective Lagrangian approach based on Wilson renormalization group. In the most interesting case of gauge theories with spontaneous symmetry breaking, the approach, combined with the quantum action principle, leads to a rather simple proof to all orders.

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1 Introduction

Most of the conceptual understanding of the renormalization of quantum field theories via the Wilson renormalization group can be directly retrieved from the works of the same Wilson [1]. Having obtained in dependence of a cut-off $\Lambda$ a flow of physically equivalent lagrangians (i.e. with the same Green functions), he realizes that the renormalizability of the theory relies on the existence of an infrared attracting surface, parametrized by the relevant coupling (i.e. the renormalizable interaction, in the language of quantum field theory). To implement this approach Wilson proposed to trade the functional integration over the high momentum modes for a functional differential equation governing the effective Lagrangian $L$. This equation, which translates the indeipendence of the partition function on the floating cut-off $\Lambda$, gives rise to an infinite system of first order differential equations in $\Lambda$ for the coefficients (vertices) of the expansion of $L$ in power series of the fields of the theory.

In spite of the appeal of this more intuitive approach no progress was made in this direction. The real investigation of the structure of the field theory has been pursued along the old and complex diagrammatic way; the proof of the perturbative renormalizability of the theory and of the most important theorems in quantum field theory (the Wilson operator expansion, the action principle and so on) is worked out in the BHPZ framework, for example.

It is only few years ago that Polchinski [2] has shown how to exploit such a differential formulation. A recursive, perturbative method of solution, within an inductive approach, allowed him to derive appropriate bounds on the effective vertices of the 4-D Euclidean scalar theory and then to prove the existence of an infrared three dimensional critical surface, i.e. renormalizability. Remarkably, this method avoids completely the hard problems of the standard approach, above all that of the overlapping divergences. This has prompted a renewed interest in the Wilson- Polchinski approach: extensions to QED, to SU(2) Yang-Mills theory [3] [4] and a thorough reformulation of the whole approach [5]. The technical simplicity of the method makes it worth to explore applications to general questions as well as to specific issues of QFT and to come up to a more intuitive and deep understanding ([6] and references therein).

The purpose of this article is to discuss the problem of the decoupling of a field of very large mass in this more general contest. The method seems, indeed, particularly suitable for the discussion of such a problem since it is mainly a low energy affair. All the results we present exist already in the literature, but their proof at all order in perturbation theory is only given in a heuristic way or simply sketched, making appeal to thecumbersome BHP for the more rigorous demonstrations. We want to show how to derive rigorously all these results in the Polchinski scheme, without facing graphical analysis or other standard complications.

The well understood state of the art for the decoupling problem is the fol-
lowing: if we study a physical problem in which one of the particle has a very large mass \( M \) we say that it decouples if at low energy its effect is suppressed by powers of \( M \). In the analysis of the physics of the light particles, it enters through an internal heavy line in the Feynman graphs: decoupling is expected if these diagrams are suppressed; more precisely we require that their effects, if not \( O(1/M) \), can be absorbed into finite renormalization of masses and coupling constants. For theories without spontaneous symmetry breaking, the decoupling of heavy particles was demonstrated by Appelquist and Carazzone [7]. In theories with spontaneous symmetry breaking, however, we have decoupling only when we increase a mass by increasing a dimensional parameter (for example a vacuum expectation value). We will discuss all these topics more diffusely in the next sections.

This paper is organized as follows: in section 2 we sketch the Polchinski renormalization scheme, giving a self-contained proof of renormalizability of a scalar field. In section 3,4 we recall the general facts about decoupling and reobtain the Appelquist-Carazzone theorem; the part of the analysis related to dimensional arguments is very simple, for the gauge problem we choose to work in the BRS approach, invoking the powerful action principle. We give also a brief discussion of the general structure of the first order correction to decoupling. In section 5 we discuss the physical more interesting case of decoupling in theory with spontaneous symmetry breaking, and we present a largely model independent analysis.

## 2 The Polchinski scheme

In this section we want to discuss the exact renormalization group equations, the renormalization scheme based on them, and the perturbative proof of the ultraviolet finiteness of the euclidean scalar field. [2]

In order to compute any Green function of a field theory one needs a regularization procedure of the ultraviolet divergences. Let’s consider a scalar field in four euclidean dimensions, with kinetic part

\[
\frac{1}{2} \int \frac{d^4p}{(2\pi)^4} \phi(p)(p^2 + m^2)\phi(-p)K^{-1}(p^2/\Lambda^2)
\]

(1)

where \( K(p^2/\Lambda^2) \) is a \( C^\infty \) function which vanishes rapidly at infinity and assumes the value 1 for \( p^2 < \Lambda^2 \). We have regularized the theory by eliminating from the propagator the modes greater than a certain cut-off.

From a complementary point of view, we may think of \( \Lambda \) not merely as a regulator to avoid divergences but also as a physical scale at which to compute physical quantities. In the Wilson approach to the renormalization group we start from an initial lagrangian at the scale \( \Lambda_0 \) and define an effective lagrangian obtained by integrating out frequencies between \( \Lambda_0 \) and \( \Lambda \). We can
associate with a given interaction lagrangian \( L(\phi, \Lambda) \) the partition function

\[
Z(J, \Lambda) = \int [d\phi] \exp \left[ - \frac{1}{2} \int \frac{dp}{(2\pi)^4} \phi(p) K^{-1}(p^2/\Lambda^2)(p^2 + m^2)\phi(-p) + L(\phi, \Lambda) + \int \frac{dp}{(2\pi)^4} J(p) \phi(-p) \right]
\] (2)

where we integrate over the remaining (low energies) modes; it’s obvious that \( Z(\Lambda) \) contains all the physical informations of the theory and is independent of the scale \( \Lambda \).

More generally, without reference to the Wilson point of view, we suppose to have an interaction lagrangian at a certain scale \( \Lambda \) and the associated partition function. We want to probe the physics under a certain physical scale \( \Lambda_R \), so we take the source of the following form: \( J(p) = 0 \) for \( p^2 > \Lambda_R^2 \) (for an alternative point of view in which \( \Lambda_R \) is zero, see \[4\]).

We obtain a flow in the space of the theories by imposing that the partition function does not change when we vary the cut-off. It is easy to verify \[2\] that, if the interaction lagrangian satisfies the following evolution equation,

\[
\Lambda \partial_\Lambda L(\phi, \Lambda) = - \frac{1}{2} \int dp(2\pi)^4 \Lambda \frac{\partial_\Lambda K(p^2/\Lambda^2)}{p^2 + m^2} \left\{ \frac{\delta^2 L}{\delta \phi(p}\delta \phi(-p)} + \frac{\delta L}{\delta \phi(p)} \frac{\delta L}{\delta \phi(-p)} \right\}
\] (3)

it follows that \( \Lambda \partial_\Lambda Z(J, \Lambda) = 0 \). In this way we obtain a trajectroy in the space of lagrangians which have the same Green functions for momenta below \( \Lambda_R \), and hence have the same physical content at low energy. As shown in the paper of Polchinski, this flow is strongly attracted by an infrared three-dimensional surface, parametrized by the coefficients of the relevant (i.e. power counting renormalizable) interaction. If we expand the lagrangian in the following manner (assuming for the theory the symmetry \( \phi \rightarrow -\phi \), preserved by the flow)

\[
L(\phi, \Lambda) = \sum_{m=1}^{\infty} \frac{1}{(2m)!} \int \frac{dp_1...dp_{2m}}{(2\pi)^{8m-4}} L_{2m}(p_1, ..., p_{2m}; \Lambda) \delta^4(p_1+...+p_{2m})\phi(p_1)\cdots\phi(p_{2m})
\] (4)

and define also the dimensionless vertices of the theory

\[
L_{2m}(p_1, ..., p_{2m}; \Lambda) = \Lambda^{2m-4}
\] (5)

we obtain the flow equation in the following form

\[
(\Lambda \partial_\Lambda + 4 - 2m)A_{2m}(p_1, ..., p_{2m}; \Lambda) =
- \frac{1}{2} \sum_{l=1}^{m} Q(P, m, \Lambda) A_{2l}(-P, p_1, ..., p_{2l-1}; \Lambda) \times
A_{2m-2l+2}(P, p_{2l}, ..., p_{2m}; \Lambda)|_{(p=p_1+...+p_{2l-1})} + Perm.
- \frac{1}{2} \int \frac{dp}{(2\pi \Lambda)^4} Q(p, m, \Lambda) A_{2m+2}(p, -p, p_1, ..., p_{2m}; \Lambda)
\] (6)
where
\[ Q(p, \Lambda, m^2) = \frac{1}{(p^2 + m^2)} \Lambda^3 \frac{\partial}{\partial \Lambda} K(p^2/\Lambda^2) \] (7)
and where we sum over all the possible combinations of 2l-1 impulses \((p_1, ..., p_{2l-1})\) out of \((p_1, ..., p_{2m})\).

It might seem somewhat irrelevant to study this evolution equation, for we already know the solution in terms of an effective lagrangian in which we have integrated out the high frequencies down to \(\Lambda\); this corresponds to compute a first functional integral over the high modes. More precisely \[3\] \[4\] \[5\], if we start at the scale \(\Lambda_0\) from the lagrangian \(S(\Lambda_0)\), it is easy to show that the evolved lagrangian can be written as the functional generator of the connected and amputated Feynman diagrams corresponding to the vertices of \(S(\Lambda_0)\) and the following propagator
\[ \Delta(\Lambda, p) = \frac{K(p^2/\Lambda_0^2) - K(p^2/\Lambda^2)}{p^2 + m^2} \] (8)
So the vertices \(L_{2m}\) are exactly the Feynman graphs of the theory described. However, this diagrammatic representation is noteworthy but technically cumbersome. Arguments which refers to it necessarily suffer from all the complications of the graphical approach, while the direct use of the equation overcomes all these problems. We will refer to this explicit solution when the intuition will need it and when the understanding of certain aspects will be enhanced from this point of view.

The use of the equation (8) in the study of the non perturbative aspects of quantum field theory is still unclear, while the perturbative solution has been well studied \[2\]. We will outline, in the following, its main features with emphasis on the technical aspects which are needed for our problem.

Up to now, we have the equations for a flow of lagrangians, parametrized by the cut-off \(\Lambda\), which give the same answer for the Green functions; the flow starts from the initial lagrangian at \(\Lambda_0\) and ends at a certain physical scale \(\Lambda_R\). The simultaneously presence of three different cut-off might confuse the reader: as it will be soon clear, \(\Lambda_0\) must be considered the regulating cut-off to be removed, \(\Lambda_R\) the energy scale at which we impose the renormalization conditions, and \(\Lambda\) is only an interpolating parameter, an independent variable in a differential equation.

We define for the general lagrangian (4) the "relevant" parameters:
\[ \rho_1(\Lambda) = -L_2(0, 0; \Lambda); \quad \rho_2(\Lambda) = -\frac{1}{8} \frac{\delta^2}{\delta^2 p} L_2(p, -p; \Lambda)|_{p=0}; \quad \rho_3(\Lambda) = -L_4(0, 0, 0, 0; \Lambda) \] (9)
they are exactly the terms of positive or null mass dimension in the lagrangian (i.e. the coefficients of \(\phi^2, (\partial \phi)^2, \phi^4\)); note that we don’t distinguish between marginal and relevant terms in Wilson sense.

We want now to construct a flow of lagrangians which converges to the renormalized \(\lambda \phi^4\) theory. We impose that, at the initial scale \(\Lambda_0\), the la-
grangian consists only of relevant terms, so we have a bare lagrangian of the form:

\[ \int d^4x \left( \frac{1}{2} (\partial \phi)^2 + \frac{m^2}{2} \phi^2 + \frac{\rho_1(\Lambda_0)}{2} \phi^2 + \frac{\rho_2(\Lambda_0)}{2} (\partial \phi)^2 + \frac{\rho_3(\Lambda_0)}{4!} \phi^4 \right) \]  

(10)

We want to stress the interpretation of this expression as a bare lagrangian regularized with a cut-off \( \Lambda_0 \); so we have

\[ \rho_1(\Lambda_0) = \delta m^2, \quad \rho_2(\Lambda_0) = Z - 1, \quad \rho_3(\Lambda_0) = \lambda_0 Z^2 \]  

(11)

We impose now as a second condition (at all effects a renormalization condition) that at low scale \( \Lambda_R \) the relevant parameters assume the values:

\[ \rho_1(\Lambda_R; \Lambda_0, \rho_0) = \rho_2(\Lambda_R; \Lambda_0, \rho_0) = 0, \quad \rho_3(\Lambda_R; \Lambda_0, \rho_0) = \lambda_R \]  

(12)

We seek for a flow which satisfies both the conditions in perturbation theory, i.e. when we consider the vertex \( A_{2m} \) expanded in power of the renormalized coupling \( \lambda_R \); whereas a non perturbative solution is not guaranteed, a perturbative solution always exists. The geometric spirit of the renormalizability theorem consists in what follows: when we increase the cut-off \( \Lambda_0 \) imposing the previous conditions, the \( \Lambda_R \) lagrangian is forced to converge to the critical surface, with an error of the order \( (\Lambda_R/\Lambda_0)^2 \). The complete proof of this statement can be found in [2] [3]. Since the theory at the point \( (0,0,\Lambda_R) \) on the critical surface is finite (like every theory with a cut-off) and since the lagrangians on the same trajectory have the same low energy Green functions, we conclude that we can take the limit \( \Lambda_0 \to \infty \) obtaining a finite theory.

This method can be used to prove the UV-finiteness of all the power counting renormalizable quantum field theories; we will sketch the proof in the simple case of \( \lambda \phi^4 \), but, since the general argument is essentially based on dimensional considerations, extensions to more complicated theories is straightforward but for chiral theories. The general analysis must be done perturbatively by expanding all the vertices in formal power series of \( \hbar \); the only difference in the proof is that the induction must be done increasing the number of external legs in the vertices and paying attention to the correct \( \hbar \) power of the parameters. We want only to note that the equation \( (3) \) involves a not well-defined functional integral, so we have to introduce an infrared cut-off which reduce the problem to a finite numbers of Fourier modes. The problem is easily by-passed observing that the equation for the vertices is perturbatively well-defined in every case. The thermodynamic limit defines a consistent quantum field theory. For a more precise discussion of all these points we refer to reference [3] and we will pursue a more qualitative discussion following Polchinski’s presentation.

We want now to give a review of the proof of the finiteness of the vertices \( A_{2m} \) in the limit \( \Lambda_0 \to \infty \).
We will simplify the argument by taking a function $K(p^2/\Lambda^2)$ which vanishes for $p \geq 2\Lambda$. We introduce the norm
\[
\|f(p_1, \ldots, p_m)\| = \max_{p_i^2 \leq 4\Lambda^2} |f(p_1, \ldots, p_m; \Lambda)|
\] (13)
it is straightforward, then, to verify that
\[
\int \frac{dp}{(2\pi)^2} |Q(p, \Lambda, m)| < C\Lambda^4
\]
\[
\left\| \frac{\partial^n}{\partial p^m} Q(p, \Lambda, m) \right\| < D_n \frac{1}{\Lambda^4}
\] (14)
with $C$ and $D_n$ constants independent of $m$.

In the perturbative theory we have the great simplification that $A^{(0)}_{2m} = 0$ and, at order $r$ in $\lambda R$, $A^{(r)}_{2m} = 0$ for $m > r + 1$. This is easily understood if we observe that at order $r$ a diagram has at most $2r+2$ external legs. This fact suggests an induction approach to the perturbative solution to the equations (13): $A^{(s)}_{2m}$ can be computed when we know the vertices at the lower perturbative orders and those of the same order with more external legs; since they are in a finite number, for the previous observation, this double inductive procedure will lead to the complete solution of the equation. In the following of this paper all the proofs, also without explicit declaration, will be by induction.

**Lemma 1** At order $r$ in $\lambda R$
\[
\| \partial^p A^{(r)}_{2m}(p_1, \ldots, p_{2m}; \Lambda) \| \leq \frac{1}{\Lambda^p} P(\ln \frac{\Lambda}{\Lambda R})
\] (15)
with $P$ a non negative coefficients polinomial and the derivation with respect with the momenta is of order $p$. The precise form of the derivative operators is not necessary for the argument.

Proof: notice, first of all,
\[
\int dx \frac{P^{(m)}(\ln x)}{x^q} = -\frac{P^{(m)}(\ln x)}{x^q}, \quad q > 0
\] (16)
(with $P, P_1 \ldots$ positive coefficient polynomials of order $m$), hence:
\[
\int_0^\Lambda \frac{dN}{N^q} P^{(m)} \left( \frac{\Lambda}{N^q} \right) \left( \frac{\Lambda^q}{\Lambda R} \right) \leq \bar{P}^{(m)} \left( \ln \frac{\Lambda}{\Lambda R} \right), \quad q > 0
\] (17)
We assume that the equation (13) is true for $A^{(s)}_{2n}$ with $n > m$ up to the perturbative order $s-1$ and proceed downward in $m$. The first step of the double induction is satisfied because $A^{(0)}_{2m} = 0$ and $A^{(s)}_{2m} = 0$ for $m > s + 1$. By considering the norm of the perturbative version of equation (13) we have
\[
\| (\Lambda \partial_\Lambda + 4 - 2m) A^{(s)}_{2m}(\Lambda) \| \leq \sum_{t=1}^{s-1} \sum_{t=1}^{s-t} \| A^{(t)}_{2l} \| \| A^{(s-t)}_{2m-2l+2} \| + \| A^{(s)}_{2m-2} \|
\] (18)
where we have used equation \( \text{(14)} \) and the fact that the function \( Q \) forces the momenta in the range \([\Lambda, 2\Lambda]\). In the following we shall always ignore positive coefficients in these bounds. The same equation for derived vertices can be deduced by using again eq. \( \text{(14)} \), where it’s needed. We stress that the only effect of a momentum derivative is a factor \( 1/\Lambda \). By the induction hypothesis

\[
\|(\Lambda \partial_{\Lambda} + 4 - 2m)\partial^p A^{(s)}_{2m}(\Lambda)\| \leq \frac{1}{\Lambda^p} P(ln\Lambda/\Lambda_R) \tag{19}
\]

For \( m \geq 3 \), \( A^{(s)}_{2m}(\Lambda_0) = 0 \). Integrating the previous formula between \( \Lambda \) and \( \Lambda_0 \), we obtain

\[
\|\partial^p A^{(s)}_{2m}(\Lambda)\| \leq \frac{1}{\Lambda^p} \int_{\Lambda_0}^{\Lambda} \frac{dN}{N} F \left( \ln \frac{\Lambda}{\Lambda_R} \right) \left( \frac{\Lambda}{\Lambda_R} \right)^{p+2m-4} \leq \frac{1}{\Lambda^p} \bar{P} \left( \ln \frac{\Lambda}{\Lambda_R} \right) \tag{20}
\]

where we can use eq. \( \text{(17)} \) because \( p + 2m - 4 > 0 \). For the same reason eq. \( \text{(13)} \) is valid for \( m = 2, p \geq 1 \). The case \( p = 0 \)

\[
|\Lambda \partial_{\Lambda} A^{(s)}_4(0, 0, 0, 0; \Lambda)| \leq P(ln\Lambda/\Lambda_R) \tag{21}
\]

must be integrated from \( \Lambda_R \) because we don’t know \( A_4(0; \Lambda_0) \), which doesn’t vanish,

\[
|A^{(s)}_4(0, \Lambda)| \leq |A^{(s)}_4(0; \Lambda_R)| + |A^{(s)}_4(0; \Lambda) - A^{(s)}_4(0; \Lambda_R)| = \delta^{s1}
\]

\[
+ \int_{\Lambda_R}^{\Lambda} \frac{dN}{\Lambda_R} F \left( \ln \frac{\Lambda}{\Lambda_R} \right) \leq \delta^{s1} + P \left( \ln \frac{\Lambda}{\Lambda_R} \right) \int_{\Lambda_R}^{\Lambda} \frac{dN}{\Lambda_R} = \bar{P} \left( \ln \frac{\Lambda}{\Lambda_R} \right) \tag{22}
\]

Since \( A^{(s)}_4(p_1, \ldots, p_4; \Lambda) \) can be reconstructed via Taylor theorem from \( A^{(s)}_4(0; \Lambda) \) and \( \partial^2 A^{(s)}_4(p - 1, \ldots, p - 4; \Lambda) \), \( \text{(15)} \) is proved for \( A^{(s)}_4 \).

It is straightforward to complete the proof in the case \( m = 1 \) following the prescriptions to integrate downward the irrelevant terms \( (m = 1, p > 2) \) from \( \Lambda_0 \) where we know that the vertex \( A_2 \) is zero, and upward the relevant terms \( (p=0,2 \text{ at zero momentum}) \) from \( \Lambda_R \) where we know the corresponding values (the renormalization conditions). The result for the whole vertex is obtained via a Taylor expansion based on these informations.

We refer, once more, to the original paper of Polchinski for the detailed discussion of the approaching to the critical surface; here we want only to indicate how to obtain from these bounds the proof of \( \Lambda_0 \) (i.e. ultraviolet) finiteness of our theory.

The very same bounds seem to indicate that at a finite scale the vertices are \( \Lambda_0 \) finite, because the coefficients in the polynomials are positive numbers independent of any dimensional parameter of the theory. More precisely, the explicit solution of the flow equation can be obtained by induction from the following formula for \( m \geq 3 \),

\[
L^{(s)}_{2m}(\Lambda) = \frac{1}{2} \int_{\Lambda_0}^{\Lambda} dN' \left( \sum_{l=1}^{m-1} \sum_{t=1}^{s-l} \frac{dK/d\Lambda}{P^2 + m^2 l^2} L^{(l)}_{2l}(\Lambda') L^{(s-l)}_{2m-2l+2}(\Lambda') \right)_{P=p_1+\ldots+p_{2l-1}} + \text{Perm.} \cdot \frac{1}{2} \int_{\Lambda_0}^{\Lambda} dN' \int \frac{dp}{(2\pi)^4} \frac{dK/d\Lambda}{p^2 + m^2} L^{(s)}_{2m+2}(\Lambda') \tag{23}
\]
We suppose, as induction hypothesis, that for $\Lambda_0 \to \infty$ $L_2^{(t)}$, $t < s$ and $L_2^{(s)}$, $p > m$ exist. It’s straightforward to bound uniformly in $\Lambda_0$ the previous integrands with convergent expressions, deducing the $\Lambda_0$-finitness of $L_2^{(s)}$.

With $\Lambda_0$ finite we can obtain the previous formula obtaining the finiteness of $\partial^p L_2 m$ for $m = 2, p \geq 1, m = 1, p \geq 2$. The result in the relevant cases is again easier since the integration is over the finite range $[\Lambda R, \Lambda]$. The use of Taylor theorem completes the proof.

3 Decoupling for theory without spontaneous symmetry breaking

If we work at a scale of energy such that some particles of our theory cannot be produced, are we able to detect experimentally their existence? In general, for renormalizable theories without spontaneous symmetry breaking a decoupling theorem can be proved: when the heavy mass $M$ is much larger than all the light masses and scales of observation, we can compute all the scattering amplitudes of light particles from an effective light theory obtained with a (light) parameter redefinition. Since the physical quantities have to be fixed from the experiments, such redefinition is undetectable. More precisely, if $M$ is much greater than all the light masses, the external momenta and the renormalization scale, we have that the total light 1PI functions can be computed from an effective lagrangian to $O(1/M^\alpha)$ precision, i.e.

$$\Gamma_n(p_i, g, m, M) = Z^{-n/2}\Gamma_n(p_i, g^*, m^*) + O(1/M^\alpha)$$

(24)

where $g^*, m^*$ represent the redefined set of light couplings and masses. The explicit form of decoupling is very sensitive to the kind of renormalization we use. Naively, we expect that every graph with an internal heavy line is suppressed by inverse powers of $M$: this is not true, of course, due to the existence of renormalization and counterterms which can depend in an unexpected way on $M$. What is true is that all these effects can be reabsorbed in coupling constant, mass and wave function redefinitions [8]. The case in which these graphs are in fact driven to zero and such renormalization of the parameters is not needed is called manifest decoupling. The typical scheme in which the decoupling is manifest is the BHPZ [8][9]; for example if a graph is already finite with ultraviolet degree $-\delta$ ($\delta > 0$) we expect that it behaves as $1/M^\delta$. On the other hand, if the graph diverges, the effect of the renormalization is to subtract the appropriate order of the Taylor expansion, leading to the same expression derivated as many times as to leave it with a negative UV degree, to which we apply the previous reasoning.

Recalling the representation of vertices as Green functions given in section 2 and the form of the renormalization conditions, we recognize many analogies of this scheme with the BHPZ scheme, when we consider our vertices as the amplitudes which must be renormalized in the latter one. An explicit
computation of the lower orders vertices by the flow equations confirms this point of view: it’s very simple, for example, to recognize the expression of the two or four point functions in terms of Feynman graphs renormalized by subtraction of the Taylor series. We want to show now in a simple model that in the Polchinski scheme decoupling exists, as expected, and it is manifest.

We work in a theory of two scalar fields interacting with a quartic potential,

\[ L = \frac{1}{2}((\partial \phi)^2 + m^2 \phi^2) + (1/2)((\partial \chi)^2 + M^2 \chi^2) + \lambda \phi^4 + gR \phi^2 \chi^2 + kR \phi^2 \chi^2 \]  

(25)

Deriving the obvious generalization of eq. (6) with respect to \( M \) and denoting

\[ W_{2m,2n} = M \frac{\partial A_{2m,2n}}{\partial M} \]  

(26)

where the index \( m \) refers to light and \( n \) to heavy fields, we obtain

\[
\left( \Lambda \frac{\partial}{\partial \Lambda} + 4 - 2m - 2n \right) W_{2m,2n} = - \sum_{l=1}^{m} \sum_{p=0}^{n} \frac{\Lambda^3 dK/d\Lambda}{P^2 + M^2} A_{2l,2p} W_{2m-2l+2,2n-2p} \\
- \frac{1}{2} \int \frac{dp}{(2\pi)^4 \Lambda^2 p^2 + m^2} W_{2m+2,2n} \\
- \sum_{l=0}^{m} \sum_{p=1}^{n} \frac{\Lambda^3 dK/d\Lambda}{P^2 + M^2} A_{2l,2p} W_{2m-2l,2n-2p+2} - \frac{1}{2} \int \frac{dp}{(2\pi)^4 \Lambda^2 p^2 + M^2} W_{2m,2n+2} \\
+ \sum_{l=0}^{m} \sum_{p=1}^{n} \frac{\Lambda^3 dK/d\Lambda}{(P^2 + M^2)^2} M^2 A_{2l+2,2p} A_{2m-2l+2,2n-2p+2} \\
+ \int \frac{dq}{(2\pi)^4 \Lambda^2 (q^2 + M^2)^2} M^2 A_{2m+2n+2}
\]  

(27)

We want to show that

\[ \| \partial^p W_{2m,2n}^{(r)}(\Lambda) \| \leq \frac{1}{\Lambda^p} \left( \frac{\Lambda}{M} \right)^{2-\epsilon} P \left( \ln \frac{\Lambda}{\Lambda_R} \right), \quad \text{for } \epsilon \in (0, 1] \]  

(28)

where the coefficients of the polynomials can depend on \( \epsilon \) and one cannot take the limit \( \epsilon \to 0 \).

We work again by induction, assuming eq. (28) true at order \( t < s \) and at order \( s \) for a total number of legs greater than \( 2(m + n) \). Obviously, \( W_{2m,2n}^{(0)} = 0 \) and \( W_{2m,2n}^{(s)} = 0 \) for \( m + n > s + 1 \).

For the \( M \) propagator we will make use of the bound

\[ \left| \int \frac{dp}{(2\pi)^4 \Lambda^2 p^2 + M^2} \frac{dK/d\Lambda}{(P^2 + M^2)^2} \right| \leq D(\epsilon) \left( \frac{\Lambda}{M} \right)^{2-\epsilon}, \quad \epsilon \in [0, 1] \]  

(29)

Using eq. (14) for the homogeneous part of the equation, the bounds (28) for the inhomogeneous one, and the bounds on the \( A \) vertices we have proved in the previous section, we obtain:

\[ \| (\Lambda \partial_{\Lambda} + 4 - 2m - 2n) \partial^p W_{2m,2n}^{(s)}(\Lambda) \| \leq \frac{1}{\Lambda^p} \left( \frac{\Lambda}{M} \right)^{2-\epsilon} P \left( \ln \frac{\Lambda}{\Lambda_R} \right) \]  

(30)
For \( m + n \geq 3 \), \( W_{2m,2n}^{(s)}(\Lambda_0) = 0 \); the upward integration gives:

\[
\| \partial^p W_{2m,2n}^{(s)}(\Lambda) \| \leq \frac{1}{\Lambda^p} \left( \frac{\Lambda}{M} \right)^{2-\epsilon} \int_\Lambda^{\Lambda_0} \frac{d\Lambda'}{\Lambda'} P \left( \frac{\Lambda}{\Lambda'} \right)^{p+2m+2n-6+\epsilon} \leq \frac{1}{\Lambda^p} \left( \frac{\Lambda}{M} \right)^{2-\epsilon} \bar{P} \left( \ln \frac{\Lambda}{\Lambda_R} \right)
\]  

(31)

We note that for this kind of proof the parameter \( \epsilon \) is unavoidable. The rest of the proof is identical to that of the lemma: we integrate upward the relevant terms, the value of the \( W \)'s at this scale being zero. We stress that the proof relies on the fact that the inhomogeneous term in the equation controls the behaviour of the vertices and that the initial conditions don't destroy it: the rest of the demonstration is analogous to the one for the \( A_{2m} \) vertices.

We find

\[
\| A_{2m,2n}^{(s)}(\Lambda, \Lambda_0; m, M) - A_{2m,2n}^{(s)}(\Lambda, \Lambda_0; m, M') \| = \\
\| \int_M^{M'} dM W_{2m,2n}^{(s)}(\Lambda, \Lambda_0; m, M) \| \leq P(\ln \frac{\Lambda}{\Lambda_R})(2 - \epsilon) \left| \left( \frac{\Lambda}{M'} \right)^{2-\epsilon} - \left( \frac{\Lambda}{M} \right)^{2-\epsilon} \right|
\]

(32)

For the uniformity of the bounds, we can take the limit \( \Lambda_0 \to \infty \). Chauchy criterium guarantees also the existence of the \( M \to \infty \).

We can compare our bound with the results of J.Ambjorn [10] for the amplitudes of Feynman graphs in the BHPZ scheme when \( M \) is sent to \( \infty \), which read

\[
|F(p_i, m, M)| \leq C(\epsilon) \frac{1}{M^{2\nu-\epsilon}}, \quad \text{for } \epsilon \in (0, 1)
\]

(33)

where \( \nu \geq 1 \). The result is obtained via the parametric representation of Feynman graph, with much more work.

We must again prove that \( A_{2m,0}^{(M \to \infty)} \) agrees with the result of the theory with only light fields. \( A_{2m,0} \) satisfies the equation

\[
(\Lambda \partial_{\Lambda} + 4 - 2m)A_{2m,0}^{(s)} = -\frac{1}{2} \sum_{l=1}^{m} \sum_{t=1}^{s-1} Q(P, m; \Lambda) A_{2l,0}^{(s-t)} A_{2m-2l+2,0}^{(s-t)} - \frac{1}{2} \int \frac{dp}{(2\pi \Lambda)^4} Q(p, M; \Lambda) A_{2m+2,0}^{(s)} - \frac{1}{2} \int \frac{dp}{(2\pi \Lambda)^4} Q(p, M; \Lambda) A_{2m+2,0}^{(s)}
\]

(34)

The equation can be integrated to give the explicit solution. Using the bounds proved above and the same tricks of the previous arguments, it is easy to show that we can take the limit \( M \to \infty \) under the sign of integrals and that, in this limit, the third integral vanishes; so the \( A_{2m,0}^{(M \to \infty)} \) satisfies the same equations of \( A_{2m,0} \) (when we write only one index in \( A \) we refer to the light theory) and, since, at the lowest order, they coincide, they coincide at all orders (the solution is uniquely determined by recursion from the initial conditions).
We can also investigate the structure of $O(1/M^2)$ corrections to light fields vertices. Working in electrodynamics, Kazama and Yao \[11\] find for the corrections the following structure

$$\Gamma_n(g, G, m, M) = \Gamma_n(g, m) + M^{-2} \sum_i C_i(g, G, \ln(m/M)) \Gamma_n(O_i, g, m) \tag{35}$$

where $g$ is the light coupling constant, $G$ the heavy one, $O_i$ are integrated local composite operators of dimension $\leq 6$ and $C_i$ are universal coefficients, calculated via a kind of Callan-Symanzik equation. We are interested in deducing this structure from the flow equation as a valid alternative to the BHPZ combinatorial formulas used by these authors.

First of all we must review some facts on the renormalization of composite operators.

We introduce a source $\epsilon$ for the composite operator and expand the lagrangian in the following way:

$$S_0(\phi, \Lambda) + \epsilon S_1(\phi, \Lambda) + \epsilon^2 S_2(\phi, \Lambda) + ... \tag{36}$$

where $S_1$ is the running operator insertion, $S_2$ is needed for double insertions and so on.

The equation of flow for the whole $S$ implies for $S_1$ a linearized equation, which, for the vertices $O_{2m}$, (analogous to $L_{2m}$) reads schematically:

$$\partial_\Lambda O_{2m} = \sum L_{2l} O_{2m-2l+2} + \int O_{2m+2} \tag{37}$$

$S_1$ is determined at low energy by its relevant parameters. For more details we refer to \[12, 5\]. For example, for the renormalization of $\phi^6$, we specify the following renormalization conditions

$$Q_6(0, ..., 0; \Lambda_R) = 1 \text{ and } \eta_i(\Lambda_R) = 0; \text{ at } \Lambda_0 \text{ only the } \eta_i \neq 0 \tag{38}$$

where $Q_{2m}$ are the adimensional counterparts of $O_{2m}$ and $\eta_i$ are the operator of dimension $\leq 6$ allowed by symmetry. We see that the counterterms must be chosen among all the operator of dimension $\leq 6$, i.e, as we know, they mix under renormalization. We finally note that now $O_{2m}^{(s)}$, $m > s + 3$ and $O_{2m}^{(0)} \neq 0$.

To exploit the counterpart of Kazama-Yao result we consider again two scalar fields coupled by a quartic interaction of the form $k_R \phi^2 \chi^2$. Note that we can always choose composite operator $O_1$ (of dimension 6) and dimensionless functions $C_i(m, M, \Lambda_R, \lambda_R, k_R)$ such that the quantities

$$C_{2m}(p_i; \Lambda) = \frac{1}{2} M^2 \partial_M L_{2m,0}(p_i; \Lambda) - \sum_i C_i(m, M, \Lambda_R, \lambda_R, k_R) O_{2m}^i(p_i; 0; \Lambda) \tag{39}$$

vanish on the relevant parameters (dim $\leq 6$) at $\Lambda_R$. For example

$$C_{\phi^6} = -1/2M^2 \partial_M L_{6,0}(0; \Lambda_R) \tag{40}$$
We want to show that $|C_{2m}^{(s)}(\Lambda_R)| \to 0$ for $M \to \infty$ which is the analogous of eq. (35). Notice that we recover the result of Kazama and Yao for the vertices of our theory and not for the real Green functions. The analogy doesn’t break for we can consistently regard the $\Lambda_R$ vertices as the connected Green functions of our theory, renormalized according to the prescription of the flow equations; as we have already noted, in this way one produces a sort of BHPZ regularization.

The rest of this section is devoted to the proof of this statement; the reader not interested in the technical details can skip the proof and go to the next section. We want only to stress that the final structure is dictated by dimensional analysis. The form of the insertion of composite operators is dictated by the canonical dimension of the quantity considered: the factor $1/M^2$ raises the reference dimension to 6, and in order to use the standard arguments we must introduce a quantity which has compatible initial data: the only way is to subtract a suitable 6-dimension operator.

First of all notice that $L^{(s,1)}_{2m,0}(\Lambda) = 0$ (we denote with the pair $(s,1)$ the perturbative order in $\lambda_R$ and $k_R$ separately); this follows from the graphical representation of the vertex: containing only one $k_R$ the diagram is necessarily divergent and it vanishes by renormalization. So $C_i$ are at least of second order in $k_R$. Moreover it is easy to show that

$$\|L^{(p,q)}_{2m,0}(\Lambda)\| \leq (\Lambda/M)^{2-\epsilon} (1/\Lambda^{2m-4}) P(ln\Lambda/\Lambda_R), \quad q \geq 1$$

$$\|L^{(p,q)}_{2m,2}\| \leq (\Lambda/M)^{2-\epsilon} (1/\Lambda^{2m-2}) P(ln\Lambda/\Lambda_R), \quad q > 1$$

so they vanish when $M \to \infty$; of course it is essential for the proof that they have null initial condition.

We already know that $C_i$ and so $C_{2m}(\Lambda_R)$ are of second order in $k_R$. Let’s show that

$$\|C_{2m}^{(p,q)}\| \leq \left(\frac{\Lambda}{M}\right)^{2-\epsilon} \frac{1}{\Lambda^{2m-6}} P\left(ln\frac{\Lambda}{\Lambda_R}\right) \left(\frac{M}{\Lambda_R}\right)^{\epsilon'}, \quad q \geq 1, \quad M > \Lambda_R, \quad \epsilon, \epsilon' \in (0, 1)$$

(42)

Due to the uniformity of these bounds, the limit $\Lambda_0 \to \infty$ is then immediate.

The $C_{2m}$ satisfy

$$\frac{\partial}{\partial \Lambda} C_{2m} = -\sum \frac{dK/d\Lambda}{P^2 + m^2} L_{2l,0} C_{2m-2l+2} - \frac{1}{2} \int \frac{dp}{(2\pi)^d} \frac{dK/d\Lambda}{p^2 + m^2} C_{2m+2} - \frac{1}{2} \int \frac{dp}{(2\pi)^d} \frac{dK/d\Lambda}{p^2 + M^2} M^2 \partial_M L_{2m,2} - \frac{1}{2} \int \frac{dp}{(2\pi)^d} \frac{dK/d\Lambda}{p^2 + M^2} M^4 L_{2m,2} + \sum C_i \frac{dK/d\Lambda}{P^2 + m^2} (L_{2l,0} - L_{2l}) O_{2m-2l+2}$$

(43)

Now $C_{2m}^{(p,0)} = 0$ because $C_i = O$ and $\partial_M L_{2m,0}^{(p,0)} = 0$ (it doesn’t contain heavy lines). In the first term on the right hand side there can appear a term $LC^{(t,1)}$, not covered by the induction hypothesis; however L will be of the
form $L^{(\geq 1)}$ for which we the bounds (41) hold. But we need also the weaker bound
\[ \| \partial^p C_{2m}^{(t,1)} \| \leq \frac{1}{\Lambda^{2m-6}} P \left( \frac{\ln \Lambda}{\Lambda_R} \right) \frac{1}{\Lambda^p} \left( \frac{M}{\Lambda} \right)^\epsilon \]
which follows from the definition of $C_{2m}$ and from the bounds over the $M$-derivative of $L_{2m}$.

We use $|C_i| \leq D(m/\Lambda_R)\epsilon'$,
\[ \| L_{2l,0} - L_{2l} \| = \| \int_M^\infty \frac{dM}{M} M \partial M L_{2l,0} \| \leq C(\epsilon) \frac{P(\ln \Lambda/\Lambda_R)}{\Lambda^{2l-4}} \left( \frac{\Lambda}{M} \right)^{2-\epsilon} \]
\[ \| \partial^p O_{2m}^{(i,s)} \| \leq \frac{1}{\Lambda^p} \frac{P(\ln \Lambda/\Lambda_R)}{\Lambda^{2m-6}} \]
we obtain
\[ \| \partial_\Lambda \partial^p C_{2m}^{(p,q)} \| \leq \frac{1}{\Lambda} P \left( \frac{\ln \Lambda}{\Lambda_R} \right) \frac{1}{\Lambda^{2m-6}} \left( \frac{\Lambda}{M} \right)^{2-\epsilon} \left( \frac{M}{\Lambda_R} \right)^\epsilon', \quad q > 1, M > \Lambda_R \]
the first step of induction is obtained from the same equation (43). The rest of the proof is straightforward: for $m \geq 4$ (irrelevant terms for the quantities considered) we integrate downward from $\Lambda_0$, and in the relevant case upward from $\Lambda_R$, where we have suitably chosen the $C_i$ in such a way to have null initial conditions, instead of a complicated and unknown $M$ dependence.

4 Gauge theories

We have proved so far that the theory of interacting scalar fields exhibits decoupling; the generalization to arbitrary power-counting renormalizable theories with global symmetry groups is straightforward, because the proof is essentially based on the dimensional analysis. The problems in gauge theories is the same of their renormalizability: it is simple to construct an UV finite quantum theory but in general if we don’t use an invariant regularization or clever renormalization conditions the final theory will have lost the explicit symmetry embodied in the Ward identities, which we consider the natural implementation of the concept of symmetry in the quantum theory.

A higher derivative gauge invariant regularization which allows the use of Polchinski arguments is presented in the work of Warr [12], whereas a dimensional regularization for the flow is still lacking. As usual the regularized theory satisfies the Ward identities and what we have to prove is that it’s possible to specify the renormalization conditions (i.e. the low energy data) in such a way that the cut-off removal is innocuous. We prefer to use the combination of the quantum action principle [18] or BRS approach [13], which we briefly review here.

In the gauge fixed lagrangian
\[ L = -\frac{1}{4g^2} tr F_{\mu\nu} F^{\mu\nu} + L_{MAT} - \frac{1}{2\alpha} tr (\partial_\mu A^\mu)^2 + \frac{1}{\alpha} tr c M \sigma \]
(47)
where \( M(,) = \partial^2 - i\partial_\mu[A^\mu,] \) we have lost the gauge invariance but we have gained the invariance under the BRS transformations

\[
\begin{align*}
\Delta A_\mu &= \partial_\mu \overline{c} + i[\overline{c}, A_\mu] \\
\delta \phi &= -i\overline{c} T^i \phi \\
\Delta \overline{c} &= (1/2)f_{ijk} \overline{c}_i \overline{c}_j \\
\Delta c &= \partial A_
u
\end{align*}
\] (48)

These are nilpotent transformations so that also the lagrangian in which we introduce the inert sources for the composite fields \( \Delta A, \Delta \phi, \Delta c \)

\[
L' = -\frac{1}{4g^2} tr F_{\mu\nu} F^{\mu\nu} + L_{MAT} - \frac{1}{2\alpha} tr(\partial_\mu A^\mu)^2 + \frac{1}{\alpha} trc M\overline{c} + tr(\rho^\mu \Delta A_\mu + U \Delta c) + Y^a \Delta \phi_a
\] (49)

is BRS invariant. To renormalize a gauge theory means to construct a quantum extension \( \Gamma \) of the lagrangian (effective action) which has the gauge symmetry expressed by the Slavnov-Taylor identities \( \Delta \Gamma(A, \phi, c, c; \rho, U, Y) = 0 \).

At the classical level this identity specifies completely the lagrangian up to a redefinition of parameters and wavefunctions \([13]\).

\( \Delta^2 \Gamma = 0 \) implies that \( \Gamma \) depends only on \( \eta_\mu = \rho_\mu - (1/\alpha)\partial_\mu c \). Defining

\[
\Gamma(A, \phi, \overline{c}, \eta, Y, U) = \Gamma + \frac{1}{2\alpha} \int d\xi (\partial A^i)^2
\] (50)

the Slavnov-Taylor identities read

\[
\Delta(\Gamma) = \frac{1}{2} B_{\gamma} \Gamma = 0
\] (51)

where \( B_\gamma \) is the linear operator

\[
B_\gamma = \int dx \left\{ \partial_\gamma \overline{\eta}_\mu \partial A^\mu_i \overline{A}^{i\mu} + \partial_\gamma \overline{\eta}_\mu \partial A^\mu_i \overline{A}^{i\mu} + \partial_\gamma \overline{\eta}_\mu \partial \phi_a \partial \phi_a + \partial_\gamma \overline{\eta}_\mu \partial \phi_a \partial \phi_a + \partial_\gamma \overline{\eta}_\mu \partial \phi_a \partial \phi_a + \partial_\gamma \overline{\eta}_\mu \partial \phi_a \partial \phi_a \right\}
\] (52)

which satisfies

\[
B_\gamma B_\gamma = 0 \quad (53)
\]

\[
B_\gamma B_\gamma = 0 \quad i f \quad B_\gamma B_\gamma = 0 \quad (54)
\]

At order zero \( \Gamma_0 \) is nothing but the classical action \( \int d^4x L' \), and the linear operator \( b = B_{\Gamma_0} \) satisfies

\[
b \Gamma_0 = 0, \quad b^2 = 0
\] (55)

On the fields \( A, \phi, \overline{c} \) it corresponds to the BRS transformation.

We begin by regularizing the theory by a cut-off; the power of the BRS approach is that it is independent of the type of regularization. The Polchinski scheme produces an UV finite theory which, of course, has the global symmetry but not the gauge one and which exhibits decoupling.
To fix the ideas, let us consider a Yang-Mills theory minimally coupled with massive scalar fields, and let us send the mass to $\infty$.

The powerful action principle, which can be proved in the Polchinski scheme ([5]) (see the next section), states that the possible violations of the Ward identities are integrated insertions of a local operator of dimension $\leq 5$ and of Faddeev-Popov number 1
\[ \Delta(\Gamma) = \Delta \Gamma \]  
(56)

If we can reabsorb the anomaly $\Delta$ in local counterterms in the action we have a renormalizable gauge invariant quantum theory. Suppose, by induction, that $\Delta$ vanishes up to $h^{n-1}$ order,
\[ \Delta(\Gamma) = \frac{1}{2} B \Gamma = \Delta + O(h\Delta) \]  
(57)

$\Delta$ is a globally invariant field polynomial, independent and moreover is a b-closed functional. In fact to $h^n$ order
\[ 0 = B^2 \Delta = b\Delta + O(h\Delta) \]  
(58)

The purpose is to classify all the possible anomalies; because $b^2 = 0$ all $\Delta = b\Delta'$ are possible anomalies; $\Delta'$ are of dimension 4 and FP number 0 so they can be obviously reabsorbed in the lagrangian by defining $L' = L - \Delta'$,
\[ \Delta(\Gamma') = \Delta(\Gamma) - \Delta(\Delta') + O(h^{n+1}) = O(h^{n+1}) \]  
(59)

The theory is now renormalizable and anomaly free to this order in $h$. We can repeat the reasoning order by order. If the theory is not chiral, one can prove, in a purely algebraic way, that b-exact functionals are all the possible solutions of $b\Delta = 0$; it is obviously a cohomological problem. [13]

We have so found a quantum gauge extension of our Yang-Mills theory; obviously we have to change the renormalization conditions by adding new counterterms. We have again a freedom: it is easy to verify that adding to the lagrangian the general solution of $b\Delta = 0$ in the space of four dimensional functionals, we don’t break the symmetry; this freedom corresponds to reparametrize the initial bare classical lagrangian and we can use it to impose some renormalization conditions.

From this point of view the extension of the decoupling theorem to gauge theory does not present problems. We work by induction in $h$, supposing to have a symmetric and decoupled theory; we renormalize it in the Polchinski scheme at order $h^n$ and we obtain a decoupled theory but with an anomaly $\Delta(\Gamma) = \tilde{a} + O(ha)$. Looking at the left hand side, we see that $\tilde{a}$ has $M$-finite coefficients which coincide with the anomaly of the decoupled theory if we take the limit $M \to \infty$; when reabsorbed in the action they have again the same property, establishing the symmetry at this order without violating manifest decoupling.

We have so proved the Appelquist-Carazzone theorem. Obviously if we want to consider the question of the effective lagrangian in a different renormalization scheme we must renormalize coupling constants and wavefunctions.
5 Theory with spontaneous symmetry breaking

So far we have proved that a general renormalizable theory without spontaneous symmetry breaking exhibits decoupling. In the case of SSB we have two ways of increasing a mass, either with a dimensional parameter (typically a vacuum expectation value), in which case we have decoupling [14], or with a dimensionless one (a coupling constant) in which case the decoupling theorem fails. A typical example of the last situation is the limit for $\lambda \to \infty$ of the linear $\sigma$-model which gives rise to the non renormalizable non linear $\sigma$-model. For definiteness, we consider the following lagrangian

$$\frac{1}{2}(\partial h)^2 + \frac{1}{2}(\partial H)^2 + \frac{m^2}{2} h^2 + \frac{M^2}{2} H^2 + \lambda_1 h^4 + \lambda_2 H^4 + \lambda_3 (h \times H)^2 + \lambda_4 h^2 H^2$$

with $h$ and $H$ in the vectorial representation of SO(3). The group is completely broken in the orthogonal directions $(0,v,0)$ and $(V,0,0)$ [15] and the parameter $v$ and $V$ take the role of the two masses $m$ and $M$ as free parameters; we want to send $V$ to $\infty$. As already mentioned, we expect decoupling.

To study the decoupling in a scalar theory, it has been proposed [16] a variation of the $\overline{MS}$ scheme where heavy graphs, too large in power of $M$, are eliminated ad hoc by local counterterms. This scheme is probably only suitable for the study of the decoupling problems. A complete proof of decoupling also in gauge theories has been provided [17]. Another proof is presented by Kazama and Yao [14] working in the traditional BHPZ scheme.

We first note that in the Polchinski approach we have an immediate problem: all the bounds we have given were derived by assuming that every dimensional parameter appearing in the lagrangian is lower than the scale $\Lambda_R$, otherwise the bounds on the vertices will be violated even at the classical level. However the breaking introduces trilinear terms of the form $M \chi \eta^2$ (where $h = v + \eta, H = V + \chi$). We have to modify the renormalization rules; a simple way is to employ a solution similar to the Chang-Das one [16].

If we are able to renormalize the theory and obtain vertices of order $M^n$, where $n$ is the number of heavy legs, we conclude, by power counting, that also the Green functions (which we can evaluate at the scale $\Lambda_R$ where the integration of the Feynman diagram are finite) with $n$ heavy lines are of order $M^n$. In particular, this implies the result that the Green functions with only external light fields are $M$-finite.

These considerations and the lagrangian form in the fields $\eta, \chi$ suggest the bound

$$\|L_{m,n}\| \leq \left( \frac{M}{\Lambda} \right)^n \Lambda^{-m-n} P \left( \frac{ln \Lambda}{\Lambda_R} \right)$$

We use hereafter an induction in $h$ so, at fixed order in $h$, we must increase the number of external legs. At tree level these bounds are verified. Assuming their validity at order $n-1$ we discover easily that, at order $n$, they are again
verified for the irrelevant parameters. But for the relevant ones we obtain the bound:

$$\|\partial_\Lambda L_{m,n}\| \leq P\left(\frac{M}{\Lambda}\right)^{(n)} \Lambda^{4-m-n} P\left(ln\frac{\Lambda}{\Lambda_R}\right)$$  \hspace{1cm} (62)

to be integrated between \(\Lambda\) and \(\Lambda_R\); here \(P\) denotes a polynomial of maximum degree \(n\). The terms which have total negative powers of \(\Lambda\) invalidate the usual argument: we are forced to treat them as irrelevant and to integrate them downward from \(\Lambda_0\). In other words, we explicitly choose the renormalization conditions (i.e. the low energy data) to cancel the unwanted terms, with a strong analogy with the method of Chang and Das. However, in our approach, it is obvious that we introduce only counterterms of dimension 4, while the previous authors must refer to a modified version of the forest formula and the relative proof.

We have obtained a decoupled theory. What’s about gauge theories? We consider the same model with SO(3) gauge group \([14]\). The use of t’Hooft gauge fixing \((\partial A^a - (\eta, T^a v) - (\chi, T^a V))^2\) distinguishes between heavy and low sector. We don’t precise further the model, because our argument is largely detail-independent.

Evaluating the anomalous Slavnov identities at the scale \(\Lambda_R\), we learn that the general term in the anomaly with \(n\) heavy fields is \(O(M^{n+1})\). For our purpose this is not enough; luckily, we can employ again the action principle.

The quantum action principle \([18]\) states, roughly speaking, that a classical symmetry can be violated at the quantum level, i.e. for the generating functionals \(Z\) and \(\Gamma\), at most by a local insertion of specified dimension. It was first proved in the BHPZ framework with a lot of technical difficulties, and then for other renormalization scheme. In the Polchinski approach, the proof is very simple \([5]\), just because the method is a sort a legitimation of the heuristic functional integral arguments.

Working again with a volume cut-off (before the thermodynamic limit) we can make the change of variables in the partition function

$$\phi_p \rightarrow \phi_p + \sum_q \epsilon_{p-q} P_{0,q} (\phi)$$  \hspace{1cm} (63)

after the introduction of the source \(\eta\) for the composite operator \(P\) in the action \(L_0\), obtaining

$$\sum_{p,q} \epsilon_{p} J_{-q} \partial_{\phi, p} Z = \sum_q \epsilon_q \Delta_q Z$$  \hspace{1cm} (64)

where

$$\Delta_p = \sum_p (\phi_{-p} P(p) + \partial_{\phi, p} L_0) P_{0,p-q} - \partial_{\phi, p} P_{0,p-q}$$  \hspace{1cm} (65)

We can repeat this change of variable at each scale \(\Lambda\). The first hand side of the equation for \(Z\) is UV finite so also the insertion operator \(\Delta\) is finite. This is the expression of the quantum action principle. The first non vanishing
order of $\Delta$ is the anomaly $a$ which appears in $\Delta(\Gamma) = a + O(\hbar a)$; despite the appearance, it is finite and scale independent. If we substitute the vertices bounds in the explicit expression of $\Delta$ we find terms of the form

$$\left( \frac{M}{\Lambda} \right)^n \Lambda^{4-m-n} \times \text{polynomials} \quad (66)$$

Evaluating this expression at the scale $\Lambda_0$ we find that only terms with power of $M$ lower than 6 can survive in the anomaly.

So we have only few terms to check. They have dimension 5 and FP number 1. Possible terms of the type $M^4\tau$ in the anomaly violate the bound $M^{n+1}$; $M^3\tau(.)$ produces, once reabsorbed, linearterms, which we can eliminate by adjusting $V$ and $v$ to give the right vacuum. The remaining terms must be checked model by model; having a number of free parameters to adjust order by order, we can control them. For example, in the specific model indicated, we have some further symmetries. The theory and the BRS transformation are invariant under $\chi \to -\chi, M \to -M, K \to -K$. It exists also an index symmetry [4] which prevents many vertices. So, by a simple check, we are left only with the light fields quadratic terms. The normalization conditions are surely enough to eliminate them. We notice also that for models with less symmetries, the Ward identities will relate some of the unknown terms to the already bounded ones, leaving a few number of independent terms, which we can control with the renormalization conditions.

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