Semilinear Hyperbolic Equations in Curved Spacetime

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Abstract

This is a survey of the author’s recent work rather than a broad survey of the literature. The survey is concerned with the global in time solutions of the Cauchy problem for matter waves propagating in the curved spacetimes, which can be, in particular, modeled by cosmological models. We examine the global in time solutions of some class of semilinear hyperbolic equations, such as the Klein-Gordon equation, which includes the Higgs boson equation in the Minkowski spacetime, de Sitter spacetime, and Einstein & de Sitter spacetime. The crucial tool for the obtaining those results is a new approach suggested by the author based on the integral transform with the kernel containing the hypergeometric function.

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1 Introduction

This survey is concerned with the global in time solutions of the Cauchy problem for matter waves propagating in the curved spacetimes, which can be, in particular, modeled by cosmological models. We are motivated by the significant importance of the qualitative description of the global solutions of the partial differential equations arising in the cosmological problems for understanding of the structure of the universe and fundamental particles physics. On the other hand, the physical implications of the mathematical results given here are out of the scope of this paper. More precisely, in this survey we examine the global in time solutions of some class of semilinear hyperbolic equations, and, in particular, the Klein-Gordon equation, which includes the Higgs boson equation in the Minkowski spacetime, de Sitter spacetime, and Einstein & de Sitter spacetime. The Higgs boson plays a fundamental role in unified theories of weak, strong, and electromagnetic interactions [45].

The Klein-Gordon equation arising in relativistic physics and, in particular, general relativity and cosmology, as well as, in more recent quantum field theories, is a covariant equation that is considered in the curved pseudo-Riemannian manifolds. (See, e.g., Birrell and Davies [7], Parker and Toms [34], Weinberg [45].) The latest astronomical observational discovery that the expansion of the universe is speeding supports the model of the expanding universe that is mathematically described by the manifold with a metric tensor depending on time and spatial variables.
The homogeneous and isotropic cosmological models possess highest symmetry, which makes them more amenable to rigorous study. Among them, FLRW (Friedmann-Lemaître-Robertson-Walker) models are mentioned, which have the flat metric of the slices of constant time. The FLRW spacetime metric can be written in the form
\[ ds^2 = -dt^2 + a^2(t)(dx^2 + dy^2 + dz^2), \]
with an appropriate scale factor \( a(t) \). (See, e.g., [26, 36, 43].) In particular, the metric in de Sitter spacetime in the Lemaître-Robertson coordinates [33, 43] has this form with the cosmic scale factor \( a(t) = e^t \). The time dependence of the function \( a(t) \) is determined by the Einstein field equations for gravity with the cosmological constant \( \Lambda \),
\[ R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi GT_{\mu\nu} - \Lambda g_{\mu\nu}. \]
The unknown of principal importance in the Einstein equations is a metric \( g \). It comprises the basic geometrical feature of the gravitational field, and consequently explains the phenomenon of the mutual gravitational attraction of substance.

The metric of the Einstein & de Sitter universe (EdeS universe) is a particular member of the FLRW metrics
\[ ds^2 = -dt^2 + a^2(t)\left[ \frac{dr^2}{1 - Kr^2} + r^2d\Omega^2 \right], \]
where \( K = -1, 0, \) or \(+1\), for a hyperbolic, flat or spherical spatial geometry, respectively. The Einstein & de Sitter model of the universe is the simplest non-empty expanding model with the line-element
\[ ds^2 = -dt^2 + a_0^2t^{4/3}(dx^2 + dy^2 + dz^2) \]
in comoving coordinates [10]. It was first proposed jointly by Einstein & de Sitter (the EdeS model) [15]. The observations of the microwave radiation fit in with this model [14]. The result of this case also correctly describes the early epoch, even in a universe with curvature different from zero [9, Sec. 8.2]. Even though the EdeS spacetime is conformally flat, its causal structure is quite different from asymptotically flat geometries. In particular, and unlike Minkowski or Schwarzschild spacetimes, the past particle horizons exist. The EdeS spacetime is a good approximation to the large scale structure of the universe during the matter dominated phase, when the averaged (over space and time) energy density evolves adiabatically and pressures are vanishingly small, as, e.g., immediately after inflation. This justifies why such a metric is adopted to model the collapse of overdensity perturbations in the early matter dominated phase that followed inflation.

The matter waves in the spacetime are described by the function \( \phi \), which satisfies equations of motion. In the model of universe with curved spacetime the equation for the scalar field with potential function \( V \) is the covariant wave equation
\[ \Box g \phi = V'(\phi) \quad \text{or} \quad \frac{1}{\sqrt{|g|}} \frac{\partial}{\partial x^i} \left( \sqrt{|g|} g^{ik} \frac{\partial \phi}{\partial x^k} \right) = V'(\phi), \]
with the usual summation convention. Written explicitly in the Lemaître-Robertson coordinates in the de Sitter spacetime it, in particular, for
\[ V'(\phi) = -\mu^2 \phi + \lambda |\phi|^{p-1} \phi, \quad p > 1, \]
has the form
\[ \phi_{tt} + n \phi_t - e^{-2t} \Delta \phi = \mu^2 \phi - \lambda |\phi|^{p-1} \phi, \quad (1.1) \]
where $\mu > 0$ and $\lambda > 0$. Here $\Delta$ is the Laplace operator on the flat metric, $\Delta := \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$.

The equation for the Higgs real-valued scalar field in the de Sitter spacetime is a special case of (1.1) when $p = 3$, $n = 3$:

$$\phi_{tt} + 3\phi_t - e^{-2t} \Delta \phi = \mu^2 \phi - \lambda \phi^3.$$

Scalar fields play a fundamental role in the standard model of particle physics, as well as its possible extensions. In particular, scalar fields generate spontaneous symmetry breaking and provide masses to gauge bosons and chiral fermions by the Brout-Englert-Higgs mechanism [18] using a Higgs-type potential [27].

In the spacetime with the constant metric tensor $g$ the differential operator in the equation contains only the second-order derivatives. For the equation

$$\phi_{tt} - \Delta \phi = \mu^2 \phi - \lambda |\phi|^{p-1} \phi$$

the existence of a weak global solution in the energy space is known (see, e.g., Proposition 3.2 [23]) under certain conditions. The equation

$$\phi_{tt} - \Delta \phi = \mu^2 \phi - \lambda \phi^3$$

(1.2)

for the Higgs scalar field in the Minkowski spacetime has the time-independent flat solution

$$\phi_N(x) = \frac{\mu}{\sqrt{\lambda}} \tanh \left( \frac{\mu}{\sqrt{2}} N \cdot (x - x_0) \right), \quad N, x_0, x \in \mathbb{R}^3.$$

The unit vector $N$ defines the direction of the propagation of the wave front. This solution, after Lorentz transformation, gives rise to a traveling solitary wave of the form

$$\phi_{N,v}(x,t) = \frac{\mu}{\sqrt{\lambda}} \tanh \left( \frac{\mu}{\sqrt{2}} [N \cdot (x - x_0) \pm v(t - t_0)] \frac{1}{\sqrt{1 - v^2}} \right), \quad N, x_0, x \in \mathbb{R}^3,$$

$t \geq t_0$, if $0 < v < 1$, where $v$ is the initial velocity. The set of zeros of the solitary wave $\phi = \phi_{N,v}(x,t)$, that is, the set given by $N \cdot (x - x_0) \pm v(t - t_0) = 0$, is the moving boundary of the wall. The existence of standing waves $\phi = \exp(i\omega t)v(x)$, which are exponentially small at infinity $|x| = \infty$, and of corresponding solitary waves for the equation (1.2) with $\mu^2 < 0$ and $\lambda < 0$ is known (see, e.g., [11]).

The covariant linear wave equation in the Einstein & de Sitter spacetime written in the coordinates is

$$\left( \frac{\partial}{\partial t} \right)^2 \psi - t^{-4/3} \sum_{i=1,2,3} \left( \frac{\partial}{\partial x^i} \right)^2 \psi + \frac{2}{t} \frac{\partial}{\partial t} \psi = f,$$

(1.3)

In this survey we investigate the initial value problem for this equation and give the representation formulas for the fundamental solutions in the case of arbitrary dimension $n \in \mathbb{N}$ of the spatial variable $x \in \mathbb{R}^n$. The equation (1.3) is strictly hyperbolic in the domain with $t > 0$. On the surface $t = 0$ its coefficients have singularities that make the study of the initial value problem difficult. Then, the speed of propagation is $t^{-2/3}$ for every $t \in \mathbb{R} \setminus \{0\}$. The classical works on the Tricomi and Gellerstedt equations (see, e.g., [8, 10, 13, 46]) appeal to the singular Cauchy problem for the Euler-Poisson-Darboux equation, and to the Asgeirsson mean value theorem when handling a high-dimensional case.
2 Method of Investigation. Integral Transform

We suggested in [48] a novel approach to study second order hyperbolic equations with variable coefficients. That approach avoids explicit appeal to the Fourier integral operators, and it seems to be more immediate than the one that uses the Euler-Poisson-Darboux equation. It is used in a series of papers [48]-[56], [21] to investigate in a unified way several equations such as the linear and semilinear Tricomi and Tricomi-type equations, Gellerstedt equation, the wave equation in EdeS spacetime, the wave and the Klein-Gordon equations in the de Sitter and anti-de Sitter spacetimes. The listed equations play an important role in the gas dynamics, elementary particle physics, quantum field theory in curved spaces, and cosmology. For all above mentioned equations, we have obtained among other things, fundamental solutions, representation formulas for the initial-value problem, $L_p - L_q$-estimates, local and global solutions for the semilinear equations, blow up phenomena, sign-changing phenomena, self-similar solutions and number of other results.

More precisely, in that method the solution $v = v(x, t; b)$ to the Cauchy problem

$$v_{tt} - \Delta v = 0, \quad (t, x) \in \mathbb{R}^{1+n}, \quad v(x, 0; b) = \varphi(x, b), \quad v_t(x, 0) = 0, \quad x \in \mathbb{R}^n,$$

(2.1)

with the parameter $b \in B \subseteq \mathbb{R}$ is utilized. Denote that solution by $v_\varphi = v_\varphi(x, t; b)$. There are well-known explicit representation formulas for the solution of the last problem. (See, e.g., [39].) In particular, if $\varphi$ is independent of the second variable $b$, then $v_\varphi(x, t; b)$ does not depend on $b$ and we write $v_\varphi(x, t)$.

The starting point of that approach [48] is the Duhamel’s principle, which we revise in order to prepare the ground for generalization. Our first observation is that we obtain the following representation

$$u(x, t) = \int_{t_0}^{t} d\tau \int_{0}^{t-\tau} w_f(x, z; \tau) dz,$$

(2.2)

of the solution of the Cauchy problem $u_{tt} - \Delta u = f(x, t)$ in $\mathbb{R}^{n+1}$, and $u(x, t_0) = 0, u_t(x, t_0) = 0 \quad \text{in} \quad \mathbb{R}^n$, where the function $w_f = w_f(x; t; \tau)$ is the solution of the problem 2.1. This formula allows us to solve problem with the source term if we solve the problem for the same equation without source term but with the first initial datum.

The second observation is that in (2.2) the upper limit $t - \tau$ of the inner integral is generated by the propagation phenomena with the speed which equals to one. In fact, that is a distance function between the points at time $t$ and $\tau$.

Our third observation is that the solution operator $G : f \mapsto u$ can be regarded as a composition of two operators. The first one

$$\mathcal{WE} : f \mapsto w$$

is a Fourier Integral Operator (FIO), which is a solution operator of the Cauchy problem with the first initial datum for wave equation in the Minkowski spacetime. The second operator

$$\mathcal{K} : w \mapsto u$$

is the integral operator given by (2.2). We regard the variable $z$ in (2.2) as a “subsidiary time”. Thus, $G = \mathcal{K} \circ \mathcal{WE}$ and we arrive at the diagram:
Based on this diagram, we generated a class of operators for which we obtained explicit representation formulas for the solutions. That means also that we have representations for the fundamental solutions of the partial differential operator. In fact, this diagram brings into a single hierarchy several different partial differential operators. Indeed, if we take into account the propagation cone by introducing the distance function \( \phi(t) \), and if we provide the integral operator with the kernel \( K(t; r, b) \) as follows:

\[
K[w](x, t) = 2 \int_{t_0}^{t} db \int_{0}^{\phi(b)} K(t; r, b) w(x, r; b) dr, \quad x \in \mathbb{R}^n, \quad t > t_0,
\]

then we actually can generate new representations for the solutions of different well-known equations. Below we give some examples of the operators with the variable coefficients. (See also [54].)

1\(^{\text{st}}\) Tricomi-type equations. This operator is generated by the kernel \( K(t; r, b) = 2E(0, t; r, b) \), where the function \( E(x, t; r, b) \) is defined by

\[
E(x, t; r, b) := c_k \left( (\phi(t) + \phi(b))^2 - (x - r)^2 \right)^{-\gamma} \times F \left( \gamma, \gamma; 1; \frac{(\phi(t) - \phi(b))^2 - (x - r)^2}{(\phi(t) + \phi(b))^2 - (x - r)^2} \right),
\]

with \( \gamma := k/(2k + 2) \), \( c_k = (k + 1)^{-k/(k+1)}2^{-1/(k+1)} \), \( k \neq -1, k \in \mathbb{R} \), and the distance function is \( \phi(t) = t^{k+1}/(k+1) \), while \( F(a, b; c; \zeta) \) is the Gauss's hypergeometric function. It is proved in [48] that for the smooth function \( f = f(x, t) \), the function

\[
u(x, t) = 2c_k \int_{t_0}^{t} db \int_{0}^{\phi(b)} \left( (\phi(t) + \phi(b))^2 - r^2 \right)^{-\gamma} \times F \left( \gamma, \gamma; 1; \frac{(\phi(t) - \phi(b))^2 - r^2}{(\phi(t) + \phi(b))^2 - r^2} \right) w(x, r; b) dr, \quad t > 0,
\]

solves the Tricomi-type equation \((l = 2k \in \mathbb{N})\) (for the Tricomi equation \( l = 1 \))

\[
u_{tt} - t^l \Delta \nu = f(x, t) \quad \text{in} \quad \mathbb{R}^{n+1}_+ := \{(x, t) | x \in \mathbb{R}^n, \ t > 0\}, \quad (2.5)
\]

and takes vanishing initial values

\[
u(x, 0) = 0, \quad \nu_t(x, 0) = 0 \quad \text{in} \quad \mathbb{R}^n. \quad (2.6)
\]
The wave equation in the FLRW-models: de Sitter spacetime. In this example 
\( K(t; r, b) = 2E(0, t; r, b), \) where the function \( E(x, t; r, b) \) [52] is defined by

\[
E(x, t; r, b) := \left( (e^{-b} + e^{-t})^2 - (x - r)^2 \right)^{-\frac{1}{2}}
\]

\[
\times F\left( \frac{1}{2}, \frac{1}{2}, 1; \frac{(e^{-t} - e^{-b})^2 - (x - r)^2}{e^{-t} + e^{-b} - (x - r)^2} \right),
\]

and \( \phi(t) := 1 - e^{-t}. \) For the simplicity, in (2.7) we use the notation \( x^2 = x \cdot x = |x|^2 \) for \( x \in \mathbb{R}^n. \) It is proved in [52] that, defined by the integral transform (2.3) with the kernel (2.7) the function

\[
u(x, t) = 2 \int_0^t db \int_0^b e^{-b-e^{-t}} \left( (e^{-b} + e^{-t})^2 - r^2 \right)^{-\frac{1}{2}}
\]

\[
\times F\left( \frac{1}{2}, \frac{1}{2}, 1; \frac{(e^{-t} - e^{-b})^2 - r^2}{e^{-t} + e^{-b} - r^2} \right) w(x, r; t) dr
\]
solves the wave equation in the FLRW spaces arising in the de Sitter model of the universe (see, e.g. [33]), \( u_{tt} - e^{-2t} \Delta u = f(x, t) \) in \( \mathbb{R}^{n+1} \), and takes vanishing initial data (2.6).

3 The wave equation in the FLRW-models: anti-de Sitter spacetime The third example we obtain if we set \( K(t; r, b) = 2E(0, t; r, b), \) where the function \( E(x, t; r, b) \) is defined by (see [31])

\[
E(x, t; r, b) := \left( (e^b + e^t)^2 - (x - r)^2 \right)^{-\frac{1}{2}} F\left( \frac{1}{2}, \frac{1}{2}, 1; \frac{(e^t - e^b)^2 - (x - r)^2}{e^t + e^b - (x - r)^2} \right),
\]

while the distance function is \( \phi(t) := e^t - 1. \) In that case the function \( u = u(x, t) \) produced by the integral transform (2.3) with \( t_0 = 0 \) and the kernel (2.8), solves the wave equation in the FLRW space arising in the anti-de Sitter model of the universe (see, e.g. [33]), \( u_{tt} - e^{2t} \Delta u = f(x, t) \) in \( \mathbb{R}^{n+1}. \) Moreover, it takes vanishing initial values (2.6).

4 The wave equation in the Einstein & de Sitter spacetime If we allow negative \( k \in \mathbb{R} \) in (2.4), then we obtain another way to get new operators of the above described hierarchy. In fact, in the hierarchy of the hypergeometric functions \( F (a, b; c; \zeta) \) the simplest non-constant function is \( F (-1, -1; 1; \zeta) = 1 + \zeta. \) The exponent \( l \) leading to \( F (-1, -1; 1; \zeta) \) is exactly the exponent \( l = -4/3 \) of the wave equation (and of the metric tensor) in the Einstein & de Sitter spacetime. In that case the kernel is \( K(t; r, b) = \frac{1}{18} (9t^{2/3} + 9b^{2/3} - r^2). \) Consequently, the function

\[
u(x, t) = \int_0^t db \int_0^{3t^{1/3} - 3b^{1/3}} \frac{1}{18} \left( (3t^{1/3})^2 + (3b^{1/3})^2 - r^2 \right) w(x, r; b) dr,
\]

\( x \in \mathbb{R}^n, t > 0, \) solves (see [21]) the equation

\[
u_{tt} - t^{-4/3} \Delta u = f \quad \text{in} \quad \mathbb{R}^{n+1},
\]

and takes vanishing initial data (2.6) provided that \( w = WE(f). \) Because of the singularity in the coefficient of equation (2.10), the Cauchy problem is not well-posed. In order to obtain
a well-posed problem the initial conditions must be modified to the weighted initial value conditions

\[
\begin{align*}
\lim_{t \to 0} u(x, t) &= \varphi_0(x), \quad x \in \mathbb{R}^n, \\
\lim_{t \to 0} \left( u_t(x, t) + 3t^{-1/3} \Delta \varphi_0(x) \right) &= \varphi_1(x), \quad x \in \mathbb{R}^n.
\end{align*}
\]

The operator of equation (2.10) coincides with the principal part of (1.3). We remind that the EdeS model of the universe is the simplest non-empty expanding model (see, e.g., Section 4.3[16]). The last equation belongs to the family of the non-Fuchsian partial differential equations. There is very advanced theory of such equations (see, e.g., [32]), but according to our knowledge the weighted initial value problem suggested in [21] is the new one.

More examples on the integral transforms and representation formulas are given below.

3 Huygens’ Principle for the Klein-Gordon equation in the de Sitter spacetime

In this section we show that the Klein-Gordon equation in the de Sitter spacetime, which is the curved manifold due to the cosmological constant, obeys the Huygens’ principle only if the physical mass \( m \) of the scalar field and the dimension \( n \geq 2 \) of the spatial variable are tied by the equation \( m^2 = (n^2 - 1)/4 \). Recall (see, e.g., [25]) that a hyperbolic equation is said to satisfy Huygens’ principle if the solution vanishes at all points which cannot be reached from the support of initial data by a null geodesic, that is, there is no tail. The tails are important within cosmological context. (See, e.g., [17, 19, 24] and references therein.)

Moreover, we define the incomplete Huygens’ principle, which is the Huygens’ principle restricted to the vanishing second initial datum, and then prove that the massless scalar field in the de Sitter spacetime obeys the incomplete Huygens’ principle and does not obey the Huygens’ principle, for the dimensions \( n = 1, 3 \), only.

In quantum field theory for the massive scalar field, the equation of motion is the Klein-Gordon equation generated by the metric \( g \):

\[
\frac{1}{\sqrt{|g|}} \frac{\partial}{\partial x^i} \left( \sqrt{|g|} g^{jk} \frac{\partial \phi}{\partial x^j} \right) = m^2 \phi + V'(\phi).
\]

In physical terms this equation describes a local self-interaction for a scalar particle. In the de Sitter universe the equation for the scalar field with mass \( m \) and potential \( V \) written out explicitly in coordinates is

\[
\phi_{tt} + n\phi_t - e^{-2t} \Delta \phi + m^2 \phi = -V'(\phi).
\]

For the solution \( \Phi \) of the Cauchy problem for the linear Klein-Gordon equation

\[
\Phi_{tt} + n\Phi_t - e^{-2t} \Delta \Phi + m^2 \Phi = 0, \quad \Phi(x, 0) = \varphi_0(x), \quad \Phi_t(x, 0) = \varphi_1(x),
\]

the following formula is obtained in [52]:

\[
\Phi(x, t) = e^{-\frac{n+1}{2}t} v_{\varphi_0}(x, \phi(t)) + e^{-\frac{n}{2}t} \int_0^1 v_{\varphi_0}(x, \phi(t)s)(2K_0(\phi(t)s, t) + nK_1(\phi(t)s, t))\phi(t) ds + 2e^{-\frac{n}{2}t} \int_0^1 v_{\varphi_1}(x, \phi(t)s)K_1(\phi(t)s, t)\phi(t) ds, \quad x \in \mathbb{R}^n, \ t > 0,
\]
provided that the mass \( m \) is large, that is, \( m^2 \geq n^2/4 \). Here, \( \phi(t) := 1 - e^{-t} \) and the function \( \nu(x, t) \) is defined by \([2.7]\). Next we proceed to the definition of \( K_0(z, t) \) and \( K_1(z, t) \). (See also Section 3 \([52]\).)

We introduce the following notations. First, we define a \textit{chronological future} \( D_+(x_0, t_0) \) and a \textit{chronological past} \( D_-(x_0, t_0) \) of the point \( (x_0, t_0) \), \( x_0 \in \mathbb{R}^n, t_0 \in \mathbb{R} \), as follows:

\[
D_+(x_0, t_0) := \{(x, t) \in \mathbb{R}^{n+1}; |x - x_0| \leq \pm (e^{t_0} - e^{-t})\}
\]

We define also the \textit{characteristic conoid} (ray cone) by \( C_\pm(x_0, t_0) := \{(x, t) \in \mathbb{R}^{n+1}; |x - x_0| = \pm (e^{t_0} - e^{-t})\} \). Then, we define for \( (x_0, t_0) \in \mathbb{R}^n \times \mathbb{R} \) the function

\[
E(x, t; x_0, t_0; M) := 4^{-M} e^{M(t_0 + t)} \left( (e^{-t} + e^{-t_0})^2 - (x - x_0)^2 \right)^{-1/2 + M} \times F\left( \frac{1}{2} - M, \frac{1}{2} - M; 1; \frac{(e^{-t} - e^{-t_0})^2 - (x - x_0)^2}{(e^{-t} + e^{-t_0})^2 - (x - x_0)^2} \right),
\]

\[
E(x, t; x_0, t_0) := E(x, t; x_0, t_0; -iM),
\]

in \( D_+(x_0, t_0) \cup D_-(x_0, t_0) \), where \( F(a, b; c; \zeta) \) is the hypergeometric function. The kernels \( K_0(z, t), K_1(z, t), K_0(z, t; M), \) and \( K_0(z, t; M) \) are defined by

\[
K_0(z, t; M) := 4^{-M} e^{Mt} ((1 + e^{-t})^2 - z^2)^M \frac{1}{[(1 - e^{-t})^2 - z^2]\sqrt{(1 + e^{-t})^2 - z^2}} \times \left[ (e^{-t} - 1 + M(e^{-2t} - 1 - z^2)) F\left( \frac{1}{2} - M, \frac{1}{2} - M; 1; \frac{(1 - e^{-t})^2 - z^2}{(1 + e^{-t})^2 - z^2} \right) \right. \\
+ (1 - e^{-2t} + z^2) \left( \frac{1}{2} + M \right) F\left( -\frac{1}{2} - M, \frac{1}{2} - M; 1; \frac{(1 - e^{-t})^2 - z^2}{(1 + e^{-t})^2 - z^2} \right) \right],
\]

\[
K_1(z, t; M) := 4^{-M} e^{ Mt} ((1 + e^{-t})^2 - z^2)^{-1/2 + M} \times F\left( \frac{1}{2} - M, \frac{1}{2} - M; 1; \frac{(1 - e^{-t})^2 - z^2}{(1 + e^{-t})^2 - z^2} \right).
\]

For \([3.3]\) we set \( M = \sqrt{m^2 - n^2/4} \).

For the case of small mass, \( m^2 \leq n^2/4 \), a similar formula is obtained in \([53]\). More precisely, if we denote \( M = \sqrt{n^2/4 - m^2} \), and define

\[
K_0(z, t) := K_0(z, t; -iM), \quad K_1(z, t) := K_1(z, t; -iM).
\]

Then for the solution \( \Phi \) of the Cauchy problem \([3.2]\), there is a representation

\[
\Phi(x, t) = e^{-\frac{n-1}{2}t} \nu_0(x, \phi(t)) + e^{-\frac{n}{2}t} \int_0^1 \nu_0(x, \phi(t)s)(2K_0(\phi(t)s, t; M) + nK_1(\phi(t)s, t; M))\phi(t) ds + 2e^{-\frac{n}{2}t} \int_0^1 \nu_1(x, \phi(t)s)K_1(\phi(t)s, t; M)\phi(t) ds, \quad x \in \mathbb{R}^n, \ t > 0.
\]
According to [54], the fundamental solutions (the retarded and advanced Green functions) of the operator have similar representations.

Suppose now that we are looking for the simplest possible kernels $K_0(z,t;M)$ and $K_1(z,t;M)$ of the integral transforms. Surprisingly that perspective shades a light on the quantum field theory in the de Sitter universe and reveals a new unexpected link between the Higuchi bound [28] and the Huygens’ principle.

Indeed, in the hierarchy of the hypergeometric functions the simplest one is the constant, $F(0,0;1;\zeta) = 1$. The parameter $M$ leading to such function $F(0,0;1;\zeta) = 1$ is $M = \frac{1}{2}$, and, consequently, $m^2 = \frac{n^2 - 1}{4}$.

The next simple non-constant function of that hierarchy is $F(-1,-1;1;\zeta) = 1 + \zeta$. The parameter $M$ leading to such function is $M = \frac{3}{2}$, and, consequently, $m^2 = \frac{n^2 - 9}{4}$.

In the case of $n = 3$ the only real masses, which simplify the kernels, that is, make $F$ polynomial, are $m = \sqrt{2}$ and $m = 0$. These are exactly the endpoints of the interval $(0, \sqrt{2})$ that, in the case of $n = 3$, is known in the quantum field theory as the so-called Higuchi bound [28]. In fact, the interval $(0, \sqrt{2})$ plays a significant role in the linear quantum field theory [28], in a completely different context than the explicit representation of the solutions of the Cauchy problem. More precisely, the Higuchi bound [28] [11] [11] [12], arises in the quantization of free massive fields with the spin-2 in the de Sitter spacetime with $n = 3$. It is the forbidden mass range for spin-2 field theory in de Sitter spacetime because of the appearance of negative norm states. Thus, the point $m = \sqrt{2}$ is exceptional for the quantum fields theory in the de Sitter spacetime. In particular, for massive spin-2 fields, it is known [11] [28] that the norm of the helicity zero mode changes sign across the line $m^2 = 2$. The region $m^2 < 2$ is therefore unitarily forbidden. It is noted in [11] that all canonically normalized helicity $-0, \pm 1, \pm 2$ modes of massive graviton on the de Sitter universe satisfy the Klein-Gordon equation for a massive scalar field with the same effective mass.

In the case of $n \in \mathbb{N}$ we obtain for the physical mass several points, $m^2 = \frac{n^2}{4} - \left(\frac{1}{2} + k\right)^2$, $k = 0, 1, \ldots, \left[\frac{n-1}{2}\right]$, which make $F$ polynomial. We will call these points the knot points. For $n = 1$ only the massless field $m = 0$ has a knot point.

We state below that the largest knot point, and, in particular, the right endpoint of the Higuchi bound if $n = 3$, is the only value of the physical mass $m$, such that the solutions of the equation

$$\Phi_{tt} + n\Phi_t - e^{-2t} \Delta \Phi + m^2 \Phi = 0,$$

(3.11)

obey the Huygens’ principle, whenever the wave equation in the Minkowski spacetime does, that is, $n \geq 3$ is an odd number.
In fact, for \( m = \sqrt{n^2 - 1}/2 \) the equation by means of the transformations of coordinates and the unknown function can be reduced to the wave equation on the Minkowski spacetime. It is easily seen that the value \( m = \sqrt{n^2 - 1}/2 \) is also the conformal value, i.e., the value under which a conformal change of the metric turns the problem into one on a compact in time cylinder.

Even if the equation is not Huygensian (not tail-free for some admissible data), one might nevertheless be interested in data that produce tail-free solution.

**Definition 3.1** [58] We say that the equation obeys the incomplete Huygens’ principle with respect to the first initial datum, if the solution with the second datum \( \varphi_1 = 0 \) vanishes at all points which cannot be reached from the support of initial data by a null geodesic.

**Theorem 3.2** [58] Suppose that equation (3.11) does not obey the Huygens’ principle. Then, it obeys the incomplete Huygens’ principle with respect to the first initial datum, if and only if the equation is massless, \( m = 0 \), and either \( n = 1 \) or \( n = 3 \).

By combining Theorem 3.1 and Theorem 3.2 we arrive at the following interesting conclusion.

**Corollary 3.2** [58] Assume that the equations \( \Phi_{tt} + n\Phi_t - c_1^2 e^{-2t} \Delta \Phi + m_1^2 \Phi = 0 \) and \( \Phi_{tt} + n\Phi_t - c_2^2 e^{-2t} \Delta \Phi + m_2^2 \Phi = 0 \), where \( c_1, c_2 \) are positive numbers, obey the incomplete Huygens’ principle. Then they describe the fields with different mass, \( m_1 \neq m_2 \), (in fact, \( \sqrt{n^2 - 1}/2 \) and \( 0 \)) if and only if the dimension \( n \) is 3.

Thus, in the de Sitter spacetime the existence of two different scalar fields (in fact, with \( m = 0 \) and \( m^2 = (n^2 - 1)/4 \), which obey incomplete Huygens’ principle, is equivalent to the condition \( n = 3 \). The dimension \( n = 3 \) of the last corollary agrees with the experimental data.

### 4 Global Solutions of Semilinear System of Klein-Gordon Equations in de Sitter Spacetime

In this section we discuss global existence of small data solutions of the Cauchy problem for the semilinear system of Klein-Gordon equations in the de Sitter spacetime. Unlike the same problem in the Minkowski spacetime, we have no restriction on the order of the nonlinearity and the structure of the nonlinear term, provided that the spectrum of the mass matrix of the fields, which describes the linear interactions of the fields, is in the positive half-line and has no intersection with some open bounded interval.

A large amount of work has been devoted to the Cauchy problem for the scalar semilinear Klein-Gordon equation in the Minkowski spacetime. The existence of global weak solutions has been obtained by Jörgens [31], Segal [37, 38], Pecher [35], Brenner [6], Strauss [42], Ginibre and Velo [22, 23] for the equation

\[
u_{tt} - \Delta u + m^2 u = |u|^\alpha u.
\]

For global solvability, the exact relation between \( n \) and \( \alpha > 0 \) was finally established. More precisely, consider the Cauchy problem for the nonlinear Klein-Gordon equation

\[
u_{tt} - \Delta u = -V'(u),
\]
where $V' = V'(u)$ is a nonlinear function, a typical form of which is the sum of two powers

$$V'(u) = \lambda_0 u + \lambda |u|^\alpha u$$

with $\alpha \geq 0$ and $\lambda \geq 0$. For this equation, a conservation of energy is valid. For finite energy solutions scaling arguments suggest the assumption $\alpha < 4/(n-1)$. In [23] the existence and uniqueness of strong global solutions in the energy space $H_{(1)} \oplus L^2$ are proved for arbitrary space dimension $n$ under assumptions on $V'$ that cover the case of a sum of powers $\lambda |u|^\alpha u$ with $0 \leq \alpha < 4/(n-1)$, $n \geq 2$, and $\lambda > 0$ for the highest $\alpha$. Some of the results can be extended to the case $\alpha = 4/(n-1)$ (see, e.g. [22, 23 Sec.4]).

In this section we consider the model of interacting fields, which can be described by the system of Klein-Gordon equations with different masses, containing interaction via mass matrix and the semilinear term. The model obeys the following system

$$\Phi_{tt} + n \Phi_t - e^{-2t} \Delta \Phi + \mathbf{M}\Phi = F(\Phi). \quad (4.1)$$

Here $F$ is a vector-valued function of the vector-valued function $\Phi$. We assume that the mass matrix $\mathbf{M}$ is real-valued, diagonalizable, and it has eigenvalues $m_1^2, \ldots, m_l^2$, $i = 1, 2, \ldots, l$. By the similarity transformation with the real-valued matrix $\mathbf{M}$ that is diagonalizable. Therefore, we use the change of the unknown function as follows:

$$\Psi = e^{\frac{\Phi}{2}} \mathbf{O} \Phi, \quad \Phi = e^{-\frac{\Phi}{2}} \mathbf{O}^{-1} \Psi.$$ 

Then the system $(4.1)$ takes the form

$$\Psi_{tt} - e^{-2t} \Delta \Psi + \mathcal{M}^2 \Psi = e^{\frac{\Phi}{2}} \mathbf{O}(e^{-\frac{\Phi}{2}} \mathbf{O}^{-1} \Psi), \quad (4.2)$$

where the diagonal matrix $\mathcal{M}$, with nonnegative real part $\Re \mathcal{M} \geq 0$, is

$$\mathcal{M}^2 := \mathbf{O}\mathbf{M}\mathbf{O}^{-1} - \frac{n^2}{4} \mathbf{I}, \quad \mathbf{I} \text{ is the identity matrix.}$$

The matrix $\mathcal{M}^2$ will be called the curved mass matrix of the particles, which is also sometimes referred to as the effective mass matrix. It is convenient to use the diagonal matrix $\mathbf{M} = \text{diag}(m_1^2 - \frac{n^2}{4})$. We distinguish the following three cases: the case of large mass matrix $\mathbf{M}$ that is $\mathcal{M}^2 \geq 0$ ($m_i^2 \geq \frac{n^2}{4}$, $i = 1, 2, \ldots, l$); the case of dimensional mass matrix $\mathbf{M}$ that is $\mathcal{M}^2 = 0$ ($m_i^2 = \frac{n^2}{4}$, $i = 1, 2, \ldots, l$); and the case of small mass matrix $\mathbf{M}$ that is $\mathcal{M}^2 < 0$ ($m_i^2 < \frac{n^2}{4}$, $i = 1, 2, \ldots, l$). We also call the mass matrix $\mathbf{M}$ critical if $\mathcal{M}^2 = -\frac{1}{4} \mathbf{I}$. They lead to three different equations: the Klein-Gordon equation with the real curved mass matrix $\mathcal{M}$,

$$\Psi_{tt} - e^{-2t} \Delta \Psi + \mathcal{M}^2 \Psi = e^{\frac{\Phi}{2}} \mathbf{O}(e^{-\frac{\Phi}{2}} \mathbf{O}^{-1} \Psi);$$

the wave equation with the zero curved mass matrix

$$\Psi_{tt} - e^{-2t} \Delta \Psi = e^{\frac{\Phi}{2}} \mathbf{O}(e^{-\frac{\Phi}{2}} \mathbf{O}^{-1} \Psi);$$

and the Klein-Gordon equation with the imaginary curved mass matrix $\mathcal{M}$,

$$\Psi_{tt} - e^{-2t} \Delta \Psi - \mathcal{M}^2 \Psi = e^{\frac{\Phi}{2}} \mathbf{O}(e^{-\frac{\Phi}{2}} \mathbf{O}^{-1} \Psi).$$
Let $H_s(\mathbb{R}^n)$ be the Sobolev space. We use the notation $\| \cdot \|_{H_s(\mathbb{R}^n)}$ for both the norm of a vector valued function and for the norm of its components. To estimate the nonlinear term $F(\Phi)$ we use the following Lipschitz condition:

**Condition (L).** The function $F$ is said to be Lipschitz continuous in the space $H_s(\mathbb{R}^n)$ if there are constants $\alpha \geq 0$ and $C$ such that

$$
\| F(\Phi_1) - F(\Phi_2) \|_{H_s(\mathbb{R}^n)} \leq C \| \Phi_1 - \Phi_2 \|_{H_s(\mathbb{R}^n)} (\| \Phi_1 \|_{H_s(\mathbb{R}^n)} + \| \Phi_2 \|_{H_s(\mathbb{R}^n)})
$$

for all $\Phi_1, \Phi_2 \in H_s(\mathbb{R}^n)$. 

If $s > n/2$, then any polynomial is Lipschitz continuous in the space $H_s(\mathbb{R}^n)$. For more examples of the Lipschitz continuous in the space $H_s(\mathbb{R}^n)$ functions with low $s$ see, for example, [35], [39]. Define the complete metric space

$$X(R, s, \gamma) := \{ \Phi \in C([0, \infty); H_s(\mathbb{R}^n)) \mid \| \Phi \|_X := \sup_{t \in [0, \infty)} e^{\gamma t} \| \Phi(x, t) \|_{H_s(\mathbb{R}^n)} \leq R \}$$

with the metric

$$d(\Phi_1, \Phi_2) := \sup_{t \in [0, \infty)} e^{\gamma t} \| \Phi_1(x, t) - \Phi_2(x, t) \|_{H_s(\mathbb{R}^n)}.$$

The first result of this section is the following theorem.

**Theorem 4.1** [57] Assume that the nonlinear term $F(\Phi)$ is Lipschitz continuous in the space $H_s(\mathbb{R}^n)$, $s > n/2 \geq 1$, $\alpha > 0$, and $F(0) = 0$. Assume also that the system has a large mass matrix. Then, there exists $\varepsilon_0 > 0$ such that, for every given vector-valued functions $\varphi_0, \varphi_1 \in H_s(\mathbb{R}^n)$, such that

$$\| \varphi_0 \|_{H_s(\mathbb{R}^n)} + \| \varphi_1 \|_{H_s(\mathbb{R}^n)} \leq \varepsilon, \quad \varepsilon < \varepsilon_0,$$

there exists a global solution $\Phi \in C^1([0, \infty); H_s(\mathbb{R}^n))$ of the Cauchy problem

$$\begin{align*}
\Phi_{tt} + n\Phi_t - e^{-2t} \Delta \Phi + M\Phi &= F(\Phi), \\
\Phi(x, 0) &= \varphi_0(x), \quad \Phi_t(x, 0) = \varphi_1(x).
\end{align*}$$

(4.3)

(4.4)

The solution $\Phi(x, t)$ belongs to the space $X(2\varepsilon, s, 0)$, that is,

$$\sup_{t \in [0, \infty)} \| \Phi(x, t) \|_{H_s(\mathbb{R}^n)} < 2\varepsilon.$$

For the scalar equation this theorem implies Theorem 0.1 [55]. In fact, for the scalar equation if $F(\Phi) = \pm |\Phi|^\alpha \Phi$ or $F(\Phi) = \pm |\Phi|^{\alpha+1}$, then, according to Theorem 0.1 [55], the small data Cauchy problem is globally solvable for every $\alpha \in (0, \infty)$ if $m \in (0, \sqrt{n^2 - 1/2}) \cup [n/2, \infty)$ and the condition (L) is fulfilled.

**Conjecture 4.1** [57] The open interval $(\sqrt{n^2 - 1/2}, n/2)$ is a forbidden physical mass interval for the small data global solvability of the Cauchy problem for all $\alpha \in (0, \infty)$. 

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Consider the particular case of the scalar equation with the spatial dimension $n = 3$. In this case the interval $(\sqrt{n^2 - 1}/2, n/2)$ for the physical mass is reduced to $(\sqrt{2}, 3/2)$. The interval $(0, \sqrt{2})$ is the Higuchi bound. This is why we pay special attention to the system of equations with the mass matrix $\mathbf{M}$ which is orthogonally similar to the matrix $\frac{n^2 - 1}{4} \mathbf{I}$. We call such mass matrix $\mathbf{M}$ critical. We also call the mass matrix $\mathbf{M}$ semi-critical mass matrix if the spectrum $\sigma(\mathbf{M})$ of the mass matrix $\mathbf{M}$ is a subset of $(0, (n^2 - 1)/4]$.

For the system with the semi-critical mass matrix $\mathbf{M}$ we prove the global existence, which is not known in the critical case even for the scalar equation.

**Theorem 4.2** \cite{57} Assume that the nonlinear term $F(\Phi)$ is Lipschitz continuous in the space $H_{(s)}(\mathbb{R}^n)$, $s > n/2 \geq 1$, $\alpha > 0$, and $F(0) = 0$. Assume also that the mass matrix $\mathbf{M}$ is semi-critical, that is $\sigma(\mathbf{M}) \subset (0, (n^2 - 1)/4]$. Then, there exists $\varepsilon_0 > 0$ such that, for every given vector-valued functions $\varphi_0, \varphi_1 \in H_{(s)}(\mathbb{R}^n)$, such that

$$\|\varphi_0\|_{H_{(s)}(\mathbb{R}^n)} + \|\varphi_1\|_{H_{(s)}(\mathbb{R}^n)} \leq \varepsilon, \quad \varepsilon < \varepsilon_0,$$

there exists a global solution $\Phi \in C^1([0, \infty); H_{(s)}(\mathbb{R}^n))$ of the Cauchy problem (4.3), (4.4). The solution $\Phi(x, t)$ belongs to the space $X(2\varepsilon, s, \gamma)$, where

$$\gamma < \frac{1}{\alpha + 1} \left(\frac{n}{2} - \max \left\{\sqrt{\frac{n^2}{4} - \lambda}; \lambda \in \sigma(\mathbf{M})\right\}\right),$$

that is,

$$\sup_{t \in [0, \infty)} e^{\gamma t} \|\Phi(x, t)\|_{H_{(s)}(\mathbb{R}^n)} < 2\varepsilon.$$

Baskin \cite{3} discussed small data global solutions for the scalar Klein-Gordon equation on asymptotically de Sitter spaces, which is a compact manifold with boundary. More precisely, in \cite{3} the Cauchy problem is considered for the semilinear equation $\square_g u + m^2 u = f(u), u(x, t_0) = \varphi_0(x) \in H_{(1)}(\mathbb{R}^n), u_t(x, t_0) = \varphi_1(x) \in L^2(\mathbb{R}^n)$, where mass is large, $m^2 > n^2/4$. In Theorem 1.3 \cite{3} the existence of the global solution for small energy data is stated. (For references on the asymptotically de Sitter spaces, see \cite{2, 44}.)

## 5 The Scalar Equation. Case of Large Mass

We extract a linear part of the system (4.2) as an initial model that must be treated first. That linear system is diagonal, which allows us to restrict ourselves to one scalar equation

$$u_{tt} - e^{-2t} \Delta u + \mathcal{M}^2 u = f,$$  \hspace{1cm} (5.1)

where $\mathcal{M}$ is a non-negative number throughout this section. The equation (5.1) is strictly hyperbolic. That implies the well-posedness of the Cauchy problem for (5.1) in several function spaces. The coefficients of the equation are analytic functions and, consequently, Holmgren’s theorem implies local uniqueness in the space of distributions. Moreover, the speed of propagation is equal to $e^{-t}$ for every $t \in \mathbb{R}$. The second-order strictly hyperbolic equation (5.1) possesses two fundamental solutions resolving the Cauchy problem. They can be written in terms of Fourier integral operators \cite{29}, which give a complete description of the wave front sets of the solutions. The distance $e^{-t/|\xi|}$ between two characteristic roots of the equation (5.1) tends to zero as $t \to +\infty$. Thus, the operator is not uniformly strictly hyperbolic. The finite
integrability of the characteristic roots leads to the existence of a so-called *horizon* for that equation. The equation (5.1) is neither Lorentz invariant nor invariant with respect to scaling and that brings additional difficulties.

**Lp – Lq estimates for equation with source.** We consider the equation with \( n \geq 2 \). The solution \( u = u(x, t) \) to the Cauchy problem

\[
u_{tt} - e^{-2t} \Delta u + M^2 u = f, \quad u(x, 0) = 0, \quad u_t(x, 0) = 0,
\]

with \( f \in C^\infty(\mathbb{R}^{n+1}) \) and with vanishing initial data is given by the next expression

\[
u(x, t) = 2 \int_0^t db \int_0^{e^{-b} - e^{-t}} dr \, v_f(x, r; b) E(r, t; 0, b),
\]

where the function \( v_f(x, t; b) \) is a solution to the Cauchy problem

\[
v_{tt} - \Delta v = 0, \quad v(x, 0; b) = f(x, b), \quad v_t(x, 0; b) = 0.
\]

Thus, for the solution \( \Phi \) of the equation

\[
\Phi_{tt} + n \Phi_t - e^{-2t} \Delta \Phi + m^2 \Phi = f,
\]

due to the relation \( u = e^{\frac{n}{2}t} \Phi \), we obtain

\[
\Phi(x, t) = 2 e^{-\frac{n}{2}t} \int_0^t db \int_0^{e^{-b} - e^{-t}} dr \, e^{\frac{n}{2}b} v_f(x, r; b) E(r, t; 0, b).
\]

For the solution \( u = u(x, t) \) of the Cauchy problem (5.2) according to Corollary 9.3 [52], one has estimate

\[
\|(-\Delta)^{-s} u(x, t)\|_{L^q(\mathbb{R}^n)} \leq C_M \int_0^t \|f(x, b)\|_{L^p(\mathbb{R}^n)} e^b \left( e^{b} - e^{-t} \right)^{1+2s-n\left(\frac{1}{p} - \frac{1}{q}\right)} (1 + t - b)^{1-\text{sgn}M} db,
\]

provided that \( 1 < p \leq 2, \quad \frac{1}{p} + \frac{1}{q} = 1, \quad \frac{1}{2}(n + 1) \left( \frac{1}{p} - \frac{1}{q} \right) \leq 2s \leq n \left( \frac{1}{p} - \frac{1}{q} \right) < 2s + 1 \). Thus, for the solution \( \Phi \) (5.5) of the equation (5.4), we obtain

\[
\|(-\Delta)^{-s} \Phi(x, t)\|_{L^q(\mathbb{R}^n)} \leq C_M e^{-\frac{n}{2}t} \times \int_0^t e^{\frac{n}{2}b} \|f(x, b)\|_{L^p(\mathbb{R}^n)} e^b \left( e^{b} - e^{-t} \right)^{1+2s-n\left(\frac{1}{p} - \frac{1}{q}\right)} (1 + t - b)^{1-\text{sgn}M} db.
\]

In particular, for \( s = 0 \) and \( p = q = 2 \), we have

\[
\|\Phi(x, t)\|_{L^2(\mathbb{R}^n)} \leq C_M e^{-\frac{n}{2}t} \int_0^t e^{\frac{n}{2}b} \|f(x, b)\|_{L^2(\mathbb{R}^n)} (1 + t - b)^{1-\text{sgn}M} db.
\]

Here the rates of exponential factors are independent of \( M \) and, consequently, of the mass \( m \).
**$L^p - L^q$ estimates for equations without source.** According to Theorem 10.1 [52] the solution $u = u(x,t)$ of the Cauchy problem

$$u_{tt} - e^{-2t} \triangle u + M^2 u = 0, \quad u(x,0) = \varphi_0(x), \quad u_t(x,0) = \varphi_1(x),$$

satisfies the following $L^p - L^q$ estimate

$$\| (-\triangle)^s u(x,t) \|_{L^q(\mathbb{R}^n)} \leq C_M (1 + t)^{1 - sgnM} (1 - e^{-t})^{2s - n\left(\frac{1}{p} - \frac{1}{q}\right)} \times \left\{ e^{\frac{t}{2}} \| \varphi_0 \|_{L^p(\mathbb{R}^n)} + (1 - e^{-t}) \| \varphi_1 \|_{L^p(\mathbb{R}^n)} \right\}$$

for all $t \in (0, \infty)$, provided that $1 < p \leq 2$, $\frac{1}{p} + \frac{1}{q} = 1$, $\frac{1}{2}(n + 1) \left( \frac{1}{p} - \frac{1}{q} \right) \leq 2s \leq n \left( \frac{1}{p} - \frac{1}{q} \right) < 2s + 1$. In particular, for large $t$ we obtain the following *no decay* estimate

$$\| (-\triangle)^s u(x,t) \|_{L^q(\mathbb{R}^n)} \leq C_M (1 + t)^{1 - sgnM} \left\{ e^{\frac{t}{2}} \| \varphi_0 \|_{L^p(\mathbb{R}^n)} + \| \varphi_1 \|_{L^p(\mathbb{R}^n)} \right\}.$$ 

Thus, for the solution $\Phi$ of the Cauchy problem [32], due to the relation $u = e^{\frac{t}{2}t} \Phi$, we obtain the decay estimate

$$\| (-\triangle)^s \Phi(x,t) \|_{L^q(\mathbb{R}^n)} \leq C_M e^{-\frac{t}{2}t} (1 + t)^{1 - sgnM} (1 - e^{-t})^{2s - n\left(\frac{1}{p} - \frac{1}{q}\right)} \times \left\{ e^{\frac{t}{2}} \| \varphi_0 \|_{L^p(\mathbb{R}^n)} + (1 - e^{-t}) \| \varphi_1 \|_{L^p(\mathbb{R}^n)} \right\}$$

for all $t > 0$.

### 6 The Scalar Equation. Imaginary Curved Mass

In this section we consider the linear part of the scalar equation

$$u_{tt} - e^{-2t} \triangle u - M^2 u = -e^{-\frac{t}{2}t} V'(e^{-\frac{t}{2}t} u),$$

with $M \geq 0$. The equation (6.1) covers two important cases. The first one is the Higgs boson equation, which has $V'(\phi) = \lambda \phi^3$ and $M^2 = \mu m^2 + n^2/4$ with $\lambda > 0$ and $\mu > 0$, while $n = 3$. The second case is the case of the small physical mass, that is $0 \leq m \leq \frac{n^2}{2}$. For the last case $M^2 = \frac{n^2}{4} - m^2$. The solution $u = u(x,t)$ to the Cauchy problem

$$u_{tt} - e^{-2t} \triangle u - M^2 u = f, \quad u(x,0) = 0, \quad u_t(x,0) = 0,$$  

with $f \in C^\infty(\mathbb{R}^{n+1})$ and with vanishing initial data is given in [53] by the next expression

$$u(x,t) = 2 \int_0^t db \int_0^{e^{-b}-e^{-t}} dr \ v(x,r;b) E(r,t;0,b;M),$$

where the function $v(x,t;b)$ is a solution to the Cauchy problem for the wave equation [53].

The solution $u = u(x,t)$ to the Cauchy problem

$$u_{tt} - e^{-2t} \triangle u - M^2 u = 0, \quad u(x,0) = \varphi_0(x), \quad u_t(x,0) = \varphi_1(x),$$
with $\varphi_0, \varphi_1 \in C^\infty_0(\mathbb{R}^n)$, $n \geq 2$, can be represented (see [53]) as follows:

$$u(x, t) = e^{\frac{t}{2}} v_{\varphi_0}(x, \phi(0)) + 2 \int_0^1 v_{\varphi_0}(x, \phi(t)s) K_0(\phi(t)s, t; M) \phi(t) \, ds$$

$$+ 2 \int_0^1 v_{\varphi_1}(x, \phi(t)s) K_1(\phi(t)s, t; M) \phi(t) \, ds, \quad x \in \mathbb{R}^n, \ t > 0,$$

where $\phi(t) := 1 - e^{-t}$. Here, for $\varphi \in C^\infty_0(\mathbb{R}^n)$ and for $x \in \mathbb{R}^n$, the function $v_{\varphi}(x, t)$ is the solution of the Cauchy problem (2.1). Thus, for the solution $\Phi$ of the Cauchy problem

$$\Phi_t + n \Phi_t - e^{-2t} \Delta \Phi + m^2 \Phi = f, \quad \Phi(x, 0) = 0, \ \Phi_t(x, 0) = 0,$$

due to the relation $u = e^{\frac{t}{2}} \Phi$, we obtain with $f \in C^\infty(\mathbb{R}^{n+1})$ and with vanishing initial data the next expression

$$\Phi(x, t) = 2 e^{-\frac{n+1}{2}t} \int_0^t db \int_0^{e^{-b} - e^{-t}} dr e^{\frac{n+1}{2}b} v_f(x, r; b) E(r, t; 0, b; M), \quad (6.3)$$

where the function $v_f(x, t; b)$ is a solution to the Cauchy problem for the wave equation (5.3). Thus, for the solution $\Phi$ of the Cauchy problem (3.2), due to the relation $u = e^{\frac{t}{2}} \Phi$, we obtain (3.10). In fact, the representation formulas of this section have been used in [56] to establish sign-changing properties of the solutions of the Higgs boson equation.

**The critical case of $m^2 = (n^2 - 1)/4$.** In this case we have

$$E(x, t; x_0, 0; \frac{1}{2}) = \frac{1}{2} e^{\frac{1}{2}(t_0 + t)}, \quad E(z, t; 0; b; \frac{1}{2}) = \frac{1}{2} e^{\frac{1}{2}(b + t)};$$

while

$$K_0(z, t; \frac{1}{2}) = -\frac{1}{2} e^{\frac{1}{2}t}, \quad K_1(z, t; \frac{1}{2}) = \frac{1}{2} e^{\frac{1}{2}t}.$$ 

For the solution (5.5) of the equation (5.3) with the source term it follows

$$\Phi(x, t) = e^{-\frac{n+1}{2}t} \int_0^t e^{\frac{n+1}{2}b} v_f(x, e^{-b} - e^{-t}; b) \, db.$$

where $v(x, r; b)$ is defined by (5.3). We denote by $V_f(x, t; b)$ the solution of the problem $V_t - \Delta V = 0, \ V(x, 0) = 0, \ V_t(x, 0) = f(x, b)$. Further, for the solution $\Phi$ (5.10) of the equation without source term we have

$$\Phi(x, t) = e^{-\frac{n+1}{2}t} \left( \frac{\partial V_{\varphi_0}}{\partial t} \right) (x, 1 - e^{-t})$$

$$+ \frac{n-1}{2} e^{-\frac{n+1}{2}t} V_{\varphi_0}(x, 1 - e^{-t}) + e^{-\frac{n+1}{2}t} V_{\varphi_1}(x, 1 - e^{-t}),$$

$x \in \mathbb{R}^n, \ t > 0$, where we denote by $V_{\varphi}$ the solution of the problem $V_t - \Delta V = 0, \ V(x, 0) = 0, \ V_t(x, 0) = \varphi(x)$. Thus, in particular, we arrive at the next theorem.

**Theorem 6.1** [57] The solutions of the equation $\Phi_t + n \Phi_t - e^{-2t} \Delta \Phi + M \Phi = 0$, obey the strong Huygens’ Principle, if and only if $n \geq 3$ is an odd number and the mass matrix $M$ is the diagonal matrix $\frac{n^2 + 1}{2} I$.
Theorem 6.2 Suppose that \( m^2 = (n^2 - 1)/4 \). If \( \varphi_0 = \varphi_1 = 0 \) and \( \frac{1}{2} (n + 1) (\frac{1}{p} - \frac{1}{q}) - 1 \leq 2s \leq n (\frac{1}{p} - \frac{1}{q}) \), then for the solution \( \Phi = \Phi(x, t) \) of the equation \((5.4)\) the following estimate holds
\[
\| (\Delta)^{-s} \Phi(x, t) \|_{L^q(\mathbb{R}^n)} \leq C e^{-\frac{n+1}{2} t} \int_0^t e^{\frac{n+1}{2} s} (e^{-b} - e^{-t})^{1 + 2s - n (\frac{1}{p} - \frac{1}{q})} \| f(x, b) \|_{L^p(\mathbb{R}^n)} db, \quad t > 0.
\]

For the solution \( \Phi = \Phi(x, t) \) of the Cauchy problem \((3.2)\): if \( f \equiv 0 \), \( \varphi_0 = 0 \), and \( \frac{1}{2} (n + 1) (\frac{1}{p} - \frac{1}{q}) - 1 \leq 2s \leq n (\frac{1}{p} - \frac{1}{q}) \), then
\[
\| (\Delta)^{-s} \Phi(x, t) \|_{L^q(\mathbb{R}^n)} \leq C e^{-\frac{n+1}{2} t} (1 - e^{-t})^{1 + 2s - n (\frac{1}{p} - \frac{1}{q})} \| \varphi_1 \|_{L^p(\mathbb{R}^n)}, \quad t > 0,
\]
while if \( f \equiv 0 \), \( \varphi_1 = 1 \), and \( \frac{1}{2} (n + 1) (\frac{1}{p} - \frac{1}{q}) \leq 2s \leq n (\frac{1}{p} - \frac{1}{q}) \), then
\[
\| (\Delta)^{-s} \Phi(x, t) \|_{L^q(\mathbb{R}^n)} \leq C e^{-\frac{n+1}{2} t} (1 - e^{-t})^{2s - n (\frac{1}{p} - \frac{1}{q})} \| \varphi_0 \|_{L^p(\mathbb{R}^n)}, \quad t > 0.
\]

To complete the list of the \( L^p - L^q \) estimates we quote below results (Theorems 2.2, 2.6 from [55]), which are applicable to the scalar equation with noncritical mass. The solution \( \Phi = \Phi(x, t) \) of the Cauchy problem
\[
\dot{\Phi} + n \Phi_t - e^{-2t} \Delta \Phi \pm m^2 \Phi = 0, \quad \Phi(x, 0) = \varphi_0(x), \quad \Phi_t(x, 0) = \varphi_1(x),
\]
with either \( M = \sqrt{n^2 - m^2} \) and \( m < \sqrt{n^2 - 1}/2 \) for the case of “plus”, or \( M = \sqrt{n^2 + m^2} \) for the case of “minus”, satisfies the following \( L^p - L^q \) estimate
\[
\| (\Delta)^{-s} \Phi(x, t) \|_{L^q(\mathbb{R}^n)} \leq C_{M, n, p, q, s} e^{-\frac{n+1}{2} t} e(1 - e^{-t})^{2s - n (\frac{1}{p} - \frac{1}{q})} \| \varphi_0 \|_{L^p(\mathbb{R}^n)} + (1 - e^{-t}) \| \varphi_1 \|_{L^p(\mathbb{R}^n)},
\]
for all \( t \in (0, \infty) \), provided that \( 1 < p \leq 2, \frac{1}{p} + \frac{1}{q} = 1, \frac{1}{2} (n + 1) (\frac{1}{p} - \frac{1}{q}) \leq 2s \leq n (\frac{1}{p} - \frac{1}{q}) < 2s + 1 \).

Further, let \( \Phi = \Phi(x, t) \) be the solution of the Cauchy problem
\[
\dot{\Phi} + n \Phi_t - e^{-2t} \Delta \Phi \pm m^2 \Phi = f, \quad \Phi(x, 0) = 0, \quad \Phi_t(x, 0) = 0, \quad \tag{6.4}
\]
with either \( M = \sqrt{n^2 - m^2} \) and \( m < \sqrt{n^2 - 1}/2 \) for the case of “plus”, or \( M = \sqrt{n^2 + m^2} \) for the case of “minus”. Then \( \Phi = \Phi(x, t) \) satisfies the following \( L^p - L^q \) estimate:
\[
\| (\Delta)^{-s} \Phi(x, t) \|_{L^q(\mathbb{R}^n)} \leq C_M e^{-\frac{Mt}{2} e^{-\frac{q}{2} t} e^{-2s - n (\frac{1}{p} - \frac{1}{q})} \| f(x, b) \|_{L^p(\mathbb{R}^n)} db, \]
as well as
\[
\| (\Delta)^{-s} \Phi(x, t) \|_{L^q(\mathbb{R}^n)} \leq C_M e^{-\frac{M}{2} t} e^{-b (2s - n (\frac{1}{p} - \frac{1}{q}))} \| f(x, b) \|_{L^p(\mathbb{R}^n)} db
\]
for all \( t \in (0, \infty) \), and for the above written range of the parameters \( p, q, s \).
7 Global Existence. Small Data Solutions

The Cauchy problem (6.2) for the scalar equation was studied in [53]. For \( F(\Phi) = c|\Phi|^{\alpha+1}, c \neq 0 \), Theorem 1.1 [53], implies nonexistence of a global solution even for arbitrary small initial data \( \varphi_0(x) \) and \( \varphi_1(x) \) under some conditions on \( n, \alpha, \) and \( M \). Theorem 3.1 [55] gives the blow up result for the equation with imaginary physical mass. That theorem shows that instability of the trivial solution occurs in a very strong sense, that is, an arbitrarily small perturbation of the initial data can make the perturbed solution blowing up in finite time. If we allow large initial data, then, according to Theorem 1.2 [53], the concentration of the mass, due to the non-dispersion property of the de Sitter spacetime, leads to the nonexistence of the global solution, which cannot be recovered even by adding an exponentially decaying factor in the nonlinear term.

In this section we are going to study the global existence of solutions for the system of semilinear Klein-Gordon equations. The first step toward such result is to establish the \( L_p - L_q \)-estimates for the equation with source term. For the scalar equation this estimate is proved in [55]. Although we want to prove the global existence for two different cases, for the system with the semi-critical mass matrix and for the system of equations with the large mass matrix, the consideration can be done in the single framework.

We reduce the Cauchy problem to the integral equation. The main tool for such reduction is the fundamental solution for the interacting fields, which can be described by the system of Klein-Gordon equations containing interaction via mass matrix and the semilinear term. The model obeys the system (4.1). By the similarity transformation \( O \) the mass matrix \( M \) can be diagonalized, therefore we use a change of unknown function

\[
\Psi = O\Phi, \quad \Phi = O^{-1}\Psi,
\]

and arrive at

\[
\Psi_{tt} + nH\Psi_t - e^{-2Ht} \triangle \Psi + \tilde{M}\Psi = \tilde{F}(\Psi),
\]

where

\[
\tilde{M} := OM O^{-1} = \text{diag}\{m_1^2, \ldots, m_l^2\}, \quad \tilde{F}(\Psi) := OF(O^{-1}\Psi).
\]

Let us consider the linear diagonal system

\[
\Psi_{tt} + nH\Psi_t - e^{-2Ht} \triangle \Psi + \tilde{M}\Psi = \tilde{f}.
\]

Here \( \tilde{f} \) is a vector-valued function with the components \( f_i, i = 1, \ldots, l \). Then, the solution of the Cauchy problem for the last system with the initial conditions

\[
\Psi(x, 0) = 0, \quad \Psi_t(x, 0) = 0,
\]

is

\[
\Psi(x, t) = 2e^{-\frac{nt}{2}} \int_0^t db \int_0^e e^{-b - e^{-t}} dr e^{\frac{nrb}{2}} \tilde{E}(r, t; 0, b) \tilde{v}(x, r; b),
\]

where the components \( v_i, i = 1, \ldots, l \), of the vector-valued function \( \tilde{v}(x, t; b) \) are solutions to the Cauchy problem for the wave equation

\[
v_{tt} - \triangle v = 0, \quad v(x, 0; b) = f_i(x, b), \quad v_t(x, 0; b) = 0, \quad i = 1, \ldots, l.
\]
The kernel $\vec{E}(r, t; 0, b)$ is a diagonal matrix with the elements $E_i(r, t; 0, b)$, $i = 1, \ldots, l$, which are defined either by (2.3) with corresponding mass terms $m_i$, $i = 1, \ldots, l$, or by (3.4), in accordance with the value of mass $m_i^2 \geq n^2/4$ or $m_i^2 < n^2/4$, respectively. Then, the solution $\Psi$ of the Cauchy problem for the equation

$$
\Psi_{tt} + nH\Psi_t - e^{-2Ht} \Delta \Psi + \tilde{M}\Psi = 0
$$

with the initial conditions

$$
\Psi(x, 0) = \tilde{\psi}_0(x), \quad \Psi_t(x, 0) = \tilde{\psi}_1(x),
$$

with the vector-valued functions $\tilde{\psi}_0, \tilde{\psi}_1 \in C_0^\infty(\mathbb{R}^n)$, $n \geq 2$, can be represented as follows:

$$
\Psi(x, t) = e^{-\frac{n-1}{2}t}\tilde{\psi}_0(x, \phi(t)) + e^{-\frac{2n}{t}}\int_0^t (2\vec{K}_0(\phi(s), t) + n\vec{K}_1(\phi(s), t))\tilde{\psi}_0(x, \phi(s), \phi(t) ds
$$

$$
+ 2e^{-\frac{2n}{t}}\int_0^t \vec{K}_1(\phi(s), t)\tilde{\psi}_1(x, \phi(t), \phi(t) ds, \quad x \in \mathbb{R}^n, \ t > 0,
$$

where $\phi(t) := 1 - e^{-t}$ and the kernels $\vec{K}_0, \vec{K}_1$, are the diagonal matrices with the elements $\vec{K}_0(i, z, t)$, $i = 1, \ldots, l$, and $\vec{K}_1(i, z, t)$, which are defined either by (3.8) and (3.9) with the corresponding mass terms $m_i$, $i = 1, \ldots, l$, or by the diagonal matrices with the elements $\vec{K}_0(i, z, t; M)$, $i = 1, \ldots, l$, and $\vec{K}_1(i, z, t; M)$, which are defined by (3.6) and (3.7), in accordance with the value of mass $m_i^2 \geq n^2/4$ or $m_i^2 < n^2/4$, respectively. Here, for the vector-valued function $\tilde{\psi} \in C_0^\infty(\mathbb{R}^n)$ and for $x \in \mathbb{R}^n$, the vector-valued function $\tilde{v}(x, t)$ is a solution of the Cauchy problem $\tilde{v}_{tt} - \Delta \tilde{v} = 0, \tilde{v}(x, 0) = \tilde{\psi}(x), \tilde{v}_t(x, 0) = 0$. We study the Cauchy problem through the integral equation. To determine that integral equation we appeal to the operator

$$
\vec{G} := \vec{K} \circ \vec{W}E,
$$

where the operator $\vec{W}E$ is defined by (7.1), that is,

$$
\vec{W}E[f](x, t; b) = \tilde{v}(x, t; b),
$$

and the vector-valued function $\tilde{v}(x, t; b)$ is a solution to the Cauchy problem for the wave equation, while $\vec{K}$ is introduced either by (5.5), for the large mass matrix, or by (6.3), for the small mass matrix. Hence,

$$
\vec{G}[f](x, t) = 2e^{-\frac{2n}{t}}\int_0^t db \int_0^{e^{-b}-e^{-t}} dr e^{2b} \vec{E}(r, t; 0, b; M)\vec{W}E[f](x, r; b).
$$

Thus, the Cauchy problem (4.3), (4.4) leads to the following integral equation

$$
\Psi(x, t) = \Psi_0(x, t) + \vec{G}[\tilde{F}(\Psi)](x, t).
$$

(7.2)

Every solution $\Phi = \Phi(x, t)$ to the equation (4.3) generates the function $\Psi = \Psi(x, t)$, which solves the last integral equation with some function $\Psi_0(x, t)$, that is generated by the solution of the Cauchy problem (5.2).
Solvability of the integral equation associated with Klein-Gordon equation. We are going to apply Banach’s fixed-point theorem. In order to estimate the nonlinear term we use the Lipschitz condition ($\mathcal{L}$), which imposes some restrictions on $n$, $\alpha$, $s$. Then we consider the equation (7.2), where the vector-valued function $\Psi_0 \in C([0, \infty); L^q(\mathbb{R}^n))$ is given. The solvability of the integral equation (7.2) depends on the operator $\mathcal{G}$. We start with the case of Sobolev space $H^{(s)}(\mathbb{R}^n)$ with $s > n/2$, which is an algebra. In the next theorem the operator $\mathcal{K}$ is generated by the linear part of the equation (4.3).

**Theorem 7.1** [57] Assume that $F(\Psi)$ is Lipschitz continuous in the space $H^{(s)}(\mathbb{R}^n)$, $s > n/2$, and also that $\alpha > 0$.

(i) Let the spectrum of the mass matrix $M$ be $\{m_1^2, \ldots, m_l^2\} \subset (0, (n^2 - 1)/4]$, and $m = \min\{m_1, m_2, \ldots, m_l\}$. Then for every given function $\Psi_0(x, t) \in X(R, s, \gamma_0)$ such that

$$\sup_{t \in [0, \infty)} e^{\gamma_0 t} \|\Psi_0(x, t)\|_{H^{(s)}(\mathbb{R}^n)} < \varepsilon,$$

where $\gamma_0 \leq \frac{n}{2} - \sqrt{\frac{n^2}{4} - m^2}$,

and for sufficiently small $\varepsilon$ the integral equation (7.2) has a unique solution $\Psi(x, t) \in X(R, s, \gamma)$ with $0 < \gamma < \gamma_0/(\alpha + 1)$. For the solution one has

$$\sup_{t \in [0, \infty)} e^{\gamma t} \|\Psi(x, t)\|_{H^{(s)}(\mathbb{R}^n)} < 2 \varepsilon.

(ii) If the eigenvalues of the mass matrix are large, $\frac{n}{4} \leq m_i$, $i = 1, \ldots, l$, then for every given function $\Psi_0(x, t) \in X(R, s, 0)$ such that

$$\sup_{t \in [0, \infty)} \|\Psi_0(x, t)\|_{H^{(s)}(\mathbb{R}^n)} < \varepsilon,$$

and for sufficiently small $\varepsilon$ the integral equation (7.2) has a unique solution $\Psi(x, t) \in X(R, s, 0)$, and

$$\sup_{t \in [0, \infty)} \|\Psi(x, t)\|_{H^{(s)}(\mathbb{R}^n)} < 2 \varepsilon.

8 Asymptotic at infinity

For $\varphi \in C_0^\infty(\mathbb{R}^n)$ let $V_\varphi(x, t)$ be a solution of the Cauchy problem

$$V_{tt} - \Delta V = 0, \quad V(x, 0) = 0, \quad V_t(x, 0) = \varphi(x).$$

Denote,

$$V_{\varphi}^{(k)}(x) = \frac{(-1)^k}{k!} \left[ \left( \frac{\partial}{\partial t} \right)^k V_{\varphi}(x, t) \right]_{t=1} \in C_0^\infty(\mathbb{R}^n), \quad k = 1, 2, \ldots.$$

Then, for every integer $N \geq 1$ we have

$$V_\varphi(x, 1 - e^{-t}) = \sum_{k=0}^{N-1} V_{\varphi}^{(k)}(x) e^{-kt} + R_{V_\varphi, N}(x, t), \quad R_{V_\varphi, N} \in C^\infty,$$

where with the constant $C(\varphi)$ the remainder $R_{V_\varphi, N}$ satisfies the inequality

$$|R_{V_\varphi, N}(x, t)| \leq C(\varphi) e^{-Nt} \quad \text{for all} \quad x \in \mathbb{R}^n \quad \text{and all} \quad t \in [0, \infty).$$
Further, we introduce the polynomial in $z$ with the smooth in $x \in \mathbb{R}^n$ coefficients as follows:

$$
\Phi^{(N)}_{\text{asymp}}(x, z) = z^{n-1/2} N_{1/2} \sum_{k=0}^{N-1} \left( \frac{n-1}{2} V^{(k)}(x) - (k+1) V^{(k+1)}(x) + V^{(k)}(x) \right) z^k.
$$

where $x \in \mathbb{R}^n$, $z \in \mathbb{C}$. Thus, we arrive at the next theorem.

**Theorem 8.1** Suppose that $m = \sqrt{n^2 - 1}/2$. Then, for every integer positive $N$ the solution of the equation (3.11) with the initial values $\varphi_0, \varphi_1 \in C_0^\infty(\mathbb{R}^n)$ has the following asymptotic expansion at infinity:

$$
\Phi(x, t) = \Phi^{(N)}_{\text{asymp}}(x, e^{-t}) + O(e^{-Nt-\frac{n-1}{2}t})
$$

for large $t$ uniformly for $x \in \mathbb{R}^n$, in the sense that for every integer positive $N$ the following estimate is valid:

$$
\|\Phi(x, t) - \Phi^{(N)}_{\text{asymp}}(x, e^{-t})\|_{L^\infty(\mathbb{R}^n)} \leq C(\varphi_0, \varphi_1)e^{-Nt-\frac{n-1}{2}t} \quad \text{for large } t.
$$

Unlike to the result by Vasy [44] the last inequality does not have the logarithmic term.

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