Gravitational waves from primordial black holes collisions in binary systems

Yu N Eroshenko
Institute for Nuclear Research of the Russian Academy of Sciences, Moscow, Russia
E-mail: eroshenko@ms2.inr.ac.ru

Abstract. It was shown by (Nakamura et al. 1997), (Ioka et al. 1998), and (Sasaki et al. 2016) that primordial black holes (PBHs) binaries can form effectively at the cosmological stage of radiation dominance, and the merge of the PBHs in pairs can explain the gravitational wave burst GW150914. In this paper, the model is re-examined by considering the effect of inflationary dark matter density perturbations which produce additional tidal forces. As a result, the merge rate of PBHs binaries and the corresponding rate of the gravitational bursts can be suppressed by the factor $1.5^{2}$ in comparison with previous calculations. This rate matches the LIGO data if the PBHs constitute the $f \approx 5 \times 10^{-4} - 5 \times 10^{-3}$ fraction of dark matter.

1. Introduction
Although gravitational waves were predicted by Albert Einstein in 1916, to date their existence was shown only indirectly through the orbital changes of binary pulsar PSR B1913+16. The first direct detection was done September 14, 2015 by two laser interferometers LIGO [1]. The form of the GW150914 signal corresponds to the general relativity prediction for the merge of two black holes with masses $36M_\odot$ and $29M_\odot$, and the statistical significance of the registration is $5.1\sigma$. This result demonstrates the existence black holes in the binary systems. Such black holes can born at the massive supernova explosions as a result of the standard stellar evolution [2-3]. In particular, the low-metallicity environment boosts the massive double black holes formation [4-6]. It is possible, however, that the black holes were born in the collapse of massive pre-galactic stars, or they were formed by merge of smaller black holes in the dense star clusters.

The merge of primordial black holes (PBH) in a pairs provides the alternative explanation of GW150914 [7-9]. The possibility of PBH formation was predicted in general by [10-11], and there are many particular scenarios of their formation [12-17]. For concreteness we consider in this paper the formation of PBHs from curvature perturbation as in [12].

The PBH pair can form occasionally if two PBH born sufficiently close to each other [7]. The motion of the binary PBH is influenced by neighbouring third PBH, therefore the PBHs do not move exactly towards each other but experience tangential displacement under the influence of the tidal gravitational forces from the third PBH. The [7-8] discussed the MACHOs with masses $\approx 0.1 - 1M_\odot$, while [9] explained the signal GW150914 as the merge of PBHs with masses $\approx 30M_\odot$. In addition, the possibility of the PBHs clusters
formation in the early universe was proposed by [18], and the PBHs collisions in such clusters can produce the bursts of gravitational waves [19-20].

In this study we consider the additional source of tidal forces – the tidal forces from the usual inflationary density perturbations in dark matter. This effect was not considered previously in the calculations of the PBHs pair evolution. These additional tidal forces alter the distribution of PBHs pairs over their orbital parameters, and finally suppress the rate of gravitational bursts.

The close PBHs pairs form at the cosmological stage of the radiation dominance (RD), but they can also form later at the stage of matter domination. A halo of dark matter is accumulated around the PBHs by the secondary accretion mechanism [21]. The PBHs in the halo experience the dynamic friction, lose angular momentum and move toward the center, there they form the gravitationally bounded binary, even if they were not associated initially [22]. The further fate of this pair is uncertain, because the effectiveness of dynamic friction in this situation was not investigated.

The paper is organised as follows. In the Section 2 we mainly define the notations and repeat some calculations of [7-8] by other method. Section 2 is devoted to the new effect – the tidal forces from adiabatic density perturbations with CMB-normalised spectrum. Then in the Section 4 we calculate the resultant rate of the gravitational bursts. In the Section 5 we consider critically the possibility of the PBHs merge inside dark matter minihalos. And finally we present some conclusions in the Section 6.

2. The evolution of binaries at the stage of radiation domination

Consider the evolution of the binary PBHs at the RD-stage, \( t < t_{eq} \), where \( t_{eq} \) is the moment of matter-radiation equality. Let the scale factor of the universe \( a(t) \) is normalized as \( a(t_{eq}) = 1 \), then the density of radiation \( \rho_r = \rho_{eq} / a^4 \). Let us assume that the PBHs constitute the \( f \leq 1 \) fraction of dark matter, \( \Omega_{m} = \Omega_{m0} \). In the case \( f < 1 \) the rest of DM can be in some other form, for example in the form of weakly interacting massive particles (WIMPs). Denote by \( x \) the comoving space between the components of PBHs pair, while the average distance between PBHs is

\[
\bar{x} = n^{-1/3} = \left( \frac{M_{BH}}{f \rho_{eq}} \right)^{1/3}.
\] (1)

Let the binary form at \( t < t_{eq} \). The condition for this is [7, 9]

\[
\frac{M_{BH}}{x' a_m^3} \approx \rho_r,
\] (2)

where

\[
a_m = \frac{1}{f} \left( \frac{x'}{\bar{x}} \right)^3.
\] (3)

This occurs at the RD-stage if \( a_m < 1 \) that can be written as \( x < x' \), where

\[
x' = \left( \frac{M_{BH}}{\rho_{eq}} \right)^{1/3}.
\] (4)

The \( x' \) would be the mean distance between the PBH in the case \( f = 1 \).

According to [7,9] the semi-major axis of orbit is fixed at the time given by (3), and the minor axis is influenced by tidal forces from the third PBH, located at the comoving distance \( y < x' \). Minor axis is calculated as the product of the tidal force and the time of free fall. As a result, [9] obtained for the major
and minor semi-axes, respectively,

\[ A = \alpha a \left( \frac{x}{y} \right)^3, \quad B = \beta A \left( \frac{x}{y} \right), \]  

and the eccentricity of the orbit

\[ e = \sqrt{1 - \frac{B^2}{A^2}} = \sqrt{1 - \beta^2 \frac{x^6}{y^6}}. \]  

In the paper [9] the expressions (5) and (6) were obtained in the case \( \alpha = \beta = 1 \). The [8] introduced the additional factors \( \alpha \) and \( \beta \), and obtained them from the numerical solution of the evolution equations. In the rest of this section we solve the similar equations and in the next section we consider additional effects.

The distance of each PBH from the center of masses of the pair is \( r/2 \). We consider the scales under the cosmological horizon \( r = ct \), so we can use the Newtonian dynamics, taking into account the contribution of uniformly distributed radiation by the substitution \( \rho \to \rho + 3p e^2 \) [23]. The \( r \) evolves according to the equation

\[ \frac{d^2 (r/2)}{dt^2} = -\frac{GM_{\text{BH}}}{r^3} - \frac{8\pi G \rho (r/2)}{3}, \]  

where the last term describes the gravitational effect of uniformly distributed radiation within the sphere with radius \( r/2 \). Following the approach of [24], we do the following parametrization

\[ r = abx. \]  

The evolution of \( b \) shows the factor of displacement from the comoving distance in the homogeneous universe. If one takes into account that at the radiation-dominated stage \( a \approx t^{1/2} / t_{\text{eq}}^{1/2} \) the Eq. (7) can be rewritten as

\[ \frac{a d^2 b}{da^2} + \frac{db}{da} = -\frac{3}{4\pi b^3} \frac{x^3}{x^3}. \]  

Analogous equation was obtained in [9] by other method. The combination of the quantities in the right-hand side

\[ \delta_i = \frac{2M_{\text{BH}}}{(4\pi/3)x^3 \rho_{\text{eq}}} = \frac{3}{2\pi} \frac{x^3}{x^3} \]  

is the density perturbation produced by two PBH, so one can conclude that Eq. (9) is equivalent to the equation of [24] (taken in the limit \( a = 1 \)), which describes the evolution of spherically symmetric entropy (isocurvature) perturbations. Initial time \( t_i \) is taken when the PBH mass is negligible in comparison with the radiation mass inside the sphere, and the initial conditions at the time \( t_i \) have the form \( b = 1 \), \( db / da = 0 \). The distance between the PBHs stops growing at the time \( t_i \), when \( dr / dt = 0 \), which corresponds to the condition \( db / da = -b / a \) [24]. It is easy to see that \( \alpha = b(a(t_i)) \) in the Eq. (5).

Let us consider the evolution of small semi-axis under the influence of tidal forces from the 3rd PBH, located at the comoving distance \( y \) from the pair. We consider the transverse displacement \( \delta r \) as a small perturbation to the radial orbit, and we introduce the parametrization

\[ \delta r = a\xi. \]  

The tidal force has the form
and the equation \( \frac{d^2 \delta r}{dt^2} = F_i \) can be written as

\[
F_i = \frac{2GM_{BH}^2}{(ay)^3} r, \tag{12}
\]

The initial conditions at the time \( t_i \) are \( a\dot{\xi} = d\xi / da = 0 \).

Figure 1. Example of the orbit evolution under the influence of tidal forces from the numerical solution of Eqs. (9) and (13). The vertical and horizontal dashed lines show the doubled major and minor semi-axis of the orbit, respectively.

\[
a\frac{d^2 \xi}{da^2} + \frac{d\xi}{da} - \frac{\xi}{a} = -\frac{3}{4\pi} \frac{x^3y^2}{y^2}. \tag{13}
\]

We solve the system of equations (9), (13) numerically until the binary expands up to the maximum distance (this distance is the A major semi-axis) and then approaches twice in radius. The value of \( a\xi \) at this moment gives the size of the minor axis \( B \). We assume for simplicity that all PBH have the same masses \( M_{BH} = 30M_\odot \). An example of the orbit is shown at Figure 1. Solving the equation (9) and (13) for any \( x \) (different values of \( y \) are taken into account in (13) by the scaling of \( \xi \)) and comparing with (5), we obtain the following correction factors in (5)

\[
\alpha \approx 0.64, \quad \beta \approx 2.8. \tag{14}
\]

In our calculations these factors are independent of \( x \) with high accuracy, however [8] obtained \( \alpha \approx 0.4 \), \( \beta \approx 0.8 \) and weak dependence on the fraction \( x/y \). We don’t consider here the angle dependence (direction to the 3rd PBH). This dependence was considered in [8].

3. Tidal forces from inflationary perturbations

Neighbourhood PBHs are not the only sources of disturbing. Tidal forces can be produced also by the dark matter density perturbations, and the main contribution comes from the characteristic scale \( x' \), where dark matter mass is equal to the PBHs mass \( M \approx M_{BH} \). Really, the scale of the PBHs binary contains the mass of dark matter

\[
M \approx \rho_{\Delta x} x'^3 = M_{BH} \frac{x'^3}{x'^3}. \tag{15}
\]

If \( M < M_{BH} \) the PBHs pair forms at the RD-stage, and from (15) it follows \( x < x' \). Otherwise, the PBHs do not define the dynamics of the system, but they follow the motion of dark matter. At smaller scales \( x < x' \) the dark matter is highly disturbed by the PBHs and is involved into the motion. The tide from the smaller scales is unclear. In contrast, the larger scales produce regular tidal force, so we consider the effect of the scales \( x \geq x' \) as minimum warranted contribution. Let the density perturbation at the mass scale \( M \)
at the moment \( t_{eq} \) is \( \delta_{eq} \). R.m.s. value of perturbation is (see e.g. [25] and references therein)

\[
\sigma_{eq}(M) \approx 8.2 \times 10^{3.7(n_{s} - 1) - 3} \left( \frac{M}{M_{e}} \right)^{1-n} \left[ 1 - 0.06 \log \left( \frac{M}{M_{e}} \right)^{3} \right],
\]

where \( n_{s} = 0.9608 \pm 0.0054 \) according to the Planck data. In particular, \( \sigma_{eq}(30M_{e}) \approx 5.2 \times 10^{-3} \). At the RD-stage the adiabatic perturbation \( \delta \) evolves only logarithmically. We can write the evolution of the dark matter perturbation as follows

\[
\delta = \delta_{eq} F(a),
\]

where the function [25]

\[
F(a) = \frac{\ln \left( g(a) / \sqrt{3} \right) + C - 1}{2},
\]

\( C \approx 0.577 \) is the Euler constant, and

\[
g(a) = \frac{\pi}{2^{2/3}} \left( \frac{3}{2\pi} \right)^{1/6} \frac{ac}{M^{1/3} G^{1/2} \rho_{eq}^{1/6}}.
\]

Then the dark matter excess can be estimated as \( \Delta M \approx M \delta \), and the tidal force acting on the pair is

\[
F_{\alpha} \approx \frac{GM_{bh} \Delta M (\alpha x b)}{(\alpha x)^{3}},
\]

where \( b \) is defined by (8). The comparison

\[
\frac{F_{\alpha}}{F_{t}} = \delta_{eq} \frac{F(a)}{f} \frac{y^{2}}{x^{3}}
\]

shows that for the sufficiently isolated PBHs pairs with \( y \approx x \) the contribution from inflation perturbations can prevail over the contribution from the 3rd PBH. The additional source of the tidal forces changes the value of maximum eccentricity \( e_{*} \), calculated in [7,9].

We substitute the (20) into the equation \( d^{2} \delta r / dt^{2} = F_{\alpha} \) and solve this equation (instead of (13)) numerically for different \( x \). If we parametrise

\[
B = kA \frac{x^{3}}{x^{3}} \nu,
\]

where \( \nu = \delta_{eq} / \sigma_{eq} \), we obtain from the numerical solution that the factor \( k \) is approximately constant,

\[
k \approx 0.005.
\]

In the next Section we use these results for the calculation of the rate of gravitational bursts.

4. Rate of gravitational bursts

Let us first neglect the effect of tidal forces from dark matter inflationary perturbations, but take into account the above correction factors (14). The probability distribution for \( A \) and \( e \), obtained in [7] and [9], can be rewritten in this case as

\[
dP = \frac{3}{2} \frac{f^{3/2}}{\alpha^{3/2} \beta^{3/2}} A^{1/2} e^{(1-e^{2})^{-3/2}} dA d\alpha.
\]

This distribution is obtained by calculating the Jacobian \( \partial(x, y) / \partial(A, e) \) under the assumption of the
uniform distribution of \( x \) and \( y \leq x \) in the ranges from 0 to \( \bar{x} \). Lifetime of PBHs pair due to the radiation of gravitational waves is [9]

\[
t_r = \frac{5e^3}{512G^3M_{nt}^4} A^4 (1 - e^3)^{7/2}.
\]  

(25)

The probability that the time (25) is less than \( t \) can be obtained from (24) by integrating over the corresponding curved region in the parameter space of \( A \) and \( e \), selected by the condition \( t_c < t \):

\[
P(<t) = \frac{1}{\beta} \left[ \frac{37}{29} \left( \frac{t}{t_{max}} \right)^{37/8} - \frac{8}{29} \left( \frac{t}{t_{max}} \right)^{38/7} \right],
\]

(26)

where

\[
t_{max} = \frac{5e^5}{512G^3M_{nt}^3} \frac{\alpha^4}{\beta^{16/5}} \frac{x_{4}^4}{f^{16/5}}.
\]

(27)

In the case \( \alpha = \beta = 1 \) the distributions (24) and (26) coincide with those obtained in [7]. The rate of gravitational wave bursts near the current moment \( t_0 \) is

\[
R = \frac{\rho_c \Omega_0 f}{M_{nt}} \frac{dP(<t)}{dt} \bigg|_{t=t_0},
\]

(28)

where \( \rho_c = 9.3 \times 10^{-30} \text{ g cm}^{-3} \) is the critical density, \( \Omega_m = 0.27 \). The result of the calculation is shown at Figure 2 in comparison with the result of [9], which is reproduced with \( \alpha = \beta = 1 \). Our rate is about 40\% of [9] value.

Now let us take into account the effect of tidal forces from ordinary inflationary perturbations as it was considered in the Section 3. In this case we have the 3-dimensional probability distribution

\[
dP = \frac{18\pi^2y^2}{\bar{x}^6} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2\beta^2}} dx dy d\nu,
\]

(29)

where we consider not only positive \( \nu > 0 \) but also negative \( \nu < 0 \) perturbations. Now we have the sum of (5) and (22)

\[
B = A \left( \frac{x}{y} \right)^3 + k \frac{A x^3}{\bar{x}^3} \nu.
\]

(30)
Figure 2. The rate of gravitational waves bursts from the collision of PBHs in pairs, depending on the fraction $f$ of PBHs in dark matter. The upper solid curve 1 shows the result of the work [9]. The middle solid curve 2 was obtained according to (28) with the correction factors (14). The lower bold curve 3 shows the result of the current work. Dotted curve 4 shows the rate of bursts under the assumption of the efficiency of dynamic friction inside minihalos at the last stage of PBHs pair evolution. The horizontal dashed lines limit the area of LIGO observations, obtained in [9].

The tidal force from inflation perturbations prevails if

\[
\nu > \nu^* \equiv \frac{\beta x^3}{y^3 k}.
\] (31)

The integration of (29) in the 3D space with the condition $t_c < t$ is more complicated in this case. The result of the numerical integration is shown on Figure 2 by the curve 3. One can see that in the region $f \approx 5 \times 10^{-4} - 5 \times 10^{-3}$ the tidal forces from dark matter do not change the result significantly, but at smaller $f$ they can suppress the rate of the gravitational bursts by the factor $\approx 1.5 - 2$.

5. Binaries inside dark matter minihalos

Consider the case $f = 1$ and $\delta < 1$ in (10). The first of these conditions means that PBHs constitute only small fraction of dark matter and the rest of the dark matter consists of something else, for example, of new elementary particles. The second condition means that we consider sufficiently large region where PBHs create only small perturbation in dark matter. This perturbation at $t > t_{eq}$ evolves into the minihalo of dark matter around the PBHs pair. Similar model was proposed in [22]. Consider the evolution of perturbations at the comoving scale $x$ (the final minihalo contains the mass $M \approx 10^5 M_{BH}$ [25]). After the detachment from the cosmological expansion the perturbed region is compressed twice and virialized at the final size [26].
The mass within this radius is given by (15), so the velocity dispersion

\[ v \approx \left( \frac{GM}{r_v} \right)^{1/2} \approx \left( \frac{2GM_{BH}}{\pi x} \right)^{1/2}, \]  

and the average density

\[ \rho_u \approx \frac{3^4}{4\pi^4} \frac{x^{9/5}}{v(r)^3} \rho_{eq}. \]

PBHs experience dynamical friction, lose the orbital angular momenta and approach the minihalos centers. The change of the orbital radius obeys the equation [27]

\[ \frac{dr}{dt} = -\frac{4\pi G^2 M_{BH} \rho_u(r) Dr}{v(r)^3}, \]

where numerical constant \( D \approx 4.27 \). From Eq. (35) we obtain the characteristic time of the fall to the center

\[ t_d \approx 5.3 \rho_{eq} \left( \frac{x}{X} \right)^{15/2}. \]

PBHs come to the radius \( r_c \), within which the mass of dark matter is \( M(r) \approx M_{BH} \), and the PBHs sit on the orbit around each other. Dark matter within their orbit and the surrounding material will lead to the evolution of the orbit, but the effectiveness of dynamical friction under these conditions has not been elucidated yet, and it requires further investigation. The lifetime of the pair (25) is many orders of magnitude greater than the age of the universe \( t_0 \), therefore, for the further approach and merger the PBHs must continue to lose the orbital angular momenta. Virial velocity in the minihalo is \( v \approx 0.36 \text{ km s}^{-1} \), and the corresponding virial temperature of baryonic gas is \( T \approx m_p v^2 / (2k_B) \approx 8 \text{ K} \), where \( k_B \) is the Boltzmann constant. So the radiative cooling mechanisms do not work, and the gas does not form stars, which could give additional dynamical friction.

Consider the most optimistic case, when at the radii \( < r_c \) the dynamic friction is as effective as before and is described by (35). Then the significant fraction of the PBHs pairs can merge during the Hubble time and produce gravitational bursts. Distribution of \( x \) under the assumption of uniformity \([7, 9]\) has the simple form

\[ dP = \frac{3x^2dx}{X^3}. \]

The condition \( t_d < t \) selects the region of integration, and

\[ P(< t) \approx 64 f \left( \frac{t}{t_0} \right)^{2/5}. \]

Substituting into (28), we obtain the rate of gravitational bursts that is shown at Figure 2 by the point curve. This result was obtained under the assumption of high efficiency of dynamical friction at the last stage of evolution, so it gives the most optimistic scenario, but this model requires further elaboration.

6. Conclusion

In this work we considered the formation and evolution of the PBHs pairs and found the expected rate of LIGO gravitational bursts. Our approach is similar to one of [7-9], but the calculations are made by...
another method, which takes into account the tidal forces from adiabatic density perturbations of dark matter with CMB-normalised spectrum. These tidal forces alter the probability distribution of PBHs pairs over their orbital parameters. It turned out that the merge rate matches the LIGO signals if the PBHs matter with CMB-normalised spectrum. These tidal forces alter the probability distribution of PBHs pairs another method, which takes into account the tidal forces from adiabatic density perturbations of dark matter.

References

[1] Abbott B P et al 2016 Observation of Gravitational Waves from a Binary Black Hole Merger. *Phys. Rev. Lett.* 116 061102
[2] Grishchuk L P, Lipunov V M, Postnov K A, Prokhorov M E and Sathyaprakash B S 2001 Gravitational Wave Astronomy: in Anticipation of First Sources to be Detected *Phys. Usp.* 44 pp 1-51
[3] Bogomazov A I, Lipunov V M and Tutukov A V 2007 Evolution of close binaries and gamma-ray bursts. *Astronomy Reports* 51 pp 308-317
[4] Belczynski K et al 2010 The effect of metallicity on the detection prospects for gravitational waves *The Astrophysical Journal Letters* 715 L138-L141
[5] Belczynski K et al 2016 The first gravitational-wave source from the isolated evolution of two 40-100 M⊙ stars *Nature* 534 pp 512-515
[6] Abbott B P et al 2016 Astrophysical Implications of the Binary Black-Hole Merger GW150914 *The Astrophysical Journal Letters* 818 L22
[7] Nakamura T, Sasaki M, Tanaka T and Thorne K S 1997 Gravitational Waves from Coalescing Black Hole MACHO Binaries *The Astrophysical Journal* 487 L139-L142
[8] Ioka K, Chiba T, Tanaka T and Nakamura T 1998 Black Hole Binary Formation in the Expanding Universe — Three Body Problem Approximation. *Phys. Rev. D* 58 063003
[9] Sasaki M, Suyama T, Tanaka T and Yokoyama S 2016 Primordial black hole scenario for the gravitational wave event GW150914 *Preprint* arXiv:1603.08338
[10] Zel’’dovich Ya B and Novikov I D 1967 The Hypothesis of Cores Retarded during Expansion and the Hot Cosmological Model. *Sov. Astron.* 10 pp 602-603
[11] Hawking S 1971 Gravitationally collapsed objects of very low mass. *MNRAS* 15 pp 75-78
[12] Carr B J 1975 The primordial black hole mass spectrum. *Astrophys. J.* 201 pp 1-19
[13] Garcia-Bellido J, Linde A, and Wands D 1996 Density perturbations and black hole formation in hybrid inflation *Phys. Rev. D* 54 pp 6040-6058
[14] Jedamzik K 1997 Primordial black hole formation during the QCD epoch *Phys. Rev. D* 55 pp 5871-5875
[15] Khlopov M Yu and Polnarev A G 1980 Primordial black holes as a cosmological test of grand unification *Phys. Lett. B* 97 pp 383-387
[16] Berezin V A, Kuzmin V A and Tkachev I I 1983 Thin-wall vacuum domain evolution *Phys. Lett. B* 120 pp 91-96
[17] Dolgov A and Silk J 1993 Baryon isocurvature fluctuations at small scales and baryonic dark matter *Phys. Rev. D* 47 pp 4244-4255
[18] Rubin S G, Sakharov A S and Khlopov M Yu 2001 The Formation of Primary Galactic Nuclei during Phase Transitions in the Early Universe *JETP* 92 pp 921-929
[19] Dokuchaev V I, Eroshenko Yu N and Rubin S G 2009 Gravitational wave bursts from collisions of...
primordial black holes in clusters *Astronomy Letters* 35 pp 143-149
[20] Clesse S and Garcia-Bellido J 2016 The clustering of massive Primordial Black Holes as Dark Matter: measuring their mass distribution with Advanced LIGO *Preprint* arXiv:1603.05234
[21] Bertschinger E 1985 Self-similar secondary infall and accretion in an Einstein-de Sitter universe *Astrophys. J. Supp.* 58 pp 39-65
[22] Hayasaki K, Takahashi K, Sendouda Y and Nagataki S 2016 Rapid Merger of Binary Primordial Black Holes *Preprint* arXiv:0909.1738
[23] Mc Crea W H 1951 Relativity Theory and the Creation of Matter. *Proc. Roy. Soc. A* 206 pp 562-575
[24] Kolb E W and Tkachev I I 1994 Large Amplitude Isothermal Fluctuations And High Density Dark Matter Clumps *Phys. Rev. D* 50 pp 769-773
[25] Berezinsky V S, Dokuchaev V I and Eroshenko Yu N 2013 Formation and internal structure of superdense dark matter clumps and ultracompact minihaloes. *JCAP* 11 059
[26] Peebles P J E 1980 *The Large-Scale Structure of the Universe* (Princeton Univ. Press, Princeton)
[27] Saslaw W C 1987 *Gravitational Physics of Stellar and Galactic Sytems* (Cambridge Univ. Press, Cambridge)