Fundamental Structure of Loop Quantum Gravity

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Abstract

In recent twenty years, loop quantum gravity, a background independent approach to unify general relativity and quantum mechanics, has been widely investigated. The aim of loop quantum gravity is to construct a mathematically rigorous, background independent, nonperturbative quantum theory for Lorentzian gravitational field on four-dimensional manifold. In the approach, the principles of quantum mechanics are combined with those of general relativity naturally. Such a combination provides us a picture of, so-called, quantum Riemannian geometry, which is discrete at fundamental scale. Imposing the quantum constraints in analogy from the classical ones, the quantum dynamics of gravity is being studied as one of the most important issues in loop quantum gravity. On the other hand, the semi-classical analysis is being carried out to test the classical limit of the quantum theory.

In this review, the fundamental structure of loop quantum gravity is presented pedagogically. Our main aim is to help non-experts to understand the motivations, basic structures, as well as general results. It may also be beneficial to practitioners to gain insights from different perspectives on the theory. We will focus on the theoretical framework itself, rather than its applications, and do our best to write it in modern and precise language while keeping the presentation accessible for beginners. After reviewing the classical connection dynamical formalism of general relativity, as a foundation, the construction of kinematical Ashtekar-Isham-Lewandowski representation is introduced in the content of quantum kinematics. The algebraic structure of quantum kinematics is also discussed. In the content of quantum dynamics, we mainly introduce the construction of a Hamiltonian constraint operator and the master constraint project. At last, some applications and recent advances are outlined. It should be noted that this strategy of quantizing gravity can also be extended to obtain other background independent quantum gauge theories. There is no divergence within this background independent and diffeomorphism invariant quantization programme of matter coupled to gravity.

Keywords: loop quantum gravity, quantum geometry, quantum dynamics, background independence.

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1 Introduction

1.1 Motivation of Quantum Gravity

Nowadays, in traditional view there are four elementary interactions widely understood by community of physicists: strong interaction, weak interaction, electromagnetic interaction and gravitational interaction. The description for the former three kinds of forces is quantized in the well-known standard model. The interactions are transmitted by exchanging mediate particles. However, the last kind of interaction, gravitational interaction, is described by Einstein’s theory of general relativity, which is absolutely a classical theory which describes the gravitational field as a smooth metric tensor field on a manifold, i.e., a 4-dimensional spacetime geometry. There is no $\hbar$ and hence no discrete structure of spacetime. Thus there is a fundamental inconsistency in our current description of the whole physical world. Physicists widely accept the assumption that our world is, so called, quantized at fundamental level. So all interactions should be brought into the framework of quantum mechanics fundamentally. As a result, the gravitational field should also have "quantum structure".

Throughout the last century, our understanding of the nature has considerably improved from macroscale to microscale, including the phenomena in molecule scale, atom scale, sub-atom scale, and elementary particle scale. The standard model of particle physics coincides almost with all present experimental tests in laboratory (see e.g. [157]). However, because unimaginably large amount of energy would be needed, no one has understood how the physical process happens near the Planck scales $\ell_p \equiv \left(\frac{G\hbar}{c^3}\right)^{1/2} \sim 10^{-33}cm$ and $t_p \equiv \left(\frac{G\hbar}{c^5}\right)^{1/2} \sim 10^{-43}s$, which are viewed as the most fundamental scales. The Planck scale arises naturally in attempts to formulate a quantum theory of gravity, since $\ell_p$ and $t_p$ are unique combinations of speed of light $c$, Planck constant $\hbar$, and gravitational constant $G$, which have the dimensions of length and time respectively. The dimensional arguments suggest that at Planck scale the smooth structure of spacetime should break down, where the well-known quantum field theory is invalid since it depends on a fixed smooth background spacetime. Hence we believe that physicists should go beyond the successful standard model to explore the new physics near Planck scale, which is, perhaps, a quantum field theory without a background spacetime, and this quantum field theory should include the quantum theory of gravity. Moreover, current theoretical physics is thirsting for a quantum theory of gravity to solve at least the following fundamental difficulties.

- **Classical Gravity - Quantum Matter Inconsistency**

  The most crucial equation to perform the relation between the matter field and gravitational field is the famous Einstein field equation:

  $$ R_{\alpha\beta}[g] - \frac{1}{2} R[g] g_{\alpha\beta} = \kappa T_{\alpha\beta}[g], $$

  where the left hand side of the equation concerns spacetime geometry which has classical smooth structure, while the right hand side concerns also matter field which is fundamentally quantum mechanical in standard model. In quantum field theory the energy-momentum tensor of matter field should be an operator-valued tensor $T_{\alpha\beta}$. One possible way to keep classical geometry consistent with
quantum matter is to replace $T_{αβ}[g]$ by the expectation value $< \hat{T}_{αβ}[g] >$ with respect to some quantum state of the matter on a fixed spacetime. A primary attempt is to consider the vacuum expectation. However, in the solution of this equation the background $g_{αβ}$ has to be changed due to the non-vanishing of $< \hat{T}_{αβ}[g] >$. So one has to feed back the new metric into the definition of the vacuum expectation value etc. The result of the iterations does not converge in general [71]. This inconsistency motivates us to quantize the background geometry to arrive at an operator formula also on the left hand side of Eq. (1).

- Singularity in General Relativity

Einstein’s theory of General Relativity is considered as one of the most elegant theories in 20th century. Many experimental tests confirm the theory in classical domain [158]. However, Penrose and Hawking proved that singularities are inevitable in general spacetimes with several tempered conditions on energy and causality by the well-known singularity theorem (for a summary, see [91][155]). Thus general relativity can not be valid unrestrictedly. One naturally expects that, in extra strong gravitational field domains near the singularities, the gravitational theory would probably be replaced by an unknown quantum theory of gravity.

- Infinity in Quantum Field Theory

It is well known that there are infinity problems in quantum field theory in Minkowski spacetime. In curved spacetime, the problem of UV divergence is even more serious because of the interacting fields. Although much progress on the renormalization for interacting fields have been made [92][156], a fundamentally satisfactory theory is still far from reaching. So it is expected that some quantum gravity theory, playing a fundamental role at Planck scale, would provide a natural cut-off to cure the UV singularity in quantum field theory. The situation of quantum field theory on a fixed spacetime looks just like that of quantum mechanics for particles in electromagnetic field before the establishing of quantum electrodynamics, where the particle mechanics (actress) is quantized but the background electromagnetic field (stage) is classical. The history suggests that such a semi-classical situation is only an expedient which should be replaced by a much more fundamental and satisfactory theory.

1.2 Purpose of Loop Quantum Gravity

The research on quantum gravity theory is rather active. Many quantization programmes for gravity are being carried out (for a summary see e.g. [144]). In these different kinds of approaches, the idea of loop quantum gravity is motivated by researchers in the community of general relativity. It follows closely the thoughts of general relativity, and hence it is a quantum theory born with background independence. Roughly speaking, loop quantum gravity is an attempt to construct a mathematically rigorous, non-perturbative, background independent quantum theory of four-dimensional, Lorentzian general relativity plus all known matter in the continuum. The
project of loop quantum gravity inherits the basic idea of Einstein that gravity is fundamentally spacetime geometry. Here one believes in that the theory of quantum gravity is a quantum theory of spacetime geometry with diffeomorphism invariance (this legacy is discussed comprehensively in Rovelli’s book [122]). To carry out the quantization procedure, one first casts general relativity into the Hamiltonian formalism of a diffeomorphism invariant Yang-Mills gauge field theory with a compact internal gauge group. Thus the construction of loop quantum gravity is valid to all background independent gauge field theories. So the theory can also be called as a background independent quantum gauge field theory.

All classical fields theories, other than gravitational field, are defined on a fixed spacetime, which provides a foundation to the perturbative Fock space quantization. However general relativity is only defined on a manifold and hence is the unique background independent classical field theory, since gravity itself is the background. So the situation for gravity is much different from other fields by construction [122], namely gravity is not only the background stage, but also the dynamical actress. Such a double character for gravity leads to many difficulties in the understanding of general relativity and its quantization, since we cannot analog the strategy in ordinary quantum theory of matter fields. However, an amazing result in loop quantum gravity is that the background independent programme can even enlighten us to avoid the difficulties in ordinary quantum field theory. In perturbative quantum field theory in curved spacetime, the definition of some basic physical quantities, such as the expectation value of energy-momentum, is ambiguous and it is difficult to calculate the back-reaction of quantum fields to the background spacetime [156]. One could speculate on that the difficulty is related to the fact that the present formulation of quantum field theories is background dependent. For instance, the vacuum state of a quantum field is closed related to spacetime structure, which plays an essential role in the description of quantum field theory in curved spacetime and their renormalization procedures. However, if the quantization programme is by construction background independent and non-perturbative, it is possible to solve the problems fundamentally. In loop quantum gravity there is no assumption of a priori background metric at all and the gravitational field and matter fields are coupled and fluctuating naturally with respect to each other on a common manifold.

In the following sections, we will review pedagogically the basic construction of a completely new, background independent quantum field theory, which is completely different from the known quantum fields theory. For completeness and accuracy, we will not avoid mathematical terminologies. While, for simplicity, we will skip the complicated proofs of many important statements. One may find the missing details in the references cited. Thus our review will not be comprehensive. We refer to Ref. [144] for a more detailed exploration, Refs. [21] and [146] for more advanced topics. It turns out that in the framework of loop quantum gravity all theoretical inconsistencies introduced in the previous section are likely to be cured. More precisely, one will see that there is no UV divergence in quantum fields of matter if they are coupled with gravity in the background independent approach. Also recent works show that the singularities in general relativity can be smeared out in the symmetry-reduced models [45] [101] [50]. The crucial point is that gravity and matter are coupled and consistently quantized non-perturbatively so that the problems of classical gravity and quantum
2 Classical Framework of Connection Dynamics

2.1 Lagrangian Formalism

In order to canonically quantize the classical system of gravity, Hamiltonian analysis has to be performed to obtain a canonical formalism of the classical theory suitable to be represented on certain Hilbert space. The first canonical formalism of general relativity is the ADM formalism (Geometric dynamics) from the Einstein-Hilbert action \[155\][97], which by now has not been cast into a quantum theory rigorously. Another well-known action of general relativity is the Palatini formalism, where the tetrad and the connection are regarded as independent dynamical variables. However, unfortunately the dynamics of Palatini action is the same with the Einstein-Hilbert action for the gravitational field without fermion coupling \[4\][88]. In 1986, Ashtekar gave a formalism of true connection dynamics with a relatively simple Hamiltonian constraint, and thus opens the door to apply quantization techniques from gauge fields theory \[2\][3][123]. However, the weakness of that formalism is that the canonical variables are complex variables, which needs a complicated real section condition. Moreover, the quantization based on the complex connection could not be carried out rigorously, since the internal gauge group is noncompact. In 1995, Barbero modified the Ashtekar new variables to give a system of real canonical variables for dynamical theory of connections \[36\]. Then Holst constructed a generalized Palatini action to support Barbero’s real connection dynamics \[93\]. Although there is a free parameter (Barbero-Immirzi parameter \(\beta\)) in generalized Palatini action and the Hamiltonian constraint is more complicated than the Ashtekar one, now the generalized Palatini Hamiltonian with the real connections is widely accepted by loop theorists for the quantization program\[4\].

All the following analysis is based on the generalized Palatini formalism.

Consider an 4-manifold, \(M\), on which the basic dynamical variables in the generalized Palatini framework are tetrad \(e_I^\alpha\) and \(\text{so}(1,3)\)-valued connection \(\omega_{IJ}^\alpha\) (not necessarily torsion-free), where the capital Latin indices \(I, J, \ldots\) denote the internal \(SO(1,3)\) group and the Greek indices \(\alpha, \beta, \ldots\) denote spacetime indices. A tensor with both spacetime indices and internal indices is named as a generalized tensor. The internal space is equipped with a Minkowskian metric \(\eta_{IJ}\) (of signature 
\(-, +, +, +\)) fixed once for all, such that the spacetime metric reads:

\[
g_{\alpha\beta} = \eta_{IJ} e_I^\alpha e_J^\beta.\]

The generalized Palatini action in which we are interested is given by \[21\]:

\[
S_p[e^\beta_K, \omega_{IJ}^\alpha] = \frac{1}{2\kappa} \int_M d^4x (e_i^\alpha e^\beta_j \Omega_{ij}^\alpha + \frac{1}{2\beta} \epsilon_{KIJ} \Omega_{ij}^\alpha \tilde{\Omega}_{ij}^{K\ell}), \tag{2}
\]

\(^1\)One may take the other viewpoint that the transition from complex connection to real variables is only a mathematical convenience at the present stage, since we do not have a rigorous framework to deal with the infinite dimensional space of connections with non-compact internal group. Some researchers are working on this generalization \[72][106][107]\.
where $e$ is the square root of the determinant of the metric $g_{\alpha\beta}$, $\epsilon^{IJ}_{KL}$ is the internal Levi-Civita symbol, $\beta$ is the Barbero-Immirzi parameter, which we fix to be real, and the $so(1,3)$-valued curvature 2-form $\Omega^I_{\alpha\beta}$ of the connection $\omega^I_{\alpha\beta}$ reads:

$$
\Omega^I_{\alpha\beta} := 2\mathcal{D}_\alpha \omega^I_{\alpha\beta} = \partial_\alpha \omega^I_{\beta} - \partial_\beta \omega^I_{\alpha} + \omega^I_{\alpha} \wedge \omega^J_{\beta} \wedge \omega^J_{\beta},
$$

here $\mathcal{D}_\alpha$ denote the $so(1,3)$ generalized covariant derivative with respect to $\omega^I_{\alpha\beta}$ acting on both spacetime and internal indices. Note that the generalized Palatini action returns to the Palatini action when $\frac{1}{\beta} = 0$ and, if a complex Barbero-Immirzi parameter is assumed, gives the (anti)self-dual Ashtekar formalism when one sets $\frac{1}{\beta} = \pm i$. Moreover, besides spacetime diffeomorphism transformations, the action is also invariant under internal $SO(1,3)$ rotations:

$$(e, \omega) \mapsto (e', \omega') = (b^{-1} e, b^{-1} \omega b + b^{-1} db),$$

for any $SO(1,3)$ valued function $b$ on $M$. The gravitational field equations are obtained by varying this action with respect to $e^\alpha_I$ and $\omega^I_{\alpha\beta}$. We first study the variation with respect to the connection $\omega^I_{\alpha\beta}$. One has

$$
\delta \Omega^I_{\alpha\beta} = (d \delta \omega^I_{\alpha\beta})_{\mu\nu} + \delta \omega^I_{\alpha} \wedge \omega^J_{\beta} + \omega^I_{\alpha} \wedge \delta \omega^J_{\beta} = 2\mathcal{D}_\alpha \delta \omega^I_{\alpha\beta}
$$

by the definition of covariant generalized derivative $\mathcal{D}_\alpha$. Note that $\delta \omega^I_{\alpha\beta}$ is a Lorentz covariant generalized tensor field since it is the difference between two Lorentz connections [108][109]. One thus obtains

$$
\delta S_p = \frac{1}{2k} \int_M d^4x (e(e(e(e(\delta \Omega^I_{\alpha\beta} + \frac{1}{2\beta} \epsilon^{IJ}_{KL} \delta \Omega^K_{\alpha\beta})))) = \frac{1}{2k} \int_M (\delta \omega^I_{\alpha\beta} + \frac{1}{2\beta} \epsilon^{IJ}_{KL} \delta \omega^K_{\alpha\beta}) \mathcal{D}_\alpha [(e(e(e(e(\delta \Omega^I_{\alpha\beta} + \frac{1}{2\beta} \epsilon^{IJ}_{KL} \delta \Omega^K_{\alpha\beta})))) = 0,
$$

where we have used the fact that $\mathcal{D}_\alpha \tilde{\omega}^\alpha = \partial_\alpha \tilde{\omega}^\alpha$ for all vector density $\tilde{\omega}^\alpha$ of weight +1 and neglected the surface term. Then it gives the equation of motion:

$$
\mathcal{D}_\alpha [(\delta \Omega^I_{\alpha\beta} + \frac{1}{2\beta} \epsilon^{IJ}_{KL} \delta \omega^K_{\alpha\beta}) \mathcal{D}_\alpha [(e(e(e(e(\delta \Omega^I_{\alpha\beta} + \frac{1}{2\beta} \epsilon^{IJ}_{KL} \delta \Omega^K_{\alpha\beta})))) = 0,
$$

where $\tilde{\eta}^{\alpha\beta\gamma\delta}$ is the spacetime Levi-Civita symbol. This equation leads to the torsion-free Cartan’s first equation:

$$
\mathcal{D}_\alpha \mathcal{D}_\alpha e^I_{\alpha\beta} = 0,
$$

which means that the connection $\omega^I_{\alpha\beta}$ is the unique torsion-free Levi-Civita spin connection compatible with the tetrad $e^\alpha_I$. As a result, the second term in the action (2) can be calculated as:

$$
(e(e(e(e(\delta \Omega^I_{\alpha\beta} + \frac{1}{2\beta} \epsilon^{IJ}_{KL} \delta \Omega^K_{\alpha\beta})))) = \tilde{\eta}^{\alpha\beta\gamma\delta} R_{\alpha\beta\gamma\delta},
$$

which is exactly vanishing, because of the symmetric properties of Riemann tensor. So the generalized Palatini action returns to the Palatini action, which will certainly give the Einstein field equation.
2.2 Hamiltonian Formalism

To carry out the Hamiltonian analysis of action (2), suppose the spacetime $M$ is topologically $\Sigma \times \mathbb{R}$ for some 3-dimensional compact manifold $\Sigma$ without boundary. We introduce a foliation parameterized by a smooth function $t$ and a time-evolution vector field $v^a$ such that $v^a(dt)_a = 1$ in $M$, where $v^a$ can be decomposed with respect to the unit normal vector $n^a$ of $\Sigma$ as:

$$v^a = Nn^a + N^a,$$

here $N$ is called the lapse function and $N^a$ the shift vector\[158\][97]. The internal normal vector is defined as $n_I = n_\alpha e^{\alpha I}$. It is convenient to carry out a partial gauge fixing, i.e., fix a internal constant vector field $n_I$ with $\eta_{IJ} n^I n^J = -1$. Note that the gauge fixing puts no restriction on the real dynamics\[155\]. Then the internal vector space $V$ is decomposed with a 3-dimensional subspace $W$ orthogonal to $n_I$, which will be the internal space on $\Sigma$. With respect to the internal normal $n_I$ and spacetime normal $n_\alpha$, the internal and spacetime projection maps are denoted by $q^I$ and $q_\alpha$ respectively, where we use $i, j, k, \ldots$ to denote the 3-dimensional internal space indices and $a, b, c, \ldots$ to denote the indices of space $\Sigma$. Then an internal reduced metric $\delta_{ij}$ and a reduced spatial metric on $\Sigma$, $q_{ab}$, are obtained by these two projection maps. The two metrics are related by:

$$q_{ab} = \delta_{ij} e^a_i e^b_j,$$

where the orthonormal co-triad on $\Sigma$ is defined by $e_a^i := e_\alpha^i q^I q_\alpha a$. Now the internal gauge group $SO(1, 3)$ is reduced to its subgroup $SO(3)$ which leaves $n^I$ invariant. Finally, two Levi-Civita symbols are obtained respectively as

$$\epsilon_{ijk} := q^I q^J q^K e_{IJJK},$$

$$\eta_{abc} := q^a d^b q^c,$$

where the internal Levi-Civita symbol $\epsilon_{ijk}$ is an isomorphism of Lie algebra $so(3)$. Using the connection 1-form $\omega_a^{ij}$, one can defined two $so(3)$-valued 1-form on $\Sigma$:

$$\Gamma^i_a := \frac{1}{2} q^a_b q^j d^n q^{ij} n^J n^J \omega_a^{KL},$$

$$K^a_i := q^a_b q^j d^n q^{ij} n^J n^J,$$

where $\Gamma$ is a spin connection on $\Sigma$ and $K$ will be related to the extrinsic curvature of $\Sigma$ on shell. After the 3+1 decomposition and the Legendre transformation, action (2) can be expressed as\[93\]:

$$S_p = \int_{\Sigma} dt \int d^3 x [\delta_{ij} A_i^a A_j^a - \mathcal{H}_{a\beta}(A_i^a, P_j^a, A_i^\beta, N, N^\beta)],$$

from which the symplectic structure on the classical phase space is obtained as

$$\{A_i^a(x), P_j^a(y)\} := \delta^a_d \delta^b_c \delta^3(x - y),$$

However, there are some arguments that such a gauge fixing is a non-natural way to break the internal Lorentz symmetry (see e.g.\[130\]).
where the configuration and conjugate momentum are defined respectively by:

\[ A^i_a := \Gamma^i_a + \beta K^i_a, \]
\[ \bar{P}^i_a := \frac{1}{2\kappa^2} \eta^{abc} e_{ij} \epsilon^j_i \epsilon^k_k \equiv \frac{1}{\kappa^2} \sqrt{|\det q^a|} \epsilon^a_i, \]

here \( \det q \) is the determinant of the 3-metric \( q_{ab} \) on \( \Sigma \) and hence \( \det q = (\kappa\beta)^3 \det P \). In the definition of the configuration variable \( A^i_a \), we should emphasize that \( \Gamma^i_a \) is restricted to be the unique torsion free \( so(3) \)-valued spin connection compatible with the triad \( e^a_i \). This conclusion is obtained by solving a second class constraint in the Hamiltonian analysis \[93\]. In the Hamiltonian formalism, one starts with the fields \( (A^i_a, \bar{P}^i_a) \). Then neither the basic dynamical variables nor their Poisson brackets depend on the Barbero-Immirzi parameter \( \beta \). Hence, for the case of pure gravitational field, the dynamical theories with different \( \beta \) are simplectic equivalent. However, as we will see, the spectrum of geometric operators are modified by different value of \( \beta \), and the non-perturbative calculation of black hole entropy is compatible with Bekenstein-Hawking’s formula only for a specific value of \( \beta \) \[69\]. In addition, it is argued that the Barbero-Immerzi parameter \( \beta \) may lead to observable effects in principle when the gravitational field is coupled with fermions \[112\]. In the decomposed action \[5\], the Hamiltonian density \( \mathcal{H}_{tot} \) is a linear combination of constraints:

\[ \mathcal{H}_{tot} = \Lambda G_i + N^a C_a + NC, \]

where \( \Lambda^i = -\frac{1}{2} \epsilon^i_{jk} \omega^j_k \), \( N^a \) and \( N \) are Lagrange multipliers. The three constraints in the Hamiltonian are expressed as \[21\]:

\[ G_i = D_a \bar{P}^a_i := \partial_a \bar{P}^a_i + \epsilon_{ijk} A^j_a \bar{P}^k_i, \]
\[ C_a = \bar{P}^i_a F^i_{ab} - \frac{1 + \beta^2}{\beta} K^i_a G_i, \]
\[ C = \frac{\kappa \beta^2}{2 \sqrt{|\det q|}} \bar{P}^i_a \Gamma^{ij}_k \bar{P}^k_j F^i_{ab} - 2(1 + \beta^2)K^i_a K^j_b \]
\[ + \kappa(1 + \beta^2) \partial_a \left( \frac{\bar{P}^a_i}{\sqrt{|\det q|}} \right) G^i, \] (7)

where the configuration variable \( A^i_a \) performs as a \( so(3) \)-valued connection on \( \Sigma \) and \( F^i_{ab} \) is the \( so(3) \)-valued curvature 2-form of \( A^i_a \) with the well-known expression:

\[ F^i_{ab} := 2D_a A^i_b = \partial_a A^i_b - \partial_b A^i_a + \epsilon_{ijk} A^j_a A^k_b. \]

In any dynamical system with constraints, the constraint analysis is essentially important because they reflect the gauge invariance of the system. From the above three constraints of general relativity, one can know the gauge invariance of the theory. The Gaussian constraint \( G_i = 0 \) has crucial importance in formulating the general relativity into a dynamical theory of connections. The corresponding smeared constraint function, \( \mathcal{G}(\Lambda) := \int \mu^3 x \Lambda^i(x) G_i(x) \), generates a transformation on the phase space as:

\[ \{ A^i_a(x), \mathcal{G}(\Lambda) \} = -D_a \Lambda^i(x) \]
\[ \{ \bar{P}^i_a(x), \mathcal{G}(\Lambda) \} = \epsilon_{ijk} \sqrt{|\det q^a|} \epsilon^a_j \Lambda^i(x) \bar{P}^k(x), \]
which are just the infinitesimal versions of the following gauge transformation for the \( so(3) \)-valued connection 1-form \( \mathbf{A} \) and internal rotation for the \( so(3) \)-valued densitized vector field \( \tilde{\mathbf{P}} \) respectively:

\[
(A_a, \tilde{P}^i) \mapsto (g^{-1}A_a g + g^{-1}(dg)_{a\alpha} g^{-1}P^\alpha g).
\]

To display the meaning of the vector constraint \( C_a = 0 \), one may consider the smeared constraint function:

\[
\mathcal{V}(\tilde{N}) := \int_{\Sigma} d^3x (N^a \tilde{P}_i^a F^i_{ab} - (N^a A^i_a G_i)).
\]

It generates the infinitesimal spatial diffeomorphism by the vector field \( N^a \) on \( \Sigma \) as:

\[
\{A^i_a(x), \mathcal{V}(\tilde{N})\} = \mathcal{L}_{\tilde{N}} A^i_a(x),
\]

\[
\{\tilde{P}^i(x), \mathcal{V}(\tilde{N})\} = \mathcal{L}_{\tilde{N}} \tilde{P}^i(x).
\]

The smeared scalar constraint is weakly equivalent to the following function, which is re-expressed for quantization purpose as

\[
S(N) := \int_{\Sigma} d^3x N(x) \tilde{C}(x)
\]

\[
= \frac{\kappa \beta^2}{2} \int_{\Sigma} d^3x N \sqrt{|\det q|} \epsilon^{ij} F_{ab} - 2(1 + \beta^2) K^i_a K^j_b]. \tag{8}
\]

It generates the infinitesimal time evolution off \( \Sigma \). The constraints algebra, i.e., the Poisson brackets between these constraints, play a crucial role in the quantization programme. It can be shown that the constraints algebra of (7) has the following form:

\[
\{\mathcal{G}(\Lambda), \mathcal{G}(\Lambda')\} = \mathcal{G}([\Lambda, \Lambda']),
\]

\[
\{\mathcal{G}(\Lambda), \mathcal{V}(\tilde{N})\} = -\mathcal{G}(\mathcal{L}_{\tilde{N}} \Lambda),
\]

\[
\{\mathcal{G}(\Lambda), \mathcal{H}(N)\} = 0,
\]

\[
\{\mathcal{V}(\tilde{N}), \mathcal{V}(\tilde{N}')\} = \mathcal{V}([\tilde{N}, \tilde{N}']),
\]

\[
\{\mathcal{V}(\tilde{N}), S(M)\} = -S(\mathcal{L}_{\tilde{N}} M),
\]

\[
\{S(N), S(M)\} = -\mathcal{V}((N\partial_b M - M\partial_b N) q^{\alpha\beta})
\]

\[
-\mathcal{G}((N\partial_b M - M\partial_b N) q^{\alpha\beta} A_a))\]

\[
-(1 + \beta^2)\mathcal{G}(\frac{[\tilde{P}^i_\alpha \partial_a N, \tilde{P}^j_\beta \partial_b M]}{|\det q|}), \tag{9}
\]

where \(|\det q| q^{\alpha\beta} = \kappa^2 \beta^2 \tilde{P}^i_\alpha \tilde{P}^j_\beta \delta^{ij}\). Hence the constraints algebra is closed under the Poisson brackets, i.e., the constraints are all of first class. Note that the evolution of constraints is consistent since the Hamiltonian \( H = \int_{\Sigma} d^3x H_{tot} \) is the linear combination of the constraints functions. The evolution equations of the basic canonical pair read

\[
\mathcal{L}_t A^i_a = \{A^i_a, H\}, \quad \mathcal{L}_t \tilde{P}^i = \{\tilde{P}^i, H\}.
\]
Together with the three constraint equations, they are completely equivalent to the Einstein field equations. Thus general relativity is cast as a dynamical theory of connections with a compact structure group. Before finishing the discussion of this section, several remarks should be emphasized.

- **Canonical Transformation Viewpoint**

  The above construction can be reformulated in the language of canonical transformation, since the phase space of connection dynamics is the same as that of triad geometrodynamics. In the triad formalism the basic conjugate pair consists of densitized triad $\tilde{E}^i_a = \beta \tilde{P}^i_a$ and "extrinsic curvature" $K^i_j$. The Hamiltonian and constraints read

  \[
  \mathcal{H}_{\text{tot}} = \mathcal{H}' + NC
  \]

  \[
  G'_i = \epsilon_{ij} K^j_a \tilde{E}^a_k,
  \]

  \[
  C_a = \tilde{E}^i_a \nabla_a K^i_j, \tag{10}
  \]

  \[
  C = \frac{1}{\sqrt{\det q}} \frac{1}{2} \det q R + \tilde{E}^i_a \tilde{E}^b_j K^i_a K^j_b, \tag{12}
  \]

  where $\nabla_a$ is the $SO(3)$ generalized derivative operator compatible with triad $e^a_i$ and $R$ is the scalar curvature with respect to it. Since $\tilde{E}^i_a$ is a vector density of weight one, we have

  \[
  \nabla_a \tilde{E}^a_i = \partial_a \tilde{E}^a_i + \epsilon_{ij} \Gamma^j_a \tilde{E}^a_k = 0.
  \]

One can construct the desired Gaussian constraint by

\[
G_i := \frac{1}{\beta} \nabla_a \tilde{E}^a_i + G'_i,
\]

\[
= \partial_a \tilde{P}^a_i + \epsilon_{ij} \left( \Gamma^j_a + \beta K^j_a \right) \tilde{P}^a_j,
\]

which is weakly zero by construction. This motivates us to define the connection $A^a_i = \Gamma^a_i + \beta K^a_i$. Moreover, the transformation from the pair $(\tilde{E}^i_a, K^i_j)$ to $(\tilde{P}^i_a, A^i_j)$ can be proved to be a canonical transformation \cite{144}, i.e., the Poisson algebra of the basic dynamical variables is preserved under the transformation:

\[
\tilde{E}^a_i \mapsto \tilde{P}^a_i = \tilde{E}^a_i / \beta
\]

\[
K^i_j \mapsto A^i_j = \Gamma^i_j + \beta K^i_j.
\]

as

\[
\{ \tilde{P}^a_i(x), A^b_j(y) \} = \{ \tilde{E}^a_i(x), K^b_j(y) \} = \delta^a_b \delta^i_j \delta(x - y),
\]

\[
\{ A^a_i(x), A^b_j(y) \} = \{ K^a_i(x), K^b_j(y) \} = 0,
\]

\[
\{ \tilde{P}^a_i(x), \tilde{P}^b_i(y) \} = \{ \tilde{E}^a_i(x), \tilde{E}^b_i(y) \} = 0.
\]
• The Preparation for Quantization

The advantage of a dynamical theory of connections is that it is convenient to be quantized background independently. In the following procedure of quantization, the quantum algebra of the elementary observables will be generated by Holonomy, i.e., connection smeared on a curve, and Electric Flux, i.e., densitized triad smeared on a 2-surface. So no information of background would affect the definition of the quantum algebra. In the remainder of the paper, in order to incorporate also spinors, we will enlarge the internal gauge group to be $SU(2)$. This does not damage the prior constructions because the Lie algebra of $SU(2)$ is the same as that of $SO(3)$. Due to the well-known nice properties of compact Lie group $SU(2)$, such as the Haar measure and Peter-Weyl theorem, one can obtain the background independent representation of the quantum algebra and the spin-network decomposition of the kinematic Hilbert space.

• Analysis on Constraint Algebra

The classical constraint algebra (9) is an infinite dimensional Poisson algebra. However, it is not a Lie algebra unfortunately, because the Poisson bracket between two scalar constraints has structure function depending on dynamical variables. This character causes much trouble in solving the constraints quantum mechanically. On the other hand, one can see from Eq.(9) that the algebra generated by Gaussian constraints forms not only a subalgebra but also a 2-side ideal in the full constraint algebra. Thus one can first solve the Gaussian constraints independently. It is convenient to find the quotient algebra with respect to the Gaussian constraint subalgebra as

$$\{ V(\vec{N}), V(\vec{N}') \} = V([\vec{N}, \vec{N}']) ,$$
$$\{ V(\vec{N}), S(M) \} = -S(\mathcal{L}_M M) ,$$
$$\{ S(N), S(M) \} = -V((N\partial_a M - M\partial_a N)q^{ab}) ,$$

which plays a crucial role in solving the constraints quantum mechanically. But the subalgebra generated by the diffeomorphism constraints can not form an ideal. Hence the procedures of solving the diffeomorphism constraints and solving Hamiltonian constraints are entangled with each other. This leads to certain ambiguity in the construction of a Hamiltonian constraint operator [132] [147]. Fortunately, Master Constraint Project addresses the above two problems as a whole by introducing a new classical constraint algebra [147]. The new algebra is a Lie algebra where the diffeomorphism constraints form a 2-side ideal. We will come back to this point in the discussion on quantum dynamics of loop quantum gravity.

3 Quantum Kinematics

In this section, we will begin to quantize the above classical dynamics of connections as a background independent quantum field theory. The main purpose is to construct a suitable kinematical Hilbert space $\mathcal{H}_{kin}$ for the representation of quantum observables.
We would like to first construct the Hilbert space in a more concrete and straightforward way (constructive quantum field theory aspect \[84\]) in present section. Then we will reformulate the construction in the language of GNS-construction (algebraic quantum field theory aspect \[86\]) in the next section. It should be emphasized that both constructions are completely equivalent and can be generalized to all background independent non-perturbative Yang-Mills gauge field theories with compact gauge groups.

### 3.1 Quantum Configuration Space

In quantum mechanics, the kinematical Hilbert space is \(L^2(\mathbb{R}^3, d^3x)\), where the simple \(\mathbb{R}^3\) is the classical configuration space of free particle which has finite degrees of freedom, and \(d^3x\) is the Lebesgue measure on \(\mathbb{R}^3\). In quantum field theory, it is expected that the kinematical Hilbert space is also the \(L^2\) space on the configuration space of the field, which is infinite dimensional, with respect to some Borel measure naturally defined. However, it is often hard to define concretely a Borel measure on the classical configuration space, since the integral theory on infinite dimensional space is involved \[58\]. Thus the intuitive expectation should be modified, and the concept of quantum configuration space should be introduced as a suitable enlargement of the classical configuration space so that an infinite dimensional measure, often called cylindrical measure, can be well defined on it. The example of a scalar field can be found in the references \[21\][25]. For quantum gravity, it should be emphasized that the construction for quantum configuration space must be background independent. Fortunately, general relativity has been reformulated as a dynamical theory of \(SU(2)\) connections, which would be great helpful for our further development.

The classical configuration space for gravitational field, which is denoted by \(\mathcal{A}\), is a collection of the \(su(2)\)-valued connection 1-form field smoothly distributed on \(\Sigma\). The idea of the construction for quantum configuration is due to the concept of Holonomy.

**Definition 3.1.1:** Given a smooth \(SU(2)\) connection field \(A^i_a\) and an analytic curve \(c\) with the parameter \(t \in [0, 1]\) supported on a compact subset (compact support) of \(\Sigma\), the corresponding holonomy is defined by the solution of the parallel transport equation \[105\]

\[
\frac{d}{dt}A(c, t) = -[A^i_a e^a \tau_i] A(c, t),
\]

with the initial value \(A(c, 0) = 1\), where \(e^a\) is the tangent vector of the curve and \(\tau_i \in su(2)\) constitute an orthonormal basis with respect to the Killing-Cartan metric \(\eta(\xi, \zeta) := -2 \text{Tr}(\xi \zeta)\), which satisfy \([\tau_i, \tau_j] = e^k \tau_k\) and are fixed once for all. Thus the holonomy is an element in \(SU(2)\), which can be expressed as

\[
A(c) = \mathcal{P} \exp\left(- \int_0^1 [A^i_a e^a \tau_i] dt \right),
\]

where \(A(c) \in SU(2)\) and \(\mathcal{P}\) is a path-ordering operator along the curve \(c\) (see the footnote at p382 in \[105\]).
The definition can be well extended to the case of piecewise analytic curves via the relation:

\[ A(c_1 \circ c_2) = A(c_1)A(c_2), \tag{15} \]

where \( \circ \) stands for the composition of two curves. It is easy to see that a holonomy is invariant under the re-parametrization and is covariant under changing the orientation, i.e.,

\[ A(c^{-1}) = A(c)^{-1}. \tag{16} \]

So one can formulate the properties of holonomy in terms of the concept of the equivalent classes of curves.

**Definition 3.1.2:** Two analytic curves \( c \) and \( c' \) are said to be equivalent if and only if they have the same source \( s(c) \) (beginning point) and the same target \( t(c) \) (end point), and the holonomies of the two curves are equal to each other, i.e., \( A(c) = A(c') \) \( \forall A \in \mathcal{A} \). A equivalent class of analytic curves is defined to be an edge, and a piecewise analytic path is an composition of edges.

To summarize, the holonomy is actually defined on the set \( \mathcal{P} \) of piecewise analytic paths with compact supports. The two properties (15) and (16) mean that each connection in \( \mathcal{A} \) is a homomorphism from \( \mathcal{P} \), which is so-called a groupoid by definition [153], to our compact gauge group \( SU(2) \). Note that the internal gauge transformation and spatial diffeomorphism act covariantly on a holonomy as

\[ A(e) \mapsto g(t(e))^{-1}A(e)g(s(e)) \quad \text{and} \quad A(e) \mapsto A(\varphi \circ e), \tag{17} \]

for any \( SU(2) \)-valued function \( g(x) \) on \( \Sigma \) and spatial diffeomorphism \( \varphi \). All above discussion is for classical smooth connections in \( \mathcal{A} \). The quantum configuration space for loop quantum gravity can be constructed by extending the concept of holonomy, since its definition does not depend on an extra background. One thus obtains the quantum configuration space \( \overline{\mathcal{A}} \) of loop quantum gravity as the following.

**Definition 3.1.3:** The quantum configuration space \( \overline{\mathcal{A}} \) is a collection of all quantum connections \( A \), which are algebraic homomorphism maps without any continuity assumption from the collection of piecewise analytic paths with compact supports, \( \mathcal{P} \), on \( \Sigma \) to the gauge group \( SU(2) \), i.e., \( \overline{\mathcal{A}} := \text{Hom}(\mathcal{P}, SU(2)) \). Thus for any \( A \in \overline{\mathcal{A}} \) and edge \( e \) in \( \mathcal{P} \),

\[ A(e_1 \circ e_2) = A(e_1)A(e_2) \quad \text{and} \quad A(e^{-1}) = A(e)^{-1}. \]

The transformations of quantum connections under internal gauge transformations and diffeomorphisms are defined by Eq. (17).

\[ ^3 \text{It is easy to see that the definition of } \overline{\mathcal{A}} \text{ does not depend on the choice of local section in } SU(2)-\text{bundle, since the internal gauge transformations leave } \overline{\mathcal{A}} \text{ invariant.} \]
The above discussion on the smooth connections shows that the classical configuration space \( \mathcal{A} \) can be understood as a subset in the quantum configuration space \( \widetilde{\mathcal{A}} \). Moreover, the Giles theorem \([83]\) shows precisely that a smooth connection can be recovered from its holonomies by varying the length and location of the paths. On the other hand, it was shown in Refs. \([18][19][153]\) that the quantum configuration space \( \widetilde{\mathcal{A}} \) can be constructed via a projective limit technique and admits a natural defined topology. To make the discussion precise, we begin with a few definitions.

**Definition 3.1.4:**

1. A finite set \( \{e_1, ..., e_N\} \) of edges is said to be independent if the edges \( e_i \) can only intersect each other at their sources \( s(e_i) \) or targets \( t(e_i) \).

2. A finite graph is a collection of a finite set \( \{e_1, ..., e_N\} \) of independent edges and their vertices, i.e. their sources \( s(e_i) \) and targets \( t(e_i) \). We denote by \( E(\gamma) \) and \( V(\gamma) \) respectively as the sets of independent edges and vertices of a given finite graph \( \gamma \). \( N_\gamma \) denotes the number of elements in \( E(\gamma) \).

3. A subgroupoid \( \alpha(\gamma) \subset \mathcal{P} \) can be generated from \( \gamma \) by identifying \( V(\gamma) \) as the set of objects and all \( e \in E(\gamma) \) together with their inverses and finite compositions as the set of homomorphisms. This kind of subgroupoid in \( \mathcal{P} \) is called tame subgroupoid. \( \alpha(\gamma) \) is independent of the orientation of \( \gamma \), so the graph \( \gamma \) can be recovered from tame subgroupoid \( \alpha \) up to the orientations on the edges. We will also denote by \( N_\alpha \) the number of elements in \( E(\gamma) \) where \( \gamma \) is recovered by the tame subgroupoid \( \alpha \).

4. \( \mathcal{L} \) denotes the set of all tame subgroupoids in \( \mathcal{P} \).

One can equip a partial order relation \( \prec \) on \( \mathcal{L} \) defined by \( \alpha \prec \alpha' \) if and only if \( \alpha \) is a subgroupoid in \( \alpha' \). Obviously, for any two tame subgroupoids \( \alpha \equiv \alpha(\gamma) \) and \( \alpha' \equiv \alpha(\gamma') \) in \( \mathcal{L} \), there exists \( \alpha'' \equiv \alpha(\gamma'') \in \mathcal{L} \) such that \( \alpha, \alpha' \prec \alpha'' \), where \( \gamma'' \equiv \gamma \cup \gamma' \). Define \( \mathcal{A}_\alpha \equiv \text{Hom}(\alpha, SU(2)) \) as the set of all homomorphisms from the subgroupoid \( \alpha(\gamma) \) to the group \( SU(2) \). Note that an element \( A_\alpha \in \mathcal{A}_\alpha \) is completely determined by the \( SU(2) \) group elements \( A(e) \) where \( e \in E(\gamma) \), so that one has a bijection \( \lambda : \mathcal{A}_\alpha \rightarrow SU(2)^{N_\alpha} \), which induces a topology on \( \mathcal{A}_\alpha \) such that \( \lambda \) is a topological homomorphism. Then for any pair \( \alpha \prec \alpha' \), one can define a surjective projection map \( P_{\alpha'\alpha} \) from \( \mathcal{A}_{\alpha'} \) to \( \mathcal{A}_\alpha \) by restricting the domain of the map \( A_{\alpha'} \) from \( \alpha' \) to the subgroupoid \( \alpha \), and these projections satisfy the consistency condition \( P_{\alpha''\alpha} \circ P_{\alpha'\alpha} = P_{\alpha''\alpha'} \). Thus a projective family \( \{ \mathcal{A}_\alpha, P_{\alpha'\alpha}\}_{\alpha \prec \alpha'} \) is obtained by above constructions. Then the projective limit \( \lim_\alpha (\mathcal{A}_\alpha) \) is naturally obtained.

**Definition 3.1.5:** The projective limit \( \lim_\alpha (\mathcal{A}_\alpha) \) of the projective family \( \{ \mathcal{A}_\alpha, P_{\alpha'\alpha}\}_{\alpha \prec \alpha'} \) is a subset of the direct product space \( \mathcal{A}_\infty := \prod_{\alpha \in \mathcal{L}} \mathcal{A}_\alpha \) defined by

\[
\lim_\alpha (\mathcal{A}_\alpha) := \{ [A_\alpha]_{\alpha \in \mathcal{L}} | P_{\alpha'\alpha} A_{\alpha'} = A_\alpha, \forall \alpha \prec \alpha' \}.
\]

\(^4\)A partial order on \( \mathcal{L} \) is a relation, which is reflective (\( \alpha \prec \alpha \)), symmetric (\( \alpha \prec \alpha' \Rightarrow \alpha' \prec \alpha \)), and transitive (\( \alpha \prec \alpha', \alpha' \prec \alpha'' \Rightarrow \alpha \prec \alpha'' \)). Note that not all pairs in \( \mathcal{L} \) need to have a relation.
Note that the projection $P_{\alpha'}$ is surjective and continuous with respect to the topology of $\mathcal{A}_\alpha$. One can equip the direct product space $\mathcal{A}_\infty := \prod_{\alpha \in L} \mathcal{A}_\alpha$ with the so-called Tychonov topology. Since any $\mathcal{A}_\alpha$ is a compact Hausdorff space, by Tychonov theorem $\mathcal{A}_\infty$ is also a compact Hausdorff space. One then can prove that the projective limit, $\lim_\alpha (\mathcal{A}_\alpha)$, is a closed subset in $\mathcal{A}_\infty$ and hence a compact Hausdorff space with respect to the topology induced from $\mathcal{A}_\infty$. At last, one can find the relation between the projective limit and the prior constructed quantum configuration space $\mathcal{A}$. As one might expect, there is a bijection $\Phi$ between $\mathcal{A}$ and $\lim_\alpha (\mathcal{A}_\alpha)$ [144]:

$$\Phi : \mathcal{A} \rightarrow \lim_\alpha (\mathcal{A}_\alpha); \quad A \mapsto \{A|_{\alpha \in L}\},$$

where $A|_{\alpha}$ means the restriction of the domain of the map $A \in \mathcal{A} = \text{Hom}(P, SU(2))$. As a result, the quantum configuration space is identified with the projective limit space and hence can be equipped with the topology. In conclusion, the quantum configuration space $\mathcal{A}$ is constructed to be a compact Hausdorff topological space.

### 3.2 Cylindrical Functions on Quantum Configuration Space

Given the projective family $\{\mathcal{A}_\alpha, P_{\alpha'\alpha}\}_{\alpha \prec \alpha'}$, the cylindrical function on its projective limit $\mathcal{A}$ is well defined as follows.

**Definition 3.2.1:** Let $C(\mathcal{A}_\alpha)$ be the set of all continuous complex functions on $\mathcal{A}_\alpha$, two functions $f_\alpha \in C(\mathcal{A}_\alpha)$ and $f_{\alpha'} \in C(\mathcal{A}_{\alpha'})$ are said to be equivalent or cylindrically consistent, denoted by $f_\alpha \sim f_{\alpha'}$, if and only if $P_{\alpha''\alpha}^* f_\alpha = P_{\alpha''\alpha'}^* f_{\alpha'}$ for all $\alpha''$ such that $\alpha'' \succ \alpha, \alpha'$, where $P_{\alpha''\alpha}^*$ denotes the pullback map induced from $P_{\alpha''\alpha}$. Then the space $\text{Cyl}(\mathcal{A})$ of cylindrical functions on the projective limit $\mathcal{A}$ is defined to be the space of equivalent classes $[f]$, i.e.,

$$\text{Cyl}(\mathcal{A}) := \bigcup_{\alpha} C(\mathcal{A}_\alpha)/\sim.$$ 

One then can easily prove the following proposition by definition.

**Proposition 3.2.1:** All continuous functions $f_\alpha$ on $\mathcal{A}_\alpha$ are automatically cylindrical since each of them can generate an equivalent class $[f_\alpha]$ via the pullback map $P_{\alpha''\alpha}^*$ for all $\alpha'' \succ \alpha$, and the dependence of $P_{\alpha''\alpha}^* f_\alpha$ on the groups associated to the edges in $\alpha'$ but not in $\alpha$ is trivial, i.e., by the definition of the pull back map,

$$(P_{\alpha''\alpha}^* f_\alpha)(A(e_1), ..., A(e_{N_\alpha}), ..., A(e_{N_{\alpha'}})) = f_\alpha(A(e_1), ..., A(e_{N_\alpha})). \quad (18)$$

On the other hand, by definition, given a cylindrical function $f \in \text{Cyl}(\mathcal{A})$ there exists a suitable groupoid $\alpha$ such that $f = [f_\alpha]$, so one can identify $f$ with $f_\alpha$. Moreover, given two cylindrical functions $f, f' \in \text{Cyl}(\mathcal{A})$, by definition of cylindrical functions and the
property of projection map, there exists a common groupoid $\alpha$ and $f_\alpha, f'_\alpha \in C(\mathcal{A}_\alpha)$ such that $f = [f_\alpha]$ and $f' = [f'_\alpha]$.

Let $f, f' \in Cyl(\mathcal{A})$, there exists groupoid $\alpha$ such that $f = [f_\alpha]$, and $f' = [f'_\alpha]$, then the following operations are well defined

$$f + f' := [f_\alpha + f'_\alpha], \quad ff' := [f_\alpha f'_\alpha], \quad zf := [zf_\alpha], \quad \bar{f} := [ar{f}_\alpha],$$

where $z \in \mathbb{C}$ and $\bar{f}$ denotes complex conjugate. So we construct $Cyl(\mathcal{A})$ as an Abelian $\ast$-algebra. In addition, there is a unital element in the algebra because $Cyl(\mathcal{A})$ contains constant functions. Moreover, we can well define the sup-norm for $f = [f_\alpha]$ by

$$\|f\| := \sup_{A_\alpha \in \mathcal{A}_\alpha} |f_\alpha(A_\alpha)|,$$ (19)

which satisfies the $C^*$ property $\|f\bar{f}\| = \|f\|^2$. Then $Cyl(\mathcal{A})$ is a unital Abelian $C^*$-algebra, after the completion with respect to the norm. From the theory of $C^*$-algebra, it is known that a unital Abelian $C^*$-algebra is identical to the space of continuous functions on its spectrum space via an isometric isomorphism, the so-called Gel’fand transformation (see e.g. [144]). So one has the following theorem [18][19], which finishes this section.

**Theorem 3.2.1:**
(1) The space $Cyl(\mathcal{A})$ has the structure of a unital Abelian $C^*$-algebra after completion with respect to the sup-norm.

(2) Quantum configuration space $\mathcal{A}$ is the spectrum space of completed $Cyl(\mathcal{A})$ such that $Cyl(\mathcal{A})$ is identical to the space $C(\mathcal{A})$ of continuous functions on $\mathcal{A}$.

### 3.3 Kinematical Hilbert Space

The main purpose of this subsection is to construct a kinematical Hilbert space $\mathcal{H}_{kin}$ for loop quantum gravity, which is a $L^2$ space on the quantum configuration space $\mathcal{A}$ with respect to some measure $d\mu$. There is a well-defined probability measure on $\mathcal{A}$ originated from the Haar measure on the compact group $SU(2)$, which is named as the Ashtekar-Lewandowski Measure for loop quantum gravity. Consider the simplest case where the groupoid is generated by one edge $e$ only. Then the corresponding quantum configuration space $\mathcal{A}_e$, being trivial elsewhere, is identical to the group $SU(2)$. The continuous functions on $\mathcal{A}_e$ is certainly contained in $Cyl(\mathcal{A})$. Due to the compactness of $SU(2)$, there exists a unique probability measure, namely the Haar measure on it, which is invariant under right and left translations and inverse of the group elements.

**Theorem 3.3.1**\[55\]:
Given a compact group $G$ and an automorphism $\varphi : G \to G$ on it, there exists a unique
measure $d\mu_H$ on $G$, named as Haar measure, such that:

\[
\int_G d\mu_H = 1, \tag{20}
\]

\[
\int_G f(g)d\mu_H = \int_G f(hg)d\mu_H = \int_G f(gh)d\mu_H
\]

\[
= \int_G f(g^{-1})d\mu_H = \int_G f \circ \varphi(g)d\mu_H, \tag{21}
\]

for all continuous functions $f$ on $G$ and for all $h \in G$.

Thus one equips $\mathcal{A}_e$ with the measure $\mu_e \equiv \mu_H$. Similarly, a probability measure can be defined on any graph with finite number of edges by the direct product of Haar measure, since $\mathcal{A}_\alpha = SU(2)^N$. Then for any groupoid $\alpha$, a Hilbert space is defined on $\mathcal{A}_\alpha$ as $H_\alpha = L^2(\mathcal{A}_\alpha, d\mu_\alpha) = \bigotimes_{e \in \mathcal{A}_\alpha} L^2(\mathcal{A}_e, d\mu_e)$. Moreover, the family of measures $\{\mu_\alpha\}_{\alpha \in L}$ defined on the projective family $\{\mathcal{A}_\alpha, P_{\alpha \beta}\}_{\alpha \prec \beta}$ are cylindrically consistent, since

\[
\int_{\mathcal{A}_\alpha} f_{\alpha}(A(e_1), ..., A(e_{N_e}))d\mu_\alpha = \int_{\mathcal{A}_\alpha} f_{\alpha}(A(e_1), ..., A(e_{N_e}))d\mu_\beta
\]

\[
= \int_{\mathcal{A}_\alpha} f_{\alpha}(A(e_1), ..., A(e_{N_e}))d\mu_\alpha
\]

\[
= \int_{\mathcal{A}_\alpha} f_{\alpha}d\mu_\alpha,
\]

due to Eqs. (18) and (20). Given such a cylindrically consistent family of measures $\{\mu_\alpha\}_{\alpha \in L}$, a probability measure $d\mu$ is uniquely well defined on the quantum configuration space $\mathcal{A}$, which is described precisely in the theorem below.

**Theorem 3.3.2 [144]:**

Given the projective family $\{\mathcal{A}_\alpha, P_{\alpha \beta}\}_{\alpha \prec \beta}$, whose projective limit is $\mathcal{A}$, and the cylindrically consistent family of measures $\{\mu_\alpha\}_{\alpha \in L}$ constructed from the Haar measure on the compact group, there exists a unique regular Borel probability measure $d\mu$ on the projective limit $\mathcal{A}$ such that

\[
\int_{\mathcal{A}} f d\mu = \int_{\mathcal{A}_\alpha} f_{\alpha}d\mu_\alpha, \forall f = [f_{\alpha}] \in \text{Cyl}(\mathcal{A}),
\]

which is guaranteed by proposition 3.2.1.

Then $\mathcal{A}$ is equipped with the Ashtekar-Lewandowski measure $d\mu$ and becomes a topological measure space [17][18]. This measure will help us define a state on the quantum holonomy-flux algebra for gauge field theory, which is called Ashtekar-Isham-Lewandowski state in the language of $\text{GNS}$-construction. Moreover the two important gauge invariant properties of Ashtekar-Lewandowski measure make it well suitable for
The diffeomorphism invariant gauge field theory.

**Theorem 3.3.3:**
The Ashtekar-Lewandowski measure is invariant under internal gauge transformations $g(x)$ and spatial diffeomorphisms $\varphi$, i.e.,

$$\int_{\mathcal{A}} g \circ f d\mu = \int_{\mathcal{A}} f d\mu \quad \text{and} \quad \int_{\mathcal{A}} \varphi \circ f d\mu = \int_{\mathcal{A}} f d\mu,$$

$\forall f \in \text{Cyl} (\mathcal{A})$.

**Proof:**
(1) (Internal gauge invariance)
$$\int_{\mathcal{A}} g \circ f d\mu = \int_{\mathcal{A}} g \circ f \alpha d\mu \alpha = \int_{\mathcal{A}} f \alpha d\mu \alpha = \int_{\mathcal{A}} f d\mu,$$

$\forall f = [f_\alpha] \in \text{Cyl} (\mathcal{A})$, where we used

$$g \circ f_\alpha (A(e_1), ..., A(e_{N_\alpha})) = f_\alpha (g(t(e_1))^{-1}A(e_1)g(s(e_1)), ..., g(t(e_{N_\alpha}))^{-1}A(e_{N_\alpha})g(s(e_{N_\alpha}))),$$

since Haar measure is invariant under right and left translations.

(2) (Diffeomorphism invariance)
$$\int_{\mathcal{A}} \varphi \circ f d\mu = \int_{\mathcal{A}} \varphi \circ f \alpha d\mu \alpha = \int_{\mathcal{A}} f \alpha d\mu \alpha = \int_{\mathcal{A}} f d\mu,$$

where $f_{\varphi \circ \alpha} \equiv f_\alpha (A(\varphi \circ e_1), ..., A(\varphi \circ e_{N_\alpha}))$ and we relabel $A(\varphi \circ e_i) \mapsto A(e_i)$ in the second step.

With the above constructed measure on $\mathcal{A}$, the kinematical Hilbert space $\mathcal{H}_{\text{kin}}$ is obtained straightforwardly as

$$\mathcal{H}_{\text{kin}} := L^2 (\mathcal{A}, d\mu).$$

Thus, given any $f = [f_\alpha], f' = [f'_\alpha] \in \text{Cyl} (\mathcal{A})$, the $L^2$ inner product of them is expressed as

$$\langle f | f' \rangle_{\text{kin}} := \int_{\mathcal{A}} (P^*_{\alpha' \alpha} \tilde{f}_\alpha)(P^*_{\alpha' \alpha'} f'_\alpha) d\mu_{\alpha' \alpha},$$

for any groupoid $\alpha''$ containing both $\alpha$ and $\alpha'$. It should be noted that the cylindrical functions in $\mathcal{H}_{\text{kin}}$ is dense with respect to the $L^2$ inner product, as they are dense in $C(\mathcal{A})$ with respect to the sup-norm. As a result, the kinematical Hilbert space can be viewed as the completion of $\text{Cyl}(\mathcal{A})$ with respect to the inner product (23), i.e.,

$$\mathcal{H}_{\text{kin}} = \langle \text{Cyl}(\mathcal{A}) \rangle = \langle \cup_{\alpha \in \mathcal{L}} \mathcal{H}_\alpha \rangle,$$

here the $\langle \cdot \rangle$ means the completion with respect to the inner product (23). Later we will show that $\mathcal{H}_{\text{kin}}$ is a non-separable Hilbert space. It is important to note that all the above constructions are background independent.

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3.4 Spin-network Decomposition of Kinematical Hilbert Space

Up to now, the kinematical Hilbert space $\mathcal{H}_{\text{kin}}$ for loop quantum gravity has been well defined. In this subsection, it will be shown that $\mathcal{H}_{\text{kin}}$ can be decomposed into the orthogonal direct sum of 1-dimensional subspaces. One can thus find a system of basis, named as spin-network basis, in the Hilbert space, which consists of uncountably infinite elements. So the kinematic Hilbert space is non-separable. In the following, we will do the decomposition in three steps.

- **Spin-network Decomposition on Single Edge**
  Given a groupoid of one edge $e$, which naturally associates with a group $SU(2) = \mathcal{G}_e$, the elements of $\mathcal{G}_e$ are the quantum connections only taking nontrivial values on $e$. Then we consider the decomposition of the Hilbert space $\mathcal{H}_e = L^2(\mathcal{G}_e, d\mu_e) = L^2(SU(2), d\mu_H)$, which is nothing but the space of square integrable functions on the compact group $SU(2)$ with the natural $L^2$ inner product.
  It is natural to define several operators on $\mathcal{H}_e$. First, the so-called configuration operator $\hat{f}(A(e))$ whose operation on any $\psi$ in a dense domain of $L^2(SU(2), d\mu_H)$ is nothing but multiplication by the function $f(A(e))$, i.e.,
  \[ \hat{f}(A(e))\psi(A(e)) := f(A(e))\psi(A(e)), \]
  where $A(e) \in SU(2)$. Second, given any vector $\xi \in su(2)$, it generates left invariant vector field $L(\xi)$ and right invariant vector field $R(\xi)$ on $SU(2)$ by
  \[ L(\xi)\psi(A(e)) := \frac{d}{dt}|_{t=0}\psi(\exp(t\xi)A(e)), \]
  \[ R(\xi)\psi(A(e)) := \frac{d}{dt}|_{t=0}\psi(\exp(-t\xi)A(e)), \]
  for any function $\psi \in C^1(SU(2))$. Then one can define the so-called momentum operators on $\mathcal{H}_e$ by
  \[ \hat{J}^{(L)}_i = iL^{(\tau_i)} \quad \text{and} \quad \hat{J}^{(R)}_i = iR^{(\tau_i)}, \]
  where the generators $\tau_i \in su(2)$ constitute an orthonormal basis with respect to the Killing-Cartan metric. The momentum operators have the well-known commutation relation of the angular momentum operators in quantum mechanics:
  \[ [\hat{J}^{(L)}_i, \hat{J}^{(L)}_j] = i\epsilon_{ijk}\hat{J}^{(L)}_k, \quad [\hat{J}^{(R)}_i, \hat{J}^{(R)}_j] = i\epsilon_{ijk}\hat{J}^{(R)}_k, \quad [\hat{J}^{(L)}_i, \hat{J}^{(R)}_j] = 0. \]
  Thirdly, the Casimir operator on $\mathcal{H}_e$ can be expressed as
  \[ \hat{f}^2 := \delta^{ij}\hat{J}^{(L)}_i\hat{J}^{(L)}_j = \delta^{ij}\hat{J}^{(R)}_i\hat{J}^{(R)}_j. \]

The decomposition of $\mathcal{H}_e = L^2(SU(2), d\mu_H)$ is provided by the following Peter-Weyl Theorem.
Theorem 3.4.1 [56]:
Given a compact group G, the function space $L^2(G, d\mu_H)$ can be decomposed as an orthogonal direct sum of finite dimensional Hilbert space, and the matrix elements of the equivalent classes of finite dimensional irreducible representations of G form an orthogonal basis in $L^2(G, d\mu_H)$.

Note that a finite dimensional irreducible representation of G can be regarded as a matrix-valued function on G, so the matrix elements are functions on G. Using this theorem, one can find the decomposition of the Hilbert space:

$$L^2(SU(2), d\mu_H) = \bigoplus_j [H^j \otimes H^j],$$

where $j$, labelling irreducible representations of SU(2), are the half integers, $H^j$ denotes the carrier space of the $j$-representation of dimension $2j + 1$, and $H^j$ is its dual space. The basis $\{e^i_m \otimes e^n_i\}$ in $H^j \otimes H^j$ maps a group element $g \in SU(2)$ to a matrix [\pi_{mn}^j(g)], where $m, n = -j, ..., j$. Thus the space $H^j \otimes H^j$ is spanned by the matrix element functions $\pi_{mn}^j$ of equivalent $j$-representations. Moreover, the spin-network basis can be defined.

Proposition 3.4.1 [57]
The system of spin-network functions on $H_e$, consisting of matrix elements $[\pi_{mn}^j]$ in finite dimensional irreducible representations labelled by half-integers $\{j\}$, satisfies

$$\hat{\mathbf{j}}^2 \pi_{mn}^j = j(j + 1) \pi_{mn}^j, \quad \hat{\mathbf{J}}^L \pi_{mn}^j = m \pi_{mn}^j, \quad \hat{\mathbf{J}}^R \pi_{mn}^j = n \pi_{mn}^j,$$

where $j$ is called angular momentum quantum number and $m, n = -j, ..., j$ magnetic quantum number. The normalized functions $\sqrt{2j + 1} \pi_{mn}^j$ form a system of complete orthonormal basis in $H_e$ since

$$\int_{H_e} \pi_{mn}^{j*} \pi_{mn}^j d\mu_e = \frac{1}{2j + 1} \delta_{j_1} \delta_{m_1} \delta_{n_1},$$

which is called the spin-network basis on $H_e$. So the Hilbert space on a single edge has been decomposed into one dimensional subspaces.

Note that the system of operators $\{\hat{\mathbf{j}}^2, \hat{\mathbf{J}}^R, \hat{\mathbf{J}}^L\}$ forms a complete set of commutable operators in $H_e$. There is a cyclic "vacuum state" in the Hilbert space, which is the $(j = 0)$-representation $\Omega_e = \pi^{\text{id}} = 1$, representing that there is no geometry on the edge.

Spin-network Decomposition on Finite Graph
Given a groupoid $\alpha$ generated by a graph $\gamma$ with $N$ oriented edges $e_i$ and $M$ vertices, one can define the configuration operators on the corresponding Hilbert space $H_\alpha$ by

$$\hat{f}(A(e_i)) \psi_\alpha(A(e_1), ..., A(e_N)) := f(A(e_i)) \psi_\alpha(A(e_1), ..., A(e_N)).$$
The momentum operators $J_i^{(e,v)}$ associated with an edge $e$ connecting a vertex $v$ are defined as

$$J_i^{(e,v)} := (1 \otimes \ldots \otimes \hat{J}_i \otimes \ldots \otimes 1),$$

where we set $\hat{J}_i = J_i^{(e)}$ if $v = s(e)$ and $\hat{J}_i = J_i^{(e)}$ if $v = t(e)$. Note that the choice is based on the definition of gauge transformations (17). Note also that $\hat{J}_i^{(e,v)}$ only acts nontrivially on the Hilbert space associated with the edge $e$. Then one can define a vertex operator associated with vertex $v$ in analogy with the total angular momentum operator via

$$[\hat{J}_i^2] := \delta^{ij} \hat{J}_i \hat{J}_j,$$

where

$$\hat{J}_j := \sum_{e' \accompany v} J_i^{(e',v)}.$$

Obviously, $\mathcal{H}_\alpha$ can be firstly decomposed by the representations on each edge $e$ of $\alpha$ as:

$$\mathcal{H}_\alpha = \otimes_v \mathcal{H}_e = \otimes_v [\oplus_j (\mathcal{H}_j^e \otimes \mathcal{H}_j^{e*})] = \oplus_j [\otimes_v (\mathcal{H}_j^{e(e)} \otimes \mathcal{H}_j^{e*(e)})],$$

where $j := (j_1, \ldots, j_N)$ assigns to each edge an irreducible representation of $SU(2)$, in the fourth step the Hilbert spaces associated with the edges are allocated to the vertices where these edges meet so that for each vertex $v$,

$$\mathcal{H}_j^{e(e)} \equiv \otimes_{e(v)=e} \mathcal{H}_j^e \quad \text{and} \quad \mathcal{H}_j^{e*(e)} \equiv \otimes_{e(v)=e} \mathcal{H}_j^{e*}.$$

The group of internal gauge transformations $g(v) \in SU(2)$ at each vertex is reducibly represented on the Hilbert space $\mathcal{H}_j^{e(e)} \otimes \mathcal{H}_j^{e*(e)}$ in a natural way. So this Hilbert space can be decomposed as a direct sum of irreducible representation spaces via Clebsch-Gordon decomposition:

$$\mathcal{H}_j^{e(e)} \otimes \mathcal{H}_j^{e*(e)} = \oplus_l \mathcal{H}_j^{l(e)}. \quad (26)$$

As a result, $\mathcal{H}_\alpha$ can be further decomposed as:

$$\mathcal{H}_\alpha = \oplus_j [\otimes_v (\oplus_l \mathcal{H}_j^{l(e)})] = \oplus_j [\otimes_l (\mathcal{H}_j^{l(e)} \otimes \mathcal{H}_j^{l*(e)})] = \oplus_j [\otimes_l \mathcal{H}_{\alpha,l)].$$

It can also be viewed as the eigenvector space decomposition of the commuting operators $[\hat{J}_i^2]$ (with eigenvalues $l(l+1)$) and $[\hat{J}_j^2] = \delta_{ij} \hat{J}_i \hat{J}_j$. Note that $l := (l_1, \ldots, l_M)$ assigns to each vertex of $\alpha$ an irreducible representation of gauge transformation. One may also enlarge the set of commuting operators to further refine the decomposition of the Hilbert space. Note that the subspace of $\mathcal{H}_\alpha$ with $l = 0$ is gauge invariant, since the representation of gauge transformations is trivial.
• Spin-network Decomposition of $\mathcal{H}_{\text{kin}}$

Since $\mathcal{H}_{\text{kin}}$ has the structure $\mathcal{H}_{\text{kin}} = \langle \bigcup_{\alpha \in \mathcal{L}} H_\alpha \rangle$, one may consider to construct it as a direct sum of $H_\alpha$. The construction is precisely described as a theorem below.

**Theorem 3.4.2:**

Consider assignments $\mathbf{j} = (j_1, ..., j_N)$ to the edges of any groupoid $\alpha \in \mathcal{L}$ and assignments $\mathbf{l} = (l_1, ..., l_M)$ to the vertices. The edge representation $j$ is non-trivial on each edge, and the vertex representation $l$ is non-trivial at each spurious vertex, unless it is the base point of a close analytic loop. Let $\mathcal{H}_\alpha'$ be the Hilbert space composed by the subspaces $H_\alpha$, $j$, $l$ (assigned the above conditions) according to Eq.(26). Then $\mathcal{H}_{\text{kin}}$ can be decomposed as the direct sum of the Hilbert spaces $H'_\alpha$, i.e.,

$$\mathcal{H}_{\text{kin}} = \oplus_{\alpha \in \mathcal{L}} H'_\alpha \oplus \mathbb{C}.$$ 

**Proof:**

Since the representation on each edge is non-trivial, by definition of the inner product, it is easy to see that $H'_\alpha$ and $H'_{\alpha'}$ are mutual orthogonal if one of the groupoids $\alpha$ and $\alpha'$ has at least an edge more than the other due to

$$\int_{\mathcal{A}_e} \pi_e j mn d\mu_e = \int_{\mathcal{A}_e} 1 \cdot \pi_e j mn d\mu_e = 0$$

for any $j \neq 0$. Now consider the case of the spurious vertex. An edge $e$ with $j$-representation in a graph is assigned the Hilbert space $H_e j \otimes H_e^* j$. Inserting a vertex $v$ into the edge, one obtains two edges $e_1$ and $e_2$ split by $v$ both with $j$-representations, which belong to a different graph. By the decomposition of the corresponding Hilbert space,

$$H'_j \otimes H^{j*} \otimes H'_j \otimes H^{j*} = H'_j \otimes (\oplus_{l=0,2} H'_l) \otimes H^{j*},$$

the subspace for all $l \neq 0$ are orthogonal to the space $H'_j \otimes H^{j*}$, while the subspace for $l = 0$ coincides with $H'_j \otimes H^{j*}$ since $H'_{\text{fin}} = \mathbb{C}$ and $A(e) = A(e_1)A(e_2)$. This completes the proof.

Since there are uncountable many graphs on $\Sigma$, the kinematical Hilbert $\mathcal{H}_{\text{kin}}$ is non-separable. We denote the spin-network basis in $\mathcal{H}_{\text{kin}}$ by the vacuum state $\Pi_0 \equiv \Omega = 1$ and $\Pi_s$, $s = (\gamma(s), \mathbf{j}, \mathbf{m}, \mathbf{n})$, i.e.,

$$\Pi_s := \prod_{e \in E(\gamma(s))} \sqrt{2j_e + 1} \pi_{m,n}^{j_e} (j_e \neq 0),$$

which form an orthonormal basis with the relation $\langle \Pi_s | \Pi_{s'} \rangle_{\text{kin}} = \delta_{s,s'}$. We further denote the subset $\text{Cyl}_\gamma(\mathcal{A}) \subset \text{Cyl}(\mathcal{A})$ as the linear span of the spin-network functions $\Pi_s$ for $\gamma(s) = \gamma$.  

---

3A vertex $v$ is spurious if it is bivalent and $e \circ e'$ is itself analytic edge with $e$, $e'$ meeting at $v$.  

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The spin-network basis can be used to construct the so-called spin-network representation of loop quantum gravity.

**Definition 3.4.1:** The spin-network representation is a vector space $\tilde{\mathcal{H}}$ of complex valued functions

$$\tilde{\Psi} : S \to \mathbb{C}; \ s \mapsto \tilde{\Psi}(s),$$

where $S$ is the set of the labels $s$ for the spin-network states. $\tilde{\mathcal{H}}$ is equipped with the scalar product

$$\langle \tilde{\Psi}, \tilde{\Psi}' \rangle := \sum_{s \in S} \tilde{\Psi}(s)\tilde{\Psi}'(s)$$

between square summable functions.

The relation between the Hilbert spaces $\tilde{\mathcal{H}}$ and $\mathcal{H}_{\text{kin}}$ is clarified by the following proposition [144].

**Proposition 3.4.2:**
The spin-network transformation

$$T : \mathcal{H}_{\text{kin}} \to \tilde{\mathcal{H}}; \ \Psi \mapsto \tilde{\Psi}(s) := \langle \Pi_s, \Psi \rangle_{\text{kin}}$$

is a unitary transformation with inverse

$$T^{-1}\Psi = \sum_{s \in S} \tilde{\Psi}(s)\Pi_s.$$

Thus the connection representation and the spin-network representation are "Fourier transforms" of each other, where the role of the kernel of the transform is played by the spin-network basis. Note that, in the gauge invariant Hilbert space of loop quantum gravity (see section 5.1), the Fourier transform with respect to the gauge invariant spin network basis is the so-called loop transform, which leads to the unitary equivalent loop representation of the theory [118][74][122].

### 3.5 Holonomy-Flux Algebra and Quantum Operators

The central aim of quantum kinematics for loop quantum gravity is to look for a proper representation of quantum algebra of elementary observables. In the classical theory, the basic dynamic variables are $su(2)$-valued connection field $A^a_i$ and densitized triad field $\tilde{P}^a_i$ on $\Sigma$. However, these two basic variables are not in the algebra of elementary classical observables which will be represented in the quantum theory, whence they do not have direct quantum analogs in loop quantum gravity. The elementary classical observables in our representation theory are the complex valued functions (cylindrical functions) $f_e$ of holonomies $A(e)$ along paths $e$ in $\Sigma$, and fluxes $P_i(S)$ of triad field across 2-surfaces $S$, which is defined as

$$P_i(S) := \int_S \eta_{abc} \tilde{P}^c_i,$$
where $\eta_{\mu\nu}$ is the Levi-Civita tensor density on $\Sigma$. In the simplest case where a single edge $e$ intersects a 2-surface $S$ at a point $p$, one can calculate the Poisson bracket between two functions $f(A(e))$ and $P_i(S)$ on classical phase space $M$ as

$$\{P_i(S), f(A(e))\} = \left\{ \frac{\partial}{\partial A(e)_{mn}} f(A(e)) \right\} \cdot \{P_i(S), A(e)_{mn}\}$$

$$= \left\{ \frac{\partial}{\partial A(e)_{mn}} f(A(e)) \right\} \cdot \left\{ \frac{\kappa(S, e)}{2} \right\} \cdot \left\{ \sum_k A(e)_{mk}(\tau_i)_{kn} \right\} \text{ if } p = s(e)$$

$$= \left\{ \frac{\partial}{\partial A(e)_{mn}} f(A(e)) \right\} \cdot \left\{ \sum_k A(e)_{mk}(\tau_i)_{kn} \right\} \text{ if } p = t(e),$$

where $A(e)_{mn}$ is the matrix elements of $A(e) \in SU(2)$ and

$$\kappa(S, e) = \begin{cases} 0, & \text{if } e \cap S = 0, \text{ or } e \text{ lies in } S; \\ 1, & \text{if } e \text{ lies above } S \text{ and } e \cap S = p; \\ -1, & \text{if } e \text{ lies below } S \text{ and } e \cap S = p. \end{cases}$$

Since the surface $S$ is oriented with normal $n_\mu$, "above" means $n_\mu(\partial/\partial \nu)_p > 0$, and "below" means $n_\mu(\partial/\partial \nu)_p < 0$, where $(\partial/\partial \nu)_p$ is the tangent vector of $e$ at $p$. Then as one might expect, each flux $P_i(S)$ is associated with a flux vector field $Y_i(S)$ on the quantum configuration space $\overline{\mathcal{A}}$, algebraically introduced by the cylindrically consistent action on cylindrical functions $\psi_\gamma \in Cyl_\gamma(\overline{\mathcal{A}})$ as:

$$Y_i(S) \circ \psi_\gamma(A(e))_{e \in E(\gamma)}) = \{P_i(S), \psi_\gamma(A(e))_{e \in E(\gamma)}\},$$

where $E(\gamma)$ is the collection of all edges of the graph $\gamma$. The corresponding momentum operator associated with $S$ is defined by

$$\hat{P}_i(S) := i\hbar Y_i(S) = \hbar \{P_i(S), \cdot \},$$

which is essentially self-adjoint on $\mathcal{H}_{kin}$. Its action on differentiable cylindrical functions can be expressed explicitly as

$$\hat{P}_i(S) \psi_\gamma(A(e))_{e \in E(\gamma)} = \frac{\hbar}{2} \sum_{v \in V(\gamma) \cap S} \left[ \sum_{e \at \gamma} \kappa(S, e) j_{i(e,v)} \right] \psi_\gamma(A(e))_{e \in E(\gamma)}$$

$$= \frac{\hbar}{2} \sum_{v \in V(\gamma) \cap S} \left[ \hat{j}_{S,v}^{\,(\overline{\overline{\Delta}})} - \hat{j}_{S,v}^{\,(\overline{\overline{d}}} \right] \psi_\gamma(A(e))_{e \in E(\gamma)}, \quad (27)$$

where $V(\gamma)$ is the collection of all vertices of $\gamma$, and

$$\hat{j}_{S,v}^{\,(\overline{\overline{\Delta}})} \equiv j_{e_1,v}^{(\overline{\overline{\Delta}})} + \ldots + j_{e_n,v}^{(\overline{\overline{\Delta}})},$$

$$\hat{j}_{S,v}^{\,(\overline{\overline{d}}} \equiv j_{e_1,v}^{(\overline{\overline{d}}} + \ldots + j_{e_n,v}^{(\overline{\overline{d}})}, \quad (28)$$

for the edges $e_1, \ldots, e_n$ lying above $S$ and $e_{n+1}, \ldots, e_{n+d}$ lying below $S$. Note that $\hat{j}_{i(v)}^{(\overline{\overline{d}}}$ is the momentum operator, defined in the last section, associated with an edge $e$ connecting a vertex $v$. On the other hand, it is obvious to construct configuration operators by cylindrical functions $f_\gamma \in Cyl_\gamma(\overline{\mathcal{A}})$ as:

$$\hat{f}_\gamma \psi_\gamma(A(e))_{e \in E(\gamma)} := f_\gamma(A(e))_{e \in E(\gamma)} \psi_\gamma(A(e))_{e \in E(\gamma)}.$$
Note that \( \hat{f}_\gamma \) may change the graph, i.e., \( \hat{f}_\gamma : \text{Cyl}_\gamma(\overline{\mathcal{A}}) \to \text{Cyl}_{\gamma'}(\overline{\mathcal{A}}) \). So far, the elementary operators of quantum kinematics have been well defined on \( \mathcal{H}_{\text{kin}} \). One can calculate the elementary canonical commutation relations between these operators as:

\[
[\hat{f}_e(A(e)), \hat{f}_{e'}(A(e'))] = 0,
[\hat{P}_i(S), \hat{f}_e(A(e))]
= \frac{i\hbar}{\partial A(e)_{mn}} f_e[A(A)] \cdot \left\{ \begin{array}{ll}
\frac{1}{2} \{ \sum_k A(e)_{mn} (\tau_i)_{kn} & \text{if } p = s(e) \\
- \sum_k (\tau_i)_{mk} A(e)_{kn} & \text{if } p = t(e),
\end{array} \right.
[\hat{P}_i(S), \hat{P}_{j'}(S')] f_e(A(e))
= \frac{i\hbar}{2} \kappa(S', e) \epsilon_{ij} \hat{P}_k(S) f_e(A(e)),
\]

where we assume the simplest case of one edge graphs. From the commutation relations, one can see that the commutators between momentum operators do not necessarily vanish if \( S \cap S' \neq \emptyset \). This unusual property reflects the non-commutativity of quantum Riemannian structures [24]. We conclude that the quantum algebra of elementary observables (holonomy-flux algebra) has been well represented on \( \mathcal{H}_{\text{kin}} \) background-independently. So the construction of quantum kinematics is finished. Two important remarks on the quantum kinematics are listed below.

- **Kinematical Vacuum and Polymer Representation**

  The constant function \( \Omega = 1 \) has the physical meaning of a kinematical vacuum state in Hilbert space \( \mathcal{H}_{\text{kin}} \) due to its following characters. First, \( \Omega = 1 \) is the unique state in \( \mathcal{H}_{\text{kin}} \) with maximal gauge symmetry under Yang-Mills gauge transformations and spatial diffeomorphisms. Secondly, \( \Omega = 1 \) means that there is no geometry at all on the 3-manifold \( \Sigma \), since the elementary operators \( \hat{A}(e)_{mn} \) and \( \hat{P}_i(S) \), corresponding to connections and triad fields in classical sense, have vanishing expectation values on constant function. Hence it implies that the vacuum of quantum geometry is no geometry but a bare manifold. While the constant function serves as a ground state in the kinematical Hilbert space, the low excited states (cylindrical functions) are only excited on graphs with finite edges. There is only 1-dimensional geometry living on these graphs, so the quantum geometry is polymer-like object. When one increases the amount of edges and graphs such that the graphs are densely distributed in \( \Sigma \), the quantum state is highly excited and the quantum geometry can weave the classical smooth one [33][1][99]. Because of this picture, the quantum kinematical representation which we obtain is also called polymer representation for background-independent quantum geometry.

- **Quantum Geometric Operator and Quantum Riemannian Geometry**

  The well-established quantum kinematics of loop quantum gravity is now in the status just like the Riemannian geometry before the appearance of general relativity and Einstein’s equation, which gives general relativity mathematical foundation and offers living place to the Einstein equation. Instead of classical geometric quantities, such as scalar, vector, tensor etc., the quantities in quantum
geometry are operators on the kinematical Hilbert space \( \mathcal{H}_{\text{kin}} \), and their spectrum serve as the possible values of the quantities in measurements. So far, the kinematical quantum geometric operators constructed properly in loop quantum gravity include area operators [125] [22], volume operators [19] [125] [23], length operator [139], \( \hat{Q} \) operator [100] etc. Recently there are discussions on the consistency check to different regularization approaches for volume operators with the triad operator [78] [79]. We thus will only introduce the volume operator defined by Ashtekar and Lewandowski [23], which is shown to be correct in the consistency check.

First, we define the area operator with respect to a 2-surface \( S \) by the elementary operators. Given a closed 2-surface or a surface \( S \) with boundary, we can divide it into a large number \( N \) of small area cells \( S_I \). Taking account of the classical expression of an area, we set the area of the 2-surface to be the limit of the Riemannian sum

\[
A_S := \lim_{N \to \infty} [A_S]_N = \lim_{N \to \infty} \kappa \beta \sum_{I=1}^{N} \sqrt{P_i(S_I)P_j(S_I)\delta^{ij}}.
\]

Then one can unambiguously obtain a quantum operator of area from the momentum operators \( \hat{P}_i(S) \). Given a cylindrical function \( \psi_\gamma \in \text{Cyl}(\mathcal{A}) \) which has second order derivative, the action of the area operator on \( \psi_\gamma \) is defined in the limit by requiring that each area cell contains at most only one intersecting point \( \nu \) of the graph \( \gamma \) and \( S \) as

\[
\hat{A}_S \psi_\gamma := \lim_{N \to \infty} [\hat{A}_S]_N \psi_\gamma = \lim_{N \to \infty} \kappa \beta \sum_{I=1}^{N} \sqrt{\hat{P}_i(S_I)\hat{P}_j(S_I)\delta^{ij}} \psi_\gamma.
\]

The regulator \( N \) is easy to be removed, since the result of the operation of the operator \( \hat{P}_i(S_I) \) does not change when \( S_I \) shrinks to a point. Since the refinement of the partition does not affect the result of action of \( [\hat{A}_S]_N \) on \( \psi_\gamma \), the limit area operator \( \hat{A}_S \), which is shown to be self-adjoint [22], is well defined on \( \mathcal{H}_{\text{kin}} \) and takes the explicit expression as:

\[
\hat{A}_S \psi_\gamma = 4\pi \beta \ell_p^2 \sum_{\nu \in \text{V}(\gamma \cap S)} \sqrt{(\hat{f}^{S,\nu}_{(u)})^2 - \hat{f}^{S,\nu}_{(d)}(\hat{f}^{S,\nu}_{(u)} - \hat{f}^{S,\nu}_{(d)})\delta^{ij}} \psi_\gamma,
\]

where \( \hat{f}^{S,\nu}_{(u)} \) and \( \hat{f}^{S,\nu}_{(d)} \) have been defined in Eq. (28). It turns out that for a given \( S \) one can find some finite linear combinations of spin network basis in \( \mathcal{H}_{\text{kin}} \) which diagonalize \( \hat{A}_S \) with eigenvalues given by finite sums,

\[
a_S = 4\pi \beta \ell_p^2 \sum_{I} \sqrt{2 \hat{j}^{(u)}(\hat{j}^{(u)} + 1) + 2 \hat{j}^{(d)}(\hat{j}^{(d)} + 1) - \hat{j}^{(u+d)}(\hat{j}^{(u+d)} + 1)}, \quad (29)
\]

where \( \hat{j}^{(u)}, \hat{j}^{(d)} \) and \( \hat{j}^{(u+d)} \) are arbitrary half-integers subject to the standard condition

\[
\hat{j}^{(u+d)} \in \{ |\hat{j}^{(u)} - \hat{j}^{(d)}|, |\hat{j}^{(u)} - \hat{j}^{(d)}| + 1, ..., \hat{j}^{(u)} + \hat{j}^{(d)} \}.
\]

(30)
Hence the spectrum of the area operator is fundamentally pure discrete, while its continuum approximation becomes excellent exponentially rapidly for large eigenvalues. However, in fundamental level, the area is discrete and so is the quantum geometry. One has seen that the eigenvalue of $\hat{A}_S$ does not vanish even in the case where only one edge intersects the surface at a single point, whence the quantum geometry is distributional.

The form of Ashtekar and Lewandowski’s volume operator was introduced for the first time in Ref. [19]. Then its detail properties were discussed in Ref. [23]. Given a region $R$ with a fixed coordinate system $\{x^a\}_{a=1,2,3}$ in it, one can introduce a partition of $R$ in the following way. Divide $R$ into small volume cells $C$ such that, each cell $C$ is a cube with coordinate volume less than $\epsilon$ and two different cells only share the points on their boundaries. In each cell $C$, we introduce three 2-surfaces $s = (S^1, S^2, S^3)$ such that $x^a$ is constant on the surface $S^a$. We denote this partition $(C, s)$ as $\mathcal{P}_\epsilon$. Then the volume of the region $R$ can be expressed classically as

$$V^s_R = \lim_{\epsilon \to 0} \sum_C \sqrt{|q_{C,s}|},$$

where

$$q_{C,s} = \frac{(\kappa \beta)^3}{3!} \epsilon^{ijk} \eta_{abc} P_i(S^a)P_j(S^b)P_k(S^c).$$

This motivates us to define the volume operator by naively changing $P_i(S^a)$ to $\hat{P}_i(S^a)$:

$$\hat{V}^s_R = \lim_{\epsilon \to 0} \sum_C \sqrt{|\hat{q}_{C,s}|},$$

$$\hat{q}_{C,s} = \frac{(\kappa \beta)^3}{3!} \epsilon^{ijk} \eta_{abc} \hat{P}_i(S^a)\hat{P}_j(S^b)\hat{P}_k(S^c).$$

Note that, given any cylindrical function $\psi_\gamma \in Cyl_{\gamma}(\mathcal{A})$, we require the vertexes of the graph $\gamma$ to be at the intersecting points of the triples of 2-surfaces $s = (S^1, S^2, S^3)$ in corresponding cells. Thus the limit operator will trivially exist due to the same reason in the case of the area operator. However, the volume operator defined here depends on the choice of orientations for the triples of surfaces $s = (S^1, S^2, S^3)$, or essentially, the choice of coordinate systems. So it is not uniquely defined. Since, for all choice of $s = (S^1, S^2, S^3)$, the resulting operators have correct semi-classical limit, one settles up the problem by averaging different operators labelled by different $s$ [23]. The process of averaging removes the freedom in defining the volume operator up to an overall constant $\kappa_0$. The resulting self-adjoint operator acts on any cylindrical function $\psi_\gamma \in Cyl_{\gamma}(\mathcal{A})$ as

$$\hat{V}_R \psi_\gamma = \kappa_0 \sum_{v \in V(\gamma)} \sqrt{|\hat{q}_{v,\gamma}|} \psi_\gamma,$$
where
\[ \hat{q}_{v,\gamma} = \left(8\pi\beta d^2_p\right)^{\frac{1}{48}} \sum_{e, e', e''} \epsilon(e, e', e'') \epsilon^v_{i,j,k} \hat{J}^i_{e} \hat{J}^j_{e'} \hat{J}^k_{e''}, \]

\(\epsilon(e, e', e'')\) \(\equiv\) \(\text{sgn}(\epsilon_{abc}\dot{e}^a\dot{e}^b\dot{e}^c)|_v\) with \(\dot{e}^a\) as the tangent vector of edge \(e\) and \(\epsilon_{abc}\) as the orientation of \(\Sigma\). The only unsatisfactory point in the present volume operator is the choice ambiguity of \(\kappa_0\). However, fortunately, the most recent discussion shows that the overall undetermined constant \(\kappa_0\) can be fixed to be \(\sqrt{6}\) by the consistency check between the volume operator and the triad operator [78][79].

4 Algebraic Aspects of Quantum Gauge Field Theory

In this section, we would like to reformulate the theory of loop quantum kinematics in an algebraic approach. The kinematical Hilbert space can be obtained via GNS-construction for the quantum holonomy-flux algebra. In the following, we will cast the general programme for canonical quantization into the algebraic framework. The formulation of loop quantum kinematics can then be regarded as a specific application of the general algebraic quantization programme.

4.1 General Programme for Algebraic Quantization

In the strategy of loop quantum gravity, a canonical programme is performed to quantize general relativity, which has been cast into a diffeomorphism invariant gauge field theory, or more generally, a dynamical system with constraints. The following is a summary for a general procedure to quantize a dynamical system with first class constraints.

- **Algebra of Classical Elementary Observables**
  One starts with the classical phase space \((M, \{\cdot, \cdot\})\) and \(R\) (\(R\) can be countable infinity) first-class constraints \(C_r (r = 1, \ldots, R)\) such that \(\{C_r, C_s\} = \sum_{t=1}^{R} f_{rs}^t C_t\), where \(f_{rs}^t\) is generally a function on phase space, namely, structure function of Poisson algebra. The algebra of classical elementary observables \(P\) is defined as:

**Definition 4.1.1:** The algebra of classical elementary observables \(P\) is a collection of functions \(f(m), m \in M\) on the phase space such that
\begin{enumerate}
  \item \(f(m) \in P\) should separate the point of \(M\), i.e., for any \(m \neq m'\), there exists \(f(m) \in P\), such that \(f(m) \neq f(m')\); (analogy to the \(p\) and \(q\) in \(M = T^*R\).)
  \item \(f(m), f'(m) \in P \Rightarrow \{f(m), f'(m)\} \in P\) (closed under Poisson bracket);
  \item \(f(m) \in P \Rightarrow \bar{f}(m) \in P\) (closed under complex conjugate).
\end{enumerate}

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6Thanks to the enlightening lectures given by Prof. T. Thiemann at Beijing Normal University.
7This includes the case of field theory with infinite many degree of freedom, since one can introduce the expression \(C_{\mu} = \int d^3x \phi_\mu(x)\phi_\mu(x)\), where \(\{\phi_\mu(x)\}_{\mu=1}^\infty\) forms a system of basis in \(L^2(\Sigma, d^3x)\).
So $\mathbf{P}$ forms a sub $\ast$-Poisson algebra of $C^\infty(M)$. In the case of $M = T^*\mathbb{R}$, $\mathbf{P}$ is generated by the conjugate pair $(q, p)$ with $[q, p] = 1$.

**Quantum Algebra of Elementary Observables**

Given the algebra of classical elementary observables $\mathbf{P}$, the quantum algebra of elementary observables can be constructed as follows. Consider the formal finite sequences of classical observable $(f_1, \ldots, f_n)$ with $f_k \in \mathbf{P}$. Then the operations of multiplication and involution are defined as

\[
(f_1, \ldots, f_n) \cdot (f'_1, \ldots, f'_m) := (f_1, \ldots, f_n, f'_1, \ldots, f'_m),
\]

\[
(f_1, \ldots, f_n)^\ast := (f_n, \ldots, f_1).
\]

One can define the direct sum of different sequences with different number of elements. Then the general element of the newly constructed free $\ast$-algebra $F(\mathbf{P})$ of $\mathbf{P}$, is formally expressed as $\bigoplus_{k=1}^N (f_1^{(k)}, \ldots, f_n^{(k)})$, where $f_n^{(k)} \in \mathbf{P}$. Consider the elements of the form (sequences consisting of only one element)

\[
(f + f') - (f) - (f'), \quad (zf) - z(f), \quad [(f), (f')] - i\hbar([f, f']),
\]

where $z \in \mathbb{C}$ and the canonical commutation bracket is defined as

\[
[(f), (f')] := (f) \cdot (f') - (f') \cdot (f).
\]

A 2-side ideal $\mathcal{Z}$ of $F(\mathbf{P})$ can be constructed from these elements, and is preserved by the action of involution $\ast$. One thus obtains

**Definition 4.1.2:** The quantum algebra $\mathbf{A}$ of elementary observables is defined to be the quotient $\ast$-algebra $F(\mathbf{P})/\mathcal{Z}$.

Note that the motivation to construct a quantum algebra of elementary observables is to avoid the problem of operators ordering in quantum theory so that the quantum algebra $\mathbf{A}$ can be represented on a Hilbert space without ordering ambiguity.

**Representation of Quantum Algebra**

In order to obtain a quantum theory, we need to quantize the classical observable in the dynamical system. The, so called, quantization is nothing but a $\ast$-representation map $\pi$ from the quantum algebra of elementary observable $\mathbf{A}$ to the collection of linear operators $\mathcal{L}(\mathcal{H})$ on a Hilbert Space $\mathcal{H}$. Recall that a map $\pi: \mathbf{A} \rightarrow \mathcal{L}(\mathcal{H})$ is a $\ast$-representation if and only if (1) there exists a dense subspace $\mathcal{D}$ of $\mathcal{H}$ contained in $\bigcap_{a \in \mathbf{A}} [D(\pi(a)) \cap D(\pi(a^\ast))]$ where $D(\pi(a))$ is the domain of the operator $\pi(a)$ and (2) for every $a, b \in \mathbf{A}$ and $\lambda \in \mathbb{C}$ the following conditions are satisfied in $\mathcal{D}$,

\[
\pi(a + b) = \pi(a) + \pi(b), \quad \pi(\lambda a) = \lambda \pi(a),
\]

\[
\pi(a \cdot b) = \pi(a)\pi(b), \quad \pi(a^\ast) = (\pi(a))^\dagger.
\]
Note that $\mathcal{L}(\mathcal{H})$ fails to be an algebra because the domains of unbounded operators cannot be the whole Hilbert space. However, the collection of bounded operators on some Hilbert space is really a $*$-algebra. At the level of quantum mechanics, the well-known Stone-Von Neumann Theorem concludes that in quantum mechanics, there is only one strongly continuous, irreducible, unitary representation of the Weyl algebra, up to unitary equivalence (see, for example, Ref. [113]). However, the conclusion of Stone-Von Neumann cannot be generalized to the quantum field theory because the latter has infinite many degrees of freedom (for detail, see, for example [156]). In quantum field theory, a representation can be constructed by GNS (Gel’fand-Naimark-Segal) construction for a quantum algebra of elementary observable $\mathcal{A}$, which is a unital $*$-algebra actually. The GNS construction for the representation of quantum algebra $\mathcal{A}$ is briefly summarized as follows.

**Definition 4.1.3:** Given a positive linear functional ($a$ state) $\omega$ on $\mathcal{A}$, the null space $\mathcal{N}_\omega \in \mathcal{A}$ with respect to $\omega$ is defined as $\mathcal{N}_\omega := \{a \in \mathcal{A} | \omega(a^* \cdot a) = 0\}$, which is a left ideal in $\mathcal{A}$. Then a quotient map can be defined as $[\cdot] : \mathcal{A} \rightarrow \mathcal{A}/\mathcal{N}_\omega; a \mapsto [a] := \{a + b | b \in \mathcal{N}_\omega\}$. The GNS-representation for $\mathcal{A}$ with respect to $\omega$ is a $*$-representation map: $\pi_\omega : \mathcal{A} \rightarrow \mathcal{L}(\mathcal{H}_\omega)$, where $\mathcal{H}_\omega := \langle \mathcal{A}/\mathcal{N}_\omega \rangle$ and $(\cdot)$ denotes the completion with respect to the naturally equipped well-defined inner product

$$< [a][b] >_{\mathcal{H}_\omega} := \omega(a^* \cdot b)$$

on $\mathcal{H}_\omega$. This representation map is defined by

$$\pi_\omega(a)[b] := [a \cdot b], \forall a \in \mathcal{A} \text{ and } [b] \in \mathcal{H}_\omega,$$

where $\pi_\omega(a)$ is an unbounded operator in general. Moreover, GNS representation is a cyclic representation, i.e., $\exists \Omega \in \mathcal{H}_\omega$ such that $\langle \langle \pi_\omega(a)\Omega | a \in \mathcal{A} \rangle \rangle = \mathcal{H}_\omega$ and $\Omega$ is called a cyclic vector in the representation space. In fact $\Omega_\omega := [1]$ is a cyclic vector in $\mathcal{H}_\omega$ and $\langle \langle \pi_\omega(a)\Omega_\omega | a \in \mathcal{A} \rangle \rangle = \mathcal{H}_\omega$. As a result, the positive linear functional with which we begin can be expressed as

$$\omega(a) = < \Omega_\omega | \pi_\omega(a)\Omega_\omega >_{\mathcal{H}_\omega}.$$

Thus a positive linear functional on $\mathcal{A}$ is equivalent to a cyclic representation of $\mathcal{A}$, which is a triple $(\mathcal{H}_\omega, \pi_\omega, \Omega_\omega)$. Moreover, every non-degenerate representation is an orthogonal direct sum of cyclic representations (for proof, see [59]).

So the kinematical Hilbert space $\mathcal{H}_{kin}$ for the system with constrains can be obtained by GNS-construction. In the case that there are gauge symmetries in our dynamical system, supposing that there is a group $G$ acting on $\mathcal{A}$ by automorphisms $\alpha_g : \mathcal{A} \rightarrow \mathcal{A}, \forall g \in G$, it is preferred to construct a gauge invariant representation of $\mathcal{A}$. So we require the positive linear functional $\omega$ on $\mathcal{A}$ to be gauge invariant, i.e., $\omega \circ \alpha_g = \omega$. Then the group $G$ is represented on the Hilbert space $\mathcal{H}_\omega$ as:

$$U(g)\pi_\omega(a)\Omega_\omega = \pi_\omega(\alpha_g(a))\Omega_\omega.$$
and such a representation is a unitary representation of $G$. In loop quantum gravity, it is crucial to construct a gauge invariant and diffeomorphism invariant representation for the quantum algebra of elementary observables.

- **Implementation of the Constraints**

  In the Dirac quantization programme for a system with constraints, the constraints should be quantized as some operators in a kinematical Hilbert space $\mathcal{H}_{\text{kin}}$. One then solves them at quantum level to get a physical Hilbert space $\mathcal{H}_{\text{phys}}$, that is, to find a quantum analogy $\hat{C}_r$ of the classical constraint formula $C_r$ and to solve the general solution of the equation $\hat{C}_r \Psi = 0$. However, there are several problems in the construction of the constraint operator $\hat{C}_r$.

  (i) $C_r$ is in general not in $\mathcal{P}$, so there is a factor ordering ambiguity in quantizing $C_r$ to be an operator $\hat{C}_r$.

  (ii) In quantum field theory, there are ultraviolet (UV) divergence problems in constructing operators. However, the UV divergence can be avoided in the background independent approach.

  (iii) Sometimes, quantum anomaly appears when there are structure functions in the Poisson algebra. Although classically we have $\{C_r, C_s\} = \Sigma_{r,s,t=1}^{R} f_{rs}^{t'} C_t$, it is possible that $[\hat{C}_r, \hat{C}_s] \neq i\hbar \Sigma_{r,s,t=1}^{R} f_{rs}^{t'} \hat{C}_t$ due to the ordering ambiguity between $\hat{f}_{rs}^{t'}$ and $\hat{C}_t$. If one sets $[\hat{C}_r, \hat{C}_s] = i\hbar \Sigma_{r,s,t=1}^{R} f_{rs}^{t'} (\hat{C}_t + \hat{C}_t \hat{f}_{rs}^{t'})$, for $\Psi$ satisfying $\hat{C}_r \Psi = 0$, we have

  $$[\hat{C}_r, \hat{C}_s] \Psi = \frac{i\hbar}{2} \sum_{t=1}^{R} \hat{C}_t \hat{f}_{rs}^{t'} \Psi = \frac{i\hbar}{2} \sum_{t=1}^{R} [\hat{C}_t, \hat{f}_{rs}^{t'}] \Psi.$$  

  However, $[\hat{C}_r, \hat{f}_{rs}^{t'}] \Psi$ are not necessary to equal to zero for all $r, s, t = 1...R$. If not, the problem of quantum anomaly comes out and the new quantum constraints $[\hat{C}_r, \hat{f}_{rs}^{t'}] \Psi = 0$ have to be imposed on physical quantum states, since the classical Poisson brackets $\{C_r, C_s\}$ are weakly equal to zero on the constraint surface $\tilde{M} \subset \tilde{M}$. Thus too many constraints are imposed so that the physical Hilbert space $\mathcal{H}_{\text{phys}}$ would be too small. Hence the quantum anomaly should be avoided anyway.

- **Solving the Constraints and Physical Hilbert Space**

  In general the original Dirac quantization approach can not be carried out directly, since there is usually no nontrivial $\Psi \in \mathcal{H}_{\text{kin}}$ such that $\hat{C}_r \Psi = 0$. This happens when the constraint operator $\hat{C}_r$ has "generalized eigenfunctions" rather than eigenfunctions. One then develops the so-called Refined Algebraic Quantization Programme, where the solutions of the quantum constraint can be found in the algebraic dual space of a dense subset in $\mathcal{H}_{\text{kin}}$ (see e.g. (26) (35)). The quantum diffeomorphism constraint in loop quantum gravity is solved in this approach (see section 5.2). But the situation for the scalar constraint in general relativity is so subtle that it is difficult to carry out the quantization programme straightforwardly. Recently, Thiemann proposed the method of Master
**Constraint Approach** to solve the quantum constraints \([147]\), which seems especially suitable to deal with the particular feature of the constraint algebra of general relativity. A master constraint is defined as

\[ M = \frac{1}{2} \sum_{r=1}^{R} \text{K}_{rs} \bar{C}_r C_s \]

for some real positive matrix \( \text{K}_{rs} \). Classically one has \( M = 0 \) if and only if \( C_r = 0 \) for all \( r = 1,...,R \). So quantum mechanically one may consider solving the **Master Equation**: \( \hat{M} \Psi = 0 \) to obtain the physical Hilbert space \( \mathcal{H}_{\text{phys}} \) instead of solving \( \hat{C}_r \Psi = 0, \forall r = 1,...,R \). Because the master constraint \( M \) is classically positive, one has opportunities to implement it as a self-adjoint operator on \( \mathcal{H}_{\text{kin}} \). If it is indeed the case and \( \mathcal{H}_{\text{kin}} \) is separable, one can use the direct integral representation of \( \mathcal{H}_{\text{kin}} \) associated with the self-adjoint operator \( \hat{M} \) to obtain \( \mathcal{H}_{\text{phys}} \):

\[
\mathcal{H}_{\text{kin}} \sim \int_{\mathcal{R}} d\mu(\lambda) \mathcal{H}_{\lambda}^{0},
\]

\[
<\Phi|\psi>_{\text{kin}} = \int_{\mathcal{R}} d\mu(\lambda) <\Phi|\psi>_{\mathcal{H}_{\lambda}^{0}},
\]

where \( \mu \) is a so-called spectral measure and \( \mathcal{H}_{\lambda}^{0} \) is the (generalized) eigenspace of \( \hat{M} \) with the eigenvalue \( \lambda \). The physical Hilbert space is then formally obtained as \( \mathcal{H}_{\text{phys}} = \mathcal{H}_{\lambda=0}^{0} \) with the induced physical inner product \( <\cdot|\cdot>_{\mathcal{H}_{\lambda}^{0}} \). Now the issue of quantum anomaly is represented in terms of the size of \( \mathcal{H}_{\text{phys}} \) and the existence of sufficient numbers of semi-classical states.

**Physical Observables**

We denote \( \mathcal{M} \) as the original unconstrained phase space, \( \overline{\mathcal{M}} \) as the constraint surface, i.e., \( \overline{\mathcal{M}} := \{m \in \mathcal{M}|C_r(m) = 0, \forall r = 1,...,R\} \), and \( \overline{\mathcal{M}} \) as the reduced phase space, i.e. the space of orbits for gauge transformations generated by all \( C_r \). The concept of Dirac observable is defined as the follows.

**Definition 4.1.4:**

1. A function \( O \) on \( \mathcal{M} \) is called a weak Dirac observable if and only if the function depends only on points of \( \overline{\mathcal{M}} \), i.e., \( [O,C_r]|_{\overline{\mathcal{M}}} = 0 \) for all \( r = 1,...,R \). For the quantum version, a self-adjoint operator \( \hat{O} \) is a weak Dirac observable if and only if the operator can be well defined on the physical Hilbert space.
2. A function \( \hat{O} \) on \( \overline{\mathcal{M}} \) is called a strong Dirac observable if and only if \( [O,C_r]|_{\overline{\mathcal{M}}} = 0 \) for all \( r = 1,...,R \). For the quantum version, a self-adjoint operator \( \hat{O} \) is a strong Dirac observable if and only if the operator can be defined on the kinematic Hilbert space \( \mathcal{H}_{\text{kin}} \) and \( [\hat{O},\hat{C}_r] = 0 \) in \( \mathcal{H}_{\text{kin}} \) for all \( r = 1,...,R \).

A physical observable is at least a weak Dirac observable. While Dirac observables have been found in symmetry reduced models, some even with an infinite number of degrees of freedom, it seems extremely difficult to find them in full general relativity. Moreover the Hamiltonian is a linear combination of first-class constraints. So there is no dynamics in the reduced phase space, and the meaning of time evolution of the Dirac observables becomes subtle. However, using the concepts of partial and complete observables \([121][115][122]\), a systematic
method to get Dirac observables is being developed, and the problem of time in this system with a Hamiltonian  is being addressed.

Classically, let \( f(m) \) and \( (T_r(m))^{R}_{r=1} \) be gauge non-invariant functions (partial observables) on phase space \( \mathcal{M} \), such that \( A_r \equiv \{C_r, T_r\} \) is a non-degenerate matrix on \( \mathcal{M} \). A system of classical weak Dirac observables (complete observables) \( F_{f,T} \) labelled by a collection of real parameters \( \tau \equiv \{\tau_r\}^R_{r=1} \) can be constructed as

\[
F_{f,T}^{\tau} := \sum_{\{n_1, \ldots, n_R\}} \frac{(\tau_1 - T_1)^{n_1} \cdots (\tau_R - T_R)^{n_R}}{n_1! \cdots n_R!} \tilde{X}_f^{n_1} \circ \cdots \circ \tilde{X}_f^{n_R}(f),
\]

where \( \tilde{X}_f(f) := \{\Sigma_{i=1}^R A_i^{-1} C_i, f\} \equiv (\tilde{C}_r, f) \). It can be verified that \( [\tilde{X}_r, \tilde{X}_s]_{\mathcal{M}} = 0 \) and \( [F_{f,T}^{\tau}, C_i]_{\mathcal{M}} = 0 \), for all \( r = 1 \ldots R \) (for details see [62] and [63]).

The partial observables \( (T_r(m))^{R}_{r=1} \) may be regarded as clock variables, and \( \tau_r \) is the time parameter for \( T_r \). The gauge is fixed by giving a system of functions \( (T_r(m))^{R}_{r=1} \) and corresponding parameters \( \{\tau_r\}^R_{r=1} \), namely, a section in \( \overline{\mathcal{M}} \) is selected by \( T_r(m) = \tau \) for each \( r \), and \( F_{f,T}^{\tau} \) is the value of \( f \) on the section. To solve the problem of dynamics, one assumes another set of canonical coordinates \( \{P_1, \ldots, P_{N-R}, \Pi_1, \ldots, \Pi_R; Q_1, \ldots, Q_{N-R}, T_1, \ldots, T_R\} \) by canonical transformations in the phase space \( (\mathcal{M}, \{, \}) \), where \( P_r \) and \( \Pi_r \) are conjugate to \( Q_r \) and \( T_r \) respectively. After solving the complete system of constraints \( \{C_r(P_r, Q_r, \Pi_r, T_r) = 0\}^{R}_{r=1} \), the Hamiltonian \( H_r \) with respect to the time \( T_r \) is obtained as \( H_r := \Pi_r(P_r, Q_r, T_r). \) Given a system of constants \( \{\tau_0\}^R_{r=1} \) for an observable \( f(P_r, Q_r) \) depending only on \( P_r \) and \( Q_r \), the physical dynamics is given by [62] [148]:

\[
\left( \frac{\partial}{\partial \tau_r} \right)_{\tau_0} F_{f,T}^{\tau_0}_{\{H_r, f\}, \mathcal{M}} = F_{\{H_r, f\}, \mathcal{M}}^{\tau_0} F_{f,T}^{\tau_0}_{\mathcal{M}} = [F_{\{H_r, f\}, \mathcal{M}}^{\tau_0}, F_{f,T}^{\tau_0}_{\mathcal{M}}],
\]

where \( F_{\{H_r, f\}, \mathcal{M}}^{\tau_0} \) is the physical Hamiltonian function generating the evolution with respect to \( \tau_r \). Thus one has addressed the problem of time and dynamics as a result.

*Semicalssical Analysis*

An important issue in the quantization is to check whether the quantum constraint operators have correct classical limits. This has to be done by using the kinematical semiclassical states in \( \mathcal{H}_\text{kin} \). Moreover, the physical Hilbert space \( \mathcal{H}_\text{phys} \) must contain enough semi-classical states to guarantee that the quantum theory one obtains can return to the classical theory when \( \hbar \to 0 \). The semi-classical states in a Hilbert space \( \mathcal{H} \) should have the following properties.

**Definition 4.1.5:** Given a class of observables \( \mathcal{S} \) which comprises a subalgebra in the space \( \mathcal{L}(\mathcal{H}) \) of linear operators on the Hilbert space, a family of (pure) states \( \{\omega_m\}_{m \in \mathcal{M}} \) are said to be semi-classical with respect to \( \mathcal{S} \) if and only if:

1. The observables in \( \mathcal{S} \) should have correct semi-classical limit on semi-classical
states and the fluctuations should be small, i.e.,

$$\lim_{\hbar \to 0} \left| \frac{\omega_m(\hat{a}) - a(m)}{a(m)} \right| = 0,$$

$$\lim_{\hbar \to 0} \left| \frac{\omega_m(\hat{a})^2 - \omega_m(\hat{a})^2}{\omega_m(\hat{a})^2} \right| = 0,$$

for all $\hat{a} \in S$.

(2) The set of cyclic vectors $\Omega_m$ related to $\omega_m$ via the GNS-representation $(\pi_{\omega_m}, \mathcal{H}_{\omega_m}, \Omega_{\omega_m})$ is dense in $\mathcal{H}$.

Seeking for semiclassical states are one of open issues of current research in loop quantum gravity. Recent original works focus on the construction of coherent states of loop quantum gravity in analogy with the coherent states for harmonic oscillator system [140][141][142][143][20][16].

The above is the general programme to quantize a system with constraints. In the following subsection, we will apply the programme to the theory of general relativity and restrict our view to the representation with the properties of background independence and spatial diffeomorphism invariance.

### 4.2 Algebraic Construction of Loop Quantum Kinematics

All prior constructions in section 3 bear analogy with constructive quantum field theory. In this subsection we perform the background-independent construction of algebraic quantum field theory for loop quantum gravity. First we construct the algebra of classical observables. Taking account of the future quantum analogs, we define the algebra of classical observables $P$ as the Poission $*$-subalgebra generated by the functions of holonomies (cylindrical functions) and the fluxes of triad fields smeared on some 2-surface. Namely, one can define the classical algebra in analogy with geometric quantization in finite dimensional phase space case by the so-called classical Ashtekar-Corichi-Zapata holonomy-flux $*$-algebra as the following [96].

**Definition 4.2.1**

The classical Ashtekar-Corichi-Zapata holonomy-flux $*$-algebra is defined to be a vector space $P_{ACZ} := Cyl(\mathcal{A}) \times V^C(\mathcal{A})$, where $V^C(\mathcal{A})$ is the vector space of cylindrically consistent vector fields spanned by the vector fields $\psi Y_f(S)$ and their commutators, here the smeared flux vector field $Y_f(S)$ is defined by $Y_f(S) := \{ \int_{\mathcal{A}} \frac{\text{vol}}{12} \tilde{P} f^i, \cdot \}$ for any su(2)-valued functions $f^i$ with compact supports on $S$ and $\psi$ are cylindrical functions on $\mathcal{A}$.

We equip $P_{ACZ}$ with the structure of an $*$-Lie algebra by:

1. Lie bracket $[ , ] : P_{ACZ} \times P_{ACZ} \to P_{ACZ}$ is defined by

$$[\psi, Y] := (Y \circ \psi' - \psi' \circ \psi, \{ Y, Y' \}),$$

for all $\psi, Y, \psi', Y' \in P_{ACZ}$ with $\psi, \psi' \in Cyl(\mathcal{A})$ and $Y, Y' \in V^C(\mathcal{A})$. 

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(2) Involution: \(a \mapsto \bar{a}\) \(\forall a \in P_{\text{ACZ}}\) is defined by complex conjugate of cylindrical functions and vector fields, i.e., \(\bar{a} := (\bar{\psi}, \bar{Y}) \forall (\psi, Y) \in P_{\text{ACZ}}\), where \(\bar{Y} \circ \psi := Y \circ \bar{\psi}\).

(3) \(P_{\text{ACZ}}\) admits a natural action of \(\text{Cyl}(A)\) by
\[
\psi' \circ (\psi, Y) := (\psi' \psi, \psi' Y),
\]
which gives \(P_{\text{ACZ}}\) a module structure.

The classical Ashtekar-Corichi-Zapata holonomy-flux \(*\)-algebra serves as an elementary algebra in our dynamic system of gauge field. Then one can construct the quantum algebra of elementary observables from \(P_{\text{ACZ}}\) in analogy with Definition 4.1.2.

**Definition 4.2.2**
The abstract free algebra \(F(P_{\text{ACZ}})\) of the classical \(*\)-algebra is defined by the formal direct sum of finite sequences of classical observables \((a_1, ..., a_n)\) with \(a_k \in P_{\text{ACZ}}\), where the operations of multiplication and involution are defined as
\[
(a_1, ..., a_n) \cdot (a'_1, ..., a'_m) := (a_1, ..., a_n, a'_1, ..., a'_m),
\]
\[
(a_1, ..., a_n)^* := (\bar{a}_n, ..., \bar{a}_1).
\]

A 2-sided ideal \(Z\) can be generated by the following elements,
\[
(a + a') - (a) - (a'), \quad (za) - z(a),
\]
\[
[(a), (a')] - \frac{i\hbar}{2}([a, a']),
\]
\[
((\psi, 0), a) - (\psi \circ a),
\]
where the canonical commutation bracket is defined by
\[
[(a), (a')] := (a) \cdot (a') - (a') \cdot (a).
\]

Note that the ideal \(Z\) is preserved by the involution \(*\).
The quantum holonomy-flux \(*\)-algebra is defined by the quotient \(*\)-algebra \(A = F(P_{\text{ACZ}})/Z\) which contains the unital element \(\mathbb{1} := ((1, 0))\). Note that a sup-norm has been defined by Eq. (19) for the Abelian sub-\(*\)-algebra \(\text{Cyl}(A)\) in \(A\).

For simplicity, we denote the one element sequences \(((\psi, 0))\) and \(((0, Y)) \forall \psi \in \text{Cyl}(A), Y \in \text{Cyl}(A)\) in \(A\) by \((\psi)\) and \((Y)\) respectively. In particular, for all cylindrical functions \((\psi)\) and flux vector fields \((Y_f(S))\),
\[
(\psi)^* = (\bar{\psi}) \quad \text{and} \quad (Y_f(S))^* = (Y_f(S)).
\]
Note that every element of the algebra \(A\) is a finite linear combination of elements of the form
\[
(\psi),
(\psi_1) \cdot (Y_{f_1}(S_{11})),
\]
\((\psi_2) \cdot (Y_{f_2}(S_{21})) \cdot (Y_{f_2}(S_{22}))\),
\[
\vdots \quad (\psi_k) \cdot (Y_{f_k}(S_{k1})) \cdot (Y_{f_k}(S_{k2})) \cdot \ldots \cdot (Y_{f_k}(S_{kh})),
\]
\[
\vdots
\]

Moreover, given a cylindrical function \(\psi\) and a flux vector field \(Y_f(S)\), one has the relation from the commutation relation:
\[
(Y_f(S)) \cdot (\psi) = i\hbar (Y_f(S) \circ \psi) + (\psi) \cdot (Y_f(S)). \quad (33)
\]

Then the kinematical Hilbert space \(\mathcal{H}_{kin}\) can be obtained properly via the \(GNS\)-construction for unital \(*\)-algebra \(\mathcal{A}\) in the same way as in Definition 4.1.3. By \(GNS\)-construction, a positive linear functional, i.e. a state \(\omega\) on \(\mathcal{A}\) defines a cyclic representation \((\mathcal{H}_{\omega}, \pi_\omega, \Omega_\omega)\) for \(\mathcal{A}\). In our case of quantum holonomy-flux \(*\)-algebra, the state with both internal gauge invariance and diffeomorphism invariance is defined for any \(\psi = [\psi_\omega] \in Cyl(\mathcal{A})\) and non-vanishing flux vector field \(Y_f(S) \in \mathcal{V}(\mathcal{A})\) as [96]:
\[
\omega(\psi) := \int_{\mathcal{S}U(2)^N} d\mu_H(A(e_1)) \ldots d\mu_H(A(e_N)) \psi(A(e_1), \ldots, A(e_N)),
\]
\[
\omega(a \cdot (Y_f(S))) := 0, \quad \forall a \in \mathcal{A},
\]
where we assume that \(\alpha\) contains \(N\) edges. This \(\omega\) is called Ashtekar-Isham-Lewandowski state. The null space \(N_\omega \in \mathcal{A}\) with respect to \(\omega\) is defined as \(N_\omega := \{a \in \mathcal{A} | \omega(a \cdot a^\dagger) = 0\}\), which is a left ideal. Then a quotient map can be defined as:
\[
[] : \mathcal{A} \rightarrow \mathcal{A}/N_\omega;
\]
\[
a \mapsto [a] := \{a + b | b \in N_\omega\}.
\]

The \(GNS\)-representation for \(\mathcal{A}\) with respect to \(\omega\) is a representation map: \(\pi_\omega : \mathcal{A} \rightarrow \mathcal{L}(\mathcal{H}_\omega)\) such that \(\pi_\omega(a \cdot b) = \pi_\omega(a) \pi_\omega(b)\), where \(\mathcal{H}_\omega := \langle \mathcal{A}/N_\omega \rangle = \langle Cyl(\mathcal{A}) \rangle = \mathcal{H}_{kin}\) by straightforward verification and the \(\langle \cdot \rangle\) denotes the completion with respect to the natural equipped inner product on \(\mathcal{H}_{\omega}\),
\[
< [a][b] >_{\mathcal{H}_\omega} := \omega(a \cdot b),
\]
which is equivalent to Eq. (23). The representation map \(\pi_\omega\) is defined by
\[
\pi_\omega(a)[b] := [a \cdot b], \quad \forall a \in \mathcal{A}, \text{ and } [b] \in \mathcal{H}_\omega.
\]

Note that \(\pi_\omega(a)\) is an unbounded operator in general. It is easy to verify that
\[
\pi_\omega((Y_f(S)))[(\psi)] = i\hbar [(Y_f(S) \circ \psi)]
\]
via Eq.(33). Hence \(\pi_\omega((Y_f(S)))[(\psi)]\) is identical with \(\hat{P}_f(S) = \hat{y} Y_f(S)\) on \(\mathcal{H}_{kin}\), which can be obtained in analogy with the method we employ at the beginning of section 3.5. Moreover, since \(\Omega_\omega := \{|1\}\) is a cyclic vector in \(\mathcal{H}_\omega\), the positive linear functional with which we begin can be expressed as
\[
\omega(a) = <\Omega_\omega | \pi_\omega(a) \Omega_\omega >_{\mathcal{H}_\omega}.
\]
Thus the Ashtekar-Isham-Lewandowski state $\omega$ on $A$ is equivalent to a cyclic representation $(\mathcal{H}_\omega, \pi_\omega, \Omega_\omega)$ for $A$, which is the Ashtekar-Isham-Lewandowski representation for quantum holonomy-flux $\ast$-algebra of background independent gauge field theory. One thus obtains the kinematical representation of loop quantum gravity via the construction of algebraic quantum field theory. It is important to note that the Ashtekar-Isham-Lewandowski state is the unique state on quantum holonomy-flux $\ast$-algebra $A$ invariant under internal gauge transformations and spatial diffeomorphisms, which are both automorphisms $\alpha_g$ and $\alpha_\phi$ on $A$ such that $\omega \circ \alpha_g = \omega$ and $\omega \circ \alpha_\phi = \omega$. So these gauge transformations are represented as unitary transformations on $\mathcal{H}_{\text{kin}}$, while the cyclic vector $\Omega_\omega$ is the unique state in $\mathcal{H}_{\text{kin}}$ invariant under internal gauge transformations and spatial diffeomorphisms. This is a very crucial uniqueness theorem for canonical quantization of gauge field theory [96].

**Theorem 4.2.1:** There exists exactly one internal gauge invariant and spatial diffeomorphism invariant state (positive linear functional) on the quantum holonomy-flux $\ast$-algebra. In other words, there exists a unique internal gauge invariant and spatial diffeomorphism invariant cyclic representation for the quantum holonomy-flux $\ast$-algebra, which is called Ashtekar-Isham-Lewandowski representation. Moreover, this representation is irreducible with respect to an exponential version of the quantum holonomy-flux algebra (defined in [129]), which is analogous to the Weyl algebra.

Hence we have finished the construction of kinematical Hilbert space for background independent gauge field theory and represented the quantum holonomy-flux algebra on it. Then following the general programme presented in the last subsection, we should impose the constraints as operators on the kinematical Hilbert space since we are dealing with a gauge system.

5 Quantum Gaussian and Diffeomorphism Constraints

After constructing the kinematical Hilbert space $\mathcal{H}_{\text{kin}}$ of loop quantum gravity, one should implement the constraints on it to obtain physical Hilbert space. Recalling the constraints (7) in generalized Palatini Hamiltonian for general relativity and the Poisson algebra among them, the subalgebra generated by the Gaussian constraints $G(\Lambda)$ forms a Lie algebra and a 2-sided ideal in the constraints algebra. So in this section, we first solve the Gaussian constraints independently of the other two kinds of constraints and find the solution space $\mathcal{H}_G$, which is constituted by internal gauge invariant quantum states. Then, although the subalgebra generated by the diffeomorphism constraints is not an ideal in the constraints algebra, we still would like to solve them independently of the scalar constraints for the technical convenience. At the end of this section, we will obtain the Hilbert space $\mathcal{H}_{\text{Diff}}^G$ free of Gaussian constraints and diffeomorphism constraints.

---

8The proof of this conclusion depends on the compact support property of the smear functions $f^i$ (see [96] for detail).
5.1 Implementation of Quantum Gaussian Constraint

Recall that the classical expression of Gaussian constraints reads

\[ \mathcal{G}(\Lambda) = \int_\Sigma d^3x \Lambda^i \tilde{D}_a \tilde{P}^i_a = -\int_\Sigma d^3x \tilde{P}^i_a D_a \Lambda^i \equiv -P(D\Lambda), \]

where \( D_a \Lambda^i = \partial_a \Lambda^i + \epsilon^j_{ik} A^j_a \Lambda^k \). As the situation of triad flux, the Gaussian constraints can be defined as cylindrically consistent vector fields \( Y_{DA} \) on \( \mathcal{A} \), which act on any cylindrical function \( f_\gamma \in \text{Cyl}_\gamma \) by

\[ \mathcal{G}(\Lambda) f_\gamma(e^{(A(e))}_{e \in \mathcal{E}(\gamma)}) := \mathcal{P}(D\Lambda), F_\gamma(e^{(A(e))}_{e \in \mathcal{E}(\gamma)}). \]

Then the Gaussian constraint operator can be defined in analogy with the momentum operator, which acts on \( f_\gamma \) as:

\[ \hat{\mathcal{G}}(\Lambda) f_\gamma(e^{(A(e))}_{e \in \mathcal{E}(\gamma)}) := i\hbar Y_{DA} \circ f_\gamma(e^{(A(e))}_{e \in \mathcal{E}(\gamma)}) = \hbar \sum_{v \in \mathcal{V}(\gamma)} \mathcal{P}(v) \hat{J}_v f_\gamma(e^{(A(e))}_{e \in \mathcal{E}(\gamma)}), \]

which is the generator of internal gauge transformations on \( \text{Cyl}_\gamma \). The kernel of the operator is easily obtained in terms of spin-network decomposition, which is the internal gauge invariant Hilbert space:

\[ \mathcal{H}^G = \oplus_{\alpha, j, l} \mathcal{H}^G_{\alpha, j, l} \oplus \mathbb{C}. \]

One then naturally gets the gauge invariant spin-network basis \( T_s = (\gamma(s), j_s, i_s) \) in \( \mathcal{H}^G \) as \([126, 20, 33]\):

\[ T_{s = (\gamma(j, k))} = \bigotimes_{v \in \mathcal{V}(\gamma)} i_v \bigotimes_{e \in \mathcal{E}(\gamma)} \pi_j^i(A(e)), (j_e \neq 0) \]

by assigning a non-trivial spin representation \( j \) on each edge and an invariant tensor \( i \) (intertwiner) on each vertex. In the following context \( T_s \) will also be called as spin-network states. We denote \( \text{Cyl}(\mathcal{A}/G) \) the space of finite linear combinations of gauge invariant spin-network states, which is dense in \( \mathcal{H}^G \), and \( \text{Cyl}((\mathcal{A}/\mathcal{G}) \subset \text{Cyl}(\mathcal{A}/G) \) the linear span of the gauge invariant spin network functions \( T_s \) for \( \gamma(s) = \gamma \). All Yang-Mills gauge invariant operators are well defined on \( \mathcal{H}^G \). However, the condition of acting on gauge invariant states often changes the structure of the spectrum of quantum geometric operators. For the area operator, the spectrum depends on certain global properties of the surface \( S \) (see \([21, 22]\) for details). For the volume operators, non-zero spectrum arises from at least 4-valent vertices.

5.2 Implementation of Quantum Diffeomorphism Constraint

Unlike the strategy in solving Gaussian constraint, one cannot define an operator for quantum diffeomorphism constraint as the infinitesimal generator of finite diffeomorphism transformations (unitary operators since the measure is diffeomorphism invariant) represented on \( \mathcal{H}_{\text{diag}} \). The representation of finite diffeomorphisms is a family of
unitary operators $\hat{U}_\varphi$ acting on cylindrical functions $\psi_\gamma$ by
\[ \hat{U}_\varphi \psi_\gamma := \psi_{\varphi \gamma}, \] (34)
for any spatial diffeomorphism $\varphi$ on $\Sigma$. An 1-parameter subgroup $\varphi_t$ in the group of spatial diffeomorphisms is then represented as an 1-parameter unitary group $\hat{U}_{\varphi_t}$ on $\mathcal{H}_{\text{kin}}$. However, $\hat{U}_{\varphi_t}$ is not weakly continuous, since the subspaces $\mathcal{H}^t_{\text{ff}}(\gamma)$ and $\mathcal{H}^t_{\text{ff}}(\varphi_\gamma)$ are orthogonal to each other no matter how small the parameter $t$ is. So one always has
\[ |<\psi_\gamma, \hat{U}_{\varphi_t} \psi_\gamma >_{\text{kin}}| < |<\psi_\gamma \psi_\gamma >_{\text{kin}}| = |<\psi_\gamma >_{\text{kin}}| 
eq 0, \] (35)
even in the limit when $t$ goes to zero. Therefore, the infinitesimal generator of $\hat{U}_{\varphi_t}$ does not exist. In the strategy to solve the diffeomorphism constraint, due to the Lie algebra structure of diffeomorphism constraints subalgebra, the so-called group averaging technique is employed. We now outline the procedure. First, given a colored graph (a graph $\gamma$ and a cylindrical function living on it), one can define the group of graph symmetries $GS_\gamma$ by
\[ GS_\gamma := \text{Diff}_\gamma / T\text{Diff}_\gamma, \]
where $\text{Diff}_\gamma$ is the group of all diffeomorphisms preserving the colored $\gamma$, and $T\text{Diff}_\gamma$ is the group of diffeomorphisms which trivially acts on $\gamma$. We define a projection map by averaging with respect to $GS_\gamma$ to obtain the subspace in $\text{Cyl}_\gamma$ which is invariant under the transformation of $GS_\gamma$:
\[ P_{\text{Diff}_\gamma, \psi_\gamma} := \frac{1}{n_\gamma} \sum_{\varphi \in GS_\gamma} \hat{U}_\varphi \psi_\gamma, \]
for all cylindrical functions $\psi_\gamma \in \mathcal{H}^t_{\text{ff}}(\gamma)$, where $n_\gamma$ is the number of the finite elements of $GS_\gamma$. Second, we average with respect to all remaining diffeomorphisms which move the graph $\gamma$. For each cylindrical function $\psi_\gamma \in \text{Cyl}_\gamma(\mathcal{A}/\mathcal{G})$, there is an element $\eta(\psi_\gamma)$ associated to it in the algebraic dual space $\text{Cyl}^* \text{ of } \text{Cyl}(\mathcal{A}/\mathcal{G})$, which acts on any cylindrical function $\phi_\gamma' \in \text{Cyl}_\gamma(\mathcal{A}/\mathcal{G})$ as:
\[ \eta(\psi_\gamma)[\phi_\gamma'] := \sum_{\varphi \in \text{Diff}(\Sigma)/\text{Diff}_\gamma} \phi_\gamma' \langle \hat{U}_\varphi P_{\text{Diff}_\gamma, \psi_\gamma} \phi_\gamma' >_{\text{kin}} . \]
It is well defined since, for any given graph $\gamma'$, only finite terms are non-zero in the summation. It is easy to verify that $\eta(\psi_\gamma)$ is invariant under the group action of $\text{Diff}(\Sigma)$, since
\[ \eta(\psi_\gamma)[\hat{U}_\varphi \phi_\gamma] = \eta(\psi_\gamma)[\phi_\gamma]. \]
Thus we have defined a rigging map $\eta : \text{Cyl}(\mathcal{A}/\mathcal{G}) \rightarrow \text{Cyl}^*_{\text{Diff}_\gamma}$ which maps every cylindrical function to a diffeomorphism invariant one, where $\text{Cyl}^*_{\text{Diff}_\gamma}$ is spanned by rigged spin-network functions $T_{[s]} \equiv \{ \eta(T_s) \}$, $[s] = ([\gamma], [j], i)$ associated with diffeomorphism classes $[\gamma]$ of graphs $\gamma$. Moreover a Hermitian inner product can be defined on $\text{Cyl}^*_{\text{Diff}_\gamma}$ by the natural action of the algebraic functional:
\[ \langle \eta(\psi_\gamma) \eta(\phi_\gamma) >_{\text{Diff}_\gamma} := \eta(\psi_\gamma)[\phi_\gamma]. \]
The diffeomorphism invariant Hilbert space $\mathcal{H}_{\text{Diff}}$ is defined by the completion of $\mathcal{C}^\infty_{\text{Diff}}$ with respect to the above inner product $< | >_{\text{Diff}}$. The vacuum state and diffeomorphism invariant spin-network functions $T_{[\mu]}$ form an orthonormal basis in $\mathcal{H}_{\text{Diff}}$. Finally, we have obtained the general solutions invariant under both internal gauge transformations and spatial diffeomorphisms.

In general relativity, the problem of observables is a subtle issue due to the diffeomorphism invariance. Now we discuss the operators as diffeomorphism invariant observables on $\mathcal{H}_{\text{Diff}}$. We call an operator $\hat{O} \in \mathcal{L}(H_{\text{kin}})$ a strong observable if and only if $\hat{U}_\varphi^{-1} \hat{O} \hat{U}_\varphi = \hat{O}$, $\forall \varphi \in \text{Diff}(\Sigma)$. We call it a weak observable if and only if $\hat{O}$ leaves $\mathcal{H}_{\text{Diff}}$ invariant. Then it is easy to see that a strong observable $\hat{O}$ must be a weak one. One notices that a strong observable $\hat{O}$ can first be defined on $\mathcal{H}_{\text{Diff}}$ by its dual operator $\hat{O}^*$ as

$$\hat{O}^* \Phi_{\text{Diff}}[\psi] = : \Phi_{\text{Diff}}[\hat{O}\psi],$$

then one gets

$$\hat{O}^* \Phi_{\text{Diff}}[\hat{U}_\varphi \psi] = \Phi_{\text{Diff}}[\hat{O} \hat{U}_\varphi \psi] = \Phi_{\text{Diff}}[\hat{U}_\varphi^{-1} \hat{O} \hat{U}_\varphi \psi] = \hat{O}^* \Phi_{\text{Diff}}[\psi].$$

for any $\Phi_{\text{Diff}} \in \mathcal{H}_{\text{Diff}}$ and $\psi \in H_{\text{kin}}$. Hence $\hat{O}^* \Phi_{\text{Diff}}$ is also diffeomorphism invariant. In addition, a strong observable also has the property of $\hat{O}^* \eta(\psi_\gamma \iota) = \eta(\hat{O}^\dagger \psi_\gamma)$ since, $\forall \psi_\gamma, \psi_\gamma \in H_{\text{kin}},$

$$< \hat{O}^* \eta(\psi_\gamma) | \eta(\psi_\gamma) >_{\text{Diff}} = (\hat{O}^* \eta(\psi_\gamma)) [\psi_\gamma] = \eta(\psi_\gamma) [\hat{O} \psi_\gamma]$$

$$= \sum_{\varphi \in \text{Diff}(\Sigma) / \text{Diff}_n} \sum_{\psi_\gamma \in \text{GS}_n} < \hat{U}_\varphi \hat{P}_{\text{Diff},\gamma} \psi_\gamma | \hat{O} \psi_\gamma >_{\text{kin}}$$

$$= \frac{1}{n^\gamma} \sum_{\varphi \in \text{Diff}(\Sigma) / \text{Diff}_n} \sum_{\psi_\gamma \in \text{GS}_n} < \hat{U}_\varphi \hat{U}_\varphi \psi_\gamma | \hat{O} \psi_\gamma >_{\text{kin}}$$

$$= \frac{1}{n^\gamma} \sum_{\varphi \in \text{Diff}(\Sigma) / \text{Diff}_n} \sum_{\psi_\gamma \in \text{GS}_n} < \hat{U}_\varphi \hat{U}_\varphi \hat{O} \psi_\gamma | \psi_\gamma >_{\text{kin}}$$

$$= < \eta(\hat{O}^\dagger \psi_\gamma) | \eta(\psi_\gamma) >_{\text{Diff}}.$$

Note that the Hilbert space $\mathcal{H}_{\text{Diff}}$ is still non-separable if one considers the $C^\infty$ diffeomorphisms with $n > 0$. However, if one extends the diffeomorphisms to be semi-analytic diffeomorphisms, i.e. homomorphisms that are analytic diffeomorphisms up to finite isolate points (which can be viewed as an extension of the classical concept to the quantum case), the Hilbert space $\mathcal{H}_{\text{Diff}}$ would be separable [70][21].

### 6 Quantum Dynamics

In this section, we consider the quantum dynamics of loop quantum gravity. One may first consider to construct a Hamiltonian constraint (scalar constraint) operator in $H_{\text{kin}}$ or $\mathcal{H}_{\text{Diff}}$, then attempt to find the physical Hilbert space $\mathcal{H}_{\text{phys}}$ by solving the quantum Hamiltonian constraint. However, difficulties arise here due to the special role played...
by the scalar constraints in the constraint algebra (9). Firstly, the scalar constraints do not form a Lie subalgebra. Hence the strategy of group average cannot be used directly on $\mathcal{H}_{\text{kin}}$ for them. Secondly, modulo the Gaussian constraint, there is still a structure function in the Poisson bracket between two scalar constraints:

$$\{S(N), S(M)\} = -\mathcal{V}((N\partial_b M - M\partial_b N)q^{ab}),$$

which raises the danger of quantum anomaly in quantization. Moreover, the diffeomorphism constraints do not form an ideal in the quotient constraint algebra modulo the Gaussian constraints. This fact results in that the scalar constraint operator cannot be well defined on $\mathcal{H}_{\text{Diff}}$, as it does not commute with the diffeomorphism transformations $\hat{U}_\varphi$. Thus the previous construction of $\mathcal{H}_{\text{Diff}}$ seems not much meaningful for the final construction of $\mathcal{H}_{\text{phys}}$, for which we are seeking. However, one may still first try to construct a Hamiltonian constraint operator in $\mathcal{H}_{\text{kin}}$.

### 6.1 Hamiltonian Constraint Operator

The aim in this subsection is to define a quantum operator corresponding to the Hamiltonian constraint. Its classical expression reads:

$$S(N) := \frac{\kappa\beta^2}{2} \int_{\Sigma} d^3x \tilde{P}^a_i \tilde{P}^b_j \left[ \epsilon^{ij} K^k_{ab} - 2(1 + \beta^2) K^i_{[a} K^j_{b]} \right],$$

$$= S_E(N) - 2(1 + \beta^2) T(N).$$

(37)

The main idea of the construction is to first express $S(N)$ in terms of the combination of Poisson brackets between the variables which have been represented as operators on $\mathcal{H}_{\text{kin}}$, then replace the Poisson brackets by canonical commutators between the operators. We will use the volume functional for a region $R \subset \Sigma$ and the extrinsic curvature functional defined by:

$$K := \kappa\beta \int_{\Sigma} d^3x \tilde{P}^a_i K^i_a.$$  

A key trick here is to consider the following classical identity of the co-triad $e^i_a(x)$ ([132]):

$$e^i_a(x) = \frac{(\kappa\beta)^2}{2} \eta_{abc} \epsilon^{ijk} \tilde{P}^b_j \tilde{P}^c_k (x) = \frac{2}{\kappa\beta} [A^i_a(x), V_R],$$

where $x \in R$, and the expression of the extrinsic curvature 1-form $K^i_a(x)$:

$$K^i_a(x) = \frac{1}{\kappa\beta} [A^i_a(x), K].$$

Note that $K$ can be expressed by a Poisson bracket as

$$K = \beta^{-2} \{S_E(1), V_\Sigma\}.$$  

(38)
Thus one can obtain the equivalent classical expressions of $S_E(N)$ and $T(N)$ as:

\[
S_E(N) = \frac{k\beta^2}{2} \int_{\Sigma} d^3x N \frac{\overline{p}_i \overline{p}_j}{\sqrt{\text{det}g}} \epsilon^{ij} F_{ab}
\]

\[
T(N) = \frac{k\beta^2}{2} \int_{\Sigma} d^3x N \frac{\overline{p}_i \overline{p}_j}{\sqrt{\text{det}g}} K_{ab}^{i} K_{ab}^{j}
\]

where $A_a = A^i_a \tau_i$, $F_{ab} = F^{ij}_{ab} \tau_i$, $\text{Tr}$ represents the trace of the Lie algebra matrix, and $R_a \subset \Sigma$ denotes an arbitrary neighborhood of $x \in \Sigma$. In order to quantize the Hamiltonian constraint as a well-defined operator on $\mathcal{H}_{\text{kin}}$, one has to express the classical formula of $S(N)$ in terms of holonomies $A(\epsilon)$ and other variables with clear quantum analogs. There are genuine ambiguities in this regularization procedure, which will be summarized at the end of this subset. However, the nontrivial fact is that there do exist well-defined strategies. We now introduce the original strategy proposed first by Thiemann.$^{[32]}$ Given a triangulation $T(\epsilon)$ of $\Sigma$, where the parameter $\epsilon$ describes how fine the triangulation is, and the triangulation will fill out the spatial manifold $\Sigma$ when $\epsilon \to 0$. For each tetrahedron $\Delta \in T(\epsilon)$, we use $\{s_i(\Delta)\}_{i=1,2,3}$ to denote the three outgoing oriented segments in $\Delta$ with a common beginning point $\nu(\Delta) = s_i(\Delta)$, and use $a_{ij}(\Delta)$ to denote the arc connecting the end points of $s_i(\Delta)$ and $s_j(\Delta)$. Then several loops $a_{ij}(\Delta)$ are formed by $a_{ij}(\Delta) := s_i(\Delta) \circ a_{ij}(\Delta) \circ s_j(\Delta)^{-1}$. Thus one has the identities:

\[
\{ \int_{s_i(\Delta)} A_a \delta^{\nu}_a(\Delta) , V_{R_{ahi}} \} = -A(s_i(\Delta))^{-1} [A(s_i(\Delta)), V_{R_{ahi}}] + \alpha(\epsilon), \tag{39}
\]

\[
\{ \int_{s_i(\Delta)} A_a \delta^{\nu}_a(\Delta) , K \} = -A(s_i(\Delta))^{-1} [A(s_i(\Delta)), K] + \alpha(\epsilon), \tag{40}
\]

\[
\int_{P_{ij}} F_{ab}(x) = \frac{1}{2} A(a_{ij}(\Delta))^{-1} - \frac{1}{2} A(a_{ij}(\Delta)) + \alpha(\epsilon^2), \tag{41}
\]

where $P_{ij}$ is the plane with boundary $a_{ij}$. Note that the above identities are constructed by taking account of internal gauge invariance of the final formula of Hamiltonian constraint operator. So we have the regularized expression of $S(N)$ by the Riemannian sum $^{[32]}$:

\[
S_E^*(N) = \frac{2}{3k^2\beta^3} \sum_{\Delta \in T(\epsilon)} N(\nu(\Delta)) \epsilon^{ijk} \times \text{Tr} \left( A(a_{ij}(\Delta))^{-1} A(s_i(\Delta))^{-1} [A(s_j(\Delta)), V_{R_{ahi}}] \right).
\]

\[
T^*(N) = \frac{\sqrt{2}}{6k^2\beta^3} \sum_{\Delta \in T(\epsilon)} N(\nu(\Delta)) \epsilon^{ijk} \times \text{Tr} \left( A(s_i(\Delta))^{-1} [A(s_j(\Delta)), K] A(s_j(\Delta))^{-1} [A(s_j(\Delta)), K] \times \right.
\]

44
Poisson brackets by canonical commutators, i.e., invariant under internal gauge transformations. Since all constituents in the expression have clear quantum analogs, one can quantize the regulated Hamiltonian constraint as an operator on $H_{\text{kin}}$ (or $H^{G}$) by replacing them by the corresponding operators and Poisson brackets by canonical commutators, i.e.,

$$H = \{A(s_{\Delta}), V_{R}^{a(b)}\}.$$  

such that $\lim_{\epsilon \to 0} S'(N) = S(N)$. It is clear that the above regulated formula of $S(N)$ is invariant under internal gauge transformations. Now we begin to construct the Hamiltonian constraint operator in analogy with the classical expression (42). All we should do is to define the corresponding regulated operators on different $H_{\epsilon}$ separately, then remove the regulator $\epsilon$ so that the limit operator is defined on $H_{\text{kin}}$ (or $H^{G}$) cylindrically consistently. In the following, given a vertex and three edges intersecting at the vertex in a graph $\gamma$ of $\psi \in C_{\text{yl}}(\mathcal{A}/G)$, we construct one triangulation of the neighborhood of the vertex adapted to the three edges. Then we average with respect to the triples of edges meeting at the given vertex. Precisely speaking, one can make the triangulations $T(\epsilon)$ with the following properties.

- The chosen triple of edges at each vertex in the graph $\gamma$ is embedded in a $T(\epsilon)$ for all $\epsilon$, so that the vertex $v$ of $\gamma$ where the three edges meet coincides with a vertex $v(\Delta)$ in $T(\epsilon)$.

- For every triple of segments $(e_{1}, e_{2}, e_{3})$ of $\gamma$ such that $v = s(e_{1}) = s(e_{2}) = s(e_{3})$, there is a tetrahedron $\Delta \in T(\epsilon)$ such that $v = v(\Delta) = s(\Delta)$, and $s_{i}(\Delta) \subset e_{i}$, $\forall$ $i = 1, 2, 3$. We denote such a tetrahedron as $\Delta_{e_{i},e_{j},e_{k}}^{0}$.

- For each tetrahedra $\Delta^{0}_{e_{i},e_{j},e_{k}}$, one can construct seven additional tetrahedra $\Delta^{\psi}_{e_{i},e_{j},e_{k}}$, $\forall = 1, ..., 7$, by backward analytic extensions of $s(\Delta)$ so that $U_{e_{i},e_{j},e_{k}} := \cup_{\psi = 0}^{7} \Delta^{\psi}_{e_{i},e_{j},e_{k}}$ is a neighborhood of $v$. The triangulation must be fine enough so that the neighborhoods $U(v) := \cup_{e_{i},e_{j},e_{k}} U_{e_{i},e_{j},e_{k}}(v)$ are disjoint for different vertices $v$ and $v'$ of $\gamma$. Thus for any open neighborhood $U_{v}$ of the graph $\gamma$, there exists a triangulation $T(\epsilon)$ such that $\cup_{v \in V(\gamma)} U(v) \subseteq U_{v}$.

- The distance between a vertex $v(\Delta)$ and the corresponding arcs $a_{ij}(\Delta)$ is described by the parameter $\epsilon$. For any two different $\epsilon$ and $\epsilon'$, the arcs $a_{ij}(\Delta)$ and $a_{ij}(\Delta')$ with respect to one vertex $v(\Delta)$ are semi-analytically diffeomorphic with each other.

- Taking account of all possible triangulations $T(\epsilon)$ given by different choices of the triples of edges at each vertex in $\gamma$, the integral over $\Sigma$ is replaced by the
Thus we find the regulated expression of Hamiltonian constraint operator with respect to the triangulations of \( \hat{\psi} \) for any cylindrical function \( S \).

The triangulations for the regions \( \varepsilon (\text{43}) \).

\[
\psi \in \gamma \rightarrow \gamma \psi \equiv \psi \gamma \equiv \gamma \psi \in \gamma.
\]

\( n(v) \) denotes the binomial coefficient which comes from the averaging with respect to the triples of edges meeting at given vertex \( v \). One then observes that

\[
\int_{\Sigma - U_{(v)}} = 8 \int_{\hat{\psi}_{e,\delta_3}(v)}
\]

in the limit \( \epsilon \to 0 \).

- The triangulations for the regions

\[
U(v) - U_{e_1,e_2,e_3}(v),
\]

\[
U_{\gamma} - \bigcup_{v \in V(\gamma)} U(v),
\]

\[
\Sigma - U_{\gamma},
\]

are arbitrary. These regions do not contribute to the construction of the operator, since the commutator term \( [A(s_1(\Delta)), V_{R_{i,j,k}}] \psi_{\gamma} \) vanishes for all tetrahedron \( \Delta \) in the regions \( \text{43} \).

Thus we find the regulated expression of Hamiltonian constraint operator with respect to the triangulations \( T(\epsilon) \) as \[132\]

\[
\hat{\mathcal{S}}_{e,\gamma}(N) = \frac{16}{3 \hbar k^2 \beta} \sum_{v \in V(\gamma)} \frac{N(v)}{E(v)} \sum_{v(\Delta) = v} \epsilon_{ijk} \times \text{Tr} \left( \hat{A}(s_i(\Delta))^{-1} \hat{A}(s_k(\Delta))^{-1} [\hat{A}(s_k(\Delta)), \hat{V}_{U_0}^\prime] \right),
\]

\[
\hat{\mathcal{T}}_{\gamma}(N) = -\frac{4 \sqrt{2}}{3 \hbar^3 k^4 \beta^3} \sum_{v \in V(\gamma)} \frac{N(v)}{E(v)} \sum_{v(\Delta) = v} \epsilon_{ijk} \times \text{Tr} \left( \hat{A}(s_i(\Delta))^{-1} [\hat{A}(s_i(\Delta)), \hat{K}'] \hat{A}(s_j(\Delta))^{-1} [\hat{A}(s_j(\Delta)), \hat{K}'] \times \hat{A}(s_k(\Delta))^{-1} [\hat{A}(s_k(\Delta)), \hat{V}_{U_0}^\prime] \right),
\]

\[
\hat{\mathcal{S}}(N)\psi_{\gamma} = [\hat{\mathcal{S}}_{e,\gamma}(N) - 2(1 + \beta^2)\hat{\mathcal{T}}_{\gamma}(N)]\psi_{\gamma} = \sum_{v \in V(\gamma)} N(v)\hat{\mathcal{S}}_{e,\gamma}(N)\psi_{\gamma},
\]

for any cylindrical function \( \psi_{\gamma} \in \text{Cyl}_\gamma(\mathcal{A}/\mathcal{G}) \). Note that, by construction, the operation of \( \hat{\mathcal{S}}(N) \) on \( \psi_{\gamma} \in \text{Cyl}_\gamma(\mathcal{A}/\mathcal{G}) \) is reduced to a finite combination of that of \( \hat{\mathcal{S}}_{e,\gamma} \) with
respect to different vertices of $\gamma$. Hence, for each $\epsilon > 0$, $\hat{S}^\epsilon(N)$ is a well-defined Yang-Mills gauge invariant and diffeomorphism covariant operator on $Cyl(\mathcal{A}/G)$. The family of regulated Hamiltonian constraint operators with respect to the ordered family of graphs are cylindrically consistent up to diffeomorphisms \[132\].

The last step is to remove the regulator by taking the limit $\epsilon \to 0$. However, the action of a regulated Hamiltonian constraint operator on $\psi_\gamma$ adds arcs $a_i(\Delta)$ with a $\frac{1}{\epsilon}$-representation with respect to each $v(\Delta)$ of $\mathcal{G}$, i.e., the action of the operators family $\hat{S}^\epsilon(N)$ on cylindrical functions is graph-changing. Thus $\hat{S}^\epsilon(N)$ does not converge with respect to the weak operator topology in $\mathcal{H}_{kin}$ when $\epsilon \to 0$, since different $\mathcal{H}_{un}$ with different graphs $\gamma$ are mutually orthogonal. Thus one has to define a weaker operator topology to make the operator limit meaningful. By physical motivation and the naturally available Hilbert space $\mathcal{H}_{Diff}$, the convergence of $\hat{S}^\epsilon(N)$ holds with respect to the so-called Uniform Rovelli-Smolin Topology \[124\], where one defines $\hat{S}^\epsilon(N)$ to converge if and only if $\Psi_{Diff}[\hat{S}^\epsilon(N)\phi]$ converge for all $\Psi_{Diff} \in Cyl_{Diff}^*$ and $\phi \in Cyl(\mathcal{A}/G)$. Since the value of $\Psi_{Diff}[\hat{S}^\epsilon(N)\phi]$ is actually independent of $\epsilon$ by the fifth property of the triangulations, the sequence converges to a nontrivial result $\Psi_{Diff}[\hat{S}^\epsilon_0(N)\phi]$ with arbitrary fixed $\epsilon_0 > 0$. Thus one has defined a diffeomorphism covariant, closed but non-symmetric operator, $\hat{S}(N) = \lim_{\epsilon \to 0} \hat{S}^\epsilon(N) = \hat{S}^\epsilon_0(N)$, on $\mathcal{H}_{kin}$ (or $\mathcal{H}^G$) representing the Hamiltonian constraint. Moreover, a dual Hamiltonian constraint operator $\hat{S}^\star(N)$ is also defined on $Cyl^*$ depending on a specific value of $\epsilon$

\[
(\hat{S}^\star(N)\Psi)[\phi] := \Psi[\hat{S}^\star(N)\phi],
\]

for all $\Psi \in Cyl^*$ and $\phi \in Cyl(\mathcal{A}/G)$. For $\Psi_{Diff} \in Cyl_{Diff}^* \subset Cyl^*$, one gets

\[
(\hat{S}^\star(N)\Psi_{Diff})[\phi] = \Psi_{Diff}[\hat{S}^\star(N)\phi].
\]

which is independent of the value of $\epsilon$. Several remarks on the Hamiltonian constraint operator are listed as follows.

• **Finiteness of $\hat{S}(N)$ on $\mathcal{H}_{kin}$**

In ordinary quantum field theory, the continuous quantum field is only recovered when one lets lattice spacing to approach zero, i.e., takes the continuous cutoff parameter to its continuous limit. However, this will produce the well-known infinity in quantum field theory and make the Hamiltonian operator ill-defined on the Fock space. So it seems surprising that our operator $\hat{S}(N)$ is still well defined, when one takes the limit $\epsilon \to 0$ with respect to the Uniform Rovelli-Smolin Topology so that the triangulation goes to the continuum. The reason behind it is that the cut-off parameter is essentially noneffective due to the diffeomorphism invariance of our quantum field theory. This is why there is no UV divergence in the background independent quantum gauge field theory with diffeomorphism invariance. On the other hand, from a convenient viewpoint, one may think the Hamiltonian constraint operator as an operator dually defined on a dense domain

---

9The Hamiltonian constraint operator depends indeed on the choice of the representation $j$ on the arcs $a_j(\Delta)$, which is known as one of the regularization ambiguities in the construction of quantum dynamics. For the simplicity of the theory, one often chooses the lowest label of representation $j = \frac{1}{2}$.
in $\mathcal{H}_{\text{Diff}}$. However, we will see that the dual Hamiltonian constraint operator cannot leave $\mathcal{H}_{\text{Diff}}$ invariant.

- **Implementation of Dual Quantum Constraint Algebra**

One important task is to check whether the commutator algebra (quantum constraint algebra) among the corresponding quantum operators of constraints both physically and mathematically coincides with the classical constraint algebra by substituting quantum constraint operators to classical constraint functionals and commutators to Poisson brackets. Here the quantum anomaly has to be avoided in the construction of constraint operators (see the discussion for Eq. (31)). First, the subalgebra of the quantum diffeomorphism constraint algebra is free of anomaly by construction:

$$
\hat{U}_\varphi \hat{U}_{\varphi'}^{-1} \hat{U}_{\varphi'}^{-1} = \hat{U}_{\varphi \varphi' \varphi'^{-1} \varphi'^{-1}},
$$

which coincides with the exponentiated version of the Poisson bracket between two diffeomorphism constraints generating the transformations $\varphi, \varphi' \in Diff(\Sigma)$. Secondly, the quantum constraint algebra between the dual Hamiltonian constraint operator $\hat{S}'(N)$ and the finite diffeomorphism transformation $\hat{U}_\varphi$ on diffeomorphism-invariant states coincides with the classical Poisson algebra between $V(\vec{N})$ and $S(M)$. Given a cylindrical function $\phi_\gamma$ associated with a graph $\gamma$ and the triangulations $T(\epsilon)$ adapted to the graph $\alpha$, the triangulations $T(\varphi \circ \epsilon) \equiv \varphi \circ T(\epsilon)$ are compatible with the graph $\varphi \circ \gamma$. Then we have by definition:

$$
\left( - ([\hat{S}'(N), \hat{U}_\varphi]) \Psi_{\text{Diff}} \right) [\phi_\gamma]
= (\left( [\hat{S}'(N), \hat{U}_\varphi^{-1}] \right) \Psi_{\text{Diff}})[\phi_\gamma]
= \Psi_{\text{Diff}} \hat{S}'(N) \phi_\gamma - \hat{S}'(N) \phi_{\varphi \gamma}
= \sum_{v \in V(\gamma)} \{ N(v) \} \Psi_{\text{Diff}} \hat{S}'_v \phi_\gamma - N(\varphi \circ v) \Psi_{\text{Diff}} \hat{S}'_{\varphi \gamma} \phi_{\varphi \gamma}
= \sum_{v \in V(\gamma)} \{ N(v) \} - N(\varphi \circ v) \Psi_{\text{Diff}} \hat{S}'_v \phi_\gamma
= \left( \hat{S}'(N - \varphi^* N) \right) \Psi_{\text{Diff}} \phi_\gamma.
$$

Thus there is no anomaly. However, Eq. (44) also explains why the Hamiltonian constraint operator $\hat{S}(N)$ cannot leave $\mathcal{H}_{\text{Diff}}$ invariant.

Third, we compute the commutator between two Hamiltonian constraint operators. Notice that

$$
\left[ \hat{S}(N), \hat{S}(M) \right] \phi_\gamma
= \sum_{v \in V(\gamma)} \left[ M(v) \right] \hat{S}(N) - \hat{S}(M) \right] \hat{S}_v \phi_\gamma
= \sum_{v \in V(\gamma)} \sum_{v' \in V(\gamma')} \left[ M(v) N(v') - N(v) M(v') \right] \hat{S}_v \hat{S}_{v'} \phi_\gamma,
$$

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where γ' is the graph changed from γ by the action of \( \hat{S}(N) \) or \( \hat{S}(M) \), which adds the arcs \( a_{ij}(\Delta) \) on γ, \( T(\epsilon) \) is the triangulation adapted to γ and \( T(\epsilon') \) adapted to γ'. Since the newly added vertices by \( \hat{S}_v \) is planar, they will never contributes the final result. So one has

\[
[\hat{S}(N), \hat{S}(M)] \phi_{\gamma} = \sum_{v, v' \in V(\gamma), v \neq v'} [M(v)N(v') - N(v)M(v')] \hat{S}_v' \hat{S}_v \phi_{\gamma} = \frac{1}{2} \sum_{v, v' \in V(\gamma), v \neq v'} [M(v)N(v') - N(v)M(v')] [\hat{S}_v' \hat{S}_v - \hat{S}_v \hat{S}_v'] \phi_{\gamma} = \frac{1}{2} \sum_{v, v' \in V(\gamma), v \neq v'} [M(v)N(v') - N(v)M(v')] [(\hat{U}_{v,v'} - \hat{U}_{v',v}) \hat{S}_v' \hat{S}_v] \phi_{\gamma},
\]

where we have used the facts that \([\hat{S}_v, \hat{S}_v'] = 0\) for \( v \neq v' \) and there exists a diffeomorphism \( \varphi_{v,v'} \) such that \( \hat{S}_v' \hat{S}_v' = \hat{U}_{v,v'} \hat{S}_v' \hat{S}_v' \). Obviously, we have in the Uniform Rovelli-Smolin Topology

\[ ([\hat{S}(N), \hat{S}(M)])^* \Psi_{diff} = 0 \]

for all \( \Psi_{diff} \in Cyl^*_{diff} \). As we have seen in classical expression Eq.(45), the Poisson bracket of any two Hamiltonian constraints is given by a generator of the diffeomorphism transformations. Therefore it is mathematically consistent with the classical expression that two Hamiltonian constraint operators commute on diffeomorphism states. On the other hand, it has been shown in Refs. \[73\] and \[95\] that the domain of dual Hamiltonian constraint operator can be extended to a slightly larger space (habitat) in \( Cyl^* \), whose elements are not necessary diffeomorphism invariant. While, it turns out that the commutator between two Hamiltonian constraint operators continues to vanish on the habitat, which seems to be problematic. Fortunately, the quantum operator corresponding to the right hand side of classical Poisson bracket Eq.(36) also annihilates every state in the habitat \[73\], so the quantum constraint algebra is consistent at this level. But it is not clear whether the quantum constraint algebra, especially the commutator between two Hamiltonian constraint is consistent with the classical one on some larger space in \( Cyl^* \) containing more diffeomorphism variant states. So further work on the semi-classical analysis is needed to test the classical limit of Eq.(45).

The way to do it is to look for some suitable semi-classical states for calculating the classical limit of the operators. However, due to the graph-changing property of the Hamiltonian constraint operator, the semi-classical analysis of the Hamiltonian constraint operator and the quantum constraint algebra is still an open issue so far.

- **General Regularization Scheme of Hamiltonian Constraint**

  In Ref.\[21\], a general scheme of regulation is introduced for the quantization of Hamiltonian constraint, which includes Thiemann’s regularization as a specific choice. Such a general regularization can be summarized as follows:
First, we assign a partition of $\Sigma$ into cells $\square$ of arbitrary shape. In every cell of the partition we define edges $s_j$, $j = 1, \ldots, n_s$ and loops $\beta_i$, $i = 1, \ldots, n_{\beta}$, where $n_s, n_{\beta}$ may be different for different cells. One uses $\epsilon$ to represent the scale of the cell $\square$. Then one fixes an arbitrary chosen representation $\rho$ of $SU(2)$ for the calculation of the holonomies in Eqs. (39)-(41). This structure is called a permissible classical regulator if the regulated Hamiltonian constraint expression with respect to this partition has correct limit when $\epsilon \to 0$.

Secondly, one assigns the diffeomorphism covariant property to the regulator and lets the partition adapted to the choice of the graph. Given a cylindrical function $\psi_\gamma \in Cy^*_\chi(\mathcal{A}U\mathcal{G})$, the partition is sufficiently refined so that every vertex $v \in V(\gamma)$ is contained in exact one cell of the partition. If $(\gamma, v)$ is diffeomorphic to $(\gamma', v')$ then, for every $\epsilon$ and $\epsilon'$, the quintuple $(\gamma, v, \square, (s_j), (\beta_i))$ is diffeomorphic to the quintuple $(\gamma', v', \square', (s'_j), (\beta'_i))$, where $\square$ and $\square'$ are the cells in the partitions with respect to $\gamma$ and $\gamma'$ respectively, containing $v$ and $v'$ respectively.

As a result, the Hamiltonian constraint operator in this general regularization scheme is expressed as:

$$\hat{S}_{E,\gamma}^\epsilon(N) = \sum_{v \in V(\gamma)} \frac{N(v)}{\hbar k^2 \beta} \sum_{ij} C^{ij} \text{Tr}\left(\rho[A(\beta_i)] - \rho[A(\beta_i^{-1})] \rho[A(s_j^{-1})] \rho[A(s_j)], \hat{V}\right),$$

$$\hat{\mathcal{H}}_\gamma^\epsilon(N) = \sum_{v \in V(\gamma)} \frac{iN(v)}{\hbar k^2 \beta} \sum_{ijkl} T^{ijkl} \text{Tr}\left(\rho[A(s_j^{-1})] \rho[A(s_l)], \hat{K}\right),$$

$$\hat{S}^\epsilon(N) \psi_\gamma = [\hat{S}_{E,\gamma}^\epsilon(N) - 2(1 + \beta^2) \hat{\mathcal{H}}_\gamma^\epsilon(N)] \psi_\gamma,$$

where $C^{ij}$ and $T^{ijkl}$ are fixed constants independent of the value of $\epsilon$. After removing the regulator $\epsilon$ via diffeomorphism invariance the same as we did above, one obtains a well-defined diffeomorphism covariant operator on $\mathcal{H}_{\text{kin}}$ (or $\mathcal{H}^G$) in the sense of Uniform Rovelli-Smolin Topology, or dually defines the operator on some suitable domain in $Cy^*$. Note that such a general scheme of construction exhibits that there are a great deal of freedom in choosing the regulators, so that there are considerable ambiguities in our quantization for seeking a proper quantum dynamics for gravity, which is still an open issue today.

### 6.2 Inclusion of Matter Field

The quantization technique for the Hamiltonian constraint can be generalized to quantize the Hamiltonian of matter fields coupled to gravity \[136\]. As an example, in this subsection we consider the situation of background independent quantum dynamics of a real massless scale field coupled to gravity. The coupled generalized Palatini action reads \[88\]

$$S[\rho_{\beta}, \omega^{IJ}, \phi] = S_\rho[\rho_{\beta}, \omega^{IJ}] + S_{KG}[\rho_{\beta}, \phi],$$

where

$$S_\rho[\rho_{\beta}, \omega^{IJ}] = \frac{1}{2\kappa} \int_M \sqrt{g} \epsilon_\beta \epsilon^\gamma \epsilon^\alpha \Omega_{\alpha \beta}^{IJ} + \frac{1}{2\beta} \epsilon^{IKL} \Omega_{\alpha \beta}^{KL},$$

and

$$S_{KG}[\rho_{\beta}, \phi] = \frac{1}{2\pi} \int_M \sqrt{g} \epsilon_\beta \epsilon^\gamma \epsilon^\alpha \Omega_{\alpha \beta}^{IJ} \phi.$$
where \( \lambda, N^a \) and \( N \) are Lagrange multipliers, and the three constraints in the Hamiltonian are expressed as [32][88]:

\[
G_i = D_i \bar{P}_i^a := \partial_i \bar{P}_i^a + \epsilon_{ij}^k A_i^k \bar{P}_i^a, \\
C_a = \bar{P}_i^a F^i_{ab} - A_i^a \bar{P}_i^a + \pi \partial_a \phi, \\
C = \frac{k \beta^2}{2} \bar{P}_i^a \bar{P}_j^b \epsilon^{ij}_{ab} F^i_{ab} - 2(1 + \beta^2) K^i [\alpha \beta_j K^j_i] \\
+ \frac{1}{\sqrt{\det q}} \left[ \frac{k \beta^2 \alpha_M}{2} \delta^{ij} \bar{P}_i^a \bar{P}_j^a (\partial_a \phi) \partial_b \phi + \frac{1}{2 \alpha_M} \pi^2 \right],
\]

here \( \pi \) denotes the momentum conjugate to \( \phi \):

\[
\pi := \frac{\partial L}{\partial \dot{\phi}} = \frac{\alpha_M}{N} \sqrt{\det q} \phi - N^a \partial_a \phi.
\]

Thus one has the elementary Poisson brackets

\[
\{ A_i^j(x), \bar{P}_j^a(y) \} = \delta_i^a \delta_j^j \delta(x-y), \\
\{ \phi(x), \pi(y) \} = \delta(x-y).
\]

Note that the second term of the Hamiltonian constraint (48) is just the Hamiltonian of the real scalar field.

Then we look for the background independent representation for the real scalar field coupled to gravity, following the polymer representation of the scalar field [28]. The classical configuration space, \( \mathcal{U} \), consists of all real-valued smooth functions \( \phi \) on \( \Sigma \). Given a set of finite number of points \( X = \{ x_1, \ldots, x_N \} \) in \( \Sigma \), an equivalent relation can be defined by: given two scalar field \( \phi_1, \phi_2 \in \mathcal{U} \), \( \phi_1 \sim \phi_2 \) if and only if \( \exp[i \lambda \phi_1(x_i)] = \exp[i \lambda \phi_2(x_i)] \) for all \( x_i \in X \) and all real number \( \lambda \). Since one can define a projective family with respect to the sets of points (graphs for scalar field), a projective limit \( \overline{\mathcal{U}} \), which is a compact topological space, is obtained as the quantum configuration space of scalar field. Next, we denote by \( Cyl_X(\overline{\mathcal{U}}) \) the vector space generated by finite linear combinations of the following functions of \( \phi \):

\[
T_{X, \lambda}(\phi) := \prod_{x_i \in X} \exp[i \lambda \phi(x_i)],
\]

where \( \lambda \equiv (\lambda_1, \lambda_2, \ldots, \lambda_N) \) are arbitrary non-zero real numbers assigned at each point. It is obvious that \( Cyl_X(\overline{\mathcal{U}}) \) has the structure of a *-algebra. The vector space \( Cyl(\overline{\mathcal{U}}) \) of all
cylindrical functions on $\mathcal{U}$ is defined by the linear span of $T_0 = 1$ and $T_{X,A}$. Completing $\text{Cyl} (\mathcal{U})$ with respect to the sup norm, one obtains a unital Abelian $C^*$-algebra $\text{Cyl} (\mathcal{U})$. Thus one can use the GNS structure to construct its cyclic representations. A preferred positive linear functional $\omega_0$ on $\text{Cyl} (\mathcal{U})$ is defined by

$$\omega_0 (T_{X,A}) = \begin{cases} 1 & \text{if } \lambda_j = 0 \forall j \\ 0 & \text{otherwise}, \end{cases}$$

which defines a diffeomorphism-invariant faithful Borel measure $\mu$ on $\mathcal{U}$ as

$$\int_{\mathcal{U}} d\mu (T_{X,A}) = \begin{cases} 1 & \text{if } \lambda_j = 0 \forall j \\ 0 & \text{otherwise}. \end{cases}$$ (49)

Thus one obtains the Hilbert space, $\mathcal{H}^\text{KG}_{\text{kin}} := L^2 (\mathcal{U}, d\mu)$, of square integrable functions on $\mathcal{U}$ with respect to $\mu$. The inner product can be expressed explicitly as:

$$< T_c | T_{c'} >^\text{KG}_{\text{kin}} = \delta_{c,c'},$$ (50)

where the label $c := (X, \lambda)$ is called scalar-network. As one might expect, the quantum configuration space $\mathcal{U}$ is just the Gel'fand spectrum of $\text{Cyl} (\mathcal{U})$. More concretely, for a single point set $X_0 \equiv \{ x_0 \}$, $\text{Cyl} (\mathcal{U})$ is the space of all almost periodic functions on a real line $\mathbb{R}$. The Gel'fand spectrum of the corresponding $C^*$-algebra $\text{Cyl} (\mathcal{U})$ is the Bohr compactification $\mathbb{R}_0$ of $\mathbb{R}$ [28], which is a compact topological space such that $\text{Cyl} (\mathcal{U})$ is the $C^*$-algebra of all continuous functions on $\mathbb{R}_0$. Since $\mathbb{R}$ is densely embedded in $\mathbb{R}_0$, $\mathbb{R}_0$ can be regarded as a completion of $\mathbb{R}$.

It is clear from Eq. (49) that an orthonormal basis in $\mathcal{H}^\text{KG}_{\text{kin}}$ is given by the scalar vacuum $T_0 = 1$ and the so-called scalar-network functions $T_c (\phi)$. So the total kinematical Hilbert space $\mathcal{H}^\text{kin}$ is the direct product of the kinematical Hilbert space $\mathcal{H}^\text{GR}$ for gravity and the kinematical Hilbert space for real scalar field, i.e., $\mathcal{H}^\text{kin} := \mathcal{H}^\text{GR} \otimes \mathcal{H}^\text{KG}_{\text{kin}}$. Then the spin-network state $T_{X,c} \equiv T_c (A) \otimes T_s (\phi) \in \text{Cyl} (\gamma (A) \otimes \mathcal{G}) \otimes \text{Cyl} (\mathcal{U})$ is a gravity-scalar cylindrical function on graph $\gamma (s, c) \equiv \gamma (s) \cup X(c)$. Note that generally $X(c)$ may not coincide with the vertices of the graph $\gamma (s)$. It is straightforward to see that all of these functions constitutes a orthonormal basis in $\mathcal{H}^\text{kin}$ as

$$< T_{c'} (A) \otimes T_{s'} (\phi) | T_s (A) \otimes T_c (\phi) >^\text{kin} = \delta_{c,c'} \delta_{s,s'} .$$

Note that none of $\mathcal{H}^\text{kin}$, $\mathcal{H}^\text{GR}_{\text{kin}}$ and $\mathcal{H}^\text{KG}_{\text{kin}}$ is a separable Hilbert space.

Given a pair $(x_0, \lambda_0)$, there is an elementary configuration for the scalar field, the so-called point holonomy,

$$U (x_0, \lambda_0) := \exp \{ i \lambda_0 \phi (x_0) \} .$$

It corresponds to a configuration operator $\hat{U} (x_0, \lambda_0)$, which acts on any cylindrical function $\psi (\phi) \in \text{Cyl} (\mathcal{U})$ by

$$\hat{U} (x_0, \lambda_0) \psi (\phi) = U (x_0, \lambda_0) \phi (\phi).$$ (51)
All these operators are unitary. But since the family of operators $\hat{U}(x_0, \lambda)$ fails to be weakly continuous in $\lambda$, there is no field operator $\hat{\phi}(x)$ on $\mathcal{H}_{kin}$. The momentum functional smeared on a 3-dimensional region $R \subset \Sigma$ is expressed by

$$\pi(R) := \int_R d^3x \vec{\pi}(x).$$

The Poisson bracket between the momentum functional and a point holonomy can be easily calculated to be

$$[\pi(R), U(x, \lambda)] = -i\lambda \chi_R(x)U(x, \lambda),$$

where $\chi_R(x)$ is the characteristic function for the region $R$. So the momentum operator is defined by the action on scalar-network functions $T_c = (\chi_c, \lambda)$ as

$$\hat{\pi}(R)T_c(\phi) := i\hbar[\pi(R), T_c(\phi)] = i\hbar \sum_{x \in X} \lambda_j \chi(x_j)T_c(\phi).$$

Now we can impose the quantum constraints on $\mathcal{H}_{kin}$ and consider the quantum dynamics. First, the Gaussian constraint can be solved independently of $\mathcal{H}_{kin}^{KG}$, since it only involves gravitational field. It is also expected that the diffeomorphism constraint can be implemented by the group averaging strategy in the similar way as that in the case of pure gravity. Given a spatial diffeomorphism transformation $\varphi$, an unitary transformation $\hat{U}_\varphi$ was induced by $\varphi$ in the Hilbert space $\mathcal{H}_{kin}$, which is expressed as

$$\hat{U}_\varphi T_{s=\varphi(x_0),c=\varphi(x),\lambda} = T_{\varphi s = \varphi(x_0), \varphi c = \varphi(x), \varphi \lambda}.$$

Then the differomorphism invariant spin-network functions are defined by group averaging as

$$T_{[s,c]} := \frac{1}{n_{\gamma}(s,c)} \sum_{\varphi \in Diff_f(\Sigma)Diff_f(\Sigma)} \sum_{\varphi' \in GS_{\gamma}} \hat{U}_\varphi \hat{U}_{\varphi'} T_{s,c}.$$  \hspace{1cm} (52)

where $Diff_f$ is the set of diffeomorphisms leaving the colored graph $\gamma$ invariant, $GS_{\gamma}$ denotes the graph symmetry quotient group $Diff_f/DirDiff_f$, where $DirDiff_f$ is the set of the diffeomorphisms which is trivial on the graph $\gamma$, and $n_{\gamma}$ is the number of elements in $GS_{\gamma}$. Following the standard strategy in quantization of pure gravity, an inner product can be defined on the vector space spanned by the diffeomorphism invariant spin-network functions (and the vacuum states for gravity, scalar and both respectively) such that they form an orthonormal basis as:

$$< T_{[s,c]}|T_{[s',c']} >_{diff} := T_{[s,c]}|T_{s',c' }\delta_{[s,c],[s',c']}.$$  \hspace{1cm} (53)

After the completion procedure, we obtain the expected Hilbert space of diffeomorphism invariant states for the scalar field coupled to gravity, which is denoted by $\mathcal{H}_{Diff}$. Then the only nontrivial task is the implementation of the Hamiltonian constraint $S(N)$. One thus needs to define a corresponding Hamiltonian constraint operator on $\mathcal{H}_{kin}$. While the gravitational part of

$$S(N) := \int_\Sigma d^3x NC$$

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is a well-defined operator \( \hat{S}_{GR}(N) \) by the Uniform Rovelli-Smolin Topology, the crucial point in this subsection is to define an operator corresponding to the Hamiltonian functional \( S_{KG}(N) \) of the scalar field, which can be decomposed into two parts

\[
S_{KG}(N) = S_{KG,\delta}(N) + S_{KG,\text{Kin}}(N),
\]

where

\[
S_{KG,\delta}(N) = \frac{k^2 \beta^2 \alpha_M}{2} \int_\Sigma d^3 x \frac{1}{\sqrt{\det q}} \delta^{ij} \tilde{P}_i^a(\partial_\alpha \phi) \partial_\beta \phi,
\]

\[
S_{KG,\text{Kin}}(N) = \frac{1}{2 \alpha_M} \int_\Sigma d^3 x \frac{1}{\sqrt{\det q}} \pi^2.
\]

We use the identities, for \( x \in R \)

\[
\tilde{P}_i^a = \frac{1}{2k\beta} \tilde{\eta}^{abc} e_i^a e_j^b e_j^c \quad \text{and} \quad e_i^a(x) = \frac{2}{k\beta} [A_i^a(x), V_{U_i^j}].
\]

Hence

\[
\tilde{P}_i^a(x) = \frac{2}{k^3 \beta^3} \tilde{\eta}^{abc} e_i^a [A_i^a(x), V_{U_i^j}].
\]

Then the expressions of \( S_{KG,\delta}(N) \) and \( S_{KG,\text{Kin}}(N) \) can be regulated via a point-splitting strategy

\[
S_{KG,\delta}^\epsilon(N) = \frac{k^2 \beta^2 \alpha_M}{2} \int_\Sigma d^3 x \int_\Sigma d^3 y N^{1/2}(x) N^{1/2}(y) \chi_e(x - y) \delta^{ij} \times
\]

\[
\frac{1}{\sqrt{V_{U_i^j}}} \tilde{P}_i^a(\partial_\alpha \phi(x)) \frac{1}{\sqrt{V_{U_j^a}}} \tilde{P}_j^b(\partial_\beta \phi(y))
\]

\[
= \frac{32 \alpha_M}{k^4 \beta^4} \int_\Sigma d^3 x \int_\Sigma d^3 y N^{1/2}(x) N^{1/2}(y) \chi_e(x - y) \delta^{ij} \times
\]

\[
\tilde{\eta}^{a \beta c}(\partial_\alpha \phi(x)) \text{Tr} \{T_i \{A_i^a(x), V_{U_i^j} \}, V_{U_j^c} \} \}
\]

\[
S_{KG,\text{Kin}} = \frac{1}{2 \alpha_M} \int_\Sigma d^3 x \int_\Sigma d^3 y N^{1/2}(x) N^{1/2}(y) \pi(x) \pi(y) \times
\]

\[
\int_\Sigma d^3 u \frac{\det(e^a_\mu)}{(V_{U_i^j})^{3/2}} \int_\Sigma d^3 w \frac{\det(e^a_\nu)}{(V_{U_j^c})^{3/2}} \chi_e(x - y) \chi_e(u - x) \chi_e(w - y)
\]

\[
= \frac{1}{2 \alpha_M} \frac{2^8}{9(k \beta)^6} \int_\Sigma d^3 x \int_\Sigma d^3 y N^{1/2}(x) N^{1/2}(y) \pi(x) \pi(y) \times
\]

\[
\int_\Sigma d^3 u \tilde{\eta}^{a \beta c} \text{Tr} \{T_i \{A_i^a(u), \sqrt{V_{U_i^j}} \}, V_{U_j^c} \} \}
\]

\[
= \frac{1}{2 \alpha_M} \frac{2^8}{9(k \beta)^6} \int_\Sigma d^3 x \int_\Sigma d^3 y N^{1/2}(x) N^{1/2}(y) \pi(x) \pi(y) \times
\]

\[
\int_\Sigma d^3 u \tilde{\eta}^{a \beta c} \text{Tr} \{T_i \{A_i^a(u), \sqrt{V_{U_i^j}} \}, V_{U_j^c} \} \}
\]

\[
\chi_e(x - y) \chi_e(u - x) \chi_e(w - y),
\]

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where the matrices $A_x \equiv A^1_x \tau_x \chi_x(x - y)$ is the characteristic function of a box containing $x$ with scale $\epsilon$ such that $\lim_{\epsilon \to 0} \chi_x(x - y)/\epsilon^3 = \delta(x - y)$, and $V_{U_{\epsilon}}$ is the volume of the box. Introducing a triangulation $T(\epsilon)$ of $\Sigma$ by tetrahedrons $\Delta$, we notice the following useful identities

$$\left\{ \int_{s(\Delta)} dt \ A_x \delta^2(t), V_{U_{\epsilon}}^{3/4} \right\} = -A'(s(\Delta))^{-1/2}A(s(\Delta)), V_{U_{\epsilon}}^{3/4} + o(\epsilon),$$

and

$$\int_{s(\Delta)} dt \ \partial \phi^2(t) = \frac{1}{i\lambda} \left[U(s(\Delta)), \lambda^{-1}U(t(s(\Delta)), \lambda) - U(s(\Delta), \lambda)\right] + o(\epsilon)$$

for nonzero $\lambda$, where $s(\Delta)$ and $t(s(\Delta))$ denote respectively the beginning and end points of segment $s(\Delta)$ with scale $\epsilon$ associated with a tetrahedron $\Delta$. Regulated on the triangulation, the scalar field part of the classical Hamiltonian constraint reads

$$S_{KG,\phi}(N) = \frac{4\alpha M}{9\pi^2} \sum_{\Delta \in T(\epsilon)} \sum_{\Delta \in T(\epsilon)} N^{1/2}(v(\Delta))N^{1/2}(v(\Delta')) \chi \left( \frac{v(\Delta) - v(\Delta')}{\delta} \right) \times$$

$$\chi \left( \frac{v(\Delta) - v(\Delta')}{\delta} \right) \times$$

$$\sum_{\Sigma_{\Delta}} e^{inn} \left\{ \tau_i (A(s_m(\Delta'))^{-1}) \right\} A(s_m(\Delta), V_{U_{\epsilon}}^{3/4}) \times$$

$$\chi \left( \frac{v(\Delta) - v(\Delta')}{\delta} \right) \times$$

$$\sum_{\Sigma_{\Delta}} e^{inn} \left\{ \tau_i (A(s_m(\Delta'))^{-1}) \right\} A(s_m(\Delta), V_{U_{\epsilon}}^{3/4}) \times$$

$$(54)$$

Note that the above regularization is explicitly dependent on the parameter $\lambda$, which will lead to a kind of quantization ambiguity of the real scalar field dynamics in polymer-like representation. Introducing a partition $\mathcal{P}$ of the 3-manifold $\Sigma$ into cells $\mathcal{C}$,
we can smear the essential “square roots” of $S_{KG,δ}$ and $S_{KG,Kin}$ in one cell $C$ respectively and promote them as regulated operators in $\mathcal{H}_{Kin}$ with respect to triangulations $T(\epsilon)$ depending on spin-scalar-network state $T_{x,C}$ as

$$\hat{W}^{r,C}_{x(C),i} = \sum_{v \in V(x,C)} \chi_C(v) E(v) \sum_{v(\Delta) = v} \hat{h}^{r,\Delta}_{\phi,i},$$

$$\hat{W}^{r,C}_{y(C),Kin} = \sum_{v \in V(y,C)} \chi_C(v) E(v) \sum_{v(\Delta) = v} \hat{h}^{r,\Delta}_{Kin,v},$$

(55)

where $\chi_C(v)$ is the characteristic function of the cell $C$, and

$$\hat{h}^{r,\Delta}_{\phi,i} := \frac{i}{\hbar^2} e^{\text{Im} m} \frac{1}{\lambda(v)} \hat{U}(v, \lambda(v))^{-1} [\hat{U}(s(\Delta)), \lambda(v)] \hat{U}(v, \lambda(v)) \times \text{Tr} (\tau_n \hat{A}(s_n(\Delta))^{-1} [\hat{A}(s_n(\Delta)), \hat{V}_{\lambda,1}]),$$

$$\hat{h}^{r,\Delta}_{Kin,v} := \frac{1}{(i\hbar)^3} \hat{U}(v) e^{\text{Im} m} \text{Tr} [\hat{A}(s(\Delta))^{-1} [\hat{A}(s(\Delta)), \sqrt{\hat{V}_{\lambda,1}}] \times \hat{A}(s_n(\Delta))^{-1} [\hat{A}(s_n(\Delta)), \sqrt{\hat{V}_{\lambda,1}}]].$$

(56)

Both operators in (55) and their adjoint operators are cylindrically consistent up to diffeomorphisms. Thus there are two densely defined operators $\hat{W}^{C}_{\phi,i}$ and $\hat{W}^{C}_{Kin}$ in $\mathcal{H}_{Kin}$ associated with the two consistent families of (55). We now give several remarks on their properties.

- **Removal of regulator $\epsilon$**

  It is not difficult to see that the action of the operator $\hat{W}^{r,C}_{x(C),i}$ on a spin-scalar-network function $T_{x,C}$ is graph-changing. It adds finite number of vertices with representation $\lambda(v)$ at $t(s(\Delta))$ with distance $\epsilon$ from the vertex $v$. Recall that the action of the gravitational Hamiltonian constraint operator on a spin network function is also graph-changing. As a result, the family of operators $\hat{W}^{r,C}_{y(C),i}$ also fails to be weakly converged when $\epsilon \to 0$. However, due to the diffeomorphism covariance properties of the triangulation, the limit operator can be well-defined via the uniform Rovelli-Smolin topology, or equivalently, the operator can be dually defined on diffeomorphism invariant states. But the dual operator cannot leave $\mathcal{H}_{Diff}$ invariant.

- **Quantization ambiguity**

  As a main difference of the dynamics in polymer-like representation from that in U(1) group representation (126), a continuous label $\lambda$ appears explicitly in the expression of (55). Hence there is an one-parameter quantization ambiguity due to the real scalar field. Recall that the construction of gravitational Hamiltonian constraint operator also has a similar ambiguity due to the choice of the representations $j$ of the edges added by its action. A related quantization ambiguity also appears in the dynamics of loop quantum cosmology [52].
By taking the limit $\mathcal{P} \to \Sigma$ so that $C \to \nu$, the quantum Hamiltonian constraint $\hat{S}_{KG}(N)$ of scalar field is expressed as:

$$\hat{S}_{KG,\nu(x)}(N) := \sum_{v \in V(\nu(x))} N(v)\left[ 64 \times \frac{4\alpha M}{9\kappa^2\beta^2} \hat{\rho}^{ij}(v) \hat{W}^{\nu,v}_{\phi,i} \hat{W}^{\nu,v}_{\phi,j} \right] + 8^4 \frac{16}{81\alpha M(k\beta)^6} (\hat{W}^{\nu,v}_{Kin})^\dagger \hat{W}^{\nu,v}_{Kin}, \tag{57}$$

where the operators $\hat{W}^{\nu,v}_{\phi,i}$ and $\hat{W}^{\nu,v}_{Kin}$ are the inductive limit of the consistent family $\{\hat{W}^{\nu,v}_{\phi,i}\}$ and $\{\hat{W}^{\nu,v}_{Kin}\}$, and $(\hat{W}^{\nu,v}_{\phi,i})^\dagger$ and $(\hat{W}^{\nu,v}_{Kin})^\dagger$ are their adjoint respectively. Hence the family of Hamiltonian constraint operators (57) is also cylindrically consistent up to a diffeomorphism, and the regulator $\epsilon$ can be removed via the uniform Rovelli-Smolin topology, or equivalently the limit operator dually acts on diffeomorphism invariant states as

$$\langle \hat{S}_{KG}(N)\Psi_{Diff}[f] \rangle = \lim_{\epsilon \to 0} \Psi_{Diff}[\hat{S}_{KG}(N)f], \tag{58}$$

for any $f \in Cyl(\Omega) \otimes Cyl(\mathcal{U})$. Similar to the dual of $\hat{S}_{GR}(N)$, the operator $\hat{S}_{KG}(N)$ fails to commute with the dual of finite diffeomorphism transformation operators, unless the smearing function $N(x)$ is a constant function over $\Sigma$. In fact, one can define a self-adjoint Hamiltonian operator from $\hat{S}_{KG}(1)$ for the polymer scalar field in the diffeomorphism invariant Hilbert space $\mathcal{H}_{Diff}$ [90]. From Eq. (57), it is not difficult to prove that for positive $N(x)$ the Hamiltonian constraint operator $\hat{S}_{KG}(N)$ of scalar field is positive and symmetric in $\mathcal{H}_{Kin}$ and hence has a unique self-adjoint extension [90]. Note that there is an 1-parameter ambiguity in our construction of $\hat{S}_{KG}(N)$ due to the real scalar field, which is manifested as the continuous parameter $\lambda$ in the expression of $\hat{h}^{\nu,v}_{\phi,i}$ in (56). Thus the total Hamiltonian constraint operator of scalar field coupled to gravity has been obtained as

$$\hat{S}(N) = \hat{S}_{GR}(N) + \hat{S}_{KG}(N). \tag{59}$$

Again, there is no UV divergence in this quantum Hamiltonian constraint. Recall that, in standard quantum field theory the UV divergence can only be cured by renormalization procedure, in which one has to multiply the Hamiltonian by a suitable power of the regulating parameter $\epsilon$ artificially. While, now $\epsilon$ has naturally disappeared from the expression of (59). So renormalization is not needed for the polymer-like scalar field coupled to gravity, since quantum gravity has been presented as a natural regulator. This result heightens our confidence that the issue of divergence in quantum fields theory can be cured in the framework of loop quantum gravity.

Now we have obtained the desired matter-coupled quantum Hamiltonian constraint equation

$$-\langle \hat{S}_{KG}(N)\Psi_{Diff}[f] \rangle \hat{\rho}(x,t) = \langle \hat{S}_{GR}(N)\Psi_{Diff}[f] \rangle \hat{\rho}(x,t), \tag{60}$$

Comparing it with the well-known Schrödinger equation for a particle,

$$i\hbar \frac{\partial}{\partial t} \psi(x,t) = H(\hat{x}, -i\hbar \frac{\partial}{\partial x}) \psi(x,t),$$

57
where $\psi(x, t) \in L^2(\mathbb{R}, dx)$ and $t$ is a parameter labeling time evolution, one may take the viewpoint that the matter field constraint operator $\hat{S}_{KG}^t(N)$ plays the role of $i\hbar \frac{\partial}{\partial t}$. Then $\phi$ appears as the parameter labeling the evolution of the gravitational field state. In the reverse viewpoint, gravitational field would become the parameter labeling the evolution of the quantum matter field. Note that such an idea has been successfully applied in loop quantum cosmology model to understand the quantum nature of big bang in the deep Planck regime [29][30].

### 6.3 Master Constraint Programme

Although the Hamiltonian constraint operator introduced in Section 6.1 is densely defined on $\mathcal{H}_{kin}$ and diffeomorphism covariant, there are still several problems unsettled which are listed below.

- It is unclear whether the commutator between two Hamiltonian constraint operators resembles the classical Poisson bracket between two Hamiltonian constraints. Hence it is doubtful whether the quantum Hamiltonian constraint produces the correct quantum dynamics with correct classical limit [73][95].

- The dual Hamiltonian constraint operator does not leave the Hilbert space $\mathcal{H}_{Diff}$ invariant. Thus the inner product structure of $\mathcal{H}_{Diff}$ cannot be employed in the construction of physical inner product.

- Classically the collection of Hamiltonian constraints do not form a Lie algebra. So one cannot employ group average strategy in solving the Hamiltonian constraint quantum mechanically, since the strategy depends on group structure crucially.

However, if one could construct an alternative classical constraint algebra, giving the same constraint phase space, which is a Lie algebra (no structure function), where the subalgebra of diffeomorphism constraints forms an ideal, then the programme of solving constraints would be much improved at a basic level. Such a constraint Lie algebra was first introduced by Thiemann in [147]. The central idea is to introduce the master constraint:

$$M := \frac{1}{2} \int_\Sigma d^3x \frac{|\tilde{C}(x)|^2}{\sqrt{|\det q(x)|}},$$  \hspace{1cm} (61)

where $\tilde{C}(x)$ is the scalar constraint in Eq. (8). One then gets the master constraint algebra:

\[
\begin{align*}
\{\mathcal{V}(N), \mathcal{V}(N')\} &= \mathcal{V}(\{N, N'\}), \\
\{\mathcal{V}(N), M\} &= 0, \\
\{M, M\} &= 0.
\end{align*}
\]

The master constraint programme has been well tested in various examples [64][65][66][67][68]. In the following, we extend the diffeomorphism transformations such that the Hilbert
space $\mathcal{H}_{\text{Diff}}$ is separable. This separability of $\mathcal{H}_{\text{Diff}}$ and the positivity and the diffeomorphism invariance of $\mathbf{M}$ will be working together properly and provide us with powerful functional analytic tools in the programme to solve the constraint algebra quantum mechanically. The regularized version of the master constraint can be expressed as

$$\mathbf{M}^\epsilon := \frac{1}{2} \int_{\Sigma} d^3 y \int_{\Sigma} d^3 x \chi_\epsilon(x-y) \frac{\overline{C}(y)}{\sqrt{V_{U'}}} \frac{C(x)}{\sqrt{V_{U'}}}.$$  

Introducing a partition $P$ of the 3-manifold $\Sigma$ into cells $C$, we have an operator $\hat{H}_{C,\gamma}^\epsilon$ acting on any cylindrical function $f_y \in C\text{yl}_\gamma(\mathcal{A}(\mathcal{G}))$ in $\mathcal{H}^G$ as

$$\hat{H}_{C,\gamma}^\epsilon f_y = \sum_{v \in \gamma(v)} \frac{\chi_C(v)}{E(v)} \sum_{v(\Delta) = v} \hat{v}_v^{\epsilon,\Delta} f_y,$$

via a family of state-dependent triangulations $T(e)$ on $\Sigma$, where $\chi_C(v)$ is the characteristic function of the cell $C(v)$ containing a vertex $v$ of the graph $\gamma$. Note that both $\hat{H}_{C,\gamma}^\epsilon$ and its adjoint are cylindrically consistent up to diffeomorphisms, and the expression of $\hat{v}_v^{\epsilon,\Delta}$ reads

$$\hat{v}_v^{\epsilon,\Delta} = \frac{16}{3i \hbar \kappa^2 \beta} e^{ijk} T \left\{ \hat{A}(s_i(\Delta))^{-1} \hat{A}(s_k(\Delta))^{-1} [\hat{A}(s_{\Delta}(\Delta)), \sqrt{V_{U'}}] \right\}$$

$$+ 2(1 + \beta^2) \frac{4 \sqrt{2}}{3i \hbar \kappa^2 \beta^2} e^{ijk} T \left\{ \hat{A}(s_i(\Delta))^{-1} [\hat{A}(s_{\Delta}(\Delta)), \hat{K}] \right\}$$

$$\hat{A}(s_i(\Delta))^{-1} [\hat{A}(s_{\Delta}(\Delta)), \sqrt{V_{U'}}] \hat{A}(s_k(\Delta))^{-1} [\hat{A}(s_{\Delta}(\Delta)), \hat{K}].$$

Note that $\hat{v}_v^{\epsilon,\Delta}$ is similar to that involved in the regulated Hamiltonian constraint operator in section 6.1, while the only difference is that now the volume operator is replaced by its square-root in Eq. (63). Hence the action of $\hat{H}_{C,\gamma}^\epsilon$ on $f_y$ adds arcs $a_{v(\Delta)}$ with 1/2-representation with respect to each $v(\Delta)$ of $\gamma$. Thus, for each $\epsilon > 0$, $\hat{H}_{C,\gamma}^\epsilon$ is a Yang-Mills gauge invariant and diffeomorphism covariant operator defined on $C\text{yl}_\gamma(\mathcal{A}(\mathcal{G}))$. The family of such operators with respect to different graphs is cylindrically consistent up to diffeomorphisms and hence can give a limit operator $\hat{H}_C$ densely defined on $\mathcal{H}^G$ by the uniform Rovelli-Smolin topology. The adjoint operator $(\hat{H}_{C,\gamma})^\dagger$ can be well defined in $\mathcal{H}^G$ as

$$(\hat{H}_{C,\gamma})^\dagger = \sum_{v \in \gamma(v)} \frac{\chi_C(v)}{E(v)} \sum_{v(\Delta) = v} (\hat{v}_v^{\epsilon,\Delta})^\dagger,$$

such that the limit operators $\hat{H}_C$ and $(\hat{H}_C)^\dagger$ in the uniform Rovelli-Smolin topology satisfy

$$< g_\gamma', \hat{H}_C f_y >_{\text{kin}} = < g_\gamma', (\hat{H}_{C,\gamma})^\dagger g_\gamma', f_y >_{\text{kin}} =< g_\gamma', f_y >_{\text{kin}},$$

$$< (\hat{H}_C)^\dagger g_\gamma', f_y >_{\text{kin}} =< (\hat{H}_C)^\dagger g_\gamma', (\hat{H}_C)^\dagger f_y >_{\text{kin}}. $$

59
where $\hat{H}_C$ and $(\hat{H}_C)^\dagger$ are respectively the inductive limits of $\hat{H}_{C,\gamma}$ and $(\hat{H}_{C,\gamma})^\dagger$. Then a master constraint operator, $\hat{M}$, acting on any $\Psi_{\text{Diff}} \in \mathcal{H}_{\text{Diff}}$ can be defined as [89]

$$
(\hat{M}\Psi_{\text{Diff}})[f_y] := \lim_{P \to \Sigma, \epsilon, \epsilon' \to 0} \Psi_{\text{Diff}}[\sum_{C \in P} \frac{1}{2} \hat{H}_C^\dagger(\hat{H}_C)^\dagger f_y],
$$

(66)

for any $f_y$ is a finite linear combination of spin-network function. Note that $\hat{H}_C^\dagger(\hat{H}_C)^\dagger f_y$ is also a finite linear combination of spin-network functions on an extended graph with the same skeleton of $\gamma$, hence the value of $(\hat{M}\Psi_{\text{Diff}})[f_y]$ is finite for any given $\Psi_{\text{Diff}}$. Thus $\hat{M}\Psi_{\text{Diff}}$ lies in the algebraic dual of the space of cylindrical functions. Furthermore, we can show that $\hat{M}$ leaves the diffeomorphism invariant distributions invariant. For any diffeomorphism transformation $\phi$ on $\Sigma$,

$$
(\hat{U}_\phi \hat{M}\Psi_{\text{Diff}})[f_y] = \lim_{P \to \Sigma, \epsilon, \epsilon' \to 0} \Psi_{\text{Diff}}[\sum_{C \in P} \frac{1}{2} \hat{H}_C^\dagger(\hat{H}_C)^\dagger \hat{U}_\phi f_y] = \lim_{P \to \Sigma, \epsilon, \epsilon' \to 0} \Psi_{\text{Diff}}[\sum_{C \in P} \frac{1}{2} \hat{H}_C^\dagger(\hat{H}_C)^\dagger f_y],
$$

(67)

where in the last step, we used the fact that the diffeomorphism transformation $\phi$ leaves the partition invariant in the limit $P \to \sigma$ and relabel $\phi(C)$ to be $C$. So we have the result

$$
(\hat{U}_\phi \hat{M}\Psi_{\text{Diff}})[f_y] = (\hat{M}\Psi_{\text{Diff}})[f_y].
$$

(68)

So given a diffeomorphism invariant spin-network state $T_{[s]}$, the resulted state $\hat{M}T_{[s]}$ must be a diffeomorphism invariant element in the algebraic dual of $C_{\text{yl}}(\overline{\mathcal{A}}/\overline{\mathcal{G}})$, which can be formally expressed as

$$
\hat{M}T_{[s]} = \sum_{[s_1]} c_{[s_1]} T_{[s_1]}.
$$

Thus for any $T_{s_2}$, one has

$$
\lim_{P \to \Sigma, \epsilon, \epsilon' \to 0} T_{[s]}[\sum_{C \in P} \frac{1}{2} \hat{H}_C^\dagger(\hat{H}_C)^\dagger T_{s_2}] = \sum_{[s_1]} c_{[s_1]} T_{[s_1]}[T_{s_2}],
$$

where the cylindrical function $\sum_{C \in P} \frac{1}{2} \hat{H}_C^\dagger(\hat{H}_C)^\dagger T_{s_2}$ is a finite linear combination of spin-network functions on some graphs $\gamma'$ with the same skeleton of $\gamma(s_2)$ up to finite number of arcs. Hence fixing the diffeomorphism equivalent class $[s]$, only for spin-networks $s_2$ which lie in a finite number of diffeomorphism equivalent classes the left hand side of the last equation can be non-zero. So there should also be only finite number of classes $[s_1]$ in the right hand side such that the corresponding coefficients $c_{[s_1]}$ are non-zero. As a result, $\hat{M}T_{[s]}$ is a finite linear combination of diffeomorphism invariant spin-network states and hence lies in the Hilbert space of diffeomorphism invariant states $\mathcal{H}_{\text{Diff}}$ for any $[s]$. Therefore $\hat{M}$ is densely defined on $\mathcal{H}_{\text{Diff}}$. Moreover,
given two diffeomorphism invariant spin-network functions $T_{[s_1]}$ and $T_{[s_2]}$, a straightforward calculation can give the matrix elements of $\hat{M}$ as

$$\langle T_{[s_1]} | \hat{M} | T_{[s_2]} \rangle_{Diff} = \lim_{\rho, \epsilon, \epsilon' \to 0} \sum_{C \in \mathcal{P}} \sum_{v \in \gamma(s_{[s_1]})} \frac{1}{2} \epsilon^{\epsilon'} T_{[s_2]} [ \hat{H}^e_C (\hat{H}^e_C)^+ ] T_{[s_1]} [ \hat{H}^e_v T_{s_{[s]}} ]$$

(69)

From Eq. (69) and the fact that the master constraint operator $\hat{M}$ is densely defined on $\mathcal{H}_{Diff}$, it is obvious that $\hat{M}$ is a positive and symmetric operator in $\mathcal{H}_{Diff}$. Therefore, the quadratic form $Q_M$ associated with $\hat{M}$ is closable [114]. The closure of $Q_M$ is the quadratic form of a unique self-adjoint operator $\tilde{M}$ called the Friedrichs extension of $\hat{M}$. We relabel $\tilde{M}$ to be $\hat{M}$ for simplicity. From the construction of $\tilde{M}$, the qualitative description of the kernel of the Hamiltonian constraint operator in Ref. [134] can be transcribed to describe the solutions to the equation: $\hat{M} \Psi_{Diff} = 0$. In particular, the diffeomorphism invariant cylindrical functions based on at most 2-valent graphs are obviously normalizable solutions. In conclusion, there exists a positive and self-adjoint operator $\hat{M}$ on $\mathcal{H}_{Diff}$ corresponding to the master constraint (61), and zero is in the point spectrum of $\hat{M}$.

Note that the quantum constraint algebra can be easily checked to be anomaly free. i.e.,

$$[\hat{M}, \hat{U}_\epsilon^\prime] = 0, \quad [\hat{M}, \hat{M}] = 0,$$

which is consistent with the classical master constraint algebra in this sense. As a result, the difficulty of the original Hamiltonian constraint algebra can be avoided by introducing the master constraint algebra, due to the Lie algebra structure of the latter.

It can be seen that zero is in the spectrum of $\tilde{M}$ [138], so the further task is to obtain the physical Hilbert space $\mathcal{H}_{phys}$ which is the kernel of the master constraint operator with some suitable physical inner product, and the issue of quantum anomaly is represented in terms of the size of $\mathcal{H}_{phys}$ and the existence of semi-classical states. Note that the master constraint programme can be straightforwardly generalized to include matter fields [90]. We list some open problems in master constraint programme for further research.

- **Kernel of Master Constraint Operator**

  Since the master constraint operator $\tilde{M}$ is self-adjoint, it is a practical problem to find the DID of $\mathcal{H}_{Diff}$:

  $$\mathcal{H}_{Diff} \sim \int d\mu(\lambda) \mathcal{H}_{\lambda}^0,$$

  $$\langle \Phi | \Psi \rangle_{Diff} = \int \mathcal{R} d\mu(\lambda) \langle \Phi | \Psi \rangle_{\mathcal{H}_{\lambda}^0},$$

  61
where $\mu(A)$ is the spectral measure with respect to the master constraint operator $\hat{M}$. It is expected that one can identify $\mathcal{H}_{0}^{\oplus}$ with the physical Hilbert space. However, as discussed in Ref.[64], such a prescription would be ambiguous in the case where zero is only in the continuous spectrum. Also certain physical information would be lost in the case where zero is an embedded eigenvalue. The prescription is unambiguous only if zero is an isolated eigenvalue, in which case however the whole machinery of the DID is not needed at all because $\mathcal{H}_{0}^{\oplus} \subset \mathcal{H}_{\text{Diff}}$ and the physical inner product coincides with the kinematical (differomorphism invariant) one. To cure the problem, some improved prescriptions are proposed also in Ref.[64], where one decomposes the measure with respect to the spectrum types before direct integral decomposition. Then certain ambiguities can be canceled by some physical criterions, such as, a complete subalgebra of bounded Dirac observables should be represented irreducibly as self-adjoint operators on the physical Hilbert space, and the resulting physical Hilbert space should admit a sufficient number of semiclassical states. Nonetheless, due to the complicated structure of master constraint operator, it is certainly difficult to manage the spectrum analysis and direct integral decomposition. On the other hand, in the light of the self-adjointness of the master constraint operator and the Lie-algebra structure of the constraint algebra, a formal group average strategy was introduced in Ref.[147] as another possible way to get the physical Hilbert space, which posses potential relation with the path-integral formulation. However, so far such a strategy is still formal.

• Dirac Observables

Classically, one can prove that a function $O \in C^\infty(M)$ is a weak observable with respect to the scalar constraint if and only if
\[
\{O, \{O, M\}\}_{M} = 0.
\]
We define $O$ to be a strong observable with respect to the scalar constraint if and only if
\[
\{O, M\}_{M} = 0,
\]
and to be a ultra-strong observable if and only if
\[
\{O, S(N)\}_{M} = 0.
\]
In quantum version, an observable $\hat{O}$ is a weak Dirac observable if and only if $\hat{O}$ leaves $\mathcal{H}_{\text{phys}}$ invariant, while $\hat{O}$ is now called a strong Dirac observable if and only if $\hat{O}$ commutes with the master constraint operator $\hat{M}$. Given a bounded self-adjoint operator $\hat{O}$ defined on $\mathcal{H}_{\text{Diff}}$, for instance, a spectral projection of some observables leaving $\mathcal{H}_{\text{Diff}}$ invariant, if the uniform limit exists, the bounded self-adjoint operator defined by group averaging
\[
[\hat{O}] := \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} dt \, \hat{U}(t)^{-1} \hat{O} \hat{U}(t)
\]
commutes with $\hat{M}$ and hence becomes a strong Dirac observable on the physical Hilbert space.
• Testing the Classical Limit of Master Constraint Operator

One needs to construct spatial diffeomorphism invariant semiclassical states to calculate the expectation value and fluctuation of the master constraint operator. If the results coincide with the classical values up to $\hbar$ corrections, one can go ahead to finish our quantization programme with confidence. It is encouraging that within the so-called Algebraic Quantum Gravity framework [80], the correct classical limit of a master constraint operator is recently obtained [81][82].

6.4 ADM Energy Operator of Loop Quantum Gravity

To solve the dynamical problem in loop quantum gravity, one may consider to find a suitable Hamiltonian operator, in order to settle up the problem of time. A strategy for that is to seek for an operator corresponding to the ADM energy for asymptotically flat spacetime, which equivalently takes the form [137]

$$E_{ADM} = \lim_{\partial \Sigma \to \partial \Sigma} -2\kappa \beta^2 \int_S dS \frac{n_a}{\sqrt{\det q}} \tilde{P}_a \delta^{ij} \hat{J}^i_{\Sigma} \hat{J}^j_{\Sigma},$$

(70)

where $n_a$ is the normal co-vector of a close 2-sphere $S$ and $dS$ is the coordinate volume element on $S$ induced from that of a asymptotically Cartesian coordinate system on $\Sigma$.

In Ref. [137], Thiemann quantized the ADM energy (70) to obtain a positive semi-definite and self-adjoint operator $\hat{E}_{ADM}$ as

$$\hat{E}_{ADM}f_a := 2\hbar^2 \kappa \beta^2 \sum_{v \in V(\alpha) \cap \partial \Sigma} \frac{1}{V_v} \delta^{ij} \hat{f}^i_{\Sigma} \hat{f}^j_{\Sigma} f_a,$$

which is defined on an extension of $H_{kin}$ allowing for edges without compact support (see the infinite tensor product extension of kinematical Hilbert space [143]). Since the volume operator $V_v$ commutes with the "total angular momentum" operator $[\hat{J}] = \delta^{ij} \hat{J}^i \hat{J}^j$, these two operators can be simultaneous diagonalized with respect to certain linear combinations of spin network states. The eigenvalues of $\hat{E}_{ADM}$ are of the form $2\hbar^2 \kappa \beta^2 \Sigma_{v \in V(\alpha) \cap \partial \Sigma} \lambda_v / \lambda_v$, where $\lambda_v$ is the eigenvalue of $V_v$. Thus we may think that the spin quantum numbers of spin network states are playing the role of the occupied numbers of Fock states in quantum field theory, which provide a non-linear Fock decomposition for loop quantum gravity. This motivates us to call the future quantum dynamics of loop quantum gravity as Quantum Spin Dynamics (QSD) [132][133][134][135][136][137].

Moreover, $\hat{E}_{ADM}$ trivially commutes with all constraint operators, since the gauge transformations are trivial at $\partial \Sigma$. Hence $\hat{E}_{ADM}$ is a true quantum Dirac observable. Then a meaningful time parameter can be selected by the continuous one-parameter unitary group generated by $\hat{E}_{ADM}$, which leads to a "Schrödinger equation" for QSD as:

$$i\hbar \frac{\partial}{\partial t} f = \hat{E}_{ADM} f$$
7 Applications and Advances

This section is devoted as a summary of the applications and some recent advances which are not discussed in the main content of the article. After providing a guidance for beginners to references in the current research of loop quantum cosmology, quantum black holes, and black hole entropy calculation, the basic ideas of coherent states construction will be sketched. We refer to Refs. [21], [149] and [146] for more concrete exploration. Some key open problems in the current research of loop quantum gravity will also be raised in our discussion.

7.1 Symmetric Models and Black Hole Entropy

It is well known that the most difficulty in general relativity is the singularity problem. The presence of singularities, such as the big bang and black holes, is widely believed to be a signal that classical general relativity has been pushed beyond the domain of its validity. Can loop quantum gravity at the present stage resolve the singularity problem?

As the full quantum dynamics of loop quantum gravity has not been solved completely, one then deals with the singularity problem in certain symmetric models by applying the ideas and techniques from loop quantum gravity. For simplifications, one generally freezes all but a finite number of degrees of freedom by imposing the suitable symmetry condition [55]. The symmetry-reduced models can also provide a mathematically simple arena to test the ideas and constructions in the full loop quantum gravity theory. The singularity problem was first considered in the so-called loop quantum cosmology models by imposing spatially homogeneity and (or) isotropy. The seminar work by Bojowald [45] shows that the big bang singularity is absent in loop quantum cosmology [11]. The result then leads to a new understanding on the initial condition problem in quantum cosmology [40][48]. Another remarkable result is that the loop quantum cosmological modification of Friedmann equation may cure the fine tuning problem of the inflation potential, so that the inflation can arise naturally and exit gracefully due to the quantum geometry effect [47][61]. Recently, semiclassical states are used to understand the quantum evolution of the universe across the deep Planck regime [29]. It turns out that the classical big bang is replaced by a quantum big bounce [30][31]. The predictions from loop quantum cosmology are reliable in the sense that the quantum dynamics of both the homogeneous with isotropic and with anisotropic models are proved to have correct classical limits [11][131]. Loop quantum cosmology is currently a very active research field. One may see Refs. [51] and [5] for brief overviews. For readers who want to know the fundamental structure of loop quantum cosmology, we refer to Refs. [11] and [154]. Also, a comprehensive review in this field has already appeared [52].

By imposing spatially spherical symmetry, one can study nonhomogeneous models, such as the Schwarzschild black hole [101], where the techniques from loop quantum gravity are also employed [49]. The treatment of these models is thus quite similar to that of loop quantum cosmology. It turns out that the interiors of the black holes are also singularity-free due to the quantum geometric properties [101][102][50][10]. One may further study the "end state" of the gravitational collapse of matter fields inside a black hole [54][103] and black hole evaporation [9]. One can find the basic framework
and recent results of loop quantum black hole in Ref.\[53\]. There are still appealing issues which one may consider about the quantum black holes. The investigation in this direction has just started. Besides the above noticeable models, there are also some other symmetric models, such as the Husain-Kuchar model \[26\] and static spacetimes \[98\]. which have been studied from the constructions of loop quantum gravity.

Another very puzzling issue in general relativity is the thermodynamics of black holes \[43\][37][156]. The black hole entropy formula brings together the three pillars of fundamental physics: general relativity, quantum theory and statistical mechanics. However, the formula itself is obtained by a rather hodge-podge mixture of classical and semi-classical ideas. Can one use loop quantum gravity to calculate the microscopic degrees of freedom which account for the black hole entropy?

We now turn to the black hole entropy calculation in loop quantum gravity. Recall that the definition of the event horizon of a black hole in general relativity concerns the global structure of the spacetime \[155\]. However, to account for black hole entropy by statistical calculations in loop quantum gravity, one needs to define locally the notion of a horizon, which can assume that the black hole itself is in equilibrium while the exterior geometry is not forced to be time independent. This is the so-called isolated horizon classically defined by Ashtekar et al (see Ref.\[15\] for a precise definition). It turns out that the zeroth and the first laws of black-hole mechanics can be naturally extended to type II isolated horizons \[15\][7], where the horizon geometry is axi-symmetric. If one considers the spacetimes which contain an isolated horizon as an internal boundary, the action principle and the Hamiltonian description are well defined. Note that, in contrast with the symmetry-reduced models, here the phase space has an infinite number of degrees of freedom.

In quantum kinematical setting, it is natural to begin with a total Hilbert space \[H = H_V \times H_S\], where \(H_V\) is built from suitable functions of generalized connections in the bulk and \(H_S\) from suitable functions of generalized surface connections. The horizon boundary condition can then be imposed as an operator equation on \(H\). Taking account of the structure of the surface term in the symplectic structure, this quantum boundary condition implies that \(H_S\) is the Hilbert space of a \(U(1)\) Chern-Simons theory on a punctured 2-sphere \[6\][14]. To calculate entropy, one constructs the micro-canonical ensemble by considering only the subspace of the bulk theory with a fixed area of the horizon (a similar idea was raised in an earlier paper by Rovelli \[117\]). Employing the spectrum \[29\] of the area operator in \(H_V\), a detail analysis can estimates the number of Chern-Simons surface states consistent with the given area. One thus obtains the (black hole) horizon entropy, whose leading term is indeed proportional to the horizon area \[6\]. However, the expression of the entropy agrees with the Hawking-Bekenstein formula only if one chooses a particular Barbero-Immirzi parameter \(\beta_0\) (see Ref.\[69\] for a recent discussion on the choice of \(\beta_0\)). The nontrivial fact is that this theory with fixed \(\beta_0\) can yield the Hawking-Bekenstein value of entropy of all isolated horizons, irrespective of the values of charges, angular momentum and cosmology constant, the amount of distortion or hair \[21\]. The sub-leading term has also been calculated and shown to be proportional to the logarithm of the horizon area \[94\]. Note that in the entropy calculation the quantum Gauss and diffeomorphism constraints are crucially used, while the final result is insensitive to the details of how the Hamiltonian constraint is imposed. There is an excellent review on this subject in Ref.\[21\].
7.2 Construction of Coherent States

As shown in section 6, both the Hamiltonian constraint operator $\hat{S}(N)$ and the master constraint operator $\hat{M}$ can be well defined in the framework of loop quantum gravity. However, since the Hilbert spaces $H_{kin}$ and $H_{Diff}$, the operators $\hat{S}(N)$ and $\hat{M}$ are constructed in such ways that are drastically different from usual quantum field theory, one has to check whether the constraint operators and the corresponding algebras have correct classical limits with respect to suitable semiclassical states. In order to find the suitable semiclassical states and check the classical limit of the theory, the idea of a non-normalizable coherent state defined by a generalized Laplace operator and its heat kernel was introduced for the first time in Ref.[27]. Recently, kinematical coherent states are constructed in two different approaches. One leads to the so-called complexifier coherent states proposed by Thiemann et al [140][141][142][143]. The other is promoted by Varadarajan [150][151][152] and developed by Ashtekar et al [20][16].

The complexifier approach is somehow motivated by the coherent state construction for compact Lie groups [87]. One begins with a positive function $C$ (complexifier) on the classical phase space and arrives at a "coherent state" $\psi_m$, which possibly belongs to the dual space $Cyl^*$ rather than $H_{kin}$. However, one may consider the so-called "cut-off state" of $\psi_m$ with respect to a finite graph as a graph-dependent coherent state in $H_{kin}$ [144]. By construction, the coherent state $\psi_m$ is an eigenstate of an annihilation operator coming also from the complexifier $C$ and hence has desired semiclassical properties [141][142]. We now sketch the basic idea of its construction. Given the Hilbert space $\mathcal{H}$ for a dynamical system with constraints and a subalgebra of observables $S$ in the space $L(\mathcal{H})$ of linear operators on $\mathcal{H}$, the semiclassical states with respect to $S$ are defined as Definition 2.2.5. Kinematical coherent states $\{\Psi_m\}_{m \in M}$ are semiclassical states which in addition satisfy the annihilation operator property [140][144], namely there exists certain non-self-adjoint operator $\hat{z} = \hat{a} + i\lambda\hat{b}$ with $\hat{a}$, $\hat{b} \in S$ and certain squeezing parameter $\lambda$, such that

$$\hat{z}\Psi_m = z(m)\Psi_m.$$  \hfill (71)

Note that Eq. (71) implies that the minimal uncertainty relation is saturated for the pair of elements $(\hat{a}, \hat{b})$, i.e.,

$$\Psi_m([\hat{a} - \Psi_m(\hat{a})]^2) = \Psi_m([\hat{b} - \Psi_m(\hat{b})]^2) = \frac{1}{2}|\Psi_m([\hat{a}, \hat{b}])|^2. \hfill (72)$$

Note also that coherent states are usually required to satisfy the additional peakedness property, namely for any $m \in M$ the overlap function $|<\Psi_m, \Psi_{m'}>|$ is concentrated in a phase volume $\frac{1}{2}|\Psi_m([\hat{q}, \hat{p}])|$, where $\hat{q}$ is a configuration operator and $\hat{p}$ a momentum operator. So the central stuff in the construction is to define a suitable "annihilation operator" $\hat{z}$ in analogy with the simplest case of harmonic oscillator. A powerful tool named as "complexifier" is introduced in Ref.[140] to define a meaningful $\hat{z}$ operator which can give rise to kinematical coherent states for a general quantum system.

**Definition 7.2.1:** Given a phase space $M = T^*C$ for some dynamical system with configuration coordinates $q$ and momentum coordinates $p$, a complexifier, $C$, is a positive smooth function on $M$, such that
1. $C/\hbar$ is dimensionless;
2. $\lim_{|p| \to \infty} |\text{Con}||p| = \infty$ for some suitable norm on the space of the momentum;
3. Certain complex coordinates $(z(m), \bar{z}(m))$ of $\mathcal{M}$ can be constructed from $C$.

Given a well-defined complexifier $C$ on phase space $\mathcal{M}$, the programme for constructing coherent states associated with $C$ can be carried out as the following.

- **Complex polarization**

  The condition (3) in Definition 7.3.1 implies that the complex coordinate $z(m)$ of $\mathcal{M}$ can be constructed via

  $$z(m) := \sum_{n=0}^{\infty} \frac{1}{n!} \{q, C\}_{(0)}(m),$$

  (73)

  where the multiple Poisson bracket is inductively defined by $\{q, C\}_{(0)} = q$, $\{q, C\}_{(n)} = \{\{q, C\}_{(n-1)}, C\}$. One will see that $z(m)$ can be regarded as the classical version of an annihilation operator.

- **Defining annihilation operator**

  After the quantization procedure, a Hilbert space $\mathcal{H} = L^2(C, d\mu)$ with a suitable measure $d\mu$ on a suitable configuration space $C$ can be constructed. It is reasonable to assume that $\hat{C}$ can be defined as a positive self-adjoint operator $\hat{\hat{C}}$ on $\mathcal{H}$. Then a corresponding operator $\hat{z}$ can be defined by transforming the Poisson brackets in Eq.(73) into commutators, i.e.,

  $$\hat{z} := \sum_{n=0}^{\infty} \frac{1}{n!} (i\hbar)^n [\hat{q}, \hat{C}]_{(0)} = e^{-\hat{C}/\hbar} \hat{q} e^{\hat{C}/\hbar},$$

  (74)

  which is called as an annihilation operator.

- **Constructing coherent states**

  Let $\delta_{q'}(q)$ be the $\delta$-distribution on $C$ with respect to the measure $d\mu$. Since $\hat{C}$ is assumed to be positive and self-adjoint, the conditions (1) and (2) in Definition 7.3.1 imply that $e^{-\hat{C}/\hbar}$ is a well-defined ”smoothening operator”. So it is quite possible that the heat kernel evolution of the $\delta$-distribution, $e^{-\hat{C}/\hbar}\delta_{q'}(q)$, is a square integrable function in $\mathcal{H}$, which is even analytic. Then one may analytically extend the variable $q'$ in $e^{-\hat{C}/\hbar}\delta_{q'}(q)$ to complex values $z(m)$ and obtain a class of states $\psi'_m$ as

  $$\psi'_m(q) := [e^{-\hat{C}/\hbar}\delta_{q'}(q)]_{q' \to z(m)},$$

  (75)

  such that one has

  $$\hat{z}\psi'_m(q) := [e^{-\hat{C}/\hbar}\hat{q}\delta_{q'}(q)]_{q' \to z(m)} = [q' e^{-\hat{C}/\hbar}\delta_{q'}(q)]_{q' \to z(m)} = z(m) \psi'_m(q).$$

  (76)

  Hence $\psi'_m$ is automatically an eigenstate of the annihilation operator $\hat{z}$. So it is natural to define coherent states $\psi_m(q)$ by normalizing $\psi'_m(q)$. 

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One may check that all the coherent state properties usually required are likely to be satisfied by the above complexifier coherent states $\psi_m(q)$ \cite{144}. As a simple example, in the case of one-dimensional harmonic oscillator with Hamiltonian $H = \frac{1}{2}(p^2 + \frac{1}{2}m_0^2q^2)$, one may choose the complexifier $C = p^2/(2m_0)$. It is straightforward to check that the coherent state constructed by the above procedure coincides with the usual harmonic oscillator coherent state up to a phase \cite{144}. So the complexifier coherent state can be considered as a suitable generalization of the concept of usual harmonic oscillator coherent state.

The complexifier approach can be used to construct kinematical coherent states in loop quantum gravity. Given a suitable complexifier $C$, for each analytic path $e \subset \Sigma$ one can define

$$A_C^e(e) := \sum_{n=0}^{\infty} \frac{i^n}{n!} [A(e), C]_{(n)}, \quad (77)$$

where $A(e) \in SU(2)$ is assigned to $e$. As the complexifier $C$ is assumed to give rise to a positive self-adjoint operator $\hat{C}$ on the kinematical Hilbert space $\mathcal{H}_{\text{kin}}$, one further supposes that $\hat{C}/\hbar T_s = \tau_k T_s$, where $\tau$ is a so-called classicality parameter, $[T_s(A)]$, form a system of basis in $\mathcal{H}_{\text{kin}}$ and are analytic in $A \in \mathcal{A}$. Moreover the $\delta$-distribution on the quantum configuration space $\mathcal{A}$ can be formally expressed as $\delta_A(A) = \{T_s(A')T_s(A), (78)$

Thus by applying Eq. (75) one obtains coherent states

$$\psi^e_{AC}(A) = (e^{-C/\hbar})\delta_A(A)|_{A \rightarrow AC} = \sum_{s} e^{-\tau_s} T_s(A'C'T_s(A). \quad (78)$$

However, since there are uncountable infinite number of terms in the expression (78), the norm of $\psi^e_{AC}(A)$ would in general be divergent. So $\psi^e_{AC}(A)$ is generally not an element of $\mathcal{H}_{\text{kin}}$ but rather an distribution on a dense subset of $\mathcal{H}_{\text{kin}}$. In order to test the semiclassical limit of quantum geometric operators on $\mathcal{H}_{\text{kin}}$, one may further consider the "cut-off state" of $\psi^e_{AC}(A)$ with respect to a finite graph $\gamma$ as a graph-dependent coherent state in $\mathcal{H}_{\text{kin}}$ \cite{144}. So the key input in the construction is to choose a suitable complexifier. There are vast possibilities of choice. For example, a candidate complexifier $C$ is considered in Ref. \cite{146} such that the corresponding operator acts on cylindrical functions $f_\gamma$ by

$$(\hat{C}/\hbar)f_\gamma = \frac{1}{2} \sum_{\epsilon \in E(\gamma)} l(\epsilon) f^2_\epsilon f_\gamma, \quad (79)$$

where $f^2_\epsilon$ is the Casimir operator defined by Eq. (25) associated to the edge $\epsilon$, the positive numbers $l(\epsilon)$ satisfy $l(\epsilon \circ \epsilon') = l(\epsilon) + l(\epsilon')$ and $l(\epsilon^{-1}) = l(\epsilon)$. Then it can be shown from Eq. (77) that $A^e(C)$ is an element of $SU(2, \mathbb{C})$. So the classical interpretation of the annihilation operators is simply the generalized complex $SU(2)$ connections. It has been shown in Refs. \cite{141} and \cite{142} that the "cut-off state" of the corresponding coherent state,

$$\psi^e_{AC, \gamma}(A) = \psi^e_{AC, \gamma}(A)/||\psi^e_{AC, \gamma}(A)||, \quad (80)$$

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has desired semiclassical properties, where
\[
\psi'_{A, \gamma} (A) := \sum_{s, \gamma} e^{\frac{i}{\hbar} \sum_{x, y \in \gamma} \left( \gamma_{x} - \gamma_{y} \right) T_x(A^C) T_y(A)}. \tag{81}
\]

But unfortunately, these cut-off coherent states cannot be directly used to test the semiclassical limit of the Hamiltonian constraint operator \( \hat{S}(N) \), since \( \hat{S}(N) \) is graph-changing so that its expectation values with respect to these cut-off states are always zero! So further work in this approach is expected in order to overcome the difficulty.

Anyway, the complexifier approach provides a clean construction mechanism and manageable calculation method for semiclassical analysis in loop quantum gravity.

We now turn to the second approach. As we have seen, loop quantum gravity is based on quantum geometry, where the fundamental excitations are 1-dimensional polymer-like. On the other hand, low energy physics is based on quantum field theories which are constructed in a flat spacetime continuum. The fundamental excitations of these fields are 3-dimensional, typically representing wavy undulations on the background Minkowskian geometry. The core strategy in this approach is then to relate the polymer excitations of quantum geometry to Fock states used in low energy physics and to locate Minkowski Fock states in the background independent framework. Since quantum Maxwell field can be constructed in both Fock representation and polymer-like representation, one first gains insights from the comparison between the two representations, then generalizes the method to quantum geometry. A "Laplacian operator" can be defined on \( \mathcal{H}_{\text{kin}} \) \cite{27}, from which one may define a candidate coherent state \( \Phi_0 \), also in \( \text{Cyl}^* \), corresponding to the Minkowski spacetime. To calculate the expectation values of kinematical operators, one considers the so-called "shadow state" of \( \Phi_0 \), which is the restriction of \( \Phi_0 \) to a given finite graph. However, the construction of shadow states is subtly different from that of cut-off states.

We will only describe the simple case of Maxwell field to illustrate the ideas of construction \cite{15}, \cite{5}, \cite{21}. Following the quantum geometry strategy discussed in Sec.4, the quantum configuration space \( \mathbf{A} \) for the polymer representation of the \( U(1) \) gauge theory can be similarly constructed. A generalized connection \( \mathbf{A} \in \mathbf{A} \) assigns each oriented analytic edge in \( \Sigma \) an element of \( U(1) \). The space \( \mathbf{A} \) carries a diffeomorphism and gauge invariant measure \( \mu_0 \) induced by the Haar measure on \( U(1) \), which give rise to the Hilbert space, \( \mathcal{H}_0 := L^2(\mathbf{A}, d\mu_0) \), of polymer states. The basic operators are holonomy operators \( \mathbf{A}(e) \) labeled by 1-dimensional edges \( e \), which act on cylindrical functions by multiplication, and smeared electric field operators \( \hat{E}(g) \) for suitable test 1-forms \( g \) on \( \Sigma \), which are self-adjoint. Note that, since the gauge group \( U(1) \) is Abelian, it is more convenient to smear the electric fields in 3 dimensions \cite{21}. The eigenstates of \( \hat{E}(g) \), so-called flux network states \( N_{\alpha, \vec{n}} \), provide an orthonormal basis in \( \mathcal{H}_0 \), which are defined for any finite graph \( \alpha \) with \( N \) edges as:

\[
N_{\alpha, \vec{n}}(A) := [A(e_1)]^{n_1} [A(e_2)]^{n_2} \cdots [A(e_N)]^{n_N}, \tag{82}
\]

where \( \vec{n} \equiv (n_1, \cdots, n_N) \) assigns an integer \( n_I \) to each edge \( e_I \). The action of \( \hat{E}(g) \) on the flux network states reads

\[
\hat{E}(g) N_{\alpha, \vec{n}} = -\hbar \sum_I n_I \int_{e_I} g N_{\alpha, \vec{n}}. \tag{83}
\]
In this polymer-like representation, cylindrical functions are finite linear combinations of flux network states and span a dense subspace of $\mathcal{H}_0$. Denote $\text{Cyl}$ the set of cylindrical functions and $\text{Cyl}^*$ its algebraic dual. One then has a triplet $\text{Cyl} \subset \mathcal{H}_0 \subset \text{Cyl}^*$ in analogy with the case of loop quantum gravity.

The Schrödinger or Fock representation of the Maxwell field, on the other hand, depends on the Minkowski background metric. Here the Hilbert space is given by $\mathcal{H}_F = L^2(S', d\mu_F)$, where $S'$ is the appropriate space of tempered distributions on $\Sigma$ and $\mu_F$ is the Gaussian measure. The basic operators are connections $\hat{A}(f)$ smeared in 3 dimensions with suitable vector densities $f$ and smeared electric fields $\hat{E}(g)$. But $\hat{A}(e)$ fail to be well defined. To resolve this tension between the two representations, one proceeds as follows. Let $\vec{x}$ be the Cartesian coordinates of a point in $\Sigma = \mathbb{R}^3$. Introduce a test function by using the Euclidean background metric on $\mathbb{R}^3$,

$$f_r(\vec{x}) = \frac{1}{(2\pi)^{3/2}r^2} \exp(-|\vec{x}|^2/2r^2),$$

(84)

which approximates the Dirac delta function for small $r$. The Gaussian smeared form factor for an edge $e$ is defined as

$$X^a_{(e,r)}(\vec{x}) := \int_e ds f_r(\vec{\mathcal{E}}(s) - \vec{x})\hat{e}^a.$$  

(85)

Then one can define a smeared holonomy for $e$ by

$$A_{(r)}(e) := \exp[-i \int_{\mathbb{R}^3} X^a_{(e,r)}(\vec{x})A_a(\vec{x})],$$

(86)

where $A_a(\vec{x})$ is the $U(1)$ connection 1-form of the Maxwell field on $\Sigma$. Similarly one can define Gaussian smeared electric fields by

$$E_{(r)}(g) := \int_{\mathbb{R}^3} g_a(\vec{x}) \int_{\mathbb{R}^3} f_r(\vec{y} - \vec{x})E^a(\vec{y}).$$

(87)

In this way one obtains two Poisson brackets algebras. One is formed by smeared holonomies and electric fields with

$$\{A_{(r)}(e), A_{(r)}(e')\} = 0 = \{E(g), E(g')\}$$

(88)

$$\{A_{(r)}(e), E(g)\} = -i(\int_{\mathbb{R}^3} X^a_{(e,r)}g_a) A_{(r)}(e).$$

The other is formed by unsmeared holonomies and Gaussian smeared electric fields with

$$\{A(e), A(e')\} = 0 = \{E_{(r)}(g), E_{(r)}(g')\}$$

(89)

$$\{A(e), E_{(r)}(g)\} = -i(\int_{\mathbb{R}^3} X^a_{(e,r)}g_a) A(e).$$

Obviously, there is an isomorphism between them,

$$I_r : (A_{(r)}(e), E(g)) \mapsto (A(e), E_{(r)}(g)).$$

(90)
Using the isomorphism $I_r$, one can pass back and forth between the polymer and the Fock representations. Specifically, the image of the Fock vacuum can be shown to be the following element of $\text{Cyl}^\ast$:

$$\langle V | \rangle = \sum_{\alpha, \vec{n}} \exp[-\frac{\hbar}{2} \sum_{IJ} G_{IJ} n_I n_J] \langle N_{\alpha, \vec{n}} |,$$

(91)

where $(N_{\alpha, \vec{n}} | \in \text{Cyl}^\ast$ maps the flux network function $|N_{\alpha, \vec{n}} \rangle$ to one and every other flux network functions to zero. While the states $(N_{\alpha, \vec{n}} |$ do not have any knowledge of the underlying Minkowskian geometry, this information is coded in the matrix $G_{IJ}$ associated with the edges of the graph $\alpha$, given by

$$G_{IJ} = \int_{\epsilon_I} d\epsilon_i^a(t) \int_{\epsilon_J} d\epsilon_j^b(t') \int d^3 x \delta_{ab}(\vec{x}) [f_I(\vec{x} - \vec{\epsilon}_I(t))|\Delta|^\frac{1}{2} f(\vec{x}, \vec{\epsilon}_J(t'))],$$

(92)

where $\delta_{ab}$ is the flat Euclidean metric and $\Delta$ its Laplacian. Therefore, one can single out the Fock vacuum state directly in the polymer representation by invoking Poincaré invariance without any reference to the Fock space. Similarly, one can directly locate in $\text{Cyl}^\ast$ all coherent states as the eigenstates of the exponentiated annihilation operators.

Since $\text{Cyl}^\ast$ does not have an inner product, one uses the notion of shadow states to do semiclassical analysis in the polymer representation. From Eq. (91), the action of the Fock vacuum $(\langle V |$ on $N_{\alpha, \vec{n}}$ reads

$$\langle V | N_{\alpha, \vec{n}} \rangle = \int_{\mathcal{X}_\alpha} d\mu^0_\alpha \nabla_\alpha N_{\alpha, \vec{n}},$$

(93)

where the state $V_\alpha$ is in the Hilbert space $\mathcal{H}_\alpha$ for the graph $\alpha$ and given by

$$V_\alpha(A) = \sum_{\vec{n}} \exp[-\frac{\hbar}{2} \sum_{IJ} G_{IJ} n_I n_J] N_{\alpha, \vec{n}}(A).$$

(94)

Thus for any cylindrical functions $\varphi_\alpha$ associated with $\alpha$,

$$\langle V | \varphi_\alpha \rangle = \langle V_\alpha | \varphi_\alpha \rangle,$$

(95)

where the inner product in the right hand side is taken in $\mathcal{H}_\alpha$. Hence $V_\alpha(A)$ are referred to as “shadows” of $(\langle V |$ on the graphs $\alpha$. The set of all shadows captures the full information in $(\langle V |$. By analyzing shadows on sufficiently refined graphs, one can introduce criteria to test if a given element of $\text{Cyl}^\ast$ represents a semi-classical state [21]. It turns out that the state $(\langle V |$ does satisfy this criterion and hence can be regarded as semi-classical in the polymer representation.

The mathematical and conceptual tools gained from simple models like the Maxwell fields are currently being used to construct semiclassical states of quantum geometry. A candidate kinematical coherent state corresponding to the Minkowski spacetime has been proposed by Ashtekar and Lewandowki in the light of a “Laplacian operator” [20][21]. However, the detail structure of this coherent state is yet analyzed and there is no a priori guarantee that it is indeed a semiclassical state.
One may find comparisons of the two approaches from both sides [145][21]. It turns out that the Varadarajan’s Laplacian coherent state for polymer Maxwell field can also be derived from Thiemann’s complexifier method. However, one cannot find a complexifier to get the coherent state proposed by Ashtekar et al for loop quantum gravity. Both approaches have their own virtues and need further developments. The complexifier approach provides a clean construction mechanism and manageable calculation method, while the Laplacian operator approach is related closely with the well-known Fock vacuum state. We expect that a judicious combination of the two approaches may lead to significant progress in semiclassical analysis of loop quantum gravity.

7.3 Semiclassical Analysis and Quantum Dynamics

Although powerful tools have been developed to construct semiclassical states, the analysis of the classical limits of the Hamiltonian constraint operator and the corresponding constraints algebra has not been carried out. Although the semiclassical analysis of a master constraint operator is being carried out in the framework of Algebraic Quantum Gravity proposed by Giesel and Thiemann [80][81][82], one still needs diffeomorphism invariant coherent states in $\mathcal{H}_{\text{Diff}}$ (see Refs. [145] and [12] for recent progress in this aspect) to do semiclassical analysis of the master constraint operator in loop quantum gravity. Moreover, a crucial question of the semiclassical analysis is whether there are enough physical semiclassical states in certain unknown physical Hilbert space of loop quantum gravity, which may correspond to all classical solutions of the Einstein equation. This is the final theoretical criterion for any candidate theory of quantum gravity with general relativity as its classical limit. The physical semiclassical states are also relevant, if one wishes to use the full theory rather than symmetric models to analyze cosmology and black holes. In the matter coupled to gravity content, one would like to check whether the coupled quantum system approaches quantum field theory in curved spacetime in suitable semiclassical limit. This issue is being studied at the kinematical level [127][128].

In the light of the canonical quantization of loop quantum gravity, the so-called spin foams are devised as histories traced out by ”time evolution” of spin networks, which provide a path-integral approach to quantum dynamics [35]. One expects that the path integral can be used to compute ”transition amplitudes” and extract physical states, which may shed new light on the quantum Hamiltonian constraint and on the physical inner product. In the successful Barrett-Crane model and its various modifications [38][39], one regards classical general relativity as a topological field theory (the so-called BF theory), supplemented with an algebraic constraint. An interesting discovery in this approach is that a certain modified version of the Barrett-Crane model is equivalent to a manageable group field theory [109][111][60]. It then turns out that the sum over geometries for a fixed discrete topology is finite. For a detail exploration of spin foam models, we refer to the recent review article [110] and references therein.

Although many developments in spin foam approach are very interesting from a mathematical physics perspective, their significance to quantum gravity is still less clear [21]. An obvious weakness in most of these works is that the discrete topology is fixed, whence the the issue of summing over all topologies remains largely unexplored.
However, it is expected that a judicious comparison of methods from the canonical treatment of the Hamiltonian constraint and spin foam models may promote the research in both approaches. In fact, there are considerable attempts to calculate particle scattering amplitude in non-perturbative quantum gravity by combining the methods from the two approaches [103][44]. There are also other approaches to deal with the quantum dynamics such as, the Vassiliev knot invariants approach [40] and the “consistent discretization” approach [75][76]. Here we will not introduce their concrete ideas. One may find the detail exploration of the former in Refs. [41] and [42], and a recent summary for the latter in Ref.[77].

In summary, the full treatments of the semiclassical analysis and quantum dynamics are entangled with each other and expected to be settled together. These are the core open problems in loop quantum gravity, which are now under investigations.

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