Towards Collaborative Conceptual Exploration

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Abstract In domains with high knowledge distribution a natural objective is to create principle foundations for collaborative interactive learning environments. We present a first mathematical characterization of a collaborative learning group, a consortium, based on closure systems of attribute sets and the well-known attribute exploration algorithm from formal concept analysis. To this end, we introduce (weak) local experts for subdomains of a given knowledge domain. These entities are able to refute and potentially accept a given (implicational) query for some closure system that is a restriction of the whole domain. On this we build up a consortial expert and show first insights about the ability of such an expert to answer queries. Furthermore, we depict techniques on how to cope with falsely accepted implications and on combining counterexamples. Using notions from combinatorial design theory we further expand those insights as far as providing first results on the decidability problem if a given consortium is able to explore some target domain. Applications in conceptual knowledge acquisition as well as in collaborative interactive ontology learning are at hand.

Keywords: Formal Concept Analysis, Implications, Attribute Exploration, Collaborative Knowledge Acquisition, Collaborative Interactive Learning

1 Introduction

Collaborative knowledge bases, like DBpedia\textsuperscript{4} and Wikidata\textsuperscript{5} [16], raise the need for (interactive) collaborative tools in order to add, enhance or extract conceptual knowledge to and from those. As well, a society with highly specialized experts needs some method to make use of the collaborative knowledge of those.

\textsuperscript{4} http://wiki.dbpedia.org
\textsuperscript{5} http://www.wikidata.org

The authors are given in alphabetical order. No priority in authorship is implied.
One particular task in knowledge acquisition is to obtain concepts in a given domain which is composed of two disjoint sets, called objects and attributes, along with some relation between them. A well-known approach for this is the (classical) attribute exploration algorithm from formal concept analysis (FCA) [3, 5]. This algorithm is able to explore any domain of the kind mentioned above by consulting some domain expert. The result is a formal concept lattice, i.e., an order-theoretic lattice which contains all formal concepts discovered in the domain. It is crucial that the algorithm has access to a domain expert for the whole domain, to whom it uses a minimal number of queries (which may still be exponential in the size of input, i.e., the size of the relation between objects and attributes).

However, the availability of a domain expert is often not given in practice. Moreover, even if it exists, such an expert might not be able or willing to answer the possibly exponential number of queries. The purpose of the present work is to provide a solution in this case, at least for some of such tasks, given a certain collaborative scenario. More precisely, suppose that we have a covering $M = \bigcup_{i \in I} N_i$ of the attribute set $M$ together with a set of local experts $p_i$ on $N_i$, then we propose a consortial expert for the domain. As is easy to see, such an expert is in general less capable of handling queries than a domain expert. Nonetheless, depending on the form of $M = \{ N_i \mid i \in I \}$ our approach may still be able to answer a significant amount of non-trivial queries.

In this work we provide a first complete characterization of (weak) local experts in order to define what a consortium is, what can be explored and what next steps should be focused on. As to our knowledge, this has not been considered before in the realm of conceptual knowledge.

Here is an outline of the remainder of this paper. After giving an account of related work in Section 2, we recall basic notions from formal concept analysis and the attribute exploration algorithm in Section 3. We define the setting of a consortium in Section 4, using a small simplification in notation to mere closure systems on $M$. Subsequently we discuss our approach in Section 5, give examples in Section 5.3, following by possible extensions in Section 5.4 and a conclusion in Section 6.

2 Related work

There are several related fields that address the problem of (interactive) collaborative learning in their respective scientific languages. Based on modal logic there are various new approaches for similar problems as considered here, using epistemic and intuitionistic types. For example, Jäger and Marti [7] present a multi-agent system for intuitionistic distributed knowledge (with truth). Another example is resolving the distributed knowledge of a group as done by Ågotnes and Wáng [1]. In this work the process of distributed knowledge, i.e., knowledge distributed throughout a group, is resolved to common knowledge, i.e., knowledge that is known to all members of the group, a fact which is also known to the members of the group.
Investigations considering a more virtual approach for collaborative knowledge acquisition are, for example, presented by Stange, Nürnberg and Heyn [13], in which a collaborative graphical editor used by experts negotiates the outcome. Our approach is yet based on (basic) formal attribute exploration [5]. Of course, there are various advanced versions like adding background knowledge [3], relational exploration [12] or conceptual exploration [14]. There are also extensions of the basic exploration to treat incomplete knowledge [2, 6, 10].

In FCA one of the first considerations on cooperatively building knowledge bases is work of Martin and Eklund [9]. Previous work on collaborative interactive concept lattice modification in order to extract knowledge can be found in [15]. These concept lattice modifications are based on removing or adding attributes/objects/concepts using expert knowledge, and those operations may be used in a later version of collaborative conceptual exploration. The most recent work specifically targeting collaborative exploration is [11], raising the task of making exploration collaborative.

3 Attribute exploration and FCA basics

In this paper we utilize notions from formal concept analysis (FCA) as specified in [5]. In short, our basic data structure is a formal context \( \mathcal{K} := (G, M, I) \) with \( G \) some object set, \( M \) some attribute set, and \( I \subseteq G \times M \) an incidence relation between them. By \( \cdot' \) we denote two mappings \( \cdot': \mathcal{P}(G) \rightarrow \mathcal{P}(M) \) and \( \cdot': \mathcal{P}(M) \rightarrow \mathcal{P}(G) \), given by \( A \mapsto A' = \{ m \in M \mid \forall g \in A: (g, m) \in I \} \) for \( A \subseteq G \) and \( B \mapsto B' = \{ g \in G \mid \forall m \in B: (g, m) \in I \} \) for \( B \subseteq M \).

The set \( \mathfrak{B}(\mathcal{K}) \) is the set of all formal concepts, i.e., the set of all pairs \( (A, B) \) with \( A \subseteq G, B \subseteq M \) such that \( A' = B \) and \( B' = A \). In a formal concept \( (A, B) \) the set \( A \) is called (concept-)extent and the set \( B \) is called (concept-)intent. The set of all formal concepts can be ordered by \( (A, B) \leq (C, D) :\Rightarrow A \subseteq C \). The ordered set \( \mathfrak{B}(\mathcal{K}) \), often denoted by \( \mathfrak{B}(\mathcal{K}) \), is called the concept lattice of \( \mathcal{K} \). Furthermore, the composition \( \cdot'' \) constitutes closure operators on \( G \) and on \( M \), respectively, i.e., mappings \( \cdot'': \mathcal{P}(G) \rightarrow \mathcal{P}(G) \) and \( \cdot'': \mathcal{P}(M) \rightarrow \mathcal{P}(M) \) which are extensive, monotone and idempotent. Therefore, every formal context gives rise, through the associated closure operator, to two closure systems, one on \( G \) and one on \( M \), called the closure system of intents and extents, respectively. Each of those closure systems can be considered as an ordered set using the inclusion operator \( \subseteq \), which in turn leads to a complete lattice. Using the basic theorem of FCA [5] one may construct for any closure system \( \mathcal{X} \) on \( M \) a formal context \( \mathcal{K} \) such that the closure system \( \mathcal{X} \) is the set of concept-intents from \( \mathcal{K} \).

In the following exposition we will concentrate on the attribute set \( M \) of a formal context. We do this for brevity and clarity reasons, only. Namely, we avoid carrying all the necessary notation through the defining parts of a collaborating consortium. However, we do keep in mind that \( M \) is still a part of a formal context \( (G, M, I) \), and we rest on this classical representation, in particular, when quoting well-known algorithms from FCA.
3.1 Setting

Let $M$ be some finite (attribute) set. We fix a closure system $\mathcal{X} \subseteq \mathcal{P}(M)$, called the (target) domain or target closure system, which is the domain knowledge to be acquired. The set of all closure systems on a set $M$ constitutes a closure system itself. In turn, this means we can also find a concept lattice for this set. We depict this lattice in general in Figure 1 (right). The size of this set is enormous and only known up to $|M| = 7$. In the next subsection we recall the classical algorithm to compute the target domain for a given set of attributes $M$ using a domain expert on $M$. This algorithm employs rules between sets of attributes which we now recall. An implication is a pair $(A, B) \in \mathcal{P}(M) \times \mathcal{P}(M)$, which can also be denoted by $A \rightarrow B$. We write $\text{Imp}(M)$ for the set $\mathcal{P}(M) \times \mathcal{P}(M)$ of all implications on $M$. The implication $(A, B) \in \text{Imp}(M)$ is valid in $\mathcal{X}$ if $\forall X \in \mathcal{X}: A \subseteq X \Rightarrow B \subseteq X$.

3.2 Attribute exploration

Attribute exploration is an instance of an elegant strategy to explore the knowledge of an (unknown) domain $(G, M, I)$ using queries to a domain expert for $M$. These queries consist of validity questions concerning implications in $M$. The expert in this setting can either accept an implication, i.e., confirming that this implication is valid in the domain, or has to provide a counterexample. The following description of this algorithm is gathered from [4], a compendium on conceptual exploration methods.

Using a signature, which specifies the logical language to be used during exploration, there is a set of possible implications $\mathcal{F}$, each either valid or not in the domain. The algorithm itself uses an exploration knowledge base $(\mathcal{L}, \mathcal{E})$, with $\mathcal{L}$ being the set of the already accepted implications and $\mathcal{E}$ the set of already collected counterexamples. These can be considered in our setting as named subsets of $M$, where the name is the object name for this set. The algorithm now makes use of a query engine which draws an implication $f$ from $\mathcal{F}$ that cannot be deduced from $\mathcal{L}$ and that cannot be refuted by already provided counterexamples in $\mathcal{E}$. This implication is presented to the domain expert, who either can accept this implication, which adds $f$ to $\mathcal{L}$, or refute $f$ by a counterexample $E \subseteq M$, which adds $E$ to $\mathcal{E}$.

The crucial part here is that the domain expert has to be an expert for the whole domain, i.e., an expert for the whole attribute set $M$ and any object possible. Otherwise, the expert would not be able to provide complete counterexamples, i.e., the provided counterexamples are possibly missing attributes from $M$, or even “understand” the query. To deal with this impractical limitation algorithms for attribute exploration with partial (counter-)examples were introduced. We refer the reader to [4, Algorithm 21]. This algorithm is able to accept partial counterexamples from a domain expert.

The return value of the attribute exploration algorithm is the canonical base of all valid implications from the domain. There is no smaller set of implications...
than the canonical base for some closure operator on an (attribute) set \( M \), which is sound and complete.

In the subsequent section we provide a characterization of a consortial expert which could be utilized as such a domain expert providing incomplete counterexamples. In addition, we show a strategy for how to deal with counterexamples de-validating already accepted implications, which will be a possible outcome when consulting a consortium.

4 Consortium

In the following we continue to utilize mere closure systems on \( M \) for some domain \((G, M, I)\) and also call such a closure system itself the (target) domain \( X \), to be explored. This ambiguity is for brevity, only. Furthermore, we consider \( M \) always to be finite.

Definition 4.1 (Expert). An expert for \( X \) is a mapping \( p: \text{Imp}(M) \to X \cup \{\top\} \) such that for every \( f = (A, B) \in \text{Imp}(M) \) the following is true:

1. \( p(f) = \top \Rightarrow f \) is valid in \( X \),
2. \( p(f) = X \in X \Rightarrow A \subseteq X \land B \not\subseteq X \).

We refer to the set \( M \) also as the expert domain.

From this definition we note, for an implication \( f \in \text{Imp}(M) \), that \( p(f) \neq \top \) implies that \( f \) is not valid in \( X \), since \( p(f) = X \Rightarrow A \subseteq X \land B \not\subseteq X \). In analogy to this expert we now introduce an expert on a subset of \( M \).

Definition 4.2 (Local expert). Let \( N \subseteq M \). A local expert for \( X \) on \( N \) is a mapping \( p_N: \text{Imp}(N) \to x_N \cup \{\top\} \) with \( x_N := \{X \cap N \mid X \in X\} \) such that for every \( f = (A, B) \in \text{Imp}(N) \) there holds:

1. \( p_N(f) = \top \Rightarrow f \) is valid in \( X \),
2. \( p_N(f) = X \in x_N \Rightarrow A \subseteq X \land B \not\subseteq X \).

Observe that the set \( x_N \) is also a closure system. Despite that, the elements of \( x_N \) are not necessarily elements of \( X \). But, since \( N \subseteq M \) there is for every \( X \in x_N \) some \( \hat{X} \in X \) such that \( \hat{X} \cap N = X \).

Remark 4.3. Every expert for \( X \) provides in the obvious way a local expert for \( X \) on \( N \), for each \( N \subseteq M \). Furthermore, every local expert for \( X \) on \( N \) is a local expert for \( X \) on \( O \) for each \( O \subseteq N \).

Lemma 4.4 (Refutation by local expert). Let \( X \) be some domain with attribute set \( M \) and let \( p_N \) be a local expert for \( X \) on \( N \subseteq M \). Then for every \( f \in \text{Imp}(N) \) there holds \( p_N(f) \neq \top \Rightarrow f \) is not valid in \( X \).

Proof. If \( p_N(f) \neq \top \), then \( \exists X \in x_N : p_N(f) = X \land A \subseteq X \land B \not\subseteq X \). By definition \( \exists X \in X : X \cap N = X \). Therefore, \( A \subseteq X = X \cap N \subseteq X \) and \( B \not\subseteq X = X \cap N \), thus \( B \not\subseteq X \) as \( B \subseteq N \). \( \square \)
Example 4.5. Suppose we have a three-element attribute set \( M = \{ \text{ro}, \text{fl}, \text{ed} \} \), for the attributes “round”, “flexible” and “edible”. Regarding the objects “ball”, “sphere” and “donut” (food) we consider the following formal context.

|       | round | flexible | edible |
|-------|-------|----------|--------|
| ball  | ×     | ×        |        |
| sphere| ×     |          |        |
| donut | ×     | ×        |        |

From this we obtain as our target domain

\[ X = \{ M, \{ \text{ro}, \text{fl} \}, \{ \text{fl}, \text{ed} \}, \{ \text{fl} \}, \{ \text{ro} \}, \emptyset \}, \]

with the canonical base \( B = \{ \text{ed} \rightarrow \text{fl} \} \). Using the shortcuts \( \text{ed}^C = \{ \text{ro}, \text{fl} \} \) and \( \text{ro}^C = \{ \text{fl}, \text{ed} \} \), the concept lattice may be depicted as:

![Concept Lattice Diagram]

Now suppose that \( I = \{ a, b, c \} \), and for each \( i \in I \) we have a local expert \( p_i \) for \( X \) on \( N_i \), where \( N_a = \{ \text{ro}, \text{fl} \} \), \( N_b = \{ \text{fl}, \text{ed} \} \) and \( N_c = \{ \text{ro}, \text{ed} \} \). We name the local experts “Alice”, “Bob” and “Carol”.

Alice may be consulted for the implications \( \text{ro} \rightarrow \text{fl} \) and \( \text{fl} \rightarrow \text{ro} \), both of which she refutes. For example, to the query \( \text{ro} \rightarrow \text{fl} \) she responds (possibly having the sphere in mind) with an attribute set \( X \) containing \( \text{ro} \) but not \( \text{fl} \), i.e., \( X = \{ \text{ro} \} \), where \( \{ \text{ro} \} = X \cap \{ \text{ro}, \text{fl} \} \) and \( X \in X \). Similarly, she refutes the query \( \text{fl} \rightarrow \text{ro} \) (having the donut in mind). Moreover, local expert Bob can be consulted with the implications \( \text{fl} \rightarrow \text{ed} \), which he refutes (ball), and \( \text{ed} \rightarrow \text{fl} \), which he correctly accepts. Finally, Carol refutes both possible queries \( \text{ed} \rightarrow \text{ro} \) (donut) and \( \text{ro} \rightarrow \text{ed} \), in which case her counterexample could stem from different objects (ball or sphere).

For some applications a local expert may be too strong in terms of having either to accept an implication (vicariously for \( X \)) or refute an implication. This would require that the local expert is aware of all possible counterexamples, which is impractical.

Definition 4.6 (Local pre-expert). A local pre-expert for \( X \) on \( N \subseteq M \) is a mapping \( p_N^*: \text{Imp}(N) \rightarrow X_N \cup \{ \top \} \) such that \( \forall f = (A, B) \in \text{Imp}(N) : p_N^*(f) = X \in X_N \Rightarrow A \subseteq X \land B \not\subseteq X \).

It is obvious that a local expert is also a local pre-expert. Using this “weaker” mapping we introduce the consortial (pre-)expert, after stating what a consortial domain is and some technical result about the intersection of closed sets.
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The closure systems of the consortium

All closure systems on $N_i \subseteq M$

All closure systems on $M$

$\{N_i\}$

$\{M\}$

Figure 1. Closure system of all closure systems on $M$ (right) and on $N_i \subseteq M$ (left). The closure systems for the set of accepted implications are denoted by $X^I$ (in $M$) and by $X^I_i$ (in $N_i$), and likewise for the set of counterexamples by $X^{CE}$ and by $X^{CE}_i$.

Definition 4.7 (Consortial domain). Let $M$ be some attribute set and $X \subseteq \mathcal{P}(M)$ be the target domain. Then a family $M := \{N_i \mid i \in I\} \subseteq \mathcal{P}(M)$ for some index set $I$ is called consortial domain on $M$ if $\bigcup_{i \in I} N_i = M$.

We call $M \subseteq \mathcal{P}(M)$ a proper consortial domain if $M \not\in M$.

Lemma 4.8 (Consortial domain closed under intersection). Let $M$ be some consortial domain on $M$. If $\mathcal{M}$ is closed under intersection, then so is the set $\bigcup_{M \in \mathcal{M}} X_M$.

In the following proof as well as in the rest of this work we may often use the abbreviation $X_i := X_{N_i}$ for some $N_i \in M$ with $M$ a consortial domain and using the notation introduced in Definition 4.2.

Proof. Whenever $X \cap N_i \in X_i$ and $Y \cap N_j \in X_j$, where $X,Y \in X$, we get

$$(X \cap N_i) \cap (Y \cap N_j) = (X \cap Y) \cap (N_i \cap N_j) \in X_{N_i \cap N_j},$$

where $X \cap Y \in X$ and $N_i \cap N_j \in M$. \qed

Corollary 4.9. If $M^* := \mathcal{M} \cup \{M\}$ is a closure system, then so is $\bigcup_{M \in \mathcal{M}} X_M$.

By definition a proper consortial domain cannot be a closure system and even a consortial domain will almost never have this property, either. However, for any consortial domain $\mathcal{M}$ we can easily construct an intersection closed set using the downset operator $\downarrow \mathcal{M}$. Therefore, whenever we have a consortial domain we may consider $\downarrow \mathcal{M}$, when necessary. Hence, we always can construct a closure system $\mathcal{M}^*$ for any consortial domain $\mathcal{M}$.

In the following we may use $M^*$ to speak about $\downarrow \mathcal{M} \cup \{M\}$. 
Remark 4.10 (Closure operator \( \mathcal{M}^* \)). Since for a given consortial domain \( \mathcal{M} \) on \( M \) the set \( \mathcal{M}^* \) is a closure system, we obtain a closure operator \( \phi : \mathcal{P}(M) \rightarrow \mathcal{P}(M) \). We may address \( \phi \) simply by \( \mathcal{M}^*(\cdot) \) and the image of \( N \subseteq M \) by \( \mathcal{M}^*(N) \).

Using the just discovered closure operator we may define which queries can be answered in a consortial domain.

Definition 4.11 (Well-formed query). Let \( \mathcal{M} \) be some proper consortial domain on \( M \) and let \( f = (A, \{b\}) \in \text{Imp}(M) \). Then \( f \) is called well-formed for \( \mathcal{M} \) if \( \mathcal{M}^*(A \cup \{b\}) \neq M \), i.e., if there exists \( N_i \in \mathcal{M} \) such that \( A \cup \{b\} \subseteq N_i \).

Well-formed queries are in fact the only queries for which in a proper consortial domain the decision problem if an implication is valid can be decided. It is easy to see that for any given \( f = (A, \{b\}) \in \text{Imp}(M) \), if \( \mathcal{M}^*(A \cup \{b\}) = M \), then there is no expert domain left, therefore either the conclusion attribute or one of the premises is missing in all \( N \in \mathcal{M} \), which leads to undecidability.

Putting all those ideas together we are finally able to define our main goal.

Definition 4.12 (Consortium for \( \mathcal{X} \)). For an attribute set \( M \) and a target domain \( \mathcal{X} \) on \( M \) let \( \mathcal{M} = \{N_i \mid i \in I\} \) be a consortial domain on \( M \). A consortium for \( \mathcal{X} \) is a family \( \mathcal{C} := \{p_i\}_{i \in I} \) of local pre-experts \( p_i \) for \( \mathcal{X} \) on \( N_i \).

All comments made before about \( \mathcal{M} \) being intersection-closed are compatible with the definition of a consortium. Using Remark 4.3 we can always obtain a local pre-expert for any \( M \in \downarrow \mathcal{M} \). A consortium is able to decide the validity of any well-formed query, by definition. Therefore, a consortium gives rise to a consortial expert. As long as all queries are well-formed, a consortium can be used in-place of a domain expert.

Example 4.13. We continue with Example 4.5. On the consortial domain \( \mathcal{M} := \{N_a, N_b, N_c\} \) the three local experts form a consortium \( \mathcal{C} := \{p_a, p_b, p_c\} \) for \( \mathcal{X} \). Note that the consortium cannot decide, e.g., on the implication \( \{\text{fl}, \text{ed}\} \rightarrow \text{ro} \), since this query is not well-formed for \( \mathcal{M} \). However, if experts are able to combine their counterexamples they may refute the query (cf. Section 5.4).

Definition 4.14 (Strong consortial expert). Let \( \mathcal{C} = \{p_i\}_{i \in I} \) be a consortium for \( \mathcal{X} \) on \( M \). A strong consortial expert is a mapping \( p_C : \bigcup_{i \in I} \text{Imp}(N_i) \rightarrow \bigcup_{i \in I} \mathcal{X}_i \cup \{\top\} \) such that for every \( f = (A, B) \in \bigcup_{i \in I} \text{Imp}(N_i) \) there holds:

1. \( \exists p_i \in \mathcal{C}, p_i(f) \neq \top \Rightarrow p_C(f) \neq \top \),
2. \( p_C(f) = X \in \bigcup_{i \in I} \mathcal{X}_i \Rightarrow A \subseteq X \land B \subseteq X \).

The strong consortial expert has to respect a possible counterexample entailed in the consortium in order to be consistent with Definition 4.6, since every counterexample by a local (pre-)expert is a restriction of an element of the target closure system. In the case of having local experts in the consortium this behavior may be in conflict with Definition 4.2, since we demand that accepting an implication by a local expert implies that the implication is true in the target domain. For example, if a local expert accepts an implication and another local
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(pre-)expert refutes it, this conflict is not resolvable. Therefore, whenever a consortium does contain local experts it is mandatory that they meet a consistency property. We will introduce consistency in Section 5.2. When using a consortium of proper local pre-experts there is no implication from accepting an implication. An accepted implication may be false in the target domain.

To meet our goal of reducing the number of inquiries to the individual expert in a consortium, the proposed consortial expert from Definition 4.14 is insufficient. We need to diminish the strong requirement from checking all experts for having a counterexample. This leads to the following definition.

Definition 4.15 (Consortial expert). Let \( C = \{ p_i \}_{i \in I} \) be a consortium for \( X \) on \( M \). A consortial expert is a mapping \( p_C : \bigcup_{i \in I} \text{Imp}(N_i) \to \bigcup_{i \in I} X_i \cup \{ \top \} \) such that for every \( f = (A, B) \in \bigcup_{i \in I} \text{Imp}(N_i) \) there holds:

1. \( \exists p_i \in S, p_i(f) \neq \top \Rightarrow p_C(f) \neq \top \),
2. \( p_C(f) = X \in \bigcup_{i \in I} X_i \Rightarrow A \subseteq X \land B \not\subseteq X \).

The set \( S \subseteq C \) is a per inquiry chosen subset of local experts such that \( f \in \text{Imp}(N_i) \) for every \( p_i \in S \).

We left the just addressed expert subset vague by intention. In practice, choosing this should be possible in various ways. There is no further restriction then of choosing “qualified” experts, i.e., how the consortial expert is choosing \( S \). One obvious choice would be to consult all local (pre-)experts at once. A more clever strategy would be to consult experts covering the attributes in question having the largest attribute size to cover in general. One may also employ a cost function, which could lead to asking only less expensive experts. While using a consortial expert for exploration, an already accepted implication may be refuted later on in the exploration process. Whenever an inquiry leads to an counterexample which is also an counterexample for an already accepted implication, the set of valid implications needs to be corrected.

So far we provided neither constraints nor constructions about the decision making of a consortium, i.e., the collaboration. The most simple case, where \( M \) is a partition of \( M \) and all queries are concerned with an element of \( M \), can easily be treated: For every query the expert for the according element of \( M \) either refutes or maybe accepts. Since this case seems artificial we will investigate different approaches of “real” collaboration in the following section.

5 Exploration with consortial experts

In general, for exploring a domain using attribute exploration with partial examples one may use instead of the domain expert some (strong) consortial expert. However, there are three possible problems to deal with. First, a query may concern some implication \( f \) that is not well-formed for the consortium \( C \) that is used by the consortial expert. Second, if a consortium containing local pre-experts does accept an implication this does not necessarily imply the implication in question to be valid in the domain. Obviously, this also depends heavily on
how a consortial expert utilizes a consortium. We deal with related problems in the following subsections. Third, while choosing a subset of \( C \) the consortial expert may have missed a local pre-expert which would have been aware of a counterexample, in contrast to a strong consortial expert.

The first problem cannot be resolved by the consortial expert. When no local (pre-)expert can be consulted for some implication the only choice is to accept \( f \). However, a more suitable response would be a third type of replying like \( NULL \). Then, the exploration algorithm could cope with this problem by deferring to other questions. The attribute exploration algorithm with partial examples from [4] but could easily be adapted for this. In turn, the algorithm would only be able to return an interval of closure systems, like in Figure 1.

For the second problem one needs to repair the set of accepted implications in case a counterexample turns up later in the process. We show a method of doing so in Section 5.4. Of course, there is still the possibility that an accepted not valid implication will never be discovered as a consequence of an incapable consortium. This leads the exploration algorithm to return not the target domain but another closure system. How “close” this closure system is to the target domain, in terms of some Jaccard-like measure, is to be investigated in some future work.

The third and last problem can always be dealt with by employing a strong consortial expert. A less exhaustive method could be to incorporate statistical methods for quantifying the number of necessary experts to consult in order to obtain a low margin of error.

5.1 Correcting falsely accepted implications

A major issue while using a consortial expert for exploration is the possibility of wrongly accepting an implication. This can be dealt with on side of the exploration algorithm. While receiving a new counterexample \( O \subseteq M \) from the consortial expert the exploration algorithm has also to check if \( O \) is a counterexample to an already accepted implication in \( L \). When such an implication \( f = (A, B) \) is found, we would need to restrict the conclusion of \( f \) to a yet not disproved subset and also add implications with stronger premises that were omitted because \( f \) was (wrongly) accepted. In particular, we would replace \( f \) in \( L \) by \( A \rightarrow B \cap C \) and also add implications \( A \cup \{m\} \rightarrow B \) for \( m \in M \setminus (A \cup C) \) to \( L \).

This approach may drastically increase the size of the set of already accepted implications. Unlike the classical exploration algorithm, this modified version would return a very large set of implications instead of the canonical base. One may cope with that by utilizing [4, Algorithm 19] after every event of replacing an implication in \( L \). This algorithm takes a set of implications and returns the canonical base. After this a next query can be computed based on the so far collected set of implications and the already collected counterexamples.
5.2 Consistency

So far we characterized what local pre-experts and consortia are, by their ability to make decisions about queries. In this section we provide ideas for a consistent consortium. We start with resolving a possible conflict for consortial experts.

Definition 5.1 (Consistent experts). Let $C = \{ p_i \}_{i \in I}$ be a consortium for $\mathcal{X}$ on $M$ and let $\mathcal{C} \subseteq C$ be the set of local experts in $C$. We say that $C$ has consistent experts if for $i, j \in I$ with $p_i, p_j \in \mathcal{C}$ and for all $f \in \text{Imp}(N_i \cap N_j)$ it holds that $p_i(f) = \top \iff p_j(f) = \top$.

We call $C$ with consistent experts a consistent experts consortium.

This idea from consistent experts does still allow for different local experts to be able to refute an implication with different counterexamples. But whenever one local expert would accept an implication, any other local expert needs to do so as well. Different local (pre-)experts may have access to disjoint sets of counterexamples, by design. Furthermore, local pre-experts may not have the knowledge for all possible counterexamples in their restriction of the target domain. Therefore, accepting an implication by a local pre-expert has no implication itself. Hence, even in a consistent experts consortium it is still possible that some local experts may provide a counterexample while others do not. A stronger notion of consistency would be to forbid that.

Definition 5.2 (Consistent consortium). Let $C = \{ p_i \}_{i \in I}$ be a consortium for $\mathcal{X}$ on $M$. The consortium $C$ is consistent if for all $i, j \in I$ and for all $f \in \text{Imp}(N_i \cap N_j)$ we have that $p_i(f) = \top \iff p_j(f) = \top$.

Again, in consequence, all local pre-experts are either able to produce some, but not necessarily the same, counterexample for some implication or all do accept. We look into the possibility of combining counterexamples in Section 5.4.

5.3 Abilities and limitations of a consortium

In this section we exhibit the theoretical abilities and limitations of a consortium for determining the whole target domain of available knowledge. After clarifying some general notions and facts, we state a reconstructability result for consortia based on combinatorial designs.

Let us, as before, fix a finite (attribute) set $M$. As is well-known, any set $\mathcal{F} \subseteq \text{Imp}(M)$ of implications constitutes a closure system

$$\mathcal{X}_\mathcal{F} := \{ X \in \mathcal{P}(M) \mid \forall f = (A, B) \in \mathcal{F} : A \subseteq X \Rightarrow B \subseteq X \}.$$ 

Conversely, any closure system $\mathcal{X}$ defines its set $\mathcal{F}_\mathcal{X} \subseteq \text{Imp}(M)$ of valid implications, and we have $\mathcal{X}_{\mathcal{F}_\mathcal{X}} = \mathcal{X}$ and $\mathcal{F}_{\mathcal{X}_\mathcal{F}} = \mathcal{F}$. Now suppose that $S$ is a class of closure systems $\mathcal{X} \subseteq \mathcal{P}(M)$ on $M$ which contains the target domain. This set $S$ describes some information on the target domain we may have in advance. Suppose that $\mathcal{M} = \{ N_i \mid i \in I \}$ is a consortial domain and we have, for some
$\mathcal{X} \subseteq \mathcal{S}$, a set of local experts $p_i: \text{Imp}(N_i) \to \mathcal{X}_i \cup \{\top\}$ on $N_i \in \mathcal{M}$, so that in particular, $p_i(f) = \top$ if and only if $f$ is valid in $\mathcal{X}$. Then we consider the set

$$\mathcal{F}_M := \{ f \in \bigcup_{i \in I} \text{Imp}(N_i) \mid f \text{ is valid} \} \subseteq \mathcal{F}_X,$$

i.e., the set of all well-formed valid implications, and let $\mathcal{X}_M := \mathcal{X}_{\mathcal{F}_M}$, which is the closure system reconstructible by the consortium. Clearly, $\mathcal{X}_M \supseteq \mathcal{X}$, and from the preceding discussion we easily deduce the following result.

**Proposition 5.3** (Ability of a consortium). The consortial domain $\mathcal{M}$, together with local experts $p_i: \text{Imp}(N_i) \to \mathcal{X}_i \cup \{\top\}$ for $N_i \in \mathcal{M}$, is able to reconstruct the target domain $\mathcal{X}$ within a class $\mathcal{S}$ of closure systems on $M$ if and only if $\mathcal{Y}_M = \mathcal{X}_M$ implies $\mathcal{Y} = \mathcal{X}$, for every $\mathcal{Y} \in \mathcal{S}$.

**Example 5.4.** We illustrate these notions with two simple extreme cases.

1. Suppose that $\mathcal{X} = \{M\}$, then every implication is valid, i.e., $\mathcal{F}_X = \text{Imp}(M)$. Since every consortial domain $\mathcal{M} = \{N_i \mid i \in I\}$ has the covering property $\bigcup_{i \in I} N_i = M$, it follows that $\mathcal{X}_M = \mathcal{X}$. Hence, if $\mathcal{Y}_M = \mathcal{X}_M$, then $\mathcal{Y} = \{M\}$, i.e., the consortium is always able to reconstruct $\mathcal{X}$ in the class of all closure systems on $M$.

2. Consider the case $\mathcal{X} = \mathcal{P}(M)$ and suppose that $\mathcal{M} = \{N_i \mid i \in I\}$ is a proper consortial domain. Then for any $m \in M$ we have $M \setminus \{m\} \to \{m\} \notin \bigcup_{i \in I} \text{Imp}(N_i)$, whence $\mathcal{X}_M = \mathcal{Y}_M$ for $\mathcal{Y} = \mathcal{P}(M) \setminus \{M \setminus \{m\}\} \neq \mathcal{X}$. Thus no proper consortium is capable of reconstructing the target domain.

Let us define for a set of implications $\mathcal{F} \subseteq \text{Imp}(M)$ the premise complexity to be $c(\mathcal{F}) := \max(|A| \mid f = (A, B) \in \mathcal{F})$ if $\mathcal{F} \neq \emptyset$ and $c(\emptyset) := -1$. Also, we associate to a closure system $\mathcal{X} \subseteq \mathcal{P}(M)$ on $M$ its premise complexity by $c(\mathcal{X}) := \min\{c(\mathcal{F}) \mid \mathcal{X}_\mathcal{F} = \mathcal{X}\}$, which equals the premise complexity of its canonical base.

**Example 5.5.** For the extreme closure systems we have $c(\mathcal{P}(M)) = -1$ and $c(\{M\}) = 0$. Considering the closure system $\mathcal{X}_k := \{X \in \mathcal{P}(M) \mid |X| \leq k\} \cup \{\emptyset\}$ we see that $c(\mathcal{X}_k) = k + 1$.

Denote by $\mathcal{S}_k$ the class of all closure systems up to premise complexity $k$.

**Theorem 5.6** (Reconstructability in bounded premise complexity). A consortium of local experts on the consortial domain $\mathcal{M}$ is able to reconstruct a target domain within the class $\mathcal{S}_k$ if and only if every subset $O \subseteq M$ of size $k+1$ is contained in some $N \in \mathcal{M}$.

**Proof.** First suppose that each subset $O \subseteq M$ of size $k+1$ is contained in some $N \in \mathcal{M}$. We claim that $\mathcal{X}_M = \mathcal{X}$ for every closure system $\mathcal{X} \in \mathcal{S}_k$, whence every target domain is reconstructible within $\mathcal{S}_k$. Let $\mathcal{X} \in \mathcal{S}_k$, then there is a set $\mathcal{F}$ of implications with premise complexity $c(\mathcal{F}) \leq k$ such that $\mathcal{X} = \mathcal{X}_\mathcal{F}$. We may assume that each implication $f \in \mathcal{F}$ is of the form $f = (A, \{b\})$. By assumption there holds $\mathcal{F} \subseteq \bigcup_{N \in \mathcal{M}} \text{Imp}(N)$, so that $\mathcal{F} \subseteq \mathcal{F}_X \cap \bigcup_{N \in \mathcal{M}} \text{Imp}(N) = \mathcal{F}_M \subseteq \mathcal{F}_X$. This implies $\mathcal{X}_\mathcal{F} = \mathcal{X}_{\mathcal{F}_M}$, i.e., $\mathcal{X}_M = \mathcal{X}$, as desired.
Conversely, suppose there exists a subset $O \subseteq M$ of size $k+1$ not contained in any $N \in M$. Choose some $b \in O$, let $A := O \setminus \{b\}$, so that $|A| = k$, and consider the implication $f := (A, \{b\})$. Then we have $f \notin \bigcup_{N \in M} \text{Imp}(N)$. Now letting $X := \mathcal{P}(M)$ and $Y := X_f$ we then have distinct $X, Y \in S_k$ with $X_M = Y_M$, showing that $X$ cannot be reconstructed within $S_k$.

Suppose that $|M| = m$. Recall (cf. [8, Sec. 2.5]) that a Steiner system $S(t, n, m)$ is a collection $\{N_i \mid i \in I\} \subseteq \mathcal{P}(M)$ of $n$-element subsets $N_i \subseteq M$ such that every $t$-element subset of $M$ is contained in exactly one subset $N_i$. In light of Theorem 5.6 it is clear that the Steiner systems $S(k+1, m, n)$ are the minimal consortial domains that are able to reconstruct target domains within the class $S_k$ of bounded premise complexity $k$.

5.4 Extensions

Combining counterexamples A lack of our consortium setting so far is the inability to recognize similar counterexamples. Combining counterexamples is a powerful idea that lifts the consortium above the knowledge of the “sum” of knowledge of the local (pre-)experts.

For this we need to augment a consortium by a background ontology of counterexamples. The most simple approach would be to identify two counterexamples from two different local (pre-)experts by matching the names of the counterexamples, which the experts would need to provide as well. In basic terms of FCA, while providing counterexamples the consortial expert needs to know if the counterexamples provided by the local (pre-)experts, restricted to their attribute sets, are of the same counterexample in the domain. We motivate this extension by an example. Given we want to explore some domain about animals with the attribute set being $M = \{\text{mammal, does not lay eggs, is not poisonous}\}$ using a set of two local (pre-)experts with $N_1 = \{\text{mammal, does not lay eggs}\}$ and $N_2 = \{\text{mammal, is not poisonous}\}$. Only expert $p_1$ can be consulted for the validity of $\{\text{mammal}\} \rightarrow \{\text{does not lay eggs}\}$. Of course, $p_1$ refutes this implication by providing the set $\{\text{mammal}\}$, which he could name for example platypus\(^6\). While exploring, the next query could be $\{\text{mammal}\} \rightarrow \{\text{is not poisonous}\}$. Note that this is not answered by the counterexample of $p_1$ since $\{\text{is not poisonous}\}$ is no subset of $N_1$. The local (pre-)expert $p_2$ refutes this of course as well, by providing the counterexample $\{\text{mammal}\}$ and naming this counterexample also platypus. Instead of collecting two different counterexamples we are now able to combine those two and say $\{\text{mammal}\}$ is not just an element of $X_1$ and $X_2$ but as well an element of $X$. In turn, the set of counterexamples the exploration algorithm is using contains now a more powerful counterexample than any local expert in the consortium could have provided. There are various ways to implement this combining of counterexamples. For example, after acquiring a counterexample for an implication from some expert one may ask all experts if they are aware of this counterexample name and if they could contribute further

\(^6\) a semiaquatic egg-laying mammal endemic to eastern Australia
attributes from their local attribute set. To investigate efficient strategies to do that is referred to future work.

**Coping with wrong counterexamples** Another desirable ability for a real world consortium would being able to handle wrong counterexamples, or more generally, having a measure that reflects the trust a consortial expert has in counterexamples of particular local (pre-)experts. Our setting for a consortium is not capable of doing this. In fact, the consortium cannot refute an implication using a wrong counterexample by design, since every $X_i$ is a restriction of the target domain. Hence, all counterexamples provided by a local (pre-)expert are “true”. In order to allow for a consortium to provide wrong counterexamples, one has to detach the closure system of some expert $p_i$ from the target domain $X$. This would also extend the possibilities of treating counterexamples by the consortial expert. Resolution strategies from simple majority voting up to minimum expert trust or confidence could be used.

6 Conclusion and outlook

In this paper we gave a first characterization of how to distribute the rôle of an domain expert for attribute exploration onto a consortium of local (pre-)experts. Besides practically using this method this result may be applied to various other tasks in the realm of FCA. It is obvious that the shown approach can easily be adapted for object exploration, the dual of attribute exploration. Hence, having object and attribute exploration through a consortium, we provided the necessary tools such that collaborative concept exploration (CCE) is at reach. Since CCE relies on both kinds of exploration in an alternating manner, the logical next step is to investigate what can be explored using a consortium. In addition we showed preliminary results on how to evaluate a consortium, how to shape it, i.e., how to choose a consortium from a potentially bigger set of experts, how to treat mistakenly accepted implications and how to increase the consistency.

Further research on this could focus on formalizing the depicted extensions from Section 5.4, where the task of modifying the consortium in order to encounter and compute wrong counterexamples seems as inevitable as it is hard to do. An easier extension that increases the ability of exploring a domain seems to be a “clever” combining of counterexamples.

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