THE TOPOLOGY OF SYSTEMS OF HYPERSPACES DETERMINED BY DIMENSION FUNCTIONS

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ABSTRACT. Given a non-degenerate Peano continuum $X$, a dimension function $D : 2^X \to [0, \infty]$ defined on the family $2^X$ of compact subsets of $X$, and a subset $\Gamma \subset [0, \infty)$, we recognize the topological structure of the system $(2^X, D_{\leq}(X))_{\alpha \in \Gamma}$, where $2^X$ is the hyperspace of non-empty compact subsets of $X$ and $D_{\leq}(X)$ is the subspace of $2^X$, consisting of non-empty compact subsets $K \subset X$ with $D(K) \leq \gamma$.

1. Introduction

The problem of topological characterization (identification) of topological objects is a central problem in topology. A classical result of this sort is the Curtis-Schori Theorem [6] asserting that for each non-degenerate Peano continuum $X$ the hyperspace $2^X$ of non-empty compact subsets of $X$ endowed with the Vietoris topology is homeomorphic to the Hilbert cube $Q = [-1,1]^\omega$. At bit later, D.Curtis [3] characterized topological spaces $X$ whose hyperspace $2^X$ is homeomorphic to the pseudointerior $s = (-1,1)^\omega$ of the Hilbert cube as connected locally connected Polish nowhere locally compact spaces.

In [8] T.Dobrowolski and L.Rubin recognized the topology of the subspace $\dim_{\leq n}(Q) \subset 2^Q$ consisting of compact subsets of $Q$ having covering dimension $\leq n$. They constructed a homeomorphism $h : 2^Q \to Q^\omega$ such that $h(\dim_{\leq n}(Q)) = Q^n \times s^{\omega \setminus n}$ for all $n = \{0, \ldots, n-1\} \in \omega$. In this case it is said that the system $(2^Q, \dim_{\leq n}(Q))_{n \in \omega}$ is homeomorphic to the system $(Q^\omega, Q^n \times s^{\omega \setminus n})_{n \in \omega}$.

This result was later generalized by H.Gladdines [12] to products of Peano continua. Finally, R.Cauty [8] has characterized spaces $X$ for which the system $(2^X, \dim_{\leq n}(X))_{n \in \omega}$ is homeomorphic to $(Q^\omega, Q^n \times s^{\omega \setminus n})_{n \in \omega}$ as Peano continua whose any non-empty open subset contains compact subsets of arbitrary high finite dimension.

In [13] given a metric space $X$ the second author initiated the study of the subspace $HD_{\leq \gamma}(X) \subset 2^X$ of compact subsets of $X$ whose Hausdorff dimension is $\leq \gamma$. Unlike the (integer-valued) topological dimension, the Hausdorff dimension of a metric compactum can take on any non-negative real value $\gamma$. So, the system $(2^X, HD_{\leq \gamma}(X))_{\gamma \in [0,\infty)}$ that naturally appears in this situation is uncountable. In [14] it was proved that for a finite-dimensional cube $X = [0,1]^n$ the system $(2^X, HD_{\leq \gamma}(X))_{\gamma \in [0,n)}$ is homeomorphic to the system $(Q^\mathbb{Q}, Q^\mathbb{Q} \times s^\mathbb{Q} \setminus \gamma)_{\gamma \in [0,n)}$ (by $\mathbb{Q}$ we denote the space of rational numbers). Here for a subset $A \subset \mathbb{R}$ and a real number $\gamma$ we put

$$A_{\leq \gamma} = \{ a \in A : a \leq \gamma \}, \quad A_{\geq \gamma} = \{ a \in A : a \geq \gamma \}$$

$$A_{< \gamma} = \{ a \in A : a < \gamma \}, \quad A_{> \gamma} = \{ a \in A : a > \gamma \}.$$

Both the (topological) covering dimension and the (metric) Hausdorff dimension are particular cases of dimension functions defined as follows.

Definition 1. A function $D : 2^X \to [0, \infty]$ defined on the family $2^X$ of compact subsets of a topological space $X$ is called a dimension function if:

1. $D(\emptyset) = 0$;
2. $D$ is monotone in the sense that $D(A) \leq D(B)$ for any compact subsets $A \subset B$ of $X$;
3. $D$ is finitely additive in the sense that $D(F \cup A \cup B) \leq \max\{D(A), D(B)\}$ for any finite subset $F \subset X$ and disjoint compact subsets $A, B \subset X$.

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(4) $D$ is $\omega$-additive in the sense that each non-empty open subset $U \subset X$ contains non-empty open sets $U_n \subset U$, $n \in \omega$, such that each compact subset $K \subset \text{cl}_X(\bigcup_{n \in \omega} U_n)$ has dimension $D(K) \leq \sup_{n \in \omega} D(K \cap U_n)$.

Given a dimension function $D : 2^X \rightarrow [0, \infty]$ on $X$ and a subset $\Gamma \subset [0, \infty)$, for every $\gamma \in \Gamma$ consider the subspace
\[ D_{\leq \gamma}(X) = \{ F \in 2^X : D(F) \leq \gamma \} \]
in the hyperspace $2^X$. Our aim is to recognize the topological structure of the system $\langle 2^X, D_{\leq \gamma}(X) \rangle_{\gamma \in \Gamma}$.

In the sequel, by a $\Gamma$-system $\langle X, \gamma \rangle_{\gamma \in \Gamma}$ we shall understand a pair consisting of a set $X$ and a family $\langle \gamma \rangle_{\gamma \in \Gamma}$ of subsets of $X$, indexed by the elements of an index set $\Gamma$. Two $\Gamma$-systems $\langle X, \gamma \rangle_{\gamma \in \Gamma}$ and $\langle Y, \gamma \rangle_{\gamma \in \Gamma}$ are homeomorphic if there is a homeomorphism $h : X \rightarrow Y$ such that $h(\gamma) = \gamma$, for all $\gamma \in \Gamma$.

The following theorem describes the topological structure of the $\Gamma$-system $\langle 2^X, D_{\leq \gamma}(X) \rangle_{\gamma \in \Gamma}$ for a dimension function $D : 2^X \rightarrow [0, \infty]$ taking values in the half-line with attached infinity (that is assumed to be larger than any real number). In that theorem we shall refer to the subsets $\langle \gamma \rangle_R$ defined for $\Gamma \subset \mathbb{R}$ and $\gamma \in \Gamma$ as follows:
\[ \langle \gamma \rangle_R = \begin{cases} \langle \gamma, \inf(\Gamma_{>\gamma}) \rangle & \text{if } \gamma < \inf(\Gamma_{>\gamma}); \\ \langle \sup(\Gamma_{<\gamma}), \gamma \rangle & \text{if } \Gamma \ni \sup(\Gamma_{<\gamma}) < \gamma = \inf(\Gamma_{>\gamma}); \\ \langle \sup(\Gamma_{<\gamma}), \gamma \rangle & \text{in all other cases}. \end{cases} \]

In this definition we assume that $\sup(\emptyset) = -\infty$ and $\inf(\emptyset) = +\infty$.

**Theorem 1.** Let $X$ be a topological space and $D : 2^X \rightarrow [0, \infty]$ be a dimension function. For every subset $\Gamma \subset [0, \infty)$ the $\Gamma$-system $\langle 2^X, D_{\leq \gamma}(X) \rangle_{\gamma \in \Gamma}$ is homeomorphic to the $\Gamma$-system $(Q^\mathbb{Q}, Q_{\leq \gamma} \times s^Q_{>\gamma})_{\gamma \in \Gamma}$ if and only if
1. $X$ is a non-degenerate Peano continuum,
2. each subspace $D_{\leq \gamma}(X)$, $\gamma \in \Gamma$, is of type $G_\delta$ in $2^X$, and
3. each non-empty open set $U \subset X$ for every $\gamma \in \Gamma$ contains a compact subset $K \subset U$ with $D(K) \in \langle \gamma \rangle_R$.

First, we apply this theorem to integer-valued dimension functions. We identify each natural number $n$ with the set $\{0, \ldots, n-1\}$. Also we put $\omega = \omega \cup \{\omega\}$.

**Corollary 1.** Let $X$ be a topological space and $D : 2^X \rightarrow \omega$ be a dimension function. For every $n \in \omega$ the $n$-system $\langle 2^X, D_{\leq k}(X) \rangle_{k \in \omega}$ is homeomorphic to the $n$-system $(Q^\mathbb{Q}, Q_{\leq \gamma} \times s^Q_{\geq k})_{k \in \omega}$ if and only if
1. $X$ is a non-degenerate Peano continuum,
2. each subspace $D_{\leq k}(X)$, $k \in n$, is of type $G_\delta$ in $2^X$, and
3. each non-empty open set $U \subset X$ for every $k \in n$ contains a compact subset $K \subset U$ with $D(K) = k$.

The covering dimension $\dim_G$ and the cohomological dimension $\text{dim}_G$ for an arbitrary Abelian group $G$ are examples of integer-valued dimension functions. Therefore Corollary 1 implies the following theorem of R. Cauty [3] that was mentioned above.

**Theorem 2 (Cauty).** For any non-degenerate Peano continuum $X$ the $\omega$-systems $\langle 2^X, \dim_{\leq n}(X) \rangle_{n \in \omega}$ is homeomorphic to $\langle Q^\mathbb{Q}, Q_{\leq \gamma} \times s^Q_{\geq n} \rangle_{n \in \omega}$ if and only if each non-empty open set $U \subset X$ contains a compact subset of arbitrary finite dimension.

In [3] R. Cauty notices, that this theorem holds also for the cohomological dimension $\text{dim}_G$ or any other dimension function in the sense of [3]. It does not demand any modifications of arguments in the proof.

Applying Theorem 1 to the half-interval $\Gamma = [0, b) \subset [0, \infty)$, we obtain:

**Corollary 2.** Let $X$ be a topological space and $D : 2^X \rightarrow [0, \infty]$ be a dimension function. For every $b \in [0, \infty)$ the $[0, b)$-system $\langle 2^X, D_{\leq \gamma}(X) \rangle_{\gamma \in [0, b)}$ is homeomorphic to the $[0, b)$-system $\langle Q^\mathbb{Q}, Q_{\leq \gamma} \times s^Q_{\geq b} \rangle_{\gamma \in [0, b)}$ if and only if
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(1) X is a non-degenerate Peano continuum,
(2) each subspace \( D_\leq \gamma(X), \gamma \in [0, b) \), is of type \( G_\delta \) in \( 2^X \), and
(3) each non-empty open set \( U \subset X \) for every \( \gamma \in [0, b) \) contains a compact subset \( K \subset U \) with \( D(K) = \gamma \).

Applying Corollary 2 to the Hausdorff dimension \( \dim_H \) we obtain the following theorem whose partial case for \( X = \mathbb{I}^n \) was proved in [14].

**Theorem 3.** For a number \( b \in (0, \infty] \) and a non-degenerate metric Peano continuum \( X \) the system \( \langle 2^X, HD_\leq \gamma(X), \gamma \in [0, b) \rangle \) is homeomorphic to the system \( \langle Q^d, Q^\leq \gamma \times s^\geq \gamma, \gamma \in [0, b) \rangle \) if and only if each non-empty open subset \( U \subset X \) has Hausdorff dimension \( \dim_H(U) \geq b \).

To derive this theorem from Corollary 2 we need to check the conditions (2) and (3) for the Hausdorff dimension. The condition (2) was establised in [13] while (3) follows from the subsequent Mean Value Theorem for Hausdorff dimension, which will be proved in Section 6.

**Theorem 4.** Let \( X \) be a separable complete metric space \( X \). For every non-negative real number \( d < \dim_H(X) \) the space \( X \) contains a compact subset \( K \subset X \) of Hausdorff dimension \( \dim_H(K) = d \).

A similar Mean Value Theorem holds for topological dimension: each regular space \( X \) with finite inductive dimension \( \text{ind}(X) \) contains a closed subspace of any dimension \( k \leq \text{ind}(X) \), see [10] 1.5.1. However, (in contrast to the Hausdorff dimension) this theorem does not hold for infinite-dimensional spaces: there is an infinite-dimensional compact metrizable space \( X \) containing no subspace of positive finite dimension [10] 5.2.23.

2. Absorbing systems in the Hilbert cube

Theorem [11] is proved by the technique of absorbing systems created and developed in [7], [12]. So, in this section we start by recalling some basic information related to absorbing systems.

From now on all topological spaces are metrizable and separable, all maps are continuous. By \( \mathbb{I} \) we denote the unit interval \([0, 1]\), by \( \mathbb{Q} \) the space of rational numbers, by \( Q = [-1, 1]^\mathbb{Q} \) the Hilbert cube, by \( s = (-1, 1)^\mathbb{Q} \) its pseudointerior and by \( B(Q) \) its pseudoboundary. By a Hilbert cube we understand any topological space homeomorphic to the Hilbert cube \( Q \). In particular, for each at most countable set \( A \) the power \( Q^A \) is a Hilbert cube; \( B(Q^A) = Q^A \setminus s^A \) will stand for its pseudoboundary.

Given two maps \( f, g : X \to Y \) and a cover \( U \) of \( Y \) we write \( (f, g) \prec U \) and say that \( f, g \) are \( U \)-near if for every point \( x \in X \) there is a set \( U \in \mathcal{U} \) such that \( \{f(x), g(x)\} \subset U \).

A closed subset \( A \) of an ANR-space \( X \) is a \( Z \)-set if for each map \( f : Q \to X \) and an open cover \( \mathcal{U} \) of \( X \) there is a map \( g : Q \to X \setminus A \) such that \( (f, g) \prec \mathcal{U} \). A subset \( A \subset X \) is called a \( \sigma Z \)-set if \( A \) can be written as the countable union of \( Z \)-sets. It is known [15] that a closed \( \sigma Z \)-set in a Polish ANR-space is a \( Z \)-set. An embedding \( f : K \to X \) is called a \( Z \)-embedding if the image \( f(K) \) is a \( Z \) set in \( X \).

It is well-known that each map \( f : K \to Q \) defined on a compact space can be approximated by \( Z \)-embeddings, see [4], [15].

Let \( \Gamma \) be a set. By a \( \Gamma \)-system \( X = \langle X, X_\gamma \rangle_{\gamma \in \Gamma} \) we shall understand a pair consisting of a space \( X \) and an indexed collection \( \langle X_\gamma \rangle_{\gamma \in \Gamma} \) of subsets of \( X \). Given a map \( f : Z \to X \) and a set \( K \subset X \) let \( f^{-1}(X) = \langle f^{-1}(X), f^{-1}(X_\gamma) \rangle_{\gamma \in \Gamma} \) and \( K \cap X = \langle K \cap X, K \cap X_\gamma \rangle_{\gamma \in \Gamma} \).

From now on, \( \mathcal{E}_\Gamma \) is a fixed class of \( \Gamma \)-systems.

Generalizing the standard concept of a strongly universal pair [11 §1.7] to \( \Gamma \)-systems we get an important notion of a strongly \( \mathcal{E}_\Gamma \)-universal \( \Gamma \)-system.

**Definition 2.** A \( \Gamma \)-system \( X = \langle X, X_\gamma \rangle_{\gamma \in \Gamma} \) is defined to be strongly \( \mathcal{E}_\Gamma \)-universal if for any open cover \( \mathcal{U} \) on \( X \), any \( \Gamma \)-system \( \mathcal{C} = \langle C, C_\gamma \rangle_{\gamma \in \Gamma} \in \mathcal{E}_\Gamma \), and a map \( f : C \to X \) whose restriction \( f|B : B \to X \) to a closed subset \( B \subset C \) is a \( Z \)-embedding with \( (f|B)^{-1}(X) = B \cap \mathcal{C} \) there exists a \( Z \)-embedding \( \tilde{f} : C \to X \) such that \( (\tilde{f}, f) \prec \mathcal{U}, \tilde{f}|B = f|B \), and \( \tilde{f}^{-1}(X) = C \).

The strong universality is the principal ingredient in the notion of a \( \mathcal{E}_\Gamma \)-absorbing system, generalizing the notion of an absorbing pair, see [11 §1.6].
Lemma 1. The class $\sigma$ consisting of the sets $\langle A, A \rangle$ is strongly $\mathfrak{C}$-universal; there is a sequence $(Z_n)_{n \in \omega}$ of $Z$-sets in $X$ such that $\bigcup_{\gamma \in \Gamma} X_\gamma \subseteq \bigcup_{n \in \omega} Z_n$ and $Z_n \cap X \in \mathfrak{C}$ for all $n \in \omega$.

A remarkable feature of $\mathfrak{C}$-absorbing system in the Hilbert cube is their topological equivalence. We define two $\Gamma$-systems $(X, X_\gamma)_{\gamma \in \Gamma}$ and $(Y, Y_\gamma)_{\gamma \in \Gamma}$ to be homeomorphic if there is a homeomorphism $h : X \to Y$ such that $h(X_\gamma) = Y_\gamma$ for $\gamma \in \Gamma$.

The following Uniqueness Theorem can be proved by analogy with Theorem 1.7.6 from [1].

Theorem 5. Two $\mathfrak{C}$-absorbing $\Gamma$-systems $(X, X_\gamma)_{\gamma \in \Gamma}$ and $(Y, Y_\gamma)_{\gamma \in \Gamma}$ are homeomorphic provided $X$ and $Y$ are homeomorphic to a manifold modeled on $\mathbb{Q}$ or $s$.

By a manifold modeled on a space $E$ we understand a metrizable separable space whose any point has an open neighborhood homeomorphic to an open subset of the model space $E$.

3. Characterizing model absorbing systems

In this section, given a subset $\Gamma \subseteq \mathbb{R}$ we characterize the topology of the model $\Gamma$-system $(Q^\mathbb{Q}, Q^{\leq \gamma} \times s^{Q_{>\gamma}})_{\gamma \in \Gamma}$. In fact, it will be more convenient to work with the complementary $\Gamma$-system

$$\Sigma_\Gamma = (Q^\mathbb{Q}, Q^{\leq \gamma} \times B(Q^{Q_{>\gamma}}))_{\gamma \in \Gamma},$$

where $B(Q^{Q_{>\gamma}}) = Q^{Q_{>\gamma}} \setminus s^{Q_{>\gamma}}$. We shall prove that the latter system is $\sigma\mathfrak{C}$-absorbing for a suitable class $\sigma\mathfrak{C}_\Gamma$ of $\Gamma$-systems.

Let $\Gamma \subseteq \mathbb{R}$. Let us define a $\Gamma$-system $(A, A_\gamma)_{\gamma \in \Gamma}$ to be

- $\sigma$-compact if the space $A$ is compact while all subspaces $A_\gamma$, $\gamma \in \Gamma$, are $\sigma$-compact;
- inf-continuous if $A_\gamma = \bigcup_{\beta \in B} A_\beta$ for any subset $B \subseteq \Gamma$ with $\inf B = \gamma \in \Gamma$.

By $\sigma\mathfrak{C}$ we shall denote the class of $\sigma$-compact inf-continuous $\Gamma$-systems. Let us observe that each $\Gamma$-system $(A, A_\gamma)_{\gamma \in \Gamma} \in \sigma\mathfrak{C}_\Gamma$ is decreasing. Indeed, for any real numbers $\alpha < \beta$ in $\Gamma$ the equality $\alpha = \inf \{\alpha, \beta\}$ implies $A_\alpha = A_\alpha \cup A_\beta \supset A_\beta$.

Each $\Gamma$-system $A = (A, A_\gamma)_{\gamma \in \Gamma} \in \sigma\mathfrak{C}_\Gamma$ can be extended to the $\mathbb{R}$-system $\tilde{A} = (A, \tilde{A}_\gamma)_{\gamma \in \mathbb{R}} \in \sigma\mathfrak{C}_\mathbb{R}$ consisting of the sets

$$\tilde{A}_\gamma = \begin{cases} \bigcup_{a \in \Gamma \geq \gamma} A_a & \text{if } \sup(\Gamma_{<\gamma}) \notin \Gamma \text{ or } \gamma = \inf(\Gamma_{\geq\gamma}) ; \\ A_a & \text{if } \alpha = \sup(\Gamma_{<\gamma}) \in \Gamma \text{ and } \gamma < \inf(\Gamma_{\geq\gamma}) , \end{cases}$$

indexed by real numbers $\gamma$.

Lemma 1. The $\mathbb{R}$-system $\tilde{A} = (A, \tilde{A}_\gamma)_{\gamma \in \mathbb{R}} \in \sigma\mathfrak{C}_\mathbb{R}$ is $\sigma$-compact, inf-continuous and extends the $\Gamma$-system $A = (A, A_\gamma)_{\gamma \in \Gamma}$ in the sense that $\tilde{A}_\gamma = A_\gamma$ for all $\gamma \in \Gamma$.

Proof. To see that the $\mathbb{R}$-system $\tilde{A}$ is $\sigma$-compact, fix any real number $\gamma$. The set $\tilde{A}_\gamma$ is clearly $\sigma$-compact if $\tilde{A}_\gamma = A_\alpha$ for some $\alpha \in \Gamma$. So, we assume that $\tilde{A}_\gamma \neq A_\alpha$ for all $\alpha \in \Gamma$. In this case $\tilde{A}_\gamma = \bigcup_{a \in \Gamma \geq \gamma} A_a$ and $\inf(\Gamma_{\geq\gamma}) \notin \Gamma$. Choose any countable dense subset $D \subseteq \Gamma$ and observe that $\inf D_{\geq\gamma} = \inf(\Gamma_{\geq\gamma})$ and hence

$$\tilde{A}_\gamma = \bigcup_{a \in \Gamma \geq \gamma} A_a = \bigcup_{a \in D_{\geq\gamma}} A_a$$

is $\sigma$-compact, being the countable union of $\sigma$-compact spaces $A_a$, $\alpha \in D_{\geq\gamma}$.

Observe that for every $\gamma \in \Gamma$ we get $\gamma = \inf(\Gamma_{\geq\gamma})$ and hence $A_\gamma \subseteq \bigcup_{a \in \Gamma \geq \gamma} A_a = \tilde{A}_\gamma$. The reverse inclusion $\tilde{A}_\gamma = \bigcup_{a \in \Gamma \geq \gamma} A_a \subseteq A_\gamma$ follows from the decreasing property of the $\gamma$-system $A$. Thus $A_\gamma = \tilde{A}_\gamma$, which means that the $\mathbb{R}$-system $\tilde{A}$ extends the $\Gamma$-system $A$.

Next, we prove that the $\mathbb{R}$-system $\tilde{A}$ is decreasing. Given two real numbers $\beta < \gamma$, we need to show that $\tilde{A}_\beta \supset \tilde{A}_\gamma$. We consider four cases:
1) Both $\beta$ and $\gamma$ satisfy the first case of the definition of $\tilde{A}_\beta$ and $\tilde{A}_\gamma$:
\[
(\sup(\Gamma_{<\beta}) \notin \Gamma \text{ or } \beta = \inf(\Gamma_{\geq \beta})) \text{ and } (\sup(\Gamma_{<\gamma}) \notin \Gamma \text{ or } \gamma = \inf(\Gamma_{\geq \gamma})).
\]
In this case $\beta < \gamma$ implies $\Gamma_{\geq \beta} \supset \Gamma_{\geq \gamma}$ and thus
\[
\tilde{A}_\beta = \bigcup_{\alpha \in \Gamma_{\geq \beta}} A_\alpha \supset \bigcup_{\alpha \in \Gamma_{\geq \gamma}} A_\alpha = \tilde{A}_\gamma.
\]

2) The element $\beta$ satisfies the first case of the definition of $\tilde{A}_\beta$ while $\gamma$ satisfies the second case:
\[
(\sup(\Gamma_{<\beta}) \notin \Gamma \text{ or } \beta = \inf(\Gamma_{\geq \beta})) \text{ and } \alpha = \sup(\Gamma_{<\gamma}) \in \Gamma \text{ and } \gamma < \inf(\Gamma_{\geq \gamma}).
\]
In this case $\beta < \alpha$. Indeed, assuming conversely that $\alpha < \beta$, we get $\Gamma_{<\beta} = \Gamma_{<\gamma}$ and thus $\alpha = \sup(\Gamma_{<\beta}) \in \Gamma$, which implies that $\beta = \inf(\Gamma_{\geq \beta})$. In this case, $\alpha = \sup(\Gamma_{<\gamma}) \geq \beta$, which is a contradiction. So, $\beta \leq \alpha$ and then $\alpha \in \Gamma_{\geq \beta}$ and $\tilde{A}_\beta \supset A_\alpha = \tilde{A}_\gamma$.

3) The element $\beta$ satisfies the second case of the definition of $\tilde{A}_\beta$ while $\gamma$ satisfies the first one:
\[
\alpha = \sup(\Gamma_{<\beta}) \in \Gamma \text{ and } \beta < \inf(\Gamma_{\geq \beta}) \text{ and } (\sup(\Gamma_{<\gamma}) \notin \Gamma \text{ or } \gamma = \inf(\Gamma_{\geq \gamma})).
\]
In this case
\[
\tilde{A}_\beta = A_\alpha \supset \bigcup_{\delta \in \Gamma_{\geq \gamma}} A_\delta = \tilde{A}_\gamma.
\]

4) Both $\beta$ and $\gamma$ satisfy the second case of the definition of $\tilde{A}_\beta$ and $\tilde{A}_\gamma$:
\[
\alpha_\beta = \sup(\Gamma_{<\beta}) \in \Gamma, \quad \beta < \inf(\Gamma_{\geq \beta}), \quad \alpha_\gamma = \sup(\Gamma_{<\gamma}) \in \Gamma, \quad \gamma < \inf(\Gamma_{\geq \gamma}).
\]
In this case $\alpha_\beta \leq \alpha_\gamma$ and $\tilde{A}_\beta = A_{\alpha_\beta} \supset A_{\alpha_\gamma} = \tilde{A}_\gamma$. This completes the proof of the decreasing property of the $\mathbb{R}$-system $\tilde{A}$.

Finally, we show that the $\mathbb{R}$-system $\tilde{A}$ is inf-continuous. Fix any real number $\gamma$ and a subset $B \subset \mathbb{R}$ with $\gamma = \inf B$. We need to check that $\tilde{A}_\gamma = \bigcup_{\beta \in B} \tilde{A}_\beta$. The decreasing property of $\tilde{A}$ guarantees that $\tilde{A}_\gamma \supset \bigcup_{\beta \in B} \tilde{A}_\beta$. It remains to prove the reverse inclusion, which is trivial if $\gamma \in B$. So, we assume that $\gamma \notin B$. Two cases are possible:

1. $\sup(\Gamma_{<\gamma}) \notin \Gamma$ or $\gamma = \inf(\Gamma_{\geq \gamma})$. In this case $\tilde{A}_\gamma = \bigcup_{\alpha \in \Gamma_{\geq \gamma}} A_\alpha$. We consider three subcases:

   1a) If $\gamma = \inf(\Gamma_{\geq \gamma})$, then
   \[
   \tilde{A}_\gamma = \bigcup_{\alpha \in \Gamma_{\geq \gamma}} A_\alpha \supset \bigcup_{\alpha \in \Gamma_{\geq \gamma}} A_\alpha
   \]
   because of the inf-continuity of the system $\tilde{A}$. Given any point $a \in \tilde{A}_\gamma$, find $\alpha \in \Gamma_{\geq \gamma}$ such that $a \in A_\alpha$. Since $B \notin \gamma = \inf B$, there is a point $\beta \in B \cap (\gamma, \alpha)$. Now the definition of $\tilde{A}_\beta$ implies that $a \in A_\alpha \subset \tilde{A}_\beta \subset \bigcup_{\delta \in B} \tilde{A}_\delta$.

   1b) If $\Gamma \ni \gamma < \inf(\Gamma_{\geq \gamma})$, then we can find $\beta \in B \cap (\gamma, \inf(\Gamma_{\geq \gamma}))$ and conclude that $\tilde{A}_\gamma = A_\gamma = \tilde{A}_\beta \subset \bigcup_{\delta \in B} \tilde{A}_\delta$.

   1c) If $\Gamma \notin \gamma < \inf(\Gamma_{\geq \gamma})$, then $\sup(\Gamma_{<\gamma}) \notin \Gamma$. Choose any point $\beta \in B \cap (\gamma, \inf(\Gamma_{\geq \gamma}))$ and observe that $\Gamma_{\geq \beta} = \Gamma_{\geq \gamma}$, $\sup(\Gamma_{<\beta}) = \sup(\Gamma_{<\gamma}) \notin \Gamma$ and thus
   \[
   \tilde{A}_\gamma = \bigcup_{\alpha \in \Gamma_{\geq \gamma}} A_\alpha = \bigcup_{\alpha \in \Gamma_{\geq \beta}} A_\alpha = \tilde{A}_\beta \subset \bigcup_{\delta \in B} \tilde{A}_\delta.
   \]

2. $\alpha = \sup(\Gamma_{<\gamma}) \in \Gamma$ and $\gamma < \inf(\Gamma_{\geq \gamma})$, in which case $\tilde{A}_\gamma = A_\alpha$. Since inf $B = \gamma \notin \Gamma$, there is a point $\beta \in B \cap (\gamma, \inf(\Gamma_{\geq \gamma}))$.

   2a) If $\gamma \in \Gamma$, then $\sup(\Gamma_{<\beta}) = \gamma \in \Gamma$ and thus $\tilde{A}_\gamma = A_\gamma = \tilde{A}_\beta \subset \bigcup_{\delta \in B} A_\delta$.

   2b) If $\gamma \notin \Gamma$, then $\Gamma_{<\beta} = \Gamma_{<\gamma}$ and thus $\tilde{A}_\gamma = A_\alpha = \tilde{A}_\beta \subset \bigcup_{\delta \in B} A_\delta$. \qed

In the following theorem for every subset $\Gamma \subset \mathbb{R}$ we introduce a model $\sigma \mathcal{C}_\Gamma$-absorbing system $\Sigma_\Gamma$ in the Hilbert cube $Q^\mathbb{R}$. 

Theorem 6. For every $\Gamma \subset \mathbb{R}$ the $\Gamma$-system $\Sigma_{\Gamma} = \langle Q^Q, Q^{\leq \gamma} \times B(Q^{>\gamma}) \rangle_{\gamma \in \Gamma}$ is $\sigma \mathcal{C}_{\Gamma}$-absorbing and hence is homeomorphic to any other $\sigma \mathcal{C}_{\Gamma}$-absorbing $\Gamma$-system $(X, X_\gamma)_{\gamma \in \Gamma}$ in a Hilbert cube $X$.

Proof. First we check that the system $\Sigma_{\Gamma}$ is strongly $\sigma \mathcal{C}_{\Gamma}$-universal.

We start defining a suitable metric on the Hilbert cube $Q^Q$. Let $\nu : Q \to (0, 1)$ be any vanishing function, which means that for every $\varepsilon > 0$ the set $\{q \in Q : \nu(q) \geq \varepsilon\}$ is finite. Take any metric $d$ generating the topology of the Hilbert cube $Q$ and consider the metric

$$
\rho((x_q), (y_q)) = \max_{q \in Q} \nu(q) \cdot d(x_q, y_q)
$$
on the Hilbert cube $Q^Q$.

In order to prove the strong $\sigma \mathcal{C}_{\Gamma}$-universality of the system $\Sigma_{\Gamma}$, fix a $\Gamma$-system $\mathcal{A} = \langle A, A_\gamma \rangle_{\gamma \in \Gamma} \in \sigma \mathcal{C}_{\Gamma}$ and a map $f : A \to Q^Q$ that restricts to a $Z$-embedding of some closed subset $K \subset A$ such that $(f|K)^{-1}(\Sigma_{\Gamma}) = K \cap A$. Given $\varepsilon > 0$, we need to construct a $Z$-embedding $\tilde{f} : A \to Q^Q$ such that $\rho(\tilde{f}(x), \tilde{f}(y)) < \varepsilon$ for all $x, y \in A$.

By Lemma 11, the $\Gamma$-system $\mathcal{A}$ extends to an $\mathbb{R}$-system $\tilde{\mathcal{A}} = \langle A, A_\gamma \rangle_{\gamma \in \mathbb{R}} \in \sigma \mathcal{C}_{\mathbb{R}}$. We shall construct a $Z$-embedding $\tilde{f} : A \to Q^Q$ such that $\rho(\tilde{f}(x), \tilde{f}(y)) < \varepsilon$ and $\tilde{f}(A \setminus K) \cap \tilde{f}(K) = \emptyset$. Using the strong $\sigma \mathcal{C}_{[0]}$-universality of the pair $(Q, B(Q))$, for each $q \in Q$ we can approximate the map $f$ by a map $f' : A \to Q^Q$ such that $\rho(f'(x), f'(y)) < \varepsilon/2$ and $f'(A \setminus K) \cap f'(K) = \emptyset$.

For every $q \in Q$ let $\mathcal{P}_q : Q^Q \to Q$ denote the coordinate projection. Since $f(K)$ is a $Z$-set in $Q^Q$, we can approximate the map $f$ by a map $f' : A \to Q^Q$ such that $\rho(f'(x), f'(y)) < \varepsilon/2$ and $f'(A \setminus K) \cap f'(K) = \emptyset$. Using the strong $\sigma \mathcal{C}_{[0]}$-universality of the pair $(Q, B(Q))$, for each $q \in Q$ we can approximate the map $f_q : A \to Q$ by a map $\tilde{f}_q : A \to Q$ such that $d(\tilde{f}_q(x), \mathcal{P}_q \circ f'(x)) \leq \varepsilon \rho(f'(x), f(K))$ for all $x \in A$;

a) $\tilde{f}_q|A \setminus K$ is injective;

b) $\tilde{f}_q(A \setminus K)$ is a $Q$-set in $Q$;

c) $\tilde{f}_q(A \setminus K)$ is a $\sigma Q$-set in $Q$;

and $d(\tilde{f}_q(x), \mathcal{P}_q \circ f'(x)) \leq \varepsilon \rho(f'(x), f(K))$ for all $x \in A$.

Now consider the diagonal product $\tilde{f} = (\tilde{f}_q)_{q \in Q} : A \to Q^Q$ of the maps $\tilde{f}_q$, $q \in Q$. It follows from (a) that $\tilde{f}|K = f|K$, $\rho(\tilde{f}(x), \tilde{f}(y)) < \varepsilon$ and $\tilde{f}(A \setminus K) \cap \tilde{f}(K) = \emptyset$. Combining this fact with (b) we conclude that the map $\tilde{f} : A \to Q^Q$ is injective and hence an embedding. It follows from (c) that $\tilde{f}(A)$ is a $\sigma Q$-set in $Q^Q$ and hence a $Q$-set, see [15, 6.2.2]. Therefore, $\tilde{f}$ is a $Z$-embedding approximating the map $f$.

It remains to check that $\tilde{f}(\Sigma_{\gamma}) = A_\gamma$ for every $\gamma \in \Gamma$. Since $\tilde{f}(\Sigma_{\gamma}) \cap K = (f|K)^{-1}(\Sigma_{\gamma}) = K \cap A_\gamma$, it suffices to check that $\tilde{f}(\Sigma_{\gamma}) \setminus K = A_\gamma \setminus K$.

It follows that

$$
\tilde{f}(\Sigma_{\gamma}) \setminus K = \tilde{f}^{-1}(Q^{\leq \gamma} \times B(Q^{>\gamma})) \setminus K = \bigcup_{q \in Q^{>\gamma}} \tilde{f}_q^{-1}(B(Q)) \setminus K = \bigcup_{q \in Q^{>\gamma}} A_q \setminus K = A_\gamma \setminus K.
$$

The last equality follows from the inf-continuity of the $\mathbb{R}$-system $\tilde{\mathcal{A}} = \langle A, A_\gamma \rangle_{\gamma \in \mathbb{R}}$ because $\gamma = \inf Q^{>\gamma}$. This completes the proof of the strong $\sigma \mathcal{C}_{\Gamma}$-universality of the system $\Sigma_{\Gamma}$.

It remains to check that the $\Gamma$-system $\Sigma_{\Gamma}$ satisfies the second condition of Definition 3 of a $\sigma \mathcal{C}_{\Gamma}$-absorbing system. It is clear the $\Gamma$-system $\Sigma_{\Gamma}$ is $\sigma$-compact and decreasing. To show that it is inf-continuous, take any subset $B \subset \Gamma$ with $\gamma = \inf B \in \Gamma$. If $\gamma \in B$, then $\Sigma_{\gamma} \supset \bigcup_{\beta \in B} \Sigma_{\beta} \supset \Sigma_{\gamma}$. So, we assume that $\gamma \notin B$. Since the $\Gamma$-system $\langle Q^Q, \Sigma_{\gamma} \rangle_{\gamma \in \Gamma}$ is decreasing, we get $\Sigma_{\gamma} \supset \bigcup_{\beta \in B} \Sigma_{\beta}$. To prove the reverse inclusion, take any point $(x_q)_{q \in Q} \in \Sigma_{\gamma} = Q^{\leq \gamma} \times B(Q^{>\gamma})$ and observe that $x_q \in B(Q)$ for some $q \in Q^{>\gamma}$. Since $\gamma = \inf B$ the half-interval $[\gamma, q]$ contains a point $\beta \in B$.

Then $(x_q)_{q \in Q} \in Q^{\leq \beta} \times B(Q^{>\beta})$ and thus $(x_q)_{q \in Q} \in \Sigma_{\beta} \subset \bigcup_{\alpha \in B} \Sigma_{\alpha}$. Therefore, $\Sigma_{\Gamma} \in \sigma \mathcal{C}_{\Gamma}$.

Since each space $\Sigma_{\gamma}, \gamma \in Q$, is a $\sigma Q$-set in $Q^Q$, so is the countable union $\bigcup_{\gamma \in Q} \Sigma_{\gamma} = \bigcup_{\gamma \in Q} \Sigma_{\gamma}$. So, we can find a sequence $\langle Z_n \rangle_{n \in \omega}$ of $Z$-sets in $Q^Q$ such that

$$
\bigcup_{n \in \omega} Z_n = \bigcup_{\gamma \in Q} \Sigma_{\gamma}.
$$
It follows from $\Sigma_\Gamma \in \sigma \mathcal{C}_\Gamma$ that $Z_n \cap \Sigma_\Gamma \in \sigma \mathcal{C}_\Gamma$, which completes the proof of the $\sigma \mathcal{C}_\Gamma$-absorbing property of the system $\Sigma_\Gamma$.

By the Uniqueness Theorem 5 each $\sigma \mathcal{C}_\Gamma$-absorbing system $\langle X, X_\gamma \rangle_{\gamma \in \Gamma}$ in a Hilbert cube $X$ is homeomorphic to the $\sigma \mathcal{C}_\Gamma$-absorbing $\Gamma$-system $\Sigma_\Gamma$. \hfill \Box

4. STRONGLY UNIVERSAL SYSTEMS OF HYPERSPACES

In this section we establish an important Theorem 7 detecting strongly $\mathcal{C}_\Gamma$-universal $\Gamma$-systems in hyperspaces. In this section, $\Gamma$ is any set and $\mathcal{C}_\Gamma$ is a class of $\Gamma$-systems.

By the hypercube of a topological space $X$ we understand the space $2^X$ of nonempty compact subsets of $X$ endowed with the Vietoris topology. This topology is generated by the sub-base consisting of the sets

$$\langle V \rangle = \{ K \in 2^X : K \subset V \} \text{ and } \langle X, V \rangle = \{ K \in 2^X : K \cap V \neq \emptyset \}$$

where $V$ is an open subset of $X$. If the topology of $X$ is generated by a metric $d$, then the Vietoris topology on $2^X$ is generated by the Hausdorff metric $d_H(A, B) = \max\{ \max_{x \in A} d(a, b), \max_{x \in B} d(b, A) \}$.

In the sequel by $2^X_{\leq \omega}$ we shall denote the subspace of $2^X$ consisting of finite non-empty subsets of $X$. By [12, 15 8.4.3] for a non-degenerate Peano continuum $X$ the subset $2^X_{\leq \omega}$ is homotopy dense in $2^X$.

We recall that a subset $A$ of a topological space $X$ is homotopy dense if there is a homotopy $h : X \times [0, 1] \to X$ such that $h(x, 0) = x$ and $h(x, t) \in A$ for all $x \in X$ and $t \in (0, 1]$.

We define a subspace $\mathcal{H} \subset 2^X$ to be finitely additive if

- $A \cup F \in \mathcal{H}$ for any $A \in \mathcal{H}$ and any finite subset $F \subset X$;
- $A \cup B \in \mathcal{H}$ for any disjoint sets $A, B \in \mathcal{H}$.

The first condition implies that each finite subset of $X$ belongs to the family

$$\text{add}(\mathcal{H}) = \{ A \in 2^X : \forall B \in \mathcal{H} \ A \cup B \in \mathcal{H} \}.$$  

For a $\Gamma$-system $\mathcal{H} = \langle 2^X, \mathcal{H}_\gamma \rangle_{\gamma \in \Gamma}$ the intersection

$$\text{add}(\mathcal{H}) = \bigcap_{\gamma \in \Gamma} \text{add}(\mathcal{H}_\gamma) \cap \text{add}(2^X \setminus \mathcal{H}_\gamma)$$

will be called the additive kernel of $\mathcal{H}$.

For example, the additive kernel of the $\omega$-system $\langle 2^X, \dim_{\leq n}(X) \rangle_{n \in \omega}$ is equal to the subspace $\dim_{\leq 0}(X)$ of all zero-dimensional compact subsets of $X$. The additive kernel of the $[0, \infty)$-system $\langle 2^X, HD_{\leq \omega}(X) \rangle_{\gamma \in [0, \infty)}$ is equal to the subspace $HD_{\leq 0}(X) \subset 2^X$ consisting of subsets of $X$ with Hausdorff dimension zero.

The following technical theorem was implicitly proved by R. Cauty in [3].

**Theorem 7.** Let $X$ be a non-degenerate Peano continuum. A $\Gamma$-system $\mathcal{H} = \langle 2^X, \mathcal{H}_\gamma \rangle_{\gamma \in \Gamma}$ is strongly $\mathcal{C}_\Gamma$-universal if:

1) for every $\gamma \in \Gamma$ the subspaces $\mathcal{H}_\gamma$ and $2^X \setminus \mathcal{H}_\gamma$ are finitely additive;
2) for every non-empty open set $U \subset X$ there is a map $\xi : Q \to 2^U \cap \text{add}(\mathcal{H})$ such that for any distinct points $x, x' \in Q$ the symmetric difference $\xi(x) \Delta \xi(x')$ is infinite;
3) for any non-empty open set $U \subset X$ and any $\Gamma$-system $\mathcal{C} = \langle C, C_\gamma \rangle_{\gamma \in \Gamma} \in \mathcal{C}_\Gamma$ there is a map $\varphi : C \to 2^U$ such that $\varphi^{-1}(3\mathcal{H}) = \mathcal{C}$.

5. THE STRONG $\sigma \mathcal{C}_\Gamma$-UNIVERSALITY OF $\Gamma$-SYSTEMS OF HYPERSPACES

In this section, we detect strongly $\sigma \mathcal{C}_\Gamma$-universal systems of the form $\langle 2^X, D_{\geq \gamma}(X) \rangle_{\gamma \in \Gamma}$ where $\Gamma \subset [0, \infty)$ and $D : 2^X_{\leq \omega} \to [0, \infty]$ is a dimension function defined on the hyperspace of a non-degenerated Peano continuum $X$.

First we establish one property of dimension functions which is formally stronger that the $\omega$-additivity.

**Lemma 2.** Let $X$ be a metrizable compact space without isolated points and $D : 2^X_{\leq \omega} \to [0, \infty]$ be a dimension function. For every non-empty open set $U \subset X$ there is a disjoint sequence $(U_n)_{n \in \omega}$ of non-empty open sets of $U$ such that
(1) \((U_n)_{n \in \omega}\) converges to some point \(x_\infty \in U\), which means that each neighborhood \(O(x_\infty)\) contains all but finitely many sets \(U_n\);

(2) for any compact subsets \(K_n \subset U_n\), \(n \in \omega\), the set \(K_\infty = \{x_\infty\} \cup \bigcup_{n\in \omega} K_n\) is compact and has dimension \(D(K_\infty) \leq \sup_{n \in \omega} D(K_n)\).

Proof. Take any non-empty open subset \(V \subset X\) with \(\text{cl}(V) \subset U\). The \(\omega\)-additivity of the dimension function \(D\) yields a sequence \((V_n)_{n \in \omega}\) of open subsets of \(V\) such that for any compact subset \(K \subset \text{cl}(\bigcup_{n \in \omega} V_n)\) has dimension \(D(K) \leq \sup_{n \in \omega} D(K \cap V_n)\).

Replacing the sets \(V_n\) by their suitable subsets, we can assume that \(\text{diam}(V_n) \to 0\) as \(n \to \infty\). In each set \(V_n\) pick a point \(x_n\). Since the space \(X\) has no isolated point, we can choose the points \(x_n\), \(n \in \omega\), to be pairwise distinct. Next, replacing the sets \(V_n\) by small neighborhoods of the points \(x_n\), we can make the sets \(V_n\), \(n \in \omega\), pairwise disjoint. By the compactness of \(X\), the sequence \((x_n)_{n \in \omega}\) contains a subsequence \((x_{n_k})_{k \in \omega}\) that converges to some point \(x_\infty \in \text{cl}(V) \subset U\). Since \(\text{diam}(V_{n_k}) \to 0\), the sequence \((V_{n_k})_{k \in \omega}\) also converges to \(x_\infty\).

It is clear that the sets \(U_k = V_{n_k}\), \(k \in \omega\), have the desired properties. \(\square\)

Now we are able to prove the principal ingredient in the proof of Theorem 1. Below \(\Gamma \subset [0, \infty)\) and \(\sigma \mathcal{C}_\Gamma\) stands for the class of inf-continuous \(\sigma\)-compact \(\Gamma\)-systems.

Theorem 8. Let \(X\) be a non-degenerate Peano continuum, \(D : 2^X \to [0, \infty]\) be a dimension function, and \(\Gamma \subset [0, \infty)\). The \(\Gamma\)-system \((2^X, D_{>\infty}(X))_{\gamma \in \Gamma}\) is strongly \(\sigma \mathcal{C}_\Gamma\)-universal if and only if each non-empty open set \(U \subset X\) for every \(\gamma \in \Gamma\) contains a compact subset \(K \subset U\) with \(D(K) \in (\gamma)_\Gamma\).

Proof. To prove the “only if” part, assume that the system \(\mathcal{D} = (2^X, D_{>\infty}(X))_{\gamma \in \Gamma}\) is strongly \(\sigma \mathcal{C}_\Gamma\)-universal.

Fix any non-empty open set \(U \subset X\) and an element \(\gamma \in \Gamma\). We need to find a compact subset \(K \subset U\) with \(D(K) \in (\gamma)_\Gamma\). To prove this inclusion, consider the three cases from the definition of the set \((\gamma)_\Gamma\).

(i) If \(\gamma < \inf(\Gamma_{>\infty})\), then \(a \in A_\gamma\) and hence \(K = f(a) \in D_{>\infty}(X)\) and \(\gamma \in D(K)\). On the other hand, for every \(\alpha \in \Gamma_{>\infty}\) we get \(a \notin A_\alpha = \emptyset\) and thus \(K = f(a) \in 2^X \setminus D_{>\alpha}(X) = D_{\leq\alpha}(X)\) and \(D(K) \leq \alpha\), which implies \(D(K) \leq \inf(\Gamma_{>\infty})\). Consequently, \(D(K) \in \gamma\), \(\inf(\Gamma_{>\infty})\) is a compact subset \(K \subset U\) with \(D(K) \in (\gamma)_\Gamma\).

(ii) \(\Gamma \ni \sup(\Gamma_{<\infty}) < \gamma = \inf(\Gamma_{>\infty})\). In this case \(a \notin A_\gamma = \emptyset\) and thus \(K = f(a) \in D_{\leq\infty}(X)\). On the other hand, \(a \in A_\alpha\) where \(\alpha = \sup(\Gamma_{<\infty}) < \gamma\) and hence \(K = f(a) \in D_{>\alpha}(X)\). Consequently, \(D(K) \subset (\sup(\Gamma_{<\infty}), \gamma) = (\gamma)_\Gamma\).

(iii) If \(\gamma = \inf(\Gamma_{>\infty})\) and \(\sup(\Gamma_{<\infty})\) is equal \(\gamma\) or does not belongs to \(\Gamma\), then for every \(\alpha \in \Gamma_{<\infty}\), we get \(a \in A_\alpha\) and thus \(K = f(a) \in D_{>\alpha}(X)\) ad \(D(K) > \alpha\). Consequently, \(D(K) \geq \sup(\Gamma_{<\infty})\). On the other hand, \(a \notin A_\gamma = \emptyset\) implies \(K = f(a) \in D_{\leq\infty}(X)\) and thus \(D(K) \subset [\sup(\Gamma_{<\infty}), \gamma) = (\gamma)_\Gamma\).

To prove the “only if” part, assume that for every non-empty open set \(U \subset X\) and every \(\gamma \in \Gamma\) there is a compact subset \(K \subset U\) with \(D(K) \in (\gamma)_\Gamma\).

The strong \(\sigma \mathcal{C}_\Gamma\)-universality of the system \(\mathcal{D}\) will follow as soon as we check the conditions (1)–(3) of Theorem 7 for the class \(\sigma \mathcal{C}_\Gamma\).

1. The monotonicity of the dimension function \(D\) implies that the subspace \(D_{>\infty}(X)\) of \(2^X\) is finitively additive. The finite additivity of the complement \(D_{\leq\infty}(X) = 2^X \setminus D_{>\infty}(X)\) follows from the finite additivity of the dimension function \(D\).

2. To establish the condition (2) of Theorem 7, fix any non-empty open set \(U \subset X\). Lemma 2 yields a sequence \((U_n)_{n \in \omega}\) of non-empty open subsets of \(U\) that converge to some point \(x_\infty \in U\) and has the property that for any compact subsets \(K_n \subset U_n\) the set \(K = \{x_\infty\} \cup \bigcup_{n \in \omega} K_n\) is compact and has dimension \(D(K) \leq \sup_{n \in \omega} D(K_n)\). Each set \(U_n\) contains a topological copy of the interval \([0, 1]\), so we can find a topological embedding \(\xi_n : [-1, 1] \to U_n\).
Let \( \nu : \omega \to \omega \) be any function such that the preimage \( \nu^{-1}(n) \) of every \( n \in \omega \) is infinite. Define a map \( \xi : Q \to 2^U \) assigning to each \( \vec{t} = \langle t_n \rangle_{n \in \omega} \in Q \) the compact subset
\[
\xi(\vec{t}) = \{x_\infty\} \cup \{\alpha_n(t_n): n \in \omega\}
\]
of \( U \) having a unique non-isolated point \( x_\infty \). The equality \( D(\emptyset) = 0 \) and the finite additivity of the dimension function \( D \) implies that \( D(F) = 0 \) for each finite subset \( F \subseteq X \). The choice of the sequence \( \langle U_n \rangle \) guarantees that \( D(\xi(\vec{t})) = 0 \) and thus
\[
\xi(Q) \subseteq D_{\leq 0}(X) \subseteq \text{add}(D).
\]
The choice of the function \( \nu \) guarantees that \( \xi(\vec{t}) \Delta \xi(\vec{u}) \) is infinite for any distinct vectors \( \vec{t}, \vec{u} \in Q \).

3. To check the condition (3) of Theorem 1, fix any non-empty open set \( U \subseteq X \) and a \( \Gamma \)-system \( A = \langle A_\gamma, \gamma \in \Gamma \rangle \in \sigma\Gamma \). Each set \( A_\gamma, \gamma \in \Gamma \), being \( \sigma \)-compact, can be written as the countable union \( A_\gamma = \bigcup_{n \in \omega} A_{\gamma,n} \) of an increasing sequence \( \langle A_{\gamma,n} \rangle_{n \in \omega} \) of compact subsets of \( A \). Let \( D \) be a countable subset of \( \Gamma \) meeting each half-interval \([\gamma, \gamma + \varepsilon)\) where \( \gamma \in \Gamma \) and \( \varepsilon > 0 \).

Apply Lemma 2 to find a disjoint family \( \langle U_d \rangle_{d \in D} \) of non-empty open subsets of \( U \) such that
- \( \langle U_d \rangle_{d \in D} \) converges to some point \( x_\infty \in U \) in the sense that each neighborhood \( O(x_\infty) \) contains all but finitely many sets \( U_d, d \in D \);
- for any compact sets \( K_d \subseteq U_d \) the set \( K = \{x_\infty\} \cup \bigcup_{d \in D} K_d \) is compact and has dimension \( D(K) \leq \sup_{d \in D} D(K_d) \).

For every \( d \in D \) use Lemma 2 once more and find a disjoint family \( \langle U_{d,n} \rangle_{n \in \omega} \) of non-empty open subsets of \( U \) such that
- \( \langle U_{d,n} \rangle_{n \in \omega} \) converges to some point \( x_d \in U_d \);
- for any compact sets \( K_n \subseteq U_{d,n} \) the set \( K_d = \{x_d\} \cup \bigcup_{n \in \omega} K_n \) is compact and has dimension \( D(K_d) = \sup_{n \in \omega} D(K_n) \).

By our assumption, for every \( d \in D \) and \( n \in \omega \) we can find a compact subset \( K_{d,n} \subseteq U_{d,n} \) with \( D(K_{d,n}) \in (d)_\gamma \). Using the homotopical density of the subspace \( 2^X_{\leq \omega} \) of finite subsets in \( 2^X \), construct a map \( \kappa_{d,n} : A \to 2^X \) such that \( \kappa_{d,n}(a) = K_{d,n} \) for every \( a \in A_{d,n} \) and \( \kappa_{d,n}(a) \) is a finite subset of \( U_{d,n} \) for every \( a \in A \setminus A_{d,n} \).

Now for every \( a \in A \) and \( d \in D \) consider the compact subset
\[
\kappa_d(a) = \{x_d\} \cup \bigcup_{n \in \omega} \kappa_{d,n}(a) \subseteq U_d
\]
having dimension
\[
D(\kappa_d(a)) = \sup_{n \in \omega} D(\kappa_{d,n}(a)).
\]

The choice of the sequence \( \langle U_d \rangle_{d \in D} \) ensures that
\[
\kappa(a) = \{x_\infty\} \cup \bigcup_{d \in D} \kappa_d(a)
\]
is a compact subset of \( U \) with dimension
\[
D(\kappa(a)) = \sup_{d \in D} D(\kappa_d(a)) = \sup_{d \in D} \{D(\kappa_{d,n}(a)): n \in \omega\}.
\]

It is easy to prove that the map
\[
\kappa : A \to 2^U, \quad \kappa : a \mapsto \kappa(a),
\]
is continuous. It remains to check that \( \kappa^{-1}(D_{> \gamma}(X)) = A_\gamma \) for all \( \gamma \in \Gamma \).

If \( a \in A \setminus A_\gamma \), then for every \( d \geq \gamma \) in \( D \) the inclusion \( a \in A \setminus A_d \) implies \( \kappa_{d,n}(a) \in 2^X_{\leq \omega} \). In this case \( D(\kappa_d(a)) \leq \sup_{n \in \omega} D(\kappa_{d,n}(a)) = 0 \leq \gamma \). On the other hand, for every \( d < \gamma \) the inclusions \( D(K_{d,n}) \in (d)_\gamma \subseteq [0, \gamma] \), \( n \in \omega \), and the choice of the sequence \( \langle U_{d,n} \rangle_{n \in \omega} \) imply \( D(\kappa_d(a)) \leq \sup_{n \in \omega} D(\kappa_{d,n}(a)) \leq \gamma \).

Now the choice of the sequence \( \langle U_d \rangle_{d \in D} \) guarantees that
\[
D(\kappa(a)) \leq \sup_{d \in D} D(\kappa_{d,n}(a)) \leq \gamma
\]
and hence
\[\kappa(a) \in D_{\leq \gamma}(X) = 2^X \setminus D_{\geq \gamma}(X).\]

Now assume that \(a \in A_n\) and hence \(a \in A_{\gamma,n}\) for some \(n \in \omega\). If \(\gamma < \inf(I_{\geq \gamma})\), then \(\gamma \in D\) and \(D(K_{\gamma,n}) \in (\gamma, \inf(I_{\geq \gamma}))\). Since \(K_{\gamma,n} \subset \kappa(a)\), we conclude that \(D(\kappa(a)) \geq D(K_{\gamma,n}) > \gamma\) and thus \(\kappa(a) \in D_{\geq \gamma}(X)\).

Next, assume that \(\gamma = \inf(I_{\geq \gamma})\). In this case \(\gamma = \inf(D_{\geq \gamma})\) and hence \(A_{\gamma} = \bigcup_{d \in D_{\geq \gamma}} A_d\). It follows that \(a \in A_{d,n}\) for some \(d \in D_{\geq \gamma}\) and \(n \in \omega\). Since \(\kappa(a) \supset K_{d,n}\) and \(D(K_{d,n}) \in (d, \gamma, +\infty)\), we conclude that \(D(\kappa(a)) \geq D(K_{d,n}) > \gamma\). So, again \(\kappa(a) \in D_{\geq \gamma}(X)\).

The following characterization theorem implies Theorem 1 announced in the Introduction.

**Theorem 9.** Let \(X\) be a topological space, \(D : 2^X \rightarrow [0, \infty]\) be a dimension function, and \(\Gamma \subset [0, \infty)\) be a subset. The \(\Gamma\)-system \(\langle 2^X, D_{\geq \gamma}(X) \rangle_{\gamma \in \Gamma}\) is homeomorphic to the model \(\sigma\epsilon_\Gamma\)-absorbing \(\Gamma\)-system \(\langle Q^\mathbb{Q}, Q^\mathbb{Q}_{\leq \gamma} \times B(Q^\mathbb{Q}_{\geq \gamma}) \rangle_{\gamma \in \Gamma}\) if and only if

1. \(X\) is a non-degenerate Peano continuum,
2. each space \(D_{\geq \gamma}(X), \gamma \in \Gamma\), is \(\sigma\)-compact, and
3. each non-empty open set \(U \subset X\) for every \(\gamma \in \Gamma\) contains a compact subset \(K \subset U\) with \(D(K) \in (\gamma, \Gamma)\).

**Proof.** To prove the “only if” part, assume that the \(\Gamma\)-system \(D = \langle 2^X, D_{\geq \gamma}(X) \rangle_{\gamma \in \Gamma}\) is homeomorphic to the model \(\Gamma\)-system \(\Sigma_{\Gamma} = \langle Q^\mathbb{Q}, Q^\mathbb{Q}_{\leq \gamma} \times B(Q^\mathbb{Q}_{\geq \gamma}) \rangle_{\gamma \in \Gamma}\). Since \(2^X\) is homeomorphic to \(Q^\mathbb{Q}\), we may apply the Curtis-Shori Theorem \([\text{II}]\) and conclude that \(X\) is a non-degenerate Peano continuum.

Since each space \(\Sigma_{\gamma} = Q^\mathbb{Q}_{\leq \gamma} \times B(Q^\mathbb{Q}_{\geq \gamma}), \gamma \in \Gamma\), is \(\sigma\)-compact, so is its topological copy \(D_{\geq \gamma}(X)\).

The \(\Gamma\)-system \(D\), being homeomorphic to the model \(\sigma\epsilon_\Gamma\)-absorbing \(\Gamma\)-system \(\Sigma_{\Gamma}\), is strongly \(\Gamma\)-universal. Now Theorem \([\text{II}]\) guarantees that for every \(\gamma \in \Gamma\) each non-empty open subset \(U \subset X\) contains a compact subset \(K \subset U\) with \(D(K) \in (\gamma, \Gamma)\).

Next, we prove the “if” part. Assume that the conditions (1)–(3) are satisfied. We shall prove that the \(\Gamma\)-system \(D = \langle 2^X, D_{\geq \gamma}(X) \rangle_{\gamma \in \Gamma}\) is homeomorphic to the Hilbert cube \(Q\). By Theorem \([\text{II}]\) the \(\Gamma\)-system \(D\) is strongly \(\sigma\epsilon_\Gamma\)-universal. It is clear that this \(\Gamma\)-system is inf-continuous. By the condition (2), it is \(\sigma\)-compact. Hence \(D \in \sigma\epsilon_\Gamma\).

Let \(D \subset \Gamma\) be countable subset that meets each half-interval \([\gamma, \gamma + \varepsilon]\) where \(\gamma \in \Gamma\) and \(\varepsilon > 0\). It follows that \(\bigcup_{\gamma \in D} D_{\geq \gamma}(X) = \bigcup_{\gamma \in D} D_{\geq \gamma}(X) \subset D_{\geq 0}(X)\) is a \(\sigma\)-Z-set in \(2^X\), being a \(\sigma\)-compact subset of \(2^X\) that has empty intersection with the homotopy dense subset \(2^\mathbb{N}_\omega \subset D_{\leq 0}(X)\) on \(2^X\). So, we can find a countable sequence \(\langle Z_n \rangle_{n \in \omega}\) of \(\sigma\)-Z-sets in \(2^X\) such that \(\bigcup_{n \in \omega} Z_n \supset \bigcup_{\gamma \in \Gamma} D_{\geq \gamma}(X)\). Since \(D \in \sigma\epsilon_\Gamma\), we get \(Z_n \cap D \in \sigma\epsilon_\Gamma\) for all \(n \in \omega\). This completes the proof of the \(\sigma\epsilon_\Gamma\)-absorbing property of the \(\Gamma\)-system \(D\). Since \(2^X\) is homeomorphic to the Hilbert cube, Theorem \([\text{II}]\) ensures that \(D\) is homeomorphic to the model \(\Gamma\)-system \(\Sigma_{\Gamma}\).

6. Mean Value Theorem for Hausdorff dimension

In this section we shall prove Theorem \([\text{II}]\). First, we recall shortly the definitions of the Hausdorff measure and dimension. Given a complete separable metric space \(E\) and two non-negative real numbers \(s, \varepsilon\), consider the number
\[H^s_\varepsilon(E) = \inf_{B} \sum_{B \in B} (\text{diam}B)^s,\]
where infimum is taken over all \(\varepsilon\)-covers \(B\) of \(E\), i.e., cover of \(E\) by sets of diameter \(\leq \varepsilon\). Since \(X\) is separable, we can restrict ourselves by countable covers by closed subsets of diameter \(\leq \varepsilon\).

The limit \(H^s(E) = \lim_{\varepsilon \to 0} H^s_\varepsilon(E)\) is called the \(s\)-dimensional Hausdorff measure of \(E\). It is known that there is a unique finite or infinite number \(\dim_H(E)\) called the Hausdorff dimension of \(E\) and denoted by \(\dim_H(E)\) such that \(H^s(E) = \infty\) for all \(s < \dim_H(E)\) and \(H^s(E) = 0\) for all \(s > \dim_H(E)\), see \([9], [11]\).

Let \((X, d)\) be a separable complete metric space. Theorem \([\text{II}]\) will be proved as soon as for every positive real number \(s < \dim_H(X)\) we shall find a compact subset \(K \subset X\) with Hausdorff dimension \(\dim_H(K) = s\).
It follows from \( s < \dim_H(E) \) that \( \mathcal{H}^s(E) = \infty \) and there exists \( 0 < \delta < 1 \) with
\[
\mathcal{H}^s_\delta(E) = k_0 > \frac{\delta^s}{2^{s-1}} = \left( \frac{\delta}{2} \right)^s + \left( \frac{\delta}{4} \right)^s + \left( \frac{\delta}{8} \right)^s + \cdots .
\] (0)

We define inductively a decreasing sequence \( \{E_i\}_{i=1}^\infty \) of closed subsets of \( E \). Let \( E_1 = E \). Consider \( \mathcal{H}^s_\delta(E_1) \). Two cases are possible (taking into account the definition of Hausdorff measure):

- \( \mathcal{H}^s_\delta(E_1) = k_0 \). In this case we take \( E_2 = E_1 \).
- \( \mathcal{H}^s_\delta(E_1) > k_0 \). Therefore we can choose a closed \( \delta/2 \)-cover \( \{U_1, \ldots, U_{m_1}, \ldots\} \) of the set \( E_1 \), (without loss of generality assume that this cover is ordered so that \( \text{diam}(U_{i+1}) \leq \text{diam}(U_i) \) for all \( i \)), such that

\[
\mathcal{H}^s_\delta(E_1) \leq \sum_{i} (\text{diam}(U_i))^s < \mathcal{H}^s_\delta(E_1) + (\delta/2)^s .
\] (1)

Find a finite number \( m_1 \) such that
\[
k_0 \leq \sum_{i=1}^{m_1} (\text{diam}(U_i))^s \leq k_0 + (\delta/2)^s .
\] (2)

Then take \( E_2 = \bigcup_{i=1}^{m_1} E_1 \cap U_i \).

Now we need to estimate \( \mathcal{H}^s_\delta(E_2) \)(obviously the second case is interesting). For this we put \( E'_2 = \bigcup_{i>m_1} E_1 \cap U_i \) and note that
\[
\mathcal{H}^s_\delta(E_1) \leq \mathcal{H}^s_\delta(E_2) + \mathcal{H}^s_\delta(E'_2) .
\] (3)

On the other hand
\[
\sum_{i} (\text{diam}(U_i))^s = \sum_{i \leq m_1} (\text{diam}(U_i))^s + \sum_{i > m_1} (\text{diam}(U_i))^s .
\] (4)

Consider the real numbers \( \varepsilon_1 = \sum_{i} (\text{diam}(U_i))^s - \mathcal{H}^s_\delta(E_1) \), \( \varepsilon_2 = \sum_{i=m_1}^{m_1} (\text{diam}(U_i))^s - \mathcal{H}^s_\delta(E_2) \), \( \varepsilon_2' = \sum_{i>m_1} (\text{diam}(U_i))^s - \mathcal{H}^s_\delta(E'_2) \) and observe that \( 0 \leq \varepsilon_1 < (\delta/2)^s \) by (1), and \( \varepsilon_2, \varepsilon_2' \geq 0 \). Therefore (3) and (4) yield \( \varepsilon_1 \geq \varepsilon_2 + \varepsilon_2' \) and hence \( 0 \leq \varepsilon_2 < (\delta/2)^s \). Taking into account (2), we have:
\[
k_0 - (\delta/2)^s \leq \mathcal{H}^s_\delta(E_2) \leq k_0 + (\delta/2)^s .
\] (5)

Now denote \( \mathcal{H}^s_\delta(E_2) = k_1 \). From (0) and (5) it follows that \( 0 < k_1 < \infty \). By the definition of Hausdorff measure we have \( 0 < \mathcal{H}^s(E_2) \leq \infty \), that in turn implies \( \dim_H(E_2) \geq s \). It allows us to make the following inductive step.

Consider now \( \mathcal{H}^s_\delta(E_2) \). If \( \mathcal{H}^s_\delta(E_2) = k_1 \), then take \( E_3 = E_2 \). If \( \mathcal{H}^s_\delta(E_2) > k_1 \), then similarly to the described above we find a closed \( \delta/4 \)-cover \( \{U_1, \ldots, U_{m_2}, \ldots\} \) of \( E_2 \), such that
\[
\mathcal{H}^s_\delta(E_2) \leq \sum_{i} (\text{diam}(U_i))^s < \mathcal{H}^s_\delta(E_2) + (\delta/4)^s .
\]

Find a finite number \( m_2 \) such that
\[
k_1 \leq \sum_{i=1}^{m_2} (\text{diam}(U_i))^s \leq k_1 + (\delta/4)^s .
\]

Let \( E_3 = \bigcup_{i=1}^{m_2} E_2 \cap U_i \). As above, we can to estimate \( \mathcal{H}^s_\delta(E_3) \). We obtain:
\[
k_1 - (\delta/4)^s \leq \mathcal{H}^s_\delta(E_3) \leq k_1 + (\delta/4)^s .
\]

Or, taking into account (5):
\[
k_0 - (\delta/2)^s - (\delta/4)^s \leq \mathcal{H}^s_\delta(E_3) \leq k_0 + (\delta/2)^s + (\delta/4)^s .
\]
Again we can state that \( \dim_H(E_3) \geq s \) and continue inductive process by constructing in similar way 
\( E_4, E_5, \ldots, E_n, \ldots \), for which we obtain in general case the estimate:

\[
k_0 - (\delta/2)^s - \cdots - (\delta/2^{n-1})^s \leq H_{\delta/2^{n-1}}(E_n) \leq k_0 + (\delta/2)^s + \cdots + (\delta/2^{n-1})^s.
\]

It follows that \( E_1 \supseteq E_2 \supseteq E_3 \supseteq \ldots \) is a decreasing sequence of closed subsets of \( X \) with compact intersection \( K = \lim_{n \to \infty} E_n = \bigcap_{n=1}^{\infty} E_n \). Using the continuity of the measure \( H^s \) we obtain:

\[ H^s(K) = \lim_{n \to \infty} H^s(E_n) = \lim_{n \to \infty} \lim_{i \to \infty} H^s_{\delta/2^{n-1}}(E_n) = \lim_{n \to \infty} H^s_{\delta/2^{n-1}}(E_n). \]

Additionally using (6) we obtain the estimate:

\[ k_0 - \frac{\delta^s}{2^{s-1}} \leq H^s(F) \leq k_0 + \frac{\delta^s}{2^{s-1}}. \]

Taking into account (0) we can state that \( \dim_H(F) = s \).

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