Dynamics on geometrically finite hyperbolic manifolds with applications to Apollonian circle packings and beyond

Hee Oh

Abstract. We present recent results on counting and distribution of circles in a given circle packing invariant under a geometrically finite Kleinian group and discuss how the dynamics of flows on geometrically finite hyperbolic 3 manifolds are related. Our results apply to Apollonian circle packings, Sierpinski curves, Schottky dances, etc.

Mathematics Subject Classification (2000). Primary 37A17, Secondary 37A40

Keywords. Circles, Apollonian circle packings, geometrically finite groups, Patterson-Sullivan density

1. Introduction

Let $G$ be a connected semisimple Lie group and $\Gamma < G$ a discrete subgroup with finite co-volume. Dynamics of flows on the homogeneous space $\Gamma \backslash G$ have been studied intensively over the last several decades and brought many surprising applications in various fields notably including analytic number theory, arithmetic geometry and Riemannian geometry (see [45], [58], [12], [18], [32], [78], [41], [74], [16], [21], [22], [76], [15], [49], [33], [75], [25], [27], [26], [66], etc.) The assumption that the volume of $\Gamma \backslash G$ is finite is crucial in most developments in the ergodic theory for flows on $\Gamma \backslash G$, as many basic ergodic theorems fail in the setting of an infinite measure space. It is unclear what kind of measure theoretic and topological rigidity for flows on $\Gamma \backslash G$ can be expected for a general discrete subgroup $\Gamma$.

In this article we consider the situation when $G$ is the isometry group of the real hyperbolic space $\mathbb{H}^n$, $n \geq 2$, and $\Gamma < G$ is a geometrically finite discrete subgroup. In such cases we have a rich theory of the Patterson-Sullivan density and the structure of a fundamental domain for $\Gamma$ in $\mathbb{H}^n$ is well understood. Using these we obtain certain equidistribution results for specific flows on the unit tangent bundle $T^1(\Gamma \backslash \mathbb{H}^n)$ and apply them to prove results on counting and equidistribution for circles in a given circle packing of the plane (and also of the sphere) invariant under geometrically finite groups.

There are numerous natural questions which arise from the analogy with the finite volume cases and most of them are unsolved. We address some of them in
the last section. We remark that an article by Sarig [61] discusses related issues but for geometrically infinite surfaces.

Acknowledgement: I would like to thank Peter Sarnak for introducing Apollonian circle packings to me and for the encouragement to work on this project. I am grateful to Curt McMullen for showing me the picture of Sierpinski curve which led me to think about more general circle packings beyond Apollonian ones, as well as for many valuable discussions. I thank my collaborators Nimish Shah and Alex Kontorovich for the joint work. I also thank Marc Burger and Gregory Margulis for carefully reading an earlier draft and making many helpful comments. Finally I thank my family for their love and support always.

2. Preliminaries

We review some of basic definitions as well as set up notations. Let $G$ be the identity component of the isometry group of the real hyperbolic space $\mathbb{H}^n$, $n \geq 2$. Let $\Gamma < G$ be a torsion-free discrete subgroup. We denote by $\partial_\infty(\mathbb{H}^n)$ the geometric boundary of $\mathbb{H}^n$. The limit set $\Lambda(\Gamma)$ of $\Gamma$ is defined to be the set of accumulation points of an orbit of $\Gamma$ in $\mathbb{H}^n \cup \partial_\infty(\mathbb{H}^n)$. As $\Gamma$ acts on $\mathbb{H}^n$ properly discontinuously, $\Lambda(\Gamma)$ lies in $\partial_\infty(\mathbb{H}^n)$. Its complement $\Omega(\Gamma) := \partial_\infty(\mathbb{H}^n) - \Lambda(\Gamma)$ is called the domain of discontinuity for $\Gamma$.

An element $g \in G$ is called parabolic if it fixes a unique point in $\partial_\infty(\mathbb{H}^n)$ and loxodromic if it fixes two points in $\partial_\infty(\mathbb{H}^n)$. A limit point $\xi \in \Lambda(\Gamma)$ is called a parabolic fixed point if it is fixed by a parabolic element of $\Gamma$ and called a radial limit point (or a conical limit point or a point of approximation) if for some geodesic ray $\beta$ tending to $\xi$ and some point $x \in \mathbb{H}^n$, there is a sequence $\gamma_i \in \Gamma$ with $\gamma_i x \to \xi$ and $d(\gamma_i x, \beta)$ is bounded, where $d$ denotes the hyperbolic distance. A parabolic fixed point $\xi$ is called bounded if $\text{Stab}_{\Gamma}(\xi) \setminus (\Lambda(\Gamma) - \{\xi\})$ is compact.

The convex core $C_\Gamma$ of $\Gamma$ is defined to be the minimal convex set in $\mathbb{H}^n$ mod $\Gamma$ which contains all geodesics connecting any two points in $\Lambda(\Gamma)$. A discrete subgroup $\Gamma$ is called geometrically finite if the unit neighborhood of its convex core has finite volume and called convex co-compact if its convex core is compact. It is clear that a (resp. co-compact) lattice in $G$ is geometrically finite (resp. convex co-compact). Bowditch showed [5] that $\Gamma$ is geometrically finite if and only if $\Lambda(\Gamma)$ consists entirely of radial limit points and bounded parabolic fixed points. It is further equivalent to saying that $\Gamma$ is finitely generated for $n = 2$, and that $\Gamma$ admits a finite sided fundamental domain in $\mathbb{H}^3$ for $n = 3$. We refer to [5] for other equivalent definitions.

$\Gamma$ is called elementary if $\Lambda(\Gamma)$ consists of at most two points, or equivalently, $\Gamma$ has an abelian subgroup of finite index.

We denote by $0 \leq \delta_\Gamma \leq n - 1$ the critical exponent of $\Gamma$, that is, the abscissa of convergence of the Poincare series of $\Gamma$:

$$P_\Gamma(s) := \sum_{\gamma \in \Gamma} e^{-s d(\alpha, \gamma \alpha)}$$
where \( o \in \mathbb{H}^n \). For a non-elementary group \( \Gamma \), \( \delta \) is positive and Sullivan [71] showed that for \( \Gamma \) geometrically finite, \( \delta \) is equal to the Hausdorff dimension of the limit set \( \Lambda(\Gamma) \).

For \( \xi \in \partial_{\infty}(\mathbb{H}^n) \) and \( y_1, y_2 \in \mathbb{H}^n \), the Busemann function \( \beta_{\xi}(y_1, y_2) \) measures a signed distance between horospheres passing through \( y_1 \) and \( y_2 \) based at \( \xi \):

\[
\beta_{\xi}(y_1, y_2) = \lim_{t \to \infty} d(y_1, \xi_t) - d(y_2, \xi_t)
\]

where \( \xi_t \) is a geodesic ray toward \( \xi \).

For a vector \( u \) in the unit tangent bundle \( T^1(\mathbb{H}^n) \), we define \( u^\pm \in \partial_{\infty}(\mathbb{H}^n) \) to be the two endpoints of the geodesic determined by \( u \):

\[
u_x, o(s) := \frac{1}{\sum_{\gamma \in \Gamma} e^{-sd(o, \gamma o)}} \sum_{\gamma \in \Gamma} e^{-sd(x, \gamma o)} \delta_{\gamma o}
\]

where \( \delta_{\gamma o} \) denotes the Dirac measure at \( \gamma o \).

Consider the Laplacian \( \Delta \) on \( \mathbb{H}^n \). In the upper half-space coordinates \( \mathbb{H}^n = \{(x_1, \ldots, x_{n-1}, y) : y > 0\} \) with the metric \( \sqrt{dx_1^2 + \cdots + dx_{n-1}^2 + dy^2} \), it is given as

\[
\Delta = -y^2 \left( \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_{n-1}^2} + \frac{\partial^2}{\partial y^2} \right) + (n - 2)y \frac{\partial}{\partial y}
\]
(strictly speaking, this is the negative of the usual hyperbolic Laplacian). Sullivan showed that
\[ \phi_\Gamma(x) := |\nu_x| \]
is an eigenfunction for \( \Delta \) with eigenvalue \( \delta_\Gamma(n - 1 - \delta_\Gamma) \). From the \( \Gamma \)-invariance of the Patterson-Sullivan density \( \{\nu_x\} \), \( \phi_\Gamma \) is a function on \( \Gamma \setminus \mathbb{H}^n \). Sullivan further showed that if \( \Gamma \) geometrically finite and \( \delta_\Gamma > (n - 1)/2 \), \( \phi_\Gamma \) belongs to \( L^2(\Gamma \setminus \mathbb{H}^n) \) and is a unique (up to a constant multiple) positive eigenfunction with the smallest eigenvalue \( \delta_\Gamma(n - 1 - \delta_\Gamma) \) (cf. [73]). Combined with a result of Yau [77], it follows that \( \delta_\Gamma = n - 1 \) if and only if \( \Gamma \) is a lattice in \( G \).

**Bowen-Margulis-Sullivan measure:** Fixing the Patterson-Sullivan density \( \{\nu_x\} \), the Bowen-Margulis-Sullivan measure \( m_{\text{BMS}} \) (\([6], [46], [72]\)) is the induced measure on \( T^1(\Gamma \setminus \mathbb{H}^n) \) of the following \( \Gamma \)-invariant measure on \( T^1(\mathbb{H}^n) \):
\[
d\tilde{m}_{\text{BMS}}(u) = e^{\delta_\Gamma \beta_+ (x, \pi(u))} e^{\delta_\Gamma \beta_- (x, \pi(u))} d\nu_x(u^+) d\nu_x(u^-) dt
\]
where \( x \in \mathbb{H}^n \).

It follows from the conformality of \( \{\nu_x\} \) that this definition is independent of the choice of \( x \). The measure \( m_{\text{BMS}} \) is invariant under the geodesic flow and is supported on the non-wandering set \( \{u \in T^1(\Gamma \setminus \mathbb{H}^n) : u^+ \in \Lambda(\Gamma)\} \) of the geodesic flow. Sullivan showed that for \( \Gamma \) geometrically finite, the total mass \( |m_{\text{BMS}}| \) is finite and the geodesic flow is ergodic with respect to \( m_{\text{BMS}} \) [72]. This is a very important point for the ergodic theory on geometrically finite hyperbolic manifolds, since despite of the fact that the Liouville measure is infinite, we do have a finite measure on \( T^1(\Gamma \setminus \mathbb{H}^n) \) which is invariant and ergodic for the geodesic flow. Rudolph [60] showed that the geodesic flow is even mixing with respect to \( m_{\text{BMS}} \).

3. Counting and distribution of circles in the plane

A circle packing in the plane \( \mathbb{C} \) is simply a union of circles. As circles may intersect with each other beyond tangency points, our definition of a circle packing is more general than what is usually thought of. For a given circle packing \( \mathcal{P} \) in the plane, we discuss questions on counting and distribution of small circles in \( \mathcal{P} \). A natural size of a circle is measured by its radius. We will use the curvature (=the reciprocal of the radius) of a circle instead.

We suppose that \( \mathcal{P} \) is infinite and that \( \mathcal{P} \) is locally finite in the sense that for any \( T > 0 \), there are only finitely many circles of curvature at most \( T \) in any fixed bounded region of the plane. See Fig. [1] [6] and [8] for examples of locally finite packings.

For a bounded region \( E \) in the plane \( \mathbb{C} \), we consider the following counting function:
\[
N_T(\mathcal{P}, E) := \#\{C \in \mathcal{P} : C \cap E \neq \emptyset, \ \text{Curv}(C) < T\}
\]
where \( \text{Curv}(C) \) denotes the curvature of \( C \). The local finiteness assumption is so that \( N_T(\mathcal{P}, E) < \infty \) for any bounded \( E \). We ask if there is an asymptotic for
Consider the upper half space model $H^3 = \{(z, r) : z \in \mathbb{C}, r > 0\}$ with the hyperbolic metric given by $\sqrt{|dz|^2 + dr^2}$. An elementary but helpful observation is that if we denote by $\hat{C} \subset H^3$ the convex hull of $C$, that is, the northern hemisphere above $C$, then $N_T(P, E)$ is equal to the number of hemispheres of height at most $T^{-1}$ in $H^3$ whose boundaries lie in $P$ and intersect $E$, as the radius of a circle is same as the height of the corresponding hemisphere.

Let $\Gamma < \text{PSL}_2(\mathbb{C})$ be a geometrically finite discrete subgroup and fix a $\Gamma$-invariant Patterson-Sullivan density $\{\nu_x : x \in H^3\}$.

In order to present our theorem on the asymptotic of $N_T(P, E)$ for $P$ invariant under $\Gamma$, we introduce two new invariants associated to $\Gamma$ and $P$. The first one is a Borel measure on $\mathbb{C}$ which depends only on $\Gamma$.

**Definition 3.1.** Define a Borel measure $\omega_\Gamma$ on $\mathbb{C}$: for $\psi \in C_c(\mathbb{C})$,

$$\omega_\Gamma(\psi) = \int_{z \in \mathbb{C}} \psi(z)e^{\delta_\Gamma \beta_s(x,z+j)} \, d\nu_x(z)$$

where $j = (0,1) \in \mathbb{H}^3$ and $x \in \mathbb{H}^3$. By the conformal property of $\{\nu_x\}$, this definition is independent of the choice of $x \in \mathbb{H}^3$.

Note that $\omega_\Gamma$ is supported on $\Lambda(\Gamma) \cap \mathbb{C}$ and in particular that $\omega_\Gamma(E) > 0$ if the interior of $E$ intersects $\Lambda(\Gamma) \cap \mathbb{C}$ non-trivially. We compute:

$$d\omega_\Gamma = (|z|^2 + 1)^{\delta_\Gamma} \, d\nu_j.$$
Definition 3.2 (The $\Gamma$-skinning size of $\mathcal{P}$). For a circle packing $\mathcal{P}$ invariant under $\Gamma$, we define:

$$\text{sk}_\Gamma(\mathcal{P}) := \sum_{i \in I} \int_{s \in \text{Stab}_\Gamma(C_i^\dagger)} e^{\delta \beta_+(x, \pi(s))} \nu_x(s^+)$$

where $x \in \mathbb{H}^3$, $\{C_i : i \in I\}$ is a set of representatives of $\Gamma$-orbits in $\mathcal{P}$ and $C_i^\dagger \subset T^1(\mathbb{H}^3)$ is the set of unit normal vectors to the convex hull $\hat{C}_i$ of $C_i$. Again by the conformal property of $\{\nu_x\}$, the definition of $\text{sk}_\Gamma(\mathcal{P})$ is independent of the choice of $x$ and the choice of representatives $\{C_i\}$.

We remark that the value of $\text{sk}_\Gamma(\mathcal{P})$ can be zero or infinite in general and we do not assume any condition on $\text{Stab}_\Gamma(C_i^\dagger)$’s (they may even be trivial). By the interior of a circle $C$, we mean the open disk which is enclosed by $C$. We then have the following:

Theorem 3.3 ([51]). Let $\Gamma$ be a non-elementary geometrically finite discrete subgroup of $\text{PSL}_2(\mathbb{C})$ and let $\mathcal{P} = \bigcup_{i \in I} \Gamma(C_i)$ be an infinite, locally finite, and $\Gamma$-invariant circle packing with finitely many $\Gamma$-orbits.

Suppose one of the following conditions hold:

1. $\Gamma$ is convex co-compact;
2. all circles in $\mathcal{P}$ are mutually disjoint;
3. $\bigcup_{i \in I} C_i^\circ \subset \Omega(\Gamma)$ where $C_i^\circ$ denotes the interior of $C_i$.

For any bounded region $E$ of $\mathbb{C}$ whose boundary is of zero Patterson-Sullivan measure, we have

$$N_T(\mathcal{P}, E) \sim \frac{\text{sk}_\Gamma(\mathcal{P})}{\delta(T) \cdot |m^\text{BMS}_T|} \cdot \omega_\Gamma(E) \cdot T^{\delta(T)} \quad \text{as} \quad T \to \infty$$

and $0 < \text{sk}_\Gamma(\mathcal{P}) < \infty$.

Remark 3.4. 1. If $\Gamma$ is Zariski dense in $\text{PSL}_2(\mathbb{C})$, considered as a real algebraic group, any real algebraic curve has zero Patterson-Sullivan measure [23, Cor. 1.4]. Hence the above theorem applies to any Borel subset $E$ whose boundary is a countable union of real algebraic curves.

2. We call the complement in $\hat{\mathcal{C}}$ of the set $\bigcup_{i \in I} \Gamma(C_i^\circ)$ the residual set of $\mathcal{P}$. The condition (3) above is then equivalent to saying that $\Lambda(\Gamma)$ is contained in the residual set of $\mathcal{P}$.

3. If we denote by $H_{\infty}(j)$ the contracting horosphere based at $\infty$ in $T^1(\mathbb{H}^3)$ which consists of all upward normal unit vectors on $\mathbb{C} + j = \{(z, 1) : z \in \mathbb{C}\}$, we can alternative write the measure $\omega_\Gamma$ as follows:

$$\omega_\Gamma(\psi) = \int_{u \in H_{\infty}(j)} \psi(u^-) e^{\delta \beta_-(x, \pi(u))} \nu_x(u^-)$$
and recognize that $\omega_\Gamma$ is the projection of the conditional of the Bowen-Margulis-Sullivan measure $\tilde{m}_{BMS}$ on the horosphere $H_\infty(j)$ to $C$ via the map $u \mapsto u^-$. It is worthwhile to note that the hyperbolic metric on $C + j$ is precisely the Euclidean metric.

4. Suppose that circles in $P$ are disjoint possibly except for tangency points and that $\Lambda(\Gamma)$ is equal to the residual set of $P$. If $\infty$ is either in $\Omega(\Gamma)$ (that is, $P$ is bounded) or a parabolic fixed point for $\Gamma$, then $\delta_\Gamma$ is equal to the circle packing exponent $e_P$ given by

$$e_P = \inf \left\{ s : \sum_{C \in P} r(C)^s < \infty \right\} = \sup \left\{ s : \sum_{C \in P} r(C)^s = \infty \right\}$$

where $r(C)$ denotes the radius of $C$ \cite{54}. This extends the earlier work of Boyd \cite{7} on bounded Apollonian circle packings.

We discuss some concrete circle packings to which our theorem applies.

3.1. Apollonian circle packings in the plane. Apollonian circle packings are one of the most beautiful circle packings whose construction can be described in a very simple manner based on an old theorem of Apollonius (262-190 BC). It says that given three mutually tangent circles in the plane, there are exactly two circles which are tangent to all the three circles.
In order to construct an Apollonian circle packing, we start with four mutually tangent circles. See Fig. 2 for possible configurations. By Apollonius’ theorem, there are precisely four new circles that are tangent to three of the four circles. Continuing to repeatedly add new circles tangent to three of the circles from the previous generations, we arrive at an infinite circle packing, called an Apollonian circle packing. See Fig. 4 and 8 for examples of Apollonian circle packings where each circle is labeled by its curvature (that is, the reciprocal of its radius). There are also Apollonian packings which spread all over the plane as well as spread all over to the half plane. As circles in these packings would become enormously large after a few first generations, it is harder to draw them on paper.

There are many natural questions about Apollonian circle packings either from the number theoretic or the geometric point of view and we refer to the series of papers by Graham, Lagarias, Mallows, Wilks, and Yan especially [30] [29], and [17] as well as the letter of Sarnak to Lagarias [64] which inspired the author to work on the topic personally. Also see a more recent article [62].

To find the symmetry group of a given Apollonian packing \( \mathcal{P} \), we consider the dual circles to any fixed four mutually tangent circles (see Fig. 3 where the red dotted circles are the dual circles to the black solid circles). Inversion with respect to each dual circle fixes three circles that the dual circle crosses perpendicularly and interchanges two circles tangent to those three circles. Hence the group, say, \( \Gamma(\mathcal{P}) \), generated by the four inversions with respect to the dual circles preserves the packing \( \mathcal{P} \) and there are four \( \Gamma(\mathcal{P}) \) orbits of circles in \( \mathcal{P} \).

As the fundamental domain of \( \Gamma(\mathcal{P}) \) in \( \mathbb{H}^3 \) can be taken to be the exterior of the four hemispheres above the dual circles in \( \mathbb{H}^3 \), \( \Gamma(\mathcal{P}) \) is geometrically finite. It is known that the limit set of \( \Gamma(\mathcal{P}) \) coincides precisely with the residual set of \( \mathcal{P} \) and hence the critical exponent of \( \Gamma(\mathcal{P}) \) is equal to the Hausdorff dimension of the residual set of \( \mathcal{P} \), which is approximately

\[
\alpha = 1.30568(8)
\]
due to C. McMullen [48] (note that as any two Apollonian packings are equivalent to each other by a Moebius transformation, \( \alpha \) is independent of \( \mathcal{P} \)). In particular it
follows that \( \Gamma(\mathcal{P}) \) is Zariski dense in the real algebraic group \( \text{PSL}_2(\mathbb{C}) \) and hence we deduce the following from Theorem 3.3 and the remark following it:

**Corollary 3.5** (\[31\]). Let \( \mathcal{P} \) be an Apollonian circle packing. For any bounded region \( E \) of \( \mathbb{C} \) whose boundary is a countable union of real algebraic curves, we have

\[
N_T(\mathcal{P}, E) \sim \frac{sk_{\Gamma \mathcal{P}}(\mathcal{P})}{\alpha \cdot |m_{\Gamma \mathcal{P}}^{\text{RMS}}|} \cdot \omega_{\Gamma \mathcal{P}}(E) \cdot T^\alpha \quad \text{as } T \to \infty
\]

where \( \Gamma \mathcal{P} := \Gamma(\mathcal{P}) \cap \text{PSL}_2(\mathbb{C}) \).

**Remark 3.6.**

1. In the cases when \( \mathcal{P} \) is bounded and \( E \) is the largest disk in such \( \mathcal{P} \), and when \( \mathcal{P} \) lies between two parallel lines and \( E \) is the whole period (see Fig. 5), the above asymptotics was previously obtained in [37] with a less explicit description of the main term.

2. Corollary 3.5 applies to any triangular region \( \mathcal{T} \) (see Fig. 4) of an Apollonian circle packing.

### 3.2. More circle packings.

**3.2.1. Counting circles in the limit set \( \Lambda(\Gamma) \).** If \( \Gamma \backslash \mathbb{H}^3 \) is a hyperbolic 3 manifold with boundary being totally geodesic, then \( \Gamma \) is automatically geometrically finite [34] and \( \Omega(\Gamma) \) is a union of countably many disjoint open disks. Hence Theorem 3.3 applies to counting these open disks in \( \Omega(\Gamma) \) with respect to the curvature, provided there are infinitely many such. The picture of a Sierpinski curve in Fig. 4 is a special case of this (so are Apollonian circle packings). More precisely, if \( \Gamma \) denotes the group generated by reflections in the sides of a unique regular tetrahedron whose convex core is bounded by four \( \frac{2}{3} \) triangles and by four right hexagons, then the residual set of a Sierpinski curve in Fig. 1 coincides with \( \Lambda(\Gamma) \) (see [37] for details), and it is known to be homeomorphic to the well-known Sierpinski carpet by a theorem of Claytor [9].

![Image](image-url)
Three pictures in Fig. 5 can be found in the beautiful book *Indra’s pearls* by Mumford, Series and Wright [50] and the residual sets are the limit sets of some (geometrically finite) Schottky groups and hence our theorem applies to counting circles in those pictures.

### 3.2.2. Schottky dance.

Other kinds of examples are obtained by considering the images of Schottky disks under Schottky groups. Take $k \geq 1$ pairs of mutually disjoint closed disks $\{(D_i, D'_i) : 1 \leq i \leq k\}$ in $\mathbb{C}$ and choose Möbius transformations $\gamma_i$ which maps $D_i$ and $D'_i$ and sends the interior of $D_i$ to the exterior of $D'_i$, respectively. The group, say, $\Gamma$, generated by $\{\gamma_i : 1 \leq i \leq k\}$ is called a Schottky group of genus $k$ (cf. [42, Sec. 2.7]). The $\Gamma$-orbits of the disks nest down onto the limit set $\Lambda(\Gamma)$ which is totally disconnected. If we denote by $\mathcal{P}$ the union $\bigcup_{i=1}^{k}(\Gamma(C_i) \cup \Gamma(C'_i))$ where $C_i$ and $C'_i$ are the boundaries of $D_i$ and $D'_i$ respectively, $\mathcal{P}$ is locally finite, as the nesting disks will become smaller and smaller (cf. [50, 4.5]). The common exterior of hemispheres above the initial disks $D_i$ and $D'_i$, $1 \leq i \leq k$, is a fundamental domain for $\Gamma$ in the upper half-space model $\mathbb{H}^3$, and hence $\Gamma$ is geometrically finite. Since $\mathcal{P}$ consists of disjoint circles, Theorem 3.3 applies to $\mathcal{P}$. For instance, see Fig. 6 ([50, Fig. 4.11]). One can find many more explicit circle packings in [50] to which Theorem 3.3 applies.

### 4. Circle packings on the sphere

In the unit sphere $\mathbb{S}^2 = \{x^2 + y^2 + z^2 = 1\}$ with the Riemannian metric induced from $\mathbb{R}^3$, the distance between two points is simply the angle between the rays connecting them to the origin $o = (0,0,0)$. 
Let $\mathcal{P}$ be a circle packing on the sphere $S^2$, i.e., a union of circles. The spherical curvature of a circle $C$ in $S^2$ is given by
\[
\text{Curv}_{S}(C) = \cot(\theta(C))
\]
where $0 < \theta(C) \leq \pi/2$ is the spherical radius of $C$, that is, the half of the visual angle of $C$ from the origin $o$. We suppose that $\mathcal{P}$ is infinite and locally finite in the sense that there are only finitely many circles in $\mathcal{P}$ of spherical curvature at most $T$ for any fixed $T > 0$.

For a region $E$ of $S^2$, we set
\[
N_{T}(\mathcal{P}, E) := \# \{ C \in \mathcal{P} : C \cap E \neq \emptyset, \quad \text{Curv}_{S}(C) < T \}.
\]
We consider the Poincare ball model $\mathbb{B} = \{ x_1^2 + x_2^2 + x_3^2 < 1 \}$ of the hyperbolic 3 space with the metric $d$ given by
\[
\frac{2 \sqrt{dx_1^2 + dx_2^2 + dx_3^2}}{1-(x_1^2+x_2^2+x_3^2)}.
\]
Note that the geometric boundary of $\mathbb{B}$ is $S^2$ and that for any circle $C$ in $S^2$, we have
\[
\sin \theta(C) = \frac{1}{\cosh d(\bar{C}, o)}
\]
where $\bar{C} \subset \mathbb{B}$ is the convex hull of $C$. As both $\sin \theta$ and $\cosh d$ are monotone functions for $0 \leq \theta \leq \pi/2$ and $d \geq 0$ respectively, understanding $N_{T}(\mathcal{P}, E)$ is equivalent to investigating the number of Euclidean hemispheres on $\mathbb{B}$ meeting the ball of hyperbolic radius $T$ based at $o$ whose boundaries are in $\mathcal{P}$ and intersect $E$.

Let $G$ denote the orientation preserving isometry group of $\mathbb{B}$.
Theorem 4.1 (52). Let $\Gamma$ be a non-elementary geometrically finite discrete subgroup of $G$ and $\mathcal{P} = \bigcup_{i \in \mathcal{I}} \Gamma(C_i)$ be an infinite, locally finite, and $\Gamma$-invariant circle packing on the sphere $S^2$ with finitely many $\Gamma$-orbits.

Suppose one of the following conditions hold:

1. $\Gamma$ is convex co-compact;
2. all circles in $\mathcal{P}$ are mutually disjoint;
3. $\bigcup_{i \in \mathcal{I}} C_i^\circ \subset \Omega(\Gamma)$ where $C_i^\circ$ denotes the interior of $C_i$.

Then for any Borel subset $E \subset S^2$ whose boundary is of zero Patterson-Sullivan measure,

$$N_T(\mathcal{P}, E) \sim \frac{\text{sk}_T(\mathcal{P}) \cdot \nu_\circ(E)}{\delta_T \cdot |m_\text{HMS}|} \cdot (2T)^{\delta_T} \quad \text{as } T \to \infty$$

where $0 < \text{sk}_T(\mathcal{P}) < \infty$ is defined in Def. 3.2

5. Integral Apollonian packings: Primes and Twin primes

A circle packing $\mathcal{P}$ is called integral if the curvatures of all circles in $\mathcal{P}$ are integral. One of the special features of Apollonian circle packings is the abundant existence of integral Apollonian circle packings.

Descartes noted in 1643 (see [10]) that a quadruple $(a, b, c, d)$ of real numbers can be realized as curvatures of four mutually tangent circles in the plane (oriented
so that their interiors are disjoint) if and only if it satisfies

\[ 2(a^2 + b^2 + c^2 + d^2) - (a + b + c + d)^2 = 0. \]  

(5.1)

Usually referred to as the Descartes circle theorem, this theorem implies that if the initial four circles in an Apollonian circle packing \( \mathcal{P} \) in the plane have integral curvatures, then \( \mathcal{P} \) is an integral packing, as observed by Soddy in 1937 \cite{Soddy1937}. The Descartes circle theorem provides an integral Apollonian packing for every integral solution of the quadratic equation \( \text{(5.1)} \) and indeed there are infinitely many distinct integral Apollonian circle packings.

Let \( \mathcal{P} \) be an integral Apollonian circle packing. We can deduce from the existence of the lower bound for the non-zero curvatures in \( \mathcal{P} \) that such \( \mathcal{P} \) is either bounded or lies between two parallel lines. We assume that \( \mathcal{P} \) is primitive, that is, the greatest common divisor of curvatures is one.

Calling a circle with a prime curvature a prime and a pair of tangent prime circles a twin prime, Sarnak showed:

**Theorem 5.2** \cite{Sarnak1988}. There are infinitely many primes, as well as twin primes, in \( \mathcal{P} \).

For \( \mathcal{P} \) bounded, denote by \( \pi^p(T) \) the number of prime circles in \( \mathcal{P} \) of curvature at most \( T \), and by \( \pi_2^p(T) \) the number of twin prime circles in \( \mathcal{P} \) of curvatures at most \( T \). For \( \mathcal{P} \) congruent to the packing in Fig. 8 we alter the definition of \( \pi^p(T) \) and \( \pi_2^p(T) \) to count prime circles in a fixed period. Sarnak showed \cite{Sarnak1988} that

\[ \pi^p(T) \asymp \frac{T}{(\log T)^{3/2}}. \]

Recently Bourgain, Gamburd and Sarnak \cite{Bourgain2006} and \cite{Gamburd2006} obtained a uniform spectral gap for the family of congruence subgroups \( \Gamma(q) = \{ \gamma \in \Gamma : \gamma \equiv 1 \pmod{q} \} \), \( q \) square-free, of any finitely generated subgroup \( \Gamma \) of \( \text{SL}_2(\mathbb{Z}) \) provided \( \delta_{\Gamma} > 1/2 \). This theorem extends to a Zariski dense subgroup \( \Gamma \) of \( \text{SL}_2(\mathbb{Z}[i]) \) and its congruence subgroups over square free ideals of \( \mathbb{Z}[i] \) if \( \delta_{\Gamma} > 1 \).
Denoting by $Q$ the Descartes quadratic form

$$Q(a, b, c, d) = 2(a^2 + b^2 + c^2 + d^2) - (a + b + c + d)^2,$$

the approach in [37] for counting circles in Apollonian circle packings which are either bounded or between two parallel lines is based on the interpretation of such circle counting problem into the counting problem for $w\Gamma \cap B_{T}^{\text{max}}$ where $\Gamma < O_{Q}(\mathbb{Z})$ is the so-called Apollonian group, $w \in \mathbb{Z}^4$ with $Q(w) = 0$ and $B_{T}^{\text{max}}$ denotes the maximum norm ball in $\mathbb{R}^4$.

Using the spin double cover $\text{Spin}_Q \to \text{SO}_Q$ and the isomorphism $\text{Spin}_Q(\mathbb{R}) = \text{SL}_2(\mathbb{C})$, we use the aforementioned result of Bourgain, Gamburd and Sarnak to obtain a smoothed counting for $w\Gamma(q) \cap B_T$ with a uniform error term for the family of square-free congruence subgroups $\Gamma(q)$’s where $B_T$ is the Euclidean norm ball. This is a crucial ingredient for the Selberg’s upper bound sieve, which is used to prove the following:

**Theorem 5.3 ([37]).** As $T \to \infty$,

$$\pi^P(T) \ll \frac{T^\alpha}{\log T}, \quad \text{and} \quad \pi_2^P(T) \ll \frac{T^\alpha}{(\log T)^2}$$

where $\alpha = 1.30568(8)$ is the residual dimension of $\mathcal{P}$.

**Remark 5.4.**

1. Modulo 16, the Descartes equation (5.1) has no solutions unless two of the curvatures are even and the other two odd. In particular, there are no “triplet primes” of three mutually tangent circles, all having odd prime curvatures.

2. We can also use the methods in [37] to give lower bounds for almost primes in a packing. A circle in $\mathcal{P}$ is called $R$-almost prime if its curvature is the product of at most $R$ primes. Similarly, a pair of tangent circles is called $R$-almost twin prime if both circles are $R$-almost prime. Employing Brun’s combinatorial sieve, our methods show the existence of $R_1, R_2 > 0$ (unspecified) such that the number of $R_1$-almost prime circles in $\mathcal{P}$ whose curvature is at most $T$ is $\asymp \frac{T^\alpha}{\log T}$ and that the number of pairs of $R_2$-almost twin prime circles whose curvatures are at most $T$ is $\asymp \frac{T^\alpha}{(\log T)^2}$.

3. A suitably modified version of Conjecture 1.4 in [3], a generalization of Schinzel’s hypothesis, implies that for some $c, c_2 > 0$,

$$\pi^P(T) \sim c \cdot \frac{T^\alpha}{\log T} \quad \text{and} \quad \pi_2^P(T) \sim c_2 \cdot \frac{T^\alpha}{(\log T)^2}.$$  

The constants $c$ and $c_2$ are detailed in [24].

4. Recently Bourgain and Fuchs [2] showed that in a given bounded integral Apollonian packing $\mathcal{P}$, the growth of the number of distinct curvatures at most $T$ is at least $c \cdot T$ for some $c > 0$.

---

1 By $f(T) \asymp g(T)$, we mean $g(T) \ll f(T) \ll g(T)$. 

---
5. The spherical Soddy-Gossett theorem says (see [38]) that the quadruple \((a, b, c, d)\) of spherical curvatures of four mutually tangent circles in \(\mathcal{P}\) satisfies

\[
2(a^2 + b^2 + c^2 + d^2) - (a + b + c + d)^2 = -4.
\]

This theorem implies again that there are infinitely many integral spherical Apollonian circle packings, that is, the spherical curvature of every circle is integral. It will be interesting to have results analogous to Theorems 5.2 and 5.3 for integral spherical Apollonian packings.

6. Equidistribution in geometrically finite hyperbolic manifolds

Let \(G\) be the identity component of the group of isometries of \(\mathbb{H}^n\) and \(\Gamma < G\) be a non-elementary geometrically finite discrete subgroup.

We have discussed that the Bowen-Margulis-Sullivan measure is a finite measure on the unit tangent bundle \(T^1(\Gamma \backslash \mathbb{H}^n)\) which is mixing for the geodesic flow. Another measure playing an important role in studying the dynamics of flows on \(T^1(\Gamma \backslash \mathbb{H}^n)\) is the following Burger-Roblin measure.

Burger-Roblin measure: The Burger-Roblin measure \(m_{\text{BR}}^\Gamma\) is the induced measure on \(T^1(\Gamma \backslash \mathbb{H}^n)\) of the following \(\Gamma\)-invariant measure on \(T^1(\mathbb{H}^n)\):

\[
d\tilde{m}_{\text{BR}}^\Gamma(u) = e^{(n-1)\beta_u(x,\pi(x))} e^{\delta_\Gamma(x_0,x) - (x,\pi(x))} m_x(u^+) \nu_x(u^-) dt
\]

where \(m_x\) denotes the probability measure on the boundary \(\partial_\infty(\mathbb{H}^n)\) invariant under the maximal compact subgroup \(\text{Stab}_G(x_0)\). For any \(x\) and \(x_0 \in \mathbb{H}^n\), we have \(dm_x(\xi) := e^{-(n-1)\beta_{x_0}(x,x_0)} dm_{x_0}(\xi)\) and it follows that this definition of \(m_{\text{BR}}^\Gamma\) is independent of the choice of \(x \in \mathbb{H}^n\).

Burger [8] showed that for a convex cocompact hyperbolic surface with \(\delta_\Gamma \geq 1/2\), this is a unique ergodic horocycle invariant measure up to homothety. Roblin [59] extended Burger’s result in much greater generality, for instance, including all non-elementary geometrically finite hyperbolic manifolds.

The name of the Burger-Roblin measure was first suggested by Shah and the author in [37] and [53] in recognition of this important classification result.

We note that the total mass \(|m_{\text{BR}}^\Gamma|\) is finite only when \(\delta_\Gamma = n - 1\) (or equivalently only when \(\Gamma\) is a lattice in \(G\)) and is supported on the set \(\{u \in T^1(\Gamma \backslash \mathbb{H}^n) : u^- \in \Lambda(\Gamma)\}\).

Let \(S^1 \subset T^1(\mathbb{H}^n)\) be one of the following:

1. an unstable horosphere;

2. the oriented unit normal bundle of a codimension one totally geodesic subspace of \(\mathbb{H}^n\)
3. the set of outward normal vectors to a (hyperbolic) sphere in $\mathbb{H}^n$.

We consider the following measures on $\text{Stab}_\Gamma(S)\backslash S$: 

$$d\mu_{S1}^{\text{Leb}}(s) = e^{(n-1)\beta_+}(x,\pi(s))dm_x(s^+), \quad d\mu_{S1}^{\text{PS}}(s) = e^{\beta_+}(x,\pi(s))d\nu_x(s^+)$$

for any $x \in \mathbb{H}^n$.

Denote by $p$ the canonical projection $T^1(\mathbb{H}^n) \to T^1(\Gamma\backslash \mathbb{H}^n) = \Gamma \backslash T^1(\mathbb{H}^n)$.

**Theorem 6.1** ([53]). For $\psi \in C_c(T^1(\Gamma\backslash \mathbb{H}^n))$ and any relatively compact subset $\mathcal{O} \subset p(S^1)$ with $\mu_{S1}^{\text{PS}}(\partial(\mathcal{O})) = 0$, 

$$e^{(n-1-\delta_1)t} \int_{\mathcal{O}} (\psi(g^t(s))) d\mu_{S1}^{\text{Leb}}(s) \sim \frac{\mu_{S1}^{\text{PS}}(\mathcal{O}^*)}{\delta_1 \cdot |m_{\Gamma}\text{BMS}|} \cdot m_{\Gamma}^{\text{BR}}(\psi) \quad \text{as} \ t \to \infty$$

where 

$$\mathcal{O}^* = \{s \in \mathcal{O} : s^+ \in \Lambda(\Gamma)\}.$$ 

**Definition 6.2.** For a hyperbolic subspace $S = \mathbb{H}^{n-1} \subset \mathbb{H}^n$, we say that a parabolic fixed point $\xi \in \Lambda(\Gamma) \cap \partial_\infty(\mathbb{H}^{n-1})$ of $\Gamma$ is internal if any parabolic element $\gamma \in \Gamma$ fixing $\xi$ preserves $\mathbb{H}^{n-1}$.

Recalling the notation $\pi$ for the canonical projection from $T^1(\mathbb{H}^n)$ to $\mathbb{H}^n$, we set $S = \pi(S^1)$.

**Theorem 6.3** ([53]). We assume that the projection map $\text{Stab}_\Gamma(S)\backslash S \to \Gamma\backslash \mathbb{H}^n$ is proper. In the case when $S$ is a codimension one totally geodesic subspace, we also assume that every parabolic fixed point of $\Gamma$ in the boundary of $S$ is internal.

For $\psi \in C_c(T^1(\Gamma\backslash \mathbb{H}^n))$, 

$$e^{(n-1-\delta_1)t} \int_{p(S^1)} (\psi(g^t(s))) d\mu_{S1}^{\text{Leb}}(s) \sim \frac{\mu_{S1}^{\text{PS}}(S^1)}{\delta_1 \cdot |m_{\Gamma}\text{BMS}|} \cdot m_{\Gamma}^{\text{BR}}(\psi) \quad \text{as} \ t \to \infty$$

where 

$$S^1 = \{s \in p(S^1) : s^+ \in \Lambda(\Gamma)\}.$$ 

We have $0 \leq \mu_{S1}^{\text{PS}}(S^1) < \infty$, and $\mu_{S1}^{\text{PS}}(S^1) = 0$ may happen only when $S$ is totally geodesic.

It can be shown by combining results of [11] and [45] that in a finite volume space $\Gamma\backslash \mathbb{H}^n$, the properness of the projection map $\text{Stab}_\Gamma(S)\backslash S \to \Gamma\backslash \mathbb{H}^3$ implies that $\text{Stab}_\Gamma(S)\backslash S$ is of finite volume as well, except for the case when $n = 2$ and $S$ is a proper geodesic in $\mathbb{H}^2$ connecting two parabolic fixed points of a lattice $\Gamma < \text{PSL}_2(\mathbb{R})$.

When both $\Gamma\backslash \mathbb{H}^n$ and $\text{Stab}_\Gamma(S)\backslash S$ are of finite volume, we have $n-1 = \delta_1$ and both $m_{\Gamma}\text{BMS}$ and $m_{\Gamma}^{\text{BR}}$ are finite invariant measures and $\mu_{S1}^{\text{PS}} = \mu_{S1}^{\text{Leb}}$ (up to a constant multiple). In this case, Theorem 6.3 is due to Sarnak [63] for the closed horocycles for $n = 2$. The general case is due to Duke, Rudnick and Sarnak [14] and Eskin and McMullen [19] gave a simpler proof of Theorem 6.3 based on the
mixing property of the geodesic flow of a finite volume hyperbolic manifold. The latter proof, combined with a strengthened version of the wavefront lemma \cite{28}, also works for proving Theorem \ref{thm:6.1}. We remark that the idea of using mixing in this type of problem goes back to the 1970 thesis of Margulis \cite{46} (see also \cite{31} Appendix). Eskin, Mozes and Shah \cite{20} and Shah \cite{69} provided yet another different proofs using the theory of unipotent flows. When both $\Gamma \backslash \mathbb{H}^n$ and $\text{Stab}_\Gamma (S) \backslash S$ are of finite volume, Theorem \ref{thm:6.1} easily implies Theorem \ref{thm:6.3} but not conversely.

In the case when $S^\dagger$ is a horosphere, Theorem \ref{thm:6.1} was obtained in \cite{59}, and Theorem \ref{thm:6.3} was proved in \cite{37} when $\delta_\Gamma > (n-1)/2$ with a different interpretation of the main term.

**Remark 6.4.**
1. The condition on the internality of all parabolic fixed points of $\Gamma$ in the boundary of $S$ is crucial, as $\mu_{\text{PS}}(S^\dagger_1) = \infty$ otherwise. This can already be seen in the level of a lattice: take $\Gamma = \text{SL}_2(\mathbb{Z})$ and let $S$ be the geodesic connecting 0 and $\infty$ in the upper half space model. Then any upper triangular matrix in $\Gamma$ fixes $\infty$ but does not stabilize $S$. Indeed the length of the image of $S$ in $\Gamma \backslash \mathbb{H}^2$ is infinite.

2. In proving Theorem \ref{thm:3.3} we count circles in $\mathbb{C}$ by counting the corresponding Euclidean hemispheres in $\mathbb{H}^3$. As the Euclidean hemispheres are totally geodesic hyperbolic planes, this amounts to understanding the distribution of a $\Gamma$-orbit of a totally geodesic hyperbolic plane in $\mathbb{H}^3$. The equidistribution theorem we use here is Theorem \ref{thm:6.3} for $S$ a hyperbolic plane.

3. More classical applications of the equidistribution theorem such as Theorem \ref{thm:6.3} can be found in the point counting problems of $\Gamma$-orbits in various spaces.

For a $\Gamma$-orbit in the hyperbolic space $\mathbb{H}^n$, the orbital counting in Riemannian balls was obtained Lax-Phillips \cite{39} for $\delta_\Gamma > \frac{n-1}{2}$ and by Roblin \cite{59} in general.

Extending the work of Duke, Rudnick and Sarnak \cite{14} and of Eskin and McMullen \cite{19} for $\Gamma$ lattices, we obtain in \cite{53}, for any geometrically finite group $\Gamma$ of $G$, the asymptotic of the number of vectors of norm at most $T$ lying in a discrete orbit $w \Gamma$ of a quadric

$$F(x_1, \cdots, x_{n+1}) = y$$

for a real quadratic form $F$ of signature $(n, 1)$ and any $y \in \mathbb{R}$ (when $y > 0$, there is an extra assumption on $w$ not being $\Gamma$ strongly parabolic. See \cite{53} for details). When $y = 0$ and $n = 2, 3$, special cases of this result were obtained in \cite{35}, \cite{37} and \cite{36} under the condition $\delta_\Gamma > (n-1)/2$. Based on the Descartes circle theorem, this result in \cite{37} was used to prove Theorem \ref{thm:3.5} for the bounded Apollonian packings. In \cite{40}, a $\Gamma$-orbit in the geometric boundary is shown to be equidistributed with respect to the Patterson-Sullivan measure, extending the work \cite{25} for the lattice case.

4. For $\psi \in C_c(\Gamma \backslash \mathbb{H}^n)$, we have

$$m_{\Gamma}^{BR}(\psi) = \langle \psi, \phi_\Gamma \rangle := \int_{\Gamma \backslash \mathbb{H}^n} \psi(x) \cdot \phi_\Gamma(x) \, dm^{\text{Leb}}(x)$$
where $\phi_T(x) = |\nu_x|$ is the positive eigenfunction of the Laplace operator on $\Gamma \backslash \mathbb{H}^n$ with eigenvalue $\delta_T(n-1-\delta_T)$ and

$$dm_{\text{Leb}}(u) = e^{(n-1)\beta_u+\langle x,\pi(u)\rangle} e^{(n-1)\beta_{u-}\langle x,\pi(u)\rangle} dm_x(u^+)dm_x(u^-)dt$$

for any $x \in \mathbb{H}^n$. Hence Theorem 6.3 says that for $\psi \in C_c(\Gamma \backslash \mathbb{H}^n)$,

$$e^{(n-1-\delta_T)t} \int_{p(S)^1} \psi(p(g^1(s))) \, d\mu_{S^1}^{\text{PS}}(s) \sim \frac{\mu_{S^1}^{\text{PS}}(S^1)}{\delta_T \cdot |\mu_{\text{BMS}}^T|} \cdot \langle \psi, \phi_T \rangle \quad \text{as } t \to \infty. \quad (6.5)$$

When $\delta_T > (n-1)/2$, $\phi_T \in L^2(\Gamma \backslash \mathbb{H}^n)$ and its eigenvalue $\delta_T(n-1-\delta_T)$ is isolated in the $L^2$-spectrum of the Laplace operator $[39]$. It will be desirable to obtain a rate of convergence in (6.5) in terms of the spectral gap of $\Gamma$ in such cases. For $\Gamma$ lattices, it was achieved in $[14]$ for $p(S)$ compact and in $[11]$ in general. This was done in the case of a horosphere in $[37]$, which was the main ingredient in the proof of Theorem 5.3. It may be possible to extend the methods of $[37]$ to obtain an error term in general.

7. Further remarks and questions

Let $G$ be the identity component of the group of isometries of $\mathbb{H}^n$ and $\Gamma$ be a geometrically finite subgroup. We further assume that $\Gamma$ is Zariski dense in $G$ for discussions in this section. When we identify $\mathbb{H}^n$ with $G/K$ for a maximal compact subgroup $K$, the unit tangent bundle $\mathbb{T}^1(\mathbb{H}^n)$ can be identified with $G/M$ where $M$ is the centralizer in $K$ of a Cartan subgroup, say, $A$, whose multiplication on the right corresponds to the geodesic flow. The frame bundle of $\mathbb{H}^n$ can be identified with $G$ and the frame flow on the frame bundle is given by the multiplications by elements of $A$ on the right.

We have stated the equidistribution results in section 6 in the level of the unit tangent bundle $\mathbb{T}^1(\Gamma \backslash \mathbb{H}^n)$. As the frame bundle is a homogeneous space of $G$ unlike the unit tangent bundle, it is much more convenient to work in the frame bundle. Fortunately, as observed in $[23]$, the frame flow is mixing on $\Gamma \backslash G$ with respect to the lift from $\Gamma \backslash G/M$ to $\Gamma \backslash G$ of the Bowen-Margulis-Sullivan measure. Using this, we can extend Theorems 6.1 and 6.3 to the level of the frame bundle $\Gamma \backslash G$. It seems that the classification theorem of Burger and Roblin can also be extended: for a horospherical group $N$, any locally finite $N$-invariant ergodic measure on $\Gamma \backslash G$ is either supported on a closed $N$-orbit or the lift of the Burger-Roblin measure (we caution here that a locally finite $N$-invariant measure supported on a closed $N$-orbit need not be a finite measure unlike the $\Gamma$-lattice cases).

In analogy with Ratner’s theorem $[56], [57]$, we propose the following problems: let $U$ be a one-parameter unipotent subgroup, or more generally a subgroup generated by unipotent one parameter subgroups of $G$:

1. [Measure rigidity] Classify all locally finite Borel $U$-invariant ergodic measures on $\Gamma \backslash G$. 
2. [Topological rigidity] Classify the closures of $U$-orbits in $\Gamma \backslash G$.

We remark that as $G = \text{SO}(n,1)$ (up to a local isomorphism) in our set-up, the above topological rigidity for $\Gamma$ lattices was also obtained by Shah \cite{68, 67} based on the approach of Margulis \cite{43, 44}, and of Dani and Margulis \cite{13}.

Both questions are known for $n = 2$ due to Burger \cite{8} and Roblin \cite{59}, as in this case, there is only one unipotent one-parameter subgroup up to conjugation, which gives the horocycle flow. Shapira used them to prove equidistribution for non-closed horocycles \cite{65}.

It may be a good idea to start with a sampling case when $G = \text{SL}_2(\mathbb{C})$, $U = \text{SL}_2(\mathbb{R})$ and $\Gamma < G$ Zariski dense and geometrically finite.

1. Are there any locally finite $\text{SL}_2(\mathbb{R})$-invariant ergodic measure on $\Gamma \backslash \text{SL}_2(\mathbb{C})$ besides the Haar measure (=the $\text{SL}_2(\mathbb{C})$-invariant measures) and the $\text{SL}_2(\mathbb{R})$-invariant measures supported on closed $\text{SL}_2(\mathbb{R})$ orbits?

2. Is every non-closed $\text{SL}_2(\mathbb{R})$-orbit dense in $\Gamma \backslash \text{SL}_2(\mathbb{C})$?

It seems that the answers are no for (1) and yes for (2).

References

\[1\] Yves Benoist and Hee Oh. Effective equidistribution of $S$-integral points on symmetric varieties, Preprint \texttt{arXiv:0706.1621}, 2007.

\[2\] Jean Bourgain and Elena Fuchs. A proof of the positive density conjecture for integer Apollonian circle packings. \textit{Preprint (arXiv:1001.3894)}, 2010.

\[3\] Jean Bourgain, Alex Gamburd, and Peter Sarnak. Affine linear sieve, expanders and sum-product, 2008. \textit{To appear in Inventiones}.

\[4\] Jean Bourgain, Alex Gamburd, and Peter Sarnak. Generalization of Selberg’s theorem and Selberg’s sieve. \textit{Preprint (arXiv:0912.5021)}, 2009.

\[5\] B. H. Bowditch. Geometrical finiteness for hyperbolic groups. \textit{J. Funct. Anal.}, 113(2):245–317, 1993.

\[6\] Rufus Bowen. Periodic points and measures for Axiom A diffeomorphisms. \textit{Trans. Amer. Math. Soc.}, 154:377–397, 1971.

\[7\] David W. Boyd. The residual set dimension of the Apollonian packing. \textit{Mathematika}, 20:170–174, 1973.

\[8\] Marc Burger. Horocycle flow on geometrically finite surfaces. \textit{Duke Math. J.}, 61(3):779–803, 1990.

\[9\] Schieffelin Claytor. Topological immersion of Peanian continua in a spherical surface. \textit{Ann. of Math. (2)}, 35(4):809–835, 1934.

\[10\] H. S. M. Coxeter. The problem of Apollonius. \textit{Amer. Math. Monthly}, 75:5–15, 1968.

\[11\] S. G. Dani. On invariant measures, minimal sets and a lemma of Margulis. \textit{Invent. Math.}, 51(3):239–260, 1979.
[12] S. G. Dani. Flows on homogeneous spaces and Diophantine approximation. In Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Zürich, 1994), pages 780–789, Basel, 1995. Birkhäuser.

[13] S. G. Dani and G. A. Margulis. Orbit closures of generic unipotent flows on homogeneous spaces of $SL(3, \mathbb{R})$. Math. Ann., 286(1-3):101–128, 1990.

[14] W. Duke, Z. Rudnick, and P. Sarnak. Density of integer points on affine homogeneous varieties. Duke Math. J., 71(1):143–179, 1993.

[15] M. Einsiedler and E. Lindenstrauss. Diagonalizable flows on locally homogeneous spaces and number theory. In International Congress of Mathematicians, Vol. II, pages 1731–1759, Eur. Math. Soc., Zürich, 2006.

[16] N. D. Elkies and C. T. McMullen. Gaps in $\sqrt{n}$ mod 1 and ergodic theory. Duke Math. J., 123(1):95–139, 2004.

[17] Nicholas Eriksson and Jeffrey C. Lagarias. Apollonian circle packings: number theory. II. Spherical and hyperbolic packings. Ramanujan J., 14(3):437–469, 2007.

[18] Alex Eskin. Counting problems and semisimple groups. In Proceedings of the International Congress of Mathematicians, Vol. II (Berlin, 1998), number Extra Vol. II, pages 539–552 (electronic), 1998.

[19] Alex Eskin and C. T. McMullen. Mixing, counting, and equidistribution in Lie groups. Duke Math. J., 71(1):181–209, 1993.

[20] Alex Eskin, Shahar Mozes, and Nimish Shah. Unipotent flows and counting lattice points on homogeneous varieties. Ann. of Math. (2), 143(2):253–299, 1996.

[21] Alex Eskin and Hee Oh. Ergodic theoretic proof of equidistribution of Hecke points. Ergodic Theory Dynam. Systems, 26(1):163–167, 2006.

[22] Alex Eskin and Hee Oh. Representations of integers by an invariant polynomial and unipotent flows. Duke Math. J., 135(3):481–506, 2006.

[23] L. Flaminio and R. J. Spatzier. Geometrically finite groups, Patterson-Sullivan measures and Ratner’s rigidity theorem. Invent. Math., 99(3):601–626, 1990.

[24] E. Fuchs and K. Sanden. Some experiments with integral apollonian circle packings. Preprint, 2010.

[25] Alex Gorodnik and Hee Oh. Orbits of discrete subgroups on a symmetric space and the Furstenberg boundary. Duke Math. J., 139(3):483–525, 2007.

[26] Alex Gorodnik and Hee Oh. Rational points on homogeneous varieties and equidistribution of adelic periods (with an appendix by Borovoi), Preprint [arXiv:0803.1996], 2008.

[27] Alex Gorodnik, Hee Oh, and Nimish Shah. Integral points on symmetric varieties and Satake compactifications. Amer. J. Math., 131(1):1–57, 2009.

[28] Alex Gorodnik, Hee Oh, and Nimish Shah. Strong wavefront lemma and counting lattice points in sectors. Israel. J. Math., 176: 419–444, 2010.

[29] Ronald L. Graham, Jeffrey C. Lagarias, Colin L. Mallows, Allan R. Wilks, and Catherine H. Yan. Apollonian circle packings: number theory. J. Number Theory, 100(1):1–45, 2003.

[30] Ronald L. Graham, Jeffrey C. Lagarias, Colin L. Mallows, Allan R. Wilks, and Catherine H. Yan. Apollonian circle packings: geometry and group theory. I. The Apollonian group. Discrete Comput. Geom., 34(4):547–585, 2005.
[31] D. Kleinbock and G. A. Margulis. Bounded orbits of nonquasiunipotent flows on homogeneous spaces. In Sinai’s Moscow Seminar on Dynamical Systems, volume 171 of Amer. Math. Soc. Transl. Ser. 2, pages 141–172. Amer. Math. Soc., Providence, RI, 1996.

[32] D. Kleinbock and G. A. Margulis. Flows on homogeneous spaces and Diophantine approximation on manifolds. Ann. of Math. (2), 148(1):339–360, 1998.

[33] B. Klingler and A. Yafaev. On the Andr’e-Oort conjecture. Preprint.

[34] Sadayoshi Kojima. Polyhedral decomposition of hyperbolic 3-manifolds with totally geodesic boundary. In Aspects of low-dimensional manifolds, volume 20 of Adv. Stud. Pure Math., pages 93–112. Kinokuniya, Tokyo, 1992.

[35] Alex Kontorovich. The hyperbolic lattice point count in infinite volume with applications to sieves. Duke Math. J., 149(1):1–36, 2009.

[36] Alex Kontorovich and Hee Oh. Almost prime Pythagorean triples in thin orbits. Preprint (arXive:1001.0370), 2010.

[37] Alex Kontorovich and Hee Oh. Apollonian circle packings and closed horospheres on hyperbolic 3-manifolds. Preprint (arXive:0811.2236), 2009.

[38] Jeffrey C. Lagarias, Colin L. Mallows, and Allan R. Wilks. Beyond the Descartes circle theorem. Amer. Math. Monthly, 109(4):338–361, 2002.

[39] Peter D. Lax and Ralph S. Phillips. The asymptotic distribution of lattice points in Euclidean and non-Euclidean spaces. J. Funct. Anal., 46(3):280–350, 1982.

[40] Seon-Hee Lim and Hee Oh. On the distribution of orbits of geometrically finite hyperbolic groups on the boundary. Preprint, 2010.

[41] Alexander Lubotzky and Robert J. Zimmer. Arithmetic structure of fundamental groups and actions of semisimple Lie groups. Topology, 40(4):851–869, 2001.

[42] A. Marden. Outer circles. Cambridge University Press, Cambridge, 2007. An introduction to hyperbolic 3-manifolds.

[43] Gregory Margulis. Indefinite quadratic forms and unipotent flows on homogeneous spaces. In Dynamical systems and ergodic theory (Warsaw, 1986), volume 23 of Banach Center Publ., pages 399–409. PWN, Warsaw, 1989.

[44] Gregory Margulis. Orbits of group actions and values of quadratic forms at integral points. In Festschrift in honor of I. I. Piatetski-Shapiro on the occasion of his sixtieth birthday, Part II (Ramat Aviv, 1989), volume 3 of Israel Math. Conf. Proc., pages 127–150. Weizmann, Jerusalem, 1990.

[45] Gregory Margulis. Dynamical and ergodic properties of subgroup actions on homogeneous spaces with applications to number theory. In Proceedings of the International Congress of Mathematicians, Vol. I, II (Kyoto, 1990), pages 193–215, Tokyo, 1991. Math. Soc. Japan.

[46] Gregory Margulis. On some aspects of the theory of Anosov systems. Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2004. With a survey by Richard Sharp: Periodic orbits of hyperbolic flows, Translated from the Russian by Valentina Vladimirovna Szulikowska.

[47] C. T. McMullen. Riemann surfaces, dynamics and geometry. Course notes for Math 275: available at www.math.harvard.edu/~ctm.
[48] C. T. McMullen. Hausdorff dimension and conformal dynamics. III. Computation of dimension. *Amer. J. Math.*, 120(4):691–721, 1998.

[49] Philippe Michel and Akshay Venkatesh. Equidistribution, L-functions and ergodic theory: on some problems of Yu. Linnik. In *International Congress of Mathematicians. Vol. II*, pages 421–457. Eur. Math. Soc., Zürich, 2006.

[50] David Mumford, Caroline Series, and David Wright. *Indra’s pearls*. Cambridge University Press, New York, 2002. The vision of Felix Klein.

[51] Hee Oh and Nimish Shah. The asymptotic distribution of circles in the orbits of Kleinian groups. *Preprint*, 2010.

[52] Hee Oh and Nimish Shah. Counting visible circles on the sphere and Kleinian groups. *Preprint*, 2010.

[53] Hee Oh and Nimish Shah. Equidistribution and counting for orbits of geometrically finite hyperbolic groups. *Preprint (arXive:1001.2096)*, 2010.

[54] John R. Parker. Kleinian circle packings. *Topology*, 34(3):489–496, 1995.

[55] S.J. Patterson. The limit set of a Fuchsian group. *Acta Mathematica*, 136:241–273, 1976.

[56] Marina Ratner. On Raghunathan’s measure conjecture. *Ann. of Math. (2)*, 134(3):545–607, 1991.

[57] Marina Ratner. Raghunathan’s topological conjecture and distributions of unipotent flows. *Duke Math. J.*, 63(1):235–280, 1991.

[58] Marina Ratner. Interactions between ergodic theory, Lie groups, and number theory. In *Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Zürich, 1994)*, pages 157–182, Basel, 1995. Birkhäuser.

[59] Thomas Roblin. Ergodicité et équidistribution en courbure négative. *Mém. Soc. Math. Fr. (N.S.)*, (95):vi+96, 2003.

[60] Daniel J. Rudolph. Ergodic behaviour of Sullivan’s geometric measure on a geometrically finite hyperbolic manifold. *Ergodic Theory Dynam. Systems*, 2(3-4):491–512 (1983), 1982.

[61] Omri Sarig. Unique ergodicity for infinite measures. To appear in Proc. ICM (2010).

[62] Peter Sarnak. Integral Apollonian packings. *MAA Lecture, 2009, available at www.math.princeton.edu/~sarnak*.

[63] Peter Sarnak. Asymptotic behavior of periodic orbits of the horocycle flow and eisenstein series. *Comm. Pure Appl. Math.*, 34(6):719–739, 1981.

[64] Peter Sarnak. Letter to J. Lagarias, 2007. available at www.math.princeton.edu/~sarnak.

[65] Barbara Schapira. Equidistribution of the horocycles of a geometrically finite surface. *Int. Math. Res. Not.*, (40):2447–2471, 2005.

[66] Nimish Shah. Equidistribution of translated submanifolds on homogeneous spaces and Dirichler’s approximation theorem To appear in Proc. ICM (2010).

[67] Nimish Shah. Closures of totally geodesic immersions in manifolds of constant negative curvature. In *Group theory from a geometrical viewpoint (Trieste, 1990)*, pages 718–732. World Sci. Publ., River Edge, NJ, 1991.
[68] Nimish Shah. Uniformly distributed orbits of certain flows on homogeneous spaces. *Math. Ann.*, 289(2):315–334, 1991.

[69] Nimish Shah. Limit distributions of expanding translates of certain orbits on homogeneous spaces. *Proc. Indian Acad. Sci. Math. Sci.*, 106(2):105–125, 1996.

[70] F. Soddy. The bowl of integers and the hexlet. *Nature*, 139:77–79, 1937.

[71] Dennis Sullivan. The density at infinity of a discrete group of hyperbolic motions. *Inst. Hautes Études Sci. Publ. Math.*, (50):171–202, 1979.

[72] Dennis Sullivan. Entropy, Hausdorff measures old and new, and limit sets of geometrically finite Kleinian groups. *Acta Math.*, 153(3-4):259–277, 1984.

[73] Dennis Sullivan. Related aspects of positivity in Riemannian geometry. *J. Differential Geom.*, 25(3):327–351, 1987.

[74] E. Ullmo. Théorie ergodique et géométrie arithmétique. In *Proceedings of the International Congress of Mathematicians, Vol. II (Beijing, 2002)*, pages 197–206, Beijing, 2002. Higher Ed. Press.

[75] E. Ullmo and A. Yafaev. Galois orbits and equidistribution of special subvarieties: towards the André-Oort conjecture, 2006. Preprint.

[76] Vinayak Vatsal. Special values of $L$-functions modulo $p$. In *International Congress of Mathematicians, Vol. II*, pages 501–514. Eur. Math. Soc., Zürich, 2006.

[77] Shing Tung Yau. Harmonic functions on complete Riemannian manifolds. *Comm. Pure Appl. Math.*, 28:201–228, 1975.

[78] Akihiko Yukie. Prehomogeneous vector spaces and ergodic theory. I. *Duke Math. J.*, 90(1):123–147, 1997.