Calculus of Variations on Time Scales with Nabla Derivatives

Natália Martins

Delfim F. M. Torres

Department of Mathematics, University of Aveiro, 3810-193 Aveiro, Portugal

Abstract

We prove a necessary optimality condition of Euler-Lagrange type for variational problems on time scales involving nabla derivatives of higher-order. The proof is done using a new and more general fundamental lemma of the calculus of variations on time scales.

Key words: time scales, calculus of variations, nabla derivatives, Euler-Lagrange equations.

2000 MSC: 39A12, 49K05.

1. Introduction

The theory of time scales was born in 1988 with the work of Stephan Hilger [12], providing a rich theory that unify and extend discrete and continuous analysis [6, 7]. The study of the calculus of variations in the context of time scales has it’s beginning in 2004 with the paper [4] of Martin Bohner. The main result of [4] is an Euler-Lagrange necessary optimality equation for a first order variational problem involving delta derivatives on a time scale $\mathbb{T}$:

Theorem 1 ([4]). If $y^* \in C^2_{rd}$ is a weak local extremum of the problem

$$
L[y(\cdot)] = \int_a^b L(t, y^\sigma(t), y^\Delta(t)) \Delta t \rightarrow \text{extr}, \quad y(a) = \alpha, \quad y(b) = \beta,
$$

where $(t, u, v) \rightarrow L(t, u, v)$ is a $C^2$ function, then the Euler-Lagrange equation

$$
L_{y^\sigma}^\Delta(t, y^\sigma(t), y^\Delta(t)) = L_{y^\Delta}^\sigma(t, y^\sigma(t), y^\Delta(t))
$$

holds for $t \in [a, b]^\kappa$.

Since the pioneer work [4] of Martin Bohner, Theorem 1 has been extended in several different directions in order to analyze variational problems on time scales with: (i) non-fixed boundary conditions [13]; (ii) two independent variables [5]; (iii) higher-order delta derivatives [9]; (iv)
an invariant group of parameter-transformations [3]; (v) multiobjectives [14]; (vi) isoperimetric constraints [10]. A different direction of study, which seems of special interest to applications, in particular to economics, is given in [1] where a Euler-Lagrange type equation is obtained for a first order variational problem on time scales involving nabla derivatives instead of delta ones:

**Theorem 2 ([1]).** If a function \( y_* \in C^2 \) provides a weak local extremum of the problem

\[
\mathcal{L}[y(\cdot)] = \int_{\rho^2(a)}^{\rho^2(b)} L(t,y^{\prime}(t),y^{\prime\prime}(t))\nabla t \rightarrow \text{extr}, \quad y(\rho^2(a)) = \alpha, \ y(\rho^2(b)) = \beta,
\]

where \((t,u,v) \rightarrow L(t,u,v)\) is a \( C^2 \) function of \((u,v)\) for each \( t \in [\rho^2(a), \rho^2(b)] \subseteq T\), then \( y_* \) satisfies the Euler-Lagrange equation

\[
L^\nabla_y(t,y^\prime(t),y^{\prime\prime}(t)) = L^\rho_{y^\prime}(t,y^\prime(t),y^{\prime\prime}(t))
\]

for \( t \in [\rho(a), b] \).

The proof of Theorem 2 found in [1] has, however, some inconsistencies [8]. Moreover, the nabla problem (2) is defined in a way that is not completely analogous to the more studied and established delta problem (1) (i.e., one does not obtain (2) from (1) by simply substituting the forward jump operator \( \sigma \) by the backward jump operator \( \rho \) and the delta derivative \( \Delta \) by the nabla derivative \( \nabla \)). The main goal of this paper is to generalize the results of [1] to higher-order nabla variational problems on time scales in a consistent way with the delta theory. This is done by first proving some new fundamental lemmas of the calculus of variations on time scales that fix the inconsistencies of [1] pointed out in [8]. Compared with the delta approach followed in [9], our technique provides a simpler and more direct proof to the higher-order Euler-Lagrange equations. Moreover, our proof seems to be new even for the continuous time.

The paper is organized as follows. In Section 2 we collect all the necessary elements of the nabla calculus on time scales. In Section 3 we state and prove our results; in Section 4 we prove a new and more general fundamental lemma of the calculus of variations on time scales (Lemma 16); in §3.2 we obtain a higher-order nabla differential Euler-Lagrange equation (Theorem 17). As an example, we give the Euler-Lagrange equation for the q-calculus variational problem (Corollary 18).

### 2. Preliminary results

For a general introduction to the calculus on time scales we refer the reader to the books [6, 7]. Here we only give those notions and results needed in the sequel. More precisely, we are interested in the nabla approach to time scales [2]. As usual, \( \mathbb{R}, \mathbb{Z}, \) and \( \mathbb{N} \) denote, respectively, the set of real, integer, and natural numbers.

A **Time Scale** \( T \) is an arbitrary non empty closed subset of \( \mathbb{R} \). Thus, \( \mathbb{R}, \mathbb{Z}, \) and \( \mathbb{N} \), are trivial examples of times scales. Other examples of times scales are: \([-1, 4] \cup \mathbb{N}, h\mathbb{Z} := \{hz | z \in \mathbb{Z}\} \) for some \( h \geq 0 \), \( q^\mathbb{N} := \{q^k | k \in \mathbb{N}_0\} \) for some \( q > 1 \), and the Cantor set. We assume that a time scale \( T \) has the topology that it inherits from the real numbers with the standard topology.

The **forward jump operator** \( \sigma : T \rightarrow T \) is defined by \( \sigma(t) = \inf \{s \in T : s > t\} \) if \( t \neq \sup T \), and \( \sigma(\sup T) = \sup T \). The **backward jump operator** \( \rho : T \rightarrow T \) is defined by \( \rho(t) = \sup \{s \in T : s < t\} \) if \( t \neq \inf T \), and \( \rho(\inf T) = \inf T \).
A point \( t \in \mathbb{T} \) is called right-dense, right-scattered, left-dense and left-scattered if \( \sigma(t) = t, \sigma(t) > t, \rho(t) = t, \) and \( \rho(t) < t, \) respectively. We say that \( t \) is isolated if \( \rho(t) < t < \sigma(t), \) that \( t \) is dense if \( \rho(t) = t = \sigma(t). \) The (backward) graininess function \( \nu : \mathbb{T} \to [0, \infty) \) is defined by \( \nu(t) = t - \rho(t), \) for all \( t \in \mathbb{T}. \) Hence, for a given \( t, \) \( \nu(t) \) measures the distance of \( t \) to its left neighbor. It is clear that when \( \mathbb{T} = \mathbb{R} \) one has \( \sigma(t) = t = \rho(t), \) and \( \nu(t) = 0 \) for any \( t. \) When \( \mathbb{T} = \mathbb{Z}, \sigma(t) = t + 1, \rho(t) = t - 1, \) and \( \nu(t) = 1 \) for any \( t. \)

In order to introduce the definition of nabla derivative, we define a new set \( \mathbb{T}_\epsilon \) which is derived from \( \mathbb{T} \) as follows: if \( \mathbb{T} \) has a right-scattered minimum \( m, \) then \( \mathbb{T}_\epsilon = \mathbb{T} \setminus \{m\}; \) otherwise, \( \mathbb{T}_\epsilon = \mathbb{T}. \)

**Definition 1.** We say that a function \( f : \mathbb{T} \to \mathbb{R} \) is nabla differentiable at \( t \in \mathbb{T}_\epsilon \) if there is a number \( f^\nabla(t) \) such that for all \( \varepsilon > 0 \) there exists a neighborhood \( U \) of \( t \) (i.e., \( U = ]t - \delta, t + \delta[ \cap \mathbb{T} \) for some \( \delta > 0 \)) such that

\[
|f(\rho(t)) - f(s) - f^\nabla(t)(\rho(t) - s)| \leq \varepsilon|\rho(t) - s|, \text{ for all } s \in U.
\]

We call \( f^\nabla(t) \) the nabla derivative of \( f \) at \( t. \) Moreover, we say that \( f \) is nabla differentiable on \( \mathbb{T} \) provided \( f^\nabla(t) \) exists for all \( t \in \mathbb{T}_\epsilon. \)

**Theorem 3.** Let \( \mathbb{T} \) be a time scale, \( f : \mathbb{T} \to \mathbb{R}, \) and \( t \in \mathbb{T}_\epsilon. \) The following holds:

1. If \( f \) is nabla differentiable at \( t, \) then \( f \) is continuous at \( t. \)
2. If \( f \) is continuous at \( t \) and \( t \) is left-scattered, then \( f \) is nabla differentiable at \( t \) and

\[
f^\nabla(t) = \frac{f(t) - f(\rho(t))}{t - \rho(t)}.
\]

3. If \( t \) is left-dense, then \( f \) is nabla differentiable at \( t \) if and only if the limit

\[
\lim_{s \to t} \frac{f(t) - f(s)}{t - s}
\]

exists as a finite number. In this case,

\[
f^\nabla(t) = \lim_{s \to t} \frac{f(t) - f(s)}{t - s}.
\]

4. If \( f \) is nabla differentiable at \( t, \) then \( f(\rho(t)) = f(t) - \nu(t)f^\nabla(t). \)

**Remark 1.** When \( \mathbb{T} = \mathbb{R}, \) then \( f : \mathbb{R} \to \mathbb{R} \) is nabla differentiable at \( t \in \mathbb{R} \) if and only if

\[
f^\nabla(t) = \lim_{s \to t} \frac{f(t) - f(s)}{t - s}
\]

exists, i.e., if and only if \( f \) is differentiable at \( t \) in the ordinary sense. When \( \mathbb{T} = \mathbb{Z}, \) then \( f : \mathbb{Z} \to \mathbb{R} \) is always nabla differentiable at \( t \in \mathbb{Z} \) and

\[
f^\nabla(t) = \frac{f(t) - f(\rho(t))}{t - \rho(t)} = f(t) - f(t - 1),
\]

i.e., \( f^\nabla \) is the usual backward difference. For any time scale \( \mathbb{T}, \) when \( f \) is a constant, then \( f^\nabla = 0; \) if \( f(t) = kt \) for some constant \( k, \) then \( f^\nabla = k. \)
In order to simplify expressions, we denote the composition $f \circ \rho$ by $f^\rho$. We also use the standard notation $\rho^0(t) = t$, $\rho^n(t) = \rho(\rho^{n-1}(t))$, and $\sigma^n(t) = (\sigma \circ \sigma^{n-1})(t)$ for $n \in \mathbb{N}$.

**Theorem 4.** Suppose $f, g : \mathbb{T} \to \mathbb{R}$ are nabla differentiable at $t \in \mathbb{T}$. Then,

1. the sum $f + g : \mathbb{T} \to \mathbb{R}$ is nabla differentiable at $t$ and $(f + g)^\nabla(t) = f^\nabla(t) + g^\nabla(t)$;
2. for any constant $\alpha$, $\alpha f : \mathbb{T} \to \mathbb{R}$ is nabla differentiable at $t$ and $(\alpha f)^\nabla(t) = \alpha f^\nabla(t)$;
3. the product $fg : \mathbb{T} \to \mathbb{R}$ is nabla differentiable at $t$ and

\[
(fg)^\nabla(t) = f^\nabla(t)g(t) + f(t)g^\nabla(t).
\]

Nabla derivatives of higher order are defined in the standard way: we define the $r$th–nabla derivative $(r \in \mathbb{N})$ of $f$ to be the function $f^{\nabla r} : \mathbb{T}_r \to \mathbb{R}$, provided $f^{\nabla r-1}$ is nabla differentiable on $\mathbb{T}_r := (\mathbb{T}_r-1)_r$.

**Definition 2.** A function $F : \mathbb{T} \to \mathbb{R}$ is called a nabla antiderivative of $f : \mathbb{T} \to \mathbb{R}$ provided

\[
F^\nabla(t) = f(t), \quad \forall t \in \mathbb{T}_r.
\]

In this case we define the nabla integral of $f$ from $a$ to $b$ ($a, b \in \mathbb{T}$) by

\[
\int_a^b f(t)\nabla t := F(b) - F(a).
\]

In order to present a class of functions that possess a nabla antiderivative, the following definition is introduced:

**Definition 3.** Let $\mathbb{T}$ be a time scale, $f : \mathbb{T} \to \mathbb{R}$. We say that function $f$ is ld-continuous if it is continuous at the left-dense points and its right-sided limits exist (finite) at all right-dense points.

Some results concerning ld-continuity are useful:

**Theorem 5.** Let $\mathbb{T}$ be a time scale, $f : \mathbb{T} \to \mathbb{R}$.

1. If $f$ is continuous, then $f$ is ld-continuous.
2. The backward jump operator $\rho$ is ld-continuous.
3. If $f$ is ld-continuous, then $f^\rho$ is also ld-continuous.
4. If $\mathbb{T} = \mathbb{R}$, then $f$ is continuous if and only if $f$ is ld-continuous.
5. If $\mathbb{T} = \mathbb{Z}$, then $f$ is ld-continuous.

**Theorem 6.** Every ld-continuous function has a nabla antiderivative. In particular, if $a \in \mathbb{T}$, then the function $F$ defined by

\[
F(t) = \int_a^t f(\tau)\nabla \tau, \quad t \in \mathbb{T},
\]

is a nabla antiderivative of $f$. 

4
The set of all ld-continuous functions \( f : T \rightarrow \mathbb{R} \) is denoted by \( C_{ld}(T, \mathbb{R}) \), and the set of all nabla differentiable functions with ld-continuous derivative by \( C_{ld}^t(T, \mathbb{R}) \).

**Theorem 7.** If \( f \in C_{ld}(T, \mathbb{R}) \) and \( t \in T \), then

\[
\int_{\rho(t)}^{a} f(\tau)\nabla \tau = v(t)f(t).
\]

**Theorem 8.** If \( a, b, c \in T, a \leq c \leq b, a \in \mathbb{R}, \) and \( f, g \in C_{ld}(T, \mathbb{R}) \), then

1. \( \int_{a}^{b} (f(t) + g(t)) \nabla t = \int_{a}^{b} f(t) \nabla t + \int_{a}^{b} g(t) \nabla t; \)
2. \( \int_{a}^{b} \alpha f(t) \nabla t = \alpha \int_{a}^{b} f(t) \nabla t; \)
3. \( \int_{a}^{b} f(t) \nabla t = -\int_{b}^{a} f(t) \nabla t; \)
4. \( \int_{a}^{b} f(t) \nabla t = 0; \)
5. \( \int_{a}^{b} f(t) \nabla t = \int_{a}^{c} f(t) \nabla t + \int_{c}^{b} f(t) \nabla t; \)
6. If \( f(t) > 0 \) for all \( a < t \leq b \), then \( \int_{a}^{b} f(t) \nabla t > 0; \)
7. \( \int_{a}^{b} f^\alpha(t) g^{\nabla}(t) \nabla t = \left[ (fg)(t) \right]_{a}^{b} \int_{a}^{b} f^\alpha(t) g(t) \nabla t; \)
8. \( \int_{a}^{b} f(t) g^{\nabla}(t) \nabla t = \left[ (fg)(t) \right]_{a}^{b} \int_{a}^{b} f(t) g^\alpha(t) \nabla t. \)

The last two formulas on Theorem 8 are usually called integration by parts formulas. These formulas are used several times in this work.

**Remark 2.** Let \( a, b \in T \) and \( f \in C_{ld}(T, \mathbb{R}) \). For \( T = \mathbb{R} \), then \( \int_{a}^{b} f(t) \nabla t = \int_{a}^{b} f(t) dt \), where the integral on the right side is the usual Riemann integral. For \( T = \mathbb{Z} \), then \( \int_{a}^{b} f(t) \nabla t = \sum_{i=a+1}^{b} f(t) \) if \( a < b, \int_{a}^{b} f(t) \nabla t = 0 \) if \( a = b \), and \( \int_{a}^{b} f(t) \nabla t = -\sum_{i=b+1}^{a} f(t) \) if \( a > b \).

Let \( a, b \in T \) with \( a < b \). We define the interval \([a, b] \) in \( T \) by

\[ [a, b] := \{ t \in T : a \leq t \leq b \}. \]

Open intervals and half-open intervals in \( T \) are defined accordingly. Note that \([a, b] = [a, b] \) if \( a \) is right-dense and \([a, b] = [a, b] \) if \( a \) is right-scattered.
3. Main results

Our main objective is to establish a necessary optimality condition for problems of the calculus of variations with higher-order nabla derivatives. We formulate the higher-order variational problem with nabla derivatives as follows:

\[ L[y(\cdot)] = \int_{\sigma^{-1}(a)}^{b} L(t, y^{\sigma^{-1}}(t), \ldots, y^{\sigma^{-1}}(t), y^{\sigma^{-1}}(t)) \nabla t \to \text{extr}, \]

where \( r \in \mathbb{N} \) and \( \sigma^{-1} \) is a function such that \( \sigma^{-1}(a) = a \) and \( \sigma^{-1}(b) = b \). We assume that:

1. The admissible functions \( y \) are of class \( C^2_r([a, b] \cap T \to \mathbb{R}) : y^{\sigma^{-1}} \) is continuous on \([a, b] \cap T \)

2. \( a, b \in T, a < b \), and \([a, b] \cap T \) has, at least, \( 2r + 1 \) points (cf. Remark 12);

3. the Lagrangian \( L(t, u_0, u_1, \ldots, u_r) \) has (standard) continuous partial derivatives with respect to \( u_0, \ldots, u_r \), and partial nabla derivative with respect to \( t \) of order \( r + 1 \).

Remark 3. For \( T = \mathbb{R} \) problem (P) coincides with the classical problem of the calculus of variations with higher-order derivatives (see, e.g., [11]). Similar to \( \mathbb{R} \), the results of the paper are easily extended to the vectorial case, i.e., to the case when admissible functions \( y \) belong to \( C^{2r}([a, b], \mathbb{R}^n) \).

Remark 4. For \( r = 1 \), problem (P) provides the nabla analog of the delta problem on time scales (1) studied in [3, 4, 13, 14], thus providing, in our opinion, a better formulation than that of [1] (cf. REMARK 2). For \( r > 1 \), restrictions to the time scale \( T \) must be done in order for (P) to be well defined (cf. Remark 6 below).

We begin with some technical results that will be useful in the proof of our fundamental lemmas (cf. §3.1).

**Proposition 9.** Suppose that \( a, b \in T, a < b \), and \( f \in C_{ld}([a, b], \mathbb{R}) \) is such that \( f \geq 0 \) on \([a, b] \cap T \).

If \( \int_a^b f(t) \nabla t = 0 \), then \( f = 0 \) on \([a, b] \cap T \).

**Proof.** Suppose, by contradiction, that there exists \( t_0 \in [a, b] \) such that \( f(t_0) > 0 \). If \( t_0 \) is left-scattered, then by the properties of the integral (Theorems 7 and 8) we may conclude that

\[
\int_a^{t_0} f(t) \nabla t = \int_a^{t_0} f(t) \nabla t + \int_{t_0}^b f(t) \nabla t + \int_{t_0}^b f(t) \nabla t \\
\geq \int_{t_0}^b f(t) \nabla t = f(t_0) (t_0 - \rho(t_0)) > 0
\]

We use this proposition to prove the following result.
which leads to a contradiction. Suppose now that \( t_0 \) is left-dense. If \( t_0 \neq a \), then by the continuity of \( f \) at \( t_0 \) we may conclude that there exists a \( \delta > 0 \) such that, for all \( t \in [t_0, t_0 + \delta] \), \( f(t) > 0 \). Since
\[
\int_a^b f(t) \nabla t = \int_a^{t_0} f(t) \nabla t + \int_{t_0}^{t_0 + \delta} f(t) \nabla t + \int_{t_0 + \delta}^b f(t) \nabla t
\]
we get a contradiction (\( \delta \) may be chosen in such a way that \( t_0 - \delta > a \)). It remains to study the case when \( t_0 = a \). If \( t_0 = a \) is right-dense, then by the continuity of \( f \) at \( t_0 \) we may conclude that there exists a \( \delta > 0 \) such that, for all \( t \in [t_0, t_0 + \delta] \), \( f(t) > 0 \). Since
\[
\int_a^b f(t) \nabla t = \int_a^{t_0} f(t) \nabla t + \int_{t_0}^{t_0 + \delta} f(t) \nabla t + \int_{t_0 + \delta}^b f(t) \nabla t
\]
we obtain again a contradiction. Note that if \( t_0 = a \) and \( a \) is right-scattered, then \( a \notin [a, b] \).

**Remark 5.** In Proposition [9] we cannot conclude that \( f = 0 \) on \([a, b]\). For example, consider \( T = \{1, 2, 3, 4, 5\} \) and \( f(t) = 1 \) if \( t = 1 \) and \( f(t) = 0 \) otherwise. Clearly, \( f \) is continuous and \( f \geq 0 \) on \( T \). We have
\[
\int_1^5 f(t) \nabla t = \sum_{i=2}^5 f(t) = 0,
\]
but \( f \neq 0 \) on \([1, 5]\).

From now on we restrict ourselves to time scales \( T \) that satisfy the following condition \((H)\):
\[(H) \quad \text{for each } t \in T, (r - 1)(\rho(t) - a_1 t - a_0) = 0 \text{ for some } a_1 \in \mathbb{R}^+ \text{ and } a_0 \in \mathbb{R}.
\]

**Remark 6.** Condition \((H)\) is equivalent to \( r = 1 \) or \( \rho(t) = a_1 t + a_0 \) for some \( a_1 \in \mathbb{R}^+ \) and \( a_0 \in \mathbb{R} \). Thus, for the first order problem of the calculus of variations we impose no restriction on the time scale \( T \). For the higher-order problems (i.e., for \( r > 1 \)) such restriction on the time scale is necessary. Indeed, for \( r > 1 \) we are implicitly assuming in (P) that \( \rho \) is nabla differentiable, which is not true for a general time scale \( T \).

**Remark 7.** Let \( r > 1 \). Condition \((H)\) implies then that \( \rho \) is nabla differentiable. Hence, \( \nu \) is also nabla differentiable and \( \nu^q(t) = a_1, t \in T_{a_1} \). Also note that condition \((H)\) englobes the differential calculus \((T = \mathbb{R}, a_1 = 1, a_0 = 0)\), the difference calculus \((T = \mathbb{Z}, a_1 = 1, a_0 = -1)\), and the q-calculus \((T = [q^n : k \in \mathbb{N}_0] \text{ for some } q > 1, a_1 = \frac{1}{q}, a_0 = 0)\).

**Lemma 10.** Let \( t \in T_{a_1} \) and \( t \neq \max T \) (if the maximum exists) satisfy the property \( \rho(t) < \sigma(t) = t \). Then, the backward jump operator \( \rho \) is not nabla differentiable at \( t \).

**Proof.** We prove that \( \rho \) is not continuous at \( t \in T_{a_1} \setminus \{\max T\} \), which implies that \( \rho \) is not nabla differentiable at \( t \). We begin by proving that \( \lim_{s \to t} \rho(s) = t \). Let \( \varepsilon > 0 \) and take \( \delta = \varepsilon \). Then, for all \( s \in ]t, t + \delta[ \) we have \( |\rho(s) - t| \leq |s - t| < \delta = \varepsilon \). Since \( \rho(t) \neq t \), \( \rho \) is not continuous at \( t \).
The following simply remark is very useful for our objectives:

**Remark 8.** Since condition (H) implies for $r > 1$ that $\rho$ is nabla differentiable, it follows from Lemma [10] that for the higher-order problem $T \setminus \{\max T\}$ cannot contain points that are simultaneously right-dense and left-scattered.

**Lemma 11.** Assume hypothesis (H) and $r > 1$. If $f : T \to \mathbb{R}$ is two times nabla differentiable, then

$$f^{\nu_r}(t) = a_1 f^{\nu_1}(t), \quad t \in T_{a_1}.$$

**Proof.** From Theorem [5] we know that $f^{\nu}(t) = f(t) - \nu(t)f^{\nu}(t)$. Thus,

$$f^{\nu_r}(t) = (f(t) - \nu(t)f^{\nu}(t))^r = f^{\nu}(t) - \nu(t)f^{\nu_1}(t) = f^{\nu_1}(t) - \nu(t)f^{\nu_1}(t).$$

Since $\nu(t) = 1 - a_1$, we may conclude that $f^{\nu_r}(t) = a_1 f^{\nu_1}(t)$ for all $t \in T_{a_1}$.

The next two lemmas are very useful for the proof of our higher-order fundamental lemma of the calculus of variations on time scales (Lemma [16]).

**Lemma 12.** Assume that the time scale $T$ satisfies condition (H) and $\eta \in C^{2r}$ is such that

$$\eta^{\nu_r}(b) = 0 \quad \text{for all} \quad i \in \{0, 1, \ldots, r\}.$$

Then, $\eta^{\nu_{r-1}}(b) = 0$.

**Proof.** If $b$ is left-dense, then the result is trivial (just use Lemma [11] and the fact that $\rho(b) = b$). Suppose that $b$ is left-scattered and fix $i \in \{1, 2, \ldots, r\}$. From item 2 of Theorem [3] we then conclude that

$$\eta^{\nu_i}(b) = \left(\eta^{\nu_{i-1}}(b)\right)^r = \frac{\eta^{\nu_{i-1}}(b) - \left(\eta^{\nu_{i-1}}(b)\right)^r}{\nu(b)}.$$

Since $\eta^{\nu_i}(b) = 0$ and $\eta^{\nu_{i-1}}(b) = 0$, then

$$\left(\eta^{\nu_{i-1}}(b)\right)^r = 0.$$

Lemma [11] shows that

$$\left(\eta^{\nu_{i-1}}(b)\right)^r = \left(\frac{1}{a_1}\right)^{i-1} (\eta^{\nu_{i-1}}(b)),$$

and we conclude that $\eta^{\nu_{i-1}}(b) = 0$.

**Lemma 13.** Assume that the time scale $T$ satisfies condition (H) and $\eta \in C^{2r}$ is such that

$$\eta^{\nu_i}(\sigma^r(a)) = 0, \quad i \in \{0, 1, \ldots, r\}.$$

Then,

$$\eta^{\nu_i}(\sigma^r(a)) = 0, \quad i \in \{0, 1, \ldots, r-1\}.$$

**Proof.** If $a$ is right-dense, the result is trivial. Suppose that $a$ is right-scattered (hence, $\sigma(a)$ is left-scattered). Since $\rho$ is nabla differentiable, by Remark [8] we cannot have points that are simultaneously right-dense and left-scattered. Hence, $\sigma(a), \sigma^2(a), \sigma^3(a), \ldots, \sigma^r(a)$ are left-scattered points. By item 2 of Theorem [3] we conclude that

$$\eta^{\nu_i}(\sigma^r(a)) = \frac{\eta^{\nu_{i-1}}(\sigma^r(a)) - \eta^{\nu_{i-1}}(\sigma^r(a))}{\nu(\sigma^r(a))} = \frac{\eta^{\nu_{i-1}}(\sigma^r(a)) - \eta^{\nu_{i-1}}(\sigma^r(a))}{\nu(\sigma^r(a))}.$$
We have $\eta^{r}(\sigma\tau(a)) = 0$ and $\eta^{r-1}(\sigma\tau(a)) = 0$. Then,

$$\eta^{r-1}(\sigma^{r-1}(a)) = 0.$$  

From item 2 of Theorem 3.

$$\eta^{r-1}(\sigma^{r}(a)) = \frac{\eta^{r-2}(\sigma^{r}(a)) - (\eta^{r-2})'(\sigma^{r}(a))}{\nu(\sigma^{r}(a))} = \frac{\eta^{r-2}(\sigma^{r}(a)) - \eta^{r-2}(\sigma^{r-1}(a))}{\nu(\sigma^{r}(a))},$$

and using the hypothesis of the lemma we conclude that $\eta^{r-2}(\sigma^{r-1}(a)) = 0$. Since

$$\eta^{r-1}(\sigma^{r-1}(a)) = \frac{\eta^{r-2}(\sigma^{r-1}(a)) - (\eta^{r-2})'(\sigma^{r-1}(a))}{\nu(\sigma^{r-1}(a))} = \frac{\eta^{r-2}(\sigma^{r-1}(a)) - \eta^{r-2}(\sigma^{r-2}(a))}{\nu(\sigma^{r-1}(a))},$$

we obtain

$$\eta^{r-2}(\sigma^{r-2}(a)) = 0.$$  

Repeating recursively this process, we conclude the proof.

### 3.1. Fundamental lemmas of the calculus of variations on time scales

We now present some fundamental lemmas of the calculus of variations on time scales involving nabla derivatives. This gives answer to a problem posed in [8, §3.2]. In what follows we assume that $a, b \in \mathbb{T}, a < b$, and $\mathbb{T}$ has sufficiently many points in order for all the calculations to make sense.

**Lemma 14.** Let $f \in C([a, b], \mathbb{R})$. If

$$\int_{a}^{b} f(t)\eta^{\tilde{r}}(t)\nabla t = 0 \quad \text{for all} \quad \eta \in C^{1}([a, b], \mathbb{R}) \quad \text{such that} \quad \eta(a) = \eta(b) = 0$$

then

$$f(t) = c \quad \forall t \in [a, b]_{\mathbb{R}}$$

for some $c \in \mathbb{R}$.

**Proof.** Let $c$ be a constant defined by the condition

$$\int_{a}^{b} (f(\tau) - c) \nabla \tau = 0$$

and let

$$\eta(t) = \int_{a}^{t} (f(\tau) - c) \nabla \tau.$$  

Clearly, $\eta \in C^{1}([a, b], \mathbb{R})$ (by Theorem [8], $\eta^{\tilde{r}}(t) = f(t) - c$) and

$$\eta(a) = \int_{a}^{b} (f(\tau) - c) \nabla \tau = 0 \quad \text{and} \quad \eta(b) = \int_{a}^{b} (f(\tau) - c) \nabla \tau = 0.$$  

Observe that

$$\int_{a}^{b} (f(t) - c) \eta^{\tilde{r}}(t)\nabla t = \int_{a}^{b} (f(t) - c)^2 \nabla t.$$
Lemma 15. Let \( f, g \in C([a, b], \mathbb{R}) \). If
\[
\int_a^b (f(t) - c) \eta^\nabla(t) \nabla t = \int_a^b f(t)\eta^\nabla(t) \nabla t - c \int_a^b \eta^\nabla(t) \nabla t = 0 - c (\eta(b) - \eta(a)) = 0.
\]
Hence,
\[
\int_a^b (f(t) - c)^2 \nabla t = 0
\]
which shows, by Proposition 9, that
\[
f(t) - c = 0, \quad \forall t \in [a, b].
\]

**Lemma 16.** Let \( f, g \in C([a, b], \mathbb{R}) \). If
\[
\int_a^b (f(t)\eta^\nabla(t) + g(t)\eta^\nabla(t)) \nabla t = 0
\]
for all \( \eta \in C^1([a, b], \mathbb{R}) \) such that \( \eta(a) = \eta(b) = 0 \), then \( g \) is nabla differentiable and
\[
g^\nabla(t) = f(t) \quad \forall t \in [a, b].
\]

**Proof.** Define \( A(t) = \int_a^t f(\tau) \nabla \tau \). Then \( A^\nabla(t) = f(t) \) for all \( t \in [a, b] \) (by Theorem 9) and
\[
\int_a^b A(t)\eta^\nabla(t) \nabla t = [A(t)\eta(t)]_{t=a}^{t=b} - \int_a^b A^\nabla(t)\eta^\nabla(t) \nabla t = -\int_a^b f(t)\eta^\nabla(t) \nabla t
\]
(by property 8 of Theorem 9). Hence,
\[
\int_a^b (f(t)\eta^\nabla(t) + g(t)\eta^\nabla(t)) \nabla t = 0 \iff \int_a^b (-A(t) + g(t)) \eta^\nabla(t) \nabla t = 0.
\]
By Lemma 14, we may conclude that \( -A(t) + g(t) = c \) for all \( t \in [a, b] \) and some \( c \in \mathbb{R} \). Therefore, \( A^\nabla(t) = g^\nabla(t) \) for all \( t \in [a, b] \), proving the desired result: \( g^\nabla(t) = f(t) \) for all \( t \in [a, b] \).

**Remark 9.** If we consider \( \mathbb{T} = \mathbb{R} \) in Lemmas 14 and 15, we obtain some well known fundamental lemmas of the classical calculus of variations (see, e.g., [11, pp. 10–11]).

**Remark 10.** Lemma 14 remains true if \( f \) is of class \( C_{id} \) and the variation \( \eta \) is of class \( C_{id}^1 \). Similar observation holds for Lemma 15.

We now prove a new and more general fundamental lemma of the calculus of variations. Lemma 16 is used to prove our Euler-Lagrange equation for variational problems on time scales involving nabla derivatives of higher-order (Theorem 17).

**Lemma 16 (higher-order fundamental lemma of the calculus of variations).** Let \( \mathbb{T} \) be a time scale satisfying condition (H), and \( f_0, f_1, \ldots, f_r \in C([a, b], \mathbb{R}) \). If
\[
\int_a^b \left( \sum_{i=0}^r f_i(t) \eta^{r-i}(t) \right) \nabla t = 0
\]
for all \( \eta \in C^2_r([a, b], \mathbb{R}) \) such that
\[
\eta(\sigma^r(a)) = 0, \quad \eta(b) = 0,
\]
\[
\vdots
\]
\[
\eta^{\kappa-1}(\sigma^r(a)) = 0, \quad \eta^{\kappa-1}(b) = 0,
\]
then
\[
\sum_{i=0}^{r} (-1)^i \left( \frac{1}{d_i} \right)^{\kappa-i} f_i^{\kappa}(t) = 0, \quad t \in [a, b]_{\kappa'}.
\]

Proof. We prove the lemma by mathematical induction. If \( r = 1 \), the result is true by Lemma \[15\]. Assume now that the result is true for some \( r, r > 1 \). We want to prove that the result is then true for \( r + 1 \). Suppose that
\[
\int_{\sigma^r(a)}^{b} \left( \sum_{i=0}^{r+1} f_i(t)\eta^{\kappa-r-i}(t) \right) \nabla t = 0
\]
for all \( \eta \in C^{2(r+1)}([a, b], \mathbb{R}) \) such that \( \eta(\sigma^r(a)) = 0, \eta(b) = 0, \ldots, \eta^{\kappa-1}(\sigma^r(a)) = 0, \eta^{\kappa-1}(b) = 0 \). We must prove that
\[
\sum_{i=0}^{r+1} (-1)^i \left( \frac{1}{d_i} \right)^{\kappa-i} f_i^{\kappa}(t) = 0, \quad t \in [a, b]_{\kappa+1}.
\]

Note that
\[
\int_{\sigma^r(a)}^{b} \left( \sum_{i=0}^{r+1} f_i(t)\eta^{\kappa-r-i}(t) \right) \nabla t = \int_{\sigma^r(a)}^{b} \left( \sum_{i=0}^{r} f_i(t)\eta^{\kappa-r-i}(t) \right) \nabla t + \int_{\sigma^r(a)}^{b} f_{r+1}(t) \left( \eta^{\kappa} \right)^{\kappa}(t) \nabla t
\]
and using the integration by parts formula (item 8 of Theorem \[8\])
\[
\int_{\sigma^r(a)}^{b} f_{r+1}(t) \left( \eta^{\kappa} \right)^{\kappa}(t) \nabla t = \left[ f_{r+1}(t) \eta^{\kappa}(t) \right]^{b}_{a=\sigma^r(a)} - \int_{\sigma^r(a)}^{b} f_{r+1}(t) \left( \eta^{\kappa} \right)^{\kappa+1}(t) \nabla t.
\]
Since \( \eta^{\kappa}(\sigma^r(a)) = 0 \) and \( \eta^{\kappa}(b) = 0 \), we may conclude that
\[
\int_{\sigma^r(a)}^{b} f_{r+1}(t) \left( \eta^{\kappa} \right)^{\kappa}(t) \nabla t = - \int_{\sigma^r(a)}^{b} f_{r+1}(t) \left( \eta^{\kappa} \right)^{\kappa+1}(t) \nabla t.
\]

By Lemma \[11\]
\[
\eta^{\kappa+1}(t) = \left( \frac{1}{d_1} \right)^{\kappa+1} \eta^{\kappa}(t), \quad t \in [a, b]_{\kappa+1}.
\]
Hence,
\[
\int_{\sigma^r(a)}^{b} f_{r+1}(t) \left( \eta^{\kappa} \right)^{\kappa}(t) \nabla t = - \int_{\sigma^r(a)}^{b} f_{r+1}(t) \left( \frac{1}{d_1} \right)^{\kappa} \eta^{\kappa}(t) \nabla t.
\]
and

\[ \int_{\sigma^{(a)}}^{b} \left( \sum_{j=0}^{r} f_i(t) \eta^{r+1-i} \right) \nabla t = \]

\[ = \int_{\sigma^{(a)}}^{b} \left( \sum_{j=0}^{r} f_i(t) \eta^{r+1-i} \right) \nabla t - \int_{\sigma^{(a)}}^{b} f_i^{(1)}(t) \left( \frac{1}{\alpha_1} \right) \eta^{r} \nabla t \]

\[ = \int_{\sigma^{(a)}}^{b} \left( \sum_{j=0}^{r} f_i(t) (\eta^{r+1-i}) \right) \nabla t - \int_{\sigma^{(a)}}^{b} f_i^{(1)}(t) \left( \frac{1}{\alpha_1} \right) \eta^{r} \nabla t \]

\[ = \int_{\sigma^{(a)}}^{b} \left( \sum_{j=0}^{r} f_i(t) (\eta^{r+1-i}) \right) \nabla t - \int_{\sigma^{(a)}}^{b} f_i^{(1)}(t) \left( \frac{1}{\alpha_1} \right) \eta^{r} \nabla t \]

\[ + \int_{\sigma^{(a)}}^{b} \left[ \sum_{j=0}^{r} f_i(t) (\eta^{r+1-i}) \right] \nabla t - \int_{\sigma^{(a)}}^{b} f_i^{(1)}(t) \left( \frac{1}{\alpha_1} \right) \eta^{r} \nabla t \]

We now prove that

\[ \int_{\sigma^{(a)}}^{b} \left( \sum_{j=0}^{r} f_i(t) (\eta^{r+1-i}) \right) \nabla t - \int_{\sigma^{(a)}}^{b} f_i^{(1)}(t) \left( \frac{1}{\alpha_1} \right) \eta^{r} \nabla t \]  \( (3) \)

is equal to zero. By Theorem[7] the integral \( (3) \) is equal to

\[ \left[ \sum_{i=0}^{r-1} f_i(\sigma^{r+1}(a)) (\eta^{r+1-i}) (\sigma^{r+1}(a)) \right] \\
+ \left[ f_i(\sigma^{r+1}(a)) - f_i^{(1)}(\sigma^{r+1}(a)) \left( \frac{1}{\alpha_1} \right) (\eta^{r}) (\sigma^{r+1}(a)) \right] \nu(\sigma^{r+1}(a)). \]

For each \( i \in \{0, 1, \ldots, r\}, \)

\[ (\eta^{r+1-i}) (\sigma^{r+1}(a)) = \eta^{r+1-i} (\sigma^{r+1}(a)) \]

\[ = (\alpha_1)^{\delta r+1-i} \eta^{r+1-i} (\sigma^{r+1}(a)) \quad \text{(by Lemma[11])} \]

\[ = (\alpha_1)^{\delta r+1-i} \eta^{r} (\sigma^{r+1}(a)) \]

\[ = 0 \quad \text{(by Lemma[13])} \]

proving that the integral \( (3) \) is equal to zero. Then,

\[ \int_{\sigma^{(a)}}^{b} \left( \sum_{j=0}^{r} f_i(t) \eta^{r+1-i} \right) \nabla t = \]

\[ = \int_{\sigma^{(a)}}^{b} \left( \sum_{j=0}^{r} f_i(t) (\eta^{r+1-i}) \right) \nabla t - \int_{\sigma^{(a)}}^{b} f_i^{(1)}(t) \left( \frac{1}{\alpha_1} \right) (\eta^{r}) \nabla t. \]
Observe that,
\[ \eta^p \left( \sigma^{r+1} (a) \right) = \eta (\sigma^r (a)) = 0 \]
\[ (\eta^p)^V \left( \sigma^{r+1} (a) \right) = a_1 \eta^V (\sigma^r (a)) = 0 \]
\[ \vdots \]
\[ (\eta^p)^{V-1} \left( \sigma^{r+1} (a) \right) = (a_1)^{r-1} \eta^{V-1} (\sigma^r (a)) = 0 \]
and, by Lemma 12,
\[ \eta^p (b) = 0 \]
\[ (\eta^p)^V (b) = 0 \]
\[ \vdots \]
\[ (\eta^p)^{V-1} (b) = 0. \]

Then, by the induction hypothesis, we conclude that
\[ \sum_{i=0}^{r-1} (-1)^i \left( \frac{1}{a_1} \right)^{\frac{d-1}{2}} f_i^V (t) + (-1)^i \left( \frac{1}{a_1} \right)^{\frac{d-1}{2}} \left( f_i (t) - f_{r+1}^V (t) \left( \frac{1}{a_1} \right)^V \right) = 0, \quad t \in [a, b]_{a^{r+1}}, \]
which is equivalent to
\[ \sum_{i=0}^{r+1} (-1)^i \left( \frac{1}{a_1} \right)^{\frac{d-1}{2}} f_i^V (t) = 0, \quad t \in [a, b]_{a^{r+1}}. \]

Remark 11. The differentiability of functions \( f_0, f_1, \ldots, f_r \) is not assumed a priori.

3.2. Euler-Lagrange equations for higher-order problems

Before presenting the Euler-Lagrange equation for the variational problem (P), we introduce the following definition.

**Definition 4.** We say that \( y_* \in C^2 ([a, b], \mathbb{R}) \) is a weak local minimizer (respectively weak local maximizer) for problem (P) if there exists \( \delta > 0 \) such that
\[ L[y_*] \leq L[y] \quad \text{(respectively} \quad L[y_*] \geq L[y]) \]
for all \( y \in C^2 ([a, b], \mathbb{R}) \) satisfying the boundary conditions in (P) and
\[ \| y - y_* \|_{r, \infty} < \delta, \]
where
\[ \| y \|_{r, \infty} := \sum_{i=0}^{r} \| y^{(i)} \|_{\infty} \]
and
\[ \| y \|_{\infty} := \sup_{t \in [a, b]_{a^{r+1}}} | y(t) |. \]
Remark 12. Observe that if \([a, b]\) has 2\(r\) points, i.e.,
\[
[a, b] = \{\rho^{2i-1}(b), \rho^{2i}(b), \ldots, \rho^2(b), \rho(b), b\},
\]
then
\[
\mathcal{L}[y(\cdot)] = \int_{\rho^{2i-1}(a)}^{b} L(t, y''(t), y'^{r-1}(t), y^{r}(t)) \, dt
\]
\[
= \int_{\rho^i(b)}^{b} L(t, y''(t), y'^{r-1}(t), y^{r}(t)) \, dt
\]
\[
= \sum_{i=0}^{r-1} \int_{\rho^{i+1}(b)}^{\rho^i(b)} L(t, y''(t), y'^{r-1}(t), y^{r}(t)) \, dt
\]
\[
= \sum_{i=0}^{r-1} L(\rho^i(b), y''(\rho^i(b)), y'^{r-1}(\rho^i(b)), \ldots, y^{r}(\rho^i(b))) (\rho^i(b) - \rho^{i+1}(b)).
\]
Using the boundary conditions in (P) and the formula
\[
y^r(t) = \frac{y(t) - y(\rho(t))}{t - \rho(t)},
\]
it is then possible to calculate
\[
y(\rho^i(b)), y''(\rho^{i+1}(b)), \ldots, y^{r}(\rho^i(b))
\]
for all \(i = 0, 1, \ldots, r - 1\). Therefore, the above integral is constant for every admissible function \(y(\cdot)\). We conclude that if \([a, b]\) has only 2\(r\) points, the problem is trivial (because there is nothing to minimize or maximize). For this reason, we are assuming that \([a, b]\) has, at least, 2\(r+1\) points.

We are in conditions to prove the following first-order necessary optimality condition for problems of the calculus of variations with higher-order nabla derivatives:

Theorem 17 (Euler-Lagrange equation for problem (P)). Let \(\mathbb{T}\) be a time scale satisfying hypothesis (H), and \(a, b \in \mathbb{T}, a < b, [a, b] \cap \mathbb{T}\) containing, at least, 2\(r+1\) points. If \(y_\ast\) is a weak local extremum (minimizer or maximizer) of problem (P), then \(y_\ast\) satisfies the Euler-Lagrange equation
\[
\sum_{i=0}^{r} (-1)^i \left( \frac{1}{a_1} \right)^{\frac{\alpha_i}{a_1}} \int_{a_i}^{b_i} L_\alpha \left( t, y_\ast^{i}(t), y_\ast'^{r-i}(t), \ldots, y_\ast^{r-i}(t), y_\ast^{r}(t) \right) \, dt = 0,
\]
\(t \in [a, b] \cap \mathbb{T}\).
Proof. Suppose that \( y_\varepsilon \) is a weak local minimizer (resp. maximizer) for problem (P). Let \( \eta \in C^2(I, \mathbb{R}) \) be an admissible variation, i.e., \( \eta \) is such that \( \eta, \eta', \eta'' \) vanish at \( t = \sigma^{r-1}(a) \) and \( t = b \). Defining \( \phi : \mathbb{R} \rightarrow \mathbb{R} \) by \( \phi(\varepsilon) := L[y_\varepsilon + \varepsilon \eta] \) it is clear that \( \phi \) has a minimum (resp. maximum) at \( \varepsilon = 0 \) and, therefore, \( \phi'(0) = 0 \).

Since

\[
\phi(\varepsilon) = \int_{\sigma^{r-1}(a)}^b L(t, y_\varepsilon(t) + \varepsilon \eta(t), y'_\varepsilon(t) + \varepsilon \eta'(t), \ldots, y^{(r-1)}_\varepsilon(t) + \varepsilon \eta^{(r-1)}(t)) \, \eta(t) \, dt,
\]

then

\[
\phi'(0) = 0 \iff \int_{\sigma^{r-1}(a)}^b \left( \sum_{i=0}^{r-1} L_u (\bullet) \eta^{(i)}(t) \right) \eta(t) \, dt = 0
\]

where \( L_u \) denote the partial derivative of \( L(t, y, y', \ldots, y^{(r-1)}) \) with respect to \( y_i \) and we write, for brevity,

\[
(\bullet) = (t, y_\varepsilon(t), y'_\varepsilon(t), \ldots, y^{(r-1)}_\varepsilon(t), y^{(r)}_\varepsilon(t)).
\]

By Lemma 16 we conclude that

\[
\sum_{i=0}^{r-1} (-1)^i \left( \frac{1}{a^i} \right) L_u (\bullet) = 0, \quad t \in ([a, b], a, b],
\]

proving the intended result.

As a straight corollary to Theorem 17, we give the Euler-Lagrange equation for the higher-order variational problem of q-calculus:

**Corollary 18 (the q-calculus Euler-Lagrange equation).** Fix \( q > 1 \), \( q = [q^k : k \in \mathbb{N}_0] \). Let \( a, b \in \mathbb{T} \) such that \( [a, b] \) contains at least \( 2r + 1 \) points. If \( y_\varepsilon \) is a weak local extremum for the correspondent problem (P), then \( y_\varepsilon \) satisfies the Euler-Lagrange equation

\[
\sum_{i=0}^{r} (-1)^i q^{i-1} L_u (t, y_\varepsilon(q^{-r} t), q^{-r} y'_\varepsilon(q^{-1-r} t), \ldots, q^{-1} y^{(r-1)}_\varepsilon(q^{-1} t), y^{(r)}_\varepsilon(t)) = 0
\]

for all \( t \in [q^2 a, b] \).

**References**

[1] F. M. Atici, D. C. Biles and A. Lebedinsky, An application of time scales to economics, Math. Comput. Modelling 43 (2006), no. 7-8, 718–726.

[2] F. M. Atici and G. Sh. Guseinov, On Green’s functions and positive solutions for boundary value problems on time scales, J. Comput. Appl. Math. 141 (2002), no. 1-2, 75–99.

[3] Z. Bartosiewicz and D. F. M. Torres, Noether’s theorem on time scales, J. Math. Anal. Appl. 342 (2008), no. 2, 1220–1226.

[4] M. Bohner, Calculus of variations on time scales, Dynam. Systems Appl. 13 (2004), no. 3-4, 339–349.

[5] M. Bohner and G. Sh. Guseinov, Double integral calculus of variations on time scales, Comput. Math. Appl. 54 (2007), no. 1, 45–57.

[6] M. Bohner and A. Peterson, Dynamic equations on time scales, Birkhäuser Boston, Boston, MA, 2001.

[7] M. Bohner and A. Peterson, Advances in dynamic equations on time scales, Birkhäuser Boston, Boston, MA, 2003.
[8] R. A. C. Ferreira and D. F. M. Torres, Remarks on the calculus of variations on time scales, Int. J. Ecol. Econ. Stat. 9 (2007), no. F07, 65–73.
[9] R. A. C. Ferreira and D. F. M. Torres, Higher-order calculus of variations on time scales, in Mathematical Control Theory and Finance (Lisbon, 2007), Economics series, Springer, 2008.
[10] R. A. C. Ferreira and D. F. M. Torres, Isoperimetric problems of the calculus of variations on time scales, accepted for publication in the Proceedings of the Conference on Nonlinear Analysis and Optimization, June 18-24, 2008, Technion, Haifa, Israel. [arXiv:0805.0278 [math.OC]]
[11] I. M. Gelfand and S. V. Fomin, Calculus of variations, Revised English edition translated and edited by Richard A. Silverman, Prentice Hall, Englewood Cliffs, N.J., 1963.
[12] S. Hilger, Ein Maßkettenkalkül mit Anwendung auf Zentrumsmannigfaltigkeiten, PhD thesis, Universität Würzburg, 1988.
[13] R. Hilscher and V. Zeidan, Calculus of variations on time scales: weak local piecewise $C^1$ solutions with variable endpoints, J. Math. Anal. Appl. 289 (2004), no. 1, 143–166.
[14] A. B. Malinowska and D. F. M. Torres, Necessary and sufficient conditions for local Pareto optimality on time scales, J. Math. Sci. (N. Y.), 2009, in press. [arXiv:0801.2123 [math.OC]]