FIXED POINTS OF NILPOTENT ACTIONS ON SURFACES OF NEGATIVE EULER CHARACTERISTIC

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ABSTRACT

We prove that a nilpotent group $G$ of $C^1$ diffeomorphisms of a compact surface of negative Euler characteristic has a finite orbit whose points are global contractible fixed points of the subgroup $G_0$ of $G$ consisting of isotopic to the identity elements. In particular $G$ has a global contractible fixed point if it consists of isotopic to the identity elements.

1. Introduction

Let $S$ be a compact surface of non-vanishing Euler characteristic $\chi(S)$. A homeomorphism $f$ of $S$ always has a finite orbit. Moreover, if $\chi(S) < 0$ and $f$ is isotopic to the identity map, $f$ has a contractible fixed point, i.e. a point in $\pi(Fix(\tilde{f}))$ where $\pi: \tilde{S} \to S$ is the universal cover of $S$ and $\tilde{f}$ is the unique lift of $f$ to $\tilde{S}$ that commutes with all deck transformations. Such properties can be deduced from the behavior of the Lefschetz number of the iterates of $f$.

It is natural to try to extend the above results for cyclic groups of homeomorphisms to more general classes of groups even if we need to replace homeomorphisms with flows of vector fields or $C^1$ diffeomorphisms.

Let us consider continuous dynamical systems. Lima shows that any continuous action of $\mathbb{R}^n$ on a compact surface $S$ with $\chi(S) \neq 0$ has a global fixed point [22]. Such a result was generalized by Plante for actions of connected nilpotent Lie groups $G$ [23]. The Lie algebra $\mathfrak{g}$ of the group $G$ is a nilpotent Lie algebra of vector fields on $S$ and to find a global fixed point of $G$ is equivalent to obtain a common singular point for the vector fields in $\mathfrak{g}$.

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It makes sense to replace vector fields with diffeomorphisms \( C^1 \)-close to the identity since we can associate to them a sort of a pseudo-flow. Following this idea, Bonatti proved that an abelian group of homeomorphisms of \( S \) whose generators are sufficiently \( C^1 \)-close to the identity has a global fixed point \([5, 6]\). The analogous theorem for nilpotent groups and \( S = S^2 \) was showed by Druck, Fang and Firmo \([8]\).

It is natural to approach the problem of existence of global fixed points for a subgroup \( G \) of homeomorphisms of \( S \) by considering the subgroups induced by \( G \) in the mapping class group of certain \( G \)-invariant subsurfaces of \( S \). The Thurston decomposition is a valuable tool since the fixed point set of homeomorphisms in Thurston normal form is minimal in some sense (cf. \([19]\)). This point of view was developed by Franks, Handel and Parwani to show the existence of a finite orbit for any abelian subgroup \( G \) consisting of \( C^1 \)-diffeomorphisms \([16, 15]\). Moreover, for the case \( S = S^2 \) they prove that it is possible to find a finite orbit with at most 2 elements for groups of orientation-preserving \( C^1 \)-diffeomorphisms \([16]\) whereas for the case \( \chi(S) < 0 \) they show the existence of a global fixed point if \( G \) consists of isotopic to the identity diffeomorphisms \([15]\). In order to make the approach work, it is necessary to extend the Thurston classification to some infinite type surfaces and they obtain such an improvement for the case of abelian groups of \( C^1 \)-diffeomorphisms \([16, 15]\).

Frequently, the problem of existence of global fixed points is lifted to the universal cover of some \( G \)-invariant subsurface. Then we can apply several results in the literature to find global fixed points for subgroups of \( C^1 \)-diffeomorphisms of \( \mathbb{R}^2 \): \([10, 12]\) for the case \( C^1 \)-close to identity, \([16, 3]\) for the abelian case and \([24, 13]\) for the nilpotent case.

Since the existence of global fixed points for nilpotent subgroups has been studied for the case of continuous actions \([23]\) and groups generated by \( C^1 \)-diffeomorphisms close to identity of the sphere \([8]\), it is natural to consider the existence of global fixed points and finite orbits for nilpotent subgroups of \( C^1 \)-diffeomorphisms. The existence of a finite orbit with at most 2 elements for a nilpotent subgroup of orientation-preserving \( C^1 \)-diffeomorphisms of the sphere was proved in \([24]\), generalizing the main result in \([16]\) to the nilpotent setting. The case of the torus was treated by Firmo and the author. There exists a finite orbit if the group contains a \( C^1 \)-diffeomorphism with non-vanishing Lefschetz number \([11]\). We show that nilpotent subgroups of irrotational \( C^1 \)-diffeomorphisms of the torus have global fixed points \([14]\). This paper completes the program of studying global fixed points.
of subgroups of $C^1$-diffeomorphisms of compact surfaces by considering the case of negative Euler characteristic.

**Theorem A.** Let $G$ be a nilpotent subgroup of $\text{Diff}^1(S)$ where $S$ is a compact surface of negative Euler characteristic. Then there exists a finite orbit contained in $\text{Fix}_c(G_0)$ where $G_0$ is the subgroup of $G$ of diffeomorphisms isotopic to the identity.

**Theorem B.** Let $G$ be a nilpotent subgroup of $\text{Diff}^1_0(S)$ where $S$ is a compact surface of negative Euler characteristic. Then $\text{Fix}_c(G) \neq \emptyset$.

Let us remark that $\text{Diff}^1(S)$ and $\text{Diff}^1_0(S)$ stand for the group of $C^1$-diffeomorphisms of $S$ and its subgroup of isotopic to identity elements. Moreover, $\text{Fix}_c(G)$ denotes the set of contractible fixed points of $G$ (see Definition 3.5).

These results are generalizations of theorems of Frank, Handel and Parwani for the abelian case [15]. The outlook of the proofs is discussed in section 2.

There are other results of independent interest on the paper. For instance, under natural hypothesis we show that the groups in the descending central series of a nilpotent subgroup of $\text{Diff}^1(S)$ are isotopically trivial relative to their sets of global fixed points (Theorem 4.1). Moreover, we exhibit several results of localization of global fixed points of nilpotent subgroups of $C^1$-diffeomorphisms for the cases of nonnegative Euler characteristic (Propositions 3.2, 3.3 and 6.2).

### 2. Outlook of the paper

First, we focus on the proof of Theorem B. We can suppose that $S$ is orientable and $G$ is finitely generated without lack of generality. It suffices to consider finitely generated groups. The theorem holds for cyclic groups since the Lefschetz number $L(\text{Fix}_c(\phi), \phi)$ is equal to $\chi(S)$ and hence non-vanishing for any $\phi \in \text{Homeo}_0(S)$. Theorem B is also satisfied for abelian groups by a theorem of Franks, Handel and Parwani [15].

The group $G/G'$ is finitely generated and abelian. Section 4 is devoted to explain that the derived group $G'$ is isotopically trivial. More precisely, any $f \in G'$ satisfies that $f$ is isotopic to the identity map relative to $\text{Fix}(G')$ (Theorem 4.1). As a consequence, many of the arguments of the abelian case can be adapted to the nilpotent setting. For instance, we will see that given any $f \in G$, we have $\text{Fix}_c(G', f) \neq \emptyset$.

We follow the approach by Franks, Handel and Parwani in [15]. We are going to consider a sequence $f_1, \ldots, f_q$ of elements of $G$ such that the classes of any $e = \text{rank}(G/G')$ of them in $G/G'$ generate a finite
index subgroup of $G/G'$. The number $q$ depends on the topology of the surface $S$. Then we study isotopy classes of elements of $G$ relative to a $G$-invariant compact subset of $\bigcup_{j=1}^{q} \text{Fix}_c\langle G', f_j \rangle$ in order to find $1 \leq i_1 < \ldots < i_e \leq q$ such that $\cap_{j=i_1}^{i_e} \text{Fix}_c\langle G', f_j \rangle \neq \emptyset$. Therefore $\langle G', f_{i_1}, \ldots, f_{i_e} \rangle$ is a finite index subgroup of $G$ with a global contractible fixed point. Then, it is easy to use Theorem 3.1 to show that $G$ has also a global contractible fixed point.

Since in particular $\bigcup_{j=1}^{q} \text{Fix}_c\langle G', f_j \rangle \subset \text{Fix}(G')$, our group $G$ is abelian up to isotopy rel $\bigcup_{j=1}^{q} \text{Fix}_c\langle G', f_j \rangle$. As a consequence, we can generalize the approach in [15] to obtain a Thurston decomposition for the group $G$ relative to the set $\bigcup_{j=1}^{q} \text{Fix}_c\langle G', f_j \rangle$. Here, it arises a technical issue since the set $\text{Fix}_c\langle G', f_l \rangle$ is not open in $\bigcup_{j=1}^{q} \text{Fix}_c\langle G', f_j \rangle$ in general. We replace $\bigcup_{j=1}^{q} \text{Fix}_c\langle G', f_j \rangle$ with a $G$-invariant compact subset $K$ such that $K \cap \text{Fix}_c\langle f_l \rangle$ is an open and closed subset of $K$ for any $1 \leq l \leq q$. By construction, such a set is complete in some sense, containing representatives of the Nielsen-classes that are relevant in our approach (see Definition 5.4). Another problem is that the set $K$ is not necessarily finite. We generalize the Thurston decomposition provided in [16, 15] for the infinite type case to the nilpotent setting. The main difference is that we need Lemma 5.1 to localize fixed points of $\text{Fix}_c\langle G', f_l \rangle$ for $1 \leq l \leq q$.

Even if $G$ is abelian up to relative isotopy, such a property does not suffice to find a global contractible fixed point of $G$. We need to find and localize global fixed points of groups of the form $\text{Fix}_c\langle G', f \rangle$. This is a major difficulty to adapt the approach of Franks, Handel and Parwani and it is the subject of sections 6, 7 and 8. More precisely, we want to localize global fixed points of subgroups of $G$ in connected components $M$ of $S \setminus \mathcal{R}_e$ where $\mathcal{R}_e$ is the set of essential reducing curves of the Thurston decomposition. Such a connected component has Euler characteristic less or equal than 0. Once we obtain $M \cap K \cap \text{Fix}_c\langle f_j \rangle \neq \emptyset$ for sufficiently many indices, we can obtain global contractible fixed points for finite index subgroups of $G$ by using the properties of the Thurston decomposition. The results of section 6 localize fixed points for the case where $M$ is an annulus whereas section 8 deals with the case of negative Euler characteristic.

In order to localize points of $\text{Fix}_c\langle G', f \rangle$ in section 8 we need to show a version of Theorem 13 for the group $\langle G', f \rangle$. Since $G'$ is trivial up to isotopy rel $K$, this is a sort of cyclic case and hence simpler
than the general one. It is treated in section 7 and requires enlarging the scope of our techniques to finite type surfaces with punctures and boundary components. This is the reason behind considering such surfaces in sections 5, 6 and 7. Let us remark that the framework in section 5 (see Definition 5.1) is larger than the one described so far in this section; for instance the group \( G' \) can be replaced with certain normal subgroups \( H \) such that \( G/H \) is abelian. The reason is having a unified approach for the proofs of Theorem B and the localization result (Theorem 7.1) in section 7. Anyway, we think that the presentation of this section is easier to understand, contains all the main ideas and avoids technicalities.

Section 9 is devoted to complete the proof of the auxiliary Theorem 8.1 that implies Theorem B in a straightforward way. Theorem A is proved in section 10. Such a proof relies on applying Theorem 8.1 to a certain group of homeomorphisms.

3. Preliminaries

In this section we are going to introduce some concepts that are going to be important in the paper, concerning the properties of nilpotent groups, isotopies and Nielsen classes, and localization of global fixed points.

3.1. Nilpotent groups. We introduce here the descending central series of a group and some of its properties that will be used in the paper.

**Definition 3.1.** Let \( G \) be a group. Consider subgroups \( H \) and \( K \) of \( G \). We define \([H, K]\) as the subgroup generated by the commutators \([h, k] := hkh^{-1}k^{-1}\) where \( h \in H \) and \( k \in K \).

**Definition 3.2.** Let \( G \) be a group. We denote \( C^{(0)}(G) = G \). We define the groups \( (C^{(j)}(G))_{j \geq 0} \) in the descending central series recursively, by using the formula \( C^{(j+1)}(G) = [C^{(j)}(G), G] \) for \( j \geq 0 \). The group \( C^{(1)}(G) \) is called the derived group of \( G \) and is also denoted by \( G' \).

We say that a group \( G \) is nilpotent if there exists \( k \in \mathbb{Z}_{\geq 0} \) such that \( C^{(k)}(G) = \{1\} \). Moreover, the minimum \( k \geq 0 \) such that \( C^{(k)}(G) = \{1\} \) is called the nilpotency class of \( G \).

**Remark 3.1.** The subgroups in the descending central series of \( G \) are normal subgroups of \( G \).

**Remark 3.2.** The group \( (C^{(j)}(G), f)/C^{(j+1)}(G) \) is abelian for all \( f \in G \) and \( j \geq 0 \). Indeed, the subgroup \([C^{(j)}(G)]\) induced by \( C^{(j)}(G) \) in \([G] := \)}
$G/C^{j+1}(G)$ is contained in the center of $[G]$. Thus any subgroup $H$ of $\langle C^{(j)}(G), f \rangle$ containing $C^{(j+1)}(G)$ is a normal subgroup of $\langle C^{(j)}(G), f \rangle$.

Moreover, we get $C^{(1)}(G', f) \subset C^{(2)}(G)$. We deduce $C^{(j)}(G', f) \subset C^{(j+1)}(G)$ for any $j \geq 1$.

The next result is a distortion property of the elements of $C^{(j)}(G)$ modulo $C^{(j+1)}(G)$ for $j \geq 1$.

**Lemma 3.1.** Let

$$\phi = [g_1, h_1] \cdots [g_m, h_m] \in C^{(j)}(G)$$

where either $(g_l, h_l)$ or $(h_l, g_l)$ belong to $C^{(j-1)}(G) \times G$ for any $1 \leq l \leq m$. Then, we obtain

$$\phi^k = [g_1^k, h_1] \cdots [g_m^k, h_m^k] \alpha_k$$

for some $\alpha_k \in C^{(j+1)}(G)$ and any $k \in \mathbb{N}$.

**Proof.** Given $\psi \in G$ we denote by $[\psi]$ its class in $[G] := G/C^{(j+1)}(G)$. Notice that $[C^{(j)}(G), G] = C^{(j+1)}(G)$ implies that $[C^{(j)}(G)]$ is a subgroup of the center of $[G]$. Thus, we get

$$[\phi]^k = [[g_1], [h_1]]^k \cdots [[g_m], [h_m]]^k$$

for any $k \in \mathbb{N}$. Therefore, it suffices to show the result for $m = 1$.

Suppose that either $(g, h)$ or $(h, g)$ belongs to $C^{(j-1)}(G) \times G$. It suffices to show $[[g], [h]]^k = [[g]^k, [h]] = [[g], [h]^k]$ for any $k \in \mathbb{N}$. Let us show the first equality by induction on $k$. It clearly holds for $k = 1$. Suppose it holds for $k \geq 1$. We have

$$[[g], [h]]^{k+1} = [[g], [h]][[g], [h]]^k = [g][h][g][h]^{-1}[h][g][h]^{-1}[[g], [h]]^k = [g][g][h][g][h]^{-1}[h][g][h]^{-1}[[g], [h]]^k$$

where the third equality holds since $[[g], [h]]^k$ belongs to the center of $[G]$ and the fourth equality is a consequence of the hypothesis of induction. The proof of $[[g], [h]]^k = [[g], [h]^k]$ for any $k \in \mathbb{N}$ is analogous. □

### 3.2. Groups of homeomorphisms.

We denote by $\text{Homeo}(S)$ and $\text{Homeo}_0(S)$ the group of homeomorphisms of a manifold $S$ and its subgroup of isotopic to the identity elements respectively. Their analogues for diffeomorphisms of class $C^1$ will be denoted by $\text{Diff}^1(S)$ and $\text{Diff}^1_0(S)$. If $S$ is orientable, we can define the subgroup $\text{Homeo}_+(S)$ of $\text{Homeo}(S)$ of orientation-preserving elements and its analogue $\text{Diff}^1_+(S)$ for diffeomorphisms of class $C^1$. 
Definition 3.3. Given a subset $C$ of $S$, we denote by $\text{MCG}(S, C)$ the mapping class group rel $C$, i.e. the group of isotopy classes relative to $C$ of homeomorphisms, of $S$. The mapping class group $\text{MCG}(S)$ of $S$ is just $\text{MCG}(S, \emptyset)$.

3.2.1. Nielsen classes.

Definition 3.4. Given $f \in \text{Homeo}(S)$ we denote by $\text{Fix}(f)$ the set of fixed points of $f$. Given a subgroup $G$ of $\text{Homeo}(S)$ we define the set $\text{Fix}(G)$ of global fixed points as $\cap f \in G \text{Fix}(f)$.

Remark 3.3. Let $f, g \in \text{Homeo}(S)$. We have $\text{Fix}(gfg^{-1}) = g(\text{Fix}(f))$. As a consequence, given $G < \text{Homeo}(S)$ and a normal subgroup $H$ of $G$, the set $\text{Fix}(H)$ is $G$-invariant.

Definition 3.5. Let $f \in \text{Homeo}_0(S)$ where $S$ is a surface of negative Euler characteristic. Let $\pi : \tilde{S} \to S$ be the universal covering map of $S$. There exists a isotopy $(f_t)_{t \in [0,1]}$ in $\text{Homeo}(S)$ such that $f_0 = \text{Id}_S$ and $f_1 = f$. It can be lifted to an isotopy $(\tilde{f}_t)_{t \in [0,1]}$ in $\text{Homeo}(\tilde{S})$ such that $\tilde{f}_0 = \text{Id}_{\tilde{S}}$. The homeomorphism $\tilde{f} := \tilde{f}_1$ is called the identity lift of $f$ to $\tilde{S}$. Note that the identity lift does not depend on the isotopy since $\chi(S) < 0$. We define the set $\text{Fix}_c(f)$ of contractible fixed points of $f$ as $\text{Fix}_c(f) = \pi(\text{Fix}(\tilde{f}))$.

We define the set $\text{Fix}_c(G) = \cap f \in G \text{Fix}_c(f)$ of global contractible fixed points for any subgroup $G$ of $\text{Homeo}_0(S)$.

Remark 3.4. Suppose $\chi(S) < 0$. Let $\tilde{f}$ be the identity lift of $f \in \text{Homeo}_0(S)$. Consider a lift $\tilde{g}$ of $g \in \text{Homeo}(S)$. Since $(gf_tg^{-1})_{t \in [0,1]}$ and $(\tilde{g}f_t\tilde{g}^{-1})_{t \in [0,1]}$ are isotopies issued from the identity map, it follows that $gf_tg^{-1} \in \text{Homeo}_0(S)$ and $\tilde{g}f_t\tilde{g}^{-1}$ is the identity lift of $gf_tg^{-1}$.

Let $\tilde{H}(S)$ be the group of all lifts of elements of $\text{Homeo}(S)$. Let $\tilde{H}_0(S)$ be the group of identity lifts of elements of $\text{Homeo}_0(S)$. The previous discussion implies that $\text{Homeo}_0(S)$ and $\tilde{H}_0(S)$ are normal subgroups of $\text{Homeo}(S)$ and $\tilde{H}(S)$ respectively. In particular, given a subgroup $G$ of $\text{Homeo}(S)$, its subgroup $G_0 = G \cap \text{Homeo}_0(S)$, of isotopic to the identity elements, is normal.

Remark 3.5. Let $f \in \text{Homeo}_0(S)$, $g \in \text{Homeo}(S)$ and $\chi(S) < 0$. We obtain $\text{Fix}_c(gfg^{-1}) = g(\text{Fix}_c(f))$ by Remark 3.4. In particular, given $G < \text{Homeo}(S)$ and a normal subgroup $H$ of $G$ contained in $\text{Homeo}_0(S)$, the set $\text{Fix}_c(H)$ is $G$-invariant.

Contractible fixed points are an example of a Nielsen class.
Definition 3.6. Let \( f \in \text{Homeo}(S) \). We say that \( z, w \in S \) are \( f \)-Nielsen equivalent if there exists a path \( \gamma : [0, 1] \to S \) with \( \gamma(0) = z \), \( \gamma(1) = w \) and such that \( \gamma \) and \( f \circ \gamma \) are homotopic relative to ends.

Remark 3.6. The \( f \)-Nielsen class \( \mathcal{N}_{f,z} \) of \( z \in \text{Fix}(f) \) is \( \pi(\text{Fix}(\tilde{f})) \) where \( \pi : \tilde{S} \to S \) is the universal covering map and \( \tilde{f} \) is a lift of \( f \) to \( \tilde{S} \) such that \( z \in \pi(\text{Fix}(\tilde{f})) \).

Definition 3.7. We say that a \( f \)-Nielsen class \( \mathcal{N} \) is compactly covered if there exists a lift (and hence for every lift) \( \tilde{f} \) of \( f \) to \( \tilde{S} \) such that \( \pi(\text{Fix}(\tilde{f})) = \mathcal{N} \) and \( \text{Fix}(\tilde{f}) \) is compact.

Definition 3.8. Let \( (f_t)_{t \in [0,1]} \) be an isotopy in \( \text{Homeo}(S) \). Let \( \mathcal{N}_0 \) be a \( f_0 \)-Nielsen class and consider a lift \( \tilde{f}_0 \) of \( f_0 \) to \( \tilde{S} \) with \( \pi(\text{Fix}(\tilde{f}_0)) = \mathcal{N}_0 \). Consider an isotopy \( (\tilde{f}_t)_{t \in [0,1]} \) in \( \text{Homeo}(S) \) issued from \( \tilde{f}_0 \). We say that \( \pi(\text{Fix}(\tilde{f}_1)) \) is the \( f_1 \)-Nielsen class induced by \( \mathcal{N}_0 \) and the isotopy \( (f_t)_{t \in [0,1]} \).

Remark 3.7. The definition does not depend on the choice of the lift \( \tilde{f}_0' \) of \( f_0 \) to \( \tilde{S} \) with \( \pi(\text{Fix}(\tilde{f}_0')) = \mathcal{N}_0 \). Indeed, \( \tilde{f}_0' \) is of the form \( T\tilde{f}_0 T^{-1} \) for some deck transformation \( T \). We deduce that \( \tilde{f}_1' = T\tilde{f}_0 T^{-1} \) and hence \( \pi(\text{Fix}(\tilde{f}_1')) = \pi(T(\text{Fix}(\tilde{f}_1))) = \pi(\text{Fix}(\tilde{f}_1)) \).

Definition 3.9. Let \( G \) be a subgroup of \( \text{Homeo}(S) \) and let \( \varpi : S' \to S \) be a covering map. We say that a subgroup \( G' \) of \( \text{Homeo}(S') \) is a lift of \( G \) if any \( f' \in G' \) is a lift of some \( f \in G \) and the map \( \tau : G' \to G \), defined by \( \tau(f') = f \), is an isomorphism of groups.

Remark 3.8. Let us point out that the set \( G'' \) of all lifts of elements of \( G \) to \( S' \) is not a lift of \( G \) if \( \varpi \) is not a homeomorphism. Indeed, the kernel of \( \tau : G'' \to G \) is the group of deck transformations. A lift of a group is useful since it is nilpotent if \( G \) is nilpotent. On the other hand \( G'' \) is not nilpotent even if \( G \) is nilpotent if \( \chi(S) < 0 \), since the group of deck transformations of a surface of negative Euler characteristic is non-nilpotent.

3.3. Fixed points. In this section we introduce some results about the existence of global fixed points in the compactly covered case.

Definition 3.10. Let \( f \in \text{Homeo}(S \setminus K) \) where \( S \) is a compact surface (maybe with boundary) and \( K \) is a closed subset of \( S \setminus \partial S \). Consider \( z \in \text{Fix}(f) \). We say that \( z \) peripherally contains a puncture \( w \) rel \( K \) if there exists a path \( \gamma : [0, 1] \to S \setminus K \) such that \( \gamma(0) = z \), \( \lim_{t \to 1} \gamma(t) = w \) and \( \gamma \) is properly homotopic to \( f \circ \gamma \) relative to \( 0 \). If \( \partial \) is a boundary component of \( S \), we say that \( z \) peripherally contains \( \partial \) if \( z \) peripherally contains the puncture induced by \( \partial \) in the surface \( S \setminus \partial S \).
**Definition 3.11.** Let \( f \in \text{Homeo}_+ (\mathbb{R}^2) \) that preserves a non-empty compact set \( K \) contained in \( \mathbb{R}^2 \setminus \text{Fix}(f) \). We denote by \( P(K,f) \) the subset of \( \text{Fix}(f) \) of points that peripherally do not contain \( \infty \) rel \( K \).

In order to find compactly covered \( f|_{\mathbb{R}^2 \setminus K} \)-Nielsen classes it is interesting to consider \( P(K,f) \) since all such Nielsen classes are contained in \( P(K,f) \).

Next, we introduce some results about global fixed points of subgroups of \( \text{Diff}^1_+ (\mathbb{R}^2) \) that will be useful in the sequel. The first one is a generalization of a theorem of Franks, Handel and Parwani [16] for finitely generated abelian groups to the nilpotent setting.

**Theorem 3.1.** [13, Theorem 1] Let \( G \) be a nilpotent subgroup of \( \text{Diff}^1_+ (\mathbb{R}^2) \) that preserves a non-empty compact set. Then \( G \) has a global fixed point.

The nilpotence condition is useful in the quest of global fixed points since it is sort of a rigidity condition. For instance, properties of existence of fixed points that hold for a single element of the group can be frequently generalized to properties of existence of global fixed points. The next results illustrate this point of view.

**Corollary 3.1.** [24] Let \( G \) be a nilpotent subgroup of \( \text{Diff}^1_+ (\mathbb{R}^2) \) such that \( \text{Fix}(\phi) \) is a non-empty compact set for some \( \phi \in G \). Then \( G \) has a global fixed point.

**Proposition 3.1.** [13, Theorem 5] Let \( K_0 \) be a non-empty compact subset of \( \mathbb{R}^2 \). Consider \( f \in \text{Diff}^1_+ (\mathbb{R}^2) \) such that \( f(K_0) = K_0 \) and \( \text{Fix}(f) \cap K_0 = \emptyset \). Then there exists a lift \( \tilde{f} \) of \( f \) to the universal covering of a connected component of \( \mathbb{R}^2 \setminus K_0 \) such that \( \text{Fix}(\tilde{f}) \) is a non-empty compact set whose projection \( N \) to \( \mathbb{R}^2 \) satisfies that the Lefschetz number \( L(N,f) \) is non-zero.

**Proposition 3.2.** Let \( G = \langle H, f \rangle \) be a nilpotent subgroup of \( \text{Diff}^1_+ (S) \) where \( S \) is an orientable surface and \( H \) is a normal subgroup of \( G \). Suppose that \( K_0 \) is a compact \( f \)-invariant subset of \( S \setminus K_0 \). Assume that \( M \) is not a sphere, a torus or an annulus such that \( f \) permutes its ends. Fix a non-empty compactly covered \( f|_M \)-Nielsen class \( N \). Furthermore, suppose that any \( h \in H \) is isotopic to \( \text{Id} \) rel \( K_0 \). Then, there exists \( z \in \text{Fix}(G) \cap N \). Moreover, \( \text{Fix}(G) \cap \text{Fix}_c(H) \cap N \neq \emptyset \) holds if \( \chi(S) < 0 \).

**Proof.** Let \( \varpi : \tilde{M} \to M \) be the universal covering map. Fix a point \( \hat{w}^\sharp \) in \( \varpi^{-1}(N) \) and a lift \( \tilde{f}^\sharp \) of \( f \) such that \( \tilde{f}^\sharp (\hat{w}^\sharp) = \hat{w}^\sharp \).

We claim that \( M \) is a topological disc or \( \chi(M) < 0 \). Otherwise \( M \) is a sphere, a torus or an annulus. We just have to consider the case where
$M$ is an annulus and $f$ does not permute its ends by hypothesis. This never happens since it would contradict that $\mathcal{N}$ is compactly covered.

Given $h \in H$, consider an isotopy $(h_t)_{t \in [0,1]}$ rel $K_0$ from $h_0 = \text{Id}_S$ to $h_1 = h$. It induces an isotopy $(h_t^\sharp)_{t \in [0,1]}$ from $\text{Id}_M$ to the identity lift $h^\sharp$ of $h$ where $(h_t^\sharp)_{t \in [0,1]}$ is a lift of the isotopy $(h_t)_{t \in [0,1]}$. Notice that $h^\sharp$ does not depend on the isotopy $(h_t^\sharp)_{t \in [0,1]}$ since $M$ is a topological disc or $\chi(M) < 0$. We denote $H^\sharp = \{h^\sharp : h \in H\}$ and $G^\sharp = \langle H^\sharp, f^\sharp \rangle$.

Since $H$ is a normal subgroup of $G$, $H^\sharp$ is a normal subgroup of $G^\sharp$ by Remark 3.4. Since the map $h \rightarrow h^\sharp$ is well-defined in $H$, the group $H^\sharp$ does not contain non-trivial deck transformations. As a consequence $C^{(q)}(G) = \{\text{Id}\}$ and $H^\sharp \triangleleft G^\sharp$ imply $C^{(q)}(G^\sharp) = \{\text{Id}\}$. In particular $G^\sharp$ is nilpotent.

Since $\mathcal{N}$ is compactly covered, $\text{Fix}(f^\sharp)$ is a non-empty compact set. There exists $z^\sharp \in \text{Fix}(G^\sharp)$ by Corollary 3.1. The point $z = \varpi(z^\sharp)$ belongs to $\text{Fix}(G) \cap \mathcal{N}$.

Finally, suppose $\chi(S) < 0$. Given $h \in H$, we consider the path $\gamma_h : [0,1] \rightarrow S$ defined by $\gamma_h(t) = h_t(z)$ for $t \in [0,1]$. Since $H^\sharp$ consists of identity lifts rel $K_0$, we deduce that $[\gamma_h] = [\gamma_h] = 0$ in $\pi_1(M, z)$. Since $M \subset S$, it follows that $[\gamma_h] = 0$ in $\pi_1(S, z)$. Therefore, $z$ belongs to $\text{Fix}_c(h)$ for any $h \in H$ and thus $z \in \text{Fix}_c(H)$.

**Corollary 3.2.** Let $G = \langle H, f \rangle$ be a nilpotent subgroup of $\text{Diff}^1_+(\mathbb{R}^2)$ where $H$ is a normal subgroup of $G$. Suppose that $K_0$ is a non-empty compact subset of $\mathbb{R}^2$ such that $K_0 \cap \text{Fix}(f) = \emptyset$ and any element $h$ of $H$ is isotopic to $\text{Id}$ rel $K_0$. Then there exists $z \in \text{Fix}(G)$ such that its $f_{|\mathbb{R}^2 \setminus K_0}$-Nielsen class is compactly covered.

**Proof.** There exists a non-empty compactly covered $f_{|\mathbb{R}^2 \setminus K_0}$-Nielsen class $\mathcal{N}$ by Proposition 3.1. We are done, since $\text{Fix}(G) \cap \mathcal{N} \neq \emptyset$ by Proposition 3.2.}

Let us generalize the previous corollary, weakening the hypotheses that we require $H$ to satisfy.

**Proposition 3.3.** Let $G = \langle H, f \rangle$ be a nilpotent subgroup of $\text{Diff}^1_+(\mathbb{R}^2)$ where $H$ is a normal subgroup of $G$. Suppose that $K_0$ is a non-empty compact subset of $\mathbb{R}^2$ such that $K_0 \cap \text{Fix}(f) = \emptyset$ and $K_0 \subset \text{Fix}(H)$. Then there exists $z \in \text{Fix}(G) \cap P(K_0, f)$.

**Proof.** There exists a $f$-minimal set $K'_0$ contained in $K_0$. Since

$$\text{Fix}(G) \cap P(K'_0, f) \subset \text{Fix}(G) \cap P(K_0, f),$$

we can consider that $K_0$ is $f$-minimal up to replace it with $K'_0$. 

Suppose that any \( h \in H \) is isotopic to \( \text{Id} \) rel \( K_0 \). Then there exists \( z \in \text{Fix}(G) \) whose \( f_{|\mathbb{R}^2 \setminus K_0} \)-Nielsen class is compactly covered by Corollary 3.2 and hence \( z \in \text{Fix}(G) \cap P(K_0, f) \). So we can assume from now on that there exists an element of \( H \) that is not isotopic to \( \text{Id} \) rel \( K_0 \).

Let \( k \geq 0 \) be the maximum nonnegative integer number such that \( H \cap C^{(k)}(G) \) contains an element \( h \) that is not isotopic to \( \text{Id} \) rel \( K_0 \). Such a \( k \) exists since \( G \) is nilpotent.

Let \( \mathcal{W}(h, K_0, \emptyset) \) be the set of finite type subsurfaces \( W \) of \( \mathbb{R}^2 \cup \{ \infty \} \) such that

- \( \partial W \) is a union of pairwise disjoint essential simple closed curves of \( \mathbb{R}^2 \setminus K_0 \);
- \( K_0 \setminus W \) is finite and every connected component of \( W \) contains infinitely many points of \( K_0 \);
- There exists \( h' \in \text{Homeo}_+(\mathbb{R}^2) \) such that \( h \) is isotopic to \( h' \) rel \( K_0 \) and \( h'|_W \equiv \text{Id} \).

We say that \( W \in \mathcal{W}(h, K_0, \emptyset) \) is maximal if given \( W' \in \mathcal{W}(h, K_0, \emptyset) \) such that \( W \) is isotopic rel \( K_0 \) to a subsurface of \( W' \) then \( W \) is isotopic to \( W' \) rel \( K_0 \). There exists a unique maximal element \( W \) of \( \mathcal{W}(h, K_0, \emptyset) \) [16, Lemmas 2.6 and 2.8].

Given \( g \in G \), we have \( \eta := [g, h] \in H \cap C^{(k+1)}(G) \). Since \( ghg^{-1} = \eta h \), we obtain \( g(\mathcal{W}(h, K_0, \emptyset)) = \mathcal{W}(\eta h, K_0, \emptyset) \). By the choice of \( k \) we have that \( \eta \) is isotopic to \( \text{Id} \) rel \( K_0 \) and hence \( \mathcal{W}(\eta h, K_0, \emptyset) = \mathcal{W}(h, K_0, \emptyset) \). We deduce that \( g(W) \) is the unique maximal element of \( \mathcal{W}(h, K_0, \emptyset) \) and thus \( g(W) \) is isotopic to \( W \) rel \( K_0 \) for any \( g \in G \).

Consider the unbounded connected component \( U \) of \( \mathbb{R}^2 \setminus \partial W \). Since \( U \) is \( G \)-invariant modulo isotopy rel \( K_0 \), the compact set \( K_0 \cap U \) is \( G \)-invariant. We claim that \( K_0 \cap U = \emptyset \). Otherwise, we have \( K_0 \cap U = K_0 \) since \( K_0 \) is \( f \)-minimal. This implies \( U \subset W \) and \( W = \mathbb{R}^2 \), contradicting that \( h \) is not isotopic to \( \text{Id} \) rel \( K_0 \).

Given \( g \in G \), there exists \( \theta_g \in \text{Homeo}_+(\mathbb{R}^2) \) and an isotopy \( (g_t)_{t \in [0, 1]} \) rel \( K_0 \) such that \( g_0 = \theta_g, g_1 = g \) and \( \theta_g(U) = U \). Since \( \chi(U) < 0 \), the class \( [\theta_g] \in \text{MCG}(U) \) depends just on \( g \). Moreover \( [G] := \{ [\theta_g] : g \in G \} \) is a nilpotent subgroup of \( \text{MCG}(U) \). The group \( [G] \) is virtually abelian and finitely generated [4]. Since \( U \) is a finite type surface, it admits a Thurston decomposition [18, Lemma 2.2]. In particular, up to replace \( U \) with the unbounded connected component of \( U \setminus \mathcal{R} \) (where \( \mathcal{R} \) is the set of reducing curves of the Thurston decomposition), we can suppose that all elements of \( [G] \) are irreducible, i.e. either have finite order or are pseudo-Anosov.

There exists a model \( \Theta \) for \( [G] \). More precisely, there exists a subgroup \( \Theta \) of \( \text{Homeo}_+(U) \) such that the the natural map \( \theta \to [\theta] \) from \( \Theta \)
to MCG(U) is an injective morphism of groups whose image is equal to [G] [24, Lemma 2.1]. We can consider U as a sphere with finitely many punctures (corresponding to the boundary components of U). A nilpotent group of orientation-preserving irreducible homeomorphisms of the sphere that has a fixed point, has a second fixed point [24, Remark 2.2]. In our case ∞ ∈ Fix(Θ) and Θ fixes no other puncture of U since K_0 is f-minimal. Thus, there exists z ∈ Fix(Θ) ∩ U.

Given g ∈ G, we can suppose that (θ_g)_U ∈ Θ. Consider the universal covering M of the connected component M of R^2 \ K_0 containing U. Fix a lift ˜z of z in M. There exists a lift ˜θ_0 ∈ Homeo (M) of θ_0 such that ˜θ_0( ˜z) = ˜z. Consider the lift ˜g of g to M with ˜g = ˜g_1 where ( ˜g_1) is the lift of (g_1) such that ˜g_0 = ˜θ_0. Since χ(U) < 0, ˜g does not depend on (g_1). Thus G := { ˜g : g ∈ G} is a nilpotent subgroup of Diff_+(M).

Any non-trivial element θ ∈ Θ satisfies that the θ-Nielsen class of z has non-vanishing fixed point index, is compactly covered and does not peripherally contain points of U. In particular, given g ∈ G with [θ_g] ≠ 1, we obtain that Fix( ˜g) is a non-empty compact set and its projection to R^2 is a compactly covered g_{[R^2 \ K_0]}-Nielsen class. Since χ(U) < 0 and K_0 is f-minimal, we obtain [θ_f] ≠ 1. There exists ˜w ∈ Fix(G) by Corollary 3.1. Its projection w in R^2 is a global fixed point of G whose f_{[R^2 \ K_0]}-Nielsen class is compactly covered and hence w ∈ Fix(G) ∩ P(K_0, f).

4. IRROTATIONAL NATURE OF DERIVED GROUPS

In order to find global fixed points for a nilpotent subgroup G of Diff_+(S), we are going to work with compact G-invariant sets K contained in the derived group G'. In this section we see that any φ ∈ G' is isotopic to identity rel Fix(G') and hence rel K. This implies that the group induced by G in the group of isotopy classes rel K is an image by a group morphism of G/G' and in particular it is abelian. Such a property will be quite useful.

In this section S is an orientable compact surface. The main result of the section is next proposition.

**Theorem 4.1.** Let G be a nilpotent subgroup of Diff_+(S) such that G' is contained in Diff_0(S). Consider φ ∈ C^{(k)}(G) with k ≥ 1. Then φ is isotopic to Id rel Fix(φ) ∩ Fix(C^{(k+1)}(G)). In particular φ is isotopic to Id rel Fix(C^{(k)}(G)).

**Corollary 4.1.** Let G be a nilpotent subgroup of Diff_+(S) such that G' is contained in Diff_0(S). Consider φ in C^{(k)}(G) with k ≥ 1. Then Fix(φ) ∩ Fix(C^{(k+1)}(G)) ⊂ Fix_c(φ).
Corollary 4.2. Let $G$ be a nilpotent subgroup of $\text{Diff}_+^1(S)$ such that $G'$ is contained in $\text{Diff}_0^1(S)$. Then we obtain $\text{Fix}(C(k)(G)) = \text{Fix}_c(C(k)(G))$ for any $k \geq 1$.

In the proof of Theorem 4.1 we associate some isotopy classes to $\phi \in C(k)(G)$. We will show that they are trivial by using the nilpotence of $G$ to see that they have finite order and then next lemma.

Lemma 4.1. Let $f$ be an orientation-preserving homeomorphism of an orientable finitely punctured compact surface $S$ such that $f$ is isotopic to $\text{Id}$. Suppose that there exist a finite subset $F$ of $\text{Fix}(f)$ and $k \in \mathbb{N}$ such that $\chi(S \setminus F) < 0$ and $f^k$ is isotopic to $\text{Id}$ relative to $F$. Then $f$ is isotopic to $\text{Id}$ rel to $F$.

Proof. In order to show the result we can replace $f$ with another element in $\text{Homeo}(S)$ that is isotopic to $f$ rel $F$. Thus, we can suppose that $f|_{S \setminus F}$ is an isometry of a hyperbolic metric in $S \setminus F$ such that $f^k \equiv \text{Id}$ by Kerekhoffs solution to the Nielsen realization problem [21].

Suppose $\chi(S) < 0$. Thus $f$ is the identity map since $f$ is a finite order element of $\text{Homeo}_+(S)$ that is isotopic to $\text{Id}$.

Suppose $\chi(S) \geq 0$. Then either $S$ is a finitely punctured sphere $S^2 \setminus A$ with $\sharp A \leq 2$ or $S$ is a torus $T^2$. Suppose that we are in the former case. The result is obvious if $\sharp (A \cup F) \leq 3$ since $A \cup F \subset \text{Fix}(f)$ and the mapping class group of a sphere minus $k$ points $(k \leq 3)$ is isomorphic to the group of permutations $S_k$. On the other hand, if $\sharp (A \cup F) > 3$, $f$ is an orientation-preserving finite order homeomorphism of the sphere with more than three fixed points and thus $f \equiv \text{Id}$.

Finally, let us assume $S = T^2$. In such a case, we consider a lift $\tilde{f}$ of $f$ to the universal covering $\tilde{f}$ such that $\tilde{f}(\tilde{z}) = \tilde{z}$ where $\tilde{z}$ is a lift of a point $z \in F$. Since $f^k \equiv \text{Id}$, we obtain $\tilde{f}^k \equiv \text{Id}$. Moreover, since $f$ is isotopic to $\text{Id}$, it follows that any point in $\tilde{z} + \mathbb{Z}^2$ is contained in $\text{Fix}(\tilde{f})$. By compactifying $\mathbb{R}^2$ by adding a point at infinity, we can interpret $\tilde{f}$ as a finite order orientation-preserving homeomorphism of a sphere with infinitely many fixed points. Therefore $\tilde{f}$ is the identity map and $f$ is also the identity map. \qed

4.1. Translation numbers. In this section we study homeomorphisms $\phi \in \text{Homeo}_+(S)$ that are isotopic to a product of finitely many Dehn twists. Specifically, we examine translation numbers induced in the ideal boundary of the universal cover of a connected component of $S \setminus K$ where $K$ is a compact subset of $\text{Fix}(\phi)$. 
Let \( \phi \in \text{Homeo}_+ (S) \) where \( S \) is an orientable compact surface of negative Euler characteristic. Let \( K \) be a compact subset of \( \text{Fix}(\phi) \). Moreover, assume that there are essential simple closed curves \( \gamma_1, \ldots, \gamma_m \) in \( S \setminus K \) such that the following properties hold:

- \( \gamma_1, \ldots, \gamma_m \) are pairwise non-homotopic in \( S \setminus K \);
- There are pairwise disjoint closed annuli \( A_1, \ldots, A_m \) in \( S \setminus K \) whose core curves are \( \gamma_1, \ldots, \gamma_m \) respectively;
- \( \phi \) is isotopic to \( f \) rel \( K \), where \( f_{|S \setminus \bigcup_{i=1}^{m} A_i} \) is the identity map and \( f_{|A_l} \) is a Dehn twist for any \( 1 \leq l \leq m \).

Consider the connected component \( B \) of \( S \setminus K \) containing \( \gamma := \gamma_1 \). We denote by \( \gamma^1 \) and \( \gamma^2 \) the connected components of \( \partial A_1 \). Consider a lift \( \tilde{A}_1 \) of \( A_1 \) to the universal covering \( \tilde{B} \) of \( B \) and denote by \( \tilde{\gamma}, \tilde{\gamma}^1 \) and \( \tilde{\gamma}^2 \) the lifts of \( \gamma, \gamma^1 \) and \( \gamma^2 \) respectively, contained in \( \tilde{A}_1 \). Let \( T \) be a deck transformation associated to \( \gamma \) and such that \( T(\tilde{A}_1) = \tilde{A}_1 \).

There are two possible cases, namely \( \tilde{B} \) is an annulus or \( \chi(B) < 0 \). First, suppose that \( B \) is an annulus. Consider the prime end compactification \( \overline{B} \) of \( B \) in \( S \) (cf. [3]); it is a closed annulus. Let \( \pi : \overline{B} \to \tilde{B} \) the universal covering map. Then \( \overline{B} \) is the union of \( \tilde{B} \) and two disjoint topological lines \( \partial_1 \) and \( \partial_2 \). We can suppose that \( \partial_l \) is in the same connected component of \( \overline{B} \setminus \tilde{\gamma} \) as \( \tilde{\gamma}^l \) for \( l \in \{1, 2\} \).

Suppose \( \chi(B) < 0 \). Then \( B \) admits a complete hyperbolic metric such that if we consider the universal covering map \( \pi : \mathbb{D} \to B \), where \( \mathbb{D} \) is the hyperbolic disc, then any homeomorphism \( g \in \text{Homeo}(B) \) satisfies that any lift \( \tilde{g} \) of \( g \) to \( \mathbb{D} \) has a continuous extension to a homeomorphism of \( \overline{\mathbb{D}} = \mathbb{D} \cup S_\infty \) [2]. We stress that we do not need to suppose \( \chi(B) \neq -\infty \). Notice that \( T \) is not parabolic since otherwise \( \gamma \) is inessential in \( S \setminus K \). As a consequence we can suppose that \( \gamma \) is a geodesic.

We define the quotient \( \tilde{B} \) of \( \overline{\mathbb{D}} \setminus \text{Fix}(T) \) by the group \( \langle T \rangle \). The set \( \tilde{B} \) is a closed annulus and the natural mapping \( \tilde{\pi} : \overline{\mathbb{D}} \setminus \text{Fix}(T) \to \tilde{B} \) is its universal covering map. The axis \( \tilde{\gamma} \) of \( T \) is a lift of \( \gamma \). Moreover, \( \tilde{\gamma}^1 \) and \( \tilde{\gamma}^2 \) are simple curves such that \( \partial \tilde{\gamma}^1 = \partial \tilde{\gamma}^2 = \text{Fix}(T) \). Consider the connected component \( \tilde{E}_l \) of \( \overline{\mathbb{D}} \setminus \tilde{\gamma}^l \) that does not contain \( \tilde{\gamma} \) for \( l \in \{1, 2\} \). We define \( \partial_l = (\partial \tilde{E}_l \cap S_\infty) \setminus \text{Fix}(T) \) for \( l \in \{1, 2\} \).

Consider a lift \( \tilde{f} \) of \( f \) to \( \tilde{B} \) that commutes with \( T \). In both cases, the sets \( \tilde{\gamma}^1, \tilde{\gamma}^2, \partial_1 \) and \( \partial_2 \) are \( \tilde{f} \)-invariant. Hence, we can define translation numbers \( \rho(f, \tilde{\gamma}^1), \rho(f, \tilde{\gamma}^2), \rho(f, \partial_1) \) and \( \rho(f, \partial_2) \).

**Remark 4.1.** Let \( (I_t)_{t \in [0,1]} \) the isotopy from \( f \) to \( \phi \) rel \( K \). The isotopy \( (I_t)_{t \in [0,1]} \) and \( \tilde{f} \) induce a lift \( \tilde{\phi} \) to \( \tilde{B} \). We have \( \tilde{\phi}_{|\overline{\mathbb{D}} \setminus \tilde{B}} \equiv \tilde{f}_{|\overline{\mathbb{D}} \setminus \tilde{B}} \) if \( B \) is an
annulus by the properties of the prime end compactification. Analogously, \( \hat{\phi}_{|s_\infty} = \hat{f}_{|s_\infty} \) holds if \( \chi(B) < 0 \) (cf. [2]). In particular, we obtain \( \rho(\hat{\phi}, \partial_l) = \rho(\hat{f}, \partial_l) \) for any \( l \in \{1, 2\} \).

**Lemma 4.2.** Consider the above setting. Then \( \rho(\hat{f}, \tilde{\gamma}^1) = \rho(\hat{f}, \partial_1) \) and \( \rho(\hat{f}, \tilde{\gamma}^2) = \rho(\hat{f}, \partial_2) \).

**Proof.** Assume \( \chi(B) < 0 \) since otherwise the result is obvious. Let us show \( \rho(\hat{f}, \tilde{\gamma}^1) = \rho(\hat{f}, \partial_1) \), the proof of the other equality is analogous.

Let \( \hat{f}_1 \) be a lift of \( f \) to \( \mathbb{D} \) such that \( \hat{f}_1 \) is the identity restricted to \( \tilde{\gamma}^1 \). Since \( f \) and \( \hat{f}_1 \) commute with \( T \) and the centralizer of \( T \) in the group of deck transformations is \( \langle T \rangle \), we deduce that \( \hat{f}_1 = T^j \circ \hat{f} \) for some \( j \in \mathbb{Z} \). In particular, it suffices to show \( \rho(\hat{f}_1, \tilde{\gamma}^1) = \rho(\hat{f}_1, \partial_1) \).

Consider a path \( \beta : [0, 1] \to \overline{\mathbb{D}} \) such that \( \beta(0) \in \tilde{\gamma}^1 \), \( \beta(0, 1) \subset E_1 \) and \( \beta(1) \in \partial_1 \). Consider the set \( F = \{ t \in (0, 1) : \pi(\beta(t)) \in A_1 \cup \ldots \cup A_m \} \). If \( F \) is empty then \( \hat{f}_1 \) is the identity by restriction to \( \beta[0, 1] \) and in particular the point \( \beta(1) \) is a fixed point of \( \hat{f}_1 \). Since \( \hat{f}_1 \) has fixed points in \( \tilde{\gamma}^1 \) and \( \partial_1 \) then the translation numbers of \( \hat{f}_1 \) at \( \tilde{\gamma}^1 \) and \( \partial_1 \) are both zero.

So, we can assume \( F \neq \emptyset \). Let \( t_0 = \inf(F) \). The point \( \beta(t_0) \) belongs to a lift of a component of some \( \partial A_l \), hence \( \hat{f}_1 \) fixes the points in an axis of a covering transformation \( T' \) associated to some \( \gamma_l \). Since clearly \( T \neq T' \) then \( \hat{f}_1 \) fixes two points in \( \partial_l \) and \( \rho(\hat{f}_1, \partial_1) = 0 \). \( \Box \)

4.2. Proof of Theorem 4.1. We denote \( C^{(j)} = C^{(j)}(G) \) for simplicity. Let \( q \) be the nilpotency class of \( G \). Since \( C^{(q)} = \{ \text{Id} \} \), the result is obviously true for \( k \geq q \). Let us prove that if the result holds true for \( k = j + 1 \) and \( j \geq 1 \) then it holds true for \( k = j \). An element \( \phi \in C^{(j)} \) is of the form

\[
\phi = [g_1, h_1] \ldots [g_m, h_m]
\]

where either \((g_l, h_l)\) or \((h_l, g_l)\) belong to \( C^{(j-1)} \times G \) for any \( 1 \leq l \leq m \).

Consider the diffeomorphisms \( \alpha_k \in C^{(j+1)} \) provided by Lemma 3.1 for \( k \geq 1 \).

Consider the set \( K = \text{Fix}(\phi) \cap \text{Fix}(C^{(j+1)}) \). If \( K = \emptyset \) there is nothing to prove. Suppose that \( K \) is a nonempty set. Consider the Thurston decomposition of \( \phi \) rel to \( K \) ([24], Proposition 3.1], \( H_1 = \langle \phi, C^{(j+1)} \rangle \), \( H_2 = G \), \( K_1 = K \), \( K_2 = \emptyset \)). Indeed there exists a finite set \( \mathcal{R} \) of reducing curves in \( S \setminus K \) that are core curves of a finite set of pairwise disjoint annuli \( A \) in \( S \setminus K \) such that any \( \psi \in G \) is isotopic rel \( K \) to some \( g \in \text{Homeo}_+(S) \) satisfying

- \( g \) preserves the decomposition, i.e. \( g(\cup_{A \in A} \partial A) = \cup_{A \in A} \partial A \).
Let $M$ be a connected component of $S \setminus A$. Suppose $M \cap K$ is finite. Then $\chi(M \setminus K) < 0$ and $g_M$ has finite order or is a pseudo-Anosov homeomorphism of $M \setminus K$ if $g(M) = M$.

Suppose $\mathcal{C}(M \cap K) = \infty$. Then $g_M \equiv Id$ if $\psi \in \langle \phi, \mathcal{C}^{(j+1)} \rangle$.

By induction hypothesis, $\eta$ is isotopic to $Id \circ \text{Fix}(\mathcal{C}^{(j+2)}, \eta)$ and then relatively to the smaller set $\text{Fix}(\mathcal{C}^{(j+1)}, \phi)$ for any $\eta \in \mathcal{C}^{(j+1)}$. Let $f \in \text{Homeo}_+(S)$ be a homeomorphism isotopic to $\phi \circ \text{rel} \ K$ in Thurston normal form (see above). Consider a connected component $M_0$ of $S \setminus A$. and the class $[\phi]_{M_0} := [f|_{M_0}]$ induced in the mapping class group $\text{MCG}(M_0, M_0 \cap K)$. It does not depend on the choice of $f$ since $\mathcal{C}(M_0 \cap K) = \infty$ or $\chi(M_0 \setminus K) < 0$. Notice that $[\phi]_{M_0}$ is trivial if $M_0 \cap K$ is an infinite set. If $M_0 \cap K$ is finite then the subgroup of $\text{MCG}(M_0, M_0 \cap K)$ induced by the stabilizer of $M_0$ in $G$ (modulo isotopy rel $K$) is a nilpotent subgroup of a mapping class group of a finite type surface. Therefore it is virtually abelian [4], i.e it has a finite index normal subgroup that is abelian. In particular there exist $k \in \mathbb{N}$ such that the classes induced by $g_j^k$ and $h_j^k$ belong to the stabilizer of $M_0$ and commute for any $1 \leq j \leq m$. Since $\alpha_k$ is isotopic to $Id \circ \text{rel} \ K$, Equation (1) implies that $[\phi]_{M_0}$ is periodic. Since $\phi$ is isotopic to $Id$ and $\chi(M_0 \setminus K) < 0$, $[\phi]_{M_0}$ satisfies the hypotheses of Lemma 4.1. Therefore, we obtain $[\phi]_{M_0} \equiv Id$. We deduce that $f$ is either the identity map or a product of non-trivial Dehn twists in closed annuli $A_1, \ldots, A_m$ whose core curves $\gamma_1, \ldots, \gamma_m$ are some of the curves in $R$. Let us prove that this last situation is impossible.

Consider the connected component $B$ of $S \setminus K$ containing $\gamma = \gamma_1$. The elements of $G$ preserve the Thurston decomposition. Hence up to replace $\phi$ with an iterate (see Equation (1) in Lemma 3.1) we can suppose that $\phi$ is of the form

$$\phi = [g_1, h_1] \cdots [g_m, h_m] \alpha$$

where $g_1, h_1, \ldots, g_m, h_m$, fix $\gamma$ modulo isotopy rel $K$ and $\alpha \in \mathcal{C}^{(j+1)}$. The open set $B$ is either an annulus or it has negative Euler characteristic. In the following, we adopt the notations in subsection 4.1.

Assume that $B$ is an annulus. We have that $g_j, h_j, \alpha, \phi$ induce homeomorphisms $\hat{g}_j, \hat{h}_j, \hat{\alpha}, \hat{\phi}$ of the prime end compactification $\hat{B}$ of $B$. Since $\alpha$ is isotopic to $Id \circ \text{rel} \ K$ by induction hypothesis, there exists a lift $\tilde{\alpha}$ of $\alpha$ to $\overline{B}$ such that $\tilde{\alpha}|_{\overline{B}} \equiv Id$. We define

$$\tilde{\phi} = [\tilde{g}_1, \tilde{h}_1] \cdots [\tilde{g}_m, \tilde{h}_m] \tilde{\alpha},$$

where $\tilde{g}_j$ (resp. $\tilde{h}_j$) is a lift of $\hat{g}_j$ (resp. $\hat{h}_j$) for any $1 \leq j \leq m$. Now, assume $\chi(B) < 0$. Notice that since $\alpha$ is isotopic to $Id \circ \text{rel} \ K$, it has a
lift $\tilde{\alpha}$ to $\mathbb{D}$ such that $\tilde{\alpha}|_{S_\infty} \equiv \text{Id}$ (cf. [2]). We consider lifts $\tilde{g}_j$ of $g_j$ and $\tilde{h}_j$ of $h_j$ such that they commute with $T$ for $1 \leq j \leq m$. We define $\tilde{\phi}$ as in Equation (2).

We claim that the group $\tilde{H} := \langle \tilde{\phi}, \tilde{g}_1, \tilde{h}_1, \ldots, \tilde{g}_m, \tilde{h}_m, T \rangle$ is nilpotent. Indeed, $C(q)(\tilde{H})$ is a subgroup of the centralizer of $T$ in the group of deck transformations and hence it is contained in $\langle T \rangle$. We deduce $C(q+1)(\tilde{H}) = \{\text{Id}\}$ since $T$ belongs to the center of $\tilde{H}$. Therefore, the translation number defines a morphism from $\tilde{H}$ to $\mathbb{R}$ for $\partial_1$ and $\partial_2$. Since the translation numbers of commutators vanish and $\tilde{\alpha}|_{\partial \tilde{B}} \equiv \text{Id}$, it follows that the translation numbers of $\tilde{\phi}$ at $\partial_1$ and $\partial_2$ are both zero.

It implies $\rho(\tilde{f}, \tilde{\partial}_1) = \rho(\tilde{f}, \tilde{\partial}_2) = 0$ for some lift $\tilde{f}$ of $f$ that commutes with $T$ by Remark 4.1. This contradicts Lemma 4.2 since $f|_{A_1}$ is a non-trivial Dehn twist and thus $\rho(\tilde{f}, \tilde{\gamma}_1) \neq \rho(\tilde{f}, \tilde{\gamma}_2)$.

5. Thurston decomposition

We introduce the configuration that will be used to show Theorems [13, A, 7.1] and [8, 1].

**Definition 5.1.** Suppose that $\mathcal{G}$ is a finitely generated nilpotent subgroup $\mathcal{G}$ of $\text{Diff}_0^1(S)$, where $S$ is an orientable connected finite type surface (not necessarily compact) of negative Euler characteristic satisfying the following properties:

- $\mathcal{H}$ is a normal subgroup of $\mathcal{G}$ such that $\mathcal{G}/\mathcal{H}$ is abelian.
- There exist $e \in \mathbb{N}$ and $f_1, \ldots, f_q \in \mathcal{G}$ such that $\langle \mathcal{H}, f_{i_1}, \ldots, f_{i_e} \rangle$ is a finite index subgroup of $\mathcal{G}$ for any choice of $1 \leq i_1 < \ldots < i_e \leq q$.
- Every element of $\mathcal{H}$ is isotopic to the identity map rel $\text{Fix}_c(\mathcal{H})$.

This is the framework associated to $\mathcal{G}$, $\mathcal{H}$, $e$ and $f_1, \ldots, f_q$.

We will make $\mathcal{S} = S$, $\mathcal{G} = G$ and $\mathcal{H} = G'$ for the proof of Theorems [13, A] and [8, 1] whereas we use a different setting for the proof of Theorem [7, 1].

In this section we define a Thurston decomposition of $\mathcal{G}$ relative to a compact $\mathcal{G}$-invariant set $\mathcal{K}$. It is complicated to provide normal forms since $\mathcal{S} \setminus \mathcal{K}$ is not in general a finite type surface. In order to make up for this problem, we will require extra properties for a fixed a priori family $\langle \mathcal{H}, f_1 \rangle, \ldots, \langle \mathcal{H}, f_q \rangle$ of subgroups of $\mathcal{G}$ that help us to obtain global fixed points. Mostly, we generalize the approach of Franks, Handle and Parwani in [15] for the abelian case to the nilpotent setting.

We would like to consider the isotopy classes of elements of $G$ relative to the invariant set $Y = \cup_{j=1}^q \text{Fix}_c(\mathcal{H}, f_j)$ but this set is not easy to
handle since \(Y \cap \text{Fix}(f_j)\) is not open and closed in \(Y\). We will replace \(Y\) with a \(G\)-invariant compact subset \(K\) of \(Y\) such that \(K \cap \text{Fix}(f_j)\) is open and closed in \(K\) for any \(1 \leq j \leq q\). Moreover, in a certain sense that we explain below, \(K\) contains representatives of the \(f_j\)-Nielsen classes of elements of \(Y\) for \(1 \leq j \leq q\).

**Definition 5.2.** We say that paths \(\gamma : [0, 1] \to \mathcal{S}\) and \(\beta : [0, 1] \to \mathcal{S}\) are homotopic rel \(K\) if there exists an homotopy \(I : [0, 1] \times [0, 1] \to \mathcal{S}\) such that the following properties hold:

- \(I(0, s) = \gamma(0)\) and \(I(1, s) = \gamma(1)\) for any \(s \in [0, 1]\);
- \(I(t, 0) = \gamma(t)\) and \(I(t, 1) = \beta(t)\) for any \(s \in [0, 1]\);
- If \(I(t_0, s_0) \in K\) then \(I(t_0, s) = I(t_0, s_0)\) for any \(s \in [0, 1]\).

**Definition 5.3.** Let \(f \in \text{Homeo}(\mathcal{S})\) and \(K\) a \(f\)-invariant closed set. We say that \(z, w \in \text{Fix}(f)\) are in the same \(f\)-Nielsen class rel \(K\) if there exists a path \(\gamma : [0, 1] \to \mathcal{S}\) such that \(\gamma(0) = z\), \(\gamma(1) = w\) and \(\gamma\) is homotopic to \(f \circ \gamma\) rel \(K\).

**Remark 5.1.** A \(f|_{\mathcal{S}\setminus K}\)-Nielsen class is contained in a \(f\)-Nielsen class rel \(K\). If the latter class does not contain points of \(K\) then both Nielsen classes coincide. In particular, if \(K \cap \text{Fix}(f) = \emptyset\) then \(z, w \in \text{Fix}(f)\) are in the same \(f\)-Nielsen class rel \(K\) if and only if \(z\) and \(w\) are in the same \(f|_{\mathcal{S}\setminus K}\)-Nielsen class.

Let \(z \in \text{Fix}(f) \setminus K\). Let \(\mathcal{N}\) and \(\mathcal{N}'\) the \(f\)-Nielsen class rel \(\mathcal{K}\) and the \(f|_{\mathcal{S}\setminus \mathcal{K}}\)-Nielsen class of \(z\) respectively. In general, the equality \(\mathcal{N} \setminus \mathcal{K} = \mathcal{N}'\) does not hold. This can be checked out by considering \(f = \text{Id}\) where \(\mathcal{N} = \mathcal{S}\) and \(\mathcal{N}'\) is the connected component of \(\mathcal{S} \setminus \mathcal{K}\) containing \(z\). Anyway, for many of the situations in this paper both concepts coincide but we will make precise which one we are considering at any moment.

**Remark 5.2.** Since \(\mathcal{H}\) is a normal subgroup of \(\mathcal{G}\) and \(\mathcal{G}/\mathcal{H}\) is abelian, any subgroup of \(\mathcal{G}\) containing \(\mathcal{H}\) is normal in \(\mathcal{G}\). In particular \(\langle \mathcal{H}, f_j \rangle\) is a normal subgroup of \(\mathcal{G}\) and \(\text{Fix}_c(\mathcal{H}, f_j)\) is a \(G\)-invariant set for any \(1 \leq j \leq q\) by Remark [5.3].

**Definition 5.4.** Denote \(Y = \bigcup_{j=1}^q \text{Fix}_c(\mathcal{H}, f_j)\). We say that a compact subset \(\mathcal{K}\) of \(\mathcal{S}\) is an excellent set for the group \(\mathcal{G}\) and the subgroups \(\langle \mathcal{H}, f_1 \rangle, \ldots, \langle \mathcal{H}, f_q \rangle\) if

- \(\mathcal{K}\) is a \(G\)-invariant subset of \(Y\) and \(\mathcal{K} \cap \text{Fix}_c(f_j) \neq \emptyset\) \(\forall 1 \leq j \leq q\).
- \(\sum_{j=1}^q 1_{\text{Fix}(f_j)}\) is locally constant in \(\mathcal{K}\) where \(1_{\text{Fix}(f_j)}\) is the characteristic function of \(\text{Fix}(f_j)\).
- If \(z \in Y \cap \text{Fix}(f_{j_1}, \ldots, f_{j_l})\) there exists \(w \in K \cap \text{Fix}(f_{j_1}, \ldots, f_{j_l})\) such that \(z\) and \(w\) are in the same \(g\)-Nielsen class rel \(\mathcal{K}\) for any \(g \in \{f_{j_1}, \ldots, f_{j_l}\}\).
We say that $\mathcal{K}$ is a good (compact) set for the group $\mathcal{G}$ and the subgroups $\langle \mathcal{H}, f_1 \rangle, \ldots, \langle \mathcal{H}, f_q \rangle$ if

- $\mathcal{K} \subset \text{Fix}_c(\mathcal{H})$ is $\mathcal{G}$-invariant and $\mathcal{K} \cap \text{Fix}_c(f_j) \neq \emptyset \ \forall 1 \leq j \leq q$.
- $\sum_{j=1}^{q} 1_{\text{Fix}(f_j)}$ is locally constant in $\mathcal{K}$.
- If $z \in Y \cap \text{Fix}_c(\langle f_{j_1}, \ldots, f_{j_l} \rangle)$ there exists $w \in \mathcal{K} \cap \text{Fix}_c(\langle f_{j_1}, \ldots, f_{j_l} \rangle)$ such that $z$ and $w$ are in the same $g$-Nielsen class rel $\mathcal{K}$ for any $g \in \{ f_{j_1}, \ldots, f_{j_l} \}$.

**Remark 5.3.** It is clear that an excellent set is a good set.

**Remark 5.4.** The existence of an excellent set $\mathcal{K}$ is a consequence of Lemma 4.3 of [15]. The original statement is for commutative groups but the proof also works if all elements of $\mathcal{H}$ are isotopic to the identity rel to $Y$. Essentially, we are working with the commutative group $\mathcal{G}/\mathcal{H}$.

**Remark 5.5.** Consider an open and closed $\mathcal{G}$-invariant subset $\mathcal{K}'$ of a good set $\mathcal{K}$ for the subgroups $\langle \mathcal{H}, f_1 \rangle, \ldots, \langle \mathcal{H}, f_q \rangle$ such that $\mathcal{K}' \cap \text{Fix}_c(\mathcal{H}, f_j) = \mathcal{K} \cap \text{Fix}_c(\mathcal{H}, f_j)$ for some $1 \leq q' \leq q$ and any $1 \leq j \leq q'$. Then $\mathcal{K}'$ is a good compact set for the group $\mathcal{G}$ and the subgroups $\langle \mathcal{H}, f_1 \rangle, \ldots, \langle \mathcal{H}, f_{q'} \rangle$. We will use this remark later on and this is the reason justifying the introduction of good sets since we can not guarantee that $\mathcal{K}'$ is excellent if $\mathcal{K}$ is.

The next step is considering a Thurston decomposition for the normal subgroups $\langle \mathcal{H}, f_1 \rangle, \ldots, \langle \mathcal{H}, f_q \rangle$ of $\mathcal{G}$ relative to a good compact set $\mathcal{K}$. The situation is analogous to the abelian one.

**Definition 5.5.** Let $\mathcal{K}$ be a good compact set for the group $\mathcal{G}$ and the subgroups $\langle \mathcal{H}, f_1 \rangle, \ldots, \langle \mathcal{H}, f_q \rangle$. We say that a finite set $\mathcal{R}$ of curves is a reducing set rel $\mathcal{K}$ if $\mathcal{R}$ consists of disjoint pairwise non-homotopic simple closed curves $\mathcal{R}$ in $\mathcal{S} \setminus \mathcal{K}$ such that

- $\mathcal{R}$ is $\mathcal{G}$-invariant modulo isotopy rel $\mathcal{K}$.
- No connected component of $\mathcal{S} \setminus \mathcal{R}$ is a disk $D$ with $\sharp(D \cap \mathcal{K}) \leq 1$.
- Given a connected component $E$ of $\mathcal{S} \setminus \mathcal{R}$ with $\sharp(E \cap \mathcal{K}) = \infty$ and $E \cap \mathcal{K} \cap \text{Fix}(f_j) \neq \emptyset$ for some $1 \leq j \leq q$, there exists a homeomorphism $\phi_j : \mathcal{S} \to \mathcal{S}$ such that $(\phi_j)|_E \equiv Id$ and $f_j$ is isotopic to $\phi_j$ rel $\mathcal{K}$.

**Remark 5.6.** Consider the framework introduced in this section. The existence of a set of reducing curves for $\mathcal{G}$, the subgroups $\langle \mathcal{H}, f_1 \rangle, \ldots, \langle \mathcal{H}, f_q \rangle$ and an excellent compact set $\mathcal{K}$ is a consequence of [15, Theorem 4.4.] if $\mathcal{H}$ is trivial (and hence $\mathcal{G}$ is abelian). A key part of the proof is by contradiction, and they use the next lemma for the case where $\mathcal{H} = \{ Id \} [15, \text{Lemma 3.10}]$ to obtain points in $Y \cap \text{Fix}(f_1, \ldots, f_l)$ that
are not $f_l$-Nielsen equivalent rel $\mathcal{K}$ to any point of $\mathcal{K}$, contradicting the choice of $\mathcal{K}$. In our case, the points of $\mathcal{K}$ have to be contained in $\text{Fix}_c(\mathcal{H})$ and hence we need to replace [15, Lemma 3.10] with next lemma to obtain points in $Y \cap \text{Fix}(\mathcal{H}, f_1, \ldots, f_l)$ that are not $f_l$-Nielsen equivalent rel $\mathcal{K}$ to any point of $\mathcal{K}$, providing the contradiction. The remainder of the proof Theorem 4.4 in [15] works well if $\mathcal{G}$ is abelian up to isotopy rel $\mathcal{K}$. This is our case since the elements of $\mathcal{H}$ are isotopic to $\text{Id}$ rel $Y$ by hypothesis and $\mathcal{G}/\mathcal{H}$ is abelian.

We denote by $\pi : \tilde{S} \to S$ the universal covering map.

**Lemma 5.1.** Let $\mathcal{G} = \langle \mathcal{H}, f_0, f_1, \ldots, f_l \rangle$ be a nilpotent subgroup of $\text{Diff}^+(\mathcal{S})$ where $\mathcal{S}$ is an orientable finitely punctured compact surface with $\chi(\mathcal{S}) < 0$, $f_0 = \text{Id}$, $l \geq 1$ and $\mathcal{H}$ is a normal subgroup of $\mathcal{G}$ such that $\mathcal{G}/\mathcal{H}$ is abelian. Let $\mathcal{K}$ be a $\mathcal{G}$-invariant compact set such that every element $h$ of $\mathcal{H}$ is isotopic to $\text{Id}$ rel $\mathcal{K}$. Assume that there exist a closed disc $D$ in $\mathcal{S}$ such that $\partial D \cap \mathcal{K} = \emptyset$, $\emptyset \neq \mathcal{K} \cap D \subset \cap_{j=0}^{l-1} \text{Fix}(f_j)$, $\mathcal{K} \cap D \cap \text{Fix}(f_l) = \emptyset$ and $f_j(\partial D)$ is homotopic to $\partial D$ rel $\mathcal{K}$ for any $1 \leq j \leq l$. Consider a lift $\tilde{f}_j$ of $f_j$ to $\tilde{S}$ such that $\tilde{f}_j(\pi^{-1}(\mathcal{K}) \cap \tilde{D}_0) = \pi^{-1}(\mathcal{K}) \cap \tilde{D}_0$ for some connected component $\tilde{D}_0$ of $\pi^{-1}(D)$ and $0 \leq j \leq l$. Then there exists a global fixed point $\tilde{z}$ of the group $\langle \tilde{\mathcal{H}}, \tilde{f}_1, \ldots, \tilde{f}_l \rangle$, where $\tilde{\mathcal{H}}$ consists of the identity lifts of elements of $\mathcal{H}$, such that $z := \pi(\tilde{z})$ is not $f_l$-Nielsen equivalent rel $\mathcal{K}$ to any element of $\mathcal{K}$. In particular if $\tilde{f}_j$ is the identity lift of $f_j$ for some $1 \leq j \leq l$, $z$ belongs to $\text{Fix}_c(\mathcal{H}, f_j)$.

**Proof.** Since $\mathcal{H} \triangleleft \mathcal{G}$, the group $\mathcal{H}$ is normal in $\tilde{\mathcal{G}} := \langle \tilde{\mathcal{H}}, \tilde{f}_1, \ldots, \tilde{f}_l \rangle$. Therefore, $C^{(\psi)}(\mathcal{G}) = \{\text{Id}\}$ implies $C^{(\psi)}(\tilde{\mathcal{G}}) = \{\text{Id}\}$ and so $\tilde{\mathcal{G}}$ is nilpotent. Moreover, the elements of $\tilde{\mathcal{H}}$ are isotopic to $\text{Id}$ rel $\pi^{-1}(\mathcal{K})$.

We denote $\tilde{\mathcal{K}}_0 = \pi^{-1}(\mathcal{K}) \cap \tilde{D}_0$. Since

$$\tilde{\mathcal{K}}_0 \subset \text{Fix}(\tilde{\mathcal{H}}, \tilde{f}_1, \ldots, \tilde{f}_{l-1}) \setminus \text{Fix}(\tilde{f}_l)$$

we can apply Proposition 5.3 to $H = \langle \tilde{\mathcal{H}}, \tilde{f}_1, \ldots, \tilde{f}_{l-1} \rangle$, $f = \tilde{f}_l$ to obtain $\tilde{z} \in \text{Fix}(\tilde{\mathcal{G}}) \cap P(\tilde{\mathcal{K}}_0, \tilde{f}_l)$.

We claim that $z$ is not $f_l$-Nielsen equivalent rel $\mathcal{K}$ to $w \in \mathcal{K}$. Otherwise, there exists $\tilde{w} \in \pi^{-1}(w)$ that is $\tilde{f}_l$-Nielsen equivalent to $\tilde{z}$ rel $\pi^{-1}(\mathcal{K})$. Since $\tilde{\mathcal{K}}_0 \cap \text{Fix}(\tilde{f}_l) = \emptyset$, it follows that $\tilde{w} \in P(\tilde{\mathcal{K}}_0, \tilde{f}_l)$. Notice that

$$\text{Fix}(\tilde{f}_l) \cap \pi^{-1}(\mathcal{K}) = \text{Fix}(\tilde{f}_l) \cap (\pi^{-1}(\mathcal{K}) \setminus \tilde{\mathcal{K}}_0)$$

and thus all points in $\text{Fix}(\tilde{f}_l) \cap \pi^{-1}(\mathcal{K})$ are peripheral to $\infty$ rel $\tilde{\mathcal{K}}_0$ for $\tilde{f}_l$ contradicting $\tilde{w} \in P(\tilde{\mathcal{K}}_0, \tilde{f}_l)$. 


It is clear that \( z \in \text{Fix}_c(\mathcal{H}) \). Moreover, if \( \tilde{f}_j \) is the identity lift then \( z \in \text{Fix}_c(\mathcal{H}, f_j) \).

We can refine the previous decomposition in the connected components \( M \) of \( \mathcal{S} \setminus \mathcal{R} \) where \( M \cap \mathcal{K} \) is finite.

**Definition 5.6.** Let \( \mathcal{R} \) be a reducing set of curves for a group \( \mathcal{G} \), subgroups \( \langle \mathcal{H}, f_1 \rangle, \ldots, \langle \mathcal{H}, f_q \rangle \) and a good compact set \( \mathcal{K} \). Consider a connected component \( M \) of \( \mathcal{S} \setminus \mathcal{R} \). We define \( \mathcal{G}_M \) as the subgroup of \( \mathcal{G} \) of diffeomorphisms \( f \) such that \( f(M) \) is isotopic to \( M \) rel \( \mathcal{K} \). In particular \( \mathcal{G}_M \) contains \( \mathcal{H} \). Given \( f \in \mathcal{G}_M \) there exists a diffeomorphism \( \theta_f \) such that \( \theta_f(M) = M \) and \( f \) is isotopic to \( \theta_f \) rel \( \mathcal{K} \). Consider the group \( \hat{\mathcal{G}}_M = \langle (\theta_f)|_M : f \in \mathcal{G}_M \rangle \). Given \( f \in \mathcal{G}_M \), the element induced by \( \theta_f \) in the mapping class group of the marked surface \( (M, M \cap \mathcal{K}) \) depends only on \( f \) since either \( \sharp(M \cap \mathcal{K}) = \infty \) or \( \chi(M \setminus \mathcal{K}) < 0 \). Thus, we can define the group \( \mathcal{G}_M \) induced by \( \hat{\mathcal{G}}_M \) in the mapping class group of \( (M, M \cap \mathcal{K}) \).

**Definition 5.7.** Let \( \mathcal{K} \) be a good compact set for the group \( \mathcal{G} \) and the subgroups \( \langle \mathcal{H}, f_1 \rangle, \ldots, \langle \mathcal{H}, f_q \rangle \). We say that a finite set \( \mathcal{R} \) of simple closed curves in \( \mathcal{S} \setminus \mathcal{K} \) is a Thurston-reducing set rel \( \mathcal{K} \) if \( \mathcal{R} \) is a reducing set rel \( \mathcal{K} \) and given any connected component \( M \) of \( \mathcal{S} \setminus \mathcal{R} \) such that \( \sharp(M \cap \mathcal{K}) < \infty \), the group \( \mathcal{G}_M \) consists of irreducible (periodic or pseudo-Anosov) elements.

**Proposition 5.1.** Let \( \mathcal{K} \) be an excellent compact set for the group \( \mathcal{G} \) and the subgroups \( \langle \mathcal{H}, f_1 \rangle, \ldots, \langle \mathcal{H}, f_q \rangle \). There exists a Thurston-reducing set rel \( \mathcal{K} \).

**Proof.** There exists a reducing set \( \mathcal{R} \) of curves rel \( \mathcal{K} \) by Remark 5.6. Consider a connected component \( M \) of \( \mathcal{S} \setminus \mathcal{R} \) such that \( M \cap \mathcal{K} \) is finite. Note that the group \( \hat{\mathcal{G}}_M \) is abelian since every element of \( \mathcal{H} \) is isotopic to \( \text{Id} \) rel \( \mathcal{K} \) and \( \mathcal{G}/\mathcal{H} \) is abelian. It is known that there exists a Thurston-reducing set for an abelian subgroup of the mapping class group of a finite type surface [18, Lemma 2.2]. Thus, it suffices to add to \( \mathcal{R} \) a set of Thurston-reducing curves of \( \hat{\mathcal{G}}_M \) for any connected component \( M \) of \( \mathcal{S} \setminus \mathcal{R} \) such that \( M \cap \mathcal{K} \) is finite. □

At some points it will be convenient to consider fewer subgroups in our framework because that allows us to recover arguments of the case where \( \mathcal{K} \) is finite. We still obtain a Thurston decomposition.

**Remark 5.7.** Let \( \mathcal{K} \) be a good compact set for the group \( \mathcal{G} \) and the subgroups \( \langle \mathcal{H}, f_1 \rangle, \ldots, \langle \mathcal{H}, f_q \rangle \). Assume that there exists a reducing set \( \mathcal{R} \) of curves rel \( \mathcal{K} \). Consider a subset \( M' \) of \( \mathcal{S} \setminus \mathcal{R} \) that is a union of
connected components of $S \setminus R$. Assume that $M'$ is $G$-invariant modulo isotopy rel $K$. Let $e \leq q' \leq q$ such that

$$K'' := M' \cap K \cap (\bigcup_{j=1}^{q'} \text{Fix}_c(H, f_j))$$

is a finite set. The set $K' := (K \setminus M') \cup K''$ is a good compact set for the group $G$ and the subgroups $\langle H, f_1 \rangle, \ldots, \langle H, f_{q'} \rangle$ (Remark 5.5).

Assume that $\chi(M \setminus K'') < 0$ for every connected component $M$ of $M'$. This guarantees that $R$ still satisfies the properties in Definition 5.5. More precisely, no connected component of $S \setminus R$ is an annulus that contains no points of $K'$ or a disc containing at most one point of $K'$. In this case, we obtain a Thurston-reducing set of curves rel $K'$ for the group $G$ and the subgroups $\langle H, f_1 \rangle, \ldots, \langle H, f_{q'} \rangle$ by adding the Thurston-reducing curves of the group induced by $G$ in the mapping class group of the finite type surface $M' \setminus K''$.

**Remark 5.8.** Let $R$ be a Thurston-reducing set of curves rel $K$ for the group $G$ and the subgroups $\langle H, f_1 \rangle, \ldots, \langle H, f_{q} \rangle$ where $K$ is a good compact set. As a consequence of Definition 5.5, there exists a finite set $A$ of pairwise disjoint non-homotopic closed annuli in $S \setminus K$ such that

- every annulus of $A$ contains exactly a reducing curve in $R$ as a core curve of the annulus;
- any $g \in G$ is isotopic rel $K$ to some $\theta_g \in \text{Homeo}_+(S)$ such that $\theta_g(\partial A) = \partial A$;
- Given a connected component $M$ of $S \setminus A$ such that $M \cap K$ is an infinite set, we have $(\theta_{f_j})|_M \equiv \text{Id}$ whenever $M \cap K \cap \text{Fix}(f_j) \neq \emptyset$.

Given a connected component $M$ of $S \setminus A$ such that $M \cap K$ is finite and $g \in G$ with $\theta_g(M) = M$, we have that $(\theta_g)|_M$ is periodic or pseudo-Anosov rel $M \cap K$.

6. **Annular dynamics**

Roughly speaking, in this paper there are two ways of finding global fixed points of groups of diffeomorphisms depending on whether the Euler characteristic of a relevant subsurface is negative or not. In this section, we consider the case where such a subsurface is an annulus.

Let us introduce the setting of this section. Let $A$ be an open annulus and $\pi : \tilde{A} \to A$ its universal covering map. We denote by 0 and $\infty$ the ends of $A$. Let $K_0$ be a compact subset of $A$. Consider $g \in \text{Diff}^1_+(A)$ and a lift $\tilde{g} \in \text{Diff}^1_+(\tilde{A})$ such that $g(K_0) = K_0$, $\text{Fix}(\tilde{g}) \cap \pi^{-1}(K_0) = \emptyset$ and $g$ fixes the ends of $A$.

**Definition 6.1.** Let $z \in \text{Fix}(g)$. We say that $z$ does not belong to $P_0(K_0, g)$ if there exists a path $\gamma : [0, 1) \to A \setminus K_0$ such that
\( \gamma(0) = z \), \( \lim_{t \to 1} \gamma(t) = 0 \) and \( \gamma \) is properly homotopic to \( g \circ \gamma \) rel 0. The definition of \( P_\infty(K_0, g) \) is analogous. We define
\[
P_{0,\infty}(K_0, g) = P_0(K_0, g) \cap P_\infty(K_0, g).
\]

We want to find compactly covered \( g \)-Nielsen classes rel \( K_0 \) in \( \pi(\text{Fix}(\tilde{g})) \). Next, and as a temporary substitute we consider classes in \( P_{0,\infty}(K_0, g) \).

**Lemma 6.1.** Suppose that \( K_0 \) is the support of a \( g \)-invariant ergodic measure \( \mu \) and there exists a lift \( \tilde{g} \) of \( g \) to \( \tilde{A} \) with \( \rho_\mu(\tilde{g}) = 0 \). Furthermore, assume that \( \text{Fix}(\tilde{g}) \cap \pi^{-1}(K_0) = 0 \). Then the intersection of \( P_{0,\infty}(K_0, g) \) and the projection of \( \text{Fix}(\tilde{g}) \) is a non-empty set.

**Proof.** We denote by \( \pi : \tilde{A} \to A \) the universal covering map. Atkinson’s lemma \([1]\) implies that the set of recurrent points in \( \pi^{-1}(K_0) \) is of total measure. Let \( \tilde{z} \in \pi^{-1}(K_0) \) be a recurrent point. Given an arbitrary small neighborhood \( V \) of \( z \) in \( A \) there exists \( h \in \text{Homeo}_+(A) \) whose support is contained in \( V \) and such that \( \tilde{g} \circ \tilde{h} \) has a finite orbit \( \mathcal{O} \) contained in \( \pi^{-1}(K_0) \), where \( \tilde{h} \) is the lift of \( h \) such that \( h|_{\tilde{A} \setminus \pi^{-1}(V)} \equiv \text{Id} \).

We define \( h = \text{Id} \) if the \( \tilde{g} \)-orbit of \( \tilde{z} \) is finite. The sets \( \text{Fix}(\tilde{g}) \) and \( \text{Fix}(\tilde{g} \circ \tilde{h}) \) coincide if \( V \) is a small neighborhood of \( z \). The set \( P(\mathcal{O}, \tilde{g} \circ \tilde{h}) \) contains a point \( \tilde{w} \) (Proposition 5.3 of \([16]\), see also \([17]\)). We claim that if \( h \) is close enough to the identity in the \( C^0 \) topology then the point \( \pi(\tilde{w}) \) also belongs to \( P_{0,\infty}(K_0, g) \).

It is clear that there exists a compact subannulus \( A_K \) of \( A \) containing \( K_0 \) such that \( (\text{Fix}(g) \setminus A_K) \cap \pi(\text{Fix}(\mathcal{O}, \tilde{g} \circ \tilde{h})) \) is empty if \( V \) is a small neighborhood of \( z \). Consider a point \( w_0 \in \text{Fix}(g) \setminus P_0(K_0, g) \). Let \( \gamma \) be the path in Definition 6.1. Let \( J \) be the homotopy between \( \gamma \) and \( g \circ \gamma \). The distance between the image of \( J \) and \( K_0 \) is positive, say greater than some \( \epsilon > 0 \). Hence if \( V \) is a small neighborhood of \( z \) the point \( w_0 \) does not belong to \( P_0(\mathcal{O}, g \circ \tilde{h}) \). Therefore no lift \( \tilde{w}_0 \) of \( w_0 \) belongs to \( P(\mathcal{O}, \tilde{g} \circ \tilde{h}) \). Moreover \( \tilde{w} \) does not belong to \( P(\mathcal{O}, \tilde{g} \circ \tilde{h}) \) for any lift \( \tilde{w} \) of a point \( w \in \text{Fix}(g) \) in the neighborhood of \( w_0 \). An analogous property holds true for points in \( \text{Fix}(g) \setminus P_\infty(K_0, g) \).

Since \( (\pi(\text{Fix}(\tilde{g})) \cap A_K) \setminus P_{0,\infty}(K_0, g) \) is compact then
\[
(\pi(\text{Fix}(\tilde{g})) \cap P_{0,\infty}(K_0, g)) \cap \pi(P(\mathcal{O}, \tilde{g} \circ \tilde{h})) = \emptyset
\]
if \( V \) is a small neighborhood of \( z \). Thus we obtain that \( \pi(P(\mathcal{O}, \tilde{g} \circ \tilde{h})) \) is a non-empty set contained in \( P_{0,\infty}(K_0, g) \cap \pi(\text{Fix}(\tilde{g})) \).

We want to find and localize global fixed points of a nilpotent subgroup \( G = \langle H, f \rangle \) of \( \text{Diff}^+_1(S_0) \), where \( H \) is a normal subgroup of \( G \) consisting of isotopic to the identity elements. We will use Proposition 6.1 to obtain compactly covered \( f \)-Nielsen classes and Proposition
Definition 6.2. Let $A$ be an open annulus contained in a surface $S_0$. We denote by $\pi : \tilde{S}_0 \to S_0$ the universal covering map. Suppose that the core curve of the annulus is not null-homotopic in $S_0$; in this case the universal covering $\tilde{A}$ of $A$ is contained in $\tilde{S}_0$. Let $K_0$ be a non-empty compact subset of $A$. Consider $f \in \text{Diff}_+^1(S_0)$ such that $f$ is isotopic to some $\theta \in \text{Homeo}_+(S_0)$ rel $K_0$ such that $f(K_0) = K_0$, $\theta(A) = A$ and $\theta$ preserves the ends of $A$. Consider a lift $\tilde{f}$ of $f$ to $\tilde{S}_0$ such that $\tilde{f}(\tilde{A})$ is isotopic to $\tilde{A}$ rel $\pi^{-1}(K_0)$. Let $\mu$ be an $f$-invariant measure with support contained in $K_0$. Up to fix a generator of $H_1(A, \mathbb{Z})$, we can define the rotation number $\rho_\mu(\tilde{f})$ of $\tilde{f}$ with respect to $\mu$ in $\tilde{A}$.

Remark 6.1. Suppose $f \in \text{Diff}_0^1(S_0)$, $\chi(S_0) < 0$ and $\tilde{f}$ is the identity lift of $f$ to $\tilde{S}_0$. Then $\rho_\mu(\tilde{f})$ does not depend on the choice of $\tilde{A}$ since it commutes with deck transformations.

Proposition 6.1. Suppose that $K_0$ is the support of an $f$-invariant ergodic measure $\mu$ such that $\rho_\mu(\tilde{f}) = 0$ and $\text{Fix}(\tilde{f}) \cap (\pi^{-1}(K_0) \cap \tilde{A}) = \emptyset$. Then there exists a lift $f^2$ such that $\text{Fix}(f^2)$ is a non-empty compact set whose projection to $S_0$ is contained in the $f$-Nielsen class defined by $\tilde{f}$.

Proof. Let $\pi : \tilde{S}_0 \to S_0$ be the universal covering map. Note that since the core curve of the annulus $A$ is not null-homotopic, $\tilde{S}_0$ is not a sphere and hence $\tilde{S}_0$ is homeomorphic to $\mathbb{R}^2$. Consider the primitive deck transformation $T$ such that $T(\tilde{A}) = \tilde{A}$. We define $\tilde{S}_0$ as the quotient of $\tilde{S}_0$ by the action of the group $\langle T \rangle$. Let $\tilde{\pi} : \tilde{S}_0 \to S_0$ and $\varpi : \tilde{S}_0 \to \tilde{S}_0$ be the natural maps. The annulus $\tilde{A} := \varpi(\tilde{A})$ satisfies that $\tilde{\pi}|_{\tilde{A}} : \tilde{A} \to A$ is a homeomorphism. We denote $\tilde{K}_0 = \tilde{\pi}^{-1}(K_0) \cap \tilde{A}$. Consider the unique lift $\tilde{f}$ of $f$ to $\tilde{S}_0$ such that $\tilde{f}$ is a lift of $\tilde{f}$. We obtain $\tilde{f}(\tilde{K}_0) = \tilde{K}_0$. Let $\tilde{\mu}$ be the measure $\tilde{\pi}^*\mu$.

Since $\text{Fix}(\tilde{f}) \cap \varpi^{-1}(\tilde{K}_0) \cap \tilde{A} = \emptyset$ by hypothesis, there exists $\tilde{z} \in P_{0,\infty}(\tilde{K}_0, \tilde{f}) \cap \varpi(\text{Fix}(\tilde{f}))$ by Lemma 6.1. Let $\tilde{U}$ be the connected component of $\tilde{S}_0 \setminus \tilde{K}_0$ containing $\tilde{z}$. We claim that $\tilde{U}$ is not an annulus. Indeed, since $\tilde{K}_0 = \text{sup}(\tilde{\mu})$ and $\tilde{\mu}$ is ergodic, the set $\tilde{S}_0 \setminus \tilde{U}$ has just one connected component. If $\tilde{U}$ is an inessential annulus then $\tilde{S}_0 \setminus \tilde{U}$ has
two connected components and we get a contradiction. If \( \hat{U} \) is an essential annulus such that \( \hat{S}_0 \setminus \hat{U} \) has just one connected component then \( \hat{\gamma} \notin P_0(\hat{K}_0, \hat{f}) \) or \( \hat{\gamma} \notin P_\infty(\hat{K}_0, \hat{f}) \) and again we obtain a contradiction.

We denote by \( \pi_0 : \hat{U} \to \hat{U} \) the universal covering map. Fix a lift \( \hat{\gamma} \) of \( \hat{\gamma} \) in \( \hat{U} \). We define \( f^\circ \) as the lift of \( \hat{f} \) to \( \hat{U} \) such that \( f^\circ(\hat{\gamma}) = \hat{\gamma} \). If \( \text{Fix}(f^\circ) \) is compact we define \( f^\sharp = f^\circ \), \( D = \hat{U} \) and \( K_0' = \hat{K}_0 \). The map \( f^\sharp : D \to D \) is a lift of \( f \) by the map \( \hat{\pi} \circ \pi_0 \).

Suppose that \( \text{Fix}(f^\circ) \) is not compact. There exists a non-trivial \([\nu] \in \pi_1(\hat{U}, \hat{\gamma})\) such that \( \hat{f}[\nu] = [\nu] \). Up to replace \( \nu \) with an homotopic closed path defining the same class in \( \pi_1(\hat{U}, \hat{\gamma}) \), there exists a finite type connected subsurface \( S \) contained in \( \hat{U} \), containing \( \nu[0, 1] \) and a homeomorphism \( \theta \) of \( \hat{S}_0 \) isotopic to \( \hat{f} \) rel \( \hat{K}_0 \) such that \( \theta|_S \) has finite order (Lemma 2.12 of [16]). Moreover no boundary curve \( \alpha \) in \( \partial S \) bounds a disk contained in \( \hat{S}_0 \setminus \hat{K}_0 \).

Consider the connected components \( S^0 \) and \( S^\infty \) of \( \hat{S}_0 \setminus S \) containing the ends \( 0 \) and \( \infty \) respectively. If \( S^0 = S^\infty \) then we denote by \( \gamma \) the unique curve in \( \partial S \cap \partial S^0 \cap \partial S^\infty \). The curve \( \gamma \) is non-essential.

Suppose now \( S^0 \neq S^\infty \). The curve \( \partial S \cap \partial S^0 \) is essential. Moreover the compact set \( \hat{K}_0 \cap S^0 \) is invariant and either has measure 1 or 0. Since \( \hat{K}_0 \cap S^0 \) is a compact invariant set and \( \supp(\hat{\mu}) = \hat{K}_0 \) then either \( \hat{K}_0 \cap S^0 = \emptyset \) or \( \hat{K}_0 \cap S^0 = \hat{K}_0 \). The latter situation is impossible since it would imply \( \hat{\gamma} \notin P_\infty(\hat{K}_0, \hat{f}) \). Analogously we obtain that \( \partial S \cap \partial S^\infty \) is an essential curve and \( \hat{K}_0 \cap S^\infty = \emptyset \). Since \( S \) is connected all the boundary curves other than \( \partial S \cap \partial S^0 \) and \( \partial S \cap \partial S^\infty \) are non-essential. There are at least one such non-essential boundary curve since otherwise we have \( \hat{K}_0 = \emptyset \). Consider the homeomorphism \( \theta^\circ \) induced by \( \theta \) in the surface \( S^0 \) obtained by contracting the curves in \( \partial S \). It has two fixed points corresponding to \( \partial S \cap \partial S^0 \) and \( \partial S \cap \partial S^\infty \). Moreover since \( \theta^\circ \) is isotopic to \( \hat{f} \) rel \( \hat{K}_0 \) and the rotation number \( \rho_{\hat{\mu}}(\hat{f}) \) vanishes, it follows that all boundary curves of \( S \) correspond to fixed points of \( \theta^\circ \) in \( S^0 \). Moreover since \( \theta^\circ \) is a finite order homeomorphism of a sphere \( S^0 \) with at least three fixed points, we obtain \( \theta|_S \equiv \text{Id} \). The ergodicity of \( \hat{\mu} \) implies that \( \partial S \) contains exactly one non-essential curve \( \gamma \).

Let \( D_0 \) be the topological disc in \( \hat{S}_0 \) enclosed by \( \gamma \). Since \( \rho_{\hat{\mu}}(\hat{f}) = 0 \) the diffeomorphism \( \hat{f} \) fixes (modulo isotopy rel \( \varpi^{-1}(\hat{K}_0) \)) a lift \( \hat{D}_0 \) of \( D_0 \). Moreover \( K_0' := \hat{D}_0 \cap \varpi^{-1}(\hat{K}_0) \) is a non-empty compact \( \hat{f} \)-invariant set such that \( \pi_{\hat{K}_0} : K_0' \to K_0 \) is an homeomorphism. Hence we can lift \( \mu \) to \( K_0' \). The hypothesis \( \text{Fix}(\hat{f}) \cap (\pi^{-1}(K_0) \cap \hat{A}) = \emptyset \) implies \( \text{Fix}(\hat{f}) \cap K_0' = \emptyset \). Now we just apply Proposition 3.1 to \( \hat{f} \) and \( K_0' \).
Hence \( f^3 : D \to D \) is a lift of \( f \) by the map \( \pi \circ \pi_V \) where \( D = \tilde{V} \) and \( \pi_V : \tilde{V} \to V \) is the universal covering map of a connected component \( V \) of \( \tilde{S}_0 \setminus K_0' \). Note that \( V \) is not an annulus since otherwise it does not contain non-empty compactly covered \( \tilde{f}|_{\tilde{S}_0 \setminus K_0'} \)-Nielsen classes. Indeed, in such a case the lift of \( \tilde{f} \) commutes with every deck transformation of \( \tilde{V} \).

\( \square \)

**Remark 6.2.** We proved that the homeomorphism \( f^3 : D \to D \) is a lift of \( f \) by a map of the form \( \pi' \circ \pi_W \) where \( \pi' \) is a covering of \( S_0 \), \( f' \) is a lift of \( f \) to \( (\pi')^{-1}(S_0) \) and \( \pi_W : D \to W \) is the universal covering map of a connected component \( W \) of \( (\pi')^{-1}(S_0) \setminus K_0' \), where \( K_0' \) is a \( f^3 \)-invariant compact set contained in \( (\pi')^{-1}(K_0) \). Moreover, \( W \) is not an annulus.

**Remark 6.3.** The map \( \pi' \circ \pi_W \) is not a covering map and it is just a local homeomorphism. Anyway, since \( \pi' \) and \( \pi_W \) are covering maps we can lift \( f \) to \( (\pi')^{-1}(S_0) \) by \( \pi' \) to obtain \( f' \) and then to \( D \) by \( \pi_W \) to obtain \( f^3 \).

The next result finds and localizes fixed points.

**Proposition 6.2.** Consider the hypotheses of Proposition 6.1. Suppose that there exists a compact \( f \)-invariant subset \( K \) of \( S_0 \setminus (\overset{\circ}{A} \setminus A) \) such that \( \emptyset \neq K_0 \subset K \), \( f(A) \) is isotopic to \( A \) rel \( K \) and \( f \) preserves the ends of \( A \) modulo isotopy rel \( K \). Consider a nilpotent group \( G = \langle H, f \rangle \subset \text{Diff}^1_+(S_0) \). Suppose that any point in \( \text{Fix}(H) \cap \pi(\text{Fix}(\tilde{f})) \) is \( f \)-Nielsen equivalent rel \( K \) to a point in \( K \). Suppose further that \( H \) is a normal subgroup of \( G \) of elements isotopic to the identity rel \( K \). Then \( \text{Fix}(H) \cap \pi(\text{Fix}(\tilde{f})) \cap K \cap A \neq \emptyset \). Moreover, if \( \chi(S_0) < 0 \), we can replace \( \text{Fix}(H) \) with \( \text{Fix}_c(H) \) in the statement and then \( \text{Fix}_c(H) \cap \pi(\text{Fix}(\tilde{f})) \cap K \cap A \neq \emptyset \) holds.

**Proof.** Let \( f^3 : D \to D \) be the lift of \( f \) provided by Proposition 6.1. Consider the description of \( f^3 \) in Remark 6.2. There are two cases, namely \( (\pi')^{-1}(S_0) \) is an annulus or a topological disc. We adopt the notations in Proposition 6.1 and Remark 6.2.

First, assume that \( \tilde{S}_0 := (\pi')^{-1}(S_0) \) is an annulus. Consider the set \( \tilde{H} \) consisting of the identity lifts rel \( K \) of elements of \( H \). It is a subgroup of \( \text{Diff}(\tilde{S}_0) \) consisting of elements isotopic to \( \text{Id} \) rel \( \pi^{-1}(K) \) and in particular rel \( \hat{K}_0 \). Moreover, \( \tilde{H} \) is a normal subgroup of \( \tilde{G} := \langle \tilde{H}, \tilde{f} \rangle \) and thus \( \tilde{G} \) is nilpotent. By construction, there exists a non-empty compactly covered \( \tilde{f}|_{\tilde{S}_0 \setminus \hat{K}_0} \)-Nielsen class \( \mathcal{N} = \pi_{\tilde{V}}(\text{Fix}(f^3)) \). There exists \( \tilde{w} \in \text{Fix}(\tilde{G}) \cap \mathcal{N} \) by Proposition 3.2, let \( w = \pi(\tilde{w}) \) be its projection in \( S_0 \). Since \( w \in \text{Fix}(H) \cap \pi(\text{Fix}(\tilde{f})) \), there exists \( z \in K \) such that \( z \) is
fixed $f$-Nielsen equivalent to $w$ rel $K$ by hypothesis. In particular, we have $z \in \pi(\text{Fix}(\tilde{f}))$. There exist $\gamma : [0,1] \to S_0$ such that $\gamma(0) = w$, $\gamma(1) = z$ and a homotopy $I$ rel $K$ between $\gamma$ and $f \circ \gamma$ (cf. Definition 5.2). We define $\hat{z} = \hat{\gamma}(1)$ where $\hat{\gamma}$ is the lift of $\gamma$ to $\hat{S}_0$ such that $\hat{\gamma}(0) = \hat{w}$. Moreover, we have that the image of $I$ belongs to the same $\hat{\chi}$-class as compactly covered, we obtain $\hat{w}$ to $\text{Fix}((\hat{H})$ if we replace $\text{Fix}(\hat{H})$ with $\text{Fix}_{c}(H)$ in the statement of the proposition, we obtain $w \in \text{Fix}_{c}(H)$ by Proposition 3.2 and hence we can proceed analogously to obtain $z \in \text{Fix}_{c}(H) \cap \pi(\text{Fix}(\tilde{f})) \cap K \cap A$.

Finally, suppose that $\hat{S}_0 := (\pi^{\prime})^{-1}(S_0)$ is a topological disc. Consider the group $\hat{H}$ of identity lifts rel $K$ of elements of $H$. The group $\hat{H}$ is a normal subgroup of the nilpotent group $\hat{G} := (\hat{H}, \tilde{f})$. Since $\text{Fix}(\tilde{f}) \cap (\pi^{-1}(K_0) \cap \hat{A}) = \emptyset$, it follows that $\text{Fix}(\tilde{f}) \cap K_0 = \emptyset$. We apply Corollary 3.2 to obtain $\hat{w} \in \text{Fix}(\hat{G})$ such that its $\hat{f}_{\mid \hat{S}_0 \setminus \hat{K}_0}^\prime$-Nielsen class $\hat{N}$ is compactly covered. Let $w = \pi(\hat{w})$ and consider a point $z \in K$ such that it is $f$-Nielsen equivalent to $w$ rel $K$. We can proceed as above to show that there exists a lift $\hat{z}$ of $z$ to $\hat{S}_0$ that belongs to $\hat{N}$. Since $\hat{N}$ is compactly covered rel $K \cap A$, we deduce that $\hat{z}$ belongs to $\hat{D}_0 \cap \pi^{-1}(K)$ and then $z \in \text{Fix}(H) \cap \pi(\text{Fix}(\tilde{f})) \cap K \cap A$. Moreover, if $\chi(S_0) < 0$, we obtain $z \in \text{Fix}_{c}(H)$ by construction. 

7. Localization of fixed points in the “cyclic” case

The goal of the next two section is finding global fixed points for suitable subgroups of nilpotent subgroup of $\text{Diff}^1_0(S)$ where $S$ is a finitely punctured compact surface of negative Euler characteristic. The next theorem is useful to localize such global fixed points.

**Theorem 7.1.** Let $S$ be an orientable connected compact surface (maybe with boundary) of negative Euler characteristic. Let $G$ be a finitely generated nilpotent subgroup of $\{\phi \in \text{Homeo}_0(S) : \phi \mid S \setminus \partial S \in \text{Diff}^1_0(S \setminus \partial S)\}$ of nilpotency class $\iota$. Assume that $\phi$ is isotopic to $\text{Id}$ rel $\text{Fix}(C(k)(G))$. Let $g \in G$ and $0 \leq l < \iota$. Let $\langle G', g \rangle^l$ be a lift of $\langle G', g \rangle$ by a covering $S^g \to S \setminus (K \cup \partial S)$ where $K \subset S \setminus \partial S$ is a $g$-invariant compact set contained in $\text{Fix}(G')$, $S^g$ is a finitely punctured surface of negative Euler characteristic and the natural map $\pi_1(S^g) \to \pi_1(S)$ is injective. Suppose that $g^k$ is isotopic to
the identity and that \( \text{Fix}_c((gh)^\sharp) \) is a non-empty compact set for any \( h \in C^{(l-1)}(G) \). Then \( \text{Fix}_c\langle C^{(l-1)}(G), g \rangle^\sharp \neq \emptyset \).

We introduce the following auxiliary property:

- \( \text{B}_k \): Theorem 7.1 holds true for any finitely generated group \( G \), any nilpotency class \( \iota \) and any \( l \leq \min(k, \iota - 1) \).

We show Property \( \text{B}_k \) by induction on \( k \). Property \( \text{B}_0 \) holds by hypothesis since \( C^{(\iota)}(G) = \{ \text{Id} \} \).

7.1. Global fixed points and the framework. Later on, we will describe the particular framework (see Definition 5.1) that we will use to show Theorem 7.1. In this section, we will introduce general properties that will be used also in the proof of Theorem 7.1.

Remark 7.1. Consider the framework described in Definition 5.1. There exists an excellent compact set \( K \) for \( G \) and \( \langle H, f_1 \rangle, \ldots, \langle H, f_q \rangle \) by Remark 5.4. We fix a Thurston decomposition for \( G \) and \( \langle H, f_1 \rangle, \ldots, \langle H, f_q \rangle \) rel \( K \) by Proposition 5.1 and Remark 5.8.

Definition 7.1. We denote by \( \mathcal{R} \) and \( \mathcal{R}_e \) the sets of reducing curves and essential (in \( S \)) reducing curves respectively for the Thurston decomposition.

We are going to present several results that allow us to find global fixed points of subgroups in the framework. Then, next lemma ensures that there are global contractible fixed points of \( \mathcal{G} \). More precisely, we will apply the next result to subgroups of the form \( \langle H, f_{i_1}, \ldots, f_{i_e} \rangle \) where \( 1 \leq i_1 < i_2 < \ldots < i_e \leq q \).

Lemma 7.1. Let \( \mathcal{G} \) be a subgroup of \( \text{Diff}^1_0(S) \) where \( S \) is a connected surface of negative Euler characteristic. Consider a finite index subgroup \( \mathcal{J} \) of \( \mathcal{G} \) such that \( \text{Fix}_c(\mathcal{J}) \neq \emptyset \). Then \( \text{Fix}_c(\mathcal{G}) \neq \emptyset \).

Proof. Denote by \( \tilde{\mathcal{J}} \) and \( \tilde{\mathcal{G}} \) the groups of identity lifts of elements of \( \mathcal{J} \) and \( \mathcal{G} \) respectively to the universal covering \( \tilde{S} \) of \( S \). Fix \( z \in \text{Fix}_c(\mathcal{J}) \) and one of its lifts \( \tilde{z} \in \tilde{S} \). Since \( \tilde{z} \in \text{Fix}(\tilde{\mathcal{J}}) \) and \( \tilde{\mathcal{J}} \) is a finite subgroup of \( \tilde{\mathcal{G}} \), we deduce that the \( \tilde{\mathcal{G}} \)-orbit of \( \tilde{z} \) is finite. Thus \( \tilde{\mathcal{G}} \) preserves a non-empty compact set and as a consequence \( \tilde{\mathcal{G}} \) has a global fixed point by Theorem 3.1. Therefore, we get \( \text{Fix}_c(\mathcal{G}) \neq \emptyset \). \qed

Later on, we will need to localize the global fixed points of the subgroups of the framework in some specific regions of the Thurston decomposition that are introduced in next definition.

Definition 7.2. Let \( \mathcal{C} \) be the set of connected components of \( S \setminus \mathcal{R}_e \) that either have negative Euler characteristic or are annuli bounding
connected components of negative Euler characteristic. Let \( C_- \) be the subset of \( C \) of connected components of negative Euler characteristic. Given a connected component \( M \) of \( S \setminus \mathcal{R}_e \) we denote by \( X_M \) the set obtained by removing from \( M \) all the topological discs enclosed by curves in \( (\mathcal{R} \setminus \mathcal{R}_e) \cap M \) and the points in \( \mathcal{A} \).

### 7.2. Chasing global fixed points.

Now, we are going to study several configurations, for the global fixed points of the auxiliary subgroups in the framework, that induce global fixed points of \( G \).

#### 7.2.1. Infinite case.

**Proposition 7.1.** Let \( M \) be a connected component of \( S \setminus \mathcal{R}_e \) such that \( \#(X_M \cap \mathcal{K} \cap \text{Fix}_c(\mathcal{H}, f_b)) = \infty \) for any \( 1 \leq b \leq e \). Then \( \text{Fix}_c(\mathcal{G}) \) is a non-empty set.

**Proof.** Every element of \( \mathcal{J} := \langle \mathcal{H}, f_1, \ldots, f_e \rangle \) is isotopic rel \( \mathcal{K} \cap X_M \) to a diffeomorphism that is the identity map in \( M \) by the properties of the Thurston decomposition. Since \( X_M \cap \mathcal{K} \subset \text{Fix}_c(\mathcal{H}, f_b) \) for any \( 1 \leq b \leq e \). Thus \( X_M \cap \mathcal{K} \) is contained in \( \text{Fix}_c(\mathcal{J}) \). We obtain \( \text{Fix}_c(\mathcal{G}) \neq \emptyset \) by Lemma 7.1.

#### 7.2.2. Finite case in negative Euler characteristic.

**Proposition 7.2.** Let \( M \) be a connected component of \( S \setminus \mathcal{R}_e \) such that \( \chi(M) < 0 \). Suppose that \( \#(X_M \cap \mathcal{K} \cap \text{Fix}_c(\mathcal{H}, f_b)) < \infty \) and \( M \cap \mathcal{K} \cap \text{Fix}_c(\mathcal{H}, f_b) \neq \emptyset \) for any \( 1 \leq b \leq e \). Then \( \text{Fix}_c(\mathcal{G}) \) is a non-empty set.

**Proof.** It suffices to show \( \text{Fix}_c(\mathcal{J}_0) \neq \emptyset \), where \( \mathcal{J} = \langle \mathcal{H}, f_1, \ldots, f_e \rangle \), by Lemma 7.1. The group \( [\mathcal{J}] \) induced in \( \text{MCG}(X_M, \mathcal{K} \cap X_M) \) by \( \mathcal{J} \) is an abelian group \( ([\mathcal{H}] = 1) \) whose elements are all irreducible. Consider the group \( \mathcal{J}_0 \) of \( \mathcal{J} \) consisting of the elements whose image in \( \text{MCG}(X_M, \mathcal{K} \cap X_M) \) is trivial. Since the boundary of \( \partial M \) consists of essential curves, the universal covering \( \tilde{M} \) of \( M \) can be interpreted as a subset of \( \tilde{S} \). We denote by \( \pi : \tilde{S} \to S \) the universal covering.

We claim that \( \text{Fix}_c(\mathcal{J}_0) \) is non-empty. Suppose that there exists \( z \in X_M \cap \mathcal{K} \). Any \( \phi \in \mathcal{J}_0 \) is isotopic rel \( \{z\} \) to a map \( \theta \) that is the identity in \( M \). Since \( \chi(M) < 0 \), we obtain \( z \in \text{Fix}_c(\mathcal{J}_0) \). Obviously \( \text{Fix}(\tilde{\phi}) \) contains \( \pi^{-1}(z) \) where \( \tilde{\phi} \) is the identity lift. Analogously, if there exists a reducing curve \( \gamma \) in \( M \cap (\mathcal{R} \setminus \mathcal{R}_e) \) we obtain that \( \tilde{\phi}(\tilde{\gamma}) \) is isotopic to \( \tilde{\gamma} \) rel \( \pi^{-1}(\mathcal{K}) \) for any lift \( \tilde{\gamma} \) of \( \gamma \). Let \( D_\gamma \) be the disc in \( S \) bounded by \( \gamma \). Let \( \tilde{D}_\gamma \) be a lift of \( D_\gamma \). We have that \( \pi^{-1}(\mathcal{K}) \cap \tilde{D}_\gamma \) is a non-empty compact \( \mathcal{J}_0 \)-invariant set. We obtain \( \text{Fix}_c(\mathcal{J}_0) \neq \emptyset \) by Theorem 3.1.
Suppose that \([\mathcal{J}]\) is finite. Then \(\mathcal{J}_0\) is a finite index normal subgroup of \(\mathcal{J}\) and the result is a consequence of Lemma 7.1.

Suppose \(\sharp[\mathcal{J}] = \infty\). Since \(\mathcal{J}\) is nilpotent, its subset \(\text{Tor}(\mathcal{J})\) of finite order elements is a subgroup \([20] \text{Theorem 16.2.7}\). Hence \(\sharp[\mathcal{J}] = \infty\) implies the existence of \(1 \leq b \leq e\) such that \([f_b]\) is of infinite order and then pseudo-Anosov. Moreover \([\mathcal{J}]\) is virtually cyclic (see Lemma 2.1 of \([24]\)) so \((\mathcal{J}_0, f_b)\) is a finite index normal subgroup of \(\mathcal{J}\). It suffices to prove \(\text{Fix}_\mathcal{c}(\mathcal{J}_0, f_b) \neq \emptyset\) by Lemma 7.1. There are two cases, namely \(X_M \cap \mathcal{K} \cap \text{Fix}_\mathcal{c}(f_b) \neq \emptyset\) and \((M \setminus X_M) \cap \mathcal{K} \cap \text{Fix}_\mathcal{c}(f_b) \neq \emptyset\). The former case is easy since \(X_M \cap \mathcal{K} \cap \text{Fix}_\mathcal{c}(f_b)\) is contained in \(\text{Fix}_\mathcal{c}(\mathcal{J}_0, f_b)\). In the latter case, consider a reducing curve \(\gamma\) in \(M \cap (\mathcal{R} \setminus \mathcal{R}_e)\) such that \(D_\gamma \cap (M \setminus X_M) \cap \mathcal{K} \cap \text{Fix}_\mathcal{c}(f_b) \neq \emptyset\). The curve \(\tilde{f}_b(\tilde{\gamma})\) is isotopic to \(\tilde{\gamma}\) rel \(\pi^{-1}(\mathcal{K})\). The compact set \(\pi^{-1}(\mathcal{K}) \cap \bar{D}_\gamma\) is invariant by the identity lift of \((\mathcal{J}_0, f_b)\). Thus \(\text{Fix}_\mathcal{c}(\mathcal{J}_0, f_b)\) is non-empty by Theorem 3.1.

7.2.3. Finite irrotational case and finite isotopically trivial case.

**Proposition 7.3.** Let \(M\) be a connected component of \(\mathcal{S} \setminus \mathcal{R}_e\) such that \(\chi(\mathcal{M}) = 0\). Suppose that

\[
\sharp(X_M \cap \mathcal{K} \cap \text{Fix}_\mathcal{c}(\mathcal{H}, f_b)) < \infty \quad \text{and} \quad M \cap \mathcal{K} \cap \text{Fix}_\mathcal{c}(\mathcal{H}, f_b) \neq \emptyset
\]

for any \(1 \leq b \leq e\). Suppose further that there exists a \(\mathcal{G}\)-invariant measure \(\mu\) such that \(\text{supp}(\mu) \subset \mathcal{K} \cap M\) and \(\rho_\mu(\tilde{\phi}) = 0\) (see Definition 6.2 and Remark 6.1) for any \(\phi \in \mathcal{G}\). Then \(\text{Fix}_\mathcal{c}(\mathcal{G})\) is a non-empty set.

**Proposition 7.4.** Let \(M\) be a connected component of \(\mathcal{S} \setminus \mathcal{R}_e\) such that \(\chi(\mathcal{M}) = 0\). Suppose that

\[
\sharp(X_M \cap \mathcal{K} \cap \text{Fix}_\mathcal{c}(\mathcal{H}, f_b)) < \infty \quad \text{and} \quad M \cap \mathcal{K} \cap \text{Fix}_\mathcal{c}(\mathcal{H}, f_b) \neq \emptyset
\]

for any \(1 \leq b \leq e\). Assume that the element \([f_b]\) of \(\text{MCG}(X_M, \mathcal{K} \cap X_M)\) induced by \(f_b\) is trivial for any \(1 \leq b \leq e\). Then \(\text{Fix}_\mathcal{c}(\mathcal{G})\) is a non-empty set.

**Proof of Propositions 7.3 and 7.4.** It suffices to show \(\text{Fix}_\mathcal{c}(\mathcal{J}) \neq \emptyset\), where \(\mathcal{J} = (\mathcal{H}, f_1, \ldots, f_e)\), by Lemma 7.1. The proof is the same as in Proposition 7.2 except that we have to change slightly the second paragraph. Indeed we need to prove that in the new setting it is still true that \(X_M \cap \mathcal{K} \subset \text{Fix}_\mathcal{c}(\mathcal{J}_0)\) and that given a curve \(\gamma\) in \(M \cap (\mathcal{R} \setminus \mathcal{R}_e)\) then \(\tilde{\phi}(\tilde{\gamma})\) is isotopic to \(\tilde{\gamma}\) rel \(\pi^{-1}(\mathcal{K})\) for any lift \(\tilde{\gamma}\) of \(\gamma\) and any \(\phi \in \mathcal{J}_0\). If any of these properties does not hold true for some \(\phi \in \mathcal{J}_0\) then there exists \(k \in \mathbb{Z} \setminus \{0\}\) such that \(\rho_\mu(\tilde{\phi}, z) = k\) for any \(z \in \mathcal{K} \cap M\). This contradicts the property \(\rho_\mu(\tilde{\phi}) = 0\). Hence the properties hold true in the setting of Proposition 7.3.
Let us consider the setting in Proposition 7.4. We claim that the properties hold true for any \( \phi \in \mathcal{H} \cup \{ f_1, \ldots, f_e \} \). Otherwise we deduce \( \text{Fix}_c(\phi) \cap M \cap \mathcal{K} = \emptyset \); it contradicts that the set \( M \cap \mathcal{K} \cap \text{Fix}_c(\mathcal{H}, f_0) \) is non-empty for any \( 1 \leq b \leq e \). Hence the properties hold true for any \( \phi \in \mathcal{J}_0 \).

\[ 7.3. \textbf{Framework for the proof of Theorem 7.1} \]

The results providing global fixed points in subsection 7.2 are a fundamental ingredient of the proofs of Theorems 5.3 and 7.1. Let us focus on the latter result and how to particularize the framework in Definition 5.1 to such a situation. Let us remind the reader that our goal in this section is showing that \( B_k \) implies \( B_{k+1} \) for \( k \geq 0 \).

Assume that the hypotheses of Theorem 7.1 and \( B_k \) hold and consider \( l \leq \min(k + 1, \iota - 1) \). Consider the groups

\[ H = \mathcal{C}^{(\iota+1-l)}(G), \quad \mathcal{H} = \mathcal{C}^{(\iota+1-l)}(G)^\sharp \quad \text{and} \quad \mathcal{G} = \langle \mathcal{C}^{(\iota-l)}(G), g \rangle^\sharp. \]

Notice that every subgroup of a finitely generated nilpotent group is finitely generated (cf. [20, Theorem 17.2.2]). The group \( \langle \mathcal{C}^{(\iota-l)}(G), g \rangle^\sharp \) is a finitely generated abelian group by Remark 3.2. There exists a subgroup \( J \subset \mathcal{C}^{(\iota-l)}(G) \) such that \( H \subset J \), the group \( J/H \) is torsion free and \( \mathcal{C}^{(\iota-l)}(G)/J \) is a finite group by the fundamental theorem of finitely generated abelian groups. We denote \( e = \text{rank}(J/H) + 1 \) and \( q = -2\chi(S^e)e \). Consider a sequence

\[ h_1, \ldots, h_q \in J \]

where the group generated by \( \langle H, h_1^{-1}h_{i_1}, h_2^{-1}h_{i_2}, \ldots, h_e^{-1}h_{i_e} \rangle \) is a finite index subgroup of \( \mathcal{C}^{(\iota-l)}(G) \) for any choice of pairwise different indexes \( i_1, \ldots, i_e \) in \( \{ 1, \ldots, q \} \). Hence, \( \langle H, gh_{i_1}, \ldots, gh_{i_e} \rangle \) is a finite index subgroup of \( \mathcal{C}^{(\iota-l)}(G), g \rangle \) for any choice of \( 1 \leq i_1 < \ldots < i_e \leq q \). We denote \( g_j = gh_j, \quad f_j = g_j^\sharp \) and \( H_j = \langle H, g_j \rangle \) for \( 1 \leq j \leq q \). Any group of the form \( H_j \) is a normal subgroup of \( \mathcal{C}^{(\iota-l)}(G), g \rangle \) (Remark 3.2) that satisfies the hypotheses of \( B_k \). We deduce that \( \text{Fix}_c(H_j^e) \) is a non-empty compact set for any \( 1 \leq j \leq q \).

Note that every element of \( \mathcal{H} \) is isotopic to \( \text{Id rel } \text{Fix}(\mathcal{H}) \) by the inclusions \( K \subset \text{Fix}(G') \subset \text{Fix}(H) \) and by hypothesis. Thus, the conditions established in the framework in Definition 5.1 are satisfied. Consider an excellent compact set \( \mathcal{K} \) with respect to the group \( \mathcal{G} \), the subgroups \( H_1^e, \ldots, H_q^e \) and the set \( Y = \cup_{j=1}^q \text{Fix}_c(H_j^e) \) (Remark 5.4). We also consider a Thurston decomposition for \( \mathcal{G} \) and \( H_1^e, \ldots, H_q^e \) relative to \( \mathcal{K} \) (Proposition 5.4). We define \( S = S^e \).
7.4. Cyclic nature of $\mathcal{G}$. We keep moving forward with the proof of Theorem 7.1. Let us introduce the idea behind the results in this section. Our point of view relies on profiting on the “isotopically trivial” nature of the subgroups $\mathcal{C}^{(j)}(G)$ for $j \geq 1$ in the descending central series. As a consequence, it is reasonable to expect that many of the properties of the group $⟨\mathcal{C}^{(r-1)}(G), f⟩$ are going to be derived from the properties of $f$.

There are at most $−\chi(S)$ components of negative Euler characteristic in $S \setminus \mathcal{R}_c$. Obviously if there is no annulus in $S \setminus \mathcal{R}_c$ then the points in $\text{Fix}_c⟨\mathcal{H}, f_j⟩$ are localized in those components for $1 \leq j \leq q$.

The situation is even simpler if one of the connected components $A$ of $S \setminus \mathcal{R}_c$ is an annulus. We have $A \cap \mathcal{K} \cap \text{Fix}_c⟨\mathcal{H}, f_{j_0}⟩ \neq \emptyset$ by construction for some $1 \leq j_0 \leq q$. Notice that $A \cap \mathcal{K} \cap \text{Fix}_c⟨\mathcal{H}, f_{j_0}⟩$ is $\mathcal{G}$-invariant by the invariance of the Thurston decomposition and Remark 6.5.

Let $π : \tilde{S} → S$ be the universal covering. Let $\tilde{A}$ be one of the lifts of $A$ to $\tilde{S}$. Given $φ ∈ ⟨G', f⟩$, we denote by $φ^\sharp$ its lift to $⟨G', g⟩^\sharp$ and by $φ^\sharp : \tilde{S} → \tilde{S}$ the identity lift of $φ^\sharp$ to $\tilde{S}$. The rotation number $ρ_μ(φ^\sharp)$ of $φ^\sharp$ in $\tilde{A}$ is well-defined (Definition 6.2) and does not depend on the choice of $\tilde{A}$ (Remark 6.1).

Lemma 7.2. Let $\mathcal{K}_0$ be a non-empty compact subset of $A ∩ \mathcal{K} ∩ \text{Fix}_c(f_{j_0})$. Suppose that there exist $1 \leq j \leq q$ and a measure $μ$ such that $\text{supp}(μ)$ is contained in $\mathcal{K}_0$ and $μ$ is $f_j$-invariant. Then $ρ_μ(\tilde{f}_j) = 0$.

Proof. We can assume that $μ$ is $f_j$-ergodic by the ergodic decomposition theorem. Since $π_1(S^2) → π_1(S)$ is injective by hypothesis, the primitive deck transformation $T^\#: \tilde{S} → \tilde{S}$ preserving $\tilde{A}$ induces by construction a deck transformation $T : \tilde{S} → \tilde{S}$. We can consider that $\tilde{A}$ is contained in $\tilde{S}$. Moreover, $T$ is associated to a non-null homotopic closed curve in $S$. Since $S$ is compact, $T$ is a hyperbolic isometry of the Poincaré disc. As a consequence $\lim_{|n| → ∞} d(z, T^n(z))/n$ is equal to $κ$ where $κ$ is the length of the geodesic in the free homotopy class determined by $T$ and $d$ is the hyperbolic metric in $\tilde{S}$. We denote $β = g_{j_0}^{-1}g_j$. Since $f_j = f_{j_0}β^\sharp$ and $\mathcal{K}_0 \subset \text{Fix}_c(f_{j_0})$, we deduce $(f_j)_{π^{-1}(\mathcal{K}_0)} ≡ β^\sharp_{π^{-1}(\mathcal{K}_0)}$ and then $ρ_μ(\tilde{f}_j) = ρ_μ(\tilde{β}^\sharp)$. Since $μ$ is $f_j$-ergodic, we have $\lim_{|n| → ∞} d(z_0, \tilde{β}^n(z_0))/n = ρ_μ(\tilde{β})κ$ for almost every point $z_0$ in $A ∩ π^{-1}(\mathcal{K}_0)$ where we consider $\tilde{A} ⊂ \tilde{S}$ and $\tilde{β}$ is the identity lift of $β$ to $\tilde{S}$. Notice that $β$ belongs to $\mathcal{C}^{(r-1)}(G)$ and $z_0$ belongs to $\text{Fix}(\mathcal{C}^{(r+1-1)}(\tilde{G}))$ where $\tilde{G}$ is the group consisting of the identity lifts of elements of $G$ to $\tilde{S}$. Equation (1) implies $\lim_{|n| → ∞} d(z_0, (\tilde{β}^n(z_0))/n^2 = 0$. We deduce that $ρ_μ(\tilde{f}_j) = ρ_μ(\tilde{g}_j) = ρ_μ(\tilde{β}) = 0$. □
Corollary 7.1. Let $A$ be an annulus connected component of $\mathcal{S} \setminus \mathcal{R}_e$. Then the set $A \cap K \cap \text{Fix}_c(\mathcal{H}, f_j)$ is non-empty for any $1 \leq j \leq q$.

Proof. There exists $1 \leq j_0 \leq q$ such that $A \cap K \cap \text{Fix}_c(\mathcal{H}, f_{j_0}) \neq \emptyset$ by construction. Since $A \cap K \cap \text{Fix}_c(\mathcal{H}, f_{j_0}) \neq \emptyset$ is $\mathcal{G}$-invariant, we consider a $f_{j_0}$-invariant measure $\mu$ with $K_0 := \text{supp}(\mu)$ contained in $A \cap K \cap \text{Fix}_c(\mathcal{H}, f_{j_0})$. We obtain $\rho_\mu(\tilde{f}_j) = 0$ by Lemma 7.2. We can assume $\text{Fix}(\tilde{f}_j) \cap (\pi^{-1}(K_0) \cap \hat{A}) = \emptyset$ since otherwise there is nothing to show. Let us apply Proposition 6.2 with $f = f_{j_0}$, $\tilde{f} = \tilde{f}_j$. The set $\text{Fix}_c(\mathcal{H}) \cap \pi(\text{Fix}(\tilde{f}_j))$ coincides with $\text{Fix}_c(\mathcal{H}, f_{j_0})$. Moreover, $\mathcal{H}$ is a normal subgroup of $\langle \mathcal{H}, f_{j_0} \rangle$ because $H$ is a normal subgroup of $G$. We deduce $A \cap K \cap \text{Fix}_c(\mathcal{H}, f_{j_0}) \neq \emptyset$ by Proposition 6.2. □

Corollary 7.2. Let $A$ be an annulus connected component of $\mathcal{S} \setminus \mathcal{R}_e$. Suppose that there exists a $\mathcal{G}$-invariant measure $\mu$ whose support is contained in $A \cap K \cap \text{Fix}_c(f_{j_0})$. Then $(\rho_\mu)_G \equiv 0$.

Proof. Let us remind the reader that $\mathcal{G} = (\mathcal{C}^u(G), f)^*$. Since the group $\hat{\mathcal{G}}$ of identity lifts of elements of $\mathcal{G}$ to $\hat{\mathcal{S}}$ is nilpotent, the map $\rho_\mu : \hat{\mathcal{G}} \to \mathbb{R}$ is a morphism of groups. We know that $\rho_\mu$ vanishes in $\{\hat{f}_1, \ldots, \hat{f}_q\}$ and $\mathcal{H}$ by Lemma 7.2 and because the elements of $\mathcal{H}$ are isotopic to $\text{Id}$ rel $K$ (see the proof of Corollary 7.1). Since $\langle \mathcal{H}, \hat{f}_1, \ldots, \hat{f}_q \rangle$ is a finite index subgroup of $\hat{\mathcal{G}}$ by construction, $\rho_\mu$ vanishes over $\hat{\mathcal{G}}$. □

7.5. Proof of $B_k$ implies $B_{k+1}$. Let us consider the possible cases for the configuration of global fixed points of subgroups of the framework. First, suppose $\sharp(X_M \cap K \cap \text{Fix}_c(\mathcal{H}, f_j)) = \infty$ for some $M \in \mathcal{C}$ (see Definition 7.2) and at least $e$ values of $j \in \{1, \ldots, q\}$. We can suppose that the property holds true for $j = 1, \ldots, e$ without lack of generality. We obtain $\text{Fix}_c(\mathcal{G}) \neq \emptyset$ by Proposition 7.4.

Now suppose that one of the connected components of $\mathcal{S} \setminus \mathcal{R}_e$ is an annulus $A$. We have that $A \cap K \cap \text{Fix}_c(\mathcal{H}, f_j)$ is a non-empty set for any $1 \leq j \leq q$ by Corollary 7.1. If $\sharp(X_A \cap K \cap \text{Fix}_c(\mathcal{H}, f_j)) = \infty$ for at least $e$ values of $j$ then there is nothing to prove by the argument in the above paragraph. Up to reordering the indexes we can suppose $\sharp(X_A \cap K \cap \text{Fix}_c(\mathcal{H}, f_j)) < \infty$ for $1 \leq j \leq q - (e - 1)$. Notice that $q - e + 1 > e$ since $q = -2e\chi(\mathcal{S})$. We define

$$K_1 = \bigcup_{j=1}^e (X_A \cap K \cap \text{Fix}_c(\mathcal{H}, f_j)), \quad K = (K \setminus X_A) \cup K_1.$$ 

The set $K$ is a good compact set for the group $\mathcal{G}$ and the subgroups $\langle \mathcal{H}, f_1 \rangle, \ldots, \langle \mathcal{H}, f_e \rangle$ by Remark 5.7. The group $\mathcal{G}$ induces a group $[\mathcal{G}]$ in $\text{MCG}(X_A, X_A \cap K)$ that does not necessarily consist of irreducible elements. Anyway we can add the curves of the Thurston
we can suppose $\# c$ implies $\text{Fix}(S \setminus R)$. The connected components in $G$ curves for $K$ with $\langle H, G \rangle$ we obtain $1 \leq \langle H, G \rangle$. By the first paragraph of the proof and up to reordering the index $e$ values of $j$ and some $M \in C$. Since $\langle H, f_{j0} \rangle$ is a compact non-empty $G$-invariant set and $G$ is nilpotent there exists a $G$-invariant measure $\mu$ such that supp($\mu$) $\subset A \cap K \cap \text{Fix}_c(\langle H, f_{j0} \rangle)$. We obtain $\rho_\mu(\phi) = 0$ for any $\phi \in G$ by Corollary 7.2. Proposition 7.3 implies $\text{Fix}_c(\langle H, f_{j0} \rangle) = 0$ for any $1 \leq j \leq e$. Since $A \cap K \cap \text{Fix}_c(\langle H, f_{j0} \rangle)$ is non-empty for any $1 \leq j \leq e$ by Corollary 7.1. Fix $1 \leq j_0 \leq e$. Since $A \cap K \cap \text{Fix}_c(\langle H, f_{j0} \rangle)$ is a Thurston-reducing set $\mathcal{R}$ of curves for $G$ and $\langle H, f_{j1} \rangle, \ldots, \langle H, f_{q'} \rangle$ rel $K$ (Remark 5.7). We replace $K$ with $K$.

By the first paragraph of the proof we can suppose $\#(X_M \cap K \cap \text{Fix}_c(\langle H, f_j \rangle)) < \infty$ for all $M \in C$ and $1 \leq j \leq q'$ where $q' = q + e\chi(\mathcal{S}) = -e\chi(\mathcal{S})$. We define

$$K_1 = \bigcup_{M \in C} \bigcup_{j=1}^{q'} (X_M \cap K \cap \text{Fix}_c(\langle H, f_j \rangle)), \quad K = (K \setminus \bigcup_{M \in C} X_M) \cup K_1.$$ 

The set $K$ is a good compact set for the group $G$ and the subgroups $\langle H, f_1 \rangle, \ldots, \langle H, f_{q'} \rangle$. By adding simple closed curves contained in $(\bigcup_{M \in C} X_M) \setminus K$ to $\mathcal{R}$, we can obtain a Thurston-reducing set $\mathcal{R}$ of curves for $G$ and $\langle H, f_1 \rangle, \ldots, \langle H, f_{q'} \rangle$ rel $K$ (Remark 5.7). We replace $K$ with $K$.

We can suppose that there is no annulus in $S \setminus \mathcal{R}$ for the new Thurston decomposition since otherwise we obtain $\text{Fix}_c(\langle H, f_j \rangle) \neq \emptyset$. Since $\# \mathcal{C} = \# \mathcal{C} - \chi(\mathcal{S})$ and $q' = -e\chi(\mathcal{S})$, we get $M \cap K \cap \text{Fix}_c(\langle H, f_j \rangle) \neq \emptyset$ for $e$ values of $j$ and some $M \in C$. Then $\text{Fix}_c(\langle H, f_j \rangle) \neq \emptyset$ by Proposition 7.2 completing the proof.

8. LOCALIZATION OF FIXED POINTS IN THE GENERAL CASE

Our next goal is proving Theorem 8.1. It will be obtained as a corollary of the following theorem:

**Theorem 8.1.** Let $G$ be a finitely generated nilpotent subgroup of Homeo$_0(S)$ where $S$ is a compact connected orientable surface (maybe with boundary) of negative Euler characteristic. Suppose $\phi|_{S \setminus \partial S} \in \text{Diff}^1_0(S \setminus \partial S)$ for any $\phi \in G$. Furthermore, assume that $\phi$ is isotopic to $\text{Id}$ rel $\text{Fix}(\mathcal{C}(k)(G)) \cup \partial S$ for all $k \geq 1$ and $\phi \in \mathcal{C}(k)(G)$. Then $\text{Fix}_c(\langle H, f_j \rangle) \neq \emptyset$.

In this section we introduce another fundamental result, providing localization of global fixed points. The localization process executed in
the proof of Theorem \ref{thm:annulus} has some simplifying characteristics: either there exists one connected component of \( S \setminus \mathcal{R}_e \) that is a topological annulus (and we find global fixed points in such a component) or the number of connected components of \( S \setminus \mathcal{R}_e \) is bounded by above by \(-\chi(S)\). In order to prove Theorem \ref{thm:framework} we cannot suppose anymore that there are finitely many connected components of \( S \setminus \mathcal{R}_e \) and hence we need to localize the global fixed points in some specific ones.

8.1. Framework for the proof of Theorem \ref{thm:framework}. First, let us specify the framework introduced in Definition \ref{def:framework} fits with our current target.

Let \( \pi : \tilde{S} \to S \) be the universal covering. Given \( f \in G \), we consider the identity lift \( \tilde{f} \) of \( f \) to \( \tilde{S} \). We denote by \( \tilde{G} \) the group consisting of the identity lifts of elements of \( G \). Suppose first \( \partial S \neq \emptyset \). Fix a boundary component \( \gamma \) of \( S \). Since \( G_{|\gamma} \) is nilpotent, we obtain that \( \rho_\gamma : \tilde{G} \to \mathbb{R} \) is a morphism of groups. Moreover, there exists a \( G \)-invariant measure \( \mu \) supported in \( \gamma \).

Assume that \( \rho_\gamma(\tilde{f}) = 0 \) for any \( \tilde{f} \in \tilde{G} \). We obtain
\[
\text{Fix}_c(f) \cap \gamma = \text{Fix}(f) \cap \gamma \neq \emptyset
\]
for any \( f \in G \). Since \( \text{Fix}(f) \cap \gamma \neq \emptyset \) and \( f \) acts by translations in the connected components of \( \gamma \setminus \text{Fix}(g) \), we deduce \( \sup(\mu) \subset \text{Fix}_c(f) \cap \gamma \) for any \( f \in \tilde{G} \). In particular we obtain \( \text{Fix}_c(G) \cap \gamma \neq \emptyset \). So we can suppose that there exists \( f_0, \gamma \in G \) such that \( \rho_\gamma(\tilde{f}_0, \gamma) \neq 0 \) for any boundary component \( \gamma \) of \( S \). Moreover, since \( \rho_\gamma \) is a morphism, it is easy to see that there exists \( f_0 \in G \) such that \( \rho_\gamma(\tilde{f}_0) \neq 0 \) for any boundary component \( \gamma' \) of \( S \). For the case \( \partial S = \emptyset \) we define \( f_0 = \text{Id} \).

Since the group \( G/G' \) is commutative, there exists a normal subgroup \( J \) of \( G \) such that \( G' \subset J \), \( \#(G/J) < \infty \) and \( J/G' \) is torsion free by the fundamental theorem of finitely generated abelian groups. We can suppose \( f_0 \in J \) up to replace it with a non-trivial iterate. We denote \( e = \text{rank}(G/G') \). Consider a sequence \( f_1, \ldots, f_q \in J \) where \( q = -8\chi(S)e \) and \( \langle [f_{i_1}], \ldots, [f_{i_e}] \rangle \subset G/G' \) has rank \( e \) (and thus \( \langle G', f_{i_1}, \ldots, f_{i_e} \rangle \) is a finite index normal subgroup of \( G \)) for any choice of \( 1 \leq i_1 < \ldots < i_e \leq q \). This property is preserved if we replace \( f_1, \ldots, f_q \) with \( f_1 f_0^{k_1}, \ldots, f_q f_0^{k_e} \) for some \( k \in \mathbb{N} \) big enough (cf. \cite{15} Lemma 7.1). In particular the conditions in the framework in Definition \ref{def:framework} are satisfied for \( S = S \), \( G = G_{|S \setminus \partial S} \), \( \mathcal{H} = G_{|S \setminus \partial S} \) and on top of that we can suppose \( \rho_\gamma(\tilde{f}_j) \neq 0 \) for all boundary component \( \gamma \) of \( S \) and \( 1 \leq j \leq q \).

We denote \( H_j = \langle G', f_j \rangle_{|S \setminus \partial S} \). Since \( L(\text{Fix}_c(f_j g), f_j g) = \chi(S) \) and \( \text{Fix}_c(f_j g) \cap \partial S = \emptyset \), it follows that \( \text{Fix}_c(f_j g) \) is a non-empty compact subset of \( S \setminus \partial S \) for any \( g \in G' \). Hence we obtain \( \text{Fix}_c(H_j) \neq \emptyset \) by
Theorem 7.1 (where \( K = \emptyset \), \( S^* = S \setminus \partial S \) and the covering \( S^* \to S \setminus \partial S \) is the identity map). There exists an excellent compact set \( K \) with respect to the group \( G \), the subgroups \( H_1, \ldots, H_q \) and the set \( Y = \cup_{j=1}^q \text{Fix}_c(H_j) \) by Remark 5.3. We also consider a Thurston-reducing set \( \mathcal{R} \) of curves rel \( K \) for the group \( G \) and the subgroups \( H_1, \ldots, H_q \) (Proposition 5.1). We obtain a Thurston decomposition of \( G \) for the normal subgroups \( H_1, \ldots, H_q \) relative to \( K \) by Remark 5.8.

Consider the set \( \mathcal{C} \) of connected components of \( S \setminus \mathcal{R} \). There are at most \( -\chi(S) \) such components of negative Euler characteristic, i.e. \( \# \mathcal{C}_- \leq -\chi(S) \) (cf. Definition 7.2). Moreover, the maximal number \( I \) of disjoint essential simple closed curves that are pairwise non-homotopical satisfies \( 2I + b = -3\chi(S) \) where \( b \) is the number of boundary components of \( S \). There are no three homotopical annuli, bounding connected components of \( S \setminus \mathcal{R} \) of negative Euler characteristic (or two homotopic annuli whose core curve is homotopic to a boundary component). We deduce that there are at most \( 2I + b \) annuli in \( \mathcal{C} \) bounding components of negative Euler characteristic, i.e. \( \#(\mathcal{C} \setminus \mathcal{C}_-) \leq -3\chi(S) \).

Next, we introduce the definition of the region \( C \) in which we localize global fixed points.

**Definition 8.1.** Consider the framework introduced in this section. We define \( \mathcal{R}_e \) (resp. \( \mathcal{A}_e \)) as the set of essential curves in \( \mathcal{R} \) (resp. annuli in \( \mathcal{A} \)). We denote by \( a \) the set of connected components of \( S \setminus \mathcal{R}_e \) that are annuli. Consider the closure \( N' \) of a connected component of \( S \setminus \cup_{A \in a} \overline{A} \); it satisfies \( \chi(N') < 0 \). Let \( A'_1, \ldots, A'_r \) be the open annuli in \( S \setminus \mathcal{R}_e \) bounding \( N' \). We denote by \( A_{N,l} \) the set of annuli \( A \) of \( \mathcal{A} \) such that \( A \cap A'_l \cap \partial N' \neq \emptyset \). We define \( \mathcal{A}_N = \cup_{l=1}^r A_{N,l} \) and \( N = N' \setminus \overline{\mathcal{A}_N} \). We denote \( C = N' \cup A'_1 \cup \ldots \cup A'_r \) and \( A_l = A'_l \setminus \mathcal{A}_e \cup A_{N,l} \).

8.2. **Property C.** Our goal is finding points of \( K \cap \text{Fix}_c(f_j) \) in \( C \) (cf. Definition 8.1).

First, we introduce a result that allows to localize global fixed points in the annuli of the Thurston decomposition. Fix \( 1 \leq j \leq q \) and denote \( f = f_j \) for simplicity.

**Lemma 8.1.** Suppose that \( K_l \subset K \cap A_l \) is the support of a \( f \)-ergodic measure \( \mu_l \) for some \( 1 \leq l \leq r \) such that \( \rho_{\mu_l}(\tilde{f}) = 0 \). Then \( A_l \cap K \cap \text{Fix}_c(f) \neq \emptyset \)

Notice that \( \rho_{\mu_l}(\tilde{f}) \) is well-defined by Definition 6.2 and Remark 6.1.

**Proof.** If \( K_l \cap \text{Fix}_c(f) \neq \emptyset \) there is nothing to prove. Otherwise, we define \( A = A_l, K_0 = K_l, H = G' \). Note that \( K_0 \cap \text{Fix}_c(f) \neq \emptyset \) implies
Fix(\tilde{f}) \cap (\pi^{-1}(K_0) \cap \tilde{A}) = \emptyset. Since \mathcal{K} is a good set, the hypotheses of Proposition 6.2 hold. Therefore, we deduce \( A_l \cap \mathcal{K} \cap \text{Fix}_c(f) \neq \emptyset \). □

Next, we introduce the setup to localize global fixed points in the case of negative Euler characteristic. We denote \( X_l = X_{A_l} \) (cf. Definition 7.2). There exists an isotopy \((f_t)_{t \in [0,1]} \in \text{Homeo}^+(S)\) rel \( K \setminus N \) such that \( f = f_1 \) and \( \theta := f_0 \) satisfies \( \theta|_N = Id \), \( \theta(A_N) = A_N \) and \( \theta(X_l) = X_l \) for any \( 1 \leq l \leq r \). Moreover we can suppose that \( \theta|_{X_l} \) is irreducible for any \( l \) such that \( \sharp(X_l \cap K) < \infty \).

Consider a connected component \( \tilde{N} \) of \( \pi^{-1}(N) \). We can define \( \tilde{\theta} \) as the lift of \( \theta \) to \( \tilde{S} \) such that \( \tilde{\theta}|_{\tilde{N}} = Id \). Since \( \chi(N) < 0 \), it is the identity lift of \( \theta \).

Definition 8.2. We denote by \( \mathcal{F}_f \) the subset of \( \{1, \ldots, r\} \) consisting of indexes \( l \) such that the following properties hold:

- \( \sharp(X_l \cap \mathcal{K}) < \infty \);
- the class induced by \( \theta \) in \( \text{MCG}(X_l) \) is non-trivial or the restriction of \( \theta \) to any connected component of \( A_N \) contained in \( A_l \) is a non-trivial Dehn twist.

We denote by \( \mathcal{E}_f \) the subset of \( \{1, \ldots, r\} \setminus \mathcal{F}_f \) consisting of indexes \( l \) such that \( A_l \cap \mathcal{K} \cap \text{Fix}_c(f) = \emptyset \).

Definition 8.3. We define \( K_l = \mathcal{K} \cap A_l \) if \( l \in \mathcal{F}_f \). Given \( l \in \mathcal{E}_f \), we choose \( K_l \) as the support of a \( f \)-invariant ergodic measure \( \mu_l \) with \( K_l \subset \mathcal{K} \cap A_l \).

Remark 8.1. Given \( l \in \mathcal{E}_f \), we have \( \rho_{\mu_l}(\tilde{f}) \neq 0 \) by Lemma 8.1. Moreover, it is obvious that \( K_l \cap \text{Fix}_c(f) = \emptyset \).

We introduce the main property of this section.

Definition 8.4 (Property \( \mathcal{C} \)). Consider the setting in Definitions 8.1 and 8.2. The property holds if

\[
N \cap \mathcal{K} \cap \text{Fix}_c(f_j) \neq \emptyset
\]

whenever \( \mathcal{F}_{f_j} \cup \mathcal{E}_{f_j} = \{1, \ldots, r\} \).

We will prove Theorem 8.1 in section 9. The goal of this section is showing that Theorem 7.1 implies \( \mathcal{C} \). Let us remind the reader that we proved \( \text{Fix}_c(\mathcal{H}, f_m) \neq \emptyset \) for any \( 1 \leq m \leq q \) in section 8.1.

Remark 8.2. Since \( \text{Fix}_c(\mathcal{H}, f_m) \neq \emptyset \) and \( \mathcal{K} \) is a good set, we deduce \( \mathcal{K} \cap \text{Fix}_c(f_m) \neq \emptyset \) for any \( 1 \leq m \leq q \). In particular Property \( \mathcal{C} \) trivially holds if \( r = 0 \), i.e. if all connected components of \( S \setminus \mathcal{R}_e \) have negative Euler characteristic.
Lemma 8.2. Suppose $\mathcal{F}_f \cup \mathcal{E}_f = \{1, \ldots, r\}$. Let $\mathcal{N}$ (resp. $\mathcal{N}'$) be the $\theta|_{\mathcal{S} \setminus \cup_{l=1}^{r} K_l}$-Nielsen class (resp. $\theta'|_{\mathcal{S} \setminus \cup_{l=1}^{r} O_l}$-Nielsen class) of $N$. Then $\theta^b$ commutes exactly with the covering maps of $S^b$ that preserve $N$. The set $\mathcal{N}'$ is a union of $\theta|_{\mathcal{S} \setminus \cup_{l=1}^{r} K_l}$-Nielsen classes,

$$N \subset \mathcal{N} \subset \mathcal{N}' \subset (N \cup A_N \cup \cup_{l \in \mathcal{E}_f} A_l) \cap \text{Fix}_c(\theta)$$

and $\mathcal{N} \cap A_l = \mathcal{N}' \cap A_l$ for any $l \in \mathcal{F}_f$. The $\theta|_{\mathcal{S} \setminus \cup_{l=1}^{r} K_l}$-Nielsen class of any point in $(\mathcal{N}' \setminus N) \cap A_l$ does not peripherally contain any of the ends of $A_l$ for any $l \in \mathcal{E}_f$. Moreover, the sum of the Lefschetz numbers of the $\theta|_{\mathcal{S} \setminus \cup_{l=1}^{r} K_l}$-Nielsen classes in $\mathcal{N}'$ is less or equal than $\chi(N)$.

Proof. The property $N \subset \mathcal{N} \subset \mathcal{N}'$ is obvious.

Let us prove a book-keeping property. Fix $l \in \mathcal{E}_f$. We claim that there is no path $\gamma : [0, 1] \to A_l \setminus K_l$ such that $\gamma(0) \in \partial A_l \cap \partial N$, $\gamma(1) \in \partial A_l$ and

$$\gamma(0) \not\simeq \gamma \simeq \theta \circ \gamma : [0, 1], 0, 1 \to A_l \setminus K_l, \partial A_l, \partial A_l$$

where $\gamma(0)$ is understood as a constant path and we are considering homotopies of maps $\alpha : ([0, 1], \{0, 1\}) \to (A_l \setminus K_l, \partial A_l)$ of pairs. Otherwise we obtain $\rho_{\mu_l}(\tilde{f}) \neq 0$, contradicting Remark 8.1. The homeomorphism $\theta'$ satisfies an analogous property replacing $K_l$ with $O_l$.

Fix $l \in \mathcal{F}_f$. We have an analogous book-keeping property. Denote $\mathcal{X}_l = X_l \cup A_{N,l}$. There is no path $\gamma : [0, 1] \to \mathcal{X}_l$ for some $1 \leq l \leq r$ such that $\gamma(0) \in \partial A_l \cap \partial N$, $\gamma(1) \in \partial \mathcal{X}_l \cup \text{Fix}(\theta)$ and

$$\gamma(0) \not\simeq \gamma \simeq \theta \circ \gamma : [0, 1], 0, 1 \to \mathcal{X}_l, \partial A_l, \partial \mathcal{X}_l \cup \text{Fix}(\theta),$$

see Lemmas 1.2, 2.2 of [19] for the case $\theta|_{X_l} \neq \text{Id}$. The case where $\theta|_{X_l} \equiv \text{Id}$ and $\theta$ restricted to any connected component of $A_{N,l}$ is a non-trivial Dehn twist is simpler (note that $A_l \cap K \neq \emptyset$).
The book-keeping properties imply that no preimage of $\partial N \cup \partial C$ (see Definition [S.1]) in $S^b$ outside of $\partial N$ is $\theta^r$-invariant. Given $z \in N$, we claim that the fixed points of $\theta^r$ whose projection in $S \setminus \bigcup_{l=1}^r K_l$ is $z$ are contained in $\tilde{N}$ since otherwise we get a violation of the book-keeping properties. Therefore $\theta^r$ commutes exactly with the covering maps of $S^b$ that preserve $\tilde{N}$. Moreover, we deduce that $N'$ and $N''$ are contained in $C$ and satisfy $N \cap A_l = N' \cap A_l$ for any $l \in \mathcal{F}_f$.

We claim that $N''$ is contained in a compact subset of $S \setminus \bigcup_{l=1}^r K_l$ for any sufficiently small perturbation $\theta'$ of $\theta$ as described above. It suffices to prove the claim for $N'' \cap A_l$ and $A_l \setminus K_l$ and any $1 \leq l \leq r$.

For the case $l \in \mathcal{F}_f$, the claim is a consequence of the book-keeping property. Given $l \in \mathcal{F}_f$, it is a consequence of $\text{Fix}_c(\theta) \cap K_l = \emptyset$ (Remark [S.1] and $N'' \subset \text{Fix}_c(\theta)$). As a consequence, we obtain that $N''$ is a union of $\theta_{[S \setminus \bigcup_{l=1}^r K_l]}$-Nielsen classes and in particular $N' \subset N''$. Fix $1 \leq l \leq r$.

Analogously, we get that for any point in $N'' \cap A_l$ whose $\theta_{[S \setminus \bigcup_{l=1}^r K_l]}$-Nielsen class peripherally contains an end of $\partial A_l$ its $\theta'_{[S \setminus \bigcup_{l=1}^r K_l]}$-Nielsen class also contains peripherally the same end.

Given $l \in \mathcal{F}_f$, it is clear that $A_l \cap N'' \subset \text{Fix}_c(\theta)$. Given $l \in \mathcal{F}_f$, the book-keeping property implies $N'' \cap A_l = A_{N,l} \cap \text{Fix}_c(\theta)$.

Consider a point $z$ in $N'' \cap A_l$ with $l \in \mathcal{F}_f$. If the $\theta_{[S \setminus \bigcup_{l=1}^r K_l]}$-Nielsen class $N_z$ of $z$ peripherally contains and end in $\partial A_l \setminus \partial N$ then we obtain a violation of the book-keeping property for $\theta'$. If $N_z$ peripherally contains and end in $\partial A_l \cap \partial N$ then there exists a path $\gamma: [0,1] \to A_l \setminus K_l$ such that $\gamma(0) \in \partial A_l \cap \partial N$, $\gamma(1) = z$ and $\theta \circ \gamma \simeq \alpha^m \gamma$ rel 0, 1 in $A_l \setminus K_l$ for some $m \in \mathbb{Z}$ where $\alpha$ is a loop with point base at $\gamma(0)$, contained in $\partial A_l \cap \partial N$ and turning once in counter clock wise sense. Since $z$ belongs to $\text{Fix}_c(\theta)$ we deduce $m = 0$. Thus $z$ belongs to $N'$.

Denote $C' = N' \cup A_N \cup \bigcup_{l \in \mathcal{F}_f} X_l \cup \bigcup_{l \in \mathcal{F}_f} A_l$. Consider the Thurston decomposition of $\theta'$ in $C'$ rel $\mathcal{O} := \bigcup_{l \in \mathcal{F}_f} (X_l \cap K_l) \cup \bigcup_{l \in \mathcal{F}_f} \mathcal{O}_l$. We denote by $\mathcal{R}'$ the set of reducing curves. Up to isotopy rel $\mathcal{O}$, we claim that a connected component $\gamma$ of $\partial (\bigcup_{l=1}^r A_l) \cap \partial N$ is contained in $\mathcal{R}'$. Otherwise, $\gamma$ is contained in $\overline{\mathcal{R}_l}$ for some $l \in \mathcal{F}_f$. Moreover, up to replace $\theta'$ with another representative in its isotopy class rel $\mathcal{O}$, we can suppose that $\theta'_{|M} \equiv \text{Id}$ where $M$ is the connected component of $C' \setminus \mathcal{R}'$ that contains $N$. We deduce that the $\tilde{\theta}'$ rotation number of every point in $\mathcal{O}_l$ is 0. This contradicts the construction of $\theta'$.

There exists a standard map $\tilde{\theta} \in \text{Homeo}(C')$ (see Jiang and Guo [19]) such that $\theta'_{|C'}$ is isotopic to $\tilde{\theta}$ rel $\mathcal{O}$ and $N$ is a $\tilde{\theta}_{|C' \setminus \mathcal{O}}$-Nielsen class. The Lefschetz index $L(N, \tilde{\theta})$ is less or equal than $\chi(N)$ (see Lemma 3.6 of [19]). We deduce that $L(N', \theta') = L(N, \tilde{\theta}) \leq \chi(N)$ and then that
the sum of Lefschetz indexes of the \( \theta|_{S \setminus \bigcup_{i=1}^{r} K_i} \)-Nielsen classes contained in \( \mathcal{N}' \) is equal to \( L(\mathcal{N}', \theta') \). \( \square \)

Next, we study the \( \theta \)-Nielsen classes in \( \mathcal{N}' \setminus \mathcal{N} \) and their induced \( f \)-Nielsen classes.

**Lemma 8.3.** Consider the setting in Lemma 8.2. Let \( \mathcal{N}_z \) the \( \theta|_{S \setminus \bigcup_{i=1}^{r} K_i} \)-Nielsen class of a point \( z \) in \( (\mathcal{N}' \setminus \mathcal{N}) \cap A_\ell \) for some \( \ell \in \mathcal{E}_f \). Suppose \( L(\mathcal{N}_z, \theta) \neq 0 \). Then \( \mathcal{N}_z \) is a compactly covered \( \theta|_{S \setminus \bigcup_{i=1}^{r} K_i} \)-Nielsen class (and \( \mathcal{N}_z \) is also a compactly covered \( \theta|_{S \setminus K_i} \)-Nielsen class). Moreover, the \( f|_{S \setminus \bigcup_{i=1}^{r} K_i} \)-Nielsen class determined by \( \mathcal{N}_z \) and the isotopy \( (f_t)_{t \in [0,1]} \) is non-empty and compactly covered.

**Proof.** Let \( S^\dagger \) be the universal covering of the connected component of \( S \setminus \bigcup_{i=1}^{r} K_i \) containing \( z \). Let \( z^\dagger \) be a lift of \( z \) in \( S^\dagger \). Consider the lift \( \theta^\dagger \) of \( \theta \) to \( S^\dagger \) such that \( \theta^\dagger(z^\dagger) = z^\dagger \). The \( \theta|_{S \setminus \bigcup_{i=1}^{r} K_i} \)-Nielsen class \( \mathcal{N}_z \) does not peripherally contain any of the ends of \( A_\ell \) by Lemma 8.2. Hence in order to prove that \( \text{Fix}(\theta^\dagger) \) is compact we can replace \( S^\dagger \) with the universal covering of the connected component of \( A_\ell \setminus K_\ell \) containing \( z \). If \( \text{Fix}(\theta^\dagger) \) is not compact then there exists a path \( \gamma : [0,1] \to A_\ell \setminus K_\ell \) in \( \pi_1(A_\ell \setminus K_\ell, z) \) such that \( [\theta \circ \gamma] = [\gamma] \neq 0 \). The next three paragraphs are intended to prove that such \( [\gamma] \) does not exist.

Let us show first that \( \gamma \) is non-homotopically trivial in \( A_\ell \). Otherwise consider the universal covering \( \pi : \tilde{A}_\ell \to A_\ell \) and a lift \( \tilde{z} \) of \( z \) in \( \tilde{A}_\ell \) of \( A_\ell \). Let \( \tilde{\gamma} \) be the lift of \( \gamma \) to \( \tilde{A}_\ell \) such that \( \tilde{\gamma}(0) = \tilde{z} \); it is a closed path. Let \( B \) be the set of bounded connected components of \( \tilde{\gamma}[0,1] \) containing points of \( \pi^{-1}(K_\ell) \). The set \( \pi^{-1}(K_\ell) \cap \bigcup_{B \in B} B \) is compact. Thus there exists a finite subset \( F \) of \( \pi^{-1}(K_\ell) \) such that any \( B \in B \) contains exactly a point of \( F \). Moreover we can suppose that the rotation number of every point in \( F \) is equal to \( \rho_{\mu_\ell}(\theta) \). The path \( \tilde{\gamma} \) is homotopically trivial in \( \tilde{A}_\ell \setminus \pi^{-1}(K_\ell) \) if and only if it is homotopically trivial in \( \tilde{A}_\ell \setminus F \). The necessary condition is obvious. For the sufficient condition notice that every bounded connected component of \( \tilde{A}_\ell \setminus \pi^{-1}(K_\ell) \) is homeomorphic to a disk. For any \( B \in B \) we can consider a closed disk \( D_B \) containing \( K_\ell \cap B \). The path \( \gamma \) is homotopically trivial in \( \tilde{A}_\ell \setminus \bigcup_{B \in B} D_B \) since this is a strong deformation retract of \( \tilde{A}_\ell \setminus F \). Thus \( \gamma \) is homotopically trivial in \( \tilde{A}_\ell \setminus \pi^{-1}(K_\ell) \) since \( \pi^{-1}(K_\ell) \cap \bigcup_{B \in B} B \) is contained in \( \bigcup_{B \in B} D_B \). The property \( \rho_{\mu_\ell}(\theta) \neq 0 \) implies that \( \tilde{\gamma} \) is null-homotopic in \( \tilde{A}_\ell \setminus \theta^{-k_0}(F) \) for some \( k_0 \in \mathbb{N} \). Hence \( \theta^{k_0}(\tilde{\gamma}) \) is null-homotopic in \( \tilde{A}_\ell \setminus F \). Since \( \theta^{k_0}(\tilde{\gamma}) \) is homotopic to \( \tilde{\gamma} \) in \( \tilde{A}_\ell \setminus \pi^{-1}(K_\ell) \) and hence in \( \tilde{A}_\ell \setminus F \), we deduce that \( \tilde{\gamma} \) is null-homotopic in \( \tilde{A}_\ell \setminus F \). Therefore, \( \tilde{\gamma} \) is null-homotopic in
We obtain that $\gamma$ is null-homotopic in $A_\ell \setminus K_\ell$. This contradicts the choice of $\gamma$.

Up to replace $\gamma$ with another representative in its homotopy class there exist an isotopy $(\theta_t)_{t \in [0,1]}$ rel $K_\ell$ and a connected closed $\theta_1$-invariant subsurface $W$ containing $\gamma$, contained in $A_\ell \setminus K_\ell$, with no curve of $\partial W$ bounding a disc in $A_\ell \setminus K_\ell$ and such that $\theta_0 = \theta$ and $\theta |_{\partial W}$ is periodic (Lemma 2.12 of [16]). The set $\partial W$ contains non-essential curves; otherwise $W$ is an annulus and the ergodicity of $\mu_\ell$ implies that $\mathcal{N}_z$ peripherally contains an end of $A_\ell$. The $(\theta_s)_{\{S \setminus K_\ell\}}$-Nielsen class $\mathcal{N}_{z,s}$ induced by $\mathcal{N}_z$ and the isotopy $(\theta_t)_{t \in [0,1]}$ still does not peripherally contain any ends for any $s \in [0,1]$. Moreover $\mathcal{N}_{z,s}$ never contains points in some fixed small neighborhood of $K_\ell$ for any $s \in [0,1]$ since $\mathcal{N}_z \subset \text{Fix}_c(\theta)$, $K_\ell$ is compact and $K_\ell \cap \text{Fix}_c(\theta) = \emptyset$. Since $L(\mathcal{N}_z, \theta) \neq 0$, it follows that $L(\mathcal{N}_{z,s}, \theta_s) \neq 0$ and in particular $\mathcal{N}_{z,s} \neq \emptyset$ for any $s \in [0,1]$. Fix $z_1 \in \mathcal{N}_{z,1}$.

The set $\partial W$ contains at least an essential curve since otherwise $\gamma$ is null-homotopic in $A_\ell$. Hence $\partial W$ contains two essential curves. Consider the annulus $V$ obtained by adding to $W$ the interior of the closed disks bounded by the non-essential curves of $\partial W$. The construction implies that $K_\ell \cap V$ is a non-empty $\theta_1$-invariant set. Since $\mu_\ell$ is $\theta_1$-ergodic and $V$ is $\theta_1$-invariant, we deduce $K_\ell \subset V$. The $(\theta_t)_{\{S \setminus K_\ell\}}$-Nielsen class $\mathcal{N}_{z,1}$ does not peripherally contain the ends of $A_\ell$ and thus $\mathcal{N}_{z,1}$ is also contained in the interior $V$. Moreover $\mathcal{N}_{z,1}$ is contained in the interior of $W$, otherwise there exists a connected component $M$ of $V \setminus W$ such that $\overline{M} \cap \mathcal{N}_{z,1} \neq \emptyset$ and since $\mathcal{N}_{z,1} \subset \text{Fix}_c(\theta_1)$, we deduce that the $\theta$-translation number of the points of $M \cap K_\ell$ is equal 0. This contradicts that the rotation number of points of $K_\ell$ is non-vanishing a.e. Consider the surface $T$ obtained from $W$ by contracting the curves in $\partial W$. We obtain that $(\theta_1)_{|T}$ is a finite order homeomorphism of a sphere fixing at least three points, namely both essential curves of $\partial W$ and $z_1$. Hence we obtain $(\theta_1)_{|W} \equiv \text{Id}$. This leads to a contradiction since it implies that $\mathcal{N}_{z,1}$ peripherally contains both ends of $A_\ell$.

We proved that $\text{Fix}(\theta^t)$ is compact and that $\mathcal{N}_z$ is a compactly covered $\theta_{\{S \setminus \bigcup_{i=1}^t K_\ell\}}$ and $\theta_{\{S \setminus K_\ell\}}$-Nielsen class. Thus the connected component of $S \setminus \bigcup_{i=1}^t K_\ell$ containing $z$ is not an annulus. Let $f_t : S \to S$ be an isotopy rel $\bigcup_{i=1}^t K_\ell$ with $f_0 = \theta$ and $f_1 = f$. Let $(f_t^\dagger)_{t \in [0,1]}$ be the lift of $(f_t)_{t \in [0,1]}$ such that $f_0^\dagger = \theta^\dagger$. We define $f^\dagger = f_1^\dagger$. The $\theta_{\{S \setminus \bigcup_{i=1}^t K_\ell\}}$-Nielsen class $\mathcal{N}_z$ is contained in $\text{Fix}_c(\theta)$ and then it is compact in $S \setminus \bigcup_{i=1}^t K_\ell$. Moreover the union of the $(f_t)_{\{S \setminus \bigcup_{i=1}^t K_\ell\}}$-Nielsen classes for $t \in [0,1]$ determined by $\mathcal{N}_z$ and $(f_t)_{t \in [0,1]}$ is contained in $\bigcup_{t \in [0,1]} \text{Fix}_c(f_t)$ and thus it is a compact subset of $S \setminus \bigcup_{i=1}^t K_\ell$. Since $f_1^\dagger$ does not commute with
non-trivial covering maps of $S^1$, the set $\bigcup_{t \in [0,1]} \text{Fix}(f_t^1)$ is compact. As a consequence we deduce
\[ L(\text{Fix}(f^1), f^1) = L(\text{Fix}(\theta^1), \theta^1) = L(N_z, \theta) \neq 0 \]
and $\text{Fix}(f^1)$ is a non-empty compact set. \hfill \Box

Now, we reap the rewards of the work in Lemmas 8.2 and 8.3.

**Lemma 8.4.** Consider the setting in Lemma 8.3. Then Property C holds.

**Proof.** Let $\mathcal{N}$ be the $f|_{S \setminus \cup_{l=1}^{r} K_l}$-Nielsen class induced by $N_z$ and the isotopy $(f_t)_{t \in [0,1]}$. We have that $\mathcal{N}$ is non-empty and compactly covered by Lemma 8.3. We apply Proposition 3.2 with $z$ part in Lemma 8.3. Then Property \[ holds.

Consider the setting in Lemma 8.3. Then Property C holds.

**Proof.** Let $\mathcal{N}$ be the $f|_{S \setminus \cup_{l=1}^{r} K_l}$-Nielsen class induced by $N_z$ and the isotopy $(f_t)_{t \in [0,1]}$. We have that $\mathcal{N}$ is non-empty and compactly covered by Lemma 8.3. We apply Proposition 3.2 with $H = G'$ and $K_0 = \cup_{l=1}^{r} K_l$ to obtain $\mathcal{N} \cap \text{Fix}_c(G') \neq \emptyset$. Since $N_z \subset \text{Fix}_c(\theta)$, we deduce $\mathcal{N} \cap \text{Fix}_c(G', f) \neq \emptyset$.

Fix $z_0 \in \mathcal{N} \cap \text{Fix}_c(G', f)$. Since $\mathcal{K}$ is good, there exists a point $z_1 \in \mathcal{K} \cap \text{Fix}_c(f)$ such that $z_0$ and $z_1$ are in the same $f$-Nielsen class $\mathcal{N}$ rel $\mathcal{K}$. Since $\mathcal{N} \subset \text{Fix}_c(f)$ and $\mathcal{K} \cap \text{Fix}_c(f) = \emptyset$, we deduce that $\mathcal{N}$ is contained in the $f|_{S \setminus \cup_{l=1}^{r} K_l}$-Nielsen class of $z_0$.

Let us show that $z_1 \in \mathcal{N}$. Otherwise $z_1$ belongs to the $\theta|_{S \setminus \cup_{l=1}^{r} K_l}$-Nielsen class $N_z$ since the homeomorphisms $f$ and $\theta$ are isotopic rel $\mathcal{K} \setminus N$ by construction. Therefore, $z_1$ belongs to $\mathcal{K} \cap N_z \cap \text{Fix}_c(\theta)$. Since $N_z$ is a compactly covered $\theta|_{S \setminus \cup_{l=1}^{r} K_l}$-Nielsen class by Lemma 8.3, we deduce $z_1 \in A_{\ell}$. This shows that $A_{\ell} \cap \mathcal{K} \cap \text{Fix}_c(f) \neq \emptyset$, contradicting the definition of $E_f$. \hfill \Box

8.3. **Proof of C.** We denote $f = f_j$. We can suppose $r \geq 1$ by Remark 8.2. We choose $K_l$ for $1 \leq l \leq r$ as in Definition 8.3. Let us apply Lemma 8.2. We can suppose $L(N_z, \theta) = 0$ for any $\theta|_{S \setminus \cup_{l=1}^{r} K_l}$-Nielsen class contained in $\mathcal{N}' \setminus \mathcal{N}$ by Lemma 8.4. Thus, we get $L(\mathcal{N}, \theta) = L(\mathcal{N}', \theta') \leq \chi(N) < 0$.

Let $S^0$ be the universal covering of the connected component of $S \setminus \cup_{l=1}^{r} K_l$ containing $N$. Consider the group $J$ of covering transformations of $S^0$ preserving $\hat{N}$. Consider the covering spaces $S^\sharp$, $\hat{S}^\sharp$ associated to the spaces $S \setminus (\cup_{l=1}^{r} K_l \cup \partial S)$, $S \setminus \cup_{l=1}^{r} K_l$ respectively and the group $J$. Let $\pi^\sharp : \hat{S}^\sharp \to S \setminus \cup_{l=1}^{r} K_l$ and $\pi^\sharp : S^0 \to \hat{S}^\sharp$ be the covering maps. In particular we have $\pi_1(S^0) \simeq J$, there is a natural embedding $N \hookrightarrow \hat{S}^\sharp$ and $S^\sharp$ is homeomorphic to $N \setminus \partial N$. We define $\theta^\sharp$ as the lift of $\theta$ to $\hat{S}^\sharp$ such that $(\theta^\sharp)|_N \equiv Id$. Hence $\theta^\sharp$ is isotopic to the identity in $S^\sharp$ since $\chi(N) < 0$.

Consider $g \in \langle G', f \rangle$. There exists an isotopy $g_t : S \to S$ rel $\mathcal{K} \setminus N$ such that $(g_0)|_N \equiv Id$ and $g_1 = g$. Indeed if $g$ is of the form $f^s h$ where
If $h \in G'$ we can define $g_0 = \theta^h$. We claim that $\theta^h$ is not isotopic to \Id rel $K \setminus N$ for any $s \in \Z^+$. Otherwise, such a property contradicts $\rho_{\mu_l}(\bar{f}) \neq 0$ for $l \in \mathcal{E}_f$ if $\mathcal{E}_f \neq \emptyset$. We also obtain a contradiction for the case $\mathcal{F}_f \neq \emptyset$. More precisely, consider the surface $S'$ obtained as the interior of $N \cup A_N \cup \bigcup_{t \in \mathcal{F}_f} X_t$. The homeomorphism $\theta|_{S'}$ would induce a finite order element $[\theta|_{S'}]$ of the mapping class group of $S'$ such that $\theta|_{N \setminus \partial S} \equiv \Id$. Since $\chi(N) < 0$, we deduce $[\theta|_{S'}] = 1$, contradicting the second bullet point in Definition 8.2. As a consequence of the claim, $s$ is uniquely defined.

We define $g_0^s$ as the lift of $g_0$ to $\overline{S}$ such that $(g_0^s)|_N = Id$. We define $g^s = g_0^s$ where $(g_0^s)$ is the lift of the isotopy $(g_t) \to S$. Since $\chi(N) < 0$, it follows that $g^s$ does not depend on the isotopy. Clearly $g^s$ is isotopic to the identity in $\overline{S}$. We define $\text{Fix}_c(g^s)$ as the Nielsen identity class of $g^s$. If $g^s$ is the lift of $g$ to $S^3$ determined by $(g_t)$ such that $(g_0^s)|_N = Id$ then $\text{Fix}_c(g^s)$ is the set $\pi^2(\text{Fix}(g^s))$. The group $\langle G', f \rangle^s$ consisting of the lifts of elements of $\langle G', f \rangle$ to the connected component of $S \setminus \bigcup_{t=1}^r K_t$ containing $N$. In particular $\langle G', f \rangle^s$ is nilpotent.

We denote $N^s = \text{Fix}_c(\theta^s)$. The projection of $N^s$ in $S$ is $N$. Since $N \subset \text{Fix}_c(\theta)$ and $A_l \cap K \cap \text{Fix}_c(\theta) = \emptyset$ for any $l \in \mathcal{E}_f$, we deduce $N \cap K \cap \bigcup_{t \in \mathcal{F}_f} A_l = \emptyset$. We also get $N \cap \bigcup_{t \in \mathcal{F}_f} K_t = \emptyset$ by the book-keeping property (see third paragraph of Lemma 8.2). Thus $N$ is a compact subset of $S \setminus (K \cap \bigcup_{t=1}^r A_l)$. Moreover, it is a $\theta|_{S \setminus \bigcup_{t=1}^r K_t}$-Nielsen class that does not peripherally contain rel $\bigcup_{t=1}^r K_t$ any of the exterior ends $\partial A_l \setminus \partial N$ of the annuli $A_1, \ldots, A_r$ (Lemma 8.2). Hence we get $N \cap K \subset N$ and $N \cap \partial S = \partial N$. Since $\theta^s$ commutes exactly with the covering transformations in $J$ by Lemma 8.2, it follows that $N^s$ is compact in $S^3 \cup (\partial N \cap (\pi^2)^{-1}(\partial S))$. Moreover, we obtain $L(N^s, \theta^s) = L(N, \theta) < 0$. Consider the isotopy $f_t : S \to S$ rel $\bigcup_{t=1}^r K_t$ such that $f_0 = \theta$, $f_1 = f$. Denote by $N_t$ the $f_t$-Nielsen class of $\pi^2(\text{Fix}_c(f_t^s))$ rel $\bigcup_{t=1}^r K_t$. Notice that $\pi^2(\text{Fix}_c(f_t^s)) \subset N_t$ for any $t \in [0, 1]$. Analogously as above, we obtain that $\bigcup_{t \in [0, 1]} \text{Fix}_c(f_t^s)$ is a compact subset of $S^3 \cup (\partial N \cap (\pi^2)^{-1}(\partial S))$ and $\bigcup_{t \in [0, 1]} N_t$ is a compact subset of $[(S \setminus \partial S) \cup (\partial S \cap \partial N)] \setminus \bigcup_{t=1}^r K_t$. We denote $f^s = f_1^s$; we obtain

$$L(\text{Fix}_c(f^s), f^s) = L(N^s, \theta^s) = L(N, \theta) < 0.$$  

Consider $g = fh$ where $h \in G'$. Since $h$ is isotopic to $\Id$ rel $\bigcup_{t=1}^r K_t$, we consider the identity lift $h^s$ of $h$ defined in $\overline{S}$. The previous construction implies $g^s = f^s h^s$ and $L(\text{Fix}_c(g^s), g^s) = L(\text{Fix}_c(f^s), f^s)$. In particular $\text{Fix}_c(g^s)$ is a non-empty compact subset of $S^3 \cup (\partial N \cap (\pi^2)^{-1}(\partial S))$.  

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Notice that since \( \text{Fix}_c(f) \cap \partial S = \text{Fix}_c(g) \cap \partial S = \emptyset \), we obtain that \( \text{Fix}_c(g^4) \) is a subset of \( S^2 \).

We obtain that \( \text{Fix}_c(G', f)^4 \) is a non-empty compact subset of \( S^2 \) by Theorem 7.1. Thus, we get \( \mathcal{N}_1 \cap \text{Fix}_c(G', f) \neq \emptyset \). \( \text{Fix}_z \in \mathcal{N}_1 \cap \text{Fix}_c(f) \). Since \( K \) is good, there exists a point \( z_1 \in K \cap \mathcal{N}_1 \cap \text{Fix}_c(f) \) such that \( z_0 \) and \( z_1 \) are in the same \( f \)-Nielsen class rel \( K \).

We claim that \( z_1 \in N \). Otherwise, \( z_1 \) belongs to \( \mathcal{N} \) since \( f \) and \( \theta \) are isotopic rel \( K \setminus N \). Since \( \mathcal{N} \cap K \subset N \), we obtain a contradiction.

9. Proof of Theorem [B]

9.1. Proof of Theorem 8.1. Consider the framework introduced in section 8.1. Let us consider the connected components of \( S \setminus R_e \) (cf. Definition 7.2). We have \( \sharp \mathcal{C}_- \leq -\chi(S) \) and \( \sharp \mathcal{C} \leq -4\chi(S) \) by the discussion in section 8.1. Property \( \mathcal{C} \) guarantees that \( \text{Fix}_c(f_j) \cap K \) intersects one of these \(-4\chi(S)\) components for any \( 1 \leq j \leq q \) where \( q = -8\chi(S)e \).

Suppose there exist \( M \in \mathcal{C} \) and \( e := \text{rank}(G/G') \) indexes \( 1 \leq j \leq q \) with \( \sharp(X_M \cap K \cap \text{Fix}_c(\mathcal{H}, f_j)) = \infty \). Hence, we get \( \text{Fix}_c(G) \neq \emptyset \) by Proposition 7.1. Thus, if there are at least \(-4\chi(S)e\) indexes \( 1 \leq j \leq q \) such that there exists \( M_j \in \mathcal{C} \) with \( \sharp(X_{M_j} \cap K \cap \text{Fix}_c(\mathcal{H}, f_j)) = \infty \) then we are done by the pigeonhole principle. Therefore, and up to replace \( q \) with \(-4\chi(S)e\) and reordering the indexes, we can suppose that \( \sharp(X_M \cap K \cap \text{Fix}_c(\mathcal{H}, f_j)) < \infty \) for all \( M \in \mathcal{C} \) and \( 1 \leq j \leq q \). Denote by \( \mathcal{C}' \) the subset of \( \mathcal{C} \setminus \mathcal{C}_- \) whose elements \( M \) satisfy either \( \chi(X_M) < 0 \) or \( M \cap K \cup \bigcup_{j=1}^q \text{Fix}_c(f_j) \neq \emptyset \). We define

\[
\mathcal{K}_1 = \bigcup_{M \in \mathcal{C}} \bigcup_{j=1}^q (X_M \cap K \cap \text{Fix}_c(\mathcal{H}, f_j)), \quad \mathcal{K} = (\mathcal{K} \setminus \bigcup_{M \in \mathcal{C}_-} X_M) \cup \mathcal{K}_1.
\]

The set \( \mathcal{K} \) is a good compact set for \( G \) and \( \langle \mathcal{H}, f_1 \rangle, \ldots, \langle \mathcal{H}, f_q \rangle \). Our choice of \( \mathcal{K} \) ensures that either \( \chi(M') < 0 \) or \( \sharp(M' \cap K) \geq 1 + \chi(M') \) for any connected component \( M' \) of \( S \setminus R \). We replace \( \mathcal{K} \) with \( \mathcal{K} \). We can obtain a Thurston-reducing set of curves rel \( \mathcal{K} \) by adding suitable simple closed curves of \( (\bigcup_{M \in \mathcal{C}_-} X_M) \setminus K \) to \( R \) (Remark 7.7). Moreover, since the components in \( \mathcal{C} \) for the new set of reducing curves are contained in the old components, we still have \( \sharp(X_M \cap K \cap \text{Fix}_c(\mathcal{H}, f_j)) < \infty \) for all \( M \in \mathcal{C} \) and \( 1 \leq j \leq q \).

Assume that there are \( e \) indexes \( 1 \leq j \leq q \) and \( M \in \mathcal{C}_- \) such that \( M \cap K \cap \text{Fix}_c(\mathcal{H}, f_j) \neq \emptyset \). Thus, we get \( \text{Fix}_c(G) \neq \emptyset \) by Proposition 7.2. So, by applying the pigeonhole principle, we can suppose that \( M \cap K \cap \text{Fix}_c(\mathcal{H}, f_j) = \emptyset \) for any \( 1 \leq j \leq q \) and \( M \in \mathcal{C}_- \) up to reorganizing the indexes and replacing \( q \) with \(-3\chi(S)e \). The set \( \mathcal{K} \setminus \bigcup_{M \in \mathcal{C}_-} M \) is a good compact set for \( G \) and \( \langle \mathcal{H}, f_1 \rangle, \ldots, \langle \mathcal{H}, f_q \rangle \). Analogously as above, up to replace \( \mathcal{K} \) with \( \mathcal{K} \setminus \bigcup_{M \in \mathcal{C}_-} M \) and updating the set \( R \).
by adding new curves, we obtain a Thurston-reducing set for $G$ and $\langle \mathcal{H}, f_1 \rangle, \ldots, \langle \mathcal{H}, f_q \rangle$ rel $\mathcal{K}$. Moreover since the updated components of $\mathcal{C}$ (resp. $\mathcal{C}_-$) are contained in the old ones, we have $\mathcal{K} \cap \bigcup_{M \in \mathcal{C}} M = \emptyset$ and $\sharp (X_M \cap \mathcal{K} \cap \text{Fix}_c (\mathcal{H}, f_j)) < \infty$ for all $1 \leq j \leq q$ and $M \in \mathcal{C}$.

We claim that there exist $M \in \mathcal{C}' \setminus \mathcal{C}_-$ and $e$ values of $1 \leq j \leq q$ such that the class $[f_j]_M$ of $f_j$ in $\text{MCG}(X_M, X_M \cap \mathcal{K})$ is trivial and $M \cap \mathcal{K} \cap \text{Fix}_c (\mathcal{H}, f_j) \neq \emptyset$. Otherwise, $\sharp (\mathcal{C}' \setminus \mathcal{C}_-) \leq -3\chi(S)$ and the pigeonhole principle imply that there exists $1 \leq j_0 \leq q$ such that either

- $M \notin \mathcal{C}'$
- $[f_{j_0}]_M$ is non-trivial or
- $M \cap \mathcal{K} \cap \text{Fix}_c (\mathcal{H}, f_{j_0}) = \emptyset$

for any $M \in \mathcal{C} \setminus \mathcal{C}_-$. Given a connected component $N'$ of $S \setminus \bigcup_{A \in a_0 A}$ (see Definition 8.1), let $A'_1, \ldots, A'_r$ be the open annuli in $S \setminus \mathcal{R}_e$ bounding $N'$. As a consequence of the discussion, we obtain that any $A'_l \in \mathcal{C}'$ satisfies $l \in \mathcal{F}_{f_{j_0}}$. Moreover, $A'_l \notin \mathcal{C}'$ implies $A'_l \cap \mathcal{K} \cap \text{Fix}_c (f_{j_0}) = \emptyset$ by the choice of $\mathcal{C}'$. We deduce $\mathcal{F}_{f_{j_0}} \cup \mathcal{E}_{f_{j_0}} = \{1, \ldots, r\}$ and hence

$$\mathcal{K} \cap N' \cap \text{Fix}_c (f_{j_0}) \neq \emptyset$$

(see Definition 8.2) by Property C, contradicting $\mathcal{K} \cap \bigcup_{M \in \mathcal{C}_-} M = \emptyset$. Finally, we obtain $\text{Fix}_c (G) \neq \emptyset$ by Proposition 7.4.

9.2. End of the proof of Theorem B. As a consequence of Theorems 4.1 and 8.1 Theorem B holds if $S$ is orientable and $G$ is finitely generated.

Assume that $S$ is orientable (but $G$ is not necessarily finitely generated). Since $S$ is compact, Theorem B is a consequence of the finite intersection property for compact subsets of $S$.

Finally, suppose that $S$ is non-orientable. Let us consider the orientable double cover $\hat{S}$ of $S$. Since any $f \in G$ isotopic to $\text{Id}$ has a unique lift $\hat{f}$ isotopic to $\text{Id}$, the group $\hat{G} := \{ \hat{f} : f \in G \}$ is a lift of $G$ contained in $\text{Diff}^1_0 (\hat{S})$. The set $\text{Fix}_c (\hat{G})$ is non-empty by Theorem B for the case where $\hat{S}$ is orientable and it projects to $\text{Fix}_c (G)$.

10. Proof of Theorem A

We show the result in two steps. First, we reduce the proof to a simpler setting in section 10.1. Then we complete it in section 10.2.

10.1. Reduction step. Analogously as in subsection 9.2 we can suppose that $S$ is orientable by replacing $S$ and $G$ with its orientable double cover and the group of orientation-preserving lifts of elements of $G$ if necessary. Moreover, we can suppose that $G \subset \text{Diff}^1_+ (S)$ by replacing $G$ with its normal subgroup $G \cap \text{Diff}^1_+ (S)$ of index at most 2.
Let $\mathcal{R}$ be a minimal Thurston-reducing set of curves for the group $G$. Let $[G]$ be the group induced by $G$ in $\text{MCG}(S)$. We consider the subgroup $[G]^0$ of elements of $[G]$ that fix every reducing curve and every connected component of $S \setminus \mathcal{R}$ modulo isotopy. Consider the normal subgroup $G^0$ of $G$ consisting of the elements whose classes belong to $[G]^0$. The group $G^0$ is a finite index normal subgroup of $G$ containing $G_0$. Thus, up to replace $G$ with $G^0$, we can suppose $G = G^0$.

Let $M$ be a connected component of $S \setminus \mathcal{R}$. Given $f \in G$, there exists $\theta_f \in \text{Homeo}_+(S)$ such that $f$ is isotopic to $\theta_f$ and $\theta_f(M) = M$. Consider the group $[G]_M$ consisting of the classes of $[\theta_f]_M$ of $\theta_f|_M$ in $\text{MCG}(M)$ for $f \in G$. The class $[\theta_f]_M$ does not depend on the choice of $\theta_f$ since $\chi(M) < 0$. The group $[G]_M$ is clearly nilpotent. Moreover, it is finitely generated since such a property holds for solvable subgroups of a finitely punctured compact surface [4]. If all the elements in the center of $[G]_M$ are of finite order then $[G]_M$ is a finite group [24, p. 3, Lemma 0.1]. Hence either $[G]_M$ is finite or the center of $[G]_M$ contains a pseudo-Anosov class.

Suppose that we are in the latter case. The group $[G]_M$ has a model $\Theta$. More precisely, there exists a subgroup $\Theta$ of $\text{Homeo}_+(M)$ such that $\theta$ is irreducible for any $\theta \in \Theta$ and the natural map $\theta \to [\theta]$ from $\Theta$ to $\text{MCG}(M)$ is an isomorphism $\tau : \Theta \to [G]_M$ [24, Lemma 2.1]. Moreover, given $f \in G$, we can assume that $\theta_f \in \Theta$ up to replace $\theta_f$ with an isotopic homeomorphism. Let $\beta$ be a pseudo-Anosov element in $Z(\Theta)$ and $g \in G$ such that $\theta_g = \beta$. There exists $m \in \mathbb{N}$ such that $\text{Fix}(\beta^m)$ contains interior points of $M$. Consider the subgroup $\Theta^0$ of $\Theta$ of elements fixing all the points in the finite set $\text{Fix}(\beta^m)$. The group $\Theta^0$ is a finite index normal subgroup of $\Theta$ and it has a global fixed point $z$ in the interior of $M$. Consider a lift $\tilde{z}$ of $z$ to $\tilde{S}$. Consider the subgroup $G^0$ of elements $f \in G$ such that $\tau^{-1}([\theta_f]_M) \in \Theta^0$. Since $G_0 \subset \{f \in G : \tau^{-1}([\theta_f]_M) = 1\}$, we deduce that $G_0$ is a subgroup of $G^0$. Given $f \in G^0$, consider an isotopy $(f_t)_{t \in [0,1]}$ such that $f_0 = \theta_f$ and $f_1 = f$. We define $\tilde{f}$ as $\tilde{f}_1$ where $(\tilde{f}_t)$ is the lift of $(f_t)$ such that $\tilde{f}_0(\tilde{z}) = \tilde{z}$. The definition of $\tilde{f}$ does not depend on the isotopy since $\chi(S) < 0$. Since $\chi(M) < 0$, it is clear that $\tilde{f}$ is the identity lift of $f$ if $f$ is isotopic to the identity. The group $\tilde{G}^0 := \{\tilde{f} : f \in G^0\}$ is isomorphic to $G^0$ and thus nilpotent. Since $\beta^m$ is pseudo-Anosov, it does not commute with deck transformations of $M$ and the $\beta^m$-Nielsen class of $z$ satisfies $L(z, \beta^m) \neq 0$ and does not contain peripherally any end of $M$. As a consequence $\tilde{g}^m$ does not commute with deck transformations of $S$ and $\text{Fix}(\tilde{g}^m)$ is a non-empty compact set. We obtain $\text{Fix}(\tilde{G}^0) \neq \emptyset$. 
by Corollary 3.1. The group $G$ has a finite orbit contained in the projection of $\text{Fix}(\tilde{G}^0)$, that is contained in $\text{Fix}_c(G_0)$.

We can suppose from now on that $[G]_M$ is a finite group for any connected component $M$ of $S \setminus \mathcal{R}$. Consider the group $G^0$ of elements of $G$ such that their images in every $[G]_M$ are trivial. The group $G^0$ is a finite index normal subgroup of $G$ containing $G_0$. We can replace $G$ with $G^0$. All the elements in $G$ are isotopic to a product of $k$ Dehn twists along disjoint pairwise non-isotopic annuli, a so called Dehn twist subgroup 15. We assume that $k$ is minimal with this property. We have a natural monomorphism $\xi : \text{MCG}(G) \rightarrow \mathbb{Z}^k$. Hence, the derived group $G'$ is contained in $G_0$. If $k = 0$ the group $G$ is contained in $\text{Diff}^1(S)$. The result is a consequence of Theorem 13. We suppose $k > 0$.

10.2. Dehn twist groups. Consider a connected component $N$ of $S \setminus \mathcal{A}$ (Remark 5.5). We choose a connected component $\tilde{N}$ of $\pi^{-1}(N)$ in $\tilde{S}$. Indeed $\tilde{N}$ is a universal covering of $N$. Given $f \in G$ there exists a isotopy $f_t$ such that $(f_0)|_N \equiv Id$ and $f_1 = f$. Consider the group $J$ of covering transformations of $\tilde{S}$ preserving $\tilde{N}$. Consider the covering space $S^\dag$ associated to the space $S$ and the group $J$. Let $\pi^\dag : S^\dag \rightarrow S$ and $\pi^b : \tilde{S} \rightarrow S^\dag$ be the covering maps. In particular we have $\pi_1(S^\dag) \simeq J$, there is a natural embedding $N \hookrightarrow S^\dag$ and $S^\dag$ is homeomorphic to $N$. We define $f^\dag_0$ as the lift of $f_0$ to $S^\dag$ such that $(f^\dag_0)|_N \equiv Id$. Hence $f^\dag_0$ is isotopic to the identity in $S^\dag$. We define $f^\dag$ as $f^\dag_1$ where $(f^\dag_1)$ is the lift of $(f_1)$. Let $\tilde{f}_0$ be the lift of $f_0$ to $\tilde{S}$ such that $(\tilde{f}_0)|_{\tilde{N}} \equiv Id$. We define $\tilde{f}$ analogously as $f^\dag$. The lifts $\tilde{f}$ and $f^\dag$ are well-defined since $\chi(N) < 0$ and satisfy $\pi^b(\text{Fix}(\tilde{f})) = \text{Fix}_c(f^\dag)$ for any $f \in G$. We obtain lifts $G^\dag$ and $\tilde{G}$ of $G$ to $S^\dag$ and $\tilde{S}$ respectively.

All the elements of $G^\dag$ admit a continuous extension to $\overline{S^\dag}$. More precisely $\overline{S^\dag}$ is obtained as a quotient of $\overline{\mathbb{D}}$ by the action of $J$ where $\tilde{S}$ is identified with the Poincaré disk $\mathbb{D}$. Let $\gamma_1, \ldots, \gamma_\ell$ the circles in $\partial N$. The surface $\overline{S^\dag} \setminus N$ consists of $\ell$ closed annuli $C_1, \ldots, C_\ell$ such that $\partial C_j$ consists of two circles, namely $\partial_j \subset \partial \overline{S^\dag}$ and $\gamma_j$ for any $1 \leq j \leq \ell$. We can consider the rotation numbers $\rho_j(\tilde{f})$ associated to the homeomorphism $f_{|\partial_j}$ and the lift $\tilde{f}$ for $1 \leq j \leq \ell$. We can consider the coordinate $c_j$ of $\xi[\alpha]$ that corresponds to the annulus $A_j$ in $\mathcal{A}$ that contains $\gamma_j$. By Remark 4.1 and Lemma 4.2 we obtain $\rho_j(\tilde{f}) = \pm c_j$ where the sign depends on the choice of a generator of $H_1(A_j, \mathbb{Z})$. Since $\mathcal{R}$ is minimal, it follows that $\text{Fix}_c(G^\dag) \cap \partial \overline{S^\dag} = \emptyset$. 
Let $H$ be a finitely generated subgroup of $G$. We have that $\phi$ is isotopic to $\text{Id} \, \text{rel} \, \text{Fix}(C^{(k)}(H))$ for all $k \geq 1$ and $\phi \in C^{(k)}(H)$ by $H' \subset H_0$ and Theorem 4.1. We deduce that $\phi^\dagger$ is isotopic to $\text{Id} \, \text{rel} \, \text{Fix}(C^{(k)}(H^\dagger)) \cup (\overline{S^\dagger} \setminus S^\dagger)$ for all $k \geq 1$ and $\phi \in C^{(k)}(H)$. We obtain that $H^\dagger$ has a contractible global fixed point in $\overline{S^\dagger}$ by Theorem 8.1. The finite intersection property for compact sets implies that $G^\dagger$ has a contractible global fixed point in $S^\dagger$. Since $k \geq 1$, the discussion in the previous paragraph implies $\text{Fix}_c(G^\dagger) \cap (\overline{S^\dagger} \setminus S^\dagger) = \emptyset$ and hence $\text{Fix}_c(G^\dagger) \cap S^\dagger \neq \emptyset$.

Since $\pi^\dagger(\text{Fix}_c(G^\dagger) \cap S^\dagger) \subset \pi^\dagger(\text{Fix}_c(G_0^\dagger) \cap S^\dagger) = \text{Fix}_c(G_0)$, we obtain $\text{Fix}(G) \cap \text{Fix}_c(G_0) \neq \emptyset$.

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