Hamiltonian formalism of extended magnetohydrodynamics

H M Abdelhamid\textsuperscript{1,2}, Y Kawazura\textsuperscript{1} and Z Yoshida\textsuperscript{1}

\textsuperscript{1} Graduate School of Frontier Sciences, University of Tokyo, Kashiwanoha, Kashiwa, Chiba 277-8561, Japan
\textsuperscript{2} Physics Department, Faculty of Science, Mansoura University, Mansoura 35516, Egypt

E-mail: hamdi@ppl.k.u-tokyo.ac.jp

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Abstract
The extended magnetohydrodynamics (MHD) system, including the Hall effect and the electron inertia effect, has a Hamiltonian structure embodied by a noncanonical Poisson algebra on an infinite-dimensional phase space. A nontrivial part of the formulation is the proof of Jacobi’s identity for the Poisson bracket. We unearth a basic Lie algebra that generates the Poisson bracket. A class of similar Poisson algebra may be generated by the same Lie algebra, which encompasses the Hall MHD system and inertial MHD system.

Keywords: extended magnetohydrodynamics, Hamiltonian dynamics, Jacobi’s identity

1. Introduction

Hamiltonian formalisms help us to elucidate geometrical structures of evolving systems. For example, non-trivial center of the governing Poisson algebra (which makes the Hamiltonian system \textit{noncanonical} \cite{1}) yields Casimir invariants and foliates the phase space; the corresponding topological constraints give rise to interesting equilibrium points on leaves, forbid various instabilities, or separate different regimes creating hierarchical structure in the phase space \cite{2}.

The Hamiltonian formalism of the ideal magnetohydrodynamics (MHD) system was given for the first time by Morrison and Green \cite{3, 4}; see \cite{5, 6} for recent studies on the noncanonical properties of the Poisson bracket. The ideal MHD model often falls short of describing interesting phenomena in plasmas originating from different scale hierarchies which are scaled by ion and electron inertial lengths. For example, the electric field in the
direction of the magnetic field must vanish in the ideal MHD model, by which the topology of magnetic field lines (such as the linking numbers or writhe) are invariant. In a high-temperature (collisionless) plasma, topological change of magnetic field lines can occur in a small scale on which the electron inertia produces a finite parallel electric field, which, however, is ignored in the ideal MHD model. Many different attempts have been made to generalize the model to include small-scale effects, and formulate them as Hamiltonian systems; see [7, 8] for different Hamiltonian forms of Hall MHD, [9] for the Casimir invariants of noncanonical Hall MHD, [10] for the canonized Hamiltonian formalism of Hall MHD and its action principle delineating the limiting path to the ideal MHD system. Another important effect is due to the electron inertia, which brings about a finite parallel electric field, allowing magnetic field lines reconnect. In this direction, two-dimensional models have been intensively studied; see [11–15] as well as [16, 17] for recent developments.

The aim of this work is to formulate a Hamiltonian system of an extended MHD model which subsumes ideal MHD, Hall MHD, as well as inertial MHD models; the small-scale effects are scaled by factors representing the ion skin depth and electron skin depth. The key issue of the modeling is the reduction from the ion-electron two-fluid model by imposing the charge neutrality condition; the two-fluid model has a natural Hamiltonian formalism (for example; see [18]), which, however, is not apparent in models assuming the charge neutrality condition. In section 2, we explain the derivation of an extended MHD model from the two-fluid model. In section 3, we formulate the Hamiltonian and Poisson operator. Section 4 is devoted for the proof of Jacobi’s identity required for the Poisson bracket. In section 5, we conclude this paper with some remarks.

2. Extended MHD model

Reducing the two-fluid model of plasmas by quasineutrality condition, we obtain a system of equations governing the total mass density $\rho$, the flow velocity $\mathbf{V}$, and the magnetic field $\mathbf{B}$. By adding of the electron and ion continuity equations, we obtain the mass conservation law

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot (\rho \mathbf{V}), \quad (1)$$

where $\rho = (m_i n_i + m_e n_e) = (1 + \delta)m n$ ($m_i$ is the ion mass, $m_e (\ll m_i)$ is the electron mass, $n_i$ is the ion number density, $n_e$ is the electron number density, and $\delta = m_e/m_i$ is a small parameter; by charge neutrality, we put $n_i = n_e = n$). The formulae of the electron velocity $\mathbf{V}_e$ and the ion velocity $\mathbf{V}_i$ in terms of the plasma flow velocity $\mathbf{V}$ and the plasma current density $\mathbf{J}$ are

$$\mathbf{V}_e = \mathbf{V} - \frac{\mathbf{J}}{en(1 + \delta)} \quad \text{and} \quad \mathbf{V}_i = \mathbf{V} + \delta \frac{\mathbf{J}}{en(1 + \delta)}. \quad (2)$$

Summing the equations of motion of both components yields the momentum equation

$$\rho \left( \frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V} \cdot \nabla) \mathbf{V} \right) = -\nabla p + \mathbf{J} \times \mathbf{B} - \frac{m_e}{e} (\mathbf{J} \cdot \mathbf{V}) \frac{\mathbf{J}}{en}, \quad (3)$$

where $p = p_i + p_e$ is the pressure ($p_i$ is the ion pressure, and $p_e$ is the electron pressure), and $e$ is the charge. Notice that, the last term on the right-hand side of equation (3) is of $O(\delta)$, which

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3 The magnetic helicity is a Casimir invariant pertinent to this topological constraint. However, the constraint is much stronger than that depicted by the magnetic helicity; see [2] for the delineation of the local topological constraints by constructing hierarchical phase spaces.
plays an essential role in conservation of energy [19]. To make (3) exact up to $O(\delta^2)$, we should replace $m_e$ by $m'_e = m_e/(1 + \delta)$. The other equation determines the evolution of electron fluid momentum. Instead of using $V_e$, we write the electron inertia in terms of $J_{en}$:

$$\frac{m_e}{e^2} \left[ \frac{\partial}{\partial t} \left( \frac{J_e}{n} \right) + (V \cdot \nabla) \frac{J_e}{n} + \left( \frac{J_e}{n} \cdot \nabla \right) \frac{J_e}{n} - \left( \frac{J_e}{n} \cdot \nabla \right) \frac{J_{en}}{n} \right] - \frac{1}{en} \nabla p_e = \left[ E + V \times B - \frac{1}{en} J \times B \right],$$

(4)

where $E$ is the electric field. On the left-hand side of equation (4), we have neglected $\nabla \equiv \partial / \partial + \frac{\partial}{\partial x}$. Assuming $\omega \ll \frac{d}{c_L}$, the dimensions of dynamical variables are substantially reduced; see (8).

Summarizing the equations and normalizing variables in the standard Alfvén units, we obtain a system of governing equations

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot (\rho V),$$

(5)

$$\rho \left( \frac{\partial V}{\partial t} + (V \cdot \nabla) V \right) = -\nabla p + J \times B - d_e^2 \left( J \cdot V \right) \frac{J}{\rho},$$

(6)

$$E + V \times B = -\frac{d_i}{\rho} \nabla p_i + d_e \frac{J}{\rho} \times B$$

$$+ d_e^2 \left[ \frac{\partial}{\partial t} \left( \frac{J}{\rho} \right) + (V \cdot \nabla) \frac{J}{\rho} + \left( \frac{J}{\rho} \cdot \nabla \right) \frac{J}{\rho} \right]$$

$$- d_e d_i^2 \left( \frac{J}{\rho} \cdot V \right) \frac{J}{\rho},$$

(7)

where $d_e = c / (\omega_{pe} L)$ is the normalized electron skin depth, $d_i = c / (\omega_{pi} L)$ is the normalized ion skin depth, $\omega_{pe}$ and $\omega_{pi}$ are the electron and ion plasma frequencies ($L$ is the system size).

The above equations are coupled with the pre-Maxwell equations

$$\nabla \times E = -\frac{\partial B}{\partial t} \quad \text{and} \quad \nabla \times B = J.$$

(8)

We omit Maxwell displacement current to make the field equations consistent with the above fluid equations. There are some phenomena, however, in which charge neutrality may be broken (see [24]), then one needs to consider independent electron and ion densities.
3. Hamiltonian formalism of extended MHD

3.1. Noncanonical Hamiltonian systems

We prepare general notations to formulate Hamiltonian systems. A general Hamiltonian equation can be written as

$$\partial_t u = J(u)\partial_u H(u),$$

(9)

where the state vector $u$ is a point in a phase space (Hilbert space) $X$, $H(u)$ is a Hamiltonian (a smooth function on $X$), $\partial_u H$ is the gradient of $H$ in $X$, and $J(u)$ is a Poisson operator (co-symplectic 2-covector). In the latter discussions, the state vector $u$ is a vector-valued function on a base space $\Omega \subset \mathbb{R}^3$. The inner product of the phase space $X$ is defined by $\langle u, v \rangle = \int_\Omega u \cdot vd^3x$.

We define a bilinear form

$$\{F, G\} = \langle \partial_u F, J \partial_u G \rangle,$$

(10)

where $F$ and $G$ are scalar smooth functionals on the phase space $X$. For $[,]$ to be a Poisson bracket, we demand (1) antisymmetry $\{F, G\} + \{G, F\} = 0$, (2) Jacobi’s identity $\{E, \{F, G\}\} + \{G, \{E, F\}\} + \{F, \{E, G\}\} = 0$, and (3) Leibniz property $\{FG, E\} = F\{G, E\} + G\{F, E\}$.

The adjoint representation of Hamilton’s equation (9) reads, for an arbitrary observable $F \in C^0_c(\Omega)$,

$$\frac{d}{dt} F = \{F, H\}.$$

(11)

3.2. Poisson algebra of extended MHD

Operating $\nabla \times$ on both side of (7), assuming barotropic pressures ($\rho^{-1}\nabla p = \nabla h(\rho)$, $\rho^{-1}\nabla h = \nabla h_e(\rho)$, where $h(\rho)$ is the total enthalpy and $h_e(\rho)$ is the electron enthalpy) and using (8), we obtain a system of evaluation equations

$$\frac{\partial}{\partial t} \rho = -\nabla \cdot (\rho V),$$

(12)

$$\frac{\partial V}{\partial t} = -(\nabla \times V) \times V - V \left( h + \frac{V^2}{2} \right) + \rho^{-1}(\nabla \times B) \times B^* - d_e^2 \nabla \left( \frac{(\nabla \times B)^2}{2\rho^2} \right),$$

(13)

$$\frac{\partial B^*}{\partial t} = \nabla \times \left( \nabla \times B^* \right) - d_e V \nabla \times \left( \rho^{-1}(\nabla \times B) \times B^* \right)$$

$$+ d_e^2 V \times \left( \rho^{-1}(\nabla \times B) \times (\nabla \times V) \right),$$

(14)

where

$$B^* = B + d_e^2 \nabla \times \rho^{-1}(\nabla \times B).$$

(15)

For the simplicity, we consider a domain $S_3$ with periodic boundary conditions.

The conservation of energy of the extended MHD was studied by [16, 19]; the total energy is given as
This $\mathcal{H}$ is the natural candidate for the Hamiltonian.

To formulate the Hamiltonian system, we consider a phase space spanned by the variables $\rho, V,$ and $B^*$; we denote the state vector by $u = \left(\rho, V, B^*\right)$. Then, $B$ in $\mathcal{H}$ must be evaluated as a function of $B^*$ and $\rho$ by (15). The gradient of the Hamiltonian $\mathcal{H}$ is

$$
\nabla_{\rho} \mathcal{H} = \left(\frac{V^2}{2} + h + d_e^2 \left(\frac{(V \times B)^2}{2\rho^2}\right)\right).
$$

Now, we propose a Poisson operator for the extended MHD equations:

$$
\mathcal{J} = \begin{pmatrix}
0 & -V \\
-V & \rho^{-1}(V \times V) \circ \rho^{-1}(V \times \circ) \\
0 & \rho^{-1}(V \times \circ) \times (V \times V)
\end{pmatrix}.
$$

With the Poisson operator (17) and the Hamiltonian (16), Hamilton’s equation (11) reproduces the extended MHD equations (12), (13), and (14).

Using the periodic boundary conditions, we can easily demonstrate the antisymmetry of $\mathcal{J}$. Hence the Poisson bracket defined by this $\mathcal{J}$ satisfies antisymmetry. However, the proof of Jacobi’s identity is rather elaborating\(^4\). Leaving it for the next section, we end this section by stating the main assertion:

**Theorem 3.1 Poisson algebra of extended MHD.** We define a bilinear form

$$
\{F, G\} = \langle \partial_{\rho} F, \mathcal{J} \partial_{\rho} G \rangle.
$$

Then, $\{,\}$ is a Poisson bracket, and $C^\infty_{\rho^{-1}}(X)$ is a Poisson algebra. Providing it with a Hamiltonian $\mathcal{H}$ of (16), we obtain the extended MHD system.

4. Jacobi’s Identity

4.1. Basic algebra

We have yet to prove Jacobi’s identity for the Poisson bracket. Apparently, it is not of a Lie–Poisson type. Complexity is caused by the factor $\rho^{-1}$, as well as differential operator $V \times$

\(^4\) If we construct a bracket by a semidirect product of sub Lie algebras, we can prove that the bracket satisfies Jacobi’s identity [25–27]. Examples of ideal fluid, MHD and multihluid plasmas [28, 29], as well as Hall-MHD [7] were studied by this method. However, the present model of generalized MHD is not an example of such systems.
appearing in many places of \( J \). However, there is a basic, common permutation relation that generates the total Poisson system. We prepare the following lemma:

**Lemma 4.1.** On \( C^\infty(X) \), we define a bracket (bilinear form)

\[
[F, G]_{q,r}^p = \int_{\Omega} \left[ \rho^{-1}(\nabla \times p) \cdot \left( \partial_r F \times \partial_r G \right) - \partial_r F \cdot \left( \nabla \cdot \partial_r G \right) - \partial_r G \cdot \partial_r F + \nabla \cdot \left( \partial_r F \times \partial_r G \right) \right] d^3x,
\]

(18)

where \( p, q, \) and \( r \) are vector fields arbitrarily chosen from \( V \) or \( A^v \) (where \( A^v \) is the vector potential and is related to \( B^v \) by the relation \( B^v = \nabla \times A^v \)). This bracket satisfies an antisymmetry relation

\[
[F, G]_{q,r}^p = -[G, F]_{r,q}^p,
\]

as well as a permutation law

\[
\left[ E, [F, G]_{q,r}^p \right]_{s,t}^p + \left[ G, [E, F]_{q,s}^p \right]_{r,p}^p + \left[ F, [G, E]_{q,t}^p \right]_{r,p}^p = O(\partial^2),
\]

(19)

where \( O(\partial^2) \) denotes terms including second-order derivatives. Hence, the sum over the permutation vanishes on the modulo operation by \( \partial^2 \).

The combination of the functionals \( E, F, G \) and the corresponding state variables \( q, r, s \) is a unique aspect of this bracket. Notice that the permutation law (19) resembles Jacobi’s identity. In fact, the algebraic relation delineated by this lemma 4.1 is the root cause of Jacobi’s identity satisfied by the Poisson bracket.

(proof of Lemma 4.1) The antisymmetry is evident. To prove Jacobi’s identity, we have to calculate the functional derivative of the bracket. By the inhomogeneous factor \( \rho^{-1}(\nabla \times p) \) included in the bracket, the derivatives such as \( \partial_r[F, G]_{q,r}^p \) and \( \partial_r[F, G]_{q,r}^p \) are sums of the terms that consist of only first derivatives of \( F \) and \( G \), as well as the terms including second-order derivatives (the second-order terms are modulo-outed in (19)). The former ones are such that

\[
\partial_r[F, G]_{q,r}^p = -\rho^{-2}(\nabla \times p) \cdot \left( \partial_r F \times \partial_r G \right) + O(\partial^2),
\]

\[
\partial_r[F, G]_{q,r}^p = \nabla \times \rho^{-1}(\partial_r F \times \partial_r G) + O(\partial^2).
\]

The permutation law is given as

\[
\left[ E, [F, G]_{q,r}^p \right]_{s,t}^p + \left[ G, [E, F]_{q,s}^p \right]_{r,p}^p + \left[ F, [G, E]_{q,t}^p \right]_{r,p}^p = \int_{\Omega} \left\{ (\nabla \times p) \cdot \left[ \rho^{-1}\partial_r E \times \nabla \times \left( \partial_r F \times \rho^{-1}\partial_r G \right) \right] \right. \\
+ \left. \partial_r E \cdot \nabla \left[ (\nabla \times p) \cdot \left( \rho^{-1}\partial_r F \times \rho^{-1}\partial_r G \right) \right] \right. \\
+ (\nabla \times p) \cdot \left[ \rho^{-1}\partial_r G \times \nabla \times \left( \partial_r E \times \rho^{-1}\partial_r F \right) \right] \right. \\
+ \left. \partial_r G \cdot \nabla \left[ (\nabla \times p) \cdot \left( \rho^{-1}\partial_r E \times \rho^{-1}\partial_r F \right) \right] \right. \\
+ (\nabla \times p) \cdot \left[ \rho^{-1}\partial_r F \times \nabla \times \left( \partial_r G \times \rho^{-1}\partial_r E \right) \right] \\
+ \partial_r F \cdot \nabla \left[ (\nabla \times p) \cdot \left( \rho^{-1}\partial_r G \times \rho^{-1}\partial_r E \right) \right] \right\} d^3x + O(\partial^2).
\]

(20)
Denoting $e \equiv \partial E$, etc., equation (20) reads
\[
\left[ E, \left[ F, G \right] \right] + \left[ G, \left[ E, F \right] \right] + \left[ F, \left[ G, E \right] \right] =
\int_{\Omega} \left\{ \left( \nabla \times p \right) \cdot \left[ \rho^{-1} e \times \left( \nabla \times \left( f \times \rho^{-1} g \right) \right) \right]
\right. \\
\left. + e \cdot \nabla \left[ \left( \nabla \times p \right) \cdot \left( \rho^{-1} f \times \rho^{-1} g \right) \right] + \bigcirc \right\} d^3x + O\left( \partial^2 \right). \tag{21}
\]

where $\bigcirc$ denotes the summation over cyclic permutation of the vectors $e, f, \text{ and } g$. After integrating by parts, the integrand of $\left[ E, \left[ F, G \right] \right]$ can be written as
\[
\left( \nabla \times p \right) \cdot \left\{ \rho^{-1} e \times \left( \nabla \times \left( f \times \rho^{-1} g \right) \right) - \left( \rho^{-1} f \times \rho^{-1} g \right) \nabla \cdot e \right\}. \tag{22}
\]

The term bracketed by \{ \} can be rewritten by vector identities as
\[
\rho^{-1} e \times \left( \nabla \cdot \rho^{-1} g \right) - \rho^{-1} g \left( \nabla \cdot f \right)
\]
\[
+ \left( \rho^{-1} g \cdot \nabla \right) x - \left( f \cdot \nabla \right) \rho^{-1} g - \left( \rho^{-1} f \times \rho^{-1} g \right) \nabla \cdot e.
\]

The second term and the last term cancel by summation over permutations. To deal with the residual terms in (22), we use the symmetry of the curl operator
\[
\rho^{-1} e \times \left( \nabla \cdot \rho^{-1} g \right) + \left( \rho^{-1} g \cdot \nabla \right) f - \left( f \cdot \nabla \right) \rho^{-1} g
\]

Invoking Levi-Civita symbol, we may write
\[
\epsilon_{ijk} \partial_j \left\{ \epsilon_{ilm} \rho^{-1} e_l \left[ f_m \partial_n \left( \rho^{-1} g_n \right) + \rho^{-1} g_n \partial_m \left( \rho^{-1} f_m \right) \right] \right\}
\]
\[
= \partial_i \left\{ \rho^{-1} e_l \left[ \partial_n \left( \rho^{-1} g_n f_j \right) - f_n \partial_i \left( \rho^{-1} g_j \right) \right] - \rho^{-1} e_j \left[ \partial_n \left( \rho^{-1} g_n f_i \right) - f_n \partial_i \left( \rho^{-1} g_j \right) \right] \right\}. \tag{23}
\]

The last two terms are manipulated as
\[
-\partial_n \left( \rho^{-2} g_n f_l e_j \right) + \rho^{-1} g_n f_l \partial_n (\rho^{-1} e_j) + \partial_n \left( \rho^{-2} g_n f_l e_j \right) - \rho^{-1} g_l \partial_n (\rho^{-2} f_n e_j).
\]

Now (23) is summarized as
\[
\partial_j \partial_i \left[ \rho^{-2} \left( g_n f_l e_j - g_n f_l e_i \right) \right] + \partial_i \left[ \rho^{-1} e_l \partial_n \left( \rho^{-1} g_n f_j \right) - \rho^{-1} g_i \partial_n \left( \rho^{-1} f_n e_j \right) \right]
\]
\[
+ \partial_i \left[ \rho^{-1} g_l f_n \partial_n \left( \rho^{-1} e_j \right) - \rho^{-1} f_l e_j \partial_n \left( \rho^{-1} g_i \right) \right].
\]

each term of which cancels out by summation over the permutation. (QED)

Remark 4.2. If we choose $p = q = r = V$, the bracket (18) is the Poisson bracket of the barotropic compressible fluid:
\[
\{ F, G \} = \int_{\Omega} \left[ \rho^{-1} \left( \nabla \times V \right) \cdot \left( \partial V F \times \partial V G \right) - \partial p F \left( \nabla \cdot \partial V G \right) - \partial p F \cdot \nabla \partial V G \right] d^3x, \tag{24}
\]

where the state vector is $u = (\rho, V)$. The Poisson operator corresponding to Poisson bracket (24) is
\[
J = \begin{pmatrix}
0 & -\nabla \\
-\nabla & -\rho^{-1}(\nabla \times \mathbf{V}) \times \mathbf{e}
\end{pmatrix}
\]  

(25)

Giving a Hamiltonian

\[
\mathcal{H} := \int_{\Omega} \rho \left( \frac{\mathbf{V}^2}{2} + U(\rho) \right) d^3x,
\]

(26)

Hamilton’s equation reads

\[
\frac{\partial \rho}{\partial t} = -\nabla \cdot (\rho \mathbf{V}),
\]

(27)

\[
\frac{\partial \mathbf{V}}{\partial t} = -(\nabla \times \mathbf{V}) \times \mathbf{V} - \nabla \left( h + \frac{\mathbf{V}^2}{2} \right).
\]

(28)

4.2. Jacobi’s identity for the Poisson bracket of extended MHD

Now we complete the proof of theorem 3.1 by verifying Jacobi’s identity for the Poisson bracket

\[
\{F, G\} = -\int_{\Omega} \left\{ \left[ F_\rho \nabla \cdot G_\rho + F_\mathbf{V} \cdot \nabla G_\mathbf{V} \right] - \left[ \rho^{-1}(\nabla \times \mathbf{V}) \cdot \left( F_\mathbf{V} \times G_\mathbf{V} \right) \right] \\
- \left[ B^* \cdot \rho^{-1} F_\mathbf{V} \times (\nabla \times G_{B^*}) \right] + B^* \cdot \rho^{-1} \left( (\nabla \times F_{B^*}) \times (\nabla \times G_{B^*}) \right) \}
\]

\[
\left. \left. \left. - \partial_u^{d_2} \left( (\nabla \times \mathbf{V}) \cdot \rho^{-1} \left( (\nabla \times F_{B^*}) \times (\nabla \times G_{B^*}) \right) \right) \right\} d^3x, \right. \}
\]

(29)

where the subscripts indicate functional derivative of the functional \( F, G \) with respect to the state variables \( \rho, \mathbf{V}, B^* \), i.e \( F_\rho = \partial_\rho F \).

To examine Jacobi’s identity, we have to study the derivatives of the bracket by the state variables, which consists of two groups of terms; group (A) is the collection of terms that include second-order derivatives (such as \( F_{B^* \mathbf{V}} \)). Formally, group (A) is generated by pretending that the coefficients in the Poisson operator \( \mathcal{J} \) are independent to (or, different from) the state vector \( \mathbf{u} \). The terms of group (A) cancel out when summed up in \( \{E, \{F, G\}\} \) + \( \mathcal{O}^5 \).

Group (B) summarizes the remaining terms that are due to the derivatives of \( \mathcal{J} \) by \( \mathbf{u} \); explicitly, we have

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\[
\{ F, G \}_{\rho} \mod \partial^2 = -\rho^{-2} (\nabla \times V) \cdot (F_V \times G_V)
- \rho^{-2} B^* \cdot \left[ F_V \times \left( \nabla \times G_{B^*} \right) \right]
- \rho^{-2} B^* \cdot \left[ \left( \nabla \times F_{B^*} \right) \times G_V \right]
+ d_i \left[ \rho^{-2} B^* \cdot \left[ \left( \nabla \times F_{B^*} \right) \times \left( \nabla \times G_{B^*} \right) \right] \right]
- d_i \left[ \rho^{-2} (\nabla \times V) \cdot \left[ \left( \nabla \times F_{B^*} \right) \times \left( \nabla \times G_{B^*} \right) \right] \right].
\]

(30)

\[
\{ F, G \}_V \mod \partial^2 = \nabla \times \rho^{-1} \left( F_V \times G_V \right)
+ d_i^2 \left[ \nabla \times \rho^{-1} \left( \left( \nabla \times F_{B^*} \right) \times \left( \nabla \times G_{B^*} \right) \right) \right].
\]

(31)

\[
\{ F, G \}_{B^*} \mod \partial^2 = \rho^{-1} \left( F_V \times \left( \nabla \times G_{B^*} \right) \right)
+ \rho^{-1} \left( \left( \nabla \times F_{B^*} \right) \times G_V \right)
- d_i \left[ \rho^{-1} \left( \left( \nabla \times F_{B^*} \right) \times \left( \nabla \times G_{B^*} \right) \right) \right].
\]

(32)

In what follows, we show that the remaining group (B) terms cancel out. By (30), (31), and (32), we obtain

\[
\{ E, \{ F, G \} \} + \hat{\nabla}
= \int_{\Omega} E_V \cdot \left[ \nabla \left( \rho^{-2} (\nabla \times V) \cdot (F_V \times G_V) \right) \right]
- \rho^{-1} (\nabla \times V) \times \left[ \nabla \times \rho^{-1} (F_V \times G_V) \right] \right] d^3x
+ \int_{\Omega} E_V \cdot \left[ \nabla \left( \rho^{-2} B^* \cdot \left( F_V \times \left( \nabla \times G_{B^*} \right) \right) \right] \right] d^3x
+ \left[ \nabla \times \left( \rho^{-1} F_V \times \left( \nabla \times G_{B^*} \right) \right) \right] \times \rho^{-1} B^* \right] \right] d^3x
+ \int_{\Omega} E_V \cdot \left[ \nabla \left( \rho^{-2} B^* \cdot \left( \left( \nabla \times F_{B^*} \right) \times G_V \right) \right] \right] d^3x
+ \left[ \nabla \times \left( \rho^{-1} \left( \nabla \times F_{B^*} \right) \times \left( \nabla \times G_{B^*} \right) \right) \right] \times \rho^{-1} B^* \right] \right] d^3x
- d_i \int_{\Omega} E_V \cdot \left[ \nabla \left( \rho^{-2} B^* \cdot \left( \left( \nabla \times F_{B^*} \right) \times \left( \nabla \times G_{B^*} \right) \right) \right] \right] d^3x
+ \left[ \nabla \times \left( \rho^{-1} \left( \nabla \times F_{B^*} \right) \times \left( \nabla \times G_{B^*} \right) \right) \right] \times \rho^{-1} B^* \right] \right] d^3x
+ d_i^2 \int_{\Omega} E_V \cdot \left[ \nabla \left( \rho^{-2} (\nabla \times V) \cdot \left[ \left( \nabla \times F_{B^*} \right) \times \left( \nabla \times G_{B^*} \right) \right] \right] \right] d^3x
- \rho^{-1} (\nabla \times V) \times \left[ \nabla \times \left( \rho^{-1} \left( \nabla \times F_{B^*} \right) \times \left( \nabla \times G_{B^*} \right) \right) \right] \right] d^3x
- d_i \int_{\Omega} E_{B^*} \cdot \nabla \times \left[ \left[ \nabla \times \left( \rho^{-1} F_V \times \left( \nabla \times G_{B^*} \right) \right) \right] \right] \times \rho^{-1} B^* \right] \right] d^3x
\]
To prove Jacobi’s identity, we collect terms that have the same combinations of functional derivatives such that \((E_V, F_V, G_V), (E_V, F_V, G_B^r), \ldots, (E_B^r, F_B^r, G_B^r)\). Then, \[
\{E, \{F, G\}\} + \mathcal{O} = \int_\Omega \left\{ \begin{array}{l}
\rho^{-1}E_V \times \left( \rho^{-1}(V \times F_V) \times G_V \right) \\
+ E_V \cdot \left[ \nabla \left( \rho^{-2} (V \times V) \cdot (F_V \times G_V) \right) \right] \\
+ (V \times V) \cdot \left[ \rho^{-1} G_V \times \left( V \times \rho^{-1} (E_V \times F_V) \right) \right] \\
+ (V \times V) \cdot \left[ \rho^{-1} G_V \times \left( V \times \rho^{-1} (E_V \times F_V) \right) \right] \\
+ V \cdot \left[ \rho^{-2} (V \times V) \cdot (E_V \times F_V) \right] \\
+ F_V \cdot \left[ \rho^{-1} (V \times V) \cdot (G_V \times E_V) \right] \\
+ \int_\Omega \left\{ \begin{array}{l}
B^* \cdot \left[ \rho^{-1} (V \times E_B^r) \times \left( V \times \rho^{-1} [F_V \times G_V] \right) \right] \\
+ B^* \cdot \left[ \rho^{-1} G_V \times \left( V \times \rho^{-1} [V \times E_B^r] \times F_V \right) \right] \\
+ G_V \cdot \nabla \left[ \rho^{-2} B^* \cdot \left( V \times E_B^r \times F_V \right) \right] \\
+ F_V \cdot \nabla \left[ \rho^{-2} B^* \cdot \left( G_V \times (V \times E_B^r) \right) \right] \\
+ \int_\Omega \left\{ \begin{array}{l}
B^* \cdot \left[ \rho^{-1} E_V \times \left( V \times \rho^{-1} [F_V \times (V \times G_B^r)] \right) \right] \\
+ E_V \cdot \nabla \left[ \rho^{-2} B^* \cdot \left( F_V \times (V \times G_B^r) \right) \right]
\end{array} \right\} \right\} d^3 x
\]
\]
\[+ B^* \cdot \left[ \rho^{-1} \left( \nabla \times G_{B^*} \right) \times \left( \nabla \times \rho^{-1} \left[ E_V \times F_V \right] \right) \right]\]
\[+ B^* \cdot \left[ \rho^{-1} F_V \times \left( \nabla \times \rho^{-1} \left[ \left( \nabla \times G_{B^*} \right) \times E_V \right] \right) \right]\]
\[+ F_V \cdot \nabla \left[ \rho^{-2} B^* \cdot \left( \left( \nabla \times G_{B^*} \right) \times E_V \right) \right] \right\} d^3x
\[+ \int_{\Omega} \left\{ B^* \cdot \left[ \rho^{-1} E_V \times \left( \nabla \times \rho^{-1} \left[ \left( \nabla \times F_{B^*} \right) \times G_V \right] \right) \right]\right\}
\[+ E_V \cdot \nabla \left[ \rho^{-2} B^* \cdot \left( \left( \nabla \times F_{B^*} \right) \times G_V \right) \right]\]
\[+ B^* \cdot \left[ \rho^{-1} G_V \times \left( \nabla \times \rho^{-1} \left[ E_V \times \left( \nabla \times F_{B^*} \right) \right] \right) \right]\]
\[+ G_V \cdot \nabla \left[ \rho^{-2} B^* \cdot \left( E_V \times \left( \nabla \times F_{B^*} \right) \right) \right]\]
\[+ B^* \cdot \left[ \rho^{-1} \left( \nabla \times F_{B^*} \right) \times \left( \nabla \times \rho^{-1} \left[ G_V \times E_V \right] \right) \right] \right\} d^3x
\[- d_i \int_{\Omega} \left\{ B^* \cdot \left[ \rho^{-1} E_V \times \left( \nabla \times \rho^{-1} \left[ \left( \nabla \times F_{B^*} \right) \times \left( \nabla \times G_{B^*} \right) \right] \right) \right]\right\}
\[+ E_V \cdot \nabla \left[ \rho^{-2} B^* \cdot \left( \left( \nabla \times F_{B^*} \right) \times \left( \nabla \times G_{B^*} \right) \right) \right]\]
\[+(\nabla \times V) \cdot \left[ \rho^{-1} \left( \nabla \times G_{B^*} \right) \times \left( \nabla \times \rho^{-1} \left[ E_V \times \left( \nabla \times F_{B^*} \right) \right] \right) \right]\]
\[+(\nabla \times V) \cdot \left[ \rho^{-1} \left( \nabla \times F_{B^*} \right) \times \left( \nabla \times \rho^{-1} \left[ \left( \nabla \times G_{B^*} \right) \times E_V \right] \right) \right] \right\} d^3x
\[- d_i \int_{\Omega} \left\{ B^* \cdot \left[ \rho^{-1} \left( \nabla \times E_{B^*} \right) \times \left( \nabla \times \rho^{-1} \left[ F_V \times \left( \nabla \times G_{B^*} \right) \right] \right) \right]\right\}
\[+ B^* \cdot \left[ \rho^{-1} \left( \nabla \times G_{B^*} \right) \times \left( \nabla \times \rho^{-1} \left[ \left( \nabla \times E_{B^*} \right) \times F_V \right] \right) \right]\]
\[+ B^* \cdot \left[ \rho^{-1} F_V \times \left( \nabla \times \rho^{-1} \left[ \left( \nabla \times G_{B^*} \right) \times \left( \nabla \times E_{B^*} \right) \right] \right) \right]\]
\[+ F_V \cdot \nabla \left[ \rho^{-2} B^* \cdot \left( \left( \nabla \times G_{B^*} \right) \times \left( \nabla \times E_{B^*} \right) \right) \right] \right\} d^3x
\[+ d_i \int_{\Omega} \left\{ (\nabla \times V) \cdot \left[ \rho^{-1} \left( \nabla \times E_{B^*} \right) \times \left( \nabla \times \rho^{-1} \left[ F_V \times \left( \nabla \times G_{B^*} \right) \right] \right) \right]\right\}
\[+(\nabla \times V) \cdot \left[ \rho^{-1} \left( \nabla \times G_{B^*} \right) \times \left( \nabla \times \rho^{-1} \left[ \left( \nabla \times E_{B^*} \right) \times F_V \right] \right) \right]\]
\[+(\nabla \times V) \cdot \left[ \rho^{-1} F_V \times \left( \nabla \times \rho^{-1} \left[ \left( \nabla \times G_{B^*} \right) \times \left( \nabla \times E_{B^*} \right) \right] \right) \right]\]
To apply lemma 4.1, we rewrite (34) in terms of the bilinear form (18):

\[ \{E, \{F, G\}\} \cdot \bigcirc \]

\[ = \left[ E, [F, G]V(V, V) \right] + \left[ G, [E, F]V(V, V) \right] + \left[ F, [G, E]V(V, V) \right] + O(\delta^2) \]
the extended MHD, we present the formulation.

The Hamiltonian formulation of Hall MHD is already known \[7, 9, 10\]. To see the relation to

\[ \begin{aligned}
&+ E, [F, G]^{(\alpha)}_{A^\alpha A^\alpha} + [G, [E, F]^{(\alpha)}_{A^\alpha A^\alpha} + [F, [G, E]^{(\alpha)}_{A^\alpha A^\alpha} \]
&+ E, [F, G]^{(\alpha)}_{A^\alpha A^\alpha} + [G, [E, F]^{(\alpha)}_{A^\alpha A^\alpha} + [F, [G, E]^{(\alpha)}_{A^\alpha A^\alpha} \]
&+ E, [F, G]^{(\alpha)}_{A^\alpha A^\alpha} + [G, [E, F]^{(\alpha)}_{A^\alpha A^\alpha} + [F, [G, E]^{(\alpha)}_{A^\alpha A^\alpha} \]
&+ E, [F, G]^{(\alpha)}_{A^\alpha A^\alpha} + [G, [E, F]^{(\alpha)}_{A^\alpha A^\alpha} + [F, [G, E]^{(\alpha)}_{A^\alpha A^\alpha} \]
&- \partial_\nu \left( E, [F, G]^{(\alpha)}_{A^\alpha A^\alpha} \right)_{\nu A^\alpha} + [G, [E, F]^{(\alpha)}_{A^\alpha A^\alpha} + [F, [G, E]^{(\alpha)}_{A^\alpha A^\alpha} \]
&+ d_\nu^2 \left( E, [F, G]^{(\alpha)}_{A^\alpha A^\alpha} \right)_{\nu A^\alpha} + [G, [E, F]^{(\alpha)}_{A^\alpha A^\alpha} + [F, [G, E]^{(\alpha)}_{A^\alpha A^\alpha} \]
&- \partial_\nu \left( E, [F, G]^{(\alpha)}_{A^\alpha A^\alpha} \right)_{\nu A^\alpha} + [G, [E, F]^{(\alpha)}_{A^\alpha A^\alpha} + [F, [G, E]^{(\alpha)}_{A^\alpha A^\alpha} \]
&+ d_\nu^2 \left( E, [F, G]^{(\alpha)}_{A^\alpha A^\alpha} \right)_{\nu A^\alpha} + [G, [E, F]^{(\alpha)}_{A^\alpha A^\alpha} + [F, [G, E]^{(\alpha)}_{A^\alpha A^\alpha} \]
&+ d_\nu^2 \left( E, [F, G]^{(\alpha)}_{A^\alpha A^\alpha} \right)_{\nu A^\alpha} + [G, [E, F]^{(\alpha)}_{A^\alpha A^\alpha} + [F, [G, E]^{(\alpha)}_{A^\alpha A^\alpha} \]
&- d_\nu d_\nu^2 \left( E, [F, G]^{(\alpha)}_{A^\alpha A^\alpha} \right)_{\nu A^\alpha} + [G, [E, F]^{(\alpha)}_{A^\alpha A^\alpha} + [F, [G, E]^{(\alpha)}_{A^\alpha A^\alpha} \]
&+ O(\partial^2).
\end{aligned} \]

By lemma 4.1, only \( O(\partial^2) \) terms remain on the right-hand side vanishes. As we have
mentioned, on the other hand, \( O(\partial^2) \) vanishes in \( \{ E, \{ F, G \} \} + \mathcal{O} \). Hence, Jacobi’s identity
has been proved.

4.3. Jacobi’s identity for Hall MHD system

The Hamiltonian formulation of Hall MHD is already known \[7, 9, 10\]. To see the relation to
the extended MHD, we present the formulation.

Setting the electron skin depth \( d_e = 0 \) in the extended MHD model, we obtain the
normalized Hall MHD equations,
\[
\frac{\partial \rho}{\partial t} = -\nabla \cdot (\rho V), \\
\frac{\partial V}{\partial t} = -(\nabla \times V) \times V - \nabla \left( h + \frac{V^2}{2} \right) + \rho^{-1}(\nabla \times B) \times B, \\
\frac{\partial B}{\partial t} = \nabla \times (V \times B) - d_i \nabla \times \left( \rho^{-1}(\nabla \times B) \times B \right).
\]

The state vector is \( u = (\rho, V, B) \). The energy of the Hall MHD reduced into

\[
\mathcal{H} := \int_{\Omega} \left\{ \rho \left( \frac{V^2}{2} + U(\rho) \right) + \frac{B^2}{2} \right\} d^3x.
\]

Also, under the same condition Poisson operator becomes

\[
J_{\text{Hall}} = \begin{pmatrix} 0 & -\nabla \cdot & -\rho^{-1}(\nabla \times V) \times \phi & \rho^{-1}(\nabla \times \phi) \times B \\
-\nabla \cdot & -\rho^{-1}(\nabla \times V) \times \phi & \rho^{-1}(\nabla \times \phi) \times B & 0 \\
0 & \nabla \times (\phi \times \rho^{-1}B) & -d_i \nabla \times \left( \rho^{-1}(\nabla \times \phi) \times B \right) & 0 \end{pmatrix}.
\]

### 4.3.1. Poisson bracket and Jacobi’s identity for Hall MHD

The noncanonical Poisson bracket of the Hall MHD system is given as

\[
\{ F, G \} = -\int_{\Omega} \left\{ \left[ F_\rho \nabla \cdot G_\nu + F_\nu \cdot V G_\rho \right] + \left[ F_\nu \cdot \rho^{-1} \left( (\nabla \times V) \times G_\nu \right) \right] \\
- \left[ B \cdot \rho^{-1} \left( F_\nu \times (\nabla \times G_B) \right) + B \cdot \rho^{-1} \left( (\nabla \times F_B) \times G_\nu \right) \right] \\
+ d_i \left[ B \cdot \rho^{-1} \left( (\nabla \times F_B) \times (\nabla \times G_B) \right) \right] \right\} d^3x.
\]

The Poisson bracket of Hall MHD is rewritten as

\[
\{ F, G \} =\left[ F, G \right]_{VV} + \left[ F, G \right]_{VA} + \left[ F, G \right]_{AA} + d_i \left[ F, G \right]_{AA} \\
+ \int_{D} \left[ F_\rho V \cdot G_\nu + F_\nu \cdot V G_\rho \right] d^3x.
\]

After similar calculations as (34), we may write

\[
\{ E, \{ F, G \} \} + \bigcirc \\
= \left[ E, \left[ F, G \right]_{VV} \right]_{VV} + \left[ G, \left[ E, F \right]_{VV} \right]_{VV} + \left[ F, \left[ G, E \right]_{VV} \right]_{VV} \\
+ \left[ E, \left[ F, G \right]_{VA} \right]_{VA} + \left[ G, \left[ E, F \right]_{VA} \right]_{VA} + \left[ F, \left[ G, E \right]_{VA} \right]_{VA} \\
+ \left[ E, \left[ F, G \right]_{AA} \right]_{AA} + \left[ G, \left[ E, F \right]_{AA} \right]_{AA} + \left[ F, \left[ G, E \right]_{AA} \right]_{AA} \\
- d_i \left[ E, \left[ F, G \right]_{AA} \right]_{AA} + \left[ G, \left[ E, F \right]_{AA} \right]_{AA} + \left[ F, \left[ G, E \right]_{AA} \right]_{AA} \\
- \left[ E, \left[ F, G \right]_{VV} \right]_{VV} + \left[ G, \left[ E, F \right]_{VV} \right]_{VV} + \left[ F, \left[ G, E \right]_{VV} \right]_{VV}
\]
which, by Lemma 4.1, vanishes, proving Jacobi’s identity.

4.4. Jacobi’s identity for inertial MHD system

The inertial MHD model is obtained by setting the ion skin depth \( d_i = 0 \) in the extended MHD model:

\[
\frac{\partial \rho}{\partial t} = - \nabla \cdot (\rho \mathbf{V}),
\]

\[
\frac{\partial \mathbf{V}}{\partial t} = -(\mathbf{V} \times \mathbf{V}) \times \mathbf{V} - \nabla \left( h + \frac{V^2}{2} \right) + \rho^{-1}(\nabla \times \mathbf{B}) \times \mathbf{B}^* - d_i^2 \nabla \left( \frac{(\nabla \times \mathbf{B})^2}{2\rho^2} \right),
\]

\[
\frac{\partial \mathbf{B}^*}{\partial t} = \nabla \times \left( \mathbf{V} \times \mathbf{B}^* \right) + d_i^2 \nabla \times \left( \rho^{-1}(\nabla \times \mathbf{B}) \times (\nabla \times \mathbf{V}) \right).
\]

The energy is [19]

\[
\mathcal{F} := \int_{\Omega} \left\{ \rho \left( \frac{V^2}{2} + U(\rho) \right) + \frac{B^2}{2} + d_i^2 \frac{(\nabla \times \mathbf{B})^2}{2\rho} \right\} d^3x.
\]

With respect to the state vector \( u = (\rho, \mathbf{V}, \mathbf{B}^*) \), the Poisson operator of the inertial MHD is

\[
\mathbf{J}_{\text{inertial}} = \begin{pmatrix}
0 & -\nabla & 0 \\
-\nabla & -\rho^{-1}(\nabla \times \mathbf{V}) \times \mathbf{V} & \rho^{-1}(\nabla \times \mathbf{B}^*) \\
0 & \nabla \times (\mathbf{B}^* \times \mathbf{B}^*) & d_i^2 \nabla \times \left( \rho^{-1}(\nabla \times \mathbf{B}) \times (\nabla \times \mathbf{V}) \right)
\end{pmatrix}.
\]

4.4.1. Poisson bracket and Jacobi’s identity for inertial MHD. The Poisson bracket of the inertial MHD system is written as

\[
\{F, G\} = -\int_{\Omega} \left[ \left\{ f_\rho \mathbf{V} \cdot \mathbf{G} + f_\mathbf{V} \cdot \mathbf{V} \mathbf{G} \right\} - \left[ \rho^{-1}(\mathbf{V} \times \mathbf{V}) \cdot (F \times \mathbf{V}) \right] \\
- \left[ \mathbf{B}^* \cdot \rho^{-1}(\mathbf{F} \times (\mathbf{V} \times \mathbf{G}^*)) \right] + \mathbf{B}^* \cdot \rho^{-1}(\mathbf{F}^* \times (\mathbf{V} \times \mathbf{G}^*)) \times \mathbf{V} + \mathbf{G} \right]\right\} d^3x.
\]

\[
\{F, G\} = -\int_{\Omega} \left[ \left\{ f_\rho \mathbf{V} \cdot \mathbf{G} + f_\mathbf{V} \cdot \mathbf{V} \mathbf{G} \right\} - \left[ \rho^{-1}(\mathbf{V} \times \mathbf{V}) \cdot (F \times \mathbf{V}) \right] \\
- \left[ \mathbf{B}^* \cdot \rho^{-1}(\mathbf{F} \times (\mathbf{V} \times \mathbf{G}^*)) \right] + \mathbf{B}^* \cdot \rho^{-1}(\mathbf{F}^* \times (\mathbf{V} \times \mathbf{G}^*)) \times \mathbf{V} + \mathbf{G} \right]\right\} d^3x.
\]
The Poisson bracket can be written as
\[
\{F, G\} = [F, G]^\nu_{\nu, V} + [F, G]^{A^\nu}_{A^\nu, V} + [F, G]^{A^\nu}_{A^\nu, V} \\
+ \int_\Omega \left[ F_{\rho V} \cdot G_{V} + F_{V} \cdot V G_{\rho} \right] d^2 x.
\] (50)

After similar calculations as (34), we may write
\[
\{E, \{F, G\}\} + \bigcirc
\]
\[
= \left[ E, [F, G]^{\nu}_{\nu, V} \right]_{\nu, V}^{\nu} + \left[ G, [E, F]^{\nu}_{\nu, V} \right]_{\nu, V}^{\nu} + \left[ F, [G, E]^{\nu}_{\nu, V} \right]_{\nu, V}^{\nu} \\
+ \left[ E, [F, G]^{A^\nu}_{A^\nu, V} \right]_{A^\nu, V}^{A^\nu} + \left[ G, [E, F]^{A^\nu}_{A^\nu, V} \right]_{A^\nu, V}^{A^\nu} + \left[ F, [G, E]^{A^\nu}_{A^\nu, V} \right]_{A^\nu, V}^{A^\nu} \\
+ \left[ E, [F, G]^{A^\nu}_{A^\nu, V} \right]_{V, A^\nu}^{A^\nu} + \left[ G, [E, F]^{A^\nu}_{A^\nu, V} \right]_{V, A^\nu}^{A^\nu} + \left[ F, [G, E]^{A^\nu}_{A^\nu, V} \right]_{V, A^\nu}^{A^\nu} \\
+ d_2^2 \left[ E, [F, G]^{A^\nu}_{A^\nu, V} \right]_{A^\nu, V}^{A^\nu} + \left[ G, [E, F]^{A^\nu}_{A^\nu, V} \right]_{A^\nu, V}^{A^\nu} + \left[ F, [G, E]^{A^\nu}_{A^\nu, V} \right]_{A^\nu, V}^{A^\nu} \\
+ d_2^2 \left[ E, [F, G]^{A^\nu}_{A^\nu, V} \right]_{V, A^\nu}^{A^\nu} + \left[ G, [E, F]^{A^\nu}_{A^\nu, V} \right]_{V, A^\nu}^{A^\nu} + \left[ F, [G, E]^{A^\nu}_{A^\nu, V} \right]_{V, A^\nu}^{A^\nu} \\
+ d_2^2 \left[ E, [F, G]^{A^\nu}_{A^\nu, V} \right]_{A^\nu, A^\nu}^{A^\nu} + \left[ G, [E, F]^{A^\nu}_{A^\nu, V} \right]_{A^\nu, A^\nu}^{A^\nu} + \left[ F, [G, E]^{A^\nu}_{A^\nu, V} \right]_{A^\nu, A^\nu}^{A^\nu} \\
+ O(\partial^2).
\] (51)

As in the previous cases, \(O(\partial^2)\) terms cancels. Hence, by lemma 4.1, we obtain Jacobi’s identity.

5. Conclusions and remarks

We have formulated the Hamiltonian and Poisson bracket for the extended MHD which subsumes the Hall MHD and the inertial MHD systems. In proving Jacobi’s identity, we have unearthed an underlying algebraic relation that is represented by a generating bracket (18) satisfying an extended permutation law.

The formulated Poisson algebra has a nontrivial center (i.e., the Hamiltonian system is concannonical). The noncanonicity is a common feature of fluid/plasma systems represented by Eulerian variables. The metamorphoses of Casimir leaves, in response to the singular perturbations scaled by ion and electron skin depths.

The Poisson bracket of the extended MHD system has three independent Casimir elements:
\[ C_1 = \int_{\Omega} \rho \, d^3x, \] (52)

\[ C_2 = \int_{\Omega} B^\ast \cdot \left( V - \frac{d_i}{2d_e} A^\ast \right) d^3x, \] (53)

\[ C_3 = \int_{\Omega} \left[ B^\ast \cdot A^\ast + d_e^2 V \cdot (V \times V) \right] d^3x. \] (54)

Combining \( C_2 \) and \( C_3 \), we may define a ‘canonical helicity’

\[ C_{2,3} = 2\lambda C_2 + \frac{1}{d_e} C_3 \]

\[ = \int_{\Omega} P^\ast \cdot \left( V \times P^\ast \right) d^3x, \] (55)

where \( P^\ast = V + \lambda A^\ast \) and \( \lambda = \frac{-d_i \pm \sqrt{d_i^2 + 4d_e^2}}{2d_e^2} \).

The inertial MHD system also has three independent Casimir elements:

\[ C_1' = \int_{\Omega} \rho \, d^3x, \] (56)

\[ C_2' = \int_{\Omega} V \cdot B^\ast d^3x, \] (57)

\[ C_3' = \int_{\Omega} \left[ B^\ast \cdot A^\ast + d_e^2 V \cdot (V \times V) \right] d^3x. \] (58)

Combining \( C_2' \) and \( C_3' \), we may define a canonical helicity

\[ C_{2,3}' = \frac{2}{d_e} C_2' + \frac{1}{d_e} C_3' \]

\[ = \int_{\Omega} P^\ast \cdot \left( V \times P^\ast \right) d^3x, \] (59)

where \( P^\ast = V \pm \frac{1}{d_e} A^\ast \). The transition from the generalized MHD to the inertial MHD is, therefore, a smooth limit of \( d_i \rightarrow 0 \). However, to take the limit of \( d_i \rightarrow 0 \) (i.e., generalized MHD \( \rightarrow \) Hall MHD), and the limit of \( d_i \rightarrow 0 \) under \( d_e = 0 \) (i.e., Hall MHD \( \rightarrow \) ideal MHD; see [10]) are not that simple. We have to multiply \( C_2 \) by \( d_e^2 \) to avoid singularity. However, we cannot reproduce the cross helicity \( C_e^\ast = \int_{\Omega} V \cdot B \, d^3x \) of ideal MHD. The only path from \( C_2 \) to \( C_e^\ast \) is first \( d_i \rightarrow 0 \) and then \( d_e \rightarrow 0 \) (which is, of course, unphysical). Those limits are singular because the order of derivatives included in the Casimir invariants drops. These singular perturbations will be discussed elsewhere.

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References

[1] Morrison P J 1998 Hamiltonian description of the ideal fluid Rev. Mod. Phys. 70 467–521
[2] Yoshida Z and Morrison P J 2014 A hierarchy of noncanonical Hamiltonian systems: circulation laws in an extended phase space Fluid Dyn. Res. 46 031412 1–21
[3] Morrison P J and Greene J M 1980 Noncanonical Hamiltonian density formulation of hydrodynamics and ideal magnetohydrodynamics Phys. Rev. Lett. 45 790–4
[4] Morrison P J and Greene J M 1982 Noncanonical Hamiltonian density formulation of hydrodynamics and ideal magnetohydrodynamics Phys. Rev. Lett. 48 7596
[5] Morrison P J 1982 Poisson Brackets for fluids and plasmas Mathematical Methods in Hydrodynamics and Integrability in Dynamical Systems (American Institute of Physics Conf. Proc. No. 88) ed M Tabor and Y Treve (New York: American Institute of Physics) pp 13–46
[6] Hameiri E 2004 The complete set of Casimir constants of the motion in magnetohydrodynamics Phys. Plasmas 11 3423–31
[7] Holm D D 1987 Hall magnetohydrodynamics: conservation laws and Lyapunov stability Phys. Fluids 30 1310
[8] Sahraoui F, Belmont G and Rezeau L 2003 Hamiltonian canonical formulation of Hall-magnetohydrodynamics: toward an application to weak turbulence theory Phys. Plasmas 10 1325
[9] Kawazura Y and Hameiri E 2012 The complete set of Casimirs in Hall magnetohydrodynamics Phys. Plasmas 19 082513
[10] Yoshida Z and Hameiri E 2013 Canonical Hamiltonian mechanics of Hall magnetohydrodynamics and its limit to ideal magnetohydrodynamics J. Phys. A: Math. Theor. 46 335502
[11] Schep T J, Pegoraro F and Kuvshinov B N 1994 Generalized two-fluid theory of nonlinear magnetic structures Phys. Plasmas 1 2843
[12] Cafaro E, Grasso D, Pegoraro F, Porcelli F and Saluzzi A 1998 Invariants and geometric structures in nonlinear Hamiltonian magnetic reconnection Phys. Rev. Lett. 80 4430
[13] Tassi E, Morrison P J, Waelbroeck F L and Grasso D 2008 Hamiltonian formulation and analysis of a collisionless fluid reconnection model Plasma Phys. Control. Fusion 50 085014
[14] Tassi E, Grasso D, Pegoraro F and Morrison P J 2009 Stability and nonlinear dynamics aspects of a model for collisionless magnetic reconnection J. Plasma Fusion Res. 8 159–64
[15] Tassi E, Morrison P J, Grasso D and Pegoraro F 2010 Hamiltonian four-field model for magnetic reconnection: nonlinear dynamics and extension to three dimensions with externally applied fields Nucl. Fusion 50 034007
[16] Keramidas Charidakos I, Lingam M, Morrison P J, White R L and Wurm A 2014 Action principles for extended MHD models Phys. Plasmas 21 092118
[17] Lingam M, Morrison P J and Tassi E 2015 Inertial magnetohydrodynamics Phys. Lett. A 379 570–6
[18] Yoshida Z and Mahajan S M 2012 Duality of the Lagrangian and Eulerian representations of collective motion—a connection built around vorticity Plasma Phys. Control. Fusion 54 014003 1–9
[19] Kimmura K and Morrison P J 2014 On energy conservation of extended magnetohydrodynamics Phys. Plasmas 21 082101
[20] Sauer K, Dubinin E and McKenzie J F 2002 Wave emission by whistler oscillitons: Application to ‘coherent lion roars’ Geophys. Res. Lett. 29 2226
[21] Dubinin E, Sauer K and McKenzie J F 2003 Nonlinear stationary whistler waves and whistler solitons (oscillitons). Exact solutions J. Plasma Phys. 69 305–30
[22] McKenzie J F, Dubinin E, Sauer K and Doyle T B 2004 The application of the constants of motion to nonlinear stationary waves in complex plasmas: a unified fluid dynamic viewpoint J. Plasma Phys. 70 431–62
[23] Webb G M, Ko C M, Mace R L, McKenzie J F and Zank G P 2008 Integrable, oblique travelling waves in quasi-charge-neutral two-fluid plasmas Nonlinear Process. Geophys. 15 179–208
[24] Verheest F, Cattaert T, Dubinin E, Sauer K and McKenzie J F 2004 Whistler oscillitons revisited: the role of charge neutrality? Nonlinear Process. Geophys. 11 447–52
[25] Marsden J E, Ratnai T and Weinstein A 1984 Semidirect products and reduction in mechanics Trans. Am. Math. Soc. 281 147–77
[26] Marsden J E and Weinstein A 1982 The Hamiltonian structure of the Maxwell-Vlasov equations Physica D 4 394–406
[27] Marsden J E and Weinstein A 1983 Coadjoint orbits, vortices, and Clebsch variables for incompressible fluids *Physica D* **7** 305–23

[28] Holm D D and Kupershmidt B A 1983 Poisson brackets and Clebsch representations for magnetohydrodynamics, multifluid plasmas, and elasticity *Physica D* **6** 347–63

[29] Spencer R G and Kaufman A N 1982 Hamiltonian structure of two-fluid plasma dynamics *Phys. Rev. A* **25** 2437