BIFURCATIONS IN A CLASS OF POLYCYCLES INVOLVING TWO SADDLE-NODES ON A MÖBIUS BAND

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Abstract. In this paper we study the bifurcations of a class of polycycles, called lips, occurring in generic three-parameter smooth families of vector fields on a M"obius band. The lips consists of a set of polycycles formed by two saddle-nodes, one attracting and the other repelling, connected by the hyperbolic separatrices of the saddle-nodes and by orbits interior to both nodal sectors. We determine, under certain genericity hypotheses, the maximum number of limit cycles that may bifurcate from a graphic belonging to the lips and we describe its bifurcation diagram.

1. Introduction

Let $M^2$ be a smooth 2-dimensional manifold. Let $X_\mu : M^2 \times \Lambda \to TM^2$ be a smooth family of vector fields on $M^2$, depending on $p$-parameters, represented by $\mu \in \Lambda$, where $\Lambda$ is a neighborhood of the origin in $\mathbb{R}^p$.

An oriented polycycle of a vector field, $X_0$, in a 2-dimensional manifold is a cyclically ordered union of singular points, called vertices $A_0$, $A_1$, \ldots, $A_n = A_0$ (some of them may coincide) and different orbits $\gamma_0, \ldots, \gamma_{n-1}$ endowed with their natural orientation, such that for any $i = 0, \ldots, n-1$ the orbit $\gamma_i = \gamma_i(t)$ tends to the point $A_i$ as $t \to -\infty$ and tends to $A_{i+1}$ as $t \to +\infty$. These orbits are called connections, or arcs of the polycycle.

A saddle-node of multiplicity 2 of a vector field $X_0$ in $M^2$ is a singular point $p_0$ of $X_0$ at which its linearization has only one zero eigenvalue and the restriction of $X_0$ to a center manifold has the form

$$(ax^2 + \cdots) \frac{\partial}{\partial x}, \ a \neq 0, \ x \in \mathbb{R},$$
and the dots denote higher order terms. If the non-zero real eigenvalue is positive, the saddle-node is called repelling and, if it is negative, the saddle-node is called attracting.

In the first (resp. second) case the basin of repulsion (resp. attraction) of the saddle-node, called the nodal sector, is a half-plane bordered by the strong unstable (resp. strong stable) separatrices and the singularity. Also the basin of attraction (resp. repulsion) is a curve coincident to half the central manifold, called the hyperbolic separatrix, since it is the common boundary of the hyperbolic sectors of the saddle-node.

We call lips the set of polycycles which consists of two saddle-nodes (one repelling and one attracting) connected by the hyperbolic separatrices and by orbits interior to both nodal sectors (see Figure 1).

**Figure 1.** Lips on a Möbius band.
It is known that limit cycles are born, i.e., bifurcate, from limit periodic sets that are closed invariant subsets of the plane, eventually containing arcs of nonisolated singularities of vector fields. But since we study only generic families, with all singularities isolated, such limit periodic sets can be only polycycles. The question of the number of limit cycles which can bifurcate from a polycycle occurring in a generic finite-parameter family of vector fields, is closely related to the Hilbert 16th Problem (see [9] for more details).

In recent years several authors have investigated the bifurcation of limit cycles from a set of polycycles. See, for example, [4], [6], [7], [8] and references therein.

This work focuses on the bifurcation diagram of the lips in the case where the normal bundle of the polycycles is non-orientable, i.e., diffeomorphic to a Möbius band. The orientable case was studied in [8]. The bifurcation diagram is a stratification of the parameter space such that different strata correspond topologically distinct phase portraits of the vector field.

The set of polycycles of lips type occur persistently in three-parameter families of vector fields in $\mathcal{M}^2$, because we need two parameters to unfold the saddle-nodes and one parameter to bifurcate the connection of separatrices of the hyperbolic sectors of the saddle-nodes. The main tools used to describe the bifurcation diagram of the lips is a normal form in a neighborhood of a saddle-node given by Theorem 1 (see Section 2). This normal form simplifies the problem. In fact, for the case where there are singular points in the unfolding of a saddle-node, it is trivial. See Theorem 2. Therefore the case of interest corresponds to the situation where there are not singular points, i.e., to the bifurcation of limit cycles.

If $X_\nu(x,y)$, $\nu = (\epsilon, \delta, \lambda) \in \mathbb{R}^3$, is a smooth 3-parameter family of vector fields on $\mathcal{M}^2$ such that for $\nu = 0$ $X_0$ has a set of polycycles of lips type, then for some values of $\nu$ we can define transition maps on the transversal sections $\Gamma_{1,2}^\pm$ (see Figure 1). We compose these maps to obtain the Poincaré map $\Delta_\nu : \Gamma_1^+ \rightarrow \Gamma_1^+$. Thus, the study of the bifurcations of limit cycles from the lips is reduced to the investigation of the bifurcations of periodic points of 3-parameter families of real maps. In fact, when studying the bifurcations of limit cycles in a Möbius band, we deal with bifurcations of fixed points and points of period 2 of the Poincaré map and their variations in function of the parameters. Isolated fixed points (resp. 2-periodic points) are in one-to-one correspondence with limit cycles that appear after perturbation of the lips and intersect $\Gamma_1^+$ only once (resp. twice). Moreover, simple
fixed and periodic points correspond to hyperbolic cycles, double fixed points correspond to semistable cycles, etc.

There are two possible types of bifurcations:

1. splitting of multiple fixed and periodic points, and
2. escaping of fixed and periodic points through the boundary points of $\Gamma_1^+$.

Therefore the bifurcation surface of the Poincaré map, i.e. the surface in the parameter space where the number of periodic and fixed points change, is the union of four surfaces, $\Sigma_1$, $\Sigma_2$, and $\Sigma_\pm$. On $\Sigma_1$ we have multiple fixed points, and on $\Sigma_2$, we have multiple 2-periodic points. Now, on $\Sigma_+$ (resp. $\Sigma_-$) there is at least one periodic point equal to the positive extreme of $\Gamma_1^+$ (resp. the negative extreme of $\Gamma_1^+$). In terms of bifurcations of the original system, the union of the surfaces $\Sigma_1$ and $\Sigma_2$ corresponds to the splitting of multiple limit cycles, while the union $\Sigma_+ \cup \Sigma_-$ corresponds to cycles escaping from the domain where the system is considered.

We will see in Section 8 that to determine the periodic points of the Poincaré map is equivalent to determine the roots of an equation

\[
\varphi_\delta(y, p, q) = 0,
\]

where the parameters $(\delta, p, q)$ are introduced by the blow-up $(\epsilon, \delta, \lambda) \mapsto \Phi(\epsilon, \delta, \lambda) = (\delta, p, q)$ defined in Section 6. This blow-up takes the point $\nu = 0$ into the half plane $\mathbb{R}_3^{(\delta, p, q)} \cap \{p > 0, \ \delta = 0\}$. The function in equation (1) is essentially $\Delta_\nu - y$, in the blown-up coordinates.

For each fixed $\delta$ we have a surface $S_\delta \in \mathbb{R}_3^{y, p, q}$ determined by equation (1). As we are interested in the multiple roots of equation (1), we must characterize the projection in the $(p, q)$-plane of the curve $C_\delta$ given by equations

\[
\varphi_\delta(y, p, q) = 0, \quad \frac{\partial \varphi_\delta}{\partial y}(y, p, q) = 0.
\]

The characterization of this curve is given in Sections 7 and 8. In the parameter space $\mathbb{R}_3^{(\delta, p, q)}$, the surface $\Sigma = \Sigma_1 \cup \Sigma_2$ is described by the property that its intersection with the plane $\delta =$constant is equal to the trace of the projection of the curve $C_\delta$ in the $(p, q)$-plane. The main result of this work, Theorem 15 of Section 9, formalizes this discussion and shows that $\Sigma = \Sigma_1 \cup \Sigma_2$ in $\mathbb{R}_3^{(\delta, p, q)}$ is diffeomorphic to the cylinder over $\Lambda_0$ the projection of $C_0$ in the $(p, q)$-plane with axis parallel to $\delta$ axis (see Figure 3). Analogously, we characterize the surfaces $\Sigma_\pm$, see Theorem 15.
The results in this paper can be regarded as an extension to a Möbius band of the work of Kotova and Stanzo [8] carried out for the orientable case, that is when the normal bundle of the polycycles are diffeomorphic to a cylinder. The main difference between the two cases is that in the nonorientable one we have to study also the 2-periodic points of the Poincaré map as well as the flip fixed points (i.e. with negative derivative) corresponding to the one-sided periodic orbits. The nonorientable case has the additional complication of presenting the flip bifurcations codimensions 1 and 2 (see [2] and [5]).

This paper is organized as follows. Section 2 is devoted to review some preliminary results pertinent to the saddle-nodes and their normal forms. Section 3 describes the bifurcation diagram when there is at least one singularity in the unfolding of the lips. In Section 4 is studied a change of coordinates which greatly simplifies the expression of the return map. Section 6 is devoted to the definition of the blow-up \( \Phi \) in the parameters, opening the origin to the \((p, q)\)-plane. In Sections 7 and 8 are characterized the bifurcation surfaces \( \Sigma_1 \) and \( \Sigma_2 \). The main theorem of this paper, presenting a synthesis of the bifurcation diagram is proved in Section 9 (Theorem 15). Section 10 studies the cyclicity of an individual polycycle of the lips (Theorem 17).

2. Preliminaries

In this section we present the tools to study the bifurcation diagram of the lips.

Let \( X_\mu(x, y), \mu = (\mu_1, \mu_2, \mu_3) \in \mathbb{R}^3 \) be a smooth 3-parameter family of vector fields on \( M^2 \) such that for \( \mu = 0 \), \( X_0 \) has a set of polycycles of lips type.

We denote by \( O_1 \) and \( O_2 \) the saddle-nodes of \( X_0(x, y) \). In what follows we assume that the orientation is chosen as in Figure 1. As we can see between the nodal sectors of saddle-nodes \( O_1 \) and \( O_2 \) form a region filled out by arcs of the lips. On the topological boundary of this region, other types of separatrix connections involving singular points outside the lips may occur. In this paper, however, we do not investigate effects here caused by this circumstance at the boundary; instead we choose a transversal section intersecting certain connections and consider their saturation by the flow curves, assuming that all of them are also connections of the lips. When the parameters of the family change, we consider only the orbits intersecting the same transversal section, which is assumed to be independent of the parameters. This restriction produces the bifurcations when a limit cycle leaves the domain under consideration.
To describe the bifurcation diagram of the lips we need the following result that is proved in \[7\].

**Theorem 1.** Let \(X_\mu, \mu \in \mathbb{R}^3\), be a generic 3-parameters smooth family of vector fields in a two manifold \(M^2\), such that \(X_0\) has a saddle-node of multiplicity 2 in the origin \((0,0)\). Then the family \(X_\mu\) may be reduced by a finitely smooth change of coordinates, time rescaling and parameter change to the following normal forms

\[
\begin{align*}
\dot{x} &= (x^2 + \epsilon)(1 + a(\mu) x)^{-1}, \quad \epsilon = \epsilon(\mu) \\
\dot{y} &= \pm y.
\end{align*}
\]

Let us give the explicit genericity assumption for the family in the previous theorem. The family \(X_\mu\) intersects transversally at the point \(X_0\) the surface of vector fields having degenerate singular points. This means that if we consider the center manifold of the local family \(X_\mu\) and write the restriction of the family to this manifold as

\[
\begin{align*}
\dot{x} &= f(x, \mu), \quad \mu = 0, \quad (x, \mu) \in \mathbb{R} \times \mathbb{R}^3, \quad f(0,0) = 0,
\end{align*}
\]

then

\[
\left. \frac{\partial f(0, \mu)}{\partial \mu} \right|_{\mu=0} \neq 0.
\]

The smoothness of the normalizing chart in the phase variables and the parameters in the previous theorem is arbitrarily high, but finite; the smoothness may be increased after shrinking the domain of the normalizing chart.

In order to investigate the bifurcations of the lips, we introduce the system of parameters \(\nu = (\epsilon, \delta, \lambda)\). We will suppose that the hyperbolic eigenvalues of saddle-nodes \(O_1\) and \(O_2\) are real and have opposite signs. If \(\mu = (\mu_1, \mu_2, \mu_3)\) are the original parameters and the lips occur for \(\mu = 0\), then, according to Theorem \[1\] in a neighborhood of the saddle-node \(O_1\) there exist local coordinates in which the family of vector fields has the form

\[
\begin{align*}
\dot{x} &= (x^2 + \epsilon)(1 + a_1(\mu) x)^{-1}, \\
\dot{y} &= y, \quad \epsilon = \epsilon(\mu).
\end{align*}
\]

Let \(\Gamma^\pm_1\) be two transversal sections to the flow that are given in the canonical chart (i.e where expression \[3\] holds). Without loss of generality we can suppose that

\[
\Gamma^\pm_1 = \{(x, y) : x = \pm 1, |y| \leq 1\}.
\]

The transversal section \(\Gamma^-\) is the entrance gate: all orbits crossing it enter the neighborhood of the singularity \(O_1\). The other section is the exit gate in the same sense. Clearly, for \(\epsilon > 0\) the derivative \(\dot{x}\)
is positive; therefore each orbit starting on $\Gamma_1^-$ will intersect $\Gamma_1^+$ at a certain point; we denote this transition map by

$$\Delta_{\nu,1} : \Gamma_1^- \to \Gamma_1^+.$$  

In the same manner, there exist a normalizing chart around $O_2$, in which the family has the form

$$\begin{align*}
\dot x &= (x^2 + \delta)(1 + a_2(\mu)x)^{-1}, \\
\dot y &= -y, \quad \delta = \delta(\mu).
\end{align*}$$

We take the transversal sections

$$\Gamma_{2}^\pm = \{(x,y) : x = \pm 1, |y| \leq 1\},$$

and similarly as in the previous case we obtain the transition map

$$\Delta_{\nu,2} : \Gamma_2^- \to \Gamma_2^+.$$  

Note that this map is defined only for $\delta > 0$, although the transversal sections $\Gamma_{2}^\pm$ are well defined for all small $\delta$.

Besides the two maps $\Delta_{\nu,1}, \Delta_{\nu,2}$, there are two regular transition maps along connections depending on parameters. In the normalizing charts they can be written as

$$\begin{align*}
\Delta_{\nu,1} : \Gamma_1^- \to \Gamma_2^+, \\
\Delta_{\nu,2} : \Gamma_2^- \to \Gamma_1^+.
\end{align*}$$

Since for $\mu = 0$ the points $O_i$ are connected along the hyperbolic separatrices, i.e. $g_0(0) = 0$ and $g'_\nu(0) < 0$ (orientation reversing). Denote by $\lambda$ the relative displacement of the separatrices (see Figure 1),

$$\lambda = g_\nu(0).$$

In order to proceed further, we need an additional genericity assumptions. We will require, from now on, that the Jacobian of the map $(\mu_1, \mu_2, \mu_3) \mapsto (\epsilon, \delta, \lambda)$ to be nonvanishing,

$$\det \left( \frac{\partial \nu}{\partial \mu} \right)_{\mu=0} \neq 0.$$  

If this condition is satisfied, we will describe the bifurcation diagram for the lips in terms of the new parameters $\nu$ rather than $\mu$: the above genericity assumption guarantees that the bifurcation diagram in the original parameter space will be diffeomorphic to the one obtained in Theorem 15.

Now we will determine expressions for the maps $\Delta_{\nu,1}, \Delta_{\nu,2}$ defined above. In fact, by explicit integration of the normal forms, which has separated variables, it follows that

$$\Delta_{\nu,1}(y) = C_1(\epsilon)^{-1}y \text{ and } \Delta_{\nu,2} = C_2(\delta)y.$$
with $C_1(\epsilon) \to 0$ when $\epsilon \to 0$ and $C_2(\delta) \to 0$ when $\delta \to 0$. More precisely,

$$(7) \quad C_1(\epsilon) = \exp\left(-\frac{2}{\sqrt{\epsilon}} \arctan \frac{1}{\sqrt{\epsilon}}\right) \text{ and } C_2(\delta) = \exp\left(-\frac{2}{\sqrt{\delta}} \arctan \frac{1}{\sqrt{\delta}}\right).$$

3. Description of the bifurcation diagram outside the positive quadrant $\epsilon > 0, \delta > 0$

The complete description of the bifurcation diagram in the domain of parameters where there is at least one singular point of the vector field, is given by the following theorem.

**Theorem 2.** The bifurcation diagram in the intersection of a small neighborhood of the origin in the space $\mathbb{R}^3(\epsilon, \delta, \lambda)$ with the set $\{\epsilon \leq 0\} \cup \{\delta \leq 0\}$, consists of twelve components, corresponding to topologically nonequivalent phase portraits differing by the type of singular points and the existence of connections between them:

1. $\epsilon = \delta = \lambda = 0$, two saddle-nodes connected by a separatrix,
2. $\epsilon = \delta = 0, \lambda \neq 0$, two saddle-nodes without connection,
3. $\epsilon < 0, \delta < 0, \lambda = 0$, two saddles connected by a separatrix, one stable and one unstable node,
4. $\epsilon < 0, \delta < 0, \lambda \neq 0$, two saddles without connection, a stable and an unstable node,
5. $\epsilon < 0, \delta = 0, \lambda = 0$, saddle and saddle-node connected by a separatrix, and also a stable node,
6. $\epsilon < 0, \delta = 0, \lambda \neq 0$, saddle, saddle-node and a stable node without connection,
7. $\epsilon < 0, \delta > 0$, a saddle and a stable node,
8. $\epsilon = 0, \delta < 0, \lambda = 0$, a saddle and a saddle-node connected by a separatrix, and also an unstable node,
9. $\epsilon = 0, \delta < 0, \lambda \neq 0$, a saddle, a saddle-node and an unstable node without connections,
10. $\epsilon = 0, \delta > 0$ or $\epsilon > 0, \delta = 0$, a saddle-node,
11. $\epsilon > 0, \delta < 0$, saddle and an unstable node.

The proof immediately follows from the local normal forms near the singularities $O_{1,2}$.

4. Admissible change of coordinates

A $k$-admissible change of coordinates is a local family of $C^k$-diffeomorphisms

$$\Phi_\mu(x, y) = (\xi_\mu(x, y), \zeta_\mu(x, y))$$

such that
(1) $\Phi_\mu$ is defined in a connected neighborhood $U \subset \mathbb{R}^2$ of the origin;

(2) $\Phi_\mu$ preserve the axes;

(3) $\Phi_\mu$ preserve the normal form obtained in Theorem 1;

(4) the map $(x, y, \mu) \mapsto \Phi_\mu(x, y)$ is of class $C^k$.

Let $\Phi_{\nu, 1}$ be a $k$-admissible change of coordinates defined in a neighborhood of the saddle-node $O_1$. Assume that the transversal sections $\Gamma_1^\pm$ belong both to the domain and to the image of $\Phi_{\nu, 1}$. We have that $\Phi_{\nu, 1}$ generates two local $C^k$ diffeomorphisms, $\varphi_{\nu, 1}(y), \psi_{\nu, 1}(y)$, such that if $\Delta_{\nu, 1}$ is the transition map from $\Gamma_1^-$ to $\Gamma_1^+$ in the coordinates $(\xi_{\nu, 1}, \zeta_{\nu, 1})$, which are defined for $\epsilon > 0$, then

$$\Delta_{\nu, 1} \circ \varphi_{\nu, 1} = \psi_{\nu, 1} \circ \Delta_{\nu, 1},$$

where $\Delta_{\nu, 1}$ is the transition map from $\Gamma_1^-$ to $\Gamma_1^+$ in the coordinates $(x, y)$. More precisely, $\varphi_{\nu, 1}$ represents the transition function from $\Gamma_1^- = \{ (x, y) : x = -1, |y| \leq 1 \}$ to the section $\Gamma_1^- = \{ (\xi_{\nu, 1} , \zeta_{\nu, 1}) : \xi_{\nu, 1} = -1, |\zeta_{\nu, 1}| \leq 1 \}$.

In the same way, $\psi_{\nu, 1}$ represents the transition function from $\Gamma_1^+ = \{ (x, y) : x = 1, |y| \leq 1 \}$ to the section $\Gamma_1^+ = \{ (\xi_{\nu, 1} , \zeta_{\nu, 1}) : \xi_{\nu, 1} = 1, |\zeta_{\nu, 1}| \leq 1 \}$.

We call the $C^k$ local family of diffeomorphism $\varphi_{\nu, 1}$, obtained above, the entrance family associated to the saddle-node $O_1$ and we call exit family associated to the saddle-node $O_1$, the other $C^k$ local family of diffeomorphisms $\psi_{\nu, 1}$, obtained above.

In the same way, we denote by $\varphi_{\nu, 2}$ the entrance family associated to the saddle-node $O_2$ and by $\psi_{\nu, 2}$ the exit family associated to the saddle-node $O_2$, defined by a $k$-admissible change of coordinates $\Phi_{\nu, 2}$ in a neighborhood of saddle-node $O_2$.

**Proposition 3.** Let $\varphi_{\nu, 1}, \psi_{\nu, 2}$, be $C^k$ local increasing families of diffeomorphisms such that $\varphi_{\nu, 1}(0) = \psi_{\nu, 2}(0) = 0$. Then there exist $k$-admissible changes of coordinates $\Phi_{\nu, i}, i = 1, 2$, such that the entrance family associated to saddle-node $O_1$ and the exit family associated to saddle-node $O_2$ are $\varphi_{\nu, 1}$, and $\psi_{\nu, 2}$, respectively.

The proof of this proposition can be found in [6] page 53.

**Theorem 4.** There exist $k$-admissible changes of coordinates, such that in these coordinates the map $g_\nu$ can be written in the form

$$g_\nu(y) = -y + \lambda.$$

**Proof.** Let $\varphi_{\nu, 1}$ and $\psi_{\nu, 2}$ be families of diffeomorphisms satisfying the hypothesis of Proposition 3. Then, there exist $k$-admissible changes of coordinates $\Phi_{\nu, 1}(x, y) = (\xi_{\nu, 1}, \zeta_{\nu, 1})$ and $\Phi_{\nu, 2}(x, y) = (\xi_{\nu, 2}, \zeta_{\nu, 2})$ such that $\varphi_{\nu, 1}$ is the entrance family associated to saddle-node $O_1$ and $\psi_{\nu, 2}$ is the
exit family associated to saddle-node $O_2$. In admissible coordinates the transition map $g_{\nu} : \Gamma_2^+ \to \Gamma_1^-$, defined in (5), is given by

$$\tilde{g}_{\nu} = \varphi_{\nu,1} \circ g_{\nu} \circ (\psi_{\nu,2})^{-1},$$

i.e.,

$$\tilde{g}_{\nu} \circ \psi_{\nu,2} = \varphi_{\nu,1} \circ g_{\nu}.$$

We want to obtain that $\tilde{g}_{\nu}(\zeta_{\nu,2}) = -\zeta_{\nu,2} + \lambda$, where $g_{\nu}(0) = \lambda$. Therefore, by the expression above, we have to choose $\varphi_{\nu,1}$ and $\psi_{\nu,2}$ such that the equation

$$-\psi_{\nu,2}(y) + g_{\nu}(0) = \varphi_{\nu,1}(g_{\nu}(y))$$

has a solution. Hence, it is sufficient to take $\varphi_{\nu,1}(y) = y$ and $\psi_{\nu,2}(y) = -g_{\nu}(y) + g_{\nu}(0)$. □

5. Limit cycles bifurcating from the lips

In the non-trivial part of the bifurcation diagram, where there are no singular points, different strata of the diagram correspond to different numbers and position of limit cycles. From this point on, it is helpful to have in mind Figure 1 and the notation in Section 2.

The Poincaré map $\Delta_{\nu} : \Gamma_1^+ \to \Gamma_1^+$ for each $\nu$ is the composition,

$$\Delta_{\nu} = f_{\nu}^{-1} \circ \Delta_{\nu,2}^{-1} \circ g_{\nu}^{-1} \circ \Delta_{\nu,1}^{-1},$$

considered as the composition of one-dimensional maps depending on the parameters. Note that in the Möbius band the limit cycles that bifurcate from the lips correspond to the fixed points of periods 1 and 2 of $\Delta_{\nu}$. Therefore, the equations that determine limit cycles are

$$\Delta_{\nu}(y) = y \quad \text{and} \quad \Delta_{\nu}^2(y) = y.$$

In fact, the solutions of $\Delta_{\nu}(y) = y$ are contained in the set of solutions of $\Delta_{\nu}^2(y) = y$. The equations of first and second return fixed points can be rewritten in the form

$$\begin{align*}
\Delta_{\nu,2}^{-1} \circ g_{\nu}^{-1} \circ \Delta_{\nu,1}^{-1}(y) &= f_{\nu}(y), \\
f_{\nu}^{-1} \circ \Delta_{\nu,2}^{-1} \circ g_{\nu}^{-1} \circ \Delta_{\nu,1}^{-1}(y) &= \Delta_{\nu,1} \circ g_{\nu} \circ \Delta_{\nu,2} \circ f_{\nu}(y).
\end{align*}$$

Introducing the new parameters

$$p = \frac{C_1(\epsilon)}{C_2(\delta)} \quad \text{and} \quad q = \frac{\lambda}{C_2(\delta)},$$

by expressions (6), (7) and (9), (10) becomes

$$\begin{align*}
-\frac{1}{p} f(y) + \frac{q}{p} &= f_1(y, \nu), \\
-\frac{1}{p} f(y) + \frac{q}{p} &= f_2(y, \nu),
\end{align*}$$

(11) \quad (12)
where
\[ f(y) = f_0(y), \quad r_1(y, \nu) = f_\nu(y) - f_0(y) \]
and
\[ r_2(y, \nu) = f_\nu \left( -\frac{1}{p} f_\nu(y) + \frac{q}{p} \right) - f_0 \left( -\frac{1}{p} f_0(y) + \frac{q}{p} \right). \]

Note that \( r_1(y, \nu) \to 0 \) and \( r_2(y, \nu) \to 0 \) in \( C^k \)-norm on \([-1, 1]\) when \( \nu \to 0 \), since \( f_\nu \) is \( C^k \)-smooth.

As in the previous section, the transversal section \( \Gamma_1^+ \) is simply the interval \([-1, 1]\). Thus we need to investigate the equations (11) and (12) for \( y \in [-1, 1] \). When studying the bifurcations of limit cycles, we in fact deal with bifurcations of roots of (12) on \([-1, 1]\), and their variations in function of the parameters. Isolated roots of equation (12) on the segment \([-1, 1]\) are in one-to-one correspondence with limit cycles intersecting \( \Gamma_1^+ \) that appear after perturbation of the lips. Moreover, simple roots correspond to hyperbolic cycles, double roots correspond to semistable cycles, etc.

There are two possible types of bifurcations:

1. splitting of multiple roots, and
2. escaping of a root through the boundary points of \( \Gamma_1^+ \).

Therefore the bifurcation surface of equation (12), i.e. the surface in the parameter space where the number of roots change, is the union of four surfaces, \( \Sigma_1, \Sigma_2 \) and \( \Sigma_{\pm} \). On \( \Sigma_1 \) we have roots of (11) that are multiple roots of (12) and on \( \Sigma_2 \) we have multiple roots of (12). Now, on \( \Sigma_+ \) (resp. \( \Sigma_- \)) there is at least one root equal to 1 (resp. -1).

In terms of bifurcations in the original system, the surface \( \Sigma_1 \cup \Sigma_2 \) corresponds to the splitting of a multiple limit cycles, while the union \( \Sigma_+ \cup \Sigma_- \) correspond to cycles escaping from the domain where the system is considered.

Fix \( P > \max_{y \in [-1, 1]} f'(y) \) and consider the subset in \( R^3_\nu \) for which \( 0 < p < P \). In this domain the equations (11) and (12) can be regarded as small perturbations of the equations

\[ -py + q = f(y), \tag{13} \]
\[ -py + q = f \left( -\frac{1}{p} f(y) + \frac{q}{p} \right). \tag{14} \]

6. The second reparametrization

Let \( V \subset R^3_\nu \) be a small neighborhood of the origin and, denote by \( V^+ \) the intersection \( V \cap \{ \epsilon > 0, \delta > 0 \} \). Without loss of generality we may assume that \( V^+ \) is a small cube with the edges parallel to the coordinate axes.
Consider the reparametrization map $\Phi : V^+ \to \mathbb{R}^{3+}_{(\delta,p,q)} = \mathbb{R}^3_{(\delta,p,q)} \cap \{p > 0\}$ defined by the formula

$$(\delta, \epsilon, \lambda) \mapsto \Phi(\delta, \epsilon, \lambda) = (\delta, p, q) = \left(\delta, \frac{C_1(\epsilon)}{C_2(\delta)}, \frac{\lambda}{C_2(\delta)}\right).$$

**Lemma 5.** The map $\Phi$ defined above has the following properties:

(a) The domain $\Phi(V^+)$ is unbounded. The half plane $poq^+ = \mathbb{R}^3_{(\delta,p,q)} \cap \{\delta = 0\}$ belongs to the boundary of $\Phi(V^+)$. \\
(b) The inverse map $\Phi^{-1}$ is defined on $\Phi(V^+)$ and extends by continuity to $poq^+$. The extended map is continuously differentiable in $\delta$ at $\delta = 0$ and takes the half plane $poq^+$ into the point $\nu = 0$. \\
(c) For any compact set $D \subset \mathbb{R}^2_{(p,q)}$ there exist $\delta_D > 0$ such that the cylinder $Z_D = (0, \delta_D) \times D$ belongs to $\Phi(V^+)$. \\
(d) For any point $A \in \mathbb{R}^2_{(p,q)}$ the extended map $\Phi^{-1}$ takes the semi-interval $[0, \delta_A) \times A$ into a part of a curve tangent at $\nu = 0$ to the line $\delta = \epsilon, \lambda = 0$.

The proof of this lemma can be found in [8] page 186.

**Corollary 6.** For any compact set $D \subset \mathbb{R}^2_{(p,q)}$, the map $\Phi^{-1}$ takes the cylinder $Z_D$ into a narrow horn with vertex at $\nu = 0$, tangent to the line $\delta = \epsilon, \lambda = 0$; the projection of this horn to the plane $\delta \epsilon \lambda$ has an opening of the order $\delta^2$, and the projection to the plane $\delta \epsilon \nu$ has an exponentially small opening of the order $\exp\left(-\frac{\pi}{\sqrt{\delta}}\right)$. 

We say that the horn $\Phi^{-1}$ correspond to the compact set $D$.

**Corollary 7.** There exists a compact set $D \subset \mathbb{R}^2_{(p,q)}$ such that the surfaces $\Sigma_1$ and $\Sigma_2$ lie inside the corresponding horn.

**Proof.** The equation (12) has multiple roots if its roots satisfy the equation

$$p^2 = f'(\frac{1}{p} f(y) + \frac{q}{p}) f'(y) + (r_2)'_y(y, \nu).$$

Now, as $f$ is a smooth increasing diffeomorphism, we have that $\min_{y \in [-1,1]} f'(y) > 0$. Hence, fix positive constants

$$P_0 < \min_{y \in [-1,1]} f'(y), \quad P_1 > \max_{y \in [-1,1]} f'(y)$$

and

$$Q > \max_{y \in [-1,1]} f'(y) + \max_{y \in [-1,1]} |f(y)|.$$ 

Choose $D = [P_0, P_1] \times [-Q, Q]$. Then for $(p, q) \notin D$ the equation (12) has only simple roots. Hence, $\Sigma_1 \cup \Sigma_2 \subset \Phi^{-1}(Z_D)$. □
Using Lemma 5 we can perform a new parametrization \((\delta, \epsilon, \lambda) \mapsto (\delta, p, q)\) so that equations (11), (12) become

\begin{align}
- py + q &= f(y) + \tilde{r}_1(y, \delta, p, q), \\
- py + q &= f \left(-\frac{1}{p} f(y) + \frac{q}{p}\right) + \tilde{r}_2(y, \delta, p, q),
\end{align}

where \(\tilde{r}_i(y, \delta, p, q) = r_i(y, \Phi^{-1}(\delta, p, q)), i = 1, 2\).

From Lemma 5 and the properties of \(r_i\) the following properties of \(\tilde{r}_i\) easily follow.

**Lemma 8.** The functions \(\tilde{r}_i\) can be extended continuously to the half plane \(poq^+\), such that \(\tilde{r}_i(x, 0, p, q) = 0\) and for \(\delta > 0\), \(\tilde{r}_i\) is of the class \(C^k\) on \(\Phi(V^+)\) and its extension to the half plane \(poq^+\) is of class \(C^1\). Moreover, when \(\delta \to 0\), any partial derivatives of \(\tilde{r}_i\), of order less than or equal to \(k\), converge uniformly to 0 in \([-1, 1] \times [P_0, P_1] \times [-Q, Q]\).

### 7. Characterization of surface \(\Sigma_1\)

From the previous section, to characterize the surfaces \(\Sigma_1\) and \(\Sigma_2\), we need to study the equations (15) and (16). First, we give the description of surface \(\Sigma_1\) in the space of parameters \((\delta, p, q)\).

The surface \(\Sigma_1\) on the space of parameters \(\mathbb{R}^3_{(\delta, p, q)}\) is the set of points \((\delta, p, q)\) such that to these values of parameters the fixed points of the first return \(\Delta_{\nu}(y)\), or equivalently the roots of (15), are multiple roots of \(\Delta_{\nu}(y) - y\), or equivalently are multiple roots of equation (16). Hence, to determine \(\Sigma_1\) we have to study the following system of equations

\[
\Delta_{\nu}(y) - y = 0, \\
\frac{\partial \Delta_{\nu}}{\partial y}(y) - 1 = 0,
\]

or the equivalently (by the chain rule) the system

\[
\Delta_{\nu}(y) - y = 0, \\
\frac{\partial \Delta_{\nu}}{\partial y}(y) + 1 = 0.
\]

(17)

Note that the roots of \(\Delta_{\nu}(y) - y\) are all simple. Moreover, for each value of the parameters \((\delta, p, q)\) corresponds a unique root of \(\Delta_{\nu}(y) - y\), i.e the Poincaré map has only one fixed point for each value of the parameters \((\delta, p, q)\). In fact, if \(y\) is a multiple root of \(\Delta_{\nu}(y) - y\), i.e. \(\frac{\partial \Delta_{\nu}}{\partial y}(y) = 1\), then by (15) we have that \(f'(y) = -p - (\tilde{r}_1)'_y(y, \delta, p, q)\), but this is a contradiction, because \(f'(y) > 0\), \(p > 0\) and \((\tilde{r}_1)'_y \to 0\), when \(\delta \to 0\).
Now, as $\Delta_\nu(y) = f^{-1}_{\Phi^{-1}(\delta,p,q)}(-py + q)$, it follows from (15) that system (17) becomes

$$\begin{align*}
- py + q &= f(y) + \tilde{r}_1(y, \delta, p, q), \\
\frac{df}{dy}(y) + (\frac{\tilde{r}_1}{y})'_y(y, \delta, p, q) &= p.
\end{align*}$$

Define the function $\phi_\delta$ by

$$\phi_\delta(y, p, q) = -py + q - f(y) - \tilde{r}_1(y, \delta, p, q).$$

For $\delta = 0$, $\tilde{r}_1(y, 0, p, q) = 0$, and as the gradient of $\phi_0$ does not vanish, it follows that $\phi_0 = 0$ is a regular surface in $\mathbb{R}^3_{(y,p,q)}$. Hence, for $\delta$ sufficiently small, $\phi_\delta = 0$ is also a regular surface in $\mathbb{R}^3_{(y,p,q)}$. Therefore, for $\delta$ fix small enough, system (18) determine a curve $C_\delta$ on $\mathbb{R}^3_{(y,p,q)}$ and the projection $(y, p, q) \mapsto (p, q)$ is a curve $L_\delta$ in $\mathbb{R}^2_{(p,q)}$ without singular points and self-intersections, because $(\phi_\delta)'_y(y, p, q) \neq 0$. In fact, for $\delta = 0$, from (18) it follows that

$$L_0(y) = (f'(y), f'(y)y + f(y)).$$

The following result is a straightforward consequence of the previous arguments.

**Theorem 9.** The surface $\Sigma_1$ is a horn in $\mathbb{R}^3_{(\epsilon,\delta,\lambda)}$ defined in the following way. Consider in the half plane $\mathbb{R}^2_{(p,q)} \cap \{p > 0\}$ the trace of curve $C_\delta$ on $\mathbb{R}^3_{(y,p,q)}$ and the embedding of $\mathbb{R}^2_{(p,q)}$ into $\mathbb{R}^2_{(\delta,p,q)}$ as part of the plane $\delta = 0$. Let $Z_1$ be the cylinder in $\Phi(V^+)$ over $L_0$, with the axis parallel to $\delta$ axis of height $\delta_0$. Then for sufficiently small $\delta_0$ the “blown-up horn”

$$Z = \Phi(\Sigma_1 \cap V^+)$$

is diffeomorphic to $Z_1$. The diffeomorphism taking $Z$ into $Z_1$ preserve the foliation $\delta = \text{const}$, is $C^1$-smooth in $\delta$, and its difference from the identity map on the fiber $\delta = \text{const}$ is of the order $O(\delta)$.

8. **Characterization of surface $\Sigma_2$**

Recall that $\Sigma = \Sigma_1 \cup \Sigma_2$ is the bifurcation surface for limit cycles, that is, the surface in the parameter space on which the number of cycles change due to splitting and disappearance of multiple cycles or, what is the same, the surface where the number of roots of equation (16) change.

Consider the function $\psi$ defined by

$$\psi(y, \delta, p, q) = -py + q - f\left(-\frac{1}{p}f(y) + \frac{q}{p}\right) - \tilde{r}_2(y, \delta, p, q).$$
The surface $\Sigma$ in the space of parameters $\mathbb{R}^3_{(\delta, p, q)}$ is the projection on $\mathbb{R}^3_{(\delta, p, q)}$ of the manifold determined by the system

$$\psi(y, \delta, p, q) = 0,$$
$$\psi'_y(y, \delta, p, q) = 0.$$  

Let $\varphi_\delta(y, p, q) = \psi(y, \delta, p, q)$. Note that $\varphi_0(y, p, q) = -py + q - f\left(-\frac{1}{p}f(y) + \frac{q}{p}\right)$. We want to study the apparent contour $\Lambda_0$ of the surface $\varphi_0 = 0$, i.e. the projection on $\mathbb{R}^2_{(p, q)}$ of the curve determined by the system

$$\varphi_0(y, p, q) = 0,$$
$$(\varphi_0)'_y(y, p, q) = 0.$$  

The curve determined by the system above is said to be the horizon of the surface $\varphi_0 = 0$. Note that, the curve $L_0$ defined by (19) belongs the apparent contour of surface $\varphi_0 = 0$. In fact, the curve $(y, f'(y), f'(y)y + f(y))$ is the parametrization of a piece of the horizon of $\varphi_0 = 0$, correspondent to surface $\Sigma_1$ and so $(f'(y), f'(y)y + f(y))$ is a piece of apparent contour of $\varphi_0 = 0$.

**Proposition 10.** For a generic function $f$ the set

$$A = \left\{ y : \frac{\partial^3 \varphi_0}{\partial y^3}(y, f'(y), f'(y)y + f(y)) = 0 \right\}$$

is finite and for all $y \in A$

$$\frac{\partial^3 \varphi_0}{\partial y^3}(y, f'(y), f'(y)y + f(y)) \neq 0.$$  

Moreover, the determinant of the Jacobian matrix of the map $((\varphi_0)'_y(y, p, q), (\varphi_0)^{'''}_{y y y}(y, p, q))$ with respect the parameters $p, q$ is different from zero at points $(y, f'(y), f'(y)y + f(y))$ with $y \in A$.

**Proof.** The proof follows from Thom’s Transversality Theorem. Details are given for the first part; the second one follows analogously.

We have that

$$\varphi_0(y, f'(y), f'(y)y + f(y)) = \frac{\partial \varphi_0}{\partial y}(y, f'(y), f'(y)y + f(y)) = 0,$$
$$\frac{\partial^2 \varphi_0}{\partial y^2}(y, f'(y), f'(y)y + f(y)) = 0,$$
$$\frac{\partial^3 \varphi_0}{\partial y^3}(y, f'(y), f'(y)y + f(y)) = -2f'''(y) + 3\frac{f''(y)^2}{f'(y)},$$
$$\frac{\partial^4 \varphi_0}{\partial y^4}(y, f'(y), f'(y)y + f(y)) = \frac{f''(y)}{f'(y)} \left(-2f'''(y) + 3\frac{f''(y)^2}{f'(y)}\right),$$

It follows from the above that the determinant of the Jacobian matrix of the map $((\varphi_0)'_y(y, p, q), (\varphi_0)^{'''}_{y y y}(y, p, q))$ with respect the parameters $p, q$ is different from zero at points $(y, f'(y), f'(y)y + f(y))$ with $y \in A$.
and
\[
\frac{\partial^5 \varphi_0}{\partial y^5}(y, f'(y), f'(y)y + f(y)) = -2f^{(5)}(y) + \frac{15}{f'(y)}f^{(4)}(y)f''(y) - \frac{5}{f'(y)^2}f'''(y)f''^2(y) - \frac{10}{f'(y)}f''''(y)f''^2(y).
\]
Note that, \(\frac{\partial^3 \varphi_0}{\partial y^3}(y, f'(y), f'(y)y + f(y)) = 0\) implies that
\[
f'''(y) = \frac{3}{2}f''(y),
\]
\(\frac{\partial^4 \varphi_0}{\partial y^4}(y, f'(y), f'(y)y + f(y)) = 0\) and
\[
\frac{\partial^5 \varphi_0}{\partial y^5}(y, f'(y), f'(y)y + f(y)) = -2f^{(5)}(y) + \frac{15f'''(y)}{f'(y)}\left(f^{(4)}(y) - \frac{2f''(y)^3}{f'(y)^2}\right).
\]
Now, consider the 5-jet
\[j^5 f : \mathbb{R} \to J^5(\mathbb{R}, \mathbb{R})\]
of \(f\). The space \(J^5(\mathbb{R}, \mathbb{R})\) may be identified with \(\mathbb{R}^7\) and the jet \(j^5 f\) with the map
\[y \mapsto (y, f(y), f'(y), f''(y), f'''(y), f^{(4)}(y), f^{(5)}(y)),\]
from \(\mathbb{R}\) into \(\mathbb{R}^7\); then to prove the proposition we apply Thom’s Transversality Theorem (see [3]) to the submanifolds of codimensions 1 and 2 in \(\mathbb{R}^7\) consisting of elements of the form \((y, a, b, c, d, e, g)\) with \(d = 3c^2/(2b)\) and \(d = 3c^2/(2b)\), \(-2g + 15c/b(e - 2e^3/b^2) = 0\) respectively. Denoting the respective submanifolds by \(V\) and \(W\), we have that for a generic function \(f\), \(j^5 f^{-1}(V)\) is a discrete set and \(j^5 f^{-1}(W)\) is empty. \(\square\)

Now we will characterize the shape of the apparent contour in the neighborhood of a point \(y_0 \in A\), where \(A\) is the set defined in Proposition 10. These points are called 2-codimension flips. Without loss of generality, we can suppose that \(y_0 = 0\). Hence, from Proposition 10 and by standard theory about flip bifurcation (see [2] and [5] for more details), we have that for a generic function \(f\) after a conjugacy by a local change of coordinates \(\varphi_0\) takes the form
\[\varphi_0(y, p, q) = py + qy^3 - y^5 + O_{(p,q)}(||y||^6).\]
Near the origin the surface \(\varphi_0 = 0\) has the same shape as \(G(y, p, q)/y = 0\), where \(G(y, p, q) = py + qy^3 - y^5\). In fact, there exists a \(C^1\) local diffeomorphism mapping the apparent contour \(\Lambda_0\) of \(\varphi_0 = 0\) onto the corresponding one for \(G = 0\). Thus, we have the following result.
Figure 2. Flip bifurcation of codimension 2.

**Proposition 11.** For a generic function $f$, the shape of the apparent contour $\Lambda_0$ of $\varphi_0 = 0$, in a neighborhood of a point which belongs to set $A$, is given by Figure 2.

The next result gives the shape of apparent contour $\Lambda_0$ of the surface $\varphi_0 = 0$.

**Proposition 12.** For a generic function $f$, the apparent contour $\Lambda_0$ of $\varphi_0 = 0$ is a curve having as singularities a finite number of ordinary cusps and codimension-2 flips, its self-intersections are transversal and occur only at smooth arcs. Moreover, $\Lambda_0$ does not have more than one self-intersections at a unique point, there are no self-intersections at the endpoints of $\Lambda_0$ and they are not singularities.

**Proof.** The proof of this proposition is a straightforward consequence of Multijet Transversality Theorem (see [3]). We will prove only the first statement. The other statements are proved in an analogous way.

From Preposition 11 it follows that the set of singularities of $\Lambda_0$ contains a finite number of codimension-2 flips. Now to show that the rest of singularities of $\Lambda_0$ is constituted of a finite number of ordinary cusps, we will do a new parametrization.

Consider the diffeomorphism $(y, p, q) \mapsto (y, p, pw + f(y))$. In the new variables $(y, p, w)$ we have that $\varphi_0$ is written as

$$\varphi_0(y, p, w) = f(w) - pw - f(y) + py.$$  \hspace{1cm} (20)

Note that when $y = w$ we have the case studied in Proposition 11 which correspond to curve $L_0$ given in (19).

By [2], to show that the rest of singularities of $\Lambda_0$ consists on a finite number of ordinary cusps, we must prove that for a generic function $f$,
the set
\[ A = \left\{ (y, p, w) : y \neq w \text{ and } \varphi_0 = \frac{\partial \varphi_0}{\partial y} = \frac{\partial^2 \varphi_0}{\partial y^2} = 0 \right\} \]
is finite. Furthermore, if \((y, p, w) \in A\) then
\[\frac{\partial^3 \varphi_0}{\partial y^3} \left( \frac{\partial \varphi_0}{\partial p} \frac{\partial^2 \varphi_0}{\partial yw} - \frac{\partial \varphi_0}{\partial w} \frac{\partial^2 \varphi_0}{\partial yp} \right) \bigg|_{(y,p,w)} \neq 0.\]

In fact, as \(w = (q - f(y))/p\) by (20) it follows that
\[\frac{\partial \varphi_0}{\partial y}(y, p, w) = f'(w)f'(y) - p^2,\]
\[\frac{\partial^2 \varphi_0}{\partial y^2}(y, p, w) = f''(w)f'(y)^2 - f'(w)f''(y)p\]
and
\[\frac{\partial^3 \varphi_0}{\partial y^3}(y, p, w) = f'''(w)f'(y)^3 - 3f''(w)f''(y)f'(y)p + f''(y)f'(w)p^2.\]
Note that if \((y, p, w)\) is a zero of \((\varphi_0)'_y\), then by (21) we have that \(p = \sqrt{f'(w)f'(y)}\), remember that \(p > 0\). Hence (20), (22) and (23) becomes
\[\varphi_0(y, p, w) = f(w) - f(y) + \sqrt{f'(w)f'(y)}(y - w),\]
\[\frac{\partial^2 \varphi_0}{\partial y^2}(y, p, w) = f''(w)f'(y)^2 - f'(w)f''(y)\sqrt{f'(w)f'(y)}\]
and
\[\frac{\partial^3 \varphi_0}{\partial y^3}(y, p, w) = f'''(w)f'(y)^3 - 3f''(w)f''(y)f'(y)\sqrt{f'(w)f'(y)} + f''(y)f'(w)f'(y)^2.\]
Now, consider the 2-multijet of order 3
\[ j^3_{(2)} f : \Delta_{(2)}(\mathbb{R}) \to J^3_{(2)}(\mathbb{R}, \mathbb{R}) \]
of \(f\), where \(\Delta_{(2)}(\mathbb{R}) \subset \mathbb{R}^2\) denote the set of the par \((y, w)\) with \(y \neq w\) and \(J^3_{(2)}(\mathbb{R}^2, \mathbb{R})\) denote the space of 2-multijets of order 3 of functions from \(\mathbb{R}\) into \(\mathbb{R}\) (see [3]). The space \(J^3_{(2)}(\mathbb{R}, \mathbb{R})\) may be identified with a subset of \(\mathbb{R}^{10}\) and the 2-multijet \(j^3_{(2)} f\) with the restriction of the map
\[(y, w) \mapsto (y, w, f(y), f(w), f'(y), f'(w), f''(y), f''(w), f'''(y), f'''(w)),\]
from \(\mathbb{R}^2\) into \(\mathbb{R}^{10}\); then to prove that \(A\) is finite and \((\varphi_0)''_{yy} \neq 0\) we apply Multijet Transversality Theorem (see [3]) to the submanifolds of codimensions 2 and 3 in \(\mathbb{R}^{10}\) consisting (by (24), (25) and (26) of
elements of the form \((y, w, a, b, c, d, e, g, k, l)\) with \(b - a + \sqrt{dc(y - w)} = 0\), \(gc^2 - de\sqrt{dc} = 0\) and \(b - a + \sqrt{dc(y - w)} = 0\), \(gc^2 - de\sqrt{dc} = 0\), \(lc^3 - 2gc\sqrt{dc} + kd^2c = 0\) respectively. Denoting the respective submanifolds by \(V\) and \(W\), since \(\Delta_2(\mathbb{R})\) has dimension 2, we have that for a generic function \(f\), \(j^3 f^{-1}(V)\) is a discrete set and \(j^3 f^{-1}(W)\) is empty. In the analogous way we can prove that if \((y, p, w) \in A\) then \((\varphi_0)_{yw} - (\varphi_0)'_{yw}(y, p, w) \neq 0\). This proves our claim and finishes the proof of the proposition.

Note that, by the Proposition 12 the apparent contour \(\Lambda_\delta\) of the surface \(\varphi_\delta = 0\) is diffeomorphic to \(\Lambda_0\). This follows from the fact that the curve \(\Lambda_0\) is structurally stable, i.e. the curve \(\Lambda_0\) does not change its topological structure by small perturbations. Hence the next result follows directly (see Lemma 8, page 197 of [8] for a similar result).

**Lemma 13.** The apparent contour \(\Lambda_\delta\) of the surface \(\varphi_\delta = 0\) for a generic function \(f\) is diffeomorphic to \(\Lambda_0\) and the diffeomorphism smoothly depends on \(\delta\) as \(\delta \to 0\).

Thus, as in the previous section we have a similar theorem that characterize the shape of the surface \(\Sigma_2\).

**Theorem 14.** The surface \(\Sigma_2\) is a horn in \(\mathbb{R}^3_{(\epsilon, \delta, \lambda)}\) defined in the following way. Consider in the half plane \(\mathbb{R}^2_{(p, q)} \cap \{p > 0\}\) the trace of curve \(\Lambda_0 \setminus L_0\) and the embedding of \(\mathbb{R}^2_{(p, q)}\) into \(\mathbb{R}^3_{(\delta, p, q)}\) as part of the plane \(\delta = 0\). Let \(Z_2\) be the cylinder in \(\Phi(V^+)\) over \(\Lambda_0 \setminus L_0\) with the axis parallel to \(\delta\) axis of the height \(\delta_0\). Then for sufficiently small \(\delta_0\) the “blown-up horn”

\[
Z = \Phi(\Sigma_1 \cap V^+)
\]

is diffeomorphic to \(Z_2\). The diffeomorphism taking \(Z\) into \(Z_1\) preserve the foliation \(\delta = \text{const}\), is \(C^1\)-smooth in \(\delta\), and its difference from the identity map on the fiber \(\delta = \text{const}\) is if order \(O(\delta)\).

9. Bifurcation diagram for the lips on a Möbius band

In this section we give the precise local description of the bifurcation diagram for the equation (16). As it was explained in Section 5 this diagram consists of four parts, \(\Sigma_1\), \(\Sigma_2\), \(\Sigma_+\) and \(\Sigma_-\).

**Theorem 15.** Let \(\varphi_\delta(y, p, q)\) be the map defined in Section 8.

1. The surface \(\Sigma_1 = \Sigma_1 \cup \Sigma_2\) is a horn in \(\mathbb{R}^3_{(\epsilon, \delta, \lambda)}\) defined in the following way. Consider in the half plane \(\mathbb{R}^2_{(p, q)} \cap \{p > 0\}\) the apparent contour \(\Lambda_0\) of surface \(\varphi_0 = 0\) and the embedding of \(\mathbb{R}^2_{(p, q)}\) into \(\mathbb{R}^3_{(\delta, p, q)}\) as part of the plane \(\delta = 0\). Let \(Z_{\Lambda_0}\) be the
cylinder in $\Phi(V^+)$ over $\Lambda_0$ with the axis parallel to $\delta$ axis of height $\delta_0$. Then for sufficiently small $\delta_0$ the “blown-up horn”

$$Z = \Phi(\Sigma \cap V^+)$$

is diffeomorphic to $Z_{\Lambda_0}$. The diffeomorphism taking $Z$ into $Z_{\Lambda_0}$ preserve the foliation $\delta = \text{const}$, is $C^1$-smooth in $\delta$, and its difference from the identity map on the fiber $\delta = \text{const}$ is of order $O(\delta)$ (see Figure 3).

(2) Consider in the half plane $\mathbb{R}^3_{(\delta,p,q)} \cap \{ p > 0 \text{ and } \delta = \text{const} \}$ the curves $l_0^\pm$ determined by the equations $\varphi_\delta(\pm 1, p, q) = 0$. In the coordinates $(\delta, p, q)$, the boundary surfaces $\Sigma_+$ and $\Sigma_-$ are regular surfaces diffeomorphic to cylinders in $\Phi(V^+)$ over $l_0^\pm$ with the axis parallel to $\delta$ axis of height $\delta_0$ (for sufficiently small $\delta_0$). This diffeomorphism preserve the foliation $\delta = \text{const}$, is $C^1$-smooth in $\delta$, and its difference from the identity map on the fiber $\delta = \text{const}$ is of order $O(\delta)$.

(3) The intersection of the boundary of $\Sigma$ with the layer $0 < \delta < \delta_0$ belongs to $\Sigma_+$ and $\Sigma_-$. At points of this intersection, the surface $\Sigma$ is tangent to either $\Sigma_+$ or $\Sigma_-$. 

Proof. Let $\varphi_\delta(y, p, q)$ be the map defined in the previous section.

The first statement of theorem follows from Theorems 9 and 14.

For to prove the second assertion of the theorem we consider only the surface $\Sigma_+$. The study of the surface $\Sigma_-$ is exactly analogous. The surface $\Sigma_+$ is described by the equation

$$\varphi_\delta(1, p, q) = 0.$$ 

Now, consider the diffeomorphism $(y, p, q) \mapsto (y, p, pw + f(y))$. In the new variables $(y, p, w)$ we have that the equation $\varphi_0(1, p, q) = 0$ is written as

$$f(w) - pw - f(1) + p = 0.$$ 

We can write equation (27) as

$$p = \int_0^1 f'(1 + s(w - 1))ds.$$ 

Thus the curve determined by equation (27) is the graphic of a function of the form $p = g(w)$. Hence the curve $l_0^\pm$ is regular and has no self-intersections. Therefore for a fixed $\delta (0 < \delta < \delta_0)$, the curve determined by the equation $\varphi_\delta(1, p, q) = 0$ is regular and is diffeomorphic to $l_0^\pm$. This prove the second statement of the theorem.

The values of the parameters for which the equation of limit cycles has a multiple root at $y = \pm 1$ belong both to the boundary of $\Sigma$ and to one of surfaces $\Sigma_{\pm}$. Let us show that at that point the surfaces
are tangent to each other. Consider the case $y = 1$, the other case is analogous. The previous claim is equivalent to saying that the surface $Z$ is tangent to $\Phi(\Sigma)$. It is sufficient to show that the apparent contour $\Lambda_\delta$ ($0 < \delta < \delta_0$) of the surface $\varphi_\delta = 0$ is tangent to the curve $\Phi(\Sigma) \cap \{ \delta = \text{const} \}$ at the endpoint of $\Lambda_\delta$ corresponding to $y = 1$. In the variables $(y, p, w)$, the curve $\Phi(\Sigma) \cap \{ \delta = \text{const} \}$ is described by the equation

$$\varphi_\delta(1, p, w) = 0.$$ 

Therefore everywhere on such curve we have

$$\frac{dp}{dw} = \frac{(\varphi_\delta)_p'(1, p, w)}{(\varphi_\delta)_w'(1, p, w)}.$$ 

Note that $(\varphi_\delta)_p'(1, p, w) \neq 0$. On the similar way, in the variables $(y, p, w)$, the smooth parts of $\Lambda_\delta$ are graphs of smooth functions of the form $p = h(w)$, where

$$h'(w) = \frac{(\varphi_\delta)_w(y, p, w)}{(\varphi_\delta)_p'(y, p, w)}.$$ 

This means that the two curves touch each other, hence the surfaces $\Sigma$ and $\Sigma_+$ are tangent. Thus the proof of this theorem is complete. $\Box$

**Remark 16.** The local behavior of the number of limit cycles in the bifurcation diagram established in theorem 15 is made more explicit is as follows.

We can consider in the parameter space $(\delta, p, q)$ only the curves determined by the intersection of the bifurcation surfaces with the plane $\delta = \text{constant}$, i.e. the curves $\Lambda_\delta$ and $l^\pm_\delta$. 

**Figure 3.** The bifurcation horn.
Near a cuspidal point of $\Lambda_\delta$ that corresponds to a root of multiplicity 3 of the equation $\varphi_\delta = 0$, the number of limit cycles decreases by 2 when passing from the inside to the outside of the cusp (see Figure 4(a)).

Near the endpoints of the curve $\Lambda_\delta$, i.e. at the intersection points of $\Lambda_\delta$ with the curves $l_\delta^\pm$ that correspond to a double root at an extreme point $y = \pm 1$, there are three local connected components of $\mathbb{R}^2_{(p,q)} \setminus \{\Lambda_\delta \cup l_\delta^\pm\}$; one is a piece of half space, the other is locally convex and the third one is a “thin” horn. The number of limit cycles decreases by 1 when we move from the “thin” component to the half space as well as when we go from the half space to the locally convex component (see Figure 4(c)).

Near a flip point of $\Lambda_\delta$ that corresponds to a root of multiplicity 5 of $\varphi_\delta = 0$, the behavior is analogous to the one just described (see Figure 4(b)).

10. Finite Cyclicity on a Möbius Lips

Cyclicity of a polycycle in a family of vector fields depending of parameters is the maximal number of limit cycles generated by this polycycle and corresponding to a parameter value close to the one with the polycycle.

Let $f$ and $g$ be two smooth real functions. We say that $f$ and $g$ are affine equivalent if there exist an affine function $\alpha(x) = b_1 x + b_2$ such that $f \circ \alpha = \alpha^{-1} \circ g$.

The study of the cyclicity of the lips in the Möbius band is similar the study of two subsequent lips, i.e. with four saddle-nodes. This can be seen by considering the double orientable covering of the Möbius band. In [4] the cyclicity of a polycycle that belongs to two subsequent lips is determined. We can use a similar idea to obtain the following result.

**Theorem 17.** Let $X_\mu$, $\mu \in \mathbb{R}^3$, be a generic $C^\infty$ 3-parameter family of vector fields in a Möbius band $M^2$, such that $X_0$ has a set of polycycles
of lips type. Consider the family of diffeomorphisms $f_\nu$ of class $C^k$ defined in (5). Suppose that $f_0$ and $f_0^{-1}$ satisfies the following generic conditions at some point $y_0$:

(i) the jets $J^n_{y_0}f_0, J^n_{y_0}f_0^{-1}$ of $f_0$ and $f_0^{-1}$ at the point $y_0$ are nonaffine maps to some order $n \leq k$,

(ii) the jets $J^n_{y_0}f_0, J^n_{y_0}f_0^{-1}$ of $f_0$ and $f_0^{-1}$ at the point $y_0$ are not affine equivalent through orientation reversing affine maps for some order $n \leq k$.

Then the polycycle of the lips passing through a point $y_0$ has cyclicity $\leq n$, where $n \leq k$ is the minimal order of the jet of the two functions $f_0$ and $f_0^{-1}$ at the point $y_0$ on which we can check the genericity conditions (i) and (ii).

Proof. By (12), we can write the equation that determine the limit cycles in the following form

$$f_\nu^{-1}(-py + q) = -\frac{1}{p}f_\nu(y) + \frac{q}{p}.$$ 

Now, we define the displacement map $V_\nu : \Gamma^+_1 \to \mathbb{R}$ by $V_\nu(y) = f_\nu^{-1}(-py + q) + \frac{1}{p}f_\nu(y) - \frac{q}{p}$.

Without loss of generality we can suppose that $y_0 = 0$. Then, we will prove that for some $n \leq k$, $V_\nu^{(n)}(0) \neq 0$ for all $\nu \neq 0$ small enough.

Consider the map

$$V_{(p,q)}(y) = f_0^{-1}(-py + q) + \frac{1}{p}f_0(y) - \frac{q}{p}$$

with $p > 0$. By the hypotheses of the theorem the system of equations

$$V_{(p,q)}^{(m)}(0) = (-p)^m(f_0^{-1})^{(m)}(q) + \frac{1}{p}f_0^{(m)}(0) = 0, \quad m = 1, \ldots, k,$$

has no solutions on the variables $p, q$. Therefore, there exist $n \leq k$ such that $V_{(p,q)}^{(n)}(0) \neq 0$ for all $p > 0$ and $q$ sufficiently small. Hence $V_\nu^{(n)}(0) \neq 0$ for all $\nu \neq 0$ small enough. This implies that in a small neighborhood of the polycycle passing by $y_0$ at most $n$ limit cycles can bifurcate from it. The theorem is proved. \qed

The proof of Theorem 17 is inspired in Theorem 3.3 and Main Lemma 3.4 of [4].
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