On an extension of the Iwatsuka model

Matěj Tušek

Department of Mathematics, Faculty of Nuclear Sciences and Physical Engineering, Czech Technical University in Prague, Trojanova 13, 120 00 Prague 2, Czech Republic

E-mail: tusekmat@fjfi.cvut.cz

Received 11 January 2016, revised 4 May 2016
Accepted for publication 1 July 2016
Published 17 August 2016

Abstract
We prove absolute continuity for an extended class of two-dimensional magnetic Hamiltonians that were initially studied by Iwatsuka. In particular, we add an electric field that is translation invariant in the same direction as the magnetic field. As an example, we study the effective Hamiltonian for a thin quantum layer in a homogeneous magnetic field.

Keywords: Iwatsuka model, spectral analysis, quantum waveguides, magnetic Hamiltonians

1. Introduction

Consider a charged, spin-less massive particle in a plane subject to electric and magnetic fields that are both invariant with respect to the translations in the y-direction. So, if we denote these fields \( W \) and \( B \), respectively, they are functions \( W(x), B(x) \) of \( x \) alone. Within the realm of non-relativistic quantum mechanics, the dynamics of the particle is governed by the following Hamiltonian:

\[
H = \frac{-\partial^2}{\partial x^2} + (-i\partial_y + A_y(x))^2 + W(x),
\]

where

\[
A_y(x) = \int_0^x B(r)dr.
\]

Here, we chose the Landau (asymmetric) gauge, set the reduced Planck constant and the ratio of the elementary charge to the speed of light equal to 1, and fixed the particle mass as \( \frac{1}{2} \). The self-adjointness of this operator will be discussed at the beginning of section 2.

It has long been conjectured [1] that in cases without an electric field \( (W = 0) \), \( H \) is purely absolutely continuous, i.e. \( \sigma(H) = \sigma_{ac}(H) \), as soon as \( B \) is non-constant. This conjecture was motivated by the seminal paper by A Iwatsuka [2]. In acknowledgement of his achievement, the model described by (1) bears his name. He proved the absolute continuity of \( H \) under the following additional pair of assumptions:
AS1: \( B \in C^\infty(\mathbb{R}; \mathbb{R}) \), and there exist constants \( M_\pm \), such that \( 0 < M_- \leq B \leq M_+ \).

AS2: and either of the following hold:

**AS2a:**
\[
\limsup_{x \to -\infty} B(x) < \liminf_{x \to +\infty} B(x)
\]

or
\[
\limsup_{x \to +\infty} B(x) < \liminf_{x \to -\infty} B(x)
\]

**AS2b:** \( B \) is constant for all sufficiently large \( |x| \) but non-constant on \( \mathbb{R} \), and there exists \( x_0 \), such that
\[
\limsup_{x \to \pm x_0} B(x) = \liminf_{x \to \pm x_0} B(x)
\]

In fact, AS2b may be relaxed to:

**AS2c:** \( B \) is non-constant and there exists a point \( x_0 \), such that for all \( x \neq x_0 \),
\[
\limsup_{x \to x_0} B(x) = \liminf_{x \to x_0} B(x)
\]
as was proved by Mântoiu and Purice [3]. Note that there is also some overlap of AS2c with AS2a.

Another nice result concerning a variation of the magnetic field that is compactly supported (in the \( x \)-variable) was given by Exner and Kovářík [4]. They proved that \( H \) is purely absolutely continuous if

AS3: \( B(x) = B_0 + b(x) \), where \( B_0 > 0 \) and \( b \) is bounded, piecewise continuous and compactly supported.

AS4: and either of the following hold:

**AS4a:** \( b \) is non-zero and does not change sign.

**AS4b:** let \( [a_l, a_r] \) be the smallest closed interval that contains \( \text{supp } b \); we have \( c, \delta > 0 \) and \( m \in \mathbb{N} \), such that \( |b(x)| \geq c(x - a_l)^m \) or \( |b(x)| \geq c(a_r - x)^m \) for all \( x \in [a_l, a_l + \delta] \) or \( x \in (a_r - \delta, a_r] \), respectively.

In this paper we generalize AS2a to the case where the electric field is switched on, relaxing AS1 simultaneously. First, we will need some notation to lighten the text. For any \( f \in L^\infty(\mathbb{R}; \mathbb{R}) \), let us define

\[
\bar{f}_+ = \sup_{t \in (a_l, +\infty)} \text{ess inf } f(t) \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \Quad...
modified to work under only slightly stricter assumptions than those of theorem 1.1. In particular, one needs the derivative of $W$ to be in $L^\infty$.

Iwatsuka’s strategy may be described as follows. First, decompose $H$ into the direct integral of the one-dimensional operators with a purely discrete spectrum. Then show that these fiber operators form an analytic family with simple and non-constant eigenvalues with respect to the quasi-momentum parameter. Although our proof, as well as all the proofs of the above-mentioned results, follow this strategy, they differ in the method used when proving the last step; i.e., the non-constancy of the eigenvalues. Let us stress that this very step is typically the most difficult to prove. Whereas Iwatsuka needed some estimates on the growth of the eigenfunctions to show that the asymptotic behaviour of the eigenvalues in $\pm \infty$ is determined by that of the magnetic field, we derive the asymptotic behaviour of the eigenvalues directly using a comparison argument based on the minimax principle combined with a norm-resolvent convergence result.

Iwatsuka’s model with a non-zero electric field ($W \neq 0$) of a particular type has been studied before in [5]. There, it was proved that $H_L + \omega^2 x^2$ remains purely absolutely continuous under a perturbation that is either a bounded function of an $x$-variable only or a bounded periodic function of a $y$-variable only. Here, $H_L$ is the Landau Hamiltonian ((1) with $W = 0$ and constant $B \neq 0$) and $\omega > 0$. For $H = H_L + W$, where $W = W(x)$ is a non-decreasing non-constant bounded function, it was proved that $\sigma(H)$ has a band structure and is purely absolutely continuous [6]. The asymptotic distribution of the discrete spectrum of $H$ under a bounded perturbation of a constant sign that decays at infinity was investigated in the same paper. The same problem with $W = W(x)$ now being periodic was addressed in [7]. In this case, the absolute continuity of $H$ was only demonstrated below a fixed but arbitrarily large energy when the magnetic field was strong enough. The asymptotic distribution of the eigenvalues in the spectral gaps was also studied for the case of the Iwatsuka Hamiltonian, essentially satisfying $\text{AS1}$ and $\text{AS2a}$, with an additional electric potential that either decays power-like at infinity or is compactly supported [8].

The transport properties of the Iwatsuka model are also of continuous interest. A non-constant translation invariant magnetic field acts as a magnetic barrier and gives rise to the so-called edge currents that are quantized [9]. When the barrier is sharp, the current-carrying states are well localized and stable with respect to the various magnetic and electric perturbations [10, 11]. The latter paper deals solely with one of the configurations when the magnetic field is constant on each of the two complementary half-planes (the so-called magnetic steps). The analysis of the magnetic steps is completed in [12]. They have been studied before from a physicist’s point of view in [13].

Finally, to demonstrate the richness of the topic, let us mention that a three-dimensional version of the Iwatsuka model was studied in [14], and random analogues of the Iwatsuka model were examined in [15].

The paper is organized as follows. In section 2, we introduce operator (1) properly and prove that its spectrum is purely absolutely continuous under our assumptions. The proof relies on an abstract operator convergence result that is proved separately in section 4. Section 3 is devoted to a significant example that comes from the realm of the so-called quantum waveguides. A particle confined in a very thin curved layer in an ambient constant magnetic field is effectively subjected to a non-constant magnetic field that is given by the projection into the direction normal to the layer [16]. Moreover, the non-trivial curvature gives rise to an additional attractive scalar potential. The interplay between the magnetic and electric fields (that may be, in the discussed example, expressed solely in terms of geometric quantities) is reflected in our sufficient condition (3).
2. Absolute continuity of the Hamiltonian

2.1. Direct integral decomposition

By $H$, we mean the closure of $\dot{H}$ given by

$$H = -\partial_x^2 + (-i\partial_y + A_y(x))^2 + W(x)$$

$$\text{Dom}(H) = C_0^\infty(\mathbb{R}^2) \subset L^2(\mathbb{R}^2),$$

where $W \in L^\infty(\mathbb{R}; \mathbb{R})$ and $A_y$ is given by (2) with $B \in L^\infty(\mathbb{R}; \mathbb{R})$. $H$ is self-adjoint [17] (we refer to this paper whenever essential self-adjointness is mentioned) and commutes with the translations in the $y$-direction. In [2], it was demonstrated that $H$ is unitarily equivalent to the direct integral in $L^2(\mathbb{R}; L^2(\mathbb{R}))$ of the self-adjoint operators $\{H[\xi], \xi \in \mathbb{R}\}$ with the following action

$$H[\xi] = -d_x^2 + (\xi + A_y(x))^2 + W(x).$$

For any $\xi \in \mathbb{R}$, $C_0^\infty(\mathbb{R})$ is a core of $H[\xi]$. To prove the absolute continuity of $H$, it is sufficient to show that [18, theorem XIII.86]

- The family $\{H[\xi]\} \xi \in \mathbb{R}$ is analytic in $\xi$.
- For all $\xi \in \mathbb{R}$, $H[\xi]$ has compact resolvent.
- If we number the eigenvalues of $H[\xi]$ in a strictly increasing order as $\lambda_n[\xi]$, $n \in \mathbb{N}$, then every $\lambda_n[\xi]$ is simple and no $\lambda_n[\xi]$ is constant in $\xi$.

For any $\xi_0 \in \mathbb{R}$, we may write

$$H[\xi] = H[\xi_0] + p_\xi,$$

where the quadratic form $p_\xi$ reads

$$p_\xi(\psi) = (\xi - \xi_0)^2||\psi||^2 + 2(\xi - \xi_0)(\psi, (\xi_0 + A_y)\psi). \quad (5)$$

For all $\delta > 0$, one easily gets

$$|p_\xi(\psi)| \leq (\xi - \xi_0)^2(1 + \delta^{-1})||\psi||^2 + \delta ||(\xi_0 + A_y)\psi||^2$$

$$\leq (\xi - \xi_0)^2(1 + \delta^{-1})||\psi||^2 + \delta \langle \psi, H[\xi_0]\psi \rangle + \delta \langle \psi, W\psi \rangle$$

$$\leq ((\xi - \xi_0)^2(1 + \delta^{-1}) + \delta ||W||_\infty)||\psi||^2 + \delta \langle \psi, H[\xi_0]\psi \rangle,$$

where $W_\cdot$ stands for the negative part of $W$.

Hence, $p_\xi$ is infinitesimally form bounded by $H[\xi_0]$. This, together with (5), implies that $H[\xi]$ forms an analytic family of type (B). In particular $H[\xi]$ is an analytic family in the sense of Kato [18], which proves the first point.

Assuming either (4) or the first part of (3), we deduce that $\lim_{x \to \pm \infty} A_y(x) = +\infty$. This implies the compactness of the resolvent of $H[\xi]$ [18, theorem XIII.67], i.e., the second condition. Consequently, the spectrum of $H[\xi]$ is purely discrete. Moreover, mimicking the proofs of [2, proposition 3.1, lemma 2.3(i)] (using the results of [19, section 16] and the simple observation that any continuous regular distribution with a non-negative weak derivative almost everywhere is non-decreasing everywhere), one may infer that all the eigenvalues of $H[\xi]$ are simple. Therefore, the first part of the third condition holds true, too.
Remark 2.1. With these results in hand we may conclude that the singular continuous component in the spectrum of $H$ is empty [20].

The second part of the third condition is easy to verify under assumption (4). If $B_+ > 0 \land \overline{B}_- < 0$ then $\lim_{\xi \to \pm \infty} A_\xi(x) = +\infty$. Using the minimax principle we obtain $\lim_{\xi \to -\infty} \lambda_\xi[\xi] = +\infty$. If $B_- > 0 \land \overline{B}_- < 0$ then $\lim_{\xi \to \pm \infty} A_\xi(x) = -\infty$ and $\lim_{\xi \to -\infty} \lambda_\xi[\xi] = +\infty$.

The rest of this section is devoted to the verification of the third condition under assumption (3). In particular, we always assume that $\overline{B}_\pm > 0$.

2.2. Some auxiliary results

2.2.1. Estimate on the potential

Lemma 2.2. Let (3) hold and $\varepsilon \in (0, \min \{B_+, B_-\} / 2)$. Then $\xi_\varepsilon \in \mathbb{R}$ exists such that for all $\xi < \xi_\varepsilon$,

$$V_{\xi_\varepsilon}(x) + W_+ - \varepsilon \leq (\xi + A_\xi(x))^2 + W(x) \leq V_{\xi_\varepsilon}(x) + W_+ + \varepsilon \quad \text{(a.e. x).}$$

Similarly, $\bar{\xi}_\varepsilon \in \mathbb{R}$ exists such that for all $\xi > \bar{\xi}_\varepsilon$,

$$V_{\bar{\xi}_\varepsilon}(x) + W_- - \varepsilon \leq (\xi + A_\xi(x))^2 + W(x) \leq V_{\bar{\xi}_\varepsilon}(x) + W_- + \varepsilon \quad \text{(a.e. x).}$$

Here,

$$V_{\xi_\varepsilon}(x) := \begin{cases} (B_+ - 2\varepsilon)^2(x - x_\xi)^2 & \text{for } x \geq -K_\xi \\ ((B_{\text{min}} - \varepsilon)(x + K_\xi) + (B_+ - 2\varepsilon)(-K_\xi - x_\xi))^2 & \text{for } x < -K_\xi \\ \end{cases}$$

$$V_{\bar{\xi}_\varepsilon}(x) := \begin{cases} (B_- + 2\varepsilon)^2(x - x_\bar{\xi})^2 & \text{for } x \geq -K_\bar{\xi} \\ ((B_{\text{max}} + \varepsilon)(x + K_\bar{\xi}) + (B_- + 2\varepsilon)(-K_\bar{\xi} - x_\bar{\xi}))^2 & \text{for } x < -K_\bar{\xi} \\ \end{cases}$$

$$V_{\xi_\varepsilon}(x) := \begin{cases} (B_- - 2\varepsilon)^2(x - x_\xi)^2 & \text{for } x \leq K_\xi \\ ((B_{\text{min}} - \varepsilon)(x - K_\xi) + (B_- - 2\varepsilon)(K_\xi - x_\xi))^2 & \text{for } x > K_\xi \\ \end{cases}$$

$$V_{\bar{\xi}_\varepsilon}(x) := \begin{cases} (B_+ + 2\varepsilon)^2(x - x_\bar{\xi})^2 & \text{for } x \leq K_\bar{\xi} \\ ((B_{\text{max}} + \varepsilon)(x - K_\bar{\xi}) + (B_+ + 2\varepsilon)(K_\bar{\xi} - x_\bar{\xi}))^2 & \text{for } x > K_\bar{\xi}, \\ \end{cases}$$

where $B_{\text{min}} = \min \{B_+, B_-\}$, $B_{\text{max}} = \max \{B_+, B_-\}$, $K_\varepsilon > 0$ is introduced below, and $x_\xi$ is the unique solution of $(\xi + A_\xi(x)) = 0$. (Uniqueness for all $\xi$ with sufficiently large $|\xi|$ is proved below, too.)

Proof. We will only prove the first inequality in (6). The remaining inequalities may be deduced in a similar manner.

Since $\overline{B}_+ > 0$, $B > \varepsilon$ almost everywhere outside a compact subset of $\mathbb{R}$. Moreover, $A_\xi$ is absolutely continuous and $A_\xi' = B$ (a.e. x). In particular, we have $\lim_{x \to \pm \infty} A_\xi(x) = \pm \infty$ and $(\xi + A_\xi(x)) = 0$ has a unique solution for all $\xi$ with sufficiently large $|\xi|$. Let us denote this solution by $x_\xi$. Clearly, $\lim_{\xi \to -\infty} x_\xi = +\infty$.

For a given $\varepsilon$, there exists $K_\varepsilon > 0$ such that almost everywhere on $(K_\varepsilon, +\infty)$,

$$B_+ - \varepsilon < B < \overline{B}_+ + \varepsilon, \quad W_+ - \varepsilon < W < W_+ + \varepsilon,$$
and almost everywhere on \((-\infty, -K)\),
\[
B_\varepsilon - \varepsilon < B < B_\varepsilon + \varepsilon, \quad W_\varepsilon - \varepsilon < W < W_\varepsilon + \varepsilon.
\]
Let us stress that the choice of \(K\) depends solely on \(\varepsilon\). Now, we restrict ourselves to all \(\xi\) that are sufficiently negative so that \(\xi > K\), and we estimate
\[
\begin{align*}
\xi + A_\varepsilon(x) > (B_\varepsilon + 2\varepsilon)(x - x_\xi) & \geq 0 \quad \text{for } x \geq x_\xi \quad (7) \\
\xi + A_\varepsilon(x) < (B_\varepsilon + 2\varepsilon)(x - x_\xi) & < 0 \quad \text{for } x < (K, x_\xi) \quad (8) \\
W(x) > W_\varepsilon - \varepsilon & \quad \text{for } x \geq K\varepsilon. \quad (9)
\end{align*}
\]
We also have
\[
\sup_{x \in (K, K\varepsilon)} |A_\varepsilon(x) - A_\varepsilon(K\varepsilon)| \leq 2K\varepsilon \|B\|_\infty, \quad \text{ess sup}_{x \in (-\infty, K\varepsilon)} |W(x) - (W_\varepsilon - \varepsilon)| < +\infty.
\]
Using these estimates together with the following observation,
\[
\lim_{\xi \to -\infty} ((\xi + A_\varepsilon(K)) - (B_\varepsilon + 2\varepsilon)(K\varepsilon - x_\xi)) = -\infty,
\]
we infer that for all sufficiently negative \(\xi\), not only \(\xi + A_\varepsilon(x) < (B_\varepsilon + 2\varepsilon)(x - x_\xi) < 0\) but
\[
(B_\varepsilon + 2\varepsilon)^2(x - x_\xi)^2 + W_\varepsilon - \varepsilon < (\xi + A_\varepsilon(x))^2 + W(x) \quad (10)
\]
on \((-K\varepsilon, K\varepsilon)\). Finally, by a similar reasoning, there exists \(\xi_\varepsilon\) such that for all \(\xi < \xi_\varepsilon, x_\xi > K\varepsilon\) and (10) hold together with
\[
((\min(B_\varepsilon, B_-) - \varepsilon)(x + K\varepsilon) + (B_\varepsilon + 2\varepsilon)(-K\varepsilon - x_\xi))^2 + W_\varepsilon - \varepsilon < (\xi + A_\varepsilon(x))^2 + W(x) \quad (11)
\]
on \((-\infty, -K)\).
Putting (7)–(11) together we arrive at
\[
\sum_{-\varepsilon}(x) + W_\varepsilon - \varepsilon < (\xi + A_\varepsilon(x))^2 + W(x) \quad \text{for } x \in \mathbb{R}.
\]

\(\square\)

\subsection{2.2.2. Abstract convergence result}

\textbf{Theorem 2.3.} Let \([A(\alpha), \alpha \in (-\infty, +\infty)]\) be a one-parametric family of lower-bounded self-adjoint operators on \(L^2(\Omega)\), where \(\Omega \subset \mathbb{R}^n\) is open, with the following properties:

i. \(C_0^\infty(\Omega)\) is a core of \(A(\alpha)\) for all \(\alpha \in (-\infty, +\infty)\).

ii. There exist \(C > 0\) and \(K, \alpha_0 \in \mathbb{R}\) such that for all \(\alpha \geq \alpha_0, CA[+\infty] + K \leq A(\alpha)\).

iii. For any compact set \(K \subset \Omega\), there exists \(\alpha_0\) such that for all \(\alpha \geq \alpha_0, A(\alpha)|_{C(\Omega)} = A[+\infty]|_{C(\Omega)}\).

iv. \((A[+\infty])\) has compact resolvent.

Then, for any \(z \in \text{Res}(A[+\infty])\) and \(\varepsilon > 0\), there exists \(\alpha_\varepsilon, \varepsilon\) such that for all \(\alpha > \alpha_\varepsilon, z \in \text{Res}(A(\alpha))\) and
\[
\|(A(\alpha) - z)^{-1} - (A[+\infty] - z)^{-1}\| < \varepsilon.
\]
The proof is given separately in section 4.

\subsection{2.2.3. Comparison operators}

Let \(\omega, \tilde{\omega} > 0\) and \(x_0, \alpha \in \mathbb{R}\). The following differential operators on \(L^2(\mathbb{R})\)
\[ H_{\omega,\alpha}[x] = \begin{cases} -d_x^2 + \omega^2(x - \alpha)^2 & \text{for } x \geq x_0 \\ -d_x^2 + (\omega(x - x_0) + \omega(x_0 - \alpha))^2 & \text{for } x < x_0 \end{cases} \]

defined on \( C_0^\infty(\mathbb{R}) \) are essentially self-adjoint. We will denote their closures by \( H_{\omega,\alpha}[\alpha] \) and \( H_{\omega,\alpha} \), respectively. Let us introduce a unitary transform \( U_\alpha : \psi(x) \mapsto \psi(x - \alpha) \). Then

\[ H_{\omega,\alpha}[\alpha] = U_\alpha H_{\omega,\alpha}[\alpha] U_\alpha^*, \quad H_{\omega,\alpha}[\alpha] = U_\alpha \tilde{H}_{\omega,\alpha} U_\alpha^* \]

with

\[ \tilde{H}_{\omega,\alpha}[\alpha] = \begin{cases} -d_x^2 + \omega^2x^2 & \text{for } x \geq x_0 - \alpha \\ -d_x^2 + (\omega(x + \alpha - x_0) + \omega(x_0 - \alpha))^2 & \text{for } x < x_0 - \alpha \end{cases} \]

\[ \tilde{H}_{\omega} = -d_x^2 + \omega^2x^2. \]

Note that \( \tilde{H}_{\omega} \) is just the harmonic oscillator Hamiltonian whose spectrum is very well known to be formed only by simple eigenvalues \( \{\gamma_n\} \). Due to unitary equivalence, \( \sigma(H_{\omega,\alpha}[\alpha]) = \sigma(\tilde{H}_{\omega}). \)

If we set \( \tilde{H}_{\omega,[\alpha]} \equiv \tilde{H}_{\omega} \), then the family \( \{\tilde{H}_{\omega,[\alpha]}, \alpha \in (-\infty, +\infty)\} \) fulfills the assumptions of theorem 2.3. In particular, for all \( \alpha > x_0, \)

\[ \min \left\{ 1, \frac{\omega}{\omega} \right\} \tilde{H}_{\omega} \leq \tilde{H}_{\omega,[\alpha]}, \]

and so the operator family obeys (ii) of the theorem. (Note that if \( \omega < 0 \), then (ii) would not be fulfilled.) Therefore, for any \( \mu \in \text{Res}(\tilde{H}_{\omega}), \) we have

\[ \lim_{\alpha \rightarrow +\infty} \| (\tilde{H}_{\omega,[\alpha]} + \mu)^{-1} - (\tilde{H}_{\omega} + \mu)^{-1} \| = 0. \]

Due to the unitarity of \( U_\alpha \) we also have

\[ \lim_{\alpha \rightarrow +\infty} \| (H_{\omega,\alpha}[\alpha] + \mu)^{-1} - (H_{\omega} + \mu)^{-1} \| = 0. \]

Since the norm-resolvent convergence implies the convergence of eigenvalues [21], this yields

**Proposition 2.4.** Let \( \sigma_n, n \in \mathbb{N} \), be the \( n \)th eigenvalue of \( H_{\omega,\alpha}[\alpha] \), then

\[ \lim_{\alpha \rightarrow +\infty} \sigma_n = (2n - 1)\omega. \]

### 2.3. Proof of theorem 1.1

Let \( H_{\xi,\xi}[\xi] \) and \( H_{\xi,\xi}[\xi] \) be closures of \( H_{\xi,\xi}[\xi] \) and \( H_{\xi,\xi}[\xi] \), respectively, that are defined on \( C_0^\infty(\mathbb{R}) \) by

\[ H_{\xi,\xi}[\xi] = -d_x^2 + V_{\xi,\pm} + W_{\pm} - \varepsilon \]
\[ H_{\xi,\xi}[\xi] = -d_x^2 + \nabla_{\xi,\pm} + W_{\pm} + \varepsilon. \]

Then \( H_{\xi,\xi}[\xi], H_{\xi,\xi}[\xi] \) are self-adjoint and have the structure of the comparison operator of section 2.2.3, with \( x_\xi \) being the free parameter instead of \( \alpha \). (For the case \( \xi \rightarrow +\infty \), we have \( x_\xi \rightarrow -\infty \), and so the results of section 2.2.3 must be modified in an obvious manner.)
By lemma 2.2, for all $\xi < \xi_\varepsilon$,
\[
H_{\varepsilon,1}[\xi] \leq H[\xi] \leq H_{\varepsilon,1}[\xi],
\]
and for all $\xi > \xi_\varepsilon$,
\[
H_{\varepsilon,-1}[\xi] \leq H[\xi] \leq H_{\varepsilon,-1}[\xi].
\]
If we now apply the minimax principle together with proposition 2.4, we obtain
\[
\liminf_{\xi \to \pm \infty} \lambda_n[\xi] \leq (B_{\pm} + 2\varepsilon)(2n - 1) + W_\pm + \varepsilon.
\]
Since $\varepsilon$ may be arbitrarily small,
\[
\liminf_{\xi \to \pm \infty} \lambda_n[\xi] \leq (B_{\pm} + 2\varepsilon)(2n - 1) + W_\pm.
\]
Therefore if for all $n \in \mathbb{N}$, either
\[
B_{\pm}(2n - 1) + W_\pm < B_{\pm}(2n - 1) + W_\pm,
\]
or
\[
B_{\pm}(2n - 1) + W_\pm < B_{\pm}(2n - 1) + W_\pm,
\]
then every $\lambda_n[\xi]$ is non-constant in the $\xi$-variable. This may be rewritten as (3).

3. Example of an effective Hamiltonian for a thin curved quantum layer in a homogeneous magnetic field

The quantum layers are important representatives of the so-called quantum waveguides that have been extensively studied over the last several decades. See the recent monograph [22] for an immense list of references. The quantum waveguides in a magnetic field that will be of particular interest to us were examined, e.g., in [16, 23–28]. In this section, we will derive a sufficient condition for the absolute continuity of the effective Hamiltonian for a very thin curved quantum layer in an ambient homogeneous magnetic field.

Let $\Sigma$ be a $y$-translation invariant surface in $\mathbb{R}^3$ given by the following parametrization:
\[
\mathcal{S}_0(s, y) = (x(s), y, z(s))
\]
with $s, y \in \mathbb{R}$. Here the functions $x$ and $z$ are assumed to be $C^3$-smooth and such that $\dot{x}(s)^2 + \dot{z}(s)^2 = 1$. The latter condition means that the curve $\Gamma : s \mapsto (x(s), z(s))$ in the $xz$ plane is parametrized by the arc length measured from some reference point on the curve. Therefore, the curvature $\kappa$ of $\Gamma$ is given by
\[
\kappa(s)^2 = \dot{x}(s)^2 + \dot{z}(s)^2
\]
and a unit normal vector to $\Sigma$ may be chosen as follows,
\[
n(s, y) \equiv n(s) = (-\dot{z}(s), 0, \dot{x}(s)).
\]
If we view $\Sigma$ as a Riemannian manifold, then the metric induced by the immersion $\mathcal{L}_0$ reads

$$(g_{\mu\nu}) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$ 

Let $a > 0$ and $I := (-1, 1)$. Define a layer $\Omega$ of the width $2a$ constructed along $\Sigma$ as the image of

$$\mathcal{L} : \mathbb{R}^2 \times I \rightarrow \mathbb{R}^3 : \{(s, y, u) \mapsto \mathcal{L}_0(s, y) + aun(s)\}.$$ 

We always assume $a < \|\kappa\|_{\infty}^4$ and that $\Omega$ does not intersect itself. Under these conditions, $\mathcal{L}$ is a diffeomorphism onto $\Omega$ as one can see, e.g., from the formula for the metric $G$ (induced by the immersion $\mathcal{L}$) on $\Omega$ that reads

$$(G_{ij}) = \begin{pmatrix} g_{11} & 0 \\ 0 & a^2 \end{pmatrix}, \quad (G_{\mu\nu}) = \begin{pmatrix} f_a(s, u)^2 & 0 \\ 0 & 1 \end{pmatrix},$$

where $f_a(s, u) = (1 - aun(s))$.

We start with the magnetic Laplacian on $\Omega$ subject to the Dirichlet boundary condition,

$$Q(-\Delta_{\mathcal{L}, A}) = \mathcal{H}^d_{\mathcal{L}, \Omega}(\Omega, dx dy dz),$$

with a special choice of vector potential, $A = B_0(0, x, 0)$, $B_0 > 0$, that corresponds to the magnetic field $B = (0, 0, B_0)$. Employing the diffeomorphism $\mathcal{L}$, we may identify $-\Delta_{\mathcal{L}, A}$ with a self-adjoint operator $\hat{H}$ on $L^2(\mathbb{R}^2 \times I, d\Omega)$ with the following action (understood in the form sense)

$$\hat{H}_\Omega = -f_a(s, u)^{-1}\partial_s f_a(s, u)^{-1}\partial_s + (-i\partial_y + \hat{A}_2(s, u))^2 - a^{-2}f_a(s, u)^{-1}\partial_s f_a(s, u)\partial_u,$$

where $\hat{A} = (D\mathcal{L})^T A$ or $\mathcal{L} = (0, \hat{A}_2, 0)$ with $\hat{A}_2(s, u) = B_0(s - aun(s))$.

Involving a unitary transform $U : L^2(\mathbb{R}^2 \times I, d\Omega) \rightarrow L^2(\mathbb{R}^2 \times I, d\Sigma du)$, $\psi \mapsto a^{-1/2}f_a^{1/2}\psi$, we arrive at a unitarily equivalent operator defined again in the form sense as

$$\hat{H}_\Omega = U\hat{H}_\Omega U^{-1} = -\partial_s f_a(s, u)^{-2}\partial_s + (-i\partial_y + \hat{A}_2(s, u))^2 - a^{-2}\partial_u^2 + V(s, u)$$

where

$$V(s, u) = -\frac{1}{4f_a(s, u)^2} - \frac{1}{2f_a(s, u)^3} - \frac{5}{4}\frac{a^2u^2\kappa(s)^2}{f_a(s, u)^4}.$$

(We have to strengthen our regularity assumptions on $\Sigma$ to give a meaning to the second derivative of the curvature.)

It was proved in [16] that for all $k$ large enough,

$$\| (\hat{H}_\Omega - (\pi/2a)^2 + k)^{-1} - (h_{\text{eff}} + k)^{-1} \oplus 0 \| = \mathcal{O}(a)$$

as $a \rightarrow 0_+$, with

$$h_{\text{eff}} = -\partial_x^2 + (-i\partial_y + B_0x(s))^2 - \frac{1}{4}\kappa^2(s)$$

acting on $L^2(\mathbb{R}^2, dx dy)$. If we assume that $\kappa$ is bounded, then $h_{\text{eff}}$ is essentially self-adjoint on $C^\infty(\mathbb{R}^2)$.

Clearly, $h_{\text{eff}}$ is of the form (1). Theorem 1.1 immediately yields

**Proposition 3.1.** $h_{\text{eff}}$ is purely absolutely continuous if $\kappa \in L^\infty$ and either

$$k_+ > 0 \land \hat{k} \geq \hat{x} \land (\hat{k}^2 + (\hat{\kappa} + \hat{x})^2 < 4B_0(\hat{k} - \hat{x})).$$
or
\[ \hat{x}_+ > 0 \land \hat{x}_- < 0. \]
(The claim remains valid if we interchange the ± indices everywhere.) In particular, it is purely absolutely continuous if
\[
\lim_{s \to +\infty} \kappa(s) = 0 \text{ and } \lim_{s \to -\infty} \hat{x}(s) = \lim_{s \to -\infty} \hat{x}(s).
\]

4. Appendix: proof of theorem 2.3

For our proof we will need two auxiliary results. The first one is due to Cazacu and Krejčiřík [29]; we present it here in a refined form.

**Lemma 4.1 (Cazacu, Krejčiřík [29]).** Let \( \{R_d\}_{d \in \mathbb{R}} \) be a family of bounded operators and \( R \) be a compact operator on some Hilbert space. If for all sequences \( (f_n) \) with properties \( \|f_n\| = 1 \) and \( f_n \xrightarrow{w} f \), any real sequence \( (d_n) \) such that \( \lim_{n \to +\infty} d_n = +\infty \), and any \( \varepsilon > 0 \), there exists a subsequence \( (n_k) \) such that \( \lim_{k \to +\infty} \|R_{d_{n_k}} f_{n_k} - Rf\| < \varepsilon \), then
\[
\lim_{d \to +\infty} \|R_d - R\| = 0.
\]

**Lemma 4.2.** Let \( \{A[a]\} \) be as in theorem 2.3 and \( \mu \) be such that \( (A[a] + \mu) \geq 1 \) for all \( a \in [a_0, +\infty] \). (\( \mu \) with this property exists, due to (ii) and semiboundness of \( (A[+\infty]) \).) Then, for any sequence of functions \( (f_n) \) such that \( \|f_n\| = 1 \) and \( f_n \xrightarrow{w} f \), any real sequence \( (d_n) \) \( \lim_{n \to +\infty} d_n = +\infty \), and any \( \varepsilon > 0 \), there exists a subsequence \( (n_k) \) such that
\[
\lim_{k \to +\infty} \|(A[d_{n_k}] + \mu)^{-1} f_{n_k} - (A[+\infty] + \mu)^{-1} f\| < \varepsilon.
\]

**Proof.** Let \( u_n \) be uniquely defined by
\[
(A[d_n] + \mu)u_n = f_n.
\]
Due to the hypothesis (i) of theorem 2.3, we may construct sequences \( (\tilde{u}_n) \) and \( (\tilde{f}_n) \) with properties
\[
\tilde{u}_n \in C_0^\infty(\Omega), \quad (A[d_n] + \mu)\tilde{u}_n = \tilde{f}_n, \quad \|\tilde{u}_n - u_n\| < \varepsilon, \quad \|\tilde{f}_n - f_n\| < \varepsilon,
\]
for all \( n \in \mathbb{N} \).

Using the hypothesis (ii) we obtain
\[
C \langle (A[+\infty] + \mu)^{1/2} \tilde{u}_n, (A[+\infty] + \mu)^{1/2} \tilde{u}_n \rangle + (K + \mu(1 - C)) \|\tilde{u}_n\|
\leq \langle (A[d_n] + \mu)^{1/2} \tilde{u}_n, (A[d_n] + \mu)^{1/2} \tilde{u}_n \rangle = \langle \tilde{u}_n, \tilde{f}_n \rangle \leq \|\tilde{u}_n\| \|\tilde{f}_n\| = \|\tilde{u}_n\|(1 + \varepsilon).
\]
Since \( \|\tilde{u}_n\| \leq \|(A[d_n] + \mu)^{-1}\| \|\tilde{f}_n\| \leq (1 + \varepsilon) \), we deduce from here that \( (\tilde{u}_n) \) is bounded in the topology of \( (A[+\infty] + \mu)^{1/2} \). Consequently, there is a weakly convergent subsequence \( (\tilde{u}_{n_k}) \) with respect to this topology, whose limit will be denoted by \( \tilde{u} \).

Consider \( v \in C_0^\infty(\Omega) \). Then we have
\[
\langle (A[d_{n_k}] + \mu)^{1/2} v, (A[d_{n_k}] + \mu)^{1/2} \tilde{u}_{n_k} \rangle = \langle v, \tilde{f}_{n_k} \rangle,
\]
J. Phys. A: Math. Theor. 49 (2016) 365205 M Tušek

10
which, due to the hypothesis (iii) and the Urysohn lemma implies that for all $k$ large enough,

$$\langle (A+[\infty])+\mu \rangle^{1/2}v, (A+[\infty])+\mu \rangle^{1/2}u_{n_k} = \langle v, \tilde{f}_{n_k} \rangle. \quad (12)$$

Since $(\tilde{f}_{n_k})$ is bounded, it has a weakly convergent subsequence, say $(\tilde{f}_{n_k})$. We will abuse the notation a little and just write $n_k$ instead of $n_{k_n}$. Now, in the limit $k \to +\infty$, (12) yields

$$\langle (A+[\infty])+\mu \rangle^{1/2}v, (A+[\infty])+\mu \rangle^{1/2}u = \langle v, \tilde{f} \rangle. \quad (13)$$

Since $C_0^\infty(\Omega)$ is a core of $(A+[\infty])+\mu$ and any core of a self-adjoint operator is a form core too, (13) implies that $\tilde{u} \in \text{Dom}(A+[\infty])$ and $(A+[\infty])+\mu)\tilde{u} = \tilde{f}$. Moreover, $(A+[\infty])+\mu)\tilde{u}_{n_k} = \tilde{f}_{n_k} \to \tilde{f}$ as $k \to +\infty$, which yields $(A+[\infty])+\mu)\tilde{u}_{n_k} \to (A+[\infty])+\mu)\tilde{u}$. Finally, by the compactness of $(A+[\infty])+\mu)^{-1}$,

$$\lim_{k \to +\infty} \|\tilde{u}_{n_k} - \tilde{u}\| = \lim_{k \to +\infty} \|(A[d_{n_k}]) + \mu)^{-1}\tilde{f}_{n_k} - (A+[\infty])+\mu)^{-1}\tilde{f}\| = 0.$$

Coming back to the untilded sequences, for all $k$ large enough, we obtain

$$\| (A[d_{n_k}]) + \mu)^{-1}\tilde{f}_{n_k} - (A+[\infty])+\mu)^{-1}\tilde{f}\| \leq \| (A[d_{n_k}]) + \mu)^{-1}\tilde{f}_{n_k} - (A+[\infty])+\mu)^{-1}\tilde{f}\| + \| (A+[\infty])+\mu)^{-1}\tilde{f}\| \to +\infty$$

The limit of the first term is zero, the second term is bounded by $\varepsilon$, and the third term is bounded by $\varepsilon^2$, too, since $\|\tilde{f} - f\| \to +\infty$.

**Proof of theorem 2.3.** For $\varepsilon = -\mu$ with $\mu$ specified in lemma 4.2, the theorem follows immediately. For $\varepsilon \in \text{Res}(A+[\infty])$, we use formula (21, (3.10) of chapter 4) to extend the resolvent estimate.

**Remark 4.3** (Alternative proof of theorem 2.3). During the peer review process, one of the reviewers proposed an alternative proof that does not actually require the $L^2$-setting of theorem 2.3. It is based on some of the properties of the collectively compact operator sequences. A set of operators is called collectively compact if and only if the union of the images of the unit ball is precompact [30]. The hypotheses (ii) and (iv) of theorem 2.3 imply that $(A[\alpha] - i)^{-1}$, $\alpha \geq \alpha_0$, is collectively compact. Indeed, let $\mathcal{B}$ be the closed unit ball (in $L^2(\Omega)$) and $M = \bigcup_{\alpha \geq \alpha_0} (A[\alpha] - i)^{-1}\mathcal{B}$. If we take $\psi \in M$, then there exists $\alpha \geq \alpha_0$ such that $\psi = (A[\alpha] - i)^{-1}\varphi$ for some $\varphi \in L^2(\Omega)$ with $\|\varphi\| \leq 1$. Using the functional calculus, we infer that $\| (A[\alpha] - i)^{-1}\| \leq 1$ and $\|A(A - i)^{-1}\| \leq 1$. Consequently, $\|\psi\| \leq 1$ and $\| (A[\alpha] - i)^{-1}\| \|A(A - i)^{-1}\| \|\varphi\| \leq 1$. If we introduce sets $\mathcal{A}_{\alpha_0,\psi} = \{ \psi \in \mathcal{A}(A[\alpha]) : \|\psi\| \leq 1, \langle \psi, A[\alpha] \psi \rangle \leq 1 \}$, we may write $\psi \in \mathcal{A}_{\alpha_0,\psi}$. By (ii) of theorem 2.3, $\psi \in \mathcal{A}_{\alpha_0,\psi}$. We conclude that $M \subset \mathcal{A}_{\alpha_0,\psi}$. The latter set is compact due to (iv) of theorem 2.3 and [18, theorem XIII.64]. Therefore, $M$ is precompact.

Using the hypotheses (i) and (iii) of theorem 2.3 together with [21, corollary VIII.1.6], we deduce that $(A[\alpha] - i)^{-1} \to (A+[\infty]) - i)^{-1}$ as $\alpha \to +\infty$. By [30, theorem 3.4 and proposition 2.1], the strong-resolvent convergence together with the collective compactness imply the norm-resolvent convergence, i.e., we have $\lim_{\alpha \to +\infty} \|(A[\alpha] - i)^{-1} - (A+[\infty]) - i)^{-1}\| = 0$. 

11
Acknowledgments

The author wishes to express his thanks to P Exner for drawing his attention to the Iwatsuka model and to D Krejčířik for pointing out some of his results that were useful in section 4. Last but not least, the author gratefully acknowledges the many useful suggestions from one of the reviewers that helped improve the manuscript significantly. The work has been supported by grant no. 13–11058S of the Czech Science Foundation (GAČR).

References

[1] Cycon H L, Froese R G, Kirsch W and Simon B 1987 Schrödinger Operators with Application to Quantum Mechanics and Global Geometry (Berlin Heidelberg: Springer-Verlag)
[2] Iwatsuka A 1985 Examples of absolutely continuous Schrödinger operators in magnetic fields Publ. RIMS, Kyoto Univ. 21 385–401
[3] Măntoiu M and Purice R 1997 Some propagation properties of the Iwatsuka model Comm. Math. Phys. 188 691–708
[4] Exner P and Kovářík H 2000 Magnetic strip waveguides J. Phys. A 33 3297–331
[5] Exner P, Joye A and Kovářík H 2001 Magnetic transport in a straight parabolic channel J. Phys. A 34 9733–52
[6] Bruneau V, Miranda P and Raikov G 2011 Discrete spectrum of quantum Hall effect Hamiltonians I: Monotone edge potentials J. Spec. Theo. 1 237–72
[7] Miranda P and Raikov G 2012 Discrete spectrum of quantum Hall effect Hamiltonians II: Periodic edge potentials Asymptot. Anal. 79 325–45
[8] Miranda P 2015 Eigenvalue asymptotics for a Schrödinger operator with non-constant magnetic field along one direction Anal. H. Poincaré 17 1713
[9] Dombrowski N, Germinet F and Raikov G 2011 Quantization of edge currents along magnetic barriers and magnetic guides Ann. H. Poincaré 12 1169–97
[10] Hislop P D and Soccorsi E 2015 Edge states induced by Iwatsuka Hamiltonians with positive magnetic fields J. Math. Anal. Appl. 422 594–624
[11] Dombrowski N, Hislop P D and Soccorsi E 2014 Edge currents and eigenvalue estimates for magnetic barrier Schrödinger operators Asymptot. Anal. 89 331–63
[12] Hislop P D, Popoff N, Raymond N and Sundqvist M P 2016 Band functions in the presence of magnetic steps Math. Models Methods Appl. Sci. 26 161–84
[13] Reijniers J and Peeters F M 2000 Snake orbits and related magnetic edge states J. Phys.: Condens. Matter 12 9771–86
[14] Yafaev D 2008 On spectral properties of translationally invariant magnetic Schrödinger operators Anal. H. Poincaré 9 181–207
[15] Leschke H, Warzel S and Weichlein A 2006 Energetic and dynamic properties of a quantum particle in spatially random magnetic fields with constant correlations along one direction Ann. H. Poincaré 7 335–63
[16] Krejčířík D, Raymond N and Tušek M 2015 The magnetic Laplacian in shrinking tubular neighbourhoods of hypersurfaces J. Geom. Anal. 25 2543–64
[17] Leinfelder H and Simader C G 1981 Schrödinger operators with singular magnetic potentials Mathematische Zeitschrift 176 1–9
[18] Reed M and Simon B 1978 Methods of Modern Mathematical Physics IV (New York: Academic Press)
[19] Naimark M A 1968 Linear Differential Operators. Part II (London: George G. Harrap & Co.)
[20] Filonov N and Sobolev A V 2006 Absence of the singular continuous component in the spectra of analytic direct integrals J. Math. Sci. 136 3826–31
[21] Kato T 1995 Perturbation Theory for Linear Operators 2nd edn (Berlin Heidelberg: Springer-Verlag)
[22] Exner P and Kovářík H 2015 Quantum Waveguides (Switzerland: Springer International)
[23] Grushin V V 2008 Asymptotic behavior of the eigenvalues of the Schrödinger operator in thin closed tubes Math. Notes 83 463–77
[24] Krejčířík D and Raymond N 2014 Magnetic effects in curved quantum waveguides Anal. H. Poincaré 15 1993–2024
[25] Bedoya R, Oliveira C R and Verri A A 2014 Complex gamma-convergence and magnetic Dirichlet Laplacian in bounded thin tubes *J. Spec. Theo.* 4 621–42
[26] Ekholm T and Kovařík H 2005 Stability of the magnetic Schrödinger operator in a waveguide *Commun. PDE* 30 539–65
[27] de Oliveira G 2014 Quantum dynamics of a particle constrained to lie on a surface *J. Math. Phys.* 55 092106
[28] Briet P, Raikov G and Soccorsi E 2008 Spectral properties of a magnetic quantum Hamiltonian on a strip *Asymptot. Anal.* 58 127–55
[29] Cazacu C and Krejčiřík D 2016 The Hardy inequality and the heat equation with magnetic field in any dimension *Commun. PDE* 41 1056–88
[30] Anselone P M and Palmer T W 1968 Spectral analysis of collectively compact strongly convergent operator sequences *Pacific J. Math.* 25 423–31