Oblivious Algorithms for the Maximum Directed Cut Problem

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Abstract

This paper introduces a special family of randomized algorithms for Max DICUT that we call oblivious algorithms. Let the bias of a vertex be the ratio between the total weight of its outgoing edges and the total weight of all its edges. An oblivious algorithm selects at random in which side of the cut to place a vertex v, with probability that only depends on the bias of v, independently of other vertices. The reader may observe that the algorithm that ignores the bias and chooses each side with probability 1/2 has an approximation ratio of 1/4, whereas no oblivious algorithm can have an approximation ratio better than 1/2 (with an even directed cycle serving as a negative example). We attempt to characterize the best approximation ratio achievable by oblivious algorithms, and present results that are nearly tight. The paper also discusses natural extensions of the notion of oblivious algorithms, and extensions to the more general problem of Max 2-AND.

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1 Introduction

Given a directed graph $G = (V, E, w)$ where $w : E \to \mathbb{R}^+$ is a weight function, and a set $S \subseteq V$, the weight of the directed cut defined by $S$ is the sum of $w((u, v))$ with $u \in S$, $v \notin S$. The goal of the maximum directed cut (Max DICUT) problem is finding a set such that the weight of the respective directed cut is as large as possible. The relative weight of a cut is the weight of the cut divided by the sum of the weights of all edges.

The maximum cut (Max CUT) problem is a similar problem; $G$ is undirected and the cut contains those edges with exactly one endpoint in $S$. Max CUT can be seen as a restriction of Max DICUT with two additional conditions: $(u, v) \in E$ iff $(v, u) \in E$ and every two antisymmetric edges have the same weight. Except in section 1.1, the term “cut” will mean directed cut, all graphs will be directed graphs, and, unless stated otherwise, all graphs will be weighted graphs.

Given a set of boolean variables $V$, a 2-AND formula is a set of clauses $C$, where each clause is a conjunction of two different literals (where a literal is a variable with either positive or negative polarity). Given a nonnegative weight function $w : C \to \mathbb{R}^+$ over the clauses, the weight of an assignment for the variables is the sum of weights of satisfied clauses. Max 2-AND is the problem of finding an assignment with maximum weight in a 2-AND formula.

Max DICUT is a special case of the Max 2-AND: Given a graph $G = (V, E, w)$, the set of variables will be $V$ and each edge will define a constraint that is true iff the first vertex is selected (the corresponding variable is true) and the second vertex is not selected (the corresponding variable is false).

**Definition 1.1.** An edge $(u, v)$ is an inedge for $v$ and an outedge for $u$. The outweight of a vertex is the sum of the weight of its outedges and the inweight of a vertex is the sum of the weight of its inedges.

**Definition 1.2.** The bias of a vertex is its outweight divided by the sum of its outweight and its inweight. The bias of a variable is the weight of the clauses in which it appears positively divided by the total weight of the clauses it appears in.

An oblivious algorithm for Max DICUT selects each vertex to be in $S$ with some probability that depends only on its bias and the selection of each vertex is independent of whether other vertices are selected. Similarly, an oblivious algorithm for Max 2-AND selects each variable to be true with some probability that depends only on its bias. The selection function of an oblivious algorithm is the function that maps a vertex’s (or variable’s) bias to the probability it is selected. All vertices must use the same selection function.

Note that a selection function uniquely determines an oblivious algorithm, so there will be no distinction between them in the text.

It will be assumed that the probabilities of selecting a vertex (or a variable) are antisymmetric. That is, if $f$ is the selection function of an oblivious algorithm then for all biases
$x \in [0, 1], f(x) + f(1 - x) = 1$, or equivalently, $f(1 - x) = 1 - f(x)$. This assumption seems natural, since with it, oblivious algorithms are invariant to reversing the direction of all edges of the graph. The assumption will be used in Section 2 and to get a better upper bound on the approximation ratio of oblivious algorithms.

The approximation ratio of an oblivious algorithm on a specific graph is the expected weight of the cut produced by the algorithm divided by the weight of the optimal cut. The approximation ratio of an oblivious algorithm is the infimum of the approximation ratios on all graphs. The approximation ratio of an oblivious algorithm for max 2-AND is defined similarly. The approximation ratio of an oblivious algorithm will be used as a measure for the quality of the algorithm.

An oblivious algorithm with positive approximation ratio must be random. Otherwise, in a graph where all neighborhoods look the same, such as a cycle, all vertices will belong to $S$ or no vertices will belong to $S$, so the weight of the cut will be 0.

We are primarily interested in oblivious algorithms, but we will also discuss two ways of using finite sets of oblivious algorithms. One is a mixed oblivious algorithm, that is, choosing an algorithm to use from the set according to some (fixed) probability distribution. The other is max of oblivious algorithms, that is, using all the algorithms in the set to generate cuts and outputting the cut with the maximal weight.

The approximation ratio for a mixed algorithm is its expected approximation ratio (where expectation is taken both over the choice of oblivious algorithm from the set, and over the randomness of the chosen oblivious algorithm).

There are two natural ways to define the approximation ratio of a max algorithm: either using $maxexp$ – the maximum (over all oblivious algorithms in the set) of the expected weight of the cut, or using $expmax$ – the expectation of the weight of the maximum cut. Observe that maxexp cannot be better than expmax, but expmax can be better than maxexp. For example, assume the set is a multiset containing a single algorithm multiple times. Then, expmax is equal to the approximation ratio of the algorithm, but maxexp may be better. However, it will be shown that the worst case approximation ratio when using expmax is the same as maxexp.

### 1.1 Related work

Our notion of oblivious algorithms can be viewed as a restricted special case of the notion of local algorithms used in distributed computing, which have been studied due to their simplicity, running time, and other useful characteristics [20].

The uniformly random algorithm selects each vertex (or sets each variable to true) independently with probability $\frac{1}{2}$. It gives a $\frac{1}{4}$ approximation to Max 2-AND and a $\frac{1}{2}$ approximation to Max CUT. There are algorithms that use semidefinite programming to achieve about 0.874 approximation to Max 2-AND [17] and about 0.878 approximation to Max CUT [11]. Assum-
ing the Unique Games Conjecture, these algorithms are optimal for Max CUT \cite{15, 19}, and nearly optimal for Max 2-AND (which under this assumption is hard to approximate within 0.87435 \cite{5}). Earlier NP-Hardness results are \(\frac{11}{12}\) for Max 2-AND and \(\frac{16}{17}\) for Max CUT \cite{13}.

Trevisan \cite{21} shows how to get \(\frac{1}{2}\) approximation to Max 2-AND using randomized rounding of a linear program. Halperin and Zwick \cite{12} show simple algorithms that achieve \(\frac{2}{5}\) and \(\frac{9}{20}\) approximation ratios, and a combinatorial algorithm that finds a solution to the previous linear program.

Bar-Noy and Lampis \cite{8} present an online version of Max DICUT for acyclic graphs. Vertices are revealed in some order (respecting the order defined by the graph), along with their inweight, outweight, and edges to previously revealed vertices, and based on this information alone they are placed in either side of the cut. They show that an algorithm achieving an approximation ratio of \(\frac{2}{3\sqrt{3}}\) is optimal against an adaptive adversary. They also show that derandomizing the uniformly random algorithm gives an approximation ratio of \(\frac{1}{3}\). Oblivious algorithms can be used in online settings, and in fact, they do not require the graph to be acyclic and do not require edges to previously revealed vertices to be given. The reason why the approximation ratios in the current manuscript are better than \(\frac{2}{3\sqrt{3}}\) is that our approximation ratio is computed against an oblivious adversary.

Alimonti shows a local search algorithm that achieves an approximation ratio of \(\frac{1}{4}\) for Max 2-AND \cite{1}, and uses non-oblivious local search to achieve a \(\frac{2}{5}\) approximation \cite{2}.

Alon et al. \cite{3} show that the minimal relative weight of a maximum directed cut in acyclic unweighted graphs is \(\frac{1}{4} + o(1)\). Lehel, Maffray and Preissmann \cite{16} study the minimal weight of a maximum directed cut (in unweighted graphs) where the indegree or outdegree of all vertices is bounded. They show that the smaller the degree the larger the maximum cut. If the indegree or outdegree is 1 for all vertices, the minimal relative weight is \(\frac{1}{3}\). If the graph also has no directed triangles, the minimal relative weight is \(\frac{2}{5}\).

Feige, Mirrokni and Vondrak \cite{10} show an algorithm that achieves a \(\frac{2}{5}\) approximation to any nonnegative submodular function (and directed cut is a special case of a submodular function).

1.2 Our results

The main results of the paper are theorems 1.8 and 1.10 that show that there is an oblivious algorithm that achieves an approximation ratio of 0.483, but no oblivious algorithm can achieve an approximation ratio of 0.4899. In the process of proving these theorems, a few other interesting results are shown.

Max DICUT is a special case of Max 2-AND, and hence approximation algorithms for Max 2-AND apply to Max DICUT as well. The following theorem shows a converse when oblivious algorithms are concerned.
Theorem 1.3. Given any antisymmetric selection function \( f \), the approximation ratio of the corresponding oblivious algorithm for Max 2-AND is the same as that for Max DICUT.

Hence our results concerning oblivious algorithms for Max DICUT extend to Max 2-AND. We remark that for general approximation algorithms, it is not known whether Max 2-AND can be approximated as well as Max DICUT (see [6] for example).

The following rather standard proposition justifies the use of expected approximation ratio as the measure of quality of a randomized approximation algorithm. It also holds for mix and max algorithms, and implies that \text{expmax} is not better than \text{maxexp} in the worst case.

**Proposition 1.4.** Given a graph \( G \) and \( \epsilon > 0 \), there is another graph \( G_\epsilon \) such that with high probability, for any oblivious algorithm, with high probability, the weight of the cut produced by running the algorithm on \( G_\epsilon \) is close to the expected weight of the cut on \( G \) up to \( \pm \epsilon \), and the weight of the optimal cut of both graphs is the same.

**Proof (Sketch).** This proposition follows immediately from the law of large numbers and a graph composed of many disjoint copies of the original graph (with the weight of edges normalized to be the same as for the original graph).

When using the same set of algorithms, the \text{max} algorithm is not worse than any \text{mixed} algorithm. The following theorem shows that the converse holds for some mixed algorithm.

**Theorem 1.5.** Given a finite set of algorithms, there is a mixed algorithm over the set such that the worst case approximation ratio is as good as that of the \text{max} algorithm of the set.

\( f : [0,1] \to [0,1] \) is a **step function** if there are \( 0 = z_0 < z_1 < \cdots < z_n < z_{n+1} = 1 \) such that \( f \) is constant on \((z_i, z_{i+1})\). We first show a simple step function that has an approximation ratio that is better than the trivial oblivious algorithm. Unlike the proof of theorem 1.8, which is computer assisted, we show a complete analytic proof of the following result.

**Theorem 1.6.** There is step function with three steps such that the corresponding oblivious algorithm has approximation ratio \( \frac{3}{8} \).

We will primarily consider step functions because any function can be approximated using a step function, in the sense that the step function will have an approximation ratio that is worse by at most an arbitrarily small constant (that depends on the width of the steps). In addition, we will show how to compute the approximation ratio of any step function.

**Theorem 1.7.** Given a selection function that is a step function with \( m \) steps, the approximation ratio of the corresponding oblivious algorithm can be computed as the solution of a linear program with \( O(m) \) constraints and \( O(m^2) \) variables.
Using a linear program to find the approximation ratio of an algorithm is referred to as *Factor Revealing Linear Programs* and was used to find the approximation ratio of algorithms for facility location [14], $k$-set cover [4], and buffer management with quality of service [7]. It was also used to find the best function to use in an algorithm for matching ads to search results [18].

**Theorem 1.8.** There is an oblivious algorithm with a step selection function that achieves an approximation ratio of at least 0.483.

We provide a computer assisted proof of Theorem 1.8 using the linear programming approach of Theorem 1.7.

The family of linear programs can be used to find the best oblivious algorithm, up to some additive factor.

**Theorem 1.9.** Given $n \in \mathbb{N}$, there is an algorithm that uses time $\text{poly}(n) n^n$ to find the best oblivious algorithm up to an additive factor of $O\left(\frac{1}{n}\right)$.

A trivial upper bound on the approximation ratio of every oblivious algorithm is $\frac{1}{2}$. For a directed even cycle, the maximum cut has relative weight $\frac{1}{2}$, whereas an oblivious algorithm can capture at most one quarter of the edges, in expectation. We improve this upper bound on the approximation ratio and show that the function from Theorem 1.8 is very close to being optimal.

**Theorem 1.10.** There is a weighted graph for which the approximation ratio of any oblivious algorithm (with an antisymmetric selection function) is less than 0.4899.

Since the upper bound is shown by a single graph, the bound holds not only for a single oblivious algorithm, but also for mixed and max algorithms.

Analyzing the approximation ratios of oblivious algorithms on weighted and unweighted graphs is practically the same. The proof of Proposition 1.11 follows standard arguments (see [9], for example) and appears in Section C in the appendix.

**Proposition 1.11.** For every oblivious algorithm the approximation ratio is the same for weighted and unweighted graphs.

Theorem 1.10 uses the fact that selection functions are antisymmetric. One might think that this is what prohibits us from reaching an approximation ratio of $\frac{1}{2}$. However, even selection functions that are not antisymmetric cannot achieve an approximation ratio of $\frac{1}{2}$, or arbitrarily close to $\frac{1}{2}$.

**Theorem 1.12.** There is a constant $\gamma > 0$ such that any oblivious algorithm, even one not using an antisymmetric selection function, has an approximation ratio at most $\frac{1}{2} - \gamma$.  

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2 2-And versus Directed Cut

In this section we prove Theorem 1.3. The theorem follows from the next lemma:

Lemma 2.1. Given an instance of Max 2-AND, \( \varphi \), there is a graph \( G_\varphi \), such that the approximation ratio of an oblivious algorithm on \( \varphi \), using a selection function \( f \), is not worse than an oblivious algorithm on \( G_\varphi \), using the same selection function.

Proof. Consider an instance \( \varphi = (V, C, w) \) of Max 2-AND. We will create a directed graph \( G_\varphi = (V', E, w') \). \( V' = \{ x, \bar{x} | x \in V \} \), the set of all literals. For any clause \( c \in C, c = y \land z \) (where \( y, z \) are literals) there are two edges in \( E \): one from the vertex \( y \) to the vertex corresponding to the negation of \( z \) and another from the vertex \( z \) to the vertex corresponding to the negation of \( y \). Each of these edges has weight \( \frac{1}{2}w(c) \).

Every assignment for \( \varphi \) can be transformed to a cut for \( G_\varphi \) of the same weight, trivially, by selecting all (and only) literals (as vertices in the graph \( G_\varphi \)) that are true in the assignment. Hence the optimal cut weighs at least as much as the optimal assignment. Note, however, that the converse does not hold. For example, for the following set of clauses: \( \{ x \land y, \bar{x} \land y, x \land \bar{y}, \bar{x} \land \bar{y} \} \) the weight of the optimal assignment is 1, whereas the optimal cut in the graph has weight 2. (Select \( x \) and \( \bar{x} \), a selection that does not correspond to an assignment.)

The expected weight of an assignment for \( \varphi \) is equal to the expected weight of a cut in \( G_\varphi \), when using oblivious algorithms with the same selection function. Note that the bias of a vertex is equal to the bias of the corresponding literal (where the bias of a negation of a variable is one minus the bias of the variable). Thus, the respective probabilities are equal. Hence, the probability of any clause being satisfied is equal to the probability of each of the two edges generated from the clause being in the cut. Since the weight of the edges is one half of the weight of the clause, and due to the linearity of expectation, the claim follows. \( \square \)

See Section A in the appendix for some remarks on the above proof.

3 Mix versus Max

In this section we will prove Theorem 1.5. We first present an example that illustrates the contents of the theorem.

The uniformly random algorithm selects every vertex independently with probability \( \frac{1}{2} \).

Proposition 3.1. The uniformly random algorithm has an approximation ratio of \( \frac{1}{4} \).

Proof. An edge is expected to be in the cut with probability \( \frac{1}{4} \) (each vertex of the edge in one side and in the correct direction) and the weight of the cut is at most all the edges. \( \square \)
The greedy algorithm selects a vertex if the outweight is larger than the inweight (for equal weights the selection can be arbitrary).

**Proposition 3.2.** If the relative weight of the maximal cut is $1 - \epsilon$, then the greedy algorithm produces a cut of relative weight at least $1 - 2\epsilon$.

**Proof.** Consider a maximum cut in the graph of relative weight $1 - \epsilon$. An endpoint of an edge is said to be misplaced by an algorithm if it is an outedge not placed in $S$ or an inedge that is placed in $S$. An edge is not in the cut if at least one of its endpoints is misplaced. The relative weight of endpoints not in the optimal cut is $2\epsilon$.

Now, estimate the relative weight of edges not in the cut produced by the greedy algorithm, using the edges’ endpoints. The greedy algorithm minimizes the weight of the endpoints not in the cut, but every endpoint may correspond to an edge. Since the estimate is at most $2\epsilon$, the relative weight of the edges is at most $2\epsilon$.

Let us consider the max of the uniformly random algorithm and the greedy algorithm. The approximation ratio is $\frac{2}{5}$: when the weight of the maximal cut is at most $\frac{5}{8}$ of the edges, the uniformly random algorithm will give an approximation ratio of at least $\frac{2}{5}$ and when at most $\frac{3}{8}$ are not in the cut, the greedy algorithm will give an approximation ratio of at least $\frac{2}{5}$ ($\frac{1 - 2\epsilon}{1 - \epsilon}$ is a decreasing function).

This approximation ratio is optimal:

Let us now consider a mixed algorithm using the two algorithms. Let $1 - \epsilon$ be the relative weight of the cut. A mixed algorithm using the greedy algorithm with probability $\gamma$ and the uniformly random otherwise will give a cut of relative weight $(1 - \gamma) \frac{1}{4} + \gamma (1 - 2\epsilon)$.

For $\gamma = \frac{1}{5}$, the mixed algorithm gives an approximation ratio of $\frac{2}{5}$.

The equality of the approximation ratios of the max and mixed algorithms is not accidental. Define a two player zero sum game: Player A (for algorithm) has a finite set of pure strategies corresponding to oblivious algorithms. Player G (for graph) has pure strategies corresponding to all graphs. When both players use pure strategies, player A is given a payoff equal to the approximation ratio of the algorithm (corresponding to the selected strategy by A) on the graph (corresponding to the selected strategy by G). A max algorithm (using the maxexp notion of approximation ratio) is the same as allowing player A to select a pure
strategy after player G has chosen a pure strategy. A mixed algorithm is the same as allowing player A to use a mixed strategy. By the minimax theorem, the best mixed strategy gives the same payoff as having first G choose a mixed strategy (a distribution over graphs), and then A chooses the best pure strategy against this distribution. Now the key observation showing equality (up to arbitrary precision) between mixed and max is that every mixed strategy for G can be approximated by a pure strategy of G. Player G can choose a single graph instead of a distribution of graphs: By losing at most $\epsilon$ of the payoff (for any $\epsilon > 0$), it can be assumed that the distribution over the graphs is rational and finitely supported. That is, the mixed strategy is $\left(\frac{p_1}{M}, \ldots, \frac{p_n}{M}\right)$, where $p_i, M \in \mathbb{N}$ and $\frac{p_i}{M}$ is the probability of selecting the graph $G_i$.

Construct $G^*$ from a disjoint union of $G_i$ (for $1 \leq i \leq n$), and multiply the weights of the edges of the copy of $G_i$ in $G^*$ by the inverse of the weight of the optimal cut in $G_i$ times $p_i$ (so that the weight of the optimal cut in $G^*$ is 1). On $G^*$, no pure strategy of A gives an approximation ratio better than $\beta + \epsilon$ (where $\beta$ is the value of the game). Hence, given a set of oblivious algorithms, a max algorithm is not better (up to arbitrary precision) than the best mixed algorithm.

Note that a mixed algorithm (over a set of oblivious algorithms) is not an oblivious algorithm. We do not know if there are mixed algorithms with worst case approximation ratios better than those for oblivious algorithms.

### 4 An oblivious algorithm with $\frac{3}{8}$ approximation ratio

In this section we will prove Theorem 1.6.

There is another way to “mix” between the greedy and uniform algorithm. Consider the family of selection functions $f_\delta$, for $0 < \delta < \frac{1}{2}$, where

$$f_\delta(x) = \begin{cases} 0 & 0 < x < \delta \\ \frac{1}{2} & \delta \leq x \leq 1 - \delta \\ 1 & 1 - \delta < x < 1 \end{cases}$$

Fix $\delta$ and consider a graph $G$. Divide its vertices into two sets: $U$ (unbalanced), which will contain all vertices with bias at most $\delta$ or more than $1 - \delta$, and $B$ (balanced), the rest of the vertices. We will modify (by adding vertices and removing and adding edges) the graph to have a simpler structure while making sure that the expected weight of the cut will not increase (when using $f_\delta$), the weight of the optimal cut will not decrease, and that the biases of all original vertices will remain the same. Divide $U$ further into $U^+$, the set of vertices of

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1The argument is a bit more delicate because G has infinitely many pure strategies. A form of the Minimax theorem holds also in this case since the payoffs are bounded (see for example Theorem 3.1 in [22]). For any $\epsilon > 0$ there is a value $\beta$, such that A has a mixed strategy with payoff at least $\beta - \epsilon$, and G has a mixed strategy limiting the payoff to be at most $\beta$. 

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bias more than $1 - \delta$, and $U^-$, the set of vertices of bias less than $\delta$. Consider an edge $(u, v)$ between $U^+$ to $U^-$ (in either direction). Add a new vertex, $w$ and two new edges $(u, w)$ and $(w, v)$, both with the same weight of the edge $(u, v)$ and remove the edge $(u, v)$.

After the transformation of $G$, there are no edges between $U^+$ and $U^-$. By normalizing the weight of the graph, we may assume that the sum of weights of edges is 1. Let $p^+$ be the weight of endpoints in $U^+$, $p^-$ the weight of endpoints in $U^-$, so there are $2 - p^+ - p^-$ endpoints in $B$. Let $\gamma^- p^-$ be the weight of endpoints misplaced by $f_\delta$ in $U^-$ and let $\gamma^+ p^+$ be the weight of endpoints misplaced by $f_\delta$ in $U^+$. Let $a^p$ be the weight of edges inside $U^+$, $b^p$ the weight of edges from $B$ to $U^+$, and $c^p$ the weight of edges from $U^+$ to $B$. Since all of the edges from $U^+$ to $B$ are not misplaced, we have $c = 1 - 2\gamma^+ + \epsilon^+$, where $0 \leq \epsilon^+ \leq \gamma^+$. We also have $2a + b + c = 1$ and $a + b = \gamma^+$, so $a + c = 1 - \gamma^+$, $a = \gamma^+ - \epsilon^+$ and $b = \epsilon^+$. Similarly, we have $(1 - 2\gamma^- + \epsilon^-) p^-$ edges from $U^-$ to $B$ and $\epsilon^- p^-$ edges from $B$ to $U^+$. The expected weight of the edges in the cut (produced cut by $f_\delta$) inside $B$ is

$$\frac{2 - p^- - p^+ - (1 - 2\gamma^- + 2\epsilon^-) p^- - (1 - 2\gamma^+ + 2\epsilon^+) p^+}{8}.$$

The expected weight of the edges in the cut from $U$ to $B$ is

$$\frac{(1 - 2\gamma^- + \epsilon^-) p^- + (1 - 2\gamma^+ + \epsilon^+) p^+}{2}.$$

The optimal cut must misplace at least $\delta$ of the endpoints in $B$ and endpoints of weight $\gamma^+ p^+ \gamma^- p^-$ in $U$, so the optimal cut has weight of at most

$$1 - \frac{\delta (2 - p^- - p^+) + \gamma p^+ + \gamma^- p^-}{2}.$$

Setting $\epsilon^- = 0 = \epsilon^+$ lowers the expected weight of the cut without affecting the weight of the bound for the optimal cut, so we will assume this is the case. Define $p = p^+ + p^-$ and $\gamma$ such that $\gamma p = \gamma^+ p^+ - \gamma^- p^-$. $2\gamma p$ is the weight of misplaced endpoints in $U$ so $\gamma \leq \delta$. The approximation ratio is no worse than

$$\frac{1 + (1 - 3\gamma) p}{4 (1 - \delta) + 2p (\delta - \gamma)}.$$

Setting $\delta = \frac{1}{3}$ and differentiating the approximation ratio shown according to $\gamma$ gives numerator (the denominator is positive for $p \geq 0, \gamma \leq \delta$)

$$-3p \left( \frac{8}{3} + 2p \left( \frac{1}{3} - \gamma \right) \right) - 2p (1 + (1 - 3\gamma) p) =$$
For $0 \leq \gamma \leq \delta$ and $p \geq 0$, the derivative is non-positive, so maximizing $\gamma$ will give the lowest approximation ratio. For $\gamma = \delta = \frac{1}{3}$, the approximation ratio is at least $\frac{3}{8}$.

A graph with two vertices $X,Y$ with edge of weight $\frac{2}{3}$ from $X$ to $Y$ and an edge of weight $\frac{1}{3}$ from $Y$ to $X$ shows $\frac{3}{8}$ to be an upper bound on the approximation ratio of $f_{\frac{1}{3}}$.

We remark that a slightly larger value of $\delta$ can give an approximation ratio better than 0.375, and in fact better than 0.39. This can be verified using the linear programming approach of Theorem 1.7.

5 Finding approximation ratios via linear programs

Proof of Theorem 1.7. For a given step function $f$, we present a linear program that constructs a graph with the worst possible approximation ratio for the oblivious algorithm that uses $f$ as a selection function.

Suppose that the set of discontinuity points of the step function $f$ is $0 = z_0 \leq z_1 \leq z_2 \leq \cdots \leq z_{n-1} \leq z_n = 1$. An isolated point (that is neither left continuous nor right continuous) is counted as two discontinuity points, for a reason that will become apparent later. In the graph produced by the LP, a certain subset $S$ of vertices will correspond to the optimal cut in the graph, $T_i$ corresponds to the set of vertices in $S$ with bias between $z_{i-1}$ and $z_i$, and $T_{i+n}$ corresponds to the set of vertices not in $S$ with bias between $z_{i-1}$ and $z_i$ (A vertex with bias $z_i$ for some $i$ can be chosen arbitrarily to be in one of the sets). We assume that the weights of the edges are normalized such that the weight of the cut corresponding to $S$ is 1. The variable $e_{ij}$ denotes the weight of the edges from the set $T_i$ to the set $T_j$. Let $l, u : \{1..2n\} \to \{1..n\}$ be such that $T_i$ contains the set of vertices of biases between $z_{l(i)}$ and $z_{u(i)}$ ($l(i) < u(i)$).

We have the following constraints:

- $\sum_{i \leq n \atop j > n} e_{ij} = 1$ - The weight of the cut is 1.
- $\forall i \ z_{l(i)} \sum_j (e_{ij} + e_{ji}) \leq \sum_j e_{ij} \leq z_{u(i)} \sum_j (e_{ij} + e_{ji})$ - The (average) bias of the vertices is correct.
- $\forall i, j \ e_{ij} \geq 0$ - The weight of edges must be nonnegative.

Note that $e_{ii}$ appears twice in $\sum_j (e_{ij} + e_{ji})$, since it contributes to both outweight and inweight.

Let $p_i = p_{i+n} = f\left(\frac{z_{l(i)} + z_{u(i)}}{2}\right)$ be the probability of selecting a vertex in the sets $T_i$ and $T_{i+n}$. (Here we used the convention that isolated points $z_i$ appear twice.) The expected weight of the cut is $\sum_{i,j} p_i (1 - p_j) e_{ij}$, and this is the approximation ratio of the oblivious algorithm on
the graph if the cut corresponding to $S$ is optimal. Minimizing $\sum_{i,j} p_i (1 - p_j) e_{ij}$ subject to the constraints gives a graph on which $f$ attains its worst approximation ratio. There are some minor technicalities involved in formally completing the proof, and the reader is referred to Section 13 in Appendix for these details. 

The linear program produces a graph that shows the worst approximation ratio of the selection function used. The dual of the linear program will produce a witness that the approximation ratio of the selection function is not worse than the value output. Every constraint of the dual has at most five variables, so it is technically possible to check that the approximation ratio is correct. However, even in the case of the function family from Section 16 where a function has three steps, the linear program has 15 constraints and 36 variables and the dual has 15 variables and 36 constraints. Hence, verifying the results manually becomes tedious. It is possible to decrease the number of variables in the primal by using symmetrization of the graph (adding another graph with inverted edges), but this will complicate the description of the linear program and will increase the probability of errors in the programming, while only reducing the number of variables by half.

We implemented the linear program above. To gain confidence in our implementation, we checked the linear program on the step function form Theorem 1.6 and indeed got that the approximation ratio is $\frac{3}{8}$. We also checked the program on the uniformly random algorithm and got $\frac{1}{4}$ approximation ratio, as expected.

We now prove theorem 1.8. Define $f(x)$ to be 0 for $x < 0.25$, 1 for $x > 0.75$, for any $0 \leq i < 100$, if $0.25 + 0.005i < x < 0.25 + 0.005(i + 1)$, $f(x) = 0.005 + 0.01i$, $f(\frac{1}{2}) = \frac{1}{2}$, and right or left continuous on all other points. Using the corresponding linear program, as defined in theorem 1.7, we have determined that the approximation ratio of $f$ is more than 0.4835 but not more than 0.4836 (according to the values of the primal and the dual), as claimed in Theorem 1.8. $f$ can be seen as a discretized version of the function $g(x) = \max\{0, \min\{1, 2(x - \frac{1}{2}) + \frac{1}{2}\}\}$, and we believe that the approximation ratio of $g$ is slightly better. In principle, it is possible to show this, using a finer discretized version of the function. However, it is too time consuming to check this, so we did not do it.

The proof of Theorem 1.9 is based on a standard discretization argument. See Section 13 in the appendix for more details.

6 An upper bound on oblivious approximation ratios

To prove Theorem 1.10 we construct two weighted graphs, $G_1$ and $G_2$. To get a good approximation ratio for $G_1$, the probability of selecting a vertex with bias $\frac{5}{9}$ needs to be close to $1/2$, whereas for $G_2$ it needs to be far from $1/2$. Combining the two graphs gives a single graph that upper bounds the approximation ratio of any oblivious algorithm. We remark that a
linear program similar to that of Section 5 assisted us in constructing \( G_1 \) and \( G_2 \).

Example 6.1. \( G_1 \) is the following weighted graph:

\[
\begin{array}{c}
A & \quad \quad & B & \quad \quad & C \\
| & c & | & c & | \\
A' & \quad \quad & B' & \quad \quad & C' \\
\end{array}
\]

where \( c_2 = c^2 - 1 \).

Note that:

- The bias of \( A \) and \( A' \) is \( \frac{c}{c+1} \).
- The bias of \( B \) and \( B' \) is \( \frac{1}{2} \).
- The bias of \( C \) and \( C' \) is \( \frac{1}{c+1} \).
- There is a cut of weight \( 2c^2 \) by selecting \( A, B, \) and \( C \).

Let \( \alpha \) be the probability of selecting a vertex with bias \( \frac{c}{c+1} \) for some oblivious algorithm (then the probability of selecting a vertex with bias \( \frac{1}{c+1} \) is \( 1 - \alpha \)). Then the expected value of a solution produced by the algorithm is

\[
2\alpha (1 - \alpha)(1 + c) + \left( \alpha + \frac{1}{4} \right) (c^2 - 1)
\]

And the approximation ratio is most

\[
\frac{2\alpha (1 - \alpha)(1 + c) + \left( \alpha + \frac{1}{4} \right) (c^2 - 1)}{2c^2}
\]

Example 6.2. \( G_2 \) is the following weighted graph:

\[
\begin{array}{c}
D & \quad \quad & E \\
| & c & | \\
1 & \quad \quad & \frac{1}{c+1} \\
E' & \quad \quad & F' \\
\end{array}
\]

where \( c_1 = c - 1 \).

Note that:

- The bias of \( D \) is \( \frac{c}{c+1} \).
- The bias of \( E \) and \( E' \) is \( \frac{1}{2} \).
- The bias of \( F' \) is \( \frac{1}{c+1} \).
There is a cut of weight $2c$ by selecting $D$ and $E$.

Let $\alpha$ be the probability of selecting the vertex $D$ (and $1-\alpha$ is the probability of selecting the vertex $F'$).

The expected weight of the cut is

$$c\alpha + \frac{c-1}{4} + 1 - \alpha$$

The approximation ratio is

$$\frac{1 + (\alpha + \frac{1}{4}) (c - 1)}{2c}$$

Consider a graph composed of one copy of $G_1$ and three copies of $G_2$. The approximation ratio is at most

$$\frac{2\alpha (1 - \alpha) (1 + c) + (\alpha + \frac{1}{4}) (c^2 - 1) + 3 (\alpha + \frac{1}{4}) (c - 1)}{2c^2 + 6c}$$

which, for fixed $c$, is a parabola with a maximal point.

For $c = 1.25$, the approximation ratio is

$$\frac{213 + 372\alpha - 288\alpha^2}{680}$$

the maximum is achieved at $\alpha = \frac{31}{48}$, and the value there is $\frac{5393}{10560} < 0.4899$. Hence, no algorithm based on oblivious algorithms (maximum of several oblivious algorithms or choosing one to use according to some distribution) can achieve better approximation ratio and this graph proves Theorem 1.10.

The proof of Theorem 1.12 appears in Section D in the appendix.

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A Some remarks on 2-AND versus DICUT

Despite the fact that the reduction from 2-AND to DICUT does not preserve the weight of an optimal solution (a cut may have larger weight than the weight of any assignment), it is possible to use a more generalized version of an oblivious algorithm to generate only such cuts that can be transformed to valid assignments. Instead of selecting \( x \) and \( \bar{x} \) to be in \( S \) independently, choose \( x \) to be in \( S \) according to the selection function and set \( \bar{x} \) to be in \( S \) iff \( x \notin S \). The probability of \( x \) and \( \bar{x} \) to be in \( S \) is the same as before, and since there are no edges between them, the approximation ratio is the same, due to the linearity of expectation.

This can be generalized further: Instead of choosing vertices independently, divide any graph into disjoint independent sets. Fix a selection function \( f \). In each set, the marginal probability of a vertex being in \( S \), will be the same as dictated by \( f \) (however, the choices inside each independent set need not be independent). Then, due to the linearity of expectation, the approximation ratio is the same.
For example, if the graph is $G_\varphi$, and the selection function decides that all vertices will be in $S$ with probability $\frac{1}{2}$, by choosing both $x$ and $\bar{x}$ to be in $S$ or both not in $S$, we can find a cut with the property that $x \in S$ iff $\bar{x} \in S$, with the same expected weight of a valid assignment.

B Notes on linear programs

Here we provide some details required for a formal proof of Theorem 1.7.

Suppose that $r$ is the minimum value of the linear program and that $\forall i e_{ii} = 0$ for the optimal solution. Define a vertex for each $T_i$ and an edge $(T_i, T_j)$ with weight $e_{ij}$ for all $i, j$. This is a graph with the property that $f$ achieves an approximation ratio $r$ if no vertex has bias $z_i$ for some $i$. However, this is a minor obstacle; add a vertex with total weight $\epsilon > 0$ arbitrarily small, with edges to or from all vertices with biases exactly $z_i$, so that their biases will change slightly and the probability of selecting the vertex $T_i$ will be $p_i$. The infimum of the approximation ratios on the graphs (as $\epsilon \to 0$) will be $r$. Now, assume that for some $i$’s $e_{ii} > 0$. Construct the previous graph (without self loops). For $i$ such that $e_{ii} > 0$, split the vertex $T_i$ into two vertices, $A_i$ and $B_i$. Every edge with an endpoint of $T_i$ will be split to two edges, each with half the weight such that one will have endpoint $A_i$ instead of $T_i$, and the other will have endpoint $B_i$ instead of $T_i$. Add the edges $(A_i, B_i)$ and $(B_i, A_i)$, each with weight $\frac{e_{ii}}{2}$. All the constraints hold for the graph.

Proof Sketch of Theorem 1.9. Consider $\mathcal{F}$, the family of $n^{n+1}$ antisymmetric step functions that are constant on each of the $2n$ intervals of width $\frac{1}{2n}$ of the unit interval, and the value on each of those intervals is of the form $\frac{k}{n}$ with $k \in \mathbb{N}$, and are left or right continuous between the intervals. Also, in order to be antisymmetric, $\forall f \in \mathcal{F} f \left( \frac{1}{2} \right) = \frac{1}{2}$. As a corollary from the proof of the linear program, left or right continuity of a step function does not change the approximation ratio, so there are indeed only $n^{n+1}$ functions (due to antisymmetry) to consider when looking at the approximation ratio. Using $n^{n+1}$ linear programs (time $\text{poly} \left( n \right) n^n$) it is possible to find the function with the best approximation ratio from the set.

It is possible that the best function from the set is not the best possible selection function. However, it is close. Suppose that the best selection function is a step function that is constant on the same intervals, but may have any value on those intervals. Let $g$ be such a step function, and let $f$ be the closest function (in $\ell_\infty$ distance) from $\mathcal{F}$. Then, the probability an edge being in the cut when using $g$ instead of $f$ is at most $O \left( \frac{1}{n} \right)$ larger, so the approximation ratio of $f$ is at most $O \left( \frac{1}{n} \right)$ lower than $g$.

Now, fix any selection function $h$. Let $g$ be a step function that is constant on each of the $2n$ intervals of width $\frac{1}{2n}$ of the unit interval such that for all $k \in \mathbb{N}$ with $k \leq 2n$, $g \left( \frac{k}{2n} \right) = h \left( \frac{k}{2n} \right)$. Given any “bad” graph for $g$, by adding new edges and a single vertex, it
can be transformed into a “bad” graph for \( h \), and the approximation ratio will be at most \( O \left( \frac{1}{n} \right) \) lower. Thus, the approximation ratio of \( g \) is at most \( O \left( \frac{1}{n} \right) \) lower than \( h \).

Therefore, the approximation ratio of the best function from \( F \) has approximation ratio worse by at most \( O \left( \frac{1}{n} \right) \) than any oblivious algorithm.

\[ \square \]

C Weighted versus unweighted graphs

Lemma C.1. Any weighted graph \( G \) with rational weights can be transformed to an unweighted graph \( G' \) such that for any oblivious algorithm the approximation on \( G' \) will not be better than the approximation ratio on \( G \).

Proof. Let \( G = (V, E, w) \). Define \( W \) to be \( \max_{e \in E} w(e) \). Define \( w' \) to be \( \frac{w}{W} \). \((V, E, w')\) is a weighted graph with rational weights, the maximal weight is 1, and the approximation ratio is the same as the approximation ratio for \( G \). There are \( w_e, M \in \mathbb{N} \) such that \( w'(e) = \frac{w_e}{M} \) for all \( e \in E \).

Let \( V' \) be composed of \( M \) copies of \( V \). \( v \in V \) will be identified with \( \{v_1, \ldots, v_M\} \subseteq V \). For every \( e = (v, u) \in E \), create a \( w_e \)-regular bipartite graph between \( \{v_1, \ldots, v_M\} \) and \( \{u_1, \ldots, u_M\} \) in \( E' \) (directed towards \( \{u_1, \ldots, u_M\} \)).

\( G' = (V', E') \) satisfies the condition of the lemma.

We can now prove Proposition 1.11. Given an arbitrary selection function and arbitrary \( \epsilon > 0 \), the proof of Theorem 1.9 shows that the selection function can be replaced by a step function that is continuous on all irrational values, and the approximation ratio deteriorates by at most \( \epsilon \). Any weighted graph \( G \) has finitely many vertices and biases, and since the selection function is continuous at all irrational values, \( G \) can be transformed to \( G' \) with only rational weights and an approximation ratio higher by at most \( \epsilon \) (by changing each irrational weight by a small value). Letting \( \epsilon \) tend to 0 proves Proposition 1.11.

D Selection functions that are not antisymmetric

Proof of Theorem 1.12. Let \( G = (V, E, w) \) and create \( G' = (V, E', w') \) from \( G \) by inverting all edges. That is, \( E' = \{(v, u) | (u, v) \in E\} \) and \( w'((v, u)) = w((u, v)) \). Let \( G'' \) be the disjoint union of these graphs. Consider a selection function \( f \) that is not antisymmetric. Let \( g(x) = \frac{f(x) + 1 - f(1 - x)}{2} \). \( g \) is antisymmetric. Let \( (u, v) \) be an edge, where the bias of \( u \) is \( r_u \) and the bias of \( v \) is \( r_v \). The probability of selecting \( (u, v) \) when using \( g \) is \( g(r_u) \) \((1 - g(r_v)) \) or \( \frac{1}{2} (f(r_u) + 1 - f(1 - r_u)) (f(1 - r_v) + 1 - f(r_v)) \). The expected weight contributed by \( (u, v) \) and \( (u, v) \) when using \( g \) is

\[ \frac{1}{2} (f(r_u) + 1 - f(1 - r_u)) (f(1 - r_v) + 1 - f(r_v)) \]
and when using $f$ the expected weight is $f(r_u) (1 - f(r_v)) + f(1 - r_v) (1 - f(1 - r_u))$. The advantage of using $g$ over $f$ is

$$
\frac{1}{2} (1 - f(r_u) - f(1 - r_u)) (1 - f(r_v) - f(1 - r_v))
$$

which is positive if $\forall z f(z) + f(1 - z) \geq 1$ or $\forall z f(z) + f(1 - z) \leq 1$.

Recall that the proof of Theorem 1.10 is based on a graph whose vertices have biases $\frac{c}{c+1}$, $c+1$, and $\frac{1}{c+1}$. Hence if $f(\frac{1}{2}) = \frac{1}{2}$, the upper bound holds for $f$, regardless of the antisymmetry of $f$.

If $f(\frac{1}{2}) = \frac{1}{2} + \delta$, since $|1 - f(x) - f(1 - x)| \leq 1$, the approximation ratio can increase by at most $\delta$ times the weight of all edges (compared to using the antisymmetric version of the function). However, the approximation ratio for an even cycle will be $\frac{1}{2} - 2\delta^2$. Therefore, there is $\gamma > 0$ such that no approximation better than $\frac{1}{2} - \gamma$ can be achieved to Max DICUT using oblivious algorithms, even if the selection function is not antisymmetric. \qed