On Star Coloring of Several Corona Graphs

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Abstract. Let $G$ be a simple graph with vertex set $V(G)$ and edge set $E(G)$. A vertex coloring of $G$ is called a star coloring of $G$ if any of the paths of 4 order are bicolored. The minimum number of colors required for a star coloring of $G$ is denoted by $\chi_s(G)$. The corona product of simple graphs $G$ of order $m$ and $H$ of order $n$ is graph $G \circ H$ with vertex set $V(G \circ H) = \{v_i | i = 1, 2, \ldots, m\} \cup \{v_j | j = 1, 2, \ldots, n\}$, in which $v_i$ is adjacent to every vertex of $H$ if and only if, $v_i \in V(G), v_j \in V(H)$. According to the existing graph dyeing literature, it has become a very important technical means to study the graph dyeing problem by using the graph structure operation. Therefore, it is of great significance to study the star coloring of graphs for studying the acyclic coloring and distance coloring of graphs, the study has strong application background and great theoretical value for computing graphs.

In this paper, we find the upper bound of $\chi_s(G \circ H)$ and the exact values of $\chi_s(G \circ H)$ of the corona product $G \circ H$ of two graphs $G$ and $H$ as: $\chi_s(G \circ H) \leq \chi_s(G) + \chi_s(H)$; $\chi_s(P_m \circ H) = \chi_s(H) + 2$; $\chi_s(K_{1,m} \circ H) = \chi_s(H) + 2$; $\chi_s(C_n \circ H) = \chi_s(H) + 2$, where $n \neq 5$.

1. Introduction
A great deal of graph-theoretical research has been conducted on star coloring since they were introduced in the early seventies by Grünbaum [1,2,3]. There are many kinds of product graphs, we will introduce the definition of the corona product of graphs. This kind of product was introduced by Harary and Frucht in 1970[4]. The reader is referred to [5,6,7,8,9,10] for some known results on corona product graphs.

All graphs considered here are undirected and simply graphs. We use standard terminology and notation of graph theory. Let $\sigma$ be a color of vertex, an vertex coloring of a graph $G$ is that no adjacent vertices receive the same color in $G$. The minimum number of colors required for an vertex coloring of $G$ is denoted by $\chi(G)$. An vertex coloring of a graph $G$ is said to be star coloring if any of the paths
of 4 order are bicolored. The minimum number of colors required for a star coloring of \( G \) is denoted by \( \chi_s(G) \). This coloring has been studied in many papers (see for example [2,11,12,13]).

The problem of graph coloring belongs to the problem of graph labeling, which is mainly to label the elements of the graph, namely the edges or vertices according to certain rules. In real life, many problems such as resource allocation, time arrangement and work allocation can be transformed or reduced to graph coloring problems. In fact, the problem of graph coloring belongs to the problem of limited classification. In the concept of graph coloring, normal dyeing is a limited minimum classification, and any color class is an independent set. Due to the lack of systematic research tools, it is difficult to determine the star coloring of any intention, and the concept of star coloring was put forward relatively late, with few relevant results. There is a relationship between the number of star coloring of a given graph and the number of acyclic coloring and the number of distance coloring: the number of acyclic coloring of a graph is less than the number of star coloring of the graph, and the number of star coloring of the graph is less than the number of distance coloring of the graph. According to the existing graph dyeing literature, it has become a very important technical means to study the graph dyeing problem by using the graph structure operation. Therefore, it is of great significance to study star coloring for graphs, and it is of great theoretical value to study the graphs with strong application background and operational graphs.

Graph products are interesting and useful in many situations[14]. Kowsalya Venkatesan et al. [15] gave the star chromatic number for the corona graph of path with complete graph on the same order, path with cycle on the same order, path on order \( n \) with star graph on order 1, path on order \( n \) with bipartite on order 1, and corona graph of star graph on order 1 with complete graph on order \( n \). Literature [14] mainly studies the star chromatic number of the crown product of a special graph of the same order. In this paper, the results are generalized to obtain the star chromatic number of \( m \) order path, \( n \) order star and any simple graph, respectively. At the same time, the star chromatic number of \( m \) order circle and any simple graph is studied, and the corresponding dyeing method is given.

Some of the following basic results will be used very often in this paper.

**Lemma 1.1 (Fertin at al.[13])** Let \( C_{2n} \) be a cycle with \( n \geq 3 \) vertices, if \( n = 5 \), then, \( \chi_s(C_{2n}) = 4 \), otherwise, \( \chi_s(C_{2n}) = 3 \).

**Lemma 1.2 (Fertin at al.[13])** Let \( K_{n,m} \) be a complete bipartite graph. Then, \( \chi_s(K_{n,m}) = \min\{m,n\} + 1 \).

**Lemma 1.3 (Fertin at al.[13])** Let \( K_m \) be a complete graph. Then, \( \chi_s(K_m) = m \).

### 2. Main results

Let \( m \) and \( n \) be two integers, and \( H \) is a simple graph of order \( n \geq 2 \). \( m \) mod \( n \) is denoted by \( (m) \) in this paper. Let \( G \) and \( H \) be two simple graphs, where \( |V(G)| = m \geq 2 \), \( |V(H)| = n \geq 2 \), and \( \chi_s(H) = t \). Note the vertex of \( G \) is \( V(G) = \{x_0, x_1, \cdots, x_{m-1}\} \), the vertex set of \( H \) is \( V(H) = \{y_0, y_1, \cdots, y_{n-1}\} \), and the vertex set of \( H_i \) for each copy of \( H \) in \( G \circ H \) is \( V(H_i) = \{y_0, y_1, \cdots, y_{n-1}\} \), where \( i = 0,1, \cdots, m-1 \). Let \( H'_i = K'_i \circ H \), which \( V(H'_i) = \{x_i\} \cup V(H'_i) \), \( V(K'_i) = \{x_i\} \), then the edge set of \( G \circ H \) is \( E(G \circ H) = E(G) \cup \left( \bigcup_{i=0}^{m-1} E(H'_i) \right) \).

**Theorem 2.1** \( \chi_s(G \circ H) \leq \chi_s(G) + \chi_s(H) \).

**Proof.** Let \( \chi_s(G) = s \), \( \chi_s(H) = r \). To prove the theorem, we only show that \( G \circ H \) has an \((s+r)\)-star-coloring. Now, a star coloring of \( G \circ H \) is constructed in two steps: first, the copy of \( G \) in \( G \circ H \)
is colored with $s$ colors, which is denoted as $\sigma_1$; secondly, use another $r$ new colors to conduct star coloring for each copy $H_i$ of $H$ in $G \circ H$, which is denoted as $\sigma_2$. Merge $\sigma_1$ with $\sigma_2$, denoted as $\sigma$.

Obviously, $\sigma$ is a $(s+r)$-coloring of $G \circ H$.

Now Let us prove that any of the paths of 4 order in $G \circ H$ are bicolored. Let $P_4$ be any path of four order in $G \circ H$, obviously, either $|V(P_4) \cap V(G)| = 0$ or 4, also $1 \leq |V(P_4) \cap V(G)| \leq 3$. For the former, since $\sigma_1$ and $\sigma_2$ is limit $G$ and $H$ cope of star-coloring, respectively, then $P_4$ is bicolored; For the latter, since $\sigma_1$ and $\sigma_2$ have different colour, thus $P_4$ 2colour paths, we can obtain

$$\chi_s(G \circ H) \leq \chi_s(G) + \chi_s(H).$$

We know Theorem 2.1, the upper bound is reachable, assume that $G$ is star equality hold in theorem, see theorem for specific proof method at theorem 2.2. In order to prove theorem 2.2, introduce the following lemma.

**Lemma 2.1** $\chi_s(P_2 \circ H) = \chi_s(H) + 2$.

**Proof.** When $G$ is $P_2$, let $P_2 = x_1x_2$, $\chi_s(H) = r$. It is easy to prove $\chi_s(P_2 \circ H) \geq r + 2$. Assume that $\chi_s(P_2 \circ H) \leq r + 1$ and $\sigma$ is a $(r+1)$-star-coloring of $P_2 \circ H$, this lead to a contradiction.

Let $\sigma(x_1) = a, \sigma(x_2) = b$, obviously, $a \neq b$. From $\chi_s(H) = r$ know, vertex coloring number of $H_1^*$ and $H_2^*$ is $r + 1$, meanwhile, let vertex $y_{k_0}^1$ and $y_{k_0}^2$, then $\sigma(y_{k_0}^1) = b, \sigma(y_{k_0}^2) = a$, where $k_0 \in \{0, 1, \ldots, n-1\}$, we can obtain $P_2 \circ H$ contain 2 colour paths of four order, contradict with the assume.

To prove $\chi_s(P_2 \circ H) \leq r + 2$, the coloring is constructed in two steps as follow $\sigma$: first, coloring the copy of $P_2$ in $P_2 \circ H$ in two colors; moreover, use another $r$ new colors to conduct star coloring for each copy $H_i$ of $H$ in $P_2 \circ H$. Obviously, $\sigma$ is a $(r+2)$-star-coloring of $P_2 \circ H$.

**Theorem 2.2** $\chi_s(K_{1,m} \circ H) = \chi_s(H) + 2$.

**Proof.** When $G$ is $K_{1,m}$ of $m+1$ order, let $K_{1,m}$ the set of vertex is $\{x_0, x_1, \ldots, x_m\}$, where $x_0$ is vertex of degree is $m$, other vertices degree is 1, and $\chi_s(H) = r$. By lemma 2.1 can prove $\chi_s(K_{1,m} \circ H) \geq r + 2$. To prove $\chi_s(K_{1,m} \circ H) \leq r + 2$, the coloring is constructed in two steps as follow $\sigma$: first, use two colors to star coloring for the copy of $K_{1,m}$ in $K_{1,m} \circ H$; moreover, use another $r$ new colors to conduct star coloring for each copy $H_i$ of $H$ in $K_{1,m} \circ H$. Obviously, $\sigma$ is a $(r+2)$-star-coloring of $K_{1,m} \circ H$.

Thus, $\chi_s(K_{1,m} \circ H) = \chi_s(H) + 2$. That is, $\chi_s(G \circ H) = \chi_s(G) + \chi_s(H)$.

Theorem 2.2 can generalize the result in proposition 1.5 as follows: for any integer $m, n \geq 2$, we have $\chi_s(K_{1,m} \circ K_n) = n + 2$.

For some particular graph, the theorem 2.1 the inequality is strictly true, let $G$ is path or circle, $\chi_s(G \circ H) = \chi_s(G) + \chi_s(H) - 1$, see theorem for specific proof method 2.3-2.4.

**Theorem 2.3** $\chi_s(P_m \circ H) = \chi_s(H) + 2$.

**Proof.** When $G$ is $P_m$, let $P_m = x_0x_1 \ldots x_{m-1}$, $\chi_s(H) = r$. From lemma 2.1 can prove $\chi_s(P_m \circ H) \geq r + 2$. To prove $\chi_s(P_m \circ H) \leq r + 2$, the coloring is constructed in two steps as follow
Let \( \sigma(x_i) = (i)_{r+2} \), where \( i = 0, 1, \ldots, m - 1 \); moreover, use the color set \( \{(i + j)_{r+2} \mid 0 \leq j \leq r \} \) to conduct star coloring for each copy \( H_i \) of \( H \) in \( P_m \circ H \). Obviously, \( \sigma \) is proper coloring. Now let us prove that \( \sigma \) is star coloring.

Let \( P_4 \) be any path of four order in \( P_m \circ H \), when \( |V(P_4) \cap V(P_m)| = 4 \) or 0, obviously, \( P_4 \) is not 2-colour path; when \( |V(P_4) \cap V(P_m)| = 3 \), we know that any three successive vertices of a copy of \( P_m \) in \( P_m \circ H \) have different colors, then, \( P_4 \) is bicolored; when \( |V(P_4) \cap V(P_m)| = 2 \), \( P_4 \) has two vertices on each copy of \( H \) prime of \( P_m \circ H \). For the former, let \( \sigma(x_i) = (i)_{r+2} \neq (i + k)_{r+2} \), where \( 1 \leq k \leq r \), then \( P_4 \) there are at least 3 vertices with different colors, that is, \( P_4 \) is bicolored. For the latter, assume that \( j = i + 1 \), then \( \sigma(x_i) = (i)_{r+2} \), \( \sigma(x_j) = (i + 1)_{r+2} \), any vertex \( x'_k \) color is \( (i + k + 1)_{r+2} \) of \( H_j \), where \( 1 \leq k \leq r \), obviously, three vertices have different color, that is, \( P_4 \) is bicolored; when \( |V(P_4) \cap V(P_m)| = 1 \), since the number of colors on the three successive vertices is at least two, and \( \sigma(x_i) \) is different from others, then \( P_4 \) is bicolored.

Thus, \( \chi_s(P_m \circ H) = \chi_s(H) + 2 \). Then, \( \chi_s(G \circ H) = \chi_s(G) + \chi_s(H) - 1 \).

The results in [15] can be generalized as follows by theorem 2.3 and lemma 1.1-1.3: for any integer \( m, n \), we have \( \chi_s(P_m \circ K_n) = n + 2 \); if \( n \neq 5 \), \( \chi_s(P_m \circ C_n) = 5 \); \( \chi_s(P_m \circ K_{1,n}) = 4 \); for any integer \( n_1, n_2 \geq 2 \), \( \chi_s(P_m \circ K_{n_1,n_2}) = \min\{n_1, n_2\} + 3 \).

**Theorem 2.4** \( \chi_s(C_m \circ H) = \chi_s(H) + 2 \), where \( m \neq 5 \).

**Proof.** When \( G \) is circle \( C_m \) of \( m \) order, let \( C_m = x_0 x_1 \cdots x_{m-1} x_0 \), \( \chi_s(H) = r \). From lemma 2.1 can prove \( \chi_s(C_m \circ H) \geq r + 2 \). To prove \( \chi_s(C_m \circ H) \leq r + 2 \), the coloring is constructed in two steps as follow \( \sigma \) : first, use color 0, 1, 2 to conduct star coloring for each copy \( H_i \) of \( H \) in \( C_m \circ H \), the coloring method is similar to theorem 5 in literature [3]; moreover, use color 3, 4, \cdots, \( r + 1 \) and \( \sigma(x_i) + 1 \), to conduct star coloring for each copy \( H_i \) of \( H \) in \( C_m \circ H \). Respectively, it is easy to prove \( \sigma \) is star coloring.

Thus, \( \chi_s(C_m \circ H) = \chi_s(H) + 2(m \neq 5) \). That is, \( \chi_s(G \circ H) = \chi_s(G) + \chi_s(H) - 1 \).

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