ZHANG’S INEQUALITY FOR LOG-CONCAVE FUNCTIONS

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Abstract. Zhang’s reverse affine isoperimetric inequality states that among all convex bodies $K \subseteq \mathbb{R}^n$, the affine invariant quantity $|K|^{n-1} |\Pi^*(K)|$ (where $\Pi^*(K)$ denotes the polar projection body of $K$) is minimized if and only if $K$ is a simplex. In this paper we prove an extension of Zhang’s inequality in the setting of integrable log-concave functions, characterizing also the equality cases.

1. Introduction

Given a convex body (compact, convex with non-empty interior) $K \subseteq \mathbb{R}^n$, its polar projection body $\Pi^*(K)$ is the unit ball of the norm given by

$$
\|x\|_{\Pi^*(K)} := |x| |P_x K|, \quad x \in \mathbb{R}^n
$$

where $|\cdot|$ denotes both the Lebesgue measure (in the suitable space) and the Euclidean norm, and $P_x K$ is the orthogonal projection of $K$ onto the hyperplane orthogonal to $x$. The Minkowski functional of a convex body $K$ containing the origin is defined, for every $x \in \mathbb{R}^n$, as

$$
\|x\|_K := \inf \{\lambda > 0 \mid x \in \lambda K\} \in [0, \infty].
$$

It is a norm if and only if $K$ is centrally symmetric.

The expression $|K|^{n-1} |\Pi^*(K)|$ is affine invariant and the extremal convex bodies are well known: Petty’s projection inequality \cite{P} states that the (affine class of the) $n$-dimensional Euclidean ball, $B_n^2$, is the only maximizer and Zhang’s inequality \cite{Z1} (see also \cite{GZ} and \cite{AJV}) proves that the (affine class of the) $n$-dimensional simplex $\Delta$ is the only minimizer. That is, for any convex body $K \subseteq \mathbb{R}^n$,

$$
\frac{2^n}{n^n} = |\Delta|^{n-1} |\Pi^*(\Delta)| \leq |K|^{n-1} |\Pi^*(K)| \leq |B_n^2|^{n-1} |\Pi^*(B_n^2)| = \left(\frac{|B_2^2|}{|B_2^{n-1}|}\right)^n.
$$

In recent years, many relevant geometric inequalities have been extended to the more general context of log-concave functions, i.e., functions $f : \mathbb{R}^n \to [0, \infty)$ of the form $f(x) = e^{-v(x)}$ with $v : \mathbb{R}^n \to (-\infty, \infty]$ a convex function. The set of all log-concave and integrable functions in $\mathbb{R}^n$ will be denoted by $\mathcal{F}(\mathbb{R}^n)$. This family of functions contains the set of convex bodies via the natural injections $K \to \chi_K$ (where $\chi_K$ denotes the characteristic function of $K$) or $K \to e^{-\|\cdot\|_K}$ (when $K$ is a convex body containing the origin). We refer the reader to \cite{KM} or \cite{C} and the references therein for a quick introduction on this topic.

The aim of this paper is to provide an extension of Zhang’s inequality for every $f \in \mathcal{F}(\mathbb{R}^n)$. For that matter, we define (see \cite{AGJV}, \cite{KM}) for every $f \in \mathcal{F}(\mathbb{R}^n)$

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the (centrally symmetric) polar projection body of \( f \), denoted \( \Pi^*(f) \), as
\[
\|x\|\Pi^*(f) = 2|x| \int_{x^+} P_{x^+} f(y) dy, \quad x \in \mathbb{R}^n
\]
where \( P_{x^+} f : x^+ \to [0, \infty) \) is the shadow of \( f \), i.e., \( P_{x^+} f(y) := \max_{s \in \mathbb{R}} f(y + s\frac{x}{|x|}) \).

Petty’s projection inequality was extended by Zhang, see [Z2], to compact domains and it was shown to be equivalent to the so called affine Sobolev inequality, which for log-concave functions in the suitable Sobolev space takes the following form: For every \( f \in \mathcal{F}(\mathbb{R}^n) \cap W^{1,1}(\mathbb{R}^n) = \{ f \in \mathcal{F}(\mathbb{R}^n) \mid \frac{\partial f}{\partial x_i} \in L^1(\mathbb{R}^n), \forall i \} \)
\[
\|f\|_{e^{-\|x\|}} \Pi^*(f) \|^{1\over n} \leq \frac{|B^n_2|}{2|B^n_2 - 1|},
\]
with equality if and only if \( f \) is the characteristic function of an ellipsoid.

Our extension of Zhang’s inequality has the following form:

**Theorem 1.1.** Let \( f \in \mathcal{F}(\mathbb{R}^n) \). Then,
\[
\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \min \{ f(y), f(x) \} \ dy dx \leq 2^n n! \|f\|_\infty \|f\|_1^{n+1} |\Pi^*(f)|.
\]
Moreover, if \( \|f\|_\infty = f(0) \) then equality holds if and only if \( \frac{f(x)}{\|f\|_\infty} = e^{-\|x\|_\Delta} \) for some \( n \)-dimensional simplex \( \Delta \) containing the origin.

**Remark 1.** The inequality in Theorem 1.1 is affine invariant, i.e., the inequality does not change under compositions of \( f \) with affine transformations of \( \mathbb{R}^n \).

**Remark 2.** Theorem 1.1 extends Zhang’s inequality. Indeed, if \( f(x) = e^{-\|x\|_K} \) for some convex body \( K \) containing the origin, then
\[
\Pi^*(f) = \frac{1}{2(n-1)!} \Pi^*(K), \quad \|f\|_1 = n! |K|,
\]
and
\[
\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \min \{ f(y), f(x) \} \ dy dx = \int_{\mathbb{R}^{2n}} e^{-\max(\|x\|_K, \|y\|_K)} \ dy dx
\]
\[
= \int_{\mathbb{R}^{2n}} e^{-\|x,y\|_K} \ dy dx = (2n)! |K|^2.
\]
Thus, Theorem 1.1 yields
\[
\frac{(2n)}{n^n} \leq |K|^{n-1} |\Pi^*(K)|.
\]

**Remark 3.** Sharp lower and upper bounds of the left hand side of the inequality in Theorem 1.1 in terms of the integral of \( f \) are known (see Lemmas 2.3 and 2.9 in [AAGJv], respectively).

The paper is structured as follows. In Section 2 we introduce some notation and preliminary results. A crucial role in the proof of Theorem 1.1 will be played by the functional form of the covariogram function, that we shall denote \( g \), associated to any \( f \in \mathcal{F}(\mathbb{R}^n) \). Recall that in the geometric setting the covariogram function of a convex body \( K \) is given by \( \mathbb{R}^n \ni x \to |K \cap (x + K)| \). In this section we shall define and study the basic properties of its functional version \( g \).

In Section 3 we prove the inequality in Theorem 1.1. The proof will rely on the following two facts: First, we will show that the polar projection body of
$f \in \mathcal{F}(\mathbb{R}^n)$ can be expressed in terms of dilations of the level sets of $g$. This can be seen as an extension of the corresponding geometric result (see [S] Theorem 1 and [AJV] Propositions 4.1 and 4.3) where the polar projection body of a convex body appears as the limit of suitable dilations of the level sets of the covariogram function. Second, we will prove a sharp relation (by inclusion) between the level sets of $g$ and a convex body in the celebrated family of bodies introduced by Ball (cf. [B], Pg. 74). The proof of such inclusion, that we state in full generality, follows ideas from [KM] Lemmas 2.1 and 2.2.

In Section 4 we characterize the equality cases. We first show that the equality in Theorem 1.1 holds if and only if the function $g$ is log-linear on every 1-dimensional linear ray starting at 0, i.e., for every $x \in \mathbb{R}^n$ and every $\lambda \in [0, 1]$, $g(\lambda x) = g(0)^{1-\lambda}g(x)^{\lambda}$ and then prove that such condition implies that $f$ has to be as in the statement of the theorem.

2. Notation and preliminaries

$S^{n-1}$ is the Euclidean unit sphere in $\mathbb{R}^n$ and $\sigma$ denotes the uniform probability measure on $S^{n-1}$. If the origin is in the interior of $K$, the function $\rho_K: S^{n-1} \to [0, +\infty)$ given by $\rho_K(u) = \sup\{\lambda \geq 0 \mid \lambda u \in K\}$ is the radial function of $K$. It extends to $\mathbb{R}^n \setminus \{0\}$ via $t\rho_K(tu) = \rho_K(u)$, for any $t > 0, u \in S^{n-1}$. The volume of $K$ is given by

$$|K| = |B_2^n| \int_{S^{n-1}} \rho_K^t(u) d\sigma(u)$$

and the boundary of $K$ will be denoted by $\partial K$.

Throughout the paper, $f: \mathbb{R}^n \to [0, +\infty)$ will always denote a (non identically null) log-concave integrable function. Recall that such $f$ is said to be log-concave if it can be written as $f = e^{-v}$ where $v: \mathbb{R}^n \to (-\infty, +\infty]$ is convex or, equivalently if for every $x, y \in \mathbb{R}^n, 0 < \lambda < 1$, $f(\lambda x + (1-\lambda)y) \geq (f(x))^{\lambda} (f(y))^{1-\lambda}$. It is well known that $f$ is then continuous in the interior of its support, $\text{int}(\text{supp}f)$, and we denote its supremum as $\|f\|_\infty$. Since Theorem 1.1 does not depend on the values of $f$ on the boundary of $\text{supp} f$ we shall assume, without loss of generality, that $f$ is continuous on its support. For any $t \in [0, \infty)$ we denote the level sets of $f$ by

$$K_t(f) = \{x \in \mathbb{R}^n : f(x) \geq e^{-t}\|f\|_\infty\} = \{x \in \mathbb{R}^n : v(x) \leq t\}.$$

Since we assume that $f$ is continuous on its support, $K_t(f)$ is a convex body for all $t > 0$. These and other basic facts on convex bodies and log-concave functions used in the paper can be found in [BGVV].

We will use the following definition of the polar projection body of $f$ which involves level sets, equivalent to the one stated in the introduction (see [AGJV] Proposition 4.1).

**Definition 2.1.** Let $f \in \mathcal{F}(\mathbb{R}^n)$. The *polar projection body of $f$, denoted as $\Pi^*(f)$, is the unit ball of the norm given by*

$$\|x\|_{\Pi^*(f)} := 2\|x\|_\infty \int_0^\infty |P_{x^\perp} K_t(f)| e^{-t} dt = 2\|f\|_\infty \int_0^\infty \|x\|_{\Pi^*(K_t(f))} e^{-t} dt.$$

**Remark 4.** If $f = \chi_K$ is the characteristic function of a convex body $K$, then $\Pi^*(f) = \frac{1}{2} \Pi^*(K)$ and, as mentioned in Remark 2 if $f(x) = e^{-\|x\|_K}$, then $\Pi^*(f) = \frac{1}{2(n-1)!} \Pi^*(K)$. 

We start by associating a function $g$ to any function $f \in \mathcal{F}(\mathbb{R}^n)$. Such function can be regarded as the functional version of the covariogram functional. We collect its properties in the following lemma.

**Lemma 2.1.** Let $f \in \mathcal{F}(\mathbb{R}^n)$. Then the function $g : \mathbb{R}^n \to \mathbb{R}$ defined by

$$
g(x) := \int_0^\infty e^{-t}|K_t(f) \cap (x + K_t(f))|dt = \int_{\mathbb{R}^n} \min \left\{ \frac{f(y)}{\|f\|_{\infty}}, \frac{f(y - x)}{\|f\|_{\infty}} \right\} dy
$$

is even, log-concave, $0 \in \text{int}(\text{supp} \ g)$ with $\|g\|_{\infty} = g(0) = \int_0^\infty e^{-t}|K_t(f)|dt = \int_{\mathbb{R}^n} \frac{f(x)}{\|f\|_{\infty}} dx > 0$, and $\int_{\mathbb{R}^n} g(x)dx = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \min \left\{ \frac{f(y)}{\|f\|_{\infty}}, \frac{f(y - x)}{\|f\|_{\infty}} \right\} dy \ dx$.

**Proof.** By Fubini’s theorem, for any $x \in \mathbb{R}^n$ we have

$$
\int_0^\infty e^{-t}|K_t(f) \cap (x + K_t(f))|dt = \int_{\mathbb{R}^n} \int_0^\infty e^{-t}\log\min \left\{ \frac{f(y)}{\|f\|_{\infty}}, \frac{f(y - x)}{\|f\|_{\infty}} \right\} dt dy.
$$

Consequently, using Fubini’s theorem and a change of variables,

$$
\int_{\mathbb{R}^n} g(x)dx = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \min \left\{ \frac{f(y)}{\|f\|_{\infty}}, \frac{f(x)}{\|f\|_{\infty}} \right\} dy \ dx.
$$

In order to prove the log-concavity of $g$, let $x_1, x_2 \in \mathbb{R}^n$, $t_1, t_2 \in [0, \infty)$, $0 \leq \lambda \leq 1$ and write $x = (1 - \lambda)x_1 + \lambda x_2$ and $t = (1 - \lambda)t_1 + \lambda t_2$. Clearly,

$$
K_t \cap (x + K_t) \supseteq (1 - \lambda)(K_{t_1} \cap (x_1 + K_{t_1})) + \lambda(K_{t_2} \cap (x_2 + K_{t_2})).
$$

By Brunn-Minkowski inequality, [BGVV Theorem 1.2.1],

$$
|K_t \cap (x + K_t)| \geq |K_{t_1} \cap (x_1 + K_{t_1})|^{1-\lambda}|K_{t_2} \cap (x_2 + K_{t_2})|^{\lambda}.
$$

Thus,

$$
e^{-t}|K_t \cap (x + K_t)| \geq (e^{-t_1}|K_{t_1} \cap (x_1 + K_{t_1})|)^{1-\lambda}(e^{-t_2}|K_{t_2} \cap (x_2 + K_{t_2})|)^{\lambda}
$$

and, by Prekopa-Leindler inequality, [BGVV Theorem 1.2.3],

$$
g((1 - \lambda)x_1 + \lambda x_2) = \int_0^\infty e^{-t}|K_t \cap ((1 - \lambda)x_1 + \lambda x_2 + K_t)|dt \geq \left( \int_0^\infty e^{-t}|K_t \cap (x_1 + K_t)|dt \right)^{1-\lambda} \left( \int_0^\infty e^{-t}|K_t \cap (x_2 + K_t)|dt \right)^{\lambda} = g(x_1)^{1-\lambda}g(x_2)^{\lambda}.
$$

Now, for any $t \in [0, \infty)$, $K_t(f) \cap (x + K_t(f)) = x + (K_t(f) \cap (-x + K_t(f)))$ and so $|K_t(f) \cap (x + K_t(f))| = |K_t(f) \cap (-x + K_t(f))|$. Therefore $g(x) = g(-x)$. Consequently, $\|g\|_{\infty} = g(0)$ and its value is

$$
g(0) = \int_0^\infty e^{-t}|K_t(f)|dt = \int_{\mathbb{R}^n} \frac{f(x)}{\|f\|_{\infty}} dx > 0.
$$

Finally, there exists $\varepsilon > 0$ such that if $|x| < \varepsilon$ then $K_t(f) \cap (x + K_t(f))$ is a non-empty convex body and has positive volume. Thus, if $|x| < \varepsilon$ then for every $t > 1$ $K_t(f) \cap (x + K_t(f))$ has positive volume and then $g(x) > 0$. Thus $0 \in \text{int}(\text{supp} g)$.
Let $g \in \mathcal{F}(\mathbb{R}^n)$ be a function such that $g(0) > 0$ and let $p > 0$. The following important family of convex bodies was introduced by K. Ball in [B] pg. 74. We denote

$$ \tilde{K}_p(g) := \left\{ x \in \mathbb{R}^n : \int_0^\infty g(rx)r^{p-1}dr \geq \frac{g(0)}{p} \right\}. $$

It follows from the definition that the radial function of $\tilde{K}_p(g)$ is given by

$$ \rho_{\tilde{K}_p(g)}(u) = \frac{1}{g(0)} \int_0^\infty pr^{p-1}g(rx)dr. $$

We will make use of the following well-known relation between the Lebesgue measure of $\tilde{K}_n(g)$ and the integral of $g$.

**Lemma 2.2** ([B]). Let $g \in \mathcal{F}(\mathbb{R}^n)$ be such that $g(0) > 0$. Then

$$ |\tilde{K}_n(g)| = \frac{1}{g(0)} \int_{\mathbb{R}^n} g(x)dx. $$

**Proof.** Integrating in polar coordinates we have that

$$ |\tilde{K}_n(g)| = |B_2^n| \int_{S^{n-1}} \rho_{\tilde{K}_n(g)}(u)d\sigma(u) = |B_2^n| \int_{S^{n-1}} \frac{n}{g(0)} \int_0^\infty r^{n-1}g(rx)drd\sigma(u) $$

$$ = \frac{1}{g(0)} \int_{\mathbb{R}^n} g(x)dx. $$

\[ \square \]

3. **Proof of the inequality in Theorem 1.1**

We split the main idea of the proof in two parts. We first prove, Lemma 3.1, that $\Pi^*(f)$ equals the intersection of suitable dilations of the level sets $K_t(g)$, with $g$ as defined in the previous section. We then show a sharp relation between Ball’s convex body $\tilde{K}_n(g)$ and the level set $K_t(g)$, see Lemma 3.2. Such a relation holds not only for the covariogram function $g$ but for a larger class of log-concave functions.

**Lemma 3.1.** Let $f \in \mathcal{F}(\mathbb{R}^n)$ and let $g : \mathbb{R}^n \to \mathbb{R}$ be the function

$$ g(x) = \int_0^\infty e^{-t}|K_t(f) \cap (x + K_t(f))|dt. $$

Then for every $0 < \lambda_0 < 1$

$$ \bigcap_{0 < \lambda < \lambda_0} \frac{K_{-\log(1-\lambda)}(g)}{\lambda} = 2\|f\|_1 \Pi^*(f). $$

**Proof.** For any $0 < \lambda < 1$ the convex body $\frac{K_{-\log(1-\lambda)}(g)}{\lambda}$ equals

$$ \left\{ x \in \mathbb{R}^n : \int_0^\infty e^{-t}|K_t(f) \cap (\lambda x + K_t(f))|dt \geq (1 - \lambda) \int_0^\infty e^{-t}|K_t(f)|dt \right\}. $$

or equivalently

$$ \left\{ x \in \mathbb{R}^n : \int_0^\infty e^{-t}\frac{|K_t(f)| - |K_t(f) \cap (\lambda x + K_t(f))|}{\lambda}dt \leq \int_0^\infty e^{-t}|K_t(f)|dt \right\}. $$

Since $|K_t(f)| - |K_t(f) \cap (\lambda x + K_t(f))| \leq \lambda |x||P_xK_t(f)|$, then

$$ \int_0^\infty e^{-t}\frac{|K_t(f)| - |K_t(f) \cap (\lambda x + K_t(f))|}{\lambda}dt \leq |x| \int_0^\infty e^{-t}|P_xK_t(f)|dt $$
Lemma 3.2. Let we provide information on the equality case as it shall be used in the next Section. In that paper a similar result was stated with an interest on large values of $t$. In the following lemma we prove the aforementioned inclusion between the level sets of a function $g$ that for every $0$, we have that if

\[ \|x\|_{\Pi^*} = \frac{\|x\|_{\Pi^*}}{2\|f\|_{\infty}} \]

and we have that if

\[ \|x\|_{\Pi^*} \leq 2\|f\|_{\infty} \int_0^\infty e^{-t} |K_t(f)|dt = 2\int_{\mathbb{R}^n} f(x)dx \]

then $x \in K_{\log(1-\lambda)}(g)/\lambda$. Thus

\[ 2\|f\|_{\Pi^*} \subseteq \frac{K_{\log(1-\lambda)}(g)}{\lambda}, \quad \text{for all } 0 < \lambda < 1. \]

On the other hand, for any $0 < \lambda < 1$ and any $x \in \mathbb{R}^n$,

\[ \int_0^\infty e^{-t} \left[ |K_t(f)| - |K_t(f) \cap (\lambda|x| + K_t(f))| \right]dt \geq |x| \int_0^\infty e^{-t} |P_{x^+}(K_t(f) \cap (\lambda x + K_t(f)))|dt \]

and then, since $\int_0^\infty e^{-t} |P_{x^+}(K_t(f) \cap (\lambda x + K_t(f)))|dt$ decreases in $\lambda$ and

\[ \sup_{\lambda \in (0,1)} |x| \int_0^\infty e^{-t} |P_{x^+}(K_t(f) \cap (\lambda x + K_t(f)))|dt = \left| x \int_0^\infty e^{-t} |P_{x^+}K_t(f)|dt \right| = \frac{\|x\|_{\Pi^*}}{2\|f\|_{\infty}} \]

we have that if $\|x\|_{\Pi^*} > 2\|f\|_{\infty} \int_0^\infty e^{-t} |K_t(f)|dt = 2\|f\|_1$, there exists $\lambda_1 > 0$ such that for every $0 < \lambda \leq \lambda_1$

\[ \int_0^\infty e^{-t} \left[ |K_t(f)| - |K_t(f) \cap (\lambda|x| + K_t(f))| \right]dt > \int_0^\infty e^{-t} |K_t(f)|dt \]

and then $x \notin \frac{K_{\log(1-\lambda)}(g)}{\lambda}$ if $0 < \lambda \leq \lambda_1$. \hfill \Box

In the following lemma we prove the aforementioned inclusion between the level sets of a function $g \in \mathcal{F}(\mathbb{R}^n)$ and the convex body $\bar{K}_n(g)$. We follow ideas from [KM]. In that paper a similar result was stated with an interest on large values of $t$. Here we shall be interested on small values of $t$. In the second part of the Lemma we provide information on the equality case as it shall be used in the next Section.

Lemma 3.2. Let $g \in \mathcal{F}(\mathbb{R}^n)$ be such that $0 \in \text{int}(\text{suppg})$, $g(0) = \|g\|_{\infty} > 0$. Then for every $0 \leq t \leq \frac{\pi}{\epsilon}$,

\[ \frac{t}{(n!)^{1/\pi}} \bar{K}_n(g) \subseteq K_t(g). \]

Moreover, for any $0 < t \leq \frac{\pi}{\epsilon}$ there is equality if and only if $g$ is log-linear on every 1-dimensional linear ray starting at 0 and, furthermore, if $g$ is not log-linear on the 1-dimensional linear ray starting at 0 spanned by $u \in S^{n-1}$, there exists $\epsilon > 0$ such that for every $0 < t \leq \frac{\pi}{\epsilon}$

\[ \frac{t}{(n!)^{1/\pi}} (\rho_{\bar{K}_n(g)}(u) + \epsilon) \leq \rho_{K_t(g)}(u). \]
Proof. We can assume without loss of generality that \( g(0) = 1 \). Otherwise consider \( \frac{g}{g(0)} \). Write \( g(x) = e^{-v(x)} \) for some convex function \( v \) and fix \( u \in S^{n-1} \).

For any \( q > 0 \) the function

\[
\phi(r) = v(ru) - q \log r
\]

is strictly convex in \((0, \infty)\) and \( \omega(r) := v(ru) \) is non-decreasing and convex on \([0, \infty)\). Since \( r^q e^{-v(ru)} \) is integrable on \([0, \infty)\) and takes value 0 at 0, we have

\[
\lim_{r \to \infty} \phi(r) = \lim_{r \to 0^+} \phi(r) = \infty
\]

and consequently \( \phi \) attains a unique minimum at some number \( r_0 = r_0(g) \) and the lateral derivatives of \( \phi \) verify \( \phi'_-(r_0) \leq 0 \) and \( \phi'_+(r_0) \geq 0 \). This implies that the lateral derivatives of \( \omega(r) \) at \( r_0 \) verify \( \omega'_-(r_0) \leq \frac{\omega_0}{r_0} \) and \( \omega'_+(r_0) \geq \frac{\omega_0}{r_0} \). Notice that if \( \omega \) is linear then necessarily \( \omega(r) = \omega(r_0) + \frac{\omega_0}{r_0} (r - r_0) \).

Notice also that \( r \to \omega(r_0) + \frac{\omega_0}{r_0} (r - r_0) \) is a supporting line of \( \omega \) at \( r_0 \) and by convexity, \( \omega(r) \geq \omega(r_0) + \frac{\omega_0}{r_0} (r - r_0) \) for every \( r \in [0, \infty) \). Thus,

\[
n \int_0^\infty r^{n-1} g(ru) dr = n \int_0^\infty r^{n-1} e^{-\omega(r)} dr \leq n e^{-\omega(r_0)} \int_0^\infty e^{q(ru)} dr
\]

\[
= n e^{-\omega(r_0)} \left( \frac{r_0}{q} \right)^n \int_0^\infty r^{n-1} e^{-r} dr = g(r_0 u) \frac{e^{qe^n}}{q^n} \nu_0^n.
\]

Moreover, the previous inequality is equality if and only if \( \omega(r) = \omega(r_0) + \frac{\omega_0}{r_0} (r - r_0) \) for every \( r \in [0, \infty) \).

Consequently, for any \( q > 0 \) and since \( \frac{\rho_{\nu}}{K_n(g)}(u) = n \int_0^\infty r^{n-1} g(ru) dr \)

\[
\frac{q}{e^{\frac{q}{n}} (n!)^\frac{1}{n}} \rho_{\nu}(g)(u) \leq g(r_0 u)^{\frac{1}{n}} \nu_0,
\]

with equality if and only if \( \omega(r) = \omega(r_0) + \frac{\omega_0}{r_0} (r - r_0) \) for every \( r \in [0, \infty) \). On the other hand, since \( \omega \) is convex, \( \omega'_+(r) \leq \frac{\omega_0}{r_0} \) if \( r < r_0 \) and then

\[
\omega(r_0) = \omega(0) + \int_0^{r_0} \omega'_+(r) dr \leq v(0) + q = q,
\]

thus \( g(r_0 u) \geq e^{-q} \), with equality if and only if \( \omega'_+(r) = \frac{\omega_0}{r_0} \) for every \( r \in [0, r_0) \) and therefore \( \omega(r) = \omega(r_0) + \frac{\omega_0}{r_0} (r - r_0) \) for every \( r \in [0, r_0] \). Now, the definition of the log-concavity of \( g \) applied to 0 and \( r_0 u \) yields

\[
g \left( g(r_0 u)^{\frac{1}{n}} r_0 u \right) \geq g(r_0 u) \frac{g(r_0 u)^{\frac{1}{n}}}{e^{g(r_0 u)^{\frac{1}{n}} \log g(r_0 u)}} \geq \frac{g(r_0 u)^{\frac{1}{n}}}{e^{g(r_0 u)^{\frac{1}{n}}}} \log g(r_0 u)
\]

with equality if and only if \( \omega(r) = v(ru) \) is linear in \([0, r_0]\) and then \( \omega(r) = \omega(r_0) + \frac{\omega_0}{r_0} (r - r_0) \) for every \( r \in [0, r_0] \). Since the function \( x \to x^{\frac{1}{n}} \log x \) attains its minimum at \( x = e^{-n} \) and is increasing in the interval \([e^{-n}, \infty)\), if \( 0 < q \leq n \)

\[
g \left( g(r_0 u)^{\frac{1}{n}} r_0 u \right) \geq e^{-qe^{-\frac{1}{n}}},
\]

with equality if and only if \( \omega(r) \) is linear in \([0, r_0]\) and \( g(r_0 u) = e^{-q} \), which occurs if and only if \( \omega(r) = \omega(r_0) + \frac{\omega_0}{r_0} (r - r_0) \) for every \( r \in [0, r_0] \), that is, \( g(r_0 u)^{\frac{1}{n}} r_0 u \in K_{qe^{-\frac{1}{n}}}(g) \) and \( g(r_0 u)^{\frac{1}{n}} r_0 u \in \partial K_{qe^{-\frac{1}{n}}}(g) \) if and only if \( \omega(r) = \omega(r_0) + \frac{\omega_0}{r_0} (r - r_0) \) for every \( r \in [0, r_0] \).
Since this is valid for any \( u \in S^{n-1} \),
\[
\frac{q e^{-\frac{u}{n}}}{(n!)^2} K_n(g) \subseteq K \left( q e^{-\frac{u}{n}}(g) \right),
\]
with equality if and only if for every \( u \in S^{n-1} \), \( \omega(r) = \omega(r_0) + \frac{q}{r_0} (r - r_0) \) for every \( r \in [0, \infty) \), i.e., \( \omega(r_u) \) is linear for every \( u \in S^{n-1} \). Finally, observe that when \( x \in (0, n) \), the function \( x \to x e^{-\frac{u}{n}} \) takes every value in \((0, \frac{n}{r})\) thus, for every \( 0 \leq t \leq \frac{n}{r} \)
\[
\frac{t}{(n!)^2} K_n(g) \subseteq K_t(g)
\]
and for any \( t \in (0, \frac{n}{r}) \) there is equality if and only if \( v(r_u) \) is linear for every \( u \in S^{n-1} \), i.e., for every \( x \in \mathbb{R}^n \) and every \( \lambda \in [0, 1] \), \( g(\lambda x) = g(0)^{1-\lambda} g(x)^{\lambda} \).

In order to establish the furthermore part, we are given some \( u \in S^{n-1} \). We first need to prove the following

**Claim:** The function \( q \to r_0(q) \) is continuous in \((0, \infty)\) and is bounded around \( 0 \).

Indeed, we first consider a sequence \((q_k)_{k=1}^\infty \) converging to \( q \in (0, \infty) \) so that there is a subsequence \((r_0(q_k))_{k=1}^\infty \) converging to some \( \tau \). We have
\[
\omega(r_0(q_k)) - q_k \log(r_0(q_k)) \leq \omega(r_0(q)) - q_k \log(r_0(q))
\]
and taking limits, \( \omega(\tau) - q \log(\tau) \leq \omega(r_0(q)) - q \log(r_0(q)) \). Therefore, \( \tau = r_0(q) \).

If, on the other hand, \( q > 0 \) and the subsequence \((r_0(q_k))_{k=1}^\infty \) tends to \( \infty \) then, since \( q_k \leq M \) for every \( k \in \mathbb{N} \) and some \( M > 0 \), we have that
\[
\omega(r_0(q_k)) - M \log(r_0(q_k)) + (M - q_k) \log(r_0(q_k)) = \omega(r_0(q_k)) - q_k \log(r_0(q_k))
\]
leading to a contradiction, since the left hand side of the inequality tends to \( \infty \). Thus, both the inferior limit and the superior limit of \( r_0(q_k) \) are equal to \( r_0(q) \) and we have proven continuity in \((0, \infty)\). Finally, if \((q_k)_{k=1}^\infty \) is a sequence converging to \( 0 \) and some subsequence \((r_0(q_k))_{k=1}^\infty \) tends to \( \infty \), we would have that for every \( r \in [0, \infty) \)
\[
\omega(r_0(q_k)) + \frac{q_k}{r_0(q_k)} (r - r_0(q_k)) \leq \omega(r),
\]
leading to a contradiction since the left hand side of this inequality tends to \( \infty \). This finishes the proof of the Claim.

As a consequence, if \((q_k)_{k=1}^\infty \) converges to \( q \in [0, \infty) \), we have that the sequence \((\frac{q}{r_0(q_k)})_{k=1}^\infty \) is bounded, since \( \frac{q}{r_0(q_k)} \leq \omega'(r_0(q_k)) \).

Now, assume that there is no \( \tau > 0 \) verifying that for every \( 0 < q \leq n \)
\[
n \int_0^\infty r^{n-1} e^{-(\omega(r_0)+\frac{q}{r_0}(r-r_0))} dr - n \int_0^\infty r^{n-1} e^{-\omega(r)} dr \geq \tau.
\]
Then we can find a sequence \((q_k)_{k=1}^\infty \) (and if necessary extract from it further subsequences which we denote in the same way) so that
\[
\lim_{k \to \infty} n \int_0^\infty r^{n-1} \left( e^{-(\omega(r_0(q_k))+\frac{q_k}{r_0(q_k)}(r-r_0(q_k)))} - e^{-\omega(r)} \right) dr = 0,
\]
\( q_k \) converges to some \( q \in [0, n] \), \( r_0(q_k) \) converges to some \( \tau \in [0, \infty) \) and \( \frac{q_k}{r_0(q_k)} \) converges to some \( \alpha \in [0, \infty) \). Therefore, since for every \( r \in [0, \infty) \)
\[
\omega(r_0(q_k)) + \frac{q_k}{r_0(q_k)}(r - r_0(q_k)) \leq \omega(r),
\]
we have that for every \( r \in [0, \infty), \omega(\tau) + \alpha(r - \tau) \leq \omega(r) \) and, since by Fatou’s lemma
\[
0 \leq n \int_0^\infty r^{n-1} \left(e^{-(\omega(\tau) + \alpha(r-\tau))} - e^{-\omega(r)}\right)\,dr 
\leq \lim_{k \to \infty} n \int_0^\infty r^{n-1} \left(e^{-(\omega(r_0(q_k)) + \frac{q_k}{r_0(q_k)}(r - r_0(q_k)))} - e^{-\omega(r)}\right)\,dr = 0,
\]
we have that for every \( r \in [0, \infty), \omega(r) = \omega(\tau) + \alpha(r - \tau) \) and so \( \omega \) is linear.

Therefore if \( \omega \) is not linear, there exists \( \tau > 0 \) such that for every \( 0 < q \leq n, \)
\[
n \int_0^{\infty} r^{n-1} g(ru)dr + \tau \leq g(r_0u)\frac{q \cdot n_1}{q^n r_0^n}
\]
and so, for some \( \varepsilon > 0 \) and every \( 0 < q \leq n \)
\[
\frac{q}{e^{\frac{n}{(n!)^{\frac{1}{n}}}}} (\rho_{\tau_n,g}(u) + \varepsilon) \leq \frac{q}{e^{\frac{n}{(n!)^{\frac{1}{n}}}}} (\rho_{K_{n,t}}(u) + \tau) \leq g(r_0u)^{\frac{1}{n}} r_0.
\]

Now, we continue as in the proof of the first part of the Lemma. If for some \( u \in S^{n-1} \) we assume that \( \omega \) is not linear, then there exists \( \varepsilon > 0 \) such that for every \( 0 < q \leq n \)
\[
\frac{q e^{\frac{n}{(n!)^{\frac{1}{n}}}}} (\rho_{\tau_n,g}(u) + \varepsilon) \leq \rho_{K_{n,t}}(u),
\]
and consequently, for every \( 0 < t \leq \frac{n}{\varepsilon} \)
\[
\frac{t}{(n!)^{\frac{1}{n}}} (\rho_{\tau_n,g}(u) + \varepsilon) \leq \rho_{K_{n,t}}(u)
\]

\( \square \)

Remark 5. The inclusion in the lemma above cannot be extended in general to the whole range of \( t \in [0, n]. \) If \( f = \chi_K \) is the characteristic function of a convex body, \( \overline{K_n}(f) = K \) and we would have, taking \( t = n, \frac{n}{(n!)^{\frac{1}{n}}} \overline{K_n}(f) \subseteq K_n(f) \) then, by using Stirling’s formula, it would imply for large values of \( n \) that \( \frac{2}{\pi} K \subseteq K \) which is trivially false.

On the other hand, using the same ideas as above we can obtain a more general result, namely, for every \( p > 0 \) and \( 0 \leq t \leq \frac{p}{\varepsilon}, \frac{t}{\Gamma(1+p)\pi} \overline{K_p}(g) \subseteq K_t(g). \)

Now we can prove the main result of the paper. In this section we prove the inequality.

**Proof of the inequality of Theorem** [4.7] Let us consider the function \( g : \mathbb{R}^n \to \mathbb{R} \) defined by
\[
g(x) = \int_0^{\infty} e^{-t} |K_t(f) \cap (x + K_t(f))|dt.
\]
By Lemma 3.1 and Lemma 3.2 (since \( g(0) = \|g\|_\infty > 0 \) by Lemma 2.1), for any \( 0 < \lambda_0 < 1 - e^{-\frac{\pi}{2}} \) we have
\[
2\|f\|_1 \Pi^* (f) = \int_{0<\lambda<\lambda_0} K_{-\log(1-\lambda)}(g) \frac{d\lambda}{\lambda} \geq \int_{0<\lambda<\lambda_0} -\log(1-\lambda) \frac{d\lambda}{(n!)^{\frac{\pi}{2}} \lambda} \K_n(g).
\]
Since \( h(\lambda) := -(\log(1-\lambda))/\lambda \) is increasing in \( \lambda \in (0,1) \), and \( \lim_{\lambda \to 0^+} h(\lambda) = 1 \), then
\[
2\|f\|_1 \Pi^* (f) \geq \frac{1}{(n!)^{\frac{\pi}{2}}} \K_n(g).
\]
Taking Lebesgue measure we obtain that
\[
2^n \|f\|_n \| \Pi^* (f) \| \geq \frac{1}{n!} |\K_n(g)|.
\]
One can conclude the result as a direct consequence of Lemmas 2.1 and 2.2. \( \square \)

4. CHARACTERIZATION OF THE EQUALITY IN THEOREM 1.1

In this section we characterize the equality case in Theorem 1.1. First we will show that if there is equality in the theorem for a function \( f \) attaining its maximum at the origin, then the associated covariogram function \( g \) has to be log-linear in every 1-dimensional ray starting at 0. Second we will prove that such condition implies that \( \|f\|_\infty = e^{-\|f\|_\Delta} \) for some simplex \( \Delta \) containing the origin.

Lemma 4.1. Let \( f \in \mathcal{F}(\mathbb{R}^n) \) be such that \( \|f\|_\infty = f(0) \) and let \( g : \mathbb{R}^n \to \mathbb{R} \) be
\[
g(x) = \int_0^\infty e^{-t}\|K_t(f) \cap (x + K_t(f))\|dt.
\]
If \( f \) attains equality in Theorem 1.1, then for every \( x \in \mathbb{R}^n \) and every \( \lambda \in [0,1] \),
\[
g(\lambda x) = g(0)^{1-\lambda} g(x)^\lambda.
\]

Proof. Assume that the statement is not true. Then, by Lemma 3.2 there exists \( u \in S^{n-1} \) and \( \varepsilon > 0 \) such that for any \( 0 < \lambda_0 < 1 - e^{-\frac{\pi}{2}} \) and any \( 0 < \lambda < \lambda_0 \)
\[
-\log(1-\lambda) \leq \frac{1}{\lambda(n!)^{\frac{\pi}{2}}} (\rho\K_n(g)(u) + \varepsilon) \leq \frac{1}{\lambda(n!)^{\frac{\pi}{2}}} (\rho\K_n(g)(u)).
\]
Consequently, for such \( u \)
\[
\rho_{\cap_{0<\lambda<\lambda_0} \frac{\K_{-\log(1-\lambda)}(g)(u)}{\lambda}} = \inf_{0<\lambda<\lambda_0} \frac{\rho_{\K_{-\log(1-\lambda)}(g)}(u)}{\lambda} \geq \inf_{0<\lambda<\lambda_0} \frac{-\log(1-\lambda)}{\lambda(n!)^{\frac{\pi}{2}}} (\rho\K_n(g)(u) + \varepsilon) = \frac{1}{(n!)^{\frac{\pi}{2}}} (\rho\K_n(g)(u) + \varepsilon),
\]
and then
\[
2\|f\|_1 \Pi^* (f) = \int_{0<\lambda<\lambda_0} K_{-\log(1-\lambda)}(g) \frac{d\lambda}{\lambda} \geq \frac{1}{(n!)^{\frac{\pi}{2}}} \K_n(g)
\]
and the volume of the left-hand side convex body is strictly greater than the volume of the right-hand side convex body. \( \square \)

The next lemma shows that if \( g \) is log-linear in every 1-dimensional linear ray starting from 0 then \( \|f\|_\infty = e^{-\|f\|_\Delta} \) for some simplex \( \Delta \) containing the origin.
Lemma 4.2. Let $f \in \mathcal{F}(\mathbb{R}^n)$ be such that $\|f\|_\infty = f(0)$ and let $g : \mathbb{R}^n \to \mathbb{R}$ be

$$g(x) = \int_0^\infty e^{-t} |K_t(f) \cap (x + K_t(f))| dt.$$  

Then, for every $x \in \mathbb{R}^n$ and every $\lambda \in [0, 1]$, $g(\lambda x) = g(0)^{1-\lambda} g(x)^{\lambda}$ if and only if $\frac{g(x)}{\|g\|_{\infty}} = e^{-\|x\|_{\Delta}}$ with $\Delta$ a simplex containing the origin.

Proof. The condition $g(\lambda x) = g(0)^{1-\lambda} g(x)^{\lambda}$ for every $x \in \mathbb{R}^n$ and every $\lambda \in [0, 1]$ implies that $g(x) \neq 0$ for every $x \in \mathbb{R}^n$, since $g$ is continuous at 0, as 0 is in the interior of the support of $g$. In the following, in order to ease the notation, we will denote $K_t = K_t(f)$ for every $t \in [0, \infty)$. For $x \in \mathbb{R}^n$ and $t \in [0, \infty)$, we define

$$h_x(t) = e^{-t} |K_t \cap (x + K_t)|.$$  

Notice that for any $x \in \mathbb{R}^n$, any $t_1, t_2 \in [0, \infty)$ and any $\lambda \in [0, 1]$

$$K_{(1-\lambda)t_1 + \lambda t_2} \cap (x + K_{(1-\lambda)t_1 + \lambda t_2}) \supseteq (1-\lambda)(K_{t_1} \cap (x + K_{t_1})) + \lambda(K_{t_2} \cap (x + K_{t_2})).$$  

Therefore, by Brunn-Minkowski inequality

$$(1) \ |K_{(1-\lambda)t_1 + \lambda t_2} \cap (x + K_{(1-\lambda)t_1 + \lambda t_2})| \geq |K_{t_1} \cap (x + K_{t_1})|^{1-\lambda}|K_{t_2} \cap (x + K_{t_2})|^\lambda$$

and

$$h_x((1-\lambda)t_1 + \lambda t_2) \geq (e^{-t_1} |K_{t_1} \cap (x + K_{t_1})|)^{1-\lambda} (e^{-t_2} |K_{t_2} \cap (x + K_{t_2})|)^\lambda = h_x(t_1)^{1-\lambda} h_x(t_2)^\lambda.$$  

Consequently, since $h_x$ is log-concave and integrable in $[0, \infty)$, for any $s \in [0, \infty)$, the set $\{t \in [0, \infty) : h_x(t) \geq s\}$ is either empty or a closed interval. Let us remark that for each $x \in \mathbb{R}^n$, since $g(x) \neq 0$, the function $h_x(t)$ is not identically 0 and it attains its maximum at a unique point, since if $\|h_x\|_{\infty} = h_x(t_1) = h_x(t_2)$, with $t_1 < t_2$, then we have that for every $\lambda \in [0, 1]$ $h_x((1-\lambda)t_1 + \lambda t_2) \geq \|h_x\|_{\infty}$, so $h_x((1-\lambda)t_1 + \lambda t_2) = \|h_x\|_{\infty}$. Therefore, we have equality in (1) and by the characterization of the equality cases in Brunn-Minkowski inequality (see, for instance AGM Section 1.2) $K_{t_2} \cap (x + K_{t_2})$ is a translation of $K_{t_1} \cap (x + K_{t_1})$ and they have the same volume. Thus $h_x(t_2) < h_x(t_1)$, which contradicts the fact that the maximum is attained both at $t_1$ and $t_2$. Notice also that for any $s \in [0, \infty)$ and any $\lambda \in [0, 1]$, if $t_1 \in \{t \in [0, \infty) : h_0(t) \geq s\|h_0\|_{\infty}\}$ and $t_2 \in \{t \in [0, \infty) : h_x(t) \geq s\|h_x\|_{\infty}\}$, then calling $t = (1-\lambda)t_1 + \lambda t_2$, since

$$K_t \cap (\lambda x + K_t) \supseteq (1-\lambda)K_{t_1} + \lambda(K_{t_2} \cap (x + K_{t_2})),$$

by Brunn-Minkowski inequality

$$|K_t \cap (\lambda x + K_t)| \geq |K_{t_1}|^{1-\lambda}|K_{t_2} \cap (x + K_{t_2})|^\lambda$$

and

$$h_{\lambda x}(t) = e^{-t} |K_t \cap (\lambda x + K_t)| \geq (e^{-t_1} |K_{t_1}|)^{1-\lambda}(e^{-t_2} |K_{t_2} \cap (x + K_{t_2})|)^\lambda = h_0(t_1)^{1-\lambda} h_x(t_2)^\lambda \geq s\|h_0\|_{\infty}^{1-\lambda}\|h_x\|_{\infty}^\lambda.$$  

Consequently, for any $x \in \mathbb{R}^n$ $s \in [0, \infty)$ and $\lambda \in [0, 1]$

$$(1-\lambda)\{t \in [0, \infty) : h_0(t) \geq s\|h_0\|_{\infty}\} \supseteq \{t \in [0, \infty) : h_0(t) \geq s\|h_0\|_{\infty}\} + \lambda \{t \in [0, \infty) : h_x(t) \geq s\|h_x\|_{\infty}\}.$$  

The last sets are non-empty for every $s \in [0, 1]$. Thus, for any $x \in \mathbb{R}^n$ and $\lambda \in [0, 1]$

$$\frac{g(\lambda x)}{\|h_0\|_{\infty}^{1-\lambda}\|h_x\|_{\infty}^\lambda} = \int_0^\infty |\{t \in [0, \infty) : h_{\lambda x}(t) \geq s\|h_0\|_{\infty}^{1-\lambda}\|h_x\|_{\infty}^\lambda\}| ds.$$
Let us remark that for any $t \in [0, \infty)$ and for any $h_x(t) \geq s\|h_0\|_\infty^{1-\lambda}\|h_x\|_\infty^{\lambda}$, since by assumption $g(\lambda x) = g(0)^{1-\lambda}g(x)^{\lambda}$, all the inequalities in the last chain of inequalities are equalities and then

- $\|h_{\lambda x}\|_\infty = \|h_0\|_\infty^{1-\lambda}\|h_x\|_\infty^{\lambda}$,
- The following two sets are equal for very $s \in [0, 1]$

\[ \{ t \in [0, \infty) : h_{\lambda x}(t) \geq s\|h_0\|_\infty^{1-\lambda}\|h_x\|_\infty^{\lambda} \} = (1-\lambda)\{ t \in [0, \infty) : h_0(t) \geq s\|h_0\|_\infty \} + \lambda\{ t \in [0, \infty) : h_x(t) \geq s\|h_x\|_\infty \}, \]

- $\int_0^\infty \frac{h_0(t)}{\|h_0\|_\infty} dt = \int_0^\infty \frac{h_x(t)}{\|h_0\|_\infty} dt$, or equivalently $\frac{g(0)}{\|h_0\|_\infty} = g(x)/\|h_0\|_\infty$.

Notice that, since the sets in the second condition are intervals, if we call

- $t_1(s) = \min\{ t \in [0, \infty) : h_0(t) \geq s\|h_0\|_\infty \}$
- $T_1(s) = \sup\{ t \in [0, \infty) : h_0(t) \geq s\|h_0\|_\infty \}$
- $t_2(s, x) = \min\{ t \in [0, \infty) : h_x(t) \geq s\|h_x\|_\infty \}$
- $T_2(s, x) = \sup\{ t \in [0, \infty) : h_x(t) \geq s\|h_x\|_\infty \}$,

the second condition is equivalent to

- $h_{\lambda x}(t_1(s)) = s\|h_0\|_\infty^{1-\lambda}\|h_x\|_\infty^{\lambda}$
- $h_{\lambda x}(T_1(s)) = s\|h_0\|_\infty^{1-\lambda}\|h_x\|_\infty^{\lambda}$

and, since $h_0(t_1(s)) = s\|h_0\|_\infty$, $h_x(t_2(s, x)) = s\|h_x\|_\infty$, $h_0(T_1(s)) = s\|h_0\|_\infty$, and $h_x(T_2(s, x)) = s\|h_x\|_\infty$, the last two equalities imply that for any $s \in [0, 1]$ there is equality in

\[ |K_(1-\lambda)t_1(s)+\lambda t_2(s, x)\cap(\lambda x+K_(1-\lambda)t_1(s)+\lambda t_2(s, x))| \geq |K_t_1(s)|^{1-\lambda}|K_t_2(s, x)\cap(x+K_t_2(s, x))| \]

and for any $s \in [0, 1]$ there is equality in

\[ |K_(1-\lambda)t_1(s)+\lambda t_2(s, x)\cap(\lambda x+K_(1-\lambda)t_1(s)+\lambda t_2(s, x))| \geq |K_t_1(s)|^{1-\lambda}|K_t_2(s, x)\cap(x+K_t_2(s, x))| \].

This, implies that for any $x \in \mathbb{R}^n$, any $\lambda \in [0, 1]$ and any $s \in [0, 1]$

- $K_{t_2(s, x)}\cap(x+K_{t_2(s, x)})$ is a translation of $K_{t_1(s)}$ and also $K_(1-\lambda)t_1(s)+\lambda t_2(s, x)$

\[ (\lambda x + K_(1-\lambda)t_1(s)+\lambda t_2(s, x)) \] is a translation of $K_t_1(s)$,

- $K_{t_2(s, x)}\cap(x+K_{t_2(s, x)})$ is a translation of $K_{t_1(s)}$ and also $K_(1-\lambda)t_1(s)+\lambda t_2(s, x)$

\[ (\lambda x + K_(1-\lambda)t_1(s)+\lambda t_2(s, x)) \] is a translation of $K_t_1(s)$.

Let us remark that for $s = 1$, since the function $h_x$ attains its maximum at a unique point and the value of this maximum is strictly positive, $t_1(1) = T_1(1)$ and $t_2(1, x) = T_2(1, x)$ for every $x \in \mathbb{R}^n$. Notice also that for any $x \in \mathbb{R}^n$ and any $\lambda \in [0, 1]$, since
Therefore, for every $f$ and for every $\lambda K$ since the last inclusion is an equality and $\lambda x \not\in K$ for every $K$ is a translation of $x$ and any $t$ and any $\lambda t (x) = \lambda (t_2(0, x) - t_1(0)) = \lambda (t_2(1, x) - t_1(1))$

Consequently, since for any $x \in \mathbb{R}^n$

Thus for every $x \in \mathbb{R}^n$, the difference $t_2(s, x) - t_1(s)$ does not depend on $s$ and $t_2(s, x) - t_1(s) = t_2(1, x) - t_1(1)$ for every $s \in [0, 1]$. Then, we have that for every $x \in \mathbb{R}^n$ and any $t_0 \in [0, 1]$

2. Fixing $t_0 \in [0, 1)$ we have that for every $x \in \partial(K_{t_0} - K_{t_0})$ inf $\{t \in [0, \infty) : x \in K_{t_0} - K_{t_0}\} = t_0$. Otherwise, inf $\{t \in [0, \infty) : x \in K_{t_0} - K_{t_0}\} \leq t_1 < t_0$. Since $t_0 \not\in K_{t_0} - K_{t_0}$ for every $\lambda > 1$, we have that inf $\{t \in [0, \infty) : \lambda x \in K_{t_0} - K_{t_0}\} \geq t_0$, and thus

3. For every $\lambda > 1$, a contradiction since $t_1 < t_0$. Thus, for any $\lambda \in [0, 1], \lambda x \in \partial(K_{\lambda t_0} - K_{\lambda t_0})$ and then for every $t \in [0, \infty)$ and any $\lambda \in [0, 1]$

4. Since $f$ is log-concave and $\|f\|_{\infty} = f(0)$ we have that $0 \in K_{t}$ for every $t \in [0, \infty)$ and for every $t \in [0, \infty)$ and any $\lambda \in [0, 1] \lambda K_{t} \subseteq K_{\lambda t}$ and then

Since the last inclusion is an equality and $\lambda K_{t} \subseteq K_{\lambda t}$ we obtain that $\lambda K_{t} = K_{\lambda t}$. Therefore, for every $t \in [0, \infty)$ $K_{t} = tK_{1}$ and $\frac{f(x)}{\|f\|_{\infty}} = e^{-\|x\|_{K_{1}}}$. Let us call $K := K_{1}$.

Since $t_1(0) = 0$ and for every $x \in \mathbb{R}^n$

where $\|\cdot\|_{\infty}$ denotes the infinity norm.
we have that for every $x \in \mathbb{R}^n$

$$g(x) = g(0) \frac{\|h_x\|_\infty}{\|h_0\|_\infty} = g(0)e^{-\langle t_2(0,x) - t_1(0) \rangle} = g(0)e^{-\|x\|K - \kappa}.$$ 

Therefore,

$$\int_{\mathbb{R}^n} g(x) = (n!)^2 |K||K - K|.$$

On the other hand, if $\frac{\|f(x)\|_\infty}{\|f\|_\infty} = e^{-\|x\|K}$

$$\int_{\mathbb{R}^n} g(x) dx = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \min\{e^{-\|x\|K}, e^{-\|y\|K}\} dy dx$$

$$= \int_{\mathbb{R}^{2n}} e^{-\|(x,y)\|K \times K} dy dx$$

$$= (2n)! |K|^2.$$

Thus

$$\left(\frac{2n}{n}\right) |K| = |K - K|$$

and, since we have equality in Rogers-Shephard inequality $\text{RS}$, $K$ is a simplex and $0 \in K = K_1$. □

Lemma 4.1 and Lemma 4.2 together characterize the equality case.

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