Research Article

Some Algebraic Properties of a Class of Integral Graphs Determined by Their Spectrum

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Let \( \Gamma = (V, E) \) be a graph. If all the eigenvalues of the adjacency matrix of the graph \( \Gamma \) are integers, then we say that \( \Gamma \) is an integral graph. A graph \( \Gamma \) is determined by its spectrum if every graph cospectral to it is in fact isomorphic to it. In this paper, we investigate some algebraic properties of the Cayley graph \( \Gamma = \text{Cay}(\mathbb{Z}_n, S) \), where \( n = p^m \) (\( p \) is a prime integer and \( m \in \mathbb{N} \)) and \( S = \{ a \in \mathbb{Z}_n | (a, n) = 1 \} \). First, we show that \( \Gamma \) is an integral graph. Also, we determine the automorphism group of \( \Gamma \). Moreover, we show that \( \Gamma \) and \( K_n \backslash \Gamma \) are determined by their spectrum.

1. Introduction

The graphs in this paper are simple, undirected, and connected. We always assume that \( \Gamma \) denotes the complement graph of \( \Gamma \). The eigenvalues of a graph \( \Gamma \) are the eigenvalues of the adjacency matrix of \( \Gamma \). The spectrum of \( \Gamma \) is the list of the eigenvalues of the adjacency matrix of \( \Gamma \) together with their multiplicities, and it is denoted by \( \text{Spec}(\Gamma) \); see [1]. If all the eigenvalues of the adjacency matrix of the graph \( \Gamma \) are integers, then we say that \( \Gamma \) is an integral graph. The notion of integral graphs was first introduced by Harary and Schwenk in 1974; see [2]. In general, the problem of characterizing integral graphs seems to be very difficult. There are good surveys in this area; see [3]. For more results depending on the integral graphs and their applications in engineering networks, see [4–6]. For any vertex \( v \) of a connected graph \( \Gamma \), we denote the set of vertices of \( \Gamma \) at distance \( r \) from \( \Gamma \) by \( \Gamma_r(v) \). Then, we have

\[
\Gamma_r(v) = \{ u \in V(\Gamma) | d(u, v) = r \},
\]

where \( d(u, v) \) denotes the distance in \( \Gamma \) between the vertices \( u \) and \( v \) and \( r \) is a nonnegative integer not exceeding \( d \), the diameter of \( \Gamma \). It is clear that \( \Gamma_0(v) = \{ v \} \), and \( V(\Gamma) \) is partitioned into the disjoint subsets \( \Gamma_0(v), \ldots, \Gamma_d(v) \), for each \( v \) in \( V(\Gamma) \). The graph \( \Gamma \) is called distance regular with diameter \( d \) and intersection array \( \{ b_0, \ldots, b_{d-1}; c_1, \ldots, c_d \} \) if it is regular of valency \( k \) and, for any two vertices \( u \) and \( v \) in \( \Gamma \) at distance \( r \), we have \( |\Gamma_r(v) \cap \Gamma_r(u)| = b_r \) (\( 0 \leq r \leq d - 1 \)), and \( |\Gamma_{r-1}(v) \cap \Gamma_r(u)| = c_r \) (\( 1 \leq r \leq d \)).

The intersection numbers \( c_r, b_r \), and \( a_r \) satisfy \( a_r = k - b_r - c_r \) (\( 0 \leq r \leq d \)), where \( a_r \) is the number of neighbours of \( u \) in \( \Gamma_r(v) \). Let \( G \) be a finite group and let \( H \) be a subset of \( G \) such that it is closed under taking inverses and does not contain the identity. A Cayley graph \( \Gamma = \text{Cay}(G, H) \) is the graph whose vertex set and edge set are defined as follows:

\[
V(\Gamma) = G;
\]
\[
E(\Gamma) = \{ \{ x, y \} | x^{-1}y \in H \}. \tag{2}
\]

It is well known that if \( \Gamma \) is a distance regular graph with valency \( k \), diameter \( d \), adjacency matrix \( A \), and intersection array

\[
\{ b_0, b_1, \ldots, b_{d-1}; c_1, c_2, \ldots, c_d \}, \tag{3}
\]

then the tridiagonal \( (d + 1) \times (d + 1) \) matrix,
2. Definitions and Preliminaries

**Definition 1** (see [7, 21]). Let $\Gamma$ be a graph with automorphism group $\text{Aut}(\Gamma)$. We say that $\Gamma$ is a vertex transitive graph if, for all vertices $x, y$ of $\Gamma$, there is an automorphism $\theta$ in $\text{Aut}(\Gamma)$ satisfying $\theta(x) = y$. Also, we say that $\Gamma$ is distance transitive graph if, for all vertices $u, v, x, y$ of $\Gamma$ such that $d(u, v) = d(x, y)$, there is an automorphism $\theta$ in $\text{Aut}(\Gamma)$ satisfying $\theta(u) = x$ and $\theta(v) = y$.

**Theorem 1** (see [22]). Let $\Gamma$ be a graph such that it contains $k$ components $\Gamma_1, \ldots, \Gamma_k$. If, for any $i \in I = \{1, \ldots, k\}$, we have $\Gamma_i \cong \Gamma_1$, then $\text{Aut}(\Gamma) \cong \text{Aut}(\Gamma_1)^{wr} \text{Sym}(k)$, where the wreath product is defined.

**Definition 2** (see [23]). Let $\Gamma_1 \cup \Gamma_2$ denote the disjoint union of graphs $\Gamma_1$ and $\Gamma_2$. The join $\Gamma_1 \vee \Gamma_2$ is the graph obtained from $\Gamma_1 \cup \Gamma_2$ by joining every vertex of $\Gamma_1$ with every vertex of $\Gamma_2$. A multicone graph is defined to be the join of a clique and a regular graph.

**Theorem 2** (see [9]). If $\Gamma$ is a distance regular graph with diameter $d$ and girth $g$ satisfying one of the following properties, then every graph cospectral with $\Gamma$ is also distance regular, with the same parameters as $\Gamma$:

(i) $g \geq 2d - 1$

(ii) $g \geq 2d - 2$ and $\Gamma$ is bipartite

**Proposition 1** (see [9]). For regular graphs, being DS (or not DS) is equivalent for the adjacency matrix, the adjacency matrix of the complement, and the Laplacian matrix.

**Proposition 2** (see [9]). The following graph and its complement, which have at most four eigenvalues, are regular DS graphs:

(i) The disjoint union of $k$ copies of a strongly regular DS graph.

**Theorem 3** (see [24]). Let $\Gamma_1$ and $\Gamma_2$ be two graphs with the Laplacian spectrum $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ and $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_m$, respectively. Then, the Laplacian spectrum of $\Gamma_1 \vee \Gamma_2$ is $n + m, m + \lambda_1, m + \lambda_2, \ldots, m + \lambda_{n-1}, n + \mu_1, n + \mu_2, \ldots, n + \mu_{m-1}, 0$.

**Theorem 4** (see [1]). Let $\Gamma$ be a graph on $n$ vertices. Then, $n$ is a Laplacian eigenvalue of $\Gamma$ if and only if $\Gamma$ is the join of two graphs.

**Lemma 1** (see [1]). A connected graph $\Gamma$ has exactly one positive eigenvalue if and only if it is a complete multipartite graph.
3. Main Results

Theorem 5. Let $\Gamma = \text{Cay}(\mathbb{Z}_m, S)$ be the Cayley graph on the cyclic group $\mathbb{Z}_m$, where $n = p^m$ ($p$ is a prime integer and $m \in \mathbb{N}$) and $S = \{a \in \mathbb{Z}_m | (a, n) = 1\}$. Then,

$$\text{Aut}(\Gamma) \equiv \text{Sym}(p^{m-1})\text{wr}_{1}\text{Sym}(p),$$

where $I = \{1, 2, \ldots, p\}.$

Proof. Let $V(\Gamma) = \{1, \ldots, n\}$ be the vertex set of $\Gamma$. Note that if $m = 1$, then the result immediately follows. Because in this case, $\Gamma \equiv K_p$, where $K_p$ is the complete graph on $p$ vertices, in the sequel, we assume that $m \geq 2$. Let $T = \langle p \rangle = \{kp | 0 \leq k \leq p^m - 1\}$ be the subgroup of the group $\mathbb{Z}_n$ of order $p^m - 1$. It is clear that $T$ and every coset of $T$ represent an independent set in the graph $\Gamma$. In fact, if $T + a$ is a coset of $T$ in the group $\mathbb{Z}_n$ such that $T + T + a = \emptyset$, then $a$ and $p$ are coprime and hence we have $a \in S$. It follows that every coset of $\Gamma$ is a clique of order $p^m - 1$ in the complement of the graph $\Gamma$. Thus, $\Gamma$ contains $p$ disjoint components $\Gamma_1, \Gamma_2, \ldots, \Gamma_p$ such that $\Gamma_i \equiv K_{p^m-1} (1 \leq i \leq p)$, where $K_{p^m-1}$ is the complete graph on $p^m - 1$ vertices. It follows that $\Gamma \equiv pK_{p^m-1}$. Hence, by Theorem 1, $\text{Aut}(\Gamma) \equiv \text{Aut}(K_{p^m-1})\text{wr}_{1}\text{Sym}(p) = \text{Sym}(p^{m-1})\text{wr}_{1}\text{Sym}(p)$. On the other hand, it is well known that, for any graph $\Gamma$, $\text{Aut}(\Gamma) = \text{Aut}(\Gamma)$; see [1].

Proposition 3. Let $\Gamma = \text{Cay}(\mathbb{Z}_m, S)$ be the Cayley graph on the cyclic group $\mathbb{Z}_m$, where $n = p^m$ ($p$ is a prime integer and $m \in \mathbb{N}$) and $S = \{a \in \mathbb{Z}_m | (a, n) = 1\}$. Then $\Gamma$ is a distance transitive graph.

Proof. Suppose that $u, v, x, y$ are vertices of $\Gamma$ such that $d(u, v) = d(x, y) = r$, where $r$ is a nonnegative integer not exceeding $d$, the diameter of $\Gamma$. So $d(u, v) = d(x, y) = 1$ or 2, since we now have the diameter of $\Gamma$ as $d = 2$. In the following cases, we show that $\Gamma$ is a distance transitive graph.

Case 1. If $d(u, v) = d(x, y) = 2$, then $u^{-1}v \notin S$ and $x^{-1}y \notin S$. Also, vertices $u, v$ and $x$ are adjacent in the complement $\Gamma$ of $\Gamma$; also two vertices $x$ and $y$ are adjacent in the complement $\Gamma$ of $\Gamma$. By Theorem 5, we know that $\Gamma$ contains $p$ components $\Gamma_1, \Gamma_2, \ldots, \Gamma_p$ such that, for any $i \in \{1, 2, \ldots, p\}$, $\Gamma_i \equiv K_{p^m-1}$. Therefore, $\Gamma \equiv pK_{p^m-1}$. If $u = x$, then $u, v$ and $y$ lie in a clique of graph $\Gamma$, hence we may assume that $\theta = (uv) \in \text{Aut}(\Gamma) = \text{Aut}(\Gamma)$, so $\theta(u) = x$ and $\theta(v) = y$. If $u \neq x$ and $v \neq y$, then $u, v$ lie in a clique of graph $\Gamma$, say $\Gamma_j$; also, $x, y$ lie in a clique of graph $\Gamma$, say $\Gamma_j$, where $\Gamma_j \neq \Gamma_1, \Gamma_j \neq \Gamma_p$. Hence, we may assume that $\theta = (ux)(vy) \in \text{Aut}(\Gamma) = \text{Aut}(\Gamma)$. Thus, $\theta(u) = x$ and $\theta(v) = y$.

Case 2. If $d(u, v) = d(x, y) = 1$, then we can show that there is an automorphism $\theta$ in $\text{Aut}(\Gamma)$ such that $\theta(u) = x$ and $\theta(v) = y$.

Proposition 4. Let $\Gamma = \text{Cay}(\mathbb{Z}_m, S)$ be the Cayley graph on the cyclic group $\mathbb{Z}_m$, where $n = p^m$ ($p$ is a prime integer and $m \in \mathbb{N}$) and $S = \{a \in \mathbb{Z}_m | (a, n) = 1\}$. Then $\Gamma$ is an integral graph.

Proof. It is well known that if $\Gamma$ is a distance transitive graph, then $\Gamma$ is also distance regular; see [21]. Now, let $V(\Gamma) = \{1, 2, \ldots, n\}$ be the vertex set of $\Gamma$. Consider the vertex $v = n$ in $V(\Gamma)$; then $\Gamma_0(v) = \{v\}$, $\Gamma_1(v) = \{a \in V(\Gamma) | (a, n) = 1\}$, and $\Gamma_2(v) = \{a \in V(\Gamma) | (a, n) \neq 1\}$. Let $u$ be the vertex in $V(\Gamma)$ such that $d(u, v) = 0$, then $u = v = n$ and $|\Gamma_1(v) \cap \Gamma_1(u)| = n - p^m$. Hence, $\theta_0 = n - p^m$ and, by definition of distance regularity of graph, we have $\theta_0 = (n - p^m) - \theta_0 = 0$. Also, if $u$ in $V(\Gamma)$ and $d(u, v) = 1$, then two vertices $u, v$ are adjacent in $\Gamma$, $|\Gamma_1(v) \cap \Gamma_1(u)| = n - p^m - 1$. Hence, $c_1 = 1, b_1 = p^m - 1 - 1$, and $a_1 = (n - p^m - 1) - b_1 - c_1 = n - 2p^m - 1$. Finally, if $u \in V(\Gamma)$ and $d(u, v) = 2$, then two vertices $u, v$ are not adjacent in $\Gamma$, $|\Gamma_1(v) \cap \Gamma_1(u)| = n - p^m - 1$. Hence, $c_2 = n - p^m - 1$ and $a_2 = (n - p^m) - (n - p^m - 1) = 0$. Thus, the intersection array of $\Gamma$ is $[n - p^m - 1, p^m - 1 - 1; 1, n - p^m - 1]$. Therefore, the tridiagonal $(3) \times (3)$ matrix,

$$
\begin{bmatrix}
0 & b_0 & 0 \\
1 & a_1 & b_1 \\
0 & c_2 & a_2
\end{bmatrix} = \begin{bmatrix}
0 & n - p^m - 1 \\
1 & n - 2p^m - 1 & p^m - 1 - 1 \\
0 & n - p^m - 1 & 0
\end{bmatrix},
$$

determines all the eigenvalues of $\Gamma$. It is clear that all the eigenvalues of $\Gamma$ are $n - p^m$, $0$, $-p^m$, and their multiplicities are $1, n - p$, $p - 1$, respectively. Thus, $\Gamma$ is an integral graph.

Corollary 1. Let $\Gamma = \text{Cay}(\mathbb{Z}_m, S)$ be the Cayley graph on the cyclic group $\mathbb{Z}_m$, where $n = p^m$ ($p$ is a prime integer and $m \in \mathbb{N}$) and $S = \{a \in \mathbb{Z}_m | (a, n) = 1\}$. Then the adjacency spectrum of $\Gamma$ is $\{n - p^m, 0^{(n - 1)}, (-p^m)^{(p - 1)}\}$.

Theorem 6. Let $\Gamma = \text{Cay}(\mathbb{Z}_m, S)$ be the Cayley graph on the cyclic group $\mathbb{Z}_m$, where $n = p^m$ ($p$ is a prime integer and $m \in \mathbb{N}$) and $S = \{a \in \mathbb{Z}_m | (a, n) = 1\}$. Then $\Gamma$ is a DS graph with respect to its adjacency spectrum.

Proof. We know that if $p$ is even prime integer, then $\Gamma$ is isomorphic to the bipartite graph $K_{p^m-1,1, p^m-1}$, and hence the result immediately follows.

Now, let $p$ be an odd prime integer; then, $\Gamma$ is not bipartite graph. In particular, $g \geq 2 d - 1$, because the diameter of $\Gamma$ is 2 and the girth of $\Gamma$ is 3. Hence, by Theorem 2, every graph cospectral with $\Gamma$ is also distance regular, with the same parameters as $\Gamma$. Because by Proposition 3 we know that $\Gamma$ is a distance regular graph, $\Gamma$ is a DS graph with respect to its adjacency spectra. Because, by Proposition 2, $\Gamma$ contains disjoint union of $p$ copies of the strongly regular DS graph $K_{p^m-1}$ in addition to the graph $\Gamma$ and its complement, which have at most four eigenvalues.

Proposition 5. Let $\Pi$ be a graph cospectral with the maticone graph $K_n, \forall \Gamma$ with respect to its adjacency matrix
spectrum, where $\Gamma = \text{Cay}(\mathbb{Z}_m, S)$, which is defined as before. Then $\Pi$ is a bigraph. Also,

$$\text{Spec}(\Pi) = \left\{ 0^{(n-p)}, (-p^{m-1})^{(p-1)}, \left(\frac{M + \sqrt{M^2 + 4N}}{2}, \frac{M - \sqrt{M^2 + 4N}}{2}\right) \right\}, \quad (7)$$

where $M = n - 1 + p^m - p^{m-1}$ and $N = p^m + p^{m-1} - p - p^{m-1}$.

**Proof.** We can deduce the following from Theorem 2.1.8 in [25] and Theorem 2.1 in [26].

**Theorem 7.** Consider the multicone graph $K_v \Delta \Gamma$, where $\Gamma = \text{Cay}(\mathbb{Z}_m, S)$, which is defined as before. Then $K_v \Delta \Gamma$ is DS with respect to its adjacency matrix spectrum.

**Proof.** In the following, we proceed by induction on the number of vertices in $K_v$. Let $K_v$ have one vertex and let $\Pi$ be a graph cospectral with the multicone graph $K_v \Delta \Gamma$ with respect to its adjacency matrix spectrum. By Proposition 5, it is easy to see that $\Pi$ has one vertex of degree $p^m$, say $j$. Hence, if Spec $(\Pi - j) = \text{Spec}(\Gamma)$, then $\Pi - j \equiv \Gamma$. Because, by Theorem 6, we know that $\Gamma$ is DS graph with respect to its adjacency matrix spectrum, $\Pi \equiv K_v \Delta \Gamma$. We assume inductively that this claim holds for $K_v$; that is, if $\Pi_1$ is a graph cospectral with the multicone graph $K_v \Delta \Gamma$ with respect to its adjacency matrix spectrum, then $\Pi_1 \equiv K_v \Delta \Gamma$. We show that the claim is true for $K_{v+1}$; that is, if $\Pi$ is a graph cospectral with the multicone graph $K_{v+1} \Delta \Gamma$ with respect to its adjacency matrix spectrum, then $\Pi \equiv K_{v+1} \Delta \Gamma$. It is obvious that $\Pi$ has one vertex and $p^m + v$ edges more than $\Pi_v$. On the other hand, by Proposition 5, we know that $\Pi_v$ has $v$ vertices of degree $p^m + v - 1$ and $p^m$ vertices of degree $p^m - p^{m-1} + 1$, and also $\Pi_v$ has $v + 1$ vertices of degree $p^m + v$ and $p^m$ vertices of degree $p^m - p^{m-1} + v + 1$. So, we must have $\Pi \equiv K_{v+1} \Delta \Pi_1$. Now, by assuming induction, we conclude that $\Pi \equiv K_{v+1} \Delta \Pi$ and complete the proof.

**Theorem 8.** Consider the complement $\overline{K_v \Delta \Gamma}$ of multicone graph $K_v \Delta \Gamma$ with respect to its adjacency spectrum, where $\Gamma = \text{Cay}(\mathbb{Z}_m, S)$, which is defined as before. Then $\overline{K_v \Delta \Gamma}$ is a DS graph.

**Proof.** By Theorem 5, we know that $\Gamma$ contains $p$ components $\Gamma_1, \Gamma_2, \ldots, \Gamma_p$, such that $\Gamma_i \equiv K_{p^{m-1}} \Pi_1$. In addition, the adjacency matrix spectrum of $\Gamma$ is

$$\left\{ \left( p^{m-1} - 1 \right)^{(p)}, -1 \left( p^{m-1} - p \right) \right\}. \quad (8)$$

Also, the adjacency matrix spectrum of $\overline{K_v}$ is $\{0^{(0)}\}$. Thus, the adjacency matrix spectrum of $\Gamma \cup \overline{K_v}$ is

$$\left\{ \left( p^{m-1} - 1 \right)^{(p)}, -1 \left( p^{m-1} - p \right), 0^{(0)} \right\}. \quad (9)$$

On the other hand, it is not hard to see that $\overline{\Gamma \cup K_v} \equiv K_v \Delta \Gamma$. Let $\Pi$ be a graph cospectral with the complement $K_v \Delta \Gamma$ of multicone graph $K_v \Delta \Gamma$ with respect to its adjacency spectrum; then,

$$\text{Spec}(\Pi) = \text{Spec}(K_v \Delta \Gamma) = \left\{ (p^{m-1} - 1)^{(p)} - 1 (p^{m-1} - p), 0^{(0)} \right\}. \quad (10)$$

It is easy to prove that $\Pi$ cannot be regular, since regularity of a graph can be determined by its spectrum. Also, we show that $\Pi$ is disconnected graph. Suppose to the contrary that $\Pi$ is connected; hence, by Lemma 1, $\Pi$ is complete multipartite graph, contradicting the adjacency spectrum of $\Pi$. Thus, $\Pi$ is disconnected graph. Therefore, we conclude that $\overline{K_v \Delta \Gamma}$ is DS with respect to its adjacency spectrum.

**Proposition 6.** Consider the multicone graph $K_v \Delta \Gamma$, where $\Gamma = \text{Cay}(\mathbb{Z}_m, S)$, which is defined as before. Then $K_v \Delta \Gamma$ is DS with respect to its Laplacian spectrum.

**Proof.** By Theorem 3, the Laplacian matrix spectrum of $K_v \Delta \Gamma$ is

$$\left\{ (n + v)^{(p+1)}, (n + v - p^{m-1})^{(n-p)}, 0 \right\}. \quad (11)$$

We proceed by induction on the number of vertices in $K_v$. If $v = 1$, there is nothing to prove. We assume inductively that this claim holds for $K_v$; that is, if Spec $(L(\Pi_v)) = \text{Spec}(L(K_v \Delta \Gamma))$, then $\Pi_v \equiv K_v \Delta \Gamma$, where $\Pi_v$ is a graph cospectral with the multicone graph $K_v \Delta \Gamma$ with respect to its Laplacian spectrum. We show that the claim is true for $K_{v+1}$; that is, if

$$\text{Spec}(L(\Pi)) = \text{Spec}(L(K_{v+1} \Delta \Gamma)) = \left\{ (n + v + 1)^{(p+1)}, (n + v + 1 - p^{m-1})^{(n-p)}, 0 \right\}, \quad (12)$$

then $\Pi \equiv K_{v+1} \Delta \Gamma$, where $\Pi$ is a graph cospectral with the multicone graph $K_{v+1} \Delta \Gamma$ with respect to its Laplacian spectrum. By Theorem 4, we know that $\Pi_1$ and $\Pi$ are join of two graphs, because $n + v$ and $n + v + 1$ are eigenvalues of $\Pi_1$ and $\Pi$, respectively. In addition, $\Pi$ has one vertex of degree $n + v$ more than $\Pi_1$, say $j$; hence, Spec $(L(\Pi - j)) \equiv \text{Spec}(L(K_v \Delta \Gamma))$, and, by assuming induction, $\Pi - j \equiv K_v \Delta \Gamma$. Thus, it can be concluded that $\Pi \equiv K_{v+1} \Delta \Gamma$. \qed
4. Conclusion

In this paper, we computed the adjacency spectrum of a class of integral graphs, denoted by $\Gamma = \text{Cay}(Z_n, S)$, where $n = p^m$ ($p$ is a prime integer and $m \in \mathbb{N}$) and $S = \{a \in Z_n | (a, n) = 1\}$. Indeed, by using the theory of distance regular graphs, it is shown that the adjacency spectrum of $\Gamma$ is $(n - p^{m-1}, 0^{(n-p)}, (-p^{m-1})^{(p-1)})$, where the superscripts give the multiplicities of eigenvalues with multiplicity greater than one. Moreover, it is shown that the Cayley graphs $\Gamma$ and $K_n \setminus \Gamma$ are determined by their spectrum. Note that this class of graphs is a special subclass of integral circulants, and hence clearly not only is this class of graphs mathematically applicable, but also it is used in the design of engineering networks.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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