HIGH-DIMENSIONAL
CONFORMALLY RECURRENT MANIFOLDS

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ABSTRACT. Conformally recurrent pseudo-Riemannian manifolds of dimension \( n \geq 5 \) are investigated. The Weyl tensor may be represented as a Koulkarni-Nomizu product involving a symmetric tensor and the recurrence vector. If the recurrence vector is a closed form, the Ricci and two other tensors are Weyl compatible. If the recurrence vector is non-null, a covariantly constant symmetric tensor exists, with geometric implications. If the metric is Lorentzian, a null recurrence vector makes the Weyl tensor of algebraic type II_\( d \) or higher in the Bel-Debever-Ortaggio classification, while a time-like recurrence vector makes the Weyl tensor purely electric.

1. INTRODUCTION

Manifolds with a recurrent curvature tensor and generalisations were investigated by many geometers and are the subject of a vast literature. The recurrent Riemann tensor was first studied in dimension \( n = 3 \) by Ruse in 1949 [37], and then by Walker [44] (see also Chaki [5], Kaigorodov [19], Khan [20]). Its relationship with plane waves in general relativity was discussed by Sciama [40]. Soon after Patterson [31] introduced Ricci-recurrent spaces. Riemannian manifolds with recurrent Weyl curvature tensor (i.e. conformally recurrent manifolds) were first studied by Adati and Miyazawa [1], and generalised to pseudo-Riemannian manifolds by Roter [34, 35, 36] and later by Suh and Kwon [42]. Mc Lenaghan and Leroy [28] and Mc Lenaghan and Thompson [29] considered Lorentzian manifolds (space-times) with complex recurrent Weyl tensor. They showed that such spaces belong to types D or N of Petrov’s classification, and obtained the expression of the metric in the case of real recurrence vector. Conformally recurrent space-times were also studied by Hall [14, 15].

Definition 1.1. A \( n \)-dimensional pseudo-Riemannian manifold is conformally recurrent, \((CR)_n\), if the conformal curvature (Weyl) tensor\(^\dagger\) satisfies the condition:

\[
\nabla_i C_{jklm} = \alpha_i C_{jklm}
\]

with a non-zero covector field \( \alpha_i \) named recurrence vector.

\(^\dagger\)The components of the Weyl tensor are [32]:

\[
C_{jklm} = R_{jklm} + \frac{1}{n-2} (\delta_j^m R_{kl} - \delta_k^m R_{jl} + \delta_l^m R_{jk} - R_{kml} g_{jl}) - R \frac{\delta_j^m g_{kl} - \delta_k^m g_{jl}}{(n-1)(n-2)}
\]

where \( R_{kl} = -R_{mkl}^m \) is the Ricci tensor and \( R = g^{ij} R_{ij} \) is the curvature scalar.
The definition has two straightforward well known consequences. The first one is an equation for the recurrence vector: \( \nabla_i C^2 = 2 \alpha_i C^2 \), where \( C^2 = C_{jklm}C^{jklm} \).

Then, if \( C^2 \neq 0 \), the recurrence vector is the gradient of a scalar:

\[ \alpha_i = \frac{1}{2} \nabla_i \log |C^2| \]  

The second consequence is the identity:

\[ [\nabla_i, \nabla_j] C_{klmn} = (\nabla_i \alpha_j - \nabla_j \alpha_i) C_{klmn} \]  

If the vector field \( \alpha_i \) is closed then \( [\nabla_i, \nabla_j] R_{klmn} = 0 \), i.e. the manifold is Weyl semi-symmetric.

We recall that a manifold is semi-symmetric if \( [\nabla_i, \nabla_j] R_{klmn} = 0 \), a property that implies Weyl semi-symmetry. The Riemannian semi-symmetric manifolds were classified by Szabo [43]. Grycak proved that for pseudo-Riemannian manifolds of dimension \( n \geq 5 \), semi-symmetry and Weyl semi-symmetry are equivalent [13]. For \( n = 4 \) the equivalence does not hold in general, but was established for Lorentzian manifolds by Eriksson and Senovilla, who also showed that such spaces are Petrov types N, D, O [12].

Senovilla also proved the remarkable result that semi-symmetric manifolds are of constant curvature at “generic” points, i.e. where the Riemann matrix \( R_{ijkl} \) is invertible. At such points it is: \( R_{ijkl} = R(g_{ik}g_{jl} - g_{jk}g_{il})/(n - 1) \) [39].

In \( n \geq 5 \), by the above equivalence, Weyl semi-symmetry implies Ricci semi-symmetry, \( [\nabla_i, \nabla_j] R_{kl} = 0 \). A summation over cyclic permutations of indices \( ijk \) gives that the Ricci tensor is Riemann-compatible [8]:

\[ R_{im}R_{jkl}^{\ i} + R_{jm}R_{kld}^{\ m} + R_{km}R_{ijd}^{\ m} = 0 \]  

The notion of K-compatible tensor, where \( K \) is Riemann’s or Weyl’s or a generalized curvature tensor, was introduced by us in [23]. Its various geometric implications were investigated in [24] for the Riemann tensor and in [25] for the Weyl tensor. In particular we showed that Weyl and Riemann compatibility are equivalent for the Ricci tensor.

Manifolds whose Ricci tensor is Weyl compatible are termed “Weyl compatible manifolds” [27]. Several such manifolds, which include Robertson-Walker space-time, are discussed in [10]; another example is Gödel’s metric (11), th.2).

In this paper we present new results about (CR)\(_n\) manifolds and the classification of the Weyl tensor on Lorentzian (CR)\(_n\) manifolds with \( n \geq 5 \).

In Section 2 we specialize the second Bianchi identity for the Weyl tensor to an algebraic one, which involves the recurrence vector. In \( n \geq 5 \) a contraction gives that the tensor \( \alpha_i \alpha_j \) is Weyl compatible.

In Section 3 we obtain a representation of the Weyl tensor in terms of the symmetric tensor \( E_{il} = C_{ijkl} \alpha^j \alpha^k / \alpha^2 \) and the recurrence vector. We show, with a Lorentzian metric \( n \geq 5 \) and null recurrence vector \( (\alpha^2 = 0) \), that the algebraic type of the Weyl tensor is at least type II\(_d\) in the high-dimensional Bel-Debever-Ortaggio classification [30]. This extends the result of Mc Lenaghan and Leroy [28] valid in \( n = 4 \). If \( \alpha^2 < 0 \), Weyl’s tensor is purely electric, according to the definition given in [18].

In Section 4 we study the tensor \( \Gamma_{ij} = C^{ijkl}C_{jklm} \). We show that it is Riemann compatible and obtain a representation for it, valid in \( n \geq 5 \) and \( \alpha^2 \neq 0 \), which highlights a covariantly constant \((0, 2)\) symmetric tensor, \( \nabla_k h_{ij} = 0 \). Based on
results by [39] and [16], the property implies that a (CR)$_n$ Lorentzian manifold is either decomposable or it admits a unique null covariantly constant vector field.

In Section 5 we show that the Ricci tensor is Riemann compatible, in the case that $\alpha$ is closed and $n \geq 5$.

In Section 6 we construct and study a new simple example of (CR)$_n$ manifold.

In Section 7 we consider the special dimension $n = 4$. We show that if a Lorentzian (CR)$_4$ space-time is Petrov’s type N with respect to a null vector $k$ and if the recurrence vector is closed, then $k$ is an eigenvector of the Ricci tensor and other tensors built with the recurrence vector. We show that in a Lorentzian (CR)$_4$ space-time the Ricci tensor is Weyl compatible.

It is assumed that the manifolds are connected, Hausdorff, with non-degenerate metric ($n$-dimensional pseudo-Riemannian manifolds) and that $\nabla_j g_{kl} = 0$. Where necessary we specialize to a metric with signature $n - 2$ (Lorentzian manifolds).

2. An algebraic identity

Adati and Miyazawa proved that in a (CR)$_n$ manifold, if $\alpha_p C_{jkl}^p = 0$ then either $\alpha$ is a null vector or the Weyl tensor is zero ([I], th.3.3). The proof was based on a general identity for the Weyl tensor ([I] eq.3.7, [26]) which is exploited here:

**Lemma 2.1.** If $C_{ijkl}$ is the Weyl tensor on a pseudo-Riemannian manifold, then:

$$\nabla_i C_{jklm} + \nabla_j C_{kim} + \nabla_k C_{ijlm} = \frac{1}{n-3} \nabla_p (g_{jm} C_{kil}^p + g_{km} C_{ijl}^p$$

$$ + g_{im} C_{jkl}^p + g_{kl} C_{ijm}^p + g_{jl} C_{ikm}^p).$$

On a (CR)$_n$ with recurrence vector $\alpha_i$, the differential identity translates into an algebraic one:

$$\alpha_i C_{jklm} + \alpha_j C_{kim} + \alpha_k C_{ijlm} = \frac{\alpha^p}{n-3} (g_{jm} C_{kil}^p + g_{km} C_{ijl}^p$$

$$ + g_{im} C_{jkl}^p + g_{kl} C_{ijm}^p + g_{jl} C_{ikm}^p).$$

While contractions with the metric tensor get nowhere, there are two useful contractions with the recurrence vector:

**Proposition 2.2.** On a $n \geq 5$ (CR)$_n$ pseudo-Riemannian manifold with recurrence vector $\alpha_i$, the tensor $\alpha_i \alpha_m$ is Weyl compatible:

$$\frac{n-4}{n-3} \left[ \alpha_i \alpha_m C_{jklm} + \alpha_j \alpha_m C_{kil}^m + \alpha_k \alpha_m C_{ijlm} \right] = 0.$$ 

**Proof.** Contraction of (5) with $\alpha^m$ cancels some terms and gives the equation. □

**Proposition 2.3.**

$$\alpha^2 C_{jklm} = -\frac{n-2}{2n-3} \alpha^p \left( \alpha_j C_{lmkp} + \alpha_k C_{mljp} + \alpha_m C_{jklp} + \alpha_l C_{jjmp} \right)$$

$$+ \frac{\alpha^2 \alpha^i}{n-3} (-g_{jm} C_{kilp} + g_{km} C_{ijlp} - g_{il} C_{jkm} + g_{jl} C_{ikm}),$$

$$0 = \frac{n-4}{n-3} \left[ \alpha^2 \alpha^p C_{kilm} - \alpha^j \alpha^p (\alpha_k C_{jjmp} - \alpha_l C_{jkm}) \right].$$
Proof. The contraction of (5) with $\alpha^i$ gives
\[
\alpha^2 C_{jklm} + \alpha^p (\alpha_j C_{lmkp} + \alpha_k C_{mljp}) - \frac{\alpha^p}{n-3} (\alpha_m C_{jklp} + \alpha_l C_{kjmp})
\]
\[
= \frac{\alpha^p \alpha^i}{n-3} (-g_{jm} C_{iklp} + g_{km} C_{ijlp} - g_{kl} C_{ijmp} + g_{jl} C_{ikmp}),
\]
symmetrization in the exchange of the pairs $jk$ and $lm$ gives the first result, while anti-symmetrization gives an equation equivalent to (6). A further contraction of (6) gives (8), i.e. double contractions in terms of single ones. □

3. A representation of the Weyl tensor in $n \geq 5$

As the formulae suggest, the dimension $n = 4$ is special, and will be discussed separately, in the last section. For $n \geq 5$ and $\alpha^2 \neq 0$, relations (7) and (8) yield a representation of the Weyl tensor in terms of the recurrence vector and the symmetric tensor
\[
E_{il} = \frac{\alpha^j \alpha^k}{\alpha^2} C_{ijkl}.
\]
The properties $E_{ij} = E_{ji}$, $E_{il} \alpha^l = 0$ and $E_{ii} = 0$ are obvious consequences of the general properties of the Weyl tensor.

**Theorem 3.1.** On a pseudo-Riemannian (CR) manifold of dimension $n > 4$ with $\alpha^2 \neq 0$, the Weyl tensor has the form:
\[
C_{jklm} = \frac{1}{n-3} \left[ g_{mk} E_{jl} - g_{mj} E_{kl} + g_{jl} E_{km} - g_{kl} E_{jm} \right]
+ \frac{n-2}{n-3} \left[ \frac{\alpha_j \alpha_m}{\alpha^2} E_{kl} - \frac{\alpha_k \alpha_m}{\alpha^2} E_{jl} + \frac{\alpha_k \alpha_l}{\alpha^2} E_{jm} - \frac{\alpha_j \alpha_l}{\alpha^2} E_{km} \right].
\]

Proof. Eq.(4) is:
\[
\alpha^2 C_{jklm} = -\frac{1}{n-3} \alpha^p (\alpha_j C_{lmkp} + \alpha_k C_{mljp} + \alpha_m C_{kjlp} + \alpha_l C_{jkmp})
+ \frac{\alpha^2}{n-3} (-g_{jm} E_{kl} + g_{km} E_{jl} - g_{kl} E_{jm} + g_{jl} E_{km}).
\]
The terms $\alpha^p C_{jklp}$ are expressed in terms of $E$ by means of (8):
\[
\alpha^l C_{kjm} = \alpha_k E_{jm} - \alpha_j E_{km},
\]
and the representation is obtained. □

**Remark 3.2.** The representation (10) of the Weyl tensor is the Koukarni-Nomizu product [4] of the tensors $\frac{1}{n-3}[g_{jk} - (n-2)\alpha_j \alpha_k / \alpha^2]$ and $E_{ml}$.

Contraction of (10) by $C_{jklm}$ gives:
\[
C^2 = 4 \frac{n-2}{n-3} E^2.
\]

**Proposition 3.3.** On a (CR) manifold with $\alpha^2 \neq 0$, the tensor $E_{ij}$ is Weyl compatible:
\[
E_{im} C_{jkl}^m + E_{jm} C_{kil}^m + E_{km} C_{ijl}^m = 0.
\]
Proof. From the representation (10) we evaluate

\[ E_{im}C_{jkl}^m = \frac{1}{n-3} (E_{ik}E_{jl} - E_{ij}E_{kl} + g_{jl}E_{km}E_{i}^m - g_{kl}E_{im}E_{j}^m) \]

\[ + \frac{n-2}{n-3} \left[ \frac{\alpha_k \alpha_l}{\alpha^2} E_{im}E_{j}^m - \frac{\alpha_j \alpha_l}{\alpha^2} E_{km}E_{i}^m \right] ; \]

the sum over cyclic permutations of indices \( ijk \) cancels all terms in the right-hand side, and Weyl compatibility is proven. \( \square \)

If \( \alpha^2 = -1 \), the tensor \( E_{ij} \) is referred to as the “electric tensor” associated to the Weyl tensor. For Lorentzian manifolds, the notions of electric and magnetic parts of the Weyl tensor (see [2]) were recently extended to \( n > 4 \) by Herwik et al. [18].

Given a time-like vector \( u \), with \( u^2 = -1 \), and the tensors \( h_{kl} = g_{kl} + u_k u_l \) and \( E_{kl} = u^j u^m C_{ijklm} \), the Weyl tensor is decomposed as \( C = C_+ + C_- \), with electric and magnetic components:

\[
(C_+)_{ijk}^m = h^{jr} h^{ks} h^{tl} u_m C_{rstu} + 4 u^j u^k C_{ijklm}^k \\
(C_-)_{ijk}^m = 2 h^{jr} h^{ks} C_{rspi}(u_m) + 2 u^j u^k C_{ijklm}^k + 2 u^j u^k C_{ijklm}^k h_{sm}.
\]

Theorem 3.5 in ref. [18] states that the Weyl tensor is purely electric \( (C_- = 0) \) if and only if \( u_i u_m C_{ijklm}^m + u_j u_m C_{ijklm}^m + u_k u_m C_{ijklm}^m = 0 \). Because of eq. (10) we may assert:

**Proposition 3.4.** On \((CR)_n\) Lorentzian manifolds of dimension \( n \geq 5 \) with time-like recurrence vector \( (\alpha^2 < 0) \) the Weyl tensor is purely electric.

We now obtain the algebraic classification of the Weyl tensor on Lorentzian \((CR)_n\) manifolds with \( n > 4 \), thus extending the result by Mc Lenaghan and Leroy [28] in \( n = 4 \). It is based on the Bel-Debever classification of Weyl tensors on Lorentzian manifolds with \( n > 4 \) by Ortaggio [30].

**Lemma 3.5.** Let \( b_{ij} \) be a Weyl compatible tensor such that there are non-collinear vectors \( X, Y \) such that \( b_{ij}X^i = 0 \) and \( b_{ij}Y^i = 0 \). If \( Z \) is a non-zero vector and \( Z^i = b^i_j U^j \) for some vector \( U \), then: \( X^i Y^j Z^k C_{ijkl} = 0 \).

**Proof.** Multiply the Weyl compatibility relation by \( X^i Y^j U^k; \)

\[
0 = X^i Y^j U^k [b_{lj} m_{ijklm} + b_{mj} m_{ijklm} + b_{k} m_{ijklm}] = X^i Y^j (U^k b_{lj} m^{ijklm}) C_{ijkl}.
\]

**Proposition 3.6.** On a \( n \geq 5 \) \((CR)_n\) Lorentzian manifold with null recurrence vector, \( \alpha^2 = 0 \), the Weyl tensor is algebraically special at least as type \( \text{II}_d \) of the Bel-Debever-Ortaggio classification.

**Proof.** Let’s introduce the locally null-frame \( \{\ell, n, m_{(2)}, \ldots, m_{(n-1)}\} \) where \( \ell = \alpha \) is null, \( n \) is another null real vector such that \( g_{ij} \ell^i n^j = 1 \), \( m_{(a)} \) are space-like real vectors such that \( g_{kl} m_{(a)}^l m_{(b)}^k = \delta_{ab} \), \( g_{kl} m_{(a)}^l \ell^k = 0 \) and \( g_{kl} m_{(a)}^k n^l = 0 \).

The tensor \( b_{ij} = \ell_i \ell_j \) is Weyl compatible and has the properties: \( \ell^i b_{ij} = 0 \), \( n^i b_{ij} = \ell_j \), \( m_{(a)}^i b_{ij} = 0 \). Therefore, by the preceding lemma:

\[
\ell^i m_{(a)}^i \ell^j C_{ijklm} = 0 \quad \text{and} \quad m_{(a)}^i m_{(b)}^j \ell^k C_{ijklm} = 0.
\]
The following null scalars

\[ \ell^i \ell^j a_{ij} = C_{0a0b} = 0 \]
\[ \ell^i n^j b_{ij} = C_{0a01} = 0 \]
\[ m^i a^j c_{ij} = C_{a0bc} = C_{ab} = 0 \]
\[ m^i n^j d_{ij} = C_{ab01} = C_{01ab} = 0 \]

characterise the Weyl tensor as type II\(_d\) in the classification by Ortaggio.

4. **The Tensor Gamma**

The algebraic identity (5) may be contracted with the Weyl tensor itself. Interesting properties are obtained for the symmetric tensor \( \Gamma_{ij} = C_{iklm} C_{jklm} \). The tensor is recurrent, \( \nabla_j \Gamma_{kl} = 2 \alpha_j \Gamma_{kl} \), with trace \( \Gamma_{kk} = C_2 \).

**Proposition 4.1.** If the recurrence vector \( \alpha_j \) is closed, then \( \Gamma_{ij} \) is Riemann compatible:

\[ \Gamma_{im} R_{jkl}^m + \Gamma_{jm} R_{kil}^m + \Gamma_{km} R_{ijl}^m = 0 \]  \( \text{(14)} \)

It also commutes with the Ricci tensor, and it is Weyl compatible.

**Proof.** A derivative of the recurrence property gives \( \nabla_i \nabla_j \Gamma_{kl} = 2(\nabla_i \alpha_j) \Gamma_{kl} + 4\alpha_i \alpha_k \Gamma_{kl} \). The antisymmetric part \([\nabla_i, \nabla_j] \Gamma_{kl} = 2(\nabla_i \alpha_j - \nabla_j \alpha_i) \Gamma_{kl} \) becomes, by the Ricci identity,

\[ R_{ijk} \Gamma_{lm} + R_{ijl} \Gamma_{km} = 2(\nabla_i \alpha_j - \nabla_j \alpha_i) \Gamma_{kl}. \]

If \( \alpha \) is closed: \( R_{ijk} \Gamma_{lm} + R_{ijl} \Gamma_{km} = 0 \). Summation on cyclic permutations of \( ijk \) and the first Bianchi identity give Riemann compatibility. Contraction with \( g^{il} \) gives \( \Gamma_{im} R_{jkl}^m - \Gamma_{il} R_{jkm}^m = 0 \). Riemann compatibility implies that \( \Gamma \) is also Weyl compatible [25]. □

Contraction of eq. (14) by \( C^{qklm} \) gives the identity:

\[ \alpha_i \Gamma_{ij} - \alpha_j \Gamma_{ij} = \alpha \Gamma_{ij} = -2 \alpha_i \Gamma_{ij} + \frac{2}{n-3} \alpha \left( C_{kij} \Gamma_{ij} - C_{kjil} \Gamma_{ij} \right) \]  \( \text{(15)} \)

Two interesting statements follow:

**Proposition 4.2.** The recurrence vector is an eigenvector of \( \Gamma \):

\[ \Gamma_{ij} \alpha^j = C^2 \frac{n-3}{2} \frac{n-2}{n-2} \alpha^i \]  \( \text{(16)} \)

**Proof.** Contract eq. (15) with \( \delta^i _{q} \). □

**Proposition 4.3.** If \( n > 4 \) and for \( \alpha^2 \neq 0 \), it is

\[ \Gamma_{ij} = \frac{C^2}{2} \left( \frac{n-4}{n-3} \frac{\alpha \alpha^q}{\alpha^2} + \frac{n-1}{n-3} E_i E_j \right) + \frac{C^2}{2} \frac{1}{(n-2)(n-3)} \delta_{ij}^q \]  \( \text{(17)} \)

**Proof.** Contraction of (15) with \( \alpha_j \), and the properties \( E_{ij} \alpha^j = 0 \) and (12) give:

\[ \Gamma_{ij} = 2 \frac{n-2}{n-3} E_{mi} E_{mj} - 2 \frac{n-3}{E_{kl} C_{qkl}^m + C^2 \frac{\alpha \alpha^q}{\alpha^2} \alpha^i \alpha^j } \]
The second term is evaluated:
\[
C_{jklm}E^{kl} = \frac{C^2}{4} \left[ -\frac{g_{jm}}{n-2} + \frac{\alpha_j\alpha_m}{\alpha^2} \right] + \frac{2}{n-3} E_{j}^{\ l} E_{lm}
\]
and the representation of \(\Gamma\) follows. \(\square\)

**Corollary 4.4.** In \(n > 4\), since \(\nabla_r C^2 = 2\alpha_r C^2\), the following covariant derivative is zero:
\[
\nabla_r \left[ \frac{\alpha_i \alpha^q}{\alpha^2} + \frac{n-1}{n-2} E_i^\ q E^q \right] = 0
\]

For a Lorentzian manifold the existence of a covariantly constant symmetric tensor \(h_{ij}\) not proportional to the metric implies the reducibility of the metric. In \(n = 4\) this result was proven by Hall [16] (see also [41]). It was then extended to \(n > 4\) by Senovilla ([39], Lemma 3.1). In summary, the lemma states that if \(\nabla_r h_{ij} = 0\) then either the manifold is decomposable or there exists a unique and null covariantly constant vector field.

In the present case we conclude:

**Corollary 4.5.** Consider a Lorentzian \((CR)\) manifold with \(n > 4\). If \(\alpha^2 \neq 0\) then either the metric is decomposable into orthogonal parts, \(g_{ij} = \alpha_i\alpha_j/\alpha^2 + g'_{ij}\), or there is a unique covariantly constant null vector field.

### 5. Weyl compatible tensors

We show that \((CR)\) pseudo-Riemannian manifolds are endowed with Weyl compatible tensors which involve the Ricci tensor, other than \(\alpha_i\alpha_j\). The construction is based on a general identity for the Weyl tensor on pseudo-Riemannian manifolds [22, 24]:

**Lemma 5.1.**

\[
\nabla_i \nabla_m C_{jkl}^\ m + \nabla_j \nabla_m C_{kil}^\ m + \nabla_k \nabla_m C_{ijl}^\ m = -\frac{n-3}{n-2} \left( R_{im} C_{jkl}^\ m + R_{jm} C_{kil}^\ m + R_{km} C_{ijl}^\ m \right).
\]

It is a consequence of the following identities by Lovelock for the Riemann and Ricci tensors [21]:

\[
\nabla_i \nabla_j R_{jkl}^\ m + \nabla_j \nabla_k R_{jkl}^\ m + \nabla_k \nabla_i R_{jkl}^\ m = -R_{im} R_{jkl}^\ m - R_{jm} R_{kil}^\ m - R_{km} R_{ijl}^\ m,
\]
\[
R_{im} R_{jkl}^\ m + R_{jm} R_{kil}^\ m + R_{km} R_{ijl}^\ m = R_{im} C_{jkl}^\ m + R_{jm} C_{kil}^\ m + R_{km} C_{ijl}^\ m.
\]

**Proposition 5.2.** On a \(n \geq 4\) dimensional \((CR)\) pseudo-Riemannian manifold with recurrence vector \(\alpha_i\), the tensor
\[
\nabla_i \alpha_j + \nabla_j \alpha_i + 2\alpha_i\alpha_j + R_{ij}
\]
is Weyl compatible, and:
\[
\left[ \nabla_i \alpha_m - \nabla_m \alpha_i + \frac{n-4}{n-2} R_{im} \right] C_{jkl}^\ m + \text{cyclic} = 0
\]

**Proof.** Because of the recurrence property [11], it is: \(\nabla_i \nabla_m C_{jkl}^\ m = \nabla_i (\alpha_m C_{jkl}^\ m) = (\nabla_i \alpha_m + \alpha_i \alpha_m) C_{jkl}^\ m\). Eq. (19) becomes:
\[
\left[ \nabla_i \alpha_m + \alpha_i \alpha_m + \frac{n-3}{n-2} R_{im} \right] C_{jkl}^\ m + \text{cyclic} = 0.
\]
The covariant divergence of (5) and the property $\nabla^m(\alpha^p C_{jkmp}) = 0$ give:

$$\left[\nabla_m \alpha_i + \frac{n-4}{n-3} \alpha_m \alpha_i - \frac{1}{n-3} \nabla_i \alpha_m\right] C_{jkl}^m + \text{cyclic} = 0.$$  

Linear combinations of the two cyclic sums on $ijk$ give the final statements. 

**Corollary 5.3.** In a CR$_n$ pseudo-Riemannian manifold with $n \geq 5$ and closed recurrence vector $\alpha_i$ the symmetric tensors $\alpha_i \alpha_j$, $\nabla_i \alpha_j$ and $R_{ij}$ are Weyl compatible. If the Ricci tensor is Weyl compatible, it is also Riemann compatible, by (20):

$$R_{lm} R_{jkl}^m + R_{jm} R_{kli}^m + R_{km} R_{ijl}^m = 0.$$  

### 6. Examples of (CR)$_n$ manifolds

The first example of (CR)$_n$ was given by Roter in [33] and further discussed in [34, 35]: $g_{ij}dx^i dx^j = Q(dx^1)^2 + k_{\mu\nu} dx^\mu dx^\nu + 2dx^1 dx^n, 1 < \mu, \nu < n$, where $Q = [A(x^1)]_{p\mu} + k_{\mu\nu} x^\mu x^\nu$ with tensors $p$ and $k$ subject to restrictions. The manifold is both conformally recurrent and Ricci-recurrent, possibly with different recurrence vectors. Another example was proposed by Derdzinski [9]: he studied the metric $g_{11} = -2e, g_{ij} = \exp F_i(x^1, x^2)$ if $i + j = n + 1$, and $g_{ij} = 0$ otherwise, with certain functions $F_i$, and periodicity conditions.

Here we give another example. Consider the metric:

$$g_{ij}dx^i dx^j = p(x^1)q(x^3)(dx^1)^2 + 2dx^1 dx^2 + (dx^3)^2 + \ldots + (dx^n)^2,$$

The non-zero Christoffel symbols (up to symmetries) are: $\Gamma^i_{21} = \frac{1}{2} p'(x^1) q(x^3)$, $\Gamma^2_{13} = -\Gamma^3_{11} = \frac{1}{2} pq''$. It follows that the non-zero components of the Riemann tensor are: $R_{1313} = \frac{1}{2} pq''$, and the only non-zero component of the Ricci tensor is $R_{11} = \frac{1}{2} pq''$. The curvature scalar is zero.

In this frame of coordinates it is easily proven that the manifold is Ricci-recurrent, $\nabla_i R_{jk} = \alpha_i R_{jk}$, with recurrence vector

$$\alpha_i = \left(\frac{p'(x^1)}{p(x^1)}, 0, \frac{q'''(x^3)}{q''(x^3)}, 0, \ldots, 0\right).$$

By similar calculations it is shown that the manifold is recurrent with the same recurrence vector: $\nabla_i R_{jkm} = \alpha_i R_{jkm}$. Therefore the Weyl tensor is recurrent, i.e. the manifold is a (CR)$_n$. Since $\alpha$ is a gradient, by (3) the metric is Weyl semi-symmetric (for the same reason it is Ricci semi-symmetric and semi-symmetric). The form of the metric gives $\alpha^2 = (q''/q'')^2 \geq 0$.

The simple form of the Riemann and Ricci tensors in the defining frame (22), implies the tensorial identity $\alpha^k \alpha^m R_{jklm} = \alpha^2 R_{jl}$. Since the curvature scalar is zero, it gives $\alpha^k \alpha^m R_{km} = 0$.

If $\alpha^2 = 0$ then $\alpha = (p'/p, 0, 0, \ldots)$ and $R_{ij} \propto \alpha_i \alpha_j$ in the defining frame. It follows that the Ricci tensor is rank-one: $R_{ij} = \frac{1}{2} \lambda \alpha_i \alpha_j, \lambda = (pq'')(p'/p)^{-2}$ has the following consequences: $\alpha^{\mu} R_{jklm} = \nabla^{\mu} R_{jklm} = \nabla_k R_{jlm} - \nabla_l R_{kjm} = \alpha_k R_{jlm} - \alpha_l R_{kjm} = 0$ and, after simple calculations, $\alpha^{\mu} C_{jklm} = 0$. For a Lorentzian metric ($|pq| > 2$), the last result characterizes the manifold as type $II_{\text{ab}}$ in the Bel-Debever-Ortaggio classification [30].
7. Some results in \( n = 4 \)

A result of corollary 5.3 can be obtained also in \( n = 4 \), in case of closed recurrence vector, \( \nabla_i \alpha_j = \nabla_j \alpha_i \).

**Proposition 7.1.** On a Lorentzian \((CR)_4\) with closed recurrence vector, the Ricci tensor is both Riemann and Weyl compatible.

*Proof.* Eq. (23) shows that the manifold is Weyl semi-symmetric whenever the recurrence vector is closed. The Lemma 2 in paper [22] guarantees that \( n = 4 \) Lorentzian Weyl semi-symmetric manifolds are also Ricci semi-symmetric: \( [\nabla_i, \nabla_j] R_{kl} = 0 \). In Sect. 3 of ref. [22], it is proven that Ricci semi-symmetry implies that the Ricci tensor is Riemann compatible (and then Weyl compatible).

In the Bel-Debever classification of Weyl tensors on \( n = 4 \) Lorentzian manifolds [3, 6, 17, 38], a manifold is type \( N \) if \( \delta_{ijkl} k^m = 0 \) for some null vector field \( k \).

In the same way, multiplication by \( A_j B^s \) gives \( A^2 C^{ki} tr = 0 \) and \( B^2 C^{ki} tr = 0 \). Then \( A_j \) and \( B_j \) are orthogonal null vectors; lemma 7.5 implies that they are proportional.

**Proposition 7.5.** In a 4-dimensional Lorentzian manifold of type \( N \) with respect to a null vector \( k \), if \( b_{ij} \) is a Weyl compatible tensor, then \( k \) is an eigenvector of \( b_{ij} \).

*Proof.* By transvecting \( b_{im} C_{jkl}^m + b_{jm} C_{kil}^m + b_{km} C_{ijl}^m = 0 \) with \( k^i \) we get \( B^m C_{jkl}^m = 0 \), where \( B_m = k^i b_{im} \). The condition for type-\( N \) is: \( k_m C_{jkl}^m = 0 \). By Lemma 7.3 the vector \( B_m \) is null and proportional to \( k_m \), i.e. \( k^i b_{im} = \lambda k_m \).

**Proposition 7.6.** Let \( M \) be a \((CR)_4\) Lorentzian manifold of type \( N \) with respect to the null vector \( k \). Then \( k \) is an eigenvector of \( \nabla_i \alpha_m + \nabla_m \alpha_i + 2 \alpha_i \alpha_m + R_{im} \). In addition, if \( \alpha_i \) is closed, then \( k \) is an eigenvector of the Ricci tensor.

*Proof.* By prop. 5.2 the tensor \( \nabla_i \alpha_m + \nabla_m \alpha_i + 2 \alpha_i \alpha_m + R_{im} \) is Weyl-compatible, then \( k \) is an eigenvector of it. If \( \alpha \) is closed, the Ricci tensor is Weyl compatible by proposition 7.1 and \( k \) is an eigenvector for it.
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