On the Finite Orthogonality of $q$-Pseudo-Jacobi Polynomials

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Abstract: Using the Sturm–Liouville theory in $q$-difference spaces, we prove the finite orthogonality of $q$-Pseudo Jacobi polynomials. Their norm square values are then explicitly computed by means of the Favard theorem.

Keywords: $q$-Pseudo Jacobi Polynomials; Sturm–Liouville problems; $q$-difference equations; finite sequences of $q$-orthogonal polynomials

1. Introduction

For $\alpha, \beta > -1$, the Jacobi polynomials are defined as [1]

$$p_n^{(\alpha, \beta)}(x) = \frac{(-1)^n}{2^n n!} (1-x)^{-\alpha} (1+x)^{-\beta} \frac{d^n}{dx^n} \left\{ (1-x)^{\alpha} (1+x)^{\beta} (1-x^2)^n \right\},$$

(1)

Another representation of Jacobi polynomials is as [2,3]

$$p_n^{(\alpha, \beta)}(x) = \frac{(\alpha + 1)_n}{n!} {}_2F_1 \left( \begin{array}{c} -n, n + \alpha + \beta + 1 \\ \alpha + 1 \end{array} \right| \frac{1-x}{2} \right),$$

(2)

where

$$(a)_k := \prod_{j=0}^{k-1} (a + j), \quad (a)_0 := 1,$$

(3)

and

$$_2F_1 \left( \begin{array}{c} a_1, \ldots, a_r \\ b_1, \ldots, b_s \end{array} \right| z \right) := \sum_{k=0}^{\infty} \frac{(a_1)_k \cdots (a_r)_k}{(b_1)_k \cdots (b_s)_k} \frac{z^k}{k!},$$

(4)

in which $a_1, a_2, \ldots, a_r, b_1, b_2, \ldots, b_s, z \in \mathbb{C}$ and $b_1, \ldots, b_s \neq 0, -1, -2, \ldots, -(k - 1)$.

The weight function corresponding to Jacobi polynomials is known in statistics as the shifted beta distribution

$$w(x; \alpha, \beta) = (1-x)^{\alpha} (1+x)^{\beta}, \quad x \in [-1, 1].$$

An interesting subclass of Jacobi polynomials is when $\alpha = -u + iv$ and $\beta = -u - iv$ for $i^2 = -1$ in (2), so that the real polynomials

$$p_n^{(u,v)}(x) = (-i)^n p_n^{(-u+iv, -u-iv)}(ix),$$

(5)
satisfy the equation

\[(1 + x^2) f''_n(x) + 2((1 - u)x + v)f'_n(x) - n(n - 2u + 1) f_n(x) = 0.\]  

(6)

It is proved in [4] that \( \{ f_n^{(u,v)}(x) \} \) are finitely orthogonal with respect to the weight function

\[w(x; u, v) = (1 + x^2)^{-u} \exp(2v \arctan x),\]

on \((-\infty, \infty)\) and can be explicitly represented in form of hypergeometric functions as

\[f_n^{(u,v)}(x) = \frac{(-2i)^n (1 - u + iv)_n}{(n - 2u + 1)_n} \, \binom{-n, n - 2u + 1}{1 - u + iv} \left( 1 - \frac{ix}{2} \right).\]

The so-called \(q\)-polynomials have found many applications in Eulerian series and continued fractions [3], \(q\)-algebras and quantum groups [5–7], and \(g\)-oscillators [8–10]. See also [11,12] in this regard.

It has been acknowledged that the theory of \(q\)-special functions is essentially based on the relation

\[\lim_{q \to 1} \frac{1 - q^a}{1 - q} = a.\]

Hence, a basic number in \(q\)-calculus is defined as

\([a]_q = \frac{1 - q^a}{1 - q}.\]

There is a \(q\)-analogue of the Pochhammer symbol (3) (called \(q\)-shifted factorial) as

\[(a; q)_k := \prod_{j=0}^{k-1} (1 - aq^j), \quad (a; q)_0 := 1.\]

Moreover we have

\[(a; q)_\infty = \prod_{k=0}^{\infty} (1 - aq^k) \quad \text{for} \quad 0 < |q| < 1,\]

and

\[(a_1, a_2, \ldots, a_m; q)_\infty = (a_1; q)_\infty (a_2; q)_\infty \cdots (a_m; q)_\infty.\]  

(7)

There exist several \(q\)-analogs of classical hypergeometric orthogonal polynomials that are known as basic hypergeometric orthogonal polynomials [3].

In the present work, using the Sturm–Liouville theory in \(q\)-difference spaces, we prove that a special case of big \(q\)-Jacobi polynomials is finitely orthogonal on \((-\infty, \infty)\). The big \(q\)-Jacobi polynomials are defined as

\[P_n(x; a, b, c; q) = \, _3\phi_2 \left( \begin{array}{c} q^{-n}, a b q^{a+1}, x \\ a, q \end{array} \middle| q, q \right),\]  

(8)

where

\[r\phi_s \left( \begin{array}{c} a_1, \ldots, a_r \\ b_1, \ldots, b_s \end{array} \middle| q; z \right) := \sum_{k=0}^{\infty} \frac{(a_1; q)_k \cdots (a_r; q)_k \cdot z^k}{(b_1; q)_k \cdots (b_s; q)_k (q; q)_k} (-1)^{k} q^\frac{k(k-1)}{2} \right)^{1+s-r},\]  

(9)

is known as the basic hypergeometric series.

Again, \(a_1, a_2, \ldots, a_r, b_1, b_2, \ldots, b_s, z \in \mathbb{C}\) and \(b_1, b_2, \ldots, b_s \neq 1, q^{-1}, q^{-2}, \ldots, q^{-1-k}\).

Notice that [3] (p. 15)

\[\lim_{q \to 1} r\phi_s \left( \begin{array}{c} a_1, \ldots, a_r \\ b_1, \ldots, b_s \end{array} \middle| q; (q - 1)^{1+s-r} z \right) = r\phi_s \left( \begin{array}{c} a_1, \ldots, a_r \\ b_1, \ldots, b_s \end{array} \middle| z \right).\]  

(10)

On the other side, if we set \(c = 0, a = q^\beta\) and \(b = q^\beta \) in (8) and then let \(q \to 1\), we find the Jacobi polynomials (2) as

\[\lim_{q \to 1} P_n(x; q^\alpha, q^\beta, 0; q) = \frac{P_n^{(\alpha, \beta)}(2x - 1)}{P_n^{(\alpha, \beta)}(1)}.\]
Moreover, by referring to (8), one can define another family of big $q$-Jacobi polynomials [13] with four free parameters as

$$P_n^{(a,b,c,d)}(x; a, b, c, d; q) = P_n(qac^{-1}x;a, b, -ac^{-1}d; q) = \frac{\phi_2}{\phi_2(aq, -qac^{-1}d; q, q)},$$

which yields

$$P_n(x; a, b, c, d) = P_n(-q^{-1}c^{-1}x; a, b, -ac^{-1}, 1, q).$$

Because a particular case of Jacobi polynomials (5) are called the pseudo Jacobi polynomials, it is reasonable to similarly consider a special case of big $q$-Jacobi polynomials preserving the limit relation as $q \to 1$. This means that the $q$-pseudo Jacobi polynomials will be derived by substituting

$$a = iq\frac{1}{2}(a-b), \quad b = -iq\frac{1}{2}(a+b), \quad c = iq\frac{1}{2}(b-d) \quad \text{and} \quad d = -iq\frac{1}{2}(a-d)$$

in a special case of the polynomials (8) as

$$P_n(cx; c, d/a, c/a, q; q) \quad \text{where} \quad a, b, c, d \in \mathbb{C} \quad \text{and} \quad (ab)/(qcd) > 0,$$

so that

$$\lim_{q \to 1} P_n(iq\frac{1}{2}(-a+b)x; -q^{-a}, -q^{-c}, q^{-a+b}; q) = \frac{f_n^a(x; i)}{f_n^a(i; i)}.$$  

Therefore, the $q$-pseudo Jacobi polynomials are defined as

$$\begin{align*}
\int_{0}^{\infty} (u, v; q) = P_n(iq\frac{1}{2}(-a+b)x; -q^{-a}, -q^{-c}, q^{-a+b}; q) = \frac{\phi_2}{\phi_2(aq, -qac^{-1}d; q, q)},
\end{align*}$$

(11)

The main aim of this paper is to apply a $q$-Sturm–Liouville theorem in order to obtain a finite orthogonality for the real polynomials (11) on $(-\infty, \infty)$, which is a new contribution in the literature.

A regular Sturm–Liouville problem of continuous type is a boundary value problem of the form

$$\frac{d}{dx} \left( K(x) \frac{dy(x)}{dx} \right) + \lambda w(x) y(x) = 0, \quad (K(x) > 0, \ w(x) > 0),$$

(12)

which is defined on an open interval, say $(\gamma_1, \gamma_2)$ with the boundary conditions

$$\begin{align*}
\alpha_1 y(\gamma_1) + \beta_1 y'(\gamma_1) = 0 \quad \text{and} \quad \alpha_2 y(\gamma_2) + \beta_2 y'(\gamma_2) = 0,
\end{align*}$$

(13)

where $\alpha_1$, $\alpha_2$ and $\beta_1$, $\beta_2$ are constant numbers and $K(x)$, and $w(x)$ in (12) are to be assumed continuous functions for $x \in [\gamma_1, \gamma_2]$. The function $w(x)$ is called the weight or density function.

Let $y_n$ and $y_m$ be two eigenfunctions of Equation (12). According to the Sturm–Liouville theory [14], they have an orthogonality property with respect to the weight function $w(x)$ under the given condition (13), so that we have

$$\int_{\gamma_1}^{\gamma_2} w(x) y_n(x) y_m(x) dx = \left( \int_{\gamma_1}^{\gamma_2} w(x) y_n^2(x) dx \right) \delta_{m,n},$$

(14)

in which

$$\delta_{m,n} = \begin{cases} 0 & (n \neq m), \\ 1 & (n = m). \end{cases}$$

There are generally two types of orthogonality for relation (14), i.e. infinitely orthogonality and finitely orthogonality. In the finite case, one has to impose some constraints on $n$, while in the infinite case, $n$ is free up to infinity [4].
By referring to the differential Equation (6), it is proved in [4] that
\[
\int_{-\infty}^{\infty} (1 + x^2)^{-u} \exp \left( 2v \arctan x \right) f_n^{(u,v)}(x) f_m^{(u,v)}(x) dx = \frac{2\pi n! 2^{2n+1-2u} \Gamma(2u-n)}{(2u-2n-1)\Gamma(u-n+iv)\Gamma(u-n-iv)} \delta_{m,n},
\]
where \( \delta_{m,n} \) is the well-known Kronecker delta. The integral as an inverse of the eigenfunctions of the first\(q\)-difference operator 
\[
\phi(x) = x^n + \cdots
\]
is the well-known gamma function. Similarly, \(q\)-orthogonal functions can be solutions of a \(q\)-Sturm-Liouville problem in the form [15]
\[
D_q (K(x;q)D_q y_n(x;q)) + \lambda_n w(x;q) y_n(x;q) = 0, \quad (K(x;q) > 0, w(x;q) > 0),
\]
where
\[
D_q f(x) = \frac{f(qx) - f(x)}{(q-1)x} \quad (x \neq 0, q \neq 1),
\]
and (15) satisfies a set of boundary conditions like (13). This means that if \(y_n(x;q)\) and \(y_m(x;q)\) are two eigenfunctions of the \(q\)-difference Equation (15), they are orthogonal with respect to a weight function \(w(x;q)\) on a discrete set [16].

Let \(\phi(x)\) and \(\psi(x)\) be two polynomials of degree at most 2 and 1, respectively, as
\[
\begin{align*}
\phi(x) &= ax^2 + bx + c \\
\psi(x) &= dx + e
\end{align*}
\]
for \(a, b, c, d, e \in \mathbb{C}, \ d \neq 0\).

If \(\{y_n(x;q)\}_n\) is a sequence of polynomials that satisfies the \(q\)-difference equation [3]
\[
\phi(x) D_q^2 y_n(x;q) + \psi(x) D_q y_n(x;q) + \lambda_n w(x;q) y_n(x;q) = 0,
\]
where
\[
D_q^2(f(x)) = \frac{f(q^2x) - (1 + q)f(qx) + qf(x)}{q(q-1)^2x^2},
\]
\(\lambda_n, q \in \mathbb{C}\) and \(q \in \mathbb{R} \setminus \{-1, 0, 1\}\), then the following orthogonality relation holds
\[
\int_{p_1}^{p_2} w(x;q) y_n(x;q) y_m(x;q) d_q x = \left( \int_{p_1}^{p_2} w(x;q) y_n^2(x;q) d_q x \right) \delta_{n,m},
\]
in which
\[
\int_{p_1}^{p_2} f(t) d_q t = (1-q) \sum_{j=0}^{\infty} q^j \left( \rho_2 f(q^j \rho_2) - \rho_1 f(q^j \rho_1) \right),
\]
and \(w(x;q)\) is a solution of the Pearson \(q\)-difference equation
\[
D_q \left( w(x;q) \phi(q^{-1} x) \right) = w(qx;q) \psi(x).
\]
Note that \(w(x;q)\) is assumed to be positive and \(w(q^{-1} x;q) \phi(q^{-2} x)\) for \(k \in \mathbb{N}\) must vanish at \(x = \rho_1, \rho_2\).

If \(P_n(x) = x^n + \cdots\) is a monic solution of Equation (16), the eigenvalue \(\lambda_n, q\) is explicitly derived as
\[
\lambda_n = -\frac{|n| q}{q^2} (a[n-1]q + d).
\]

The \(q\)-integral as an inverse of the \(q\)-difference operator [3,17,18] is defined as
\[
\int_{0}^{x} f(t) d_q t = (1-q) x \sum_{j=0}^{\infty} q^j f(q^j x) \quad (x \in \mathbb{R})
\]
provided that the series converges absolutely. Furthermore, we have
\[
\int_0^\infty f(t) d_q t = (1 - q) \sum_{n=-\infty}^{\infty} q^n f(q^n),
\]
and
\[
\int_{-\infty}^{\infty} f(t) d_q t = (1 - q) \sum_{n=-\infty}^{\infty} q^n (f(q^n) + f(-q^n)).
\]

2. Finite Orthogonality of $q$-Pseudo Jacobi Polynomials

Let us consider the following $q$-difference equation
\[
(q^2 - u^2 x^2 + 2 \sin \left( \frac{u}{2} \ln q \right) x + 1) D_q^2 y_n(x; q)
+ \left( \frac{q^u - q^{2-u}}{1 - q} x - 2 \sin \left( \frac{u}{2} \ln q \right) (q^{1-u} - q^u) \right) D_q y_n(x; q) + \lambda_{n,q}^* y_n(x; q) = 0,
\]
with
\[
\lambda_{n,q}^* = -\frac{|n|}{q^n} \left( q^{2-u}|n| + q^u - q^{2-u} \right),
\]
for $n = 0, 1, 2, \ldots$ and $q \in \mathbb{R} \setminus \{-1, 0, 1\}$.

It is clear that
\[
\lim_{q \to 1} \lambda_{n,q}^* = -n(n - 2u + 1),
\]
gives the same eigenvalues as in the continuous case (6).

**Theorem 1.** Let \( \{f_n^{(u,v)}(x; q)\}_n \) defined in (11) be a sequence of polynomials that satisfies the $q$-difference Equation (18). Subsequently, we have
\[
\int_{-\infty}^{\infty} w^{(u,v)}(x; q) f_n^{(u,v)}(x; q) f_m^{(u,v)}(x; q) d_q x = \left( \int_{-\infty}^{\infty} w^{(u,v)}(x; q) \left( f_n^{(u,v)}(x; q) \right)^2 d_q x \right) \delta_{n,m},
\]
where $N < u - \frac{1}{2}$ for $N = \max\{m, n\}$ and the positive function $w^{(u,v)}(x; q)$ is a solution of the Pearson-type $q$-difference equation
\[
D_q \left( w^{(u,v)}(x; q) \right) \left( q^2 - u^2 x^2 + 2 \sin \left( \frac{u}{2} \ln q \right) x + 1 \right)
= \left( \frac{q^u - q^{2-u}}{1 - q} x - 2 \sin \left( \frac{u}{2} \ln q \right) (q^{1-u} - q^u) \right) w^{(u,v)}(x; q),
\]
which is equivalent to
\[
\frac{w^{(u,v)}(x; q)}{w^{(u,v)}(qx; q)} = \frac{q^u x^2 - 2q^u \sin \left( \frac{u}{2} \ln q \right) x + 1}{q^{-u} x^2 + 2q^{-u} \sin \left( \frac{u}{2} \ln q \right) x + 1}.
\]

**Proof.** First, according to [3] and referring to (7) it is not difficult to verify that
\[
w^{(u,v)}(x; q) = \frac{\left( iq^{(u+iv)/2} x, -iq^{(u+iv)/2} x; q \right)_\infty}{\left( iq^{(-u+iv)/2} x, -iq^{(-u+iv)/2} x; q \right)_\infty}
= x^{-2u} \frac{\left( -iq^{(-u+iv)/2} x^{-1}, iq^{(-u+iv)/2} x^{-1}; q^{-1} \right)_\infty}{\left( -iq^{(u-iv)/2} x^{-1}, iq^{(u-iv)/2} x^{-1}; q^{-1} \right)_\infty},
\]
is a solution of Equation (19).

Now, if Equation (18) is written in the self-adjoint form
\[
D_q \left( w^{(u,v)}(x; q) \right) \left( q^2 - u^2 x^2 + 2 \sin \left( \frac{u}{2} \ln q \right) x + 1 \right) D_q f_n^{(u,v)}(x; q) + \lambda_{n,q}^* w^{(u,v)}(qx; q) f_n^{(u,v)}(x; q) = 0,
\]

and for \( m \) as
\[
D_q \left( w^{(n,p)}(x;\gamma) \left( q^2 - u x^2 + 2 \sin \left( \frac{\gamma}{2} \ln q \right) x + 1 \right) D_q f_m^{(u,p)}(x;\gamma) \right) + \lambda_{m,q} w^{(u,p)}(q x;\gamma) f_m^{(u,p)}(q x;\gamma) = 0,
\]
by multiplying (21) by \( f_m^{(u,p)}(q x;\gamma) \) and (22) by \( f_n^{(u,p)}(q x;\gamma) \) and subtracting each other we get
\[
(\lambda_{m,q} - \lambda_{n,q}) w^{(u,p)}(x;\gamma) f_m^{(u,p)}(x;\gamma) = q^2 D_q \left( w^{(u,p)}(q^{-1} x;\gamma) \left( q^2 - u x^2 + 2 \sin \left( \frac{\gamma}{2} \ln q \right) x + 1 \right) D_q f_m^{(u,p)}(q^{-1} x;\gamma) \right) - q^2 D_q \left( w^{(u,p)}(q^{-1} x;\gamma) \left( q^2 - u x^2 + 2 \sin \left( \frac{\gamma}{2} \ln q \right) x + 1 \right) D_q f_m^{(u,p)}(q^{-1} x;\gamma) \right) f_n^{(u,p)}(q x;\gamma).
\]
Hence, \( q \)-integration by parts on both sides of (23) over \((-\infty, \infty)\) yields
\[
(\lambda_{m,q} - \lambda_{n,q}) \int_{-\infty}^\infty w^{(u,p)}(x;\gamma) f_m^{(u,p)}(x;\gamma) f_n^{(u,p)}(x;\gamma) d_q x = q^2 \int_{-\infty}^\infty \left\{ D_q \left( w^{(u,p)}(q^{-1} x;\gamma) \left( q^2 - u x^2 + 2 \sin \left( \frac{\gamma}{2} \ln q \right) x + 1 \right) D_q f_m^{(u,p)}(q^{-1} x;\gamma) \right) f_m^{(u,p)}(x;\gamma) \right\} d_q x - q^2 \int_{-\infty}^\infty \left\{ D_q \left( w^{(u,p)}(q^{-1} x;\gamma) \left( q^2 - u x^2 + 2 \sin \left( \frac{\gamma}{2} \ln q \right) x + 1 \right) D_q f_m^{(u,p)}(q^{-1} x;\gamma) \right) f_n^{(u,p)}(x;\gamma) \right\} d_q x
\]
\[
	imes \left( D_q f_n^{(u,p)}(q^{-1} x;\gamma) f_m^{(u,p)}(x;\gamma) - D_q f_m^{(u,p)}(q^{-1} x;\gamma) f_n^{(u,p)}(x;\gamma) \right) \right|_{-\infty}^\infty
\]
Because
\[
\max \deg \{ D_q f_n^{(u,p)}(q^{-1} x;\gamma) f_m^{(u,p)}(x;\gamma) - D_q f_m^{(u,p)}(q^{-1} x;\gamma) f_n^{(u,p)}(x;\gamma) \} = m + n - 1,
\]
the left-hand side of (24) is zero if
\[
\lim_{x \to \infty} w^{(u,p)}(q^{-1} x;\gamma) \left( q^2 - u x^2 + 2 \sin \left( \frac{\gamma}{2} \ln q \right) x + 1 \right) x^{m+n-1} = 0.
\]
By taking max \( \{ m, n \} = N \), relation (25) would be equivalent to
\[
\lim_{x \to \infty} \frac{(-i q^{-2} x^{-1}, q^{-2} x^{-1}; q^{-1})_\infty}{\left(-i q^{-2} x^{-1}, q^{-2} x^{-1}; q^{-1}\right)_\infty} x^{2N-2u+1} = 0.
\]
Note that (26) is valid if and only if
\[
2N + 1 - 2u < 0 \quad \text{or} \quad N < u - \frac{1}{2}.
\]
Therefore, the right-hand side of (24) tends to zero and
\[
\int_{-\infty}^\infty w^{(u,p)}(x;\gamma) f_m^{(u,p)}(x;\gamma) f_n^{(u,p)}(x;\gamma) d_q x = 0,
\]
if and only if \( m \neq n \) and \( N < u - \frac{1}{2} \) for \( N = \max \{ m, n \} \).

**Corollary 1.** The finite polynomial set \( \{ f_n^{(u,p)}(x;\gamma) \}_{n=0}^{\frac{N-u-\frac{1}{2}}{2}} \) is orthogonal with respect to the weight function (20) on \((-\infty, \infty)\).

### 2.1. Computing the Norm Square Value

According to (17), because \( f_n^{(u,p)}(x;\gamma) \) is a particular case of the big \( q \)-Jacobi polynomials, it satisfies the recurrence relation [3]
\[ f_{n+1}(u; v; q) = (x - c_n(u, v; q)) f_n(u; v; q) - d_n(u, v; q) f_{n+1}(u; v; q), \]

with the initial terms
\[ f_0(u; v; q) = 1, \quad f_1(u; v; q) = x + \frac{2 \sin \left( \frac{u}{2} \ln q \right)(1 - q)(q^{2u/2} + q^{1+u/2})}{(q^u - q^{2u})}, \]

where
\[
c_n(u, v; q) = \frac{2 \sin \left( \frac{u}{2} \ln q \right)q^u}{(q^u - q^{2u-2})(q^u - q^{2u+2})} \times \{(q^u - q^{u-1})(q^{u-2u/2}|_{|q}|(1 + q) + (q^{2u-2} + q^{1+u/2})) - q^{n+1-u}(1 - q^{n+1})(q^{1-u/2} + q^{u/2})\},
\]

and
\[
d_n(u, v; q) = \frac{(q^{n+1} - q^{u-u})(q^{n} - q^{u})}{(1 - q)^2(q^n - q^{2n-u-1})(q^n - q^{2n-u+1})} \times \left\{ 4 \sin^2 \left( \frac{u}{2} \ln q \right)q^{u-1-u/2}(1 - q) \left( 1 + q - q^2 + q^n - q^{1+n} - q^{-1} \right) - q^{n-u-1}(1 - q - q^2) - q^{-1}(1 - q) \right\} - (q^{4n-2n} + 2q^{2n} + q^{2n}).
\]

Now, by applying the Favard theorem [19] for the monic type of polynomials (11), we get
\[
\int_{-\infty}^{\infty} \omega(u, v)(x; q) f_n(u; v; q) f_n(x; q) d_q x = \left( \mu_0 \prod_{k=1}^{n} d_k(u, v; q) \right) \delta_{m,n},
\]

where
\[
\mu_0 = \int_{-\infty}^{\infty} \frac{(iq^{(u+i)v}/2 x_q - iq^{(u+i)v}/2 x_q)_{\infty}}{\theta(z - q x_q)_{\infty} \theta(z x_q)_{\infty} \theta(z w_q)_{\infty}}, \quad (27)
\]

in which
\[
\theta(x; q) = (x, q/x; q)_{\infty}.
\]

Therefore, it is enough to replace \( z = -1, z = 1 \) in (27) to finally obtain
\[
\mu_0 = \frac{(q, q^{u-iv} - q^{u}, -q^{u}, q^{u+iv}; q)_{\infty}}{(q^{2u-1}; q)_{\infty}} \times \frac{(-1, -q^{-iv} - q^{u+1}; q)_{\infty}}{(-iq^{1-ix}, iq^{1-ix}, -iq^{1-ix}, iq^{1-ix}, -iq^{1-ix}, iq^{1-ix}, iq^{1-ix}, iq^{1-ix}; q)_{\infty}}.
\]

For example, the set \( \{ f_n^{(21,1)}(x; q) \}_{n=0}^{20} \) is a finite sequence of q-orthogonal polynomials that satisfies the orthogonality relation
\[
\int_{-\infty}^{\infty} \frac{(iq^{21-i}/2 x_q - iq^{21+i}/2 x_q)_{\infty}}{(-iq^{(21-i)/2} x_q - iq^{(21+i)/2} x_q)_{\infty}} f_n^{(21,1)}(x; q) f_n^{(21,1)}(x; q) d_q x = \left( \frac{(q, q^{21-iv} - q^{21}, -q^{21}, q^{21+iv}; q)_{\infty}}{(q^{21}; q)_{\infty}} \right) \delta_{m,n}
\]

\( \iff m, n < 20, \)
where
\[
d_k(21, 1; q) = \frac{(q^{k+1} - q^{2k+1})(q^{21} - q^{2-21})}{(1-q)^2(q^{k-1} - q^{2k-2})(q^{k-1} - q^{2k-20})} \\
\times \left\{ 4 \sin^2 \left( \frac{1}{2} \ln q \right) q^{k-23/2} \left( 1 - q \right) \left( 1 + q - q^2 + q^{21} - q^{22} - q^{k-1} \right) \left( 1 - q^{k-20} (1 + q - q^2) - q^{k+1} (1-q) \right) \right. \\
- \left. (q^{4k-42} + 2q^{2k} + q^{42}) \right\}. 
\] (28)

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