ON REGULARITY PROPERTIES AND APPROXIMATIONS
OF VALUE FUNCTIONS FOR STOCHASTIC
DIFFERENTIAL GAMES IN DOMAINS

N.V. KRYLOV

Abstract. We prove that for any constant $K \geq 1$, value functions for
time homogeneous stochastic differential games in the whole space
can be approximated up to a constant over $K$ by value functions whose
second order derivatives are bounded by a constant times $K$.
On the way of proving this result we prove that the value functions for
stochastic differential games in domains and in the whole space admit
estimates of their Lipschitz constants in a variety of settings.

1. Introduction

In this paper we prove that for any constant $K \geq 1$, value functions for
time homogeneous stochastic differential games in the whole space can be
approximated up to a constant over $K$ by the value functions whose second
order derivatives are bounded by a constant times $K$ (see Theorem 2.4 and
Remark 2.4). In terms of the corresponding Isaacs equations the approximation is done in such a way that the equations are modified only for large
values of the derivatives of the value functions. Such approximation of sto-
chastic games can be useful while evaluating the value functions numerically
because one can expect that approximations might be more accurate if the
approximating function is more regular.

Two main tools are used. One is the stochastic dynamic principle with
randomized stopping times and another is based on estimates of the Lips-
chitz constants of the value functions.

The dynamic programming principle we use is proved in [10] and origi-
nated in the work by Fleming and Souganidis [3] (see also Kovats [5] and
Święch [11]).

Here we concentrate on proving the Lipschitz continuity of the value func-
tions for time homogeneous stochastic differential games in domains and in
the whole space and on proving the above mentioned approximation result,
which is a particular case of a conjecture from [8].

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There is an enormous literature treating smoothness properties for controlled diffusion processes or, from analytical point of view, for fully nonlinear equations under convexity assumptions. We are going to focus only on stochastic differential games for which there is not much known concerning the regularity of the value function in more or less general case.

Ishii and Lions in [4] prove the Lipschitz continuity for viscosity solutions of fully nonlinear uniformly nondegenerate equations. Earlier Trudinger in [12] proved that the first derivatives are, actually, Hölder continuous. The same result under somewhat more restrictive assumptions can be found in the book [2] by Caffarelli and Cabré. Further results on Lipschitz continuity, still for uniformly nondegenerate case, with sharp constants are contained and referred to in Vitolo [13].

We deal with global and local estimates only for the Isaacs equations in contrast with the more general equations in the above mentioned references, which reduce to the Isaacs equations only if the equation is determined by the so-called boundedly inhomogeneous functions. Our methods are also different from the methods of the above cited articles where the authors rely on the theory of viscosity solutions. Our solutions are given as value functions of stochastic differential games and we use probabilistic methods, with the main tool being based on different probabilistic representations for the value functions at different points. This is very close to using the so-called quasiderivatives of solutions of stochastic equations in the theory of controlled diffusion processes, which can be traced down starting from [6]. We could also use quasiderivatives in this article but it would require more work and what we are actually using can be called the method of quasidifferences. In the author’s opinion the methods of this article can be also applied to proving interior first derivatives estimates for degenerate equations similar to those in [14] when the boundary data are only Lipschitz continuous and processes are not uniformly nondegenerate.

Even though our stochastic differential games are assumed to be uniformly nondegenerate, one of our main results, Theorem 2.3, is about estimates of the Lipschitz constant independent of the constant of nondegeneracy. The author is not aware of any analytical proof of it. The only results similar to the above mentioned one, the author is aware of, are contained in Barles [1]. We discuss them in detail in Remark 3.3.

We also prove two estimates which do depend on the constant of nondegeneracy: one is global, Theorem 2.1 and another is local, Theorem 2.2. These results are much weaker than the ones in [12]. The emphasis here is to show that probabilistic methods can use nondegeneracy in an efficient way. Of course, Theorem 2.3 contains Theorem 2.1 the proof of the latter is given just because it is short, instructive, and requires less machinery.

The main results of the paper are stated in Section 2. Section 3 contains their discussion. In Section 4 we show that the value function admits very many representations. In Section 5 we prove auxiliary results aimed at estimating the difference of value function at close points when different
probabilistic representations are taken for those points. The result of Section 4 in a very rough form is used in Section 6 to prove Theorem 2.1. In Section 7 we prove Theorem 2.2 about interior estimates. A very short Section 8 contains the proof of Theorem 2.3 about estimates independent of the constant of nondegeneracy. It is short because the main ideas are given before in Section 4. In the final again short Section 9 we prove Theorem 2.4.

2. Main results

Let \( \mathbb{R}^d = \{ x = (x_1, \ldots, x_d) \} \) be a \( d \)-dimensional Euclidean space and let \( d_1 \geq d \) be an integer. Denote by \( \mathcal{O} \) the set of \( d_1 \times d_1 \) orthogonal matrices, fix an integer \( k \geq 1 \) and assume that we are given separable metric spaces \( A \) and \( B \) and let, for each \( \alpha \in A, \beta \in B, \) and \( p \in \mathbb{R}^k \), the following functions on \( \mathbb{R}^d \times \mathbb{R}^d \) be given:

(i) matrix-valued \( \sigma^{\alpha\beta}(p, x) = (\sigma^{\alpha\beta}_{ij}(p, x)) \),
(ii) \( \mathcal{O} \)-valued function \( P^{\alpha\beta}(x, y) \), \( \mathbb{R}^k \)-valued function \( p^{\alpha\beta}(x, y) \), and real-valued function \( r^{\alpha\beta}(x, y) \),
(iii) \( \mathbb{R}^d \)-valued \( \beta^{\alpha\beta}(p, x) = (\beta^{\alpha\beta}_{i}(p, x)) \),
(iv) real-valued functions \( c^{\alpha\beta}(p, x) \geq 0, f^{\alpha\beta}(p, x), \) and \( g(x) \).

Define
\[
a^{\alpha\beta}(p, x) := (1/2)\sigma^{\alpha\beta}(p, x)(\sigma^{\alpha\beta}(p, x))^*.
\]

Also set
\[
(\sigma, a, b, c, f)^{\alpha\beta}(x) = (\sigma, a, b, c, f)^{\alpha\beta}(0, x)
\]
and note that for our first main result, Theorem 2.1 only these values of \( \sigma, a, b, c, f \) are relevant and the parameters \( r, p, P \) are not present. These parameters are important in Theorem 2.3.

Fix some constants \( K_0, K_1 \in [0, \infty) \), and \( \delta_0 \in (0, 1] \).

Assumption 2.1. (i) The functions \( (\sigma, a, b, c, f)^{\alpha\beta}(p, x) \) and \( p^{\alpha\beta}(x, y) \) are continuous with respect to \( \beta \in B \) for each \( (\alpha, p, x, y) \) and continuous with respect to \( \alpha \in A \) uniformly with respect to \( \beta \in B \) for each \( (p, x, y) \). Furthermore, they are Borel measurable functions of \( (p, x, y) \) for each \( (\alpha, \beta) \) and they are bounded by \( K_0 \).

(ii) The functions \( r^{\alpha\beta}(x, y) \) and \( P^{\alpha\beta}(x, y) \) are bounded by constant \( K_0 \), they are Borel measurable with respect to all variables, and along with \( p^{\alpha\beta}(x, y) \) they are Lipschitz continuous with respect to \( x \) with Lipschitz constant \( K_1 \), and
\[
r^{\alpha\beta}(x, x) \equiv 1, \quad p^{\alpha\beta}(x, x) \equiv 0, \quad P^{\alpha\beta}(x, x) \equiv I,
\]
where \( I \) is the \( d_1 \times d_1 \)-identity matrix. The function \( p^{\alpha\beta}(x, y) \) is uniformly continuous with respect to \( y \) uniformly with respect to \( (\alpha, \beta, x) \).

(iii) The functions \( \sigma^{\alpha\beta}(p, x), \beta^{\alpha\beta}(p, x), c^{\alpha\beta}(p, x), \) and \( f^{\alpha\beta}(p, x) \) are Lipschitz continuous with respect to \( (p, x) \) with Lipschitz constant \( K_1 \). We have \( \|g\|_{C^2(\mathbb{R}^d)} \leq K_1 \).
(iv) For any $\alpha \in A$, $\beta \in B$, $x, \lambda \in \mathbb{R}^d$, and $p \in \mathbb{R}^k$ we have

$$a_{ij}^{\alpha \beta}(p, x)\lambda_i \lambda_j \geq \delta_0|\lambda|^2.$$

The reader understands, of course, that the summation convention is adopted throughout the article.

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space, let $\{\mathcal{F}_t, t \geq 0\}$ be an increasing filtration of $\sigma$-fields $\mathcal{F}_t \subset \mathcal{F}$ such that each $\mathcal{F}_t$ is complete with respect to $\mathcal{F}$, $P$, and let $w_t, t \geq 0$, be a standard $d_1$-dimensional Wiener process given on $\Omega$ such that $w_t$ is a Wiener process relative to the filtration $\{\mathcal{F}_t, t \geq 0\}$.

The set of progressively measurable $A$-valued processes $\alpha_t = \alpha_t(\omega)$ is denoted by $\mathfrak{A}$. Similarly we define $\mathfrak{B}$ as the set of $B$-valued progressively measurable functions. By $\mathfrak{A}$ we denote the set of $\mathfrak{B}$-valued functions $\beta(\cdot)$ on $\mathfrak{A}$ such that, for any $T \in (0, \infty)$ and any $\alpha^1, \alpha^2 \in \mathfrak{A}$ satisfying

$$P(\alpha^1_t = \alpha^2_t \text{ for almost all } t \leq T) = 1,$$

we have

$$P(\beta(\alpha^1_t) = \beta(\alpha^2_t) \text{ for almost all } t \leq T) = 1.$$

For $\alpha \in \mathfrak{A}$, $\beta \in \mathfrak{B}$, and $x \in \mathbb{R}^d$ introduce $x_t^{\alpha, \beta}$ as a unique solution of the Itô equation

$$x_t = x + \int_0^t \sigma^{\alpha, \beta}(x_s) \, dw_s + \int_0^t b^{\alpha, \beta}(x_s) \, ds. \quad (2.2)$$

and denote

$$\phi_t^{\alpha, \beta} = \int_0^t c^{\alpha, \beta}(x_s^{\alpha, \beta}) \, ds.$$

Next, fix a domain $D \subset \mathbb{R}^d$, define $\tau^{\alpha, \beta}$ as the first exit time of $x_t^{\alpha, \beta}$ from $D$ ($\tau^{\alpha, \beta} = \infty$ if $D = \mathbb{R}^d$), and introduce

$$v(x) = \inf_{\beta \in \mathfrak{B}} \sup_{\alpha \in \mathfrak{A}} E_x^{\alpha, \beta}[\int_0^T f(x_t)e^{-\phi_t} \, dt + g(x_T)e^{-\phi_T}], \quad (2.3)$$

where the indices $\alpha$, $\beta$, and $x$ at the expectation sign are written to mean that they should be placed inside the expectation sign wherever and as appropriate, that is

$$E_x^{\alpha, \beta}[\int_0^T f(x_t)e^{-\phi_t} \, dt + g(x_T)e^{-\phi_T}]$$

$$:= E\left[\int_0^T f(x_t^{\alpha, \beta})e^{-\phi_t} \, dt + \int_0^T f^{\alpha, \beta}(x_t^{\alpha, \beta})e^{-\phi_t} \, dt\right].$$

Observe that $v(x) = g(x)$ in $\mathbb{R}^d \setminus D$.

Our first main result is the following.
Theorem 2.1. Under the above assumptions also suppose that either \( D \) is bounded and satisfies the uniform exterior ball condition, or \( D = \mathbb{R}^d \) and there is a constant \( \delta_1 > 0 \) such that \( e^{\alpha\beta}(x) \geq \delta_1 \).

Then \( v \) is Lipschitz continuous in \( \mathbb{R}^d \) with Lipschitz constant depending only on \( D, K_0, K_1, \delta_0, \) and \( \delta_1 \).

The above setting and notation follow [10] and, as there, we convince ourselves that the definition of \( v \) makes sense and \( v \) is bounded.

Here is a result about interior smoothness of \( v \).

**Theorem 2.2.** Let \( D \) be bounded and in Assumption 2.1 (iii) replace the requirement \( \|g\|_{C^2(\mathbb{R}^d)} \leq K_1 \) with the requirement that \( g \) be continuous. Then \( v \) is Lipschitz continuous on any compact set \( \Gamma \subset D \).

Our next result is about Lipschitz continuity of \( v \) with constant independent of \( \delta_0 \). As usual in this case we need

**Assumption 2.2.** There exists a \( \delta_1 \in (0, 1] \) such that for any \( \alpha \in A, \beta \in B, x \in \mathbb{R}^d, \) and \( p \in \mathbb{R}^k \) we have

\[
e^{\alpha \beta}(p, x) \geq \delta_1.
\]

**Remark 2.1.** Assume that \( D \) lies in the ball of radius \( R \) centered at the origin. For \( \mu > 0 \) define \( \Psi(x) = \cosh(\mu R) - \cosh(\mu |x|) + 2 \). It is easy to check that for \( \mu \) large enough depending only on \( \delta_0, K_0, \) and \( d \) the function \( \Psi \) is infinitely differentiable on \( \mathbb{R}^d, \Psi \geq 2 \) on \( D \), and \((L^{\alpha \beta} + e^{\alpha \beta})\Psi \leq -1 \) on \( D \) for all \( \alpha, \beta \). This is a so-called global barrier for \( D \).

We modify it for \( |x| \geq R \) in such a way that it will be still infinitely differentiable on \( \mathbb{R}^d \), have bounded derivatives, and be such that \( \Psi \geq 1 \) on \( \mathbb{R}^d \). We keep the same notation for the modified function. By Remark 2.3 of [10] if we construct \( \tilde{v} \) from

\[
\begin{align*}
\tilde{\sigma}^{\alpha \beta}(x) &= \Psi^{1/2}(x)\sigma^{\alpha \beta}(x), \\
\tilde{b}^{\alpha \beta}(x) &= \Psi(x)\tilde{b}^{\alpha \beta}(x) + 2\sigma^{\alpha \beta}(x)D\Psi(x), \\
\tilde{c}^{\alpha \beta}(x) &= -L^{\alpha \beta}\Psi(x), \\
\tilde{f}^{\alpha \beta}(x) &= f^{\alpha \beta}(x), \\
\tilde{g}(x) &= \Psi^{-1}(x)g(x),
\end{align*}
\]

where \( D\Psi \) is the gradient of \( \Psi \) (a column vector), in the same way as \( v \) was constructed from the original \( \sigma, b, c, f, \) and \( g \), then \( \tilde{v} = \Psi^{-1}v \). This shows that without restricting generality we could have supposed that Assumption 2.2 is satisfied from the very beginning.

Introduce

\[
\begin{align*}
\tilde{\sigma}^{\alpha \beta}(x, y) &= r^{\alpha \beta}(x, y)\sigma^{\alpha \beta}(x, y)P^{\alpha \beta}(x, y), \\
(\tilde{a}, \tilde{b}, \tilde{c}, \tilde{f})^{\alpha \beta}(x, y) &= [r^{\alpha \beta}(x, y)]^2(a, b, c, f)^{\alpha \beta}(x, y),
\end{align*}
\]

and for unit \( \xi \in \mathbb{R}^d \) introduce a convex function \( \|\sigma\|^2_\xi \) on the set of \( d \times d_1 \) matrices by

\[
\|\sigma\|^2_\xi := \|\sigma\|^2 - |\xi^* \sigma|^2 = \|(I - \xi^* \xi)\sigma\|^2, \quad \|\sigma\|^2 = \sum_{i,j} \sigma_{ij}^2,
\]

(2.4)

where \( I \) is the unit \( d \times d \) matrix.
Assumption 2.3. For all $\alpha \in A$, $\beta \in B$, and $x, y \in \mathbb{R}^d$
\[ \delta_1^{-1} \geq r^{\alpha \beta}(x, y) \geq \delta_1. \]

Assumption 2.4. There exist constants $\delta \geq 2\delta_1$, $\varepsilon_0 > 0$, and $\mu \geq 1$ such that for all $\alpha \in A$, $\beta \in B$, and $x, y \in \mathbb{R}^d$, for which $|x - y| \leq \varepsilon_0$, we have
\[ \|\tilde{\sigma}^{\alpha \beta}(x, y) - \sigma^{\alpha \beta}(y)\|^2 + 2\langle x - y, \hat{b}^{\alpha \beta}(x, y) - \hat{b}^{\alpha \beta}(y) \rangle \]
\[ \leq 2(c^{\alpha \beta}(y) - \delta)|x - y|^2 + 4\mu\langle x - y, a^{\alpha \beta}(x)(x - y) \rangle, \quad (2.5) \]
where $\xi = (x - y)/|x - y|$.  

Remark 2.2. If $d = 1$, then for any $d \times d_1$-matrix $\sigma$ and unit $\xi \in \mathbb{R}^d$, we have $\|\sigma\| = |\xi^*\sigma|$, so that in that case the term involving $\sigma$ in (2.5) disappears. Also notice that if $\sigma$ and $b$ are independent of $p$ and $r \equiv 1$, $p \equiv 0$, and $P \equiv I$, then $(\hat{a}, \hat{\sigma}, \hat{b})^{\alpha \beta}(x, y) = (a, \sigma, b)^{\alpha \beta}(x)$, and condition (2.5) becomes
\[ \|\sigma^{\alpha \beta}(x) - \sigma^{\alpha \beta}(y)\|^2 + 2\langle x - y, b^{\alpha \beta}(x) - b^{\alpha \beta}(y) \rangle \]
\[ \leq 2(c^{\alpha \beta}(y) - \delta)|x - y|^2 + 4\mu\langle x - y, a^{\alpha \beta}(x)(x - y) \rangle, \quad (2.6) \]
which is satisfied with any $\delta$ on the account of choosing a sufficiently large $\mu$ (depending on $\delta_0$ and $K_1$) since $\sigma$ and $b$ are Lipschitz continuous. Therefore, Theorem 2.1 is a particular case of Theorem 2.3. It is also worth noting that if $d = 1$, condition (2.6) is satisfied with $\mu = 0$ when $b^{\alpha \beta}(x)$ are decreasing functions of $x$ and $c^{\alpha \beta} \geq \delta$.

In Section 3 we give more examples when one can check Assumption 2.3.

Introduce
\[ H(p, x, u, (u_{ij})), \quad H(p, x, u, (u_{ij})) = \sup_{\alpha \in A} \inf_{\beta \in B} [a^{\alpha \beta}_{ij}(p, x)u_{ij} + b^{\alpha \beta}_{ij}(p, x)u_i + c^{\alpha \beta}(p, x)u + f^{\alpha \beta}(p, x)]. \]

Assumption 2.5. The set of $(x, u, (u_i), (u_{ij}))$ such that
\[ H(p, x, u, (u_i), (u_{ij})) \leq 0 \quad (2.7) \]
is independent of $p$ and the same is true if we reverse the sign of the inequality.

Theorem 2.3. Under the above assumptions also suppose that either $D = \mathbb{R}^d$, or $D$ is bounded and there exists a nonnegative function $G \in C^{0,1}(\overline{D}) \cap C^{1,0}_{\text{loc}}(\overline{D})$ such that $G = 0$ on $\partial D$ and
\[ L^{\alpha \beta}G(p, x) := a^{\alpha \beta}_{ij}(p, x)D_{ij}G(x) + b^{\alpha \beta}_{ij}(p, x)D_iG(x) - c^{\alpha \beta}(p, x)G(x) \leq -1 \]
in $D$ for any $p$, where $D_i = \partial/\partial x_i$, $D_{ij} = D_iD_j$.

Then $v$ is Lipschitz continuous in $\mathbb{R}^d$ with Lipschitz constant independent of $\delta_0$. 


Remark 2.3. If $D$ is bounded and satisfies the uniform exterior ball condition, the function $G$ always exists since the operators $L^{\alpha \beta}$ are uniformly nondegenerate, have bounded coefficients and $c^{\alpha \beta} \geq 0$. However, the proof of this well-known fact relies on the uniform nondegeneracy and gives a function $G$ depending on $\delta_0$. The reader understands that there are plenty of cases when this assumption is satisfied even for degenerate operators.

Finally, we state one more result, which was, actually, the main motivation of writing the whole series consisting of [9], [10], and the present article. We take $D = \mathbb{R}^d$ and suppose that all above assumptions are satisfied and $\sigma, b, c, f$ are independent of $p$.

Set

$$A_1 = A$$

and let $A_2$ be a separable metric space having no common points with $A_1$.

**Assumption 2.6.** The functions $\sigma^{\alpha \beta}(x), b^{\alpha \beta}(x), c^{\alpha \beta}(x)$, and $f^{\alpha \beta}(x)$ are also defined on $A_2 \times B \times \mathbb{R}^d$ in such a way that they are independent of $\beta$ and satisfy Assumptions 2.1 (i), (iii), (iv) with the same constants $K_0, K_1$ and, of course, with $A_2$ in place of $A$.

Define

$$\hat{A} = A_1 \cup A_2.$$ 

Then we introduce $\hat{A}$ as the set of progressively measurable $\hat{A}$-valued processes and $\hat{B}$ as the set of $\mathbb{B}$-valued functions $\beta(\alpha \cdot)$ on $\hat{A}$ such that, for any $T \in [0, \infty)$ and any $\alpha^1, \alpha^2 \in \hat{A}$ satisfying

$$P(\alpha^1_t = \alpha^2_t \text{ for almost all } t \leq T) = 1,$$

we have

$$P(\beta_t(\alpha^1) = \beta_t(\alpha^2) \text{ for almost all } t \leq T) = 1.$$

For a constant $K \geq 0$ set

$$v_K(x) = \inf_{\beta \in \hat{B}} \sup_{\alpha \in \hat{A}} v_{K}^{\alpha, \beta(\alpha \cdot)}(x),$$

where

$$v_{K}^{\alpha, \beta}(x) = E_x^{\alpha, \beta} \int_0^\infty f_K(x) e^{-\phi_t} dt =: v^{\alpha, \beta}(x) - K E_x^{\alpha, \beta} \int_0^\gamma I_{\alpha \cdot \in A_2} e^{-\phi_t} dt,$$

$$f_K^{\alpha, \beta}(x) = f^{\alpha, \beta}(x) - K I_{\alpha \in A_2}.$$ 

The above formula extends $v^{\alpha, \beta}(x)$, initially defined for $\alpha \in \mathfrak{A}$ and $\beta \in \mathfrak{B}$, on the set $\hat{A} \times \mathfrak{B}$. Of course, (2.3) is preserved with $\tau = \infty$ and no $g$ involved.

**Theorem 2.4.** There is a constant $N$ such that $|v_K(x) - v(x)| \leq N/K$ for all $x \in \mathbb{R}^d$ and $K \geq 1$. 

Remark 2.4. In one of the main cases of interest $v_K$ turns out to have second-order derivatives bounded by a constant times $K$ if $K \geq 1$ (see Section 7 in [10]). From the point of view of finite-difference approximations it should be easier to approximate “smooth” functions $v_K$ than $v$. However, the author has no idea how to prove a fact similar to Theorem 2.4 for finite-difference equations.

In this connection it would be very interesting to find any proof of Theorem 2.4 not using probability theory, of course, defining $v_K$ and $v$ as viscosity solutions of the corresponding Isaacs equations.

3. Comments and examples

Remark 3.1. Let $\sigma$ and $b$ be independent of $\alpha$ and $\beta$ and consider a particular case where $d_1 = d$ and equation (2.2) is

$$x_t = x + \int_0^t \sigma(x_s) \, dw_s,$$

where $\sigma$ is an $O$-valued Lipschitz continuous function. Then the left-hand side of (2.5) vanishes for $r \equiv 1$ and $P(x, y) = \sigma^*(x) \sigma^*(y)$. Of course, this is not a big surprise since $x_t$ is just a Brownian motion starting at $x$. Still one can see that the parameters $P$ take care of rotations of the increments of the original Wiener process and, basically show that (2.5) is a condition on $a$ rather than $\sigma$.

Remark 3.2. The function $v$ will not change if we change $\sigma, b, c, f$ outside $D$. In connection with this it is worth noting that in Assumption 2.4 we may restrict $x$ and $y$ to $D_{\varepsilon_0}$ which is the $\varepsilon_0$ neighborhood of $D$. Indeed, if only thus restricted Assumption 2.4 is satisfied we could just change $c$ outside $D$ so that it will be bigger than the original one and become any large constant outside $D_{\varepsilon_0}$. Then Assumption 2.4 will be satisfied in the form it is stated.

Remark 3.3. For later discussion we show that Assumption 2.4 can be replaced with a slightly more transparent one. We will be only concerned with Assumption 2.4 leaving other assumptions aside.

Denote by $S_k$ the set of $d_1 \times d_1$ skew-symmetric matrices and assume that for each $\alpha \in A$, $\beta \in B$, and $\xi \in \mathbb{R}^d$ the following functions on $\mathbb{R}^k$ are also given: $S_k$-valued function $\Theta^{\alpha \beta}(x, \xi)$, $k \times d$ matrix-valued function $p^{\alpha \beta}(x)$, and $\mathbb{R}^d$-valued function $r^{\alpha \beta}(x)$.

For a differentiable function $u(p, x)$ and $\xi \in \mathbb{R}^d$ introduce

$$\partial_{\xi} u^{\alpha \beta}(x) = \xi_i u_{x_i}(0, x) + (p^{\alpha \beta}(x) \xi_j) u_{x_j}(0, x).$$

Also denote Conv $(D)$ the open convex hull of $D$.

Assumption 3.1. (i) For $|\xi| \leq 1$ the above functions are bounded by $K_0$ and $\Theta^{\alpha \beta}(x, y)$ is a linear function of $y$ (in particular $\Theta^{\alpha \beta}(x, 0) = 0$).

(ii) For any $\alpha \in A$ and $\beta \in B$ the functions $\sigma^{\alpha \beta}(p, x)$ and $b^{\alpha \beta}(p, x)$ are continuously differentiable with respect to $(p, x) \in \mathbb{R}^k \times \mathbb{R}^d$ and their first order derivatives are bounded by $K_1$. Furthermore, their derivatives
are uniformly continuous with respect to \((p, x)\) uniformly with respect to \((\alpha, \beta) \in A \times B\).

(iii) There are constants \(\mu \geq 1\) and \(\delta \geq 2\delta_1\) such that for any unit \(\xi \in \mathbb{R}^d\) and \((\alpha, \beta, x) \in A \times B \times \text{Conv}(D)\) we have

\[
\|\partial_\xi \sigma^{\alpha\beta}(x) + (r^{\alpha\beta}(x), \xi)\sigma^{\alpha\beta}(x) + \sigma^{\alpha\beta}(x)\Theta^{\alpha\beta}(x, \xi)\|^2_{\xi} \\
+ 2\langle \xi, \partial_t b^{\alpha\beta}(x) + 2(r^{\alpha\beta}(x), \xi)b^{\alpha\beta}(x) \rangle \leq 2(c^{\alpha\beta}(x) - \delta_1 - \delta) + 4\mu\langle \xi, a^{\alpha\beta}(x)\xi \rangle.
\]

(3.1)

Introduce

\[
r^{\alpha\beta}(x, y) = 1 + \langle r^{\alpha\beta}(y), x - y \rangle, \quad p^{\alpha\beta}(x, y) = p^{\alpha\beta}(y)(x - y),
\]

\[
P^{\alpha\beta}(x, y) = \exp \Theta^{\alpha\beta}(y, x - y).
\]

We claim that there exists an \(\varepsilon_0 > 0\), depending only on \(K_0, K_1, \delta_1, d\), and the moduli of continuity in \((p, x)\) of the derivatives of \(\sigma^{\alpha\beta}(p, x)\) and \(b^{\alpha\beta}(p, x)\) with respect to \((p, x)\), such that Assumption 2.4 is satisfied with \(x, y\) restricted to \(D\).

To prove the claim fix \(y \in D\) and a unit \(\xi \in \mathbb{R}^d\) and for \(t \geq 0\) introduce \(x(t) = y + t\xi\), so that (2.5) becomes

\[
\|\sigma^{\alpha\beta}(x(t), y) - \sigma^{\alpha\beta}(y)\|^2_{\xi} \leq 2(c^{\alpha\beta}(y) - \delta) + 4\mu\langle \xi, a^{\alpha\beta}(x(t))\xi \rangle t^2,
\]

(3.3)

which we want to prove for \(t \in (0, \varepsilon_0]\). For simplicity of notation we will drop the superscripts \(\alpha, \beta\) in a few lines below.

Observe that

\[
\tilde{\sigma}(x(t), y) - \sigma(y) = \int_0^t \xi_t \tilde{\sigma}_{x_t}(x(s), y) ds,
\]

where

\[
\xi_t \tilde{\sigma}_{x_t}(x(s), y) = \langle r(y), \xi \rangle \sigma(sp(y)\xi, x(s)) P(x(s), y)
\]

\[
+ r(x(s), y)\left[\xi_t \sigma_{x_t}(sp(y)\xi, x(s)) + (p(y)\xi)\xi \sigma_{p_t}(sp(y)\xi, x(s))\right] P(x(s), y)
\]

\[
+ r(x(s), y)\sigma(sp(y)\xi, x(s)) \Theta(y, \xi) P(x(s), y)
\]

\[
= : \langle r(y), \xi \rangle \sigma(y) + \partial_\xi \sigma(y) + \sigma(y)\Theta(y, \xi) + R(s),
\]

and \(R(s)\) is introduced by the above equality.

Owing to the convexity of function \([\Sigma_{2.4}\] and Assumption \([\Sigma_{2.4}\]) there exists an \(\varepsilon_0 > 0\) such that for all \(t \in (0, \varepsilon_0]\) and all values of other arguments we have

\[
\|\tilde{\sigma}^{\alpha\beta}(x(t), y) - \sigma^{\alpha\beta}(y)\|^2_{\xi} - 4\mu\langle \xi, a^{\alpha\beta}(x(t))\xi \rangle t^2 \\
\leq t^2\|\partial_\xi \sigma^{\alpha\beta}(y) + (r^{\alpha\beta}(y), \xi)\sigma^{\alpha\beta}(y) + \sigma^{\alpha\beta}(y)\Theta^{\alpha\beta}(y, \xi)\|^2_{\xi} \\
- 4\mu\langle \xi, a^{\alpha\beta}(y)\xi \rangle t^2 + t^2\delta_1.
\]
It is even easier to prove that, by reducing $\varepsilon_0$ if necessary, we have that for $t \in (0, \varepsilon_0]$ and all values of other arguments,

$$t\langle \xi, \hat{b}^{\alpha\beta}(x(t), y) - b^{\alpha\beta}(y) \rangle$$

$$\leq t^2\langle \xi, \partial_\xi b^{\alpha\beta}(y) + 2r^{\alpha\beta}(y), \xi \rangle b^{\alpha\beta}(y) + t^2\delta_1.$$  

Hence, by assumption, the left-hand side of (3.3) is less than

$$t^2[2(c^{\alpha\beta}(y) - \delta_1 - \delta) + 4\mu(\xi, a^{\alpha\beta}(y)\xi)] + 2t^2\delta_1,$$

which is the right-hand side of (3.3).

**Remark 3.4.** Consider the case that $\sigma$ and $b$ are independent of $\alpha$ and $\beta$. Let $d = 1$, $c > 0$, and $D = (-1, 1)$. Assume that $a = a_0 + \delta_0$, where $a_0 \geq 0$. In that case, as it follows from the arguments in Remarks 2.2 and 3.3, we do not need to assume that $\sigma'$ is continuous. We still assume that $a$, $b'$, and $c$ are continuous. Then by Remark 2.2 Assumption 2.4 is satisfied with $\mu$ depending on $\delta_0$ among other things.

However, assume additionally that at every point $x \in [-2, 2]$ where

$$a_0(x) = b(x) = 0$$

we have

$$b'(x) < c(x).$$  

We claim that then Assumption 2.4 is satisfied with $x, y$ restricted to $[-2, 2]$ with some $\delta, \delta_1, \varepsilon_0$, and $\mu$ independent of $\delta_0$ and hence, by Remark 3.3 valid in case $D = (-1, 1)$.

To prove the claim, we use Remark 3.3 and observe that for $r = -nb/2$, $\delta_1 + \delta = 1/n$, $\mu = n$ and $|\xi| = 1$ conditions (3.1) is satisfied if

$$b'(x) \leq c(x) - \frac{1}{n} + n(a_0(x) + |b(x)|^2).$$  

Suppose that for any $n = 1, 2, \ldots$ we can find a point $x_n \in [-2, 2]$ at which the inequality converse to (3.5) holds. Then we can extract from the sequence $x_n$ a subsequence that converges to an $x_0 \in [-2, 2]$. Clearly, for large $n$

$$a_0(x_n) + |b(x_n)|^2 \leq Nn^{-1},$$

where $N = \sup b' + 1$. Therefore, $a_0(x_0) + |b(x_0)|^2 = 0$ and

$$b'(x_n) \geq c(x_n) - 1/n, \quad b'(x_0) \geq c(x_0).$$

We have obtained a contradiction to (3.4), so inequality (3.5) holds in $[-2, 2]$ for some $n$ independent of $\delta_0$ thus proving our claim.

**Example 3.1.** Consider the one-dimensional equation

$$\delta_0v'' + bvv' - v = 0$$  

on $[-1, 1]$ with data 1 at $\pm 1$, where constant $b > 0$. This is, of course, a simple example of the Isaacs equation in a differential “game” with the
value function \( v \). Here the assumption stated in Theorem 2.3 concerning \( G \) is satisfied with \( G(x) = (1 - x^2) \max(1, 1/(2b)) \).

If we assume that the solution \( v = v_{\delta_0} \) admits an estimate of its Lipschitz constant independent of \( \delta_0 \), then, as is easy to understand, say from the probabilistic representation of \( v_{\delta} \), the function

\[
v_0(x) = E e^{-\tau_x}
\]

would be Lipschitz continuous, where \( \tau_x \) is the first exit time of the solution of

\[
x_t = x + \int_0^t bx_s ds
\]

from \((-1, 1)\). Since \( x_t = x e^{bt} \), \( \tau_x = -b^{-1} \ln |x| \) for \( |x| < 1 \) and \( v_0(x) = |x|^{1/b} \), which is Lipschitz continuous only if \( b \leq 1 \).

This example shows that in the situation of Remark 3.4, if one has \( b'(x) > c(x) \) at at least one point at which \( a_0(x) = b(x) = 0 \), the assertion of Theorem 2.3 may be no longer true. In this respect requiring condition \( (3.4) \) at those points is close to be optimal and it is, actually, necessary for \( v \) to be continuously differentiable.

**Remark 3.5.** Barles in [1] derived first-order derivatives estimates for viscosity solutions of nonlinear equations

\[
H(x, u, Du, D^2 u) = 0
\]

in domains, where \( Du = (D_i u) \) is the gradient of \( u \) and \( D^2 u = (D^2_{ij} u) \) is its Hessian. Our value functions are viscosity solutions of the corresponding Isaacs equations. This is proved, for instance, in [3] on the basis of the dynamic programming principle. The Isaacs equations in this paper are included in the framework of [1] and many of the equations in [1] do not fit into our scheme. Yet it is worth comparing our conditions with the ones from [1] in the simplest example of linear equations with

\[
H(x, u_0, u', u'') = a_{ij}(x)u''_{ij} + b_i(x)u'_i - c(x)u_0 + f(x),
\]

for which solutions have probabilistic representations (with no \( \alpha \) and \( \beta \) involved).

One of the assumptions in [1] reads as follows: For any \( R > 0 \) and all large enough \( L \)

\[
c \sum_{i=1}^d |u'_i|^2 + g \text{tr } u'' uu'' = [u''_{ij}D_k a_{ij}u''_{ij} + u''_{ij}D_k b_i(x)u'_i - u''_{ij}D_k c(x)u_0 + u''_{ij}D_k f(x)] \geq h,
\]

where \( g, h > 0 \) are some constants > 0, provided that

\[
|u_0| \leq R, \quad \sum_{i=1}^d |u'_i|^2 \geq L, \quad H(x, u_0, u', u'') = 0, \quad u''_{ij}u'_{ij} = 0 \quad \forall i.
\]
If \( c \equiv 0, b \equiv 0, \) and both \( f \) and \( Df \) vanish at a point \( x_0, \) so that \( H(x_0, 0) = 0, \) then for \( u'' = 0 \) inequality (3.7) at \( x_0 \) becomes \( 0 \geq h, \) which cannot hold even in the one-dimensional case. Therefore, the one-dimensional equation

\[
D^2 u + x^2 = 0
\]

in \((-1, 1)\) with zero boundary condition does not fit in the scheme of [1].

Equation

\[
\delta_0 D^2 u + (b_1 x + b_0) Du - cu + x^2 = 0
\]

in \((-1, 1)\) with zero boundary condition and constant \( c > 0, b_0, b_1 \) does not fit in either if \( c \leq b_1. \)

Indeed, if we take \( x = 0, u'' = 0, u_0 = 0, \) and \( u' \) bigger by magnitude than \( L, \) (3.7) becomes

\[
(c - b_1)|u'|^2 \geq h,
\]

which for large \( |u'| \) can only hold if \( b_1 < c. \) Remark 3.4 shows that one always has an estimate of the Lipschitz constant of \( v. \) This estimate is even independent of \( \delta_0, \) provided that either \( \min_{[-1,1]} |b_1 x + b_0| \geq 0 \) or \( b_1 < c. \)

It looks like the methods of [1] are not adapted to use uniform nondegeneracy and even in the above examples lead to the requirement that \( c \) be sufficiently large.

In [1] the author also claims that interior or local estimates can be obtained “by truncation arguments” but, as far as the author of the present article is aware, there is no evidence to date to support this claim.

Remark 3.6. Above we saw that the parameters \( \mu, r, \) and \( P \) can play a role while checking Assumption 2.4. We now show how the external parameters \( p \) can be used. Here we consider the situation in which \( \sigma, b, c, \) and \( f \) depend only on \( x \) and \( \alpha \) so that we are dealing with controlled diffusion processes rather than differential games. Our interest is in obtaining estimates independent of \( \delta_0 \) and, therefore, from the start in this remark we focus on degenerate processes.

Let \( A = \mathbb{R} \) and consider a one-dimensional process defined by the equation

\[
x_t = x + \int_0^t \sigma(x_s) dw_s + \int_0^t \tanh(x_s + 2 \cos \alpha_s) ds,
\]

(3.9)

where \( w_t \) is a one-dimensional Wiener process, \( \sigma(x) \) is a smooth nonnegative even function satisfying \( \sigma(x) > 0 \) for \( x \in (1, 3) \) and vanishing outside \((1, 3)\) (and \( \alpha_t \) is a progressively measurable \( A \)-valued process). We also take a sufficiently regular function \( c(x) \geq \delta_2 \) (independent of \( \alpha \) and \( \beta \)), where \( \delta_2 > 0, \) and take \( D = \mathbb{R}. \)

If we want to satisfy (3.1) for \( |x| \not\in [1, 3] \) with \( r(x) = 0 \) (and \( \Theta \equiv 0 \) for having no other options) and some \( \delta \)'s we obviously need to have

\[
c(x) > 1 \text{ for } |x| \leq 1, \quad c(x) > \cosh^{-2}(|x| - 2) \text{ for } |x| \geq 3. \tag{3.10}
\]

The inequalities in (3.10) extend for \( |x| \not\in (1 + \varepsilon, 3 - \varepsilon) \) with some \( \varepsilon > 0 \) and one can find \( \mu \geq 1 \) such that (3.11) is satisfied (with some \( \delta \)'s) for
$|x| \in (1 + \varepsilon, 3 - \varepsilon)$ with $r(x) = 0$. Therefore, if we do not use parameter $r$, then (3.1) reduces to (3.10).

However, if we take

$$r^\alpha(x) = -2I_{|x + 2\cos \alpha| > \varepsilon} \sinh^{-1}(2x + 4\cos \alpha),$$

(3.11)

then the left-hand side of (3.1) becomes

$$2I_{|x + 2\cos \alpha| \leq \varepsilon} \cosh^{-2}(x + 2\cos \alpha) \leq 2I_{|x + 2\cos \alpha| \leq \varepsilon},$$

and for $|x| \notin (1 + \varepsilon, 3 - \varepsilon)$ this is strictly less than $2c(x)$ if

$$c(x) > 1 \quad \text{for} \quad |x| \leq 1 + \varepsilon.$$ (3.12)

Hence, with the so specified $r^\alpha$ condition (3.1) reduces to (3.12), which is a significant improvement over (3.10).

Next we take $f$ independent of $\alpha$, say $f \equiv 1$, and instead of

$$b^\alpha(p, x) = \tanh(x + 2\cos \alpha),$$

consider

$$b^\alpha(p, x) = \tanh(x + 2\cos(\alpha + p)),$$

where $p \in \mathbb{R}$. Obviously, Assumption 2.5 is satisfied.

Take $r^\alpha(x)$ from (3.11) and

$$p^\alpha(x) = (1/2)I_{|x + 2\cos \alpha| \leq \varepsilon}I_{|\sin \alpha| > \varepsilon} \sin^{-1} \alpha.$$ (3.13)

Then the left-hand side of (3.1) becomes

$$2I_{|x + 2\cos \alpha| \leq \varepsilon} \cosh^{-2}(x + 2\cos \alpha) - 2I_{|x + 2\cos \alpha| \leq \varepsilon} \sinh^{-1}(2x + 4\cos \alpha)$$

$$= 2I_{|x + 2\cos \alpha| \leq \varepsilon} \cosh^{-2}(x + 2\cos \alpha) \leq 2I_{|x + 2\cos \alpha| \leq \varepsilon} \cosh^{-2}(x + 2\cos \alpha)$$

and the latter is zero if $|x| \leq 1 + \varepsilon$ and $\varepsilon$ is sufficiently small. Thus adding $p^\alpha(x)$ into the picture eliminates condition (3.12) entirely and there is nothing more than $c(x) \geq \delta_2$ required of $c(x)$ in order for (3.1) to be satisfied with $r^\alpha(x)$ from (3.11) and $p^\alpha(x)$ from (3.13).

By the way, the Isaacs (Bellman) equation in this case is

$$a(x)D^2v(x) + (Dv(x))\tanh[x + 2\text{sign}(Dv(x))] - c(x)v(x) + f(x) = 0,$$

where $a = (1/2)\sigma^2$. This equation suggests a different representation of the value function with $A = \{\pm 1\}$ when using parameters $p$ becomes unnecessary (and impossible) but using $r$ will suffice. In this connection it is worth mentioning that much more sophisticated use of the external parameters $p$ can be found in [6].
4. ON EQUIVALENT REPRESENTATIONS OF VALUE FUNCTIONS

Here we suppose that Assumptions 2.1, 2.2, 2.3 and 2.5 are satisfied.

**Assumption 4.1.** There exists a nonnegative $G \in C(\bar{D}) \cap C^2_{loc}(D)$ such that $G = 0$ on $\partial D$ (if $D \neq \mathbb{R}^d$) and

$$L^{\alpha \beta}G(p, x) \leq -1$$

in $D$ for all $p \in \mathbb{R}^k$, $\alpha \in A$, and $\beta \in B$.

Suppose that we are also given an $\mathbb{R}^{d_1}$-valued function $\pi^{\alpha \beta}(x, y)$ defined for $x, y \in \mathbb{R}^d$, $\alpha \in A$, and $\beta \in B$, which is bounded by $K_0$, Borel measurable, and Lipschitz continuous with respect to $x$ with Lipschitz constant $K_1$.

Then for $\alpha \in A$, $\beta \in B$, $x, y \in \mathbb{R}^d$ introduce $y^{\alpha \beta}_t = y^{\alpha \beta}_t, x, y$ as a unique solution of the Itô equation

$$y_t = y + \int_0^t \sigma^{\alpha \beta}(s) \, dw_s + \int_0^t b^{\alpha \beta}(s) \, ds$$

(4.1)

and introduce $x^{\alpha \beta, x, y}_t$ as a unique solution of the Itô equation (recall that $\sigma, b, c, \hat{f}$ are introduced before Assumption 2.3)

$$x_t = x + \int_0^t \hat{\sigma}^{\alpha \beta}(s, x_s, y_s) \, dw_s + \int_0^t (\hat{b} - \hat{\sigma} \pi)^{\alpha \beta}(s, x_s, y_s) \, ds,$$  

(4.2)

where, of course, $y_s = y^{\alpha \beta, x, y}_s$. We emphasize that (4.1) has a unique solution since its coefficients are Lipschitz continuous in $y$ and are bounded and for given $y$, equation (4.2) has a unique solution since its coefficients are Lipschitz continuous in $x$ and are bounded. It follows that, in terminology of [9], the system (4.1)-(4.2) satisfies the usual hypothesis (although the coefficients in (4.2) may not be Lipschitz continuous with respect to the $y$ variable).

With the above $y_s$ and $x_s = x^{\alpha \beta, x, y}_s$ also define

$$\phi^{\alpha \beta, x, y}_t = \int_0^t \sigma^{\alpha \beta}(s, x_s, y_s) \, ds$$

and for $z \in \mathbb{R}$ introduce $z^{\alpha \beta, x, y, z}_t$ as a unique solution of

$$z_t = z + \int_0^t z_s \, [\pi^{\alpha \beta}(s, x_s, y_s)]^\ast \, dw_s.$$  

(4.3)

Next, for $X = (x, y, z)$, $x, y \in \mathbb{R}^d$, $z \in \mathbb{R}$ denote

$$x^{\alpha \beta}_t = x^{\alpha \beta, x, y}_t, \quad y^{\alpha \beta}_t = y^{\alpha \beta, x, y}_t, \quad \phi^{\alpha \beta}_t = \phi^{\alpha \beta, x, y}_t \quad \text{and} \quad X^{\alpha \beta}_t = (x_t, y_t, z_t) \alpha \beta,$$

fix a number $M \in (1, \infty)$, for $X = (x, y, z)$ define $\tau^{\alpha \beta}_t X$ as the first exit time of $(x, z)\alpha \beta, X$ from $D \times (M^{-1}, M)$, and set

$$v^{\alpha \beta}(X) = E^\alpha_Y \left[ \int_0^{\tau} \hat{f}(X_t) e^{-\phi_t} \, dt + z_{\tau} v(x_{\tau}) e^{-\phi_{\tau}} \right].$$
where \( \hat{f}^{\alpha \beta}(x, y, z) = z \hat{f}^{\alpha \beta}(x, y) \) and \( v \) is taken as in Theorem 2.1 and is at least bounded and continuous according to the results of [10] and owing to Assumption 4.1. Finally, introduce
\[
v(X) = \inf_{\beta \in \mathcal{B}} \sup_{\alpha \in \mathcal{A}} v^{\alpha \beta}(X).\]

The fact that \( v^{\alpha \beta}(X) \) and \( v(X) \) are well defined and bounded will be seen from the proof of the following.

**Theorem 4.1.** Under the above notation for \( X = (x, y, z) \) we have
\[
v(X) = z v(x). \tag{4.4}
\]
Furthermore, if we are given stopping times \( \gamma^{\alpha \beta, X} \leq \tau^{\alpha \beta, X} \), then
\[
rv(x) = \inf_{\beta \in \mathcal{B}} \sup_{\alpha \in \mathcal{A}} E_X^{\alpha \beta} \left[ \int_0^\tau \hat{f}(X_t)e^{-\phi_t} dt + z \gamma v(x) e^{-\phi_\gamma} \right]. \tag{4.5}
\]

Proof. Introduce
\[
(a, \sigma, b, c, f)^{\alpha \beta}(x, y) = (a, \sigma, b, c, f)(p^{\alpha \beta}(x, y), x) \tag{4.6}
\]
(specifying the value of \( p \) transforms the letters to their boldface options). Also denote by \( \mathcal{P} \) the set of triples \( \tilde{p} = (r, \pi, P) \), where \( r \in [\delta_1, \delta_1^{-1}], \pi \in \mathbb{R}^{d_1} \) with \( |\pi| \leq K_0 \), and \( P \in \mathcal{O} \). For \( \tilde{p} = (r, \pi, P) \in \mathcal{P} \) define
\[
\tilde{\sigma}^{\alpha \beta}(\tilde{p}, x, y) = r \sigma^{\alpha \beta}(x, y) P, \quad \tilde{b}^{\alpha \beta}(\tilde{p}, x, y) = r^2 b^{\alpha \beta}(x, y) - r \sigma^{\alpha \beta}(x, y) P \pi, \quad \tilde{c}^{\alpha \beta}(\tilde{p}, x, y, z) = r^2 c^{\alpha \beta}(x, y), \quad \tilde{f}^{\alpha \beta}(\tilde{p}, x, y, z) = r^2 f^{\alpha \beta}(x, y)
\]
and also write
\[
r = r(\tilde{p}), \quad \pi = \pi(\tilde{p}), \quad P = P(\tilde{p}).
\]
We thus freed the coefficients of (4.2) of the particular value \( s \) of \( r, \pi, P \).

For each \( \tilde{p} \in \mathcal{P} \) there is a natural operator \( \tilde{L}^{\alpha \beta} \) acting on smooth functions \( u(x, y, z) \) and mapping them to
\[
\tilde{L}^{\alpha \beta} u(\tilde{p}, x, y, z)
\]
associated with the matrix of second-order coefficients
\[
\frac{1}{2} \begin{pmatrix}
\tilde{\sigma}^{\alpha \beta}(\tilde{p}, x, y) \\
\sigma^{\alpha \beta}(y) \\
z \pi(\tilde{p})
\end{pmatrix}^* \begin{pmatrix}
\tilde{\sigma}^{\alpha \beta}(\tilde{p}, x, y) \\
\sigma^{\alpha \beta}(y) \\
z \pi(\tilde{p})
\end{pmatrix},
\]
the drift term
\[
\begin{pmatrix}
\tilde{b}^{\alpha \beta}(\tilde{p}, x, y) \\
\tilde{b}^{\alpha \beta}(y) \\
0
\end{pmatrix},
\]
and the zeroth-order (killing) coefficient \(-\tilde{c}^{\alpha \beta}(\tilde{p}, x, y, z)\). Introduce \( \tilde{p} = (1, 0, I) \) and
\[
\tilde{L}^{\alpha \beta} u(x, y, z) = \tilde{L}^{\alpha \beta} u(\tilde{p}, x, y, z), \quad \tilde{f}^{\alpha \beta}(x, y, z) = \tilde{f}^{\alpha \beta}(\tilde{p}, x, y, z).
\]
We also need the operator $L$ acting on functions $u(x, y)$ by the formula

$$L^{\alpha\beta}u(x, y) = a_{ij}^{\alpha\beta}(x, y)D_{ij}u(x, y) + b_i^{\alpha\beta}(x, y)D_iu(x, y) - c^{\alpha\beta}(x, y)u(x, y),$$

(no differentiation with respect to $y$ is involved). Notice that, if $u = u(x)$ is a smooth function on $\mathbb{R}^d$ and $u(x, y, z) := zu(x)$, then as is easy to check

$$\bar{L}^{\alpha\beta}\tilde{u}(\tilde{p}, x, y, z) = z\tau^2(\tilde{p})\bar{L}^{\alpha\beta}u(x, y, z) = z\tau^2(\tilde{p})L^{\alpha\beta}u(x, y). \quad (4.7)$$

One of consequences of Assumption [4.1] and (4.7) is that in $D \times \mathbb{R}^d \times (M^{-1}, M)$ we have

$$\bar{L}^{\alpha\beta}\bar{G}(\tilde{p}, x, y, z) \leq -1$$

for all $\tilde{p}$, where $\bar{G}(x, y, z) = M\delta_1^2 zG(x)$. In particular, this implies that $v^{\alpha\beta}(X)$ and $v(X)$ are well defined and are bounded.

Next, fix $x_0 \in D$, $y_0 \in \mathbb{R}^d$, and set

$$\bar{p}_t^{\alpha\beta} = (\tau, \pi, p)^{\alpha\beta}(x, y)_{t}^{x_0, y_0},$$

As is easy to see, $\bar{p}_t^{\alpha\beta}$ is a control adapted process in terminology of [3] (see Remark 2.3 there). For $\alpha, \in B$ and $\beta, \in B$ consider the following system of Itô’s equations

$$d\bar{x}_t = \sigma^{\alpha\beta}_t(\bar{p}_t^{\alpha\beta}, \bar{x}_t, \bar{y}_t) dw_t, \quad d\bar{y}_t = \sigma^{\alpha\beta}_t(\bar{y}_t) dt, \quad d\bar{z}_t = \bar{z}_t \nu^{\alpha\beta}(\bar{p}_t^{\alpha\beta}) dw_t.$$ (4.8)

Its solution with initial condition $X = (x, y, z)$ will be denoted by

$$\bar{X}_t^{\alpha\beta}X = (\bar{x}_t, \bar{y}_t, \bar{z}_t).$$

Observe that by uniqueness

$$\bar{X}_t^{\alpha\beta}X_{t_0, y_0, z} = \bar{X}_t^{\alpha\beta}X_{t_0, y_0, z} \quad (4.9)$$

for any $z$. Also define

$$\bar{\phi}_t^{\alpha\beta}X = \int_0^t e^{\alpha\beta}_s(\bar{p}_s^{\alpha\beta}, \bar{X}_s^{\alpha\beta}X) dt,$$

$$\bar{v}(X) = \inf_{\beta \in B} \sup_{\alpha \in A} E^X_{\alpha} \mathbb{E}^{\beta\alpha} \left[ \int_0^\tau f(\tilde{p}_t, \tilde{X}_t) e^{-\phi_t} dt + \bar{z}_t v(\bar{x}_t) e^{-\phi_t} \right],$$

where $\bar{\tau}^{\alpha\beta}X$ is the first exit time of $\bar{X}_t^{\alpha\beta}X$ from $D \times \mathbb{R}^d \times (M^{-1}, M)$.

It turns out that, in the terminology of [3], for any $C^2_\text{loc}(D)$ function $u = u(x)$, the function $zu(x)$ is $p$-insensitive in $D$ relative to $(z\tau^2(\tilde{p}), \bar{L}^{\alpha\beta})$. This follows from the fact that, if $X \in D$, then by Itô’s formula and (4.7), for $t < \bar{\tau}^{\alpha\beta}X$,

$$d(u(\bar{X}_t^{\alpha\beta}X)z_t^{\alpha\beta}X e^{-\phi_t^{\alpha\beta}X}) = e^{-\phi_t^{\alpha\beta}X} \tau^2(\bar{p}_t^{\alpha\beta})(\bar{L}^{\alpha\beta}u)(\bar{X}_t^{\alpha\beta}X, \bar{y}_t^{\alpha\beta}X, z_t^{\alpha\beta}X) dt + dm_t,$$

where $m_t$ is a local martingale starting at zero, and $z\tau^2(\tilde{p}) \in [M^{-1}\delta_1^2, M\delta_1^{-2}]$. 
Furthermore, it turns out that equation (4.7) and Assumption 2.5 also imply that for smooth \( u = u(x) \), if at a particular point \( x \) it holds that
\[
J(x) := \sup_{\alpha \in A} \inf_{\beta \in B} \left[ a_{ij}^{\alpha \beta}(x) D_{ij} u(x) + b_i^{\alpha \beta}(x) D_i u(x) - c^{\alpha \beta}(x) u(x) + f^{\alpha \beta}(x) \right] \leq 0,
\]
then with the same \( x \), any \( y \), and \( z > 0 \) we also have
\[
I(x, y, z) := \sup_{\alpha \in A} \inf_{\beta \in B} \left[ L^{\alpha \beta} \tilde{u}(x, y, z) + \tilde{f}^{\alpha \beta}(x, y, z) \right] \leq 0,
\]
where \( \tilde{u}(x, y, z) := z u(x) \). Indeed, since
\[
J(x) = \sup_{\alpha \in A} \inf_{\beta \in B} \left[ a_{ij}^{\alpha \beta}(x, x) D_{ij} u(x) + b_i^{\alpha \beta}(x, x) D_i u(x) - c^{\alpha \beta}(x, x) u(x) + f^{\alpha \beta}(x, x) \right],
\]
the inequality \( J(x) \leq 0 \) implies by Assumption 2.5 that
\[
\sup_{\alpha \in A} \inf_{\beta \in B} \left[ a_{ij}^{\alpha \beta}(x, y) D_{ij} u(x) + b_i^{\alpha \beta}(x, y) D_i u(x) - c^{\alpha \beta}(x, y) u(x) + f^{\alpha \beta}(x, y) \right] \leq 0,
\]
and it only remains to notice that the left-hand side is just \( z^{-1} I(x, y, z) \). Similarly, \( J(x) \geq 0 \) implies that \( I(x, y, z) \geq 0 \).

These facts combined imply by Theorems 2.3 and 3.1 of [10] that for all \( x \in D, y \in \mathbb{R}^d \), and \( z \in [M^{-1}, M] \) we have
\[
\hat{v}(x, y, z) = z v(x)
\]
and, for any stopping times \( \gamma \in \alpha, \beta, X \leq x^{\alpha, \beta, X} \),
\[
z v(x) = \inf_{\beta \in B} \sup_{\alpha \in A} E_X^{\alpha, \beta} \left[ \int_0^T \tilde{f}(\tilde{p}_t, \tilde{X}_t) e^{-\tilde{\phi}_t} dt + \tilde{\gamma} v(\tilde{X}_\gamma) e^{-\tilde{\phi}_\gamma} \right]. \quad (4.10)
\]

By (4.9) for \( X_0 = (x_0, y_0, z_0), z_0 \in [M^{-1}, M] \), we have
\[
X^{\alpha, \beta, X_0} = X^{\alpha, \beta, X_0}, \quad \tilde{f}^{\alpha, \beta}(\tilde{p}_t^{\alpha, \beta}, \tilde{X}_t^{\alpha, \beta, X_0}) = \tilde{f}(X_t^{\alpha, \beta, X_0}),
\]
so that \( v(x_0, y_0, z_0) = \tilde{v}(x_0, y_0, z_0) \). It follows that (4.11) holds at \( (x_0, y_0, z_0) \in D \). Outside \( D \) the equality is obvious. Finally, (4.5) follows from (4.10) and the theorem is proved.

Remark 4.1. One of assumptions in Theorems 2.3 and 3.1 of [10] is that the coefficients satisfy Assumption 2.1 (i) without \( p^{\alpha \beta}(x, y) \) there. Since \( p \) is involved in (4.6) we needed to include it in Assumption 2.1 (i) in contrast with the parameters \( r^{\alpha \beta}(x, y) \) and \( P^{\alpha \beta}(x, y) \). The same reasons caused the last requirement in Assumption 2.1 (ii). Recall that in Theorems 2.3 and 3.1 of [10] the coefficients of Itô equations are not supposed to be Lipschitz but rather uniformly continuous.
5. Estimating the Difference of Solutions of Stochastic Equations Whose Coefficients Are Close

Suppose that on \( \Omega \times (0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d \) we are given the following functions:

- \( d \times d_1 \) matrix-valued \( \sigma_t(x, y) \), \( \mathbb{R}^d \)-valued \( b_t(x, y) \), and real-valued functions \( c_t(x, y) \geq \delta_1, f_t(x, y) \), where \( \delta_1 > 0 \) is a fixed constant.

Introduce

\[
(s_t, b_t, c_t, f_t)(x) = (s_t, b_t, c_t, f_t)(x, x), \quad a_t(x) = (1/2)\sigma_t\sigma_t^*(x).
\]

**Assumption 5.1.**

(i) All the above functions are measurable with respect to the product of \( \mathcal{F} \) and Borel \( \sigma \)-algebras on \( (0, \infty), \mathbb{R}^d \), and \( \mathbb{R}^d \), they are progressively measurable as functions of \( (\omega, t) \) for each \( (x, y) \).

(ii) All the above functions are bounded by a constant \( K_0 \).

(iii) For any \( t > 0, x', x'', y \in \mathbb{R}^d \), and

\[
\eta_t = (s_t, b_t)(x, y), \quad \xi_t = (s_t, b_t)(x, y),
\]

we have

\[
|\xi_t(x', y) - \xi_t(x'', y)| + |\eta_t(x') - \eta_t(x'')| \leq K_1 |x' - x''|,
\]

where \( K_1 \) is a fixed constant. Also there exists a constant \( \varepsilon_0 > 0 \) such that for any \( t > 0 \) and \( x, y \in \mathbb{R}^d \) with \( |x - y| \leq \varepsilon_0 \) we have

\[
|c_t(x, y) - c_t(y)| + |f_t(x, y) - f_t(y)| \leq K_1 |x - y|.
\]

Observe that Assumption 5.1(iii) implies, in particular, that \(|b_t(x, y) - b_t(y)| \leq K_1 |x - y|\).

**Assumption 5.2.** There exist constants \( \mu \geq 1 \) and \( \delta \geq 2\delta_1 \) such that for all \( x, y \in \mathbb{R}^d \) satisfying \(|x - y| \leq \varepsilon_0 \) we have

\[
R_t(x, y) := \|\sigma_t(x, y) - \sigma_t(y)\|_\xi^2 + 2(x - y, b_t(x, y) - b_t(y)) - 4\mu(x - y, a_t(x)(x - y)) \leq 2(c_t(y) - \delta)\|x - y\|^2,
\]

where \( \xi = (x - y)/|x - y| \).

Fix a unit \( \xi \in \mathbb{R}^d \) and for \( \varepsilon \in [0, \varepsilon_0] \) introduce \( x^\varepsilon_t \) as a unique solution of

\[
x_t = \varepsilon \xi + \int_0^t \sigma_s(x_s, y_s) \, dw_s + \int_0^t [b_s(x_s, y_s) - 2\mu a_s(x_s)(x_s - y_s)] \, ds,
\]

where \( y_s \) is a unique solution of

\[
y_t = \int_0^t \sigma_s(y_s) \, dw_s + \int_0^t b_s(y_s) \, ds.
\]

Observe that owing to uniqueness

\[
x^\varepsilon_t = y_t.
\]

For \( \varepsilon > 0 \) define

\[
\xi^\varepsilon_t = \frac{1}{\varepsilon}(x^\varepsilon_t - x^0_t), \quad \phi_t = \int_0^t c_s(x^0_t) \, ds
\]
and for $\lambda > 0$ let

$$\kappa_{\varepsilon}(\lambda) = \inf\{t \geq 0 : |x_t^\varepsilon - x_t^0| \geq \lambda\}.$$  

Notice that $\kappa_{\varepsilon}(\lambda) = 0$ if $\lambda \leq \varepsilon$ and start with the following.

**Lemma 5.1.** For any $\lambda \in (0, \varepsilon_0]$

$$J_\varepsilon := E \int_0^{\kappa_{\varepsilon}(\lambda)} |\xi_t^\varepsilon| e^{-\phi_t + \delta t/2} dt \leq 2/\delta,$$  

(5.2)

$$I_\varepsilon := E \sup_{t < \kappa_{\varepsilon}(\lambda)} |\xi_t^\varepsilon| e^{-\phi_t + \delta t/2} \leq N,$$  

(5.3)

where $N$ is a constant depending only on $K_1$ and $\delta$.

Proof. We have

$$d\xi_t^\varepsilon = \varepsilon^{-1}[\sigma_t(x_t^\varepsilon, x_t^0) - \sigma_t(x_t^0)] dw_t$$  

$$+ \varepsilon^{-1}[b_t(x_t^\varepsilon, x_t^0) - b(x_t^0) - 2\mu_a(x_t^\varepsilon)(x_t^\varepsilon - x_t^0)] dt,$$  

(5.4)

where the magnitudes of the coefficients of $dw_t$ and $dt$ are dominated by constants times $|\xi_t^\varepsilon|$. This allows us to use Itô’s formula (cf. the proof of Theorem 5.8.7 of [7]) and obtain that $(0/0 := 0)$

$$d||\xi_t^\varepsilon||e^{-\phi_t + \delta t/2} = \frac{1}{2|x_t^\varepsilon - x_t^0|^2}[R_t(x_t^\varepsilon, x_t^0) - 2(c_t(x_t^0) - \delta/2)|x_t^\varepsilon - x_t^0|^2]e^{-\phi_t + \delta t/2} dt$$  

$$+ S_t(x_t^\varepsilon, x_t^0)e^{-\phi_t + \delta t/2} dw_t,$$

where

$$S_t(x_t^\varepsilon, x_t^0) = \frac{1}{|\xi_t^\varepsilon|^2} [\sigma_t(x_t^\varepsilon, x_t^0) - \sigma_t(x_t^0)].$$

By assumption, for $t < \kappa_{\varepsilon}(\lambda)$ we have

$$R_t(x_t^\varepsilon, x_t^0) - 2(c_t(x_t^0) - \delta/2)|x_t^\varepsilon - x_t^0|^2 \leq -\delta|x_t^\varepsilon - x_t^0|^2.$$  

It follows that for $t < \kappa_{\varepsilon}(\lambda)$

$$d||\xi_t^\varepsilon||e^{-\phi_t + \delta t/2} \leq -(\delta/2)||\xi_t^\varepsilon||e^{-\phi_t + \delta t/2} dt + \varepsilon^{-1}S_t(x_t^\varepsilon, x_t^0)e^{-\phi_t + \delta t/2} dw_t.$$  

(5.5)

In particular, (5.2) holds. Furthermore,

$$|\varepsilon^{-1}S_t(x_t^\varepsilon, x_t^0)| \leq K_1||\xi_t^\varepsilon||$$  

(5.6)

and by Davis’s inequality

$$I_\varepsilon \leq 3K_1E\left(\int_0^{\kappa_{\varepsilon}(\lambda)} ||\xi_t^\varepsilon||^2 e^{-2\phi_t + 2\delta t} dt\right)^{1/2}$$  

$$\leq 3K_1E\left(\sup_{s < \kappa_{\varepsilon}(\lambda)} ||\xi_s^\varepsilon|| e^{-\phi_s + 2\delta s/2}\right)^{1/2}\left(\int_0^{\kappa_{\varepsilon}(\lambda)} ||\xi_t^\varepsilon|| e^{-\phi_t + \delta t/2} dt\right)^{1/2} \leq NI_\varepsilon^{1/2}J_\varepsilon^{1/2},$$

which, due to (5.2), proves (5.3) and the lemma.

**Corollary 5.2.** For $\lambda > 0$ we have

$$Ee^{-\phi_{\kappa_{\varepsilon}(\lambda)} + \kappa_{\varepsilon}(\lambda)\delta/2}I_{\kappa_{\varepsilon}(\lambda) < \infty} \leq N\varepsilon/\lambda.$$  

Indeed, if \( \lambda \leq \varepsilon \), the estimate is obvious since \( \kappa_\varepsilon(\lambda) = 0 \) and for \( \lambda > \varepsilon \)
\[
\lambda e^{-\phi_{\kappa_\varepsilon}(\lambda) + \kappa_\varepsilon(\lambda)\delta/2}I_{\kappa_\varepsilon(\lambda) < \infty} = \varepsilon E|\xi_\varepsilon| e^{-\phi_{\kappa_\varepsilon}(\lambda) + \kappa_\varepsilon(\lambda)\delta/2}I_{\kappa_\varepsilon(\lambda) < \infty} \leq N\varepsilon.
\]

**Remark 5.1.** If \( \delta \geq K_1^2 \), then it follows from (5.5) and (5.6) that for \( t < \kappa_\varepsilon(\lambda) \) we have
\[
d|\xi_\varepsilon|^2 e^{-2\phi_t + \delta t} \leq dm_t,
\]
where \( m_t \) is a local martingale. Hence, for any stopping time \( \gamma \leq \kappa_\varepsilon(\lambda) \),
\[
E|\xi_\gamma|^2 e^{-2\phi_\gamma + \delta \gamma} \leq 1.
\]

Psychologically, the condition \( \delta \geq K_1^2 \) may look artificial. However, in the proof of Theorem 2.2 the parameter \( \delta \) will be, basically, sent to infinity.

Next introduce
\[
\pi_s(x, y) = \mu \sigma^s(x)(x - y)
\]
and introduce \( \rho^\varepsilon_t \) as a unique solution of
\[
\rho_t = 1 + \int_0^t \rho_s \pi_s(x_s^\varepsilon, x_0^\varepsilon) \, dw_s + \int_0^t \rho_s [c_s(x_s^\varepsilon) - c_s(x_s^\varepsilon, x_0^\varepsilon)] \, ds.
\]
Take a constant \( M > 1 \) and define
\[
\gamma_\varepsilon(M)
\]
as the first exit time of \( \rho^\varepsilon_t \) from \((M^{-1}, M)\).

Recall that \( c \geq \delta_1 \).

**Lemma 5.3.** There exists \( \lambda_1 \in (0, \varepsilon_0) \), depending only on \( \varepsilon_0, K_0, K_1 \), and \( \delta_1 \) and there exists a constant \( N \), depending only on \( K_1 \) and \( \delta_1 \), such that for \( \lambda = \lambda_1/\mu \) and \( \mu \geq 1 \) we have
\[
I := E \sup_{t < \gamma_\varepsilon(M) \wedge \kappa_\varepsilon(\lambda)} |\rho^\varepsilon_t - 1| e^{-\phi_t + \delta_1 t/2} \leq N(M \mu^2 + 1)^{1/2} \delta_1^{-1/2} \varepsilon.
\]

**Proof.** Denote \( C_t(x^\varepsilon_t, x^\varepsilon_0) = c_t(x^\varepsilon_t) - c_t(x^\varepsilon_t, x^\varepsilon_0) \) and \( \eta_t = (\rho^\varepsilon_t - 1)^2 \). Then
\[
d\eta_t = 2(\rho^\varepsilon_t - 1)\rho^\varepsilon_t \pi_t(x^\varepsilon_t, x^\varepsilon_0) \, dw_t + 2(\rho^\varepsilon_t - 1)\rho^\varepsilon_t C_t(x^\varepsilon_t, x^\varepsilon_0) \, dt + |\rho^\varepsilon_t|^2 |\pi_t(x^\varepsilon_t, x^\varepsilon_0)|^2 \, dt,
\]

\[
d\eta_t e^{-2\phi_t + \delta_1 t} = e^{-2\phi_t + \delta_1 t} \big[ 2\eta_t C_t(x^\varepsilon_t, x^\varepsilon_0) + 2(\rho^\varepsilon_t - 1)C_t(x^\varepsilon_t, x^\varepsilon_0) \big] \, dt + dm_t,
\]
where \( m_t \) is a local martingale starting at zero and, for \( t < \gamma_\varepsilon(M) \), the expression in the square brackets is less than
\[
\eta_t \big[ 2C_t(x^\varepsilon_t, x^\varepsilon_0) + \delta_1/2 + |\pi_t(x^\varepsilon_t, x^\varepsilon_0)|^2 - (2c_t(x^\varepsilon_0) - \delta_1) \big] \big[ 2(\rho^\varepsilon_t - 1)C_t(x^\varepsilon_t, x^\varepsilon_0) + (2M - 1)|\pi_t(x^\varepsilon_t, x^\varepsilon_0)|^2 \big].
\]
We have that \( |G_t| \leq K_1|x^\varepsilon_t - x^\varepsilon_0|, |\pi_t| \leq \mu K_0|x^\varepsilon_t - x^\varepsilon_0|, c \geq \delta_1, \) and \( \mu \geq 1 \) and, therefore, one can find \( \lambda_1 \in (0, \varepsilon_0) \) such that, for \( \lambda = \lambda_1/\mu \) and \( t < \kappa_\varepsilon(\lambda) \),
\[
2C_t(x^\varepsilon_t, x^\varepsilon_0) + \delta_1/2 + |\pi_t(x^\varepsilon_t, x^\varepsilon_0)|^2 - (2c_t(x^\varepsilon_0) - \delta_1) \leq 0
\]
and then
\[ d\eta e^{-2\phi_t + \delta_1 t} \leq N_1(M\mu^2 + 1)\varepsilon^2|\xi_t|e^{-2\phi_t + \delta_1 t} dt + dm_t. \]

Hence, for any bounded stopping time \( \tau \) it holds that
\[ E\eta_{\tau \wedge \gamma_\varepsilon(M) \wedge \kappa_e(\lambda)}e^{-2\phi_t \tau \wedge \gamma_\varepsilon(M) \wedge \kappa_e(\lambda)} + \delta_1(\tau \wedge \gamma_\varepsilon(M) \wedge \kappa_e(\lambda)) \]
\[ \leq N_1(M\mu^2 + 1)\varepsilon^2E \int_0^{\tau \wedge \gamma_\varepsilon(M) \wedge \kappa_e(\lambda)} |\xi_t|^2e^{-2\phi_t + \delta_1 t} dt, \]
which owing to well-known properties of such inequalities (see, for instance, Theorem 3.6.8 [7]) implies that
\[ E\sup_{t \leq \gamma_\varepsilon(M) \wedge \kappa_e(\lambda)} \eta_t^{1/2} e^{-\phi_t + \delta_1 t/2} \]
\[ \leq 3N_1(M\mu^2 + 1)^{1/2}\varepsilon E \int_0^{\gamma_\varepsilon(M) \wedge \kappa_e(\lambda)} |\xi_t|^2e^{-2\phi_t + \delta_1 t} dt)^{1/2}. \]

Owing to (5.3) and the assumption that \( \delta \geq 2\delta_1 \), the last expectation is dominated by
\[ N_1 \left( \int_0^{\infty} e^{(\delta_1 - \delta)t} dt \right)^{1/2} \leq N\delta^{-1/2}. \]

The lemma is proved.

**Corollary 5.4.** There is a constant \( N \), depending only on \( K_1 \) and \( \delta_1 \), such that for any \( M \geq 2 \) and \( \lambda = \lambda_1/\mu \)
\[ Ee^{-\phi_t \gamma_\varepsilon(M) \wedge \kappa_e(\lambda)} \leq N[\mu + (M\mu^2 + 1)^{1/2}\delta^{-1/2}]\varepsilon. \] (5.8)

To prove (5.8), it suffices to notice that
\[ Ee^{-\phi_t \gamma_\varepsilon(M) \wedge \kappa_e(\lambda)} I_{\gamma_\varepsilon(M) < \kappa_e(\lambda)} \leq M(M - 1)^{-1}E|\rho_t^\varepsilon(M) - 1|e^{-\phi \gamma_\varepsilon(M) I_{\gamma_\varepsilon(M) < \kappa_e}} \]
\[ \leq M(M - 1)^{-1}E \sup_{t \leq \gamma_\varepsilon(M) \wedge \kappa_e(\lambda)} |\rho_t^\varepsilon - 1|e^{-\phi t} \]
and then use Corollary 5.2 and recall that \( c \geq \delta_1 \).

Now for \( \lambda = \lambda_1/\mu, \varepsilon \in (0, \varepsilon_0] \), and \( M \geq 2 \) take a stopping time
\[ \tau \leq \gamma_\varepsilon(M) \wedge \kappa_e(\lambda). \]

Also take a function \( g_t(x) \), which is measurable in \((\omega, t, x)\) and such that \( |g| \leq K_0 \) and introduce
\[ v^\varepsilon = E\left[ \int_0^\tau z_t^\varepsilon f(x_t^\varepsilon, x_0^\varepsilon) e^{-\phi_t} dt + z_t^\varepsilon g_t(x_t^\varepsilon) e^{-\phi_t} \right], \]
where
\[ \phi_t^\varepsilon = \int_0^t c_s(x_s^\varepsilon, x_0^\varepsilon) ds \]
and \( z_t^\varepsilon \) is defined as a unique solution of
\[ z_t = 1 + \int_0^t z_s\pi_s^\varepsilon(x_s^\varepsilon, x_0^\varepsilon) dw_s. \]
Finally, define
\[ v^0 = E \left[ \int_0^\tau f(x_t^0) e^{-\phi_t} \, dt + g_\tau(x_0^0) e^{-\phi_\tau} \right]. \]

**Theorem 5.5.** Suppose that there is a constant \( N_0 \) such that
\[ E|g_\tau(x_\tau^\varepsilon) - g_\tau(x_\tau^0)|e^{-\phi_\tau} I_{x<\gamma(M)\wedge \alpha(\lambda)} \leq N_0 \varepsilon. \] (5.9)
Then there exists a constant \( N \), depending only on \( K_0, K_1 \), and \( \delta_1 \), such that for \( \lambda = \lambda_1/M \) we have
\[ |v^\varepsilon - v^0| \leq N_0 \varepsilon + N[\mu + (M\mu^2 + 1)^{1/2} \delta^{-1/2} + \delta^{-1}] \varepsilon. \]

Proof. First notice that
\[ z_t^\varepsilon e^{-\phi_t} = \rho_t^\varepsilon e^{-\phi_t}, \]
so that
\[ \left| \int_0^\tau [z_t^\varepsilon f(x_t^\varepsilon, x_t^0) e^{-\phi_t} - f(x_t^0) e^{-\phi_t}] \, dt \right| \leq I_{\varepsilon} + J_{\varepsilon}, \]
where
\[ I_{\varepsilon} = \int_0^\tau |\rho_t^\varepsilon - 1| |f(x_t^\varepsilon, x_t^0)| e^{-\phi_t} \, dt, \]
\[ J_{\varepsilon} = \int_0^\tau |f(x_t^\varepsilon, x_t^0) - f(x_t^0)| e^{-\phi_t} \, dt. \]
By Lemma 5.3
\[ EI_{\varepsilon} \leq NE \sup_{s \leq \tau} |\rho_s^\varepsilon - 1| e^{-\phi_s + \delta_1 s/2} \int_0^\infty e^{-\delta_1 t/2} \, dt \]
\[ \leq N(M\mu^2 + 1)^{1/2} \delta^{-1/2} \varepsilon. \]
By Lemma 5.1
\[ EJ_{\varepsilon} \leq N\varepsilon E \int_0^\tau |\xi_t^\varepsilon| e^{-\phi_t} \, dt \leq N\varepsilon/\delta. \]
Next
\[ E|z_t^\varepsilon g_\tau(x_\tau^\varepsilon) e^{-\phi_t} - g_\tau(x_\tau^0) e^{-\phi_\tau}| = E|\rho_t^\varepsilon g_\tau(x_\tau^\varepsilon) - g_\tau(x_\tau^0)| e^{-\phi_t} \]
\[ \leq K_0 E|\rho_t^\varepsilon - 1| e^{-\phi_t} + E|g_\tau(x_\tau^\varepsilon) - g_\tau(x_\tau^0)| e^{-\phi_\tau}, \]
where the first term is estimated as above and, owing to (5.9), the second term is dominated by
\[ N_0 \varepsilon + E|g_\tau(x_\tau^\varepsilon) - g_\tau(x_\tau^0)| e^{-\phi_\tau} I_{x=\gamma(M)\wedge \alpha(\lambda)} \]
\[ \leq N_0 \varepsilon + 2K_0 E e^{-\phi_{x<\gamma(M)\wedge \alpha(\lambda)}} \leq N_0 \varepsilon + N[\mu + (M\mu^2 + 1)^{1/2} \delta^{-1/2}] \varepsilon. \]
with the second inequality following from Corollary 5.4. The theorem is proved.
6. Proof of Theorem 2.1

According to Remark 2.1 in the proof of Theorem 2.1 we may assume that \(c^{\alpha\beta}(x) \geq \delta_1\).

First, we estimate the Lipschitz constant of \(v\) on the boundary when \(D \neq \mathbb{R}^d\).

**Lemma 6.1.** Let \(D\) be bounded and satisfy the uniform exterior ball condition. Let \(x \in \mathbb{R}^d\) and \(y \not\in D\). Then there is a constant \(N\) depending only on \(D, K_0, \) and \(\|g\|_{C^2(\mathbb{R}^d)}\), such that
\[
|v(x) - v(y)| \leq N|x - y|.
\]

**Proof.** If \(x \not\in D\), then
\[
|v(x) - v(y)| = |g(x) - g(y)| \leq N|x - y|.
\]
Therefore in the rest of the proof we assume that \(x \in D\). Then observe that by Itô’s formula we have
\[
v(x) = g(x) + \inf_{\beta \in \mathcal{B}} \sup_{\alpha \in \mathcal{A}} E_x^\alpha \beta(\cdot) \int_0^T [Lg(x_t) + f(x_t)] e^{-\phi_t} dt.
\]

(6.1)

It is well known that, in light of the boundedness of \(L^{\alpha\beta}g + f^{\alpha\beta}\) and \(D\) and the uniform exterior ball condition, the expectations in (6.1) by magnitude are dominated by a constant times \(\text{dist}(x, \partial D) \leq |x - y|\). This proves the lemma since \(v(y) = g(y)\) and \(|g(x) - g(y)| \leq N|x - y|\).

**Proof of Theorem 2.1.** In Section 4 take \(r \equiv 1, \ p \equiv 0, \ P \equiv I, \ \pi^{\alpha\beta}(x, y) = (\sigma^{\alpha\beta}(x))^* (x - y), \)
where the constant \(\mu \geq 1\) is chosen to be such that (5.1) with \(\delta = 1\) holds for all \(\alpha, \beta \in \mathcal{A}, x, y\). This is possible since \(\sigma\) and \(b\) are Lipschitz continuous and \(a\) is uniformly nondegenerate. In Section 4 we required \(\pi^{\alpha\beta}(x, y)\) to be bounded and Lipschitz continuous with respect to \(x\). Since we will be only concerned with its values for \(|x - y| \leq 1\), we can appropriately modify the above \(\pi^{\alpha\beta}(x, y)\) for \(|x - y| \geq 1\) keeping the same notation.

Then for a unit \(\xi \in \mathbb{R}^d, \ v \geq 0, \ \alpha, \beta \in \mathcal{A},\) and \(\beta \in \mathcal{B}\) introduce \(x_t^{\alpha\beta}(\epsilon)\) as a unique solution of
\[
x_t = \epsilon \xi + \int_0^t \sigma^{\alpha\beta}(x_s) dw_s + \int_0^t \left[ b^{\alpha\beta}(x_s) - \sigma^{\alpha\beta}(x_s) \pi^{\alpha\beta}(x_s, y_s) \right] ds,
\]
where
\[
y_s = x_s^{\alpha\beta}(\epsilon).
\]

Next introduce
\[
\phi_t^{\alpha\beta}(\epsilon) = \int_0^t c^{\alpha\beta}(x_s^{\beta\alpha}(\epsilon)) ds
\]
and let $z_t^{\alpha, \beta, 0}(\varepsilon)$ be a unique solution of

$$z_t = 1 + \int_0^t z_s [\pi^{\alpha, \beta, 0}(x_s^{\alpha, \beta, 0}(\varepsilon), x_s^{\alpha, \beta, 0}(0))]^* dw_s.$$  

Keeping in mind that $\mu$ is already fixed, set $\delta_1 := \varepsilon_1 = 1$, take $\lambda$ from Lemma 5.3, fix $\varepsilon \in (0, 1]$, and introduce

$$\tau_\varepsilon^{\alpha, \beta, 0} = \inf \{ t \geq 0 : x_t^{\alpha, \beta, 0}(\varepsilon) \not\in D \},$$

$$\gamma_\varepsilon^{\alpha, \beta, 0} = \inf \{ t \geq 0 : x_t^{\alpha, \beta, 0}(\varepsilon) \in \phi_t^{\alpha, \beta, 0}(0) - \phi_t^{\alpha, \beta, 0}(\varepsilon) \not\in (1/2, 2) \},$$

$$\kappa_\varepsilon^{\alpha, \beta, 0} = \inf \{ t \geq 0 : |x_t^{\alpha, \beta, 0}(\varepsilon) - x_t^{\alpha, \beta, 0}(0)| \geq \lambda \},$$

$$\gamma_\varepsilon^{\alpha, \beta, 0} = \tau_\varepsilon^{\alpha, \beta, 0} \wedge \tau_0^{\alpha, \beta, 0} \wedge \kappa_\varepsilon^{\alpha, \beta, 0} \wedge \gamma_\varepsilon^{\alpha, \beta, 0}.$$  

By Theorem 4.1

$$v(\varepsilon \xi) = \inf_{\beta \in \mathbb{B}} \sup_{\alpha \in \mathbb{A}} \int_0^\gamma z_t(\varepsilon) f(x_t(\varepsilon)) e^{-\phi_t(\varepsilon)} dt + z_\gamma(\varepsilon) v(x_\gamma(\varepsilon)) e^{-\phi_\gamma(\varepsilon)}.$$  

(6.2)

Next we fix $\alpha \in \mathbb{A}$ and $\beta \in \mathbb{B}$ and in Section 5 use the functions

$$(\sigma_t, b_t, c_t, f_t)(x, y) = (\sigma, b, c, f)^{\alpha \beta}(x).$$

Observe that in the expectation

$$E_0^{\alpha, \beta} \left[ \int_0^\gamma z_t(\varepsilon) f(x_t(\varepsilon)) e^{-\phi_t(\varepsilon)} dt + z_\gamma(\varepsilon) v(x_\gamma(\varepsilon)) e^{-\phi_\gamma(\varepsilon)} \right]$$

one can replace $x_t^{\alpha, \beta, 0}(\varepsilon)$ with $x_t^\varepsilon$ since both satisfy the same equation on $[0, \gamma^{\alpha, \beta, 0}]$ and by Theorem 5.5 we get that

$$\left| E_0^{\alpha, \beta} \left[ \int_0^\gamma z_t(\varepsilon) f(x_t(\varepsilon)) e^{-\phi_t(\varepsilon)} dt + z_\gamma(\varepsilon) v(x_\gamma(\varepsilon)) e^{-\phi_\gamma(\varepsilon)} \right] \right|$$

$$\leq N\varepsilon + E_0^{\alpha, \beta} |v(x_\gamma(\varepsilon)) - v(x_\gamma(0))| e^{-\phi_\gamma} I_{\gamma < \gamma^{\alpha, \beta, 0}},$$  

(6.3)

If $t = \gamma^{\alpha, \beta, 0} < \gamma^{\alpha, \beta, 0} \wedge \kappa_\varepsilon^{\alpha, \beta, 0}$, then $(D \neq \mathbb{R}^d$ and at least one of $x_t^{\alpha, \beta, 0}(\varepsilon)$ and $x_t^{\alpha, \beta, 0}(0)$ is outside $D$ and by Lemma 6.1 we obtain

$$E_0^{\alpha, \beta} |v(x_\gamma(\varepsilon)) - v(x_\gamma(0))| e^{-\phi_\gamma} I_{\gamma < \gamma^{\alpha, \beta, 0}} \leq N E_0^{\alpha, \beta} |x_\gamma(\varepsilon) - x_\gamma(0)| e^{-\phi_\gamma} I_{\gamma < \gamma^{\alpha, \beta, 0}}$$

$$= N\varepsilon E_0^{\alpha, \beta} |\xi_\gamma(\varepsilon)| e^{-\phi_\gamma} I_{\gamma < \gamma^{\alpha, \beta, 0}} \leq N\varepsilon E_0^{\alpha, \beta} \sup_{t < \kappa_\varepsilon} |\xi_t(\varepsilon)| e^{-\phi_t},$$

where $\xi_t^{\alpha, \beta, 0}(\varepsilon) = x_t^{\alpha, \beta, 0}(\varepsilon) - x_t^{\alpha, \beta, 0}(0)$. By using Lemma 5.1, 6.3, and the fact that $\alpha$ and $\beta$ in the above argument are arbitrary, we see that $|v(\varepsilon \xi) - v(0)| \leq N\varepsilon$. Similarly one proves that $|v(x + \varepsilon \xi) - v(x)| \leq N\varepsilon$ for any $x$, which is what we need. The theorem is proved.
7. Proof of Theorem 2.2

In contrast with Section 6, where we used \( \delta = 1 \), here \( \delta \) will be chosen large. We begin with the following.

**Lemma 7.1.** Let \( D \) be a bounded domain satisfying the uniform exterior ball condition and let \( \|g\|_{C^2(\mathbb{R}^d)} < \infty \). For \( R \in (0, 1] \) let \( B_R = \{ x : |x| \leq R \} \). Assume that for an \( R \) we have \( B_R \subset D \) and denote by \( L_R \) the Lipschitz constant of \( v \) in \( B_R \) (finite by Theorem 2.7). Finally assume that \( |v| \leq K_0 \) in \( B_R \).

Then for any \( \delta \geq K_1^2 + 4K_0^2 + 2 \) we have

\[
\lim_{x \to 0} \frac{|v(x) - v(0)|}{|x|} \leq N\delta R^{-1} + Ne^{-\nu \sqrt{\delta} L_R},
\]

where \( N \) and \( \nu > 0 \) depend only on \( d \), \( K_0 \), \( K_1 \), and \( \delta_0 \).

Proof. First suppose that \( R = 1 \). Observe that by the dynamic programming principle

\[
v(x) = \inf_{\beta \in \mathfrak{B}} \sup_{\alpha \in \mathfrak{A}} E_x^{\alpha, \beta(\alpha)} \left[ \int_0^{\tau_1} f(x_t) e^{-\phi_t} \, dt + v(x_{\tau_1}(\varepsilon)) e^{-\phi_{\tau_1}} \right],
\]

where \( \tau_1^{\alpha, \beta, \varepsilon} \) is the first exit time of \( x_t^{\alpha, \beta, \varepsilon} \) from \( B_1 \).

Remark 2.1 allows us to rewrite (7.2) by using a global barrier for \( B_1 \) for a slightly modified \( v \). Obviously, if we can prove (7.1) with \( R = 1 \) for such modification, then we will have it also for the original function. Hence, concentrating on (7.1) and the case \( R = 1 \), without losing generality we may assume that \( c^{\alpha, \beta} \geq 1 \).

Set \( \mu = \delta_0^{-1} \delta + N_0 \), where \( N_0 \) depending only on \( K_1 \), \( \delta_0 \), and \( d \) is chosen in such a way that (5.1) is satisfied with

\[(\sigma_1, b_1)(x, y) = (\sigma, b)^{\alpha_1 \beta}(x)\]

for all \( \alpha \in \mathfrak{A}, \beta \in \mathfrak{B}, x, y, \) and \( \delta > 0 \).

We use the notation from the proof of Theorem 2.1 in Section 6 and write (5.2) with

\[
\gamma^{\alpha, \beta, 0} = \tau_1^{\alpha, \beta, 0}(\varepsilon) \land \tau_1^{\alpha, \beta, 0}(0) \land \kappa^{\alpha, \beta, 0} \land \gamma^{\alpha, \beta, 0},
\]

where \( \tau_1^{\alpha, \beta, 0}(\varepsilon) \) is the first exit time of \( x_t^{\alpha, \beta, 0}(\varepsilon) \) from \( B_1 \).

As in the proof of Theorem 2.1 by Theorem 5.5 (with \( \tau = \gamma^{\alpha, \beta, 0} \) there) we get that (recall that \( M = 2 \) and \( \mu \) is of order \( \delta \) if \( \delta \geq 1 \))

\[
|v(\varepsilon\xi) - v(0)| \leq N\delta \varepsilon + S_\varepsilon,
\]

where \( N \) depends only on \( K_0 \), \( K_1 \), and \( \delta_0 \) (recall that \( \delta_1 = 1 \) and)

\[
S_\varepsilon := \sup_{\alpha, \beta} E_0^{\alpha, \beta} |v(x_{\tau_1}(\varepsilon)) - v(x_{\tau_1}(0))| e^{-\phi_{\tau_1}} I_{\tau_1(\varepsilon) \land \tau_1(0) < \gamma^{\alpha, \beta, 0} \land \kappa^{\alpha, \beta, 0}}
\]

\[
\leq \varepsilon L_1 \sup_{\alpha, \beta} E_0^{\alpha, \beta} \xi_{\tau_1(\varepsilon) \land \tau_1(0) (\varepsilon)} e^{-\phi_{\tau_1}(\varepsilon) \land \tau_1(0)} I_{\tau_1(\varepsilon) \land \tau_1(0) < \kappa^{\alpha, \beta, 0}}.
\]
Observe that for any $T > 0$ by Lemma \[5.1\] and Remark \[5.1\] ($\delta \geq K_0^2$)
\[
E_0^{\alpha, \beta} |\xi_{\tau_1}(\varepsilon)\rangle e^{-\phi_{\tau_1}(\varepsilon)} I_{\tau_1(\varepsilon) < \varepsilon} = E_0^{\alpha, \beta} |\xi_{\tau_1}(\varepsilon)\rangle e^{-\phi_{\tau_1}(\varepsilon)} I_{\tau_1(\varepsilon) < \varepsilon \wedge T} + E_0^{\alpha, \beta} |\xi_{\tau_1}(\varepsilon)\rangle e^{-\phi_{\tau_1}(\varepsilon)} I_{\tau_1(\varepsilon) < \varepsilon} I_{\tau_1(\varepsilon) \geq T}
\leq (E_0^{\alpha, \beta} I_{\tau_1(\varepsilon) < T})^{1/2} + e^{-\delta T/2} E_0^{\alpha, \beta} \sup_{t < \kappa_\varepsilon} |\xi_t(\varepsilon)\rangle e^{-\phi_t + \delta t/2}
\leq N e^{-\delta T/2} + (E_0^{\alpha, \beta} I_{\tau_1(\varepsilon) < T})^{1/2}.
\]
Similarly,
\[
E_0^{\alpha, \beta} |\xi_{\tau_1(0)}(\varepsilon)\rangle e^{-\phi_{\tau_1}(0)} I_{\tau_1(0) < \varepsilon} \leq N e^{-\delta T/2} + (E_0^{\alpha, \beta} I_{\tau_1(0) < T})^{1/2}.
\]
One knows that if the starting point of a diffusion process with coefficients bounded by $K_0$ is in the ball of radius $\varepsilon < 1/2$, then the probability that the process will exit from $B_1$ before time $T$ is less than $N \exp(-\nu/T)$ if $K_0 T \leq 1/2$, where $N$ and $\nu$ depend only on $K_0$ and $d$. This result is easily obtained by using the McKeen estimate (see, for instance, Corollary IV.2.9 of \[7\]) for each coordinate of the process from which one subtracts the drift term. Hence (with another $\nu$)
\[
S_\varepsilon \leq \varepsilon L_1 (N e^{-\delta T/2} + N e^{-\nu/T}).
\]
For $T = \delta^{-1/2}$ (so that $K_0 T \leq 1/2$ since $\delta \geq 4K_0^2$) we get that (yet with another $\nu$)
\[
S_\varepsilon \leq \varepsilon L_1 N e^{-\nu \sqrt{\delta}}
\]
and the result follows in case $R = 1$.

Once \[7.1\] is proved for $R = 1$, for $R \in (0, 1)$ it follows by using dilations (see Remark 2.5 of \[10\]), which allow us to keep the constants $\delta_0, K_0$, and $K_1$ (actually, after dilations the constant $K_1$ can be taken even smaller than the original one). The lemma is proved.

**Proof of Theorem 2.2** First suppose that $\|g\|_{C^2(\mathbb{R}^d)} < \infty$ and that for an $R_0 > 0$ we have $B_{2R_0} \subset D$. Estimate \[7.1\] can be applied to any point rather than only 0 and it shows that for any $R' < R'' \leq 2R_0$ and $\delta \geq K_1^2 + 4K_0^2 + 2$ we have
\[
L_{R'} \leq N \delta / (R'' - R') + N_1 e^{-\nu \sqrt{\delta}} L_{R''}.
\]
We apply this inequality to $R' = R_n$ and $R'' = R_{n+1}$, where $R_n$, $n \geq 1$ are defined by
\[
R_n = R_0 + R_0 \sum_{i=1}^{n} \frac{\chi}{i^2}
\]
and $\chi$ is such that $R_n \to 2R_0$ as $n \to \infty$. We also take and fix $\delta \geq K_1^2 + 4K_0^2 + 2$ so large that $N_1 e^{-\nu \sqrt{\delta}} \leq 1/2$. Then for a constant $N_0$ depending only on $\delta_0, K_0, K_1$, and $d$ and all $n \geq 0$ we get that
\[
L_{R_n} \leq N_0 R_0^{-1} (n + 1)^2 + 2^{-1} L_{R_{n+1}},
2^{-n} L_{R_n} \leq 2^{-n} N_0 R_0^{-1} (n + 1)^2 + 2^{-\left(n+1\right)} L_{R_{n+1}},
\]
\[
\sum_{n=0}^{\infty} 2^{-n} L_{R_n} \leq N_0 R_0^{-1} \sum_{n=0}^{\infty} 2^{-n} (n+1)^2 + \sum_{n=0}^{\infty} 2^{-(n+1)} L_{R_{n+1}}
\]

and \( L_{R_0} \leq N_0 IR_0^{-1} \), where

\[
I = 2 \sum_{n=1}^{\infty} 2^{-n} n^2.
\]

One can do the same estimate for any ball inside \( D \) not necessarily centered at the origin and this yields the desired result in case \( \|g\|_{C^2(\mathbb{R}^d)} < \infty \). In the general case where \( g \) is only continuous it suffices to use appropriate approximations of it by smooth functions. The theorem is proved.

8. Proof of Theorem 2.3

First of all we point out that the assertion of Lemma 6.1 continues to hold true with only one difference that \( N \) depends only on \( K_0 \), \( G \), \( d \), and \( \|g\|_{C^2(\mathbb{R}^d)} \). The proof remains the same with Itô’s formula showing that the expectations in (6.1) are bounded by \( N G(x) \). The remaining arguments follow the ones from Section 6 almost word for word.

In Section 1 for \( |x - y| \leq 1 \) take

\[
\pi^{\alpha\beta}(x, y) = \mu(\sigma^{\alpha\beta}(y))s(x - y)
\]

and extend it appropriately for \( |x - y| > 1 \).

Then for a unit \( \xi \in \mathbb{R}^d \), \( \varepsilon \geq 0 \), \( \alpha \in \mathcal{A} \), and \( \beta \in \mathcal{B} \) introduce \( x^{\alpha\beta}_t(\varepsilon) \) as a unique solution of equation (4.2) with initial condition \( \xi \varepsilon \) and

\[
y_s = x^{\alpha\beta}_s.
\]

Observe that \( x^{\alpha\beta}_t(0) = x^{\alpha\beta}_t \). Then define \( z^{\alpha\beta}_t(\varepsilon) \), \( r^{\alpha\beta}_t \), \( \gamma^{\alpha\beta}_t \), \( \kappa^{\alpha\beta}_t \), and \( \gamma^{\alpha\beta}_t \) in the same way as in Section 6 and use Theorem 4.1 to get that

\[
v(\varepsilon \xi) = \inf_{\beta \in \mathcal{B}} \sup_{\alpha \in \mathcal{A}} E_0^{\alpha\beta} \left[ z^{\gamma}(\varepsilon) v(x^{\gamma}(\varepsilon)) e^{-\phi(\varepsilon)} \right]
\]

\[
+ \int_0^{\gamma} z^{\gamma}(\varepsilon) \dot{f}(x^{\gamma}(\varepsilon), x(0)) e^{-\phi(\varepsilon)} dt,
\]

where

\[
\phi^{\alpha\beta}_t(\varepsilon) = \int_0^t c^{\alpha\beta}(x^{\alpha\beta}_s(\varepsilon), x^{\alpha\beta}_s(0)) ds.
\]

Fix \( \alpha \in \mathcal{A} \) and \( \beta \in \mathcal{B} \) and in Section 5 use the functions

\[
(\sigma_t, b_t, c_t, f_t)(x, y) = (\hat{\sigma}, b, \hat{c}, \hat{f})^{\alpha\beta}_t(x, y).
\]

Observe that Assumption 5.2 is satisfied owing to Assumption 2.3.

Furthermore, for \( t \leq \gamma^{\alpha\beta} \) the processes \( x^{\gamma}_t \) and \( y_t \) coincide with \( x^{\alpha\beta}_t(\varepsilon) \) and \( x^{\alpha\beta}_t(0) \), respectively, since they satisfy the same equations, respectively. It follows that in the expectation

\[
E_0^{\alpha\beta} \left[ \int_0^{\gamma} z^{\gamma}(\varepsilon) f(x^{\gamma}(\varepsilon), x(0)) e^{-\phi(\varepsilon)} dt + z^{\gamma}(\varepsilon) v(x^{\gamma}(\varepsilon)) e^{-\phi(\gamma)(\varepsilon)} \right]
\]
one can replace $x^\alpha_2(\varepsilon)$ with $x^\varepsilon$ and by Theorem 5.5 we get that
\[
|E_0^{\alpha, \beta} \left[ \int_0^\gamma z_t(\varepsilon)f(x_t(\varepsilon), x_t(0))e^{-\phi_t(\varepsilon)} dt + z_\gamma(\varepsilon)v(x_\gamma(\varepsilon))e^{-\phi_\gamma(\varepsilon)} \right] - E_0^{\alpha, \beta} \left[ \int_0^\gamma f(x_t)e^{-\phi_t} dt + v(x_\gamma)e^{-\phi_\gamma} \right] \leq N \varepsilon + E_0^{\alpha, \beta} |v(x_\gamma(\varepsilon)) - v(x_\gamma(0))|e^{-\phi_\gamma} I_{\gamma < \gamma \vee \kappa \varepsilon}, \tag{8.1}
\]

If $t = \gamma^{\alpha, \beta} \varepsilon < \gamma^{\alpha, \beta} \vee \kappa \varepsilon$, then at least one of $x^{\alpha, \beta}_t(\varepsilon)$ and $x^{\alpha, \beta}_t(0)$ is outside $D$ and by Lemma 6.1 we obtain
\[
E_0^{\alpha, \beta} |v(x_\gamma(\varepsilon)) - v(x_\gamma(0))|e^{-\phi_\gamma} I_{\gamma < \gamma \vee \kappa \varepsilon} \leq NE_0^{\alpha, \beta} |x_\gamma(\varepsilon) - x_\gamma(0)|e^{-\phi_\gamma} I_{\gamma < \gamma \vee \kappa \varepsilon}
= \varepsilon E_0^{\alpha, \beta} |\xi_\gamma(\varepsilon)|e^{-\phi_\gamma} I_{\gamma < \gamma \vee \kappa \varepsilon} \leq \varepsilon E_0^{\alpha, \beta} \sup_{\xi \in \kappa \varepsilon} |\xi_\gamma(\varepsilon)|e^{-\phi_\gamma},
\]
where $v^{\alpha, \beta}_t(\varepsilon) = x^{\alpha, \beta}_t(\varepsilon) - x^{\alpha, \beta}_t(0)$. By using Lemma 5.1 (8.1) and the fact that $\alpha$ and $\beta$ in the above argument are arbitrary we see that $|v(\varepsilon \xi) - v(0)| \leq N \varepsilon$. Similarly one proves that $|v(x + \varepsilon \xi) - v(x)| \leq N \varepsilon$ for any $x$, which is what we need. The theorem is proved.

9. PROOF OF THEOREM 2.3

Obviously $v \leq v_K$. To estimate $v_K - v$ from above define
\[
d_K = \sup_{x \in \mathbb{R}^d}(v_K - v), \quad \lambda = \sup_{\alpha, \beta, x} c^{\alpha, \beta}(x).
\]

By the dynamic programming principle (see Theorem 3.1 in [10])
\[
v_K(x) = \inf_{\beta \in B} \sup_{\alpha \in A} E_x^{\alpha, \beta} \left[ v_K(x_1)e^{-\lambda} + \int_0^1 \{f_K + (\lambda - c)v\}(x_t)e^{-\lambda} dt \right].
\]

Observe that
\[
e^{-\lambda} + \int_0^1 [\lambda - c^{\alpha, \beta}(x_t)e^{-\lambda} dt \leq e^{-\lambda} + \int_0^1 (\lambda - \delta_1)e^{-\lambda} dt =: \kappa < 1.
\]

Hence,
\[
v_K(x) \leq \inf_{\beta \in B} \sup_{\alpha \in A} E_x^{\alpha, \beta} \left[ v(x_1)e^{-\lambda} + \int_0^1 \{f_K + (\lambda - c)v\}(x_t)e^{-\lambda} dt \right] + \kappa d_K.
\]

Now take a sequence $x^n$ maximizing $v_K - v$ and take $\beta^n \in B$ such that
\[
v(x^n) \geq \sup_{\alpha \in A} E_{x^n}^{\alpha, \beta^n} \left[ \int_0^1 (f + (\lambda - c)v)(x_t)e^{-\lambda} dt + e^{-\lambda} v(x_1) \right] - 1/n. \quad (9.1)
\]

Also define $\pi \alpha = \alpha$ if $\alpha \in A_1$ and $\pi \alpha = \alpha^*$ if $\alpha \in A_1$, where $\alpha^*$ is a fixed element of $A_1$, and find $\alpha^n \in A$ such that
\[
v_K(x^n) \leq E_{x^n}^{\alpha^n, \beta^n} \left[ v(x_1)e^{-\lambda} + \int_0^1 \{f_K + (\lambda - c)v\}(x_t)e^{-\lambda} dt \right] + \kappa d_K + 1/n
\]
\begin{equation}
E^\alpha_n \beta^\alpha(\pi \alpha^n) [v(x_1)e^{-\lambda} + \int_0^1 \{f + (\lambda - c)v\}(x_t)e^{-\lambda t} dt] - KR_n + \kappa d_K + 1/n,
\end{equation}

where

\[R_n = E \int_0^1 e^{-\lambda t} I_{\alpha^n_t \in A_2} dt.\]

By Lemma 5.3 of [10] for any \(\alpha \in \hat{\mathfrak{A}}, \beta \in \mathfrak{B},\) and \(x \in \mathbb{R}^d\) we have

\[E \sup_{t \leq 1} |x_t^{\alpha, \beta x} - x_t^{\alpha, \beta x}| \leq N(E_x^{\alpha, \beta} \int_0^1 e^{-t} I_{\alpha^n_t \in A_2} dt)^{1/2},\]

where the constant \(N\) depends only on \(K_0, K_1,\) and \(d.\) We use this and since \(c, f, v\) are Lipschitz continuous, we get from (9.2) and (9.1)

\[v_K(x^n) + (K - N_0)R_n \leq E^\alpha_n \beta^\alpha(\pi \alpha^n) [v(x_1)e^{-\lambda} + \int_0^1 \{f + (\lambda - c)v\}(x_t)e^{-\lambda t} dt] + \kappa d_K + 1/n + NR_n^{1/2} \leq v(x^n) + \kappa d_K + 2/n + NR_n^{1/2},\]

where the constant \(N_0\) depends only on the supremums of \(c, v,\) and \(f.\) Hence

\[v_K(x^n) - v(x^n) - \kappa d_K + (K - N_0)R_n \leq 2/n + NR_n^{1/2}.\]

When \(n\) is large enough, \(v_K(x^n) - v(x^n) - \kappa d_K \geq 0\) because of the way we chose \(x^n\) and the fact that \(\kappa < 1.\) It follows that for \(n\) large enough

\[(K - N_0)R_n \leq 2/n + NR_n^{1/2},\]

which for \(K \geq 2N_0 + 1\) implies that \(KR_n \leq 4/n + NR_n^{1/2},\) so that, if \(KR_n \geq 8/n,\) then \(KR_n \leq NR_n^{1/2}\) and \(R_n \leq N/K^2.\) Thus,

\[R_n \leq 8/(nK) + N/K^2,\]

which after coming back to (9.3) finally yields

\[(1 - \kappa)d_K = \lim_{n \to \infty} [v_K(x^n) - v(x^n)] - \kappa d_K \leq N/K,\]

and the theorem is proved.

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E-mail address: krylov@math.umn.edu

127 Vincent Hall, University of Minnesota, Minneapolis, MN, 55455