A Canonical Characterization of the Family of Barriers in General Graphs

Nanao Kita
Keio University, Yokohama, Japan
kita@a2.keio.jp

Abstract. Given a graph, a barrier is a set of vertices determined by the Berge formula—the min-max theorem characterizing the size of maximum matchings. The notion of barriers plays important roles in numerous contexts of matching theory, since barriers essentially coincides with dual optimal solutions of the maximum matching problem. In a special class of graphs called the elementary graphs, the family of maximal barriers forms a partition of the vertices; this partition was found by Lovász and is called the canonical partition. The canonical partition has produced many fundamental results in matching theory, such as the two ear theorem. However, in non-elementary graphs, the family of maximal barriers never forms a partition, and there has not been the canonical partition for general graphs. In this paper, using our previous work, we give a canonical description of structures of the odd-maximal barriers—a class of barriers including the maximal barriers—for general graphs; we also reveal structures of odd components associated with odd-maximal barriers. This result of ours can be regarded as a generalization of Lovász’s canonical partition.

1 Introduction

A matching of a graph $G$ is a set of edges no two of which have common vertices. A matching of cardinality $|V(G)|/2$ (resp. $|V(G)|/2 - 1$) is called a perfect matching (resp. a near-perfect matching). We call a graph factorizable if it has at least one perfect matching. Now let $G$ be a factorizable graph. An edge $e \in E(G)$ is called allowed if there is a perfect matching containing $e$. Let $\hat{M}$ be the union of all the allowed edges of $G$. For each connected component $C$ of the subgraph of $G$ determined by $\hat{M}$, we call the subgraph of $G$ induced by $V(C)$ as factor-connected component or factor-component for short. The set of all the factor-components of $G$ is denoted by $\mathcal{G}(G)$. Therefore, a factorizable graph is composed of factor-components and some edges joining between different factor-components. A factorizable graph with exactly one factor-component is called elementary.

Matching theory is of central importance in graph theory and combinatorial optimization, with numerous practical applications [1]. In matching theory, the notion of barriers plays significant roles. Given a graph, we call a connected component of it with an odd (resp. even) number of vertices odd component (resp. even component). Given $X \subseteq V(G)$ of a graph $G$, we denote
as \( q_G(X) \) the number of odd components that the graph resulting from deleting \( X \) from \( G \) has; we denote the cardinality of a maximum matching of \( G \) as \( \nu(G) \). There is a min-max theorem called the Berge formula \[2\] that for any graph \( G \), \( |V(G)| - 2\nu(G) = \max\{q_G(X) - |X| : X \subseteq V(G)\} \). A set of vertices that attains the maximum in the right side of the equation is called a barrier. Roughly speaking, barriers essentially coincide with dual optimal solutions of the maximum matching problem, and decompose graphs so that one can see the structures of maximum matchings. However, compared to numerous results on maximum matchings, “much less is known about barriers \[2\].”

There is a structure of elementary graphs called the canonical partition; Kotzig first introduced it as the equivalence classes of a certain equivalence relation, and later Lovász reformulated it from the point of view of barriers, stating that the family of maximal barriers forms a partition of the vertices in elementary graphs. This reformulation by Lovász has produced many fundamental properties in matching theory such as the two ear theorem \[1,2\], and the brick decomposition or the tight cut decomposition, and underlies polyhedral studies of matching theory; see the survey article \[3\].

However, in non-elementary graphs, the family of maximal barriers never forms a partition of the vertices, and there has not been known the counterpart structure of Lovász’s canonical partition for general graphs. In this paper, therefore, we reveal canonical structures of maximal barriers and obtain a generalization of Lovász’s canonical partition for general graphs; here, our previous work on canonical structures of general factorizable graphs \[4,5\], the generalized cathedral structure (see Section 2.4), serves as a language to describe barriers. (Actually, we work on a wider notion called odd-maximal barriers; see Section 2.3.) In \[4,5\], we defined an equivalence relation and introduced a generalization of the canonical partition based on Kotzig’s formulation: the generalized canonical partition. In this paper, we show that it can be also regarded as a generalization based on Lovász’s formulation, stating that the family of equivalence classes of the generalized canonical partition are “atoms” that constitute (odd-)maximal barriers in general graphs (which shall be introduced in Section 3). We also reveal the structure of odd components associated with (odd-)maximal barriers.

Because the canonical partition and the notion of barriers are important, we are sure that our result will produce many applications in matching theory. There has been known a close relationship between algorithms in matching theory, barriers, and canonical structure theorems \[1,2\]; therefore, our result will have algorithmic applications. Lovász’s canonical partition has been the foundation in the study of polyhedral aspects of matchings; therefore, our results will make a contribution to this field. So far we have already obtained some consequences \[6\] on the optimal ear-decomposition \[7\].
2 Preliminaries

2.1 Definitions and Some Preliminary Facts

In this paper we mostly observe those given by Schrijver [8] for standard definitions and notations. We list here those additional or somewhat non-standard.

Hereafter for a while let $G$ be a graph. We denote the vertex set and the edge set of $G$ as $V(G)$ and $E(G)$. Now for a while let $X \subseteq V(G)$. We define the contraction of $G$ by $X$ as the graph obtained by contracting $X$ into one vertex, and denote it as $G/X$. For simplicity, we identify vertices, edges, subgraphs of $G/X$ with those of $G$ naturally corresponding to them. The subgraph of $G$ induced by $X$ is denoted by $G[X]$. We denote by $G - X$ the graph $G[V(G) \setminus X]$.

The set of edges that has one end vertex in $X \subseteq V(G)$ and the other end in $Y \subseteq V(G)$ is denoted as $E_G[X, Y]$. We denote $E_G[X, V(G) \setminus X]$ as $\delta_G(X)$. We define the set of neighbors of $X$ as the set of vertices in $V(G) \setminus X$ that are adjacent to vertices in $X$, and denote as $N_G(X)$. We sometimes denote $E_G[X, Y]$, $\delta_G(X)$, $N_G(X)$ as just $E[X, Y]$, $\delta(X)$, $N(X)$ if their meanings are apparent from contexts. Given two graphs $G_1$ and $G_2$, we denote by $G_1 + G_2$ the graph $(V(G_1) \cup V(G_2), E(G_1) \cup E(G_2))$. Let $\hat{G}$ be a supergraph of $G$. For $e = uv \in E(\hat{G})$, $G - e$ means the graph $(V(G), E(G) \setminus \{e\})$. For a set of edges $F = \{e_i\}_{i=1}^k$, $G - F$ means the graph $G - e_1 - \cdots - e_k$.

In many contexts, we often regard a subgraph $H$ of $G$ as a vertex set $V(H)$. For example, $G/H$ means $G/V(H)$. We treat paths and circuits as graphs. For a path $P$ and $x, y \in V(P)$, $xPy$ means the subpath of $P$ whose end vertices are $x$ and $y$.

We say a matching $M$ of $G$ exposes $v \in V(G)$ if $\delta(v) \cap M = \emptyset$, otherwise say it covers $v$. For a matching $M$ of $G$ and $u \in V(G)$ covered by $M$, $u^c$ denotes the vertex to which $u$ is matched by $M$. For $X \subseteq V(G)$, $M_X$ denotes $M \cap E(G[X])$.

Hereafter for a while let $M$ be a matching of $G$. For a subgraph $Q$ of $G$, which is a path or circuit, we call $Q$ $M$-alternating if $E(Q) \setminus M$ is a matching of $Q$. Let $P$ be an $M$-alternating path of $G$ with end vertices $u$ and $v$. If $P$ has an even number of edges and $M \cap E(P)$ is a near-perfect matching of $P$ exposing only $v$, we call it an $M$-balanced path from $u$ to $v$. We regard a trivial path, that is, a path composed of one vertex and no edges as an $M$-balanced path. If $P$ has an odd number of edges and $M \cap E(P)$ (resp. $E(P) \setminus M$) is a perfect matching of $P$, we call it $M$-saturated (resp. $M$-exposed).

We say a path $P$ of $G$ is an ear relative to $X \subseteq V(G)$ if both end vertices of $P$ are in $X$ while internal vertices are not. We also call a circuit an ear relative to $X$ if exactly one vertex of it is in $X$. For simplicity, we call the vertices of $V(P) \cap X$ end vertices of $P$, even if $P$ is a circuit. For an ear $P$ of $G$ relative to $X$, we call it an $M$-ear if $P - X$ is an $M$-saturated path. Given an ear $P$ and $Y \subseteq V(G)$, we say $P$ is through $Y$ if $P$ has some internal vertices in $Y$.

Hereafter in this section we present some basic properties used explicitly or implicitly throughout this paper. These are easy to see and the succeeding two propositions are well-known and might be folklores. A graph is called factor-
critical if each deletion of an arbitrary vertex results in an empty graph or a factorizable graph.

**Proposition 1 (folklore).** Let $M$ be a near-perfect matching of a graph $G$ that exposes $v \in V(G)$. Then, $G$ is factor-critical if and only if for any $u \in V(G)$ there exists an $M$-balanced path from $u$ to $v$.

Given a graph $G$ and $X \subseteq V(G)$, we denote the vertices contained in the odd components of $G - X$ as $D_X$, and $V(G) \setminus X \setminus D_X$ as $C_X$. The next proposition can be easily observed by the Berge formula.

**Proposition 2 (folklore).** Let $G$ be a factorizable graph, and $X \subseteq V(G)$ be a barrier of $G$. Then for any perfect matching $M$ of $G$,

(i) each vertex of $X$ is matched to a vertex of $D_X$,

(ii) for each component $K$ of $G[D_X]$, $M_K$ is a near-perfect matching of $K$, accordingly $|\delta(K) \cap M| = 1$,

(iii) $M$ contains a perfect matching of $G[C_X]$, and

(iv) no edge in $E[X, C_X]$ nor $E(G[X])$ is allowed.

Now let $G$ be a factorizable graph. We say $X \subseteq V(G)$ is separating if any $H \in \mathcal{G}(G)$ satisfies $V(H) \subseteq X$ or $V(H) \cap X = \emptyset$. The next one is easy to see by the definitions.

**Proposition 3.** Let $G$ be a factorizable graph, and let $X \subseteq V(G)$. Then, the following four properties are equivalent:

(i) The set $X$ is separating.

(ii) The set $X$ is an empty set, or there exist $H_1, \ldots, H_k \in \mathcal{G}(G)$ such that $X = V(H_1) \cup \cdots \cup V(H_k)$.

(iii) For any perfect matching $M$ of $G$, $\delta(X) \cap M = \emptyset$.

(iv) For any perfect matching $M$ of $G$, $M_X$ forms a perfect matching of $G[X]$.

**Proposition 4.** Let $G$ be a factorizable graph, and $M$ be a perfect matching of $G$. Let $X \subseteq V(G)$ be such that $M_X$ forms a perfect matching of $G[X]$, and $P$ be an $M$-alternating path.

(i) If $X \cap V(P)$ has no vertex exposed by $M_P$, then each connected component of $P \setminus X$ is an $M$-saturated path.

(ii) If both end vertices of $P$ are in $X$, then each connected component of $P - E(G[X])$ is an $M$-ear relative to $X$.

**Proposition 5.** Let $G$ be a factorizable graph, and $M$ be a perfect matching of $G$. Let $P$ and $Q$ be $M$-alternating paths. If $P$ and $Q$ intersect only with their internal vertices, then each connected component of $P \cap Q$ is an $M$-saturated path.

**Proposition 6.** Let $G$ be a factorizable graph and $M$ be a perfect matching of $G$, and let $u, v \in V(G)$. Then, the following two properties are equivalent:
The graph $G - u - v$ is factorizable.

Proposition 7. Let $G$ be a factorizable graph and $M$ be a perfect matching of $G$, and let $u, v \in V(G)$. If there is an $M$-alternating circuit $C$ with $u, v \in V(C)$, then $u$ and $v$ are contained in the same factor-component of $G$.

2.2 The Dulmage-Mendelsohn Decomposition

Factor-components of a bipartite factorizable graph are known to have the following partially ordered structure:

Theorem 1 (The Dulmage-Mendelsohn Decomposition [2, 9–12]). Let $G = (A, B; E)$ be a bipartite factorizable graph, and let $G(G) = \{G_i\}_{i \in I}$. Let $A_i := A \cap V(G_i)$ and $B_i := B \cap V(G_i)$ for each $i \in I$. Then, there exists a partial order $\preceq_A$ on $G(G)$ such that for any $i, j \in I$,

(i) $E[B_j, A_i] \neq \emptyset$ yields $G_j \preceq_A G_i$, and
(ii) if $G_j \preceq_A H \preceq_A G_i$ yields $G_i = H$ or $G_j = H$ for any $H \in G(G)$, then $E[B_j, A_i] \neq \emptyset$.

We call this decomposition of $G$ into a poset the Dulmage-Mendelsohn decomposition (in short, the DM-decomposition), and each element of $G(G)$, in this context, a DM-component. The DM-decomposition is uniquely determined by a graph, up to the choice of roles of color classes. In this paper, we call the DM-decomposition of $G = (A, B; E)$ as in Theorem 1 the DM-decomposition with respect to $A$.

Proposition 8 (Dulmage & Mendelsohn [9, 12]). Let $G = (A, B; E)$ be a bipartite factorizable graph, and $M$ be a perfect matching of $G$. Let $G_1, G_2 \in G(G)$, and let $u \in A \cap V(G_1)$, $v \in A \cap V(G_2)$, and $w \in B \cap V(G_2)$. Then there is an $M$-balanced path from $u$ to $v$ if and only if $G_1 \preceq_A G_2$; additionally, there is an $M$-saturated path between $u$ to $w$ if and only if $G_1 \preceq_A G_2$.

2.3 Our Aim

Given an elementary graph $G$, we say $u \sim v$ for $u, v \in V(G)$ if $u = v$ holds or $G - u - v$ is not factorizable. Kotzig [13–15] found that $\sim$ is an equivalence relation. Later Lovász redefined it:

Theorem 2 (Lovász [2]). Let $G$ be an elementary graph. Then, the family of maximal barriers forms a partition of $V(G)$. Additionally, this partition coincides with the equivalence classes by $\sim$.

1This is different from the one in [4, 5]. Though it is sometimes presented as a theorem for general bipartite graphs, we introduce it as one for bipartite factorizable graphs.

2For more details on the statements in this section, see Appendix.
This partition by the maximal barriers is called the canonical partition. As we mention in Section 1, it plays fundamental and significant roles in matching theory. On the other hand, as for non-elementary graphs, the family of maximal barriers never forms a partition of the vertices (see [2]). The question remains: how all the maximal barriers exist and what is the counterpart in general graphs? Therefore, we are going to investigate it. Actually, we work on a wider notion: odd-maximal barriers.  

Definition 1. Let $G$ be a graph. A barrier $X \subseteq V(G)$ is called an odd-maximal barrier if it is a barrier which is maximal with respect to $X \cup D_X$, i.e., no $Y \subseteq D_X$ with $Y \neq \emptyset$ satisfies that $X \cup Y$ is a barrier of $G$.

Odd-maximal barriers have some nice properties (see [16,17]): First, a maximal barrier is an odd-maximal barrier. Second, for elementary graphs, the notion of maximal barriers and the notion of odd-maximal barriers coincide. Hence, it seems reasonable to work on the odd-maximal barriers.

Given a graph $G$, we define $D(G)$ as the set of vertices that can be respectively exposed by maximum matchings, $A(G)$ as $N(D(G))$ and $C(G)$ as $V(G) \setminus (D(G) \cup A(G))$. There is a well-known theorem stating that $A(G)$ forms a barrier with special properties, called the Gallai-Edmonds structure theorem [2]. Actually, with the Gallai-Edmonds structure theorem and the theorem by Király [16], we can see that it suffices to work on factorizable graphs:

Proposition 9 (see also Király [16]). Let $G$ be a graph. A set of vertices $S \subseteq V(G)$ is an odd-maximal barrier of $G$ if and only if it is a disjoint union of $A(G)$ and an odd-maximal barrier of the factorizable subgraph $G[C(G)]$.

Given the above facts, in this paper we give canonical structures of odd-maximal barriers in general factorizable graphs that can be regarded as a generalization of Lovász’s canonical partition, aiming to contribute to the foundation of matching theory.

2.4 The Generalized Cathedral Structure

In this section we are going to introduce the canonical structure theorems of factorizable graphs, which shall serve as a language to describe odd-maximal barriers. They are composed of three parts: a partially ordered structure on the factor-components (Theorem 4), a generalization of the canonical partition (Theorem 5), and a relationship between these two (Theorem 6).  

Definition 2. Let $G$ be a factorizable graph, and let $G_1, G_2 \in \mathcal{G}(G)$. We say $X \subseteq V(G)$ is a critical-inducing set for $G_1$ to $G_2$ if $X$ is separating, $V(G_1) \cup V(G_2) \subseteq X$ holds, and $G[X]/G_1$ is factor-critical. Additionally, we say $G_1 \triangleleft G_2$ if there is a critical-inducing set for $G_1$ to $G_2$. 

3 This is identical to those Király calls strong barriers [16], however we call it in the different way so as to avoid the confusion with the notion of strong end by Frank [7].

4 All the statements in [5] can be also found in [4].
Theorem 3 (Kita [4,5]). For any factorizable graph $G$, $\triangleleft$ is a partial order on $\mathcal{G}(G)$.

Definition 3. Let $G$ be a factorizable graph. For $u, v \in V(G)$ we say $u \sim_G v$ if $u$ and $v$ are contained in the same factor-component of $G$, and $G - u - v$ is NOT factorizable.

Theorem 4 (Kita [4,5]). For any factorizable graph $G$, $\sim_G$ is an equivalence relation on $V(G)$.

As you can see by the definition, if $G$ is an elementary graph then $\sim$ and $\sim_G$ coincide. Therefore, we call the equivalence classes by $\sim_G$, i.e. $V(G)/\sim_G$, the generalized canonical partition or just the canonical partition, and denote by $\mathcal{P}(G)$. For each $H \in \mathcal{G}(G)$, we define $\mathcal{P}_G(H) := \{S \in \mathcal{P}(G) : S \subseteq V(H)\}$; then, $\mathcal{P}_G(H)$ forms a partition of $V(H)$, since by the definition each equivalence class is respectively contained in one of the factor-components. Note that $\mathcal{P}_G(H)$ is always a refinement of $\mathcal{P}(H)$, which equals to $\mathcal{P}_H(H)$.

For each $H \in \mathcal{G}(G)$, we denote the family of the upper bounds of $H$ in the poset $(\mathcal{G}(G), \triangleleft)$ as $\mathcal{U}_G(H)$, and $\mathcal{U}_G(H) \setminus \{H\}$ as $\mathcal{U}_G(H)$. Moreover, we denote the vertices contained in $\mathcal{U}_G(H)$ as $\mathcal{U}_G^+(H)$; i.e., $\mathcal{U}_G^+(H) := \bigcup_{H' \in \mathcal{U}_G(H)} V(H')$. We also denote $\mathcal{U}_G^*(H) \setminus V(H)$ as $\mathcal{U}_G(H)$. Actually, the next theorem states that each strict upper bound of $H \in \mathcal{G}(G)$ in $(\mathcal{G}(G), \triangleleft)$ is respectively “assigned” to some $S \in \mathcal{P}_G(H)$:

Theorem 5 (Kita [4,5]). Let $G$ be a factorizable graph, and let $H \in \mathcal{G}(G)$. For each connected component $K$ of $G[U_G(H)]$, there exists $S_K \in \mathcal{P}_G(H)$ such that $N(K) \cap V(H) \subseteq S_K$.

Based on Theorem 3 we define $\mathcal{U}_G(S)$ as follows: $H' \in \mathcal{U}_G(S)$ if and only if $H \triangleleft H'$ and $H \neq H'$ holds and there exists a connected component $K$ of $G[U(H)]$ with $N(K) \cap V(H) \subseteq S$ such that $V(H') \subseteq V(K)$. Additionally, we denote the vertices contained in $\mathcal{U}_G(S)$ by $\mathcal{U}_G(S)$; i.e., $\mathcal{U}_G(S) := \bigcup_{H' \in \mathcal{U}_G(S)} V(H')$. We also define $\mathcal{U}_G^*(S) := \mathcal{U}_G(S) \cup S$. Regarding these eight notations we sometimes omit the subscripts “$G$” if they are apparent from the contexts. Note that $\bigcup_{T \in \mathcal{P}_G(H)} U(T) = U(H)$.

We call the canonical structures of factorizable graphs given by Theorems 3 [4] and 5 the generalized cathedral structures or just the cathedral structures. Now let us add some propositions used later in this paper:

Proposition 10 (Kita [4,5]). Let $G$ be a factorizable graph, and let $H \in \mathcal{G}(G)$. Then, $G[U^+(H)]/H$ is factor-critical, so is each block of it.

Proposition 11 (Kita [4,5]). Let $G$ be a factorizable graph and $M$ be a perfect matching of $G$, and let $H \in \mathcal{G}(G)$. Let $P$ be an $M$-ear relative to $H$.

(i) Let $H' \in \mathcal{G}(G)$. If $P$ is through $H'$, then $H \triangleleft H'$.

(ii) The end vertices $u, v \in V(H)$ of $P$ satisfies $u \sim_G v$. 


3 A Generalization of Lovász’s Canonical Partition

3.1 Our Main Result

Our main result is the following:

**Main Theorem.** Let $G$ be a factorizable graph, and $X \subseteq V(G)$ be an odd-maximal barrier of $G$. Then, $X$ is a disjoint union of some members of $\mathcal{P}(G)$; namely, there exists $S_1, \ldots, S_k \in \mathcal{P}(G)$ such that $X = S_1 \cup \cdots \cup S_k$. Additionally, odd components of $G - X$ have structures as follows: $D_X = (U^*(G_1) \setminus U^*(S_1)) \cup \cdots \cup (U^*(G_k) \setminus U^*(S_k))$, where $G_i \in \mathcal{G}(G)$ is such that $S_i \in \mathcal{P}_G(G_i)$ for each $i \in \{1, \ldots, k\}$.

This theorem states that in general graphs the equivalence classes of the generalized canonical partition are the “atoms” that constitute odd-maximal barriers, and that odd components associated to odd-maximal barriers are also described canonically by the generalized cathedral structure. As we see in previous sections, among two formulations of the canonical partition of elementary graphs, the generalization of the canonical partition introduced in [4,5] is attained based on Kotzig’s formulation; here we show it is as well a generalization based on Lovász’s formulation.

This theorem is an immediate corollary of Theorem 8, and the rest of this paper is to prove Theorem 8. We shall prove it by examining the reachability of alternating paths from two viewpoints—regarding odd-maximal barriers and regarding the generalized cathedral structure—and showing their equivalence. Let us mention an additional property used later in this paper.

**Proposition 12 (Király [16]).** A barrier $X \subseteq V(G)$ of a graph $G$ is odd-maximal if and only if all the odd components of $G - X$ are factor-critical.

3.2 Barriers vs. Alternating Paths

In this subsection we introduce some lemmas on the reachability of alternating paths regarding odd-maximal barriers. Given an odd-maximal barrier $X$ of a factorizable graph $G$, we generate a bipartite graph, thus canonically decompose $X \cup D_X$ and state the reachability using the DM-decomposition as a language. This technique of generating a bipartite graph has been known [2,7] and essences of ideas are found there. However, we first reveal it thoroughly to obtain Proposition 17 and Theorem 6.

**Proposition 13 (folklore).** Let $G$ be a factorizable graph, and $M$ be a perfect matching of $G$. If $X \subseteq V(G)$ is a barrier, then for any $u, v \in X$ there is no $M$-saturated path between $u$ and $v$.

**Proof.** Suppose the claim fails, namely, there is an $M$-saturated path, say $P$, between vertices $u$ and $v$. Then, $M \triangle E(P)$, i.e., $(M \setminus E(P)) \cup (E(P) \setminus M)$, forms a perfect matching of $G - u - v$; accordingly, $G - u - v$ is factorizable. Now recall that since $X$ is a barrier of the factorizable graph $G$, $G - X$ has
exactly $|X|$ odd components by the definition of barriers. Therefore, the graph $(G-u-v)-(X\setminus\{u,v\})$, which equals to $G-X$, also has $|X|$ odd components; this means by the Berge formula that $G-u-v$ is not factorizable, a contradiction. \hfill \blacksquare

**Proposition 14 (folklore).** Let $G$ be a factorizable graph, $M$ be a perfect matching, and $X$ be a barrier of $G$. Then, for any $x \in X$ and $y \in C_X$, there is no $M$-saturated path between $x$ and $y$ nor $M$-balanced path from $x$ to $y$.

**Proof.** Suppose otherwise, that is, there is a path $P$ which is $M$-saturated between $x$ and $y$ or $M$-balanced from $x$ to $y$. Trace $P$ from $x$; let $z$ be the first vertex we encounter that is in $C_X$, and let $w$ be the last vertex in $X \cup D_X$ we encounter if we trace $xPz$ from $x$. Apparently, $w \in X$ and $wz \in E(xPz)$ hold, and by Proposition 2 $wz \notin M$ holds. Therefore, $xPw$ is an $M$-saturated path between $x$ and $w$, contradicting Proposition 13. \hfill \blacksquare

**Proposition 15 (folklore).** Let $G$ be a factorizable graph, $M$ be a perfect matching, and $X \subseteq V(G)$ be an odd-maximal barrier. Then, for any $u \in X$ and $v \in X \cup C_X$ there is no $M$-saturated path between $u$ and $v$.

**Proof.** This is immediate by Proposition 13 and Proposition 14. \hfill \blacksquare

**Definition 4.** Let $G$ be a graph, $X \subseteq V(G)$, and $K_1, \ldots, K_l$ be the odd components of $G-X$. We denote the bipartite graph resulting from deleting the even components of $G-X$, removing the edges whose vertices are all contained in $X$, and contracting each $K_i$, where $i=1, \ldots, l$, respectively into one vertex, as $H_G(X)$. Namely, $H_G(X) := (G-C_X-E(G[X]))/K_1/\cdots/K_l$.

The next proposition is easily seen by Propositions 2 and 12 and enables us to discuss Proposition 17 and so on.

**Proposition 16 (might be a folklore).** Let $G$ be a factorizable graph and $X$ be an odd-maximal barrier of $G$. If $M \subseteq E(G)$ is a perfect matching of $G$, then $M \cap \delta(X)$ forms a perfect matching of $H_G(X)$. Conversely, if $M'$ is a perfect matching of $H_G(X)$, there is a perfect matching $M$ of $G$ such that $M' = M \cap \delta(X)$.

**Proof.** The first claim follows by Proposition 2. For the second claim, first note that $G[C_X]$ is factorizable by Proposition 2 and let $N$ be a perfect matching of $G[C_X]$. By Proposition 12 the odd components $K_1, \ldots, K_l$ of $G-X$ are each factor-critical. For each $i=1, \ldots, l$ let $M_i$ be a near-perfect matching of $K_i$ exposing only the vertex covered by $M'$. Then, $N \cup M' \cup \bigcup_{i=1}^l M_i$ forms a desired perfect matching. \hfill \blacksquare

The next proposition shows that the reachabilities of alternating paths are equivalent between $G$ and $H_G(X)$, which, with Proposition 8 derives Theorem 9 immediately.
Proposition 17. Let $G$ be a factorizable graph, $X \subseteq V(G)$ be an odd-maximal barrier of $G$, and $K := \{K_i\}_{i=1}^l$ be the family of odd components of $G - X$, where $l = |X|$. Let $M$ be a perfect matching of $G$, and $M'$ be the perfect matching of $H_G(X)$ such that $M' = M \cap \delta(X)$. Let $u, v \in X$, and $w \in V(K)$, where $K \in K$, and let $w_K$ be the contracted vertex of $H_G(X)$ corresponding to $K$.

(i) Then, for any $M$-balanced path (resp. $M$-saturated path) $P$ of $G$ from $u$ to $v$ (resp. between $u$ and $w$), $P' = P/K_1/\cdots/K_l$ is an $M'$-balanced path (resp. $M'$-saturated path) of $H_G(X)$ from $u$ to $v$ (resp. between $u$ and $w_K$).

(ii) Conversely, for any $M'$-balanced path (resp. $M'$-saturated path) $P'$ from $u$ to $v$ in $H_G(X)$ (resp. between $u$ and $w_K$), there is an $M$-balanced path (resp. $M$-saturated path) $P$ from $u$ to $v$ in $G$ (resp. between $u$ and $w$) such that $P' = P/K_1/\cdots/K_l$.

Proof. For [i], we first prove the case where $P$ is an $M$-balanced path. Let $u = x_1, \ldots, x_l = v$ be the vertices of $X \cap V(P)$, and suppose, without loss of generality, they appear in this order if we trace $P$ from $u$. For each $i = 1, \ldots, l'$, let $L_i \in K$ be such that $x_i' \in V(L_i)$, which is well-defined by Proposition 2 and let $z_i$ be the contracted vertex of $H_G(X)$ corresponding to $L_i$. Note that by Proposition 2

Claim 1. if $x_i \neq x_j$, then $L_i \neq L_j$, accordingly, $z_i \neq z_j$.

We are going to prove a bit refined statement,

$$P'$$ is an $M'$-balanced path from $u$ to $v$, with $V(P') = \{x_i\}_{i=1}^{l'} \cup \{z_i\}_{i=1}^{l'} \setminus \{z_{l'}\}$,

by induction on $k$, where $|E(P)| = 2k$.

If $k = 0$, then the statement is obviously true. Let $k > 0$ and suppose the claim is true for $0, \ldots, k - 1$. By the definitions, the internal vertices of $uP_x$ are contained in $L_1$. By Proposition 2 $\delta(L_i) \cap M = \{uu'\}$. Thus, if we trace $uP_x$ from $u$ then the last edge is not in $M$, which means $uP_x$ is an $M$-balanced path from $u$ to $x_2$. Accordingly, $x_2Pv$ is also an $M$-balanced path, from $x_2$ to $v$.

Note that $P'_{x_1} := uP_x/K_1/\ldots/K_l$ is apparently an $M$-balanced path, since $E(P_{x_1}) = \{u2, z_2x_2\}$. Therefore, if $x_2 = v$ then the claim follows. Hence hereafter we prove the case where $x_2 \neq v$. By the induction hypothesis, $P_{x_2}' := x_2Pv/K_1/\ldots/K_l$ is an $M'$-balanced path of $H_G(X)$, whose vertices are $\{x_2, \ldots, x_l = v\} \cup \{2_2, \ldots, 2_{l-1}\}$. Thus, $V(uP_{x_2}) \cap V(x_2Pv) = \{x_2\}$ by Claim 1 accordingly, $P' = P'_{x_1} + P_{x_2}'$ is an $M'$-balanced path of $H_G(X)$ from $u$ to $v$ with $V(P') = \{x_i\}_{i=1}^{l'} \cup \{z_i\}_{i=1}^{l'} \setminus \{z_{l'}\}$. The other case where $P$ is an $M$-saturated path can be proved by similar arguments.

For [ii], we first prove the case where $P'$ is an $M'$-balanced path of $H_G(X)$.

Since it is apparently true if $u = v$, we prove the case where $u \neq v$. Let $u = x_0, y_0, \ldots, x_p, y_p, x_{p+1} = v$ be the vertices of $P'$, and suppose they appear in this order if we trace $P'$ from $u$. Note that $x_i \in X$ for each $i = 0, \ldots, p + 1$ and that $y_i$ be a contracted vertex corresponding to an odd component of $G - X$, say $L_i$, for each $0, \ldots, p$. For each $i = 0, \ldots, p$, let $y_i^1, y_i^2 \in V(L_i)$ be such that
Given a factorizable graph $G$ and an odd-maximal barrier $X$, we denote the DM-decomposition of $H_G(X)$ with respect to $X$ as just the DM-decomposition of $H_G(X)$. In this case, we sometimes denote $\preceq_X$ as just $\preceq$, omitting the subscript “$X$”.

**Definition 5.** Let $G$ be a factorizable graph, and $X$ be an odd-maximal barrier of $G$. Let $D$ be a DM-component of $H_G(X)$, whose vertices in $V(D) \setminus X$ are the contracted vertices resulting from some odd components of $G - X$, say $K_1, \ldots, K_i$, where $I \leq |X|$. We say $\tilde{D}$ is the expansion of $D$ if it is the subgraph of $G$ induced by $(V(D) \cap X) \cup \bigcup_{i=1}^I V(K_i)$.

The next proposition is a basic observation on expansions.

**Proposition 18.** Let $G$ be a factorizable graph, and $X$ be an odd-maximal barrier of $G$. Let $D_1, \ldots, D_k$ be the DM-components of $H_G(X)$. For each $i = 1, \ldots, k$, let $\tilde{D}_i$ be the expansion of $D_i$. Then,

(i) $\{V(\tilde{D}_i)\}_{i=1}^k$ forms a partition of $X \cup D_X$,
(ii) $V(\tilde{D}_i)$ is separating, accordingly $\tilde{D}_i$ is factorizable,
(iii) $X \cap V(\tilde{D}_i)$ is an odd-maximal barrier of $\tilde{D}_i$, and
(iv) $H_{\tilde{D}_i}(X \cap V(\tilde{D}_i))$ is isomorphic to $D_i$, for each $i = 1, \ldots, k$.

**Proof.** Since the DM-components of $H_G(X)$ give the partition of the vertices of it, the first half of (ii) apparently follows from the definitions. For the first half of (ii), suppose that $V(\tilde{D}_i)$ is not separating, equivalently by Proposition 4, that there is a perfect matching $M$ of $G$ such that $\delta(\tilde{D}_i) \cap M \neq \emptyset$. Then, by Proposition 10, $M' := M \cap \delta(X)$ forms a perfect matching of $H_G(X)$ satisfying $\delta(D_i) \cap M' \neq \emptyset$, a contradiction. Therefore, $V(\tilde{D}_i)$ is separating; accordingly, $\tilde{D}_i$ is factorizable, and we are done for (ii).

By the definition, $D_i \setminus X$ is composed of $|X \cap V(\tilde{D}_i)|$ number of odd-components, each of which is factor-critical by Proposition 12. Therefore, $V(\tilde{D}_i) \cap X$ is an odd-maximal barrier of $\tilde{D}_i$ by the statement (ii) and Proposition 12 again. Thus, we are done for (iii). The statement (iv) is apparent from the definitions. □

**Theorem 6.** Let $G$ be a factorizable graph, $X$ be an odd-maximal barrier, and $M$ be a perfect matching of $G$. Let $u, v, w \in X$, and $w \in D_X$, and for each $\alpha = u, v, w$ let $D_\alpha$ be the DM-component of $H_G(X)$ whose expansion $\tilde{D}_\alpha$ contains $\alpha$. Then, there is an $M$-balanced path from $u$ to $v$ (resp. an $M$-saturated path from $u$ to $w$) in $G$ if and only if $D_u \leq D_v$ (resp. $D_u \leq D_w$).
Proof. First note that $\hat{D}_\alpha$ is well-defined by Proposition 18. Now the claim is immediate from Proposition 8 and Proposition 17. $\square$

Lemma 1. Let $G = (A, B; E)$ be a bipartite factorizable graph, $M$ be a perfect matching of $G$, and $D_1, D_2$ be DM-components of $G$ with $D_1 \preceq_A D_2$. Then, for any $u \in V(D_1) \cap A$ and $v \in V(D_2) \cap B$, any $M$-saturated path $P$ between $u$ and $v$ traverses $A \cap V(D_2)$.

Proof. Apparently $vv' \in E(P)$ and $v' \in V(D_2) \cap A$; therefore, the claim follows. $\square$

The following lemma is obtained by Propositions 17 and 18, and Theorem 6.

Lemma 2. Let $G$ be a factorizable graph, $X$ be an odd-maximal barrier, and $M$ be a perfect matching of $G$. Let $\hat{D}_1$ and $\hat{D}_2$ be the subgraphs of $G$ which are respectively the expansions of DM-components $D_1$ and $D_2$ such that $D_1 \preceq D_2$. Then, for any $u \in X \cap V(\hat{D}_1)$ and $w \in V(\hat{D}_2) \setminus X$, any $M$-saturated path $P$ between $u$ and $w$ traverses $X \cap V(\hat{D}_2)$.

Proof. Let $K_1, \ldots, K_l$, where $l = |X|$, be the odd components of $G - X$. By Proposition 17, $P' := P/K_1/\cdots/K_l$ is an $M'$-saturated path, where $M' = M \cap \delta(X)$, whose end vertices are respectively in $X \cap V(D_1)$ and $V(\hat{D}_2) \setminus X$. Therefore, $P'$ traverses $X \cap V(D_2)$ by Lemma 1, which means $P$ traverses $X \cap V(\hat{D}_2)$. $\square$

3.3 Canonical Structures of Odd-maximal Barriers

In this subsection we examine the reachability of alternating paths regarding the cathedral structure and derive the main theorem.

Proposition 19 (implicitly stated in [4, 5]). Let $G$ be a factorizable graph, and let $H \in G(G)$ and $S \in \mathcal{P}_G(H)$. Then, $G[U^*(S)]/S$ is factor-critical.

Proof. This is a mere restatement of Proposition [10]. $\square$

The next lemma is obtained by Proposition [10] and Proposition [1].

Lemma 3. Let $G$ be a factorizable graph and $M$ be a perfect matching of $G$, and let $H \in G(G)$ and $S \in \mathcal{P}_G(H)$. Then, for any $x \in U^*(S)$, there is an $M$-balanced path from $x$ to some vertex $y \in S$, whose vertices except $y$ are contained in $U(S)$.

Proof. $M_{U(S)}$ forms a near-perfect matching of $G' := G[U^*(S)]/S$ exposing only the contracted vertex $s$ corresponding to $S$, and by Proposition 19, $G'$ is factor-critical. Therefore, by Proposition 1 there is an $M_{U(S)}$-balanced path from any $x \in U^*(S)$ to $s$, which corresponds to a desired path in $G$. Thus, the claim follows. $\square$

Immediately by Theorem 5 we can see the next proposition:
Proposition 20. Let $G$ be a factorizable graph and $M$ be a perfect matching of $G$, and let $H \in \mathcal{G}(G)$. A set of vertices $S \subseteq V(H)$ is a member of $\mathcal{P}_G(H)$ if and only if it is a maximal subset of $V(H)$ satisfying that there is no $M$-saturated path between any two vertices of it.

Proof. This follows easily from Theorem 4 and Proposition 6.

The next one is by Proposition 11 and Lemma 3.

Lemma 4. Let $G$ be a factorizable graph and $M$ be a perfect matching of $G$, and let $H \in \mathcal{G}(G)$ and $S \in \mathcal{P}_G(H)$. Then, for any $s \in S$ and $x \in U(s)$, there is no $M$-saturated path between $s$ and $x$ nor $M$-balanced path from $s$ to $x$.

Proof. Suppose the claim fails, that is, there is a path $P$ that is $M$-balanced from $s$ to $x$ or $M$-saturated between $s$ and $x$. Trace $P$ from $s$ and let $y$ be the first vertex we encounter that is in $U(s)$. Trace $sPy$ from $y$ and let $z$ be the first vertex we encounter that is in $V(H)$. Then, since $V(H)$ and $U(s)$ are separating, $zPy$ is an $M$-exposed path by Proposition 3. Consequently $sPz$ is an $M$-saturated path between $s$ and $z$, which means $z \notin S$ by Proposition 20.

On the other hand, by Lemma 3 there is an $M$-balanced path $Q$ from $y$ to some vertex $t \in S$ whose vertices except $t$ are contained in $U(s)$. Therefore, $zPy + yQt$ is an $M$-ear relative to $H$, whose end vertices are $z$ and $t$; this contradicts Proposition 11 since $z \not\sim_G t$.

The next one, Lemma 5, is rather easy to see by Proposition 11 and combining it with Lemma 3; we can obtain Lemma 6.

Lemma 5. Let $G$ be a factorizable graph and $M$ be a perfect matching of $G$. Let $H \in \mathcal{G}(G)$, and let $u, v \in V(H)$ be such that $u \not\sim_G v$. Let $P$ be an $M$-saturated path between $u$ and $v$ such that $E(P) \setminus E(H) \neq \emptyset$, and let $P_1, \ldots, P_l$ be the components of $P - E(H)$. Let $S_0, S_{l+1} \in \mathcal{P}_G(H)$ be such that $u \in S_0$ and $v \in S_{l+1}$. Then,

(i) two end vertices of $P_i$ belong to the same member of $\mathcal{P}_G(H)$, say $S_i$,
(ii) $P_i$ is, except its end vertices, contained in $U(S_i)$ for each $i = 1, \ldots, l$, and
(iii) for any $i, j \in \{0, \ldots, l + 1\}$ with $i \neq j$, $S_i \neq S_j$.

Proof. By Proposition 11, $P_i$ is an $M$-ear relative to $H$ for each $i = 1, \ldots, l$; therefore, (i) follows by Proposition 11. Thus, (ii) follows by Proposition 11. For (iii) let the end vertices of $P_i$ be $x_i$ and $y_i$ for each $i = 1, \ldots, l$. Without loss of generality, we can assume that the vertices $u =: y_0, x_1, y_1, \ldots, x_l, y_l, x_{l+1} := v$ appear in this order if we trace $P$ from $u$. Then, for any $i, j$ with $0 \leq i < j \leq l+1$, $y_i P x_j$ forms an $M$-saturated path between $y_i \in S_i$ and $x_j \in S_j$. Thus we have $S_i \neq S_j$ by Proposition 20; this means (iii), and we are done.

Lemma 6. Let $G$ be a factorizable graph and $M$ be a perfect matching of $G$. Let $H \in \mathcal{G}(G)$, and let $S, T \in \mathcal{P}_G(H)$ be such that $S \neq T$. Then, for any $s \in S$ and $t \in U^+(T)$, there is an $M$-saturated path $P$ between $s$ and $t$, which is contained in $U^+(H) \setminus U(S)$. 

13
Proof. By Lemma 3 there is an $M$-balanced path $P_1$ from $t$ to a vertex $x \in T$ whose vertices except $x$ are contained in $U(T)$. By Proposition 20 there is an $M$-saturated path $P_2$ between $s$ and $x$. By Lemma 5, $V(P_2)$ is contained in $U^*(H) \setminus U(S) \setminus U(T)$; accordingly, $V(P_1) \cap V(P_2) = \{x\}$. Hence, $P := P_1 + P_2$ is an $M$-saturated path between $s$ and $t$, contained in $U^*(H) \setminus U(S)$. □

Lemma 6 immediately yields the following: Lemma 7.

**Theorem 7.** Let $G$ be a factorizable graph, $M$ be a perfect matching of $G$, and $u, v \in V(G)$ be such that $G - u - v$ is not factorizable. If there are $M$-balanced paths respectively from $u$ to $v$ and from $v$ to $u$, then $u$ and $v$ are in the same factor-component of $G$.

**Proof.** Let $P$ be an $M$-balanced path from $u$ to $v$, and $Q$ be an $M$-balanced path from $v$ to $u$. Let $x_0, x_1, \ldots$ be the sequence of vertices in $V(P) \cap V(Q)$ defined by the following procedure:

1. $x_0 := v$; $i := 0$;
2. **while** $x_i \neq u$ **do**
3. trace $x_iQu$ from $x_i$ and let $x_{i+1}$ be the first vertex we encounter that is in $V(uPx_i) \setminus \{x_i\}$;
4. $i + +$.
5. **end while**

Note that this procedure surely stops in finite time (since each repetition of the while-loop $x_i$ draw nearer to $u$) and returns $v = x_0, x_1 = u$ for some $l \geq 0$. Note also the next claim, which is easy to see by the definition.

**Claim 2.** (i) Tracing $P$ from $u$, we encounter $x_1, \ldots, x_0$ in this order.
(ii) For each $i = 0, \ldots, l-1$, $uP x_i$ and $x_i Q x_{i+1}$ have only $\{x_i, x_{i+1}\}$ as common vertices.
(iii) For each $i = 0, \ldots, l$, $uP x_i$ and $vQ x_i$ has only $x_i$ as a common vertex.

**Proof.** By the definition procedure, for each $i = 0, \ldots, l - 1$, $x_{i+1}$ is located on $P$ nearer to $u$ than $x_i$ is; this yields [i]. The statement [ii] is also apparent from the definition.

For [iii] note that $vQ x_i = x_0Q x_1 + \cdots + x_{i-1}Q x_i$. Therefore it suffices to prove that for each $0 \leq j \leq i - 1$ $uP x_j$ and $x_jQ x_{j+1}$ have at most $x_i$ as a common vertex; this holds true, since $V(uP x_j) \cap V(x_jQ x_{j+1}) = \{x_j, x_{j+1}\}$ by [ii] and $V(uP x_j) \subseteq V(uP x_j) \setminus \{x_j, \ldots, x_{i-1}\}$ by [i]. □

**Claim 3.** For each $i = 0, \ldots, l - 1$, $x_i Q x_{i+1}$ is an $M$-balanced path from $x_i$ to $x_{i+1}$. For each $i = 0, \ldots, l$, $uP x_i$ and $vQ x_i$ are $M$-balanced paths from $u$ to $x_i$ and from $v$ to $x_i$, respectively.
Proof. We give it by the induction on $i$. If $i = 0$ then both of the claims are rather trivially true, and if $i = l$ then the second claim is trivially true. Therefore let $0 < i < l$ and suppose the claims are true for $i-1$. Since $vQx_{i-1}+1$ is an $M$-balanced path from $v$ to $x_{i-1}$ by the induction hypothesis, $x_{i-1}Qu$ is an $M$-balanced path from $x_{i-1}$ to $u$. Additionally, since $uP_{x_{i-1}}$ is an $M$-balanced path from $u$ to $x_{i-1}$ by the induction hypothesis, it follows that $x_{i-1}Pv$ and the definition procedure yields that $x_{i-1}Qx_{i}$ is an $M$-balanced path from $x_{i-1}$ to $x_{i}$. Therefore, we have that $vQx_{i} = vQx_{i-1}+1x_{i-1}Qx_{i}$ is also an $M$-balanced path, from $v$ to $x_{i}$.

Now note that $uP_{x_{i}} + x_{i}Qv$ forms a path, since they have only $x_{i}$ as a common vertex by Claim 2. Suppose that $uP_{x_{i}}$ is an $M$-saturated path between $u$ and $x_{i}$. Then, $uP_{x_{i}} + x_{i}Qv$ is an $M$-saturated path between $u$ and $v$. This contradicts Proposition 13. Therefore, $uP_{x_{i}}$ is an $M$-balanced path from $u$ to $x_{i}$, and we are done.

Since Claim 4 says $uP_{x_{i}}$ is an $M$-balanced path for each $i = 0, \ldots, l$, it follows by Claim 2 that $x_{i}P_{x_{i+1}}$ is an $M$-balanced path from $x_{i+1}$ to $x_{i}$ for each $i = 0, \ldots, l - 1$. Therefore, $x_{i}Q_{x_{i+1}}$ and $x_{i+1}P_{x_{i}}$ forms an $M$-alternating circuit, since they have only $\{x_{i}, x_{i+1}\}$ as common vertices by Claim 2. Therefore, by Proposition 4, $x_{i}$ and $x_{i+1}$ are contained in the same factor-component of $G$ for each $i = 0, \ldots, l - 1$. This yields that $u$ and $v$ are contained in the same factor-component.

Now we are ready to prove the main theorem, combining up the results in this section.

**Theorem 8.** Let $G$ be a factorizable graph, and $X$ be an odd-maximal barrier of $G$. Let $D_1, \ldots, D_k$ be the DM-components of $H_G(X)$. Let $\hat{V}_1, \ldots, \hat{V}_k$ be the partition of $X \cup D_X$ such that for each $i = 1, \ldots, k$, $\hat{D}_i := G[\hat{V}_i]$ is the expansion of $D_i$. Then, for each $i = 1, \ldots, k$, $S_i := X \cap \hat{V}_i$ coincides with a member of $\mathcal{P}_G(H_i)$ for some $H_i \in \mathcal{G}(G)$, and $\hat{V}_i$ coincides with $U^+(H_i) \setminus U(S_i)$.

**Proof.** Note that such a partition of $X \cup D_X$ surely exists by Proposition 18. Let $M$ be a perfect matching of $G$. Let $i \in \{1, \ldots, k\}$.

**Claim 4.** There is no $M$-saturated path between any two vertices of $S_i$.

**Proof.** This is immediate from Proposition 18.

**Claim 5.** $S_i$ is contained in the same factor-component of $G$, say $H_i$.

**Proof.** Take $u, v \in S_i$ arbitrarily. Note first that there is no $M$-saturated path between $u$ and $v$, by Claim 4. Additionally, there are $M$-balanced paths from $u$ to $v$ and from $v$ to $u$ respectively, which is immediate from Theorem 6 and Proposition 8. Therefore by Theorem 7, $u$ and $v$ are contained in the same factor-component. Thus, we have the claim.

Since $\hat{V}_i$ is separating by Proposition 18.
Remark 1. If \( \hat{V} \) is contained in \( \hat{V}_i \), then by Claim 8 and 9, we have

\[ U \cap \hat{V}_i \]

Therefore, Theorem 8 can be regarded as a generalization of

\[ V \]

Theorem 2.

Proof. Note that \( M_{\hat{V}} \) is a perfect matching of \( \hat{D}_i \), \( S_i \) is an odd-maximal barrier of \( \hat{D}_i \), and \( H_{\hat{D}_i}(S_i) \) is a factorizable bipartite graph with exactly one DM-component by Proposition 18. Thus, by applying Theorem 6 to \( \hat{D}_i \), \( M_{\hat{V}} \), and \( S_i \), there is an \( M \)-saturated path between any \( u \in S_i \) and any \( v \in \hat{V}_i \setminus S_i \), which is contained in \( \hat{V}_i \).

By combining Claims 5 and 6 we obtain that \( S_i \) is a maximal subset of \( V(H_i) \) such that there is no \( M \)-saturated path between any two vertices of it. Hence, by Proposition 20, \( S_i \in \mathcal{P}_{G}(H_i) \) holds.

Claim 8. \( \hat{V}_i \supseteq U^*(H_i) \setminus U(S_i) \).

Proof. Take \( y \in U^*(H_i) \setminus U(S_i) \) arbitrarily. If \( y \in S_i \), then of course \( y \in \hat{V}_i \). Hence hereafter let \( y \in U^*(H_i) \setminus U(S_i) \), and let \( T \in \mathcal{P}_{G}(H_i) \setminus \{S_i\} \) be such that \( y \in U^*(T) \).

Let \( u \in S_i \). There is an \( M \)-saturated path \( P \) between \( u \) and \( y \) by Lemma 6. Hence, by Proposition 15, \( y \in D_{\hat{V}} \). Therefore, there exists \( j \in \{1, \ldots, k\} \) such that \( y \in \hat{V}_j \). By Theorem 6 and Proposition 8, \( D_i \leq D_j \).

If \( i \neq j \), then by Lemma 2 \( P \) has some internal vertices which belong to \( S_j \). However, by Proposition 13, there is no \( M \)-saturated path between any two vertices respectively in \( S_i \) and \( S_j \), and of course \( V(P) \cap S_j \) is disjoint from \( S_i \). This contradicts Lemma 7. Hence, we obtain \( i = j \); accordingly, \( U^*(H_i) \setminus U(S_i) \) is contained in \( \hat{V}_i \).

Claim 9. \( \hat{V}_i \subseteq U^*(H_i) \setminus U(S_i) \).

Proof. Let \( z \in \hat{V}_i \setminus V(H_i) \). By Claim 7 there is an \( M \)-saturated path \( P \) between \( z \) and some vertex of \( S_i \) which is contained in \( \hat{V}_i \). Trace \( P \) from \( z \) and let \( w \) be the first vertex we encounter that is in \( V(H_i) \). Since \( V(H_i) \) is separating, \( zPw \) is an \( M \)-balanced path from \( z \) to \( w \) by Proposition 8. In \( \hat{D}_i/H_i \), \( zPw \) corresponds to an \( M \)-balanced path from \( z \) to the contracted vertex \( h \), corresponding to \( H_i \). Obviously, \( M \) contains a near-perfect matching of \( \hat{D}_i/H_i \) exposing only \( h \).

Therefore, \( \hat{D}_i/H_i \) is factor-critical by Proposition 11. Accordingly, \( \hat{V}_i \) is contained in \( U^*(H_i) \). Additionally, by Claim 7 again and Lemma 4 we can see that \( \hat{V}_i \) is disjoint from \( U(S_i) \) and that \( \hat{V}_i \) is contained in \( U^*(H_i) \setminus U(S_i) \).

Thus, by Claims 8 and 9 we have \( \hat{V}_i = U^*(H_i) \setminus U(S_i) \).

Remark 1. If \( G \) in Theorem 8 is elementary, then \( k = 1 \) and \( \hat{V}_i = V(G) \), which follows by Propositions 6 and 18. Therefore, in this case, Theorem 8 claims that \( \mathcal{P}(G) \) is the family of (odd-) maximal barriers; namely, Theorem 8 coincides with Theorem 7. Therefore, Theorem 8 can be regarded as a generalization of Theorem 2.
Remark 2. Let \( G \) be a factorizable graph. For an arbitrary vertex \( x \in V(G) \), take a maximal barrier of \( G - x \), say \( X \). Then, \( X \cup \{x\} \) is a maximal barrier of \( G \); namely, for any vertex \( x \) there is an odd-maximal barrier that contains \( x \). Therefore, for any \( S \in \mathcal{P}(G) \), there exists an odd-maximal barrier that contains \( S \).

Remark 3. With Király [16], if \( G \) is a non-factorizable graph, then \( \{A(G)\} \cup \mathcal{P}(G[C(G)]) \) are the “atoms” that constitute odd-maximal barriers. For each odd-maximal barrier \( X \), the odd components of \( G - X \) are the components of \( G[D(G)] \) and the odd components of \( G[C(G)] - (X \setminus A(G)) \); here \( G[C(G)] \) forms a factorizable graph and \( X \setminus A(G) \) is an odd-maximal barrier.

4 A Slightly More Efficient Algorithm to Compute the Cathedral Structure

Hereafter we denote by \( n \) and \( m \) the number of vertices and edges (resp. arcs) of input graph (resp. digraph), respectively. Note that factorizable graphs satisfy \( m = \Omega(n) \) and accordingly \( O(n + m) = O(m) \).

In [4, 5], we show that the partial order \( \prec \) and the generalized canonical partition can be computed in \( O(nm) \) time if there input a factorizable graph. The algorithm is composed of three stages, each of which is \( O(n) \) times iteration of \( O(m) \) time procedure of growing alternating trees. It first computes the factor-components, then computes \( \prec \) and \( \mathcal{P}(G) \) respectively.

With the results in this paper, we present another \( O(nm) \) time algorithm to compute them. The upper bound of its time complexity is the same as the known one, however the factor-components, \( \prec \), and \( \mathcal{P}(G) \) are here computed simultaneously. Thus, it has some possibility of exhibiting a bit more efficiency.

Theorem 9 (Micali & Vazirani [18], Vazirani [19]). A maximum matching of a graph can be computed in \( O(\sqrt{mn}) \) time.

Theorem 10 (Edmonds [20], Tarjan [21], Gabow & Tarjan [22]). Let \( G \) be a graph with \( m = \Omega(n) \) and suppose we are given a perfect matching of \( G \). Then, \( D(G), \ A(G), \) and \( C(G) \) can be computed in \( O(m) \) time.

Theorem 11 (Dulmage & Mendelsohn [9–12]). For any bipartite factorizable graph \( G \), the Dulmage-Mendelsohn decomposition of \( G \) can be computed in \( O(m) \) time.

Proposition 21 (folklore, see [12]). Let \( D \) be a digraph, and \( \mathcal{D} \) be the set of strongly-connected components of \( D \). For \( D_1, D_2 \in \mathcal{D} \) we say \( D_1 \to D_2 \) if for any \( u \in V(D_1) \) and any \( v \in V(D_2) \) there is a dipath from \( u \) to \( v \). Then, \( \to \) is a partial order on \( \mathcal{D} \).

Proposition 22 (folklore, see [8]). For any digraph \( D \), the strongly connected components of \( D \) can be computed in \( O(n + m) \) time.
Below is the new algorithm, Algorithm 1:

Require: a factorizable graph $G$
Ensure: the generalized canonical partition $\mathcal{P}(G)$ and the digraph $\text{Aux}(G)$ representing $(G(G), \triangleleft)$

1: compute a perfect matching $M$ of $G$;
2: $U := V(G)$; initialize $f : V(G) \rightarrow \{0, 1\}$ by 0;
3: $A := \emptyset$; $\mathcal{P}(G) := \emptyset$;
4: while $U \neq \emptyset$ do
5: choose $u \in U$;
6: compute $X := A(G - u) \cup \{u\}$;
7: compute the DM-decomposition of $H_G(X)$;
8: for all DM-component $D$ of $H_G(X)$ do
9: let $S := X \cap V(D)$; choose arbitrary $v \in S$;
10: if $f(v) = 0$ then
11: $\mathcal{P}(G) := \mathcal{P}(G) \cup \{S\}$;
12: let $\hat{D} \subseteq G$ be the expansion of $D$;
13: for all $x \in S$ do
14: for all $y \in V(\hat{D}) \setminus X$ do
15: $A := A \cup \{(x, y)\}$;
16: end for
17: $U := U \setminus \{x\}$; $f(x) := 1$;
18: end for
19: end if
20: end for
21: end while
22: output $\mathcal{P}(G)$;
23: $\text{Aux}(G) := (V(G), A)$; decompose $\text{Aux}(G)$ into its strongly-connected components and output it; STOP.

Proposition 23. While Algorithm 1 is running,

(i) $X = A(G - u) \cup \{u\}$ of Line [7] is an odd-maximal barrier of $G$,
(ii) $S$ defined at Line [9] coincides with a member of $\mathcal{P}(G)$, and
(iii) $V(\hat{D}) \setminus X$ at Line [14] coincides with $cU(S)$.

Proof. The statement [i] follows by a simple counting argument. Therefore, [ii] and [iii] follows by Theorem 8. □

Lemma 8. Let $G$ be a factorizable graph and $\text{Aux}(G) = (V(G), A)$ be the digraph obtained by inputting $G$ to Algorithm 1. Let $H_1, H_2 \in G(G)$, $u \in V(H_1)$, and $v \in V(H_2)$.

(i) If $(u, v) \in A$, then $H_1 \triangleleft H_2$.
(ii) If there exists a dipath from $u$ to $v$ in $\text{Aux}(G)$, then $H_1 \triangleleft H_2$.

Given $H \in G(G)$ and $S \in \mathcal{P}(G(H))$, we denote $U^*(H) \setminus U^*(S)$ as $cU(S)$.
Proof. The arc \((u, v)\) is added to \(A\) only at Line 15 if \(u \in X \cap V(\hat{D})\) and \(v \in V(\hat{D}) \setminus X\). Thus \([1]\) follows by Proposition 23. Hence \([4]\) follows by the transitivity of \(\prec\).

Lemma 9. Let \(G\) be a factorizable graph and \(Aux(G) = (V(G), A)\) be the digraph obtained by inputting \(G\) to Algorithm 1. Let \(H_1, H_2 \in \mathcal{G}(G)\) be such that \(H_1 \prec H_2\). Then, for any \(u \in V(H_1)\) and \(v \in V(H_2)\), there is a dipath from \(u\) to \(v\) in \(Aux(G)\).

Proof. Let \(S \in \mathcal{P}_G(H_1)\) be such that \(u \in S\). First suppose that \(v \in ^cU(S)\). Then, \((u, v)\) is added to \(A\) at Line 15 when \(X \cap V(D)\) of Line 13 coincides with \(S\), which surely occurs by Proposition 23. Hence, the claim holds for this case.

Now suppose the other case that \(v \in U^*(S)\). Take \(T \in \mathcal{P}_G(H_1) \setminus \{S\}\) and \(w \in T\) arbitrarily. The arc \((u, w)\) is added to \(A\) at Line 15 when \(S\) coincides with \(X \cap V(D)\) of Line 13 so is the arc \((w, v)\) when \(T\) coincides with \(X \cap V(D)\). Therefore the dipath \(uw + vw\) satisfies the claim for this case, and we are done.

Theorem 12. Let \(G\) be a factorizable graph and \(Aux(G) = (V(G), A)\) be the digraph obtained by inputting \(G\) to Algorithm 1. Then, \(H \in \mathcal{G}(G)\) holds if and only if there is a strongly-connected component \(D\) of \(Aux(G)\) with \(V(H) = V(D)\). Additionally, for any \(H_1, H_2 \in \mathcal{G}(G)\), \(H_1 \prec H_2\) holds if and only if \(D_1 \to D_2\), where \(D_i\) is the strongly-connected component of \(Aux(G)\) with \(V(H_i) = V(D_i)\), for each \(i = 1, 2\).

Proof. Combining Lemmas 8 and 9, we immediately obtain the following claim:

Claim 10. \(H_1 \prec H_2\) holds if and only if for any \(u \in V(H_1)\) and any \(v \in V(H_2)\) there is a dipath from \(u\) to \(v\) in \(Aux(G)\).

Therefore, we are done by Proposition 21.

Theorem 13. Given a factorizable graph \(G\), the poset \((\mathcal{G}(G), \prec)\) and the generalized canonical partition \(\mathcal{P}(G)\) can be computed in \(O(nm)\) time by Algorithm 1.

Proof. The correctness follows by Proposition 23 and Theorem 12.

Hereafter we prove the complexity. Line 1 costs \(O(\sqrt{nm})\) time by Theorem 9. Line 2 costs \(O(n)\) time, and Line 3 costs \(O(1)\) time. Lines 4 to 7 cost \(O(m)\) time per each iteration of the while-loop in Line 4. As the while-loop in Line 4 is repeated \(O(n)\) times, they cost \(O(nm)\) time over the whole algorithm.

Each operations in Lines 8 to 10 costs \(O(1)\) time per iteration, and they are iterated \(O(n^2)\) time over the whole computation; therefore, they cost \(O(n^2)\) time.

Note that \(f(v) = 0\) at Line 10 holds true for at most \(n\) times. Therefore, Lines 11 and 12 cost \(O(n)\) time. The number of repetition of Lines 13 to 19 is bounded by \(|A| = O(n^2)\). Therefore, the operations there costs \(O(n^2)\) over the algorithm.

\(\square\)
References

1. Carvalho, M.H., Cherian, J.: An $O(VE)$ algorithm for ear decompositions of matching-covered graphs. ACM Transactions on Algorithms 1(2) (2005) 324–337
2. Lovász, L., Plummer, M.D.: Matching Theory. AMS Chelsea Publishing (2009)
3. Carvalho, M.H., Lucchesi, C.L., Murty, U.S.R.: The matching lattice. In Reed, B., Sales, C.L., eds.: Recent Advances in Algorithms and Combinatorics. Springer-Verlag (2003)
4. Kita, N.: A partially ordered structure and a generalization of the canonical partition for general graphs with perfect matchings. CoRR abs/1205.3816 (2012)
5. Kita, N.: A partially ordered structure and a generalization of the canonical partition for general graphs with perfect matchings. In: Proceedings of The 23rd International Symposium on Algorithms and Computation (ISAAC 2012). (2012) 85–94
6. Kita, N.: A generalization of the Dulmage-Mendelsohn decomposition for general graphs. preprint
7. Frank, A.: Conservative weightings and ear-decompositions of graphs. Combinatorica 13(1) (1993) 65–81
8. Schrijver, A.: Combinatorial Optimization: Polyhedra and Efficiency. Springer-Verlag (2003)
9. Dulmage, A.L., Mendelsohn, N.S.: Coverings of bipartite graphs. Canadian Journal of Mathematics 10 (1958) 517–534
10. Dulmage, A.L., Mendelsohn, N.S.: A structure theory of bipartite graphs of finite exterior dimension. Transactions of the Royal Society of Canada, Section III 53 (1959) 1–13
11. Dulmage, A.L., Mendelsohn, N.S.: Two algorithms for bipartite graphs. Journal of the Society for Industrial and Applied Mathematics 11(1) (1963) 183–194
12. Murota, K.: Matrices and matroids for systems analysis. Springer-Verlag (2000)
13. Kotzig, A.: Z teórié konéétné grafof s lineéarnym faktorem. I (in slovak). Mathematica Slovaca 9(2) (1959) 73–91
14. Kotzig, A.: Z teórié konéétné grafof s lineéarnym faktorem. II (in slovak). Mathematica Slovaca 9(3) (1959) 136–159
15. Kotzig, A.: Z teórié konéétné grafof s lineéarnym faktorem. III (in slovak). Mathematica Slovaca 10(4) (1960) 205–215
16. Király, Z.: The calculus of barriers. Technical Report TR-9801-2, ELTE (1998)
17. Kita, N.: A canonical characterization of the family of barriers in general graphs. CoRR abs/1212.5960 (2012)
18. Micall, S., Vazirani, V.V.: An $O(\sqrt{|V|}\cdot|E|)$ algorithm for finding maximum matching in general graphs. In: Proceedings of the 21st Annual IEEE Symposium on Foundations of Computer Science. (1980) 17–27
19. Vazirani, V.V.: A theory of alternating paths and blossoms for proving correctness of the $O(\sqrt{V}E)$ general graph maximum matching algorithm. Combinatorica 14 (1994) 71–109
20. Edmonds, J.: Paths, trees and flowers. Canadian Journal of Mathematics 17 (1965) 449–467
21. Tarjan, R.E.: Data Structures and Network Algorithms. Society for Industrial and Applied Mathematics (1983)
22. Gabow, H.N., Tarjan, R.E.: A linear-time algorithm for a special case of disjoint set union. Journal of Computer and System Sciences 30 (1985) 209–221
Appendix: Backgrounds on Odd-maximal Barriers

Here we are going to explain more details on odd-maximal barriers which are omitted in Section 2.3. Readers familiar with matching theory might skip this section.

Maximal Barriers vs. Odd-maximal Barriers

As we mention in Section 2.3, for elementary graphs, the notion of maximal barriers and the notion of odd-maximal barriers are equivalent. This fact is easy to see using known properties; we are going to show it in the following. The next two propositions are to see Proposition 26:

Proposition 24 (see [2] or [16]). Let $G$ be a graph and $X \subseteq V(G)$ be an odd-maximal barrier of $G$. Then, $X$ is a maximal barrier if and only if $C_X = \emptyset$.

Proof. The necessity part is obvious by the definition. For the sufficiency part, let $C_X \neq \emptyset$ and take $u \in C_X$ arbitrarily. Then $X \cup \{u\}$ is also a barrier of $G$, contradicting $X$ being a maximal one. $\square$

Proposition 25 (see [2] or [16]). Let $G$ be an elementary graph and $X$ be a barrier of $G$. Then, $C_X = \emptyset$.

Proof. If $C_X \neq \emptyset$, then since no the edges of $E[X, C_X]$ are allowed as stated in Proposition 2, we can see that $G$ is not elementary, a contradiction. $\square$

Proposition 26. For an elementary graph $G$, if $X \subseteq V(G)$ is an odd-maximal barrier then it is also a maximal barrier.

Proof. This is by combining Proposition 24 and Proposition 25. $\square$

Since maximal barriers are apparently odd-maximal barriers by the definitions, now we have that these two notions are equivalent for elementary graphs by Proposition 26.

Why It Suffices to Work on Factorizable Graphs

The following statements, leading to Proposition 9, show that in order to know canonical structures of odd-maximal barriers in general graphs, it suffices to work on factorizable graphs.

Proposition 27 (folklore, see [2] or [16]). Let $G$ be a graph, $X \subseteq V(G)$ be a barrier of $G$, and $Y \subseteq V(G)$ be such that $X \subseteq Y$. Then, $Y$ is a barrier of $G$ if and only if $Y \setminus X$ is a union of barriers of some connected components of $G - X$.

Given a graph $G$, we define $D(G)$ as the set of vertices that can be respectively exposed by maximum matchings, $A(G)$ as $N(D(G))$ and $C(G)$ as $V(G) \setminus (D(G) \cup A(G))$. There is a well-known theorem stating that $A(G)$ forms a barrier with special properties, called the Gallai-Edmonds structure theorem [2]; the next one is a part of it.
Proposition 28. Let $G$ be a graph. Then, $A(G)$ is an odd-maximal barrier of $G$ such that $D_{A(G)} = D(G)$ and $C_{A(G)} = C(G)$.

Additionally, Király shows that $A(G)$ is the minimum odd-maximal barriers in any graph $G$.

Theorem 14 (Király [16]). Let $G$ be a graph, and $X \subseteq 2^{V(G)}$ be the family of the odd-maximal barriers of $G$. Then, $\bigcap_{X \in \mathcal{X}} X = A(G)$.

Therefore, combining up Proposition 27 and Proposition 28 and Theorem 14 we can see the following:

Proposition 29. Let $G$ be a graph. A set of vertices $S \subseteq V(G)$ is an odd-maximal barrier of $G$ if and only if it is a disjoint union of $A(G)$ and an odd-maximal barrier of the factorizable subgraph $G[C(G)]$. Now let $S$ be an odd-maximal barrier. Then, the odd components of $G - S$ are the components of $G[D(G)]$ and the odd components of $G[C(G)] - (S \setminus A(G))$.

Therefore, we can see that to obtain the structure of odd-maximal barriers and the odd components associated with them in general graphs, it suffices to investigate factorizable graphs.