LEHMER’S PROBLEM FOR ARBITRARY GROUPS

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Abstract. We consider the problem whether for a group $G$ there exists a constant $\Lambda(G) > 1$ such that for any $(r,s)$-matrix $A$ over the integral group ring $\mathbb{Z}G$ the Fuglede-Kadison determinant of the $G$-equivariant bounded operator $L^2(G)^r \to L^2(G)^s$ given by right multiplication with $A$ is either one or greater or equal to $\Lambda(G)$. If $G$ is the infinite cyclic group and we consider only $r = s = 1$, this is precisely Lehmer’s problem.

0. Introduction

Lehmer’s problem is the question whether the Mahler measure of a polynomial with integer coefficients is either one or bounded from below by a fixed constant $\Lambda > 1$. If one views the polynomial as an element in the integral group ring $\mathbb{Z}[G]$ ring of $\mathbb{Z}$, then its Mahler measure agrees with the Fuglede-Kadison determinant of the $\mathbb{Z}$-equivariant bounded operator $r_A^{(2)} : L^2(\mathbb{Z}) \to L^2(\mathbb{Z})$ given by right multiplication with $p$. This suggests to consider for any group $G$ the following generalization.

Let $A$ be an $(r,s)$-matrix over the integral group ring $\mathbb{Z}G$. We propose to study the problem whether there is a constant $\Lambda(G) > 1$ such that the Fuglede-Kadison determinant of $r_A^{(2)} : L^2(G)^r \to L^2(G)^s$ is either one or larger or equal to $\Lambda(G)$. If we only allow $r = s = 1$, we denote such a constant by $\Lambda_1(G)$. If we consider only the case $r = s$ or the case $r = s = 1$ and additionally require that $r_s^{(2)} : L^2(G)^r \to L^2(G)^s$ is a weak isomorphism, or, equivalently, is injective, we denote such a constant by $\Lambda_u(G)$ or $\Lambda_1^u(G)$. Lehmer’s problem is equivalent to the question whether $\Lambda_1(\mathbb{Z}) > 1$ holds.

For obvious reasons we have $\Lambda(G) \leq \Lambda_u(G)$, $\Lambda_1(G) \leq \Lambda_1^u(G)$, $\Lambda(G) \leq \Lambda_1(G)$, and $\Lambda_u(G) \leq \Lambda_1^u(G)$. Since for a group $G$ which contains $\mathbb{Z}$ as a subgroup we have $\Lambda_1^u(G) \leq \Lambda_1^u(\mathbb{Z}) = \Lambda_1(\mathbb{Z})$ by Lemma 5.1, we see that a counterexample to Lehmer’s problem would imply $\Lambda(G) = \Lambda_u(G) = \Lambda_1(G) = \Lambda_1^u(G) = 1$ for any group which contains $\mathbb{Z}$ as subgroup. Hence all the discussions in this paper are more or less void if a counterexample to Lehmer’s problem exists what is not known and fortunately not expected to be true.

If there is no upper bound on the order of finite subgroups of $G$, then $\Lambda_1^u(G) = 1$ by Remark 12.1. Indeed, there is a finitely presented group $G$ with $\Lambda(G) = \Lambda_u(G) = \Lambda_1(G) = \Lambda_1^u(G) = 1$, see Example 12.2. Therefore we will concentrate on torsionfree groups.

The most optimistic scenario would be that for any torsionfree group $G$ all the constants $\Lambda(G)$, $\Lambda_u(G)$, $\Lambda_1(G)$ and $\Lambda_1^u(G)$ can be chosen to be the Mahler measure $M(L)$ of Lehmer’s polynomial $L(z) := z^{10} + z^9 - z^7 - z^6 - z^5 - z^4 - z^3 + 1$. But this is not the case in general, there is a hyperbolic closed 3-manifold, the Week’s manifold $W$ satisfying $\Lambda_u(\pi_1(W)) < \Lambda(\mathbb{Z})$, see Example 11.2.
We will not make any new contributions to Lehmer’s problem in this article. However, we think that it is interesting to put Lehmer’s problem, which itself is already very interesting and has many intriguing connections to number theory, topology and geometry, in a more general context. Moreover, we will give some evidence for the hope that \( \Lambda(G) > 1 \) or even \( \Lambda(G) \geq \Lambda_1(\mathbb{Z}) \) holds for some torsionfree groups \( G \). Namely, we will show in Theorem 8.1 that \( \Lambda(\mathbb{Z}^d) = \Lambda_\omega(\mathbb{Z}^d) = \Lambda_1(\mathbb{Z}^d) = \Lambda_\omega(\mathbb{Z}^d) \) holds for all natural numbers \( d \) and that this value is actually independent of \( d \). Moreover, we will explain in Section 7 how to use approximation techniques to potentially extend the class of groups for which the problem above has a positive answer.

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## 1. Lehmer’s problem

Let \( p(z) \in \mathbb{C}[\mathbb{Z}] = \mathbb{C}[z, z^{-1}] \) be a non-trivial element. Its Mahler measure is defined by

\[
M(p) := \exp \left( \int_{S^1} \ln(|p(z)|) d\mu \right).
\]

By Jensen’s equality we have

\[
\exp \left( \int_{S^1} \ln(|p(z)|) d\mu \right) = |c| \cdot \prod_{\substack{i=1,2,...,r \atop |a_i| > 1}} |a_i|,
\]
Theorem 2.1 (Mahler measure and Fuglede-Kadison determinant over a group $G$ to be the infimum of the set of Fuglede-Kadison determinants

\[ \det_{\Lambda}(G) = \inf \{ \det_{\Lambda}(p) : p \in \Lambda \} \]

\[ \det_{\Lambda}(G) = \inf \{ \det_{\Lambda}(p) : p \in \Lambda \} \]

Problem 1.3 (Lehmer’s Problem). Does there exist a constant $\Lambda > 1$ such that for all non-trivial elements $p(z) \in \mathbb{Z}[z] = \mathbb{Z}[z, z^{-1}]$ with $M(p) \neq 1$ we have

\[ M(p) \geq \Lambda. \]

Remark 1.4 (Lehmer’s polynomial). There exists a candidate for which the minimal Mahler measure is attained, namely, Lehmer’s polynomial

\[ L(z) := z^{10} + z^9 - z^7 - z^6 - z^5 - z^3 + z + 1. \]

It is conceivable that for any non-trivial element $p \in \mathbb{Z}[z]$ with $M(p) > 1$ holds.

\[ M(p) \geq M(L) = 1.17628 \ldots \]

For a survey on Lehmer’s problem we refer for instance to \[13, 14, 15, 22].

2. THE MAHLER MEASURE AS FUGLEDE-KADISON DETERMINANT

The following result is proved in \[26, (3.23) \text{ on page } 136\]. We will recall the Fuglede-Kadison determinant and its basic properties in the Appendix \[13\].

Theorem 2.1 (Mahler measure and Fuglede-Kadison determinants over $\mathbb{Z}$). Consider an element $p = p(z) \in \mathbb{C}[z] = \mathbb{C}[z, z^{-1}]$. It defines a bounded $\mathbb{Z}$-equivariant operator $r^{(2)}_p : L^2(\mathbb{Z}) \to L^2(\mathbb{Z})$ by multiplication with $p$. Suppose that $p$ is not zero.

Then the Fuglede-Kadison determinant $\det_{\Lambda}(\mathbb{Z})(r^{(2)}_p)$ of $r^{(2)}_p$ agrees with the Mahler measure, i.e.,

\[ \det_{\Lambda}(\mathbb{Z})(r^{(2)}_p) = M(p). \]

Notice the identification of the Fuglede-Kadison determinant with the Mahler measure holds also for non-trivial elements $p$ in $\mathbb{C}[z^d] = \mathbb{C}[z^{i_1}, z^{i_2}, \ldots, z^{i_s}]$, where $d$ is any natural number, see \[26, \text{ Example } 3.13 \text{ on page } 128\].

3. LEHMER’S PROBLEM FOR ARBITRARY GROUPS

Given a group $G$, we consider $L^2(G)$ as a Hilbert space with the obvious isometric linear $G$-action from the left and write an element in $L^2(G)^r := \bigoplus_{i=1}^r L^2(G)$ as a row $(x_1, x_2, \ldots, x_r)$ for $x_i \in L^2(G)$, in other words as a $(1,r)$-matrix. Given a $A$ in $M_{r,s}(\mathbb{Z}G)$ or $M_{r,s}(\mathbb{C}G)$, we obtain by right multiplication with $A$ a bounded $G$-equivariant operator

\[ (3.1) \quad r^{(2)}_A : L^2(G)^r \to L^2(G)^s, \quad (x_i)_{i=1,2,\ldots,r} \mapsto \left( \sum_{k=1}^r x_k \cdot a_{kj} \right)_{j=1,2,\ldots,s}. \]

Notice that with these conventions we have $r^{(2)}_{AB} = r^{(2)}_B \circ r^{(2)}_A$ for $A \in M_{r,s}(\mathbb{Z}G)$ and $B \in M_{s,t}(\mathbb{C}G)$.

Definition 3.2 (Lehmer’s constant of a group). Define Lehmer’s constant of a group $G$

\[ \Lambda(G) \in [1, \infty) \]

to be the infimum of the set of Fuglede-Kadison determinants

\[ \det_{\Lambda}(\mathbb{Z})(r^{(2)}_A : L^2(G)^r \to L^2(G)^s), \]
A better behavior under approximation, see Section 7. The problem is that for a
differentials are given by \((r, s)\) and we denote the corresponding infimum by
\[ \Lambda_1(G) \in [1, \infty). \]

If we only allow \((1,1)\)-matrices \(A\) with \(\det_{\mathcal{N}(G)}(r_A) > 1\), we denote the corre-
sponding infimum by
\[ \Lambda_1^w(G) \in [1, \infty). \]

If we only allow \((r,r)\)-matrices \(A\) for any natural number \(r\) such that \(r_A^{(2)} : L^2(G)^r \to L^2(G)^r\) is a weak iso-

morphism, or, equivalently, is injective, and \(\det_{\mathcal{N}(G)}(r_A) > 1\), we denote the corre-
sponding infimum by
\[ \Lambda_1^w(G) \in [1, \infty). \]

Obviously we have
\[ \Lambda(G) \leq \Lambda^w(G) \leq \Lambda^w_1(G); \]
\[ \Lambda(G) \leq \Lambda_1(G) \leq \Lambda_1^w(G). \]

A priori there is no obvious relation between \(\Lambda^w(G)\) and \(\Lambda_1(G)\).

**Problem 3.3** (Lehmer’s problem for arbitrary groups). For which groups \(G\) is
\(\Lambda(G) > 1\), \(\Lambda_1(G) > 1\), \(\Lambda^w(G) > 1\) or \(\Lambda_1^w(G) > 1\) true?

For amenable groups this problem is already considered in [3, Question 4.7]. See
also [3, 9].

**Remark 3.4** (\(\Lambda_1(Z)\) and Lehmer’s problem). In view of Theorem 2.1.1 we see that
Lehmer’s Problem 1.3 is equivalent to the question whether \(\Lambda_1(Z) > 1\). In view
of Remark 1.4 one would expect that \(\Lambda_1(Z)\) is the Mahler measure of Lehmer’s
polynomial. We conclude \(\Lambda_1(Z) = \Lambda^w_1(Z) = \Lambda(Z) = \Lambda^w(Z)\) from Theorem 5.1.

**Remark 3.5** (Why matrices and why \(\Lambda(G)\)?). We are also interested besides \(\Lambda_1(G)\)
in the numbers \(\Lambda^w(G), \Lambda^w_1(G)\) and \(\Lambda(G)\) for the following reasons. There is the
notion of \(L^2\)-torsion, see for instance [25, Chapter 3], which is essential defined
in terms of the Fuglede-Kadison determinants of the differentials of the \(L^2\)-chain
complex of the universal covering of a finite CW-complex or closed manifold. These
differentials are given by \((r,s)\)-matrices over \(ZG\), where \(r\) and \(s\) can be any natural
numbers. Therefore it is important to consider matrices and not only elements in
\(ZG\). Moreover, these differentials are not injective in general.

Another reason to consider matrices is the possibility to consider restriction to
a subgroup of finite index since this passage turns a \((1,1)\)-matrix into a matrix of
the size \([G : H],[G : H]\).

One advantage of \(\Lambda(G)\) or \(\Lambda_1(G)\) in comparison with \(\Lambda^w(G)\) or \(\Lambda^w_1(G)\) is the
better behavior under approximation, see Section 7. The problem is that for a
square matrix \(A\) over \(G\) such that \(r_A^{(2)}\) is a weak isomorphism, the operator \(r_A^{(2)}\)
is not necessarily again a weak isomorphism, if we have a not necessarily injective
group homomorphism \(p: G \to Q\) and \(p(A)\) is the reduction of \(A\) to a matrix over
\(Q\).

**Remark 3.6** (Dobrowolski’s estimate). Dobrowolski [12] shows for a monic poly-
nomial \(p(z)\) with \(p(0) \neq 0\) which is not a product of cyclotomic polynomials
\[ M(p) \geq 1 \frac{1}{a \exp(bk^k)}, \]
where \(k\) is the number of non-zero coefficients of \(p\) and \(a\) and \(b\) are given constants.
Remark 4.2 (Status of the Determinant Conjecture) Theorem 5], [26, Section 13.2 on pages 459 ff], [30, Theorem 1.21].

\[ \text{L with integer coefficients. This is expected to be true for the Fuglede-Kadison determinant for all groups, namely, there is the} \]

Conjecture 4.1 (Determinant Conjecture). Let \( G \) be a group. Then for any \( A \in M_{r,s}(\mathbb{Z}G) \) the Fuglede-Kadison determinant of the morphism \( r_A^{(2)} : L^2(G)^r \to L^2(G)^s \) given by right multiplication with \( A \) satisfies

\[ \det_{N(G)}(r_A^{(2)}) \geq 1. \]

Remark 4.2 (Status of the Determinant Conjecture). The following is known for the class \( \mathcal{D} \) of groups for which the Determinant Conjecture \[4.4\] is true, see [13, Theorem 5], [26, Section 13.2 on pages 459 ff], [30, Theorem 1.21].

1. Amenable quotient
   Let \( H \subset G \) be a normal subgroup. Suppose that \( H \in \mathcal{D} \) and the quotient \( G/H \) is amenable. Then \( G \in \mathcal{D} \);

2. Colimits
   If \( G = \text{colim}_{i \in I} G_i \) is the colimit of the directed system \( \{G_i \mid i \in I\} \) of groups indexed by the directed set \( I \) (with not necessarily injective structure maps) and each \( G_i \) belongs to \( \mathcal{D} \), then \( G \) belongs to \( \mathcal{D} \);

3. Inverse limits
   If \( G = \text{lim}_{i \in I} G_i \) is the limit of the inverse system \( \{G_i \mid i \in I\} \) of groups indexed by the directed set \( I \) and each \( G_i \) belongs to \( \mathcal{D} \), then \( G \) belongs to \( \mathcal{D} \);

4. Subgroups
   If \( H \) is isomorphic to a subgroup of a group \( G \) with \( G \in \mathcal{D} \), then \( H \in \mathcal{D} \);

5. Quotients with finite kernel
   Let \( 1 \to K \to G \to Q \to 1 \) be an exact sequence of groups. If \( K \) is finite and \( G \) belongs to \( \mathcal{D} \), then \( Q \) belongs to \( \mathcal{D} \);

6. Sofic groups belong to \( \mathcal{D} \).

The class of sofic groups is very large. It is closed under direct and free products, taking subgroups, taking inverse and direct limits over directed index sets, and under extensions with amenable groups as quotients and a sofic group as kernel. In particular it contains all residually amenable groups. One expects that there exists non-sofic groups but no example is known. More information about sofic groups can be found for instance in [14] and [29].

Remark 4.3 (Invertible matrices and the Determinant Conjecture). Let \( G \) be a group. Consider a matrix \( A \in \text{Gl}_r(\mathbb{Z}G) \). Then we get from Theorem \[13,18\] \[1\]

\[ \det_{N(G)}(r_A^{(2)}) \cdot \det_{N(G)}(r_{A^{-1}}^{(2)}) = 1. \]

If \( G \) satisfies the Determinant Conjecture \[4.1\] we get

\[ (4.4) \quad \det_{N(G)}(r_A) = 1 \quad \text{for} \quad A \in \text{Gl}_r(\mathbb{Z}G). \]

The argument appearing in the proof of [27, Theorem 6.7 (2)] shows that the \( K \)-theoretic Farrell-Jones Conjecture for \( \mathbb{Z}G \) also implies \[4.4\].
5. Basic properties of Lehmer’s constant for arbitrary groups

Lemma 5.1.  (1) If $H$ is a subgroup of $G$, then
\[
\Lambda(G) \leq \Lambda(H);
\]
\[
\Lambda_1(G) \leq \Lambda_1(H);
\]
\[
\Lambda^w(G) \leq \Lambda^w(H);
\]
\[
\Lambda^t(G) \leq \Lambda^t(H);
\]

(2) If $H \subseteq G$ has finite index, then
\[
\Lambda(H)^{G:H} \leq \Lambda(G);
\]
\[
\Lambda^w(H)^{G:H} \leq \Lambda^w(G);
\]

(3) We have
\[
\Lambda(\{1\}) = \Lambda_1(\{1\}) = \Lambda^w(\{1\}) = \Lambda^t(\{1\}) = 2,
\]
and
\[
\Lambda^w(\mathbb{Z}/2) = \Lambda^w(\mathbb{Z}/2) = \sqrt{3};
\]
\[
\Lambda(\mathbb{Z}/2) = \Lambda_1(\mathbb{Z}/2) = \sqrt{2};
\]

(4) If $G$ is finite and $|G| \geq 3$, we get
\[
2^{|G|^{-1}} \leq \Lambda(G), \Lambda^w(G), \Lambda_1(G), \Lambda^t(G) \leq (|G| - 1)^{|G|^{-1}};
\]

(5) Let $G$ be a group. Then
\[
\Lambda(G) = \inf \{ \Lambda(H) \mid H \subseteq G \text{ finitely generated subgroup} \};
\]
\[
\Lambda_1(G) = \inf \{ \Lambda_1(H) \mid H \subseteq G \text{ finitely generated subgroup} \};
\]
\[
\Lambda^w(G) = \inf \{ \Lambda^w(H) \mid H \subseteq G \text{ finitely generated subgroup} \};
\]
\[
\Lambda^t(G) = \inf \{ \Lambda^t(H) \mid H \subseteq G \text{ finitely generated subgroup} \}.
\]

Proof. (1) Consider $A \in M_{r,s}(\mathbb{Z}H)$. Let $i : H \to G$ be the inclusion. By applying the ring homomorphism $\mathbb{Z}H \to \mathbb{Z}G$ induced by $i$ to the entries of $A$, we obtain a matrix $i_*A \in M_{r,s}(\mathbb{Z}G)$. Then we get
\[
\det_N(r^{(2)}_{i_*A}) = \det_N(r^{(2)}_A)
\]
from Theorem 3.3.18 (3) and hence $\Lambda(H) \leq \Lambda(G)$ and $\Lambda_1(H) \leq \Lambda_1(G)$.

If $r = s$ and $r^{(2)}_A$ is injective, then $r^{(2)}_A$ is injective because of [20] Lemma 1.24 (3) on page 30. This implies $\Lambda^w(G) \leq \Lambda^w(H)$ and $\Lambda^t(G) \leq \Lambda^t(H)$.

(2) Consider a matrix $A \in M_{r,s}(\mathbb{Z}G)$. We have introduced the bounded $G$-equivariant operator $r^{(2)}_A : L^2(G)^r \to L^2(G)^s$ in (5.1). Let $i^*r^{(2)}_A : i^*L^2(G)^r \to i^*L^2(G)^s$ be the bounded $H$-equivariant operator obtained by restricting the $G$-action to an $H$-action. Since $[G : H]$ is finite, there is an $H$-equivariant isometric isomorphism of Hilbert spaces from $L^2(H)^{|G:H|}$ to $i^*L^2(G)$. Hence for an appropriate matrix $B \in M_{r,[G:H],s,[G:H]}(\mathbb{Z}H)$ the bounded $H$-equivariant operator $i^*r^{(2)}_A : i^*L^2(G)^r \to i^*L^2(G)^s$ can be identified with $r^{(2)}_B : L^2(H)^{r_{[G:H]}} \to L^2(H)^{s_{[G:H]}}$. We conclude from Theorem 3.3.18 (4)
\[
\det_N(r^{(2)}_B) = \det_N(i^*r^{(2)}_A) = \det_N(r^{(2)}_A)^{|G:H|}.
\]
This implies $\Lambda(G)^{|G:H|} \geq \Lambda(H)$.

If $r = s$ and $r^{(2)}_A$ is injective, then $r^{(2)}_A$ is injective. This implies $\Lambda^w(G)^{|G:H|} \geq \Lambda^w(H)$.

(3) We have $\Lambda(\{1\}) = \Lambda_1(\{1\}) = \Lambda^w(\{1\}) = \Lambda^t(\{1\}) = 2$, since for any $A \in$
$M_{r,s}(Z) = M_{r,s}(Z[\{1\})$ the Fuglede-Kadison determinant $\det_{N(1)}(r^{(2)}_A)$ is a non-trivial integer by the argument in the proof of [25] Lemma 13.12 on page 459.

Consider $A \in M_{r,s}(Z[Z/2])$. It induces a $Z[Z/2]$-homomorphism $r_A: Z[Z/2] \to Z[Z/2]$. There exists an obvious short exact sequence of $Z[Z/2]$-modules $0 \to Z^- \to Z[Z/2] \to Z^+ \to 0$, where $Z$ is the underlying abelian group of $Z^\pm$ and the generator of $Z/2$ acts by $\pm \text{id}$ on $Z^\pm$. We obtain a commutative diagram of endomorphisms of finitely generated free $Z$-modules

$$
\begin{array}{ccccccc}
0 & \to & Z[Z/2]^r & \otimes_{Z[Z/2]} Z^- & \to & Z[Z/2]^r & \otimes_{Z[Z/2]} Z^+ & \to & 0 \\
\quad & \quad & \downarrow r_A & \otimes_{Z[Z/2]} \text{id}_{Z^-} & \quad & \downarrow r_A & \otimes_{Z[Z/2]} \text{id}_{Z^+} & \quad & \quad \\
0 & \to & Z[Z/2]^r & \otimes_{Z[Z/2]} Z^- & \to & Z[Z/2]^r & \otimes_{Z[Z/2]} Z^+ & \to & 0 
\end{array}
$$

This implies

$$
\det_{Z}(r_A) = \det_{Z}(r_A \otimes_{Z[Z/2]} \text{id}_{Z^-}) \cdot \det_{Z}(r_A \otimes_{Z[Z/2]} \text{id}_{Z^+}) .
$$

Since $Z^+ \otimes_{Z} F_2$ and $Z^- \otimes_{Z} F_2$ are isomorphic as $F_2[Z/2]$-modules, we get

$$
\det_{F_2}(r_A \otimes_{Z[Z/2]} \text{id}_{Z^-} \otimes_{Z} \text{id}_{F_2}) = \det_{F_2}(r_A \otimes_{Z[Z/2]} \text{id}_{Z^+} \otimes_{Z} \text{id}_{F_2}) .
$$

Since the reduction to $F_2$ of $\det_{Z}(r_A \otimes_{Z[Z/2]} \text{id}_{Z^\pm})$ is $\det_{F_2}(r_A \otimes_{Z[Z/2]} \text{id}_{Z^\pm} \otimes_{Z} \text{id}_{F_2})$, we conclude

$$
\det(r_A \otimes_{Z[Z/2]} \text{id}_{Z^-}) = \det(r_A \otimes_{Z[Z/2]} \text{id}_{Z^+}) \mod 2 .
$$

This implies that $\det_{Z}(r_A)$ is odd or divisible by four. In particular $|\det_{Z}(r_A)|$ is different from 2. This implies that for any matrix $A \in M_{r,s}(Z[Z/2])$ for which $r_A: Z[Z/2] \to Z[Z/2]$ is injective, we have $|\det_{Z}(r_A)| = 1$ or $|\det_{Z}(r_A)| \geq 3$.

One easily checks that $|\det_{Z}(r_{t+1}): Z[Z/2] \to Z[Z/2]| = 3$. Since for any matrix $A \in M_{r,s}(Z[Z/2])$ with injective $r^{(2)}_A: L^2(Z/2)^r \to L^2(Z/2)^r$ we have $|\det_{N(Z/2)}(r_A)| = \sqrt{\det_{Z}(r_A)}$ and $A^w(Z/2) = A^w\sqrt{Z/2}$, we conclude $A^w\sqrt{Z/2} = A^w\sqrt{Z/2}$.

Since we get $|\det_{N(Z/2)}(r_{t+1}): L^2(Z/2) \to L^2(Z/2)| = \sqrt{2}$ from Example 13.16 and Theorem 13.18 (4), we get $A_{\Lambda}(Z/2) \leq A_{\Lambda}(Z/2) \leq \sqrt{2}$. We conclude $\sqrt{2} \leq A(Z/2)$ from $\Lambda(\{1\}) = 2$ and assertion (2). This implies $A(Z/2) = A_{\Lambda}(Z/2) = \sqrt{2}$.

1. We conclude from assertions (2) and (3) for the finite group $G$

$$
\Lambda(G) \geq |G|^{-1} .
$$

Consider the norm element $N_G := \sum_{g \in G} n^{-1}g$. Let $e \in G$ be the unit element. Put $x = N_G - e \in ZG$. We have a canonical $CG$-decomposition $CG = C \oplus V$, where $C$ is the trivial $G$-representation and $V$ is a direct sum of irreducible $G$-representations with $V^G = 0$. Then $\Lambda_N: CG \to CG$ is the direct sum of $|G| \cdot \text{id}: C \to C$ and $0: V \to V$. Hence $r_\cdot : CG \to CG$ is the direct sum of $(|G| - 1) \cdot \text{id}_C: C \to C$ and of $- \text{id}_V: V \to V$. This implies that $r_\cdot : CG \to CG$ is a $C$-isomorphism and

$$
\det_{C}(r_\cdot : CG \to CG) = |G| - 1 .
$$

We conclude from by Example 13.16

$$
\det_{N(G)}(r^{(2)}_x) = (|G| - 1)^{|G|^{-1}} .
$$

Since $|G| \geq 3$ and hence $(|G| - 1)^{|G|^{-1}}$ is different from 1, we get

$$
\Lambda_1(G) \leq \Lambda^w(G) \leq (|G| - 1)^{|G|^{-1}} .
$$
We obtain
\[
\Lambda(G) \geq \inf\{\Lambda(H) \mid H \subseteq G \text{ finitely generated subgroup}\};
\]
\[
\Lambda_1(G) \geq \inf\{\Lambda_1(H) \mid H \subseteq G \text{ finitely generated subgroup}\};
\]
\[
\Lambda^w(G) \geq \inf\{\Lambda^w(H) \mid H \subseteq G \text{ finitely generated subgroup}\};
\]
\[
\Lambda^C_1(G) \geq \inf\{\Lambda^C_1(H) \mid H \subseteq G \text{ finitely generated subgroup}\},
\]
from assertion (1).

Consider any matrix \( A \in M_{r,s}(\mathbb{Z}G) \). Let \( H \) be the subgroup of \( G \) which is generated by the finite set consisting of those elements \( g \in G \) for which for at least one entry in \( A \) the coefficient of \( g \) is non-trivial. Then \( H \subseteq G \) is finitely generated and \( A = i_\ast B \) for some matrix \( B \in M_{r,s}(\mathbb{Z}H) \) for the inclusion \( i : H \to G \). We get
\[
\det_{\Lambda(G)}(r^{(2)}_B) = \det_{\Lambda(H)}(r^{(2)}_A)
\]
from Theorem \([13, 18] \). This implies
\[
\Lambda(G) \leq \inf\{\Lambda(H) \mid H \subseteq G \text{ finitely generated subgroup}\};
\]
\[
\Lambda_1(G) \leq \inf\{\Lambda_1(H) \mid H \subseteq G \text{ finitely generated subgroup}\};
\]
If \( r = s \) and \( r^{(2)}_A \) is injective, then also \( r^{(2)}_H \) is injective. Hence we get
\[
\Lambda^w(G) \leq \inf\{\Lambda^w(H) \mid H \subseteq G \text{ finitely generated subgroup}\};
\]
\[
\Lambda^C_1(G) \leq \inf\{\Lambda^C_1(H) \mid H \subseteq G \text{ finitely generated subgroup}\}.
\]
This finishes the proof of Lemma \([5, 1]\).

**Example 5.2** (Finite cyclic group of odd order). Let \( n \) be an odd natural number. Then we get for the finite cyclic group \( \mathbb{Z}/n \) the equality
\[
(5.3) \quad \Lambda(\mathbb{Z}/n) = \Lambda_1(\mathbb{Z}/n) = \Lambda^w(\mathbb{Z}/n) = \Lambda^C_1(\mathbb{Z}/n) = 2^{n-1}.
\]
Namely, let \( t \in \mathbb{Z}/n \) be a generator. Consider the element \( t + 1 \). Then the \( \mathbb{Z}/\mathbb{Z}/n \)-homomorphism \( r_{t+1} : \mathbb{Z}/\mathbb{Z}/n \to \mathbb{Z}/\mathbb{Z}/n \) defines after forgetting the \( \mathbb{Z}/n \)-action an \( \mathbb{Z} \)-automorphism of \( \mathbb{Z}^n \) given by the matrix
\[
B[n] = \begin{pmatrix}
1 & 0 & 0 & \cdots & 0 & 0 & 1 \\
1 & 1 & 0 & \cdots & 0 & 0 & 0 \\
0 & 1 & 1 & \cdots & 0 & 0 & 0 \\
0 & 0 & 1 & \cdots & 0 & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 1 & 1 & 0 \\
0 & 0 & 0 & \cdots & 0 & 1 & 1
\end{pmatrix}
\]
Since \( n \) is odd, we compute by developing after the first row \( \det(B[n]) = 2 \). Hence \( r_{t+1}^{(2)} \) is injective and we get from Example \([13, 15] \)
\[
\det_{\Lambda(\mathbb{Z}/n)}(r_{t+1}^{(2)}) = 2^{n-1}.
\]
This together with Lemma \([5, 1] \) implies \([5, 3] \).

**Remark 5.4** (Computation for finite abelian groups). Lind \([23] \), Definition 1.1] has introduced a Lehmer constant for compact abelian groups. If \( G \) is a finite abelian group, then his constant agrees with \( \ln(\Lambda^C_1(G)) \) for the number \( \Lambda^C_1(G) \) introduced in Definition \([5, 2] \). Lind gives some precise values and some estimates for \( \ln(\Lambda^C_1(G)) \) for finite abelian groups which were considerably improved by Kaiblinger \([18] \) for finite cyclic groups.

It is not hard to show that \( \Lambda^C_1(G) = \Lambda^w(G) \) holds for finite abelian \( G \). In general we have \( \Lambda_1(\mathbb{Z}/n) \neq \Lambda^C_1(\mathbb{Z}/n) \) and \( \Lambda(\mathbb{Z}/n) \neq \Lambda^w(\mathbb{Z}/n) \), see Lemma \([5, 1] \).
Computations for finite dihedral groups can be found in [2].

6. TORSIONFREE ELEMENTARY AMENABLE GROUPS

Throughout this section let $G$ be an amenable group for which $\mathbb{Q}G$ has no non-trivial zero-divisor. Examples for $G$ are torsionfree elementary amenable groups, see [20] Theorem 1.2, [24] Theorem 2.3. Then $\mathbb{Q}G$ has a skewfield of fractions $S^{-1}\mathbb{Q}G$ given by the Ore localization with respect to the multiplicative closed subset $S$ of non-trivial elements in $\mathbb{Q}G$, see [20] Example 8.16 on page 324.

The proof of the next result is analogous to the one in [27], Lemma 7.10, if we take $F = \mathbb{Q}$ and $V$ to be trivial 1-dimensional representation there and notice that $G$ satisfies the Determinant Conjecture [11] by Remark [12], and hence the Fuglede-Kadison determinants of both $r_A^{(2)}$ and $r_B^{(2)}$ are well-defined real numbers.

**Lemma 6.1** (Estimate in terms of minors). Consider a matrix $A$ over $\mathbb{Q}G$. Let $B$ be a quadratic submatrix of $A$ of maximal size $k$ such that the map $r_B: \mathbb{Q}G^k \to \mathbb{Q}G^k$ is injective.

Then:

1. The rank of $A$ over the skew field $S^{-1}\mathbb{Q}G$ is $k$;
2. The morphism $r_B^{(2)}: L^2(G)^k \to L^2(G)^k$ is a weak isomorphism;
3. We have

$$\det_{\mathcal{N}(G)}(r_B^{(2)}) \geq \det_{\mathcal{N}(G)}(r_A^{(2)}).$$

We get directly from Lemma 6.1

**Theorem 6.2.** Let $G$ be a torsionfree amenable group whose group ring $\mathbb{Q}G$ has no non-trivial zero-divisor, e.g., a torsionfree elementary amenable group. Then we get

$$\Lambda(G) = \Lambda^\times(G).$$

Next want to define a homomorphism

$$\Delta: K_1(S^{-1}\mathbb{Q}G) \to \mathbb{R}^{\geq 0}$$

as follows. Consider any natural number $r$ and a matrix $A \in GL_r(S^{-1}\mathbb{Q}G)$. We can choose $a \in \mathbb{Q}G$ with $a \neq 0$ such that $A[a] = (a \cdot I_r) \cdot A$ belongs to $M_{r,r}(\mathbb{Q}G)$, where $(a \cdot I_r)$ is the diagonal $(r,r)$-matrix whose entries on the diagonal are all equal to $a$. Since $G$ satisfies the Determinant Conjecture [11] by Remark [12], the Fuglede-Kadison determinants of both $r_A[a]: L^2(G)^r \to L^2(G)^r$ and $r_a: L^2(G)^r \to L^2(G)^r$ are well-defined real numbers. If $[A]$ denotes the class represented by $A$ in $K_1(S^{-1}\mathbb{Q}G)$, we want to define

$$\Delta([A]) := \frac{\det_{\mathcal{N}(G)}(r_A^{(2)}[a])}{\det_{\mathcal{N}(G)}(r_a^{(2)} I_r)}.$$

Notice for the sequel that $r_A[a]: L^2(G)^r \to L^2(G)^r$ and $r_a: L^2(G)^r \to L^2(G)^r$ are weak isomorphisms by Lemma 6.1. The proof that that this is a well-defined homomorphism of abelian groups can be found in [27] on the pages following (7.14), take $F = \mathbb{Q}$ and $V$ to be the trivial 1-dimensional representation there.

There is a Dieudonné determinant for invertible matrices over a skewfield $K$ which takes values in the abelianization of the group of units of the skewfield $K^\times/[K^\times, K^\times]$ and induces an isomorphism, see [31] Corollary 4.3 in page 133]

$$\det_D : K_1(K) \xrightarrow{\cong} K^\times/[K^\times, K^\times].$$
The inverse

\[ \iota: K^\times/[K^\times, K^\times] \xrightarrow{\cong} K_1(K) \]

send the class of a unit to the class of the corresponding $(1,1)$-matrix. In the sequel $K$ is chosen to be $S^{-1}QG$.

The next result follows from [27] Lemma 7.23.

**Lemma 6.6.** Consider any matrix $A \in M_{r,r}(QG)$. Then

1. The following statements are equivalent:
   a. $r_A: QQ^r \to QQ^r$ is injective;
   b. $r_A: S^{-1}QQ^r \to S^{-1}QQ^r$ is injective;
   c. $r_A: S^{-1}QQ^r \to S^{-1}QQ^r$ is bijective, or, equivalently $A$ becomes invertible over $S^{-1}QG$;
   d. $r_A^2: L^2(QG)^r \to L^2(QG)^r$ is injective;
   e. $r_A^2: L^2(QG)^r \to L^2(QG)^r$ is a weak isomorphism;

2. If one of the equivalent conditions above is satisfied then $r_A^2$ is a weak isomorphism of determinant class and we get the equation

\[ \det_{N(G)}(r_A^2) = \Delta \circ \iota(\det_D(A)), \]

where the homomorphisms $\Delta$ and $\iota$ have been defined in (6.3) and (6.5).

In particular $\det_{N(G)}(r_A^2)$ agrees with the quotient $\frac{\det_{N(G)}(r_A^2)}{\det_{N(G)}(r_y^2)}$ for two appropriate elements $x, y \in QG$ with $x, y \neq 0$.

On the first glance Lemma 6.6 (2) seems to be enough to show $\Lambda^w(G) = \Lambda^w_1(G)$ but this is not the case since we would need to replace $\frac{\det_{N(G)}(r_A^2)}{\det_{N(G)}(r_y^2)}$ by $\frac{\det_{N(G)}(r_A^2)}{\det_{N(G)}(r_y^2)}$.

The following example illustrates why we do not know whether this is true in general.

**Example 6.7.** If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is a $(2,2)$-matrix over a skewfield $K$, its Dieudonné determinant in $K^\times/[K^\times, K^\times]$ is defined to be the class of $-cb$ if $a = 0$ and to be the class of $ad - ac - b$ otherwise. It can happen that the matrix $A$ lives over $QG$, but the obvious representative of the Dieudonné determinant does not. The following example is due to Peter Linnell. Let $G$ be the metabelian group

\[ \mathbb{Z}/4 \mathbb{Z} = \langle x, y \mid x^4 = 1, y^2 = 1, y^{-1}x_iy = x_{i+1} \text{ for all } i, j \in \mathbb{Z} \rangle. \]

Then we have $QG \subset L^1(G) \subset \mathcal{U}(G)$, and division ring of quotients for $QG$ is contained in $\mathcal{U}(G)$, where $\mathcal{U}(G)$ is the algebra of affiliated operators. Consider the element $2 - x_0 \in QG$. Then $(2 - x_0)y(2 - x_0)^{-1}$ is not contained in $QG$ by the following observation. This element is the same as $y(1 - x_1/2)(1 - x_0/2)^{-1}$ and now we work inside $L^1(G)$, so we get $y(1 - x_1/2)(1 + x_0/2 + x_0^2/4 + \cdots)$. So the Dieudonné determinant of the matrix $A = \begin{pmatrix} 2 - x_0 & 1 \\ y & 0 \end{pmatrix}$ is represented by the element $(2 - x_0)y(2 - x_0)^{-1}$ which is not contained in $QG$ although all entries of $A$ belong to $QG$.

**Remark 6.8.** If in the situation of Lemma 6.6 the group $G$ happens to be abelian, then the Dieudonné determinant reduces to the standard determinant $\det_{QG}$ for the commutative ring $QG$ and it has the property that for a square matrix $A$ over $QG$ its value $\det_{QG}(A)$ is an element in $QG$. Moreover, we can replace in Lemma 6.6 (2) the fraction $\frac{\det_{N(G)}(r_A^2)}{\det_{N(G)}(r_y^2)}$ by $\frac{\det_{N(G)}(r_A^2)}{\det_{N(G)}(r_y^2)}$ for some $x \in QG$ with $x \neq 0$.

We conclude from Lemma 6.6 and Remark 6.8.
Lemma 6.9. We have $\Lambda^u(\mathbb{Z}^d) = \Lambda^v(\mathbb{Z}^d)$.

Remark 6.10. Example 6.7 does not rule out the possibility that for every square matrix $A$ over $\mathbb{Q}G$, which is invertible over $S^{-1}\mathbb{Q}G$, there exists a non-trivial element $u \in \mathbb{Q}G$ such that the Dieudonné determinant of $A$ regarded as invertible matrix over the skewfield $S^{-1}\mathbb{Q}G$ in the abelian group $S^{-1}\mathbb{Q}G^\times/[S^{-1}\mathbb{Q}G^\times, S^{-1}\mathbb{Q}G^\times]$ is represented by $u$. We neither have a proof for this claim nor a counterexample. This question is also interesting in connection with the $L^2$-polytope homomorphism appearing in [15, Section 3.2]. Moreover, a positive answer implies [19, Theorem 5.14].

7. Approximation

7.1. Approximation Conjecture for Fuglede-Kadison determinants. We have the following conjecture which was formulated as a question in [26, Question 13.52 on page 478].

Conjecture 7.1 (Approximation Conjecture for Fuglede-Kadison determinants). Let $G$ be a group together with an inverse system $\{G_i \mid i \in I\}$ of normal subgroups of $G$ directed by inclusion over the directed set $I$ such that $\bigcap_{i \in I} G_i = \{1\}$. Put $Q_i := G/G_i$. Consider any matrix $A \in M_{r,s}(\mathbb{Z}G)$. Denote by $A_i \in M_{r,s}(\mathbb{Z}Q_i)$ the reduction of $A$ to $\mathbb{Z}Q_i$ coming from the projection $G \to Q_i$.

Then we get for the Fuglede-Kadison determinants

$$\det_{N(\mathbb{G})}(r^{(2)}_A : L^2(G)^r \to L^2(G)^S) = \lim_{i \in I} \det_{N(\mathbb{Q}_i)}(r^{(2)}_{A_i} : L^2(Q_i)^r \to L^2(Q_i)^S).$$

Unfortunately, the status of Conjecture 7.1 is very poor, it does follow for virtually cyclic groups $G$ from the special case proved in [26, Lemma 13.53 on page 478], but we are not aware of a proof for a group which is not virtually cyclic. Nevertheless there is hope that Conjecture 7.1 is true for torsionfree groups.

Remark 7.2 (Integer coefficients are necessary). There are counterexamples to Conjecture 7.1 if one replaces the coefficients in $\mathbb{Z}$ by coefficients in $\mathbb{C}$, see [26, Example 13.69 on page 481]. This is in contrast to Theorem 7.1.

Conjecture 7.1 has the following interesting consequence.

Theorem 7.3 (Consequence of the Approximation Theorem Conjecture for Fuglede-Kadison determinants). Let $G$ be a group together with an inverse system $\{G_i \mid i \in I\}$ of normal subgroups of $G$ directed by inclusion over the directed set $I$ such that $\bigcap_{i \in I} G_i = \{1\}$. Put $Q_i := G/G_i$. Assume that each group $Q_i$ satisfies the Determinant Conjecture 7.1. Moreover, suppose that $G$ satisfies the Approximation Conjecture for Fuglede-Kadison determinants 7.1. Then

1. We have

$$\Lambda(G) \geq \limsup_{i \in I} \Lambda(Q_i);$$

$$\Lambda_1(G) \geq \limsup_{i \in I} \Lambda_1(Q_i);$$

2. Suppose that for any element $A \in M_{r,s}(\mathbb{Z}G)$ there exists a constant $\beta(A) > 0$ and an index $i_0(A) \in I$ such that the implication $\dim_{N(\mathbb{Q}_i)}(\ker(r^{(2)}_{A_i})) > 0 \implies \dim_{N(\mathbb{Q}_i)}(\ker(r^{(2)}_{A_i})) \geq \beta(A)$ holds for all $i \in I$ with $i \geq i_0(A)$. (We will recall the notion of the von Neumann dimension $\dim_{N(\mathbb{G})}$ in Appendix 15.)
Then we have
\[ \Lambda^w(G) \geq \limsup_{i \in I} \Lambda^w(Q_i); \]

(3) Suppose that for any element \( A \in M_{1,1}(\mathbb{Z}G) \) there exists a constant \( \beta_1(A) > 0 \) and an index \( i_0(A) \in I \) such that the implication \( \dim_{\mathcal{N}(Q_i)}(\ker(r_A^{(2)})) > 0 \implies \dim_{\mathcal{N}(Q_i)}(\ker(r_A^{(2)})) \geq \beta_1(A) \) holds for all \( i \in I \) with \( i \geq i_0(A) \). Then we have
\[ \Lambda^w_1(G) \geq \limsup_{i \in I} \Lambda^w_1(Q_i); \]

Proof. This is obvious.

(2) and (3) Consider a matrix \( A \in M_{r,s}(\mathbb{Z}G) \) such that \( r_A^{(2)} \) is injective, or, equivalently, \( \dim_{\mathcal{N}(G)}(\ker(r_A^{(2)})) = 0 \). We conclude from [26, Theorem 13.19 (2) on page 461]
\[ 0 = \dim_{\mathcal{N}(G)}(\ker(r_A^{(2)})) = \limsup_{i \in I} \dim_{\mathcal{N}(Q_i)}(\ker(r_A^{(2)})). \]

Hence there exists \( i_0 \) such that \( \dim_{\mathcal{N}(Q)}(\ker(r_A^{(2)})) = 0 \) holds for \( i \geq i_0 \) and hence that \( r_A^{(2)} \) is injective for \( i \geq i_0 \).

Remark 7.4 (Atiyah Conjecture). A version of the Atiyah Conjecture says for a group \( G \) for which there exists a natural number \( D \) such that the order of any finite subgroup of \( G \) divides \( D \) that for any element \( A \in M_{r,s}(\mathbb{Z}G) \) we get
\[ D \cdot \dim_{\mathcal{N}(G)}(\ker(r_A^{(2)}): L^2(G)^r \to L^2(G)^s) \in \mathbb{Z}. \]

If \( G \) happens to be torsion free, we can choose \( D = 1 \) and get the implication \( \dim_{\mathcal{N}(G)}(\ker(r_A^{(2)}): L^2(G)^r \to L^2(G)^s) \in \mathbb{Z} \).

Suppose that there exists a natural number \( D \) such that for every \( i \in I \) and every finite subgroup \( H \subseteq Q_i \), the order \( |H| \) divides \( D \). Then the implication \( \dim_{\mathcal{N}(Q_i)}(\ker(r_A^{(2)})) > 0 \implies \dim_{\mathcal{N}(Q_i)}(\ker(r_A^{(2)})) \geq \beta(A) \) appearing in assertions (2) and (3) of Theorem 7.3 is automatically satisfied if each group \( Q_i \) satisfies the Atiyah Conjecture above, just take \( \beta(A) := \frac{1}{D} \).

A survey on the Atiyah Conjecture and the groups for which it is known to be true can be found in [26, Chapter 10]. We mention that the Atiyah Conjecture holds for \( G \) if \( G \) is elementarily amenable and there is a upper bound on the orders of the finite subgroups of \( G \).

We conclude from Theorem 7.3 and Remark 7.4.

Theorem 7.5 (Residually torsionfree elementary amenable groups). Let \( G \) be a residually torsionfree elementary amenable group in the sense that we can find a sequence of in \( G \) normal subgroups \( G = G_0 \supseteq G_1 \supseteq G_2 \supseteq \cdots \) such that \( G/G_n \) is torsionfree elementary amenable for \( n = 0, 1, 2, \ldots \) and \( \bigcap_{n \geq 0} G_n = \{1\} \). Suppose that \( G \) satisfies the Approximation Conjecture \( \frac{1}{2} \) for Fuglede-Kadison determinants.

Then
\[ \Lambda^w(G) \geq \limsup_{i \in I} \Lambda^w(Q_i); \]
\[ \Lambda^w_1(G) \geq \limsup_{i \in I} \Lambda^w_1(Q_i). \]

Example 7.6 (Examples of (virtually) residually torsionfree nilpotent). Free groups are examples of residually torsionfree nilpotent groups, see [23, § 2].

Let \( M \) be a compact orientable irreducible 3-manifold whose boundary is empty or is a disjoint union of incompressible tori and whose fundamental group \( \pi \) is
infinite and not solvable. Then its fundamental group $\pi$ is virtually residually torsionfree nilpotent, see for instance [11, page 84].

This implies that $\Lambda(\pi) > 1$ if $\pi$ satisfies the Approximation Conjecture 4.1 and $\Lambda^w(H) = \Lambda(Z)$ holds for any torsionfree nilpotent group. Notice that we cannot conclude $\Lambda(\pi) = \Lambda(Z)$ since we only know that $\pi$ is virtually residually torsionfree nilpotent and not that $\pi$ is residually torsionfree nilpotent.

Remark 7.7. In order to apply Theorem 7.3 one needs to know Conjecture 4.1 which we have already discussed above and also have some information for $\Lambda^w(H)$, for nilpotent groups. Not much is known for these groups. Not even the three-dimensional Heisenberg group is fully understood. See for instance [10, Section 5].

7.2. Sub-Approximation Theorem. At least we can prove an inequality in a situation which is more general than the case of a normal chain considered in Subsection 7.1.

Given a matrix $A \in M_{r,s}(\mathbb{C}G)$, we will in the sequel denote by $A^r$ the element in $M_{r,r}(\mathbb{C}G)$, whose $(i,j)$-th entry is $a^r_{ij}$, where for an element $x = \sum_{g \in G} \lambda_g g \in \mathbb{C}G$ we denote by $x^r$ the element $\sum_{g \in G} \lambda_g^{-1} g^{-1} \in \mathbb{C}G$. With this convention the adjoint $(r_A^2)^r$ of the bounded operator $r_A^2 : L^2(G)^r \rightarrow L^2(G)^r$ is $r_A^2 : L^2(G)^r \rightarrow L^2(G)^r$.

Theorem 7.8 (Sub-Approximation Theorem). Let $G$ be a group. Suppose that $I$ is a directed set and we have a collection of groups $\{Q_i \mid i \in I\}$ together with group homomorphisms $q_i : G \rightarrow Q_i$ for each $i \in I$. Given $A \in M_{r,s}(\mathbb{Z}G)$, denote by $A_i \in M_{r,s}(\mathbb{Z}Q_i)$ the reduction of $A$ to $\mathbb{Z}Q_i$, coming from the projection $G \rightarrow Q_i$. Suppose:

- For any finite subset $F \subseteq G$ there exists an index $i_0(F) \in I$ such that for all $i \geq i_0(F)$ and $f \in F$ the implication $q_i(f) \neq e \Rightarrow f \neq e$ holds, where $e$ denotes the unit element in $G$ and $Q_i$; (This is automatically satisfied if there is an inverse system $\{G_i \mid i \in I\}$ of normal subgroups of $G$ directed by inclusion over the directed set $I$ such that $\bigcap_{i \in I} G_i = \{1\}$, $Q_i = G/G_i$ and $q_i : G \rightarrow Q_i$ is the projection.)

- Each group $Q_i$ satisfies the Determinant Conjecture 4.1.

Then for any element $A \in M_{r,s}(\mathbb{Z}G)$ we have

$$\det N(G)(r_A^2) \geq \limsup_{i \in I} \det N(Q_i)(r_{A_i}^2).$$

Proof. We conclude from Theorem 13.18 [35, 35]

$$\det N(G)(r_A) = \sqrt{\det N(G)(r_{AA^*})};$$

$$\det N(Q_i)(r_{A_i}^2) = \sqrt{\det N(Q_i)(r_{A_i A_i^*})}.$$

Hence we can assume without loss of generality that $r = s$ and that $r_A^2 : L^2(G)^r \rightarrow L^2(G)^r$ and $r_{A_i}^2 : L^2(Q_i)^r \rightarrow L^2(Q_i)^r$ are positive operators, otherwise replace $A$ by $AA^*$. We want to apply [26, Theorem 13.19 on page 461] in the case, where $G_i$ in the notation of [26, Theorem 13.19 on page 461] is $Q_i$, $A_i \in M_{r,s}(\mathbb{Z}Q_i)$ is the matrix above and $tr_r = tr_N(G)$. Then the claim does not follow directly from the assertions in [26, Theorem 13.19 on page 461] but from the inequality

$$\ln \left( \det N(G)(r_A^2) \right) \geq \limsup_{i \in I} \ln \left( \det N(Q_i)(r_{A_i}^2) \right)$$

appearing at the very end of the proof of [26, Theorem 13.19 on page 465]. It remains to check all the assumptions appearing in [26, Theorem 13.19 on page 461].
Choose a real number $K$ satisfying
\[ K \geq \sqrt{(2r - 1) \cdot r \cdot \max \{|A_{j,k}|1 \mid 1 \leq j \leq r, 1 \leq k \leq s\}}. \]

Then we also have
\[ K \geq \sqrt{(2r - 1) \cdot r \cdot \max \{||(A_{j,k})_{j,k}|1 \mid 1 \leq j \leq r, 1 \leq k \leq s\}}. \]

This implies the inequalities \(|r^{(2)}_{A_i}| \leq K\) and \(|r^{(2)}_{A_i}| \leq K\) for the operator norms of \(r^{(2)}_{A_i}\) and \(r^{(2)}_{A_i}\).

Consider a polynomial \(p\) with real coefficients. Let \(F\) be the finite set of elements \(g \in G\) for which there exists a natural number \(j\) with \(1 \leq j \leq r\) such that the coefficient of \(g\) of the \((j,j)\)-th entry of \(p(A)\) is different from zero. Choose \(i_0 \in I\) such for all \(i \geq i_0\) and \(f \in F\) the implication \(q_i(f) \neq e \implies f \neq e\) holds. This implies
\[ \text{tr}_{N(G)}(p(A)) = \text{tr}_{N(Q_i)}(p(A_i)) \quad \text{for } i \geq i_0. \]

In particular we get
\[ \text{tr}_{N(G)}(p(A)) = \lim_{i \to I} \text{tr}_{N(Q_i)}(A_i). \]

This finishes the proof of Theorem 7.8 \(\square\)

**Remark 7.9** (Sub-Approximation and Lehmer’s problem). Theorem \(7.8\) looks more promising than Theorem \(7.3\) if one is interested in the conclusion \(1\) of Theorem \(7.3\) only. Theorem \(7.8\) applies to more general systems of group homomorphisms \(g \to Q_i\) than in Theorem \(7.3\) and it does not need in contrast to Theorem \(7.3\) the condition that \(G\) satisfies the Approximation Conjecture for Figled-Kadison determinants \(7.3\) but only requires only a milder version to imply the same conclusion as in Theorem \(7.3\) \(1\).

Namely, we have additionally to assume that \(\det_{N(G)}(r^{(2)}_{A_i}) > 1\) implies the existence of an index \(i_0\) such that \(\det_{N(Q_i)}(r^{(2)}_{A_i}) > 1\) holds for \(i \geq i_0\), because then \(\det_{N(Q_i)}(r^{(2)}_{A_i}) > \Lambda(Q_i)\) holds for \(i \in I\) with \(i \geq i_0\) and hence the inequality \(\limsup_{i \in I} \det_{N(Q_i)}(r^{(2)}_{A_i}) \geq \limsup_{i \in I} \Lambda(Q_i)\) is true.

Or we have additionally to assume that \(\limsup_{i \in I} \det_{N(Q_i)}(r^{(2)}_{A_i}) = 1\) and that there exists a number \(\Lambda > 1\) and an index \(i_0\) such that \(\Lambda(Q_i) \geq \Lambda\) holds for \(i \geq i_0\), because then either \(\limsup_{i \in I} \det_{N(Q_i)}(r^{(2)}_{A_i}) = 1\) or the inequality \(\limsup_{i \in I} \det_{N(Q_i)}(r^{(2)}_{A_i}) \geq \Lambda\) holds.

**Remark 7.10** (Approximating Mahler measures for polynomials in several variables by polynomials in one variable). There is a case, where the inequality in Theorem \(7.8\) becomes an equality with \(\limsup\) replaced by \(\lim\). Namely, let \(p(z_1, z_2)\) be a non-trivial polynomial with complex coefficients in two variables \(z_1\) and \(z_2\). For a natural number \(k\) let \(p(z, z^k)\) be the polynomial with complex coefficients in one variable \(z\) obtained from \(p(z_1, z_2)\) by replacing \(z_1 = z\) and \(z_2 = z^k\) in \(p(z_1, z_2)\).

This corresponds to the homomorphism \(q_k: \mathbb{Z}^2 \to \mathbb{Z}, (n_1, n_2) \mapsto n_1 + k \cdot n_2\). If \(k\) is large enough, then \(p(z, z^k)\) is again non-trivial. We have the formula
\[ \det_{N(\mathbb{Z}^2)}(r^{(2)}_{p(z_1, z_2)}): L^2(\mathbb{Z}^2) \to L^2(\mathbb{Z}^2) = \lim_{k \to \infty} \det_{N(\mathbb{Z}^2)}(r^{(2)}_{p(z, z^k)}): L^2(\mathbb{Z}) \to L^2(\mathbb{Z})). \]

Its proof can be found in [1 Appendix 3]. The corresponding formula for a non-trivial polynomial \(p(z_1, z_2, \ldots, z_d)\) with complex coefficients in \(d\)-variables \(z_1, z_2, \ldots, z_d\)
\[ \det_{N(\mathbb{Z}^d)}(r^{(2)}_{p(z_1, z_2, \ldots, z_d)}): L^2(\mathbb{Z}^d) \to L^2(\mathbb{Z}^d) = \lim_{k_2 \to \infty} \lim_{k_3 \to \infty} \ldots \lim_{k_d \to \infty} \det_{N(\mathbb{Z}^d)}(r^{(2)}_{p(z_1, z_2, \ldots, z_d^k)}): L^2(\mathbb{Z}) \to L^2(\mathbb{Z}). \]
is proved in [3] Appendix 4] and [21] Theorem 2].

Consider a natural number \( d \geq 2 \). For natural numbers \( k_2, k_3, \ldots, k_d \) define a group homomorphism

\[
q(k_2, k_3, \ldots, k_d) : \mathbb{Z}^d \to \mathbb{Z}, \quad (a_1, a_2, \ldots, a_d) \mapsto a_1 + \sum_{i=2}^{d} k_i \cdot a_i.
\]

It induces a ring homomorphism \( \hat{q}(k_2, k_3, \ldots, k_d) : \mathbb{C}[\mathbb{Z}^d] \to \mathbb{C}[\mathbb{Z}] \). Given a matrix \( A \), let

\[
A[k_2, \ldots, k_d] \in M_{r,s}(\mathbb{C}[\mathbb{Z}])
\]

be the matrix obtained from \( A \) by applying the ring homomorphism \( \hat{q}(k_2, k_3, \ldots, k_d) \) to each entry. If \( p = p(z_1, z_2, \ldots, z_d) \) is polynomial and we regard it as element in \( M_{1,1}(\mathbb{C}[\mathbb{Z}^d]) \), then \( p[k_2, \ldots, k_d] \) is the \((1,1)\)-matrix over \( \mathbb{C}[\mathbb{Z}] \) given by the polynomial \( p(z_1^{k_1}, z_2^{k_2}, \ldots, z_d^{k_d}) \). Notice that for any element \( p \in \mathbb{C}[\mathbb{Z}^d] = \mathbb{C}[z_1^{\pm 1}, \ldots, z_d^{\pm 1}] \) we can find a monomial \( z_1^{n_1} \cdot z_2^{n_2} \cdot \cdots \cdot z_d^{n_d} \) such that \( p_0 := z_1^{n_1} \cdot z_2^{n_2} \cdot \cdots \cdot z_d^{n_d} \cdot p \) is a polynomial in variables \( z_1, \ldots, z_d \) and we have \( \det_{\mathbb{C}[\mathbb{Z}^d]}(r^{(p)}_{(d)}) = \det_{\mathbb{C}[\mathbb{Z}]}(r^{(d)}_{(A)}) \).

Hence we conclude from the iterated limit appearing in Remark 7.10 that for a \((1,1)\)-matrix \( A \) over \( \mathbb{C}[\mathbb{Z}] \) we have

\[
(7.11) \quad \det_{\mathbb{C}[\mathbb{Z}]}(r^{(A)}_{(2)} : L^2(\mathbb{Z}^d) \to L^2(\mathbb{Z}^d)) = \lim_{k_2 \to \infty} \lim_{k_3 \to \infty} \cdots \lim_{k_d \to \infty} \det_{\mathbb{C}[\mathbb{Z}]}(r^{(2)}_{[k_2, \ldots, k_d]} : L^2(\mathbb{Z}) \to L^2(\mathbb{Z})).
\]

We want to extend this to matrices of arbitrary finite size. For this purpose the following formula is useful, which reduces the computation of the Fuglede-Kadison determinant \( \det_{\mathbb{C}[\mathbb{Z}^d]}(r^{(A)}_{(2)} \sqcup r^{(A)}_{(B)}) \) for \( A \in M_{r,s}(\mathbb{C}[\mathbb{Z}^d]) \) to the special case, where \( r = s \) and \( r^{(A)}_{(2)} \) is injective, and finally to the case \( r = s = 1 \).

**Lemma 7.12.** Consider a matrix \( A \in M_{r,s}(\mathbb{C}[\mathbb{Z}^d]) \). Then there exists an integer \( q \geq 0 \) and a matrix \( B \in M_{q,s}(\mathbb{C}[\mathbb{Z}^d]) \) such that the sequence \( 0 \to L^2(\mathbb{Z}^d)^q \xrightarrow{r^{(A)}_{(2)}} L^2(\mathbb{Z}^d)^r \xrightarrow{\sqcup r^{(B)}_{(2)}} L^2(\mathbb{Z}^d)^s \to 0 \) is weakly exact, i.e., \( r^{(B)}_{(2)} \) is injective and the closure of the image of \( \sqcup r^{(B)}_{(2)} \) is the kernel of \( r^{(A)}_{(2)} \). For any such choice we get for the matrices \( D_1 = B^*B + AA^* \) in \( M_{r,q}(\mathbb{C}[\mathbb{Z}^d]) \) and \( D_2 = BB^* \) in \( M_{q,s}(\mathbb{C}[\mathbb{Z}^d]) \):

1. The operators \( \sqcup r^{(D_1)}_{(2)} : L^2(\mathbb{Z}^d)^r \to L^2(\mathbb{Z}^d)^r \) and \( \sqcup r^{(D_2)}_{(2)} : L^2(\mathbb{Z}^d)^q \to L^2(\mathbb{Z}^d)^q \) are injective;
2. We have

\[
\det_{\mathbb{C}[\mathbb{Z}]}(r^{(A)}_{(2)}) = \frac{\det_{\mathbb{C}[\mathbb{Z}]_1}(r^{(D_1)}_{(2)})}{\det_{\mathbb{C}[\mathbb{Z}]}(r^{(D_2)}_{(2)})}
\]

3. The elements \( \det_{\mathbb{C}[\mathbb{Z}]}(D_1) \) and \( \det_{\mathbb{C}[\mathbb{Z}]}(D_2) \) in \( \mathbb{C}[\mathbb{Z}^d] \) are different from zero and we get

\[
\det_{\mathbb{C}[\mathbb{Z}]}(r^{(D_1)}_{(2)}) = \det_{\mathbb{C}[\mathbb{Z}]}(r^{(D_2)}_{(2)}), \quad \det_{\mathbb{C}[\mathbb{Z}]}(r^{(D_1)}_{(2)}) = \det_{\mathbb{C}[\mathbb{Z}]}(r^{(D_2)}_{(2)}).
\]

**Proof.** Let \( \mathbb{C}[\mathbb{Z}^d]_{(0)} \) be the quotient field of \( \mathbb{C}[\mathbb{Z}^d] \). Let \( q \) be the \( \mathbb{C}[\mathbb{Z}^d]_{(0)} \)-dimension of the kernel of \( r_{A} : \mathbb{C}[\mathbb{Z}^d]_{(0)} \to \mathbb{C}[\mathbb{Z}^d]_{(0)} \). We can choose a matrix \( B \in M_{q,r}(\mathbb{C}[\mathbb{Z}^d]_{(0)}) \) such that the sequence of \( \mathbb{C}[\mathbb{Z}^d]_{(0)} \)-modules \( 0 \to \mathbb{C}[\mathbb{Z}^d]_{(0)} \xrightarrow{r_{A}} \mathbb{C}[\mathbb{Z}^d]_{(0)} \xrightarrow{\sqcup r_{B}} \mathbb{C}[\mathbb{Z}^d]_{(0)} \to \mathbb{C}[\mathbb{Z}^d]_{(0)} \) is exact. By possibly multiplying each entry of \( B \) with the same non-trivial element in \( \mathbb{C}[\mathbb{Z}^d] \), we can additionally arrange that \( B \) lies...
We conclude from [26] Lemma 1.34 (1) on page 35 that the following sequence is weakly exact

$$0 \to L^2(\mathbb{Z}^d) q \to L^2(\mathbb{Z}^d) r \to L^2(\mathbb{Z}^d)^s.$$  

We consider it as a chain complex of Hilbert $\mathcal{N}(\mathbb{Z}^d)$-modules concentrated in dimensions 2, 1 and 0. Then its first and second $L^2$-homology vanishes. The Laplace operators of it in dimensions 1 and 2 is given by $r_{D_1}$ and $r_{D_2}$. We conclude from [26] Lemma 3.39 on page 145 applied in the case where we replace $L^2(\mathbb{Z}^d)^r$ by the closure of the image of $r_{A}^{(2)}$ that $r_{D_1}^{(2)}$ and $r_{D_2}^{(2)}$ are weak isomorphisms and in particular injective.

We conclude from Theorem [13.18 (3)] and from [26, Lemma 3.30 on page 140] applied to the chain complex of Hilbert modules above

\[
\ln \left( \det \mathcal{N}(\mathbb{Z}^d)(r_{A}^{(2)}) \right) = \ln \left( \det \mathcal{N}(\mathbb{Z}^d)(r_{B}^{(2)}) \right) - \frac{1}{2} \cdot \left( 2 \cdot \ln \left( \det \mathcal{N}(\mathbb{Z}^d)(r_{D_1}^{(2)}) \right) - \ln \left( \det \mathcal{N}(\mathbb{Z}^d)(r_{D_1}^{(2)}) \right) \right) = \frac{1}{2} \cdot \ln \left( \det \mathcal{N}(\mathbb{Z}^d)(r_{D_1}^{(2)}) \right) - \ln \left( \det \mathcal{N}(\mathbb{Z}^d)(r_{D_1}^{(2)}) \right) + \frac{1}{2} \cdot \ln \left( \det \mathcal{N}(\mathbb{Z}^d)(r_{D_1}^{(2)}) \right) = \ln \left( \frac{\det \mathcal{N}(\mathbb{Z}^d)(r_{D_1}^{(2)})}{\det \mathcal{N}(\mathbb{Z}^d)(r_{D_1}^{(2)})} \right).
\]

This follows from assertion (1) and Lemma [6.10] and Remark [6.8]. This finishes the proof of Lemma 7.12.

**Theorem 7.13** (Approximating Mahler measures for matrices over $\mathbb{C}[\mathbb{Z}^d]$ by matrices over $\mathbb{C}[\mathbb{Z}]$). Consider a matrix $A \in M_{r,s}(\mathbb{C}[\mathbb{Z}^d])$. Then we get

\[
\det \mathcal{N}(\mathbb{Z}^d)(r_{A}^{(2)}): L^2(\mathbb{Z}^d)^r \to L^2(\mathbb{Z}^d)^r
\]

\[
= \lim_{k_2 \to \infty} \cdots \lim_{k_d \to \infty} \det \mathcal{N}(\mathbb{Z})(r_{A[k_2,\ldots,k_d]}^{(2)}): L^2(\mathbb{Z})^r \to L^2(\mathbb{Z})^r.
\]

**Proof.** Because of Lemma 7.12 we can choose for $A$ an integer $q \geq 0$ and a matrix $B \in M_{q,r}(\mathbb{C}[\mathbb{Z}^d])$ such that the sequence $0 \to L^2(\mathbb{Z})^q \xrightarrow{r_{B}^{(2)}} L^2(\mathbb{Z})^r \xrightarrow{r_{A}^{(2)}} L^2(\mathbb{Z})^* \to L^2(\mathbb{Z})^*$ is weakly exact. Put $D_1 = B^*B + AA^*$ in $M_{r,r}(\mathbb{C}[\mathbb{Z}^d])$ and $D_2 = BB^*$ in $M_{q,q}(\mathbb{C}[\mathbb{Z}^d])$ as in in Lemma 7.12. We conclude from Lemma 7.12 that $\det \mathcal{Z}[\mathbb{Z}^d](D_1)$ and $\det \mathcal{Z}[\mathbb{Z}^d](D_2)$ in $\mathbb{C}[\mathbb{Z}^d]$ are different from zero and we get

\[d_{t_{\mathcal{Z}^d}}(r_{D_1}^{(2)}) = \det \mathcal{N}(\mathbb{Z}^d)(r_{D_1}^{(2)}), \]

\[d_{t_{\mathcal{Z}^d}}(r_{D_2}^{(2)}) = \det \mathcal{N}(\mathbb{Z}^d)(r_{D_2}^{(2)}), \]

\[d_{t_{\mathcal{Z}^d}}(r_{D_3}^{(2)}) = \det \mathcal{N}(\mathbb{Z}^d)(r_{D_3}^{(2)}).\]

Fix $i = 1,\ldots,d - 1$. Let $b_i$ be the maximum over the norms of those integers $n_i$ for which there exists a monomial of the form $z_1^{n_1}z_2^{n_2}\cdots z_d^{n_d}$ such that its coefficients in the description of $\det \mathcal{Z}[\mathbb{Z}^d](D_1)$ and $\det \mathcal{Z}[\mathbb{Z}^d](D_2)$ as a sum of such monomials is non-trivial. Define $c_i = 2 \cdot \sum_{j=1}^{b_i} b_i$ for $i = 1,2,\ldots,(d - 1)$. 


Consider any sequence of natural numbers $k_1, k_2, \ldots, k_d$ satisfying $k_2 > c_1, k_3 > c_2 \cdot k_2, k_4 > c_3 \cdot k_3, \ldots, k_d > c_d-1 \cdot k_{d-1}$. Let $l = 1, 2$. Then we get for any two $d$-tuples $(m_1, m_2, \ldots, m_d)$ and $(n_1, n_2, \ldots, n_d)$ for which there exists a monomial of the form $z_1^{m_1} z_2^{m_2} \cdots z_d^{m_d}$ and $z_1^{n_1} z_2^{n_2} \cdots z_d^{n_d}$ such that its coefficients in the description of $\det_{\mathbb{Z}^l}(D_1)$ as a sum of such monomials is non-trivial,

$$q(k_1, k_3, \ldots, k_d)(m_1, m_2, \ldots, m_d) = q(k_1, k_3, \ldots, k_d)(n_1, n_2, \ldots, n_d) \implies (m_1, m_2, \ldots, m_d) = (n_1, n_2, \ldots, n_d).$$

This implies that $\det_{\mathbb{Z}^l}(D_1)[k_2, \ldots, k_d]$ is not zero. One easily checks

$$\det_{\mathbb{Z}^l}(D_1)[k_2, \ldots, k_d] = \det_{\mathbb{Z}^l}(D_1)[k_2, \ldots, k_d];$$

$$\det_{\mathbb{Z}^l}(D_2)[k_2, \ldots, k_d] = \det_{\mathbb{Z}^l}(D_2)[k_2, \ldots, k_d];$$

$$D_1[k_2, \ldots, k_d] = B[k_2, \ldots, k_d]^* B[k_2, \ldots, k_d] + A[k_2, \ldots, k_d] A[k_2, \ldots, k_d]^*;$$

$$D_2[k_2, \ldots, k_d] = B[k_2, \ldots, k_d] B[k_2, \ldots, k_d]^*. $$

We conclude that $\det_{\mathbb{Z}^l}(D_1[k_2, \ldots, k_d])$ and $\det_{\mathbb{Z}^l}(D_2[k_2, \ldots, k_d])$ are non-trivial. Lemma 6.6(4) implies that $r_{D_1[k_2, \ldots, k_d]}^{(2)}: L^2(\mathbb{Z})^r \rightarrow L^2(\mathbb{Z})^r$ and $r_{D_2[k_2, \ldots, k_d]}^{(2)}: L^2(\mathbb{Z})^q \rightarrow L^2(\mathbb{Z})^q$ are weak isomorphism. This implies that the sequence $0 \rightarrow L^2(\mathbb{Z})^r \xrightarrow{r_{D_2[k_2, \ldots, k_d]}^{(2)}} L^2(\mathbb{Z})^q$ is weakly exact since we can view it as a chain complex of Hilbert $\mathcal{N}(\mathbb{Z})$-modules, its Laplace operator in dimension 1 and 2 is given by the weak isomorphisms $r_{D_1[k_2, \ldots, k_d]}^{(2)}$ and $r_{D_2[k_2, \ldots, k_d]}^{(2)}$ and we can apply [26, Lemma 3.39 on page 145] applied in the case where we replace $L^2(\mathbb{Z})^q$ by the closure of the image of $r_{A[k_2, \ldots, k_d]}^{(2)}$. We conclude from Lemma 7.12 applied to $A[k_2, \ldots, k_d]$ and $B[k_2, \ldots, k_d]$

$$\det_{\mathcal{N}(\mathbb{Z})}(r_{A[k_2, \ldots, k_d]}^{(2)}): L^2(\mathbb{Z})^r \rightarrow L^2(\mathbb{Z})^*$$

$$\det_{\mathcal{N}(\mathbb{Z})}(r_{B[k_2, \ldots, k_d]}^{(2)}): L^2(\mathbb{Z})^q \rightarrow L^2(\mathbb{Z})^*$$

provided that $k_2 > c_1, k_3 > c_2 \cdot k_2, k_4 > c_3 \cdot k_3, \ldots, k_d > c_{d-1} \cdot k_{d-1}$. This implies

$$(7.17) \quad \lim_{k_2 \rightarrow \infty} \lim_{k_3 \rightarrow \infty} \ldots \lim_{k_d \rightarrow \infty} \det_{\mathcal{N}(\mathbb{Z})}(r_{A[k_2, \ldots, k_d]}^{(2)}): L^2(\mathbb{Z})^r \rightarrow L^2(\mathbb{Z})^*)$$

$$= \lim_{k_2 \rightarrow \infty} \lim_{k_3 \rightarrow \infty} \ldots \lim_{k_d \rightarrow \infty} \det_{\mathcal{N}(\mathbb{Z})}(r_{B[k_2, \ldots, k_d]}^{(2)}): L^2(\mathbb{Z})^q \rightarrow L^2(\mathbb{Z})^*.$$
This implies
\[
\begin{aligned}
\det_{\mathcal{N}(Z^d)}(r^{(2)}_{\det_{\mathbb{Z}^d}(D_1)}): L^2(\mathbb{Z}^d) & \to L^2(\mathbb{Z}^d) \\
\det_{\mathcal{N}(Z^d)}(r^{(2)}_{\det_{\mathbb{Z}^d}(D_2)}): L^2(\mathbb{Z}^d) & \to L^2(\mathbb{Z}^d)
\end{aligned}
\]
\[
= \lim_{k_2 \to \infty} \lim_{k_3 \to \infty} \ldots \lim_{k_4 \to \infty} \det_{\mathcal{N}(Z^d)}(r^{(2)}_{\det_{\mathbb{Z}^d}(D_1[k_2], \ldots, k_4]}): L^2(\mathbb{Z}^d) \to L^2(\mathbb{Z}^d)
\]
Putting (7.16), (7.17) and (7.18) together yields the desired equality
\[
\det_{\mathcal{N}(Z^d)}(r^{(2)}_{A^d[k_2], \ldots, k_4]}): L^2(\mathbb{Z}^d) \to L^2(\mathbb{Z}^d).
\]
This finishes the proof of Theorem 7.13. \qed

8. Finitely generated free abelian groups

**Theorem 8.1** (Finitely generated free abelian groups). Let \(d \geq 1\) be an integer. Then we have:
\[
\Lambda_1(Z^d) = \Lambda^w_1(Z^d) = \Lambda^w(Z^d) = \Lambda(Z^d) = \Lambda^w_1(Z) = \Lambda^w(Z) = \Lambda(Z^d).
\]
In particular \(\Lambda^w_1(Z) > 1\) implies \(\Lambda(Z^d) > 1\).

**Proof.** Theorem 8.1 follows after we have shown the following equalities and inequalities
\[
(8.2) \quad \Lambda(Z^d) = \Lambda(Z);
(8.3) \quad \Lambda^w_1(Z^d) = \Lambda_1(Z^d);
(8.4) \quad \Lambda^w(Z^d) = \Lambda^w(Z);
(8.5) \quad \Lambda(Z^d) = \Lambda^w(Z^d).
\]
Equality (8.2) follows from Theorem 7.13.
Equation (8.3) follows from the conclusion of Lemma 6.9 (1) that for \(a \in \mathbb{Z}[Z^d]\) with \(a \neq 0\) the map \(r^{(2)}_A: L^2(\mathbb{Z}^d) \to L^2(\mathbb{Z}^d)\) is injective.
Equation (8.4) has already been proved in Lemma 6.9.
Equation (8.5) is a special case of Theorem 6.2. \qed

9. Non abelian free groups

We cannot prove anything for non-abelian free groups so far, but we want at least to list some good properties and explain the main problem. Throughout this section \(F\) denotes a non-abelian free group.

A ring \(R\) is called *semifir* if every finitely generated submodule of a free \(R\)-module is free and for a free module any two basis have the same cardinality. For any field \(K\) the group ring \(K[F]\) is a semifir [7 Corollary on page 68], [11]. Moreover, \(\mathbb{C}G\) embeds into a skewfield \(\mathcal{D}(F)\) and the inclusion is \(\Sigma(\mathbb{C}G \subseteq \mathcal{U}(G))-\)inverting as explained in [26 Section 10.3.6 and Lemma 10.82 on page 408].

A problem is that for a matrix \(A \in M_{r,s}(\mathbb{C}F)\) it can happen that \(r_A^*: \mathbb{C}G^r \to \mathbb{C}G^s\) is injective but \(r^{(2)}_A: L^2(\mathbb{C}G^r) \to L^2(\mathbb{C}G^s)\) is not injective. An example comes for the free group \(F_r\) in \(r \geq 2\) generators \(s_1, s_2, \ldots, s_r\) from the \(\mathbb{C}[F_r]\)-chain complex of the universal covering of \(\bigvee_{i=1}^r S^1\) whose first differential \(r^{(2)}_A: \mathbb{C}[F_r]^r \to \mathbb{C}[F_r]\) is given by the matrix \(A\) which is the transpose of \((s_1 - 1, s_2 - 1, \ldots, s_r - 1)\). The map \(r_A\)
is injective since the universal covering is 1-connected. The induced $F$-equivariant bounded operator $r^{(2)}_A : L^2(F_r) \to L^2(F_r)$ is not injective since Lemma \ref{lem:bdd_op} implies
\[ \dim_{N(F_r)}(\ker(r^{(2)}_A)) \geq \dim_{N(F_r)}(L^2(F_r)) - \dim_{N(F_r)}(L^2(F_r)) = r - 1 > 0. \]
Actually we have $\dim_{N(F_r)}(\ker(r^{(2)}_A)) = r - 1$.

10. Lehmer’s problem for torsionfree groups

Theorem \ref{thm:lehmer} leads to state the following version of Lehmer’s problem for torsionfree groups.

**Problem 10.1** (Lehmer’s problem for torsionfree groups.) For which torsionfree group $G$ does
\[ \Lambda^w(G) > 1 \]
hold?

11. 3-manifolds

Let $M$ be a closed hyperbolic 3-manifold with fundamental group $\pi$. Then the $L^2$-torsion of its universal covering satisfies
\[ -\rho^2(\tilde{M}) = \frac{1}{6\pi} \cdot \text{vol}(M) \]
where $\text{vol}(M)$ is the volume, see for instance \cite{20} Theorem 3.152 on page 187. There is a natural number $n$ and a $(n, n)$-matrix $A$ with entries in $\mathbb{Z}\pi$ such that $r^{(2)}_A : L^2(\pi)^n \to L^2(\pi)^n$ is a weak isomorphisms and its Fulgede-Kadison determinant satisfies
\[ -\rho^2(\tilde{M}) = \ln(\det(r^{(2)}_A)). \]
This follows from the argument in the proof of \cite{25} Theorem 2.4. We conclude
\[ \det(\Lambda^w(\pi)(r^{(2)}_A) = \exp \left( \frac{1}{6\pi} \cdot \text{vol}(M) \right). \]
Hence we get
\[ (11.1) \quad \Lambda^w(\pi) \leq \exp \left( \frac{1}{6\pi} \cdot \text{vol}(M) \right). \]

**Example 11.2** (Weeks manifold). There is a closed hyperbolic 3-manifold $W$, the so called Weeks manifold, which is the unique closed hyperbolic 3-manifold with smallest volume, see \cite{16} Corollary 1.3. Its volume is between 0.942 and 0.943. Hence we get from (11.1) the estimate
\[ \Lambda^w(\pi) \leq \exp \left( \frac{1}{6\pi} \cdot 0.943 \right) \leq 1,06 \]
This implies $\Lambda^w(\pi) < \Lambda_1(\mathbb{Z}) = \Lambda^w_1(\mathbb{Z})$.

12. Lehmer’s problem for groups with torsion

**Remark 12.1** (Bound on the order of finite subgroups). Let $G$ be a group with $\Lambda^w(G) > 1$. Let $H \leq G$ be any finite subgroup. We conclude from Lemma \ref{lem:finite_subgroup} and \ref{lem:finite_subgroup} that $\Lambda^w_1(G) \leq \Lambda^w_1(H) \leq (|H| - 1)|H|^{-1}$ holds if $|H| \geq 3$. Since we have $\lim_{m \to \infty} (m - 1)m^{-1} = 1$, there is a constant $C > 1$ depending only on $\Lambda^w_1(G)$ such that $|H| \leq C$ holds.
The Fourier transform yields an isometric $k$-by-sending $f$ to $T$ subset of measure zero. Define the ring $T$ bounded measurable functions $C > 20 L$.

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13. Appendix: $L^2$-invariants

13.1. Group von Neumann algebras. Denote by $L^2(G)$ the Hilbert space $L^2(G)$ consisting of formal sums $\sum_{g \in G} \lambda_g \cdot g$ for complex numbers $\lambda_g$ such that $\sum_{g \in G} |\lambda_g|^2 < \infty$. This is the same as the Hilbert space completion of the complex group ring $\mathbb{C} G$ with respect to the pre-Hilbert space structure for which $G$ is an orthonormal basis. Notice that left multiplication with elements in $G$ induces an isometric $G$-action on $L^2(G)$. Given a Hilbert space $H$, denote by $B(H)$ the $C^*$-algebra of bounded (linear) operators from $H$ to itself, where the norm is the operator norm and the involution is given by taking adjoints.

Definition 13.1 (Group von Neumann algebra). The group von Neumann algebra $\mathcal{N}(G)$ of the group $G$ is defined as the algebra of $G$-equivariant bounded operators from $L^2(G)$ to $L^2(G)$

$$\mathcal{N}(G) := B(L^2(G))^G.$$ 

In the sequel we will view the complex group ring $\mathbb{C} G$ as a subring of $\mathcal{N}(G)$ by the embedding of $\mathbb{C}$-algebras $\rho_g : \mathbb{C} G \to \mathcal{N}(G)$ which sends $g \in G$ to the $G$-equivariant operator $r_g : L^2(G) \to L^2(G)$ given by right multiplication with $g^{-1}$.

Remark 13.2 (The general definition of von Neumann algebras). In general a von Neumann algebra $\mathcal{A}$ is a sub-$\ast$-algebra of $B(H)$ for some Hilbert space $H$, which is closed in the weak topology and contains id: $H \to H$. Often in the literature the group von Neumann algebra $\mathcal{N}(G)$ is defined as the closure in the weak topology of the complex group ring $\mathbb{C} G$ considered as $\ast$-subalgebra of $B(L^2(G))$. This definition and Definition 13.1 agree, see [17, Theorem 7.2 on page 434].

Example 13.3 (The von Neumann algebra of a finite group). If $G$ is finite, then nothing happens, namely $\mathbb{C} G = L^2(G) = \mathcal{N}(G)$.

Example 13.4 (The von Neumann algebra of $\mathbb{Z}^d$). In general there is no concrete model for $\mathcal{N}(G)$. However, for $G = \mathbb{Z}^d$, there is the following illuminating model for the group von Neumann algebra $\mathcal{N}(\mathbb{Z}^d)$. Let $L^2(T^d)$ be the Hilbert space of equivalence classes of $L^2$-integrable complex-valued functions on the $n$-dimensional torus $T^d$, where two such functions are called equivalent if they differ only on a subset of measure zero. Define the ring $L^\infty(T^d)$ by equivalence classes of essentially bounded measurable functions $f : T^d \to \mathbb{C}$, where essentially bounded means that there is a constant $C > 0$ such that the set $\{ x \in T^d | |f(x)| \geq C \}$ has measure zero. An element $(k_1, \ldots, k_n)$ in $\mathbb{Z}^d$ acts isometrically on $L^2(T^d)$ by pointwise multiplication with the function $T^d \to \mathbb{C}$, which maps $(z_1, z_2, \ldots, z_n)$ to $z_1^{k_1} \cdots z_n^{k_n}$. The Fourier transform yields an isometric $\mathbb{Z}^d$-equivariant isomorphism $L^2(\mathbb{Z}^d) \xrightarrow{\sim} L^2(T^d)$. Hence $\mathcal{N}(\mathbb{Z}^d) = B(L^2(T^d))^{\mathbb{Z}^d}$. We obtain an isomorphism of $C^*$-algebras

$$L^\infty(T^d) \xrightarrow{\sim} \mathcal{N}(\mathbb{Z}^d)$$

by sending $f \in L^\infty(T^d)$ to the $\mathbb{Z}^d$-equivariant operator $M_f : L^2(T^d) \to L^2(T^d)$, $g \mapsto g \cdot f$, where $(g \cdot f)(x)$ is defined by $g(x) \cdot f(x)$.
13.2. The von Neumann dimension. An important feature of the group von Neumann algebra is its trace.

Definition 13.5 (Von Neumann trace). The von Neumann trace on $\mathcal{N}(G)$ is defined by
\[
\text{tr}_{\mathcal{N}(G)} : \mathcal{N}(G) \to \mathbb{C}, \quad f \mapsto \langle f(e), e \rangle_{L^2(G)},
\]
where $e \in G \subseteq L^2(G)$ is the unit element.

Definition 13.6 (Finitely generated Hilbert module). A finitely generated Hilbert $\mathcal{N}(G)$-module $V$ is a Hilbert space $V$ together with a linear isometric $G$-action such that there exists an isometric linear $G$-embedding of $V$ into $L^2(G)^r$ for some natural number $r$. A map of Hilbert $\mathcal{N}(G)$-modules $f : V \to W$ is a bounded $G$-equivariant operator.

Definition 13.7 (Von Neumann dimension). Let $V$ be a finitely generated Hilbert $\mathcal{N}(G)$-module. Choose a matrix $A = (a_{i,j}) \in M_{r,r}(\mathcal{N}(G))$ with $A^2 = A$ such that the image of the $G$-equivariant bounded operator $r_N^{(2)} : L^2(G)^r \to L^2(G)^r$ given by $A$ is isometrically $G$-isomorphic to $V$. Define the von Neumann dimension of $V$ by
\[
\dim_{\mathcal{N}(G)}(V) := \sum_{i=1}^r \text{tr}_{\mathcal{N}(G)}(a_{i,i}) \in [0, \infty).
\]

The von Neumann dimension $\dim_{\mathcal{N}(G)}(V)$ depends only on the isomorphism class of the Hilbert $\mathcal{N}(G)$-module $V$ but not on the choice of $r$ and the matrix $A$. The von Neumann dimension $\dim_{\mathcal{N}(G)}(V)$ is faithful, i.e., $\dim_{\mathcal{N}(G)}(V) = 0 \Leftrightarrow V = 0$ holds for any finitely generated Hilbert $\mathcal{N}(G)$-module $V$. It is weakly exact in the following sense, see [26, Theorem 1.12 on page 21].

Lemma 13.8. Let $0 \to V_0 \xrightarrow{i} V_1 \xrightarrow{p} V_2 \to 0$ be a sequence of finitely generated Hilbert $\mathcal{N}(G)$-modules. Suppose that it is weakly exact, i.e., $i$ is injective, the closure of $i$ is the kernel of $p$ and the image of $p$ is dense. Then
\[
\dim_{\mathcal{N}(G)}(V_1) = \dim_{\mathcal{N}(G)}(V_0) + \dim_{\mathcal{N}(G)}(V_2).
\]

Example 13.9 (Von Neumann dimension for finite groups). If $G$ is finite, then $\dim_{\mathcal{N}(G)}(V)$ is $\frac{1}{|G|}$-times the complex dimension of the underlying complex vector space $V$.

Example 13.10 (Von Neumann dimension for $\mathbb{Z}^d$). Let $X \subset T^d$ be any measurable set and $\chi_X \in L^\infty(T^d)$ be its characteristic function. Denote by $M_{\chi_X} : L^2(T^d) \to L^2(T^d)$ the $\mathbb{Z}^d$-equivariant unitary projection given by multiplication with $\chi_X$. Its image $V$ is a Hilbert $\mathcal{N}(\mathbb{Z}^d)$-module with $\dim_{\mathcal{N}(\mathbb{Z}^d)}(V) = \text{vol}(X)$.

13.3. Weak isomorphisms. A bounded $G$-equivariant operator $f : L^2(G)^r \to L^2(G)^s$ is called a weak isomorphism if and only if it is injective and its image is dense. If there exists a weak isomorphism $L^2(G)^r \to L^2(G)^s$, then we must have $r = s$ by Lemma 13.8. The following statements are equivalent for a bounded $G$-equivariant operator $f : L^2(G)^r \to L^2(G)^s$, see [26, Lemma 1.13 on page 23]:

(1) $f$ is a weak isomorphism;
(2) Its adjoint $f^*$ is a weak isomorphism;
(3) $f$ is injective;
(4) $f$ has dense image;
(5) The von Neumann dimension of the closure of the image of $f$ is $r$.

Consider a matrix $A \in M_{r,s}(\mathbb{C}G)$. If $r_N^{(2)} : L^2(G)^r \to L^2(G)^s$ is injective, then the $\mathbb{C}G$-homomorphism $r_N : \mathbb{C}G^r \to \mathbb{C}G^s$ is injective. The converse is not true in general but in the special case that $G$ is amenable, this follows from [26, Theorem 6.24 on page 249 and Theorem 6.37 on page 259].
13.4. The Fuglede-Kadison determinant.

**Definition 13.11** (Spectral density function). Let $f : V \to V$ be a morphisms of finitely generated Hilbert $\mathcal{N}(G)$-modules. Denote by $\{E_{\lambda}^{f,f} \mid \lambda \in \mathbb{R}\}$ the (right-continuous) family of spectral projections of the positive operator $f^*f$. Define the spectral density function of $f$ by

$$F_f : \mathbb{R} \to [0, \infty) \; \lambda \mapsto \dim_{\mathcal{N}(G)}(\text{im}(E_{\lambda}^{f,f})).$$

The spectral density function is monotone non-decreasing and right-continuous.

**Example 13.12** (Spectral density function for finite groups). Let $G$ be finite and $f : U \to V$ a be a map of finitely generated Hilbert $\mathcal{N}(G)$-modules, i.e., of finite-dimensional unitary $G$-representations. Then $F(f)$ is the right-continuous step function whose value at $\lambda$ is the sum of the complex dimensions of the eigenspaces of $f^*f$ for eigenvalues $\mu \leq \lambda^2$ divided by the order of $G$; or, equivalently, the sum of the complex dimensions of the eigenspaces of $|f|$ for eigenvalues $\mu \leq \lambda$ divided by the order of $G$.

**Example 13.13** (Spectral density function for $\mathbb{Z}^d$). Let $G = \mathbb{Z}^d$. In the sequel we use the notation and the identification $\mathcal{N}(\mathbb{Z}^d) = \mathcal{L}^\infty(T^d)$ of Example 13.4. For $f \in \mathcal{L}^\infty(T^d)$ the spectral density function $F(M_f)$ of $M_f : L^2(T^d) \to L^2(T^d)$ sends $\lambda$ to the volume of the set $\{ z \in T^d \mid |f(z)| \leq \lambda \}$. Let $dF$ be the unique measure on the Borel $\sigma$-algebra on $\mathbb{R}$ which satisfies $dF([a,b]) = F(b) - F(a)$ for $a < b$. Then define the Fuglede-Kadison determinant

$$\det_{\mathcal{N}(G)}(f) \in [0, \infty)$$

to be the positive real number

$$\det_{\mathcal{N}(G)}(f) = \exp \left( \int_{0+}^\infty \ln(\lambda) \; dF \right)$$

if the Lebesgue integral $\int_{0+}^\infty \ln(\lambda) \; dF$ converges to a real number and by 0 otherwise.

Notice that in the definition above we do not require that the source and domain of $f$ agree or that $f$ is injective or that $f$ is surjective. Our conventions imply that the Fuglede-Kadison operator of the zero operator $0 : V \to V$ is 1.

**Example 13.15** (Fuglede-Kadison determinant for finite groups). To illustrate this definition, we look at the example where $G$ is finite. We essentially get the classical determinant $\det_{\mathcal{C}}$. Namely, let $\lambda_1, \lambda_2, \ldots, \lambda_r$ be the non-zero eigenvalues of $f^*f$ with multiplicity $\mu_i$. Then one obtains, if $\overline{f^*f}$ is the automorphism of the orthogonal complement of the kernel of $f^*f$ induced by $f^*f$,

$$\det_{\mathcal{G}(G)}(f) = \exp \left( \sum_{i=1}^r \frac{\mu_i}{|G|} \ln(\sqrt[|\mathcal{G}|]{\lambda_i}) \right) = \prod_{i=1}^r |\lambda_i|^{\frac{\mu_i}{|\mathcal{G}|}} = \det_{\mathcal{C}}(\overline{f^*f})^{\frac{1}{|\mathcal{G}|}}.$$

where $\det_{\mathcal{C}}(\overline{f^*f})$ is put to be 1 of $f$ is the zero operator and hence $\overline{f^*f}$ is id: $\{0\} \to \{0\}$. If $f : \mathbb{C}G^m \to \mathbb{C}G^m$ is an automorphism, we get

$$\det_{\mathcal{G}(G)}(f) = |\det_{\mathcal{C}}(f)|^{\frac{1}{|\mathcal{G}|}}.$$

**Example 13.16** (Fuglede-Kadison determinant for $(2,2)$-matrices over the trivial group). Consider $A \in M_{2,2}(\mathbb{C})$. Let $A^*$ be the conjugate transpose of $A$ and denote
by $\text{tr}_Z(AA^\ast)$ the trace of $AA^\ast$. We conclude from Example 13.15 for the trivial group $\{1\}$

$$\det_{N(\{1\})}(r_A : \mathbb{C}^2 \to \mathbb{C}^2) = \begin{cases} |\det_C(A)| & \text{if } \det_C(A) \neq 0; \\ \sqrt{\text{tr}_Z(AA^\ast)} & \text{if } \det_C(A) = 0 \text{ and } A \neq 0; \\ 1 & \text{if } A = 0. \end{cases}$$

**Example 13.17** (Fuglede-Kadison determinant for $N(\mathbb{Z}^d)$). Let $G = \mathbb{Z}^d$. We use the identification $N(\mathbb{Z}^d) = L^\infty(T^d)$ of Example 13.13. For $f \in L^\infty(T^m)$ we conclude from Example 13.13

$$\det_{N(\mathbb{Z}^d)}(M_f : L^2(T^d) \to L^2(T^d)) = \exp\left(\int_{T^d} \ln(|f(z)|) \cdot \chi_{\{u \in S^1 | f(u) \neq 0\}} \, d\text{vol}_2\right)$$

using the convention $\exp(-\infty) = 0$.

Let $i : H \to G$ be an injective group homomorphism. Let $V$ be a finitely generated Hilbert $N(H)$-module. There is an obvious pre-Hilbert structure on $\mathcal{G}G \otimes_\mathcal{G}H V$ for which $G$ acts by isometries since $\mathcal{G}G \otimes_\mathcal{G}H V$ as a complex vector space can be identified with $\bigoplus_{G/H}$ $V$. Its Hilbert space completion is a finitely generated Hilbert $N(G)$-module and denoted by $i_*M$. A morphism of finitely generated Hilbert $N(H)$-modules $f : V \to W$ induces a map of finitely generated Hilbert $N(G)$-modules $i_*f : i_*V \to i_*W$.

The following theorem can be found with proof in [26] Theorem 3.14 on page 128 and Lemma 3.15 (4) on page 129.

**Theorem 13.18** (Fuglede-Kadison determinant).

1. Let $f : U \to V$ and $g : V \to W$ be morphisms of finitely generated Hilbert $N(G)$-modules such that $f$ has dense image and $g$ is injective. Then

$$\det_{N(G)}(g \circ f) = \det_{N(G)}(f) \cdot \det_{N(G)}(g);$$

2. Let $f_1 : U_1 \to V_1$, $f_2 : U_2 \to V_2$ and $f_3 : U_2 \to V_1$ be morphisms of finitely generated Hilbert $N(G)$-modules such that $f_1$ has dense image and $f_2$ is injective. Then

$$\det_{N(G)} \left( \begin{array}{cc} f_1 & f_3 \\ 0 & f_2 \end{array} \right) = \det_{N(G)}(f_1) \cdot \det_{N(G)}(f_2);$$

3. Let $f : U \to V$ be a morphism of finitely generated Hilbert $N(G)$-modules. Then

$$\det_{N(G)}(f) = \det_{N(G)}(f^*);$$

4. Let $i : H \to G$ be the inclusion of a subgroup of finite index $[G : H]$. Let $i^*f : i^*U \to i^*V$ be the morphism of finitely generated Hilbert $N(H)$-modules obtained from $f$ by restriction. Then

$$\det_{N(H)}(i^*f) = \det_{N(G)}(f)^{[G:H]};$$

5. Let $i : H \to G$ be an injective group homomorphism and let $f : U \to V$ be a morphism of finitely generated Hilbert $N(H)$-modules. Then

$$\det_{N(G)}(i_*f) = \det_{N(H)}(f).$$

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