On the Data Augmentation Algorithm for Bayesian Multivariate Linear Regression with Non-Gaussian Errors

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December 2015

Abstract

Let \( \pi \) denote the intractable posterior density that results when the likelihood from a multivariate linear regression model with errors from a scale mixture of normals is combined with the standard non-informative prior. There is a simple data augmentation algorithm (based on latent data from the mixing density) that can be used to explore \( \pi \). Hobert et al. (2015) recently performed a convergence rate analysis of the Markov chain underlying this MCMC algorithm in the special case where the regression model is univariate. These authors provide simple sufficient conditions (on the mixing density) for geometric ergodicity of the Markov chain. In this note, we extend Hobert et al.’s (2015) result to the multivariate case.

1 Introduction

Let \( Y_1, Y_2, \ldots, Y_n \) be independent \( d \)-dimensional random vectors from the multivariate linear regression model

\[
Y_i = \beta^T x_i + \Sigma^{\frac{1}{2}} \varepsilon_i,
\]

where \( x_i \) is a \( p \times 1 \) vector of known covariates associated with \( Y_i \), \( \beta \) is a \( p \times d \) matrix of unknown regression coefficients, \( \Sigma^{\frac{1}{2}} \) is an unknown scale matrix, and \( \varepsilon_1, \ldots, \varepsilon_n \) are iid errors from a scale

\textbf{Key words and phrases.} Drift condition, Geometric ergodicity, Heavy-tailed distribution, Markov chain Monte Carlo, Minorization condition, Scale mixture
mixture of multivariate normals; that is, from a density of the form

\[ f_H(\varepsilon) = \int_0^\infty \frac{u^d}{(2\pi)^{d/2}} \exp \left\{ -\frac{u}{2} \varepsilon^T \varepsilon \right\} h(u) \, du, \]

where \( h \) is the density function of some positive random variable. We shall refer to \( h \) as a mixing density. For example, when \( h \) is the density of a Gamma \((\nu/2, \nu/2)\) random variable, then \( f_H \) becomes the multivariate Student’s \( t \) density with \( \nu > 0 \) degrees of freedom, which, aside from a normalizing constant, is given by

\[ \left[ 1 + \nu^{-1} \varepsilon^T \varepsilon \right]^{-\frac{d+\nu}{2}}. \]

Let \( Y \) denote the \( n \times d \) matrix whose \( i \)th row is \( Y_i^T \), and let \( X \) stand for the \( n \times p \) matrix whose \( i \)th row is \( x_i^T \), and, finally, let \( \varepsilon \) represent the \( n \times d \) matrix whose \( i \)th row is \( \varepsilon_i^T \). Using this notation, we can state the \( n \) equations in (1) more succinctly as follows

\[ Y = X \beta + \varepsilon \Sigma^\frac{1}{2}. \]

Let \( y \) and \( y_i \) denote the observed values of \( Y \) and \( Y_i \), respectively.

Consider a Bayesian analysis of the data from the regression model (1) using an improper prior on \((\beta, \Sigma)\) that takes the form \( \omega(\beta, \Sigma) \propto |\Sigma|^{-a} I_{S_d}(\Sigma) \) where \( S_d \subset \mathbb{R}^{d(d+1)/2} \) denotes the space of \( d \times d \) positive definite matrices. Taking \( a = (d+1)/2 \) yields the standard non-informative prior for multivariate location scale problems. The joint density of the data from model (1) is, of course, given by

\[ f(y|\beta, \Sigma) = \prod_{i=1}^n \int_0^\infty \frac{u^d}{(2\pi)^{d/2}|\Sigma|^2} \exp \left\{ -\frac{u}{2} (y_i - \beta^T x_i)^T \Sigma^{-1} (y_i - \beta^T x_i) \right\} h(u) \, du. \]

Define

\[ m(y) = \int_{S_d} \int_{\mathbb{R}^{p+d}} f(y|\beta, \Sigma) \omega(\beta, \Sigma) \, d\beta \\ d\Sigma. \]

The posterior distribution is proper precisely when \( m(y) < \infty \), and is given by

\[ \pi(\beta, \Sigma|y) = \frac{f(y|\beta, \Sigma) \omega(\beta, \Sigma)}{m(y)}. \]

Let \( \Lambda \) denote the \( n \times (p+d) \) matrix \((X : y)\). The following conditions are necessary for propriety:

(\(N1\)) \( \text{rank}(\Lambda) = p + d \);

(\(N2\)) \( n > p + 2d - 2a \).
We assume throughout that \((N1)\) and \((N2)\) hold.

There is a well-known DA algorithm that can be used to explore the intractable posterior \(\pi(\beta, \Sigma | y)\) (see, e.g., Liu [1996]). In order to state this algorithm, we must introduce some additional notation. For \(z = (z_1, \ldots, z_n)\), let \(Q\) be an \(n \times n\) diagonal matrix whose \(i\)th diagonal element is \(z_i^{-1}\). Also, define \(\Omega = (X^T Q^{-1} X)^{-1}\) and \(\mu = (X^T Q^{-1} X)^{-1} X^T Q^{-1} y\). We shall assume throughout the paper that the mixing density \(h\) satisfies the following condition
\[
\int_1^{\infty} u^\frac{d}{2} h(u) du < \infty.
\]
We will refer to this as “condition \(M\).” Finally, define a parametric family of univariate density functions indexed by \(s \geq 0\) as follows
\[
\psi(u; s) = c(s) u^\frac{d}{2} e^{-\frac{su^2}{2}} h(u),
\]
where \(c(s)\) is the normalizing constant. When \(h\) is a standard density, \(\psi\) often turns out to be one as well, but even in non-standard cases, it’s typically straightforward to make draws from \(\psi(\cdot, s)\) [Hobert et al. 2015]. The DA algorithm calls for draws from the inverse Wishart (IW\(_d\)) and matrix normal (N\(_{p,d}\)) distributions. The precise forms of the densities are given in the Appendix. We now present the DA algorithm. If the current state of the DA Markov chain is \((\beta_m, \Sigma_m) = (\beta, \Sigma)\), then we simulate the new state, \((\beta_{m+1}, \Sigma_{m+1})\), using the following three-step procedure.

**Iteration \(m+1\) of the DA algorithm:**

1. Draw \(\{Z_i\}_{i=1}^n\) independently with \(Z_i \sim \psi\left( \cdot; (\beta^T x_i - y_i)^T \Sigma^{-1} (\beta^T x_i - y_i) \right)\), and call the result \(z = (z_1, \ldots, z_n)\).

2. Draw
\[
\Sigma_{m+1} \sim \text{IW}_d\left(n - p + 2a - d - 1, \left( y^T Q^{-1} y - \mu^T \Omega^{-1} \mu \right)^{-1} \right).
\]

3. Draw \(\beta_{m+1} \sim \text{N}_{p,d} (\mu, \Omega, \Sigma_{m+1})\)

Denote the DA Markov chain by \(\Phi = \{ (\beta_m, \Sigma_m) \}_{m=0}^{\infty}\), and its state space by \(X := \mathbb{R}^{p \times d} \times S_d\). For positive integer \(m\), let \(k^m : X \times X \rightarrow (0, \infty)\) denote the \(m\)-step Markov transition density (Mtd) of \(\Phi\), so that if \(A\) is a measurable set in \(X\),
\[
P \left( (\beta_m, \Sigma_m) \in A \mid (\beta_0, \Sigma_0) = (\beta, \Sigma) \right) = \int_A k^m((\beta', \Sigma') | (\beta, \Sigma)) \ d\beta' \ d\Sigma'.
\]
(The precise form of \( k^1 \) is given in Section 2.) If there exist \( M : X \to [0, \infty) \) and \( \rho \in [0, 1) \) such that, for all \( m \),

\[
\int_{S_d} \int_{\mathbb{R}^p \times d} \left| k^m (\beta, \Sigma|\tilde{\beta}, \tilde{\Sigma}) - \pi (\beta, \Sigma|y) \right| d\beta d\Sigma \leq M (\tilde{\beta}, \tilde{\Sigma}) \rho^m,
\]

then the chain \( \Phi \) is \textit{geometrically ergodic}. (The quantity on the left-hand side of (3) is the total variation distance between the posterior distribution and the distribution of \((\beta_m, \Sigma_m)\) conditional on \((\beta_0, \Sigma_0) = (\tilde{\beta}, \tilde{\Sigma})\).) The main contribution of this paper is to demonstrate that \( \Phi \) is geometrically ergodic as long as \( h \) converges to zero at the origin at an appropriate rate. This result is important from a practical perspective because geometric ergodicity guarantees the existence of the central limit theorems that form the basis of all the standard methods of calculating valid asymptotic standard errors for MCMC-based estimators (see, e.g., Roberts and Rosenthal (1998), Jones and Hobert (2001) and Flegal et al. (2008)).

We now define three classes of mixing densities based on behavior near the origin, and this will allow us to provide a formal statement of our main result. Let \( h : \mathbb{R}_+ \to [0, \infty) \) be a mixing density, where \( \mathbb{R}_+ := (0, \infty) \). If there is a \( \delta > 0 \) such that \( h(u) = 0 \) for all \( u \in (0, \delta) \), then we say that \( h \) is \textit{zero near the origin}. Now assume that \( h \) is strictly positive in a neighborhood of 0 (i.e., \( h \) is not zero near the origin). If there exists a \( c > -1 \) such that

\[
\lim_{u \to 0} \frac{h(u)}{u^c} \in (0, \infty),
\]

then we say that \( h \) is \textit{polynomial near the origin} with power \( c \). Finally, if for every \( c > 0 \), there exists an \( \eta_c > 0 \) such that the ratio \( \frac{h(u)}{u^c} \) is strictly increasing in \((0, \eta_c)\), then we say that \( h \) is \textit{faster than polynomial near the origin}. As shown in Hobert et al. (2015) (henceforth, HJ&K), all of the standard parametric families with support \((0, \infty)\) are either polynomial near the origin, or faster than polynomial near the origin. Here is our main result.

**Theorem 1.** Let \( h \) be a mixing density that satisfies condition \( \mathcal{M} \). Then the DA Markov chain is geometrically ergodic if \( h \) is zero near the origin, or if \( h \) is faster than polynomial near the origin, or if \( h \) is polynomial near the origin with power \( c > \frac{n-p+2a-d-1}{2} \).

**Remark 1.** Theorem[7] is the multivariate version of HJ&K’s univariate result. However, because the parametrization used in HJ&K is slightly different than that used here, setting \( d = 1 \) in Theorem[7] does not yield HJ&K’s result. In particular, whereas we parametrize our model in terms of \( \Sigma \), which is the natural parametrization in the multivariate setting, HJ&K use \( \sqrt{\Sigma} \). Hence, we cannot
directly compare the two results in the case \( d = 1 \) since our hyperparameter \( a \) has a different meaning than theirs. To put the two models on the same footing when \( d = 1 \), we would have to change our prior to \( \omega^* (\beta, \Sigma) \propto |\Sigma|^{-\frac{a+1}{2}} I_{d_a}(\Sigma) \). Note that, if we set \( d = 1 \) and replace \( a \) by \( (a + 1)/2 \) in Theorem 1, then we do indeed recover HJ&K’s result.

**Remark 2.** Fix \( \nu > 0 \) and suppose that the mixing density is \( \text{Gamma}(\nu/2, \nu/2) \), which is clearly polynomial near the origin with power \( \nu/2 - 1 \). It follows from Theorem 1 that the DA Markov chain is geometrically ergodic as long as \( \nu > n - p + 2a - d + 1 \). In particular, when \( a = (d + 1)/2 \), we need \( \nu > n - p + 2 \). This special case of Theorem 1 was established in Roy and Hobert (2010).

### 2 Proof of the main result

In order to formally define the Markov chain that the DA algorithm simulates, we must introduce the latent data model. Suppose that, conditional on \( (\beta, \Sigma) \), \( \{(Y_i, Z_i)\}_{i=1}^n \) are iid pairs such that

\[
Y_i | Z_i = z_i \sim N_d(\beta^T x_i, \Sigma/z_i)
\]

\[
Z_i \sim h.
\]

Let \( z = (z_1, \ldots, z_n) \) and denote the joint density of \( \{Y_i, Z_i\}_{i=1}^n \) by \( \tilde{f}(y, z | \beta, \Sigma) \). It’s easy to see that \( \int_{\mathbb{R}^n} \tilde{f}(y, z | \beta, \Sigma) \, dz = f(y | \beta, \Sigma) \). Thus, if we define

\[
\pi(\beta, \Sigma, z | y) = \frac{\tilde{f}(y, z | \beta, \Sigma) \omega(\beta, \Sigma)}{m(y)},
\]

then we have \( \int_{\mathbb{R}^n} \pi(\beta, \Sigma, z | y) \, dz = \pi(\beta, \Sigma | y) \) which is the posterior (target) density. From here on, to simplify notation, we will write \( \pi(\beta, \Sigma, z) \) instead of \( \pi(\beta, \Sigma, z | y) \). The Mtd of the chain underlying the DA algorithm is given by

\[
k(\beta, \Sigma | \beta, \Sigma) = \int_{\mathbb{R}^n} \pi(\beta, \Sigma | z) \pi(z | \beta, \Sigma) \, dz,
\]

where \( \pi(\beta, \Sigma | z) \) and \( \pi(z | \beta, \Sigma) \) are conditional densities associated with \( \pi(\beta, \Sigma, z) \). The precise forms of these densities can be gleaned from the statement of the DA algorithm given in the Introduction. (Note that the algorithm exploits the representation \( \pi(\beta, \Sigma | z) = \pi(\beta | \Sigma, z) \pi(\Sigma | z) \).) An argument used in Section 2 of HJ&K can be used here to show that \( k \) is strictly positive on \( X \times X \), and this implies that (when the posterior is proper) the DA Markov chain is Harris recurrent. We now use a standard drift and minorization argument to develop a sufficient condition for geometric ergodicity of \( \Phi \).
Proposition 1. Suppose that there exist \( \lambda \in \left[ 0, \frac{1}{n-p+2a-1} \right] \) and \( L \in \mathbb{R} \) such that
\[
\frac{\int_0^\infty u^{d/2} e^{-u/2} h(u) \, du}{\int_0^\infty u^{d/2} e^{-u/2} h(u) \, du} \leq \lambda s + L
\]
for every \( s \geq 0 \). Then the DA Markov chain is geometrically ergodic.

Proof. We will prove the result by establishing a drift condition and an associated minorization condition, as in Rosenthal’s (1995) Theorem 12. Our drift function, \( V : \mathbb{R}^{p \times d} \times S_d \to \mathbb{R} \), is as follows
\[
V(\beta, \Sigma) = \sum_{i=1}^n (y_i - \beta^T x_i) \Sigma^{-1} (y_i - \beta^T x_i).
\]

Part I: Minorization. Fix \( l > 0 \) and define
\[
B_l = \{ (\beta, \Sigma) : V(\beta, \Sigma) \leq l \}.
\]

We will construct \( \epsilon \in (0, 1) \) and a density function \( f^* : \mathbb{R}^{p \times d} \times S_d \to [0, \infty) \) (both of which depend on \( l \)) such that, for all \( (\tilde{\beta}, \tilde{\Sigma}) \in B_l \),
\[
k(\beta, \Sigma | \tilde{\beta}, \tilde{\Sigma}) \geq \epsilon f^*(\beta, \Sigma).
\]

This is the minorization condition. It suffices to construct \( \epsilon \in (0, 1) \) and a density function \( \hat{f} : \mathbb{R}^n_+ \to [0, \infty) \) such that, for all \( (\tilde{\beta}, \tilde{\Sigma}) \in B_l \),
\[
\pi(z | \tilde{\beta}, \tilde{\Sigma}) \geq \epsilon \hat{f}(z).
\]

Indeed, if such an \( \hat{f} \) exists, then for all \( (\tilde{\beta}, \tilde{\Sigma}) \in B_l \), we have
\[
k(\beta, \Sigma | \tilde{\beta}, \tilde{\Sigma}) = \int_{\mathbb{R}^n_+} \pi(\beta, \Sigma | z) \pi(z | \tilde{\beta}, \tilde{\Sigma}) \, dz \geq \epsilon \int_{\mathbb{R}^n_+} \pi(\beta, \Sigma | z) \hat{f}(z) \, dz = \epsilon f^*(\beta, \Sigma).
\]

We now build \( \hat{f} \). Define \( \tilde{r}_i = (y_i - \tilde{\beta}^T x_i)^T \tilde{\Sigma}^{-1} (y_i - \tilde{\beta}^T x_i) \), and note that
\[
\pi(z | \tilde{\beta}, \tilde{\Sigma}) = \prod_{i=1}^n \psi(z_i; \tilde{r}_i) = \prod_{i=1}^n c(\tilde{r}_i) z_i^{d/2} e^{-\frac{\tilde{r}_i z_i}{2}} h(z_i).
\]

Now, for any \( s \geq 0 \), we have
\[
c(s) = \frac{1}{\int_0^\infty u^{d/2} e^{-u/2} h(u) \, du} \geq \frac{1}{\int_0^\infty u^{d/2} h(u) \, du}.
\]
By definition, if \((\tilde{\beta}, \tilde{\Sigma}) \in B_l\), then \(\sum_{i=1}^n \tilde{r}_i \leq l\), which implies that \(\tilde{r}_i \leq l\) for each \(i = 1, \ldots, n\). Thus, if \((\tilde{\beta}, \tilde{\Sigma}) \in B_l\), then for each \(i = 1, \ldots, n\), we have
\[
z_i^2 e^{-\frac{\tilde{r}_i z_i}{2}} h(z_i) \geq z_i^2 e^{-\frac{\tilde{r}_i z_i}{2}} h(z_i).
\]
Therefore,
\[
\pi(z|\tilde{\beta}, \tilde{\sigma}) \geq \left[ \int_0^\infty u^d h(u) du \right]^{-n} \prod_{i=1}^n \frac{z_i^2 e^{-\frac{\tilde{r}_i z_i}{2}} h(z_i)}{\int_0^\infty u^d e^{-\frac{\tilde{r}_i z_i}{2}} h(u) du}
\]
\[
= \left[ \int_0^\infty u^d e^{-\frac{\tilde{r}_i z_i}{2}} h(u) du \right] \prod_{i=1}^n \frac{z_i^2 e^{-\frac{\tilde{r}_i z_i}{2}} h(z_i)}{\int_0^\infty u^d e^{-\frac{\tilde{r}_i z_i}{2}} h(u) du}
\]
\[
\pi(z|\tilde{\beta}, \tilde{\sigma}) \geq e^\tilde{f}(z).
\]
Hence, our minorization condition is established.

**Part II: Drift.** To establish the required drift condition, we need to bound the expectation of \(V(\beta_{m+1}, \Sigma_{m+1})\) given that \((\beta_m, \Sigma_m) = (\tilde{\beta}, \tilde{\Sigma})\). This expectation is given by
\[
\int_{S_d} \int_{\mathbb{R}_p} V(\beta, \Sigma) k(\beta, \Sigma|\tilde{\beta}, \tilde{\Sigma}) d\beta d\Sigma
\]
\[
= \int_{\mathbb{R}_n^+} \left\{ \int_{S_d} \left[ \int_{\mathbb{R}_p} V(\beta, \Sigma) \pi(\beta|\Sigma, z) d\beta \right] \pi(\Sigma|z) d\Sigma \right\} \pi(z|\tilde{\beta}, \tilde{\Sigma}) dz .
\]
Calculations in Roy and Hobert’s (2010) Section 4 show that
\[
\int_{S_d} \left[ \int_{\mathbb{R}_p} V(\beta, \Sigma) \pi(\beta|\Sigma, z) d\beta \right] \pi(\Sigma|z) d\Sigma \leq (n - p + 2a - 1) \sum_{i=1}^n \frac{1}{z_i}.
\]
It follows from (4) that
\[
\int_{\mathbb{R}_n^+} \left\{ \int_{S_d} \left[ \int_{\mathbb{R}_p} V(\beta, \Sigma) \pi(\beta|\Sigma, z) d\beta \right] \pi(\Sigma|z) d\Sigma \right\} \pi(z|\tilde{\beta}, \tilde{\Sigma}) dz
\]
\[
\leq (n - p + 2a - 1) \int_{\mathbb{R}_n^+} \left[ \sum_{i=1}^n \frac{1}{z_i} \right] \pi(z|\tilde{\beta}, \tilde{\sigma}) dz
\]
\[
= (n - p + 2a - 1) \sum_{i=1}^n c(\tilde{r}_i) \int_0^\infty u^d e^{-\frac{\tilde{r}_i u}{2}} h(u) du
\]
\[
\leq (n - p + 2a - 1) \left( \lambda \sum_{i=1}^n \tilde{r}_i + nL \right)
\]
\[
= \lambda(n - p + 2a - 1)V(\tilde{\beta}, \tilde{\Sigma}) + (n - p + 2a - 1)nL
\]
\[
= \lambda V(\tilde{\beta}, \tilde{\sigma}) + L',
\]
where $\lambda' := \lambda(n - p + 2a - 1) \in [0, 1)$ and $L' := (n - p + 2a - 1)nL$. Since the minorization condition holds for any $l > 0$, we can appeal to Rosenthal’s (1995) Theorem 12 to get the result. This completes the proof.

Suppose that $g$ is a mixing density. For a positive integer $d$, we say that $g$ satisfies Condition $A_d$ with $\lambda \in [0, \infty)$ if there exists $k_\lambda \in \mathbb{R}$ such that

$$
\frac{\int_0^\infty u^{d-2} e^{-\frac{su}{2}} g(u) \, du}{\int_0^\infty u^d e^{-\frac{su}{2}} g(u) \, du} \leq \lambda s + k_\lambda
$$

for every $s \geq 0$. The following result can be proven using the results in HJ&K’s Section 4.

**Theorem 2.** Suppose that $g$ is a mixing density such that $\int_1^\infty \sqrt{u} g(u) \, du < \infty$. If $g$ is either zero near the origin or faster than polynomial near the origin, then $g$ satisfies Condition $A_1$ with any $\lambda > 0$. If $g$ is polynomial near the origin with power $c > -\frac{1}{2}$, then $g$ satisfies Condition $A_1$ with any $\lambda > \frac{1}{2c+1}$.

We use this result to prove the following:

**Corollary 1.** Fix a positive integer $d$ and suppose that $g$ is a mixing density with $\int_1^\infty u^{\frac{d}{2}} g(u) \, du < \infty$. If $g$ is either zero near the origin or faster than polynomial near the origin, then $g$ satisfies Condition $A_d$ with any $\lambda > 0$. If $g$ is polynomial near the origin with power $c > -\frac{1}{2}$, then $g$ satisfies Condition $A_d$ with any $\lambda > \frac{1}{2c+d}$.

**Proof.** Let $g^*(u)$ be the mixing density that is proportional to $u^{\frac{d-1}{2}} g(u)$. Then $\int_1^\infty \sqrt{u} g^*(u) \, du < \infty$, so Theorem 2 applies to $g^*$, and

$$
\frac{\int_0^\infty u^{\frac{d-1}{2}} e^{-\frac{su}{2}} g^*(u) \, du}{\int_0^\infty u^d e^{-\frac{su}{2}} g^*(u) \, du} = \frac{\int_0^\infty u^{\frac{d-2}{2}} e^{-\frac{su}{2}} g(u) \, du}{\int_0^\infty u^d e^{-\frac{su}{2}} g(u) \, du}.
$$

(5)

Now, it’s easy to see that, if $g$ is zero near the origin, then so is $g^*$, and if $g$ is faster than polynomial near the origin, then so is $g^*$. Hence, if $g$ is either zero near the origin or faster than polynomial near the origin, then by Theorem 2, $g^*$ satisfies Condition $A_1$ with any $\lambda > 0$, which implies (by (5)) that $g$ satisfies Condition $A_d$ with any $\lambda > 0$. Finally, assume that $g$ is polynomial near the origin with power $c > -\frac{1}{2}$. Then $g^*$ is polynomial near the origin with power $(2c + d - 1)/2 > -\frac{1}{2}$. Theorem 2 implies that $g^*$ satisfies Condition $A_1$ with any $\lambda > 1/(2c + d)$. It follows from (5) that $g$ satisfies Condition $A_d$ with any $\lambda > 1/(2c + d)$. □
**Proof of Theorem** Assume that $h$ is zero near the origin or faster than polynomial near the origin. Corollary implies that $h$ satisfies Condition $A_d$ with any $\lambda > 0$, and it follows from Proposition that the DA Markov chain is geometrically ergodic. Now assume that $h$ is polynomial near the origin with power $c > \frac{n-p+2a-d-1}{2}$, which is strictly positive by (N2). Corollary implies that $h$ satisfies Condition $A_d$ with any $\lambda > \frac{1}{2c+d}$, and since

$$\frac{1}{2c+d} < \frac{1}{n-p+2a-1},$$

Proposition implies that the DA Markov chain is geometrically ergodic.

**Appendix**

**Matrix Normal Distribution** Suppose $Z$ is an $r \times c$ random matrix with density

$$f_Z(z) = \frac{1}{(2\pi)^{\frac{rc}{2}} |A|^\frac{r}{2} |B|^\frac{c}{2}} \exp \left( -\frac{1}{2} \text{tr} \left\{ A^{-1}(z - \theta)B^{-1}(z - \theta)^T \right\} \right),$$

where $\theta$ is an $r \times c$ matrix, $A$ and $B$ are $r \times r$ and $c \times c$ positive definite matrices. Then $Z$ is said to have a matrix normal distribution and we denote this by $Z \sim \mathcal{N}_{r,c}(\theta, A, B)$ (Arnold, 1981, Chapter 17).

**Inverse Wishart Distribution** Suppose $W$ is a $d \times d$ random positive definite matrix with density

$$f_W(w) = \frac{|w|^{-\frac{m+d+1}{2}}}{\left( \frac{m}{2} \right)^{\frac{d(d-1)}{4}} |\Theta|^{\frac{m}{2}} \prod_{i=1}^{d} \Gamma \left( \frac{1}{2} (m+1-i) \right) I_{S_d}(W)},$$

where $m > d - 1$ and $\Theta$ is a $d \times d$ positive definite matrix. Then $W$ is said to have an inverse Wishart distribution and this is denoted by $W \sim \mathcal{IW}_{d}(m, \Theta)$.

**Acknowledgment.** The second author was supported by NSF Grant DMS-15-11945.

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