Maximal $L_p$--$L_q$ regularity for the quasi-steady elliptic problems

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Abstract. In this paper, we consider maximal regularity for the vector-valued quasi-steady linear elliptic problems. The equations are the elliptic equation in the domain and the evolution equations on its boundary. We prove the maximal $L_p$--$L_q$ regularity for these problems and give examples that our results are applicable. The Lopatinskii–Shapiro and the asymptotic Lopatinskii–Shapiro conditions are important to get boundedness of solution operators.

1. Introduction

We consider the vector-valued quasi-steady problems of the following

$$
\begin{align*}
\eta u + A(t, x, D)u &= f(t, x) \quad (t \in J, \ x \in G), \\
\partial_t \rho + B_0(t, x, D)u + C_0(t, x, D_G)\rho &= g_0(t, x) \quad (t \in J, \ x \in \Gamma), \\
B_j(t, x, D)u + C_j(t, x, D_G)\rho &= g_j(t, x) \quad (t \in J, \ x \in \Gamma, \ j = 1, \ldots, m), \\
\rho(0, x) &= \rho_0(x) \quad (x \in \Gamma),
\end{align*}
$$

(1)

where $\eta > 0$, $J \subset [0, T]$ is a finite interval, and $G \subset \mathbb{R}^n$ ($n \geq 2$) is a bounded or an exterior domain with the boundary $\Gamma$. The functions $f, \{g_j\}_{j=0}^m, \rho_0$ are given data, and the functions $u$ and $\rho$ are unknown functions. $(A, B_j, C_j)$ are differential operators with order $(2m, m_j, k_j)$, respectively. The aim of this paper is to obtain maximal $L_p$--$L_q$ regularity of these equations. More precisely, we characterize the data space $X \times \prod_{j=0}^m Y_j \times \pi Z_\rho$ and the solution space $Z_u \times Z_\rho$ such that these spaces are isomorphism.

This quasi-steady problems are considered as the linearized equations for the various nonlinear equations, e.g., free boundary problems. One of the successful methods to solve the free boundary problems is the transformation from time-varying domain to fixed domain. After we use this transformation, the equation has an unknown function called a height function on the boundary, and the equation on the boundary has time derivative of order one. If the original equation has a time derivative in an interior domain, the transformed equation also has a time derivative in a domain. On the other hand, if there is no time derivative in the original equation in the domain, the

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transformed equation does not have a time derivative. Usually, the derived equation is also nonlinear, but the linearized equation corresponds to the relaxation type or the quasi-steady type. The first one corresponds to the first derivative in the interior equation and it has already been considered in the paper [1]. As far as we know, the second one has not considered yet. Therefore, we consider these problems in this paper.

The paper is organized as follows. In Sect. 2, basic function spaces and assumptions for \( A, B_j \) and \( C_j \) including smoothness are introduced. Then, our main result is stated. In Sect. 3, basic notions of operator theory, e.g., operator-valued multiplier theorems and \( H^\infty \)-calculus, are introduced for the reader’s convenience. In Sect. 4, we first consider (1) under \( G = \mathbb{R}^n \) with the differential operators having no lower-order terms and constant coefficients. The problem is first reduced into the case of \( f = 0 \) and \( \rho_0 = 0 \). Then, the partial Laplace–Fourier transform is applied to get the solution formula of Fourier multiplier type. In this step, the Lopatinskii–Shapiro condition (LS) is frequently used. Operator-valued Fourier multiplier theorem due to Weis [13] and the operator-valued \( H^\infty \)-functional calculus due to Kalton–Weis [9] are applied to the solution operator to obtain its maximal regularity of the solutions. Here, the asymptotic Lopatinskii–Shapiro (ALS) conditions are also needed. By perturbation and localization procedure, our maximal regularity result for the full problem of (1) is proved.

2. Main results

Let us introduce notation to give our main results and state our theorem. Let \( \mathbb{N} \) be a set of positive integer and \( \mathbb{N}_0 = \mathbb{N} \cup \{0\} \). Differential operators in (1) are given by

\[
A(t, x, D) := \sum_{|\alpha| \leq 2m} a_\alpha(t, x) D^\alpha,
\]

\[
B_j(t, x, D) := \sum_{|\beta| \leq m_j} b_{j\beta}(t, x) D^\beta,
\]

\[
C_j(t, x, D_\Gamma) := \sum_{|\gamma| \leq k_j} c_{j\gamma}(t, x) D_\Gamma^\gamma,
\]

where \( m \) is a positive integer, \( m_j \in \mathbb{N}_0 \cap [0, 2m) \), \( k_j \in \mathbb{N}_0 \) for \( j = 0, \ldots, m \). The symbols \( D \), respectively, \( D_\Gamma \) mean \(-i \nabla\), respectively, \(-i \nabla_\Gamma\), where \( \nabla \) denotes the gradient in \( G \) and \( \nabla_\Gamma \) the surface gradient on \( \Gamma \). We assume that all boundary operators \( B_j \) and at least one \( C_j \) are non-trivial. The order \( k_j \) is defined by \(-\infty \) when \( C_j = 0 \). The unknown functions \( u(t, x), \rho(t, x) \) belong to Hilbert spaces \( E \) and \( F \). Note that the case \( E = F = \mathbb{C}^N (N \in \mathbb{N}) \) is allowed. For the coefficients of the above differential operators, \( a_\alpha(t, x), b_{j\beta}(t, x) \in B(E) \), \( c_{j\gamma}(t, x) \in B(F, E) \) for \( j = 1, \ldots, m \), and \( b_{0\beta}(t, x) \in B(E, F) \) and \( c_{0\gamma}(t, x) \in B(F) \). Let \( 1 < p, q < \infty \). We would like to find
the maximal $L_p$–$L_q$ regularity solutions, i.e.,

$$u \in Z_u := L_p(J; W^{2m}_q(G; E)), $$

then we should assume

$$f \in X := L_p(J; L_q(G; E)).$$

Since we expect the regularity of $g_j$ is the same as $B_j u$,

$$g_0 \in Y_0 := L_p(J; W^{2mk_0}_q(\Gamma; F)), $$

$$g_j \in Y_j := L_p(J; W^{2mk_j}_q(\Gamma; E)) \ (j = 1, \ldots, m) $$

with

$$\kappa_j := 1 - \frac{m_j}{2m} - \frac{1}{2mq} \ (j = 0, \ldots, m)$$

from the trace theorem. Thus, the solution class which $\rho$ belongs to should be

$$\rho \in W^1_p(J; W^{2mk_0}_q(\Gamma; F)) \cap \bigcap_{j=0}^{m} L_p(J; W^{k_j+2mk_j}_q(\Gamma; F))$$

$$=: W^1_p(J; W^{2mk_0}_q(\Gamma; F)) \cap L_p(J; W^{l+2mk_0}_q(\Gamma; F))$$

from the differential structure of Eq. (1), where $l := \max_{j=0,\ldots,m} l_j$ with $l_j = k_j - m_j + m_0$. We always assume $l \geq 0$ in this paper. It can be expected by the trace theorem that

$$\rho_0 \in \pi Z_\rho := B_{q^p}^{l(1-1/p)+2mk_0}(\Gamma; F).$$

Under these settings and assumptions (E), (SA), (SB), (SC), (LS) and (ALS) introduced later, we shall show the solution operator is an isomorphism between the data $(f, \{g_j\}_{j=0}^{m}, \rho_0) \in X \times \prod_{j=0}^{m} Y_j \times \pi Z_\rho$ and the solution $(u, \rho) \in Z_u \times Z_\rho$.

First we assume normal ellipticity of $A$ as usual. The subscript # denotes the principal part of the corresponding operator, e.g., $A_{#}(t, x, D) = \sum_{|\alpha|=2m} a_\alpha(t, x) D^\alpha$.

(E) (Ellipticity of the interior symbol) For all $t \in J, x \in \overline{G}$ in the case $G$ is a bounded domain, $x \in \overline{G} \cup \{\infty\}$ in the case $G$ is an exterior domain, and for all $\xi \in \mathbb{R}^n$ satisfying $|\xi| = 1$, we assume normal ellipticity for $A(t, x, \xi)$ with an angle less than $\pi/2$, and thus

$$\sigma(A_{#}(t, x, \xi)) \subset \mathbb{C}_+ := \{z \in \mathbb{C} \mid \Re z > 0\}.$$

Here $\sigma(A_{#}(t, x, \xi))$ denotes the spectrum of the bounded operator $A_{#}(t, x, \xi) \in \mathcal{B}(E)$.

Next, we introduce conditions of smoothness to the coefficients of $A, B_j$ and $C_j$. These conditions allow us to use localization and perturbation argument.
(SA) For $|\alpha| = k \leq 2m - 1$, there exists $r_{\alpha} \geq q$ with $\frac{n}{r_{\alpha}} \leq 2m - k$ such that

$$a_{\alpha} \in L_{\infty}(J; L_{r_{\alpha}}(G; \mathcal{B}(E))).$$

For $|\alpha| = 2m$, assume

$$a_{\alpha} \in BUC(J \times \overline{G}; \mathcal{B}(E)).$$

In the case $G$ is exterior domain, we impose the condition that the asymptotic state at infinity $a_{\alpha}(t, \infty) := \lim_{|x| \to \infty, x \in G} a_{\alpha}(t, x)$ exists and is bounded uniformly with respect to $t \in J$ for all $|\alpha| = 2m$.

(SB) Let $\mathcal{E}_0 := \mathcal{B}(E, F)$ and $\mathcal{E}_j := \mathcal{B}(E)$ for $j = 1, \ldots, m$. For each $j = 0, \ldots, m$ and $|\beta| = k \leq m_j - 1$, there exist $s_{j\beta}, r_{j\beta} \geq q$ with $\frac{n-1}{r_{j\beta}} \leq m_j - k$, $\frac{n-1}{\tau_{j\beta}} \leq 2m - k - 1/q$ such that

$$b_{j\beta} \in L_{\infty}(J; (L_{s_{j\beta}} \cap B_{r_{j\beta}, q}^{2m\kappa_0})(\Gamma; \mathcal{E}_j)).$$

For $|\beta| = m_j$, assume

$$b_{j\beta} \in BUC(J \times \Gamma; \mathcal{E}_j).$$

(SC) Let $\mathcal{F}_0 := \mathcal{B}(F)$ and $\mathcal{F}_j := \mathcal{B}(F, E)$ for $j = 1, \ldots, m$. For each $j = 0, \ldots, m$ and $|\gamma| = k \leq k_j - 1$, there exist $t_{j\gamma}, \tau_{j\gamma} \geq q$ and $s_{j\gamma}^c, r_{j\gamma}^c \geq q$ with $\frac{l}{t_{j\gamma}} + \frac{n-1}{s_{j\gamma}^c} \leq l - k + m_j - m_0$ and $\frac{l}{\tau_{j\gamma}} + \frac{n-1}{r_{j\gamma}^c} \leq l - k + 2m\kappa_0$ such that

$$c_{j\gamma} \in L_{t_{j\gamma}}(J; L_{s_{j\gamma}^c}(\Gamma; \mathcal{F}_j)) \cap L_{\tau_{j\gamma}}(J; B_{r_{j\gamma}^c, q}^{2m\kappa_0}(\Gamma; \mathcal{F}_j)).$$

For $|\beta| = k_j$, assume

$$c_{j\gamma} \in BUC(J \times \Gamma; \mathcal{F}_j).$$

The following two conditions are needed to get the formula of solution operator and ensure their boundedness.

(LS)(Lopatinskii–Shapiro conditions) For each fixed $t \in J$ and $x \in \Gamma$, we freeze the coefficients of differential operator at $(t, x)$. We rewrite Eq. (1) in coordinates associated with $x$ so that the positive part of $x_n$-axis has the direction of the inner normal at $x$ after a transformation and a rotation. For all $\eta > 0$, $(\lambda, \xi') \in (\Sigma_\theta \times \mathbb{R}^{n-1}) \setminus (0, 0)$ (where $\theta > \pi/2$) and $\{h_j\}_{j=0}^{m} \subset F \times E^m$, the ODEs on the half line $\mathbb{R}_+ = (0, \infty)$ given by

$$\begin{cases}
\eta v(y) + A_{\theta}(t, x, \xi', D_y)v(y) = 0 & (y > 0), \\
B_{0\#}(t, x, \xi', D_y)v(0) + (\lambda + C_{0\#}(t, x, \xi'))\sigma = h_0, \\
B_{j\#}(t, x, \xi', D_y)v(0) + C_{j\#}(t, x, \xi')\sigma = h_j & (j = 1, \ldots, m)
\end{cases}$$

(2)
admit a unique solution \((v, \sigma) \in C_0^{2m}(\mathbb{R}_+; E) \times F\), where

\[
C_0^{2m}(\mathbb{R}_+; E) = \left\{ v \in C^{2m}(\mathbb{R}_+; E) : \lim_{y \to \infty} v(y) = 0 \right\}.
\]

To obtain the maximal \(L_p-L_q\) regularity, we need another type of Lopatinskii–Shapiro condition which ensures boundedness of the symbol of the solution operator.

(ALS) (Asymptotic Lopatinskii–Shapiro conditions) For each fixed \(t \in J\) and \(x \in \Gamma\) we rewrite Eq. (1) by the same way as above. For all \(\eta > 0, \xi \in \mathbb{R}^{n-1}\) and \(\{h_j\}_{j=1}^m \in F \times E^m\),

\[
\eta v(y) + A_{\#}(t, x, \xi', D_y)v(y) = 0 \quad (y > 0),
\]

\[
B_{j\#}(t, x, \xi', D_y)v(0) = h_j \quad (j = 1, \ldots, m)
\]

admit a unique solution \(v \in C_0^{2m}(\mathbb{R}_+; E)\). For all \((\lambda, \xi') \in (\Sigma_0 \cup \{0\}) \times \mathbb{R}^{n-2}\) \((\theta > \pi/2)\), all \(\{h_j\}_{j=0}^m \in F \times E^m\) the ordinary differential equations in \(\mathbb{R}_+\) given by

\[
A_{\#}(t, x, \xi', D_y)v(y) = 0 \quad (y > 0),
\]

\[
B_{0\#}(t, x, \xi', D_y)v(0) + (\lambda + \delta_{l,0}C_{0\#}(t, x, \xi'))\sigma = h_0,
\]

\[
B_{j\#}(t, x, \xi', D_y)v(0) + \delta_{l,l}C_{j\#}(t, x, \xi')\sigma = h_j \quad (j = 1, \ldots, m)
\]

admit a unique solution \((v, \sigma) \in C_0^{2m}(\mathbb{R}_+; E) \times F\). Here \(\mathbb{R}^{n-2} := \{\xi' \in \mathbb{R}^{n-1} ; |\xi'| = 1\}\) and \(\delta_{i,j}\) is the Kronecker delta, \(\delta_{i,j} = 1\) for \(i = j\) and \(\delta_{i,j} = 0\) for \(i \neq j\). Moreover, we assume the following elliptic equations. For \(\xi' \in \mathbb{R}^{n-2}\),

\[
A_{\#}(t, x, \xi', D_y)v(y) = 0 \quad (y > 0),
\]

\[
B_{j\#}(t, x, \xi', D_y)v(0) = h_j \quad (j = 1, \ldots, m)
\]

admit a unique solution \(v \in C_0^{2m}(\mathbb{R}_+; E)\), respectively. We are now in the position to state our main results.

**Theorem 1.** Let \(J = [0, T]\), \(G \subset \mathbb{R}^n\) be a domain with a compact boundary \(\Gamma = \partial G\) of class \(C^{2m+l-m_0}\), \(1 < p, q < \infty\) and \(E\) and \(F\) be Hilbert spaces. Assume assumptions (E), (SA), (SB), (SC), (LS) and (ALS) hold. Then, there exist positive constants \(\eta_0, C\) and \(C_T\), if \(\eta \geq \eta_0\), for

\[
(f, \{g_j\}_{j=0}^m, \rho_0) \in X \times \prod_{j=0}^m Y_j \times \pi Z_\rho,
\]

(1) admits a unique solution \((u, \rho) \in Z_u \times Z_\rho\) such that

\[
\|u\|_{Z_u} + \|\rho\|_{Z_\rho} \leq C\|f\|_X + C\sum_{j=0}^m \|g_j\|_{Y_j} + C_T\|\rho_0\|_{\pi Z_\rho},
\]
3. Preliminaries

In this section, notation, notion, basic tools of vector-valued harmonic analysis are introduced. For Banach spaces $X$ and $Y$, $B(X; Y)$ denotes the set of bounded linear operators from $X$ to $Y$. $H^\infty(\Sigma_\phi)$ denotes the set of bounded holomorphic functions on a sector

$$\Sigma_\phi := \left\{ re^{i\psi} \in \mathbb{C} \setminus \{0\} : r > 0, \ |\psi| < \phi \right\}.$$ 

For a Banach space $X$, $H^\infty(\Sigma_\phi; X)$ is the set of $X$-valued bounded holomorphic functions on $\Sigma_\phi$ for $0 < \phi < \pi$ equipped with the norm

$$\|f\|_{H^\infty(\Sigma_\phi)} := \sup_{\lambda \in \Sigma_\phi} |f(\lambda)|.$$ 

$L_p(\Omega; X)$ $(1 \leq p \leq \infty)$ and $W^s_q(\Omega; X)$ $(s \in \mathbb{R}, \ 1 \leq q \leq \infty)$ are the $X$-valued Lebesgue space and the Sobolev space on $\Omega$. $\mathcal{F}$ and $\mathcal{F}^{-1}$ are Fourier transform and its inverse transform, respectively. Especially, we denote $\mathcal{F}_{x'}$ by the partial Fourier transform with respect to $x'$-variable. $\mathcal{L}$ and $\mathcal{L}^{-1}$ are Laplace transform and its inverse transform, respectively.

**Definition 1.** A Banach space $X$ is said to be of class $\mathcal{HT}$ if the Hilbert transform $H$ defined by

$$Hf(t) := \frac{1}{\pi} \lim_{R \to \infty} \int_{|s| \leq R} f(t-s) \frac{ds}{s}$$

is bounded on $L_p(\mathbb{R}; X)$ for some $p \in (1, \infty)$. When $X$ is of the class $\mathcal{HT}$, then $L_p(J; X)$ is also of class $\mathcal{HT}$.

**Definition 2.** Let $X$ and $Y$ be Banach spaces. A family of operators $T \subset B(X; Y)$ is said to be $\mathcal{R}$-bounded, if there exists a constant $C > 0$ and $p \in [1, \infty)$ such that, for each positive integer $N$, $\{T_i\}_{i=1}^N \subset T$, $\{x_i\}_{i=1}^N \subset X$ and for all independent symmetric $\{-1, 1\}$-valued random variables $\varepsilon_i$ on a probability space $(\Omega, \mathcal{A}, \mu)$, the inequality

$$\left\| \sum_{i=1}^N \varepsilon_i T_i x_i \right\|_{L_p(\Omega; Y)} \leq C \left( \sum_{i=1}^N \varepsilon_i x_i \right) \left\| x_i \right\|_{L_p(\Omega; X)}$$  \hspace{1cm} (3)

holds. We denote by $\mathcal{RT}$ the infimum constant of $C$ which (3) holds.

It is known that if (3) holds for some $p \in [1, \infty)$, then (3) holds for all $p \in [1, \infty)$. Note that uniformly bounded family of operators on Hilbert spaces is always $\mathcal{R}$-bounded.
Definition 3. A Banach space $X$ is said to have property $(\alpha)$ if there exists a constants $C > 0$ such that

$$\left| \sum_{i,j=1}^{N} \alpha_{ij} \varepsilon_{i} \varepsilon'_{j} x_{ij} \right|_{L^{2}(\Omega \times \Omega'; X)} \leq C \left| \sum_{i,j=1}^{N} \varepsilon_{i} \varepsilon'_{j} x_{ij} \right|_{L^{2}(\Omega \times \Omega'; X)}$$

for all $\alpha_{ij} \in \{-1, 1\}, \{x_{ij}\}_{i,j=1}^{N} \subset X$, positive integer $N$, and all symmetric independent $\{-1, 1\}$-valued random variables $\varepsilon_{i}$ (respectively, $\varepsilon'_{j}$) on a probability space $(\Omega, \mathcal{A}, \mu)$ (respectively, $(\Omega', \mathcal{A}', \mu')$). $\mathcal{H}(\alpha)$ denotes the class of Banach spaces which belong to $\mathcal{H}$ and have property $(\alpha)$.

Note that Hilbert space is of the class $\mathcal{H}(\alpha)$ and all closed subspaces of $L^{p}(G)$ have property $(\alpha)$.

Proposition 1. (Operator-valued Fourier multiplier theorem of Lizorkin type) (see [11]) Let $1 < p < \infty$, $X$ and $Y$ be Banach spaces of the class $\mathcal{H}(\alpha)$. Let $\mathcal{M} \subset C(n)(\mathbb{R}^{n}\{0\}; B(X; Y))$ be a family of multipliers such that

$$\mathcal{R} \left\{ \xi^{\alpha} \partial_{\xi}^{\alpha} m(\xi) : \xi \in \mathbb{R}^{n}\{0\}, \alpha \in \{0, 1\}^{n}, m \in \mathcal{M} \right\} =: C_{L} < \infty.$$  

Then, $\mathcal{F}^{-1} m \mathcal{F} \in B(L^{p}(\mathbb{R}^{n}; X); L^{p}(\mathbb{R}^{n}; Y))$. Moreover,

$$\mathcal{R} \left\{ \mathcal{F}^{-1} m \mathcal{F} ; m \in \mathcal{M} \right\} \leq CC_{L},$$

for some constant $C = C(p, n, X, Y)$.

We define a class of holomorphic functions vanishing at the origin and infinity by

$$H_{0}^{\infty}(\Sigma_{\phi}) = \left\{ f \in H^{\infty}(\Sigma_{\phi}) : |f(\lambda)| \leq C|\chi(\lambda)|^{\varepsilon} \text{ for some } C > 0, \varepsilon > 0 \right\},$$

where $1 < \phi < \pi$ and $\chi(\lambda) = \lambda/(1 + \lambda)^{2}$. Let $0 < \phi_{A} < \phi < \pi$ and $A$ be a sectorial operator with spectral angle $\phi_{A}$ and $f \in H_{0}^{\infty}(\Sigma_{\phi})$. We define $f(A)$ via the Cauchy formula

$$f(A) := \frac{1}{2\pi i} \int_{\partial \Sigma_{\phi}} (\lambda - A)^{-1} f(\lambda) d\lambda.$$  

It is called that a sectorial operator $A : D(A) \subset X \to Y$ with spectral angle $\phi_{A}$ have a bounded $H^{\infty}$-calculus if there exists a constant $C > 0$

$$\| f(A) \|_{B(X; Y)} \leq C \| f \|_{H^{\infty}(\Sigma_{\phi})}$$

holds for all $f \in H_{0}^{\infty}(\Sigma_{\phi})$, $(\phi > \phi_{A})$. Sectorial operators satisfying (4) have an extended calculus $f(A)$ for $f \in H^{\infty}(\Sigma_{\phi})$ by the canonical way, and this extension is uniquely determined.
**Definition 4.** Let $X$ be a Banach space. Let $0 < \phi_A < \pi$ and $A$ be a sectorial operator on $X$ with spectral angle $\phi_A$ admitting a bounded $H^\infty$-calculus. $A$ is said to have a $R$-bounded $H^\infty$-calculus if

$$\left\{ h(A) : h \in H^\infty(\Sigma_\phi), |h|_{H^\infty(\Sigma_\phi)} \leq 1 \right\}$$

is $R$-bounded for some $\phi \geq \phi_A$. Such an operator is denoted by $A \in R H^\infty(X)$. We denote by $\phi_{R H^\infty}$ the infimum of $\phi$ which (5) holds.

Let us introduce the Kalton–Weis theorem, which gives a sufficient condition for boundedness of joint functional calculus and is used to show boundedness of solution operator in this paper, see, e.g., [4, 9–11].

**Lemma 1.** (Kalton–Weis Theorem) (see [4, 9–11]) Let $X$ be a Banach space of the class $\mathcal{H}T(\alpha)$, $A$ be a sectorial operator with spectral angle $\phi_A$ admitting a bounded $H^\infty$-calculus and $\mathcal{F}$ be a family of operators satisfying $\mathcal{F} \subset H^\infty(\Sigma_\phi; B(X))$ for $\phi > \phi_A$. Assume each $F \in \mathcal{F}$ commute with the resolvent of $A$, i.e.,

$$F(\lambda)(\mu - A)^{-1} = (\mu - A)^{-1} F(\lambda),$$

and

$$\left\{ F(z) : z \in \Sigma_\phi, F \in \mathcal{F} \right\}$$

is $R$-bounded. Then, there exist a constant $C > 0$ such that

$$\mathcal{R}(\mathcal{F}(A)) \leq C \mathcal{R} \left\{ F(z) : z \in \Sigma_\phi, F \in \mathcal{F} \right\}.$$

Lemma 1 also implies each operator admitting bounded $H^\infty$-calculus belongs to $R H^\infty$ provided that $X$ is of class $\mathcal{H}T(\alpha)$.

4. Solvability in the maximal regularity space

4.1. Reduction to $f = 0$ and $\rho_0 = 0$

We first consider our problem on the half space $\mathbb{R}^n_+$ and assume the differential operators have constant coefficients without lower order. Let $E_{\mathbb{R}^n}$ be the zero extension operator from $L^p(J; L^q(\mathbb{R}^n_+; E))$ to $L^p(J; L^q(\mathbb{R}^n_+; E))$. It follows from the Mikhlin theorem that there exist a unique solution $u_\ast$ to

$$\eta u + Au = E_{\mathbb{R}^n} f \quad (t \in J, x \in \mathbb{R}^n),$$

for $f \in L^p(J; L^q(\mathbb{R}^n_+; E))$ such that

$$\eta \|u_\ast\|_{L^p(J; L^q(\mathbb{R}^n_+; E))} + \|u_\ast\|_{L^p(J; W^{2q}_q(\mathbb{R}^n_+; E))} \leq C \|f\|_{L^p(J; L^q(\mathbb{R}^n_+; E))}.$$ 

Let $\rho_0 \in B^2_{q, p}(\mathbb{R}^{n-1}; F)$. Then, we also find a unique solution $\rho_\ast$ via maximal regularity of to $(\eta - \Delta)^{1/2}$ that
\[
\partial_t \rho + (\eta - \Delta)^{1/2} \rho = 0 \quad (t \in \mathbb{R}_+, x \in \mathbb{R}^{n-1}), \\
\rho(0) = \rho_0 \quad (x \in \mathbb{R}^{n-1})
\]

such that

\[
\|\rho_*\|_{W^{1, \infty}(\mathbb{R}^{n-1}; W^{2m_0}_q(\mathbb{R}^{n-1}; F))} \leq C \|\rho_0\|_{B^{2m_0+\frac{1}{1/p}}(\mathbb{R}^{n-1}; F)}.
\]

For the solution \((u, \rho)\) to (1), if we put \((\tilde{u}, \tilde{\rho}) := (u - u_*, \rho - \rho_*)\), then \((\tilde{u}, \tilde{\rho})\) satisfies

\[
\eta\tilde{u} + \mathcal{A}\tilde{u} = 0 \quad (t \in J, x \in G), \\
\partial_t \tilde{\rho} + B_0\tilde{u} + C_0\tilde{\rho} = g_0 - (\partial_t \rho_* + B_0 u_* + C_0 \rho_*) \quad (t \in J, x \in \Gamma), \\
B_j \tilde{u} + C_j \tilde{\rho} = g_j - (B_j u_* + C_j \rho_*) \quad (t \in J, x \in \Gamma, j = 1, \ldots, m), \\
\tilde{\rho}(0, x) = 0 \quad (x \in \Gamma).
\]

Note that \(g_0 - (\partial_t \rho_* + B_0 u_* + C_0 \rho_*) \in Y_0\) and \(g_j - (B_j u_* + C_j \rho_*) \in Y_j\).

Conversely, the solution of the original equations is given by \((u, \rho) := (\tilde{u} + u_*, \tilde{\rho} + \rho_*)\). Thus, it suffice to consider the case of \(f = 0\) and \(\rho_0 = 0\) from now on.

4.2. Partial Fourier transform and solution formula on the half space

We continue to consider the case of the half space and assume differential operators having constant coefficients without lower order terms. Assume that \((u, \rho)\) are solutions to (1) with \(f = 0\) and \(\rho_0 = 0\). Put

\[
v = \mathcal{L}_t \mathcal{F}_x' u, \quad \sigma = \mathcal{L}_t \mathcal{F}_x' \rho.
\]

Then, \((v, \sigma)\) satisfy

\[
\begin{align*}
\eta v + \mathcal{A}(\xi', D_x) v &= 0, \\
B_0(\xi', D_x) v(0) + (\lambda + C_0(\xi')) \sigma &= h_0, \\
B_j(\xi', D_x) v(0) + C_j(\xi') \sigma &= h_j \quad (j = 1, \ldots, m),
\end{align*}
\]

where \(h_j = \mathcal{L}_t \mathcal{F}_x' g_j\) for \(j = 0, \ldots, m\).

The Lopatinskii–Shapiro condition (LS) ensures, for each \((\lambda, \xi') \in (\Sigma_0 \times \mathbb{R}^{n-1}) \setminus \{(0, 0)\} (\theta > \pi/2)\) and for any \(\{h_j\}_{j=0}^m \in F \times E^m\), there exists a unique solution

\[
v \in C^2_0(\mathbb{R}_+; E), \quad \sigma \in F
\]

to (7). We derive the solution operator of Fourier multiplier type and show its boundedness. As [1,3,11], we construct the solution formula. By definition of \(\mathcal{A}\) and \(B_j\),
\[ A(\xi', D_y) = \sum_{k=0}^{2m} a_k(\xi')D_y^{2m-k}, \quad B_j(\xi', D_y) = \sum_{k=0}^{m_j} b_{jk}(\xi')D_y^{m_j-k}, \]

where \( a_k(\xi') \) and \( b_{jk}(\xi') \) are homogeneous of degree \( k \). Set

\[ \mu = (\eta + |\xi'|^2)^{1/2m}, \quad b = |\xi'|/\mu, \quad \zeta = \xi'/\mu, \quad a = \eta/\mu^{2m} \]

and \( w := (w_1, \ldots, w_{2m})^T \) for

\[ w_k := \left( \frac{1}{\mu} D_y \right)^{k-1} v \quad (k = 1, \ldots, 2m). \]

Note that \((\mu, \zeta, a) \in [\eta^{1/2m}, \infty) \times B_{\mathbb{R}^{n-1}}(0;1) \times (0, 1]\), where \( B_{\mathbb{R}^d}(c; r) \) is the \( d \)-dimensional open ball with center \( c \) and radius \( r \). Then, the first equation of (7) is equivalent to

\[ D_y w = \mu A_0(\zeta, a) w, \quad (8) \]

where

\[ A_0(\zeta, a) := \begin{pmatrix} 0 & I & 0 & \cdots & 0 \\ 0 & 0 & I & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & I \\ c_{2m} & c_{2m-1} & \cdots & c_2 & c_1 \end{pmatrix}, \]

and

\[ c_j := c_j(\zeta) := -a_{j-1} a_j(\zeta) \quad (j = 1, \ldots, 2m - 1), \]

\[ c_{2m} := c_{2m}(\zeta, a) := -a_{2m-1}(a_{2m}(\zeta) + a). \]

Actually, it follows from the first equation of (7) and definition of \( w_j \) that

\[ (1/\mu)D_y w_{2m} = -a_{0}^{-1} \left( a_{2m}(\xi'/\mu) + \eta/\mu^{2m} \right) w_1 - \sum_{k=2}^{2m} a_{0}^{-1} a_{2m-k+1}(\xi'/\mu)w_k. \]

Thus, we find (8) from the definition of \( w \). Moreover, (8) implies

\[ w(y) := e^{iy A_0(\zeta, a)} w(0) \quad (y \geq 0). \]

We write \( w(0) = w|_{y=0} \) for simplicity. The functions \( w(y) \) have to be determined so that tends to zero at infinity. This is guaranteed by

\[ P_+(\zeta, a) w_0 = 0, \]

where \( P_+(\zeta, a) \in \mathcal{B}(E^{2m}) \) is the associated positive spectral projection with \( i A_0(\zeta, a) \). Note that each spectrum of \( i A_0(\zeta, a) \) does not lie on the imaginary axis.
and $P_+$ is holomorphic and bounded uniformly in $(\zeta, a)$ by the Lopatinskii–Shapiro condition since $(\zeta, a)$ run a compact set away from $(0, 0)$. Supremum of real part of negative spectrum of $iA(\zeta, a)$ is less than zero and infimum of real part of positive spectrum is larger than zero, this facts imply $e^{\mu t A_0(\zeta, a)} y w(0) \to 0$ for $w(0)$ satisfying $P_+(\zeta, a) w(0) = 0$ as $y \to \infty$. See [1,11] for details of the above discussion.

Let $w := \mathcal{F}_y L_t h$ for $h \in L_p(J; W^{2m-1/p}_q(\mathbb{R}^{n-1}; E^{2m}))$. Define the canonical extension of functions from the boundary to the half space by

$$T h := \mathcal{F}_y^{-1} \left[ e^{2m \mu t A_0(\zeta, a)} (I - P_+(\zeta, a)) w \right].$$

(Boundedness of this extension operator is ensured by the following Proposition 2.

Proposition 2. Let $1 < p, q < \infty$, $\eta \in \Sigma_\varphi$ for small $\varphi > 0$ and $J = \mathbb{R}_+$. Then, there exists a constant $C > 0$ such that

$$\|Th\|_{L_p(J; L^q_2(\mathbb{R}^{n-1}; E^{2m}))} \leq C \|h\|_{L_p(J; W^{2m-1/p}_q(\mathbb{R}^{n-1}; E^{2m}))},$$

for $h \in L_p(J; W^{2m-1/p}_q(\mathbb{R}^{n-1}; E^{2m}))$.

Proof. See [1, Sect. 7].

Put $w^0 = w(0)$. Let us continue to seek the solution formula of $(w^0, \sigma)$. Since

$$B_j v = \sum_{k=0}^{m_j} b_{jk}(\xi') \mu^{m_j-k} w_{m_j-k+1} = \sum_{k=0}^{m_j} b_{jk}(\xi) \mu^{m_j} w_{m_j-k+1}$$

and

$$C_j(\xi') \sigma = C_j(\xi) \mu^k \sigma$$

the second and the third equations of (7) are equivalent to

$$B_0(\xi) w^0 + \left\{ \lambda \mu^{-m_0} + C_0(\xi) \mu^{-m_0+k_0} \right\} \sigma = \mu^{-m_0} h_0,$$

$$B_j(\xi) w^0 + C_j(\xi) \mu^{-m_j+k_j} \sigma = \mu^{-m_j} h_j \quad (j = 1, \ldots, m),$$

$$P_+(\zeta, a) w^0 = 0,$$

where $B_j(\xi) := (b_{jm_j}(\xi), \ldots, b_{j0}, 0, \ldots, 0)$ for $j = 0, \ldots, m$. Note that by the assumptions on (E) and (LS) Eq. (11) admit a unique solution

$$(w^0, \sigma) \in E^{2m} \times F$$

for each $(\lambda, \zeta) \in (\Sigma_\theta \times \mathbb{R}^{n-1}) \backslash \{(0, 0)\}$ ($\theta > \pi/2$) and $\{h_j\}_{j=0}^m \in F \times E^m$. Introducing

$$\sigma^0 := (\lambda + \mu^l) \mu^{-m_0} \sigma, \quad h := (h^0_j)_{j=0}^m := (\mu^{-m_j} h_j)_{j=0}^m,$$
We rewrite (11) into

\[ B_0(\xi)w^0 + \frac{\lambda + C_0(\xi)\mu_0^0}{\lambda + \mu^0} \sigma^0 = h_0^0, \]

\[ B_j(\xi)w^0 + \frac{C_j(\xi)\mu_j^0}{\lambda + \mu^0} \sigma^0 = h_j^0 (j = 1, \ldots, m), \]

Thus, it follows

\[ B_0(\xi)w^0 + \frac{\nu + C_0(\xi)\eta^{-l_0/l_0} a^{-l_0}}{\nu + 1} \sigma^0 = h_0^0, \]

\[ B_j(\xi)w^0 + \frac{C_j(\xi)\eta^{-l_j/l_j} a^{-l_j}}{\nu + 1} \sigma^0 = h_j^0 (j = 1, \ldots, m), \]

\[ P_+(\xi, \tilde{a}^2m)w^0 = 0. \]

for \( \nu = \lambda/\mu^0 \) and \( \tilde{a} = a^{1/2m} \). We write the solution to (13) as

\[ w^0 := M_0^0(\xi, \tilde{a}, \nu)h, \quad \sigma^0 := M_\sigma^0(\xi, \tilde{a}, \nu)h. \]

Set \( Y_E := L_p(J; W_{q}^{2m-1/p}(\mathbb{R}^{n-1}; E)) \) and \( Y_F := L_p(J; W_{q}^{2m-1/p}(\mathbb{R}^{n-1}; F)) \).

**Proposition 3.** Let \( 1 < p, q < \infty, \eta > 0 \) and \( G = \mathbb{R}^n_+ \). Assume assumptions (E), (LS) and (ALS) hold. Then, there exist a positive constant \( C > 0 \), it holds that

\[ \| \mathcal{L}_\lambda^{-1}\mathcal{F}_E^{-1} \left[ \left( M_0^0, M_\sigma^0 \right) \mathcal{F}_E^* \mathcal{L}_\xi \right] \|_{\mathcal{B}(Y_F \times Y_{\tilde{m}}; Y_{E}^{2m} \times Y_{F})} \leq C. \]

**Proof.** Analyticity of \( (M_0^0, M_\sigma^0) \) on a open set

\[ D_\xi \times D_\tilde{a} \times \Sigma_\theta \subseteq B_{\mathbb{R}^{n-1}}(0; 1) \times (0, 1] \times \Sigma_\theta (\theta > \pi/2) \] is guaranteed by (LS) and analyticity of \( B_j, C_j \) for \( j = 0, \ldots, m \). Boundedness of \( (M_0^0, M_\sigma^0) \) is equivalent to the solvability for \( \xi \in \mathbb{B}_{\mathbb{R}^{n-1}}(0; 1), \tilde{a} \in [0, 1] \) and \( \nu \in \Sigma_\theta \cup \{ \infty \} (\theta > \pi/2) \). The solvability of \( (M_0^0, M_\sigma^0) \) in the case of \( \mu \neq \infty \) and \( \lambda \neq \infty \) is guaranteed by (LS). We need to control behavior of \( (M_0^0, M_\sigma^0) \) on \( \mu \) and \( \lambda \) at infinity. Let us consider the case of \( |\mu| \to \infty \) or \( |\lambda| \to \infty \). We find

\[ \eta^{-l_0/l_0} a^{-l_0} \to \begin{cases} 0 & \text{if } |\lambda|/|\mu| \to \infty, \\ \delta_{l_0} / c^{l_0} & \text{if } \lambda/\mu \to c, \end{cases} \]

and

\[ \nu \to \begin{cases} 1 / c & \text{if } |\lambda|/|\mu| \to \infty, \\ \xi / c & \text{if } \lambda/\mu \to c, \end{cases} \]

for some \( c \in \Sigma_\theta \cup \{ 0 \} \). Let us consider the case (i) \( (|\lambda|/|\mu|) \to \infty \). The limit problem of this case is
\begin{align}
B_0(\zeta)w^0 + \sigma^0 &= h_0^0, \\
B_j(\zeta)w^0 &= h_j^0 \quad (j = 1, \ldots, m), \\
P_+(\zeta, a_*)w^0 &= 0.
\end{align}

(14)

for some $a_* \in [0, 1]$ which is the limit of $\tilde{a}^{2m}$. If $\mu$ tend to infinity at the same time, i.e., $a_* = 0$, this system corresponds to the following problem; for all $\{h_j^0\}_{j=0}^m \in F \times E^m$ and for any $\xi' \in S^{n-2}$,

\begin{align}
\mathcal{A}(\xi', D_y)v(y) &= 0 \quad (y > 0), \\
B_0(\xi', D_y)v^0 + \sigma^0 &= h_0^0, \\
B_j(\xi', D_y)v^0 &= h_j^0 \quad (j = 1, \ldots, m),
\end{align}

admits a unique solution $v \in C^{2m}_0(\mathbb{R}_+; E)$, which is guaranteed by the third asymptotic Lopatinskii–Shapiro condition. On the other hand, if $\mu$ is still finite, i.e., $a_* \in (0, 1]$, the corresponding problem is given by

\begin{align}
\eta v(y) + \mathcal{A}(\xi', D_y)v(y) &= 0 \quad (y > 0), \\
B_0(\xi', D_y)v^0 + \sigma^0 &= h_0^0, \\
B_j(\xi', D_y)v^0 &= h_j^0 \quad (j = 1, \ldots, m),
\end{align}

for all $\eta > 0, \xi' \in \mathbb{R}^{n-1}$ and $\{h_j^0\}_{j=0}^m \in F \times E^m$. This problem is solvable by the first asymptotic Lopatinskii–Shapiro condition. Next we consider case (ii) ($\lambda / \mu \rightarrow c \in \Sigma_{\theta} \cup \{0\}$). In these cases, $a_* = 0$ and the limit problem is

\begin{align}
B_0(\zeta)w^0 + \frac{c + \delta_{l,l_0}C_0(\zeta)}{\frac{c+1}{\delta_{l,l_j}C_j(\zeta)}}\sigma^0 &= h_0^0, \\
B_j(\zeta)w^0 + \frac{c + \delta_{l,l_j}C_j(\zeta)}{\frac{c+1}{\delta_{l,l_j}C_j(\zeta)}}\sigma^0 &= h_j^0 \quad (j = 1, \ldots, m), \\
P_+(\zeta, 0)w^0 &= 0.
\end{align}

(15)

To ensure solvability of this problem, it is enough to impose the following the condition; for all $\{h_j\}_{j=0}^m \in F \times E^m$ and for any $\lambda \in \Sigma_{\theta}$ and $\xi' \in S^{n-2}$,

\begin{align}
\mathcal{A}(\xi', D_y)v(y) &= 0 \quad (y > 0), \\
B_0#(\xi', D_y)v_0 + (\lambda + \delta_{l,l_0}C_{0#}(\xi'))\sigma &= h_0, \\
B_j#(\xi', D_y)v_0 + \delta_{l,l_j}C_{j#}(\xi')\sigma &= h_j \quad (j = 1, \ldots, m)
\end{align}

admits a unique solution $(v, \sigma) \in C^{2m}_0(\mathbb{R}_+; E) \times F$. This condition is nothing but the second asymptotic Lopatinskii–Shapiro condition. We find from the above discussion that $M_w^0$ and $M_{\sigma}^0$ is bounded holomorphic on $D_\zeta \times D_\alpha \times \Sigma_{\theta}$ ($\theta > \pi / 2$). Moreover,

$$\{(M_w^0, M_{\sigma}^0)(\zeta, \tilde{a}, v) : (\zeta, \tilde{a}, v) \in \overline{D_\zeta} \times \overline{D_\tilde{a}} \times \Sigma_{\theta}\}$$
is $\mathcal{R}$-bounded since $E$ and $F$ are Hilbert spaces. Set $M^0 = (M^0_w, M^0_\sigma)$ and
\[
L_1 = -i \nabla'(\eta + (\Delta')^m)^{-1/2m},
\]
\[
L_2 = \eta^{1/2m} (\eta + (\Delta')^m)^{-1/2m},
\]
\[
L_3 = \partial_t (\eta + (\Delta')^m)^{-1/2m}.
\]
Then, boundedness and analyticity of $M^0$ with respect to $\zeta$ and $a$ lead
\[
\mathcal{R} \left\{ \xi^{\alpha} \frac{\partial}{\partial \xi^0} M^0 \left( \frac{\xi'}{1 - (\eta + (\xi')^2)^{1/2m}}, \frac{\xi'}{(\eta + (\xi')^2)^{1/2m}}, \nu \right) : \xi' \in \mathbb{R}^{n-1} \setminus \{0\}, \alpha \in \{0, 1\}^{n-1}, \nu \in \Sigma_\theta \right\} < \infty.
\]
Thus, the operator-valued Fourier multiplier theorem implies
\[M^0(L_1, L_2, \nu) \in \mathcal{B}(Y_F \times Y_E; Y_E^{2m} \times Y_F)\]
and $\{M^0(L_1, L_2, \nu) : \nu \in \Sigma_\theta \}$ is $\mathcal{R}$-bounded on $\mathcal{B}(Y_F \times Y_E; Y_E^{2m} \times Y_F)$. Finally, because of analyticity of $M^0(L_1, L_2, \cdot)$, we can use the Kalton–Weis theorem to find $M^0(L_1, L_2, L_3) \in \mathcal{B}(Y_F \times Y_E; Y_E^{2m} \times Y_F)$.

We find from Propositions 2, 3 and
\[
\frac{d}{dt} + (\eta + (\Delta')^m)^{1/2m} \in \text{Isom}(W_p^1(J; W_q^{2m-1/p}(\mathbb{R}^{n-1}; F)) \cap L_p(J; W_q^{l+2m-1/p}(\mathbb{R}^{n-1}; F)); Y_F),
\]
\[
(\eta + (\Delta')^m)^{m_0/2m} \in \text{Isom}(W_p^1(J; W_q^{2m-1/p}(\mathbb{R}^{n-1}; F)) \cap L_p(J; W_q^{l+2m-1/p}(\mathbb{R}^{n-1}; F)); Z_p),
\]
\[
(\eta + (\Delta')^m)^{m_j/2m} \in \text{Isom}(Y_F; Y_0)
\]
\[
(\eta + (\Delta')^m)^{m_j/2m} \in \text{Isom}(Y_F; Y_j) \quad (j = 1, \ldots, m),
\]
that
\[
\|u(0)\|_{Y_F} + \left\| \frac{d}{dt} + (\eta + (\Delta')^m)^{1/2m} \right\|_{Y_F} \left( \eta + (\Delta')^m \right)^{m_0/2m} \rho \|_{Y_F} \leq C \left( \| \eta + (\Delta')^m \|_{Y_F} \right) g_0 \|_{Y_F} + \sum_{j=1}^m \| \eta + (\Delta')^m \|_{Y_F} g_j \|_{Y_F},
\]
where $u(0) := u|_{y=0}$. This leads the desired maximal $L_p$ regularity
\[
\eta \|u\|_X + \|u\|_{Z_a} + \|\rho\|_{Z_\rho} \leq C \left( g_0 \|y_0\| + \sum_{j=1}^m g_j \|y_j\| \right),
\]
for some $C > 0$. We finish proving Theorem 1 in the case of the half space.
4.3. The case of a domain with a compact boundary

Let us consider the case of a bounded domain $G$ and an exterior. The proof is based on (i) solving the case of variable coefficient with lower-order terms and (ii) localization procedure and coordinate transform. Since this method is well known, we do not give a detail of the proof, see [1,3,10,11] for example. We show only outline of the proof. Note that conditions (E), (LS) and (ALS) are invariant under the coordinate transform. First we give estimates for lower-order terms.

**Proposition 4.** Let $a_\alpha, b_{j\beta}, c_{j\gamma}$ satisfy (SA), (SB) and (SC), then there exists $C > 0$ such that

\[
\|a_\alpha D^\alpha u\|_X \leq C \|a_\alpha\|_{L_\infty(J;L_r(G))}\|u\|_{Z_u}, \quad (|\alpha| \leq 2m - 1)
\]

\[
\|b_{j\beta} D^\beta u\|_{J} \leq C \|b_{j\beta}\|_{L_\infty(J;L_{s_{j\beta}}^G(B_{r_{j\beta},q}^2(G)))}\|u\|_{Z_u}, \quad (|\beta| \leq m_j - 1)
\]

\[
\|c_{j\gamma} D_{j\gamma}^{\rho} \|_{J} \leq C \|c_{j\gamma}\|_{L_{s_{j\gamma}}^G(J;L_{s_{j\gamma}}^G(B_{r_{j\gamma},q}^2(G)))}\|\rho\|_{Z_{\rho}}, \quad (|\gamma| \leq k_j - 1)
\]

**Proof.** First, for each $|\alpha| = k \leq 2m - 1$, the assumption (SA) derives

\[
\|a_\alpha D^\alpha u\|_{L_q^r(G)} \leq \|a_\alpha\|_{L_{r_\alpha}(G)}\|D^\alpha u\|_{L_{r\alpha}^r(G)} \leq \|a_\alpha\|_{L_{r_\alpha}(G)}\|u\|_{W_{q}^{2m-1}(G)},
\]

where $1/q = 1/r_\alpha + 1/r_\alpha'$ and we use the embedding $W_{q}^{2m}(G) \hookrightarrow W_{r_\alpha'}^{k}(G)$. This means

\[
\|a_\alpha D^\alpha u\|_X \leq \|a_\alpha\|_{L_\infty(J;L_{r_\alpha}(G))}\|u\|_{Z_u}.
\]

Second, for each $|\beta| = k \leq m_j - 1$, we find from paraproduct formula, definition of Besov spaces on a domain and the assumption (SB) that

\[
\|b_{j\beta} D^\beta u\|_{W_{q}^2(B_{r_{j\beta},q}^2(G))} \leq C \|b_{j\beta}\|_{L_{s_{j\beta}}^G(B_{r_{j\beta},q}^2(G))}\|D^\beta u\|_{L_{s_{j\beta}}^r(G)}
\]

\[
\leq C \|b_{j\beta}\|_{L_{s_{j\beta}}^G(B_{r_{j\beta},q}^2(G))}\|u\|_{W_{q}^{2m-1/2}(G)},
\]

where $1/q = 1/s_{j\beta} + 1/s_{j\beta}' = 1/r_{j\beta} + 1/r_{j\beta}'$ and we used the embeddings

\[
\text{tr}|\Gamma u \in W_{q}^{2m-1/2}(G) \hookrightarrow (B_{s_{j\beta},q}^{k+2m}(G) \cap W_{r_{j\beta}'}^{k}(G))
\]

Moreover, we have $\text{tr}|\gamma u \in L_p(J; W_{q}^{2m-1/2}(G))$ for $u \in Z_u$. This means

\[
\|b_{j\beta} D^\beta u\|_{J} \leq C \|b_{j\beta}\|_{L_\infty(J;L_{s_{j\beta}}^G(\cap W_{r_{j\beta}'}^{k}(G)))}\|u\|_{Z_u}.
\]
At last, for each $|γ| = k \leq k_j - 1$, it follows from the same way as for (16) that
\[
\|c_{jγ} D_Γ^γ ρ\|_{W_q^j (Γ)} ≤ C \left( \|c_{jγ}\|_{L_{jγ}^c (Γ)} \|D_Γ^γ ρ\|_{B^{2m}_{jγ} (Γ)} + \|c_{jγ}\|_{B^{2m}_{jγ} (Γ)} \|D_Γ^γ ρ\|_{L_{jγ}^c (Γ)} \right)
\]
where $1/q = 1/s_{jγ}^c + 1/s_{jγ}^c = 1/r_{jγ}^c + 1/r_{jγ}^c$. Integral in time and use Hölder’s inequality,
\[
\|c_{jγ} D_Γ^γ ρ\|_{Y_j} ≤ C \|c_{jγ}\|_{L_{jγ}^c (J; L_{jγ}^c (Γ))} \|D_Γ^γ ρ\|_{L_{jγ}^c (J; B^{2m}_{jγ} (Γ))} + C \|c_{jγ}\|_{L_{jγ}^c (J; B^{2m}_{jγ} (Γ))} \|D_Γ^γ ρ\|_{L_{jγ}^c (J; L_{jγ}^c (Γ))}
\]
where $1/p = 1/t_{jγ} + 1/t_{jγ} = 1/τ_{jγ} + 1/τ_{jγ}'$. Here, we use the mixed derivative theorems
\[
Z_ρ = W_p^1 (J; W_{q}^{2mκ_0} (Γ)) \cap L_p (J; W_{q}^{l+2mκ_0} (Γ)) = \bigcap_{0 ≤ s ≤ 1} W_p^s (J; W_{q}^{l(1-s)+2mκ_0} (Γ)).
\]
The assumption (SC) ensures the existence of $s \in [0, 1]$ such that
\[
W_p^s (J; W_{q}^{l(1-s)+2mκ_0} (Γ)) \hookrightarrow L_{τ_{jγ}'}^1 (J; B^{k+2mκ_j}_{s_{jγ}^c} (Γ)), L_{τ_{jγ}}^1 (J; W_{r_{jγ}^c}^{k} (Γ)),
\]
respectively. This means
\[
\|c_{jγ} D_Γ^γ ρ\|_{Y_j} ≤ C \|c_{jγ}\|_{L_{jγ}^c (J; L_{jγ}^c (Γ)) \cap L_{τ_{jγ}^c} (J; B^{2m}_{jγ} (Γ))} \|ρ\|_{Z_ρ}.
\]

Proposition 5. Let $J = [0, T]$, $G = \mathbb{R}^n_u$ and $Γ = \partial G$, $1 < p, q < ∞$ and $E$ and $F$ be separable Hilbert spaces. Let assumptions (E), (SA), (SB), (SC), (LS) and (ALS) hold. Assume $A, B_j$ and $C_j$ are given by
\[
A(t, x, D) = A_{#} (D) + A_{\text{small}} (t, x, D) + A_{\text{low}} (t, x, D),
B_j (t, x, D) = B_{#j} (D) + B_{j\text{small}} (t, x, D) + B_{j\text{low}} (t, x, D),
C_j (t, x', D_Γ) = C_{#j} (D_Γ) + C_{j\text{small}} (t, x', D_Γ) + C_{j\text{low}} (t, x', D_Γ),
\]
where the equation with $A_{#}, B_{#j}$ and $C_{#j}$ satisfy (LS) and (ALS), $A_{\text{low}}, B_{j\text{low}}$ and $C_{j\text{low}} (j = 0, \ldots, m)$ are lower-order terms and
\[
\|A_{\text{small}} (t, x, D) u\|_X ≤ δ\|u\|_{Z_u},
\|B_{j\text{small}} (t, x, D) u\|_{Y_j} ≤ δ\|u\|_{Z_u} \quad (j = 0, \ldots, m),
\]
\[ \|c_j^{\text{small}}(t, x, D)\rho\|_{Y_j} \leq \delta \|\rho\|_{Z_\rho} \quad (j = 0, \ldots, m), \]

for sufficiently small \( \delta > 0 \). Then, there exist positive constants \( \eta_0 > 0 \), \( C \) and \( C_T \), for

\[ (f, \{g_j\}_{j=0}^m, \rho_0) \in X \times \prod_{j=0}^m Y_j \times \pi Z_\rho, \]

if \( \eta \geq \eta_0 \), Eq. (1) admits a unique solution \((u, \rho) \in Z_u \times Z_\rho\) such that

\[ \eta \|u\|_X + \|u\|_{Z_u} + \|\rho\|_{Z_\rho} \leq C \|f\|_X + C \sum_{j=0}^m \|g_j\|_{Y_j} + C_T \|\rho_0\|_{\pi Z_\rho}. \]  

(17)

**Proof.** Assume \(|J|\) is small, where \(|J|\) is the length of \(J\). Clearly,

\[ \|u\|_{D(A)} \leq \|u\|_{D(A_{\text{low}})}, \|u\|_{D(B)} \leq \|u\|_{D(B_{\text{low}})} \text{ and } \|u\|_{D(C)} \leq \|u\|_{D(C_{\text{low}})}. \]

Thus, if we take \( \eta > 0 \) sufficiently large, we find from the space-time Sobolev embedding, which enables us to estimate lower-order terms as small perturbation since \(|J|\) is small, and the Neumann series argument that can be estimated \(A^\text{small}, A^\text{low}, B^\text{small}, B^\text{low}, C^\text{small}\) and \(C^\text{low}\) as relatively small perturbations. For \(J\) with arbitrary finite length, we can divide \(J\) into finite short intervals. For these short intervals, we can apply the same argument as above step by step to get (17). \(\square\)

Now we prove Theorem 1. For the sake of simplicity, we consider the case of bounded domains. The case of exterior domains is treated by a similar way. Temporarily, we assume \(|J|\) is small. Let \(\delta > 0\) be small. Let us introduce an open covering of \(G\) such that

\[ G \subset \bigcup_{k=0}^N U_k \]

\[ U_k = B(x_k, \delta), \quad x_k \in G \quad (k = 0, \ldots, M) \]

\[ U_k = B(x_k, \delta), \quad x_k \in \partial G \quad (k = M+1, \ldots, N) \]

for some \(M\) and \(N\). We also introduce a partition of unity \(\{\varphi_j\}_{j=0}^N\) satisfying

\[ \varphi_j \in C_0^\infty(\mathbb{R}^n), \quad 0 \leq \varphi_j \leq 1, \quad \text{spt } \varphi_j \subset U_j, \quad \sum_{j=0}^N \varphi_j \equiv 1 \text{ on } \overline{G} \]

Suppose \((u, \rho) \in Z_u \times Z_\rho\) be a solution to (1) with \(\rho_0 = 0\), which is without loss of generality. For \(k \geq M + 1\), we apply the canonical coordinate transform, which is denoted by \(\Phi_k\), from \(U_k\) to local neighborhood of the half space so that \(U_k \cap \Gamma^+\) is flat. Since coefficients are continuous, if we take \(\delta > 0\) be sufficiently small beforehand, we can extend coefficients to the half space and write these extended coefficients as \(a^k_{\alpha}, b^k_{j\beta}\) and \(c^k_{j\gamma}\) to the half space so that

\[ \|a^k_{\alpha} - a_{\alpha}(0, x_k)\|_{L_\infty(J \times \mathbb{R}^n_+; B(E))}, \quad \|b^k_{j\beta} - b_{j\beta}(0, x_k)\|_{L_\infty(J \times \mathbb{R}^{n-1}; E_j)}, \]
are sufficiently small. Put \((u_k, \rho_k, f_k, g_{j,k}) = (\phi_k u, \phi_k \rho, \phi_k f, \phi_k g_j)\) for 
k = 0, \ldots, N. Then, for \(k = M + 1, \ldots, N\), \((u_k, \rho_k)\) satisfies
\[
\begin{align*}
\eta u_k + A_\# u_k &= F_k(f_k, u) \quad (t \in J, x \in G), \\
\partial_t \rho_k + B_0 \# u_k + C_0 \# \rho_k &= G_{0,k}(g_{0,k}, u, \rho) \quad (t \in J, x \in \Gamma), \\
B_j \# u_k + C_j \# \rho_k &= G_{j,k}(g_{j,k}, u, \rho) \quad (t \in J, x \in \Gamma) \quad (j = 1, \ldots, m), \\
\rho_k(0) &= 0 \quad (x \in \Gamma),
\end{align*}
\]  
(18)
for
\[
\begin{align*}
F_k(f_k, u) &= f_k - \varphi_k(A - A_\#) u + [A, \varphi_k] u, \\
G_{0,k}(g_{0,k}, u, \rho) &= \varphi_k(B_0 \# - B_0) u + [B_0, \varphi_k] u, \\
&\quad - \varphi_k(C_0 \# - C_0) \rho + [C_0, \varphi_k] \rho, \\
G_{j,k}(g_{j,k}, u, \rho) &= \varphi_k(B_j \# - B_j) u + [B_j, \varphi_k] u, \\
&\quad - \varphi_k(C_j \# - C_j) \rho + [C_j, \varphi_k] \rho \quad (j = 1, \ldots, m),
\end{align*}
\]
where \([\cdot, \cdot]\) is the commutator. For \(k = 0, \ldots, M\), we can solve
\[
\eta u_k + A_\# u_k = F_k(f_k, u) \quad (t \in J, x \in \mathbb{R}^n)
\]
and find from Proposition 4 and the regularity estimate of elliptic operators that
\[
\eta \|u_k\|_X + \|u_k\|_{Z_u} \leq C_1 \|f\|_X + \epsilon \|u\|_{Z_u} + C_\epsilon \|u\|_X.
\]  
(19)
for sufficiently small \(\epsilon > 0\). Put \(\rho_k = 0\) for \(k = 0, \ldots, M\). In the case of 
k = M + 1, \ldots, N, we apply coordinate transform by \(\Phi_k\) to (18). The transformed problem is solvable by Proposition 5. Pulling buck the solution, we obtain the solution to (18) such that
\[
\eta \|u_k\|_X + \|u_k\|_{Z_u} + \|\rho_k\|_{Z_\rho} \leq C \|f\|_X + C \sum_{j=0}^m \|g_j\|_{Y_j} + \epsilon \|u\|_{Z_u} + C_\epsilon \|u\|_X + C |J|^{\alpha} \|\rho\|_{Z_\rho},
\]  
(20)
for some \(\alpha > 0\). \(|J|^{\alpha}\) appears because \(F_k(f_k, u) - f_k\) has only lower-order differential terms. We denote by \(S^k : X \times Y_0 \times \cdots \times Y_m \rightarrow Z_u \times Z_\rho\) the solution operator of (18) with \(\rho_0 = 0\), i.e., \((u_k, \rho_k) = S^k(F_k, G_{0,k}, \ldots, G_{m,k})\). On the other hand, it follows that
\[(u, \rho) = \sum_{k=0}^{N} (u_k, \rho_k) = \sum_{k=0}^{N} S^k (F_k(f_k, u), G_{0,k}(g_{0,k}, u, \rho), \ldots, G_{m,k}(g_{m,k}, u, \rho)) \]
\[= \sum_{k=0}^{N} S^k (F_k(f_k, 0), G_{0,k}(g_{0,k}, 0, 0), \ldots, G_{m,k}(g_{m,k}, 0, 0)) \]
\[+ \sum_{k=0}^{N} S^k (F_k(0, u), G_{0,k}(0, u, \rho), \ldots, G_{m,k}(0, u, \rho)) \]
\[=: T(f, g_0, \ldots, g_m) - R(u, \rho), \quad (21)\]

where we write restriction of \((u_k, \rho_k)\) on \(U_k\) as \((u_k, \rho_k)\) for simplicity.

Equations (19) and (20) imply
\[\|R(u, \rho)\|_{Z_u \times Z_\rho} \leq \frac{1}{2} \|u\|_{Z_u} + C|J|^\gamma \|\rho\|_{Z_\rho}.\]

for some \(\gamma > 0\). Let \(S_0 : X \times Y_0 \times \cdots \times Y_m \to Z_u \times Z_\rho\) be the solution operator of (1) with \(\rho_0 = 0\), i.e., \((u, \rho) = S_0(f, g_0, \ldots, g_m)\). It follows from (21)

\[S_0(f, g_0, \ldots, g_m) = T(f, g_0, \ldots, g_m) - R(S_0(f, g_0, \ldots, g_m)). \quad (22)\]

Then, if we take \(|J|\) small and \(\eta > 0\) large, we can use the Neumann series argument to get \(S_0 = (Id + R)^{-1} T\) and

\[\|S_0\|_{B(X \times Y_0 \times \cdots \times Y_m; Z_u \times Z_\rho)} \leq C.\]

For \(J\) with arbitrary finite length, we can divide \(J\) into finite short interval. For this short intervals, the same argument as above also works.

5. Examples

In this section, we give some examples for our problems. We especially focus on checking the Lopatinskii–Shapiro and asymptotic Lopatinskii–Shapiro conditions. Throughout this section, we assume \(E = F = \mathbb{C}\) and write the outer unit normal on the boundary by \(\nu\).

Example 5.1.

\[
\begin{aligned}
\eta u - \Delta u &= f \quad (t \in J, \ x \in G), \\
\partial_t u + \partial_t \rho &= g_0 \quad (t \in J, \ x \in \Gamma), \\
u u - \rho &= 0 \quad (t \in J, \ x \in \Gamma), \\
\rho(0, x) &= \rho_0(x) \quad (x \in \Gamma). 
\end{aligned}
\quad (23)
\]
The equation of the Lopatinskii–Shapiro condition is
\[
\begin{cases}
(\eta + |\xi'|^2 - \partial^2_y) v(y) = 0 \quad (y > 0), \\
-\partial_y v(0) + \lambda \sigma = h_0, \\
v(0) - \sigma = h_1.
\end{cases}
\]

The solution of the first equation \(C_0(\mathbb{R}^+; E)\) is given by
\[v(y) = e^{-\sqrt{\eta + |\xi'|^2}} v(0) = e^{-\mu y} v(0)\] for \(\mu = (\eta + |\xi'|^2)^{1/2}\). The boundary conditions lead to the equation
\[
\left( \sqrt{\eta + |\xi'|^2} \begin{array}{cc} \lambda \\ 1 \end{array} \begin{array}{c} v(0) \\ \sigma \end{array} = \begin{array}{c} h_0 \\ h_1 \end{array} \right).
\]

We see that the determinant of the matrix is \(-\lambda - \sqrt{\eta + |\xi'|^2} \neq 0\) for \(\eta > 0\), \((\lambda, \xi') \in (\Sigma_\theta \times \mathbb{R}^{n-1}) \setminus \{(0, 0)\} \quad (\theta > \pi/2)\). Therefore, the Lopatinskii–Shapiro condition is satisfied.

The equation of the first asymptotic Lopatinskii–Shapiro condition is
\[
\begin{cases}
(\eta + |\xi'|^2 - \partial^2_y) v(y) = 0 \quad (y > 0), \\
v(0) = h_1.
\end{cases}
\]

for \(\eta > 0\) and \(\xi' \in \mathbb{R}^{n-1}\). The solution to this ODE is uniquely determined by \(v(y) = e^{-\lambda y} h_1\).

The equation of the second asymptotic Lopatinskii–Shapiro condition is
\[
\begin{cases}
(|\xi'|^2 - \partial^2_y) v(y) = 0 \quad (y > 0), \\
-\partial_y v(0) + \lambda \sigma = h_0, \\
v(0) - \sigma = h_1.
\end{cases}
\]

for \((\lambda, \xi') \in (\Sigma_\theta \cup \{0\}) \times S^{n-2} \quad (\theta > \pi/2)\). The equation of the first equation implies \(v(y) = e^{-|\xi'| y} v(0)\), and thus \(-\partial_y v(0) = |\xi'| v(0)\). Since the determinant of the matrix
\[
\begin{pmatrix}
|\xi'| & \lambda \\
1 & -1
\end{pmatrix}
\]
is never zero by the choice of \((\lambda, \xi')\),
the equation of the third asymptotic Lopatinskii–Shapiro condition is
\[
\begin{cases}
(|\xi'|^2 - \partial^2_y) v(y) = 0 \quad (y > 0), \\
v(0) = h_1
\end{cases}
\]
for \(\xi' \in S^{n-2}\). This equation is uniquely determined by \(v(y) = e^{-|\xi'| y} h_1\). Thus, the Lopatinskii–Shapiro and asymptotic Lopatinskii–Shapiro conditions are satisfied, and (23) is solvable in the maximal regularity space.
Example 5.2.  

\[
\begin{cases}
\eta u - \Delta u = f & (t \in J, \ x \in G), \\
\partial_t u + \partial_t \rho - \Delta \Gamma \rho = g_0 & (t \in J, \ x \in \Gamma), \\
u - \rho = g_0 & (t \in J, \ x \in \Gamma), \\
\rho(0, x) = \rho_0(x) & (x \in \Gamma). 
\end{cases}
\]  

The equation of the Lopatinskii–Shapiro condition is  

\[
\begin{cases}
(\eta + |\xi'|^2 - \partial_y^2)v(y) = 0 & (y > 0) \\
- \partial_y v(0) + \lambda \sigma + |\xi'|^2 v = h_0 \\
v(0) - \sigma = h_1.
\end{cases}
\]  

We find \(v(y) = e^{-\mu y}v(0)\) for \(\mu = (\eta + |\xi'|^2)^{1/2}\) and  

\[
\det \begin{pmatrix} \mu & \lambda + |\xi'|^2 \\ 1 & -1 \end{pmatrix} \neq 0
\]  

for \((\lambda, \xi') \in (\Sigma_\theta \times \mathbb{R}^{n-1}) \setminus \{(0, 0)\} (\theta > \pi/2)\). Thus, Lopatinskii–Shapiro condition is satisfied.  

Let us check asymptotic Lopatinskii–Shapiro conditions. The equations of the first and third asymptotic Lopatinskii–Shapiro conditions are  

\[
\begin{cases}
(\eta + |\xi'|^2 - \partial_y^2)v(y) = 0 & (y > 0) \\
v(0) - \sigma = h_1
\end{cases}
\]  

for \(\xi' \in \mathbb{R}^{n-1}\) and  

\[
\begin{cases}
(|\xi'|^2 - \partial_y^2)v(y) = 0 & (y > 0) \\
v(0) - \sigma = h_1
\end{cases}
\]  

for \(\xi' \in S^{n-2}\). By the same way as above, we find this equation is uniquely solvable.  

The equation of the second asymptotic Lopatinskii–Shapiro condition is  

\[
\begin{cases}
(|\xi'|^2 - \partial_y^2)v(y) = 0 & (y > 0) \\
- \partial_y v(0) + \lambda \sigma + |\xi'|^2 v = h_0 \\
v(0) = h_1.
\end{cases}
\]  

\(v(y)\) is determined by the first and third equations and \(\sigma\) is uniquely determined by the second equation for \((\lambda, \xi')\).  

Example 5.3. The third example is the quasi-steady Stefan problem with surface tension, which is also called Mullins–Sekerka model (or Hele–Shaw model with surface tension). The details are in [5,6]. The equations transformed from original equations with free boundary to a fixed domain are quasi-linear equations as follows.
\[
\begin{align*}
\mathcal{A}(\rho)v &= f \quad (t \in J, \ x \in G), \\
\partial_t \rho + \mathcal{B}(\rho)v &= g_0 \quad (t \in J, \ x \in \Gamma), \\
v - \sigma H(\rho) &= 0 \quad (t \in J, \ x \in \Gamma), \\
\rho(0, x) &= \rho_0(x) \quad (x \in \Gamma).
\end{align*}
\] (25)

Here \( v \) is the temperature, \( \rho \) is the height function, \( \sigma \) is a positive constant, called surface tension, and \( \mathcal{A}, \mathcal{B}, H \) are the differential operators. In this example, we consider the very simplified linearized problems;

\[
\begin{align*}
(\eta - \Delta) v &= f \quad (t \in J, \ x \in G), \\
\partial_t \rho + \partial_v v &= g_0 \quad (t \in J, \ x \in \Gamma), \\
v + \Delta \Gamma \rho &= g_1 \quad (t \in J, \ x \in \Gamma), \\
\rho(0, x) &= \rho_0(x) \quad (x \in \Gamma).
\end{align*}
\]

We are able to prove two types of Lopatinskii–Shapiro conditions by the similar calculations as before. Therefore, we get the maximal \( L_p-L_q \) regularity of this Eq. (25).

**Example 5.4.** The forth example is the Cahn–Hilliard equations with the dynamic boundary condition and surface diffusion

\[
\begin{align*}
\eta u + \Delta^2 u &= f \quad (t \in J, \ x \in G), \\
\partial_t u + \partial_v \rho - \Delta \Gamma \rho &= g_0 \quad (t \in J, \ x \in \Gamma), \\
\partial_v \Delta u &= g_1 \quad (t \in J, \ x \in \Gamma), \\
u - \rho &= g_2 \quad (t \in J, \ x \in \Gamma), \\
\rho(0, x) &= \rho_0(x) \quad (x \in \Gamma).
\end{align*}
\] (26)

The equation of the Lopatinskii–Shapiro condition is

\[
\begin{align*}
(\eta + (|\xi'|^2 - \partial_y^2)^2)v(y) &= 0 \quad (y > 0), \\
-\partial_y v(0) + (\lambda + |\xi'|^2)\sigma &= h_0, \\
-\partial_y(|\xi'|^2 - \partial_y^2)u(0) &= h_1, \\
v(0) - \sigma &= h_2.
\end{align*}
\]

The solution of the first equation which belongs to \( C_0(\mathbb{R}^+; E) \) is \( v(y) = C_1 e^{-z_1y} + C_2 e^{-z_2y} \) with \( z_{1,2} := \sqrt{|\xi|^2 \pm \eta^{1/2}i} \) and \( C_{1,2} \in \mathbb{C} \). Note that the real parts of \( z_1 \) and \( z_2 \) are nonnegative. The boundary conditions lead to

\[
\begin{pmatrix}
z_1 \\
z_2 \\
1
\end{pmatrix}
\begin{pmatrix}
z_1(z_1^2 - |\xi'|^2) & z_2(z_2^2 - |\xi'|^2) & \lambda + |\xi'|^2 \\
1 & 1 & -1
\end{pmatrix}
\begin{pmatrix}
C_1 \\
C_2 \\
\sigma
\end{pmatrix}
= 
\begin{pmatrix}
h_0 \\
h_1 \\
h_2
\end{pmatrix}.
\]

We see that the determinant of the matrix is \( i\eta^{1/2}((\lambda + |\xi'|)(z_1 + z_2) + 2z_1z_2) \), and this is not zero for \( \eta > 0, (\lambda, \xi') \in (\Sigma_\theta \times \mathbb{R}^{n-1}) \setminus \{(0, 0)\} \) \((\theta > \pi/2)\). Therefore Lopatinskii–Shapiro condition is satisfied.
Let us check asymptotic Lopatinskii–Shapiro conditions.

\[
\begin{cases}
\eta v(y) + (|\xi'|^2 - \partial_{\xi y}^2) v(y) = 0 \quad (y > 0) \\
\partial_y (|\xi'|^2 - \partial_{\xi y}^2) v(0) = h_1 \\
v(0) = h_2.
\end{cases}
\]

The solution is of the form \( v(y) = C_1 e^{z_1 y} + C_2 e^{z_2 y} \). Since

\[
\det \begin{pmatrix}
|\xi'|^2 - z_1^2 & -|\xi'|^2 - z_2^2 \\
1 & 1
\end{pmatrix} = -i \eta^{1/2} \left( \frac{2|\xi'|^2}{z_1 + z_2} + z_1 + z_2 \right) \neq 0
\]

for \( \xi' \in \mathbb{R}^{n-1} \), the first asymptotic Lopatinskii–Shapiro condition is satisfied.

The equation of the second asymptotic Lopatinskii–Shapiro condition is

\[
\begin{cases}
(|\xi'|^2 - \partial_{\xi y}^2) v(y) = 0 \quad (y > 0) \\
-\partial_y v(0) + \lambda \sigma + |\xi'|^2 \sigma = h_0 \\
\partial_y (|\xi'|^2 - \partial_{\xi y}^2) v(0) = h_1 \\
v(0) = h_2.
\end{cases}
\]

The solution is of the form \( v(y) = C_1 e^{-|\xi'| y} + C_2 ye^{-|\xi'| y} \). Since

\[
\det \begin{pmatrix}
|\xi'| & -1 & \lambda + |\xi'|^2 \\
0 & 2|\xi'|^2 & 0 \\
1 & 0 & 0
\end{pmatrix} = -2|\xi'|(\lambda + |\xi'|^2) \neq 0
\]

for \((\lambda, \xi') \in (\Sigma_\theta \cup \{0\}) \times S^{n-2} (\theta > \pi/2)\), it holds.

The equation of the third asymptotic Lopatinskii–Shapiro condition is

\[
\begin{cases}
(|\xi'|^2 - \partial_{\xi y}^2) v(y) = 0 \quad (y > 0) \\
\partial_y (|\xi'|^2 - \partial_{\xi y}^2) v(0) = h_1 \\
v(0) = h_2.
\end{cases}
\]

We find from the same way as above this equation admits a unique solution for \( \xi' \in S^{n-2} \).

**Example 5.5.** The sixth example is the quasi-steady Cahn–Hilliard equations with permeable wall conditions, while the equations which have time derivative in a domain are in [7,8]. This example is not treated in the paper [3].

\[
\begin{cases}
\eta u + \Delta^2 u = f \quad (t \in J, \ x \in G), \\
\partial_t \rho - \partial_x \Delta u - \Delta \rho = g_0 \quad (t \in J, \ x \in \Gamma), \\
-\Delta u + \Delta \rho = g_1 \quad (t \in J, \ x \in \Gamma), \\
u - \rho = g_2 \quad (t \in J, \ x \in \Gamma), \\
\rho(0, x) = \rho_0(x) \quad (x \in \Gamma).
\end{cases}
\]

(27)

As we have seen the paper [8], we get the standard LS condition as follows. First equation derives \( v(y) = C_1 e^{z_1 y} + C_2 e^{z_2 y} \) with
\[ z_{1,2} = -\sqrt{|\xi'|^2 + (-1)^k - 1\sqrt{-\eta}} \quad (k = 1, 2), \]

and the boundary equations are
\[
\begin{pmatrix}
  z_1\sqrt{-\eta} - z_2\sqrt{-\eta} \lambda + |\xi'|^2 \\
  -\sqrt{-\eta} \sqrt{-\eta} - |\xi'|^2 \\
  1 & 1 & -1
\end{pmatrix}
\begin{pmatrix}
  C_1 \\
  C_2 \\
  \sigma
\end{pmatrix}
= \begin{pmatrix}
  h_0 \\
  h_1 \\
  h_2
\end{pmatrix}
\]

so that the determinant of the coefficient matrix
\[ -\sqrt{-\eta}[2(\lambda + |\xi'|^2) - |\xi'|^2(z_1 + z_2) + \sqrt{-\eta}(z_1 - z_2)] \]
is never zero.

Thus, we need to check the asymptotic LS conditions. Since the first one is easily checked, we see the second equation. For the second equations with \( l = 3, l_0 = 2, l_1 = l_2 = 3 \), we consider the equations
\[
\begin{cases}
 (-1 + \partial_y^2) v(y) = 0 & (y > 0), \\
 \partial_y(-1 + \partial_y^2) v(0) + \lambda \sigma = h_0 \\
 (1 + \partial_y^2) v(0) - \sigma = h_1 \\
 v(0) - \sigma = h_2
\end{cases}
\]

We substitute \( v(y) = (C_1 + C_2 y)e^{-y} \) in the boundary equations, then
\[
\begin{pmatrix}
  0 & 2 & \lambda \\
  0 & 2 & -1 \\
  1 & 0 & -1
\end{pmatrix}
\begin{pmatrix}
  C_1 \\
  C_2 \\
  \sigma
\end{pmatrix}
= \begin{pmatrix}
  h_0 \\
  h_1 \\
  h_2
\end{pmatrix}.
\]

The determinant of the coefficient matrix \(-2 - 2\lambda\) is never zero. The third asymptotic LS conditions is similar to the second one, so we omit it. Therefore, we are able to get the maximal \( L_p-L_q \) regularity results of Eq. (27).

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