CURVATURE ESTIMATES FOR SUBMANIFOLDS IMMERSED INTO HOROBALLS AND HOROCYLINDERS

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ABSTRACT. We prove mean curvature estimates and a Jorge-Koutroufiotis type theorem for submanifolds confined into either a horocylinder of $N \times L$ or a horoball of $N$, where $N$ is a Cartan-Hadamard manifold with pinched curvature. Thus, these submanifolds behave in many respects like submanifolds immersed into compact balls and into cylinders over compact balls. The proofs rely on the Hessian comparison theorem for the Busemann function.

1. INTRODUCTION

The first result of the paper is the following theorem which extends, with a different and somewhat more direct argument, one of the theorems in [2]. We explicitly note that the result remains true when the fibre $L$ degenerates to a 0-dimensional point and, hence, the horocylinder reduces to a horoball.

**Theorem 1.** Let $(N, g_N)$ be an $n$-dimensional Cartan-Hadamard manifold with sectional curvature satisfying $-B \leq \text{Sec}_N \leq -A < 0$ and let $(L, g_L)$ be any complete $l$-dimensional Riemannian manifold with sectional curvature $\text{Sec}_L \geq -C^2$, where $A, B, C$ are positive constants. If $f : \Sigma \hookrightarrow N \times L$ is an $m$-dimensional properly immersed submanifold with $f(M)$ inside a horocylinder of $N \times L$ and $m > l$ then the mean curvature vector $H$ of the immersion satisfies

$$\sup_{\Sigma} |H| \geq \frac{m-l}{m} A.$$ 

The second result is the following sectional curvature lower estimate in the spirit of the classical theorem by Jorge-Koutroufiotis, [6]. We point out that, although it is stated for submanifolds in a horoball, one can prove a version for submanifolds contained in a horocylinder of $N \times L$, where $\text{Sec}_L \geq -B$, under suitably modified assumptions on the dimensions.

**Theorem 2.** Let $(N, g_N)$ be an $n$-dimensional Cartan-Hadamard manifold with sectional curvature satisfying $-B \leq \text{Sec}_N \leq -A < 0$ where $A, B$ are positive constants. Let $f : \Sigma \hookrightarrow N$ be an $m$-dimensional submanifold properly immersed with

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f(M) inside a horoball of N and \( n \leq 2m - 1 \) then the sectional curvature of \( \Sigma \) satisfies the estimate
\[
\sup_{\Sigma} \text{Sec}_\Sigma \geq A - B.
\]

As the geometric setting suggests, the main tool to obtain the results is the analysis of the Busemann function in Cartan-Hadamard manifolds.

2. Busemann Functions in Cartan-Hadamard Spaces

Throughout this section we let \( (N, g_N) \) be a Cartan-Hadamard manifold, i.e., a simply connected, complete Riemannian manifold of non-positive sectional curvature. Further assumptions on \( N \) will be introduced when required. First, we are going to collect some of the basic differentiable properties of the Busemann function of \( N \) with respect to a fixed geodesic ray. Since we are not aware of any specific reference we decided to provide fairly detailed proofs.

Let \( \sigma : [0, +\infty) \to N \) be a geodesic ray issuing from \( \sigma(0) = o \). Recall that, by its very definition, the Busemann function of \( N \) with respect to \( \sigma \) is the function \( b_\sigma : N \to \mathbb{R} \) defined by
\[
b_\sigma(x) = \lim_{t \to +\infty} b_{\sigma(t)}(x)
\]
where, for any fixed \( t \geq 0 \),
\[
b_{\sigma(t)}(x) = r_{\sigma(t)}(x) - r_{\sigma(t)}(o) = r_{\sigma(t)}(x) - t.
\]
Here and below, \( r_p(x) = d(p, x) \) denotes the distance function from a point \( p \). In some sense, the Busemann function measures the distance of the points of \( N \) from the point \( \sigma(+\infty) \) in the ideal boundary \( \partial N \). Since \( t \mapsto -b_{\sigma(t)}(x) \) is monotone decreasing and bounded by \( |b_{\sigma(t)}(x)| \leq r_o(x) \), the limit \( b_\sigma(x) \) exists and is finite. Moreover, the convergence is uniform on compacts by Dini’s theorem. Clearly, by Gauss Lemma, each \( b_{\sigma(t)} \) is 1-Lipschitz and, therefore, so is also \( b_\sigma \). In particular, \( b_\sigma \) is differentiable a.e. Actually, in the special case of Cartan-Hadamard manifolds it was proved by P. Eberlein, [5], that \( b_\sigma \) is a function of class \( C^2 \). To begin with we observe that the gradient \( \nabla b_\sigma \) of the Busemann function can be obtained via a limit procedure from \( \nabla b_{\sigma(t)} \) as \( t \to +\infty \).

**Lemma 3 (limit of gradients).** Assume that the sectional curvature of \( N \) is bounded, namely, there exists \( B > 0 \) such that \( -B \leq \text{Sec}_N \leq 0 \). Then, for every \( x \in N \),
\[
\lim_{t \to +\infty} \nabla b_{\sigma(t)}(x) = \nabla b_\sigma(x)
\]
and the convergence is locally uniform.

**Proof.** By the Hessian comparison theorem, we know that, having fixed a compact ball \( K \subset N \), there exist \( T = T(K) > 0 \) and \( C = C(K, B) > 0 \) such that
\[
|\text{Hess} b_{\sigma(t)}| \leq C,
\]
for every \( x \in K \) and for every \( t \geq T \). It follows that for any sequence \( \{t_k\} \to +\infty \) the corresponding sequence of gradients \( \{\nabla b_{\sigma(t_k)}\} \) is eventually equi-continuous on \( K \). Since it is equi-bounded as observed above, we deduce that there exists a
subsequence \( \{ \nabla b_{\sigma(t_k)} \} \) that converges uniformly on \( K \) to a continuous vector field \( \xi \) on \( K \). On the other hand, the sequence \( \{ \nabla b_{\sigma(t_k)} \} \) converges weakly to \( \nabla b_{\sigma} \) on compact sets. Indeed, if \( V \) is a smooth, compactly supported vector field, then, by dominated convergence,
\[
\langle \nabla b_{\sigma(t_k)}, V \rangle = -\int_M b_{\sigma(t_k)} \text{div} V \to -\int_M b_{\sigma} \text{div} V = \langle \nabla b_{\sigma}, V \rangle,
\]
as claimed. It follows that
\[
\xi = \nabla b_{\sigma}
\]
a.e. on \( K \) and, in fact, everywhere on \( K \) by continuity. Since the selected sequence \( \{t_k\} \) was arbitrary, the required conclusion follows. \( \square \)

In the above proof the conclusion is obtained using the weak definition of the gradient, which behaves well under limits, together with the fact the weak gradient agrees with the classical gradient for sufficiently regular functions. A similar trick will be used in the next result where we deduce a comparison principle for the Hessian of the Busemann function. Recall that a function \( h : N \to \mathbb{R} \) is said to satisfy the differential inequality
\[
\text{Hess} h \leq \mathcal{T}
\]
in the sense of distributions, where \( \mathcal{T} \) is a symmetric 2-tensor, if the integral inequality
\[
\int_N h \{ \text{div} (V \text{div} V) - \text{div} D_V V \} \leq \int_N \mathcal{T} (V, V)
\]
holds for every smooth compactly supported vector field \( V \). Clearly, in case \( h \) is of class \( C^2 \), a double integration by parts shows that the distributional inequality is equivalent to
\[
\int_N \text{Hess} h (V, V) \leq \int_N \mathcal{T} (V, V)
\]
which in turn, by a standard continuity argument (together with a suitable choice of the vector field \( V \)), is equivalent to the usual pointwise inequality.

**Lemma 4** (Hessian comparison). Assume that the sectional curvatures of \( N \) satisfy
\[
-B \leq \text{Sec}_N \leq -A
\]
for some constants \( B \geq A > 0 \). Then
\[
\sqrt{A} (g_N - db_{\sigma} \otimes db_{\sigma}) \leq \text{Hess} b_{\sigma} \leq \sqrt{B} (g_N - db_{\sigma} \otimes db_{\sigma}),
\]
in the sense of quadratic forms.

**Proof.** Let us show how to prove the upper bound of \( \text{Hess} b_{\sigma} \). Obviously, the lower bound can be obtained using exactly the same arguments. By the Hessian comparison theorem, having fixed a ball \( B_R \) of \( N \), we find \( T > 0 \) such that, for every \( t \geq T \),
\[
\text{Hess} b_{\sigma(t)} \leq \sqrt{B} \text{coth} \left( r_{\sigma(t)} \sqrt{B} \right) (g_N - db_{\sigma(t)} \otimes db_{\sigma(t)}), \text{ on } B_R.
\]
In particular, this inequality holds in the sense of distribution, namely, for every vector field $V$ compactly supported in $B_R$, it holds
\[
\int_N b_\sigma(t) \{ \text{div}(V \text{div} V) - \text{div} D_V V \} \leq \int_N \left\{ |V|^2 - \langle \nabla b_\sigma(t), V \rangle^2 \right\}.
\]
Evaluating this latter along a sequence $\{t_k\} \to +\infty$, using Lemma 3, and applying the dominated convergence theorem we deduce that the integral inequality
\[
\int_N b_\sigma \{ \text{div}(V \text{div} V) - \text{div} D_V V \} \leq \int_N \left\{ |V|^2 - \langle \nabla b_\sigma, V \rangle^2 \right\}
\]
holds for every $V \in \chi_c(B_R)$. To conclude we now recall that this is equivalent to the pointwise inequality
\[
\text{Hess} b_\sigma \leq \sqrt{B} (g_N - db_\sigma \otimes db_\sigma),
\]
on $B_R$. □

We remark that a version of the above lemma was also observed without proof in [4].

**Corollary 5.** Keeping the notation and the assumptions of Proposition 4 let $u : N \to \mathbb{R}$ be the smooth function defined by
\[
u(x) = e^{\sqrt{A} b_\sigma(x)}.
\]
Then
\[
A u \cdot g_N \leq \text{Hess} (u) \leq \sqrt{AB} u \cdot g_N.
\]

3. **Proof of the main theorems**

Theorem 1 easily follows from the next more general result since a bounded mean curvature submanifold properly immersed into a complete ambient manifold of bounded sectional curvature is stochastically complete. See for instance [9].

Recall that a (generalized solid) horocylinder of $N \times L$ is the region defined by
\[
\mathcal{B}_{\sigma,R}^{nl} = \{ b_\sigma \leq R \} \times L,
\]
where $R > 0$ and $b_\sigma$ is the Busemann function with respect to the ray $\sigma$.

**Theorem 6.** Let $(N, g_N)$ be an n-dimensional Cartan-Hadamard manifold with sectional curvature satisfying $-B \leq \text{Sec}_N \leq -A$ and let $\sigma : [0, +\infty) \to N$ be a ray of $N$. Let $(L, g_L)$ be any l-dimensional Riemannian manifold and let $f = (f_N, f_L) : \Sigma \hookrightarrow N \times L$ be an m-dimensional isometric immersion with mean curvature vector field $H$. Assume that $m > l$. If $\Sigma$ is stochastically complete and $f(\Sigma)$ is contained in the horocylinder $\mathcal{B}_{\sigma,R}^{nl}$, then
\[
\sup_{\Sigma} |H| \geq \frac{m-l}{m} \sqrt{A}.
\]
Proof. Let

\[ w = u \circ f_N : \Sigma \rightarrow \mathbb{R}_{>0} \]

where \( u : N \rightarrow \mathbb{R} \) is the smooth function defined in Corollary 5. By the composition law for the Laplacians we have

\[ \Delta w = \text{tr}_\Sigma \{ \text{Hess} (u) (d f_N \otimes d f_N) \} + du (\text{tr}_\Sigma D d f_N). \]

On the other hand, by Corollary 5,

\[ \text{Hess} (u) \geq Au \cdot g_N \]

so that

\[ \Delta w \geq w \{ A \text{tr}_\Sigma (g_N) \} \]

\[ = w \{ A \text{tr}_\Sigma f_N^* (g_N - m |H|) \}. \]

Since \( f^* g_{N \times L} = g_\Sigma \) then

\[ \text{tr}_\Sigma f_N^* (g_N) = m - \text{tr}_\Sigma f_L^* (g_L) \]

\[ \geq m - l, \]

and from the above we conclude that

\[ (1) \quad \Delta w (x) \geq mw (x) \left\{ A \frac{m - l}{m} - \sup_\Sigma |H| (x) \right\}. \]

Now we apply the weak maximum principle for the Laplacian, [8], to get

\[ 0 \geq m \sup_\Sigma w \left\{ \frac{m - l}{m} A - \sup_\Sigma |H| (x) \right\}, \]

as required. \( \square \)

Remark 7. By applying the strong maximum principle to (1) we can also obtain directly the following touching principle.

Let \( f : \Sigma \hookrightarrow N \times L \) be a complete, immersed submanifold, where \( N \) is a Cartan-Hadamard manifold of pinched negative curvature and \( L \) is any complete Riemannian manifold. Let us assume that (a) the mean curvature \( H \) of the immersion satisfies \( |H| \leq A (m - l) / m \) and that (b) \( f (\Sigma) \) is contained inside the horocylinder \( \mathcal{B}_R^{m,1} \).

If \( f (\Sigma) \cap \partial \mathcal{B}_R^{m,1} \neq 0 \), then \( f (\Sigma) = \partial \mathcal{B}_R^{m,1} \).

Remark 8. In a different direction, if \( f = f_N : \Sigma \rightarrow \mathcal{B}_R^{n,0} = \{ b_\alpha \leq R \} \subset N \) is a proper immersion into a horoball of \( N \) and

\[ \sup_\Sigma |H| (x) < A \]

then \( w \) is a bounded exhaustion function that violates the weak maximum principle at infinity. By Theorem 32 in [3] it follows that the essential spectrum of \( \Sigma \) is empty.

The estimates for the Hessian of the Busemann function allows us to obtain also the Jorge-Koutroufiotis type result stated in Theorem 2. This result gives a further indication of the phenomenon according to which submanifolds of non-compact horoballs behave in many respects like submanifolds of compact balls.
Proof (of Theorem 2). The proof follows closely the arguments in [1]. For every $k$ consider the function $h_k : \Sigma \to \mathbb{R}$

$$h_k = w - \frac{1}{k} \left[ \log(\rho_N \circ f + 1) + 1 \right],$$

where, as above $w = e^{\sqrt{A} h_\sigma \circ f}$ and $\rho_N$ denotes the Riemannian distance in $N$ from an origin $o$ in the complement of $\mathcal{B}_0^{\sigma, R}$. Since $f(\Sigma)$ is contained in a horoball, the first summand is bounded above, and since the $f$ is proper in $N$, the second summand tends to infinity at infinity. It follows that for every $k$, $h_k$ attains an absolute maximum at a point $x_k$ where

$$\text{Hess} h_k = \text{Hess} w - \frac{1}{k} \text{Hess} \left[ \log(\rho_N \circ f + 1) + 1 \right] \leq 0$$

in the sense of quadratic forms. Now, according to our previous computations, for all vectors $X_k \in T_{x_k} \Sigma$,\n
\begin{equation}
(2) \quad \text{Hess} w(X_k, X_k) \geq \sqrt{A} w(x_k) \left( \sqrt{A} |X_k|^2 - |\text{II}(X_k, X_k)| \right),
\end{equation}

where $\text{II}$ is the second fundamental form of the immersion. On the other hand, by the Hessian comparison theorem,

$$\text{Hess} \rho_N \leq \sqrt{B} \coth(\sqrt{B} \rho_N) (g_N - d\rho_N \otimes d\rho_N),$$

and after some computation we obtain

$$\text{Hess} \left[ \log(\rho_N \circ f + 1) + 1 \right] (X_k, X_k) \leq \frac{1}{\rho_N(f(x_k)) + 1} \left\{ \sqrt{B} \coth(\sqrt{B} \rho_N(f(x_k))) |X_k|^2 + |\text{II}(X_k, X_k)| \right\}.$$\n
Combining the two inequalities and rearranging we conclude that

$$|\text{II}(X_k, X_k)| \left( \sqrt{A} w(x_k) + \frac{1}{k|\rho_N(f(x_k)) + 1|} \right) \geq \left( A w(x_k) - \frac{\sqrt{B} \coth(\sqrt{B} \rho_N(f(x_k)))}{k|\rho_N(f(x_k)) + 1|} \right) |X_k|^2$$\n
Now notice that $w(x_k)$ is positive and bounded away from 0. Indeed, if $\bar{x}$ is a point such that $\rho_N(f(\bar{x})) = \min_{\Sigma} \rho_N(f(x))$, then for every $k$ we have

$$h_k(x_k) \geq h_k(\bar{x})$$

and therefore

$$w(x_k) \geq w(\bar{x}) + \frac{1}{k} \left\{ \log[\rho_N(f(x_k)) + 1] - \log[\rho_N(f(\bar{x})) + 1] \right\} \geq w(\bar{x}).$$
Since $\rho_N(f(x_k))$ is also bounded away from zero, it follows that for every sufficiently large $k$, and every non zero $X_k \in T_{x_k} \Sigma,$

$$|\langle X_k, X_k \rangle| \geq \frac{\sqrt{B} \coth(\sqrt{B} \rho_N(f(x_k)))}{k \rho_N(f(x_k)) + 1} |X_k|^2 \sqrt{A w(x_k) + 1} \cdot \frac{1}{k \rho_N(f(x_k)) + 1} \cdot \frac{1}{k \rho_N(f(x_k)) + 1}$$

In particular, $\langle X_k, X_k \rangle > 0$ for every sufficiently large $k$ and every $X_k \in T_{x_k} \Sigma \setminus \{0\}$ and we may apply Otzuki’s lemma (see, e.g., [7], p. 28) to find unit tangent vectors $X_k$ and $Y_k$ such that $\langle X_k, X_k \rangle = \langle Y_k, Y_k \rangle$ and $\langle X_k, Y_k \rangle = 0$. The required conclusion now follows from Gauss equations as in the original proof:

$$\text{Sec}_\Sigma(X_k, Y_k) = \text{Sec}_\Sigma(df X_k, df Y_k) + g_N(\langle II(X_k, X_k), II(Y_k, Y_k) \rangle) - |\langle II(X_k, Y_k) \rangle|^2 \geq -B + A + o(k^{-1}).$$

Again as in the classical proof, we note that the conclusion of the theorem follows directly from (2) if we assume that $\text{Scal}_\Sigma \geq -G(\rho_\Sigma)$ where the function $G$ is positive, non-decreasing and $G^{-1/2}$ is integrable at infinity. Indeed, assuming that $\text{Sec}_\Sigma$ is bounded above, for otherwise the conclusion holds trivially, then $\text{Sec}_\Sigma$ is bounded below by a multiple of $-G$ and the Omori-Yau maximum principle for the Hessian holds on $\Sigma$ ([9]). Thus there exists a sequence $x_k$ such that $w(x_k) \to \sup_\Sigma w$ and

$$\text{Hess} w(x_k) \leq k^{-1} g_\Sigma,$$

which together with (2) allows to conclude as in the last part of the above proof.

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