Phase diagrams of soluble multi-spin glass models

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Abstract

We include $p$-spin interactions in a spherical version of a soluble mean-field spin-glass model proposed by van Hemmen. Due to the simplicity of the solutions, which do not require the use of the replica trick, we are able to carry out a detailed investigation of a number of special situations. For $p \geq 3$, there appear first-order transitions between the paramagnetic and the ordered phases. In the presence of additional ferromagnetic interactions, we show that there is no stable mixed phase, with both ferromagnetic and spin-glass properties.

1 Introduction

The attempts to account for the physical properties of disordered systems have been a continuous source of models and problems in statistical mechanics. It is known that a mean-field $p$-spin-glass model, with $p \geq 3$, displays a first-order transition to a glassy phase, and non-trivial dynamical behavior. Although models with $p$-spin interactions may be far from representing real glasses, it has been noted that the dynamical mean-field equations do resemble the corresponding mode-coupling expressions for glassy systems [1].

In some recent publications, a ferromagnetic term, as well as additional multi-spin interactions, have been added to the simple spin-glass $p$-spin model [2,3,4]. These new systems display continuous and first-order transitions, and a glassy ferromagnetic region with pronounced reentrant borders. However, the search for the thermodynamic solutions of these mean-field models requires the use of the subtleties of the replica method. The replica-symmetric solutions have to be supplemented by a one-step replica-symmetry breaking scheme in order to characterize the glassy ferromagnetic phase.

In this paper, we decided to investigate the thermodynamic behavior of highly
simplified versions of disordered multi-spin mean-field models. The idea consists in the inclusion of \( p \)-spin interactions, with \( p \geq 3 \), in a slightly more general version of a spin-glass model proposed by van Hemmen about twenty years ago [5]. This simple van Hemmen model, which does not require the use of the replica method, is obviously unable to reproduce the rich (ultrametric) structure of the spin-glass phase [6]. However, it does give a reasonable account of the main features of the phase diagrams, including the location of phase regions and transition lines. In this paper, we further simplify the calculations by assuming spherical instead of Ising spin variables.

In Section 2, we review the van Hemmen spin-glass model with spherical spin variables. In Section 3, we introduce multi-spin interactions in the van Hemmen model, and make a number of contacts with recent work. Section 4 is devoted to some conclusions.

2 The van Hemmen model

The van Hemmen spin-glass model, with the inclusion of a ferromagnetic term, is given by the Hamiltonian

\[
H = -\frac{J_0}{N} \sum_{1 \leq i < j \leq N} \sigma_i \sigma_j - \sum_{1 \leq i < j \leq N} J_{ij} \sigma_i \sigma_j - H \sum_{i=1}^{N} \sigma_i, \tag{1}
\]

where \( J_0 > 0 \),

\[
J_{ij} = \frac{1}{2N} J \left[ \xi^{(1)}_i \xi^{(2)}_j + \xi^{(2)}_i \xi^{(1)}_j \right], \tag{2}
\]

\( J > 0 \), and \( H \) is an external field. The set of independent, identically distributed random variables \( \{ \xi^{(\alpha)}_i \} \), for \( i = 1, 2, ..., N \) and \( \alpha = 1, 2 \), associated with a double-delta probability distribution,

\[
p \left( \xi^{(\alpha)}_i \right) = \frac{1}{2} \delta \left( \xi^{(\alpha)}_i - 1 \right) + \frac{1}{2} \delta \left( \xi^{(\alpha)}_i + 1 \right), \tag{3}
\]

is supposed to mimic the presence of disorder and competition in real spin glasses. In the spherical version of this model, the spin variables are real numbers, \( -\infty \leq \sigma_i \leq \infty \), for all \( i = 1, 2, ..., N \), with the spherical constraint,

\[
\sum_{i=1}^{N} \sigma_i^2 = N. \tag{4}
\]

Using more convenient variables,

\[
m = \frac{1}{N} \sum_{i=1}^{N} \sigma_i, \quad m_\alpha = \frac{1}{N} \sum_{i=1}^{N} \xi^{(\alpha)}_i \sigma_i, \tag{5}
\]
for \( \alpha = 1, 2 \), and discarding irrelevant terms in the thermodynamic limit, we can rewrite the Hamiltonian, given by Eq. (1), in the form

\[
H = -\frac{1}{2}J_0 N m^2 - \frac{1}{2}J N m_1 m_2 - H N m. \tag{6}
\]

Given a configuration of the random variables, the partition function may be written as

\[
Z = \text{Tr} \exp \left\{ \frac{1}{2} \beta J_0 N m^2 + \frac{1}{2} \beta J N m_1 m_2 + \beta H N m \right\}, \tag{7}
\]

where \( \beta \) is the inverse of temperature, and the trace should take into account the spherical constraint.

We now rewrite the partition function in the form

\[
Z = \int_{-\infty}^{+\infty} dm \int_{-\infty}^{+\infty} dm_1 \int_{-\infty}^{+\infty} dm_2 \Omega (m, m_1, m_2) \\
\times \exp \left\{ \frac{1}{2} \beta J_0 N m^2 + \frac{1}{2} \beta J N m_1 m_2 + \beta H N m \right\}, \tag{8}
\]

where

\[
\Omega = \text{Tr} \prod_{i=1}^{N} \left( \int_{-\infty}^{+\infty} d\sigma_i \right) \delta \left( \sum_{i=1}^{N} \sigma_i^2 - N \right) \delta \left( m - \frac{1}{N} \sum_{i=1}^{N} \sigma_i \right) \\
\times \delta \left( m_1 - \frac{1}{N} \sum_{i=1}^{N} \xi_i^{(1)} \sigma_i \right) \delta \left( m_2 - \frac{1}{N} \sum_{i=1}^{N} \xi_i^{(2)} \sigma_i \right). \tag{9}
\]

In the thermodynamic limit, we use an integral representation for the delta functions and invoke the law of large numbers in order to obtain the asymptotic result \[7\]

\[
\Omega \sim \exp \left\{ \frac{N}{2} \left[ 1 + \ln (2\pi) + \ln \left( 1 - m^2 - m_1^2 - m_2^2 \right) \right] \right\}, \tag{10}
\]

which leads to the partition function

\[
Z \sim \int_{-\infty}^{+\infty} dm \int_{-\infty}^{+\infty} dm_1 \int_{-\infty}^{+\infty} dm_2 \exp \left[ -\beta N f (m, m_1, m_2) \right], \tag{11}
\]

where

\[
f = -\frac{1}{2} \left( J_0 m^2 + J m_1 m_2 \right) - H m - \frac{1}{2\beta} \left[ 1 + \ln (2\pi) + \ln \left( 1 - m^2 - m_1^2 - m_2^2 \right) \right]. \tag{12}
\]

The thermodynamic solutions come from the stable minima of this free-energy functional.
Fig. 1. Phase diagram of the spherical van Hemmen model. Solid lines correspond to zero external field, while dashed lines are obtained for $H_R/J = 1/4$. Thin (thick) lines indicate second (first) order transitions. The labels correspond to the paramagnetic (PARA), ferromagnetic (FERRO), and to the “spin-glass” (SG) phases.

In zero external field, $H = 0$, it is easy to draw the phase diagram of Figure 1, in terms of $T = (\beta J)^{-1}$ and $r = J_0/J$. Besides the disordered paramagnetic phase ($m = m_1 = m_2 = 0$), there is a “spin-glass” ($m = 0$, and $m_1, m_2 \neq 0$) and a ferromagnetic phase ($m \neq 0$, and $m_1 = m_2 = 0$). In the ferromagnetic region, $m^2 = 1 - T/r$; in the spin-glass region, $m_1^2 = m_2^2 = 1/2 - T$. The paramagnetic borders are lines of continuous phase transitions. Mixed phases (with $m \neq 0$ and $m_1, m_2 \neq 0$) are restricted to a line of coexistence at $r = J_0/J = 1/2$ (which also corresponds to the limits of stability of the ferromagnetic and spin-glass solutions). Similar topological features are also present in the phase diagram of the much more elaborate Sherrington-Kirkpatrick (SK) mean-field model of a spin glass.

In the presence of a random field, the Hamiltonian of the van Hemmen model is written as

$$\begin{align*}
H &= -\frac{J_0}{N} \sum_{1 \leq i < j \leq N} \sigma_i \sigma_j - \sum_{1 \leq i < j \leq N} J_{ij} \sigma_i \sigma_j - \sum_{i=1}^{N} H_i \sigma_i, \\
&= \left(1 - \frac{1}{2}\delta(H_i - H_R) + \frac{1}{2}\delta(H_i + H_R)\right)
\end{align*}$$

where $\{H_i\}$, for $i = 1, 2, \ldots, N$, is a set of quenched independent, identically distributed random variables, given by the probability distribution

$$p_H(H_i) = \frac{1}{2}\delta(H_i - H_R) + \frac{1}{2}\delta(H_i + H_R).$$

Introducing the new variable

$$q = \frac{1}{N} \sum_{i=1}^{N} \frac{H_i}{H_R} \sigma_i.$$
it is straightforward to obtain the free-energy functional
\[
f = -\frac{1}{2} \left( J_0 m^2 + Jm_1 m_2 \right) - H_R q - \frac{1}{2\beta} \left[ 1 + \ln (2\pi) + \ln \left( 1 - m^2 - m_1^2 - m_2^2 \right) \right].
\]
(16)

In the $T - r$ phase diagram, there is a depression of the paramagnetic lines, which still meet at $r = 1/2$ (see Figure 1). For small values of $h_R = H_R / J \ll 1$, we have the following asymptotic forms of the paramagnetic critical lines: (i) $T = 1/2 - 2h_R^2$, at the spin-glass border; (ii) $T = r - h_R^2/r$, at the paramagnetic-ferromagnetic border. There is no tricritical point with spherical spin variables. The same qualitative depression of the paramagnetic borders is also present in the phase diagram of an SK model in a random field [8].

3 Multi-spin interactions

The inclusion of $p$-spin interactions, with $p \geq 2$, leads to a natural generalization of the van Hemmen model. Let us write the spin Hamiltonian
\[
H = -\sum_{1 \leq i_1 < i_2 < \ldots < i_p \leq N} J_{i_1 \ldots i_p} \sigma_{i_1} \ldots \sigma_{i_p},
\]
(17)

with
\[
J_{i_1 \ldots i_p} = \frac{J_p}{p!N^{p-1}} \sum_{(\alpha_1 \ldots \alpha_p)} \xi_{i_1}^{(\alpha_1)} \xi_{i_2}^{(\alpha_2)} \ldots \xi_{i_p}^{(\alpha_p)},
\]
(18)

where the sum is over the permutations of the $p$ patterns, $\alpha_1, \ldots, \alpha_p$. In analogy with the $p = 2$ case, we have enlarged the set of random variables, $\{\xi_i^{(\alpha)}\}$, which are still given by the probability distribution of Eq. (3). With this choice of patterns, the $p$-spin Hamiltonian can be written as
\[
H = -\frac{1}{p!} J_p N m_1 \ldots m_p,
\]
(19)

from which we obtain the free-energy functional
\[
f = -\frac{1}{p!} J_p m_1 \ldots m_p - \frac{1}{2\beta} \left[ 1 + \ln (2\pi) + \ln \left( 1 - \sum_{\alpha=1}^{p} m_{\alpha}^2 \right) \right].
\]
(20)

For $p \geq 3$, it is easy to show that there is a first-order transition between a paramagnetic disordered phase ($m_1 = m_2 = \ldots = 0$) and an ordered “spin-glass” phase ($m_1^2 = m_2^2 = \ldots = 0$). The existence of a first-order transition in these $p$-spin models, which leads to spinodal lines and may be responsible for peculiar dynamical phenomena, is usually taken as an important contact with the behavior of real glasses.
These calculations can be easily extended to a much larger class of models. Let us consider blocks of \( p \)-spin and \( r \)-spin interactions, as well as ferromagnetic \( n \)-spin interactions and a random external field. A sufficiently general spin Hamiltonian may be written as

\[
H = -\sum_{1 \leq i_1 < i_2 < \ldots < i_p \leq N} J_{i_1 \ldots i_p} \sigma_{i_1} \ldots \sigma_{i_p} - \sum_{1 \leq i_1 < i_2 < \ldots < i_r \leq N} J_{i_1 \ldots i_r} \sigma_{i_1} \ldots \sigma_{i_r} - 
\]

\[
- \frac{J_0}{n!N^{n-1}} \left[ \sum_{i=1}^N \sigma_i \right]^n - \sum_{i=1}^N H_i \sigma_i, 
\]

with

\[
J_{i_1 \ldots i_p} = \frac{J_p}{p!N^{p-1}} \sum_{(\alpha_1 \ldots \alpha_p)} \xi_{\alpha_1}^{(\alpha_1)} \xi_{\alpha_2}^{(\alpha_2)} \ldots \xi_{\alpha_p}^{(\alpha_p)},
\]

and

\[
J_{i_1 \ldots i_r} = \frac{J_r}{r!N^{r-1}} \sum_{(\alpha_{p+1} \ldots \alpha_{p+r})} \xi_{\alpha_1}^{(\alpha_{p+1})} \xi_{\alpha_2}^{(\alpha_{p+2})} \ldots \xi_{\alpha_r}^{(\alpha_{p+r})},
\]

where the sums are over all permutations of \( \alpha_1 \ldots \alpha_p \) and \( \alpha_{p+1} \ldots \alpha_{p+r} \), and the independent random variables are still given by the probability distributions of equations (3) and (14).

Using the definitions of \( m \), \( m_\alpha \), and \( q \), given by Eqs. (5) and (15), for \( \alpha = 1, \ldots, p+r \), it is easy to rearrange the Hamiltonian in the more convenient form

\[
H = -\frac{1}{p!} J_p N m_1 \ldots m_p - \frac{1}{r!} J_r m_{p+1} \ldots m_{p+r} - \frac{1}{n!} J_0 N m^n - H_R N q. 
\]

In the thermodynamic limit, assuming finite values of the parameters \( p \), \( r \), and \( n \), we can write the free-energy functional

\[
f = -\frac{1}{p!} J_p m_1 \ldots m_p - \frac{1}{r!} J_r m_{p+1} \ldots m_{p+r} - \frac{1}{n!} J_0 m^n - H_R q
- \frac{1}{2\beta} \left[ 1 + \ln (2\pi) + \ln \left( 1 - m^2 - q^2 - \sum_{\alpha=1}^{p+r} m_\alpha^2 \right) \right].
\]

We now consider some particular situations:

(A) A model with (uniform) ferromagnetic interactions \( J_{02} \) and \( J_{04} \) involving blocks of \( n = 2 \) and \( n = 4 \) spins, in the presence of a random field, leads to the free-energy functional

\[
f_u = -\frac{1}{2} J_{02} m^2 - \frac{1}{4!} J_{04} m^4 - q H_R - \frac{1}{2\beta} \left[ 1 + \ln (2\pi) + \ln \left( 1 - m^2 - q^2 \right) \right].
\]
Fig. 2. Phase diagram of the mean-field spherical model with ferromagnetic $J_{02}$ and $J_{04}$ couplings in zero external field. Thin (thick) lines indicate second (first) order transitions, while $P_{tr}$ marks the tricritical point.

Fig. 3. Phase diagram of the mean-field spherical model with ferromagnetic $J_{02}$ and $J_{04}$ couplings in a random field of strength $H_R$. The surface corresponds to a paramagnetic boundary, and the thick curve is a line of tricritical points, separating a region of first-order (large $r$) from a region of second-order (small $r$) phase transitions.

In zero random field, $H_R = 0$, the phase diagram, in terms of temperature, $T = (\beta J_{02})^{-1}$, versus the ratio of ferromagnetic uniform interactions, $r = J_{04}/J_{02}$, displays a line of second-order transitions that turns into a first-order boundary at a tricritical point, $J_{04}/J_{02} = 6$ (see Figure 2). The existence of first and second-order transitions at the mean-field level gives an additional
motivation to pursue the investigation of the spin-glass versions of this model. The random field introduces just a shift of the tricritical point. The global phase diagram, in terms of $T/J_{04}/J_{02}$, and the strength of the random field, $H_{R}/J_{02}$, as shown in Figure 3, displays a line of tricritical points.

(B) If we assume $r = 0$ and $n = 2$, in zero field, it is possible to make contact with some recent calculations for a spherical version of a $p$-spin Sherrington-Kirkpatrick (SK) spin-glass model with the addition of a simple ferromagnetic term. According to these calculations [2,3], for $p \geq 3$, the phase diagram of the spherical $p$-spin SK model displays a glassy ferromagnetic phase, besides the usual paramagnetic, spin glass, and pure ferromagnetic phases. Also, there is a pronounced reentrance of the border between the two ferromagnetic phases.

In the context of the van Hemmen models, we write the free-energy functional

$$f = -\frac{1}{p!} J_p m_1 \ldots m_p - \frac{1}{2} J_0 m^2 - \frac{1}{2\beta} \left[ 1 + \ln (2\pi) + \ln \left( 1 - m^2 - \sum_{\alpha=1}^{p} m_{\alpha}^2 \right) \right].$$

We now set $p = 3$, which gives typical results for $p > n$. The minimization of $f$ leads to the phase diagram shown in Figure 4, in terms of temperature, $T = 1/(\beta J_3)$, and the ratio of interactions, $r = J_0/J_3$. The paramagnetic phase $(m = m_1 = m_2 = m_3 = 0)$ is stable for $T > r$. The ferromagnetic phase $(m^2 = 1 - T/r, m_1 = m_2 = m_3 = 0)$ is stable for $T < r$. The critical line between the paramagnetic and the ferromagnetic phases ends at a critical endpoint (at $T = r = 0.03278...$). The “spin-glass phase” $(m = 0, m_1^2 = m_2^2 = m_3^2 \neq 0)$ is limited by a first-order boundary (at $T = 0.03278...$, for small values of $r$; and at $r = 1/9\sqrt{3}$ for $T = 0$). The extremization of the free energy $f$ also leads to a “mixed ferromagnetic solution”, $m \neq 0$, with $m_1^2 = m_2^2 = m_3^2 \neq 0$, which might be the analogue of the glassy ferromagnetic phase of the $p$-spin SK models. However, it is easy to show that the free energy of this solution is always larger than the free energy of the uniform ferromagnetic phase. For this $p = 3$ example, at any temperature, $f$ (ferro) $-$ $f$ (mixed ferro) $= -18r^3 < 0$. Indeed, it is easy to obtain analogous results for $p > 3$.

4 Conclusions

We have introduced multi-spin interactions in the van Hemmen spin-glass model with spherical spin variables. Using standard techniques, we write analytic expressions for a self-averaged free energy of a model Hamiltonian including blocks of disordered and uniform interactions, in the presence of random fields. From this free energy, it is easy to draw some phase diagrams, in order to compare with results for the corresponding SK models.
Fig. 4. Phase diagram of the spherical van Hemmen model with ferromagnetic pair interactions and random \( p = 3 \) couplings. Thin (thick) lines indicate second (first) order transitions.

In particular, for \( p = 3 \) and two-spin uniform ferromagnetic interactions, we obtain a phase diagram with first-order transitions between spin-glass and either paramagnetic or ferromagnetic phases, and a second-order ferro-paramagnetic transition line, which ends at a critical endpoint. In contrast to the calculations for the analogous SK model, we show that there is no possibility of appearance of a ferromagnetic phase of spin-glass character.

References

[1] L. Cugliandolo, Dynamics of glassy systems, in Slow Relaxations and Nonequilibrium Dynamics in Condensed Matter, vol. 77 of Les Houches – École d’Été de Physique Théorique, edited by J. Barrat, M. V. Feigelman, J. Kurchan, and J. Dalibard (Springer, New York, 2003); also online at arXiv: cond-mat/0210312.

[2] J. A. Hertz, David Sherrington, and Th. M. Nieuwenhuizen, Phys. Rev. E60, R2460 (1999).

[3] Peter Gillin and David Sherrington, J. Phys. A: Math. Gen. 33, 3081 (2000).

[4] A. Crisanti and L. Leuzzi, Phys. Rev. Lett. 93, 217203 (2004).

[5] J. L. van Hemmen, Phys. Rev. Lett. 49, 409 (1982); J. L. van Hemmen, A. C. D. van Enter, and J. Canisius, Z. Phys. B50, 311 (1983).

[6] T. C. Choy and D. Sherrington, J. Phys. C17, 739 (1984).

[7] T. A. S. Haddad, A. P. Vieira, and S. R. Salinas, Physica A342, 76 (2004).

[8] R. F. Soares, F. D. Nobre, and J. R. L. De Almeida, Phys. Rev. B50, 6151 (1994).