Tipping Point Dynamics: A Universal Formula

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Abstract

Sudden and unexpected disruptive phenomena like current French “gilets jaunes” movement, Arab springs, Trump and Brexit victories, put at stake the feasibility of their eventual forecasting. Here we claim that such unpredictable social events are indeed predictable, provided one can build the opinion landscape to identify the underlying dynamics. We derive a universal mathematical formula for the temporal evolution of opinion distribution among a heterogeneous population. It allows identifying the various tipping points and attractors triggering opinion flows thus unveiling the hidden mechanisms behind sudden upheavals like minority spreading and majority collapse. With our formula, the final outcome of an opinion dynamics can be predicted with substantial flexibility in the parameters evaluations, if the proper identification of relevant attractors are made. This opens the path to a large spectrum of real applications including voting outcomes, market shares and societal trends. This positioning provides a new ground to predict opinion dynamics outcomes and envision competing strategies to win a public debate.

1. Introduction

Last decade has witnessed a series of upheavals, which have created profound and unexpected reshaping of the world settings politically, economically and militarily. Among others stand the Arab Spring, the Brexit victory and Trump election. Independently of their respective content, they all share the fact that they broke out as total surprises to all analysts, experts, pundits. Even poll predictions, when available, turned wrong. That prompts the question to determine if these events were a series of “very rare accidents”, which by definitions are unpredictable, or on the contrary, if they were the deterministic outcomes of well defined processes, which means that at least in principle, they could have been predicted.

For four decades, sociophysics has embraced the second option, based on the assumption that opinion dynamics obeys discoverable universal quantitative laws [1–3]. If proven right, such an assumption constitutes a major challenging stake for the future of citizens, public policies and policy makers. The search for interaction laws among individuals has started with the 1982 paper, which investigated the asymmetrical mechanisms by which a strike breaks out [4].

Among the numerous existing models of opinion dynamics [5–15], stands the earlier Galam model [16–21], which has established a promising frame leading to a unifying frame [22]. Galam model has yielded several successful predictions of unexpected events like 2016 Brexit victory [18, 19, 23] and Trump election [24]. It also has predicted the 2005 French rejection of the European constitution project [25].

The Galam model incorporates several different psychological traits for agents, respectively floaters, inflexibles [26–33] and contrarians [34–38]. It also incorporates a tie breaking effect driven by the existence of hidden prejudices for even update group sizes [16–18]. Some combinations of both inflexibles and contrarians have studied [39–41]. A great deal of works have been tackling the issue [42–50]. However up to now, update groups were restricted to size 2, 3 and 4. Moreover, it has not been possible to derive a universal update equation which incorporates all of those items simultaneously for any size of update groups.

In this paper, we are able to overcome above restrictions, and derive a mathematical formula for opinion update, which includes, at once, any combination of proportions of the various types of agents, floaters, contrarians, inflexibles...
with any value of prejudice distribution and for group updates of arbitrary size. On this basis, the phase diagram of the various outcomes of opinion dynamics is obtained in a parameter space of six dimensions.

Having the formula at hands, we performed the analysis of its fixed points to discover an opinion landscape, sparse with a series of attractors and tipping points. Studying their respective stabilities, we could evaluate the critical values of parameters which delimit the various basins of attractions. Building the corresponding phase diagram reveals the existence of a rich variety of non-linearities and singularities. Sudden upheavals like minority spreading and majority collapse are thus given a rationale, unveiling the hidden mechanisms behind the occurrence of unexpected shifts in the distribution of opinions.

In particular, the approach to large group sizes, where the effects of inflexible agents disappear with simple majority rule holding, is found to affect the results only very slowly, showing the insights obtained from group sizes three and four remain intact for larger group sizes.

We point out the existence of the phase structure in the model, which is common to all group size. In addition, to have the update equation for any group size $r$ makes possible to extend it to an update equation, which accounts for any combination of group sizes, making the model further realistic.

Moreover, identifying the complete phase structure in the parameter space enables a robust ground for the predictions of the dynamics, since what matters is to determine in which basin of attraction the dynamics is taking place, rather then the precise values of parameters, giving us substantial flexibility in the parameters evaluations.

We thus have considerably upgraded the predicting power of our model for all expected and unexpected opinion dynamics outcomes, which should have real-world applications in voting outcomes, market shares and societal trends. Next step to test the validity of our universal update formula will be by making a series of predictions of real events and with an emphasize on setting tools to evaluate the actual values of the model parameters. While more progress has still to be made to get a robust predictive tool, we are nearing a breaching point.

The paper is organized as follows. In section 2, we outline the derivation of the universal formula of opinion update in Galam model. In section 3, we write down concrete expressions for the evolution equation for some values of $r$. In section 4, we analyzed the mathematical structure of the obtained universal formula through fixed points of the dynamics. In section 5, we explore some aspects of the six-dimensional parameter space of the the model, and points out the existence of critical points. The results are illustrated with several numerical examples. We summarize and conclude the paper with the final sixth section.

2. The universal formula in six dimensional parameter space

We consider a population with $N$ agents each capable of taking two states 1 and 0, respectively representing two exclusive opinions $A$ and $B$. At a given time $t$ the corresponding proportions of agents holding $A$ and $B$ are denoted $p_t$ and $(1-p_t)$. To make legitimate the use of proportions and probabilities we focus on cases with $N > 100$. Choosing an initial time $t = 0$, we investigate the time evolution of $p_0$ driven by informal discussions among agents at successive discrete time steps with $p_0 \rightarrow p_1 \rightarrow p_2 \rightarrow ...$

To account for this complicated and unknown process we use the Galam Dynamical Model, which monitors the opinion dynamics by a sequential iterated scheme. To implement one scheme, agents are first randomly distributed within small groups of size $r$. Then, within each group a majority rule is applied to update locally agent opinions. One agent, one vote is applied to determine the actual local majority. However, to account for the heterogeneity of psychological traits among agents, not all of them obeys the majority rule by shifting opinion if holding the local minority choice. Three types of agents, floaters, inflexibles and contrarians, are considered. After local updates are performed, all groups are dismantled and agents are reshuffled thus erasing the local correlations which were created by applying the local majority rules.

- The floaters vote to determine the local majority and afterwards the ones holding the minority opinion shift to adopt the majority opinion. However, in case of a tie at a local even group, floaters adopt the choice in tune with the leading prejudice among the group members. This prejudice driven tie breaking is activated unconsciously by the agents. Accordingly, at a tie opinion $A$ is chosen with probability $k$ and opinion $B$ with probability $(1-k)$. The value of $k$ satisfies $0 \leq k \leq 1$ and is a function of the distribution of prejudices within the social group.
• The inflexibles also vote to determine the local majority but contrary to the floaters, in case they hold the minority choice they stick to it. They are stubborn and keep on their own pre-decided choice whatever the local majority is. Accordingly an A-inflexible stays in state 1 and a B-inflexible in 0. The same holds true at a tie. Their respective proportions denoted \( a \) and \( b \) are external fixed parameters with the constraints \( 0 \leq a + b \leq 1 \).

• The contrarians are floaters, who once groups are dismantled, decide individually to shift opinion to oppose the group majority they contributed to. The attitude is independent of the majority choice, either \( A \) or \( B \). The proportion of contrarians among floaters is denoted \( c \) satisfying \( 0 \leq c \leq 1 \). The value of \( c \) is also a fixed external parameter with the case \( c > \frac{1}{2} \) corresponding to a situation where a majority of floaters systematically flips to the other opinion creating an on going alternative shifting between \( A \) and \( B \). The proportion of floaters \( A \) and \( B \) being \( (1-a-b) \), the proportion of contrarians among all agents is given by \( \tilde{c} = (1-a-b)c \).

At time \( t \) once a scheme is completed with \( p_t \rightarrow p_{t+1} \), another one is performed to get \( p_{t+1} \rightarrow p_{t+2} \) and so forth till the debate ends with an eventual vote at a given time, which determines the total number of updates to be implemented. The frequency of updates is a function of the intensity of the on going debate. For any real situation the debate initial proportion \( p_0 \) is evaluated using polls.

Let us now evaluate the function \( P_{a,b,c,k}^{(r)} \) yielding \( p_{t+1} \) from \( p_t \), given the values of \( (a, b, c, k) \) with,

\[
p_{t+1} = P_{a,b,c,k}^{(r)}(p_t).
\]

However, since the contrarian phenomenon is mathematically identical to a random flipping of floaters with probability \( c \) among reshuffled agents, we can decoupled the \( c \) effect from the local update writing,

\[
p_{t+1} = (1-c)\left[P_{a,b,k}^{(r)}(p_t) - a\right] + c\left[1 - P_{a,b,k}^{(r)}(p_t) - b\right].
\]

which simplifies as,

\[
p_{t+1} = (1 - 2c)P_{a,b,k}^{(r)}(p_t) + c(1 + a - b).
\]

To evaluate \( P_{a,b,k}^{(r)} \) we note that in a given configuration of a group of size \( r \) with \( (r - \mu) \) opinion \( A \) and \( \mu \) agents holding opinion \( B \), the contributions to \( P_{a,b,k}^{(r)} \) result from two different families. One family includes contributions with a majority of \( A \) and equality of \( A \) and \( B \) (for even size at a tie with probability \( k \)) while the other family corresponds to contributions with a minority of \( A \) and equality of \( A \) and \( B \) (for even size at a tie with probability \( 1 - k \)). These families are denoted respectively \( P_{a,b,c}^{(r,\mu)} \) and \( Q_{a,b,c}^{(r,\mu)} \) under the constraint \( \mu \leq \left\lfloor \frac{r}{2} \right\rfloor \) where \( \lfloor x \rfloor \) is the integer part of \( x \). Eq. (1) can be rewritten as,

\[
P_{a,b,k}^{(r)}(p_t) = \sum_{\mu=0}^{\left\lfloor \frac{r}{2} \right\rfloor} \left\{ K_{r,k}^{(r,\mu)} P_{a,b}^{(r,\mu)}(p_t) + K_{1-k}^{(r,\mu)} Q_{a,b}^{(r,\mu)}(p_t) \right\}
\]

with,

\[
K_{r,k}^{(r,\mu)} = 1 + (k - 1)\delta_{r,2\mu} \quad = 1 \quad (r \neq 2\mu) \\
= k \quad (r = 2\mu),
\]

which allows to factorize the \( k \) dependence.

The quantity \( P_{a,b}^{(r,\mu)}(p) \), which accounts for the contribution to \( A \) after one update from configurations with \( (r - \mu) \) agents with opinion \( A \) and \( \mu \) agents with opinion \( B \), can be decomposed as the sum of contributions with \( r \) \( A \)-inflexibles and \( r \) \( B \)-inflexibles, where integer \( r \) runs from 0 to \( (r - \mu) \) and \( \beta \) from 0 to \( \mu \) giving (See Fig.3 left),

\[
P_{a,b}^{(r,\mu)}(p) = \sum_{\mu=0}^{r-\mu} \sum_{\beta=0}^\mu P_{a,b,a,\beta}^{(r,\mu)}(p)
\]
Since \( \mu \leq \frac{r}{2} \) (A majority or tie), contributions to \( p_{t+1} \) from Eq. (6) come from all agents except \( B \)-inflexibles minus contrarians after the update yielding,

\[
p^{(r,\mu)}_{a,b,a,b}(p) = \frac{r}{\mu} \left( \frac{r - \mu}{\mu} \right) (1 - p - b)^{r-\mu}(p - \alpha)^{r-\mu-\alpha} b^0 a^\alpha \times \left( \frac{r - \mu}{r} \right).
\]  

(7)

Similarly, the quantity \( Q^{(r,\mu)}_{a,b}(p) \), which accounts for the contribution to \( A \) from configurations with \( \mu \) opinion \( A \) and \( (r - \mu) \) \( B \), can be decomposed as the sum of contributions with \( \alpha = 0, ..., \mu \) \( A \)-inflexibles and \( \beta = 0, ..., (r - \mu) \) \( B \)-inflexibles giving (See Fig. 1 right),

\[
Q^{(r,\mu)}_{a,b}(p) = \sum_{\alpha=0}^{\mu} \sum_{\beta=0}^{r-\mu} Q^{(r,\mu)}_{a,b,a,b}(p)
\]  

(8)

Since \( \mu \leq \frac{r}{2} \) (A minority or tie), contributions to \( p_{t+1} \) from Eq. (8) come only from \( A \)-inflexibles after the update yielding,

\[
Q^{(r,\mu)}_{a,b,a,b}(p) = \frac{r}{\mu} \left( \frac{r - \mu}{\mu} \right) (1 - p - b)^{r-\mu}(p - \alpha)^{r-\mu-\alpha} b^0 a^\alpha \times \left( \frac{\alpha}{r} \right).
\]  

(9)

With the use of the formulae \((x + y)^n = \sum_{j=0}^{n} \binom{n}{j} x^{n-j} y^j \) and \( n(y + x)^n-1 = \sum_{j=0}^{n} \binom{n}{j} x^{n-j} y^j \), the summation over \( \alpha \) and \( \beta \) leads us to,

\[
p^{(r,\mu)}_{a,b}(p) = \frac{r}{\mu} p^{r-\mu}(1 - p)^\mu \left[ 1 - \frac{\mu}{r(1 - p)b} \right].
\]  

(10)

and

\[
Q^{(r,\mu)}_{a,b}(p) = \frac{r}{\mu} p^{r}(1 - p)^{r-\mu} \left[ \frac{\mu}{rp} \right].
\]  

(11)

With the use of identities

\[
\sum_{\mu=0}^{[\frac{r}{2}]} K^{(r,\mu)}_{1-k} \left( \frac{r}{\mu} \right) p^{\mu}(1 - p)^{r-\mu} + \sum_{\mu=0}^{[\frac{r}{2}]} K^{(r,\mu)}_{k} \left( \frac{r}{\mu} \right) p^{r-\mu}(1 - p)^\mu = (p + 1 - p)^r = 1,
\]  

(12)
and
\[
\sum_{\mu=0}^{[\frac{r}{\mu}]} K^{(\mu)}_k \left( \frac{r}{\mu} \right) p^\mu (1 - p)^{r-\mu} + \sum_{\mu=0}^{[\frac{r}{\mu}]} K^{(\mu)}_{1-k} \left( \frac{r}{\mu} \right) \frac{\mu}{rp} p^{-\mu} (1 - p)^{r-\mu}
\]
\[
= \frac{1}{rp} \sum_{\mu=0}^{[\frac{r}{\mu}]} \partial_\eta (\eta p + 1 - p)^r \bigg|_{\eta=1} = (p + 1 - p)^r = 1,
\]
we arrive at the compact expression of the evolution function \( P^{(r)}_{a,b,k} \) for general group size \( r \) in the form,
\[
P^{(r)}_{a,b,k}(p) = \Pi_0^{(r)}(p,k) + a \Pi_1^{(r)}(p,k) - b \Pi_2^{(r)}(p,k),
\]
where
\[
\Pi_0^{(r)}(p,k) = \sum_{\mu=0}^{[\frac{r}{\mu}]} \left( \frac{r}{\mu} \right) K^{(\mu)}_k \left( \frac{r}{\mu} \right) p^\mu (1 - p)^{r-\mu}
\]
\[
\Pi_1^{(r)}(p,k) = \sum_{\mu=0}^{[\frac{r}{\mu}]} \left( \frac{r}{\mu} \right) K^{(\mu)}_{1-k} \left( \frac{r}{\mu} \right) \frac{\mu}{rp} p^{-\mu} (1 - p)^{r-\mu}
\]
\[
\Pi_2^{(r)}(p,k) = \sum_{\mu=0}^{[\frac{r}{\mu}]} \left( \frac{r}{\mu} \right) K^{(\mu)}_k \left( \frac{r}{\mu} \right) \frac{\mu}{r(1 - p)} p^{-\mu} (1 - p)^{r-\mu}.
\]
Note the relation \( \Pi_2^{(r)}(p,k) = \Pi_1^{(r)}(p,k) - 1 \). Also note the relation
\[
P^{(r)}_{a,b}(p) + Q^{(r)}_{b,a}(1 - p) = \left( \frac{r}{\mu} \right) p^{-\mu} (1 - p)^{r-\mu}.
\]
which guarantees the covariance of \( P^{(r)}_{a,b,k}(p) \) with respect to the renaming of opinions A and B,
\[
P^{(r)}_{a,b,k}(p) + P^{(r)}_{b,a,k}(1 - p) = 1.
\]
At this stage to get the complete update Equation we use Eq. \( (19) \) to get,
\[
P^{(r)}_{a,b,c,k}(p) = (1 - 2c) \left[ \Pi_0^{(r)}(p,k) + a \Pi_1^{(r)}(p,k) - b \Pi_2^{(r)}(p,k) \right] + c(1 + a - b).
\]
which allows calculating the opinion dynamics for any set of parameters \( (a,b,c,k) \) given any group size \( r \) for any initial support \( p_0 \) for opinion A. We thus have calculated a universal formula \( c \) to predict opinion dynamics in a parameter space of six dimensions.

We now go one step further to make our formula further realistic by considering a distribution of size with \( r = 1, 2, ..., L \) where \( L \) is the larger update size. Usually, in most social situations \( L \) is around 4, 5 or 6. The size weight \( w_r \) of groups of size \( r \) has to be evaluated from observation under the constraint \( \sum_{r=1}^{L} w_r = 1 \). Including the size \( r = 1 \) allows to consider agents who do not take part in local discussions during each update. Then, Eq. \( (20) \) becomes,
\[
P^{(r)}_{a,b,c,k}(p) = (1 - 2c) \sum_{r=1}^{L} w_r \left[ \Pi_0^{(r)}(p,k) + a \Pi_1^{(r)}(p,k) - b \Pi_2^{(r)}(p,k) \right] + c(1 + a - b).
\]
3. Examples of evolution equation with several values of \( r \)

To make our universal update equation more concrete we list explicit expressions for some selective values \( r = 3, 4, 5, 6, 7, 8, 9, 10, 11 \). For \( r = 3 \)

\[
\begin{align*}
\Pi_0^{(3)}(p,k) &= p^2(3 - 2p) \\
\Pi_1^{(3)}(p,k) &= (1 - p)^2 \\
\Pi_2^{(3)}(p,k) &= p^2, 
\end{align*}
\]

\( r = 4 \)

\[
\begin{align*}
\Pi_0^{(4)}(p,k) &= p^4(4 - 3p) + 6kp^2(1 - p)^2 \\
\Pi_1^{(4)}(p,k) &= (1 - p)^4(1 + 2p) - 3kp(1 - p)^2 \\
\Pi_2^{(4)}(p,k) &= p^3 + 3kp^2(1 - p), 
\end{align*}
\]

\( r = 5 \)

\[
\begin{align*}
\Pi_0^{(5)}(p,k) &= p^5(10 - 15p + 6p^2) \\
\Pi_1^{(5)}(p,k) &= (1 - p)^5(1 + 3p) \\
\Pi_2^{(5)}(p,k) &= p^3(4 - 3p), 
\end{align*}
\]

\( r = 6 \)

\[
\begin{align*}
\Pi_0^{(6)}(p,k) &= p^6(14 - 24p + 10p^2) + 20kp^3(1 - p)^3 \\
\Pi_1^{(6)}(p,k) &= (1 - p)^6(1 + 3p + 6p^2) - 10kp^3(1 - p)^3 \\
\Pi_2^{(6)}(p,k) &= p^4(5 - 4p) + 10kp^3(1 - p)^3, 
\end{align*}
\]

\( r = 7 \)

\[
\begin{align*}
\Pi_0^{(7)}(p,k) &= p^7(35 - 84p + 70p^2 - 20p^3) \\
\Pi_1^{(7)}(p,k) &= (1 - p)^7(1 + 4p + 10p^2) \\
\Pi_2^{(7)}(p,k) &= p^6(15 - 24p + 18p^2), 
\end{align*}
\]

\( r = 8 \)

\[
\begin{align*}
\Pi_0^{(8)}(p,k) &= p^8(56 - 140p + 120p^2 - 35p^3) + 70kp^4(1 - p)^4 \\
\Pi_1^{(8)}(p,k) &= (1 - p)^8(1 + 4p + 10p^2 + 20p^3) - 35kp^4(1 - p)^4 \\
\Pi_2^{(8)}(p,k) &= p^7(21 - 35p + 15p^2) + 35kp^4(1 - p)^4, 
\end{align*}
\]

\( r = 9 \)

\[
\begin{align*}
\Pi_0^{(9)}(p,k) &= p^9(126 - 420p + 540p^2 - 315p^3 + 70p^4) \\
\Pi_1^{(9)}(p,k) &= (1 - p)^9(1 + 5p + 15p^2 + 35p^3) \\
\Pi_2^{(9)}(p,k) &= p^8(56 - 140p + 120p^2 - 35p^3), 
\end{align*}
\]

\( r = 10 \)

\[
\begin{align*}
\Pi_0^{(10)}(p,k) &= p^{10}(210 - 720p + 945p^2 - 560p^3 + 126p^4) + 252kp^5(1 - p)^5 \\
\Pi_1^{(10)}(p,k) &= (1 - p)^{10}(1 + 5p + 15p^2 + 35p^3 + 70p^4) - 126kp^5(1 - p)^5 \\
\Pi_2^{(10)}(p,k) &= p^9(84 - 216p + 189p^2 - 56p^3) + 126kp^5(1 - p)^5, 
\end{align*}
\]
\( r = 11 \)

\[
\Pi_0^{(11)}(p,k) = p^6(462 - 1980p + 3465p^2 - 3080p^3 + 1386p^4 - 252p^5)
\]

\[
\Pi_1^{(11)}(p,k) = (1 - p)^3(1 + 6p + 21p^2 + 56p^3 + 126p^4)
\]

\[
\Pi_2^{(11)}(p,k) = p^6(210 - 720p + 945p^2 - 560p^3 + 126p^4).
\] (30)

4. Fixed points of the dynamics

The dynamics implemented by repeated applications of Eq. (20) exhibits a rather large spectrum of different scenarios within the six dimensional space spanned by \( 0 \leq a \leq p \leq 1, 0 \leq b \leq 1 - p \leq 1, 0 \leq c \leq 1, a + b + c \leq 1, 0 \leq k \leq 1 \) and \( 1 \leq r \leq \infty \). The phase diagram landscape is shaped by the various attractors and tipping surfaces, which are solutions of the fixed point equation,

\[
p^* = P^{(r)}_{a,b,c,d}(p^*).
\] (31)

It is of interest to mention that Eq. (31) is a polynomial of degree \( r \) in \( p \) and thus exhibits \( r \) solutions of which no more than three are real and contained within the \( 0 - 1 \) range. This assessment results from playing with the equation and hand waving arguments but a mathematical proof is still on hold.

In case of three fixed points \( p_0^*, p_1^*, p_1^* \), in ascending order. The smallest \( p_0^* \leq \frac{1}{2} \) is stable and represents \( B \) majority final state, while the largest \( p_0^* \geq \frac{1}{2} \), also stable, represents \( A \) majority final state. The medium valued \( p_1^* \) is unstable, and acts as a separator of the basins of attraction to \( p_0^* \) and \( p_1^* \), it is a tipping point of the dynamics.

![Figure 2: Examples of the evolution function \( P^{(r)}_{a,b,c,d}(p) \) around critical value \( a = a_r \) for \( b = c = 0 \). The group size is chosen to be \( r = 7 \).](image)

![Figure 3: Examples of the evolution function \( P^{(r)}_{a,b,c,d}(p) \) around triple critical value \( a = b = a_r \) for \( c = 0 \). The group size is chosen to be \( r = 7 \).](image)

Although making a visual representation is impossible in six dimensions, being interested in the evolution of an initial value \( p_0 \) given fixed values of \( (a, b, c, k, r) \), the operative use of the phase diagram is to select two-dimensional slices showing the evolution of \( p_0 \) as a function of repeated updates with \( p_0 \to p_1 \to \ldots \to p_n \) where \( n \) is the number of iterations. With respect to prediction about a real event, what matters is to determine if \( p_n > \frac{1}{2} \) (opinion \( A \) victory), \( p_n < \frac{1}{2} \) (opinion \( A \) failure) or \( p_n \approx \frac{1}{2} \) (hung outcome).
An alternative practical use of the phase diagram is to extract the two dimensional slices, which shows the function $p_{t+1} = P_{a,b,c,k}(p_t)$ for a fixed set of values $(a, b, c, k, r)$. These curves display the eventual attractors and tipping points underlying the dynamics from which predictions can be made. These points are located at the crossing of $p_{t+1} = P_{a,b,c,k}(p_t)$ and the diagonal $p_{t+1} = p_t$.

Indeed all "slices" share the common property of having at least one single attractor for the dynamics. In addition, series of slices exhibits one additional attractor and a tipping point located between the two attractors. For those cases, varying some of the parameters $(a, b, c, k)$ may lead either to have one attractor and the tipping point to coalesce at critical values yielding then a single attractor dynamics. The other scheme is having the two attractors to merge at the tipping point to produce another single attractor dynamics. To have a single attractor dynamics implies one identified opinion is certain to win whatever its initial support is. That supposes the debate or the campaign duration lasts enough time to cross the $\frac{1}{2}$, i.e., 50% of the ballots for an election. Fig. 2 and Fig. 3 shows a series of illustrating cases of above two scenarios.

5. Exploring the phase diagram

From Eq. (31) when $b = c = 0$, increasing the parameter $a$ makes $p_0^*$ and $p_s^*$ merge into single value at $a = a_c$, and then disappear to make $p_1^*$ the sole final state of the system (See Fig. 4). The critical value $a$ for $b = 0$, which we call $a_c(c)$, is obtained from

$$p^* - P_{a(c),b,c,k}(p^*) = 0$$

$$1 - \partial p^* P_{a(c),b,c,k}(p^*) = 0$$

In particular, for $c = 0$, the critical value $a_c(0)$, which we simply call $a_c$, and its associated $p_c^*$ are obtained from

$$\Pi_0^{(d)}(p_c^*, k)\Pi_1^{(d)}(p_c^*, k) - \Pi_0^{(d)}(p_c^*, k)\Pi_1^{(d)}(p_c^*, k) + \Pi_0^{(d)}(p_c^*, k) - \Pi_0^{(d)}(p_c^*, k) = 0$$

$$a_c = \frac{p_c^* - \Pi_0^{(d)}(p_c^*, k)}{\Pi_1^{(d)}(p_c^*, k)}$$

With small but non-zero $b$, the system goes through transition between two stable fixed point phase and single fixed point phase, when $a$ is varied, at a critical value higher than $a_c$. Above certain value of $b$, the system goes from

![Figure 4: An example of the phase diagram of the parameter space $(a, b)$. The group size here is chosen to be $r = 9$. The regions marked as $A$, $B$, and $A/B$ represent parameter values with which the system converges unconditionally to $A$ majority, unconditionally to $B$ majority, and either $A$ or $B$ majority depending on the initial configuration, respectively.](image)
$B$ majority single fixed-point phase to $A$ majority single fixed-point phase without passing through three fixed-point phase (See Fig. 4). The transition is characterized by triple critical point $a = b = a_t(c)$

$$p_t^* = F_{a(c),a(c),c}^{(r)}(p_t^*) = 0$$  \hspace{2cm} (36)$$

$$1 - \partial_p F_{a(c),a(c),c}^{(r)}(p_t^*) = 0$$  \hspace{2cm} (37)$$

limiting ourselves to $c = 0$ for now, the triple point $a_t = a_t(0)$ and its associated $p_t^*$ are obtained as $p_t^* = \frac{1}{2}$ and

$$a_t = \frac{1 - \Pi_0^{(r)}(\frac{1}{2},k)}{2\Pi_1^{(r)}(\frac{1}{2},k)}$$  \hspace{2cm} (38)$$

It is instructive to draw the phase diagram on $\{a, b\}$ plane. There is a region of small $a$ and $b$ in which final majority can go either way depending on their initial support. For large value of $a$ and/or $b$, final majority is predetermined due to the strong influence of inflexibles. When $a$ or $b$ exceeds $a_t$, two inflexible-dominated regions are placed next to each other without intermediate region of floater-determinability. An example of $r = 9$ is shown in Fig. 4.

The $r$-dependence of $a_c$ and $a_t$ can be seen in Fig. 5. We list just some of them: For $r = 3$, we have $a_c = 0.1714$ and $a_t = 0.25$, and for $r = 5$, $a_c = 0.2104$ and $a_t = 0.2917$. For $r = 7$, we have $a_c = 0.2358$ and $a_t = 0.3167$, and for $r = 9$, $a_c = 0.2452$ and $a_t = 0.3339$.

Figure 5: Critical parameters $a_c$ and $a_t$ as functions of group size $r$

Figure 6: Phase diagram on $\{a, b\}$ plane for $r = 9$. Left for $c = 0.1$ and right for $c = 0.2$. The regions marked as $A$, $B$, and $A/B$ represent parameter values with which the system converges unconditionally to $A$ majority, unconditionally to $B$ majority, and either $A$ or $B$ majority depending on the initial configuration, respectively.
Overall, approach to $r = \infty$ limit, $a_r = a_t = 0.5$ is very slow, and for reasonably small group size, $r \approx 10$, $a_r \approx \frac{1}{4}$ and $a_t \approx \frac{1}{2}$ holds, which are closer to $r = 3$ case than $r = \infty$. Even at $r \approx 50$, we have surprisingly small $a_r \approx \frac{1}{3}$ and $a_t \approx \frac{1}{2}$.

It is again instructive to look at the phase diagram on $\{a, b\}$ plane with different values of $c$ (Fig. 6). It should be now very clear, that, for all group size $r$, the effect of contrarians to the final majority formation is to decrease the role of floaters and increase the power of inflexibles. This fact, which has originally been found in $r = 3$ example, turns out to be a generic feature of the Galam model.

6. Summary

In this paper, we have obtained a universal mathematical formula for the temporal evolution of agents following the Galam opinion dynamics with arbitrary group size updates in a parameter space of six dimensions. These dimensions include the respective proportions of inflexibles (stubbornness), the proportion of contrarians and the mean value of shared prejudices.

The formula has allowed the opinion landscape building, which is found to be sparsed with a series of attractors and tipping points. The associated critical parameters has revealed the existence of a rich variety of nonlinearities and singularities in the various opinion flows. Sudden upheavals like minority spreading and majority collapse are thus given a rationale unveiling the hidden mechanisms behind the occurrence of unexpected and sudden shifts in the distribution of opinions like with the current French "gilets jaunes" movement, the Arab Spring as well as the Brexit victory and Trump election. The time dependence of the corresponding phenomena is also exhibited.

The analysis on the formula fixed points and critical parameters reveals that the approach to large group size, where the effects of inflexible agents disappear and simple majority rule holds, is very slow. This guarantees that the insights obtained from group size three remain intact for larger group sizes. For even sizes although the prejudice effect weakens with increasing size, it also stays effective for large group sizes.

The phase diagram on parameter space have been studied for one specific case showing the existence of crucial critical points $a_c$ and $a_t$ which corner and branch out the lines separating the inflexibles-dominant and floaters-determinable phases. It is pointed out that the admixture and increase of contrarians effectively assist the dominance of inflexibles, and reduce the parameter region in which initial composition of floaters can determine the outcome of majority opinion formation.

It is important to reaffirm that the increase of the group size $r$ induces modest but clear increase of both $a_c$ and $a_t$. This implies that, when the setting of discussion gets more “democratic”, a larger number of determined minority is necessary to impose its will on the majority, or identically, a larger number of enlightened minority is necessary for its persuasion of the mass.

Getting the complete phase diagram of the dynamics of opinions yields a robust forecasting frame. In addition, the predictions are robust since what matters is no longer the precise estimates of the parameters, but the determination of basin of attraction in which the dynamics is taking place thus providing substantial flexibility in the parameters evaluation. Identifying the corresponding relevant attractor allows envisioning unexpected opinion dynamics trends ahead of their occurrences.

To conclude, we have obtained a ready-to-use universal formula to make predictions for practically any situations for which aggregation of individual choices is a decisive input. To forecast an event, the requirement is a fair estimate of the parameters value $(a, b, c, k, r)$ and a knowledge of the initial value $p_0$, which is readily obtained from polls. Our results provide a new ground to predict opinion dynamics outcomes for a large spectrum of applications including voting outcomes, market shares and societal trends.

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