KÄHLER FINSLER METRICS
AND CONFORMAL DEFORMATIONS*

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ABSTRACT
The conformal properties of complex Finsler metrics are studied. We first give a characterization of a compact complex Finsler manifold to be globally conformal Kähler. By considering the total holomorphic curvature and total Ricci curvature in the volume preserved conformal classes, we then study the variational properties of Kähler Finsler metrics. By studying the spectral properties of two average metrics, the stabilities of critical Kähler Finsler metrics are verified. Finally, a Yamabe type problem for mean holomorphic Ricci curvature is considered, and a partial existence result is obtained.

1. Introduction
Searching for the notion of the “best” metric on a manifold is a central problem in geometry. In the Riemannian realm, the canonical ones are Yamabe metrics, Einstein metrics, etc. In complex geometry, one is led to extremal metrics, Kähler Einstein metrics, etc. During the past decades, there is a bundle of results on the “best” Finsler metrics, such as Einstein Finsler metrics, Yamabe Finsler metrics, etc. (cf. [1, 3, 7] and references therein). Complex Finsler metrics are a natural generalization of Hermitian metrics. Since the most often used intrinsic (depending only on the complex structure) metrics are generally Finsler ones (such as the Kobayashi metric and Carathéodory metric), it is one hot issue to develop the theory of complex Finsler geometry. In this paper, we will study some canonical complex Finsler metrics in a conformal class. The manifolds considered in this paper are of the complex dimension $n \geq 2$.

The concept of Kähler Finsler metrics is introduced by M. Abate and G. Patrizio in [1]. The global properties of Kähler Finsler spaces are well studied. The Hodge decomposition theorem is proved by C. Zhong and T. Zhong [19]. Later, J. Han and the second author study the existence of harmonic maps [10]. Recently, S. Yin and X. Zhang obtained the comparison theorems in [18], and J. Li and C. Qiu study Wu’s theorem on a strongly convex Kähler Finsler manifold in [15].
The first goal of this paper is to study the existence of Kähler Finsler metrics in a conformal class. Let $M$ be an $n$-dimensional compact complex space with a complex Finsler metric $G$, whose volume preserved conformal class is denoted by $[G]$. It is natural to ask whether there exists a Kähler Finsler metric in $[G]$. The uniqueness is easy to obtain.

**Theorem 1.1:** In the volume preserved conformal class $[G]$, there exists at most one Kähler Finsler metric.

In order to get the existence of Kähler Finsler metrics in $[G]$, we should work on Kähler Finsler manifolds. A manifold $M$ is called a **Kähler Finsler manifold** if it admits a Kähler Finsler metric.

**Theorem 1.2:** Let $M$ be a compact Kähler Finsler manifold, and $G$ be an arbitrary complex Finsler metric (not necessarily Kählerian) on $M$. Then, there exists a Kähler Finsler metric in $[G]$ if and only if the horizontal torsion of $G$ is reducible and the real part of its mean horizontal torsion is closed.

The second goal of this paper is to understand the curvature behavior of a Kähler Finsler metric in its conformal class. Applying the integration along the fiber of the projectivized tangent bundle over $M$, we introduce the mean holomorphic curvature $\kappa = \kappa(z)$ (see (5.18)) and the mean holomorphic Ricci curvature $\rho = \rho(z)$ (see (6.8)). By considering the following two total curvature functionals

\[
K(G) = \int_M \kappa \, d\mu_M, \quad R(G) = \int_M \rho \, d\mu_M,
\]

we obtain the following result.

**Theorem 1.3:** Let $G$ be a Kähler Finsler metric on a compact complex manifold.

(i) $G$ is a critical point of $K$ in $[G]$ if and only if $\kappa = \text{const}$. Moreover, $G$ is stable if and only if $\kappa \leq \lambda_1^K$.

(ii) $G$ is a critical point of $R$ in $[G]$ if and only if $\rho = \text{const}$. Moreover, $G$ is stable if and only if $\rho \leq \lambda_1^g$. 

Here $\lambda_1^h$ and $\lambda_1^g$ are the first eigenvalues of the weighted Laplacian of the metric measure spaces $(M, h, d\mu_M)$ and $(M, g, d\mu_M)$ respectively, where the induced metrics $h$ and $g$ are given by (5.29) and (6.14). We shall remark that the total holomorphic curvature was first considered by J. Bland and M. Kalka and the variation formula was obtained in [4].

A Kähler Finsler metric is said to be Einstein if its holomorphic Ricci curvature is constant. One can immediately get the following corollary.

**Corollary 1.4:** On a compact complex manifold, a Kähler Einstein Finsler metric with non-positive holomorphic Ricci curvature is a stable critical point of $\mathcal{R}$ in its volume preserved conformal class.

It is noteworthy that, in [8], H. Feng, K. Liu and X. Wan study the Finsler-Einstein bundles over a Kähler manifold by introducing a Donaldson type functional.

The last goal of this paper is to consider a Yamabe type problem. For a complex Finsler metric which is not necessarily Kählerian, the $\vartheta$-mean holomorphic Ricci curvature $\rho_\vartheta$ is introduced (see (6.9)). We then study the existence of conformal metrics with constant $\rho_\vartheta$. In the real Finsler geometry, a similar problem is considered in [7] for “C-convex” metrics. It is interesting that the C-convexity is not needed in the complex realm. Precisely, by introducing the conformal invariants $Y(G)$ and $C(G)$ (see (7.4) and (7.11) respectively), we prove the following existence theorem.

**Theorem 1.5:** Let $(M, G)$ be a compact complex Finsler manifold with complex dimension $n$. It always holds that

$$Y(G) \cdot C(G) \leq \frac{\sigma_{2n}}{2n - 2}$$

where $\sigma_{2n}$ is the best Sobolev constant. If $Y(G) \cdot C(G) < \frac{\sigma_{2n}}{2n - 2}$, then there exists a metric with constant $\rho_\vartheta$ in the conformal class $[G]$.

The contents of this paper are arranged as follows. In §2, we give a brief overview of complex Finsler metrics and the Kähler condition. In §3, we introduce the integration along the fiber of the projectivized tangent bundle. In §4, the notions of locally conformal Kähler and globally conformal Kähler are given, and Theorems 1.1 and 1.2 are proved. In §5, we consider the functional $\mathcal{K}$ and obtain the first part of Theorem 1.3. In §6, the functional $\mathcal{R}$ is studied and the second part of Theorem 1.3 is obtained. In the last section, the Yamabe type problem is considered and Theorem 1.5 is verified.
2. Complex Finsler metrics

Let $M$ be a complex manifold with $\dim_{\mathbb{C}} M = n$, and $T'M$ be the holomorphic tangent bundle. The points of $T'M$ will be denoted by $(z, v)$ where $v = v^i \partial/\partial z^i \in T^i_z M$, and thus $(z^i; v^i)$ forms a local holomorphic coordinate system of $T'M$. Let us denote the slit holomorphic tangent bundle $T'M \setminus \{0\}$ by $\tilde{M}$.

A complex Finsler metric on $M$ is a continuous function $G : T'M \to [0, +\infty)$ satisfying

(I) $G(z, v) \geq 0$, where the equality holds if and only if $v = 0$;

(II) $G(z, v) \in C^\infty(\tilde{M})$;

(III) $G(z, \lambda v) = \lambda \overline{\lambda} G(z, v)$ for $\lambda \in \mathbb{C}^* = \mathbb{C} \setminus \{0\}$;

(IV) the Levi matrix $(G_{ij})_{n \times n} := (\frac{\partial^2 G}{\partial z^i \partial \overline{z}^j})_{n \times n}$ is positive definite on $\tilde{M}$.

The last condition is called the strongly pseudo-convexity of $G$. The pair $(M, G)$ is called a complex Finsler manifold. Throughout this paper, all the manifolds are connected with dimension $n \geq 2$, and assumed to be compact while the integrals are taken.

By putting

\[ N^i_j = G^{jk}_{ik} \partial \overline{k} \partial_j G \]

where $(G^{jk}_{ik})_{n \times n} = (G_{ij})_{n \times n}^{-1}$ and

\[ \partial_i := \frac{\partial}{\partial z^i}, \quad \partial_j := \frac{\partial}{\partial \overline{z}^j}, \quad \partial_i := \frac{\partial}{\partial v^i}, \quad \partial_j := \frac{\partial}{\partial \overline{v}^j}, \]

the horizontal vectors and vertical covectors can be defined by

\[ \delta_i = \frac{\delta}{\delta z^i} := \partial_i - N^k_i \partial_k, \quad \delta v^i := dv^i + N_k^i dz^k. \]

The complexified (co)tangent bundle has the following horizontal and vertical decomposition:

\[ T_C \tilde{M} = \mathcal{H} \oplus \overline{\mathcal{H}} \oplus \mathcal{V} \oplus \overline{\mathcal{V}}, \quad T_C^* \tilde{M} = \mathcal{H}^* \oplus \overline{\mathcal{H}}^* \oplus \mathcal{V}^* \oplus \overline{\mathcal{V}}^* \]

where $\mathcal{H} = \text{span}\{\delta_i\}$, $\mathcal{V} = \text{span}\{\delta_{\overline{z}^i}\}$, $\mathcal{H}^* = \text{span}\{dz^i\}$ and $\mathcal{V}^* = \text{span}\{\delta v^i\}$.

Therefore, the operators $\partial$, $\overline{\partial}$ and $d$ on $\tilde{M}$ can be decomposed into

\[ \partial = \partial_{\mathcal{H}} + \partial_{\mathcal{V}}, \quad \overline{\partial} = \overline{\partial}_{\mathcal{H}} + \overline{\partial}_{\mathcal{V}}, \quad d = d_{\mathcal{H}} + d_{\mathcal{V}} = (\partial_{\mathcal{H}} + \overline{\partial}_{\mathcal{H}}) + (\partial_{\mathcal{V}} + \overline{\partial}_{\mathcal{V}}). \]

The collection of smooth sections of $(\wedge^p \mathcal{H}^*) \wedge (\wedge^q \overline{\mathcal{H}}^*) \wedge (\wedge^r \mathcal{V}^*) \wedge (\wedge^s \overline{\mathcal{V}}^*)$ is denoted by $A^{p, q; r, s}(\tilde{M})$, and each element of $A^{p, q; r, s}(\tilde{M})$ is called a $(p, q; r, s)$-form of $\tilde{M}$. The elements in $A^{p, q; 0, 0}(\tilde{M})$ are called horizontal $(p, q)$-forms.
The space of \((l,m)\)-forms is clearly

\[
A^{l,m}(\tilde{M}) = \bigoplus_{p+r=l, q+s=m} A^{p,q;r,s}(\tilde{M}).
\]

The **Kähler form** (fundamental form) of a complex Finsler metric \(G\) is a horizontal \((1,1)\)-form defined by

\[
\omega_{\mathcal{H}} = \sqrt{-1} G_{ij}^j (z,v) dz^i \wedge d\bar{z}^j.
\]

For a Hermitian metric, \(\omega_{\mathcal{H}} = \sqrt{-1} G_{ij}^j (z) dz^i \wedge d\bar{z}^j\) is independent of \(v\) and is a \((1,1)\)-form living on the base manifold \(M\). Generally, \(\omega_{\mathcal{H}}\) lives on \(\tilde{M}\).

**Definition 2.1** ([1, 6]): A complex Finsler metric \(G\) is said to be **Kähler** if and only if \(d_{\mathcal{H}} \omega_{\mathcal{H}} = 0\). In this case, \(G\) is called a **Kähler Finsler metric**.

The Kähler condition is equivalent to the symmetricity of the Chern–Finsler connection. In fact, equipping the vertical bundle \(V\) with an inner product \(G\) where

\[
G(X,Y) = X_i \bar{Y}^j G_{ij}^j (z,v)
\]

for any \(X,Y \in V_{(z,v)}\), the Chern–Finsler connection is just the Hermitian connection of the Hermitian bundle \((V,G)\), and thus the connection 1-forms \((\omega^i_j)\) can be written as

\[
\omega^i_j = G^{ki} \partial G_{jk} = \Gamma_{j,k}^i dz^k + C_{jk}^i \delta v^k,
\]

where

\[
\Gamma_{j,k}^i = G^{i\dot{k}} \delta \bar{k} G_{j\bar{l}}, \quad C_{jk}^i = G^{i\dot{k}} \dot{k} \bar{G}_{j\bar{l}}.
\]

The **horizontal torsion** is defined by

\[
\theta = \theta_{ki}^m dz^k \wedge dz^i \otimes \delta_m = (\Gamma_{i,k}^m - \Gamma_{k,i}^m) dz^k \wedge dz^i \otimes \delta_m.
\]

We call \(\vartheta = \vartheta_{k} dz^k = \theta_{km} dz^k\) the **mean horizontal torsion**.

A direct computation gives

\[
\partial_{\mathcal{H}} \omega_{\mathcal{H}} = \frac{\sqrt{-1}}{2} (\Gamma_{i,k}^m - \Gamma_{k,i}^m) G_{m\bar{j}} dz^k \wedge dz^i \wedge d\bar{z}^j.
\]

**Lemma 2.1**: A complex Finsler metric is Kähler if and only if \(\theta = 0\), i.e.,

\[
\Gamma_{i,k}^m = \Gamma_{k,i}^m.
\]
3. Integrations on the projectivized bundle

In this section, we will introduce several notions of integration on the projectivized tangent bundle $\pi : P(\tilde{M}) \to M$ where $P(\tilde{M}) := \tilde{M}/\mathbb{C}^*$, of which each fiber is biholomorphic to $\mathbb{CP}^{n-1}$. The complexified bundles $T_C(P(\tilde{M}))$ and $T^*_C(P(\tilde{M}))$ also have the horizontal and vertical decomposition as (2.4). We shall adopt the same notion $H$, $V$ etc., though the vertical sub-bundle is $(n-1)$-dimensional in this case. The notations $A_{p,q}^{r,s}(P(\tilde{M}))$ and $A_{l,m}^n(\tilde{M})$ have similar definitions with $A_{p,q}^{r,s}(\tilde{M})$ and $A_{l,m}^n(\tilde{M})$ respectively.

Being aware of $G_{i\bar{j}}(z, \lambda v) = G_{i\bar{j}}(z, v)$, the Kähler form $\omega_H$ actually lives on $P(\tilde{M})$. We have another $(1,1)$-form $\sqrt{-1}\partial\bar{\partial}\log G$ which has no mixed part. Considering $v$ as the homogeneous coordinate of $P(\tilde{M})$, it turns out that

$$(3.1) \quad \sqrt{-1}\partial\bar{\partial}\log G = \omega_V - \Theta$$

where

$$(3.2) \quad \omega_V = \sqrt{-1}(\log G)_{i\bar{j}} \delta v^i \wedge \delta \bar{v}^j, \quad (\log G)_{i\bar{j}} = \hat{\partial}_i \hat{\partial}_{\bar{j}}(\log G)$$

and $\Theta$ is the Kobayashi curvature ([11])

$$(3.3) \quad \Theta = \frac{\sqrt{-1}}{G} K_{i\bar{j}} dz^i \wedge d\bar{z}^j, \quad K_{i\bar{j}} = -\partial_i \partial_{\bar{j}} G + G^{k\bar{m}} (\partial_i \hat{\partial}_{\bar{m}} G)(\partial_{\bar{j}} \hat{\partial}_k G).$$

The pull-back

$$i_z^* \omega_V = \sqrt{-1}(\log G)_{i\bar{j}} dv^i \wedge d\bar{v}^j$$

is the Fubini–Study metric on $P_z := \pi^{-1}(z)$, where $i_z : P_z \to P(\tilde{M})$ is the inclusion. Together with $\omega_H$, the Sasaki type metric on $P(\tilde{M})$ is defined as

$$(3.4) \quad \omega_{P(\tilde{M})} := \omega_V + \omega_H.$$

The invariant volume form can be given by

$$(3.5) \quad d\mu_{P(\tilde{M})} := \frac{\omega_V^{n-1}}{(n-1)!} \wedge \frac{\omega_H^n}{n!}.$$
Lemma 3.1 ([19]): We have 
\[ \delta_i d\mu_{P(\tilde{M})} = \Gamma_j^i d\mu_{P(\tilde{M})} \]
and its conjugate form 
\[ \delta_\bar{i} d\mu_{P(\tilde{M})} = \Gamma_j^\bar{i} d\mu_{P(\tilde{M})}, \]
where “\( \cdot \)” is the interior derivative.

Denote \( A^{l,m}(M) \) the space of \((l,m)\)-forms on \( M \). Given \( l, m \geq 0 \), putting \( l^* = l + (n - 1) \) and \( m^* = m + (n - 1) \), the integration along the fiber is a map \( \pi_*: A^{l^*,m^*}(P(\tilde{M})) \to A^{l,m}(M) \) which is defined as follows:

\[
(3.6) \quad (\pi_* \phi)|_z(X_1, \ldots, X_l, Y_1, \ldots, Y_m) := \int_{\mathbb{P}_z} i^*_z[\phi(\tilde{X}_1, \ldots, \tilde{X}_l, \tilde{Y}_1, \ldots, \tilde{Y}_m, \ldots)]
\]

where \( X_i, Y_j \in T_z^*M \) and \( \tilde{X}_i, \tilde{Y}_j \) are their lifts. The RHS of (3.6) is independent of the lifts, and one may use the horizontal ones. Moreover, one can see that \( \pi_*(A^{p,q,r,s}(P(\tilde{M}))) = 0 \) if \( r \neq n-1 \) or \( s \neq n-1 \), since \( \mathbb{P}_z \) is \((n-1)\)-dimensional.

Lemma 3.2 (cf. § 6 of [5]): For the bundle \( \pi: P(\tilde{M}) \to M \), given \( \phi \in A(P(\tilde{M})) \) and \( \alpha \in A(M) \), the integration along the fiber \( \pi_* \) satisfies

(i) \( d(\pi_* \phi) = \pi_*(d\phi) \);
(ii) \( \pi_*((\pi^* \alpha) \wedge \phi) = \alpha \wedge \pi_* \phi \).

If \( M \) is compact in addition, then

(iii) \( \int_M \alpha \wedge \pi_* \phi = \int_{P(\tilde{M})} \pi_* \alpha \wedge \phi \).

Applying the above lemma, one can obtain the constancy of the volumes of each fiber which was first discovered by R. Yan.

Theorem 3.3 ([17]): Assuming that \((M, G)\) is a complex Finsler manifold, the volume of each fiber

\[ \text{vol}(\mathbb{P}_z) := \pi_*\left(\frac{\omega^{n-1}}{(n-1)!}\right)|_z \]

is a constant.

Proof. Recall \( \pi_\* \phi = 0 \) if the vertical part of \( \phi \) is not full. Thus

\[ \pi_* \omega^{n-1} = \pi_* (\sqrt{-1}\partial \bar{\partial} \log G)^{n-1} \]

by (3.1). Hence

\[ d(\text{vol}(\mathbb{P}_z)) = d\left( \pi_* \left( \frac{\sqrt{-1}\partial \bar{\partial} \log G)^{n-1}}{(n-1)!} \right) \right) = \pi_* \left( d\left( \frac{\sqrt{-1}\partial \bar{\partial} \log G)^{n-1}}{(n-1)!} \right) \right) = 0. \]

By the connectedness of \( M \), the volumes of each fiber are constant. \( \blacksquare \)

The same technique will give the following rigid result.
Theorem 3.4 ([2]): If $M$ admits a Kähler Finsler metric, then it admits a Kähler Hermite metric.

Proof. Let $F$ be a Kähler Finsler metric. Consider the form
\[
\omega_M := \pi_*(\omega_H \wedge (\sqrt{-1} \partial \bar{\partial} \log G)^{n-1}) = \pi_*(\omega_H \wedge (\omega_V - \Theta)^{n-1}) = \pi_*(\omega_H \wedge \omega_V^{n-1}).
\]
Since $d_H \omega_H = 0$, by Lemma 3.2 We have
\[
d\omega_M = \pi_*(d \omega_H) \wedge (\sqrt{-1} \partial \bar{\partial} \log G)^{n-1}
= \pi_*(d \omega_H) \wedge (\sqrt{-1} \partial \bar{\partial} \log G)^{n-1}
= \pi_*\left(d \omega_H \wedge \sum_{k=0}^{n-1} C_k^{n-1} \omega_V^k \wedge (\Theta)^{n-1-k}\right).
\]
Recall that $\dim_{\mathbb{C}} \mathbb{P}_{\mathbb{C}^{n-1}} = n - 1$. For $k < n - 1$, the vertical part of
\[
(d \omega_H) \wedge \omega_V^k \wedge (\Theta)^{n-1-k}
\]
is not full, thus
\[
\pi_*((d \omega_H) \wedge \omega_V^k \wedge (\Theta)^{n-1-k}) = 0.
\]
For $k = n - 1$, the vertical part of $(d \omega_H) \wedge \omega_V^{n-1}$ overflows. Hence $d \omega_M = 0$. One can deduce the positivity of $\omega_M$ from $\omega_M(X, \bar{X}) = \pi_*(\omega_H(X, \bar{X}) \cdot \omega_V^{n-1})$.

At the end of this section, let us give the definition of the induced volume form on $M$.

Definition 3.1: The induced volume form of $M$ is defined by
\[
d\mu_M := \pi_*(d\mu_{\mathbb{P}(\mathcal{M})}).
\]
In other words,
\[
(3.7) \quad \int_M f(z)d\mu_M = \int_{\mathbb{P}(\mathcal{M})} f(z)d\mu_{\mathbb{P}(\mathcal{M})}
\]
for any function $f \in C^\infty(M)$.

Remark 3.1: In other literature, the induced volume form may be divided by a constant and refer to \(\frac{1}{\text{vol}(\mathbb{C}^{n-1})} \pi_*(d\mu_{\mathbb{P}(\mathcal{M})})\) or \(\frac{1}{\text{vol}(\mathbb{F}_z)} \pi_*(d\mu_{\mathbb{P}(\mathcal{M})})\).
4. Conformal Kähler metrics

Let $G$ be a complex Finsler metric on $M$. A **conformal transformation** of $G$ is a change $G \mapsto e^f G$ where $f = f(z)$ is a smooth real function on $M$. We denote $e^f G$ by $\hat{G}$, and the notations of the quantities of $\hat{G}$ shall wear a hat, e.g., $\hat{\mathcal{H}}$ is the horizontal sub-bundle with respect to $\hat{G}$ and $\hat{\omega}_{\hat{\mathcal{H}}}$ is the Kähler form of $\hat{G}$. One can easily check that

\begin{align}
\hat{G}_{i\bar{j}} &= e^f G_{i\bar{j}}, \quad \hat{\omega}_{\hat{\mathcal{H}}} = e^f \omega_{\mathcal{H}}, \\
\hat{N}^i_j &= N^i_j + f_j v^i, \quad \hat{\Gamma}^i_{k,j} = \Gamma^i_{k,j} + f_j \delta^i_k,
\end{align}

where $f_j := \partial_j f$. Thus

\begin{align}
\hat{\delta}^i_j &= \delta^i_j - f_j v^i \partial v^j, \quad \hat{\delta} v^i = \delta v^i + v^i \partial f.
\end{align}

Since $v^m \hat{\delta}_m G_{i\bar{j}} = 0$ by the homogeneity of $G$, we see that

\begin{align}
\partial_{\hat{\mathcal{H}}} \hat{\omega}_{\hat{\mathcal{H}}} &= \partial_{\mathcal{H}}(\sqrt{-1} e^f G_{i\bar{j}} dz^i \wedge d\bar{z}^j) \\
&= \hat{\delta}_k(\sqrt{-1} e^f G_{i\bar{j}}) dz^k \wedge dz^i \wedge d\bar{z}^j \\
&= e^f f_k dz^k \wedge \omega_{\mathcal{H}} + \sqrt{-1} e^f (\delta_k G_{i\bar{j}} - f_k v^m \delta_{m} G_{i\bar{j}}) dz^k \wedge dz^i \wedge d\bar{z}^j \\
&= e^f (\partial f \wedge \omega_{\mathcal{H}} + \partial_{\mathcal{H}} \omega_{\mathcal{H}})
\end{align}

and thus

\begin{align}
d_{\hat{\mathcal{H}}} \hat{\omega}_{\hat{\mathcal{H}}} &= e^f (df \wedge \omega_{\mathcal{H}} + \partial_{\mathcal{H}} \omega_{\mathcal{H}}).
\end{align}

One can obtain the uniqueness of the Kähler Finsler metric in a conformal class by (4.5). Indeed, a stronger result can be proved. A Finsler metric is said to be **weakly Kähler** if $d_{\mathcal{H}} \omega_{\mathcal{H}}(\cdot, \chi, \bar{\chi}) = 0$ where $\chi = v^i \delta_i$ (cf. [1]). We can show the uniqueness of the weakly Kähler Finsler metric in a conformal class.

**Theorem 4.1:** In the conformal class of a complex Finsler metric, there exists at most one weakly Kähler metric up to homotheties.

**Proof.** By (4.3), one can see that $\hat{\chi} - \chi$ is vertical. Thus (4.5) gives

\[ d_{\hat{\mathcal{H}}} \hat{\omega}_{\hat{\mathcal{H}}}(\cdot, \hat{\chi}, \bar{\chi}) = e^f (df \wedge \omega_{\mathcal{H}} + \partial_{\mathcal{H}} \omega_{\mathcal{H}})(\cdot, \chi, \bar{\chi}). \]

If $e^f G$ and $e^g G$ are both weakly Kähler, then

\[ d(f - g) \wedge \omega_{\mathcal{H}}(\cdot, \chi, \bar{\chi}) = 0 \]
which is equivalent to

$$(f_i - g_i)G = (f_m - g_m)v^m \partial_i G.$$  

Taking the derivative with respect to $\bar{v}^j$, we get

$$(f_k - g_k)\bar{v}^k(f_m - g_m)v^m G_{ij} = (f_k - g_k)\bar{v}^k(f_i - g_i)\partial_j G$$

$$= (f_i - g_i)(f_k - g_k)\bar{v}^k \partial_j G$$

$$= (f_i - g_i)(f_j - g_j)G.$$  

One can easily see that the RHS and LHS have different rank unless $d(f-g) = 0$. Therefore, $e^f G$ and $e^g G$ are homothetic if they are both weakly Kähler.

At present, let us consider the existence of a Kähler Finsler metric in the conformal class of a complex Finsler metric. In other words, we shall consider the solvability of the equation

$$(4.6) \quad df \wedge \omega_H + d_H \omega_H = 0.$$  

A Finsler manifold $(M, G)$ is said to be **globally conformal Kähler** if and only if there exists a global defined function $f \in C^\infty(M)$ such that $e^f G$ is a Kähler Finsler metric. We give the following definition for local solutions.

**Definition 4.1** (cf. [16]): A complex Finsler manifold $(M, G)$ is said to be **locally conformal Kähler** if and only if there exists an open cover $\{ U_\alpha \}$ endowed with smooth functions $f_\alpha : U_\alpha \to \mathbb{R}$ such that $e^{f_\alpha} G$ is a Kähler Finsler metric on $U_\alpha$.

By Theorem 4.1, one can see that $d(f_\alpha - f_\beta) = 0$ on $U_\alpha \cap U_\beta$ whenever it is nonempty. Thus we obtain a globally defined real 1-form $\varphi \in A^1(M)$ such that $\varphi|_{U_\alpha} = df_\alpha$.

Additionally, we have

$$(4.7) \quad \varphi \wedge \omega_H + d_H \omega_H = 0, \quad d\varphi = 0.$$  

Such an equation was considered by H. Lee [14]. A real 1-form $\varphi \in A^1(M)$ satisfying (4.7) is called a **Lee form** of $(M, G)$. Thus, if $(M, G)$ is locally conformal Kähler, then $(M, G)$ admits a Lee form. Conversely, given a Lee form $\varphi$, locally we have $\varphi = df_\alpha$ by the Poincaré Lemma, and hence $e^{f_\alpha} G$ is a Kähler Finsler metric.
Lemma 4.2: A complex Finsler metric $G$ is locally conformal Kähler if and only if $(M, G)$ admits a Lee form.

On a simply connected manifold, a Lee form is (globally) $d$-exact. Hence, a simply connected, locally conformal Kähler manifold is globally conformal Kähler. Moreover, following I. Vaisman [16], we can prove the following rigid theorem.

Theorem 4.3: Let $(M, G)$ be a compact, locally conformal Kähler Finsler manifold. Then $(M, G)$ is globally conformal Kähler if and only if $M$ admits a Kähler Finsler metric.

Proof. We prove the sufficiency. Let $\phi$ be a Lee form of $(M, G)$. We will show that there exists a global function $f \in C^\infty(M)$ such that $\phi = df$. Decompose $\phi$ into $(1, 0)$ and $(0, 1)$ types $\phi = \phi' + \phi''$ where $\phi'' = \overline{\phi'}$. Put $\phi = \sqrt{-1}(\phi' - \phi'')$ which is again a real 1-form. We have

$$d\phi = \sqrt{-1}(d\phi' - d\phi'') = 2\sqrt{-1}d\phi' = 2\sqrt{-1}\partial\phi'$$

by $d\phi = \partial\phi' + (\bar{\partial}\phi' + \partial\phi'') + \bar{\partial}\phi'' = 0$. Thus $d\phi$ is a real exact $(1, 1)$-form.

On the other hand, since $M$ admits a Kähler Finsler metric, we have a Kähler Hermitian metric on $M$ by Theorem 3.4. Hence, the $\partial\bar{\partial}$-lemma holds on the compact manifold $M$. Thus, there exists a global real function $f \in C^\infty(M)$ such that

$$\bar{\partial}\phi' = \partial\bar{d}f.$$ 

Let us consider the metric $\hat{G} = e^f G$. Putting $\hat{\phi} = \phi - df$, by (4.5) and (4.7) we have

$$\hat{\phi} \wedge \hat{\omega}_{\hat{H}} + d_{\hat{H}}\hat{\omega}_{\hat{H}} = (\phi - df) \wedge e^f \omega_{H} + e^f (df \wedge \omega_{H} + d_{H}\omega_{H}) = 0.$$ 

Therefore $\hat{\phi}$ is a Lee form of $(M, \hat{G})$. Write $\hat{\phi} = \hat{\phi}' + \hat{\phi}''$ into $(1, 0)$ and $(0, 1)$ types. By (4.9) we have

$$\bar{\partial}\hat{\phi}' = \partial(\phi' - \partial f) = 0.$$ 

Thus $\hat{\phi}' = \hat{\phi}_i dz^i$ is a holomorphic 1-form. Noting $\overline{\phi''} = \phi'$, (4.10) is equivalent to

$$\hat{\phi}_i \hat{G}_{j\bar{k}} + \delta_i \hat{G}_{j\bar{k}} = \hat{\phi}_j \hat{G}_{i\bar{k}} + \delta_j \hat{G}_{i\bar{k}}.$$
With the help of (2.8), contracting the above equation with $G^{jk}$, one shall reach
\begin{equation}
(n - 1)\hat{\varphi}_i = \hat{\Gamma}_{i,m}^m - \hat{\Gamma}_{m,i}^m.
\end{equation}

By Lemma 3.1 and (4.11), we finally get
\begin{equation}
0 = \int_{\mathcal{P}(\tilde{M})} d(\hat{\varphi}_i \tilde{G}^{ij}) d\hat{\mu}_{\mathcal{P}(\tilde{M})} = \int_{\mathcal{P}(\tilde{M})} \hat{\delta}_j(\hat{\varphi}_i \tilde{G}^{ij}) d\hat{\mu}_{\mathcal{P}(\tilde{M})} + \int_{\mathcal{P}(\tilde{M})} \hat{\varphi}_i \tilde{G}^{ij} d(\hat{\delta}_j d\hat{\mu}_{\mathcal{P}(\tilde{M})})
\end{equation}
\begin{equation}
= \int_{\mathcal{P}(\tilde{M})} \hat{\varphi}_i \hat{\delta}_j(\tilde{G}^{ij}) d\hat{\mu}_{\mathcal{P}(\tilde{M})} + \int_{\mathcal{P}(\tilde{M})} \hat{\varphi}_i \tilde{G}^{ij} \hat{\Gamma}_{m,j} d\hat{\mu}_{\mathcal{P}(\tilde{M})}
\end{equation}
\begin{equation}
= -(n - 1) \int_{\mathcal{P}(\tilde{M})} \hat{\varphi}_i \tilde{G}^{ij} \hat{\delta}_j d\hat{\mu}_{\mathcal{P}(\tilde{M})}
\end{equation}
which implies $\hat{\varphi}' = \hat{\varphi}_i dz^i = 0$. Hence, $\hat{\varphi} = 0$ and $\varphi = df$.

Theorem 4.3 tells us that equation (4.6) is globally solvable if and only if it is locally solvable, if the compact manifold $M$ admits a Kähler Finsler metric. Recalling the definitions of the horizontal torsion $\theta$, equation (4.6) can be expressed in the form
\begin{equation}
f_k \delta_j^i - f_j \delta_k^i = \Gamma_{k,i}^i - \Gamma_{j,k}^i = \theta_{j,k}^i.
\end{equation}
The trace of (4.15) gives
\begin{equation}
(n - 1)f_j = -\vartheta_j
\end{equation}
where the $\vartheta_j$’s are the components of the mean horizontal torsion $\vartheta$.

**Theorem 4.4:** Let $M$ be a compact manifold admitting a Kähler Finsler metric. Then, a complex Finsler metric $G$ on $M$ is globally conformal Kähler if and only if
(i) the horizontal torsion is reducible, that is it has the form
\[\theta_{j,k}^i = \frac{1}{n - 1}(\vartheta_j \delta_k^i - \vartheta_k \delta_j^i)\]
where $\vartheta_j = \theta_{j,m}^m$;
(ii) and $d(\vartheta + \bar{\vartheta}) = 0$. 

Proof. One can easily get the necessity by (4.15) and (4.16). Conversely,

\[ d(\vartheta + \bar{\vartheta}) = \partial \vartheta + \partial \bar{\vartheta} + \bar{\partial} \vartheta + \bar{\partial} \bar{\vartheta} = 0 \]

implies \( \bar{\partial} \nu \bar{\vartheta} = (\dot{\partial} j \bar{\vartheta} i) \delta \bar{\nu}^j \land d \bar{z}^i = 0 \). Thus \( \vartheta_i = \vartheta_i(z) \) is independent of \( v \), and \( \vartheta \) must be a 1-form living on the base manifold \( M \). Then by the Poincaré Lemma, \( (n - 1)df = -(\vartheta + \bar{\vartheta}) \) is locally solvable on \( M \), which implies \( (n - 1)\partial f = -\vartheta \) locally. Together with (i), we get (4.15). Finally, (4.15) is globally solvable by Theorem 4.3.

5. Total holomorphic curvature

In this section, we will consider the total holomorphic curvature in the conformal classes. Let us recall the definition of the curvature forms. The curvature forms \( \Omega^i_j := \bar{\partial} \omega^i_j \) of the Chern–Finsler connection can be divided into four parts, namely, \( h\bar{h}-, v\bar{h}-, h\bar{v}- \) and \( v\bar{v}- \) curvatures. By (2.7), the \( h\bar{h}- \) curvature has the form

\[
(5.1) \quad h\bar{h}-\text{component of } \Omega^i_j = R^i_j,km dz^k \land d\bar{z}^m = (-\delta_{\bar{m}}^i \Gamma^i_{j,k} - C^i_{js} \delta_{\bar{m}}^s N^s_k) dz^k \land d\bar{z}^m .
\]

Putting \( R^i_j,km := G^i_{\bar{l}} R^i_j,\bar{k}m \), a direct computation gives (cf. [1])

\[
(5.2) \quad R^i_j,km v^j \bar{v}^l = K_{km} .
\]

where \( K_{km} \) is the Kobayashi curvature given in (3.3). The holomorphic curvature is defined by

\[
(5.3) \quad K(z, v) := \frac{1}{G^2} R^i_j,km v^j \bar{v}^l v^k \bar{v}^m = \frac{1}{G^2} K_{ij} v^i \bar{v}^j .
\]

We define the total holomorphic curvature of a compact complex Finsler manifold \((M, G)\) by setting

\[
(5.4) \quad \mathcal{K}(G) = \int_{\mathcal{P}(\tilde{M})} K(z, v) d\mu_{\mathcal{P}(\tilde{M})} .
\]

In order to consider the above functional in the volume preserved conformal class

\[
(5.5) \quad [G] = \{ e^f G \mid f \in C^\infty(M), \text{vol}(M, e^f G) = \text{vol}(M, G) \} ,
\]

let us give a divergence lemma.
Lemma 5.1: Given $\alpha = \alpha_i dz^i \in A^{1,0;0,0}(\mathbb{P}(\tilde{M}))$, we have

$$(5.6) \quad d(\alpha_i G^{ij} \delta_j \cdot d\mu_{\mathbb{P}(\tilde{M})}) = G^{ij}(\alpha_i \bar{\partial}_j + \alpha_i \bar{\partial}_j) d\mu_{\mathbb{P}(\tilde{M})},$$

$$(5.7) \quad d\left(\frac{1}{G} \alpha_i v^i \cdot \bar{\omega} d\mu_{\mathbb{P}(\tilde{M})}\right) = \frac{1}{G} \left(\alpha_i v^i \bar{\partial}_j v^j + \alpha_i v^i \cdot \bar{\partial}_j \bar{v}^j\right) d\mu_{\mathbb{P}(\tilde{M})},$$

and their conjugate forms, where $\alpha_{i,j} := \delta_{j} \alpha_i$ and $\chi = v^j \delta_j$.

Proof. The proof of (5.6) is similar to the proof of (4.14). For (5.7), applying $\delta_{j} G = 0$, $v^j \Gamma_{i,j} = N^i_j$ and Lemma 3.1, we get

$$d\left(\frac{1}{G} \alpha_i v^i \cdot \bar{\omega} d\mu_{\mathbb{P}(\tilde{M})}\right) = \delta_j \left(\frac{1}{G} \alpha_i v^i \bar{v}^j\right) d\mu_{\mathbb{P}(\tilde{M})} + \frac{1}{G} \alpha_i v^i v^j d(\delta_j \cdot d\mu_{\mathbb{P}(\tilde{M})})$$

$$= \frac{1}{G} \left(\alpha_i \delta_{j} v^i v^j - \alpha_i v^i \cdot N^j_j + \alpha_i v^i \cdot v^j \Gamma_{i,j}^m d\mu_{\mathbb{P}(\tilde{M})}\right)$$

$$= \frac{1}{G} \left(\alpha_i \delta_{j} v^i v^j + \alpha_i v^i \cdot \bar{\partial}_j \bar{v}^j\right) d\mu_{\mathbb{P}(\tilde{M})}.$$ 

The conjugate forms of (5.6) and (5.7) are obviously true. 

Let us give the relations of the curvatures of two conformal related metrics. Putting $\hat{G} = e^{f(z)} G$, by (4.1)–(4.3), we get $\hat{\omega} = e^{f} \omega$ and

$$(5.8) \quad \hat{\omega}_\mathbb{V} = \sqrt{-1}(\log G)_{ij}(\delta v^i + v^i \partial f) \wedge (\delta \bar{v}^j + \bar{v}^j \bar{\partial} f) = \omega_\mathbb{V}$$

where we use $(\log G)_{ij} v^i = (\log G)_{ij} \bar{v}^j = 0$ for the last equality. Thus the fiber volume $\text{vol}(\mathbb{P}_z)$ is a conformal invariant, and

$$(5.9) \quad d\hat{\mu}_{\mathbb{P}(\tilde{M})} = e^{nf} d\mu_{\mathbb{P}(\tilde{M})}, \quad d\hat{\mu}_{\mathbb{M}} = e^{nf} d\mu_{\mathbb{M}}.$$ 

Recalling (3.3), one can obtain

$$\hat{K}_{ij} = e^{f} (K_{ij} - f_{ij} G)$$

where $f_{ij} = \partial_i \partial_j f$. Invariantly, it says that

$$(5.11) \quad \hat{\Theta} = \Theta - \sqrt{-1} \partial \bar{\partial} f.$$ 

Moreover, one can get

$$(5.12) \quad \hat{K} = e^{-f} \left( K - \frac{1}{G} f_{ij} v^i \bar{v}^j \right).$$
Now by considering a family of conformal deformations $e^{f(t,z)}G$ with the initial data $f(0,z) = 0$, one can find

$$\frac{d}{dt}K(e^f G) = \frac{d}{dt} \int_{\mathbb{P}(\hat{M})} e^{(n-1)f} (K - \frac{1}{G} f\bar{v}^i\bar{v}^j) d\mu_{\mathbb{P}(\hat{M})}$$

(5.13)

$$= \int_{\mathbb{P}(\hat{M})} (n-1)f' e^{(n-1)f} (K - \frac{1}{G} f\bar{v}^i\bar{v}^j) d\mu_{\mathbb{P}(\hat{M})}$$

$$- \int_{\mathbb{P}(\hat{M})} \frac{1}{G} e^{(n-1)f} f' \bar{v}^i\bar{v}^j d\mu_{\mathbb{P}(\hat{M})},$$

where $f' = \frac{\partial}{\partial t} f$.

Denoting $f'(0,z) := \nu(z)$, and substituting $f(0,z) = 0$ into (5.13), it turns out that

(5.14) $$\frac{d}{dt}K(e^f G)|_{t=0} = \int_{\mathbb{P}(\hat{M})} \left( (n-1)\nu K - \frac{1}{G} \nu_i \bar{v}^i \bar{v}^j \right) d\mu_{\mathbb{P}(\hat{M})}. $$

Taking $\alpha_i = \nu_i$ in (5.7), we get

(5.15) $$\int_{\mathbb{P}(\hat{M})} \frac{1}{G} \nu_i \bar{v}^i \bar{v}^j d\mu_{\mathbb{P}(\hat{M})} = - \int_{\mathbb{P}(\hat{M})} \frac{1}{G} \nu_i v^i \bar{v}^j d\mu_{\mathbb{P}(\hat{M})}. $$

Then taking $\alpha_i = \nu \partial_i$ in (5.7), its conjugate form gives

(5.16) $$\int_{\mathbb{P}(\hat{M})} \frac{1}{G} \nu_i v^i \bar{\partial}_j \bar{v}^j d\mu_{\mathbb{P}(\hat{M})} = - \int_{\mathbb{P}(\hat{M})} \frac{1}{G} \nu (|\partial_i v^i|^2 + \bar{\partial}_i \bar{v}^j) d\mu_{\mathbb{P}(\hat{M})}. $$

Noting that $\nu_i \bar{v}^i \bar{v}^j$ is real, we obtain

(5.17) $$\frac{d}{dt}K(e^f G)|_{t=0} = \int_{\mathbb{P}(\hat{M})} \nu \left( (n-1)K - \frac{1}{G} (|\partial_i v^i|^2 + \Re(\partial_i \bar{v}^j)) \right) d\mu_{\mathbb{P}(\hat{M})}. $$

At this point, let us define the mean holomorphic curvature $\kappa$ by

(5.18) $$\kappa = \pi_*(Kd\mu_{\mathbb{P}(\hat{M})})/\pi_*(d\mu_{\mathbb{P}(\hat{M})}) $$

which is a real function on $M$, and call

(5.19) $$\kappa_\vartheta = \pi_*(\left( K - \frac{1}{G(n-1)} (|\partial_i v^i|^2 + 2\Re(\partial_i \bar{v}^j)) \right) d\mu_{\mathbb{P}(\hat{M})})/\pi_*(d\mu_{\mathbb{P}(\hat{M})}) $$

the $\vartheta$-mean holomorphic curvature. By (5.7), one can see that

$$\int_{\mathbb{P}(\hat{M})} (|\partial_i v^i|^2 + 2\Re(\partial_i \bar{v}^j)) d\mu_{\mathbb{P}(\hat{M})} = 0.$$
Recalling $\pi_*(d\mu_{\mathbb{P}(\tilde{M})}) = d\mu_M$, we obtain various representations of $K(G)$:

\begin{equation}
(5.20) \quad \int_M \kappa \vartheta d\mu_M = \int_M \kappa d\mu_M = \int_M \pi_*(Kd\mu_{\mathbb{P}(\tilde{M})}) = \int_{\mathbb{P}(\tilde{M})} Kd\mu_{\mathbb{P}(\tilde{M})} = K(G).
\end{equation}

Since $\pi_*(\nu \phi) = \nu \pi_*(\phi)$ for any $\phi \in A(\mathbb{P}(\tilde{M}))$, the formula (5.17) becomes

\begin{equation}
(5.21) \quad \frac{d}{dt} K(e^{tf}G)|_{t=0} = (n-1) \int_M \nu \kappa \vartheta d\mu_M.
\end{equation}

Assuming $e^{tf}G$ in the volume preserved class $[G]$, we have

\[ 0 = \frac{d}{dt} \int_M e^{nf}d\mu_M = n \int_M f'e^{nf}d\mu_M. \]

At $t = 0$, it reads as

\[ 0 = \int_M \nu d\mu_M. \]

Thus a critical point satisfies

\begin{equation}
(5.22) \quad \int_M \nu \kappa \vartheta d\mu_M = 0 \quad \text{where} \quad \int_M \nu d\mu_M = 0.
\end{equation}

Denoting the average

\[ \bar{\kappa} = \frac{1}{\text{vol}(M)} \int_M \kappa \vartheta d\mu_M, \]

it is equivalent to

\begin{equation}
(5.23) \quad \int_M \nu (\kappa \vartheta - \bar{\kappa} \vartheta) d\mu_M = 0 \quad \text{where} \quad \int_M \nu d\mu_M = 0.
\end{equation}

Taking $\nu = \kappa \vartheta - \bar{\kappa} \vartheta$, it becomes

\begin{equation}
(5.24) \quad \int_M (\kappa \vartheta - \bar{\kappa} \vartheta)² d\mu_M = 0.
\end{equation}

**Theorem 5.2:** Let $M$ be a compact complex manifold. A metric $G$ is a critical point of $\int_M \kappa \vartheta d\mu_M$ in its volume preserved conformal class $[G]$ if and only if $\kappa = \text{const}$. If $G$ is a Kähler Finsler metric, then $\kappa = \text{const}$. 
In particular, a Kähler Finsler metric with constant holomorphic curvature is critical in the volume preserved conformal class. Next, let us consider the stability of a critical Kähler Finsler metric. The second variation is

\[
\frac{d^2}{dt^2} K(e^f G) = \int_{\mathbb{M}} (n-1) f'' e^{(n-1)f} \left( K - \frac{1}{G} f_{ij} v^i \bar{v}^j \right) d\mu_{\mathbb{M}(\tilde{M})}
\]

\[
+ \int_{\mathbb{M}(\tilde{M})} (n-1)^2 f' e^{(n-1)f} \left( K - \frac{1}{G} f_{ij} v^i \bar{v}^j \right) d\mu_{\mathbb{M}(\tilde{M})}
\]

\[
- \int_{\mathbb{M}(\tilde{M})} \frac{2(n-1)}{G} f' e^{(n-1)f} f_{ij} v^i \bar{v}^j d\mu_{\mathbb{M}(\tilde{M})}
\]

\[
- \int_{\mathbb{M}(\tilde{M})} \frac{1}{G} e^{(n-1)f} f_{ij} v^i \bar{v}^j d\mu_{\mathbb{M}(\tilde{M})}
\]

where \( f' = \frac{\partial}{\partial t} f \) and \( f'' = \frac{\partial^2}{\partial t^2} f \). At \( t = 0 \), denoting \( f''(0, z) = \psi(z) \) and recalling that \( f(0, z) = 0 \) and \( f'(0, z) = \nu(z) \), we get

\[
\frac{d^2}{dt^2} K(e^f G)|_{t=0} = \int_{\mathbb{M}(\tilde{M})} (n-1) \psi K d\mu_{\mathbb{M}(\tilde{M})} + \int_{\mathbb{M}(\tilde{M})} (n-1)^2 \nu^2 K d\mu_{\mathbb{M}(\tilde{M})}
\]

\[
- \int_{\mathbb{M}(\tilde{M})} \frac{2(n-1)}{G} \nu f_{ij} v^i \bar{v}^j d\mu_{\mathbb{M}(\tilde{M})}
\]

\[
- \int_{\mathbb{M}(\tilde{M})} \frac{1}{G} \psi f_{ij} v^i \bar{v}^j d\mu_{\mathbb{M}(\tilde{M})}
\]

Since \( G \) is a Kähler Finsler metric, the torsion \( \vartheta \) vanishes. Taking \( \alpha_i = \nu \nu_i \) in (5.7), we get

\[
\int_{\mathbb{M}(\tilde{M})} \frac{1}{G} \nu \nu_{ij} v^i \bar{v}^j d\mu_{\mathbb{M}(\tilde{M})} = - \int_{\mathbb{M}(\tilde{M})} \frac{1}{G} \nu_j \nu_i v^i \bar{v}^j d\mu_{\mathbb{M}(\tilde{M})}
\]

while taking \( \alpha_i = \psi_i \) it leads to

\[
\int_{\mathbb{M}(\tilde{M})} \frac{1}{G} \psi_i v^i \bar{v}^j d\mu_{\mathbb{M}(\tilde{M})} = 0.
\]

By defining an induced Hermitian metric \( h \)

\[
h^{i\bar{j}} := \pi_* \left( \frac{2}{G} v^i \bar{v}^j d\mu_{\mathbb{M}(\tilde{M})} \right) / \pi_* \left( d\mu_{\mathbb{M}(\tilde{M})} \right),
\]

equation (5.26) becomes

\[
\frac{d^2}{dt^2} K(e^f G)|_{t=0} = (n-1) \int_M (h^{i\bar{j}} \nu_i \nu_{\bar{j}} + (\psi + (n-1)\nu^2) \kappa) d\mu_M.
\]
Let us recall that
\begin{equation}
0 = \frac{d^2}{dt^2} |_{t=0} \int_M e^{n f} d\mu_M = \int_M n(\psi + n\nu^2) d\mu_M.
\end{equation}

Thus, by the constancy of \(\kappa\), finally we have
\begin{equation}
\frac{d^2}{dt^2} \mathcal{K}(e^f G)|_{t=0} = (n - 1) \int_M (h^{ij} \nu_i \nu_j - \nu^2 \kappa) d\mu_M
\end{equation}
where
\[\int_M \nu d\mu_M = 0.\]

We call \(G\) a stable critical metric of \(\mathcal{K}\) if the above second variation is nonnegative.

**Theorem 5.3:** In a volume preserved conformal class of a compact complex Finsler manifold, a critical Kähler Finsler metric of the functional \(\int_M \kappa d\mu_M\) is stable if and only if the constant mean holomorphic curvature satisfies \(\kappa \leq \lambda_1^h\), where \(\lambda_1^h\) is given by
\begin{equation}
\lambda_1^h := \inf \left\{ \frac{\int_M h^{ij} \phi_i \phi_j d\mu_M}{\int_M \phi^2 d\mu_M} \mid \phi \in C^\infty(M), \int_M \phi d\mu_M = 0 \right\}.
\end{equation}

**Remark 5.1:** The number \(\lambda_1^h\) is precisely the first eigenvalue of the weighted Laplacian of the metric measure space \((M, h, d\mu_M)\) (cf. [9]).

### 6. Total holomorphic Ricci curvature

In this section, we will consider the Ricci curvature of a complex Finsler metric. The **holomorphic Ricci curvature** of \(G\) is defined as
\begin{equation}
\text{Ric}(z, v) = \frac{1}{G} G^{ij} R_{km,ij} v^k \bar{v}^m = \frac{1}{G} G^{ij} K_{ij}.
\end{equation}

Kobayashi introduced an analogous quantity for complex Finsler vector bundles in [12], and named it the mean curvature.

The **total holomorphic Ricci curvature** of a compact complex Finsler space \((M, G)\) is given by
\begin{equation}
\mathcal{R}(G) = \int_{\mathbb{P}(\tilde{M})} \text{Ric}(z, v) d\mu_{\mathbb{P}(\tilde{M})}.
\end{equation}
Denoting $\hat{G} = e^f G$ again, one can deduce

$$\hat{\text{Ric}} = e^{-f}(\text{Ric} - G^{i\bar{j}} f_{i\bar{j}})$$

from (5.10). By a similar calculation of §5, we have

$$\frac{d}{dt} e^{f G}|_{t=0} = \int_{\mathfrak{P}(\hat{M})} (n-1)\nu \text{Ric} - \int_{\mathfrak{P}(\hat{M})} G^{i\bar{j}} \nu_{i\bar{j}} d\mu_{\mathfrak{P}(\hat{M})}.$$  

Taking $\alpha = \nu_i dz^i$, one can deduce from (5.6) that

$$-\int_{\mathfrak{P}(\hat{M})} G^{i\bar{j}} \nu_i \bar{\nu}_{i\bar{j}} d\mu_{\mathfrak{P}(\hat{M})} = \int_{\mathfrak{P}(\hat{M})} \nu G^{i\bar{j}} (\bar{\nu}_{i\bar{j}} + \bar{\nu}_{j\bar{i}}) d\mu_{\mathfrak{P}(\hat{M})}.$$  

Since the expression is real, we obtain

$$\frac{d}{dt} e^{f G}|_{t=0} = \int_{\mathfrak{P}(\hat{M})} \nu ((n-1)\text{Ric} - (\|\bar{\nu}\|_G^2 + \mathfrak{Re}(\bar{\nu}_{i\bar{j}} G^{i\bar{j}}))) d\mu_{\mathfrak{P}(\hat{M})}.$$  

Let us define the **mean holomorphic Ricci curvature** $\rho$ by

$$\rho = \pi^* (\text{Ric} d\mu_{\mathfrak{P}(\hat{M})}) / \pi^* (d\mu_{\mathfrak{P}(\hat{M})})$$

which is again a real function on $M$. We call

$$\varphi_{\varphi} = \pi^* \left( \left( \text{Ric} - \frac{1}{(n-1)}(\|\varphi\|_G^2 + \mathfrak{Re}(\varphi_{i\bar{j}} G^{i\bar{j}})) \right) d\mu_{\mathfrak{P}(\hat{M})} \right) / \pi^* (d\mu_{\mathfrak{P}(\hat{M})})$$

the $\varphi$-**mean holomorphic Ricci curvature**. By (5.6), one can see that

$$\int_{\mathfrak{P}(\hat{M})} (\|\varphi\|_G^2 + \mathfrak{Re}(\varphi_{i\bar{j}} G^{i\bar{j}})) d\mu_{\mathfrak{P}(\hat{M})} = 0$$

and thus

$$\int_M \varphi_{\varphi} d\mu_M = \int_M \rho d\mu_M = \int_M \pi^* (\text{Ric} d\mu_{\mathfrak{P}(\hat{M})}) = \int_{\mathfrak{P}(\hat{M})} \text{Ric} d\mu_{\mathfrak{P}(\hat{M})} = \mathcal{R}(G).$$

By the definition of $\varphi_{\varphi}$, the first variation formula (6.7) becomes

$$\frac{d}{dt} \mathcal{R}(e^{f G})|_{t=0} = (n-1) \int_M \nu \varphi_{\varphi} d\mu_M.$$
**Theorem 6.1:** Let $M$ be a compact complex manifold. A metric $G$ is a critical point of $\int_M \rho \vartheta d\mu_M$ in its volume preserved conformal class $[G]$ if and only if $\rho = \text{const}$. If $G$ is a Kähler Finsler metric, then $\rho = \text{const}$.

Let $G$ be a critical Kähler Finsler metric. We shall give its second variation formula. Similarly to §5, we have

$$d^2 dt^2 \mathcal{R}(e^f G)|_{t=0} = \int_{\mathcal{P}(\tilde{M})} (n-1) \psi \text{Ric} d\mu_{\mathcal{P}(\tilde{M})} + \int_{\mathcal{P}(\tilde{M})} (n-1)^2 \nu^2 \text{Ric} d\mu_{\mathcal{P}(\tilde{M})} - \int_{\mathcal{P}(\tilde{M})} 2(n-1) \nu \nu_i \xi^j G^{ij} d\mu_{\mathcal{P}(\tilde{M})}$$

$$= \int_M (n-1) \psi \rho d\mu_M + \int_M (n-1)^2 \nu^2 \rho d\mu_M$$

$$+ \int_{\mathcal{P}(\tilde{M})} 2(n-1) \nu \nu_i \xi^j G^{ij} d\mu_{\mathcal{P}(\tilde{M})}.$$  

(6.13)

Let us define another **induced Hermitian metric** $g$ by

$$g^{i\bar{j}} := \pi_* (2G^{i\bar{j}} d\mu_{\mathcal{P}(\tilde{M})}) / \pi_* (d\mu_{\mathcal{P}(\tilde{M})}).$$

(6.14)

By (5.31), we have

$$d^2 dt^2 \mathcal{R}(e^f G)|_{t=0} = (n-1) \int_M (g^{i\bar{j}} \nu_i \nu_{\bar{j}} - \nu^2 \rho) d\mu_M,$$

where $\int_M \nu d\mu_M = 0$. Finally, we can state the **stability** of a critical Kähler Finsler metric of the functional $\mathcal{R}$.

**Theorem 6.2:** In a volume preserved conformal class of a compact complex Finsler manifold, a critical Kähler Finsler metric of the functional $\mathcal{R} = \int_M \rho d\mu_M$ is stable if and only if the constant mean holomorphic Ricci curvature satisfies $\rho \leq \lambda_{1}^g$, where $\lambda_{1}^g$ is defined by

$$\lambda_{1}^g := \inf \left\{ \int_M g^{i\bar{j}} \phi_i \phi_{\bar{j}} d\mu_M / \int_M \phi^2 d\mu_M | \phi \in C^\infty(M), \int_M \phi d\mu_M = 0 \right\}.$$  

(6.16)

**Remark 6.1:** The number $\lambda_{1}^g$ is the first eigenvalue of the weighted manifold $(M, g, d\mu_M)$.

We adopt Kobayashi’s notion of Finsler Einstein bundles ([12]) and give the following definition of Kähler Finsler metrics.
Definition 6.1: A Kähler Finsler metric with constant holomorphic Ricci curvature is called a **Kähler Einstein Finsler metric**.

By this definition, one can immediately get the following corollary.

**Corollary 6.3:** On a compact complex manifold, a Kähler Einstein Finsler metric with non-positive holomorphic Ricci curvature is a stable critical point of $\mathcal{R}$ in its volume preserved conformal class.

### 7. A Yamabe type problem

In this section, we shall study the existence of complex Finsler metrics with constant $\rho_\theta$ in the volume preserved conformal class $[G]$ on a compact manifold. Through the variational approach (cf. [7, 13]), we can get the existence of metrics with $\rho_\theta = \text{const}$.

Let us write the conformal change in the form $\hat{G} = \phi^{\frac{2}{n-1}}G$, where $\phi$ is a positive function and $n$ is the complex dimension of $M$. Consider the following Yamabe type functional of a compact complex Finsler manifold $(M, G)$

\[
\mathcal{R}(\phi) = \frac{1}{\text{vol}^{1-\frac{1}{n}}(M, \phi^{\frac{2}{n-1}}G)}\mathcal{R}(\phi^{\frac{2}{n-1}}G).
\]

Using Lemma 5.1, (6.9) and (6.14), we have

\[
\int_{\mathcal{P}(\hat{M})} \hat{\text{Ric}} d\hat{\mu}_{\mathcal{P}(\hat{M})}
= \int_{\mathcal{P}(\hat{M})} \left( \phi^2 \text{Ric} + \frac{2}{n-1} (\phi_i \phi_j - \phi_i \phi_j G^{ij}) \right) d\mu_{\mathcal{P}(\hat{M})}
\]

\[
= \int_{\mathcal{P}(\hat{M})} \left( \phi^2 \text{Ric} + \frac{1}{n-1} (4G^{ij} \phi_i \phi_j - \phi^2 \|\theta\|_G^2 - \phi^2 \mathcal{R}(\nabla \phi_i \partial_j G^{ij}) \right) d\mu_{\mathcal{P}(\hat{M})}
\]

\[
= \int_{M} \left( \frac{2}{n-1} g^{ij} \phi_i \phi_j + \phi^2 \rho_\theta \right) d\mu_{M}.
\]

In the real expression, $g^{ij} \phi_i \phi_j$ is $\frac{1}{4} \|d\phi\|_g^2$, thus the Yamabe type functional (7.1) is of the form

\[
\mathcal{R}(\phi) = \frac{\int_{M} \left( \frac{1}{2(n-1)} \|d\phi\|_g^2 + \phi^2 \rho_\theta \right) d\mu_{M}}{\left( \int_{M} \phi^{-\frac{2n}{n-1}} d\mu_{M} \right)^{\frac{n-1}{n}}}.
\]
By Hölder’s inequality, one can get \( R(\phi) \geq -\left( \int_M |\rho|^n d\mu_M \right)^{1/n} \), thus we can define a **conformal invariant** as

\[
Y(G) = \inf_{0 < \phi \in C^\infty(M)} R(\phi).
\]

The **energy** of \( \phi \) is given by

\[
E(\phi) = \int_M \left( \frac{1}{2(n-1)} \|d\phi\|^2_g + \phi^2 \rho \right) d\mu_M
\]

and the \( L^q \)-norm is defined as \( \|\phi\|_q = \left( \int_M |\phi|^q d\mu_M \right)^{1/q} \). By putting \( p = \frac{2n}{n-1} \), we have

\[
R(\phi) = \frac{E(\phi)}{\|\phi\|^2_p}.
\]

Since \( C^\infty(M) \) is dense in the Sobolev space \( W^{1,2}(M) \), \( \mathcal{R}(|\phi|) \leq \mathcal{R}(\phi) \) and \( \mathcal{R}(\lambda \phi) = \mathcal{R}(\phi) \) for \( \lambda > 0 \), we see that

\[
Y(G) = \inf_{\phi \in W^{1,2}} \mathcal{R}(\phi) = \inf_{\|\phi\|_p = 1} E(\phi).
\]

The Euler–Lagrange equation of the minimizer with \( \|\phi\|_p = 1 \) is

\[
L\phi := \frac{1}{2(n-1)} \Delta_g \phi + \frac{1}{2(n-1)} (d\phi, d\log \tau)_g - \phi \rho = -Y(G) \phi^{p-1},
\]

where \( \Delta_g \) is the Laplacian of the induced Hermitian metric \( g \) and \( \tau = \frac{d\mu_M}{d\mu_g} \).

Note that the real dimension of \( M \) is \( m = 2n \), therefore

\[
p = \frac{2n}{n-1} = \frac{2m}{m-2}
\]

is the critical exponent of the Sobolev embedding theorem. Following Yamabe, let us consider the disturbed functional

\[
\mathcal{R}_t(\phi) = \frac{E(\phi)}{\|\phi\|_t^2}, \quad 2 \leq t \leq p = \frac{2n}{n-1},
\]

whose infimum is denoted by \( Y_t \). The Euler–Lagrange equation of the minimizer of \( \mathcal{R}_t(\phi) \) with \( \|\phi\|_t = 1 \) is

\[
L\phi = -Y_t \phi^{t-1}.
\]

By the regularity theory, for any \( t < p \) there exists a smooth and positive minimizer \( \phi_t \) of \( \mathcal{R}_t \) with \( \|\phi_t\|_t = 1 \) (cf. [7, Lemma 5.2] or [13, Proposition 4.2]). In other words, for any \( 2 \leq t < p \) we have a smooth and positive function \( \phi_t \) that satisfies

\[
L\phi_t = -Y_t \phi_t^{t-1}.
\]
At this point, we shall consider the limit when $t \to p^+$. Henceforth, let us assume the initial metric $G$ has unit volume $\text{vol}(M, G) = 1$.

**Lemma 7.1** (cf. [13, Lemma 4.3]): Given $\text{vol}(M, G) = 1$, we have

1. if $Y_p < 0$, then $\limsup_{t \to p^-} Y_t \leq Y_p = Y(G)$;
2. if $Y_p \geq 0$, then $\lim_{t \to p^-} Y_t = Y_p = Y(G)$.

As we did in [7], let us introduce another conformal invariant

\begin{equation}
C(G) = \sup_{x \in M} \left[ \frac{d\mu_G}{d\mu_M} \right]^{1/n}.
\end{equation}

By Definition 3.1 and (6.14), when $G$ is Hermitian, we have

$$C(G) = \frac{1}{2\text{vol}(\mathbb{CP}^{n-1})^{1/n}}$$

which can be considered as the normalizing factor of $Y(G)$. Then we have a Sobolev inequality.

**Lemma 7.2** (cf. [7, Lemma 5.4]): Let $(M, G)$ be a compact complex Finsler manifold. Then for any $\epsilon > 0$, there exists $C_\epsilon$ such that

\begin{equation}
\|w\|_p^2 \leq \frac{(1 + \epsilon)C(G)}{\sigma_{2n}} \int_M \|dw\|_g^2 d\mu_M + C_\epsilon \int_M w^2 d\mu_M
\end{equation}

where $\sigma_{2n}$ is the best Sobolev constant on $\mathbb{R}^{2n}$ satisfying

\begin{equation}
\sigma_{2n} \left( \int_{\mathbb{R}^{2n}} |f|^p dx \right)^{2/p} \leq \int_{\mathbb{R}^{2n}} \|df\|^2 dx.
\end{equation}

**Proof.** Recalling $\tau = d\mu_M/d\mu_g$, let us put $\tilde{g}_{ij} = \tau^{1/n} g_{ij}$. It turns out that $d\mu_{\tilde{g}} = d\mu_M$ and thus (cf. [13, Theorem 2.3])

$$\|w\|_p^2 \leq \frac{(1 + \epsilon)}{\sigma_{2n}} \int_M \|dw\|_{\tilde{g}}^2 d\mu_{\tilde{g}} + C_\epsilon \int_M w^2 d\mu_{\tilde{g}}.$$

We can deduce (7.12) from $\|dw\|_{\tilde{g}}^2 = \tau^{-1/n} \|dw\|_g^2 \leq C(G) \|dw\|_g^2$.

According to Lemmas 7.1 and 7.2, by a similar argument of [13, Proposition 4.4], one can obtain the following uniform $L^{p_0}$ estimate.

**Lemma 7.3:** If $Y(G) \cdot C(G) < \frac{\sigma_{2n}}{2n-2}$, then there exists $t_0 < p$ and $p_0 > p$ such that $\phi_t(t_0 \leq t < p)$ are uniformly bounded in $L^{p_0}$.
Finally, the regularity theory gives that \( \{ \phi_t \} \) are uniformly bounded in \( C^{2,\alpha}(M) \). Then \( \phi_t \to \phi \) in \( C^2(M) \) for some \( t_i \to p \), and the limit gives
\[-L \phi \leq Y(G) \phi^{p-1}, \| \phi \|_p = 1 \text{ and } \Re(\phi) \leq Y(G).\]
Hence \( \Re(\phi) = Y(G) \) by the definition of \( Y(G) \). Moreover, the minimizer \( \phi \) satisfies \(-L \phi = Y(G) \phi^{p-1}\) and then \( \phi \) is smooth and positive.

**Theorem 7.4:** Let \((M, G)\) be a compact complex Finsler manifold. If \( Y(G) \cdot C(G) < \frac{\sigma_{2n}}{2n^2} \), then there exists a smooth positive function \( \phi \) such that \( \Re(\phi) = Y(G) \). In this case, there exists a metric \( \hat{G} \) in the conformal class \([G]\) such that \( \hat{\rho} = \text{const.} \).

To conclude, we shall give the following upper bound theorem.

**Theorem 7.5:** For any compact complex Finsler manifold \((M, G)\),
\[ Y(G) \cdot C(G) \leq \frac{2\sigma_{2n}}{n-1}. \]

**Proof.** The proof is similar to the real case given in [7]. Recall that \( m = 2n \) is the real dimension of \( M \). It is well-known that the function
\[ u_\epsilon := \left( \frac{\epsilon}{\epsilon^2 + r^2} \right)^{\frac{m-2}{2}}, \quad r = |x|, \quad \epsilon > 0 \]
achieves the best Sobolev constant on the Euclidean space \( \mathbb{R}^m \) and satisfies
\[ \partial_r u_\epsilon = -(m-2)\frac{r}{\epsilon^2 + r^2} u_\epsilon, \quad \Delta_{\mathbb{R}^m} u_\epsilon = -m(m-2) u_\epsilon^{p-1} \]
which imply
\[ \int_{B(R)-B(\rho)} |du_\epsilon|^2 dx = m(m-2) \int_{B(R)-B(\rho)} u_\epsilon^p dx \]

\[ + (2-m) \omega_{m-1} \epsilon^{m-2} \left[ \frac{R^m}{(\epsilon^2 + R^2)^{m-1}} - \frac{\rho^m}{(\epsilon^2 + \rho^2)^{m-1}} \right] \]
where \( B(R) = \{ x : |x| < R \} \) and \( \omega_{m-1} = \text{vol}(S^{m-1}) \).

Hence the Sobolev constant satisfies
\[ \sigma_{2n} = \sigma_m = \frac{\int_{\mathbb{R}^m} |du_\epsilon|^2 dx}{\left( \int_{\mathbb{R}^m} u_\epsilon^p dx \right)^{\frac{2}{p}}} = m(m-2) \left( \int_{\mathbb{R}^m} u_\epsilon^p dx \right)^{\frac{2}{p}}. \]
Moreover, we have

\begin{equation}
\int_{B(\rho)} |du_\epsilon|^2 dx \leq m(m-2) \int_{B(\rho)} u_\epsilon^p dx < \sigma_m \left( \int_{B(\rho)} u_\epsilon^p dx \right)^{\frac{2}{p}}
\end{equation}

and

\begin{equation}
\int_{B(\rho)} u_\epsilon^p dx = \omega_{m-1} \int_0^\rho \left( \frac{\epsilon}{\epsilon^2 + r^2} \right)^m r^{m-1} dt = \omega_{m-1} \int_0^{\rho/\epsilon} \frac{t^{m-1}}{(1+t^2)^m} dt.
\end{equation}

Let \( \eta = \eta(r) \) be a radial cutoff function on \( \mathbb{R}^m \), such that \( 0 \leq \eta \leq 1 \), \( \eta|_{B(1)} = 1 \), \( \eta|_{\mathbb{R}^m - B(2)} = 0 \), and \( |d\eta| = |\partial_r \eta| \leq 2 \). Putting \( \eta_\rho := \eta \left( \frac{r}{\rho} \right) \) for \( \rho > 0 \), we have \( 0 \leq \eta_\rho \leq 1 \), \( \eta_\rho|_{B(\rho)} = 1 \), \( \eta_\rho|_{\mathbb{R}^m - B(2\rho)} = 0 \), and \( |d\eta_\rho| = |\partial_r \eta_\rho| \leq \frac{2}{\rho} \).

Consider the test function

\[ \varphi := \eta_\rho u_\epsilon \quad \text{for} \quad \epsilon << \rho. \]

Recall that

\[ \tau = d\mu_M/d\mu_g \quad \text{and} \quad \tilde{g} = \tau^{\frac{1}{n}} g = \left[ d\mu_M/d\mu_g \right]^{\frac{1}{n}} g. \]

Let us pick a point \( x_0 \in M \) such that \( C(G) = \sup_{x \in M} \tau^{-1/n}(x) = \tau^{-1/n}(x_0) \), and take a normal coordinate system of \( \tilde{g} \) centered at \( x_0 \). By the continuity, we have

\[ \tau^{-1/n}(x) \leq \frac{1}{C(G)} + \delta(\rho), \quad x \in B(2\rho) \]

where \( \delta(\rho) \to 0 \) when \( \rho \to 0 \). Suppose \( 2\rho \) is less than the injectivity radius of \( x_0 \) with respect to \( \tilde{g} \). The test function \( \varphi = \eta_\rho u_\epsilon \) can be considered as a globally defined function on \( M \). We will give the estimate of \( \mathcal{R}(\varphi) = \frac{E(\varphi)}{\|\varphi\|^p_p} \).

Applying the relations between \( \tilde{g} \) and \( g \), we have

\[ E(\varphi) \leq \frac{1}{2n-2} \int_M \tau^{\frac{1}{n}} \| d\varphi \|_\tilde{g}^2 d\mu_{\tilde{g}} + c_1 \int_M \varphi^2 d\mu_{\tilde{g}}. \]

Assume \( (1 - c_2 |x|) dx \leq d\mu_{\tilde{g}} \leq (1 + c_2 |x|) dx \) in \( B(2\rho) \). By the Hölder inequality and (7.18), one gets the estimate

\[ \int_M \varphi^2 d\mu_{\tilde{g}} \leq (1 + 2c_2 \rho) \int_{B(2\rho)} u_\epsilon^2 dx \leq c_3 \left( \int_{B(2\rho)} u_\epsilon^p dx \right)^\frac{2}{p} \rho^2 \leq c_4 \rho^2. \]
Next, we give the estimate of the term
\[ \int_{B(2\rho)} \tau^{1/n} \|d\varphi\|^2_{\tilde{g}} \, d\tilde{\mu}_{\tilde{g}} \leq \left( \frac{1}{C(G)} + \delta(\rho) \right) \int_{B(2\rho)} \|d\varphi\|^2_{\tilde{g}} \, d\tilde{\mu}_{\tilde{g}}. \]

Since the space is locally Euclidean, one can obtain
\[ \int_M \|d\varphi\|^2_{\tilde{g}} \, d\mu_{\tilde{g}} \leq (1 + 2c_2\rho) \int_{B(2\rho)} |\partial_r \varphi|^2 \, dx \]
\[ = (1 + 2c_2\rho) \left[ \int_{B(\rho)} |\partial_r u_\epsilon|^2 \, dx + \int_{B(2\rho) - B(\rho)} |\partial_r (\eta_\rho u_\epsilon)|^2 \, dx \right]. \]

The first term can be estimated by (7.17). For the second term, we see from (7.15) that
\[ \int_{B(2\rho) - B(\rho)} |\partial_r (\eta_\rho u_\epsilon)|^2 \, dx \leq \frac{8}{\rho^2} \int_{B(2\rho) - B(\rho)} u_\epsilon^2 \, dx + 2 \int_{B(2\rho) - B(\rho)} |\partial_r u_\epsilon|^2 \, dx \]
\[ \leq c_5 \left( \int_{B(2\rho) - B(\rho)} u_\epsilon^p \, dx \right)^{\frac{2}{p}} + c_5 \int_{B(2\rho) - B(\rho)} u_\epsilon^p \, dx \]
\[ + c_5 \rho^{2-m} \epsilon^{m-2}. \]

Being aware of (7.18), we see that
\[ (1 + 2c_2\rho) \int_{B(2\rho) - B(\rho)} |\partial_r (\eta_\rho u_\epsilon)|^2 \, dx \leq \frac{c_6 \epsilon^{m-2}}{\rho^{m-2}}. \]

On the other hand, for any \( \epsilon < \rho < \frac{1}{2c_2}, \)
\[ (7.19) \quad \left( \int_M \varphi^p d\mu_M \right)^{\frac{2}{p}} = \left( \int_M \varphi^p d\tilde{\mu}_{\tilde{g}} \right)^{\frac{2}{p}} \geq (1 - c_2\rho)^{\frac{2}{p}} \left( \int_{B(\rho)} u_\epsilon^p \, dx \right)^{\frac{2}{p}} \geq c_7. \]

Together with (7.16)–(7.19), we reach
\[ \mathcal{R}(\varphi) \leq \left( \frac{1}{C(G)} + \delta(\rho) \right) \left( \frac{1 + 2c_2\rho}{(1 - c_2\rho)^{\frac{2}{p}}} \frac{\sigma_{2n}}{2n - 2} + \frac{c_6 \epsilon^{n-2}}{c_7 \rho^{n-2}} \right) + \frac{c_1 c_4}{c_7} \rho^2. \]

By letting \( \epsilon \to 0 \) and \( \rho \to 0, \) we see that \( Y(G) \leq \frac{1}{C(G)} \cdot \frac{\sigma_{2n}}{2n - 2}. \]

Remark 7.1: The same procedure can be used to study the existence of metrics with constant \( \kappa_{\tilde{g}}. \)
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