Moment Ratios for Absorbing-State Phase Transitions

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(March 24, 2022)

Abstract

We determine the first through fourth moments of the order parameter, and various ratios, for several one- and two-dimensional models with absorbing-state phase transitions. We perform a detailed analysis of the system-size dependence of these ratios, and confirm that they are indeed universal for three models — the contact process, the A model, and the pair contact process — belonging to the directed percolation universality class. Our studies also yield a refined estimate for the critical point of the pair contact process.

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I. INTRODUCTION

Testing the universality hypothesis, and establishing the universality class of the critical points exhibited by various spin models and field theories has been a major preoccupation of statistical physics for some time [1]. In this ongoing project, the utility of studying a variety of universal quantities beyond the critical exponents, such as amplitude ratios and scaling functions, is well established. In particular, obtaining values for cumulant ratios of the order parameter has proven an efficient method for identifying the universality class [2]. In many cases, Binder’s “reduced fourth cumulant,” \( q_4 \equiv 1 - \langle \rho^4 \rangle / 3\langle \rho^2 \rangle \) (here \( \rho \) represents the order parameter and the brackets denote a stationary average), is remarkably insensitive to finite-size effects, permitting one to determine the universality class with a modest computational effort [3].

While critical phenomena in nonequilibrium systems have been under intensive study for a good twenty years, and controversies regarding the universality classes have frequently arisen, moment ratios have not, to our knowledge, been applied in this study. Our aim in this paper is to extend the method to nonequilibrium models. We focus on one- and two-dimensional models exhibiting an absorbing-state phase transition (i.e., between an active phase and one admitting no escape or evolution) [4]; all belong to the universality class of directed percolation [5,6]. In the following section we define the models. Sections III and IV present our results for one- and two-dimensional models, respectively. Section V contains a brief summary.

II. MODELS

The models considered here are all examples of interacting particle systems: Markov processes whose state space is a set of particle configurations on a lattice, with transitions involving local processes of particle creation or annihilation [7–9]. In the contact process (CP) [10], each site of the hypercubic lattice \( \mathbb{Z}^d \) is either vacant or occupied by a particle. Particles are created at vacant sites at rate \( \lambda n / 2d \), where \( n \) is the number of occupied nearest-neighbors, and are annihilated at unit rate, independent of the surrounding configuration. The order parameter is the particle density \( \rho \); it vanishes in the vacuum state, which is absorbing. As \( \lambda \) is increased beyond \( \lambda_c \), there is a continuous phase transition from the vacuum to an active steady state; for \( \lambda > \lambda_c \), \( \rho \sim (\lambda - \lambda_c)^\beta \). The A model was devised as a simplified description of surface catalysis [11]; in the present notation we may define it as a generalized CP in which the creation rate (at a vacant site) is \( \lambda \) for \( n > 0 \), i.e., as long as the site has at least one occupied neighbor. Since creation occurs more readily in the A model than in the CP, the critical creation rate \( \lambda_c \) is smaller (1.7417 for the A model versus 3.2978 for the CP, in one dimension), but the two models (and indeed the whole family of generalized contact processes), share the same critical behavior [12], namely, that of directed percolation (DP).

The pair contact process (PCP) has a somewhat more complicated dynamics [13,14]; transitions only occur in the presence of a nearest-neighbor pair of particles. Specifically, in one-dimension, if sites \( i \) and \( i+1 \) are both occupied, then the particles at these sites mutually annihilate at rate \( p \leq 1 \), while at rate \( 1 - p \) they create a new particle at either \( i-1 \) or \( i+2 \) (chosen at random, each with probability 1/2), provided the chosen site is vacant.
The PCP exhibits an active phase for \( p < p_c \); above this value the system falls into one of an infinite number of absorbing configurations. (Any arrangement of particles devoid of nearest-neighbor pairs is absorbing.) While the presence of infinitely many absorbing configurations is associated with nonuniversal dynamics, the static critical behavior, which concerns us here, again falls in the DP class \([14,17]\).

In equilibrium spin systems the magnetization (or its projection along a chosen direction), takes positive and negative values with equal likelihood, but in the present case the order parameter is non-negative. This means that we can study odd as well as even moments of the order parameter. In the following sections we report the first through fourth moments, \( m_1, \ldots, m_4 \), where \( m_n \equiv \langle \rho^n \rangle \), and the ratios \( m_4/m_2, K_2/m_1^2 \) and \( K_4/K_2^2 \), where \( K_n \) is the \( n \)-th cumulant of the order parameter, in particular,

\[
K_2 = m_2 - m_1^2
\]  

and

\[
K_4 = m_4 - 4m_3m_1 - 3m_2^2 + 12m_2m_1^2 - 6m_1^4
\]

The argument for universality amongst moment or cumulant ratios follows the same lines as in equilibrium \([3]\). We first note that at the critical point, \( m_1 \simeq AL^{-\beta/\nu_\perp} \), where \( \beta \) is the order parameter critical exponent, \( \nu_\perp \) is the critical exponent governing the correlation length, \( L \) is the linear extent of the system, and \( A \) is a nonuniversal constant. From finite-size scaling, we have that at the critical point, the probability density for \( \rho \) satisfies \( P(\rho; L; \lambda_c) = P(\rho/m_1) \simeq (L^{\beta/\nu_\perp}/A)P(\rho L^{\beta/\nu_\perp}/A) \), where \( P \) is a universal scaling function. It follows that \( m_n \simeq A^n L^{-n\beta/\nu_\perp} I_n \), where

\[
I_n = \int_0^\infty u^n P(u) du
\]

is model- and size-independent. Thus ratios of the form \( m_n/(m_1^i m_2^j) \) are universal for \( ir + js = n \), as are cumulant ratios such as \( K_4/K_2^2 \).

**III. RESULTS FOR ONE-DIMENSIONAL MODELS**

**A. The Contact Process**

We studied the one-dimensional CP on periodic lattices of \( L = 20, 40, \ldots, 320 \) sites, at the critical point, \( \lambda = \lambda_c = 3.297848 \) \([13]\). The sample sizes range from \( 2 \times 10^6 \) for \( L = 20 \) trials to \( 2 \times 10^5 \) for \( L = 320 \); the maximum duration of each sample realization extends from 400 to 16000. All trials start from a fully occupied lattice; we extract moments from the results for the surviving sample, when it has relaxed to the quasi-stationary state.

The CP density moments are listed in Table I; cumulant ratios are listed in Table II. From the latter we see that \( K_2/m_1^2 \) converges more rapidly than \( K_4/K_2^2 \); between \( L = 160 \) and 320 the value of the former changes by about 0.8%, the latter by about 2%. The ratio \( m_4/m_2^2 \) is remarkably stable, changing by only about 0.3% between the two largest \( L \) values. Similar trends are seen for the A model and the PCP (see below).
We also determined moment ratios for off-critical values of $\lambda$; in Fig. 1 we plot $K_2/m_1^2$ versus $\lambda$. As $L$ increases, the point at which the curves intersect rapidly approaches $\lambda_c$. (We find that the crossing points $\lambda_{cr}(L, 2L)$ for each pair of successive $L$ values follows $\lambda_{cr}(L, 2L) \approx \lambda_{cr}(\infty) + \text{const.} / L^2$, with $\lambda_{cr}(\infty) = 3.29785(8)$, consistent with the best available estimate for $\lambda_c$.)

B. The A Model

We studied the one-dimensional A model on periodic lattices of $L = 20, 40, \ldots, 320$ sites, at the critical point, $\lambda = \lambda_c = 1.74173$ [13]. Sample sizes range from $2 \times 10^6$ for $L = 20$ to $2 \times 10^5$ for $L = 320$; the maximum duration of each sample realization extends from from 200 ($L = 20$) to $2 \times 10^4$ ($L = 320$). The density moments are listed in Table III; cumulant ratios are listed in Table IV. The latter are seen to take values very similar to those found for the CP.

C. Pair Contact Process

In this model, the order parameter $\rho$ is the density of nearest-neighbor particle pairs. We studied the one-dimensional PCP on periodic lattices of $L = 20, 40, \ldots, 640$ sites, for $p = 0.07708, 0.07709, 0.07710, \text{and } 0.07712$. The sample sizes range from $10^7$ for $L = 20$ to $2 \times 10^5$ for $L = 640$; the maximum duration of each realization extends from from 200 to $10^4$. Previous work yielded $p_c = 0.0771(1)$ [13,14], but we decided to try to sharpen this estimate. To this end we analyzed the scaling of the lifetime, $\tau(p, L)$, defined as follows: Starting with all sites occupied, the probability that a trial survives (remains active) until time $t$ decays $\sim \exp[-t/\tau(p, L)]$. At the critical point, the lifetime has a power-law dependence on system size, $\tau(p_c, L) \sim L^{\nu_\parallel/\nu_\perp}$, while for $p \neq p_c$ deviations from a power law are seen. In Fig. 2 we plot $\ln t^* = \ln \tau(p_c, L) - (\nu_\parallel/\nu_\perp) \ln L$ versus $\ln L$, using $\nu_\parallel/\nu_\perp = 1.5822$, the value for the DP class in 1+1 dimensions [17]. It is evident that the data for $p = 0.07709$ are consistent with a power law, while those for the other $p$ values are not, allowing us to conclude that $p_c = 0.077090(5)$. The order parameter moments — for $p = p_c = 0.07709$ — are listed in Table V; cumulant ratios are given in Table VI.
TABLES

TABLE I. Density moments for the critical CP. Numbers in parentheses denote statistical uncertainties in the last figure(s).

| $L$ | $m_1$          | $m_2$  | $m_3$  | $m_4$   |
|-----|---------------|--------|--------|---------|
| 20  | 0.47427(7)    | 0.26000(6) | 0.15654(6) | 0.10075(5) |
| 40  | 0.39754(5)    | 0.18399(4) | 0.09392(3) | 0.05144(3) |
| 80  | 0.33367(8)    | 0.13011(5) | 0.05608(3) | 0.02599(2)  |
| 160 | 0.28001(10)   | 0.09181(5) | 0.03333(3) | 0.01302(1)  |
| 320 | 0.23505(9)    | 0.06477(4) | 0.01977(2) | 0.006503(5) |

TABLE II. Cumulants and ratios for the critical CP.

| $L$ | $K_2$        | $K_2/m_1^2$ | $K_4/K_2^2$ | $m_4/m_2^2$ |
|-----|--------------|-------------|-------------|-------------|
| 20  | 0.03507(12)  | 0.1559(12)  | -0.641(4)   | 1.490(1)    |
| 40  | 0.02596(1)   | 0.1643(1)   | -0.577(2)   | 1.520(2)    |
| 80  | 0.01877(1)   | 0.1686(2)   | -0.544(3)   | 1.535(2)    |
| 160 | 0.01341(1)   | 0.1710(2)   | -0.525(3)   | 1.545(3)    |
| 320 | 0.009517(3)  | 0.1723(3)   | -0.515(4)   | 1.550(2)    |

TABLE III. Density moments for the critical A model.

| $L$ | $m_1$          | $m_2$  | $m_3$  | $m_4$   |
|-----|---------------|--------|--------|---------|
| 20  | 0.43792(10)   | 0.22144(8) | 0.12305(5) | 0.07316(3) |
| 40  | 0.36672(3)    | 0.15648(2) | 0.07366(2) | 0.03721(1) |
| 80  | 0.30770(3)    | 0.11059(2) | 0.04395(2) | 0.01878(2) |
| 160 | 0.25812(5)    | 0.07800(3) | 0.02609(1) | 0.009398(5) |
| 320 | 0.21691(4)    | 0.05513(3) | 0.01552(1) | 0.004707(4) |
### TABLE IV. Cumulants and ratios for the critical A model.

| $L$ | $K_2$       | $K_2/m_1^2$ | $K_4/K_2^2$ | $m_4/m_2^2$ |
|-----|-------------|-------------|-------------|-------------|
| 20  | 0.02972(6)  | 0.1550(4)   | -0.608(4)   | 1.492(2)    |
| 40  | 0.021995(4) | 0.1636(1)   | -0.566(1)   | 1.520(1)    |
| 80  | 0.015914(4) | 0.1681(1)   | -0.538(1)   | 1.536(1)    |
| 160 | 0.011375(5) | 0.1707(1)   | -0.522(2)   | 1.545(2)    |
| 320 | 0.008082(5) | 0.1718(2)   | -0.512(8)   | 1.549(3)    |

### TABLE V. Density moments for the critical PCP.

| $L$ | $m_1$       | $m_2$       | $m_3$       | $m_4$       |
|-----|-------------|-------------|-------------|-------------|
| 20  | 0.51654(8)  | 0.31178(4)  | 0.20815(4)  | 0.14927(4)  |
| 40  | 0.43101(3)  | 0.21800(2)  | 0.12228(2)  | 0.07390(1)  |
| 80  | 0.36123(3)  | 0.15327(2)  | 0.07215(1)  | 0.03663(1)  |
| 160 | 0.30308(4)  | 0.10789(1)  | 0.04262(1)  | 0.01815(1)  |
| 320 | 0.25448(4)  | 0.07604(1)  | 0.02521(1)  | 0.009010(3) |
| 640 | 0.21373(10) | 0.05363(3)  | 0.01493(2)  | 0.004480(5) |

### TABLE VI. Cumulants and ratios for the critical PCP.

| $L$ | $K_2$       | $K_2/m_1^2$ | $K_4/K_2^2$ | $m_4/m_2^2$ |
|-----|-------------|-------------|-------------|-------------|
| 20  | 0.044921(3) | 0.16836(6)  | -0.66102(3) | 1.5356(8)   |
| 40  | 0.032230(4) | 0.17349(4)  | -0.55536(3) | 1.5550(5)   |
| 80  | 0.022779(5) | 0.17457(7)  | -0.5152(5)  | 1.5593(8)   |
| 160 | 0.016038(6) | 0.1746(1)   | -0.5014(4)  | 1.559(1)    |
| 320 | 0.011279(5) | 0.1742(1)   | -0.496(2)   | 1.558(1)    |
| 640 | 0.007949(5) | 0.1740(3)   | -0.497(3)   | 1.558(4)    |
In Fig. 3 we plot the ratios $K_2/m_1^2$ versus $L^{-1}$ for each of the three one-dimensional models. For the CP and the A model we observe a very similar, nearly linear approach to a limit. (Least-squares quadratic fits of $K_2/m_1^2$ versus $L^{-1}$ yield limiting values of 0.1736 for the CP, 0.1732 for the A model, and 0.1738 for the PCP.) The PCP ratio initially approaches its limit more rapidly than the CP and A model do, overshoots the limit, and, for large $L$, approaches the limit from above. Thus it would appear that the dominant correction to scaling for the size-dependence of $P(\rho, L)$ is different in the PCP than in the CP and the A model. We conclude that for the DP class in 1+1 dimensions, $K_2/m_1^2 = 0.1735(5)$, with the uncertainty figure a subjective assessment based on the degree of regularity of the data.

The ratios $m_4/m_2^2$ in the three one-dimensional models are plotted versus $L^{-1}$ in Fig. 4. As before, the CP and A model exhibit very similar trends, and the PCP has a nonmonotonic approach to its apparent limit. Quadratic fits yield limiting values of 1.554, for the CP and the A model, and 1.558 for PCP. This suggests a value of 1.556(3) for the DP class in 1+1 dimensions. Our estimates for the cumulant ratio, $K_4/K_2^2$, are considerably more scattered. Quadratic fits to the data for for $L \geq 40$ yield limiting values of -0.505, -0.503, and -0.493 for the CP, A model, and PCP, respectively. We estimate the value of this ratio as -0.50(1) for the DP class in 1+1 dimensions.

IV. CONTACT PROCESS IN TWO DIMENSIONS

In this section we report results for the two dimensional CP at the critical point, $\lambda = \lambda_c = 1.6488 \ [10]$. We studied systems of $L \times L$ sites with $L$ ranging from 10 to 160, with a sample size ranging from $3 \times 10^6$ for $L = 10$ to $10^6$ for $L = 160$. The maximum duration of a trial runs from 1000 for $L = 10$ to $10^5$ for $L = 160$. We calculate moments from a sample of $10^4$ maximum-duration trials for $L = 10$ to $L = 80$, and of 5000 such trials for the largest $L$. Density moments and ratios are given in Tables VII and VIII, respectively. The results indicate that $K_2/m_1^2 = 0.3264(5)$ for the DP class in 2+1 dimensions. As in one dimension, $m_4/m_2^2$ is particularly stable, and here takes the value 2.094(8). Our results for $K_4/K_2^2$ are less precise; we conclude that this ratio takes a value of $-0.088(4)$.

| L  | $m_1$         | $m_2$           | $m_3$            | $m_4$            |
|----|---------------|-----------------|------------------|------------------|
| 10 | 0.15706(6)    | 0.03228(2)      | 0.00781(1)       | 0.002120(3)      |
| 20 | 0.09027(6)    | 0.01078(1)      | 0.001522(2)      | 0.0002416(4)     |
| 40 | 0.05200(3)    | 0.003587(3)     | 0.0002929(3)     | 0.00002693(4)    |
| 80 | 0.03003(2)    | 0.001197(2)     | 0.0000564(1)     | 0.00000300(1)    |
| 160| 0.01736(1)    | 0.0003998(5)    | 0.00001090(2)    | 0.335(1) $\times 10^{-6}$ |
TABLE VIII. Cumulants and ratios for the critical CP in two dimensions.

| $L$ | $K_2$   | $K_2/m_1^2$ | $K_4/K_2^2$ | $m_4/m_2^2$ |
|-----|---------|-------------|-------------|-------------|
| 10  | 0.00761(1) | 0.3085(3)   | -0.169(3)   | 2.034(6)    |
| 20  | 0.002631(2) | 0.3229(4)   | -0.115(3)   | 2.079(8)    |
| 40  | 0.0008832(7) | 0.3267(4)   | -0.090(4)   | 2.093(7)    |
| 80  | 0.0002946(3) | 0.3269(4)   | -0.089(5)   | 2.095(8)    |
| 160 | 0.0000983(1) | 0.3264(5)   | -0.088(4)   | 2.094(8)    |

V. SUMMARY

We have determined order parameter moments and cumulant ratios for one and two dimensional models in the directed percolation universality class. In one dimension, we studied three different models: the contact process, the closely-related A model, and the pair contact process. Our results lead to the estimates: $K_2/m_1^2 = 0.1735(5)$, $m_4/m_2^2 = 1.556(3)$ and $K_4/K_2^2 = -0.50(1)$, for the DP class in 1+1 dimensions. We also derived an improved estimate, $p_c = 0.077090(5)$, for the critical point of the one-dimensional pair contact process. In two dimensions we restricted our attention to the basic contact process, obtaining $K_2/m_1^2 = 0.3264(5)$, $m_4/m_2^2 = 2.094(8)$, and $K_4/K_2^2 = -0.088(4)$. 
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Figure Captions

FIG. 1. The ratio $K_2/m_1^2$ versus creation rate $\lambda$ in the one-dimensional CP. System sizes $L = 20, 40, ..., 320$, in order of increasing steepness.

FIG. 2. Scaled lifetime $t^* \equiv \tau/L^{\nu_\parallel/\nu_\perp}$ versus system size $L$ in the PCP. Diamonds: $p = 0.07708$; $\circ$: 0.07709; squares: 0.07710; triangles: 0.07712.

FIG. 3. $K_2/m_1^2$ versus $1/L$ for one-dimensional models at their critical points. $\circ$: CP; $+: A$ model; $\times$: PCP.

FIG. 4. Same as Fig. 3, but for $m_4/m_2^2$. 
FIG. 3
FIG. 4

The graph shows a plot of $m_4/m_2^2$ versus $1/L$.
\( \ln L \)

\( \ln t^* \)

FIG. 2
FIG. 1

![Graph](image-url)