Two-Level Fingerprinting Codes

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Abstract—We introduce the notion of two-level fingerprinting and traceability codes. In this setting, the users are organized in a hierarchical manner by classifying them into various groups: for instance, by dividing the distribution area into several geographic regions, and collecting users from the same region into one group. Two-level fingerprinting and traceability codes have the following property: As in traditional (one-level) codes, when given an illegal copy produced by a coalition of users, the decoder identifies one of the guilty users if the coalition size is less than a certain threshold \( t \). Moreover, even when the coalition is of a larger size \( s > t \), the decoder still provides partial information by tracing one of the groups containing a guilty user.

We establish sufficient conditions for a code to possess the two-level traceability property. In addition, we also provide constructions for two-level fingerprinting codes and characterize the corresponding set of achievable rates.

I. INTRODUCTION

In order to protect copyrighted digital content against unauthorized distribution or piracy, several combinatorial schemes have been proposed in the literature (see [4] for a survey). In this paper, we focus on two such techniques: fingerprinting codes [5] and traceability codes [6].

The owner (distributor) of the content hides a unique mark called a fingerprint in each licensed copy bought by a user. The collection of fingerprint assignments is referred to as a code. If a naive user distributes a copy of his fingerprinted content illegally, then the pirated copy can easily be traced back to the guilty user. However, if a group of users (pirates) form a coalition to detect the fingerprints and modify/erase them to create an illegal copy, then tracing a guilty user becomes a non-trivial task.

Fingerprinting and traceability codes assign fingerprints in such a way that given an illegal copy, the distributor can use a tracing algorithm to identify at least one of the pirates as long as the coalition size does not exceed a certain threshold \( t \), which is a parameter of the problem. However, if the coalition size exceeds this threshold, the output of the tracing algorithm can be useless.

To overcome this weakness, we formalize the notion of multi-level fingerprinting codes, which are inspired by error-correcting codes with unequal error protection used in communications problems (cf. for instance Bassalygo et al. [3]). We focus on the simplest case of two-level fingerprinting codes in this paper, but the concepts introduced apply to an arbitrary number of protection levels.

In this setting, the users are organized in a hierarchical manner, for instance, according to geographical location. The distribution area is divided into several regions, and users from the same region are collected into one group. The two-level fingerprinting codes studied in this paper have the following property: As in traditional (one-level) codes, the tracing algorithm determines at least one of the guilty users if the coalition size is at most \( t \). Moreover, even when a larger number \( s > t \) of pirates participate, the algorithm provides partial information by retrieving the index of a group that contains a member of the pirate coalition.

Formal definitions are available in Section II. In Section III we obtain sufficient conditions for two-level traceability codes. Finally, we provide constructions for two-level fingerprinting codes and analyze the achievable rates in Section IV.

II. PROBLEM STATEMENT

Consider the problem where the content is to be distributed to \( M_1 M_2 \) users organized in \( M_1 \) groups, each of which contains \( M_2 \) users. Assume that there is some ordering of the groups, and of the users within each group. Thus, any user \( u \) is identified by a pair of indices \( u = (u_1, u_2) \in [M_1] \times [M_2] \), where the notation \([n]\) stands for the set \( \{1, \ldots, n\} \). For a user \( u = (u_1, u_2) \), let \( \mathcal{G}(u) \) be its group index, i.e., \( \mathcal{G}(u) = u_1 \).

The distributor hides a distinctive fingerprint in each legal copy. The fingerprints are assumed to be distributed inside the host message so that their location is unknown to the users. The location of the fingerprints is the same for all users.

Let \( n \) denote the length of the fingerprints. Let \( Q \) denote an alphabet of (finite) size \( q \), usually taken to be \( \{0, \ldots, q - 1\} \) with modulo \( q \) addition. An \((n, M)_q\) (one-level) code \((C, D)\) is a pair of encoding and decoding mappings \( C: [M] \to Q^n \), \( D: Q^n \to [M] \cup \{0\} \), where the decoder output 0 signifies a decoding failure. For convenience, we sometimes abuse terminology by calling the range of \( C \) a code, and use the same notation \( C \) for it.

The distributor’s strategy of assigning fingerprints to users may be either deterministic or randomized as explained in the following subsections. Randomization can potentially increase the number of users that can be supported for a given fingerprint length at the cost of a small error probability.

Notation: Throughout we will denote random variables (r.v.’s) by capital letters and their realizations by lower case letters. The Hamming distance between vectors \( x, y \) will be written as \( d_H(x, y) \), while \( |x| \) denotes the Hamming weight of \( x \). If \( \mathcal{X} \) is a set of vectors, we abbreviate \( \min_{x \in \mathcal{X}} d_H(x, y) \) as \( d_H(\mathcal{X}, y) \). We will denote the \( q \)-ary entropy function by \( h(x) = -x \log_q x/(q - 1) - (1 - x) \log_q (1 - x) \). For two functions \( f(n), g(n) \), we write \( f(n) \sim g(n) \) if \( \lim_{n \to \infty} n^{-1} \log(f(n)/g(n)) = 0 \).
A. Deterministic Codes

An \((n, M_1, M_2)_q\) two-level code \((C, D_1, D_2)\) is a triple consisting of one encoding and two decoding mappings
\begin{align}
C &: [M_1] \times [M_2] \rightarrow \mathbb{Q}^n, \\
D_1 &: \mathbb{Q}^n \rightarrow [M_1] \cup \{0\}, \\
D_2 &: \mathbb{Q}^n \rightarrow ([M_1] \times [M_2]) \cup \{0\},
\end{align}
with 0 signifying a decoding failure. A two-level deterministic assignment of fingerprints is given by the encoding mapping \(C\) of such a two-level code. The rate pair of an \((n, M_1, M_2)_q\) two-level code is defined as
\[(R_1, R_2) := \left(\frac{1}{n} \log_q M_1, \frac{1}{n} \log_q M_2\right).\]

A coalition of users is an arbitrary subset of \([M_1] \times [M_2]\). Members of the coalition are commonly referred to as pirates. A coalition \(U\) has access to the collection of fingerprints, namely \(C(U)\), that are assigned to it. Let \(U\) be a coalition of \(t\) users and suppose \(C(U) = \{x_1, \ldots, x_t\}\). In order to conceal their identities from the distributor, the coalition’s members attempt to create a pirated copy with a modified fingerprint \(y \in \mathbb{Q}^n\). We assume that the code \((C, D_1, D_2)\) is public and can be used by the pirates in designing their attack.

Note that although the fingerprint locations are not available to the pirates, they may detect some of these locations by comparing their copies for differences and modify the detected positions. Coordinate \(i\) of the fingerprints is called undetectable for the coalition \(U\) if \(x_{1i} = x_{2i} = \cdots = x_{ti}\) and is called detectable otherwise. The set of forgeries that can be created by the coalition in this manner is called the envelope and is given by:
\[E(x_1, \ldots, x_t) = \{y \in \mathbb{Q}^n \mid y_i \in \{x_{1i}, \ldots, x_{ti}\}, \forall i \in [n]\}.\]

Given a pirated copy with a forged fingerprint, the distributor performs tracing based on \(D_1\) and \(D_2\) to locate one of the pirates. The decoder \(D_2\) attempts to trace the exact identity of one of the pirates, while \(D_1\) focuses only on locating a group containing at least one of the pirates.

In order to extend the notion of traceability to two-level codes, let us consider the case where the tracing is accomplished using minimum distance (MD) decoding. Specifically, we take
\begin{align}
D_2(y) &= \arg \min_{u \in [M_1] \times [M_2]} d_H(C(u), y), \\
D_1(y) &= \mathcal{G}(D_2(y)).
\end{align}
If the minimum distance above is attained for multiple users, the decoder \(D_2\) outputs any one of the closest users. This leads us to the notion of two-level traceability codes in the deterministic setting.

Definition 2.1: A two-level code \(C\) has \((t_1, t_2)\)-traceability property (or is \((t_1, t_2)\)-TA) where \(t_1 > t_2\) if:
(a) For any coalition \(U\) of size at most \(t_2\) and any \(y \in E(C(U))\), the decoding result \(D_2(y) \in U\).
(b) For any coalition \(U\) of size at most \(t_1\) and any \(y \in E(C(U))\), the decoding result \(D_1(y) \in \mathcal{G}(U)\).

We observe that an \((n, M_1, M_2)_q\) two-level code which is \((t_1, t_2)\)-TA has the \(t_2\)-TA property when viewed as an \((n, M_1M_2)_q\) one-level code; moreover, for coalitions of the larger size \(t_1\), one of the groups containing a pirate is closer to the forgery compared to the remaining groups. In this paper, we examine sufficient conditions under which a two-level code has the \((t_1, t_2)\)-traceability property.

B. Randomized Codes

A randomized strategy to assign fingerprints is defined as the following random experiment. The distributor has a family of \((n, M_1, M_2)_q\) two-level codes \(\{(C_k, D_{1k}, D_{2k}), k \in K\}\), where \(K\) is a finite set of “keys”. The distributor chooses one of the keys according to a probability distribution \(\{\pi(k), k \in K\}\). If the key \(k\) is selected, then fingerprints are assigned according to \(C_k\) and tracing is done using \(D_{1k}\) and \(D_{2k}\). The code resulting from this random experiment is called a (two-level) randomized code and is denoted by \((C, D_1, D_2)\).

Following the standard convention in cryptography of the system design being publicly available, we allow the users to have knowledge of the family of codes \(\{(C_k, D_{1k}, D_{2k})\}\) and the distribution \(\pi(\cdot)\), while the exact key choice is kept secret by the distributor.

Consider a coalition \(U\) of size \(t\). Any attack by the coalition can be modeled as a randomized strategy \(V(\cdot, \ldots, \cdot)\), where \(V(y|x_1, \ldots, x_t)\) gives the probability that the coalition creates \(y\) given that it observes the fingerprints \(x_1, \ldots, x_t\). Our interest is in a special class of strategies which satisfy the restrictions 2 in creating a forgery. A strategy \(V\) is called admissible if
\[V(y|x_1, \ldots, x_t) = 0 \text{ for all } y \notin E(x_1, \ldots, x_t).\]
Let \(V_t\) denote the class of admissible strategies.

Denote the random forgery generated by \(U\) using the strategy \(V\) by \(Y_{C,U,V}\). The distributor, on observing the forged fingerprint, employs the decoders \(D_{1k}\) and \(D_{2k}\) while using the key \(k\). For a given coalition \(U\) and strategy \(V\), we define the following error probabilities:
\begin{align}
e_1(C, D_1, U, V) &= P[D_1|Y_{C,U,V} \neq \mathcal{G}(U)], \\
&= E_{y:D_1k(y) \notin \mathcal{G}(U)} V(y|C_k(U)), \\
e_2(C, D_2, U, V) &= P[D_2|Y_{C,U,V} \neq \mathcal{G}(U)], \\
&= E_{y:D_2k(y) \notin \mathcal{G}(U)} V(y|C_k(U)),
\end{align}
where the expectation is over the r.v. \(K\) with distribution \(\pi(k)\).

Definition 2.2: A randomized code \((C, D_1, D_2)\) is said to be a \((t_1, t_2)\)-fingerprinting with \(\varepsilon\)-error where \(t_1 > t_2\) if:
(a) For any coalition \(U\) of size at most \(t_2\) and any admissible strategy \(V\), the error probability \(e_2(C, D_2, U, V) \leq \varepsilon\).
(b) For any coalition \(U\) of size at most \(t_1\) and any admissible strategy \(V\), the error probability \(e_1(C, D_1, U, V) \leq \varepsilon\).

We observe that an \((n, M_1, M_2)_q\) two-level code which is \((t_1, t_2)\)-fingerprinting has the \(t_2\)-fingerprinting property when
viewed as an \((n, M_1M_2)\) one-level code; in addition, for the larger size-\(t_1\) coalitions, the tracing algorithm can locate a group containing one of the pirates with high probability.

A rate pair \((R_1, R_2)\) is said to be achievable for \(q\)-ary \((t_1, t_2)\)-fingerprinting if there exists a sequence of \((n, q^{t_1R_1}, q^{t_2R_2})\) randomized codes that are \((t_1, t_2)\)-fingerprinting with error probability \(\varepsilon_n\) such that

\[
\lim_{n \to \infty} \varepsilon_n = 0, \quad \lim \inf_{n \to \infty} R_{n} = R_i, \quad i = 1, 2.
\]

The goal of this paper is to investigate constructions of two-level fingerprinting codes and to characterize the corresponding set of achievable rate pairs.

**Remark 2.3:**
1. If an \((n, M_1, M_2)_q\) two-level code is \((t_1, t_2)\)-fingerprinting (resp., TA), then choosing any single user from every group forms an \((n, M_1)_q\) one-level code that is \(t_1\)-fingerprinting (resp., TA).
2. If an \((n, M_1, M_2)_q\) one-level code is \(t_1\)-fingerprinting (resp., TA), then for any \(t_2 < t_1\), it can also be treated as a \((n, M_1, M_2)_q\) two-level code that is \((t_1, t_2)\)-fingerprinting (resp., TA).

### III. Traceability Codes

It is known \([6]\) that a one-level code of length \(n\) is \(t\)-TA if the distance between any pair of fingerprints is strictly greater than \(n(1-1/t^2)\). We wish to obtain an analogous result for the case of two-level codes.

For a given two-level code \(C\), we define the following minimum distances:

\[
d_1(C) := \min_{\mathbf{u}, \mathbf{v} \in [M_1] \times [M_2]} d_H(C(\mathbf{u}), C(\mathbf{v})),
\]

\[
d_2(C) := \min_{\mathbf{u}, \mathbf{v} \in [M_1] \times [M_2]} d_H(C(\mathbf{u}), C(\mathbf{v})).
\]

### IV. Fingerprinting Codes

For \(w \in [n]\), denote \(S_{w,n} := \{x \in Q^n : |x| = w\}\). For \(R_1, R_2 \in [0, 1]\), define \(M_{1n} = [q^{nR_1}], M_{2n} = [q^{nR_2}]\). Fix \(\omega \in [0, 1]\). We take \(n\) such that \(\omega = \omega_n\) is an integer and construct an \((n, M_{1n}, M_{2n})_q\) two-level randomized code \((c_n, D_{1n}, D_{2n})\) as follows.

For \(i \in [M_{1n}]\), pick vectors \(R_i\) independently and uniformly at random from \(Q^n\). We will refer to the \(R_i\)'s as “centers”.

Choose \(S_{ij}, (i, j) \in [M_{1n}] \times [M_{2n}]\), independently and uniformly at random from \(S_{w,n}\). Generate \(M_{1n}M_{2n}\) fingerprints

\[
X_{ij} = R_i + S_{ij}, \quad (i, j) \in [M_{1n}] \times [M_{2n}]
\]

and assign \(X_{ij}\) as the fingerprint for user \((i, j)\).

Once the fingerprints are assigned, tracing is based on the MD decoder \([3]\). The MD decoder may be sub-optimal in general; however, it is amenable in our construction.

In the following subsections, we analyze the error probability and characterize the achievable rate pairs for the above construction. The lemmas below will be useful in the analysis.

**Lemma 4.1:** Let \(S\) have a uniform distribution on \(S_{w,n}\). Then, for \(l \in [n]\) and \(a \in Q \setminus \{0\}\), \(P\{S_l = a\} = \omega/(q-1)\). Moreover, the r.v.'s \(\{S_l, l \in [n]\}\) are asymptotically pairwise independent.

**Lemma 4.2:** Fix \(p \in [0, 1]\) and \(\varepsilon > 0\). For \(l \in [n]\), let \(Z_l\) be a Bernoulli r.v. with \(P\{Z_l = 1\} = p\), and let \(\{Z_l, l \in [n]\}\) be pairwise independent. Then, with \(Z := \sum_{l \in [n]} Z_l\), we have

\[
P\{Z \notin [n(p-\varepsilon), n(p+\varepsilon)]\} \leq \frac{p(1-p)}{\varepsilon^2 n}.
\]

**Notation:** For a coalition \(U = \{u^1, \ldots, u^t\}\), we denote the realizations of \(X_{u^1}, X_{u^2}, \ldots, X_{u^t}\) by \(x_1, x_2, \ldots, x_t\) respectively, with \(x_i = r_i + s_i, i \in [t]\). Let \(z \in Q^t\) be a vector. Denote by \(s_z(x_1, \ldots, x_t)\) the number of columns equal to \(z^t\) in the matrix whose rows are \(x_1, \ldots, x_t\).

### A. \((t, 1)\)-fingerprinting

First, we consider the \((2, 1)\)-fingerprinting property. This is the simplest case of two-level fingerprinting that goes beyond the known techniques for one-level codes. Although coalitions of size 1 are trivial to handle for one-level fingerprinting, it is still non-trivial to construct a \((2, 1)\)-fingerprinting code.

**Theorem 4.3:** For any \(\omega \in [0, (q-1)/2q]\), the randomized code \((c_n, D_{1n}, D_{2n})\) is \((2, 1)\)-fingerprinting with error probability decaying to 0 if

\[
R_1 < 1 - h((q-1)/2q + \omega),
\]

\[
R_2 < h(\omega).
\]
Consider the inner probability term $V$ the analysis for the two cases is similar, we only consider the users are in the same group or they are in different groups. 

Proof: (of Theorem 4.3) Size-1 coalitions: Let $u = (u_1, u_2)$ be the pirate. For size-1 coalitions, the envelope is degenerate as it consists of only the user’s own fingerprint. Now,

$$
e_2(C_n, D_{2n}^2, u) = P[\exists u' \neq u : X_{u'} = X_u] 
\leq P[\exists u' \neq u : u_1' = u_1, X_{u'} = X_u] + P[\exists u' \neq u : u_1' \neq u_1, X_{u'} = X_u]$$

$$\leq P[\exists u_1' \neq u_2 : S_{u_1u_2} = S_{u_1u_2}] + P[\exists u_1' \neq u : d_H(R_{u_1'}, X_u) \leq w]$$

$$\leq q^{R_1} P[D_{2n}^2, u_1u_2 = S_{u_1u_2}] + q^{R_1} P[d_H(R_{u_1'}, X_u) \leq w]$$

where (a) is due to the fact that if the fingerprint of another user matches with the pirate’s fingerprint, then the corresponding center is within distance $w$ from the pirate’s fingerprint, and (b) follows from the union bound. Consequently, the error probability for size-1 coalitions approaches 0 if $R_2 < h(\omega)$ and $R_1 < 1 - h(\omega)$.

Size-2 coalitions: There are two possibilities: either both users are in the same group or they are in different groups. It turns out that the latter case is the dominant one. Since the analysis for the two cases is similar, we only consider the latter case below.

Let $U = \{u^1, u^2\}$ be such a coalition. For any strategy $V \in \nu_2$, we have

$$e_1(C_n, D_{2n}^2, U, V) = \sum_{r_1, r_2, s_1, s_2} P[r_1, r_2, s_1, s_2] \sum_{y} V(y|x_1, x_2)$$

$$\times P[D_{2n}^2, x \in G(U) \mid r_1, r_2, s_1, s_2].$$

Consider the inner probability term

$$P[D_{2n}^2, x \in G(U) \mid r_1, r_2, s_1, s_2]$$

$$\leq P[\exists u' \neq U : u_1' \neq G(U), d_H(X_{u'}, y) \leq d_H(\{x_1, x_2\}, y)]$$

where we have exploited the independence in the construction in (a), and (b) follows because if the fingerprint of another user is within distance $d$ from $y$, then the corresponding center is within $d + w$ from $y$. For $\epsilon > 0$, define

$$T_n^\epsilon := \{r_1, r_2, s_1, s_2 : \exists a \in Q, s_{(a,a)}(x_1, x_2) \in I_n(1/q^2, \epsilon/q)\}.$$ 

Observe that $X_{u_1}$ and $X_{u_2}$ are independent and uniformly distributed over $Q^m$. Therefore, using Lemma 4.2, it is a simple matter to show that $P[D_{2n}^2, X_{u_1}, X_{u_2} \notin T_n^\epsilon]$ decays to 0 as $n \to \infty$. Now, take any $(r_1, r_2, s_1, s_2) \in T_n^\epsilon$ and $y \in E(x_1, x_2)$. The number of undetectable positions in $\{x_1, x_2\}$ is at least $n(1/q - \epsilon)$, implying that $d_H(\{x_1, x_2\}, y) \leq 1/q - \epsilon$. Thus, in this case

$$q^{nR_1} P[d_H(R_{u_1'}, y) \leq d_H(\{x_1, x_2\}, y) + w]$$

$$\leq q^{nR_1} P[d_H(R_{u_1'}, y) \leq \frac{n}{m} \left(1 - \frac{1}{q} + \epsilon\right) + w]$$

where $m$ is the number of pirates.

Substituting the above in (8) and taking $\epsilon \to 0$, we conclude that the error probability for size-2 coalitions approaches 0 if (6) holds.

We now extend the techniques to larger coalitions.

Theorem 4.4: For any $\omega$ such that $\frac{t-1}{t} \leq \frac{1}{1-q^{-1}} + \omega \leq \frac{1}{q^{-1}}$, the randomized code $(C_n^t, D_{1n}^t, D_{2n}^t)$ is $(t, 1)$-fingerprinting with error probability decaying to 0 if

$$R_1 < 1 - h\left(\frac{t-1}{t} \left(1 - \frac{1}{q^t-1}\right) + \omega\right),$$

(9)

$$R_2 < h(\omega).$$

(10)

Proof: Size-1 coalitions: For a single pirate $u$, the analysis in Theorem 4.3 proves that the probability of decoding error approaches 0 if $R_2 < h(\omega)$ and $R_1 < 1 - h(\omega)$.

Size-$t$ coalitions: It can be shown that the case where the $t$ pirates are in distinct groups is the dominant one. Once this is shown, we use exactly the same arguments as in the case of size-2 coalitions in Theorem 4.3. We finally obtain that the error probability for coalitions of size $t$ approaches 0 if (9) holds.

Remark 4.5: A sufficiently large alphabet is required in order for an $\omega$ satisfying $\frac{t-1}{t} \leq \frac{1}{1-q^{-1}} \leq \frac{1}{q^{-1}}$ to exist. For instance, it suffices to take $q \geq t+1$.

B. $(t, 2)$-fingerprinting

Let $q \geq 3$. For $\omega, \gamma, \alpha, \beta \in [0, 1]$, with $\alpha \leq 1 - \gamma$, $\beta \leq \gamma$, $\alpha + \beta \leq \omega$, $\omega - \alpha \leq \gamma$, let

$$\phi(\omega, \gamma, \alpha, \beta) := (1 - \gamma)h\left(\frac{\alpha}{1-\gamma}\right) + (\gamma - \beta)h\left(\frac{\omega - \alpha - \beta}{\gamma - \beta}\right) + \gamma h\left(\frac{\beta}{\gamma}\right) + (\omega - \alpha) \log_q\left(\frac{q-2}{q-1}\right) - \beta \log_q(q-2).$$
Let
\[ \delta_1(\omega) = \frac{1}{2} \left( 1 - (1 - \omega)^2 - \frac{\omega^2}{q - 1} \right), \]
\[ \delta_2(\omega) = \frac{1}{2} \left( 1 - \frac{1}{q} \right), \]
\[ f_1(\omega) = \max_{\gamma, \alpha, \beta} \varphi(\omega, \gamma, \alpha, \beta), \]
\[ f_2(\omega) = \max_{\gamma, \alpha, \beta} \varphi(\omega, \gamma, \alpha, \beta). \]

**Theorem 4.6:** Let \( q \geq 3 \). For any \( \omega \) such that \( \frac{t}{t-1}(1 - \frac{1}{q}) + \omega \leq \frac{q-1}{q} \), the randomized code \( (C_n^\omega, D_{1n}^\omega, D_{2n}^\omega) \) is \((t, 2)\)-fingerprinting with error probability decaying to 0 if
\[ R_1 < 1 - \left( \frac{t}{t-1}(1 - \frac{1}{q^2-1}) + \omega \right), \]
(11)
\[ R_2 < h(\omega) - \max(f_1(\omega), f_2(\omega)). \]
(12)

**Proof:** Size-\( t \) coalitions are handled in the same way as in Theorem 4.4.

**Size-2 coalitions:** There are two possibilities depending on whether the pirates belong to the same group or not. We sketch the case where they are in different groups below. The other case is analyzed similarly.

Consider a coalition \( \mathcal{U} = \{u^1, u^2\} \), where the users are in different groups, and let \( V \subseteq V_2 \) be an admissible strategy. We have
\[ e_2(C_n^\omega, D_{2n}^\omega, U, V) \]
\[ = \sum_{r_1, r_2, s_1, s_2} P[r_1, r_2, s_1, s_2] \sum_y V(y|x_1, x_2) \times P[D_{2n}^\omega(y) \notin U|r_1, r_2, s_1, s_2]. \]
(13)

Now,
\[ [D_{2n}^\omega(y) \notin U] = E_1 \cup E_2 \cup E_3, \]
where, the events \( E_1, E_2, E_3 \) are formed of those \( u^i \notin U \) that satisfy \( d_H(X_{u^i}, y) \leq d_H(x_1, x_2, y) \) and the conditions \( u^i_1 = u^1_1, u^i_1 = u^2_1, u^i_2 \notin G(U) \), respectively. The error event \( E_3 \) was already analyzed in Theorem 4.4 and its conditional probability approaches 0 if \( \delta_4 \) holds. We consider \( E_1 \) below. The analysis for \( E_2 \) is identical by symmetry.

\[ P[E_1|r_1, r_2, s_1, s_2] \]
\[ = P[\exists u \notin u^1_1 : d_H(r_1 + S_{u_1}u_2, y) \leq d_H(x_1, x_2, y)] \]
\[ \leq q^{nR_2}P[d_H(r_1 + S_{u_1}u_2, y) \leq d_H(x_1, x_2, y)] \]
\[ = q^{nR_2}P[d_H(S_{u_1}u_2, \gamma, \beta) \leq d_H(x_1, x_2, y)] \]
(14)

where \( \gamma' = y + r_1 \in \mathcal{E}(s_1, r_1 + x_2) \). In this case, we use Lemmas 4.1 and 4.2 to show that
\[ T_n^\omega := \left\{ (r_1, r_2, s_1, s_2) : \left( \begin{array}{c} \text{for } a, a' \in Q \not\in \{0\} \end{array} \right) \right. \]
is the typical set. For simplicity, we have omitted \( \varepsilon \) and will use the approximate relations \( \simeq, \lesssim, \gtrsim \) in its place. Now, take any \( (r_1, r_2, s_1, s_2) \in T_n^\omega \) and \( \gamma' \in \mathcal{E}(s_1, r_1 + x_2) \). The number of undetectable positions in \( (s_1, r_1 + x_2) \) is \( \simeq n/q \), while the number of coordinates where both symbols are non-zero is \( \simeq n\omega(q - 1)/q \). This implies \( d_H(s_1, r_1 + x_2, y) \lesssim n\delta_2(\omega) \) and \( n\omega(q - 1)/q \lesssim |y'| \lesssim n(1 - (1 - \omega)/q) \).

Let \( |y'| = \gamma n \), where \( \gamma \in [0, 1] \). Then
\[ P[d_H(S_{u_1}u_2, \gamma) \leq n\delta_2(\omega)] \equiv q^{-nE(\omega, \gamma)}, \]
where
\[ E(\omega, \gamma) = h(\omega) - \max_{\alpha, \beta} \varphi(\omega, \gamma, \alpha, \beta). \]

Since \( \gamma \) can be chosen by the pirates such that \( \omega \simeq \gamma \simeq 1 - \frac{\omega}{q^2 - 1} \), by substituting the above in (14), we conclude that the conditional probability of \( E_1 \) and \( E_2 \) approaches 0 if \( R_2 < h(\omega) - f_2(\omega) \). Similarly, we obtain \( R_2 < h(\omega) - f_1(\omega) \) when the pirates are in the same group.

Let us show that the rate region thus defined is not trivial. Given \( \omega \) and \( \gamma \), the maximizing values of the other arguments of \( \varphi \) are \( \alpha = \omega(1 - \gamma) \) and \( \beta = \omega/(q - 1) \), so
\[ \varphi(\omega, \gamma, \alpha, \beta) \leq h(\omega) - \gamma \omega \left( \log_q \frac{q - 1}{q - 2} + \log_q \frac{q - 2}{q - 1} \right). \]
Consequently, we get \( \max(f_1(\omega), f_2(\omega)) \leq h(\omega) - D \), where
\[ D = D(\omega) = \omega^2 \left( \log_q \frac{q - 1}{q - 2} \right) \text{ and } D(\omega) > 0 \]
for all \( \omega > 0 \). This shows that the r.h.s. of (12) is positive. By Remark 4.3 the r.h.s. of (11) is also positive if \( q \geq t + 1 \) and \( \frac{t}{t-1}(1 - \frac{1}{q^2}) + \omega < \frac{q-1}{q^2} \). This calculation can be further refined because of the additional constraints on the parameters \( \alpha, \beta, \gamma \) mentioned above.

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