The geometry of $\mathbb{C}^2$ equipped with Warren’s metric

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Abstract

The aim of this note is to describe the geometry of $\mathbb{C}^2$ equipped with a Kähler metric defined by Warren in [4]. It is shown that with that metric $\mathbb{C}^2$ is a flat manifold. Explicit formulae for geodesics and volume of geodesic ball are also computed. Finally, a family of similar flat metrics is constructed.

1 Introduction

In his note [4] Warren showed an interesting construction of an entire solution to unimodular complex Monge-Ampère equation in $\mathbb{C}^2$ that depends only on three real parameters. This solution is a special case of non-convex solution to Donaldson’s equation, i.e.

$$u_{tt} \Delta u - |\nabla u|^2 = 1, \quad (t, x) \in \mathbb{R} \times \mathbb{R}^{n-1}.$$ 

As showed by Donaldson in [3], for $n = 3$ a multiple of a solution to above problem must also satisfy unimodular complex Monge-Ampère equation in complex dimension 2 with one ’artificial’ imaginary parameter.

Let $(z, w) = (x + iy, u + iv) \in \mathbb{C}^2$. Then Warren’s solution is

$$u(z, w) = 4|z|^2e^{Re(w)} + e^{-Re(w)}.$$ 

This plurisubharmonic function produces the Kähler metric $g = \frac{\partial^2 u}{\partial z \partial \bar{z}}$ on $\mathbb{C}^2$, such that det$(g) = 1$. Explicitly

$$g = \begin{pmatrix} \frac{\partial^2 u}{\partial z \partial \bar{z}} & \frac{\partial^2 u}{\partial z \partial w} \\ \frac{\partial^2 u}{\partial w \partial \bar{z}} & \frac{\partial^2 u}{\partial w \partial w} \end{pmatrix} = \begin{pmatrix} 4e^u & 2ze^u \\ 2ze^u & |z|^2e^u + \frac{1}{4}e^{-u} \end{pmatrix},$$

with inverse

$$\begin{pmatrix} 4e^{-u} + |z|^2e^u & -2ze^u \\ -2ze^u & 4e^u \end{pmatrix}.$$ 

The construction of this metric was based on a method for constructig non-polynomial solution to $k$-Hessian equations given by Warren in [5].
In the real case, it was proved by Jörgens, Calabi and Pogorelov that for \( n \geq 2 \), the only convex solutions to unimodular Monge-Ampere equation on the whole \( \mathbb{R}^n \) are quadratic functions. It is well known that similar property for entire plurisubharmonic solutions to complex Monge-Ampere equation does not hold in \( \mathbb{C}^n \). However, a question posed by Calabi in [2] whether a Kähler metric given by a complex Hessian of such a solution is flat is still open.

In this note the geometry of \((\mathbb{C}^2, g)\) is studied. First, the curvature tensor is computed directly using Christoffel symbols, showing that \( g \) is flat. Then the geodesic equations are solved, allowing one to describe geometry of geodesic balls. Finally, a family of similar flat metrics on \( \mathbb{C}^2 \) is constructed by slight generalization of Warren’s argument.

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2 Curvature

For any Kähler metric \( g \) the formulae for Christoffel symbols simplify (see [1]) to

\[
\Gamma^\alpha_{\beta\gamma} = \frac{\partial g_{\bar{\nu}\gamma}}{\partial z_\beta} g^{\bar{\nu} \alpha}, \quad \forall \alpha, \beta, \gamma \in \{1, \ldots, n\},
\]

where \( g_{\bar{\nu}\gamma} \) are the coefficients of the metric and \( g^{\bar{\nu} \alpha} \) are coefficients of the inverse. This makes the computation of Christoffel symbols for the metric \( g \) straightforward:

\[
\begin{align*}
\Gamma^z_{zz} &= 0, \\
\Gamma^z_{zw} &= \Gamma^z_{wz} = 2e^u \left( \frac{1}{4} e^{-u} + |z|^2 e^u \right) + z e^u (-2z e^u) = \frac{1}{2}, \\
\Gamma^w_{zw} &= ze^u \left( \frac{e^{-u}}{4} + |z|^2 e^u \right) - 2ze^u \left( \frac{|z|^2 e^u}{2} - \frac{e^{-u}}{8} \right) = \frac{z}{2}, \\
\Gamma^w_{zz} &= 0, \\
\Gamma^w_{wz} &= (2ze^u)2e^u + 4e^u z e^u = 0, \\
\Gamma^w_{ww} &= (-2ze^u)ze^u + 4e^u \left( \frac{|z|^2 e^u}{2} - \frac{e^{-u}}{8} \right) = -\frac{1}{2}.
\end{align*}
\]

The curvature tensor coefficients again simplify. For Kähler metrics, the only non-zero coefficients can be

\[
R^\delta_{\alpha\beta\gamma} = -\frac{\partial \Gamma^\delta_{\alpha\gamma}}{\partial z_\beta}, \quad R^\delta_{\alpha\beta\gamma} = \frac{\partial \Gamma^\delta_{\beta\gamma}}{\partial z_\alpha}, \quad \forall \alpha, \beta, \gamma \in \{1, \ldots, n\}
\]

and their conjugates. So, clearly the curvature of \( g \) vanishes.
3 Geodesics and Incompleteness

With Christoffel symbols computed, the geodesic equations in complex co-
ordinates are

\[
\frac{d^2 z}{dt^2} + \frac{dz}{dt} \frac{dw}{dt} + z \left( \frac{dw}{dt} \right)^2 = 0
\]

\[
\frac{d^2 w}{dt^2} - \frac{1}{2} \left( \frac{dw}{dt} \right)^2 = 0.
\]

(1)

The order of first equation can be reduced by plugging second equation
into it, thus producing

\[
\frac{dz}{dt} + z \frac{dw}{dt} = C,
\]

where \(C\) is a constant. Written in real coordinates equation (1) is

\[
u'' = \left( u' \right)^2 - \left( v' \right)^2,
\]

\[v'' = u'v'.\]

It can be solved given initial velocity vector \((U, V)\) at starting point
\((u_0, v_0)\). The second equation is a derivative of logarithm, so it reduces to

\[v' = V e^{u-u_0}.\]

Now, after putting this into first equation and making a substitution
\(\omega(u) = (u')^2\) one gets

\[u' = \pm \sqrt{(U^2 + V^2)e^{u-u_0} - V^2e^{2u-2u_0}},\]

where the sign depends on the sign of initial velocity component \(U\) and
time. Since this equation is autonomous it integrates to

\[u(t) = u_0 - \log \left( \frac{U^2 + V^2}{4} t^2 - Ut + 1 \right),\]

but it might not be defined for every \(t > 0\). If \(U \leq 0\), then the formula holds
for every \(t > 0\). For \(U > 0, V \neq 0\) the formula also holds but this time the
velocity \(u'\) switches sign at time \(t_0 = 2U/(U^2 + V^2)\) since then \(u'\) vanishes
and \(u''\) is negative. For \(U > 0\) and \(V = 0\) the curve \(u(t)\) reaches infinity
at \(t_0\), independently of the initial point. This shows incompleteness of the
metric \(g\), already established by Warren in [4].

Equipped with explicit formula for \(u\), one integrates \(v'\) to

\[v(t) = 2 \arctan \left( \frac{U}{V} \right) - 2 \arctan \left( \frac{U - \sqrt{U^2 + V^2} t}{V} \right) + v_0,\]

3
provided $V \neq 0$, otherwise $v(t) = v_0$.

Now, the equation for the first component of the geodesic is linear

$$\frac{dz}{dt} + z \frac{dw}{dt} = C$$

for some constant $C$ depending on the initial velocity. The solution then is

$$z(t) = e^{-w} \left[ (Z + z_0W) \int_0^t e^w + z_0 e^{w_0} \right],$$

Where $Z = X + iY, W = U + iV$ are the initial velocities and $z_0, w_0$ are starting points. Denote by $Q(t) = \frac{|W|^2 t^2 - Ut + 1}{4} = e^{u_0 - u}$ and $\vartheta(t) = v(t) - v_0$, then

$$z(t) = (Z + z_0W)[Q(t) \cos \vartheta(t) - iQ(t) \sin \vartheta(t)] \int_0^t \frac{\cos \vartheta(s) + i \sin \vartheta(s)}{Q(s)} ds + z_0[Q(t) \cos \vartheta(t) - iQ(t) \sin \vartheta(t)].$$

Then formulae for sine and cosine of arctangent yield

$$\cos \vartheta(s) = Q^{-1}(s) \left( \frac{U^2 - V^2}{4} s^2 - Us + 1 \right),$$

$$\sin \vartheta(s) = V Q^{-1}(s) \left( s - \frac{U}{2} s^2 \right),$$

so the integral is

$$\int_0^t e^{w - w_0} = Q^{-1}(t) \left[ t - \frac{U}{2} t^2 + i \frac{V}{2} t^2 \right] = Q^{-1}(t) \left[ 1 - \frac{|W|^2}{2} t^2 \right].$$

Finally,

$$z(t) = \left[ \frac{W^2}{4} t^2 - Wt + 1 \right] \left[ (Z + z_0W) Q^{-1}(t) \left( 1 + \frac{|W|^2}{2} t^2 \right) + z_0 \right]$$

$$= [Z + z_0W] \left[ t - \frac{W}{2} t^2 \right] + z_0 \left[ \frac{W^2}{4} t^2 + 1 \right]$$

$$= Z \left[ t - \frac{W}{2} t^2 \right] + z_0 \left[ - \frac{W^2}{4} t^2 + 1 \right] = \left[ 1 - \frac{W}{2} t \right] z_0 \left[ Z + \frac{z_0W}{2} \right] t.$$  

4 Geometry of geodesic balls

Since the metric is flat, one knows by the Frobenius theorem that around each point $z_0 \in (\mathbb{C}^2, g)$ there is a neighbourhood $U_{z_0}$ and a map $\varphi_{z_0} : U_{z_0} \rightarrow (\mathbb{C}^2, E)$ ($E$ is the usual Euclidean metric) that is an isometry onto its image.
Since \( \det(g) = 1 \), those two facts imply that for any \( z_0 \), the volume of geodesic ball \( B_g(z_0, r) \) is the same as volume of Euclidean ball of radius \( r \), for \( r \) sufficiently small.

In the case of the metric \( g \) more is true. Let \( \exp_g(z_0, \cdot) \) denote the exponential map from the metric ball \( \tilde{B}(z_0, r) = \{ Z \in T_{z_0}C^2 \mid g_{z_0}(Z, Z) < r^2 \} \) to the geodesic ball \( B_g(z_0, r) \) for fixed, small radius \( r \). Now if \( ^t g^{-1/2}(z_0, \cdot) \) is the transpose of the positive square root of matrix \( g^{-1} \) at point \( z_0 \) then \( \tilde{B}(z_0, r) = ^t g^{-1/2}(z_0, B_E(0, r)) \) and since \( \det(g) = 1 \), same is true for \( ^t g^{-1/2} \). So this reduces the volume of ball \( B_g(z_0, r) \) to formula

\[
\int_{B_g} d\text{vol}_g = \int_{B_E} \text{jac}(\exp_g(z_0, ^t g^{-1/2}(x))) d\text{vol}_E,
\]

and the Jacobian can be computed, since the formulae for the geodesics are given explicitly. Namely, if \((e_1, e_2)\) is the canonical basis of \( C^2 \), then in new variables \( X + iY = Z = ^t g^{-1/2}(z, e_1); U + iV = W = ^t g^{-1/2}(z, e_2) \) the equation for metric ball at \( z \) becomes \( |Z|^2 + |W|^2 \leq r^2 \). Computing Jacobian of exponential map in this variables reduces to computing derivatives of geodesics with respect to variables \((X, Y, U, V)\) at time \( t = 1 \) thus revealing that

\[
\det
\begin{pmatrix}
\begin{array}{cccc}
2 - U & V & . & . \\
-2V & 2 - U & . & . \\
0 & 0 & 2 - U & -V \\
0 & 0 & 2Q(1) & 2 - U \\
\end{array}
\end{pmatrix}
= 1.
\]

So up until \( U = 2 \) the geodesic ball has the same volume as the Euclidean ball. At \( U = 2 \) the entries in the matrix above are undefined for \( V = 0 \), since then \( Q(1) = 0 \).

That means the set of directions \( N \) for which the geodesics reach infinity in finite time is negligible, namely \( N = \{(X, Y, U, 0) \in TC^2 \mid U > 0 \} \). Any geodesic with intial velocity \( V \neq 0 \) or \( U \leq 0 \) can be extended indefinitely at any initial point and the set \( B_g(\cdot, r) \setminus N \) will have the same volume as euclidean \( r \)-ball for any \( r \). Here the set \( B_g(\cdot, r) \setminus N \) is understood as set of constant \((\leq r)\) speed geodesics with the same initial point and with initial velocities in complement of \( N \).

5 Generalization

The orginal approach of Warren was based on the problem of finding non-polynomial solutions to \( k \)-Hessian equations. There, given \((x, y) \in \mathbb{R}^{n-1} \times \mathbb{R} \)
one was looking for a function $h(y)$, such that function $u(x, y) = \|x\|^2 \exp(y) + h(y)$ satisfies

$$\sigma_k(D^2 u) = 1, \quad 2k - 1 \leq n,$$

with $\sigma_k$ being $k$-th symmetric polynomial. For $n = 3$ and $k = 2$ with $h(y) = e^{-y}/4$ this gave a solution to Donaldson’s equation and the metric $g$ under consideration. The aim of this section is to generalize the construction of $g$. This is done by noticing that leaving the quadratic part in the potential $u$ and changing $\exp(y)$ and $h(y)$ can still produce flat metrics with unit determinant under suitable assumptions.

**Proposition 1.** Let $h(z)$ be the coefficient of a flat metric on $\mathbb{C}$, then metric $K_h$ defined as

$$K_h = \begin{pmatrix} h^{-1} & \frac{z}{z} \frac{\partial h^{-1}}{\partial w} \\ \frac{\partial h^{-1}}{\partial w} & |z|^2 \left( \frac{\partial^2 h^{-1}}{\partial w \partial \bar{w}} + h \right) \end{pmatrix}$$

is a flat Kähler metric on $\mathbb{C}^2$ such that $\det K_h = 1$.

**Proof.** The proof of Kählerness follows directly from computation. Similarly for the determinant:

$$\det(K_h) = 1 + |z|^2 \left( h^{-1} \frac{\partial^2 h^{-1}}{\partial w \partial \bar{w}} - \left| \frac{\partial h^{-1}}{\partial w} \right|^2 \right) = 1 + |z|^2 (h^{-2} R(h^{-1})) = 1,$$

where $R(h^{-1})$ is curvature of $h^{-1}$, which is also zero.

Let $\Gamma^a_{\beta\gamma}$ denote the Christoffel symbols of $K_h$. Except for $\Gamma^z_{ww}$ and $\Gamma^w_{ww}$ the computation is straightforward, showing that every symbol is holomorphic. For the remaining two one notices that since $h \in C^\infty(\mathbb{C}, (0, \infty))$ it can be represented as $h = e^f$ for some smooth $f$, then flatness implies that $f$ satisfies

$$\Delta f = 0,$$

thus $f$ is harmonic. Then

$$\Gamma^z_{ww} = ze^{-f} \left( \left( \frac{\partial f}{\partial w} \right)^2 - \frac{\partial^2 f}{\partial w^2} \right) \left( |z|^2 e^{-f} \left| \frac{\partial f}{\partial w} \right|^2 + e^f \right) +$$

$$\left( |z|^2 e^{-f} \frac{\partial f}{\partial w} \left( \frac{\partial^2 f}{\partial w^2} - \left( \frac{\partial f}{\partial w} \right)^2 \right) + e^f \frac{\partial f}{\partial w} \right) \left( ze^{-f} \frac{\partial f}{\partial w} \right) =$$

$$z \left( 2 \left( \frac{\partial f}{\partial w} \right)^2 - \frac{\partial^2 f}{\partial w^2} \right),$$

$$\Gamma^w_{ww} = ze^{-f} \left( \left( \frac{\partial f}{\partial w} \right)^2 - \frac{\partial^2 f}{\partial w^2} \right) \left( ze^{-f} \frac{\partial f}{\partial w} \right) +$$

$$ze^{-f} \frac{\partial f}{\partial w} \left( \frac{\partial^2 f}{\partial w^2} - \left( \frac{\partial f}{\partial w} \right)^2 \right) =$$

$$ze^{-f} \frac{\partial f}{\partial w} \left( \frac{\partial f}{\partial w} \right)^2 - ze^{-f} \frac{\partial f}{\partial w} \left( \frac{\partial^2 f}{\partial w^2} \right) +$$

$$ze^{-f} \frac{\partial f}{\partial w} \left( \frac{\partial f}{\partial w} \right)^2.$$
\[
\left|z\right|^2 e^{-f} \frac{\partial f}{\partial w} \left( \frac{\partial^2 f}{\partial w^2} - \left( \frac{\partial f}{\partial w} \right)^2 \right) + e^{f} \frac{\partial f}{\partial w} \right) e^{-f} = \frac{\partial f}{\partial w}.
\]

Both symbols are holomorphic and $K_h$ is flat.

Now, taking $h = \frac{1}{4} e^{-\text{Re}(w)}$, or equivalently $f = -\text{Re}(w) - \log 4$ produces Warren’s metric. This choice of $h$ also explains incompleteness of $g$. Namely, consider $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ equipped with Euclidean metric. This manifold is incomplete with geodesics being line segments. Let $\mathbb{C}$ be the universal cover of $\mathbb{C}^*$ with projection $\exp(z)$. Then $\mathbb{C}$ can be equipped with a metric $h'$ that will turn $\exp(z)$ into a local isometry, that is, for $X, Y \in T_z \mathbb{C}$ let

\[\langle X, Y \rangle^h_{z} := \exp(2\text{Re}(z)) \langle X, Y \rangle^\mathbb{C}.
\]

Then $h'$ is incomplete, since $\exp(z)$ isometrically maps lines $\text{Im}(z) = \text{const.}$ into radial lines in $\mathbb{C}^*$ which are geodesics. One gets $h$ just by linear change of coordinates. The only thing left to notice is that $\{0\} \times \mathbb{C}$ is a submanifold of $(\mathbb{C}^2, g)$ that is isometric to $(\mathbb{C}, h)$.

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