The Complex Lagrangian Germ
and the Canonical Operator

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Abstract

We give a manifestly invariant definition of the Lagrangian complex germ with the
minimal degree of accuracy required to define the canonical operator. The equivalence
with the traditional definition is proved, and the canonical operator is constructed in
new terms. A new form of the quantization condition is given, in which the volume
form is assumed to be defined on the universal covering of the Lagrangian manifold
rather than on the manifold itself. This allows one to solve a wider class of eigenvalue
problems.

0 Introduction

The complex WKB method, originally developed in [8, 9], is a method for constructing
asymptotic solutions to $1/h$-(pseudo)differential equations. In the simplest case, it deals
with asymptotic expansions of the form

$$
\psi(x, h) = \exp \left\{ \frac{i}{h} S(x) \right\} \sum h^k \varphi_k(x) + O(h^N),
$$

(0.1)

where $S$ and $\varphi_k$ are smooth functions and $\text{Im} \ S \geq 0$.

The following observation is of crucial importance:

$$
\varphi(x) \exp \left\{ \frac{i}{h} S(x) \right\} = O(h^s)
$$

whenever $\varphi(x) = O((\text{Im} \ S)^r)$, and $\partial \text{Im} \ S / \partial x = O((\text{Im} \ S)^{1/2})$. Thus, expansions in powers
of $\text{Im} \ S$ and $\partial \text{Im} \ S / \partial x$ in the amplitude result in expansions in powers of $h$ in (0.1), and
the Hamilton-Jacobi equation $H(x, \partial S/\partial x) = 0$ makes sense and can be solved even for
nonanalytic Hamiltonians, since it can be understood as an asymptotic expansion in powers
of $\partial (\text{Im} \ S) / \partial x$. It turns out that the minimum accuracy in specifying $S(x)$ which still
allows one to obtain asymptotic expansions of solutions modulo $O(h^N)$ with arbitrary $N$ is
$O((\text{Im} \ S)^{3/2})$. Accordingly, the accuracy in specifying the associated Lagrangian manifold
$L : \{(x, p) \mid p = \partial S / \partial x\}$ is $O(\text{Im} \ S)$. This minimum accuracy condition was followed in
but an unnecessary assumption was made, namely, that there is an underlying real Lagrangian manifold invariant with respect to the real part of the Hamiltonian vector field. For this reason the results of [9] apply mainly to the Cauchy problem and cannot be used directly in solving general eigenvalue problems.

Various subsequent expositions of the complex germ theory (another name for the complex WKB method) either fail to satisfy the minimum accuracy condition [6, 7, 12, 13, 14, 15], or deal only with the special case of the complex germ over an isotropic manifold (\(\text{Im} S_2 = 0\) on some submanifold and \(\text{Im} S_2\) has a nondegenerate Hessian in the transversal direction) [1, 2, 10, 22], or fail to provide an invariant geometric description [24, 25].

It was proposed in [5] to describe the Lagrangian manifold \(L\) by using the ideal generated by \(p_i - \partial S/\partial x_i, i = 1, \ldots, n\), in the spirit of algebraic geometry. However, the impact of the damping factor \(\exp\{-\text{Im} S/h\}\) on the geometry was not investigated in [5], and hence the canonical operator cannot be constructed on the basis of these results.

Here we give a definition of Lagrangian asymptotic manifolds resembling that in [25] but incorporating the minimum accuracy condition. We prove its equivalence to the traditional definition and construct the canonical operator in these new terms. A new form of the quantization condition is given, in which the volume form is assumed to be defined on the universal covering of the Lagrangian manifold rather than on the manifold itself. This allows one to solve a wider class of eigenvalue problems.

We also suggest a new approach of defining the dissipation in the canonical charts as the minimum of a dissipation globally defined on the phase space. The canonical operator to the accuracy of \(O(h^\infty)\) is defined in \(\S\) 5 with the help of the so-called \(V\)-objects, originally introduced in the real case in [9] and closely related to the famous Atiyah group operators.

The first four sections have been written by V. P. Maslov and V. E. Nazaikinskii, and the fifth section by V. P. Maslov and V. L. Dubnov.

## 1 Asymptotic manifolds

Throughout this section, \(M\) will be an \(n\)-dimensional differential real manifold. By \(\mathcal{C}^\infty(M)\) we denote the sheaf of germs of complex-valued \(\mathcal{C}^\infty\) functions on \(M\). Unless otherwise specified, vector fields, differential forms, etc., are allowed to have complex-valued coefficients.

We are not too pedantic about the distinction between sections and elements of sheaves. Occasionally, we may write something like \(f \in \mathcal{C}^\infty(M)\) instead of the formally correct \(f \in \Gamma(U, \mathcal{C}^\infty(M))\). However, such liberties are generally harmless (since the sheaves we consider are fine) and the set \(U\) is always obvious from the context.

### 1.1 Dissipations and dissipation ideals

**Definition 1.1.** A **dissipation** on \(M\) is a smooth nonnegative function \(D : M \to \mathbb{R}\). Two dissipations \(D_1\) and \(D_2\) are said to be **equivalent**, \(D_1 \sim D_2\), if locally (in the vicinity of any point \(m_0 \in M\)) we have

\[
c_1 D_1(m) \leq D_2(m) \leq c_2 D_2(m)
\]

with some positive constants \(c_1\) and \(c_2\).

**Definition 1.2.** Let \(D\) be a dissipation on \(M\). Consider the sheaf of ideals \(\mathcal{D} \subset \mathcal{C}^\infty(M)\) such that for any \(m \in M\) the stalk \(\mathcal{D}_m\) is the set of germs \(f \in \mathcal{C}^\infty(M)\) of functions \(\tilde{f}\) satisfying the estimate

\[
|\tilde{f}| \leq cD
\]
with some constant \( c \geq 0 \). The sheaf \( \mathcal{D} \) is called the **dissipation ideal** associated with \( D \).

Obviously, a dissipation ideal depends only on the equivalence class of the corresponding dissipation, rather than on the dissipation itself.

Let \( \mathcal{D} \) be a dissipation ideal on \( M \). The locus \( \text{loc}(\mathcal{D}) \) (i.e., the set of common zeros of all sections of \( \mathcal{D} \)) will be denoted by \( \Gamma \) (or by \( \Gamma_\mathcal{D} \) if there is any risk of confusion). Equivalently, \( \Gamma \) can be characterized as the support of the quotient sheaf \( \mathcal{C}^\infty(M)/\mathcal{D} \). It is obvious that

\[
\Gamma = \{ m \in M \mid D(m) = 0 \}
\]

for any dissipation \( D \) associated with \( \mathcal{D} \).

Given a dissipation ideal \( \mathcal{D} \subset \mathcal{C}^\infty(M) \), for any \( s \geq 0 \) we construct an ideal \( \mathcal{D}^s \) as follows. Let \( D \) be some dissipation associated with \( \mathcal{D} \). We define the stalk \( \mathcal{D}^s_m \) to be the set of germs \( \tilde{f} \in \mathcal{C}^\infty_m(M) \) of functions \( \tilde{f} \) satisfying the estimate

\[
|\tilde{f}| \leq cD^s
\]

for some constant \( c \geq 0 \). Obviously, \( \mathcal{D}^s \supset \mathcal{D}^k \) for \( s \leq k \); furthermore, \( \mathcal{D}^s\mathcal{D}^k \subset \mathcal{D}^{s+k} \), but the inclusion is strict in general, i.e., \( \mathcal{D}^s\mathcal{D}^k \neq \mathcal{D}^{s+k} \). Although the definition of \( \mathcal{D}^s \) makes sense for any \( s \geq 0 \), we mainly use the ideals \( \mathcal{D}^s \) for \( s = N/2 \), where \( N \) is a positive integer.

The reason for introducing half-integer values of \( s \) in our considerations is clear from the following lemma.

**Lemma 1.3.** Let \( \mathcal{D} \) be a dissipation ideal on \( M \). If \( f \in \mathcal{D}^s \) for some \( s \geq 1/2 \) and if \( X \) is a smooth vector field on \( M \), then \( Xf \in \mathcal{D}^{s-1/2} \).

**Proof.** The statement of the lemma is equivalent to saying that

\[
\left| \frac{\partial^{|\alpha|} f(x)}{\partial x^\alpha} \right| \leq c_\alpha (D(x))^{s-|\alpha|/2}, \quad |\alpha| \leq 2s,
\]

whenever \( |f(x)| \leq c(D(x))^s \) (here \( c \) and \( c_\alpha \) are constants). The proof of the latter statement can be found in [24], pp. 20–23.

The following ideals are sometimes useful, as well as \( \mathcal{D}^s \). Set

\[
\mathcal{D}^s = \bigcap_{\epsilon > 0} \mathcal{D}^{s-\epsilon}, \quad s > 0, \quad \mathcal{D} \equiv \mathcal{D}^1.
\]

Obviously, for any \( k, s \geq 0 \) with \( s > k \), we have \( \mathcal{D}^s \subset \mathcal{D} \subset \mathcal{D}^k \), and if \( X \) is a smooth vector field on \( M \), then \( X\mathcal{D} \subset \mathcal{D}^{s-1/2} \) for any \( s > 1/2 \). It can happen that not all vector fields behave that badly. Very frequently we shall use vector fields for which the ideals \( \mathcal{D}^s \) are invariant, that is, \( X\mathcal{D}^s \subset \mathcal{D}^s \) for any \( s \).

Let \( \mathcal{D} \) be a dissipation ideal on \( M \), let \( D \) be some dissipation associated with \( \mathcal{D} \), and let \( X \) be a smooth vector field on \( M \).

**Lemma 1.4.** i) Suppose that \( XD \in \mathcal{D} \). Then \( X\mathcal{D}^s \subset \mathcal{D}^s \) for any \( s > 0 \).

ii) The inclusion \( XD \in \mathcal{D} \) (and even \( XD = 0 \)) does not imply \( XD^s \subset \mathcal{D}^s \) in general.
Let us take $\bar{N}$ left-hand side in the last equation is $O$ separately. By Taylor's formula, for sufficiently small $t$ that $X$ is

Let us estimate each term on the right-hand side in this equation. The first term (we omit the argument $m$) is $O(D)$, and the second term is $O(D^{1/(r+1)}D^{1-\varepsilon}) = O(D)$ by the hypothesis of the lemma. Next, we have $X^{k-1}(XD) = O(D^{1-\varepsilon-(k-1)/r})$ by the inductive assumption (note that $1 - (k-1)/r > 0$ for $k \leq r$ and that $\varepsilon$ is arbitrarily small); hence, each term in the sum is

$$O(D^{k/(r+1)+1-\varepsilon-(k-1)/r}) = O(D^{1-\varepsilon+(r-k+1)/(r(r+1))}) = O(D)$$

since $k \leq r$ and $\varepsilon$ is arbitrarily small. We conclude that $D(g^{tD^{1/(r+1)}(m)}(m)) = O(D(m))$.

Now let $f \in D^s$. Again by Taylor’s formula,

$$f(g^{tD^{1/(r+1)}(m)}(m)) = \sum_{k=0}^{N-1} \frac{t^k}{k!} D^{k/(r+1)}(m)(X^k f)(m) + O(D(m)^{N/(r+1)}).$$

Let us take $N$ to be the least positive integer such that $N/(r+1) \geq s$. By the preceding, the left-hand side in the last equation is $O(D^s(g^{tD^{1/(r+1)}(m)}(m)) = O(D^s(m))$, and we see that

$$\sum_{k=0}^{N-1} \frac{t^k}{k!} D^{k/(r+1)}(m)(X^k f)(m) = O(D^s(m))$$

for any sufficiently small $t$. Let us consider the last equation for $N$ distinct values $t = t_1, \ldots, t = t_N$. Since the Vandermonde determinant

$$\begin{vmatrix} 1 & t_1 & \cdots & t_1^{N-1} \\ \cdots & \cdots & \cdots & \cdots \\ 1 & t_N & \cdots & t_N^{N-1} \end{vmatrix}$$

is nonzero, it follows that $D^{k/(r+1)}X^k f = O(D^s)$; in particular, for $k = 1$, we obtain $X f = O(D^{s-1/(r+1)})$, as desired.

ii) Consider the following example:

$$M = \mathbb{R}^2 \ni x = (x_1, x_2); \quad D(x) = \exp(-1/x_2^2); \quad X = \frac{\partial}{\partial x_1};$$

$$f(x) = D(x) \sin(x_1/x_2) \quad (D(x) = f(x) = 0 \text{ for } x_2 = 0).$$
Proof (see [11, 24]). We use local coordinates \((\phi, \text{Remark})\).

\[ D = 0 \text{ and } f = O(D), \text{ but } Xf = \frac{1}{x_2} \cos(x_1/x_2)D(x) \neq O(D). \] The lemma is proved. \[ \square \]

If \(X\) is a vector field satisfying the conclusion of Lemma 1.4 i), then we say that the dissipation ideal is invariant with respect to \(X\).

**Definition 1.5.** Let \(m_0 \in \Gamma\), and let \(U \subset M\) be a neighborhood of \(m_0\). A smooth mapping \(\phi : U \to M\) is called an almost-identity diffeomorphism if \(\phi\) is a diffeomorphism of \(U\) onto \(\phi(U)\) and \(\text{dist}(\phi(m), m) \leq c(D(m))^{1/2}, m \in U\), where \(D\) is some dissipation associated with \(\mathcal{D}, c > 0\), and \(\text{dist}(\cdot, \cdot)\) is the distance function induced by some Riemannian metric on \(M\).

**Remark.** Note that \(\phi(m) = m\) for any \(m \in \Gamma\).

**Lemma 1.6.** Let \(\phi\) be an almost-identity diffeomorphism near \(m_0 \in \Gamma\). Then \(\phi^{-1}\) is also an almost-identity diffeomorphism, each of the ideals \(\mathcal{D}_s\) is invariant by \(\phi\), i.e., \(\phi^*(\mathcal{D}_s) = \mathcal{D}_s\), and if \(\psi\) is another almost-identity diffeomorphism near \(m_0\), then so is \(\phi \circ \psi\).

**Proof** (see [11, 24]). We use local coordinates \((x_1, \ldots, x_n)\) on \(M\) near \(m_0\). By Hadamard’s lemma,

\[ D(\phi(x)) = D(x) + \left\langle \frac{\partial D(x)}{\partial x}, \phi(x) - x \right\rangle + \left\langle \phi(x) - x, B(x)(\phi(x) - x) \right\rangle, \]

where \(B(x)\) is a smooth matrix function and \(\langle z, w \rangle = z_1 w_1 + \cdots + z_n w_n\). By the assumption of the lemma, \(\|\phi(x) - x\| \leq cD(x)^{1/2}\), and by Lemma 1.3

\[ \left| \frac{\partial D(x)}{\partial x} \right| \leq c_1 D(x)^{1/2}. \]

Substituting the last two estimates into (1.1) yields

\[ D(\phi(x)) \leq c_2 D(x). \]

To prove the reverse inequality, let us apply Hadamard’s lemma to the identity \(x = \phi(\phi^{-1}(x))\). We obtain

\[ x - \phi(x) = A(x)(\phi^{-1}(x) - x), \]

where \(A(x)\) is a smooth matrix function and \(A(x_0) = (\partial \phi/\partial x)(x_0)\) (here \(x_0\) is the coordinate image of \(m_0\)), since \(\phi(x_0) = x_0\). Consequently, \(\text{det} A(x) \neq 0\) for \(x\) close to \(x_0\), and we obtain the estimate

\[ \|\phi^{-1}(x) - x\| \leq c_3 D(x)^{1/2} \]

by multiplying both sides in (1.2) by \(A(x)^{-1}\). Hence, \(\phi^{-1}(x)\) is an almost-identity diffeomorphism, and the inequality

\[ D(x) \leq c_4 D(\phi(x)) \]

follows from the above reasoning by symmetry.

The statement concerning the composition \(\phi \circ \psi\) is trivial, and so the lemma is proved. \[ \square \]

In what follows we shall sometimes use the notion of “complex coordinates” on the manifold \(M\) (cf. [11, 24]). Suppose that \(F_1, \ldots, F_n \in C^\infty(M), n = \text{dim} M\), are smooth complex-valued functions on \(M\). We say that \(F_1, \ldots, F_n\) form a complex coordinate system in a neighborhood of a point \(x_0 \in M\) if the differentials \(dF_1, \ldots, dF_n\) are linearly independent at \(x_0\). Then for any function \(f \in C^\infty(M)\) we have

\[ df = a_1 dF_1 + \cdots + a_n dF_n \]
near $x_0$, where the functions $a_1, \ldots, a_n \in C^\infty(M)$ are uniquely determined. They are referred to as the partial derivatives of $f$ with respect to $F_1, \ldots, F_n$ and denoted by

$$a_j = \frac{\partial f}{\partial F_j}, \quad j = 1, \ldots, n;$$

this coincides with the usual definition if $F_1 = x_1, \ldots, F_n = x_n$ is a usual coordinate system on $M$. The derivatives thus defined retain such familiar properties as

$$\frac{\partial^2 f}{\partial F_j \partial F_j} = \frac{\partial^2 f}{\partial F_{i} \partial F_{i}}$$

(this follows readily from the identity $d^2 = 0$) and

$$\frac{\partial f}{\partial F_j} = \sum_{k=1}^{n} \frac{\partial f}{\partial Q_k} \frac{\partial Q_k}{\partial F_j}$$

for any coordinate system $Q_1, \ldots, Q_n$ (complex or real). One can often safely think of a function $f \in C^\infty(M)$ as $f = f(F_1, \ldots, F_n)$, where $F_1, \ldots, F_n$ are complex coordinates; if a dissipation ideal $D$ is given on $M$ and there is a system of functions $\Delta F_1, \ldots, \Delta F_n \in D^{1/2}$, then the asymptotic substitution operator is defined

$$(\sigma_{F \to F + \Delta F}^{(N)}f)(x) \simeq \sum_{|\alpha|=0}^{N} (\Delta F)^{\alpha} \frac{\partial^{n} f}{\partial F^{\alpha}}, \quad \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}_+^n.$$

This operator represents the $N$th partial sum of the formal Taylor series for $f(F_1 + \Delta F_1, \ldots, F_n + \Delta F_n)$ and possesses many elegant properties for which we refer the reader to [11], [14], and [24]. Here we only note that

$$\sigma_{F \to F + \Delta F}^{(N+l)}(f) - \sigma_{F \to F + \Delta F}^{(N)}(f) = O(D^{(N+1)/2})$$

for $l > 0$ and that many of our formulas arise from the specialization of the asymptotic substitution operator to $N = 1$ or $N = 2$.

### 1.2 Asymptotic submanifolds. Global definition

The only kind of asymptotic manifolds used in this paper are asymptotic submanifolds, and so it is in this case that we give detailed definitions, lemmas, theorems, etc. However, it may be instructive at least to sketch the intrinsic definitions.

Let $M$ be an $n$-dimensional manifold, and let $D$ be a dissipation ideal on $M$. Take some $s \geq 1$. The sheaf

$$\mathcal{A} = (C^\infty(M)/D)^s|_{\Gamma_D}$$

is a sheaf of local rings on $\Gamma = \Gamma_D$. Thus, $(\Gamma, \mathcal{A})$ is a ringed space, which will serve as a local model in our definition. Namely, an $s$-asymptotic manifold of dimension $n$ is a ringed space $(\mathcal{T}, \mathcal{B})$, where $\mathcal{T}$ is a Hausdorff locally compact space and $\mathcal{B}$ is a sheaf of local rings on $\mathcal{T}$ such that for any point $\gamma_0 \in \mathcal{T}$ there exist a neighborhood $U(\gamma_0) \in \Gamma$, a dissipation ideal $D$ on $\mathbb{R}^n$, and a mapping $\tau : U(\gamma_0) \to \mathbb{R}^n$ such that

1. there is a neighborhood $U \subset \mathbb{R}^n$ of the set $\tau(U(\gamma_0))$ such that $U \cap \Gamma_D = \tau(U(\gamma_0))$;
(b) \( \tau \) is a homeomorphism of \( U(\gamma_0) \) into \( U \cap \Gamma_D \) (the latter set is equipped with the topology inherited from \( \mathbb{R}^n \));

(c) there is an isomorphism of sheaves \( B|_{U(\gamma_0)} \simeq \gamma_0^{-1}(C^\infty(M)/D^s) \).

Then the construction goes along standard lines, under various auxiliary assumptions (cf. [24]). However, the approach adopted here is more straightforward; namely, we directly proceed to asymptotic submanifolds.

**Definition 1.7.** Let \( s \geq 1 \). An \( s \)-asymptotic submanifold of codimension \( k \) in \( M \) is a pair \( L = (\mathcal{D}, \mathcal{J}) \), where \( \mathcal{D} \) is a dissipation ideal on \( M \) and \( \mathcal{J} \subset C^\infty(M) \) is an ideal such that

(i) \( \mathcal{D}^s \subset \mathcal{J} \subset \mathcal{D}^{1/2} \); \hspace{1cm} (1.3)

(ii) in a neighborhood of each point \( m_0 \in \Gamma = \Gamma_D \) the ideal \( \mathcal{J} \) is generated by \( \mathcal{D}^s \) and by \( k \) functions \( f_1, \ldots, f_k \) such that the differentials \( df_1, \ldots, df_k \) are linearly independent at \( m_0 \) (in this case we say that \( \mathcal{J} \) is \( \mathcal{D}^s \)-nondegenerate of rank \( k \), and \( f_1, \ldots, f_k \) are referred to as generators modulo \( \mathcal{D}^s \) or simply generators of \( \mathcal{J} \)).

Obviously, the number \( k \) in condition (ii) is independent of the choice of generators. Indeed, if \( g_1, \ldots, g_l \) is another system of generators, then

\[
g_j = \sum_{r=1}^{k} a_{jr} f_r + \varphi_j, \quad j = 1, \ldots, l,
\]

where \( a_{jr} \) are smooth functions and \( \varphi_j \in \mathcal{D}^s \). Since \( f_1(m_0) = \cdots = f_k(m_0) = g_1(m_0) = \cdots = g_l(m_0) = 0 \) and \( d\varphi_j(m_0) = 0 \), it follows that

\[
dg_j(m_0) = \sum_{r=1}^{k} a_{jr}(m_0) df_r(m_0), \quad j = 1, \ldots, l,
\]

and hence \( l \leq k \) (the differentials \( dg_j \) are assumed to be linearly independent). By symmetry, \( k \leq l \), and so in fact \( k = l \).

As usual, the *dimension* of \( L \) is defined by the formula \( \dim L = \dim M - k \).

**Example 1.8.** Let \( L \subset M \) be an ordinary submanifold of codimension \( k \). Let us show that it can be interpreted naturally as an \( s \)-asymptotic submanifold for any \( s \).

We introduce a dissipation on \( M \) by setting

\[
D(x) = (\text{dist}(x, L))^2,
\]

where \( \text{dist}(\cdot, \cdot) \) is the distance in some metric on \( M \) (the function (1.4) should be smooth; the metric can always be chosen so as to satisfy this condition). Next, we set

\[
\mathcal{J} = \mathcal{J}_L = \{ f \in C^\infty(M) \mid f|_L \equiv 0 \};
\]

that is, \( \mathcal{J} \) is the defining ideal of \( L \).

It is easy to verify that \( \mathcal{J} = \mathcal{D}^{1/2} \) and that condition (ii) of Definition 1.7 is also satisfied. In what follows we chiefly use 1-asymptotic submanifolds and refer to them simply as asymptotic submanifolds.
1.3 Asymptotic submanifolds. Nonparametric local description

If $L \subset M$ is a $k$-codimensional submanifold that is diffeomorphically projected on the coordinate plane $(x_{k+1}, \ldots, x_n)$ in a coordinate system $(x_1, \ldots, x_n)$ about a point $m_0 \in L$, then locally (in a neighborhood of $m_0$), $L$ can be described by equations of the form

$$x_1 = g_1(x_{k+1}, \ldots, x_n), \ldots, x_k = g_k(x_{k+1}, \ldots, x_n),$$

(1.5)

where $g_1, \ldots, g_k$ are smooth functions. Note that the defining ideal $J$ is generated by the functions $x_1 - g_1(x_{k+1}, \ldots, x_n), \ldots, x_k - g_k(x_{k+1}, \ldots, x_n)$ in this case. A description similar to (1.5) holds for asymptotic submanifolds. Let us study this description in detail. Since our considerations are purely local, we can assume that $M = \mathbb{R}^n$ with the coordinates $x = (x', x'')$, where $x' = (x_1, \ldots, x_k), x'' = (x_{k+1}, \ldots, x_n)$.

We obtain a local description of an asymptotic submanifold by allowing the functions $g_1, \ldots, g_k$ in (1.5) to take complex values. More precisely, let $d(x'')$ be a dissipation on $\mathbb{R}^{n-k}_x$, and let $g_1(x''), \ldots, g_k(x'')$ be smooth complex-valued functions such that

$$|\text{Im} g_j(x'')| \leq cd(x'')^{1/2}, \quad j = 1, \ldots, k.$$  

(1.6)

Set

$$D(x) = d(x'') + \sum_{j=1}^{k} |x_j - g_j(x'')|^2$$

(1.7)

and let $J \subset C^\infty(M)$ be the ideal generated by $D$ and by the $k$ functions $x_1 - g_1(x''), \ldots, x_k - g_k(x'')$. Then the pair $(D, J)$ is an asymptotic submanifold in $M$, since all requirements in Definition 1.7 are obviously satisfied.

**Remark 1.9.** If we replace $d(x'')$ by an equivalent dissipation $\tilde{d}(x'')$ and take smooth functions $\tilde{g}_1(x''), \ldots, \tilde{g}_k(x'')$ such that

$$\tilde{g}_j(x'') - g_j(x'') = O(d(x''))), \quad j = 1, \ldots, k,$$

then the new data $(\tilde{d}, \tilde{g}_1, \ldots, \tilde{g}_k)$ define the same asymptotic submanifold $(D, J)$.

Let us now proceed to the inverse problem.

**Theorem 1.10.** 1.10 (Implicit Function Theorem for asymptotic submanifolds). Let $L = (D, J)$ be an asymptotic $k$-codimensional submanifold in $M$, let $x_0 \in \Gamma$, and suppose that for some system of generators $f_1(x), \ldots, f_k(x)$ of the ideal $J$, the condition

$$\det \frac{\partial (f_1(x), \ldots, f_k(x))}{\partial (x_1, \ldots, x_k)} \neq 0$$

(1.8)

is satisfied at the point $x_0$. Then

(a) the same condition is satisfied for any system of generators of $J$;

(b) there exist a dissipation $d(x'')$ defined in a neighborhood of $x''_0$ and smooth functions $g_1(x''), \ldots, g_k(x'')$ such that

$$\tilde{D}(x) = d(x'') + \sum_{j=1}^{k} |x_j - g_j(x'')|^2$$
is a dissipation associated with $\mathcal{D}$,

$$|\text{Im } g_j(x'')| \leq c d(x'')^{1/2}, \quad j = 1, \ldots, k,$$

and $\mathcal{J}$ is generated by $\mathcal{D}$ and by the functions

$$x_1 - g_1(x''), \ldots, x_k - g_k(x'')$$

in a neighborhood of $x_0$;

(c) the dissipation $d(x'')$ is determined uniquely modulo the equivalence relation introduced in Definition 1.1, and the functions $g_j(x'')$ are determined uniquely modulo $O(d(x''))$;

(d) any function $\Phi(x) \in C^\infty(\mathcal{M})$ can be represented in the following form in a neighborhood of $x_0$:

$$\Phi(x) = \varphi(x'') + \eta(x),$$

where $\eta(x) \in \mathcal{J}$; the function $\varphi(x'')$ is uniquely determined modulo $O(d(x''))$.

Proof. (a) This assertion is obvious.

(b) Let $D(x)$ be a dissipation associated with $\mathcal{D}$ in a neighborhood of $x_0$. First, let us prove that

$$\det \frac{\partial^2 D}{\partial x' \partial x}(x_0) \neq 0. \quad (1.9)$$

To this end, consider the function $F(x) = |f_1(x)|^2 + \cdots + |f_k(x)|^2$. This function has the following properties:

$$0 \leq F(x) \leq cD(x); \quad F(x_0) = D(x_0) = 0; \quad \frac{\partial^2 F}{\partial x' \partial x}(x_0) = \sum_{r=1}^k \left( \frac{\partial f_r(x_0)}{\partial x_j} \frac{\partial f_r(x_0)}{\partial x_j} + \frac{\partial \overline{f_r}(x)}{\partial x_j} \frac{\partial f_r(x)}{\partial x_i} \right) \quad (1.10)$$

(the bar denotes complex conjugation). It follows from the last equation in (1.10) that for any $\xi \in \mathbb{R}^k$ we have

$$\left( \xi, \frac{\partial^2 F}{\partial x' \partial x}(x_0) \xi \right) = 2 \sum_{r=1}^k \left| \left( \frac{\partial f_r}{\partial x'}(x_0) \xi \right) \right|^2 \geq c_1 |\xi|^2 \quad (1.11)$$

since the vectors $(\partial f_r/\partial x')(x_0)$ form a basis in $\mathbb{C}^k$. Furthermore, it follows from the first two equations in (1.10) that

$$\frac{\partial^2 D}{\partial x' \partial x}(x_0) \geq \frac{1}{c} \frac{\partial^2 F}{\partial x' \partial x}(x_0).$$

By (1.11), the matrix $(\partial^2 F/\partial x' \partial x')(x_0)$ is positive definite, and so, a fortiori, is the matrix $(\partial^2 D/\partial x' \partial x')(x_0)$. In particular, $\det(\partial^2 D/\partial x' \partial x')(x_0) \neq 0$.

Let us now consider the equation

$$\frac{\partial D}{\partial x'}(x', x'') = 0. \quad (1.12)$$

We have $(\partial D/\partial x')(x_0', x_0'') = 0$ and $\det(\partial^2 D/\partial x' \partial x')(x_0', x_0'') \neq 0$. By the implicit function theorem, Eq. (1.12) has a unique smooth solution $x' = x'(x'')$ in a neighborhood of $x_0$, and this solution is obviously the solution to the minimization problem

$$D(x', x'') \rightarrow \min, \quad x'' \text{ fixed}, \quad (x', x'') \text{ lies in a neighborhood of } x_0. \quad (1.13)$$
Set
\[ d(x'') \overset{\text{def}}{=} D(x'(x''), x''). \] (1.14)

By construction, we have
\[ d(x'') \leq D(x', x'') \] (1.15)
in a neighborhood of \( x_0 \), and moreover,
\[ D(x) \sim d(x'') + (x' - x'(x''))^2. \] (1.16)

Let us now find the functions \( g_j(x'') \). The functions must satisfy the system
\[ f(x) = C(x)(x' - g(x'')) + \Phi(x), \] (1.17)
where \( C(x) \) is a smooth \( k \times k \) matrix function,
\[ f(x) = t(f_1(x), \ldots, f_k(x)), \quad g(x'') = t(g_1(x''), \ldots, g_k(x'')) \]
and \( \Phi(x) \in \mathcal{D} \). We seek \( g(x'') \) in the form
\[ g(x'') = x'(x'') + h(x''), \] (1.18)
where \( h(x'') = O(d(x'')^{1/2}) \) are new unknown functions. By Morse’s lemma, we have
\[ f(x) = f(x'(x''), x'') + \frac{\partial f}{\partial x'}(x'(x''), x'')(x' - x'(x'')) + \langle x' - x'(x''), \Psi(x)(x' - x'(x'')) \rangle, \] (1.19)
where \( \Psi(x) \) is smooth. Substituting Eqs. (1.18) and (1.19) into Eq. (1.17) we obtain
\[ f(x'(x''), x'') + \frac{\partial f}{\partial x'}(x'(x''), x'')(x' - x'(x'')) = C(x)(x' - x'(x'')) - C(x)h(x'') + O(D(x)). \] (1.20)

We can satisfy Eq. (1.20) by setting
\[ C(x) = \frac{\partial f}{\partial x'}(x'(x''), x'') \] (1.21)
(in particular, \( C(x) \) is independent of \( x' \)) and
\[ h(x'') = -\left[ \frac{\partial f}{\partial x'}(x'(x''), x'') \right]^{-1} f(x'(x''), x'') \] (1.22)
(note that \( (\partial f/\partial x')(x) \) is invertible near \( x_0 \) by (1.8)). Since \( f(x'(x''), x'') = O(d(x'')^{1/2}) \), the same is true of \( h(x'') \). The matrix \( C(x) \) is invertible, and it follows from (1.17) that \( x' - g(x'') \) is a system of generators of \( \mathcal{J} \). Finally,
\[ |x' - g(x'')|^2 \leq |x' - x'(x'')|^2 + |h(x'')|^2 \leq CD(x); \]
in conjunction with (1.16), this implies that \( D(x) \sim D(x) \).

(c) Let \( \tilde{g}_1(x''), \ldots, \tilde{g}_k(x'') \) be functions such that \( x'_1 - \tilde{g}_1(x''), \ldots, x'_k - \tilde{g}_k(x'') \) is a system of generators of \( \mathcal{J} \). Then there exists a nondegenerate matrix \( \mathcal{E}(x) \) such that
\[ x' - g(x'') = \mathcal{E}(x)(x' - \tilde{g}(x'')) + O(D). \] (1.23)
Since
\[ \mathcal{E}(x) = \mathcal{E}(x'(x''), x'') + L(x)(x' - x'(x'')) , \]
we can assume that \( \mathcal{E}(x) \) is independent of \( x' \) in Eq. (1.23), i.e.,
\[ x' - g(x'') = \mathcal{E}(x'')(x' - \tilde{g}(x'')) + O(D). \tag{1.24} \]
By differentiating (1.24) with respect to \( x' \), we obtain \( \mathcal{E}(x'') = I + O(D^{1/2}) \), where \( I \) is the identity matrix, so that
\[ g(x'') - \tilde{g}(x'') = O(D(x)) , \]
and, by minimizing with respect to \( x' \), we obtain
\[ g(x'') - \tilde{g}(x'') = O(d(x'')). \tag{1.25} \]

Let us prove that \( d(x'') \) is unique modulo equivalence; to this end, we take some dissipation \( D \) associated with \( D \) and note that the conditions imposed in item (b) imply
\[
e c \left( d(x'') + \sum_{j=1}^{k} |x_j - \text{Re } g_j(x'')|^2 + \sum_{j=1}^{k} |\text{Im } g_j(x'')|^2 \right) \leq D(x)
\]
\[
\leq C \left( d(x'') + \sum_{j=1}^{k} |x_j - \text{Re } g_j(x'')|^2 + \sum_{j=1}^{k} |\text{Im } g_j(x'')|^2 \right) \leq C_1 \left( d(x'') + \sum_{j=1}^{k} |x_j - \text{Re } g_j(x'')|^2 \right),
\]
where \( c, C, C_1 \) are some positive constants, the last inequality being due to the fact that \( \text{Im } g_j(x'') = O(d(x'')) \). Let us pass to the minimum over \( x' \) in these inequalities and discard the nonnegative terms under the summation signs on the left-hand side. We obtain
\[
cd(x'') \leq \min_{x'} D(x', x'') \leq C_1 d(x''),
\]
which implies that any function \( d(x'') \) satisfying the conditions of item (b) is equivalent to \( \min_{x'} D(x', x'') \).

d) Let \( \Phi(x) \in C^\infty(M) \). In the preceding notation we have
\[
\Phi(x) = \Phi(x', x'') = \Phi(x'(x''), x'') + (x' - x'(x'')) \Phi_x'(x'(x''), x'') + O(D)
\]
\[
= \Phi(x'(x''), x'') + [(g(x'') - x'(x'')) + (x' - g(x''))] \Phi_x'(x'(x''), x'') + O(D);
\]
thus, the desired identity holds with
\[
\varphi(x'') = \Phi(x'(x''), x'') + (g(x'') - x'(x'')) \Phi_x'(x'(x''), x'');
\]
if some other function \( \varphi_1(x'') \) satisfies the same identity, then \( \varphi(x'') - \varphi_1(x'') \in \mathcal{J} \), and hence
\[
\varphi(x'') - \varphi_1(x'') = \sum_{j=1}^{k} a_j(x', x'')(x_j - g_j(x'')) + O(D)
\]
\[
= \sum_{j=1}^{k} a_j(x'(x''), x'')(x_j - g_j(x'')) + O(D).
\]
Differentiating both sides with respect to \( x' \) yields \( a_j(x'(x''), x'') = O(D^{1/2}) \); hence, \( \varphi(x'') - \varphi_1(x'') = O(D) \), and it remains to minimize the right-hand side over \( x' \).

The theorem is thereby proved. \( \square \)
A trivial analog of Theorem 1.10 holds for complex coordinates.

**Theorem 1.10′.** Let \( L = (\mathcal{D}, \mathcal{J}) \) be an asymptotic submanifold of codimension \( k \) in \( M \), and let \((F_1, \ldots, F_k, F_{k+1}, \ldots, F_n) = (F', F'')\) be a complex coordinate system in a neighborhood of a point \( x_0 \in \Gamma \). Suppose that

\[
\det \frac{\partial (f_1, \ldots, f_k)}{\partial (F_1, \ldots, F_k)}(x_0) \neq 0.
\]

Then there exist functions \( \Phi_1, \ldots, \Phi_k \) such that

(i) \( F_1 - \Phi_1, \ldots, F_k - \Phi_k \) generate \( \mathcal{J} \);

(ii) \( \partial \Phi_i / \partial F_j \in \mathcal{D}, \ i,j = 1, \ldots, k. \)

Condition (ii) states that the functions \( \Phi_i \) “do not depend” on \( F_j, \ i,j = 1, \ldots, k. \)

**Proof.** Assuming summation over repeated indices from 1 to \( k \), we set

\[
\Phi_i = F_i - \left( \frac{\partial f}{\partial F'} \right)^{-1} f_s + \frac{1}{2} f_t A^i_{ts} f_s,
\]

where

\[
A^i_{ts} = \left( \frac{\partial f}{\partial F'} \right)^{-1} \left( \frac{\partial f}{\partial F'} \right)^{-1} \left( \frac{\partial f}{\partial F'} \right)^{-1} \frac{\partial^2 f_m}{\partial F_r \partial F_t}.
\]

then

\[
\frac{\partial \Phi_i}{\partial F_s} = \frac{1}{2} \left\langle f, \frac{\partial A^i}{\partial F_s} f \right\rangle \in \mathcal{D}, \quad i, s = 1, \ldots, k;
\]

and

\[
F' - \Phi = \left( \frac{\partial f}{\partial F'} \right)^{-1} f - \frac{1}{2} \left\langle f, Af \right\rangle
\]

is obviously a system of generators of \( \mathcal{J} \). The theorem is proved. \( \square \)

### 1.4 Asymptotic submanifolds. Parametric local description

There is still another way to describe asymptotic submanifolds, namely, by using equations describing the “embedding” of this manifold in \( M \). Let \( U \subset \mathbb{R}^m \) be some domain, and let a dissipation \( d(\alpha), \alpha \in U \), be given on \( U \). Suppose that we have a set of functions

\[
X(\alpha) = (X_1(\alpha), \ldots, X_n(\alpha)) \quad (1.26)
\]

defined on \( U \) such that the following conditions are satisfied:

\[
i) \quad \text{rank}_\mathbb{C} \left( \frac{\partial X_1}{\partial \alpha}, \ldots, \frac{\partial X_n}{\partial \alpha} \right) = m, \quad (1.27)
\]

\[
ii) \quad \text{Im} X_i(\alpha) = O(d(\alpha)^{1/2}). \quad (1.28)
\]

In particular, we have \( m \leq n \), and only the case \( m < n \) is of interest.

Let \((x_1, \ldots, x_n)\) be a coordinate system about some point \( m_0 \in M \). Then we can use the vector function \( (1.26) \) to define an asymptotic submanifold in \( M \) near \( m_0 \) as follows. Let \( \alpha_0 \in \Gamma_d \), and let \( x_0 = X(\alpha_0) \) (note that \( x_0 \) is necessary real). Set

\[
D(x, \alpha) = d(\alpha) + \sum_{i=1}^n |x_i - X_i(\alpha)|^2 \quad (1.29)
\]
and consider the functions
\[ \varphi_i(x, \alpha) = x_i - X_i(\alpha), \quad i = 1, \ldots, n. \] (1.30)

The pair \((\mathcal{D}, \mathcal{J})\), where \(\mathcal{D}\) is the ideal corresponding to the dissipation (1.29) and \(\mathcal{J}\) is the ideal generated by \(\mathcal{D}\) and the functions (1.30), is obviously an asymptotic submanifold in \(M \times U\) near \((x_0, \alpha_0)\). Let us construct the projection of this submanifold on \(M\) (this is possible by virtue of condition (1.27), but the reader should be careful to keep in mind that the construction is purely local).

We proceed as follows. By condition (1.27), we have
\[ \frac{\partial^2 D(x, \alpha)}{\partial \alpha \partial \alpha} \bigg|_{x=x_0, \alpha=\alpha_0} > 0 \] (1.31)
and hence the same is true in a neighborhood of \((x_0, \alpha_0)\). Furthermore, we have
\[ \frac{\partial D}{\partial \alpha}(x_0, \alpha_0) = 0, \] (1.32)
and by the implicit function theorem there exists a smooth vector function
\[ \alpha = \alpha(x), \quad \alpha(x_0) = \alpha_0, \] (1.33)
that satisfies Eq. (1.32) in a neighborhood of \(x_0\). Set
\[ \tilde{D}(x) = D(x, \alpha(x)). \] (1.34)

Obviously, \(\tilde{D}(x) \leq D(x, \alpha)\) in a neighborhood of \((x_0, \alpha_0)\). Furthermore, set
\[ \tilde{\mathcal{J}} = \{ f(x) \mid f(x) \otimes 1(\alpha) \in \mathcal{J} \}. \] (1.35)
It is easy to see that the pair \((\tilde{\mathcal{D}}, \tilde{\mathcal{J}})\), where \(\tilde{\mathcal{D}}\) is the ideal generated by \(\tilde{\mathcal{D}}\) (1.34), is an asymptotic submanifold in \(M\).

### 1.5 Asymptotic mappings

Let \(L = (\mathcal{D}, \mathcal{J})\) be a \(k\)-codimensional asymptotic submanifold in \(M\), and let \(f : M \to N\) be a diffeomorphism. Then the image \(\tilde{L} = f(L)\) can be defined in a natural manner as follows. We set \(\tilde{\mathcal{D}} = (f^{-1})^* \mathcal{D}\) and \(\tilde{\mathcal{J}} = (f^{-1})^* \mathcal{J}\), where \((f^{-1})^*\) acts elementwise, that is, \((f^{-1})^* (A) = \{(f^{-1})^* \varphi, \, \varphi \in A\}\). Then \(\tilde{L} = (\tilde{\mathcal{D}}, \tilde{\mathcal{J}})\). There is still another description of \(f(L)\). In the Cartesian product \(M \times N\) consider the submanifold
\[ \text{graph } f = \{(x, y) \in M \times N \mid y = f(x)\}. \]

The associated asymptotic submanifold \((\mathcal{D}_f, \mathcal{J}_f)\) in \(M \times N\) (cf. Example 1.8) can be described locally as follows. Let \((x_1, \ldots, x_n)\) and \((y_1, \ldots, y_n)\) be local coordinate systems on \(M\) and \(N\), respectively, and let \(f\) be given by the functions
\[ y_1 = f_1(x_1, \ldots, x_n), \]
\[ \cdots \quad \cdots \quad \cdots \]
\[ y_n = f_n(x_1, \ldots, x_n) \]
in the coordinates. Then $D_f$ corresponds to the dissipation $D_f(x, y) = \sum (y_j - f_j(x))^2$ and $J_f$ is the ideal generated by $y_1 - f_1(x), \ldots, y_n - f_n(x)$. It is an easy exercise to verify that if $D$ is a dissipation corresponding to $\mathcal{D}$, then $\tilde{D}$ is associated with the dissipation

$$\tilde{D}(y) = \min_x \{D(x) + D_f(x, y)\}$$

and that $\tilde{J}$ can be described as

$$\tilde{J} = \{\varphi(x, y) \in J + J_f \mid \varphi \text{ is independent of } x\}.$$ 

Indeed, first of all, note that if $D_1(x, y)$ and $D_2(x, y)$ are equivalent dissipations on $M \times N$, then the dissipations $d_1(y) = \min_x D_1(x, y)$ and $d_1(y) = \min_x D_2(x, y)$ are also equivalent; to observe this, it suffices to apply $\min_x$ to the inequalities $cD_1(x, y) \leq D_2(x, y) \leq CD_1(x, y)$.

Thus, in the definition of $\tilde{D}(y)$ we can safely replace $D(x) + D_f(x, y)$ by

$$D(x, y) = D(x) + \frac{1}{2} \sum (x_j - g_j(y))^2,$$

where $x = g(y)$ is the inverse of the mapping $y = f(x)$.

Consider any point $x_0 \in \Gamma_{D(x)}$ and set $y_0 = f(x_0)$. The mapping

$$\sigma(x) = x + \frac{\partial D(x)}{\partial x}$$

is an almost-identity diffeomorphism in a neighborhood of $x_0$. Indeed,

$$\sigma(x) - x = \frac{\partial D(x)}{\partial x} = O(\sqrt{D(x)}),$$

and

$$\frac{\partial \sigma}{\partial x} \bigg|_{x = x_0} = I + \frac{\partial^2 D}{\partial x \partial x} \bigg|_{x = x_0} > 0,$$

which implies that $\det \frac{\partial \sigma}{\partial x} \neq 0$ near $x_0$. For $y$ close to $y_0$, the point

$$y(x) = \arg \min_x \left\{D(x) + \frac{1}{2} \sum (x_j - g_j(y))^2\right\}$$

is determined from the equation

$$\frac{\partial}{\partial x} \left\{D(x) + \frac{1}{2} \sum (x_j - g_j(y))^2\right\} = \sigma(x) - g(y) = 0.$$

That is, $x = \sigma^{-1}(g(y))$ and

$$\tilde{D}(y) = D(\sigma^{-1}(g(y))) + \frac{1}{2} ||\sigma^{-1}(g(y)) - g(y)||^2.$$

By using Lemma 1.6, we easily obtain $\tilde{D}(y) \sim D(q(y))$. Furthermore, $\varphi(y) \in \tilde{J}$ if and only if $\varphi(f(x)) \in J$. We have

$$\varphi(y) = \varphi(f(x)) + C(x, y)(y - f(x))$$

(1.36)
by Hadamard's lemma (here \( C(x, y) = (C_1(x, y), \ldots, C_n(x, y)) \) is a smooth vector function). If \( \varphi(f(x)) \in \mathcal{J} \), then it follows from (1.36) that
\[
\varphi(y) \in \{ \psi(x, y) \in \mathcal{J} + \mathcal{J}_f \mid \psi \text{ is independent of } x \}.
\] (1.37)

Conversely, let (1.37) be true; then for some \( a(x) \in \mathcal{J} \) we have
\[
\varphi(y) = a(x) + b(x, y)(y - f(x)),
\]
and we find that \( a(x) = \varphi(f(x)) \) by setting \( y = f(x) \) in the last equation.

These considerations motivate the following definition.

**Definition 1.11.** Let \( M \) and \( N \) be two manifolds of the same dimension \( n \), and let \( G = (D_G, J_G) \) be an \( n \)-dimensional asymptotic submanifold in \( M \) such that the following conditions are satisfied:

(i) for any \( x \in M \) there is at most one point \( y \in N \) such that \( (x, y) \in \Gamma_G \equiv \Gamma_{D_G} \), and vice versa;

(ii) for any \( (x_0, y_0) \in \Gamma_G \) the Jacobians
\[
\det \frac{\partial(f_1, \ldots, f_n)}{\partial(x_1, \ldots, x_n)} \quad \text{and} \quad \det \frac{\partial(f_1, \ldots, f_n)}{\partial(y_1, \ldots, y_n)}
\]
are nonzero for a certain (and hence for any) system \( f_1, \ldots, f_n \) of generators of the ideal \( J_G \).

Then \( G \) is called a \textit{(graph of) asymptotic diffeomorphism} from \( N \) into \( M \) and is denoted \( G : N \to M \).

We are interested in the action of asymptotic diffeomorphisms on asymptotic submanifolds.

**Theorem 1.12.** Let \( L = (D, J) \) be an asymptotic submanifold of \( N \), and let \( G = (D_G, J_G) \) be an asymptotic diffeomorphism from \( N \) into \( M \). Let \( D \) and \( D_G \) be dissipations associated with \( D \) and \( D_G \). Suppose that the set
\[
\tilde{\Gamma} = \{ x \in M \mid \exists y \in N : y \in \Gamma_D \text{ and } (x, y) \in \Gamma_{D_G} \}
\]
is nonempty and define a dissipation \( \tilde{D}(x) \) on \( M \) in the vicinity of \( \tilde{\Gamma} \) by the formula
\[
\tilde{D}(x) = \min_y \{ D(y) + D_G(x, y) \}
\]
(the minimum is taken over some neighborhood of \( \Gamma_D \)). Let \( \tilde{D} \) be the dissipation ideal associated with \( \tilde{D}(x) \), and set
\[
\tilde{J} = \{ \varphi(x, y) \in J + J_f \mid \varphi \text{ is independent of } y \}.
\]
Then \( \tilde{L} = (\tilde{D}, \tilde{J}) \) is an asymptotic submanifold in \( M \) and \( \dim \tilde{L} = \dim L \).
Proof. Let \((x_0, y_0) \in \Gamma_{D_G}\) and \(y_0 \in \Gamma_D\). We have
\[
\frac{\partial^2}{\partial y \partial y} \{D(y) + D_G(x, y)\} = \frac{\partial^2 D}{\partial y \partial y} + \frac{\partial^2 D_G}{\partial y \partial y} > 0
\]
at \((x_0, y_0)\), and so
\[
y(x) = \arg \min_y \{p(y) + D_G(x, y)\}
\]
is a smooth mapping in the vicinity of \(x_0\) and \(y(x_0) = y_0\).

By the implicit function theorem (Theorem 1.10), we can choose a set of generators of \(\mathcal{J}_G\) of the form
\[
f_j(x, y) = y_j - g_j(x), \quad j = 1, \ldots, n.
\]
For brevity, in what follows we write
\[
Q(x, y) = D(y) + D_G(x, y).
\]
Let \(\psi_1(y), \ldots, \psi_k(y)\) be a system of generators of the ideal \(\mathcal{J}\). Set
\[
\Xi_j(x, y) = \psi_j(y) + \frac{\partial \psi_j(y)}{\partial y}(g(x) - y),
\]
\[
\tilde{\psi}_j(x) = \psi_j(y(x)) + \frac{\partial \psi_j(y(x))}{\partial y}(g(x) - y(x)) = \Xi_j(x, y(x)).
\]
Obviously, \(\Xi_j(x, y) \in \mathcal{J} + \mathcal{J}_G\). Furthermore, we have
\[
\frac{\partial \Xi_j}{\partial y}(x, y) = \frac{\partial^2 \psi_j(y)}{\partial y^2}(g(x) - y).
\]
Consequently,
\[
\tilde{\psi}_j(x) - \Xi_j(x, y) = (y(x) - y) \frac{\partial^2 \psi_j(y)}{\partial y^2}(g(x) - y) + O(\|y(x) - y\|^2).
\]
Since \(\partial^2 Q/\partial y \partial y > 0\), it follows that \(\|y(x) - y\|^2 = O(Q)\) and thus \(\tilde{\psi}_j(x) \in \mathcal{J} + \mathcal{J}_G\); since \(\tilde{\psi}_j(x)\) is independent of \(x\), we see that \(\tilde{\psi}_j(x) \in \tilde{\mathcal{J}}\).

Next, \(\theta(x) \in \tilde{\mathcal{J}}\) if and only if
\[
\theta(x) = \sum b_l(y)\psi_l(y) + \sum a_j(x, y)(y_j - g_j(x)) + O(Q(x, y)),
\]
where \(b_l(y)\) and \(a_j(x, y)\) are smooth functions.

Set
\[
\tilde{\theta}(x) = \sum \left[ b_l(y(x)) + \frac{\partial b_l(y(x))}{\partial y}(g(x) - y(x)) \right] \tilde{\psi}_l(x).
\]
Then
\[
\theta(x) - \tilde{\theta}(x) = \sum b_l(y)[\psi_l(y) - \tilde{\psi}_l(x)]
\]
\[
+ \sum \left[ b_l(y) - b_l(y(x)) - \frac{\partial b_l(y(x))}{\partial y}(y(x) - y(x)) \right] \tilde{\psi}_l(x) + \sum a_j(x, y)(y_j - g_j(x))
\]
\[
= \sum b_l(y)[\psi_l(y) - \Xi_l(x, y)]
\]
\[
+ \sum \left[ b_l(y) - b_l(y) - \frac{\partial b_l(y)}{\partial y}(g(x) - y) \right] \Xi_l(x, y) + a(x, y)(y(x) - g(x)) + O(Q(x, y))
\]
\[
= \sum b_l(y) \frac{\partial \psi_l(y)}{\partial y}(y - g(x)) + \sum \frac{\partial b_l(y)}{\partial y}(y - g(x)) \psi_l(y) + a(x, y)(y - g(x)) + O(Q(x, y))
\]
\[
= c(x, y)(y - g(x)) + O(Q(x, y)) = c(x, y)(y - g(x)) + O(Q(x, y)).
\]
Since the left-hand side of the last equation is independent of \( y \), it follows by differentiation with respect to \( y \) that \( c(x, y(x)) = O(Q^{1/2}) \). Therefore

\[
\Theta(x) = \tilde{\Theta}(x) + O(Q(x, y)) = \tilde{\Theta}(x) + O(\tilde{D}(x))
\]

and we see that \( \tilde{\psi}_1(x), \ldots, \tilde{\psi}_l(x) \) is a system of generators of the ideal \( \tilde{J} \). Furthermore, we have

\[
d\tilde{\psi}_j(x) = \left[ \frac{\partial \Xi_j}{\partial x} + \frac{\partial \Xi_j}{\partial y} \frac{\partial y_j}{\partial x} \right]_{y=y(x)} dx
\]

\[
= \left[ \frac{\partial \psi_j}{\partial y}(y(x)) \frac{\partial g(x)}{\partial x} + \frac{\partial^2 \psi_j}{\partial y^2}(y(x))(g(x) - y(x)) \right] dx.
\]

At \((x_0, y_0)\) we have

\[
d\tilde{\psi}_j = \frac{\partial \psi_j}{\partial y}(y_0) \frac{\partial g}{\partial x}(x_0) dx
\]

and, since \( d\psi_j \) are linearly independent and \((\partial g/\partial x)(x_0)\) is a nondegenerate matrix, it follows that \( d\tilde{\psi}_j \) are linearly independent. Finally, from the formula defining \( \tilde{\psi}_j(x) \) we obtain

\[
\tilde{\psi}_j(x) = O(Q(x, y(x))^{1/2}) = O(\tilde{D}(x)^{1/2}),
\]

and \( \tilde{J} \subset \tilde{D}^{1/2} \). The inclusion \( \tilde{D} \subset \tilde{J} \) is obvious.

The theorem is proved.

The asymptotic manifold \( \tilde{L} \) constructed in Theorem 1.12 will be denoted \( \tilde{L} = G(L) \).

**Theorem 1.13.** Let \( G : M_1 \to M_2 \) and \( H : M_2 \to M_3 \) be asymptotic diffeomorphisms. Then

(i) for any asymptotic submanifold \( L \) in \( M_1 \) we have

\[
H(G(L)) = (H \circ G)(L),
\]

where \( H \circ G : M_1 \to M_3 \) is the asymptotic diffeomorphism defined as follows. Let \( G = (\mathcal{D}_G, \mathcal{J}_G) \) and \( H = (\mathcal{D}_H, \mathcal{J}_H) \), and let \( D_G(x, y) \) and \( D_H(y, z) \) be dissipations associated with \( \mathcal{D}_G \) and \( \mathcal{D}_H \), respectively. Then \( \mathcal{D}_{H \circ G} \) is the dissipation ideal corresponding to the dissipation

\[
\mathcal{D}_{H \circ G}(x, z) = \min_y \{ D(x, y) + D(y, z) \},
\]

and

\[
\mathcal{J}_{H \circ G} = \{ \varphi(x, y, z) \in \mathcal{J}_H + \mathcal{J}_G \mid \varphi \text{ is independent of } y \};
\]

(ii) the composition of asymptotic diffeomorphisms thus defined is associative, \((G \circ H) \circ K = G \circ (H \circ K)\).

We omit the proof of Theorem 1.13, since it is purely technical and contains no new ideas as compared with the preceding theorem.
2 Objects on asymptotic manifolds

2.1 Functions, vector fields, and differential forms

Let $L = (\mathcal{D}, \mathcal{J})$ be a $k$-codimensional asymptotic submanifold in $M$. We set

$$\mathcal{C}^\infty(L) = \mathcal{C}^\infty(M)/\mathcal{J}$$

and

$$\mathcal{C}^\infty_{(1)}(L) = \mathcal{C}^\infty(M)/\mathcal{D}^{1/2} = \mathcal{C}^\infty(L)/(\mathcal{D}^{1/2}/\mathcal{J}).$$

The sheaf $\mathcal{C}^\infty(L)$ is called the sheaf of smooth functions on $L$. The reason for introducing the sheaf $\mathcal{C}^\infty_{(1)}(L)$ will be clarified later on. We shall also make extensive use of the sheaves

$$\mathcal{C}^\infty(L) = \mathcal{C}^\infty(M)/\mathcal{J}, \quad \mathcal{C}^\infty_{(1)}(L) = \mathcal{C}^\infty(M)/\mathcal{D}^{1/2},$$

where $\mathcal{J} = \mathcal{J} + \mathcal{D}$.

There is an obvious restriction map $i^* : \mathcal{C}^\infty(M) \to \mathcal{C}^\infty(L)$ and also a projection $\pi : \mathcal{C}^\infty(L) \to \mathcal{C}^\infty_{(1)}(L)$, such that $\mathcal{C}^\infty_{(1)}(L)$ possesses the natural structure of a $\mathcal{C}^\infty(L)$-module. Similar mappings are defined for the “circled” spaces.

**Definition 2.1.** A vector field on $L$ is a derivation $X : \mathcal{C}^\infty(L) \to \mathcal{C}^\infty_{(1)}(L)$, that is, a linear mapping such that

$$X(fg) = X(f)\pi(g) + \pi(f)X(g)$$

for any $f, g \in \mathcal{C}^\infty(L)$.

**Lemma 2.2.** Let $Y : \mathcal{C}^\infty(M) \to \mathcal{C}^\infty(M)$ be a vector field on $M$, and suppose that $Y\mathcal{J} \subset \mathcal{D}^{1/2}$. Then $Y$ correctly defines a vector field on $L$.

**Proof.** By the hypotheses of the lemma, $Y$ factors through the natural projections

$$\mathcal{C}^\infty(M) \to \mathcal{C}^\infty(M)/\mathcal{J}, \quad \mathcal{C}^\infty(M) \to \mathcal{C}^\infty(M)/\mathcal{D}^{1/2}$$

and thus gives rise to a vector field on $L$. $\square$

**Definition 2.3.** A vector field $Y$ on $L$ is said to be geometric if it is obtained as a restriction of some vector field on $M$ (i.e., by the method described in Lemma 2.2). The sheaf of geometric vector fields on $L$ will be denoted $\text{Vect}(L)$.

Obviously, $\text{Vect}(L)$ is a $\mathcal{C}^\infty_{(1)}(L)$-module, and we have

$$\text{Vect}(L) = \text{Vect}_{L}(M)/\mathcal{D}^{1/2},$$

where $\text{Vect}_{L}(M)$ is the sheaf of vector fields satisfying the conditions of Lemma 2.2 (such fields are said to be tangent to $L$). If $X \in \text{Vect}_{L}(M)$ and $m_0 \in \Gamma$, then the vector $X(m_0) \in T_{m_0}M$ is obviously well defined; such vectors will be called tangent vectors to $L$ at $m_0$. The space of tangent vectors will be denoted by $T_{m_0}M$. The following lemma shows that there are “sufficiently many” geometric fields on $L$.

**Lemma 2.4.** Let $L = (\mathcal{D}, \mathcal{J})$ be a $k$-codimensional asymptotic submanifold in an $n$-dimensional manifold $M$. Then in a neighborhood of any point $m_0 \in \Gamma$ there exist exactly $n - k$ geometric vector fields on $L$ linearly independent over $\mathcal{C}^\infty_{(1)}(L)$.
Proof. Let us use the local description given in Theorem 1.10. Thus, we can assume that the ideal \( \mathcal{J} \) is generated by \( \mathcal{D} \) and by the \( k \) functions \( x_i - g_i(x'') \), \( i = 1, \ldots, k \), where \( x = (x', x'') \), \( x' = (x_1, \ldots, x_k) \), \( x'' = (x_{k+1}, \ldots, x_n) \).

By the same theorem, an arbitrary function \( f(x) = f(x', x'') \in C^\infty(M) \) can be represented in the form

\[
    f(x', x'') = f_0(x'') + \sum_{i=1}^{k} (x_i - g_i(x'')) f_i(x'') + \eta(x), \quad (2.1)
\]

where

\[
    f_0(x'') = f(\text{Re} g'(x''), x'') + \sum_{j=1}^{n} \left( \text{Im} g'(x''), \frac{\partial f}{\partial x'}(\text{Re} g'(x''), x'') \right),
\]

\[
    f_i(x'') = \frac{\partial f}{\partial x_i}(\text{Re} g'(x''), x''),
\]

and \( \eta(x) \in \mathcal{D} \). Set

\[
    X_j = \frac{\partial}{\partial x_j} - \sum_{i=1}^{k} \frac{\partial g_i(x'')}{\partial x_j} \frac{\partial}{\partial x_i}, \quad j = k+1, \ldots, n.
\]

Then \( X_j(x_i - g_i(x)) = 0, \quad i = 1, \ldots, k \), so that the operators \( X_j \) give rise to geometric vector fields on \( L \). Next,

\[
    X_j f(x', x'') = \frac{\partial f_0(x'')}{\partial x_j} + O(D^{1/2}), \quad j = 1, \ldots, k.
\]

Let \( a_j(x) \), \( j = k+1, \ldots, n \), be functions such that \( \sum_{j=k+1}^{n} a_j(x) X_j \) is the zero vector field on \( L \). Then

\[
    \sum_{j=k+1}^{n} a_j(x) \frac{\partial f_0(x'')}{\partial x_j} = O(D^{1/2})
\]

for any smooth function \( f_0(x'') \). By choosing \( f_0(x'') = x_j \), we see that \( a_j(x) \in \mathcal{D}^{1/2}, \ j = 1, \ldots, n \), that is, the \( a_j(x) \) generate zero elements in \( C^\infty_{(1)}(L) \).

Thus, the fields \( X_j \) are linearly independent over \( C^\infty_{(1)}(L) \). Let us now prove that any system of \( n - k + 1 \) vector fields is linearly dependent over \( C^\infty_{(1)}(L) \). This statement is obvious from the representation \( (2.1) \). Indeed, let \( X_1, \ldots, X_{s+1} \) be such a system (here \( s = n - k \)); then \( X_i(x', x'') = X_i f_0(x'') \) in \( C^\infty_{(1)}(L) \) and \( X_i f_0(x'') = Y_i f_0(x'') \), where \( Y_i \) are some \( s \)-dimensional vector fields depending on the parameters \( x' \). However, the linear dependence of \( Y_i \) over \( C^\infty(M) \) is obvious, and hence the statement of the lemma follows.

\[ \square \]

Remark. Lemma 2.4 can be restated as follows: \( \text{Vect}(L) \) is a locally free \( C^\infty_{(1)}(L) \)-module of rank \( \dim L \).

Definition 2.5. Let \( X \) be a vector field tangent to \( L, X \in \text{Vect}_L(M) \), and suppose that the dissipation ideal \( \mathcal{D} \) is invariant by \( X \) and \( X \mathcal{J} \subset \mathcal{J} \). Then we say that \( L \) is strongly invariant with respect to \( X \) (or \( X \) is a strong tangent field to \( L \)).

If \( X \) is a strong tangent field to \( L \), then \( X \) acts as a derivation of the sheaves

\[
    X : \mathcal{C}^\infty(L) \to \mathcal{C}^\infty(L) \quad \text{and} \quad X : \mathcal{C}^\infty_{(1)}(L) \to \mathcal{C}^\infty_{(1)}(L).
\]
2.2 Differential forms

Definition 2.6. A differential 1-form on $L$ is a $C^\infty_{(1)}(L)$-linear functional $\omega : \text{Vect } L \to C^\infty_{(1)}(L)$.

The sheaf of differential 1-forms on $L$ will be denoted by $\Lambda^1(L)$; by virtue of the preceding results, $\Lambda^1(L)$ is a locally free $C^\infty_{(1)}(L)$-module of rank $\dim L$.

There is an obvious mapping $d : C^\infty(L) \to \Lambda^1(L)$; it is given by the formula $df(X) \overset{\text{def}}{=} X(f)$; one can prove that $\Lambda^1(L)$ is generated over $C^\infty(L)$ by elements of the form $df$.

Furthermore, we have the commutative diagram

\[
\begin{array}{ccc}
C^\infty(M) & \overset{d}{\longrightarrow} & \Lambda^1(M) \\
\downarrow & & \downarrow i^* \\
C^\infty(L) & \overset{d}{\longrightarrow} & \Lambda^1(L)
\end{array}
\]

where the left vertical arrow is the natural projection and

\[i^* \omega(X) = \omega(\tilde{X})\]

for any $\omega \in \Lambda^1(M)$ and any $x \in \text{Vect}(L)$, where $\tilde{X} \in \text{Vect}_L(M)$ is a representative of $X$. Since $X \in D^{1/2} \text{Vect}(M)$ implies $\omega(X) \in D^{1/2}$, it follows that $i^*$ is well defined.

Definition 2.7. A differential $s$-form on $L$ is an alternating $C^\infty_{(1)}(L)$-polylinear mapping

\[\omega : \text{Vect}(L) \times \cdots \times \text{Vect}(L) \to C^\infty_{(1)}(L).\]

We note that the mapping $i^* : \Lambda^k(M) \to \Lambda^k(L)$ is well defined for any $k$.

We shall be mainly interested in $m$-forms, where $m = \dim L$. Nondegenerate $m$-forms will be referred to as volume forms. In this case, the following assertion is valid.

Lemma 2.8. Let $\omega$ be a differential $s$-form on an $s$-dimensional asymptotic submanifold $L$ in $M$. Then in a neighborhood of each point $m_0 \in \Gamma$ the form $\omega$ is uniquely determined by its value on an arbitrary $s$-tuple $(X_1, \ldots, X_s)$ of linearly independent vector fields near $m_0$.

The proof is obvious.

Definition 2.9. Let $L = (\mathcal{D}, \mathcal{J})$ be an $s$-dimensional asymptotic submanifold in $M$ and let $m_0 \in \Gamma$. A (complex) coordinate system on $L$ in a neighborhood of $m_0$ is an $s$-tuple $(Q_1, \ldots, Q_s)$ of elements of $C^\infty(L)$ such that $dQ_1 \wedge \cdots \wedge dQ_s \neq 0$ at $m_0$ (or, which is the same, the differentials $dQ_1, \ldots, dQ_s$ are linearly independent near $m_0$).

Sometimes we shall consider representatives $\tilde{Q}_1, \ldots, \tilde{Q}_s$ of $Q_1, \ldots, Q_s$ in $C^\infty(M)$; these will also be referred to as local coordinates on $M$.

Since $dQ_1, \ldots, dQ_s$ are linearly independent, we have a unique decomposition

\[df = a_1 dQ_1 + \cdots + a_s dQ_s\]

for any $f \in C^\infty(L)$. The coefficients $a_j \in C^\infty_{(1)}(L)$ are denoted $a_j = \partial f / \partial Q_j$ and are referred to as the partial derivatives of $f$ with respect to $Q_j$. 
Proposition 2.10. Let \( L = (\mathcal{D}, \mathcal{J}) \) be a \( k \)-codimensional submanifold in \( M \), and let \((Q_{k+1}, \ldots, Q_n)\) be a local coordinate system on \( L \). Then

(a) \( \partial/\partial Q_{k+1}, \ldots, \partial/\partial Q_n \in \text{Vect}(L) \);

(b) if \( F_{k+1}, \ldots, F_n \) are arbitrary representatives of \( Q_{k+1}, \ldots, Q_n \) in \( \mathcal{C}^\infty(L) \), then we can complete \(( F_{k+1}, \ldots, F_n)\) to a coordinate system on \( M \) such that for any \( \varphi \in \mathcal{C}^\infty(L) \) the following conditions are satisfied:

(b1) \( \partial \Phi / \partial F_j = \partial \varphi / \partial Q_j \), \( j = k+1, \ldots, n \), in \( \mathcal{C}^\infty(\mathcal{J}_1) \) for any representative \( \Phi \in \mathcal{C}^\infty(M) \) of \( \varphi \);

(b2) there exists a representative \( \tilde{\Phi} \in \mathcal{C}^\infty(M) \) of \( \varphi \) such that \( \partial \Phi / \partial F_j \in \mathcal{D} \), \( j = 1, \ldots, k \).

Proof. Let \((F_1, \ldots, F_k)\) be a system of generators of \( \mathcal{J} \), and let \( m \in \Gamma \). Then \( dF_i(\xi) = 0 \), \( i = 1, \ldots, k \), for any \( \xi \in T_m \Gamma \). Since \( dF_1 \wedge \cdots \wedge dF_k \neq 0 \) and since \( dF_{k+1} \wedge \cdots \wedge dF_n \neq 0 \) at \( m \), \((F_1, \ldots, F_n)\) is a coordinate system on \( M \) in a neighborhood of \( m \). Let \( \Phi \in \mathcal{C}^\infty(M) \) be an arbitrary representative of \( \varphi \in \mathcal{C}^\infty(L) \); then

\[
\frac{d\Phi}{d\Phi} = a_1 dF_1 + \cdots + a_k dF_k + a_{k+1} dF_{k+1} + \cdots + a_n dF_n.
\]

Since \( F_1, \ldots, F_k \in \mathcal{J} \), the first \( k \) terms lie in the kernel of \( i^* : \Lambda^1(M) \to \Lambda^1(L) \), and (b1) is proved. Furthermore, we have \( \partial F_j / \partial F_s = 0 \) for \( j \leq k < s \). Since \( F_1, \ldots, F_k \) span \( \mathcal{J} \) modulo \( \mathcal{D} \), it follows that \( \partial / \partial F_s(\mathcal{J}) \subset D^{1/2} \), \( s = k+1, \ldots, n \), that is, the field \( \partial / \partial F_s \) is tangent to \( L \). We see that \( \partial / \partial Q_s \) are geometric vector fields on \( L \), generated by \( \partial / \partial F_s \), and (a) is proved. Finally, we set

\[
\tilde{\Phi} = \Phi - \sum_{j=1}^k F_j \frac{\partial \Phi}{\partial F_j} + \frac{1}{2} \sum_{j,s=1}^k F_j F_s \frac{\partial^2 \Phi}{\partial F_j \partial F_s}.
\]

Then \( \tilde{\Phi} - \Phi \in \mathcal{J} \) and \( \partial \tilde{\Phi} / \partial F_j \in \mathcal{D} \), which implies (b2). The proposition is proved. \( \square \)

Definition 2.11. Let \( \omega \) be a volume form on an \( s \)-dimensional asymptotic submanifold \( L \), and let \( Q_1, \ldots, Q_s \) be coordinates on \( L \). The function

\[
\frac{D \omega}{DQ} = \omega \left( \frac{\partial}{\partial Q_1}, \ldots, \frac{\partial}{\partial Q_s} \right) \in C^\infty(\mathcal{J}_1)
\]

is called the density of \( \omega \) in the coordinates \( Q_1, \ldots, Q_s \).

Let \( X \in \text{Vect} M \) be a strong tangent field to a \( k \)-codimensional asymptotic submanifold \( L \). Let \((Q_{k+1}, \ldots, Q_n)\) be an arbitrary coordinate system on \( L \), and let \((F_1, \ldots, F_n)\) be any coordinate system on \( M \) constructed in the proof of Proposition 2.10 (that is, \((F_1, \ldots, F_k)\) is a \( k \)-tuple of generators of \( \mathcal{J} \) and \( F_{k+1}, \ldots, F_n \) are representatives of \((Q_{k+1}, \ldots, Q_n)\)). Then

\[
X = \sum_{j=1}^n a_j (\partial / \partial F_j),
\]

and the strong tangency condition in particular implies that \( a_j \in \mathcal{J} \), \( j = 1, \ldots, k \). Set

\[
\text{div}_Q X = \sum_{j=k+1}^n \frac{\partial a_j}{\partial F_j}.
\]

Proposition 2.12. (a) \( \text{div}_Q X \) is a well-defined element of \( \mathcal{C}^\infty(\mathcal{J}_1) \).

(b) If \( \tilde{Q} = (\tilde{Q}_{k+1}, \ldots, \tilde{Q}_n) \) is another coordinate system on \( L \), then

\[
\text{div}_{\tilde{Q}} X = \text{div}_Q X + X \left( \ln \frac{\partial \tilde{Q}}{\partial Q} \right).
\]
Proof. First, let us establish that the class of (2.2) in \( \mathcal{C}_{(1)}(L) \) does not depend on the choice of \( F_1, \ldots, F_k \); to this end, let \( S_i = \sum_{j=1}^{k} A_{ij} F_j + O(\mathcal{D}), \ i = 1, \ldots, k, \) be another set of generators of \( J \), and set \( S_i = F_i, i = k+1, \ldots, n. \) We have

\[
\frac{\partial}{\partial F_j} = \sum_{l=1}^{k} \frac{\partial S_l}{\partial F_j} \frac{\partial}{\partial S_l} = \left\{ \begin{array}{ll}
\sum_{l=1}^{k} A_{lj} \frac{\partial}{\partial S_l} + O(\mathcal{D}^{1/2}), & 1 \leq j \leq k, \\
\sum_{l=k+1}^{n} A_{lj} \frac{\partial}{\partial S_l} + O(\mathcal{D}^{1/2}), & k+1 \leq j \leq n.
\end{array} \right.
\]

(2.4)

Consequently,

\[
X = \sum_{j=1}^{k} A_{ij} \frac{\partial}{\partial S_j} + \sum_{l=k+1}^{n} a_i \frac{\partial}{\partial S_l} + O(\mathcal{D}^{1/2}).
\]

Now we have

\[
\sum_{j=k+1}^{n} \frac{\partial a_j}{\partial F_j} = \sum_{j=k+1}^{n} \frac{\partial a_j}{\partial S_j},
\]
as desired.

Let us now fix \( F_1, \ldots, F_k \) and consider some representatives \((S_{k+1}, \ldots, S_n)\) of the coordinate system \((\bar{Q}_{k+1}, \ldots, \bar{Q}_n)\). This time, we set \( S_i = F_i, i = 1, \ldots, k. \) We now have

\[
\frac{\partial}{\partial F_j} = \sum_{l=1}^{n} \frac{\partial S_l}{\partial F_j} \frac{\partial}{\partial S_l} = \left\{ \begin{array}{ll}
\sum_{l=1}^{n} \frac{\partial S_l}{\partial F_j} \frac{\partial}{\partial S_l} + \sum_{l=k+1}^{n} \frac{\partial S_l}{\partial F_j} \frac{\partial}{\partial S_l}, & 1 \leq j \leq k, \\
\sum_{l=k+1}^{n} \frac{\partial S_l}{\partial F_j} \frac{\partial}{\partial S_l}, & k+1 \leq j \leq n,
\end{array} \right.
\]

(2.5)

and so

\[
X = \sum_{l=1}^{k} a_i \frac{\partial}{\partial S_l} + \sum_{l=k+1}^{n} \sum_{j=1}^{n} \frac{\partial S_l}{\partial F_j} \frac{\partial a_j}{\partial S_l}.
\]

We have

\[
\text{div} \bar{Q} X = \text{div}_F X - \sum_{j=1}^{k} \frac{\partial a_j}{\partial F_j}, \quad \text{where} \quad \text{div}_F X = \sum_{j=1}^{n} \frac{\partial a_j}{\partial F_j}.
\]

Similarly,

\[
\text{div} \bar{Q} X = \text{div}_S X - \sum_{j=1}^{k} \frac{\partial a_j}{\partial S_j} = \text{div}_S X - \sum_{j=1}^{k} \frac{\partial a_j}{\partial F_j} + O(\mathcal{D}^{1/2})
\]

by virtue of (2.5), since \( a_j \in \mathcal{J} \) for \( j = 1, \ldots, k \) and \( \partial/\partial S_l \) is tangent to \( L \) for \( l = k+1, \ldots, n. \) Thus,

\[
\text{div} \bar{Q} X - \text{div}_Q X = \text{div}_S X - \text{div}_F X + O(\mathcal{D}^{1/2}) = X \left( \ln \det \frac{\partial S}{\partial F} \right) + O(\mathcal{D}^{1/2})
\]

(the last equality is valid by Sobolev’s lemma; e.g., see [11]). Since \( S_i = F_i \) for \( i = 1, \ldots, k, \) it follows that

\[
\det \frac{\partial S}{\partial F} = \det \frac{\partial (S_{k+1}, \ldots, S_n)}{\partial (F_{k+1}, \ldots, F_n)}.
\]

Thus, the class of \( \det \partial S/\partial F \) in \( \mathcal{C}_{(1)}(L) \) is \( \det \partial \bar{Q}/\partial Q \), and since \( X \) is a strong tangent field, it follows that the class of \( X(\ln \det \partial S/\partial F) \) in \( \mathcal{C}_{(1)}(L) \) is well defined and is equal to \( X(\ln \det \partial \bar{Q}/\partial Q). \) We have thus arrived at (2.3); by taking \( \bar{Q} = Q \) in (2.3), we see that \( \text{div} \bar{Q} X = \text{div} Q X \), i.e., the definition of \( \text{div} Q X \) is independent of the choice of \( F_{k+1}, \ldots, F_n. \) Proposition 2.12 is proved. \( \square \)
Now let \( \dim L = s \), and let \( \omega \in \Lambda^s(L) \) be a volume form on \( L \). In an arbitrary system of local coordinates \( Q_1, \ldots, Q_s \) on \( L \) we have
\[
\omega = \frac{D\omega}{DQ} dQ_1 \wedge \cdots \wedge dQ_s.
\]

**Definition 2.13.** Let \( X \) be a strong tangent field to \( L \). We define the Lie derivative of \( \omega \) along \( X \) by setting
\[
\mathcal{L}_X \omega = X\left(\frac{D\omega}{DQ}\right) + \frac{D\omega}{DQ} \text{div}_Q X \ dQ_1 \wedge \cdots \wedge dQ_s.
\]

**Lemma 2.14.** Equation (2.6) specifies a well-defined element \( \mathcal{L}_X \omega \in \tilde{\Lambda}^s(L) \).

**Proof.** We need to show that the form (2.6) is independent of the choice of the coordinates \( (Q_1, \ldots, Q_s) \). If \( (\tilde{Q}_1, \ldots, \tilde{Q}_s) \) is another system of coordinates on \( L \), we have \( (D\tilde{Q}/DQ = \det \partial \tilde{Q}/\partial Q) \)
\[
X\left(\frac{D\omega}{DQ}\right) + \frac{D\omega}{DQ} \text{div}_Q X = X\left(\frac{D\omega}{D\tilde{Q}}\right) + \frac{D\omega}{D\tilde{Q}} \left(\text{div}_\tilde{Q} X + \left(\frac{D\tilde{Q}}{DQ}\right)^{-1} X\left(\frac{D\tilde{Q}}{DQ}\right)\right)
\]
\[
= \left(\frac{D\tilde{Q}}{DQ}\right)^{-1} X\left(\frac{D\omega}{DQ}\right) + \frac{D\omega}{DQ} \left(\frac{D\tilde{Q}}{DQ} X + \left(\frac{D\tilde{Q}}{DQ}\right)^{-1} X\left(\frac{D\tilde{Q}}{DQ}\right)\right)
\]
\[
= \frac{DQ}{DQ} \left(\frac{D\omega}{DQ} + \frac{D\omega}{DQ} \text{div}_Q X\right) + \frac{DQ}{DQ} \left(\frac{D\omega}{DQ}\right) \left(\frac{D\tilde{Q}}{DQ} \text{div}_\tilde{Q} X + \left(\frac{D\tilde{Q}}{DQ}\right)^{-1} X\left(\frac{D\tilde{Q}}{DQ}\right)\right)
\]
However, the terms in the square brackets cancel out, and we obtain the desired result. Lemma 2.14 is proved. \( \square \)

**Definition 2.15.** A volume form \( \omega \in \Lambda^s(L) \) is said to be invariant with respect to a strong tangent vector field \( X \) if \( \mathcal{L}_X \omega = 0 \).

In the sequel we also need the following technical result.

**Lemma 2.16.** Let \( m_0 \in \Gamma, f \in C^\infty(\Gamma/L), f(m_0) \neq 0 \) (note that the value of \( f \) at \( m_0 \) is well defined). Then the square root \( \sqrt{f} \) is a well-defined element of \( C^\infty(\Gamma/L) \) in a neighborhood of \( m_0 \).

**Proof.** Let \( f_1, f_2 \in C^\infty(M) \) be two representatives of \( f \). Then \( f_1(m_0) = f_2(m_0) \neq 0 \), \( f_1 - f_2 \in D^{1/2} \). We have
\[
\sqrt{f_2} = \sqrt{f_1 + f_2 - f_1} = \sqrt{f_1} \sqrt{1 + f_2 - f_1} = \sqrt{f_1} + O(D^{1/2}).
\]
The lemma is proved. \( \square \)

### 2.3 Bundles and connections

**Definition 2.17.** Let \( L \) be an asymptotic submanifold in \( M \), and let \( E \) be a linear space over \( \mathbb{C} \). A vector bundle with fiber \( E \) over \( L \) is a \( C^\infty(L) \)-module \( \mathcal{E} \) on \( M \) locally isomorphic to \( C^\infty(L) \otimes \mathcal{E} \).
Remark. What we have defined is in fact an analog of the sheaf of germs of sections of a vector bundle.

Example. The “tangent bundle” $\text{Vect}(L)$ and the “cotangent bundle” $\Lambda^1(L)$ are $s$-dimensional vector bundles over $L$ (here $s = \dim L$).

If $\mathcal{E}$ is a vector bundle with fiber $E$ over $L$, then we introduce the sheaves

$$
\mathcal{E}(1) = \mathcal{E}/\mathcal{D}^{1/2}\mathcal{E}, \quad \mathcal{E} = \mathcal{E}/\mathcal{D}\mathcal{E}, \quad \mathcal{E}(1) = \mathcal{E}/\mathcal{D}^{1/2}
$$

(note that the action of $\mathcal{D}^{1/2}, \mathcal{D},$ and $\mathcal{D}^{1/2}$ on $\mathcal{E}$ is naturally defined). These sheaves are $C_\infty(L), \mathcal{C}(L),$ and $C_\infty(L)$-modules, respectively, and there are natural homomorphisms

$$
\mathcal{E} \rightarrow \mathcal{E} \rightarrow \mathcal{E}(1) \rightarrow \mathcal{E}(1)
$$

over the homomorphisms of sheaves of rings

$$
C_\infty(L) \rightarrow \mathcal{C}(L) \rightarrow C_\infty(L) \rightarrow C_\infty(L).
$$

Let $\pi : F \rightarrow M$ be a vector bundle with fiber $E$ over $M$. Consider the sheaf $\mathcal{F}$ of germs of sections of $F$. If $L = (\mathcal{D}, \mathcal{J})$ is an asymptotic submanifold in $M$, then we can define the pullback of $\mathcal{F}$ on $L$ by setting

$$
\mathcal{E} = i^* \mathcal{F} = \mathcal{F}/\mathcal{J} \mathcal{F}
$$

(note that $i^*$ in (2.7) symbolizes the pullback by the “embedding” $i = L \hookrightarrow M$).

Any vector bundle $F$ over $M$ is a subbundle of some trivial bundle $M \times B$, where $B$ is a vector space over $\mathbb{C}$, and hence can be specified by a smooth projection-valued mapping

$$
\Pi : M \rightarrow \text{End}(B),
$$

the range of $\Pi(x)$ being the fiber of $F$ over $x \in M$ (if the space $B$ is infinite-dimensional, the case which is important in applications, then one should be very careful about the differentiability conditions to be imposed on $\Pi$; in any case we assume that the range of $\Pi$ is finite-dimensional).

The smooth sections of $F$ are the mappings $u : M \rightarrow B$ such that $\Pi(x)u(x) = u(x)$ for any $x \in M$. Let $\mathcal{E}$ be the pullback (2.7). We shall briefly discuss the nonparametric local description of $\mathcal{E}$.

Let $x = (x'; x'') = (x_1, \ldots, x_k; x_{k+1}, \ldots, x_n)$ be a local coordinate system on $M$, let $\mathcal{D}$ be the dissipation ideal associated with the dissipation

$$
D(x) = d(x'') + \|x' - g(x'')\|^2,
$$

and let $\mathcal{J}$ be the ideal generated by $\mathcal{D}$ and by the functions $x_i - g_i(x'')$, $i = 1, \ldots, k$ (cf. Theorem 1.10). Set

$$
x'(x'') = \arg\min_{x'} D(x).
$$

By following the proof of Theorem 1.10, it is easy to establish that $\Pi(x)$ and $u(x)$ can be represented in the form

$$
\Pi(x) = \tilde{\Pi}(x'') + \Pi_1(x), \quad u(x) = \tilde{u}(x'') + u_1(x),
$$
where $\Pi_1(x) \in \mathcal{J} \text{End}(B, B)$, $u_1(x) \in \mathcal{J}\mathcal{C}^\infty(M, B)$, and $\tilde{\Pi}$ and $\tilde{u}$ are unique modulo $O(d(x''))$.

The explicit form of $\tilde{\Pi}$ and $\tilde{u}$ is given by

$$
\tilde{\Pi}(x'') = \Pi(x'(x''), x'' + [g(x'') - x'(x'')] \frac{\partial\Pi}{\partial x'}(x'(x''), x'');
$$

$$
\tilde{u}(x'') = u(x'(x''), x'' + [g(x'') - x'(x'')] \frac{\partial u}{\partial x'}(x'(x''), x'').
$$

(2.9)

The objects (2.9) will be referred to as the local representatives of $\Pi$ and $u$ in the coordinates $x''$. A straightforward calculation yields

$$
\tilde{\Pi}^2 = \tilde{\Pi} + O(d), \quad \tilde{\Pi}\tilde{u} = \tilde{u} + O(d).
$$

If we regard the local representatives as classes modulo $O(d)$ rather than functions, then we have $\tilde{\Pi}^2 = \tilde{\Pi}$ and $\tilde{\Pi}\tilde{u} = \tilde{u}$.

In the following we shall make some use of connections and covariant derivatives.

**Definition 2.18.** Let $E$ be a vector bundle over $L$. A connection $\partial$ on $E$ is a $\mathbb{C}$-linear mapping

$$
\partial : E \to E \otimes \Lambda^1(L)
$$

such that for any $f \in C^\infty(L)$ and any $\varphi \in E$ we have

$$
\partial(f \varphi) = f \partial \varphi + \varphi \otimes df
$$

(2.10)

(we write $f$ instead of $\pi(f)$ on the right-hand side in (2.10)).

Let $X$ be a vector field on $L$. Then the covariant derivative $\nabla_X\varphi$ of a section $\varphi \in E$ is defined as follows:

$$
\nabla_X\varphi \overset{\text{def}}{=} \partial_X(\varphi).
$$

This is well defined, since $\partial \varphi \in E \otimes \Lambda^1(L)$ and can be applied to $X$ with respect to the second factor of the tensor product.

Let $F \subset M \times B$ be a subbundle of the trivial bundle $M \times B$, and let $\Pi : M \to \text{End} B$ be the corresponding projection family.

The bundle $F$ is equipped with the natural Levi-Civit`a connection $\partial = \Pi d$. It is easy to see that this connection factors through the natural projections, so that we obtain a connection $\tilde{\partial} = i^*\partial$ on the pullback $E = i^*F$ (2.7). Obviously, in the local coordinates $x''$ we have $\partial = \tilde{\Pi}\tilde{d}$, where $\tilde{d}$ is the differential with respect to the local coordinates.

### 3 Positive asymptotic Lagrangian submanifolds

We begin by recalling, without proof, the notion and the main points concerning Lagrangian asymptotic manifolds as defined in [24]. Then we devise a new definition in the spirit of the approach outlined in §1 and §2 and show that the two approaches are equivalent. This will help us save space by resorting to some proofs that have already been published.

But first of all, let us introduce some notation.

In this section we deal with asymptotic submanifolds in $\mathbb{R}^{2n}$. The coordinates in $\mathbb{R}^{2n}$ will be denoted by $(p, q) = (p_1, \ldots, p_n, q_1, \ldots, q_n)$. We assume that $R^n$ is equipped with the standard symplectic structure

$$
\omega^2 = dp \wedge dq \equiv \sum_{i=1}^n dp_i \wedge dq_i.
$$

(3.1)
The following notation, in fact standard in the literature on the canonical operator, will be used freely. Let \( I = \{i_1, \ldots, i_k\} \subset \{1, \ldots, n\} \) be an arbitrary subset. Then by \( \overline{I} \) we denote its complement \( \overline{I} = \{1, \ldots, n\} \setminus I = \{i_{k+1}, \ldots, i_n\} \), by \( |I| \) the cardinality \( |I| = k \), and if \( \xi = (\xi_1, \ldots, \xi_n) \) is an \( n \)-vector, then \( \xi_I \) is used to denote the \( k \)-vector \( (\xi_{i_1}, \ldots, \xi_{i_k}) \) and \( \xi_{\overline{I}} = (\xi_{i_{k+1}}, \ldots, \xi_{i_n}) \). Furthermore, we feel free to write \( p_I dq_I \) for \( \sum_{i \in I} p_i dq_i \) etc; however, unless otherwise specified, summation is never assumed in matrices of second partial derivatives; thus, \( \partial^2 \Phi / \partial q_I \partial q_I \) may stand for the matrix \( (\partial^2 \Phi / \partial q_i \partial q_j)_{i,j \in I} \) rather than for its trace; we even sometimes write \( \xi_I (\partial^2 \Phi / \partial x_I \partial x_I) \xi_I \) to denote

\[
\sum_{i,j \in I} \xi_i \xi_j \frac{\partial^2 \Phi}{\partial x_i \partial x_j},
\]

but if misunderstanding is likely to occur, then the less ambiguous notation

\[
\left< \xi_I, \frac{\partial^2 \Phi}{\partial x_I \partial x_I} \eta_I \right>
\]

is used; here \( \langle \cdot, \cdot \rangle \) is the standard bilinear pairing of vectors in \( \mathbb{C}^{|I|} \).

Finally, \( dp_I \wedge dq_{\overline{I}} \) stands for \( (-1)^\sigma dp_{i_1} \wedge \cdots \wedge dp_{i_k} \wedge dq_{i_{k+1}} \wedge \cdots \wedge dq_{i_n} \), where \( \sigma \) is the parity of the permutation \( (i_1, \ldots, i_n) \); the effect is as if the factors were arranged in the ascending order of the subscripts. The subscript \( I \) is usually omitted altogether if \( I = \{1, \ldots, n\} \).

For any \( I \subset \{1, \ldots, n\} \) we define a transformation \( \gamma_I : \mathbb{R}^{2n} \to \mathbb{R}^{2n} \) by setting

\[
\gamma_I(p,q) = \left( (p_I, -q_I), (q_I, p_I) \right).
\]

Note that \( \gamma_I^* \omega^2 = \omega^2 \) and \( \gamma^*(pdq) = p_I dq_I - q_I dp_I \).

### 3.1 Lagrangian asymptotic manifolds: one of the traditional definitions

In this subsection we follow [24] and [25] with minor alterations as to notation and the form of presentation. However, there is one significant difference: here, as well as in §1 and §2, we deal only with asymptotic Lagrangian submanifolds in the first approximation (c-Lagrangian structures in the terminology of [25]); the accuracy \( O(D^\infty) \) is actually redundant and is not used. We omit all proofs, which can be extracted from [24] and [25].

**Definition 3.1.** A Lagrangian chart is a quintuple \( r = (U, d, P, Q, W) \), where \( U \subset \mathbb{R}^n \) is a domain and \( d : U \to \mathbb{R}^n \); \( P, Q : U \to \mathbb{C}^n \), \( W : U \to \mathbb{C} \) are smooth functions such that

i) \( \text{rank}_C (\frac{\partial P(\alpha)}{\partial \alpha}, \frac{\partial Q(\alpha)}{\partial \alpha}) = n \) for \( \alpha \in \Gamma_d \);

ii) \( \text{Im} P = O(d^{1/2}), \text{Im} Q = O(d^{1/2}), \text{Im} W = O(d) \);

iii) \( (P, Q) : \Gamma_d \to \mathbb{R}^n \) is a topological embedding (note that \( (P, Q)|_{r_d} \) are real by ii));

iv) \( dW = P dQ + O(d) \).

We denote \( \Gamma(r) = \{ (p, q) \in \mathbb{R}^{2n} \mid p = P(\alpha), q = Q(\alpha) \text{ for some } \alpha \in \Gamma_\alpha \} \); the set \( \Gamma(r) \) is called the zero image of the chart \( r \). The function \( W(\alpha) \) is called the action in the chart \( r \).

**Definition 3.2.** Two Lagrangian charts

\[
r = (U, d, P, Q, W) \quad \text{and} \quad \tilde{r} = (\tilde{U}, \tilde{d}, \tilde{P}, \tilde{Q}, \tilde{W})
\]

are said to be consistent if for any two points \( \alpha_0 \in \Gamma_d \) and \( \tilde{\alpha}_0 \in \Gamma_{\tilde{d}} \) such that \( (P(\alpha_0), Q(\alpha_0)) = (\tilde{P}(\tilde{\alpha}_0), \tilde{Q}(\tilde{\alpha}_0)) \) there exists a neighborhood \( V \subset U \) of \( \alpha_0 \), a neighborhood \( \tilde{V} \subset \tilde{U} \) of \( \tilde{\alpha}_0 \), and
a diffeomorphism \( V \leftrightarrow \tilde{V} \), \( \alpha_0 \rightarrow \tilde{\alpha}_0 \), such that (under the identification of \( \alpha \) with \( \tilde{\alpha} \) by this diffeomorphism)

i) \( d \) and \( \tilde{d} \) define the same dissipation ideal \( \mathfrak{d} \);

ii) \( P - \tilde{P} \in d^{1/2} \), \( Q - \tilde{Q} \in d^{1/2} \), and \( (P - \tilde{P}) dQ = (\tilde{Q} - Q) dP + O(d) \);

iii) \( \tilde{W} - W = (1/2)(P + \tilde{P}, \tilde{Q} - Q) + O(d^{3/2}) + c \), where \( c \) is some constant.

**Definition 3.3.** A Lagrangian asymptotic manifold \( L \) in \( \mathbb{R}^{2n} \) is a collection of the following data: a closed subset \( \Gamma = \Gamma_L \subset \mathbb{R}^{2n} \) (the support of \( L \)) and a family \( \{r_a\}_{a \in A} \) of pairwise consistent Lagrangian charts (an atlas of \( L \)) such that \( \Gamma(r_a) \) is a relatively open subset in \( \Gamma \) for each \( a \in A \) and \( \cup_{a \in A} \Gamma(r_a) = \Gamma \).

One does not distinguish Lagrangian asymptotic manifolds with equivalent atlases (two atlases are said to be equivalent if their union is itself a valid atlas), and in what follows we assume that the atlas in Definition 3.3 is maximal (i.e., is the union of all atlases in an equivalence class).

If \((p, q) \in \Gamma(r_a)\), then we say that \( r_a \) is a chart in a neighborhood of the point \((p, q)\) on \( L \), or that the chart \( \Gamma_a \) covers the point \((p, q)\).

**Definition 3.4.** Let \( I \subset \{1, \ldots, n\} \). A Lagrangian chart \( V = (U, d, P, Q, W) \) is said to be \( I \)-nonsingular if \((Q_I(\alpha), P_I(\alpha)) = \alpha \). An \( I \)-nonsingular chart with \( I = \{1, \ldots, n\} \) is merely said to be nonsingular without mentioning \( I \). The function \( S_I(q_I, p_I) = W(q_I, p_I) - p_IQ_I(q_I, p_I) \) is called the \( I \)-phase in \( r \).

**Lemma 3.5.** Let \( L \) be a Lagrangian asymptotic manifold in \( \mathbb{R}^{2n} \). Then each point \((p, q) \in \Gamma_L \) is covered by an \( I \)-nonsingular chart for some \( I \subset \{1, \ldots, n\} \).

The proof is immediate from the following lemma.

**Lemma (on local coordinates).** Let \( V \subset \mathbb{C}_{\xi,\eta}^{2n} \) be a complex Lagrangian plane (that is, \( \dim V = n \) and \( d\xi \wedge d\eta|_V = 0 \)). Then there exists a subset \( I \subset \{1, \ldots, n\} \) such that \((\xi_I, \eta_I)\) is a coordinate system on \( V \) (that is, the differentials \((d\xi_I|_V, d\eta_I|_V)\) are linearly independent).

The proof can be found in [9], p. 369, and elsewhere.

The atlas consisting of \( I \)-nonsingular charts with various \( I \) will be called the canonical covering.

Let a point \((p_0, q_0) \in \Gamma_L \) be covered by an \( I \)-nonsingular chart \( r_I \) and a \( K \)-nonsingular chart \( r_K \) for some \( I, K \subset \{1, \ldots, n\} \). Let us write out the formula relating the corresponding \( I \)-phase and \( K \)-phase \( S_K \). By applying the transformation \( \gamma_K \), we can reduce the problem to the case \( K = \emptyset \).

We denote the chart \( r_K \) simply by \( r \) and the phase \( S_K \) simply by \( S \). In this notation,

\[
S_I(q_I, p_I) = \left\{ S(q) - q_Ip_I - \frac{1}{2} \left( p_I - \frac{\partial S}{\partial q_I} \right)^{-1} \left( p_I - \frac{\partial S}{\partial q_I} \right) \right\}_{q_I = q_I(q_I, p_I)} + O(d_I^{3/2}),
\]

where \( q_I = q_I(q_I, p_I) \) is an arbitrary smooth mapping such that \( q_I(q_0I, q_{\sigma}) = q_{\sigma} \) and \( p_I - \frac{\partial S}{\partial q_I} \) is an arbitrary smooth mapping such that \( q_I(q_0I, q_{\sigma}) = q_{\sigma} \) and \( p_I - \frac{\partial S}{\partial q_I} \).

**Definition 3.6.** A Lagrangian asymptotic manifold \( L \) is said to be positive\(^1\) if for any \( I \subset \{1, \ldots, n\} \) and any \( I \)-nonsingular chart on \( L \), the function \( \text{Im} S_I(q_I, p_I) \) is equivalent to \( d(q_I, p_I) \) in a sufficiently small neighborhood of \( \Gamma_d \), that is, the dissipativity inequality

\[
\text{cd}(p_I, q_I) \leq \text{Im} S_I(q_I, p_I) \leq \text{Cd}(p_I, q_I)
\]

\(^1\)We prefer this term to the term “dissipative” used in [25] and some other papers. Maybe “nonnegative” would be even a better choice, but we use “positive.”
is valid with some positive constants $c$ and $C$ in a neighborhood of each point of $\Gamma_d$.

**Lemma 3.7.** Let $(p_0, q_0) \in \Gamma(r_I) \cap \Gamma(r_K)$, where $r_I$ and $r_K$ are an $I$- and a $K$-nonsingular chart on $L$, respectively. Then the dissipativity inequality is valid for $\text{Im} S_I$ in the chart $r_I$ in a neighborhood of $(q_{0I}, p_{0I})$ if and only if it is valid for $\text{Im} S_K$ in the chart $r_K$ in a neighborhood of $(q_{0K}, p_{0K})$.

The proof can be found, say in [9], p. 386, or [25], p. 104; however, later on in this paper we shall give an independent proof based on a lemma that will also prove useful when we shall consider canonical transformations.

### 3.2 Lagrangian asymptotic manifolds as asymptotic manifolds: local description

Given a Lagrangian asymptotic manifold $L$ in the sense of Definition 3.3, it is easy to interpret $L$ as an asymptotic manifold in the sense of §1 and §2. Namely, let a Lagrangian chart $r = (U, d, P, Q, W)$ be given. The quadruple $(U, d, P, Q)$ determines an $n$-dimensional asymptotic submanifold $L = (D, J)$ in $\mathbb{R}^{2n}_{p,q}$ in the standard way (parametric local description, see §1.4): we set

$$
\hat{D}(p, q, \alpha) = d(\alpha) + \|p - P(\alpha)\|^2 + \|q - Q(\alpha)\|^2,
$$

$$
\hat{J} = \hat{D} + \{p_1 - P_1(\alpha), \ldots, p_n - P_n(\alpha), q_1 - Q_1(\alpha), \ldots, q_n - Q_n(\alpha)\},
$$

(3.5)

where $\hat{D}$ is the dissipation ideal generated by $D(p, q, \alpha)$; then we find

$$
D(p, q) = \min_{\alpha} \hat{D}(p, q, \alpha),
$$

consider the dissipation ideal $D$ associated with $D$, and set

$$
J = \{f(p, q, \alpha) \in \hat{J} \mid f \text{ is independent of } \alpha\}.
$$

**Lemma 3.8.** (a) The manifold $L$ is involutive in the sense that $\{J, J\} \subset J$, where $\{\cdot, \cdot\}$ is the standard Poisson bracket corresponding to the symplectic structure (3.1).

(b) Consistent Lagrangian charts determine the same asymptotic submanifold on their intersection.

**Remark.** Note that the converse of Lemma 3.8 (b) is not true, since condition iii) in Definition 3.2 does not follow from i) and ii) (nor does the very existence of a function $W$ satisfying condition iii) in Definition 3.1 follow from conditions i) and ii) in that definition).

For this reason, we must retain the phase $W_I$, i.e., incorporate it in the new definition of Lagrangian asymptotic manifold to be devised; this is done in the next subsection.

**Proof of Lemma 3.8.** Let us prove (b). Let $r$ and $\tilde{r}$ be two consistent charts. Without loss of generality we can assume that $V = U$ and $\tilde{V} = \tilde{U}$. It readily follows from condition ii) in Definition 3.2 that $\hat{D}(p, q, \alpha)$ and

$$
\hat{\tilde{D}}(p, q, \alpha) = d(\alpha) + \|p - \tilde{P}(\alpha)\|^2 + \|q - \tilde{Q}(\alpha)\|^2
$$

are equivalent; hence, so are

$$
D(p, q) = \min_{\alpha} \hat{D}(p, q, \alpha) \quad \text{and} \quad \tilde{D}(p, q) = \min_{\alpha} \hat{\tilde{D}}(p, q, \alpha).
$$
The ideal \( \hat{\mathcal{J}} = \hat{\mathcal{D}} + \{ p_1 - P_1(\alpha), \ldots, p_n - P_n(\alpha), q_1 - Q_1(\alpha), \ldots, q_n - Q_n(\alpha) \} \) is obviously different from \( \hat{\mathcal{J}} \); however, the ideal \( \mathcal{J} \) coincides with the ideal

\[
\mathcal{J} = \{ f(p,q,\alpha) \in \hat{\mathcal{J}} \mid f \text{ is independent of } \alpha \}.
\]

Indeed, let \( f(p,q) \in \mathcal{J} \). Then

\[
f(p,q) = A(\alpha, p, q)(p - P(\alpha)) + B(\alpha, p, q)(q - Q(\alpha)) + O(\hat{\mathcal{D}}).
\]

Differentiating \( f(p,q) \) with respect to \( \alpha \) yields

\[
A \frac{\partial P}{\partial \alpha} + B \frac{\partial Q}{\partial \alpha} = O(\hat{\mathcal{D}}^{1/2}).
\]

It follows from Lemma 3.6 that \( \det(\partial Q_1/\partial \alpha, \partial P_\tau/\partial \alpha) \neq 0 \) for some \( I \subset \{1, \ldots, n\} \); by applying \( \gamma_I \) we can assume without loss of generality that \( I = \{1, \ldots, n\} \). Then \( A(\partial P/\partial Q) + B = O(\hat{\mathcal{D}}^{1/2}) \). Next,

\[
f(p,q) = A(p - \hat{P}) + B(q - \hat{Q}) + A(\hat{P} - p) + B(\hat{Q} - Q) + O(\hat{\mathcal{D}}).
\]

It follows from Definition 3.1, iv) that \( \partial P/\partial Q - \partial P/\partial Q = O(d^{1/2}) \) and from Definition 3.2 that

\[
\hat{P} - P = ^t\partial P/\partial Q(\hat{Q} - Q) + O(d) = \frac{\partial P}{\partial Q}(\hat{Q} - Q) + O(d).
\]

Thus,

\[
A(\hat{P} - P) + B(\hat{Q} - Q) = \left( A \frac{\partial P}{\partial Q} + B \right)(\hat{Q} - Q) = O(\hat{\mathcal{D}}),
\]

and we see that \( f(p,q) \in \mathcal{J} \). By symmetry, \( \mathcal{J} \subset \mathcal{J} \), so that \( \mathcal{J} = \hat{\mathcal{J}} \), and item (b) is proved. It follows from (b) that it suffices to verify (a) for the case in which the chart \( r \) is \( I \)-nonsingular for some \( I \subset \{1, \ldots, n\} \) (and even for \( I = \{1, \ldots, n\} \)).

In an \( I \)-nonsingular chart the manifold \( L = (\mathcal{D}, \mathcal{J}) \) can be described more explicitly as follows.

**Lemma 3.9.** Let \( r = (U,d,P,Q,W) \) be an \( I \)-nonsingular chart. Then the corresponding Lagrangian manifold \( L = (\mathcal{D}, \mathcal{J}) \) is given by

i) the dissipation

\[
D(p,q) = d(q_I, p_\tau) + \left\| p_I - \frac{\partial S_I}{\partial q_I} \right\|^2 + \left\| q_\tau + \frac{\partial S}{\partial p_\tau} \right\|^2;
\]

ii) the ideal

\[
\mathcal{J} = \mathcal{D} + \left\{ p_I - \frac{\partial S_I}{\partial q_I}, p_s - \frac{\partial S}{\partial q_s} \right\}.
\]

Here \( S_I(q_I, p_\tau) \) is the \( I \)-phase in the \( I \)-nonsingular chart on \( L \).

The proof is by straightforward computation using Definition 3.1 and Eq. (3.5).

We can now finish the proof of Lemma 3.8. Assuming \( I = \{1, \ldots, n\} \), we have \( \mathcal{J} = D + \{ p - \partial S/\partial q \} \), and involutivity follows readily, since for the Poisson bracket we have

\[
\left\{ p_i - \frac{\partial S}{\partial q_i}, p_s - \frac{\partial S}{\partial q_s} \right\} = \frac{\partial^2 S}{\partial q_i \partial q_s} - \frac{\partial^2 S}{\partial q_s \partial q_i} = 0.
\]

Lemma 3.8 is proved.

Thus, to any Lagrangian asymptotic manifold in the sense of Definition 3.3 we have assigned an asymptotic manifold in the sense of §1. However, the inverse correspondence is not clear as yet; to guarantee its existence, we must first incorporate phases in the definition; this will be done in the next subsection.
3.3 Global definition

We shall sometimes use the “complex coordinates” \((z, \bar{z})\) on \(\mathbb{R}^{2n}\), where \(z = (z_1, \ldots, z_n)\), \(\bar{z} = (\bar{z}_1, \ldots, \bar{z}_n)\), and

\[
  z_j = q_j - ip_j, \quad \bar{z}_j = q_j + ip_j, \quad j = 1, \ldots, n.
\]

Let \(\mathcal{D}\) be a dissipation ideal in \(C^\infty(\mathbb{R}^{2n})\), and let \(\Gamma = \text{loc}(\mathcal{D})\) be the set of its zeros. Furthermore, let \(U \subset \mathbb{R}^{2n}\) be a sufficiently small neighborhood of \(\Gamma\), and let \(\pi: \tilde{U} \to U\) be the universal covering over \(U\). Then \(\tilde{U}\) is a simply connected manifold. The mapping \(\pi\) is a local diffeomorphism, and so we can freely use the same coordinates in \(U\) and in \(\tilde{U}\).

Furthermore, the ideal \(\pi^*(\mathcal{D})\) is well defined in \(C^\infty(\tilde{U})\); for brevity, it will be denoted by the same letter \(\mathcal{D}\).

**Definition 3.10.** A \(z\)-action is an element \(\Phi \in C^\infty(\tilde{U})/\mathcal{D}^{3/2}\) that satisfies the following three conditions.

i) Let \(m_0 \in \Gamma\) be an arbitrary point, and let \(\Phi_1(p, q)\) and \(\Phi_2(p, q)\) be two branches of \(\Phi\) defined in a neighborhood of \(m_0\). Then

\[
  \Phi_1(p, q) - \Phi_2(p, q) = \Phi_1(m_0) - \Phi_2(m_0) + O(\mathcal{D}^{3/2})
\]

in a neighborhood of \(m_0\). In other words, the values of \(\Phi\) on any two sheets of the covering \(\pi\) differ by a constant modulo \(O(\mathcal{D}^{3/2})\).

ii) There exists a vector function \(Z^*(p, q) = (Z_1(p, q), \ldots, Z_n(p, q)) \in C^\infty(U)/\mathcal{D}\) such that

\[
  d\Phi = \frac{1}{2i} Z^* dz + O(\mathcal{D})
\]

(note that \(Z^*\) is a function on \(U\) rather than on \(\tilde{U}\), which is not surprising in view of condition i)).

iii) The function \(Z^*\) satisfies the condition \(\bar{z}_j - Z^*_j \in \mathcal{D}^{1/2}, j = 1, \ldots, n\).

Suppose that a dissipation ideal \(\mathcal{D}\), a covering \(\pi: \tilde{U} \to U\), and a \(z\)-action \(\Phi\) are given. We shall now construct the corresponding Lagrangian asymptotic manifold.

Set \(\mathcal{J} = \mathcal{D} + \{\bar{z}_1 - Z^*_1, \ldots, \bar{z}_n - Z^*_n\}\). Then \(L = (\mathcal{D}, \mathcal{J})\) is obviously an asymptotic manifold of dimension \(n\).

Furthermore, \(\{\mathcal{J}, \mathcal{J}\} \subset \mathcal{J}\) (this can easily be proved by straightforward computation), and so in a neighborhood of each point of \(\Gamma\) the functions \((q_I, p_I)\) for some \(I \subset \{1, \ldots, n\}\) are coordinates on \(L\). By \(U_I\) we denote the projection of this neighborhood on the coordinate plane \((q_I, p_I)\).

Let us construct the corresponding Lagrangian charts. By applying the transformation \(\gamma_I\), we can always assume that we are in a nonsingular chart. Set

\[
  Q(p, q) = \frac{Z^* + z}{2}, \quad P(p, q) = \frac{Z^* - z}{2}.
\]

Then \(Z^* = Q + iP, z = Q - iP\). We define

\[
  W = \Phi + \frac{\langle P, Q \rangle}{2} - \frac{P^2 + Q^2}{4i}. \quad (3.6)
\]
Then straightforward computation shows that
\[ dW = P \, dQ. \]  \hfill (3.7)

Furthermore, since \( dQ_1, \ldots, dQ_n \) are linearly independent, we can complete them by some differentials \( dF_1, \ldots, dF_n \) to form a basis of differentials on \( \mathbb{R}^{2n} \) in a neighborhood of \( (p_0, q_0) \). By differentiating (3.7), we obtain
\[ dP = \mathcal{E} \, dQ + \mu \, dF, \]
where \( \mathcal{E} = O(D^{1/2}) \), \( \mu = O(D^{1/2}) \), and \( \mathcal{E} \) is independent of the choice of \( F_1, \ldots, F_n \) modulo \( O(D^{1/2}) \). Let
\[ p(q) = \arg \min_p D(p, q). \]

We set
\[ S(q) = \left\{ W + \langle P, q - Q \rangle + \frac{1}{2} \langle q - Q, \mathcal{E}(q - Q) \rangle \right\} \bigg|_{p=p(q)}. \]  \hfill (3.8)

The corresponding formulas for \( I \neq \{1, \ldots, n\} \) read
\[ W_I = W - P_I Q_T, \]
\[ S_I(q_I, p_T) = \left\{ W_I + \langle P_I, q_I - Q_I \rangle - \langle Q_T, p_T - P_I \rangle \right. \]
\[ \left. + \frac{1}{2} \left( \langle q_I - Q_I, P_I - P_T \rangle, \mathcal{E}_I(q_I - Q_I) \right) \right\} \bigg|_{p_I=q_I, p_T=p_T}, \]  \hfill (3.9)

where the matrix function \( \mathcal{E}_I \) is defined from the condition
\[ d(P_I, Q_T) = \mathcal{E}_I d(Q_I, -P_T) + O(D^{1/2}) \]
and
\[ (p_I(q_I, p_T), q_T(q_I, p_T)) = \arg \min_{p_I, q_T} D(p, q). \]

Furthermore, we set
\[ d_I(q_I, p_T) = \min_{p_I, q_T} D(p, q). \]  \hfill (3.10)

**Lemma 3.11.** (a) The function \( S_I(q_I, p_T) \) does not depend modulo \( O(d_I^{3/2}) \) on the choice of the representative of \( \Phi \in C^\infty(\bar{U})/D^{3/2} \) in \( C^\infty(\bar{U}) \).
(b) The quintuple
\[ r_I = \left( U_I, d_I, \left( p_T, \frac{\partial S_I}{\partial q_I} \right), \left( q_I, -\frac{\partial S_I}{\partial p_T} \right), S_I \right) \]
is an \( I \)-nonsingular Lagrangian chart associated with the asymptotic manifold \( (\mathcal{D}, \mathcal{J}) \).
(c) All Lagrangian charts described in (b) are pairwise consistent.

**Proof.** (a) We can assume that \( I = \{1, \ldots, n\} \). Equation (3.6) for \( W \) can be rewritten as follows:
\[ W = \Phi + \frac{1}{8i}((Z^*)^2 - z^2 - 2Z^*). \]
Let \( \tilde{\Phi} = \Phi + O(D^{3/2}) \). It follows that \( P - \tilde{P} = O(D) \), \( Q - \tilde{Q} = O(D) \), and \( \mathcal{E} - \tilde{\mathcal{E}} = O(D^{1/2}) \).
Furthermore,
\[
\tilde{W} - W = \Phi - \Phi + \frac{1}{8i}(\tilde{Z}^* - Z^* - 2z(\tilde{Z}^* - Z^*)) = \frac{1}{8i}(\tilde{Z}^* - Z^*)(\tilde{Z}^* + Z^* - 2z) = \frac{1}{8i}(\tilde{Q}^* - Q) \times 2i(\tilde{P} + P) = \frac{1}{2}(\tilde{P} + P)(\tilde{Q} - Q) = P(\tilde{Q} - Q) + O(D^{3/2}).
\]

Further, we obtain
\[
S = W + \langle P, q - Q \rangle + \frac{1}{2}\langle q - Q, \mathcal{E}(q - Q) \rangle,
\]
\[
\tilde{S} = \tilde{W} + \langle \tilde{P}, q - \tilde{Q} \rangle + \frac{1}{2}\langle q - \tilde{Q}, \mathcal{E}(q - \tilde{Q}) \rangle,
\]
\[
\tilde{S} - S = \tilde{W} - W + \langle P, Q - \tilde{Q} \rangle + O(D^{3/2}) = O(D^{3/2}),
\]
and (a) is proved.

(b) Again we assume that \( I = \{1, \ldots, n\} \). Then what we need to prove is that the functions \( p - \partial S/\partial q \) generate the same ideal as \( z - Z^* \). This can be proved by straightforward computation.

(c) This can be verified by straightforward computation. Lemma 3.11 is proved. \( \square \)

To prove that the traditional description of positive Lagrangian asymptotic manifolds is equivalent to that via the \( z \)-action, it remains to explain how to reconstruct the \( z \)-action from the phases. The answer is given by the following lemma.

**Lemma 3.12.** Let \( L \) be a positive Lagrangian manifold, and let \( S(q) \) be a nonsingular phase on \( L \). Then the function
\[
\Phi(p, q) = S(q) - \frac{1}{2}pq + \frac{q^2}{4i} - \frac{(\partial S/\partial q)^2}{4i} - \frac{1}{4i}\left\langle p - \frac{\partial S}{\partial q}, (1 - i\frac{\partial^2 S}{\partial q \partial q})^{-1}\left(1 + i\frac{\partial^2 S}{\partial q \partial q}\right)\left(p - \frac{\partial S}{\partial q}\right)\right\rangle
\]
is the \( z \)-action on \( L \).

The proof is by straightforward computation.

Note that for positive Lagrangian manifolds the matrix \( (1 - i(\partial^2 S/\partial q \partial q)) \) is always nonsingular, and positivity is essential here. The formulas for constructing \( \Phi(p, q) \) from an \( I \)-nonsingular phase \( S_I(q_I, p_I) \) are obtained by applying \( \gamma_I \).

Let us now give the independent proof (promised above) of the fact that positivity is preserved in transition from one \( I \)-nonsingular chart to another (Lemma 3.7). It suffices to consider the case in which \( I = \{1, \ldots, n\} \) and \( K = \emptyset \) (the variables \( x_K \) and \( p_K \) can be regarded as parameters). Then, in view of the transition formula (3.3), Lemma 3.8 is a consequence of the following general statement.

**Lemma 3.13.** Let \( F(p, q) = F_1(p, q) + iF_2(p, q) \) be a smooth function satisfying the conditions \( F_2(p, q) \geq 0, F_2(p_0, q_0) = 0, (\partial F/\partial q)(p_0, q_0) = 0 \), and
\[
\det \frac{\partial^2 F}{\partial q \partial q}(p_0, q_0) \neq 0.
\]
Also let
\[
D(p, q) = F_2(p, q) + \left\| \frac{\partial F}{\partial q}(p, q) \right\|^2.
\]
Then
\( D(p, q) \mapsto \min_q \) (the minimum is taken over a small neighborhood of \( q_0 \))

has a unique, smooth solution \( q = q(p) \) for \( p \) close to \( p_0 \), and \( q(p_0) = q_0 \).

(b) Let \( d(p) = D(p, q(p)) \), and set
\[
\tilde{F}(p) = \left\{ F(p, q) - \frac{1}{2} \left( \frac{\partial F}{\partial q}(p, q), \left( \frac{\partial^2 F}{\partial q \partial q}(p, q) \right)^{-1} \frac{\partial F}{\partial q}(p, q) \right) \right\}_{q=q(p)}.
\] (3.11)

Then there exist nonnegative constants \( c \) and \( C \) such that
\[
qc \leq \tilde{F}_2(p) \leq C d(p), \tag{3.12}
\]

where \( \tilde{F}_2(p) = \text{Im} \tilde{F}(p) \) is the imaginary part of \( \tilde{F}(p) \).

The proof is given in the appendix.

Lemma 3.8 follows from Lemma 3.13 by setting \( F(p, q) = S(q) - pq \).

### 3.4 Volume forms and the quantization condition

As we established in §3.3, a positive Lagrangian asymptotic manifold is given by the following data: a closed subset \( \Gamma \subset \mathbb{R}^{2n} \), a dissipation ideal \( \mathcal{D} \) with \( \Gamma_{\mathcal{D}} = \Gamma \), the universal covering \( \pi : \tilde{U} \to U \) over a small neighborhood of \( \Gamma \), and a \( z \)-action \( \Phi(p, q) \) defined on \( \tilde{U} \). These data uniquely determine the Lagrangian manifold \( L = (\mathcal{D}, \mathcal{J}) \) itself, and the \( I \)-nonsingular phases in the charts of the canonical cover are given by formulas (3.9). Note that the canonical cover is in fact a cover of \( \tilde{U} \) rather than of \( U \); that is, the associated objects (\( I \)-nonsingular phases) depend on the choice of the sheet of \( \tilde{U} \).

We assume that a volume form \( \mu \) is given on \( L \). Since the form \( dz_1 \wedge \cdots \wedge dz_n \) determines a nonzero element in \( \Lambda^n(L) \), we can specify \( \mu \) by choosing a fixed function \( a(p, q) \in C^\infty(\mathbb{R}^{2n}) \) such that
\[
\mu = i^*(a(q, p) \, dz_1 \wedge \cdots \wedge dz_n).
\]

Of course only the class of \( a(p, q) \) in \( C^\infty_{\text{loc}}(\mathbb{R}^{2n}) = C^\infty(\mathbb{R}^{2n}) / D^{1/2} \) is of interest.

We shall assume that the function \( a(p, q) \) is defined on \( \tilde{U} \) rather than on \( U \) (that is, the measure is defined on the universal covering over \( L \) rather than on \( L \) itself).

Let \( (p_0, q_0) \in \Gamma \), let \( V \subset U \) be a connected simply connected neighborhood of \( (p_0, q_0) \), and let \( V_1 \) and \( V_2 \) be two connected components of \( \pi^{-1}(V) \subset U \). By the definition of the \( z \)-action, we have
\[
\Phi_1(p, q) - \Phi_2(p, q) = \Phi_1(p_0, q_0) - \Phi_2(p_0, q_0) + O(D^{3/2}), \tag{3.13}
\]

where \( \Phi_i = \Phi|_{V_i}, i = 1, 2 \).

Let \( a_i(p, q) = a(p, q)|_{V_i}, i = 1, 2 \).

Since \( \tilde{U} \) is simply connected, the expression
\[
\text{Var} \ln a = \ln a_2 - \ln a_1 = \ln(a_2/a_1) \tag{3.14}
\]
is well defined in \( V \). Indeed, let us arbitrarily choose the branch of \( \ln a_1 \); then the branch of \( \ln a_2 \) is uniquely determined by the condition that \( \ln a \) be continuous on \( \tilde{U} \). The arbitrary multiple of \( 2\pi \) cancels in (3.14), and \( \text{Var} \ln a \) is well defined.
Definition 3.14. A Lagrangian asymptotic manifold with $z$-action $\Phi$ and measure $\mu = i^*adz_1 \wedge \cdots \wedge dz_n$ is said to satisfy the quantization condition if for any $(p_0, q_0) \in \Gamma$ and any two connected components $V_1$ and $V_2$ of $\pi^{-1}(V)$, where $V$ is a small neighborhood of $(p_0, q_0)$, we have
\[
\Phi_1(p_0, q_0) - \Phi_2(p_0, q_0) + \frac{i}{2}(\text{Var} \ln a)(p, q) = O(D^{1/2}) + 2\pi l,
\]
where $l \in \mathbb{Z}$ is an arbitrary integer.

Condition (3.15) can be interpreted in two different ways.

First, we can regard it as a condition imposed on the admissible values of $h$, which selects a sequence $h_l \to 0$.

Alternatively, if the Lagrangian manifold depends on parameters such as energy, condition (3.15) selects admissible values of these parameters.

The following lemma is obvious.

**Lemma 3.15.** Suppose that $\Gamma$ is arcwise connected; and let $\gamma_1, \ldots, \gamma_s$ be a fundamental system of cycles on $\Gamma$. The quantization condition (3.15) is satisfied if and only if

(a) $a_2/a_1$ is constant modulo $O(D^{1/2})$ for any branches $a_1$ and $a_2$ of $a$;

(b) $\text{Var}_{\gamma_i} \left[ \frac{1}{h} \Phi + \frac{i}{2} \ln a \right] \in 2\pi \mathbb{Z}, \quad i - 1, \ldots, s,$

where $\text{Var}_{\gamma} f$ is the variation of a function $f : \tilde{U} \to \mathbb{C}$ along a lift of a closed path $\gamma \subset U$.

Condition (3.16) will also be referred to as the quantization condition.

**Remark.** For each $i$ Eq. (3.16) gives infinitely many conditions, since $\text{Var}_{\gamma_i}$ may depend on the choice of the lift of $\gamma$. However, if $\Gamma$ itself is a submanifold (necessarily isotropic), as is the case in [10], then
\[
\text{Var}_{\gamma_i} \Phi = \oint_{\gamma_i} p \, dq
\]
and does not depend on the choice of the lift; and furthermore, if $\Gamma$ is a closed trajectory of a Hamiltonian vector field and the measure $\mu$ is invariant with respect to that field, then $\text{Var}_{\gamma_i}(\ln a)$ is also independent of the lift and can be expressed via the Floquet exponents for the variational system along this trajectory.

### 3.5 Positive canonical transformations

Consider the space $\mathbb{R}^{4n} = \mathbb{R}^{2n}_{(p,q)} \oplus \mathbb{R}^{2n}_{(\xi,x)}$ equipped with the symplectic form
\[
\Omega^2 = dp \wedge dq - d\xi \wedge dx.
\]

Let $\Lambda = (\Delta, \mathcal{M})$ be a positive Lagrangian manifold in $\mathbb{R}^{4n}$ with $z$-action $\Psi$ and suppose that $\Lambda$ is “diffeomorphically projected” on $\mathbb{R}^{2n}_{(p,q)}$ and $\mathbb{R}^{2n}_{(\xi,x)}$ in the following sense:

(a) $\Gamma$ is simply connected;

(b) the projections of $\Gamma_{\Lambda}$ on $\mathbb{R}^{2n}_{(p,q)}$ and on $\mathbb{R}^{2n}_{(\xi,x)}$ are homeomorphisms onto their images;

(c) $(p, q)$ and $(\xi, x)$ are coordinate systems on $L$. 


Lemma 3.17. Let $g = (\Lambda, \Psi)$ be a phase $(\text{the last equivalence is due to the fact that } \text{Im} S)$. Formally, we write $g : \mathbb{R}^{2n}_{(\xi,x)} \to \mathbb{R}^{2n}_{(\xi,x)}$.

Let $L = (\mathcal{D}, \mathcal{J})$ be a positive Lagrangian manifold in $\mathbb{R}^{2n}_{(\xi,x)}$ with $z$-action $\Phi$. Set

$$g[\mathcal{D}](p, q) = \min_{x, \xi} \left( D(x, \xi) + \delta(x, \xi, p, q) \right),$$

where $D$ and $\delta$ are some dissipations associated with $\mathcal{D}$ and $\Delta$, respectively, and

$$g[\mathcal{J}] = \{ f(x, \xi, p, q) \in \mathcal{M} + \mathcal{J} \mid f \text{ is independent of } (x, \xi) \}.$$

Let $g[\mathcal{D}]$ be the dissipation ideal corresponding to the function $g[\mathcal{D}]$.

**Lemma 3.18.** $g[L] = (g[\mathcal{D}], g[\mathcal{J}])$ is a positive Lagrangian manifold with $z$-action

$$g[\Phi](q, p) = \left\{ F(x, \xi, p, q) - \frac{1}{2} \left\langle \frac{\partial F}{\partial (x, \xi)}, \frac{\partial^2 F}{\partial (x, \xi) \partial (x, \xi)} \right\rangle \right\}_{x = x(q,p), \xi = \xi(q,p)}, \tag{3.18}$$

where

$$F(x, \xi, q, p) = \Phi(x, \xi) + \Psi(x, \xi, p, q) \tag{3.19}$$

and

$$(x(q, p), \xi(q, p)) = \arg \min_{(x, \xi)} \mathcal{D}(x, \xi) + \Delta(x, \xi, p, q). \tag{3.20}$$

**Proof.** The proof is purely technical, and we omit lengthy calculations; the only point worth nothing is that positivity is preserved. To avoid using too many subscripts, let us consider the particular case in which $L$ is covered by a $\otimes$-nonsingular chart and hence determined by a phase $S(\xi)$, while $\Lambda$ is determined by a phase $S_1(\xi, q)$. Then the nonsingular phase $S_2(x)$ for $g(L)$ can be obtained as follows:

$$S_2(q) = \left\{ F(q, \xi) - \frac{1}{2} \left\langle F_q(q, \xi), \left( \frac{\partial^2 F}{\partial \xi \partial \xi} \right)^{-1} F_p(q, \xi) \right\rangle \right\}_{\xi = \xi(q)}, \tag{3.21}$$

where $F(q, \xi) = S_1(\xi, q) + S(\xi)$ and

$$\xi(q) = \arg \min_{\xi, x} \left\{ \text{Im} S_1(\xi, q) + \text{Im} S(q) + \left\| x + \frac{\partial S(\xi)}{\partial \xi} \right\|^2 + \left\| x - \frac{\partial S_1(\xi, q)}{\partial \xi} \right\|^2 \right\}.$$

The dissipation on $g(L)$ in the nonsingular chart is

$$d(q) = \min_{\xi, x} \left\{ \text{Im} S_1(\xi, q) + \text{Im} S(q) + \left\| x + \frac{\partial S(\xi)}{\partial \xi} \right\|^2 + \left\| x - \frac{\partial S_1(\xi, q)}{\partial \xi} \right\|^2 \right\}.$$

It is easy to see that $d(q)$ is equivalent to

$$d_1(q) = \min \left\{ \text{Im} S_1 + \text{Im} S_2 + \left\| \text{Re} \frac{\partial S_1}{\partial \xi} + \text{Re} \frac{\partial S}{\partial \xi} \right\|^2 + \left\| \text{Im} \frac{\partial S_1}{\partial \xi} \right\|^2 + \left\| \text{Im} \frac{\partial S}{\partial \xi} \right\|^2 \right\}$$

$$\simeq \min \left\{ \text{Im}(S_1 + S) + \left\| \frac{\partial(S_1 + S)}{\partial \xi} \right\|^2 \right\}$$

(the last equivalence is due to the fact that $\text{Im} S_1 \geq 0$ and $\text{Im} S_2 \geq 0$).

It remains to apply Lemma 3.13.

**Lemma 3.18.** *Positive canonical transformations preserve quantization conditions.*

**Proof.** This is obvious, since the $z$-action on each sheet undergoes the same additive correction, and the density of the volume form undergoes the same multiplicative correction.
4 The canonical operator

In this section we construct the first-approximation canonical operator on a quantized Lagrangian asymptotic manifold with volume form. This material is quite traditional (e.g., see [9, 10, 11, 25, 14, 7]), and we are rather brief on the subject; our main goal is to relate the traditional construction to the new definition of Lagrangian asymptotic manifold given in §3.

4.1 Original objects

We assume that a positive asymptotic Lagrangian manifold with z-action and with a volume form is given. That is, we have the following collection of objects:

a) a closed subset \( \Gamma \subset \mathbb{R}^{2n} \), assumed to be arcwise connected;

b) a dissipation ideal \( \mathcal{D} \subset C^\infty(\mathbb{R}^{2n}) \) with \( \Gamma_{D} = \Gamma \);

c) a small tubular neighborhood \( U \supset \Gamma \) and the universal covering \( \pi : \tilde{U} \rightarrow U \);

d) a z-action \( \Phi \in C^\infty(\tilde{U}) \) satisfying the conditions of Definition 3.10;

e) the corresponding asymptotic manifold \( L = (\mathcal{D}, \mathcal{J}) \);

f) a volume form \( \mu = i^*(a(p, q) \, dz_1 \wedge \cdots \wedge dz_n) \) on \( \pi^{-1}(L) \).

We fix some branch of \( \ln a(p, q) \) on \( \tilde{U} \). We assume that the quantization condition (3.15) is satisfied.

Furthermore, we choose a canonical cover \( \tilde{U} = \cup_j \tilde{U}_j \), where each \( \tilde{U}_j \) is a connected simply connected domain such that for some \( I = I(j) \subset \{1, \ldots, n\} \) the functions \( (q_I, p_T) \) can be chosen as coordinates on \( L \) in \( U_j = \pi(\tilde{U}_j) \). Although it may well happen that \( I(j) = I(k) \) for some \( j \neq k \), we shall use the notation \( \tilde{U}_I, U_I \) instead of \( \tilde{U}_j, U_j \) with \( I(j) = I \); this will not lead to any misunderstanding.

For each canonical chart \( \tilde{U}_I \) the I-phase \( S_I(q_I, p_T) \) is defined in the projection of \( U_I \) on the coordinate \( (q_I, p_T) \)-plane by Eq. (3.9) and the dissipation \( d_I(q_I, p_T) \) by Eq. (3.10).

Furthermore, the form \( \mu|_{\tilde{U}_I} \) can be rewritten in the coordinates \( (q_I, p_T) \) as follows:

\[
\mu = a_I(q_I, p_T) \, dq_I \wedge dp_T,
\]

where

\[
a_I(q_I, p_T) = (-i)^n a(p, q) \big|_{p_I = \pi_I(q_I, p_T), q_T = \pi_T(q_I, p_T)} \det \frac{\partial(q_I - iP_I, p_T + iQ_T)}{\partial(q_I, p_T)} + O(d_I^{1/2}); \quad (4.1)
\]

here

\[
(p_I(q_I, p_T), q_T(q_I, p_T)) = \arg \min_{p_I, q_T} \mathcal{D}(p, q), \quad P_I = \frac{\partial S_I}{\partial q_I}, \quad Q_T = -\frac{\partial S_I}{\partial p_T}.
\]

We choose a continuous branch of \( \ln a_I(q_I, p_T) \) as follows. Since \( S_{I2} \) is nonnegative,

\[
B(q_I, p_T, r) = \det \frac{\partial(q_I - i\tau P_I, p_T + i\tau Q_T)}{\partial(q_I, p_T)} \neq 0 \quad \forall \tau \geq 0
\]

(e.g., see [14]). We have \( b(q_I, p_T, 0) = 1 \) and set \( \ln B(q_I, p_T, 0) = 0 \). Then, by continuity, \( \ln b \) is uniquely determined for all \( \tau > 0 \). We set

\[
\ln a_I(q_I, p_T) = \ln a(p, q) \big|_{p_I = \pi_I(q_I, p_T), q_T = \pi_T(q_I, p_T)} + \ln b(q_I, p_T, 1) - \frac{i\pi}{2} |T| + O(d_I^{1/2}), \quad (4.2)
\]

where \( \ln a(p, q) \) is the fixed branch of the logarithm. Equation (4.2) specifies \( \ln a_I(q_I, p_T) \) uniquely.
4.2 Local canonical operator

Let $\tilde{U}_I$ be some canonical operator chart. The local canonical operator

$$K_I : C^\infty(L) \rightarrow H_h(\mathbb{R}^n)$$

acts from the space

$$C^\infty_0(L) = \{ \varphi \in C^\infty(L) \mid \text{supp } \varphi \text{ is a compact subset in } U_I \}$$

to the Fréchet space

$$H_h(\mathbb{R}^n) = \bigcap_{k=0}^{\infty} H^k_h(\mathbb{R}^n),$$

$$H^k_h(\mathbb{R}^n) = \{ f(q,h), \, q \in \mathbb{R}^n, \, h \in (0,1] \mid \sup_{h} \| (1 - h^2 \Delta + x^2)^{k/2} f \|_{L^2(\mathbb{R}^n)} < \infty \}$$

(here $\Delta = \sum_{i=1}^{n} \partial^2 / \partial q_i^2$ is the Laplace operator) according to the formula

$$[K_I \varphi](q) = \left( \frac{i}{2\pi h} \right)^{\frac{n}{2}} \int_{\mathbb{R}^n} e^{i(h)(S_I(q_I, p_I) + p_I q_I)} \varphi_I(q_I, p_I) \sqrt{a_I(q_I, p_I)} \, dp_I, \quad (4.3)$$

where $\varphi_I(q_I, p_I)$ is the $(q_I, p_I)$-coordinate representative of $\varphi$, i.e., $\varphi(q, p) - \varphi_I(q_I, p_I) \in \mathcal{J}$.

**Lemma 4.1.** The operator (4.3) is well defined as an operator $f$ from $C^\infty(L, U_I)$ to $C^\infty_0(L)/H^1_h(\mathbb{R}^n)$ (i.e., the image of $K_I \varphi$ in the quotient space does not depend on the ambiguity in the choice of representatives of $\varphi_I, a_I$, and $S_I$).

4.3 Global canonical operator and the commutation theorem

Recall that we assume the quantization condition (3.15) to be satisfied. Then the following assertion is valid.

**Theorem 4.2.** For any $\varphi \in C^\infty(L, U_I) \cap C^\infty(L, U_K)$ we have

$$K_I \varphi = K_K \varphi \quad \text{in} \quad H_h(\mathbb{R}^n)/H^{1/2}_h(\mathbb{R}^n).$$

Let $\{ e_I(p, q) \}$ be a partition of unity subordinate to the canonical cover.

We can now introduce the canonical operator as the operator

$$K : C^\infty_0(L) \rightarrow H_h(\mathbb{R}^n)/H^{1/2}_h(\mathbb{R}^n)$$

given by the formula

$$K \varphi = \sum_I K_I(e_I \varphi). \quad (4.4)$$

Obviously, the canonical operator is independent of the choice of the partition of unity (by Theorem 4.2). Similarly, for any $\varepsilon > 0$, we can define the canonical operator

$$K : C^\infty_0(L) \rightarrow H_h(\mathbb{R}^n)/H^{1/2-\varepsilon}_h(\mathbb{R}^n). \quad (4.5)$$
Theorem 4.3. (Commutation theorem). (a) Let \( H(q,p) \) be an arbitrary symbol (i.e., a function satisfying the estimates

\[
\left| \frac{\partial^{\alpha+\beta} H}{\partial x^\alpha \partial p^\beta}(q,p) \right| \leq C_{\alpha\beta}(1 + |q| + |p|)^m,
\]

where \( m \) is independent of \( \alpha \) and \( \beta \). Then

\[
H\left( \frac{\partial}{\partial x}, -\frac{1}{\partial x} \right) K\varphi = K[(i^*H)\varphi],
\]

where \( i^*H \) is the restriction of \( H \) on \( L, \) that is, the image of \( H \in C^\infty(\mathbb{R}^{2n}) \) under the natural projection \( C^\infty(\mathbb{R}^{2n}) \to C^\infty(L). \)

(b) Let, in addition, \( i^*H = 0 \), and suppose that \( L \) is strongly invariant with respect to the Hamiltonian vector field \( V(H) \). Then

\[
\frac{i}{\hbar} H\left( \frac{\partial}{\partial x}, -\frac{1}{\partial x} \right) K\varphi
\]

is a well-defined element in \( H_\hbar(\mathbb{R}^n)/h^{1/2-\varepsilon}H_\hbar(\mathbb{R}^n) \) for any \( \varepsilon > 0 \), and we have

\[
\frac{i}{\hbar} H\left( \frac{\partial}{\partial x}, -\frac{1}{\partial x} \right) K\varphi = KP\varphi \quad \text{in} \quad H_\hbar(\mathbb{R}^n)/h^{1/2-\varepsilon}H_\hbar(\mathbb{R}^n),
\]

where the transport operator \( P \) is given by

\[
P = V(H) - \frac{1}{2} i^* \left( \sum_{j=1}^n \frac{\partial^2 H}{\partial p_j \partial q_j} \right) + \frac{1}{2} \frac{\mathcal{L}_V(H)\mu}{\mu}
\]

(note that the Lie derivative \( \mathcal{L}_V(H)\mu \) is well defined since \( L \) is strongly invariant with respect to \( V(H) \)).

4.4 Canonical operator for equations with operator-valued symbol

Let \( H : \mathbb{R}^{2n}_{(p,q)} \to \text{Op}(\mathcal{H}) \) be an operator-valued Hamiltonian (see [9]) such that \( H(p,q) \) is a self-adjoint operator for each \((p,q) \in \mathbb{R}^{2n} \), and let \( \lambda(p,q) \) be an isolated eigenvalue of constant multiplicity \( s \). Then associated with \((p,q) \in \mathbb{R}^{n} \) is the corresponding \( s \)-dimensional eigenspace \( E(p,q) \) of \( H(p,q) \), and we have a vector bundle \( E \to \mathbb{R}^{2n}_{(p,q)} \) whose fiber over \((p,q) \) is \( E(p,q) \). Let a quantized Lagrangian asymptotic manifold \( L \) with \( z \)-action be given; then the bundle \( \mathcal{E} = i^*(E) \) over \( L \) is well defined, where \( i \) is the “embedding” \( L \hookrightarrow \mathbb{R}^{2n} \), and we can define the canonical operator

\[
\mathcal{K}_\mathcal{E} : C^\infty_0(\mathcal{L}; \mathcal{E}) \to H_\hbar(\mathbb{R}^n, \mathcal{H})
\]

acting from the space of sections of \( \mathcal{E} \) over \( L \) into the Fréchet space \( H_\hbar(\mathbb{R}^n, \mathcal{H}) \) of \( \mathcal{H} \)-valued functions on \( \mathbb{R}^n_q \) with the topology defined by the family of norms

\[
\|f\|_l = \|(1 - h^2\Lambda + q^2)^{l/2}f\|_{L^2(\mathbb{R}^n_q; \mathcal{H})}, \quad l = 0, 1, 2, \ldots,
\]

where \( \Lambda = \sum_{j=1}^n \frac{\partial^2}{\partial p_j^2} \).
as follows. We define the local canonical operator by formula (4.3), where \( \varphi_I \) now takes values in \( \mathcal{H} \), and the global canonical operator is defined by formula (4.4). Furthermore, we may well consider the canonical operator

\[
K: C_0^\infty(L, \mathcal{H}) \to H_\hbar(\mathbb{R}^n, \mathcal{H})
\]

acting from the space of arbitrary smooth compactly supported functions with values in \( \mathcal{H} \) by the same formulas. Then \( K \) is the restriction of \( K \) to \( C_0^\infty(L, \mathcal{E}) \).

Lemma 4.1, Theorem 4.2, and Theorem 4.3 (a) remain valid in this situation without any modifications. However, instead of Theorem 4.3 (b) we have the following assertion.

**Theorem 4.4.** Suppose that the Lagrangian asymptotic manifold \( L \) is strongly invariant with respect to the Hamiltonian vector field \( V(\lambda) \) corresponding to the eigenvalue \( \lambda \) and that \( i^*\lambda = 0 \). Then for any \( \varphi \in C_0^\infty(L; \mathcal{E}) \) the function \((i/\hbar)K\varphi \) is a well-defined element of the quotient space

\[
H_\hbar(\mathbb{R}^n, \mathcal{H})/B,
\]

where

\[
B = \{ \psi \in H_\hbar(\mathbb{R}^n, \mathcal{H}) \mid \psi = K(i^*H)\eta \text{ for some } \eta \in C_0^\infty(L, \mathcal{H}) \}.
\]

In the quotient space (4.10) we have

\[
\frac{i}{\hbar}K\varphi = K\mathcal{P}\varphi,
\]

where the transport operator \( \mathcal{P} \) has the form

\[
\mathcal{P} = \nabla_{V(\lambda)} - \frac{1}{2}i^*\left(\sum_{j=1}^n \frac{\partial^2 \lambda}{\partial p_j \partial q_j}\right) + M + \frac{L_{V(\lambda)}^\mu}{\mu}.
\]

Here \( \nabla_{V(\lambda)} \) is the covariant derivative along \( V(\lambda) \) with respect to the Levi-Civit\'a connection \( \partial \) on \( \mathcal{E} \) associated with the operator of orthogonal projection onto the fibers of \( \mathcal{E} \) in \( \mathcal{H} \), and the homomorphism \( M: \mathcal{E} \to \mathcal{E} \) has the form \( M = (dF, V(\lambda)) \) for some other homomorphism \( F: \mathcal{E} \to \mathcal{E} \) (cf. [4]). In a local frame \( \chi_1(p,q), \ldots, \chi_s(p,q) \) in \( E \) (and hence in \( \mathcal{E} \)) we have

\[
(\nabla_{V(\lambda)})_{\mu \nu} = V(\lambda)\delta_{\mu \nu} + (\chi_\nu, \dot{\chi}_\mu) \text{ (here } \dot{\chi}_\mu = V(\lambda)\chi_\mu) \text{),}
\]

\[
M_{\mu \nu} = \sum_{i=1}^n \left( \chi_\nu, \left( \frac{\partial H}{\partial p_i} - \frac{\partial \lambda}{\partial p_i} \right) \frac{\partial \chi_\mu}{\partial x_i} \right), \quad \nu, \mu = 1, \ldots, l.
\]

Since \( M \) is a total derivative, it does not contribute to the spectrum of the transport operator. However, the term \((\chi_\nu, \dot{\chi}_\mu)\) does contribute; its contribution along closed trajectories is an element of the holonomy group of the connection \( \delta \) and is known as Berry’s phase [3, 18]. If the connection is flat, Berry’s phase can be incorporated in the quantization conditions in an obvious way.

### 5 Canonical operator modulo \( 0(\hbar^\infty) \)

#### 5.1 Asymptotic functions

Here we introduce asymptotic (with respect to a small positive parameter \( \hbar \)) functions of the two following kinds:
i) $h, \mathcal{D}$-asymptotic functions on an asymptotic manifold; these functions may be regarded as meromorphic in $h$ if taken modulo $h^N$, $N \in \mathbb{N}$;

ii) $h$-asymptotic functions on $\mathbb{R}^n$ that may have an “essential singularity” in $h$; in fact, we are interested in rapidly oscillating $h$-asymptotic functions.

First we recall some definitions. Let $(E_n)_{n \in \mathbb{Z}}$ be a filtration of a vector space $E$. Then for any $e \in E$ we denote

$$\text{ord} e = \sup \{ n \in \mathbb{Z} \mid e \in E_n \}.$$  

**Definition 5.1.** An asymptotic series (with respect to the filtration $(E_n)$) is a formal sum $\sum_{j=0}^{\infty} e_j$, where $e_j \in E$ satisfy the condition $\lim \text{ord} e_j = +\infty$. Two asymptotic series $e' = \sum_{j=0}^{\infty} e'_j$ and $e'' = \sum_{j=0}^{\infty} e''_j$ are said to be equivalent, $e' \sim e''$, if

$$\lim_{N \to \infty} \text{ord} \sum_{j=0}^{N} (e'_j - e''_j) = +\infty.$$  

**Definition 5.2.** Let $A(E)$ be the set of all asymptotic series with respect to a given filtration of $E$. Elements of the quotient set $A(E)/\sim$ are called asymptotic elements of $E$. If $E$ is a space of functions, then asymptotic elements will be called asymptotic functions.

Given $x_0 \in \mathbb{R}^n$, let $E(x_0)$ be the space of germs at $x_0$ of $C^\infty$-smooth complex-valued functions smoothly depending on a strictly positive small parameter $h$, and define a filtration of $E(x_0)$ by setting

$$E_m(x_0) = \{ f \in E(x_0) \mid f^{(\alpha)} = O(h^{m-k}) \text{ for any partial derivative } f^{\alpha} \text{ of order } |\alpha| = k \}.$$  

(5.1)

For example, $e^{(i/h)S}h^m u \in E_m(x_0)$ whenever $S$ and $u$ are germs at $x_0$ of smooth functions independent of $h$ and $\text{Im} \ S \geq 0$.

**Definition 5.3.** An $h$-asymptotic function is a section of the sheaf $\mathcal{A}$ over $\mathbb{R}^n$ such that the stalk $\mathcal{A}_{x_0}$ consists of all the asymptotic elements with respect to the filtration (5.1).

Let $L = (\mathcal{D}, \mathcal{J})$ be an $\infty$-asymptotic submanifold in some manifold $M$. We define the space $E$ as the set of functions $f \in C^\infty(L) \otimes C^\infty((0, \infty))$ satisfying the condition $h^k f(m, h) \in C^\infty(L) \otimes C^\infty([0, \infty))$ for some $k \in \mathbb{Z}$. Let us consider the following filtration of $E$:

$$E_s = \{ e \in E \mid e(m, h) = \sum_{j} e_j(m) h^j + h^k \bar{e}(m, h) \text{ for some } e_j \in \mathcal{D}^{s-j}, \ k \geq s/2, \ \bar{e} \in C^\infty(L) \otimes C^\infty([0, \infty)) \};$$  

(5.2)

in (5.2) we assume that $j$ runs over a finite subset of $\mathbb{Z}$ (this subset depends on $e$).

The reason for using the filtration (5.2) is clear from the following lemma.

**Lemma 5.4.** Let $f \in E$. Then the following two relations are equivalent:

i) $f \in E_s$;
ii) in the vicinity of any point $m_0 \in \Gamma_D$ the inequality

$$|\exp(-D(m)/h)f(m,h)| \leq ch^{s/2}$$

holds for some constant $c$ (here $D$ is a dissipation associated with $\mathcal{D}$).

Proof. The statement of the lemma is equivalent to saying that for any smooth function $g(x,h)$, any dissipation $D(x)$ in a neighborhood of $x_0$, and any $s \in \mathbb{Z}$ the two following relations are equivalent:

1) $|\partial^i g/\partial h^j|_{h=0} = O(D(s/2)^{-j})$ for $j = 0, 1, 2, \ldots$;
2) $|\exp(-D(x)/h)g(x,h)| \leq ch^{s/2}$ for some constant $c$.

The proof of the latter statement can be found, for example, in [24].

Definition 5.5. Asymptotic elements of $E$ corresponding to the filtration (5.2) will be called $(h,\mathcal{D})$-asymptotic functions on $L$.

5.2 $V$-objects on asymptotic Lagrangian submanifolds equipped with measure

Suppose there is a positive asymptotic Lagrangian submanifold $L$ in $\mathbb{R}^{2n}$ equipped with a measure $\mu$, and let $\mathcal{D}$ be the associated dissipation ideal on $\mathbb{R}^{2n}$. In this subsection we shall introduce some sheaf $\mathcal{V}$ over the locus $\Gamma$ of the ideal $\mathcal{D}$. Locally, in a chart, this sheaf can be regarded as the “bundle” of $(h,\mathcal{D})$-asymptotic functions on $L$.

Let $m_0 \in \Gamma$ be an $I$-nonsingular point in $L$, $S_I$ and let $a_I$ be well-defined modulo $\mathcal{D}^\infty$ functions which represent respectively the branches of the $I$-phase and the density of the measure on the same sheet of the simply connected covering of a neighborhood of $\Gamma$. Then the triple $T = (I, S_I, a_I)$ will be called a trivializator (for the “bundle” $\mathcal{V}$ near $m_0$). Given a trivializator $T$, an $\infty$-asymptotic manifold $L_\infty$ (with $L = L_\infty/\mathcal{D}$) is locally defined, and we can identify the stalk $\mathcal{V}_{m_0}$ with the space of germs at $m_0$ of $(h,\mathcal{D})$-asymptotic functions on $L_\infty$.

Further, the latter can be identified with the space $L_I(m_0)$ of germs at $(q^I_T, p^I_T)$ of $(h,\mathcal{D}_I)$-asymptotic functions on $\mathbb{R}^{\alpha}_{(q^I_T, p^I_T)}$, where $q^I = Q(m_0)$, $p^I = P(m_0)$, and $\mathcal{D}_I$ is the dissipation ideal on $\mathbb{R}^{\alpha}_{(q^I_T, p^I_T)}$ corresponding to the nonparametric local description of $L$: if $\mathcal{D}$ is induced by a dissipation $D$, then $\mathcal{D}_I$ is associated with $d(q_I, p_T) = \min_{(q_I, p_T)} D(p, q)$.

Remark. Note that any manifold $M$ equipped with a dissipation ideal $\mathcal{D}$ can be regarded as an 0-codimensional $\infty$-asymptotic submanifold $(\mathcal{D}, \mathcal{D}^\infty)$ in itself.

Now, to complete the definition of the sheaf $\mathcal{V}$, it is sufficient to fix a certain family of gluing isomorphisms $V^m_{T,T'} : L_I(m_0) \to L_{I'}(m_0)$ for each trivializator $T = (I, S_I, a_I), T' = (I', S'_I, a'_I)$ near $m_0$. Naturally, these isomorphisms are assumed to satisfy the conditions

$$V^m_{T,T} = \text{id}, \quad V^m_{T',T''} \circ V^m_{T,T'} = V^m_{T,T''}.$$

We call $V^m_{T,T'}$ the transition operators. They will be chosen later (see Eq. (5.4)).

Definition 5.6. A section of the sheaf $\mathcal{V}$ is called a V-object on $(L, \mu)$.
5.3 The canonical operator on $V$-objects

Suppose that we have a positive asymptotic Lagrangian submanifold $L$ in $\mathbb{R}^{2n}_{(p,q)}$ with a measure $\mu$ together with standard agreed-upon phase arguments (defined modulo $4\pi$) of its densities relative to the Lagrangian coordinates $(q_I,p_T)$. Let $V$ be the corresponding sheaf of $V$-objects. For each $m \in \Gamma$ and any trivializer $T = (I,S_I,a_I)$ near $m$ we define the precanonical operator $\bar{K}_{m,T} : V_m \to A_{Q(m)}$

\[(K_{m,T}\varphi)(q) = \left(\frac{i}{2\pi h}\right)^{|T|} \int \exp \left(\frac{i}{h}(S_I(q_I,p_T) + p_Tq_I)\right) \sqrt{a_I(q_I,p_T)}\varphi(q_I,p_T) dp_T, \quad (5.3)\]

where $e$ is a smooth cutoff function equal to 1 near $P_T(m)$ and to 0 near infinity, while $\varphi(q_I,p_T)$ is an $(h,d)$-asymptotic function representing the germ $\varphi \in L_I(m) = V_m$. Note that the right-hand side of (5.3) does not depend on the choice of $e$ modulo $O(h^\infty)$.

Assume now that the asymptotic Lagrangian submanifold $L$ satisfies the quantization condition. Then the transition operators $V^m_{T,T'}$ in the definition of the sheaf $V$ can be uniquely chosen so that the precanonical operator $\bar{K}_{m,T}$ does not depend on $T$. In other words, the transition operators are defined by the equation

\[\left(\frac{i}{2\pi h}\right)^{|T|} \int \exp \left(\frac{i}{h}(S_I(q_I,p_T) + p_Tq_I)\right) \sqrt{a_I(q_I,p_T)}\varphi(q_I,p_T) dp_T = \left(\frac{i}{2\pi h}\right)^{|T'|} \int \exp \left(\frac{i}{h}(S'_I(q_I,p_T) + p_Tq_I)\right) \sqrt{a'_I(q_I,p_T)}\varphi'(q_I,p_T) dp_T \quad (5.4)\]

near $q = Q(m)$ for any $(h,d_I)$-asymptotic function $\varphi$ supported near $(Q_I(m),P_T(m))$.

**Lemma 5.7.** There exist operators $V^m_{I,T'}$ such that (5.4) holds.

**Proof.** Applying the Fourier transform from $q_T$ to $p_T$ and using the canonical transformation $\gamma'_T$, we can assume without loss of generality that $I' = \{1,2,\ldots,n\}$.

The statement is trivial in the case $T = \emptyset$. Consider the case $|T| > 0$. \hfill $\square$

Denote the left-hand side of (5.4) by $(K\varphi)(q)$. Then $(K\varphi)(q)$ can be expanded relative to the $h$-asymptotic filtration by using the saddle-point method, say, in the form of the quantum bypassing focuses operation introduced in [9], Sec. 1 of Chap. V. Using our notation, we can formulate the result as follows. Consider the $I$-nonsingular Lagrangian chart $r = (U_I,d_I,P'_I,Q'_I,W')$:

\[Q_I(\alpha) = \alpha_I, \quad P_T'(\alpha) = \alpha_T, \quad Q_T'(\alpha) = -\frac{\partial S_I(\alpha)}{\partial \alpha_T}, \quad P_I'(\alpha) = \frac{\partial S_I(\alpha)}{\partial \alpha_I}, \quad W(\alpha) = S_I(\alpha) + \alpha_TQ'_T(\alpha).\]

Let $y = (y_1,\ldots,y_n)$ be nonsingular coordinates on $U_I$, i.e., $y - Q' = O(d^{1/2})$. Denote by $\Phi$ the $y$-phase on $U_I$ (cf. [25], §1):

\[\Phi = W + \frac{\partial W}{\partial Q}(y - Q') + \frac{1}{2}\left<y - Q', \frac{\partial^2 W}{\partial Q'\partial Q'}(y - Q')\right>\]

Then

\[(K\varphi)(y(\alpha)) = \exp \left\{\frac{i}{h}\Phi(\alpha)\right\} \sqrt{a(\alpha)(v\varphi)(\alpha)}\]
for some local (near $\alpha_0 = (Q_I(m), P_I(m))$) automorphism $v$ of the sheaf of $(h, d_I)$-asymptotic functions on $U$, where $\tilde{a}$ is the density of the measure $a_I(\alpha) \, d\alpha_1 \wedge \cdots \wedge d\alpha_n$ with respect to the complex coordinates $Q'$ on $U$. Now, to complete the proof, it is sufficient to verify that

$$a \circ y - \tilde{a} = O(d_I^{1/2}),$$  \hspace{1cm} (5.5)$$

$$S \circ y - \Phi = O(d_I^{3/2}).$$  \hspace{1cm} (5.6)$$

We consider the nonsingular Lagrangian chart $r'' = \{U, d, P''_I, Q''_I, S\}$, where $Q''_I(\beta) = \beta$, $P''_I(\beta) = \partial S(\beta) / \partial \beta$. Here $U$ is a neighborhood of $Q(m)$. Then, in some neighborhoods of the corresponding images of the point $m$, the charts $r'$ and $r''$ are equivalent, $y : \alpha \mapsto \beta$ being a diffeomorphism identifying their domains. Hence $d_I \circ y^{-1}$ is equivalent to $d$, and

$$\Phi \circ y^{-1} - S \circ \text{id} = O(d^{3/2})$$

as demonstrated in [25], so that (5.6) holds.

Finally, we note that (5.5) holds, since the function $\tilde{a} / a_I$ is a representative of

$$\frac{D(Q_I, P_I)}{DQ} \in C^\infty(1),$$

and the same can be said about $(a \circ y) / a_I$. The lemma is proved.

**Definition 5.8.** Given a quantized positive asymptotic Lagrangian submanifold $L \subset \mathbb{R}^n_p \times \mathbb{R}^n_q$ with measure $\mu$, let us denote by $V_0$ the space of $V$-objects with compact supports. We define the **canonical operator** $K : V_0 \rightarrow A_0$, where $A_0$ is the space of $h$-asymptotic functions on $\mathbb{R}^n$ as follows: the germ $K\varphi$ at the point $x \in Q(\Gamma)$ is equal to $\sum_m K_m \{\varphi\}_m$, where $m$ runs over the set $\{m \in \text{supp } \varphi \mid Q(m) = x\}$, and $\{\varphi\}_m$ is the germ of $\varphi$ at $m$. We denote by $K_m$ the precanonical operators, which are independent of the choice of trivializators under our definition of the transition operators.

**5.4 Commutation of the canonical operators with Hamiltonians**

Throughout this subsection, $\mathcal{H}(p, q, h)$ will be a real function smooth on $\mathbb{R}^{2n} \times [0, \varepsilon]$ (the Hamiltonian function) satisfying the following condition: there exists a positive integer $k$ such that for any multiindex $\alpha = (\alpha_1, \ldots, \alpha_{2n})$

$$\left(\frac{\partial}{\partial z}\right)^{\alpha} \mathcal{H}(z, h) = O(|z|^k) \quad \text{as} \quad z \to \infty;$$

here $z = (p, q) \in \mathbb{R}^{2n}$. The pseudodifferential operator

$$\mathcal{H}\left(-i h \frac{\partial}{\partial x}, \frac{2}{\partial x}\right)$$

will be referred to as the **Hamiltonian** corresponding to the symbol $\mathcal{H}$. We assert that Hamiltonians are in agreement with $h$-asymptotic filtration (for example, see [9] or [25]). Hence Hamiltonians can and will be interpreted as linear operators in the space of $h$-asymptotic functions.

We start from a version of the commutation formula for a complex bounded exponential and a Hamiltonian [25]:

$$\mathcal{H}\left(-i h \frac{\partial}{\partial x}, \frac{2}{\partial x}\right) \circ \exp \left(i h S(x)\right) = \exp \left(i h S(x)\right) \circ \hat{\mathcal{H}},$$  \hspace{1cm} (5.7)
where \( \hat{H} \) is a linear operator in the space \( A_0 \) of finite \((h, \mathcal{D})\)-asymptotic functions on \( \mathbb{R}^n \), where \( \mathcal{D} \) is the dissipation ideal induced by the imaginary part of \( S \).

Example. Let \( n = 1 \),

\[ H(p, x) = \frac{p^2 q^2}{2} + q. \]

It is easy to verify that

\[
\hat{H} = \exp \left( -\frac{i}{h} S(x) \right) \circ H \left( -ih \frac{\partial}{\partial x}, x \right) \circ \exp \left( \frac{i}{h} S(x) \right)
\]

\[
= \mathcal{H} \left( -ih \frac{\partial}{\partial x} + \frac{\partial S}{\partial x}, x \right) = x^2 \left( \frac{dS}{dx} \right)^2 + x - ihx^2 \left( \frac{d^2 S}{dx^2} + \frac{dS}{dx} \frac{d}{dx} \right) - \frac{h^2}{2} x^2 \frac{d^2}{dx^2}.
\]

We see immediately that the operator \( \hat{H} \) does not decrease the order of a function with respect to the \((h, \mathcal{D})\)-asymptotic filtration and does not enlarge its support. Hence \( \hat{H} \) acts on \( A_0 \).

In general, the operator \( \hat{H} \) on the right-hand side of (5.7) is described by an operator series of the form

\[
\hat{H} = \sum_{k=0}^{\infty} \sum_{k \leq |\alpha| \leq 2k \land |\beta| \leq k} (-ih)^k \mathcal{H}_{k,\alpha,\beta}.
\]

(5.8)

If the symbol \( \mathcal{H} \) is independent of \( h \), then the operators \( \mathcal{H}_{k,\alpha,\beta} \) have the form

\[
\mathcal{H}_{k,\alpha,\beta} = \frac{\partial^{|\alpha| \mathcal{H}}}{\partial p^\alpha \partial x^\beta} \left( \frac{\partial S}{\partial x}, x \right) P_{k,\alpha,\beta}(S) \left( \frac{\partial}{\partial x} \right)^\beta,
\]

(5.9)

where \( P_{k,\alpha,\beta} \) are nonlinear differential operators and \((\partial^{|\alpha| \mathcal{H}}/\partial p^\alpha)(\partial S/\partial x, x)\) stands for the Taylor expansion with respect to the imaginary part of \( \partial S/\partial x \). Note that \( P_{k,\alpha,\beta} \) can be calculated by using the fact that they are independent of the symbol \( \mathcal{H} \), choosing symbols in a special way. Specifically, we have

\[
P_{000}(S) = 1,
\]

\[
P_{1\alpha\beta}(S) = \begin{cases} 
0 & \text{for } |\alpha| = 1, \quad |\beta| = 0 \quad \text{or} \quad |\alpha| = 2, \quad |\beta| = 1, \\
\langle \alpha, \beta \rangle & \text{for } |\alpha| = |\beta| = 1, \\
\frac{1}{\alpha!} & \text{for } |\alpha| = 2, \quad |\beta| = 0.
\end{cases}
\]

Further, it is easy to see that \( P_{k,\alpha,\beta} = 0 \) for \( 2k < |\alpha| + |\beta| \), and

\[
P_{k,\alpha,\beta} = \begin{cases} 
0 & \text{for } |\alpha| = |\beta| = k, \quad \alpha \neq \beta, \\
\frac{1}{\alpha!} & \text{for } \alpha = \beta, \quad |\alpha| = k.
\end{cases}
\]

Finally, in the case when \( \mathcal{H} \) depends on \( h \), we have

\[
\mathcal{H}_{k,\alpha,\beta} = \sum_{l=0}^{k} i^l \mathcal{H}_{k-l,\alpha,\beta}^{(l)},
\]

(5.10)

where

\[
\mathcal{H}^{(l)} = \left. \frac{1}{l!} \frac{\partial^l \mathcal{H}}{\partial h^l} \right|_{h=0}.
\]
Denote by $F_{x\rightarrow \xi_T}$ the $H^{-1}$-Fourier transformation with respect to the $7$th group of coordinates:

$$F_{x\rightarrow \xi_T}u(x) = (2\pi i\hbar)^{-\overline{7}/2} \int \exp \left\{ -i \frac{1}{\hbar} \xi_T x_T \right\} u(x) \, dx_T,$$

and let $F_{\xi_T \rightarrow x_T}^{-1}$ denote its inverse. Then

$$F_{x\rightarrow \xi_T} \circ \mathcal{H} \left( -i \hbar \frac{\partial}{\partial x}, \frac{2}{\hbar}, h \right) \circ F_{\xi_T \rightarrow x_T}^{-1} = \mathcal{H} \left( -i \hbar \frac{\partial}{\partial x_T}, \xi_T \right) \circ \mathcal{H} \left( x, \hbar \frac{\partial}{\partial \xi_T}, h \right).$$

For the pseudodifferential operator on the right-hand side, there is a commutation formula with a complex exponential, similar to that for the Hamiltonian

$$\mathcal{H} \left( -i \hbar \frac{\partial}{\partial x}, \frac{2}{\hbar}, h \right).$$

Thus, we obtain the following commutation formula for a Hamiltonian and the composition of the multiplication operator by a complex exponential with the Fourier transformation (see [9], Sec. 2 in Chap. V)

$$\mathcal{H} \left( -i \hbar \frac{\partial}{\partial x}, \frac{2}{\hbar}, h \right) \circ F_{\xi_T \rightarrow x_T}^{-1} \circ e^{(i/h)S(x, \xi_T)} = F_{\xi_T \rightarrow x_T}^{-1} \circ e^{(i/h)S(x, \xi_T)} \mathcal{H}_I, \quad (5.11)$$

where

$$\mathcal{H}_I = \sum_{k=0}^{\infty} \sum_{|\alpha|=k} \sum_{|\beta|=\gamma} \sum_{|\delta|=0} \left\{ (-i \hbar)^k \frac{\beta_\gamma!}{\gamma!} \delta! \left( \frac{\partial}{\partial q_\gamma p_\delta} \right)^{\alpha+\delta} \left( \frac{\partial}{\partial p_\gamma} \right)^\gamma \mathcal{H} \right\} \times \left( \left( \frac{\partial S_1}{\partial x_I}, \xi_T \right), \left( x_I, -i \frac{\partial S_1}{\partial x_I} \right) \right) \left( \left( \frac{\partial S_2}{\partial x_I}, -i \frac{\partial S_2}{\partial x_I} \right), \left( x_I, -i \frac{\partial S_2}{\partial x_I} \right) \mathcal{H}_I \mathcal{H}_I \right),$$

where $S_1 = \text{Re } S, S_2 = \text{Im } S,$ and $P_{k\alpha\beta}^I$ are some nonlinear differential operators independent of $\mathcal{H}$. (One can easily obtain the formula for Hamiltonians depending on $h$.)

Now we are ready to commute a Hamiltonian with a canonical operator.

**Theorem 5.9.** Given a symbol $\mathcal{H}(p,q)$ and a quantized positive Lagrangian submanifold $L$ equipped with a measure $\mu$, there is an operator $\mathcal{P}_L$ acting on finite $V$-objects such that

$$\mathcal{H} \left( -i \hbar \frac{\partial}{\partial x}, \frac{2}{\hbar}, h \right) (K\varphi)(x) = (K\mathcal{P}_L \varphi)(x).$$

**Proof.** Commute the Hamiltonian with a precanonical operator. Let $T = (I, S_I, a_I)$ be a trivializer near a point $m \in \Gamma$ and let $\varphi(x_I, \xi_T)$ be the germ of a $(h, d_I)$-asymptotic function at $(Q_I(m), P_T(m))$. By formula (5.11) we have

$$\mathcal{H} \left( -i \hbar \frac{\partial}{\partial x}, \frac{2}{\hbar}, h \right) \tilde{K}_{m,T} \varphi = F_{\xi_T \rightarrow x_T}^{-1} \exp \left\{ \frac{i}{\hbar} S_I(x_I, \xi_T) \right\} \tilde{K}_{m,T} \varphi(x_I, \xi_T).$$

It follows that

$$\mathcal{H} \left( -i \hbar \frac{\partial}{\partial x}, \frac{2}{\hbar}, h \right) \tilde{K}_{m,T} = \tilde{K}_{m,T} \mathcal{P}_{m,T},$$

Theorem 5.9 is proved.
where
\[ \mathcal{P}_{m,T} = \frac{1}{\sqrt{a_I}} \circ \hat{\mathcal{H}}_I \circ \sqrt{a_I}. \]

If we interpret a precanonical operator as a local homomorphism of sheaves: \( \mathcal{V}_m \to \mathcal{A}_{Q(m)} \), then it is a monomorphism independent of the choice of a trivializator. Hence there is a morphism \( \mathcal{P} \) of the sheaf \( \mathcal{V} \) to itself that, under the local trivialization of \( \mathcal{V} \) determined by \( T \), identifies \( \mathcal{P}_m \) with \( \mathcal{P}_{m,T} \). This implies the required statement.

### 5.5 The transport operator

Here we consider a quantized positive asymptotic Lagrangian submanifold with a measure \( (L, \mu) \) strongly invariant with respect to the Hamiltonian vector field \( \mathcal{V}_H \) corresponding to a Hamiltonian function \( \mathcal{H}(p,q) \). In this case we call \( \mathcal{P}_H = (i/h)\mathcal{P}_H \) the transport operator.

To describe the transport operator we need the following definition.

**Definition 5.10.** [24] Let \( L = (\mathcal{D}, \mathcal{J}) \) be an \( \infty \)-asymptotic submanifold in a manifold \( M \), and let the operator series \( \kappa = \sum_{j=0}^{\infty} h^{t_j}A_j \), where \( A_j \) are linear operators from \( C^\infty(L) \) to itself, represent an endomorphism of the space of \((h, D)\)-asymptotic functions on \( L \). We say \( \kappa \) is a **perturbator** if the following conditions hold:

a) \( h^{t_j}A_j \) does not decrease order relative to the \((h, D)\)-asymptotic filtration for any \( j \);

b) if \( t_j \leq 0 \), then \( h^{t_j}A_j \) increases the order.

Perturbators are valid in perturbation theory for equations with \((h, D)\)-asymptotic functions because of the following result.

**Proposition 5.11.** ([24], p. 84) Let \( \kappa \) be a perturbator on \( L = (\mathcal{D}, \mathcal{J}) \). Then \( \text{id} - \kappa \) is an isomorphism of the space of \((h, D)\)-asymptotic functions on \( L \), and its inverse is defined by the operator series \( \sum_{r=0}^{\infty} \kappa^r \).

**Theorem 5.12.** The transport operator has the form
\[ \mathcal{P}_H = \mathcal{V}_H - \frac{1}{2} \text{tr} \left( \frac{\partial^2 \mathcal{H}}{\partial p \partial q} \right)_{L} + \kappa, \]

where \( \mathcal{V}_H \) is the Hamiltonian vector field associated with \( \mathcal{H} \), and \( \kappa \) is a perturbator depending on the choice of a trivializator for \( \mathcal{V} \)-objects.

**Proof.** Without loss of generality let us take some nonsingular trivializator \((S, a)\). By using the commutation formula for a Hamiltonian with an exponential, we obtain the following local expression for \( \mathcal{P}_H \):
\[ \mathcal{P}_H = \frac{1}{\sqrt{a}} \left( \frac{\partial \mathcal{H}}{\partial p} \left( \frac{\partial S}{\partial x}, x \right), \frac{\partial}{\partial x} \right) \sqrt{a} + \frac{1}{2} \text{tr} \left( \frac{\partial^2 \mathcal{H}}{\partial p^2} \left( \frac{\partial S}{\partial x}, x \right) \frac{\partial^2 S}{\partial x^2} \right) + \tilde{\kappa}, \]

where \( \tilde{\kappa} \) is a perturbator. It is not difficult to show that the invariance of \( \mu \) with respect to \( \mathcal{V}_H \) implies the following result (similar to Liouville's theorem):
\[ \mathcal{V}_H \frac{1}{a} = \frac{1}{a} \text{tr} \left( \frac{\partial^2 \mathcal{H}}{\partial p^2} \left( \frac{\partial S}{\partial x}, x \right) \frac{\partial^2 S}{\partial x^2} + \frac{\partial^2 \mathcal{H}}{\partial p \partial q} \left( \frac{\partial S}{\partial x}, x \right) \right) + O(d^{1/2-\varepsilon}). \]
Furthermore, the operator
\[ \langle \frac{\partial H}{\partial p} \left( \frac{\partial S}{\partial x}, x \right), \frac{\partial}{\partial x} \rangle \]
represents \( V_{\mathbb{H}} \) in our coordinates, and we obtain
\[ \langle \frac{\partial H}{\partial p} \left( \frac{\partial S}{\partial x}, x \right), \frac{\partial}{\partial x} \rangle \frac{1}{\sqrt{a}} = \frac{1}{2\sqrt{a}} \text{tr} \left( \frac{\partial^2 H}{\partial p^2} \left( \frac{\partial S}{\partial x}, x \right) \frac{\partial^2 S}{\partial x^2} + \frac{\partial^2 H}{\partial p \partial q} \left( \frac{\partial S}{\partial x}, x \right) \right) + O(d^{1/2-\varepsilon}). \quad (5.13) \]
Substituting (5.13) into (5.12) completes the proof. \( \square \)

**Appendix. Proof of Lemma 3.13**

We write
\[ \mathcal{E}(q) = \left( \frac{\partial^2 F(p, q)}{\partial q_i \partial q_j} \right)_{i,j=1}^n \]
Set
\[ D_{\mu}(p, q) = F_2(p, q) + \frac{\mu}{2} \left\| \frac{\partial F(p, q)}{\partial q} \right\|^2 \]
(where \( \mu > 0 \) will be chosen later) and consider the problem
\[ D_{\mu}(p, q) \to \min_q \]
also set
\[ \tilde{F}_{\mu}(p) = \left\{ F(p, q) - \frac{1}{2} \left( \frac{\partial F(p, q)}{\partial q}, \left( \frac{\partial^2 F(p, q)}{\partial q_i \partial q_j} \right)^{-1} \frac{\partial F(p, q)}{\partial q} \right) \right\} \Bigg|_{q=q(p)}, \quad (A.1) \]
where \( q = q(p) \) is the solution to the minimization problem (A.1). For \( \mu = 2 \) we obtain the functions (3.10), (3.11). First of all, let us prove that problem (A.1) has a unique solution, which is smooth, in a sufficiently small neighborhood of \((p_0, q_0)\). To this end, let us calculate the first and the second derivatives of the function \( D_{\mu}(p, q) \). We have (denoting the derivatives by subscripts, omitting the arguments, and denoting \( D_{\mu} \) simply by \( D \) and \( F_{\mu} \) by \( F \)):
\[ D = F_2 + \frac{\mu}{2} ||F_{1q}||^2 + \frac{\mu}{2} ||F_{2q}||^2, \]
\[ D_q = F_{2q} + \mu F_{1qq} F_{1q} + \mu F_{2qq} F_{2q} = (I + \mu \mathcal{E}_2) F_{2q} + \mu \mathcal{E}_1 F_{2q}, \]
\[ D_{qq} = F_{2qq} + \mu F_{1qq} F_{1qq} + \mu F_{2qq} F_{2qq} + \cdots = \mathcal{E}_2 + \mu \mathcal{E}_1^2 + \mu \mathcal{E}_2^2 + \cdots, \quad (A.3) \]
where \( I \) is the identity matrix and the dots stand for terms linear in \( F_{1q} \) and \( F_{2q} \). At the point \((p_0, q_0)\) we have \( F_{1q} = F_{2q} = D_q = 0 \), and the matrix \( D_{qq} = \mathcal{E}_2 + \mu \mathcal{E}_1^2 + \mu \mathcal{E}_2^2 \) is positive definite. Indeed, \( \mathcal{E}_1 \) and \( \mathcal{E}_2 \) are symmetric and \( \mathcal{E}_2 \) is positive semidefinite, and so for any vector \( \xi \in \mathbb{R}^n \) we have
\[ \langle \xi, D_{qq} \xi \rangle = \langle \xi, \mathcal{E}_2 \xi \rangle + \mu \langle \mathcal{E}_1 \xi, \mathcal{E}_1 \xi \rangle + \mu \langle \mathcal{E}_2 \xi, \mathcal{E}_2 \xi \rangle = \langle \xi, \mathcal{E}_2 \xi \rangle + \mu \langle \mathcal{E} \xi, \mathcal{E} \xi \rangle \geq \mu ||\xi||^2 ||\mathcal{E}^{-1}||^{-2} \]
(here \( \langle \cdot, \cdot \rangle \) is the \( \mathbb{C}^n \) inner product). It follows that \( D_{qq} \) is positive definite in a neighborhood of the point \((p_0, q_0)\), and so the solution to problem (A.1) is unique and smooth and is determined by the equation \( D_q = 0 \). Set
\[ d(p) = d_{\mu}(p) = D(p, q(p)). \]
Then, obviously,
\[ F_2(p, q(p)) \leq d(p) \quad \text{and} \quad \left| \frac{\partial F_2}{\partial q}(p, q(p)) \right| \leq \text{const} \, d(p)^{1/2}. \]

It is quite obvious that, if we choose various \( \mu > 0 \), then the resultant functions \( d_\mu(p) \) will be equivalent to each other; furthermore, \( q_\mu(p) - q_\nu(p) = O(d(p)^{1/2}) \), and hence, writing \( \Delta q = q_\mu(p) - q_\nu(p) \), we have
\[ \tilde{F}_\mu(p) - \tilde{F}_\nu(p) = F_q sq + \frac{1}{2} \langle \Delta q, F_{qq} \Delta q \rangle - F_q \Delta q - \frac{1}{2} \langle \Delta q, F_{qq} \Delta q \rangle + O(\|F_q\|^3 + \|\Delta q\|^3) = O(d(p)^{3/2}). \]

Consequently, it suffices to prove the desired inequality for \( \tilde{F}_\mu(p) \) with \( \mu \) arbitrarily small.

Let us calculate the imaginary part of the function \( \tilde{F}(p) = \tilde{F}_\mu(p) \) given by Eq. (A.2). In our shorthand notation, \( \tilde{F}(p) = \{F - \frac{1}{2}(F_q, \mathcal{E}^{-1} F_q)\}_{q=q(p)} \).

Let \( \mathcal{E}^{-1} = A + iB \), where \( A \) and \( B \) are symmetric matrices with real entries. Then
\[ (A + iB)(\mathcal{E}_1 + i\mathcal{E}_2) = I \quad \text{(the identity matrix)}, \]
and so
\[ A\mathcal{E}_1 - B\mathcal{E}_2 = I, \quad B\mathcal{E}_1 + A\mathcal{E}_2 = 0. \tag{A.4} \]

Let us prove that for any \( \xi \in \mathbb{R}^n \)
\[ \langle \xi, B\xi \rangle \leq \varphi \|\xi\|^2, \tag{A.5} \]
where \( \varphi = \varphi(p, q) \) is a continuous function such that \( \varphi(p_0, q_0) = 0 \). Since
\[ \langle \xi, B\xi \rangle = \langle \xi, B(p_0, q_0)\xi \rangle + \langle \xi, [B - B(p_0, q_0)]\xi \rangle, \]
the estimate (A.5) will follow with \( \varphi(q) = \|B - B(p_0, q_0)\| \) if we prove that \( \langle \xi, B(p_0, q_0)\xi \rangle \leq 0 \) for any \( \xi \in \mathbb{R}^n \), or, equivalently \( \text{Im} \langle \xi, \mathcal{E}^{-1} \xi \rangle \leq 0 \) (we omit the argument \( (p_0, q_0) \)) for any \( \xi \in \mathbb{R}^n \). Let us take \( \xi = i\mathcal{E}\eta, \eta = \eta_1 + i\eta_2, \eta_1, \eta_2 \in \mathbb{R}^n \). Furthermore,
\[ \text{Im} \langle \xi, \mathcal{E}^{-1} \xi \rangle = -\text{Im} \langle \eta, \mathcal{E}\eta \rangle = -\langle \eta_1, \mathcal{E}_2\eta_1 \rangle - \langle \eta_2, \mathcal{E}_2\eta_2 \rangle - 2\langle \eta_2, \mathcal{E}_1\eta_1 \rangle. \]

Since \( \xi \in \mathbb{R}^n \), it follows that \( \mathcal{E}_2\eta_1 = \mathcal{E}_2\eta_2 \), and we obtain \( \text{Im} \langle \xi, \mathcal{E}^{-1} \xi \rangle = -\langle \eta_1, \mathcal{E}_2\eta_1 \rangle - \langle \eta_2, \mathcal{E}_2\eta_2 \rangle \leq 0 \) (recall that \( \mathcal{E}_2 \geq 0 \)).

Let us now write
\[ \tilde{F}_2(p) = \left\{ F_2 - \frac{1}{2} \left\{ \langle F_{1q}, BF_{1q} \rangle - \langle F_{2q}, BF_{2q} \rangle + 2\langle F_{1q}, AF_{2q} \rangle \right\} \right\}_{q=q(p)}. \tag{A.6} \]

For \( q = q(p) \) from (A.3) we have
\[ F_{2q} = -(1 + \mu\mathcal{E}_2)^{-1}\mu\mathcal{E}_1 F_{1q} = -\mu\mathcal{E}_1 F_{1q} + O(\mu^2), \tag{A.7} \]
where the \( O(\mu^2) \) estimate is uniform whenever \( q(p) \) lies in any given bounded neighborhood of \( q_0 \) and \( \|p - p_0\| < \varepsilon(\mu) \), where \( \varepsilon(\mu) > 0 \) for \( \mu > 0 \).

Let us substitute (A.7) into (A.6) and take into account the first equation in (A.4). We obtain
\[ \tilde{F}_2 = F_2 - \frac{1}{2} \left[ \langle F_{1q}, BF_{1q} \rangle - 2\mu \langle F_{1q}, (I + B\mathcal{E}_2) F_{1q} \rangle \right] + O(\mu^2) \]
\[ = F_2 - \frac{1}{2} \langle F_{1q}, BF_{1q} \rangle + \mu \langle F_{1q}, F_{1q} \rangle - \mu \langle BF_{1q}, \mathcal{E}_2 F_{1q} \rangle + \mathcal{F}, \]
where
\[ |\mathcal{F}| \leq C \mu^2 \| F_{1q} \|^2 \] (A.8)
whenever \( q = q(p) \) lies in a sufficiently small neighborhood of \( q_0 \), and \( \| p - p_0 \| \leq \varepsilon(\mu) \); the constant \( C \) is independent of \( \mu \) and \( p \). Set \( \bar{B}(p, q) = B(p, q) - \varphi(p, q) \). Then \( \bar{B} \leq 0 \) and
\[
\bar{F}_2 = F_2 - \frac{1}{2} \langle f_{1q}, \bar{B} F_{1q} \rangle + \left( \mu - \frac{\varphi}{2} \right) \langle f_{1q}, F_{1q} \rangle - \mu \langle \bar{B} F_{1q}, \mathcal{E}_2 F_{1q} \rangle - \mu \varphi \langle F_{1q}, \mathcal{E}_2 F_{1q} \rangle + \mathcal{F}.
\]
Since the matrix \( \bar{B} \) is symmetric and nonpositive, we have
\[
\| \bar{B} F_{1q} \|^2 \leq -C_1 \langle f_{1q}, \bar{B} F_{1q} \rangle;
\]
Furthermore, for any \( \lambda > 0 \) we have
\[
|\langle \bar{B} F_{1q}, \mathcal{E}_2 F_{1q} \rangle| \leq \frac{1}{2} \lambda \| \bar{B} F_{1q} \|^2 + \frac{\| \mathcal{E}_2 \|^2}{\lambda} \| F_{1q} \|^2 \leq -\frac{C_1 \lambda}{2} \langle f_{1q}, \bar{B} F_{1q} \rangle + \frac{\| \mathcal{E}_2 \|^2}{\lambda} \| F_{1q} \|^2.
\]
By combining (A.8) with the subsequent two estimates, we obtain
\[
\bar{F}_2 \leq F_2 - \frac{1}{2} (1 - C_1 \lambda \mu) \langle f_{1q}, \bar{B} F_{1q} \rangle + \left( \mu - \frac{\varphi}{2} - \frac{\mu \| \mathcal{E}_2 \|^2}{\lambda} - \mu \varphi \| \mathcal{E}_2 \| - C \mu^2 \right) \| F_{1q} \|^2. \tag{A.9}
\]
Now set \( \lambda = \varphi \sup \| \mathcal{E}_2 \|^2 \) and \( \varphi_0 = 1/\varphi \sup \| \mathcal{E}_2 \|^2 \). The coefficient of \( \| F_{1q} \|^2 \) in (A.9) will not be less than \( \mu/2 - C \mu^2 - \varphi/2 \). Set \( \mu = \min\{1/(C_1 \lambda), 1/(4 C)\} \). Then the coefficient of \( \langle f_{1q}, \bar{B} F_{1q} \rangle \) in (A.9) will be nonpositive, and the coefficient of \( \| F_{1q} \|^2 \) will be greater than or equal to \( \mu/4 - \varphi/2 \). Now that \( \mu \) is fixed, we can choose a neighborhood of \( p_0 \) small enough to have \( \varphi(q(p)) < \mu/4 \) for \( p \) in that neighborhood. For these \( p \) we obtain
\[
\bar{F}_2(p) \geq \{ F_2 + c_3 \| F_{1q} \|^2 \}_{q=q(p)} \geq \{ F_2 + c_4 \| F_{1q} \|^2 \}_{q=q(p)}; \quad c_4 > 0,
\]
in view of (A.7).
Thus,
\[
c_5 \, dp \leq \bar{F}_2(p) \leq c_6 \, dp
\]
in a small neighborhood of \( p_0 \) (the right inequality is obvious from (A.2)).
The lemma is proved.

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