ON THE LIE-SOLVABILITY OF NOVIKOV ALGEBRAS

Kaisar Tulenbaev†, Ualbai Umirbaev‡, and Viktor Zhelyabin¶

Abstract. We prove that any Novikov algebra over a field of characteristic $\neq 2$ is Lie-solvable if and only if its commutator ideal $[N, N]$ is right nilpotent. We also construct examples of infinite-dimensional Lie-solvable Novikov algebras $N$ with non nilpotent commutator ideal $[N, N]$.

Mathematics Subject Classification (2020): 17D25, 17B30, 17B70
Key words: Novikov algebra, Lie-solvability, nilpotency

1. Introduction

An algebra $N$ over a field $K$ is called a Novikov algebra if it satisfies the following identities:

1. $(x, y, z) = (y, x, z)$,
2. $(xy)z = (xz)y$,

where $(x, y, z) = (xy)z - x(yz)$ is the associator of elements $x, y, z$.

Recall that any algebra satisfying the identity (1) is called left-symmetric. Left-symmetric algebras are Lie-admissible, i.e., every left-symmetric algebra $L$ becomes a Lie algebra with respect to the commutator $[x, y] = xy - yx$. This Lie algebra is denoted by $L^(-)$ and is called the commutator algebra of $L$.

Left-symmetric algebras arise in many areas of mathematics and physics [3]. The defining identities of Novikov algebras first appeared in the study of Hamiltonian operators in the formal calculus of variations by I.M. Gelfand and I.Ya. Dorfman [8]. These identities played a crucial role in the classification of linear Poisson brackets of hydrodynamical type by A.A. Balinskii and S.P. Novikov [1].

In 1987 E.I. Zelmanov [27] proved that any finite dimensional simple Novikov algebra over a field $K$ of characteristic zero is one-dimensional. V.T. Filippov [6] constructed a wide class of simple Novikov algebras of characteristic $p \geq 0$. J.M. Osborn [14, 15, 16] and X. Xu [25, 26] continued the study of simple finite dimensional algebras over fields

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of positive characteristic and simple infinite dimensional algebras over fields of characteristic zero. A complete classification of finite dimensional simple Novikov algebras over algebraically closed fields of characteristic \( p > 2 \) is given in [25].

E.I. Zelmanov also proved that if \( N \) is a finite dimensional right nilpotent Novikov algebra then \( N^2 \) is nilpotent [27]. In 2001 V.T. Filippov [7] proved that any left-nil Novikov algebra of bounded index over a field of characteristic zero is nilpotent. A.S. Dzhumadildaev and K.M. Tulenbaev [5] proved that any right-nil Novikov algebra of bounded index \( n \) is right nilpotent if the characteristic \( p \) of the field \( K \) is 0 or \( p > n \). In 2001 V.T. Filippov [7] proved that any left-nil Novikov algebra of bounded index over a field of characteristic zero is nilpotent. A.S. Dzhumadildaev and K.M. Tulenbaev [5] proved that any right-nil Novikov algebra of bounded index \( n \) is right nilpotent if the characteristic \( p \) of the field \( K \) is 0 or \( p > n \). In 2020 I. Shestakov and Z. Zhang proved [22] that for any Novikov algebra \( N \) over a field the following conditions are equivalent:

(i) \( N \) is solvable;
(ii) \( N^2 \) is nilpotent;
(iii) \( N \) is right nilpotent.

U.U. Umirbaev and V.N. Zhelyabin proved [24, 29] that any \( \mathbb{Z}_n \)-graded Novikov algebra with solvable 0-component is solvable.

It is well known [9] that if \( L \) is a finite dimensional solvable Lie algebra over a field \( K \) of characteristic zero then \([L, L]\) is nilpotent. In 1973 Yu.P. Razmyslov proved [19] that over a field \( K \) of characteristic zero \([L, L]\) is nilpotent for any algebra \( L \) from any proper subvariety of the variety of algebras generated by the simple three dimensional Lie algebra \( \text{sl}_2(K) \). There was a long standing conjecture about solvable algebras of the variety of algebras generated by the Witt algebra \( W_1 \).

Conjecture 1. If \( L \) is a solvable algebra of the variety of algebras generated by the Witt algebra \( W_1 \), then is it true that \([L, L]\) is nilpotent?

This conjecture was proven to be not true by A. Mishchenko [13] in 1988.

The variety of Lie algebras generated by the Witt algebra \( W_1 \) is closely related to the variety of Novikov algebras. Let \( K[x] \) be the algebra of all polynomials in one variable \( x \) over a field \( K \). Consider \( K[x] \) as a differential algebra with derivation \( \partial = \frac{\partial}{\partial x} \). Then \( K[x] \) is a simple differential algebra over a field of characteristic zero. With respect to the product

\[ f \circ g = fg' \]

the vector space \( K[x] \) becomes a Novikov algebra. We denote this algebra by \( L_1 \). The construction described above is called the Gelfand-Dorfman construction for Novikov algebras. Recently, L.A. Bokut, Y. Chen, and Z. Zhang [2] proved that any Novikov algebra over a field of characteristic zero is a subalgebra of a Novikov algebra obtained from some differential algebra by the Gelfand-Dorfman construction.

Notice that \( K[x] \) becomes a Lie algebra with respect to the product

\[ [f, g] = fg' - gf'. \]

This algebra is a well known Witt algebra \( W_1 \). This construction of Lie algebras is also studied by many specialists [17, 18, 20]. In this case the differential enveloping algebra of a Lie algebra is called the Wronskian enveloping algebra [18]. Although there are many interesting results, the class of Lie algebras embeddable into their Wronskian enveloping algebras is not described yet.
Conjecture 2. A Lie algebra over a field of characteristic zero is embeddable into its Wronskian enveloping algebra if and only if it belongs to the variety of algebras generated by the Witt algebra $W_1$.

Notice that the commutator algebra of $L_1$ is the Witt algebra $W_1$. For this reason we call $L_1$ the Novikov-Witt algebra [10]. This is the first algebra in the list of left-symmetric Witt algebras $L_n$ [23]. The variety of Novikov algebras is generated by the Novikov-Witt algebra $L_1$ in characteristic zero [12]. The identities of the Witt algebras $W_n$ are studied mainly by Yu.P. Razmyslov [21] and the identities of the left-symmetric Witt algebras $L_n$ are studied in [11].

We say that a Novikov algebra $N$ is Lie-solvable if the Lie algebra $N^{(-)}$ is solvable. It is known that every finite dimensional Novikov algebra over a field is Lie-solvable [4]. Recently Z. Zhang and T.G. Nam [31] proved that if a Novikov algebra is Lie-nilpotent then its ideal generated by all commutators $[a, b]$ is nilpotent.

This paper is devoted to the study of Lie-solvable Novikov algebras. We noticed that the space of commutators $[N, N]$ of a Novikov algebra $N$ is an ideal of $N$ over a field of characteristic $\neq 2$. We prove that a Novikov algebra $N$ over a field of characteristic $\neq 2$ is Lie-solvable if and only if $[N, N]$ is right nilpotent. Using Mishchenko’s example [13], we constructed examples of Lie-solvable Novikov algebras with non nilpotent $[N, N]$.

The right nilpotency of $[N, N]$ for Lie-solvable Novikov algebras means that Conjecture 1 was not baseless. This property just cannot be expressed in the language of Lie algebras. Notice that if $[N, N]$ is right nilpotent then $[N, N][N, N]$ is nilpotent by the above mentioned result of I. Shestakov and Z. Zhang [22]. This fact suggests us to formulate the following weaker version of Conjecture 1.

**Conjecture 3.** If $L$ is a solvable algebra of the variety of algebras generated by the Witt algebra $W_1$, then is it true that $[[L, L], [L, L]]$ is nilpotent?

The paper is organized as follows. In Section 2 we give some identities, construction of ideals, and recall some definitions. Sections 3 is devoted to the proof of the main result on the right nilpotency of $[N, N]$. Examples of Novikov algebras $N$ with non nilpotent $[N, N]$ are given in Section 4.

### 2. Identities, ideals, and some definitions

As we mentioned above, any left-symmetric algebra is Lie-admissible, i.e., satisfies the Jacobi identity

$$[[x, y], z] + [[y, z], x] + [[z, x], y] = 0.$$ (3)

Moreover, in the class of Novikov algebras this identity splits into the identities

$$[x, y]z + [y, z]x + [z, x]y = 0$$ (4)

and

$$x[y, z] + y[z, x] + z[x, y] = 0.$$ (5)

Indeed,

$$[x, y]z + [y, z]x + [z, x]y = (xy)z - (yx)z + (yz)x - (zy)x + (zx)y - (xz)y = 0.$$
by (2). This proves (4). Using (3) and (4) we also get (5).

It is useful to write the identity (2) in the form
\[ x[y, z] = (x, z, y) - (x, y, z). \]

Any nonassociative algebra satisfies (see [30]) the identity
\[ [xy, z] - x[y, z] - [x, z]y = (x, y, z) - (x, z, y) + (z, x, y). \]
Using this and (6) we get
\[ (z, x, y) = [xy, z] - [x, z]y. \]

The identities (11) and (2) easily imply that
\[ (xy, z, t) = (x, z, t)y \]
and, consequently,
\[ (x, yz, t) = (x, y, t)z. \]

Recall that any nonassociative algebra also satisfies (see [30]) the identity
\[ (x, y, zt) = (x, yz, t) - (x, z, t)y + x(y, z, t) + (x, y, z)t, \]
Then (8), (9), and (10) give that
\[ (x, yz, t) = (x, y, t)z - (x, z, t)y + x(y, z, t) + (x, y, z)t. \]

It is well known that if \( I \) and \( J \) are ideals of a Novikov algebra \( N \), then \( IJ \) is an ideal of \( N \).

**Lemma 1.** [28] In any Novikov algebra \( N \) the space of associators \( (N, N, N) \) is an ideal of \( N \).

**Proof.** The space \( (N, N, N) \) is a right ideal by (8) or (9). Applying (10) or (11), we get that \( (N, N, N) \) is also a left ideal. \( \square \)

**Lemma 2.** Any Novikov algebra over a field of characteristic \( \neq 2 \) satisfies the identities
\[ (a, b, x) = \frac{1}{2}([ax, b] - [a, bx]), \]
\[ [a, b]x = \frac{1}{2}([ax, b] + [a, bx]), \]
and
\[ x[a, b] = [[x, a], b] + [a, [x, b]] + \frac{1}{2}([ax, b] + [a, bx]). \]

**Proof.** Applying once the identity (2), we get
\[ (a, b, x) = (ab)x - a(bx) = (ab)x - [a, bx] - (bx)a \]
\[ = (ab)x - [a, bx] - (ba)x = [a, b]x - [a, bx]. \]

Consequently,
\[ (b, a, x) = [b, a]x - [b, ax]. \]
By (1), we get
\[ 2(a, b, x) = [a, b]x - [a, bx] + [b, a]x - [b, ax] \]
and
\[ [a, b]x - [a, bx] = [b, a]x - [b, ax]. \]
Consequently,
\[ 2(a, b, x) = [ax, b] - [a, bx] \]
and
\[ 2[a, b]x = [ax, b] + [a, bx], \]
which imply (12) and (13), respectively.

Using (3) and (13), we get
\[ x[a, b] = \frac{1}{2}([ax, b] + [a, bx]) + [a, b]x = [[a, x], b] + [a, [x, b]] + \frac{1}{2}([ax, b] + [a, bx]), \]
i.e., (14) holds. \(\square\)

**Corollary 1.** Let \( N \) be a Novikov algebra over a field of characteristic \( \neq 2 \). Then the following statements are true:

(i) If \( I \) and \( J \) are right ideals of \( N \) then \([I, J]\) is a right ideal of \( N\);

(ii) If \( I \) and \( J \) are ideals of \( N \) then \([I, J]\) is an ideal of \( N\).

At the end of this section we recall the definitions of solvable, nilpotent, and right nilpotent algebras.

Let \( A \) be an arbitrary algebra. The powers of \( A \) are defined inductively by \( A^1 = A \) and
\[ A^m = \sum_{i=1}^{m-1} A^i A^{m-i} \]
for all positive integers \( m \geq 2 \). The algebra \( A \) is called *nilpotent* if \( A^m = 0 \) for some positive integer \( m \).

The right powers of \( A \) are defined inductively by \( A^{[1]} = A \) and \( A^{[m+1]} = A^{[m]} A \) for all integers \( m \geq 1 \). The algebra \( A \) is called *right nilpotent* if there exists a positive integer \( m \) such that \( A^{[m]} = 0 \). In general, the right nilpotency of an algebra does not imply its nilpotency. This is also true in the case of Novikov algebras.

**Example 1.** [27] Let \( N = Fa + Fb \) be a vector space of dimension 2. The product on \( N \) is defined as
\[ ab = b, a^2 = b^2 = ba = 0. \]
It is easy to check that \( N \) is a right nilpotent Novikov algebra, but not nilpotent.

The derived powers of \( A \) are defined by \( A^{(0)} = A, A^{(1)} = A^2 \), and \( A^{(m)} = A^{(m-1)} A^{(m-1)} \) for all positive integers \( m \geq 2 \). The algebra \( A \) is called *solvable* if \( A^{(m)} = 0 \) for some positive integer \( m \). Every right nilpotent algebra is solvable, and, in general, the converse is not true. But every solvable Novikov algebra is right nilpotent [22].

A Novikov algebra \( N \) is called *Lie-solvable* if the Lie algebra \( N^{(-)} \) is solvable.
3. Lie-solvable Novikov algebras

A Novikov algebra $N$ is called \textit{Lie-metabelian} if it satisfies the identity

\[(x, y), [z, t] = 0.\]

In any algebra we denote by $x_1x_2\ldots x_k$ the right normed product $(\ldots(x_1x_2)\ldots)x_k$ of elements $x_1, x_2, \ldots, x_k$.

**Lemma 3.** Any Lie-metabelian Novikov algebra $N$ over a field $K$ of characteristic $\neq 2$ satisfies the identity

\[(x, y), [z, t], s = 0.\]

**Proof.** Using (15) and (13), we immediately get

\[(a, b)x, [y, z] = 0.\]

Then the identities (15) and (2) imply that

\[(x, y), [z, t], a, b = 0.\]

**Corollary 2.** The ideals $(N, N, N)$ and $[N, N]$ of a Novikov algebra $N$ over a field $K$ of characteristic $\neq 2$ are associative and commutative and $(N, N, N) \subseteq [N, N]$.

**Proof.** Notice that $(N, N, N)$ is an ideal of $N$ by Lemma 1 and $[N, N]$ is an ideal of $N$ by Corollary 1. The identity (12) implies that $(N, N, N) \subseteq [N, N]$. The identities (15) and (16) imply that $[N, N]$ is an associative and commutative algebra.

**Lemma 4.** Any Lie-metabelian Novikov algebra $N$ over a field $K$ of characteristic $\neq 2$ satisfies the identities

\[(x, [y, z], t)[a, b] = 0,\]

\[(x, y), [z, t], a, b = 0,\]

and

\[[x, y][z, t](a, b, c) = 0.\]

**Proof.** The identities (14) and (16) imply that

\[(x[a, b], [y, z], t) = 0,\]

Then, by (8),

\[(x, [y, z], t)[a, b] = (x[a, b], [y, z], t) = 0,\]

i.e., (17) holds. Using (8), (11), and (17), we get

\[(x, y)[z, t], a, b = (x, y), a, b)[z, t] = (a, [x, y], b)[z, t] = 0,\]

i.e., (18) also holds. By (10),

\[x, y][z, t](a, b, c) = ([x, y][z, t], a, bc)\]

\[ - ([x, y][z, t], ab, c) + ([x, y][z, t], a, b) - ([x, y][z, t], a, b)c.\]

Using (18), from this we get (19). □
Lemma 5. Let $N$ be a Lie-metabelian Novikov algebra over a field $K$ of characteristic $\neq 2$. Then $(N, N, N)^3 = [N, N]^4 = 0$.

Proof. The identity $(19)$ implies that $(N, N, N)^2(N, N, N) = 0$ since $(N, N, N) \subseteq [N, N]$ by Corollary 2. Consequently, $(N, N, N)^3 = 0$ since $(N, N, N)$ is associative.

Notice that $N[N, N] \subseteq (N, N, N)$ by (6). Consequently, $[N, N]^2 \subseteq (N, N, N)$. Then (19) implies that $[N, N]^2 [N, N]^2 = 0$. This gives $[N, N]^4 = 0$ since $[N, N]$ is an associative algebra. \hfill \Box

Theorem 1. Let $N$ be a Lie-solvable Novikov algebra over a field of characteristic $\neq 2$. Then the ideal $[N, N]$ is right nilpotent.

Proof. Let $N$ be a Lie-solvable Novikov algebra with Lie-solvable index $n$. We prove the statement of the theorem by induction on $n$. By Lemma 5, this is true for $n = 2$. Suppose that $n \geq 3$. Then $[N, N]$ is a Lie-solvable Novikov algebra with Lie-solvable index $n - 1$.

By the induction hypothesis $[[N, N], [N, N]]$ is a right nilpotent ideal of $N$. Notice that $[N, N]^4 \subseteq [[N, N], [N, N]]$. Consequently, $[N, N]$ is a solvable ideal of $N$. Recall that every solvable Novikov algebra is right nilpotent [22]. Therefore $[N, N]$ is a right nilpotent ideal of $N$. \hfill \Box

4. Lie-solvable Novikov algebras with non-nilpotent commutator ideal

Let $K[x]$ be the polynomial algebra over a field $K$ of characteristic zero in one variable $x$. Recall that the Witt algebra $W_1$ is the Lie algebra of all derivations of $K[x]$. Any element of $W_1$ can be written in the form

$$ f \partial, $$

where $f \in K[x]$ and $\partial = \frac{\partial}{\partial x}$. The vector space of $W_1$ with respect to the product

$$ f \partial \circ g \partial = fg' \partial $$

becomes a Novikov algebra [8]. This algebra is denoted by $L_1$ and is called the Novikov-Witt algebra [10]. The elements

$$ e_n = x^{n+1} \partial, n \geq -1, $$

form a linear basis of $L_1$ and

$$ e_i \circ e_j = (j + 1)e_{i+j} $$

for all $i, j \geq -1$. Consequently,

$$ L_1 = \bigoplus_{i \geq -1} Ke_i $$

is a graded algebra.

Set

$$ R = Ke_{-1} \oplus Ke_0. $$

Notice that $R$ is a subalgebra of $L_1$. The left and right actions of elements of $R$ on $L_1$ are naturally defined since $R$ is a subalgebra of $L_1$. We denote an isomorphic copy of this $R$-bimodule $L_1$ by $M$ and assume $e_i$ corresponds to $f_i \in M$ for all $i$. This means that

$$ M = \bigoplus_{i \geq -1} Kf_i $$
and
\[ e_i \circ f_j = (j + 1)f_{i+j}, \quad f_j \circ e_i = (i + 1)f_{j+i} \]
for all \( i = -1, 0 \) and \( j \geq -1 \).

Since \( R \) is a subalgebra of \( L_1 \) it follows that \( L_1 \) is a Novikov bimodule over \( R \), i.e., \( M \) is a Novikov \( R \)-bimodule. By definition this means that the space
\[ N = R \oplus M \]
with the product
\[ (r_1 + m_1)(r_2 + m_2) = r_1 \circ r_2 + r_1 \circ m_2 + m_1 \circ r_2, \]
for all \( r_1, r_2 \in R \) and \( m_1, m_2 \in M \), is a Novikov algebra. Recall that \( N \) is called the zero split extension of \( R \) by \( M \).

**Proposition 1.** The Novikov algebra \( N \) is Lie-solvable of index 3 over a field of characteristic zero and \([N, N]\) is not nilpotent.

**Proof.** Obviously,
\[ [N, N] = [R, R] \oplus [R, M] \]
and \([R, R] = Ke_{-1}\). Moreover,
\[ [R, M] = M \]
since
\[ [e_{-1}, f_j] = (j + 1)f_{j-1} \]
for all \( j \). Consequently,
\[ [N, N] = Ke_{-1} \oplus M. \]
Obviously, \([N, N]\) is not left nilpotent since \( e_{-1}M = M \). Furthermore,
\[ [[N, N], [N, N]] = M \]
and, consequently, \( N \) is Lie-solvable of index 3. \( \square \)

The Lie algebra \( N^{-} \) coincides with Mishchenko’s example from [13].

Notice that \([N, N]\) is not nilpotent over fields of positive characteristic. In order to adopt this example to the case of positive characteristic, we consider another basis
\[ E_i = \frac{1}{(i+1)!}x^{i+1}\partial \]
of the space of \( L_1 \). Recall that binomial coefficients are defined by
\[ \binom{n}{k} = \frac{n!}{k!(n-k)!} \]
for all integers \( n \geq k \geq 0 \). For convenience of notation we set \( \binom{n}{k} = 0 \) if \( n < k \). Then
\[ E_i \circ E_j = \binom{i+j+1}{i+1} E_{i+j} \]
(20)
for all \( i, j \geq -1 \).
Denote by $L$ the abstract algebra over a field $K$ of arbitrary characteristic with a linear basis
\begin{equation}
E_{-1}, E_0, E_1, \ldots, E_k, \ldots
\end{equation}
and with multiplication defined by (20).

**Lemma 6.** The algebra $L$ is a Novikov algebra.

**Proof.** Let $S$ be the free $\mathbb{Z}$-module with a free basis (21). We turn $S$ into a $\mathbb{Z}$-algebra by (20). If the characteristic of $K$ is zero, then $S_K = S \otimes_{\mathbb{Z}} K$ is a free $K$-module with a linear basis (21) since $S$ and $K$ are both free $\mathbb{Z}$-modules. Consequently, $S$ embeds into $S \otimes_{\mathbb{Z}} K$. Then $K$-algebras $L$, $S_K$, and $L_1$ are isomorphic by construction. Consequently, $L$ is a Novikov algebra over $K$ and $S$ is a Novikov algebra over $\mathbb{Z}$.

Assume that the characteristic of the field $K$ is $p > 0$. Obviously, $S_1 = S \otimes_{\mathbb{Z}} (\mathbb{Z}/p\mathbb{Z}) = S/(p\mathbb{Z})S$ is a free $\mathbb{Z}/p\mathbb{Z}$-module with a basis (21). Since $S$ is a Novikov algebra over $\mathbb{Z}$ it follows that $S_1$ is a Novikov algebra over $\mathbb{Z}/p\mathbb{Z}$. This implies that $S_K = S \otimes_{\mathbb{Z}} K = S_1 \otimes_{\mathbb{Z}/p\mathbb{Z}} K$ is a Novikov algebra over $K$ with a linear basis (21). Obviously, $L \simeq S_K$. □

Let $R = KE_{-1} \oplus KE_0$ be the two dimensional subalgebra of $L$. Consider $L$ as an $R$-bimodule. Denote by $M$ an isomorphic copy of $R$-bimodule $L$ and denote by $F_i$ the images of $E_i$ in $M$ for all $i$. Then

\begin{equation}
M = \oplus_{i \geq -1} KF_i
\end{equation}

and

\begin{align*}
E_i \circ F_j &= \binom{i + j + 1}{i + 1} F_{i+j}, \\
F_j \circ E_i &= \binom{i + j + 1}{j + 1} F_{j+i}
\end{align*}

for all $i = -1, 0$ and $j \geq -1$.

Then the zero split extension

\begin{equation}
N = R \oplus M
\end{equation}

of $R$ by $M$ is a Novikov algebra.

**Proposition 2.** The Novikov algebra $N$ is Lie-solvable of index $3$ over an arbitrary field $K$ and $[N, N]$ is not nilpotent.

**Proof.** Obviously, $[R, R] = KE_{-1}$ and $[R, M] = M$ since $E_{-1} \circ F_j = F_{j-1}$ for all $j \geq -1$. Consequently,

\begin{equation}
[N, N] = KE_{-1} \oplus M.
\end{equation}
Obviously, $[N, N]$ is not left nilpotent since $E_{-1}M = M$. Furthermore,

$$[[N, N], [N, N]] = M$$

and, consequently, $N$ is Lie-solvable of index 3. □

Acknowledgments

This research is supported by the Russian Science Foundation (project 21-11-00286) and by the grants of the Ministry of Education and Science of the Republic of Kazakhstan (projects AP08855944 and AP09261086).

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