The Use of Multidimensional RAS Method in Input-Output Matrix Estimation

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Abstract: In national accounts, relations between industries are analyzed using input-output tables. In the Czech Republic these tables are compiled in a five year period. For the remaining years tables must be estimated. Typically, this is done by the RAS method which takes the structure between industries from the last known table and adjusts it to the current industry consumption totals. This approach can also be used for more detailed tables, e.g. quarterly and regional tables. However, the regular RAS method does not ensure that the sums of quarterly/regional tables are equal to the annual/national table. For this problem we propose multidimensional RAS method which besides input and output totals also ensures quarterly or regional totals. Our analysis of quarterly and regional input-output structure shows that the quarterly/regional totals condition should not be omitted and the multidimensional RAS method produces more precise estimations than the regular RAS method. The application of these approaches to regional Isard model shows that the multidimensional RAS has a 6.65% greater accuracy than the classical RAS. Finally, the proposed multidimensional RAS method can also assure consistency during the construction of national symmetric input-output tables themselves.

Keywords: Input-output analysis, RAS method, Iterative proportional fitting procedure, regional tables, quarterly tables.

1 Introduction

In economics the relationships between different branches of national economy can be described using input-output analysis. It was invented by Wassily Leontief who received the Nobel prize for this contribution. The input-output analysis plays a major role in national accounts. The key aspect of this analysis is the input-output table (or the input-output matrix). It shows how output of one industry is input to another industry. More about input-output analysis can be found for example in [Miller and Blair 2009].

In empirical work the situation when only some parts of input-output matrix are known arises quite often. There are many methods for updating technical coefficients in input-output tables. These methods can be classified to several categories (Goga 2009).
The first class consists of expert methods such as Cross impact analysis and Delphi method (Chenery and Clark, 1959). The second class includes extrapolation methods which estimate trends and forecast future values of coefficients (Miller and Blair, 2009). Finally, the third class comprises of formal methods such as Successive correction metody and RAS method (Miller and Blair, 2009; Goga, 2009).

The RAS method is used in a situation when only row and column sums of desired input-output table are known. The table is than estimated from an older fully-known input-output table in a way that the resulting table is consistent with given row and column sums. The origin of RAS method lies in the iterative proportional fitting procedure introduced by Deming and Stephan (1940). It was used for estimation of cell values of contingency tables. The earliest use of the RAS algorithm for estimation of input-output matrices includes the work of Stone and Brown (1962) and Bacharach (1965). Since then the RAS method has gained popularity and was further improved for example by Lecomber (1975), Cole (1992), Gilchrist and ST Louis (1999), Junius and Oosterhaven (2003), Lenzen et al. (2009) and Temurshoev et al. (2013).

In this paper we present another extension to the RAS algorithm, the multidimensional RAS. This method extends the two-dimensional input-output matrices to multidimensional matrices. It can be used for example when the national input-output matrix is divided into regional matrices or when annual matrix is divided into quarterly matrices. Both of these problems add a third dimension to the RAS method. We can also combine these divisions resulting in four-dimensional RAS method.

The rest of this paper is structured as follows. In Section 2 we describe the general input-output framework and multidimensional extension of RAS method. We follow with practical applications to regional tables in Section 3, quarterly tables in Section 4 and domestic/imported tables in Section 5. We conclude this paper with discussion of proposed method and its applications in Section 6.

### 2 The RAS Method

First, we describe the input-output industry-by-industry table. When we categorize the economy into $n$ industries the input-output table has the following structure. Variables $x_{i,j}$ represent the flow from industry $i$ to industry $j$. Variables $u_{i}^{[1]}$ are the total flow from industry $i$ to all industries while variables $u_{j}^{[2]}$ are the total flow from all industries to industry $j$. The total resources are denoted as $p_{i}$, the final demand as $v_{i}^{[1]}$ and the value added as $v_{j}^{[2]}$. The relationships between these variables can be described by output

| input-output table | intermediate consumption | final demand | total resources |
|---------------------|--------------------------|--------------|----------------|
| $x_{1,1}$   $\cdots$   $x_{1,n}$ | $u_{1}^{[1]}$ | $v_{1}^{[1]}$ | $p_{1}$ |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |
| $x_{n,1}$   $\cdots$   $x_{n,n}$ | $u_{n}^{[1]}$ | $v_{n}^{[1]}$ | $p_{n}$ |
| $u_{1}^{[2]}$   $\cdots$   $u_{n}^{[2]}$ | $v_{1}^{[2]}$   $\cdots$   $v_{n}^{[2]}$ | $p_{1}$   $\cdots$   $p_{n}$ |

Table 1: The structure of input-output table.
equations
\[ p_i = u_i^1 + v_i^1, \quad u_i^1 = \sum_{j=1}^{n} x_{i,j} \quad \text{for} \quad i = 1, \ldots, n \quad (1) \]
and input equations
\[ p_j = u_j^2 + v_j^2, \quad u_j^2 = \sum_{i=1}^{n} x_{i,j} \quad \text{for} \quad j = 1, \ldots, n. \quad (2) \]

In this paper we focus mostly on intermediate consumption. We can use matrix and vector notation

\[
\begin{align*}
X &= \begin{pmatrix}
  x_{1,1} & \cdots & x_{1,n} \\
  \vdots & \ddots & \vdots \\
  x_{n,1} & \cdots & x_{n,n}
\end{pmatrix}, &
\mathbf{u}^{[1]} &= \begin{pmatrix}
  u_1^1 \\
  \vdots \\
  u_n^1
\end{pmatrix}, &
\mathbf{u}^{[2]} &= \begin{pmatrix}
  u_1^2 \\
  \vdots \\
  u_n^2
\end{pmatrix}, \\
\mathbf{p} &= \begin{pmatrix}
  p_1 \\
  \vdots \\
  p_n
\end{pmatrix}, &
\mathbf{v}^{[1]} &= \begin{pmatrix}
  v_1^1 \\
  \vdots \\
  v_n^1
\end{pmatrix}, &
\mathbf{v}^{[2]} &= \begin{pmatrix}
  v_1^2 \\
  \vdots \\
  v_n^2
\end{pmatrix}.
\end{align*}
\]

The purpose of the classical RAS method is to modify elements of some nonnegative matrix in a way that is consistent with some predefined row and column totals. In national accounts it is used for estimation of input-output tables. The iterative algorithm starts with the original matrix \( M \). In one iteration it multiplies rows in a way that their totals would be the same as some desired row totals \( \mathbf{u}^{[1]} \) and then it multiplies columns in a way that their totals would be the same as some desired column totals \( \mathbf{u}^{[2]} \). However, the multiplication of columns leads to violation of row totals and vice versa so the multiple iterations must be performed. It can be shown that the algorithm indeed converges to a matrix \( X \) with row totals \( \mathbf{u}^{[1]} \) and column totals \( \mathbf{u}^{[2]} \) if the row and column totals are consistent (i.e. the sum of row totals is equal to the sum of column totals).

2.1 Formulation of Multidimensional RAS Problem

Traditionally, the RAS method (and even its modifications presented in Section 2.3) deals only with two dimensional input-output tables. We take more general approach and formulate the RAS problem in multidimensional case. As we show in sections 3–5 the Multidimensional RAS method (DRAS) can significantly improve estimation and ensure consistency of input-output coefficients over classical two-dimensional RAS method.

Let \( M \) be the initial \( D \)-dimensional matrix of the form
\[ M = (m_{n_1, \ldots, n_D}) \in \mathbb{R}^{N_1 \times \cdots \times N_D}_{\geq 0} \quad (4) \]
and \( \mathbf{U}^{[d]}, d = 1, \ldots, D \) the margin \((D - 1)\)-dimensional matrices of the form
\[ \mathbf{U}^{[d]} = (u_{n_1, \ldots, n_{d-1}, n_{d+1}, \ldots, n_D}) \in \mathbb{R}^{N_1 \times \cdots \times N_{d-1} \times N_{d+1} \times \cdots \times N_D}_{\geq 0}. \quad (5) \]
We require that margin matrices comply with
\[
\sum_{n_d=1}^{N_d} U_{n_1 \ldots, n_d \ldots, n_{e-1}, n_{e+1} \ldots, n_D}^{[e]} = \sum_{n_e=1}^{N_e} U_{n_1 \ldots, n_{d-1}, n_{d+1} \ldots, n_{e-1}, n_{e+1} \ldots, n_D}^{[d]} \quad \forall \ n_1, \ldots, n_{d-1}, n_{d+1}, \ldots, n_{e-1}, n_{e+1}, \ldots, n_D \quad \forall \ d = 1, \ldots, e \quad \forall \ e = 2, \ldots, D.
\]

We look for a \( D \)-dimensional matrix \( X \) of the form
\[
X = (x_{n_1 \ldots, n_D}) \in \mathbb{R}_{\geq 0}^{N_1 \times \ldots \times N_D}
\]
which is consistent with conditions
\[
\sum_{n_1=1}^{N_1} x_{n_1 \ldots, n_D} = u_{n_2 \ldots, n_D}^{[1]} \quad \forall \ n_2, \ldots, n_D
\]
\[
\vdots
\]
\[
\sum_{n_D=1}^{N_D} x_{n_1 \ldots, n_D} = u_{n_1 \ldots, n_{D-1}}^{[D]} \quad \forall \ n_1, \ldots, n_{D-1}
\]
and also satisfies the structure conservation
\[
\frac{m_{i_1^1 \ldots, i_D^1} \cdots m_{i_1^D \ldots, i_D^D}}{m_{j_1^1 \ldots, j_D^1} \cdots m_{j_1^D \ldots, j_D^D}} = \frac{x_{i_1^1 \ldots, i_D^1} \cdots x_{i_1^D \ldots, i_D^D}}{x_{j_1^1 \ldots, j_D^1} \cdots x_{j_1^D \ldots, j_D^D}} \quad \forall \ I_k, J_k \quad \forall \ k = 1, \ldots, D
\]
where
\[
I_k = (i_k^1, \ldots, i_k^D), \quad i_k^i \in \{1, \ldots, N_k\}
\]
and
\[
J_k = \pi_k(I_k) = (\pi_k^1(I_k), \ldots, \pi_k^D(I_k)) = (j_k^1, \ldots, j_k^D)
\]
is a permutation of \( I_k \).

### 2.2 Multidimensional RAS Algorithm

The RAS method is an iterative algorithm in which we progressively compute matrices \( X^{(0)}, X^{(1)}, \ldots, X^{(7)} \). We start by setting \( X^{(0)} = M \). In each iteration \( t \geq 1 \) we perform calculations of matrices \( X^{(t)[d]} \), \( d = 1, \ldots, D \) corresponding to adjusting to each margin totals. We denote \( X^{(t)[0]} = X^{(t-1)} \). Then for \( d = 1, \ldots, D \) we compute elements of \( X^{(t)[d]} \) as
\[
x_{n_1 \ldots, n_D}^{(t)[d]} = x_{n_1 \ldots, n_D}^{(t)[d-1]} \frac{\sum_{i=1}^{N_d} x_{n_1 \ldots, n_{d-1}, i, n_{d+1} \ldots, n_D}^{(t)[d-1]}}{\sum_{i=1}^{N_d} x_{n_1 \ldots, n_{d-1}, i, n_{d+1} \ldots, n_D}^{(t)[d-1]}} \quad \forall \ n_1, \ldots, n_D.
\]
At the end of iteration we set \( X^{(t)} = X^{(t[D)]} \). In this algorithm we simply adjust the matrix in dimension order \( 1, 2, \ldots, D \), but with more dimensions in the RAS method we
have more possible orders of adjusting over specific dimension. However, our simulations and data analysis show that there is no difference between any fixed order of dimensions or even random order in each iteration. We can stop the algorithm based on one of the following conditions:

- **Predefined number of iterations** $T$ is reached.

- A distance between matrices $X(t)$ and $X(t-1)$ is smaller than predefined threshold value $e$. The distance can be measured for example by Frobenius norm giving us the termination condition

$$\sqrt{\sum_{n_1=1}^{N_1} \cdots \sum_{n_D=1}^{N_D} (x_{n_1,\ldots,n_D}^{(t)} - x_{n_1,\ldots,n_D}^{(t-1)})^2} < e. \quad (13)$$

- A distance between the margin totals of $X(t)$ and desired margin totals $U[d]$ is smaller than predefined threshold value $f$. Again, we can use quadratic function giving us the termination condition

$$\sqrt{\sum_{d=1}^{D} \left( \sum_{n_1=1}^{N_1} \cdots \sum_{n_{d-1}=1}^{N_{d-1}} \sum_{n_{d+1}=1}^{N_{d+1}} \cdots \sum_{n_D=1}^{N_D} (x_{n_1,\ldots,n_D}^{(t)} - u[d]_{n_1,\ldots,n_{d-1},n_{d+1},\ldots,n_D})^2 \right)} < f. \quad (14)$$

If possible solution exists (i.e. there are not too many zeros in the original matrix $M$), the algorithm converges to desired solution

$$X = \lim_{t \to \infty} X(t). \quad (15)$$

More on complexity and effective implementation of the algorithm can be found in Jiroušek and Přeučil (1995). The usage of this multidimensional algorithm for different problems is discussed later in Section 2.4.

### 2.3 Overview of Some Other RAS Extensions

In this section we briefly describe generalizations of classical two-dimensional RAS method for uncertainty of data, constraints on matrix elements, nonnegative matrix elements and conflicting information. The more detailed overview of RAS extensions as well as the history of development of RAS method can be found in Lahr et al. (2004).

With uncertainty of the preliminary estimates deals the Modified RAS (MRAS) method by Lecomber (1975). Case studies were presented by Allen (1974) and Allen and Lecomber (1975). Gilchrist and ST Louis (1999) proposed a three-stage extension of classical RAS algorithm called the Three-Stage RAS (TRAS). It allows to add restrictions on arbitrary subsets of matrix elements in addition to row and column margins. The Generalized RAS (GRAS) algorithm proposed by Junius and Oosterhaven (2003) generalizes the RAS method by allowing matrices with some negative elements. Temurshoev et al. (2013) improves this method by allowing some rows or columns to consist only of nonpositive elements. The Konfliktfreies RAS (KRAS) method proposed by Lenzen et al. (2009) incorporates features of TRAS, MRAS and GRAS and further generalizes the RAS algorithm for the case of conflicting data.
2.4 Alternative Formulations and Names

The RAS method is known under many other names in different applications. Originally, the \textit{iterative proportional fitting procedure (IPFP)} was used for estimation of contingency table cell values. It is also known as \textit{Biproportional fitting} or \textit{Iterative proportional scaling} in statistics and \textit{Matrix raking} or \textit{Matrix scaling} in computer sciences. The IPFP method is also closely related to \textit{Entropy maximization}. The IPFP was first introduced by \cite{Deming:1940}. It was stated in this paper that the algorithm minimizes Pearson’s $\chi^2$ statistic, but as shown by \cite{Stephan:1942} this is not correct. \cite{Fienberg:1970} proved the convergence of the algorithm using differential geometry. A proof built on non-trivial measure theorems and entropy minimization was presented by \cite{Csiszar:1975}. The solution of the IPFP algorithm is the same as of the RAS method.

The multidimensional IPFP method was applied to three-way and multiway contingency tables by \cite{Deming:1943, Darroch:1962, Ireland:1968, Bishop:1969}. \cite{Johnston:1993} discussed entropy maximization and IPFP in geographical context with illustration on British voting patterns. \cite{Gange:1995} applied multidimensional IPFP algorithm to generate multivariate binary data with fixed marginal distributions. \cite{Lovelace:2015} used the IPFP method for spatial microsimulation.

3 Regional Input-Output Matrices

In this section we use the proposed DRAS method for estimation of regional input-output tables of the Czech Republic. We use industry-by-industry tables with total of 82 industries. The source of annual national input-output tables of the Czech Republic is \cite{CzechStatisticalOffice:2016}. We have national input-output table and row and column margins for regional tables. Our goal is to estimate the input-output table for each region. The Czech Republic is split into 14 regions: Prague (R1), Central Bohemia (R2), South Bohemia (R3), Plzeň (R4), Karlovy Vary (R5), Ústí nad Labem (R6), Liberec (R7), Hradec Králové (R8), Pardubice (R9), Vysočina (R10), South Moravia (R11), Olomouc (R12), Zlín (R13) and Moravia-Silesia (R14). The region R1 is the capital city. We estimate the three-dimensional matrix $X$ of $N_1 = 82$ input industries, $N_2 = 82$ output industries and $N_3 = 14$ regions with total of 94 136 elements.

3.1 Difference of Regional Sums

We can estimate the regional matrices by classical RAS method from the structure of national matrix and regional margins. However, this approach does not ensure that all regional matrices estimated in this way will add up to the national matrix. The sum of some elements over all regions can be lower or higher than the real value of national table. In this section we compare the sum of regional tables estimated by classical RAS algorithm and the real national table. In Figure\[1\] we can see that for some elements the relative deviation can exceed 100 \%. If we use multidimensional RAS method instead, this deviations will be always 0, because this is exactly what the third dimension in algorithm ensures. The results from Figure\[1\] motivate us for using the DRAS method for more consistent estimates.
Figure 1: Relative deviations of real national input-output table of the Czech Republic and the sum of regional classical estimates.
3.2 Comparison of Regional Estimates with Real Matrices

As we have the real regional tables available we can compare them to the classical and multidimensional estimates. First, we use the Frobenius norm

\[ FN(X, Y) = \sqrt{\sum_{n_1=1}^{N_1} \sum_{n_2=1}^{N_2} (x_{n_1,n_2} - y_{n_1,n_2})^2} \]  \hspace{1cm} (16)

for comparing matrices \( X \) and \( Y \). The results are shown in Figure 2. We can see that the difference between the real tables and multidimensional estimate is much lower than the difference between the real tables and classical estimates. The highest deviation of multidimensional estimate from real table id for the region R1. This is not suprising result as R1 is the capital city Prague and its economy structure differs from the rest of the country.

Next we compare the Leontief inverse of estimates and real tables. The Leontief inverse or the total requirements matrix is defined as

\[ L = (I - A)^{-1}, \]  \hspace{1cm} (17)

where \( I \) is identity matrix and \( A \) is matrix of technical coefficients given by

\[ A = (a_{n_1,n_2})_{n_1=1}^{N_1},_{n_2=1}^{N_2}, \quad a_{n_1,n_2} = \frac{x_{n_1,n_2}}{a_{n_2}}. \]  \hspace{1cm} (18)

This will give us some idea about prediction ability of estimations because the Leontief inverse is often used to predict total resources from final consumption using equation

\[ p = Le^{[l]}. \]  \hspace{1cm} (19)

Again, we compare the matrices by Frobenius norm given by (16). In Figure 3 we can see that the Leontief inverse of multidimensional RAS estimate is closer to the real matrix than the classical estimate. However, the difference is not as significant as in the case of original non-inverted matrices.
3.3 Isard’s Inter-Regional Input-Output Model

As an analytical application, we use Isard’s inter-regional input-output model (IRIO). These models are mainly used to analyze the local effects of national economic policies. They are also used for the analysis of environmental issues in the context of individual sectors in the Czech Republic. Finally, they form the data basis for more advanced models such as DSGE discussed in Bouakez et al. (2009) and Bouakez et al. (2014).

The advantage of Isard’s IRIO model is the fact that this approach allows us to analyze so-called backward relationships in the model. Given the complexity of this approach we only summarize the basic facts. More about the IRIO model can be found in Isard (1960). This model is disaggregated national model which allows to analyze effects at both the individual industry level as well as regional unit level. It can be used in particular to analyze the impact of economic policy.

Our model is designed for 14 regions of the Czech Republic and 82 industries. The data source of regional input-output tables is University of Economics, Prague (2016) constructed according to the methodology of Sixta et al. (2016), Sixta and Fischer (2015), Sixta and Vltavská (2016). However, these data do not contain the detailed allocation of inter-regional production flows. For this reason the used data are constructed according to Safr (2016).

Using the methodology of Miller and Blair (2009) and derived procedure for the inter-regional model based on the Czech Republic data of Safr and Vltavská (2016) we construct inter-regional model for the Czech Republic. The model is built entirely on the basis of data sources.

We also construct two additional models. In these models we assume that we do not know the regional matrix describing the use of industry imports (for each region). This missing data source is used for the allocation of inter-regional production flows into the intermediate consumption in other regions. Then, using a national matrix of regional flows (which is known and represents the sum of regional input-output matrices
|          | Real | RAS  | DRAS |
|----------|------|------|------|
| Real     | 0    | 53 696 | 50 346 |
| RAS      | 53 696 | 0    | 48 782 |
| DRAS     | 50 346 | 48 782 | 0    |

Table 2: Total Frobenius norm for classical RAS and multidimensional RAS approach compared to real data in Isard’s Inter-Regional Input-Output Model.

Figure 4: Regional Frobenius norm for classical RAS and multidimensional RAS approach compared to real data in Isard’s Inter-Regional Input-Output Model.

As we can see in Table 2, the multidimensional RAS shows less difference to the model constructed on the basis of real data. The classical RAS approach is overestimated by 6.65% compared to the multidimensional RAS. For a detailed look, we decompose Frobenius norm to specific regions in Figure 4.

Based on Figure 4, we can not assume that one method is systematically better or worse than the other in each region. Yet the multidimensional RAS method achieves better results in this test and one can assume that it can rather prevent extreme values than the classical RAS. For this reason, the multidimensional RAS method is more suitable tool for the construction of data.

for the use of imports) we estimate regional matrices using classical RAS method and multidimensional RAS method.

The evaluation of results is performed on total intermediate consumption matrix for all regions and industries (14 regions each with 82 industries times 14 regions each with 82 industries). As in previous sections, we calculate the Frobenius norm. The method (classical or multidimensional RAS) which is more similar to the model results on the basis of real data is more effective.
4 Quarterly Input-Output Matrices

In similar fashion as in Section 3 we can analyze the difference between the multidimensional RAS method and the classical RAS method, only this time the third dimension represents division into 4 quarters (Q1-Q4). We have annual input-output table and row and column margins for quarterly tables for the years 2013 and 2014. We estimate the three-dimensional matrix $X$ of $N_1 = 82$ input industries, $N_2 = 82$ output industries and $N_3 = 4$ quarters with total of 26 896 elements.

4.1 Difference of Quarterly Sums

We estimate the quarterly matrices by classical RAS method from the known structure of annual matrices and quarterly margins. Just as in the case of regional matrices in Section 3.1 this approach does not ensure that the total of quarterly matrices will be equal to the corresponding annual matrix. In Figure 5 we have the deviations of elements for the year 2013 and in Figure 6 for the year 2014. Both years show very similar results. The deviation of some elements can exceed 10%, which is not as high as in regional tables in Figure 1 but still very significant. This can be caused by the fact that in regional case we split the matrix into 14 matrices, but in quarterly case we split it only into 4 matrices. We can see in Figure 5 as well as in Figure 6 that some industries exhibit very strong quarterly deviations in input and output perspective.

4.2 Comparison of Quarterly Estimates

Since we do not have the real quarterly tables available, our analysis of quarterly tables is limited. However the comparison of the multidimensional RAS estimates and the classical RAS estimates can give us some idea of quarterly behaviour since in the case of regional comparison in Figure 2 the difference between the multidimensional RAS estimates and the real matrices is almost identical to the difference between the multidimensional RAS estimates and the classical RAS estimates. Figure 7 shows us the Frobenius norm (16) of this difference for the years 2013 and 2014. In both years the difference increases in each subsequent quarter suggesting that the deviation from annual structure is higher closer to the end of the year. However without the real quarterly matrices this cannot be verified.

5 Domestic/Imported Input-Output Matrices

Multidimensional RAS can be used in the same way as classical RAS with additional higher dimensions. A typical example is the previously mentioned quarterly and regional input-output tables. Another application is the construction of symmetric input-output tables themselves. The third dimension then represents the division into domestic product use and imported product use.

5.1 Difference of Domestic/Imported Sums

The input-output tables are typically designed as follows: Firstly, the total table (TOT) including both the domestic use and imported use is constructed. Then the table for domestic use (DOM) and the table for imported use (IMP) are constructed from TOT.
Figure 5: Relative deviations of the real 2013 input-output table of the Czech Republic and the sum of quarterly classical estimates.
Figure 6: Relative deviations of the real 2014 input-output table of the Czech Republic and the sum of quarterly classical estimates.

Figure 7: The Frobenius norm of the difference between classical and multidimensional quarterly estimates.
matrix using classical RAS. However, here a problem arises. Although this approach ensures that the sums of rows and the sums of columns of DOM and IMP total matrix are equal to the sums of TOT matrix, the elements of DOM and IMP total matrix can be different from the elements of TOT matrix. This inconsistency causes that the old national TOT matrix is discarded and the new total matrix is created as the sum of DOM and IMP matrices. The Frobenius norm of the differences between TOT matrices and the sums of DOM and IMP matrices for the years 1995, 2000, 2005 and 2010 are presented in Table 3.

### Table 3: The Frobenius norm of the differences between real input-output tables of the Czech Republic and the sums of domestic use and imported use estimates

| Year | Frobenius norm |
|------|----------------|
| 1995 | 5              |
| 2000 | 376            |
| 2005 | 12 072         |
| 2010 | 15 010         |

Figure 8: The Frobenius norm of the difference between classical and multidimensional domestic use and imported use estimates.

5.2 Comparison of Domestic/Imported Estimates

Next, we investigate the difference between classical and multidimensional estimates. As we can see from Figure 8, neither IMP matrix nor DOM matrix estimated by multidimensional RAS exhibits significant deviations from classical RAS. Furthermore, the data can not conclude that this method has a major impact on the input-output structure. This is caused by the fact that the structure of DOM and IMP is very similar to the structure of TOT. Using proposed multidimensional RAS method does not lead to a decrease in accuracy while ensuring consistency with the original national input-output table. For these reasons we consider the proposed method to be appropriate for this application.


6 Discussion

We propose the extension of the RAS algorithm for multiple dimensions. This extension can be beneficial when we estimate more matrices that are related to each other. We use the multidimensional RAS method for estimation of regional, quarterly and domestic/imported input-output industry-by-industry tables of the Czech Republic. All these applications add a third dimension to the RAS problem. We show that the estimated matrices given by the proposed multidimensional RAS method are better than the classical estimates in terms of accuracy and consistency with overall input-output structure. The application of the multidimensional RAS method to the inter-regional input-output model also shows better results than the classical RAS method. The test of the difference between classical and multidimensional RAS applied to domestic use tables and imported use tables shows that proposed multidimensional RAS method does not create major differences yet it ensures consistency of the third dimension.

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