Geometric quantum discord of a Jaynes–Cummings atom and an isolated atom

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Abstract
We studied the geometric quantum discord of a quantum system consisting of a Jaynes–Cummings (JC) atom, a cavity and an isolated atom. The analytical expressions of the geometric quantum discord for two atoms, every atom with a cavity and the total system were obtained. We showed that the geometric quantum discord is not always zero when the entanglement falls to zero for a two-atom subsystem; the geometric measurement of the quantum discord of the total system developed periodically with a single frequency if the initial two-atom state was not entangled, otherwise, it oscillated with two or four frequencies according to whether the cavity was initially empty or not, respectively.

Keywords: Jaynes–Cummings model, quantum correlation, quantum discord, geometric quantum discord

(Some figures may appear in colour only in the online journal)

1. Introduction

The cavity quantum electrodynamics (CQED) system is one of the fundamental subjects of quantum mechanics and the quantum information theory because atoms can be used to store quantum information and photons are suitable for the transfer of quantum information [1]. The study of the CQED system has attracted much research interest. Many authors have investigated a system composed of several atoms which were trapped in cavities, but these cavities were connected by optical fibers or were isolated from each other. Some authors have proposed various schemes to generate useful quantum states [1–5] or realize quantum gates [6–8] in the quantum information process and quantum computation.

On the other hand, some authors have studied the properties of entanglement of the CQED system. Yu and Eberly investigated two entangled qubits, which individually interact with vacuum noise. They showed that the sudden death of entanglement between atom A in its cavity and the isolated atom B appears when the cavity lies in the nonzero number state. Furthermore, the entanglement resurrection can occur after a period of time, which is not dependent on the degree of entanglement of the initial state [11]. More recently, Lu examined a CQED system, which was comprised of three JC two-level atoms resonantly interacting with three cavities that were coupled by two optical fibers [12]. Their results demonstrate that the entanglement between non-adjacent atoms or that which occurs between adjacent cavities has a nonlinear relation with the increase of the atom-cavity coupling coefficient, but the entanglement between non-adjacent cavities is strengthened and the entanglement between adjacent atoms is weakened with the increase of the atom-cavity coupling constant. These studies revealed some important properties of entanglement or other correlation, but only for the subsystem involved in the study. We do not know the total correlation properties of these CQED systems; for subsystems containing a cavity, the...
concurrency has not yet been given. To remedy these defects, this paper will employ geometric quantum discord (GQD) to measure the correlation character of all the subsystems and total systems for a CQED system. For simplicity, we chose the JC system with an isolated atom in [11].

This paper is arranged as follows. In section 2, we provide a brief review of the geometric measure of quantum discord. In section 3, we calculate the GQD for bipartite subsystems. In section 4, we give the GQD of the total system and discuss the monogamy of the system. Finally, we summarize the paper in section 5.

2. Brief review of the geometric measure of quantum discord

Before starting, we give a brief review of the geometric measure of quantum discord. Quantum discord is a quantum-versus-classical paradigm for correlations [13–15] and is not in the entanglement-versus-separability framework [16, 17]. The quantum discord of a bipartite state \( \rho \) on a system \( H^a \otimes H^b \) with marginals \( \rho^a \) and \( \rho^b \) can be expressed as [18, 19]

\[
Q(\rho) = \min_{\Pi^k} \left\{ I(\rho) - I(\Pi^k(\rho)) \right\}. \tag{1}
\]

Here the minimum is over von Neumann measurements (one-dimensional orthogonal projectors summing to the identity); \( \Pi^k = \{\Pi^k_a\} \) is a positive operator acting on the subsystem \( a \); the label \( k \) distinguishes the different outcomes of this measurement) on the subsystem \( a \), and

\[
\Pi^k(\rho) = \sum_k (\Pi^k_a \otimes I^b) \rho (\Pi^k_a \otimes I^b) \tag{2}
\]

is the resulting state after the measurement. \( I(\rho) = S(\rho^a) + S(\rho^b) - S(\rho) \) is the quantum mutual information, \( S(\rho) = -\text{tr} \ln \rho \) is the von Neumann entropy, and \( I^b \) is the identity operator on \( H^b \).

The calculation of the quantum discord involves a difficult optimization procedure. It is generally hard to obtain analytical results except for a few families of two-qubit states [20–28]. Huang has proved that computing the quantum discord is nondeterministic polynomial time (NP)-complete: the running time of any algorithm for computing the quantum discord is believed to grow exponentially with the dimension of the Hilbert space. Therefore, computing the quantum discord in a quantum system even with moderate size is impossible in practice [29]. In order to overcome this problem, Dakić et al proposed the following geometric measure of quantum discord [30]:

\[
D(\rho) = \min_{\chi} \| \rho - \chi \|_{2}, \tag{3}
\]

where \( \chi \) is the zero-discord states [i.e. \( Q(\chi) = 0 \)], the minimum is over the set of \( \chi \) and \( \| A \|_{2} = \sqrt{\text{tr}(A^A)} \) is the Frobenius or Hilbert–Schmidt norm. The density operator of any two-qubit state can be expressed as

\[
\rho = \frac{1}{4} \left( I^a \otimes I^b + \sum_{i=1}^{3} (x_i \sigma_i \otimes I^b + I^a \otimes y_i \sigma_i) + \sum_{i,j=1}^{3} t_{ij} \sigma_i \otimes \sigma_j \right), \tag{4}
\]

where \( \{\sigma_i, i = 1, 2, 3\} \) denote the Pauli spin matrices and \( I^a(X = A, B) \) is a \( 2 \times 2 \) unit matrix for the qubit \( X \). Then, the geometric measure of the quantum discord of any two-qubit state is evaluated as

\[
D(\rho) = \frac{1}{4} \left( \| x \|^2 + \| T \|^2 - k_{\text{max}} \right), \tag{5}
\]

where \( x := (x_1, x_2, x_3)^T \) is a column vector, \( \| x \|^2 = \sum_i x_i^2 \), \( x_i = \text{tr}(\rho (\sigma_i \otimes I^b)) \), \( T = (t_{ij}) \) is a matrix and \( t_{ij} = \text{tr}(\rho (\sigma_i \otimes \sigma_j)) \), \( k_{\text{max}} \) is the largest eigenvalue of matrix \( xx^T + TT^T \).

Since Dakić et al proposed the geometric measure of quantum discord, many authors extended Dakić’s results to the general bipartite states. Luo and Fu evaluated the geometric measure of the quantum discord for an arbitrary state and gave a tight lower bound for the geometric discord of arbitrary bipartite states [31]. They also proved the original definition of the geometric measure of quantum discord (3) can be expressed as

\[
D(\rho) = \min_{\Pi^a} \| \rho - \Pi^a(\rho) \|_{2}^2, \tag{6}
\]

where the minimum is over von Neumann measurements \( \Pi^a = \{\Pi^k_a\} \) on system \( H^a \), and \( \Pi^a(\rho) = \sum_k (\Pi^k_a \otimes I^b) \rho (\Pi^k_a \otimes I^b) \).

Recently, a different tight lower bound for the geometric discord of arbitrary bipartite states was given by Rana and Parashar [32], and Hassan et al [33] independently. Alternatively, Girolami et al found an explicit expression of the geometric discord for a two-qubit system and extended it to \( 2 \otimes d \)-dimensional systems [34]. Tufarelli et al also gave another formula of the geometric discord for a qubit-qudit system, which is available to \( 2 \otimes d \)-dimensional systems including \( d = \infty \) [35]. It worth noting that the authors of [34, 35] added a normalization factor of 2 to the definition of the geometric measure of quantum discord in equation (3) to ensure the maximum value of the geometric discord of the Bell states is 1. Then, for a quantum state \( \rho_{AB} \) with \( A \) being a qubit and \( B \) being an arbitrary (finite or infinite) \( d \)-dimensional system, they defined a vector \( v = \text{Tr}_{B}(\rho_{AB} \sigma) \) and derived the normalized geometric discord

\[
D_{G}(\rho_{AB}) = \text{tr}(S) - \lambda_{\text{max}}(S), \tag{7}
\]

with \( S = \text{tr}_{B}(vv^T) \) and the symbol ‘tr’ in \( \text{tr}(S) \) denoting the usual trace of a matrix. In the following sections, we are going to use this algorithm to calculate the geometric discord of a quantum system consisting of a JC atom and an isolated atom.
3. GQD of bipartite subsystems

The Hamiltonian of a quantum system, which we are considering, can be written as \((\hbar = 1)\) [36]:

\[
H_{\text{tot}} = \frac{\omega}{2} \sigma_z^A + \frac{\omega}{2} \sigma_z^B + g \left( a^\dagger \sigma_z^A + a \sigma_z^A \right) + va^\dagger a, \tag{8}
\]

where \(\sigma_z^x (x = A, B)\) is the third Pauli matrix for the atom \(x\); \(\sigma_z^a\) are the atomic raising and lowering operators and \(a^\dagger (a)\) is the creation (annihilation) operator of the cavity field; \(\omega\) is the transition frequency of the two-level atom \(A\) and \(B\); \(g\) is the coupling strength for the interaction between atom \(A\) and the cavity \(C\); \(v\) is the frequency of the cavity field. Obviously, atom \(B\) interacts neither with atom \(A\) nor with cavity \(C\) (see figure 1). We can prepare the two atoms initially in an entangled pure state and cavity \(C\) in the Fork state \(|a_i\rangle\). The initial state of the total system can be written as

\[
|\Psi_0\rangle = \left( \cos \alpha |1_A 0_B\rangle + \sin \alpha |0_A 1_B\rangle \right) \otimes |n_C\rangle
= \cos \alpha |1_A 0_B n_C\rangle + \sin \alpha |0_A 1_B n_C\rangle. \tag{9}
\]

The state of the system at time \(t\) can be expressed as

\[
|\Psi(t)\rangle = x_1(t)|1_A 0_B n_C\rangle + x_2(t)|0_A 1_B n_C\rangle + x_3(t)|0_A 0_B (n + 1)C\rangle + x_4(t)|1_A 1_B (n - 1)C\rangle. \tag{10}
\]

For simplicity, we only consider the case of detuning \(\Delta = \omega - \nu = 0\). The solution of the Schrödinger equation with Hamiltonian (8) is [36]

\[
x_1(t) = e^{-i\omega t} \cos \left( g \sqrt{n + 1} t \right) \cos \alpha,
\]

\[
x_2(t) = e^{-i\omega t} \cos \left( g \sqrt{n} t \right) \sin \alpha,
\]

\[
x_3(t) = -ie^{-i\omega t} \sin \left( g \sqrt{n + 1} t \right) \cos \alpha,
\]

\[
x_4(t) = -ie^{-i\omega t} \sin \left( g \sqrt{n} t \right) \sin \alpha. \tag{11}
\]

The density operator of the system \(ABC\) is

\[
\rho_{ABC} = \left[ x_4(t)^* x_1(t) |0_A 0_B (n - 1)C\rangle \langle 0_A 0_B (n - 1)C| \right.
+ x_4(t)^* x_3(t) |0_A 0_B (n - 1)C\rangle \langle 0_A 0_B (n - 1)C| \right)
+ x_4(t)^* x_2(t) |1_A 0_B n_C\rangle \langle 0_A 0_B (n - 1)C| \right)
+ x_4(t)^* x_1(t) |0_A 0_B (n - 1)C\rangle \langle 0_A 0_B n_C| \right)
+ \left[ x_3(t)^* |0_A 0_B n_C\rangle \langle 0_A 0_B n_C| \right.
+ x_3(t)^* x_2(t) |1_A 0_B n_C\rangle \langle 0_A 0_B n_C| \right)
+ x_3(t)^* x_1(t) |0_A 0_B (n + 1)C\rangle \langle 0_A 0_B (n + 1)C| \right)
+ x_3(t)^* x_4(t) |0_A 0_B (n - 1)C\rangle \langle 1_A 0_B n_C| \right)
+ \left[ x_2(t)^* |1_A 0_B (n + 1)C\rangle \langle 1_A 0_B (n + 1)C| \right. \tag{12}
\]

Taking the trace over the cavity \(C\), we obtain the reduced density operator \(\rho_{AB}\) between the two atoms,

\[
\rho_{AB} = \left[ x_4(t)^* |0_A 0_B\rangle \langle 0_A 0_B| \right.
+ x_4(t)^* x_1(t) |0_A 0_B\rangle \langle 0_A 0_B| \right)
+ x_4(t)^* x_3(t) |0_A 0_B\rangle \langle 0_A 0_B| \right)
+ x_4(t)^* x_2(t) |0_A 0_B\rangle \langle 0_A 0_B| \right)
+ \left[ x_3(t)^* |0_A 0_B\rangle \langle 0_A 0_B| \right. \tag{13}
\]

Recall that Pauli spin matrices can be expressed by Dirac notation,

\[
\sigma_z^a = |0_a\rangle \langle 0_a| + |1_a\rangle \langle 1_a|, \tag{14a}
\]

\[
\sigma_x^a = i \left( |1_a\rangle \langle 0_a| - |0_a\rangle \langle 1_a| \right), \tag{14b}
\]

\[
\sigma_y^a = |0_a\rangle \langle 0_a| - |1_a\rangle \langle 1_a|. \tag{14c}
\]

where \(\sigma_z^x (x = 1, 2, 3)\) is a Pauli spin matrix expressed by basis vectors \(|0_a\rangle\) and \(|1_a\rangle\) of qubit \(a (a = A, B)\). We can now use equation (7) to calculate \(D_{cl}(\rho_{AB})\). We first calculate the vector \(\nu\) and obtain

\[
\nu = \text{tr}_A(\rho_{AB} \cdot \sigma_z^A) = \left\{ x_2(t)^* x_3(t) |1_A\rangle \langle 0_B| + x_3(t)^* x_2(t) |0_B\rangle \langle 1_B|, \right.
\]

\[
i \left( x_2(t)^* x_3(t) |1_A\rangle \langle 0_B| - x_3(t)^* x_2(t) |0_B\rangle \langle 1_B| \right),
\]

\[- x_2(t)^* |0_B\rangle \langle 0_B| + x_3(t)^* |0_B\rangle \langle 0_B|. \tag{15}
\]
According to $S = tr_0(\rho vv^T)$, we get a diagonal matrix with diagonal elements

$$S = \{ 2 |x_1(t)x_2(t)|^2, 2 |x_1(t)x_3(t)|^2, (|x_1(t)|^2 - |x_5(t)|^2)^2 + (|x_2(t)|^2 - |x_4(t)|^2)^2 \}. \tag{16}$$

Obviously, matrix $S$ has three eigenvalues

$$\lambda_1 = \lambda_2 = 2|x_1(t)x_2(t)|^2, \quad \lambda_3 = (|x_1(t)|^2 - |x_5(t)|^2)^2 + (|x_2(t)|^2 - |x_4(t)|^2)^2. \tag{17}$$

We finally get the GQD of $\rho_{AB}$

$$D_G(\rho_{AB}) = 4 |x_1(t)x_2(t)|^2 + (|x_1(t)|^2 - |x_5(t)|^2)^2$$
$$- \left( |x_2(t)|^2 - |x_4(t)|^2 \right)^2 + \left( \lambda_1 \right)^2 - \lambda_3} \times 2 \left( |x_1(t)x_3(t)|^2 + |x_2(t)x_4(t)|^2 \right). \tag{18}$$

To reveal the properties of $D_G(\rho_{AB})$ with time $t$ and parameter $\alpha$ and $n$, we plot $D_G(\rho_{AB})$ as a function of $t$ for some typical values of $\alpha$ and $n$ in figure 2.

Comparing figure 2 with figure 2 in [36] let us find some interesting features of the geometric measurements of the quantum discord. Figure 2 of [36] showed that the sudden death of entanglement between atom A and B can occur when the cavity lies in the nonzero number state. Our figure 2 shows that though the geometric measurements of quantum discord have a roughly similar behavior with the entanglement, the geometric measurements of quantum discord however are not always zero in the time interval for the zero entanglement. This confirms that even some separable states can still contain nonclassical correlation [37] and also that the quantum discord is a reliable indicator of the quantum nature of the correlations [18].

Using the same procedure used above, we can obtain $D_G(\rho_{AC})$ and $D_G(\rho_{BC})$ as follows,

$$D_G(\rho_{AC}) = |x_1(t)|^2 + |x_2(t)|^2 + |x_3(t)|^4 + |x_4(t)|^4$$
$$- 2 |x_1(t)x_2(t)|^2 + 4 \left( |x_1(t)x_3(t)|^2 + |x_2(t)x_4(t)|^2 \right)$$
$$- \max \left[ |x_1(t)|^4 + |x_2(t)|^4 + |x_3(t)|^4 \right]$$
$$+ \left( |x_3(t)|^4 - |x_4(t)|^4 \right)^2,$$
$$\times 2 \left( |x_1(t)x_3(t)|^2 + |x_2(t)x_4(t)|^2 \right). \tag{19}$$

$$D_G(\rho_{BC}) = |x_1(t)|^2 + |x_2(t)|^2 + |x_3(t)|^4 + |x_4(t)|^4$$
$$- 2 |x_1(t)x_2(t)|^2 + 4 \left( |x_2(t)x_3(t)|^2 + |x_1(t)x_4(t)|^2 \right)$$
$$- \max \left[ |x_1(t)|^4 + |x_2(t)|^4 \right]$$
$$+ \left( |x_3(t)|^4 - |x_4(t)|^4 \right)^2,$$
$$\times 2 \left( |x_2(t)x_3(t)|^2 + |x_1(t)x_4(t)|^2 \right). \tag{20}$$

We also plot $D_G(\rho_{AC})$ and $D_G(\rho_{BC})$ as functions of $t$ for some typical values of $\alpha$ and $n$ in figure 3 and figure 4 respectively.

Figures 2, 3 and 4 show that the dependence on the degree of entanglement of the initial two-atom state (scaled by $\alpha$) of the amplitude, with which $D_G(\rho_{AB}), D_G(\rho_{AC})$ and $D_G(\rho_{BC})$ vibrate, are different. The vibrating or quasi-vibrating amplitude of $D_G(\rho_{AB})$ and $D_G(\rho_{BC})$ decrease with the decrease of the degree of entanglement of the initial two-atom state ($\alpha$ decreases from $\pi/4$ to 0 or increases from $\pi/4$ to $\pi/2$), especially when, $D_G(\rho_{AB}) = D_G(\rho_{BC}) = 0$ for $\alpha = 0$. The situation for $D_G(\rho_{AC})$ is contrary. The vibrating or quasi-vibrating amplitude of $D_G(\rho_{AC})$ increases with the decrease of $\alpha$.
the degree of entanglement of the initial two atomic states and $D_{G}(\rho_{AC})$ is not always zero for $\alpha = 0$. The reason why $D_{G}(\rho_{AB}), D_{G}(\rho_{BC})$ and $D_{G}(\rho_{AC})$ have different behaviors with $\alpha$, can be explained as follows. First, atom $A$ and $B$ become initially entangled and atom $A$ interacts with cavity $C$. On the other hand, though atom $B$ does not interact directly with cavity $C$, it correlates with cavity $C$ later by the interaction between $A$ and $C$. So, Atom $A$ and $B$ become entangled, and atom $B$ and cavity $C$ also become entangled in the development of the system. Therefore, the vibrating amplitudes of $D_{G}(\rho_{AB})$ and $D_{G}(\rho_{BC})$ increase with the increase of the degree of entanglement of the initial two-atom state. The reason why the vibrating amplitude of $D_{G}(\rho_{AC})$ decreases with the increase of the degree of entanglement of the initial two-atom state is that $D_{G}(\rho_{AB})$ and $D_{G}(\rho_{AC})$ satisfy Coffman–Kundu–Wootters (CKW) monogamy inequality (26), which will be discussed in the next section, and from equation (26), we know that when $D_{G}(\rho_{AB})$ increases, $D_{G}(\rho_{AC})$ naturally decreases.

4. GQD of the total system and the monogamy

To calculate the geometric measure of the quantum discord for the total system $\rho_{ABC}$, according to equation (6), we make a von Neumann measurement on atom $A$, $\Pi^A(\rho_{ABC}) = \Pi^A_+\rho_{ABC}\Pi^A_+ + \Pi^A_-\rho_{ABC}\Pi^A_-$, where

$$\Pi^A_{\pm} = \frac{I \pm \Pi \cdot \sigma^A}{2}$$

are projective measurements on the single qubit $A$ and $\Pi = \{\alpha, \beta, \gamma\}$ with $\alpha^2 + \beta^2 + \gamma^2 = 1$. Now, after a tedious and direct calculation and simplification, we obtain

$$\| \rho_{ABC} - \Pi^A(\rho_{ABC}) \|^2$$

$$= \text{tr}\left( (\rho_{ABC} - \Pi^A(\rho_{ABC}))^2 \right)$$

$$= \frac{1}{2} \left[ 1 - \left( |x_1(t)|^2 + |x_2(t)|^2 + |x_3(t)|^2 + |x_4(t)|^2 \right)^2 \right].$$

(22)
It is obvious that when $\gamma = \pm 1, \| \rho_{ABC} - \Pi^4(\rho_{ABC}) \|^2$ gets its minimum values. Therefore,

$$D_G(\rho_{ABC}) = 2 \min_{\Pi^4} (\| \rho_{ABC} - \Pi^4(\rho_{ABC}) \|^2)$$
$$= 1 - (1 - 2 |x_2(t)|^2 - 2 |x_3(t)|^2)^2$$
$$= 4 \left( |x_1(t)|^2 + |x_4(t)|^2 \right)$$
$$\times \left( |x_2(t)|^2 + |x_3(t)|^2 \right).$$

(23)

In the above equation, we have written it in a more symmetrical form in the last step by using $|x_1(t)|^2 + |x_2(t)|^2 + |x_3(t)|^2 + |x_4(t)|^2 = 1$. To show the evolution of $D_G(\rho_{ABC})$ with time $t$, we plot it as a function of $t$ for different $\alpha$ and $n$ in figure 5.

Figure 5 shows that $D_G(\rho_{ABC})$ vibrates with time $t$. When $\alpha = 0$, $D_G(\rho_{ABC})$ vibrates with a single frequency, otherwise, it has multi-frequency vibration. In order to get a better physical insight into the above phenomenon we take into account the power spectrum of $D_G(\rho_{ABC})$. Considering the frequency $\omega$ and time $t$ are positive, we use the Fourier transformation

$$FD_G(\omega) = \frac{1}{\sqrt{2\pi}} \int_0^{\infty} D_G(\rho_{ABC}) e^{i\omega t} dt$$

(24)

and make a log–log plot (a log–log plot effectively generates a curve in which log[f] is plotted against log[x], but with tick marks indicating the original values of f and x) of $FD_G(\omega)$ for $n = 3$ and $n = 0$ in figure 6. If we plot $FD_G(\omega)$ in the same way as in figure 6 for $n = 1, 2, 4, 5 \ldots$, we shall obtain similar graphics as represented in figure 6(a). These figures clearly show that the $FD_G(\omega) \sim \omega$ curves only have one very sharp peak for $\alpha = 0$, which is independent of $n$, but have four very sharp peaks for $n > 0$ and two very sharp peaks for $n = 0$ when $\alpha \neq 0$.

Getting the geometric discord of the state $\rho_{ABC}$ also enables us to study the monogamy of this state. The monogamy is an important property of a tripartite system. A correlation measure $Q$ is monogamous if and only if the following CKW monogamy inequality

$$Q_{A|BC} \geq Q_{AB} + Q_{AC}$$

(25)
holds for any tripartite state $\rho_{ABC}$ [39].

Using equations (18), (19) and (23), we obtain

$$D_G(\rho_{ABC}) - D_G(\rho_{AB}) - D_G(\rho_{AC}) = $$

$$\begin{cases} 
-2\left[|x_1(t)|^4 + |x_2(t)|^4 - 2|x_3(t)|^2|x_4(t)|^2\right] \\
-2|x_3(t)|^2\left(|x_2(t)|^2 + |x_3(t)|^2\right) \\
+\left(|x_3(t)|^2 - |x_4(t)|^2\right)^2 \\
\geq -2\left[|x_1(t)|^4 + |x_2(t)|^4\right] \\
-2|x_2(t)x_4(t)|^2 - 2 - |x_1(t)|^2|x_3(t)|^2 \\
-\left[\left(|x_3(t)|^2 - |x_5(t)|^2\right)^2 + \left(|x_5(t)|^2 - |x_4(t)|^2\right)^2\right] \\
+\left(|x_3(t)|^2 - |x_4(t)|^2\right)^2 \right) = 4|x_3(t)x_4(t)|^2 \geq 0, \\
\text{for } 2|x_1(t)|^2|x_2(t)|^2 \geq \left(|x_1(t)|^2\right)^2 \\
|\frac{x_3(t)}{|x_3(t)|}|^2 \left(|x_2(t)|^2 - |x_4(t)|^2\right)^2; \\
4|x_3(t)x_4(t)|^2 \geq 0, \\
\text{for } 2|x_1(t)|^2|x_2(t)|^2 \leq \left(|x_1(t)|^2\right)^2 \\
-|x_3(t)|^2 \left(|x_2(t)|^2 - |x_4(t)|^2\right)^2. \\
\end{cases}$$

(26)

In the above equation, we have replaced $2|x_1(t)|^2|x_2(t)|^2$ by $\left(|x_1(t)|^2 - |x_3(t)|^2\right)^2 + \left(|x_2(t)|^2 - |x_4(t)|^2\right)^2$, which is not greater than $2|x_1(t)|^2|x_2(t)|^2$, after the first greater-than-equal symbol on the right side of the large semi-brace. Equation (26) shows the inequality $D_G(\rho_{ABC}) \geq D_G(\rho_{AB}) + D_G(\rho_{AC})$ holds in the present situation.

Using the same procedure employed in deriving equation (23) and (18), we can get

$$D_G(\rho_{BAC}) = 2\min_{\Pi^B}\left(\|\rho_{ABC} - \Pi^B(\rho_{ABC})\|^2\right)$$

$$= 4\left(|x_1(t)|^2 + |x_3(t)|^2\right)^2\left(|x_2(t)|^2 + |x_4(t)|^2\right).$$

(27)

$$D_G(\rho_{BA}) = |x_1(t)|^4 + |x_2(t)|^4 + |x_3(t)|^4 + |x_4(t)|^4$$

$$+ 2|x_2(t)|^2\left(2|x_2(t)|^2 - |x_4(t)|^2\right)$$

$$- 2|x_2(t)|^2|x_3(t)|^2$$

$$-\text{Max}\left[2|x_2(t)|^2,|x_3(t)|^2\right]^4$$

$$+\left|x_2(t)^4 + |x_3(t)|^4 + |x_4(t)|^4\right.$$

$$- 2|x_2(t)|^2|x_3(t)|^2 - 2|x_1(t)|^2|x_4(t)|^2\right].$$

(28)

Combining above two equations and equation (20), we obtain

$$D_G(\rho_{BAC}) - D_G(\rho_{BA}) - D_G(\rho_{BC})$$

$$= \begin{cases} 
4\left|x_3(t)x_4(t)\right|^2 \geq 0, \\
\text{for } |x_1(t)|^4 + |x_2(t)|^4 + |x_3(t)|^4 + |x_4(t)|^4 \\
\geq 2\left(|x_2(t)|^2|x_3(t)|^2 + |x_1(t)|^2|x_2(t)|^2 \\
+ |x_3(t)|^2|x_4(t)|^2\right); \\
4\left|x_3(t)|^2 + |x_3(t)|^2\right)^2\left(|x_2(t)|^2 \\
+ |x_4(t)|^2\right)^2 \\
- 2\left(|x_1(t)|^4 + |x_2(t)|^4 \\
+ |x_3(t)|^4 + |x_4(t)|^4 \right. \\
\geq 4\left(|x_1(t)|^2 + |x_3(t)|^2\right)\left(|x_2(t)|^2 + |x_4(t)|^2\right) \\
- 2\left(|x_2(t)|^2|x_3(t)|^2 + |x_1(t)|^2|x_2(t)|^2 \\
+ |x_3(t)|^2|x_4(t)|^2\right)^2 \\
\geq 2\left(|x_1(t)|^2 + |x_3(t)|^2\right)\left(|x_2(t)|^2 + |x_4(t)|^2\right) \geq 0, \\
\text{for } |x_1(t)|^4 + |x_2(t)|^4 + |x_3(t)|^4 \\
+ |x_4(t)|^4 \right.$$

$$< 2\left(|x_2(t)|^2\left|x_3(t)|^2 + |x_1(t)|^2|x_2(t)|^2 \\
+ |x_3(t)|^2|x_4(t)|^2\right)^2. \end{cases}$$

(29)

Therefore, we can conclude that the geometric quantum discord of the state $\rho_{ABC}$ is monogamous when von Neumann measurements acting on the qubit $A$ or $B$ was carried out.

To conclude this section, it is worth pointing out that because up to now no approaches and methods to calculate the geometric measurements of quantum discord for a tripartite system have been reported, we have calculated the $D_G(\rho_{ABC})$ based on the original definition of the geometric measurements of quantum discord. However, there are at least three alternative approaches to accomplish this task. Firstly, we can generalize the formula $S = \text{Tr}_B(\rho^{AB})$ to $S = \text{Tr}_{BC}(\rho^{BB}) = \text{Tr}_C(\text{Tr}_B(\rho^{BB}))$, but keep equation (7) unchanged. Secondly, we can consider subsystem $BC$ as a four-level system when $n \geq 1$ and as a three-level system when $n = 0$. Specifically, we let $|0\psi\rangle = |0\psi_{BC}\rangle$, $|1\psi\rangle = |1\psi(n - 1)\rangle$, $|2\psi\rangle = |1\psi_{BC}\rangle$, $|3\psi\rangle = |0\psi(n + 1)\rangle$, then the wave-function of the system can be written as

$$|\psi_{4x}(t)\rangle = x_1(t)|0\Psi\rangle + x_2(t)|0\Psi\rangle$$

$$+ x_3(t)|0\Psi\rangle + x_4(t)|0\Psi\rangle. \quad (30)$$

(30)

We can now treat this equivalent $2 \times 4$ system by using the methods used in section 2. Thirdly, we can further rewrite
as
\[
|\Psi(t)\rangle = \sqrt{|x_1(t)|^2 + |x_4(t)|^2} |1_x 0_x\rangle \\
+ \sqrt{|x_2(t)|^2 + |x_3(t)|^2} |0_x 1_x\rangle ,
\]
where
\[
|0_x\rangle = \frac{x_1(t) |0_y n_c\rangle + x_4(t) |1_y (n-1) c\rangle}{\sqrt{|x_1(t)|^2 + |x_4(t)|^2}}, \\
|1_x\rangle = \frac{x_2(t) |1_y n_c\rangle + x_3(t) |0_y (n+1) c\rangle}{\sqrt{|x_2(t)|^2 + |x_3(t)|^2}}.
\]

The corresponding density matrix is
\[
\begin{pmatrix}
0 & 0 & \sqrt{\left(|x_1(t)|^2 + |x_4(t)|^2\right)\left(|x_2(t)|^2 + |x_3(t)|^2\right)} & 0 \\
0 & 0 & \sqrt{\left(|x_1(t)|^2 + |x_4(t)|^2\right)\left(|x_2(t)|^2 + |x_3(t)|^2\right)} & 0 \\
\sqrt{\left(|x_1(t)|^2 + |x_4(t)|^2\right)\left(|x_2(t)|^2 + |x_3(t)|^2\right)} & \sqrt{\left(|x_1(t)|^2 + |x_4(t)|^2\right)\left(|x_2(t)|^2 + |x_3(t)|^2\right)} & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\]

Equation (31) indicates that the total system ABC is equivalent to a two-qubit system AX. So we can now use all methods applicable to any two-qubit system to calculate the geometric measurements of quantum discord for the system AX. In particular, equation (33) shows that the equivalent system AX is an X-state, therefore, we can directly use the formula of X-states for the geometric measurements of quantum discord in [38] to obtain the geometric measurements of quantum discord of the total system. Of course, the results obtained by using above three methods are the same as those of equation (23). We stress that to calculate the geometric measurements of quantum discord of complex quantum systems, the first one of the above three approaches can be used for any multipartite quantum system that includes at least one qubit subsystem; the latter two are applicable to any pure quantum states that also contain at least one qubit subsystem.

5. Summary

In summary, we have calculated the geometric measurements of quantum discord for the subsystems AB, AC and BC and the whole system ABC. Our results demonstrate that the geometric measurement of quantum discord can show better evolutionary behavior of a quantum system especially when the entanglement is zero. So, the geometric measurement of quantum discord surpasses the entanglement to describe quantum correlation. In addition, we obtained the analytical expressions of the geometric measurement of quantum discord for \(2 \times n\) \((n = 0, 1, 2, 3, \ldots)\) subsystems AC and BC, which is contrasted with the situation in [36] where though the expressions of the negativity for subsystems AC and BC were given when \(n > 0\), the expressions of the concurrence for the same subsystems were not reported. Furthermore and more importantly, we studied the geometric measurement of quantum discord for the total system ABC and found that the correlation of the system ABC developed periodically if the initial state of the two atoms was not entangled; it oscillates with two or four frequencies according to whether \(n = 0\) or \(n > 0\), respectively. Finally, we put forward three alternative methods to calculate the geometric measurement of quantum discord of a tripartite quantum system including at least one qubit subsystem. The first one is an extension of Tufarelli’s approach. The other two make any tripartite quantum pure states including at least one qubit subsystem equivalent to qubit-qudit or two-qubit states. These approaches will greatly simplify the calculation of various measurements of quantum correlation of a quantum system including at least one qubit subsystem.

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