Random dynamics and thermodynamic limits for polygonal Markov fields in the plane

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Abstract: We construct random dynamics on collections of non-intersecting planar contours, leaving invariant the distributions of length- and area-interacting polygonal Markov fields with V-shaped nodes. The first of these dynamics is based on the dynamic construction of consistent polygonal fields, as presented in the original articles by Arak (1982) and Arak & Surgailis (1989, 1991), and it provides an easy-to-implement Metropolis-type simulation algorithm. The second dynamics leads to a graphical construction in the spirit of Fernández, Ferrari & Garcia (1998,2002) and it yields a perfect simulation scheme in a finite window from the infinite-volume limit. This algorithm seems difficult to implement, yet its value lies in that it allows for theoretical analysis of thermodynamic limit behaviour of length-interacting polygonal fields. The results thus obtained include the uniqueness and exponential α-mixing of the thermodynamic limit of such fields in the low temperature region, in the class of infinite-volume Gibbs measures without infinite contours. Outside this class we conjecture the existence of an infinite number of extreme phases breaking both the translational and rotational symmetries.

Keywords: Polygonal Markov fields, random dynamics, Metropolis simulation, perfect simulation, thermodynamic limit, phase transitions

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1 Introduction

An example of a planar Markov field with polygonal realisations was first introduced in Arak (1982). The original Arak process in a bounded open convex set $D$ is constructed as briefly sketched below. We define the family $\Gamma_D$ of admissible polygonal configurations on $D$ by taking all the finite planar graphs $\gamma$ in $D \cup \partial D$, with straight-line segments as edges, such that

(P1) the edges of $\gamma$ do not intersect,

(P2) all the interior vertices of $\gamma$ (lying in $D$) are of degree 2,

(P3) all the boundary vertices of $\gamma$ (lying in $\partial D$) are of degree 1,

(P4) no two edges of $\gamma$ are colinear.

In other words, $\gamma$ consists of a finite number of disjoint polygons, possibly nested and chopped off by the boundary. Further, for a finite collection $(l) = (l_i)_{i=1}^n$ of straight lines intersecting $D$ we write $\Gamma_D(l)$ for the family of admissible configurations $\gamma$ with the additional properties that $\gamma \subseteq \bigcup_{i=1}^n l_i$ and $\gamma \cap l_i$ is a single interval of a strictly positive length for each $l_i, i = 1, \ldots, n$, possibly with some isolated points added. Let $\Lambda_D$ be the restriction to $D$ of a homogeneous Poisson line process $\Lambda$ with intensity measure given by the standard isometry-invariant Lebesgue measure $\mu$ on the space of straight lines in $\mathbb{R}^2$. One possible construction of $\mu$ goes by identifying a straight line $l$ with the pair $(\rho, \varphi) \in [0, \pi) \times \mathbb{R}$, where $(\rho \sin(\varphi), \rho \cos(\varphi))$ is the vector orthogonal to $l$, and joining it to the origin, and then by endowing the parameter space $[0, \pi) \times \mathbb{R}$ with the usual Lebesgue measure. With the above notation, the polygonal Arak process $\mathcal{A}_D$ on $D$ arises as the Gibbsian modification of the process induced on $\Gamma_D$ by $\Lambda_D$, with the Hamiltonian given by the double total edge length, that is to say

$$\mathbb{P}(\mathcal{A}_D \in G) = \frac{\mathbb{E} \sum_{\gamma \in \Gamma_D(\Lambda_D) \cap G} \exp(-2 \text{length}(\gamma))}{\mathbb{E} \sum_{\gamma \in \Gamma_D(\Lambda_D)} \exp(-2 \text{length}(\gamma))}$$ (1)

for all $G \subseteq \Gamma_D$ Borel measurable, say with respect to the usual Hausdorff distance topology, see Section 4 in Arak & Surgailis (1989). The Arak process has a number of remarkable properties. It is exactly solvable (an explicit formula for the partition function is available), consistent ($\mathcal{A}_D$ coincides in distribution with the restriction of $\mathcal{A}_C$ to $D$ for $C \supseteq D$) and enjoys a two-dimensional Markov property stating that the conditional behaviour of the process in an
open bounded domain depends on the exterior configuration only through arbitrarily close neighbourhoods of the boundary, see ibidem. These nice features are shared by a much broader class of processes, so-called consistent polygonal Markov fields, introduced and investigated in detail in Arak & Surgailis (1989, 1991). Arak, Clifford & Surgailis (1993) introduce an alternative point-rather than line-based representation of these models. Our description below specialises for the standard Arak process $A_D$. For a given point configuration $\bar{x} = \{x_1, ..., x_n\} \subseteq D \cup \partial D$ denote by $\Gamma_D(\bar{x})$ the family of admissible configurations $\gamma$ whose vertex set coincides with $\bar{x}$. Write $\Pi_D$ for the Poisson point process in $D \cup \partial D$ with the intensity measure given by the area element on $D$ and by the length element on $\partial D$. By Theorem 1 ibidem (see also (2.6) there) the Arak process $A_D$ coincides with the Gibbsian modification of the process on $\Gamma_D$ induced by $\Pi_D$ with the Hamiltonian

$$\Phi(\gamma) := 2 \text{length}(\gamma) + \sum_{e \in E(\gamma)} \log \text{length}(e) - \sum_{x \in V(\gamma)} \log |\sin \phi_x|,$$

where $E(\gamma)$ and $V(\gamma)$ are, respectively, the edge and vertex sets of $\gamma$ while $\phi_x$ stands for the angle between the edges meeting in $x \in D$ and for the angle between the edge and the tangent to $\partial D$ at $x$ if $x \in \partial D$. This means that

$$P(A_D \in G) = \frac{\mathbb{E} \sum_{\gamma \in \Gamma_D(\Pi_D) \cap G} \exp(-\Phi(\gamma))}{\mathbb{E} \sum_{\gamma \in \Gamma_D(\Pi_D)} \exp(-\Phi(\gamma))}$$

(3)

for all Borel $G \subseteq \Gamma_D$. The third equivalent description of polygonal Markov fields is available in terms of equilibrium evolution of one-dimensional particle systems, tracing the polygonal realisations of the process in two-dimensional time-space. This description, usually referred to as the dynamic representation and introduced already in the original Arak work (1982), turned out to be very useful in establishing the essential properties of the models. Below, we discuss the dynamic representation for the Arak process, see Section 4 in Arak & Surgailis (1989). We interpret the open convex domain $D$ as a set of time-space points $(t, y) \in D$, with $t$ referred to as the time coordinate and with $y$ standing for the spatial coordinate of a particle at the time $t$. In this language, a straight line segment in $D$ stands for a piece of the time-space trajectory of a freely moving particle. For a straight line $l$ non-parallel to the time axis and crossing the domain $D$ we define in the obvious way its entry point to $D$, $\text{in}(l, D) \in \partial D$ and its exit point $\text{out}(l, D) \in \partial D$. We choose the time-space birth coordinates for the new particles according to a homogeneous intensity $\pi$ Poisson point process in $D$ (interior birth sites)
superposed with a Poisson point process on the boundary (boundary birth sites) with the intensity measure

\[ \kappa(B) = \mathbb{E} \text{card}\{l \in \Lambda, \text{ in}(l, D) \in B\}, \ B \subseteq \partial D. \]  \hspace{1cm} (4)

Each interior birth site emits two particles, moving with initial velocities \( v' \) and \( v'' \) chosen according to the joint distribution

\[ \theta(dv', dv'') := \pi^{-1} |v' - v''| (1 + v'^2)^{-3/2} (1 + v''^2)^{-3/2} dv'dv''. \] \hspace{1cm} (5)

This can be shown to be equivalent to choosing the directions of the straight lines representing the space-time trajectories of the emitted particles according to the distribution of the typical angle between two lines of \( \Lambda \), see Sections 3 and 4 in Arak & Surgailis (1989) and the references therein. Each boundary birth site \( x \in \partial D \) yields one particle with initial speed \( v \) determined according to the distribution \( \theta_x(dv) \) identified by requiring that the direction of the line entering \( D \) at \( x \) and representing the time-space trajectory of the emitted particle be chosen according to the distribution of a straight line \( l \in \Lambda \) conditioned on the event \{ \( x = \text{in}(l, D) \) \}.

All the particles evolve independently in time according to the following rules.

\textbf{(E1)} Between the critical moments listed below each particle moves freely with constant velocity so that \( dy = vdt \),

\textbf{(E2)} When a particle touches the boundary \( \partial D \), it dies,

\textbf{(E3)} In case of a collision of two particles (equal spatial coordinates \( y \) at some moment \( t \) with \( (t, y) \in D \)), both of them die,

\textbf{(E4)} The time evolution of the velocity \( v_t \) of an individual particle is given by a pure-jump Markov process so that

\[ \mathbb{P}(v_{t+dt} \in du \mid v_t = v) = q(v, du)dt \]

for the transition kernel

\[ q(v, du) := |u - v| (1 + u^2)^{-3/2} du dt. \]

It has been proven (see e.g. Lemma 4.1 in Arak & Surgailis (1989)) that with the above construction of the interacting particle system, the time-space trajectories
traced by the evolving particles coincide in distribution with the Arak process \( \mathcal{A}_D \). Moreover, a much broader class of consistent polygonal Markov fields admit analogous dynamic representations, possibly enhanced to allow vertices of higher degrees (3 and 4), see ibidem. The question of characterising the class of all polygonal Markov fields admitting dynamic representation is far from being trivial and a conjectured description of this class has been provided in Arak, Clifford & Surgailis (1993).

The above dynamic construction of the Arak process makes it very suitable for simulation. However, in the present paper we focus our interest on the family of processes \( \hat{\mathcal{A}}_{D}^{[\alpha,\beta]} \), \( \alpha, \beta \in \mathbb{R} \), arising as the Ising-like length- and area-interacting Gibbsian modifications of \( \mathcal{A}_D \). To this end we colour the original Arak process \( \mathcal{A}_D \) as follows. Requiring that the polygonal contours of \( \mathcal{A}_D \) stand for interfaces between black- and white-coloured regions in \( D \) leaves us almost surely with two possible ways of colouring \( D \) in black and white, arising from each other by a simple colour flip. We choose one of these colourings at random, with probability \( 1/2 \), thus obtaining a coloured version of \( \mathcal{A}_D \), denoted in the sequel by \( \hat{\mathcal{A}}_D \). The family of all admissible coloured polygonal configurations in \( D \), carrying information not only about the planar contours it consists of, but also about the associated colouring, will be denoted by \( \hat{\Gamma}_D \).

With this notation and terminology we define the (coloured) processes \( \hat{\mathcal{A}}_{D}^{[\alpha,\beta]} \) by

\[
\frac{d\mathcal{L}(\hat{\mathcal{A}}_{D}^{[\alpha,\beta]})}{d\mathcal{L}(\mathcal{A}_D)}(\hat{\gamma}) := \frac{\exp(-\mathcal{H}_{D}^{[\alpha,\beta]}(\hat{\gamma}))}{\mathbb{E}\exp(-\mathcal{H}_{D}^{[\alpha,\beta]}(\mathcal{A}_D))}, \quad \hat{\gamma} \in \hat{\Gamma}_D, \tag{6}
\]

with \( \mathcal{L}(\cdot) \) denoting the law of the argument random object and with

\[
\mathcal{H}_{D}^{[\alpha,\beta]}(\hat{\gamma}) := \alpha A(\text{black}[\hat{\gamma}]) + \beta \text{length}(\hat{\gamma}), \tag{7}
\]

where \( \text{black}[\hat{\gamma}] \) is the black-coloured region in \( D \) for \( \hat{\gamma} \) while \( A(\cdot) \) stands for the area measure. We also write \( \mathcal{A}_{D}^{[\alpha,\beta]} \) for the contour ensemble of \( \hat{\mathcal{A}}_{D}^{[\alpha,\beta]} \), with the colours 'forgotten' and, likewise, \( \gamma \) for the colourless version of \( \hat{\gamma} \in \hat{\Gamma}_D \). Note that using the symmetry between black and white and possibly flipping the colours, whenever convenient we may assume without loss of generality that \( \alpha \geq 0 \) (and we do so in the proof of Theorem 3 below).

Observe that the modifications of the type (6) fall into the general setting considered by Arak & Surgailis (1989) only for \( \beta \geq 0 \), see Corollary 4.1 there. However, we find it natural to admit also negative \( \beta \)'s since there is no obvious infinite temperature non-interacting field available as the reference object for
polygonal Markov fields. Consequently, in the sequel we will abuse the language by referring to large positive values of $\beta$ as to the low temperature region, and to small, possibly negative $\beta$'s as to the high temperature regime. For $\beta < 0$ one has to check that the partition function $\mathbb{E} \exp \left( -\mathcal{H}_D^{[\alpha,\beta]}(\hat{A}_D) \right)$ is finite. In Corollary 2 we show that this is indeed the case and, consequently, the definition (6) is correct for all $\beta \in \mathbb{R}$. Clearly, there are no such problems for $\alpha$, since the overall black or white area is deterministically bounded by $A(D)$. It should be emphasised though that at present we are able to establish the existence of the thermodynamic limit only for $\beta > 0$, see Theorem 3.

Models of the type (6) have recently found interest in the physical literature, see Nicholls (2001). In particular, it has been argued that they exhibit a phase transition similar to that of the planar Ising model, with the low temperature phase admitting only finite contour nesting (as rigorously shown in Nicholls (2001)), and with the high temperature phase conjectured (not yet proven) to exhibit infinite contour nesting.

Below, we shall also consider versions of the above models with empty boundary conditions, arising by conditioning the original model on the event of there being no vertices on the boundary, so that

\[
\mathcal{L} \left( \hat{A}_D^{[\alpha,\beta]}(\emptyset) \right) := \mathcal{L} \left( \hat{A}_D^{[\alpha,\beta]} \bigg| A_D^{[\alpha,\beta]} \cap \partial D = \emptyset \right).
\]

In particular,

\[
\hat{A}_D^{[\alpha,\beta]}(\emptyset) := \hat{A}_D^{[0,0]}.
\]

Likewise, we shall consider versions of these models with black (or white) boundary conditions given by

\[
\mathcal{L} \left( \hat{A}_D^{[\alpha,\beta]}(\text{black(white)}) \right) := \mathcal{L} \left( \hat{A}_D^{[\alpha,\beta]} \bigg| A_D^{[\alpha,\beta]} \cap \partial D = \emptyset, \partial D \text{ is black (white)} \right)
\]

with

\[
\hat{A}_D^{[\alpha,\beta]}(\text{black(white)}) := \hat{A}_D^{[0,0]}(\text{black(white)}).
\]

As a direct conclusion from (6) we get

\[
\frac{d\mathcal{L}(\hat{A}_D^{[\alpha,\beta]}_{bd})}{d\mathcal{L}(\hat{A}_D^{[\alpha,\beta]}_{bd})}(\hat{\gamma}) = \frac{\exp \left( -\mathcal{H}_D^{[\alpha,\beta]}(\hat{\gamma}) \right)}{\mathbb{E} \exp \left( -\mathcal{H}_D^{[\alpha,\beta]}(\hat{A}_D^{[\alpha,\beta]}_{bd}) \right)}, \quad \hat{\gamma} \in \hat{\Gamma}_D, \ \gamma \cap \partial D = \emptyset
\]

for $bd \in \{\emptyset, \text{black, white}\}$. Observe that, unlike the unconditioned finite-volume fields $\hat{A}_D^{[\alpha,\beta]}(\emptyset, \alpha \neq 0)$, the conditioned fields with monochromatic boundary conditions are well defined also for non-convex bounded open $D$ with piecewise
smooth boundary. Indeed, take any bounded open convex set $D'$ containing $D$ and set $A^{[\alpha, \beta]}_{D|bd}$, $bd \in \{\text{black, white}\}$, to coincide with $A^{[\alpha, \beta]}_{D'}$ conditioned on the event that no edge hits $\partial D$ and that the colour on $\partial D$ agrees with that specified by $bd$. The Markov property of polygonal fields (see Arak & Surgailis (1989)) implies that this construction does not depend on the choice of $D'$. Note that this argument does not apply for the empty boundary condition $bd = \emptyset$, unless $\alpha = 0$.

The purpose of this paper is to construct for $\alpha, \beta \in \mathbb{R}$ a family of random dynamics on $\hat{\Gamma}_D$ which leave the distribution of $A^{[\alpha, \beta]}_D$ invariant. This yields simulating algorithms for $A^{[\alpha, \beta]}_D$, both of the Metropolis type and of perfect type in the spirit of Fernández, Ferrari & Garcia (1998,2002). While the Metropolis algorithm is given for all $\alpha, \beta \in \mathbb{R}$ and can be readily implemented (which is a subject of our work in progress), the perfect scheme is restricted to $\alpha = 0$ and seems more difficult to implement, yet its value lies mainly in that it provides important theoretical information about the thermodynamic limit behaviour of $A^{[0, \beta]}_D$ in the low temperature region (for large $\beta$) and in that it can be used to simulate in finite windows directly from the thermodynamic limit. The finite volume dynamics are discussed in the next Section 2. In Section 3 we discuss infinite-volume thermodynamic limits of polygonal fields and establish their existence. For $\alpha = 0$ and $\beta$ large enough one of our dynamics, constructed in Subsection 2.2 below, admits an infinite-volume extension and, as mentioned above, it yields a perfect simulation scheme which enables us to show in Section 4 that for $A^{[0, \beta]}_D$ there exists exactly one thermodynamic limit without infinite chains, as made specific below, and that this limit is isometry invariant as well as exponentially $\alpha$-mixing. In particular, it follows that in the class of infinite-volume measures without infinite chains there exist exactly two extremal infinite-volume Gibbs measures for $A^{[0, \beta]}_D$, the black-dominated and white-dominated phase, corresponding to the same contour distribution. In this context it should be noted that this simple picture does not seem to extend to the whole simplex of infinite-volume Gibbs measures for $A^{[0, \beta]}_D$: we conjecture the existence and sketch, in Section 3 below, a tentative construction of an infinite number of infinite-volume states admitting infinite chains and breaking both the translational and rotational symmetry.

As already mentioned above, the implementation of the algorithms described in this paper is a subject of our current work in progress. It should be emphasised that an algorithm for simulating polygonal Markov fields, very different
than ours, has already been given in the literature by Clifford & Nicholls (1994).

2 Finite volume dynamics

Below we construct two families of random dynamics which leave invariant the
laws of the Gibbs-modified polygonal random fields $\hat{A}_D^{[\alpha,\beta]}$ in a bounded open
convex domain $D \subseteq \mathbb{R}^2$. First of these dynamics, leading to a practically feasible
and easy to implement Metropolis-type simulation algorithm, is based on the
dynamic representation of the Arak process. The second one relies mainly on
the point- and line-based representation of general polygonal Markov fields and,
after some additional work, leads to a graphical construction and a perfect
algorithm discussed in Section 4. We postpone the proof of the finiteness of the
partition function in (6) to Corollary 2 below.

2.1 Disagreement loop birth and death dynamics

An important concept below will be that of a disagreement loop, borrowed from
Schreiber (2004), Section 2.2. This arises from the dynamic construction of the
Arak process as provided by the evolution rules (E1-4) with the corresponding
birth rules, see (4) and (5).

Suppose that we observe a particular realisation $\gamma \in \Gamma_D$ of the colourless
basic Arak process $A_D$ and that we modify the configuration by adding an
extra birth site $x_0$ to the existing collection of birth sites for $\gamma$, while keeping
the evolution rules (E1-4) for all the particles, including the two newly
added ones if $x_0 \in D$ and the single newly added one if $x_0 \in \partial D$. Denote the
resulting new random (colourless) polygonal configuration by $\gamma \oplus x_0$.
A simple
yet crucial observation is that for $x_0 \in D$ the symmetric difference $\gamma \triangle \gamma \oplus x_0$ is
almost surely a single loop (a closed polygonal curve), possibly self-intersecting
and possibly chopped off by the boundary. Indeed, this is seen as follows. The
leftmost point of the loop $\gamma \triangle \gamma \oplus x_0$ is of course $x_0$. Each of the two new particles
$p_1, p_2$ emitted from $x_0$ move independently, according to (E1 – 4), each giving
rise to a disagreement path. The initial segments of such a disagreement path
correspond to the movement of a particle, say $p_1$, before its annihilation in the
first collision. If this is a collision with the boundary, the disagreement path
gets chopped off and terminates there. If this is a collision with a segment
of the original configuration $\gamma$ corresponding to a certain old particle $p_3$, the
new particle $p_1$ dies but the disagreement path continues along the part of the
trajectory of \( p_3 \) which is contained in \( \gamma \) but not in \( \gamma \oplus x_0 \). At some further moment \( p_3 \) dies itself in \( \gamma \), touching the boundary or killing another particle \( p_4 \) in \( \gamma \). In the second case, however, this collision only happens for \( \gamma \) and not for \( \gamma \oplus x_0 \) so the particle \( p_4 \) survives (for some time) in \( \gamma \oplus x_0 \) yielding a further connected portion of the disagreement path for \( p_1 \), which is contained in \( \gamma \oplus x_0 \) but not in \( \gamma \) etc. A recursive continuation of this construction shows that the disagreement path initiated by \( p_1 \) consists alternately of connected polygonal subpaths contained in \( [\gamma \oplus x_0] \setminus \gamma \) (call these positive parts) and in \( \gamma \setminus [\gamma \oplus x_0] \) (call these negative parts). Note that this disagreement path is self-avoiding and, in fact, it can be represented as the graph of some piecewise linear function \( t \mapsto y(t) \). Clearly, the same applies for the disagreement path initiated by \( p_2 \).

An important observation is that whenever two positive or two negative segments of the two disagreement paths hit each other, both disagreement paths die at this point and the disagreement loop closes (as opposed to intersections of segments of distinct signs which do not have this effect). Obviously, if the disagreement loop does not close in the above way, it gets eventually chopped off by the boundary. We shall write \( \Delta^{\oplus}[x_0;\gamma] = \gamma \Delta[\gamma \oplus x_0] \) to denote the (random) disagreement loop constructed above. It remains to consider the case \( x_0 \in \partial D \), which is much simpler because there is only one particle emitted and so \( \Delta^{\oplus}[x_0;\gamma] = \gamma \Delta[\gamma \oplus x_0] \) is a single self-avoiding polygonal path eventually chopped off by the boundary. We abuse the language calling such \( \Delta^{\oplus}[x_0;\gamma] \) a (degenerate) disagreement loop as well.

Likewise, a disagreement loop arises if we remove one birth site \( x_0 \) from the collection of birth sites of an admissible polygonal configuration \( \gamma \in \Gamma_D \), while keeping the evolution rules for all the remaining particles. We write \( \gamma \ominus x_0 \) for the configuration obtained from \( \gamma \) by removing \( x_0 \) from the list of the birth sites, while the resulting random disagreement loop is denoted by \( \Delta^{\ominus}[x_0;\gamma] \) so that \( \Delta^{\ominus}[x_0;\gamma] = \gamma \Delta[\gamma \ominus x_0] \).

With the above terminology we are in a position to describe a random dynamics on the coloured configuration space \( \hat{\Gamma}_D \), which leaves invariant the law of the basic Arak process \( \hat{A}_D \). Particular care is needed, however, to distinguish between the notion of time considered in the dynamic representation of the Arak process as well as throughout the construction of the disagreement loops above, and the notion of time to be introduced for the random dynamics on \( \hat{\Gamma}_D \) constructed below. To make this distinction clear we shall refer to the former as to the representation time (r-time for short) and shall keep for it the notation
while the latter will be called the *simulation time* (s-time for short) and will
be consequently denoted by \( s \) in the sequel.

Consider the following pure jump birth and death type Markovian dynamics
on \( \hat{\Gamma}_D \).

**(DL:birth)** With intensity \( \pi \, dx + \kappa \, (dx) \) set \( \gamma_{s+ds} := \gamma_s \oplus x \) for \( \kappa \) as in (4),
then construct \( \hat{\gamma}_{s+ds} \) by randomly choosing, with probability \( 1/2 \),
either of the two possible colourings for \( \gamma_{s+ds} \).

**(DL:death)** For each birth site \( x \) in \( \gamma_s \) with intensity 1 set \( \gamma_{s+ds} := \gamma_s \ominus x \),
then construct \( \hat{\gamma}_{s+ds} \) by randomly choosing, with probability \( 1/2 \),
either of the two possible colourings for \( \gamma_{s+ds} \).

If none of the above updates occurs we keep \( \hat{\gamma}_{s+ds} = \hat{\gamma}_s \). It is convenient to
perceive the above dynamics in terms of generating random disagreement loops \( \lambda \) and setting \( \gamma_{s+ds} := \gamma_s \triangle \lambda \), with the loops of the type \( \Delta^\oplus[\cdot,\cdot] \) corresponding
to the rule *(DL:birth)* and \( \Delta^\ominus[\cdot,\cdot] \) to the rule *(DL:death)*.

As an direct consequence of the dynamic representation of the Arak process \( \hat{A}_D \) we obtain

**Proposition 1** The distribution of the Arak process \( \hat{A}_D \) is the unique invariant
law of the dynamics given by *(DL:birth)* and *(DL:death)*. The resulting
stationary process is reversible. Moreover, for any initial distribution of \( \hat{\gamma}_0 \) the laws of the random polygonal fields \( \hat{\gamma}_s \) converge in variational distance to the
law of \( \hat{A}_D \) as \( s \to \infty \).

The uniqueness and convergence statements in the above proposition require a
short justification. They both follow by the observation that, in finite volume,
regardless of the initial state, the process \( \hat{\gamma}_s \) spends a non-null fraction of time
in the state 'black' (no contours, the whole domain \( D \) coloured black). Indeed,
this observation allows us to conclude the required uniqueness and convergence
by a standard coupling argument.

Below, we show that the laws of the Gibbs-modified polygonal fields \( \hat{A}_D^{[\alpha,\beta]} \)
arise as the unique invariant distributions for appropriate modifications of the
reference dynamics *(DL:birth)*, *(DL:death)*. The main change is that the
birth and death updates are no more performed unconditionally, they pass an
acceptance test instead and are accepted with certain state-dependent probabili-
ties, upon failure of the acceptance test the update is discarded. For \( a \geq 0, b \geq 0 \)
and \( \alpha + a \geq 0, \beta + b \geq 0 \) consider the following dynamics
(DL : birth[α, β; a, b]) With intensity \([πdx + κ(dx)]ds\) do

- put \(δ := γ_s ⊕ x\),
- construct \(\hat{δ}\) by randomly choosing, with probability 1/2, either of the two possible colourings for \(δ\),
- accept \(\hat{δ}\) with probability
  \[
  \exp \left( -αA \left( \text{black}[\hat{δ}] \setminus \text{black}[\hat{γ}_s] \right) - β \text{length}(δ \setminus γ_s) \right) 
  \exp \left( -aA \left( \text{black}[\hat{δ}] \Delta \text{black}[\hat{γ}_s] \right) - b \text{length}(δ \Delta γ_s) \right),
  \]
- if accepted, set \(\hat{γ}_{s+ds} := \hat{δ}\), otherwise keep \(\hat{γ}_{s+ds} := \hat{γ}_s\).

(DL : death[α, β; a, b]) For each birth site \(x\) in \(γ_s\) with intensity 1 do

- put \(δ := γ_s ⊖ x\),
- construct \(\hat{δ}\) by randomly choosing, with probability 1/2, either of the two possible colourings for \(δ\),
- accept \(\hat{δ}\) with probability
  \[
  \exp \left( -αA \left( \text{black}[\hat{δ}] \setminus \text{black}[\hat{γ}_s] \right) - β \text{length}(δ \setminus γ_s) \right) 
  \exp \left( -aA \left( \text{black}[\hat{δ}] \Delta \text{black}[\hat{γ}_s] \right) - b \text{length}(δ \Delta γ_s) \right),
  \]
- if accepted, set \(\hat{γ}_{s+ds} := \hat{δ}\), otherwise keep \(\hat{γ}_{s+ds} := \hat{γ}_s\).

In analogy with its original reference form (DL:birth), (DL:death), the above dynamics should be thought of as generating random disagreement loops \(λ\) and setting \(γ_{s+ds} := γ_Δλ\) provided \(λ\) passes the acceptance test. It should be emphasised that the random disagreement loops above are generated according to the dynamic representation of the original Arak process \(A_D\). The following theorem justifies the above construction.

**Theorem 1** For each \(a ≥ 0, b ≥ 0\) and \(α + a ≥ 0, β + b ≥ 0\) the law of the Gibbs-modified Arak process \(\hat{A}_D^{[α, β]}\) is the unique invariant distribution of the dynamics (DL : birth[α, β; a, b]), (DL : death[α, β; a, b]). The resulting stationary process is reversible. For any initial distribution of \(γ_0\) the laws of the random polygonal fields \(\hat{γ}_s\) converge in variational distance to the law of \(\hat{A}_D^{[α, β]}\) as \(s → ∞\).
Theorem 1 can be easily concluded from Proposition 1 by a straightforward check of the detailed balance conditions. We chose, however, to provide below a geometric proof of this result for the case $\alpha, \beta \geq 0$, revealing, in our opinion, the geometric intuition underlying the dynamics (a similar proof can be provided for $\alpha < 0$ or $\beta < 0$ as well). Note that the reason for introducing the additional parameters $a$ and $b$ with the possibility that $a > 0, b > 0, \alpha + a > 0$ and $\beta + b > 0$ was to gain direct control over the diameter of the region affected by a single update, which decays exponentially in the current dynamics. The control of the diameter of the affected region is a condition sine qua non for possible infinite volume extensions of the $\{\text{DL : } \ldots[\alpha, \beta; a, b]\}$ dynamics, which is the subject of our current work in progress. Clearly, we could also have chosen another standard set of acceptance probabilities conforming to the detailed balance conditions, e.g. we could accept a transition $\hat{\gamma}_s \mapsto \hat{\gamma}_s + ds := \delta$ with probability $\min \left(1, \exp(\mathcal{H}_{D}^{[\alpha, \beta]}(\hat{\gamma}_s) - \mathcal{H}_{D}^{[\alpha, \beta]}(\delta)) \right)$ and a direct check of the detailed balance conditions, based on Proposition 1, would show that the law of $\hat{A}_{D}^{[\alpha, \beta]}$ is invariant with respect to such a dynamics. However, in this dynamics, in general we cannot efficiently control the size of the region affected in a single update.

Versions of the disagreement loop birth and death dynamics can be easily constructed which leave invariant the distributions of the polygonal fields $\hat{A}_{D}^{[\alpha, \beta]}|\emptyset (\hat{A}_{D}^{[\alpha, \beta]}|\text{black}, \hat{A}_{D}^{[\alpha, \beta]}|\text{white})$ with empty (black,white) boundary conditions respectively. To this end, we modify accordingly the dynamics $\{\text{DL : birth}[\alpha, \beta; a, b]\}$ and $\{\text{DL : death}[\alpha, \beta; a, b]\}$ by discarding all the updates which make the contour collection $\gamma_s$ hit the boundary, and for the monochromatic black or white boundary condition, in addition, upon an update we do not pick the colouring by random but we choose the unique one compatible with the boundary condition. Denoting the so constructed dynamics by $\{\text{DL}_0 : \ldots[\alpha, \beta; a, b]\}$, $\{\text{DL}_{\text{black}} : \ldots[\alpha, \beta; a, b]\}$ and $\{\text{DL}_{\text{white}} : \ldots[\alpha, \beta; a, b]\}$ respectively, we immediately conclude the following corollary from Theorem 1.

**Corollary 1** For each $a \geq 0, b \geq 0$ and $\alpha + a \geq 0, \beta + b \geq 0$, the law of the Gibbs-modified Arak process $\hat{A}_{D}^{[\alpha, \beta]}|\emptyset (\hat{A}_{D}^{[\alpha, \beta]}|\text{black}, \hat{A}_{D}^{[\alpha, \beta]}|\text{white})$ is the unique invariant distribution of the dynamics $\{\text{DL}_0 : \ldots[\alpha, \beta; a, b]\}$ $\{\text{DL}_{\text{black}} : \ldots[\alpha, \beta; a, b]\}$ or $\{\text{DL}_{\text{white}} : \ldots[\alpha, \beta; a, b]\}$ respectively. The resulting stationary processes are reversible. For any initial distribution of $\hat{\gamma}_0$ the laws of the random polygonal fields $\hat{\gamma}_s$ converge in variational distance to the law of $\hat{A}_{D}^{[\alpha, \beta]}|\emptyset (\hat{A}_{D}^{[\alpha, \beta]}|\text{black}, \hat{A}_{D}^{[\alpha, \beta]}|\text{white})$ respectively, as $s \to \infty$. 

We believe that a very similar dynamics could be used to simulate length- and area-interacting modifications of more general consistent polygonal Markov fields admitting the dynamic representation as discussed in Arak & Surgailis (1989, 1991) and Clifford, Arak & Surgailis (1993). The only change would be an appropriate redefinition of the operations $\Delta^\oplus[\cdot;\cdot]$ and $\Delta^\ominus[\cdot;\cdot]$, and the resulting disagreement field would no more be a single loop.

2.2 Contour birth and death dynamics

As already mentioned, unlike the previous one, the dynamics discussed in this subsection is constructed in a much narrower setting, restricted to colourless contour configurations which do not hit the boundary, and it is meant to leave invariant the distributions of $A_{D|\emptyset}^{[0,\beta]}$. Recall from the discussion following (10) that in this setting we can take $D$ to be an arbitrary bounded open set in $\mathbb{R}^d$, with piecewise smooth boundary, we do not need convexity. The approach developed in this section leads to a simulation algorithm discussed in Section 4 below, which, though perfect, seems to be practically infeasible due to non-constructive description of the intensity measure of contour births. However, its value lies in that its infinite volume extension provides important theoretical information about the thermodynamic limit $A_{D|\emptyset}^{[0,\beta]}$, yielding in particular the uniqueness of the thermodynamic limit for $\beta$ large enough. Observe that the dynamics constructed in this section could be in principle also used directly for Metropolis sampling, yet the previous disagreement loop dynamics seems much better suited for this particular purpose.

To proceed, we consider the space $C_D$ consisting of all closed polygonal contours in $D$ which do not touch the boundary $\partial D$. For a given point configuration $\bar{x} := \{x_1, \ldots, x_n\}$ we denote by $C_D(\bar{x})$ the family of those polygonal contours in $C_D$ which belong to $\Gamma_D(\bar{x})$, i.e. whose vertex set coincides with $\bar{x}$. We construct the so-called free contour measure $\Theta_D$ on $C_D$ by putting for $C \subseteq C_D$ measurable, say, with respect to the Borel $\sigma$-field generated by the Hausdorff distance topology,

$$\Theta_D(C) := \int_{\text{Fin}(D)} \sum_{\theta \in C \cap C_D(\bar{x})} \exp(-\Phi(\theta)) \nu^*(d\bar{x})$$

(11)

with the Hamiltonian $\Phi$ as in (2), with $\text{Fin}(D)$ standing for the family of finite point configurations in $D$ and where $\nu^*$ is the measure on $\text{Fin}(D)$ given by $d\nu^*(\bar{x}) := dx_1 \ldots dx_n$. In order to provide an alternative line- rather than
point-based expression for $\Theta_D$, for a given finite configuration $(l) := (l_1, \ldots, l_n)$ of straight lines intersecting $D$ denote by $C_D(l)$ the family of those polygonal contours in $C_D$ which belong to $\Gamma_D(l)$. Then we have, see e.g. (3.8) in the proof of Theorem 1 in Arak, Clifford & Surgailis (1993),

$$\Theta_D(C) = \int_{\text{Fin}(L[D])} \sum_{\theta \in C \cap C_D(l)} \exp(-2\text{length}(\theta)) d\mu^*((l))$$  \hspace{1cm} (12)

with $\text{Fin}(L[D])$ standing for the family of finite line configurations intersecting $D$ and where $\mu^*$ is the measure on $\text{Fin}(L[D])$ given by $d\mu^*((l_1, \ldots, l_n)) := d\mu(l_1) \ldots d\mu(l_n)$ with $\mu$ defined in the discussion preceding (1).

For $\beta \in \mathbb{R}$ we consider the exponential modification $\Theta^{[\beta]}_D$ of the free measure $\Theta_D$, given by

$$\Theta^{[\beta]}_D(d\theta) := \exp(-\beta \text{length}(\theta)) \Theta_D(d\theta).$$  \hspace{1cm} (13)

It is easily seen that the total mass $\Theta^{[\beta]}_D(C_D)$ is always finite. Indeed, using (12), taking into account that the length of a line segment in $D$ can be at most $\text{diam}(D)$ and recalling that, by standard integral geometry, $M := \mu(\{l \mid l \cap D \neq \emptyset\}) \leq \text{length}(\partial \text{conv}(D))$ we conclude that

$$\Theta^{[\beta]}_D(C_D) \leq \sum_{k=0}^{\infty} \frac{M^k \exp(k|\beta| \text{diam}(D))}{k!} \leq \exp[\text{length}(\partial \text{conv}(D)) \exp(|\beta| \text{diam}(D))] < \infty.$$  \hspace{1cm} (14)

Let $\mathcal{P}_{\Theta^{[\beta]}_D}$ be the Poisson point process on $C_D$ with intensity measure $\Theta^{[\beta]}_D$. It then follows directly by (11), by the point-based representation (3) and by (8) that for all $\beta \in \mathbb{R}$ for which the partition function $\mathbb{E} \exp\left(-\mathcal{H}^{[\alpha,\beta]}_D(A_D|\emptyset)\right)$ in (10) is finite (in fact, we show that this holds for all $\beta \in \mathbb{R}$ in Corollary 2 below), the polygonal field $A^{[0,\beta]}_{D|\emptyset}$ coincides in distribution with the union of contours in $\mathcal{P}_{\Theta^{[\beta]}_D}$ conditioned on the event that they are disjoint so that

$$\mathcal{L}\left(A^{[0,\beta]}_{D|\emptyset}\right) = \mathcal{L}\left(\bigcup_{\theta \in \mathcal{P}_{\Theta^{[\beta]}_D}} \theta \mid \forall \theta, \theta' \in \mathcal{P}_{\Theta^{[\beta]}_D} \theta \neq \theta' \Rightarrow \theta \cap \theta' = \emptyset\right),$$  \hspace{1cm} (15)

where the conditioning is well defined in view of (14). In particular, taking into account (1) and (12), we have for all $\beta$ where (10) makes sense

$$\mathbb{P}\left(\forall \theta, \theta' \in \mathcal{P}_{\Theta^{[\beta]}_D} \theta \neq \theta' \Rightarrow \theta \cap \theta' = \emptyset\right) = \mathbb{E} \sum_{\delta \in \Gamma_D(\emptyset)} \exp(-2|\beta| \text{length}(\delta))$$

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with \( \Gamma_{D|\emptyset} \) standing for the family of admissible polygonal configurations in \( D \) which do not touch \( \partial D \). It easily follows that the law of \( A_{D|\emptyset}^{[0,\beta]} \) is invariant and reversible with respect to the following contour birth and death dynamics \((\gamma_s)_{s \geq 0}\) on \( \Gamma_{D|\emptyset} \).

**\( (C : \text{birth}[\beta]) \)** With intensity \( \Theta_D^{[\beta]}(d\theta)ds \) do

- Choose a new contour \( \theta \),
- If \( \theta \cap \gamma_s = \emptyset \), accept \( \theta \) and set \( \gamma_{s+ds} := \gamma_s \cup \theta \),
- Otherwise reject \( \theta \) and keep \( \gamma_{s+ds} := \gamma_s \).

**\( (C : \text{death}[\beta]) \)** With intensity 1 for each contour \( \theta \in \gamma_s \) remove \( \theta \) from \( \gamma_s \) setting \( \gamma_{s} := \gamma_s \setminus \theta \).

It is worth noting that, should we accept all the new-coming contours without the disjointness test in the above dynamics, we would get the Poisson contour process \( \mathcal{P}_{\Theta_D} \) as the stationary state.

Observing that the process \( \gamma_s \) constructed above spends a non-null fraction of time in the state \( \emptyset \) and using a standard coupling argument we are led to

**Theorem 2** The law of the Gibbs-modified Arak process \( A_{D|\emptyset}^{[0,\beta]} \) is the unique invariant distribution of the dynamics \((C : \text{birth}[\beta]), (C : \text{death}[\beta])\). The resulting stationary process is reversible. For any initial distribution of \( \gamma_0 \) the laws of random polygonal fields \( \gamma_s \) converge in variational distance to the law of \( A_{D|\emptyset}^{[0,\beta]} \) as \( s \to \infty \).

All our results in this section were conditional on the partition function in (10) being finite. We claim here that this holds for all \( \beta \in \mathbb{R} \). Indeed, since \( \beta = 0 \) clearly satisfies this condition as corresponding to the basic Arak process \( A_{D|\emptyset} \), Theorem 2 can be used for \( \beta = 0 \). The dynamics \((C)\) above implies that the empty-boundary Arak process \( A_{D|\emptyset} \) is stochastically dominated (in the sense of inclusion) by the union of contours in \( \mathcal{P}_{\Theta_D} \), see Corollary 5 below. In particular, by (14), for all \( \alpha, \beta \in \mathbb{R} \)

\[
\mathbb{E} \exp \left( -\mathcal{H}_D^{[\alpha,\beta]}(A_{D|\emptyset}) \right) \leq \exp(|\alpha| A(D)) \mathbb{E} \exp \left( |\beta| \sum_{\theta \in \mathcal{P}_{\Theta_D}} \text{length}(\theta) \right) = \\
\exp(|\alpha| A(D)) \exp (-\Theta_D(C_D)) \exp \left( \Theta_D^{[\beta]}(C_D) \right) < \infty.
\]
By an appropriate redefinition of \( \Theta_D \) admitting edges chopped off by the boundary, the same argument can be repeated for \( A_{D|\emptyset} \) replaced by \( A_D \). Thus, we have proven

**Corollary 2** For each bounded open domain \( D \subseteq \mathbb{R}^2 \) both the partition functions \( \mathbb{E} \exp \left( -H_D^{[\alpha, \beta]}(\hat{A}_D) \right) \) in (6) and \( \mathbb{E} \exp \left( -H_D^{[\alpha, \beta]}(\hat{A}_{D|\emptyset}) \right) \) in (10) are finite for all \( \alpha, \beta \in \mathbb{R} \).

### 3 Thermodynamic limit

The purpose of this section is to define the notion of thermodynamic limit for the considered polygonal fields and to establish its existence (cf. Surgailis (1991)).

For a smooth closed simple (non-intersecting) curve \( c \) in \( D \) by the trace of a polygonal configuration \( \hat{\gamma} \) on \( c \), denoted in the sequel by \( \hat{\gamma} \wedge c \), we mean the knowledge of

- intersection points and intersection directions of \( \hat{\gamma} \) with \( c \),
- colouring of points of \( c \).

This concept can be formalised in various compatible ways, yet we keep the above informal definition in hope that it does not lead to any ambiguities while allowing us to avoid unnecessary technicalities. For convenience we assume that no edge of \( \hat{\gamma} \) is tangent to \( c \), which can be ensured with probability 1 in view of the smoothness of \( c \).

Fix \( \alpha, \beta \in \mathbb{R} \). In view of the Gibbsian representations (1), (3) and (6) we easily check that for each \( c \) as above and with \( \hat{\theta} \) standing for a trace on \( c \) there exists a stochastic kernel \( \hat{A}_{D|\emptyset}^{[\alpha, \beta]}(\cdot | \hat{\theta}) \) with the property that

\[
L_{\text{Int } c} \left( \hat{A}_{D|\emptyset}^{[\alpha, \beta]} | A_D^{[\alpha, \beta]} \wedge c = \hat{\theta} \right) = L_{\text{Int } c} \left( \hat{A}_{D|\emptyset}^{[\alpha, \beta]} | A_D^{[\alpha, \beta]} \wedge c = \hat{\theta} \right) = \hat{A}_{\text{Int } c}^{[\alpha, \beta]}(\cdot | \hat{\theta})
\]

(16)

for all bounded open and convex \( D \supseteq \text{Int } c \) and for \( \text{bd} \in \{\text{black}, \text{white}\} \), where \( L_{\text{Int } c} \) denotes the law of the argument random element restricted to \( \text{Int } c \) (the interior of \( c \)).

Consider the family \( \Gamma_{\mathbb{R}^2} \) of whole-plane admissible polygonal configurations, determined by \( (P1), (P2) \) and \( (P4) \) \((P3) \) is meaningless in this context\) and by the requirement of local finiteness (any bounded set is hit by at most a finite number of edges). Let \( \hat{\Gamma}_{\mathbb{R}^2} \) be the corresponding collection of black-and-white coloured whole-plane admissible polygonal configurations. It is natural to
define the family $\mathcal{G}(\hat{A}^{[\alpha,\beta]})$ of infinite volume Gibbs measures (thermodynamic limits) for $\hat{A}^{[\alpha,\beta]}$ as the collection of all probability measures on $\hat{\Gamma}_{\mathbb{R}^2}$ with the accordingly distributed random element $\hat{A}$ satisfying
\[ L_{\text{Int}c}(\hat{A}|\hat{A} \wedge c = \hat{\theta}) = A^{[\alpha,\beta]}_{\text{Int}c}(\cdot|\hat{\theta}). \] (17)

In addition, we shall consider the family $\mathcal{G}_\tau(\hat{A}^{[\alpha,\beta]})$ of isometry invariant measures in $\mathcal{G}(\hat{A}^{[\alpha,\beta]})$. Using an appropriate relative compactness argument much along the same lines as in Schreiber (2004) we will readily get the existence of at least one isometry-invariant thermodynamic limit for each $\beta > 0$.

**Theorem 3** For all $\alpha \in \mathbb{R}$ and $\beta > 0$, the family $\mathcal{G}_\tau(\hat{A}^{[\alpha,\beta]})$ is non-empty.

Note that for $\alpha = 0$ and $\beta$ large enough this statement follows also by Theorem in Surgailis (1991).

In the sequel, we will establish certain uniqueness results for the thermodynamic limit in the low temperature region within a particular class of infinite-volume measures without infinite contours. However, we do conjecture that for $\alpha = 0$ outside this class there exists an infinite number of extreme infinite-volume phases breaking both the rotational and translational symmetries. We briefly and informally sketch their tentative construction. For the increasing sequence of squares $(-n, n)^2, n = 1, 2, \ldots$ we consider a sequence of boundary conditions arising by requiring that a large number $C(n)$ of edges hit the left-hand side of $(-n, n)^2$ (with the intersection points located more or less uniformly over the edge), the same number of edges intersect the opposite right-hand side, but no edges hit the upper and lower sides. We believe that by choosing an appropriate growth rate for $C(n)$ we can assure that the resulting sequence of polygonal fields on $(-n, n)^2$ is uniformly tight (e.g. in the topology discussed in the proof of Theorem 3) and the accumulation points of this sequence are thermodynamic limits for $A^{[0,\beta]}$ with infinite number of infinite left-to-right polygonal chains. Moreover, the expected number of such chains hitting a disk of radius 1 should exhibit untempered growth to infinity with the distance of the centre of the disk from the origin. We conjecture that such untempered thermodynamic limits should exist even for $\beta = 0$ where, in the language of the dynamic time-space construction of the basic Arak process, one could, roughly speaking, have an infinite-density cloud of particles born at the time $-\infty$. Such constructions are possible due to the fact that, under very rapid edge density growth with the distance from the origin, one can enforce the situation where
the influence of the boundary conditions on $\partial(-n,n)^2$ competes on equal rights or even dominates the stabilising bulk effects within $(-n,n)^2$. Clearly, such phenomena cannot show up in the stationary regime, see Schreiber (2004) for a discussion.

4 Perfect simulation from thermodynamic limit and exponential mixing

The purpose of the section is to study the contour birth and death dynamics of Subsection 2.2 in context of the perfect infinite-volume simulation scheme as developed by Fernández, Ferrari & Garcia (1998,2002). This approach is valid only for sufficiently large $\beta$. It yields a perfect algorithm for simulating thermodynamic limits in finite windows and it allows us as well to conclude certain uniqueness and mixing results for the thermodynamic limit in low temperature regime.

To this end, we observe first that for all bounded open sets $D$ with piecewise smooth boundary the free contour measures $\Theta_D$ as defined in (11) arise as the respective restrictions to $C_D$ of the same measure $\Theta = \Theta_{\mathbb{R}^2}$ on $C := \bigcup_{n=1}^{\infty} C(-n,n)^2$, in the sequel referred to as the infinite volume free contour measure. Indeed, this follows easily by the observation that $\Theta_{D_1}$ restricted to $C_{D_2}$ coincides with $\Theta_{D_2}$ for $D_2 \subseteq D_1$. In the same way we construct the infinite-volume exponentially modified measures $\Theta[\beta] = \Theta[\beta]_{\mathbb{R}^2}$. The following result, which is related to the Lemma in the Appendix of Nicholls (2001), will be crucial for our further purposes as stating exponential decay of the measure $\Theta[\beta]$ with respect to the contour size.

**Lemma 1** For $\beta \geq 2$ we have

$$\Theta[\beta](\{\theta \mid dx \in \text{Vertices}(\theta), \text{length}(\theta) > R\}) \leq 4\pi \exp(-|\beta - 2|R)dx.$$  

Moreover, there exists a constant $\varepsilon > 0$ such that, for $\beta \geq 2$,

$$\Theta[\beta](\{\theta \mid 0 \in \text{Int} \theta, \text{length}(\theta) > R\}) \leq \exp(-|\beta - 2 + \varepsilon/2|R + o(R)).$$  

We note that, in view of (15) in Section 2.2, a standard Peierls-type argument can be applied to conclude from Lemma 1 that there is no infinite contour nesting for $\mathcal{A}^{[0,\beta]}$ whenever $\beta \geq 2$.  

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The approach of Fernández, Ferrari & Garcia (1998, 2002) specialised for our purposes relies on the following graphical construction, briefly sketched below, see ibidem for further details. Choose $\beta \geq 2$ large enough, as specified below. Define $\mathcal{F}(\mathcal{C})$ to be the space of countable and locally finite collections of contours from $\mathcal{C}$, with the local finiteness requirement meaning that at most a finite number of contours can hit a bounded subset of $\mathbb{R}^2$. Observe that $\mathcal{F}(\mathcal{C}) \subseteq \Gamma_{\mathbb{R}^2}$ (there is no equality since $\mathcal{F}(\mathcal{C})$ contains only bounded closed contours while $\Gamma_{\mathbb{R}^2}$ also admits infinite polygonal chains). On the $s$-time-space $\mathbb{R} \times \mathcal{F}(\mathcal{C})$ we construct the stationary \textit{unconstrained} (free) contour birth and death process $(\varrho_s)_{s \in \mathbb{R}}$ with the birth intensity measure given by $\Theta[\beta]$ and with the death intensity 1. Note that \textit{unconstrained} or \textit{free} means here that every new-born contour is accepted regardless of whether it hits the union of already existing contours or not, moreover we admit negative time here, letting $s$ range through $\mathbb{R}$ rather than $\mathbb{R}_+$. Observe also that we need the birth measure $\Theta[\beta]$ to be finite on the sets $\{\theta \in \mathcal{C} \mid \theta \cap A \neq \emptyset\}$ for all bounded Borel $A \subseteq \mathbb{R}^2$ in order to have the process $(\varrho_s)_{s \in \mathbb{R}}$ well defined on $\mathbb{R} \times \mathcal{F}(\mathcal{C})$. By Lemma 1 this is ensured whenever $\beta \geq 2$. It is easily seen that, for each $s \in \mathbb{R}$, $\varrho_s$ coincides in distribution with the whole-plane Poisson contour process $\mathcal{P}_{\Theta[\beta]}$.

To proceed, for the free process $(\varrho_s)_{s \in \mathbb{R}}$ we perform the following \textit{trimming} procedure. We place a directed connection from each $s$-time-space instance of a contour showing up in $(\varrho_s)_{s \in \mathbb{R}}$ and denoted by $\theta \times [s_0, s_1)$, with $\theta$ standing for the contour and $[s_0, s_1)$ for its lifespan, to all $s$-time-space contour instances $\theta' \times [s'_0, s'_1)$ with $\theta' \cap \theta \neq \emptyset$, $s'_0 \leq s_0$ and $s'_1 > s_0$. In other words, we connect $\theta \times [s_0, s_1)$ to those contour instances which may have affected the acceptance status of $\theta \times [s_0, s_1)$ in the \textit{constrained} contour birth and death dynamics (C) as discussed in Subsection 2.2. These connections yield directed chains of $s$-time-space contour instances, we call them the \textit{ancestor chains} in the sequel. Following Fernández, Ferrari & Garcia (2002) the union of all ancestor chains stemming from a given contour instance is referred to as its \textit{clan of ancestors}. Using Lemma 1 combined with a general technique of stochastic domination by subcritical multitype branching processes as discussed in detail in Fernández, Ferrari & Garcia (1998, 2002), for $\beta$ large enough we can ensure that all such clans of ancestors are a.s. finite and that a single clan size has exponentially decaying tail [i.e. the probability that the clan size exceeds $R$ is of order $O(\exp(-cR))$ for some $c > 0$]. In this case we can uniquely determine the acceptance status of all the clan members: contour instances with no
ancestors are a.s. accepted, which automatically and uniquely determines the acceptance status of all the remaining members of the clan by recursive application of the inter-contour exclusion rule. Discarding the unaccepted contour instances leaves us with an s-time-space representation of a stationary evolution \((\gamma_s)_{s \in \mathbb{R}}\) on \(\mathcal{F}(C) \subseteq \Gamma_{\mathbb{R}^2}\). The graphical construction and the argument in Fernández, Ferrari & Garcia (1998, 2002) specialised to our setting yield

**Theorem 4** Choose \(\beta \geq 2\) large enough so that all the ancestor clans in the above graphical construction are a.s. finite and a single clan size exhibits exponentially decaying tail. Then

1. the \(\mathcal{F}(C)\)-valued process \((\gamma_s)_{s \geq 0}\) given above is well-defined, stationary and reversible,

2. the stationary distribution \(\mathcal{L}(\gamma_0)\) on \(\mathcal{F}(C) \subseteq \Gamma_{\mathbb{R}^2}\) is isometry invariant and belongs to \(\mathcal{G}_\tau(A^{[0,\beta]}_\mathbb{R})\),

3. the dynamics of \((\gamma_s)_{s \in \mathbb{R}}\) is an infinite-volume extension of the contour birth and death dynamics \((C)\) as introduced in Section 2.2, i.e. \((\gamma_s)_{s \in \mathbb{R}}\) is a Markov process on \(\mathcal{F}(C)\) with the infinitesimal generator

\[
[L[\beta]F](\eta) := \int_C [F(\eta \cup \{\theta\}) - F(\eta)]d\Theta[\beta](\theta) + \sum_{\theta \in \eta} [F(\eta \setminus \{\theta\}) - F(\eta)]
\]

for \(\eta \in \mathcal{F}(C)\) and bounded \(F : \mathcal{F}(C) \rightarrow \mathbb{R}\) such that \(F(\eta)\) depends only on \(\eta \cap D\) for some bounded convex set \(D\),

4. \((\gamma_s)_{s \in \mathbb{R}}\) exhibits exponential s-time-space \(\alpha\)-mixing in that there exists \(c > 0\) such that

\[
\sup_{\mathcal{E}_1 \in \mathcal{F}_B(x,1) \times [s_0,s_1]} \sup_{\mathcal{E}_2 \in \mathcal{F}_B(y,1) \times [s'_0,s'_1]} |\mathbb{P}(\mathcal{E}_1 \cap \mathcal{E}_2) - \mathbb{P}(\mathcal{E}_1)\mathbb{P}(\mathcal{E}_2)| \leq e^{-c[\text{dist}(x,y) + \text{dist}([s_0,s_1],[s'_0,s'_1])]} \]

whenever \(\text{dist}(x,y)\) is sufficiently large, with \(\mathcal{F}_B(x,1) \times [s_0,s_1]\) standing for the \(\sigma\)-field generated by the restriction of \((\gamma_s)_{s \in \mathbb{R}}\) to the s-space-time region \(B(x,1) \times [s_0,s_1]\), where \(B(x,1)\) is the disk of radius 1 centred at \(x \in \mathbb{R}^2\),

5. consequently, the stationary distribution \(\mathcal{L}(\gamma_0)\) exhibits exponential spatial \(\alpha\)-mixing.
It is worth noting that even if \( \beta \) is not large enough to ensure a.s. finiteness of ancestor clans, a weaker version of the above graphical construction can be provided as soon as the birth intensity measure \( \Theta^{[\beta]} \) is finite on \( \{ \theta \in C \mid \theta \cap A \neq \emptyset \} \) for all bounded \( A \subseteq \mathbb{R}^2 \), which is the case whenever \( \beta \geq 2 \) by Lemma 1. To this end we restrict the s-time to \( \mathbb{R}_+ \) and choose an initial condition, which is an \( \mathcal{F}(C) \)-valued random element independent of the free birth and death process of the graphical construction. The birth and death process here is also restricted to positive times in that there are no contours born or alive before the s-time 0, in other words the birth and death process starts with the initial state \( \emptyset \) at s-time 0, consequently it is no more stationary. In this context the local finiteness of \( \Theta^{[\beta]} \) allows us to conclude that for each contour instance \( \theta \times [s_0, s_1) \), \( s_0, s_1 > 0 \), the expected cardinality of its ancestor clan extending down to the s-time 0 is finite, consequently the clan is a.s. finite (note that it could extend to an infinite clan through negative s-times in the original graphical construction). Thus, with the initial state given, the acceptance status of each contour instance is uniquely determined by the inter-contour exclusion rule. This leads us to

**Corollary 3** With \( \beta \geq 2 \), for each \( \mathcal{F}(C) \)-valued initial condition \( \gamma_0 \) there exists a Markov process \( (\gamma_s)_{s \geq 0} \) on \( \mathcal{F}(C) \) with infinitesimal generator given by (20).

In the remaining part of the present section we will not use Corollary 3 and, unless otherwise stated, we shall assume that \( \beta \) stays within the region of validity of the original graphical construction preceding Theorem 4. We denote by \( \mu^{[\beta]} \) the infinite-volume stationary distribution \( \mathcal{L}(\gamma_0) \) arising in this graphical construction. Observe that the fact that \( \mu^{[\beta]} \) is concentrated on \( \mathcal{F}(C) \) means that it contains no infinite polygonal chains – all the contours are bounded and closed. Below, we show that, with the assumptions of Theorem 4, \( \mu^{[\beta]} \) is in fact the unique element of \( \mathcal{G}(A^{[0, \beta]}) \) concentrated on \( \mathcal{F}(C) \), although we conjecture that \( \mathcal{G}(A^{[0, \beta]}) \) is non-empty as argued in Section 3. To proceed with our argument we consider finite-volume versions of the above graphical construction, with the infinite-volume birth intensity measure \( \Theta^{[\beta]} \) replaced by its finite volume restrictions \( \Theta_D^{[\beta]} \) for bounded and open \( D \) with piecewise smooth boundary. Clearly, the graphical construction yields then a version of the finite-volume contour birth and death dynamics (C) as discussed in Subsection 2.2. For each \( D \) denote the resulting finite-volume stationary process on \( \mathcal{F}(C_D) \) by \( (\gamma_s^D)_{s \in \mathbb{R}} \). Write also \( (\phi_s^D) \) for the corresponding free contour birth and death process. Note that this finite-volume construction is valid for all \( \beta \in \mathbb{R} \), even though in this section it is only used for \( \beta \) as in Theorem 4. In view of Theorem
we see that $\gamma_s^D$ coincides in distribution with $\mathcal{A}^{[0,\beta]}_D|\emptyset$ for all $s \in \mathbb{R}$. Moreover, it is easily seen that $\varrho_s^D$ coincides in distribution with $\mathcal{P}_{\Theta[D]}$ for all $s \in \mathbb{R}$. From the construction, Lemma 1 and the general theory developed in Fernández, Ferrari & Garcia (1998, 2002) it follows that

**Proposition 2** With $\beta$ as in Theorem 4 the finite-volume graphical constructions for different $D \subseteq \mathbb{R}^2$ and the infinite-volume graphical construction can be coupled on a common probability space so that there exists $c > 0$ with

$$\mathbb{P}\left(\gamma_s^{D_1} \cap B(x,1) \neq \gamma_s^{D_2} \cap B(x,1)\right) \leq \exp(-c \min(\text{dist}(x, \partial D_1), \text{dist}(x, \partial D_2)))$$

for bounded $D_1, D_2 \subseteq \mathbb{R}^2$, for $x$ sufficiently far from $\partial D_1$ and $\partial D_2$ and for all $s \in \mathbb{R}$. Moreover,

$$\mathbb{P}\left(\gamma_s^D \cap B(x,1) \neq \gamma_s \cap B(x,1)\right) \leq \exp(-c \text{dist}(x, \partial D))$$

for bounded $D \subseteq \mathbb{R}^2$, for $x$ far enough from $\partial D$ and for all $s \in \mathbb{R}$.

Taking into account that, by the construction and by the results of Section 2.2, $\gamma_s^D$ coincides in distribution with $\mathcal{A}^{[0,\beta]}_D|\emptyset$, and that for each contour collection in $\mathcal{F}(C)$ every bounded region can be surrounded by a smooth curve which does not hit any of the contours, we can use the Markov property of the considered polygonal fields combined with Proposition 2 to conclude the claimed property

**Corollary 4** For $\beta$ as in Theorem 4 the measure $\mu^{[\beta]}$ is the only element of $\mathcal{G}(\mathcal{A}^{[0,\beta]})$ concentrated on $\mathcal{F}(C)$.

For $\beta$ as in Theorem 4, using Lemma 1 we easily conclude that the number of contours in $\varrho_0$ surrounding a given point is a.s. finite. Consequently, the number of contours surrounding a given point in $\gamma_0$ is a.s. finite as well, whence there is no infinitesimal contour nesting. Thus, we observe a unique infinite connected region surrounding finitely nested contour collections. Colouring this region black or white gives rise to two distinct phases, respectively black- and white-dominated. There are no other extreme phases without infinite chains in the coloured model, because their corresponding colourless contour ensembles have to coincide with $\mu^{[\beta]}$.

The last important conclusion of the graphical construction, based on the above-made observations that almost surely $\gamma_s \subseteq \varrho_s$, $\gamma_s^D \subseteq \varrho_s^D$ and that $\gamma_s \xrightarrow{d} \mathcal{A}^{[0,\beta]}_D$, $\gamma_s^D \xrightarrow{d} \mathcal{A}^{[0,\beta]}_D|\emptyset$, $\varrho_s \xrightarrow{d} \mathcal{P}_{\Theta[D]}$ and $\varrho_s^D \xrightarrow{d} \mathcal{P}_{\Theta[D]}$, is the following stochastic domination statement.

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Corollary 5  The Poisson contour process $P_{\Theta[\alpha]}$ stochastically dominates (in the sense of inclusion of contour collections) the polygonal field $A^{[0,\beta]}$. Likewise, for each bounded $D$ with piecewise smooth boundary, the Poisson process $P_{\Theta_D[\beta]}$ stochastically dominates the finite-volume polygonal field $A_{\beta}^{[0,\beta]}$.

5  Proofs

5.1  Proof of Theorem 1

In order to provide a geometrical and intuitive proof of the theorem we construct an auxiliary model. For $r > 0$ define $\hat{A}^{[\alpha,\beta;r]}_D$ to be the Gibbsian modification of $\hat{A}^D$ with the Hamiltonian

$$H^{[\alpha,\beta;r]}_D(\hat{\gamma}) := r^{-1} \beta A([\gamma + M B(r)] \cap D) + \alpha A([\text{black}\hat{\gamma} + M B(r)] \cap D),$$

with $+M$ standing for the usual Minkowski addition and with $B(r)$ denoting the radius $r$ disk in $\mathbb{R}^2$, centred in 0. It is easily seen that, for each $\hat{\gamma} \in \hat{\Gamma}_D$,

$$\lim_{r \to 0} H^{[\alpha,\beta;r]}_D(\hat{\gamma}) = H^{[\alpha,\beta]}_D(\hat{\gamma}).$$

(22)

so that $H^{[\alpha,\beta;r]}_D$ is an approximation of $H^{[\alpha,\beta]}_D$ for small $r$. Take $\Pi^{[\alpha+a]}$, $\Pi^{[r^{-1}(\beta+b)]}$, $\Pi^{[a]}$ and $\Pi^{[r^{-1}b]}$ to be independent homogeneous Poisson point processes on $D$, jointly independent of $\hat{A}^D$, with respective intensities $\alpha + a$, $r^{-1}(\beta + b)$, $a$ and $r^{-1}b$. We claim that $\hat{A}^{[\alpha,\beta;r]}_D$ coincides in distribution with $\hat{A}^D$ conditioned jointly with $\Pi^{[\alpha+a]}$, $\Pi^{[r^{-1}(\beta+b)]}$, $\Pi^{[a]}$ and $\Pi^{[r^{-1}b]}$ on the event $E[\alpha, \beta; a, b; r]$ that the following conditions are simultaneously satisfied

- $\Pi^{[r^{-1}(\beta+b)]} \cap [\gamma + M B(r)] = \emptyset,$
- $\Pi^{[r^{-1}b]} \subseteq [\gamma + M B(r)],$
- $\Pi^{[a]} \cap [\text{black}\hat{\gamma} + M B(r)] = \emptyset,$
- $\Pi^{[a]} \subseteq [\text{black}\hat{\gamma} + M B(r)] = \emptyset,$

so that

$$\mathcal{L}(\hat{A}^{[\alpha,\beta;r]}_D) = \mathcal{L}\left(\hat{A}^D | E[\alpha, \beta; a, b; r]\right).$$

(23)

Indeed, for a given $\hat{\gamma} \in \hat{\Gamma}_D$ the probability of the event $E[\alpha, \beta; a, b; r]$ is

$$\mathbb{P}(E[\alpha, \beta; a, b; r] | \hat{\gamma}) = \exp \left( -r^{-1} [\beta + b] A([\gamma + M B(r)] \cap D) \right)$$

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\[
\exp \left( -r^{-1}b[A(D) - A(\gamma + M B(r)) \cap D] \right)
\]
\[
\exp \left( -[\alpha + a]A([\text{black}(\hat{\gamma}) + M B(r)] \cap D) \right)
\]
\[
\exp \left( -a[A(D) - A([\text{black}(\hat{\gamma}) + M B(r)] \cap D)] \right) =
\]
\[
\exp \left( -\mathcal{H}_D^{[\alpha, \beta]}(\hat{\gamma}) \right) \exp \left( -[a + r^{-1}b]A(D) \right),
\]
which yields (23) by the definition of \( \hat{A}_D^{[\alpha, \beta; \gamma]} \).

To proceed, we construct an auxiliary Markovian dynamics which leaves invariant the joint distribution of \( \hat{A}_D, \Pi^{[r^{-1}(\beta+b)]}, \Pi^{[\alpha+a]}, \Pi^{[r^{-1}]} \) and \( \Pi^{[a]} \), and makes the resulting stationary process reversible. To this end, set

\[
\hat{\gamma}_0 := \hat{A}_D, \quad \pi_0^\alpha := \Pi^{[\alpha+a]}, \quad \pi_0^\beta := \Pi^{[\beta+b]}, \quad \pi_0^a := \Pi^{[a]}, \quad \pi_0^b := \Pi^{[b]}
\]

and let the quintuple \( (\hat{\gamma}_s, \pi_s^\alpha, \pi_s^\beta, \pi_s^a, \pi_s^b)_{s \geq 0} \) evolve according to the following rules, applied independently for each component,

(Aux1) \( \hat{\gamma}_s \) evolves according to (DL: birth) and (DL: death),

(Aux2) \( \pi_s^\alpha, \pi_s^\beta, \pi_s^a \) and \( \pi_s^b \) evolve according to a birth and death process with death intensity 1 and with birth intensities \( \alpha + a, r^{-1}(\beta+b), a \) and \( r^{-1}b \) respectively.

The above invariance and reversibility statements follow as direct consequences of Proposition 1. Thus, we conclude that the joint distribution of \( (\hat{A}_D, \Pi^{[r^{-1}(\beta+b)]}, \Pi^{[\alpha+a]}, \Pi^{[r^{-1}]} \) and \( \Pi^{[a]} \)) conditioned on the event \( E[\alpha, \beta; a, b; r] \), is invariant and reversible with respect to the following Markovian dynamics, arising from (Aux1) and (Aux2) by adding an appropriate acceptance test to be passed only by admissible updates:

(B1) Choose an update \( (\hat{\delta}, \theta^\alpha, \theta^\beta, \theta^a, \theta^b) \) for \( (\hat{\gamma}_{s+ds}, \pi_{s+ds}^\alpha, \pi_{s+ds}^\beta, \pi_{s+ds}^a, \pi_{s+ds}^b) \) according to the rules (Aux1), (Aux2).

(B2) Accept the update, setting

\[
(\hat{\gamma}_{s+ds}, \pi_{s+ds}^\alpha, \pi_{s+ds}^\beta, \pi_{s+ds}^a, \pi_{s+ds}^b) := (\hat{\delta}, \theta^\alpha, \theta^\beta, \theta^a, \theta^b),
\]

provided the following conditions are satisfied

- \( \theta^\beta \cap [\delta + M B(r)] = \emptyset \),
- \( \theta^b \subseteq [\delta + M B(r)] \),

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\(\theta^a \cap [\text{black}(\hat{\delta}) +_MB(r)] = \emptyset,\)
\(\theta^a \subseteq [\text{black}(\hat{\delta}) +_MB(r)],\)

(B3) Otherwise discard the update, keeping
\[ (\hat{\gamma}_{s+ds}, \pi^\alpha_{s+ds}, \pi^\beta_{s+ds}, \pi^a_{s+ds}, \pi^b_{s+ds}) := (\hat{\gamma}_s, \pi^\alpha_s, \pi^\beta_s, \pi^a_s, \pi^b_s). \]

Consequently, in view of (23), the first component \(\hat{\gamma}_s\) under the above stationary dynamics (B1 – 3), with the initial distribution at \(s = 0\) given by the joint law of \(\hat{\mathcal{A}}_D, \Pi^{[r^{-1}(\beta + b)]}, \Pi^{[\alpha + a]}, \Pi^{[r^{-1}b]}, \Pi^{[a]}\) conditioned on the event \(E[\alpha, \beta; a, b; r]\), coincides in distribution with \(\hat{\mathcal{A}}_{D}^{[\alpha; \beta; r]}\) for all \(s \in \mathbb{R}_+\). Moreover, the conditional distributions of the remaining components given \(\hat{\gamma}_s\) are also readily determined. Indeed, \(\pi^\alpha_s\) is just a homogeneous Poisson point process on \(D \setminus [\text{black}(\hat{\gamma}_s) +_MB(r)]\) with intensity \(a + a\), \(\pi^\beta_s\) is an intensity \(r^{-1}(\beta + b)\) homogeneous Poisson point process on \(D \setminus [\gamma_s +_MB(r)]\), \(\pi^a_s\) is a homogeneous Poisson point process on \(\text{black}(\hat{\gamma}_s) +_MB(r)\) of intensity \(a\) while \(\pi^b_s\) is a homogeneous Poisson point process on \(\gamma_s +_MB(r)\) with intensity \(r^{-1}b\). All four components \(\pi^\alpha_s, \pi^\beta_s, \pi^a_s, \pi^b_s\) are jointly independent given \(\hat{\gamma}_s\). Consequently, we observe that if we integrate out the Poisson components \(\pi^\alpha, \pi^\beta, \pi^a\) and \(\pi^b\), the polygonal field component \(\hat{\gamma}_s\) turns out to evolve according to the following dynamics (see Subsection 2.1 for the notation):

(DL : birth[\(\alpha, \beta; a, b; r]\]) With intensity \([\pi dx + \kappa(dx)]ds\) do

- put \(\delta := \gamma_s \odot x,\)
- construct \(\hat{\delta}\) by randomly choosing, with probability 1/2, either of the two possible colourings for \(\delta,\)
- accept \(\hat{\delta}\) with probability

\[
\exp \left( -[\alpha + a]A \left( [\text{black}(\hat{\delta}) +_MB(r)] \setminus [\text{black}(\hat{\gamma}_s) +_MB(r)] \right) \right) \\
\exp \left( -r^{-1}[\beta + b]A([\delta +_MB(r)] \setminus [\gamma_s +_MB(r)]) \right) \\
\exp \left( -aA \left( [\text{black}(\hat{\gamma}_s) +_MB(r)] \setminus [\text{black}(\hat{\delta}) +_MB(r)] \right) \right) \\
\exp \left( -r^{-1}bA([\gamma_s +_MB(r)] \setminus [\delta +_MB(r)]) \right) = \\
\exp \left( -\alpha A \left( [\text{black}(\hat{\delta}) +_MB(r)] \setminus [\text{black}(\hat{\gamma}_s) +_MB(r)] \right) \right) \\
\exp \left( -\beta r^{-1}A([\delta +_MB(r)] \setminus [\gamma_s +_MB(r)]) \right)
\]
\[
\exp \left( -aA \left( [\text{black}(\hat{\delta}) + M B(r)] \Delta [\text{black}(\hat{\gamma}_s) + M B(r)] \right) \right) \\
\exp \left( -br^{-1}A([\delta + M B(r)] \Delta [\gamma_s + M B(r)]) \right),
\]

- if accepted, set \( \hat{\gamma}_{s+ds} := \hat{\delta} \), otherwise keep \( \hat{\gamma}_{s+ds} := \hat{\gamma}_s \).

**DL : death(\alpha, \beta; a, b, r)**

For each birth site \( x \) in \( \gamma_s \) with intensity 1 do

- put \( \hat{\delta} := \gamma_s \ominus x \),
- construct \( \hat{\delta} \) by randomly choosing, with probability \( 1/2 \), either of the two possible colourings for \( \delta \),
- accept \( \hat{\delta} \) with probability

\[
\exp \left( -\alpha A \left( [\text{black}(\hat{\delta}) + M B(r)] \setminus [\text{black}(\hat{\gamma}_s) + M B(r)] \right) \right) \\
\exp \left( -\beta r^{-1}A([\delta + M B(r)] \setminus [\gamma_s + M B(r)]) \right) \\
\exp \left( -aA \left( [\text{black}(\hat{\delta}) + M B(r)] \Delta [\text{black}(\hat{\gamma}_s) + M B(r)] \right) \right) \\
\exp \left( -br^{-1}A([\delta + M B(r)] \Delta [\gamma_s + M B(r)]) \right),
\]

- if accepted, set \( \hat{\gamma}_{s+ds} := \hat{\delta} \), otherwise keep \( \hat{\gamma}_{s+ds} := \hat{\gamma}_s \).

Thus, the distribution of \( \hat{A}^{[\alpha, \beta; r]}_D \) is invariant and reversible with respect to the above dynamics. Moreover, it is easily seen that the acceptance probabilities in the rules (DL : birth(\alpha, \beta; a, b, r)) and (DL : death(\alpha, \beta; a, b, r)) converge to these in (DL : birth(\alpha, \beta; a, b)) and (DL : death(\alpha, \beta; a, b)) as \( r \to 0 \). Taking into account (22) and letting \( r \to 0 \) we get by a standard continuity argument that \( \hat{A}^{[\alpha, \beta]}_D \) is invariant and reversible with respect to the dynamics (DL : birth(\alpha, \beta; a, b)) and (DL : death(\alpha, \beta; a, b)).

To complete the proof of Theorem 1 it suffices now to establish the remaining uniqueness and convergence statements. These follow, however, along the same lines as in Proposition 1, by the observation that, in finite volume, regardless of the initial state, the process \( \hat{\gamma}_s \) spends a non-null fraction of time in the state 'black' (no contours, the whole domain \( D \) coloured black) and by a standard application of coupling argument. The proof is complete. \( \square \)
5.2 Proof of Theorem 3

Following the ideas of Schreiber (2004) it is convenient to consider the family \( \Gamma_{R^2} \) of admissible configurations in the plane embedded into the space \( G_{R^2} \) of locally finite non-negative Borel measures on \( R^2 \), by identifying a configuration \( \hat{\gamma} \in \Gamma_{R^2} \) with the measure

\[
M_{\hat{\gamma}}(U) := \text{length}(\gamma \cap U) + A(\text{black}(\hat{\gamma}) \cap U) + N(\gamma \cap U)
\]

for Borel \( U \subseteq R^2 \), with \( N(\gamma \cap U) \) standing for the number of vertices of \( \gamma \) falling into \( U \). Endow the space \( G_{R^2} \) with the vague topology defined as the weakest one to make continuous the mappings \( \mu \mapsto \int f \, d\mu \) for all continuous \( f \) with bounded support. Observe that in general \( \Gamma_D \nsubseteq \Gamma_{R^2} \) for \( G \subset R^2 \) due to the presence of edges chopped off by the boundary. Therefore, in order to have our embedding defined also for finite-volume configurations, we agree to put \( M_{\hat{\gamma}}(D^c) := 0 \) for all \( \hat{\gamma} \in \Gamma_D \).

Note that only the internal vertices of finite-volume configurations are counted in \( N(\cdot) \).

Consider the sequence \( ((-n, n)^2)_{n=1}^{\infty} \) of growing open squares in \( R^2 \). By the properties of the basic Arak process, see Section 4 in Arak & Surgailis (1989) and Section 2.1 in Schreiber (2004), it immediately follows that there exists a finite constant \( C \) with

\[
E M_{A^{[\alpha,\beta]}_{(-n,n)^2}}((-n, n)^2) \leq CA((-n, n)^2)
\]

for all \( n \geq 1 \). We will show that the above conclusion can be extended for arbitrary \( \alpha \in \mathbb{R} \) and \( \beta > 0 \) in that there exists \( C^{[\alpha,\beta]} < \infty \) with

\[
E M_{A^{[\alpha,\beta]}_{(-n,n)^2}}((-n, n)^2) \leq C^{[\alpha,\beta]} A((-n, n)^2).
\]

Below, we assume without loss of generality that \( \alpha \geq 0 \), which can be done in view of the colour-flip symmetry. Observe first that, in view of (6),

\[
\frac{\partial}{\partial h} \mathcal{H}_{(-n,n)^2}^{[\alpha,\beta]}(A^{[h \alpha, h \beta]}_{(-n,n)^2}) = - \text{Var} \left( \mathcal{H}_{(-n,n)^2}^{[\alpha,\beta]}(A^{[h \alpha, h \beta]}_{(-n,n)^2}) \right) < 0
\]

with \( \mathcal{H}_{(-n,n)^2}^{[\alpha,\beta]} \) as in (7). Consequently, taking into account that the area term in the Hamiltonian \( \mathcal{H}_{(-n,n)^2}^{[\alpha,\beta]} \) is bounded by \( \alpha A((-n, n)^2) \) and that the Hamiltonian is always positive, we conclude by (25) that also the expectation of the edge length term in the Hamiltonian admits an area-order upper bound. It remains to show that this is also the case for the number of vertices – we sketch the argument omitting standard technical details. To this end, we take advantage of the
dynamic representation (as discussed in the introduction of this paper and in Section 4 of Arak & Surgailis (1989)) to conclude that for the basic Arak process $A_{(-n,n)^2}$ the number of internal left-extreme vertices (with the corresponding sharp angle lying to the right of the vertex) is $\text{Po}(\pi A_{(-n,n)^2})$, where $\text{Po}(\tau)$ stands for Poisson-distributed random variable with mean $\tau$. The same applies for the number of internal right-extreme, upper-extreme and lower-extreme vertices (recall that we do not count the boundary vertices here). Consequently, the overall number of internal vertices $N(A_{(-n,n)^2})$ is stochastically bounded by $4 \text{Po}(4\pi n^2)$ and has its mean of area order, not greater than $16\pi n^2$. In view of the representation (6) and taking into account that the Hamiltonian $\mathcal{H}^{[\alpha,\beta]}_{(-n,n)^2}$ is always positive since $\alpha \geq 0$, we conclude that, for all $\mathbf{K} > 0$,

$$P\left(N(\hat{A}^{[\alpha,\beta]}_{(-n,n)^2}) > 4\mathbf{K}\right) \leq \frac{P(\text{Po}(4\pi n^2) > \mathbf{K})}{\mathbb{E} \exp(-\mathcal{H}^{[\alpha,\beta]}_{(-n,n)^2}(\hat{A}^{[\alpha,\beta]}_{(-n,n)^2}))}. \quad (27)$$

Recall that Poisson distributions exhibit superexponentially decaying tails

$$P(\text{Po}(4\pi n^2) > \mathbf{K}) \leq \exp\left(-\frac{K}{8\pi n^2}\right), \quad K \geq 64\pi n^2,$$

see Shorack & Wellner (1986), p. 485. Moreover, the negative logarithm of the denominator in (27) exhibits at most area-order growth, which is due to the easily verified finiteness of the free energy density for $\mathcal{H}^{[\alpha,\beta]}_{(-n,n)^2}$

$$\liminf_{n \to \infty} \frac{1}{(2n)^2} \log \mathbb{E} \exp\left(-\mathcal{H}^{[\alpha,\beta]}_{(-n,n)^2}(\hat{A}^{[\alpha,\beta]}_{(-n,n)^2})\right) > -\infty.$$

Consequently, the required area-order bound for $\mathbb{E}N(\hat{A}^{[\alpha,\beta]}_{(-n,n)^2})$ follows now from (27) by a direct calculation. This completes the verification of (26).

To proceed with the proof of the theorem, consider the sequence $(M^{[\alpha,\beta]}_n)_{n=1}^{\infty}$ of $G_{\mathbb{R}^2}$-valued random elements with laws given by

$$\mathcal{L}(M^{[\alpha,\beta]}_n) := \frac{1}{4\pi(2n)^2} \int_{[0,2\pi)} \int_{(-n,n)^2} \mathcal{L} \left( [\tau_x \circ R_\phi] M^{[\alpha,\beta]}_{(-n,n)^2} \right) dx d\phi + \frac{1}{4\pi(2n)^2} \int_{[0,2\pi)} \int_{(-n,n)^2} \mathcal{L} \left( [\Sigma \circ \tau_x \circ R_\phi] M^{[\alpha,\beta]}_{(-n,n)^2} \right) dx d\phi, \quad (28)$$

where $\tau_x$ stands for the standard translation operator $\tau_x \mu(U) := \mu(U + x)$ while $R_\phi$, $\phi \in [0, 2\pi)$ is the rotation by angle $\phi$ around 0 and $\Sigma$ is the reflection with respect to some fixed axis passing through the origin. By (26) it follows that

$$\mathbb{E}M^{[\alpha,\beta]}_n(U) < \infty$$
for all bounded \( U \subseteq \mathbb{R}^2 \). Applying Corollary A2.6.V in Daley & Vere-Jones (1988) we conclude that the sequence of random measures \((M_n^{[\alpha,\beta]})_{n=1}^{\infty}\) is uniformly tight in \(G_{\mathbb{R}^2}\) and, consequently, it contains a subsequence converging in law to some \(M_\infty\) corresponding to a whole-plane polygonal field \(\hat{A}_\infty^{[\alpha,\beta]}\). In view of (28) it is clear that
\[
\mathcal{L}(\hat{A}_\infty^{[\alpha,\beta]}) \in \mathcal{G}_r(\hat{A}^{[\alpha,\beta]})
\]
which completes the proof of the theorem. \(\square\)

5.3 Proof of Lemma 1

By the definition (13) of the \(\beta\)-tilted contour measure \(\Theta^{[\beta]}\) it is enough to establish the assertion of the lemma for the henceforth assumed case \(\beta := 2\). In order to establish (18) define the continuous-time random walk \(Z_t\) in \(\mathbb{R}^2\) with the following transition mechanism

- between critical events specified below move in a constant direction with speed 1,
- with intensity given by 4 times the covered length element update the movement direction, choosing the angle \(\phi \in (0, 2\pi)\) between the old and new direction according to the density \(|\sin(\phi)|/4\),

We start the random walk \(Z_t\) at a given point \(x\) and with a given initial velocity vector. Moreover, we choose the loop-closing angle \(\phi^* \in (0, 2\pi)\) according to the density \(|\sin(\phi)|/4\) and we draw an infinite loop-closing half-line \(l^*\) starting at \(x\) and forming the angle \(\phi^*\) with the initial velocity vector. Let \(\hat{Z}_t\) be the random walk \(Z_t\) killed whenever hitting its past trajectory or the loop-closing line \(l^*\). The directed nature of the random walk trajectories as constructed above requires considering for each contour \(\theta\) two oriented instances \(\theta^{\rightarrow}\) (clockwise) and \(\theta^{\leftarrow}\) (anti-clockwise). We claim that for \(x \in \mathbb{R}^2\) and \(\theta \in \mathcal{C}\) with \(x \in \text{Vertices}(\theta)\) we have

\[
8\pi dx e^{-4\text{length}(e^*)} \mathbb{P}\left(\hat{Z}_t \text{ reaches } l^* \text{ and the resulting contour falls into } d\theta^{\rightarrow}\right) = \Theta^{[2]}(d\theta),
\]

where \(e^*\) stands for the last segment of \(\theta^{\rightarrow}\) counting from \(x\) as the initial vertex, which is to coincide with the segment of the loop-closing line \(l^*\) joining its
intersection point with $\tilde{Z}_t$ to $x$. Clearly, the same relation holds then for $\theta^\leftarrow$, hence adding versions of (29) for $\theta^\to$ and $\theta^\leftarrow$, which amounts to taking into account two possible directions in which the random walk can move along $\theta$, will yield $2\Theta^{(\beta)}(d\theta)$ on the RHS. The relation (18) will easily follow by using the trivial upper bound 1 for the probability on the LHS of (29).

To establish (29), we observe that the probability element

$$P\left(\tilde{Z}_t \text{ reaches } l^\ast \text{ and the resulting contour falls into } d\theta^\to\right)$$

is exactly

$$\frac{1}{4[\mu \times \mu]\{(l,l^\ast) \mid l \cap l^\ast \in dx\}} \exp(-4 \operatorname{length}(\theta \setminus e^\ast)) \prod_{i=1}^{k} d\mu(l[e_i]),$$

where $e_1, \ldots, e_k$ are all segments of $\theta$ including $e^\ast$, while $l[e_i]$ stands for the straight line determined by $e_i$. Indeed,

- the prefactor $\frac{1}{4[\mu \times \mu]\{(l,l^\ast) \mid l \cap l^\ast \in dx\}}^{-1}$ comes from the choice of the lines containing respectively the initial segment of $\theta^\to$ [counting from $x$] and $l^\ast$, as well as from the choice between two equiprobable directions on each of these lines,

- for the remaining segments we use the fact that, for any given straight line $l_0$, $\mu(\{(l \mid l \cap l_0 \in dl, \angle(l,l_0) \in d\phi\}) = |\sin \phi| dl d\phi$ with $dl$ standing for the length element on $l_0$ and with $\angle(l_0, l)$ denoting the angle between $l$ and $l_0$, see Proposition 3.1 in Arak & Surgailis (1989) as well as the argument justifying the dynamic representation in Section 4 ibidem. Note that the direction update intensity was set to 4 to coincide with $\int_{0}^{2\pi} |\sin \phi| = 4$.

To get the required relation (29) it is now enough to use (30), recall the definition of $\Theta$ and observe that $[\mu \times \mu]\{(l,l^\ast) \mid l \cap l^\ast \in dx\} = 2\pi dx$ as follows by standard integral geometry. This completes the proof of (18).

To proceed, let $\tilde{Z}_t$ be the random walk $Z_t$ killed whenever hitting its past trajectory, but not when hitting the loop-closing half-line $l^\ast$. Define

$$\varepsilon := - \lim_{T \to \infty} \frac{1}{T} \log P(\tilde{\tau} > T),$$

where $\tilde{\tau}$ is the lifetime of $\tilde{Z}_t$ or, in other words, the first moment when $Z_t$ hits its past trajectory. The existence of the limit in (31) follows by a standard superadditivity argument, see Section 1.2 in Madras & Slade (1993), and in fact
\( \varepsilon \) can be regarded as the connective constant for the self-avoiding version of the random walk \( Z_t \), see ibidem. It is easily checked that \( \varepsilon > 0 \) since during each unit time of its evolution the walk \( Z_t \) has a certain positive probability of hitting its past trajectory, uniformly bounded away from 0 through time. To establish (19) observe that, as in the argument above,

\[
\Theta^{[2]}(\{ \theta \mid dx \in \text{Vertices}(\theta), \text{length}(\theta) > R \}) \leq \\
4\pi dx \mathbb{P} \left( \tilde{Z}_t \text{ survives up to time } R/2 \right) \leq 4\pi dx \mathbb{P}(\tilde{\tau} > R/2). \tag{32}
\]

The required relation (19) follows now by (32), (31) and by the observation that

\[
\Theta^{[2]}(\{ \theta \mid 0 \in \text{Int } \theta, \text{length}(\theta) > R \}) \leq \\
\sum_{k=0}^{\infty} \Theta^{[2]}(\{ \theta \mid \text{Vertices}(\theta) \cap [B(0, k+1) \setminus B(0, k)] \neq \emptyset, \text{length}(\theta) > \max(R, k) \}).
\]

The proof is complete. \( \square \)

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