The Spectral Theory of Tensors
(Rough Version)

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The spectral theory of tensors is an important part of numerical multi-linear algebra, or tensor computation [48, 76, 90].

It is possible that the ideas of eigenvalues of tensors had been raised earlier. However, it was in 2005, the papers of Lim and Qi initiated the rapid developments of the spectral theory of tensors. In 2005, independently, Lim [57] and Qi [70] defined eigenvalues and eigenvectors of a real symmetric tensor, and explored their practical application in determining positive definiteness of an even degree multivariate form. This work extended the classical concept of eigenvalues of square matrices, forms an important part of numerical multi-linear algebra, and has found applications or links with automatic control, statistical data analysis, optimization, magnetic resonance imaging, solid mechanics, quantum physics, higher order Markov chains, spectral hypergraph theory, Finsler geometry, etc, and attracted attention of mathematicians from different disciplines. After six years’ developments, we may classify the spectral theory of tensors to 18 research topics. Before describing these 18 research topics in four groups, we state the basic definitions and properties of eigenvalues of tensors.

A. Basic Theory

An $n$-dimensional homogeneous polynomial form of degree $m$, $f(x)$, where $x \in \mathbb{R}^n$, is equivalent to the tensor product of a symmetric $n$-dimensional tensor $A = (a_{i_1, \ldots, i_m})$ of order $m$, and the rank-one tensor $x^m$:

$$f(x) \equiv Ax^m := \sum_{i_1, \ldots, i_m=1}^n a_{i_1, \ldots, i_m} x_{i_1} \cdots x_{i_m}.$$}

The tensor $A$ is called symmetric as its entries $a_{i_1, \ldots, i_m}$ are invariant under any permutation of their indices. The tensor $A$ is called positive definite (semidefinite) if
$f(x) > 0$ ($f(x) \geq 0$) for all $x \in \mathbb{R}^n, x \neq 0$. When $m$ is even, the positive definiteness of such a homogeneous polynomial form $f(x)$ plays an important role in the stability study of nonlinear autonomous systems via Liapunov’s direct method in *Automatic Control* [66]. For $n$ big and $m \geq 4$, this issue is a hard problem in mathematics.

In 2005, Qi [70] defined eigenvalues and eigenvectors of a real symmetric tensor, and explored their practical applications in determining positive definiteness of an even degree multivariate form.

By the tensor product, $\mathcal{A}x^{m-1}$ for a vector $x \in \mathbb{R}^n$ denotes a vector in $\mathbb{R}^n$, whose $i$th component is

$$(\mathcal{A}x^{m-1})_i = \sum_{i_2, \ldots, i_m=1}^n a_{i,i_2,\ldots,i_m}x_{i_2}\cdots x_{i_m}.$$ 

We call a number $\lambda \in \mathbb{C}$ an eigenvalue of $\mathcal{A}$ if it and a nonzero vector $x \in \mathbb{C}^n$ are solutions of the following homogeneous polynomial equation:

$$(\mathcal{A}x^{m-1})_i = \lambda x_i^{m-1}, \quad \forall i = 1, \cdots, n.$$ (1)

and call the solution $x$ an eigenvector of $\mathcal{A}$ associated with the eigenvalue $\lambda$. We call an eigenvalue of $\mathcal{A}$ an H-eigenvalue of $\mathcal{A}$ if it has a real eigenvector $x$. An eigenvalue which is not an H-eigenvalue is called an N-eigenvalue. A real eigenvector associated with an H-eigenvalue is called an H-eigenvector.

The resultant of (1) is a one-dimensional polynomial of $\lambda$. We call it the characteristic polynomial of $\mathcal{A}$.

We have the following conclusions on eigenvalues of an $m$th order $n$-dimensional symmetric tensor $\mathcal{A}$:

**Theorem 1. (Qi 2005)**

(a). A number $\lambda \in \mathbb{C}$ is an eigenvalue of $\mathcal{A}$ if and only if it is a root of the characteristic polynomial $\phi$.

(b). The number of eigenvalues of $\mathcal{A}$ is $d = n(m-1)^{n-1}$. Their product is equal to $\det(\mathcal{A})$, the resultant of $\mathcal{A}x^{m-1} = 0$.

(c). The sum of all the eigenvalues of $\mathcal{A}$ is

$$(m-1)^{n-1}\text{tr}(\mathcal{A}),$$

where $\text{tr}(\mathcal{A})$ denotes the sum of the diagonal elements of $\mathcal{A}$.

(d). If $m$ is even, then $\mathcal{A}$ always has H-eigenvalues. $\mathcal{A}$ is positive definite (positive semidefinite) if and only if all of its H-eigenvalues are positive (nonnegative).

(e). The eigenvalues of $\mathcal{A}$ lie in the following $n$ disks:

$$|\lambda - a_{i,i_2,\ldots,i_m}| \leq \sum\{|a_{i_2,\ldots,i_m}| : i_2, \ldots, i_m = 1, \cdots, n, \{i_2, \ldots, i_m\} \neq \{i, \cdots, i\}\},$$

for $i = 1, \cdots, n$. 

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The resultant of $Ax^{m-1} = 0$, denoted as $\text{det}(A)$ in this theorem, is called the symmetric hyperdeterminant of $A$. When $m \geq 3$, it is different from the hyperdeterminant, introduced by Cayley [12] in 1845, and studied by Gelfand, Kapranov and Zelevinsky [28] in 1994. When $m = 2$, both the hyperdeterminant and the symmetric hyperdeterminant are the same as the determinant in the classical sense.

This theorem extended the classic theory of matrix eigenvalue to tensors. However, eigenvalues and H-eigenvalues are not invariant under orthogonal transformation. This is needed in mechanics and physics. Hence, in [70], another kind of eigenvalues of tensors was defined.

Suppose that $A$ is an $m$th order $n$-dimensional symmetric tensor. We say a complex number $\lambda$ is an E-eigenvalue of $A$ if there exists a complex vector $x$ such that

\[
\begin{cases}
Ax^{m-1} = \lambda x, \\
x^T x = 1.
\end{cases}
\]

In this case, we say that $x$ is an E-eigenvector of the tensor $A$ associated with the E-eigenvalue $\lambda$. If an E-eigenvalue has a real E-eigenvector, then we call it a Z-eigenvalue and call the real E-eigenvector a Z-eigenvector.

When $m$ is even, the resultant of

\[
Ax^{m-1} - \lambda(x^T x)^{\frac{m-2}{2}} x = 0
\]

is a one dimensional polynomial of $\lambda$ and is called the E-characteristic polynomial of $A$. We say that $A$ is regular if the following system has no nonzero complex solutions:

\[
\begin{cases}
Ax^{m-1} = 0, \\
x^T x = 0.
\end{cases}
\]

Let $P = (p_{ij})$ be an $n \times n$ real matrix. Define $B = P^m A$ as another $m$th order $n$-dimensional tensor with entries

\[
b_{i_1,i_2,\ldots,i_m} = \sum_{j_1,j_2,\ldots,j_m=1}^n p_{i_1j_1}p_{i_2j_2}\cdots p_{i_mj_m}a_{j_1,j_2,\ldots,j_m}.
\]

If $P$ is an orthogonal matrix, then we say that $A$ and $B$ are orthogonally similar.

**Theorem 2. (Qi 2005)**

We have the following conclusions on E-eigenvalues of an $m$th order $n$-dimensional symmetric tensor $A$:

(a). When $A$ is regular, a complex number is an E-eigenvalue of $A$ if and only if it is a root of its E-characteristic polynomial.

(b). Z-eigenvalues always exist. An even order symmetric tensor is positive definite if and only if all of its Z-eigenvalues are positive.
(c). If $\mathbf{A}$ and $\mathbf{B}$ are orthogonally similar, then they have the same $E$-eigenvalues and $Z$-eigenvalues.

(d). If $\lambda$ is the $Z$-eigenvalue of $\mathbf{A}$ with the largest absolute value and $x$ is a $Z$-eigenvector associated with it, then $\lambda x^m$ is the best rank-one approximation of $\mathbf{A}$, i.e.,

$$\|\mathbf{A} - \lambda x^m\|_F = \sqrt{\|\mathbf{A}\|_F^2 - \lambda^2} = \min\{\|\mathbf{A} - \alpha u^m\|_F : \alpha \in \mathbb{R}, u \in \mathbb{R}^n, \|u\|_2 = 1\},$$

where $\| \cdot \|_F$ is the Frobenius norm.

Theorem 2 (d) indicates that $Z$-eigenvalues play an important role in the best rank-one approximation. This also implies that $Z$-eigenvalues are significant in practice. The best rank-one approximation of higher order tensors has extensive engineering and statistical applications, such as Statistical Data Analysis [25, 47, 106].

Eigenvalues, $H$-eigenvalues, $E$-eigenvalues and $Z$-eigenvalues were extended to non-symmetric tensors in [72]. When $m$ is odd, the $E$-characteristic polynomial was defined in [72] as the resultant of

$$\mathcal{A}x^{m-1} - \lambda x_0^{m-2} x, \quad x^\top x = x_0^2.$$  

Independently, Lim also defined eigenvalues for tensors in [57]. Lim defined eigenvalues for general real tensors in the real field. The $l^2$ eigenvalues of tensors defined by Lim are $Z$-eigenvalues of Qi [70], while the $l^k$ eigenvalues of tensors defined by Lim are $H$-eigenvalues in Qi [70]. Notably, Lim proposed a multilinear generalization of the Perron-Frobenius theorem based upon the notion of $l^k$ eigenvalues ($H$-eigenvalues) of tensors.

The calculation of eigenvalues is NP-hard with respect to $n$ when $m \geq 3$. In the case of $Z$-eigenvalues, when $n$ is small, we may solve polynomial system [2] by elimination methods, such as the Gröbner method. When $n = 2$ or 3, we may eliminate $\lambda$ in $\mathcal{A}x^{m-1} = \lambda x$ first. See [77]. When $n$ is not small, we may use the power method. This method only can get one eigenvalue and cannot guarantee it is the largest or the smallest eigenvalue. See [49].

B. Eigenvalues of Nonnegative Tensors, Higher-Order Markov Chains and Spectral Hypergraph Theory

This group include four research topics.

(1) The Largest $H$-Eigenvalue of a Nonnegative Tensor. We call $\rho(\mathcal{A})$ the spectral radius of tensor $\mathcal{A}$ if

$$\rho(\mathcal{A}) = \max\{|\lambda| : \lambda \text{ is an eigenvalue of } \mathcal{A}\},$$

where $|\lambda|$ denotes the modulus of $\lambda$. A tensor $\mathcal{A} = (a_{i_1\ldots i_m})$ is called nonnegative, if $a_{i_1\ldots i_m} \geq 0$, for all $i_1, \ldots, i_m = 1, \ldots, n$. In the study of the Perron-Frobenius theorem for nonnegative tensors and related algorithms, several classes of (nonnegative) tensors
were generalized to (nonnegative) tensors, some new classes of (nonnegative) tensors were introduced. Among these, Chang, Pearson and Zhang \[15\] extended irreducible matrices to irreducible tensors; Pearson \[69\] extended essentially positive matrices to essentially positive tensors; Chang, Pearson and Zhang \[16\] extended primitive matrices to primitive tensors; Friedland, Gaubert and Han \[28\] introduced weakly irreducible tensors and weakly primitive tensors; Zhang and Qi \[102\] introduced weakly positive tensors; Hu, Huang and Qi \[38\] introduced strictly nonnegative tensors and described the relations among corresponding nonnegative tensors.

Now we may state the Perron-Frobenius theorem for nonnegative tensors as follows.

**Theorem 3. (The Perron-Frobenius Theorem for Nonnegative Tensors)** If \(A\) is a nonnegative tensor of order \(n\) and dimension \(n\), then \(\rho(A)\) is an eigenvalue of \(A\) with a nonnegative eigenvector \(x \in \mathbb{R}^{n+}\). (Yang and Yang \[96\])

If furthermore \(A\) is strictly nonnegative, then \(\rho(A) > 0\). (Hu, Huang and Qi \[38\])

If furthermore \(A\) is weakly irreducible, then \(\rho(A)\) is an eigenvalue of \(A\) with a positive eigenvector \(x \in \mathbb{R}^{n++}\). (Friedland, Gaubert and Han \[28\])

Suppose that furthermore \(A\) is irreducible. If \(\lambda\) is an eigenvalue with a nonnegative eigenvector, then \(\lambda = \rho(A)\). (Chang, Pearson and Zhang \[15\])

In this case, if there are \(k\) distinct eigenvalues of modulus \(\rho(A)\), then the eigenvalues are \(\rho(A)e^{i2\pi j/k}\), where \(j = 0, \ldots, k-1\). The number \(k\) is called the cyclic index of \(A\). (Yang and Yang \[96\])

If moreover \(A\) is primitive, then its cyclic number is 1. (Chang, Pearson and Zhang \[16\])

If \(A\) is essentially positive or even-order irreducible, then the unique positive eigenvalue is real geometrically simple. (Pearson 2010) (Yang and Yang \[94\])

Chang, Pearson and Zhang \[15\] extended the well-known Collatz minimax theorem for irreducible nonnegative matrices to irreducible nonnegative tensors. Based upon this, Ng, Qi and Zhou proposed an algorithm for calculating \(\rho(A)\) in \[64\]. For a weakly irreducible nonnegative tensor \(A\), this algorithm generates two sequence \(\{\lambda_k\}\) and \(\{\tilde{\lambda}_k\}\) such that

\[\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_0 = \rho(A) \leq \cdots \leq \tilde{\lambda}_2 \leq \tilde{\lambda}_1.\]

The convergence of the NQZ algorithm was established by Chang, Pearson and Zhang \[16\] under primitivity, and by Friedland, Gaubert and Han \[28\] under weak primitivity. The linear convergence of the NQZ algorithm was established by Zhang and Qi \[102\].

Further developments in this topic can be found in \[13, 61, 86, 87, 98, 99, 101, 103, 104, 111, 112, 113\].
paper. A higher-order Markov chain model is used to fit the observed data through the calculation of higher-order transition probabilities:

$$0 \leq p_{k_1k_2...k_m} = \text{Prob}(X_t = k_1 \mid X_{t-1} = k_2, \ldots, X_{t-m+1} = k_m) \leq 1 \quad (3)$$

where

$$\sum_{k_1=1}^{n} p_{k_1k_2...k_m} = 1. \quad (4)$$

Thus, we have an $m$-order $n$-dimensional nonnegative tensor $P$ consisting of $n^m$ entries in between 0 and 1:

$$P = (p_{k_1k_2...k_m}), \quad 1 \leq k_1, k_2, \ldots, k_m \leq n.$$ Let the probability distribution at time $t$ be $x^{(t)} \in S_n$, where $S_n = \{x \in \mathbb{R}^n : x \geq 0, \sum_{j=1}^{n} x_j = 1\}$. Then we have

$$x^{(t)} = Px^{(t-1)} \cdots x^{(t-m+1)} := \left(\sum_{k_2...k_m=1}^{n} p_{k_2...k_m}x_{k_2}^{(t-1)} \cdots x_{k_m}^{(t-m+1)}\right)_{i=1}^{n} \in S_n \quad (5)$$

for $t = m + 1, \ldots$. Assume that

$$\lim_{t \to \infty} x^{(t)} = x^*.$$ (6)

Then we have $x^* \in S_n$ and we may call $x$ as the stationary probability distribution of the higher-order Markov chain. We see $x^*$ satisfies the following tensor equation:

$$P x^{m-1} = x \in S_n.$$ (7)

Li and Ng [55] showed that if $P$ is irreducible, then a positive solution of (7) exists. Then also give sufficient conditions for uniqueness of the solution of the fixed point problem (7). Under the same conditions, Li and Ng [56] established linear convergence of the power method

$$y^{(k+1)} = P \left(y^{(k)}\right)^{m-1} \quad (8)$$

Li and Ng [56] established linear convergence of the power method (8), but has not proved the convergence of the higher order Markov chain [53], i.e., (6). For second-order Markov chain, under some conditions, Hu and Qi [41] established the convergence (6).

The fixed point problem (7) is different from the eigenvalue problem (1). It is somewhat connected with the Z-eigenvalue problem of a special class of nonnegative tensors. Unlike the largest H-eigenvalue of a nonnegative tensor, the largest Z-eigenvalues of a nonnegative tensor are not unique in general [15]. The recent paper of Chang, Pearson and Zhang [17] reveals some similarities as well as differences between the Z-eigenvalues and H-eigenvalues of a nonnegative tensor.
(3) Spectral Hypergraph Theory. Lim [57] pointed out that a potential application of the largest eigenvalue of a nonnegative tensor is on hypergraphs. A graph can be described by its adjacency matrix. The properties of eigenvalues of the adjacency matrix of a graph are related to the properties of the graph. This is the topic of spectral graph theory. A hypergraph can be described by a \((0, 1)\)-symmetric tensor, which is called its adjacency tensor. Are the properties of the eigenvalues of the adjacency tensor of a hypergraph related to the properties of the hypergraph? Recently, several papers appeared in spectral hypergraph theory [9, 10, 24, 40, 54]. It is interesting that in [9, 10, 24], H-eigenvalues are used, while in [40, 54], Z-eigenvalues are used. This shows that there are more to explore in this topic. We expect a comprehensive spectral hypergraph theory will emerge eventually.

(4) The Relation with Nonnegative Tensor Factorization. Nonnegative tensor factorization is an important topic in the tensor decomposition community [22, 84]. What is the relation between the spectral theory of nonnegative tensors with the nonnegative tensor factorization topic? Currently, it is still blank in the literature.

C. Eigenvalues of General Tensors: Theory
This group include six research topics.

(1) The Number of E-Eigenvalues, the Number of Real Eigenvalues, E-Characteristic Polynomial. In [65], Ni, Qi, Wang and Wang showed that the degree of the E-characteristic polynomial of an even order tensor is bounded by
\[ d(m, n) = \frac{(m - 1)^n - 1}{m - 2}, \]
where \(m\) is the order and \(n\) is the dimension, \(m\) is even and \(m \geq 4\). The definition of E-eigenvalues in [70] is not a strict generalization of eigenvalues of a square matrix, as it excludes the case that \(x^\top x = 0\). In [11], Cartwright and Sturmfels gave a precise definition by using equivalence classes, and showed the number of the equivalent classes is
\[ d(m, n) = \frac{(m - 1)^n - 1}{m - 2}, \]
for any \(m > 2\), echoing the result of [65].

In [14], Chang, Pearson and Zhang unified the definitions of H-eigenvalues, Z-eigenvalues and D-eigenvalues, and showed that for a real \(n\)-dimensional \(m\)th order symmetric tensor, there are at least \(n\) H-/Z-/D-eigenvalues. Existence of real eigenvalues of real tensors was further studied in [105].

In [50], Li, Qi and Zhang studied the properties of E-characteristic polynomials.

(2) Symmetric Hyperdeterminant or E-determinant. In [24], Cooper and Dutle gave explicit expression of symmetric hyper-determinants of some special classes of hypergraphs for arbitrary \(n\). This shows that for some special cases, symmetric hyper-determinants are computable. In [50], Li, Qi and Zhang showed that symmetric hyper-
determinants are invariant under orthogonal transformation. In \cite{36}, symmetric hyper-determinants are called E-determinants. By the definition, \( \det(\mathcal{A}) = 0 \) if and only if \( \mathcal{A}x^{m-1} = 0 \) has a nonzero solution. Hu, Huang, Ling and Qi \cite{36} showed that if \( \det(\mathcal{A}) \neq 0 \), then any nonhomogeneous polynomial system with \( \mathcal{A} \) as its leading coefficient tensor has a solution. This shows that like determinants are closely related to the solution theory of linear systems, hyper-determinants are closely related to the solution theory of polynomial systems. More research is needed in this topic.

(3) Singular Values, Symmetric Embedding and Eigenvalue Inclusion. Singular values of rectangular tensors were studied in \cite{18, 19, 57, 98, 101, 111}. Symmetric embedding, which links singular values of rectangular values, and eigenvalues of square tensors, was studied in \cite{20, 82}.

In \cite{51, 52}, the authors give a number of eigenvalue inclusion theorems, including a Taussky-type boundary result and a Brauer eigenvalue inclusion theorem for tensors, and their applications.

(4) Waring Decomposition Related Problems. A Waring decomposition of a (homogeneous) polynomial \( f \) is a minimal sum of powers of linear forms expressing \( f \). Under certain conditions, such a decomposition is unique. In \cite{67}, Oeding and Ottaviani discussed some algorithms to compute the Waring decomposition, which are linked to the equations of certain secant varieties and to eigenvectors of tensors. In particular they explicitly decomposed a cubic polynomial in three variables as the sum of five cubes (Sylvester Pentahedral Theorem).

(5) Third-Order Tensors as Linear Operators. In \cite{8, 30, 45}, special operations were introduced to regard third-order tensors as linear operations. With more complicated manipulations, this approach may also be extended to higher-order tensors. The cost of this approach is that the original tensor operations are given up.

(6) Successive Symmetric Best Rank-One approximation, the Best Rank-One Approximation Ratio, and the Symmetric Rank Conjecture. Denote the space of all real \( m \)-th order \( n \)-dimensional symmetric space as \( S_{m,n} \). Suppose that \( \mathcal{A} \in S_{m,n} \). Let \( \mathcal{A}^{(0)} \equiv \mathcal{A} \). In general, suppose that \( \mathcal{A}^{(k)} \in S_{m,n} \). Let \( \lambda_k \) be the Z-eigenvalue of \( \mathcal{A}^{(k)} \) with the largest absolute value, and \( x^{(k)} \) be the corresponding Z-eigenvector. Then we define

\[
\mathcal{A}^{(k+1)} = \mathcal{A}^{(k)} - \lambda_k \left( x^{(k)} \right)^m.
\]

Wang and Qi \cite{92} proved that

\[
\mathcal{A} = \sum_{k=0}^{\infty} \lambda_k \left( x^{(k)} \right)^m,
\]

and called this the successive symmetric best rank-one approximation of \( \mathcal{A} \).

Qi \cite{73} studied further the convergence of \eqref{9}. Denote the largest absolute value of the Z-eigenvalues of \( \mathcal{A} \in S_{m,n} \) as \( \rho_Z(\mathcal{A}) \). When \( m = 2 \), \( \rho_Z(\mathcal{A}) \) is actually the spectral
radius, i.e., the 2-norm of $A$. In $S_{m,n}$, $\rho_Z(A)$ is still a norm of $A$. Denote the Frobenius norm of $A$ by $\|A\|_F$. Then

$$\text{App}(S_{m,n}) := \min \left\{ \frac{\rho_Z(A)}{\|A\|_F} : A \in S_{m,n} \right\} > 0.$$ 

Qi [73] called it the best rank-one approximation ratio of $S_{m,n}$. Actually, we have

$$0 < \text{App}(S_{m,n}) < 1.$$ 

Suppose that $A^{(k)} \in S_{m,n}$. Let $\lambda_k$ be the Z-eigenvalue of $A^{(k)}$ with the largest absolute value, and $x^{(k)}$ be the corresponding Z-eigenvector. Then we have

$$\|A^{(k+1)}\|_F^2 \leq \|A^{(k)}\|_F^2 \left[ 1 - \text{App}(S_{m,n})^2 \right].$$

This not only proves (9), but also gives its convergence rate.

Let

$$\mu_{m,n} = \frac{1}{\sqrt{n^{m-1}}}.$$ 

If $m = 2k$ is even, then let $A^{(m,n)} \in S_{m,n}$ and $\bar{\mu}_{m,n}$ be defined by

$$A^{(m,n)}x^m = (x^\top x)^k$$

and

$$\bar{\mu}_{m,n} = \frac{1}{\|A^{(m,n)}\|}.$$ 

If $m = 2k + 1$ is odd, then let $A^{(m,n)} \in S_{m,n}$ and $\bar{\mu}_{m,n}$ be defined by

$$A^{(m,n)}x^m = (x^\top x)^k \left( \sum_{i=1}^n x_i \right)$$

and

$$\bar{\mu}_{m,n} = \frac{\sqrt{n}}{\|A^{(m,n)}\|}.$$ 

The following upper and lower bounds for $\text{App}(S_{m,n})$ were established in [73].

**Theorem 4.** The value $\mu_{3,n}$ is a positive lower bound for $\text{App}(S_{3,n})$.

On the other hand, the value $\bar{\mu}_{m,n}$ is an upper bound for $\text{App}(S_{m,n})$ for $m = 2, 3, \ldots$. We have

$$\frac{1}{\sqrt{n}} = \mu_{2,n} = \text{App}(S_{2,n}) = \bar{\mu}_{2,n} = \frac{1}{\sqrt{n}},$$

$$\frac{1}{n} = \mu_{3,n} \leq \text{App}(S_{m,n}) \leq \bar{\mu}_{3,n} = \sqrt{\frac{6}{n + 5}},$$

and

$$\text{App}(S_{4,n}) \leq \bar{\mu}_{4,n} = \sqrt{\frac{3}{n^2 + 2n}}.$$
To improve this theorem, Qi [73] made a conjecture that $\rho_Z(\mathcal{A})$ is equal to
$$\sigma(\mathcal{A}) := \max \left\{ |Ax^{(1)} \cdots x^{(m)}| : (x^{(i)})^\top x^{(i)} = 1, \forall i = 1, \cdots, m \right\}.$$ This conjecture was proved by Zhang, Ling and Qi [108]. This improved Theorem 4 that $\mu_{m,n}$ is a positive lower bound for $\text{App}(S_{m,n})$ for all $m$.

One referee of [108] pointed out the above result can be regarded the first step to prove the symmetric rank conjecture of Comon, Golub, Lim and Mourrain [23].

Let $\mathcal{A}$ be an $m$th-order $n$-dimensional tensor. We may have a rank-one tensor decomposition of $\mathcal{A}$:
$$\mathcal{A} = \sum_{i=1}^r \alpha_i x^{(i,1)} \cdots x^{(i,m)},$$
where $x^{(i,j)}$ are $n$-dimensional vectors, $\alpha_i$ are numbers. The minimum value of $r$ is called the rank of $\mathcal{A}$.

If $\mathcal{A}$ is symmetric, than we may have a symmetric rank-one tensor decomposition of $\mathcal{A}$:
$$\mathcal{A} = \sum_{i=1}^r \alpha_i (x^{(i)})^m,$$
where $x^{(i)}$ are $n$-dimensional vectors, $\alpha_i$ are numbers. The minimum value of $r$ is called the symmetric rank of $\mathcal{A}$.

It is conjectured by Comon, Golub, Lim and Mourrain [23] that for a symmetric tensor, its symmetric rank is the same as its rank. We call this conjecture the symmetric rank conjecture.

D. Eigenvalues of General Tensors: Links with Optimization, Numerical Analysis and Geometry

This group include four research topics.

(1) Bi-quadratical Optimization and Spherical Optimization. Bi-quadratical optimization and spherical optimization were studied in [7, 33, 59, 60, 63, 85, 93, 107, 108, 110]. Bi-quadratical optimization is linked with the M-eigenvalue problem [?, ?, 59, 93], while spherical optimization is linked with the Z-eigenvalue problem. They are NP-hard in general. The approach in [7, 33, 59, 60, 63, 85, 107, 108] is to use semi-definite programming to find a lower bound for the minimization problem. Symmetric spherical optimization was studied in [108, 110]. As stated in the last section, Zhang, Ling and Qi [108] proved that for $\mathcal{A} \in S_{m,n}$,
$$\rho_Z(\mathcal{A}) \equiv \max \left\{ |Ax^m| : x^\top x = 1 \right\}$$
$$= \sigma(\mathcal{A}) \equiv \max \left\{ |Ax^{(1)} \cdots x^{(m)}| : (x^{(i)})^\top x^{(i)} = 1, \forall i = 1, \cdots, m \right\}.$$ This reveals some deep insight property of symmetric spherical optimization.
(2) **Space Tensor Conic Programming.** In the study of higher order diffusion tensor imaging, positive semi-definite tensor models arise [37, 80, 81], where the tensor is positive definite and of dimension three. We call three dimensional tensors **space tensors.** Space tensors appear in physics and mechanics. They are real physical entities. The positive semi-definite space tensors of the same order form a convex cone. The optimization problem involving positive semi-definite space tensors is called the **space tensor conic programming** problem [79]. In [54], based on the analysis of the semismoothness properties of the maximum Z-eigenvalue function, a generalized Newton method is proposed to solve the space tensor conic linear programming problem. In [39], a sequential SDP method is proposed to solve the space tensor conic linear programming problem.

The problem to identify a space tensor is positive (semi-)definite or not is equivalent to finding the minimum Z-eigenvalue of an $m$-th order tensor with dimension three. As we discussed in the basic theory section, this problem can be solved by elimination methods. It is not NP-hard. This approach requests to solving a one variable polynomial equation of degree generically $N = \frac{1}{2}(m + 1)(m + 2)$. When $m$ is big, the solution procedure is unstable. Hence, new algorithms are needed and this research topic is interesting.

(3) **Characterizing the Limiting Behaviour of Newton’s Method.** Dupont and Scott [26] studied the angular orientations of convergent iterates generated by Newton’s method in multiple space dimensions. They showed that the Newton iteration can be interpreted as a fixed-point algorithm for solving a tensor eigenproblem. They gave an extensive computational analysis of this tensor eigenproblem in two dimensions. In a large fraction of cases, the tensor eigenproblem has a discrete number of solutions to which the Newton directions converge quickly, but there is also a large fraction of cases in which the behavior is more complicated. Dupont and Scott contrasted the angular orientations of iterates generated by Newton’s method with the corresponding directions of the continuous Newton algorithm.

(4) **Geometry related Problems.** Qi [71] studied the orthogonal classification problem of real hypersurfaces given by an equation of the form

$$ S = \{ x \in \mathbb{R}^n : f(x) = Ax^m = c \}, $$

where $A$ is a symmetric tensor, through the rank, the Z-eigenvalues and the asymptotic directions.

Balan [1, 2, 3, 9] showed that there is a close relationship between Finsler metrics of Berwald-Moor type, and the spectral theory of tensors. Balan used the results of [70, 71] to analyze spectral properties of m-root models, geometric relevance of spectral equations, degeneracy sets, asymptotic rays, base indices and based rank one approximation of various Berwald-Moor tensors, Chernov tensors and Bogoslovski tensors. A further investigation on the geometrical meanings of eigenvalues of higher order tensors is needed.
E. Eigenvalues of General Tensors: Links with Image Science, Solid Mechanics and Quantum Physics

This group includes four research topics.

(1) **Higher Order Diffusion Tensor Imaging.** Diffusion tensor imaging (DTI) is the most popular magnetic resonance imaging model [5]. In the DTI model, eigenvalues of the diffusion tensors play a fundamental role as they are the main invariants of these tensors, and thus can be used in the measures of DTI [5]. However, DTI is known to have a limited capability in resolving multiple fibre orientations within one voxel. This is mainly because the probability density function for random spin displacement is non-Gaussian in the confining environment of biological tissues and, thus, the modeling of self-diffusion by a second order tensor breaks down. Hence, researchers presented various higher order diffusion tensor imaging models to overcome this problem [44, 62, 68, 88, 89]. For the measures of these higher order diffusion tensor imaging models, the spectral theory of tensors naturally play an important role [6, 21, 32, 37, 42, 43, 75, 78, 80, 81, 109]. In particular, Qi, Wang and Wu [78] proposed D-eigenvalues to study the diffusion kurtosis model [44, 62]; Bloy and Verma [6] studied the underlying fiber directions from the diffusion orientation distribution function by using Z-eigenvalues; Qi, Yu and Wu [81], Hu, Huang, Ni and Qi [37], and Qi, Yu and Xu [80] studied positive semi-definite higher order diffusion tensor imaging models.

(2) **Image Authenticity Verification Problem.** In [100], Zhang, Zhou and Peng generalized the definition of D-eigenvalues, and introduced the gradient skewness tensor which involves a three-order tensor derived from the skewness statistic of gradient images. They found out that the skewness value of oriented gradients of an image can measure the directional characteristic of illumination, and the local illumination detection problem for an image can be abstracted as solving the largest D-eigenvalue of gradient skewness tensors. Their method presented excellent results in a class of image authenticity verification problem, which is to distinguish real and flat objects in a photograph.

(3) **Elasticity Tensors in Solid Mechanics.** In solid mechanics, the elasticity tensor $\mathbf{A} = (a_{ijkl})$ is partially symmetric in the sense that for any $i, j, k, l$, we have $a_{ijkl} = a_{kijl} = a_{ilkj}$. We say that they are strongly elliptic if and only if

$$f(x, y) \equiv \mathbf{A}_{xy}xy \equiv \sum_{i,j,k,l=1}^{n} a_{ijkl}x_{i}y_{j}x_{k}y_{l} > 0,$$

for all unit vectors $x, y \in \mathbb{R}^{n}, n = 2$ or $3$. For an isotropic material, some inequalities have been established to judge the strong ellipticity [46, 91]. In [31, 74], Dai, Han and Qi studied conditions for strong ellipticity and introduced M-eigenvalues for the ellipticity tensor $\mathbf{A}$. They showed that M-eigenvalues always exist and the strong ellipticity condition holds if and only if the smallest M-eigenvalue of the elasticity tensor is positive.

(4) **Quantum Entanglement Problem.** The entanglement problem is to determine whether a quantum state is separable or inseparable (entangled), or to check
whether an $mn \times mn$ symmetric matrix $A \succeq 0$ can be decomposed as a convex combination of tensor products of $n$ and $m$ dimensional vectors. It has fundamental importance in quantum science and has attracted much attention since the pioneer work of Einstein, Podolsky and Rosen [27] and Schrödinger [83]. The typical optimization problem in quantum entanglement problem is like the following [34]:

$$\max \left\{ |Ax^{(1)} \cdots x^{(m)}| : |x^{(i)}| = 1, \text{ for } i = 1, \cdots, m \right\},$$

where $A$ is an $m$th-order $(n_1, \cdots, n_m)$-dimensional complex tensor, $x^{(i)} \in C^{n_i}$ for $i = 1, \cdots, m$. The singular value form of this optimization problem is

$$A x^{(1)} \cdots x^{(i-1)} x^{(i+1)} \cdots x^{(m)} = \lambda x^{(i)}, \quad |x^{(i)}| = 1, \text{ for } i = 1, \cdots, m.$$  

The optimization problem and the singular value problem involve complex and conjugate numbers. They are different but have some similarities with what we studied before. With their strong physics background, this topic will be a new fertile land of our research.

The International Conference on The Spectral Theory of Tensors will be held at Chern Research Institute, Nankai University, Tianjin, China, during May 30 - June 2, 2012. Its website is:

http://www.nim.nankai.edu.cn/activites/conferences/hy20120530/index.htm

Two special issues on the Spectral Theory of Tensors are being edited [58, 95].

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