The construction of the portrait of the correlation dimension of an attractor in the boundary layer of Earth’s atmosphere

A V Dmitrenko¹,²

¹Department of Thermal Physics, National Research Nuclear University «MEPhI», Kashirskoye shosse 31, Moscow, 115409, Russian Federation
²Department of Thermal Engineering, Russian University of Transport «MIIT», Obraztsova Street 9, Moscow, 127994, Russian Federation

AVDmitrenko@mephi.ru, ammsv@yandex.ru

Abstract. The construction of the portrait of the correlation dimension of the attractor in the boundary layer of Earth’s atmosphere on the basis of the new theory of stochastic hydrodynamics is presented. This theory is based on the theory of stochastic equations of continuum laws and equivalence of measures between random and deterministic movements. Using these equations a new formula for the correlation dimension of the attractor is applying for conditions in the boundary layer of Earth’s atmosphere.

1. Introduction
The research of studies [1-37] had allowed in articles [38-51] to discover the physical regularity of the equivalence of measures and to create the systems of the stochastic equations for the determination onset of turbulence in isothermal and non-isothermal and flows. Using these new dependences in paper [52,53] the result of calculation of the correlation dimension in the boundary layer on the flat plate and in the round pipe were presented also. Here these new formulas for the correlation dimension of the attractor are applying for conditions in the boundary layer of Earth’s atmosphere. This is very important, because the existing methods and the resulting relationships for determining the dimension of the attractor, in essence, require in fact the repeated carrying out of the numerous and laborious experimental and theoretical studies of hydrodynamic turbulence that have been carried out in the last hundred years. Also this very important, because the new formulas allow us to apply the already known experimental results for atmospheric turbulence to determine the dimension of the attractor in space. The new results for the construction of the space-time portrait of the correlation dimension of the attractor for conditions in the boundary layer of Earth’s atmosphere permits to verify results of so-called direct numerical integration of the system of Navier-Stokes equations.

2. The set of equations
Stochastic equations of conservation, which were derived in [38-53] take the form:
the equation of mass (continuity)

\[ \frac{d (\rho)_{\text{col}}}{d\tau} + \frac{(\rho)_{\text{st}}}{\tau_{\text{cor}}} - \frac{d (\rho)_{\text{st}}}{d\tau} = 0, \]

the momentum equation
\[
\frac{d(\rho u_i)}{dt}^{\text{col st}} = \text{div}(\tau _{i,j})^{\text{col st}} + \text{div}(\tau _{i,j})^{\text{st}} - \frac{(\rho U)^{\text{st}}}{\tau_{cor}} \frac{d(\rho U)^{\text{st}}}{dt} + F_{i}^{\text{col st}} 1 + F_{i}^{\text{st}} 1 ,
\]  
and the energy equation
\[
\frac{dE}{dt}^{\text{col st}} = \frac{d(\rho T)}{dt} + \text{div}(\tau _{i,j})^{\text{col st}} + \text{div}(\tau _{i,j})^{\text{st}} - \frac{E^{\text{st}}}{\tau_{cor}} \frac{dE^{\text{st}}}{dt} + (u_{i}F)^{\text{col st}} 1 + (u_{i}F)^{\text{st}} 1 .
\]  

Here, \( \rho, \dot{\Upsilon}, u_i, u_j, \mu, \tau, \epsilon_{ij} \) are the density; the velocity vector; the velocity components in directions \( x_n, x_j, x_i \) (i, j, l = 1, 2, 3); the dynamic viscosity; the time; and stress tensor \( \tau_{ij} = P + \sigma_{ij} , \delta_{ij}=1 \) if \( i=j \), \( \delta_{ij}=0 \) for \( i \neq j \). \( P \) is the pressure of liquid or gas; \( \lambda \) is the thermal conductivity; \( c_p \) and \( c_v \) are the specific heat at constant pressure and volume, respectively; \( F \) is the external force, and
\[
\sigma_{ij} = \mu \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) - \delta_{ij} \left( \frac{2}{3} \mu \right) \frac{\partial u_l}{\partial x_l} .
\]  
Then for the non-isothermal motion of the medium, using the definition of equivalency measures between deterministic and random process in the critical point, the sets of stochastic equations of energy, momentum, and mass are defined for the next space-time areas: 1) the beginning of the generation (index 1,0 or 1 ); 2) generation (index 1,1); 3) diffusion (1,1,1) and 4) the dissipation of the turbulent fields. These results provide an opportunity to introduce the concept of the correlator, which is defined for the potential physical quantities and combinations (N, M). This correlator, in its structure, will determine the possible range of motion in space depending on the different combinations of (M, N) and the corresponding values that determine the correlation interval of space-time. According to [12,13], the correlator in space-time is
\[
\lim_{m_i \rightarrow m_c ; r_i \rightarrow r_c ; \Delta \tau_i \rightarrow \tau_c} \left( D_{N,M} (m_i \rightarrow m_c ; r_i \rightarrow r_c ; \Delta \tau_i \rightarrow \tau_c) \right) = 0 ,
\]  
\[
D_{N,M} (m_i ; r_c ; \tau_c) = \sum_{i} \lim_{m_i \rightarrow m_c} \lim_{r_i \rightarrow r_c} \lim_{\Delta \tau_i \rightarrow \tau_c} \left( m(T^M Z^* \cap T^N Y^*) - R_{1}^{T^M Z^* T^N Y^*} m(T^M Z^*) \right) .
\]  
Subscript \( j \) denotes the parameters \( m_{ij} \) (j = 3 means mass, momentum, and energy). For the case of the binary intersections, it was written that \( X = Y + Z + W \). Here subscripts \( \ll \) or \( \gg \) refer to critical point \( r(x_m, \tau_c) \) or \( r_c \); the space-time point of the beginning of the interaction between the deterministic field and random field that leads to turbulence. In addition, subsets \( Y, Z, W \) are called extended in \( X \). Then Applying the obtained result to calculate second-order correlations, we determined the dependence for the correlation dimension.

### 3. The dimension of an attractor

Using equations (1)-(5) in the paper [52] was shown that the Haussdorff’s dimension can be defined as the new formula
\[
d_H \leq 1 + 1.2 \left[ C(C(Re)^2 + C(Re)^3) \right]^{n/4} .
\]  
Also for incompressible flows, the number of degrees of freedom \( N \) was defined as
\[
N \leq 1 + 1.05 \left[ C(C(Re)^2 + C(Re)^3) \right]^{n/4} .
\]  
The dependence for the turbulent Reynolds number is \( \text{Re}_t = \sqrt{u_i u_j L_m / \nu} \). Here \( L_m \) is the mixing path, \( u_i, u_j \) are the pulsational components of the velocity vector \( u \) in \( i, j \) directions.
Dependences (6), (7) can be considered as the upper limit of the number of degrees of freedom $N$ (the number of frequencies) at the considered point of the hydrodynamic flow, in case of the two-dimensional $n = 2$ or three-dimensional $n = 3$ flow. In this connection, it seems necessary to give a number of examples on the calculation of the Haussdorf's dimension and the estimation of the number of degrees of freedom in various types of hydrodynamic flows using the experimental data already obtained for them. First of all, we note that the relations (6), (7) include "uncertainty" with respect to the number of dimension "$n" - that is, two or three-dimensional Navier-Stokes equations can describe the flow of the medium. This uncertainty does not seem problematic, based on a priori knowledge of the three-dimensionality of hydrodynamic turbulence. Also in [52] was shown that turbulent Reynolds number $Re_i$ in formulas (6), (7) being the function of statistical moment in the calculating point of the flow may be written as the function of turbulent Reynolds number of statistical moments of the initial fluctuation $Re_{st}$:

$$Re_i = \left(\sqrt{\left<E_{ij}\right>_{st} L_{st0}/v}\right)\sqrt{Re_{st0}(L_{x_i}/L_{st0})} = Re^{1.5}_{st0}(L_{x_i}/L_{st0}).$$

(8)

So we have the new formula for Haussdorf's dimension written as a function of the initial fluctuation in the flow

$$d_H \leq 1 + 1.2 \left[ C_1 (Re^{1.5}_{st0}(L_{x_i}/L_{st0}))^2 + C_0 (Re^{1.5}_{st0}(L_{x_i}/L_{st0}))^{3/4} \right]^{n/4}$$

(9)

and the number of degrees of freedom $N$ (the number of frequencies) at the considered point of the hydrodynamic flow in the case of the two-dimensional $n=2$ or three-dimensional $n=3$ flow. For incompressible flows, the number of degrees of freedom $N$ is

$$N \leq 1 + 1.05 \left[ C_1 (Re^{1.5}_{st0}(L_{x_i}/L_{st0}))^2 + C_0 (Re^{1.5}_{st0}(L_{x_i}/L_{st0}))^{3/4} \right]^{n/4}.$$ 

(10)

$L_{x_i}$ is the scale of turbulence, so that $L_y$ on $x_2 = y$, or $L_{x_i}$, $x_1 = x$. Here, $x_1$ and $x_2$ are coordinates along and normal to Earth surface.

4. The calculation of the attractor dimension

An estimate of the correlation dimension in the boundary layer of the atmosphere is given here primarily for the purpose of demonstrating, perhaps a priori, the expected identity of the distribution of the dimension of the attractor in the boundary layers of the hydrodynamic flows. But first of all, it should be noted that the numerical value of the dimension estimate is largely determined by the experimental values of long-term observations, taking into account the spatial inversion of atmospheric fluxes and the absolute values of wind speed characteristic for each specific locality.

However, published data allow, in a first approximation, to give both a qualitative and a quantitative estimate of the distribution of the correlation dimension. Obviously, in the most general case, we should speak of a spatial picture in the distribution. Here, the data that determine the distribution of turbulent characteristics from altitude above the Earth's surface will be used. The experimental data [54] of radar observations of pulsations of the horizontal velocity component as a function of the distance from the surface are in general a complex layered distribution associated with the structure of the local (for a given region) air temperature profile and the averaged wind speed.

In the surface layer of the atmosphere, when the distance from the earth $Z$ is much smaller than the Obukhov-Monin scale $L_{OM}$, an "indifferent" (neutral) stratification is realized, so the turbulence characteristics in this region have distributions practically unaffected by the unevenness of the temperature field. In addition, as shown by experimental studies [54-56], despite the variety of temperature regimes (night / day, winter / summer) affecting the measured values of turbulent characteristics obtained, nevertheless, it was possible to obtain "average temperature profiles" of the pulsation profiles components of the velocities and scale of turbulence-the mixing path of $L_{mix}$. 

So we have the new formula for Haussdorf's dimension written as a function of the initial fluctuation in the flow

$$d_H \leq 1 + 1.2 \left[ C_1 (Re^{1.5}_{st0}(L_{x_i}/L_{st0}))^2 + C_0 (Re^{1.5}_{st0}(L_{x_i}/L_{st0}))^{3/4} \right]^{n/4}$$

(9)

and the number of degrees of freedom $N$ (the number of frequencies) at the considered point of the hydrodynamic flow in the case of the two-dimensional $n=2$ or three-dimensional $n=3$ flow. For incompressible flows, the number of degrees of freedom $N$ is

$$N \leq 1 + 1.05 \left[ C_1 (Re^{1.5}_{st0}(L_{x_i}/L_{st0}))^2 + C_0 (Re^{1.5}_{st0}(L_{x_i}/L_{st0}))^{3/4} \right]^{n/4}.$$ 

(10)

$L_{x_i}$ is the scale of turbulence, so that $L_y$ on $x_2 = y$, or $L_{x_i}$, $x_1 = x$. Here, $x_1$ and $x_2$ are coordinates along and normal to Earth surface.

4. The calculation of the attractor dimension

An estimate of the correlation dimension in the boundary layer of the atmosphere is given here primarily for the purpose of demonstrating, perhaps a priori, the expected identity of the distribution of the dimension of the attractor in the boundary layers of the hydrodynamic flows. But first of all, it should be noted that the numerical value of the dimension estimate is largely determined by the experimental values of long-term observations, taking into account the spatial inversion of atmospheric fluxes and the absolute values of wind speed characteristic for each specific locality.

However, published data allow, in a first approximation, to give both a qualitative and a quantitative estimate of the distribution of the correlation dimension. Obviously, in the most general case, we should speak of a spatial picture in the distribution. Here, the data that determine the distribution of turbulent characteristics from altitude above the Earth's surface will be used. The experimental data [54] of radar observations of pulsations of the horizontal velocity component as a function of the distance from the surface are in general a complex layered distribution associated with the structure of the local (for a given region) air temperature profile and the averaged wind speed.

In the surface layer of the atmosphere, when the distance from the earth $Z$ is much smaller than the Obukhov-Monin scale $L_{OM}$, an "indifferent" (neutral) stratification is realized, so the turbulence characteristics in this region have distributions practically unaffected by the unevenness of the temperature field. In addition, as shown by experimental studies [54-56], despite the variety of temperature regimes (night / day, winter / summer) affecting the measured values of turbulent characteristics obtained, nevertheless, it was possible to obtain "average temperature profiles" of the pulsation profiles components of the velocities and scale of turbulence-the mixing path of $L_{mix}$. 

So we have the new formula for Haussdorf's dimension written as a function of the initial fluctuation in the flow

$$d_H \leq 1 + 1.2 \left[ C_1 (Re^{1.5}_{st0}(L_{x_i}/L_{st0}))^2 + C_0 (Re^{1.5}_{st0}(L_{x_i}/L_{st0}))^{3/4} \right]^{n/4}$$

(9)

and the number of degrees of freedom $N$ (the number of frequencies) at the considered point of the hydrodynamic flow in the case of the two-dimensional $n=2$ or three-dimensional $n=3$ flow. For incompressible flows, the number of degrees of freedom $N$ is

$$N \leq 1 + 1.05 \left[ C_1 (Re^{1.5}_{st0}(L_{x_i}/L_{st0}))^2 + C_0 (Re^{1.5}_{st0}(L_{x_i}/L_{st0}))^{3/4} \right]^{n/4}.$$ 

(10)

$L_{x_i}$ is the scale of turbulence, so that $L_y$ on $x_2 = y$, or $L_{x_i}$, $x_1 = x$. Here, $x_1$ and $x_2$ are coordinates along and normal to Earth surface.
Obviously, the dependence of these characteristics on the absolute values of the velocity exists. However, we emphasize once again that the data presented here are nothing more than "averages", in a broad sense, distributions obtained in the course of continuous and numerous measurements and used in solving scientific and practical problems of meteorology [54-56]. Therefore, such experimental distributions can be used as estimates here. The experiments considered were characterized by Richardson numbers $R_i \sim 1$, such values were determined with an almost isothermal stratification of the layer up to 100 meters, so here we observed a dependence close to the Karman relation for the scale of turbulence. It should also be noted that the values of the boundary layer in the surface atmosphere are calculated to the values $(x_2)_0 = Z_0$, which depend on the surface roughness called the "underlying surface" in hydrometeorology (desert, steppe, forest, hilly or rocky surfaces, mountains). In the presented experiments, the characteristics considered are determined for a plain surface ($Z_0 = 2$ - 4 m). The results of calculating the correlation dimension in accordance with the data for the boundary layer of the Earth's atmosphere show that near a surface of the order of 2 meters, the estimate of the correlation dimension can be $\sim 10^9$. Outside the boundary layer, the estimate of the dimension of the atmospheric attractor is $\sim 10^{13}$. Calculated portrait of the correlation dimension of the attractor in the boundary layer of Earth's atmosphere in accordance with the data [54-56] is presented on Figure 1, where $\delta$ is the thickness of the boundary layer of Earth’s atmosphere, and $x_2$ is the transverse coordinate.

![Figure 1](image_url)

**Figure 1.** Calculated portrait of the correlation dimension of the attractor in the boundary layer of Earth’s atmosphere in accordance with data [54-56].
5. Conclusions
Based on new stochastic theory of hydrodynamic turbulence, a new dependence is presented for the
construction of the portrait of the correlation dimension of the attractor in the boundary layer of
Earth’s atmosphere. Also new dependencies are allowing us to determine the dimension of the
attractor as a function of the parameters of the initial fluctuations which always are in the air flows
Earth’s atmosphere. The results of calculating the correlation dimension in accordance with the data
for the boundary layer of the Earth’s atmosphere show that near a surface of the order of 2 meters, the
estimate of the correlation dimension can be ~ 10^7. Outside the boundary layer, the estimate of the
dimension of the atmospheric attractor is ~ 10^11.

Acknowledgements
This work was supported by the program of increasing the competitive ability of National Research
Nuclear University MEPhI (agreement with the Ministry of Education and Science of the Russian
Federation of August 27, 2013, Project No.02.a03.21.0005).

References
[1] Kolmogorov A N 1940 Dokl. Akad. Nauk 26 (1) 6
[2] Kolmogorov A N 1958 Dokl. Akad. Nauk 119 No. 5 861
[3] Kolmogorov A N 1959 Dokl. Akad. Nauk 124 No. 4 754
[4] Kolmogorov A N 2004 Usp. Mat. Nauk 59 (1) (355) 5
[5] Landau L D 1944 Dokl. Akad. Nauk 44 (8) 339
[6] Lorenz E N 1963 J. Atmos. Sci. 20 130
[7] Ruelle D and Takens F 1971 Commun. Math. Phys. 20 p 167
[8] Klimontovich Yu L 1989 Usp. Fiz. Nauk 158 B1 p 59
[9] Arnol’d V I 1990 Theory of Catastrophes (Nauka: Moscow)
[10] Haller G 1999 Chaos Near Resonance (Springer: Berlin)
[11] Orzag S A and Kells L C 1980 J. Fluid Mech. 96 (1) p 159
[12] Babin A V and Vishik M I 1982 Russian Math. Surveys, 37:3 195
[13] Vishik M I and Chepyzhov V V 2011 Russian Math. Surveys, 66:4 637
[14] Ladyzhenskaya O A 1975 J. Soviet Math., 3 458
[15] Vishik M I, Zelik S V and Chepyzhov V V 2013 Sb. Math., 204:1 1
[16] Landau L D., Lifshits E F 1959 Fluid mechanics (Perg. Press Oxford London)
[17] Constantin P, Foais C, Temam R 1988 Physica D 30 284
[18] Babin A V and Vishik M I 1983 Russian Math. Surveys, 38:4, 151
[19] Vishik M I, Komech A I 1983 Tr. Mosk. Mat. Obs., 46 3
[20] Packard N H, Crutchfield J P, Farmer J D, Shaw R S 1980 Phys.Rev. Lett. 45 N9 712
[21] Malraison B, Berge P, Dubois M 1983 J. Physique-Lett. 44 , P.L897
[22] Procaccia I, Grassberger P 1983 Phys.Rev. Lett. 50 346
[23] Procaccia I, Grassberger P 1983 Phys. Rev. A. 28 N4, 2591
[24] Gromov P R, Zobin F B, Rabinovich M I, Reiman AM, Sushchik M M 1987 DAN 292 N2, 284
[25] Rabinovich M I, Reiman AM, Sushchik M M and etc. 1987 JETP Letters. 13 (16) 987
[26] Brandstater A., Swift J. Harry L. Swinney and etc. 1983 Phys. Rev. let. 51 N16 1442
[27] Priymak V G 2013 Dok. Phys. 58:10 457
[28] Davidson P A 2004 Turbulence (Oxford Univ. Press) p 678
[29] Millionshchikov M D 1969 Turbulent flow in boundary layers and in pipes (Nauka: Moscow)
[30] Schlichting H 1968 Boundary-Layer Theory (6th Edition McGraw-Hill)
[31] Monin A S and Yaglom A M 1971 Statistical Fluid Mechanics (M.I.T. Press)
[32] Hinze J O 1975 Turbulence (McGraw-Hill)
[33] Dmitrenko A V 2008 Fundamentals of heat and mass transfer and hydrodynamics of single-
phase and two-phase media. Critical integral statistical methods and direct numerical
simulation. (Galleya print: Moscow) p 398 URL: http://search.rsl.ru/ru/catalog/record/6633402
[34] Dmitrenko A V 1997 33th AIAA/ASME/SAE/ASEE AIAA Paper97-2911; doi: 10.2514/6.1997-2911; http://arc.aiaa.org/doi/abs/10.2514/6.1997-2911.
[35] Dmitrenko A V 1998 34th AIAA/ASME/SAE/ASEE AIAA Paper98-3444; doi:10.2514/6.1998-3444; http://arc.aiaa.org/doi/abs/10.2514/6.1998-3444
[36] Dmitrenko A V 1993 Aviats. Tekh. No.1 39p
[37] Dmitrenko A V 1986 Proc.11th Conf. Young Scientists Moscow Physicotechnical Institute Part 2 Moscow (1986) pp 48–52 Deposited at VINITI 08.08.86 No 5698-V86
[38] Dmitrenko AV 2013 J.of Eng. Phys.and Thermophys.88(6)1569 doi.org/10.1007/s10891-017-1685-8
[39] Dmitrenko A V 2017 J.of Phys.:Conf.Series 1009 doi:10.1088/1742-6596/1009/1/012017
[40] Dmitrenko A V 2019 Contin. Mechan. and Thermod. doi.org/10.1007/s00161-019-00792-0
[41] Venichenko N K 1976 Turbulence in free atmosphere [in Russian] (L: Gidrometeoizdat) p 287
[42] Vorontsov P A 1966 Turbulence and vertical currents in the boundary layer atmosphere [in Russian] (L: Hydrometeorological publishing house) p 297
[43] Tatarsky V I 1967 Propagation of waves in a turbulent atmosphere [in Russian] (M:Nauka) p 548