Generalized information entropies in nonextensive quantum systems: The interpolation approach

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(September 20, 2009)

Abstract

We discuss the generalized von Neumann (Tsallis) entropy and the generalized Fisher information (GFI) in nonextensive quantum systems, by using the interpolation approximation (IA) which has been shown to yield good results for the quantal distributions within $O(q - 1)$ and in high- and low-temperature limits, $q$ being the entropic index [H. Hasegawa, Phys. Rev. E 80 (2009) 011126]. Three types of GFIs which have been proposed so far in the nonextensive statistics, are discussed from the viewpoint of their metric properties and the Cramér-Rao theorem. Numerical calculations of the $q$-and temperature-dependent Tsallis entropy and GFIs are performed for the electron band model and the Debye phonon model.

PACS No.: 05.30.-d, 89.70.Cf, 05.30.Fk, 05.30.Jp

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1 Introduction

The Boltzmann-Gibbs-Shannon-von Neumann entropy and the Fisher information play central roles as information measures in classical and quantum statistics (for review see [1], relevant references therein). The Boltzmann-Gibbs-Shannon-von Neumann entropy represents a global measure of ignorance, while the Fisher information expresses a local measure of a positive amount of information [1]. The Fisher information has the two properties: (i) it expresses the metric tensor for the neighboring points in the Riemannian space spanned by probability distributions or density matrices, and (ii) it provides the lower bound of unbiased estimation errors in the Cramér-Rao theorem (CRT). The Fisher information has been employed for a study of efficiency of information transmission and its decodings.

In the last decade, much progress has been made in the nonextensive statistics initiated by Tsallis [2], who proposed the generalized entropy (called the Tsallis entropy) defined by

\[ S_q = \frac{k_B}{q-1} \left[ 1 - \int p_q(x)^q \, dx \right] \quad \text{for classical case,} \]

\[ = \frac{k_B}{q-1} (1 - Tr \hat{\rho}_q^q) \quad \text{for quantum case.} \]  

Here \( q \) denotes the entropic index, \( k_B \) the Boltzmann constant, \( p_q(x) \) the probability distribution and \( \hat{\rho}_q \) the density matrix. The Tsallis entropy is a one-parameter generalization of the Boltzmann-Gibbs-Shannon-von Neumann entropy, to which it reduces in the limit of \( q \to 1.0 \). The Tsallis entropy is non-additive in a sense that for \( p(A \cup B) = p(A) p(B) \) or \( \hat{\rho}(A \cup B) = \hat{\rho}(A) \otimes \hat{\rho}(A) \), we obtain

\[ S_q(A \cup B) = S_q(A) + S_q(B) + \frac{(1-q)}{k_B} S_q(A) S_q(B), \]  

\[ \neq S_q(A) + S_q(B) \quad \text{for } q \neq 1. \]  

The Tsallis entropy is super-extensive (sub-extensive) for \( q < 1 \) (\( q > 1 \)), and \( q - 1 \) expresses the degree of the nonextensivity. The nonextensive statistics has been widely applied to various subjects in physics, chemistry, information science, biology and economics [3].

At the moment, three types of generalized Fisher informations (GFIs) have been proposed in the nonextensive statistics [4]-[10],

\[ g_{\theta_n \theta_m} = q Tr \left[ \hat{\rho}_q \left( \hat{\rho}_q^{-1} \frac{\partial \hat{\rho}_q}{\partial \theta_n} \right) \left( \hat{\rho}_q^{-1} \frac{\partial \hat{\rho}_q}{\partial \theta_m} \right) \right], \]  

\[ G_{\theta_n \theta_m} = Tr \left[ \hat{P}_q \left( \hat{P}_q^{-1} \frac{\partial \hat{P}_q}{\partial \theta_n} \right) \left( \hat{P}_q^{-1} \frac{\partial \hat{P}_q}{\partial \theta_m} \right) \right], \]  

\[ \tilde{g}_{\theta_n \theta_m} = Tr \left[ \hat{P}_q \left( \hat{P}_q^{-1} \frac{\partial \hat{P}_q}{\partial \theta_n} \right) \left( \hat{P}_q^{-1} \frac{\partial \hat{P}_q}{\partial \theta_m} \right) \right], \]  

where \( \hat{P}_q (= \hat{\rho}_q^q / Tr \hat{\rho}_q^q) \) expresses the escort density matrix and \( \theta_n \) a parameter specifying \( \hat{\rho}_q \): the classical case of Eqs. (5)-(7) is obtainable if we read \( \hat{\rho}_q \to p_q(x) \).
and $\text{Tr} \rightarrow \int dx$ [see Eqs. \((85)-(87)\)]. In the limit of $q \rightarrow 1.0$, the three GFIs given by Eqs. \((5), (6)\) and \((7)\) reduce to the conventional expression,

$$g_{\theta_n\theta_m} = \text{Tr} \left[ \hat{\rho}_1 \left( \frac{\partial \ln \hat{\rho}_1}{\partial \theta_n} \right) \left( \frac{\partial \ln \hat{\rho}_1}{\partial \theta_m} \right) \right]. \quad (8)$$

The GFI in Eq. \((5)\) which is derived with the use of the generalized Kullback-Leibler divergence \([11, 12, 13]\), expresses the metric tensor of the first properties (i) of the Fisher information discussed above. The GFI in Eq. \((5)\), however, is not applicable to the CRT for variance and covariance of physical quantities averaged over the escort density matrix. The GFI given by Eq. \((6)\) \([6, 7]\) preserves the second properties (ii) of the lower bound in the CRT, although it does not have the metric properties (i). The GFI given by Eq. \((6)\) \([8, 9, 10]\) has both the properties (i) and (ii). Details of the three GFIs and a comparison among them will be discussed in this paper (Secs. III and V).

The Tsallis entropy and generalized Fisher information (GFI) in nonextensive classical systems have been considerably studied in \([4, 6, 8, 9, 10, 14-19]\). In recent years, the generalized von Neumann (quantum Tsallis) entropy has been investigated in the bipartite spin-1/2 \([20]\), 1d spin-1/2 \([21]\), 2d bosonic systems \([21]\) and Hubbard dimers \([22]\). The purpose of this paper is to discuss the Tsallis entropy as well as the GFI in nonextensive quantum systems. Physical quantities in nonextensive quantum systems are evaluated by the trace over the escort density matrix \([23]\), which may be formally expressed in exact integral representations \([24, 25]\). Their actual evaluations are, however, tedious and difficult because they involve self-consistent calculations of expectation values of the energy and number of particles. This is the case in a calculation of the generalized Bose-Einstein and Fermi-Dirac distributions (referred to $q$-BED and $q$-FDD, respectively) in nonextensive quantum statistics. Quite recently it has been pointed out that this difficulty may be overcome when we adopt the interpolation approximation (IA) which yields good results within the $O(q-1)$ and in the high- and low-temperature limits \([26]\). Indeed the $q$-BED and $q$-FDD in the IA are given in simple analytic expressions, which have been successfully applied to nonextensive quantum systems such as black-body radiation, Bose-Einstein condensation, BCS superconductivity and metallic ferromagnetism \([27]\). The three GFIs which are formally expressed in exact integral representations, may be much simplified when we adopt the IA \([26]\).

The paper is organized as follows. In Sec. II, we briefly explain the exact and interpolation approaches to nonextensive quantum statistics \([26]\). In Sec. III, the Tsallis entropy and the three GFIs in the IA are discussed. Numerical calculations are reported in Sec. IV for the electron band model and for the Debye phonon model. Discussion and conclusion are presented in Sec. V, where a comparison among the three GFIs is made. The CRT for the three GFIs in \([8, 9, 10, 7]\) is discussed also for the $q$-Gaussian distribution in the nonextensive classical statistics.
2 Exact and interpolation approaches

2.1 Exact approach

We first obtain the optimum density matrix \( \hat{\rho}_q \), applying the maximum entropy method (MEM) with the optimum Langange multiplier (OLM) \([23]\) to the generalized von Neumann (Tsallis) entropy given by Eq. (2) under the constraints given by

\[
\begin{align*}
Tr \hat{\rho}_q &= 1, \\
Tr \{ \hat{\rho}_q^q \hat{N} \} &= Tr \hat{\rho}_q^q N_q, \\
Tr \{ \hat{\rho}_q^q \hat{H} \} &= Tr \hat{\rho}_q^q E_q,
\end{align*}
\]

where \( N_q \) and \( E_q \) denote expectation values of the number operator (\( \hat{N} \)) and the Hamiltonian (\( \hat{H} \)), respectively. The OLM-MEM leads to the density matrix given by \([23]\)

\[
\hat{\rho}_q = \frac{1}{X_q} \hat{w},
\]

with

\[
\begin{align*}
X_q &= Tr \hat{w}, \\
N_q &= \frac{1}{X_q} Tr \{ \hat{w}^q \hat{N}\}, \\
E_q &= \frac{1}{X_q} Tr \{ \hat{w}^q \hat{H}\}, \\
\hat{w} &= [(1 - (1 - q)\beta) (\hat{H} - \mu \hat{N} - E_q + \mu N_q)]^{1/(1-q)},
\end{align*}
\]

where \( \beta \) stands for the inverse temperature and \( \mu \) the chemical potential (Fermi level). The escort density matrix \( \hat{P}_q \) is expressed by

\[
\hat{P}_q = \frac{\hat{\rho}_q^q}{c_q} = \frac{1}{X_q} \hat{w}^q,
\]

where we have employed the relation,

\[
c_q = Tr \hat{\rho}_q^q = X_q^{1-q}.
\]

By using the formulae for the gamma function \( \Gamma(z) \) given by

\[
x^{-s} = \frac{1}{\Gamma(s)} \int_0^\infty u^{s-1} e^{-xu} du \quad \text{for } \Re s > 0, \quad (17)
\]

\[
x^s = \frac{i}{2\pi} \Gamma(s+1) \int_C (-t)^{-s-1} e^{-xt} dt \quad \text{for } \Re s > 0, \quad (18)
\]
we may express Eqs. (11)-(13) as integrals along the real axis for $q > 1.0$ and in the complex plane for $q < 1.0$ as follows [24, 25, 26]:

$$X_q = \begin{cases} \int_0^\infty G \left( u; \frac{1}{q-1}, \frac{1}{(q-1)\beta} \right) Y_1(u) \, du & \text{for } q > 1, \\ \frac{i}{2\pi} \oint_C H \left( t; \frac{1}{1-q}, \frac{1}{(1-q)\beta} \right) Y_1(-t) \, dt & \text{for } q < 1, \end{cases} \quad (19)$$

$$N_q = \begin{cases} \frac{1}{X_q} \int_0^\infty G \left( u; \frac{q}{q-1}, \frac{1}{(q-1)\beta} \right) Y_1(u) N_1(u) \, du & \text{for } q > 1, \\ \frac{i}{2\pi X_q} \oint_C H \left( t; \frac{q}{1-q}, \frac{1}{(1-q)\beta} \right) Y_1(-t) N_1(-t) \, dt & \text{for } q < 1, \end{cases} \quad (20)$$

$$E_q = \begin{cases} \frac{1}{X_q} \int_0^\infty G \left( u; \frac{q}{q-1}, \frac{1}{(q-1)\beta} \right) Y_1(u) E_1(u) \, du & \text{for } q > 1, \\ \frac{i}{2\pi X_q} \oint_C H \left( t; \frac{q}{1-q}, \frac{1}{(1-q)\beta} \right) Y_1(-t) E_1(-t) \, dt & \text{for } q < 1, \end{cases} \quad (21)$$

where

$$Y_1(u) = e^{u(E_q - \mu N_q)} \Xi_1(u), \quad (22)$$

$$\Xi_1(u) = Tr e^{-u(H - \mu \hat{N})} = e^\pm \sum_k \ln[1 \pm e^{-u(\epsilon_k - \mu)}], \quad (23)$$

$$N_1(u) = \sum_k f_1(\epsilon_k, u), \quad (24)$$

$$E_1(u) = \sum_k f_1(\epsilon_k, u) \epsilon_k, \quad (25)$$

$$f_1(\epsilon, u) = \frac{1}{e^{u(\epsilon - \mu)} \mp 1}, \quad (26)$$

$$G(u; a, b) = b^a u^{-a-1} e^{-bu}, \quad (27)$$

$$H(t; a, b) = \Gamma(a+1)b^a (-t)^{-a-1} e^{-bt}. \quad (28)$$

Upper and lower signs in Eqs. (23) and (26) are applied to boson and fermion, respectively, and $C$ denotes the Hankel path in the complex plane [24, 25]. Note that $N_q$ and $E_q$ are obtained by Eqs. (19)-(21) in a self-consistent way. Such self-consistent calculations have been made for the electron band model and the Debye phonon model in [26].

2.2 Interpolation approach

Self-consistent calculations including $N_q$ and $E_q$ are rather tedious and difficult. In order to overcome this difficulty, we have proposed the IA [26], assuming that

$$\frac{Y_1(u)}{X_q} = \frac{1}{X_q} e^{u(E_q - \mu N_q)} \Xi_1(u) = 1, \quad (29)$$
in Eqs. (20) and (21). Then they are expressed by

\[
N_q = \begin{cases} 
\int_0^\infty G \left( u; \frac{q}{q-1}, \frac{1}{(q-1)\beta} \right) N_1(u) \, du & \text{for } q > 1, \\
\frac{i}{2\pi} \int_C H \left( t; \frac{q}{1-q}, \frac{1}{(1-q)\beta} \right) N_1(-t) \, dt & \text{for } q < 1,
\end{cases}
\]  
\tag{30}

\[
E_q = \begin{cases} 
\int_0^\infty G \left( u; \frac{q}{q-1}, \frac{1}{(q-1)\beta} \right) E_1(u) \, du & \text{for } q > 1, \\
\frac{i}{2\pi} \int_C H \left( t; \frac{q}{1-q}, \frac{1}{(1-q)\beta} \right) E_1(-t) \, dt & \text{for } q < 1.
\end{cases}
\]  
\tag{31}

Equations (20) and (21) are alternatively expressed by

\[
N_q = \sum_k f_q(\epsilon_k, \beta),
\]  
\tag{32}

\[
E_q = \sum_k f_q(\epsilon_k, \beta) \epsilon_k,
\]  
\tag{33}

where the quantal distribution \( f_q(\epsilon_k, \beta) \) is given by

\[
f_q(\epsilon_k, \beta) = \begin{cases} 
\int_0^\infty G \left( u; \frac{q}{q-1}, \frac{1}{(q-1)\beta} \right) f_1(\epsilon_k, u) \, du & \text{for } q > 1, \\
\frac{i}{2\pi} \int_C H \left( t; \frac{q}{1-q}, \frac{1}{(1-q)\beta} \right) f_1(\epsilon_k, -t) \, dt & \text{for } q < 1.
\end{cases}
\]  
\tag{34}

With the use of Eq. (34), the analytic expression of the \( q \)-BED in the IA is given by [26]

\[
f_q(\epsilon, \beta) = \sum_{n=0}^{\infty} \left[ e_q^{- (n+1) x} \right]^q \quad \text{for } 0 < q < 3,
\]  
\tag{35}

where \( x = \beta (\epsilon - \mu) \) and \( e_q^x \) expresses the \( q \)-exponential function defined by

\[
e_q^x = \left[ 1 + (1 - q)x \right]_+^{1/(1-q)},
\]  
\tag{36}

with \([y]_+ = y \Theta(y)\) and \( \Theta(y) \) is the Heaviside function.

Similarly, the analytic expression of the \( q \)-FDD in the IA is given by [26]

\[
f_q(\epsilon, \beta) = \begin{cases} 
F_q(x) & \text{for } \epsilon > \mu, \\
\frac{1}{2} & \text{for } \epsilon = \mu, \\
1.0 - F_q(|x|) & \text{for } \epsilon < \mu,
\end{cases}
\]  
\tag{37}

with

\[
F_q(x) = \sum_{n=0}^{\infty} (-1)^n [e_q^{-(n+1)x}]^q \quad \text{for } 0 < q < 3,
\]  
\tag{38}

where \( x = \beta (\epsilon - \mu) \). \( f_q^{IA}(\epsilon, \beta) \) given by Eqs. (35) and (37) reduces to \( f_1(\epsilon, \beta) \) in the limit of \( q \to 1.0 \) where \( e_q^x \to e^x \).
3 Information Entropies

3.1 Tsallis entropy

By using Eqs. (2) and (16), we obtain

\[ S_q = k_B \frac{1 - c_q}{q - 1} = k_B \ln_q(X_q), \]  

where \( X_q \) is given by Eq. (19), or alternatively by

\[ X_q = \begin{cases} 
\int_0^\infty G_\beta(u; \frac{1}{q-1}, \frac{1}{(q-1)\beta}) e^{\beta \sum_k (\epsilon_k - \mu) f_q(\epsilon_k, \beta) + \sum_k \ln[1 \pm f_1(\epsilon_k, \beta)]} \, du, & \text{for } q > 1, \\
\frac{i}{2\pi} \int_\mathcal{C} H_\beta(t; \frac{1}{1-q}, \frac{1}{(1-q)\beta}) e^{-\beta \sum_k (\epsilon_k - \mu) f_q(\epsilon_k, \beta) + \sum_k \ln[1 \pm f_1(\epsilon_k, \beta)]} \, dt & \text{for } q < 1,
\end{cases} \]

\[ \ln_q(x) = \frac{x^{1-q} - 1}{1 - q}. \]

In the limit of \( q \to 1.0 \) where \( \ln_q(x) \to \ln(x) \), Eq. (40) reduces to

\[ X_1 = e^{\beta \sum_k (\epsilon_k - \mu) f_1(\epsilon_k, \beta) + \sum_k \ln[1 \pm f_1(\epsilon_k, \beta)]}, \]

which yields the well-known expression of the quantum Boltzmann-Gibbs entropy given by

\[ S_1 = k_B \ln(X_1) = k_B \beta \left( \sum_k (\epsilon - \mu) f_1(\epsilon_k, \beta) + \sum_k [1 \pm f_1(\epsilon_k, \beta)] \right), \]

\[ = -k_B \left( \sum_k f_1 \ln f_1 + \sum_k (1 \pm f_1) \ln(1 \pm f_1) \right). \]

3.2 Generalized Fisher information matrix

3.2.1 \( g_{\theta_n \theta_m} \)

The distance between two operators \( \hat{\rho} \) and \( \hat{\sigma} \) in the Riemann space is defined by

\[ D_q(\hat{\rho} \parallel \hat{\sigma}) = K_q(\hat{\rho} \parallel \hat{\sigma}) + K_q(\hat{\sigma} \parallel \hat{\rho}), \]  

with the generalized Kullback-Leibler divergence \( K_q(\hat{\rho} \parallel \hat{\sigma}) \) [11, 12],

\[ K_q(\hat{\rho} \parallel \hat{\sigma}) = \frac{1}{1-q} (Tr \hat{\rho} - Tr \{ \hat{\rho}^q \hat{\sigma}^{1-q} \}), \]

which is in conformity with the Tsallis entropy given by Eq. (2). The distance between the neighboring operators of \( \hat{\rho}_q(\{\theta_n\}) \) (\( \equiv \hat{\rho}_q \)) and \( \hat{\rho}_q(\{\theta_n + \delta \theta_n\}) \) (\( \equiv \hat{\rho}_q' \)) is expressed by

\[ D_q(\hat{\rho}_q \parallel \hat{\rho}_q') \approx \sum_{nm} g_{\theta_n \theta_m} \delta \theta_n \delta \theta_m. \]
where the GFI of \( g_{\theta_n \theta_m} \) is given by \[5\]

\[
 g_{\theta_n \theta_m} = q \left\langle \left( \frac{\partial \ln \hat{\rho}_q}{\partial \theta_n} \right) \left( \frac{\partial \ln \hat{\rho}_q}{\partial \theta_m} \right) \right\rangle_q, \tag{46}
\]

the bracket \( \langle \hat{Q} \rangle_q \) denoting the expectation value of an operator \( \hat{Q} \) over \( \hat{\rho}_q \):

\[
 \langle \hat{Q} \rangle_q = Tr \{ \hat{\rho}_q \hat{Q} \}. \tag{47}
\]

The GFI given by Eq. \( (46) \) has a clear geometrical meaning expressing the metric between the adjacent density matrices in the Riemannian space spanned by the OLM density matrices of \( \{ \hat{\rho}_q \} \). Equations \( (44) \) and \( (46) \) are quantum extensions of the counterparts in the nonextensive classical statistics \[4\].

For \( (\theta_1, \theta_2) = (\beta, \beta \mu) \), Eqs. \( (10) \) and \( (46) \) yield

\[
 g_{11} = \left( \frac{q}{X_q} \right) Tr \{ \hat{\omega}^{2q-1}(\hat{H} - E_q)^2 \}, \tag{48}
\]

\[
 g_{22} = \left( \frac{q}{X_q} \right) Tr \{ \hat{\omega}^{2q-1}(\hat{N} - N_q)^2 \}, \tag{49}
\]

\[
 g_{12} = g_{21} = - \left( \frac{q}{X_q} \right) Tr \{ \hat{\omega}^{2q-1}(\hat{H} - E_q)(\hat{N} - N_q) \}. \tag{50}
\]

### 3.2.2 \( G_{\theta_n \theta_m} \)

Naudts proposed the quantum GFI given by \[7\]

\[
 G_{\theta_n \theta_m} = \left[ \left( \hat{P}_q^{-1} \frac{\partial \hat{\rho}_q}{\partial \theta_n} \right) \left( \hat{P}_q^{-1} \frac{\partial \hat{\rho}_q}{\partial \theta_m} \right) \right]_q, \tag{51}
\]

where the bracket \( [\hat{Q}]_q \) denotes the expectation value given by

\[
 [\hat{Q}]_q = Tr \{ \hat{\rho}_q \hat{Q} \}. \tag{52}
\]

When we consider the expectation value of \( [\hat{A}]_q \) for an operator of \( \hat{A} \) and \( \theta_n = \theta_m = \theta \), the generalized CRT is shown to be expressed by \[7\]

\[
 (\hat{A} - [\hat{A}]_q)^2 \geq \frac{(\partial A')^2}{G_{\theta \theta}}, \tag{53}
\]

where

\[
 A' = \langle \hat{A} \rangle_q. \tag{54}
\]

Note that \( A' \) is an average over \( \hat{\rho}_q \). The CRT given by Eq. \( (53) \) may be derived as follows \[7\]. Taking the derivative of \( A' \) with respect to \( \theta \) and using the relation: \( \partial \hat{\rho}_q / \partial \theta = 0 \), we obtain

\[
 \frac{\partial A'}{\partial \theta} = Tr \{ \hat{\rho}_q \frac{\partial \hat{\rho}_q}{\partial \theta} (\hat{A} - [\hat{A}]_q) \}, \tag{55}
\]

\[
 = Tr \{ \hat{P}_q \left( \hat{P}_q^{-1} \frac{\partial \hat{\rho}_q}{\partial \theta} \right) (\hat{A} - [\hat{A}]_q) \}. \tag{56}
\]
Employing the Cauchy-Schwartz inequality, we obtain
\[
\left( \frac{\partial A'}{\partial \theta} \right)^2 \leq \text{Tr} \{ \hat{P}_q \left( \hat{P}_q^{-1} \frac{\partial \hat{P}_q}{\partial \theta} \right)^2 \} \text{Tr} \{ \hat{P}_q (\hat{A} - [\hat{A}]_q)^2 \},
\]
which leads to Eq. (53).

It is straightforward to extend the method mentioned above to the case of \((\hat{A}_1, \hat{A}_2) = (\hat{H}, \hat{N})\) and \((\theta_1, \theta_2) = (\beta, \beta\mu)\). The generalized CRT is expressed by
\[
V \geq C^T G^{-1} C \equiv D, \tag{58}
\]
where the covariance matrix \(V\) is given by
\[
V = \begin{pmatrix}
\left[ (\hat{H} - E_q)^2 \right]_q & \left[ (\hat{H} - E_q)(\hat{N} - N_q) \right]_q \\
\left[ (\hat{H} - E_q)(\hat{N} - N_q) \right]_q & \left[ (\hat{N} - N_q)^2 \right]_q
\end{pmatrix}.
\]
Calculations using Eqs. (15) and (51) yield elements of the Fisher information matrix \(G\) given by
\[
G_{11} = \left[ (\hat{H} - E_q)^2 \right]_q = V_{11}, \tag{59}
\]
\[
G_{22} = \left[ (\hat{N} - N_q)^2 \right]_q = V_{22}, \tag{60}
\]
\[
G_{12} = G_{21} = -\left[ (\hat{H} - E_q)(\hat{N} - N_q) \right]_q = -V_{12} = -V_{21}, \tag{61}
\]
and those of \(C\) expressed by
\[
C_{11} = \frac{\partial \langle \hat{H} \rangle_q}{\partial \beta} = -\left[ \hat{H}(\hat{H} - E_q) \right]_q, \tag{62}
\]
\[
C_{22} = \frac{\partial \langle \hat{N} \rangle_q}{\partial (\beta\mu)} = \left[ \hat{N}(\hat{N} - N_q) \right]_q, \tag{63}
\]
\[
C_{12} = \frac{\partial \langle \hat{H} \rangle_q}{\partial (\beta\mu)} = \left[ \hat{H}(\hat{N} - N_q) \right]_q, \tag{64}
\]
\[
C_{21} = \frac{\partial \langle \hat{N} \rangle_q}{\partial \beta} = -\left[ \hat{N}(\hat{H} - E_q) \right]_q. \tag{65}
\]
A simple calculation leads to
\[
D = V, \tag{66}
\]
which implies that the CRT given by Eq. (58) is satisfied with an equal sign.
The GFI given by $$\tilde{g}_{\theta_n \theta_m} = \left[ \left( \frac{\partial \ln \hat{P}_q}{\partial \theta_n} \right) \left( \frac{\partial \ln \hat{P}_q}{\partial \theta_m} \right) \right]_q,$$ (67)
provides the lower bound of unbiased estimates in the CRT as shown in the following. For the expectation value of $$A = \hat{A}_q$$, we obtain
$$\frac{\partial A}{\partial \theta} = Tr \{ \hat{P} \left( \hat{P}^{-1} \frac{\partial \hat{P}}{\partial \theta} \right) (\hat{A} - \hat{A}_q) \}.$$ (68)
$$= Tr \{ \hat{P} \left( \hat{P}^{-1} \frac{\partial \hat{P}}{\partial \theta} \right) (\hat{A} - \hat{A}_q) \}.$$ (69)

By using the Cauchy-Schwartz inequality, we then obtain the CRT given by
$$[(\hat{A} - \hat{A}_q)^2]_q \geq \frac{\left( \frac{\partial A}{\partial \theta} \right)^2}{\tilde{g}_{\theta \theta}}.$$ (70)

With the use of Eq. (15) and (67), the generalized CRT for the case of $$(\hat{A}_1, \hat{A}_2) = (\hat{H}, \hat{N})$$ and $$(\theta_1, \theta_2) = (\beta, \beta \mu)$$ is expressed by
$$\mathbf{V} \geq \tilde{C}^T \tilde{g}^{-1} \tilde{C} \equiv \tilde{D},$$ (71)
where elements of $$\tilde{g}$$ are given by
$$\tilde{g}_{11} = \left( \frac{q^2}{X_q} \right) Tr \{ \hat{w}^{3q-2} (\hat{H} - E_q)^2 \},$$ (72)
$$\tilde{g}_{22} = \left( \frac{q^2}{X_q} \right) Tr \{ \hat{w}^{3q-2} (\hat{N} - N_q)^2 \},$$ (73)
$$\tilde{g}_{12} = \tilde{g}_{21} = - \left( \frac{q^2}{X_q} \right) Tr \{ \hat{w}^{3q-2} (\hat{H} - E_q)(\hat{N} - N_q) \},$$ (74)
and those of $$\tilde{C}$$ are given by
$$\tilde{C}_{11} = \frac{\partial E_q}{\partial \beta} = - \left( \frac{q}{X_q} \right) Tr \{ \hat{w}^{2q-1} \hat{H} (\hat{H} - E_q) \},$$ (75)
$$\tilde{C}_{22} = \frac{\partial N_q}{\partial (\beta \mu)} = \left( \frac{q}{X_q} \right) Tr \{ \hat{w}^{2q-1} \hat{N} (\hat{N} - N_q) \},$$ (76)
$$\tilde{C}_{12} = \frac{\partial E_q}{\partial (\beta \mu)} = \left( \frac{q}{X_q} \right) Tr \{ \hat{w}^{2q-1} \hat{H} (\hat{N} - N_q) \},$$ (77)
$$\tilde{C}_{21} = \frac{\partial N_q}{\partial \beta} = - \left( \frac{q}{X_q} \right) Tr \{ \hat{w}^{2q-1} \hat{N} (\hat{H} - E_q) \}.$$ (78)

It is noted that the GFI given by Eq. (67) is nothing but a quantum extension of that proposed in nonextensive classical statistics [8, 9, 10].
Table 1: A comparison among three GFIs for $\theta_n = \theta_m = \theta \ (q \neq 1.0)$

| GFI | Metric | CRT | Refs. |
|-----|--------|-----|-------|
| $g_{\theta\theta} = q \text{Tr}\{\hat{\rho}_q^{-1}(\partial\hat{\rho}_q/\partial\theta)^2\}$ | Eq. (45) | NA | [4, 5] |
| $G_{\theta\theta} = \text{Tr}\{\hat{P}_q^{-1}(\partial\hat{P}_q/\partial\theta)^2\}$ | NA | Eq. (53) | [6, 7] |
| $\tilde{g}_{\theta\theta} = \text{Tr}\{\hat{P}_q^{-1}(\partial\hat{P}_q/\partial\theta)^2\}$ | Eq. (79) | Eq. (70) | [8, 9, 10] |

CRT: Cramér-Rao theorem, NA: not applicable

We may show that $\tilde{g}_{\theta_n\theta_m}$ given by Eq. (67) denotes the metric tensor for the neighboring escort density matrices of $\hat{P}_q(\{\theta_n\}) (\equiv \hat{P}_q)$ and $\hat{P}_q(\{\theta_n + \delta\theta_n\}) (\equiv \hat{P}_q')$ as given by

$$D_1(\hat{P}_q||\hat{P}_q') \cong \sum_{nm} \tilde{g}_{\theta_n\theta_m} \delta\theta_n \delta\theta_m, \quad (79)$$

where $D_1(\hat{\rho}||\hat{\sigma})$ stands for the distance given by Eqs. (43) and (44) with $q = 1.0$.

3.2.4 A comparison among $g$, $G$ and $\tilde{g}$

In the preceding subsections, we have discussed the properties of $g$, $G$ and $\tilde{g}$. A comparison among the three GFIs is made in Table 1. We note that $g$ expresses the metric in the Riemann space spanned by density matrices $(\hat{\rho}_q)$, but $G$ does not. In contrast, $G$ provides the lower bound for the CRT, to which $g$ is not applicable. On the contrary, $\tilde{g}$ denotes the distance between the escort density matrices $(\hat{P}_q)$ and it also satisfies the CRT. It is shown that $G$ provides a better bound for the inequality in the CRT than $\tilde{g}$: $V = D \geq \tilde{D}$.

When we adopt the exact transformation with the use of formulae for the gamma function given by Eqs. (17) and (18), equations for the GFIs may be expressed as integrals along real axis for $q \geq 1.0$ or in the complex plane for $q < 1.0$. Expressions of the GFIs for $q \geq 1.0$ with the IA [Eq. (29)] are summarized in the Appendix. It is now possible for us to numerically calculate the Tsallis entropy $S_q$ and the GFI matrices as functions of $q$ and temperature. We will report such numerical calculations for the electron band model and the Debye phonon model in the following section.

4 Numerical calculations

4.1 Electron band model

We employ a band model for electrons with a uniform density of state given by [26]

$$\rho(\epsilon) = (1/2W) \Theta(W - |\epsilon|), \quad (80)$$
where $W$ denotes a half of the total band width. We adopt $N = 0.5$, for which
$\mu = 0.0$ independently of the temperature because of the adopted symmetric density of
states given by Eq. (80).

Figure 1 shows the temperature dependence of $E_q$ for various $q$ calculated self-
consistently with the use of Eqs. (19)-(21). With increasing $q$ from unity, $E_q$ at
higher temperatures is decreased, although that at lower temperatures is increased
(see also the inset of Fig. 1 of Ref. [10]). We have calculated the average energy also
by using Eq. (31) in the IA. The ratio of $E_{IA}^q/E_q (\equiv \lambda)$ is plotted in the inset of Fig.
1. For $q = 1.5$, for example, this ratio is changed from unity at low temperatures
($k_B T / W \sim 0.005$) to about 0.991 at high temperatures ($k_B T / W \sim 1.0$). For $1.0 < q < 1.5$, an agreement between $E_q$ and $E_{IA}^q$ is much better.

The temperature dependence of $S_q$ for various $q$ is shown in Fig. 2. With increas-
ing $q$ from unity, the temperature dependence of $S_q$ at low temperatures becomes
more significant but its saturated value at high temperatures become smaller.

Solid, chain and dotted curves in Fig. 3 show temperature dependence of $G_{ii}$, $g_{ii}$
and $\tilde{g}_{ii} (i = 1, 2)$, respectively, for $q = 1.1$: results for $q = 1.0$ are plotted by dashed
curves. Note that $V_{ii} = G_{ii}$ [Eqs. (59) and (60)] which implies that the temperature
dependence of the variance $V_{ii}$ is the same as that of $G_{ii}$. All the GFIs show a similar
temperature dependence. A closer inspection, however, shows that there are some
differences between them. In particular, with increasing $q$ from unity, $g_{ii}$ and $\tilde{g}_{ii}$ are
increased while $G_{ii}$ is decreased.

4.2 Debye phonon model

We adopt the Debye model whose phonon density of states is given by [26]

$$\rho(\omega) = A \omega^2 \quad \text{for} \ 0 < \omega \leq \omega_D,$$

(81)

where $A = 9N_a/w_D^3$, $N_a$ denotes the number of atoms, $\omega$ the phonon frequency and
$\omega_D$ the Debye cutoff frequency.

Figure 4 shows the temperature dependence of $E_q$ for various $q$ (with $\mu = 0.0$),
which is self-consistently calculated by Eqs. (19)-(21). It shows that $E_q$ is larger for
larger $q$. The inset of Fig. 4 shows the ratio of $E_{IA}^q/E_q (\equiv \lambda)$ where $E_{IA}^q$ is calculated
within the IA [Eq. (31)]. Although the ratio is not good at $0.02 \lesssim T/T_D \lesssim 0.5$,
it becomes better at $1 \lesssim T/T_D \lesssim 2$, where $T_D$ signifies the Debye temperature
($k_B T_D = \hbar \omega_D$).

Figure 5 shows the temperature dependence of $S_q$. We note that for larger $q$, the
temperature dependence at low temperatures becomes more steep and its saturated
value at high temperatures becomes smaller.

Solid, chain and dotted curves in Fig. 6 show the temperature dependence of $G_{11}$, $g_{11}$
and $\tilde{g}_{11}$, respectively, for $q = 1.1$: results for $q = 1.0$ are shown by dashed
curves. Note that the temperature dependence of the variance $V_{11}$ is the same as that
of $G_{11}$ [Eq. (59)]. All the GFIs show a similar temperature dependence, although
with increasing $q$, magnitudes of $G_{ii}$ and $g_{ii}$ becomes larger while that of $\tilde{g}_{ii}$ becomes
smaller.
Filled and open marks in Fig. 7 show the \( q \) dependence of \( D \) and \( \tilde{D} \), respectively, for the Debye phonon model: note that \( D = V \) for the GFI in [7]. It is shown that the ratio of \( \tilde{D}/V \), which is unity for \( q = 1.0 \), is gradually reduced with increasing \( q \).

5 Conclusion and discussion

It is instructive to make a comparison among \( g \), \( G \) and \( \tilde{g} \) for the \( q \)-Gaussian distribution in classical nonextensive statistics. For given mean \( \langle \mu_q \rangle \) and variance \( \langle \sigma^2_q \rangle \), the \( q \)-Gaussian distribution \( p_q(x) \) and its escort distribution \( P_q(x) \) are expressed by [10]

\[
p_q(x) = \frac{1}{Z_q} \left[ 1 - (1-q) \frac{(x - \mu_q)^2}{2\nu\sigma^2_q} \right]^{1/(1-q)}, \quad (82)
\]

\[
P_q(x) = \frac{1}{\nu Z_q} \left[ 1 - (1-q) \frac{(x - \mu_q)^2}{2\nu\sigma^2_q} \right]^{q/(1-q)}, \quad (83)
\]

with

\[
Z_q = \begin{cases} 
\left( \frac{2\nu\sigma^2_q}{q-1} \right)^{1/2} B \left( \frac{1}{2}, \frac{1}{q-1} - \frac{1}{2} \right) & \text{for } 1 < q < 3, \\
\sqrt{2\pi\sigma_q} & \text{for } q = 1, \\
\left( \frac{2\nu\sigma^2_q}{1-q} \right)^{1/2} B \left( \frac{1}{2}, \frac{1}{1-q} + 1 \right) & \text{for } 0 < q < 1,
\end{cases} \quad (84)
\]

where \( \nu = (3 - q)/2 \) and \( B(a,b) \) stands for the Beta function. The GFIs given by Eqs. (46), (51) and (67) in the classical case are expressed by

\[
g_{ij} = q \int p_q(x) \left( \frac{1}{p_q(x)} \frac{\partial p_q(x)}{\partial \theta_i} \right) \left( \frac{1}{p_q(x)} \frac{\partial p_q(x)}{\partial \theta_j} \right) dx, \quad (85)
\]

\[
G_{ij} = \int P_q(x) \left( \frac{1}{P_q(x)} \frac{\partial P_q(x)}{\partial \theta_i} \right) \left( \frac{1}{P_q(x)} \frac{\partial P_q(x)}{\partial \theta_j} \right) dx, \quad (86)
\]

\[
\tilde{g}_{ij} = \int P_q(x) \left( \frac{1}{P_q(x)} \frac{\partial P_q(x)}{\partial \theta_i} \right) \left( \frac{1}{P_q(x)} \frac{\partial P_q(x)}{\partial \theta_j} \right) dx. \quad (87)
\]

Averages over \( p_q(x) \) and \( P_q(x) \) are expressed by \( \langle \hat{Q} \rangle_q \) and \( [\hat{Q}]_q \), respectively.

For \( (A_1, A_2) = ([x]_q, [(\delta x)^2]_q) \) and \( (\theta_1, \theta_2) = (\mu_q, \sigma^2_q) \) where \( \delta x = x - \mu_q \), we obtain [10]

\[
g = \begin{pmatrix} \frac{1}{\sigma^2_q} & 0 \\ 0 & \frac{3-q}{4\sigma^2_q} \end{pmatrix},
\]

\[
\tilde{g} = \begin{pmatrix} \frac{q(q+1)}{(3-q)(2q-1)\sigma^2_q} & 0 \\ 0 & \frac{(q+1)}{4(2q-1)\sigma^2_q} \end{pmatrix}.
\]
\[ V = \begin{pmatrix} \sigma_q^2 & 0 \\ 0 & \frac{4\sigma_q^4}{(5-3q)} \end{pmatrix}, \]
\[ \tilde{C} = \begin{pmatrix} \frac{\partial[x_q]}{\partial \mu_q} & \frac{\partial[x_q]}{\partial \sigma_q^2} \\ \frac{\partial[(\delta x)^2]_q}{\partial \mu_q} & \frac{\partial[(\delta x)^2]_q}{\partial \sigma_q^2} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \]

The CRT is then expressed by
\[ V \geq \tilde{C}^T \tilde{g}^{-1} \tilde{C} \equiv \tilde{D} = \begin{pmatrix} \frac{(3-q)(2q-1)\sigma_q^2}{q(q+1)} & 0 \\ 0 & \frac{4(2q-1)\sigma_q^4}{(q+1)} \end{pmatrix}. \]

On the contrary, for the GFI of \( G \) \([6, 7]\) with \((A_1', A_2') = (\langle x \rangle_q, \langle (\delta x)^2 \rangle_q)\) and \((\theta_1, \theta_2) = (\mu_q, \sigma_q^2)\), we obtain
\[ G = \begin{pmatrix} \frac{1}{\sigma_q^2} & 0 \\ 0 & \frac{(3-q)^2}{4(5-3q)\sigma_q^4} \end{pmatrix}, \]
\[ C = \begin{pmatrix} \frac{\partial x_q}{\partial \mu_q} & \frac{\partial x_q}{\partial \sigma_q^2} \\ \frac{\partial[(\delta x)^2]_q}{\partial \mu_q} & \frac{\partial[(\delta x)^2]_q}{\partial \sigma_q^2} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{3-q}{5-3q} \end{pmatrix}. \]

The CRT is expressed by
\[ V \geq C^T G^{-1} C \equiv D = \begin{pmatrix} \sigma_q^2 & 0 \\ 0 & \frac{4\sigma_q^4}{(5-3q)} \end{pmatrix}, \]
which shows that the CRT is satisfied with an equal sign.

Solid curves in Figs 8(a) and 8(b) show \( D_{11}/\sigma_q^2 \) and \( D_{22}/\sigma_q^4 \), respectively, dotted curves expressing \( V_{11}/\sigma_q^2 \) and \( V_{22}/\sigma_q^4 \). Chain curves in Figs 8(a) and 8(b) show \( \tilde{D}_{11}/\sigma_q^2 \) and \( \tilde{D}_{22}/\sigma_q^4 \), respectively. For a comparison, we show by the dashed curve in Fig. 8(b), \( 1/g_{22}\sigma_q^4 \) \([10]\). These figures clearly show that \( G \) and \( \tilde{g} \) preserve the CRT and that \( g \) is not applicable to the CRT because \( V_{22} < 1/g_{22} \) for \( q < 1 \) in Fig. 8(b). The \( q \) dependence of \( \tilde{D}/V \) in Figs 8(a) and 8(b) is similar to that for Debye phonon model shown in Fig. 7.

Adopting the factorization approximation (FA) to a calculation of the grandcanonical partition function, Büyükkılıç, Demirhan and Gülec \([28]\) derived the generalized quantal distribution given by
\[ f_q^{FA}(\epsilon) = \frac{1}{\{e_q[-\beta(\epsilon - \mu)]\}^{-1} + 1}. \]
where the upper (lower) sign is applied to $q$-BED ($q$-FDD). It has been shown that $f_q^{FA}(\epsilon_k)$ may be alternatively derived by applying the variational condition to the entropy $S_q^{FA}$ given by [29]

$$S_q^{FA} = - \sum_k \{[f_q^{FA}(\epsilon_k)]^q \ln_q [f_q^{FA}(\epsilon_k)] \mp [1 \pm f_q^{FA}(\epsilon_k)]^q \ln_q [1 \pm f_q^{FA}(\epsilon_k)]\}, \tag{89}$$

with the constraints:

$$\sum_k [f_q^{FA}(\epsilon_k)]^q = N,$n
$$\sum_k [f_q^{FA}(\epsilon_k)]^q \epsilon_k = E.$n

Quite recently the $q$-BED and $q$-FDD given by Eq. (88) in the FA are criticized based on the exact approach [26]. It has been pointed out that the $O(q-1)$-order contribution in the FA does not agree with that of the exact approach and that its $q$-FDD yields inappropriate results even qualitatively. This criticism is applied also to the expression for the FA entropy given by Eq. (89).

To summarize, we have discussed the generalized von Neumann (Tsallis) entropy and the GFI in nonextensive quantum systems, by using the IA [26]. Numerical calculations of the $q$- and temperature-dependent information entropies have been performed for the electron band model and the Debye phonon model. A comparison among the three GFIs (Table 1) has shown that for the CRT in the nonextensive statistics, we have to employ $G$ [7] or $\tilde{g}$ [8, 9, 10] rather than $g$ [5] which has a geometrical meaning derived from the generalized Kullback-Leibler divergence. Although our present discussion has been confined to $q \geq 1.0$ for nonextensive quantum systems, it is necessary to extend our study to the case of $q < 1.0$, which is our future subject.

Acknowledgments

This work is partly supported by a Grant-in-Aid for Scientific Research from the Japanese Ministry of Education, Culture, Sports, Science and Technology.

A The generalized Fisher information in the IA

By using formulae for the gamma function given by Eqs. (17) and (18) and adopting the IA given by Eq. (29), we obtain following expressions for the GFIs.

Elements of $g$ in Eqs. (48)-(50) for $q \geq 1.0$ are expressed by

$$g_{11} = q \int_0^{\infty} G \left( u; \frac{2q - 1}{q - 1}, \frac{1}{(q - 1)\beta} \right) \left[ E_1^{(2)}(u) - 2E_qE_1(u) + E_q^2 \right] du, \tag{A1}$$
\begin{align*}
g_{22} & = q \int_0^\infty G \left( u; \frac{2q - 1}{q - 1}, \frac{1}{(q - 1)\beta} \right) \left[ N_1^{(2)}(u) - 2N_qN_1(u) + N_q^2 \right] du, \\
g_{12} & = g_{21} = -q \int_0^\infty G \left( u; \frac{2q - 1}{q - 1}, \frac{1}{(q - 1)\beta} \right) \\
& \times \left[ D_1^{(2)}(u) - E_qN_1(u) - N_qE_1(u) + E_qN_q \right] du, \
\end{align*}

\text{with}

\begin{align*}
N_1^{(2)}(u) & = \left[ N_1(u) \right]^2 + \sum_k f_1(\epsilon_k, u) \left[ 1 \pm f_1(\epsilon_k, u) \right], \\
E_1^{(2)}(u) & = \left[ E_1(u) \right]^2 + \sum_k f_1(\epsilon_k, u) \left[ 1 \pm f_1(\epsilon_k, u) \right] \epsilon_k^2, \\
D_1^{(2)}(u) & = E_1(u)N_1(u) + \sum_k f_1(\epsilon_k, u) \left[ 1 \pm f_1(\epsilon_k, u) \right] \epsilon_k,
\end{align*}

where the upper (lower) sign is applied to boson (fermion). Relevant results for \( q < 1.0 \) may be obtainable with a proper modification.

Elements of matrix of \( G \) in Eqs. (59)-(61) are expressed by

\begin{align*}
G_{11} & = \left[ \hat{H}^2 \right]_q - E_q^2, \\
G_{22} & = \left[ \hat{N}^2 \right]_q - N_q^2, \\
G_{12} & = G_{21} = - \left( \left[ \hat{H}\hat{N} \right]_q - E_qN_q \right),
\end{align*}

\text{with}

\begin{align*}
\left[ \hat{H}^2 \right]_q & = \int_0^\infty G \left( u; \frac{q}{q - 1}, \frac{1}{(q - 1)\beta} \right) E_1^{(2)}(u) du, \\
\left[ \hat{N}^2 \right]_q & = \int_0^\infty G \left( u; \frac{q}{q - 1}, \frac{1}{(q - 1)\beta} \right) N_1^{(2)}(u) du, \\
\left[ \hat{H}\hat{N} \right]_q & = \int_0^\infty G \left( u; \frac{q}{q - 1}, \frac{1}{(q - 1)\beta} \right) D_1^{(2)}(u) du.
\end{align*}

Elements of matrix of \( \tilde{g} \) in Eqs. (72)-(74) are expressed by

\begin{align*}
\tilde{g}_{11} & = q \int_0^\infty G \left( u; \frac{3q - 2}{q - 1}, \frac{1}{(q - 1)\beta} \right) \left[ E^{(2)}(u) - 2E_qE_1(u) + E_q^2 \right] du, \\
\tilde{g}_{22} & = q \int_0^\infty G \left( u; \frac{3q - 2}{q - 1}, \frac{1}{(q - 1)\beta} \right) \left[ N^{(2)}(u) - 2N_qN_1(u) + N_q^2 \right] du, \\
\tilde{g}_{12} & = g_{21} = -q \int_0^\infty G \left( u; \frac{3q - 2}{q - 1}, \frac{1}{(q - 1)\beta} \right) \\
& \times \left[ D^{(2)}(u) - E_qN_1(u) - N_qE_1(u) + E_qN_q \right] du.
\end{align*}
In the limit of $q \to 1.0$, all the GFIs reduce to

\[
\begin{align*}
    g_{11} &= G_{11} = \tilde{g}_{11} = \frac{1}{\langle \hat{H}^2 \rangle_1 - \langle \hat{H} \rangle_1^2}, \\
    g_{22} &= G_{22} = \tilde{g}_{22} = \frac{1}{\langle \hat{N}^2 \rangle_1 - \langle \hat{N} \rangle_1^2}, \\
    g_{12} &= G_{21} = \tilde{g}_{12} = \frac{1}{\langle \hat{H} \hat{N} \rangle_1 - \langle \hat{H} \rangle_1 \langle \hat{N} \rangle_1}.
\end{align*}
\]  

(A16)

\(g_{22}\) = \(G_{22}\) = \(\tilde{g}_{22}\) = \(1\) \(\langle \hat{N}^2 \rangle_1 - \langle \hat{N} \rangle_1^2\), \(g_{12}\) = \(G_{21}\) = \(\tilde{g}_{12}\) = \(1\) \(\langle \hat{H} \hat{N} \rangle_1 - \langle \hat{H} \rangle_1 \langle \hat{N} \rangle_1\).

\(g_{11}\) = \(G_{11}\) = \(\tilde{g}_{11}\) = \(1\) \(\langle \hat{H}^2 \rangle_1 - \langle \hat{H} \rangle_1^2\).

\(g_{12}\) = \(G_{21}\) = \(\tilde{g}_{12}\) = \(1\) \(\langle \hat{H} \hat{N} \rangle_1 - \langle \hat{H} \rangle_1 \langle \hat{N} \rangle_1\).

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Figure 1: (Color online) The temperature dependence of $E_q$ of the electron band model for $q = 1.0$ (dashed curves), $q = 1.1$ (chain curves), $q = 1.2$ (dotted curves), $q = 1.3$ (solid curves) and $q = 1.5$ (double-chain curve), the inset showing the ratio of $\lambda = E_q^{TA}/E_q$.

Figure 2: (Color online) The temperature dependence of the Tsallis entropy $S_q$ of the electron band model for $q = 1.0$ (the dashed curve), $q = 1.1$ (the chain curve), $q = 1.2$ (the dotted curve), $q = 1.3$ (the solid curve) and $q = 1.5$ (the double-chain curve).

Figure 3: (Color online) The temperature dependence of the $(i, i)$ component $(i = 1, 2)$ of $G$ (solid curves), $g$ (chain curves) and $\tilde{g}$ (dotted curves) for $q = 1.1$ of the electron band model, results for $q = 1.0$ being plotted by dotted curves for a comparison.

Figure 4: (Color online) The temperature dependence of $E_q$ of the Debye phonon model for $q = 1.0$ (dashed curves), $q = 1.1$ (chain curves), $q = 1.2$ (dotted curves), $q = 1.3$ (solid curves) and $q = 1.5$ (double-chain curves), the inset showing the ratio of $\lambda = E_q^{TA}/E_q$.

Figure 5: (Color online) The temperature dependence of the Tsallis entropy $S_q$ of the Debye phonon model for $q = 1.0$ (the dashed curve), $q = 1.1$ (the chain curve), $q = 1.2$ (the dotted curve), $q = 1.3$ (the solid curve) and $q = 1.5$ (the double-chain curve).

Figure 6: (Color online) The temperature dependence of the $(i, i)$ component $(i = 1, 2)$ of $G$ (solid curves), $g$ (chain curves) and $\tilde{g}$ (dotted curves) for $q = 1.1$ of the Debye phonon model, results for $q = 1.0$ being plotted by dashed curves for a comparison.

Figure 7: (Color online) The $q$ dependence of $D_{11}$ (filled circles), $D_{22}$ (filled squares), $D_{12}$ (filled triangles), $\hat{D}_{11}$ (open circles), $\hat{D}_{22}$ (open squares) and $\hat{D}_{12}$ (open triangles) with $T/T_D = 1.0$ for the Debye phonon model: note that $V_{ij} = D_{ij} (i, j = 1, 2)$. 

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Figure 8: (Color online) The $q$ dependence of (a) $V_{11}/\sigma_q^2$ (the dotted curve), $D_{11}/\sigma_q^2$ (the solid curve), $\tilde{D}_{11}/\sigma_q^2$ (the chain curve) and $(1/g_{11}\sigma_q^2)$ (the dashed curve), and (b) that of $V_{22}/\sigma_q^4$ (the dotted curve), $D_{22}/\sigma_q^4$ (the solid curve), $\tilde{D}_{22}/\sigma_q^4$ (the chain curve) and $1/g_{22}\sigma_q^4$ (the dashed curve): note that $V_{11} = D_{11} = 1/g_{11}$ in (a) and $V_{22} = D_{22}$ in (b).
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