ANALYZING WIRELESS COMMUNICATION NETWORK VULNERABILITY WITH HOMOLOGICAL INVARIANTS

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ABSTRACT

This article explains how sheaves and homology theory can be applied to simplicial complex models of wireless communication networks to study their vulnerability to jamming. It develops two classes of invariants (one local and one global) for studying which nodes and links present more of a liability to the network’s performance when under attack.

Index Terms— wireless network; simplicial complex; sheaf; persistent homology; relative homology

1. INTRODUCTION

Wireless communication networks are vulnerable to adversarial jamming because they are based on a broadcast medium. However, the problems of designing attack and defense strategies for jamming have been neglected areas of study [4]. Even if the network’s nodes and environment remains fixed, which links present the network with the greatest liability? We will study two classes of networks: (1) a fixed network in which all nodes communicate by broadcasting into a single channel, or (2) a fixed network of base station nodes that communicate with dedicated mobile terminals using a single broadcast resource. We specifically do not address wired networks, since the vulnerability of wired networks is better understood.

2. CONTRIBUTIONS

This article argues that appropriate tools for addressing these problems come from algebraic topology, and makes three main contributions to the study of a wireless communication network’s physical vulnerability:

1. It develops two simplicial complex models of communication networks. This specifically generalizes graph-based models and permits more refined, higher-dimensional resource contention to be captured.

2. On these models, the article outlines a sheaf-theoretic model of network activity and interference, which provides a disciplined way to explain the possible non-interfering configurations of active nodes.

3. Finally, the article pioneers vulnerability assessment using persistent and relative homology. These two techniques yield novel global and per-node (respectively) vulnerability assessments.

3. HISTORICAL CONTEXT

We assume that the proper operation of the network under study requires every node to be able to communicate with every other node, possibly through a relay. Following [1, 2], we will declare that the network is vulnerable at a collection of links if their removal results in a disconnected network. Although this is a fairly drastic mode of failure (the loss of a few links can dramatically reduce the network capacity [3]), it is a failure that is easy to describe.

Although vulnerability to purely random failures have been extensively studied, Commander et al. [4] observe that vulnerability to an adversarial jammer has received little attention. Xu et al. [5] was one of the few works discussing jammers, from both the perspective of the attacker and defender.

Although the use of graph theory is well-established in studying resource conflict in wireless networks [6, 7, 8, 9], we are inspired by the detailed survey [12], which states, “Solving generalized NUM [network utility maximization] problems over networks with randomly varying topology remains an under-explored area with little known results on models or methodologies.” Along these lines, graph theory plays a central role in identifying critical nodes [3], those through which a substantial amount of traffic passes. Because identifying these nodes is computationally difficult [11, 10], our (more naïve) definition of vulnerability as causing disconnectedness avoids a combinatorial explosion.

The tools of topology have been used more extensively in sensor networks. Our simplicial complexes are inspired by the work of [13, 14, 15, 16]. Jamming and defense problems for sensor networks were discussed in [17, 18, 19]. Finally,
we observe that communication network capacity has been successfully studied in [20] using the tools of sheaf theory.

4. TWO SIMPLICIAL MODELS OF WIRELESS NETWORKS

We will take the number of path components of a network as a measure of its health: fewer components is better. (If the network is given a topological structure \( X \), as will be done in the next section, it is well known that the zeroth homology group \( H_0(X) \) specifies the path components of \( X \).) If there are multiple path components, there exist pairs of nodes which cannot communicate with each other, even through relays. We also make the following single channel assumption: if a link connected to a node is jammed, then that node cannot receive transmissions from any other node.

Definition 1. A wireless network vulnerability is its susceptibility to becoming disconnected when a single source of interference is present.

This article presents techniques for identifying these vulnerabilities within a simplicial model of the network. We begin with a few preliminary definitions that are relevant to simplicial complexes.

Definition 2. An abstract simplicial complex \( X \) on a set \( A \) is a collection of ordered subsets of \( A \) that is closed under the operation of taking subsets.

- We call each element of \( X \) a cell. A cell with \( k + 1 \) elements is called a \( k \)-cell, though we usually call a 0-cell a vertex and a 1-cell an edge.
- If \( a, b \) are cells with \( a \subset b \), we say that \( a \) is a face of \( b \), and that \( b \) is a coface of \( a \). A cell of \( X \) that has no cofaces is called a facet.
- The closure \( \text{cl} \ S \) of a set \( S \) of cells in \( X \) is the smallest abstract simplicial complex that contains \( S \). Observe that \( \text{cl} \ S \subset X \).
- The star \( \text{star} \ S \) of a set \( S \) of cells in \( X \) is the set of all cells that have at least one face in \( S \). Although \( \text{star} \ S \subset X \), it is typically not an abstract simplicial complex.

Suppose a radio network is active in a spatial region \( R \), which we model as being a topological space. Let the network consist of a collection of nodes \( N = \{n_i\} \subset R \) that communicate through a single-channel, broadcast resource. An open set \( U_i \) is associated to each node \( n_i \) that represents its transmitter coverage region. (See Figure 1.) For each node \( n_i \), a continuous function \( s_i : U_i \to \mathbb{R} \) represents its signal level at each point in \( U_i \). For simplicity, we assume that there is a global threshold \( T \) for accurately decoding the transmission from any node.

Two simplicial complexes are useful for describing the network: the interference complex that describes interference received by mobile terminals communicating with each node, and the link complex that describes the possible communication links between nodes.

Definition 3. The interference complex \( I = I(N, U, s, T) \) consists of all subsets of \( N \) of the form \( \{i_1, \ldots, i_n\} \) for which

\[
U_{i_1} \cap \cdots \cap U_{i_n} \text{ contains a point } x \text{ for which } s_{i_k}(x) > T \text{ for all } k = 1, \ldots, n.
\]

Briefly, the interference complex describes the lists of transmitters that when transmitting will result in at least one mobile receiver location receiving multiple signals simultaneously. (Without the constraint on the decoding threshold, the interference complex reduces to the well-known Čech complex [21].)

Proposition 4. Each facet of the interference complex corresponds to a maximal collection of nodes that mutually interfere.

Proof. Let \( c \) be a cell of the interference complex. Then \( c \) is a collection of nodes whose footprints have a nontrivial intersection. By construction, the decoding threshold is exceeded for all nodes at some point \( x \) in this intersection. Therefore, if any two nodes in \( c \) transmit simultaneously, they will interfere at \( x \). If \( c \) is a facet, it is contained in no larger cell, so it is clearly maximal. \( \square \)

Definition 5. The link graph is the following collection of subsets of \( N \):

1. \( \{n_i\} \in N \) for each node \( n_i \), and
2. \( \{n_i, n_j\} \in N \) if \( s_i(n_j) > T \) and \( s_j(n_i) > T \).

The link complex \( L = L(N, U, s, T) \) is the clique complex of the link graph, which means that it contains all elements of the form \( \{i_1, \ldots, i_n\} \) whenever this set is a clique in the link graph.

Proposition 6. Each facet in the link complex is a maximal set of nodes that can communicate directly with one another (with only one transmitting at a time).

Proof. Let \( c \) be a cell of the link complex. By definition, for each pair of nodes, \( i, j \in c \) implies that \( s_i(n_j) > T \) and \( s_j(n_i) > T \). Therefore, \( i \) and \( j \) can communicate with one another. \( \square \)

Corollary 7. Facets of the interference and link complexes represent common broadcast resources.

Example 8. Figure 1 shows a network with three nodes. The coverage regions are shown at left for a given threshold \( T \). Since there are nonempty pairwise intersections between the
Fig. 1. Transmitter coverage regions (left), the associated interference complex (middle), and link complex (right)

coverage regions, but there is no common point of intersection for all three nodes, the interference complex (middle) contains no 2-cells. However, since no pair of nodes can actually communicate, the link complex consists of three isolated vertices.

5. INTERFERENCE FROM A TRANSMISSION

The interference caused by a transmission impacts the usability of the network outside of the transmission’s immediate vicinity. This section builds a consistent definition of the region of influence of a node or a link within the network. To justify this definition, we use a local model that describes which configurations of nodes can transmit simultaneously.

Definition 9. Suppose that $X$ is a simplicial complex (such as an interference or link complex) whose set of vertices is $N$. Consider the following assignment $T$ of additional information to capture which nodes are transmitting and decodable:

1. To each cell $c \in X$, assign the set

   $$T(c) = \{ n \in N : \text{there exists a cell } d \in X \text{ with } c \subset d \text{ and } n \in d \} \cup \{ \perp \}$$

   of nodes that have a coface in common with $c$, along with the symbol $\perp$. We call $T(c)$ the stalk of $T$ at $c$.

2. To each pair $c \subset d$ of cells, assign the restriction function

   $$T(c \subset d)(n) = \begin{cases} n & \text{if } n \in T(d) \\ \perp & \text{otherwise} \end{cases}$$

For instance, if $c \in X$ is a cell of a link complex, $T(c)$ specifies which nearby node is transmitting and decodable, or $\perp$ if none are. The restriction functions relate the decodable transmitting nodes at the nodes to which nodes are decodable along an attached wireless link. Similarly, if $c \in X$ is a cell of an interference complex, $T(c)$ also specifies which nearby node is transmitting, and effectively locks out any interfering transmissions from other nodes.

Definition 10. The assignment $T$ is called the transmission sheaf and is an example of a cellular sheaf — a mathematical object that stores local data. The theory of sheaves explains how to extract consistent information, which in the case of networks consists of nodes that whose transmissions do not interfere with one another.

A section of $T$ supported on a subset $Y \subset X$ is an assignment $s : Y \to N$ so that for each $c \subset d$ in $Y$, $s(c) \in T(c)$ and $T(c \subset d)(s(c)) = s(d)$. A section supported on $X$ is called a global section.

Specifically, global sections are complete lists of nodes that can be transmitting without interference.

Example 11. Figure 2 shows a network with three nodes, labeled 1, 2, and 3. When node 1 transmits, node 2 receives. Because node 2 is busy, its link to node 3 must remain inactive (right top). When node 2 transmits, both nodes 1 and 3 receive (right middle). The right bottom diagram shows a local section that cannot be extended to the cell marked with a blank. This corresponds to the situation where nodes 1 and 3 attempt to transmit but instead cause interference at node 2.

Example 12. Observe that in either of the simplicial complex models shown in Figure 1, only one of the nodes may transmit at a time.

Definition 13. Suppose that $s$ is a global section of $T$. The active region associated to a node $n \in X$ in $s$ is the set

$$\text{active}(s, n) = \{ a \in X : s(a) = n \},$$

which is the set of all nodes that are currently waiting on $n$ to finish transmitting.

Lemma 14. The active region of a node is a connected, closed subcomplex of $X$ that contains $n$.

Proof. Consider a cell $c \in \text{active}(s, n)$. If $c$ is not a vertex, then there exists a $b \subset c$; we must show that $b \in \text{active}(s, n)$. Since $s$ is a global section $T(b \subset c)s(b) = s(c) = n$. Because $s(c) \neq \perp$, the definition of the restriction function

Fig. 2. A link complex (left top), sheaf $T$ (left bottom), and three sections (right). The restrictions are shown with arrows, global section when node 1 transmits (right top), global section when node 2 transmits (right middle), and a local section with nodes 1 and 3 attempting to transmit, interfering at node 2 (right bottom)
The active region of a node is independent of the global section. More precisely, if \( r \) and \( s \) are global sections of \( T \) and the active regions associated to \( n \in X \) are nonempty in both, then active\((s, n) = \text{active}(r, n)\).

**Proof.** Without loss of generality, we need only show that active\((s, n) \subseteq \text{active}(r, n)\). If \( c \in \text{active}(s, n) \), there must be a cell \( d \in X \) that has both \( n \) and \( c \) as faces. Now \( s(n) \), \( r(n) = n \) by Lemma 14, which means that \( r(d) = T(n \subseteq d) r(n) = n \). Therefore, since \( \text{active}(r, n) \) is closed, this implies that \( c \in \text{active}(r, n) \).

**Definition 17.** Because of the Lemmas, we call the star over an active region associated to a node \( n \) the region of influence. The region of influence of a node can be written as a union

\[
\text{roi}(n) = \bigcup_{f \in F} \text{star cl } f,
\]

for a collection \( F \) of facets, so we call the star over a closure of a facet the region of influence of that facet.

**Corollary 18.** The complement of the region of influence of a facet is a closed subcomplex.

**6. GLOBAL VULNERABILITY ASSESSMENT USING PERSISTENT HOMOLOGY**

Based on the previous section, we assume that interfering with a link removes it and its region of influence from the network. If the strength of the interference is unknown, it is more appropriate to assume that there is a filtration \( \emptyset = L_0 \subseteq L_1 \subseteq \ldots \) of open subcomplexes of the network \( X \) that represent an increasingly large collection of links being disrupted by the interference. At one end of the filtration \( L_0 = \emptyset \), in which the network is unaffected. At the other end, there can be severe disruptions of the network.

Observe that if \( R_1 < R_2 \), then \( L_{R_1} \subseteq L_{R_2} \). This induces a homomorphism \( H_k(X \setminus \text{roi } L_{R_1}) \to H_k(X \setminus \text{roi } L_{R_2}) \).

**Definition 19.** Because of this, we usually think of elements of \( H_k(X \setminus \text{roi } L_{R_1}) \) as being part of a larger \( k \)-th persistent homology group \( H_k(X \setminus \text{roi } L_{R_1}) \) associated to the interference filtration \( L_{R_1} \). Following standard practice [22, 23], an element of \( a \in H_k(X \setminus \text{roi } L_{R_1}) \) is said to be born at \( R_1 \) if \( a \in H_k(X \setminus \text{roi } L_{R_1}) \) is nontrivial, and its preimage is trivial in all \( H_k(X \setminus \text{roi } L_{R_2}) \) with \( R_2 < R_1 \). Conversely, it is said to die at \( R_1 \) if its image in \( H_k(X \setminus \text{roi } L_{R_2}) \) is zero for all \( R_2 > R_1 \). The difference between its birth and death is called its age.

Elements of the persistent homology group with larger ages represent “more significant” topological features in the data, because they are more robust to perturbations [24].

We considered the link complex of a random network \( N \subseteq \mathbb{R}^2 \) (Figure 3) and two interference sources located at \( a_1, a_2 \in \mathbb{R}^2 \). To simulate varying levels of interference, the filtration \( L_{R_i} = \{n \in N : d(n, a_i) < R \} \) was computed according to the Euclidean distance function \( d \). Given this information, Perseus [25] was used to compute the 0-th persistent homology of the remaining network for each of the two jammers. This software decomposes the persistent homology group into its set of generators, and displays the birth and death times corresponding to each.

Examination of the results (Figure 4) shows that the number of significant generators (farther from the diagonal) of \( H_0(X \setminus \text{roi } L_{R_1}) \) depends strongly on the position of the jammer. In the case where the jammer is located near the narrow portion of the network at the center, there is a significant persistent generator. Communication from one half of the network to the other is impossible when the jammer is active. For the other case, when the jammer is located away from the narrow portion of the network, there are no significant generators. For a large range of interference strengths, the network remains connected. Because of this, the presence of more and
Suppose that Theorem 21. remains connected. But if removed from the network shown in Figure 5. If the region of influence of a facet L removes its region of influence from the network, the nodes that are faces of this facet cannot communicate, but other portions of the network may become disconnected, too. This section develops a precise, theoretical bound on the number of components a network is split into under the influence of interference.

For example, consider the two networks in Figure 5. Attacking a facet L in a straight-line (left) network X results in a network X \ roi L that is disconnected, but in a ring network Y (right) does not disconnect the network.

**Example 20.** A facet L is marked in each of the two networks shown in Figure 5. If the region of influence of L is removed from the network in the network X at left, it becomes disconnected. But if removed from the network Y at right, it remains connected.

**Theorem 21.** Suppose that X is either a link or interference complex, and that L is a facet of X. If X is connected and roi L is a proper subset of X, the number rank \( H_1(X, X \setminus \text{roi } L) \) + 1 is an upper bound on the number of connected components that an attack on L cuts the network into. When \( H_1(X) \) is trivial, that upper bound is attained.

**Proof.** By Corollary 18, the portion of the network that is not directly disrupted by the attack (the complement of L’s region of influence) is closed. Observe that this region is the simplicial complex \( Y = X \setminus \text{roi } L \). Consider the long exact sequence associated to the pair \((X, Y)\), which is

\[
\cdots \rightarrow H_1(Y) \rightarrow H_1(X) \rightarrow H_1(X, Y) \rightarrow H_0(Y) \rightarrow H_0(X) \rightarrow H_0(X, Y) \rightarrow 0.
\]

\( H_k(X, Y) \cong \tilde{H}_k(Y/X) \) follows via [21, Prop. 2.22]. This means that \( H_0(Y) = 0 \) and \( H_0(X) \cong \mathbb{Z} \) because X is connected. Thus the long exact sequence reduces to

\[
\cdots \rightarrow H_1(Y) \rightarrow H_1(X) \rightarrow H_1(X, Y) \xrightarrow{\delta} H_0(Y) \xrightarrow{\delta} \mathbb{Z} \rightarrow 0.
\]

The number of connected components of the network during the attack is rank \( H_0(Y) \). Because of the last term in the long exact sequence, this is at least 1. If \( H_1(X) = 0 \) then \( H_1(Y) = 0 \) also, so the kernel of the homomorphism \( \delta \) is precisely the image of the monomorphism \( f \). Hence rank \( H_1(X, Y) \) + 1 \( = \) rank \( H_0(Y) \) as claimed.

On the other hand, if \( H_1(X) \) is not trivial, then \( H_1(Y) \) may or may not be trivial, depending on exactly where L happens to fall. By exactness,

\[
\ker g = \text{rank } H_0(Y) - 1 = \text{image } f = \text{rank } H_1(X, Y) - \ker f
\]

where the rank-nullity theorem for finitely generated abelian groups applies in the last equality. The kernel of \( \delta \) may be as large as rank \( H_1(X) \), but it may be smaller. Thus, we can claim only that

\[
\text{rank } H_0(Y) \leq \text{rank } H_1(X, Y) + 1.
\]

Because of the excision property for relative homology, \( H_1(X; X \setminus \text{roi } L) \) only detects the immediate vicinity of the link, and is a generalization of vertex degree to higher dimensional complexes.

**Example 22.** If the simplicial complex for a network is a tree, then attacking a node with degree larger than 2 will result in a network with more than two connected components.
Fig. 6. An interference complex that is a hollow tetrahedron (left) for which \( H_1(X, X \setminus \text{roi } L) \) and \( H_1(X, X \setminus \text{roi } F) \) are trivial.

Fig. 7. Even though the graph for network \( X \) has nontrivial \( H_1(X) \), attacking facet \( L \) results in a disconnected network.

**Example 23.** Figure 6 shows an example of a network for which the interference complex is 2-dimensional. In this case, \( H_1(X, X \setminus \text{roi } L) \) and \( H_1(X, X \setminus \text{roi } F) \) are trivial. Although each vertex in the link graph has degree 3, Theorem 21 implies that the removal of the region of influence for an edge or a face from the network results in a single connected component.

**Example 24.** The presence of loops farther from the link can provide back-up connections in the event of an attack, though these are not measured by \( H_1(X; X \setminus \text{roi } L) \). Therefore, the presence of a nontrivial \( H_1(X) \) appears to be protective (but is not a guarantee) of network robustness (see Figure 7).

### 8. NEXT STEPS

This article showed that a wireless communication network’s vulnerability to jamming can be analyzed using tools from algebraic topology. It showed that simplicial complexes, sheaves, and persistent homology provide differing kinds of information about the network. Our simplicial complex model of a network can include higher dimensional data, but our analysis treated low-dimensional homological data exclusively. Therefore, it is reasonable to continue our analysis into higher dimensions, aiming to assess decision thresholds and to study the higher homology groups themselves.

The network vulnerability invariant from relative homology has been studied extensively in the context of local homology groups. However, this mathematical tool has not been applied to networks, so it is worthwhile to examine the impact of the other maps within the long exact sequence, to identify other useful invariants. Finally, we expect to extend the formalism to timeseries data within a network intending to understanding protocol-level jamming.

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