Bent waveguides for matter-waves: supersymmetric potentials and reflectionless geometries

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Non-zero curvature in a waveguide leads to the appearance of an attractive quantum potential which crucially affects the dynamics in matter-wave circuits. Using methods of supersymmetric quantum mechanics, pairs of bent waveguides are found whose geometry-induced potentials share the same scattering properties. As a result, reflectionless waveguides, dual to the straight waveguide, are identified. Strictly isospectral waveguides are also found by modulating the depth of the trapping potential. Numerical simulations are used to demonstrate the efficiency of these approaches in tailoring and controlling curvature-induced quantum-mechanical effects.

Waveguides with non-zero curvature are basic constituents of matter-wave circuits in atom chip technology\(^{1,2}\), as well as its ion\(^3\), molecular\(^4\), and electron\(^5\) counterparts. Their relevance is further enhanced by the development of flexible techniques to create optical waveguides for ultracold gases. In this context, waveguide trapping potentials can be engineered by a variety of methods including the time-averaging painted potential technique\(^6\), the use of an intensity mask\(^7,8\), and holographic methods, in particular, digital holography\(^9\). Circular ring traps have attracted a considerable amount of attention\(^6,10–15\), most recently to study Josephson junction dynamics\(^16,17\). Other curved waveguides have also been engineered, such as a stadium-shaped potential trap\(^18,19\).

The propagation of matter-waves in bent waveguides generally differs from that in straight waveguides due to the appearance of a purely attractive local quantum potential of geometrical origin\(^20–22\). Under tight-transverse confinement, the magnitude of this curvature-induced potential (CIP) is proportional to the square of the curvature of the waveguide, and affects both the single-particle and many-body physics of the confined matter-waves\(^23–29\). As a result, the scattering properties of a curved waveguide are modified, e.g., by the appearance of bound states\(^29\). Advances in the design of bent waveguides, in which curvature-induced effects are tailored and suppressed, are required for the miniaturization of matter-wave circuits. It is to this problem that we turn our attention.

Results

In this manuscript we design bent waveguides for matter-wave circuits free from spurious quantum mechanical effects associated with CIPs. Three novel ideas are presented: (i) Exploiting the interplay of geometry and supersymmetry in quantum mechanics, we relate pairs of waveguides whose CIPs are isospectral and share the same scattering properties. (ii) We then identify waveguides which are reflectionless for coherent matter-waves at all energies. (iii) Furthermore, we show that by tailoring the depth of the waveguide trap, it is possible to cancel the CIP, rendering the dynamics of the guided matter-waves equivalent to that in straight waveguides.

Let us consider the dynamics of matter-waves confined in a tight waveguide whose axis follows the curve \(\gamma\), parametrized as a function of the arc length \(q_1\), by the vector \(\mathbf{r} = \mathbf{r}(q_1)\), with tangent \(\mathbf{t}(q_1)\). We start by recalling the fundamental theorem of curves which asserts that a curve is completely determined, up to its position in space, by its curvature \(\kappa\) and torsion \(\tau\). Indeed, the expressions \(\kappa = \kappa(q_1)\) and \(\tau = \tau(q_1)\) constitute the natural intrinsic equations of a curve. A parametrization of the curve can be obtained by integration of the Frenet-Serret equations

\[
\begin{align*}
\frac{dq_1}{\lambda} = \begin{pmatrix}
\mathbf{t} \\
\mathbf{n} \\
\mathbf{b}
\end{pmatrix} = \begin{pmatrix}
0 & \kappa & 0 \\
-\kappa & 0 & \tau \\
0 & -\tau & 0
\end{pmatrix} \begin{pmatrix}
\dot{t} \\
\dot{n} \\
\dot{b}
\end{pmatrix},
\end{align*}
\]
where \( \mathbf{n} = (d\mathbf{t}/dq_1)/\kappa \) (provided \( d\mathbf{t}/dq_1 \neq 0 \)) and \( \mathbf{b} = \mathbf{i} \times \mathbf{n} \) are the principal normal and binormal unit vectors, and the curvature and torsion at the point of arc length \( q_1 \) are defined as \( \kappa(q_1) = |d\mathbf{t}(q_1)/dq_1| \) and \( \tau(q_1) = -\mathbf{n} \cdot d\mathbf{b}/dq_1 \), respectively. Let \( (q_2, q_3) \) be the transverse local coordinates and consider a transverse confining potential \( U_\perp(q_2, q_3) \) such that in the limit of tight confinement \( \lambda \to \infty \) the particle is bounded to \( \gamma \). Under dimensional reduction, the purely-attractive CIP emerges\(^{21-24,26} \)

\[
\mathcal{V}(q_1) = -\frac{h^2}{8m} \kappa^2(q_1). 
\]

(2)

This result is independent of \( U_\perp \) and holds in particular under an isotropic transverse harmonic confinement \( U_\perp = m\omega_0^2 (q_2^2 + q_3^2)/2 \) with ground state width \( \sigma_0 = \hbar/(m\omega_0) \)^\(^{28,29} \). The conditions for the dimensional reduction to be valid explicitly read

\[
\kappa_0 \ll 1, \quad |\kappa'|/\sigma_0 \ll |\kappa|, \quad |\kappa''|/\sigma_0 < \kappa^2, 
\]

(3)

where primes denote derivatives with respect to \( q_1 \).

We next pose the problem of identifying pairs of waveguides with the same scattering properties, and engineering a waveguide which is isotropic transverse harmonic confinement \( U_\perp \) and has the same scattering properties, and engineering a waveguide which is purely-attractive CIP emerges\(^ {21–24,26} \), where \( \kappa_\perp \) satisfying \( \kappa_\perp = -2m\Phi^2 + \sqrt{2m}\Phi') \), that we shall refer to as SUSY partner curves. This set can be extended to include curves associated with a family of shape-invariant potentials\(^{34} \), as discussed in Methods, or using higher-order SUSY QM\(^{36} \).

Why are waveguides along these curves of interest? The main physical feature of SUSY partner curves is that they exhibit the same scattering properties, a distinguishing feature directly inherited from the fundamental theorem of curves, the shape and length of a planar curve is completely determined by its (single-valued and continuous) curvature. It follows that the SUSY partner potentials \( V_\pm \) are associated with the family of curves \( \gamma_\pm \) (with curvatures \( \kappa_\pm \) satisfying \( \kappa_\pm = -2m\Phi_\pm^2 + \sqrt{2m}\Phi_\pm') \), and consider the scattering states of momentum \( k \) and energy \( E = \hbar^2 k^2/(2m) \), with reflection and transmission amplitudes \( R_\pm(k) \) and \( T_\pm(k) \), respectively. It follows that \( R_\pm(k) = \Phi_\pm + i\hbar k/\sqrt{2m} \Phi_\pm \) and \( T_\pm(k) = \Phi_\pm - i\hbar k/\sqrt{2m} \Phi_\pm \) (where \( k = [2m(E - \Phi_\pm^2)]^{1/2}/\hbar \) and \( k' = [2m(E - \Phi_\pm^2)]^{1/2}/\hbar \) is the reflection as well as the transmission probabilities are the same for SUSY partner curves. Further, the Hamiltonians \( H_\pm \) associated with SUSY partner curves are isospectral, except for the lowest energy level of \( H_- \) with zero-energy, which is absent in the spectrum of \( H_+ \).

### Design of reflectionless curves

CIPs are of attractive character and as a result can lead to quantum reflection\(^ {36-39} \). The dynamics of a guided matter-wave on a bent waveguide is generally affected by the curvature. We next illustrate the power of the SUSY partner waveguides in designing reflectionless curves. An obvious instance where the CIP vanishes is that of an infinite straight waveguide, with \( \kappa_- = 0 \) and superpotential \( \Phi = A \tanh q_2 \) with \( A > 0 \). This configuration is of relevance to guided atom lasers\(^ {40,41} \), and we wish to mimic it in bent waveguides. SUSY QM allows us to find SUSY partners which are reflectionless. In this case, \( V_+(q_1) \) is given by the modified Pöschl-Teller potential \( V_{\text{curv}}(q_1) = -\hbar^2 \sqrt{v/(v+1) \tanh \pi v q_1} \)\(^ {32} \), so that the curvature of the SUSY \( \gamma_+ \) curve reads

\[
\kappa_+(q_1) = 2\sqrt{v(v+1) \tanh \pi v q_1},
\]

(9)

where \( v \) is a positive integer. Provided that the dimensional reduction is valid, the transmission probability for a waveguide with curvature \( (9) \) and arbitrary \( v \), is given by \( |T_+(q_1)^2| = \frac{\mu^2}{1 + \mu^2} \) with \( \mu = \frac{\sinh(\pi k/2)}{\sin \pi v} \) and \( k = \sqrt{2mE/\hbar} \). Such a waveguide becomes reflectionless for integer values of \( v \). Different reflectionless...
waveguides with the curvature (9) and whose realization might be achieved using an extended version of the torsion of the curvature, or by considering a nonzero torsion exploiting the invariance of the CIP with respect to changes in the number of multiple points increases with the magnitude of the curvature. A variety of waveguides are plotted in Fig. 1, where it is shown that the number of multiple points increases with the magnitude of the curvature. A simple waveguide without junctions can be engineered in a non-planar waveguide, with non-zero torsion $\tau$.

Following Shabat43 and Spiridonov 44. The reflectionless character of the SUSY curves associated with (9). Figure 1 (lower panels) shows the reflectionless curves corresponding to a Sukumar potential supporting two bound states with $\eta_1 = 1$ and $\eta_2 = 3/2$, where multiple points in 1(c) are avoided by a non-zero torsion ($\tau = 20q_0^{-1}$) in 1(d). The axis of the associated waveguide follows the curve $(x(s), y(s), z(s))$ with (squared) curvature $k^2(s)$ and arc length $q_1 = \sqrt{1 + \tau^2 s}$. At variance with (9), the relative angle between the asymptotes can be tuned by adjusting the value of $\eta_2$ relative to $\eta_1$, which will allow for the engineering of reflectionless bends through a range of desired angles. Further examples of reflectionless waveguides can be found by using the infinite family of reflectionless potentials discussed by Shabat43 and Spiridonov 44. The reflectionless character of the SUSY waveguides becomes apparent in the dynamics of guided matter waves. Figure 2 shows an elongated Gaussian beam being guided in a bent waveguide with curvature given by (9). $\gamma_{+}$ is asymptotically flat for $q_1 \to \pm \infty$. For a general non-integer value of $v$, the traveling beam is substantially reflected off the bent region. For integer $v$ there exists a delocalized critical bound-state with zero energy and the waveguide becomes reflectionless for all scattering energies. However, the degree of bending increases with $v$. As a result, reflectionless waveguides provide a remarkable counterexample to the common expectation that the reflection probability increases with the degree of bending of the waveguide. In addition, the numerical simulations correspond to the propagation in a waveguide with finite transverse width, for which the explicit form of the curvature induced potential is more complex than that in Eq. (2) used to design the reflectionless SUSY waveguide, and where excitations of the transverse waveguide modes are possible. The fact that despite the finite transverse width the waveguide remains reflectionless illustrates the robustness of its design against imperfections. We also note that the reflectivity of the Pöschl-Teller potential changes only gradually as $\nu$ departs from an integer value.

Figure 2 | Scattering dynamics in bent waveguides. Sequence of snapshots of the time-evolution of the density profile of a wavepacket along a planar bent waveguide with the curvature (9) and $v = 1/2$ (left), and $v = 1$ (right), as in Fig. 1(b). Generally, the wavepacket is split by the CIP into a transmitted component and a reflected component. Despite the high degree of bending shown in the inset, whenever $v$ is an integer, the waveguide becomes reflectionless and exhibits unit transmission probability for all energies of the impinging matter-wave beam. The color coding varies from white to red as the probability density increases. The dimensions of each waveguide image are 908$a_0$ × 470$a_0$ and the time interval between successive images is 1920/100. The initial wavepacket has FWHM = 235$a_0$ and momentum $(1/32)na_0$, $a_0$, and $\alpha = 1/8$.
Figure 3 | Canceling out the curvature-induced potential. Elliptical waveguide potentials of increasing eccentricity (top) and corresponding ground state densities (middle). Bottom row shows ground state densities when the CIP is compensated by modulating the depth of the trap. The dimensionless density profile \( n(q_1) \) is scaled up by a factor 10, the perimeter of the ellipse is \( L = 150 \sigma_0 \) and the plotted area is \( 80 \sigma_0 \times 50 \sigma_0 \).

\( \gamma \) (both either open or closed, and without multiple points), with CIPs \( \gamma'_c(q_1) \) and \( \gamma'_g(q_1) \), respectively. Under the consistency conditions (3), it is then possible to make \( \gamma \) isospectral to \( \gamma \) by modulating the depth of the waveguide potential, i.e., by creating a potential barrier of the form \( U(q_1) = -[\gamma'_c(q_1) - \gamma'_g(q_1)] \). A natural case is that in which \( \gamma'_c(q_1) \) either vanishes or is an irrelevant constant energy shift. \( U(q_1) \) is then the potential required to flatten out the depth of the global potential of \( \gamma \). In addition, the acceleration of the guided matter-waves towards the region of high-curvature is prevented. To explore in detail this possibility, we consider an elliptical trap \( 20,29,45 \), associated with the curve \( r(u) = (a \cos u, b \sin u) \), with \( a \approx b > 0 \), and circumference \( L \). The CIP in an elliptical trap reads

\[
\gamma'(u) = -\frac{h^2}{8m} \frac{a^2 b^2}{(b^2 \cos^2 u + a^2 \sin^2 u)^2}.
\] (11)

The eccentricity of an ellipse is defined by \( \varepsilon = \left[ 1 - \left( b/a \right)^2 \right]^{1/2} \in [0,1] \) and can be used to quantify the deformation from a circle (for which \( a = b, \varepsilon = 0 \)). For a ring of radius \( a = b (\gamma) \), with \( \varepsilon = 0 \), the curvature is \( \kappa(q_1) = 1/a \) and the CIP becomes constant, and the ground state density profile is uniform along the arc length \( q_1 \). For \( \gamma \) with \( \varepsilon > 0 \), the CIP comes into play and creates two attractive double wells, centered around the points with higher curvature \( q_1 = [0,L/2] \) (\( b < a \)) and with the minimum value \(-\frac{h^2}{8m} \frac{a^2}{b^4}\). The extent to which geometry-induced effects can be cancelled out by painting a barrier \( U(q_1) = -\gamma'_c(q_1) \) is illustrated in Fig. 3. Such cancellation is effective as long as the consistency conditions for the dimensional reduction hold, which ceases to be the case as \( \varepsilon \) is increased while the transverse width \( \sigma_n \) remains fixed.

The ground state density profile is a fairly robust quantity, but we note that this compensation is efficient as well for dynamical processes involving all spectral properties of the waveguide. Consider

\( t/\tau_R \)

\( q_1 \)

Figure 4 | Curvature-induced suppression of temporal Talbot oscillations. Time evolution of the density profile \( n(q_1,t) = \int dq_\perp n(q_1,q_\perp,t) \) of an initially tightly-localized wavepacket released in a two-dimensional elliptical waveguide. (a) For a ring trap \( \varepsilon = 0 \) \( n(q_1,t) \) exhibits Talbot oscillations as a result of the quadratic dispersion relation (left). Two Talbot oscillations are displayed. (b) Whenever \( \varepsilon > 0 \), the CIP lifts the degeneracies in the spectrum and suppresses Talbot oscillations \( (\varepsilon = 0.9) \). (c) The CIP can be cancelled out by modulating the depth of the trap \( (\varepsilon = 0.9) \). \( L = 150 \sigma_0 \) in all cases and hence the revival time \( \tau_R = mL^2/(nh) \) is constant for different values of \( \varepsilon \).
the time evolution of the density profile of an initially localized wavepacket released in the elliptical trap, displayed in Fig. 4. For a ring trap, where $\varepsilon = 0$, the evolution of the density profile $n(q_1, t)$ weaves a highly structured interference pattern with "scars" in the plane $(q_1, t)$, known as a "quantum carpet". Such quantum carpets exhibit a temporal analogue of the Talbot effect: in wave optics, the near-field diffraction pattern of a wave incident upon a periodic grating is characterized by a spatial periodicity. The quantum dynamics of an initially localised wavepacket which is released in a two-dimensional ring trap exhibits a periodic revival of the initial state, with period $T_\pi = mL^2/(2\pi\hbar)$ (the temporal analogue of optical Talbot oscillations). The reconstruction of the density profile at $t = 0$ can be traced back to the quadratic dispersion relation of the trap and the degeneracies it entails. This phenomenon has been experimentally observed in a variety of systems. In a two-dimensional elliptical waveguide, the spectrum is modified and the dispersion relation ceases to be quadratic. For $\varepsilon > 0$, the geometry-induced potential lifts the degeneracy in the spectrum, leading to the suppression of Talbot oscillations. Nonetheless, the dynamics corresponding to a ring trap can be effectively recovered in an elliptical trap with $\varepsilon > 0$ after compensating the depth of the waveguide potential. The reapparence of Talbot oscillations in compensated elliptical waveguides signals the isospectral properties with respect to the ring trap, illustrating the suppression of curvature-induced effects.

**Discussion**

The dynamics of matter waves in bent waveguides is severely distorted by the appearance of an attractive curvature-induced quantum potential. As wave matter circuits shrink in size and atomic velocities must be reduced to maintain single mode propagation, curvature-induced potentials impose practical limitations on minimum velocities, and methods to reduce their effects are needed. Using methods of supersymmetric quantum mechanics, we have introduced a framework to design sets of bent waveguides which share the same scattering properties. As a relevant example, an infinite family of supersymmetric quantum mechanics, we have introduced a framework to design sets of bent waveguides which share the same scattering properties. As a relevant example, an infinite family of bending waveguides which share the same curvature properties.

**Methods**

Let us consider the case in which the superpotential depends on a collective set of parameters $\Phi = \Phi(q_1, a_1)$. The partner potentials $\hat{\gamma}$ are shape-invariant if they are related by $\hat{\gamma}(q_1, a_1) = \hat{\gamma}(q_1, a_1) + R(a_1)$ where the residual term $R(a_1)$ is independent of $q_1$ and $a_1 = R(a_1)$ is a new set of parameters obtained form $a_1$ by the action of the function $f$. By iteration, one can construct the series of Hamiltonians $H_0 = a_0 + H_1 + H_2$, such that $H_n = H_0 + \sum_{i=1}^{n} R(a_i)$, with $a_0 = f(a_0)$, i.e., obtained by the function of the Hamiltonian iterated $k$ times. It follows that the squared curvatures of the SUSY partner curves $\{\nu, \gamma, \gamma, \mu\}$ are related by a constant shift $\nu_i^\gamma(\gamma_1) = \nu_i^\gamma(\gamma_1) - \frac{8m}{\hbar^2} R(a_1) = \sum_{i=1}^{n} \nu_i^\gamma(\gamma_1) - \frac{8m}{\hbar^2} \sum_{i=1}^{n} R(a_i)$. (12)

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Acknowledgments
It is a pleasure to thank X.-W. Guan, E. Passemar, M. Pons, N. Sinitsyn and A. Szameit for stimulating discussions. This research is supported by the U.S Department of Energy through the LANL/LDRD Program and a LANL J. Robert Oppenheimer fellowship (AD).

Author contributions
A.D.C. and A.S. initiated the project. A.D.C. developed the theoretical analysis. M.B. and A.D.C. carried out the numerical simulations. All authors contributed to the analysis and interpretation of the numerical data and the preparation of the manuscript.

Additional information
Competing financial interests: The authors declare no competing financial interests.

How to cite this article: del Campo, A., Boshier, M.G. & Saxena, A. Bent waveguides for matter-waves: supersymmetric potentials and reflectionless geometries. Sci. Rep. 4, 5274; DOI:10.1038/srep05274 (2014).

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