Research Article

Mass Spectrum of Mesons via the WKB Approximation Method

Ekwevugbe Omugbe, 1 Omosede E. Osafile, 1 and Michael C. Onyeaju 2

1Department of Physics, Federal University of Petroleum Resources, Effurun, Delta State, Nigeria
2Theoretical Physics Group, Department of Physics, University of Port Harcourt, Port Harcourt, Nigeria

Correspondence should be addressed to Ekwevugbe Omugbe; omugbeekwevugbe@gmail.com

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In this paper, we demonstrated that the multiple turning point problems within the framework of the Wentzel-Kramers-Brillouin (WKB) approximation method can be reduced to two turning point one for a nonsymmetric potential function by using an appropriate Pekeris-type approximation scheme. We solved the Schrödinger equation with the Killingbeck potential plus an inversely quadratic potential (KPIQP) function. The special cases of the modeled potential are discussed. We obtained the energy eigenvalues and the mass spectra of the heavy \( \frac{Q}{C^2} \) and heavy-light \( \frac{Q}{C^2}q \) mesons systems. The results in this present work are in good agreement with the results obtained by other analytical methods and available experimental data in the literature.

1. Introduction

In the Quark model, a meson is a hadronic subparticle which consists of the quark and its antiquark, and the reduced mass is dominated by the light quark mass [1]. The meson system is mediated by the strong interactions which are described by the theory of quantum chromo-dynamics (QCD) [2]. The heavier mesons also known as quarkonia \( (QQ) \) are the constituents of the heavy quarks such as the bottom \( (b) \) and charmed \( (c) \) and are considered as nonrelativistic bound systems described by the Schrodinger wave equation (SE) [1, 2]. However, the case is different for the light \( (uds) \) quark interaction systems \( (q\bar{q}) \) which are described by relativistic equations [1, 2]. Also, the heavy-light meson \( (Q\bar{q}) \) system bound states have been studied using both the relativistic and nonrelativistic quark models [1, 2]. The bound state solutions to the wave equations under the quark-antiquark interaction potential functions such as the Killingbeck or the Cornell potentials have attracted much research interest in atomic and high energy physics [3–26]. The Killingbeck potential comprises the sum of the Cornell plus the Harmonic oscillator potentials, while the Cornell potential is the sum of the Coulomb plus linear potentials. The Cornell potential and its extended forms have been extensively solved with the SE [3–22], semirelativistic equation [23], and also with the relativistic equation [24–26]. The solution of the wave equation with some potentials are exactly solvable for \( l = 0 \), whereas other potentials are insoluble and nontrivial for any arbitrary angular momentum quantum number \( (l \neq 0) \). In this case, numerical methods and approximate analytical techniques are required to obtain the solutions of the quantum system of choice [27, 28]. Owing to the nontrivial mathematical properties of the quark-antiquark interaction potentials, several analytical techniques and approximate methods have been used to calculate the energy and mass spectra. The standard methods used in the literature are the Nikoforov Uvarov (NU) method [12–17, 29], asymptotic iterative method (AIM) [30], the Laplace transformation method (LTM) [18], artificial neural network method (ANN) [19], and the analytical exact iterative method (AEIM) [11]. The authors in [12–17] obtained the mass spectrum of the quark-antiquark interaction system using an appropriate Pekeris-type approximation scheme to deal with the orbital centrifugal energy barrier. In this present work, motivated by their approach, the same Pekeris-type approximation scheme is extended to the WKB formalism for the first time to the best of our knowledge. It is well known that the multiple turning points problems within the framework of the WKB approximation method can be reduced to the standard two turning points problem for symmetric potential functions, for
example, the isotropic harmonic oscillator and the molecular pseudoharmonic potential with the proper coordinate transformation [31–33]. This approach allows us to obtain the exact bound state solutions of the SE for any arbitrary $l$ quantum numbers in the physical axis $(0 < r_1 < r_2)$, where $r_1, r_2$ are the classical turning points.

In this present work, we studied the nonrelativistic quark model under the interaction of the nonsymmetric KPIQP. The main focus of this paper is to obtain the mass spectra of the heavy meson such as charmonium ($c\bar{c}$), bottomonium ($b\bar{b}$), bottom-charmed ($b\bar{c}$), and the heavy-light meson such as the charmed-strange ($c\bar{s}$) system via the WKB approximation method.

The KPIQP has the form [15, 30] of

$$V(r) = Ar^2 + Br - \frac{C}{r} + \frac{D}{r^2},$$

(1)

where $A, B, C$, and $D$ are constant potential parameters.

If we set $A = B = D = 0$, the KPIQP reduces to the Coulomb potential used in the description of the hydrogenic atom.

$$V(r) = -\frac{C}{r}$$

(2)

The KPIQP reduces to the Cornell potential if we set the constants ($A = D = 0$)

$$V(r) = Br - \frac{C}{r}, B, C > 0.$$  

(3)

where $C$ is a coupling constant and $B$ is a linear confinement parameter.

The SE spectrum generated by the Cornell potential can be used in the investigation of the masses and decay widths of charmonium states [8, 19]. The coulombic term arises from one gluon exchange between the quark and its antiquark and dominates at short distances [25]. While the linear term, which is supported by lattice QCD measurements, dominates at large distances [19]. The addition of an inversely quadratic potential (IQP) and the harmonic oscillator (HO) term to the Cornell potential improves the behavior of the potential in the region $r \rightarrow 0$. Furthermore, it leads to improved results as compared to the Cornell potential [30].

The remaining part of the paper is organized as follows. In section two, the synopsis of the WKB approximation formalism is presented. Section three contains the analytical solution of the bound states of the SE generated by the KPIQP. The special cases of the obtained energy eigenvalue are discussed. In section four, we present the numerical results of the masses of charmonium ($c\bar{c}$), bottomonium ($b\bar{b}$), bottom-charmed ($b\bar{c}$), and charmed-strange ($c\bar{s}$) mesons. Furthermore, we compared the WKB mass spectra with the ones obtained by other analytical methods and available experimental data. The paper is concluded in section five.

2. WKB Approximation Formalism

The WKB approximation method is an effective tool initially proposed to find approximate solutions to the one dimensional time independent SE in the limiting case of large radial quantum numbers. It is used to obtain the finite wave function and energy eigenvalues of potentials of interest [20, 31–43]. The method can be used to study quantum tunneling rates in a potential barrier, resonance behavior in a continuum, the exponential decay of an unstable system [43], and the quasi-normal nodes of a black hole and quantum cosmology [44]. The method fails at the classical turning point where the momentum vanishes. This failure is one of the major problems associated with the method. The difficulty can be circumvented using the connection formula [42, 43]. The method accuracy varies markedly for the ground and other low lying states depending on the potential function [39]. Also, the leading order WKB approximation scheme does not yield an exact eigenvalue of the radial SE [31]. To overcome this problem, the orbital centrifugal barrier term $(l+1)$ in the radial SE has to be replaced with the term $(l+1/2)^2$. This modification is known as the Langer correction [40]. Sergeenko [38] stated that the Langer correction regularizes the WKB wave function at the origin and ensures the correct asymptotic behaviour at large radial quantum numbers. Also, the centrifugal barrier contribution of the effective potential does not vanish for the s-wave case $(l=0)$ which makes the SE nontrivial for some potential functions [31, 45], hence the need to use an appropriate approximation scheme.

The three-dimensional time independent SE with a reduced mass $\mu$ and wave-function $\psi(r, \theta, \phi)$ is given as

$$\frac{\hbar^2}{2\mu} \left[ \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2}{\partial \phi^2} \right] \psi(r, \theta, \phi) + V(r) \psi(r, \theta, \phi) = E\psi(r, \theta, \phi).$$

(4)

With the help of the method of separation of variables, we can obtain the radial SE by using the transformation $\psi(r, \theta, \phi) = R(r) Y(\theta, \phi) / r$

$$\frac{d^2R(r)}{dr^2} + \frac{2\mu}{\hbar^2} \left[ E - V(r) - \frac{(l+1/2)^2\hbar^2}{2\mu r^2} \right] R(r) = 0$$

(5)

where the effective potential is given as

$$V_{eff}(r) = V(r) + \frac{(l+1/2)^2\hbar^2}{2\mu r^2}$$

(6)

We can rewrite Eq. (5) as

$$\left[ -\frac{i\hbar}{dr} \right]^2 R(r) = 2\mu \left[ E - V(r) - \frac{(l+1/2)^2\hbar^2}{2\mu r^2} \right] R(r).$$

(7)
From Eq. (7), the classical momentum is given as

\[ P(r) = \left\{ 2\mu \left[ E - V(r) - \frac{(l + 1/2)^2 \hbar^2}{2\mu r^2} \right] \right\}^{1/2} \tag{8} \]

The standard WKB quantization condition [38, 39] for two turning points \((r_1, r_2)\) problem is given as

\[ \int_{r_1}^{r_2} P(r)dr = \pi \hbar \left( n + \frac{1}{2} \right) \cdot r_1 < r < r_2 \quad n = 0, 1, 2 \ldots \tag{9} \]

The turning points are gotten from Eq. (8) by setting \(P(r) = 0\).

The semiclassical wave function in the leading \(\hbar\) approximation has the form

\[ \psi_{WKB}^{(r)}(r) = \frac{N}{\sqrt{P(r)}} \exp \left[ \pm \frac{i}{\hbar} \int P(r)dr \right]. \tag{10} \]

### 3. Solution of the Radial Schrödinger Equation

In this section, we obtain the bound state solution of the SE by substituting Eq. (1) into the WKB standard quantization condition given by Eq. (9)

\[ \int_{r_1}^{r_2} \left\{ 2\mu \left[ E - Ar^2 - Br + C - \frac{D}{r^2} - \frac{(l + 1/2)^2 \hbar^2}{2\mu r^2} \right] \right\}^{1/2} \]
\[ = n\hbar \left( n + \frac{1}{2} \right) \cdot r_1 < r < r_2, \quad n = 0, 1, 2 \ldots \tag{11} \]

In this case, the classical momentum is given as

\[ P(r) = \left\{ 2\mu \left[ E - Ar^2 - Br + C - \frac{D}{r^2} - \frac{(l + 1/2)^2 \hbar^2}{2\mu r^2} \right] \right\}^{1/2} \tag{12} \]

Equation (12) will produce four turning points \(r_1, r_2, r_3, r_4\) and \(r_4\) of which two of the points must lie in the real axis \((r > 0)\) where \(P(r) = 0\). The multiple turning point problem with the potential in Eq (1) is beset with some mathematical challenges. (i) The turning points of the polynomial equation in Eq. (12) would have to be determined by some means of algebra. (ii) The evaluation of the integral in Eq. (11) is not so easy to solve analytically even without the centrifugal barrier term.

To obtain the analytical solution of the WKB quantization integral in Eq. (11), we used the Pekeris-type approximation scheme by setting \(r = 1/y\).

Changing the variable from \(r\) to \(y\), we obtained

\[ -\sqrt{2\mu} \int_{y_1}^{y_2} \sqrt{E - \frac{A}{y^2} - \frac{B}{y} + Cy - \frac{(l + 1/2)^2 \hbar^2 y^2}{2\mu}} dy \]
\[ = n\hbar \left( n + \frac{1}{2} \right), \quad y_1 < y < y_2, \quad n = 0, 1, 2 \ldots \tag{13} \]

Next, we expanded the terms \(A/y^2\) and \(B/y\) in power series form to the second-order around \(r_0(\delta = 1/r_0)\) which is assumed to be the characteristic radius of a meson [12–17].

If we let \(x = y - \delta\) and expand around \(x = 0\), we obtained

\[ A \quad \frac{A}{\delta^2} = \frac{A}{\delta^2} \left( 1 + \frac{x}{\delta} \right)^2 \approx \frac{A}{\delta^2} \left( 1 - \frac{2x}{\delta} + \frac{3x^2}{\delta^2} \right) \tag{14} \]

In the same vein, we obtained the expansion for \(B/y\) as

\[ B \quad \frac{B}{\delta} = \frac{B}{\delta} \left( 3 - \frac{3y}{\delta} + \frac{y^2}{\delta^2} \right) \tag{15} \]

On substituting Eqs. (14) and (15) into (13), we obtained

\[ -\sqrt{2\mu} \int_{y_1}^{y_2} \sqrt{E - \frac{A}{y^2} - \frac{B}{y} + Cy - \frac{(l + 1/2)^2 \hbar^2 y^2}{2\mu}} dy \]
\[ = n\hbar \left( n + \frac{1}{2} \right), \quad y_1 < y < y_2, \quad n = 0, 1, 2 \ldots \tag{16} \]

where

\[ N = \left( D + \frac{3A}{\delta^2} + \frac{B}{\delta^2} + \frac{(l + 1/2)^2 \hbar^2}{2\mu} \right) \tag{17} \]

\[ M = \left( C + \frac{8A}{\delta^2} + \frac{3B}{\delta^2} \right) \tag{18} \]

\[ -T = \left( E - \frac{6A}{\delta^2} - \frac{3B}{\delta^2} \right) \tag{19} \]

Equation (16) can further be simplified as

\[ -\sqrt{2\mu N} \int_{y_1}^{y_2} \sqrt{C + By - y^2} \]
\[ = n\hbar \left( n + \frac{1}{2} \right), \quad y_1 < y < y_2, \quad n = 0, 1, 2 \ldots \tag{20} \]

where

\[ B = \frac{M}{N} \tag{21} \]
The eigenvalue equation has the form of a molecular Kratzer-type 

\[ \text{solution} \text{ of Eq. (20).} \]

\[ \text{the quadratic equation in the squared root of the integrand} \]

\[ \text{of quarkonia and the heavy-light meson system.} \]

4. Discussion

In this section, we present some special cases of the energy eigenvalues of the KPIQP and also obtain the mass spectra of quarkonia and the heavy-light meson system.

If we set \( A = B = D = 0 \) and \( C = Z e^2 \) in Eq. (31), we immediately retrieved the Coulomb’s energy eigenvalue equation given as

\[ E_{nl} = \frac{2\mu}{\hbar^2} \left[ \frac{\left( (8A/\delta^3) + C + (3B/\delta^1) \right)}{(2n + 1) + \sqrt{8\mu/\hbar^2} \left( (3A/\delta^4) + (B/\delta^3) + D + ((l + 1/2)2\hbar^2/2\mu) \right)} \right]^2 + \frac{6A}{\delta^2} + \frac{3B}{\delta} \]  

(31)

By comparing Eqs. (23) and (26), implies that

\[ \frac{\sqrt{2\mu N} \left[ 1/2(y_2 + y_1) - \sqrt{y_2 y_1} \right]}{\sqrt{y_2 y_1}} = h \left( n + \frac{1}{2} \right). \]

(27)

Furthermore, with the help of Eqs. (24) and (25), we can write the sum and products of the turning points as

\[ (y_2 + y_1) = B = \frac{M}{N} \]

(28)

\[ y_2 y_1 = C = \frac{T}{N} \]

(29)

substituting Eqs. (28) and (29) into (27), we obtained the expression

\[ T = \frac{\mu}{2} \left[ \frac{M}{\hbar(n + 1/2) + \sqrt{2\mu N}} \right]^2 \]

(30)

Finally, using the respective notations \( T, M, \) and \( N \) given in Eqs. ((17)–(19)), we obtained the energy eigenvalue expression for the KPIQP as

\[ E_{nl} = -\frac{2\mu}{\hbar^2} \left[ \frac{C}{(2n + 1) + \sqrt{8\mu/\hbar^2} \left( (l + 1/2)^2\hbar^2/2\mu) \right)} \right]^2 \]

(33)

If we let \( A = D = 0 \), then Eq. (31) reduces to the approximate energy expression of the Cornell potential,

\[ E_{nl} = -\frac{2\mu}{\hbar^2} \left[ \frac{C + (3B/\delta^1)}{(2n + 1) + \sqrt{8\mu/\hbar^2} \left( (l + 1/2)^2\hbar^2/2\mu) \right)} \right]^2 + \frac{3B}{\delta} \]

(34)

In order to compute the mass spectra of meson systems, we use the meson mass relation [12]

\[ M_{nl} = m_q + m_{\bar{q}} + E_{nl} \]

(35)
Where $m_q$ and $m_{\bar{q}}$ are the respective quark and antiquark masses.

By substituting Eq. (34) into (35), the mass spectrum of the meson systems for any arbitrary radial and angular momentum quantum numbers becomes

$$M_{nl} = m_q + m_{\bar{q}} - \frac{2\mu_{q\bar{q}}}{\hbar^2} \left[ (2n+1) + \sqrt{(8\mu/\hbar^2)((C+3\delta B^2)/(B/\delta^2) + ((l+1/2)^2\hbar^2/2\mu))} \right] + \frac{3B}{\delta}$$

(36)

Where $m_{q\bar{q}}$ is the reduced mass of the quark-antiquark systems given by the relation

$$\mu_{q\bar{q}} = \frac{m_q m_{\bar{q}}}{m_q + m_{\bar{q}}}$$

(37)

With the numerical support of the MAPLE package, we generated the masses of $(Q\bar{Q})$ and $(Q\bar{q})$ mesons systems by fitting Eq. (36) with experimental data given in Tables 1–4. For the charmonium meson, we substituted the experimental data for the 1S, 2S, 3S states into Eq. (36). This enabled us to determine the $C$ and $\delta$ parameters by solving three algebraic equations. The masses for other excited states are obtained and compared with the results obtained by other analytical methods including available experimental data. Generally, the mass spectra increase as the adjacent quantum levels increase. These trends are in consonance with the results reported in the literature [11, 15, 18, 19, 30], including available experimental data [60–62]. Also, the results for the charmed-strange meson tabulated in Table 4 are in excellent agreement with the ones obtained in Refs. [15, 30] and experimental data such as the excited 1D-state [62]. Furthermore, in Table 3, the mass spectrum of the bottom-charmed meson is very close to the ones obtained with the artificial neural network method [19] and also to the 2S state experimental data [60] indicating an improvement compared to the other methods.

5. Conclusion

The energy eigenvalue equation of the SE with the KPIQP has been obtained with elegance. We demonstrated that the multiple turning point problems can be truncated to two turning points one for a nonsymmetric potential function by using an appropriate Pekeris-type approximation scheme to the deal with the orbital centrifugal barrier energy. This approximation scheme allows us to solve the WKB integral analytically for any angular momentum quantum number. We computed the masses of mesons systems by using the Cornell potential as a model. The approximate mass spectra of the meson systems obtained in this present work are in excellent agreement

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**Table 1: Charmonium (cc) spectra in GeV ($m_c = 1.209 \text{GeV}$, $B = 0.202 \text{GeV}^2$, $C = 1.213$, $\delta = 0.236 \text{GeV}$).**

| State | Present work | AEIM [11] | NU [15] | AIM [30] | LTM [18] | ANN [19] | Exp. [60] |
|-------|--------------|-----------|---------|----------|----------|----------|----------|
| 1S    | 3.098        | 3.095481  | 3.095   | 3.096    | 3.0963   | 3.098    | 3.097    |
| 2S    | 3.689        | 3.567354  | 3.685   | 3.686    | 3.5681   | 3.688    | 3.686    |
| 3S    | 4.041        | 4.039226  | 4.040   | 4.275    | 4.0400   | 4.029    | 4.039    |
| 4S    | 4.266        | 4.511089  | 4.262   | 4.865    | 4.5119   |          |          |
| 1P    | 3.262        | 3.567735  | 3.258   | 3.214    | 3.5687   | 3.516    | 3.516    |
| 2P    | 3.784        | 4.039607  | 3.779   | 3.773    | 4.0406   | 3.925    | 3.927    |
| 1D    | 3.515        | 4.039683  | 3.510   | 3.412    | 4.0407   | 3.779    | 3.779    |

**Table 2: Bottomonium (bb) spectra in GeV ($m_b = 4.823 \text{GeV}$, $B = 0.202 \text{GeV}^2$, $C = 1.664$, $\delta = 0.361 \text{GeV}$).**

| State | Present work | AEIM [11] | NU [15] | AIM [30] | LTM [18] | ANN [19] | Exp. [60] |
|-------|--------------|-----------|---------|----------|----------|----------|----------|
| 1S    | 9.461        | 9.74473   | 9.460   | 9.460    | 9.745    | 9.460    | 9.460    |
| 2S    | 10.023       | 10.02315  | 10.022  | 10.023   | 10.023   | 10.026   | 10.023   |
| 3S    | 10.365       | 10.30158  | 10.360  | 10.585   | 10.3016  | 10.354   | 10.355   |
| 4S    | 10.588       | 10.58000  | 10.580  | 11.148   | 10.580   | 10.572   | 10.579   |
| 1P    | 9.608        | 10.02406  | 9.609   | 9.492    | 10.0246  | 9.891    | 9.899    |
| 2P    | 10.110       | 10.30248  | 10.109  | 10.038   | 10.3029  | 10.258   | 10.260   |
| 1D    | 9.841        | 10.30266  | 9.846   | 9.551    | 10.3032  | 10.156   | 10.164   |
with the results obtained by the other analytical methods and thus reinforce the exactness of the leading order WKB approximation method. Finally, our results may be useful to future experimental works and also can be applied in the study of molecular structures and interactions of diatomic molecules.

**Data Availability**

The results in this present work were numerically obtained from the theoretical calculations without taking any data. Therefore, no data is used in our manuscript.

**Conflicts of Interest**

The authors declare that they have no conflicts of interest.

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**References**

[1] H. Mutuk, “Mass Spectra and Decay Constants of Heavy-Light Mesons: A Case Study of QCD Sum Rules and Quark Model,” *Advances in High Energy Physics*, vol. 2018, Article ID 8095653, 8 pages, 2018.

[2] M. N. Sergeenko, “Light and Heavy Mesons in The Complex Mass Scheme,” 2019, https://arxiv.org/abs/1909.10511.

[3] E. Syahroni, A. Suparmi, and C. Cari, “The determination of the spectrum energy on the model of DNA-protein interactions using WKB approximation method,” *Journal of Physics: Conference Series*, vol. 795, article 012027, 2017.

[4] C. Cari, A. Suparmi, M. Yunianto, and B. N. Pratiwi, “Supersymmetric approach for Killingbeck radial potential plus non-central potential in Schrodinger equation,” *Journal of Physics: Conference Series*, vol. 776, article 012092, 2016.

[5] R. K. Roychoudhury and Y. P. Varshni, “Shifted 1/N expansion and exact solutions for the potential $V(r) = -Zr + gr + \lambda r^2$,” *Journal of Physics A: Mathematical and General*, vol. 21, no. 13, pp. 3025–3034, 1988.

[6] M. Abu-Shady, “Heavy Quarkonia and Mesons in the Cornell potential with harmonic oscillator potential in the N-dimensional Schrödinger equation,” *International Journal of Applied Mathematics and Theoretical Physics*, vol. 2, no. 2, pp. 16–20, 2016.

[7] H. Ciftci and H. F. Kisoglu, “Nonrelativistic Arbitrary $I$-States of Quarkonium through Asymptotic Iteration Method,” *Advances in High Energy Physics*, vol. 2018, Article ID 4549705, 7 pages, 2018.

[8] R. L. Hall and N. Saad, “Schrödinger spectrum generated by the Cornell potential,” *Open Physics*, vol. 13, no. 1, pp. 83–89, 2015.

[9] H. S. Chung, J. Lee, and D. Kang, “Cornell potential parameters for s-wave heavy quarkonia,” *Journal of the Korean Physical Society*, vol. 52, no. 4, pp. 1151–1154, 2008.

[10] A. Vega and J. Flores, “Heavy quarkonium properties from Cornell potential using variational method and supersymmetric quantum mechanics,” *Pramana*, vol. 87, no. 5, article 73, 2016.

[11] E. M. Khokha, M. Abu-Shady, and T. A. Abdel-Karim, “Quarkonium masses in the N-dimensional space using the analytical exact iteration method,” *International Journal of Theoretical and Applied Mathematics*, vol. 2, no. 2, pp. 86–92, 2016.

[12] S. M. Kuchin and N. V. Maksimenko, “Theoretical estimation of spin- average mass spectra of heavy quarkonia and Bc Mesons,” *Universal Journal of Physics and Application*, vol. 1, no. 3, pp. 295–298, 2013.
Advances in High Energy Physics

[13] A. F. Al-Jamel and H. Widyan, "Heavy quarkonium mass spectra in a Coulomb field plus quadratic potential using Nikiforov-Uvarov method," Applied Physics Research, vol. 4, no. 3, 2012.

[14] A. I. Ahmadov, C. Aydin, and O. Uzun, "Bound state solution of the Schrödinger equation at finite temperature," Journal of Physics: Conference Series, vol. 1194, article 012001, 2019.

[15] M. Abu-Shady, T. A. Abdel-Karim, and S. Y. Ezz-Alarab, "Masses and thermodynamic properties of heavy mesons in the non-relativistic quark model using the Nikiforov-Uvarov method," Journal of the Egyptian Mathematical Society, vol. 27, no. 1, 2019.

[16] H. Mansour and A. Gamal, "Bound State of Heavy Quarks Using a General Polynomial Potential," Advances in High Energy Physics, vol. 2018, Article ID 7269657, 7 pages, 2018.

[17] E. Omugbe, "Non-relativistic eigensolutions of molecular and heavy quarkonia interacting potentials via the Nikiforov Uvarov method," Canadian Journal of Physics, 2020.

[18] M. Abu-Shady, T. A. Abdel-Karim, and E. M. Khokha, "Exact Solution of the N-dimensional Radial Schrödinger Equation via Laplace Transformation Method with the Generalized Cornell Potential," SciFed Journal of Quantum Physics, vol. 2, no. 1, 2018.

[19] H. Mutuk, "Cornell potential: A Neural Network approach," Advances in High Energy Physics, vol. 2019, Article ID 3105373, 9 pages, 2019.

[20] R. N. Faustov, V. O. Galkin, A. V. Tatarintsev, and A. S. Vshivtsev, "Algebraic approach to the spectral problem for the Schrödinger equation with power potentials," International Journal of Modern Physics A, vol. 15, no. 2, pp. 209–226, 2002.

[21] D. Kang and E. Won, "Precise numerical solutions of potential problems using the Crank–Nicolson method," Journal of Computational Physics, vol. 227, no. 5, pp. 2970–2976, 2008.

[22] M. Seetharaman, S. Raghavan, and S. S. Vasan, "Analytic WKB energy expression for the linear plus coulomb potential," Journal of Physics A: Mathematical and General, vol. 16, no. 3, pp. 455–462, 1983.

[23] S. Hassanabadi, A. A. Rajabi, and S. Zarrinkamar, "Cornell and Kratzer potentials within the semirelativistic treatment," Modern Physics Letters A, vol. 27, no. 10, article 1250057, 2012.

[24] V. V. Rubish, V. Y. Lazur, O. K. Reity, S. Chalupka, and M. Salak, "The WKB method for the Dirac equation with the vector and scalar potentials," Czechoslovak Journal of Physics, vol. 54, no. 9, pp. 897–919, 2004.

[25] S. M. Ilkhdair, "Relativistic bound states of spinless particle by the Cornell potential model in external fields," Advances in High Energy Physics, vol. 2013, Article ID 491648, 10 pages, 2013.

[26] M. Abu-Shady, "Analytic solution of Dirac equation for extended Cornell potential using the Nikiforov-Uvarov method," Boson Journal of Modern Physics, vol. 1, no. 1, pp. 16–19, 2015.

[27] P. C. W. Davies and D. S. Betts, Quantum Mechanics, Chapman and Hall, London, 2nd edition, 1994.

[28] B. J. Hazarika and D. K. Choudhury, "Isgur–Wise function in a QCD-inspired potential model with WKB approximation," Pramana, vol. 88, no. 3, 2017.

[29] A. F. Nikiforov and V. B. Uvarov, Special Functions of Mathematical Physics, Birkhauser, Basel, 1988.

[30] R. Rani, S. B. Bhardwaj, and F. Chand, "Mass Spectra of Heavy and Light Mesons Using Asymptotic Iteration Method," Communications in Theoretical Physics, vol. 70, no. 2, p. 179, 2018.

[31] M. N. Sergeenko, "Zeroth WKB approximation in quantum mechanics," 2002, https://arxiv.org/abs/quant-ph/0206179.

[32] E. Omugbe, O. E. Osafiele, and I. B. Okon, "WKB Energy Expression for the Radial Schrödinger Equation with a Generalized Pseudoharmonic Potential," Asian Journal of Physical and Chemical Sciences, vol. 8, no. 2, pp. 13–20, 2020.

[33] M. N. Sergeenko, "Quasiclassical analysis of three-dimensional Schrödinger's equation and its solution," Modern Physics Letters A, vol. 15, no. 2, pp. 83–100, 2011.

[34] B. I. Ita, H. Louis, O. U. Akakuru et al., "Approximate solution to the Schrodinger equation with Manning-Rosen plus a class of Yukawa potential via the WKBJ approximation method," Bulgaria Journal of Physics, vol. 45, pp. 323–333, 2018.

[35] I. I. Benedict, H. Louis, N. I. Nelson et al., "Approximate Instanton States solutions to the Schrodinger equation with Manning-Rosen plus Hellmann potential via WKB approximation scheme," Sri Lankan Journal of Physics, vol. 19, no. 1, pp. 37–45, 2018.

[36] H. Louis, B. I. Ita, P. I. Amos et al., "WKB solution for inversely quadratic Yukawa potential plus inversely Hellmann potential," World Journal of Applied Physics, vol. 2, no. 4, pp. 109–112, 2017.

[37] M. N. Sergeenko, "Quantum fluctuations of the angular momentum and energy of the ground state," Modern Physics Letters A, vol. 13, no. 1, pp. 33–37, 2011.

[38] M. N. Sergeenko, "Semiclassical wave equation and exactness of the WKB method," Physical Review A, vol. 53, no. 6, pp. 3798–3804, 1996.

[39] M. Hruska, W. Y. Keung, and U. Sukhatme, "Accuracy of semiclassical methods for shape-invariant potentials," Physical Review A, vol. 55, no. 5, pp. 3345–3350, 1997.

[40] R. E. Langer, "On the connection formulas and the solutions of the wave equation," Physical Review, vol. 51, no. 8, pp. 669–676, 1937.

[41] U. Sukhatme and A. Pagnamenta, "Finite eigenfunctions in the WKB approximation," American Journal of Physics, vol. 59, no. 10, pp. 944–947, 1991.

[42] D. J. Griffiths, Introduction to Quantum Mechanics, Prentice Hall Inc., Upper Saddle River, NJ USA, 1995.

[43] E. Merzbacher, Quantum Mechanics, John Wiley and Sons Inc., New York, 3rd edition, 1998.

[44] F. Lu, B. Lv, P. Wang, and H. Yang, "WKB approximation for a deformed Schrödinger-like equation and its applications to quasinormal modes of black holes and quantum cosmology," Nuclear Physics B, vol. 937, pp. 502–532, 2018.

[45] E. Omugbe, "Non-relativistic Energy Spectrum of the Deng-Fan Oscillator via the WKB Approximation Method," Asian Journal of Physical and Chemical Sciences, vol. 8, no. 1, pp. 26–36, 2020.

[46] H. Akcay and R. Sever, "Analytical solutions of Schrödinger equation for the diatomic molecular potentials with any angular momentum," Journal of Mathematical Chemistry, vol. 50, no. 7, pp. 1973–1987, 2012.

[47] S. M. Ilkhdair and R. Sever, "Exact quantization rule to the Kratzer-type potentials: an application to the diatomic molecules," Journal of Mathematical Chemistry, vol. 45, no. 4, pp. 1137–1152, 2009.
[48] S. Ikhdair and R. Sever, “Exact solutions of the radial Schrödinger equation for some physical potentials,” *Open Physics*, vol. 5, no. 4, pp. 516–527, 2007.

[49] K. J. Oyewumi, “Realization of the spectrum generating algebra for the generalized Kratzer potentials,” 2010, https://arxiv.org/abs/1008.1515v1.

[50] R. Rani, S. B. Bhardwaj, and F. Chand, “Bound state solutions to the Schrödinger equation for some diatomic molecules,” *Pramana*, vol. 91, no. 4, 2018.

[51] C. Berkdemir, “Application of the Nikiforov-Uvarov Method in Quantum Mechanics,” in *Theoretical Concepts of Quantum Mechanics*, M. R. Pahlavani, Ed., Intech Open, 2012.

[52] B. I. Ita, H. Louis, O. I. Michael, and N. I. Nelson, “Solutions of the Schrödinger Equation for the Superposed Screened Coulomb plus Kratzer Fues Potential Using the WKB Approximation Method,” *Discovery*, vol. 54, no. 276, pp. 447–452, 2018.

[53] K. J. Oyewumi, “Analytical solutions of the Kratzer-Fues potential in an arbitrary number of dimensions,” *Foundations of Physics Letters*, vol. 18, no. 1, pp. 75–84, 2005.

[54] M. Molski and J. Konarski, “Modified Kratzer-Fues formula for rotation-vibration energy of diatomic molecules,” *Acta Physica Polonica A*, vol. 82, no. 6, pp. 927–936, 1992.

[55] C. Tezcan and R. Sever, “A General Approach for the Exact Solution of the Schrodinger Equation,” 2009, https://arxiv.org/abs/0807.2304v2.

[56] P. G. Hajigeorgiou, “Exact analytical expressions for diatomic rotational and centrifugal distortion constants for a Kratzer-Fues oscillator,” *Journal of Molecular Spectroscopy*, vol. 235, no. 1, pp. 111–116, 2006.

[57] R. L. Hall and N. Saad, “Smooth transformations of Kratzer’s potential in N dimensions,” *The Journal of Chemical Physics*, vol. 109, no. 8, pp. 2983–2986, 1998.

[58] O. Bayrak, I. Boztosun, and H. Ciftci, “Exact analytical solutions to the Kratzer potential by the asymptotic iteration method,” *International Journal of Quantum Chemistry*, vol. 107, no. 3, pp. 540–544, 2007.

[59] S. Ikhdair and R. Sever, “On solutions of the Schrödinger equation for some molecular potentials: wave function ansatz,” *Open Physics*, vol. 6, no. 3, pp. 697–703, 2008.

[60] C. Patrignani and Particle Data Group, “Review of particle physics,” *Chinese Physics C*, vol. 40, no. 10, article 100001, 2016.

[61] K. A. Olive and Particle Data Group, “Review of Particle Physics,” *Chinese Physics C*, vol. 38, no. 9, article 090001, 2014.

[62] S. Godfrey and K. Moats, 2015, https://arxiv.org/abs/1409.0874v3.