Young’s Inequality in Semifinite von Neumann Algebras

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Abstract

This paper formulates Young-type inequalities for singular values (or $s$-numbers) and traces in the context of von Neumann algebras. In particular, it shown that if $\text{tr} (\cdot)$ is a faithful semifinite normal trace on a semifinite von Neumann algebra $M$ and if $p$ and $q$ are positive real numbers for which $p^{-1} + q^{-1} = 1$, then, for all positive operators $a, b \in M$, $\text{tr} (|ab|) \leq p^{-1}\text{tr} (a^p) + q^{-1}\text{tr} (b^q)$, with equality holding (in the cases where $p^{-1}\text{tr} (a^p) + q^{-1}\text{tr} (b^q) < \infty$) if and only if $b^q = a^p$.

1 Introduction

Young’s inequality (see, for example, page 17 of [10]) asserts that if $p$ and $q$ are positive real numbers for which $p^{-1} + q^{-1} = 1$, then $|\lambda\mu| \leq p^{-1}|\lambda|^p + q^{-1}|\mu|^q$, for all complex numbers $\lambda$ and $\mu$, and equality holds if and only if $|\mu|^q = |\lambda|^p$. Several generalisations of Young’s inequality whereby $\lambda$ and $\mu$ are replaced by Hilbert space operators—or by singular values, norms, or traces of operators—are known [1, 5, 6, 14]. The present paper adds to these results by formulating new Young-type inequalities in the context of von Neumann algebras.

In what follows, $N$ shall denote an arbitrary von Neumann algebra of operators acting on a complex infinite-dimensional Hilbert space $H$. By $M$
we denote a semifinite von Neumann algebra and $\text{tr}(\cdot)$ is assumed to be a faithful semifinite normal trace on $M$. The cone of positive operators in $N$ and the projection lattice in $N$ are denoted by $N^+$ and $\mathcal{P}(N)$ respectively. For any $z \in N$, $|z|$ denotes $(z^*z)^{1/2}$, the unique positive square root of $z^*z$. The notation $e \sim f$, for $e, f \in \mathcal{P}(N)$, shall indicate that $e$ and $f$ are Murray–von Neumann equivalent, which is to say that $e = v^*v$ and $f = vv^*$ for some $v \in N$. If $x \in N$, then $R[x]$ denotes the range projection for $x$ (that is, the projection in $N$ whose range is the closure of the range of $x$.) The spectral resolution of the identity of $a \in N^+$ is denoted by $p^a$, yielding

$$a = \int_0^\infty s \, dp^a(s) .$$

Murray and von Neumann, in their 1936 paper [15], introduced generalised singular values, or $s$-numbers, for operators in semifinite von Neumann algebras. Years later, in the 1980s, interest in majorisation and operator inequalities—some of which is chronicled in [4] and [17]—led to careful studies of singular values by Fack [7], Fack and Kosaki [8], Hiai and Nakamura [12], and others, in the operator algebra context.

The singular values $\mu_z(t)$ of $z \in M$ are defined for each $t \in \mathbb{R}_0^+$, where $\mathbb{R}_0^+$ is the set of nonnegative real numbers, by the equation

$$\mu_z(t) = \inf \{ \|ze\| : e \in \mathcal{P}(M), \text{tr}(1 - e) \leq t \} .$$

The trace of $|x|$, for $x \in M$, is recovered from the singular values of $x$ by way of the equation

$$\text{tr}(|x|) = \int_0^\infty \mu_x(t) \, dt .$$

Thus, inequalities between each of the singular values of a pair of positive operators necessarily imply an inequality between the traces of these operators. In consequence, one aims to formulate operator inequalities at the level of singular values, if possible.

The singular value inequality to be established in the present paper is inequality (1) below. Inequality (1) was first established by Ando [1] for finite-dimensional $M$, whereas the cases of equality, again for $M$ of finite dimension, were analysed by Hirzallah and Kittaneh [13].

**Theorem 1.1** Assume that $M$ is a semifinite von Neumann algebra and that $\text{tr}(\cdot)$ is a faithful semifinite normal trace on $M$. Let $p, q \in \mathbb{R}^+$ satisfy }
\( p^{-1} + q^{-1} = 1 \). Then, for every \( x, y \in M \),
\[
\mu_{|xy|^*}(t) \leq \mu_{p^{-1}|x|^p + q^{-1}|y|^q}(t), \quad \text{for all } t \in \mathbb{R}_0^+ .
\] (1)

If \( \text{tr} (1) < \infty \), then equality holds in the Young inequality (1), for some \( x, y \in M \), if and only if \( |y|^q = |x|^p \).

Young inequalities in traces can also be formulated, leading to the following result for \( C^* \)-algebras.

**Theorem 1.2** Assume that \( A \) is a unital \( C^* \)-algebra and that \( \tau \) is a faithful tracial state on \( A \). Let \( p, q \in \mathbb{R}^+ \) satisfy \( p^{-1} + q^{-1} = 1 \). Then, for every \( a, b \in A^+ \),
\[
\tau(|ab|) \leq p^{-1}\tau(a^p) + q^{-1}\tau(b^q) .
\] (2)

Equality holds in the Young inequality (2), for some \( a, b \in A^+ \), if and only if \( b^q = a^p \).

The proofs of Theorems 1.1 and 1.2 make extensive use of various properties of singular values; these properties are review below. Further details can be found in [7].

For each \( z \in M \), the function \( \mu_z : \mathbb{R}_0^+ \to \mathbb{R}_0^+ \) is nonincreasing and continuous on the right. Moreover, \( \mu_z = \mu_{z^*} = \mu_{|z|^*} \) and, consequently, for any \( x, y \in M \),
\[
\mu_{|xy|^*} = \mu_{|yx|^*} .
\] (3)

In addition, for every \( w_1, w_2 \in M \) and \( t \in \mathbb{R}_0^+ \),
\[
\mu_{w_1zw_2}(t) \leq \|w_1\| \|w_2\| \mu_z(t) .
\] (4)

The dependence of \( \mu_z \) on \( z \) is as follows: if \( z_1, z_2 \in M \), then, for all \( t \in \mathbb{R}_0^+ \),
\[
|\mu_{z_1}(t) - \mu_{z_2}(t)| \leq \|z_1 - z_2\| .
\] (5)

The singular values of positive operators are especially well behaved. If \( h \in M^+ \), then
\[
\mu_h(t) = \min \left\{ s \in \mathbb{R}_0^+ : \text{tr} \left( p^h(s, \infty) \right) \leq t \right\}.
\] (6)

Alternatively, one can employ a variational principle to evaluate \( \mu_h(t) \):
\[
\mu_h(t) = \inf \{ \sup \{ \langle h\xi, \xi \rangle : \xi \in \text{ran } e, \|\xi\| = 1 \} : e \in P(M), \text{tr} (1 - e) \leq t \} .
\]
Finally, we shall also require the following continuous functional calculus (Proposition 1.6 of [7]). If \( \psi : \mathbb{R}_0^+ \to \mathbb{R}_0^+ \) is an increasing continuous function such that \( \psi(0) = 0 \), then

\[
\mu_{\psi(h)}(t) = \psi(\mu_h(t)), \text{ for all } t \in \mathbb{R}_0^+.
\] (7)

Throughout, the notation \( a \leq b \), for hermitian operators \( a, b \in \mathcal{N} \), refers to the Löwner partial order, namely \( a \leq b \) if and only if \( \langle a \xi, \xi \rangle \leq \langle b \xi, \xi \rangle \) for all \( \xi \in H \).

## 2 Inequalities

Ando proved in [1] that, for operators on finite-dimensional Hilbert spaces, Young’s inequality holds at the level of singular values. This Young-type inequality for singular values was later extended to compact operators in [6]. Theorem 2.3 below represents the most general form of Ando’s original result. However, the proof of Theorem 2.3 and other results herein rest upon a core result concerning a compressed form of Young’s inequality. This “compression lemma” was also first established in finite dimensions by Ando [1], but it holds in arbitrary von Neumann algebras as well [6].

**Lemma 2.1 (Compression Lemma)** Assume that \( p \in (1, 2], q = (1 - p^{-1})^{-1}, a, b \in \mathbb{N}^+, \) and \( b \) is invertible. If \( f_s = R[b^{-1}p^{ab}((s, \infty))] \), for \( s \in \mathbb{R}_0^+ \), then

\[
sf_s \leq f_s \left( p^{-1}a^p + q^{-1}b^q \right) f_s \quad \text{and} \quad f_s \sim p^{ab}((s, \infty)).
\]

**Proof.** Except for the claim that \( f_s \sim p^{ab}((s, \infty)) \), the rest of the lemma is precisely Proposition 2.3 of [6]. To prove that \( f_s \sim p^{ab}((s, \infty)) \), it is enough to prove the following general proposition: if \( e, f \in \mathcal{P}(\mathbb{N}) \), \( b \in \mathbb{N}^+ \) is invertible, and \( f = R[b^{-1}e] \), then \( e \sim f \). To this end, if \( b^{-1}e = v|b^{-1}e| \) is the polar decomposition of \( b^{-1} \), then \( v^*v = R[eb^{-1}] = e \) and \( vv^* = R[b^{-1}e] = f \), whence \( e \sim f \). \( \square \)

**Lemma 2.2** If \( f \in \mathcal{P}(M) \), then \( \mu_f(t) = 1 \) for all \( t < tr(f) \) and \( \mu_f(t) = 0 \) for all \( t \geq tr(f) \).

**Proof.** If \( t \geq tr(f) \), then set \( e = 1 - f \) to obtain \( tr(1 - e) \leq t \) and \( 0 = \|f(1 - f)\| = \|fe\| \geq \mu_f(t) \geq 0 \), which shows that \( \mu_f(t) = 0 \).
Assume now that \( t < \text{tr}(f) \). If \( e \in \mathcal{P}(M) \) satisfies \( \text{tr}(1 - e) \leq t \), then \( \text{tr}(1 - e) < \text{tr}(f) \). By Kaplansky’s Lemma,

\[
e - (e \land f) \sim (e \lor f) - f \leq 1 - f,
\]
and so if it were true that \( e \land f = 0 \), then we would have \( \text{tr}(e) \leq \text{tr}(1 - f) \), or equivalently \( \text{tr}(1 - e) \geq \text{tr}(f) \), in contradiction to \( \text{tr}(1 - e) < \text{tr}(f) \). Thus, it must be that \( e \land f \neq 0 \), and so \( \|fe\| = 1 \). This proves that \( \|fe\| = 1 \) for all \( e \in \mathcal{P}(M) \) that satisfy \( \text{tr}(1 - e) \leq t \); hence, \( \mu_f(t) = 1 \).

The general form of Ando’s theorem \([1]\) can now be established.

**Theorem 2.3 (Young’s Inequality in Singular Values)** If \( p \) and \( q \) are positive real numbers for which \( p^{-1} + q^{-1} = 1 \), and if \( x, y \in M \) and \( t \in \mathbb{R}_0^+ \), then

\[
\mu_{|xy^*|}(t) \leq \mu_{p^{-1}|x|^p + q^{-1}|y|^q}(t).
\]

(8)

**Proof.** We begin by showing that the proof can be reduced to the case of positive operators.

If \( y = w|y| \) is the polar decomposition of \( y \) in \( M \), then the proof of Proposition 4.1 in \([3]\) demonstrates that

\[
|x| = w|y| = w^*.
\]

(9)

By \( \|w\| \leq 1 \) and property (4) of \( \mu_z \), we have that

\[
\mu_{|xy^*|}(t) \leq \mu_{|x||y|}(t).
\]

(10)

Thus, in setting \( a = |x| \) and \( b = |y| \), it is sufficient to prove that

\[
\mu_{|ab|}(t) \leq \mu_{p^{-1}a^p + q^{-1}b^q}(t).
\]

(11)

Therefore, we shall prove that inequality (11) holds for all \( a, b \in M^+ \).

So, we assume henceforth that \( a, b \in M^+ \). We assume, further, that \( p \in (1, 2] \) and that \( b \in M^+ \) is invertible. The assumption on \( p \) entails no loss of generality because if inequality (11) holds for \( 1 < p \leq 2 \), then in cases where \( p > 2 \) the conjugate \( q \) satisfies \( q < 2 \), and so

\[
\mu_{|ab|}(t) = \mu_{|ba|}(t) \leq \mu_{q^{-1}b^q + p^{-1}a^p}(t),
\]

5
where the equality $\mu_{|ab|} = \mu_{|ba|}$ is obtained from property (3) of the function $\mu_z$. The assumption that $b \in M^+$ be invertible is also no loss in generality, for if $b$ is not invertible, then consider $b_\varepsilon = b + \varepsilon I$, an invertible element for which $\|b_\varepsilon - b\| \to 0$ as $\varepsilon \to 0^+$. Property (5) of the function $\mu_z$ implies that
\[
\lim_{\varepsilon \to 0^+} \mu_{|ab_\varepsilon|} (t) = \mu_{|ab|} (t),
\]
for every $t \in \mathbb{R}^+_0$. Thus, if inequality (11) holds for invertible elements, then for every $t$,
\[
\mu_{|ab|} (t) = \lim_{\varepsilon \to 0^+} \mu_{|ab_\varepsilon|} (t) \leq \lim_{\varepsilon \to 0^+} \mu_{p^{-1}a^p + q^{-1}b^q} (t) = \mu_{p^{-1}a^p + q^{-1}b^q} (t).
\]
Hence, it is enough to already assume that $b \in M^+$ is invertible.

Invoke Lemma 2.1 to obtain, for $s \in \mathbb{R}^+_0$ and $f_s = R[b^{-1}p^{(s, \infty)}]$,
\[
fs \leq f_s (p^{-1}a^p + q^{-1}b^q) f_s
\]
and $f_s \sim p^{(s, \infty)}$. Consequently, $t (f_s) = \text{tr} (p^{(s, \infty)})$.

Now fix $t \in \mathbb{R}^+_0$ and let $\zeta = \mu_{|ab|} (t)$. If $\zeta = 0$, then inequality (11) holds trivially. Therefore, assume that $\zeta > 0$. Suppose that $\varepsilon > 0$ satisfies $(\zeta - \varepsilon) > 0$. By (3),
\[
\zeta = \min \{ s \in \mathbb{R} : \text{tr} (p^{(s, \infty)}) \leq t \}.
\]
With $s = \zeta$, Lemma 2.1 yields $f_{\zeta} \sim p^{(\zeta, \infty)}$. Hence,
\[
\text{tr} (f_{\zeta}) = \text{tr} (p^{(\zeta, \infty)}) \leq t.
\]
Thus,
\[
0 \leq (\zeta - \varepsilon) < \zeta \implies \text{tr} (f_{\zeta - \varepsilon}) > t.
\]
Now replace in $s$ with $\zeta - \varepsilon$ in inequality (12) to obtain the inequality
\[
(\zeta - \varepsilon) f_{\zeta - \varepsilon} \leq f_{\zeta - \varepsilon} (p^{-1}a^p + q^{-1}b^q) f_{\zeta - \varepsilon}.
\]
By (3), inequality (13) yields
\[
(\zeta - \varepsilon) \mu_{f_{\zeta - \varepsilon}} (t) \leq \mu_{p^{-1}a^p + q^{-1}b^q} (t) \quad \text{for all } t \in \mathbb{R}^+_0.
\]
By Lemma 2.2, $\text{tr} (f_{\zeta - \varepsilon}) > t$ implies that $\mu_{f_{\zeta - \varepsilon}} (t) = 1$. Therefore, inequality (14) can be rewritten as
\[
\zeta \leq \mu_{p^{-1}a^p + q^{-1}b^q} (t) + \varepsilon.
\]
Because \( \mu_{|ab|}(t) = \zeta \) and because the inequality above is true for every \( \varepsilon > 0 \),
\( \mu_{|ab|}(t) \leq \mu_{p^{-1}a^{p}+q^{-1}b^{q}}(t) \), which completes the proof. \( \square \)

Theorem 2.3 does not hold, in general, if \( \mu_{|xy^{*}|}(t) \) is replaced by \( \mu_{|xy|}(t) \) on the left hand side. A counterexample in \( 2 \times 2 \) matrices can be found on p. 263 of [4].

As noted in the introduction, Young’s inequality in singular values automatically leads to a Young-type inequality for traces.

Corollary 2.4 (A Tracial Young Inequality) If \( p \) and \( q \) are positive real numbers for which \( p^{-1} + q^{-1} = 1 \), then, for all \( x,y \in M \),
\[
\text{tr}(|x^{*}y^{*}|) \leq p^{-1}\text{tr}(|x|^{p}) + q^{-1}\text{tr}(|y|^{q}), \tag{15}
\]
where \( x^{i} \in \{ x, x^{*}, |x|, |x^{*}| \}, y^{i} \in \{ y, y^{*}, |y|, |y^{*}| \} \).

Proof. Theorem 2.3 and the integral representation of traces imply that
\[
\text{tr}(|xy^{*}|) = \int_{0}^{\infty} \mu_{|xy^{*}|}(t) \, dt \\
\leq \int_{0}^{\infty} \mu_{p^{-1}|x|^{p}+q^{-1}|y|^{q}}(t) \, dt \\
= \text{tr}(p^{-1}|x|^{p} + q^{-1}|y|^{q}).
\]
Thus, what remains to be shown is that the right hand side of the inequality does not change if \( x \) and \( y \) are replaced by \( x^{i} \) and \( y^{i} \). That is, we need only show that \( \text{tr}(|x^{i}|^{p}) = \text{tr}(|x|^{p}) \) and that \( \text{tr}(|y^{i}|^{q}) = \text{tr}(|y|^{q}) \). It is enough to consider the case of \( x \). The identity \( \text{tr}(x^{*}x) = \text{tr}(xx^{*}) \) implies that \( \text{tr}((x^{*}x)^{k}) = \text{tr}((xx^{*})^{k}) \) for all positive integers \( k \). Thus, by functional calculus, \( \text{tr}(\psi(|x|)) = \text{tr}(\psi(|x^{*}|)) \), for all continuous functions \( \psi : \mathbb{R}_{0}^{+} \rightarrow \mathbb{R}_{0}^{+} \)—and in particular for \( \psi(t) = t^{p} \). \( \square \)

An elegant and far-reaching theory of majorisation in von Neumann algebras was developed by Hiai in [11]. If \( x, y \in M \), then we write \( \mu(x) \prec_{w} \mu(y) \) to denote that the singular values of \( x \) are weakly majorised by the singular values of \( y \). That is,
\[
\mu(x) \prec_{w} \mu(y) \text{ if and only if } \int_{0}^{s} \mu_{x}(t) \, dt \leq \int_{0}^{s} \mu_{y}(t) \, dt \text{ for all } s \in \mathbb{R}_{0}^{+}.
\]
The following result (Proposition 4.3 of [7]) concerning weak majorisation is fundamental: \( \mu(xy) \prec_w \mu(x)\mu(y) \). That is,

\[
\int_0^s \mu_{xy}(t) \, dt \leq \int_0^s \mu_x(t)\mu_y(t) \, dt \quad \text{for all } s \in \mathbb{R}_0^+.
\] (16)

Of course, Theorem 2.3 implies that \( \mu(|xy^*|) \prec_w \mu(p^{-1}|x|^p + q^{-1}|y|^q) \), for all \( x, y \in M \).

In addition to weak majorisation, it is useful to consider the spectral pre-order, which can be formulated in arbitrary von Neumann algebras. For \( a, b \in N^+ \), we write \( a \prec_{sp} b \) to indicate that \( p^a(s, \infty) \) is Murray-von Neumann equivalent to a subprojection of \( p^b(s, \infty) \), for every \( s \in \mathbb{R}_0^+ \).

We now apply a little of the theory of majorisation to the singular value Young inequality to obtain Theorem 2.5 below, which shows that, in finite factors, Young’s inequality in singular values implies a Young inequality in the spectral pre-order and a Young inequality in the Löwner partial order after a correction by a doubly stochastic map. (A positive linear map \( \Phi : M \to M \) is doubly stochastic if \( \Phi(1) = 1 \) and \( \text{tr} (\Phi(h)) = \text{tr} (h) \) for all \( h \in M^+ \).

**Theorem 2.5** Assume that \( p \) and \( q \) are positive real numbers for which \( p^{-1} + q^{-1} = 1 \), and let \( x, y \in M \), where \( M \) is a (semifinite) factor.

1. \( |xy^*| \prec_{sp} p^{-1}|x|^p + q^{-1}|y|^q \), and

2. if \( M \) is finite, then \( |xy^*| \leq p^{-1}\Phi(|x|^p) + q^{-1}\Phi(|y|^q) \) for some doubly stochastic positive linear map \( \Phi : M \to M \).

**Proof.** Because \( M \) is a factor, if \( e, f \in \mathcal{P}(M) \), then \( e \) is equivalent to a subprojection of \( f \) or vice versa. If we consider this fact with the spectral projections

\[
p^{xy^*}(s, \infty) \quad \text{and} \quad p^{p^{-1}|x|^p + q^{-1}|y|^q}(s, \infty),
\]

for all \( s \in \mathbb{R}_0^+ \), then the Young inequality \( \mu_{|xy^*|}(t) \leq \mu_{p^{-1}|x|^p + q^{-1}|y|^q}(t) \), (Theorem 2.3) yields \( |xy^*| \prec_{sp} p^{-1}|x|^p + q^{-1}|y|^q \), which proves the first assertion.

If \( M \) is finite, then Young’s inequality \( \mu_{|xy^*|}(t) \leq \mu_{p^{-1}|x|^p + q^{-1}|y|^q}(t) \), for all \( t \in \mathbb{R}_0^+ \), and Theorem 4.7 of [11] imply that \( |xy^*| = \Psi(p^{-1}|x|^p + q^{-1}|y|^q) \), for some positive linear map \( \Psi : M \to M \) for which \( \text{tr} (\Psi(a)) \leq \text{tr} (a) \), for all \( a \in M^+ \). Proposition 4.3 of [11] completes the argument: because \( M \) is
finite, there is a doubly stochastic positive linear map \( \Phi : M \to M \) such that 
\[ \Psi(a) \leq \Phi(a), \text{ for all } a \in M^+. \]

It would be interesting to know whether the doubly stochastic positive linear map \( \Phi : M \to M \) could in fact be chosen to be an automorphism. (This is the case if \( M \) is a factor of type I, as the spectral theorem and Theorem 2.3 demonstrate.)

In the case \( p = q = 2 \), Young’s inequality is the arithmetic–geometric mean inequality, which, if \( M \) were taken to be \( \mathbb{C} \), would be in the form 
\[ \alpha \beta \leq \frac{1}{2} (\alpha^2 + \beta^2), \]
for positive real numbers \( \alpha, \beta \). Sometimes, however, one wants the arithmetic–geometric mean inequality in its more traditional (equivalent) form: 
\[ \sqrt{\alpha \beta} \leq \frac{1}{2} (\alpha + \beta). \]

Such a formulation extends to non-commutative \( M \) in Theorem 2.6 below as weak majorisation and as a tracial inequality. (See, also, [3] for a formulation in unitarily-invariant norms.)

**Theorem 2.6 (Arithmetic–Geometric Mean Inequality)** If \( a, b \in M^+ \), then
\[ \mu(|ab|^{1/2}) \prec_w \frac{1}{2} (\mu(a) + \mu(b)) \]
and
\[ \text{tr}(|ab|^{1/2}) \leq (\text{tr}(a)\text{tr}(b))^{1/2} \leq \frac{1}{2} (\text{tr}(a) + \text{tr}(b)). \]

**Proof.** For any \( h \in M^+ \), let \( \Lambda_h : (0, \text{tr}(1)) \to \mathbb{R}^+ \) denote the function
\[ \Lambda_h(s) = \exp \left( \int_0^s \log \mu_h(t) \, dt \right). \]
Because \( \mu_{h^{1/2}}(t) = \sqrt{\mu_h(t)} \) for all \( t \geq 0 \), by the functional calculus [7], it follows that \( \Lambda_{h^{1/2}}(s) = \sqrt{\Lambda_h(s)} \) for all \( s \in (0, \text{tr}(1)) \). Furthermore, Theorem 2.3 of [7] indicates that \( \Lambda_{ab^{1/2}}(s) \leq \Lambda_a(s)\Lambda_b(s) \) and so \( \Lambda_{|ab|^{1/2}}(s) \leq \Lambda_{a^{1/2}}(s)\Lambda_{b^{1/2}}(s) \). Using this last inequality and the equations
\[ \Lambda_{a^{1/2}}(s)\Lambda_{b^{1/2}}(s) = \exp \left( \int_0^s (\log \mu_{a^{1/2}}(t) + \log \mu_{b^{1/2}}(t)) \, dt \right) \]
\[ = \exp \left( \int_0^s \log \sqrt{\mu_a(t)\mu_b(t)} \, dt \right), \]
we see that, for all \( s \in (0, \text{tr}(1)) \),
\[ \int_0^s \log \mu_{|ab|^{1/2}}(t) \, dt \leq \int_0^s \log \sqrt{\mu_a(t)\mu_b(t)} \, dt. \quad (17) \]
Each of the integrands in (17) above is nonincreasing, and so if \( f \) is any increasing convex function, then for all \( s \in (0, \text{tr}(1)) \) (by majorisation theory [10]),
\[
\int_0^s f \left( \log \mu_{|ab|^{1/2}}(t) \right) \, dt \leq \int_0^s f \left( \log \sqrt{\mu_a(t)\mu_b(t)} \right) \, dt.
\]
In particular, inequality (18) holds for the function \( f(r) = e^r \). Hence,
\[
\int_0^s \mu_{|ab|^{1/2}}(t) \, dt \leq \int_0^s \sqrt{\mu_a(t)\mu_b(t)} \, dt
\]
\[
\leq \left( \int_0^s \mu_a(t) \, dt \right)^{1/2} \left( \int_0^s \mu_b(t) \, dt \right)^{1/2}
\]
\[
\leq \frac{1}{2} \int_0^s \mu_a(t) \, dt + \frac{1}{2} \int_0^s \mu_b(t) \, dt.
\]
(The second inequality above is the Cauchy–Schwarz inequality.) □

Theorem 2.6 above is proved with considerably less effort than what Theorem 2.3 requires. Some other related tracial inequalities are just as readily established; for example, the Fenchel–Young inequality. To describe what is involved, assume that \( F : \mathbb{R}_0^+ \to \mathbb{R}_0^+ \) is a convex function and that \( F^* : \Gamma_F \to \mathbb{R}_0^+ \) is the Fenchel conjugate [9] of \( F \). That is, \( \Gamma_F \) is a convex subset of \( \mathbb{R} \) and
\[
F^*(r) = \sup_{t \in \Gamma_F} (rt - F(t)).
\]
The Fenchel–Young inequality is \( \alpha \beta \leq F(\alpha) + F^*(\beta) \), for all \( \alpha \in \mathbb{R}_0^+ \), \( \beta \in \Gamma_F \). (If \( p > 1 \) and \( F(t) = p^{-1}t^p \), then the Fenchel conjugate is \( F^*(s) = q^{-1}s^q \), where \( p^{-1} + q^{-1} = 1 \), and \( \Gamma_F = \mathbb{R}_0^+ \). Thus, the Fenchel–Young inequality implies the Young inequality under study here.) If \( a, b \in M^+ \), then the weak majorisation relation \( \mu(|ab|) \prec_w \mu(a)\mu(b) \) suggests that one can apply the Fenchel–Young inequality pointwise to the products \( \mu_a(t)\mu_b(t) \) to obtain a Fenchel–Young inequality in traces:
\[
\text{tr}(|ab|) \leq \text{tr}(F(a)) + \text{tr}(F^*(b)),
\]
for all \( a, b \in M^+ \) for which \( \mathbb{R}_0^+ \cap \Gamma_F \) contains the spectrum of \( b \).

## 3 Cases of Equality

There are few results about cases of equality in operator inequalities; however, Young’s inequality is somewhat of an exception. A characterisation
of equality in the singular value Young inequality (Theorem 2.3) was given for finite-dimensional $M$ in [13], whereas a characterisation of equality (for elements of finite trace) in the tracial Young inequality (Corollary 2.4) was given in [2] for the $I\infty$-factor $B(H)$. Theorems 3.1 and 3.2 below add to these results by characterising the cases of equality in the Young inequalities when $M$ is an arbitrary semifinite von Neumann algebra.

**Theorem 3.1 (Equality in Traces)** Let $p$ and $q$ be positive real numbers such that $p^{-1} + q^{-1} = 1$, and assume that $a, b \in M^+$ satisfy $\text{tr}(a) < \infty$ and $\text{tr}(b) < \infty$. Then

$$\text{tr}(|ab|) = p^{-1}\text{tr}(a^p) + q^{-1}\text{tr}(b^q)$$  \hspace{1cm} (19)

if and only if $b^q = a^p$.

**Proof.** It is clear that equation (19) holds if $b^q = a^p$, and so we focus on the converse.

First, observe that if $r \in \mathbb{R}_0^+$ and $r > 1$, and if $h \in M^+$ has finite trace, then so does $h^r$. Indeed,

$$\text{tr}(h) = \int_0^\infty \mu_h(t) \, dt < \infty$$

implies that $\mu_h(t) \to 0^+$ as $t \to \infty$. Thus, for sufficiently large $t$, $[\mu_h(t)]^r \leq \mu_h(t)$. Hence, using $\mu_{h^r}(t) = [\mu_h(t)]^r$ by (7), $\text{tr}(h^r) < \infty$.

Assume that equation (19) holds for some $a, b \in M^+$. Inequality (16) states that

$$\text{tr}(|ab|) = \int_0^\infty \mu_{ab}(t) \, dt \leq \int_0^\infty \mu_a(t)\mu_b(t) \, dt .$$

Thus, using that equality (19) holds,

$$p^{-1}\text{tr}(a^p) + q^{-1}\text{tr}(b^q) = \int_0^\infty \left( p^{-1}\mu_{a^p}(t) + q^{-1}\mu_{b^q}(t) \right) \, dt \leq \int_0^\infty \mu_a(t)\mu_b(t) \, dt .$$

By Young’s inequality,

$$\mu_a(t)\mu_b(t) \leq p^{-1}[\mu_a(t)]^p + q^{-1}[\mu_b(t)]^q = p^{-1}\mu_{a^p}(t) + q^{-1}\mu_{b^q}(t)$$  \hspace{1cm} (20)
for every \( t \in \mathbb{R}_0^+ \). Therefore,
\[
\int_0^\infty \mu_a(t)\mu_b(t) \, dt = \int_0^\infty \left( p^{-1}\mu_{ap}(t) + q^{-1}\mu_{bq}(t) \right) \, dt.
\]
This shows, when coupled with (20), that
\[
\mu_a(t)\mu_b(t) = p^{-1}\mu_{ap}(t) + q^{-1}\mu_{bq}(t)
\]
for almost all \( t \in \mathbb{R}_0^+ \). However, as the nonincreasing functions \( \mu_z \), for \( z \in M \), are right continuous, \( \mu_a(t)\mu_b(t) = p^{-1}\mu_{ap}(t) + q^{-1}\mu_{bq}(t) \) for all \( t \in \mathbb{R}_0^+ \). But these are cases of equality in Young’s inequality, and so \( [\mu_b(t)]^q = [\mu_a(t)]^p \) for all \( t \in \mathbb{R}_0^+ \). That is, by again using (7),
\[
\mu_{bq}(t) = \mu_{ap}(t) \text{ for all } t \in \mathbb{R}_0^+.
\]
With \( t = 0 \), this equation implies that
\[
\|b\| = \|a\|^\frac{p}{q}.
\]  
(21)
Integration over all \( t \) yields
\[
\text{tr}(a^p) = \int_0^\infty \mu_{ap}(t) \, dt = \int_0^\infty \mu_{bq}(t) \, dt = \text{tr}(b^q).
\]  
(22)
A similar argument to the one above shows that \( \mu_{ab}(t) = \mu_{p^{-1}a^p+q^{-1}b^q}(t) \) for all \( t \in \mathbb{R}_0^+ \). The reasons for this are: \( \mu_{ab}(t) \leq \mu_{p^{-1}a^p+q^{-1}b^q}(t) \) for all \( t \in \mathbb{R}_0^+ \) and
\[
\int_0^\infty \mu_{ab}(t) \, dt = \text{tr}(ab) = p^{-1}\text{tr}(a^p) + q^{-1}\text{tr}(b^q)
\]
\[
= \int_0^\infty \mu_{p^{-1}a^p+q^{-1}b^q}(t) \, dt,
\]
whence \( \mu_{ab}(t) = \mu_{p^{-1}a^p+q^{-1}b^q}(t) \) for almost all \( t \in \mathbb{R}_0^+ \).
Hence, we have thus far proved that, for every \( t \in \mathbb{R}_0^+ \),
\[
\mu_{ab}(t) = \mu_{p^{-1}a^p+q^{-1}b^q}(t) = p^{-1}\mu_{ap}(t) + q^{-1}\mu_{bq}(t).
\]  
(23)
The remainder of the proof carried out in cases. The method of the first case, in particular, is inspired by an approach of Hirzallah and Kittaneh [13].
Case 1: \((p = q = 2)\) Assume that \(p = q = 2\). Equation (23) becomes

\[
\mu_{|ab|}(t) = \mu_{\frac{1}{2}(a^2 + b^2)}(t), \text{ for all } t \in \mathbb{R}_0^+, \text{ implying that } \text{tr}(\frac{1}{4}(a^2 + b^2)^2) = \text{tr}(|ab|^2) < \infty \text{ by the functional calculus (7).}
\]

The equation

\[
\text{tr} \left( \frac{1}{4}(a^2 + b^2)^2 - \frac{1}{4}(a^2 - b^2)^2 - a^2b^2 \right) = 0
\]
is readily verified by expanding the left hand side and it yields

\[
\frac{1}{4}\text{tr} \left( (a^2 - b^2)^2 \right) + \text{tr}(a^2b^2) = \text{tr} \left( \frac{1}{4}(a^2 + b^2)^2 \right) = \text{tr}(|ab|^2).
\]

As \(\text{tr}(a^2b^2) = \text{tr}(|ab|^2)\), the equation above holds only if \(\text{tr}((a^2 - b^2)^2) = 0\). By the faithfulness of the trace and the uniqueness of positive square roots, it follows that \(b = a\).

Case 2: \((p < q)\) Assume that \(p < q\); then necessarily \(q > 2 > p\). We first aim to show that

\[
\mu_{|ab|}(t) \leq \mu_{a^{p/2}b^{q/2}}(t)\|a\|^{1-\frac{p}{2}}\|b\|^{1-\frac{q}{2}}, \text{ for all } t \in \mathbb{R}_0^+. \tag{24}
\]

As \(1 - \frac{q}{2} < 0\), the operator \(b^{1-\frac{q}{2}}\) exists only if \(b\) is invertible. Therefore, to prove (24) we shall assume that \(b\) is invertible. This assumption entails no loss in generality, for if \(b\) were not invertible, then we could replace \(b\) by \(b_\varepsilon = b + \varepsilon 1\), which is invertible and which satisfies \(\mu_{z^zb}(t) \rightarrow \mu_{z^zb}(t)\), for every \(z \in M\) and \(t \in \mathbb{R}_0^+\), as \(\varepsilon \rightarrow 0^+\). Thus, (24) is achieved for noninvertible \(b\) as a limiting case of (24) using invertible \(b_\varepsilon\). Factor \(ab\) as \(ab = a^{1-\frac{p}{2}}b^{\frac{q}{2}}b^{1-\frac{q}{2}}\).

Inequality (4) shows, therefore, that

\[
\mu_{|ab|}(t) = \mu_{ab}(t) = \mu_{a^{1-\frac{p}{2}}b^{\frac{q}{2}}b^{1-\frac{q}{2}}}(t) \leq \|a\|^{1-\frac{p}{2}}\mu_{a^{p/2}b^{q/2}}(t)\|b\|^{1-\frac{q}{2}},
\]

thereby proving (24).

Now let \(a_1 = a^{p/2}\) and \(b_1 = b^{q/2}\). Because \(\text{tr}(a^p) = \text{tr}(b^q)\) (equation (22)), we have that

\[
\text{tr}(|a_1b_1|) \leq \frac{1}{2}\text{tr}(a_1^2) + \frac{1}{2}\text{tr}(b_1^2)
\]

\[
= \frac{1}{2}\text{tr}(a^p) + \frac{1}{2}\text{tr}(b^q)
\]

\[
= p^{-1}\text{tr}(a^p) + q^{-1}\text{tr}(b^q)
\]

\[
= \text{tr}(|ab|).
\]

13
On the other hand, inequality (24) and equation (21) yield
\[ \mu_{|ab|}(t) \leq \mu_{a^{p/2}b^{q/2}}(t)\|a\|^{1-\frac{p}{2}}\|b\|^{1-\frac{q}{2}} = \mu_{|a_1b_1|}(t)\|a\|^{1+\frac{p}{2}-p} = \mu_{|a_1b_1|}(t). \]
Thus, upon integrating, we have \( \text{tr} (|ab|) \leq \text{tr} (|a_1b_1|) \), making inequality (24) an equality:
\[ \text{tr} (|a_1b_1|) = \frac{1}{2} (\text{tr} (a_1^2) + \text{tr} (b_1^2)). \]
Hence, from what was proved in Case 1, we conclude that \( b_1 = a_1 \) and, as a result, that \( b^q = a^p \), as desired.

Case 3: \((p > q)\) This case is handled in analogous way to the case \( p < q \). \( \square \)

The characterisation of equality in the tracial Young inequality leads to a similar result for singular values.

**Theorem 3.2 (Equality in Singular Values)** Let \( p \) and \( q \) be positive real numbers for which \( p^{-1} + q^{-1} = 1 \). Assume that \( x,y \in M \) satisfy \( \text{tr} (|x|) < \infty \) and \( \text{tr} (|y|) < \infty \). Then
\[ \mu_{|xy^*|}(t) = \mu_{p^{-1}|x|^{p}+q^{-1}|y|^q}(t), \text{ for all } t \in \mathbb{R}_0^+, \]
if and only if \( |y|^q = |x|^p \).

**Proof.** Integration leads to
\[
\text{tr} (|xy^*|) = \int_0^\infty \mu_{|xy^*|}(t) \, dt = \int_0^\infty \mu_{p^{-1}|x|^{p}+q^{-1}|y|^q}(t) \, dt = p^{-1}\text{tr} (|x|^p) + q^{-1}\text{tr} (|y|^q). \tag{26}
\]
Let \( y = w|y| \) be the polar decomposition of \( y \). Then
\[
\begin{align*}
(i) & \quad w\|x||y||w^* = |xy^*| \\
(ii) & \quad w^*w \|x||y\| = \|x||y\| \\
(iii) & \quad \text{tr} \left( \|x||y\| \right) = p^{-1}\text{tr} (|x|^p) + q^{-1}\text{tr} (|y|^q).
\end{align*}
\]
Equation (i) above is proved in Proposition 4.1 in \( [6] \); equation (ii) is from the polar decomposition; and equation (iii) follows from (i), (ii), and (26). Hence, by Theorem 3.1, \( |y|^q = |x|^p \). \( \square \)
4 Tracial Inequalities in C*-algebras

By way of standard arguments with the Gelfand–Naimark–Segal representation of states, the tracial Young inequalities extend to C*-algebras.

**Theorem 4.1** If $A$ is a C*-algebra, if $\tau$ is a faithful tracial state on $A$, and if $p$ and $q$ are positive real numbers for which $p^{-1} + q^{-1} = 1$, then, for all $a, b \in A^+$,

1. $\tau(|ab|) \leq p^{-1}\tau(a^p) + q^{-1}\tau(b^q)$, where equality holds if and only if $b^q = a^p$, and

2. $\tau(|ab|^\frac{1}{2}) \leq \sqrt{\tau(a)\tau(b)} \leq \frac{1}{2}(\tau(a) + \tau(b))$.

**Proof.** Let $\tau$ be a faithful trace on $A$ and write $\tau$ in its GNS representation $\tau(x) = \langle \pi_\tau(x)\xi_\tau, \xi_\tau \rangle$, for $x \in A$, where $\pi_\tau : A \rightarrow B(H_\tau)$ is a *-representation of $A$ on a Hilbert space $H_\tau$ and $\xi_\tau \in H_\tau$ is a unit cyclic vector for $\pi_\tau(A)$. If $M = \pi_\tau(A)''$ (the double commutant), then there is a faithful normal trace $\text{tr}(\cdot)$ on $M$ such that $\text{tr}(\pi_\tau(x)) = \tau(x)$, for all $x \in A$ (Proposition V.3.19 of [16]). Thus, the results concerning tracial Young-type inequalities in semifinite von Neumann algebras apply to $A$ and $\tau$ through $\text{tr}(\cdot)$.

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