A cavity-QED toolbox for quantum magnetism

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The recent experimental observation of spinor self-ordering of ultracold atoms in optical resonators has set the stage for the exploration of emergent magnetic orders in quantum-gas–cavity systems. Based on this platform, we introduce a generic scheme for the implementation of quantum spin Hamiltonians composed of various kinds of couplings, including Heisenberg and Dzyaloshinskii-Moriya interactions. These interactions with spatially modulated coupling coefficients are mediated by cavity photons. Specifically, our model is comprised of a two-component Bose-Einstein condensate, driven by two classical pump lasers and coupled to a single dynamic mode of a linear cavity in a double Λ scheme. We exemplify the emergence of an achiral antiferromagnetic domain-wall phase and a chiral spin-spiral order beyond critical laser strengths. The transition between these two phases can be observed in a single experimental setup by tuning the reflectivity of a mirror. We also discuss straightforward extensions of our scheme for the implementation of other classes of spin Hamiltonians.

Introduction.—Quantum magnetism plays a crucial role in many phenomena in condensed matter physics [1], including for instance high-temperature superconductivity [2] and spin liquids [3]. In materials, there exist different forms of interactions between electronic spins. The Heisenberg interaction, originating from the isotropic quantum exchange interaction between electrons, favors ferromagnetic (FM) or antiferromagnetic (AFM) ordering [4]. The more exotic Dzyaloshinskii-Moriya (DM) interaction [5–7], a relativistic antisymmetric exchange interaction originating from spin-orbit coupling, favors chiral states such as spin spirals and skyrmions [8–11].

When the DM interaction is strong enough, it can lead to a chiral ground states [12–15], with potential applications in spintronics [16]. In addition, when frustration is introduced by lattice geometry or by competing interactions, magnetic orders are suppressed and highly entangled phases with nonlocal excitations and topological orders can emerge.

That said, it is not an easy task to modify the strength and nature of spin-spin interactions in materials, putting stringent constraints on controlled experimental explorations of complex spin models as well as technological applications. Therefore, the notion of simulating quantum magnetism using cleaner and more controllable systems has emerged and attracted a lot of attention, with experimental implementations using ultracold atoms [17–29], molecules [30], and ions [31, 32].

Another promising route for simulating quantum magnetism with atoms is to utilize cavity-mediated long-range spin-spin interactions, which do not require extremely low temperatures to come into play [33–38]. The first step in this direction has been taken very recently by the experimental realization of density and spin self-ordering with ultracold bosonic atoms inside an optical cavity [39, 40], and with thermal atoms near a retroreflecting mirror [41]. These experiments have basically realized a long-range Heisenberg model, with an emergent domain-wall AFM order.

Motivated by these recent experimental progresses and considering the state of the art, in this Letter we demonstrate how to engineer a variety of spin models in the framework of cavity QED. We consider a two-component Bose-Einstein condensate (BEC), which is driven by two pump lasers and coupled to a single dynamic mode of a standing-wave optical cavity in a double Λ scheme as depicted in Fig. 1. By adiabatic elimination of the atomic excited states and the cavity field, we derive an effective long-range spin-1/2 Hamiltonian with spatially modulated coupling coefficients. The spatial modulations are

![FIG. 1. Schematic view of a transversely pumped multicomponent BEC in 2D inside a cavity. The inset depicts the internal atom-photon coupling scheme in a double Λ configuration. The first (second) pump laser solely induces the transition $\uparrow \leftrightarrow 1$ ($\downarrow \leftrightarrow 2$) with the Rabi frequency $\Omega_1(\mathbf{r})$ [$\Omega_2(\mathbf{r})$], while the transitions $\downarrow \leftrightarrow 1$ and $\uparrow \leftrightarrow 2$ are coupled to the cavity mode with the identical strength $\mathcal{G}(x)$.](image-url)
resulted from the interference among different electromagnetic modes, and thus depend crucially on the spatial profiles of the two pump lasers and the cavity-mode function.

We show that the resultant quantum spin models are not restricted to only Heisenberg-type interactions as considered so far [33–36, 39, 40], but can also include DM-type interactions. For two standing-wave pump lasers in the transverse direction as the experiment of Ref. [40], the spin Hamiltonian reduces to a Heisenberg Hamiltonian with an infinite-range, spatially periodic coupling constant. The photon-mediated Heisenberg-type interaction induces an achiral, AFM domain-wall spin order beyond a critical laser strength as shown in Fig. 3. At the end we also discuss straightforward extensions of our scheme to implement general spin models characterized by additional competing Heisenberg and DM interactions, which would pave the way for the realization of exotic magnetic orders, such as skyrmions, and frustrated states in atom-cavity systems.

**Model.**—Consider four-level ultracold bosonic atoms trapped in two dimensions (2D) in the x–y plane inside a linear cavity and illuminated in this plane by two classical pump lasers as depicted in Fig. 1. The transition $\uparrow \leftrightarrow 1$ ($\downarrow \leftrightarrow 2$) is driven by the first (second) pump laser with the position-dependent Rabi coupling $\Omega_1(r)$ [$\Omega_2(r)$], where $r = (x, y)$. On the other hand, the transitions $\downarrow \leftrightarrow 1$ and $\uparrow \leftrightarrow 2$ are coupled to a single (initially empty) cavity mode with the same coupling strength $\mathcal{G}(x) = G_0 \cos(k_c x)$. In this double $\Lambda$ configuration, $\tau = \{\downarrow, \uparrow\}$ are the relevant (e.g., Zeeman or hyperfine pseudo) spin states and $\{1, 2\}$ are some auxiliary electronic excited states, with energies $\{\omega_{1\downarrow} = 0, \omega_{2\uparrow}, \omega_{1\uparrow}, \omega_{2\downarrow}\}$. For the moment we do not specify the spatial profiles of the Rabi couplings $\Omega_{1,2}(r)$; two examples will be given later. However, their frequencies $\{\omega_1, \omega_2\}$ are assumed to be in close resonance with the cavity frequency $\omega_c = c k_c$. The pump and cavity frequencies are all far red detuned from the atomic transition frequencies in that $\Delta_1 \equiv (\omega_{1\downarrow} + \omega_{2\uparrow})/2 - \omega_1$ and $\Delta_2 \equiv \omega_{1\uparrow} - \omega_2$ are large. Nonetheless, two-photon Raman transitions are close to resonant: $\omega_c - \omega_{1\downarrow} \approx \omega_{1\uparrow} - \omega_2 \approx \omega_{2\uparrow}$.

After adiabatic elimination of the atomic excited states [36], in the rotating frame of the pump lasers the system is described by the many-body Hamiltonian $\hat{H} = -\hbar \Delta \hat{a}^\dagger \hat{a} + \int \hat{\Psi}^\dagger(r) \hat{H}_0 \hat{\Psi}(r) \, dr$, where $\Delta \equiv (\omega_1 + \omega_2)/2 - \omega_c$, and $\hat{a}$ and $\hat{\Psi} = (\hat{\Psi}_1, \hat{\Psi}_2)^T$ are the photonic and two-component atomic bosonic annihilation field operators, respectively. The single-particle Hamiltonian density reads,

$$\hat{H}_0 = \left( \begin{array}{cc} \frac{\mathcal{G}^2}{\Delta^2} + \hbar \delta + \hat{V}_1(r) & \hbar \hat{\Omega}_1(r) \\ \hbar \hat{\Omega}_1^\dagger(r) & \frac{\mathcal{G}^2}{\Delta^2} + \hbar \delta + \hat{V}_1(r) \end{array} \right),$$

(1)

with the potential operators $\hat{V}_1(r) = \hbar \Delta \hat{a}^\dagger \mathcal{G}(x)^2/\Delta^2 + \hbar |\Omega_1(r)|^2/\Delta_1$, $\hat{V}_2(r) = \hbar \Delta \hat{a}^\dagger \mathcal{G}(x)^2/\Delta_1 + \hbar |\Omega_2(r)|^2/\Delta_2$, and the two-photon Raman-Rabi coupling operator $\hat{\Omega}_1(r) = \Omega_1^\dagger(r) \hat{a} \mathcal{G}(x)/\Delta_1 + \hat{a}^\dagger \mathcal{G}^\ast(r) \Omega_2(r)/\Delta_2$. Here $M$ is the atomic mass, $p = (p_x, p_y)$ is the atomic momentum operator, and $\delta \equiv \omega_1 - (\omega_{p1} - \omega_{p2})/2 + B_{ext}$ being an external magnetic field to tune the internal atomic levels.

In the Born-Oppenheimer (or adiabatic) approximation, the cavity-field operator can be replaced by its steady-state value

$$\hat{a}_{ss} = \int \mathcal{G}^\ast(x) \left[ \frac{\Delta_1}{\Delta^2} \hat{\Omega}_1(s) \hat{\Sigma}_+(r) + \frac{\Delta_1}{\Delta^2} \hat{\Omega}_2(s) \hat{\Sigma}_-(r) \right] dr,$$

(2)

where $\hat{\Sigma}_+(r) = \hat{\psi}_1^\dagger(r) \hat{\psi}_2^\dagger(r)$ is the density operator for component $\tau$, $\hat{\Sigma}_-(r) = \hat{\psi}_1^\dagger(r) \hat{\psi}_2^\dagger(r)$ are the local spin raising and lowering operators, and $\kappa$ is the cavity-field decay rate. This approximation is justified for large $\Delta_\tau$ and/or $\kappa$, for which the time scale of the cavity-field evolution is much shorter than the atomic external and (pseudospin) internal time scales.

**Effective long-range spin-spin interactions.**—The cavity field can be thus integrated out by formally substituting the steady-state photonic field operator (2) in the Hamiltonian $\hat{H}$, yielding an effective atom-only Hamiltonian. This effective atomic Hamiltonian consists of a (local) single-particle part for the center-of-mass motion, plus a long-range interaction part for the spin degree of freedom [42].

$$\hat{H}_{\text{spin}} = \int \left\{ \sum_{\beta=x,y} J_{\text{Heis}}(r',r) \hat{\sigma}_{\beta}(r') \hat{\sigma}_{\beta}(r) + J_{\text{DM}}(r', r) \left[ \hat{\sigma}_x(r') \hat{\sigma}_y(r) + \hat{\sigma}_y(r') \hat{\sigma}_x(r) \right] + J_{\text{int}}(r',r) \left[ \hat{\sigma}_z(r') \hat{\sigma}_z(r) + \hat{\sigma}_y(r') \hat{\sigma}_y(r) \right] \right\} dr dr' + \int B_z(r) \hat{\sigma}_z(r) \, dr,$$

(3)

where $\hat{\sigma}(r) = \hat{\Psi}^\dagger(r) \sigma \hat{\Psi}(r)$ is the local pseudospin operator ($\sigma$ is the vector of Pauli matrices). The first line in Eq. (3) corresponds to the $x$ and $y$ components of a Heisenberg-type interaction $\hat{\sigma}(r') \cdot \hat{\sigma}(r)$. The second line corresponds to the $z$ component of a DM-type interaction $\hat{\sigma}(r') \times \hat{\sigma}(r)$. The third line is cross couplings between $x$ and $y$ components of the spins, which will be referred to as the cross-spin interactions in what follows. Finally, the last line serves as a local magnetic bias field along the $z$ axis. The coupling coefficients are position dependent and are related to the
cavity-mode function and the pump-field spatial profiles as \( B_z = \hbar^2/2 + \hbar\Omega_z(\mathbf{r})^2/2\Delta_t - \hbar\Omega_\gamma(\mathbf{r})^2/2\Delta_\gamma \), \( J^{x/y}_{\text{Heis}} = \Re(c_1) \pm \Re(c_2) \), \( J^{\text{DM}}_c = -\Im(c_1) \), and \( J^{x/y}_c = -\Im(c_2) \) with
\[
c_1(\mathbf{r}', \mathbf{r}) = \left[ \frac{1}{\Delta_c^2} \Omega_1(\mathbf{r}')\Omega_1^*(\mathbf{r}) + \frac{1}{\Delta_c^2} \Omega_2(\mathbf{r}')\Omega_2^*(\mathbf{r}) \right] \times \frac{2\hbar}{\mathcal{G}(x')\mathcal{G}(x)},
\]
\[
c_2(\mathbf{r}', \mathbf{r}) = \left[ \frac{1}{\Delta_c^2} \Omega_2(\mathbf{r}')\Omega_1^*(\mathbf{r}) + \frac{1}{\Delta_c^2} \Omega_1(\mathbf{r}')\Omega_2^*(\mathbf{r}) \right] \times \frac{2\hbar}{\mathcal{G}(x')\mathcal{G}(x)}.
\]
Here we have assumed, without loss of generality, that \( \mathcal{G}_m = \mathcal{G}_0^* \) is real and have introduced \( \Delta_c \equiv \Delta_c + i\kappa = \int \mathcal{G}(x)|\hat{n}_z(\mathbf{r})/\Delta_1 + \hat{n}_z(\mathbf{r})/\Delta_2|\,d\mathbf{r} \) for the shorthand. In the absence of the cavity-mode decay \( \kappa = 0 \), the cavity-mediated interactions are infinite range as long as the laser waists are much larger than the atomic cloud size as in here.

The Heisenberg interactions favor parallel or antiparallel — depending on the sign of \( J^{x/y}_{\text{Heis}} \) — alignment of the spins. On the other hand, the DM and cross-spin interactions favor chiral spin-spiral texture [42]. The interplay between these interactions results in spin canting, which for dominant DM and cross-spin interactions can lead to a state with chiral magnetic order.

Let us now consider two concrete examples for the spatial profiles of the lasers, and examine the resulting spin Hamiltonian (3) and corresponding magnetic phases. We consider the thermodynamic limit, where quantum fluctuations become negligible (due to the infinite range of the interactions [43]) and one can replace the photonic and atomic field operators with their corresponding quantum averages: \( \hat{a} \to \langle \hat{a} \rangle \equiv \alpha \) and \( \hat{\Psi} \to \langle \hat{\Psi} \rangle \equiv \Psi = (\psi_\uparrow, \psi_\downarrow)^\top \). In this limit, we look for self-consistent solutions of the mean-field Hamiltonian corresponding to Eq. (1), endowed with steady-sate field amplitude \( \alpha_{\omega} \), corresponding to Eq. (2).

**Domain-wall AFM phase.** — The first example we consider is the recent experiment of Ref. [40], where the pump lasers are both standing waves \( \Omega_{1,2}(y) = \Omega_{01,02}\cos(k_y y) \). The amplitudes \( \Omega_{01} \) and \( \Omega_{02} \) are assumed to be real, with the balanced Raman condition \( \gamma_\parallel = \Omega_{01}\Omega_{02}/\Delta_1 = \Omega_{01}\Omega_{02}/\Delta_2 \) as in the experiment. In this case, the coefficients \( c_1 \) and \( c_2 \) are identical (i.e., \( c_1 = c_2 \)) and both real. Therefore, all the long-range spin-spin interactions vanish except the \( x \) component of the Heisenberg Hamiltonian, \( \int J^{x}_{\text{Heis}}(\mathbf{r}', \mathbf{r})\hat{s}_x(\mathbf{r}')\hat{s}_x(\mathbf{r})\,d\mathbf{r}' \) with the periodically modulated coupling strength \( J^{x}_{\text{Heis}} \propto \Delta_c \cos(k_x x') \cos(k_y y') \cos(k_z z) \) [42]. Assuming \( \Delta_c < 0 \) and \( \mathbf{r} - \mathbf{r}' = (m_x x + m_y y)/2 \) with \( m_x, m_y \) being integers, \( J^{x}_{\text{Heis}}(\mathbf{r}', \mathbf{r}) \) is positive (i.e., AFM) when \( m_x + m_y \) is odd, and it is negative (i.e., FM) when \( m_x + m_y \) is even.

Below a threshold value of the pump-laser strength \( \eta_{\text{AFM}}^{-} \), the steady state is a spin-polarized normal state, where all the atoms are in the spin-up or spin-down state, depending on the bias field \( B_z \), with uniform atomic density and no photon in the cavity. By increasing the laser strength \( \eta_\parallel \) above \( \eta_{\text{AFM}}^{-} \), the system becomes unstable toward an ordered phase with square density pattern, checkerboard AFM domain-wall spin texture, and finite cavity-photon number. The \( Z_2 \) symmetry of the system corresponding to the transformation \( x \to x + \lambda_c/2 \) and \( \hat{a} \to -\hat{a} \) is spontaneously broken on the onset of the self-ordering phase transition at \( \eta_\parallel = \eta_{\text{AFM}}^{+} \). Different spin domains are separated by domain-wall lines, which are 1D topological defects. A typical self-ordered AFM domain-wall spin texture is illustrated in Fig. 2, where the projection of the normalized local spin \( \hat{s}(\mathbf{r}) \) on the \( \hat{s}_x-\hat{s}_z \) plane is displayed as a function of \( \mathbf{r} \) for two standing-wave pump lasers along the \( y \) direction. The color code indicates the spin angle with respect to the \( \hat{s}_x \) axis, \( \varphi = \tan^{-1}(\hat{s}_y/\hat{s}_x) \). Inside each domain, all spins are oriented with an angle \( \varphi_0 \approx 0.081\pi \) or \( \pi - \varphi_0 \), while in boundaries they rotate quite rapidly. The parameters are set to \( (\Delta_c, \kappa, \eta_{\text{AFM}}^{+}, \delta) = (-10.5, -1, -0.1)\omega_c \) and \( \sqrt{N}\eta_\parallel = -3.8\omega_c \), with the self-consistent field amplitude \( \alpha_{\omega}/\sqrt{N} \approx 0.22 \). Here \( N \) is the total particle number and \( \omega_c \equiv \hbar k_0^2/2M \).

**Spin-spiral phase.** — Let us now consider a straightforward, though crucial, modification of the experiment
of Ref. [40], where the driving lasers are assumed to be counterpropagating running waves along the y direction, \( \Omega_{1,2}(y) = \Omega_{01,02} e^{\pm i k_c y} \). As before, the amplitudes \( \Omega_{01} \) and \( \Omega_{02} \) are taken to be real, with the balanced Raman condition. The coefficients \( c_1 \) and \( c_2 \) are now both complex and different from one another, \( c_1 \neq c_2 \). Therefore, all the long-range spin-spin interactions are present in the effective Hamiltonian (3). All the couplings have the same \( \Re(\Delta_z) \cos(k_c x') \cos(k_c x) \) position dependence along the \( x \) direction, but different modulations along the \( y \) direction: \( \Delta_{\text{Heis}} \propto \cos(k_c y') \cos(k_c y), \Delta_{\text{Heis}}' \propto \sin(k_c y') \sin(k_c y), J_{\text{DM}} \propto -\sin k_c (y' - y) \), and \( J_{\text{yy}} \propto \sin k_c (y' + y) \) [42].

Below the laser-strength threshold \( \eta_{\text{spiral}} \), the system is again in the spin-polarized normal state. By increasing the pump strength \( \eta_0 \) above \( \eta_{\text{spiral}} \), the spin-spin interactions make the normal state unstable toward a magnetically self-ordered state with finite photon number in the cavity. The DM and cross-spin interactions result in an emergent transverse, conical spin-spiral state [14, 15], also known as magnetic soliton [8]. The spirals solely appear in the \( x-y \) plane as the DM interaction has only the \( z \) component and the cross-spin terms only couple the \( x \) and \( y \) components of the spins. The Heisenberg interactions favor magnetic domains as before. The steady state exhibits stripe-density patterns along the \( y \) direction at minima of the cavity potential \( x_{m_z} = m_x \lambda_c / 2 \) (recall that the Stark shifts along the \( y \) direction \( |\Omega_{1,2}(y)|^2 / \Delta_{1,2} \) are constant in this case).

Figure 3 illustrates a typical self-ordered chiral spin-spiral state. The spin does a full \( 2\pi \) rotation in the \( \tilde{s}_x-\tilde{s}_y \) plane over one wave length \( \lambda_c \) along the \( y \) direction due to the DM and cross-spin interactions. This is clearly evident in Fig. 3(b), which shows the projection of \( \tilde{s}(r) \) in the \( \tilde{s}_x-\tilde{s}_y \) plane as a function of \( r \). This can be understood by re-examining the sum of the DM and cross-spin interactions, which is proportional to

\[
\Re(\Delta_z) \int \cos(k_c x') \cos(k_c x) \left[ \cos(k_c y') \sin(k_c y) \tilde{s}_x(r') \tilde{s}_y(r) + \sin(k_c y') \cos(k_c y) \tilde{s}_y(r') \tilde{s}_x(r) \right] dr' dr.
\]

Along a stripe in the \( y \) direction (i.e., \( x' = x \neq m_x \lambda_c / 4 \)), the coupling coefficients change smoothly between negative and positive values; therefore, in order to minimize these interactions the spin rotates fully in the \( \tilde{s}_x-\tilde{s}_y \) plane along the \( y \) axis [44]. For adjacent density stripes (i.e., \( x' = x \pm \lambda_c / 2 \)), the spin spirals are shifted by \( \lambda_c / 2 \) along the \( y \) axis due to the \( x \)-modulation of the couplings which introduces an extra minus sign [45]. These are also compatible with the Heisenberg interactions. The discrete choice of the origin for the spin spirals is fixed at the onset of the self-ordering phase transition at \( \eta_0 = \eta_{\text{spiral}} \) through the spontaneous \( \mathbb{Z}_2 \) symmetry breaking process. Here the continuous \( U(1) \) screw symmetry of the system along the \( y \) direction is broken explicitly by fixing the phases of the lasers. Figure 3(c) shows the projections of the normalized spin in the \( \tilde{s}_x-\tilde{s}_z \) and \( \tilde{s}_y-\tilde{s}_z \) planes, where the existence of magnetic domains due to the Heisenberg...
interactions are visible. Note that the magnetic domains in the $\tilde{s}_x, \tilde{s}_z$ and $\tilde{s}_y, \tilde{s}_z$ planes are shifted by $\lambda_s/4$ along the $y$ axis with respect to each other, consistent with the $y$ dependence of $J^{\gamma}_{\text{Heis}}$ given above [42].

Concluding remarks.—We have shown that a variety of long-range spin models can be implemented using driven atoms coupled to an optical cavity, by properly choosing the driving electromagnetic modes. As concrete illustrations, we demonstrated the emergence of the AFM domain-wall and chiral spin-spiral orders. The transition between these two magnetic phases can be explored in a single state-of-the-art experimental setup [39, 40], by continuously tuning the reflectivity of retroreflectors which create standing-wave pumps from running-wave lasers.

As possible extensions of our scheme, we mention that general laser configurations $\Omega_{1,2}(r)$, where $r$ is not restricted only to the $y$ axis, would allow to implement more complex spatial modulations of the spin-spin interactions. This feature, together with the tuneability of the interaction range in multimode cavities [46], should introduce frustration [3]. Another generalization is to apply an extra pump laser with the same polarization as the cavity mode, in order to further induce the transitions $\downarrow \leftrightarrow 1$ and $\uparrow \leftrightarrow 2$. This would result in long-range spin-spin interactions of the form $s_x(r')s_x(r), s_y(r')s_y(r), s_z(r')s_z(r)$, and $s_y(r')s_z(r)$, yielding additional components of the Heisenberg and DM interactions. This should allow the realization of topological chiral phases such as skyrmions.

It is also worth mentioning that the linear cavity can be replaced by a ring cavity with running-wave modes, providing another complex degree of freedom to tune the cavity-mediated spin-spin interactions with a continuous $U(1)$ spatial symmetry [47–49]. As the proposed setups can be implemented in several laboratories with straightforward modifications to the state of the art, the framework of cavity QED appears to be one of the most promising candidates for the controlled simulation of quantum magnetism.

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SUPPLEMENTAL MATERIAL

Here we present the details of the adiabatic elimination of the cavity field and the derivation of the effective spin Hamiltonian.

ADIABATIC ELIMINATION OF THE CAVITY MODE

The steady-state cavity field operator,
\[ \hat{a}_{ss} = \frac{1}{\Delta_c} \int \mathcal{G}^*(x) \left[ \frac{1}{\Delta_1} \Omega_1(r) \hat{s}_-(r) + \frac{1}{\Delta_2} \Omega_2(r) \hat{s}_+(r) \right] \, dr, \]
with \( \Delta_c \equiv \Delta_c + i \kappa - \int |\mathcal{G}(x)|^2 [\hat{n}_4(r)/\Delta_1 + \hat{n}_1(r)/\Delta_2] \, dr \) for the shorthand, can be substituted in the many-body Hamiltonian \( \hat{H} \) to yield an effective atomic Hamiltonian,
\[ \hat{H}_\text{eff} = \int \hat{\Psi}^\dagger(r) \left[ \frac{\mathbf{p}^2}{2M} + \frac{\hbar^2}{2\Delta_1} |\Omega_1(r)|^2 + \frac{\hbar^2}{2\Delta_2} |\Omega_2(r)|^2 \right] I_{2 \times 2} \hat{\Psi}(r) \, dr + \hat{H}_\text{spin}. \]

The Hamiltonian \( \hat{H}_\text{spin} \) is an effective long-range spin Hamiltonian stemmed from photon-mediated interactions and it reads,
\[ \hat{H}_\text{spin} = 2\hbar \left\{ -\frac{4}{\Delta_1^2 \Delta_2} \mathcal{G}^*(x') \mathcal{G}(x) \Omega_1(r') \Omega_1^*(r) \left[ \hat{s}_{-x}(r') \hat{s}_{-y}(r) + \hat{s}_{-y}(r') \hat{s}_{-x}(r) \right] \right. \]
\[ + \frac{1}{\Delta_1 \Delta_2} \left[ \frac{1}{\Delta_c} \mathcal{G}^*(x') \mathcal{G}(x) \Omega_2(r') \Omega_2^*(r) \left[ \hat{s}_{-x}(r') \hat{s}_{-y}(r) - \hat{s}_{-y}(r') \hat{s}_{-x}(r) \right] \right. \]
\[ \left. - \frac{1}{\Delta_1^2} \mathcal{G}^*(x') \mathcal{G}(x) \Omega_1(r') \Omega_1^*(r) \left[ \hat{s}_{-x}(r') \hat{s}_{-y}(r) - \hat{s}_{-y}(r') \hat{s}_{-x}(r) \right] \right\}. \]
Here we have assumed, without loss of generality, that $G_0 < 0$ for our used parameters in the main text.

\[ - \frac{1}{\Delta_1 \Delta_2} \Im \left[ \frac{1}{\Delta_c} \mathcal{G}^*(x') \mathcal{G}(x) \Omega_2(r') \Omega_1(r) + \frac{1}{\Delta_c} \mathcal{G}(x') \mathcal{G}^*(x) \Omega_2^*(r') \Omega_1(r) \right] \left[ \hat{s}_x(r') \hat{s}_y(r) + \hat{s}_y(r') \hat{s}_x(r) \right] \right] d \mathbf{r} d \mathbf{r}' + \int \left[ \frac{\hbar \delta}{2} + \frac{\hbar}{2 \Delta_1} |\Omega_1(r)|^2 - \frac{\hbar}{2 \Delta_2} |\Omega_2(r)|^2 \right] \hat{s}_z(r) d \mathbf{r} + \sum_{j=0}^{3} O \left( \frac{1}{\Delta_1 r_j \Delta_2 r_j} \right). \] (S3)

The terms arising from the free cavity Hamiltonian $-\hbar \Delta_c \hat{a} \hat{a}^\dagger$ and the cavity potentials $\hbar \hat{a} \hat{a}^\dagger |\mathcal{G}(x)|^2 / \Delta_{1,2}$, as well as from the non-commutation of the steady-state cavity field operator (S1) with the atomic field operators, have been discarded. The spin Hamiltonian (S3) can be recast in a more compact way,

\[ H_{\text{spin}} = \int \left\{ J_H^{x y}(r', r) \hat{s}_x(r') \hat{s}_x(r) + J_H^{y x}(r', r) \hat{s}_y(r') \hat{s}_y(r) + J_{\text{DM}}(r', r) \left[ \hat{s}_x(r') \hat{s}_y(r) - \hat{s}_y(r') \hat{s}_x(r) \right] \right\} d \mathbf{r} d \mathbf{r}' + \int B_z(r) \hat{s}_z(r) d \mathbf{r}, \] (S4)

where $B_z = \hbar \delta / 2 + \hbar |\Omega_1(r)|^2 / 2 \Delta_1 - \hbar |\Omega_2(r)|^2 / 2 \Delta_2$, $J_H^{x y} = \Re(c_1) \pm \Re(c_2)$, $J_{\text{DM}} = -\Im(c_1)$, and $J_{\text{DM}} = -\Im(c_2)$ with

\[ c_1(r', r) = 2 \hbar \mathcal{G}(x') \mathcal{G}(x) \left[ \frac{1}{\Delta_1^2} \Omega_1(r') \Omega_1^*(r) + \frac{1}{\Delta_2^2} \Omega_2(r') \Omega_2^*(r) \right] , \]

\[ c_2(r', r) = \frac{2 \hbar}{\Delta_1 \Delta_2} \mathcal{G}(x') \mathcal{G}(x) \left[ \frac{1}{\Delta_c} \Omega_1(r') \Omega_1^*(r) + \frac{1}{\Delta_c^*} \Omega_2(r') \Omega_2^*(r) \right]. \] (S5)

Here we have assumed, without loss of generality, that $\mathcal{G}_0 = \mathcal{G}_0^*$ is real.

For standing-wave pump lasers along the $y$ direction $\Omega_1(y) = \Omega_0 \cos(k_c y)$ and $\Omega_2(y) = \Omega_0 \cos(k_c y)$ with the balanced Raman condition $\eta_0 \equiv \mathcal{G}_0 \Omega_0 / \Delta_1 = \mathcal{G}_0 \Omega_0 / \Delta_2$, where the amplitudes $\Omega_0$ and $\Omega_0$ are both real, one obtains

\[ c_1(r', r) = c_2(r', r) = \frac{4 \hbar |\Delta_c| \eta_0^2}{|\Delta_c|^2} \cos(k_c x') \cos(k_c x) \cos(k_c y') \cos(k_c y), \] (S6)

which is real. Therefore, the only nonzero long-range spin-spin coupling is the $x$ component of the Heisenberg interaction $\int J_H^{x}(r', r) \hat{s}_x(r') \hat{s}_x(r) d \mathbf{r} d \mathbf{r}'$, with the position-dependent coupling energy $J_H^{x} = 2c_1$; see Fig. S1.
On the other hand for counterpropagating running-wave pumps along the $y$ axis $\Omega_1(y) = \Omega_{01} e^{ik_x y}$ and $\Omega_2(y) = \Omega_{02} e^{-ik_x y}$ with the balanced Raman condition $\eta_0 \equiv \eta_0^0 \Omega_{01}/\Delta_1 = \eta_0^0 \Omega_{02}/\Delta_2$, where the amplitudes $\Omega_{01}$ and $\Omega_{02}$ are both real again, we obtain

$$c_1(r', r) = \frac{4\Re(\Delta_c)\eta_0^2}{|\Delta_c|^2} \cos(k_x x') \cos(k_x x) e^{ik_x(y' - y)},$$

$$c_2(r', r) = \frac{4\Re(\Delta_c)\eta_0^2}{|\Delta_c|^2} \cos(k_x x') \cos(k_x x) e^{-ik_x(y' + y)},$$

which both are complex. Hence, all the long-range spin-spin interactions are present with the position-dependent periodic couplings

$$J_{\text{Heis}}^h(r', r) = \frac{8\Re(\Delta_c)\eta_0^2}{|\Delta_c|^2} \cos(k_x x') \cos(k_x x) \cos(k_x y') \cos(k_x y).$$
\[
J_{\text{Heis}}^y (r', r) = \frac{8\hbar \Re(\tilde{\Delta}_c) \eta_0^2}{|\tilde{\Delta}_c|} \cos(k_c x') \cos(k_c x) \sin(k_c y') \sin(k_c y),
\]
\[
J_{\text{DM}}^y (r', r) = -\frac{4\hbar \Re(\tilde{\Delta}_c) \eta_0^2}{|\tilde{\Delta}_c|} \cos(k_c x') \cos(k_c x) \sin(k_c (y' - y)),
\]
\[
J_{c^y}^x (r', r) = \frac{4\hbar \Re(\tilde{\Delta}_c) \eta_0^2}{|\Delta_c|} \cos(k_c x') \cos(k_c x) \sin(k_c (y' + y)).
\] (S8)

The spatial modulations of these spin-coupling coefficients are plotted in Fig. S2 as a function of \( r \) for fixed \( r' \). Note that both the DM and cross-spin interactions favor chiral spin-spiral textures.