QCD in terms of gauge-invariant dynamical variables

Hans-Peter Pavel
Institut für Kernphysik, TU Darmstadt, D-64289 Darmstadt, Germany
Bogoliubov Laboratory of Theoretical Physics, JINR Dubna, Russia
E-mail: hans-peter.pavel@physik.tu-darmstadt.de

For a complete description of the physical properties of low-energy QCD, it might be advantageous to first reformulate QCD in terms of gauge-invariant dynamical variables, before applying any approximation schemes. Using a canonical transformation of the dynamical variables, which Abelianises the non-Abelian Gauss-law constraints to be implemented, such a reformulation can be achieved for QCD. The exact implementation of the Gauss laws reduces the colored spin-1 gluons and spin-1/2 quarks to unconstrained colorless spin-0, spin-1, spin-2 and spin-3 glueball fields and colorless Rarita-Schwinger fields respectively. The obtained physical Hamiltonian can then be rewritten into a form, which separates the rotational from the scalar degrees of freedom, and admits a systematic strong-coupling expansion in powers of $\lambda = g^{-2/3}$, equivalent to an expansion in the number of spatial derivatives. The leading-order term in this expansion corresponds to non-interacting hybrid-glueballs, whose low-lying masses can be calculated with high accuracy by solving the Schrödinger-equation of the Dirac-Yang-Mills quantum mechanics of spatially constant physical fields (at the moment only for the 2-color case). Due to the presence of classical zero-energy valleys of the chromomagnetic potential for two arbitrarily large classical glueball fields (the unconstrained analogs of the well-known constant Abelian fields), practically all glueball excitation energy is expected to go into the increase of the strengths of these two fields. Higher-order terms in $\lambda$ lead to interactions between the hybrid-glueballs and can be taken into account systematically using perturbation theory in $\lambda$. 
1. Introduction

The QCD action

\[
S[A, \psi, \bar{\psi}] = \int d^4x \left[ -\frac{1}{4} F_{\mu\nu}^a F^{a\mu\nu} + \bar{\psi} \left( i \gamma^\mu D_\mu - m \right) \psi \right]
\]  

(1.1)

is invariant under the \( SU(3) \) gauge transformations \( U[\omega(x)] \equiv \exp(i \omega_a \tau_a / 2) \)

\[
\psi^a(x) = U[\omega(x)] \psi(x), \quad A^a_{\mu}(x) \tau_a / 2 = U[\omega(x)] \left( A^a_{\mu}(x) \tau_a / 2 + i / g \partial_\mu \right) U^{-1}[\omega(x)].
\]

(1.2)

Introducing the chromoelectric \( E_i^a \equiv F_{i0}^a \) and chromomagnetic \( B_i^a \equiv \frac{1}{2} \epsilon_{ijk} F_{jk}^a \) and noting that the momenta conjugate to the spatial \( A_{ai} \) are \( \Pi_{ai} = -E_{ai} \), one obtains the canonical Hamiltonian

\[
H_C = \int d^3x \left[ \frac{1}{2} E_{ai}^2 + \frac{1}{2} B_{ai}^2(A) - g A_{ai} j_{ia}(\psi) + \bar{\psi} \left( \gamma^\mu \partial_\mu + m \right) \psi - g A_{ai} (D_i(A)_{ab} E_{bi} - \rho_a(\psi)) \right],
\]

(1.3)

with the covariant derivative \( D_i(A)_{ab} \equiv \delta_{ab} \partial_i - g f_{abc} A_{ci} \) in the adjoint representation.

Exploiting the time dependence of the gauge transformations (1.2) to put (see e.g. [1])

\[
A_{a0} = 0, \quad a = 1, \ldots, 8 \quad \text{(Weyl gauge)},
\]

(1.4)

and quantising the dynamical variables \( A_{ai}, -E_{ai}, \psi_{\tau r} \) and \( \psi^*_{\tau r} \) in the Schrödinger functional approach by imposing equal-time (anti-) commutation relations (CR), e.g. \( -E_{ai} = -i \partial / \partial A_{ai} \), the physical states \( \Phi \) have to satisfy both the Schrödinger equation and the Gauss laws

\[
H \Phi = \int d^3x \left[ \frac{1}{2} E_{ai}^2 + \frac{1}{2} B_{ai}^2(A) - A_{ai} j_{ia}(\psi) + \bar{\psi} \left( \gamma^\mu \partial_\mu + m \right) \psi \right] \Phi = E \Phi, \quad \Phi = 0, \quad a = 1, \ldots , 8.
\]

(1.5)

The Gauss law operators \( G_a(x) \) are the generators of the residual time independent gauge transformations in (1.2), satisfying \( [G_a(x), H] = 0 \) and \( [G_a(x), G_b(y)] = i f_{abc} G_c(x) \delta(x - y) \).

Furthermore, \( H \) commutes with the angular momentum operators

\[
J_i = \int d^3x \left[ -\epsilon_{ijk} A_{aj} E_{ak} + \Sigma_i(\psi) + \text{orbital parts} \right], \quad i = 1, 2, 3.
\]

(1.7)

The matrix element of an operator \( O \) is given in the Cartesian form

\[
\langle \Phi' | O | \Phi \rangle \sim \int dA \, d\bar{\psi} \, d\psi \, \Phi'(A, \bar{\psi}, \psi) \, O(A, \bar{\psi}, \psi).
\]

(1.8)

The spectrum of Eqn.(1.5)-(1.6) for the case of Yang-Mills quantum mechanics of spatially constant gluon fields, has been found in [2] for \( SU(2) \) and in [3] for \( SU(3) \), in the context of a weak coupling expansion in \( g^2/3 \), using the variational approach with gauge-invariant wave-functional automatically satisfying (1.6). The corresponding unconstrained approach, a description in terms of gauge-invariant dynamical variables via an exact implementation of the Gauss laws, has been considered by many authors (o.a. [1],[4],[5],[10], and references therein) to obtain a non-perturbative description of QCD at low energy, as an alternative to lattice QCD.

I shall first discuss in Section 2 the unphysical, but technically much simpler case of 2-colors, and then show in Section 3 how the results can be generalised to \( SU(3) \).
2. Unconstrained Hamiltonian formulation of 2-color QCD

2.1 Canonical transformation to adapted coordinates

Point transformation from the $A_{ai}, \psi_\alpha$ to a new set of adapted coordinates, the 3 angles $q_j$ of an orthogonal matrix $O(q)$, the 6 elements of a pos. definite symmetric $3 \times 3$ matrix $S$, and new $\psi'_\beta$

$$A_{ai}(q,S) = O_{ak}(q)S_{ki} - \frac{1}{2g}E_{abc} (O(q) \partial_i O^T(q))_{bc}, \quad \psi_\alpha(q, \psi') = U_{\alpha\beta}(q) \psi_\beta,$$

where the orthogonal $O(q)$ and the unitary $U(q)$ are related via $O_{ab}(q) = \frac{1}{2}\text{Tr}(U^{-1}(q)\tau_a U(q)\tau_b)$. Equ. (2.1) is the generalisation of the (unique) polar decomposition of $A$ and corresponds to

$$\chi(A) = \epsilon_{ijk}A_{jk} = 0 \quad ("\text{symmetric gauge}).$$

Preserving the CR, we obtain the old canonical momenta in terms of the new variables

$$-E_{ai}(q,S,p,P) = O_{ak}(q) \left[ P_{ki} + \epsilon_{kll}D^{-1}_{ll}(S) \left( \Omega^{-1}_{kj}(q)p_j + \rho_i(\psi') + D_a(S)\lambda_m P_{mn} \right) \right].$$

In terms of the new canonical variables the Gauss law constraints are Abelianised,

$$G_a\Phi \equiv O_{ak}(q)\Omega^{-1}_{ki}(q)p_i\Phi = 0 \quad \Leftrightarrow \quad \frac{\delta}{\delta q_i}\Phi = 0 \quad (\text{Abelianisation}),$$

and the angular momenta become

$$J_i = \int d^3x \left[ -2\epsilon_{ijk}S_{mk} + \Sigma_i(\psi') + \rho_i(\psi') + \text{orbital parts} \right].$$

Equ.(2.4) identifies the $q_i$ with the gauge angles and $S$ and $\psi'$ as the physical fields. Furthermore, from Equ.(2.3) follows that the $S$ are colorless spin-0 and spin-2 glueball fields, and the $\psi'$ colorless reduced quark fields of spin-0 and spin-1. Hence the gauge reduction corresponds to the conversion "color $\rightarrow$ spin". The obtained unusual spin-statistics relation is specific to SU(2).

2.2 Physical quantum Hamiltonian

According to the general scheme [1], the correctly ordered physical quantum Hamiltonian in terms of the physical variables $S_{ik}(x)$ and the canonically conjugate $P_{ik}(x) \equiv -i\delta / \delta S_{ik}(x)$ reads [8]

$$H(S,P) = \frac{1}{2} \mathcal{J}^{-1} \int d^3x P_{ai} \mathcal{J} P_{ai} + \frac{1}{2} \int d^3x \left[ B^2_{ai}(S) - S_{ai} j_{ia}(\psi') + \psi'(\gamma_0 \partial_i + m) \psi' \right]$$

$$- \mathcal{J}^{-1} \int d^3x \int d^3y \left\{ \left( D_a(S)\lambda_m P_{mn} + \rho_a(\psi') \right)(x) \mathcal{J} \right\} \left( x, a \right) \delta^{-2}(S) \langle y b \rangle \left( D_j(S)\lambda_n P_{nj} + \rho_b(\psi') \right)(y),$$

with the Faddeev-Popov (FP) operator

$^*D_{ki}(S) \equiv \epsilon_{kmn}D_j(S)_{nl} = \epsilon_{kli}\partial_i - g(S_{kl} - \delta_{kl}trS)$,

and the Jacobian $\mathcal{J} \equiv \det|\mathcal{J}|$. The matrix element of a physical operator $O$ is given by

$$\langle \Psi'|O|\Psi \rangle \propto \int_{S \text{pos.def.}} \int_{\psi'} \prod_x \left[ dS(x)d\psi'(x)d\psi(x) \right] \mathcal{J} \mathcal{J}^{*} \left[ S, \psi', \psi \right] O \left[ S, \psi', \psi \right].$$

The inverse of the FP operator and hence the physical Hamiltonian can be expanded in the number of spatial derivatives, equivalent to a strong coupling expansion in $\lambda = g^{-2/3}$. 

Hans-Peter Pavel
2.3 Coarse-graining and strong coupling expansion of the physical Hamiltonian in $\lambda = g^{-2/3}$

Introducing an UV cutoff $a$ by considering an infinite spatial lattice of granulas $G(n,a)$ at $x = an$ ($n \in \mathbb{Z}^3$) and averaged variables

$$S(n) := \frac{1}{a^3} \int_{G(n,a)} dx \, S(x),$$

and discretised spatial derivatives, the expansion of the Hamiltonian in $\lambda = g^{-2/3}$ can be written

$$H = \frac{g^{2/3}}{a} \left[ \mathcal{H}_0 + \lambda \sum_{\alpha} \mathcal{V}^{(\alpha)}(n) + \lambda^2 \left( \sum_{\beta} \mathcal{V}^{(\beta)}(n) + \sum_{\gamma} \mathcal{V}^{(\beta \gamma)}(n) \right) + \mathcal{O}(\lambda^3) \right].$$

(10.10)

The "free" Hamiltonian $H_0 = (g^{2/3}/a) \mathcal{H}_0 + H_m = \sum_n H_0^{QM}(n)$ is the sum of the Hamiltonians of Dirac-Yang-Mills quantum mechanics of constant fields in each box, and the interaction terms $\mathcal{V}^{(\alpha)}, \mathcal{V}^{(\beta \gamma)}$ lead to interactions between the granulas.

2.4 Zeroth-order: Dirac-Yang-Mills quantum mechanics of spatially constant fields

Transforming to the intrinsic system of the symmetric tensor $S$, with Jacobian $\sin \beta \prod_{i<j} (\phi_i - \phi_j)$,

$$S = R^T (\alpha, \beta, \gamma) \text{diag}(\phi_1, \phi_2, \phi_3) R(\alpha, \beta, \gamma),$$

the "free" Hamiltonian in each box (volume $V$) takes the form

$$H_0^{QM} = \frac{g^{2/3}}{V^{1/3}} \left[ \mathcal{H}^G + \mathcal{H}^D + \mathcal{H}^C \right] + \frac{1}{2} m \left[ \left( \bar{\psi}_L^{(i)} \psi_R^{(i)} + \sum_{i=1}^{3} \bar{\psi}_L^{(i)} \psi_R^{(i)} \right) + h.c. \right],$$

(12.12)

with the glueball part $\mathcal{H}^G$, the minimal-coupling $\mathcal{H}^D$, and the Coulomb-potential-type part $\mathcal{H}^C$

$$\mathcal{H}^G = \frac{1}{2} \sum_{i,j,k} \left( - \frac{\partial^2}{\partial \phi_i^2} - \frac{2}{\phi_j - \phi_k} \left( \phi_i \frac{\partial}{\partial \phi_i} - \phi_j \frac{\partial}{\partial \phi_j} \right) + \left( \bar{\phi}_i - \bar{\phi}_j \right)^2 \phi_j^2 + \phi_k^2 \right),$$

(12.13)

$$\mathcal{H}^D = \frac{1}{2} \left( \bar{\psi}^{(i)}_L \psi^{(i)}_R \right) + \frac{1}{2} \sum_{i,j,k} \left( \phi_i - \phi_j \right) \left( \bar{\psi}^{(i)}_L \psi^{(i)}_R \right),$$

(12.14)

$$\mathcal{H}^C = \sum_{i,j,k} \frac{\bar{\psi}^{(i)}_L \psi^{(i)}_R}{(\phi_j + \phi_k)^2},$$

(12.15)

and the total spin

$$J_i = R_{ij}(R) \bar{\xi}_j, \quad [J_i, H] = 0.$$  

(12.16)

The matrix elements become

$$\langle \Phi_1 | G | \Phi_2 \rangle = \int d\alpha \sin \beta d\beta d\gamma \int_{0<\phi_1<\phi_2<\phi_3} \cdots \int d\psi \, d\psi' \, \Phi^*_1 G \Phi_2.$$  

The l.h.s. of Fig.1 shows the $0^+$ energy spectrum of the lowest pure-gluon (G) and quark-gluon (QG) cases for one quark-flavor which can be calculated with high accuracy using the variational approach. The energies of the quark-gluon ground state and the sigma-antisigma excitation are lower than that of the lowest pure-gluon state. This is due to a large negative contribution from $\langle \mathcal{H}^D \rangle$, in addition to the large positive $\langle \mathcal{H}^G \rangle$, while $\langle \mathcal{H}^C \rangle \simeq 0$ (see [?] for details).

Furthermore, as a consequence of the zero-energy valleys "$\phi_1 = \phi_2 = 0, \phi_3$ arbitrary" of the classical magnetic potential $B^3 = \phi_2^2 + \phi_1^2 + \phi_1 \phi_2$, practically all glueball excitation-energy results from an increase of expectation value of the "constant Abelian field" $\phi_3$ as shown for the pure-gluon case on the r.h.s. of Fig.1 (see [?] for details).
QCD in terms of gauge-invariant dynamical variables

Hans-Peter Pavel

Figure 1: L.h.s.: Lowest energy levels for the pure-gluon (G) and the quark-gluon case (QG) for 2-colors and one quark flavor. The energies of the quark-gluon ground state and the sigma-antisigma excitation are lower than that of the lowest pure-gluon state. R.h.s. (for pure-gluon case and setting $V \equiv 1$): $\langle \phi_3 \rangle$ is raising with increasing excitation, whereas $\langle \phi_1 \rangle$ and $\langle \phi_2 \rangle$ are practically constant, independent of whether spin-0 (dark boxes) or spin-2 states (open circles).

2.5 Perturbation theory in $\lambda$ and coupling constant renormalisation in the IR

Including the interactions $V(\partial)$, $V(\Delta)$ using 1st and 2nd order perturbation theory in $\lambda = g - 2/3$ give the result [8] (for pure-gluon case and only including spin-0 fields in a first approximation)

$$E_{\text{vac}}^+ = \mathcal{N} \frac{g^{2/3}}{a} \left[ 4.1167 + 29.894 \lambda^2 + O(\lambda^3) \right],$$

$$E_{1}^{(0)+}(k) - E_{\text{vac}}^+ = \left[ 2.270 + 13.511 \lambda^2 + O(\lambda^3) \right] \frac{g^{2/3}}{a} + 0.488 \frac{a}{g^{2/3}} k^2 + O((a^2 k^2)^2),$$

for the energy of the interacting glueball vacuum and the spectrum of the interacting spin-0 glueball. Lorentz invariance demands $E = \sqrt{M^2 + k^2} \simeq M + \frac{1}{2 M} k^2$, which is violated in this 1st approximation by a factor of 2. In order to get a Lorentz invariant result, $J = L + S$ states should be considered including also spin-2 states and the general $\gamma^{(\partial \Delta)}$.

Independence of the physical glueball mass

$$M = \frac{g_0^{2/3}}{a} \left[ \mu + c g_0^{-4/3} \right]$$

of box size $a$, one obtains

$$\gamma(g_0) \equiv a \frac{d}{da} g_0(a) = \frac{3}{2} \frac{g_0 \mu + c g_0^{-4/3}}{\mu - c g_0^{-4/3}}$$

which vanishes for $g_0 = 0$ (pert. fixed point) or $g_0^{4/3} = -c/\mu$ (IR fixed point, if $c < 0$). For $c > 0$

for $c > 0$:

$$g_0^{2/3} (Ma) = \frac{Ma}{2 \mu} + \sqrt{\left( \frac{Ma}{2 \mu} \right)^2 - \frac{c}{\mu}}, \quad a > a_c := 2 \sqrt{c/\mu}/M$$

My (incomplete) result $c_1^{(0)}/\mu_1^{(0)} = 5.95$ suggests, that no IR fixed points exist. critical coupling $g_0^{2/3} = 14.52$ and $a_c \sim 1.4$ fm for $M \sim 1.6$ GeV.
3. Symmetric gauge for SU(3)

Using the idea of minimal embedding of $su(2)$ in $su(3)$ by Kihlberg and Marnelius \[3\]

\[
\begin{align*}
\tau_1 &:= \lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, & \tau_2 &:= -\lambda_5 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}, & \tau_3 &:= \lambda_7 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\
\tau_4 &:= \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, & \tau_5 &:= \lambda_4 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, & \tau_6 &:= \lambda_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\
\tau_7 &:= \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, & \tau_8 &:= \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}
\end{align*}
\tag{3.1}
\]

such that the corresponding non-trivial non-vanishing structure constants, \[c_{\frac{8}{2}, \frac{8}{2}} = i c_{abc} \frac{8}{2} \] have at least one index \(i \in \{1, 2, 3\}\), the symmetric gauge, Eq. (2.2), can be generalised to SU(3) \[5, 10\].

\[
\chi_a(A) = \sum_{b=1}^{8} \sum_{i=1}^{3} c_{abi} A_{bi} = 0, \quad a = 1, \ldots, 8 \quad ("symmetric gauge" \text{ for SU}(3)).
\tag{3.2}
\]

Carrying out the coordinate transformation \[10\]

\[
A_{ak}(q_1, \ldots, q_8, \vec{S}) = O_{a\bar{a}}(q) \tilde{S}_{\bar{a}k} - \frac{1}{2g} c_{abc} \left(O(q) \partial_k O^T(q)\right)_{be}, \quad \psi_a(q_1, \ldots, q_8, \psi^{RS}) = U_{a\bar{a}}(q) \psi^{RS}_{\bar{a}}
\]

\[
\tilde{S}_{\bar{a}k} \equiv \left( \begin{array}{c}
S_{\bar{a}k} \\
\frac{S_{\bar{a}k}}{S_{\bar{a}k}}
\end{array} \right) = \\
\left( \begin{array}{ccc}
W_0 & X_3 - W_3 & X_2 + W_2 \\
X_3 + W_3 & W_0 & X_1 - W_1 \\
X_2 - W_2 & X_1 + W_1 & W_0 \\
-\frac{\sqrt{3}}{2} Y_1 - \frac{1}{2} W_1 & \frac{\sqrt{3}}{2} Y_2 - \frac{1}{2} W_2 & W_3 \\
-\frac{\sqrt{3}}{2} Y_1 + \frac{1}{2} W_1 & \frac{\sqrt{3}}{2} Y_2 + \frac{1}{2} W_2 & -\frac{1}{2} Y_2 & Y_3
\end{array} \right)
\]

\[c_{\bar{a}b\bar{k}} \tilde{S}_{\bar{a}k} = 0, \quad (3.3)
\]

an unconstrained Hamiltonian formulation of QCD can be obtained. The existence and uniqueness of (3.3) can be investigated by solving the 16 equs.

\[
\tilde{S}_{\bar{a}i} \tilde{S}_{\bar{a}j} = A_{ai} A_{aj} (6 \text{ equs.}) \quad \land \quad d_{\bar{a}\bar{b}\bar{c}} \tilde{S}_{\bar{a}i} \tilde{S}_{\bar{b}j} \tilde{S}_{\bar{c}k} = d_{abc} A_{ai} A_{bj} A_{ck} (10 \text{ equs.})
\tag{3.4}
\]

for the 16 components of \(\tilde{S}\) in terms of 24 given components \(A\).

Analysing the Gauss law operators and the unconstrained angular momentum operators in terms of the new variables in analogy to the 2-color case, it can be shown that the original constrained 24 colored spin-1 gluon fields \(A\) and the 12 colored spin-1/2 quark fields \(\psi\) (per flavor) reduce to 16 physical colorless spin-0, spin-1, spin-2, and spin-3 glueball fields (the 16 components of \(S\)) and a colorless spin-3/2 Rarita-Schwinger field \(\psi^{RS}\) (12 components per flavor). As for the 2-color case, the gauge reduction converts color \(\rightarrow\) spin, which might have important consequences for low energy Spin-Physics. In terms of the colorless Rarita-Schwinger fields the \(\Delta^{++}(3/2)\) could have the spin content \((+3/2, +1/2, -1/2)\) in accordance with the Spin-Statistics-Theorem.
Transforming to the intrinsic system of the embedded upper part $S$ of $\tilde{S}$ (see [10] for details)

$$S = R^T(\alpha, \beta, \gamma) \text{diag}(\phi_1, \phi_2, \phi_3) R(\alpha, \beta, \gamma), \ \wedge \ X_i \rightarrow x_i, \ Y_i \rightarrow y_i, \ \wedge \ \psi^{\text{RS}} \rightarrow \tilde{\psi}^{\text{RS}},$$ (3.5)

one finds that the magnetic potential $B^2$ has the zero-energy valleys ("constant Abelian fields")

$$B^2 = 0 : \ \phi_3 \text{ and } y_3 \text{ arbitrary} \ \wedge \ \text{all others zero} \ \ (3.6)$$

Hence, practically all glueball excitation-energy should result from an increase of expectation values of these two "constant Abelian fields", in analogy to $SU(2)$. Furthermore, at the bottom of the valleys the important minimal-coupling-interaction of $\tilde{\psi}^{\text{RS}}$ (analogous to (2,14)) becomes diagonal

$$\mathcal{H}_{\text{diag}} = \frac{1}{2} \tilde{S}_L^{(1,4)^+} \left[ (\phi_3 \lambda_3 + y_3 \lambda_8) \otimes \sigma_3 \right] \tilde{S}_L^{(1,4)} - \frac{1}{2} \tilde{S}_R^{(1,4)^+} \left[ \sigma_3 \otimes (\phi_3 \lambda_3 + y_3 \lambda_8) \right] \tilde{S}_R^{(1,4)}. \ (3.7)$$

Due to the difficulty of the FP-determinant (see [10]), precise calculations are not possible yet. Note, however, that in one spatial dimension the symmetric gauge for $SU(3)$ reduces to

$$A^{(1d)} = \begin{pmatrix} 0 & 0 & A_{13} \\ 0 & 0 & A_{23} \\ 0 & 0 & A_{33} \\ 0 & 0 & A_{43} \\ 0 & 0 & A_{53} \\ 0 & 0 & A_{63} \\ 0 & 0 & A_{73} \\ 0 & 0 & A_{83} \end{pmatrix} \rightarrow \tilde{S}^{(1d)} = \tilde{S}^{(1d)}_{\text{intrinsic}} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \ (3.8)$$

which consistently reduces the Equ.(3.4) for given $A_3$ to

$$\phi_3^2 + y_3^2 = A_{33} A_{33} \ \wedge \ \phi_3^2 y_3 - 3 y_3^3 = d_{abc} A_{33} A_{b3} A_{c3} \ (3.9)$$

with 6 solutions separated by zero-lines of the FP-determinant ("Gribov-horizons"). Exactly one solution exists in the "fundamental domain" $0 < \phi_3 < \infty \wedge \phi_3/\sqrt{3} < y_3 < \infty$, and we can replace

$$\int_{-\infty}^{\infty} \prod_{d=1}^{8} dA_{d3} \rightarrow \int_0^{\infty} d\phi_3 \int_{\phi_3/\sqrt{3}}^{\infty} dy_3 \ \phi_3^2 (\phi_3^2 - 3 y_3^3)^2 \alpha \int_0^{\infty} dr \int_{\pi/6}^{\pi/2} d\psi \cos^2(3\psi). \ (3.10)$$

For two spatial dimensions, one can show that (putting in Equ.(3.3) $W_1 = X_1, W_2 = -X_2$)

$$A^{(2d)} = \begin{pmatrix} A_{11} & A_{12} & 0 \\ A_{21} & A_{22} & 0 \\ A_{31} & A_{32} & 0 \\ A_{41} & A_{42} & 0 \\ A_{51} & A_{52} & 0 \\ A_{61} & A_{62} & 0 \\ A_{71} & A_{72} & 0 \\ A_{81} & A_{82} & 0 \end{pmatrix} \rightarrow \tilde{S}^{(2d)} = \tilde{S}^{(2d)}_{\text{intrinsic}} = \begin{pmatrix} \phi_1 & 0 & 0 \\ 0 & \phi_2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \ (3.11)$$

consistently reduces (3,4) to a system of 7 equ. for 8 physical fields (incl. rot.-angle $\gamma$), which, adding as an 8th equ. $(d_{abc} A_{b1} \tilde{S}_1)^2 = (d_{abc} A_{b1} A_{c2})^2$, can be solved numerically for randomly generated $A^{(2d)}$, again yielding solutions separated by horizons. Restricting to a fundamental domain

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dA_{a1} dA_{b2} \rightarrow \int d\gamma \int d\phi_1 d\phi_2 (\phi_1 - \phi_2) \int_{R_1(\phi_1, \phi_2)}^{\infty} dx_1 dx_2 dx_3 \int_{R_2(x_1, x_2, x_3, \phi_1, \phi_2)}^{\infty} dy_1 dy_2 \mathcal{F}.$$
Due to the difficulty of the FP-determinant, I have, however, not yet succeeded in a satisfactory
description of the regions $R_1$ and $R_2$. For the general case of three dimensions, I have found several
solutions of the Equ.\[5,4\] numerically for a randomly generated $A$, but to write the corresponding
unconstrained integral over a fundamental domain is a difficult, but I think solvable, future task.

4. Conclusions

Using a canonical transformation of the dynamical variables, which Abelianises the non-
Abelian Gauss-law constraints to be implemented, a reformulation of QCD in terms of gauge in-
vARIANT dynamical variables can be achieved. The exact implementation of the Gauss laws reduces
the colored spin-1 gluons and spin-1/2 quarks to unconstrained colorless spin-0, spin-1, spin-2 and
spin-3 glueball fields and colorless Rarita-Schwinger fields respectively. The obtained physical
Hamiltonian admits a systematic strong-coupling expansion in powers of $\lambda = g^{-2/3}$, equivalent
to an expansion in the number of spatial derivatives. The leading-order term in this expansion
Corresponds to non-interacting hybrid-glueballs, whose low-lying masses can be calculated with
high accuracy (at the moment only for the unphysical, but technically much simpler 2-color case)
by solving the Schrödinger-equation of Dirac-Yang-Mills quantum mechanics of spatially constant
fields. Higher-order terms in $\lambda$ lead to interactions between the hybrid-glueballs and can be taken
into account systematically, using perturbation theory in $\lambda$, allowing for the study of the difficult
questions of Lorentz invariance and coupling constant renormalisation in the IR.

References

[1] N.H. Christ and T.D. Lee, Operator ordering and Feynman rules in gauge theories, Phys. Rev. D 22
(1980) 939.
[2] M. Lüscher and G. Münster, Weak coupling expansion of the low-lying energy values in the SU(2)
gauge theory on a torus, Nucl. Phys. B 232 (1984) 445.
[3] P. Weisz and V. Ziemann, Weak coupling expansion of the low-lying energy values in SU(3) gauge
theory on a torus, Nucl. Phys. B 284 (1987) 157.
[4] A. Kihlberg and R. Marnelius, Properties of Yang-Mills theories with gauge-fixing conditions on the
field strength, Phys. Rev. D 26 (1982) 2003.
[5] B. Dahmen and B. Raabe, Unconstrained SU(2) and SU(3) Yang-Mills classical mechanics, Nucl.
Phys. B 384 (1992) 352.
[6] A.M. Khvedelidze and H.-P. Pavel, Unconstrained Hamiltonian formulation of SU(2) gluodynamics,
Phys. Rev. D 59 (1999) 105017 [hep-th/9808102].
[7] H.-P. Pavel, SU(2) Yang-Mills quantum mechanics of spatially constant fields, Phys. Lett. B 648
(2007) 97 [hep-th/0701283].
[8] H.-P. Pavel, Expansion of the Yang-Mills Hamiltonian in spatial derivatives and glueball spectrum,
Phys. Lett. B 685 (2010) 353 [arXiv:0912.5465 [hep-th]].
[9] H.-P. Pavel, SU(2) Dirac-Yang-Mills quantum mechanics of spatially constant quark and gluon fields,
Phys. Lett. B 700 (2011) 265 [arXiv:1104.1576 [hep-th]].
[10] H.-P. Pavel, Unconstrained Hamiltonian formulation of low energy SU(3) Yang-Mills quantum theory,
arXiv: 1205.2237 [hep-th] (2012).