Generalized finite difference approach verification on circular plates

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Abstract. The best-known method, finite differences (abbreviated in text as - FDM), consists of replacing each derivative by a difference algebraic quotient in the classic formulation. It is that discretization method which simple to code and economic to compute. In a sense, a finite difference formulation offers a more direct approach to the numerical solution of partial differential equations (abbreviated in text as – PDE) than does a method based on other formulations. The main drawback of the finite difference methods is the flexibility. Standard finite difference methods requires more regularity of the solution and triangulation (e.g. uniform meshing). Difficulties also arise in imposing boundary conditions.

Nowadays, FDM predominates the numerical solutions of PDE not less in pair with the method of finite elements (abbreviated in text as - FEM).

In This paper, an advanced static structural-deflection analysis procedure of circular plates through generalized (modernized) finite difference method is presented. The generalized approach considers a lot of parameters been regarded-less by the classical one. The thing which negatively affects the accuracy (convergence to the exact solution values) and consequently; tendency of results it generates. Besides, the newly presented can be easily applied to many cases that the classical even fails to solve. A theoretical mathematical statement been reviewed containing all the necessary explanatories, graphs and indications followed by verification testing problems simultaneously with a comparison been held among variety of results generated via different techniques. The verification comparison indicates the powerfulness of the innovated approach relative to the classical one and the easiness and precision it offers in relevant to the others.

Keywords
Generalized finite difference method (approach) - Numerical method(s) - Partial differential equations (PDE) - Applied mathematics - Circular plate(s) static analysis

1. Introduction
The finite differences technique is one of the most widely adopted methods for boundary value problems numerical solutions. In this method;

1.1. The differential equation governing the problem is replaced by a difference equation through replace each of the derivatives in the differential equation with an appropriate difference-quotient approximation. The particular difference quotient is chosen to maintain a known error as minimum as possible.
1.2. The continuous region (domain), where the differential equation applies, is substituted by a set of discrete algebraic difference equations much more easy (especially for machinery computing) to solve.

2. Theory
This section briefs the pre-quest theoretical base on both Mathematical and structural-mechanical (applicable) necessary phases

2.1. Mathematical Base

2.1.1. Classical FDM \((3)(4)(5)(7)(8)\)

As a direct explanation of a finite difference method for solving a differential equation, the second order ordinary differential equation (abbreviated in text as \(-ODE\) of the 2\textsuperscript{nd} order) to be considered,

\[
\frac{d^2u}{dx^2} = u'' = f(x) \quad \text{For} \quad 0 < x < 1
\]

With some known boundary conditions

\[
u(0) = \alpha; \quad u(1) = \beta \quad (2)
\]

The function \(f(x)\) is specified and \(u(x)\) to be determined in the interval \((0 < x < 1).\) This problem is so called a two-point boundary value problem since boundary conditions are known at two distinct points. This problem is so simple that we can solve it explicitly \([\text{integrate } f(x) \text{ twice and choose the two constants of integration so that the boundary conditions are satisfied}].\) But studying finite difference methods for this simple equation will reveal some of the essential features of all such analysis, particularly the relation of the global error to the local truncation error and the use of stability in making this connection.

An attempt to compute a grid function consisting of values \(U_0, U_1, \ldots, U_m, U_{m+1}\) where \(U_j\) to approximate the solution \(u(x_j).\) Here \(x_j = j \cdot \lambda\) and \(\lambda = 1 / (m+1)\) is the mesh spacing ; the distance between grid points (nodes). Out of the boundary conditions, its well-known that \(U_0 = \alpha\) and \(U_{m+1} = \beta\) and so on hands; \((m)\) unknown values \(U_1, \ldots, U_m\) to compute . If \(u''(x)\) be replaced in \(Equation (1)\) by the centred difference approximation

\[
\bar{\partial}^2 U_j = \frac{1}{\lambda^2}(U_{j-1} - 2U_j + U_{j+1})
\]

Then, a set of algebraic equations to be obtained such that;

\[
\frac{1}{\lambda^2}(U_{j-1} - 2U_j + U_{j+1}) = f(x_j) \quad \text{For} \quad j = 1, 2, \ldots, m
\]

(4)

Note that the first equation \((j = 1)\) involves the value \(U_0 = \alpha\) and the last equation \((j = m)\) involves the value \(U_{m+1} = \beta.\) We have a linear system of \((m)\) equations for the \((m)\) unknowns, which can be written in the form

\[
AU = F
\]

(5)

Where \(U\) is the vector of unknowns \(U = [U_1, U_2, \ldots, U_m]^T\) and

\[
A = \frac{1}{\lambda^2}
\left(
\begin{array}{cccc}
-2 & 1 & & \\
1 & -2 & 1 & \\
& \ddots & \ddots & \\
1 & -2 & 1 & \\
& & 1 & -2
\end{array}
\right), \quad F = \begin{pmatrix} f(x_1) - (\alpha / \lambda^2) \\ f(x_2) \\ \vdots \\ f(x_{m-1}) - (\beta / \lambda^2) \\ f(x_m) \end{pmatrix}
\]

(6)

\[\dagger\text{Numbers shown in such parentheses refer to the corresponding number in the List of References at the end of this paper.}\]
This tri-diagonal linear system is non-singular and can be easily solved for U from any right hand side F neglecting the error resulting out O (λ²) 

2.1.2. Generalized (Modernized) FDM

The generalized procedure to be rebuilt adopting the previous (Classical Sec.2.1.1) one on bases of replacing the Classical Equation (3) with the Generalized Equation (7) down shown as;

\[-\lambda^2 \cdot \partial^2 U_j + \frac{\lambda^2}{2} \Delta \partial^2 U_j = U_{j-1} - 2U_j + U_{j+1} + \Delta U_j + \lambda \cdot \partial U_j\]  

(7)

Thus, the solution set of algebraic equations to be expressed such follows;

\[U_{j-1} - 2U_j + U_{j+1} + \Delta U_j + \lambda \cdot \partial U_j = -\lambda^2 \cdot f(x_j) + \frac{\lambda^2}{2} \Delta f(x_j) \quad \text{For} \quad j = 1, 2, ..., m\]  

(8)

2.2. Applying the Generalized FDM on Circular plates deflection analysis:

A circular plate is considered to be a domain (polar type), that categorized under a problem described by a model which easily simulated (solved) by Laplacian operator (Equation of Laplace of 2nd order-elliptic type) on a polar domain Ω = {0 < r; θ < 2π} that can be conveniently written in polar coordinates as in (Figure 1).

\[\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r} \frac{\partial^2 u}{\partial \theta^2} = f(r, \theta)\]  

(9)

Taking into consideration, Euler–Bernoulli equation concerns the topic of deformation analysis on structural elements, it describes the correlation between the function (U) ↔ its 2nd derivative (\(\partial^2 U\) )\(^{(1)}\) thus we’ve:

\[\text{[Deflection}(W) \leftrightarrow \text{Applied Bending Moment}(M)] \quad ; \quad \partial^2 W = \frac{M}{D}\]  

\[\text{[Bending Moment}(M) \leftrightarrow \text{Applied Load}(P)] \quad ; \quad \partial^2 M = -P\]

Classical procedure \(^{(2)}\)(9)(10) approximates the terms of Equation (9) numerically so that it can be expressed as:
\[
\left( \frac{\partial^2 u}{\partial t^2} \right)_j = \left( \frac{1}{\lambda^2} - \frac{1}{2\lambda \cdot r_j} \right) U_j + \left( \frac{U_m}{(r_j \cdot \theta^\text{per})^2} - 2 \left( \frac{1}{\lambda^2} + \frac{1}{(r_j \cdot \theta^\text{rad})^2} \right) \right) U_j + \left( \frac{1}{\lambda^2} \right) U_j
\] (9.1)

Substituting with (W) for the function \((U)\) & with \((M)\) for 2\textsuperscript{nd} derivative \((\partial^2 U)\) getting:
\[
\frac{M_j}{D} = \left( \frac{1}{\lambda^2} - \frac{1}{2\lambda \cdot r_j} \right) W_j + \left( \frac{W_m}{(r_j \cdot \theta^\text{per})^2} - 2 \left( \frac{1}{\lambda^2} + \frac{1}{(r_j \cdot \theta^\text{rad})^2} \right) \right) W_j + \left( \frac{1}{\lambda^2} \right) W_j
\] (9.1.1)

Also, Substituting with \((M)\) for the function \((U)\) & with \((-P)\) for 2\textsuperscript{nd} derivative \((\partial^2 U)\) getting:
\[
-P_j = \left( \frac{1}{\lambda^2} - \frac{1}{2\lambda \cdot r_j} \right) M_j + \left( \frac{M_m}{(r_j \cdot \theta^\text{per})^2} - 2 \left( \frac{1}{\lambda^2} + \frac{1}{(r_j \cdot \theta^\text{rad})^2} \right) \right) M_j + \left( \frac{1}{\lambda^2} \right) M_j
\] (9.1.2)

While, as per the proposed generalized procedure expands the terms of \textit{Equation (9)} such as:
\[
\left[ \frac{(\partial^2 U^I) + (\partial^2 U^II) + (\partial^2 U^III) + (\partial^2 U^IV)}{4} \right] + \left[ \frac{(\Delta \partial^2 U^{(I \rightarrow III)} + \Delta \partial^2 U^{(II \rightarrow IV)} + \Delta \partial^2 U^{(II \rightarrow I)} + \Delta \partial^2 U^{(IV \rightarrow III)})}{8} \right] = \left[ \frac{U_n}{(r_j \cdot \theta^\text{rad})^2} + \frac{1}{\lambda} \left( \frac{1}{\lambda^2} - \frac{1}{2r_j} \right) \right] U_k + \frac{1}{\lambda} \left( \frac{1}{\lambda^2} \right) U_j + \frac{1}{\lambda} \left( \frac{1}{\lambda^2} \right) U_j + \frac{1}{\lambda} \left( \frac{1}{\lambda^2} \right) U_j
\] (9.2)

Same wise, Substituting with (W) for the function \((U)\) & with \((M)\) for 2\textsuperscript{nd} derivative \((\partial^2 U)\) (neglecting some non-effective terms simplifying L.H.S calculations) getting:
\[
\left[ \frac{M_j}{D} = \left( \frac{1}{\lambda^2} - \frac{1}{2\lambda \cdot r_j} \right) W_k + \left( \frac{W_m}{(r_j \cdot \theta^\text{rad})^2} - 2 \left( \frac{1}{\lambda^2} + \frac{1}{(r_j \cdot \theta^\text{rad})^2} \right) \right) W_j + \left( \frac{1}{\lambda^2} \right) W_j
\] (9.2.1)

Once more, Substituting with \((M)\) for the function \((U)\) & with \((-P)\) for 2\textsuperscript{nd} derivative \((\partial^2 U)\) getting:
\[
\left[ \frac{\Delta P^{(I \rightarrow III)} + \Delta P^{(II \rightarrow IV)} + \Delta P^{(II \rightarrow I)} + \Delta P^{(IV \rightarrow III)}}{8} \right] = \left[ \frac{\partial U}{\partial t} \right]
\] (9.2.2)

3. Numerical verification
In this section, a couple of verification problems have been accurately chosen to contrast the difference between the classical & the proposed innovated technique
3.1. As per that shown in (Figure 2), a steel circular plate \((E=2.1x10^5 \text{kg/cm}^2, \nu=\frac{1}{3})\) of a dimensions \((a=100 \text{cm}, t=2.5 \text{cm})\) with a \textit{clamped perimeter} loaded on a cyclic symmetry pattern such that the loaded zones are subjected to \((P=4 \text{kg/cm}^2)\).
In here, verification action as a main task explained through evaluating the out-coming deflections values precision in comparison to that resulting-out of other techniques.

| Node №‡ | Deflection in mm values (+ down) Via | Modernized FDM | Classical FDM | FEM |
|---------|-------------------------------------|----------------|---------------|-----|
| 0       | 5.732                               | 7.611          | 8.031         |
| 1       | 12.687                              | 13.673         | 11.25         |
| 2       | 10.011                              | 10.477         | 10.00         |
| 3       | 4.139                               | 4.266          | 3.75          |
| 5       | 5.623                               | 8.944          | 7.50          |
| 6       | 3.699                               | 6.258          | 4.01          |
| 7       | 1.367                               | 2.415          | 1.25          |
| 9       | 3.227                               | 4.214          | 6.25          |
| 10      | 1.573                               | 2.039          | 3.760         |
| 11      | 0.438                               | 0.564          | 0.75          |

‡ Nodes (4,8,12) are deflection less

3.2. As per that shown in (Figure 3), a Reinforced concrete circular plate \((E=1.9 \times 10^5 \text{kg/cm}^2, \nu=\frac{1}{3})\) of a dimensions \((a=140\text{cm}, t=12\text{cm})\) with a hinged perimeter and a line load so that the load forms a circle of 70cm radius from the plate’s center \((0)\) with \((Q=20\text{kg/cm})\).

In here, verification action is explained not only through evaluating the out-coming deflections values precision in comparison to that resulting-out of the analytical technique, but also, through demonstrating a type of problems which the classical technique can’t solve.

| Node №‡ | Deflection in cm values (+ down) Via | Modernized FDM | Classical FDM | Analitical(11) |
|---------|-------------------------------------|----------------|---------------|---------------|
| 0       | 0.2391                              | 0.2060         |
| 1       | 0.2292                              | Not            | 0.1910        |
| 2       | 0.1741                              | applicable     | 0.1470        |
| 3       | 0.0929                              |                | 0.0782        |

‡ Node (4) is a deflection less node.

4. Conclusion

The new proposed algorithm that based on a generalization (modernization) of finite difference equations; clearly verified for circular plates calculations.

Looking backward to the stage of verification (item 3), focusing accurately; figuring out that the super advantage of that proposal doesn’t only conclude the precision of the results it comes with; compared to that outcomes of other similar techniques and how far it converges the exact solution (item 3.1), but also the solution equations new terms which consider straining actions never been considered by the classical FDM, the thing which allows that new technique to analyse plates subjected to such actions that classical fails to solve (item 3.2).

Finally, it’s not hard to solve problems using numerical techniques especially the above proposed one, just a matter of getting the continuous right hand side of an original differential equation and its...
derivatives, discredited. A small number of divisions (segments); also, high accuracy speaks up about the possibility of using the new developed method together besides the method of finite elements...

Nomenclature

\[ U_j \]

\[ \frac{d^j u}{d\eta^j} = u^{(j)} = \partial^j U \]

\[ \left( \Delta \frac{d^j u}{d\eta^j} \right)_{\alpha \rightarrow \beta} = \left( \Delta u^{(j)}(\alpha \rightarrow \beta) \right)_{\beta \rightarrow \alpha} \]

\[ P_j, W_j, M_j \]

\[ \Delta P_j^{(\alpha \rightarrow \beta)}, \Delta W_j^{(\alpha \rightarrow \beta)}, \Delta M_j^{(\alpha \rightarrow \beta)} \]

\[ Q_j^{(\alpha \rightarrow \beta)} \]

\[ \phi_j^{(\alpha \rightarrow \beta)} \]

\[ \lambda, r_j, \theta \]

\[ D = \left( \frac{E1^3}{12(1-\nu^2)} \right) \]

\[ E, \nu, t, \alpha \]

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