Four – dimensional models for control system typical units

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Abstract. This study examines the possibility of applying different forms of complex numbers and quaternions for control system typical units modeling. The hypothesis is that the current understanding of development may be expanded by making use of different mathematical objects and approaches. As a result four-dimensional general models have been obtained.

1. Introduction
Complex numbers are widely used for analysis, synthesis and evaluation of automatic control systems. It is usually implied that the «complex number» is the number expressed in the form of \( z = a + ib \), where \( a \) and \( b \) are real numbers and \( i \) satisfies \( i^2 = -1 \). In [1] the mathematical means for two-dimensional control systems were designed and the matrix representation for complex numbers was applied for the case when \( i \) satisfies \( i^2 = -1 \). Nevertheless there are two complex number systems which are less known, but could be useful for control systems development. These are parabolic complex numbers and hyperbolic complex numbers. For the sake of accuracy it should be mentioned that the complex numbers when \( i \) satisfies \( i^2 = -1 \) are called elliptic numbers [2].

2. Method description
The parabolic complex numbers are defined to be numbers of the form \( z = a + ib \) where \( a \) and \( b \) are real numbers and \( i \) satisfies \( i^2 = 0 \). Hyperbolic numbers are the numbers of the form \( z = a + ib \) where \( a \) and \( b \) are real numbers and \( i \) satisfies \( i^2 = 1 \).

Throughout the paper the following matrix representation for any complex number will be used [1]:

\[
\begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}
\begin{bmatrix}
a & b \\
-b & a
\end{bmatrix}
\]

(1)

Here the entry \( E \) is the unit matrix for the real number in the real part of the complex number, \( I \) is the unit matrix for the imaginary part of the complex number. Hence an algebraic form for the complex number, they are the first elements of the first row of \( z \) matrix or the down-up elements of the second column. Matrix definition for hyperbolic complex numbers (\( i^2 = 1 \)) as follows:

\[
\begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}
\begin{bmatrix}
a & b \\
b & a
\end{bmatrix}
\]

(2)

for parabolic complex numbers (\( i^2 = 0 \)).

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Then for some control system presented by the following linear differential equation with constant coefficients and zero initial conditions as [3]:

$$\frac{d^n x}{dt^n} + a_{n-1} \frac{d^{n-1} x}{dt^{n-1}} + \ldots + a_1 \frac{dx}{dt} + a_0 x(t) = b_m \frac{d^m y}{dt^m} + \ldots + b_0 y(t)$$

(4)

an input signal \(y(t)\) is a complex valued time function:

$$y(t) = y_1(t) + i y_2(t)$$

(5)

An output \(x(t)\) is the complex valued time function as well:

$$x(t) = x_1(t) + i x_2(t)$$

(6)

Based on the mathematical means developed in [2], a transfer function for the hyperbolic complex numbers can be written:

$$W(p) = \begin{bmatrix} W_1(p) & W_2(p) \\ W_2(p) & W_1(p) \end{bmatrix}$$

(7)

Here \(W_1, W_2\) are the matrix-complex-valued elements of the transfer function \(W\).

The structural scheme for the system with the matrix-complex-valued transfer function is shown below:

![Figure 1. Generalized matrix-complex-valued control system for the hyperbolic numbers.](image)

As well as in the case of elliptic numbers the developed system is full-connected. As a result direct cross-connections have transfer functions equal in values and signs. While elliptic numbers provide transfer functions with opposite signs. But it is important for the system shown in figure 1 that \(W_1 \neq W_2\).

For the parabolic numbers the matrix-complex-valued transfer function is:

$$W(p) = \begin{bmatrix} W_1(p) & W_2(p) \\ 0 & W_1(p) \end{bmatrix}$$

(8)

The structural scheme for such system is:

![Figure 2. Generalized matrix-complex-valued control system for the parabolic numbers.](image)
Figure 2 demonstrates that there is another new type of connection such as positive connection for the imaginary channel.

3. Application of quaternion algebra for typical control system modeling

As the quaternions system is a system extending the complex numbers, logically to explore quaternions applicability for the control systems modeling.

Let there will be again some control system presented by the following linear differential equation with constant coefficients and zero initial conditions [3]:

\[
a_{m_{m}} \frac{d^{n}x_{m}(t)}{dt^{n}} + a_{m-1,m} \frac{d^{n-1}x_{m}(t)}{dt^{n-1}} + \ldots + a_{1,m} \frac{dx_{m}(t)}{dt} + a_{0,m} x_{m}(t) = -b_{km} \frac{d^{n}y_{m}(t)}{dt^{n}} \ldots + b_{0,m}y(t)
\]

where m is the channel number \((m = 0,1,2,3)\), \(y_{m}(t)\) is the m-channel input, \(x_{m}(t)\) is the m-channel output; n, k are the orders of derivation.

Input signal \(y(t)\) is a quaternion function:

\[
y(t) = y_{1}(t) + iy_{2}(t) + jy_{3}(t) + ky_{4}(t)
\]

Output signal \(x(t)\) is also the quaternion function:

\[
x(t) = x_{1}(t) + ix_{2}(t) + jx_{3}(t) + kx_{4}(t)
\]

These equations (10), (11) mean that one-dimensional system with hyper complex numbers as input and output functions in the set of hyper complex variables will be four-dimensional for the real variables functions, composing four hypercomplex functions of input and four hypercomplex functions of output.

For the sake of precision let both the input and the output orders of derivation be equal to n and to k respectively. Then (9) can be presented as symbolic equation [4-5]:

\[
A_{m}(p) x_{m}(t) = B_{m}(p) y_{m}(t)
\]

\[
A_{m}(p) = T_{1m}^{n} p^{n} + T_{2m}^{n-1} p^{n-1} + \ldots + T_{n-1,m} p^{1} + 1, \quad B_{m}(p) = \tau_{1m}^{k} p^{k} + \ldots + \tau_{(k-1)m} p^{1} + 1
\]

are the symbolic polynomials for m-channel input and output, \(T_{nm}\), \(\tau_{nm}\) are time constants for m-channel, for \(T_{1m}, T_{2m}, T_{3m}, T_{4m}, \tau_{1m}, \tau_{2m}, \tau_{3m}, \tau_{4m}\) index is equal to 1 when the coefficient is for the real part of the variable, otherwise index is the imaginary part of the variable.

The product of (12) is the symbolic equation system:

\[
A_{0}(p)x_{0}(t) = B_{0}(p)y_{0}(t),
A_{1}(p)x_{1}(t) = B_{1}(p)y_{1}(t),
A_{2}(p)x_{2}(t) = B_{2}(p)y_{2}(t),
A_{3}(p)x_{3}(t) = B_{3}(p)y_{3}(t)
\]

Or, represented as matrix:

\[
A(p)X(t) = B(p)Y(t)
\]

Matrices \(A(p)\) and \(B(p)\) describe four-channel control system with independent channels, but the purpose of this paper is full cross-connected control system, provided by quaternions. Using quaternions for matrix \(A(p)\) and matrix \(B(p)\) we obtain 16-elements matrices for both \(A(p)\) and \(B(p)\) respectively.

The product of (14) has a form:

\[
X(t) = A^{-1}(p)B(p)Y(t) = W(p)Y(t)
\]

\[
X(t) \text{ is the output signal, } W(p) = A^{-1}(p)B(p) \text{ is a quaternion matrix of transfer function of given system:}
\]

\[
W(p) = \begin{bmatrix}
    W_{0}(p) & -W_{1}(p) & -W_{2}(p) & -W_{3}(p) \\
    W_{1}(p) & W_{0}(p) & -W_{3}(p) & W_{2}(p) \\
    W_{2}(p) & W_{3}(p) & W_{0}(p) & -W_{1}(p) \\
    W_{3}(p) & -W_{2}(p) & W_{1}(p) & W_{0}(p)
\end{bmatrix}
\]

Therefore the structural scheme for such system as follows:
As seen from figure 3 quaternions really provide full cross-connected system. There are a lot of possibilities to study such systems, for example an obvious dependence between input parameters: it is because of cross-connections any change of input will affect the whole system, but not just one particular element. But before such analyses being carried on, it is worth saying that there is not only represented quaternions form exists. There is an interesting case when quaternion function is the quaternion with scalar part equal to zero. Then equations (10) and (11) will take form as shown below:

\[ y(t) = iy_2(t) + jy_3(t) + ky_4(t) \]  

(17)

and \( y_j(t) = 0 \).

Hence the output function:

\[ x(t) = ix_2(t) + jx_3(t) + kx_4(t) \]  

(18)

and \( x_j(t) = 0 \).

The result of (17) and (18) is the quaternion matrix of transfer function:

\[
W(p) = \begin{bmatrix}
0 & -W_1(p) & -W_2(p) & -W_3(p) \\
W_1(p) & 0 & -W_3(p) & W_2(p) \\
W_2(p) & W_3(p) & 0 & -W_1(p) \\
W_3(p) & -W_2(p) & W_1(p) & 0
\end{bmatrix}
\]  

(19)
Although the number of elements has been decreased the system is still full-connected, it means that the system has the same features as the system shown in figure 3. System provided by (19) may be useful if the real part of input signal is constant. But it is the question for the further study.

**Figure 4.** Generalized quaternion-matrix control system when scalar part is 0.

In addition there are such number systems as pseudo-quaternions, degenerated pseudo-quaternions, degenerated quaternions and doubly degenerated quaternions [6]. These number systems have the same features as the “regular” quaternions, hence as abovementioned elliptic, hyperbolic and parabolic complex numbers. They have such features as the complex conjugate of the complex number \( z \) is \( \bar{z} \); there is a modulus \(|z|\) and for \(|z|^2 = z\bar{z}\) the result is the real number and so on.

Algebraic form of pseudo-quaternion can be represented as:

\[
z = a + ib + ec + fd
\]  
(20)

| Table 1. Pseudo-quaternion multiplication. |
|----|----|----|
| i  | e  | f  |
| i  | -1 | f  | -e |
| e  | -f | 1  | -i |
| f  | e  | i  | 1  |
algebraic form of degenerated quaternion is:
\[ z = a + ib + c + \eta d \]  
(21)

| Table 2. Degenerated quaternion multiplication. |
|-----|-----|-----|
| i   | -1  | \eta |
| c   | -\eta | 0   |
| \eta | c   | 0   |

algebraic form of degenerated pseudo-quaternion is:
\[ z = a + eb + c + \zeta d \]  
(22)

| Table 3. Degenerated pseudo-quaternion multiplication. |
|-----|-----|-----|
| e   | 0   | \zeta |
| c   | -\zeta | 0   |
| \zeta | 0   | 0   |

algebraic form of doubly degenerated quaternion is:
\[ z = a + eb + \eta c + \zeta d \]  
(23)

| Table 4. Doubly degenerated pseudo-quaternion multiplication. |
|-----|-----|-----|
| e   | 0   | \zeta |
| \eta | -\zeta | 0   |
| \zeta | 0   | 0   |

As an example here is the control system based on pseudo-quaternions. Obtaining of matrix representation form is given in [1], so it is just for the sake of comprehension the transfer function matrix is represented as follows:
\[ W(p) = \begin{bmatrix} W_0(p) & -W_1(p) & -W_2(p) & W_3(p) \\ W_1(p) & W_0(p) & -W_3(p) & -W_2(p) \\ W_2(p) & W_3(p) & W_0(p) & -W_1(p) \\ W_3(p) & -W_2(p) & W_1(p) & W_0(p) \end{bmatrix} \]  
(24)

However not all quaternion number systems can be represented in such way. For example degenerated quaternions have no matrix representation. According to [7] quaternions can be represented in the form:
\[ Q = a_0 E + a_1 I + a_2 J + a_3 K \]  
(25)

here E, I, J, and K are unit quaternions: \(1 \rightarrow E, \ i \rightarrow I, \ j \rightarrow J, \ k \rightarrow K.\)

\[ E = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad I = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad J = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad K = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \]  
(26)

For matrix representation the determinants of matrices E, I, J, and K must be equal to 1. But the determinant of matrix J is zero. Matrix determinant for matrix K is also equal to zero. This means the impossibility of matrix representation for degenerated quaternions. But theoretically there is a potentiality to apply them for modeling of specialized control systems.

Although there are three types of biquaternions corresponding to elliptic, parabolic and hyperbolic complex numbers, they provide 8-dimensional systems and will be considered in the following papers.
4. Conclusion
The described mathematical method and hypercomplex numbers may be applied for multiply input-output control systems modeling, providing more possibilities. Although for practical applications there are problems of stability, controllability and observability of such systems to be studied first.

5. References
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