SOME REMARKS ON YAMABE SOLITONS

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Abstract. In this paper we have obtained evolution of some geometric quantities on a compact Riemannian manifold $M^n$ when the metric is a Yamabe soliton. Using these quantities we have obtained bound on the soliton constant. We have proved that the commutator of two soliton vector fields with the same metric in a given conformal class produces a Killing vector field. Also it is shown that the soliton vector field becomes a geodesic vector field if and only if the manifold is of constant curvature.

1. Introduction

Let $M^n(n \geq 3)$ be a compact Riemannian manifold without boundary with a Riemannian metric $g$. The Yamabe flow, introduced by (1, 2) is a way of deforming the initial metric $g$ in a given conformal class by means of the PDE

$$\frac{\partial g}{\partial t} = -(R - r)g,$$

where $R$ is the scalar curvature and $r$ is the average of $R$

$$i.e \quad r = \frac{\int_M R \, dv_g}{\int_M dv_g}$$

Similar to the Ricci soliton, Yamabe solitons are special solution of (1.1) moving purely by diffeomorphism and homothety and also arises as the blow-up limit of (1.1). More precisely a complete Riemannian metric $g$ is called a Yamabe soliton if it satisfies

$$\mathcal{L}_V g_{ij} = 2(-c + R)g_{ij},$$

where $V$ is the soliton vector field and $c$ is the soliton constant. If $c$ is a function in space, then $g$ is said to be almost Yamabe soliton.

The soliton is shrinking, steady and expanding according as $c < 0$, $= 0$ and $> 0$ respectively. After the introduction in section 1, section 2 deals with some evolution equation of the Christoffel symbols, curvature tensor, Ricci tensor and scalar curvature. Section 3 is devoted with the study commutator of two Yamabe solitons. Finally, we have obtained the condition of soliton vector field to be a geodesic vector field in the last section.

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2. Evolution of Geometric Quantities

Lemma 2.1 If \((g, V, c)\) be a Yamabe soliton on a Riemannian manifold \(M^n\), then

\[(i) \mathcal{L}_V \Gamma^h_{ij} = (\nabla_j R) \delta^h_i + (\nabla_i R) \delta^h_j - \nabla^h R g_{ij},\]

\[(ii) \mathcal{L}_V R^h_{kji} = (\nabla_k \nabla_i R) \delta^h_j - (\nabla_j \nabla_i R) \delta^h_k + (\nabla_j \nabla^h R) g_{ik} - (\nabla_k \nabla^h R) g_{ij},\]

\[(iii) \mathcal{L}_V R_{ij} = g^{jk} \nabla_k \nabla_i R - \Delta R g_{ij} + \nabla_j \nabla^i R - n \nabla_j \nabla_i R,\]

\[(iv) \mathcal{L}_V R = 2(1 - n) \Delta R,\]

where \(R^h_{kji}\) and \(R\) are respectively the curvature tensor, Ricci tensor and scalar curvature of \(M\).

Proof: Using equation (1.2) in the formula
\[
\mathcal{L}_V \Gamma^h_{ij} = \frac{1}{2} g^{hl}[\nabla_j (\mathcal{L}_V g_{il}) + \nabla_i (\mathcal{L}_V g_{jl}) - \nabla_l (\mathcal{L}_V g_{ij})]
\]
we have,

\[
\mathcal{L}_V \Gamma^h_{ij} = g^{hl}[(\nabla_j R) g_{il} + (\nabla_i R) g_{jl} - (\nabla_l R) g_{ij}] = (\nabla_j R) \delta^h_i + (\nabla_i R) \delta^h_j - \nabla^h R g_{ij}.
\]

This is the evolution of the Christoffel symbol or more precisely the connection.

Having this hand and using commutation formula
\[
\nabla_k (\mathcal{L}_V \Gamma^h_{ij}) - \nabla_j (\mathcal{L}_V \Gamma^h_{ik}) = \mathcal{L}_V R^h_{kji}
\]

we have

\[
\nabla_k (\mathcal{L}_V \Gamma^h_{ij}) - \nabla_j (\mathcal{L}_V \Gamma^h_{ik}) = \mathcal{L}_V R^h_{kji}
\]

Thus we get the evolution equation of curvature tensor along Yamabe soliton.

Again contracting (2.2) we have,

\[
\mathcal{L}_V R_{ij} = g^{jk} \nabla_k \nabla_i R - \Delta R g_{ij} + \nabla_j \nabla^i R - n \nabla_j \nabla_i R.
\]

This gives the evolution equation for the Ricci tensor.

Again contracting (2.3) we have,

\[
\mathcal{L}_V R = \Delta R - n \Delta R + \Delta R - n \Delta R = 2(1 - n) \Delta R.
\]

This completes the proof of the lemma.

Now we can state our main theorem as the following:

Theorem 2.1. Let \(M(g, V, c)\) be a compact Yamabe soliton. If the scalar curvature is bounded below by some positive constant \(\alpha\), then we have the same lower bound for the soliton constant and the soliton is expanding.
Proof. From (2.4) we have,
\[(2.5)\quad g(\nabla R, V) = (1 - n)\Delta R.\]
Since \(M\) is compact, using divergence theorem we have,
\[(2.6)\quad \int_M g(\nabla R, V) dv_g = 0.\]
Now note that,
\[(2.7)\quad \text{div}(RV) = \nabla_i(RV^i) = g(\nabla R, V) + R \text{div} V.\]
and integrate (2.7) over \(M\) in order to get,
\[(2.8)\quad \int_M [g(\nabla R, V) + R \text{div} V] dv_g = 0.\]
Now contracting equation (1.2) we obtain
\[(2.9)\quad \text{div} V = n(-c + R).\]
Using (2.9) and (2.6) in (2.8) we have,
\[\int_M R \text{div} X \ dv_g = 0,\]
\[i.e\quad \int_M R(-c + R)dv_g = 0,\]
\[i.e\quad c = \frac{\int_M R^2 \ dv_g}{\int_M R \ dv_g}.\]
If \(R \geq \alpha\) for some \(\alpha > 0\), then we have \(c \geq \alpha\). This proves our theorem.

3. Commutator of Soliton vector fields

We now consider two distinct Yamabe solitons with the same background Riemannian metric and consider commutator of there flow vector fields. More precisely we prove that

**Theorem 3.1.** Let \(M(g, V_1, c_1)\) and \(M(g, V_2, c_2)\) be two distinct non-trivial Yamabe solitons. Then \([V_1, V_2]\) is a Killing vector field.

**Proof.** Since \((g, V_1, c_1)\) and \((g, V_2, c_2)\) are Yamabe solitons, we have
\[(3.1)\quad \begin{cases} \mathcal{L}_{V_1}g = 2(R - c_1)g, \\ \mathcal{L}_{V_2}g = 2(R - c_2)g. \end{cases}\]
Since the Yamabe solitons are unique, without loss of generality we may assume \(c_1 < c_2\). From (3.1) we have,
\[\mathcal{L}_{V_1 - V_2}g = 2(c_2 - c_1)g.\]
This shows that $X = V_1 - V_2$ is a homothetic vector field. Now,

$$\mathcal{L}_{[V_1, V_2]g} = \mathcal{L}_X \mathcal{L}_{V_2} g - \mathcal{L}_{V_2} \mathcal{L}_X g = 2\mathcal{L}_X (R - c_2) g - 2\mathcal{L}_{V_2} (c_2 - c_1) g = 4(R - c_2)(c_2 - c_1) g - 4(c_2 - c_1)(R - c_2) g = 0.$$  

i.e $[V_1, V_2]$ is Killing. This proves the proposition.

**Remark 1.** The theorem is also true for almost Yamabe solitons.

### 4. Yamabe solitons and Geodesic vector fields

In this section we have studied solitons and obtained the condition for which its potential vector field becomes geodesic vector field in the following sense.

**Definition 4.1.** A vector field $X$ on a Riemannian manifold $M$ is called a geodesic vector field if $\square X = 0$, where $\square$ is an operator acting on a smooth vector field $X^i$, given in local coordinates by

$$\square X^i = -(g^{jk} \nabla_j \nabla_k X^i + R^i_j X^j).$$

We now prove the following:

**Theorem 4.1.** If $(g, V, c)$ is a Yamabe soliton on $M$, then $V$ is a geodesic vector field if and only if $M$ is of constant scalar curvature.

**Proof.** Tracing equation (1.2) we have,

$$\nabla_i V^i = (R - c)n.$$  

Taking covariant derivative of (4.2) we have

$$\nabla_j \nabla_i V^i = n \nabla_j R.$$  

Again writing (1.2) as

$$\nabla_j V^i + \nabla^i V_j = 2(R - c)\delta^i_j$$

and taking covariant derivative it we get

$$\nabla_i \nabla_j V^i + \nabla_i \nabla^i V_j = 2\nabla_j R.$$  

Now subtracting (4.3) from (4.4) we have,

$$\nabla_i \nabla_j V^i - \nabla_j \nabla^i V_i + \nabla_i \nabla^i V_j = (2 - n) \nabla_j R.$$  

Now applying Ricci formula we have

$$\square V = (n - 2) \nabla_j R.$$  

This proves the theorem.

**Remark 2.** As for a surface, the notion of Ricci soliton and Yamabe soliton coincides, it follows from (4.5) that the soliton vector field is a geodesic vector field.
References

[1] Hamilton, R. S., *The Ricci flow on surfaces*, Mathematics and general relativity, Contemp. Math., 71 (1988), 237-262.

[2] Hamilton, R. S., *Lectures on geometric flows*, unpublished manuscript, 1989.

[3] Yano, K., *Integral formulas in Riemannian geometry*, Marcel Dekker Inc., 1970.

[4] Yano, K. and Nagano, T., *On geodesic vector fields in a compact orientable Riemannian space*, Commun. Math. Helv. 35 (1961), 55-64.

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