Counting Points on Curves over Families in Polynomial Time

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The purpose of this note is to apply the result of [9] to obtain an algorithm for counting points on curves over finite fields, for curves that belong to families. By a family of curves we mean curves that are specified by equations of the same form, in an ambient projective space of the same dimension, and have the same degree and genus. We will be more precise below. The algorithm will be polynomial time in the size of the finite field, with the degree of the polynomial depending on the family.

The main result of [9] was the following generalization to Abelian varieties over finite fields of the algorithm of Schoof [10] for elliptic curves over finite fields.

**Theorem A.** Let $A$ be an Abelian variety over a finite field $\mathbb{F}_q$, given explicitly as a projective variety with an explicit addition law. Then one can compute the characteristic polynomial of the Frobenius endomorphism of $A$ in time bounded by $B_1 (\log q)^{B_2}$, where $B_1, B_2$ depend only on the embedding space dimension of $A$, the number of equations defining $A$ and the addition law, and their degrees.

Let us remark at this point that the above is not a generalization of Schoof’s result in the strict sense: the elliptic curve case of the above does not reduce to Schoof’s algorithm. In particular, we require for input a more comprehensive description of the group law (see [8, section 3]). The group law on an elliptic curve is traditionally specified by a set of rational functions defined on an open subset of $E \times E$; another set of rational functions gives the group law on the complement of that open set in $E \times E$. Such a specification suffices for Schoof. We require that several such sets of rational functions be given, each determining the group law on an open subset of $E \times E$, and such that these open subsets cover $E \times E$. See Silverman [12, Remark 3.6.1] for a discussion of how to obtain further charts of the addition law in the elliptic curve case.

In order to apply our algorithm to count points on curves over finite fields, we must first construct the Jacobian variety of the curve, and the group law upon it in the above sense. In the applications in [9], this was done following the construction of Chow [3]. Applying Chow’s construction to a curve $C$ defined over $\mathbb{Q}$, one obtains in explicit form the Jacobian $J$. One can then obtain the Jacobian of the reduction of $C$ modulo primes $p$ (for almost all $p$) by reducing $J$. These reductions are all defined by equations of the same form.

These applications were thus of the type considered here: the family of curves being the reductions mod $p$ of a fixed curve over $\mathbb{Q}$. They depended on the compatibility of Chow’s construction with reduction.

However, Chow’s construction is also compatible with other specializations; the following is a theorem of Igusa [7], who also analysed the situation in which the specialization of the curve acquires some limited singularities. A still more general statement of the universality of the Jacobian construction is due to Grothendieck [6]. A convenient reference is Milne [8, section 8]. The statement below is sufficient for our purposes.
Theorem B. Let \( C \) and \( C' \) be two irreducible curves in \( \mathbb{P}^n \) such that \( C' \) is a specialization of \( C \) over \( K \). Let \( J \) and \( J' \) be the completed generalized Jacobian varieties of \( C \) and \( C' \) respectively. Suppose that \( C \) and \( C' \) have the same arithmetic genus, and that the same reference integer is used in the constructions of the Jacobian varieties. Then \( J \) and \( J' \) have the same ambient space. If \( C \) is nonsingular, and if \( C' \) is either nonsingular or has one and only one ordinary double point, then \( J' \) is the unique specialization of \( J \) over the specialization \( C \to C' \) with reference to \( K \). \( \Box \)

To state a result on the form of the equations defining the Jacobian, we begin with a definition.

**Definition.** A *curve presentation* is a finite collection of forms (in the variables \( X_i \)) in
\[
\mathbb{Z}[a_1, \ldots, a_m][X_0, \ldots, X_n]
\]
where the \( a_j \) are indeterminates, with the property that if the \( a_j \) are considered as independent transcendental elements over \( \mathbb{Q} \), the projective algebraic set determined by the forms is an absolutely irreducible nonsingular curve. We refer to the degree and genus of this curve as the degree and genus of the presentation.

**Remark.** The restriction that the \( a_j \) be independent indeterminates is not necessary. One can allow them to be connected by some arbitrary algebraic relations. This gives a family of curves corresponding to the points of some algebraic set, \( V \), given as the zeros of an ideal \( \Lambda \). The stated definition takes \( V \) to be affine \( m \)-space.

If the ideal \( \Lambda \) is prime then we require that the curve determined by the forms over the field of fractions of \( \mathbb{Z}[a_j]/\Lambda \) (that is, the generic point) be an absolutely irreducible nonsingular curve. If \( V \) is reducible, we require the above at all constituent primes. For simplicity we will assume that our presentations are irreducible; reducible presentations can be dealt with by first finding (in a finite amount of time [11]) the constituent irreducible presentations.

Associated with a curve presentation \( C \) is a finite collection \( \Delta(C) \) of polynomials \( \Delta_i \),
\[
\Delta_i \in \mathbb{Z}[a_1, \ldots, a_m],
\]
with the following property: if the \( a_j \) are specialized to values in an arbitrary field \( K \), the projective algebraic set determined by the forms of the presentation will be an irreducible nonsingular curve, having the degree and genus of the presentation, except possibly when the polynomials \( \Delta_i \) all vanish. We call \( \Delta(C) \) the *discriminant* of the presentation.

To build the identity element of the Jacobian variety we need some rational divisor on the curve presentation. The identity element will be a suitable multiple of this divisor. This should be specified as being a point (with coordinates in \( \mathbb{Z}[a_1, \ldots, a_m] \)) in some fixed symmetric product of the curve presentation. For simplicity we will assume that it is given as the intersection of the curve with a hyperplane.

**Definition.** A *rational divisor presentation* \( D \) associated to a curve presentation \( C \) is a linear form in \( \mathbb{Z}[a_1, \ldots, a_m][X_0, \ldots, X_n] \) with the property that when the \( a_j \) as considered as independent transcendental elements over \( \mathbb{Q} \), the hyperplane determined by the form properly intersects the curve.
Associated with a curve presentation and a rational divisor presentation is a discriminant $\Delta(C, D)$ with the property that when the $a_i$ are specialized to values in an arbitrary field, the specialization of the hyperplane is a hyperplane (that is, not all the coefficients vanish), and this hyperplane properly intersects the specialization of the curve, except possibly when the polynomials comprising the discriminant all vanish.

**Definition.** An Abelian variety presentation $\mathbf{A}$ consists of the following. Let $a_j$ be indeterminates.

1. A finite collection of forms $F_i(X)$ in $\mathbb{Z}[a_1, \ldots, a_m][X_0, \ldots, X_n]$;
2. A finite collection of $n + 1$-tuples of polynomials

$$G^{(r)}(X, Y), \ r = 1, \ldots, R$$

$$G^{(r)}(X, Y) = (G^{(r)}_0(X, Y), \ldots, G^{(r)}_n(X, Y)),$$

where $X = (X_0, \ldots, X_n), Y = (Y_0, \ldots, Y_n)$ and the $G^{(r)}_i$ are homogeneous of the same degree in each system of variables.
3. An $n$-tuple $E$ of elements of $\mathbb{Z}[a_1, \ldots, a_m]$, not all zero.

These objects should further have the property that when the $a_j$ are considered as independent transcendental elements over $\mathbb{Q}$, the forms $F_i$ determine a nonsingular variety $A$, the $n$-tuple $E$ point of this variety, and the collection of $n + 1$-tuples $G^{(r)}$ a group law $A \times A \to A$ with the point $E$ as identity element. By this last condition we mean that each $n + 1$-tuple should define the group law on an open subset of $A \times A$, and that these open subsets should together cover $A \times A$.

Here again we will allow the $a_j$ to be connected by some algebraic relations. We will insist that the corresponding ideal of relations be prime.

Associated with an Abelian variety presentation is a discriminant $\Delta(\mathbf{A})$ comprising a finite number of polynomials in $\mathbb{Z}[a_1, \ldots, a_m]$. It has the property that if the $a_j$ are specialized to values in an arbitrary field, the corresponding algebraic set is an abelian variety, with $E$ a point of this variety, with group law determined by the forms $G^{(r)}$ in the above sense, and with $E$ as the identity element, except possibly when the constituent polynomials of the discriminant all vanish.

Given two systems of polynomials $\Delta_1$ and $\Delta_2$, the product $\Delta_1 \Delta_2$ will denote the system of generators for the product of the ideals. We can now state an equational version of Chow’s construction and Igusa’s theorem.

**Theorem C.** Let $\mathbf{C}$ be a curve presentation, and $\mathbf{D}$ an associated rational divisor presentation. Then we can construct an Abelian variety presentation $\mathbf{J}$ and a finite non-empty collection of polynomials $\Delta^*$ with the following property. Let the $a_j$ be specialized to values in an arbitrary field in such a way that the polynomials $\Delta^* \Delta(C) \Delta(C, D)$ do not all vanish. Then the Abelian variety presentation determines the Jacobian variety of the curve.

We call $\mathbf{J}$ the Jacobian presentation associated to the curve presentation $\mathbf{C}$, and the rational divisor presentation $\mathbf{D}$. It is completely determined by the curve presentation, the rational divisor presentation, and the choice of an auxiliary positive integer in the construction. The discriminant $\Delta(\mathbf{J})$ thus divides $\Delta^* \Delta(C) \Delta(C, D)$. 
Proof. Chow’s construction obtains equations for the variety underlying the Jacobian, and for the graph of the group law, in universal form for the presentation that specialize whenever the curve and rational divisor specialize appropriately. We desire a rational parametrization of this graph. Initially, we construct such a parametrization over $\mathbb{Z}[a_j]$. The forms $G_i^{(r)}$ we construct will give a complete description of the group law on $J \times J$ over $\mathbb{Q}(a_1, \ldots, a_m)$ when the $a_j$ are taken to be independent transcendental elements; it follows that the forms $G_i^{(r)}$ have no common zeros on $J \times J$. By the nullstellensatz, the ideal generated by the $G_i^{(r)}$ and the $F_i$ over $\mathbb{Z}[a_j]$ contains a non-zero element of $\mathbb{Z}[a_j]$; moreover, by the elimination theorem, there are a finite number of elements of $\mathbb{Z}[a_j]$ whose vanishing is a necessary and sufficient condition for the existence of a common zero (in the $X_i$) of $G_i^{(r)}$ and the $F_i$. This non-empty collection of non-zero polynomials gives the additional discriminant $\Delta^*$ of the theorem.

We thus have a presentation of the Jacobian that presents the Jacobian in a form suitable for the application of theorem A, except on the zero set of $\Delta^*$, a lower dimensional set. On the zero set of $\Delta^*$ (but outside that of $\Delta$) we have a presentation of the variety, and the graph of the group law. We can determine ([11]) the associated primes of $\Delta^*$. On each, we can consider the Abelian variety over the corresponding field of fractions, and construct a parametrization of the graph of the group law. Repeating the argument of the above proof, the forms we construct will continue to give a complete description of the group law when specialized outside a set of lower dimension, determined by some resultant system. In this way we can construct a finite number of Abelian variety presentations and systems of discriminants such that, for any specialization of the curve and divisor, outside their discriminants, one (at least) of the Abelian variety presentations will present the Jacobian of the curve. (Since the irreducible components of a given algebraic set will intersect, a given specialization of the $a_j$ might correspond to more than one of the Abelian variety presentations.)

We thus conclude the following result.

**Theorem D.** Let $C$ be a curve presentation, and $D$ an associated rational divisor presentation. Let $\Delta$ be $\Delta(C) \Delta(C, D)$. Then we can construct a finite number of Abelian variety presentations $J_1, \ldots, J_r$, corresponding to prime ideals $\Lambda_i$ in $\mathbb{Z}[a_j]$, and with discriminants $\Delta_i$, with the following property. Let the $a_j$ be specialized to values in any field in such a way that the polynomials comprising $\Delta$ do not all vanish. Then one (at least) of the Abelian variety presentations $J_i$ will have the property that the $a_j$ belong to the corresponding set $V(\Lambda_i)$, and the polynomials comprising the corresponding discriminant $\Delta_i$ do not all vanish. Further, any of the presentations with the above property will determine the Jacobian variety of the curve.

**Remark.** Let us raise the question of whether the above stratification procedure is necessary: perhaps the kind of data we desire describing the group law can be obtained universally, but we do not see how to do this.

More generally, suppose that $V$ and $W$ each represent smooth, complete varieties, parametrized by points of another variety (outside some discriminant), and determined by equations in a universal way. Suppose further that we have a universal presentation of
the graph of a morphism $V \to W$. Can we get a universal system of rational functions determining this morphism?

If the discriminant of the original presentation of the curve is of codimension 1, one might expect that the discriminant $\Delta^*$ already coincides with $\Delta$.

Let us now eliminate the divisor presentation from the data. Given a curve presentation, we can find some rational divisor presentation that generically intersects the curve properly. We now proceed by a similar process of stratification to determine divisors on the lower dimensional presentations. We thus get a version of theorem D with no rational divisor in the hypothesis.

**Theorem E.** Let $C$ be a curve presentation with discriminant $\Delta$. Then we can construct a finite number of Abelian variety presentations $J_1, \ldots, J_s$, corresponding to prime ideals $\Lambda_i$ in $\mathbb{Z}[a_j]$, and with discriminants $\Delta_i$, with the following property. Let the $a_j$ be specialized to values in any field in such a way that the polynomials comprising $\Delta$ do not all vanish. Then one (at least) of the Abelian variety presentations $J_i$ will have the property that the $a_j$ belong to the set corresponding to $\Lambda_i$, but not to the set corresponding to $\Delta_i$. Further, any of the presentations with the above property will determine the Jacobian variety of the curve.

Combining with theorem A yields the following algorithm.

**Theorem F.** Let $C$ be a curve presentation of degree $d$ and genus $g$. There exist positive integers $B_1$, $B_2$, and a deterministic algorithm, depending only on the presentation, with the following property. Let the $a_j$ be specialized to values in a finite field $\mathbb{F}_q$ such that the polynomials comprising $\Delta(C)$ do not all vanish. Then the algorithm computes the Zeta function of the curve over the finite field $\mathbb{F}_q$, and hence in particular the number of rational points on the curve over $\mathbb{F}_q$ in time bounded by $B_1 (\log q)^{B_2}$.

As an example of the above we will consider the hyperelliptic families

$$y^2 = f(x) = a_0 x^d + a_1 x^{d-1} + \cdots + a_d$$

where $d \geq 3$ is an integer. This is not a presentation according to our definition, since the equation is not homogeneous, and the plane curve is singular (if $d \geq 5$). However, a (completed) projective embedding is easily given, in universal form (see Silverman [11, exercise 2.14]). The curve is nonsingular of degree $d$ and genus $g$, where $d - 2 \leq 2g < d$ for a given specialization of the $a_j$ provided that the discriminant $\Delta(d)$ of the polynomial $f(x)$ is non-vanishing. In this case a rational divisor presentation is easily found with discriminant 1. Hence we obtain the following result.

**Theorem G.** Let $d \geq 3$ be an integer. There exist positive integers $B_1, B_2$, and a deterministic algorithm, depending only on $d$, with the following property. Let $f(x)$ have coefficients $a_j$ in a finite field $\mathbb{F}_q$, such that the discriminant $\Delta(d)$ does not vanish. Then the algorithm computes the number of points on the corresponding hyperelliptic curve over $\mathbb{F}_q$ in time bounded by $B_1 (\log q)^{B_2}$. 

A random polynomial time algorithm in the genus 2 case has been given by Adleman and Huang [1] as part of their random polynomial time primality test.

Let us also remark that the above, and all the algorithms presented here are of purely theoretical interest (if any) at the present time: not only because the algorithm of theorem A is impractical, but because the construction of the Jacobian in the desired form is impractical. As remarked earlier, this is generally avoided even in the elliptic curve case.

As a further application we can give a uniform version of theorem C of [9]. For this, we consider initially a family of projective plane curves, generically absolutely irreducible, of a certain genus and degree. We allow a finite number of singularities. All these properties will be preserved by specializations outside an appropriate discriminant. Such a family is specified by a single form $H(X, Y, Z)$ in $\mathbb{Z}[a_1, \ldots, a_m][X, Y, Z]$, with some other relations on the $a_j$ given by a (prime) ideal $\Lambda$ in $\mathbb{Z}[a_j]$.

**Theorem H.** Let $H$ be a family of plane curves, with discriminant $\Delta$. There is a deterministic polynomial time algorithm operating as follows. Taking as input a specialization of the $a_j$ in an arbitrary finite field $\mathbb{F}_q$ such that the polynomials in $\Lambda$ all vanish, but not all those in $\Delta$, the algorithm counts the number of points on the curve $H(X, Y, Z) = 0$ in $\mathbb{P}^2(\mathbb{F}_q)$.

**Proof.** We show how to obtain a finite number of curve presentations from the planar family. We begin by resolving the generic curve of the family. This gives a smooth projective curve presentation, outside of a set of lower dimension determined by an appropriate discriminant. After finding the constituent primes of the zero set of this discriminant, one resolves the corresponding curves. In this way, one produces a finite number of curve presentations that suffice to resolve every curve in the family. We can now apply theorem F. The number of rational points will be the same except for a correction due to the singular points. The correction is easily made in polynomial time, as it entails counting points in a zero dimensional algebraic set with a fixed description (see [9]).

One can state a similar theorem for families of affine plane curves. The extra accounting for the points at infinity is easily accomplished in polynomial time.

We now make some concluding remarks. Apart from using simpler input data, Schoof is able to reduce the computations needed to calculations involving univariate polynomials (essentially by operating on the Kummer variety). The algorithm of theorem A does ideal-theoretic calculations with polynomials in many variables. In this sense our result might be paraphrased as the assertion that Schoof’s idea of computing using the $\ell$-adic representations is strong enough to succeed without further assistance from simplifications in the ideal-theoretic computations.

Nevertheless one would like to investigate such simplifications in the higher dimensional cases, especially the genus or dimension 2 case that is the next simplest after elliptic curves. Explicit equations for Jacobian varieties of hyperelliptic curves of genus two, together with group laws (seemingly not in the comprehensive form we require) are now available, due to Grant [5] (degree 5), and Cassels and Flynn [2], [4] (degree 6). These provide a starting point.

Returning to theoretical questions, it is natural to pursue an algorithm of the type of theorem E applicable to a family containing all curves of a given genus.
In order to avoid subtleties connected with the moduli space, it might be desirable to work instead with a family that represents most isomorphism classes of curves, each a roughly equal number of times. Thus we could (say) leave out hyperelliptic curves \((g \geq 3)\), and also allow curves with automorphisms to be under-represented. The object would be a family of curves imitating the moduli space in a statistical sense: with respect to the distribution of coefficients of Zeta functions over finite fields. A similar question arises for Abelian varieties of a given dimension.

**Note (April 2005).** This paper was written in (March) 1991, and I have posted it “as was”. At a later point I intend to revise it and update references. My thanks to Claus Diem for his suggestions. At the time the paper was written, I was affiliated with the Department of Mathematics of Columbia University (New York).

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