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ABSTRACT
In this paper, the apparent elasticity method is used to derive the transverse effective electromechanical coupling coefficient $K_e$ of a piezoelectric rectangular thin plate with an arbitrary aspect ratio $G$ (defined as the ratio of the two transverse dimensions). A unified formula $K_e$ with regard to $G$ is obtained, and the dependence of $K_e$ on $G$ is presented and discussed. The results show that, at $G = 1$ (square thin plate), $K_e$ drops to 0 at the low-frequency branch, whereas it reaches its maximum value at the high-frequency branch. It is proposed that the maximum value of $K_e$ corresponds to the planar vibration mode. In addition, thin plates made of the PZT-4 ceramic with different aspect ratios were fabricated and measured to check the validity of the above method. For the low-frequency branch of $K_e$, the experimental results were consistent with the trend of the theoretical values. For the high-frequency branch of $K_e$ within the range of $0.2 \leq G \leq 0.55$ and $1.82 \leq G \leq 5$, however, the experimental results exhibited large deviations compared with the theoretical values owing to the negative influence of disturbing resonances. In contrast, the experimental results within the range of $0.6 \leq G \leq 1.67$ were in good agreement with the theoretical values and reached a maximum value at $G = 1$. It is suggested that the piezoelectric plates with $0.6 \leq G \leq 1.67$ should be preferred for practical applications due to their undisturbed high-frequency resonances with high values of $K_e$.

I. INTRODUCTION
In the past few decades, piezoelectric materials have been extensively employed in underwater acoustic transducers, ultrasonic transducers, vector sensors, hydrophones, actuators, energy-harvesting applications, etc. As one of the most important parameters of piezoelectric materials, the electromechanical coupling coefficient is closely related to transducer bandwidth and electromechanical conversion efficiency. Therefore, researchers pay close attention to the study of this parameter.

According to different operation conditions, the electromechanical coupling coefficient can be divided into the following two forms: the piezoelectric material coupling coefficient $K_m$ and the effective coupling coefficient $K_e$. Under static conditions, the coupling coefficient of the whole vibrator is equal to that of a small volume of the material owing to the uniformity of the energy distribution. In this case, the electromechanical coupling coefficient is defined as $K_m$ and can be calculated using Eq. (1). However, under dynamic conditions, especially around the resonance frequency, the coupling coefficient is affected not only by the piezoelectric properties of the material itself but also by the vibration modes, where the electromechanical coupling coefficient is denoted $K_e$ and can be determined from Eq. (2).
In Eq. (1), $U_m$, $U_e$, and $U_d$ are the mutual, elastic, and dielectric energies, respectively. In Eq. (2), $C_0$ is the clamped capacitance and $C_m$ is the motional capacitance in the lumped-parameter equivalent circuit around the resonance. A number of studies\textsuperscript{15–17} have derived expressions for $K_m$ and $K_e$ for various vibrations. However, these results were obtained for the vibrators with ideal dimensions. For instance, the longitudinal vibration mode requires a long bar with the longitudinal dimension much larger than the radial or transverse dimension. In contrast, in most applications, the dimension differences in all directions are not so large that strong coupling inevitably occurs. Accurate solutions of these complex vibrations from a rigorous theoretical derivation are difficult to obtain. Although finite element simulation can assist in analyzing the complex coupling vibrations, depending solely on finite element simulation is not beneficial to gaining a deeper physical insight and may result in confusion in some applications without theoretical guidance. Therefore, a simple, approximate theoretical method that can provide a solution with acceptable accuracy to the complex vibration would be helpful. Kim et al.\textsuperscript{18} have derived the formulas of the piezoelectric material coupling coefficient $K_m$ as a function of the aspect ratio for various vibration modes. Compared to $K_m$, however, the effective coupling coefficient $K_e$ provides a more accurate description of the electromechanical conversion capability of a vibration mode around the resonance where the transducer can output the maximum energy into the medium.

The transverse vibration mode of a piezoelectric rectangular thin plate and provide more accurate predictions for the high-frequency branch is discussed. These results were obtained for the vibrators with ideal dimensions. For instance, the longitudinal vibration mode requires a long bar with the longitudinal dimension much larger than the radial or transverse dimension. In contrast, in most applications, the dimension differences in all directions are not so large that strong coupling inevitably occurs. Accurate solutions of these complex vibrations from a rigorous theoretical derivation are difficult to obtain. Although finite element simulation can assist in analyzing the complex coupling vibrations, depending solely on finite element simulation is not beneficial to gaining a deeper physical insight and may result in confusion in some applications without theoretical guidance. Therefore, a simple, approximate theoretical method that can provide a solution with acceptable accuracy to the complex vibration would be helpful. Kim et al.\textsuperscript{18} have derived the formulas of the piezoelectric material coupling coefficient $K_m$ as a function of the aspect ratio for various vibration modes. Compared to $K_m$, however, the effective coupling coefficient $K_e$ provides a more accurate description of the electromechanical conversion capability of a vibration mode around the resonance where the transducer can output the maximum energy into the medium.

The transverse vibration mode of a piezoelectric rectangular thin plate has been employed in many applications, such as the ultrasonic array transducer,\textsuperscript{19–21} the macrofiber composite transducer,\textsuperscript{22} the broadband underwater acoustic transducer,\textsuperscript{23} and the piezoelectric actuator.\textsuperscript{24,25} The aspect ratio of the thin plates used in these applications plays a significant role in determining the electromechanical characteristics. In this study, the transverse effective coupling coefficient $K_e$ of a piezoelectric rectangular thin plate with an arbitrary aspect ratio $G$ was investigated using the apparent elasticity method,\textsuperscript{26} from which the complete electromechanical equivalent circuit of the transverse coupling vibration was obtained. From this circuit, the aspect ratio dependence of the resonance frequency $f_r$ was calculated for both the low- and high-frequency branches. Furthermore, $K_e$ was derived according to Eq. (2) by the lumped-parameter equivalent circuits around the resonances. In this study, not only the aspect ratio dependence of $K_e$ but also its maximum value is depicted, and the corresponding vibration mode at the resonance where the transducer can output the maximum energy into the medium.

II. THEORETICAL METHOD AND DERIVATION

A. Theoretical method for solving the coupling vibration

The transverse vibration of a piezoelectric rectangular thin plate is schematically illustrated in Fig. 1. The thin plate made of the PZT-4 ceramic is poled along the 3-direction, and the electrode surfaces are perpendicular to the 3-direction. The variables $l_1$, $l_2$, and $l_3$ are the respective dimensions of the 1-, 2-, and 3-directions, and the aspect ratio is defined as $G = l_1/l_2$. When the harmonic driving voltage $V$ is applied to the electrodes, the thin plate will vibrate along the 1- and 2-directions. It should be noted that, for the thin plate, the effect of the thickness vibration can be neglected because its resonance frequency of the 3-direction is far from the transverse vibration frequencies of the 1- and 2-directions. When $l_1$ is much larger than $l_2$ ($G > 1$) or $l_2$ is much larger than $l_1$ ($G < 1$), one-dimensional theory\textsuperscript{3,14} can be used to study the transverse vibration. However, if $l_1$ is comparable to $l_2$, the transverse vibration of the thin plate becomes very complicated, especially around $l_1 = l_2$ ($G = 1$).

The apparent elasticity method is an approximate analytical method\textsuperscript{26–28} by which the complex vibration of a thin plate can be simplified as a coupling between the two transverse vibrations of the 1- and 2-directions. All shear stresses and deformations are ignored. The mechanical coupling coefficient $n_m$, defined as $T_1/T_2$, is introduced in the theory at the beginning of the derivation ($T_1$ and $T_2$ are the axial stresses of the 1- and 2-directions) and divides the piezoelectric constitutive equation into two transverse vibration parts for the 1- and 2-directions. Then, the equivalent circuits of the two directions are derived from the piezoelectric constitutive equation and the motion equation, respectively, and both of them constitute the complete equivalent circuit of the coupling vibration. It should be noted that the two transverse vibrations are not independent, and they are coupled together by the mechanical coupling coefficient $n_m$. All derived results obtained by this method are related to $n_m$, such as the resonance frequency and the effective coupling coefficient $K_e$, and thus, the corresponding calculated results take account of the coupling between the transverse vibrations since these results are affected by the value of $n_m$.

B. Electromechanical equivalent circuit analysis

Based on the apparent elasticity method, the equation sets determining the electromechanical equivalent circuits of the 1- and 2-directions can be derived (see Appendix A for further details) from their piezoelectric constitutive equations and motion equations and can be expressed as

\begin{equation}
K_m = \frac{U_m}{\sqrt{U_e U_d}}, \quad (1)
\end{equation}

\begin{equation}
K_e = \frac{C_m}{C_0 + C_m}. \quad (2)
\end{equation}
where the corresponding impedances are

\[
Z_1 = Z_2 = jpc_1 l_1 b \tan \left( \frac{k_1 l_1}{2} \right) \left( 1 + \frac{1}{n_m} \right),
\]

\[
Z_3 = \frac{pc_1 l_1 b}{j \sin(k_1 l_1)} \left( 1 + \frac{1}{n_m} \right),
\]

\[
Z_4 = Z_5 = jpc_2 l_1 b \tan \left( \frac{k_2 l_2}{2} \right) \left( 1 + n_m \right),
\]

\[
Z_6 = \frac{pc_2 l_2 b}{j \sin(k_2 l_2)} \left( 1 + n_m \right).
\]

In equation sets (3) and (4), the first equations of both sets are called the electrical part equations, in which \( I_1 \) and \( I_2 \) are the current components of the 1- and 2-directions, \( j \) is the imaginary unit, \( \omega \) is the angular frequency, \( C_{01} \) and \( C_{02} \) denote the clamped capacitances, \( N_1 \) and \( N_2 \) are the electromechanical conversion factors of the transformers in the equivalent circuit, and \( V_{1a}, V_{1b}, V_{2a}, \) and \( V_{2b} \) are the boundary vibration velocities. The other two equations in equation sets (3) and (4) represent the vibration equations of the mechanical part, in which \( F'_{1a}, F'_{1b}, F'_{2a}, \) and \( F'_{2b} \) are the equivalent external forces at the boundaries, \( \rho \) is the density of the thin plate, \( c_1 \) and \( c_2 \) are the equivalent sound velocities, and \( k_1 \) and \( k_2 \) are the equivalent wavenumbers. Based on equation sets (3) and (4), the equivalent circuits of the transverse vibrations with respect to the 1- and 2-directions are drawn in red and blue, respectively, in Fig. 2, and both of them constitute the complete electromechanical equivalent circuit of the transverse coupling vibration. The energy transformation between electrical excitation and mechanical vibration is accomplished by the transformers. All of the variables with respect to the transverse vibration in Fig. 1 can be expressed in the equivalent circuit, such as the driving voltage \( V \) and the boundary vibration velocities \( v_{1a}, v_{1b}, v_{2a}, \) and \( v_{2b} \). In the mechanical part, vibration velocities and external forces act like currents and voltages, and thus, the impedances \( Z_1 \sim Z_6 \) can be considered the resistances.

When the vibrations are free at both ends of the 1- and 2-directions, the boundary conditions are \( F'_{1a} = F'_{1b} = 0 \) and \( F'_{2a} = F'_{2b} = 0 \). In this case, according to the equivalent circuit, the expressions for the admittances \( Y_1 \) and \( Y_2 \) of the two directions can be obtained as follows:

\[
Y_1 = j \omega c_3 l_2 b \frac{e^{k_2 l_2 l_2}}{l_2} \left[ 1 - K_{31m}^2 - K_{33m}^2 \tan(k_2 l_2/2) \right],
\]

\[
Y_2 = j \omega c_3 l_2 b \frac{e^{k_2 l_2 l_2}}{l_2} \left[ 1 - K_{32m}^2 - K_{33m}^2 \tan(k_2 l_2/2) \right],
\]

where \( K_{31m} \) and \( K_{33m} \) are the material coupling coefficients.

The resonance frequency can be calculated from Eqs. (5a) and (5b), where \( \omega = \Delta/2 \) is the one-dimensional sound velocity, and \( V_{12} = \sqrt{c_1 l_1} \) represents the Poisson ratio. It should be noted that, for the horizontal isotropic ceramic PZT-4, the resonance frequency of the smaller-dimension direction should be located at the high-frequency branch. Hence, the low-frequency and high-frequency branches represent the respective responses of the 2- and 1-directions when \( G \) is less than 1 (\( l_1 < l_2 \)), and for \( G \) greater than 1 (\( l_1 > l_2 \)), the responses of the 2- and 1-directions correspond to the high-frequency and low-frequency branches, respectively.

\[
(f l_1)_H = \omega G \sqrt{2(1 + G^2 - \Delta)},
\]

\[
(f l_1)_L = \omega G \sqrt{2(1 + G^2 + \Delta)},
\]

\[
(n_m)_H = \sqrt{(1 - G^2)l_2}/2v_{12},
\]

\[
(n_m)_L = \sqrt{(1 - G^2 - \Delta)l_2}/2v_{12},
\]

where \( \Delta = G^4 + 2(2v_{12}^2 - 1)G^2 + 1 \).
Two lumped-parameter equivalent circuits for the 1- and 2-directions around their resonances are introduced via Eqs. (5a) and (5b) and are shown in Fig. 3. The motional capacitances (see Appendix B for the detailed derivations) can be written as

\[ C_{m1} = \frac{8d_{31}^2 (1 + 1/n_{m})l_3 l_2}{\pi^2 k_b}, \quad (10a) \]
\[ C_{m2} = \frac{8d_{31}^2 (1 + n_{m})l_1 l_2}{\pi^2 s_{y} l_3}, \quad (10b) \]

and the motional inductances can be written as

\[ L_{m1} = \frac{\rho s_{31}^2 l_1 l_2}{8d_{31}^2 (1 + 1/n_{m})l_3}, \quad (11a) \]
\[ L_{m2} = \frac{\rho s_{31}^2 l_1 l_3}{8d_{31}^2 (1 + n_{m})l_2}. \quad (11b) \]

Combining Eqs. (2), (10a), and (10b) and the lumped-parameter equivalent circuits, the high-frequency branch \( K_{hf} \) and the low-frequency branch \( K_{lf} \) of the effective electromechanical coupling coefficient are composed of piecewise functions as follows:

\[ K_{lf} = \begin{cases} 
\left( \frac{C_{01}}{C_{02} + C_{00}} \right)^{1/2} & (G < 1), \\
\left( \frac{C_{01}}{C_{00}} \right)^{1/2} & (G \geq 1), 
\end{cases} \quad (12a) \]
\[ K_{hf} = \begin{cases} 
\left( \frac{C_{02}}{C_{00} + C_{01}} \right)^{1/2} & (G < 1), \\
\left( \frac{C_{02}}{C_{00}} \right)^{1/2} & (G \geq 1), 
\end{cases} \quad (12b) \]

for which the detailed expressions of \( C_{01} \) and \( C_{02} \) can be found in Appendix A.

### III. NUMERICAL RESULTS AND ANALYSIS

In order to investigate the dependence of \( f_r \) and \( K_e \) on the aspect ratio \( G \), the dimension of the 1-direction is fixed (\( l_1 = 1 \) mm) and the dimension of the 2-direction is gradually changed from 0.1 mm to 10 mm (\( l_2 = 0.1\text{–}10 \) mm), which makes \( G \) change from 0.1 to 10. The thickness \( l_3 \) should be much less than \( l_1 \) and \( l_2 \) (\( l_3 = 0.01 \) mm). The thin plates are made of the PZT-4 ceramic, and the material parameters are listed in Table I.

Using Eqs. (8a) and (8b), the dependence of the resonance frequency constant branches \( (f_r)_1 \) and \( (f_r)_2 \) on the aspect ratio \( G \) is shown in Fig. 4. As previously discussed, when \( G \) is less than 1 (\( l_1 < l_2 \)), the low-frequency and high-frequency branches correspond to the respective resonances of the 2- and 1-directions, and for \( G \) greater than 1 (\( l_1 > l_2 \)), the labels of the low-frequency and high-frequency branches should be exchanged. Linear variations of the curves can be found within the range of \( G < 0.5 \) or \( G > 2 \). However, when \( G \) tends to 1, these curves exhibit nonlinear characteristics due to the coupling effect between the transverse vibrations of the 1- and 2-directions.

The aspect ratio dependence of the two branches \((n_{m})_1\) and \((n_{m})_2\), according to Eqs. (9a) and (9b) is shown in Fig. 5. When \( G \to 0 \), \((n_{m})_1 \to 0\), \((n_{m})_2 \to 1/(\nu_{12})\) and \((n_{m})_2 \to -\infty\), \((n_{m})_1 \to \nu_{12}\) as \( G \) tends to \( \infty \). In particular, when \( G = 1\), \((n_{m})_1 = -1\) and \((n_{m})_2 = 1\). It should be noted that \( T_1/T_2 < 0 \) and \( T_1/T_2 > 0 \) correspond to the low- and high-frequency branches, respectively. In general, the tensile stress is positive and the compressive stress is negative. At the

### Table I. Material parameters of the PZT-4 ceramic.

| Parameters | Value |
|------------|-------|
| \( s_{11}^E \) | \( 12.3 \times 10^{-12} \text{ m}^2/\text{N} \) |
| \( s_{12}^E \) | \( -4.05 \times 10^{-12} \text{ m}^2/\text{N} \) |
| \( s_{13}^E \) | \( -5.31 \times 10^{-12} \text{ m}^2/\text{N} \) |
| \( s_{33}^E \) | \( 15.5 \times 10^{-12} \text{ m}^2/\text{N} \) |
| \( d_{31} \) | \( -123 \times 10^{-12} \text{ C/N} \) |
| \( d_{33} \) | \( 289 \times 10^{-12} \text{ C/N} \) |
| \( e_{33}^T \) | \( 1.15 \times 10^{-8} \text{ F/m} \) |
| \( \rho \) | \( 7500 \text{ kg/m}^3 \) |
| \( Q_m \) | \( 500 \) |
high-frequency branch, $T_1$ and $T_2$ have the same sign, which means that the stresses of the 1- and 2-directions are simultaneously tensile or compressive. Similarly, at the low-frequency branch, if the stress of the 1-direction is tensile, the stress of the 2-direction must be compressive. From this, it is concluded that the vibrations of the 1- and 2-directions are out of phase at the low-frequency branch, whereas they are in phase at the high-frequency branch. One can see that the phase relation at the high-frequency resonance seems to be contrary to the Poisson effect. It should be noted that the Poisson ratio cannot be used to predict the dynamic vibration.

Moreover, when the aspect ratio $G \to 0$ ($l_1 \ll l_2$) or $G \to \infty$ ($l_1 \gg l_2$), the vibration can be considered its one-dimensional mode. For example, after substituting $m_n = (n_m)_l = -\infty$ into the expressions with respect to $s_{21}, c_1, K_{31m}$, etc., in Appendix A, we can obtain $s_2 = s_2^{11}, c_1 = (1/\rho s_2^{11})^{1/2}$, and the square of the piezoelectric material coupling coefficient $K^2_{31m} = d_{31}^2/\epsilon_3^{k_3}k_3^{k_3} = k_3^{k_3}$, which are the same as the definitions in the IEEE Standard for the low-frequency length extensional vibration of the 1-direction.

The aspect ratio dependence of the effective electromechanical coupling coefficient $K_e$ is calculated according to Eqs. (12a) and (12b) and is shown in Fig. 6, in which the curves are symmetrical on both sides at $G = 1$ due to the horizontal isotropy of the PZT-4 ceramic. It can be seen that, at $G = 1$, $K_e$ is equal to 0 for the low-frequency branch, whereas it reaches a maximum value at the high-frequency branch. Focusing on the low-frequency branch first, for the extreme state of $G \to \infty$ (or $G \to 0$), the transverse vibration can be considered its one-dimensional mode, and the material coupling coefficient $K_{31m}$ expressed by Eq. (A7) is

$$K_{31m} = d_{31}^2/\sqrt{s_{21}^2c_1} = k_{31},$$

which is absolutely the definition of the length extensional vibration in the IEEE Standard. The effective coupling coefficient $K_e = 0.296$ according to Eq. (12a) is exactly the same as the one-dimensional result calculated by

$$K_e^2 = 1 + \frac{1}{(\pi^2/8)(1 - k_{31}^2)/k_{31}^2}. \tag{13}$$

It is important that the value of $K_e$ of the low-frequency branch reduces to 0 rapidly when $G$ approaches 1. This phenomenon can be expounded from various aspects. Initially, at $G = 1$, $n_m = (n_m)_l = -1$ results in $C_{m1} = C_{m2} = 0$, and thus, there is no energy stored in the mechanical form, which leads to a complete loss of the total input energy calculated by Eq. (12a). Additionally, the electromechanical conversion factors $N_1$ and $N_2$ (detailed expressions for $N_1$ and $N_2$ can be found in Appendix A) in the electromechanical equivalent circuit will vanish for this special state. Therefore, electrical excitation cannot be transformed into mechanical vibration. Finally, considering the relation of $K_e \leq K_m$ and $K_m = 0$ [Eq. (A7)] when $n_m = -1$, we have

$$0 \leq K_e \leq K_m = 0, \tag{14}$$

and the result $K_e = 0$ can be captured.

Another significant phenomenon is that $K_e$ reaches its maximum value at $G = 1$ for the high-frequency branch. This can be explained in a way similar to that for the low-frequency branch. When $n_m = (n_m)_1 = 1$, the motional capacitances $C_{m1}$ and $C_{m2}$ and the electromechanical conversion factors $N_1$ and $N_2$ are all maximum, which contribute to the value of $K_e$ being maximum. Furthermore, substituting $n_m = 1$ into Eq. (A7) yields

$$K_{31m}^2 = 2K_{31}^2/(1 - \nu_{12}) = k_{31}^2, \tag{15}$$

where $k_p$ is the material planar electromechanical coupling coefficient. Taking into account the relation $n_m = T_1/T_2 = 1$, the mechanical boundary condition $T_1 = T_2$ exactly corresponds to the planar vibration. For the PZT-4 ceramic, $k_p = 0.58$, which is a little higher than the value of $K_e = 0.524$ calculated in this study. This is reasonable as $K_e$ is less than or equal to $K_m$.

Moreover, for the low-frequency branch, the relative error of $K_e$ between the one-dimensional theory results calculated by Eq. (13)
and the coupling theory results from Eq. (12a) has already reached 5% at $G = 0.42$ or $G = 2.38$, and the relative error is even greater when $0.42 \leq G \leq 2.38$. Therefore, in this study, coupling theory can provide a more accurate prediction of $K_e$ in the range of $0.42 \leq G \leq 2.38$.

IV. EXPERIMENTAL RESULTS AND DISCUSSION

An experiment was conducted to validate the theoretical method proposed in this study by measuring the resonance frequency $f_e$ and the effective coupling coefficient $K_e$. All of the thin plates with different aspect ratios, e.g., $G = 0.2$ ($l_1 = 20$ mm, $l_2 = 100$ mm), $G = 0.5$ ($l_1 = 40$ mm, $l_2 = 80$ mm), $G = 1$ ($l_1 = 40$ mm, $l_2 = 40$ mm), and $G = 5$ ($l_1 = 100$ mm, $l_2 = 20$ mm), were obtained by cutting large PZT-4 thin plates ($100$ mm $\times$ $40$ mm $\times$ $3$ mm). As the mechanical quality factor $Q_m$ of the PZT-4 ceramic is sufficiently high, $K_e$ can be calculated from the following equation:

$$K_e^2 = \frac{(f_e^2 - f_r^2)}{f_r^2}, \quad (16)$$

where the resonance frequency $f_r$ and antiresonance frequency $f_a$ were obtained from the admittance spectra measured by an impedance analyzer (IM3570, HIOKI, Japan).

The experimental results for $f_e$ and $K_e$ are plotted in Figs. 4 and 6, respectively. It can be seen that the agreement between the experimental and theoretical values for $f_r$ is very good. However, some deviations can be observed when the experimental results for $K_e$ are compared with the theoretical values. Here, we only discuss the cases where $G \leq 1$ as the curves of $K_e$ are symmetrical in Fig. 6. Focusing on the low-frequency branch first, with the increase in $G$ from 0.2 to 1, the experimental results for $K_e$ decrease gradually from 0.285 to 0.0, which is consistent with the trend of the theoretical values. For the high-frequency branch of $K_e$, the experimental results within the range of $0.6 \leq G \leq 1$ are in good agreement with the theoretical values and reach a maximum value 0.461 at $G = 1$. However, the experimental results for $K_e$ in the circle ($0.2 \leq G \leq 0.55$) show large deviations from the corresponding theoretical values. In order to explain this inconsistency, four representative admittance spectra are selected from the experimental results and are shown in Fig. 7, in which the resonance frequencies are marked by the perpendicular dotted lines. The theoretical values of the admittance spectra calculated by Eqs. (5a) and (5b) are also shown in Fig. 7 for comparison.

It can be observed in Fig. 7(a) that an unwanted disturbing resonance (in the circle) appears near the high-frequency resonance in the experimental results and leads to the reduction in $K_e$ calculated by Eq. (16). An even worse case is that the negative influence of the disturbing resonances on the high-frequency resonances is irregular within the range of $0.2 \leq G \leq 0.55$ (or $1.82 \leq G \leq 5$). Thus, the experimental results for $K_e$ for the high-frequency resonances are disordered, as shown in the circles in Fig. 6. The results suggest that the thin plates with aspect ratios of $0.6 \leq G \leq 1.67$ should be preferred for further applications due to the fact that their high-frequency resonances are undisturbed.

Another important finding of Fig. 7 is that the resonance frequencies and the antiresonance frequencies of the 2-direction approach each other with increasing $G$, which results in the reduction in $K_e$ calculated by Eq. (16). Therefore, for the low-frequency branch in Fig. 6, the experimental results for $K_e$ are gradually diminished as $G$ tends to 1. In Fig. 7(d), the low-frequency resonance of the 2-direction disappears at $G = 1$ owing to the fact that $f_e = f_r$, and thus, $K_e$ equals 0. The mechanism of how the 2-direction resonance disappears is disclosed in our work, but this phenomenon can easily be ignored in simulation and experimental investigations. In contrast, the resonance frequencies and antiresonance frequencies of the 1-direction gradually separate when $G$ approaches 1, which makes $K_e$ of the high-frequency branch increase, and the planar vibration mode is located at the resonance of the 1-direction in Fig. 7(d).

![Fig. 7. Admittance curves of various thin plates with different aspect ratios: (a) $l_1 = 40$ mm, $l_2 = 72.7$ mm, $G = 0.55$; (b) $l_1 = 40$ mm, $l_2 = 57.2$ mm, $G = 0.7$; (c) $l_1 = 40$ mm, $l_2 = 44$ mm, $G = 0.91$; (d) $l_1 = 40$ mm, $l_2 = 40$ mm, $G = 1$ (solid lines show the experimental results, blue and red dashed lines show the theoretical values of the 2- and 1-directions, 1 and 2 mark the two resonances of the experimental results, and the 2-direction resonance disappears at $G = 1$).](image-url)
The vibration is pure, and a disturbing resonance cannot exist around it; the experimental value of the effective coupling coefficient \( K_e \) reaches a maximum value of 0.461.

Additionally, it is worth noting that there are some discrepancies between the theoretical values and the experimental results for the admittance spectra of all thin plates in Fig. 7. Compared with the experimental results, the theoretical values of the 2-direction resonances are located at lower frequencies, whereas the frequencies of the 1-direction resonances are all higher than the experimental results. The differences between them become greater when the aspect ratio \( G \) tends to 1. A primary explanation for these differences is that the apparent elasticity method used in this paper is an approximate analytical method. In theoretical derivations, the complex transverse vibration has been simplified as the coupling vibration between the two transverse directions and all shear stresses and deformations are not considered, which produce differences between the theoretical values and the experimental results. Despite all this, in the admittance spectra in Fig. 7, the relative errors of the resonance frequency \( f_i \) between the experimental results and the theoretical values are less than 5% for all plates. In general, these differences are acceptable, and the theoretical values can provide useful predictions for practical applications.

V. CONCLUSION

Based on a rectangular thin plate of the PZT-4 ceramic with an arbitrary aspect ratio \( G \), unified formulas for the transverse effective electromechanical coupling coefficient \( K_e \) and resonance frequency \( f_i \) have been derived by the apparent elastic method. The aspect ratio dependence of \( K_e \) and \( f_i \) are presented and discussed. The results show that, at \( G = 1 \), the value of \( K_e \) of the low-frequency branch is equal to 0, and thus the thin plate has no capability for electromechanical conversion. However, the \( K_e \) value of the high-frequency branch attains the maximum value corresponding to the planar vibration mode of the square thin plate. Moreover, coupling theory has advantages over one-dimensional theory and can calculate \( f_i \) and \( K_e \) for rectangular thin plates with arbitrary aspect ratios. This method is also effective for other widely used PZT ceramics having the same crystal symmetry as the PZT-4 ceramic. Finally, an experiment was carried out to estimate the validity of the theoretical method used in this paper. It was confirmed that the formulas obtained can provide effective predictions for the resonance frequencies and effective electromechanical coupling coefficients for the transverse vibrations of rectangular thin plates with arbitrary aspect ratios.

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APPENDIX A: DERIVATION OF THE ELECTRICAL AND MECHANICAL PART EQUATIONS FOR THE 1- AND 2-DIRECTIONS

As is shown in Fig. 1, the transverse electric field components \( E_1 \) and \( E_2 \) are all 0 when the voltage \( V = V_0 e^{j\omega t} \) is applied to the electrodes. The axial stress component \( T_3 \) is assumed to be 0 for a thin plate, and all shear stress components are ignored, i.e., \( T_3 = T_4 = T_5 = T_6 = 0 \). Accordingly, the piezoelectric constitutive equations \(^{13} \) of the thin plate are simplified significantly and listed as follows:

\[
\begin{align*}
S_1 &= s_{31} T_1 + s_{31} E_3, \quad (A1a) \\
S_2 &= s_{52} T_1 + s_{52} E_3, \quad (A1b) \\
D_3 &= d_{33} T_1 + d_{33} E_3. \quad (A1c)
\end{align*}
\]

where \( s_{ij} \) and \( d_{ij} \) denote the elastic compliance constants and the piezoelectric constants \( (i, j = 1, 2, 3) \). \( T_n \) and \( S_n \) are the stress and strain components \( (n = 1, 2) \), \( E_3 \) and \( D_3 \) represent the electric field and the electric displacement along the poling direction, and \( \varepsilon_{33} \) is the dielectric constant.

The piezoelectric equations of the 1- and 2-directions can be obtained after substituting \( n_m = T_1/T_2 \) into Eqs. (A1a)–(A1c) and are shown as Eqs. (A2a), (A2b), (A3a), and (A3b), respectively,

\[
\begin{align*}
S_1 &= s_X T_1 + d_{31} E_3, \quad (A2a) \\
D_3 &= d_{31} (1 + 1/n_m) T_1 + \varepsilon_{33} E_3, \quad (A2b) \\
S_2 &= s_Y T_2 + d_{33} E_3, \quad (A3a) \\
D_3 &= d_{33} (1 + n_m) T_2 + \varepsilon_{33} E_3. \quad (A3b)
\end{align*}
\]

Here, \( s_X = s_{31} + s_{52}/n_m \) and \( s_Y = s_{11} + s_{52} n_m \) are called the apparent elasticity constants.

The motion equations of the two directions are

\[
\begin{align*}
\rho \frac{\partial^2 \xi_1}{\partial t^2} &= \frac{\partial T_1}{\partial x}, \quad (A4a) \\
\rho \frac{\partial^2 \xi_2}{\partial t^2} &= \frac{\partial T_2}{\partial y}, \quad (A4b)
\end{align*}
\]

where \( \rho \) is the density of the plate and \( \xi_1 \) and \( \xi_2 \) are the respective displacement functions of the 1- and 2-directions. For harmonic vibration, the displacement function is expressed as \( \xi = \xi_0(x) e^{j\omega t} \) and the vibration velocity is \( \gamma = \partial \xi / \partial t = j\omega \xi_0(x) \), where \( \xi_0(x) \) is the amplitude, \( j \) is the imaginary unit, \( \omega \) is the angular frequency of the vibration, and \( e^{j\omega t} \) is the time-dependent factor. For simplicity, the time-dependent factor is omitted in the following derivations. The electrical and mechanical part equations of the 1-direction are derived in detail, while those of the 2-direction are given directly, following the same steps.

1. Derivation of the electrical part equation

Using Eq. (A2a), we have

\[
T_1 = \frac{S_1 - d_{31} E_3}{s_X}. \quad (A5)
\]

Substituting Eq. (A5) into Eq. (A2b), the expression for \( D_3 \) can be obtained as

\[
D_3 = \frac{d_{31} (1 + 1/n_m)}{s_X} S_1 + \varepsilon_{33} E_3. \quad (A6)
\]
For harmonic excitation, the current component $I_c$ can be written as

$$I_1 = dQ/dt = jωQ = jω \int_0^h \int_0^l D_3 \, dx \, dy,$$

where $Q$ is the surface charge on the electrode. The value of the electric displacement $D_3$ is equal to the surface density of the free charge on the electrode. According to this, in Eq. (A8), the charge $Q$ on the electrode can be computed from the area integral of $D_3$. Combining the relation $S_1 = ∂ξ_1/∂x$ and the boundary conditions,

$$v_1|_{x=0} = jωξ_1|_{x=0} = -v_{1a}, \quad v_1|_{x=l_1} = jωξ_1|_{x=l_1} = v_{1b},$$

$I_1$ can be obtained as the following equation after substituting Eq. (A6) into Eq. (A8):

$$I_1 = jωC_0 V + N_1 (v_{1a} + v_{1b}),$$

where $C_0 = \frac{\varepsilon_3}{\varepsilon_3 l_2 l_3}$ denotes the clamped capacitance, $N_1 = d_{31} l_2 (1 + 1/n_m)/s_x$ is the electromechanical conversion factor of the equivalent circuit, and $V = E l_3$ is the driving voltage.

### 2. Derivation of the mechanical part equations

Substituting Eq. (A5) into Eq. (A3), we have

$$\frac{∂^2 ξ_1}{∂t^2} = c_1^2 \frac{∂^2 ξ_1}{∂x^2},$$

where $c_1 = (1/ρs_x)^{1/2}$ is the equivalent sound velocity. For harmonic excitation, Eq. (A11) can be transformed to

$$\frac{∂^2 ξ_1}{∂t^2} = -k_1^2 ξ_1,$$

where $k_1 = ω/c_1$ is the equivalent wavenumber. The general solution of Eq. (A12) is

$$ξ_1 = A \sin(k_1 x) + B \cos(k_1 x).$$

Taking into account the boundary condition given by Eq. (A9), we get

$$A = \frac{v_{1b} + v_{1a} \cos(k_1 l_1)}{jω \sin(k_1 l_1)}; \quad B = \frac{v_{1a}}{jω}.$$ 

Then, Eq. (A13) can be rewritten as

$$ξ_1 = -v_{1a} \sin[k_1 (l_1 - x)] + v_{1b} \sin(k_1 x).$$

Combining Eq. (A14) with Eq. (A5), the external forces $F_{1a}$ and $F_{1b}$ at the boundaries can be expressed as

$$F_{1a} = -(T_{1\mid x=0}) l_2 l_3 = -\frac{1}{s_x} \left( \frac{∂ξ_1}{∂x} \right)_{x=0} l_2 l_3 + \frac{d_{31}}{s_x} l_3 E_3,$$

$$F_{1b} = -(T_{1\mid x=l_1}) l_2 l_3 = -\frac{1}{s_x} \left( \frac{∂ξ_1}{∂x} \right)_{x=l_1} l_2 l_3 + \frac{d_{31}}{s_x} l_3 E_3,$$

where $T_{1\mid x=0}$ and $T_{1\mid x=l_1}$ are the axial stresses at the two ends of the thin plate, and the products of the stresses and side area, $(T_{1\mid x=0}) l_2 l_3$ and $(T_{1\mid x=l_1}) l_2 l_3$, represent the inner forces at the two ends generated by the vibration. At the boundaries, the inner forces should have identical amplitudes to the external forces, but their directions are opposite. Substituting Eq. (A14) into Eqs. (A15a) and (A15b) yields

$$F_{1a} = -\frac{ρ c_1 l_2 l_3}{\sin(k_1 l_1)} \left( 1 + \frac{1}{n_m} \right) (v_{1a} + v_{1b}),$$

$$F_{1b} = -\frac{ρ c_1 l_2 l_3}{\sin(k_1 l_1)} \left( 1 + \frac{1}{n_m} \right) (v_{1b} + N_1 V),$$

which are called the mechanical part equations. Here, $F_{1a} = F_{1a}(1 + 1/n_m)$ and $F_{1b} = F_{1b}(1 + 1/n_m)$ are the equivalent external forces.

The electrical part equation and the mechanical part equations of the 2-direction are expressed as

$$I_2 = jωC_0 V + N_2 (v_{2a} + v_{2b}),$$

$$F_{2a} = -\frac{ρ c_2 l_2 l_3}{\sin(k_2 l_2)} \left( 1 + n_m \right) (v_{2a} + v_{2b}),$$

$$F_{2b} = -\frac{ρ c_2 l_2 l_3}{\sin(k_2 l_2)} \left( 1 + n_m \right) (v_{2b} + N_2 V),$$

where $C_{02} = \frac{ε_3^2}{ε_3 (1 - K_{23m}^2)/l_3}$ is the clamped capacitance, in which $K_{23m} = d_{31}^2 (1 + n_m) / (ε_{3Z} s_x)$ is the square of the piezoelectric material coupling coefficient, $N_2 = d_{31} l_2 (1 + n_m)/s_x$ is the electromechanical conversion factor, $c_2 = (1/ρs_y)^{1/2}$ is the equivalent sound velocity, $k_2 = ω/c_2$ is the equivalent wavenumber, $v_{2a}$ and $v_{2b}$ are the boundary vibration velocities at $y = 0$ and $y = l_2$, and $F_{2a} = F_{2a}(1 + n_m)$ and $F_{2b} = F_{2b}(1 + n_m)$ are the equivalent external forces.

### APPENDIX B: DERIVATION OF $C_m$, $L_m$ AND $C_m$ IN THE LUMPED-PARAMETER EQUIVALENT CIRCUITS AROUND THE RESONANCES OF THE 1- AND 2-DIRECTIONS

The derivations of $C_{m1}$ and $L_{m1}$ for the 1-direction are given in detail, while those for the 2-direction are given directly and are obtained in a similar way. From Eq. (5a), the admittance of the 1-direction can be divided into two parts, i.e., the static admittance $Y_{01}$ and the motional admittance $Y_{m1}$,

$$Y_1 = Y_{01} + Y_{m1},$$

where

$$Y_{01} = jωε_{31} l_2 (1 - K_{13m}^2)/l_3 = jωC_{01},$$
Comparing Eq. (B7) with Eq. (B11),

\[ \tan(k_1 l_1/2) = \tan\left(\frac{\pi}{2} \frac{\omega}{\omega_1} \right) \]

where \( \omega_1 = \pi c_1 / l_1 \) is the resonance angular frequency. Around the resonance, we have \( \Delta \omega \to 0 \) and \( \omega \to \omega_1 \). Substituting Eq. (B4) into \( \tan(k_1 l_1/2) \) gives

\[ \tan(k_1 l_1/2) = \tan\left(\frac{\pi}{2} \frac{\omega}{\omega_1} \right) = \tan\left[\frac{\pi}{2} \left(1 + \frac{\Delta \omega}{\omega_1}\right)\right] = -1/\tan\left(\frac{\pi \Delta \omega}{2 \omega_1}\right). \]

Around the resonance \( (\Delta \omega \to 0) \), Eq. (B5) can be written as

\[ \tan(k_1 l_1/2) \approx -\frac{1}{\pi \Delta \omega / \omega_1}. \]

and then the motional admittance \( Y_{m1} \) is given as

\[ Y_{m1} \approx -j \omega_1 \frac{4 \pi \epsilon_0 c_1 l_1^2}{\pi^2 l_1^2} \frac{\omega_1}{\Delta \omega}. \]

\[ Y_{m1} = \frac{1}{Z_{m1}} = \frac{1}{j \left(\omega L_{m1} - 1/\omega C_{m1}\right)}. \]

In the lumped-parameter equivalent circuit, \( C_{m1} \) and \( L_{m1} \) satisfy

\[ \omega_1^2 = \frac{1}{L_{m1} C_{m1}}. \]

Substituting Eq. (B9) into Eq. (B8), we have

\[ Y_{m1} = -j \omega \sqrt{C_{m1}} \frac{\omega_1}{\omega + \omega_1} (\omega - \omega_1). \]

Around the resonance \( (\Delta \omega \to 0) \) and \( \omega \to \omega_1 \), Eq. (B10) can be written as

\[ Y_{m1} \approx -j \omega \sqrt{C_{m1}} \frac{\omega_1}{2 \Delta \omega}. \]

Comparing Eq. (B7) with Eq. (B11), \( C_{m1} \) can be obtained as

\[ C_{m1} = \frac{8 d_{31}^2 (1 + 1/n_m) h_1 l_1}{\pi^2 s_4 l_1}. \]

According to Eq. (B9), \( L_{m1} \) is given as

\[ L_{m1} = \frac{\rho s_4^2 h_1 l_1}{8 d_{31}^2 (1 + 1/n_m) l_1^2}. \]

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