Macroscopic discrete modelling of stochastic reaction-diffusion equations on a periodic domain

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Abstract

Dynamical systems theory provides powerful methods to extract effective macroscopic dynamics from complex systems with slow modes and fast modes. Here we derive and theoretically support a macroscopic, spatially discrete, model for a class of stochastic reaction-diffusion partial differential equations with cubic nonlinearity. Dividing space into overlapping finite elements, a special coupling condition between neighbouring elements preserves the self-adjoint dynamics and controls interelement interactions. When the interelement coupling parameter is small, an averaging method and an asymptotic expansion of the slow modes show that the macroscopic discrete model will be a family of coupled stochastic ordinary differential equations which describe the evolution of the grid values. This modelling shows the importance of subgrid scale interaction between noise and spatial diffusion and provides a new rigorous approach to constructing semi-discrete approximations to stochastic reaction-diffusion partial differential equations.

1 Introduction

In modelling complex systems we often desire to model the effective macroscopic dynamics. Some cases can be extracted from the full, or microscopic,
description by methods such as averaging, invariant manifold reduction and homogenization \[4, 13\] e.g.

Stochastic partial differential equations (SPDEs) are widely studied in modeling, analyzing, simulating and predicting complex phenomena in many fields of nonlinear science \[2, 5, 15, 23\] e.g.]. Recently, macroscopic reduction for dissipative SPDEs, with two widely separated timescales, has been studied by the dynamical systems theory of stochastic invariant manifolds \[8, 9, 17\] e.g. and by averaging methods \[19, 20\] e.g.]. Moreover, invariant manifold theory also applies to generate a macroscopic discrete model of deterministic and stochastic dissipative PDEs \[7, 10, 12\] e.g., the so-called holistic finite differences. Roberts \[11\] recently extended the approach to ensure macroscopic discrete models preserve important self-adjoint properties of the fine scale dynamics for deterministic systems.

Here we address the macroscopic discrete modelling of dissipative SPDEs and develop a novel rigorous approach. We consider reaction-diffusion in a one dimensional spatial domain driven by a noise which is white in time and with spatial structure. Let the non-dimensional spatial interval \(I = [0, L]\) with length \(L > 0\), and let \(L^2(I)\) be the Lebesgue space of square integrable real valued functions on \(I\). Consider the following non-dimensional stochastic reaction-diffusion equation for a stochastic field \(u(x, t)\), of period \(L\) in space \(x\),

\[
\begin{align*}
\partial_t u &= \partial_{xx} u + \alpha(u - u^3) + \sigma \partial_t W \quad \text{on } I, \quad (1) \\
u(0, t) &= u(L, t) \quad \text{and} \quad u_x(0, t) = u_x(L, t), \quad (2)
\end{align*}
\]

where \(W(x, t)\) is an \(L^2(I)\) valued \(Q\)-Wiener process defined on a complete probability space \((\Omega, \mathcal{F}, P)\) which is detailed in the next section.

The spatial domain \(I\) is divided into \(M\) elements and consequently a set of \(M\) fields are defined, one on each of these elements. In order to preserve the self-adjoint property of the linear operator defined on the elements we impose special interelement coupling conditions with a strength parametrised by \(\gamma\).

The system dynamics are expanded in the interelement coupling \(\gamma\) so that based upon the case of weak coupling, that is, small \(\gamma > 0\), an asymptotic approximation and an averaging method derive a family of coupled stochastic differential equations (SDEs) which describe the evolution of grid values; that is, the amplitude of the system on each element. These SDEs are a discrete stochastic model of the continuous space stochastic system \((1) - (2)\). Our discrete SDE model highlights the macroscopic influence of ‘subgrid’ interactions between noise and spatial diffusion in the SPDE.

The simplest conventional finite difference approximation of the SPDE \((1)\) on a regular grid in \(x\), say \(X_j = jh\) for some constant grid spacing \(h\), is

\[
dU_j = \frac{1}{h^2}(U_{j+1} - 2U_j + U_{j-1}) dt + \alpha(U_j - U_j^3) dt + \sigma dW_j \quad (3)
\]
where \( U_j(t) \) is the grid value of the field \( u(x,t) \) at the grid points \( X_j \), and similarly \( W_j(t) = W(X_j, t) \). However, our analysis herein recommends that a more accurate closure incorporates stochastic influences from the neighbour grids as in the Ito system of SDEs

\[
\begin{align*}
    dU_j &= \frac{1}{h^2} (U_{j+1} - 2U_j + U_{j-1}) dt + (\hat{\alpha} U_j - \alpha U_j^3) dt + \sigma \, dW_j \\
    &+ 3\sqrt{2}U_j d\tilde{W}_j + \frac{\sigma}{4} \left( dW_{j+1} - 2dW_j + dW_{j-1} \right). 
\end{align*}
\]

(4)

The second terms in the last line of the above SDE system reflect interaction between noise and spatial diffusion. Terms in \( \tilde{W}_j \) and \( \hat{\alpha} \) are due to the microscopic, subgrid scale, stochastic interactions discussed in Sections 5.

In order to generate a macroscopic discrete model we divide the domain into finite overlapping elements and choose special coupling boundary conditions, Section 2. Such interelement coupling rules were first introduced by Roberts [11] to construct spatially discrete models of deterministic dynamics. One important property of this interelement coupling is the preservation of self-adjoint symmetry in the underlying spatial dynamics. Moreover, the strength of the coupling is parametrised by \( \gamma \), \( 0 \leq \gamma \leq 1 \): when the coupling parameter \( \gamma \) is small, the coupling is weak and the system separates into ‘uninterestingly’ decaying fast parts and the relevant slow parts, Section 5. Then an averaging method [19] derives a reduced model which describes the evolution of the overall amplitude of (1)–(2) on the whole domain. Further, an analysis on eigenfunctions obtains a reduced model describing the evolution of the local amplitude on each element. This model is the macroscopic discrete approximation to (1)–(2) expressed in (4).

In this approach one difficulty is to construct from the original spatio-temporal noise \( W(x, t) \) a Wiener process \( W_j^\gamma(x, t) \) on each element. A natural method is expanding \( W(x, t) \) by the eigenfunctions of the linear operator \( \mathcal{L}_\gamma \) on each element, Section 2, which has analogues in the method of finite elements [16]. This construction shows that for small \( h \) and small coupling \( \gamma \), the stochastic force on the first mode of \( \mathcal{L}_\gamma \) approximates the grid values \( W(X_j, t) \). Moreover, analysis of the eigensystem of \( \mathcal{L}_\gamma \) for small \( \gamma \) shows the slow parts of the system dominate: the fast parts converge to quasi-equilibrium with rate \( 1/h^2 \). Then by this and the construction of \( W_\gamma(x, t) \), the macroscopic discrete reduced model is proved to be consistent to the stochastic reaction-diffusion equations (1)–(2) as the element size \( h \to 0 \).

Another difficulty is that the linear operator \( \mathcal{L}_\gamma \), defined in (10), varies with the coupling parameter \( \gamma \). So the continuity of the linear operator \( \mathcal{L}_\gamma \) in \( \gamma \) is needed. Section 4 argues that the graph convergence of \( \mathcal{L}_\gamma \) as \( \gamma \to 0 \) ensures the continuity of eigenfunctions and eigenvalues in \( \gamma \), then an asymp-
totic expansion of the first eigenfunction in $\gamma$ shows that the grid value is the amplitude of the system on the element. Then by averaging and the expansion of the first eigenfunction of $L_{\gamma}$, the macroscopic reduced model is in the first eigenspace of $L_{\gamma}$, which is varying with respect to $\gamma$. So last we project the reduced model to the first eigenspace of $L_{0}$, the basic mode, to derive the macroscopic model. In this approach one interesting phenomenon is that the effect of noise in the subgrid scale fast modes is transmitted into the macroscopic slow modes by the projection. Numerical simulations confirm such transmittal [10].

2 Overlapping finite elements and coupling boundary conditions

This section divides the spatial domain $I$ into $M$ overlapping elements with grid spacing $h$. Let the $j$th element

$$I_{j} = [X_{j} - h, X_{j} + h]$$

with grid points $X_{j} = jh$ on each element $I_{j}$, $j = 1, 2, \ldots, M$. Here for the periodic boundary condition we use the notation $X_{j \pm M} = X_{j}$. Let $u_{j}(x)$ denote the field on the element $I_{j}$, $j = 1, 2, \ldots, M$. Denote by $f(u) = -u^{3}$.

Then, modified from the SPDE (1), consider the following system of SPDES defined on elements $I_{j}$, $j = 1, 2, \ldots, M$,

$$\partial_{t}u_{j}^{\gamma}(x, t) = \partial_{xx}u_{j}^{\gamma}(x, t) + \alpha\gamma^{2}u_{j}^{\gamma}(x, t) + \alpha f(u_{j}^{\gamma}(x, t)) + \sigma\partial_{t}W_{j}^{\gamma}(x, t) \quad \text{on} \; I_{j}, \; (5)$$

with the following interelement coupling conditions on the fields parametrised by $\gamma$, and with $\gamma' + \gamma = 1$,

$$u_{j}^{\gamma}(X_{j \pm 1}, t) = \gamma' u_{j}^{\gamma}(X_{j}, t) + \gamma u_{j \pm 1}^{\gamma}(X_{j \pm 1}, t), \quad j = 1, 2, \ldots, M, \; (6)$$

and coupling of the first spatial derivative, denoted by subscript $x$,

$$u_{j,x}^{\gamma}(X_{j-1}, t) - u_{j,x}^{\gamma}(X_{j+1}, t) + \gamma u_{j-1,x}^{\gamma}(X_{j}, t) - \gamma u_{j+1,x}^{\gamma}(X_{j}, t) - \gamma' u_{j,x}^{\gamma}(X_{j-1}, t) + \gamma' u_{j,x}^{\gamma}(X_{j+1}, t) = 0, \quad j = 1, 2, \ldots, M, \; (7)$$

with, to account for $L$-periodicity of solutions,

$$u_{j \pm M}^{\gamma}(x \pm L, t) = u_{j}^{\gamma}(x, t) \quad (8)$$

$$u_{j,\pm M,x}^{\gamma}(x \pm L, t) = u_{j,x}^{\gamma}(x, t). \; (9)$$
The coupling parameter $\gamma$ controls the flow of information between the two adjacent elements: when the coupling $\gamma = 0$, adjacent elements are decoupled; when $\gamma = 1$, the system is full coupled and (5)–(7) is equivalent to the dynamics of the physical stochastic reaction-diffusion equation (1)–(2), see Section 3.

The noise fields $W_j(x, t)$ defined on each element $I_j$, $j = 1, \ldots, M$, are infinite dimensional Wiener process which are detailed later from $W(x, t)$.

Related but different interelement coupling boundary conditions empowered an earlier exploration of the non-self-adjoint interaction between noise, nonlinear advection and spatial diffusion in discretely modelling the stochastic Burgers’ equation [10].

For our purposes, first we introduce a mathematical framework for system (5)–(7). Let $H_j = L^2(I_j)$ be the set of all square integrable function on $I_j$ and $V_1 = H^1(I_j)$. Denote by $H = \prod_{j=1}^{M} H_j$, $V = \prod_{j=1}^{M} V_j$ and $H^\alpha = \prod_{j=1}^{M} H^\alpha_j$, $\alpha > 0$.

Here $H^\alpha_j$ denotes the usual Sobolev space $W^{2,\alpha}(I_j)$ [14]. Define the inner product $\langle \cdot, \cdot \rangle$ on $H$ as the sum of the element integrals

$$\langle u, v \rangle = \sum_{j=1}^{M} \left[ \int_{X_{j-1}}^{X_j} u_j(x)v_j(x) \, dx + \int_{X_j}^{X_{j+1}} u_j(x)v_j(x) \, dx \right]$$

for any $u, v \in H$ with $u = (u_j)$ and $v = (v_j)$. And denote by $\| \cdot \|_0$ the product $L^2$-norm on space $H$. For any $\alpha \in \mathbb{Z}^+$ denote the semi-norm on $H^\alpha$ as

$$\| u \|_\alpha^2 = \sum_{j=1}^{M} \| \partial_x^\alpha u_j \|_0^2, \quad u = (u_j) \in H^\alpha.$$  

For the system (5)–(7) we introduce the family of functional spaces

$$\mathcal{H}_\gamma = \{(u_j) \in H : u_j(X_j \pm 1, t) = \gamma' u_j(X_j, t) + \gamma u_{j \pm 1}(X_{j \pm 1}, t), \quad j = 1, 2, \ldots, M, \quad \text{with } L\text{-periodicty (8)} \}$$

and the subspaces

$$\mathcal{V}_\gamma = \{(u_j) \in V \cap \mathcal{H}_\gamma : u_{j,x}(X_j \pm 1, t) + \gamma u_{j-1,x}(X_j, t) - \gamma u_{j,x}(X_j, t) - \gamma' u_{j \pm 1,x}(X_{j \pm 1}, t) + \gamma' u_{j+1,x}(X_{j+1}, t) = 0, \quad j = 1, 2, \ldots, M, \quad \text{with } L\text{-periodicty (9)} \}.$$

Then define the second order differential operator $\mathcal{L}_\gamma : D(\mathcal{L}_\gamma) \subset \mathcal{V}_\gamma \to \mathcal{H}$ by

$$\mathcal{L}_\gamma u = \left( \frac{\partial^2 u_j}{\partial x^2} \right) \quad \text{for all } u = (u_j) \in D(\mathcal{L}_\gamma).$$  

(10)
By a basic calculation [11], $-\mathcal{L}_\gamma$ is a self-adjoint second order operator. A direct calculation yields that for any $u = (u_j) \in D(\mathcal{L}_\gamma)$

$$\langle -\mathcal{L}_\gamma u, u \rangle = \sum_{j=1}^{M} \|u_j\|_1^2 \geq 0 \quad (11)$$

which establishes the positivity of operator $-\mathcal{L}_\gamma$. Then there are coupling dependent eigenfunctions $\{ (e_{j,k}^\gamma(x)) \}_{k=0}^\infty$ which form a standard orthonormal system in space $\mathcal{H}_\gamma$ and a sequence of real numbers $0 < \lambda_0(\gamma) \leq \lambda_1(\gamma) \leq \cdots$ such that

$$\mathcal{L}_\gamma(e_{j,k}^\gamma) = -\lambda_k(\gamma)(e_{j,k}^\gamma), \quad k = 0, 1, \ldots, \quad 0 < \gamma \leq 1. \quad (12)$$

Moreover, $\mathcal{L}_\gamma$ is an infinitesimal generator of a $C_0$ semigroup; denote this semigroup by $\{ S_\gamma(t) \}_{t \geq 0}$. By the positivity of $-\mathcal{L}_\gamma$ we define $(-\mathcal{L}_\gamma)^\alpha$ for any exponent $\alpha > 0$ by

$$(-\mathcal{L}_\gamma)^\alpha u = \sum_k [\lambda_k(\gamma)]^\alpha (u_{j,k}e_{j,k}^\gamma)$$

for $u = (u_j) = (\sum_k u_{j,k}e_{j,k}^\gamma)$. Denote by $\mathcal{H}_\gamma^\alpha = D((-\mathcal{L}_\gamma)^\alpha)$ and define the semi-norm $\| \cdot \|_{\alpha,\gamma}$ in space $\mathcal{H}_\gamma^\alpha$ by

$$\|u\|_{\alpha,\gamma} = \|(-\mathcal{L}_\gamma)^{\alpha/2}u\|_0, \quad u \in \mathcal{V}_\gamma. \quad (13)$$

By the same calculation as (11), for $\alpha \in \mathbb{Z}^+ \cup \{0\}$,

$$\|u\|_{\alpha,\gamma} = \|u\|_\alpha, \quad u \in \mathcal{H}_\gamma^\alpha. \quad (13)$$

Given a complete probability space $(\Omega, \mathcal{F}, \{ \mathcal{F}_t \}_{t \geq 0}, \mathbb{P})$, define the $L^2(I)$ valued $Q$-Wiener process

$$W(x, t) = \sum_{k=0}^\infty \sqrt{q_k} \beta_k(t)e_k(x)$$

where $\{ \beta_k(t) \}_{k}$ are mutually independent standard Brownian motions and $\{ e_k(x) \}_{k}$ is a standard basis of $L^2(I)$ with $e_0(x) = \sqrt{1/L}$ and for $k \geq 1$

$$e_k(x) = \begin{cases} \sqrt{\frac{2}{L}} \cos \frac{2m\pi x}{L}, & k = 2m, \\ \sqrt{\frac{2}{L}} \sin \frac{2m\pi x}{L}, & k = 2m - 1. \end{cases}$$

Moreover, assume that the Wiener process is sufficiently well-behaved that

$$\sum_{k=0}^\infty kq_k < \infty. \quad (14)$$
Now we define the $\mathcal{H}_\gamma$-valued Wiener process $(W_j(\gamma)(x,t))$ on all the overlapping elements in the following series form

$$W_j(\gamma)(x,t) = \gamma \sum_{l=0}^{\infty} \sqrt{q_{j,l}^h} \beta_j(t) e_{j,l}^\gamma(x), \quad (15)$$

where $\{\beta_j\}_{k=0}^{\infty}$ are mutually independent standard Brownian motions on $(\Omega, \mathcal{F}, \mathbb{P})$ and there are $q_{j,l} \in \mathbb{R}$ such that

$$q_{j,l}^h = q_{j,l} h$$

Here for each element $j = 1, \ldots, M$

$$\sqrt{q_{j,l}^h} \beta_j(t) e_{j,l}^\gamma(x) = \langle W(x,t), e_{j,l}^\gamma(x) \rangle e_{j,l}^\gamma(x)/\|e_{j,l}^\gamma\|_0.$$

Moreover, by the assumption (14)

$$\sum_{l=0}^{\infty} \lambda_l(\gamma) q_{j,l}^h < B < \infty, \quad (16)$$

where the bound $B$ is independent of the coupling parameter $\gamma \in (0, 1]$.

**Remark 1.** The definition of $W(\gamma)(x,t)$ is similar to the definition of that in the finite element method [16, 24, 25].

Now for fixed $h > 0$ and for any $T > 0$,

$$W(\gamma)(x,t) = (W_j(\gamma)(x,t)) \in C^{1/2}(0, T; \mathcal{H}_\gamma^2).$$

and $\{W(\gamma)(x,t)\}_{0 < \gamma \leq 1}$ is compact in space $C(0, T; \mathcal{H}_\gamma^\alpha)$ for $\alpha < 2$ for almost all $\omega \in \Omega$. Then for almost all $\omega \in \Omega$, the following limit

$$\tilde{W}(x,t) = (\tilde{W}_j(x,t)) = \lim_{n \to \infty} (W_{\gamma_n}(x,t))$$

is well defined in space $C(0, T; \mathcal{H})$ for some $\gamma_n \to 1$ as $n \to \infty$. Moreover, by the coupling conditions we have almost surely

$$W_j(X_{j\pm 1}, t) = W_{j\pm 1}(X_{j\pm 1}, t), \quad j = 1, 2, \ldots, M,$$

with notations $W_0(0, t) = W_M(X_M, t)$ and $W_{M+1}(X_{M+1}, t) = W_1(X_1, t)$.

**Remark 2.** By the analysis on eigenfunctions $(e_{j,k}^\gamma(x))_k$ in Section 4, the above limit of $W(\gamma)(x,t)$ in space $C(0, T; \mathcal{H})$ is unique in the sense of distribution for any sequence $\gamma_n \to 1$. Further, the distribution of $W_j(x,t)$ coincides with that of $W_{j\pm 1}(x,t)$ on the common overlapping domain.
Finally, we explore the linear operator \( L_\gamma \) as \( \gamma \to 0 \), denoted by \( L_0 \). Define
\[
L_0 u = \left( \frac{\partial^2 u_j}{\partial x^2} \right), \quad u = (u_j) \in D(L_0),
\]
with the ‘insulating’ version of the coupling conditions (5)–(7), \( j = 1, \ldots, M \),
\[
\begin{align*}
    u_j(X_j^{\pm}) &= u_j(X_j^{-}) = u_j(X_j^{+}), \\
    u_{j,x}(X_j^{+}) - u_{j,x}(X_j^{-}) + u_{j,x}(X_{j-1}) - u_{j,x}(X_{j+1}) &= 0.
\end{align*}
\]
By a standard computation \[11\], the spectrum of \(-L_0\), \( \{\lambda_k\}_{k=0}^\infty \), is
\[
\{0, \pi^2/h^2, 4\pi^2/h^2(\text{triple}), 9\pi^2/h^2, 16\pi^2/h^2(\text{triple}), \ldots, k^2\pi^2/h^2, \ldots\}.
\]
Denote the corresponding orthonormal standard eigenmodes on each element by \( \{e_{j,k}\}_{k=0}^\infty \), \( j = 1, \ldots, M \), then \( e_{j,0}(x) = 1/\sqrt{h} \) and \( e_{j,k}(x) \) \( k \ge 1 \) are
\[
\begin{align*}
    \left\{ \frac{1}{\sqrt{h}} \sin \frac{\pi(x-x_j)}{h}, \quad \frac{1}{\sqrt{h}} \cos \frac{2\pi(x-x_j)}{h}, \quad \frac{1}{\sqrt{h}} \sin \frac{2\pi(x-x_j)}{h}, \quad \frac{1}{\sqrt{h}} \sin \frac{4\pi(x-x_j)}{h}, \quad \frac{1}{\sqrt{h}} \sin \frac{8\pi(x-x_j)}{h}, \quad \frac{1}{\sqrt{h}} \sin \frac{16\pi(x-x_j)}{h} \right\}, \\
    \ldots
\end{align*}
\]

3 Limit system for full coupling

Now we show that for full coupling, that is, as \( \gamma \to 1 \), equations (5)–(7) generates a model for the dynamics of the original physical stochastic reaction-diffusion equation \[1\]–\[2\].

This is followed by a discussion similar to the case of Dirichlet boundary conditions \[2\], here we just state the result and omit the detailed proof.

**Theorem 3.** Assume bound \[16\] and \( u^\gamma(0) \in H^2 \) with \( \|u^\gamma(0)\|_2 \le C_0 \) which is independent of \( \gamma \) and \( u^\gamma(0) \to u^0 \) in \( H \) as \( \gamma \to 1 \). Then for any \( T > 0 \), \( u^\gamma \) converges to \( u \) in distribution as \( \gamma \to 1 \) in space \( C(0,T;H) \) where \( u = (u_j) \) solves
\[
\partial_t u_j = \partial_{xx} u_j + \alpha u_j^3 + \alpha f(u_j) + \sigma \partial_t W_j \quad \text{on} \quad I_j
\]
with \( u(0) = u^0 \) and
\[
u_j(X_{j-1},t) = u_{j+1}(X_{j+1},t), \quad u_j(x,t) = u_{j,M}(x,t).
\]

To prove the above result needs some energy estimates on the solutions \( u^\gamma(x,t) \). By the definition of \( L_\gamma \), the SPDEs (5)–(7) takes the following abstract from
\[
du^\gamma(t) = \left[ L_\gamma u^\gamma(t) + \alpha \gamma^2 u^\gamma(t) + \alpha F\left(u^\gamma(t)\right) \right] dt + \sigma dW^\gamma(t),
\]
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\[ u^\gamma(0) = (u^\gamma_j(0)), \quad (20) \]

where \( u^\gamma(t) = (u^\gamma_j(t)) \) and \( F(u^\gamma) = (f(u^\gamma_j)) \). Writing the temporal dependence explicitly, in a mild sense we have

\[
\begin{align*}
\dot{u}^\gamma(t) &= \mathcal{S}_\gamma(t)u^\gamma(0) + \alpha \int_0^t \mathcal{S}_\gamma(t-s) \left[ \gamma^2 u^\gamma(s) + F(u^\gamma(s)) \right] ds \\
&\quad + \sigma \int_0^t \mathcal{S}_\gamma(t-s) dW^\gamma(s).
\end{align*}
\]

Then by a standard semigroup approach [6], for any \( u^\gamma(0) \in \mathcal{H}_\gamma \) and any \( T > 0 \), (20) has a unique mild solution \( u^\gamma(t) \in C(0,T;\mathcal{H}^*) \cap L^2(0,T;\mathcal{V}^*) \).

Letting

\[ z^\gamma(t) = \sigma \int_0^t \mathcal{S}_\gamma(t-s) dW^\gamma(s), \]

we have the following lemma.

**Lemma 4.** Assume boundedness \([16]\). For any \( T > 0 \) and \( q > 0 \) there is a positive constant \( C_q(T) \) such that

\[
\mathbb{E} \sup_{0 \leq t \leq T} \| z^\gamma(t) \|^q_{2,\gamma} \leq C_q(T).
\]

**Proof.** Since \(-\mathcal{L}_\gamma\) is positive and self-adjoint,

\[
\| \mathcal{S}_\gamma(t) \|_\mathcal{L} \leq 1, \quad t \geq 0.
\]

Then the result follows from the stochastic factorization formula [6]. \(\square\)

Now define the difference \( w^\gamma = u^\gamma - z^\gamma \), then

\[
dw^\gamma(t) = [\mathcal{L}_\gamma w^\gamma(t) + \alpha \gamma^2 w^\gamma(t) + \alpha F(u^\gamma)] dt, \quad w^\gamma(0) = u^\gamma(0).
\]

By the standard energy estimate to stochastic reaction-diffusion equations with more general nonlinearity [19] we have

**Lemma 5.** Assume \( u^\gamma(0) \in \mathcal{H}_\gamma \), then for any \( T > 0 \), there is a positive constant \( C_T > 0 \) such that for any \( p \in \mathbb{Z}^+ \)

\[
\mathbb{E}\|\nabla u^\gamma\|_{C(0,T;\mathcal{H}^*)} + \mathbb{E}\|\nabla u^\gamma\|_{L^p(0,T;\mathcal{V}^*)} + \mathbb{E}\|\partial_t u^\gamma\|_{L^2(0,T;\mathcal{H}^{-1})} \leq C_T (1 + \| u^\gamma(0) \|^2_0).
\]

We show that \( \{ \mathcal{D}(u^\gamma) \\}_{\gamma} \), the distribution of \( u^\gamma \) in space \( C(0,T;\mathcal{H}) \), is tight. For this we need the following lemma by Simon [13].
Lemma 6. Assume $E$, $E_0$ and $E_1$ are Banach spaces such that $E_1 \subset E_0$, the interpolation space $(E_0, E_1)_{\theta, 1} \subset E$ with $\theta \in (0, 1)$ and $E \subset E_0$ with $\subset$ and $\subset$ denoting continuous and compact embedding respectively. Suppose $p_0, p_1 \in [1, \infty]$ and $T > 0$, such that

\[ X \text{ is a bounded set in } L^{p_1}(0, T; E_1) \]

and

\[ \partial X := \{ \partial v : v \in X \} \text{ is a bounded set in } L^{p_0}(0, T; E_0). \]

Here $\partial$ denotes the distributional derivative. If $1 - \theta > 1/p_0$ with

\[ \frac{1}{p_\theta} = \frac{1 - \theta}{p_0} + \frac{\theta}{p_1}, \]

then $X$ is relatively compact in $C(0, T; E)$.

By the above lemma, and noticing the relation (13), we have the following theorem.

Theorem 7. Assume (16) and $u^\gamma(0) \in \mathcal{H}_\gamma$ with $\|u^\gamma(0)\|_0 \leq C_0$ which is independent of $\gamma$. For any $T > 0$, $\mathcal{D}(u^\gamma)$ is tight in $C(0, T; \mathcal{H})$.

Similarly if $u^\gamma(0) \in \mathcal{H}^2_\gamma$ with $\|u^\gamma(0)\|_{2, \gamma} \leq C_0$ which is independent of $\gamma$, for any $T > 0$ there is a positive constant $C_T > 0$ such that

\[ \mathbb{E} \sup_{0 \leq t \leq T} \|u^\gamma(t)\|_{2, \gamma} \leq C_T. \]

Then by the embedding of $H^2(I) \subset C^1(I)$ [14],

\[ \mathbb{E} \left( \left| \frac{\partial u^\gamma}{\partial x}(X_{j+1}, t) \right| \right) \leq C_T, \quad 0 \leq t \leq T. \quad (21) \]

By the above estimates we can treat the boundary value in passing to the limit $\gamma \to 1$ of full coupling.

By Theorem 7, for any $\kappa > 0$ there is a compact set $K_\kappa \subset C(0, T; \mathcal{H})$ such that

\[ \mathbb{P}\{u^\gamma \in K_\kappa\} \geq 1 - \kappa. \]

Then there is a function $u \in C(0, T; \mathcal{H})$ and a subsequence $\gamma_n \to 1$ as $n \to \infty$, such that in probability

\[ u^{\gamma_n} \to u \quad \text{as } n \to \infty. \]

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Now we determine the equation solved by the limit $u$. Define a test function $\varphi \in C^\infty_0(0, L)$ and define

$$\varphi_j = \varphi|_{I_j}.$$ 

Then by the boundary conditions we have in the variational form for the system (5)–(7)

$$\langle u^\gamma(t), \varphi \rangle = \langle u^\gamma(0), \varphi \rangle + \int_0^t \langle \mathcal{L}_\gamma u^\gamma(s) + \alpha \gamma_n^2 u^\gamma(s), \varphi \rangle \, ds$$

$$+ \int_0^t \langle \alpha F(u^\gamma(s)), \varphi \rangle \, ds + \int_0^t \langle \sigma dW^\gamma(s), \varphi \rangle$$

$$= \langle u^\gamma(0), \varphi \rangle - \sum_j \int_0^t \int_{X_{j-1}}^{X_{j+1}} \frac{\partial u_j^\gamma(s)}{\partial x} \frac{\partial \varphi_j}{\partial x} \, dx \, ds$$

$$+ \sigma \sum_j \int_0^t \int_{X_{j-1}}^{X_{j+1}} dW_j^\gamma(s) \varphi_j \, dx$$

$$+ \alpha \gamma_n^2 \sum_j \int_0^t \int_{X_{j-1}}^{X_{j+1}} u_j^\gamma(s) \varphi_j \, dx \, ds + \alpha \sum_j \int_0^t \int_{X_{j-1}}^{X_{j+1}} f(u^\gamma_j(s)) \varphi_j \, dx \, ds$$

$$- \gamma_n' \sum_j \int_0^t \left[ \frac{\partial u_j^\gamma(X_j, s)}{\partial x} - \frac{\partial u_j^\gamma(X_{j+1}, s)}{\partial x} \right] \varphi_j(X_j) \, ds$$

$$+ \gamma_n' \sum_j \left[ \frac{\partial u_j^\gamma(X_{j-1}, s)}{\partial x} - \frac{\partial u_j^\gamma(X_{j+1}, s)}{\partial x} \right] \varphi_j(X_j) \, ds. \tag{22}$$

Then by estimate (21), letting $n \to \infty$, that is $\gamma_n' \to 0$, the last two terms disappear. Notice that $f(u^\gamma_j) \to f(u_j)$ weakly in space $L^2(0, T; L^2)$ and by the assumption on $W^\gamma$ we have, by passing to the limit $n \to \infty$,

$$\langle u(t), \varphi \rangle = \langle u(0), \varphi \rangle - \sum_j \int_0^t \int_{X_{j-1}}^{X_{j+1}} \frac{\partial u_j(s)}{\partial x} \frac{\partial \varphi_j}{\partial x} \, dx \, ds$$

$$+ \alpha \sum_j \int_0^t \int_{X_{j-1}}^{X_{j+1}} (u_j(s) - u_j^3(s)) \varphi \, dx \, ds$$

$$+ \int_0^t \langle \sigma d\tilde{W}(s), \varphi \rangle \tag{23}$$

with $\tilde{W}(t) = (W_j(t))$ which is well defined by Remark 2. Then a density argument yields that $u = (u_j)$ solves the following stochastic equations

$$\partial_t u_j = [\partial_{xx} u_j + \alpha u_j - \alpha (u_j)^3] \, dt + \sigma \partial_t W_j \quad \text{on } I_j,$$
with coupling boundary conditions

\[ u_j(X_{j\pm 1}, t) = u_{j\pm 1}(X_{j\pm 1}, t), \quad u_j(x, t) = u_{j\pm M}(x \pm L, t). \]

**Remark 8.** By the boundary condition (19) and Remark 2, the distributions of \( u_j \) and \( u_{j+1} \) in space \( C(0, T; L^2(X_j, X_j+h)) \) coincides.

Now define

\[ u(x, t) = u_j(x, t), \quad x \in [X_j, X_j+h] \]

and an \( L^2(I) \)-valued Wiener process \( \overline{W}(x, t) \) as

\[ \overline{W}(x, t) = W_j(x, t), \quad x \in [X_j, X_j+h]. \]

Then \( u(x, t) \) solves the stochastic reaction-diffusion equation (1)–(2) with the noise term \( W(t) \) replaced by \( \overline{W}(t) \) without changing the distribution. So (5)–(7) recovers the original system (1)–(2), in distribution, in the limit of full coupling, as \( \gamma \to 1 \).

### 4 Amplitudes on the elements

We derive a discrete macroscopic approximation to the system of SPDEs (5)–(7) based upon small coupling parameter \( \gamma > 0 \). By the analysis on operator \( L_0 \) in section 2, for \( \gamma = 0 \) the dominant mode is \( (e_{0j}) \), so by hyperbolicity we expect that for small \( \gamma > 0 \), the dominant mode is \( (e_{0j}) \). This is followed by the analysis on the continuity of \( \{(e_{jk})\}_k \) and \( \lambda_k(\gamma) \) on coupling parameter \( \gamma \). Further the asymptotic expansion for \( (e_{0j}) \) in \( \gamma \) shows that the grid value approximates the amplitude on each element.

For this we study the continuity properties of \( L_\gamma \) as coupling \( \gamma \to 0 \). We use variational convergence for operators [1]. For any subsequence \( \gamma_n \) with \( \gamma_n \to 0 \) as \( n \to \infty \), we introduce the G-convergence for \( L_{\gamma_n} \).

**Definition 9 (G-convergence).** Operator \( L_{\gamma_n} \) is said to be graph-convergent (G-convergent) to \( L_0 \) as \( n \to \infty \) if for every \( (u, v) \) with \( v = L_0 u \), there exists a sequence \( (u^n, v^n) \) with \( v^n = L_{\gamma_n} u^n \) such that \( u^n \to u \) strongly in \( V \) and \( v^n \to v \) strongly in \( V^* \), the dual space of \( V \).

Now for any \( u = (u_j(x)) \in V_0 \), denote by \( v = L_0 u \). First we choose bounded set \( \{v^n\} \subset H_{\gamma_n} \) such that \( v^n \to v \) in the dual space \( V^* \). Then solve the following equation

\[ L_{\gamma_n} u^n = v^n. \tag{24} \]

By the relation (13), \( \{u^n\} \) is bounded in \( H^2 \) which yields that \( \{u^n\} \) is compact in \( V \). Then there is a subsequence, which we still denote by \( \{u^n\} \), that
converges to \( \tilde{u} \) in \( V \) as \( n \to \infty \). Multiplying testing function \( \varphi \in C_0^\infty(0, L) \) on both sides of (24) and passing to the limit \( n \to \infty \), we have

\[
\mathcal{L}_0 \tilde{u} = v
\]

which yields that \( u = \tilde{u} \) by the uniqueness of the solution to (25). Then we have \( \mathcal{L}_{\gamma_n} \) is G-convergent to \( \mathcal{L}_0 \).

Now we draw the following result on the continuity of eigenvalues and eigenfunctions in coupling \( \gamma \) [21].

**Theorem 10.**

\[
\lim_{\gamma \to 0} \lambda_k(\gamma) = \lambda_k = -k^2 \pi^2 / h^2, \quad k = 0, 1, 2, \ldots .
\]

Let \( m_k \) be the multiplicity of \( \lambda_k(\gamma) \), the sequence of subspaces \( L_{\gamma_k} \) of dimension \( m_k \) generated by \((\gamma_j^1, \ldots , \gamma_j^m)\) converges in \( H \) to the eigenspace of \( \mathcal{L}_0 \) corresponding to \( \lambda_k = -k^2 \pi^2 / h^2 \).

**Remark 11.** Such convergence of the eigenspaces is called Mosco-convergence [1]. In the sense of this convergence, \((\gamma_j^l, \ldots , \gamma_j^{l'})\) may converges to \((\gamma_j^l, 1 \leq l \neq l' \leq m_k)\).

By the above result for small coupling \( \gamma > 0 \), we write for each element \( j = 1, \ldots , M \),

\[
e_j^0(x) = e_j^0(x) + \tilde{e}_j^0(x)
\]

with

\[
\tilde{e}_j^0(x) \to 0 \quad \text{as} \quad \gamma \to 0.
\]

Next we show that \( \tilde{e}_{j,0}(x) \) has an expansion to \( \gamma^2 \) terms in the coupling parameters:

\[
\frac{\partial^2}{\partial x^2} \tilde{e}_{j,0}(x) = \lambda_0(\gamma) \tilde{e}_{j,0}(x) + \lambda_0(\gamma) / \sqrt{2h},
\]

with boundary condition (6)–(7). Now if we account for the coupling and boundary condition by

\[
\tilde{e}_{j,0}(x) = e_{j,0}(X_j) + \tilde{e}_{j,0}(x)
\]

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\[
\frac{\partial^2}{\partial x^2} \tilde{e}_{j,0}(x) = \lambda_0(\gamma) \tilde{e}_{j,0}(x) + \lambda_0(\gamma) / \sqrt{2h},
\]

with boundary condition (6)–(7). Now if we account for the coupling and boundary condition by

\[
\tilde{e}_{j,0}(x) = e_{j,0}(X_j) + \tilde{e}_{j,0}(x)
\]
with boundary condition
\[
\bar{e}_{j,0}(X_{j\pm1}) = \bar{e}_{j,0}(X_j) = 0.
\]

But, by (7) we also have coupling in the derivatives:
\[
\partial_x e^\gamma_{j,0}(X_j) - \partial_x e^\gamma_{j,0}(X_j^+) + \gamma \partial_x e^\gamma_{j-1,0}(X_j) - \gamma \partial_x e^\gamma_{j+1,0}(X_j)
\]
\[
= 2\frac{\gamma^2}{h} [e^\gamma_{j+1,0}(X_{j+1}) - 2e^\gamma_{j,0}(X_j) + e^\gamma_{j-1,0}(X_{j-1})] , \quad j = 1, 2, \ldots, M ,
\]

Then by the above boundary condition, equation (26) and the fact \( \bar{e}^\gamma_{j,0}(x) \to 0 \) as \( \gamma \to 0 \), we have
\[
\lambda_0(\gamma) = \mathcal{O}(\gamma^2) \quad \text{as} \quad \gamma \to 0
\]

which implies that \( \bar{e}^\gamma_{j,0}(x) \) have an expansion to \( \gamma^2 \) terms in the coupling parameter.

Assume we have the following asymptotic expansion for each element, \( j = 1, \ldots, M \),
\[
e^\gamma_{j,0}(x) = e^\gamma_{j,0}(X_j) + \gamma F_{j,1}^\gamma(x) + \gamma^2 F_{j,2}^\gamma(x) + F_{j,3}^\gamma(x) \quad (27)
\]
where \( F_{j,k}^\gamma(x) = \mathcal{O}(\gamma^3) \). By \( \lambda_0(0) = 0 \) and the coupling boundary condition (8)–(10), \( F_{j,k}^\gamma \), \( k = 1, 2 \), are \( k \)th order polynomial in \( x \). Then also by the boundary condition (8)–(10) we have
\[
F_{j,1}^\gamma(x) = \begin{cases} 
\frac{e^\gamma_{j,0}(X_j) - e^\gamma_{j-1,0}(X_{j-1})}{h}(x - X_j), & X_{j-1} \leq x \leq X_j, \\
\frac{e^\gamma_{j+1,0}(X_{j+1}) - e^\gamma_{j,0}(X_j)}{h}(x - X_j), & X_j \leq x \leq X_{j+1},
\end{cases} \quad (28)
\]
and
\[
F_{j,2}^\gamma(x) = \begin{cases} 
A_j(x - X_j)(x - X_{j-1}), & X_{j-1} \leq x \leq X_j, \\
A_j(x - X_j)(x - X_{j+1}), & X_j \leq x \leq X_{j+1},
\end{cases} \quad (29)
\]
with
\[
A_j = \frac{e^\gamma_{j-1,0}(X_{j-1}) - 2e^\gamma_{j,0}(X_j) + e^\gamma_{j+1,0}(X_{j+1})}{2h^2}.
\]

The above asymptotic expansion shows that for small coupling \( \gamma > 0 \), the first mode is dominating and the grid value \( u^\gamma_j(X_j, t) \) approximates the amplitude of the field \( u^\gamma(x, t) \) on the element \( I_j \).
5 Macroscopic models for small coupling

By the asymptotic expansion in the previous section we derive a discrete macroscopic approximation model to (5)–(7) for small coupling \( \gamma > 0 \). For this we first apply an averaging method to reduce (5)–(7) onto the slow mode \((e_{\gamma,0}^\gamma)\).

We split \((u_j^\gamma)\) into slow part and fast part. Define map \(P_0^\gamma\) on \( H_\gamma^\gamma \) to \( H_\gamma^{P_0^\gamma} \)
\[ P_0^\gamma(v_j^\gamma) = (\langle v_j^\gamma, e_{\gamma,0}^\gamma(x) \rangle ||e_{\gamma,0}^\gamma(x)||^2_0) \quad \text{and} \quad P_1^\gamma = I - P_0^\gamma, \]
where \(I\) is the identity operator on \( H_\gamma^\gamma \). And for no coupling, \(\gamma = 0\), write \( P_0^0 = P_0^\gamma \) and \( P_1^0 = P_1^\gamma \). Then denote by \((u_j^\gamma(x, t))\) the solution to system (5)–(7) and make the following expansion
\[ (u_j^\gamma(x, t)) = \sum_{k=0}^{\infty} a_k^\gamma(t)(e_{j,k}^\gamma(x)). \]
Now define the slow part and fast part, respectively,
\[ U_j^\gamma(x, t) = (a_0^\gamma(t)e_{j,0}^\gamma(x)) \quad \text{and} \quad V_j^\gamma(x, t) = (u_j^\gamma(x, t)) - U_j^\gamma(x, t). \]
Then we have that these satisfy the coupled SPDEs
\[ dU_j^\gamma = \left[ L_j^\gamma U_j^\gamma + \alpha \gamma^2 U_j^\gamma + \alpha P_0^\gamma F(U_j^\gamma, V_j^\gamma) \right] dt + \sigma \gamma dB_j^\gamma, \]
\[ dV_j^\gamma = \left[ L_j^\gamma V_j^\gamma + \alpha \gamma^2 V_j^\gamma + \alpha P_1^\gamma F(U_j^\gamma, V_j^\gamma) \right] dt + \sigma \gamma dB_j^\gamma, \]
where
\[ B_j^\gamma(t) = \left( \sqrt{q_{j,0}^\gamma \beta_{j,0}(t)} \right) \quad \text{and} \quad B_j^\gamma(t) = \left( \sum_{k=1}^{\infty} \sqrt{q_{j,k}^\gamma \beta_{j,k}(t)} e_{j,k}^\gamma(x) \right). \]

By the analysis of Section 4, \( \lambda_0(\gamma) = O(\gamma^2) \) as \( \gamma \to 0 \), then for small coupling \( \gamma > 0 \), (30)–(31) have completely separated time scales. Thus an averaging approach applies to derive a macroscopic reduced system over the time scale \( \gamma^{-2} T \) for any \( T > 0 \) [19]. For this introduce a slow time scale \( t' = \gamma^2 t \) and small fields
\[ U_j^\gamma(t) = \gamma \tilde{U}_j^\gamma(\gamma^2 t) \quad \text{and} \quad V_j^\gamma(t) = \gamma \tilde{V}_j^\gamma(\gamma^2 t), \]
then on the slow time scale \( t' \)
\[ d\tilde{U}_j^\gamma(t') = \left[ \gamma^{-2} L_j^\gamma \tilde{U}_j^\gamma + \alpha \tilde{U}_j^\gamma + \alpha P_0^\gamma F(\tilde{U}_j^\gamma, \tilde{V}_j^\gamma) \right] dt' + \sigma \gamma^{-1} dB_j^\gamma(t'), \]
\[ d\tilde{V}^{\gamma}(t') = \left[ \gamma^{-2} \mathcal{L}_\gamma \tilde{V}^{\gamma} + \alpha \tilde{V}^{\gamma} + \alpha P_1 F(\tilde{U}^{\gamma}, \tilde{V}^{\gamma}) \right] dt' + \sigma \gamma^{-1} d\tilde{B}_1^{\gamma}(t'). \]

Here \( \tilde{B}_0^{\gamma}(t') = \gamma B_0^{\gamma}(\gamma^{-2}t') \) and \( \tilde{B}_1^{\gamma}(t') = \gamma B_1^{\gamma}(\gamma^{-2}t') \) are Wiener processes with the same distributions as those of \( B_0^{\gamma}(t') \) and \( B_1^{\gamma}(t') \), respectively, due to the scaling properties of the Wiener process.

Let \( \tilde{\eta}^{\gamma}(t') = (\tilde{\eta}_{j0}^{\gamma}(t')) \in P_1^{\gamma} \mathcal{H} \) be the unique stationary solution of the following linear equation

\[ d\tilde{\eta}^{\gamma}(t') = \gamma^{-2} \mathcal{L}_\gamma \tilde{\eta}^{\gamma}(t') \, dt' + \sigma \gamma^{-1} d\tilde{B}_1^{\gamma}(t'). \quad (33) \]

Then by an energy estimate and almost the same discussion to that by Wang and Roberts [19], for any \( T \geq 0 \), there is a positive constant \( C_T \) such that

\[ \sup_{0 \leq t' \leq T} \mathbb{E} \| \tilde{V}^{\gamma}(t') - \tilde{\eta}^{\gamma}(t') \|_0 \leq \gamma^2 C_T \left( \mathbb{E} \| \tilde{\eta}^{\gamma}(0) \|_0 + \| (u_{j0}^{\gamma}(t)) \|_0^6 \right). \quad (34) \]

We have no explicit expressions of \( (e_{jk}^{\gamma}(x)), \, k \geq 1 \), so for our purpose we derive another approximation for \( \tilde{V}^{\gamma}(t') \).

**Lemma 12.** Assume bound (16). For any \( T > 0 \), there is positive constant \( C_T \) such that

\[ \sup_{0 \leq t' \leq T} \mathbb{E} \| \tilde{V}^{\gamma}(t') - \tilde{\eta}(t') \|_0 \leq \gamma C_T \left( \mathbb{E} \| \tilde{\eta}^{\gamma}(0) \|_0 + \mathbb{E} \| \tilde{\eta}(0) \|_0 + \| (u_{j0}(0)) \|_0^6 \right), \]

where \( \tilde{\eta}(t') = (\tilde{\eta}_{j0}(t')) \) is the unique stationary solution of the following linear equation

\[ d\tilde{\eta}(t') = \gamma^{-2} \mathcal{L}_0 \tilde{\eta}(t') \, dt' + \sigma \gamma^{-1} d\tilde{B}_1(t') \]

and

\[ \tilde{B}_1(t') = \sum_{k=1}^{\infty} \left( \gamma \sqrt{\beta_{j,k}(\gamma^{-2}t')} e_{j,k}(x) \right) \]

with distributions independent of coupling parameter \( \gamma \).

**Proof.** Expand \( \tilde{\eta}^{\gamma}(t) \) and \( \tilde{\eta}(t) \) by the eigenfunctions of \( \mathcal{L}_\gamma \) and \( \mathcal{L}_0 \) respectively as

\[ \tilde{\eta}^{\gamma}(x, t') = \sum_{k=1}^{\infty} \tilde{\eta}_{k}^{\gamma}(x, t') \quad \text{and} \quad \tilde{\eta}(x, t') = \sum_{k=1}^{\infty} \tilde{\eta}_{k}(x, t'), \]

where

\[ \tilde{\eta}_{k}(x, t') = \langle \tilde{\eta}^{\gamma}(t'), (e_{j,k}(x)) \rangle (e_{j,k}(x))/\| (e_{j,k}) \|_0^2; \]

\[ \tilde{\eta}_{k}(x, t') = \langle \tilde{\eta}(t'), (e_{j,k}(x)) \rangle (e_{j,k}(x))/M. \]
Then for $k \geq 1$

$$d\tilde{\eta}_k(t') = -\frac{1}{\gamma^2} \lambda_k(\gamma)\tilde{\eta}_k(t')dt' + \frac{\sigma}{\gamma} d\tilde{B}_{1,k}(t')$$

and

$$d\hat{\eta}_k(t') = -\frac{1}{\gamma^2} \lambda_k(\gamma)\hat{\eta}_k(t')dt' + \frac{\sigma}{\gamma} d\tilde{B}_{1,k}(t')$$

with

$$\tilde{B}_{1,k}(t') = \gamma \left( \sqrt{q_{j,k}\beta_{j,k}(\gamma^{-2}t')e_{j,k}(x)} \right),$$

$$\tilde{B}_1(t') = \gamma \left( \sqrt{q_{j,k}\beta_{j,k}(\gamma^{-2}t')e_{j,k}(x)} \right).$$

Using the Itô formula and the stationary property of $\tilde{\eta}_k$ and $\hat{\eta}_k$, there is a positive constant $C$ such that for $k \geq 1$

$$E\|\tilde{\eta}_k(t')\|_0 \leq C \quad \text{and} \quad E\|\hat{\eta}_k(t')\|_0 \leq C.$$  \hspace{1cm} (35)

Define the difference $z_k(t') = \tilde{\eta}_k(t') - \hat{\eta}_k(t')$ which solves

$$dz_k(t') = -\frac{1}{\gamma^2} [\lambda_k z_k(t') + (\lambda_k(\gamma) - \lambda_k)\tilde{\eta}_k(t')]dt' + \frac{\sigma}{\gamma} d\left[\tilde{B}_{1,k}(t') - \tilde{B}_{1,k}(t')\right],$$

and hence

$$z_k(t') = e^{-\lambda_k t'/\gamma^2} z_k(0) + \frac{\lambda_k - \lambda_k(\gamma)}{\gamma^2} \int_0^{t'} e^{-\lambda_k(t'-s)/\gamma^2} \tilde{\eta}_k(s)ds$$

$$+ \frac{\sigma}{\gamma} \int_0^{t'} e^{-\lambda_k(t'-s)/\gamma^2} d\left[\tilde{B}_{1,k}(s) - \tilde{B}_{1,k}(s)\right].$$

Then by the analysis of Section 4 on $\lambda_k(\gamma)$, $k \geq 1$, assumption (16) and the estimates (35) we have

$$\sup_{0 \leq t' \leq T} E\|\tilde{\eta}(t') - \hat{\eta}(t')\|_0 \leq \gamma C_T E \left( \|\tilde{\eta}(0)\|_0 + \|\hat{\eta}(0)\|_0 \right).$$

Thus by (34), the proof is complete. \hspace{1cm} \Box

Now by (32) and the above result, for $V^\gamma$ on the original time scale,

$$\sup_{0 \leq t \leq T} E\|V^\gamma(t) - \gamma \tilde{\eta}(\gamma^{-2}t)\|_0 \leq \gamma^2 C_T \left( E\|\tilde{\eta}(0)\|_0 + E\|\hat{\eta}(0)\|_0 + \|(u_0(0))\|_0 \right).$$

\hspace{1cm} (36)
Moreover,
\[ \mathbb{E}\tilde{\eta}^2(t) = \sigma^2 \sum_{k=1}^{\infty} \frac{1}{2\lambda_k} (q^h_{j,k} e^2_{j,k}(x)). \] (37)

For \( \tilde{U}^\gamma \) we follow an averaging approach [19] which yields the following averaged equation
\[
d\bar{U}^\gamma(t') = [\gamma^{-2} \mathcal{L}_\gamma \tilde{U}^\gamma(t') + \alpha \tilde{U}^\gamma(t') + \alpha P_0^\gamma \bar{F}(\tilde{U}^\gamma(t'))] dt' + \sigma \gamma^{-1} d\tilde{B}^\gamma_0(t') \quad (38)
\]
where \( \tilde{U}^\gamma = (\tilde{u}^\gamma_{j,0}) \) and
\[
\bar{F}(-) = \mathbb{E}[F(\cdot, \tilde{\eta}(\omega))].
\]

By the definition of \( f \) and that \( \tilde{\eta} \) is Gaussian with zero mean, we have
\[
\bar{F}(\bar{U}^\gamma) = -\left( (\bar{u}^\gamma_{j,0})^3 + 3\bar{u}^\gamma_{j,0} \mathbb{E}\tilde{\eta}^2_{j,0} \right).
\]

Moreover, by a deviation argument [19, 20], stochastic effects in these subgrid scale fast modes are fed into the slow modes by the nonlinear interaction. So we have the following averaged equation plus deviation
\[
d\bar{U}^\gamma(t') = [\gamma^{-2} \mathcal{L}_\gamma \tilde{U}^\gamma + \alpha \tilde{U}^\gamma + \alpha P_0^\gamma \bar{F}(\tilde{U}^\gamma)] dt' + \sigma \gamma^{-1} d\tilde{B}^\gamma_0(t') \quad (39)
\]
with, for fixed \( \tilde{U}^\gamma \),
\[
\bar{Q}(\bar{U}^\gamma) = 2\mathbb{E} \int_0^\infty P_0^\gamma \left[ F(\bar{U}^\gamma + \tilde{\eta}(s)) - \bar{F}(\bar{U}^\gamma) \right] \otimes P_0^\gamma \left[ F(\bar{U}^\gamma + \tilde{\eta}(0)) - \bar{F}(\bar{U}^\gamma) \right] ds,
\]
and \( \tilde{\beta}(t) = (\tilde{\beta}_j(t)) \) is an \( M \) dimensional standard Brownian motion. For any \( T > 0 \) and any \( \kappa > 0 \), there is a positive constant \( C_{\kappa,T} \) such that
\[
\mathbb{P}\left\{ \sup_{0 \leq t' \leq T} |\tilde{U}^\gamma(t') - \bar{U}^\gamma(t')| \leq \gamma^{1+\kappa} C_{\kappa,T} \right\} \geq 1 - \kappa.
\]

Then for the original system (5), by (32), we have the following reduced equation
\[
dU^\gamma_0(t) = [\mathcal{L}_\gamma U^\gamma_0(t) + \alpha \gamma^2 U^\gamma_0(t) + \alpha P_0^\gamma \bar{F}(U^\gamma_0(t)) ] dt \\
+ \sigma \gamma dB^\gamma_0(t) + \alpha \gamma^3 \sqrt{\bar{Q}(\gamma^{-1} U^\gamma_0(t))} d\tilde{\beta}(t), \quad (40)
\]
where \( \tilde{\beta}(t) = (\tilde{\beta}_j(t)) = \gamma^{-1} \tilde{\beta}(\gamma^2 t) \) is the scaled \( M \) dimensional standard Brownian motion. Then for any \( \kappa > 0 \), there is a constant \( C_{\kappa,T} > 0 \) such that
\[
\mathbb{P}\left\{ \sup_{0 \leq t \leq \gamma^{-2} T} |U^\gamma(t) - U^\gamma_0(t)| \leq \gamma^{2+\kappa} C_{\kappa,T} \right\} \geq 1 - \kappa.
\]
Now having the reduced system (40), we use the approximation to the amplitude on each element to derive a further approximate model. Write

\[ U_0^\gamma(x, t) = (u_{j,0}^\gamma(x, t)) \]

and define

\[ (U_j(t)) = a_j^\gamma(t) \left( e_{j,0}^\gamma(X_j) \right) . \]

Then by (27) for \((u_{j,0}^\gamma(x, t))\) we have the following asymptotic expansion

\[
u_{j,0}^\gamma(x, t) = a_j^\gamma(t)e_{j,0}^\gamma(X_j) + \gamma a_j^\gamma(t)F_{j,1}^\gamma(x) + \gamma^2 a_j^\gamma(t)F_{j,2}^\gamma(x) + \mathcal{O}(\gamma^3) \]

\[
\begin{cases}
U_j(t) + \frac{\gamma}{h}(U_j(t) - U_{j-1}(t))(x - X_j) \\
+ \frac{\sigma^2}{2h^2}(U_{j-1}(t) - 2U_j(t) + U_{j+1}(t))(x - X_j)(x - X_{j-1}) \\
+ \mathcal{O}(\gamma^2), \quad X_{j-1} \leq x \leq X_j, \\
U_j(t) + \frac{\gamma}{h}(U_{j+1}(t) - U_j(t))(x - X_j) \\
+ \frac{\sigma^2}{2h^2}(U_{j-1}(t) - 2U_j(t) + U_{j+1}(t))(x - X_j)(x - X_{j+1}) \\
+ \mathcal{O}(\gamma^2), \quad X_j \leq x \leq X_{j+1},
\end{cases}
\]

Putting (41) into (10) yields

\[
dU_j(t) = \frac{\gamma^2}{h^2} (U_{j+1}(t) - 2U_j(t) + U_{j-1}(t)) dt + \gamma^2 \hat{\alpha}_j U_j(t) - \alpha U_j^3(t) \\
+ \sigma \gamma \sqrt{q_{j,0}} e_{j,0}^\gamma(x) d\beta_j(t) + \alpha \gamma^3 \sqrt{2Q_j} U_j(t) e_{j,0}^\gamma(x) d\beta_j(t) \\
+ \mathcal{O}(\gamma^3, \alpha^2),
\]

where

\[
\hat{\alpha}_j = \alpha - 3\alpha \sigma^2 \sum_{k=1}^{\infty} \frac{q_{j,k}}{2\lambda_k}(e_{j,0}(X_j))^2,
\]

\[
Q_j = \int_0^\infty \mathbb{E} \left[ \langle \eta_{j,0}^2(s) - \mathbb{E}\eta_{j,0}^2, e_{j,0} \rangle \langle \eta_{j,0}^2(0) - \mathbb{E}\eta_{j,0}^2, e_{j,0} \rangle \right] ds .
\]

Here we use the approximation of \(e_{j,0}(x)\) to \(e_{j,0}^\gamma(x)\) for small \(\gamma\).

Notice that system (42) is not a complete discrete model because the noise terms are still described on the mode \((e_{j,0}^\gamma(x))\). In order to give a discrete approximating model for small coupling \(\gamma\), we explore the evolution of the amplitude of the basic mode \((e_{j,0}(x))\). For this we project \((U_j)\) onto the basic space \(E_0\) spanned by \((e_{j,0}(x))\). However, the fast modes \(V^\gamma\) have a nonzero projection in basic space \(E_0\); because of the complicated expression for \((e_{j,k}^\gamma(x))\), we choose to project \(\gamma \tilde{\eta}\), which approximates \(V^\gamma\) up to error of \(\mathcal{O}(\gamma^2)\). For this we first project \(\gamma \tilde{\eta}\) onto \((e_{j,0}^\gamma(x))\), then project to \(E_0\).

Notice that for small coupling \(\gamma\), \(\frac{1}{\gamma} \tilde{\eta}(t')\) behaves as a noise process. By a martingale approach [5, 19, 22] we have the following lemma.
Lemma 13. Assume bound \( (16) \). Then
\[
\sigma \sum_{k=1}^{\infty} \sqrt{\frac{1}{\lambda_k}} \left( \sqrt{q_{j,k}^h} \tilde{e}_{j,k}(x) \tilde{\beta}_{j,k}(t') \right) \quad \text{as } \gamma \to 0,
\]
where \( (\tilde{\beta}_{j,k}(t')) \), \( k = 1, 2, \ldots \), are mutually independent standard \( M \) dimensional Brownian motion in time scale \( t' \).

Then on the right-hand sides of \((12)\) there are additional noise forcing terms when projected to the basic space \( E_0 \): namely
\[
\gamma \sigma \sum_{k=1}^{\infty} \sqrt{\frac{q_{j,k}^h}{\lambda_k}} \left( \tilde{e}_{j,k}(x), e_{j,0}(x) \right) d\tilde{\beta}_{j,k}(t)
\]
where \( (\tilde{\beta}_{j,k}(t)) = (\gamma^{-1}\tilde{\beta}_{j,k}(\gamma^2 t)) \), \( k = 1, 2, \ldots \), are mutually independent scalar standard Brownian motions. By the expansion of \( e_{j,0}(x) \), \((27)\) and the expression of \( (e_{j,k}(x)) \),
\[
\left( e_{j,k}(x), e_{j,0}(x) \right) = \gamma \left( e_{j,k}(x), F_{j,1}^\gamma(x) \right) + O(\gamma^2).
\]
Then define
\[
\tilde{\beta}_{j,0}^\gamma(t) = \sum_{k=1}^{\infty} \sqrt{\frac{q_{j,k}^h}{\lambda_k}} \left( e_{j,k}(x), F_{j,1}^\gamma(x) \right) \tilde{\beta}_{j,k}(t),
\]
and for \( j = 1, 2, \ldots , M \), \( i = -1, 0, 1 \)
\[
B_{j,i}(t) = \sqrt{\frac{q_{j,0}^h}{q_{j,0}^h}} \tilde{\beta}_{j,0}(t) e_{j-i,0}(X_{j-i}), \quad \hat{B}_{j,i}(t) = \tilde{\beta}_{j,0}^\gamma(t) e_{j-i,0}(X_{j-i}),
\]
\[
\tilde{B}_{j,i}(t) = \sqrt{q_{j}^\gamma} \tilde{\beta}_{j}(t) e_{j-i,0}(X_{j-i}).
\]
Projecting the above system onto the basic space \( E_0 \) and by \((37)\) we then have the following macroscopic discrete approximation model to the SPDE \((5)-(7)\) for small coupling \( \gamma > 0 \)
\[
dU_j(t) = \frac{\gamma^2}{h^2} (U_{j-1}(t) - 2U_j(t) + U_{j+1}(t)) \ dt + \gamma^2 \hat{\alpha}_j U_j(t) - \alpha U_j^3(t)
\]
\[
+ \sigma \gamma dB_{j,0}(t) + \sigma \gamma dB_{j,0}^\gamma(t) + 3\gamma^2 \sqrt{2} U_j(t) d\hat{B}_{j,0}(t)
\]
\[
+ \frac{\sigma \gamma^2}{4} (dB_{j,1}(t) - 2dB_{j,0}(t) + dB_{j,-1})
\]
\[
+ \frac{\sigma \gamma^3}{4} \left( d\hat{B}_{j,1}(t) - 2d\hat{B}_{j,0}^\gamma(t) + d\hat{B}_{j,-1}^\gamma \right)
\]
Furthermore by (28), \( F_{j,1}^{\gamma} = \mathcal{O}(\gamma) \). Then a truncation of (43) to errors \( \mathcal{O}(\gamma^3, \alpha^2) \) and evaluating at full coupling \( \gamma = 1 \) yields the following macroscopic discrete system of SDEs, \( j = 1, \ldots, M \),

\[
dU_j(t) \approx \frac{1}{k^2} (U_{j-1}(t) - 2U_j(t) + U_{j+1}(t)) dt + (\dot{\alpha}_j U_j(t) - \alpha U_j^3(t)) dt \\
+ \sigma dB_{j,0}(t) + 3\sqrt{2}U_j(t)dB_{j,0}(t) \\
+ \frac{\sigma}{4} (dB_{j,1}(t) - 2dB_{j,0}(t) + dB_{j,-1}).
\]

(44)

This system of SDEs reduces to the system (4) discussed in the Introduction as a discrete model of the reaction-diffusion SPDE (1).

### 6 Consistency of macroscopic discrete model

Next we study the consistency of the macroscopic discrete model (44) by the definition of the Wiener processes \( W\gamma(x, t) \) in Section 2, as \( h \), the size of each element, converges to zero.

By the properties of \( e_{j,k}(x) \) and \( e_{j,k}^\gamma(x) \) for small coupling \( \gamma > 0 \) in section 2 and section 4 respectively, we have

\[
\sqrt{2q_{j,0}^h} \beta_{j,0}(t) e_{j,0}^\gamma(x) = \langle W(x, t), e_{j,0}^\gamma(x) \rangle e_{j,0}^\gamma(x) / ||e_{j,0}^\gamma||_0 \\
= \sum_{k=0}^\infty \sqrt{q_k} \beta_k(t) \langle e_k(x), e_{j,0}^\gamma(x) \rangle e_{j,0}^\gamma(x) / ||e_{j,0}^\gamma||_0 \\
= \sum_{k=0}^\infty \sqrt{q_k} \beta_k(t) \langle e_k(x), e_{j,0}^\gamma(X_j) + \gamma F_{j,1}(x) + \gamma^2 F_{j,2}(x) + F_{j,3}^\gamma(x) \rangle e_{j,0}^\gamma(x) / ||e_{j,0}^\gamma||_0 \\
= \sum_{k=0}^\infty \sqrt{q_k} \beta_k(t) \langle e_k(x), e_{j,0}(X_j) + \gamma F_{j,1}(x) + \gamma^2 F_{j,2}(x) + F_{j,3}^\gamma(x) \rangle e_{j,0}(x) + \mathcal{O}(\gamma) \\
= \sqrt{\frac{q_0}{L}} \beta_0(t) + \sum_{m=1}^\infty \sqrt{\frac{2q_{2m}}{L}} \beta_{2m}(t) \frac{L}{2m\pi} \frac{1}{2h} \left[ \sin \frac{2m\pi}{L} X_{j+1} - \sin \frac{2m\pi}{L} X_{j-1} \right] \\
+ \sum_{m=1}^\infty \sqrt{\frac{2q_{2m-1}}{L}} \beta_{2m-1}(t) \frac{L}{2m\pi} \frac{1}{2h} \left[ \cos \frac{2m\pi}{L} X_{j-1} - \cos \frac{2m\pi}{L} X_{j+1} \right] \\
+ \mathcal{O}(h, \gamma)
\]
\[
\sqrt{q_{h,j,l}}(t) e^\gamma_{j,l}(x) = \langle W(x, t), e^\gamma_{j,0}(x) \rangle_{e^\gamma_{j,l}(x)} + O(h, \gamma),
\]

(46)

Theorem 14. The macroscopic discrete model (44) is consistent to the original stochastic reaction-diffusion equation (1) – (2).

Proof. Following the exact same discussion by Roberts et al. [12] for deterministic systems, we just consider the stochastic terms and \( \hat{\alpha}_j \).

By the definition of \( \hat{\alpha}_j \), since \( q^h_{j,l} = O(h) \) and noticing that \( \lambda_k = k^2 \pi^2 / h^2 \), then we have

\[
\hat{\alpha}_j = \alpha + O(h^2), \quad h \to 0.
\]

For the stochastic terms, notice that \( \tilde{\eta}_k \) converges to zero in mean square with speed \( \exp\{-\lambda_k\} = \exp\{-k^2 \pi^2 / h^2\} \) as \( h \to 0 \). Then since \( q^h_{j,l} = O(h) \), we have \( Q_j = O(h^2) \) as \( h \to 0 \). Then by the definition of \( W^\gamma(x, t) \) and analysis (45) – (46) the macroscopic discrete model (44) is consistent up error \( O(h) \) to the following stochastic reaction-diffusion equation

\[
\begin{align*}
p_t u(x, t) &= \partial_{xx} u(x, t) + \alpha(u(x, t) - u^3(x, t)) + \sigma B(x, t), \quad x \in [0, L] \\
u(0, t) &= u(L, t) \quad t \geq 0
\end{align*}
\]

where \( B(x, t) \) distributes the same as \( W(x, t) \). This completes the proof. \( \square \)

7 Conclusion

Stochastic averaging is an effective method to extract macroscopic dynamics from SPDEs with separated time scale [19] – [20]. Here by applying the stochastic averaging and dividing the spatial domain into overlapping finite sized elements with special interelement coupling boundary conditions (6) – (7), we derive a macroscopic discrete model (44) for stochastic reaction-diffusion partial differential equations (11) with periodic boundary conditions. The most
important property of such interelement coupling boundary conditions is preserving the self-adjoint symmetry which is often so important in application [12]. Furthermore, by the choice of stochastic forcing on each element, this coupling boundary conditions also assures the consistency for vanishing element size, section 6.

Moreover, the final discrete model (44), which is different from the usual finite difference approximation model (3), shows the importance of the sub-grid scale interaction between noise and spatial diffusion and provides a new rigorous approach to constructing semi-discrete approximations to stochastic reaction-diffusion SPDEs.

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References

[1] H. Attouch, Variational Convergence for Functions and Operators, Pitman Publishing Limited, London, 1984.

[2] W. E, X. Li & E. Vanden-Eijnden, Some recent progress in multiscale modeling, Multiscale modelling and simulation, Lect. Notes Comput. Sci. Eng., 39, 3–21, Springer, Berlin, 2004.

[3] P. Imkeller & A. Monahan (Eds.). Stochastic Climate Dynamics, a Special Issue in the journal Stoch. and Dyna., 2(3), 2002.

[4] D. Givon, R. Kupferman & A. Stuart, Extracting macroscopic dynamics: model problems and algorithms, Nonlinearity, 17 (2004), R55–R127. http://dx.doi.org/10.1088/0951-7715/17/6/R01

[5] H. Kesten & G. C. Papanicolaou, A limit theorem for turbulent diffusion, Commun. Math. Phys., 65(1979), 79–128. http://dx.doi.org/10.1007/BF01428989

[6] G. Da Prato & J. Zabczyk, Stochastic Equations in Infinite Dimensions, Cambridge University Press, 1992.

[7] A. J. Roberts, A holistic finite difference approach models linear dynamics consistently, Mathematics of Computation, 72(241) (2002), 247–262. http://www.ams.org/mcom/2003-72-241/S0025-5718-02-01448-5
[8] A. J. Roberts, A step towards holistic discretisation of stochastic partial differential equations, In Jagoda Crawford and A. J. Roberts, editors, Proc. of 11th Computational Techniques and Applications Conference CTAC-2003, volume 45 (2003), C1–C15. http://anziamj.austms.org.au/V45/CTAC2003/Roberts.

[9] A. J. Roberts, Resolving the multitude of microscale interactions accurately models stochastic partial differential equations, LMS J. Computation and Maths, 9 (2006), 193–221. http://www.lms.ac.uk/jcm/9/lms2005-032

[10] A. J. Roberts, Subgrid and interelement interactions affect discretisations of stochastically forced diffusion. In Wayne Read, Jay W. Larson, and A. J. Roberts, editors, Proc. of 13th Biennial Computational Techniques and Applications Conference, CTAC-2006, volume 48 (2006), C168–C187. http://anziamj.austms.org.au/ojs/index.php/ANZIAMJ/article/view/36

[11] A. J. Roberts, Choose interelement coupling to preserve self-adjoint dynamics in multiscale modelling and computation, 2008. Technique Report, http://arxiv.org/abs/0811.0688

[12] Tony MacKenzie & A.J. Roberts, Holistic discretisation ensures fidelity to dynamics in two spatial dimensions, 2009. Technique Report.

[13] J. Simon, Compact sets in the space $L^p(0,T;B)$, Ann. Mat. Pura Appl., 146 (1987), 65–96. http://dx.doi.org/10.1007/BF01762360

[14] R. A. Adams & J. J. F. Fournier, Sobolev Spaces, Academic Press, 2003.

[15] S. Engblom, L. Ferm, A. Hellander & P. Löstedt, Simulation of stochastic reaction diffusion processes on unstructured meshes, Technical report, http://arXiv.org/abs/0804.3288, 2008.

[16] V. Thomée, Galerkin Finite Element Methods for Parabolic Problems, Springer–Verlag, Berlin, 1997.

[17] W. Wang & J. Duan, A dynamical approximation for stochastic partial differential equations, J. Math. Phys. 48, (2007), http://dx.doi.org/102701--14.10.1063/1.2800164

[18] W. Wang & J. Duan, Homogenized dynamics of stochastic partial differential equations with dynamical boundary conditions, Comm. Math. Phys. 275 (2007), 163–186. http://dx.doi.org/10.1007/s00220-007-0301-8
[19] W. Wang & A. J. Roberts, Macroscopic reduction for stochastic reaction-diffusion equations, Technical report, 2008. 
http://arxiv.org/abs/0812.1837

[20] W. Wang & A. J. Roberts, Average and deviation for slow–fast stochastic partial differential equations, Technical report, 2008. 
http://arxiv.org/abs/0904.1462

[21] W. Wang & A. J. Roberts, Macroscopic discrete modelling of stochastic reaction-diffusion equations, Technical report, 2009.

[22] H. Watanabe, Averaging and fluctuations for parabolic equations with rapidly oscillating random coefficients, Probab. Th. & Rel. Fields, 77 (1988), 359–378. http://dx.doi.org/10.1007/BF00319294

[23] E. Waymire & J. Duan (Eds.), Probability and Partial Differential Equations in Modern Applied Mathematics. IMA Volume 140, Springer-Verlag, New York, 2005.

[24] Y. B. Yan, Galerkin finite element methods for stochastic parabolic partial differential equations, SIAM J. Numer. Anal, 43 (2005), 1363–1384.

[25] Q. Du & T. Zhang, Numerical approximation of some linear stochastic partial differential equations, SIAM J. Numer. Anal, 40 (2005), 1421–1445.