RATIONAL CURVES ON CALABI-YAU THREEFOLDS:
VERIFYING MIRROR SYMMETRY PREDICTIONS

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Dedicated to Professor Stein Arild Strømme

Abstract. In this paper, the numbers of rational curves on general complete intersection Calabi-Yau threefolds in complex projective spaces are computed up to degree six. The results are all in agreement with the predictions made from mirror symmetry.

Contents

1. Introduction 1
1.1. The main result 2
2. Gromov-Witten invariants 3
2.1. Moduli spaces of stable maps 4
2.2. Virtual fundamental classes 5
2.3. Defining Gromov-Witten invariants 5
3. Localization of $\overline{M}_{0,0}(\mathbb{P}^r, d)$ 6
4. Lines on hypersurfaces 11
5. Rational curves on quintic threefolds 13
6. Rational curves on complete intersections 15
Acknowledgements 19
References 19

1. Introduction

In [2], the string theorists Candelas, de la Ossa, Green and Parkes predicted the numbers $n_d$ of degree $d$ rational curves on a general quintic hypersurface in 4-dimensional projective space. By similar methods, Libgober and Teitelbaum [18] predicted the numbers of rational curves on the remaining general Calabi-Yau threefolds in projective spaces. The computation of the physicists is based on arguments from topological quantum field
theory and mirror symmetry. These predictions went far beyond anything algebraic geometry could prove at the time and became a challenge for mathematicians to understand mirror symmetry and to find a mathematically rigorous proof of the physical predictions. The process of creating a rigorous mathematical foundation for mirror symmetry is still far from being finished. For important aspects concerning this question, we refer to Morrison [19], Givental [10, 11], and Lian, Liu, and Yau [17]. For an introduction to the algebro-geometric aspects of mirror symmetry, see Cox and Katz [3]. To verify the physical predictions by algebro-geometric methods, we have to find a suitable moduli space of rational curves separately for every degree $d$, and express the locus of rational curves on a general Calabi-Yau threefold as a certain zero-dimensional cycle class on the moduli space. We then need in particular to evaluate the degree of a given zero-dimensional cycle class. This is possible, in principle, whenever the Chow ring of the moduli space is known. Unfortunately, in general it is quite difficult to describe the Chow ring of a moduli space. Alternatively, Bott’s formula allows us to express the degree of a certain zero-dimensional cycle class on a moduli space endowed with a torus action in terms of local contributions supported on the components of the fixed point locus. The local contributions are of course much simpler than the whole space. By these methods, algebraic geometers checked the results of physicists up to degree 4: $n_1 = 2875$ is classically known, $n_2 = 609250$ has been shown by Katz [12], $n_3 = 317206375$ by Ellingsrud and Strømme [6, 7], $n_4 = 242467530000$ by Kontsevich [16].

In this paper, we present a strategy for computing Gromov-Witten invariants using the localization theorem and Bott’s formula. This method is based on the work of Kontsevich [16]. As an insightful example, the numbers of rational curves on general complete intersection Calabi-Yau threefolds in complex projective spaces are computed up to degree six. The results are all in agreement with the predictions made from mirror symmetry in [2, 18]. All computations have been implemented in the computer algebra system SINGULAR [4]. The code is available from the author upon request or at https://github.com/hiepdang/Singular.

1.1. The main result. Let

$$X = \bigcap_{i=1}^{k} X_i \subset \mathbb{P}^{k+3}$$

be a smooth complete intersection threefold, where $X_i$ is a smooth hypersurface of degree $d_i \geq 2$ for all $i = 1, \ldots, k$. In this case, we will say that $X$ has type $(d_1, \ldots, d_k)$. If the canonical bundle $K_X$ on $X$ is trivial, then $X$ is called Calabi-Yau. By the adjunction formula [8 Example 3.2.12], we have

$$K_X \cong O \left( \sum_{i=1}^{k} d_i - (k + 4) \right).$$
Suppose that $X$ is Calabi-Yau. Then we obtain

$$\sum_{i=1}^{k} d_i = k + 4.$$ 

Without loss of generality, we assume that $d_1 \geq \cdots \geq d_k \geq 2$. Thus we have

$$2k \leq \sum_{i=1}^{k} d_i = k + 4,$$

so $k \leq 4$. Therefore there are precisely five possibilities.

1. If $k = 1$, then $d_1 = 5$, and thus $X$ is a quintic threefold in $\mathbb{P}^4$.
2. If $k = 2$, then there will be two possibilities as follows:
   (a) $d_1 = 4$, $d_2 = 2$. Then $X$ is a complete intersection of type $(4, 2)$ in $\mathbb{P}^5$.
   (b) $d_1 = d_2 = 3$. Then $X$ is a complete intersection of type $(3, 3)$ in $\mathbb{P}^5$.
3. If $k = 3$, then $d_1 = 3$, $d_2 = d_3 = 2$. In this case, $X$ is a complete intersection of type $(3, 2, 2)$ in $\mathbb{P}^6$.
4. If $k = 4$, then $d_1 = d_2 = d_3 = d_4 = 2$. In this case, $X$ is a complete intersection of type $(2, 2, 2, 2)$ in $\mathbb{P}^7$.

The main result of this paper, which agrees with the mirror symmetry computation, is the following theorem:

**Theorem 1.1.** Let $d$ be an integer with $1 \leq d \leq 6$. The numbers of rational curves of degree $d$ on the general complete intersection Calabi-Yau threefolds are given by Table 1.

| $d$ | $(5)$ | $(4, 2)$ | $(3, 3)$ | $(3, 2, 2)$ | $(2, 2, 2, 2)$ |
|-----|-------|----------|----------|-------------|----------------|
| 1   | 2875  | 1280     | 1053     | 720         | 512            |
| 2   | 609250| 92288    | 52812    | 22428       | 9728           |
| 3   | 317206375 | 15655168 | 6424326  | 1611504     | 416256         |
| 4   | 242467530000 | 3883902528 | 1139448384 | 168199200    | 25703936       |
| 5   | 2293058888887625 | 1190923282176 | 249787892583 | 21676931712   | 1957983744     |
| 6   | 248249742118022000 | 417874605342336 | 62660964509532 | 3195557904564 | 170535923200   |

**Table 1.** The numbers of rational curves on Calabi-Yau threefolds.

Compare with [2, Table 4] and [18, Table 1] for the numbers obtained by mirror symmetry.

2. **Gromov-Witten invariants**

Gromov-Witten invariants have their origin in physics. There are several ways to define Gromov-Witten invariants using algebraic geometry or symplectic geometry. For details of the definitions and properties of Gromov-Witten invariants, we refer to [3, Chapter 7]. Roughly speaking, Gromov-Witten invariants are the intersection numbers
of cycle classes in the rational Chow rings of moduli spaces of stable maps subject to certain geometric conditions on the curves. In order to define Gromov-Witten invariants in algebraic geometry, first of all we need to recall some basic facts about moduli spaces of stable maps and virtual fundamental classes.

2.1. Moduli spaces of stable maps. In [15], Kontsevich introduced the notion of moduli spaces of stable maps. In this section, we define and state some results about these moduli spaces. We only concentrate on the case of rational curves (genus zero), but some results are still true in the general case. An introduction to this subject was given by Fulton and Pandharipande in [9].

Let $X$ be a smooth projective variety and fix a homology class $\beta \in H_2(X, \mathbb{Z})$. An $n$-pointed map is a morphism $f : C \to X$, where $C$ denotes a nodal curve with $n$ distinct marked points that are smooth on $C$. An $n$-pointed map $f : C \to X$ is called stable if its group of automorphisms is finite. We denote by $M_{0,n}(X, \beta)$ the set of all $n$-pointed stable maps from a rational curve $C$ to $X$ such that $f_*(C) = \beta$.

**Theorem 2.1.** [9, Theorem 1] The moduli space $\overline{M}_{0,n}(X, \beta)$ has a structure of a projective scheme.

We say that $X$ is convex if for every morphism $f : \mathbb{P}^1 \to X$, \[ H^1(\mathbb{P}^1, f^*(T_X)) = 0, \] where $T_X$ is the tangent bundle on $X$. The following theorem concerns the convex case.

**Theorem 2.2.** [9, Theorem 2] Let $X$ be a convex variety and $\beta \in A_1(X)$. Then $\overline{M}_{0,n}(X, \beta)$ is a normal projective variety of pure dimension \[ \dim(X) + \int_\beta c_1(T_X) + n - 3. \]

For each marked point $p_i$, there is a natural map \[ e_i : \overline{M}_{0,n}(X, \beta) \to X \]
\[ (C; p_1, \ldots, p_n; f) \mapsto f(p_i) \]
which is called the evaluation map at $p_i$. These morphisms play an important role to relate the geometry of of $\overline{M}_{0,n}(X, \beta)$ to the geometry of $X$.

In case $X = \mathbb{P}^r$, we can write $\beta = d\ell$, where $\ell$ is the cycle class of a line. The moduli space $\overline{M}_{0,n}(\mathbb{P}^r, d\ell)$ is an orbifold which is the quotient of a smooth variety (see [3, Section 7.1.1]). By Theorem 2.2, $\overline{M}_{0,n}(\mathbb{P}^r, d\ell)$ is also a normal projective variety of dimension $rd + r + d + n - 3$. We will write $\overline{M}_{0,n}(\mathbb{P}^r, d)$ in place of $\overline{M}_{0,n}(\mathbb{P}^r, d\ell)$. For more details on the construction and properties of $\overline{M}_{0,n}(\mathbb{P}^r, d)$, we refer to [14, Chapter 2].

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1A nodal curve is a compact connected curve with at most nodes as singularities. For convenience, a curve is always assumed to be nodal in this paper.
2.2. Virtual fundamental classes. For details of the construction and properties of virtual fundamental classes we refer to [3 Section 7.1.4]. Naturally, the virtual fundamental class \([\overline{M}_{0,n}(X,\beta)]^{\text{virt}}\) is an element of the Chow group of \(\overline{M}_{0,n}(X,\beta)\). In many cases, we have

\[[\overline{M}_{0,n}(X,\beta)]^{\text{virt}} = [\overline{M}_{0,n}(X,\beta)]\]

for instance, in the case when \(X\) is a projective space, and more generally a convex variety.

Example 2.3. In this example, we construct the virtual fundamental class for the quintic hypersurface in \(\mathbb{P}^4\). Let \(X \subset \mathbb{P}^4\) be a smooth quintic hypersurface, and let \(d\) be a positive integer. The inclusion \(X \subset \mathbb{P}^4\) induces a natural embedding

\[i : \overline{M}_{0,0}(X, d\ell) \to \overline{M}_{0,0}(\mathbb{P}^4, d),\]

where \(\ell\) is the class of a line. Let \(\mathcal{V}_d\) be the rank \(5d + 1\) vector bundle on the stack \(\overline{M}_{0,0}(\mathbb{P}^4, d)\) whose fiber over a stable map \(f : C \to \mathbb{P}^4\) is \(H^0(C, f^*\mathcal{O}_{\mathbb{P}^4}(5))\). One can show that \(\overline{M}_{0,0}(X, d\ell)\) is the zero locus of a section \(s\) of \(\mathcal{V}_d\). Moreover, we have

\[[\overline{M}_{0,0}(X, d\ell)]^{\text{virt}} = s^*[C_Z Y],\]

where \(Z = \overline{M}_{0,0}(X, d\ell), Y = \overline{M}_{0,0}(\mathbb{P}^4, d)\), and \(C_Z Y\) is the normal cone to \(Z\) in \(Y\). For details of the definition and properties of normal cones we refer to [8 Section 2.5].

2.3. Defining Gromov-Witten invariants.

Definition 2.4. Let \(X\) be a smooth projective variety and let \(\beta \in A_1(X)\) be a cycle class. The Gromov-Witten invariant of \(\beta\) associated with the cycle classes \(\gamma_1, \ldots, \gamma_n \in A^*(X)\) is the rational number defined by

\[I_{n,\beta}(\gamma_1, \ldots, \gamma_n) := \int_{\overline{M}_{0,n}(X,\beta)} e_1^*(\gamma_1) \cup \cdots \cup e_n^*(\gamma_n) \cup [\overline{M}_{0,n}(X,\beta)]^{\text{virt}},\]

where \(\cup\) is the cup product. Note that we have to use the cup product since the moduli space \(\overline{M}_{0,n}(X,\beta)\) is a singular variety. For the definition of the cup product, we refer to [14 Section 4.1.2].

In case \(n = 0\), we denote the Gromov-Witten invariant of the cycle class \(\beta \in A_1(X)\) by

\[I_{\beta} := \int_{\overline{M}_{0,0}(X,\beta)} [\overline{M}_{0,0}(X,\beta)]^{\text{virt}},\]

that is the degree of the virtual fundamental class \([\overline{M}_{0,0}(X,\beta)]^{\text{virt}}\).

Example 2.5. Let \(X \subset \mathbb{P}^4\) be a smooth quintic hypersurface. We have shown in Example 2.3 that the virtual fundamental class \([\overline{M}_{0,0}(X, d\ell)]^{\text{virt}}\) is a cycle class on \(\overline{M}_{0,0}(X, d\ell)\). By Definition 2.4 we have the Gromov-Witten invariant

\[I_{d\ell} = \int_{\overline{M}_{0,0}(X,d\ell)} [\overline{M}_{0,0}(X,d\ell)]^{\text{virt}}.\]
By [3, Lemma 7.1.5], we have
\[ i_*(\overline{M}_{0,0}(X, df)|^{\text{virt}}) = c_{5d+1}(\mathcal{V}_d) \in A^*(\overline{M}_{0,0}(\mathbb{P}^4, d)). \]

This will lead to a nice formula of the form
\[ I_d = \int_{\overline{M}_{0,0}(\mathbb{P}^4, d)} c_{5d+1}(\mathcal{V}_d). \]

In Section 5, we will present how to compute this integral using the localization of \( \overline{M}_{0,0}(\mathbb{P}^4, d) \) and Bott's formula.

3. Localization of \( \overline{M}_{0,0}(\mathbb{P}^r, d) \)

The natural action of \( T = (\mathbb{C}^*)^{r+1} \) on \( \mathbb{P}^r \) induces an action of \( T \) on \( \overline{M}_{0,0}(\mathbb{P}^r, d) \) by the composition of the action with stable maps. The fixed point loci and their equivariant normal bundles were determined for \( \overline{M}_{0,0}(\mathbb{P}^r, d) \) by Kontsevich in [16].

We denote the set of fixed points by \( \overline{M}_{0,0}(\mathbb{P}^r, d)^T \). A fixed point of the \( T \)-action on \( \overline{M}_{0,0}(\mathbb{P}^r, d) \) corresponds to a stable map \((C, f)\) where each irreducible component \( C_i \) of \( C \) is either mapped to a fixed point of \( \mathbb{P}^r \) or a multiple cover of a coordinate line. Each node of \( C \) and each ramification point of \( f \) is also mapped to a fixed point of \( \mathbb{P}^r \). Thus each fixed point \((C, f)\) of the \( T \)-action on \( \overline{M}_{0,0}(\mathbb{P}^r, d) \) can be associated with a graph \( \Gamma \).

Let \( q_0, \ldots, q_r \) be the fixed points of \( \mathbb{P}^r \) under the torus action.

1. The vertices of \( \Gamma \) are in one-to-one correspondence with the connected components \( C_i \) of \( f^{-1}\{q_0, \ldots, q_r\} \), where each \( C_i \) is either a point or a non-empty union of irreducible components of \( C \).

2. The edges of \( \Gamma \) correspond to irreducible components \( C_e \) of \( C \) which are mapped onto some coordinate line \( l_e \) in \( \mathbb{P}^r \).

The graph \( \Gamma \) has the following labels: Associate to each vertex \( v \) the number \( i_v \) defined by \( f(C_v) = q_{i_v} \). Associate to each edge \( e \) the degree \( d_e \) of the map \( f|_{C_e} \). The connected components of \( \overline{M}_{0,0}(\mathbb{P}^r, d)^T \) are naturally labelled by equivalence classes of connected graphs \( \Gamma \). Furthermore, the following conditions must be satisfied:

1. If an edge \( e \) connects \( v \) and \( v' \), then \( i_v \neq i_{v'} \), and \( l_e \) is the coordinate line joining \( q_{i_v} \) and \( q_{i_{v'}} \).
2. \( \sum e d_e = d \).

We denote the number of edges connected to \( v \) by \( \text{val}(v) \). The stable maps associated with the fixed graph \( \Gamma \) define a subspace
\[ \overline{M}_\Gamma \subset \overline{M}_{0,0}(\mathbb{P}^r, d). \]

Fix \((C, f) \in \overline{M}_\Gamma\). For each vertex \( v \) such that the component \( C_v \) of \( C \) is one dimensional, \( C_v \) has \( \text{val}(v) \) special points. The data consisting of \( C_v \) plus these \( \text{val}(v) \) points forms a
stable curve, giving an element of $\overline{\mathcal{M}}_{0,\text{val}(v)}$. Using the data of $\Gamma$, we can construct a map

$$\psi_{\Gamma} : \prod_{v : \text{dim } C_v = 1} \overline{\mathcal{M}}_{0,\text{val}(v)} \longrightarrow \overline{\mathcal{M}}_{\Gamma}.$$  

See [3, Section 9.2.1] for more details of this construction. Note that $\text{val}(v)$ can be defined for all vertices $v$ and that $\text{dim } C_v = 1$ if and only if $C_v$ contains a component of $C$ contracted by $f$ if and only if $\text{val}(v) \geq 3$. We define

$$F_{\Gamma} = \prod_{v : \text{dim } C_v = 1} \overline{\mathcal{M}}_{0,\text{val}(v)}.$$  

If there are no contracted components, then we let $F_{\Gamma}$ be a point. The map $\psi_{\Gamma}$ is finite. There is a finite group of automorphisms $A_{\Gamma}$ acting on $F_{\Gamma}$ such that the quotient space is $\overline{\mathcal{M}}_{\Gamma}$. The group $A_{\Gamma}$ fits into an exact sequence

$$0 \longrightarrow \prod_{e} \mathbb{Z}/d_e \mathbb{Z} \longrightarrow A_{\Gamma} \longrightarrow \text{Aut}(\Gamma) \longrightarrow 0,$$

where $\text{Aut}(\Gamma)$ is the group of automorphisms of $\Gamma$ which preserve the labels. We denote $a_{\Gamma}$ by the order of $A_{\Gamma}$. This number appears in the denominator of the formula in [3, Corollary 9.1.4].

**Example 3.1.** Let us describe the fixed point components of the natural action of $T$ on $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^r, 1)$. The possible graphs are of the following type:

$$\Gamma_{i,j} : \begin{array}{c}
\begin{array}{c}
i \end{array} \\
\begin{array}{c}
\overset{1}{\longrightarrow}
\end{array} \\
\begin{array}{c}
\Downarrow
\end{array} \\
\begin{array}{c}
\downarrow j
\end{array}
\end{array}$$  

In these labelled graphs, we have added the labels $i, j \in \{0, 1, \ldots, r\} \text{ with } i \neq j$ below the vertices and the degree 1 above the edge. For each graph, it is easy to see that $|\text{Aut}(\Gamma_{i,j})| = 2$, hence we have $a_{\Gamma_{i,j}} = 2$. For example, in the case of $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^4, 1)$, there are 20 graphs.

**Example 3.2.** Let us describe the fixed point components of the natural action of $T$ on $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^r, 2)$. The possible graphs are of the following types:

$$\Gamma_1 : \begin{array}{c}
\begin{array}{c}
i \end{array} \\
\begin{array}{c}
\overset{2}{\longrightarrow}
\end{array} \\
\begin{array}{c}
\downarrow j
\end{array}
\end{array} \quad \Gamma_2 : \begin{array}{c}
\begin{array}{c}
i \end{array} \\
\begin{array}{c}
\overset{1}{\longrightarrow}
\end{array} \\
\begin{array}{c}
\overset{1}{\longrightarrow}
\end{array} \\
\begin{array}{c}
\downarrow j \\
\downarrow \big| \big| \\
\big| \big| \\
\big| \big| \\
\downarrow k
\end{array}
\end{array}$$

Note that $i, j, k \in \{0, 1, \ldots, r\}$, and the vertices of an edge must have different labels. For each type, it is easy to see that $|\text{Aut}(\Gamma_i)| = 2$, hence $a_{\Gamma_1} = 4$ and $a_{\Gamma_2} = 2$. For instance, if $r = 1$, then there will be 4 graphs. If $r = 4$, then there will be 100 graphs.

**Example 3.3.** In the case of $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^r, 3)$, the possible graphs have the following types:

$$\Gamma_1 : \begin{array}{c}
\begin{array}{c}
i \end{array} \\
\begin{array}{c}
\overset{3}{\longrightarrow}
\end{array} \\
\begin{array}{c}
\downarrow j
\end{array}
\end{array} \quad \Gamma_2 : \begin{array}{c}
\begin{array}{c}
i \end{array} \\
\begin{array}{c}
\overset{2}{\longrightarrow}
\end{array} \\
\begin{array}{c}
\overset{1}{\longrightarrow}
\end{array} \\
\begin{array}{c}
\downarrow j \\
\downarrow \big| \\
\big| \\
\downarrow k
\end{array}
\end{array}$$
Note that \(i, j, k, h \in \{0, 1, \ldots, r\}\), and the vertices of an edge must have different labels. It is easy to see that \(|\text{Aut}(\Gamma_1)| = 2, |\text{Aut}(\Gamma_2)| = 1, |\text{Aut}(\Gamma_3)| = 2, \text{ and } |\text{Aut}(\Gamma_4)| = 6\). Hence \(a_{\Gamma_1} = 6, a_{\Gamma_2} = a_{\Gamma_3} = 2, \text{ and } a_{\Gamma_4} = 6\).

**Example 3.4.** In the case of \(\overline{M}_{0,0}(\mathbb{P}^r, 4)\), the possible graphs have the following types:

\[
\begin{align*}
\Gamma_1 & : \quad \begin{array}{c}
\text{i} \\
\text{j} \\
\text{k} \\
\text{h}
\end{array}, \quad \begin{array}{c}
\text{4} \\
\text{j} \\
\text{k} \\
\text{h}
\end{array}, \quad \begin{array}{c}
\text{3} \\
\text{j} \\
\text{k} \\
\text{h}
\end{array}, \quad \begin{array}{c}
\text{2} \\
\text{j} \\
\text{k} \\
\text{h}
\end{array} \\
\Gamma_2 & : \quad \begin{array}{c}
\text{i} \\
\text{j} \\
\text{k} \\
\text{h}
\end{array}, \quad \begin{array}{c}
\text{2} \\
\text{j} \\
\text{k} \\
\text{h}
\end{array}, \quad \begin{array}{c}
\text{1} \\
\text{1} \\
\text{1} \\
\text{1}
\end{array}, \quad \begin{array}{c}
\text{2} \\
\text{j} \\
\text{k} \\
\text{h}
\end{array}, \quad \begin{array}{c}
\text{2} \\
\text{j} \\
\text{k} \\
\text{h}
\end{array}
\end{align*}
\]

Note that \(i, j, k, h, m \in \{0, 1, \ldots, r\}\), and the vertices of an edge must have different labels. For these graphs, we have \(|\text{Aut}(\Gamma_1)| = |\text{Aut}(\Gamma_2)| = |\text{Aut}(\Gamma_3)| = |\text{Aut}(\Gamma_5)| = |\text{Aut}(\Gamma_6)| = |\text{Aut}(\Gamma_7)| = |\text{Aut}(\Gamma_8)| = 2, |\text{Aut}(\Gamma_2)| = |\text{Aut}(\Gamma_4)| = 1\), and \(|\text{Aut}(\Gamma_9)| = 24\). Hence \(a_{\Gamma_1} = 8, a_{\Gamma_2} = 3, a_{\Gamma_4} = a_{\Gamma_7} = a_{\Gamma_8} = 2, a_{\Gamma_5} = a_{\Gamma_6} = 4, \text{ and } a_{\Gamma_9} = 24\).

**Example 3.5.** In the case of \(\overline{M}_{0,0}(\mathbb{P}^r, 5)\), in addition to the analogues of the graphs above, we have the following graphs:

\[
\begin{align*}
\Gamma_1 & : \quad \begin{array}{c}
\text{i} \\
\text{j} \\
\text{k} \\
\text{h} \\
\text{m} \\
\text{n}
\end{array}, \quad \begin{array}{c}
\text{1} \\
\text{1} \\
\text{1} \\
\text{1} \\
\text{1} \\
\text{1}
\end{array}, \quad \begin{array}{c}
\text{i} \\
\text{j} \\
\text{k} \\
\text{h} \\
\text{m} \\
\text{n}
\end{array}, \quad \begin{array}{c}
\text{i} \\
\text{j} \\
\text{k} \\
\text{h} \\
\text{m} \\
\text{n}
\end{array}, \quad \begin{array}{c}
\text{1} \\
\text{1} \\
\text{1} \\
\text{1} \\
\text{1} \\
\text{1}
\end{array} \\
\Gamma_2 & : \quad \begin{array}{c}
\text{i} \\
\text{j} \\
\text{k} \\
\text{h} \\
\text{m} \\
\text{n}
\end{array}, \quad \begin{array}{c}
\text{1} \\
\text{1} \\
\text{1} \\
\text{1} \\
\text{1} \\
\text{1}
\end{array}
\end{align*}
\]
Note that $i, j, k, h, m, n \in \{0, 1, \ldots, r\}$, and the vertices of an edge must have different labels. For these graphs, we have $a_{\Gamma_1} = a_{\Gamma_2} = a_{\Gamma_3} = 2, a_{\Gamma_4} = 6, a_{\Gamma_5} = 8$, and $a_{\Gamma_6} = 120$.

**Example 3.6.** In the case of $\overline{M}_{0,0}(\mathbb{P}^r, 6)$, in addition to the analogues of the graphs above, we have the following graphs:
Note that $i,j,k,h,m,n,p \in \{0,1,\ldots,r\}$, and the vertices of an edge must have different labels. For these graphs, we have $a_{\Gamma_1} = a_{\Gamma_2} = a_{\Gamma_7} = 2, a_{\Gamma_3} = 1, a_{\Gamma_4} = a_{\Gamma_8} = 6, a_{\Gamma_5} = 4, a_{\Gamma_6} = 8, a_{\Gamma_9} = 12, a_{\Gamma_{10}} = 24$, and $a_{\Gamma_{11}} = 720$.

In order to apply Bott’s formula to $M_{0,0}(\mathbb{P}^r, d)$, we need a formula for computing the $T$-equivariant Euler class of the normal bundle $N_\Gamma$ of $M_\Gamma$. We define a flag $F$ of a graph to be a pair $(v,e)$ such that $v$ is a vertex of $e$. Put $i(F) = v$ and let $j(F)$ be the other vertex of $e$. Set

$$\omega_F = \frac{\lambda_{i(F)} - \lambda_{j(F)}}{d_e}.$$ 

This corresponds to the weight of a torus action on the tangent space of the component $C_e$ of $C$ at the point $p_F$ lying over $i_v$. If $v$ is a vertex with $\text{val}(v) = 1$, then we denote by $F(v)$ the unique flag containing $v$. If $\text{val}(v) = n \geq 2$, then we denote by $F_1(v), F_2(v), \ldots, F_n(v)$ the $n$ flags containing $v$. We also denote by $v_1(e)$ and $v_2(e)$ the two vertices of an edge $e$.

**Theorem 3.7** (Kontsevich, [16]). The $T$-equivariant Euler class of the normal bundle $N_\Gamma$ is a product of contributions from the vertices and edges. More precisely, we have

$$e^T_\Gamma(N_\Gamma) = e^v_\Gamma e^e_\Gamma,$$

where $e^v_\Gamma$ and $e^e_\Gamma$ are defined by the following formulas:

$$e^v_\Gamma = \prod_v \left( \prod_{j \neq i_v} (\lambda_{i_v} - \lambda_j) \right)^{1-\text{val}(v)} \left( \sum_i \omega_{i(v)}^{-1} \right)^{3-\text{val}(v)} \prod_i \omega_{i(F_i(v))},$$

$$e^e_\Gamma = \prod_e \left( \frac{(-1)^{d_e} (d_e)!^2 (\lambda_{i_{v_1}(e)} - \lambda_{v_2}(e))^{2d_e}}{d_e^{2d_e}} \prod_{a,b \in \mathbb{N}, a+b=d_e, k \neq i_{v_1}(e)} (a\lambda_{i_{v_1}(e)} + b\lambda_{i_{v_2}(e)}) \frac{d}{d_e} \right).$$

**Proof.** A proof of this theorem can be found in [16] Sections 3.3.3 and 3.3.4. \qed
The formulas in Theorem 3.7 give an effective way to compute $e^T(N_\Gamma)$. We will work out the nontrivial examples of these formulas in the next section.

4. Lines on hypersurfaces

In this section, we reconsider the problem of counting lines on a general hypersurface of degree $d = 2r - 3$ in $\mathbb{P}^r$. This problem was considered in [5, Chapter 8] using classical Schubert calculus. In this section, we use the localization of $\overline{M}_{0,0}(\mathbb{P}^r, 1)$ and Bott’s formula for solving the problem.

Given a general hypersurface $X$ of degree $d = 2r - 3$ in $\mathbb{P}^r$ and the class $\ell \in H_2(X, \mathbb{Z})$, we compute the Gromov-Witten invariant

$$I^{(r)}_\ell = \int_{\overline{M}_{0,0}(\mathbb{P}^r, 1)} [c_{2r-2}(\mathcal{E})]^{\text{virt}}.$$

By arguments similar to those in Example 2.5, we have the following formula:

$$I^{(r)}_\ell = \int_{\overline{M}_{0,0}(\mathbb{P}^r, 1)} c_{2r-2}(\mathcal{E}),$$

where $\mathcal{E}$ is the vector bundle on $\overline{M}_{0,0}(\mathbb{P}^r, 1)$ whose fiber at a stable map $f : C \to \mathbb{P}^r$ is $H^0(C, f^*O_{\mathbb{P}^r}(2r - 3))$.

By Example 3.1, there are $r(r + 1)$ graphs $\Gamma_{i,j}$ corresponding to the fixed point components of the natural action of $T$ on $\overline{M}_{0,0}(\mathbb{P}^r, 1)$. By arguments similar to those in the paper of Kontsevich (see [16], Example 2.2) and using Corollary 9.1.4 in [3], we obtain:

$$I^{(r)}_\ell = \int_{\overline{M}_{0,0}(\mathbb{P}^r, 1)} c_{2r-2}(\mathcal{E}) = \sum_{\Gamma_{i,j}} \frac{c^{T}_{2r-2}(\mathcal{E}|_{\Gamma_{i,j}})}{a_{\Gamma_{i,j}} \epsilon^T(N_{\Gamma_{i,j}})},$$

where $c^{T}_{2r-2}(\mathcal{E}|_{\Gamma_{i,j}})$ is defined by the following formula:

$$c^{T}_{2r-2}(\mathcal{E}|_{\Gamma_{i,j}}) = \prod_{e} \left( \prod_{a,b \in N, a+b=(2r-3)d_e} \frac{a \lambda_{v_1(e)} + b \lambda_{v_2(e)}}{d_e} \right) \prod_{u} \left( \prod_{v \neq u} (2r-3) \lambda_{v} \right)^{1-\text{val}(v)}.$$

Since each graph $\Gamma_{i,j}$ has two vertices and one edge, we have $d_e = 1$, $i_{v_1(e)} = i$, $i_{v_2(e)} = j$, and val$(v) = 1$. Thus we have

$$c^{T}_{2r-2}(\mathcal{E}|_{\Gamma_{i,j}}) = \prod_{a,b \in N, a+b=2r-3} (a \lambda_i + b \lambda_j).$$

For computing $\epsilon^T(N_{\Gamma_{i,j}})$, we use the formulas coming from Theorem 3.7. More precisely, we have

$$\epsilon^{w}_{\Gamma_{i,j}} = \frac{1}{(\lambda_i - \lambda_j)(\lambda_j - \lambda_i)},$$

$$\epsilon^{c}_{\Gamma_{i,j}} = - (\lambda_i - \lambda_j)^2 \prod_{k \neq i,j} (\lambda_i - \lambda_k)(\lambda_j - \lambda_k).$$
Thus
\[ e^T(N_{\Gamma_{i,j}}) = e^e_{\Gamma_{i,j}} e^e_{\Gamma_{i,j}} = \prod_{k \neq i, j} (\lambda_i - \lambda_k)(\lambda_j - \lambda_k). \]

Since \( a_{\Gamma_{i,j}} = 2 \) for all \( \Gamma_{i,j} \), we have

\[ I_{r}^{(r)} = \sum_{0 \leq i, j \leq r, i \neq j} \frac{\prod_{a,b \in \mathbb{N}, a+b=2r-3} (a\lambda_i + b\lambda_j)}{2 \prod_{k \neq i, j} (\lambda_i - \lambda_k)(\lambda_j - \lambda_k)}. \]

By \([4, \text{Example 2.1.15}]\), the moduli space \( \overline{M}_{0,0}(\mathbb{P}^r, 1) \) is isomorphic to the Grassmannian \( G(2, r+1) \) of lines in \( \mathbb{P}^r \). In this case, using arguments similar to those in \([3, \text{Example 7.1.3.1}]\), we construct the virtual fundamental class \( \overline{M}_{0,0}(X, \ell)_{\text{virt}} \) as follows. Let \( S \) be the tautological subbundle on the Grassmannian \( G(2, r+1) \). The fiber \( S_{\ell} \) at a line \( \ell \) is the 2-dimensional subspace of \( \mathbb{C}^{r+1} \) whose projectivization is \( \ell \). An equation for \( X \) gives a section \( s \) of the vector bundle \( \text{Sym}^{2r-3} S^\vee \). Then \( \overline{M}_{0,0}(X, \ell) \) is the zero locus of \( s \). This contribution produces the cycle class
\[ s^*[C_Z Y] \in A_0(\overline{M}_{0,0}(X, \ell)), \]
where \( C_Z Y \) is the normal cone of \( Z = \overline{M}_{0,0}(X, \ell) = Z(s) \subset Y = G(2, r+1) \). By arguments similar to those in \([3, \text{Example 7.1.5.1}]\), this class is the virtual fundamental class \( \overline{M}_{0,0}(X, \ell)_{\text{virt}} \). Then \([3, \text{Lemma 7.1.5}]\) implies that
\[ i_*(\overline{M}_{0,0}(X, \ell)_{\text{virt}}) = c_{2r-2}(\text{Sym}^{2r-3} S^\vee) \in A^*(G(2, r+1)), \]
where \( i : \overline{M}_{0,0}(X, \ell) \hookrightarrow G(2, r+1) \) is an embedding. This will lead to the following formula:

\[ I_{r}^{(r)} = \int_{G(2, r+1)} c_{2r-2}(\text{Sym}^{2r-3} S^\vee). \]

In summary, we have the following theorem:

**Theorem 4.1.** Let \( X \subset \mathbb{P}^r \) be a general hypersurface of degree \( 2r - 3 \). The number of lines on \( X \) is given by

\[ \sum_{0 \leq i < j \leq r} \frac{\prod_{a,b \in \mathbb{N}, a+b=2r-3} (a\lambda_i + b\lambda_j)}{\prod_{k \neq i, j} (\lambda_i - \lambda_k)(\lambda_j - \lambda_k)}. \]

**Proof.** By \([3, \text{Chapter 8}]\), the number of lines on \( X \) is finite and can be computed by the following formula:

\[ \text{number of lines on } X = \int_{G(2, r+1)} c_{2r-2}(\text{Sym}^{2r-3} S^\vee). \]

The induced action on \( \overline{M}_{0,0}(\mathbb{P}^r, 1) \) gives us formula \([3]\). By \([4]\), we get the desired formula. \( \square \)
Note that all of the $\lambda_i$ cancel as they must, and the answer is computed to be the number of lines on $X$, which depends only on $n$. Some numbers of lines are listed in Table 2.

| $d$ | $r$ | Numbers of lines |
|-----|-----|------------------|
| 1   | 2   | 1                |
| 3   | 3   | 27               |
| 5   | 4   | 2875             |
| 7   | 5   | 698005           |
| 9   | 6   | 305093061        |
| 11  | 7   | 210480374951     |
| 13  | 8   | 210776836330775  |
| 15  | 9   | 289139638632755625 |
| 17  | 10  | 520764738758073845321 |
| 19  | 11  | 1192221463356102320754899 |
| 21  | 12  | 3381929766320534635615064019 |
| 23  | 13  | 11643962664020516264785825991165 |
| 25  | 14  | 47837786502063195088311032392578125 |
| 27  | 15  | 231191601420598135249236900564098773215 |

Table 2. The numbers of lines on a general hypersurface of degree $d = 2r - 3$ in $\mathbb{P}^r$.

5. RATIONAL CURVES ON QUINTIC THREEFOLDS

Let $X \subset \mathbb{P}^4$ be a general quintic threefold and $d$ be a positive integer. An important question in enumerative geometry is how many rational curves of degree $d$ there are on $X$. It has already been mentioned that the string theorists can compute the numbers $n_d$ of rational curves of degree $d$ on a general quintic threefold via topological quantum field theory. By the algebro-geometric methods, the mathematicians checked these results up to degree 4. We here present how to use the localization of $\overline{M}_{0,0}(\mathbb{P}^4, d)$ and Bott’s formula for verifying these results up to degree 6.

Alternatively, we need to compute the Gromov-Witten invariant of class $\beta = d\ell \in H_2(X, \mathbb{Z})$, that is

$$N_d = \int_{\overline{M}_{0,0}(\mathbb{P}^4, d)}^{\text{virt}} 1.$$ 

By Example 2.5, we have

$$N_d = \int_{\overline{M}_{0,0}(\mathbb{P}^4, d)} c_{5d+1}(\mathcal{N}_d).$$

By [3, Corollary 9.1.4] and the arguments as in [16, Example 2.2], we have

$$N_d = \int_{\overline{M}_{0,0}(\mathbb{P}^4, d)} c_{5d+1}(\mathcal{N}_d) = \sum_{\Gamma} \frac{c_{5d+1}(\mathcal{N}_d|\Gamma)}{a_\Gamma e^{\ell}(N_\Gamma)},$$
where the sum runs over all graphs corresponding the fixed point components of the natural action of $T$ on $\overline{M}_{0,0}([\mathbb{P}^4, d])$, and $c^T_{5d+1}(\mathcal{V}_d|\mathcal{R})$ is defined by the following formula

$$c^T_{5d+1}(\mathcal{V}_d|\mathcal{R}) = \prod_{e} \left( \prod_{a,b \geq 0, a+b=5d_e} \frac{a \lambda_{i_{v_1}(e)} + b \lambda_{i_{v_2}(e)}}{d_e} \right) \prod_{v} (5\lambda_{i\nu})^{1-\text{val}(v)}.$$ 

We illustrate the computation of the numbers $N_1, N_2, N_3, N_4, N_5, N_6$ by SINGULAR as follows:

```plaintext
> LIB "schubert.lib";
> ring r = 0,x,dp;
> for (int d=1;d<=6;d++) {rationalCurve(d);}

2875
4876875/8
8564575000/27
15517926796875/64
229305888887648
248249742157695375
```

By [3, Example 7.4.4.1], for $1 \leq d \leq 9$, the numbers $N_d$ are related with the numbers $n_d$ by the following formula:

$$N_d = \sum_{k|d} \frac{n_d/k}{k^3}.$$ 

In particular, we have

$$N_1 = n_1, N_2 = \frac{n_1}{8} + n_2, N_3 = \frac{n_1}{27} + n_3,$$

$$N_4 = \frac{n_1}{64} + \frac{n_2}{8} + n_4, N_5 = \frac{n_1}{125} + n_5, N_6 = \frac{n_1}{216} + \frac{n_2}{27} + \frac{n_3}{8} + n_6.$$ 

Thus we have

$$n_1 = N_1 = 2875,$$

$$n_2 = N_2 - \frac{n_1}{8} = \frac{4876875}{8} - \frac{2875}{8} = 609250,$$

$$n_3 = N_3 - \frac{n_1}{27} = \frac{8564575000}{27} - \frac{2875}{27} = 317206375,$$

$$n_4 = N_4 - \frac{n_2}{8} - \frac{n_1}{64} = \frac{15517926796875}{64} - \frac{609250}{8} - \frac{2875}{64} = 242467530000,$$

$$n_5 = N_5 - \frac{n_1}{125} = \frac{229305888887648}{125} - \frac{2875}{125} = 229305888887625,$$

$$n_6 = N_6 - \frac{n_3}{8} - \frac{n_2}{27} - \frac{n_1}{216}$$

$$= \frac{248249742157695375}{8} - \frac{317206375}{27} - \frac{609250}{216} - \frac{2875}{216} = 248249742118022000.$$
The results are in agreement with [2, Table 4] up to degree 6.

6. Rational curves on complete intersections

Using the method applied to the quintic hypersurfaces in $\mathbb{P}^4$, we can also deal with any complete intersection of hypersurfaces. In Section 1.1, we have already shown that there are five complete intersection Calabi-Yau threefolds of type $(d_1, \ldots, d_k)$ in $\mathbb{P}^{k+3}$. In this section, we present how to compute the numbers of rational curves of degree $d$ on general complete intersection Calabi-Yau threefolds using the localization of moduli spaces of stable maps and Bott’s formula.

Let $X$ be a general complete intersection Calabi-Yau threefold of type $(d_1, \ldots, d_k)$ in $\mathbb{P}^{k+3}$ and let $d$ be a positive integer. We need to compute the Gromov-Witten invariant of the cycle class $d\ell \in A_1(X)$, that is

\begin{equation}
N_d^{(d_1, \ldots, d_k)} = \int_{\overline{M}_{0,0}(X, d\ell)} \left[ \overline{M}_{0,0}(X, d\ell) \right]^{\text{virt}}.
\end{equation}

By arguments similar to those in [3, Example 7.1.5.1], we construct the virtual fundamental class $\left[ \overline{M}_{0,0}(X, d\ell) \right]^{\text{virt}}$ as follows. Let $\mathcal{V}_d^{(d_1, \ldots, d_k)}$ be the vector bundle on the moduli space $\overline{M}_{0,0}(\mathbb{P}^{k+3}, d)$ whose fiber over a stable map $f : C \to \mathbb{P}^{k+3}$ is $H^0(C, f^*(\mathcal{O}_{\mathbb{P}^{k+3}}(d_1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^{k+3}}(d_k)))$.

In this case, $\overline{M}_{0,0}(X, d\ell)$ is the zero locus of a general section $s$ of $\mathcal{V}_d^{(d_1, \ldots, d_k)}$. Moreover, we have

$\left[ \overline{M}_{0,0}(X, d\ell) \right]^{\text{virt}} = s^*[C_Y],$

where $Z = \overline{M}_{0,0}(X, d\ell), Y = \overline{M}_{0,0}(\mathbb{P}^{k+3}, d)$, and $C_Y$ is the normal cone to $Z$ in $Y$.

By [3, Lemma 7.1.5], we obtain

\begin{equation}
i_* \left[ \overline{M}_{0,0}(X, d\ell) \right]^{\text{virt}} = c_{(k+4)d+k}(\mathcal{V}_d^{(d_1, \ldots, d_k)}) \in A^*(\overline{M}_{0,0}(\mathbb{P}^{k+3}, d)),
\end{equation}

where $i : \overline{M}_{0,0}(X, d\ell) \hookrightarrow \overline{M}_{0,0}(\mathbb{P}^{k+3}, d)$ is an embedding. This implies the following formula:

\begin{equation}
N_d^{(d_1, \ldots, d_k)} = \int_{\overline{M}_{0,0}(\mathbb{P}^{k+3}, d)} c_{(k+4)d+k}(\mathcal{V}_d^{(d_1, \ldots, d_k)}).
\end{equation}

In other words, we have

\begin{equation}
\mathcal{V}_d^{(d_1, \ldots, d_k)} = \bigoplus_{i=1}^k \mathcal{V}_d^{(d_i)},
\end{equation}

where $\mathcal{V}_d^{(d_i)}$ is the vector bundle on the moduli space $\overline{M}_{0,0}(\mathbb{P}^{k+3}, d)$ whose fiber over a stable map $f : C \to \mathbb{P}^{k+3}$ is $H^0(C, f^*(\mathcal{O}_{\mathbb{P}^{k+3}}(d_i)))$. Note that $\mathcal{V}_d^{(5)}$ is $\mathcal{V}_d$ in the quintic case. Therefore, we have the following formula:

\begin{equation}
N_d^{(d_1, \ldots, d_k)} = \int_{\overline{M}_{0,0}(\mathbb{P}^{k+3}, d)} \prod_{i=1}^k c_{d_id+1}(\mathcal{V}_d^{(d_i)}).
\end{equation}
Using Corollary 9.1.4 in [3], we have

\[ N_d^{(d_1,\ldots,d_k)} = \sum_{\Gamma} \frac{\prod_{i=1}^{k} c_{d_i,d+1}^T(N_d^{(d_i)}|\Gamma)}{a_T e^T(N_\Gamma)}, \]

where the sum runs over all graphs corresponding to the fixed point components of a torus action on \( M_{0,0}(\mathbb{P}^{k+3},d) \). For each graph \( \Gamma \) and each \( i \), the \( T \)-equivariant Chern class \( c_{d_i,d+1}^T(N_d^{(d_i)}|\Gamma) \) is computed by the following formula:

\[ c_{d_i,d+1}^T(N_d^{(d_i)}|\Gamma) = \prod_e \left( \prod_{a,b \in \mathbb{N}, a+b=d, d_e} \frac{a\lambda_{v_1(e)} + b\lambda_{v_2(e)}}{d_e} \right) \prod_v (d_i \lambda_{i_v})^{1-\text{val}(v)}, \]

where the products run over all the edges and vertices of \( \Gamma \).

We illustrate the computation of the numbers \( N_1^{(4,2)} \), \( N_1^{(3,3)} \), \( N_1^{(3,2,2)} \), \( N_1^{(2,2,2,2)} \) by SINGULAR as follows:

```plaintext
> LIB "schubert.lib";
> ring r = 0,x,dp;
> list l = list(4,2),list(3,3),list(3,2,2),list(2,2,2,2);
> for (int i=1;i<=4;i++) {rationalCurve(1,l[i]);}
```

```plaintext
1280
1053
720
512
```

These results are the numbers \( n_1^{(4,2)} = 1280 \), \( n_1^{(3,3)} = 1053 \), \( n_1^{(3,2,2)} = 720 \), \( n_1^{(2,2,2,2)} = 512 \) of lines on the general complete intersection Calabi-Yau threefolds. They agree with [13, Table 1.3].

The numbers \( N_2^{(4,2)} \), \( N_2^{(3,3)} \), \( N_2^{(3,2,2)} \), \( N_2^{(2,2,2,2)} \) are computed as follows:

```plaintext
> for (int i=1;i<=4;i++) {rationalCurve(2,l[i]);}
```

```plaintext
92448
423549/8
22518
9792
```

The numbers \( n_2^{(4,2)} = 92448 \), \( n_2^{(3,3)} = 423549/8 \), \( n_2^{(3,2,2)} = 22518 \), \( n_2^{(2,2,2,2)} = 9792 \), of conics on the general complete intersection Calabi-Yau threefolds, respectively, are computed as follows:

\[ n_2^{(4,2)} = N_2^{(4,2)} - \frac{n_1^{(4,2)}}{8} = 92448 - \frac{9288}{8} = 92288, \]

\[ n_2^{(3,3)} = N_2^{(3,3)} - \frac{n_1^{(3,3)}}{8} = \frac{423549}{8} - \frac{1053}{8} = 52812, \]

\[ n_2^{(3,2,2)} = N_2^{(3,2,2)} - \frac{n_1^{(3,2,2)}}{8} = \frac{22518}{8} - \frac{720}{8} = 22428, \]
\[ n_2^{(2,2,2,2)} = N_2^{(2,2,2,2)} - \frac{n_1^{(2,2,2,2)}}{8} = 9792 - \frac{512}{8} = 9728. \]

The numbers \( N_3^{(4,2)}, N_3^{(3,3)}, N_3^{(3,2,2)}, N_3^{(2,2,2,2)} \) are computed as follows:

```cpp
> for (int i=1;i<=4;i++) {rationalCurve(3,1[i]);}
422690816/27
6424365
4834592/3
11239424/27
```

The numbers \( N_3^{(4,2)}, N_3^{(3,3)}, N_3^{(3,2,2)}, N_3^{(2,2,2,2)} \) of conics on the general complete intersection Calabi-Yau threefolds of types \((4, 2), (3, 3), (3, 2, 2), (2, 2, 2), (2, 2, 2, 2)\), respectively, are computed as follows:

\[
\begin{align*}
n_3^{(4,2)} &= N_3^{(4,2)} - \frac{n_1^{(4,2)}}{27} = \frac{422690816}{27} - \frac{1280}{27} = 15655168, \\
n_3^{(3,3)} &= N_3^{(3,3)} - \frac{n_1^{(3,3)}}{27} = \frac{6424365}{27} - \frac{1053}{27} = 6424326, \\
n_3^{(3,2,2)} &= N_3^{(3,2,2)} - \frac{n_1^{(3,2,2)}}{27} = \frac{4834592}{27} - \frac{720}{27} = 1611504, \\
n_3^{(2,2,2,2)} &= N_3^{(2,2,2,2)} - \frac{n_1^{(2,2,2,2)}}{27} = \frac{11239424}{27} - \frac{512}{27} = 416256.
\end{align*}
\]

These numbers agree with [7, Theorem 1.1].

The numbers \( N_4^{(4,2)}, N_4^{(3,3)}, N_4^{(3,2,2)}, N_4^{(2,2,2,2)} \) are computed as follows:

```cpp
> for (int i=1;i<=4;i++) {rationalCurve(4,1[i]);}
3883914084
72925120125/64
672808059/4
25705160
```

The numbers \( n_4^{(4,2)}, n_4^{(3,3)}, n_4^{(3,2,2)}, n_4^{(2,2,2,2)} \) of conics on the general complete intersection Calabi-Yau threefolds of types \((4, 2), (3, 3), (3, 2, 2), (2, 2, 2), (2, 2, 2, 2)\), respectively, are computed as follows:

\[
\begin{align*}
n_4^{(4,2)} &= N_4^{(4,2)} - \frac{n_2^{(4,2)}}{8} - \frac{n_1^{(4,2)}}{64} = \frac{3883914084}{8} - \frac{92288}{8} - \frac{1280}{64} = 3883902528, \\
n_4^{(3,3)} &= N_4^{(3,3)} - \frac{n_2^{(3,3)}}{8} - \frac{n_1^{(3,3)}}{64} = \frac{72925120125}{8} - \frac{52812}{8} - \frac{1053}{64} = 1139448384, \\
n_4^{(3,2,2)} &= N_4^{(3,2,2)} - \frac{n_2^{(3,2,2)}}{8} - \frac{n_1^{(3,2,2)}}{64} = \frac{672808059}{8} - \frac{22428}{8} - \frac{720}{64} = 168199200, \\
n_4^{(2,2,2,2)} &= N_4^{(2,2,2,2)} - \frac{n_2^{(2,2,2,2)}}{8} - \frac{n_1^{(2,2,2,2)}}{64} = \frac{25705160}{8} - \frac{9728}{8} - \frac{512}{64} = 25703936.
\end{align*}
\]

The numbers \( N_5^{(4,2)}, N_5^{(3,3)}, N_5^{(3,2,2)}, N_5^{(2,2,2,2)} \) are computed as follows:
> for (int i=1;i<=4;i++) {rationalCurve(5,l[i]);}
29773082054656/25
31223486573928/125
541923292944/25
244747968512/125

The numbers $n_5^{(4,2)}$, $n_5^{(3,3)}$, $n_5^{(3,2,2)}$, $n_5^{(2,2,2,2)}$ of conics on the general complete intersection Calabi-Yau threefolds of types $(4, 2)$, $(3, 3)$, $(3, 2, 2)$, $(2, 2, 2, 2)$, respectively, are computed as follows:

$$n_5^{(4,2)} = N_5^{(4,2)} - \frac{n_1^{(4,2)}}{125} = 417874607302656 - \frac{92288}{27} - \frac{1280}{216} = 417874605342336,$$

$$n_5^{(3,3)} = N_5^{(3,3)} - \frac{n_1^{(3,3)}}{125} = 501287722516269/8 - \frac{52812}{27} - \frac{1053}{216} = 62660964509532,$$

The numbers $N_6^{(4,2)}$, $N_6^{(3,3)}$, $N_6^{(3,2,2)}$, $N_6^{(2,2,2,2)}$ are computed as follows:

$$n_6^{(4,2)} = N_6^{(4,2)} - \frac{n_3^{(4,2)}}{8} - \frac{n_2^{(4,2)}}{27} - \frac{n_1^{(4,2)}}{216} = 417874607302656 - \frac{15655168}{8} - \frac{92288}{27} - \frac{1280}{216} = 417874605342336,$$

$$n_6^{(3,3)} = N_6^{(3,3)} - \frac{n_3^{(3,3)}}{8} - \frac{n_2^{(3,3)}}{27} - \frac{n_1^{(3,3)}}{216} = 501287722516269/8 - \frac{6424326}{8} - \frac{52812}{27} - \frac{1053}{216} = 62660964509532,$$
\[
\begin{align*}
n_6^{(3,2,2)} &= N_6^{(3,2,2)} - \frac{n_3^{(3,2,2)}}{8} - \frac{n_2^{(3,2,2)}}{27} - \frac{n_1^{(3,2,2)}}{216} \\
&= 3195558106836 - \frac{1611504}{8} - \frac{22428}{27} - \frac{720}{216} \\
&= 319557904564, \\
n_6^{(2,2,2,2)} &= N_6^{(2,2,2,2)} - \frac{n_3^{(2,2,2,2)}}{8} - \frac{n_2^{(2,2,2,2)}}{27} - \frac{n_1^{(2,2,2,2)}}{216} \\
&= \frac{511607926784}{3} - \frac{416256}{8} - \frac{9728}{27} - \frac{512}{216} \\
&= 170535923200.
\end{align*}
\]

The results are all in agreement with \[18\, \text{Table 1}\] up to degree 6.

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