Systems of Gibbons-Tsarev type
and integrable 3-dimensional models

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Abstract We review the role of Gibbons-Tsarev-type systems in classification of integrable multi-dimensional hydrodynamic-type systems. Our main observation is an universality of Gibbons-Tsarev-type systems. We also construct explicitly a wide class of 3-dimensional hydrodynamic-type systems corresponding to the simplest possible Gibbons-Tsarev-type system.

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1 Introduction

Integrable equations play an important role in both Mathematics and Physics. Unfortunately, rigorous and universal definition of integrability applicable in all situations does not exist. Different viewpoints on the integrability can be found in [1, 2]. It is well known that integrable 2-dimensional PDEs
\[ u_t = F(u, u_x, \ldots, u_{n_x}) \]  
like the KdV-equation, for any N possess families of exact solutions depending on arbitrary constants \( c_1, \ldots, c_N \). All these finite-gap and solitonic-type solutions can be constructed by so-called ODE-reductions. A pair of compatible N-component systems of ODEs
\[ r^i_x = f^i(r^1, \ldots, r^N), \quad r^i_t = g^i(r^1, \ldots, r^N), \quad i = 1, \ldots, N \]
is called an ODE-reduction of (1.1) if there exists a function \( U(r^1, \ldots, r^N) \) such that \( u = U(r^1(x, t), \ldots, r^N(x, t)) \) satisfies (1.1) for any solution \( r^1(x, t), \ldots, r^N(x, t) \) of (1.2). It is clear that the solution \( u \) depends on \( N \) arbitrary parameters being initial values for (1.2) at generic point. The existence of special ODE-reductions for arbitrary \( N \) can be chosen as a criterion of integrability for equation (1.1). For example, one can assume that (1.1) admits a series of differential ODE-constraints
\[ u_{m,x} = G_m(u, u_x, \ldots, u_{m-1,x}), \]
where \( m \) is arbitrary. Clearly, equation (1.1) and the ODE-constraint can be rewritten as a pair of compatible dynamical systems with respect to \( x \) and \( t \). Another example of ODE-reductions is provided by Dubrovin’s equations [3]. However, in the 2-dimensional case there exist more efficient and constructive integrability criteria, like the existence of higher local symmetries or conservation laws (see [4] and references therein).

If the number \( d \) of independent variables is greater than 2, then higher \textbf{local} symmetries for integrable models do not exist (for some generalization of the symmetry approach to the case of non-local symmetries see [5]). In such a situation the existence of \( N \)-component reductions can be regarded as one of the most powerful methods of searching for new integrable models. Notice that one has to consider for the reductions some compatible systems of PDEs of dimension \( \leq d - 1 \) instead of ODEs (1.2).

In [6] this approach has been systematically applied to some classes of 3-dimensional systems of the form
\[ \sum_{j=1}^{n} a_{ij}(u) u_{j,t} + \sum_{j=1}^{n} b_{ij}(u) u_{j,y} + \sum_{j=1}^{n} c_{ij}(u) u_{j,x} = 0, \quad i = 1, \ldots, n + k, \]
where \( u = (u_1, \ldots, u_n) \), and \( k \geq 0 \). Pairs of compatible diagonal semi-Hamiltonian (see formula (2.12)) hydrodynamic-type systems of the form
\[ r^i_x = v^i(r^1, \ldots, r^N), \quad r^i_y = w^i(r^1, \ldots, r^N)r^i_x \]
i = 1, 2, …. N,
have been taken for reductions. According to definition of reductions, the corresponding solutions of \((1.3)\) are determined by some functions \(U^i(r_1, ..., r_N), i = 1, ..., n\) converting any solution of \((1.4)\) to a solution of \((1.3)\). In hydrodynamics such solutions describe nonlinear interaction of \(N\) planar waves. Sometimes they are called \(N\)-phase solutions.

Clearly, the general solution of \((1.4)\) contains \(N\) arbitrary functions of one variable. It turns out that functions \(v^i, w^i\) in the reduction \((1.4)\) may contain additional functions of one variables as functional parameters and the number of these functions is not greater then \(N\). In \([6]\) the existence of hydrodynamic reductions \((1.4)\) locally parameterized by \(N\) functions of one variables, where \(N\) is arbitrary, was proposed as a criterion of integrability for systems \((1.3)\). The corresponding \(N\)-phase solutions depend on \(2N\) arbitrary functions of one variables.

Usually the integrability of systems \((1.3)\) is associated with a representation of \((1.3)\) as commutativity conditions for a pair of vector fields \([7]\). For systems that admit the pseudopotential representation \([8, 9, 10, 11]\) these vector fields are Hamiltonian whereas for some integrable models the vector fields have more complicated structure. Moreover, for some systems the vector fields depend on a spectral parameter. Thus it is very difficult to choose any constructive class of the vector fields covering all known examples and to propose an universal definition of integrability based on the commutativity of vector fields. The same problems arise with definitions of integrability given in terms of dispersionless Lax or zero-curvature representations.

Quite the contrary, the hydrodynamic reduction approach is universal. This means that all integrable models known by now admit the hydrodynamic reductions. All notions of this approach can be rigorously defined (see Section 2). It was demonstrated in \([6]\) that the existence of hydrodynamic reductions can be algorithmically verified for a given system \((1.3)\) and what is more can be efficiently used for classification of integrable cases.

Families of systems \((1.4)\) parameterized by \(N\) functions of one variables can be described in terms of the so-called systems of Gibbons-Tsarev type (GT-type systems). The GT-type systems play a crucial role in the approach to integrability based on the hydrodynamic reductions.

**Definition.** A compatible system of PDEs of the form
\[
\partial_i p_j = f(p_i, p_j, u_1, ..., u_n) \partial_i u_1, \quad i \neq j, \ i, j = 1, ..., N, \\
\partial_i u_m = g_m(p_i, u_1, ..., u_n) \partial_i u_1, \quad m = 2, ..., n, \ i = 1, ..., N, \\
\partial_i \partial_j u_1 = h(p_i, p_j, u_1, ..., u_n) \partial_i u_1 \partial_j u_1, \quad i \neq j, \ i, j = 1, ..., N
\] (1.5)
is called \(n\)-fields GT-type system. Here \(p_1, ..., p_N, u_1, ..., u_n\) are functions of \(r^1, ..., r^N, \ N \geq 3\) and \(\partial_i = \frac{\partial}{\partial r_i}\). Notice that the compatibility conditions give rise to a system of functional equations for the functions \(f, g_k, h\) and these equations don’t depend on \(N\).

**Example 1 \([10]\).** The system
\[
\partial_i p_j = \frac{p_j(p_j - 1)}{p_i - p_j} \partial_i u_1, \quad \partial_i u_m = \frac{u_m(u_m - 1)}{p_i - u_m} \partial_i u_1, \quad m = 2, ..., n, 
\] (1.6)
\[ \partial_i \partial_j u_1 = \frac{2p_i p_j - p_i - p_j}{(p_i - p_j)^2} \partial_i u_1 \partial_j u_1, \quad i, j = 1, \ldots, N, \quad i \neq j \] 

(1.7)
is an \( n \)-field GT-type system for any \( n, N \). □

The original Gibbons-Tsarev system \[12\] is a degeneration of \((1.6), (1.7)\). A wide class of integrable systems \((1.3)\) related to \((1.6), (1.7)\) is described in \[10\]. An elliptic version of this GT-type system and the corresponding integrable 3-dimensional systems were proposed in \[11\].

**Definition.** Two GT-type systems are called *equivalent* if they are related by a transformation of the form

\[ p_i \to \lambda(p_i, u_1, \ldots, u_n), \quad i = 1, \ldots, N, \] 

(1.8)

\[ u_m \to \mu_m(u_1, \ldots, u_n), \quad m = 1, \ldots, n. \] 

(1.9)

**Remark 1.** Our Definitions and formulas \((1.5), (1.8), (1.8)\) admit a coordinates-free interpretation. Let \( M \) be a bundle with one-dimensional fiber \( E \) and \( n \)-dimensional base \( F \). Then each of \( p_i \) is a coordinate on \( E \) and \( u_1, \ldots, u_n \) are some coordinates on \( F \). Thus we obtain a notion of a GT-type structure on \( M \). It is likely that there exists a canonical GT-type structure on the natural bundle over the moduli space \( M_g \) of genus \( g \) algebraic curves. Here \( E \) is a curve corresponding to a point in \( M_g \). We will not use coordinates-free language in this paper.

For generic GT-type systems the functions \( f, h \) have a pole at \( p_i = p_j \). However, there exist GT-type systems holomorphic at \( p_i = p_j \).

**Example 2.** The system

\[ \partial_i p_j = 0, \quad \partial_i u_m = g_m(p_i) \partial_i u_1, \quad \partial_i \partial_j u_1 = 0 \] 

(1.10)
is an \( n \)-field GT-type system for any \( n, N \) and any functions \( g_m(x) \). Notice that only special choice of the functions \( g_m(x) \) gives rise to pairs of compatible semi-Hamiltonian systems \((1.4)\).

□

In this paper we study systems \((1.3)\) related to GT-type systems of the form \((1.10)\). The main motivation is the following observation. We examined the 3-dimensional travel wave reductions for known examples of integrable \( d \)-dimensional systems with \( d > 3 \) and found that the GT-type systems corresponding to these reductions are equivalent to \((1.10)\) with rational functions \( g_k \). We believe that this observation gives us an algorithm for constructing of new interesting examples of integrable multi-dimensional systems.

The paper is organized as follows. In Section 2.1 following \[6\], we describe the hydrodynamic reduction method and show that any integrable system \((1.3)\) is related to a GT-type system. Section 2.2 is devoted to the return way from GT-type systems to integrable 3-dimensional systems. Moreover, we present all known to us GT-type systems and give a new interpretation of results obtained in \[10, 11\].

In Section 3 we consider GT-type systems \((1.10)\) with rational functions \( g_m(x) \). Using an algorithm described in Section 2.2, we construct the corresponding families of compatible pairs.
of hydrodynamic-type 2-dimensional systems, and finally 3-dimensional systems of the form \((1.3)\) with arbitrary \(n, k\), whose hydrodynamic reductions are given by our 2-dimensional systems. It turns out that all these 3-dimensional systems possess pseudopotential representations with a spectral parameter. In the generic case, the coefficients of the 3-dimensional systems are expressed in terms of exponents of \(u_i\). Degenerations considered in subsection 3.2 involves polynomials in addition to the exponents. In the case of small \(n\) and \(k\) some of our systems are equivalent to known dispersionless equations of second order. In particular, the generic system corresponding to \(n = 3, k = 1\) is equivalent to the dispersionless Hirota equation

\[
a_1 Z_x Z_{yt} + a_2 Z_y Z_{2t} + a_3 Z_t Z_{xy} = 0, \quad a_1 + a_2 + a_3 = 0.
\]

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2 The GT-type systems and integrability

2.1 The method of hydrodynamic reductions

Recall the definitions of the hydrodynamic reduction method and the corresponding criteria of integrability for 3-dimensional hydrodynamic-type systems [6].

**Definition.** An \((1+1)\)-dimensional hydrodynamic-type system of the form

\[
\dot{r}_i^r = \lambda_i^{r_1, \ldots, r_N} r_i^r, \quad i = 1, \ldots, N,
\]

is called semi-Hamiltonian if the following relation holds

\[
\partial_j \frac{\partial \lambda_m^r}{\lambda_i^r - \lambda_m^r} = \partial_i \frac{\partial \lambda_m^r}{\lambda_j^r - \lambda_m^r}, \quad i \neq j \neq m.
\]

Semi-Hamiltonian systems have infinitely many symmetries and conservation laws of hydrodynamic type [14].

**Definition.** A hydrodynamic reduction of a system \((1.3)\) is defined by a pair of compatible semi-Hamiltonian hydrodynamic-type systems

\[
\dot{r}_i^r = \lambda_i^{r_1, \ldots, r_N} r_i^r, \quad \dot{r}_i^y = \mu_i^{r_1, \ldots, r_N} r_i^y, \quad i = 1, \ldots, N,
\]

and by functions \(u_1(r_1, \ldots, r_N), \ldots, u_n(r_1, \ldots, r_N)\) such that for each solution of \((2.13)\) the functions

\[
u_1 = u_1(r_1, \ldots, r_N), \ldots, u_n = u_n(r_1, \ldots, r_N)
\]

satisfy \((1.3)\).
According to [6] a system (1.3) is called integrable if it possesses as many hydrodynamic reductions as possible. Namely, substituting (2.14) into (1.3), eliminating $t$- and $y$-derivatives via (2.13), and equating coefficients at $r^l_x$ to zero, we obtain

$$\sum_{j=1}^{n} a_{ij}(u) \lambda^l \partial_l u_j + \sum_{j=1}^{n} b_{ij}(u) \mu^l \partial_l u_j + \sum_{j=1}^{n} c_{ij}(u) \partial_j u_j = 0, \quad i = 1, \ldots, n+k, \quad l = 1, \ldots, N. \quad (2.15)$$

For each fixed $l$ this is a linear overdetermined system for $n$ unknowns $\partial_l u_1, \ldots, \partial_l u_n$, whose coefficients do not depend on $l$. This linear system must have non-zero solution so all its $n \times n$ minors must be equal to zero. These minors are polynomials in $\lambda^l, \mu^l$ independent on $l$. We assume that this system of polynomial equations is equivalent to one equation

$$P(\lambda^l, \mu^l) = 0 \quad (2.16)$$

(otherwise $\lambda^l, \mu^l$ are fixed and we have not sufficiently many reductions). Equation (2.16) defines the so-called dispersion algebraic curve. Let $p$ be a coordinate on this curve. Then (2.16) is equivalent to equations

$$\lambda^l = F(p_i, u_1, \ldots, u_n), \quad \mu^l = G(p_i, u_1, \ldots, u_n)$$

for some functions $F, G$. Assume that for generic $p$ the linear system (2.15) has one solution up to proportionality. Solving this system, we obtain

$$\partial_i u_m = g_m(p_i, u_1, \ldots, u_n) \partial_i u_1, \quad m = 2, \ldots, n, \quad i = 1, \ldots, N \quad (2.17)$$

for some functions $g_m$. Rewrite (2.13) in the form

$$r^i_t = F(p_i, u_1, \ldots, u_n) r^i_x, \quad r^i_y = G(p_i, u_1, \ldots, u_n) r^i_x, \quad i = 1, \ldots, N. \quad (2.18)$$

It is easy to see that the compatibility conditions for (2.18) have the form

$$\frac{\partial_i F(p_j)}{F(p_i) - F(p_j)} = \frac{\partial_i G(p_j)}{G(p_i) - G(p_j)}. \quad (2.19)$$

Here we omit arguments $u_1, \ldots, u_n$ in $F, G$. From (2.19) we can find $\partial_i p_j$ in the form

$$\partial_i p_j = f(p_i, p_j, u_1, \ldots, u_n) \partial_i u_1, \quad i \neq j, \quad i, j = 1, \ldots, N. \quad (2.20)$$

Finally, the compatibility conditions $\partial_i \partial_j u_m = \partial_j \partial_i u_m$ give rise to

$$\partial_i \partial_j u_1 = h(p_i, p_j, u_1, \ldots, u_n) \partial_i u_1 \partial_j u_1, \quad i \neq j, \quad i, j = 1, \ldots, N. \quad (2.21)$$

Collecting equations (2.15), (2.20), (2.21) together, we obtain a system of the form (1.3). Since we want to have as many reductions as possible, we assume that this system is in involution (i.e. fully compatible). In this case the family of hydrodynamic reductions (2.18) locally depends on $N$ functions in one variable.
2.2 From GT-type systems to integrable models

In the classification works [6] the authors start from a class of systems (1.3) with fixed small \( n \) and \( k \), calculate the corresponding GT-type system and derive integrability conditions for (1.3) from the compatibility conditions for the GT-type system.

In this section we trace the return way and show how to construct wide classes of integrable systems (1.3) with arbitrary \( n \) and \( k \) starting from a given GT-type system. We also describe our previous results [10, 11] from this point of view.

A list of known one-field GT-type systems is given by the following examples.

**Example 3.** Let \( P(x) = a_3x^3 + a_2x^2 + a_1x + a_0 \). Then

\[
\partial_{ij}u = \frac{K_2(p_i, p_j)u^2 + K_1(p_i, p_j)u + K_0(p_i, p_j)}{\partial^2_{u}(p_i - p_j)^2} \partial_i u \partial_j u,
\]

\[
\partial_i p_j = \frac{P(p_j)(u - p_i)}{P(u)(p_i - p_j)} \partial_i u, \quad i, j = 1, ..., N, \quad i \neq j,
\]

where

\[
K_2(p_i, p_j) = 2a_3(p_i - p_j)^2,
\]

\[
K_1(p_i, p_j) = -a_3(p_i^2p_j + p_i p_j^2) + a_2(p_i^2 + p_j^2 - 4p_ip_j) - a_1(p_i + p_j) - 2a_0,
\]

\[
K_0(p_i, p_j) = 2a_3p_i^2p_j^2 + a_2(p_i^2p_j + p_i p_j^2) + a_1(p_i^2 + p_j^2) + a_0(p_i + p_j)
\]

is an one-field GT-type system.

Using transformations of the form

\[
u \rightarrow au + b, \quad p \rightarrow ap_i + b
\]

one can put the polynomial \( P \) to one of the canonical forms: \( P(x) = x(x-1), P(x) = x \), or \( P(x) = 1 \). \( \Box \)

Note that in the case \( P(x) = x(x - 1) \) we return to the Example 1 with \( n = 1 \).

**Example 4.** Let

\[
\theta(z, \tau) = \sum_{\alpha \in \mathbb{Z}} (-1)^\alpha e^{2\pi i (\alpha z + \frac{\alpha(\alpha + 1)}{2} \tau)}, \quad \rho(z, \tau) = \frac{\theta(z, \tau)}{\theta(z, \tau)}.
\]

Then

\[
\partial_\alpha p_\beta = \frac{1}{2\pi i} \left( \rho(p_\alpha - p_\beta) - \rho(p_\alpha) \right) \partial_\alpha \tau,
\]

\[
\partial_\alpha \partial_\beta \tau = -\frac{1}{\pi i} \rho'(p_\alpha - p_\beta) \partial_\alpha \tau \partial_\beta \tau,
\]

where \( \alpha, \beta = 1, ..., N, \quad \alpha \neq \beta \), is an one-field GT-type system. \( \Box \)
It turns out that if we add the following equations:

$$\partial_i u_m = f(p_i, u_m, u) \partial_i u_1, \quad m = 2, ..., n$$

to any one-field GT-type system

$$\partial_i p_j = f(p_i, p_j, u_1) \partial_i u_1, \quad \partial_i \partial_j u_1 = h(p_i, p_j, u_1) \partial_i u_1 \partial_j u_1,$$

then the system of PDEs thus obtained is in involution. One can obtain \(n\)-fields GT-type system for any \(n\) in this way. We call this procedure regular extension. For example, in the case of Example 4 the regular extension is given by

$$\partial_\alpha u_\beta = \frac{1}{2\pi i} \left( \rho(p_\alpha - u_\beta) - \rho(p_\alpha) \right) \partial_\alpha \tau, \quad \beta = 1, ..., n - 1.$$  

The Example 1 is a regular extension of Example 3 with \(P(x) = x(x - 1)\). As far as we know, the regular extensions of Examples 3, 4 are the only GT-type systems appeared in the literature. In this paper we investigate the simplest possible GT-type system from Example 2 and obtain the corresponding systems of the type (1.3) which are probably new.

A basic object associated with a given GT-type system is a pair of compatible (1+1)-dimensional hydrodynamic-type systems of the form (2.18). One should solve the functional equation (2.19) in order to find all possible functions \(F, G\). Notice that the derivatives in (2.19) are supposed to be calculated by virtue of the GT-type system.

The existence of non-constant solutions \(F, G\) for the functional equation (2.19) is an additional condition, which we impose on the GT-type system. For instance, in the case of Example 2 the solutions exist not for any functions \(g_m\).

The following statement can be proved straightforwardly.

**Proposition 1.** Any one-field GT-type system having a non-constant solution of the form \(F(p, u), G(p, u)\) for the functional equation (2.19) is equivalent to one described in Example 3. □

In [10] we found the following solutions \(F, G\) for the \(n\)-field GT-type system in Example 1:

**Proposition 2.** Fix \(s_1, ..., s_{n+2} \in \mathbb{C}\). Consider the following compatible overdetermined system of linear PDEs:

$$\frac{\partial^2 h}{\partial u_i \partial u_k} = \frac{s_j}{u_j - u_k} \cdot \frac{\partial h}{\partial u_k} + \frac{s_k}{u_k - u_j} \cdot \frac{\partial h}{\partial u_j}, \quad i, j = 1, ..., n, \quad j \neq k,$$

and

$$\frac{\partial^2 h}{\partial u_i \partial u_j} = -\left(1 + \sum_{k=1}^{n+2} s_k \right) \frac{s_j}{u_j (u_j - 1)} \cdot h + \frac{s_j}{u_j (u_j - 1)} \sum_{k \neq j} u_k (u_k - 1) \cdot \frac{\partial h}{\partial u_k} +$$

$$\left(\sum_{k \neq j}^{n} \frac{s_k}{u_j - u_k} + \frac{s_j + s_{n+1}}{u_j} + \frac{s_j + s_{n+2}}{u_j - 1}\right) \cdot \frac{\partial h}{\partial u_j}.$$
It is easy to show that the vector space $\mathcal{H}$ of all solutions is $n + 1$-dimensional. For any $h \in \mathcal{H}$ we put

$$S(h, p) = \sum_{1 \leq i \leq n} u_i(u_i - 1)(p - u_1)\ldots(p - u_n)h_{u_i} + (1 + \sum_{1 \leq i \leq n+2} s_i)(p - u_1)\ldots(p - u_n)h.$$ 

Clearly, $S$ is a polynomial of degree $n$ in $p$.

Let $h_1, h_2, h_3$ be linearly independent elements of $\mathcal{H}$. Then

$$F = \frac{S(h_1, p)}{S(h_3, p)}, \quad G = \frac{S(h_2, p)}{S(h_3, p)}$$

(2.22)

satisfy the functional equation (2.19) for reductions.

In the case when the degree of $S$ is less than $n$ the solutions are given by more complicated determinant formulas (see [10]). □

For the regular extensions of the Example 4 solutions of the functional equation (2.19) are given by the same formula (2.22), where

$$S(h, p) = \sum_{1 \leq \alpha \leq n} \frac{\theta(u_{\alpha})\theta(p - u_{\alpha} - \eta)}{\theta(u_{\alpha} + \eta)\theta(p - u_{\alpha})}h_{u_{\alpha}} - (s_1 + \ldots + s_n)\frac{\theta'(0)\theta(p - \eta)}{\theta(\eta)\theta(p)}h.$$ 

Here $\eta = s_1 u_1 + \ldots + s_n u_n + r\tau + \eta_0$, where $s_1, \ldots, s_n, r, \eta_0$ are arbitrary constants and $h(u_1, \ldots, u_n, \tau)$ is a solution of the following elliptic hypergeometric system:

$$h_{u_{\alpha}u_{\beta}} = s_\beta \left( \rho(u_{\beta} - u_{\alpha}) + \rho(u_{\alpha} + \eta) - \rho(u_{\beta}) - \rho(\eta) \right)h_{u_{\alpha}} + s_\alpha \left( \rho(u_{\alpha} - u_{\beta}) + \rho(u_{\beta} + \eta) - \rho(u_{\alpha}) - \rho(\eta) \right)h_{u_{\beta}},$$

$$h_{u_{\alpha}u_{\alpha}} = s_\alpha \sum_{\beta \neq \alpha} \left( \rho(u_{\alpha}) + \rho(\eta) - \rho(u_{\alpha} - u_{\beta}) - \rho(u_{\beta} + \eta) \right)h_{u_{\beta}} + \left( \sum_{\beta \neq \alpha} s_\beta \rho(u_{\alpha} - u_{\beta}) + (s_\alpha + 1)\rho(u_{\alpha} + \eta) + s_\alpha \rho(-\eta) + (s_0 - s_\alpha - 1)\rho(u_{\alpha}) + 2\pi i r \right)h_{u_{\alpha}} - s_0 s_\alpha(\rho'(u_{\alpha}) - \rho'(\eta))h,$$

$$h_{\tau} = \frac{1}{2\pi i} \sum_{\beta} \left( \rho(u_{\beta} + \eta) - \rho(\eta) \right)h_{u_{\beta}} - \frac{s_0}{2\pi i} \rho'(\eta)h.$$

Given a GT-type system and a solution $F, G$ of the functional equation (2.19) for reduction, one can easily construct an integrable system of the form (1.3). Integer $k$ is called the defect of the system.
Lemma 1. Consider the linear space $V$ of functions in $p$ spanned by
\[
\{F(p, u_1, ..., u_n)g_j(p, u_1, ..., u_n), G(p, u_1, ..., u_n)g_j(p, u_1, ..., u_n), g_j(p, u_1, ..., u_n); \quad j = 1, ..., n\}. 
\]
Here by definition $g_1 = 1$. Then the system of the form (1.3) with reductions (2.18) consists of $l$ equations iff $V$ is $(3n - l)$-dimensional. Moreover, the coefficients of (1.3) are defined by relations:
\[
\sum_{j=1}^{n} \left(a_{ij}(u)F(p, u_1, ..., u_n) + b_{ij}(u)G(p, u_1, ..., u_n) + c_{ij}(u)g_j(p, u_1, ..., u_n)\right) = 0, \quad i = 1, ..., n + k. 
\]

An explicit form of integrable systems (1.3) corresponding to Examples 3, 4 can be found in [10, 11].

3 Weakly nonlinear 3-dimensional systems

For generic GT-type systems the functions $f$, $h$ have poles at $p_i = p_j$. However, there exist GT-type systems holomorphic at $p_i = p_j$. We call integrable system (1.3) weakly nonlinear if the corresponding GT-type system is holomorphic at $p_i = p_j$. It is possible to check that if $k = 0$, then any 2-dimensional system describing travel wave solutions $u = u(c_1 x + c_2 y + c_3 t, c_4 x + c_5 y + c_6 t)$ for weakly nonlinear 3-dimensional system (1.3) is a weakly nonlinear 2-dimensional system in the sense of [13].

Example 5. Consider the following 3-dimensional system (see [6]):
\[
v_t + av_x + pv_y + qw_y = 0, \quad w_t + bw_x + rv_y + sw_y = 0, \quad (3.23)
\]
where
\[
a = w, \quad b = v, \quad r = \frac{P(w)}{w - v}, \quad q = \frac{P(v)}{v - w}, \\
s = \frac{P(v)}{w - v} + \frac{1}{3}P'(v), \quad p = \frac{P(w)}{v - w} + \frac{1}{3}P'(w).
\]
Here $P$ is an arbitrary polynomial of degree three.

The corresponding GT-type system is given by
\[
\partial_i p_j = \frac{P(w)}{(w - v)P(v)} p_j^2 p_i + \left(\frac{1}{w - v} + \frac{P'(v)}{P(v)}\right) p_j p_i - \left(\frac{1}{v - w} + \frac{P'(w)}{P(w)}\right) p_j - \frac{P(v)}{(v - w)P(w)},
\]
\[
\partial_i v = p_i \partial_i w, \\
\partial_i \partial_j w = \left(\frac{P(w)}{(v - w)P(v)} p_i p_j + \frac{1}{v - w} + \frac{P'(w)}{P(w)}\right) \partial_i w \partial_j w. 
\]
This GT-type system is polynomial in $p_i, p_j$ and therefore the corresponding 3-dimensional system is weakly nonlinear. It is possible to verify that this GT-type system is equivalent to

$$\partial_i p_j = 0, \quad \partial_i u_2 = p_i \partial_i u_1, \quad \partial_i \partial_j u_1 = 0.$$  

□

It was mentioned in [6] that the system (3.23) possesses a hydrodynamic-type Lax representation depending on a spectral parameter. It turns out that this is a general property of 3-dimensional systems corresponding to GT-type systems of the form (1.10).

**Proposition 3.** Let $F(p, u_1, ..., u_n), G(p, u_1, ..., u_n)$ be a solution of the functional equation (2.19) for a GT-type system (1.10). Then the corresponding 3-dimensional system admits the Lax representation

$$\psi_t = F(\xi, u_1, ..., u_n) \psi_x, \quad \psi_y = G(\xi, u_1, ..., u_n) \psi_x,$$

where $\xi$ is a spectral parameter. □

### 3.1 Generic case

Using our observation that the GT-type system from Example 5 is equivalent to (1.10) with rational functions $g_m$, we generalize Example 5 to the case of arbitrary $n$ and $k$.

Consider the $(n+1)$-field GT-type system (1.10) with $g_m = M_m/M$, where $M, M_1, ..., M_{n+1}$ are generic polynomials of degree $n$. Suppose that $M$ has pairwise distinct roots $\lambda_0, \lambda_1, ..., \lambda_n$. Then up to equivalence the GT-type system can be written as

$$\partial_i p_j = 0, \quad \partial_i u_m = \frac{\lambda_m - \lambda_0}{p_i - \lambda_m} \partial_i w, \quad \partial_i \partial_j w = 0 \quad (3.24)$$

with fields denoted by $u_1, ..., u_n, w$.

Let $H_n$ be the linear space of functions in $u_1, ..., u_n$ spanned by $1, e^{u_1}, ..., e^{u_n}$. For any function $g = a_0 + a_1 e^{u_1} + ... + a_n e^{u_n} \in H_n$ we put

$$S_n(g, p) = \frac{a_0}{p - \lambda_0} + \sum_{i=1}^{n} \frac{a_i e^{u_i}}{p - \lambda_i}.$$  

For $k \in \mathbb{N}$ such that $0 < k < n - 1$ we fix functions $h_1, ..., h_k \in H_n$, where $h_i = b_{i,0} + b_{i,1} e^{u_1} + ... + b_{i,n} e^{u_n}$, and define

$$S_{n,k}(g, p) = \det \begin{pmatrix} S_n(g, p) & S_n(h_1, p) & ... & S_n(h_k, p) \\ g & h_1 & ... & h_k \\ a_{n-k+2} & b_{1,n-k+2} & ... & b_{k,n-k+2} \\ ........ & .... & .... & ........ \\ a_n & b_{1,n} & ... & b_{k,n} \end{pmatrix}. \quad (3.25)$$
By definition, $S_{n,0}(g, p) = S_{n}(g, p)$.

**Proposition 4.** Let $g_1$, $g_2$, $g_3$ be linearly independent elements of $H_n$. Then for any $0 \leq k < n - 1$ the functions

$$F = \frac{S_{n,k}(g_1, p_i)}{S_{n,k}(g_3, p_i)}, \quad G = \frac{S_{n,k}(g_2, p_i)}{S_{n,k}(g_3, p_i)}$$

satisfy the functional equation (2.19) for hydrodynamic reductions. □

To find an explicit form of the corresponding 3-dimensional systems we note that

$$\sum_{i=1}^{n} (A_i u_{i,t_1} + B_i u_{i,t_2} + C_i u_{i,x}) = 0$$

is an equation from the 3-dimensional system iff

$$\sum_{i=1}^{n} \frac{\lambda_i - \lambda_0}{p - \lambda_i} \left( A_i S_{n,k}(g_1, p) + B_i S_{n,k}(g_2, p) + C_i S_{n,k}(g_3, p) \right) = 0$$

as function in $p$. Let $g_i = a_{i,0} + a_{i,1} e^{a_{i,1}} + \ldots + a_{i,n} e^{a_{i,n}}$, $i = 1, 2, 3$.

If $k = 0$, then the corresponding 3-dimensional system reads as follows:

$$\sum_{1 \leq j \leq n, j \neq i} (a_{2,1}a_{3,j} - a_{2,j}a_{3,i}) e^{a_{j}} \frac{u_{i,t_1} - u_{j,t_1}}{\lambda_i - \lambda_j} + (a_{2,j}a_{3,0} - a_{3,j}a_{2,0}) \frac{u_{i,t_1}}{\lambda_i - \lambda_0} +$$

$$\sum_{1 \leq j \leq n, j \neq i} (a_{3,i}a_{1,j} - a_{3,j}a_{1,i}) e^{a_{j}} \frac{u_{i,t_2} - u_{j,t_2}}{\lambda_i - \lambda_j} + (a_{3,j}a_{1,0} - a_{1,j}a_{3,0}) \frac{u_{i,t_2}}{\lambda_i - \lambda_0} +$$

$$\sum_{1 \leq j \leq n, j \neq i} (a_{1,i}a_{2,j} - a_{1,j}a_{2,i}) e^{a_{j}} \frac{u_{i,x} - u_{j,x}}{\lambda_i - \lambda_j} + (a_{1,j}a_{2,0} - a_{2,j}a_{1,0}) \frac{u_{i,x}}{\lambda_i - \lambda_0} = 0,$$

where $i = 1, \ldots, n$.

If $k > 0$, then the corresponding 3-dimensional system reads as follows:

$$\sum_{1 \leq j \leq n-k+1, j \neq i} \left( \Delta_i(g_2) \Delta_j(g_3) - \Delta_j(g_2) \Delta_i(g_3) \right) e^{a_{j}} \frac{u_{i,t_1} - u_{j,t_1}}{\lambda_i - \lambda_j} + \left( \Delta_i(g_2) \Delta_0(g_3) - \Delta_0(g_2) \Delta_i(g_3) \right) \frac{u_{i,t_1}}{\lambda_i - \lambda_0} +$$

$$\sum_{1 \leq j \leq n-k+1, j \neq i} \left( \Delta_i(g_3) \Delta_j(g_1) - \Delta_j(g_3) \Delta_i(g_1) \right) e^{a_{j}} \frac{u_{i,t_2} - u_{j,t_2}}{\lambda_i - \lambda_j} + \left( \Delta_i(g_3) \Delta_0(g_1) - \Delta_0(g_3) \Delta_i(g_1) \right) \frac{u_{i,t_2}}{\lambda_i - \lambda_0} +$$

$$\sum_{1 \leq j \leq n-k+1, j \neq i} \left( \Delta_i(g_1) \Delta_j(g_2) - \Delta_j(g_1) \Delta_i(g_2) \right) e^{a_{j}} \frac{u_{i,x} - u_{j,x}}{\lambda_i - \lambda_j} + \left( \Delta_i(g_1) \Delta_0(g_2) - \Delta_0(g_1) \Delta_i(g_2) \right) \frac{u_{i,x}}{\lambda_i - \lambda_0} = 0,$$
where \( i = 1, \ldots, n - k + 1 \) and

\[
\sum_{j=1}^{n-k+1} e^{u_j} \Delta_j(g_r) u_{j,t_s} = \sum_{j=1}^{n-k+1} e^{u_j} \Delta_j(g_s) u_{j,t_r},
\]

\[
\sum_{j=1}^{n-k+1} \Delta_j(g_r) e^{u_j} u_{i,t_s} - u_{j,t_s} - \Delta_0(g_r) \frac{u_{i,t_s}}{\lambda_i - \lambda_j} + \Delta_0(g_r) \frac{u_{j,t_s}}{\lambda_i - \lambda_j} = \sum_{j=1}^{n-k+1} \Delta_j(g_s) e^{u_j} u_{i,t_r} - u_{j,t_r} + \Delta_0(g_s) \frac{u_{i,t_r}}{\lambda_i - \lambda_j} + \Delta_0(g_s) \frac{u_{j,t_r}}{\lambda_i - \lambda_j},
\]

where \( i = n - k + 2, \ldots, n \). Here \( r, s = 1, 2, 3; t_3 = x \) and

\[
\Delta_j(g) = \det \begin{pmatrix}
g & h_1 & \ldots & h_k \\
a_j & b_{1,j} & \ldots & b_{k,j} \\
a_{n-k+2} & b_{1,n-k+2} & \ldots & b_{k,n-k+2} \\
\ldots & \ldots & \ldots & \ldots \\
a_n & b_{1,n} & \ldots & b_{k,n}
\end{pmatrix}
\]

for \( j = 1, \ldots, n, g = a_0 + a_1 e^{u_1} + \ldots + a_n e^{u_n}, h_1 = b_{1,0} + b_{1,1} e^{u_1} + \ldots + b_{1,n} e^{u_n}, \ldots, h_k = b_{k,0} + b_{k,1} e^{u_1} + \ldots + b_{k,n} e^{u_n} \). Note that this equations are linearly dependent and the system is equivalent to a system with \( n + k \) linearly independent equations.

**Proposition 5.** If \( k = 0 \), then the corresponding system possesses the following \( n + 1 \) conservation laws of hydrodynamic type:

\[
\begin{vmatrix}
S_n^{reg}(g_2, \lambda_i) & S_n^{reg}(g_3, \lambda_i) \\
a_{2,i} e^{-u_i} & a_{3,i} e^{-u_i}
\end{vmatrix}_{t_1} + \begin{vmatrix}
S_n^{reg}(g_3, \lambda_i) & S_n^{reg}(g_1, \lambda_i) \\
a_{3,i} e^{-u_i} & a_{1,i} e^{-u_i}
\end{vmatrix}_{t_2} + \begin{vmatrix}
S_n^{reg}(g_1, \lambda_i) & S_n^{reg}(g_2, \lambda_i) \\
a_{1,i} e^{-u_i} & a_{2,i} e^{-u_i}
\end{vmatrix}_{x} = 0,
\]

where \( i = 1, \ldots, n \) and

\[
\begin{vmatrix}
S_n^{reg}(g_2, \lambda_0) & S_n^{reg}(g_3, \lambda_0) \\
a_{2,0} & a_{3,0}
\end{vmatrix}_{t_1} + \begin{vmatrix}
S_n^{reg}(g_3, \lambda_0) & S_n^{reg}(g_1, \lambda_0) \\
a_{3,0} & a_{1,0}
\end{vmatrix}_{t_2} + \begin{vmatrix}
S_n^{reg}(g_1, \lambda_0) & S_n^{reg}(g_2, \lambda_0) \\
a_{1,0} & a_{2,0}
\end{vmatrix}_{x} = 0.
\]

Here

\[
S_n^{reg}(g, \lambda_i) = \left( S_n(g, p) - \frac{a_i e^{u_i}}{p - \lambda_i} \right)_{p=\lambda_i}
\]

and

\[
S_n^{reg}(g, \lambda_0) = \left( S_n(g, p) - \frac{a_0}{p - \lambda_0} \right)_{p=\lambda_0}.
\]

**Proposition 6.** If \( k > 0 \), then the corresponding system possesses the following \( 3k \) conservation laws of hydrodynamic type:

\[
\left( \frac{\Delta(g_r, h_1, \ldots, h_k)}{\Delta(h_1, \ldots, h_k)} \right)_{t_s} = \left( \frac{\Delta(g_s, h_1, \ldots, h_k)}{\Delta(h_1, \ldots, h_k)} \right)_{t_r},
\]

(3.28)
where \( i = 1, \ldots, k \), \( r, s = 1, 2, 3 \), \( t_3 = x \) and
\[
\Delta(f_1, \ldots, f_k) = \det \begin{pmatrix} f_1 & \ldots & f_k \\ f_{1, u_n - k + 2} & \ldots & f_{k, u_n - k + 2} \\ \vdots & \ddots & \vdots \\ f_{1, u_n} & \ldots & f_{k, u_n} \end{pmatrix}.
\]

**Remark 2.** It is likely that for \( k > 0 \) the corresponding 3-dimensional system possesses additional \( n + 1 \) conservation laws of hydrodynamic type.

**Remark 3.** Proposition 6 allows us to define functions \( z_1, \ldots, z_k \) such that
\[
\frac{\Delta(g_r, h_1, \ldots, h_k)}{\Delta(h_1, \ldots, h_k)} = z_{i, t_r}
\]
for all \( i = 1, \ldots, k \) and \( r = 0, 1, 3 \). See [10] or [11] for further discussion.

### 3.2 Degenerations

Our constructions of GT-type systems and functions \( S_{n,k} \) are valid in the case of pairwise distinct roots \( \lambda_0, \ldots, \lambda_n \). In this section we study degenerations of the GT-type systems described in Section 3.1.

Define polynomials \( P_i(u_1, u_2, \ldots) \) as coefficients of the following Taylor expansion
\[
\exp(\varepsilon u_1 + \varepsilon^2 u_2 + \ldots) = 1 + P_1\varepsilon + P_2\varepsilon^2 + \cdots.
\]

In particular,
\[
P_1 = u_1, \quad P_2 = u_2 + \frac{1}{2}u_1^2, \quad P_3 = u_3 + u_1u_2 + \frac{1}{6}u_1^3.
\]

Denote the partial sums \( 1 + \sum_{i=1}^k P_i\varepsilon^i \) by \( Q_k(\varepsilon) \). By definition, \( P_0 = Q_0(\varepsilon) = 1 \).

**Degeneration 1.** This degeneration corresponds to the case \( \lambda_0 \neq \lambda_1 = \ldots = \lambda_n \). Consider the following \( (n + 1) \)-field GT-type system with fields \( u_1, \ldots, u_n, w \):
\[
\partial_ip_j = 0, \quad \partial_i u_m = \left( \frac{\lambda_1 - \lambda_0}{(p_i - \lambda_1)^m} + \frac{1}{(p_i - \lambda_1)^{m+1}} \right) \partial_i w, \quad \partial_i \partial_j w = 0.
\]

For any vector \( (a_0, a_1, \ldots, a_n) \) define
\[
g = a_0 + e^{u_1} \sum_{i=1}^n a_i P_{i-1}
\]
and
\[
S_n(g, p) = \frac{a_0}{p - \lambda_0} + e^{u_1} \sum_{i=1}^n a_i Q_{i-1}(p - \lambda_1).
\]
**Degeneration 2.** This degeneration corresponds to the case $\lambda_0 = \lambda_1 = \cdots = \lambda_n$. Consider the following $(n + 1)$-field GT-type system:

$$
\partial_i p_j = 0, \quad \partial_i u_m = \frac{1}{(p_i - \lambda_0)^m} \partial_i w, \quad \partial_i \partial_j w = 0. \quad (3.31)
$$

For any vector $(a_0, a_1, \ldots, a_n)$ define

$$
g = \sum_{i=0}^{n} a_i P_i.
$$

and

$$
S_n(g, p) = \sum_{i=0}^{n} a_i Q_i(p - \lambda_0) \frac{(p - \lambda_0)^{i+1}}{(p - \lambda_0)^{i+1}}.
$$

Combining these degenerations, one obtains the general case. Let $\lambda_0, \ldots, \lambda_l$ be pairwise distinct roots of multiplicities $n_0 + 1, n_1, \ldots, n_l$ correspondingly. Note that $n_0 + \cdots + n_l = n$. Consider the following $(n + 1)$-field system with fields $u_{0,1}, \ldots, u_{0,n_0}, u_{1,1}, \ldots, u_{1,n_1}, \ldots, u_{l,1}, \ldots, u_{l,n_l}$ $w$:

$$
\partial_i p_j = 0, \quad \partial_i u_{0,m} = \frac{1}{(p_i - \lambda_0)^m} \partial_i w, \quad \partial_i u_{s,m} = \left(\frac{\lambda_s - \lambda_0}{(p_i - \lambda_s)^m} + \frac{1}{(p_i - \lambda_s)^{m+1}}\right) \partial_i w, \quad \partial_i \partial_j w = 0. \quad (3.32)
$$

For any vector $(a_{0,0}, a_{0,1}, \ldots, a_{0,n_0}, a_{1,1}, \ldots, a_{1,n_1}, \ldots, a_{l,n_l})$ define

$$
g = \sum_{i=0}^{n_0} a_{0,i} P_i + \sum_{s=1}^{l} \sum_{i=1}^{n_s} a_{s,i} e^{u_s,i} P_{i-1}
$$

and

$$
S_n(g, p) = \sum_{i=0}^{n_0} a_{0,i} Q_i(p - \lambda_0) \frac{(p - \lambda_0)^{i+1}}{(p - \lambda_0)^{i+1}} + \sum_{s=1}^{l} \sum_{i=1}^{n_s} a_{s,i} e^{u_s,i} Q_{i-1}(p - \lambda_s) \frac{(p - \lambda_s)^{i}}{(p - \lambda_s)^{i}}.
$$

Let $k > 0$. Fix vectors $(b_{0,0,j}, b_{0,1,j}, \ldots, b_{0,n_0,j}, b_{1,1,j}, \ldots, b_{1,n_1,j}, \ldots, b_{l,n_l,j})$, $j = 1, \ldots, k$. Let

$$
g = \sum_{i=0}^{n_0} a_{0,i} P_i + \sum_{s=1}^{l} \sum_{i=1}^{n_s} a_{s,i} e^{u_s,i} P_{i-1}
$$

and similarly

$$
h_j = \sum_{i=0}^{n_0} b_{0,i,j} P_i + \sum_{s=1}^{l} \sum_{i=1}^{n_s} b_{s,i,j} e^{u_s,i} P_{i-1}, \quad j = 1, \ldots, k.
$$

Define $S_{n,k}(g, p)$ by

$$
S_{n,k}(g, p) = \det \begin{pmatrix}
S_n(g, p) & S_n(h_1, p) & \cdots & S_n(h_k, p) \\
g & h_1 & \cdots & h_k \\
g v_{n-k+2} & h_{1,v_n-k+2} & \cdots & h_{k,v_n-k+2} \\
& \cdots & \cdots & \cdots \\
g v_n & h_{1,v_n} & \cdots & h_{k,v_n}
\end{pmatrix} \quad (3.33)
$$
where \((v_1, ..., v_n) = (u_{0,1}, ..., u_{0,n_0}, u_{1,1}, ..., u_{1,n_1}, ..., u_{l,1}, ..., u_{l,n_l})\). Then Proposition 4 holds. The explicit form of the corresponding 3-dimensional systems can be calculated using the general recipe.

### 4 Summary and discussion: toward a classification of integrable 3-dimensional hydrodynamic-type systems

We summarize our previous remarks as the following project of classification of integrable 3-dimensional hydrodynamic-type systems:

1. Classify (up to equivalence (1.8), (1.9)) all GT-type systems (1.5) possessing a non-trivial solution of the functional equation (2.19).

2. For each such GT-type system classify all solutions of the functional equation (2.19).

3. For each GT-type system and solution of (2.19) construct the corresponding 3-dimensional system (see Lemma 1).

This approach has the following advantages:

- **a.** It turns out that GT-type systems are universal: for each GT-type system there exist several families of the corresponding 3-dimensional systems depending on essential parameters. Moreover, it is likely that for each genus \(g = 0, 1, ...\) there exists an unique GT-type system (with one field for \(g = 0, 1\) and with \(3g - 3\) fields for \(g > 1\)) such that any generic 3-dimensional system with the dispersion curve of genus \(g\) corresponds to this GT-type system or its regular extension.

- **b.** Integrable systems (1.3) are defined up to arbitrary point transformations. Since the equivalence problem for systems (1.3) is highly non-trivial, it is not easy to find the simplest coordinates in which a given 3-dimensional system has the simplest form. As the rule, the simplest coordinates for the GT-type system are at the same time the most natural coordinates for the corresponding system (1.3). For example, each of the variables \(p_i, u\) from the Example 3 admits natural interpretation as a coordinate on \(CP^1\). Similarly, each of the variables \(p_i\) from the Example 4 can be naturally interpreted as a coordinate on an elliptic curve and \(\tau\) as a modular parameter of this curve. We expect that for \(g > 1\) there exists a canonical GT-type system with \(3g - 3\) fields being coordinates on the moduli space of genus \(g\) curves. This GT-type system should have (an analog of) regular extensions with additional fields being points on this curve. One could find this GT-type system by study of hydrodynamic reductions for 3-dimensional systems from [9]. On the other hand, taking into account the results related to \(g = 0, 1\) we expect that there are much more 3-dimensional systems corresponding to the canonical GT-type system, than it was found in [9].

The main disadvantage of the approach outlined above is that the realization of the item 1 is very hard for \(n > 1\). We hope to address this classification problem later. In this paper we
consider the simplest possible GT-type system of the form (1.10) and demonstrate that even in this case there exists a rich family of interesting integrable 3-dimensional systems associated with (1.10). The coefficients of these systems are quasi-polynomials. Note that slightly more complicated GT-type systems from Examples 3 and 4 require generalized hypergeometric functions for description of the corresponding 3-dimensional systems.

References

[1] What is Integrability? ed. V.E. Zakharov, Springer series in Nonlinear Dynamics, 321 pages, 1991, ISBN-13: 978-0387519647.

[2] Integrability, ed. A.V. Mikhailov, Lecture Notes in Physics, 767, Springer, 339 pages, 2009, ISBN: 978-3-540-88110-0.

[3] B.A. Dubrovin, Geometry of 2D topological field theories. In Integrable Systems and Quantum Groups, Lecture Notes in Math. 1620 (1996), 120–348.

[4] Mikhailov A. V., Sokolov V. V. Symmetries of differential equations and the problem of Integrability, in the book [2], 19–88.

[5] Mikhailov, A. V.; Yamilov, R. I, Towards classification of (2 + 1)-dimensional integrable equations. Integrability conditions. I, J. Phys. A 31, (1998), no. 31, 6707–6715.

[6] E.V. Ferapontov, K.R. Khusnutdinova, On integrability of (2+1)-dimensional quasilinear systems, Comm. Math. Phys. 248 (2004) 187-206, E.V. Ferapontov, K.R. Khusnutdinova, The characterization of 2-component (2+1)-dimensional integrable systems of hydrodynamic type, J. Phys. A: Math. Gen. 37(8) (2004) 2949 - 2963.

[7] S.V. Manakov and P.M. Santini, Inverse scattering problem for vector fields and the Cauchy problem for the heavenly equation, Phys. Letters A, 359, (2006) 613619.

[8] V.E. Zakharov, Benney’s equations and quasi-classical approximation in the inverse problem method, Funct. Anal. Appl., 14 No. 2 (1980) 89-98. V.E. Zakharov, On the Benney’s Equations, Physica 3D (1981) 193-200.

[9] I.M. Krichever, The τ-function of the universal Whitham hierarchy, matrix models and topological field theories, Comm. Pure Appl. Math., 47 (1994), no. 4, 437–475.

[10] A. Odesskii, V. Sokolov, Integrable pseudopotentials related to generalized hypergeometric functions, arXiv:0803.0086

[11] A. Odesskii, V. Sokolov, Integrable pseudopotentials related to elliptic curves, to appear in Theoretical and Mathematical Physics, arXiv:0810.3879
[12] J. Gibbons, S.P. Tsarev, Reductions of Benney’s equations, Phys. Lett. A, 211 (1996) 19-24. J. Gibbons, S.P. Tsarev, Conformal maps and reductions of the Benney equations, Phys. Lett. A, 258 (1999) 263-270.

[13] E.V. Ferapontov, Integration of weakly nonlinear hydrodynamic systems in Riemann invariants, Physics Letters A 158, (1991) 112–118.

[14] S.P. Tsarev, On Poisson brackets and one-dimensional Hamiltonian systems of hydrodynamic type, Soviet Math. Dokl., 31 (1985) 488–491. S.P. Tsarev, The geometry of Hamiltonian systems of hydrodynamic type. The generalized hodograph method, Math. USSR Izvestiya, 37 No. 2 (1991) 397–419. 1048–1068.