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Let $X^*_0(k, n, s)$ denote the sum of all multiple zeta-star values of weight $k$, depth $n$ and height $s$. Kaneko and Ohno conjectured that, for any positive integers $m, n, s$ with $m, n \geq s$, the difference

$$(-1)^m X^*_0(m+n+1, n+1, s) - (-1)^n X^*_0(m+n+1, m+1, s)$$

can be expressed as a polynomial of zeta values with rational coefficients. We give a proof of this conjecture.

1. Introduction

Given a sequence $k = (k_1, \ldots, k_n)$ of positive integers with $k_1 > 1$, the weight $\text{wt}(k)$, depth $\text{dep}(k)$ and height $\text{ht}(k)$ are defined by

$$\text{wt}(k) = k_1 + \cdots + k_n, \quad \text{dep}(k) = n, \quad \text{ht}(k) = \#\{i \mid k_i \geq 2\},$$

respectively. For such a sequence $k$, there are two well-studied real numbers: the multiple zeta value $\zeta(k)$, defined by

$$\zeta(k) = \zeta(k_1, \ldots, k_n) = \sum_{m_1 > \cdots > m_n \geq 0} \frac{1}{m_1^{k_1} \cdots m_n^{k_n}},$$

and the multiple zeta-star value $\zeta^*(k)$, defined by

$$\zeta^*(k) = \zeta^*(k_1, \ldots, k_n) = \sum_{m_1 \geq \cdots \geq m_n \geq 1} \frac{1}{m_1^{k_1} \cdots m_n^{k_n}}.$$

We call the values $\zeta(k)$ and $\zeta^*(k)$ with weight $\text{wt}(k)$, depth $\text{dep}(k)$ and height $\text{ht}(k)$.

The well-known Ohno–Zagier relation [2001] is a class of relations about the sums of multiple zeta values of fixed weight, depth and height. For integers $k, n, s$ with $k \geq n + s$ and $n \geq s \geq 1$, we denote by $X_0(k, n, s)$ the sum of all multiple zeta values of weight $k$, depth $n$ and height $s$. The Ohno–Zagier relation says that

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More explicitly, Ohno and Zagier gave the generating function expression
\[
\sum_{\substack{k \geq n+s \\ n \geq s \geq 1}} X_0(k, n, s) u^{k-n-s} v^{n-s} t^{s-1} = \frac{1}{uv-t} \left\{ 1 - \exp\left( \sum_{n=2}^{\infty} \frac{\zeta(n)}{n} (u^n + v^n - \alpha^n - \beta^n) \right) \right\},
\]
where \(\alpha, \beta\) are determined by \(\alpha + \beta = u + v\) and \(\alpha \beta = uv - t^2\). In [Li 2010], we showed that the Ohno–Zagier relation can be deduced from the regularized double shuffle relation. In [Li 2008], we generalized the concept height to \(i\)-height, studied sums of multiple zeta values of fixed weight, depth and general height, and expressed a kind of generating function of these sums in terms of generalized hypergeometric functions.

Similarly, we denote by \(X_0^*(k, n, s)\) the sum of all multiple zeta-star values of weight \(k\), depth \(n\) and height \(s\) for integers \(k, n, s\) with \(k \geq n+s\) and \(n \geq s \geq 1\). The authors of [Aoki et al. 2008] considered a generating function \(8^*_{0}(u, v, t)\) of sums \(X_0^*(k, n, s)\), where
\[
8^*_{0}(u, v, t) = \sum_{\substack{k \geq n+s \\ n \geq s \geq 1}} X_0^*(k, n, s) u^{k-n-s} v^{n-s} t^{2s-2}.
\]
It was proved there that \(8^*_{0}(u, v, t)\) can be expressed by a special value of the generalized hypergeometric function \(3 \, F_2\) as
\[
8^*_{0}(u, v, t) = \frac{1}{(1-v)(1-\beta)} \, 3 \, F_2\left( \begin{array}{c} 1-\beta, 1-\beta+u, 1 \\ 2-v, 2-\beta \end{array} ; 1 \right),
\]
where \(\alpha, \beta\) are determined by \(\alpha + \beta = u + v\) and \(\alpha \beta = uv - t^2\), and the generalized hypergeometric function \(3 \, F_2\) is defined as (see [Bailey 1935])
\[
3 \, F_2\left( \begin{array}{c} \alpha_1, \alpha_2, \alpha_3 \\ \beta_1, \beta_2 \end{array} ; z \right) = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n (\alpha_2)_n (\alpha_3)_n}{n! (\beta_1)_n (\beta_2)_n} z^n,
\]
with the Pochhammer symbol \((a)_n\) given by
\[
(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)} = \begin{cases} 1 & \text{if } n = 0, \\ a(a+1) \cdots (a+n-1) & \text{if } n > 0. \end{cases}
\]
Since the generating function \(8^*_{0}(u, v, t)\) is represented by \(3 \, F_2\) as in (1), it is expected that in general \(X_0^*(k, n, s)\) can’t be written as a polynomial of zeta values.
with rational coefficients. While in [2010] Kaneko and Ohno considered some kind of duality of multiple zeta-star values, and proposed the following conjecture.

**Conjecture** [Kaneko and Ohno 2010]. For any positive integers \(m, n, s\) satisfying \(m, n \geq s\), we have

\[
(-1)^m X^*_0(m + n + 1, n + 1, s) - (-1)^n X^*_0(m + n + 1, m + 1, s) \\
\in \mathbb{Q}[\zeta(2), \zeta(3), \zeta(5), \ldots].
\]

Kaneko and Ohno showed that this is true for \(s = 1\). Using the result of [Aoki et al. 2008] about the generating function \(8^*_0(u, v, 0)\), Yamazaki [2010] gave another proof of this case. Note that the Kaneko–Ohno theorem for their conjecture in the case \(s = 1\) can be restated as

\[
(2) \quad \frac{u}{u - 1} \Phi^*_0(-u, v, 0) - \frac{v}{v - 1} \Phi^*_0(-v, u, 0) \\
= \frac{1}{u - 1} \Gamma(u + v) \Gamma(v) \left( \frac{\Gamma(v)}{\Gamma(u)} \left( (\Gamma(v) \Gamma(1 - v))^2 - (\Gamma(u) \Gamma(1 - u))^2 \right) \right).
\]

The purpose of this paper is to give a proof of the Kaneko–Ohno conjecture. In fact, similarly to (2), we give an expression of \(u \Phi^*_0(-u, v, t) - v \Phi^*_0(-v, u, t)\) by gamma functions in Theorem 2.2. Our proof is based on the expression of \(8^*_0(u, v, t)\) given in [Aoki et al. 2008], and hence is similar to the one of [Yamazaki 2010] for the special case \(s = 1\).

In Section 2, we state our main result and give some corollaries. In Section 3, we prepare a result about generalized hypergeometric series \(3\,F_2\). In the last section, we give the proof of the main theorem.

### 2. Statement of the main result

**Main theorem.** As in Section 1, we denote by \(X^*_0(k, n, s)\) the sum of all multiple zeta-star values of weight \(k\), depth \(n\) and height \(s\) for integers \(k, n, s\) with \(k \geq n + s\) and \(n \geq s \geq 1\). Let \(\Phi^*_0(u, v, t)\) be the generating function defined by

\[
\Phi^*_0(u, v, t) = \sum_{k \geq n + s, n \geq s \geq 1} X^*_0(k, n, s) u^{k-n-s} v^{n-s} t^{2s-2}.
\]

For variables \(u, v, t\), we define \(a\) and \(b\) by the conditions \(a + b = -u + v\) and \(ab = -uv - t^2\). Equivalently, we have

\[
a, b = \frac{-u + v \pm \sqrt{(u + v)^2 + 4t^2}}{2}.
\]

After that we define the function \(A(u, v, a, b)\) by

\[
(3) \quad A(u, v, a, b) = \frac{1}{2\pi} \left\{ \cos \frac{\pi u}{\sin \pi v} - \cos \frac{\pi v}{\sin \pi u} + \cos \pi (a - b) (\cot \pi u - \cot \pi v) \right\}.
\]
Note that \( A(u, v, a, b) = A(u, v, b, a) \), which shall play an important role in the proof of our main theorem. We can express \( A(u, v, a, b) \) by gamma functions as in the following lemma.

**Lemma 2.1.** We have

\[
A(u, v, a, b) = \frac{1}{\Gamma(u+a) \Gamma(1-u-a)} \left( \frac{\Gamma(v) \Gamma(1-v)}{\Gamma(a) \Gamma(1-a)} + \frac{\Gamma(u) \Gamma(1-u)}{\Gamma(b) \Gamma(1-b)} \right),
\]

\[
A(u, v, a, b) = \frac{1}{\Gamma(u+b) \Gamma(1-u-b)} \left( \frac{\Gamma(v) \Gamma(1-v)}{\Gamma(b) \Gamma(1-b)} + \frac{\Gamma(u) \Gamma(1-u)}{\Gamma(a) \Gamma(1-a)} \right).
\]

**Proof.** Equation (5) follows from (4) and the fact \( A(u, v, a, b) = A(u, v, b, a) \).

Using the well-known reflection formula

\[
\Gamma(s) \Gamma(1-s) = \frac{\pi}{\sin \pi s},
\]

we find that the right-hand side of Equation (4) becomes

\[
\frac{\sin \pi (u+a)}{\pi} \left( \frac{\sin \pi a}{\sin \pi v} + \frac{\sin \pi b}{\sin \pi u} \right),
\]

which is equal to

\[
\frac{1}{2\pi} \left( \frac{\cos \pi u - \cos \pi (v+a-b)}{\sin \pi v} + \frac{\cos \pi (u+a-b) - \cos \pi v}{\sin \pi u} \right).
\]

Now it is easy to finish the proof.

The main theorem of this paper is this:

**Theorem 2.2.** We have

\[
u \Phi_0^*(-u, v, t) - v \Phi_0^*(-v, u, t)
= \frac{u - v}{ab} + A(u, v, a, b) \frac{\Gamma(a) \Gamma(1-a) \Gamma(b) \Gamma(1-b) \Gamma(u+a) \Gamma(u+b)}{\Gamma(u) \Gamma(v)}.
\]

**Some remarks.** By the definition of the generating function \( \Phi_0^*(u, v, t) \), it is easy to see that

\[
u \Phi_0^*(-u, v, t) - v \Phi_0^*(-v, u, t)
= \sum_{\substack{m,n \geq 1 \\text{s.t.} \\ s \\ s \geq 1}} (-1)^s \left( (-1)^m X_0^*(m+n+1, n+1, s) - (-1)^n X_0^*(m+n+1, m+1, s) \right)
\times u^{m+1-s} v^{n+1-s} t^{2s-2}
+ \sum_{n \geq s \geq 1} (-1)^{n+s} X_0^*(n+s, s, s) (u^{n+1-s} - v^{n+1-s}) t^{2s-2}.
\]
Since we have the expansion
\[\Gamma(1-x) = \exp\left(\gamma x + \sum_{n=2}^{\infty} \frac{\zeta(n)}{n} x^n\right),\]
where \(\gamma\) is Euler’s constant, we know that Theorem 2.2 indeed implies the Kaneko–Ohno conjecture.

**Corollary 2.3.** For any positive integers \(m, n, s\) with \(m, n \geq s\), the difference
\[
(-1)^m X^*_0(m + n + 1, n + 1, s) - (-1)^n X^*_0(m + n + 1, m + 1, s)
\]
can be expressed as a polynomial of zeta values with rational coefficients.

Taking into account to the second term of the right-hand side of (7), we have another corollary.

**Corollary 2.4.** For any positive integers \(k, s\) with \(k \geq 2s\), the sum \(X^*_0(k, s, s)\) can be expressed as a polynomial of zeta values with rational coefficients.

Note that this is an immediate consequence of the symmetric sum formula for multiple zeta-star values (see [Hoffman 1992, Theorem 2.1]).

Letting \(t = 0\) in Theorem 2.2, we can derive (2). In fact, in this case, we can assume that \(a = -u\) and \(b = v\). For \(A(u, v, a, b)\), we use the equivalent equation (4). Then using Theorem 2.2, we get (2).

3. A result about generalized hypergeometric series \(\mathfrak{F}_2\)

To prove the main theorem of this paper, we introduce the following result.

**Proposition 3.1.** For \(a, b, c \in \mathbb{C}\) with sufficient small real parts, we have
\[
\mathfrak{F}_2\left(\begin{array}{c} a, b, c \\ a+b, 1+c \end{array} ; 1 \right) = \frac{\Gamma(a+b)\Gamma(1+c)\Gamma(1+c-a-b)}{\Gamma(a)\Gamma(b)\Gamma(1+c-a)\Gamma(1+c-b)} (\psi(1+c-b) - \psi(a) - \psi(b) - \gamma) \\
- \frac{\Gamma(a+b)\Gamma(1+c)\Gamma(1+c-a-b)}{\Gamma(a)\Gamma(b)\Gamma(1+c-a)\Gamma(1+c-b)} \sum_{n=1}^{\infty} \frac{(a)_n(b)_n}{n! (1+c-b)_n},
\]
where \(\psi(x) = \Gamma'(x)/\Gamma(x)\) is the digamma function.

To save space, from now on we will denote the special value
\[
\mathfrak{F}_2\left(\begin{array}{c} \alpha_1, \alpha_2, \alpha_3 \\ \beta_1, \beta_2 \end{array} ; 1 \right)
\]
by \(\mathfrak{F}_2\left(\begin{array}{c} \alpha_1, \alpha_2, \alpha_3 \\ \beta_1, \beta_2 \end{array} \right)\).

To prove the proposition, we need two transformation formulas. The first one,
from [Bailey 1935, Section 3.8, (1), p. 21], is

$$3 F_2 \left( \begin{array}{c} \alpha_1, \alpha_2, \alpha_3 \\ \beta_1, \beta_2 \end{array} \right) = \frac{\Gamma(\beta_1) \Gamma(\beta_2 - \alpha_1 - \alpha_2)}{\Gamma(\beta_1 - \alpha_1) \Gamma(\beta_1 - \alpha_2)} 3 F_2 \left( \begin{array}{c} \alpha_1, \alpha_2, \alpha_2 - \alpha_3 \\ \alpha_1 + \alpha_2 - \beta_1 + 1, \beta_2 \end{array} \right)$$

$$+ \frac{\Gamma(\beta_1) \Gamma(\beta_2) \Gamma(\alpha_1 + \alpha_2 - \beta_1) \Gamma(\beta_1 + \beta_2 - \alpha_1 - \alpha_2 - \alpha_3)}{\Gamma(\alpha_1) \Gamma(\alpha_2) \Gamma(\beta_2 - \alpha_3) \Gamma(\beta_1 + \beta_2 - \alpha_1 - \alpha_2)} 3 F_2 \left( \begin{array}{c} \beta_1 - \alpha_1, \beta_1 - \alpha_2, \beta_1 + \beta_2 - \alpha_1 - \alpha_2 - \alpha_3 \\ \beta_1 - \alpha_1 - \alpha_2 + 1, \beta_1 + \beta_2 - \alpha_1 - \alpha_2 \end{array} \right),$$

provided that $\text{Re}(\beta_1 + \beta_2 - \alpha_1 - \alpha_2 - \alpha_3) > 0$ and $\text{Re}(\alpha_3 - \beta_1 + 1) > 0$. The second one, from [Bailey 1935, Examples 7, p. 98] is

$$3 F_2 \left( \begin{array}{c} \alpha_1, \alpha_2, \alpha_3 \\ \beta_1, \beta_2 \end{array} \right) = \frac{\Gamma(\beta_2) \Gamma(\beta_1 + \beta_2 - \alpha_1 - \alpha_2 - \alpha_3)}{\Gamma(\beta_2 - \alpha_3) \Gamma(\beta_1 + \beta_2 - \alpha_1 - \alpha_2)} 3 F_2 \left( \begin{array}{c} \beta_1 - \alpha_1, \beta_1 - \alpha_2, \alpha_3 \\ \beta_1, \beta_1 + \beta_2 - \alpha_1 - \alpha_2 \end{array} \right),$$

provided that $\text{Re}(\beta_1 + \beta_2 - \alpha_1 - \alpha_2 - \alpha_3) > 0$ and $\text{Re}(\beta_2 - \alpha_3) > 0$.

Proof of Proposition 3.1. Taking a parameter $\varepsilon$ such that $|\varepsilon|$ is sufficient small, we have

$$3 F_2 \left( \begin{array}{c} a, b, c \\ a+b, 1+c \end{array} \right) = \lim_{\varepsilon \to 0} 3 F_2 \left( \begin{array}{c} a, b, c \\ a+b+\varepsilon, 1+c-\varepsilon \end{array} \right).$$

Now we consider the series

$$3 F_2 \left( \begin{array}{c} a, b, c \\ a+b+\varepsilon, 1+c-\varepsilon \end{array} \right).$$

Applying (9), we get

$$3 F_2 \left( \begin{array}{c} a, b, c \\ a+b+\varepsilon, 1+c-\varepsilon \end{array} \right) = \frac{\Gamma(a+b+\varepsilon) \Gamma(\varepsilon)}{\Gamma(a+\varepsilon) \Gamma(b+\varepsilon)} 3 F_2 \left( \begin{array}{c} a, b, 1-\varepsilon \\ 1-\varepsilon, 1+c-\varepsilon \end{array} \right)$$

$$+ \frac{\Gamma(a+b+\varepsilon) \Gamma(1+c-\varepsilon) \Gamma(-\varepsilon)}{\Gamma(a) \Gamma(b) \Gamma(1-\varepsilon) \Gamma(1+c)} 3 F_2 \left( \begin{array}{c} a+\varepsilon, b+\varepsilon, 1 \\ 1+\varepsilon, 1+c \end{array} \right).$$

To the first $3 F_2$-series in the right-hand side of (11), we apply the Gaussian summation formula (see [Bailey 1935, Section 1.3, (1)])

$$\sum_{n=0}^{\infty} \frac{(\alpha_1)_n (\alpha_2)_n}{n! (\beta)_n} = \frac{\Gamma(\beta) \Gamma(\beta-\alpha_1-\alpha_2)}{\Gamma(\beta-\alpha_1) \Gamma(\beta-\alpha_2)}.$$
for \( \text{Re}(\beta - \alpha_1 - \alpha_2) > 0 \), and apply (10) to the second \( 3F_2 \)-series in the right-hand side of (11), we obtain

\[
3F_2 \left( \frac{a, b, c}{a+b+\varepsilon, 1+c-\varepsilon} \right) = \frac{\Gamma(1+\varepsilon) \Gamma(a+b+\varepsilon) \Gamma(1+c-\varepsilon) \Gamma(1+c-a-b-\varepsilon)}{\varepsilon \Gamma(a+\varepsilon) \Gamma(b+\varepsilon) \Gamma(1+c-a-\varepsilon) \Gamma(1+c-b-\varepsilon)} - \frac{\Gamma(a+b+\varepsilon) \Gamma(1+c-\varepsilon) \Gamma(1+c-a-b-\varepsilon)}{\Gamma(a) \Gamma(b) \Gamma(1+c-a-\varepsilon) \Gamma(1+c-b)} \frac{\Gamma(a+b+\varepsilon) \Gamma(1+c-\varepsilon) \Gamma(1+c-a-b-\varepsilon)}{\Gamma(a) \Gamma(b) \Gamma(1+c-a-\varepsilon) \Gamma(1+c-b)} \sum_{n=1}^{\infty} \frac{(a+\varepsilon)n(1-b)_n}{(n+\varepsilon)n!(1+c-b)_n}.
\]

Finally, let \( \varepsilon \) go to 0 to finish the proof. For the first two lines of the above expression, we use l'Hôpital's rule and the fact that \( \psi(1) = -\gamma \).

\[\square\]

4. Proof of the main theorem

In this section, we prove Theorem 2.2.

Lemma 4.1. Let \( \alpha \) and \( \beta \) be determined by \( \alpha + \beta = u + v \) and \( \alpha \beta = uv - t^2 \). We have

\[
\Phi_0^*(u, v, t) = \frac{\Gamma(\beta - \alpha) \Gamma(1-\beta) \Gamma(v) \Gamma(1-v) \sum_{n=0}^{\infty} \frac{(\alpha)_n(1-\beta)_n}{n!(1+\alpha-\beta)_n} \frac{\alpha - u}{n + \alpha - u}}{\Gamma(1-\alpha) \Gamma(1+u-\alpha) \Gamma(1+\alpha-u)} + \frac{\Gamma(\alpha - \beta) \Gamma(1-\alpha) \Gamma(v) \Gamma(1-v) \sum_{n=0}^{\infty} \frac{(\beta)_n(1-\alpha)_n}{n!(1+\beta-\alpha)_n} \frac{\beta - u}{n + \beta - u}}{\Gamma(1-\beta) \Gamma(1+u-\beta) \Gamma(1+\beta-u)}.
\]

Proof. A result of Aoki, Kombu, and Ohno [Aoki et al. 2008] about the generating function \( \Phi_0^*(u, v, t) \) gives that
\[ \Phi_0^*(u, v, t) = \frac{\Gamma(\beta - \alpha) \Gamma(1 - v)}{\Gamma(1 - \alpha) \Gamma(1 + u - \alpha)} \int_0^1 s^{-\beta} (1 - s)^{v-1} 2F_1 \left( \frac{\alpha, \alpha - u}{1 + \alpha - \beta}; s \right) ds \\
+ \frac{\Gamma(\alpha - \beta) \Gamma(1 - v)}{\Gamma(1 - \beta) \Gamma(1 + u - \beta)} \int_0^1 s^{-\alpha} (1 - s)^{v-1} 2F_1 \left( \frac{\beta, \beta - u}{1 + \beta - \alpha}; s \right) ds. \]

Here \( 2F_1 \left( \frac{a, b}{c}; s \right) \) is the Gaussian hypergeometric function, given by

\[ 2F_1 \left( \frac{a, b}{c}; s \right) = \sum_{n=0}^\infty \frac{(a)_n (b)_n}{n! (c)_n} s^n. \]

Hence, we have

\[ \int_0^1 s^{-\beta} (1 - s)^{v-1} 2F_1 \left( \frac{\alpha, \alpha - u}{1 + \alpha - \beta}; s \right) ds = \sum_{n=0}^\infty \frac{(\alpha)_n (\alpha - u)_n}{n! (1 + \alpha - \beta)_n} \int_0^1 s^{-\beta} (1 - s)^{v-1} ds \\
= \sum_{n=0}^\infty \frac{(\alpha)_n (\alpha - u)_n}{n! (1 + \alpha - \beta)_n} \frac{\Gamma(1 + n - \beta) \Gamma(v)}{\Gamma(1 + n + v - \beta)}. \]

Now it is easy to finish the proof. \( \square \)

Recall that we have defined \( a \) and \( b \) by \( a + b = -u + v \) and \( ab = -uv + t^2 \). Using Lemma 4.1, we immediately get the following result.

**Lemma 4.2.** We have

\[ u \Phi_0^*(-u, v, t) - v \Phi_0^*(-v, u, t) = F(u, v, a, b) + F(u, v, b, a), \]

where \( F(u, v, a, b) \) is defined by

\[ \frac{\Gamma(b-a)}{\Gamma(1-u-a) \Gamma(1+u+a)} \times \left( \frac{u \Gamma(v) \Gamma(1-v) \Gamma(1-b)}{\Gamma(1-a)} \sum_{n=0}^\infty \frac{(a)_n (1-b)_n}{n!(1+a-b)_n} \frac{u+a}{n+u+a} \\
- \frac{v \Gamma(u) \Gamma(1-u) \Gamma(1+a)}{\Gamma(1+b)} \sum_{n=0}^\infty \frac{(1+a)_n (-b)_n}{n!(1+a-b)_n} \frac{u+a}{n+u+a} \right). \]

Since we have

\[ \sum_{n=0}^\infty \frac{(a)_n (1-b)_n}{n!(1+a-b)_n} \frac{u+a}{n+u+a} = \sum_{n=0}^\infty \frac{(a)_n (-b)_n}{n!(a-b)_n} \frac{(a-b)(u+a)(n-b)}{-b(n+a-b)(n+u+a)}, \]

and

\[ \frac{n-b}{(n+a-b)(n+u+a)} = -a \frac{1}{u+b} + b \frac{1}{n+u+a}, \]
we get
\[
\sum_{n=0}^{\infty} \frac{(a)_n (1 - b)_n}{n!(1 + a - b)_n} \frac{u + a}{n + u + a} = \frac{a(u + a) \Gamma(1 + a - b)}{b(u + b) \Gamma(1 + a) \Gamma(1 - b)} - \frac{v(a - b)}{b(u + b)} 3F_2\left(\frac{a - b, u + a}{a - b, 1 + u + a}\right).
\]

In the above, we have used Gaussian summation formula for Gaussian hypergeometric function at unit argument. Similarly, we have
\[
\sum_{n=0}^{\infty} \frac{(1 + a)_n (-b)_n}{n!(1 + a - b)_n} \frac{u + a}{n + u + a} = \frac{b(u + a) \Gamma(1 + a - b)}{a(u + b) \Gamma(1 + a) \Gamma(1 - b)} + \frac{u(a - b)}{a(u + b)} 3F_2\left(\frac{a - b, u + a}{a - b, 1 + u + a}\right).
\]

Hence we get the following lemma.

**Lemma 4.3.** We have
\[
F(u, v, a, b) = F_1(u, v, a, b) + F_2(u, v, a, b),
\]
where
\[
F_1(u, v, a, b) = \frac{(u + a) \Gamma(b - a) \Gamma(1 + a - b)}{(u + b) \Gamma(1 - u - a) \Gamma(1 + u + a)} \left(\frac{u \Gamma(v) \Gamma(1 - v)}{b \Gamma(a) \Gamma(1 - a)} - \frac{v \Gamma(u) \Gamma(1 - u)}{a \Gamma(b) \Gamma(1 - b)}\right),
\]
\[
F_2(u, v, a, b) = \frac{uv(a - b) \Gamma(b - a)}{(u + b) \Gamma(1 - u - a) \Gamma(1 + u + a)} \times \left(\frac{\Gamma(v) \Gamma(1 - v) \Gamma(-b)}{\Gamma(1 - a)} - \frac{\Gamma(u) \Gamma(1 - u) \Gamma(a)}{\Gamma(1 + b)}\right) 3F_2\left(\frac{a - b, u + a}{a - b, 1 + u + a}\right).
\]

Now we compute \(F_1(u, v, a, b) + F_1(u, v, b, a)\) and \(F_2(u, v, a, b) + F_2(u, v, b, a)\).

**Lemma 4.4.** The sum \(F_1(u, v, a, b) + F_1(u, v, b, a)\) equals
\[
\frac{u - v}{ab} + \frac{(a - b)uv}{ab(u + a)(u + b)} \Gamma(b - a) \Gamma(1 + a - b) A(u, v, a, b).
\]

**Proof.** Using the reflection formula for gamma function, we see that
\[
F_1(u, v, a, b) + F_1(u, v, b, a) = \Gamma(b - a) \Gamma(1 + a - b)
\times \left\{\frac{\sin \pi(u + a)}{\pi(u + b)} \left(\frac{u \sin \pi a}{b \sin \pi v} - \frac{v \sin \pi b}{a \sin \pi u}\right) - \frac{\sin \pi(u + b)}{\pi(u + a)} \left(\frac{u \sin \pi b}{a \sin \pi v} - \frac{v \sin \pi a}{b \sin \pi u}\right)\right\}.
\]
The term in braces in this expression is

\[
\begin{align*}
&\frac{1}{2\pi(u+b)} \left( u(\cos \pi u - \cos \pi v \cos (a - b) + \sin \pi v \sin (a - b)) \right. \\
&\quad - \frac{v(\cos \pi u \cos (a - b) - \sin \pi u \sin (a - b) - \cos \pi v)}{a \sin \pi u} \\
&\quad - \frac{1}{2\pi(u+a)} \left( u(\cos \pi u - \cos \pi v \cos (b - a) + \sin \pi v \sin (b - a)) \right. \\
&\quad \left. - \frac{v(\cos \pi u \cos (b - a) - \sin \pi u \sin (b - a) - \cos \pi v)}{b \sin \pi u} \right)
\end{align*}
\]

Picking up the common factors, and noting the identities

\[
\begin{align*}
&\frac{1}{b(u+b)} - \frac{1}{a(u+a)} = \frac{v(a-b)}{ab(u+a)(u+b)}, \\
&\frac{1}{a(u+b)} - \frac{1}{b(u+a)} = \frac{u(b-a)}{ab(u+a)(u+b)}, \\
&\frac{u}{b(u+b)} + \frac{v}{a(u+b)} + \frac{u}{a(u+a)} + \frac{v}{b(u+a)} = \frac{2(v-u)}{ab},
\end{align*}
\]

we see that (12) becomes

\[
\frac{uv(a-b)}{ab(u+a)(u+b)} A(u, v, a, b) + \frac{(v-u) \sin \pi (a-b)}{ab\pi},
\]

which finishes the proof. \(\square\)

**Lemma 4.5.** The sum \( F_2(u, v, a, b) + F_2(u, v, b, a) \) equals

\[
\frac{(b-a)uv}{ab(u+a)(u+b)} \Gamma(b-a) \Gamma(1+a-b) A(u, v, a, b) \\
+ A(u, v, a, b) \frac{\Gamma(a) \Gamma(1-a) \Gamma(b) \Gamma(1-b) \Gamma(u+a) \Gamma(u+b)}{\Gamma(u) \Gamma(v)}.
\]

**Proof.** Applying Proposition 3.1 to the \( _3 F_2 \)-series in \( F_2(u, v, a, b) \), we find that \( F_2(u, v, a, b) \) becomes

\[
\frac{(a-b)\Gamma(a-b)\Gamma(b-a)\Gamma(u+b)}{\Gamma(u)\Gamma(v)\Gamma(1-u-a)} \left( \frac{\Gamma(v) \Gamma(1-v)}{\Gamma(a) \Gamma(1-a)} - \frac{\Gamma(u) \Gamma(1-u)}{\Gamma(-b) \Gamma(1+b)} \right) \\
\times \left( \psi(1+v) - \psi(a) - \psi(-b) - \gamma - \sum_{n=1}^{\infty} \frac{(a)_n(1+b)_n}{nn!(1+v)_n} \right),
\]
which is just
\[
A(u, v, a, b) \frac{\Gamma(b-a) \Gamma(1+a-b) \Gamma(u) \Gamma(u+b)}{\Gamma(u) \Gamma(v)} \
\times \left( \psi(1+v) - \psi(a) - \psi(-b) - \gamma - \sum_{n=1}^{\infty} \frac{(a)_n (1+b)_n}{nn!(1+v)_n} \right).
\]

Hence, using the equality \( A(u, v, a, b) = A(u, v, b, a) \), we find that
\[
F_2(u, v, a, b) + F_2(u, v, b, a) \
= A(u, v, a, b) \frac{\Gamma(b-a) \Gamma(1+a-b) \Gamma(u) \Gamma(u+b)}{\Gamma(u) \Gamma(v)} \
\times \left\{ \sum_{n=1}^{\infty} \left( \frac{(1+a)_n (b)_n}{nn!(1+v)_n} - \frac{(a)_n (1+b)_n}{nn!(1+v)_n} \right) + \psi(b) - \psi(-b) + \psi(-a) - \psi(a) \right\}.
\]

It is easy to see that
\[
\sum_{n=1}^{\infty} \left( \frac{(1+a)_n (b)_n}{nn!(1+v)_n} - \frac{(a)_n (1+b)_n}{nn!(1+v)_n} \right) = b-a \sum_{n=1}^{\infty} \frac{(a)_n (b)_n}{n!(1+v)_n},
\]
which equals
\[
\frac{b-a}{ab} \frac{\Gamma(1+u) \Gamma(1+v)}{\Gamma(1+u+a) \Gamma(1+u+b)} + \frac{1}{b} - \frac{1}{a}
\]
by the Gaussian summation formula. Applying the formulas
\[
\psi(-x) - \psi(x) - \frac{1}{x} = \pi \cot \pi x,
\]
and
\[
\pi \cot \pi a - \pi \cot \pi b = \frac{\Gamma(a) \Gamma(1-a) \Gamma(b) \Gamma(1-b)}{\Gamma(b-a) \Gamma(1+a-b)},
\]
we finish the proof.

Proof of Theorem 2.2. The theorem follows from Lemmas 4.2, 4.3, 4.4 and 4.5.

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