Stochastic reachability of a target tube: Theory and computation

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Abstract

Given a discrete-time stochastic system and a time-varying sequence of target sets, we consider the problem of maximizing the probability of the state evolving within this tube under bounded control authority. This problem subsumes existing work on stochastic viability and terminal hitting-time stochastic reach-avoid problems. Of special interest is the stochastic reach set, the set of all initial states from which the probability of staying in the target tube is above a desired threshold. This set provides non-trivial information about the safety and the performance of the system. In this paper, we provide sufficient conditions under which the stochastic reach set is closed, compact, and convex. We also discuss an underapproximative interpolation technique for stochastic reach sets. Finally, we propose a scalable, grid-free, and anytime algorithm that computes a polytopic underapproximation of the stochastic reach set and synthesizes an open-loop controller using convex optimization. We demonstrate the efficacy and scalability of our approach over existing techniques using three numerical simulations — stochastic viability of a chain of integrators, stochastic reach-avoid computation for a satellite rendezvous and docking problem, and stochastic reachability of a target tube for a Dubin’s car with a known turning rate sequence.

Index terms— Stochastic reachability, stochastic optimal control, optimization, convex optimization, chance constraint optimization, stochastic systems, constrained control

1 Introduction

Guarantees of safety or performance are crucial for a wide range of applications, including robotics, biomedical applications, and spacecraft applications [1–9]. Stochastic reachability analysis of discrete-time stochastic dynamical systems provides a mathematical framework to obtain probabilistic guarantees. The problem of stochastic reachability of a target tube is concerned with the computation of the maximum probability of staying within a target tube (a collection of time-stamped target sets) using an admissible controller when starting from a given initial state. This generalization subsumes existing work on stochastic viability and terminal hitting-time stochastic reach-avoid problems [1,2], and builds on our prior work in stochastic reachability [3,4].

The motivation for this work arises from the following question: what initial states of a stochastic dynamical system can be driven to stay within a target tube with a desired likelihood, while respecting the given bounds on control authority? For example, in a spacecraft rendezvous problem [3,6,7], the relative dynamics of the docking spacecraft with respect to the station can be modeled as a stochastic system because of model uncertainties. To ensure accurate sensing, the docking spacecraft must remain in a
line-of-sight cone (the target tube) and approach the origin (location of the station). For such safety-critical and expensive applications, it is crucial to identify the set of “good” initial locations from which rendezvous can be ensured with a probability of success above a desired threshold, while respecting the dynamics and limits on actuation. We also require synthesis of admissible controllers. In this paper, we focus on the set theoretic properties of the “good” initial states. We also propose tractable computational approaches to compute this set and design corresponding optimal controllers.

The problem of robust reachability of a target tube, in which the state of an uncertainty-perturbed system must be steered to lie within a target tube, despite a non-stochastic uncertainty that takes on values in a bounded set, has been discussed in [10,11]. This work utilizes computational geometry to construct the reach sets, and has been extended for the case of stochastic reach-avoid problems in [7]. However, its reliance on vertex-facet enumeration precludes computation on problems with large time horizons or small sets in the target tube. Alternatively, synthesizing (sub)optimal controllers for a stochastic optimal control problem using a receding horizon framework has been well-studied in stochastic model predictive control [12]. See [13,14] for recent surveys on this topic. However, these approaches do not tackle the problem of characterizing the set of “good” initial states, which is the main focus of this paper.

A dynamic programming approach, similar to [1,2], may be used as a theoretical and computational framework to address the stochastic reachability problem. This approach defines optimal value functions which map the states to their maximal reach probability. The superlevel sets of these functions, the stochastic reach sets, are the set of “good” initial states, i.e., the set of initial states from which the system may be driven to stay within the target tube with a probability greater than a given threshold. For stochastic reach-avoid problems, sufficient conditions have been proposed for the well-posedness of the stochastic reach-avoid problem and the existence of an optimal Markov policy [4,15,17]. In this paper, we propose two different sets of sufficient conditions for the well-posedness of the more general problem of stochastic reachability of a target tube for nonlinear time-varying systems. We also propose sufficient conditions for closedness, compactness, and convexity of the associated stochastic reach sets, extending our previous investigations into stochastic reach-avoid sets for LTI systems [3,4].

We can compute the maximal reach probability and the reach sets for low-dimensional systems using dynamic programming by gridding the state space [18]. Researchers have focused on alleviating this curse of dimensionality via approximate dynamic programming [19,20], Gaussian mixtures [19], particle filters [6,20], convex chance-constrained optimization [6], Fourier transforms and open-loop controllers [3,4], set-theoretic (Lagrangian) approaches [7], and semi-definite programming [16,21]. Our prior work proposed a scalable and grid-free algorithm to generate polytopic underapproximation of the stochastic reach-avoid sets for stochastic LTI systems using convex optimization, Fourier transforms, and open-loop controllers for problems [3,4]. This approach enabled, for the first time, the verification of systems as high as 40-dimensional. In this paper, we extend the open-loop controller-based underapproximation, presented in [3,4], to the more general problem of stochastic reachability of a target tube, and extend our algorithm to compute the polytopic underapproximation of the corresponding stochastic reach sets.

The main contributions of this paper are:

1. characterization of sufficient conditions under which the stochastic reachability problem of target tubes is well-defined, and the stochastic reach sets are closed, compact, and convex,

2. an underapproximative interpolation technique for stochastic reach sets, and

3. design of a scalable, grid-free, and anytime algorithm that utilizes convex optimization to provide an open-loop controller-based underapproximation.

Anytime algorithms provide a valid solution, even if terminated early. Our underapproximative interpolation technique can be exploited in real-time applications as stochastic reach sets at a desired threshold may be generated by simply interpolating among a small set of stochastic reach sets, pre-computed for a priori thresholds. We demonstrate our open-loop controller-based polytopic underapproximation on three numerical examples — stochastic viability of a chain of integrators, stochastic reach-avoid computation
for a satellite rendezvous and docking problem, and stochastic reachability of a target tube for a Dubin’s car with a known turning rate sequence.

The rest of this paper is organized as follows. Section 2 describes the stochastic reachability problem and relevant properties from probability theory and real analysis. Section 3 presents sufficient conditions for various properties of the stochastic reach set (see Table 1, pg. 11). Specifically, we formulate four different conditions to guarantee existence, closedness, compactness, and convexity of the stochastic reach sets (see Figure 3, pg. 8). We also discuss the underapproximative interpolation guarantee afforded by the convexity results. Section 4 presents the open-loop controller-based underapproximation and a scalable, grid-free, and anytime algorithm (Algorithm 1, pg. 16) to synthesize probabilistically safe open-loop controllers and polytopic underapproximations of the stochastic reach set using convex optimization. We demonstrate the proposed algorithm on several numerical examples in Section 5. We conclude and provide directions for future work in Section 6.

2 Preliminaries and problem formulation

We denote the Borel $\sigma$-algebra by $\mathcal{B}()$, a discrete-time time interval which inclusively enumerates all integers in between $a$ and $b$ for $a, b \in \mathbb{N}$ and $a \leq b$ by $\mathbb{N}_{[a,b]}$, random vectors with bold case, and non-random vectors with an overline. The indicator function of a non-empty set $E$ is denoted by $1_E(y)$, such that $1_E(y) = 1$ if $y \in E$ and is zero otherwise. We denote the affine hull and the convex hull of a set $E$ by $\text{affine}(E)$ and $\text{conv}(E)$, respectively.

2.1 Real analysis and probability theory

The relative interior of a set $E \subseteq \mathbb{R}^n$ is defined as

$$\text{relint}(E) = \{ \overline{x} \in \mathbb{R}^n : \exists r > 0, \text{Ball}(\overline{x}, r) \cap \text{affine}(E) \subseteq E \}$$

where $\text{Ball}(\overline{x}, r)$ denotes a ball in $\mathbb{R}^n$ centered at $\overline{x}$ and of radius $r$ with respect to any Euclidean norm [22, Sec. 2.1.3]. The relative interior of a set is always non-empty, while the interior of a low-dimensional set embedded in a high-dimensional space is empty (no open-ball exists such that it is a complete subset of the set). The relative boundary is $\partial E = \text{closure}(E) \setminus \text{relint}(E)$. From the Heine-Borel theorem [23, Thm 12.5.7], $E$ is compact if and only if it is closed and bounded.

A function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is upper semi-continuous (u.s.c.) if its superlevel sets $\{ \overline{x} \in \mathbb{R}^n : f(\overline{x}) \geq \alpha \}$ for every $\alpha \in \mathbb{R}$ are closed [24, Defs. 2.3 and 2.8]. A function $f$ is lower semi-continuous (l.s.c.) if $-f$ is u.s.c., and l.s.c. functions have closed sublevel sets $\{ \overline{x} \in \mathbb{R}^n : f(\overline{x}) \leq \alpha \}$ for every $\alpha \in \mathbb{R}$. A function is continuous if and only if it is both u.s.c. and l.s.c. The indicator function of a closed set is u.s.c.

A non-negative function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is log-concave if $\log f$ is concave with $\log 0 \triangleq -\infty$ [22, Sec. 3.5.1]. Many standard distributions are log-concave, for example, Gaussian, uniform, and exponential [22, Eg. 3.40]. The indicator function of a convex set is log-concave (See [22, Eg. 3.1 and Sec. 3.1.7]). Since log-concave functions are quasiconcave [22, Sec. 3.5.1], their superlevel sets are convex.

A random vector $\mathbf{y}$ is a measurable transformation defined in the probability space $(\Omega, \mathcal{Y}, \mathbb{P})$ with sample space $\Omega$, $\sigma$-algebra $\mathcal{Y}$, and probability measure $\mathbb{P}$ over $\mathcal{Y}$. In this paper, we will consider only Borel-measurable random vectors, $\mathbf{y} : \mathbb{R}^p \rightarrow \mathbb{R}^p$ with $\Omega = \mathbb{R}^p$ and $\mathcal{Y} = \mathcal{B}(\mathbb{R}^p)$. Recall that semicontinuous functions are Borel-measurable [23, Lem. 18.5.8]. For $N \in \mathbb{N}$, a random process is a sequence of random vectors $\{\mathbf{y}_k\}_{k=0}^{N}$ where the random vectors $\mathbf{y}_k$ are defined in the probability space $(\mathbb{R}^p, \mathcal{B}(\mathbb{R}^p), \mathbb{P})$. The random vector $\mathbf{Y} = [y_0 \ y_1 \ \cdots \ y_N]^T$ is defined in the probability space $(\mathbb{R}^{p(N+1)}, \mathcal{B}(\mathbb{R}^{p(N+1)}), \mathbb{P}_Y)$, with $\mathbb{P}_Y$ induced from $\mathbb{P}$. See [25,26] for details.
Figure 1: The target tube $\mathcal{T} = \{T_k\}_{k=0}^N$, the stochastic evolution of (1) under a maximal reach policy $\pi^*$, and the stochastic reach set $\mathcal{L}_0^{\pi^*}(\alpha, \mathcal{T})$ (12) for $\alpha = \frac{2}{3}$. Problem (8) subsumes the terminal hitting-time reach-avoid problem [1] $(\forall k \in \mathbb{N}_{[0,N-1]}, T_k = S, T_N = R)$ and the viability problem [2, 18] $(\forall k \in \mathbb{N}_{[0,N]}, T_k = S)$ for Borel safe $S$ and Borel terminal set $R$.

2.2 System description

Consider the discrete-time nonlinear time-varying system,

$$x_{k+1} = f_k(x_k, u_k, w_k)$$

with state $x_k \in \mathcal{X} \subseteq \mathbb{R}^n$, input $u_k \in \mathcal{U} \subseteq \mathbb{R}^m$, disturbance $w_k \in \mathcal{W} \subseteq \mathbb{R}^p$, time-varying nonlinear function $f_k : \mathcal{X} \times \mathcal{U} \times \mathcal{W} \to \mathcal{X}$, an initial state $x_0 \in \mathcal{X}$, and a time horizon of interest $N \in \mathbb{N}, N > 0$. We assume the input space $\mathcal{U}$ to be compact. We model the disturbance process $\{w_k\}_{k=0}^{N-1}$ in (1) as an independent, time-varying random process. Specifically, we associate with the random vector $w_k$, a probability space $(\mathcal{W}, \mathcal{B}(\mathcal{W}), \mathbb{P}_{w_k})$ and a probability density function $\psi_{w,k}$ for each $k \in \mathbb{N}_{[0,N-1]}$. The concatenated disturbance random vector $W = [w_0^\top \ w_1^\top \ldots \ w_{N-1}^\top]^\top$ is defined in the probability space $(\mathcal{W}^N, \mathcal{B}(\mathcal{W}^N), \mathbb{P}_W)$ with $\mathbb{P}_W = \prod_{k=0}^{N-1} \mathbb{P}_{w,k}$. We require the nonlinear function $f$ to be Borel-measurable, which guarantees that the state $\{x\}_{k=1}^N$ is a random process by [26, Sec. 1.4, Thm. 4]. Two special cases of (1) are

1. affine-perturbed nonlinear time-varying systems,

$$x_{k+1} = g_k(x_k, u_k) + w_k$$

where $g_k : \mathcal{X} \times \mathcal{U} \to \mathcal{X}$ is a nonlinear function defined for $k \in \mathbb{N}_{[0,N-1]}$, and

2. linear time-varying systems,

$$x_{k+1} = A_k x_k + B_k x_k + w_k$$

where $A_k \in \mathbb{R}^{n \times n}$ and $B_k \in \mathbb{R}^{n \times m}$ are the time-varying state and input matrices defined for $k \in \mathbb{N}_{[0,N-1]}$. For (3), the state space is $\mathcal{X} = \mathbb{R}^n$.

The system (1) can be equivalently described by a controlled Markov process with a stochastic kernel that is a time-varying Borel-measurable function $Q_k : \mathcal{B}(\mathcal{X}) \times \mathcal{X} \times \mathcal{U} \to [0, 1]$. The stochastic kernel assigns a probability measure on the Borel space $(\mathcal{X}, \mathcal{B}(\mathcal{X}))$ for $x_{k+1}$, parameterized by the current state $x_k$ and current action $u_k$, i.e., for any $G \in \mathcal{B}(\mathcal{X})$, $x \in \mathcal{X}$, and $u \in \mathcal{U}$,

$$\mathbb{P}_x \{x_{k+1} \in G | x_k = x, u_k = u\} = \mathbb{P}_{w,k} \{f_k(x, u, w_k) \in G\} = \int_G Q_k(dy|x, u).$$

(4)
By \((4)\), for any bounded Borel-measurable \(h : \mathcal{X} \rightarrow \mathbb{R}\),
\[
\int_{\mathcal{X}} h(\overline{y})Q_k(d\overline{y}|\overline{x}, \overline{u}) = \int_{\mathcal{X}} h(f_k(\overline{x}, \overline{u}, \overline{w}))\psi_{w,k}(\overline{w})d\overline{w}. \tag{5}
\]
In some cases, \(Q_k\) may also be explicitly expressed in terms of \(\psi_{w,k}\),
\[
Q_k(d\overline{y}|\overline{x}, \overline{u}) = \left\{ \begin{array}{ll}
\psi_{w,k}(\overline{y} - g_k(\overline{x}, \overline{u}))d\overline{y}, & \text{for } (2), \\
\psi_{w,k}(\overline{y} - A_k\overline{x} - B_k\overline{u})d\overline{y}, & \text{for } (3).
\end{array} \right. \tag{6}
\]
We define a Markov policy \(\pi = (\mu_0, \mu_1, \ldots, \mu_{N-1}) \in \mathcal{M}\) as a sequence of universally measurable state-feedback laws \(\mu_k : \mathcal{X} \rightarrow \mathcal{U}\) \([4\text{ Defn. } 2]\). The random vector \(\mathbf{X} = [x_1^\top \ x_2^\top \ \ldots \ x_N^\top]^\top\), defined in \((\mathcal{X}^N, \mathcal{B}(\mathcal{X}^N), \mathbb{P}_{\mathbf{X}}^{\pi})\) has a probability measure \(\mathbb{P}_{\mathbf{X}}^{\pi}\) defined using \(Q_k\) \([27\text{ Prop. } 7.45]\). Borel-measurable functions are universally measurable \([27\text{ Defn. } 7.20]\).

### 2.3 Stochastic reachability of a target tube

We define the target tube as \(\mathcal{T} = \{T_k\}_{k=0}^N, T_k \in \mathcal{B}(\mathcal{X})\). These are pre-determined time-stamped sets of states that are deemed safe at each time instant within the time horizon. Define the reach probability of a target tube, \(r^{\pi}_{\mathbf{x}_0}(\mathcal{T})\), for known \(\mathbf{x}_0\) and \(\pi\), as the probability that the execution with policy \(\pi\) lies within the target tube \(\mathcal{T}\) for the entire time horizon. Similarly to \([1,2]\),
\[
r^{\pi}_{\mathbf{x}_0}(\mathcal{T}) = \mathbb{P}_{\mathbf{X}}^{\pi,\mathbf{x}_0} \{ \forall k \in \mathbb{N}_{[0,N)}, \ x_k \in T_k \}. \tag{7}
\]
Motivated by \([1\text{ Def. } 10]\), we define a Markov policy \(\pi^*\) as a maximal reach policy when it is the optimal solution of \((8)\),
\[
r^{\pi^*}_{\mathbf{x}_0}(\mathcal{T}) = \sup_{\pi \in \mathcal{M}} r^{\pi}_{\mathbf{x}_0}(\mathcal{T}). \tag{8}
\]
The solution of \((8)\) may be characterized via dynamic programming, a straightforward extension of stochastic reachability \([1\text{ Thm. } 11]\) and viability \([2\text{ Thm. } 2]\). Define \(V^*_k : \mathcal{X} \rightarrow [0,1], k \in \mathbb{N}_{[0,N]}\), by the backward recursion for \(\pi \in \mathcal{X}\),
\[
V^*_N(\overline{x}) = 1_{\mathcal{T}_N}(\overline{x}) \tag{9a}
\]
\[
V^*_k(\overline{x}) = \sup_{\pi \in \mathcal{U}} 1_{\mathcal{T}_k}(\overline{x}) \int_{\mathcal{X}} V^*_{k+1}(\overline{y})Q_k(d\overline{y}|\overline{x}, \overline{u}). \tag{9b}
\]
Then, the optimal value to \((8)\) is \(r^{\pi^*}_{\mathbf{x}_0}(\mathcal{T}) = V^*_0(\mathbf{x}_0)\) for every \(\mathbf{x}_0 \in \mathcal{X}\). The optimal value function \(V^*_0(\mathbf{x}_0)\) assigns to each initial state \(\mathbf{x}_0 \in \mathcal{X}\) the maximal reach probability for the given target tube, and these maps are not probability density functions themselves (they don’t integrate to 1 over \(\mathcal{X}\)). By construction,
\[
0 \leq V^*_k(\overline{x}) \leq 1_{\mathcal{T}_k}(\overline{x}), \quad \forall \overline{x} \in \mathcal{X}. \tag{10}
\]
For \(\alpha \in [0,1]\), we define the superlevel sets of \(V^*_k(\cdot)\) as,
\[
\mathcal{L}^\pi_k(\alpha, \mathcal{T}) = \{\overline{x} \in \mathcal{X} : V^*_k(\overline{x}) \geq \alpha\}. \tag{11}
\]
Of special interest is the superlevel set of \(V^*_0(\cdot)\), the \(\alpha\)-level stochastic reach set,
\[
\mathcal{L}^\pi_0(\alpha, \mathcal{T}) = \{\overline{x} \in \mathcal{X} : r^{\pi^*}_{\mathbf{x}_0}(\mathcal{T}) \geq \alpha\}. \tag{12}
\]
Here, \(\mathcal{L}^\pi_0(\alpha, \mathcal{T})\) is the set of states which satisfies the objective of staying within the given target tube with a probability greater than or equal to \(\alpha\). From \([7]\), \(\mathcal{L}^\pi(0, \mathcal{T}) = \mathcal{X}\).
Lemma 1. If \( \alpha > 0 \), then \( L_k^\pi(\alpha, \mathcal{T}) \subseteq T_k, \forall k \in \mathbb{N}_{[0,N]} \). Additionally, bounded \( T_k \) implies bounded \( L_k^\pi(\alpha, \mathcal{T}) \) for any \( k \in \mathbb{N}_{[0,N]} \) and \( \alpha > 0 \).

Proof. For any \( \pi \in L_k^\pi(\alpha, \mathcal{T}) \), \( V_k^\pi(\pi) \geq \alpha \). By \([10]\), we have \( 1_{T_k}(\pi) \geq V_k^\pi(\pi) \geq \alpha > 0 \Rightarrow \pi \in T_k \). The boundedness of \( L_k^\pi(\alpha, \mathcal{T}) \) follows by definition \([23], \text{Defn. 12.5.3}\].

Figure 1 illustrates the definition of the target tube \( \mathcal{T} \) and the stochastic reach set \( L_0^\pi(\alpha, \mathcal{T}) \) \([12]\). Problem \((8)\) defines the problem of stochastic reachability of a target tube, and it subsumes existing work done on stochastic viability and stochastic reach-avoid problems \([1,2,18]\). For \( \mathcal{T} = \{S\}_{k=0}^N, r_{\pi_0}^\rho(\mathcal{T}) \) and \( L_0^\pi(\alpha, \mathcal{T}) \) is the maximal probabilistic safety probability and maximally probabilistic safe set (stochastic viability set) respectively \([2,18]\). For \( \mathcal{T} = \{(S)_{k=0}^{N-1}, T\}, r_{\pi_0}^\rho(\mathcal{T}) \) and \( L_0^\pi(\alpha, \mathcal{T}) \) is the maximal terminal hitting-time reach-avoid probability and the terminal hitting-time stochastic reach-avoid set respectively \([1]\).

Illustrative example: Consider the following one-dimensional system,

\[
x_{k+1} = x_k + u_k + w_k
\]  

with state \( x_k \in \mathbb{R} \), input \( u_k \in [-1,1] \), and disturbance \( w_k \sim \mathcal{N}(0,0.001) \). We consider the stochastic reachability of a target tube \( \mathcal{T} = \{-\gamma^k, \gamma^k\}_{k=0}^N \) with \( \gamma = 0.6 \) and time horizon \( N = 5 \). Using a step size of 0.01, the dynamic programming solution \([9]\) is shown in Figure 2. As prescribed by \([9a]\), we set \( V_0^\pi(x) = 1_{T_0}(x) \), and compute \( V_k^\pi(\cdot) \) using the backward recursion \([9b]\) over a grid of \( \{-1, -0.99, \ldots, 0.99, 1\} \). The 0.8-level stochastic reach set is given by the superlevel set of \( V_0^\pi(\cdot) \) at 0.8. From \( V_k^\pi(\cdot) \) shown in Figure 2 we observe \( 0 \leq V_k^\pi(\pi) \leq 1_{T_k}(\pi), \forall \pi \in \mathcal{X} \) \([10]\) and Lemma 1 \( L_k^\pi(\alpha, \mathcal{T}) \subseteq T_k, \forall k \in \mathbb{N}_{[0,N]} \) and \( \alpha > 0 \).

2.4 Problem formulation

In this paper, we study the set properties of the stochastic reachability problem of a target tube, and propose tractable approaches to compute the reach set and synthesize controllers. We will seek exact representations or underapproximations of \( L_k^\pi(\alpha, \mathcal{T}) \) and \( r_{\pi_0}^\rho(\mathcal{T}) \) as opposed to their overapproximations \([16]\), in order to remain conservative regarding our safety assessment.

Problem 1. Provide sufficient conditions under which:

1. a maximal Markov policy to solve \((8)\) exists,
2. \( V_k^\pi(\cdot) \) is Borel-measurable, u.s.c., and log-concave, and
3. the \( \alpha \)-superlevel set of \( V_0^\pi(\cdot), L_k^\pi(\alpha, \mathcal{T}) \), is closed, compact, and convex,

for every \( k \in \mathbb{N}_{[0,N]} \) and \( \alpha \in [0,1] \).
Problem 2. Provide sufficient conditions under which an underapproximative interpolation of $L_k^\pi^*(\alpha, \mathcal{T})$ can be constructed, given $L_k^\pi^*(\alpha_1, \mathcal{T})$ and $L_k^\pi^*(\alpha_2, \mathcal{T})$ with $\alpha \in [\alpha_1, \alpha_2]$ and $\alpha_1, \alpha_2 \in [0, 1]$ for any $k \in \mathbb{N}_{[0,N]}$.

Problem 3. Propose a scalable, grid-free, and anytime algorithm to compute a open-loop controller-based polytopic underapproximation to $L_k^\pi^*(\alpha, \mathcal{T})$, $\forall k \in \mathbb{N}_{[0,N]}$, $\forall \alpha \in [0, 1]$.

Problem 3.a. Characterize sufficient conditions under which:

1. the open-loop controller-based underapproximation to (8) is well-posed and convex,
2. the $\alpha$-superlevel set of $W_0^*(\cdot), K_0^\rho^*(\alpha, \mathcal{T})$, is convex and compact for $\alpha \in (0, 1]$, and
3. the underapproximative interpolation technique, described in Problem 2, holds for $K_0^\rho^*(\cdot, \mathcal{T})$.

Problem 3.b. Demonstrate that the restriction of admissible policies for (8) to open-loop controllers yields an underapproximation $W_0^* : \mathcal{X} \to [0, 1]$ to the maximal reach probability obtained via (8).

3 Properties of the stochastic reachability problem (8)

In this section, we will address Problem 1. We describe the relationship between various assumptions introduced in Section 3 in Figure 3.

3.1 Existence and measurability: Borel assumption

Sufficient conditions for the existence of an optimal Markov policy and the Borel-measurability of the optimal value functions have been formulated for reach-avoid problems [27, Sec. 8.3], [4, 15–17]. These results impose continuity requirements on the stochastic kernel (Definition 1) and utilize a measurable selection theorem [28, Thm. 2] to obtain the desired existence and measurability results. We now present straightforward extensions of these results to the more general problem of stochastic reachability of a target tube (8).

Definition 1. (Continuity of stochastic kernels) Let $\mathcal{H}$ be the set of all bounded and Borel-measurable functions $h : \mathcal{X} \to \mathbb{R}$. A stochastic kernel $Q_k(\cdot | \pi, \bar{u})$ is said to be:

a. input-continuous, if $\int_{\mathcal{X}} h(\bar{y})Q(d\bar{y}|\pi, \bar{u})$ is continuous over $\mathcal{U}$ for each $\pi \in \mathcal{X}$ for any $h \in \mathcal{H}$, and
b. continuous, if $\int_{\mathcal{X}} h(\bar{y})Q(d\bar{y}|\pi, \bar{u})$ is continuous over $\mathcal{X} \times \mathcal{U}$ for any $h \in \mathcal{H}$.

Recall that a function is said to be continuous if and only if its image of every sequence in its domain is also a convergent sequence [23, Thm. 13.4]. Since continuity over product spaces imply continuity over individual spaces [23, Lem. 13.2.1], continuous stochastic kernels are input-continuous. In other words, Definition 1b imposes a stronger requirement on the stochastic kernel $Q_k(\cdot | \pi, \bar{u})$ than Definition 1a.

Assumption 1 (Borel).

a. $f_k$ is Borel-measurable over $\mathcal{X} \times \mathcal{U} \times \mathcal{W}$, $\forall k \in \mathbb{N}_{[0,N-1]}$,
b. $\mathcal{U}$ is compact,
c. $\mathcal{T} = \{T_k\}_{k=0}^N$ such that $T_k \subseteq \mathcal{X}$ are Borel $\forall k \in \mathbb{N}_{[0,N]}$, and
d. $Q_k$ in (4) is input-continuous (Definition 1b).
Figure 3: Various assumptions introduced in Section 3, and the resulting set properties (italicized) of $L^*_k(\alpha, T)$.

In Theorem 1 and Proposition 1, we generalize the existence, measurability, and continuity results presented in [16, Props. 1 and 2] to the stochastic reachability problem of a target tube for a system described by a time-varying stochastic kernel. Note that unlike [16, Prop. 2], the structure in (8) permits exact characterization of where $V^*_k(\cdot)$ may be discontinuous in Proposition 1. Our proofs, provided in Appendices A.1 and A.2, are similar in structure to [16, Prop. 1 and 2] [4, Thm. 1], and exploit the u.s.c. property of the objective of (9b) afforded by Definition 1a.

**Theorem 1.** Under Assumption 1,

a. $V^*_k(\cdot), \forall k \in \mathbb{N}[0,N]$ is Borel-measurable, and

b. $\pi^*$ exists, and consists of Borel-measurable maps $\mu^*_k(\cdot), \forall k \in \mathbb{N}[0,N-1]$.

Since continuity implies u.s.c., $\int_{\mathcal{X}} V^*_{k+1}(\overline{y})Q_k(d\overline{y}|\overline{x},\mu^*_k(\overline{x}))$ is u.s.c. over $\mathcal{U}$ for every $\overline{x} \in \mathcal{X}$ and $k \in \mathbb{N}[0,N-1]$. Thus, the set

$$\mathcal{U}_k(\overline{x},\lambda) = \left\{ \overline{u} \in \mathcal{U} : \int_{\mathcal{X}} V^*_{k+1}(\overline{y})Q_k(\overline{y}|\overline{x},\overline{u})d\overline{y} \geq \lambda \right\}$$

is closed for every $\lambda \in \mathbb{R}$. Since $\mathcal{U}$ is compact (Assumption 1) and $\mathcal{U}_k(\overline{x},\lambda)$ is closed, $\mathcal{U}_k(\overline{x},\lambda)$ is compact for every $\overline{x} \in \mathcal{X}, k \in \mathbb{N}[0,N-1]$, and $\lambda \in \mathbb{R}$ [23, Thm. 12.5.10a]. The compactness of $\mathcal{U}_k(\overline{x},\lambda)$ for every $\overline{x} \in \mathcal{X}, k \in \mathbb{N}[0,N-1]$, and $\lambda \in \mathbb{R}$ is another well-known sufficient condition for the existence of Markov policy (see [2 Thm. 1] [1 Thm. 11] [27 Lem. 3.1]).

**Proposition 1.** Under Assumption 1, if $Q_k$ is continuous, then

a. $\int_{\mathcal{X}} V^*_{k+1}(\overline{y})Q_k(d\overline{y}|\overline{x},\mu^*_k(\overline{x}))$, $\forall k \in \mathbb{N}[0,N-1]$ is continuous over $\mathcal{X}$, and

b. $V^*_k(\cdot), \forall k \in \mathbb{N}[0,N-1]$ is piecewise-continuous over $\mathcal{X}$ where the discontinuities, if any, is restricted to the relative boundary of the target sets $\partial T_k$.

By Proposition 1 if for some $k \in \mathbb{N}[0,N-1]$, the target set $T_k = \mathcal{X}$, then $V^*_k(\cdot)$ is continuous over $\mathcal{X}$ for that particular $k$. For reachability problems that do not have safety constraints at $k = 0$ ($T_0 = \mathcal{X}$), $V^*_0(\cdot)$ is continuous over $\mathcal{X}$, presuming the restrictions specified in Assumption 1 and continuous $Q_k$.

Assumptions 1, 2, and 3 impose requirements on the stochastic reachability problem that are easy to ensure. Based on [27, Sec. 8.3], Lemma 2 provides a set of sufficient conditions that guarantees Assumption 1.
Lemma 2. Given an affine-perturbed nonlinear system (2) with \( g_k(\cdot, \cdot) \) continuous in \( U \) for each \( \pi \in \mathcal{X} \) and \( k \in \mathbb{N}_{[0,N-1]} \); if the disturbance PDF \( \psi_{w,k} \) is continuous over \( \mathcal{W} \), then the stochastic kernel \( Q_k \) defined by (6) is input-continuous.

Lemma 2 applies to linear systems (3) as well [4, Lem. 2]. If \( g_k \) is continuous over \( \mathcal{X} \times U \) for each \( k \in \mathbb{N}_{[0,N-1]} \), then we have continuous (as opposed to input-continuous) \( Q_k \).

3.2 Existence and compactness: Closed assumption

In this section, we consider Assumption 2 to provide an alternative set of sufficient conditions to guarantee existence of an optimal Markov policy to solve (8).

Assumption 2 (Closed).

a. \( f_k \) is continuous over \( \mathcal{X} \times \mathcal{U} \times \mathcal{W} \), \( \forall k \in \mathbb{N}_{[0,N-1]} \),

b. \( \mathcal{X} \) is closed.

c. \( \mathcal{U} \) is compact,

d. \( \mathcal{T} = \{ T_k \}_{k=0}^N \) such that \( T_k \subseteq \mathcal{X} \) are closed \( \forall k \in \mathbb{N}_{[0,N]} \), and

The key difference between Assumptions 1 and 2 is the relaxation (replacement) of Assumption 1, the continuity requirements on \( Q_k \), with stricter requirements on \( f_k \), \( \mathcal{X} \), and \( \mathcal{T} \). Note that Assumption 1 imposes restrictions on \( \psi_{w,k} \) but not on \( \mathcal{T} \), whereas Assumption 2 imposes restrictions on \( \mathcal{T} \) but not on \( \psi_{w,k} \). Hence, we do not expect either of these assumptions to subsume the other (see Figure 3).

For Assumption 2, Theorem 2 guarantees the existence of an optimal Markov policy and u.s.c. optimal value functions. In contrast to the proof of Theorem 1, the proof of Theorem 2 uses Proposition 2 (proof is given in Appendix A.3) to guarantee that the objective of (9b) is u.s.c. and then uses [27, Prop. 7.33] to guarantee that the optimal value functions are u.s.c. Note that Proposition 2 does not impose any restrictions on the stochastic kernel \( Q_k \).

Proposition 2. Suppose Assumptions 3 and 3 holds. For every bounded, non-negative, and u.s.c. function \( h : \mathcal{X} \to \mathbb{R} \), \( \int_{\mathcal{X}} h(\overline{y}) Q_k(d\overline{y} | \overline{x}, \overline{u}) \), \( \forall k \in \mathbb{N}_{[0,N-1]} \) is u.s.c. over \( \mathcal{X} \times \mathcal{U} \).

Theorem 2. Under Assumption 2

a. \( V_k^*(\cdot) \), \( \forall k \in \mathbb{N}_{[0,N]} \) is u.s.c. over \( \mathcal{X} \),

b. \( \pi^* \) exists, and it consists of Borel-measurable maps \( \mu_k^*(\cdot) \), \( \forall k \in \mathbb{N}_{[0,N-1]} \), and

c. \( L_k^+(\alpha, \mathcal{T}) \), \( \forall k \in \mathbb{N}_{[0,N]} \), \( \forall \alpha \in [0,1] \) is closed.

Proof. Since \( T_k \) and \( \mathcal{X} \) are closed, \( 1_{T_k}(\cdot) \), \( \forall k \in \mathbb{N}_{[0,N]} \) is u.s.c. over \( \mathcal{X} \). Hence, \( V_k^*(\cdot) \) is u.s.c. over \( \mathcal{X} \).

Consider the base case \( k = N - 1 \). Due to closedness of \( T_N \), \( V_N^*(\cdot) \) is u.s.c., and \( V_N^*(\cdot) \) is bounded and non-negative by (10). Hence, \( \int_{\mathcal{X}} V_N^*(\overline{y}) Q_{N-1}(d\overline{y} | \overline{x}, \overline{u}) \) is u.s.c. over \( \mathcal{X} \times \mathcal{U} \) by Proposition 2. By a selection result for semicontinuous cost functions [27, Prop. 7.33] and compactness of \( \mathcal{U} \), an optimal Borel-measurable input map \( \mu_{N-1}^*(\cdot) \) exists and \( \int_{\mathcal{X}} V_N^*(\overline{y}) Q_{N}(d\overline{y} | \overline{x}, \mu_{N-1}^*(\overline{x})) \) is u.s.c. over \( \mathcal{X} \). Since upper semicontinuity is preserved under multiplication [29, Props. B.1], \( V_{N-1}^*(\cdot) \) is u.s.c. over \( \mathcal{X} \) by (9b).

For the case \( k = t \) with \( t \in \mathbb{N}_{[0,N-2]} \), assume for induction that \( V_{t+1}^*(\cdot) \) is u.s.c.. By the same arguments as above, a Borel-measurable \( \mu_{t}^*(\cdot) \) exists and \( V_{t}^*(\cdot) \) is u.s.c., completing the proof for a) and b).

Upper semi-continuity of \( V_k^*(\cdot) \), \( \forall k \in \mathbb{N}_{[0,N]} \), implies that \( L_k^+(\alpha, \mathcal{T}) \), \( \forall k \in \mathbb{N}_{[0,N]} \) is closed for \( \alpha \in [0,1] \).

Proposition 3. Under Assumption 3 if \( T_k \) is bounded (and thereby compact) for some \( k \in \mathbb{N}_{[0,N]} \), then \( L_k^+(\alpha, \mathcal{T}) \), \( \forall \alpha \in (0,1] \), is compact.
Proof. Follows from Heine-Borel theorem, and closedness (Theorem 2c) and boundedness (Lemma 1) of $L^*_k(\alpha, \mathcal{T})$.

By Proposition 3, if $\mathcal{T}_0$ is bounded (and thereby compact), then $L^*_0(\alpha, \mathcal{T})$, $\forall \alpha \in (0,1]$ is compact, under Assumption 2. Further, if $\mathcal{X}$ is bounded, then $L^*(0, \mathcal{T}) = \mathcal{X}$ is compact.

Remark 1. The stochastic reachability problem of a target tube (8) is well-posed under Assumptions 1 or 2.

3.3 Convexity: Convex assumption

With existence conditions established for Assumptions 1 and 2, we now focus on establishing sufficient conditions under which $L^*_0(\alpha, \mathcal{T})$ is convex.

Assumption 3 (Convex).

a. System dynamics are linear (3) and $\mathcal{X} = \mathbb{R}^n$,
b. $\mathcal{U}$ is convex and compact,
c. Either
   1) $Q_k$ is input-continuous $\forall k \in \mathbb{N}_{[0,N-1]}$, OR
   2) $\mathcal{T}_k$ is closed $\forall k \in \mathbb{N}_{[0,N]}$,
d. $T_k$ is convex $\forall k \in \mathbb{N}_{[0,N]}$, and
e. $\psi_{w,k}$ is a log-concave PDF.

Under Assumption 3a, 3b, and 3c, the optimization problems in (9b) are well-defined and an optimal Markov policy exists (see Remark 1). We will use Proposition 4 in the proof of Theorem 3 to guarantee that the objective of (9b) is log-concave (similar to the role played by Proposition 2 in the proof of Theorem 2). The proof of Proposition 4 is given in Appendix A.4.

Proposition 4. Suppose Assumption 3 holds and $\mathcal{U}$ is convex. For every log-concave, Borel-measurable, and non-negative function $h : \mathcal{X} \rightarrow \mathbb{R}$, $\int_{\mathcal{X}} h(\varpi)Q_k(d\varpi|\varpi, u)$, $\forall k \in \mathbb{N}_{[0,N-1]}$, is log-concave over $\mathcal{X} \times \mathcal{U}$.

Theorem 3. Under Assumption 3.

a. $V^*_k(\cdot)$, $\forall k \in \mathbb{N}_{[0,N]}$ is log-concave over $\mathcal{X}$, and
b. $L^*_k(\alpha, \mathcal{T})$, $\forall k \in \mathbb{N}_{[0,N]}, \forall \alpha \in [0,1]$ is convex.

Proof. The proof of the log-concavity of $V^*_k(\cdot)$ is similar to Theorem 2. The convexity of $\mathcal{T}_k$, $\forall k \in \mathbb{N}_{[0,N]}$ ensures that their respective indicator functions are log-concave. The log-concavity of $V^*_k(\cdot)$, $\forall k \in \mathbb{N}_{[0,N]}$ follows from Proposition 4, the fact that log-concavity is preserved under partial supremum over convex sets and multiplication [22, Secs. 3.2.5 and 3.5.2], and the convexity of $\mathcal{U}$.

Log-concavity of $V^*_k(\cdot)$, $\forall k \in \mathbb{N}_{[0,N]}$ (via quasiconcavity) implies that $L^*_k(\alpha, \mathcal{T})$, $\forall k \in \mathbb{N}_{[0,N]}$ is convex for $\alpha \in [0,1]$ [22, Sec. 3.5].

Remark 2. With Theorem 3, we have also shown that the dynamic programming solution (9) to the stochastic reachability problem of a target tube (8) under Assumption 3 is a series of convex optimization problems.
| Property for $k \in \mathbb{N}_{[0,N]}$ | $f_k, \forall k \in \mathbb{N}_{[0,N-1]}$ over $\mathcal{X}$ | $\mathcal{X}$ (Borel) | $\mathcal{U}$ (Compact) | $\mathcal{Q}_k, \forall k \in \mathbb{N}_{[0,N]}$ (Borel-measurable) | Result |
|---------------------------------|-----------------|-----------------|-----------------|---------------------------------|--------|
| Measurability                   | Measurable      |                 |                 |                                 |        |
| Piecewise continuity            |                 |                 |                 |                                 |        |
| Upper semi-continuity           | Closed $\forall \alpha \in [0,1]$ | Continuous | Closed | Closed | Thm. 1 |
|                                 | Compact $\forall \alpha \in (0,1)$ |                 |                 |                                 | Prop. 1 |
| Log-concavity                   | Convex $\forall \alpha \in [0,1]$ | Linear (3) | Convex | Convex | Thm. 2 |
|                                 |                 |                 |                 | Input-continuous | Thm. 3 |

Table 1: Sufficient conditions for various properties of $V^*_k(\cdot)$ and $\mathcal{L}^*_k(\cdot)$. See [18, Thm. 2] for Lipschitz continuity of $V^*_k(\cdot)$.

### 3.4 Polytopic representation of $\mathcal{L}^*_k(\alpha, \mathcal{T})$: Convex and compact assumption

Polytopic underapproximation of convex and compact sets has been well studied [22, Ex. 2.25] [30, Ch. 2]. An underapproximative polytopic representation is provided by the convex hull of a set of points within the set [22, Sec. 2.1.4], and compactness ensures that this underapproximation can be made tight by identifying the extreme points [30, Thm 2.6.16]. Recall that a point $\bar{x} \in \mathcal{E}$ is an extreme point of the set $\mathcal{E}$ if and only if the only way to express $\bar{x}$ as a convex combination $(1-\theta)\bar{y} + \theta \bar{z}$, such that $\bar{y}, \bar{z} \in \mathcal{E}$, $\bar{y} \neq \bar{z}$, and $0 < \theta < 1$, is by taking $\bar{y} = \bar{z} = \bar{x}$ [30, Ch. 2]. Theorem 3 and Proposition 3 together guarantee convex and compact $\mathcal{L}^*_k(\alpha, \mathcal{T})$.

For ease of discussion, we formulate Assumption 4 to combine the requirements of Assumptions 2 and 3.

**Assumption 4** (Convex and compact).

- **a.** System dynamics are linear (3) and $\mathcal{X} = \mathbb{R}^n$,
- **b.** $\mathcal{U}$ is convex and compact,
- **c.** $\mathcal{T}_k$ is convex and compact $\forall k \in \mathbb{N}_{[0,N]}$, and
- **d.** $\psi_{w_k}$ is a log-concave PDF.

**Theorem 4.** Under Assumption 4, $\mathcal{L}^*_k(\alpha, \mathcal{T}), \forall k \in \mathbb{N}_{[0,N]}, \forall \alpha \in (0,1]$ is convex and compact.

**Proof.** Follows from Proposition 3 and Theorem 3.

**Remark 3.** Assumption 4 enables tight polytopic representation of $\mathcal{L}^*_k(\alpha, \mathcal{T}), \forall k \in \mathbb{N}_{[0,N]}, \alpha \in (0,1]$.

By Proposition 3 if for every $\alpha \in (0,1]$, we require only $\mathcal{L}^*_0(\alpha, \mathcal{T})$ be convex and compact, then Assumption 4 may be relaxed to the following requirements: 1) $\mathcal{T}_0$ is convex and compact, and 2) $\mathcal{T}_k, \forall k \in \mathbb{N}_{[1,N]}$ is convex and closed.

### 3.5 Underapproximative interpolation

Next, we address Problem 2 using Theorem 5 under Assumption 4. Theorem 5 states that given the polytopic representations of $\mathcal{L}^*_k(\alpha_1, \mathcal{T})$ and $\mathcal{L}^*_k(\alpha_2, \mathcal{T})$, we can compute the convex combination of the vertices of these polytopes using a specific weight $\theta$ to construct a polytopic underapproximation of $\mathcal{L}^*_k(\beta, \mathcal{T}), \beta \in [\alpha_1, \alpha_2]$.
Theorem 5. Suppose Assumption 4 holds and let \( k \in \mathbb{N}_{[0,N]} \) and \( K \in \mathbb{N}, K > 0 \). Given \( \alpha_1, \alpha_2 \in (0, 1] \), let \( \alpha_1 < \alpha_2 \), \( \bar{x}_1^{(i)}, \ldots, \bar{x}_1^{(K)} \in \mathcal{L}_k^{\pi*}(\alpha_1, \mathcal{T}) \) and \( \bar{x}_2^{(i)}, \ldots, \bar{x}_2^{(K)} \in \mathcal{L}_k^{\pi*}(\alpha_2, \mathcal{T}) \).

For any \( \beta \in [\alpha_1, \alpha_2] \), \( \text{conv}_{i \in \mathbb{N}_{[1,K]}}(\bar{y}^{(i)}) \subseteq \mathcal{L}_k^{\pi*}(\beta, \mathcal{T}) \) where

\[
\bar{y}^{(i)} = \theta \bar{x}_1^{(i)} + (1 - \theta) \bar{x}_2^{(i)}, \quad \forall i \in \mathbb{N}_{[1,K]}, \quad \text{and}
\]

\[
\theta = \frac{\log(\alpha_2) - \log(\beta)}{\log(\alpha_2) - \log(\alpha_1)} \in [0, 1].
\]

Proof. By definition of \( \bar{x}_1^{(i)}, \bar{x}_2^{(i)} \), \( V_k^*(\bar{x}_1^{(i)}) \geq \alpha_1 > 0 \) and \( V_k^*(\bar{x}_2^{(i)}) \geq \alpha_2 > 0 \) for every \( i \in \mathbb{N}_{[1,K]} \). Note that for \( \theta \) defined by \( (15) \), \( \theta \in [0, 1] \) and \( \beta = \alpha_1 \alpha_2^{(1-\theta)} \).

Since \( x^\theta \) for \( x > 0 \) and \( \theta \in [0, 1] \) is nondecreasing, we have \( (V_k^*(\bar{x}_1^{(i)}))^\theta \geq \alpha_1^\theta > 0 \), \( (V_k^*(\bar{x}_2^{(i)}))^{(1-\theta)} \geq \alpha_2^{(1-\theta)} > 0 \), and \( (V_k^*(\bar{x}_1^{(i)}))^\theta (V_k^*(\bar{x}_2^{(i)}))^{(1-\theta)} \geq \alpha_1^\theta \alpha_2^{(1-\theta)} \) by [23, Prop. 5.4.7e]. By log-concavity of \( V_k^*(\cdot) \) (Theorem 3) and the definition of \( \bar{y}^{(i)} \) in \( (14) \), we have, for every \( i \in \mathbb{N}_{[1,K]} \),

\[
V_k^*(\bar{y}^{(i)}) = V_k^*(\theta \bar{x}_1^{(i)} + (1 - \theta) \bar{x}_2^{(i)})
\]

\[
\geq (V_k^*(\bar{x}_1^{(i)}))^\theta (V_k^*(\bar{x}_2^{(i)}))^{(1-\theta)}
\]

\[
\geq \alpha_1^\theta \alpha_2^{(1-\theta)} = \beta.
\]

Hence, \( \bar{x}^{(1)}, \ldots, \bar{x}^{(K)} \in \mathcal{L}_k^{\pi*}(\beta, \mathcal{T}) \). The proof is completed by noting that the convex hull of a finite collection of points in a convex set is contained in the set [22, Sec. 2.1.4].

We summarize the sufficient conditions for existence, measurability, continuity, and log-concavity of \( V_k^*(\cdot) \) and closedness, compactness, and convexity of \( \mathcal{L}_k^{\pi*}(\alpha, \mathcal{T}) \) in Table 1. Table 2 and Theorem 5 addresses Problem 1.

4 Underapproximative stochastic reachability of a target tube using open-loop controllers

In stochastic reachability problems, we are typically interested in either an exact computation or an underapproximation. In safety problems, we do not want to overestimate our probability of safety, while underestimating the probability of safety is potentially useful. This trend holds for the stochastic reach set computation as well. In this section, we use open-loop controllers to compute an underapproximation to maximal reach probability (8) and the stochastic reach set (11), discuss its compactness and convexity properties, and propose a scalable, grid-free, and anytime algorithm to compute the stochastic reach set.

4.1 Formulation of the optimization problem

In [4,6], the authors proposed a tractable solution to the stochastic reach-avoid problem by restricting the search for the optimal control policy to open-loop control policies. An open-loop policy \( \rho : \mathcal{X} \rightarrow \mathcal{U}^N \) provides a sequence of inputs \( \rho(\bar{x}_0) = [\bar{u}_0^T \bar{u}_1^T \ldots \bar{u}_{N-1}^T]^T \) for every initial condition \( \bar{x}_0 \). Note that all actions of this policy are contingent only on the initial state, and not the current state, as in a Markov policy. The random vector describing the extended state \( \bar{X} \), under the action of \( \bar{U} = \rho(\bar{x}_0) \), lies in the probability space \( \mathcal{X}^N, \mathcal{B}(\mathcal{X}^N), \mathbb{P}^{\bar{x}_0,\bar{U}}_{\bar{X}} \) with \( \mathbb{P}^{\bar{x}_0,\bar{U}}_{\bar{X}} \) defined using \( Q_k \) [27, Prop. 7.45]. Under \( \rho(\cdot) \), the reach probability is given by

\[
r^\rho_{\bar{x}_0}(\mathcal{T}) \triangleq 1_{\mathcal{T}_0}(\bar{x}_0) \mathbb{P}^{\bar{x}_0,\bar{U}}_{\bar{X}} \left\{ \bar{X} \in \bigcap_{k=1}^{N} \mathcal{T}_k \right\}.
\]
The probability measure \( \mathbb{P}_{\tau_0, \mathcal{U}} \) in \([16]\) is linked to the forward stochastic reach probability measure \([31, 32]\). For linear systems, \( \mathbb{P}_{\tau_0, \mathcal{U}} \) can be computed for arbitrary disturbances using Fourier transforms. Denoting the optimal open-loop controller by \( \rho^* \), we define \( W_0^*(\cdot) : \mathcal{X} \to [0,1] \) as,

\[
W_0^*(\tau_0) \triangleq r_{\tau_0}^{\rho^*}(\mathcal{T}) = \sup_{\rho(\tau_0) = \mathcal{U} \in \mathcal{U}^N} r_{\tau_0}^{\rho}(\mathcal{T}).
\]  

(17)

where \( W_0^* : \mathcal{X} \to [0,1] \) is the maximal reach probability attained by evolving \( \rho \) from \( \tau_0 \), when restricted to open-loop controllers. Similarly to \((9)\), we define \( W_k : \mathcal{X} \times \mathcal{U}^{N-k} \to [0,1], \forall k \in \mathbb{N}_{[0,N-1]} \),

\[
W_{k+1}(\tau, \mathcal{U}_{k+1}) = 1_{\tau_{N-1}}(\tau) \int_{\mathcal{X}} 1_{\tau_k}(\bar{y}) Q_k(\bar{y}|\tau, \mathcal{U}_{N-1})
\]

(18a)

\[
W_k(\tau, \mathcal{U}_{k:N}) = 1_{\tau_k}(\tau) \int_{\mathcal{X}} W_{k+1}(\bar{y}, \mathcal{U}_{k+1:N}) Q_k(\bar{y}|\tau, \mathcal{U}_k)
\]

(18b)

\[
W_0^*(\tau_0) = \sup_{\mathcal{U} \in \mathcal{U}^N} W_0(\tau_0, \mathcal{U})
\]

(18c)

where \( \mathcal{U}_{k:N} = \{ \tau_k^1 \tau_{k+1}^1 \cdots \tau_N^1 \}^\top \in \mathcal{U}^{N-k}, \forall k \in \mathbb{N}_{[0,N-1]} \), \( \mathcal{U} = \mathcal{U}_0, \mathcal{U} \in \mathcal{U}^N \), and \( \mathcal{U}_{N-1:N} = \pi_{N-1} \in \mathcal{U} \).

In contrast to \( V_k^* \) in \((9)\), \( W_k(\cdot) \) are not optimal value functions, as there is no optimization. Since \( r_{\tau_0}^{\rho^*}(\mathcal{T}) = W_0(\tau_0, \mathcal{U}) \) for \( \rho(\tau_0) = \mathcal{U} \in \mathcal{U}^N \), the optimization problems \((17)\) and \((18c)\) are equivalent. Similarly to \((12)\), we define the \( \alpha \)-superlevel set of \( r_{\tau_0}^{\rho^*}(\mathcal{T}) \) as \( \mathcal{K}_0^\alpha(\alpha, \mathcal{T}) \),

\[
\mathcal{K}_0^\alpha(\alpha, \mathcal{T}) = \{ \tau_0 \in \mathcal{X} : r_{\tau_0}^{\rho^*}(\mathcal{T}) \geq \alpha \}.
\]

(19)

Theorem 6 addresses Problem 3.1 by carrying forward all the results in Section 3 for the open-loop controller-based underapproximations \((17)\) and \((19)\).

**Theorem 6.**

a. Under Assumption 1 or Assumption 2 \((17)\) is well-posed. Under Assumption 1, \( W_0^*(\cdot) \) is Borel-measurable, and under Assumption 2, \( W_0^*(\cdot) \) is u.s.c..

b. Under Assumption 1 \((17)\) is a log-concave optimization problem, \( W_0(\cdot, \cdot) \) is log-concave over \( \mathcal{X} \times \mathcal{U}^N \), and \( W_0^*(\cdot) \) is log-concave over \( \mathcal{X} \).

c. Under Assumption 4, \( \mathcal{K}_0^\alpha(\alpha, \mathcal{T}) \), \( \forall \alpha \in (0,1] \) is convex and compact.

d. Under Assumption 4, \( \mathcal{K}_0^\alpha(\alpha, \mathcal{T}) \), \( \forall \alpha \in (0,1] \) can be underapproximated by interpolating the vertices of polytopic (underapproximative) representations of \( \mathcal{K}_0^\alpha(\alpha_1, \mathcal{T}) \) and \( \mathcal{K}_0^\alpha(\alpha_2, \mathcal{T}) \), as described in Theorem 5.

**Proof.**

**Proof of a** with Assumption 1: Similarly to the proof of Theorem 1 we can show by induction and Definition 1a that \( W_0(\tau_0, \mathcal{U}) \) is continuous (and thereby u.s.c.) in \( \mathcal{U}^N \) for every \( \tau_0 \in \mathcal{X} \) when \( Q_k \) is input-continuous. Hence, by \([28, \text{Thm. 2}]\), we know that \((18c)\) (and thereby \((17)\)) is well-posed, and an optimal Borel-measurable open-loop controller \( \rho^* \) exists and \( W_0^*(\cdot) \) is Borel-measurable.

**Proof of a** with Assumption 2: Similarly to the proof of Theorem 2 we can show by induction and Proposition 2 that \( W_0(\tau_0, \mathcal{U}) \) is u.s.c. in \( \mathcal{X} \times \mathcal{U}^N \) and is bounded and nonnegative, when \( \mathcal{T}_k \) is closed and \( f_k \) is continuous. Hence, by \([27, \text{Prop. 7.33}]\), we know that \((18c)\) (and thereby \((17)\)) is well-posed, and an optimal Borel-measurable open-loop controller \( \rho^* \) exists, and \( W_0^*(\cdot) \) is u.s.c..

**Proof of b**: Similarly to the proof of Theorem 3 we can show by induction and Proposition 3 that \( W_0(\tau_0, \mathcal{U}) \) is log-concave in \( \mathcal{X} \times \mathcal{U}^N \) when \( \mathcal{T}_k \) is convex and \( \psi_{\mathcal{U}, k} \) is log-concave. Note that \( \mathcal{U}^N \) is convex since \( \mathcal{U} \) is convex \([22, \text{Sec. 2.3.2}]\). Hence, \((18c)\) (and thereby \((17)\)) is a log-concave optimization. Since partial supremum over convex sets preserves log-concavity \([22, \text{Sec. 3.2.5, 3.5}]\), \( W_0^*(\cdot) \) is log-concave over \( \mathcal{X} \).
Proof of (a): From Proposition 4 and the fact that Assumption 4 is a special case of Assumptions 2 and 3 (see Figure 3), $K_0^\alpha(\alpha, \mathcal{T})$, $\forall \alpha \in [0, 1]$ is convex and closed by (19). Similar to Lemma 1, we note that $K_0^\alpha(\alpha, \mathcal{T}) \subseteq T_0$. The compactness assumption of $T_0$ in Assumption 4 completes the proof, as in Proposition 3.

Proof of (d): From Proposition 4 and the discussion in Section 3.4, we know that polytopic underapproximations exist for $K_0^\alpha(\alpha, \mathcal{T})$. The proof, similar to Theorem 5, follows from the log-concavity of $W_0^\alpha$.

We next address Problem 3.b using Theorem 7. We first show that the value functions $W_k(\cdot)$ are underapproximations of the optimal value functions $V_k^\alpha(\cdot)$ in Proposition 5.

**Proposition 5.** Under Assumption 1 or 2, $W_k(\bar{x}, \bar{u}; N) \leq V_k^\alpha(\bar{x})$, $\forall k \in \mathbb{N}, \forall \bar{u}, \bar{N} \in \mathcal{U}^{N-k}$ and $\forall \bar{x} \in \mathcal{X}$.

Proof. (By induction) We first prove the base case $k = N - 1$, i.e., $W_{N-1}(\bar{x}, \bar{u}; N-1) \leq V_{N-1}^\alpha(\bar{x})$. From (9a) and (18a), $W_N(\bar{x}) = V_N^\alpha(\bar{x})$, $\forall \bar{x} \in \mathcal{X}$. By [23] Prop. 19.3.3d, for every $(\bar{x}, \bar{u}) \in \mathcal{X} \times \mathcal{U}$, we have $\int_{\mathcal{X}} W_N(\bar{y})Q_N(\bar{d}\bar{y}; \bar{x}, \bar{u}) = \int_{\mathcal{X}} V_N^\alpha(\bar{y})Q_N(\bar{d}\bar{y}; \bar{x}, \bar{u}) \leq \sup_{\bar{x} \in \mathcal{X}} \int_{\mathcal{X}} V_N^\alpha(\bar{y})Q_N(\bar{d}\bar{y}; \bar{x}, \bar{u})$. By (9b), (18b), and the fact that indicator functions are non-negative, we have $W_{N-1}(\bar{x}, \bar{u}; N) \leq V_{N-1}^\alpha(\bar{x})$ for every $\bar{x} \in \mathcal{X}, \forall \bar{u}, \bar{N} \in \mathcal{U}$.

Assume for induction, the case $k = t$ (i.e., $W_{t+1}(\bar{x}, \bar{u}; t+1) \leq V_{t+1}^\alpha(\bar{x})$). By [23] Prop. 19.3.3c, for every $(\bar{x}, \bar{u}; t) \in \mathcal{X} \times \mathcal{U}^{N-t}$, we have $\int_{\mathcal{X}} W_{t+1}(\bar{y}, \bar{u}; t+1)Q_t(\bar{d}\bar{y}; \bar{x}, \bar{u}) = \int_{\mathcal{X}} V_{t+1}^\alpha(\bar{y})Q_t(\bar{d}\bar{y}; \bar{x}, \bar{u}) \leq \sup_{\bar{x} \in \mathcal{X}} \int_{\mathcal{X}} V_{t+1}^\alpha(\bar{y})Q_t(\bar{d}\bar{y}; \bar{x}, \bar{u})$. The proof is completed by (9b), (18b), and the fact that indicator functions are non-negative.

We require the assumptions of Assumption 1 or 2 to ensure that (17) is well-posed (Theorem 4b).

**Theorem 7.** Under Assumption 1 or 2, $W_0^\alpha(\bar{x}) \leq V_0^\alpha(\bar{x})$, $\forall \bar{x} \in \mathcal{X}$, and $K_0^\alpha(\alpha, \mathcal{T}) \subseteq L_0^\alpha(\alpha, \mathcal{T})$, $\forall \alpha \in [0, 1]$.

Proof. By Proposition 3 we know that $W_0(\bar{x}, \bar{U}) \leq V_0^\alpha(\bar{x})$ for every $\bar{x} \in \mathcal{X}$ and $\bar{U} \in \mathcal{U}$. By (18c) and the definition of the supremum, we have $W_0(\bar{x}) = \sup_{\bar{U} \in \mathcal{U}} W_0(\bar{x}, \bar{U}) \leq V_0^\alpha(\bar{x})$ Consequently, we have $K_0^\alpha(\alpha, \mathcal{T}) \subseteq L_0^\alpha(\alpha, \mathcal{T})$, $\forall \alpha \in [0, 1]$ by (12) and (19).

4.2 Construction of a polytopic underapproximation of $K_0^\alpha(\alpha, \mathcal{T})$ under Assumption 4

Given a finite set $\mathcal{D} \subset \mathcal{X}$ consisting of linearly independent direction vectors $\bar{d}_i$, we propose Algorithm 1 to compute a polytopic underapproximation of $K_0^\alpha(\alpha, \mathcal{T})$ in three steps (Figure 4):

1. find $\bar{x}_{\text{max}} = \arg \max_{\bar{x} \in \mathcal{X}} W_0^\alpha(\bar{x})$; if $W_0^\alpha(\bar{x}_{\text{max}}) < \alpha$, then $K_0^\alpha(\alpha, \mathcal{T}) = \emptyset$; else, continue,

2. obtain relative boundary points of the set $K_0^\alpha(\alpha, \mathcal{T})$ via line searches from $\bar{x}_{\text{max}}$ along the directions $\bar{d}_i \in \mathcal{D}$, and

3. compute the convex hull of the computed relative boundary points to obtain a polytope $K_0^\alpha(\alpha, \mathcal{T}, \mathcal{D})$.

By [22] Sec. 2.1.4, we have $K_0^\alpha(\alpha, \mathcal{T}, \mathcal{D}) \subseteq K_0^\alpha(\alpha, \mathcal{T})$. We have equality when all of the extreme points of $K_0^\alpha(\alpha, \mathcal{T})$ are discovered by this approach (possible for a polytopic $K_0^\alpha(\alpha, \mathcal{T})$) [30] Thm. 2.6.16]. The steps 2) and 3) are enabled by the compactness and convexity of $K_0^\alpha(\alpha, \mathcal{T})$ [3] Prop. 4] [30] Ch. 2. Algorithm 1 solves Problem 3.
4.2.1 Compute $x_{\text{max}}$ that maximizes $W_0^*(\bar{x})$

We solve the following optimization problem

$$\begin{align*}
\text{maximize} & \quad \beta \\
\text{subject to} & \quad \bar{x} \in \mathcal{T}_0, \quad U \in \mathcal{U}^N, \quad \beta \in [0, 1], \\
& \quad W_0(\bar{x}, U) \geq \beta, \quad \beta \geq \alpha
\end{align*}$$

(20)

We denote the optimal solution of (20) as $\bar{x}_{\text{max}} \in \mathcal{X}$ (the maximizer of $W_0^*(\cdot)$), the associated optimal open-loop controller $U_{\text{max}} \in \mathcal{U}^N$, and the highest value of maximal reach probability $\beta^* = W_0^*(\bar{x}_{\text{max}})$ with $W_0(\cdot, \cdot)$ given by (18) at $k = 0$. The optimization problem (20) is (18) written in the epigraph form [22, Eq. 4.11], with an additional constraint of $W_0(\bar{x}, U) \geq \alpha$.

By Theorem 6b, applying log($\cdot$) to the constraint $W_0(\bar{x}, U) \geq \beta$ converts (20) into a convex problem. The formulation of (20) ensures that if it is infeasible, then $\mathcal{K}_0^\rho^*(\alpha, \mathcal{T}, \mathcal{D})$ and $\mathcal{K}_0^\rho^*(\alpha, \mathcal{T})$ are empty. We cannot conclude that $\mathcal{L}_0^\rho^*(\alpha, \mathcal{T})$ is empty, because of the underapproximative nature of our approach (Theorem 7).

4.2.2 Compute relative boundary points of $\mathcal{K}_0^\rho^*(\alpha, \mathcal{T})$ via line searches

To compute the relative boundary points of $\mathcal{K}_0^\rho^*(\alpha, \mathcal{T})$, we must solve for each $i \in \mathbb{N}_{[1, |\mathcal{D}|]}$

$$\begin{align*}
\text{maximize} & \quad \theta_i, U_i \\
\text{subject to} & \quad U_i \in \mathcal{U}^N, \quad \theta_i \in \mathbb{R}, \quad \theta_i \geq 0, \\
& \quad W_0(\bar{x}_{\text{max}} + \theta_i \bar{d}_i, U_i) \geq \alpha
\end{align*}$$

(21)

We denote the optimal solution of (21) as $\theta_i^*$ and $U_i^*$. The optimization problem (21) is also known as a line search problem, an integral component of the gradient descent algorithms [22, Sec. 9.3]. It may be solved exactly via convex optimization (by Theorem 6b and [22, Sec. 3.2.2]), or approximatively via backtracking line search [22, Pg. 465]. Note that, by construction, $W_0(\bar{x}_{\text{max}} + \theta_i^* \bar{d}_i, U_i^*) = \alpha$, and for any $\epsilon > 0$, $W_0(\bar{x}_{\text{max}} + (\theta_i^* + \epsilon) \bar{d}_i, U_i^*) < \alpha$. Hence, $\bar{x}_{\text{max}} + \theta_i^* \bar{d}_i \in \partial \mathcal{K}_0^\rho^*(\alpha, \mathcal{T})$, and the optimal open-loop controller from this relative boundary point is $\rho^*(\bar{x}_{\text{max}} + \theta_i^* \bar{d}_i) = U_i^*$ [3, Prop. 4].

4.2.3 Construction of $\mathcal{K}_0^\rho^*(\alpha, \mathcal{T}, \mathcal{D})$

If (20) has a solution, we have $W_0^*(\bar{x}_{\text{max}}) \geq \alpha$, and we construct the polytope $\mathcal{K}_0^\rho^*(\alpha, \mathcal{T}, \mathcal{D})$ via the convex hull of the computed relative boundary points $\bar{x}_{\text{max}} + \theta_i^* \bar{d}_i, \forall i \in \mathbb{N}_{[1, |\mathcal{D}|]}$. Since $\mathcal{K}_0^\rho^*(\alpha, \mathcal{T})$ is convex and...
compact, and \( x_{\text{max}} + \theta_i d_i \in \partial K_0^\rho(\alpha, \mathcal{T}) \), we have \( K_0^\rho(\alpha, \mathcal{T}, \mathcal{D}) \subseteq K_0^\rho(\alpha, \mathcal{T}) \) [22]. On the other hand, if (20) is infeasible, then \( K_0^\rho(\alpha, \mathcal{T}) \) is empty, which implies \( K_0^\rho(\alpha, \mathcal{T}, \mathcal{D}) \) is empty.

**Algorithm 1** Compute polytopic \( K_0^\rho(\alpha, \mathcal{T}, \mathcal{D}) \subseteq L_0^\rho(\alpha, \mathcal{T}) \)

**Input:** Probability threshold \( \alpha \), set of direction vectors \( \mathcal{D} \)

**Output:** \( K_0^\rho(\alpha, \mathcal{T}, \mathcal{D}) \subseteq K_0^\rho(\alpha, \mathcal{T}) \subseteq L_0^\rho(\alpha, \mathcal{T}) \)

1. Solve (20) to compute \( x_{\text{max}} \)
2. If \( W_0(x_{\text{max}}) \geq \alpha \) then
   3. For \( d_i \in \mathcal{D} \) do
      4. Solve (21) to compute a relative boundary point \( x_{\text{max}} + \theta_i d_i \) and its optimal open-loop controller \( \rho^*(x_{\text{max}} + \theta_i d_i) \)
   5. End for
3. \( K_0^\rho(\alpha, \mathcal{T}, \mathcal{D}) \leftarrow \text{conv}_{i \in \mathbb{N}[1,|\mathcal{D}|]}(x_{\text{max}} + \theta_i d_i) \)
4. Else
   5. \( K_0^\rho(\alpha, \mathcal{T}, \mathcal{D}) \leftarrow \emptyset \)
   6. End if

### 4.3 Implementation of Algorithm 1

Algorithm 1 is an anytime algorithm, as interrupting the convex hull of the solutions of (21) for an arbitrary subset of direction vectors in \( \mathcal{D} \) also provides a valid underapproximation. Additionally, Algorithm 1 is parallelizable since the computations along each of the direction \( d_i \) are independent.

The choice of \( \mathcal{D} \) influences the quality (in terms of volume) of underapproximation provided by Algorithm 1. One strategy is to choose the vectors in \( \mathcal{D} \) that are spaced far apart initially, and then increase \( |\mathcal{D}| \) by sampling appropriate directions to tighten the underapproximation as appropriate, at the cost of increased computational time.

Denoting the computation times to solve (20) and (21) as \( t_{\text{max}} \) and \( t_{\text{line}} \) respectively, the computation time for Algorithm 1 is \( O(t_{\text{max}} + t_{\text{line}}|\mathcal{D}|) \). Since (20) and (21) are convex problems, globally optimal solutions are assured with (potentially) low \( t_{\text{max}} \) and \( t_{\text{line}} \). However, the joint chance constraint \( W_0(\cdot, \cdot) \geq \alpha \) is not solver-friendly, since we do not have a closed-form expression for \( W_0(\cdot, \cdot) \), or an exact reformulation into a conic constraint. In Section 4.5 (see Table 2), we discuss computationally efficient methods to enforce this constraint under some additional assumptions (Assumption 5).

The memory requirements of Algorithm 1 grow linearly with \( |\mathcal{D}| \) and are independent of the system dimension. In contrast, dynamic programming requires an exponential number of grid points in memory, leading to the curse of dimensionality [18]. Algorithm 1 is grid-free and recursion-free, and it scales favorably with the system dimension, as compared to dynamic programming. Thus, Algorithm 1 can verify and synthesize controllers for high-dimensional systems that were previously not verifiable.

### 4.4 Open-loop controller synthesis via solutions of (21)

As a side product of Algorithm 1, specifically of solving (21), we obtain open-loop controllers for the vertices (extreme points) of \( K_0^\rho(\alpha, \mathcal{T}, \mathcal{D}) \). Since any non-extreme point in \( K_0^\rho(\alpha, \mathcal{T}, \mathcal{D}) \) can be written as a convex combination of the vertices of \( K_0^\rho(\alpha, \mathcal{T}, \mathcal{D}) \) [30] Ch. 2], we can use this information to synthesize open-loop controllers for the non-extreme points. We note that, for a given non-extreme initial state of interest, the corresponding convex combination of the optimal open-loop controllers at the vertices serves as a good initial guess to solve (17).
4.5 Gaussian linear time-varying systems with polytopic input space and polytopic target tube

Assumption 5. We presume conditions of Assumption 4, polytopic $U$, Gaussian $w_k \sim \mathcal{N}(\mu_{w,k}, C_{w,k})$, $\mu_{w,k} \in \mathbb{R}^p, C_{w,k} \in \mathbb{R}^{p \times p}$ and polytopic $T_k$, $\forall k \in \mathbb{N}_{[0,N]}$.

Under this assumption, the concatenated disturbance random vector is $W \sim \mathcal{N}(\mu_{W}, C_{W})$, where $\mu_W = [\mu_{w,0}^\top, \ldots, \mu_{w,N-1}^\top]^\top \in \mathbb{R}^{pN}$ and $C_W = \text{blkdiag}(C_{w,0}, \ldots, C_{w,N-1}) \in \mathbb{R}^{pN \times pN}$, with $\text{blkdiag}(\cdot)$ indicating block diagonal matrix construction. Due to the linearity of the system (3), $X$ is also Gaussian [25, Sec. 9.2]. Given an initial state $x_0 \in X$ and an open-loop vector $U \in U^N$, $X \sim \mathcal{N}(\mu_X, C_X)$, (22a)

$$
\mu_X = \mathcal{A}x_0 + HU + GW,
$$

(22b)

$$
C_X = GCWG^\top,
$$

(22c)

where $X = \mathcal{A}x_0 + HU + GW$. The matrices $\mathcal{A}, H, G$ account for how the dynamics (3) influence the mean and the covariance of $X$ (see [33] for the definitions).

In Algorithm 1 to solve (20) and (21), we need an efficient way to enforce the constraint $W_0(\pi, U) \geq \alpha$ for any $\pi \in X$, $U \in U^N$ and $\alpha \in [0, 1]$. Under Assumption 5, $W_0(\pi_0, U)$ is the integration of a Gaussian PDF over a polytope [4]. We consider three approaches to enforce the constraint $W_0(\pi_0, U) \geq \alpha$.

1. Convex chance constraints: An underapproximative reformulation via Boole’s inequality and Gaussian random vector properties [6]. A sufficient condition for convexity of this reformulation requires $\alpha \in [0.5, 1]$ (see [6,34]).

2. Sampling: Mixed integer-linear reformulation via scenarios drawn from $P_{\pi_0, U}$ that optionally satisfy the reach objective (stay within the target tube) [6].

3. Fourier transform: An approximative reformulation via a numerical “noisy” Monte Carlo simulations-based evaluation of $W_0(\pi, U)$. We rely on gradient-free optimization techniques [35] to optimize the resulting “noisy” optimization problem [3]. For a Gaussian $X$ (22), we use Genz’s algorithm to evaluate $W_0(\pi, U)$ via quasi-Monte Carlo simulations and Cholesky decomposition [36,37].

Table 2 compares the implementation of the constraint $W_0(\pi, U) \geq \alpha$ using these approaches. Figure 5 summarizes the conservativeness introduced at different stages of the open-loop controller-based underapproximation when chance constrained reformulation is used.

5 Numerical results

All computations were performed using MATLAB on an Intel Xeon CPU with 3.4GHz clock rate and 32 GB RAM. We used Stochastic Reachability Toolbox SReachTools [39], a MATLAB toolbox for verification and controller synthesis of stochastic linear systems to perform all the simulations. SReachTools uses MPT3 [40] for computational geometry and CVX [41] for parsing convex problems. We used Gurobi [42] as the backend solver for the chance constrained approach, and MATLAB’s patternsearch as the nonlinear solver for the Fourier transform approach.
Enforce $W_0(\cdot) \geq \alpha$

| Approximation | Implementation |
|---------------|----------------|
| Tighten the constraint $W_0(\cdot) \geq \alpha$ via Boole’s inequality | Nonlinear solver [6] or iterative linear programs [34]; Requires $\alpha \geq 0.5$ for convexity of $\{\mathcal{U} : W_0(\pi_0, \mathcal{U}) \geq \alpha\}$ |
| Approximation quality improves with increase in the number of samples $N_s$ | Mixed-integer linear program [6, 38]; Exponential compute cost w.r.t. $N_s$ [38] |
| Approximation to a desired tolerance | Nonlinear solver that can handle noisy objectives [35]; Use Genz’s algorithm [37] to evaluate $W_0(\cdot)$ |

Table 2: Enforcing $W_0(\pi_0, \mathcal{U}) \geq \alpha$ under Assumption 5.

Figure 5: Underapproximative steps taken to compute the polytopic underapproximation $K_0^\pi(\alpha, \mathcal{T}, \mathcal{D})$ of the stochastic reach set $\mathcal{L}_0^\pi(\alpha, \mathcal{T})$ (Section 4.5) via chance constraints. At a higher computation cost, the Fourier transforms-based approach can remove the last underapproximative step in some cases (Tab. 4 and Fig. 9).
Figure 6: Stochastic reachability for a double integrator: a) stochastic viable sets $L^\pi_0(\alpha, \mathcal{T})$ (contours) and their polytopic underapproximations $K^\rho_0(\alpha, \mathcal{T}, \mathcal{D})$ for $\alpha \in \{0.6, 0.9\}$; b) the true sets $(L^\pi_0(\alpha, \mathcal{T})$ and $K^\rho_0(\alpha, \mathcal{T}, \mathcal{D}))$ at $\alpha = 0.85$ and their underapproximative interpolations (Thms. 5 and 6d) using their counterparts for $\alpha \in \{0.6, 0.9\}$.

### 5.1 Integrator chain: Interpolation & scalability demonstration

Consider a chain of integrators,

$$
\begin{align*}
    x_{k+1} &= \begin{bmatrix}
        1 & N_s & \frac{1}{2}N_s^2 & \cdots & \frac{1}{(n-1)!}N_s^{n-1} \\
        0 & 1 & N_s & \cdots & \\
        \vdots & & \ddots & \ddots & \\
        0 & 0 & 0 & \cdots & N_s \\
        0 & 0 & 0 & \cdots & 1
    \end{bmatrix} x_k \\
    &\quad + \begin{bmatrix}
        \frac{1}{n!}N_s^n & \cdots & \frac{1}{2}N_s & N_s
    \end{bmatrix}^\top u_k + w_k
\end{align*}
$$

with state $x_k \in \mathbb{R}^n$, input $u_k \in \mathcal{U} = [-1, 1]$, a Gaussian disturbance $w_k \sim \mathcal{N}(0, 0.01I_2)$, sampling time $N_s = 0.1$, and time horizon $N = 5$. Here, $I_n$ refers to the $n$-dimensional identity matrix and $\bar{0}_n$ is the $n$-dimensional zero vector.

#### 5.1.1 2D system

Consider the terminal time problem with $\mathcal{T}_i = [-1, 1]^2 i \in \mathbb{N}_{[0,N-1]}$ and $\mathcal{T}_N = [-0.5, 0.5]^2$. We compare $K^\rho_0(\alpha, \mathcal{T}, \mathcal{D})$ obtained using Algorithm 1 with $|\mathcal{D}| = 32$ and the set $L^\pi_0(\alpha, \mathcal{T})$ obtained via grid-based dynamic programming [18]. We discretized the state space and the input space in steps of 0.05.

Figure 6a shows that, in general, Algorithm 1 provides a good underapproximation of the true stochastic viable set for a double integrator. The advantage of using state-feedback $\pi^*$ over an open loop controller $\rho^*$ is seen in the underapproximation “gaps” between the polytopes (Theorem 7). Figure 6b shows that the (interpolated) polytopic underapproximation obtained at $\alpha = 0.85$ using the polytopic representations of $L^\pi_0(\alpha, \mathcal{T})$ and $K^\rho_0(\alpha, \mathcal{T})$ for $\alpha \in \{0.6, 0.9\}$ (Thms. 5 and 6d) approximates the true sets well.

Table 3 provides the computation times. As expected, the grid-free nature of Algorithm 1 coupled with the convexity and compactness properties established in Sections 3 and 4 and the underapproximative guarantee (Theorem 7) provides significant speed-up for underapproximative verification and controller synthesis. The interpolation approach (Thms. 5 and 6d) took significantly lower computation time while producing a good underapproximation. We can now perform real-time verification by computing a few stochastic reach sets offline and then performing appropriate interpolations.
Table 3: Computation time (in seconds) for verification of a chain of integrators.

| Algorithm        | n = 2 (|D| = 32) | n = 40 (|D| = 6) |
|------------------|----------------|-----------------|
| Interpolate $K^0_0(\alpha, \mathcal{T}, \mathcal{D})$ | 19.36 19.28 18.94 | 6138.7 6101.2 6101.2 |
| Dynamic programming | – 0.037 – | – 0.06 – |
| Interpolate $L^0_0(\alpha, \mathcal{T})$ | 47.37 | Not possible |

Figure 7: Stochastic reach-avoid analysis for a chain of integrators ($n = 40$): a) the polytopic underapproximations of the stochastic reach-avoid sets $K^0_0(\alpha, \mathcal{T}, \mathcal{D})$ for $\alpha \in \{0.6, 0.9\}$; b) the $K^0_0(\alpha, \mathcal{T}, \mathcal{D})$ at $\alpha = 0.85$ and its tight underapproximative interpolation (Thm. 6d) using $K^0_0(\alpha, \mathcal{T}, \mathcal{D})$ for $\alpha \in \{0.6, 0.9\}$.

For illustration, we consider initial states $x_0 = [x_1 x_2 0 \ldots 0]^\top$.

### 5.1.2 40D System

To demonstrate scalability, consider the terminal time problem with $n = 40$ and $T_i = [-10,10]^{40}, i \in \mathbb{N}_{0,N-1}$ and $T_N = [-8,8]^{40}$. Due to the curse of dimensionality, we can not use dynamic programming. We compute $K^0_0(\alpha, \mathcal{T}, \mathcal{D})$ for $n = 40$ for $\alpha \in \{0.6,0.85,0.9\}$ and also demonstrate the underapproximative interpolation for this high-dimensional system. Figure 7a shows a 2D slice of $K^0_0(\alpha, \mathcal{T}, \mathcal{D})$ that verifies $x_0$ of the form $[x_1 x_2 x_3 0 \ldots 0]^\top \in \mathbb{R}^{40}$. The difference in volume between the underapproximative interpolation (Theorem 6d) and $K^0_0(\alpha, \mathcal{T}, \mathcal{D})$ at $\alpha = 0.85$ is negligible (1.124 via MPT3 [40]), as seen in Figure 7b. To the best of our knowledge, this is the largest stochastic LTI system verified to date through a stochastic reachability formulation.

### 5.2 Spacecraft rendezvous problem: Comparison with existing methods

We consider two spacecraft in the same elliptical orbit. One spacecraft, referred to as the deputy, must approach and dock with another spacecraft, referred to as the chief, while remaining in a line-of-sight cone, in which accurate sensing of the other vehicle is possible. The relative dynamics are described by the Clohessy-Wiltshire-Hill (CWH) equations [43] with additive stochastic noise to account for model uncertainties,

$$\ddot{x} - 3\omega x - 2\omega \dot{y} = m_d^{-1} F_x, \quad \ddot{y} + 2\omega \dot{x} = m_d^{-1} F_y. \quad (24)$$

The chief is located at the origin, the position of the deputy is $x, y \in \mathbb{R}$, $\omega = \sqrt{\mu/R_0^3}$ is the orbital frequency, $\mu$ is the gravitational constant, and $R_0$ is the orbital radius of the spacecraft. See [6,7] for further details.
We define the state as $z = [x, y, \dot{x}, \dot{y}] \in \mathbb{R}^4$ and input as $u = [F_x, F_y] \in U \subseteq \mathbb{R}^2$. We discretize the dynamics (24) in time to obtain the discrete-time LTI system,

$$z_{k+1} = Az_k + Bu_k + w_k$$

with $w_k \in \mathbb{R}^4$ a Gaussian i.i.d. disturbance, with $\mathbb{E}[w_k] = 0$, $\Sigma = \mathbb{E}[w_k w_k^\top] = 10^{-4} \times \text{diag}(1, 1, 5 \times 10^{-4}, 5 \times 10^{-4})$. Given $u_M \in \mathbb{R}, u_M > 0$, we define the input space as $U = [-u_M, u_M]^2$ and the target tube for a time horizon of $N = 5$,

$$T_5 = \{ z \in \mathbb{R}^4 : |z_1| \leq 0.1, -0.1 \leq z_2 \leq 0, |z_3| \leq 0.01, |z_4| \leq 0.01 \},$$

and

$$T_i = \{ z \in \mathbb{R}^4 : |z_1| \leq z_2, \max(|z_3|, |z_4|) \leq 5u_M \}, i \in \mathbb{N}[0,4].$$

We consider two verification (terminal time) problems as done in [3,6,7]:

P1) initial velocity $\dot{x} = \dot{y} = 0$ km/s and input bound $u_M = 0.1$,

P2) initial velocity $\dot{x} = \dot{y} = 0.01$ km/s and input bound $u_M = 0.01$.

We are interested in solving these stochastic reach-avoid problems, by computing $K_{\rho}^0(\alpha, \mathcal{T})$, at $\alpha = 0.8$.

We solve the terminal time problem conservatively using Algorithm 1 (via Fourier transform and chance constraint approaches) and Lagrangian methods [7]. Both of these problems are intractable via dynamic programming [1]. Exploiting the convexity and compactness results from Section 4, Algorithm 1 performs significantly faster than the grid-based implementation of chance constraints approach proposed in [6]. Figures 8 and 9 show a slice of the stochastic reach-avoid underapproximations for both the verification problems. Computational times are summarized in Table 4.

The Lagrangian method [7] utilizes computational geometry to underapproximate the stochastic reach-avoid set and searches in the space of closed-loop controllers. However, it relies on the vertex-facet enumeration problem, which fails for large time horizons, small target sets, or small safe sets. While this approach fails to compute a set for P2, it is slightly faster than Algorithm 1 (see Table 4) for P1. The

| Underapproximative method | Algorithm 1 for $K_{\rho}^0(\alpha, \mathcal{T})$ | Lagrangian [7] |
|---------------------------|------------------------------------------|-----------------|
| Figure 8 ($\dot{x} = \dot{y} = 0, u_M = 0.1$) | 16.87 ($|\mathcal{D}| = 32$) | 940.45 ($|\mathcal{D}| = 8$) | 14.5 |
| Figure 9 ($\dot{x} = \dot{y} = u_M = 0.01$) | 18.24 ($|\mathcal{D}| = 32$) | 2029.51 ($|\mathcal{D}| = 8$) | – |

Table 4: Computation times of various methods in seconds. Dynamic programming [18] is not possible for 4D systems.
Figure 9: Underapproximative verification and open-loop controller synthesis for spacecraft rendezvous problem for non-zero initial velocity. Monte Carlo simulations (10^5 scenarios with five randomly chosen trajectories displayed) show a simulated reach probability of 0.81; the chance constraint estimate was 0.8.

associated set subsumes the polytopes computed using Algorithm 1 since Lagrangian method searches for a closed-loop controller in a conservative manner.

Within Algorithm 1, the implementation using chance constraints (via risk allocation [6,34]) outperforms the implementation via Fourier transforms (Genz’s algorithm and MATLAB’s patternsearch [4,35,37]) in terms of computational time. The computational efficiency of chance constraint approach results from its use of a series of linear programs [34]. The Fourier transform approach evaluates $W_0(\cdot)$ using Genz’s algorithm (quasi-Monte Carlo simulation) and solves the problems (20) and (21) using MATLAB’s patternsearch and bisection [3]. The Fourier transform approach does not have additional conservativeness (due to Boole’s inequality), as compared to chance constraint approach (see Figure 9). However, due to the noisy nature of the optimization problem, the line search in the Fourier transform approach may terminate prematurely (see Figure 8).

5.3 Dubin’s car with a known turning rate sequence: Demonstration on LTV systems and target tube

We consider the problem of driving a Dubin’s car under a known turning rate sequence while staying within a target tube. The linear time-varying dynamics with additive disturbance describing the position of the car is given by

$$x_{k+1} = x_k + \begin{bmatrix} T_s \cos(\theta_0 + T_s \sum_{i=0}^{k-1} \omega_i) \\ T_s \sin(\theta_0 + T_s \sum_{i=0}^{k-1} \omega_i) \end{bmatrix} u_k + \eta_k$$

with $x_k \in \mathbb{R}^2$ as the two-dimensional position of the car, $u_k \in [0, u_{\text{max}}]$ as the heading velocity, sampling time $T_s$, known initial heading $\theta_0$, time horizon $N$, known sequence of turning rates $\{\omega_k\}_{k=0}^{N-1}$, and a Gaussian random process $\eta_k \sim \mathcal{N}(\mu_\eta, \Sigma_\eta)$. The dynamics (26) are obtained using the observation that when the turning rate sequence and the initial heading are known, one can a priori construct the resulting sequence of heading angles. For a fixed heading velocity, $u_k = \nu$, $\forall k \in \mathbb{N}_{[0,N]}$, for some $\nu \in \mathbb{R}$, let $\{c_k\}_{k=0}^{N-1}$ be the resulting nominal trajectory of (26).

We choose the parameters of the problem as $N = 50$, $T_s = 0.1$, $\omega_k = 0.5$, $\forall k \in \mathbb{N}_{[0,N-1]}$, $\theta_0 = \frac{\pi}{4}$, $\nu = 10$, $u_{\text{max}} = \frac{2\nu}{T_s}$, $\mu_\eta = 0$, and $\Sigma_\eta = 10^{-3}I_2$. We are interested in the 0.8-level stochastic reach set of the target tube $T_k = \text{Box}(c_k, 0.5 \exp \left(\frac{-k}{N_c}\right))$, $\forall k \in \mathbb{N}_{[0,N]}$ where $N_c = 100$ is the decay time constant.

Algorithm 1 (chance constrained approach) solves this problem in 137.43 seconds for $|D| = 16$. In contrast to Section 5.2, the conservativeness introduced by Boole’s inequality is more severe — the simulated maximal reach probability is 0.15 above the chance constrained estimate. Due to the size of
the state space involved, dynamic programming is not feasible. One may use time and state-dependent gridding to compute an approximate solution.

6 Conclusions and future work

In this paper, we have characterized the properties of the stochastic reachability problem of a target tube. We have analyzed four different assumptions that guarantee existence and closed, compact, and convex stochastic reach set. Further, using the established convexity properties, we have demonstrated how an underapproximation of a desired stochastic reach set may be obtained from given reach sets via interpolation. Finally, we propose a scalable, grid-free, and anytime algorithm that provides an open-loop controller-based polytopic underapproximation of the stochastic reach set.

In future, we plan to investigate the exact relation between Assumptions 1 and 2, relax the linearity requirement in Assumption 3 for convexity, and extend the open-loop controller-based underapproximation to linear-feedback controllers.

A Omitted proofs

A.1 Theorem 1

(By induction) Since $T_{N-1}, T_N$ are Borel sets and indicator functions are bounded, $1_{T_{N-1}}(\cdot)$ and $1_{T_N}(\cdot)$ are bounded and Borel-measurable. The Borel-measurability and boundedness of $V^*_{N}(\cdot)$ follows from (9a).

Consider the base case $k = N-1$. Since $V^*_{N}(\cdot)$ is Borel-measurable (by above) and bounded (by (10)), $\int_{\mathcal{X}} V^*_{N}(\tilde{\pi})Q_{N}(d\tilde{\pi}|\mathcal{X}, \pi)$ is continuous over $\mathcal{U}$ for each $\bar{x} \in \mathcal{X}$ by Definition [8]. Since continuity implies upper semi-continuity [27, Lem. 7.13 (b)] and $\mathcal{U}$ is compact, an optimal Borel-measurable input map $\mu_{N-1}^*(\cdot)$ exists and $\int_{\mathcal{X}} V^*_{N}(\tilde{\pi})Q_{N}(d\tilde{\pi}|\mathcal{X}, \mu_{N-1}^*(\pi))$ is Borel-measurable over $\mathcal{X}$ by [28, Thm. 2]. Finally, $V^*_{N-1}(\cdot)$ is Borel-measurable since the product operator preserves Borel-measurability [23, Cor. 18.5.7].

For the case $k = t$ with $t \in \mathbb{N}_{[0,N-2]}$, assume for induction that $V^*_{t+1}(\cdot)$ is Borel-measurable. By the same arguments as above, a Borel-measurable $\mu_t^*(\cdot)$ exists and $V^*_{t}(\cdot)$ is Borel-measurable, completing the proof.

A.2 Proposition 1

Proof of a): Since continuous stochastic kernels are input-continuous, we have for every $k \in \mathbb{N}_{[0,N]}$, $V^*_{k}(\cdot)$ is Borel-measurable by Theorem 1 and bounded by (10). By Definition [8], $\int_{\mathcal{X}} V^*_{k+1}(\tilde{\pi})Q_{k}(d\tilde{\pi}|\mathcal{X}, \pi)$ is continuous over $\mathcal{X} \times \mathcal{U}$ for every $k \in \mathbb{N}_{[0,N-1]}$. By (10) and [23, Prop. ], $\int_{\mathcal{X}} V^*_{k+1}(\tilde{\pi})Q_{k}(d\tilde{\pi}|\mathcal{X}, \pi)$ is bounded and nonnegative. By [27, Prop. 7.32], we know that $\int_{\mathcal{X}} V^*_{k+1}(\tilde{\pi})Q_{k}(d\tilde{\pi}|\mathcal{X}, \mu_k^*(\pi))$ is l.s.c. and u.s.c. over $\mathcal{X}$, implying its continuity.
Proof of b): For every \(k \in \mathbb{N}_{[0,N]}\), every \(\varpi \in \text{int}(T_k)\), and any sequence \(\varpi_i \xrightarrow{i \to \infty} \varpi\) where \(\varpi_i \in \mathcal{X}\), there exists \(i_0 \in \mathbb{N}\) such that \(\forall i \geq i_0, \varpi_i \in T_k\). This implies \(1_{T_k}(\varpi_i) = 1_{T_k}(\varpi) = 1, \forall i \geq i_0\), implying the continuity of \(1_{T_k}(\cdot)\), \(\forall k \in \mathbb{N}_{[0,N]}\) over int\((T_k)\). Since multiplication of continuous functions are continuous \([23, \text{Cor. 13.2.3a}]\), \(V_k^*(\varpi)\) is continuous over int\((T_k)\) by \([9b]\) and Proposition \([10]\). By construction, \(V_k^*(\varpi) = 0\) for every \(\varpi \in \text{int}(\mathcal{X} \setminus T_k)\), implying \(V_k^*(\varpi)\) is trivially continuous over int\((\mathcal{X} \setminus T_k)\). Hence, \(V_k^*(\cdot)\) is piecewise-continuous over \(\mathcal{X}\), with discontinuities if any restricted to the relative boundary of \(T_k\). \(\square\)

A.3 Proposition \([2]\)

By \([5]\), we can rewrite \(\int_{\mathcal{X}} h(\varpi)Q_k(d\varpi|\varpi, \bar{v}) = \int_{\mathcal{X}} h(f_k(\varpi, \bar{v}, \bar{w}))\psi_{w,k}(\bar{w})d\bar{w}\). Note that \(h(f_k(\varpi, \bar{v}, \bar{w}))\) is u.s.c. over \(\mathcal{X} \times \mathcal{U} \times \mathcal{W}\) by Assumption \([24]\) and the fact that u.s.c. function of a continuous function is u.s.c. \([44, \text{Ex. 1.4}]\), and the assumption that \(h(\cdot)\) is u.s.c.. Additionally, \(h(f_k(\varpi, \bar{v}, \bar{w}))\) is bounded and non-negative since \(h(\cdot)\) is bounded and non-negative. If \(L \in \mathbb{R}\) is an upper bound of \(h(\cdot)\), then \(L - h(f_k(\varpi, \bar{v}, \bar{w}))\) is non-negative and l.s.c over \(\mathcal{X} \times \mathcal{U}\) for every \(\bar{w} \in \mathcal{W}\). By Borel-measurability of \(h\), \(h(f_k(\varpi, \bar{v}, \bar{w}))\) is a non-negative random vector defined on \((h(\mathcal{X}), \mathcal{B}(h(\mathcal{X})))\). From Fatou’s lemma \([26, \text{Sec. 6.2, Thm. 2.1}]\) and the fact that \(L - h(f_k(\varpi, \bar{v}, \bar{w}))\) is l.s.c, Borel-measurable, and non-negative, we have

\[
\liminf_i \int_{\mathcal{X}} (L - h(f_k(\varpi, \bar{v}, \bar{w})))\psi_{w,k}(\bar{w})d\bar{w} \\
\geq \int_{\mathcal{X}} \liminf_i (L - h(f_k(\varpi, \bar{v}, \bar{w})))\psi_{w,k}(\bar{w})d\bar{w}, \\
\geq \int_{\mathcal{X}} (L - h(f_k(\varpi, \bar{v}, \bar{w})))\psi_{w,k}(\bar{w})d\bar{w}.
\]

By the linearity properties of the Lebesgue integral on \([27, 23, \text{Prop. 19.2.6c}]\),

\[
\limsup_i \int_{\mathcal{X}} h(f_k(\varpi, \bar{v}, \bar{w}))\psi_{w,k}(\bar{w})d\bar{w} \\
\leq \int_{\mathcal{X}} h(f_k(\varpi, \bar{v}, \bar{w}))\psi_{w,k}(\bar{w})d\bar{w},
\]

which completes the proof. \(\square\)

A.4 Proposition \([4]\)

Similarly to Proposition \([2]\), we show the log-concavity of \(\int_{\mathcal{X}} h(\bar{v})Q_k(d\bar{v}|\varpi, \bar{v}) = \int_{\mathcal{X}} h(A_k\varpi + B_k\bar{u} + \bar{w})\psi_{w,k}(\bar{w})d\bar{w}\) \(\forall k \in \mathbb{N}_{[0,N-1]}\) over \(\mathcal{X} \times \mathcal{U}\). Note that compositions of log-concave functions with affine functions preserve log-concavity \([22, \text{Sec. 3.2.2}]\). Hence, \(h(A_k\varpi + B_k\bar{u} + \bar{w})\) is log-concave over \(\mathcal{X} \times \mathcal{U} \times \mathcal{W}\). Since multiplication and partial integration preserves log-concavity \([22, \text{Sec. 3.5.2}]\), we conclude that \(\int_{\mathcal{X}} h(\bar{v})Q_k(d\bar{v}|\varpi, \bar{v})\), \(\forall k \in \mathbb{N}_{[0,N-1]}\) is log-concave over \(\mathcal{X} \times \mathcal{U}\). \(\square\)

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