Objective material barriers to the transport of momentum and vorticity

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We derive a theory for material surfaces that maximally inhibit the diffusive transport of a dynamically active vector field, such as the linear momentum, angular momentum or vorticity, in three-dimensional unsteady flows. Along these evolving material surfaces (Lagrangian active barriers), the time-averaged change of diffusive flux generated by the active vector field vanishes. Such active barriers turn out to evolve from streamsurfaces of an associated objective, steady and incompressible vector field: the time-averaged pullback of the viscous stress terms appearing in the evolution equation of the active vector field. Instantaneous limits of these active Lagrangian barriers represent objective Eulerian active barriers in the flow. We show that Lagrangian and Eulerian active barriers coincide with level curves of vorticity-related quantities in all planar Navier–Stokes flows. Such active barriers also coincide with classic invariant manifolds known from chaotic advection studies in all three-dimensional, directionally steady Beltrami flows. In more general flows, however, active barriers are given by previously undocumented surfaces which we can extract numerically. We illustrate these results by visualizing active material and instantaneous barriers in several unsteady flows, including three-dimensional wall-bounded turbulence simulations.

\textbf{Key words:} Transport, mixing, coherent structures, turbulence, invariant manifolds, flow visualization

\section{1. Introduction}

The geometry of fluid transport is often the simplest to capture through barriers that inhibit the redistribution of the transported quantity. Accordingly, transport barriers are routinely invoked in discussions of transport in classical fluid dynamics (Ottino 1989), geophysics (Weiss & Provenzale 1989), reactive flows (Rosner 2000) and plasma fusion (Dinklage 2005).

Despite their broadly recognized significance, however, transport barriers have remained loosely defined and little understood. Perhaps the only generally agreed definition is the one for purely advective transport in two-dimensional, incompressible, time-periodic flows. For such flows, MacKay, Meiss & Percival (1984) define transport barriers as invariant curves of the corresponding stroboscopic map (or Poincaré map) for fluid

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particle motions. This definition extends to three-dimensional steady flows, identifying advective transport barriers as two-dimensional surfaces whose intersection with a section transverse to the flow is an invariant curve for the Poincaré map defined for that section (Ottino 1989, MacKay 1994). In many three-dimensional flows, however, trajectories may rarely or never return to the physically relevant Poincaré sections, such as the cross-stream sections of pipe flows. In that case, all streamsurfaces are viable candidates for advective transport barriers, but no systematic procedure has been developed to identify those streamsurfaces that reveal the overall transport geometry most efficiently.

More generally, any material surface blocks material transport completely, and hence advective transport barriers are ill-defined in the absence of additional requirements. A typical additional requirement for barriers has been their material coherence over extended periods, leading to the notion of Lagrangian coherent structures (Haller 2015). Without a universally accepted notion of material coherence, however, different coherence definitions continue to coexist and highlight different material surfaces as advective transport barriers (Hadjighasem et al. 2017).

As a more recent development, Haller, Karrasch & Kogelbauer (2018) formalize the definition of transport barriers for passively advected diffusive scalars. They argue that for small enough diffusivities, advective transport through any non-material surface is always larger than diffusive transport. This prompts them to seek transport barriers in incompressible flows as material surfaces that inhibit the diffusive transport of a weakly diffusive scalar more than neighboring material surfaces do. They solve this variational problem for two-dimensional flows, obtaining explicit differential equations for material barriers to diffusive transport. A more recent extension of these results by the same authors (Haller Karrasch & Kogelbauer 2019) covers passively advected scalar fields with dynamically constrained initial conditions, such as the scalar vorticity in two-dimensional flows. Katsanoulis et al. (2019) use these results to define and locate two-dimensional vortex boundaries as outermost closed barriers to the diffusive transport of vorticity. All these variational approaches to transport barriers are objective (observer-independent), but do not cover barriers to the transport of active vector fields, such as linear and angular momentum, or the vorticity in three dimensions. In addition, some observed barriers to the redistribution of vorticity are only captured by this approach in the limit of infinitely long extraction times (cf. Appendix A for an example).

More broadly speaking, a general approach to barriers to the transport of dynamically active quantities in fluid flows has been lacking. Perhaps the only notable exception is the work of Meyers & Meneveau (2013), who propose a thoughtful extension of the concept of a streamtube (viewed as barrier to mass transport in steady flows) to the transport of active scalars in unsteady flows. Assuming a statistically stationary, incompressible Navier–Stokes flow with a known mean velocity field, Meyers and Meneveau consider the evolution equation for the scalar field of interest and formally identify all terms that can be written as the divergence of some vector field. They then define the sum of all such vector fields to be the flux vector associated with the original scalar field and point out that any tube tangent to this flux vector field has zero flux with respect to that vector field.

While highly insightful, this approach has several heuristic elements. The construct depends on the frame of reference, as well as on the choice of an initial direction in which scalar transport is considered. The flow data is assumed statistically stationary, and a well-defined mean velocity field is assumed to be available. The flux vector introduced in this fashion is nonunique: any divergence-free vector field could be added to it. In addition, the flux vector differs from the classic momentum and energy flux that it purports to represent. All these features of the approach prevent the detection of notable
observed barriers to vorticity redistribution already in simple two-dimensional flows, as our motivating example shows in Appendix A.

In the present work, we seek to fill the gaps in previous approaches by identifying material barriers to the viscosity-driven redistribution of dynamically active vector fields in general unsteady flows. The material nature of these barriers will ensure, on the one hand, observability in flow visualization by particles or dye, and, on the other hand, the independence of the barriers from observers. A major challenge to overcome is the dependence of generic active vector fields, such as momentum, vorticity and their fluxes, on the observer and on the choice of a reference direction. Another challenge is to define a flux notion that contains a well-identifiable objective component.

We address these challenges by adopting the classic, mathematical notion of vector-field flux, which has been broadly used for the vorticity but not for the momentum in fluid mechanics. This flux is still frame-dependent, but its change along trajectories has an objective component arising from Cauchy stresses. Accordingly, we define transport barriers as material surfaces along which the temporal average of this viscous flux-change vanishes. Physically, these barriers are material surfaces to which viscous forces are tangent on average. As a consequence for two-dimensional flows, active material barriers to linear momentum transport turn out to be level curves of the time-averaged Lagrangian vorticity, whereas active material barriers to momentum transport turn out to be level curves of the vorticity-increment function along trajectories. We also obtain that for all three-dimensional, directionally steady Bernoulli solutions of the Navier–Stokes equation, active barriers to momentum and vorticity transport coincide with classically documented advective transport barriers, i.e., material invariant manifolds of the velocity field.

In more general flows, our results reveal previously undetected objective barriers to the viscosity-driven redistribution of active vector fields. As a consequence, our approach also yields a unified, objective theory of vortex boundaries as barriers to diffusive vorticity transport both in two-dimensional and in three-dimensional flows. We illustrate all these results on canonical flow examples, a two-dimensional turbulence simulation and a three-dimensional wall-bounded turbulence simulation.

2. Set-up

Consider a smooth, three-dimensional velocity field \( \mathbf{v}(\mathbf{x}, t) \), known at spatial locations \( \mathbf{x} \in U \subseteq \mathbb{R}^3 \) in a bounded set \( U \) and at times \( t \in [t_1, t_2] \). Trajectories generated by \( \mathbf{v} \) are solutions of the differential equation \( \dot{\mathbf{x}} = \mathbf{v}(\mathbf{x}, t) \). We denote the time-\( t \) position of a trajectory starting from \( \mathbf{x}_0 \) at time \( t_0 \) by \( \mathbf{x}(t; t_0, \mathbf{x}_0) \). The flow map induced by \( \mathbf{v} \) is defined as the mapping \( F^t_{t_0}: \mathbf{x}_0 \mapsto \mathbf{x}(t; t_0, \mathbf{x}_0) \).

A material surface \( \mathcal{M}(t) \subset U \) is a time-dependent two-dimensional manifold transported by the flow map from its initial position \( \mathcal{M}_0 := \mathcal{M}(t_0) \) as

\[
\mathcal{M}(t) = F^t_{t_0} [\mathcal{M}(t_0)] .
\]  

Let \( \mathbf{u}(\mathbf{x}, t) \) be another smooth vector field defined on the same spatiotemporal domain \( U \times [t_0, t_1] \). We will primarily be interested in \( \mathbf{u} \) fields that are active vector fields, i.e., functions of \( \mathbf{v} \). The simplest physical examples of such active velocity fields are the momentum \( \mathbf{u} = \rho \mathbf{v} \), with \( \rho(\mathbf{x}, t) \) denoting the density of the medium whose velocity field is \( \mathbf{v}(\mathbf{x}, t) \), and the vorticity \( \mathbf{u} = \mathbf{\omega} = \nabla \times \mathbf{v} \) of \( \mathbf{v} \). Both of these examples of \( \mathbf{u} \) are frame-dependent (non-objective) vector fields, as they do not transform properly under general frame changes of the form

\[
\mathbf{x} = Q(t)\mathbf{y} + b(t), \quad QQ^T = I, \quad Q(t) \in SO(3), \quad b(t) \in \mathbb{R}^3 ,
\]  

(2.2)
where both \( \mathbf{Q}(t) \) and \( \mathbf{b}(t) \) are smooth in time. Indeed, evaluating the definition of these vectors in the \( \mathbf{y} \) frame gives transformed vector fields \( \mathbf{u}(\mathbf{y},t) \) for which

\[
\mathbf{u}(\mathbf{y},t) \neq \mathbf{Q}^T(t)\mathbf{u}(x,t).
\] (2.3)

It is, therefore, a challenge to describe the transport of such a \( \mathbf{u} \) through a material surface in an intrinsic, observer-independent fashion.

Assuming Fickian-type diffusion for \( \mathbf{u} \), its evolution will generally be governed by a partial differential equation of the form

\[
\dot{\mathbf{u}}(x,t) = \mathbf{h}(x,t,\mathbf{u},\mathbf{v}) + \nu \mathbf{g}(x,t,\mathbf{v}), \quad \partial_t \mathbf{h} = 0,
\] (2.4)

where \( \dot{\mathbf{u}} = \partial_t \mathbf{u} + (\nabla \mathbf{u}) \mathbf{u} \) denotes the material derivative of \( \mathbf{u} \). The constant \( \nu > 0 \) is an appropriate parameter characterizing the overall magnitude of Cauchy-stress-related forces relevant for \( \mathbf{u} \). The specific form of \( \mathbf{g}(x,t,\mathbf{v}) \) will depend on the constitutive law relevant for the fluid. The term \( \mathbf{h} \) will contain the remaining body forces, generally including inertial terms impacting the evolution of \( \mathbf{u} \) in the given frame. These terms are assumed to have no explicit dependence on \( \nu \). In contrast, \( \mathbf{g} \) is a generalized diffusive force governing the diffusive component of the evolution of the active vector field \( \mathbf{u} \). For instance, when \( \mathbf{u} \) is the linear momentum of the flow, then \( \mathbf{g} \) is the resultant force arising from Cauchy stresses, and when \( \mathbf{u} \) is the vorticity, then \( \mathbf{g} \) is the curl of the same resultant force (cf. section 6). Both \( \mathbf{h} \) and \( \mathbf{g} \) may contain differential operators and hence are generally unbounded functions of their arguments even on bounded spatial domains.

We finally assume that \( \mathbf{g} \) is an objective vector field, i.e., under the observer change, we have

\[
\mathbf{g}(\mathbf{y},t,\dot{\mathbf{u}},\dot{\mathbf{v}}) = \mathbf{Q}^T(t)\mathbf{g}(x,t,\mathbf{u},\mathbf{v}).
\] (2.5)

With its assumed dependence on inertial effects, the vector field \( \mathbf{h} \) will not be objective.

### 3. Active transport through material surfaces

By classic vector analysis, the instantaneous flux of a vector field \( \mathbf{u} \) through a material surface \( \mathcal{M}(t) \) is defined as

\[
\Phi(\mathcal{M}(t)) = \int_{\mathcal{M}(t)} \mathbf{u}(x,t) \cdot \mathbf{n}(x,t)dA,
\] (3.1)

where \( \mathbf{n}(x,t) \) is a smoothly oriented unit normal vector field for \( \mathcal{M}(t) \) (see, e.g., Kirkwood 2018). The best known example of this flux in fluid mechanics is the vorticity flux with \( \mathbf{u} = \dot{\omega} = \nabla \times \mathbf{v} \), but a similar algebraic flux can be defined for other active vector fields, such as the linear momentum \( \mathbf{u} = \rho \mathbf{v} \) or the angular momentum \( \mathbf{u} = \mathbf{r} \times \rho \mathbf{v} \).

Because of the property (2.3) and by the transformation formula

\[
\mathbf{n}(x,t) = \mathbf{Q}(t)\hat{\mathbf{n}}(\mathbf{y},t)
\] (3.2)

for unit normals, the flux \( \Phi \) under the frame-change (2.2) becomes

\[
\tilde{\Phi}(\tilde{\mathcal{M}}(t)) = \int_{\tilde{\mathcal{M}}(t)} \tilde{\mathbf{u}} \cdot \tilde{\mathbf{n}}dA = \int_{\mathcal{M}(t)} \mathbf{u} \cdot \mathbf{Q}^T \mathbf{n}dA = \int_{\mathcal{M}(t)} \mathbf{Q}\tilde{\mathbf{u}} \cdot \mathbf{n}dA \neq \Phi(\mathcal{M}(t)).
\]

Therefore, the flux \( \Phi(\mathcal{M}(t)) \) as a scalar function of \( \mathcal{M}(t) \), is not objective and hence its specific value does not carry any intrinsic meaning in unsteady fluid flows, which have no |

| specifically, \( \mathbf{v} = \mathbf{Q}^T(\mathbf{v}(x,t) - \dot{\mathbf{Q}}y - \dot{\mathbf{b}}) \) and \( \dot{\omega} = \mathbf{Q}^T(\mathbf{\omega} - \dot{\mathbf{q}}) \), where the vorticity of the frame change, \( \dot{\mathbf{q}} \), is defined by the requirement that \( \frac{1}{2}\dot{\mathbf{q}} \times \mathbf{e} = \dot{\mathbf{Q}}\mathbf{Q}^T\mathbf{e} \) for all vectors \( \mathbf{e} \in \mathbb{R}^3 \).
distinguished frame of reference (Lugt 1979). Specifically, the classic notion of a vortex tube is not objective: observers rotating relative to each other will pronounce different surfaces to be vortex tubes. We also note that if \( \mathbf{u} \) is divergence-free, \( U \) is simply connected and \( \mathcal{M}(t) \) is a closed surface, such as a torus or a sphere, then \( \Phi(\mathcal{M}(t)) \) is always zero by the divergence theorem. This cancellation prevents any systematic assessment of the permeability of closed surfaces under divergence-free active vector fields, such as vorticity or linear momentum, in incompressible flows.

All this suggests that we need to work more on \( \Phi(\mathcal{M}(t)) \) to turn it into a viable (observer-independent) measure of transport of \( \mathbf{u}(x, t) \) through a material surface. As the value of the flux \( \Phi \) is frame-dependent, we will focus instead on the change of \( \Phi \) along trajectories normalized by the elapsed time, given by \([\Phi(\mathcal{M}(t_1)) - \Phi(\mathcal{M}(t_0))] / (t_1 - t_0)\). Moreover, we will isolate the explicitly diffusion-dependent part of this time-normalized flux by defining the transport functional

\[
\psi_{t_0}^{t_1}(\mathcal{M}_0) = \partial_{\nu} \frac{\Phi(\mathcal{M}(t_1)) - \Phi(\mathcal{M}(t_0))}{t_1 - t_0} \bigg|_{\nu=0}
\]

(3.3)

associated with the vector field \( \mathbf{u} \). The argument of this functional is the initial position \( \mathcal{M}_0 \) of a material surface \( \mathcal{M}(t) \) through which \( \psi_{t_0}^{t_1} \) measures the Cauchy-stress-related component of the total flux change, normalized by the length of the time interval \([t_0, t_1]\). This quantity can be viewed purely as a function of \( \mathcal{M}_0 = \mathcal{M}(t_0) \), because later positions of the material surface \( \mathcal{M}(t) \) are fully determined by \( \mathcal{M}_0 \) through the relationship (2.1).

An equivalent expression for \( \psi_{t_0}^{t_1} \) is given by (cf. (B3) of Appendix B)

\[
\psi_{t_0}^{t_1} = \frac{1}{t_1 - t_0} \int_{t_0}^{t_1} \int_{\mathcal{M}(t)} \mathbf{g}(x, t, \mathbf{u}, \mathbf{v}) \cdot \mathbf{n}(x, t) dA dt,
\]

which shows \( \psi_{t_0}^{t_1} \) to be the time-averaged flux of the generalized objective force term \( \mathbf{g}(x, t, \mathbf{u}, \mathbf{v}) \) over the evolving material surface \( \mathcal{M}(t) \). This flux is small if \( \mathbf{g}(x, t, \mathbf{u}, \mathbf{v}) \) is, on average, nearly tangent to \( \mathcal{M}(t) \).

To obtain a more explicit formula for the transport functional yet keep our notation simple, we let \( \mathbf{w}(x) \) denote the temporal average of a vector field \( \mathbf{w}(x, t) \) over \([t_0, t_1]\). We will also denote by \((\mathbf{F}_{t_0}^t)^* \mathbf{w}\) the classic pull-back of a vector field \( \mathbf{w}(x, t) \) under the flow map \( \mathbf{F}_{t_0}^t \) to the initial configuration at \( t_0 \):

\[
(\mathbf{F}_{t_0}^t)^* \mathbf{w}(x_0, t) = \left[ \nabla \mathbf{F}_{t_0}^t(x_0) \right]^{-1} \mathbf{w}(\mathbf{F}_{t_0}^t(x_0), t).
\]

(3.4)

With this notation, we have the following result:

**Proposition 1.** Under the assumptions (2.4)-(2.5) on the active vector field \( \mathbf{u} \), the transport functional \( \psi_{t_0}^{t_1} \) of \( \mathbf{u} \) can be calculated as

\[
\psi_{t_0}^{t_1}(\mathcal{M}_0) = \int_{\mathcal{M}_0} \mathbf{b}_{t_0}^{t_1}(x_0) \cdot \mathbf{n}_0 dA_0,
\]

(3.5)

with the objective Lagrangian vector field

\[
\mathbf{b}_{t_0}^{t_1}(x_0) := \det \nabla \mathbf{F}_{t_0}^t(\mathbf{F}_{t_0}^t)^* \mathbf{g}.
\]

As a consequence, the transport functional is objective.

**Proof.** See Appendix B.

\( \ddagger \) i.e., a cylindrical surface with pointwise zero vorticity flux \( \omega(x, t) \cdot \mathbf{n}(x, t) \), which implies \( \Phi = 0 \) on the surface for the choice \( \mathbf{u} = \omega \).
Figure 1. Qualitative meaning of the time-$t_0$ position of a material barrier to the transport of a dynamically active vector field $u(x,t)$, whose associated barrier vector field is $b_{t_0}^i(x_0) = \det \nabla F_{t_0}^i (F_{t_0}^1) g$. Here the objective forcing term $g(x,t,u,v)$ arises from the constitutive law associated with the fluid, as seen in the evolution equation (2.4) for $u(x,t)$.

Proposition 1 shows that $\psi_{t_0}^{t_1}(M_0)$ can be calculated as the (algebraic) flux of the Lagrangian vector field $b_{t_0}^i(x_0)$ through the initial surface $M_0$. Following MacKay (1994), we also define the geometric flux of this vector field through $M_0$ as

$$\Psi_{t_0}^{t_1}(M_0) = \int_{M_0} |b_{t_0}^i(x_0) \cdot n_0| \, dA_0.$$  \hfill (3.6)

The geometric flux cannot vanish due to global cancellations, and hence is a better measure of the overall permeability (non-invariance) of a surface under a vector field than the algebraic flux.

4. Material barriers to active transport

Next we seek initial positions of transport barriers as material surfaces $M_0$ that render the algebraic flux, $\psi_{t_0}^{t_1}(M_0)$, zero because the integrand in (3.4) vanishes pointwise along them. In other words, we look for barriers as material surfaces that are global minimizers of the geometric flux $\Psi_{t_0}^{t_1}$.

We note from (3.4) that $\Psi_{t_0}^{t_1}(M_0)$ can only vanish if $M_0$ is everywhere tangent to $b_{t_0}^i(x_0)$. Physically, therefore, the barrier surfaces we seek are material surfaces to which the generalized, objective force term $g(x,t,u,v)$ is tangent on average, and hence causes only tangential shrinking or stretching to those surfaces on average (cf. Fig. 1). Consequently, if we denote by $s \in \mathbb{R}$ a parametrization of the streamlines $x_0(s)$ of the vector field $b_{t_0}^i(x_0)$, then any two-dimensional streamsurface of the three-dimensional autonomous differential equation,

$$x_0' = b_{t_0}^i(x_0)$$  \hfill (4.1)

is a perfect barrier with respect to the averaged flux $\psi_{t_0}^{t_1}$, i.e., experiences globally minimal geometric flux $\Psi_{t_0}^{t_1}(M_0)$. For this reason, we refer to (4.1) as the active barrier equation, and to $b_{t_0}^i(x_0)$ as the corresponding barrier vector field. By the objectivity of the vector field $b_{t_0}^i(x_0)$, the barrier equation (4.1) is objective. Indeed, after a frame change of the form (2.2), we obtain the transformed barrier equation $\tilde{y}_0' = \tilde{b}_{t_0}^i(y_0)$.

Any smooth curve of initial conditions for (4.1), however, generates a two-dimensional streamsurface of trajectories for (2.2). Out of these infinitely many barrier candidates, we would like to find barrier surfaces with an observable impact on the overall transport of the active vector $u$. To this end, we define active transport barriers as follows:

Definition 1. An active transport barrier to the diffusive transport of the vector field $u$ over the time interval $[t_0,t_1]$ is a material surface $B(t) \subset U$ whose initial position $B_0 = \ldots$
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**Figure 2.** Possible geometries of material barriers to active transport in three-dimensional, incompressible flows. Curves with arrows indicate qualitative sketches of trajectories of the barrier equation (4.1) with $t_1 \geq t_0$, for which these surfaces are structurally stable, two-dimensional invariant manifolds.

The local impact of an active transport barrier can be quantified by the increase in the geometric flux $\Psi_{t_0}^{t_1}(\mathcal{M}_0)$ from zero after small, localized perturbations of a barrier $\mathcal{B}_0$. Probing this increase in transport under local deformation of $\mathcal{B}_0$ is equivalent to assessing the increase in transport under local perturbations to its surface normal. As the surface normal remains of unit norm, an $\epsilon$ perturbation of the integrand of $\Psi_{t_0}^{t_1}$ yields

$$|b_{t_0}(x_0) \cdot n(x_0)| = |b_{t_0}(x_0) \cdot (n_0(x_0) + n_1(x_0))| = |c b_{t_0}(x_0) \cdot n_1(x_0)|.$$ 

Therefore, normalizing the maximal transport growth with the length of the perturbation gives the active barrier sensitivity (ABS) field

$$\text{ABS}_{t_0}^{t_1}(x_0) = \lim_{\epsilon \to 0} \max_{|n_1|=1} \frac{|b_{t_0}(x_0) \cdot n(x_0)|}{|c n_1(x_0)|} = |b_{t_0}(x_0)|. \quad (4.2)$$

This scalar diagnostic field can also be used independently, without locating invariant manifolds of (4.1), to identify regions in $U$ that are candidates for containing impactful barriers.
5. Instantaneous barriers to transport

Our treatment of active barriers has so far been fundamentally Lagrangian, targeting barriers as material surfaces that globally minimize the transport functional $\psi_{t_0}^{t_1}$. Taking the $t_1 \to t_0 \equiv t$ limit in our arguments, however, is also possible, leading to the identification of material surfaces that act instantaneously as barriers to active transport over infinitesimally short time intervals. Accordingly, these instantaneous active-flux minimizing surfaces, $B(t)$, are structurally stable, two-dimensional invariant manifolds of the differential equation

$$\dot{x} = b_t(x) = g(x, t, u, v),$$

(5.1)

the instantaneous limit of the barrier equation (4.1). Such barriers are Eulerian, i.e., can be calculated from instantaneous velocity data without any Lagrangian advection, yet inherit the objectivity of Lagrangian barriers. These instantaneous barriers, therefore, extend to notion of objective Eulerian coherent structures (Serra & Haller 2016) and instantaneous diffusion barriers (Haller, Karrasch & Kogelbauer 2018, 2019) to the context of active transport of vector fields.

In the following section, we derive specific barrier equations for different active vector fields. In each case, the instantaneous limits of these equations can directly be obtained by replacing $F_{t_0}^{t_1}$ with the identity map, and by omitting the averaging operation in time.

6. Active barrier equations for momentum and vorticity

We consider general (compressible and possibly non-Newtonian) flows, for which the momentum equation takes the form

$$\rho \dot{v} = -\nabla p + f + \nu \nabla \cdot \sigma,$$

(6.1)

where $\rho(x, t)$ and $p(x, t)$ denote the density and the pressure fields; $\sigma(x, t) \in \mathbb{R}^{3 \times 3}$ is the deviatoric part of the Cauchy stress tensor and hence is a symmetric tensor of zero trace; and $f(x, t) \in \mathbb{R}^3$ is the vector of body forces (see, e.g., Byron, Stewart, & Lightfoot 2007). Here $\nu > 0$ is a parameter, representing the overall magnitude of Cauchy–stress-related terms for bookkeeping purposes; its actual value will be immaterial to our discussion and hence it may be a dimensional number. We now consider various active vector fields associated with such a fluid and write our their corresponding barrier equation, when applicable.

6.1. Barriers to linear momentum transport

We first recall that the classic expression for momentum flux through a control surface is $\rho (v \cdot n) v$, where $n$ is the unit surface normal and $v$ is the velocity relative to the surface. This relative velocity is zero along any material surface $\mathcal{M}(t)$, so the classic momentum flux is always zero in our context. For this reason, we seek active barriers to momentum transport in the sense of Section 4.

Setting $u = \rho v$, we can rewrite eq. (6.1) as

$$\dot{u} = -\dot{\rho} v + f - \nabla p + \nu \nabla \cdot \sigma.$$

(6.2)

The first two terms on the right-hand side depend on the frame and hence are not objective. The third term (containing the pressure gradient and the density) is objective but represents conservative terms as opposed to dissipative terms. The last term in eq. (6.2), in contrast, leads to diffusion of linear momentum and is objective, given that the deviatoric stress tensor, its divergence and the density are all objective (Gurtin, Fried
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Therefore, if the active vector \( u \) is the linear momentum, then the \( g \) function in our general formulation (2.4) is \( g = \nabla \cdot \sigma \). Accordingly, the barrier equations (4.1) and (5.1) for linear momentum transport become

\[
\begin{align*}
x_0' &= \det \nabla F(t_0)(F(t_0))' \nabla \cdot \sigma (x_0), \\
x' &= \nabla \cdot \sigma (x, t).
\end{align*}
\]

Specifically, in the case of incompressible, unit-density \((\rho \equiv 1)\) Navier–Stokes flows, we have a dynamic viscosity \( \mu \), a kinematic viscosity \( \nu \) and the constitutive law \( \nu \nabla \cdot \sigma = \mu \Delta v = \nu \Delta v \) in the general momentum equation (6.2); we also observe that \( \det \nabla F(t_0) = 1 \) holds by incompressibility. We then obtain the following:

**Theorem 1.** For incompressible, uniform-density Navier–Stokes flows, the material and instantaneous barrier equations (4.1) and (5.1) for linear momentum take the specific forms

\[
\begin{align*}
x_0' &= (F(t_0))' \Delta v(x_0), \\
x' &= \Delta v(x, t).
\end{align*}
\]

Note that (6.4) are three-dimensional, steady, incompressible dynamical systems with respect to the evolutionary variable \( s \in \mathbb{R} \), and hence can be analyzed by tools developed for incompressible flows in the chaotic advection literature (Aref et al. 2017). The possible active barriers arising from such an analysis are those shown in Fig. 2.

In order to solve for trajectories of (6.4) accurately over a domain \( U \) with boundary \( \partial U \), one must be aware of any special boundary condition that \( b(t_0)(x_0) \) may have to satisfy along \( \partial U \). We assume for simplicity that the flow is incompressible and \( \partial U \) is a flat, no-slip wall. Then in orthogonal \((x_1, x_2, x_3)\) coordinates (with \( x_3 \) normal to the wall), the Navier–Stokes equation (6.1) along the wall takes the form

\[
\begin{pmatrix}
0 \\
0 \\
0
\end{pmatrix} = \nu \Delta v + \begin{pmatrix}
f_1 - \partial_{x_1} p \\
f_2 - \partial_{x_2} p \\
f_3 - \partial_{x_3} p
\end{pmatrix}.
\]

Therefore, if the wall-normal pressure gradient balances out external body forces along \( \partial U \) (as is often assumed in CFD simulations), then \( \Delta v \) satisfies a no-penetration boundary condition along the no-slip boundary \( \partial U \) for any incompressible Navier–Stokes velocity field \( v \). Given that such a boundary \( \partial U \) is invariant under the flow map \( F(t_0) \), we obtain that the pull-back of \( \Delta v \) under the flow map must also be tangent to the boundary. Consequently, any no-slip boundary \( \partial U \) with a vanishing wall-normal resultant force is an invariant manifold for both barrier equations in (6.11). Under physically admissible perturbations of \( v \), the boundary \( \partial U \) remains unchanged, and hence the boundary is also an active barrier by definition, as long as the wall-normal pressure gradient balances out the external body forces along \( \partial U \).

6.2. Barriers to angular momentum transport

To analyze angular momentum barriers, we take the cross product of eq. (6.1) with a vector \( r = x - \hat{x} \) where \( \hat{x} \in U \) marks a fixed reference point. Setting then \( u = r \times \rho v \), we obtain an evolution equation for \( u \) in the form

\[
\dot{u} = (x - \hat{x}) \times \rho v - (x - \hat{x}) \times \nabla p + (x - \hat{x}) \times f + \nu (x - \hat{x}) \times \nabla \cdot \sigma,
\]
implying that the \( g \) function in our general formulation (2.4) is \( g = (x - \hat{x}) \times \nabla \cdot \sigma \). Under a frame-change of the form (2.2), we can rewrite \( g \) as

\[
g = (x - \hat{x}) \times \nabla \cdot \sigma = Q(t) (y - \hat{y}) \times \hat{\nabla} \cdot \hat{\sigma} = \tilde{g},
\]

where we have used the objectivity of \( \sigma \). We conclude from (6.7) that the objectivity condition (2.5) is satisfied for this choice of \( g \), and hence our formulation is applicable. We therefore obtain, as in the case of linear momentum, the following:

**Theorem 2.** For incompressible, uniform-density Navier–Stokes flows, the material and instantaneous barrier equations (4.1) and (5.1) for angular momentum take the specific form

\[
x_0' = (F_{to}^t)^* (x_0),
\]

\[
x' = (x - \hat{x}) \times \Delta v (x, t).
\]

These equations again define three-dimensional, steady, incompressible flows with respect to the time-like independent variable \( s \in \mathbb{R} \). As in the case of barriers to the transport of linear momentum, we find that in the presence of zero wall-normal resultant force, eq. (6.6) implies any no-slip wall \( \partial U \) to be an invariant manifold for the two flows in (6.8).

### 6.3. Barriers to vorticity transport

The classic definition of vorticity flux along a material surface is given by \( \Phi \) in formula (3.1), with the substitution \( u = \omega \). Our approach to vorticity transport, in contrast, involves the transport functional \( \psi_{\omega} \). To compute the corresponding barrier equation, we take the curl of eq. (6.1) and use the relation \( \nabla \times \nabla p = 0 \) to obtain the general vorticity transport equation

\[
\dot{\omega} = (\nabla v) u - (\nabla \cdot v) u + \frac{1}{\rho^2} \nabla \rho \times \nabla p + \nabla \times \left( \frac{f}{\rho} \right) + \nu \nabla \times \left( \frac{\nabla \cdot \sigma}{\rho} \right).
\]

Consequently, for the case of vorticity transport, the \( g \) function in our general formulation (2.4) is \( g = \nabla \times \left( \frac{\nabla \sigma}{\rho} \right) \). Following the derivation of the transformation formula for vorticity under an observer change (2.2) (see, e.g., Truesdell & Rajagopal 2009), we obtain that \( g = Q(t) \tilde{g} \). Therefore, the objectivity condition (2.5) is satisfied for this choice of \( g \), and hence our formulation is applicable. Therefore, the barrier equations (4.1) and (5.1) for vorticity transport becomes

\[
x_0' = \det F_{to}^t (x_0),
\]

\[
x' = \nabla \times \left( \frac{\nabla \cdot \sigma (x, t)}{\rho (x, t)} \right).
\]

Specifically, as in the case of linear and angular momentum barriers, we obtain:

**Theorem 3.** For incompressible, uniform-density Navier–Stokes flows, the material and instantaneous barrier equations (4.1) and (5.1) for vorticity take the specific form

\[
x_0' = (F_{to}^t)^* \Delta \omega (x_0),
\]

\[
x' = \Delta \omega (x, t).
\]

As in the case of linear and angular momenta, the barrier equations (6.11) define three-
dimensional, steady, incompressible flows with respect to the evolutionary variable \( s \in \mathbb{R} \), and hence can be analyzed accordingly.

As for boundary conditions for trajectories of the equations (6.11) along a no-slip flat boundary \( \partial U \) in the incompressible case with \( \rho_0(x) \equiv 1 \), the vorticity-transport equation along the wall \( \partial U \) takes the form

\[
\begin{pmatrix}
\frac{D}{Dt} (\partial_{x_3} v_3 - \partial_{x_2} v_2) \\
\frac{D}{Dt} (\partial_{x_2} v_3 - \partial_{x_1} v_1) \\
0
\end{pmatrix} = \nu \Delta \omega + \begin{pmatrix}
\partial_{x_2} f_3 - \partial_{x_3} f_2 \\
\partial_{x_3} f_1 - \partial_{x_1} f_3 \\
\partial_{x_1} f_2 - \partial_{x_2} f_1
\end{pmatrix}.
\]

Consequently, whenever the curl of non-potential body forces is normal to a no-slip boundary \( \partial U \), the vector field \( \Delta \omega \) satisfies a no-penetration boundary condition along \( \partial U \). As we have already noted in relation to formula (6.5), this in turn implies that \( \partial U \) is an invariant manifold for the two flows in (6.11).

7. Active transport barriers in special classes of flows

In order to illustrate the feasibility of the active barriers we have constructed, we now identify them in classes of explicit Navier–Stokes solutions, with details of the calculations relegated to the Appendices B and C. The general conclusion from the calculations below is that the material and instantaneous barriers we have defined return the observed barriers to the redistribution of active vector fields in simple, explicitly known solutions of the 2D and 3D unsteady Navier–Stokes equations. This gives convincing support to applying the same approach to more complex flows.

7.1. 2D Navier–Stokes flows viewed as 3D Navier–Stokes flows with symmetry

We define the planar variable \( \hat{x} = (x_1, x_2) \in \mathbb{R}^2 \) and assume that a solution of the 3D incompressible Navier–Stokes equation is of the form

\[
v(x, t) = (\hat{v}(\hat{x}, t), w(\hat{x}, t)) , \\
p(x, t) = p(\hat{x}, t), \\
x = (\hat{x}, x_3) \in \mathbb{R}^3,
\]

with the two-dimensional velocity field \( \hat{v}(\hat{x}, t) \) and the scalar functions \( w(\hat{x}, t) \) and \( p(\hat{x}, t) \) (see, e.g., Majda & Bertozzi 2002). Under this 2D-symmetry ansatz, substitution of \( v \) and \( p \) into the 3D Navier–Stokes equation gives the equations

\[
\begin{align*}
\partial_t \hat{v} + (\nabla_{\hat{x}} \hat{v}) \cdot \hat{v} &= -\frac{1}{\rho} \nabla_{\hat{x}} p + \nu \Delta_{\hat{x}} \hat{v}, \\
\partial_t w + \nabla_{\hat{x}} w \cdot \hat{v} &= \nu \Delta_{\hat{x}} w,
\end{align*}
\]

with the subscript \( \hat{x} \) referring to the two-dimensional version of the differential operators involved. Therefore, the symmetry ansatz (7.1) for a 3D Navier-Stokes solution is valid if \( w(\hat{x}, t) \) is chosen as a solution of the advection-diffusion equation appearing in (7.2). This advection-diffusion equation, however, coincides with the two-dimensional vorticity transport equation, which is solved by

\[
w(\hat{x}, t) = \hat{\omega}(\hat{x}, t),
\]

where \( \hat{\omega}(\hat{x}, t) \) is the scalar vorticity field of the 2D Navier–Stokes solution \( \hat{v}(\hat{x}, t) \). In the following, we will choose the third component of \( v \) as in (7.3), in which case active barriers to momentum transport in (7.1) can be located using the following result:

**Theorem 4.** Time-\( t_0 \) positions of material active barriers to linear momentum transport over \([t_0, t_1]\) in incompressible 2D Navier–Stokes flows are structurally stable level
curves of the time-averaged Lagrangian vorticity \( \hat{\omega}(\hat{x}, t) \). Consequently, instantaneous active barriers to linear momentum transport at time \( t \) are structurally stable level curves of the vorticity \( \hat{\omega}(\hat{x}, t) \).

Proof. See Appendix C.

While streamlines in general are not objective, the streamlines of the vorticity \( \hat{\omega}(\hat{x}, t) \) and of the time-averaged Lagrangian vorticity \( \hat{\omega}(\hat{x}, t) \) are objective (see, e.g., Ogden 1984). This is consistent with the more general result (B4) we have already obtained for the objectivity of all active barriers.

In contrast to Theorem 1, active barriers to vorticity transport in (7.1) can be captured as follows:

**Theorem 5.** The time-\( t_0 \) positions of Lagrangian active barriers to vorticity transport over \([t_0, t_1]\) in 2D Navier–Stokes flows are structurally stable level sets of the Lagrangian vorticity-change function \( \hat{\omega}(\hat{x}, t) \) — \( \hat{\omega}(\hat{x}, t) \) for material derivatives \( \frac{D}{Dt} \hat{\omega}(\hat{x}, t) \), or equivalently, of the vorticity Laplacian \( \Delta \hat{\omega}(\hat{x}, t) \).

Proof. See Appendix C.

While vorticity is not objective, the level curves of the change of vorticity along trajectories is objective. This follows directly from the objectivity of the barrier equations but can also be verified directly using the definition of objectivity.

The implementation of Theorems 4-5 for material barriers to momentum and vorticity transport in two dimensions is straightforward. One computes the averaged Lagrangian vorticity \( \hat{\omega}(\hat{x}, t) \) and the Lagrangian vorticity-change function \( \hat{\omega}(\hat{x}, t) \) — \( \hat{\omega}(\hat{x}, t) \), respectively, along particle released from an initial grid, then identify structurally stable level curves of these functions to obtain the appropriate barrier families. Such structurally stable curves include families of closed level curves, homoclinic level curves, and heteroclinic level curves connecting invariant flow boundaries. To locate instantaneous active barriers to momentum and vorticity transport, one identifies the structurally stable level curves of \( \hat{\omega}(\hat{x}, t) \) and its material derivative \( \frac{D}{Dt} \hat{\omega}(\hat{x}, t) \), respectively.

**Example 1.** We consider the spatially doubly-periodic Navier–Stokes flow family described by Majda & Bertozzi (2002) in the form

\[
\hat{v}(\hat{x}, t) = e^{-4\pi^2\ell^2 t} v_0(\hat{x}), \quad p(\hat{x}, t) = e^{-4\pi^2\ell^2 t} p_0(\hat{x}),
\]

\[
\hat{v}_0(\hat{x}) = \sum_{\|k\|^2 = \ell} \left( \begin{array}{c} a_k k_2 \sin(2\pi k \cdot \hat{x}) - b_k k_2 \cos(2\pi k \cdot \hat{x}) \\ -a_k k_1 \sin(2\pi k \cdot \hat{x}) + b_k k_1 \cos(2\pi k \cdot \hat{x}) \end{array} \right), \quad (7.5)
\]

where \( \hat{v}_0(\hat{x}) \) and \( p_0(\hat{x}) \) solve the steady planar Euler equation for some positive integer \( \ell \)† In that case, we have

\[
\Delta v = \left( \begin{array}{c} \Delta x \hat{v} \\ \Delta \hat{\omega} \end{array} \right) = \left( \begin{array}{c} -4\pi^2\ell e^{-4\pi^2\ell^2 t} \hat{v}_0(\hat{x}) \\ \Delta \hat{\omega}(\hat{x}, t) \end{array} \right).
\]

† This flow family contains our motivating example (A.1) in Appendix A with the choice \( k_1 = 0, \ell = k_2 = 1, a_{(1,0)} = b_{(1,0)} = a_{(0,1)} = 0 \) and \( b_{(0,1)} = a \) if we let \( x_2 \to -x_2 \).
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One can verify by direct substitution that $e^{-4\pi^2\ell \nu t} \hat{\mathbf{v}}_0 \left( \hat{\mathbf{F}}^t_{t_0} (\hat{x}_0) \right)$ is a solution of the equation of variations \( \tilde{\xi} = e^{-4\pi^2\ell \nu t} \nabla_{\hat{x}} \hat{\mathbf{v}}_0(\hat{x}(t)) \xi \) (whose fundamental matrix solution is $\nabla_{\hat{x}} \hat{\mathbf{F}}^t_{t_0} (x_0)$) for the differential equation $\hat{x} = e^{-4\pi^2\ell \nu t} \hat{\mathbf{v}}_0(\hat{x})$. As a consequence, we have

\[
\left[ \nabla_{\hat{x}_0} \hat{\mathbf{F}}^t_{t_0} (\hat{x}_0) \right]^{-1} e^{-4\pi^2\ell \nu t} \hat{\mathbf{v}}_0 \left( \hat{\mathbf{F}}^t_{t_0} (\hat{x}_0) \right) = e^{-4\pi^2\ell \nu t_0} \hat{\mathbf{v}}_0 (\hat{x}_0),
\]

and hence, by Theorem 3, the material and instantaneous barrier equations for linear momentum take the specific form

\[
\hat{x}'_0 = e^{-4\pi^2\ell \nu t_0} \hat{\mathbf{v}}_0 (\hat{x}_0),
\]

\[
x'_{03} = A(\hat{x}_0, t_1, t_0),
\]

\[
\hat{x}' = e^{-4\pi^2\ell \nu t} \hat{\mathbf{v}}_0 (\hat{x}),
\]

\[
x'_3 = A(\hat{x}, t, t),
\]

for an appropriate function $A(\hat{x}_0, t_1, t_0)$. Therefore, both material and instantaneous barriers to linear momentum transport in the 2D Navier–Stokes flow family \( \hat{\mathbf{v}}_0 (\hat{x}_0) \) are structurally stable streamlines of the steady velocity field $\hat{\mathbf{v}}_0 (\hat{x}_0)$.

As for vorticity barriers in this example, note that

\[
\hat{\omega} = \partial_{x_1} v_2 - \partial_{x_2} v_1 = -2\pi \ell e^{-4\pi^2\ell \nu t} \hat{\omega}_0, \quad \hat{\omega}_0 (\hat{x}) = \sum_{|k|=\ell} a_k \cos (2\pi k \cdot \hat{x}) + b_k \sin (2\pi k \cdot \hat{x}).
\]

As the steady part of the vorticity field solves the steady planar Euler equation, trajectories of $\hat{\mathbf{v}}(\hat{x}, t)$ remain confined to the steady streamlines of $\hat{\mathbf{v}}_0 (\hat{x}_0)$. Since these trajectories also conserve the vorticity $\hat{\omega}_0$ of the inviscid limit of the flow, the change in vorticity $\hat{\omega}(\hat{x}, t)$ along trajectories of $\hat{\mathbf{v}}(\hat{x}, t)$ can be written as

\[
\hat{\omega} \left( \hat{\mathbf{F}}^{t_1}_{t_0} (\hat{x}_0), t_1 \right) - \hat{\omega} (\hat{x}_0, t_0) = -2\pi \ell \left( e^{-4\pi^2\ell \nu t_1} - e^{-4\pi^2\ell \nu t_0} \right) \hat{\omega}_0 (\hat{x}_0).
\]

Therefore, level curves of the vorticity change along trajectories coincide with those of the inviscid vorticity $\hat{\omega}_0 (\hat{x}_0)$, which are in turn just the streamlines of $\hat{\mathbf{v}}_0 (\hat{x}_0)$. Finally, we have

\[
\Delta \hat{\omega} (\hat{x}, t) = 8\pi^3 \ell^2 e^{-4\pi^2\ell \nu t} \hat{\omega}_0 (\hat{x}),
\]

and hence the level curves of $\Delta \hat{\omega} (\hat{x}, t)$ also coincide with those of $\hat{\mathbf{v}}_0 (\hat{x})$.

We conclude that both material and instantaneous active barriers to vorticity and linear momentum transport coincide with the streamlines of $\hat{\mathbf{v}}_0 (\hat{x}_0)$. In particular, we obtain the correct active barrier distributions that we inferred for our motivational example \( A1 \) in Appendix A, which is part of the solution family \( \hat{\mathbf{v}}_0 (\hat{x}_0) \). Importantly, one obtains this frame-indifferent conclusion from any finite-time (or even instantaneous) analysis of the flow.

### 7.2. Directionally steady Beltrami flows

A number of explicitly known solutions of the 3D incompressible Navier–Stokes equations are strong Beltrami flows, i.e., satisfy

\[
\omega(x, t) = k(t)v(x, t)
\]
for some scalar function $k(t)$ (see Majda & Bertozzi 2002). By definition, for any incompressible strong Beltrami flow, we obtain
\[
\Delta v = \frac{1}{k} \Delta \omega = -k^2 v,
\]
\[
\Delta \omega = \nabla (\nabla \cdot \omega) - \nabla \times (\nabla \times \omega) = -k^3 v.
\]
Recall that if a steady Euler flow is non-Bernoulli, then it is integrable (Arnold & Keshin 1998). Therefore, only Bernoulli solutions can have complicated dynamics in steady Euler flows.

We call an unsteady strong Beltrami flow with velocity field $v(x,t)$ a directionally steady Beltrami flow if
\[
v(x,t) = \alpha(t) v_0(x), \quad \omega(x,t) = \nabla \times v(x,t) = k(t) \alpha(t) v_0(x)
\]
hold for some continuously differentiable scalar function $\alpha(t)$. Note that any steady strong Beltrami flow $v_0(x)$ (which necessarily admits $k(t) = k = \text{const.}$) solves the steady Euler equation and generates a directionally steady Beltrami solution $v(x,t) = \exp\left(\frac{-\nu k^2 t}{2}\right) v_0(x)$ for the unsteady Navier–Stokes equation under conservative forcing (cf. Majda and Bertozzi 2002). For all directionally steady Bernoulli flows, we obtain the following simple result on active barriers:

**Theorem 6.** Both material and instantaneous active barriers to the diffusive transport of linear momentum and vorticity in directionally steady Beltrami flows coincide exactly with structurally stable 2D invariant manifolds of the steady component $v_0(x)$ of the velocity field. These in turn coincide with 2D invariant manifolds of $v(x,t)$ defined in (D1).

**Proof.** See Appendix D.

Theorem 6 shows that invariant manifolds for Lagrangian particle motions in directionally steady Bernoulli flows coincide with material and instantaneous active barriers to linear momentum and vorticity transport. This again agrees with one’s intuition, as observed mass transport barriers in these flows are expected to coincide with barriers to vorticity and momentum transport, given the special structure of the flow. Remarkably, as in the case of 2D flows analyzed in the previous section, these barriers already arise from an arbitrarily-short-time analysis of these unsteady flows.

**Example 2.** Examples of directionally steady Beltrami flows include the Navier-Stokes flow family (Ethier & Steinman 1994)
\[
v(x,t) = e^{-\nu t^2} v_0(x), \quad v_0(x) = -a \begin{pmatrix} e^{ax_1} \sin (ax_2 \pm dx_3) + e^{ax_3} \cos (ax_1 \pm dx_2) \\ e^{ax_2} \sin (ax_3 \pm dx_1) + e^{ax_1} \cos (ax_2 \pm dx_3) \\ e^{ax_3} \sin (ax_1 \pm dx_2) + e^{ax_2} \cos (ax_3 \pm dx_1) \end{pmatrix}
\]
and the viscous, unsteady version of the classic ABC flow $v_0(x)$ (Dombre et al. 1986), given by
\[
v(x,t) = e^{-\nu t} v_0(x), \quad v_0(x) = \begin{pmatrix} A \sin x_3 + C \cos x_2 \\ B \sin x_1 + A \cos x_3 \\ C \sin x_2 + B \cos x_1 \end{pmatrix}.
\]

For both of these flows, Theorem 6 guarantees that all material and instantaneous active barriers to momentum and vorticity transport coincide with structurally stable, two-dimensional invariant manifolds of the flow generated by the steady velocity field $v_0(x)$. Such manifolds can be captured via their intersections with Poincaré sections,
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Figure 3. Instantaneous and Lagrangian barriers to vorticity and momentum barriers for the unsteady ABC flow \((7.8)\), visualized as two-dimensional invariant manifolds of three Poincaré maps associated with the steady component \(v_0(x)\) of the full velocity field \(v(x, t)\).

with these intersections appearing as one-dimensional invariant curves of the associated Poincaré map, as first illustrated by Dombre et al. (1986) for one cross-section of the ABC flow. A more complete set of Poincaré maps along three orthogonal planes is shown in Fig. 3, which reveals a wealth of two-dimensional open and closed surfaces that are exact barriers to the transport of momentum and vorticity for the unsteady, temporally aperiodic velocity field \((7.8)\) in the sense of Definition 1. Further studies revealing the same hyperbolic and invariant manifolds in the steady ABC flow using finite-time Lyapunov exponents (FTLE) and the polar rotation angle (PRA) were given by Haller (2001) and Farazmand & Haller (2016), each emphasizing different classes of barriers from the complete collection revealed in Fig. 3. FTLE and PRA are generally usable structure detection tools along any cross section of an unsteady flow, whereas Poincaré maps are only defined for trajectories returning to the same cross section of a steady or time-periodic flow.

8. Practical implementation on velocity data sets

All applications of our main results in Theorems 1-3 require the analysis of an associated 3D steady, incompressible flow whose velocity field depends on the Laplacian of the original velocity or vorticity fields. In direct numerical simulations of the Navier–Stokes equation, these Laplacians can be computed internally during the simulation with high accuracy, as our numerical results in Section 9.2 will show.

Taking second and third spatial derivatives of velocity field from an already finalized numerical simulation or experiment, however, is challenging. For this reason, it is often more accurate to work with the original material derivatives arising in our definition of active tensor, rather than with the Laplacians of the velocity and the vorticity. More specifically, if we let \(a(x, t) = \dot{v}(x, t)\) denote the Lagrangian particle acceleration along fluid trajectories, then using the general momentum equation \((6.1)\), the active barrier equations \((6.3)\) and \((5.1)\) for linear momentum can be rewritten as

\[
\begin{align*}
\dot{x}'_0 &= \det \nabla F_{t_0}^t (F_{t_0}^t)^* [\rho a + \nabla p - f] (x_0), \\
\dot{x}' &= \rho(x, t)a(x, t) + \nabla p(x, t) - f(x, t),
\end{align*}
\]

where we have dropped the \(1/\nu\) factor from the equations, as it does not affect trajectories and invariant manifolds in these autonomous systems.
Similarly, the most general active barrier equations (6.10) for vorticity can be rewritten as

\[
\begin{align*}
    \mathbf{x}'_0 &= \det \nabla \mathbf{F}_{t_0} (\mathbf{F}_{t_0}^t)^* \nabla \times \left[ \mathbf{a} + \frac{1}{\rho} (\nabla p - \mathbf{f}) \right] (\mathbf{x}_0), \\
    \mathbf{x}' &= \nabla \times \left[ \mathbf{a}(\mathbf{x},t) + \frac{1}{\rho(\mathbf{x},t)} (\nabla p(\mathbf{x},t) - \mathbf{f}(\mathbf{x},t)) \right],
\end{align*}
\]

where we have again dropped the $1/\nu$ factor from the equations. In particular, for incompressible, constant density, Newtonian fluids subject only to potential body forces, the material and instantaneous barrier equations for vorticity in (8.2) simplify to

\[
\begin{align*}
    \mathbf{x}'_0 &= (\mathbf{F}_{t_0}^t)^* \nabla \times \mathbf{a}(\mathbf{x}_0), \\
    \mathbf{x}' &= \nabla \times \mathbf{a}(\mathbf{x},t).
\end{align*}
\]

These latter equations only involve one spatial derivative of the Lagrangian acceleration field, $\mathbf{a}(\mathbf{x},t)$, which can be obtained from high-resolution numerical or experimental data via the temporal differentiation of the velocity vector along trajectories.

As already mentioned, Poincaré maps are the most straightforward tools for identifying structurally stable codimension-one invariant manifolds of the various 3D steady, incompressible barrier equations we have derived. Such a Poincaré (or first-return) map generally does not preserve the standard two-dimensional area, but preserves a general area form, which makes the Poincaré map a two-dimensional symplectic map. One-dimensional invariant curves of two-dimensional symplectic maps satisfy the only available formal definition of advective transport barriers by MacKay, Meiss & Percival (1984), as we noted in the Introduction. Structurally stable invariant curves of 2D symplectic maps involve stable and unstable manifolds of hyperbolic fixed points and KAM curves, i.e., nested families of closed curves satisfying non-resonance and twist-conditions (Arnold 1978).

Poincaré maps give an indication of the overall location of such curves, but several other frame-indifferent techniques have also been developed to target such curves as coherent structures. Such techniques include various Lagrangian coherent structure detection tools reviewed in Haller (2015), the polar rotation angle (PRA) diagnostic (Farazmand & Haller 2016) and Lagrangian-averaged vorticity deviation (LAVD) diagnostic (Haller et al. 2016). Visualizations of these diagnostics tend to suppress invariant manifolds with weaker repulsion/stretching rate, which can be prevented for steady flows by normalizing the right-hand sides of the vector field in the barrier equations to 1, keeping only the pointwise orientation of the original barrier field. This normalization preserves all trajectories and invariant manifolds of a steady flow away from its zeros, yet ensures a more even trajectory spread for invariant manifold detection purposes.

All active barriers block the underlying diffusive flux-change of vorticity or momentum completely, but influential barriers display high active barrier sensitivity (ABS) values, with the ABS field defined for all (unnormalized) barrier equations as the norm of their right-hand side (cf. eq. (4.2)). As a diagnostic procedure, therefore, one may also simply plot the ABS field for a given barrier equation to obtain a rough idea of the location of the most influential barriers.

We finally note that while the barrier equations we have derived in Section 6 are invalid for inviscid flows, they are nevertheless formally computable for such flows. In contrast,

† The PRA is not objective when applied directly to analyze 3D fluid flows, as those have non-objective velocity fields. The PRA is, however, an objective diagnostic for the barrier equations, whose right-hand side is an objective vector field, as we have seen.
the formulations eqs. (8.1)-(8.3) all have vanishing right-hand sides in the inviscid limit, correctly reflecting that all material surfaces are barriers to the viscous transport of momentum and vorticity in such flow.

9. Active barriers in specific unsteady flows

9.1. Two-dimensional turbulence

Here we evaluate our results on active barriers in a two-dimensional turbulence simulation over a spatially periodic domain. Obtained from a pseudo-spectral code applied to the two-dimensional, incompressible Navier-Stokes equation, this velocity data set was already analyzed by Katsanoulis et al. (2019), who located vortex boundaries as barriers to the diffusive transport of vorticity in this data set using the theory of constrained diffusion barriers from Haller, Karrasch & Kogelbauer (2019).

At Reynolds number \( Re = \nu^{-1} = 5 \times 10^4 \), the spatial coordinates are resolved in the data set using 1024\(^2\) Fourier modes with 2/3 dealiasing. The domain is \([0, 2\pi] \times [0, 2\pi]\) with periodic boundary conditions. For all the Eulerian and Lagrangian computations to follow, we used a slightly oversampled grid of 1100 \(\times\) 1100 points in the interest of accurate particle advection. For the Lagrangian calculations, we advected trajectories over the time interval \([0, 50]\). Whenever a numerical integration scheme was required, i.e., trajectory advection, integration of the corresponding barrier field and computation of Lagrangian acceleration, the Runge–Kutta algorithm of MATLAB (i.e., ode45) was used. This algorithm uses an adaptive time stepping such that the relative and absolute errors were below \(10^{-6}\) (\(10^{-8}\) for the acceleration computation).

9.1.1. Vorticity barriers

To locate material barriers to vorticity transport, we used Theorem 5 with select barriers shown in red over the corresponding ABS field in the left subplot of Fig. 4. To obtain this image, we used a box filter to smooth out the vorticity-change function \( \left( \hat{\omega} \left( \hat{F}_{t_{1}t_{0}} (\hat{x}_{0}), t_{1} \right) - \hat{\omega} (\hat{x}_{0}, t_{0}) \right) / (t_{1} - t_{0}) \). For better visibility, only those contour lines of this function are shown (in red) that are closed and have a length of at least 0.5. The plots reveal several vortical material barriers to vorticity transport, as well as more intricate barriers in the background flow.

To compute instantaneous barriers to vorticity transport from Theorem 5, we plotted the level sets of \([\nabla \times a (\hat{x}_{0}, t_{0})]_{3} = \frac{\partial}{\partial t} \hat{\omega} (\hat{x}_{0}, t_{0})\), with \(a = v_{t} + (\nabla v) v\) denoting the Lagrangian acceleration (cf. Section 8), in the right subplot of Fig. 4 under the same filtering criteria as in the left subplot.

9.1.2. Linear momentum barriers

To detect material barriers to linear momentum transport over the same \([0, 50]\) time interval, we use the identity \(\Delta v = -\nabla \times \omega\) to obtain the velocity Laplacian for this incompressible flow. After smoothing the vorticity field via the same box filter used for vorticity barriers, we use Theorem 4 to extract material and instantaneous barriers to momentum transport. Figure 5 shows the outermost members of families of the corresponding level curves with a length greater than 0.5 and less than 2.0 for visual clarity. A comparison with Fig. 4 reveals that the vorticity and momentum barriers tend to be similar in vortical regions but may differ significantly in the background turbulence region.
We consider now a three-dimensional turbulent flow of a Newtonian fluid in a doubly periodic channel. The code employed in the present study is based on the mixed-discretization parallel solver of the incompressible Navier–Stokes equations in the wall-normal velocity and vorticity formulation developed by Luchini & Quadrio (2006). The equations of motions are discretized via a Fourier–Galerkin approach along the two statistically homogenous streamwise $x$ and spanwise $z$ directions. Fourth-order compact finite differences Lele (1992) based on a five-point computational molecule are adopted in the wall-normal direction $y$.

The governing equations are integrated forward in time at constant pressure gradient with a partially-implicit approach, combining the three-step low-storage Runge–Kutta (RK3) scheme with the implicit Crank–Nicolson scheme for the viscous terms. The friction Reynolds number is $Re_f = u_f h / \nu = 200$, based on the friction velocity $u_f$, the channel half height $h$ and the kinematic viscosity $\nu$, which corresponds to a bulk Reynolds number $Re_b = U_b h / \nu = 3177$, where $U_b$ is the mean velocity across the channel. The computational domain is $L_x = 4\pi h$ long and $L_z = 2\pi h$ wide. The number of Fourier modes is 256 both in the streamwise and spanwise direction; the number of points in the wall-normal direction is 256, unevenly spaced in order to decrease the grid size near the walls. The corresponding spatial resolution in the homogeneous directions is $\Delta x^+ = 9.8$ and $\Delta z^+ = 4.9$ (without accounting for the additional modes required for dealiasing according to the 3/2 rule); the wall-normal resolution increases from $\Delta y^+ = 0.4$ near the walls to $\Delta y^+ = 2.6$ at the centerline, while the temporal resolution is kept constant and equals $\Delta t^+ = 0.063$. The superscript + denotes nondimensionalization in viscous units, i.e. with $u_f$ and $\nu$.

We compute the right-hand sides of both the instantaneous and the material barrier equations for momentum and vorticity obtained in Section 6. We then use the FTLE
Figure 5. Left: Time $t_0 = 0$ positions of material barriers to linear momentum transport over the time interval $[0, 50]$ in the 2D turbulent flow data. The barriers are computed as level curves of the time-averaged Lagrangian vorticity $\tilde{\omega} \left( \mathbf{F}_{t_0}^t (\mathbf{x}_0), t \right)$ (red). Right: Instantaneous barriers to the active transport of linear momentum at time $t_0 = 0$ in the same dataset, computed as level curves of the vorticity field (red). For both plots, the curves shown are the outermost members of the families of level curves and the scalar field in the background is the corresponding ABS field.

and the PRA diagnostics (Haller 2001, Farazmand & Haller 2016) to locate structurally stable, two-dimensional hyperbolic and elliptic invariant manifolds in the incompressible, steady barrier ODEs (8.1) and (8.3). By our main results (Theorems 1 and 3), these invariant manifolds will be barriers to the transport of linear momentum and vorticity in the precise sense given in Definition 1.

As noted in Section 8, we compute the right-hand sides of the relevant barrier equations on the fly, using the same discrete differential operators that we use in our direct numerical simulation. The computation of material barriers requires, in addition, the simultaneous advection of Lagrangian tracers in the flow and the evaluation of the time-averaged pull-back of the objective $\mathbf{g}(\mathbf{x}, t, \mathbf{u}, \mathbf{v})$ vector field, defined in general by eq. (2.4), for both momentum and vorticity. We also perform these operations while solving for the velocity field $\mathbf{v}$ to guarantee sufficient temporal resolution yet minimize the required storage space. We transform transformed $\mathbf{g}(\mathbf{x}, t, \mathbf{u}, \mathbf{v})$ into physical space with a three-fold oversampling in both statistically homogeneous directions. We then use a total of approximately 151 million Lagrangian tracers seeded at each grid point in space. The evolving positions of these tracers, which are images of their initial positions under the flow map $\mathbf{F}_{t_0}^t$, are advanced in time by integrating the $\mathbf{v}$ field with the same RK3 algorithm adopted for advancing the governing equations. The vectors $\mathbf{v}$ and $\mathbf{g}$ are evaluated at the particle position through a sixth-order, three-dimensional Lagrangian interpolation (van Hinsberg et al. 2012, Pitton et al. 2012) of the underlying vector field, which reduces to fourth-order only between the wall and the first grid point above it. We have made all quantities nondimensional via $U_b$ and $h$ unless stated otherwise.

**Objective barriers to the transport of linear momentum**

Figures 6-7 show the FTLE- and PRA-based visualizations of instantaneous and material barriers to momentum transport, highlighting their intersections with the two-dimensional cross-section located at the half-length of the channel. The range for the
Figure 6. Time $t_0 = 0$ positions of instantaneous (Eulerian) barriers to linear momentum transport, highlighted by the FTLE field (upper plot) and the PRA field (lower plot) of the second momentum-barrier equation in (8.1). FTLE values range within the interval $[0, 400]$ and PRA values range within the interval $[0, 3.25]$. The horizontal axis represents the spanwise $z$ coordinate; the vertical axis represents the vertical $y$ coordinate in the channel.

The streamline parametrization $x_0(s)$ of the barrier equations was $s \in [0, 0.01]$ for both the Eulerian and Lagrangian computations. This short range already reveals an abundance of invariant manifolds via the FTLE and PRA diagnostics in the autonomous barrier ODEs (8.1) and (8.3) due to the large norm of their right-hand sides. Increasing the range of $s$ further reveals smaller and smaller scales in these barriers which gradually deteriorates their visibility. Several open (or hyperbolic) barriers are visible as ridges of the FTLE fields. Ridges wrapping around closed regions suggest vortical (or elliptic) barriers to transport, which are in turn confirmed as level sets of the corresponding PRA field.

Both the the Eulerian and the Lagrangian results reveal an objective signature of a large, distinguished flow region in the middle of the channel. The interior of this region displays practically no discernible barriers, while the exterior or the region displays an abundance of open and closed barriers to linear momentum transport. Other smaller regions with practically no interior transport barriers coexist with the large region in the middle. The boundaries of all these distinguished regions are formed by the envelopes of filamented open (hyperbolic) transport barriers, which are finite-time generalizations of infinite-time classic stable and unstable manifolds. Unlike in previous approaches, however, these finite-time invariant manifolds are constructed here as perfect barriers to active transport, rather than as Lagrangian coherent structures acting as backbones of advected fluid-mass patterns (cf. Haller 2015).

We also compare our results with those obtained from a common visualization tool for coherent vortical structures in wall-bounded turbulence via level surfaces of the $\lambda_2(x, t)$ field (Jeong & Hussein 1995). This scalar field is defined as the instantaneous intermediate eigenvalue of the tensor field $\mathbf{S}^2(x, t) + \mathbf{W}^2(x, t)$, where $\mathbf{S}$ is the rate-of-strain tensor (the symmetric part of the velocity gradient) and $\mathbf{W}$ is the spin tensor (the skew-symmetric part of the velocity gradient). Here we choose the level set defined by $\lambda_2^+(x, t) = -0.02$, with the $+$ superscript referring to the usual scaling in wall units. This choice follows the
Objective material barriers to the transport of momentum and vorticity

Figures 7-10 show the corresponding FTLE- and PRA-based visualizations of instantaneous and material barriers to vorticity transport within the same two-dimensional cross-section of the channel. The range for the streamline parametrization $x_0(s)$ of the barrier equations was $s \in [0, 0.0001]$ for both the Eulerian and Lagrangian computations. This is significantly shorter than the range we used to explore momentum barriers. The reason is that the barrier equations for vorticity have right-hand sides that are orders of magnitude larger due to the additional spatial derivative involved in their definition. Figures 9-10 highlight structures that are similar, but not identical, to the momentum barriers shown in Figs. 6-7. The similarity is especially notable between the Lagrangian momentum and vorticity barriers (lower subplots of Figs. 7 and 7), where the Lagrangian averaging appears to compensate for the differences arising from the different orders of spatial differentiations involved in obtaining their Eulerian counterparts in the upper subplots in the same figures.

Finally, in Fig. 11 we highlight Eulerian vorticity barriers and compare them with the same $\lambda_2^+$ level-sets that we have already shown in Fig. 8. The lower inset again shows a portion of the flow where the $\lambda_2^+$ surface is locally consistent with some of the
Figure 8. Intersections of instantaneous momentum barrier surfaces with three different planes at time \( t = 0 \), highlighted by ridges of the FTLE field of the second equation in (8.1). Some trajectories of this Eulerian barrier equation are also shown for reference (in pink). Superimposed are the components of the instantaneous level set \( \lambda_2^+(x, t) = -0.02 \), which is traditionally used for vortex identification in turbulent channel flows. Instances of similarities (lower inset), false negatives (upper inset) and false positives suggested by the \( \lambda_2^+ \) level sets can be found.

Figure 9. Time \( t_0 = 0 \) positions of instantaneous (Eulerian) barriers to vorticity transport, highlighted by the FTLE field (upper plot) and the PRA field (lower plot) of the momentum second barrier equation in (8.1). FTLE values range within the interval \([0, 3.5 \times 10^4]\) and PRA values range within the interval \([0, 3.25]\). All other details are the same as for the instantaneous momentum barriers in Fig. 6.

objectively defined tubular barriers to vorticity transport. The higher inset shows cases of false positives for the detection of vorticity-transport barriers via the \( \lambda_2^+ \) parameter.
10. Conclusions

We have used the classic definition of the vector-field flux to extend the broadly accepted notion of vorticity flux to the flux of more general dynamically active vector fields, such as linear and angular momentum. We then sought material surfaces across which the time-averaged change of this generalized flux due to viscous effects vanishes pointwise. This procedure was motivated by our aim to develop a frame-independent measure of permeability of experimentally observable material surfaces under the viscous forces responsible for the evolution of a dynamically active vector field. Active material barriers are global minimizers of this objective measure.
We have obtained that in incompressible Navier–Stokes flows, active material barriers to transport evolve from structurally stable stream-surfaces of an associated Lagrangian vector field. This vector field is the time-averaged pull-back of viscous forces responsible for the change of the active vector field along material surfaces. Physically, instantaneous limits of these material barriers to linear momentum are surfaces to which viscous forces are tangent. Similarly, instantaneous limits to active barriers to vorticity are surfaces tangent to the curl of viscous forces.

We have shown that material and instantaneous active barriers in two-dimensional Navier-Stokes flows and three-dimensional directionally steady Beltrami flows coincide with readily inferable or classically known material barriers to advective transport. For more general flows, however, the barriers obtained from our approach are previously undocumented material surfaces with a well-defined, globally minimal permeability property under the active vector field of interest. We have illustrated this on two- and three-dimensional numerical simulations of unsteady Navier–Stokes flows.

Instantaneous active barriers provide an objective and parameter-free alternative to currently used, observer-dependent flow-visualization tools, such as level surfaces of the velocity norm, of the velocity components and of the $\lambda_2$-parameter. Undoubtedly, the implementation of these tools is appealingly simple via automated level-surface visualization packages. Yet such evolving surfaces are observer-dependent and non-material, thereby lacking any experimental verifiability. In addition, beyond the simplicity of generating surfaces as level sets of a scalar, the physical meaning of such level sets is unclear.

Locating material or instantaneous active barriers for linear momentum, angular momentum and vorticity requires the identification of robust (structurally stable) stream-surfaces in the appropriate steady and incompressible barrier equations. For flows on bounded invariant domains, this can be often achieved with globally defined Poincaré maps. For more general flows, FTLE, PRA and other LCS diagnostics can be applied to the active barrier equations we have derived. An automated identification of all structurally stable streamsurfaces in those equations, however, is still an unsolved problem in numerical flow visualization (cf. Peikert & Sadlo 2009 for work in this direction). Further research should clearly address this important open problem in order to complete the fully automated extraction of active barriers in generic three-dimensional flows.

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Declaration of Interests

The authors declare no conflict of interest.

Appendix A. A motivating example

A simple example underlying the challenges of defining barriers to momentum and vorticity transport is a planar, unsteady Navier-Stokes vector field representing a decaying horizontal shear-jet between two horizontal no-slip boundaries at $x_2 = \pm \frac{1}{4}$ (see. Fig. 12). The corresponding velocity and scalar vorticity fields are
Normalized by their instantaneous global maxima, the normalized linear momentum \( \rho v^0 = (\cos 2\pi x_2, 0) \) and vorticity \( \omega^0 = \sin 2\pi x_2 \) are both constant in time. There is, therefore, no structural reorganization in the topology of the momentum and vorticity fields. Instead, for all times, horizontal lines act as level curves for both the horizontal momentum and the vorticity, forming material barriers between higher and lower values of these scalars.

Similarly, over a finite time interval \([t_0, t_1]\), Haller et al. (2019) obtain an ODE family describing the time \( t_0 \) position of uniform barriers to the diffusive (passive) transport of the scalar vorticity. With the constants

\[
A = \frac{a^2}{\nu} \sin 2y_0 \left[ \frac{1}{2} e^{-2\nu t_1} + \frac{1}{2} e^{-2\nu t_0} - e^{-\nu(t_1+t_0)} \right], \quad B = a \left( e^{-\nu t_0} - e^{-\nu t_1} \right),
\]

and the vector field \( \bar{q}^{t_1}_{t_0}(x_0) = \frac{1}{2
u(t_1-t_0)} \left( \begin{array}{c} A \sin 2y_0 \\ B \cos y_0 \end{array} \right) \), the ODE family describing the time \( t_0 \) position of uniform constrained barriers is given by

\[
x_0 = \frac{1}{2\nu(t_1-t_0)} \left\{ \sqrt{|\bar{q}^{t_1}_{t_0}(x_0)|^2 - T_0^2} \left( \begin{array}{c} A \sin 2y_0 \\ B \cos y_0 \end{array} \right) + \frac{T_0}{|\bar{q}^{t_1}_{t_0}(x_0)|^2} \left( \begin{array}{c} B \cos y_0 \\ -A \sin 2y_0 \end{array} \right) \right\}
\]

for some value of the transport density constant \( T_0 \in \mathbb{R} \). For the choice

\[
T_0 = |\bar{q}^{t_1}_{t_0}(x_0)|_{y_0=0} = \frac{B}{2\nu(t_1-t_0)},
\]

the ODE (A2) becomes

\[
x_0'_{y_0=0} = \frac{B}{2\nu(t_1-t_0)} \left( \begin{array}{c} B \\ 0 \end{array} \right) \parallel \Omega \bar{q}^{t_1}_{t_0}(x_0)|_{y_0=0}, \tag{A4}
\]

showing that \( y_0 = 0 \) is an invariant line for equation (A2) for the parameter value \( T_0 \) selected as in (A3). Consequently, the jet core at \( y_0 = 0 \) is a uniform, constrained barrier to vorticity diffusion along which the pointwise diffusive transport of vorticity is equal to (A2). Choosing the constant \( T_0 = 0 \) in (A2) gives

\[
x_0' = \frac{1}{2\nu a(t_1-t_0)|\bar{q}^{t_1}_{t_0}(x_0)|} \left( \begin{array}{c} A \sin 2y_0 \\ B \cos y_0 \end{array} \right),
\]

for which \( y_0 = \pm \pi/2 \) are invariant lines, and hence the channel walls at \( y_0 = \pm \pi/2 \) are perfect constrained barriers to diffusive transport. Therefore, the variational theory of Haller et al. (2019) identifies the jet core at \( x_2 = 0 \) and the upper and lower walls as
we conclude that

is given by considering vorticity transport as an active, horizontal lines, which is suboptimal, given that these horizontal barriers prevail already in any finite-time observation of the vorticity field. The objective of the present paper is to strengthen these results by considering vorticity transport as an active, vectorial transport problem, rather than a passive scalar transport problem.

For the same flow (A1), the momentum flux vector field of Meyers & Meneveau (2013) is given by $F_\omega (x) = ae^{-4\pi^2 \nu^2 t} (\cos^2 2\pi x_2, 4\nu \pi \sin 2\pi x_2)$. The $x_2 = 0$ jet core is an integral curve of this vector field, correctly conveying the fundamental role of the jet core in blocking vertical momentum transfer. All other integral curves of $F_\omega (x)$, however, curl either upwards or downwards, running into the two horizontal walls perpendicularly. The latter curves do not delineate observable structures governing the rearrangement of momentum within this flow, not even in the limit of $t \to \infty$.

Appendix B. Proof of Proposition 1

Using the surface-element deformation formula $n dA = \det \nabla F_{t_0} \left[ \nabla F_{t_0} \right]^{-T} n_0 dA_0$ from continuum mechanics (see, e.g., Gurtin, Fried & Anand 2013), we first observe that

$$\Phi (M(t_1)) - \Phi (M(t_0)) = \int_{M(t_1)} u(x, t_1) \cdot n(x, t_1) dA - \int_{M(t_0)} u(x, t_0) \cdot n(x, t_0) dA$$

$$= \int_{t_0}^{t_1} \left[ \frac{D}{Dt} \int_{M(t)} u(x, t) \cdot n(x, t) dA \right] dt$$

$$= \int_{t_0}^{t_1} \left[ \frac{D}{Dt} \int_{M(t_0)} u(F_{t_0}, t) \cdot \det \nabla F_{t_0} \left[ \nabla F_{t_0} \right]^{-T} n_0 dA_0 \right] dt$$

$$= \int_{t_0}^{t_1} \left[ \int_{M(t_0)} \nabla (F_{t_0}^t, t) \cdot \det \nabla F_{t_0} \left[ \nabla F_{t_0} \right]^{-T} n_0 dA_0 \right] dt$$

$$+ \int_{t_0}^{t_1} \left[ \int_{M(t_0)} u(F_{t_0}^t, t) \cdot \frac{D}{Dt} \left\{ \det \nabla F_{t_0} \left[ \nabla F_{t_0} \right]^{-T} \right\} n_0 dA_0 \right] dt.$$ (B1)

Recalling that (see, e.g., Gurtin, Fried & Anand 2013)

$$\frac{D}{Dt} \nabla F_{t_0} (x_0) = \nabla \nu \left( F_{t_0} (x_0), t \right) \nabla F_{t_0} (x_0),$$

$$\frac{D}{Dt} \det \nabla F_{t_0} (x_0) = \det \nabla F_{t_0} (x_0) \int_{t_0}^{t} \nabla \cdot \nu \left( F_{t_0} (x_0), s \right) ds,$$

we conclude that

$$\frac{D}{Dt} \left\{ \det \nabla F_{t_0} \left[ \nabla F_{t_0} \right]^{-T} \right\} = \left( \frac{D}{Dt} \det \nabla F_{t_0} \left[ \nabla F_{t_0} \right]^{-T} \right) = \det \nabla F_{t_0} \int_{t_0}^{t} \nabla \cdot \nu \left( F_{t_0} (x_0), s \right) ds \left[ \nabla F_{t_0} \right]^{-T}$$

$$= \left( \int_{t_0}^{t} \nabla \cdot \nu \left( F_{t_0} (x_0), s \right) ds \right) \det \nabla F_{t_0} \left[ \nabla F_{t_0} \right]^{-T}$$

$$- \det \nabla F_{t_0} \nabla \nu \left[ \nabla F_{t_0} \right]^{-T}.$$
Using this last expression, we can rewrite eq. (B1) as

\[
\Phi (\mathcal{M}(t_1)) - \Phi (\mathcal{M}(t_0)) = \int_{t_0}^{t_1} \int_{\mathcal{M}(t)} \dot{u}(x,t) \cdot n(x,t) dA dt \\
+ \int_{t_0}^{t_1} \int_{\mathcal{M}(t)} (\int_{t_0}^{t} \nabla \cdot v(x,s) ds) u(x,t) \cdot n(x,t) dA dt \\
- \int_{t_0}^{t_1} \int_{\mathcal{M}(t)} u(x,t) \cdot \nabla v^T(x,t) n(x,t) dA dt. \tag{B2}
\]

Using assumption (2.4) on the material derivative \(\dot{u}(x,t)\), we observe that only the first integral in the formula (B2) has explicit dependence on \(\nu\). We then substitute that formula into the definition (3.3) to obtain

\[
\psi^{t_1}_{t_0}(\mathcal{M}_0) = \frac{1}{t_1 - t_0} \int_{t_0}^{t_1} \int_{\mathcal{M}(t)} g(x,t,u,v) \cdot n(x,t) dA dt \\
= \int_{t_0}^{t_1} \left[ \int_{\mathcal{M}(t)} g(x,t,u,v) \cdot \det \nabla F^t_{t_0} [\nabla F^t_{t_0}]^{-T} n_0 dA_0 \right] dt \\
= \left[ \int_{\mathcal{M}(t)} \int_{t_0}^{t_1} \det \nabla F^t_{t_0} [\nabla F^t_{t_0}]^{-1} g \left( F^t_{t_0}, t, u \left( F^t_{t_0}, t \right), v \left( F^t_{t_0}, t \right) \right) \cdot n_0 dA_0 \right] dt, \tag{B3}
\]

which proves the formulas (3.4)-(3.5). The vector field \(b^{t_1}_{t_0}(x_0)\) defined in (3.5) is objective in the Lagrangian sense (cf. Ogden 1984), because under assumption (2.5), an observer change of the form (2.2) gives

\[
b^{t_1}_{t_0} = \frac{\det \nabla F^t_{t_0} (F^t_{t_0})^* g}{\det \nabla F^t_{t_0} [\nabla F^t_{t_0}]^{-1} Q(t) \tilde{g}} \\
= \frac{\det \nabla F^t_{t_0} [\nabla F^t_{t_0}]^{-1} Q(t) \tilde{g}}{\det Q(t) \tilde{\nabla F^t_{t_0}} Q^T(t_0) Q(t_0) \tilde{\nabla F^t_{t_0}} Q^T(t_0)} \\
= \frac{\det \tilde{\nabla F^t_{t_0}} Q(t_0) \tilde{\nabla F^t_{t_0}}^{-1} \tilde{g}}{\det Q(t_0) \tilde{\nabla F^t_{t_0}} Q^T(t_0) Q(t_0) \tilde{\nabla F^t_{t_0}} Q^T(t_0)} \\
= Q(t_0) \tilde{b}^{t_1}_{t_0}. \tag{B4}
\]

Appendix C. Proofs of Theorems 4 and 5

C.1. Proof of Theorem 4

For a Navier-Stokes velocity field \(v\) of the form (7.1)-(7.3), we have

\[
\Delta v(x,t) = \begin{pmatrix} \Delta x \tilde{v} \\ \Delta x \tilde{\omega} \end{pmatrix}.
\]
Therefore,

\[
(F^t_{t_0})^* \Delta \mathbf{v}(x_0) = \left[ \nabla_{x_0} F^t_{t_0}(x_0) \right]^{-1} \begin{pmatrix} \Delta \hat{\omega} & \Delta \hat{\mathbf{v}} \\ \Delta \hat{\mathbf{v}} & \Delta \hat{\omega}(\hat{x}, t) \end{pmatrix}
\]

\[
= \begin{pmatrix}
\nabla_{x_0} \hat{F}^t_{t_0}(\hat{x}) & 0 \\
\int_t^{t_0} \nabla_{x_0} \hat{\omega} \left( \hat{F}^s_{t_0}(\hat{x}), s \right) ds & 1
\end{pmatrix} \begin{pmatrix} \Delta \hat{\omega} \\ \Delta \hat{\omega}(\hat{x}, t) \end{pmatrix}
\]

\[
= \begin{pmatrix}
\nabla_{x_0} \hat{F}^t_{t_0}(\hat{x}) \Delta \hat{\omega} \\
\int_t^{t_0} \nabla_{x_0} \hat{\omega} \left( \hat{F}^s_{t_0}(\hat{x}), s \right) ds \cdot \Delta \hat{\mathbf{v}} + \Delta \hat{\omega}(\hat{x}, t)
\end{pmatrix}.
\]  

(C1)

With these expressions, the barrier equation (4.1) becomes

\[
\hat{x}'_0 = \left( F^t_{t_0} \right)^* \Delta \hat{\mathbf{v}}(\hat{x}_0),
\]

\[
x'_{03} = A(\hat{x}_0, t_1, t_0),
\]

(C2)

for an appropriate smooth function \(A(\hat{x}_0, t_1, t_0)\). Two-dimensional invariant manifolds of this dynamical system are of the form \(\{x_0(s)\}_{s \in \mathbb{R}}\) × \(\mathbb{R}\), i.e., topological products of trajectories of the \(x_0\)-component of the \((7, 6)\) with a line in the \(x_{03}\) direction. As trajectories \(\{x_0(s)\}_{s \in \mathbb{R}}\) are contained in the streamlines of the steady 2D velocity field \(F^t_{t_0}\), instantaneous barriers to momentum transport are, structurally stable streamlines of the vector field \(\Delta \hat{\mathbf{v}}(\hat{x}, t)\). By incompressibility, we have

\[
\Delta \hat{\mathbf{v}} = \begin{pmatrix}
\frac{\partial^2 v_1}{\partial x_1^2} + \frac{\partial^2 v_1}{\partial x_2^2} & \frac{\partial^2 v_2}{\partial x_1 \partial x_2} + \frac{\partial^2 v_2}{\partial x_2 \partial x_1} \\
\frac{\partial^2 v_2}{\partial x_1 \partial x_2} + \frac{\partial^2 v_2}{\partial x_2^2} & \frac{\partial^2 v_1}{\partial x_1^2} + \frac{\partial^2 v_1}{\partial x_2^2}
\end{pmatrix} = \begin{pmatrix}
\frac{\partial^2 v_2}{\partial x_1 \partial x_2} - \frac{\partial^2 v_2}{\partial x_2 \partial x_1} & \frac{\partial^2 v_1}{\partial x_1^2} - \frac{\partial^2 v_1}{\partial x_2^2}
\end{pmatrix} = \begin{pmatrix}
\frac{\partial^2 v_2}{\partial x_1 \partial x_2} & \frac{\partial^2 v_1}{\partial x_1^2}
\end{pmatrix},
\]

(C3)

and hence these streamlines are structurally stable level curves of the streamfunction \(\hat{\omega}(\hat{x}, t)\), as claimed.

Using formula (C3) and the canonical symplectic matrix \(J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\), we also find that

\[
\Delta \hat{\mathbf{v}} \left( F^t_{t_0}(x_0), t \right) = J \nabla_{x_0} \hat{\omega} \left( F^t_{t_0}(x_0), t \right) = J \left[ \nabla_{x_0} F^t_{t_0}(x_0) \right]^{-T} \nabla_{x_0} \hat{\omega} \left( F^t_{t_0}(x_0), t \right),
\]

where \(\nabla_{x_0} \hat{\omega} \left( F^t_{t_0}(x_0), t \right)\) denotes the derivative of the Lagrangian vorticity \(\omega \left( F^t_{t_0}(x_0), t \right)\) with respect to the initial condition \(x_0\). This last equation implies

\[
\left( F^t_{t_0} \right)^* \Delta \hat{\mathbf{v}}(x_0) = \left[ \nabla_{x_0} \hat{F}^t_{t_0}(x_0) \right]^{-1} \Delta \hat{\mathbf{v}} \left( F^t_{t_0}(x_0), t \right)
\]

\[
= \left[ \nabla_{x_0} \hat{F}^t_{t_0}(x_0) \right]^{-1} J \left[ \nabla_{x_0} \hat{F}^t_{t_0}(x_0) \right]^{-T} \nabla_{x_0} \hat{\omega} \left( F^t_{t_0}(x_0), t \right)
\]

\[
= \det \left[ \nabla_{x_0} \hat{F}^t_{t_0}(x_0) \right]^{-1} J \nabla_{x_0} \hat{\omega} \left( F^t_{t_0}(x_0), t \right) = J \nabla_{x_0} \hat{\omega} \left( F^t_{t_0}(x_0), t \right),
\]

given that \(\det \left[ \nabla_{x_0} \hat{F}^t_{t_0}(x_0) \right]^{-1} = 1\) holds by incompressibility. Here, we have also used the fact that for any constants \(a, b, c, d \in \mathbb{R}\) satisfying \(ad - bc = 1\), we have

\[
\begin{pmatrix}
a b \\
c d
\end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} a & c \\ b & d \end{pmatrix} = \begin{pmatrix} 0 & ad - bc \\ bc - ad & 0 \end{pmatrix}.
\]

(C4)
Consequently, we have
\[
\left( \hat{F}_{t_0}^t \right)^* \Delta x \hat{v}(\hat{x}_0) = J \nabla \hat{\omega} \left( \hat{F}_{t_0}^t (\hat{x}_0), t \right),
\]
and hence the averaged Lagrangian vorticity \( \hat{\omega} \left( \hat{F}_{t_0}^t (\hat{x}_0), t \right) \) acts as a Hamiltonian (or stream function) for the \( \hat{x}_0 \)-component of eq. (C2). In summary, initial positions of material barriers to momentum transport are level curves of the time-averaged Lagrangian vorticity \( \omega \left( \hat{F}_{t_0}^t (x_0), t \right) \), as claimed.

**C.2. Proof of Theorem 5**

For \( v \) defined in (7.1) and (7.3), the full vorticity of the 3D flow is given by
\[
\omega(x, t) = (\partial_{x_2} \hat{\omega}(\hat{x}, t), -\partial_{x_1} \hat{\omega}(\hat{x}, t), \hat{\omega}(\hat{x}, t)),
\]
implying
\[
\Delta \omega = \begin{pmatrix}
\partial_{x_2} \Delta \hat{\omega} \\
-\partial_{x_1} \Delta \hat{\omega} \\
\Delta \hat{\omega}
\end{pmatrix}.
\]
In all \( x_3 = \text{const.} \) planes, therefore, the vector field \( \Delta \omega \) admits the same reduced Hamiltonian dynamics, with the Hamiltonian \( H = \Delta \hat{\omega} = \frac{1}{\nu} \frac{D}{Dt} \hat{\omega} \) acting as the stream function in that plane. With the notation \( J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \), we use the calculations in (C1) to obtain
\[
\left( \hat{F}_{t_0}^t \right)^* \Delta \omega(x_0) = \begin{pmatrix}
\nabla \Delta \hat{v}(\hat{x}_0) J \nabla \frac{D}{Dt} \hat{\omega} \left( \hat{F}_{t_0}^t (\hat{x}_0), t \right) \\
\int_{t}^{t_0} \nabla \Delta \hat{v}(\hat{x})(s) \cdot J \nabla \frac{D}{Dt} \hat{\omega} \left( \hat{F}_{t_0}^t (\hat{x}_0), t \right) + \frac{1}{\nu} \frac{D}{Dt} \hat{\omega} \left( \hat{F}_{t_0}^t (\hat{x}_0), t \right)
\end{pmatrix}.
\]
As a consequence, the first two components of the vorticity barrier equation (6.11) are
\[
\hat{x}'_0 = \nabla \Delta \hat{v}(\hat{x}_0) J \nabla \frac{D}{Dt} \hat{\omega} \left( \hat{F}_{t_0}^t (\hat{x}_0), t \right)
= \nabla \Delta \hat{v}(\hat{x}_0) J \left[ \nabla \Delta \hat{v}(\hat{x}_0) \right]^T \nabla \Delta \hat{\omega} \left( \hat{F}_{t_0}^t (\hat{x}_0), t \right). \tag{C5}
\]
Using formula (C4) again, we obtain from (C5) that 2D Lagrangian vorticity diffusion barriers must satisfy
\[
\hat{x}'_0 = J \nabla x_0 H_{t_0}^{t_1}(\hat{x}_0), \quad H_{t_0}^{t_1}(\hat{x}_0) = \frac{\hat{\omega} \left( \hat{F}_{t_0}^t (\hat{x}_0), t_1 \right) - \hat{\omega}(\hat{x}_0, t_0)}{t_1 - t_0}, \tag{C6}
\]
with \( H_{t_0}^{t_1}(\hat{x}_0) \) playing the role of a Hamiltonian for the two-dimensional \( \hat{x}_0 \)-component of the full material barrier equation, which is therefore of the general form
\[
\hat{x}'_0 = J \nabla x_0 H_{t_0}^{t_1}(\hat{x}_0), \\
x'_{03} = B(\hat{x}_0, t_1, t_0), \tag{C7}
\]
for an appropriate scalar-valued function \( G_{t_0}^{t_1}(\hat{x}_0) \). As trajectories \( \{ \hat{x}_0(s) \}_{s \in \mathbb{R}} \) are contained in the level curves of the Hamiltonian \( H_{t_0}^{t_1}(\hat{x}_0) \), we obtain the statement of Theorem 5 using the definition of \( H_{t_0}^{t_1} \) from (C6).
Appendix D. Proof of Theorem 6

To identify barrier equations for directionally steady Beltrami flows, note that the flow map for the particle motion ODE

$$\dot{x} = \alpha(t)v_0(x)$$  \hspace{1cm} (D1)

of any such flow can be computed from the flow map $G_{t_0}^\tau(x_0)$ of the autonomous ODE $x' = v_0(x)$ as

$$F_{t_0}^t(x_0) = G_{t_0}^{\tau(t)}(x_0) = G_{t_0}^{\int_{t_0}^t \alpha(s) \, ds}(x_0),$$  \hspace{1cm} (D2)

as one verifies by direct substitution of this $F_{t_0}^t(x_0)$ into (D1). Since $x' = v_0(x)$ is a solution of its equation of variations, i.e.,

$$v(G_{t_0}^\tau(x_0)) = \nabla G_{t_0}^\tau(x_0) v(x_0).$$

This implies

$$v(G_{t_0}^{\int_{t_0}^t \alpha(s) \, ds}(x_0)) = \nabla G_{t_0}^{\int_{t_0}^t \alpha(s) \, ds}(x_0) v(x_0),$$

from which we conclude, after multiplication by $\alpha(t)$ and using the definition of $F_{t_0}^t(x_0)$ in (D2), that

$$[\nabla F_{t_0}^t(x_0)]^{-1}(\alpha(t)v(F_{t_0}^t(x_0))) = \alpha(t)v(x_0).$$  \hspace{1cm} (D3)

As a consequence of the relation (D3), for a directionally steady, strong Beltrami flow, the linear momentum barrier equation (6.4) takes the specific form

$$x'_0 = b_{t_0}^{\tau_1} = -(F_{t_0}^t)^* \nabla \times (\nabla \times v) = -(F_{t_0}^t)^* \alpha(t)k^2(t)v_0(x)$$

$$= -\frac{1}{t_1 - t_0} \int_{t_0}^{t_1} k^2(t) \left[\nabla v_0(F_{t_0}^t(x_0))\right]^{-1}(\alpha(t)v_0(F_{t_0}^t(x_0))) \, dt$$

$$= -\frac{\int_{t_0}^{t_1} k^2(t)\alpha(t) \, dt}{t_1 - t_0} v_0(x_0).$$

Note that all invariant manifolds of this barrier equation coincide with invariant manifolds of the particle motion (D1) of the directionally steady Bernoulli flow defined by (D1), which proves the statement of Theorem 6 for linear momentum barriers.

With the relation (D3), the vorticity barrier equation (6.11) for directionally steady Beltrami flows takes the specific form

$$x'_0 = b_{t_0}^{\tau_1} = (F_{t_0}^t)^* \Delta \omega = -(F_{t_0}^t)^* \nabla \times (\nabla \times \omega) = -(F_{t_0}^t)^* \alpha(t)k^3(t)v_0(x),$$

$$= -\frac{1}{t_1 - t_0} \int_{t_0}^{t_1} k^3(t) \left[\nabla v_0(F_{t_0}^t(x_0))\right]^{-1}(\alpha(t)v_0(F_{t_0}^t(x_0))) \, dt$$

$$= -\frac{\int_{t_0}^{t_1} k^3(t)\alpha(t) \, dt}{t_1 - t_0} v_0(x_0).$$

Again, all invariant manifolds of this barrier equation coincide with invariant manifolds of the particle motion (D1) of the directionally steady Bernoulli velocity field $v(x,t)$, which proves the statement of Theorem 6 for vorticity barriers.

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