One-fluid Relativistic Magnetohydrodynamic Equations for a Two-fluid Plasma with the Landau–Lifshitz Radiation Reaction Force

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Received 2018 April 19; revised 2018 October 18; accepted 2018 October 18; published 2018 December 3

Abstract
We derive a set of one-fluid relativistic magnetohydrodynamic equations including the Landau–Lifshitz radiation reaction force based on a relativistic two-fluid plasma. These equations could be used in situations where the spatiotemporal scales of the plasma’s motion are sufficiently large.

Key words: acceleration of particles – accretion, accretion disks – magnetohydrodynamics (MHD)

1. Introduction
Recent observations show that there is a strong magnetic field of several hundred Gauss around the supermassive black hole residing in the center of the Milky Way (Eatough et al. 2013). The magnetic field around the black hole can be produced by the dynamo mechanism of an accretion disk made of plasma orbiting a black hole (Brandenburg et al. 1995; Hawley et al. 1996; Punsly 2001). It is clear that a neutron star can have huge surface magnetic fields of up to $10^{14}$ G (Duncan & Thompson 1992; Usov 1992; Paczyński 1992; Thompson & Duncan 1995, 1996; Vasish & Gotthelf 1997), and many neutron stars are found in a binary system in which the neutron star can accrete material from the companion star (Bhattacharya & van den Heuvel 1991). The presence of such a magnetic field can affect a test charge particle’s motion significantly. The magnetic field could cause a drastic effect on the evolution of an accretion disk around a neutron star (Hayashi et al. 1996; Miller & Stone 1997), or a stellar-mass black hole accreting material from its companion star in a binary system (Koide et al. 2002), or around a supermassive black hole in an active galactic nucleus (AGN; Yuan & Narayan 2014). Results from both magnetohydrodynamics and general relativistic magnetohydrodynamics (RMHD) simulations show that the magnetized stellar-mass black hole or supermassive black hole in an AGN accretes material from the accretion disk due to magnetorotational instability (Gammie et al. 2003; Yuan & Narayan 2014), and accretion materials flow onto the neutron star along the magnetic field line (Hayashi et al. 1996; Miller & Stone 1997; Romanova et al. 2002, 2012; Kato et al. 2004; Li et al. 2014).

In these high-energy accretion disks, it is suggested that the disks are made of ion–electron plasma (Ford et al. 1994). The plasma disk is often described by RMHD equations without the effects of Landau–Lifshitz radiation reaction, which may play an important role when the spatiotemporal scales of the plasma’s motion are sufficiently large. Aside from the particle-in-cell method (Chen & Beloborodov 2014; Philippov & Spitkovsky 2014; Belyaev 2015; Cerutti et al. 2015; Philippov et al. 2015a, 2015b; Cerutti et al. 2016), which incorporates radiation reaction forces, applications of radiation reaction to the fluid dynamics perspective have been studied by Tam & Kiang (1979) and Berezhiani et al. (2004, 2008). Here, with the one-fluid description using a relativistic two-fluid approximation of plasma consisting of positively and negatively charged particles, we derive the one-fluid RMHD equations of a two-fluid plasma in which the Landau–Lifshitz radiation reaction force is included.

2. Equations Derived
In this work, the background spacetime is assumed to be Minkowski spacetime defined by $ds^2 = -d\sigma^2 + (dx^1)^2 + (dx^2)^2 + (dx^3)^2 = \eta_{\mu\nu} dx^\mu dx^\nu$, where $(x^0, x^1, x^2, x^3) = (t, x, y, z)$. We set $c = 1$, $\epsilon_0 = 1$, $\mu_0 = 1$ which represent the speed of light, the dielectric constant, and the magnetic permeability in vacuum, and all are unity.

To derive the relativistic one-fluid equations of a two-fluid plasma with the radiation reaction force, we define the average and difference variables similarly to Koide (2009). The two-fluid plasma consists of positively charged particles, each with charge $e$ and mass $m_+$, and of negatively charged particles, each with charge $-e$ and mass $m_-$. Here we list the four-velocity and four-current density as follows:

$$U^\mu = \frac{1}{\rho}(m_+ u_+^\mu + m_- u_-^\mu),$$

$$J^\mu = e(n_+ u_+^\mu - n_- u_-^\mu),$$

where $m = m_+ + m_-$ and $\rho = n_+ m_+ + n_- m_-$. The generalized RMHD equations for a two-fluid plasma with the Maxwell equations are given as follows (Koide 2009):

$$\partial_\nu(\rho U^\nu) = 0,$$

$$\partial_\nu \left( h \left( U^\nu U^\nu + \frac{\mu}{q^2} J^\nu J^\nu \right) \right) = -\partial\mu p + J^\nu F_{\nu\lambda}^\lambda,$$

$$\frac{1}{q} \partial_\nu \left[ \frac{\mu h}{q} (U^\nu J^\nu + J^\nu U^\nu) - \frac{2\mu \Delta \mu h}{q^2} J^\nu J^\nu \right] = \frac{1}{2q} \partial\nu (\Delta \mu p - \Delta \rho) + \left( U^\nu - \frac{\Delta \mu h}{q} \right) F_{\nu\lambda}^{\lambda\nu}$$

$$- \eta [J^\mu + Q (1 + \Theta) E_{\mu\nu}],$$

$$\nabla_{\nu} F^{\nu\mu} = 0,$$

$$\nabla_{\nu} F_{\nu\mu} = J^\mu,$$

where $\mu = m_+ m_- / m^2$; $\Delta \mu = (m_+ - m_-) / m$; $p = p_+ + p_-$; $\Delta \rho = p_+ - p_-$; $q = ne; n = \frac{m}{e}$; $F_{\mu\nu}$ is the electromagnetic field tensor; $*F_{\mu\nu}$ is the dual tensor density of $F_{\mu\nu}$; $\nu = 0, 1, 2, 3$; $Q = U^\nu J_\nu$; and $\Theta$ is the thermal energy exchange rate from the
negatively charged fluid to the positively charged one (see Appendix A of Koide 2009 for details). Before deriving the main equations, we begin with the dynamical equation for a charged particle of the mass \( m \) with the electric charge \( e \) in a background magnetic field. This dynamical equation reads

\[
m \frac{du^\mu}{ds} = eF^{\mu\nu}u_\nu + g^\mu, \tag{8}
\]

where \( u_\mu \) is the four-velocity and \( \mu = 0, 1, 2, 3 \).

The radiation reaction force in the Lorentz–Abraham–Dirac form is given by

\[
g^\mu = \frac{2e^2}{3m} \left[ \frac{d^2u^\mu}{ds^2} - \nu^\mu \nu v^\nu \frac{d^2u^\nu}{ds^2} \right]. \tag{9}
\]

We follow Landau and Lifshitz (LL; Landau & Lifshitz 1975) assuming that \( g^\mu \) is small compared with the Lorentz force in the instantaneous rest frame of the charged particle, allowing us to express Equation (9) as follows:

\[
g^\mu = \frac{2e^2}{3m} \times \left\{ \frac{\partial F_{\mu\nu}}{\partial x_\lambda} u_\mu u_\nu - \frac{e}{m} [F_{\mu\nu} F_{\rho\sigma} u^\rho (F_{\nu\sigma} u_\sigma)] - (F_{\mu\lambda} u^\lambda)(F_{\nu\lambda} u_\lambda) u^\nu \right\}, \tag{10}
\]

where the first term is negligible compared to the other terms. The reason is that the force, called the Frenkel force (Walser & Keitel 2001), acting on a charged particle with a spin degree of freedom moving in an external electromagnetic field—in the case of a plane wave, for example—is quite larger than the first term in Equation (10). In the regime where the classical equation of motion is adopted without the Frenkel force from quantum effects, therefore, it is customary to ignore the first term in Equation (10) (Cerutti et al. 2016; Tamburini et al. 2010). Then, Equation (8) becomes

\[
m \frac{du^\mu}{ds} = eF^{\mu\nu}u_\nu - \frac{2e^2}{3m} \times \left[ F^{\mu\nu} F_{\rho\sigma} u^\rho (F_{\nu\sigma} u_\sigma) u^\nu \right]. \tag{11}
\]

When inserting the radiation reaction force into the generalized RMHD equations, we can obtain the total radiation reaction force on the two species of the two-fluid plasma. To do so, we must get \( u^\mu_{\pm} \) and \( n_{\pm} \), that is to say, 10 unknowns. Equations (1) and (2) are only eight equations. To solve this, we can combine two additional equations as follows:

\[
\gamma_{\pm} = u^0_{\pm}, \tag{12}
\]

where

\[
\gamma_{\pm} = \frac{1}{\sqrt{1 - \left( \frac{\gamma^0_{\pm}}{c} \right)^2 + \left( \frac{\gamma^1_{\pm}}{c} \right)^2 + \left( \frac{\gamma^2_{\pm}}{c} \right)^2}}.
\]

Using Equations (1) and (2), we get

\[
n_{\pm} u^0_{\pm} = \frac{1}{m} \rho U^0_{\pm} + \frac{m_{\pm} e}{e} J^0_{\pm}. \tag{13}
\]

Combined with Equation (12), we can get \( u^\mu_{\pm} \) and \( n_{\pm} \) as follows:

\[
n_{\pm} = \gamma_{\pm} \frac{\left( \rho U^0_{\pm} \pm \frac{m_{\pm} e}{e} J^0_{\pm} \right)^2 - \sum_{n=1}^3 \left( \rho U^n_{\pm} \pm \frac{m_{\pm} e}{e} J^n_{\pm} \right)^2}{m}, \tag{14}
\]

\[
u_{\pm} = \gamma_{\pm} \frac{\left( \rho U^0_{\pm} \pm \frac{m_{\pm} e}{e} J^0_{\pm} \right)^2 - \sum_{n=1}^3 \left( \rho U^n_{\pm} \pm \frac{m_{\pm} e}{e} J^n_{\pm} \right)^2}{m}.
\]

Then the radiation reaction force of the two species in the fluid element is

\[
J^{\mu}_{RR\pm} = -\frac{2n_{\pm} e^4}{3m^2_{\pm}} \left[ F^{\mu\rho} F_{\rho\lambda} u^\lambda_{\pm} - (F_{\rho\lambda} u^\lambda_{\pm}) (F^{\sigma\nu} u_{\sigma\nu}) u^\mu_{\pm} \right]. \tag{16}
\]

Inserting the fluid element’s radiation reaction force, Equation (16), into the momentum density equation, Equation (4), we get the momentum density equation with the radiation reaction force as

\[
\partial_{\nu} \left[ \frac{h}{q} \left( U^{\mu} U^\nu + \frac{\mu}{q} F^{\mu\nu} \right) \right] = -\partial_{\mu} p + J^{\mu}_{RR\nu},
\]

\[
-2n_{\pm} e^4 \left[ F^{\mu\rho} F_{\rho\lambda} u^\lambda_{\pm} - (F_{\rho\lambda} u^\lambda_{\pm}) (F^{\sigma\nu} u_{\sigma\nu}) u^\mu_{\pm} \right], \tag{17}
\]

The generalized RMHD equations for the ion–electron plasma with only the electron’s radiation reaction force (the ion’s radiation reaction force is negligible due to the relatively large \( m_+ \)) are given as follows in the limit \( m \approx m_+ \gg m_- \) and assuming \( \Theta = 0 \):

\[
\partial_{\nu} \left( \rho U^{\nu} \right) = 0, \tag{18}
\]

\[
\partial_{\nu} (h U^{\nu} U^\nu) = -\partial_{\mu} p + J^{\mu}_{F^{\mu\nu}} - \frac{2n_{-} e^4}{3m^2_{-}} \times \left[ F^{\mu\rho} F_{\rho\lambda} u^\lambda_{-} - (F_{\rho\lambda} u^\lambda_{-}) (F^{\sigma\nu} u_{\sigma\nu}) u^\mu_{-} \right], \tag{19}
\]

\[
\frac{m_{-} \partial_{\nu}}{mq} \left[ \frac{h}{q} \left( U^{\mu} J^\nu + J^{\mu} U^\nu \right) \right] = \frac{1}{q} \partial_{\mu} p_{-} + \left( U^{\nu} - \frac{1}{q} J^{\nu} \right) F_{\nu\lambda} - \frac{1}{q} J^{\nu} + Q U^{\nu}, \tag{20}
\]

\[
\partial_{\nu} F^{\mu\nu} = 0, \tag{21}
\]

\[
\partial_{\nu} J^{\mu\nu} = J^{\mu\nu}. \tag{22}
\]

The generalized RMHD equations for a pair plasma with the radiation reaction force, where \( m_+ = m_- = m_e \) (\( m_e \) is the mass of electron), are given by

\[
\partial_{\nu} \left( \rho U^{\nu} \right) = 0, \tag{23}
\]

\[
\partial_{\nu} \left[ \frac{h}{q} \left( U^{\mu} J^\nu + J^{\mu} U^\nu \right) \right] = -\partial_{\mu} p + J^{\mu}_{F^{\mu\nu}} - \frac{2n_{+} e^4}{3m^2_{+}} \times \left[ F^{\mu\rho} F_{\rho\lambda} u^\lambda_{+} - (F_{\rho\lambda} u^\lambda_{+}) (F^{\sigma\nu} u_{\sigma\nu}) u^\mu_{+} \right], \tag{24}
\]

\[
\frac{1}{4q} \partial_{\nu} \left[ \frac{h}{q} \left( U^{\mu} J^\nu + J^{\mu} U^\nu \right) \right] = \frac{1}{2q} \partial_{\nu} \Delta p + U^{\nu} J^{\mu\nu} - \frac{1}{q} J^{\nu} + Q \left( 1 + \Theta \right) U^{\nu} \tag{25}
\]

\[
\partial_{\nu} F^{\mu\nu} = 0, \tag{26}
\]
When the inertia of the current density, the thermal electromotive force, and the Hall effect are ignored in Equations (20), Equations (18)–(22) change to the standard RMHD equations with the radiation reaction force:

\[ \partial_t (\rho U^\mu) = 0, \]

\[ \partial_t (\hbar U^\mu U^\nu) = -\partial_t p + J^\mu F^\nu - \frac{2n_e e^4}{3m_e^3}, \]

\[ \times [F^\rho_{\lambda\mu} \partial_{\lambda} U^\nu - (F^\rho_{\lambda\mu} \partial_{\lambda} U^\nu) U^\rho], \]

\[ U^\mu F^\nu_{\mu} = \gamma [U^\mu + QU^\mu].\]

The Landau–Lifshitz radiation reaction force for an electron in Equation (10) is valid when (Cerutti et al. 2012)

\[ \frac{\gamma_e B}{B_e} \ll 1, \]

that is to say, the magnetic field in the electron’s rest frame should not exceed the classical critical magnetic field strength \( B_e = m_e^2 c^3/e^3 \approx 6 \times 10^{15} \text{ G} \), above which the Larmor radius of an electron in the electron’s rest frame \( m_e c^2/eB \) becomes smaller than \( \hbar/\gamma \). But quantum effects could be important at even smaller magnetic fields, \( B_{\text{QED}} = m_e^2 c^3/\hbar e = \alpha_F B_e \approx 4.4 \times 10^{13} \text{ G} \). Thus, Equation (33) should be updated to (Cerutti et al. 2012)

\[ \frac{\gamma_e B}{B_{\text{QED}}} \ll 1. \]

In the four-dimensional form, Equation (10) can be written as

\[ g^0 = \frac{2e^4}{3m_e^2} u^0 \{E \cdot (E + v \times B) \}

- \frac{2e^4}{3m_e^2} u^0 \{(E + v \times B)^2 - (v \cdot E)^2\}, \]

\[ g_i = \frac{2e^4}{3m_e^2} u^0 \{E \times B + (B \times (B \times v)) + E(v \cdot E) \}

- \frac{2e^4}{3m_e^2} u^0 \{(E + v \times B)^2 - (v \cdot E)^2\}, \]

where \( i = 1, 2, 3, v = (v^1, v^2, v^3) \), and \( v' = \frac{v}{\gamma} \).

Thus, the ratio of the radiation reaction force and the Lorentz force acting on an electron is (Cerutti et al. 2012)

\[ \left| \frac{F_{\text{RR}}}{\left| F_i \right|} \right| \sim \frac{\gamma_e^2 B}{B_e} = \alpha_F \gamma_e \frac{\gamma_e B}{B_{\text{QED}}}. \]

Taking the accretion of rotating gas onto a newly born stellar-mass black hole in a gamma-ray burst for instance, the magnetic field can reach up to \( 10^{15} \text{ G} \) and \( \gamma \sim 10^2\text{–}10^3 \). When we take \( B = 10^8 \text{ G} \) and \( \gamma = 10^3 \), then \( \frac{\gamma_B}{B_{\text{QED}}} \approx 0.00227 \) from Equation (34), but \( \left| \frac{F_{\text{RR}}}{\left| F_i \right|} \right| \sim \frac{\gamma_B}{B_c} \approx 0.016 \) from Equation (37).

When we take \( B = 10^{10} \text{ G} \) and \( \gamma = 10^3 \), then \( \frac{\gamma_B}{B_{\text{QED}}} \approx 0.227 \) and \( \left| \frac{F_{\text{RR}}}{\left| F_i \right|} \right| \sim \frac{\gamma_B}{B_c} \approx 1.6 \). From the above, the radiation reaction force acting on an electron in a plasma could have a significant effect on the dynamics of the electron, further having a significant effect on the current density in the plasma. There exist situations wherein an electron with ultrarelativistic velocity experiences a Landau–Lifshitz radiation reaction force that is comparable to the Lorentz force in the laboratory frame, while being much smaller in the electron’s instantaneous rest frame (Landau & Lifshitz 1975).

Expanding Equations (3) and (17) and Equations (6)–(7) including the Landau–Lifshitz radiation reaction force acting on arbitrary masses of positively and negatively charged particles, \( m_+ \) and \( m_- \), combined with Equations (35) and (36), and assuming the ideal RMHD regime, we get the following one-fluid RMHD equations for a two-fluid plasma:

\[ \frac{\partial (\rho U^\mu)}{\partial t} + \nabla \cdot (\gamma \rho U^\mu) = 0, \]

\[ \frac{\partial R}{\partial t} + \nabla \cdot F = \frac{2n_e e^4}{3m_e^2} u^0 \{E \times B + (B \times (B \times v)) + E(v \cdot E)\}

- \frac{2n_e e^4}{3m_e^2 (1 - v^2)} \{ (E + v \times B)^2 - (v \cdot E)^2 \}

+ \frac{2n_e e^4}{3m_e^2} u^0 \{E \times B + (B \times (B \times v)) + E(v \cdot E)\}

- \frac{2n_e e^4}{3m_e^2 (1 - v^2)} \{ (E + v \times B)^2 - (v \cdot E)^2 \}, \]

\[ \frac{\partial B}{\partial t} = -\nabla \times E, \]

\[ \nabla \cdot E = \rho_e, \]

\[ \nabla \cdot B = 0, \]

\[ J + \frac{\partial E}{\partial t} = \nabla \times B, \]

\[ E + v \times B = 0, \]

\[ u_{\pm} = \frac{\left( \rho \gamma \pm \frac{m_{\pm}}{e} J \right)}{\sqrt{\left( \rho \gamma \pm \frac{m_{\pm}}{e} J \right)^2 - \sum_{n=1}^{3} (\rho \gamma v^n \pm \frac{m_{\pm}}{e} J)^2}}. \]
\[ u_±^0 = \frac{(\gamma \rho \pm m_c \rho c)^2}{\sqrt{2 \sum_{n=1}^{\infty} \left( (\gamma \rho)^{2n} \pm \frac{m_c}{\rho} \right)}}, \] (47)

\[ \nu_± = \frac{\rho \gamma v \pm m_c \gamma J}{\sqrt{2 \sum_{n=1}^{\infty} \left( (\gamma \rho)^{2n} \pm \frac{m_c}{\rho} \right)}}, \] (48)

\[ n_± = \frac{(\gamma \rho \pm m_c \rho c)^2}{\sqrt{2 \sum_{n=1}^{\infty} \left( (\gamma \rho)^{2n} \pm \frac{m_c}{\rho} \right)}}, \] (49)

where \( R = \gamma^2 (e + P) - \frac{\mu (e + P) J}{(\rho^2 c^2)} + E \times B; F = \left[ (P + \frac{B^2 + E^2}{2}) \right] + 2 \gamma^2 (e + P) - \frac{\mu (e + P) J}{(\rho^2 c^2)} - BB - EE \],

\[ \varepsilon = \gamma^2 (e + P) + \frac{\mu (e + P) J}{(\rho^2 c^2)} - P + \frac{B^2 + E^2}{2}, \] where \( \gamma = \frac{1}{\sqrt{1 - \left( \frac{v^2}{c^2} \right)}} \) is the Lorentz factor; and \( e = \rho + \frac{\mu}{\gamma^2}, \) where \( \Gamma \) is the specific heat ratio.

To get a simplified set of the above equations, we substitute Equations (46)–(49) into Equations (39)–(40), and we get the equations in Appendix A. Noting that \( m_+ = m_+ - m_0 \) is the normalized reduced mass, and \( \Delta \mu_+ = -m_0 \Delta \mu + m_0^2 / m_0 \) is the normalized mass difference, we substitute these into Equations (71)–(72) in Appendix A, we could get the equations with \( m_+ \), \( \mu \), and \( \Delta \mu \) instead of \( m_+ \) and \( \mu \) in Appendix B.

The one-fluid RMHD description of two-fluid plasmas with the Landau–Lifshitz radiation reaction force has been accomplished. The one-fluid description of such two-fluid plasmas with the Landau–Lifshitz radiation reaction force in curved spacetime can also be achieved through the same method described above.

To compare the Lorentz force with the Landau–Lifshitz radiation reaction force acting on the fluid element in extreme astrophysical situations—a jet along the magnetic field line in a gamma-ray burst, for example—we first get the Landau–Lifshitz radiation reaction force from the right side of Equation (71) in Appendix A with the conditions that the relativistic ion–electron plasma is neutral and that the flow is parallel to the magnetic field line, which is perpendicular to the current density for simplicity (see Appendix C for the detailed derivation. Here, for the sake of quantitative analysis, we conduct the derivation without setting \( c = 1 \):)

\[ F_{L+} = \frac{2}{3} r_e^0 \left[ \left( B \times \left( \frac{J}{c} \times B \right) \right) + \left( \frac{v}{c} \times B \right) \left( \frac{v}{c} \times B \right) \right] - \frac{2}{3} n e^2 \gamma^2 \left( \frac{J \times B}{c n} \right)^2 - \left( \frac{J \times B}{c n} \right) \left( \frac{v}{c} \times \frac{J \times B}{c n} \right). \] (50)

where \( r_e = \frac{e^2}{m_e c^2}, n = \frac{\gamma n_r}{m}, \) for ion–electron plasma, and \( c \) is the speed of light.

In ultrarelativistic conditions, the second term is larger than the first due to \( \gamma^2 \) in Equation (50), and we find that the first term can be reduced to

\[ \frac{2}{3} B_c \left( \frac{J \times B}{c} \right) \left( \frac{v - \gamma v}{c} \right), \] (51)

where \( B_c = m_e^2 e^4 / c^3 \).

Then, the ratio of formula (51) to the Lorentz force is \( \frac{2B}{3B_c} \). The value of the second term is reduced to

\[ \frac{2}{3} n e^2 \gamma^2 \left( \frac{J \times B}{c n} \right)^2, \] (52)

and its ratio to the Lorentz force is

\[ \frac{2}{3} e \gamma^2 |jB| \sim \frac{2}{3} e \gamma^2 \left( \frac{v - \gamma v}{c} \right), \] (53)

where we assume \( \gamma \approx \gamma_i \), and the current density is perpendicular to the magnetic field for simplicity, and \( v \) and \( \gamma_i \) are the velocity and Lorentz factor of electron fluid element, respectively.

We find from Equation (53) that if \( v - \gamma_i v \) in the fluid element is mildly relativistic, the radiation reaction force can be comparable to the Lorentz force when \( B = 10^{10} G \) and \( \gamma = 10^3 \), for example.

In the work by Cerutti et al. (2013), in which the radiation reaction force is included to investigate particle acceleration in ultrarelativistic pair plasma reconnection using the particle-in-cell method, with the initial \( E_0 = 0, B_0 = 5 mG \), and background particles with Lorentz factor from \( \gamma = 4 \times 10^7 \) to \( \gamma = 4 \times 10^8 \) with a power-law index of 2, we get the value of the Landau–Lifshitz radiation reaction force acting on the positive and negative particles in the fluid element, \( \frac{2}{3} n e^2 \gamma^2 B_0^2 \).

With the Lorentz force whose value is \( n_e B_0 \) for the positive and negative particles in the fluid element, then we get the ratio of the radiation reaction force to the Lorentz force, \( \frac{2}{3} n e^2 \gamma^2 \sim \frac{10^2}{2} \) for both positive and negative particles from the fluid element when taking \( \gamma = 4 \times 10^8 \). During the pair plasma evolution in Cerutti et al. (2013), the Lorentz factor of the background particle can reach \( 10^9 \), which may show that the radiation reaction force acting on positive or negative particles in the fluid element can be comparable to the Lorentz force acting on it.

By considering the relativistic fluid in the rest frame, we investigate waves propagating in a rest plasma embedded in a uniform magnetic field \( B_0 = (0, 0, B_0) \). Then, we linearize these equations by separating the variables into background fields with subscripts 0 and perturbations with tilde symbols proportional to \( \exp(\mathbf{k} \cdot \mathbf{r} - \omega t) \), where the wave vector \( k = (k_0, 0, 0) \) and \( \omega \) is the frequency. The background conditions are \( v_0 = 0, \rho_0 = 0, E_0 = 0, \) and \( J_0 = 0 \), and we assume the plasma is neutral everywhere. Then, the linearized equations of perturbations are given by

\[ \frac{\partial}{\partial t} \tilde{\rho} + \rho_0 \nabla \cdot \tilde{v} = 0, \] (54)

\[ h_0 \frac{\partial}{\partial t} \tilde{v} = -c^2 \nabla \tilde{\rho} + c J \times B_0 + \frac{2}{3} \tilde{\rho} \gamma_e^2 B_0 \times (J \times B_0), \] (55)

\[ \nabla \cdot \tilde{J} = 0, \] (56)
\[
\begin{align*}
\frac{\partial}{\partial t} \vec{B} &= -c \nabla \times \vec{E}, \\
\vec{J} + \frac{\partial}{\partial t} \vec{E} &= c \nabla \times \vec{B}, \\
(\rho_0/\rho_0)(\vec{p}/\vec{p}) &= \Gamma, \\
\vec{E} + \frac{\vec{\nu}}{c} \times \vec{B}_0 &= 0,
\end{align*}
\]

where we set \( \Gamma = \frac{4}{3} \).

Substituting perturbations proportional to \( \exp(ik \cdot x - \omega t) \) into the above equations, we get
\[
-\omega \vec{p} + i \rho_0 k \cdot \vec{\nu} = 0,
\]
\[
-\omega \vec{h}_0 \vec{\nu} = -i k c^2 \vec{p} + c \vec{J} \times \vec{B}_0 + \frac{2 \gamma^2 c}{3} e \vec{B}_0 \times (\vec{J} \times \vec{B}_0),
\]
\[
\vec{J} = c \vec{k} \times \vec{B} + \frac{1}{c} i \omega \vec{B},
\]
\[
\vec{E} = \frac{1}{c} k \vec{K} \times \vec{B},
\]
\[
\vec{E} + \frac{\vec{\nu}}{c} \times \vec{B}_0 = 0.
\]

Using \( k \times \vec{B} \) combined with Equation (64), we get \( \vec{B} \). Substituting \( \vec{E} \) from Equation (67) and \( \vec{B} \) into Equation (65) yields \( \vec{J} \). Substituting \( \vec{p} \) from Equation (61) into Equation (66), we get \( \vec{p} \). Then, substituting \( \vec{J} \) and \( \vec{p} \) into Equation (62), we get the dispersion relation
\[
\omega^2 = \frac{(\Gamma \rho_0 c^2 + B_0^2 c^2)(h_0 + B_0^2)}{(h_0 + B_0^2)^2 + \left(\frac{2 \gamma^2}{3} e\right) B_0^6} k^2
\]
and the normalized phase velocity
\[
(v_{ph})^2 = \left(\frac{\omega}{c k_c}\right)^2 = \frac{(\Gamma \rho_0 + B_0^2)(h_0 + B_0^2) + \left(\frac{2 \gamma^2}{3} e\right) B_0^6}{(h_0 + B_0^2)^2 + \left(\frac{2 \gamma^2}{3} e\right) B_0^6}.
\]

Note that \( \Gamma \rho_0 + \frac{B_0^2}{h_0 + B_0^2} < 1 \), so we find that
\[
\frac{(\Gamma \rho_0 + \frac{B_0^2}{h_0 + B_0^2})(h_0 + B_0^2) + \left(\frac{2 \gamma^2}{3} e\right) B_0^6}{(h_0 + B_0^2)^2 + \left(\frac{2 \gamma^2}{3} e\right) B_0^6} > \Gamma \rho_0 + \frac{B_0^2}{h_0 + B_0^2}
\]
meaning that the normalized phase velocity with the Landau–Lifshitz radiation reaction force is larger than that without it.

3. Discussion

The one-fluid RMHD equations using a relativistic two-fluid approximation of plasma consisting of positively and negatively charged particles in which the Landau–Lifshitz radiation reaction force is included provide a detailed description containing a self-consistent expression of the Landau–Lifshitz radiation reaction force acting on the charged particles in the plasma. The equations derived in this work is a natural generalization of modern RMHD and could be used in many situations in relativistic plasmas.

The radiation reaction force has been ignored in most previous studies due to its small value compared to the Lorentz force, and thus, is not expected to have a major effect on the dynamics of plasma. This assumption may not be so suitable for situations in some astrophysical conditions where the spatiotemporal scales of the plasma’s motion are so sufficiently large that the effect of radiation reaction could not be ignored, like the work by Cerutti et al. (2013) in which the radiation reaction force is included to investigate particle acceleration in ultrarelativistic pair plasma reconnection using the particle-in-cell method. In astrophysical conditions such as those present in a gamma-ray burst, the inner region of the accretion disk around a neutron star or black hole, or the inner region of AGNs where plasma motion is relativistic and the background magnetic field is large, the radiation reaction may not be negligible; then, the one-fluid RMHD description of two-fluid plasmas with the Landau–Lifshitz radiation reaction force derived in this work could be applied to these astrophysical conditions for simulations. Here, from a qualitative point of view, we may find some distinctive phenomena from extreme astrophysical conditions when using the equations in this work. Consider an electron moving in a stable orbit around a black hole immersed in a large magnetic field perpendicular to the orbital plane of the electron. Due to the radiation reaction force, the kinetic energy of the electron is decreased. Depending on the orientation of the Lorentz force on the electron with respect to the black hole, the radiation reaction can lead to the charged particle falling into the black hole when the direction of the Lorentz force is toward the black hole; otherwise, the orbit of the electron will remain bounded while oscillations are decaying when the Lorentz force is pointing out of the black hole (Tursunov et al. 2018). When considering the accretion disk around such a black hole, we can expect that the accretion rate of material flowing into the black hole with radiation reaction will be different from that without radiation reaction. The numerical simulation using equations derived in this work applied to these astrophysical conditions will be conducted in the future.

We are very grateful to the anonymous referee for the instructive comments, which improved the content of the paper. This work is supported by the National Key Research and Development Program of China (No. 2017YFA0402703). This work has been supported by the National Science Foundations of China (Nos. 11373024, 11233003, and 11873032).

Appendix A

When we substitute Equations (46)–(49) into Equations (39)–(40), we get the following relativistic magnetohydrodynamic equations containing the Landau–Lifshitz radiation reaction force with \( m_e \) and \( m_— \):
\[ \frac{\partial \mathbf{R}}{\partial t} + \nabla \cdot \mathbf{F} = \frac{2e^4}{3m^2} \left( \rho \gamma - \frac{m_e}{e} \rho_e \right) \]
\[ \times \left( E \times B + \left( B \times \left( \mathbf{B} \times \left( \frac{\rho \gamma v - \frac{m_e}{e} \mathbf{J}}{\rho \gamma - \frac{m_e}{e} \rho_e} \right) \right) \right) \right) \]
\[ + E \left( \frac{\rho \gamma v - \frac{m_e}{e} \mathbf{J}}{\rho \gamma - \frac{m_e}{e} \rho_e} \right) \cdot E \right\} \]
\[ - \frac{2e^4}{3m^2} \left( \rho \gamma - \frac{m_e}{e} \rho_e \right) \]
\[ \times \left( \left\{ E + \left( \frac{\rho \gamma v - \frac{m_e}{e} \mathbf{J}}{\rho \gamma - \frac{m_e}{e} \rho_e} \right) \times B \right\}^2 - \left( \frac{\rho \gamma v - \frac{m_e}{e} \mathbf{J}}{\rho \gamma - \frac{m_e}{e} \rho_e} \right) \cdot E \right\} \]
\[ + \frac{2e^4}{3m^2} \left( \rho \gamma + \frac{m_e}{e} \rho_e \right) \]
\[ \times \left( E + \left( \frac{\rho \gamma v - \frac{m_e}{e} \mathbf{J}}{\rho \gamma + \frac{m_e}{e} \rho_e} \right) \times B \right) \]
\[ - \left( \frac{\rho \gamma v + \frac{m_e}{e} \mathbf{J}}{\rho \gamma + \frac{m_e}{e} \rho_e} \right) \cdot E \right\} \left\{ \frac{\rho \gamma + \frac{m_e}{e} \rho_e}{\rho \gamma + \frac{m_e}{e} \rho_e} \cdot E \right\} \] (71)

\[ \frac{\partial E}{\partial t} + \nabla \cdot \mathbf{E} = \rho_e, \] (73)
\[ \nabla \cdot \mathbf{B} = 0, \] (74)
\[ \frac{\partial B}{\partial t} = -\nabla \times \mathbf{E}, \] (75)
\[ \mathbf{J} + \frac{\partial \mathbf{E}}{\partial t} = \nabla \times \mathbf{B}, \] (76)
\[ \mathbf{E} + \nu \times \mathbf{B} = 0, \] (77)

where \( m_+ = \frac{m_{\Delta \mu} + \sqrt{m_{\Delta \mu}^2 + 4m^2\mu}}{2} \) and \( m_- = \frac{m_{\Delta \mu} - \sqrt{m_{\Delta \mu}^2 + 4m^2\mu}}{2} \).

**Appendix B**

When we substitute \( m_+ \) and \( m_- \) into Equations (71)–(77) in Appendix A with the relation \( m = m_+ + m_- \), \( \mu = \frac{m_- m_+}{m} \), and \( \Delta \mu = m_--m_+ \), then we get the following relativistic magnetohydrodynamic equations containing the Landau–Lifshitz radiation reaction force with \( m, \mu, \) and \( \Delta \mu \):

\[ \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0, \]

\[ \frac{\partial \mathbf{v}}{\partial t} + \nabla \cdot \mathbf{F} = \frac{2e^4}{3m} \left\{ \frac{(1 - 2\mu)\rho \gamma}{m^2 \mu^2} \right\} \]
\[ \times \left[ E \times B + B \times (B \times \mathbf{v}) + (\mathbf{v} \times \mathbf{E}) \right] \]
\[ - \frac{\Delta \mu (1 - \mu)}{m \mu^2 \varepsilon} \left\{ \rho_e (E \times B) + B \times (B \times J) + (J \cdot E) \right\} \]
\[ - \frac{2e^4}{3m^2 \mu^2} \left\{ (a_1 b_1 + a_2 b_2 m^2 \mu^2) - a_3 b_3 m^2 \mu \right\} \]
\[ \times [\rho \gamma v (1 - 2\mu) - \frac{J}{\mu} ((\Delta \mu m (1 - \mu))] \]
\[ + (a_3 b_2 m^2 \mu - a_1 b_3) \]
\[ \times \left[ \rho \gamma v ((m - 2\mu) - \frac{J}{\mu} ((m - 2\mu) - 2m^2 \mu)] \right] \]
\[ + (a_3 b_1 - a_2 b_3 m^2 \mu) [\rho \gamma v (m - 2\mu) - \frac{J}{\mu} ((m - 2\mu) - 2m^2 \mu)] \]
\[ + a_2 b_1 \left[ \frac{\rho \gamma v ((m - 2\mu) - 2m^2 \mu - \frac{J}{\mu} ((m - 2\mu) - 2m^2 \mu)] + \frac{J}{\mu} ((m - 2m^2 \mu + m^2 \mu)] \right] \]
\[ + a_2 b_1 m^2 \mu^2 \left[ \frac{2\rho \gamma v - \frac{J}{\mu} m \Delta \mu}{[a_1^2 + a_1 a_2 (m^2 - 2m^2 \mu)] - a_1 a_3 m \Delta \mu + a_2 a_3 m^2 \Delta \mu + a_2^2 m^4 \mu^2 - a_2^2 m^2 \mu} \right] \].
\[
\frac{\partial \varepsilon}{\partial t} + \nabla \cdot \mathbf{R} = \frac{2e^4}{3m_i^3 c^4} E \\
\cdot [\rho_\gamma (E + \mathbf{v} \times \mathbf{B})(1 - 2\mu) - \frac{\rho_b E + \mathbf{J} \times \mathbf{B}}{e} (\Delta \mu m (1 - \mu))]
\]
\[
- \frac{2e^4}{3m_i^3 c^4} \left( (a_1 b_1 + a_2 b_2 m^4 \mu^2 - a_3 b_3 m^2 \mu) \right)
\times [\rho_\gamma (1 - 2\mu) - \frac{\rho_e}{e} (\Delta \mu m (1 - \mu))]
\]
\[
+ (a_3 b_2 - a_2 b_3) [\rho_\gamma (\Delta \mu m (1 - \mu)) - \frac{\rho_e}{e} (m^2 - 2m^2 \mu)]
\]
\[
+ \int_{a_1}^{a_1^2} \left( [2\rho_\gamma - \frac{\rho_e}{e} m \Delta \mu] \right) / [a_1^2 + a_2 a_3 (m^2 - 2m^2 \mu)]
\]
\[
- a_3 a_3 \Delta \mu + a_2 a_3 m^2 \mu + a_2^2 m^2 \mu - a_3^2 m^2 \mu).
\]

\[
\nabla \cdot \mathbf{E} = \rho_e,
\]
\[
\nabla \cdot \mathbf{B} = 0,
\]
\[
\frac{\partial \mathbf{B}}{\partial t} = - \nabla \times \mathbf{E},
\]
\[
\mathbf{J} + \frac{\partial \mathbf{E}}{\partial t} = \nabla \times \mathbf{B},
\]
\[
\mathbf{E} + \mathbf{v} \times \mathbf{B} = 0,
\]

where
\[
a_1 = \rho^2 \gamma^2 - \rho^2 \gamma^2 \nu^2
\]
\[
a_2 = \frac{\rho_e^2}{e^2} - J^2
\]
\[
a_3 = 2\rho_\gamma \rho_e - 2\rho_\gamma \nu \cdot \mathbf{J}
\]
\[
b_1 = \rho^2 \gamma^2 (E + \mathbf{v} \times \mathbf{B})^2 - \rho^2 \gamma^2 (\nu \cdot \mathbf{E})^2
\]
\[
b_2 = (\rho_b E + \mathbf{J} \times \mathbf{B})^2 - (\mathbf{J} \cdot \mathbf{E})^2
\]
\[
b_3 = 2\rho^2 (E + \mathbf{v} \times \mathbf{B}) \cdot (\rho_b E + \mathbf{J} \times \mathbf{B}) - 2\rho_\gamma (\nu \cdot \mathbf{E}) (\mathbf{J} \cdot \mathbf{E}).
\]

**Appendix C**

The Landau–Lifshitz radiation reaction force from the right side of Equation (71) in Appendix A acting on an ion–electron fluid element is (the radiation reaction force acting on ion is ignored since \( m_i \gg m_e \)). Here, we conduct the derivation without setting \( c = 1 \).

\[
F_{\text{LL}} = \frac{2e^4}{3m_e c^4} \frac{\rho_\gamma - \frac{m_e}{e} \rho_e}{m} \times \mathbf{E} + \mathbf{B} + \frac{1}{c} \left( B \times \left( B \times \left( \frac{\rho_\gamma v - m_e J}{e \rho_\gamma - \frac{m_e}{e} \rho_e} \right) \right) \right)
\]
\[
+ \frac{1}{c} E \left( \frac{\rho_\gamma v - \frac{m_e}{e} J}{e \rho_\gamma - \frac{m_e}{e} \rho_e} \cdot E \right)
\]
\[
- \frac{2e^4}{3m_e c^4} \left( \frac{\rho_\gamma v - \frac{m_e}{e} J}{e \rho_\gamma - \frac{m_e}{e} \rho_e} \right) \times \left( E + \left( \frac{1}{c} \frac{\rho_\gamma v - \frac{m_e}{e} J}{e \rho_\gamma - \frac{m_e}{e} \rho_e} \right) \times \mathbf{B} \right)^2
\]
\[
- \frac{1}{c^2} \left( \frac{\rho_\gamma v - \frac{m_e}{e} J}{e \rho_\gamma - \frac{m_e}{e} \rho_e} \right)^2.
\]

When we assume the plasma is neutral, \( \rho_e = 0 \), then the above equation changes to

\[
F_{\text{LL}} = \frac{2e^4}{3m_e c^4} \frac{\rho_\gamma}{m} \left( \mathbf{E} \times \mathbf{B} + \frac{1}{c} \left( B \times \left( B \times \left( \frac{\rho_\gamma v - m_e J}{e \rho_\gamma} \right) \right) \right) \right)
\]
\[
+ \frac{1}{c} E \left( \frac{\rho_\gamma v - \frac{m_e}{e} J}{e \rho_\gamma} \cdot \mathbf{E} \right)
\]
\[
- \frac{2e^4}{3m_e c^4} \left( \frac{\rho_\gamma v - \frac{m_e}{e} J}{e \rho_\gamma} \right) \times \left( E + \left( \frac{1}{c} \frac{\rho_\gamma v - \frac{m_e}{e} J}{e \rho_\gamma} \right) \times \mathbf{B} \right)^2
\]
\[
- \frac{1}{c^2} \left( \frac{\rho_\gamma v - \frac{m_e}{e} J}{e \rho_\gamma} \right)^2,
\]

In ideal relativistic magnetohydrodynamics, \( \mathbf{E} = -\frac{1}{c} \mathbf{v} \times \mathbf{B} \). Substituting this equation into the above, we get

\[
F_{\text{LL}} = \frac{2e^2}{3c^2} \left( \left( \mathbf{B} \times \left( \frac{\mathbf{J}}{ce} \times \mathbf{B} \right) \right) + \left( \frac{\mathbf{v}}{c} \times \mathbf{B} \right) \left( \frac{\mathbf{v}}{c} \cdot \left( \frac{\mathbf{J}}{ce} \times \mathbf{B} \right) \right) \right)
\]
\[
- \frac{2}{3} \frac{m_e^2 \gamma^2}{m} \left( \frac{1 - \frac{1}{c^2} (\mathbf{v} - \frac{\mathbf{J}}{c e})^2}{\left( \mathbf{v} - \frac{\mathbf{J}}{c e} \right)} \right) \times \left( \mathbf{J} \times \mathbf{B} \right)^2 - \left( \mathbf{J} \times \mathbf{B} \right) \cdot \mathbf{v} \left( \mathbf{v} \cdot \left( \frac{\mathbf{J}}{c e} \right) \right),
\]

where \( r_e = \frac{e^2}{m_e c^2} \), \( n = \frac{\gamma}{m} \approx \frac{\gamma}{m_e} \) for ion–electron plasma, and \( c \) is the speed of light.
When the current density is assumed to be perpendicular to the magnetic field, which is parallel to the velocity of the plasma, we get
\[ \frac{1}{1 - \frac{1}{c^2}v^2} < \frac{1}{1 - \frac{1}{c^2}(v - \frac{1}{c}J)^2}, \]
that is to say, \( \gamma^2 < \gamma_c^2 \). Here, we take \( \gamma_c > 1 \), and then the above equation becomes
\[
F_{LL} = \frac{2}{3} \frac{r}{c^2} \left[ \left( B \times \left( \frac{J}{c} \times B \right) \right) + \left( \frac{v}{c} \times B \right) \left( \frac{v}{c} \cdot \left( \frac{J}{c} \times B \right) \right) \right]
- \frac{2}{3} \frac{m c^2 \gamma^2}{1 - \frac{1}{c^2}v^2} \left[ \left( \frac{J}{c} \times B \right) \right] \left( \left( \frac{J}{c} \times B \right) \cdot \frac{v}{c} \right)^2
\]
\times \left( \frac{v}{c} - \frac{J}{c} \right).

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