Povzner–Wienholtz-Type Theorems for Sturm–Liouville Operators with Singular Coefficients

Vladimir Mikhailets¹,² · Andrii Goriunov² · Volodymyr Molyboga²

Abstract
We introduce and investigate symmetric operators $L_0$ associated in the complex Hilbert space $L^2(\mathbb{R})$ with a formal differential expression

$$l[u] := -(pu')' + qu + i((ru)' + ru')$$

under minimal conditions on the regularity of the coefficients. They are assumed to satisfy conditions

$$q = Q' + s; \quad \frac{1}{\sqrt{|p|}}, \frac{Q}{\sqrt{|p|}}, \frac{r}{\sqrt{|p|}} \in L^2_{loc}(\mathbb{R}), \quad s \in L^1_{loc}(\mathbb{R}),$$

where the derivative of the function $Q$ is understood in the sense of distributions, and all functions $p, Q, r, s$ are real-valued. In particular, the coefficients $q$ and $r'$ may be Radon measures on $\mathbb{R}$, while function $p$ may be discontinuous. The main result of the paper are two sufficient conditions on the coefficient $p$ which provide that the operator $L_0$ being semi-bounded implies it being self-adjoint.
1 Introduction

The problem of symmetric operators being self-adjoint is one of the main problems in the theory of differential operators and serves as the basis for the analysis of their spectral properties and scattering problems (see, e.g. [1–5]). Investigation of this problem for the Sturm–Liouville and Schrödinger operators in the spaces $L^2(\mathbb{R}^n)$ is inspired by the problems of mathematical physics and has numerous applications. The results that were obtained for this problem in the case of regular coefficients are rather complete. Hartman [6] and Rellich [7] were the first to show that boundedness from below of the operator generated by the Sturm–Liouville differential expression

$$l(u) := -(pu')' + qu$$

in the Hilbert space $L^2(\mathbb{R})$ together with the integral condition on the function $p > 0$ (which is obviously satisfied if $p \equiv 1$) are sufficient for the minimal operator to be self-adjoint.

Povzner [8] and later independently Wienholtz [9] established that preminimal operator

$$L_{00}u = -\Delta u + qu, \quad u \in C^\infty_0(\mathbb{R}^n)$$

being semi-bounded implies it being essentially self-adjoint in the space $L^2(\mathbb{R}^n)$, if function $q$ is real-valued and continuous. Later conditions on the regularity of potential $q$ were significantly weakened in papers [10–13].

A more general result regarding the self-adjointness of second-order symmetric elliptic operators with smooth coefficients was received by Berezanskii [14, ch. VI.1.7]. It states that the semi-bounded differential operator $L_{00}$ defined on $C^\infty_0(\mathbb{R}^n)$ is essentially self-adjoint in the Hilbert space $L^2(\mathbb{R}^n)$, if the condition of a globally finite rate of propagation is satisfied, that is, each solution to a hyperbolic differential equation

$$u_{tt} + Lu = 0$$

which has compact support for $t = 0$ has compact support for any $t > 0$. These results were developed further in papers [15–17], see also [18] and the bibliography therein.

At the same time, due to important applications there arose and in the recent years has been increasing an interest towards differential operators with singular coefficients which are not locally summable functions (see [19–21] and the references therein). Analysis of such operators presents significant mathematical difficulties, since their domain does not allow an explicit description and may not contain smooth functions.
other then zero. Moreover, the correct definition of such operators in the case of singular coefficients is a non-trivial problem which may be of considerable interest for the theory of differential operators and its applications.

In the present paper we solve this problem, applying an approach based on the theory of quasi-differential operators (see [22–25]). This approach is natural from an analytical point of view, since it allows to use the techniques and methods of the theory of ordinary differential equations. Furthermore, we show that the minimal operator $L_0$ generated by the formal differential expression $l$ is symmetric in space $L^2(\mathbb{R})$. Therefore, the question naturally arises about the conditions for it to be self-adjoint. Substantial answer to it is provided by two new theorems, which are formulated in Sect. 2. The proofs of these theorems are given in Sect. 4 of this paper. Some auxiliary statements are outlined in Sect. 3.

## 2 Povzner–Wienholtz-Type Results

We consider operators generated by the differential Sturm–Liouville expression

$$l[u] := -(pu')' + qu + i((ru')' + ru')$$

with real coefficients $p$, $q$ and $r$ given on $\mathbb{R}$. If these coefficients are regular enough, then the mapping

$$L_{00} : u \mapsto l[u], \quad u \in C_0^\infty(\mathbb{R})$$

defines a densely defined in the complex Hilbert space $L^2(\mathbb{R})$ preminimal symmetric operator $L_{00}$. Therefore, the question arises whether the closure of this operator $L_0 := (L_{00})^\sim$ is self-adjoint. A large number of papers are devoted to this problem (see, e.g. references in [26]). For instance, Hartman [6] and Rellich [7] established that if operator $L_{00}$ is bounded from below and

$$r \equiv 0, \quad 0 < p \in C^2(\mathbb{R}), \quad q \text{ is piecewise continuous on } \mathbb{R},$$

and the following condition is satisfied

$$\int_0^\infty p^{-1/2}(t)dt = \int_{-\infty}^0 p^{-1/2}(t)dt = \infty,$$

then the operator $L_0$ corresponding to $l$ is self-adjoint. In the paper [27] the conditions on the regularity of the coefficients of $l$ were weakened:

$$r \equiv 0, \quad 0 < p \text{ is locally Lipschitz, } \quad q \in L_{loc}^2(\mathbb{R}).$$

Another sufficient condition for the operator $L_0$ to be self-adjoint was obtained in [28]. It may be written in the form

$$\|p\|_{L^\infty(-\rho,-\rho/2)}, \|p\|_{L^\infty(\rho/2,\rho)} = O(\rho^2), \quad \rho \to \infty.$$
Here the coefficients of (1) satisfy conditions
\[ r \equiv 0, \quad 0 < p \in W_{2, loc}^1(\mathbb{R}), \quad q \in L_{loc}^1(\mathbb{R}). \]

Examples show that conditions (2) and (3) are independent (see [28]).

We will assume throughout what follows that the assumptions
\[ q = Q' + s, \quad \frac{1}{\sqrt{|p|}} \frac{Q}{\sqrt{|p|}}, \frac{r}{\sqrt{|p|}} \in L_{loc}^2(\mathbb{R}), \quad s \in L_{loc}^1(\mathbb{R}) \] (4)

hold, where the derivative of function \( Q \) is understood in the sense of distributions and all the coefficients \( p, Q, s, r \) are real-valued functions.

We propose to consider the operators generated by the formal differential expression (1) as quasi-differential operators, which are defined applying compositions of differential operators with locally summable coefficients. These operators are defined using the Shin–Zettl matrix function specifically chosen to correspond to the coefficients of \( l \) (see [22–25]).

In our case it has the form
\[ A(x) = \begin{pmatrix} \frac{Q+ir}{p} & \frac{1}{p} \\ -\frac{Q^2+2r^2}{p} + s - \frac{1}{p} \end{pmatrix} \] (5)

and due to our assumptions belongs to the class \( L_{loc}^1(\mathbb{R}, \mathbb{C}^{2 \times 2}) \).

It can be used to define corresponding quasi-derivatives as follows:
\[ u[0] := u, \quad u[1] := pu' - (Q + ir)u, \]
\[ u[2] := (u[1])' + \frac{Q - ir}{p} u[1] + \left( \frac{Q^2 + r^2}{p} - s \right) u. \] (6)

Formal differential expression (1) may now be defined as quasi-differential:
\[ l[u] := -u[2], \quad \text{Dom}(l) := \left\{ u : \mathbb{R} \to \mathbb{C} \mid u, u[1] \in \text{AC}_{loc}(\mathbb{R}) \right\}. \]

This definition is motivated by the fact that
\[ \langle -u[2], \varphi \rangle = \langle -(pu')' + qu + i((ru')' + ru'), \varphi \rangle \quad \forall \varphi \in C_0^\infty(\mathbb{R}) \]
in the sense of distributions.

We define for the quasi-differential expression \( l \) the operators \( L \) and \( L_{00} \) as:
\[ Lu := l[u], \quad \text{Dom}(L) := \left\{ u \in L^2(\mathbb{R}) \mid u, u[1] \in \text{AC}_{loc}(\mathbb{R}), l[u] \in L^2(\mathbb{R}) \right\}, \]
\[ L_{00}u := Lu, \quad \text{Dom}(L_{00}) := \left\{ u \in \text{Dom}(L) \mid \text{supp} \, u \text{ is compact} \right\}. \]
The operators \( L \) and \( L_{00} \) are maximal and preminimal operators for expression \( l \) respectively. Their definitions coincide with the classical ones if the coefficients \( l \) are sufficiently smooth. Below in Sect. 3 we will show that the operator \( L_{00} \) is densely defined in \( L^2(\mathbb{R}) \) and is symmetric.

Let us formulate the main results of the paper in the form of two theorems. The first of them is a natural generalization of the above-mentioned result of Hartman and Rellich.

**Theorem 1** Let the coefficients of the formal differential expression (1) satisfy the assumptions (4) and also

(i) \( p \in W^1_{2,\text{loc}}(\mathbb{R}), \quad p > 0, \)

(ii) \( \int_{-\infty}^{0} p^{-1/2}(t) d t = \int_{0}^{\infty} p^{-1/2}(t) d t = \infty. \)

Then, if operator \( L_{00} \) is bounded from below, then it is essentially self-adjoint and \( L^*_{00} = L = L^* \).

For the case \( p \equiv 1, r \equiv 0 \) Theorem 1 was previously established in [29].

In the second theorem, additional conditions on the coefficient \( p \) are imposed not on the entire axis, but only on a sequence of finite intervals. However, outside of these intervals the function \( p \) may vanish and be discontinuous.

**Theorem 2** Let the assumptions (4) are satisfied and the operator \( L_{00} \) is bounded from below. Suppose the sequence of intervals \( \Delta_n := [a_n, b_n] \) exists such that

\[-\infty < a_n < b_n < \infty, \quad b_n \to -\infty, \quad n \to -\infty, \quad a_n \to \infty, \quad n \to \infty,\]

where the coefficients \( p \) satisfy the additional conditions

(i) \( p_n := p|\Delta_n| \in W^1_2(\Delta_n), \quad p_n > 0; \)

(ii) \( \exists C > 0 : \quad p_n(x) \leq C|\Delta_n|^2, \quad n \in \mathbb{Z}, \) where \(|\Delta_n|\) is the length of interval \( \Delta_n \).

Then operator \( L_{00} \) is essentially self-adjoint and \( L^*_{00} = L = L^* \).

For the case \( p \equiv 1, r \equiv 0 \) necessary and sufficient conditions for semi-boundedness of operator \( L_{00} \) were obtained in [30] and some sufficient conditions in [31].

### 3 Preliminary Results

Let us formulate some auxiliary statements that are used in the proof of Theorems 1 and 2. Some of them may be of interest by themselves. Let \( \Delta = (\alpha, \beta) \) be a finite interval in \( \mathbb{R} \).

Consider the Sturm–Liouville operators generated by the formal differential expression (1) in the complex Hilbert space \( L^2(\Delta) \). Our assumptions (4) imply that

\[ q = Q' + s, \quad \frac{1}{\sqrt{|p|}}, \quad \frac{Q}{\sqrt{|p|}}, \quad \frac{r}{\sqrt{|p|}} \in L^2(\Delta), \quad s \in L^1(\Delta). \]
Therefore, the Shin–Zettl matrix of the form (5) is correctly defined, which defines
 corresponding quasi-derivatives by the formulas (6).

Let us now consider a quasi-differential expression \( l_{\Delta} \) on the interval \( \Delta \)

\[
l_{\Delta}[u] := -u^{[2]}, \quad \text{Dom}(l_{\Delta}) := \left\{ u \in L^2(\Delta) \mid u, u^{[1]} \in AC(\Delta) \right\},
\]
and the corresponding minimal and maximal operators

\[
L_{\Delta}u := l_{\Delta}[u], \quad \text{Dom}(L_{\Delta}) := \left\{ u \in L^2(\Delta) \mid u, u^{[1]} \in AC(\Delta), l_{\Delta}[u] \in L^2(\Delta) \right\},
\]

\[
L_{0,\Delta}u := L_{\Delta}u, \quad \text{Dom}(L_{0,\Delta}) := \left\{ u \in \text{Dom}(L_{\Delta}) \mid u^{[j]}(\alpha) = u^{[j]}(\beta) = 0, \quad j = 0, 1 \right\},
\]

**Lemma 1** (Theorems 9, 10 [22]) Operator \( L_{0,\Delta} \) has a domain which is dense in the
space \( L^2(\Delta) \) and

\[
(\text{i}) \quad (L_{0,\Delta})^* = L_{\Delta},
\]

\[
(\text{ii}) \quad L_{0,\Delta} = (L_{\Delta})^*.
\]

Let us now pass from the finite interval \( \Delta \) to the entire axis.

**Theorem 3** The following statements hold for operators \( L \) and \( L_{0,0} \).

1°. Operator \( L_{0,0} \) has a domain which is dense in the Hilbert space \( L^2(\mathbb{R}) \).

2°. The equality holds:

\[
(L_{0,0})^* = L.
\]

3°. Operator \( L \) is closed and preminimal operator \( L_{0,0} \) is closable.

4°. The domain of the minimal operator \( L_0 \) may be represented as

\[
\text{Dom}(L_0) = \left\{ u \in \text{Dom}(L) \mid [u, v]_{-\infty}^{\infty} = 0 \quad \forall v \in \text{Dom}(L) \right\},
\]

where

\[
[u, v](t) \equiv [u, v] := u(t)v^{[1]}(t) - u^{[1]}(t)v(t),
\]

\[
[u, v]^b_a := [u, v](b) - [u, v](a), \quad -\infty \leq a \leq b \leq \infty.
\]

5°. Operator \( L_0 \) is symmetric.

6°. Let \( p, 1/p \in L^\infty(\Delta) \). Then

\[
\text{i}) \quad Q, r \in L^2(\Delta), \quad \text{ii}) \quad u \in \text{Dom}(L) \Rightarrow u|_{\Delta} \in W^1_2(\Delta).
\]

In particular, if \( p, 1/p \in L^\infty_{loc}(\mathbb{R}) \), then

\[
\text{i}) \quad Q, r \in L^2_{loc}(\mathbb{R}), \quad \text{ii}) \quad \text{Dom}(L) \subset W^1_{2,loc}(\mathbb{R}).
\]
$\mathcal{L}_0^0$. Let $p, 1/p \in L^\infty_{loc}(\mathbb{R})$. Then

$$(L_0 u, u)_{L^2(\mathbb{R})} = \int_{\mathbb{R}} p|u'|^2\,dx - \int_{\mathbb{R}} Qd|u|^2 + \int_{\mathbb{R}} s|u|^2\,dx, \quad u \in \text{Dom}(L_0).$$

Before proving Theorem 3, we prove several statements we will need.

**Lemma 2** For arbitrary functions $u, v \in \text{Dom}(L)$ on a finite interval $[a, b]$ the relations hold:

$$\int_a^b l[u]v\,dx - \int_a^b ul[v]\,dx = [u, v]_a^b. \quad (7)$$

**Proof** The statement is proved by a direct computation applying integration by parts:

$$\begin{align*}
\int_a^b l[u]v\,dx - \int_a^b ul[v]\,dx &= \int_a^b \left( -\frac{Q - ir}{p} u[1]v - \frac{Q}{p} u v + suv \right)\,dx \\
& \quad + \int_a^b \left( u[1]v' + \frac{Q + ir}{p} u[1]v + \frac{Q}{p} u v - suv \right)\,dx \\
& = \left( u[1]v - u[1]v \right)_a^b + \int_a^b \left( u[1]v' - \frac{Q - ir}{p} u[1]v - u[1]v + \frac{Q + ir}{p} u[1]v \right)\,dx \\
& = [u, v]_a^b + \int_a^b \left( \frac{1}{p} u[1]v - \frac{1}{p} u[1]v \right)\,dx = [u, v]_a^b.
\end{align*}$$

The proof is complete. $\square$

**Lemma 3** For arbitrary functions $u, v \in \text{Dom}(L)$ the following limits exist and are finite:

$$[u, v](-\infty) := \lim_{t \to -\infty} [u, v](t), \quad [u, v](\infty) := \lim_{t \to \infty} [u, v](t).$$

**Proof** Let us fix the number $b$ in the equality (7) and pass in it to the limit as $a \to -\infty$. Since, by the assumptions of the Lemma, the functions $u, v, l[u], l[v] \in L^2(\mathbb{R})$, then limit $[u, v](-\infty)$ exists and is finite. The existence of the limit $[u, v](\infty)$ is proved similarly. $\square$

**Proof of Theorem 3** The proofs of assertions $1^0 - 4^0$ are similar to the proofs for the case of the semiaxis $[22, 32]$. They rely on similar properties of operators on a finite interval. For the convenience of the reader, we present these proofs in full.

1^0. Let the function $h \in L^2(\mathbb{R})$ be orthogonal to $\text{Dom}(L_0)$. Let us show that $h \equiv 0$.

Let $\Delta$ be an arbitrary fixed finite interval.

Let us define by the expression $l$ the quasi-differential operators $L_{0\Delta}$ and $L_\Delta$. The space $L^2(\Delta)$ may be embedded into $L^2(\mathbb{R})$, assuming that outside the interval $\Delta$
function \( u \in L^2(\Delta) \) is equal to zero. Thus, domain \( \text{Dom}(L_{0\Delta}) \) of the operator \( L_{0\Delta} \) becomes a part of \( \text{Dom}(L) \). Moreover, the function \( u \in \text{Dom}(L_{0\Delta}) \) extended in such a way belongs to \( \text{Dom}(L_{00}) \).

Thus, the function \( h \) will be orthogonal to \( \text{Dom}(L_{0\Delta}) \). Due to Lemma 1 the domain \( \text{Dom}(L_{0\Delta}) \) is dense in the space \( L^2(\Delta) \). Therefore \( h|_{\overline{\Delta}} = 0 \) almost everywhere.

As the interval \( \Delta \subset \mathbb{R} \) is arbitrary, this implies that \( h = 0 \) almost everywhere on \( \mathbb{R} \).

The proof of assertion 1) is complete.

2. Due to assertion 1) for the operator \( L_{00} \) the adjoint operator \( (L_{00})^* \) exists.

Let \( u \in \text{Dom}(L_{00}) \), \( v \in \text{Dom}(L) \). Then applying the Lagrange identity from Lemma 2 we get:

\[
(L_{00}u, v)_{L^2(\mathbb{R})} = (u, L_0v)_{L^2(\mathbb{R})}.
\]

Hence \( L \subset (L_{00})^* \). Let us prove the reverse inclusion.

Let \( v \) be an arbitrary element from \( \text{Dom}((L_{00})^*) \) and let \( \Delta = (\alpha, \beta) \) be a fixed finite interval in \( \mathbb{R} \). Then

\[
\left((L_{00})^*v, u\right)_{L^2(\mathbb{R})} = (v, L_{00}u)_{L^2(\mathbb{R})} \quad \forall u \in \text{Dom}(L_{0\Delta}).
\]

Since \( u|_{\mathbb{R}\setminus \overline{\Delta}} = 0 \), then inner products may be expressed as integrals over the interval \( \overline{\Delta} \), i.e. they are inner products in \( L^2(\Delta) \) and

\[
\left(((L_{00})^*v)_{\Delta}, u\right)_{L^2(\Delta)} = (v_{\Delta}, L_{0\Delta}u)_{L^2(\Delta)} \quad \forall u \in \text{Dom}(L_{0\Delta}),
\]

where \(((L_{00})^*v)_{\Delta}, v_{\Delta}\) are the restrictions of corresponding functions onto interval \( \overline{\Delta} \).

Equality (8) due to Lemma 1 implies that

\[
((L_{00})^*v)_{\Delta} = L_{\Delta}v_{\Delta} = (l[v])_{\Delta}.
\]

Since the interval \( \Delta \) is arbitrary it follows that

\[
v \in \text{Dom}(L), \quad (L_{00})^*v = l[v] = Lv,
\]

which we needed to prove. \( \square \)

3. Immediately follows from assertion 2.

4. Since the operator \( L_0 \) is closed and \( (L_0)^* = L \), then obviously \( \text{Dom}(L_0) \) consists of those and only those functions \( u \in \text{Dom}(L) \) that satisfy relation

\[
(u, Lv)_{L^2(\mathbb{R})} = (Lu, v)_{L^2(\mathbb{R})} \quad \forall v \in \text{Dom}(L),
\]

which due to Lagrange identity is equivalent to assertion 4.

The proof of assertion 4 is complete.

5. Lemma 2 implies that operator \( L_{00} \) is symmetric. Therefore operator \( L_0 \) being its closure is also symmetric.
Let the conditions of the assertion be satisfied. They immediately imply that \( \sqrt{|p|} \in L^\infty_{loc}(\mathbb{R}) \). Due to assumptions (4) functions \( \frac{Q}{\sqrt{|p|}}, \frac{r}{\sqrt{|p|}} \in L^2_{loc}(\mathbb{R}) \). Therefore, if \( p \in L^\infty(\Delta) \) then

\[
\sqrt{|p|} \cdot \frac{Q}{\sqrt{|p|}} = Q \in L^2(\Delta), \quad \sqrt{|p|} \cdot \frac{r}{\sqrt{|p|}} = r \in L^2(\Delta).
\]

Let, further, \( u \in \text{Dom}(L) \). Definition of the domain \( \text{Dom}(L) \) implies that

\[
p\left(u' - \frac{Q + ir}{p} u\right) = pu' - (Q + ir)u \in AC_{loc}(\mathbb{R}) \subset L^2_{loc}(\mathbb{R}).
\]

Therefore, taking into account that \( \frac{1}{p} \in L^\infty(\Delta) \) we get:

\[
\frac{1}{p} \cdot pu' = u' \in L^2(\Delta), \quad \text{i.e.} \quad u_\Delta \in W^1_2(\Delta).
\]

The proof of assertion 60 is complete.

70. Considering that, due to our assumptions \( Q, r \in L^2_{loc}(\mathbb{R}) \) and \( \text{Dom}(L_{00}) \subset W^1_{2,comp}(\mathbb{R}) \), for \( u \in \text{Dom}(L_{00}) \) we have:

\[
(L_{00}u, u)_{L^2(\mathbb{R})} = \int_\mathbb{R} \left(-u^{(1)} u' - \frac{Q - ir}{p} u^{(1)} u - \frac{Q^2 + r^2}{p} |u|^2 + s|u|^2\right) \, dx =
\]

\[
= \int_\mathbb{R} \left(u^{(1)} u' - \frac{Q - ir}{p} u^{(1)} u - \frac{Q^2 + r^2}{p} |u|^2 + s|u|^2\right) \, dx =
\]

\[
= \int_\mathbb{R} \left(p|u'|^2 - Q(u' u + u u') + s|u|^2\right) \, dx = \int_\mathbb{R} p|u'|^2 \, dx - \int_\mathbb{R} Qd|u|^2 + \int_\mathbb{R} s|u|^2 \, dx.
\]

The proof of assertion 70 and therefore Theorem 3 are complete.

4 Proofs of Theorems 1 and 2

To prove Theorems 1 and 2 we need two more auxiliary statements regarding local regularity of functions from domains of preminimal and maximal operators.

**Lemma 4** Let the initial assumptions about the coefficients (4) be satisfied and suppose \( p > 0 \) and \( p \in W^1_{2,loc}(\mathbb{R}) \). Then for an arbitrary function \( \varphi \in W^2_{2,comp}(\mathbb{R}) \) and any function \( u \in \text{Dom}(L) \) we have \( \varphi u \in \text{Dom}(L_{00}) \).

**Proof** Conditions \( p \in W^1_{2,loc}(\mathbb{R}) \) and \( p > 0 \) imply that continuous on \( \mathbb{R} \) function \( p \) is locally separated from zero. Therefore function \( \frac{1}{p} \in C(\mathbb{R}) \) and is locally bounded on
\[ Q, r \in L^2_{\text{loc}}(\mathbb{R}), \quad \text{Dom}(L) \subset W^1_{2,\text{loc}}(\mathbb{R}). \] (9)

Due to (9) we have:

(i) \( \varphi u \in AC_{\text{comp}}(\mathbb{R}), \)
(ii) \((\varphi u)[1] = (p(\varphi u)' - (Q + ir)\varphi u) = p\varphi' u + \varphi u[1] \in AC_{\text{comp}}(\mathbb{R}), \)
(iii) \(l[\varphi u] = \varphi l[u] - \varphi' u[1] - p' \varphi' u - p\varphi'' u - (Q - ir)\varphi' u \in L^2_{\text{comp}}(\mathbb{R}).\)
(iv) \(\text{supp} \varphi u \) is a compact set.

Thus, \( \varphi u \in \text{Dom}(L_{00}). \)

Therefore, the proof of the lemma is complete. \( \square \)

**Lemma 5** Let the assumptions (4) about the coefficients of \( l \) be satisfied, suppose numbers \( a' < a < b < b' \) and positive functions \( p|[a',a] \in W^1_2([a',a]), p|[b,b'] \in W^1_2([b,b']), \) and suppose function \( \varphi \in C^2(\mathbb{R}) \) is such that

(i) \( 0 \leq \varphi(x) \leq 1, \quad x \in \mathbb{R}, \)
(ii) \( \varphi(x) = \begin{cases} 1, & x \in [a, b]; \\ 0, & x \in (\infty, a') \cup (b', \infty). \end{cases} \)

Then for every function \( u \in \text{Dom}(L) \) we have \( \varphi u \in \text{Dom}(L_{00}). \)

**Proof** Assumptions of Lemma imply that function \( p \) is continuous and separated from zero on the set \([a', a] \cup [b, b']\). Therefore function \( \frac{1}{p} \) is bounded on the compact set \([a', a] \cup [b, b']\) due to assertion 60 of Theorem 3. Considering that \( \varphi'(x) = 0, \quad x \in (\infty, a') \cup [a, b] \cup (b', \infty) \) we have:

(i) \( \varphi u \in AC_{\text{comp}}(\mathbb{R}), \)
(ii) \((\varphi u)[1] = (p(\varphi u)' - (Q + ir)\varphi u) = p\varphi' u + \varphi u[1] \in AC_{\text{comp}}(\mathbb{R}), \)
(iii) \(l[\varphi u] = \varphi l[u] - \varphi' u[1] - p' \varphi' u - p\varphi'' u - (Q - ir)\varphi' u \in L^2_{\text{comp}}(\mathbb{R}).\)
(iv) \(\text{supp} \varphi u \) is a compact set.

Thus, \( \varphi u \in \text{Dom}(L_{00}) \) and the proof is complete. \( \square \)

**Proof of Theorem 1** Without loss of generality, we assume that

\[ (L_0 u, u)_{L^2(\mathbb{R})} \geq (u, u)_{L^2(\mathbb{R})}, \quad u \in \text{Dom}(L_0). \] (10)

To prove that operator \( L_0 \) is self-adjoint, it is sufficient (and necessary) to show that every function that satisfies conditions

\[ Lv = 0, \quad v \in \text{Dom}(L) \] (11)
is equal to zero almost everywhere.

Let equality (11) hold. We choose a function sequence \( \{ \varphi_n \}_{n \in \mathbb{N}} \) so that following conditions are satisfied:

(i) \( 0 \leq \varphi_n(x) \leq 1, \)
(ii) \( \varphi_n(x) = 1, \ x \in [-n, n]; \)
(iii) \( \text{supp} \varphi_n \subset [-n - 1, n + 1]; \)
(iv) \( |\varphi'_n| \leq K, \) where the constant \( K \) does not depend on \( n. \)

Consider now the function

\[
\rho(x) := \begin{cases} 
- \int_x^0 p^{-1/2}(t) \, dt, & x \in (-\infty, 0], \\
\int_0^x p^{-1/2}(t) \, dt, & x \in [0, \infty). 
\end{cases}
\]

Then, taking into account condition (ii) of Theorem 1 the composition \( \varphi_n(\rho) \) belongs to the space \( W_{2, \text{comp}}^2(\mathbb{R}), \ n \in \mathbb{N}. \) Therefore, functions \( \varphi_n(\rho)v \) satisfy inequality (10).

After substitution we obtain:

\[
K^2 \int_{n \leq |\rho(x)| \leq n + 1} |v(x)|^2 \, dx \geq \int_{|\rho(x)| \leq n} |v(x)|^2 \, dx.
\]

Taking into account that \( v \in L^2(\mathbb{R}) \) and passing to the limit in the last inequality, we obtain that \( v = 0. \)

The proof is complete. \( \square \)

**Proof of Theorem 2** Without loss of generality, we assume that the intervals \( \Delta_n = [a_n, b_n] \) do not intersect and that \( b_{-1} < 0 \) and \( a_1 > 0. \) Let us show that every function \( v \) that satisfies the equality \( Lv = 0 \) is equal to 0 almost everywhere on \( \mathbb{R}. \)

Let \( \{ \varphi_n \}_{n \in \mathbb{N}} \) be a sequence of real infinitely differentiable functions with following properties:

(i) \( 0 \leq \varphi_n(x) \leq 1, x \in \mathbb{R}, n \in \mathbb{N}; \)
(ii) \( \varphi_n(x) = 1, x \in [b_{-n}, a_n]; \)
(iii) \( \text{supp} \varphi_n \subset (a_{-n}, b_n); \)
(iv) \( \exists K > 0: |\varphi'_n(x)| \leq \begin{cases} 
K|\Delta_{-n}|^2, & \text{if } x < 0, \\
K|\Delta_n|^2, & \text{if } x > 0. 
\end{cases} \)

Due to Lemma 5 function \( \varphi_nv \in \text{Dom}(L_{00}) \subset \text{Dom}(L_0) \) and \( \text{supp} \varphi'_n \subset \Delta_{-n} \cup \Delta_n, \ n \in \mathbb{N}. \)

\[
(L_{00}\varphi_nv, \varphi_nv)_{L^2(\mathbb{R})} = \int_{\mathbb{R}} p(\varphi'_n)^2 |v|^2 \, dx + \int_{\mathbb{R}} p\varphi_n\varphi'_n(v\overline{v}' - v'\overline{v}) \, dx. \tag{12}
\]

Due to our assumptions

\[
\text{Re} \int_{\mathbb{R}} p\varphi_n\varphi'_n(v\overline{v}' - v'\overline{v}) \, dx = 0,
\]
Therefore, from inequality (10) we have that
\[ \int_{b_{-n}}^{a_n} |v|^2 \, dx \leq CK^2 \int_{\Delta_{-n} \cup \Delta_n} |v|^2 \, dx, \]  
(13)
where the constant \( C > 0 \) is the one from assumption (i.i) of Theorem 2.
Taking into account that \( v \in L^2(\mathbb{R}) \) and passing in inequality (13) to the limit as \( n \to \infty \) we conclude that \( v = 0 \).
The proof is complete. \( \square \)

Similar results can be proved for symmetric Sturm–Liouville operators defined on the semiaxis.

References

1. Reed, M., Simon, B.: Methods of Modern Mathematical Physics. II. Fourier Analysis, Self-Adjointness. Academic Press, London (1975)
2. Shubin, M.: Essential self-adjointness for semi-bounded magnetic Schrödinger operators on non-compact manifolds. J. Funct. Anal. 186(1), 92–116 (2001)
3. Albeverio, S., Kostenko, A., Malamud, M.: Spectral theory of semibounded Sturm–Liouville operators with local interactions on a discrete set. J. Math. Phys. 51(10), 102102, 24 (2010)
4. Hryniv, R.O., Myktyuk, Ya., V.: Self-adjointness of Schrödinger operators with singular potentials. Methods Funct. Anal. Topol. 18(2), 152–159 (2012)
5. Kostenko, A., Malamud, M., Nicolussi, M.: Glazman–Povzner–Wienholtz theorem on graphs. Adv. Math. 395, 108158, 30 (2022)
6. Hartman, P.: Differential equations with non-oscillatory eigenfunctions. Duke Math. J. 15(3), 697–709 (1948). https://doi.org/10.1215/S0012-7094-48-01559-2
7. Rellich, F.: Halbbeschränkte gewöhnliche Differentialoperatoren zweiter Ordnung. Math. Ann. 122, 343–368 (1951)
8. Povzner, A.Ya.: The expansion of arbitrary functions in eigenfunctions of the operator \(-\Delta u + cu\). Mat. Sbornik N.S. 32, 109–156 (1953) (in Russian) (translation in Am. Math. Soc. Trans. 60(2), 1–49 (1967))
9. Wienholtz, E.: Halbbeschränkte partielle Differentialoperatoren zweiter Ordnung vom elliptischen Typus. Math. Ann. 135, 50–80 (1958)
10. Hinz, A.M.: Regularity of solutions for singular Schrödinger equations. Rev. Math. Phys. 4(1), 95–161 (1992). https://doi.org/10.1142/S0129055X92000054
11. Schmincke, U.-W.: Proofs of Povzner–Wienholtz type theorems on selfadjointness of Schrödinger operators by means of positive eigensolutions. Bull. Lond. Math. Soc. 23(3), 263–266 (1991). https://doi.org/10.1112/blms/23.3.263
12. Simader, C.G.: Essential self-adjointness of Schrödinger operators bounded from below. Math. Z. 159(1), 47–50 (1978). https://doi.org/10.1007/BF01174567
13. Simader, C.G.: Remarks on essential self-adjointness of Schrödinger operators with singular electrostatic potential. J. Reine Angew. Math. 431, 1–6 (1992). https://doi.org/10.1515/crll.1992.431.1
14. Berezanskii, Ju.: Expansions in Eigenfunctions of Selfadjoint Operators. Translations of Mathematical Monographs, vol. 17. American Mathematical Society, Providence (1968)
15. Berezanskii, Yu.M., Samoilenko, V.G.: On the self-adjointness of differential operators with finitely or infinitely many variables, and evolution equations. Russ. Math. Surv. 36(5), 1–62 (1981). https://doi.org/10.1070/RM1981v036n05ABEH003029
16. Orochko, Yu.B.: The hyperbolic equation method in the theory of operators of Schrödinger type with a locally integrable potential. Russ. Math. Surv. 43(2), 51–102 (1988). https://doi.org/10.1070/RM1988v043n02ABEH001728
17. Rofe-Beketov, F.S.: Necessary and sufficient conditions for a finite propagation rate for elliptic operators. Ukrl. Math. J. 37(5), 547–549 (1985). https://doi.org/10.1007/BF01061187
18. Braverman, M., Milatovic, O., Shubin, M.: Essential self-adjointness of Schrödinger type operators on manifolds. Russ. Math. Surv. 57(4), 641–692 (2002). https://doi.org/10.1070/RM2002v057n04ABEH000532
19. Albeverio, S., Gesztesy, F., Hoegh-Krohn, R., Holden, H.: Solvable Models in Quantum Mechanics, 2nd edn. AMS Chelsea Publishing, Providence (2005)
20. Eckhardt, J., Teschl, G.: Sturm–Liouville operators with measure-valued coefficients. J. Anal. Math. 120(1), 151–224 (2013). https://doi.org/10.1007/s11854-013-0018-x
21. Mikhailets, V., Sobolev, A.: Common eigenvalue problem and periodic Schrödinger operators. J. Funct. Anal. 165(1), 150–172 (1999). https://doi.org/10.1006/jfan.1999.3406
22. Zettl, A.: Formally self-adjoint quasi-differential operator. Rocky Mt. J. Math. 5(3), 453–474 (1975). https://doi.org/10.1216/RMJ-1975-5-3-453
23. Gorinov, A., Mikhailets, V.: Regularization of singular Sturm–Liouville equations. Methods Funct. Anal. Topol. 16(2), 120–130 (2010)
24. Gorinov, A., Mikhailets, V., Pankrashkin, K.: Formally self-adjoint quasi-differential operators and boundary value problems. Electron. J. Differ. Equ. 2013(101), 1–16 (2013)
25. Eckhardt, J., Gesztesy, F., Nichols, R., Teschl, G.: Weyl–Titchmarsh theory for Sturm–Liouville operators with distributional potentials. Opusc. Math. 33(3), 467–563 (2013). https://doi.org/10.7494/OpMath.2013.33.3.467
26. Zetlì, A.: Sturm–Liouville Theory. American Mathematical Society, Providence (2005)
27. Stetkaer-Hansen, H.: A generalization of a theorem of Wienholtz concerning essential selfadjointness of singular elliptic operators. Math. Scand. 19, 108–112 (1966). https://doi.org/10.7146/math.scand.a-10798
28. Clark, S., Gesztesy, F.: On Povzner–Weinholtz-type self-adjointness results for matrix-valued Sturm–Liouville operators. Proc. R. Soc. Edinb. Sect. A 133(4), 747–758 (2003). https://doi.org/10.1017/S0308210500002651
29. Mikhailets, V., Molyboga, V.: Remarks on Schrödinger operators with singular matrix potentials. Methods Funct. Anal. Topol. 19(2), 161–167 (2013)
30. Mikhailets, V., Murach, A., Novikov, V.: Localization principles for Schrödinger operator with a singular matrix potential. Methods Funct. Anal. Topol. 23(4), 367–377 (2017)
31. Mikhailets, V., Molyboga, V.: Schrödinger operators with measure-valued potentials: semiboundness and spectrum. Methods Funct. Anal. Topol. 24(3), 240–254 (2018)
32. Naimark, M.: Linear Differential Operators. Harrap, London (1968)

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Springer Nature or its licensor (e.g. a society or other partner) holds exclusive rights to this article under a publishing agreement with the author(s) or other rightsholder(s); author self-archiving of the accepted manuscript version of this article is solely governed by the terms of such publishing agreement and applicable law.