BOUNDNESS IN A TWO SPECIES ATTRACTION-REPULSION CHEMOTAXIS SYSTEM WITH TWO CHEMICALS

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Abstract. This paper deals with a class of attraction-repulsion chemotaxis systems in a smoothly bounded domain. When the system is parabolic-elliptic-parabolic-elliptic and the domain is $n$-dimensional, if the repulsion effect is strong enough then the solutions of the system are globally bounded. Meanwhile, when the system is fully parabolic and the domain is either one-dimensional or two-dimensional, the system also possesses a globally bounded classical solution.

1. Introduction. Chemotaxis refers to the guided migration of cells under the guidance of chemical gradients. For bacteria, chemotaxis guides cells to obtain nutrients and away from harmful substances, which is crucial for a variety of biological processes. Chemotaxis has been confirmed in patterning of the slime mold Dictyostelium, embryonic morphogenesis, wound healing, tumor invasion, and so on (see [25] and references therein). Continuous models have been developed to describe this process.

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globally bounded classical solution under the conditions that \( \min \) \( \tau_0 \)

that the finite-time blow-up occurs at \( \Omega \)

and Xiang [18] established global boundedness of solutions for three cases, \( \bar{R} \)

\( \eta > \chi \)

repulsion prevails over attraction in the sense that \( \bar{R} \)

and Wang [30] showed that the system is globally well-posed in high dimensions if \( \eta > \chi \)

decay. For chemotaxis models like (1), one of the most important problems is the

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system (1) can also describe the quorum sensing effect in the chemotactic process ([26]). The first equation implies that the cell movement

repellent, respectively. Here \( \Omega \)

subject to the homogeneous Neumann boundary conditions, where \( u \)

as activated microglia, \( v \) and \( w \) stand for chemical concentrations of attractant and repellent, respectively. Here \( \Omega \subseteq \mathbb{R}^n \) \( n \geq 1 \) is a smoothly bounded domain; \( \tau \in \{0,1\} \) and all the other parameters \( \bar{R}, \bar{u}, \bar{v}, \gamma, \delta \) are positive constants; \( f(u) \)

measures the respective strengths of the attraction and repulsion. The second and third equations

state that chemoattractant and chemorepellent are produced by cells and undergo decay. For chemotaxis models like (1), one of the most important problems is the global boundedness and finite/infinite time blow-up of solutions.

For (1), when \( \tau = 0 \) and there is no source for microglia (i.e., \( f \equiv 0 \)), Tao and Wang [30] showed that the system is globally well-posed in high dimensions if repulsion prevails over attraction in the sense that \( \bar{R} > \bar{u} \). Assuming that \( \bar{R} > \bar{u} \), Guo et al. [9] proved that the solutions are globally bounded when \( \|u_0\|_{L^1(\Omega)} < \frac{4\pi}{\bar{u}_0 - \bar{R}} \) and the finite-time blow-up may occur at \( x_0 \in \partial \Omega \) under the condition \( \int_{\Omega} u_0(x) \|x - x_0\|^2 dx \) is sufficiently small provided \( \|u_0\|_{L^1(\Omega)} > \frac{4\pi}{\bar{u}_0 - \bar{R}} \) for some \( \Omega \subseteq \mathbb{R}^2 \). Subsequently, Viglialoro [37] provided an explicit lower bound for the blow-up time estimated in terms of \( \|u_0\|_{L^1(\Omega)} \). Meanwhile, Yu et al. [42] pointed out that the finite-time blow-up occurs at \( x_0 \in \Omega \) under the condition \( \int_{\Omega} u_0(x) \|x - x_0\|^2 dx \) is sufficiently small provided \( \|u_0\|_{L^1(\Omega)} > \frac{4\pi}{\bar{u}_0 - \bar{R}} \) with \( \bar{R} > \bar{u} \). When \( \tau = 0 \) and the logistic source satisfies \( f(s) \leq a - bs^n \) for all \( s \geq 0 \) and some \( \eta > 0 \), Li and Xiang [18] established global boundedness of solutions for three cases, \( \bar{R} < \bar{u} \) with \( \eta \geq 1 \), \( \bar{R} < \bar{u} \) with \( \eta = 2 \) (or \( \eta = 2, \text{properly large} \)), and \( \bar{R} = \bar{u} \) with \( \eta > \frac{1}{2} (\sqrt{n^2 + 4n} - n + 2) \). Later on, Xu and Zheng [41] weakened the condition \( \eta > \frac{1}{2} (\sqrt{n^2 + 4n} - n + 2) \) to \( \eta > \frac{2^n + 2}{n^2 + 2} \) in the case of \( \bar{R} = \bar{u} \). Moreover, Li and Wang [19] showed that the solutions are globally bounded when \( \eta = \frac{3}{2} \) and \( \Omega \subseteq \mathbb{R}^4 \) for the balance case \( \bar{R} = \bar{u} \), and thus partially improved the result in [41]. When \( \tau = 1 \) and \( f \equiv 0 \), Tao and Wang [30] showed that the system is globally well-posed in two dimensions if \( \bar{R} > \bar{u} \) and \( \beta = \delta \), which was improved by Liu and Tao in [20] by removing \( \beta = \delta \). Recently, Jin and Wang [12] showed that if \( \frac{\bar{R}}{\bar{u}_0} \geq \max \left\{ \frac{\bar{R}}{\bar{u}_0}, \frac{\bar{R}}{\bar{u}_0}, \frac{\bar{R}}{\bar{u}_0} \right\} \), then (1) admits a unique global solution which converges to the constant steady state \( (\bar{u}_0, \frac{\bar{R}}{\bar{u}_0}, \frac{\bar{R}}{\bar{u}_0}) \) as \( t \to \infty \), where \( \bar{u}_0 = \frac{1}{|\Omega|} \int_{\Omega} u_0 \). Assuming that \( \tau = 1 \) and \( f \) is logistic, Li et al. [14] proved that (1) possesses a unique globally bounded classical solution under the conditions that \( \min \{\beta, \delta\} \geq \frac{1}{2} \) and \( \mu \geq \max \{\frac{\bar{R}}{\bar{u}_0} \bar{u}_0 + \frac{\bar{R}}{\bar{u}_0} \delta \} \), \( \eta \geq \frac{3}{2} \) and \( \Omega \subseteq \mathbb{R}^3 \) is a smoothly bounded convex domain and \( f(s) = s - \mu s^{1+\kappa} \) for some \( \mu > 0 \) and \( \kappa \geq 1 \). Moreover, whenever \( u_0 \neq 0 \) and for any \( \kappa \in \mathbb{N} \), the solution approaches the steady state \( (\frac{1}{\mu})^{\frac{1}{\kappa}}, (\frac{1}{\mu})^{\frac{1}{\kappa}}, (\frac{1}{\mu})^{\frac{1}{\kappa}} \) as \( t \to \infty \). Later on, Li et al. [15] showed that the system possesses a unique
globally bounded classical solution if \( k > 1 \) or \( k = 1 \) with \( \mu > C_n \mu^* \) for some \( \mu^* \) and positive \( C_n \) when \( \Omega \subset \mathbb{R}^n \) (\( n \geq 1 \)) and \( f(s) = \kappa s - \mu s^{1+k} \) with \( \kappa \in \mathbb{R} \), \( \mu > 0 \), and \( k \geq 1 \). Moreover, when \( \kappa < 0 \) (respectively, \( \kappa = 0 \)), the corresponding solution of the system decays to \((0,0,0)\) exponentially (respectively, algebraically), and when \( \kappa > 0 \), the solution converges to \( \left( (\frac{\chi}{\mu})^\frac{1}{k}, \frac{\chi}{\mu} \left( \frac{n}{\mu} \right)^\frac{1}{k}, \frac{\chi}{\mu} \left( \frac{n}{\mu} \right)^\frac{1}{k} \right) \) exponentially if \( \mu \) is large enough. Recently, there have been some advances on nonlinear variants involved in diffusivity and chemosensitivity as well as signal production. To name a few, see [35, 48, 44, 16, 13, 10].

In nature, populations always interact with each other. Studies have confirmed that interactions of several populations via chemotactic mechanisms play an important role in various biological processes [11, 27]. So far, existing literature is scarce that interactions of several populations via chemotactic mechanisms play an important role in various biological processes. Therefore, recently, there have been some advances on nonlinear variants involved in diffusivity and chemosensitivity as well as signal production. To name a few, see [35, 48, 44, 16, 13, 10].

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This inspires us to consider the following system,

\[
\begin{aligned}
&u_t = \Delta u + \chi \nabla \cdot (u \nabla z) - \xi \nabla \cdot (u \nabla v), \\
&\tau v_t = \Delta v - v + \omega, \\
&\omega_t = \Delta \omega + \chi \nabla \cdot (\omega \nabla v) - \xi \nabla \cdot (\omega \nabla z), \\
&\tau z_t = \Delta z - z + u,
\end{aligned}
\]

(3)

where \( \tau \in \{0, 1\} \) and \( \chi, \tilde{\chi}, \xi, \tilde{\xi} > 0 \). System (3) describes a situation of self-generated chemoattractant which is chemoattractant for the other. The goal of this paper is to establish the global existence and boundedness of classical solutions of system (3). Precisely, we obtain the following main results.

**Theorem 1.1.** Let \( \Omega \subseteq \mathbb{R}^n \) \( (n \geq 1) \) be a smoothly bounded domain and all the other parameters \( \chi, \tilde{\chi}, \xi, \tilde{\xi} \) be positive constants. Then for any positive initial data \( (u_0, \omega_0) \in (C^0(\bar{\Omega}))^2 \), if \( \min \{\chi, \tilde{\chi}\} > \xi + \tilde{\xi} \), system (3) with \( \tau = 0 \) possesses a global classical solution \( (u, v, \omega, z) \) which is determined by the properties that

\[
\|u\|_{L^\infty(\Omega)} + \|v\|_{L^\infty(\Omega)} + \|\omega\|_{L^\infty(\Omega)} + \|z\|_{L^\infty(\Omega)} \leq C \quad \text{for all } t \geq 0.
\]

Theorem 1.1 indicates that strong repulsive effect is conducive to the global existence of solutions.

**Theorem 1.2.** Let \( \Omega \subseteq \mathbb{R} \) be a bounded open interval and all the other parameters \( \chi, \tilde{\chi}, \xi, \tilde{\xi} \) be positive constants. Then for any positive initial data \( (u_0, \omega_0, \tau, \tau_0) \in (W^{1,2}(\Omega))^4 \), system (3) with \( \tau = 1 \) possesses a global classical solution \( (u, v, \omega, z) \) which is uniquely determined by the properties that

\[
\|u\|_{W^{1,2}(\Omega)} + \|v\|_{W^{1,2}(\Omega)} + \|\omega\|_{W^{1,2}(\Omega)} + \|z\|_{W^{1,2}(\Omega)} \leq C \quad \text{for all } t \geq 0.
\]

**Theorem 1.3.** Let \( \Omega \subseteq \mathbb{R}^2 \) be a smoothly bounded domain and all the other parameters \( \chi, \tilde{\chi}, \xi, \tilde{\xi} \) be positive constants. Assume that positive initial data \( (u_0, \omega_0, \tau, \tau_0) \in C^0(\bar{\Omega}) \times W^{1,q}(\Omega) \times C^0(\bar{\Omega}) \times W^{1,q}(\Omega) \) with \( q \in (2, \infty) \) satisfy

\[
\|u_0\|_{L^1(\Omega)} < \frac{4}{C_{GN}[1 + (\chi^2 + \xi^2)]} \quad \text{and} \quad \|\omega_0\|_{L^1(\Omega)} < \frac{4}{C_{GN}[1 + (\tilde{\chi}^2 + \tilde{\xi}^2)]},
\]

where \( C_{GN} \) is a positive constant from the Gagliardo-Nirenberg interpolation inequality (14). Then system (3) with \( \tau = 1 \) possesses a global classical solution \( (u, v, \omega, z) \) which is determined by the properties that

\[
\begin{aligned}
&\{(u, \omega) \in \left( C^0(\bar{\Omega} \times [0, \infty)) \right) \cap C^{2,1}(\Omega \times (0, \infty))\}^2, \\
&\{(v, z) \in \left( C^0(\bar{\Omega} \times [0, \infty)) \right) \cap C^{2,1}(\Omega \times (0, \infty)) \cap L^\infty_{loc}([0, \infty); W^{1,q}(\Omega))\}^2,
\end{aligned}
\]
and which is such that $u,v,\omega,z > 0$ in $\Omega \times (0,\infty)$. Moreover, the solution $(u,v,\omega,z)$ is bounded in the sense that there exists positive constant $C$ such that

$$
\|u\|_{L^\infty(\Omega)} + \|v\|_{W^{1,q}(\Omega)} + \|\omega\|_{L^\infty(\Omega)} + \|z\|_{W^{1,q}(\Omega)} \leq C \quad \text{for all } t \geq 0.
$$

Theorem 1.3 tells us that small initial mass is beneficial to the global existence of solutions. Moreover, we guess that the methods used in the proof of Theorems 1.2-1.3 can also be applied to system (2).

Before giving the proof of the main results in Sections 3–5, we first show the local existence of a classical solution to (3) and provide some preliminary results. The paper ends with a brief summary and discussion.

2. Local existence of solutions and preliminaries. As far as Theorem 1.1 is concerned, the proof of the local existence of solutions in time is similar to those in [30, 33], which is achieved by employing a fixed point theorem. We can give the following result directly.

**Proposition 2.1.** Let $\Omega \subseteq \mathbb{R}^n$ ($n \geq 1$) be a smoothly bounded domain and all the other parameters $\chi, \tilde{\chi}, \xi, \tilde{\xi}$ be positive constants. Then for positive initial data $(u_0, v_0, \omega_0, z_0) \in C^0(\Omega) \times C^1(\Omega) \times C^0(\bar{\Omega}) \times C^1(\Omega)$, system (3) with $\tau = 0$ admits a local-in-time positive classical solution $(u, v, \omega, z) \in (C^0(\Omega \times [0,T_{\max}))) \cap C^{2,1}(\bar{\Omega} \times (0,T_{\max}))^4$. Here $T_{\max}$ denotes the maximum existence time. Moreover if $T_{\max} < \infty$, then

$$
\limsup_{t \to T_{\max}^-} (\|u\|_{L^\infty(\Omega)} + \|v\|_{L^\infty(\Omega)} + \|\omega\|_{L^\infty(\Omega)} + \|z\|_{L^\infty(\Omega)}) = \infty.
$$

For Theorem 1.2, the proof of the local existence of solutions in time is similar to that in [31], which is achieved by the standard theory on evolution systems of parabolic type. We also give the following result directly.

**Proposition 2.2.** Let $\Omega \subseteq \mathbb{R}$ be a bounded open interval and all the other parameters $\chi, \tilde{\chi}, \xi, \tilde{\xi}$ be positive constants. Then for any positive initial data $(u_0, v_0, \omega_0, z_0) \in (W^{1,\infty}(\Omega))^4$, there exist $T_{\max} \in (0,\infty]$ and positive functions $(u, v, \omega, z)$ in $\Omega \times (0,T_{\max})$ uniquely determined by

$$
(u, v, \omega, z) \in (C^0([0,T_{\max}]); W^{1,2}(\Omega)) \cap C^{2,1}(\bar{\Omega} \times (0,T_{\max}))^4,
$$

which solve (3) with $\tau = 1$ in the classical sense in $\Omega \times (0,T_{\max})$, and if $T_{\max} < \infty$, then for all $q \in (1,\infty)$,

$$
\limsup_{t \to T_{\max}^-} (\|u\|_{W^{1,q}(\Omega)} + \|v\|_{W^{1,q}(\Omega)} + \|\omega\|_{W^{1,q}(\Omega)} + \|z\|_{W^{1,q}(\Omega)}) = \infty.
$$

While for Theorem 1.3, the proof of the local existence of solutions in time is similar to that in [3], which is achieved by applying Banach’s fixed point theorem.

**Proposition 2.3.** Let $\Omega \subseteq \mathbb{R}^n$ ($n \geq 1$) be a smoothly bounded domain and all the other parameters $\chi, \tilde{\chi}, \xi, \tilde{\xi}$ be positive constants. Then for any positive initial data $(u_0, v_0, \omega_0, z_0) \in C^0(\Omega) \times W^{1,q}(\Omega) \times C^0(\bar{\Omega}) \times W^{1,q}(\Omega)$ with $q \in (n,\infty)$, there exist $T_{\max} \in (0,\infty]$ and positive functions $u, v, \omega, z$ in $\Omega \times (0,T_{\max})$ determined by

$$
\begin{align*}
\left\{ (u,\omega) \in (C^0(\Omega \times [0,T_{\max}])) \cap C^{2,1}(\bar{\Omega} \times (0,T_{\max})) \right\}^2,
\end{align*}
$$

$$
\begin{align*}
\left\{ (v,z) \in (C^0(\Omega \times [0,T_{\max}])) \cap C^{2,1}(\bar{\Omega} \times (0,T_{\max})) \cap L^\infty(\Omega; W^{1,q}(\Omega)) \right\}^2,
\end{align*}
$$

which solve (3) with $\tau = 1$ in the classical sense in $\Omega \times (0,T_{\max})$, and if $T_{\max} < \infty$, then

$$
\limsup_{t \to T_{\max}^-} (\|u\|_{L^\infty(\Omega)} + \|v\|_{W^{1,q}(\Omega)} + \|\omega\|_{L^\infty(\Omega)} + \|z\|_{W^{1,q}(\Omega)}) = \infty.
$$
The following result gives some fundamental properties of solutions to problem (3).

**Lemma 2.4.** Assume that the positive initial data \((u_0, v_0) \in C^0(\bar{\Omega}) \times C^0(\bar{\Omega})\) when \(\tau = 0\), while positive initial data \((u_0, v_0, \omega_0, z_0) \in (W^{1,\infty}(\Omega))^4\) or \((u_0, v_0, \omega_0, z_0) \in C^0(\bar{\Omega}) \times W^{1,q}(\Omega) \times C^0(\bar{\Omega}) \times W^{1,q}(\Omega)\) with \(q \in (n, \infty)\) when \(\tau = 1\). Then

\[
\| u(t) \|_{L^1(\Omega)} = \| u_0 \|_{L^1(\Omega)}, \quad \| v(t) \|_{L^1(\Omega)} = \max \{ \| v_0 \|_{L^1(\Omega)}, \| \omega_0 \|_{L^1(\Omega)} \}, \\
\| \omega(t) \|_{L^1(\Omega)} = \| \omega_0 \|_{L^1(\Omega)}, \quad \| z(t) \|_{L^1(\Omega)} = \max \{ \| z_0 \|_{L^1(\Omega)}, \| u_0 \|_{L^1(\Omega)} \}.
\]

**Proof.** Integrating each equation in system (3) over \(\Omega\) yields the desired results. \(\square\)

Next, the following result is trivial, which can be completed by using well-known smoothing properties of the Neumann heat semigroup and appropriately modifying the proof in [3, Lemma 3.2].

**Proposition 2.5.** Let \(\Omega \subset \mathbb{R}^n\) \((n \geq 1)\) be a smoothly bounded domain and all the other parameters \(\lambda, \tilde{\lambda}, \tilde{\xi}\) be positive constants. Assume positive initial data \((u_0, \omega_0) \in C^0(\bar{\Omega}) \times C^0(\bar{\Omega})\) and \((v_0, z_0) \in W^{1,q}(\Omega) \times W^{1,q}(\Omega)\) with \(q \in (n, \infty)\) as well as \((u, v, \omega, z)\) is a positive classical solution of (3) when \(\tau = 1\) in \(\Omega \times (0, T)\) with \(T \in (0, \infty)\). If there exist \(M > 0\) and \(p \geq 1\) with \(p > \frac{q}{2}\) such that

\[
\| u \|_{L^p(\Omega)} \leq M \quad \text{and} \quad \| \omega \|_{L^p(\Omega)} \leq M \quad \text{for all} \quad t \in (0, T),
\]

then we can find a positive constant \(C\) fulfilling

\[
\| u \|_{L^\infty(\Omega)} + \| v \|_{W^{1,q}(\Omega)} + \| \omega \|_{L^\infty(\Omega)} + \| z \|_{W^{1,q}(\Omega)} \leq C \quad \text{for all} \quad t \in (0, T).
\]

We also recall the Gagliardo-Nirenberg interpolation inequality, which will be used frequently in the proof of our main results.

**Lemma 2.6 ([6]).** Let \(l\) and \(k\) be two integers satisfying \(l \in [0, k)\). Suppose that \(q, r \in [1, \infty)\), \(p > 0\), and \(a \in \left[ \frac{1}{k}, 1 \right]\) such that

\[
\frac{1}{p} - \frac{1}{n} = a \left( \frac{1}{q} - \frac{k}{n} \right) + (1 - a) \frac{1}{r}.
\]

Then, for any \(\phi \in W^{k,q}(\Omega) \cap L^r(\Omega)\), there exist two positive constants \(c_1\) and \(c_2\) depending only on \(\Omega, q, k, r,\) and \(n\) such that

\[
\| D^l \phi \|_{L^p(\Omega)} \leq c_1 \| D^k \phi \|_{L^q(\Omega)}^a \| \phi \|_{L^r(\Omega)}^{1-a} + c_2 \| \phi \|_{L^r(\Omega)}
\]

with the following exception: If \(q \in (1, \infty)\) and \(k - l - \frac{n}{q}\) is a non-negative integer, then (11) holds only for \(a \in \left[ \frac{1}{k}, 1 \right]\). Here \(D^k \phi\) is expressed as Fréchet derivative of order \(k\). In particular, when \(n = 2\), we have

\[
\| \phi \|_{L^4(\Omega)}^4 \leq C_{GN} \| \nabla \phi \|_{L^2(\Omega)}^2 \| \phi \|_{L^2(\Omega)}^2 + \| \phi \|_{L^2(\Omega)}^4 \quad \text{for} \quad \phi \in W^{1,2}(\Omega),
\]

where \(C_{GN}\) is some positive constant only depending on \(\Omega\).

We also need the following generalization of the Gagliardo-Nirenberg inequality for the general case of \(r > 0\), which extends the standard case with \(r \geq 1\) in [5].

**Lemma 2.7 ([32, Lemma A.5]).** Let \(\Omega \subset \mathbb{R}^2\) be a bounded domain with smooth boundary. Then there exists \(C > 0\) such that, for each \(\eta > 0\), one can pick \(C_\eta > 0\) with the property that

\[
\| \phi \|_{L^3(\Omega)}^3 \leq \eta \| \nabla \phi \|_{L^2(\Omega)}^2 \| \ln \phi \|_{L^1(\Omega)} + C \| \phi \|_{L^1(\Omega)}^3 + C_\eta \quad \text{for} \quad \phi \in W^{1,2}(\Omega).
\]
3. **Proof of Theorem 1.1.** We first prove Theorem 1.1, which is based on the superposition argument, Agmon-Douglis-Nirenberg $L^p$ estimates, Sobolev embedding, and existing Moser-type iterative method [30].

**Proof of Theorem 1.1.** We first define the following parameters

$$
\epsilon_1 = \frac{(k + 1)A}{3k\chi}, \quad \epsilon = \frac{k(k+1)A}{3k\chi}, \quad \eta = \frac{A(k+1)(k-1)}{3k},
$$

$$
\epsilon_2 = \frac{(k + 1)B}{3k\chi}, \quad \tilde{\epsilon} = \frac{k(k+1)B}{3k\chi}, \quad \tilde{\eta} = \frac{B(k+1)(k-1)}{3k},
$$

where $A = \chi - (\xi + \tilde{\xi})$ and $B = \tilde{\chi} - (\xi + \tilde{\xi})$. Later on, we will employ the above parameters in (24).

Multiplying the first equation of (3) by $u^{k-1}$ ($k > \max\{\frac{n}{2}, 1\}$) and integrating by parts over $\Omega$, with the assistance of the fourth equation of (3), we can deduce that

$$
\frac{1}{k} \frac{d}{dt} \int_{\Omega} u^k + \frac{4(k - 1)}{k^2} \int_{\Omega} |\nabla u^k|^2 = -\chi(k - 1) \int_{\Omega} u^{k-1} \nabla u \cdot \nabla z + \xi(k - 1) \int_{\Omega} u^{k-1} \nabla u \cdot \nabla v
$$

$$
= \frac{\chi(k - 1)}{k} \int_{\Omega} u^k \Delta z - \frac{\xi(k - 1)}{k} \int_{\Omega} u^k \Delta v
$$

$$
= \frac{\chi(k - 1)}{k} \int_{\Omega} u^k z - \frac{\chi(k - 1)}{k} \int_{\Omega} u^{k+1} + \frac{\xi(k - 1)}{k} \int_{\Omega} u^k \omega - \frac{\xi(k - 1)}{k} \int_{\Omega} u^k v.
$$

In order to estimate the integrals $\int_{\Omega} u^k z$ and $\int_{\Omega} u^k \omega$ on the right-hand side of (16), we first apply Young’s inequality to obtain

$$
\int_{\Omega} u^k z \leq \frac{k\epsilon_1}{k + 1} \int_{\Omega} u^{k+1} + \frac{\epsilon_{1-k}}{k + 1} \int_{\Omega} z^{k+1}, \int_{\Omega} u^k \omega \leq \frac{k}{k + 1} \int_{\Omega} u^{k+1} + \frac{1}{k + 1} \int_{\Omega} \omega^{k+1}.
$$

Inserting the above estimates into (16) gives us

$$
\frac{1}{k} \frac{d}{dt} \int_{\Omega} u^k + \frac{4(k - 1)}{k^2} \int_{\Omega} |\nabla u^k|^2 \leq \frac{\chi(k - 1)\epsilon_1}{k + 1} \int_{\Omega} u^{k+1} + \frac{\chi(k - 1)\epsilon_{1-k}}{k(k + 1)} \int_{\Omega} z^{k+1} - \frac{\chi(k - 1)}{k} \int_{\Omega} u^{k+1} \int_{\Omega} \omega^{k+1} + \frac{\xi(k - 1)}{k + 1} \int_{\Omega} u^{k+1} \int_{\Omega} \omega^{k+1}
$$

$$
+ \frac{\xi(k - 1)}{k + 1} \int_{\Omega} u^{k+1} \int_{\Omega} \omega^{k+1} - \frac{\xi(k - 1)}{k} \int_{\Omega} u^k v.
$$

The same argument leads to

$$
\frac{1}{k} \frac{d}{dt} \int_{\Omega} \omega^k + \frac{4(k - 1)}{k^2} \int_{\Omega} |\nabla \omega^k|^2 \leq \frac{\tilde{\chi}(k - 1)\epsilon_2}{k + 1} \int_{\Omega} \omega^{k+1} + \frac{\tilde{\chi}(k - 1)\epsilon_{2-k}}{k(k + 1)} \int_{\Omega} u^{k+1} - \frac{\tilde{\chi}(k - 1)}{k} \int_{\Omega} \omega^{k+1} \int_{\Omega} \omega^{k+1}
$$

$$
+ \frac{\tilde{\xi}(k - 1)}{k + 1} \int_{\Omega} \omega^{k+1} + \frac{\tilde{\xi}(k - 1)}{k(k + 1)} \int_{\Omega} u^{k+1} - \frac{\tilde{\xi}(k - 1)}{k} \int_{\Omega} \omega^k z.
$$
Adding up (17) and (18) produces
\[
\frac{1}{k} \frac{d}{dt} \int_{\Omega} u^k + \frac{1}{k} \frac{d}{dt} \int_{\Omega} \omega^k + \frac{4(k-1)}{k^2} \int_{\Omega} |\nabla \omega|^2 + \frac{4(k-1)}{k^2} \int_{\Omega} |\nabla u|^2 \leq -\frac{k-1}{k} \left[ \chi - (\xi + \tilde{\xi}) \right] \int_{\Omega} u^{k+1} + \frac{\chi(k-1)\epsilon_1}{k+1} \int_{\Omega} u^{k+1} + \frac{\chi(k-1)\epsilon_1^2}{k(k+1)} \int_{\Omega} \omega^{k+1} \quad (19)
\]

Now we show that the integrals \( \int_{\Omega} \omega^{k+1} \) and \( \int_{\Omega} \omega^{k+1} \) can be controlled by \( \eta_1 \int_{\Omega} u^{k+1} + c_1 \) and \( \eta_2 \int_{\Omega} u^{k+1} + c_2 \), respectively, for sufficiently small \( \eta_1 > 0 \) and \( \eta_2 > 0 \) as well as some constants \( c_1 > 0 \) and \( c_2 > 0 \). Note that \( z \) solves
\[
\begin{align*}
-\Delta z + z &= u, \\
\frac{\partial z}{\partial \nu} &= 0, \\
(x, t) &\in \Omega \times (0, T_{\text{max}}), \\
(x, t) &\in \partial \Omega \times (0, T_{\text{max}}).
\end{align*}
\]

Applying the Agmon-Douglis-Nirenberg \( L^p \) estimates [1, 2] on linear elliptic equations with the (zero) Neumann boundary condition, we derive that there exists some positive constant \( c_3 \) such that
\[
\|z\|_{W^{2,k}(\Omega)} \leq c_3 \|u\|_{L^{k}(\Omega)} \quad \text{for} \ t \in (0, T_{\text{max}}).
\]

We interpolate the Gagliardo-Nirenberg inequality (11) and (21) to obtain some positive constants \( c_2 \) and \( c_3 \) such that
\[
\int_{\Omega} \omega^{k+1} = \|z\|_{L^{k+1}(\Omega)} \leq c_2 \|D^2 z\|_{L^k(\Omega)} \|z\|^{(k+1)(1-\theta)}_{L^1(\Omega)} + c_2 \|z\|_{L^{k}(\Omega)} \leq c_3 \|u\|^{(k+1)\theta}_{L^k(\Omega)} + c_3,
\]

where \( \theta = \frac{1-\frac{1}{k}}{1+\frac{1}{k}} = \frac{n\epsilon^2}{(k+1)(n+2k-2)} \in (0, 1) \). Furthermore, it is easy to check that \((k+1)\theta < k\). We then apply the Young’s inequality to further obtain
\[
\int_{\Omega} \omega^{k+1} \leq c_3 \|u\|^{(k+1)\theta}_{L^k(\Omega)} + c_3 \\
\leq c_3 \|u\|_{L^k(\Omega)}^{(k+1)\theta} + 1 + c_3 \\
= c_3 \int_{\Omega} u^k + 2c_3 \\
\leq c_3 \epsilon \int_{\Omega} u^{k+1} + c_4(\epsilon).
\]

Similarly, we can get
\[
\int_{\Omega} v^{k+1} \leq \tilde{c}_3 \epsilon \int_{\Omega} \omega^{k+1} + \tilde{c}_4(\epsilon).
\]

Here \( \tilde{c}_3 > 0 \) is a constant. Inserting (22) and (23) into (19) as well as applying Young’s inequality to the integrals \( \int_{\Omega} u^k \) and \( \int_{\Omega} \omega^k \), we derive
\[
\frac{1}{k} \frac{d}{dt} \int_{\Omega} u^k + \frac{1}{k} \frac{d}{dt} \int_{\Omega} \omega^k + \frac{1}{k} \int_{\Omega} u^k + \frac{1}{k} \int_{\Omega} \omega^k \leq -\frac{k-1}{k} \left[ \chi - (\xi + \tilde{\xi}) \right] \int_{\Omega} u^{k+1} \\
+ \frac{\chi(k-1)\epsilon_1}{k+1} \int_{\Omega} u^{k+1} + \frac{\chi(k-1)\epsilon_1^2 c_3\epsilon}{k(k+1)} \int_{\Omega} u^{k+1} + \frac{\eta_1}{k+1} \int_{\Omega} u^{k+1}
\]
Inserting (26) and (27) into (16) produces
\[
\frac{\chi(k-1)c_1^k}{k(k+1)} + \frac{\eta^{-k}[\Omega]}{k(k+1)} - \frac{k-1}{k} [\tilde{\chi} - (\xi + \tilde{\xi})] \int_{\Omega} \omega^{k+1} \\
+ \frac{\tilde{\chi}(k-1)c_2^k}{k+1} + \frac{\tilde{\chi}(k-1)c_3^k}{k(k+1)} + \frac{\tilde{\eta}^{-k}[\Omega]}{k(k+1)} \int_{\Omega} \omega^{k+1} \\
+ \frac{\tilde{\chi}(k-1)c_4^k}{k(k+1)} + \frac{\tilde{\eta}^{-k}[\Omega]}{k(k+1)}.
\]  
(24)

Thanks to the assumption \(\min\{\chi, \tilde{\chi}\} > \xi + \tilde{\xi}\), we deduce that there exists some positive constant \(c_5\) such that
\[
d \frac{d}{dt} \left\{ \int_{\Omega} u^k + \int_{\Omega} \omega^k \right\} \\
\leq \int_{\Omega} u^k + \int_{\Omega} \omega^k \leq c_5.
\]  
(25)

By an ODE comparison argument, (25) leads to
\[
\int_{\Omega} u^k + \int_{\Omega} \omega^k \leq \max \left\{ \int_{\Omega} u_0^k + \int_{\Omega} \omega_0^k, c_5 \right\}.
\]

Recall that \(z\) satisfies (20) and \(v\) satisfies
\[
\begin{align*}
-\Delta v + v &= \omega, \quad (x, t) \in \Omega \times (0, T_{\max}), \\
\frac{\partial v}{\partial n} &= 0, \quad (x, t) \in \partial \Omega \times (0, T_{\max}).
\end{align*}
\]

Applying the Agmon-Douglis-Nirenberg \(L^p\) estimates [1, 2] on linear elliptic equations with the (zero) Neumann boundary condition along with the Sobolev embedding [8]
\[
W^{2,n+1}(\Omega) \hookrightarrow C^1(\bar{\Omega}),
\]
we can derive that there exist positive constants \(c_6, c_7, c_8\) such that
\[
\|\nabla v\|_{L^\infty(\Omega)} \leq \|v\|_{C^1(\bar{\Omega})} \leq c_6 \|v\|_{W^{2,n+1}(\Omega)} \leq c_7 \|\omega\|_{L^{n+1}(\Omega)} \leq c_8
\]  
(26)

and
\[
\|\nabla z\|_{L^\infty(\Omega)} \leq \|z\|_{C^1(\bar{\Omega})} \leq c_6 \|z\|_{W^{2,n+1}(\Omega)} \leq c_7 \|u\|_{L^{n+1}(\Omega)} \leq c_8.
\]  
(27)

Inserting (26) and (27) into (16) produces
\[
\frac{1}{k} \frac{d}{dt} \int_{\Omega} u^k \leq -\frac{4(k-1)}{k^2} \int_{\Omega} \left| \nabla u^k \right|^2 + (k-1) \int_{\Omega} u^{k-1} \nabla u \cdot (\Omega \nabla v - \chi \nabla z) \\
\leq -\frac{4(k-1)}{k^2} \int_{\Omega} \left| \nabla u^k \right|^2 + (k-1)(\chi + \xi) c_8 \int_{\Omega} u^{k-1} |\nabla u|.
\]

Based on the Moser-type iterative method [30, Theorem 2.1], we obtain
\[
\sup_{(0,T_{\max})} \|u\|_{L^\infty(\Omega)} < \infty \quad \text{and} \quad \sup_{(0,T_{\max})} \|\omega\|_{L^\infty(\Omega)} < \infty.
\]

According to the well-known elliptic maximum principle, we get
\[
\|v(\cdot, t)\|_{L^\infty(\Omega)} \leq \|\omega(\cdot, t)\|_{L^\infty(\Omega)} < \infty \quad \text{and} \quad \|z(\cdot, t)\|_{L^\infty(\Omega)} \leq \|u(\cdot, t)\|_{L^\infty(\Omega)} < \infty.
\]

As described above, we deduce
\[
\sup_{(0,T_{\max})} \left( \|u\|_{L^\infty(\Omega)} + \|v\|_{L^\infty(\Omega)} + \|\omega\|_{L^\infty(\Omega)} + \|z\|_{L^\infty(\Omega)} \right) < \infty.
\]

With the aid of Proposition 2.1, we get \(T_{\max} = \infty\). This completes the proof. \(\Box\)

4. Proof of Theorem 1.2. Inspired by [31], we need some preparations to prove Theorem 1.2. We divide this section into the following three subsections.
4.1. Estimates for \( \|v_x\|_{L^q(\Omega)} \) and \( \|z_x\|_{L^q(\Omega)} \). We will employ well-known smoothing properties of the Neumann heat semigroup to deduce the boundedness for \( \|v_x\|_{L^q(\Omega)} \) and \( \|z_x\|_{L^q(\Omega)} \) by acting on the second and fourth equations in (3). Under the condition that \( q \) is finite, we shall use the fact that the spatial framework is one-dimensional, i.e. \( \Omega \subseteq \mathbb{R}^1 \).

**Proposition 4.1.** Let \( M = \int_{\Omega} u_0 + \int_{\Omega} \omega_0 \) and \( q \in (1, \infty) \). Then for any positive initial data \((u_0, v_0, \omega_0, z_0) \in (W^{1,\infty}(\Omega))^4\), there exist positive constants \( K(M, q) \) and \( C = C(v_0, z_0) \) such that the positive classical solution \((u, v, \omega, z) \) of (3) with \( \tau = 1 \) has the properties that for all \( t \in (0, T_{\text{max}}) \),

\[
\|v_x\|_{L^q(\Omega)} \leq K(M, q) + Ce^{-t} \quad \text{and} \quad \|z_x\|_{L^q(\Omega)} \leq K(M, q) + Ce^{-t}.
\]  

**Proof.** By means of the properties of the Neumann heat semigroup \((e^{t\sigma \Delta})_{t \geq 0}\) on \( \Omega \) with \( \Delta = -(\gamma \Delta)_{x}(\Omega) \), we can fix positive \( c_i(q) \) \((i = 1, 2)\) such that for all \( t > 0 \),

\[
\|\partial_x e^{t\sigma \Delta} \phi\|_{L^q(\Omega)} \leq c_1(q)\|\phi\|_{W^{1,\infty}(\Omega)} \quad \text{for all } \phi \in W^{1,\infty}(\Omega)
\]  

and

\[
\|\partial_x e^{t\sigma \Delta} \phi\|_{L^q(\Omega)} \leq c_2(q)(1 + t^{-1} + \frac{1}{q})\|\phi\|_{L^1(\Omega)} \quad \text{for all } \phi \in C^0(\bar{\Omega}).
\]  

Representing \( v \) according to variation-of-constants formula yields

\[
v = e^{t(\sigma - \Delta)} v_0 + \int_0^t e^{(t-s)(\sigma - \Delta)} \omega \text{ for all } t \in (0, T_{\text{max}}).
\]  

Combining (29) with (30), we can estimate

\[
\|v_x\|_{L^q(\Omega)} \leq c_1(q)\|v_0\|_{W^{1,\infty}(\Omega)} + c_2(q) \int_0^t e^{-(t-s)}[1 + (t-s)^{-1} + \frac{1}{q^2}]\|\omega\|_{L^1(\Omega)}
\]

\[
\leq c_1(q)\|v_0\|_{W^{1,\infty}(\Omega)} + c_2(q)\|\omega_0\|_{L^1(\Omega)} \int_0^t e^{-(t-s)}[1 + (t-s)^{-1} + \frac{1}{q^2}]
\]

\[
\leq 2(1 + q)c_2(q)\|\omega_0\|_{L^1(\Omega)} + c_1(q)\|v_0\|_{W^{1,\infty}(\Omega)} e^{-t}, \quad \text{for } t \in (0, T_{\text{max}}).
\]  

Here it is easy to check

\[
\int_0^t e^{-(t-s)}[1 + (t-s)^{-1} + \frac{1}{q^2}]ds
\]

\[
= \int_0^t e^{-s}[1 + s^{-1} + \frac{1}{q^2}]ds
\]

\[
\leq \int_0^\infty e^{-s}[1 + s^{-1} + \frac{1}{q^2}]ds
\]

\[
= \int_0^\infty e^{-s}ds + \int_0^\infty e^{-s} \cdot s^{-1} + \frac{1}{q^2} < 2 + 2q.
\]  

The similar argument as that for \( v \) leads to

\[
\|z_x\|_{L^q(\Omega)} \leq 2(1 + q)c_2(q)\|u_0\|_{L^1(\Omega)} + c_1(q)\|z_0\|_{W^{1,\infty}(\Omega)} e^{-t}
\]  

for all \( t \in (0, T_{\text{max}}) \). Then the desired results follow immediately. \(\square\)
4.2. Estimates for \( \|u\|_{L^\infty(\Omega)}, \|v\|_{L^\infty(\Omega)}, \|\omega\|_{L^\infty(\Omega)}, \text{ and } \|z\|_{L^\infty(\Omega)} \). With the estimates (28) at hand, one can employ Gagliardo-Nirenberg inequality and Young’s inequality to get the estimates for \( \int_\Omega u^2, \int_\Omega v^2, \int_\Omega \omega^2, \int_\Omega z^2 \). Then we will employ them to get the estimates for \( \|u\|_{L^\infty(\Omega)}, \|v\|_{L^\infty(\Omega)}, \|\omega\|_{L^\infty(\Omega)}, \|z\|_{L^\infty(\Omega)} \), which are very important in obtaining (59).

**Proposition 4.2.** Let \( M = \int_\Omega u_0 + \int_\Omega \omega_0 \). Then for any positive initial data \((u_0, v_0, \omega_0, z_0) \in (W^{1,\infty}(\Omega))^4\), there exist positive constants \( K(M), C = C(u_0, v_0, \omega_0, z_0) \), and \( \alpha \) such that the positive classical solution \((u, v, \omega, z)\) of (3) with \( \tau = 1 \) has the properties that for all \( t \in (0, T_{\max}) \),

\[
\int_\Omega u^2 \leq K(M)u_0^2 + Ce^{-\alpha t}, \quad \int_\Omega \omega^2 \leq K(M)\omega_0^2 + Ce^{-\alpha t},
\]

\[
\int_\Omega v^2 \leq K(M)\omega_0^2 + Ce^{-\alpha t}, \quad \int_\Omega z^2 \leq K(M)\omega_0^2 + Ce^{-\alpha t},
\]  

(35)

and for all \( t \in (0, T_{\max} - \zeta) \),

\[
\int_t^{t+\zeta} \int_\Omega u^2 \leq K(M)u_0^2 + Ce^{-\alpha t}, \quad \int_t^{t+\zeta} \int_\Omega \omega^2 \leq K(M)\omega_0^2 + Ce^{-\alpha t},
\]

\[
\int_t^{t+\zeta} \int_\Omega v^2 \leq K(M)\omega_0^2 + Ce^{-\alpha t}, \quad \int_t^{t+\zeta} \int_\Omega z^2 \leq K(M)\omega_0^2 + Ce^{-\alpha t}.
\]  

(36)

Here \( \zeta = \min\{1, \frac{1}{2}T_{\max}\} \).

**Proof.** Employing Proposition 4.1 to \( q = 2 \), we know that there exist positive constants \( c_1 = c_1(v_0, z_0) \) and \( k_1(M) \) such that for all \( t \in (0, T_{\max}) \),

\[
\|v_x\|_{L^2(\Omega)} \leq k_1(M) + c_1e^{-t} \text{ and } \|z_x\|_{L^2(\Omega)} \leq k_1(M) + c_1e^{-t},
\]  

(37)

where without loss of generality we may assume that \( k_1(M) \geq 1 \). Moreover, with the aid of the Gagliardo-Nirenberg inequality and Young’s inequality we can choose \( c_i > 0 \) \((i = 2, 3, 4)\) such that

\[
\|\phi\|_{L^\infty(\Omega)} \leq c_2\|\phi_x\|_{L^2(\Omega)}^{\frac{1}{2}}\|\phi\|_{L^1(\Omega)}^{\frac{1}{2}} + c_2\|\phi\|_{L^1(\Omega)} \quad \text{for all } \phi \in W^{1,2}(\Omega)
\]  

(38)

and

\[
\frac{1}{2}\|\phi\|_{L^2(\Omega)}^2 \leq \frac{1}{4}\|\phi_x\|_{L^2(\Omega)}^2 + c_3\|\phi\|_{L^1(\Omega)}^2 \quad \text{for all } \phi \in W^{1,2}(\Omega),
\]  

(39)

as well as

\[
c_2\chi_1|\Omega|^{\frac{1}{4}}ab \leq \frac{1}{8}a^2 + c_4b^6 \quad \text{for all } a \geq 0 \text{ and } b \geq 0.
\]  

(40)
Meanwhile, employing Young’s inequality, it is easy to obtain immediately from (42) that
\[
\leq c_2(\chi + \xi) \left[ \|u_x\|_{L^2(\Omega)}^\frac{3}{2} + 1 \right] \|u\|_{L^1(\Omega)} \cdot \|u\|_{L^1(\Omega)} \left[ k_1(M) + c_1 e^{-t} \right]
\]
\[
+ \frac{1}{4} \|u_x\|_{L^2(\Omega)}^2 + c_3 \|u\|_{L^1(\Omega)}^2.
\]
Moreover, according to (40), it is easy to verify
\[
c_2(\chi + \xi) |\Omega|^{\frac{3}{2}} u_0^\frac{1}{2} \|u_x\|_{L^2(\Omega)} \left[ k_1(M) + c_1 e^{-t} \right]
\]
\[
\leq \frac{1}{8} \left( \|u_x\|_{L^2(\Omega)}^\frac{3}{2} \right)^2 + c_4 u_0^2 \left[ k_1(M) + c_1 e^{-t} \right]^6.
\]
Meanwhile, employing Young’s inequality, it is easy to get
\[
c_2(\chi + \xi) |\Omega| u_0 \|u_x\|_{L^2(\Omega)} \left[ k_1(M) + c_1 e^{-t} \right]
\]
\[
\leq \frac{1}{8} \left( \|u_x\|_{L^2(\Omega)}^\frac{3}{2} \right)^2 + 2c_5^2(\chi + \xi)^2 |\Omega|^2 u_0^2 \left[ k_1(M) + c_1 e^{-t} \right]^2.
\]
Since $k_1(M) \geq 1$, we have
\[
\left[ k_1(M) + c_1 e^{-t} \right]^2 \leq \left[ k_1(M) + c_1 e^{-t} \right]^6 \leq 32 \left[ k_1^6(M) + c_1^6 e^{-6t} \right] \quad \text{for all } t \in (0, T_{\max}).
\]
With the above estimates at hand, we rewrite (41) as
\[
\frac{d}{dt} \int_{\Omega} u^2 + \int_{\Omega} u^2 + \int_{\Omega} u_x^2 \leq k_2(M) u_0^2 + c_5 e^{-6t},
\]
where
\[
k_2(M) = 2 \left[ 32c_4 k_1^6(M) + 64c_2^2(\chi + \xi)^2 |\Omega|^2 k_1^6(M) + c_3 |\Omega|^2 \right]
\]
and
\[
c_5 = \left[ 32c_4^2 c_4 u_0^2 + 64c_2^2(\chi + \xi)^2 |\Omega|^2 c_4^2 u_0^2 \right].
\]
Taking $\alpha \in (0, 1)$, we see $e^{-6t} < e^{-\alpha t}$ and thus
\[
\frac{d}{dt} \int_{\Omega} u^2 + \int_{\Omega} u^2 + \int_{\Omega} u_x^2 \leq k_2(M) u_0^2 + c_5 e^{-\alpha t}.
\]
Employing \cite[Lemma A.1]{31}, it can be obtained immediately from (42) that
\[
\int_{\Omega} u^2 \leq \left( \int_{\Omega} u_0^2 + \frac{c_5}{1 - \alpha} \right) e^{-\alpha t} + k_2(M) u_0^2 \quad \text{for all } t \in (0, T_{\max}).
\]
Integrating (42) yields
\[ \int_t^{t+\zeta} \int_{\Omega} u_x^2 \leq \int_{\Omega} u^2 + k_2(M)\bar{u}_0^2 + c_\bar{e} e^{-\alpha t} \text{ for all } t \in (0, T_{\text{max}} - \zeta). \quad (44) \]

Following the same procedure for \( u \) yields
\[ \int_{\Omega} \omega^2 \leq \left( \int_{\Omega} \omega_0^2 + \frac{c_5}{1 - \alpha} \right) e^{-\alpha t} + \bar{k}_2(M)\omega_0^2 \text{ for all } t \in (0, T_{\text{max}}), \quad (45) \]
and
\[ \int_t^{t+\zeta} \int_{\Omega} \omega_x^2 \leq \int_{\Omega} \omega^2 + \bar{k}_2(M)\omega_0^2 + c_\bar{e} e^{-\alpha t} \text{ for all } t \in (0, T_{\text{max}} - \zeta), \quad (46) \]
where
\[ \bar{k}_2(M) = 2[32c_4k_5^2(M) + 64c_2^2(\chi + \bar{\xi})^2|\Omega|^2\bar{k}_1^2(M) + c_3|\Omega|^2] \]
and
\[ c_\bar{e} = 2[32c_0^2c_4\omega_0^2 + 64c_2^2(\chi + \bar{\xi})^2|\Omega|^2\omega_0^2]. \]

Testing the second equation in (3) by \( v \) yields
\[ \frac{d}{dt} \int_{\Omega} v^2 + 2 \int_{\Omega} v^2 + 2 \int_{\Omega} v_x^2 = 2 \int_{\Omega} v\omega \leq \int_{\Omega} v^2 + \int_{\Omega} \omega^2. \quad (47) \]
Rewriting the above estimates leads to
\[ \frac{d}{dt} \int_{\Omega} v^2 + 2 \int_{\Omega} v^2 + 2 \int_{\Omega} v_x^2 \leq \int_{\Omega} \omega^2. \quad (48) \]

Employing [31, Lemma A.1] once again to (45), we see
\[ \int_{\Omega} v^2 \leq \left( \int_{\Omega} v_0^2 + \frac{1}{1 - \alpha} \int_{\Omega} \omega_0^2 + \frac{c_5}{(1 - \alpha)^2} \right) e^{-\alpha t} + \bar{k}_2(M)\omega_0^2, \quad t \in (0, T_{\text{max}}). \quad (49) \]

By direct integration of (47) we obtain for all \( t \in (0, T_{\text{max}} - \zeta) \),
\[ \int_t^{t+\zeta} \int_{\Omega} v_x^2 \leq \left( \int_{\Omega} v_0^2 + \frac{2 - \alpha}{1 - \alpha} \int_{\Omega} \omega_0^2 + \frac{\bar{c}_3(2 - \alpha)}{(1 - \alpha)^2} \right) e^{-\alpha t} + 2\bar{k}_2(M)\omega_0^2. \quad (50) \]
Applying the same procedure to \( z \) results in for all \( t \in (0, T_{\text{max}}) \),
\[ \int_{\Omega} z^2 \leq \left( \int_{\Omega} z_0^2 + \frac{1}{1 - \alpha} \int_{\Omega} u_0^2 + \frac{\bar{c}_3}{(1 - \alpha)^2} \right) e^{-\alpha t} + k_2(M)\bar{u}_0^2 \quad (51) \]
and for \( t \in (0, T_{\text{max}} - \zeta) \),
\[ \int_t^{t+\zeta} \int_{\Omega} z_x^2 \leq \left( \int_{\Omega} z_0^2 + \frac{2 - \alpha}{1 - \alpha} \int_{\Omega} u_0^2 + \frac{\bar{c}_3(2 - \alpha)}{(1 - \alpha)^2} \right) e^{-\alpha t} + 2k_2(M)\bar{u}_0^2. \quad (52) \]

As described above, (43)-(46) and (48)-(51) imply (35)-(36).

Combining Propositions 4.1 and 4.2, we can now once more apply heat semigroup estimates to improve \( L^2(\Omega) \) norm in (35) to the following \( L^\infty(\Omega) \) norm.

**Proposition 4.3.** Let \( M = \int_{\Omega} u_0 + \int_{\Omega} \bar{u}_0 \). Then for any positive initial data \((u_0, v_0, \omega_0, z_0) \in (W^{1,\infty}(\Omega))^4 \), there exist positive constants \( K(M), C = C(u_0, v_0, \omega_0, z_0) \), and \( \alpha \) such that the positive classical solution \((u, v, \omega, z) \) of (3) with \( \tau = 1 \) has the properties that for all \( t \in (0, T_{\text{max}}) \),
\[ \|u\|_{L^\infty(\Omega)} \leq K(M)\bar{u}_0 + Ce^{-\alpha t}, \|\omega\|_{L^\infty(\Omega)} \leq K(M)\omega_0 + Ce^{-\alpha t}, \|v\|_{L^\infty(\Omega)} \leq K(M)\bar{v}_0 + Ce^{-\alpha t}, \|z\|_{L^\infty(\Omega)} \leq K(M)\bar{z}_0 + Ce^{-\alpha t}. \]
Therefore, by the H"older inequality we get
\[ \|u\|_{L^2(\Omega)} \leq k_1(M)\bar{u}_0 + c_1e^{-\alpha_1 t}, \quad \|\omega\|_{L^2(\Omega)} \leq k_1(M)\bar{\omega}_0 + c_1e^{-\alpha_1 t} \]
and
\[ \|v_x\|_{L^4(\Omega)} \leq k_2(M) + c_2e^{-\alpha_2 t}, \quad \|z_x\|_{L^4(\Omega)} \leq k_2(M) + c_2e^{-\alpha_2 t}. \]

Therefore, by the Hölder inequality we get
\[ \|u z_x\|_{L^4(\Omega)} \leq \|u\|_{L^2(\Omega)} \|z_x\|_{L^4(\Omega)} \]
\[ \leq k_1(M)k_2(M)\bar{u}_0 + k_1(M)c_2\bar{u}_0e^{-\alpha t} + k_2(M)c_1e^{-\alpha t} + c_1c_2e^{-\alpha t} \]
\[ \leq k_1(M)k_2(M)\bar{u}_0 + c_3e^{-\alpha t}. \]

The same argument produces
\[ \|u v_x\|_{L^4(\Omega)} \leq k_1(M)k_2(M)\bar{u}_0 + c_3e^{-\alpha t}, \]
where \( \alpha = \min\{\alpha_1, \alpha_2\} \) and \( c_3 = k_1(M)c_2\bar{u}_0 + k_2(M)c_1 + c_1c_2 \).

Next, parabolic smoothing estimates \([40, 7]\) provide positive constants \( c_4 \) and \( c_5 \) such that for all \( t > 0 \),
\[ \|e^{t\Delta} \phi\|_{L^\infty(\Omega)} \leq c_4(1 + t^{-\frac{2}{3}})\|\phi\|_{L^4(\Omega)} \quad \text{for all } \phi \in C^1(\bar{\Omega}) \text{ with } \phi|_{\partial\Omega} = 0 \]
and
\[ \|e^{t\Delta} \phi\|_{L^\infty(\Omega)} \leq c_5(1 + t^{-\frac{1}{2}})\|\phi\|_{L^4(\Omega)} \quad \text{for all } \phi \in C^0(\bar{\Omega}). \]

Representing \( u \) according to the first equation results in
\[ u = e^{t(\Delta - 1)}u_0 + \chi \int_0^t e^{i(t-s)(\Delta-1)}\nabla \cdot (u \nabla z) \]
\[ - \xi \int_0^t e^{i(t-s)(\Delta - 1)}\nabla \cdot (u \nabla v) + \int_0^t e^{i(t-s)(\Delta - 1)}u. \]

It follows that
\[ \|u\|_{L^\infty(\Omega)} \leq e^{-t}\|e^{t\Delta}u_0\|_{L^\infty(\Omega)} + c_4\chi \int_0^t e^{-(t-s)}[1 + (t-s)^{-\frac{2}{3}}]\|u z_x\|_{L^4(\Omega)} \]
\[ + c_4\xi \int_0^t e^{-(t-s)}[1 + (t-s)^{-\frac{2}{3}}]\|u v_x\|_{L^4(\Omega)} \]
\[ + c_5 \int_0^t e^{-(t-s)}[1 + (t-s)^{-\frac{1}{2}}]\|u\|_{L^1(\Omega)} \quad \text{for all } t \in (0, T_{\text{max}}). \]

Here thanks to the maximum principle and the fact that \( \alpha \leq 1 \), we have
\[ e^{-t}\|e^{t\Delta}u_0\|_{L^\infty(\Omega)} \leq \|u_0\|_{L^\infty(\Omega)}e^{-t} \leq \|u_0\|_{L^\infty(\Omega)}e^{-\alpha t} \quad \text{for all } t > 0 \]
and
\[ c_5 \int_0^t e^{-(t-s)}[1 + (t-s)^{-\frac{1}{2}}]\|u\|_{L^1(\Omega)}ds \]
\[ = c_5\|u_0\|_{L^1(\Omega)} \int_0^t e^{-(t-s)}[1 + (t-s)^{-\frac{1}{2}}]ds \leq c_5c_6\|u_0\|_{L^1(\Omega)}, \quad t \in (0, T_{\text{max}}), \]
with \( c_6 = \int_0^\infty e^{-\sigma} (1 + \frac{1}{2}) = 1 + \Gamma\left(\frac{1}{2}\right) \) (Here \( \Gamma(\cdot) \) denotes the Gamma function).

Moreover, due to (53) we see

\[
\begin{align*}
c_4\chi \int_0^t e^{-(t-s)}[1 + (t-s)^{-\frac{7}{8}}] & \|uz_x\|_{L^\frac{8}{7}(\Omega)} \\
+ c_4\zeta \int_0^t e^{-(t-s)}[1 + (t-s)^{-\frac{7}{8}}] & \|uv_x\|_{L^\frac{8}{7}(\Omega)} \\
\leq c_4(\chi + \zeta)k_1(M)k_2(M)\tilde{u}_0 \int_0^t e^{-(t-s)}[1 + (t-s)^{-\frac{7}{8}}] \\
+ c_3c_4(\chi + \zeta) & \int_0^t e^{-(t-s)}[1 + (t-s)^{-\frac{7}{8}}]e^{-\alpha s},
\end{align*}
\]

where

\[
\int_0^t e^{-(t-s)}[1 + (t-s)^{-\frac{7}{8}}]ds \leq c_7 := \int_0^\infty e^{-\sigma[1 + \sigma^{-\frac{7}{8}}]} = 1 + \Gamma\left(\frac{1}{8}\right) < \infty
\]

and

\[
\int_0^t e^{-(t-s)}[1 + (t-s)^{-\frac{7}{8}}]e^{-\alpha s} ds = e^{-\alpha t} \int_0^t e^{-(1-\alpha)s}[1 + (1-\alpha)^{\frac{7}{8}}]ds \leq c_8e^{-\alpha t}, \ t > 0
\]

with \( c_8 = \int_0^\infty e^{-(1-\alpha)s}[1 + \sigma^{-\frac{7}{8}}] = \frac{1}{\alpha} + (1-\alpha)^{\frac{7}{8}}\Gamma\left(\frac{1}{8}\right) < \infty. \) Substituting (56)-(58) into (55) shows that for all \( t \in (0, T_{\text{max}}) \)

\[
\|u\|_{L^\infty(\Omega)} \leq [c_4c_7k_1(M)k_2(M)(\chi + \zeta) + c_5c_6|\Omega|] \tilde{u}_0 \\
+ [\|u_0\|_{L^\infty(\Omega)} + c_3c_4c_8(\chi + \zeta)] e^{-\alpha t}.
\]

In precisely the same manner, we infer that

\[
\|\omega\|_{L^\infty(\Omega)} \leq [c_4c_7k_1(M)k_2(M)(\tilde{\chi} + \tilde{\zeta}) + c_5c_6|\Omega|] \tilde{\omega}_0 \\
+ [\|\omega_0\|_{L^\infty(\Omega)} + c_3c_4c_8(\tilde{\chi} + \tilde{\zeta})] e^{-\alpha t}.
\]

By (31) we see

\[
\|v\|_{L^\infty(\Omega)} \leq \|e^{(\Delta-1)t}v_0\|_{L^\infty(\Omega)} + \int_0^t \|e^{(t-s)(\Delta-1)}\omega\|_{L^\infty(\Omega)} \\
\leq e^{-t}\|e^{\Delta t}v_0\|_{L^\infty(\Omega)} + c_5 \int_0^t e^{-(t-s)}[1 + (t-s)^{-\frac{7}{8}}]\|\omega\|_{L^1(\Omega)} \\
\leq \|v_0\|_{L^\infty(\Omega)}e^{-t} + c_5c_6|\Omega|\tilde{\omega}_0.
\]

The same procedure applied to \( z \) results in

\[
\|z\|_{L^\infty(\Omega)} \leq \|z_0\|_{L^\infty(\Omega)}e^{-t} + c_5c_6|\Omega|\tilde{u}_0.
\]

As described above, we complete the proof. \( \square \)

4.3. **Estimates for** \( \|u_x\|_{L^2(\Omega)}, \|v_x\|_{L^2(\Omega)}, \|w_x\|_{L^2(\Omega)}, \text{ and } \|z_x\|_{L^2(\Omega)}. \)** With the estimate (52) at hand, we will see \( u_x, v_x, w_x, \text{ and } z_x \) indeed possess the following integrability features which exceed those in Proposition 4.2.

**Proposition 4.4.** Let \( M = \int_\Omega u_0 + \int_\Omega \omega_0. \) Then for any positive initial data \((u_0, v_0, \omega_0, z_0) \in (W^{1,\infty}(\Omega))^4, \) there exist positive constants \( K(M), C = C(u_0, v_0, \omega_0, z_0), \)
and $\alpha$ such that the positive classical solution $(u, v, \omega, z)$ of (3) with $\tau = 1$ has the properties that for all $t \in (0, T_{\max})$,
\begin{align*}
&\int_\Omega u_x^2 \leq \frac{K(M)}{\zeta^2} + Ce^{-\alpha t}, \quad \int_\Omega \omega_x^2 \leq \frac{K(M)}{\zeta^2} + Ce^{-\alpha t}, \\
&\int_\Omega v_x^2 \leq \frac{K(M)}{\zeta} + Ce^{-\alpha t}, \quad \int_\Omega z_x^2 \leq \frac{K(M)}{\zeta} + Ce^{-\alpha t}.
\end{align*}
(59)

and for all $t \in (0, T_{\max} - \zeta)$,
\begin{align*}
&\int_t^{t+\zeta} \int_\Omega u_{xx}^2 \leq \frac{K(M)}{\zeta^2} + Ce^{-\alpha t}, \quad \int_t^{t+\zeta} \int_\Omega \omega_{xx}^2 \leq \frac{K(M)}{\zeta^2} + Ce^{-\alpha t}, \\
&\int_t^{t+\zeta} \int_\Omega v_{xx}^2 \leq \frac{K(M)}{\zeta} + Ce^{-\alpha t}, \quad \int_t^{t+\zeta} \int_\Omega z_{xx}^2 \leq \frac{K(M)}{\zeta} + Ce^{-\alpha t}.
\end{align*}
(60)

Here $\zeta = \min\{1, \frac{1}{2} T_{\max}\}$.

Proof. We take a constant $\alpha \in (0, 1)$ such that for given $M > 0$ there exist $\tilde{k}_i(M) > 0$ ($i = 1, 2$) and some $\tilde{c}_i = c_i(u_0, v_0, \omega_0, z_0) > 0$ ($i = 1, 2$) such that
\begin{align*}
&\int_\Omega u^2 \leq \tilde{k}_1(M) u_0^2 + \tilde{c}_1 e^{-\alpha t}, \quad \|u\|_{L^\infty(\Omega)} \leq \tilde{k}_2(M) u_0 + \tilde{c}_2 e^{-\alpha t}, \\
&\int_\Omega \omega^2 \leq \tilde{k}_1(M) \omega_0^2 + \tilde{c}_1 e^{-\alpha t}, \quad \|\omega\|_{L^\infty(\Omega)} \leq \tilde{k}_2(M) \omega_0 + \tilde{c}_2 e^{-\alpha t}.
\end{align*}
(61)

Employing $v_{xx}$ as a test function for the third equation in (3) yields
\begin{equation*}
\frac{d}{dt} \int_\Omega v_x^2 + 2 \int_\Omega v_x^2 + 2 \int_\Omega v_x^2 = 2 \int_\Omega v_x \omega_x = -2 \int_\Omega v_{xx} \omega \leq \int_\Omega \omega^2 + \int_\Omega v_{xx}^2.
\end{equation*}

We rewrite the above inequality as
\begin{equation*}
\frac{d}{dt} \int_\Omega v_x^2 + 2 \int_\Omega v_x^2 + \int_\Omega v_{xx}^2 \leq \int_\Omega \omega^2.
\end{equation*}
(62)

Moreover, it is easy to see
\begin{equation*}
\int_t^{t+\zeta} \int_\Omega \omega^2 \leq \int_t^{t+\zeta} \left[ \tilde{k}_1(M) \omega_0^2 + \tilde{c}_1 e^{-\alpha t} \right] \leq \tilde{k}_1(M) \omega_0^2 + \tilde{c}_1 e^{-\alpha t}.
\end{equation*}

By [31, Lemma A.1] we get
\begin{align*}
\int_\Omega v_x^2 \leq \left\{ \left( \int_\Omega v_x^2(0) + \frac{\tilde{c}_1(2 - \alpha)}{1 - \alpha} + \tilde{k}_1(M) \omega_0^2 \right) \frac{e^\alpha}{\zeta} + \tilde{c}_1 e^\alpha \right\} e^{-\alpha t} \\
+ \frac{\tilde{k}_1(M) \omega_0^2}{\zeta} + \tilde{k}_1(M) \omega_0^2
\end{align*}
(63)

By direct integration of (62) we see
\begin{align*}
\int_t^{t+\zeta} \int_\Omega v_{xx}^2 \leq \int_\Omega v_{xx}^2 + \tilde{k}_1(M) \omega_0^2 + \tilde{c}_1 e^{-\alpha t} \\
\leq \left\{ \left( \int_\Omega v_x^2(0) + \frac{\tilde{c}_1(2 - \alpha)}{1 - \alpha} + \tilde{k}_1(M) \omega_0^2 \right) \frac{e^\alpha}{\zeta} + \tilde{c}_1 e^\alpha + \tilde{c}_1 \right\} e^{-\alpha t} \\
+ 3 \tilde{k}_1(M) \omega_0^2.
\end{align*}
(64)
With the same argument as that for \( v \), we get
\[
\int_{\Omega} z_{x}^{2} \leq \left\{ \left( \int_{\Omega} z_{x}^{2}(0) + \bar{c}_{1}(2 - \alpha) + \bar{k}_{1}(M) \bar{u}_{0}^{2} \right) \frac{e^{\alpha t}}{\zeta} + \bar{c}_{1} e^{\alpha t} \right\} e^{-\alpha t}
\]
\[
= \left\{ \left( \int_{\Omega} z_{x}^{2}(0) + \bar{c}_{1}(2 - \alpha) + \bar{k}_{1}(M) \bar{u}_{0}^{2} \right) \frac{e^{\alpha t}}{\zeta} + \bar{c}_{1} e^{\alpha t} \right\} e^{-\alpha t}
\]
\[
= \left\{ \left( \int_{\Omega} z_{x}^{2}(0) + \bar{c}_{1}(2 - \alpha) + \bar{k}_{1}(M) \bar{u}_{0}^{2} \right) \frac{e^{\alpha t}}{\zeta} + \bar{c}_{1} e^{\alpha t} \right\} e^{-\alpha t}
\]
\[
\quad + \frac{\bar{k}_{1}(M) \bar{u}_{0}^{2}}{\zeta}.
\]

(65)

and
\[
\int_{t}^{t+\zeta} \int_{\Omega} z_{xx}^{2} \leq \int_{t}^{t+\zeta} \int_{\Omega} z_{x}^{2} + \bar{k}_{1}(M) \bar{u}_{0}^{2} + \bar{c}_{1} e^{-\alpha t}
\]
\[
\leq \left\{ \left( \int_{\Omega} z_{x}^{2}(0) + \bar{c}_{1}(2 - \alpha) + \bar{k}_{1}(M) \bar{u}_{0}^{2} \right) \frac{e^{\alpha t}}{\zeta} + \bar{c}_{1} e^{\alpha t} + \bar{c}_{1} e^{\alpha t} \right\} e^{-\alpha t}
\]
\[
\quad + \frac{3\bar{k}_{1}(M) \bar{u}_{0}^{2}}{\zeta}.
\]

Apart from that, we combine the Gagliardo-Nirenberg inequality with Young’s inequality to obtain \( c_{3} > 0 \) and \( c_{4} > 0 \) such that
\[
\|\psi_{x}\|_{L^{2}(\Omega)}^{2} \leq c_{3}\|\psi_{xx}\|_{L^{2}(\Omega)}\|\psi\|_{L^{\infty}(\Omega)} \quad \text{for all} \quad \psi \in W^{2,2}(\Omega)
\]
\[
(67)
\]
and
\[
\int_{\Omega} \psi_{xx}^{2} \leq \frac{1}{2} \int_{\Omega} \psi_{x}^{2} + c_{4} \left( \int_{\Omega} |\psi| \right)^{2} \quad \text{for all} \quad \psi \in W^{2,2}(\Omega).
\]
\[
(68)
\]

We integrate by parts the first equation in (3) and use the Cauchy-Schwarz inequality along with (67), Young’s inequality and (68) to deduce that for all \( t \in (0, T_{\text{max}}) \),
\[
\frac{d}{dt} \int_{\Omega} u_{x}^{2} + 2 \int_{\Omega} u_{xx}^{2} + \int_{\Omega} u_{x}^{2}
\]
\[
= -2\chi \int_{\Omega} u_{x} u_{xx} u_{xx} - 2\chi \int_{\Omega} u u_{xx} z_{xx} + 2\zeta \int_{\Omega} u_{x} v_{x} u_{xx} + 2\zeta \int_{\Omega} u u_{xx} v_{xx} + \int_{\Omega} u_{x}^{2}
\]
\[
= \chi \int_{\Omega} u_{x}^{2} z_{xx} - 2\chi \int_{\Omega} u u_{xx} z_{xx} - \zeta \int_{\Omega} u_{x} v_{xx} + 2\zeta \int_{\Omega} u u_{xx} v_{xx} + \int_{\Omega} u_{x}^{2}
\]
\[
\leq c_{3} \chi \|u_{x}\|_{L^{2}(\Omega)} \|u\|_{L^{\infty}(\Omega)} \|u_{xx}\|_{L^{2}(\Omega)} \|z_{xx}\|_{L^{2}(\Omega)} + 2\chi \|u\|_{L^{\infty}(\Omega)} \|u_{xx}\|_{L^{2}(\Omega)} \|z_{xx}\|_{L^{2}(\Omega)}
\]
\[
+ \zeta \|u_{x}\|_{L^{2}(\Omega)} \|u_{xx}\|_{L^{2}(\Omega)} + 2\zeta \|u\|_{L^{\infty}(\Omega)} \|u_{xx}\|_{L^{2}(\Omega)} \|v_{xx}\|_{L^{2}(\Omega)} + \int_{\Omega} u_{x}^{2}
\]
\[
\leq c_{3} \chi \|u_{xx}\|_{L^{2}(\Omega)} \|u\|_{L^{\infty}(\Omega)} \|u_{x}\|_{L^{2}(\Omega)} \|v_{xx}\|_{L^{2}(\Omega)} + 2\chi \|u\|_{L^{\infty}(\Omega)} \|u_{xx}\|_{L^{2}(\Omega)} \|z_{xx}\|_{L^{2}(\Omega)}
\]
\[
+ c_{3} \chi \|u_{xx}\|_{L^{2}(\Omega)} \|u\|_{L^{\infty}(\Omega)} \|v_{xx}\|_{L^{2}(\Omega)}
\]
\[
+ 2\zeta \|u\|_{L^{\infty}(\Omega)} \|u_{xx}\|_{L^{2}(\Omega)} \|v_{xx}\|_{L^{2}(\Omega)} + \int_{\Omega} u_{x}^{2}
\]
\[
\leq \|u_{xx}\|_{L^{2}(\Omega)}^{2} + (c_{3} + 2)^{2} \chi^{2} \|u\|_{L^{\infty}(\Omega)}^{2} \|z_{xx}\|_{L^{2}(\Omega)}^{2}
\]
\[
+ (c_{3} + 2)^{2} \chi^{2} \|u\|_{L^{\infty}(\Omega)}^{2} \|v_{xx}\|_{L^{2}(\Omega)}^{2} + c_{4} \left( \int_{\Omega} u \right)^{2}.
\]

Rewriting the above estimates produces
\[
\frac{d}{dt} \int_{\Omega} u_{x}^{2} + \int_{\Omega} u_{x}^{2} + \int_{\Omega} u_{xx}^{2} \leq (c_{3} + 2)^{2} \chi^{2} \|u\|_{L^{\infty}(\Omega)}^{2} \|z_{xx}\|_{L^{2}(\Omega)}^{2}
\]
\[
+ (c_{3} + 2)^{2} \chi^{2} \|u\|_{L^{\infty}(\Omega)}^{2} \|v_{xx}\|_{L^{2}(\Omega)}^{2} + c_{4} M^{2}.
\]
\[
(69)
\]
Moreover, along with (61) we derive the existence of positive constants $\hat{k}_3(M)$ and $c_6 = c_6(u_0, v_0, \omega_0, z_0)$ such that

$$\int_t^{t+\zeta} \left[ (c_3 + 2)^2 \chi^2 \|u\|_{L^\infty(\Omega)}^2 \|z_{xx}\|_{L^2(\Omega)}^2 + (c_3 + 2)^2 \xi^2 \|u\|_{L^\infty(\Omega)}^2 \|v_{xx}\|_{L^2(\Omega)}^2 + c_4 M^2 \right]$$

$$\leq c_5 \int_t^{t+\zeta} \left[ \hat{k}_2(M) \bar{u}_0 + \bar{c}_2 e^{-\alpha t} \right]^2 \int_\Omega z_{xx}^2 + c_5 \int_t^{t+\zeta} \left[ \hat{k}_2(M) \bar{u}_0 + \bar{c}_2 e^{-\alpha t} \right]^2 \int_\Omega v_{xx}^2 + c_4 M^2$$

$$\leq c_5 \left[ \hat{k}_2(M) \bar{u}_0 + \bar{c}_2 e^{-\alpha t} \right]^2 \left[ \int_t^{t+\zeta} \int_\Omega z_{xx}^2 + \int_t^{t+\zeta} \int_\Omega v_{xx}^2 \right] + c_4 M^2$$

$$\leq \frac{\hat{k}_3(M)}{\zeta} + c_6 e^{-\alpha t}$$

with $c_5 = (c_3 + 2)^2 (\chi^2 + \xi^2)$. By [31, Lemma A.1] once again we see that

$$\int_\Omega u_x^2 \leq \frac{\hat{k}_3(M)}{\zeta^2} + \frac{\hat{k}_3(M)}{\zeta} + c_7 e^{-\alpha t}$$

(70)

with $c_7 = \frac{\zeta}{\alpha} (\int_\Omega u_x^2(0) + \frac{c_6(2-\alpha)}{1-\alpha} + \frac{\hat{k}_3(M)}{\zeta}) + c_6 e\alpha$. By direct integration of (69) we derive

$$\int_t^{t+\zeta} \int_\Omega u_{xx}^2 \leq \int_t^{t+\zeta} \int_\Omega u_x^2 + \frac{\hat{k}_3(M)}{\zeta} + c_6 e^{-\alpha t}$$

$$< \frac{\hat{k}_3(M)}{\zeta^2} + \frac{2\hat{k}_3(M)}{\zeta} + (c_6 + c_7) e^{-\alpha t}.$$  (71)

Following the same procedure for $u$ leads to

$$\frac{d}{dt} \int_\Omega \omega_x^2 + \int_\Omega \omega_{xx}^2$$

$$\leq (c_3 + 2)^2 \chi^2 \|\omega\|_{L^\infty(\Omega)}^2 \|v_{xx}\|_{L^2(\Omega)}$$

$$\left[ \int_t^{t+\zeta} \int_\Omega \omega_{xx}^2 \right] + \frac{(c_3 + 2)^2 \xi^2 \|\omega\|_{L^\infty(\Omega)}^2 \|z_{xx}\|_{L^2(\Omega)} + c_4 \left( \int_\Omega \omega \right)^2}$$

In precisely the same manner, by [31, Lemma A.1] once more we obtain that there exist $\hat{k}_4(M) > 0$ and $\tilde{c}_i = \tilde{c}_i(u_0, v_0, \omega_0, z_0)$ ($i = 6, 7$) such that

$$\int_\Omega \omega_x^2 \leq \frac{\hat{k}_4(M)}{\zeta^2} + \frac{\hat{k}_4(M)}{\zeta} + \tilde{c}_7 e^{-\alpha t}$$

(72)

and

$$\int_t^{t+\zeta} \int_\Omega \omega_{xx}^2 \leq \int_t^{t+\zeta} \int_\Omega \omega_x^2 + \frac{\hat{k}_4(M)}{\zeta} + c_6 e^{-\alpha t}$$

$$< \frac{\hat{k}_4(M)}{\zeta^2} + \frac{2\hat{k}_4(M)}{\zeta} + (\tilde{c}_6 + \tilde{c}_7) e^{-\alpha t}.$$  (73)

As described above, the estimates (63)-(66) and (70)-(73) imply (59)-(60). Thus, we complete the proof. 

By now, with the above estimates at hand, we are ready to prove Theorem 1.2.

Proof of Theorem 1.2. According to the extensibility criterion (8), for any fixed $(u_0, v_0, \omega_0, z_0) \in (W^{1,\infty}(\Omega))^4$ we can employ Propositions 4.2 and 4.4 with $M = \int_\Omega u_0 + \int_\Omega \omega_0$ and $q = 2$ and it is easy to get that $T_{\text{max}}$ must be infinite, and thus (5) is a consequence of (35) and (59). □
5. **Proof of Theorem 1.3.** Inspired by [3], we need some preparations to prove Theorem 1.3. We first establish estimates on the integrals $\int_{\Omega} u \ln u$ and $\int_{\Omega} \omega \ln \omega$ by employing Young's inequality and Gagliardo-Nirenberg inequality.

**Proposition 5.1.** Let $\Omega \subset \mathbb{R}^2$ be a smoothly bounded domain. Assume that the positive initial data $(u_0, v_0, \omega_0, z_0) \in C^0(\Omega) \times W^{1,q}(\Omega) \times C^0(\Omega) \times W^{1,q}(\Omega)$ with $q \in (2, \infty)$ fulfills
\[
\|u_0\|_{L^1(\Omega)} < \frac{4}{C_{GN}[1 + (\chi^2 + \xi^2)]} \quad \text{and} \quad \|\omega_0\|_{L^1(\Omega)} < \frac{4}{C_{GN}[1 + (\chi^2 + \xi^2)]},
\]
where $C_{GN} > 0$ is taken from (14). Then we can find a positive constant $M_1$ such that the positive classical solution $(u, v, \omega, z)$ of (3) with $\tau = 1$ has the properties that
\[
\int_{\Omega} u \ln u < M_1, \quad \int_{\Omega} \omega \ln \omega < M_1, \quad \int_{\Omega} |\nabla v|^2 < M_1, \quad \int_{\Omega} |\nabla z|^2 < M_1. \quad (75)
\]

*Proof.* Multiplying the first equation in (3) by $\ln u$, integrating by parts over $\Omega$, and applying Young’s inequality, we get
\[
\frac{d}{dt} \int_{\Omega} u \ln u + \int_{\Omega} \frac{|\nabla u|^2}{u} = -\chi \int_{\Omega} \nabla u \cdot \nabla z + \xi \int_{\Omega} \nabla u \cdot \nabla v
\]
\[
= \chi \int_{\Omega} u \Delta z - \xi \int_{\Omega} u \Delta v
\]
\[
\leq (\chi^2 + \xi^2) \int_{\Omega} u^2 + \frac{1}{4} \int_{\Omega} |\Delta z|^2 + \frac{1}{4} \int_{\Omega} |\Delta v|^2. \quad (76)
\]

Following the same argument for $u$ produces
\[
\frac{d}{dt} \int_{\Omega} \omega \ln \omega + \int_{\Omega} \frac{|\nabla \omega|^2}{\omega} \leq (\chi^2 + \xi^2) \int_{\Omega} \omega^2 + \frac{1}{4} \int_{\Omega} |\Delta v|^2 + \frac{1}{4} \int_{\Omega} |\Delta z|^2. \quad (77)
\]

In order to remove the integrals $\int_{\Omega} |\Delta v|^2$ and $\int_{\Omega} |\Delta z|^2$ on the right-hand sides of (76) and (77), we multiply the second equation in (3) by $-\Delta v$ and apply Young’s inequality to obtain
\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega} |\nabla v|^2 + \int_{\Omega} |\nabla v|^2 \leq -\frac{1}{2} \int_{\Omega} |\Delta v|^2 + \frac{1}{2} \int_{\Omega} \omega^2. \quad (78)
\]

In the same way, one has
\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega} |\nabla z|^2 + \int_{\Omega} |\nabla z|^2 \leq -\frac{1}{2} \int_{\Omega} |\Delta z|^2 + \frac{1}{2} \int_{\Omega} u^2. \quad (79)
\]

Adding up (76)-(79) gives
\[
\frac{d}{dt} \left\{ \int_{\Omega} u \ln u + \int_{\Omega} \omega \ln \omega + \frac{1}{2} \int_{\Omega} |\nabla v|^2 + \frac{1}{2} \int_{\Omega} |\nabla z|^2 \right\}
\]
\[
+ \int_{\Omega} \frac{|\nabla u|^2}{u} + \int_{\Omega} \frac{|\nabla \omega|^2}{\omega} + \int_{\Omega} |\nabla v|^2 + \int_{\Omega} |\nabla z|^2
\]
\[
\leq \frac{1 + 2(\chi^2 + \xi^2)}{2} \int_{\Omega} u^2 + \frac{1 + 2(\chi^2 + \xi^2)}{2} \int_{\Omega} \omega^2. \quad (80)
\]
In order to eliminate the integrals \( \int_\Omega \frac{\nabla u}{u} \) and \( \int_\Omega \frac{\nabla \omega}{\omega} \) on the left-hand side of (80), we use the Gagliardo-Nirenberg inequality to obtain
\[
\int_\Omega u^2 \leq \frac{C_{GN} \|u_0\|_{L^1(\Omega)}}{4} \int_\Omega \frac{\nabla u}{u} + C_{GN} \|u_0\|_{L^1(\Omega)},
\]
\[
\int_\Omega \omega^2 \leq \frac{C_{GN} \|\omega_0\|_{L^1(\Omega)}}{4} \int_\Omega \frac{\nabla \omega}{\omega} + C_{GN} \|\omega_0\|_{L^1(\Omega)}.
\]
Substituting them into (80) and applying (74), we infer that
\[
\text{the estimates (75) follow immediately.}
\]

Let \( y(t) = \int_\Omega u \ln u + \int_\Omega \omega \ln \omega + \frac{1}{2} \int_\Omega |\nabla v|^2 + \frac{1}{2} \int_\Omega |\nabla z|^2 \). Thanks to the assumption (74), we infer that
\[
y'(t) + y(t) \leq 4\|u_0\|_{L^1(\Omega)} + 4\|\omega_0\|_{L^1(\Omega)}.
\]
It follows from the ODE comparison argument that there exists a positive constant \( C \) such that
\[
\int_\Omega u \ln u + \int_\Omega \omega \ln \omega + \frac{1}{2} \int_\Omega |\nabla v|^2 + \frac{1}{2} \int_\Omega |\nabla z|^2 \leq C,
\]
where \( C =: \max\{\int_\Omega u \ln u_0 + \int_\Omega \omega \ln \omega_0 + \frac{1}{2} \int_\Omega |\nabla v_0|^2 + \frac{1}{2} \int_\Omega |\nabla z_0|^2, 4\|u_0\|_{L^1(\Omega)} + 4\|\omega_0\|_{L^1(\Omega)}\}. \) Then (75) follows immediately.

With the estimates (75) at hand, we will employ Young’s inequality, compact embedding result, Gagliardo-Nirenberg inequality, and Poincaré inequality to obtain the estimates \( \int_\Omega u^2 \) and \( \int_\Omega \omega^2 \), which along with Proposition 2.5 leads directly to uniform-in-time boundedness of solutions.

**Proposition 5.2.** Under the assumption of Proposition 5.1, there exists a positive constant \( M_2 \) such that the positive classical solution \((u, v, \omega, z)\) of (3) with \( \tau = 1 \) has the properties that
\[
\int_\Omega u^2 < M_2, \quad \int_\Omega |\nabla v|^4 < M_2, \quad \int_\Omega \omega^2 < M_2, \quad \int_\Omega |\nabla z|^4 < M_2.
\]

**Proof.** Testing the first equation in (3) by \( u \) and Young’s inequality, we know that there exists a positive constant \( c_1 \) such that
\[
\frac{1}{2} \frac{d}{dt} \int_\Omega u^2 + \int_\Omega |\nabla u|^2 = -\chi \int \Omega u \nabla u \cdot \nabla z + \xi \int \Omega u \nabla u \cdot \nabla v \leq \frac{1}{2} \int_\Omega |\nabla u|^2 + \chi^2 \int_\Omega u^2 |\nabla z|^2 + \xi^2 \int_\Omega u^2 |\nabla v|^2.
\]
A simple rearrangement leads to

$$\frac{1}{4} \frac{d}{dt} \int_{\Omega} |\nabla v|^4 - \frac{1}{2} \int_{\partial \Omega} |\nabla v|^2 |\Delta v - v + \omega|^2 \leq \int_{\Omega} |\nabla v|^2 |\Delta v|^2 + \frac{1}{2} \int_{\partial \Omega} |\nabla v|^2 \frac{\partial |\nabla v|^2}{\partial \nu}$$

for all $t \in (0, T_{\text{max}})$, whereas using the fact that $\nabla v \cdot \nabla \Delta v = \frac{1}{2} \Delta |\nabla v|^2 - |D^2 v|^2$, from the second equation in (3) we see that

$$\frac{1}{4} \frac{d}{dt} \int_{\Omega} |\nabla v|^4 = \int_{\Omega} |\nabla v|^2 \nabla \cdot \nabla (\Delta v - v + \omega)$$

$$= -\frac{1}{2} \int_{\Omega} |\nabla |\nabla v|^2|^2 + \frac{1}{2} \int_{\partial \Omega} |\nabla v|^2 \frac{\partial |\nabla v|^2}{\partial \nu}$$

$$- \int_{\Omega} |\nabla v|^2 |D^2 v|^2 - \int_{\Omega} |\nabla v|^4 + \int_{\Omega} |\nabla v|^2 \nabla v \cdot \nabla \omega.$$  \hspace{1cm} (83)

Since $\frac{\partial |\nabla v|^2}{\partial \nu} \leq c_1 |\nabla v|^2$ on $\partial \Omega$ with some $c_1 > 0$ ([22]), by compactness of the embedding $W^{1,2}(\Omega) \to L^2(\partial \Omega)$ we obtain $c_2 > 0$ such that

$$\frac{1}{4} \int_{\partial \Omega} |\nabla v|^2 \frac{\partial |\nabla v|^2}{\partial \nu} \leq \frac{1}{4} \int_{\Omega} |\nabla |\nabla v|^2|^2 + c_2 \left( \int_{\Omega} |\nabla v|^2 \right)^2$$

$$\leq \frac{1}{4} \int_{\Omega} |\nabla |\nabla v|^2|^2 + c_2 M_1^2.$$  \hspace{1cm} (84)

Since

$$\int_{\Omega} |\nabla v|^2 |\nabla v|^2 \nabla \omega = -\int_{\Omega} \omega |\nabla v|^2 \Delta v - \int_{\Omega} \nabla v \cdot \nabla |\nabla v|^2,$$

inserting (84) and (85) into (83) leads to

$$\frac{1}{4} \frac{d}{dt} \int_{\Omega} |\nabla v|^4 \leq -\frac{1}{2} \int_{\Omega} |\nabla |\nabla v|^2|^2 + \frac{1}{4} \int_{\Omega} |\nabla |\nabla v|^2|^2$$

$$+ c_3 \left( \int_{\Omega} |\nabla v|^2 \right)^2 - \int_{\Omega} |\nabla v|^2 |D^2 v|^2 - \int_{\Omega} |\nabla v|^4$$

$$- \int_{\Omega} \omega |\nabla v|^2 \Delta v - \int_{\Omega} \nabla v \cdot \nabla |\nabla v|^2$$

$$\leq -\frac{1}{4} \int_{\Omega} |\nabla |\nabla v|^2|^2 + c_2 \left( \int_{\Omega} |\nabla v|^2 \right)^2 - \int_{\Omega} |\nabla v|^2 |D^2 v|^2$$

$$- \int_{\Omega} |\nabla v|^4 + \frac{1}{2} \int_{\Omega} \omega |\nabla v|^2 + \frac{1}{2} \int_{\Omega} |\nabla v|^2 \cdot 2 |D^2 v|^2$$

$$+ 2 \int_{\Omega} \omega |\nabla v|^2 + \frac{1}{8} \int_{\Omega} |\nabla |\nabla v|^2|^2.$$  \hspace{1cm} (86)

Here in order to remove the integral $\int_{\Omega} |\nabla v|^2 |D^2 v|^2$, we used the fact that $|\Delta v|^2 \leq n |D^2 v|^2 = 2 |D^2 v|^2$. Rewriting (86) yields

$$\frac{1}{4} \frac{d}{dt} \int_{\Omega} |\nabla v|^4 + \frac{1}{8} \int_{\Omega} |\nabla |\nabla v|^2|^2 + \int_{\Omega} |\nabla v|^4 \leq c_2 M_1^2 + \frac{5}{2} \int_{\Omega} \omega^2 |\nabla v|^2.$$  \hspace{1cm} (87)

In the same way, we infer that

$$\frac{1}{2} \frac{d}{dt} \int_{\Omega} \omega^2 + \frac{1}{2} \int_{\Omega} |\nabla \omega|^2 \leq \tilde{\chi}^2 \int_{\Omega} \omega^2 |\nabla v|^2 + \tilde{\xi}^2 \int_{\Omega} \omega^2 |\nabla z|^2$$

$$\frac{d}{dt} \int_{\Omega} |\nabla z|^4 + \int_{\Omega} |\nabla |\nabla z|^2|^2 + \int_{\Omega} |\nabla z|^4 \leq c_2 M_1^2 + \frac{5}{2} \int_{\Omega} u^2 |\nabla z|^2.$$  \hspace{1cm} (88)
Combining (82) with (87)-(89) leads to

\[
\frac{d}{dt}\left\{ \frac{1}{2} \int_{\Omega} u^2 + \frac{1}{4} \int_{\Omega} |\nabla v|^4 + \frac{1}{2} \int_{\Omega} \omega^2 + \frac{1}{4} \int_{\Omega} |\nabla z|^4 \right\} + \frac{1}{2} \int_{\Omega} |\nabla u|^2 \\
+ \frac{1}{2} \int_{\Omega} |\nabla \omega|^2 + \frac{1}{8} \int_{\Omega} |\nabla |\nabla v|^2|^2 + \frac{1}{8} \int_{\Omega} |\nabla |\nabla z|^2|^2 + \int_{\Omega} |\nabla v|^4 + \int_{\Omega} |\nabla z|^4
\leq \left( \chi^2 + \frac{5}{2} \right) \int_{\Omega} \omega^2 |\nabla v|^2 + \left( \chi^2 + \frac{5}{2} \right) \int_{\Omega} u^2 |\nabla z|^2 \\
+ \xi^2 \int_{\Omega} u^2 |\nabla v|^2 + \xi^2 \int_{\Omega} \omega^2 |\nabla z|^2 + 2c_5 M_t^2. \tag{90}
\]

We now invoke an extended interpolation inequality (15), which along with (75) ensures that for each \( \epsilon_1 > 0 \) and \( \tilde{\epsilon}_1 > 0 \) we can pick \( c_4(\epsilon_1) > 0 \) and \( c_5(\tilde{\epsilon}_1) > 0 \) such that

\[
\|u\|_{L^3(\Omega)}^3 \leq c_1 \|\nabla u\|_{L^2(\Omega)}^2 \|u \ln u\|_{L^1(\Omega)} + c_3(\epsilon_1) \left[ \|u\|_{L^1(\Omega)}^3 + 1 \right] \\
\leq \tilde{\epsilon}_1 \|\nabla u\|_{L^2(\Omega)}^2 + c_4(\tilde{\epsilon}_1) \\
= \tilde{\epsilon}_1 \int_{\Omega} |\nabla u|^2 + c_4(\tilde{\epsilon}_1). \tag{91}
\]

Meanwhile by the Gagliardo-Nirenberg inequality and (75) we have

\[
\|\nabla z\|_{L^3(\Omega)}^3 \leq c_5 \|\nabla |\nabla z|^2\|_{L^2(\Omega)} \|\nabla z\|_{L^2(\Omega)} + c_6 \|\nabla z\|_{L^2(\Omega)}^3 \tag{92}
\]

for all \( t \in (0, T_{\text{max}}) \) with some \( c_5 > 0 \), due to the Hölder inequality and Young’s inequality this implies that there exists \( c_7 > 0 \) such that

\[
\left( \chi^2 + \frac{5}{2} \right) \int_{\Omega} u^2 |\nabla z|^2 \\
\leq \left( \chi^2 + \frac{5}{2} \right) \|u\|_{L^3(\Omega)}^2 |\nabla z|^2 \|L^3(\Omega) \\
\leq \left( \chi^2 + \frac{5}{2} \right) \left\{ \tilde{\epsilon}_1 \int_{\Omega} |\nabla u|^2 + c_4(\tilde{\epsilon}_1) \right\}^\frac{1}{2} \cdot c_6^\frac{1}{2} \left\{ \int_{\Omega} |\nabla |\nabla z|^2|^2 + 1 \right\}^\frac{1}{2} \\
\leq \frac{1}{32} \left\{ \int_{\Omega} |\nabla |\nabla z|^2|^2 + 1 \right\} + c_7 \left\{ \tilde{\epsilon}_1 \int_{\Omega} |\nabla u|^2 + c_4(\tilde{\epsilon}_1) \right\}. \tag{93}
\]

In a similar manner we get that there exist positive \( c_i \) \((i = 8, 9, 10)\) such that

\[
(\chi^2 + \frac{7}{2}) \int_{\Omega} \omega^2 |\nabla v|^2 \leq \frac{1}{32} \left\{ \int_{\Omega} |\nabla |\nabla v|^2|^2 + 1 \right\} + c_8 \left\{ \tilde{\epsilon}_2 \int_{\Omega} |\nabla \omega|^2 + c_4(\tilde{\epsilon}_2) \right\}, \tag{94}
\]

\[
\xi^2 \int_{\Omega} u^2 |\nabla v|^2 \leq \frac{1}{32} \left\{ \int_{\Omega} |\nabla |\nabla v|^2|^2 + 1 \right\} + c_9 \left\{ \tilde{\epsilon}_3 \int_{\Omega} |\nabla u|^2 + c_4(\tilde{\epsilon}_3) \right\}, \tag{95}
\]

\[
\tilde{\xi}^2 \int_{\Omega} \omega^2 |\nabla z|^2 \leq \frac{1}{32} \left\{ \int_{\Omega} |\nabla |\nabla z|^2|^2 + 1 \right\} + c_{10} \left\{ \tilde{\epsilon}_4 \int_{\Omega} |\nabla \omega|^2 + c_4(\tilde{\epsilon}_4) \right\}. \tag{96}
\]
We pick $c_7\tilde{c}_1 = c_8\tilde{c}_2 = c_9\tilde{c}_3 = c_{10}\tilde{c}_4 = \frac{1}{8}$ and insert (93)-(96) into (90). A simple rearrangement leads to the existence of $c_{11} > 0$ such that

$$c_{11} \geq \frac{d}{dt} \left\{ \frac{1}{2} \int_{\Omega} u^2 + \frac{1}{4} \int_{\Omega} |\nabla u|^4 + \frac{1}{2} \int_{\Omega} \omega^2 + \frac{1}{4} \int_{\Omega} |\nabla \omega|^4 \right\}$$

$$+ \frac{1}{4} \int_{\Omega} |\nabla u|^2 + \frac{1}{4} \int_{\Omega} |\nabla \omega|^2 + \frac{1}{16} \int_{\Omega} |\nabla |\nabla \omega|^2|^2$$

$$+ \frac{1}{16} \int_{\Omega} |\nabla |\nabla z|^2|^2 + \int_{\Omega} |\nabla v|^4 + \int_{\Omega} |\nabla z|^4.$$  

Especially,

$$c_{11} \geq \frac{d}{dt} \left\{ \frac{1}{2} \int_{\Omega} u^2 + \frac{1}{4} \int_{\Omega} |\nabla u|^4 + \frac{1}{2} \int_{\Omega} \omega^2 + \frac{1}{4} \int_{\Omega} |\nabla \omega|^4 \right\}$$

$$+ \frac{1}{4} \int_{\Omega} |\nabla u|^2 + \frac{1}{4} \int_{\Omega} |\nabla \omega|^2 + \frac{1}{16} \int_{\Omega} |\nabla |\nabla \omega|^2|^2 + \frac{1}{16} \int_{\Omega} |\nabla |\nabla z|^2|^2.$$  

The Poincaré inequality and (75) provide $c_{12} > 0$ such that

$$\int_{\Omega} u^2 \leq c_{12} \left[ \int_{\Omega} |\nabla u|^2 + 1 \right], \quad \int_{\Omega} |\nabla v|^4 \leq c_{12} \left[ \int_{\Omega} |\nabla |\nabla v|^2|^2 + 1 \right],$$

$$\int_{\Omega} \omega^2 \leq c_{12} \left[ \int_{\Omega} |\nabla \omega|^2 + 1 \right], \quad \int_{\Omega} |\nabla z|^4 \leq c_{12} \left[ \int_{\Omega} |\nabla |\nabla z|^2|^2 + 1 \right].$$  

We see that $y(t) = \frac{1}{2} \int_{\Omega} u^2 + \frac{1}{4} \int_{\Omega} |\nabla v|^4 + \frac{1}{2} \int_{\Omega} \omega^2 + \frac{1}{4} \int_{\Omega} |\nabla z|^4$ satisfies

$$y'(t) + \min \left\{ 1, \frac{1}{8c_{12}}, \frac{1}{4} \right\} y(t) \leq c_{11} + \frac{1}{4}.  \quad (99)$$

By an ODE comparison argument, we infer that there exists a positive constant $M_2$ such that

$$\int_{\Omega} u^2 < M_2, \quad \int_{\Omega} |\nabla v|^4 < M_2, \quad \int_{\Omega} \omega^2 < M_2, \quad \int_{\Omega} |\nabla z|^4 < M_2.  \quad (100)$$

Therefore, we complete the proof. \hfill \Box

By now, we are ready to prove Theorem 1.3.

**Proof of Theorem 1.3.** With the estimates $\int_{\Omega} u^2 < M_2$ and $\int_{\Omega} \omega^2 < M_2$ at hand, by Proposition 2.5 we see

$$\|u\|_{L^\infty(\Omega)} + \|v\|_{W^{1,4}(\Omega)} + \|\omega\|_{L^\infty(\Omega)} + \|z\|_{W^{1,4}(\Omega)} < C.  \quad (101)$$

In view of the extensibility criterion (9), we can see $T_{\max}$ must be infinite, and thus the proof of Theorem 1.3 is completed. \hfill \Box

6. **Conclusion.** In this article, under the assumptions that species can move away from themselves by the self-produced chemicals and make directional movement according to the chemicals secreted by the others, we proposed and studied a two species chemotaxis model with attraction-repulsion, namely, (3). When the system is parabolic-elliptic-parabolic-elliptic type, strong repulsion effect ensures the global existence of solutions; meanwhile, when the system is fully parabolic, we can only obtain the existence of global bounded solutions in lower dimensional spaces.

It is necessary to point out that in Theorem 1.1, if $\min\{\chi, \tilde{\chi}\} > \xi + \tilde{\xi}$, i.e., the repulsive effect is sufficiently strong, the answers to both global existence and boundedness are yes. Conversely, what can occur? We also recall that for (2)
with $\Omega \subset \mathbb{R}^2$, the blow-up is inevitable when the initial mass is beyond certain threshold value. As a generalization of system (2), in the case of $\tau \in \{0, 1\}$ and two dimensional domains, some natural questions on (3) arise: Is there a critical mass phenomenon in (3)? Meanwhile, due to the lack of research techniques, it is very challenging to derive properties of solutions to (2) with $\tau = 1$ in higher dimensional spaces. As a result, properties of solutions to system (3) with $\tau = 1$ also remain open.

At present, there is scarce literature that clearly answered the above questions which are very interesting, and hence we leave them for future work.

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