The ground state energy of the two dimensional Ginzburg–Landau functional with variable magnetic field

K. Attar\textsuperscript{a, b}

\textsuperscript{a} Laboratoire de Mathématiques, Université Paris-Sud, Rue G. Clémenceau, Bât 425, F-91405 Orsay Cedex, France
\textsuperscript{b} Laboratoire de Mathématiques, Université Libanaise, Hadath, Beyrouth, Liban

Received 24 May 2013; accepted 13 December 2013
Available online 7 January 2014

Abstract

We consider the Ginzburg–Landau functional with a variable applied magnetic field in a bounded and smooth two dimensional domain. We determine an accurate asymptotic formula for the minimizing energy when the Ginzburg–Landau parameter and the magnetic field are large and of the same order. As a consequence, it is shown how bulk superconductivity decreases in average as the applied magnetic field increases.

© 2014 Elsevier Masson SAS. All rights reserved.

MSC: 82D55

Keywords: Superconductivity; Ginzburg–Landau; Variable magnetic field

1. Introduction

1.1. The functional and main results

We consider a bounded open simply connected set $\Omega \subset \mathbb{R}^2$ with smooth boundary. We suppose that $\Omega$ models a superconducting sample submitted to an applied external magnetic field. The energy of the sample is given by the Ginzburg–Landau functional,

$$\mathcal{E}_\kappa,H(\psi, A) = \int_\Omega \left[ |(\nabla - i \kappa H A)\psi|^2 - \kappa^2 |\psi|^2 + \frac{\kappa^2}{2} |\psi|^4 \right] dx + \kappa^2 H^2 \int_\Omega |\text{curl} A - B_0|^2 dx. \quad (1.1)$$

Here $\kappa$ and $H$ are two positive parameters; $\kappa$ (the Ginzburg–Landau constant) is a material parameter and $H$ measures the intensity of the applied magnetic field. The wave function (order parameter) $\psi \in H^1(\Omega; \mathbb{C})$ describes the superconducting properties of the material. The induced magnetic field is $\text{curl} A$, where the potential $A \in H^1_{\text{div}}(\Omega)$, with $H^1_{\text{div}}(\Omega)$ is the space defined in (1.4) below. Finally, $B_0 \in C^\infty(\overline{\Omega})$ is the intensity of the external variable magnetic field and satisfies

$$|B_0| + |\nabla B_0| > 0 \quad \text{in } \overline{\Omega}. \quad (1.2)$$

0294-1449/ – see front matter © 2014 Elsevier Masson SAS. All rights reserved.
http://dx.doi.org/10.1016/j.anihpc.2013.12.002
The assumption in (1.2) implies that for any open set \( \omega \) relatively compact in \( \Omega \) the set \( \{ x \in \omega, \ B_0(x) = 0 \} \) will be either empty, or consists of a union of smooth curves. Let \( F : \Omega \to \mathbb{R}^2 \) be the unique vector field such that
\[
\text{div} F = 0 \quad \text{and} \quad \text{curl} F = B_0 \quad \text{in} \ \Omega, \quad \nu \cdot F = 0 \quad \text{on} \ \partial \Omega.
\] (1.3)
The vector \( \nu \) is the unit interior normal vector of \( \partial \Omega \). The construction of \( F \) is recalled in Appendix A. We define the space
\[
H^1_{\text{div}}(\Omega) = \{ A = (A_1, A_2) \in H^1(\Omega)^2 : \text{div} A = 0 \text{ in } \Omega, \ A \cdot \nu = 0 \text{ on } \partial \Omega \}.
\] (1.4)
Critical points \((\psi, A) \in H^1(\Omega; \mathbb{C}) \times H^1_{\text{div}}(\Omega)\) of \( E_{\kappa, H} \) are weak solutions of the Ginzburg–Landau equations
\[
\begin{aligned}
&-(\nabla - i \kappa HA)^2 \psi = \kappa^2 (1 - |\psi|^2) \psi \quad \text{in } \Omega, \\
&-\nabla^\perp \text{curl}(A - F) = \frac{1}{\kappa H} \text{Im}(\overline{\psi}(\nabla - i \kappa HA)\psi) \quad \text{in } \Omega, \\
&\nu \cdot (\nabla - i \kappa HA)\psi = 0 \quad \text{on } \partial \Omega, \\
&\text{curl} A = \text{curl} F \quad \text{on } \partial \Omega.
\end{aligned}
\] (1.5)
Here, \( \text{curl} A = \partial_{x_1} A_2 - \partial_{x_2} A_1 \) and \( \nabla^\perp \text{curl} A = (\partial_{x_2}(\text{curl} A), -\partial_{x_1}(\text{curl} A)) \). If \( \text{div} A = 0 \), then \( \nabla^\perp \text{curl} A = \Delta A \). In this paper, we study the ground state energy defined as follows:
\[
E_g(\kappa, H) = \inf \{ E_{\kappa, H}(\psi, A) : (\psi, A) \in H^1(\Omega; \mathbb{C}) \times H^1_{\text{div}}(\Omega) \}.
\] (1.6)
More precisely, we give an asymptotic estimate which is valid in the simultaneous limit \( \kappa \to \infty \) and \( H \to \infty \) in such a way that \( \frac{H}{\kappa} \) remains asymptotically constant. The behavior of \( E_g(\kappa, H) \) involves an auxiliary function \( g : [0, \infty) \to [-\frac{1}{2}, 0] \) introduced in [9] whose definition will be recalled in (2.5) below. The function \( g \) is increasing, continuous, \( g(b) = 0 \) for all \( b \geq 1 \) and \( g(0) = -\frac{1}{2} \).

**Theorem 1.1.** Let \( 0 < \Lambda_{\min} < \Lambda_{\max} \). Under assumption (1.2), there exist positive constants \( C, \kappa_0 \) and \( \tau_0 \in (1, 2) \) such that if
\[
\kappa_0 \leq \kappa, \quad \Lambda_{\min} \leq \frac{H}{\kappa} \leq \Lambda_{\max},
\]
then the ground state energy in (1.6) satisfies
\[
\left| E_g(\kappa, H) - \kappa^2 \int_{\Omega} g \left( \frac{H}{\kappa} |B_0(x)| \right) dx \right| \leq C \kappa^{\tau_0}.
\] (1.7)

Theorem 1.1 was proved in [9] when the magnetic field is constant \( B_0(x) = 1 \). However, the estimate of the remainder is not explicitly given in [9].

The approach used in the proof of Theorem 1.1 is slightly different from the one in [9], and is closer to that in [4] which studies the same problem when \( \Omega \subset \mathbb{R}^3 \) and \( B_0 \) constant.

**Corollary 1.2.** Suppose that the assumptions of Theorem 1.1 are satisfied. Then the magnetic energy of the minimizer satisfies, for some positive constant \( C \)
\[
(\kappa H)^2 \int_{\Omega} |\text{curl} A - B_0|^2 dx \leq C \kappa^{\tau_0}.
\] (1.8)

**Remark 1.3.** The value of \( \tau_0 \) depends on the properties of \( B_0 \): we find \( \tau_0 = \frac{7}{4} \) when \( B_0 \) does not vanish in \( \Omega \) and \( \tau_0 = \frac{15}{8} \) in the general case.

**Theorem 1.4.** Suppose the assumptions of Theorem 1.1 are satisfied. There exist positive constants \( C, \kappa_0 \) and a negative constant \( \tau_1 \in (-1, 0) \) such that, if \( \kappa \geq \kappa_0 \), and \( D \) is regular set such that \( \overline{D} \subset \Omega \), then the following are true.
(1) If \((\psi, A) \in H^1(\Omega; \mathbb{C}) \times H^1_{\text{div}}(\Omega)\) is a solution of (1.5), then
\[
\frac{1}{2} \int_{\Omega} |\psi|^4 \, dx \leq -\int_{\Omega} g\left(\frac{H}{\kappa}|B_0(x)|\right) \, dx + C\kappa^\tau_1. \tag{1.9}
\]

(2) If \((\psi, A) \in H^1(\Omega; \mathbb{C}) \times H^1_{\text{div}}(\Omega)\) is a minimizer of (1.1), then
\[
\left|\int_{\Omega} |\psi|^4 \, dx + 2 \int_{\Omega} g\left(\frac{H}{\kappa}|B_0(x)|\right) \, dx \right| \leq C\kappa^\tau_1. \tag{1.10}
\]

**Remark 1.5.** The value of \(\tau_1\) depends on the properties of \(B_0\): we find \(\tau_1 = -\frac{1}{4}\) when \(B_0\) does not vanish in \(\mathcal{I}\) and \(\tau_1 = -\frac{1}{8}\) in the general case.

### 1.2. Discussion of main result

If \(\{x \in \mathcal{I}: B_0(x) = 0\} \neq \emptyset\) and \(H = b\kappa, b > 0\), then \(g\left(\frac{H}{\kappa}|B_0(x)|\right) \neq 0\) in \(D = \{x \in \Omega: \frac{H}{\kappa}|B_0(x)| < 1\}\), and \(|D| \neq 0\). Consequently, for \(\kappa\) sufficiently large, the restriction of \(\psi\) on \(D\) is not zero in \(L^2(\Omega)\). This is a significant difference between our result and the one for constant magnetic field. When the magnetic field is a non-zero constant, then (see [3]), there is a universal constant \(\mathcal{C}_0 \in (\frac{1}{2}, 1)\) such that, if \(H = b\kappa\) and \(b > \mathcal{C}_0^{-1}\), then \(\psi = 0\) in \(\overline{\Omega}\). Moreover, in the same situation, when \(H = b\kappa\) and \(1 < b < \mathcal{C}_0^{-1}\), then \(\psi\) is small everywhere except in a thin tubular neighborhood of \(\partial\Omega\) (see [6]). Our result goes in the same spirit as in [8], where the authors established under the assumption (1.2) that when \(H = b\kappa^2\) and \(b > b_0\), then \(\psi = 0\) in \(\overline{\Omega}\) (\(b_0\) is a constant).

### 1.3. Notation

Throughout the paper, we use the following notation:

- We write \(E\) for the functional \(E_{\kappa,H}\) in (1.1).
- The letter \(C\) denotes a positive constant that is independent of the parameters \(\kappa\) and \(H\), and whose value may change from a formula to another.
- If \(a(\kappa)\) and \(b(\kappa)\) are two positive constants, we write \(a(\kappa) \ll b(\kappa)\) if \(a(\kappa)/b(\kappa) \to 0\) as \(\kappa \to \infty\).
- If \(a(\kappa)\) and \(b(\kappa)\) are two functions with \(b(\kappa) \neq 0\), we write \(a(\kappa) \sim b(\kappa)\) if \(a(\kappa)/b(\kappa) \to 1\) as \(\kappa \to \infty\).
- If \(a(\kappa)\) and \(b(\kappa)\) are two positive functions, we write \(a(\kappa) \approx b(\kappa)\) if there exist positive constants \(c_1, c_2\) and \(k_0\) such that \(c_1 b(\kappa) \leq a(\kappa) \leq c_2 b(\kappa)\) for all \(\kappa \geq k_0\).
- If \(x \in \mathbb{R}\), we let \(|x|^+ = \max(x, 0)\).
- Given \(R > 0\) and \(x = (x_1, x_2) \in \mathbb{R}^2\), we denote by \(Q_R(x) = (-R/2 + x_1, R/2 + x_1) \times (-R/2 + x_2, R/2 + x_2)\) the square of side length \(R\) centered at \(x\).
- We will use the standard Sobolev spaces \(W^{k,p}\). For integer values of \(s\) these are given by
  \[W^{n,p}(\Omega) := \{u \in L^p(\Omega): D^n u \in L^p(\Omega)\text{ for all }|\alpha| \leq n\}\]
- Finally we use the standard symbol \(H^n(\Omega) = W^{n,2}(\Omega)\).

### 2. The limiting energy

#### 2.1. Two-dimensional limiting energy

Given a constant \(b \geq 0\) and an open set \(\mathcal{D} \subset \mathbb{R}^2\), we define the following Ginzburg–Landau energy
\[
G_{b,\mathcal{D}}(u) = \int_{\mathcal{D}} \left( b |(\nabla - i\sigma A_0)u|^2 - |u|^2 + \frac{1}{2} |u|^4 \right) \, dx, \quad \forall u \in H_0^1(\mathcal{D}). \tag{2.1}
\]
Here $\sigma \in \{-1, +1\}$ and $A_0$ is the canonical magnetic potential
\begin{equation}
A_0(x) = \frac{1}{2}(-x_2, x_1), \quad \forall x = (x_1, x_2) \in \mathbb{R}^2, \tag{2.2}
\end{equation}
that satisfies
\[ \text{curl } A_0 = 1 \quad \text{in } \mathbb{R}^2. \]

We write $Q_R = Q_R(0)$ and let
\begin{equation}
m_0(b, R) = \inf_{u \in H^1_0(Q_R; \mathbb{C})} G^{+1}_{b, Q_R}(u). \tag{2.3}
\end{equation}

**Remark 2.1.** As $G^{+1}_{b, D}(u) = G^{-1}_{b, D}(\bar{u})$, it is immediate that
\[ \inf_{u \in H^1_0(Q_R; \mathbb{C})} G^{-1}_{b, Q_R}(u) = \inf_{u \in H^1_0(Q_R; \mathbb{C})} G^{+1}_{b, Q_R}(u). \tag{2.4} \]

The main part of the next theorem was obtained by Sandier and Serfaty [9] and Aftalion and Serfaty [1, Lemma 2.4]. However, the estimate in (2.7) is obtained by Fournais and Kachmar [4].

**Theorem 2.2.** Let $m_0(b, R)$ be as defined in (2.3).

1. For all $b \geq 1$ and $R > 0$, we have $m_0(b, R) = 0$.
2. For any $b \in [0, \infty)$, there exists a constant $g(b) \leq 0$ such that
\[ g(b) = \lim_{R \to \infty} \frac{m_0(b, R)}{|Q_R|} = -\frac{1}{2}. \tag{2.5} \]
3. The function $[0, +\infty) \ni b \mapsto g(b)$ is continuous, non-decreasing, concave and its range is the interval $[-\frac{1}{2}, 0]$.
4. There exists a constant $\alpha \in (0, \frac{1}{2})$ such that
\[ \forall b \in [0, 1], \quad \alpha(b - 1)^2 \leq |g(b)| \leq \frac{1}{2} (b - 1)^2. \tag{2.6} \]
5. There exist constants $C$ and $R_0$ such that
\[ \forall R \geq R_0, \forall b \in [0, 1], \quad g(b) \leq \frac{m_0(b, R)}{R^2} \leq g(b) + \frac{C}{R}. \tag{2.7} \]

### 3. A priori estimates

The aim of this section is to give *a priori* estimates for solutions of the Ginzburg–Landau equations (1.5). These estimates play an essential role in controlling the errors resulting from various approximations. The starting point is the following $L^\infty$-bound resulting from the maximum principle. Actually, if $(\psi, A) \in H^1(\Omega; \mathbb{C}) \times H^1_{\text{div}}(\Omega)$ is a solution of (1.5), then
\[ \|\psi\|_{L^\infty(\Omega)} \leq 1. \tag{3.1} \]
The set of estimates below is proved in [2, Theorem 3.3 and Eq. (3.35)] (see also [7] for an earlier version).

**Theorem 3.1.** Let $\Omega \subset \mathbb{R}^2$ be bounded and smooth and $B_0 \in C^\infty(\overline{\Omega})$.

1. For all $p \in (1, \infty)$ there exists $C_p > 0$ such that, if $(\psi, A) \in H^1(\Omega; C) \times H^1_{\text{div}}(\Omega)$ is a solution of (1.5), then
\[ \|\text{curl } A - B_0\|_{W^{1, p}(\Omega)} \leq C_p \frac{1 + \kappa H + \kappa^2}{\kappa H} \|\psi\|_{L^\infty(\Omega)} \|\psi\|_{L^2(\Omega)}. \tag{3.2} \]
(2) For all \( \alpha \in (0, 1) \) there exists \( C_\alpha > 0 \) such that, if \( (\psi, A) \in H^1(\Omega, C) \times H^1_{\text{div}}(\Omega) \) is a solution of (1.5), then
\[
\|\text{curl} A - B_0\|_{C^0,\alpha(\mathbb{T}_2)} \leq C_\alpha \frac{1 + \kappa H + \kappa^2}{\kappa H} \|\psi\|_{L^\infty(\Omega)} \|\psi\|_{L^2(\Omega)}.
\] (3.3)

(3) For all \( p \in [2, \infty) \) there exists \( C > 0 \) such that, if \( \kappa > 0, H > 0 \) and \( (\psi, A) \in H^1(\Omega, C) \times H^1_{\text{div}}(\Omega) \) is a solution of (1.5), then
\[
\| (\nabla - i\kappa H A)^2 \psi \|_p \leq \kappa^2 \|\psi\|_p, \quad (3.4)
\]
\[
\| (\nabla - i\kappa H A) \psi \|_2 \leq \kappa \|\psi\|_2, \quad (3.5)
\]
\[
\|\text{curl}(A - F)\|_{W^{1,p}(\Omega)} \leq \frac{C}{\kappa H} \|\psi\|_{L^\infty(\Omega)} \|\psi\|_{L^2(\Omega)}. \quad (3.6)
\]

Remark 3.2.

(1) Using the \( W^{k,p}_s \)-regularity of the curl–div system [3, Appendix A, Proposition A.5.1], we obtain from (3.2),
\[
\|A - F\|_{W^{2,p}(\Omega)} \leq C \frac{1 + \kappa H + \kappa^2}{\kappa H} \|\psi\|_{L^\infty(\Omega)} \|\psi\|_{L^2(\Omega)}. \quad (3.7)
\]
The estimate is true for any \( p \in [2, \infty) \).

(2) Using the Sobolev embedding theorem we get, for all \( \alpha \in (0, 1) \)
\[
\|A - F\|_{C^{1,\alpha}(\mathbb{T}_2)} \leq C_\alpha \frac{1 + \kappa H + \kappa^2}{\kappa H} \|\psi\|_{L^\infty(\Omega)} \|\psi\|_{L^2(\Omega)}. \quad (3.8)
\]

(3) Combining (3.5) and (3.6) (with \( p = 2 \)) yields
\[
\|\text{curl}(A - F)\|_{L^2(\Omega)} \leq \frac{C}{H} \|\psi\|_{L^\infty(\Omega)} \|\psi\|_{L^2(\Omega)}. \quad (3.9)
\]

Theorem 3.1 is needed in order to obtain the improved \textit{a priori} estimates of the next theorem. Similar estimates are given in [7].

**Theorem 3.3.** Suppose that 0 < \( A_{\text{min}} \leq A_{\text{max}} \). There exist constants \( \kappa_0 > 1, C_1 > 0 \) and for any \( \alpha \in (0, 1), C_\alpha > 0 \) such that, if
\[
\kappa \geq \kappa_0, \quad A_{\text{min}} \leq \frac{H}{\kappa} \leq A_{\text{max}}, \quad (3.10)
\]
and \((\psi, A) \in H^1(\Omega; C) \times H^1_{\text{div}}(\Omega)\) is a solution of (1.5), then
\[
\| (\nabla - i\kappa H A) \psi \|_{C(\mathbb{T}_2)} \leq C_1 \sqrt{\kappa H} \|\psi\|_{L^\infty(\Omega)}, \quad (3.11)
\]
\[
\|A - F\|_{H^2(\Omega)} \leq C_1 \left( \|\text{curl}(A - F)\|_{L^2(\Omega)} + \frac{1}{\sqrt{\kappa H}} \|\psi\|_{L^2(\Omega)} \|\psi\|_{L^\infty(\Omega)} \right). \quad (3.12)
\]
\[
\|A - F\|_{C^{0,\alpha}(\mathbb{T}_2)} \leq C_\alpha \left( \|\text{curl}(A - F)\|_{L^2(\Omega)} + \frac{1}{\sqrt{\kappa H}} \|\psi\|_{L^2(\Omega)} \|\psi\|_{L^\infty(\Omega)} \right). \quad (3.13)
\]

**Proof.** Proof of (3.11): See [3, Proposition 12.4.4].

Proof of (3.12): Let \( a = A - F \). Since \( \text{div} a = 0 \) and \( a \cdot v = 0 \) on \( \partial \Omega \), we get by regularity of the curl–div system (see Appendix A, Proposition A.1),
\[
\|a\|_{H^2(\Omega)} \leq C \|\text{curl} a\|_{H^1(\Omega)}. \quad (3.14)
\]
The second equation in (1.5) reads as follows
\[
-\nabla \cdot \text{curl} a = \frac{1}{\kappa H} \text{Im}(\bar{\psi}(\nabla - i\kappa H A)\psi). \]
The estimates in (3.11) and (3.14) now give
\[ \|a\|_{H^2(\Omega)} \leq C \left( \|\nabla a\|_{L^2(\Omega)} + \frac{1}{\sqrt{kH}} \|\psi\|_{L^2(\Omega)} \|\psi\|_{L^\infty(\Omega)} \right). \]

Proof of (3.13): This is a consequence of the Sobolev embedding of $H^2(\Omega)$ into $C^{0,\alpha}(\overline{\Omega})$ for any $\alpha \in (0, 1)$ and (3.12). □

4. Energy estimates in small squares

If $(\psi, A) \in H^1(\Omega; \mathbb{C}) \times H^1_{\text{div}}(\Omega)$, we introduce the energy density
\[ e(\psi, A) = |(\nabla - i \kappa H A)\psi|^2 - \kappa^2 |\psi|^2 + \frac{\kappa^2}{2} |\psi|^4. \]
We also introduce the local energy of $(\psi, A)$ in a domain $D \subset \Omega$:
\[ \mathcal{E}_0(u, A; D) = \int_D e(\psi, A) \, dx. \tag{4.1} \]
Furthermore, we define the Ginzburg–Landau energy of $(\psi, A)$ in a domain $D \subset \Omega$ as follows,
\[ \mathcal{E}(\psi, A; D) = \mathcal{E}_0(\psi, A; D) + (\kappa H)^2 \int_\Omega |\nabla (A - F)|^2 \, dx. \tag{4.2} \]
If $D = \Omega$, we sometimes omit the dependence on the domain and write $\mathcal{E}_0(\psi, A)$ for $\mathcal{E}_0(\psi, A; \Omega)$. We start with a lemma that will be useful in the proof of Proposition 4.2 below. Before we start to state the lemma, we define for all $(\ell, x_0)$ such that $\overline{Q_\ell(x_0)} \subset \Omega$,
\[ \overline{B_{Q_\ell(x_0)}} = \sup_{x \in Q_\ell(x_0)} |B_0(x)|, \tag{4.3} \]
where $B_0$ is introduced in (1.2). Later $x_0$ will be chosen in a lattice of $\mathbb{R}^2$.

Lemma 4.1. For any $\alpha \in (0, 1)$, there exist positive constants $C$ and $\kappa_0$ such that if (3.10) holds, $0 < \delta < 1$, $0 < \ell < 1$, and $(\psi, A) \in H^1(\Omega; \mathbb{C}) \times H^1_{\text{div}}(\Omega)$ is a critical point of (1.1) (i.e. a solution of (1.5)), then, for any square $Q_\ell(x_0)$ relatively compact in $\Omega \cap \{|B_0| > 0\}$, there exists $\varphi \in H^1(\Omega)$, such that
\[ \mathcal{E}_0(\psi, A; Q_\ell(x_0)) \geq (1 - \delta) \mathcal{E}_0(e^{-i \kappa H \varphi}, \sigma_\ell B_{Q_\ell(x_0)} A_0(x - x_0), Q_\ell(x_0)) \]
\[ - C \kappa^2 (\delta^{-1} \ell^{2\alpha} + \delta^{-1} \ell^{4\kappa^2} + \delta) \int_{Q_\ell(x_0)} |\varphi|^2 \, dx, \tag{4.4} \]
where $\sigma_\ell$ denotes the sign of $B_0$ in $Q_\ell(x_0)$.

Proof. Construction of $\varphi$: Let $\phi_{x_0}(x) = (A(x_0) - F(x_0)) \cdot x$, where $F$ is the magnetic potential introduced in (1.3). Using the estimate in (3.13), we get for all $x \in Q_\ell(x_0)$ and $\alpha \in (0, 1),$
\[ |A(x) - \nabla \phi_{x_0} - F(x)| \leq \|A - F\|_{C^{0,\alpha}} \cdot |x - x_0|^\alpha \]
\[ \leq \frac{\|A - F\|_{C^{0,\alpha}}}{\kappa H} \cdot |x - x_0|^{\alpha}, \]
\[ \leq C \sqrt{\frac{\lambda}{\ell \kappa H}}, \tag{4.5} \]
where
\[ \lambda = (\kappa H)^2 \left( \|\nabla (A - F)\|_{L^2(\Omega)}^2 + \frac{1}{\kappa H} \|\psi\|_{L^2(\Omega)}^2 \right). \]
Using the bound \( \| \psi \|_{\infty} \leq 1 \) and the estimate in (3.9), we get
\[
\lambda \leq C \kappa^2,
\]
which implies that
\[
|A(x) - \nabla \phi(x) - F(x)| \leq C \frac{\epsilon}{H},
\]
We estimate the energy \( E_0(\psi, A; Q_\ell(x_0)) \) from below. We will need the function \( \varphi_0 \) introduced in Lemma A.3 and satisfying
\[
|F(x) - \sigma_i B_{Q_\ell(x_0)} A_0 (x - x_0) - \nabla \varphi_0(x)| \leq C \ell^2 \quad \text{in } Q_\ell(x_0).
\]
Let
\[
u = e^{-i \kappa H} \psi,
\]
where \( \varphi = \varphi_0 + \phi_{x_0} \).

**Lower bound:** We start with estimating the kinetic energy from below as follows. For any \( \delta \in (0, 1) \), we write
\[
|\langle \nabla - i \kappa H A \rangle \rangle^2 = |\langle \nabla - i \kappa H (\sigma_i B_{Q_\ell(x_0)} A_0 (x - x_0) + \nabla \varphi) \rangle \rangle^2 \leq (1 - \delta) |\langle \nabla - i \kappa H (\sigma_i B_{Q_\ell(x_0)} A_0 (x - x_0) + \nabla \varphi) \rangle \rangle^2
\]
\[
+ (1 - \delta^2) (i \kappa H)^2 (\langle A - \nabla \phi - F \rangle \psi + \langle F - \sigma_i B_{Q_\ell(x_0)} A_0 (x - x_0) - \nabla \varphi_0 \rangle \psi|^2.
\]
Using the estimates in (4.7), (A.3) and the assumption in (3.10), we get, for any \( \alpha \in (0, 1) \)
\[
|\langle \nabla - i \kappa H A \rangle \rangle^2 \leq (1 - \delta) |\langle \nabla - i \kappa H (\sigma_i B_{Q_\ell(x_0)} A_0 (x - x_0) + \nabla \varphi) \rangle \rangle^2 \nabla - C \kappa^2 (\delta^{-2} \ell^2 + \delta^{-1} \ell^2 H)^2 |\psi|^2.
\]

Remembering the definition of \( \nu \) in (4.8), then, we deduce the lower bound of \( E_0 \)
\[
E_0(\psi, A; Q_\ell(x_0)) \geq \int_{Q_\ell(x_0)} \left[ (1 - \delta) |\langle \nabla - i \kappa H (\sigma_i B_{Q_\ell(x_0)} A_0 (x - x_0)) \rangle \rangle|^2 - \kappa^2 |u|^2 + \frac{\kappa^2}{2} |u|^4 \right] dx
\]
\[
- C \kappa^2 (\delta^{-2} \ell^2 + \delta^{-1} \ell^2 H)^2 \int_{Q_\ell(x_0)} |\psi|^2 dx
\]
\[
\geq (1 - \delta) E_0(u, \sigma_i B_{Q_\ell(x_0)} A_0 (x - x_0); Q_\ell(x_0))
\]
\[
- C \kappa^2 (\delta^{-1} \ell^2 \kappa^2 + \delta^{-1} \ell^2 \kappa + \delta^{-1} \ell^2 H)^2 \int_{Q_\ell(x_0)} |\psi|^2 dx.
\]

This finishes the proof of Lemma 4.1. \( \square \)

**Proposition 4.2.** For all \( \alpha \in (0, 1) \), there exist positive constants \( C, \epsilon_0 \) and \( \kappa_0 \) such that, if (3.10) holds, \( \kappa \geq \kappa_0, \ell \in (0, \frac{1}{2}), \epsilon \in (0, \epsilon_0), \ell^2 \kappa^2 \epsilon > 1 \), \( (\psi, A) \in H^1(\Omega; C) \times H^1_{div}(\Omega) \) is a critical point of (1.1), and \( Q_\ell(x_0) \subset \Omega \cap \{|B_0| > \epsilon\} \), then
\[
\frac{1}{|Q_\ell(x_0)|} E_0(\psi, A; Q_\ell(x_0)) \geq g \left( \frac{H}{k} B_{Q_\ell(x_0)} \right) \kappa^2 - C (\ell^3 \kappa^2 + \ell^2 \kappa^2 + (\ell \kappa)^{-1} + \ell e^{-1}) \kappa^2.
\]
Here \( g(\cdot) \) is the function introduced in (2.5), and \( B_{Q_\ell(x_0)} \) is introduced in (4.3).

**Proof.** Using the inequality \( \| \psi \|_{\infty} \leq 1 \) and (4.4) to obtain
\[
E_0(\psi, A; Q_\ell(x_0)) \geq (1 - \delta) E_0(u, \sigma_i B_{Q_\ell(x_0)} A_0 (x - x_0); Q_\ell(x_0))
\]
\[
- C \kappa^2 (\delta^{-1} \ell^2 \kappa^2 + \delta^{-1} \ell^2 \kappa + \delta^{-1} \ell^2 H)^2 \int_{Q_\ell(x_0)} |\psi|^2 dx,
\]
where \( u \) is defined in (4.8).
Let
\[ b = \frac{H}{\kappa} B_{Q_t(x_0)}, \quad R = \ell \sqrt{\kappa H B_{Q_t(x_0)}}. \]  
(4.12)

Define the rescaled function
\[ v(x) = u \left( \frac{\ell}{R} x + x_0 \right), \quad \forall x \in Q_R. \]  
(4.13)

Remember that \( \sigma_\ell \) denotes the sign of \( B_0 \) in \( Q_\ell(x_0) \). The change of variable \( y = R \ell (x - x_0) \) gives
\[
E_0(u, \sigma_\ell B_{Q_\ell(x_0)} A_0(x - x_0); Q_\ell(x_0)) \\
= \int_{Q_R} \left( \left( \left( \frac{R}{\ell} \nabla_y - i \sigma_\ell \frac{R}{\ell} A_0(y) \right) v \right)^2 - \kappa^2 |v|^2 + \frac{\kappa^2}{2} |v|^4 \right) \frac{\ell}{R} dy \\
= \int_{Q_R} \left( \left( \nabla_y - i \sigma_\ell A_0 \right) v \right)^2 - \frac{\kappa}{H B_{Q_t(x_0)}} |v|^2 + \frac{\kappa}{2 H B_{Q_t(x_0)}} |v|^4 \right) dy \\
= \frac{\kappa}{H B_{Q_t(x_0)}} \int_{Q_R} b \left( \left( \nabla_y - i \sigma_\ell A_0 \right) v \right)^2 - |v|^2 + \frac{1}{2} |v|^4 \right) dy \\
= \frac{1}{b} G_{b, Q_R}^{\sigma_\ell} (v). \]  
(4.14)

We still need to estimate from below the reduced energy \( G_{b, Q_R}^{\sigma_\ell} (v) \). Since \( v \) is not in \( H^1_0(Q_R) \), we introduce a cut-off function \( \chi_R \in C^\infty_c(\mathbb{R}^2) \) such that
\[
0 \leq \chi_R \leq 1 \quad \text{in } \mathbb{R}^2, \quad \text{supp } \chi_R \subset Q_R, \quad \chi_R = 1 \quad \text{in } Q_R - 1, \quad \text{and } |\nabla \chi_R| \leq M \quad \text{in } \mathbb{R}^2. \]  
(4.15)

The constant \( M \) is universal.

Let
\[ u_R = \chi_R v. \]  
(4.16)

We have
\[
G_{b, Q_R}^{\sigma_\ell} (v) = \int_{Q_R} \left( b \left( \nabla - i \sigma_\ell A_0 \right) v \right)^2 - |v|^2 + \frac{1}{2} |v|^4 \right) dx \]
\[ \geq \int_{Q_R} b \chi_R \left( \nabla - i \sigma_\ell A_0 \right) v \right|_2^2 - |\chi_R v|^2 + \frac{1}{2} |v|^4 + (\chi_R^2 - 1) |v|^2 \right) dx \]
\[ \geq G_{b, Q_R}^{\sigma_\ell} (\chi_R v) - \int_{Q_R} (1 - \chi_R^2) |v|^2 dx - 2 \int_{Q_R} ||\nabla - i \sigma_\ell A_0|| \chi_R v, \nabla \chi_R v || dy. \]  
(4.17)

Having in mind (4.13) and (4.8), we get
\[
|\nabla_y - i \sigma_\ell A_0(y) v(y)| = \frac{\ell}{R} \left| \nabla x - i \kappa H A_0 B_{Q_t(x_0)} A_0(x - x_0) u(x) \right|. \]

Using the estimate in (3.11), (4.7) and (A.3) we get
\[
|\nabla_y - i \sigma_\ell A_0(y) v(y)| \leq \frac{\ell}{R} \left| \nabla x - i \kappa H A_0 B_{Q_t(x_0)} (A + \nabla \varphi) u(x) \right| + \frac{\kappa H \ell}{R} |(A - \sigma_\ell B_{Q_t(x_0)} A_0(x - x_0) - \nabla \varphi) u(x)| \leq C_1 \kappa (\kappa + \kappa \ell^a + \kappa^2 \ell^2). \]  
(4.18)
From the definition of $u_R$ in (4.16) and $\chi_R$ in (4.15) we get

$$|v| \leq 1.$$  \hfill (4.19)

Using (4.19), (4.18) and the definition of $\chi_R$ in (4.15), we get

$$\int_{\partial Q} \left| (\nabla - i\sigma_\ell A_0) \chi_R v, \nabla \chi_R v \right| dy \leq \frac{C_1 \ell}{R} \left( \kappa + \kappa^{2} \ell^{2} + \ell^{2} \right) \int_{Q \setminus Q_{R-1}} |\nabla \chi_R| dx \leq C_1 \left( \kappa \ell + \kappa^{2} \ell^{2} + \kappa^{2} \ell^{3} \right),$$

and

$$\int_{Q} (1 - \chi_R^2) |v|^2 dx \leq |Q \setminus Q_{R-1}| \leq R.$$ \hfill (4.21)

Inserting (4.20) and (4.21) into (4.17), we get

$$G^{\sigma_\ell}_{b, Q_R}(v) \geq G^{\sigma_\ell}_{b, Q_R}(u_R) - C_2 \left( \kappa \ell + \kappa^{2} \ell^{2} + \kappa \ell^{3} \right) \geq G^{\sigma_\ell}_{b, Q_R}(u_R) - C_2 \left( \kappa \ell (\sqrt{\ell} + 1) + \kappa^{2} \ell^{3} \right).$$

There are two cases:

Case 1: $\sigma_\ell = +1$, when $B_0 > 0$, in $Q_\ell(x_0)$.

Case 2: $\sigma_\ell = -1$, when $B_0 < 0$, in $Q_\ell(x_0)$.

In Case 1, after recalling the definition of $m_0(b, R)$ introduced in (2.3), where $b$ is introduced in (4.12) we get

$$G^{+1}_{b, Q_R}(v) \geq m_0(b, R) - C_2 \left( \kappa \ell (\sqrt{\ell} + 1) + \kappa^{2} \ell^{3} \right).$$  \hfill (4.22)

We get by collecting the estimates in (4.11)–(4.22):

$$\frac{1}{|Q_\ell(x_0)|} \mathcal{E}_0(\psi, A; Q_\ell(x_0)) \geq \frac{(1 - \delta)}{b\ell^2} \left( m_0(b, R) - C_2 \left( \kappa \ell + \kappa^{2} \ell^{3} (\epsilon + 1) \right) \right) - C \left( \delta - 1 \ell^4 \kappa^{2} + \delta - 1 \ell^2 \delta + \delta^{-1} \ell^2 \kappa^2 \right) \kappa^2 \geq \frac{(1 - \delta)}{b\ell^2} m_0(b, R) - r(\kappa),$$ \hfill (4.23)

where

$$r(\kappa) = C_3 \left( \delta^{-1} \ell^4 \kappa^{2} + \delta^{-1} \ell^2 \delta + \delta \kappa^2 + \frac{1}{b\ell^2} \left( \kappa \ell (\sqrt{\ell} + 1) + \kappa^{2} \ell^{3} \right) \right).$$ \hfill (4.24)

Theorem 2.2 tells us that $m_0(b, R) \geq R^2 g(b)$ for all $b \in [0, 1]$ and $R$ sufficiently large. Here $g(b)$ is introduced in (2.5). Therefore, we get from (4.23) the estimate

$$\frac{1}{|Q_\ell(x_0)|} \mathcal{E}_0(\psi, A; Q_\ell(x_0)) \geq \left( \frac{(1 - \delta) R^2}{b\ell^2} \right) g(b) - r(\kappa),$$ \hfill (4.25)

with $b$ defined in (4.12). By choosing $\delta = \ell$ and using that $Q_\ell(x_0) \subset \{|B_0| > \epsilon\}$, we get

$$r(\kappa) = O \left( \ell^{3} \kappa^{2} + \ell^{2} \delta^{-1} + \frac{1}{\epsilon} \right) \kappa^2.$$

This implies that

$$\frac{1}{|Q_\ell(x_0)|} \mathcal{E}_0(\psi, A; Q_\ell(x_0)) \geq b \left( \frac{H}{\kappa} \rho_{Q_\ell(x_0)} \right) \kappa^2 - C \left( \ell^{3} \kappa^{2} + \ell^{2} \delta^{-1} + (\ell \kappa \epsilon^{-1} + \ell \epsilon^{-1}) \kappa^2 \right).$$

Similarly, in Case 2, according to Remark 2.1, we get that

$$G^{-1}_{b, Q_R}(v) \geq m_0(b, R) - C_2 \left( \kappa \ell + \kappa^{2} \ell^{3} (\epsilon + 1) \right),$$

and the rest of the proof is as for Case 1. \hfill $\Box$
5. Proof of Theorem 1.1

5.1. Upper bound

Proposition 5.1. There exist positive constants \( C \) and \( \kappa_0 \) such that, if (3.10) holds, then the ground state energy \( E_g(\kappa, H) \) in (1.6) satisfies

\[
E_g(\kappa, H) \leq \kappa^2 \int_\Omega g \left( \frac{H}{\kappa} |B_0(x)| \right) dx + C\kappa^{15/8}.
\]

Proof. Let \( \ell = \ell(\kappa) \) and \( \epsilon = \epsilon(\kappa) \) be positive parameters such that \( \kappa^{-1} \ll \ell \ll 1 \) and \( \kappa^{-1} \ll \epsilon \ll 1 \) as \( \kappa \to \infty \). For some \( \beta \in (0, 1) \), \( \mu \in (0, 1) \) to be determined later, we will choose

\[
\ell = \kappa^{-\beta}, \quad \epsilon = \kappa^{-\mu}.
\]

Consider the lattice \( \Gamma_\ell := \ell \mathbb{Z} \times \ell \mathbb{Z} \) and write for \( \gamma \in \Gamma_\ell \), \( Q_{\gamma,\ell} = Q_\ell(\gamma) \). For any \( \gamma \in \Gamma_\ell \) such that \( Q_{\gamma,\ell} \subset \Omega \cap \{|B_0| > \epsilon\} \) let

\[
B_{\gamma,\ell} = \inf_{x \in Q_{\gamma,\ell}} |B_0(x)|.
\]

Let

\[
I_{\ell,\epsilon} = \{ \gamma : \overline{Q_{\gamma,\ell}} \subset \Omega \cap \{|B_0| > \epsilon\} \},
\]

\[
N = \text{Card} I_{\ell,\epsilon},
\]

and

\[
\Omega_{\ell,\epsilon} = \text{int} \left( \bigcup_{\gamma \in I_{\ell,\epsilon}} \overline{Q_{\gamma,\ell}} \right).
\]

It follows from (1.2) that

\[
N = |\Omega| \ell^{-2} + \mathcal{O}(\ell^{-2}) + \mathcal{O}(\ell^{-1}) \quad \text{as } \ell \to 0 \text{ and } \epsilon \to 0.
\]

Let

\[
b = \frac{H}{\kappa} \frac{B_{\gamma,\ell}}{\sqrt{\kappa HB_{\gamma,\ell}}}, \quad R = \ell \sqrt{\kappa HB_{\gamma,\ell}},
\]

and \( u_R \) be a minimizer of the functional in (2.1), i.e.

\[
m_0(b, R) = \int_{\Omega} \left( b \left| (\nabla - i A_0) u_R \right|^2 - |u_R|^2 + \frac{1}{2} |u_R|^4 \right) dx.
\]

We will need the function \( \varphi_\gamma \) introduced in Lemma A.3 which satisfies

\[
|F(x) - \sigma_{\gamma,\ell} B_{\gamma,\ell} A_0(x - \gamma) - \nabla \varphi_\gamma(x)| \leq C \ell^2 \quad \text{in } Q_{\gamma,\ell},
\]

where \( \sigma_{\gamma,\ell} \) is the sign of \( B_0 \) in \( Q_{\gamma,\ell} \).

We define the function

\[
v(x) = \begin{cases} 
    e^{-i \kappa H \varphi_\gamma} u_R \left( \frac{B_0(x - \gamma)}{\ell} \right) & \text{if } x \in Q_{\gamma,\ell} \subset \{ B_0 > \epsilon \}, \\
    e^{-i \kappa H \varphi_\gamma} u_R \left( \frac{B_0(x - \gamma)}{\ell} \right) & \text{if } x \in Q_{\gamma,\ell} \subset \{ B_0 < -\epsilon \}, \\
    0 & \text{if } x \in \Omega \setminus \Omega_{\ell,\epsilon}.
\end{cases}
\]
Since \( u_R \in H^1_0(Q_R) \), then \( v \in H^1(\Omega) \). We compute the energy of the configuration \((v, F)\). We get
\[
\mathcal{E}(v, F) = \int_\Omega \left( |(\nabla - i\kappa H)F| - \kappa^2 |v|^2 + \frac{\kappa^2}{2} |v|^4 \right) dx
\]
\[
= \sum_{\gamma \in I_{\ell,\epsilon}} \mathcal{E}_0(v, F; Q_{\gamma,\ell}).
\]
We estimate the term \( \mathcal{E}_0(v, F; Q_{\gamma,\ell}) \) from above and we write
\[
\mathcal{E}_0(v, F; Q_{\gamma,\ell}) = \int_{Q_{\gamma,\ell}} \left| (\nabla - i\kappa H) \left( \sigma_{\gamma,\ell} B_{\gamma,\ell} A_0(x - \gamma) + \nabla \varphi_{\gamma}(x) \right) \right| v
\]
\[
- i\kappa H (F - \sigma_{\gamma,\ell} B_{\gamma,\ell} A_0(x - \gamma) - \nabla \varphi_{\gamma}(x)) v| dx
\]
\[
\leq (1 + \delta) \int_{Q_{\gamma,\ell}} \left| (\nabla - i\kappa H) \left( \sigma_{\gamma,\ell} B_{\gamma,\ell} A_0(x - \gamma) + \nabla \varphi_{\gamma}(x) \right) \right| v| dx
\]
\[
+ C(1 + \delta^{-1}) (\kappa H)^2 \int_{Q_{\gamma,\ell}} \left| (\nabla - \sigma_{\gamma,\ell} B_{\gamma,\ell} A_0(x - \gamma) - \nabla \varphi_{\gamma}(x) \right| v^2 dx
\]
\[
\leq (1 + \delta) \mathcal{E}_0 \left( e^{-i\kappa H \varphi} v, \sigma_{\gamma,\ell} B_{\gamma,\ell} A_0(x - \gamma); Q_{\gamma,\ell} \right) + C(\delta \kappa^2 + \delta^{-1} \kappa^4 \ell^4) \int_{Q_{\gamma,\ell}} |v|^2 dx.
\]
Having in mind that \( u_R \) is a minimizer of the functional in (2.1), and using the estimate in (3.1) we get
\[
\int_{Q_{\gamma,\ell}} |v|^2 dx \leq |Q_{\gamma,\ell}|.
\]
Remark 2.1 and a change of variables give us
\[
\int_{Q_{\gamma,\ell}} \left( \left| \nabla - i\kappa H \sigma_{\gamma,\ell} (B_{\gamma,\ell} A_0(x - \gamma)) e^{-i\kappa H \varphi} \right| v| dx - \kappa^2 |v|^2 + \frac{\kappa^2}{2} |v|^4 \right) dx = \frac{m_0(b, R)}{b}.
\]
We insert this into (5.5) to obtain
\[
\mathcal{E}_0(v, F; Q_{\gamma,\ell}) \leq (1 + \delta) \frac{m_0(b, R)}{b} + C(\delta \kappa^2 + \delta^{-1} \kappa^4 \ell^4) \ell^2.
\]
We know from Theorem 2.2 that \( m_0(b, R) \leq g(b) R^2 + CR \) for all \( b \in [0, 1] \) and \( R \) sufficiently large, where \( b \) introduced in (5.3). We choose \( \delta = \ell \) in (5.6). That way we get
\[
\mathcal{E}_0(v, F; Q_{\gamma,\ell}) \leq g \left( \frac{H}{\kappa} B_{\gamma,\ell} \right) \ell^2 \kappa^2 + C \left( \frac{1}{\kappa \ell \sqrt{\epsilon} + \ell + \kappa^2 \ell^3} \right) \ell^2 \kappa^2.
\]
Summing (5.7) over \( \gamma \) in \( I_{\ell,\epsilon} \), we recognize the lower Riemann sum of \( x \rightarrow g \left( \frac{H}{\kappa} B_0(x) \right) \). By monotonicity of \( g \), \( g \) is Riemann-integrable and its integral is larger than any lower Riemann sum. Thus
\[
\mathcal{E}(v, F) \leq \left( \int_{\Omega_{\ell,\epsilon}} g \left( \frac{H}{\kappa} |B_0(x)| \right) dx \right) \kappa^2 + C \left( \frac{1}{\kappa \ell \sqrt{\epsilon} + \ell + \kappa^2 \ell^3} \right) \kappa^2.
\]
Notice that using the regularity of $\partial \Omega$ and (1.2), there exists $C > 0$ such that

$$|\Omega \setminus \Omega_{\ell,\epsilon}| = O(\ell |\partial \Omega| + C \epsilon),$$

as $\epsilon$ and $\ell$ tend to 0.

Thus, we get by using the properties of $g$ in Theorem 2.2,

$$\int_{\Omega_{\ell,\epsilon}} g\left(\frac{H}{\kappa} |B_0(x)|\right) \, dx \leq \int_{\Omega} g\left(\frac{H}{\kappa} |B_0(x)|\right) \, dx + \frac{1}{2} |\Omega \setminus \Omega_{\ell,\epsilon}|.$$

This implies that

$$\mathcal{E}(v, F) \leq \int_{\Omega} g\left(\frac{H}{\kappa} |B_0(x)|\right) \, dx + C \left(\frac{1}{\kappa \ell \sqrt{\epsilon}} + \ell + \epsilon + \kappa^2 \ell^3\right) \kappa^2.$$

We choose in (5.1)

$$\beta = \frac{3}{4} \quad \text{and} \quad \mu = \frac{1}{8}. \quad (5.11)$$

With this choice, we infer from (5.10),

$$\mathcal{E}(v, F) \leq \int_{\Omega} g\left(\frac{H}{\kappa} |B_0(x)|\right) \, dx + C_1 \kappa^{\frac{15}{8}}, \quad (5.12)$$

and

$$\ell^2 \kappa^2 \epsilon = \kappa^{\frac{3}{2}} > 1. \quad (5.13)$$

This finishes the proof of Proposition 5.1. $\square$

**Remark 5.2.** In the case when $B_0$ does not vanish in $\Omega$, $\epsilon$ disappears and $\{x \in \Omega; |B_0(x)| > 0\} = \Omega$. Consequently, the Ginzburg–Landau energy of $(v, F)$ in (4.2) satisfies

$$\mathcal{E}(v, F) \leq \int_{\Omega} g\left(\frac{H}{\kappa} |B_0(x)|\right) \, dx + \frac{1}{\kappa \ell} |B_0(x)|. \quad (5.16)$$

We now establish a lower bound for the ground state energy $E_g(\kappa, H)$ in (1.6). The parameters $\epsilon$ and $\ell$ have the same form as in (5.1).

Let

$$\mathcal{B}_{y,\ell} = \sup_{x \in Q_{y,\ell}} |B_0(x)|, \quad (5.14)$$

and

$$b_{y,\ell} = \frac{H}{\kappa} \mathcal{B}_{y,\ell}, \quad R = \ell \sqrt{\kappa H \mathcal{B}_{y,\ell}}. \quad (5.15)$$

If $(\psi, A)$ is a minimizer of (1.1), we have

$$E_g(\kappa, H) = \mathcal{E}_0(\psi, A; \Omega_{\ell,\epsilon}) + \mathcal{E}_0(\psi, A; \Omega \setminus \Omega_{\ell,\epsilon}) + (\kappa H)^2 \int_{\Omega} |\text{curl}(A - F)|^2 \, dx, \quad (5.16)$$
where, for any $D \subset \Omega$, the energy $\mathcal{E}_0(\psi, A; D)$ is introduced in (4.1). Since the magnetic energy term is positive, we may write

$$E_g(\kappa, H) \geq \mathcal{E}_0(\psi, A; \Omega_{\ell, e}) + \mathcal{E}_0(\psi, A; \Omega \setminus \Omega_{\ell, e}).$$  

(5.17)

Thus, we get by using (3.1), (3.11), and (5.9):

$$\left| \mathcal{E}_0(\psi, A; \Omega \setminus \Omega_{\ell, e}) \right| \leq \int_{\Omega \setminus \Omega_{\ell, e}} \left| (\nabla - i\kappa H A)\psi \right|^2 + \kappa^2 |\psi|^2 + \frac{\kappa^2}{2} |\psi|^4 \, dx$$

$$\leq |\Omega \setminus \Omega_{\ell, e}| \left( C_1 \kappa^2 \|\psi\|_{L^\infty(\Omega)}^2 + \kappa^2 \|\psi\|_{L^2(\Omega)}^2 + \frac{\kappa^2}{2} \|\psi\|_{L^\infty(\Omega)}^4 \right)$$

$$\leq C_2 (\ell + \epsilon) \kappa^2.$$

(5.18)

To estimate $\mathcal{E}_0(\psi, A; \Omega_{\ell, e})$, we notice that

$$\mathcal{E}_0(\psi, A; \Omega_{\ell, e}) = \sum_{\gamma \in I_{\ell, e}} \mathcal{E}_0(\psi, A; Q_{\ell, e}).$$

Using Proposition 4.2 with $\alpha = \frac{2}{3}$ and $\beta = \frac{3}{4}$ and $\mu = \frac{1}{8}$ in (5.1), we get

$$\mathcal{E}_0(\psi, A; \Omega_{\ell, e}) \geq \sum_{\gamma \in I_{\ell, e}} g \left( \frac{H}{\kappa} |B_{Q_{\ell, e}}(x_0)| \right) \ell^2 \kappa^2 - C (\ell^3 \kappa^2 + \ell^{2\alpha - 1} + (\ell \kappa)^{-1} + \ell \epsilon^{-1}) \kappa^2$$

$$\geq \kappa^2 \sum_{\gamma \in I_{\ell, e}} g \left( \frac{H}{\kappa} |B_{Q_{\ell, e}}(x_0)| \right) \ell^2 - C_1 \kappa^{\frac{15}{8}},$$

and

$$\mathcal{E}_0(\psi, A; \Omega \setminus \Omega_{\ell, e}) \geq -C_2 \kappa^{\frac{15}{8}}.$$

(5.19)

As for the upper bound, we can use the monotonicity of $g$ and recognize that the sum above is an upper Riemann sum of $g$. In that way, we get

$$\mathcal{E}_0(\psi, A; \Omega_{\ell, e}) \geq \kappa^2 \int_{\Omega_{\ell, e}} g \left( \frac{H}{\kappa} |B_0(x)| \right) \, dx - C_1 \kappa^{\frac{15}{8}}.$$

Notice that $\Omega_{\ell, e} \subset \Omega$ and that $g \leq 0$, we deduce that

$$\mathcal{E}_0(\psi, A; \Omega_{\ell, e}) \geq \kappa^2 \int_{\Omega} g \left( \frac{H}{\kappa} |B_0(x)| \right) \, dx - C_1 \kappa^{\frac{15}{8}}.$$

(5.20)

Finally, putting (5.19) and (5.20) into (5.17), we obtain

$$E_g(\kappa, H) \geq \kappa^2 \int_{\Omega} g \left( \frac{H}{\kappa} |B_0(x)| \right) \, dx - CK^{\frac{15}{8}}.$$

(5.21)

**Remark 5.3.** When $B_0$ does not vanish, the local energy in $Q_{\ell}(x_0)$ in Proposition 4.2 becomes

$$\frac{1}{|Q_{\ell}(x_0)|} \mathcal{E}_0(\psi, A; Q_{\ell}(x_0)) \geq g \left( \frac{H}{\kappa} |B_{Q_{\ell}(x_0)}| \right) \kappa^2 - C (\ell^3 \kappa^2 + \ell^{2\alpha - 1} + (\ell \kappa)^{-1}) \kappa^2.$$

Similarly, we choose $\alpha = \frac{2}{3}$ and $\ell = \kappa^{-\frac{1}{4}}$, we get

$$\mathcal{E}_0(\psi, A; \Omega \setminus \Omega_{\ell, e}) \geq -C_2 \kappa^2,$$

(5.22)
and
\[ E_0(\psi, A; \Omega) \geq \kappa^2 \int_{\Omega} g(\frac{H}{\kappa} |B_0(x)|) \, dx - C_1 \kappa^2. \]  
(5.23)

As a consequence of (5.22) and (5.23), (5.21) becomes
\[ E_g(\kappa, H) \geq \kappa^2 \int_{\Omega} g(\frac{H}{\kappa} |B_0(x)|) \, dx - C\kappa^2. \]  
(5.24)

### 5.3. Proof of Corollary 1.2

If \((\psi, A)\) is a minimizer of (1.1), we have
\[ E(\psi, A; \Omega) = E_0(\psi, A; \Omega) + (\kappa H)^2 \int_{\Omega} |\text{curl}(A - F)|^2 \, dx. \]  
(5.25)

Theorem 1.1 tells us that
\[ E(\psi, A; \Omega) \leq \kappa^2 \int_{\Omega} g(\frac{H}{\kappa} |B_0(x)|) \, dx + C\kappa^2. \]  
(5.26)

Using (5.19), (5.20), (5.22) and (5.23), we get
\[ E_0(\psi, A; \Omega) \geq \kappa^2 \int_{\Omega} g(\frac{H}{\kappa} |B_0(x)|) \, dx - C_2 \kappa^2. \]  
(5.27)

Putting (5.27) into (5.26), we get
\[ -C_2 \kappa^2 + \kappa^2 \int_{\Omega} g(\frac{H}{\kappa} |B_0(x)|) \, dx + (\kappa H)^2 \int_{\Omega} |\text{curl}(A - F)|^2 \, dx \leq \kappa^2 \int_{\Omega} g(\frac{H}{\kappa} |B_0(x)|) \, dx + C_1 \kappa^2. \]  
(5.28)

By simplification, we obtain
\[ (\kappa H)^2 \int_{\Omega} |\text{curl}(A - F)|^2 \, dx \leq C' \kappa^2. \]  
(5.29)

### 6. Local energy estimates

The object of this section is to give an estimate to the Ginzburg–Landau energy (4.2) in the open set \(D \subset \Omega\).

#### 6.1. Main statements

**Theorem 6.1.** There exist positive constants \(\kappa_0\) such that if (3.10) is true and \(D \subset \Omega\) is an open set, then the local energy of the minimizer satisfies
\[ \left| E(\psi, A; D) - \kappa^2 \int_{D} g(\frac{H}{\kappa} |B_0(x)|) \, dx \right| = o(\kappa^2). \]  
(6.1)
For all $(\ell, x_0)$ such that $Q_\ell(x_0) \subset \Omega \cap \{ |B_0| > \epsilon \}$, we define
\[
B_{Q_\ell(x_0)} = \inf_{x \in Q_\ell(x_0)} |B_0(x)|,
\] (6.2)
where $B_0$ is introduced in (1.2).

**Proposition 6.2.** For all $\alpha \in (0, 1)$, there exist positive constants $C, \epsilon_0$ and $\kappa_0$ such that if (3.10) is true, $\kappa \geq \kappa_0$, $\ell \in (0, \frac{1}{2})$, $\epsilon \in (0, \epsilon_0)$, $\ell^2 \kappa^2 \epsilon > 1$, $(\psi, A) \in H^1(\Omega; \mathbb{C}) \times H^1_{\text{div}}(\Omega)$ is a minimizer of (1.1), and $Q_\ell(x_0) \subset \Omega \cap \{ |B_0| > \epsilon \}$, then
\[
\frac{1}{|Q_\ell(x_0)|} \mathcal{E}_0(\psi, A; Q_\ell(x_0)) \leq g\left(\frac{H}{\kappa} B_{Q_\ell(x_0)}\right)\kappa^2 + C(\ell^3 \kappa^2 + \ell^{2\alpha - 1} + (\ell \kappa \sqrt{\epsilon})^{-1})\kappa^2.
\]
Here $g(\cdot)$ is the function introduced in (2.5) and $\mathcal{E}_0$ is the functional in (4.1).

**Proof.** As explained earlier in the proof of Lemma 4.1 in (4.5), we may suppose after performing a gauge transformation that the magnetic potential $A$ satisfies
\[
|A(x) - F(x)| \leq C\frac{\ell^\alpha}{H}, \quad \forall x \in Q_\ell(x_0).
\] (6.3)
Let
\[
b = \frac{H}{\kappa} B_{Q_\ell(x_0)} , \quad R = \ell \sqrt{\kappa HB_{Q_\ell(x_0)}},
\] (6.4)
and $u_R \in H^1_0(Q_R)$ be the minimizer of the functional $G_{b,R}^{1+\alpha}$ introduced in (2.1). Let $\chi_R \in C^\infty_c(\mathbb{R}^2)$ be a cut-off function such that
\[
0 \leq \chi_R \leq 1 \quad \text{in } \mathbb{R}^2, \quad \text{supp } \chi_R \subset Q_{R+1}, \quad \chi_R = 1 \quad \text{in } Q_R,
\]
and $|\nabla \chi_R| \leq C$ for some universal constant $C$.
Let $\eta_R(x) = 1 - \chi_R\left(\frac{R}{\ell}(x - x_0)\right)$ for all $x \in \mathbb{R}^2$ and $\tilde{\ell} = \ell(1 + \frac{1}{R})$.
This implies that
\[
\eta_R(x) = 0 \quad \text{in } Q_\ell(x_0),
\] (6.5)
\[
0 \leq \eta_R(x) \leq 1 \quad \text{in } Q_{R}(x_0) \setminus Q_\ell(x_0),
\] (6.6)
\[
\eta_R(x) = 1 \quad \text{in } \Omega \setminus Q_{R}(x_0).
\] (6.7)
Consider the function $w(x)$ defined as follows,
\[
w(x) = \eta_R(x) \psi(x) \quad \text{in } \Omega \setminus Q_\ell(x_0),
\]
and, if $x \in Q_\ell(x_0)$,
\[
w(x) = \begin{cases} 
eq e^{iK\psi}u_R\left(\frac{R}{\ell}(x - x_0)\right) & \text{if } Q_\ell(x_0) \subset \{ B_0 > \epsilon \} \cap \Omega, \\
eq e^{iK\psi}u_R\left(\frac{R}{\ell}(x - x_0)\right) & \text{if } Q_\ell(x_0) \subset \{ B_0 < -\epsilon \} \cap \Omega.
\end{cases}
\]
Notice that by construction, $w = \psi$ in $\Omega \setminus Q_\ell(x_0)$. We will prove that, for any $\delta \in (0, 1)$ and $\alpha \in (0, 1)$,
\[
\mathcal{E}(w, A; \Omega) \leq \mathcal{E}(\psi, A; \Omega \setminus Q_\ell(x_0)) + (1 + \delta)\frac{\ell}{bR}m_0(b, R) + r_0(\kappa)\ell^2,
\] (6.8)
and for some constant $C$, $r_0(\kappa)$ is given as follows,
\[
r_0(\kappa) = C\left(\delta + \delta^{-1} \ell^4 \kappa^2 + \delta^{-1} \ell^{2\alpha} + \frac{1}{\ell \kappa \sqrt{\epsilon}}\right)\kappa^2.
\] (6.9)
**Proof of (6.8):** With $\mathcal{E}_0$ defined in (4.1), we write
\[
\mathcal{E}_0(w, A; \Omega) = \mathcal{E}_1 + \mathcal{E}_2,
\] (6.10)
where
\[ E_1 = E_0(w, A; \Omega \setminus Q_\ell(x_0)), \quad E_2 = E_0(w, A; Q_\ell(x_0)). \] (6.11)

We estimate \( E_1 \) and \( E_2 \) from above. Starting with \( E_1 \) and using (6.7), we get
\[
E_1 = \int_{\Omega \setminus Q_\ell(x_0)} \left| (\nabla - i\kappa H A)\eta_R \psi \right|^2 - \kappa^2 |\eta_R \psi|^2 + \frac{\kappa^2}{2} |\eta_R \psi|^4 \, dx
\]
\[
= \int_{\Omega \setminus Q_\ell(x_0)} \eta_R^2 |(\nabla - i\kappa H A)\psi|^2 + |\nabla \eta_R \psi|^2 + 2R|\eta_R(\nabla - i\kappa H A)\psi, \nabla \eta_R \psi| - \kappa^2 \eta_R^2 |\psi|^2 + \frac{\kappa^2}{2} \eta_R^4 |\psi|^4 \, dx
\]
\[
= E_0(\psi, A; \Omega \setminus Q_\ell(x_0)) + R(\psi, A), \tag{6.12}
\]

where
\[
R(\psi, A) = \int_{Q_\ell(x_0) \setminus Q_\ell(x_0)} \left( (\eta_R^2 - 1) \left( |(\nabla - i\kappa H A)\psi|^2 - \kappa^2 |\psi|^2 \right) + |\psi \nabla \eta_R|^2 + \frac{\kappa^2}{2} (\eta_R^4 - 1) |\psi|^4 \right)
\]
\[
+ 2R|\eta_R(\nabla - i\kappa H A)\psi, \psi \nabla \eta_R| \, dx.
\]

Noticing that \( |Q_\ell(x_0) \setminus Q_\ell(x_0)| \leq \frac{\ell}{\sqrt{\kappa H \rho_{Q_\ell(x_0)}}} \) and using (6.6) together with the estimates in (3.1), (3.10), (3.11) and \( |\nabla \eta_R| \leq C\frac{\ell}{\sqrt{\epsilon}} \), we get
\[
|R(\psi, A)| \leq C \frac{\ell}{\sqrt{\epsilon}}. \tag{6.13}
\]

Inserting (6.13) in (6.12), we get the following estimate
\[
E_1 \leq E_0(\psi, A; \Omega \setminus Q_\ell(x_0)) + C \frac{\ell}{\sqrt{\epsilon}}. \tag{6.14}
\]

We estimate the term \( E_2 \) in (6.11). We will need the function \( \varphi_0 \) introduced in Lemma A.3 and satisfying \( |F(x) - \sigma_\ell B_{Q_\ell(x_0)} A_0(x - x_0) - \nabla \varphi_0(x)| \leq C\ell^2 \) in \( Q_\ell(x_0) \), where \( \sigma_\ell \) denotes the sign of \( B_0 \). We start with the kinetic energy term and write for any \( \delta \in (0, 1) \):
\[
E_2 = \int_{Q_\ell(x_0)} \left| (\nabla - i\kappa H (\sigma_\ell B_{Q_\ell(x_0)} A_0(x - x_0) + \nabla \varphi(x))) w \right|^2
\]
\[
- i\kappa H (A - (\sigma_\ell B_{Q_\ell(x_0)} A_0(x - x_0) + \nabla \varphi(x))) \right|^2 + \left( -\kappa^2 |w|^2 + \frac{\kappa^2}{2} |w|^4 \right) \, dx
\]
\[
\leq \int_{Q_\ell(x_0)} \left( 1 + \delta \right) \left| (\nabla - i\kappa H (\sigma_\ell B_{Q_\ell(x_0)} A_0(x - x_0) + \nabla \varphi(x))) w \right|^2 - \kappa^2 |w|^2 + \frac{\kappa^2}{2} |w|^4 \, dx
\]
\[
+ \left( 1 + \delta^{-1} \right) (\kappa H)^2 \int_{Q_\ell(x_0)} \left| (A - \nabla \varphi_0 - F) w + (F - \sigma_\ell B_{Q_\ell(x_0)} A_0(x - x_0) - \nabla \varphi_0(x)) w \right|^2 \, dx.
\]
(6.15)

Using the estimate in (6.3) together with (3.10) and (3.1), we deduce the upper bound
\[
E_2 \leq (1 + \delta) E_0(e^{-i\kappa H \varphi_0} w, \sigma_\ell B_{Q_\ell(x_0)} A_0(x - x_0); Q_\ell(x_0)) + C(\delta^{-1} \ell^{2\alpha} + \delta^{-1} \ell^4 \kappa^2 + \delta) \ell^2, \tag{6.16}
\]
where \( \alpha \in (0, 1) \).
There are two cases:

Case 1: If $B_0 > \epsilon$ in $Q_\ell(x_0)$, then $\sigma_\ell = +1$ and

\[ u(x) = \begin{cases} 
  e^{i \kappa H\psi} u_R\left( \frac{R}{\ell} (x - x_0) \right) & \text{in } Q_\ell(x_0), \\
  \eta_R(x) \psi(x) & \text{in } \Omega \setminus Q_\ell(x_0).
\end{cases} \]

The change of variable $y = \frac{R}{\ell} (x - x_0)$ and (4.12) gives us

\[
\mathcal{E}_0(e^{-i \kappa H\psi} w, \sigma_\ell B_{Q_\ell(x_0)} \mathbf{A}_0(x - x_0); Q_\ell(x_0)) = \int_{Q_\ell(x_0)} \left( \left| \nabla_y - i \frac{R}{\ell} \mathbf{A}_0(y) \right| u_R \right)^2 - \frac{\kappa^2}{2} |u_R|^4 \right) \frac{\ell}{R} dy
\]

\[
= \int_{Q_\ell(x_0)} \left( |\nabla_y - i \mathbf{A}_0(y)| u_R \right)^2 - \frac{\kappa^2}{2} |u_R|^4 \right) \frac{\ell}{R} dy
\]

\[
= \frac{\kappa}{H B_{Q_\ell(x_0)}} \int_{Q_\ell(x_0)} b \left( |\nabla_y - i \mathbf{A}_0(y)| u_R \right)^2 - |u_R|^4 \right) \frac{\ell}{R} dy
\]

\[
= \frac{1}{b} G_{b, QR}^{-1}(u_R).
\] (6.17)

where $G_{b, QR}^{-1}$ is the functional from (2.1).

Case 2: If $B_0 < -\epsilon$ in $Q_\ell(x_0)$, then $\sigma_\ell = -1$ and

\[ u(x) = \begin{cases} 
  e^{i \kappa H\psi} \bar{u}_R\left( \frac{R}{\ell} (x - x_0) \right) & \text{in } Q_\ell(x_0), \\
  \eta_R(x) \psi(x) & \text{in } \Omega \setminus Q_\ell(x_0).
\end{cases} \]

Similarly, like in Case 1, we have

\[
\mathcal{E}_0(e^{-i \kappa H\psi} w, \sigma_\ell B_{Q_\ell(x_0)} \mathbf{A}_0(x - x_0); Q_\ell(x_0)) = \frac{1}{b} G_{b, QR}^{-1}(\bar{u}_R) = \frac{1}{b} G_{b, QR}(u_R).
\] (6.18)

In both cases we see that

\[
\mathcal{E}_0(e^{-i \kappa H\psi} w, \sigma_\ell B_{Q_\ell(x_0)} \mathbf{A}_0(x - x_0); Q_\ell(x_0)) = \frac{1}{b} G_{b, QR}^{-1}(u_R) = \frac{m_0(b, R)}{b}.
\] (6.19)

Inserting (6.18) into (6.16), we get

\[
\mathcal{E}_2 \leq (1 + \delta) \frac{1}{b} m_0(b, R) + C(\delta + \delta^{-1} \ell^4 \kappa^2 + \delta^{-1} \ell^2 a) \kappa^2 \ell^2.
\] (6.20)

Inserting (6.14) and (6.19) into (6.10), we deduce that

\[
\mathcal{E}_0(\psi, A) \leq \mathcal{E}_0(\psi, A; \Omega \setminus Q_\ell(x_0)) + (1 + \delta) \frac{1}{b} m_0(b, R) + C(\delta + \delta^{-1} \ell^4 \kappa^2 + \delta^{-1} \ell^2 a) \kappa^2 \ell^2.
\] (6.21)

This proves (6.8). Now, we show how (6.8) proves Proposition 6.2. By definition of the minimizer $(\psi, A)$, we have

\[
\mathcal{E}(\psi, A) \leq \mathcal{E}(\psi, A; \Omega).
\]

Since $\mathcal{E}(\psi, A; \Omega) = \mathcal{E}(\psi, A; \Omega \setminus Q_\ell(x_0)) + \mathcal{E}_0(\psi, A; Q_\ell(x_0))$, the estimate (6.8) gives us

\[
\mathcal{E}_0(\psi, A; Q_\ell(x_0)) \leq \frac{1}{b} m_0(b, R) + r_0(\kappa),
\]

where $r_0(\kappa)$ is defined in (6.9).

Dividing both sides by $|Q_\ell(x_0)| = \ell^2$, we get

\[
\frac{1}{|Q_\ell(x_0)|} \mathcal{E}_0(\psi, A, Q_\ell(x_0)) \leq \frac{1 + \delta}{b} m_0(b, R) + C(\delta + \delta^{-1} \ell^4 \kappa^2 + \frac{1}{\ell \kappa \sqrt{a}} + \delta^{-1} \ell^2 a) \kappa^2.
\] (6.22)
The inequality in (2.7) tells us that \( m_0(b, R) \leq R^2 g(b) + CR \) for all \( b \in [0, 1] \) and \( R \) sufficiently large. We substitute this into (6.21) and we select \( \delta = \ell \), so that
\[
r_0(\kappa) = \kappa^2 O\left((\ell \sqrt{\epsilon})^{-1} + \ell^3 \kappa^2 + \ell^{2\alpha - 1}\right).
\]

Using (4.12) we get
\[
\frac{1}{|Q_{\ell}(x_0)|} \mathcal{E}(\psi, A, Q_{\ell}(x_0)) \leq \frac{(1 + \delta)R^2}{b\ell^2} g(b) + \frac{CR}{b\ell^2} + \kappa^2 O\left((\ell \sqrt{\epsilon})^{-1} + \ell^3 \kappa^2 + \ell^{2\alpha - 1}\right)
\]
\[
\leq g\left(\frac{H}{\kappa} B_{Q_{\ell}(x_0)}\right)^2 \kappa^2 + C\left((\ell \sqrt{\epsilon})^{-1} + \ell^3 \kappa^2 + \ell^{2\alpha - 1}\right)^2 \kappa^2.
\]

This establishes the result of Proposition 6.2. \( \square \)

6.2. Proof of Theorem 6.1, upper bound

The parameters \( \ell \) and \( \epsilon \) have the same form as in (5.1) and we take the same choice of \( \beta \) and \( \mu \) as in (5.11). Consider the lattice \( \Gamma_{\ell} := \ell \mathbb{Z} \times \ell \mathbb{Z} \) and write, for \( \gamma \in \Gamma_{\ell} \), \( Q_{\ell, \gamma} = Q_{\ell}(\gamma) \). For any \( \gamma \in \Gamma_{\ell} \) such that \( Q_{\ell}(\gamma) \subset \Omega \cap \{|B_0| > \epsilon\} \), let
\[
\mathcal{I}_{\ell, \epsilon}(D) = \{ \gamma : \overline{Q_{\ell, \gamma}} \subset D \cap \{|B_0| > \epsilon\} \}, \quad N = \text{Card} \mathcal{I}_{\ell, \epsilon}(D),
\]
and
\[
D_{\ell, \epsilon} = \text{int} \left( \bigcup_{\gamma \in \mathcal{I}_{\ell, \epsilon}(D)} \overline{Q_{\ell, \gamma}} \right).
\]

Notice that, by (1.2),
\[
N = |D| \ell^{-2} + O(\epsilon \ell^{-2}) + O(\ell^{-1}) \quad \text{as } \ell \to 0 \text{ and } \epsilon \to 0.
\]

If \((\psi, A)\) is a minimizer of (1.1), we have
\[
\mathcal{E}(\psi, A; D) = \mathcal{E}_0(\psi, A; D_{\ell, \epsilon}) + \mathcal{E}_0(\psi, A; D \setminus D_{\ell, \epsilon}) + (\kappa H)^2 \int_{\Omega} |\text{curl}(A - F)|^2 \, dx.
\] (6.22)

Using Corollary 1.2, we may write
\[
\mathcal{E}(\psi, A; D) \leq \mathcal{E}_0(\psi, A; D_{\ell, \epsilon}) + \mathcal{E}_0(\psi, A; D \setminus D_{\ell, \epsilon}) + C \kappa^{\tau_0}.
\] (6.23)

Here \( \tau_0 \in (1, 2) \). Notice that
\[
|D \setminus D_{\ell, \epsilon}| = O(\ell|D_{\ell, \epsilon}| + \epsilon).
\] (6.24)

We get by using (3.1) and (3.11):
\[
|\mathcal{E}_0(\psi, A; D \setminus D_{\ell, \epsilon})| \leq |D \setminus D_{\ell, \epsilon}| \left( C_1 \kappa^2 \|\psi\|^2_{L^\infty(D)} + \kappa^2 \|\psi\|^2_{L^\infty(D)} + \frac{\kappa^2}{2} \|\psi\|^4_{L^\infty(D)} \right)
\]
\[
\leq C_2 (\ell + \epsilon) \kappa^2.
\] (6.25)

To estimate \( \mathcal{E}_0(\psi, A; D_{\ell, \epsilon}) \), we notice that
\[
\mathcal{E}_0(\psi, A; D_{\ell, \epsilon}) = \sum_{\gamma \in \mathcal{I}_{\ell, \epsilon}(D)} \mathcal{E}_0(\psi, A; Q_{\ell, \gamma}).
\]

Using Proposition 6.2 and the estimates in (6.25) with \( \beta = \frac{3}{4}, \alpha = \frac{2}{3} \) and \( \mu = \frac{1}{8} \), we get
\[
\mathcal{E}_0(\psi, A; D) \leq \sum_{\gamma \in \mathcal{I}_{\ell, \epsilon}(D)} g\left(\frac{H}{\kappa} B_{Q_{\ell}(x_0)}\right) \kappa^2 \ell^2 + C\left(\ell^3 \kappa^2 + \ell^{2\alpha - 1} + (\ell \sqrt{\epsilon})^{-1} + \epsilon\right)^2 \kappa^2 + C_1 \kappa^{\tau_0}
\]
\[
\leq \kappa^2 \left( \sum_{\gamma \in \mathcal{I}_{\ell, \epsilon}(D)} g\left(\frac{H}{\kappa} B_{Q_{\ell}(x_0)}\right) \ell^2 + C_2 \kappa^{\tau_0} \right).
\]
where
\[
P_{Q_{\ell}(x_0)} = \sup_{x \in Q_{\ell}(x_0)} B_0(x).
\]

Recognizing the lower Riemann sum of \(x \mapsto g(H_\kappa B_0(x))\), and using the monotonicity of \(g\) we get
\[
E_0(\psi, A; D) \leq \kappa^2 \int_{D_\ell,\epsilon} g \left( \frac{H}{\kappa} B_0(x) \right) dx + C_2 \kappa \tau_0. \tag{6.26}
\]

Thus, we get by using (6.24) and the property of \(g\) in Theorem 2.2,
\[
\kappa^2 \int_{D_\ell,\epsilon} g \left( \frac{H}{\kappa} B_0(x) \right) dx \leq \kappa^2 \int_{D} g \left( \frac{H}{\kappa} B_0(x) \right) dx + C_3 \kappa \tau_0.
\]

This finishes the proof of the upper bound.

6.3. Lower bound

We keep the same notation as in the derivation of the upper bound. We start with (6.22) and write
\[
E(\psi, A; D) \geq E_0(\psi, A; D_\ell,\epsilon) + E_0(\psi, A; D \setminus D_\ell,\epsilon). \tag{6.27}
\]

Similarly, as we did for the Lower bound Section 5.2, we get
\[
E(\psi, A; D) \geq \kappa^2 \int_{D} g \left( \frac{H}{\kappa} B_0(x) \right) dx - C \kappa \tau_0. \tag{6.28}
\]

This finishes the proof of Theorem 6.1.

7. Proof of Theorem 1.4

7.1. Proof of (1.9)

Let \((\psi, A)\) be a solution of (1.5) and \(\tau_1 = \tau_0 - 2\). Then \(\psi\) satisfies
\[
-(\nabla - i\kappa H A)^2 \psi = \kappa^2 (1 - |\psi|^2) \psi \quad \text{in } \Omega. \tag{7.1}
\]

We multiply both sides of the equation in (7.1) by \(\overline{\psi}\) then we integrate over \(D\). An integration by parts gives us
\[
\int_{D} \left( (\nabla - i\kappa H A) \psi \right)^2 - \kappa^2 |\psi|^2 + \kappa^2 |\psi|^4 \right) dx - \int_{\partial D} v \cdot (\nabla - i\kappa H A) \psi \overline{\psi} d\sigma(x) = 0. \tag{7.2}
\]

Using the estimates (3.1), (3.10) and (3.11), we get that the boundary term which is not necessary 0 if \(D \neq \Omega\) above is \(O(\kappa)\). So, we rewrite (7.2) as follows,
\[
-\frac{1}{2} \kappa^2 \int_{D} |\psi|^4 \, dx = E_0(\psi, A; D) + O(\kappa). \tag{7.3}
\]

Using (6.28), we conclude that
\[
\frac{1}{2} \int_{D} |\psi|^4 \, dx \leq - \int_{D} g \left( \frac{H}{\kappa} B_0(x) \right) dx + C \kappa^{\tau_1}. \tag{7.4}
\]
7.2. Proof of (1.10)

If \((\psi, A)\) is a minimizer of (1.1), then (7.3) is still true. We apply in this case Theorem 6.1 to write an upper bound of \(E_0(\psi, A; D)\). Consequently, we deduce that

\[
\frac{1}{2} \int_D |\psi|^4 \, dx \geq - \int_D g \left( \frac{H}{\kappa} B_0(x) \right) \, dx - C \kappa \tau.
\]  

(7.5)

Combining the upper bound in (7.5) with the lower bound in (7.4) finishes the proof of Theorem 1.4.

Acknowledgements

This work is partially supported by a grant from Lebanese University. I would like to thank my supervisors B. Helffer and A. Kachmar for their support.

Appendix A

A.1. \(L^p\)-regularity for the curl–div system

We consider the two dimensional case. We denote, for \(k \in \mathbb{N}\), by \(W^{k,p}_{\text{div}}(\Omega)\) the space

\[
W^{k,p}_{\text{div}}(\Omega) = \{ A \in W^{k,p}(\Omega), \text{ div } A = 0 \text{ and } A \cdot \nu = 0 \text{ on } \partial \Omega \}.
\]

Then we have the following \(L^p\) regularity for the curl–div system.

Proposition A.1. Let \(1 \leq p < \infty\). If \(A \in W^{1,p}_{\text{div}}(\Omega)\) satisfies \(\text{curl } A \in W^{k,p}(\Omega)\), for some \(k \geq 0\), then \(A \in W^{k+1,p}_{\text{div}}(\Omega)\).

Proof. If \(A\) belongs to \(W^{1,p}_{\text{div}}(\Omega)\) and \(\text{curl } A \in L^p(\Omega)\), then there exists \(\psi \in W^{2,p}(\Omega)\) such that \(A = (-\partial_x \psi, \partial_y \psi), -\Delta \psi = \text{curl } A\), with \(\psi = 0\) on \(\partial \Omega\). This is simply the Dirichlet \(L^p\) problem for the Laplacian (see [2, Section A.1]). The result we need for proving the proposition is then that if \(-\Delta \psi\) is in addition in \(W^{k,p}(\Omega)\) then \(\psi \in W^{k+2,p}(\Omega)\). This is simply an \(L^p\) regularity result for the Dirichlet problem for the Laplacian which is described in [2, Section F.4].

A.2. Construction of \(\varphi_{x_0}\)

Lemma A.2. If \(B_0 \in L^2(\Omega)\), then there exists a unique \(F \in H^1_{\text{div}}(\Omega)\) such that

\[
\text{curl } F = B_0.
\]  

(A.1)

Proof. The proof is standard, see [5]. Let \(F = \begin{bmatrix} \partial_x f \\ -\partial_y f \end{bmatrix}\), where \(f \in H^2(\Omega) \cap H^1_0(\Omega)\) is the unique solution of

\[
-\Delta f = B_0 \quad \text{in } \Omega.
\]  

(A.2)

Then we deduce from the Dirichlet condition satisfied by \(f\) that \(\tau \cdot \nabla f = 0 \text{ on } \partial \Omega\) which is equivalent to \(\nu \cdot F = 0 \text{ on } \partial \Omega\). This finishes the proof of Lemma A.2.

We continue with a lemma that will be useful in estimating the Ginzburg–Landau functional.

Lemma A.3. There exists a positive constant \(C\) such that, if \(\ell \in (0, 1)\) and \(x_0 \in \Omega\) are such that \(Q_{\ell}(x_0) \subset \Omega\), then for any \(\tilde{x}_0 \in \overline{Q}_{\ell}(x_0)\), there exists a function \(\varphi_0 \in H^1(\Omega)\) such that the magnetic potential \(F\) satisfies

\[
|F(x) - \nabla \varphi_0(x) - B_0(\tilde{x}_0)A_0(x - x_0)| \leq C \ell^2 \quad (x \in Q_{\ell}(x_0)),
\]  

(A.3)

where \(B_0\) is the function introduced in (1.2) and \(A_0\) is the magnetic potential introduced in (2.2).
Proof. We use Taylor formula near $\tilde{x}_0$ to order 2 and get
\[ F(x) = F(\tilde{x}_0) + M(x - \tilde{x}_0) + O(|x - \tilde{x}_0|^2), \quad \forall x \in Q_\ell(x_0), \] (A.4)
where
\[ M = DF(\tilde{x}_0) = \begin{bmatrix} \frac{\partial F^1}{\partial x_1} |_{\tilde{x}_0} & \frac{\partial F^1}{\partial x_2} |_{\tilde{x}_0} \\ \frac{\partial F^2}{\partial x_1} |_{\tilde{x}_0} & \frac{\partial F^2}{\partial x_2} |_{\tilde{x}_0} \end{bmatrix}. \]
We can write $M$ as the sum of two matrices,
\[ M = M^s + M^{as}, \]
where $M^s = M + M^t$ is symmetric and $M^{as} = M - M^t$ is antisymmetric.
Notice that $\text{curl} F(\tilde{x}_0) = \frac{\partial F^2}{\partial x_1} |_{\tilde{x}_0} - \frac{\partial F^1}{\partial x_2} |_{\tilde{x}_0} = B_0(\tilde{x}_0).$ Consequently,
\[ M^{as} = \begin{bmatrix} 0 & -B_0/2 \\ B_0/2 & 0 \end{bmatrix}. \]
Substitution into $M$ gives as that
\[ M(x - x_0) = \nabla \phi_0(x) + B_0(\tilde{x}_0)A_0(x - x_0), \]
where $A_0(x) = \frac{1}{2}(-x_2, x_1)$ and the function $\phi_0$ is defined by
\[ \phi_0(x) = \frac{1}{2} \left( \frac{M + M^t}{2} \right)(x - x_0), (x - x_0) \].
Let $\varphi_0(x) = \phi_0(x) + (F(\tilde{x}_0) + M(x_0 - \tilde{x}_0)) \cdot x$. Substitution into (A.4) gives as that
\[ F = B_0(\tilde{x}_0)A_0(x - x_0) + \nabla \varphi_0(x) + O(|x - \tilde{x}_0|^2). \]
Notice that, if $x \in Q_\ell(x_0)$, then $|x - \tilde{x}_0| \leq \ell \sqrt{2}$. This finishes the proof of Lemma A.3. ⊓⊔

Remark A.4. We will apply this lemma by considering $\tilde{x}_0$ such that $B_0(\tilde{x}_0) = \sup_{Q_\ell(x_0)} B_0(x)$ or $B_0(\tilde{x}_0) = \inf_{Q_\ell(x_0)} B_0(x)$.

References

[1] A. Aftalion, S. Serfaty, Lowest Landau level approach in superconductivity for the Abrikosov lattice close to $H_c^2$, Sel. Math. New Ser. 13 (2) (2007) 183–202.
[2] S. Fournais, B. Helffer, Optimal uniform elliptic estimates for the Ginzburg–Landau system, in: Adventures in Mathematical Physics, in: Contemp. Math., vol. 447, Amer. Math. Soc., 2007, pp. 83–102.
[3] S. Fournais, B. Helffer, Spectral Methods in Surface Superconductivity, Prog. Nonlinear Differ. Equ. Appl., vol. 77, Birkhäuser, Boston, 2010.
[4] S. Fournais, A. Kachmar, The ground state energy of the three dimensional Ginzburg–Landau functional part I: Bulk regime, Commun. Partial Differ. Equ. 38 (2) (2013) 339–383.
[5] V. Girault, P.-A. Raviart, Finite Elements Methods for Navier–Stokes Equations, Springer, 1986.
[6] X.B. Pan, Surface superconductivity in applied magnetic fields above $HC_2$, Commun. Math. Phys. 228 (2) (2002) 327–370.
[7] X.B. Pan, Surface superconductivity in 3 dimensions, Trans. Am. Math. Soc. 356 (10) (2004) 3899–3937.
[8] X.B. Pan, K.H. Kwek, Schrödinger operators with non-degenerately vanishing magnetic fields in bounded domains, Trans. Am. Math. Soc. 354 (10) (2002) 4201–4227.
[9] S. Sandier, S. Serfaty, The decrease of bulk-superconductivity close to the second critical field in the Ginzburg–Landau model, SIAM J. Math. Anal. 34 (4) (2003) 939–956.