Computational Complexity of Normalizing Constants for the Product of Determinantal Point Processes

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Abstract

We consider the product of determinantal point processes (DPPs), a point process whose probability mass is proportional to the product of principal minors of multiple matrices, as a natural, promising generalization of DPPs. We study the computational complexity of computing its normalizing constant, which is among the most essential probabilistic inference tasks. Our complexity-theoretic results (almost) rule out the existence of efficient algorithms for this task unless the input matrices are forced to have favorable structures. In particular, we prove the following:

• Computing $\sum_S \det(A_{S,S})^p$ exactly for every (fixed) positive even integer $p$ is UP-hard and Mod_3P-hard, which gives a negative answer to an open question posed by Kulesza and Taskar (2012).

• $\sum_S \det(A_{S,S}) \det(B_{S,S}) \det(C_{S,S})$ is NP-hard to approximate within a factor of $2^{O(|I|^{1-\epsilon})}$ or $2^{O(n^{1/\epsilon})}$ for any $\epsilon > 0$, where $|I|$ is the input size and $n$ is the order of the input matrix. This result is stronger than the #P-hardness for the case of two matrices derived by Gillenwater (2014).

• There exists a $k^{O(k)} n^{O(1)}$-time algorithm for computing $\sum_S \det(A_{S,S}) \det(B_{S,S})$, where $k$ is the maximum rank of $A$ and $B$ or the treewidth of the graph formed by nonzero entries of $A$ and $B$. Such parameterized algorithms are said to be fixed-parameter tractable.

These results can be extended to the fixed-size case. Further, we present two applications of fixed-parameter tractable algorithms given a matrix $A$ of treewidth $w$:

• We can compute a $2^{\sqrt{n}}$-approximation to $\sum_S \det(A_{S,S})^p$ for any fractional number $p > 1$ in $w^{O(wp)} n^{O(1)}$ time.

• We can find a $2^{\sqrt{n}}$-approximation to unconstrained maximum a posteriori inference in $w^{O(w\sqrt{n})} n^{O(1)}$ time.

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1 Introduction

Determinantal point processes (DPPs) offer an appealing probabilistic model to compactly express negative correlation among combinatorial objects (Borodin and Rains, 2005; Macchi, 1975). Given an $n \times n$ matrix $A$, a DPP defines a probability distribution on $2^{[n]}$ such that the probability of drawing a particular subset $S \subseteq [n]$ is proportional to the principal minor $\det(A_{S,S})$. Consider the following subset selection task: given $n$ items (e.g., images Kulesza and Taskar, 2011) associated with quality scores $q_i$ and feature vectors $\phi_i$ for each $i \in [n]$, we are asked to choose a small group of high-quality, diverse items. One can construct $A$ as $A_{i,j} = q_i q_j \phi_i^\top \phi_j$, resulting in that $\det(A_{S,S})$ is the squared volume of the parallelepiped spanned by $\{q_i \phi_i\}_{i \in S}$, which balances item quality and set diversity. With the development of efficient algorithms for many inference tasks, such as normalization, sampling, and marginalization, DPPs have come to be applied to numerous machine learning tasks, e.g., image search (Kulesza and Taskar, 2011), video summarization (Gong et al., 2014), object retrieval (Affandi et al., 2014), sensor placement (Krause et al., 2008), and the Nyström method (Li et al., 2016).

One of the recent research trends is to extend or generalize DPPs to express more complicated distributions. Computing the normalizing constant (a.k.a. the partition function) for such new models is at the heart of efficient probabilistic inference. For example, we can efficiently sample a subset from partition DPPs (Celis et al., 2017), which are restricted to including a fixed number of elements from each prespecified group, by quickly evaluating their normalizing constant. Such tractability is, of course, not necessarily the case for every generalization.

In this paper, we consider a natural, (seemingly) promising generalization of DPPs involving multiple matrices. The product DPP (Π-DPP) of $m$ matrices $A_1^1, \ldots, A_m^m$ of size $n \times n$ defines the probability mass for each subset $S$ as proportional to $\det(A_{S,S}^1) \cdots \det(A_{S,S}^m)$, which can be significantly expressive: it enables us to embed some constraints in DPPs, e.g., those that are defined by partitions (Celis et al., 2017) and bipartite matching, and it contains exponentiated DPPs (Mariet et al., 2018) of an integer exponent as a special case. The computational complexity of its normalizing constant, i.e.,

$$Z_m(A^1, \ldots, A^m) \triangleq \sum_{S \subseteq [n]} \det(A_{S,S}^1) \cdots \det(A_{S,S}^m)$$

is almost nebulous, except for $m \leq 2$ (Anari and Gharan, 2017; Gillenwater, 2014; Gurvits, 2005, 2009, see Section 1.2). Our research question is thus the following:

How hard (or easy) is it to compute normalizing constants for Π-DPPs?

1.1 Our Contributions

We present an intensive study on the computational complexity of computing the normalizing constant for Π-DPPs. Our quest can be partitioned into five investigations: intractability,
inapproximability, fixed-parameter tractability, extensions to the fixed-size version, and applications. Our complexity-theoretic results in Sections 4 to 6 summarize in Table 1 (almost) rule out the existence of efficient algorithms for this computation problem unless the input matrices are forced to have favorable structures. We also demonstrate two fundamental properties of Π-DPPs (Section 3). We refer the reader to Section 2.5 for a brief introduction to complexity classes. The paragraph headings begin with a ✗ mark for negative (i.e., hardness) results and with a ✓ mark for positive (i.e., algorithmic) results.

**Contribution 1: Intractability (Section 4)**

We analyze the hardness of computing normalizing constants exactly. Computing $Z_2(\mathbf{A}, \mathbf{B}) = \sum_S \det(\mathbf{A}_{S,S}) \det(\mathbf{B}_{S,S})$ for two positive semi-definite matrices $\mathbf{A}$ and $\mathbf{B}$ is known to be #P-hard (Gillenwater, 2014).²

**Exponentiated DPPs.** Our first target is a special case where $A^i = A$ for all $i$; i.e., the probability mass for subset $S \subseteq [n]$ is proportional to $\det(A_{S,S})^p$ for some integer $p$, which includes exponentiated DPPs (E-DPPs) (Mariet et al., 2018) of an integer exponent. The diversity preference can be controlled via exponent parameter $p$: increasing $p$ prefers more diverse subsets than DPPs, while setting $p = 0$ results in a uniform distribution. The original motivation for computing of the normalizing constant $\sum_S \det(\mathbf{A}_{S,S})^p$ is the Hellinger distance between two DPPs (Kulesza and Taskar, 2012). We prove that for every (fixed) positive even integer $p = 2, 4, 6, \ldots$, it is UP-hard³ and Mod₃P-hard⁴ to compute this normalizing constant, even when $\mathbf{A}$ is a $(-1, 0, 1)$-matrix or a P-matrix (Corollary 4.2). Hence, no polynomial-time algorithm exists for it unless both INTEGER FACTORIZATION $\in$ UP and GRAPH ISOMORPHISM $\in$ Mod₃P are polynomial-time solvable. In particular, UP-hardness excludes the existence of any polynomial-time algorithm unless RP = NP (Valiant and Vazirani, 1986). Our result gives a negative answer to an open question posed by Kulesza and Taskar (2012, Section 7.2). We must emphasize that Gurvits (2005, 2009) already proved the #P-hardness of computing $\sum_S \det(\mathbf{A}_{S,S})^2$ (see Section 1.2).

**Contribution 2: Inapproximability (Section 5)**

After gaining an understanding of the hardness of exact computation, we examine the possibility of approximation. Our hope is to guess an accurate estimate; e.g., an $e^n$-approximation is possible for the case of two matrices (Anari and Gharam, 2017).

**Exponentiated Inapproximability for the Case of Three Matrices.** Unfortunately, our hopes are dashed: we prove that it is NP-hard to approximate the normalizing constant for Π-DPPs of three matrices, i.e., $Z_3(\mathbf{A}, \mathbf{B}, \mathbf{C}) = \sum_S \det(\mathbf{A}_{S,S}) \det(\mathbf{B}_{S,S}) \det(\mathbf{C}_{S,S})$, within a factor of

² #P is the class of function problems of counting the number of accepting paths of a nondeterministic polynomial-time Turing machine (NP machine), and hence it holds that NP $\subseteq$ #P.
³ UP is the class of decision problems solvable by an NP machine with at most one accepting path; it holds that P $\subseteq$ UP $\subseteq$ NP.
⁴ Mod₃P is the class of decision problems solvable by an NP machine, where the number of accepting paths is not divisible by 3.
2^O(|\mathcal{I}|^{1-\epsilon}) or 2^O(n^{1/\epsilon}) for any $\epsilon > 0$ even when $A$, $B$, and $C$ are positive semi-definite, where $|\mathcal{I}|$ is the number of bits required to represent the three matrices (Theorem 5.1). For instance, even a $2^{n^{100}}$-approximation cannot be expected in polynomial time. Moreover, unless RP = NP, approximate sampling is impossible; i.e., we cannot generate a sample (in polynomial time) from a distribution whose total variation distance from the $\Pi$-DPP defined by $A$, $B$, and $C$ is at most $\frac{1}{3}$. The same hardness results hold for the case of four or more matrices (i.e., $m \geq 4$). On the other hand, a simple guess of the number 1 is proven to be a $2^O(|\mathcal{I}|^2)$-approximation (Observation 5.2).

\section*{Approximation-Preserving Reduction from Mixed Discriminant to the Case of Two Matrices.}

We devise an approximation-preserving reduction from the mixed discriminant to the normalizing constant $Z_2(A, B)$ for the $\Pi$-DPP of two matrices (Theorem 5.4). Not only is the mixed discriminant #P-hard to compute, but no fully polynomial-time randomized approximation scheme (FPRAS)\footnote{An FPRAS is a randomized algorithm that outputs an $e^\epsilon$-approximation with probability at least $\frac{3}{4}$ and runs in polynomial time in the input size and $e^{-1}$ (see Definition 2.5).} is also currently known, and its existence is rather doubtful (Gurvits, 2005); hence, the approximation-preserving reduction tells us that $Z_2$ is unlikely to admit an FPRAS.

\section*{Contribution 3: Fixed-Parameter Tractability (Section 6)}

We now resort to parameterization, which has recently succeeded in overcoming the difficulty of machine learning problems (Eiben et al., 2019; Ganian et al., 2018). Parameterized complexity (Downey and Fellows, 2012) is a research field aiming to classify (typically, NP-hard) problems based on their computational complexity with respect to some parameters. Given a parameter $k$ that may be independent of the input size $|\mathcal{I}|$, we say that a problem is fixed-parameter tractable (FPT) if it is solvable in $f(k)|\mathcal{I}|^{O(1)}$ time for some computable function $f$. On the other hand, a problem solvable in $|\mathcal{I}|^{f(k)}$ time is slice-wise polynomial (XP). While both have polynomial runtimes for every fixed $k$, the polynomial part is dramatically different between them ($|\mathcal{I}|^{O(1)}$ versus $|\mathcal{I}|^{f(k)}$). Selecting appropriate parameters is vital to devising fixed-parameter tractability. We introduce three parameters; the first two turn out to be FPT, and the third is unlikely to be even XP.

\begin{itemize}
  \item \textbf{(1) Maximum Rank $\rightarrow$ FPT.} \textit{Rank} is a natural parameter for matrices. We can assume bounded-rank matrices for DPPs if the feature vectors $\phi_i$ are low-dimensional (Celis et al., 2018), or the largest possible subset is far smaller than the ground set size $n$; e.g., Gartrell, Paquet, and Koenigstein (2017) learned a matrix factorization of rank 15 for $n \approx 2,000$ by using real-world data. We prove that there exists an $r^{O(r)}n^{O(1)}$-time FPT algorithm for computing the normalizing constant for two $n \times n$ positive semi-definite matrices $A$ and $B$, where $r$ is the maximum rank of $A$ and $B$ (Theorem 6.1). The central idea is to decompose $A$ and $B$ into $n \times r$ rectangular matrices and then apply the Cauchy–Binet formula. Our FPT algorithm can be generalized to the case of $m$ matrices of rank at most $r$, increasing the runtime to $r^{O(mr)}n^{O(1)}$ (Theorem 6.4).
  \item \textbf{(2) Treewidth of Union $\rightarrow$ FPT.} \textit{Treewidth} (Arnborg and Proskurowski, 1989; Bertelè and Brioschi, 1972; Halin, 1976; Robertson and Seymour, 1986) is one of the most important graph-theoretic pa-
\end{itemize}
rameters; it measures the “tree-likeness” of a graph (see Definition 2.3). Many NP-hard problems on graphs have been shown to be FPT when parameterized by the treewidth (Cygan et al., 2015; Fomin and Kratsch, 2010). Informally, the treewidth of a matrix is that of the graph formed by the nonzero entries in the matrix; e.g., matrices of bandwidth \( b \) have treewidth \( \mathcal{O}(b) \). If feature vectors \( \phi_i \) exhibit clustering properties (van der Maaten and Hinton, 2008), the similarity score \( \phi_i^\top \phi_j \) between items from different clusters would be negligibly small, and such entries can be discarded to obtain a small-bandwidth matrix. In the context of change-point detection applications, Zhang and Ou (2016) observe a small-bandwidth matrix to efficiently solve maximum a posteriori inference on DPPs. We prove that there exists a \( \omega^{O(w)} n^{O(1)} \)-time FPT algorithm for computing the normalizing constant \( Z_2(A, B) \) for two matrices \( A \) and \( B \), where \( w \) is the treewidth of the union of nonzero entries in \( A \) and \( B \) (Theorem 6.5). The proof is based on dynamic programming, which is a typical approach but requires complicated procedures. Our FPT algorithm can be generalized to the case of \( m \) matrices, increasing the runtime to \( \omega^{O(mw)} n^{O(1)} \) (Theorem 6.11).

\[ \mathcal{X} \ (3) \text{ Maximum Treewidth \to Unlikely to be XP} \]. Our FPT algorithm in Theorem 6.5 implicitly benefits from the fact that \( A \) and \( B \) have nonzero entries in similar places. So, what happens if \( A \) and \( B \) are structurally different? Can we still get FPT algorithms when the parameterization is by the maximum treewidth of \( A \) and \( B \)? The answer is no: computing the normalizing constant \( Z_2(A, B) \) is \#P-hard even if both \( A \) and \( B \) have treewidth at most 3 (Theorem 6.33), implying that even XP algorithms do not exist unless \( \text{FP} = \#P \) (which is a requirement that is at least as strong as \( \text{P} = \text{NP} \)).

**Contribution 4: Extensions to Fixed-Size \( \Pi \)-DPPs (Section 7)**

We extend the complexity-theoretic results devised so far to the case of fixed-size \( \Pi \)-DPPs. Given \( m \) matrices \( A^1, \ldots, A^m \) of size \( n \times n \) and a size parameter \( k \), the \( k \)-\( \Pi \)-DPP specifies the probability mass for each subset \( S \subseteq [n] \) to be proportional to \( \det(A_{S,S}) \) only if \( |S| = k \). Thus, the normalizing constant for \( k \)-\( \Pi \)-DPP is given by

\[
\sum_{S \subseteq [n]; |S| = k} \det(A^1_{S,S}) \cdots \det(A^m_{S,S}).
\]

The special case of \( m = 1 \) coincides with \( k \)-DPPs (Kulesza and Taskar, 2011), which are among the most important extensions of DPPs because of their applications to image search (Kulesza and Taskar, 2011) and the Nyström method (Li et al., 2016). We derive complexity-theoretic results for \( k \)-\( \Pi \)-DPPs corresponding to those on \( \Pi \)-DPPs presented in Sections 4 to 6, including intractability of \( E \)-DPPs, indistinguishability of \( Z_3 \), an approximation-preserving reduction from the mixed discriminant to \( Z_2 \) (Theorem 7.1), and FPT algorithms parameterized by maximum rank and treewidth (Corollary 7.3). Further, we examine the fixed-parameter tractability of computing the normalizing constant for \( k \)-\( \Pi \)-DPPs parameterized by \( k \). It is easy to check that a brute-force algorithm runs in \( n^{O(k)} \) time, which is XP. One might further expect there to be an FPT algorithm; however, we show that computing the normalizing constant \( Z_2 \) parameterized by \( k \) is \#W[1]-hard (Theorem 7.4). Since it is a plausible assumption that \( \text{FPT} \neq \#W[1] \) in parameterized complexity (Flum and Grohe, 2004), the problem of interest is unlikely to be FPT.
Table 1: Summary of complexity-theoretic results presented in Sections 4 to 6 and in previous work. Our positive and negative results are marked with ✓ and ✗, respectively. \( Z_m(A^1, \ldots, A^m) \triangleq \sum_{S \subseteq [n]} \det(A^1_{S,S}) \cdots \det(A^m_{S,S}) \) denotes the normalizing constant for Π-DPPs, \( n \) denotes the order of input matrices, \( |I| \) is the number of bits required to represent \( A^1, \ldots, A^m \), nz is the set of nonzero entries in a matrix, and tw is the treewidth (see Section 2).

| exact/approx./parameters | \( Z_p(A, \ldots, A) \) exponentiated DPP | \( Z_2(A, B) \) | \( Z_3(A, B, C) \) | \( Z_m(A^1, \ldots, A^m) \) |
|--------------------------|------------------------------------------|----------------|-----------------|-----------------|
| exact                    | ✓ UP-hard & Mod₃P-hard                    |                |                 |                 |
|                          | (Corollary 4.2, \( p = 2, 4, 6, \ldots \)) |                |                 |                 |
| approximation            |                                           | ✓ 2\( O(|I|^{1-\epsilon}) \)-approx. is NP-hard (Theorem 5.1) | ✓ 1 is 2\( O(|I|^2) \)-approx. (Observation 5.2) |                 |
| \( m = \text{number of matrices} \) | ✓ (special case of \( \rightarrow \)) | ✓ FPT; \( r^{O(r)}n^{O(1)} \) time (Theorem 6.1) | ✓ (special case of \( \rightarrow \)) | ✓ FPT; \( r^{O(mr)}n^{O(1)} \) time (Theorem 6.4) |
| \( r = \max_{i \in [m]} \text{rank}(A^i) \) |                                           |                |                 |                 |
| \( m = \text{number of matrices} \) | ✓ (special case of \( \rightarrow \)) | ✓ FPT; \( w^{O(w)}n^{O(1)} \) time (Theorem 6.5) | ✓ (special case of \( \rightarrow \)) | ✓ FPT; \( w^{O(mw)}n^{O(1)} \) time (Theorem 6.11) |
| \( w = \text{tw(} \bigcup_{i \in [m]} \text{nz}(A^i) \text{)} \) |                                           |                |                 |                 |
| \( m = \text{number of matrices} \) | ✓ (same as \( \uparrow \)) |                                           |                 |                 |
| \( w = \max_{i \in [m]} \text{tw(} \text{nz}(A^i) \text{)} \) |                                           |                                           |                 |                 |

\( Z_3(A^1, \ldots, A^m) \) is \#P-hard (\( w \leq 3, m = 2 \)) (Theorem 6.33)
Finally, we apply the FPT algorithm to two related problems, which bypasses the complexity-theoretic barrier in the general case. Hereafter, we will denote by $w$ the treewidth of an $n \times n$ matrix $A$.

✓ **Approximation Algorithm for E-DPPs of Fractional Exponents.** The first application is approximating the normalizing constant for E-DPPs of fractional exponent $p > 1$, i.e., $\sum_{S \subseteq [n]} \det(A_{S,S})^p$. Our FPT algorithm (Theorem 6.11) does not directly apply to this case because an E-DPP is a $\Pi$-DPP only if $p$ is integer. Generally, it is possible to compute a $2^{n(p-1)}$-approximation to this quantity efficiently (see Remark 8.3), which is tight up to constant in the exponent (Ohsaka, 2021b). On the other hand, it is known that there is an FPRAS if $p < 1$ (see Section 1.2; Anari et al., 2019; Robinson et al., 2019). Intriguingly, we can compute a $2^{\sqrt{n}}$-approximation for a $P_0$-matrix $A$ in $w^{O(w)} n^{O(\frac{1}{2})}$ time (Theorem 8.1). This result is apparently strange since the accuracy of estimation improves with the value of $p$. The idea behind the proof is to compute the normalizing constant for E-DPPs of exponent $\lfloor p \rfloor$ or $\lceil p \rceil$ by exploiting Theorem 6.11, either of which ensures the desired approximation.

✓ **Subexponential Algorithm for Unconstrained MAP Inference.** The second application is unconstrained maximum a posteriori (MAP) inference on DPPs, which is equivalent to finding a principal submatrix having the maximum determinant, i.e., to compute $\operatorname{argmax}_{S \subseteq [n]} \det(A_{S,S})$. The current best approximation factor for MAP inference is $2^{O(n)}$ (Nikolov, 2015), which is optimal up to constant in the exponent (Ohsaka, 2021b, see Section 1.2). Here, we present a $w^{O(w\sqrt{n})} n^{O(\frac{1}{2})}$-time randomized algorithm that approximates unconstrained MAP inference within a factor of $2^{\sqrt{n}}$ (Theorem 8.4). In particular, if $w$ is a constant independent of $n$, we can obtain a subexponential-time and subexponential-approximation algorithm, which is a “sweet spot” between a $2^{n^{O(\frac{1}{2})}}$-time exact (brute-force) algorithm and a polynomial-time $2^{O(n)}$-approximation algorithm (Nikolov, 2015). The proof uses Theorem 6.11 to generate a sample from an E-DPP of a sufficiently large exponent $p = O(\sqrt{n})$, for the desired guarantee of approximation with high probability.

### 1.2 Related Work

Exponentiated DPPs (E-DPPs) of exponent parameter $p > 0$ define the probability mass for each subset $S$ as proportional to $\det(A_{S,S})^p$. A Markov chain Monte Carlo algorithm on E-DPPs for $p < 1$ is known to converge rapidly as E-DPPs are strongly log-concave as shown by Anari, Liu, Gharan, and Vinzant (2019); Robinson, Sra, and Jegelka (2019), implying an FPRAS for the normalizing constant. Mariet, Sra, and Jegelka (2018) investigate the case when a DPP defined by the $p$-th power of $A$ is close to an E-DPP of exponent $p$ for $A$. Quite surprisingly, Gurvits (2005, 2009) proves the $\#P$-hardness of exactly computing $\sum_S \det(A_{S,S})^2$ for a P-matrix $A$ before the more recent study by Gillenwater (2014); Kulesza and Taskar (2012); this result seems to be not well known in the machine learning community. Gillenwater (2014) proves that computing the normalizing constant for $\Pi$-DPPs defined by two positive semi-definite matrices is $\#P$-hard, while Anari and Gharan (2017) prove that it is approximable within a factor of $e^n$ in polynomial...
time, which is an affirmative answer to an open question posed by Kulesza and Taskar (2012). Ohsaka (2021a,b) proves that it is NP-hard to approximate the normalizing constant for E-DPPs within a factor of $2^{\beta n p}$ when $p \geq \beta^{-1}$, where $\beta = 10^{-10^{13}}$. Our study strengthens these results by showing the hardness for an E-DPP of exponent $p = 2, 4, 6, \ldots$, and the impossibility of an exponential approximation and approximate sampling for three matrices.

II-DPPs can be thought of as log-submodular point processes (Djolonga and Krause, 2014; Gotovos et al., 2015), whose probability mass for a subset $S$ is proportional to $\exp(f(S))$, where $f$ is a submodular set function. Setting $f(S) \triangleq \log \det(A_{S,S}^1) + \cdots + \log \det(A_{S,S}^m) = \log(\det(A_{S,S}^1) \cdots \det(A_{S,S}^m))$ coincides with II-DPPs. Gotovos, Hassani, and Krause (2015) devised a bound on the mixing time of a Gibbs sampler, though this is not very helpful in our case because $f$ can take $\log(0) = -\infty$ as a value.

Constrained DPPs output a subset $S$ with probability proportional to $\det(A_{S,S})$ if $S$ satisfies specific constraints, e.g., size constraints (Kulesza and Taskar, 2011), partition constraints (Celis et al., 2018), budget constraints (Celis et al., 2017), and spanning-tree constraints (Matsuoka and Ohsaka, 2021). Note that II-DPPs can express partition-matroid constraints.

Maximum a posteriori (MAP) inference on DPPs finds applications wherein we seek the most diverse subset. Unconstrained MAP inference is especially preferable if we do not (or cannot) specify in advance the desired size of the output, as in, e.g., tweet timeline generation (Yao et al., 2016), object detection (Lee et al., 2016), and change-point detection (Zhang and Ou, 2016). On the negative side, it is NP-hard to approximate size-constrained MAP inference within a factor of $2^c$ for the output size $k$ and some number $c > 0$ (Çivril and Magdon-Ismail, 2013; Di Summa et al., 2014; Koutis, 2006). Kulesza and Taskar (2012) prove an inapproximability factor of $(\frac{a}{b} - \epsilon)$ for unconstrained MAP inference for any $\epsilon > 0$, which has since been improved to $2^{\beta n}$ for $\beta = 10^{-10^{13}}$ (Ohsaka, 2021b). On the algorithmic side, Çivril and Magdon-Ismail (2009) prove that the standard greedy algorithm for size-constrained MAP inference achieves an approximation factor of $k!^2 = 2^{O(k \log k)}$, where $k$ denotes the output size. Nikolov (2015) gives an $e^k$-approximation algorithm for size-constrained MAP inference; this is the current best approximation factor. Note that invoking Nikolov’s algorithm for all $k \in [n]$ immediately yields an $e^k$-approximation for unconstrained MAP inference. The greedy algorithm is widely used in the machine learning community because it efficiently extracts reasonably diverse subsets in practice (Yao et al., 2016; Zhang and Ou, 2016). Other than the greedy algorithm, Gillenwater, Kulesza, and Taskar (2012) propose a gradient-based algorithm; Zhang and Ou (2016) develop a dynamic-programming algorithm designed for small-bandwidth matrix, which has no provable approximation guarantee.

This article is an extended version of our conference paper presented at the 37th International Conference on Machine Learning (Ohsaka and Matsuoka, 2020). It includes the following new results:

- **Section 3** proves two fundamental properties of II-DPPs: (1) we can generate a sample from II-DPPs in polynomial time if we are given access to an oracle for the normalizing constant (Theorem 3.1); (2) $\mathcal{Z}_m$ either admits an FPRAS or cannot be approximated within a factor of $2^\delta$ for any $\delta \in (0, 1)$ (Theorem 3.3).

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6We say that a set function $f : 2^{[n]} \rightarrow \mathbb{R}$ is submodular if $f(S) + f(T) \geq f(S \cup T) + f(S \cap T)$ for all $S, T \subseteq [n]$. 
• Section 5.3 presents an approximation-preserving reduction from the mixed discriminant to the normalizing constant for Π-DPPs of two matrices, rather than the polynomial-time Turing reduction presented in Ohsaka and Matsuoka (2020).

• Section 7 extends the complexity-theoretic results in Sections 4 to 6 to fixed-size Π-DPPs.

• Section 8 introduces two applications of the FPT algorithms: (1) a \( w^{O(wp)}n^{O(1)} \)-time algorithm that approximates the normalizing constant for E-DPPs of any fractional exponent \( p > 1 \) within a factor of \( 2^{\sqrt{n}} \) (Theorem 8.1); (2) a \( w^{O(\sqrt{\pi})}n^{O(1)} \)-time (randomized) algorithm that approximates unconstrained MAP inference within a factor of \( 2\sqrt{n} \) (Theorem 8.4), where \( n \) is the order of an input matrix and \( w \) is the treewidth of the matrix.

2 Preliminaries

2.1 Notations

For two integers \( m, n \in \mathbb{N} \) such that \( m \leq n \), let \([n] \equiv \{1, 2, \ldots, n\}\) and \([m..n] \equiv \{m, m+1, \ldots, n-1, n\}\). The imaginary unit is denoted \( i = \sqrt{-1} \). For a finite set \( S \) and an integer \( k \in [0..|S|] \), we write \( \binom{S}{k} \) for the family of all size-\( k \) subsets of \( S \). For a statement \( P \), \([P]\) is 1 if \( P \) is true, and 0 otherwise. The symbol \( \cup \) is used to emphasize that the union is taken over two disjoint sets. The symmetric group on \([n]\), consisting of all permutations over \([n]\), is denoted \( \mathfrak{S}_n \). We use \( \sigma : S \rightarrow T \) for two same-sized sets \( S \) and \( T \) to mean a bijection from \( S \) to \( T \), and \( \sigma|X \) for a set \( X \) to denote the restriction of \( \sigma \) to \( X \cap S \). We also define \( \sigma(X) \equiv \{\sigma(x) \mid x \in X\} \) for a set \( X \subseteq S \) and \( \sigma^{-1}(Y) \equiv \{y \mid \sigma(y) \in Y\} \) for a set \( Y \subseteq T \). For a bijection \( \sigma : S \rightarrow T \) and an ordering \( \prec \) on \( S \cup T \), the inversion number and sign of \( \sigma \) regarding \( \prec \) are defined as

\[
\text{inv}^{-}(\sigma) \equiv |\{(i, j) \mid i < j, \sigma(i) \succ \sigma(j)\}|,
\]

\[
\text{sgn}^{-}(\sigma) \equiv (-1)^{\text{inv}^{-}(\sigma)}.
\]

In particular, if \( \sigma \) is a permutation in \( \mathfrak{S}_n \), there exists \( \text{inv}(\sigma) \in \mathbb{N} \) and \( \text{sgn}(\sigma) \in \{+1, -1\} \) such that \( \text{inv}^{-}(\sigma) = \text{inv}(\sigma) \) and \( \text{sgn}^{-}(\sigma) = \text{sgn}(\sigma) \) for any ordering \( \prec \). We use \( S \prec T \) for two sets \( S \) and \( T \) to mean that \( i \prec j \) for all \( i \in S \) and \( j \in T \). For two orderings \( \prec_1 \) on a set \( S_1 \) and \( \prec_2 \) on a set \( S_2 \), we say that \( \prec_1 \) and \( \prec_2 \) agree on a set \( T \subseteq S_1 \cap S_2 \) whenever \( i \prec_1 j \) if and only if \( i \prec_2 j \) for all \( i, j \in T \). Unless otherwise specified, the base of the logarithm is 2. For a set \( S \) and element \( e \), we shall write \( S + e \) and \( S - e \) as shorthand for \( S \cup \{e\} \) and \( S \setminus \{e\} \), respectively.

We denote the \( n \times n \) identity matrix by \( I_n \) and the \( n \times n \) all-ones matrix by \( J_n \). For an \( m \times n \) matrix \( A \) and two subsets \( S \subseteq [m] \) and \( T \subseteq [n] \) of indices, we write \( A_S \) for the \( |S| \times n \) submatrix whose rows are the rows of \( A \) indexed by \( S \), and \( A_{S,T} \) for the \( |S| \times |T| \) submatrix whose rows are the rows of \( A \) indexed by \( S \) and columns are the columns of \( A \) indexed by \( T \). The determinant and permanent of a matrix \( A \in \mathbb{R}^{n \times n} \) are defined as

\[
\text{det}(A) \equiv \sum_{\sigma \in \mathfrak{S}_n} \text{sgn}(\sigma) \prod_{i \in [n]} A_{i,\sigma(i)},
\]

\[
\text{per}(A) \equiv \sum_{\sigma \in \mathfrak{S}_n} \prod_{i \in [n]} A_{i,\sigma(i)}.
\]
In particular, \( \det(A_{S,S}) \) for any \( S \subseteq [n] \) is called a principal minor. We define \( \det(A_{\emptyset,\emptyset}) \triangleq 1 \). A symmetric matrix \( A \in \mathbb{R}^{n \times n} \) is called positive semi-definite if \( x^T A x \geq 0 \) for all \( x \in \mathbb{R}^n \). A matrix \( A \in \mathbb{R}^{n \times n} \) is called a \( P \)-matrix (resp. \( P_0 \)-matrix) if all of its principal minors are positive (resp. nonnegative). A positive semi-definite matrix is a \( P_0 \)-matrix, but not vice versa. A real-valued matrix \( A \) is a \( P \)-matrix whenever it has positive diagonal entries and is row diagonally dominant (i.e., \( |A_{i,i}| > \sum_{j \neq i} |A_{i,j}| \) for all \( i \)). For a bijection \( \sigma \) from \( S \subseteq [n] \) to \( T \subseteq [n] \), we define \( A(\sigma) \triangleq \prod_{i \in S} A_{i,\sigma(i)} \).

### 2.2 Product of Determinantal Point Processes

Given a matrix \( A \in \mathbb{R}^{n \times n} \), a determinantal point process (DPP) (Borodin and Rains, 2005; Macchi, 1975) is defined as a probability measure on the power set \( 2^{[n]} \) whose probability mass for \( S \subseteq [n] \) is proportional to \( \det(A_{S,S}) \). Generally speaking, a \( P_0 \)-matrix is acceptable to define a proper probability distribution, while positive semi-definite matrices are commonly used (Gartrell et al., 2019). The normalizing constant for a DPP has the following simple closed form (Kulesza and Taskar, 2012):

\[
\sum_{S \subseteq [n]} \det(A_{S,S}) = \det(A + I_n).
\]

Hence, the probability mass for a set \( S \subseteq [n] \) is equal to \( \det(A_{S,S}) / \det(A + I_n) \). This equality holds for any (not necessarily symmetric) real-valued matrix \( A \).

This paper studies a point process whose probability mass is determined from the product of principal minors for multiple matrices. Given \( m \) matrices \( A^1, \ldots, A^m \in \mathbb{R}^{n \times n} \), the product DPP (\( \Pi \)-DPP) defines the probability mass for each subset \( S \subseteq [n] \) as being proportional to \( \det(A^1_{S,S}) \cdots \det(A^m_{S,S}) \). We use \( Z_m(A^1, \ldots, A^m) \) to denote its normalizing constant; namely,

\[
Z_m(A^1, \ldots, A^m) \triangleq \sum_{S \subseteq [n]} \prod_{i \in [m]} \det(A^i_{S,S}).
\]

In particular, we have that \( Z_1(A) = \det(A + I_n) \). Since \( \prod_{i \in [m]} \det(A^i_{S,S}) \) is easy to compute, evaluating \( Z_m \) is crucial for estimating the probability mass. Our objective in this paper is to elucidate the computational complexity of estimating \( Z_m \). We shall raise two examples of \( \Pi \)-DPPs.

**Example 2.1** (Embedding partition and matching constraints). Given a partition \( \mathcal{P} \) of \( [n] \), we can build \( A \) such that \( \det(A_{S,S}) = [S \text{ contains at most one element from each group of } \mathcal{P}] \) by defining \( A_{i,j} = [i, j \text{ belong to the same group}] \). Given a bipartite graph whose edge set is \([n]\), we can build \( A \) and \( B \) such that \( \det(A_{S,S}) \det(B_{S,S}) = [S \text{ has no common vertices}] \) (Gillenwater, 2014); such an \( S \) is called a matching.

**Example 2.2** (Exponentiated DPPs). Setting \( A^i = A \) for all \( i \in [m] \), the \( \Pi \)-DPP becomes an exponentiated DPP of exponent \( p = m \geq 1 \), which sharpens the diversity nature of DPPs (Mariet et al., 2018).

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\(^7\)We adopt the L-ensemble form introduced by Borodin and Rains (2005).
2.3 Graph-Theoretic Concepts

Here, we briefly introduce the notions and definitions from graph theory that will play a crucial role in Section 6. Let $G = (V, E)$ be a graph, where $V$ is the set of vertices, and $E$ is the set of edges. We use $(u, v)$ to denote an (undirected or directed) edge connecting $u$ and $v$. Moreover, we define the treewidth of a graph and matrix. 

**Tree decomposition** (Arnborg and Proskurowski, 1989; Bertelè and Brioschi, 1972; Halin, 1976; Robertson and Seymour, 1986) is one of the most important notions in graph theory, which captures the "tree-likeness" of a graph.

**Definition 2.3** (Robertson and Seymour, 1986, tree decomposition). A *tree decomposition* of an undirected graph $G = (V, E)$ is a pair $(T, \{X_t\}_{t \in T})$, where $T$ is a tree of which vertex $t \in T$, referred to as a *node*, is associated with a vertex set $X_t \subseteq V$, referred to as a *bag*, such that the following conditions are satisfied:

- $\bigcup_{t \in T} X_t = V$;
- for every edge $(u, v) \in E$, there exists a node $t \in T$ such that $u, v \in X_t$;
- for every vertex $v \in V$, the set $T_v = \{t \mid v \in X_t\}$ induces a connected subtree of $T$.

The *width* of a tree decomposition $(T, \{X_t\}_{t \in T})$ is defined as $\max_{t \in T} |X_t| - 1$. The *treewidth* of a graph $G$, denoted $\operatorname{tw}(G)$, is the minimum possible width among all tree decompositions of $G$.

For example, a tree has treewidth 1, an $n$-vertex planar graph has treewidth $O(\sqrt{n})$, and an $n$-clique has treewidth $n - 1$.

For an $n \times n$ square matrix $A$, we define $\operatorname{nz}(A) \triangleq \{(i, j) \mid A_{ij} \neq 0, i \neq j\}$. The treewidth of $A$, denoted $\operatorname{tw}(\operatorname{nz}(A))$ or $\operatorname{tw}(A)$, is defined as the treewidth of the graph $([n], \operatorname{nz}(A))$ formed by the nonzero entries of $A$. See Figures 1 to 3 for an example of a tree decomposition of a matrix. For example, $\operatorname{tw}(I_n) = 1$, $\operatorname{tw}(J_n) = n - 1$, and a matrix of bandwidth\(^9\) $b$ has treewidth $O(b)$. One important property of tree decompositions is that any bag $X_t$ is a *separator*: for three nodes $t, t', t''$ of $T$ such that $t$ is on the (unique) path from $t'$ to $t''$, $X_t$ separates $X_{t'} \setminus X_t$ and $X_{t''} \setminus X_t$; i.e., the submatrices $A_{X_{t'}, X_t \setminus X_t}$ and $A_{X_{t''}, X_t \setminus X_t}$ must be zero matrices. It is known that the permanent of bounded-treewidth matrices is polynomial-time computable (Courcelle et al., 2001). Though it is NP-complete to determine whether an input graph $G$ has treewidth at most $w$, there exist numerous FPT and approximation algorithms, e.g., a $w^{O(w^3)}n$-time exact algorithm (Bodlaender, 1996) and a 5-approximation algorithm having faster runtime $2^{O(w)}n$ (Bodlaender et al., 2016).

**Remark 2.4.** Our FPT algorithms parameterized by rank (Section 6.1) and by treewidth (Section 6.2) are not comparable in the sense that the identity matrix $I_n$ has rank $n$ and treewidth 1 while the all-ones matrix $J_n$ has rank 1 and treewidth $n - 1$.

2.4 Computational Models

We will introduce the notion of input size and computational model carefully since we use several *reductions* that transform an input for one problem to an input for another problem.

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\(^8\)We will refer to the vertices of $T$ as nodes to distinguish them from the vertices of $G$.

\(^9\)The bandwidth of a matrix $A$ is defined as the smallest integer $b$ such that $A_{ij} = 0$ whenever $|i - j| > b$. 

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The size of an input $I$, denoted $|I|$, is defined as the number of bits required to represent $I$. In particular, we assume that all numbers appearing in this paper are rational.\(^{10}\) The size of a rational number $x = p/q \in \mathbb{Q}$ (where $p$ and $q$ are relatively prime integers) and a rational matrix $A \in \mathbb{Q}^{m \times n}$ is defined as follows (Schrijver, 1999):

$$
\text{size}(x) \triangleq 1 + \lceil \log(|p| + 1) \rceil + \lceil \log(|q| + 1) \rceil,
$$

$$
\text{size}(A) \triangleq mn + \sum_{i \in [m], j \in [n]} \text{size}(A_{ij}).
$$

The size of a graph is defined as the size of its incidence matrix.

Selection of computational models is crucial for determining the runtime of algorithms; e.g., while multiplying two $n$-bit integers can be done in $O(n \log n \, 8^{log^* n})$ time on Turing machines (Harvey et al., 2016), we do not need this level of precision. Thus, for ease of analysis, we adopt the unit-cost random-access machine model of computation, which can perform basic arithmetic operations (e.g., add, subtract, multiply, and divide) in constant time. In other words, we will measure the runtime in terms of the number of operations. However, abusing unrealistically powerful models leads to an unreasonable conclusion (Arora and Barak, 2009, Example 16.1): “iterating $n$ times the operation $x \leftarrow x^2$, we can compute $2^{2^n}$, a $2^n$-bit integer, in $O(n)$ time.” To avoid such pitfalls, we will ensure that numbers produced during the execution of algorithms intermediately are of size $|I|^{O(1)}$.

### 2.5 Brief Introduction to Complexity Classes

Here, we briefly introduce the complexity classes appearing throughout this paper.

**Decision Problems.**

- **P**: The class of decision problems solvable by a deterministic polynomial-time Turing machine. Examples include PRIMES (Q. Is an input integer prime?) (Agrawal et al., 2004).

- **NP**: The class of decision problems solvable by a nondeterministic polynomial-time Turing machine (NP machine). Examples include SAT (Q. Is there a truth assignment satisfying an input Boolean formula?). It is widely believed that $P \neq NP$, see, e.g., Arora and Barak (2009).

- **RP**: The class of decision problems for which a probabilistic polynomial-time Turing machine exists such that (1) if the answer is “yes,” then it returns “yes” with probability at least $\frac{1}{2}$, and (2) if the answer is “no,” then it always returns “no.” Note that $P \subseteq RP \subseteq NP$, and it is suspected that $RP \neq NP$.

- **UP**: The class of decision problems solvable by an NP machine with at most one accepting path. Note that $P \subseteq UP \subseteq NP$, but it is unknown if the inclusion is strict. Examples include UNAMBIGUOUS SAT (Q. Is there a truth assignment satisfying an input Boolean formula that is restricted to have at most one satisfying assignment?) and INTEGERFACTORIZATION (Q. Is there

\(^{10}\)We can run the algorithms in Section 6 for real-valued matrices if arithmetic operations on real numbers are allowed.
a factor \( d \in [m] \) of an integer \( n \) given \( n \) and \( m \)?, for which no polynomial-time algorithms are known. The Valiant–Vazirani theorem states that if \textsc{UnambiguousSat} is solvable in polynomial-time, then \( \text{RP} = \text{NP} \) (Valiant and Vazirani, 1986).

**Counting-Related Problems.**

- **FP**: The class of functions computable by a deterministic polynomial-time Turing machine, which is a function-problem analogue of \( P \). Examples include \textsc{Determinant} (Q. Compute the determinant of an input square matrix), which is efficiently-computable via Gaussian elimination (Edmonds, 1967; Schrijver, 1999).

- **\#P**: The class of function problems of counting the number of accepting paths of an \( \text{NP} \) machine. Examples of \( \text{\#P} \)-complete problems include \textsc{Permanent} (Q. Compute the permanent of an input matrix) and \textsc{Sat} (Q. Compute the number of truth assignments satisfying an input Boolean formula). Note that \( P \neq \text{NP} \) implies \( \text{FP} \neq \text{\#P} \) (Arora and Barak, 2009).

- **\text{Mod}_k \text{P}**: The class of decision problems solvable by an \( \text{NP} \) machine, where the number of accepting paths is not divisible by \( k \). Examples include \textsc{GraphIsomorphism} (Q. Is there an edge-preserving bijection between the vertex sets of two input graphs?), which is in \( \text{Mod}_k \text{P} \) for all \( k \) (Arvind and Kurur, 2006).

**Parameterized Problems.**

- **\text{FPT}** (fixed-parameter tractable): The class of problems with parameter \( k \) solvable in \( f(k)|I|^{O(1)} \) time for some computable function \( f \), where \( |I| \) is the input size. Examples include \textsc{k-VertexCover} (Q. Is there a \( k \)-vertex set including at least one endpoint of every edge of an \( n \)-vertex graph?), for which an \( O(1.2738^k + kn) \)-time algorithm is known (Chen et al., 2010).

- **\text{XP}** (slice-wise polynomial): The class of problems with parameter \( k \) solvable in \( |I|^{f(k)} \) time for some computable function \( f \); hence it holds that \( \text{FPT} \subseteq \text{XP} \). Examples include \textsc{k-Clique} (Q. Is there a size-\( k \) complete subgraph in an \( n \)-vertex graph?), for which a brute-force search algorithm runs in \( n^{O(k)} \) time. It is suspected that \( \text{FPT} \neq \text{XP} \) in parameterized complexity (Downey and Fellows, 2012).

- **\#W[1]**: The class of function problems parameterized reducible to \( \textsc{k-Clique} \) (Q. Compute the number of \( k \)-cliques in an \( n \)-vertex graph). Note that \( \text{FPT} \subseteq \#W[1] \subseteq \text{XP} \). It is a plausible assumption in parameterized complexity (Flum and Grohe, 2004) that \( \text{FPT} \neq \#W[1] \); i.e., \( \text{\#k-Clique} \) does not admit an \( \text{FPT} \) algorithm parameterized by \( k \).

**2.6 Approximation Algorithms**

Here, we introduce some concepts related to approximation algorithms. We say that an estimate \( \hat{Z} \) is a \( \rho \)-approximation to some value \( Z \) for \( \rho \geq 1 \) if

\[
Z \leq \hat{Z} \leq \rho \cdot Z.
\]
The approximation factor $\rho$ can be a function of the input size, e.g., $\rho(n) = 2^n$; an (asymptotically) smaller $\rho$ is a better approximation factor. For a function $f : \Sigma^* \to \mathbb{R}$ and an approximation factor $\rho$, a $\rho$-approximation algorithm is a polynomial-time algorithm that returns a $\rho$-approximation to $f(I)$ for every input $I \in \Sigma^*$.

We define a fully polynomial-time randomized approximation scheme (FPRAS). The existence of an FPRAS for a particular problem means that the problem can be efficiently approximated to an arbitrary precision.

**Definition 2.5.** For a function $f : \Sigma^* \to \mathbb{R}$, a fully polynomial-time randomized approximation scheme (FPRAS) is a randomized algorithm $\text{alg}$ that takes an input $I \in \Sigma^*$ of $f$ and an error tolerance $\epsilon \in (0, 1)$ and satisfies the following conditions:

- for every input $I \in \Sigma^*$ and $\epsilon \in (0, 1)$, it holds that
  \[
  \Pr_{\text{alg}}[e^{-\epsilon} \cdot f(I) \leq \text{alg}(I) \leq e^{\epsilon} \cdot f(I)] \geq \frac{3}{4},
  \]
  where $\text{alg}(I)$ denotes $\text{alg}$’s output on $I$;\(^{11}\)

- the running time of $\text{alg}$ is bounded by a polynomial in $|I|$ and $\epsilon^{-1}$, where $|I|$ denotes the size of input $I$.

### 3 Fundamental Properties of Π-DPPs

In this section, we establish two fundamental properties of Π-DPPs, i.e., (1) exact sampling given oracle access (Theorem 3.1), and (2) an all-or-nothing nature (Theorem 3.3). These properties are common to counting problems that have self-reducibility (Jerrum, 2003).

#### 3.1 Exact Sampling Given Exact Oracle

We will show that if we are given access to an oracle that can (magically) return the value of $Z_m$ in a single step, we can generate a sample from a Π-DPP defined by any $m$ matrices in polynomial time.

**Theorem 3.1.** Suppose we are given access to an oracle that returns $Z_m$. Let $A^1, \ldots, A^m$ be $m$ $P_0$-matrices in $\mathbb{Q}^{n \times n}$. Then, there exists a polynomial-time algorithm that generates a sample from the Π-DPP defined by $A^1, \ldots, A^m$ by calling the oracle for $L \triangleq O(n^2)$ tuples of $m$ matrices. Furthermore, if $\{(A^{1, \ell}, \ldots, A^{m, \ell})\}_{\ell \in [L]}$ denotes the set consisting of the tuples of $m$ matrices for which the oracle is called (i.e., we call the oracle to evaluate $Z_m(A^{1, \ell}, \ldots, A^{m, \ell})$ for all $\ell \in [L]$), then it holds that $\text{rank}(A^{1, \ell}) \leq \text{rank}(A^i)$ and $\text{nz}(A^{1, \ell}) \subseteq \text{nz}(A^i)$ for all $i \in [m]$ and $\ell \in [L]$.

\(^{11}\)The constant $\frac{3}{4}$ in Eq. (1) can be replaced by any number in $(\frac{1}{2}, 1)$ (Jerrum et al., 1986).
The proof involves a general sampling procedure using the conditional probability, e.g., Celis et al. (2017); Jerrum (2003). For two disjoint subsets $Y$ and $N$ of $[n]$ and an element $e$ of $[n]$ not in $Y \cup N$, let us consider the following conditional probability:

$$
\Pr_{S \sim \mu} \left[ e \in S \mid Y \subseteq S, N \cap S = \emptyset \right] = \frac{\sum_{S \subseteq [n] : Y + e \subseteq S, N \cap S = \emptyset} \prod_{i \in [m]} \det(A^i_{S,S})}{\sum_{S \subseteq [n]: Y \subseteq S, N \cap S = \emptyset} \prod_{i \in [m]} \det(A^i_{S,S})}, \tag{2}
$$

where $\mu$ denotes the $\Pi$-DPP defined by $A^1, \ldots, A^m$. Eq. (2) represents the probability that we draw a sample $S$ including $e$ from $\mu$ conditioned on that $S$ contains $Y$ but does not include any element of $N$. We first prove that the conditional probability can be computed in polynomial time given access to an oracle for $Z_m$.

**Lemma 3.2.** Let $Y$ and $N$ be disjoint subsets of $[n]$ and $e$ be an element of $[n]$ not in $Y \cup N$. Given access to an oracle that returns $Z_m$, we can compute the conditional probability in Eq. (2) in polynomial time by calling the oracle $L \triangleq 4n + 2$ times. Furthermore, if $\{(A^{1,\ell}, \ldots, A^{m,\ell})\}_{\ell \in [L]}$ denotes the set consisting of the tuples of $m$ matrices for which the oracle is called, then it holds that $\text{rank}(A^{1,\ell}) \leq \text{rank}(A^i)$ and $\text{nz}(A^{1,\ell}) \subseteq \text{nz}(A^i)$ for all $i \in [m]$ and $\ell \in [L]$.

**Proof.** Fix two disjoint subsets $Y$ and $N$ of $[n]$. It is sufficient to show how to compute the denominator of Eq. (2) in polynomial time by calling the oracle $2n + 1$ times. Introduce a positive number $x \in \mathbb{Q}$ and define a matrix $X \in \mathbb{Q}^{n \times n}$ depending on the value of $x$ as follows:

$$X_{i,j} \triangleq \begin{cases} 
1 & \text{if } i, j \notin Y \cup N, \\
x & \text{if } i \in Y \text{ and } j \notin Y \cup N, \\
x & \text{if } j \in Y \text{ and } i \notin Y \cup N, \\
x^2 & \text{if } i, j \in Y, \\
0 & \text{otherwise}.
\end{cases}$$

An example of constructing $X$ in the case of $n = 6, Y = \{3, 4\}$, and $N = \{5, 6\}$ is shown below.

$$X = \begin{bmatrix}
1 & 1 & x & x & 0 & 0 \\
1 & 1 & x & x & 0 & 0 \\
x & x & x^2 & x^2 & 0 & 0 \\
x & x & x^2 & x^2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}.$$ 

Consider the matrix $A^1 \circ X$, where the symbol $\circ$ denotes the Hadamard product; namely, $(A^1 \circ X)_{i,j} = A^1_{i,j} \cdot X_{i,j}$ for each $i, j \in [n]$. It is easy to see that for each set $S \subseteq [n]$,

$$\det((A^1 \circ X)_{S,S}) = \begin{cases} 
0 & \text{if } S \cap N \neq \emptyset, \\
x^{2 |S \cap Y|} \cdot \det(A^1_{S,S}) & \text{otherwise}.
\end{cases}$$

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Further, we define a univariate polynomial $Z$ in $x$ as $$Z(x) \triangleq Z_m(A_1 \circ X, A_2, \ldots, A^m).$$

Observe that the degree of $Z$ is at most $2n$. Expanding $Z(x)$ yields

\[
Z(x) = \sum_{S \subseteq [n]} \det((A^1 \circ X)_{S,S}) \prod_{2 \leq i \leq m} \det(A^i_{S,S}) \\
= \sum_{S \subseteq [n]} x^{2|S\setminus Y|} \prod_{i \in [m]} \det(A^i_{S,S}) \\
= \sum_{X \subseteq Y} \sum_{S \subseteq [n] \setminus (N \cap Y)} x^{2|X|} \prod_{i \in [m]} \det(A^i_{S,S}) \\
= \sum_{X \subseteq Y} x^{2|X|} \sum_{S \subseteq [n] \setminus (N \cap Y)} \prod_{i \in [m]} \det(A^i_{S,S}).
\]

Therefore, the coefficient of $x^{2n}$ in $Z(x)$ is exactly equal to the desired value, i.e.,

\[
\sum_{S \subseteq [n] \setminus (N \cap Y)} \prod_{i \in [m]} \det(A^i_{S,S}).
\]

Given $Z(1), Z(2), \ldots, Z(2n+1)$, each of which is obtained by calling the oracle for $Z_m$, we can exactly recover all the coefficients in $Z(x)$ by Lagrange interpolation as desired. Observe that $\text{rank}(A^1 \circ X) \leq \text{rank}(A^1)$ and $\text{nz}(A^1 \circ X) \subseteq \text{nz}(A^1)$, which completes the proof. \hfill $\square$

**Proof of Theorem 3.1.** Our sampling algorithm is essentially equivalent to that given by Celis et al. (2017). Starting with an empty set $Y \triangleq \emptyset$, it sequentially determines whether to include each element $e \in [n]$ in $Y$ or not, by computing the conditional probability in Eq. (2) with Lemma 3.2. A precise description is presented as follows:

**Sampling algorithm for $\Pi$-DPPs given access to oracle for $Z_m$.**

- **Step 1.** initialize $Y \triangleq \emptyset$ and $N \triangleq \emptyset$.
- **Step 2.** for each element $e \in [n]$:  
  - **Step 2-1.** compute the conditional probability $p_e \triangleq \Pr[e \in S \mid Y \subseteq S, N \cap S = \emptyset]$ in Eq. (2), where $\mu$ denotes the $\Pi$-DPP defined by $A^1, \ldots, A^m$, by calling an oracle for $Z_m$ according to Lemma 3.2.  
  - **Step 2-2.** add $e$ to $Y$ with probability $p_e$; otherwise, add $e$ to $N$.
- **Step 3.** output $S \triangleq Y$ as a sample.

The above algorithm correctly produces a sample from $\mu$. The number of oracle calls is bounded by $n(4n+2) = \mathcal{O}(n^2)$ due to Lemma 3.2. The structural arguments on the matrices for which the oracle is called are obvious from Lemma 3.2. \hfill $\square$
3.2 All-or-Nothing Nature

Here, we point out that the computation of \( Z_m \) (for fixed \( m \)) either admits an FPRAS or cannot be approximated within any subexponential factor.

**Theorem 3.3.** For a fixed positive integer \( m \), either of the following two statements holds:

- there exists an FPRAS for \( Z_m \), or
- there does not exist a \( 2^n^\delta \)-approximation randomized algorithm for \( Z_m \) for any \( \delta \in (0,1) \), where \( n \) is the order of the input matrices.

**Proof.** We show that if there exists a \( 2^n^\delta \)-approximation randomized algorithm for \( Z_m \) for some \( \delta \in (0,1) \), then there exists an FPRAS for \( Z_m \). Let \( A^1, \ldots, A^m \) be \( m \) positive semi-definite matrices in \( Q^{n \times n} \), and let \( \epsilon \in (0,1) \) be an error tolerance. We also introduce a positive integer \( t \), the value of which will be determined later. For each \( i \in [m] \), we define \( A^{i(t)} \) to be an \( nt \times nt \) block diagonal matrix, each diagonal block of which is \( A^i \). Note that \( A^{i(t)} \) is positive semi-definite for all \( i \in [m] \). Then, by a simple calculation, we can expand \( Z_m(A^{1(t)}, \ldots, A^{m(t)}) \) as follows:

\[
Z_m(A^{1(t)}, \ldots, A^{m(t)}) = \sum_{S \subseteq [nt]} \prod_{i \in [m]} \det(A^{i(t)}_{S,S})
\]

\[
= \sum_{S_1 \subseteq [n]} \cdots \sum_{S_m \subseteq [n]} \prod_{i \in [m]} \det(A^{i(S_1,S_1)}_{S_1,S_1}) \cdots \det(A^{i(S_m,S_m)}_{S_m,S_m})
\]

\[
= \left( \sum_{S \subseteq [n]} \prod_{i \in [m]} \det(A^{i(S,S)}) \right)^t
\]

\[
= Z_m(A^1, \ldots, A^m)^t.
\]

Suppose now there exists a \( 2^n^\delta \)-approximation randomized algorithm for \( Z_m \) for some fixed \( \delta \in (0,1) \), where \( n \) is the order of the input matrices. Here, we can assume that the algorithm satisfies the approximation guarantee with probability at least \( \frac{1}{4} \). When invoking this approximation algorithm on \( A^{1(t)}, \ldots, A^{m(t)} \), we obtain an estimate \( \hat{Z} \) to \( Z_m(A^{1(t)}, \ldots, A^{m(t)}) \) such that

\[
Z_m(A^1, \ldots, A^m)^t \leq \hat{Z} \leq 2^{(nt)^\delta} \cdot Z_m(A^1, \ldots, A^m)^t.
\]

Taking the \( t \)-th root of both sides yields

\[
Z_m(A^1, \ldots, A^m) \leq \hat{Z}^{1/t} \leq 2^{n^\delta} \cdot Z_m(A^1, \ldots, A^m).
\]  \hspace{1cm} (3)

We now specify the value of \( t \):

\[
t = \left\lceil \frac{n^\delta}{\frac{\epsilon}{2} \cdot \log_2 e} \right\rceil^{\frac{1}{2}},
\]

the number of bits required to represent which is bounded by a polynomial in \( \log n \) and \( \log e^{-1} \) for fixed \( \delta \). The approximation factor \( 2^{n^\delta} \cdot 1 \) in Eq. (3) can be bounded as follows:

\[
2^{n^\delta} \leq 2^n \left( \frac{n^\delta}{\frac{\epsilon}{2} \cdot \log_2 e} \right)^{\frac{1}{2}} = 2^\frac{\epsilon}{2} \cdot \log_2 e = e^\epsilon/2.
\]
Our algorithm simply constructs $m$ positive semi-definite matrices $A^{1(t)}, \ldots, A^{m(t)}$, invokes a $2n^d$-approximation algorithm on them to obtain $\hat{Z}$, computes an $e^{c/2}$-approximation to $\hat{Z}^{1/t}$, denoted $\breve{Z}$, and outputs it as an estimate. This algorithm meets the specifications for FPRAS because the size of the $m$ matrices $A^{1(t)}, \ldots, A^{m(t)}$, is bounded by a polynomial in $n$ and $e^{-1}$ for fixed $\delta$, and the output $\breve{Z}$ satisfies Eq. (1) with probability at least $\frac{3}{4}$, which completes the proof.

4 Intractability of Exponentiated DPPs

We present the intractability of computing the normalizing constant for exponentiated DPPs of every positive even exponent, e.g., $Z_2(A, A)$, $Z_4(A, A, A, A)$, $Z_6(A, A, A, A, A, A)$, and so on. For a positive number $p$ and a matrix $A \in \mathbb{R}^{n \times n}$, we define

$$Z^p(A) \triangleq \sum_{S \subseteq [n]} \det(A_{S,S})^p.$$

We will prove the following theorem.

**Theorem 4.1.** Computing $Z^2(A) \mod 3$ for a matrix $A \in \mathbb{Q}^{n \times n}$ is UP-hard and Mod$_3$P-hard. The same statement holds even when $A$ is restricted to be either a $(-1, 0, 1)$-matrix or a P-matrix.

As a corollary, we can show the same hardness for every fixed positive even integer $p$ (i.e., $p$ is not in the input), thus giving a negative answer to an open question of Kulesza and Taskar (2012).

**Corollary 4.2.** For every fixed positive even integer $p$, computing $Z^p(A) \mod 3$ for either a $(-1, 0, 1)$-matrix or a P-matrix $A$ is UP-hard and Mod$_3$P-hard.

**Proof.** Since $0^p \equiv 0^2$, $1^p \equiv 1^2$, $2^p \equiv 2^2 \mod 3$ if $p$ is a positive even integer, we have that $Z^p(A) \equiv Z^2(A) \mod 3$. \qed

The proof of Theorem 4.1 relies on the celebrated results relating $Z^2$ to the permanent by Kogan (1996), who presented an efficient algorithm for computing $\text{per}(A) \mod 3$ for a matrix $A$ with rank$(AA^T - I_n) \leq 1$. In the remainder of this subsection, arithmetic operations are performed over modulo 3, and the symbol $\equiv$ means congruence modulo 3.

**Lemma 4.3** (Kogan, 1996, Lemma 2.7). Let $X$ be a matrix such that $\det(X + iI_n) \neq 0$. Then, it holds that

$$Z^2(X) \equiv \det(X + iI_n)^2 \text{per}((I_n + iX)^{-1} + I_n).$$

**Proof of Theorem 4.1.** We reduce from the problem of computing the permanent of a $(0, 1)$-matrix mod3, which is UP-hard and Mod$_3$P-hard (Valiant, 1979, Theorem 2), to the problem of computing $Z^2 \mod 3$. Let $A$ be an $n \times n$ $(0, 1)$-matrix. By Proposition 2.2 due to Kogan (1996), we compute a diagonal $(-1, 1)$-matrix $D$ in polynomial time such that $DA - I_n$ is not singular and

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per(A) \equiv \det(D) \det(DA). We then compute the matrix \( X \equiv i^{-1}((DA - I_n)^{-1} - I_n) \) by performing Gaussian elimination modulo 3. Since \( \det(X + iI_n) \neq 0 \), we have by Lemma 4.3 that per(A) \equiv \mathcal{Z}^2(X) \det(D) \det(X + iI_n)^{-2}. We transform X into a new matrix X' according to the following two cases:

- **Case (1)** \( \det(D) \det(X + iI_n)^{-2} \equiv 1 \): let X' \( \equiv X \).
- **Case (2)** \( \det(D) \det(X + iI_n)^{-2} \equiv 2 \): let X' \( \equiv \begin{bmatrix} 0 & 0 & 0 \\ 1 & i & i \\ 0 & i & i \end{bmatrix} \), where 0 is the n \times 1 zero matrix. We have that \( \mathcal{Z}^2(X') \equiv 2 \mathcal{Z}^2(X) \); note that \( \mathcal{Z}^2([1 1 1]) = -1 \).

Consequently, we always have that per(A) \equiv \mathcal{Z}^2(X'). Because the entries of X' are purely imaginary numbers by construction, we can uniquely define a real-valued matrix Y such that X' = iY. Consider the polynomial \( \mathcal{Z}^2(xy) \) for a variable x as a polynomial, i.e.,

\[
\mathcal{Z}^2(xy) = \sum_{S \subseteq [n]} x^{2|S|} \det(Y_{S,S})^2 \equiv a_0 + a_1 x + a_2 x^2
\]

for some \( a_0, a_1, a_2 \). Solving a system of linear equations

\[
\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} \equiv \begin{bmatrix} \mathcal{Z}^2(0Y) \\ \mathcal{Z}^2(1Y) \\ \mathcal{Z}^2(2Y) \end{bmatrix}
\]

and noting that \( \mathcal{Z}^2(1Y) \equiv \mathcal{Z}^2(2Y) \), we have that \( a_0 \equiv 1, a_1 \equiv 0, a_2 \equiv \mathcal{Z}^2(Y) - 1 \) and hence \( \mathcal{Z}^2(iY) \equiv 2 - \mathcal{Z}^2(Y) \). We can transform Y into a \((-1, 0, 1)\)-matrix Y' having the same permanent so that

\[
Y'_{ij} = \begin{cases} 0 & \text{if } Y_{ij} \equiv 0, \\ +1 & \text{if } Y_{ij} \equiv 1, \\ -1 & \text{if } Y_{ij} \equiv 2. \end{cases}
\]

Further, we can obtain another P-matrix Y'' defined as Y'' \( \equiv Y' + 3nI_n \). Finally, we find that per(A) \( \equiv 2 - \mathcal{Z}^2(Y') \equiv 2 - \mathcal{Z}^2(Y'') \). Accordingly, deciding whether per(A) \( \neq 0 \) is reduced to deciding whether \( \mathcal{Z}^2(Y') \neq 2 \) (and \( \mathcal{Z}^2(Y'') \neq 2 \)), in polynomial time, completing the proof.

## 5 Inapproximability for Three Matrices (and Two Matrices)

Albeit the \#P-hardness of \( \mathcal{Z}_m \) for all \( m \geq 2 \), there is still room to consider the approximability of \( \mathcal{Z}_m \); e.g., Anari and Gharan (2017) have given an \( e^n \)-approximation algorithm for \( \mathcal{Z}_2 \). Unfortunately, we show below strong inapproximability for the case of \( m \geq 3 \).

### 5.1 (Sub)exponential Inapproximability

We will show exponential inapproximability for the case of three matrices. For two probability measures \( \mu \) and \( \eta \) on \( \Omega \), the total variation distance is defined as \( \frac{1}{2} \sum_{S \subseteq \Omega} |\mu_S - \eta_S| \). The proof is reminiscent of the one on the NP-hardness of three-matroid intersection (Papadimitriou and Steiglitz, 2013).

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Theorem 5.1. For any fixed positive number \( \epsilon > 0 \), it is NP-hard to approximate \( Z_3(A, B, C) \) for three matrices \( A, B, C \) in \( \mathbb{Q}^{n \times n} \) within a factor of \( 2^{O(|\mathcal{I}|^{1/\epsilon})} \) or \( 2^{O((n^{1/\epsilon})^4)} \), where \( |\mathcal{I}| \) is the input size. Moreover, unless \( \text{RP} = \text{NP} \), no polynomial-time algorithm can generate a random sample from a distribution whose total variation distance from the \( \Pi\text{-DPP} \) defined by \( A, B, C \) is at most \( \frac{1}{3} \). The same statement holds if \( A, B, C \) are restricted to be positive semi-definite.

Proof. We will show a polynomial-time Turing (a.k.a. Cook) reduction from an NP-complete HamiltonianPath problem ([Garey and Johnson, 1979]), which, for a directed graph \( G = (V, E) \) on \( n \) vertices and \( m \) edges, asks us to find a directed simple path that visits every vertex of \( V \) exactly once (called a Hamiltonian path). Such a graph \( G \) having a Hamiltonian path is called Hamiltonian.

We construct \( m \times m \) three positive semi-definite matrices \( A, B, C \) indexed by edges in \( E \) such that \( Z_3(A, B, C) \) is "significantly" large if \( G \) is Hamiltonian. Define \( A \) and \( B \) so that \( A_{i,j} = 1 \) if edges \( i, j \) share a common head and 0 otherwise, and \( B_{i,j} = 1 \) if edges \( i, j \) share a common tail and 0 otherwise. Note that for any \( S \subseteq E \), \( \det(A_{S,S})\det(B_{S,S}) \) takes 1 if \( S \) consists of directed paths or cycles only, and 0 otherwise. Next, define \( C \) so that \( \det(C_{S,S}) = \text{Pr}_T[S \subseteq T] \) for all \( S \subseteq E \), where a random edge set \( T \) is chosen from a uniform distribution over all spanning trees in (the undirected version of) \( G \). Such \( C \) can be found in polynomial time: it in fact holds that \( C = ML^\dagger M^\top \) ([Burton and Pemantle, 1993]), where \( M \in \{-1,0,1\}^{m \times n} \) is the edge-vertex incidence matrix of \( G \), and \( L^\dagger \in \mathbb{Q}^{n \times n} \) is the Moore–Penrose inverse of the Laplacian of \( G \), which can be obtained as \( \left(L + \frac{1}{2}J_n\right)^{-1} - \frac{1}{n}J_n \) by Gaussian elimination in polynomial time ([Edmonds, 1967; Schrijver, 1999]). Since \( m \leq n^2 \), \( \det(C_{S,S}) \) for \( S \subseteq E \) is within the range between \( 2^{-n^2} \) and 1 if a spanning tree exists that contains \( S \) and 0 otherwise. It turns out that \( \det(A_{S,S})\det(B_{S,S})\det(C_{S,S}) \) for \( S \in \binom{E}{n-1} \) is positive if and only if \( S \) is a Hamiltonian path.

Redefine \( \epsilon \leftarrow [1/\epsilon]^{-1} \), which does not decrease the value of \( \epsilon \), and \( A \leftarrow \theta A \), where \( \theta \doteq 2^{n^{4/\epsilon}} \in \mathbb{N} \). Since each entry of \( A \) is an integer at most \( \theta \) and each entry of \( B \) is 1, we have that \( \text{size}(A) = O(m^2 \log(2^{n^{1/\epsilon}})) = O(n^{4/\epsilon} + 4) \) and \( \text{size}(B) = O(n^4) \). Since \( \text{size}(X^{-1}) = O(\text{size}(X)n^2) \) for any \( n \times n \) nonsingular matrix \( X \) ([Schrijver, 1999]) and \( \text{size}(L + \frac{1}{2}J) = O(n^4 \log n) \), we have that \( \text{size}(L^\dagger) = O(n^4 \log n) \), and thus \( \text{size}(C) = m^2O(n^4 \log n) = O(n^8 \log n) \). Consequently, the input size is bounded by \( |\mathcal{I}| = O(n^{4/\epsilon} + 4) + O(n^4) + O(n^8 \log n) = O(n^{4/\epsilon} + 4) \), a polynomial in \( n \) (for fixed \( \epsilon < 1 \)).

Now, we explain how to use \( Z_3 \) to decide the Hamiltonicity of \( G \). The value of \( \det(A_{S,S})\det(B_{S,S})\det(C_{S,S}) \) for edge set \( S \subseteq E \) is 0 whenever \( |S| \geq n \) or \( |S| = n - 1 \) but \( S \) is not a Hamiltonian path. Then, \( Z_3(A, B, C) \) can be decomposed into two sums

\[
\sum_{S: |S| < n-1} \det(A_{S,S})\det(B_{S,S})\det(C_{S,S}) + \sum_{S: \text{Hamiltonian}} \det(A_{S,S})\det(B_{S,S})\det(C_{S,S}).
\]

There are two cases:

- **Case (1)** if there exists (at least) one Hamiltonian path \( S^* \) in \( G \), then \( Z_3(A, B, C) \) is at least \( \theta^{|S^*|}2^{-n^2} = 2^{n^{4/\epsilon} + 1} - n^{4/\epsilon} - n^2 \).

- **Case (2)** if no Hamiltonian path exists in \( G \), then \( Z_3(A, B, C) \) is at most \( \sum_{S: |S| < n-1} \theta^n - 2 \leq 2n^22^{n^{4/\epsilon}(n-2)} = 2^{n^{4/\epsilon} + 1} - 2n^{4/\epsilon} + n^2 \).
Hence, there is an exponential gap $2^{n^{4/3}-2n^2}$ between the two cases. Since $|I|^{1-\epsilon} = O(n^{(4/3)-4\epsilon})$, a $2^{O(|I|^{1-\epsilon})}$- or $2^{O(n^{1/\epsilon})}$-approximation to $Z_3$ suffices to distinguish the two cases (for sufficiently large $n$).

Now let us prove the second argument. Assume that $G$ is Hamiltonian. Observe that a random edge set drawn from the $\Pi$-DPP defined by $A, B, C$ (denoted $\mu$) is Hamiltonian with probability at least $1 - 1/(1+2^{n^{4/3}-2n^2})$. Hence, provided a polynomial-time algorithm to generate random edge sets whose total variation distance from $\mu$ is at most $\frac{2}{3}$, we can use it to verify the Hamiltonicity of $G$ with probability at least $\frac{2}{3} - \frac{1}{1+2^{n^{4/3}-2n^2}} > \frac{1}{2}$ (whenever $n \geq 2$), implying that $\text{HAMILTONIANPath} \in \text{RP}$; hence, $\text{RP} = \text{NP}$. This completes the proof.

\[\square\]

### 5.2 Exponential Approximability

Whereas making a subexponential approximation for $Z_3$ in terms of the input size $|I|$ is hard, we show that there is a simple exponential approximation for $Z_m$ for all $m$.

**Observation 5.2.** For $m$ $P_0$-matrices $A^1, \ldots, A^m$, the number $1$ is a $2^{O(|I|^2)}$-approximation to $Z_m(A^1, \ldots, A^m)$, where $|I|$ is the input size.

**Proof.** Obviously, $Z_m$ is bounded from below by $1$, so we only have to show an upper bound. Applying Hadamard’s inequality, we find that all principal minors are at most $M^n n^{n/2}$, where $M$ is the maximum absolute entry in the $m$ matrices. Hence, we have that

$$Z_m(A^1, \ldots, A^m) = \sum_{S \subseteq [n]} \det(A^1_{\mathcal{S}, \mathcal{S}}) \cdots \det(A^m_{\mathcal{S}, \mathcal{S}})$$

$$\leq 2^m (M^n n^{n/2})^m$$

$$= 2^{n+mn} \log M + \frac{mn}{2} \log n = 2^{O(|I|^2)},$$

where the last deformation comes from the fact that $|I| \geq \log M$ and $|I| \geq mn^2$. Thus, $Z_m(A^1, \ldots, A^m)$ takes a number between $1$ and $2^{O(|I|^2)}$, which completes the proof. \[\square\]

### 5.3 Approximation-Preserving Reduction from Mixed Discriminant to Two Matrices

Finally, we present a relation between $Z_2$ and the mixed discriminant. The **mixed discriminant** of $m$ positive semi-definite matrices $K^1, \ldots, K^m \in \mathbb{R}^{m \times m}$ is defined as

$$D(K^1, \ldots, K^m) \triangleq \frac{1}{\prod_{i=1}^m} \det(x_1K^1 + \cdots + x_mK^m).$$

Mixed discriminants are known to be a generalization of permanents (Barvinok, 2016): for an $m \times m$ matrix $A$, we define $m \times m$ matrices $K^1, \ldots, K^m$ such that $K^i = \text{diag}(A_{i1}, \ldots, A_{im})$ for all $i \in [m]$; it holds that $D(K^1, \ldots, K^m) = \text{per}(A)$. Hence, computing the mixed discriminant is #P-hard. We demonstrate an approximation-preserving reduction from the mixed discriminant $D$ to $Z_2$, which means that if $Z_2$ admits an FPRAS, then so does the mixed discriminant. Since the existence of an FPRAS for the mixed discriminant is suspected to be false (Gurvits, 2005), our result implies that $Z_2$ is unlikely to have an FPRAS. We stress that Gillenwater (2014) proves the
The mixed discriminant $D$ for $m$ positive semi-definite matrices in $Q_{m \times m}$ is $AP$-reducible to $Z_2$ for two positive semi-definite matrices in $Q_{n^2 \times n^2}$. Therefore, if there exists an FPRAS for $Z_2$, then there exists an FPRAS for $D$.

**Proof.** We will construct an AP-reduction from the mixed discriminant $D$ to $Z_2$ mimicking the reduction from $D$ to spanning-tree DPPs presented by the same set of authors as this article (Matsuoka and Ohsaka, 2021). Suppose we have an FPRAS for $Z_2$. Let $K^1, \ldots, K^m$ be $m$ positive semi-definite matrices in $Q_{m \times m}$, and define $n = m^2$. Let $\varepsilon \in (0, 1)$ be an error tolerance for $D$; i.e., we are asked to estimate $D(K^1, \ldots, K^m)$ within a factor of $e^\varepsilon$. In accordance with Celis et al. (2017, Proof of Lemma 12), we first construct an $n \times n$ positive semi-definite matrix $A$ and an equal-sized partition of $[n]$, denoted $P_1, P_2, \ldots, P_m$ with $|P_1| = |P_2| = \cdots = |P_m| = m$, in polynomial time such that

$$
\sum_{S \in C} \det(A_{S,S}) = m! \ D(K^1, \ldots, K^m),
$$

where we define $C \triangleq \{S \in \binom{[n]}{m} \mid |S \cap P_i| = 1 \text{ for all } i \in [m]\}$.

Then, we construct an $n \times n$ positive semi-definite matrix $B$ as follows:

$$
B_{ij} \triangleq \begin{cases} 
1 & \text{if } i \text{ and } j \text{ belong to the same group in the partition of } [n], \\
0 & \text{otherwise.}
\end{cases}
$$

We claim the following:

**Claim 5.5.** For each subset $S \subseteq [n]$, $\det(B_{S,S})$ is 1 if no two elements in $S$ belong to the same group in the partition and 0 otherwise. In particular, we have that
• for any \( S \in \binom{[n]}{m} \), \( \det(B_{S,S}) = 1 \) if and only if \( S \in \mathcal{C} \);

• \( \det(B_{S,S}) \) is 0 whenever \( |S| > m \).

Proof of Claim 5.5. Fix a subset \( S \subseteq [n] \). If \( S \) contains two elements \( i, j \) that belong to the same group in the partition, we have that \( B_{S,\{i\}} = B_{S,\{j\}} \); hence \( \det(B_{S,S}) = 0 \). On the other hand, if \( S \) does not contain two such elements, \( B_{S,S} \) is exactly an \( S \times S \) identity matrix; hence, \( \det(B_{S,S}) = 1 \).

The following equality is a direct consequence of Claim 5.5.

\[
\sum_{S \in \binom{[n]}{m}} \det(A_{S,S}) \det(B_{S,S}) = \sum_{S \in \mathcal{C}} \det(A_{S,S}).
\]

(5)

First, we verify whether a subset \( S \in \mathcal{C} \) exists such that \( \det(A_{S,S}) > 0 \) because otherwise, we can safely declare that Eq. (5) is 0; i.e., \( D(K^1, \ldots, K^m) \) is 0 as well. Such a subset can be found (if it exists) by performing matroid intersection because \( \mathcal{I}_1 = \{ S \subseteq [n] \mid \det(A_{S,S}) > 0 \} \) forms a linear matroid and \( \mathcal{I}_2 = \{ S \subseteq [n] \mid \exists T \in \mathcal{C}, S \subseteq T \} \) forms a partition matroid. Denote the subset found by \( \tilde{S} \in \mathcal{I}_1 \cap \mathcal{I}_2 \). Next, we define a positive number \( x \in \mathbb{Q} \) as

\[
x \triangleq \frac{\det(A + I_n)}{\det(A_{\tilde{S},\tilde{S}})}\cdot \varepsilon.
\]

Note that the size of \( x \) is bounded by a polynomial in the size of \( A \) and \( \varepsilon^{-1} \). It is easy to see that for each \( S \subseteq [n] \),

\[
\det((xB)_{S,S}) = x^{|S|} \det(B_{S,S}).
\]

(6)

Since Claim 5.5 ensures that \( \det(A_{S,S}) \det(B_{S,S}) = 0 \) whenever \( |S| > m \), we can bound \( Z_2(A, xB) \) from above as follows:

\[
Z_2(A, xB) = \sum_{S \subseteq [n]} \det(A_{S,S}) \det((xB)_{S,S})
\]

\[
= \sum_{S: |S| \leq m-1} x^{|S|} \det(A_{S,S}) \det(B_{S,S}) + \sum_{S: |S| = m} x^{|S|} \det(A_{S,S}) \det(B_{S,S})
\]

\[
\leq \sum_{S: |S| \leq m-1} x^{m-1} \det(A_{S,S}) + \sum_{S \in \mathcal{C}} x^m \det(A_{S,S})
\]

\[
\leq x^m \left( \sum_{S \in \mathcal{C}} \det(A_{S,S}) \right) \left( 1 + \frac{\sum_{S: |S| \leq m-1} \det(A_{S,S})}{\sum_{S \in \mathcal{C}} \det(A_{S,S})} \frac{1}{x} \right)
\]

\[
\leq x^m \left( \sum_{S \in \mathcal{C}} \det(A_{S,S}) \right) \left( 1 + \frac{\sum_{S: |S| \leq m-1} \det(A_{S,S})}{\sum_{S \in \mathcal{C}} \det(A_{S,S})} \frac{\det(A_{\tilde{S},\tilde{S}})}{\det(A + I_n)} \frac{\varepsilon}{2} \right).
\]

Observing the fact that

\[
\sum_{S: |S| \leq m-1} \det(A_{S,S}) \leq \det(A + I_n) \quad \text{and} \quad \det(A_{\tilde{S},\tilde{S}}) \leq \sum_{S \in \mathcal{C}} \det(A_{S,S}),
\]

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we further have that
\[ Z_2(A, xB) \leq \left(1 + \frac{\epsilon}{2}\right) x^n \sum_{S \in C} \det(A_{S,S}) \]
\[ \leq e^\frac{\epsilon}{2} x^m m! D(K^1, \ldots, K^m). \quad \text{(by Eq. (4))} \]

Since \( Z_2(A, xB) \geq x^m m! D(K^1, \ldots, K^m) \), we have that
\[ D(K^1, \ldots, K^m) \leq \frac{Z_2(A, xB)}{x^m m!} \leq e^\frac{\epsilon}{2} D(K^1, \ldots, K^m). \quad (7) \]

We are now ready to describe the AP-reduction from the mixed discriminant \( D \) to \( Z_2 \).

**AP-reduction from \( D \) to \( Z_2 \).**

- **Step 1.** construct two matrices \( A, B \in Q^{n \times n} \) satisfying Eqs. (4) and (5) by following the procedure described at the beginning of the proof.

- **Step 2.** determine if there exists a subset \( S \subseteq [n] \) such that \( S \in C \) and \( \det(A_{S,S}) > 0 \) by matroid intersection in polynomial time (Edmonds, 1970). If no such subset has been found, declare that “\( D(K^1, \ldots, K^m) = 0 \)”; otherwise, denote the subset found by \( \tilde{S} \).

- **Step 3.** calculate the value of \( x \) according to Eq. (6), which can be done in polynomial time in the input size and \( \epsilon^{-1} \) because the size of \( A \) is bounded by a polynomial in the size of \( K^1, \ldots, K^m \) and the determinant can be computed in polynomial time by Gaussian elimination (Edmonds, 1967; Schrijver, 1999).

- **Step 4.** call an oracle for \( Z_2 \) on \( A \) and \( xB \) with error tolerance \( \delta = \epsilon/2 \) to obtain an \( e^{\epsilon/2} \)-approximation to \( Z_2(A, xB) \), which will be denoted by \( \hat{Z} \).

- **Step 5.** output \( \frac{\hat{Z}}{x^m m!} \) as an estimate for \( D(K^1, \ldots, K^m) \).

By Eq. (7), if the oracle meets the specifications for an FPRAS for \( Z_2 \), then the output \( \hat{Z} \) of the AP-reduction described above satisfies that
\[ e^{-\epsilon} \cdot D(K^1, \ldots, K^m) \leq \frac{\hat{Z}}{x^m m!} \leq e^\epsilon \cdot D(K^1, \ldots, K^m). \]

with probability at least \( \frac{3}{4} \). Therefore, the AP-reduction meets the specification for an FPRAS for \( D \), which completes the proof. \( \square \)

### 6 Fixed-Parameter Tractability

Here, we investigate the fixed-parameter tractability of computing \( Z_m \). Given a parameter \( k \), a problem is said to be **fixed-parameter tractable** (FPT) and **slice-wise polynomial** (XP) if it is solvable in
\( f(k) \mid I \mid^{O(1)} \) and \( \mid I \mid^{f(k)} \) time for some computable function \( f \), respectively. It should be noted that the value of \( k \) may be independent of the input size \( \mid I \mid \) and may be given by some computable function \( k = k(I) \) on input \( I \) (e.g., the rank of an input matrix). Our goal is either (1) to develop an FPT algorithm for an appropriate parameter, or (2) to disprove the existence of such algorithms under plausible assumptions.

### 6.1 Parameterization by Maximum Rank

First, let us consider the maximum rank of matrices as a parameter. The theorem below demonstrates that computing \( Z_2(A, B) \) for two positive semi-definite matrices \( A \) and \( B \) parameterized by the maximum rank is FPT.

**Theorem 6.1.** Let \( A, B \) be two positive semi-definite matrices in \( Q^{n \times n} \) of rank at most \( r \). Then, there exists an \( r^{O(r)} n^{O(1)} \)-time algorithm computing \( Z_2(A, B) \) exactly.

Before proceeding to the proof, we introduce the following technical lemma.

**Lemma 6.2.** Let \( A^1, \ldots, A^m \) be \( m \) matrices in \( Q^{n \times s} \), and \( \sigma_1, \ldots, \sigma_m \in S_s \) be \( m \) permutations over \([s]\). Then,

\[
\sum_{S \subseteq \{1,\ldots,m\}} A^1_S(\sigma_1) \cdots A^m_S(\sigma_m) = \sum_{S \subseteq \{1,\ldots,m\}} \prod_{i \in [s]} (A^1_S)_{i,\sigma_1(i)} \cdots (A^m_S)_{i,\sigma_m(i)}
\]

can be computed in \( O(msn^2) \) time.

**Proof.** The proof is based on dynamic programming. First, we define a table \( dp \) of size \( s \times n \), whose entries for each pair of \( \ell \in [s] \) and \( o \in [n] \) are defined as

\[
dp[\ell, o] = \sum_{S \subseteq \{1,\ldots,m\}} \prod_{i \in [\ell]} (A^1_S)_{i,\sigma_1(i)} \cdots (A^m_S)_{i,\sigma_m(i)}.
\]

The desired value is equal to \( \sum_{o \in [n]} dp[s, o] \). Observe that for \( \ell \in [2..s] \) and \( o \in [n] \),

\[
dp[\ell, o] = \sum_{1 \leq o' < o} dp[\ell - 1, o'] A^1_{o',\sigma_1(\ell)} \cdots A^m_{o',\sigma_m(\ell)}.
\]

\[
dp[1, o] = A^1_{o,\sigma_1(1)} \cdots A^m_{o,\sigma_m(1)}.
\]

Note that the number of bits required to express each entry is bounded by \( \log(2^n)(\text{size}(A^1) + \cdots + \text{size}(A^m)) \). Since calculating \( dp[\ell, o] \) given \( dp[\ell - 1, o'] \) for all \( o' \in [n] \) requires \( O(nm) \) arithmetic operations, standard dynamic programming fills all entries of \( dp \) within \( O(msn^2) \) arithmetic operations.

Next, we introduce the Cauchy–Binet formula.

**Lemma 6.3 (Cauchy–Binet formula).** Let \( A \) be an \( s \times r \) matrix and \( B \) be an \( r \times s \) matrix. Then, the determinant of \( AB \) is

\[
\det(AB) = \sum_{C \subseteq \{1,\ldots,s\}} \det(A_{[s],C}) \det(B_{C,[s]}).
\]

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Proof of Theorem 6.1. We decompose $A$ into two $n \times r$ rectangular matrices. For this purpose, we first compute an LDL decomposition\(^\text{13}\) $A = LDL^\top$, where $L \in \mathbb{Q}^{n \times n}$ and $D \in \mathbb{Q}^{n \times n}$ is a diagonal matrix such that $D_{i,i} = 0$ for all $i \in [r+1 .. n]$ (since the rank is at most $r$). This is always possible in polynomial time (O’Donnell and Ta, 2011) because $A$ is positive semi-definite. We further decompose $D$ into the product of an $n \times r$ matrix $C$ such that $C_{i,i} = D_{i,i}$ for all $i \in [r]$ and all the other elements are 0, and an $r \times n$ matrix $I$ such that $I_{i,i} = 1$ for all $i \in [r]$ and all the other elements are 0. Setting $U = LC \in \mathbb{Q}^{n \times r}$ and $V = LI^\top \in \mathbb{Q}^{n \times r}$, we have that $A = UV^\top$. Similarly, we decompose $B = XY^\top$, where $X$ and $Y$ are some $n \times r$ rectangular matrices in $\mathbb{Q}^{n \times r}$.

Because $\det(A_{S,S}) \det(B_{S,S}) = 0$ for all $S \subseteq [n]$ of size greater than $r$, we can expand $Z_2(A, B)$ by using the Cauchy–Binet formula as follows.

\[
Z_2(A, B) = \sum_{0 \leq s \leq r} \sum_{S \subseteq [n]} \det(U_S V_S^\top) \det(X_S Y_S^\top)
\]

\[
= \sum_{0 \leq s \leq r} \sum_{S \subseteq [n]} \det(U_{S,C_1} V_{S,C_1}^\top) \sum_{C_2 \subseteq [n]} \det(X_{S,C_2} Y_{S,C_2}^\top).
\]

Noting that $|S| = |C_1| = |C_2|$, we further expand $Z_2(A, B)$ as

\[
\sum_{0 \leq s \leq r} \sum_{C_1,C_2 \subseteq [n]} \sgn(\sigma_1) \sgn(\tau_1) \sgn(\sigma_2) \sgn(\tau_2) \times \\
\sum_{S \subseteq [n]} U_{S,C_1}(\sigma_1) V_{S,C_1}(\tau_1) X_{S,C_2}(\sigma_2) Y_{S,C_2}(\tau_2).
\]

Since $\star$ can be evaluated in $O(sn^2)$ time by Lemma 6.2, we can take the sum of $\star$ over all possible combinations of $s, C_1, C_2, \sigma_1, \tau_1, \sigma_2, \tau_2$ in $\sum_{0 \leq s \leq r} \binom{n}{s}^2 (s!)^4 O(sn^2) = O(r^4 r^2 n^2)$ time. Consequently, the overall computation time is bounded by $r^4 r^2 n^2$. \hfill $\square$

Theorem 6.1 can be generalized to the case of $m$ matrices $A^1, \ldots, A^m$; that is, the computation of $Z_m$ parameterized by the maximum rank $\max_{i \in [m]} \text{rank}(A^i)$ plus the number of matrices $m$ is FPT.

**Theorem 6.4.** For a positive integer $m$, let $A^1, \ldots, A^m$ be $m$ positive semi-definite matrices in $\mathbb{Q}^{n \times n}$ of rank at most $r$. Then, there exists an $r^{O(mr)} n^{O(1)}$-time algorithm computing $Z_m(A^1, \ldots, A^m)$ exactly.

\(^\text{13}\)We do not use the Cholesky decomposition because we must avoid the square root computation, which violates the assumption that every number appearing in this paper is rational.
Proof. Similar to the proof of Theorem 6.1, we first decompose each of \( m \) matrices into the product of two \( n \times r \) rectangular matrices, i.e., \( A^i = X^i(Y^i)^\top \) for some \( X^i, Y^i \in \mathbb{Q}^{n \times r} \) for all \( i \in [m] \), by LDL decomposition. We then expand \( Z_m(A^1, \ldots, A^m) \) as

\[
\sum_{S \subseteq \mathbb{R}} \det(A^1_{S,S}) \cdots \det(A^m_{S,S}) \\
= \sum_{0 \leq s \leq r} \sum_{S \in \mathcal{P}(S)} \det(X^1_{S,C_1}) \det((Y^1_{S,C_1})^\top) \cdots \det(X^m_{S,C_m}) \det((Y^m_{S,C_m})^\top) \\
= \sum_{0 \leq s \leq r} \sum_{C_1 \in \mathcal{P}(S)} \cdots \sum_{C_m \in \mathcal{P}(S)} \det(X^1_{S,C_1}) \cdots \det(X^m_{S,C_m}) \\
\sum_{S \in \mathcal{P}(S)} \prod_{i \in \mathbb{R}} (X^1_{S,C_1} i_{\sigma_1(i)}(Y^1_{S,C_1}) i_{\tau_1(i)}) \cdots (X^m_{S,C_m} i_{\sigma_m(i)}(Y^m_{S,C_m}) i_{\tau_m(i)}),
\]

By applying Lemma 6.2, we can calculate \( \bullet \) in \( O(msn^2) \) time; thus, the entire time complexity is bounded by

\[
\sum_{0 \leq s \leq r} \binom{r}{s} 2^{mn} (s!)^2 m O(msn^2) = O(r^{2mr}r^2n^2) = O(r^2n^2).
\]

which completes the proof. \( \square \)

6.2 Parameterization by Treewidth of Union

Now we consider the treewidth of the graph formed by the union of nonzero entries as a parameter. The following theorem demonstrates that computing \( Z_2(A, B) \) parameterized by \( \text{tw}(\text{nz}(A) \cup \text{nz}(B)) \) is FPT.

**Theorem 6.5.** Let \( A, B \) be two matrices in \( \mathbb{Q}^{n \times n} \). Then, there exists a \( w^{O(w)}/n^{O(1)} \)-time algorithm that, given a tree decomposition of the graph \( ([n], \text{nz}(A) \cup \text{nz}(B)) \) of width at most \( w \), computes \( Z_2(A, B) \) exactly.

**Remark 6.6.** To construct “reasonable” tree decompositions, we can use existing algorithms, e.g., a \( 2^{O(w)}n \)-time 5-approximation algorithm by Bodlaender, Drange, Dregi, Fomin, Lokshtanov, and Pilipczuk (2016), where \( n \) is the number of vertices and \( w \) is the treewidth. Hence, we do not need to be given a tree decomposition to use the algorithm of Theorem 6.5.

6.2.1 Design of Dynamic Programming

Our proof is based on dynamic programming upon a tree decomposition. First, we define a nice tree decomposition \( (T, \{X_t\}_{t \in T}) \) formally, which is a convenient form of tree decomposition. Think of \( T \) as a rooted tree by referring to a particular node \( r \) as the root of \( T \), which naturally introduces the notions of parent, child, and leaf.

**Definition 6.7 (Kloks, 1994, nice tree decomposition).** A tree decomposition \( (T, \{X_t\}_{t \in T}) \) rooted at \( r \) is said to be nice if
Figure 1: Matrix $A \in \mathbb{Q}^{6 \times 6}$, where 

$$
\begin{bmatrix}
* & * & * & 0 & 0 & 0 \\
* & * & * & * & 0 & 0 \\
* & * & 0 & * & * & 0 \\
0 & 0 & * & 0 & 0 & * \\
0 & * & * & * & * & * \\
0 & 0 & 0 & 0 & 0 & * \\
\end{bmatrix}
$$

where "*" denotes nonzero entries.

Figure 2: Graph $G = (V, E)$ constructed from the nonzero entries of $A$, where $V = [6]$ and $E = \text{nz}(A)$.

Figure 3: Tree decomposition $(T, \{X_t\}_{t \in T})$ of $G$. $T$ contains four nodes, and bags are of size 3; i.e., its treewidth is 2.

Figure 4: Nice tree decomposition of $G$. This decomposition is essentially identical to $(T, \{X_t\}_{t \in T})$, but this representation makes easier to develop and analyze dynamic programming algorithms.

- every leaf and the root have empty bags; i.e., $X_r = \emptyset$ and $X_\ell = \emptyset$ for every leaf $\ell$ of $T$;
- each non-leaf node is one of the following:
  - $\text{Introduce node}$: a node $t$ with exactly one child $t'$ such that $X_t = X_{t'} + \nu$ for some $\nu \notin X_{t'}$.
  - $\text{Forget node}$: a node $t$ with exactly one child $t'$ such that $X_t = X_{t'} - \nu$ for some $\nu \in X_{t'}$.
  - $\text{Join node}$: a node $t$ with exactly two children $t', t''$ such that $X_t = X_{t'} = X_{t''}$.

For a node $t$ of $T$, we define

$$
V_t = \bigcup_{t' \text{ in subtree rooted at } t} X_{t'}.
$$

In particular, it holds that $V_r = [n]$ for the root $r$, and $V_\ell = X_\ell = \emptyset$ for every leaf $\ell$ of $T$. Figures 1 to 4 show an example of a (nice) tree decomposition.

Next we design dynamic programming tables. Given a nice tree decomposition $(T, \{X_t\}_{t \in T})$ of the graph $([n], \text{nz}(A) \cup \text{nz}(B))$, we assume to be given an ordering $\prec_t$ on $V_t$ for node $t$ of $T$. 

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whose definition is deferred to Section 6.2.2. We aim to compute the following quantity for each node $t$:

$$\sum_{S \subseteq V_t \setminus X_t \atop O_{A1}, O_{A2} \subseteq X_t; |O_{A1}| = |O_{A2}|; \sigma: S \mapsto \tau \mapsto O_{B1}, O_{B2} \subseteq X_t; |O_{B1}| = |O_{B2}|; \nu \mapsto \text{sgn}_{\nu}(\sigma) \text{sgn}_{\nu}(\sigma_B) A(\sigma_A) B(\sigma_B).}$$

(8)

Recall that $A(\sigma) \equiv \prod_i A_{L^r(i)}$ for bijection $\sigma$. In particular, Eq. (8) is equal to $Z_2(A, B)$ at the root $r$ since $X_r = \emptyset$ and $V_r = [n]$. We then discuss how to group exponentially many bijections into an FPT number of bins. A configuration for node $t$ is defined as a tuple $C = (O_1, O_2, F_1, F_2, \tau, \nu)$, where

- $O_1, O_2 \subseteq X_t$ are subsets such that $|O_1| = |O_2|$
- $F_1 \subseteq O_1, F_2 \subseteq O_2$ are subsets such that $|F_1| = |F_2|$
- $\tau : O_1 \setminus F_1 \rightarrow O_2 \setminus F_2$ is a bijection;
- $\nu \in \{0, 1\}$ is the parity of inversion number.

In the remainder of this subsection, arithmetic operations on the parity of inversion number to be performed over modulo 2, and the symbol $\equiv$ means congruence modulo 2. We say that a bijection $\sigma$ is consistent with $S \subseteq V_t \setminus X_t$ and $C = (O_1, O_2, F_1, F_2, \tau, \nu)$ if

- $\sigma$ is a bijection $S \mapsto O_1 \mapsto S \mapsto O_2$;
- $F_1 = \sigma^{-1}(S) \cap O_1$ and $F_2 = \sigma(S) \cap O_2$;
- $\tau = \sigma|_{O_1 \setminus F_1}$;
- $\nu \equiv \text{inv}_{\nu}(\sigma)$.

We show that for any bijection appearing in Eq. (8), there exists a unique pair of a subset $S$ and a configuration $C$ that is consistent with the bijection.

**Lemma 6.8.** Let $S \subseteq V_t \setminus X_t$ and $O_1, O_2 \subseteq X_t$ be two subsets such that $|O_1| = |O_2|$. For any bijection $\sigma : S \mapsto O_1 \mapsto S \mapsto O_2$, there exists a unique configuration $C$ for $t$ that $\sigma$ is consistent with.

**Proof.** Since $\sigma$ is a bijection, we can let $F_1 \triangleq \sigma^{-1}(S) \cap O_1$ $F_2 \triangleq \sigma(S) \cap O_2$, $\tau \triangleq \sigma|_{O_1 \setminus F_1}$, and $\nu \triangleq \text{inv}_{\nu}(\sigma)$. Then, $\sigma$ must be consistent with $S$ and $C$. Uniqueness is obvious from the definition of configuration and consistency. \qed

Hereafter, we will use $\Sigma(S, C)$ to denote the set of all bijections consistent with a subset $S \subseteq V_t \setminus X_t$ and a configuration $C$ for a node $t$ of $T$. By Lemma 6.8, we have that

$$\big| \prod_{S \subseteq V_t \setminus X_t \atop \text{C for t}} \Sigma(S, C) = \left\{ \sigma : S \mapsto O_1 \mapsto S \mapsto O_2 \mid S \subseteq V_t \setminus X_t, O_1 \subseteq X_t, O_2 \subseteq X_t, |O_1| = |O_2| \right\}. \right.$$  

(9)

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We can thus express Eq. (8) as follows:

\[
\text{value of Eq. (8)} = \sum_{s \in V_t^\ell \setminus \{s\}} \sum_{\sigma_A \in \Sigma(s, C_A)} \sum_{\sigma_B \in \Sigma(s, C_B)} \text{sgn}_{s_A}(\sigma_A) \cdot \text{sgn}_{s_B}(\sigma_B) \cdot A(\sigma_A) \cdot B(\sigma_B)
\]

\[
= \sum_{c_A, c_B \in \mathcal{T}} (-1)^{v_A + v_B} \sum_{s \leq \{s\}_t} Y_{t,A}(s, C_A) \cdot Y_{t,B}(s, C_B),
\]

where we define \( Y_{t,A} \) and \( Y_{t,B} \) as

\[
Y_{t,A}(s, C_A) \triangleq \sum_{\sigma_A \in \Sigma(s, C_A)} A(\sigma_A),
\]

\[
Y_{t,B}(s, C_B) \triangleq \sum_{\sigma_B \in \Sigma(s, C_B)} B(\sigma_B).
\]

We now define a dynamic programming table \( dp_{t,s} \) for each node \( t \in T \) and each integer \( s \in [0..n] \) so as to store the following quantity with key \( [c_A \mid c_B] \):

\[
dp_{t,s} \left[ \begin{array}{c} c_A \\ c_B \end{array} \right] \triangleq \sum_{s \leq \{s\}_t} Y_{t,A}(s, C_A) \cdot Y_{t,B}(s, C_B).
\]

Since there are at most \( 2^{|X_i|}2^{|X_j|}2^{|X_k|}2^{|X_l|} \cdot |X_l|! \leq 16^w n! (w + 1)! \) possible configurations for node \( t \) by definition, \( dp_{t,s} \) contains at most \( w^{O(w)} \) entries. The number of bits required to represent each entry of \( dp_{t,s} \) is roughly bounded by \( \log(2^n n!) (\text{size}(A) + \text{size}(B)) = O((\text{size}(A) + \text{size}(B)) n \log n) \).

Having defined the dynamic programming table, we are ready to construct \( dp_{t,s} \) given already-filled \( dp_{t',s'} \) for children \( t' \) of \( t \) and \( s' \in [0..n] \); the proof is deferred to Section 6.2.2.

**Lemma 6.9.** Let \( t \) be a non-leaf node of \( T \), and \( s \in [0..n] \). Given \( dp_{t',s'} \) for all children \( t' \) of \( t \) and \( s' \in [0..n] \), we can compute each entry of \( dp_{t,s} \) in \( w^{O(w)} n^{O(1)} \) time.

**Proof of Theorem 6.5.** Our parameterized algorithm works as follows. Given a tree decomposition for \((|\mathcal{X}|, \text{nz}(A) \cup \text{nz}(B))\) of width at most \( w \), we transform it to a nice tree decomposition \((T, \{X_i\}_{i \in T})\) rooted at \( r \) of width at most \( w \) that has \( O(w n) \) nodes in polynomial time (Cygan et al., 2015). For every leaf \( \ell \) of \( T \), any configuration \((O_1, O_2, F_1, F_2, \tau, v)\) for \( \ell \) satisfies that \( O_1 = O_2 = F_1 = F_2 = \emptyset \) and \( \tau : \emptyset \rightarrow \emptyset \) because \( X_\ell = \emptyset \). We thus initialize \( dp_{t,s} \) so that

\[
dp_{t,s} \left[ \begin{array}{c} \emptyset, \emptyset, \emptyset, \emptyset, \emptyset \rightarrow \emptyset, v_A \\ \emptyset, \emptyset, \emptyset, \emptyset, \emptyset \rightarrow \emptyset, v_B \end{array} \right] = \begin{cases} 1 & \text{if } s = 0 \text{ and } v_A = v_B = 0, \\ 0 & \text{otherwise}. \end{cases}
\]

Then, for each non-leaf node \( t \), we apply Lemma 6.9 to fill \( dp_{t,s} \) using the already-filled \( dp_{t',s'} \) for all children \( t' \) of \( t \) in a bottom-up fashion. Completing dynamic programming, we compute \( \mathcal{Z}_2 \) using \( dp_{r,s} \) at the root \( r \) as follows:

\[
\mathcal{Z}_2(A, B) = \sum_{s \in [0..n], v_A, v_B \in \{0,1\}} (-1)^{v_A + v_B} dp_{r,s} \left[ \begin{array}{c} \emptyset, \emptyset, \emptyset, \emptyset, \emptyset \rightarrow \emptyset, v_A \\ \emptyset, \emptyset, \emptyset, \emptyset, \emptyset \rightarrow \emptyset, v_B \end{array} \right].
\]
Correctness follows from Lemmas 6.8 and 6.9. We finally bound the time complexity. Because $T$ has at most $O(wn)$ nodes, each table is of size $w^{O(w)}n^{O(1)}$, and each table entry can be computed in $w^{O(w)}n^{O(1)}$ time by Lemma 6.9, the whole time complexity is bounded by $w^{O(w)}n^{O(1)}$, thereby completing the proof.

Remark 6.10. Our dynamic programming implies that an FPT algorithm exists for permanental processes (Macchi, 1975) since it holds that

$$\sum_{S \subseteq [n]} \per(A_S) \per(B_S) = \sum_{s,t,\mu,\nu} d_{\mu,\nu} \per(s,\nu,\mu,\nu,\mu,\nu,\mu,\nu,\mu,\nu,\mu,\nu,\mu,\nu,\mu,\nu,\mu,\nu,\mu,\nu,\mu,\nu,\mu,\nu,\mu,\nu,\mu,\nu,\mu,\nu) = 0.$$ 

Theorem 6.5 can be generalized to the case of $m$ matrices $A^1, \ldots, A^m$. Computing $Z_m$ parameterized by the treewidth of $\text{nz}(A^1) \cup \cdots \cup \text{nz}(A^m)$ plus the number of matrices $m$ is FPT, whose proof is deferred to Section 6.2.3.

Theorem 6.11. For a positive integer $m$, let $A^1, \ldots, A^m$ be $m$ matrices in $Q^{n \times n}$. Then, there exists a $w^{O(mw)}n^{O(1)}$-time algorithm that, given a tree decomposition of the graph $([n], \cup_{i \in [m]} \text{nz}(A^i))$ of width at most $w$, computes $Z_m(A^1, \ldots, A^m)$ exactly.

6.2.2 Proof of Lemma 6.9

We first define an ordering $\prec_t$ on $V_t$ for each node $t$ of $T$.

Definition 6.12. An ordering $\prec_t$ on set $V_t$ for node $t$ is recursively defined as follows.

- If $t$ is a leaf: $\prec_t$ is just an ordering on the empty set $V_t = \emptyset$.
- If $t$ is an introduce node with one child $t'$ such that $X_t = X_{t'} + v$: Given $\prec_{t'}$ on set $V_{t'}$, we define $\prec_t$ on set $V_t = V_{t'} + v$ as follows:
  - $\prec_t$ and $\prec_{t'}$ agree on $V_t - v = V_{t'}$;
  - $V_t - v \prec_t \{v\}$.
- If $t$ is a forget node with one child $t'$ such that $X_t = X_{t'} - v$: Given $\prec_{t'}$ on set $V_{t'}$, we define $\prec_t$ on set $V_t = V_{t'}$ as follows:
  - $\prec_t$ and $\prec_{t'}$ agree on $V_t - v$;
  - $V_t \setminus X_t - v \prec_t \{v\}$.
- If $t$ is a join node with two children $t', t''$ such that $X_t = X_{t'} = X_{t''}$: Given $\prec_{t'}$ and $\prec_{t''}$ on set $V_{t'}$ and $V_{t''}$, respectively, we define $\prec_t$ on set $V_t = V_{t'} \cup V_{t''}$ as follows:
  - $V_t \setminus X_{t'} \prec_t V_{t'} \setminus X_{t'} \prec_t X_t$;
  - $\prec_t$ and $\prec_{t'}$ agree on $V_t \setminus X_{t'}$;
  - $\prec_t$ and $\prec_{t''}$ agree on $V_t \setminus X_{t''}$;
  - $\prec_t$ and $\prec_{t'}$ agree on $X_t = X_{t'} = X_{t''}$.
By construction, we have that $V_i \setminus X_t \prec_t X_i$ for every node $t$ of $T$. We have an auxiliary lemma that plays a role in updating the parity of inversion number.

**Lemma 6.13.** Let $C = (O_1, O_2, F_1, F_2, \tau, v)$ be a configuration for node $t$, and $\prec_x$ and $\prec_y$ be two orderings on $V_t$ such that

- $\prec_x$ and $\prec_y$ agree on $V_i \setminus X_t$ (i.e., $v \prec_x w$ if and only if $v \prec_y w$ for all $v, w \in V_i \setminus X_t$);
- $V_i \setminus X_t \prec_x X_i$ and $V_i \setminus X_t \prec_y X_i$ (i.e., $v \prec_x w$ and $v \prec_y w$ for all $v \in V_i \setminus X_t$ and $w \in X_t$).

Then, we can compute a 0-1 integer $\Delta = \Delta(C, \prec_x, \prec_y)$ in polynomial time such that $\text{inv}_{\prec_y}(\sigma) - \text{inv}_{\prec_x}(\sigma) \equiv \Delta$ for all $\sigma \in \Sigma(S, C)$ and $S \subseteq V_i \setminus X_t$.

**Proof.** Given two orderings $\prec_x$ and $\prec_y$ on $V_t$ that meet the assumption, we can construct a sequence of orderings, denoted $\prec^{(0)}, \prec^{(1)}, \ldots, \prec^{(i-1)}, \prec^{(i)}$, starting from $\prec^{(0)} = \prec^{(i-1)}$ and ending with $\prec_y = \prec^{(i)}$ such that each $\prec^{(i)}$ for $i \in [\ell]$ is obtained from $\prec^{(i-1)}$ by reversing the order of two consecutive elements of $X_i$ with regard to $\prec^{(i-1)}$; i.e., there exists a partition of $V_t$, denoted $P \cup \{v, w\} \cup Q$, where $v, w \in X_t$ and $Q \subseteq X_t$, such that

- $\prec^{(i-1)}$ and $\prec^{(i)}$ agree on $P \cup Q$;
- $P \prec^{(i-1)} \{v\} \prec^{(i-1)} \{w\} \prec^{(i-1)} Q$;
- $P \prec^{(i)} \{w\} \prec^{(i)} \{v\} \prec^{(i)} Q$.

Since it holds that

$$\text{inv}_{\prec_y}(\sigma) - \text{inv}_{\prec_x}(\sigma) = \sum_{i \in [\ell]} \text{inv}_{\prec^{(i)}}(\sigma) - \text{inv}_{\prec^{(i-1)}}(\sigma),$$

we hereafter assume the existence of a partition $P \cup \{v, w\} \cup Q$, where $v, w \in X_t$ and $Q \subseteq X_t$, such that

- $\prec_x$ and $\prec_y$ agree on $P \cup Q$;
- $P \prec_x \{v\} \prec_x \{w\} \prec_x Q$;
- $P \prec_y \{w\} \prec_y \{v\} \prec_y Q$.

Let $\sigma$ be a bijection in $\Sigma(S, C)$ for $S \subseteq V_i \setminus X_t$. In order for $\text{inv}_{\prec_x}(\sigma)$ and $\text{inv}_{\prec_y}(\sigma)$ to differ, one of the following conditions must be satisfied:

- $\sigma(v) \prec_x \sigma(w)$ and $\sigma(v) \prec_y \sigma(w)$;
- $\sigma(v) \succ_x \sigma(w)$ and $\sigma(v) \succ_y \sigma(w)$;
- $\sigma^{-1}(v) \prec_x \sigma^{-1}(w)$ and $\sigma^{-1}(v) \prec_y \sigma^{-1}(w)$;
- $\sigma^{-1}(v) \succ_x \sigma^{-1}(w)$ and $\sigma^{-1}(v) \succ_y \sigma^{-1}(w)$.

The value of $\Delta \equiv \text{inv}_{\prec_y}(\sigma) - \text{inv}_{\prec_x}(\sigma)$ can be determined based on the following case analysis:
We can determine which case \( \sigma \) falls into without looking into \( \sigma \), which completes the proof. □
We are now ready to prove Lemma 6.9. The proof is separated into the following three lemmas.

**Lemma 6.14.** Let \( t \) be an introduce node with one child \( t' \) such that \( X_t = X_{t'} + v \), and \( s \in [0 .. n] \). Given \( d_{p_{t',s}} \) for all \( s' \), we can compute each entry of \( d_{p_{t,s}} \) in \( n^{O(1)} \) time.

**Lemma 6.15.** Let \( t \) be a forget node with one child \( t' \) such that \( X_t = X_{t'} - v \), and \( s \in [0 .. n] \). Given \( d_{p_{t',s'}} \) for all \( s' \), we can compute each entry of \( d_{p_{t,s}} \) in \( w^{O(w)} n^{O(1)} \) time.

**Lemma 6.16.** Let \( t \) be a join node with two children \( t' \) and \( t'' \) such that \( X_t = X_{t'} = X_{t''} \), and \( s \in [0 .. n] \). Given \( d_{p_{t',s'}} \) and \( d_{p_{t'',s''}} \) for all \( s' \) and \( s'' \), respectively, we can compute each entry of \( d_{p_{t,s}} \) in \( w^{O(w)} n^{O(1)} \) time.

**Proof of Lemma 6.14.** Consider a bijection \( \sigma \in \Sigma(S, C) \) for \( S \subseteq V_t \setminus X_t \) and \( C = (O_1, O_2, F_1, F_2, \tau, v) \) for \( t \). Then, a restriction \( \sigma|_{V_t} \) may belong to \( \Sigma(S', C') \) for some \( S' \subseteq V_t \setminus X_{t'} \) and \( C' \) for \( t' \). We will show that such \( S' \) and \( C' \) can be determined independent of \( \sigma \).

Observe first that if \( v \in F_1 \) or \( v \notin F_2 \), then we can declare that \( Y_{t,A}(S, C) = 0 \); if this is the case, (1) any bijection \( \sigma \) in \( \Sigma(S, C) \) satisfies that \( \sigma(v) \in V_t \setminus X_t \) or \( \sigma^{-1}(v) \in V_t \setminus X_t \), while (2) \( A_{\{v\}, V_t \setminus X_t} \) and \( A_{V_t \setminus X_t, \{v\}} \) must be zero matrices by the separator property of a tree decomposition. Hereafter, we can safely assume that \( v \notin F_1 \) and \( v \notin F_2 \).

We first discuss the relation between \( \text{inv}_{<_t}(\sigma) \) and \( \text{inv}_{<_t}(\sigma|_{V_t}) \).

**Lemma 6.17.** Let \( C = (O_1, O_2, F_1, F_2, \tau, v) \) be a configuration for \( t \). Then, there exists a 0-1 integer \( \Delta = \Delta(C) \) such that \( v \equiv \text{inv}_{<_t}(\sigma|_{V_t}) + \Delta \) for any \( \sigma \in \Sigma(S, C) \) for \( S \subseteq V_t \setminus X_t \). Moreover, we can compute the value of \( \Delta \) in polynomial time.

**Proof.** Let \( \sigma \in \Sigma(S, C) \) for \( S \subseteq V_t \setminus X_t \). By Definition 6.12, we have that \( \text{inv}_{<_t}(\sigma|_{V_t}) = \text{inv}_{<_t}(\sigma|_{V_t}) \). Simple calculation yields that

\[
v \equiv \text{inv}_{<_t}(\sigma) = \left| \left\{ (u, w) \mid u \not<_t w, \sigma(u) \succ_t \sigma(w) \right\} \right| \\
= \left| \left\{ (u, w) \mid u \not<_t w, \sigma(u) \succ_t \sigma(w), \{u, w, \sigma(u), \sigma(w)\} \subseteq V_t \right\} \right| \\
\overset{\text{inv}_{<_t}(\sigma|_{V_t})}{=} \left| \left\{ (u, w) \mid u \not<_t w, \sigma(u) \succ_t \sigma(w), v \in \{u, w, \sigma(u), \sigma(w)\} \right\} \right| \equiv \text{inv}_{<_t}(\sigma|_{V_t}) + \Delta.
\]

Observing that \( V_t \not<_t \{v\} \), we can determine the value of \( \Delta \) based on the following case analysis:

- **Case (1)** \( v \notin O_1, v \notin O_2 \): Since \( \sigma = \sigma|_{V_t} \), we have that \( \Delta \equiv 0 \).

- **Case (2)** \( v \in O_1 \setminus F_1, v \notin O_2 \): Since \( \sigma(v) = \tau(v) \in X_t \), we have that
  \[
\Delta \equiv \left| \left\{ (u, v) \mid \sigma(u) \succ_t \sigma(v) \right\} \right| = \left| \left\{ w \in O_2 \mid w \succ_t \tau(v) \right\} \right|.
\]

- **Case (3)** \( v \notin O_1, v \in O_2 \setminus F_2 \): Since \( \sigma^{-1}(v) = \tau^{-1}(v) \in X_t \), we have that
  \[
\Delta \equiv \left| \left\{ \sigma^{-1}(v), w \mid \sigma^{-1}(v) \not<_t w \right\} \right| = \left| \left\{ w \in O_1 \mid w \not>_t \tau^{-1}(v) \right\} \right|.
\]
• **Case (4)** \( v \in O_1 \setminus F_1, v \in O_2 \setminus F_2, \sigma(v) = \tau(v) = v \): Since \( u \vartriangleright_u v \) and \( \tau(u) \vartriangleright_u \tau(v) \) for all \( u \in V \), we have that \( \Delta = 0 \).

• **Case (5)** \( v \in O_1 \setminus F_1, v \in O_2 \setminus F_2, \sigma(v) = \tau(v) \neq v \): We have that

\[
\Delta = |\{(u,v) \mid \sigma(u) \triangleright_u \sigma(v)\} \cup \{(\sigma^{-1}(v), w) \mid \sigma^{-1}(v) \triangleright_u w\} \cup \{(\sigma^{-1}(v), v)\}| \\
= |\{w \in O_2 \mid w \triangleright_u \tau(v)\}| + |\{w \in O_1 \mid w \triangleright_u \tau^{-1}(v)\}| + 1.
\]

We can determine which case \( C \) falls into and calculate \( \Delta \) in polynomial time, as desired.

We then define a mapping \( \mathcal{J} \) from a configuration for \( t \) to a configuration for \( t' \).

**Definition 6.18.** Let \( C = (O_1, O_2, F_1, F_2, \tau, v) \) be a configuration for \( t \) such that \( v \not\in F_1 \) and \( v \not\in F_2 \). We define \( \mathcal{J}(C) \) as follows:

---

**Definition of \( \mathcal{J}(C) \).**

Compute \( \Delta \) according to Lemma 6.17.

- **Case (1)** \( v \not\in O_1, v \not\in O_2 \): We define

\[
\mathcal{J}(C) \triangleq (O_1, O_2, F_1, F_2, \tau|_{X_u}, v - \Delta).
\]

- **Case (2)** \( v \not\in O_1 \setminus F_1, v \not\in O_2 \): We define

\[
\mathcal{J}(C) \triangleq (O_1 - v, O_2 - \tau(v), F_1, F_2, \tau|_{X_u}, v - \Delta).
\]

- **Case (3)** \( v \not\in O_1, v \in O_2 \setminus F_2 \): We define

\[
\mathcal{J}(C) \triangleq (O_1 - \tau^{-1}(v), O_1 - v, F_1, F_2, \tau|_{X_u}, v - \Delta).
\]

- **Case (4)** \( v \in O_1 \setminus F_1, v \in O_2 \setminus F_2, \tau(v) = v \): We define

\[
\mathcal{J}(C) \triangleq (O_1 - v, O_2 - v, F_1, F_2, \tau|_{X_u}, v - \Delta).
\]

- **Case (5)** \( v \in O_1 \setminus F_1, v \in O_2 \setminus F_2, \tau(v) \neq v \): We define

\[
\mathcal{J}(C) \triangleq (O_1 - v - \tau^{-1}(v), O_2 - \tau(v) - v, F_1, F_2, \tau|_{X_u}, v - \Delta).
\]

---

We claim the following:

**Claim 6.19.** For any \( C \) for \( t \), there exists unique \( C' \) for \( t' \) such that for any \( \sigma \in \Sigma(S, C) \) for \( S \subseteq V \setminus X \), it holds that \( \sigma|_{V_u} \in \Sigma(S, C') \). Moreover, \( C' = \mathcal{J}(C) \).

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Proof. The proof is immediate from Definition 6.18 and Lemma 6.17.

For any bijection $\sigma \in \Sigma(S,C)$ for $S \subseteq V_t \setminus X_t$ and $C = (O_t, O_2, F_1, F_2, \tau, \nu)$ for $t$, it holds that

$$A(\sigma) = A(\sigma|_{V_t}) \cdot \begin{cases} 1 & \text{Case (1)} \\ A_{v,\tau(v)} & \text{Case (2)} \\ A_{\tau^{-1}(v),\nu} & \text{Case (3)} \\ A_{v,\nu} & \text{Case (4)} \\ A_{v,\tau(v)} \cdot A_{\tau^{-1}(v),\nu} & \text{Case (5)} \end{cases}$$

By Claim 6.19, we have that

$$Y_{t,A}(S,C) = \sum_{\sigma \in \Sigma(S,C)} A(\sigma) = \sum_{\sigma' \in \Sigma(S,J(C))} A(\sigma') \cdot \begin{cases} 1 & \text{Case (1)} \\ A_{v,\tau(v)} & \text{Case (2)} \\ A_{\tau^{-1}(v),\nu} & \text{Case (3)} \\ A_{v,\nu} & \text{Case (4)} \\ A_{v,\tau(v)} \cdot A_{\tau^{-1}(v),\nu} & \text{Case (5)} \end{cases}$$

We have an analogue with regard to $Y_{t,B}(S,C)$. Observing that $V_t \setminus X_t = V_{t'} \setminus X_{t'}$, we finally obtain that for $C_A = (O_{A1}, O_{A2}, F_{A1}, F_{A2}, \tau_A, \nu_A)$ and $C_B = (O_{B1}, O_{B2}, F_{B1}, F_{B2}, \tau_B, \nu_B)$ for $t$,

$$dp_{t,S} \left[ \frac{C_A}{C_B} \right] = \sum_{S \in (V_t \setminus X_t)} Y_{t,A}(S,C_A) \cdot Y_{t,B}(S,C_B)$$

$$= dp_{t',S} \left[ \frac{J(C_A)}{J(C_B)} \right] \cdot \begin{cases} 1 & \text{Case (1)} \\ A_{v,\tau_A(v)} & \text{Case (2)} \\ A_{\tau_A^{-1}(v),\nu} & \text{Case (3)} \\ A_{v,\nu} & \text{Case (4)} \\ A_{v,\tau_A(v)} \cdot A_{\tau_A^{-1}(v),\nu} & \text{Case (5)} \end{cases} \begin{cases} 1 & \text{Case (1)} \\ B_{v,\tau_B(v)} & \text{Case (2)} \\ B_{\tau_B^{-1}(v),\nu} & \text{Case (3)} \\ B_{v,\nu} & \text{Case (4)} \\ B_{v,\tau_B(v)} \cdot B_{\tau_B^{-1}(v),\nu} & \text{Case (5)} \end{cases}$$

division into cases by $C_A$

division into cases by $C_B$

if $v \notin F_{A1} \cup F_{A2} \cup F_{B1} \cup F_{B2}$. Otherwise, it holds that $dp_{t,S} \left[ \frac{C_A}{C_B} \right] = 0$. Because evaluating $J(C_A)$ and $J(C_B)$ completes in $n^{O(1)}$ time, so does evaluating $dp_{t,S} \left[ \frac{C_A}{C_B} \right]$. \qed
**Proof of Lemma 6.15.** Consider a bijection $\sigma' \in \Sigma(S', C')$ for $S' \subseteq V_t' \setminus X_t'$ and $C'$ for $t'$. Since $V_t$ is equal to $V_{t'}$, $\sigma'$ may belong to $\Sigma(S, C)$ for some $S \subseteq V_t \setminus X_t$ and $C$ for $t$. We will show that if this is the case, such $S$ and $C$ can be determined independent of $\sigma'$.

We first discuss the relation between $\text{inv}_{\prec_t}(\sigma')$ and $\text{inv}_{\prec_{t'}}(\sigma')$.

**Lemma 6.20.** Let $C' = (O_1', O_2', F_1', F_2', \tau', v')$ be a configuration for $t'$. Then, there exists a 0-1 integer $\Delta = \Delta(C')$ such that $\text{inv}_{\prec_t}(\sigma') \equiv v' + \Delta$ for any $\sigma' \in \Sigma(S', C')$ for $S' \subseteq V_{t'} \setminus X_{t'}$. Moreover, we can compute the value of $\Delta$ in polynomial time.

**Proof.** The proof is a direct consequence of Lemma 6.13 since $\prec_t$ and $\prec_{t'}$ satisfy the conditions in Lemma 6.13.

We then define a mapping $\mathcal{F}$ from a set-configuration pair for $t'$ to a set-configuration pair for $t$.

**Definition 6.21.** Let $S' \subseteq V_{t'} \setminus X_{t'}$ and $C' = (O_1', O_2', F_1', F_2', \tau', v')$. Then, we define $\mathcal{F}(S', C')$ as follows:

**Definition of $\mathcal{F}(S', C')$.**

- **Case (1)** $v \notin O_1'$, $v \notin O_2'$: We define $\mathcal{F}(S', C') \triangleq (S', (O_1', O_2', F_1', F_2', \tau', v' + \Delta))$, where $\Delta$ is computed according to Lemma 6.20.

- **Case (2)** $v \in O_1'$, $v \in O_2'$: $F_1$ and $F_2$ are defined as follows:
  - **Case (2-1)** $v \in F_1'$, $v \in F_2'$: $F_1 \triangleq F_1' - v$, $F_2 = F_2' - v$.
  - **Case (2-2)** $v \notin F_1'$, $v \notin F_2'$: $F_1 \triangleq F_1' + \tau'(v) - v$, $F_2 \triangleq F_2' - v$.
  - **Case (2-3)** $v \in F_1'$, $v \notin F_2'$: $F_1 \triangleq F_1' + \tau^{-1}(v) - v$, $F_2 \triangleq F_2'$.
  - **Case (2-4)** $v \notin F_1'$, $v \notin F_2'$, $\tau(v) \neq v$: $F_1 \triangleq F_1' + \tau^{-1}(v)$, $F_2 \triangleq F_2' + \tau'(v)$.
  - **Case (2-5)** $v \notin F_1'$, $v \notin F_2'$, $\tau(v) = v$: $F_1 \triangleq F_1'$, $F_2 \triangleq F_2'$.

We define

$$\mathcal{F}(S', C') \triangleq (S' + v, (O_1' - v, O_2' - v, F_1, F_2, \tau'|_{X_t \setminus \tau^{-1}(X_t)}, v' + \Delta)),$$

where $\Delta$ is computed according to Lemma 6.20.

- **Case (3)** $v \notin O_1'$, $v \in O_2'$: $\mathcal{F}(S', C')$ is undefined.

- **Case (4)** $v \in O_1'$, $v \notin O_2'$: $\mathcal{F}(S', C')$ is undefined.

Here, we claim a kind of completeness and soundness of $\mathcal{F}$.

**Claim 6.22.** For any $S \subseteq V_t \setminus X_t$, $C$ for $t$, and $\sigma \in \Sigma(S, C)$, there exist unique $S' \subseteq V_{t'} \setminus X_{t'}$ and $C'$ for $t'$ such that $\sigma \in \Sigma(S', C')$. Moreover, $\mathcal{F}(S', C') = (S, C)$.
Definition 6.21
Lemma 6.20
and 6.23

\(\sigma\) obvious. The proof is immediate from Proof.

Claim 6.23. For any \(S\) \(\subseteq V_\tau \setminus X_\tau\), we construct \(S' \subseteq V_\tau \setminus X_\tau\) and \(C'\) for \(t'\) as follows:

**Case (1) \(v \notin S\):** We define \(S' \triangleq S\) and \(C' \triangleq (O_1, O_2, F_1, F_2, \tau, \nu, \text{inv}_{X_{t'}}(\sigma))\).

**Case (2) \(v \in S\):** \(F'_1\) and \(F'_2\) are defined as follows:

**Case (2-1) \(\sigma(v) \notin X_{t'}\), \(\sigma^{-1}(v) \notin X_{t'}\):** \(F'_1 \triangleq F_1 + v, F'_2 \triangleq F_2 + v\).

**Case (2-2) \(\sigma(v) \in X_{t'}\), \(\sigma^{-1}(v) \notin X_{t'}\):** \(F'_1 \triangleq F_1, F'_2 \triangleq F_2 - \sigma(v) + v\).

**Case (2-3) \(\sigma(v) \notin X_{t'}\), \(\sigma^{-1}(v) \in X_{t'}\):** \(F'_1 \triangleq F_1 - \sigma^{-1}(v) + v, F'_2 \triangleq F_2\).

**Case (2-4) \(\sigma(v) \in X_{t'}\), \(\sigma^{-1}(v) \in X_{t'}\), \(\sigma(v) \neq v\):** \(F'_1 \triangleq F_1 - \sigma^{-1}(v), F'_2 \triangleq F_2 - \sigma(v)\).

**Case (2-5) \(\sigma(v) \in X_{t'}\), \(\sigma^{-1}(v) \in X_{t'}\), \(\sigma(v) = v\):** \(F'_1 \triangleq F_1, F'_2 \triangleq F_2\).

We define \(S' \triangleq S - v\) and \(C' \triangleq (O_1 + v, O_2 + v, F'_1, F'_2, \sigma|_{X_{t'} \cap \sigma^{-1}(X_{t'})}, \text{inv}_{X_{t'}}(\sigma))\).

It is easy to verify that \(\sigma \in \Sigma(S', C')\) for all \(\sigma \in \Sigma(S, C)\) and \(\mathcal{H}(S', C') = (S, C)\). The uniqueness is obvious.

**Claim 6.23.** For any \(S' \subseteq V_\tau \setminus X_\tau\) and \(C'\) for \(t'\) such that \(\mathcal{H}(S', C')\) is defined, there exists unique \(S \subseteq V_\tau \setminus X_\tau\) and \(C\) for \(t\) such that any \(\sigma' \in \Sigma(S', C')\) belongs to \((S, C)\). Moreover, \(\mathcal{H}(S', C') = (S, C)\).

**Proof.** The proof is immediate from Definition 6.21 and Lemma 6.20.

Since \(\mathcal{H}\) determines a configuration for \(t\) based only on a configuration for \(t'\), we abuse the notation by writing \(\mathcal{H}(C') = C\) if there exist \(S' \subseteq V_\tau \setminus X_\tau\) and \(S \subseteq V_\tau \setminus X_\tau\) such that \(\mathcal{H}(S', C') = (S, C)\). By definition of \(\mathcal{H}\) and Claims 6.22 and 6.23, we have that for \(S \subseteq V_\tau \setminus X_\tau\) and \(C\) for \(t\),

\[
\Sigma(S, C) = \biguplus_{S' \subseteq V_\tau \setminus X_\tau, C' \text{for \(t'\) \(\mathcal{H}(C') = (S, C)\)}} \Sigma(S', C') = \begin{cases} \biguplus \Sigma(S, C') & \text{if } v \notin S, \\
\biguplus \Sigma(S - v, C') & \text{if } v \in S.
\end{cases}
\]

It thus turns out that

\[
Y_{t, A}(S, C) = \begin{cases} \sum_{C' \text{for \(t'\) \(\mathcal{H}(C') = C\)} \forall v \in O_{t'}^{c}, \forall v \in O_{t'}^{y}} Y_{t', A}(S, C') & \text{if } v \notin S, \\
\sum_{C' \text{for \(t'\) \(\mathcal{H}(C') = C\)} \forall v \in O_{t'}^{c}, \forall v \in O_{t'}^{y}} Y_{t', A}(S - v, C') & \text{if } v \in S.
\end{cases}
\]
We have an analogue regarding $Y_{t,B}$. Observing that $V_t \setminus X_t = V_t \setminus X_t - v$, we decompose $d_{p_t,s}$ into the sum over $d_{p_t,s'}$ as follows.

\[
d_{p_t,s} \left[ \begin{array}{c} C_A \\ C_B \end{array} \right] = \sum_{S \in (V_t \setminus X_t) : v \not\in S} \sum_{p \in S} Y_{t,A}(S,C_A) \cdot Y_{t,B}(S,C_B) + \sum_{S \in (V_t \setminus X_t) : v \in S} Y_{t,A}(S,C_A) \cdot Y_{t,B}(S,C_B)
\]

\[
= \sum_{S \in (V_t \setminus X_t) : v \not\in S} \sum_{p \in S} C'_A \text{ for } t' \quad C'_B \text{ for } t' \quad S' \in (V_t \setminus X_t) \quad Y_{t,A}(S',C'_A) \cdot Y_{t,B}(S',C'_B)
\]

\[
+ \sum_{S \in (V_t \setminus X_t) : v \not\in S} \sum_{p \in S} C'_A \text{ for } t' \quad C'_B \text{ for } t' \quad S \in (V_t \setminus X_t) \quad Y_{t,A}(S-v,C'_A) \cdot Y_{t,B}(S-v,C'_B)
\]

\[
= \sum_{S \in (V_t \setminus X_t) : v \not\in S} \sum_{p \in S} \sum_{C'_A \text{ for } t'} \sum_{C'_B \text{ for } t'} \sum_{S' \in (V_t \setminus X_t)} \sum_{S \in (V_t \setminus X_t)} Y_{t,A}(S',C'_A) \cdot Y_{t,B}(S',C'_B)
\]

Note that we define $d_{p_t,s-1} \left[ \begin{array}{c} C_A \\ C_B \end{array} \right] \triangleq 0$. Running through all possible combinations of $C'_A$ and $C'_B$, we can compute $d_{p_t,s} \left[ \begin{array}{c} C_A \\ C_B \end{array} \right]$ by $w^{O(w)} n^{O(1)}$ arithmetic operations.

\[\quad\]

**Proof of Lemma 6.16.** Consider a bijection $\sigma' \in \Sigma(S',C')$ for $S' \subseteq V_t \setminus X_t$ and $C'$ for $t'$ and a bijection $\sigma'' \in \Sigma(S'',C'')$ for $S'' \subseteq V_{t'} \setminus X_{t'}$ and $C''$ for $t''$. We would like to examine a new bijection $\sigma$ obtained by **concatenating** $\sigma'$ and $\sigma''$, which may belong to $\Sigma(S,C)$ for some $S \subseteq V_t \setminus X_t$ and $C$ for $t$. For this purpose, we first define the concatenation of two bijections.

**Definition 6.24.** Given two bijections $\sigma_1 : S_1 \rightarrow T_1$ and $\sigma_2 : S_2 \rightarrow T_2$ such that $|S_1| = |T_1|$ and $|S_2| = |T_2|$, we assume that

- $\sigma_1(i) = \sigma_2(i)$ for all $i \in S_1 \cap S_2$;
- $\sigma_1(i_1) \neq \sigma_2(i_2)$ for all $i_1 \in S_1 \setminus S_2$ and $i_2 \in S_2 \setminus S_1$.

Then, the **concatenation** of $\sigma_1$ and $\sigma_2$, denoted $\sigma_1 \sqcup \sigma_2$, is defined as a bijection from $S_1 \cup S_2$ to $T_1 \cup T_2$ such that

\[
(\sigma_1 \sqcup \sigma_2)(i) = \begin{cases} 
\sigma_1(i) & \text{if } i \in S_1, \\
\sigma_2(i) & \text{if } i \in S_2 \setminus S_1.
\end{cases}
\]
We first discuss the relation between $\text{inv}_{<_{\tau}}(\sigma')$, $\text{inv}_{<_{\tau'}}(\sigma'')$, and $\text{inv}_{<_{\tau}}(\sigma' \sqcup \sigma'')$.

**Lemma 6.25.** Given $s' \in [0 .. n]$, $C' = (O'_1, O'_2, F'_1, F'_2, \tau', v')$ for $t'$, $s'' \in [0 .. n]$, and $C'' = (O''_1, O''_2, F''_1, F''_2, \tau'', v'')$ for $t''$, we can compute a 0-1 integer $v$ in polynomial time such that for all $\sigma' \in \Sigma(S', C')$ with $S' \in (V'_p, X'_p)$ and $\sigma'' \in \Sigma(S'', C'')$ with $S' \in (V''_p, X''_p)$ with $\sigma' \sqcup \sigma''$ defined, $\text{inv}_{<_{\tau}}(\sigma' \sqcup \sigma'') \equiv v$.

**Proof.** Define $\sigma \triangleq \sigma' \sqcup \sigma''$. Since $\sigma' \sqcup \sigma''$ is defined, it must hold that $\tau' = \tau''$, According to the definition of $\sigma' \sqcup \sigma''$, we can expand $\text{inv}_{<_{\tau}}(\sigma' \sqcup \sigma'')$ as follows:

$$
\text{inv}_{<_{\tau}}(\sigma' \sqcup \sigma'') = \{(v, w) \mid v <_{t} w, \sigma(v) \Rightarrow_{t} \sigma(w)\}
= \{(v, w) \mid v <_{t} w, \sigma(v) \Rightarrow_{t} \sigma(w), \{v, w, \sigma(v), \sigma(w)\} \subseteq V''_t\}
+ \{(v, w) \mid v <_{t} w, \sigma(v) \Rightarrow_{t} \sigma(w), \{v, w, \sigma(v), \sigma(w)\} \subseteq V''_{t''}\}
- \{(v, w) \mid v <_{t} w, \sigma(v) \Rightarrow_{t} \sigma(w), \{v, w, \sigma(v), \sigma(w)\} \subseteq V''_t \cap V''_{t''} = X_t\}
+ \left\{(v, w) \mid v <_{t} w, \sigma(v) \Rightarrow_{t} \sigma(w), \{v, w, \sigma(v), \sigma(w)\} \cap V''_t \setminus X_t \neq \emptyset, \{v, w, \sigma(v), \sigma(w)\} \cap V''_{t''} \setminus X_{t''} \neq \emptyset \right\}.
$$

Since $\sigma'(v') = \sigma'(v') \not\in V''_t \setminus X_{t''}$ for all $v' \in V''_t \setminus X_{t''}$, and $\sigma'(v'') = \sigma''(v'') \not\in V''_t \setminus X_{t''}$ for all $v'' \in V''_t \setminus X_{t''}$, it holds that

$$
\text{inv}_{<_{\tau}}(\sigma') + \text{inv}_{<_{\tau}}(\sigma'') - \text{inv}_{<_{\tau}}(\tau')
+ |S''| \cdot (|F'_1| + |F''_1|) + |\{(v', v'') \in F'_1 \times F''_1 \mid v' \Rightarrow_{t} v''\}| + |\{(v', v'') \in F'_2 \times F''_2 \mid v' \Rightarrow_{t} v''\}|.
$$

Consequently, $\text{inv}_{<_{\tau}}(\sigma' \sqcup \sigma'')$ is equal to

$$
\text{inv}_{<_{\tau}}(\sigma') + \text{inv}_{<_{\tau}}(\sigma'') - \text{inv}_{<_{\tau}}(\tau')
+ |S''| \cdot (|F'_1| + |F''_1|) + |\{(v', v'') \in F'_1 \times F''_1 \mid v' \Rightarrow_{t} v''\}| + |\{(v', v'') \in F'_2 \times F''_2 \mid v' \Rightarrow_{t} v''\}|.
$$

By Lemma 6.13, we can compute 0-1 integers $\Delta'$ and $\Delta''$ (which are independent of $\sigma'$ and $\sigma''$) such that $\text{inv}_{<_{\tau}}(\sigma') \equiv \text{inv}_{<_{\tau'}}(\sigma') + \Delta' \equiv v' + \Delta'$ and $\text{inv}_{<_{\tau}}(\sigma'') \equiv \text{inv}_{<_{\tau'}}(\sigma'') + \Delta'' \equiv v'' + \Delta''$. Since $\text{inv}_{<_{\tau}}(\tau')$ can be computed naively and the remaining terms are easy-to-compute, we can compute (the parity of) $\text{inv}_{<_{\tau}}(\sigma' \sqcup \sigma'')$, which completes the proof.

We then define a mapping $\mathcal{J}$ from a set-configuration pair for $t'$ and a set-configuration pair for $t''$ to a set-configuration pair for $t$. 

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Definition 6.26. Let \( S' \subseteq V'_t \setminus X_{t'}, C' = (O'_1, O'_2, F'_1, F'_2, \tau', v') \) for node \( t' \), \( S'' \subseteq V''_t \setminus X_{t''} \), and \( C'' = (O''_1, O''_2, F''_1, F''_2, \tau'', v'') \) for node \( t'' \). Then, we define \( J(S', C', S'', C'') \) as follows:

**Definition of \( J(S', C', S'', C'') \).**

- **Case (1)** If the following conditions are satisfied:
  - \( O'_1 \setminus F'_1 = O''_1 \setminus F''_1 \);
  - \( O'_2 \setminus F'_2 = O''_2 \setminus F''_2 \);
  - \( F'_1 \cap F''_1 = \emptyset \);
  - \( F'_2 \cap F''_2 = \emptyset \);
  - \( \tau' = \tau'' \).

  We define
  
  \[
  J(S', C', S'', C'') \triangleq (S' \cup S'', C),
  \]
  
  where
  
  \[
  C \triangleq (O'_1 \cup O''_1, O'_2 \cup O''_2, F'_1 \cup F''_1, F'_2 \cup F''_2, \tau', v),
  \]

  where \( v \) is computed according to Lemma 6.25.

- **Case (2)** Otherwise: \( J(S', C', S'', C'') \) is undefined.

Here, we claim a kind of completeness and soundness of \( J \).

**Claim 6.27.** For any \( S \subseteq V_t \setminus X_t, C \) for \( t \), and \( \sigma \in \Sigma(S, C) \), there exist unique \( \sigma' \in \Sigma(S', C') \) and \( \sigma'' \in \Sigma(S'', C'') \) for \( S' \subseteq V'_t \setminus X_{t'}, C' \) for \( t' \), \( S'' \subseteq V''_t \setminus X_{t''} \), and \( C'' \) for \( t'' \) such that \( \sigma' \sqcup \sigma'' = \sigma \). Moreover, \( J(S', C', S'', C'') = (S, C) \).

**Proof.** Observe that \( X_t = X'_t = X''_t \) and \( V_t \setminus X_t = (V'_t \setminus X_{t'}) \cup (V''_t \setminus X_{t''}) \). Given \( S \subseteq V_t \setminus X_t, C = (O_1, O_2, F_1, F_2, \tau, v), \) and \( \sigma \in \Sigma(S, C) \), we first define \( \sigma' \triangleq \sigma|_{V'_t} \) and \( \sigma'' \triangleq \sigma|_{V''_t} \). Observe that \( \sigma' \sqcup \sigma'' = \sigma \). We then construct \( S', S'', C', C'' \) as

\[
S' \triangleq S \cap (V'_t \setminus X_{t'}), \quad S'' \triangleq S \cap (V''_t \setminus X_{t''}),
\]

\[
C' \triangleq (O'_1, O'_2, F'_1, F'_2, \tau', v'), \quad C'' \triangleq (O''_1, O''_2, F''_1, F''_2, \tau'', v'').
\]
where we further define

\[ O_1' \triangleq \sigma^{-1}(S' \cup O_2) \cap X_t, \]
\[ O_2' \triangleq \sigma(S' \cup O_1) \cap X_t, \]
\[ F_1' \triangleq \sigma^{-1}(S') \cap X_t, \]
\[ F_2' \triangleq \sigma(S') \cap X_t, \]
\[ \tau' \triangleq \sigma|_{X_t \cap \sigma^{-1}(X_t)}, \]
\[ \nu' \triangleq \text{inv}_{\prec}(\sigma'), \]
\[ \nu'' \triangleq \text{inv}_{\prec}(\sigma''). \]

It is easy to verify that \( \sigma' \in \Sigma(S',C') \) and \( \sigma'' \in \Sigma(S'',C'') \) and that \( \mathfrak{J}(S',C',S'',C'') = (S, C) \). Uniqueness is obvious.

\( \square \)

**Claim 6.28.** For any \( S' \subseteq V_{t'} \setminus X_{t'}, C' \) for \( t' \), \( S'' \subseteq V_{t''} \setminus X_{t''}, C'' \) for \( t'' \) such that \( \mathfrak{J}(S',C',S'',C'') \) is defined, there exist unique \( S \subseteq V_t \setminus X_t \) and \( C \) for \( t \) such that for any \( \sigma' \in \Sigma(S',C') \) and \( \sigma'' \in \Sigma(S'',C'') \), it holds that \( \sigma' \sqcup \sigma'' \in \Sigma(S,C) \). Moreover, \( \mathfrak{J}(S',C',S'',C'') = (S, C) \).

**Proof.** The proof is immediate from **Definition 6.26** and **Lemma 6.25**.

By **Claims 6.27** and 6.28, we have that for \( S \subseteq V_t \setminus X_t \) and \( C \) for \( t \),

\[ \Sigma(S,C) = \left\{ \sigma' \sqcup \sigma'' \mid \sigma' \in \Sigma(S',C'), \sigma'' \in \Sigma(S'',C''), \mathfrak{J}(S',C',S'',C'') = (S, C) \right\}. \]

Suppose \( \mathfrak{J}(S',C',S'',C'') = (S, C) \) for some \( S, S', S'', C, C', C'', C'' \). By **Definition 6.26**, we have that \( S = S' \cup S'', S' = S \cap (V_{t'} \setminus X_{t'}), \) and \( S'' = S \cap (V_{t''} \setminus X_{t''}) \). Further, \( \mathfrak{J} \) determines \( C \) based only on \( |S'|, C', |S''|, C'' \). We thus abuse the notation by writing \( \mathfrak{J}(s',C',s'',C'') = C \) for two integers \( s' \) and \( s'' \) if there exist \( S' \subseteq (V_{t'} \setminus X_{t'}) \) and \( S'' \subseteq (V_{t''} \setminus X_{t''}) \) such that \( \mathfrak{J}(S',C',S'',C'') = (S, C) \).

By **Claims 6.27** and 6.28 and the above discussion, we have that for \( S' \subseteq V_t \setminus X_t \), \( S'' \subseteq V_t \setminus X_t \), and \( C \) for \( t \),

\[ \Sigma(S' \cup S'',C) = \left\{ \sigma' \sqcup \sigma'' \mid \sigma' \in \Sigma(S',C'), \sigma'' \in \Sigma(S'',C''), \mathfrak{J}(|S'|,C',|S''|,C'') = C \right\}. \]

Since any bijection \( \sigma' \sqcup \sigma'' \in \Sigma(S' \cup S'',C) \) for \( C = (O_1, O_2, F_1, F_2, \tau, \nu) \) satisfies that

\[ A(\sigma' \sqcup \sigma'') = \frac{A(\sigma') \cdot A(\sigma'')}{A(\tau)}, \]

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we have that

\[ Y_{t,A}(S' \sqcup S'', C) = \sum_{\sigma \in \Sigma(S' \sqcup S'', C)} A(\sigma) \]

\[ = \sum_{\mathcal{C}' \text{ for } t', \mathcal{C}'' \text{ for } t''} \sum_{\sigma' \in \Sigma(S', \mathcal{C}')} \sum_{\mathcal{C}'' \text{ for } t''} \sum_{\sigma'' \in \Sigma(S'', \mathcal{C}'')} A(\sigma') A(\sigma'') \]

\[ = \frac{1}{A(\tau)} \sum_{\mathcal{C}' \text{ for } t', \mathcal{C}'' \text{ for } t''} \sum_{\sigma' \in \Sigma(S', \mathcal{C}')} \sum_{\mathcal{C}'' \text{ for } t''} \sum_{\sigma'' \in \Sigma(S'', \mathcal{C}'')} A(\sigma') A(\sigma'') \]

\[ = \frac{1}{A(\tau)} \sum_{\mathcal{C}' \text{ for } t', \mathcal{C}'' \text{ for } t''} \sum_{\sigma' \in \Sigma(S', \mathcal{C}')} \sum_{\mathcal{C}'' \text{ for } t''} \sum_{\sigma'' \in \Sigma(S'', \mathcal{C}'')} Y_{t',A}(S', \mathcal{C}') \cdot Y_{t'',A}(S'', \mathcal{C}''). \]

We have an analogue regarding \( Y_{t,B} \). Observing the quality that

\[ (V_s \setminus X_s)_t = \bigcup_{s',s'' \in \{0, s\}} \left\{ S' \sqcup S'' \mid S' \in \left( V_{s'} \setminus X_{s'} \right), S'' \in \left( V_{s''} \setminus X_{s''} \right) \right\}, \]

we can decompose \( dp_{t,s} \) into the sum over \( dp_{t',s'} \) and \( dp_{t'',s''} \) as follows.

\[ dp_{t,s} \begin{bmatrix} C_A \\ C_B \end{bmatrix} = \sum_{S \in \left( V_s \setminus X_s \right)} Y_{t,A}(S, C_A) \cdot Y_{t,B}(S, C_B) \]

\[ = \sum_{s',s'' \in \{0, s\}} \sum_{s' \in \left( V_{s'} \setminus X_{s'} \right)} Y_{t,A}(S' \sqcup S'', C_A) \cdot Y_{t,B}(S' \sqcup S'', C_B) \]

\[ = \frac{1}{A(\tau) \cdot B(\tau)} \sum_{s',s'' \in \{0, s\}} \sum_{s' \in \left( V_{s'} \setminus X_{s'} \right)} \left( \sum_{S'' \in \left( V_{s''} \setminus X_{s''} \right)} Y_{t',A}(S', C_A') \cdot Y_{t'',B}(S'', C_B') \right) \]

\[ = \frac{1}{A(\tau) \cdot B(\tau)} \sum_{s',s'' \in \{0, s\}} \sum_{s' \in \left( V_{s'} \setminus X_{s'} \right)} dp_{t',s'} \begin{bmatrix} C_A' \\ C_B' \end{bmatrix} \cdot dp_{t'',s''} \begin{bmatrix} C_A'' \\ C_B'' \end{bmatrix}. \]
Running through all possible combinations of \( s', s'', C'_A, C''_A, C'_B, C''_B \), we can compute \( dp_{t,s} \left[ \begin{array}{c} C_A \\ C_B \end{array} \right] \) by \( o(w) \) arithmetic operations.

### 6.2.3 Proof of Theorem 6.11

Let \( A^1, \ldots, A^m \) be \( m \) matrices in \( \mathbb{Q}^{n \times n} \) and \((T, \{X_v\}_{v \in T})\) be a nice tree decomposition of graph \((\{n\}, \bigcup_{i \in [m]} \text{nz}(A^i))\), which is of width at most \( w \) and rooted at \( r \in T \). We aim to compute the following quantity for each node \( t \):

\[
\sum_{s \subseteq V_t \setminus X_t} \prod_{i \in [m]} \sum_{O_{i,1}, O_{i,2} \subseteq X_t : |O_{i,1}| = |O_{i,2}|} \prod_{\sigma_t : S \subseteq O_{i,1} \Rightarrow S \supseteq O_{i,2}} \text{sgn}(\sigma_t) A^i(\sigma_t).
\]

(10)

In particular, Eq. (10) is equal to \( Z_m(A^1, \ldots, A^m) \) at the root \( r \). A configuration for node \( t \) is defined as a tuple \( C = (O_{i,1}, O_{i,2}, F_i, F_{i,1}, F_{i,2}, \tau_i, v_i) \) in the same manner as in Section 6.2. Due to Lemma 6.8 and Eq. (9), letting \( C_i = (O_{i,1}, O_{i,2}, F_i, F_{i,1}, F_{i,2}, \tau_i, v_i) \) for \( i \in [m] \) be a configuration for node \( t \) of \( T \), we can express Eq. (10) as follows:

\[
\sum_{c_1, \ldots, c_m \text{ for } t} (-1)^{v_1 + \ldots + v_m} \sum_{s \subseteq \{v_1, \ldots, v_m\}} \prod_{i \in [m]} Y_{t,i}(S, C_i),
\]

where for all \( i \in [m] \), we define

\[
Y_{t,i}(S, C_i) \triangleq \sum_{\sigma_t \in \Sigma(S, C_i)} A^i(\sigma_t).
\]

We then define a dynamic programming table \( dp_{t,s} \) for each \( t \in T \) and \( s \in [0 .. n] \) to store the following quantity with key \( \left[ \begin{array}{c} C_1 \\ \vdots \\ C_m \end{array} \right] \):

\[
dp_{t,s} \left[ \begin{array}{c} C_1 \\ \vdots \\ C_m \end{array} \right] \triangleq \sum_{s \subseteq \{v_1, \ldots, v_m\}} \prod_{i \in [m]} Y_{t,i}(S, C_i).
\]

By definition, \( dp_{t,s} \) contains at most \( o(w) \) entries. The number of bits required to represent each entry of \( dp_{t,s} \) is roughly bounded by \( O(\sum_{i \in [m]} \text{size}(A^i) n \log n) \).

We can easily extend the proof of Lemmas 6.14 to 6.16 for the case of \( m \) matrices as follows.

**Lemma 6.29.** Let \( t \) be an introduce node with one child \( t' \) such that \( X_t = X_{t'} + v \), and \( s \in [0 .. n] \). Given \( dp_{t',s'} \) for all \( s' \), we can compute each entry of \( dp_{t,s} \) in \( (mn)^{O(1)} \) time.
Proof. Using a mapping $\mathcal{I}$ introduced in the proof of Lemma 6.14, we have that for $m$ configurations $C_1, \ldots, C_m$ for $t$,

\[
dp_{t,s} \left[ \begin{array}{c} C_1 \\ \vdots \\ C_m \end{array} \right] = \dp_{t',s'} \left[ \begin{array}{c} \mathcal{I}(C_1) \\ \vdots \\ \mathcal{I}(C_m) \end{array} \right] \prod_{i \in [m]} \begin{cases} 1 & \text{Case (1)}, \\ A_{i,v_i(v)}^i & \text{Case (2)}, \\ A_{i,v_i(v)}^{i-1} & \text{Case (3)}, \\ A_{i,v_i(v)} & \text{Case (4)}, \\ A_{v_i(v)}^i \cdot A_{i,v_i(v)}^{i-1} & \text{Case (5)}. \end{cases} \]

(11)

Since evaluating $\mathcal{I}(C_i)$ for each $i \in [m]$ completes in $(mn)^\mathcal{O}(1)$ time, so does evaluating $\dp_{t,s}$. \hfill \Box

**Lemma 6.30.** Let $t$ be a forget node with one child $t'$ such that $X_t = X_{t'} = v$, and $s \in [0..n]$. Given $\dp_{t',s'}$ for all $s'$, we can compute each entry of $\dp_{t,s}$ in $w^{\mathcal{O}(wn)}n^{\mathcal{O}(1)}$ time.

Proof. Using a mapping $\mathcal{I}$ introduced in the proof of Lemma 6.15, for any $S \subseteq V_t \setminus X_t$, $C$ for $t$, and $i \in [m]$, we have that

\[
\Y_{i,t}(S, C) = \begin{dcases} \sum_{C' \in \mathcal{C}' \mid \mathcal{I}(C') = S} \Y_{i,t'}(S', C') & \text{if } v \not\in S, \\ \sum_{C' \in \mathcal{C}' \mid \mathcal{I}(C') = S - v} \Y_{i,t'}(S - v, C') & \text{if } v \in S. \end{dcases}
\]

Consequently, $\dp_{t,s}$ can be decomposed into two sums as follows.

\[
\dp_{t,s} \left[ \begin{array}{c} C_1 \\ \vdots \\ C_m \end{array} \right] = \sum_{C_1' \in \mathcal{C}'_1 \mid \mathcal{I}(C_1') = C_1} \cdots \sum_{C_m' \in \mathcal{C}'_m \mid \mathcal{I}(C_m') = C_m} \dp_{t',s'} \left[ \begin{array}{c} C_1' \\ \vdots \\ C_m' \end{array} \right] + \sum_{C_1' \in \mathcal{C}'_1 \mid \mathcal{I}(C_1') = C_1} \cdots \sum_{C_m' \in \mathcal{C}'_m \mid \mathcal{I}(C_m') = C_m} \dp_{t',s'} \left[ \begin{array}{c} C_1' \\ \vdots \\ C_m' \end{array} \right].
\]

Here, $C_i'$ for $i \in [m]$ denotes a tuple $(O_{i,1}', O_{i,2}', F_{i,1}', F_{i,2}', \tau_i', v_i')$. Running through all possible combinations of $C_1', \ldots, C_m'$, we can compute each entry of $\dp_{t,s}$ in $w^{\mathcal{O}(wn)}n^{\mathcal{O}(1)}$ time. \hfill \Box

**Lemma 6.31.** Let $t$ be a join node with two children $t'$ and $t''$ such that $X_t = X_{t'} = X_{t''}$, and $s \in [0..n]$. Given $\dp_{t',s'}$ and $\dp_{t'',s''}$ for all $s'$ and $s''$, respectively, we can compute each entry of $\dp_{t,s}$ in $w^{\mathcal{O}(wn)}n^{\mathcal{O}(1)}$ time.

Proof. Using a mapping $\mathcal{I}$ introduced in the proof of Lemma 6.16, for any $S' \subseteq V_{t'} \setminus X_{t'}$, $S'' \subseteq V_{t''} \setminus X_{t''}$, $C = (O_1, O_2, F_1, F_2, \tau, v)$ for $t$, and $i \in [m]$, we have that

\[
\Y_{i,t}(S' \cup S'', C) = \frac{1}{A^i(\tau)} \sum_{C' \in \mathcal{C}' \mid \mathcal{I}(C') = S', C'' \in \mathcal{C}''} \Y_{i,t'}(S', C') \cdot \Y_{i,t''}(S'', C'').
\]

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Consequently, $dp_{t,s}$ can be decomposed as follows.

$$
\begin{align*}
    dp_{t,s} \left[ \begin{array}{c}
        C_1 \\
        \vdots \\
        C_m
    \end{array} \right] &= \frac{1}{A^1(\tau) \cdots A^m(\tau)} \sum_{s',s'' \in [0,n]} \sum_{s'' + s' = s} dp_{t',s'} \left[ \begin{array}{c}
        C'_1 \\
        \vdots \\
        C'_m
    \end{array} \right] \cdot dp_{t',s''} \left[ \begin{array}{c}
        C''_1 \\
        \vdots \\
        C''_m
    \end{array} \right] \\
    &\text{for } t' \text{ with } \Delta(s',C'_1,s''C''_1) = C_1 \\
    &\text{and } \Delta(s',C'_m,s''C''_m) = C_m
\end{align*}
$$

Running through all possible combinations of $s', s'', C'_1, C'_m, \ldots, C'_m, C''_m$, we can compute each entry of $dp_{t,s}$ in $w^{O(wn)} n^{O(1)}$ time. \qed

We are now ready to describe an FPT algorithm for computing $Z_m$. Our algorithm is almost identical to that for Theorem 6.5. Given a tree decomposition for $(|n|, \bigcup_{i \in [m]} nz(A^i))$ of width at most $w$, we transform it to a nice tree decomposition $(T, \{X_t\}_{t \in T})$ rooted at $r$ of width at most $w$ that has $O(wn)$ nodes in polynomial time (Cygan et al., 2015). For every leaf $\ell$ of $T$, we initialize $dp_{t,s}$ as

$$
\begin{align*}
    dp_{t,s} \left[ \begin{array}{c}
        \emptyset, \emptyset, \emptyset, \emptyset, \emptyset \rightarrow \emptyset, v_1 \\
        \vdots \\
        \emptyset, \emptyset, \emptyset, \emptyset, \emptyset \rightarrow \emptyset, v_m
    \end{array} \right] &= \begin{cases} 
        1 & \text{if } s = 0 \text{ and } v_1 = \cdots = v_m = 0, \\
        0 & \text{otherwise.}
    \end{cases}
\end{align*}
$$

Then, for each non-leaf node $t \in T$, we apply either of Lemmas 6.29 to 6.31 to fill $dp_{t,s}$ using already-filled $dt_{t',s'}$ for all children $t'$ of $t$ in a bottom-up fashion. Completing dynamic programming, we compute $Z_m$ as follows:

$$
Z_m(A^1, \ldots, A^m) = \sum_{s,v_1,\ldots,v_m} (-1)^{v_1+\cdots+v_m} \cdot dp_{t,s} \left[ \begin{array}{c}
        \emptyset, \emptyset, \emptyset, \emptyset, \emptyset \rightarrow \emptyset, v_1 \\
        \vdots \\
        \emptyset, \emptyset, \emptyset, \emptyset, \emptyset \rightarrow \emptyset, v_m
    \end{array} \right].
$$

The correctness follows from Lemmas 6.29 to 6.31. Because $T$ has at most $O(wn)$ nodes, each table is of size $w^{O(wn)}$, and each table entry can be computed in $w^{O(wn)} n^{O(1)}$ time by Lemmas 6.29 to 6.31, the whole time complexity is bounded by $w^{O(wn)} n^{O(1)}$. \qed

### 6.3 Parameterization by Maximum Treewidth

Finally, let us take the maximum treewidth of two matrices as a parameter and refute its fixed-parameter tractability. This parameterization is preferable to the previous one because the maximum treewidth can be far smaller than the treewidth of the union. One can, for example, construct two $n \times n$ matrices $A$ and $B$ such that $\max\{\text{tw}(A), \text{tw}(B)\} = 1$ and $\text{tw}(\text{nz}(A) \cup \text{nz}(B)) = O(\sqrt{n})$ because we can “weave” two paths into a grid.

**Example 6.32.** Consider any two $16 \times 16$ matrices $A$ and $B$ whose nonzero entries $\text{nz}(A)$ and $\text{nz}(B)$ are drawn in Figure 5. Because each of $\text{nz}(A)$ and $\text{nz}(B)$ forms a path graph on 16 vertices, we have that $\max\{\text{tw}(A), \text{tw}(B)\} = 1$. On the other hand, the graph $([16], \text{nz}(A) \cup \text{nz}(B))$ forms a $4 \times 4$ grid graph, and we thus have that $\text{tw}(\text{nz}(A) \cup \text{nz}(B)) = 4$.  

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Unluckily, even when both $A$ and $B$ have treewidth at most 3, it is still \#P-hard to compute $Z_2(A, B)$. Thus, parameterization by $\max\{\text{tw}(A), \text{tw}(B)\}$ is not even XP unless FP = \#P.

**Theorem 6.33.** Let $A, B$ be two positive semi-definite $(0, 1)$-matrices such that $\text{tw}(A) \leq 3$ and $\text{tw}(B) \leq 3$. Then, computing $Z_2(A, B)$ is \#P-hard.

**Proof.** We reduce from the problem of counting all (not necessarily perfect) matchings in a bipartite graph of maximum degree 4, which is known to be \#P-complete (Vadhan, 2001). Let $H = (X \cup Y, E)$ be a bipartite graph of maximum degree 4, where $E \subseteq X \times Y$ is the set of edges between $X$ and $Y$. In accordance with Gillenwater (2014)’s reduction, we construct two positive semi-definite matrices $A, B \in \{0, 1\}^{E \times E}$ so that $A_{i,j}$ is 1 if edges $i, j$ in $E$ share a common vertex in $X$ and 0 otherwise, and $B_{i,j}$ is 1 if edges $i, j$ in $E$ share a common vertex in $Y$ and 0 otherwise. Then, we have that for any edge set $S \subseteq E$, the value of $\det(A_{S,S}) \det(B_{S,S})$ is 1 if $S$ is a matching in $H$ and 0 otherwise, that is, $Z_2(A, B)$ is exactly equal to the total number of matchings in $H$, and thus \#P-hardness follows.

By construction, $A$ and $B$ must be block-diagonal matrices, where each block is a $k \times k$ all-one matrix for some $k \in [4]$. Because the graph $([n], \text{nz}(A))$ is the disjoint union of cliques of size at most 4, whose treewidth is at most 3, it holds that $\text{tw}(A) \leq 3$. Similarly, we have that $\text{tw}(B) \leq 3$, completing the proof. \qed

## 7 Extensions to Fixed-Size Π-DPPs

In this section, we impose size constraints on Π-DPPs so as to produce a fixed-size set and analyze the normalizing constant of the resulting distribution. Formally, given $m$ $P_0$-matrices $A^1, \ldots, A^m \in \mathbb{R}^{n \times n}$ and a size parameter $k \in [0..n]$, the $k$Π-DPP is defined as a distribution whose probability mass for each subset $S \subseteq [n]$ as proportional to $\det(A^1_{S,S}) \cdots \det(A^m_{S,S}) \cdot |S| \in \binom{[n]}{k}|$. We will use $Z_{m,k}(A^1, \ldots, A^m)$ to denote its normalizing constant; namely,

$$Z_{m,k}(A^1, \ldots, A^m) \triangleq \sum_{S \in \binom{[n]}{k}} \prod_{i \in [m]} \det(A^i_{S,S}).$$

The special case of $m = 1$ coincides with $k$-DPPs (Kulesza and Taskar, 2011), the normalizing constant of which is known to be amenable to compute. We will investigate the computational
complexity of estimating $Z_{m,k}$ by taking a similar approach to the one in Sections 4 to 6. Intuitively, $Z_{m,k}$ seems harder to estimate than $Z_m$ because we have that $Z_m = \sum_{0 \leq k \leq n} Z_{m,k}$.

7.1 Intractability, Inapproximability, and Indistinguishability

First, we provide hardness results that can be thought of as size-constraint counterparts to those in Sections 4 and 5:

**Theorem 7.1.** The following results hold for $Z_{m,k}$:

- For every fixed positive even integer $p$, computing $Z_{p,k}(A, \ldots, A) \mod 3$ for either a $(-1, 0, 1)$-matrix or a $P$-matrix $A$ is $UP$-hard and $\text{Mod}_3P$-hard.

- It is $NP$-hard to determine whether $Z_{3,k}(A, B, C)$ is positive or 0 for three positive semi-definite matrices $A, B, C$ in $Q^{n \times n}$. Therefore, for any polynomial-time computable function $\rho$, $Z_{3,k}$ cannot be approximated within a factor of $\rho(|I|)$, where $|I|$ is the input size, unless $P = NP$.

- The mixed discriminant $D$ is $AP$-reducible to $Z_{2,k}$. Therefore, if there exists an FPRAS for $Z_{2,k}$, then there exists an FPRAS for $D$.

Some remarks are in order. The first statement is a direct consequence of Corollary 4.2. The second statement corresponds to Theorem 5.1; however, there is a crucial difference between $Z_{3,k}$ and $Z_3$ wherein the former cannot be approximated within any (polynomial-time computable) factor, while the latter is approximable within a factor of $2^{O(|I|^3)}$ (Observation 5.2). The third statement corresponds to Theorem 5.4, whose proof is much simpler than that of Theorem 5.4.

**Proof of Theorem 7.1.** The first statement is a direct consequence of Corollary 4.2 and the following equality for any $n \times n$ matrix $A$ and any positive integer $p$:

$$Z_p(A, \ldots, A) = \sum_{0 \leq k \leq n} Z_{p,k}(A, \ldots, A).$$  \hspace{1cm} (12)

To prove the second statement, we show a polynomial-time Turing reduction from HAMILTONIANPATH (Garey and Johnson, 1979). Let $G = (V, E)$ be a directed graph on $n$ vertices and $m$ edges. We construct $m \times m$ three positive semi-definite matrices $A, B, C \in Q^{E \times E}$ in polynomial time according to the procedure described in the proof of Theorem 5.1. In particular, the value of $\det(A_{S,S}) \det(B_{S,S}) \det(C_{S,S})$ for $S \in \binom{E}{n-1}$ is positive if and only if $S$ is a Hamiltonian path. We can thus use $Z_{3,n-1}(A, B, C)$ to decide the Hamiltonicity of $G$ as follows:

- **Case (1)** if there exists (at least) one Hamiltonian path in $G$, then $Z_{3,n-1}(A, B, C) > 0$;

- **Case (2)** if no Hamiltonian path exists in $G$, then $Z_{3,n-1}(A, B, C) = 0$.

Finally, we show a polynomial-time Turing reduction from $D$ to $Z_{2,k}$, which is sufficient to prove AP-reducibility. Let $K^1, \ldots, K^m$ be $m$ positive semi-definite matrices in $Q^{m \times m}$, and define $n = m^2$. We construct the two matrices $A, B \in Q^{n \times n}$ by following the procedure in the proof of Theorem 5.4. By Eq. (5), we have the following relation:

$$Z_{2,m}(A, B) = \sum_{S \in \binom{E}{m}} \det(A_{S,S}) \det(B_{S,S}) = m! \, D(K^1, \ldots, K^m),$$  \hspace{1cm} (13)

which completes the reduction. \qed
7.2 Fixed-Parameter Tractability

Here, we demonstrate the fixed-parameter tractability of computing \( Z_{m,k} \). First, we show that even if we are only given access to an oracle for \( Z_m \), we can recover the values of \( Z_{m,k} \) by calling the oracle polynomially many times.

**Theorem 7.2.** Let \( A^1, \ldots, A^m \) be \( m \) matrices in \( \mathbb{Q}^{n \times n} \) and \( k \) be an integer in \([n]\). Suppose we are given access to an oracle that returns \( Z_m \). Then, for all \( k \), \( Z_{m,k}(A^1, \ldots, A^m) \) can be computed in polynomial time, by calling the oracle \( L \triangleq n + 1 \) times. Furthermore, if \( \{(A^{1,\ell}, \ldots, A^{m,\ell})\}_{\ell \in [L]} \) denotes the set consisting of the tuples of \( m \) matrices for which the oracle is called, we have that \( \text{rank}(A^{i,\ell}) \leq \text{rank}(A^i) \) and \( \text{nz}(A^{i,\ell}) \subseteq \text{nz}(A^i) \) for all \( i \in [m] \) and \( \ell \in [L] \).

**Proof.** Introduce a positive number \( x \in Q \) and consider a polynomial \( Z(x) \triangleq Z_m(xA^1, A^2, \ldots, A^m) \) in \( x \). We can expand it as follows:

\[
Z_m(xA^1, A^2, \ldots, A^m) = \sum_{S \subseteq [n]} x^{|S|} \prod_{i \in [m]} \det(A^i) = \sum_{k \in [n]} x^k \cdot Z_{m,k}(A^1, \ldots, A^m).
\]

Given the values of \( Z(x) \) for all \( x \in [n+1] \), each of which can be computed by calling the oracle for \( Z_m \), we can recover all the coefficients of \( Z(x) \) by Lagrange interpolation as desired. The structural arguments are obvious.

As a corollary of Theorems 6.4, 6.11 and 7.2, we obtain respective FPT algorithms for computing \( Z_{m,k} \) parameterized by maximum rank and treewidth.

**Corollary 7.3.** For \( m \) matrices \( A^1, \ldots, A^m \) in \( \mathbb{Q}^{n \times n} \), the following fixed-parameter tractability results hold:

- there exists an \( r^{O(mr) n^{O(1)}} \)-time algorithm for computing \( Z_{m,k}(A^1, \ldots, A^m) \) for all \( k \in [n] \), where \( r = \max_{i \in [m]} \text{rank}(A^i) \) denotes the maximum rank among the \( m \) matrices;
- there exists a \( w^{O(mw) n^{O(1)}} \)-time algorithm for computing \( Z_{m,k}(A^1, \ldots, A^m) \) for all \( k \in [n] \), where \( w = \text{tw}(\bigcup_{i \in [m]} \text{nz}(A^i)) \) denotes the treewidth of the union of nonzero entries in the \( m \) matrices.

7.3 Parameterization by Output Size

Finally, we investigate the fixed-parameter tractability of the computation of \( Z_{m,k} \) parameterized by \( k \). Since a brute-force algorithm that examines all possible \( \binom{n}{k} = O(n^k) \) subsets has time complexity \( mn^{k+O(1)} \), computing \( Z_{m,k} \) parameterized by \( k \) is XP. On the other hand, we show that computing \( Z_{m,k} \) is \#W[1]-hard. Consequently, an FPT algorithm parameterized by \( k \) for computing \( Z_{m,k} \) does not exist unless FPT = \#W[1], which is suspected to be false.

**Theorem 7.4.** Let \( A, B \) be two \( n \times n \) positive semi-definite \((0,1)\)-matrices and \( k \) be a positive integer in \([n]\). Then, it is \#W[1]-hard to compute \( Z_{2,k}(A, B) \) parameterized by \( k \).

**Proof.** We show a polynomial-time parsimonious reduction from the problem of counting all (imperfect) matchings of size \( k \) in a bipartite graph \( H \), which was proven to be \#W[1]-hard by Curticapean and Marx (2014, Theorem I.2). In according with Gillenwater (2014) (cf. proof of
Theorem 6.33), we construct two matrices A and B from H in polynomial time in n, so that \( Z_{2k}(A, B) \) is equal to the total number of k-matchings in G. Because this reduction from an input G with a parameter k to an input \((A, B)\) with a parameter k is an FPT parsimonious reduction (see Flum and Grohe, 2004), the proof follows.

8 Application of FPT Algorithms to Two Related Problems

In this section, we introduce two applications of the FPT algorithm for computing \( Z_m \) parameterized by the treewidth (Theorem 6.11).

8.1 Approximation Algorithm for E-DPPs of Fractional Exponents

We first use Theorem 6.11 to estimate the normalizing constant for E-DPPs of fractional exponent \( p \). Since \( \Pi \)-DPPs include E-DPPs of exponent \( p \) only if \( p \) is an integer, fixed-parameter tractability does not apply to the fractional case. The resulting application is a \( wO(wp) \)-time parameterized algorithm that estimates \( Z^p(A) \) within a factor of \( 2^{n^{2p-1}} \).

**Theorem 8.1.** Let \( A \) be a \( P_0 \)-matrix in \( \mathbb{Q}^{n \times n} \) and \( p \) be a positive fractional number greater than 1. Then, there exists a \( wO(wp) \)-time algorithm that returns a \( 2^{n^{2p-1}} \)-factor approximation to \( Z^p(A) \).

We introduce the general equivalence of \( \ell_p \) norms, which can be derived from Hölder’s inequality.

**Lemma 8.2** (General equivalence of \( \ell_p \)-nons, see, e.g., Steele, 2004). For an n-dimensional vector \( x \) in \( \mathbb{R}^n \) and two positive real numbers \( p, q \) such that \( 0 < p < q \), the following inequality holds:

\[
\|x\|_q \leq \|x\|_p \leq \frac{n^{q-p}}{q} \cdot \|x\|_q,
\]

where \( \| \cdot \|_p \) denotes the \( \ell_p \) norm; i.e.,

\[
\|x\|_p = \left( \sum_{i \in [n]} |x_i|^p \right)^{1/p}.
\]

**Remark 8.3.** If we apply Eq. (14) to a \( 2^n \)-dimensional vector \( x \in \mathbb{Q}^{2^n} \) such that \( x_S \triangleq \text{det}(A_{S,S}) \) for each \( S \subseteq [n] \) for a \( P_0 \)-matrix \( A \in \mathbb{Q}^{n \times n} \), we have that \( Z^p(A) \leq \det(A + I)^p \leq 2^{n^{(p-1)}} Z^p(A) \) for any \( p > 1 \), which gives a \( 2^{n^{(p-1)}} \)-approximation. On the other hand, it is \( \text{NP} \)-hard to approximate \( Z^p(A) \) within a factor of \( 2^{\beta n} \) for some \( \beta > 0 \) (Ohsaka, 2021b). Therefore, \( 2^{O(pn)} \) is a tight approximation factor for the normalizing constant for E-DPPs in the general case.

**Proof of Theorem 8.1.** Fix a positive semi-definite matrix \( A \in \mathbb{Q}^{n \times n} \) and a fractional number \( p > 1 \). Since \( \lfloor p \rfloor < p < \lceil p \rceil \), we can write \( p = \lambda \lfloor p \rfloor + (1 - \lambda) \lceil p \rceil \) for some \( \lambda \in (0, 1) \). First, we derive two estimates of \( Z^p(A) \) using \( Z^{\lfloor p \rfloor}(A) \) and \( Z^{\lceil p \rceil}(A) \).
Estimate Using $Z^{[p]}$. We will show that $(Z^{[p]}(A))^\frac{p}{|p|}$ is a $2^{n\left(\frac{p}{|p|}-1\right)}$-approximation to $Z^p(A)$. By using Eq. (14), we bound $(Z^{[p]}(A))^\frac{p}{|p|}$ as follows:

$$\left(\sum_{S \subseteq [n]} \det(A_{S,S})^p\right)^\frac{1}{p} \leq \left(\sum_{S \subseteq [n]} \det(A_{S,S})^{|p|}\right)^\frac{1}{|p|} \leq (2^n)^\frac{1}{|p|} \cdot \left(\sum_{S \subseteq [n]} \det(A_{S,S})^p\right)^\frac{1}{p},$$

which immediately implies that

$$Z^p(A) \leq (Z^{[p]}(A))^\frac{p}{|p|} \leq 2^{n\left(\frac{p}{|p|}-1\right)} \cdot Z^p(A). \quad (15)$$

Estimate Using $Z^{[p]}$. We will show that $(Z^{[p]}(A))^\frac{p}{|p|}$ is a $2^{n\left(1-\frac{p}{|p|}\right)}$-approximation to $Z^p(A)$. By using Eq. (14), we bound $(Z^{[p]}(A))^\frac{p}{|p|}$ as follows:

$$(2^n)^\frac{1}{|p|} \cdot \left(\sum_{S \subseteq [n]} \det(A_{S,S})^p\right)^\frac{1}{p} \leq \left(\sum_{S \subseteq [n]} \det(A_{S,S})^{|p|}\right)^\frac{1}{|p|} \leq \left(\sum_{S \subseteq [n]} \det(A_{S,S})^p\right)^\frac{1}{p},$$

which immediately implies that

$$Z^p(A) \leq 2^{n\left(1-\frac{p}{|p|}\right)} \cdot (Z^{[p]}(A))^\frac{p}{|p|} \leq 2^{n\left(1-\frac{p}{|p|}\right)} \cdot Z^p(A). \quad (16)$$

Next, we choose the “better” of the two estimates, depending on the value of $\lambda$. Since it holds that $p = \lambda \left[\frac{p}{|p|}\right] + (1 - \lambda) \left[\frac{p}{|p|}\right] = \left[\frac{p}{|p|}\right] + 1 - \lambda$, we can simplify the exponents of the approximation factors in Eqs. (15) and (16) as follows:

$$\frac{p}{|p|} - 1 = \frac{1 - \lambda}{\left[\frac{p}{|p|}\right]} \quad \text{and} \quad 1 - \frac{p}{|p|} = \frac{\lambda}{\left[\frac{p}{|p|}\right] + 1}.$$

Observing that the linear equation $\frac{1 - \lambda}{\left[\frac{p}{|p|}\right]} = \frac{\lambda}{\left[\frac{p}{|p|}\right] + 1}$ has a unique solution $\lambda^* \triangleq \frac{\left[\frac{p}{|p|}\right] + 1}{2\left[\frac{p}{|p|}\right] + 1}$, we have the following relation:

$$\min\left\{\frac{1 - \lambda}{\left[\frac{p}{|p|}\right]}, \frac{\lambda}{\left[\frac{p}{|p|}\right] + 1}\right\} = \begin{cases} \frac{1 - \lambda}{\left[\frac{p}{|p|}\right]} & \text{if } \lambda > \lambda^* = \frac{\left[\frac{p}{|p|}\right] + 1}{2\left[\frac{p}{|p|}\right] + 1}, \\ \frac{\lambda}{\left[\frac{p}{|p|}\right] + 1} & \text{otherwise}. \end{cases}$$

Further, $\min\left\{\frac{1 - \lambda}{\left[\frac{p}{|p|}\right]}, \frac{\lambda}{\left[\frac{p}{|p|}\right] + 1}\right\}$ takes $\frac{1}{2\left[\frac{p}{|p|}\right] + 1}$ as the maximum value when $\lambda = \lambda^* \in (0, 1)$.

Our algorithm works as follows. First, we compute $Z^{[p]}(A)$ and $Z^{[p]}(A)$ in $O(\alpha n) \mu^O(1)$ time by using the FPT algorithm in Theorem 6.11. Then, if $\lambda > \lambda^*$, we output an $\alpha$-approximation to $(Z^{[p]}(A))^\frac{p}{|p|}$; otherwise we output an $\alpha$-approximation to $2^{n\left(1-\frac{p}{|p|}\right)} \cdot (Z^{[p]}(A))^\frac{p}{|p|}$, where $\alpha \triangleq 2^{\left(\frac{1}{\left[\frac{p}{|p|}\right]} - \frac{1}{\left[\frac{p}{|p|}\right] + 1}\right)n} > 1$. The output ensures an approximation factor of $\alpha \cdot 2^{\frac{n}{|p|+1}} \leq 2^{\frac{n}{|p|}}$, which completes the proof. 

\footnote{We can use, for example, the Newton–Raphson method to compute an $\alpha$-approximation to fractional exponents efficiently.}
8.2 Subexponential Algorithm for Unconstrained MAP Inference

We apply the FPT algorithm to maximum a posteriori (MAP) inference. For a positive semi-definite matrix $A \in \mathbb{Q}^{n \times n}$, unconstrained MAP inference requests that we find a subset $S \subseteq [n]$ having the maximum determinant, i.e., $\text{argmax}_{S \subseteq [n]} \det(A_{S,S})$. We say that a polynomial-time algorithm $\text{alg}$ is a $\rho$-approximation algorithm for $\rho \geq 1$ if for all positive semi-definite matrix $A \in \mathbb{Q}^{n \times n}$,

$$\det(\text{alg}(A)) \geq \left( \frac{1}{\rho} \right) \cdot \max_{S \subseteq [n]} \det(A_{S,S}),$$

where $\text{alg}(A)$ is the output of $\text{alg}$ on $A$. The approximation factor $\rho$ can be a function in the input size $|\mathcal{I}|$, and (asymptotically) smaller $\rho$ is a better approximation factor. The following theorem states that we can find a $2\sqrt{n}$-approximation to unconstrained MAP inference in $2^{O(\sqrt{n})}$ time (with high probability) provided that the treewidth of $A$ is a constant.

**Theorem 8.4.** Let $A$ be a positive semi-definite matrix of treewidth $w$ in $\mathbb{Q}^{n \times n}$. Then, there exists a $w^{O(w\sqrt{n})}n^{O(1)}$-time randomized algorithm that outputs a $2\sqrt{n}$-approximation to unconstrained MAP inference on $A$ with probability at least $1 - 2^{-n}$. In particular, if the treewidth $w$ is $O(1)$, then the time complexity is bounded by $2^{O(\sqrt{n})}$.

**Proof.** Fix a positive semi-definite matrix $A \in \mathbb{Q}^{n \times n}$ of treewidth $w$ and let $\text{OPT} \triangleq \max_{S \subseteq [n]} \det(A_{S,S})$ be the optimum value of unconstrained MAP inference on $A$. Define $p \triangleq \lfloor 2\sqrt{n} \rfloor$. Consider an E-DPP of exponent $p$ defined by $A$, whose probability mass for each subset $S \subseteq [n]$ is $\frac{\det(A_{S,S})}{Z_p^p(A)}$. Since this E-DPP coincides with the II-DPP defined by $p$ copies of $A$, by Theorems 3.1 and 6.11, we can draw a random sample $S$ from it in $w^{O(w\sqrt{n})}n^{O(1)}$ time. Observe that the event that $S$ being a $2\sqrt{n}$-approximation (which includes the case of $\det(A_{S,S}) = \text{OPT}$) occurs with probability at least $\frac{\text{OPT}^p}{Z_p^p(A)}$ and that it does not occur with probability at most

$$\sum_{S \subseteq [n]:S \text{ is not } 2\sqrt{n}\text{-approx.}} \frac{\det(A_{S,S})^p}{Z_p^p(A)} \leq \sum_{S \subseteq [n]:S \text{ is not } 2\sqrt{n}\text{-approx.}} \frac{(2^{-\sqrt{n}}\text{OPT})^p}{Z_p^p(A)} \leq \frac{2^n \cdot 2^{-p\sqrt{n}}\text{OPT}^p}{Z_p^p(A)}.$$

Hence, we have that the probability of success is at least

$$1 - \left( \frac{2^n \cdot 2^{-p\sqrt{n}}\text{OPT}^p}{Z_p^p(A)} \right) \geq 1 - 2^{n-p\sqrt{n}} \geq 1 - 2^{-n},$$

which completes the proof. \qed

**Remark 8.5.** A similar approach does not work for size-constrained MAP inference for the following reason: for a size parameter $k \in [n]$, the objective is to compute a $2^k$-approximation, i.e., a set $S \subseteq \binom{[n]}{k}$ such that $\det(A_{S,S}) \geq 2^{-\sqrt{k}} \text{argmax}_{S \in \binom{[n]}{k}} \det(A_{S,S})$. Let $p \triangleq \lfloor 2\sqrt{k} \rfloor$, and consider a fixed-size E-
DPP of exponent $p$ defined by $A$, which coincides with the $k\Pi$-DPP defined by $p$ copies of $A$. Then, a sample drawn from this $k\Pi$-DPP is a $2^{\sqrt{k}}$-approximation with probability at least

$$1 - \binom{n}{k} \cdot 2^{-p\sqrt{k}\text{OPT}^{p}} \geq 1 - O\left(\frac{n}{k}\right)^{k},$$

which can be 0.

9 Concluding Remarks and Open Questions

We studied the computational complexity of the normalizing constant for the product of determinantal point processes. Our results (almost) rule out the possibility of efficient algorithms for general cases and devised the fixed-parameter tractability. Several open questions are listed below.

- **Q1.** Can we show the intractability of computing $Z^{p}$ for $p$ which is not a positive even integer, such as $p = 3$ and $p = 1.1$? The proof strategy employed in Section 4 would no longer work.

- **Q2.** Can we develop more “practical” FPT algorithms having a small exponential factor, e.g., $f(k) = 2^{k}$? We might have to avoid enumerating bijections.

- **Q3.** Can we establish fixed-parameter tractability for parameters other than the treewidth or matrix rank, such as the cliquewidth?

- **Q4.** Can we devise an exact FPT algorithm for computing $Z^{p}$ for a fractional exponent $p$ parameterized by the treewidth?

- **Q5.** Can we remove the bounded-treewidth assumption from Theorem 8.4 to obtain a subexponential algorithm for (unconstrained) MAP inference?

The first author of this article gave a partial answer to Q1 (Ohsaka, 2021b): it is NP-hard to approximate $Z^{p}$ within an exponential factor in the order of an input matrix if $p \geq 10^{10^{13}}$.

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