SHORT COMMUNICATION
A negative answer to the question of the linearity of Tate’s Trace for the sum of two endomorphisms
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The aim of this note is to solve a problem proposed by J. Tate in 1968 by offering a counter-example of the linearity of the trace for the sum of two finite potent operators on an infinite-dimensional vector space.
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1. Introduction
Let \( k \) be a fixed ground field and \( V \) a vector space over \( k \). If we consider an endomorphism \( \varphi \) of \( V \), according to \[1\] we say that \( \varphi \) is ‘finite-potent’ if \( \varphi^n V \) is finite dimensional for some \( n \).

For a finite potent endomorphism, \( \varphi : V \rightarrow V \), a trace \( \text{Tr}_V(\varphi) \in k \) may be defined, having the following properties:

(1) If \( V \) is finite dimensional, then \( \text{Tr}_V(\varphi) \) is the ordinary trace.
(2) If \( W \) is a subspace of \( V \) such that \( \varphi W \subseteq W \), then
\[ \text{Tr}_V(\varphi) = \text{Tr}_W(\varphi) + \text{Tr}_{V/W}(\varphi). \]
(3) If \( \varphi \) is nilpotent, then \( \text{Tr}_V(\varphi) = 0 \).
(4) If \( F \) is a ‘finite-potent’ subspace of \( \text{End}(V) \) (i.e. if there exists an \( n \) such that for any family of \( n \) elements \( \varphi_1, \ldots, \varphi_n \in F \), the space \( \varphi_1 \ldots \varphi_n V \) is finite dimensional), then \( \text{Tr}_V : F \rightarrow k \) is \( k \)-linear.
(5) If \( f : V' \rightarrow V \) and \( g : V \rightarrow V' \) are \( k \)-linear and \( f \circ g \) is finite potent, then \( g \circ f \) is finite potent, and
\[ \text{Tr}_V(f \circ g) = \text{Tr}_{V'}(g \circ f). \]

Remark 1.1 Properties (1), (2) and (3) characterize traces, because if \( W \) is a finite dimensional subspace of \( V \) such that \( \varphi W \subseteq W \) and \( \varphi^n V \subseteq W \), for some \( n \), then \( \text{Tr}_V(\varphi) = \text{Tr}_W(\varphi) \).
And, since \( \varphi \) is finite potent, we may take \( W = \varphi^n V \). Note that if \( W \) and \( W' \) are two finite dimensional \( \varphi \)-invariant subspaces of \( V \), such that \( \varphi^n V \subseteq W \subseteq W' \), then \( \text{Tr}_{W'/W}(\varphi) = 0 \), and we deduce that \( \text{Tr}_W(\varphi) = \text{Tr}_{W'}(\varphi) \) in this case. In general, if \( \varphi^n V \subseteq W \) and \( \varphi^n V \subseteq W' \),

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one has that $\text{Tr}_W(\varphi) = \text{Tr}_{W \cap W'(\varphi)} = \text{Tr}_{W'}(\varphi)$. Thus, the trace $\text{Tr}_V(\varphi)$ is independent of the choice of the subspace $W$.

This trace is the main tool used by J. Tate in his elegant definition of the residue offered in [1].

An open problem has been to determine whether this trace satisfies the linearity property. In fact, in the article mentioned J. Tate wrote: ‘I doubt whether the rule

$$\text{Tr}_V \theta_1 + \text{Tr}_V \theta_2 = \text{Tr}_V (\theta_1 + \theta_2)$$

holds in general, i.e. whenever all three endomorphisms $\theta_1$, $\theta_2$ and $\theta_1 + \theta_2$ are finite-potent, although I do not know a counter-example. (If a counter-example exists at all, then there will be one with $\theta_1$ and $\theta_2$ nilpotent, because every finite-potent endomorphism is the sum of a nilpotent one and one with finite range.)’

Recently, the second author of this note has studied this problem, solving the question of the linearity with a negative answer for the case of the sum of three finite potent endomorphisms.

Indeed, in [2] Pablos Romo offered three nilpotent endomorphisms $\theta_1$, $\theta_2$ and $\theta_3$ of an infinite-dimensional $\mathbb{Z}/2\mathbb{Z}$-vector space $V$, such that $\theta_1 + \theta_2 + \theta_3$ is a finite-potent endomorphism of $V$ and

$$\text{Tr}_V (\theta_1 + \theta_2 + \theta_3) = 1.$$ 

Accordingly, in general Tate’s trace does not satisfy the linearity property.

Moreover, Argerami, Szechtmam and Tifenbach have given an alternative characterization of finite potent endomorphisms that can be used to reduce the question to a special case. They have shown in [3] that an endomorphism $\varphi$ is finite potent if and only if $V$ admits a $\varphi$-invariant decomposition $V = W_\varphi \oplus U_\varphi$ such that $\varphi|_{U_\varphi}$ is nilpotent, $W_\varphi$ is finite dimensional, and $\varphi|_{W_\varphi} : W_\varphi \rightarrow W_\varphi$ is an isomorphism. This decomposition is unique and one has that $\text{Tr}_V(\varphi) = \text{Tr}_W(\varphi|_{W_\varphi})$.

Considering $\varphi_1$ and $\varphi_2$ nilpotent endomorphisms in $V$ such that $\varphi_1 + \varphi_2 = \varphi$ is a finite potent endomorphism, they also pointed that $V$ may be viewed as a left $k(s, t)$-module where $k(s, t)$ is the algebra of polynomials in the non-commuting variables $s$ and $t$, letting $s$ act as the endomorphism $\varphi_1$ and $t$ act as $\varphi_2$.

From this characterization and the unique decomposition referred above, the question of the linearity of Tate’s trace can be reduced to a particular case of finite potent linear operators in the vector space $V = k(s, t)/I$, where $I$ is a left ideal of $k(s, t)$ such that:

- $I$ contains the two-sided ideal generated by $s^n$, $t^m$, $f(s + t)$ for some $n, m \geq 2$ and $f(x) \in k[x]$, where $f(x) = x^l g(x)$, $l \geq 2$ and $g(x) \neq 1$ is monic and relatively prime to $x$.
- If $J$ is the right ideal of $k(s, t)$ generated by $(s + t)^l$, then $V_1 = (J + I)/I$ is finite dimensional and non-trivial.
- $V$ is infinite dimensional.

The aim of this note is to give a negative answer to the question of the linearity of Tate’s trace for the sum of two endomorphisms by offering a counter-example of this property. For this counter-example we consider a nilpotent endomorphism over a space $V$ of countable dimension with diagonal entries $1, 0, 0, 0, \ldots$—exhibited in [3].
2. Counter-example of the linearity property of the Tate’s trace

Let $V$ be a vector space of countable dimension over an arbitrary ground field $k$. Let \( \{e_1, e_2, e_3, \ldots \} \) be a basis of $V$ indexed by the natural numbers. Let us consider the linear nilpotent operator $\theta_1$ of $V$ defined in [3] by:

\[
\theta_1(e_i) = \begin{cases} 
0 & \text{if } i \text{ is odd} \\
e_{i-1} & \text{if } i \text{ is even}
\end{cases}
\]

If we denote by \( \{v_j\}_{j \in \mathbb{N}} \) the basis constructed according to the following scheme:

\[
v_1 = e_2, v_2 = e_2 + e_4, v_3 = e_1 + e_4, v_{2i} = e_{2i} + e_{2i+2}, v_{2i+1} = e_{2i-1} + e_{2i+2} \text{ for all } i \geq 1,
\]

it is easy to check that the matrix associated with $\theta_1$ in the basis $\{v_j\}_{j \in \mathbb{N}}$ is:

\[
\theta_1 = \begin{pmatrix}
1 & 0 & -1 & 0 & 1 & 0 & -1 & 0 & 1 & \cdots \\
-1 & 0 & 1 & 0 & -1 & 0 & 1 & 0 & -1 & \cdots \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & -1 & -1 & 0 & 1 & 0 & -1 & 0 & 1 & \cdots \\
0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & -1 & -1 & 0 & 1 & 0 & -1 & \cdots \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}
\]

Note that the explicit expression of $\theta_1$ in that basis is:

\[
\theta_1(v_i) = \begin{cases} 
v_1 - v_2 + v_3 & \text{if } i = 1 \\
v_3 - v_4 + v_5 & \text{if } i = 2 \\
-v_1 + v_2 - v_4 + v_5 & \text{if } i = 3 \\
v_{2k+1} - v_{2k+2} + v_{2k+3} & \text{if } i = 2k \text{ for } k \geq 2 \\
(-1)^k v_1 + (-1)^{k+1} v_2 + \left[ \sum_{j=4}^{i-1} (-1)^{j+k} v_{2j} \right] - v_{i+1} + v_{i+2} & \text{if } i = 2k + 1 \text{ for } k \geq 2
\end{cases}
\]

If $\{v_j\}_{j \in \mathbb{N}}$ is again the basis of $V$ described above, let us now consider the linear operator $\theta_2$ of $V$ defined by:

\[
\theta_2(v_i) = \begin{cases} 
v_1 - v_3 & \text{if } i = 1 \\
v_1 - v_2 - v_3 + v_4 - v_5 & \text{if } i = 2 \\
v_1 - v_2 + v_4 & \text{if } i = 3 \\
v_{4k+2} & \text{if } i = 4k \text{ for } k \geq 1 \\
-i+1 & \text{if } i = 4k + 1 \text{ for } k \geq 1 \\
-v_1 + v_2 + \sum_{j=4}^{i-1} (-1)^{j-1} v_{2j} & \text{if } i = 4k + 2 \text{ for } k \geq 1 \\
0 & \text{if } i = 4k + 3 \text{ for } k \geq 1 \\
v_1 - v_2 + \left[ \sum_{j=4}^{i+1} (-1)^j v_{2j} \right] - v_{i+2} & \text{if } i = 4k + 4 \text{ for } k \geq 1
\end{cases}
\]
A computation shows that $\theta_2$ is nilpotent of order 6 and its matrix with respect to the basis $\{v_j\}_{j \in \mathbb{N}}$ is:

$$
\begin{pmatrix}
1 & 1 & 1 & 0 & -1 & 0 & 1 & 0 & -1 & \cdots \\
0 & -1 & -1 & 0 & 1 & 0 & -1 & 0 & 1 & \cdots \\
-1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 1 & 1 & 0 & -1 & 0 & 1 & 0 & -1 & \cdots \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 1 & 1 & 0 & -1 & 0 & 1 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \\
\end{pmatrix}
$$

Thus, if $\varphi = \theta_1 + \theta_2$, one has that $\varphi$ is a finite potent endomorphism of $V$, and the $\varphi$-invariant decomposition of $V$ referred to above is $V = W_\varphi \oplus U_\varphi$, with $W_\varphi = \langle v_1, v_2 \rangle$ and $U_\varphi = \langle v_r \rangle_{r \geq 3}$.

Regarding the basis $\{v_1, v_2\}$, it is clear that the isomorphism $\varphi|_{W_\varphi}$ is

$$(\theta_1 + \theta_2)|_{W_\varphi} \equiv \begin{pmatrix} 2 & 1 \\ -1 & -1 \end{pmatrix}.$$ 

Accordingly, the explicit expression of the linear operator $\varphi|_{U_\varphi}$ in the basis $\{v_r\}_{r \geq 3}$ is:

$$
\varphi|_{U_\varphi} (v_r) = \begin{cases}
v_5 & \text{if } r = 3 \\
v_{4k+1} + v_{4k+3} & \text{if } r = 4k \text{ for } k \geq 1 \\
v_{4k+3} & \text{if } r = 4k + 1 \text{ for } k \geq 1 \\
v_{4k+3} - v_{4k+4} + v_{4k+5} & \text{if } r = 4k + 2 \text{ for } k \geq 1 \\
0 & \text{if } r = 4k + 3 \text{ for } k \geq 1 
\end{cases}
$$

which is a nilpotent endomorphism of order 4, whose matrix in this basis is:

$$
(\theta_1 + \theta_2)|_{U_\varphi} \equiv \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 1 & 1 & 1 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 1 & 1 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \\
\end{pmatrix}
$$

Hence, $\theta_1$ and $\theta_2$ are nilpotent endomorphisms of $V$, and $\theta_1 + \theta_2$ is a finite potent endomorphism with $\text{Tr}_V (\theta_1 + \theta_2) = 1$. Thus, we obtain a counter-example of the linearity property of Tate’s trace for finite potent endomorphisms and we solve the above referred problem proposed by Tate in [1].
Remark 2.1 From the Argerami–Szechtman–Tifenbach characterizarion of Tate’s trace referred to above, we deduce that there exists a left ideal $I$ of $k\langle s, t \rangle$ such that:

- $I$ contains the two-sided ideal generated by $(s^2, t^6, (s + t)^4 p(s + t))$, where $p(x) = x^2 - x - 1$.
- If $J$ is the right ideal of $k\langle s, t \rangle$ generated by $(s + t)^4$, and $V_1 = (J + I)/I$, then $\dim_k V_1 = 2$, and the trace of the homothety $h_{s+t} \in \text{End}_k(V_1)$ is 1.
- $V = k\langle s, t \rangle/I$ is infinite dimensional.

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References
[1] Tate J. Residues of differentials on curves, Ann. Scient. Éc. Norm. Sup. 1, 4a série; 1968. p. 149–159.
[2] Romo Pablos F. On the linearity property of Tate’s trace. Linear Multilinear Algebra. 2007;55:323–326.
[3] Argerami M, Szechtman F, Tifenbach R. On Tate’s trace. Linear Multilinear Algebra. 2007;55:515–520.