Effective Vortex Dynamics in Superfluid Systems

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An alternative approach to the derivation of the force on a vortex based in an adiabatic approximation in the action of the superfluid system is developed. Assuming that the vortex motion is relatively slow compared with the characteristic times involved in the microscopic degrees of freedom, the effect of the superfluid and its excitations is reduced to a gauge potential, and the associated transverse force. The vortex velocity part of the transverse force is found in terms of the thermal expectation of the angular momentum of the fluid around the vortex. The excitations countercirculate the vortex reducing the effective density to that of the superfluid part only. Non-adiabatic contributions appear in this formalism as a non-Abelian gauge potential connecting different microscopic states, in particular the renormalization of the vortex mass is given by the lowest order diagonal correction to the adiabatic theory, and the long wavelength contributions found to depend only on the static structure factor for the fluid.

INTRODUCTION

Quantized vortices have been an essential part in the theory of superfluids since Onsager first stated the idea of quantized circulation in the late 40’s. In fact, vortices in general have been an important part of classical fluid mechanics for decades, in particular in the study of turbulent regimes. The Magnus force, or Kutta-Joukowski hydrodynamic lift, $F_M = \rho \kappa \times (v_V - v_{\text{fluid}})$ is well known from classical hydrodynamics and occurs whenever an object with circulation $\kappa$ around it moves through a fluid, an important application being the lift force on an airplane wing. The coupling to the fluid is also known to produce a hydrodynamic mass.

The problem of obtaining effective dynamics for adiabatic and non-adiabatic motion of vortices is highly controversial. In particular, the expression for the Magnus force at finite temperatures is still far from clear, and today (more than three decades after the first measurements by Vinen), different expressions for the Magnus force can be found in the literature. The confusion arises, in part, from different interpretations on the role played by excitations being scattered asymmetrically at the vortex, leading to a transverse force proportional to the normal fluid density $\rho_n$ and either the relative velocity $(v_n - v_V)$ or $(v_n - v_s)$, namely the Iordanskii force. The magnitude of this term must be calculated and, moreover, it is not clear whether it should be added to the Magnus force written above, with the coefficient $\rho$, or to a similar expression involving the superfluid part $\rho_s$ only. This paper analyzes the situation for neutral, homogeneous superfluids, while the case of charged superfluids and non translation invariant systems will be published elsewhere.

For fermion superfluids, Volovik and Stone have identified additional contributions from localized quasiparticles inside the vortex core. Thouless, Ao and Niu (TAN) show that no such a contribution exists for translation invariant systems, as seems to be observed for the $B$ phase of $^3$He.

Another point of controversy is the effective mass. Coupling of the vortex motion and the superfluid circulating around it renormalize the mass, and it has been argued by some authors that this renormalized mass is logarithmically divergent with the system size, while others estimate it to be finite or even zero. Recent studies by us and Demircan et al. show that the vortex mass is logarithmically dependent on the frequency, cutting off the size dependence for large systems.

In section II we develop a formalism for the effective action of a vortex moving relative to the fluid. Following TAN, this is accomplished by the introduction of a short range repulsive potential for the particles, strong enough to pin the vortex (a simple example would be a macroscopic solid wire as done in Vinen’s experiment). We follow closely the adiabatic approximation as done by Moody, Shapere and Wilczek, and arrive to a simple exact expression for the effective action involving the vortex coordinates. Expansion of this action yields the vortex velocity part of the Magnus force (VVPMF) to first order in the vortex velocity (sec. III). This force depends on the total circulation around the vortex, which is calculated in section IV. The excitations countercirculate the vortex in a way that is equivalent to stating that only the superfluid fraction has a non-vanishing circulation as stated in TAN. The VVPMF is given by $\rho_s \kappa_s \times v_V$.

The normal and superfluid velocity parts of the Magnus force (NVPMF, SVPMF) require extra work, and will be dealt with in a future publication.
Section IV deals with the second order terms, which renormalize the vortex mass. This mass renormalization depends only on the static structure factors for the fluid. We recover a logarithmically divergent mass in the zero-frequency limit.

I. ADIABATIC EFFECTIVE ACTION FOR A VORTEX

Following reference TAN, we control the motion of a vortex by means of a pinning potential, consisting in a short-range repulsive potential for the particles that form the fluid. At this stage we only require this interaction to be a pure potential $V(\mathbf{R}, \eta)$ without any velocity dependent part, where $\mathbf{R}$ is the vortex coordinate and $\eta \equiv \{r_i\}$ is the set of all particle coordinates. We follow Moody, Shapere and Wilczek, rather closely, avoiding some sign inconsistencies along the way. We can immediately write the Lagrangian of the system as

$$ L = L_{\text{bare}} + l_{\text{fluid}}(\mathbf{R}, \eta), $$

where $L_{\text{bare}}$ is the Lagrangian of the bare pinning center and $l_{\text{fluid}}$ incorporates the full Lagrangian of the fluid and the repulsive potential $V(\mathbf{R}, \eta)$. Notice that we have imposed no condition on the form of the Lagrangians, except for the type of interaction between the pinning center and the fluid particles.

The full time-evolution kernel of the system can now be written in terms of Feynman path integrals over all possible configurations:

$$ U(\mathbf{R}_f, \eta_f, t_f; \mathbf{R}_i, \eta_i, t_i) \equiv \langle \phi_{m\mathbf{R}_f, \eta_f, t_f} | \phi_{m\mathbf{R}_i, \eta_i, t_i} \rangle = \int_{\mathbf{R}_i}^{\mathbf{R}_f} \mathcal{D}[\mathbf{R}] \mathcal{D}[\eta] \exp \left\{ \frac{i}{\hbar} \int_{t_i}^{t_f} [L_{\text{bare}} + l_{\text{fluid}}(\mathbf{R}, \eta)] dt \right\}. $$

For a stationary vortex at position $\mathbf{R}$ we can also, in principle, obtain the exact eigenstates of the fluid subsystem

$$ h(\mathbf{R})\phi_n(\mathbf{R}) = \epsilon_n(\mathbf{R})\phi_n(\mathbf{R}), $$

and for any particular value of $\mathbf{R}$, any wavefunction for the fast system can be expanded in this complete basis

$$ \psi = \sum_n \phi_n(\mathbf{R})F_n(\mathbf{R}). $$

We will be interested in finding a kernel $U_{mn}(\mathbf{R}_f, \eta_f, t_f; \mathbf{R}_i, \eta_i, t_i)$ that links the final weights $F_m(\mathbf{R}_f, t_f)$ with the initial state’s $F_n(\mathbf{R}_i, t_i)$:

$$ F_m(\mathbf{R}_f, t_f) = \sum_n U_{mn}(\mathbf{R}_f, \eta_f, t_f; \mathbf{R}_i, \eta_i, t_i) F_n(\mathbf{R}_i, t_i). $$

This kernel has all the information that we need, and is diagonal in the stationary case $\dot{\mathbf{R}} = 0$. Most important is the fact that, for non-degenerate cases, off-diagonal contributions are small. This new kernel is given by

$$ U_{mn}(\mathbf{R}_f, \eta_f, t_f; \mathbf{R}_i, \eta_i, t_i) \equiv \int_{\mathbf{R}_i}^{\mathbf{R}_f} d\mathbf{R} \mathcal{D}[\mathbf{R}] \mathcal{D}[\eta] \exp \left\{ \frac{i}{\hbar} \int_{t_i}^{t_f} \left[ L_{\text{bare}} + U_{\text{fluid}} \right] dt \right\} \langle \phi_{m\mathbf{R}_f, \eta_f, t_f} | \phi_{n\mathbf{R}_i, \eta_i, t_i} \rangle, $$

$$ U_{\text{fluid}} \equiv \int_{\mathbf{R}_i}^{\mathbf{R}_f} d\mathbf{R} \mathcal{D}[\mathbf{R}] \mathcal{D}[\eta] \exp \left\{ \frac{i}{\hbar} \int_{t_i}^{t_f} l_{\text{fluid}}(\mathbf{R}, \eta) dt \right\} \langle \phi_{m\mathbf{R}_f, \eta_f, t_f} | \phi_{n\mathbf{R}_i, \eta_i, t_i} \rangle. $$

We can calculate this propagator in the usual way by dividing the time interval $[t_i, t_f]$ into $N$ segments and inserting the identity operator at each time. Taking $N \rightarrow \infty$ yields

$$ U_{mn}^{\text{fluid}} = \sum_{\Delta t = -\epsilon_m \delta_{mn}} T \exp \left\{ \frac{i}{\hbar} \int_{t_i}^{t_f} dt \left[ -\epsilon_m \delta_{mn} + i \hbar \dot{\mathbf{R}}(t) \cdot \left( \phi_{m|\mathbf{R}} \nabla_R \phi_n |\mathbf{R}\right) \right] \right\} |\phi_{\eta_i}, t_i\rangle \langle \phi_{\eta_f}, t_f|, $$

$$ = \sum_{\Delta t = -\epsilon_m \delta_{mn}} T \exp \left\{ \frac{i}{\hbar} \int_{t_i}^{t_f} dt \left[ -\epsilon_m \delta_{mn} + i \hbar \dot{\mathbf{R}}(t) \cdot \left( \phi_{m|\mathbf{R}} \nabla_R \phi_n |\mathbf{R}\right) \right] \right\} |\phi_{\eta_i}, t_i\rangle \langle \phi_{\eta_f}, t_f|. $$
where the last expression is formal way of denoting an infinite number of time ordered matrix products. The time evolution of the pinning center can now be written in terms of an effective action

$$U_{mn}(R_f, t_f; R_i, t_i) = \int_{R_i}^{R_f} D[R] T e^{i S_{\text{eff}}^{(2)}},$$

$$S_{\text{eff}}^{(2)} = \int_{t_i}^{t_f} dt [L_{\text{bare}} - \epsilon_m \delta_{mn} + i \hbar \dot{R} \cdot (\phi_m | \nabla R \phi_n)].$$

The results insofar are exact, but not very helpful. In general, the gauge-like term $i \hbar \dot{R} \cdot (\phi_m | \nabla R \phi_n)$ is non-Abelian, and we have to deal with complicated time ordered products, etc. For non-degenerate states, however, it is possible to show that non-diagonal terms vanish faster than any power of $(\tau \Delta \epsilon/\hbar)^{-1}$, with $\tau$ being a characteristic time of motion of the pinning center, and $\Delta \epsilon$ the smallest energy difference of the fluid’s dynamics (see ref. [2]). In most cases it will be possible to greatly simplify the problem by only dealing with a few degenerate or closely degenerate states.

For non-degenerate cases, transitions to other states may be neglected and we can easily calculate the effective action for a given state of the fluid to any order in the vortex velocity. In particular, the Magnus force is linear in the velocities, and effective masses correspond to quadratic terms in the action. Expanding to these orders, the effective action for the vortex motion is simply given by

$$S_n = \int_{t_i}^{t_f} dt \left[ L_{\text{bare}} - \epsilon_n + i \hbar \dot{R}(t_j) \cdot (\phi_n | \nabla R \phi_n) + \frac{1}{2} \sum_{ij} M_{nj}^{ij} \dot{R}_i \dot{R}_j \right],

M_{nj}^{ij} = 2 \hbar^2 \sum_{\ell \neq n} \frac{(\partial \phi_n | \phi_\ell) (\phi_\ell | \partial \phi_n)}{\epsilon_\ell - \epsilon_n}.
\tag{12}$$

II. VORTEX VELOCITY PART OF THE MAGNUS FORCE

From the effective action [11], we can immediately find the transverse VVPMF for any given quantum state $\phi_n$ of the fluid:

$$F_n = i \hbar \dot{R} \times (\nabla R \times (\phi_n | \nabla R \phi_n)).
\tag{13}$$

Identifying $v_V = \dot{R}$, we now perform a statistical averaging over initial states to obtain the thermally averaged VVPMF

$$F \times \hat{z} = -i \hbar v_V \sum_n \int f_n \left[ \frac{\partial}{\partial X} (\phi_n | \partial \phi_n) - \frac{\partial}{\partial Y} (\phi_n | \partial \phi_n) \right] \left[ (\partial \phi_n | \partial \phi_n) - (\partial \phi_n | \partial \phi_n) \right],$$

$$= -i \hbar v_V \sum_n \int f_n \left[ (\partial \phi_n | \partial \phi_n) - (\partial \phi_n | \partial \phi_n) \right] \left[ (\partial \phi_n | \partial \phi_n) - (\partial \phi_n | \partial \phi_n) \right],
\tag{14}$$

where $f_n$ is the probability of occupation of the state $\phi_n$. This is the familiar form of the Berry curvature [2]. For a homogeneous system it is possible to substitute $\nabla R = -\sum_i \nabla_i$, and therefore

$$F/v_V = -i \hbar \sum_{i,j} \sum_n f_n \left[ \frac{\partial \phi_n | \partial \phi_n}{\partial x_i} - \frac{\partial \phi_n | \partial \phi_n}{\partial y_j} \right] \left[ (\partial \phi_n | \partial \phi_n) - (\partial \phi_n | \partial \phi_n) \right]$$

$$= -i \hbar \hat{z} \cdot \int d^2 r [\nabla \times \nabla \rho(r', r)]_{r=r'} - i \hbar \int d^2 r \int d^2 r' [2 \nabla_1 \times \nabla_2' \Gamma(r_1', r_2'; r_1, r_2)]_{r=r'},
\tag{15}$$

The last term in the equation above vanishes, since the first line above is the commutator of the $x$ and $y$ components of the total momentum which is a one particle operator [14]. The integrand of the first term is equal to $\nabla \times (\nabla - \nabla') \rho(r, r')/2$. Application of Stokes’ theorem can be used to write the force per unit length
\[ F/v = -\frac{i\hbar}{2} \oint_{\Gamma} d\mathbf{l} \cdot [(\nabla - \nabla')\rho(r, r')]_{r=r'}. \] (16)

This result is exact, and the contour of integration \( \Gamma \) can be taken as far from the vortex as one wishes, and there are no contributions from the vicinity of the core.

For a neutral superfluid the integrand is just the momentum density \( \mathbf{j} \). At this stage one can decompose the momentum density in a two fluid picture as \( \mathbf{j} \equiv \rho_s \mathbf{v}_s + \rho_n \mathbf{v}_n \), and the vortex velocity part of the Magnus force will be given by the superfluid and normal mass densities times the superfluid and normal circulations

\[ F/v = \oint_{\Gamma} (\rho_s \mathbf{v}_s + \rho_n \mathbf{v}_n) \cdot d\mathbf{l} = \rho_s \kappa_s + \rho_n \kappa_n. \] (17)

The circulation of the superfluid \( \kappa_s \) is quantized to multiples of \( \hbar/m \), and both \( \rho_s \) and \( \rho_n \) are well defined quantities.

TAN have argued for the normal fluid to have no circulation. However, the circulation of a normal fluid is something that has to be explicitly calculated\[24\], which is done in the following section.

III. CIRCULATION OF THE NORMAL FLUID

As stated before, one cannot in general assume the value of the circulation of a regular fluid. Aeronautical engineers put up a fair amount of effort to calculate the circulation around a particular wing section. Fortunately our case is considerably simpler. We follow the method used by Landau to calculate the normal density\[25\] in order to obtain the circulation of the normal fluid. The VVPMF is then completely defined in terms of macroscopic transport parameters.

Note that the expression for the Magnus force is given in terms of the circulation, and that it can be defined by any path circling the vortex. Consider a vortex centered in a cylindrical container and “average” over all possible paths that go once around the vortex

\[ F/v_V = \frac{1}{L_z} \int_0^{L_z} dz \frac{2}{R^2} \int_0^R r dr \oint_{\Gamma(z,r)} j \cdot d\mathbf{l} = 2\pi \frac{1}{\pi R^2 L_z} \int d^3 x \hat{z} \cdot (r \times j) = 2\pi \frac{L_z}{V}. \] (18)

The Magnus force per unit length is just \( 2\pi \) times the average angular momentum \( \mathcal{L}_z \) per unit volume \( V \). At zero temperature, all the fluid is superfluid, each particle averages and angular momentum \( \bar{\hbar} \). At zero temperature (or in the ground-state), the Magnus force is simply given by

\[ F_0/v_V = 2\pi \frac{\hbar N}{V} = \rho \frac{\hbar}{m}. \] (19)

Finite temperatures can be analyzed by adding the effect of excitations. At low temperatures excitations can be dealt in a dilute gas approximation. Moreover, phonons become predominant and we may neglect rotons. At this stage one wishes to calculate the expectation value of the angular momentum of the excitations. The presence of the vortex couples the excitations to the superfluid velocity field, Doppler shifting the energy of excitations, thus increasing the occupation of countercirculating modes relative to those that circulate in the direction of the superfluid.

The expectation of the angular momentum of the excitations is simply given by thermally populating the different phonon modes (labeled in cylindrical coordinates by the radial and axial wave-numbers, and by the azimuthal angular momentum):

\[ \mathcal{L}_{\text{excit}} = \sum_{k,\ell,m} \hbar m \frac{1}{e^{h\omega/k_B T} - 1}. \] (20)

The mode frequencies can be obtained from wave equations derived from either a Feynman many-body state\[26\] or a non-linear Gross-Pitaevskii theory\[27\]. From this latter point of view, we start with a non-linear Schrödinger equation to describe a weakly interacting Bose gas, and after some algebra (Appendix A) get the following wave equation for phonons in presence of a static vortex:

\[ c^2 \nabla^2 \psi - \ddot{\psi} - \frac{\kappa_s}{\hat{r}} \hat{\phi} \cdot \nabla \dot{\psi} = 0. \] (21)

This last term decreases the frequency of phonons countercirculating the vortex. In a cylindrical geometry, the eigenstates can be labeled by the angular momentum, radial and azimuthal wave-vectors. To a good approximation the normal mode frequencies are given by:
\[ \omega \approx c \sqrt{k_r^2 + k_z^2} + m \frac{\kappa_s}{2\pi} \left( \frac{1}{r^2} \right). \] (22)

Using this information in equation (21) we get to first order in the superfluid circulation

\[ L_{\text{excit}} \approx -\frac{\hbar^2 \kappa_s}{2\pi k_B T} \sum_{k_r, k_z, m} m^2 \left( \frac{1}{r^2} \right)^2 \frac{e^{\bar{h}ck/k_B T}}{(e^{\bar{h}ck/k_B T} - 1)^2}, \] (23)

where now the sum is performed on the normal modes in absence of the vortex. We now assume that the typical inter-phonon equilibration distance is much smaller than the size of our container, which might be rather unrealistic at very low temperatures \[ \overline{e} \] but makes sense for a theoretical infinite volume, infinite time-scale limit. In this limit, equilibration occurs locally, rather than at the boundaries of the system, and we may simplify the summation over modes by a sum over uniformly distributed states in phase space \( \sum_{k_r, k_z, m} \rightarrow \int d^3r \int (2\pi)^3, m \rightarrow (r \times k)_z \) and \( \approx 1/r^2 \rightarrow 1/r^2 \):

\[ L_{\text{excit}} \approx -\frac{\hbar^2 \kappa_s}{2\pi k_B T} (\pi R^2 L_z) \int_0^{2\pi} \sin^2 \varphi_k \, d\varphi_k \int_0^\infty \, dk_r \int_{-\infty}^\infty \, dk_z \, k_r^3 \frac{e^{\bar{h}ck/k_B T}}{(e^{\bar{h}ck/k_B T} - 1)^2} = -(\pi R^2 L_z) \frac{2\pi^2 (k_B T)^4}{45} \frac{\kappa_s}{\hbar^3 e^5} \frac{\pi R}{\kappa_s}, \] (24)

which shows the excitations are indeed giving some countercirculation around the vortex. The total angular momentum is just given by the sum of the ground state (or zero-temperature) angular momentum and that of the excitations, which for \( \kappa_s = \hbar/m \) yields

\[ L = L_0 + L_{\text{excit}} = \frac{\pi R^2 L_z}{2\pi} \frac{h}{m} \left[ \rho - \frac{2\pi^2 (k_B T)^4}{45} \right] = \frac{\pi R^2 L_z}{2\pi} \frac{h}{m} \left[ \rho - \rho_n \right] = \frac{\pi R^2 L_z}{2\pi} \frac{h}{m} \rho_s, \] (25)

where the last two identities follow from identifying the second term in the brackets as the normal fluid density due to phonons \[ \overline{e} \]. This magnitude of the excitation countercirculation is equivalent to the non-circulation of the normal fluid in the two fluid model. Application of equation (18) gives the correct vortex velocity part of the Magnus force

\[ F_{\text{vortex}} = \rho_s \frac{h}{m} \hat{z} \times \hat{v}_V. \] (26)

Where the result above was calculated with \( v_s = v_n = 0 \). Next section deals with the coefficient of the force proportional to \( v_n \), that is the NVPMF.

IV. VORTEX MASS

In this section we calculate the renormalization of the vortex mass in the static limit. Vortex mass calculations have been performed starting from the hydrodynamic equations\[ \overline{e} \], and here we approach the problem from a different (albeit equivalent) point of view: from the second order correction to the adiabatic approximation in the effective action (eq. [1] and [2]). This second order adiabatic approximation yields correct results for the long wavelength contribution to the vortex mass.

For a superfluid, one can write Feynman’s many-body wavefunction\[ \overline{e} \] for a static vortex at \( R \) as

\[ |\psi\rangle \approx \prod_i e^{i\theta(r_i - R)} |\phi_0\rangle, \] (27)

where \( |\phi_0\rangle \) is the many-body ground state. In this simplified case

\[ |\partial_X \psi\rangle = \sum_i \frac{i y_i}{x_i^2 + y_i^2} |\psi\rangle = \int dr \frac{i y}{x^2 + y^2} \sum_i \delta(r - r_i) |\psi\rangle = \int dr \frac{i y}{x^2 + y^2} \rho(r) |\psi\rangle, \] (28)

where \( \rho(r) \) is the density matrix. This can be Fourier transformed to read

\[ |\partial_X \psi\rangle = -4\pi^2 \int \frac{dk'}{(2\pi)^3} \delta(k_z') \frac{k_r'}{k_r^2 (k_z'^2 + k_r'^2)} |\psi\rangle. \] (29)

It is clear that the only significant couplings in eq. (14) will be to states that differ from the “one vortex ground state” by \( \rho \), that is the one vortex plus one phonon states:

\[ |\partial_X \psi\rangle = -4\pi^2 \int \frac{dk'}{(2\pi)^3} \delta(k_z') \frac{k_r'}{k_r^2 (k_z'^2 + k_r'^2)} |\psi\rangle. \] (29)
\[ |k\rangle \approx \rho_k |\psi\rangle. \] (30)

We can readily calculate the overlap

\[
\langle k| \partial_X |\psi\rangle / \langle k|k\rangle^{1/2} = -4\pi^2 \int \frac{dk'}{(2\pi)^3} \delta(k'_z) k'_y \langle |\psi| \rho^+_k \rho_k |\psi\rangle / \langle |\psi| \rho^+_k \rho_k |\psi\rangle^{1/2} \] (31)

Under the approximations used so far the correlation in the last term can be substituted by the ground state correlation

\[
\langle |\psi| \rho^+_k \rho_k |\psi\rangle \approx \langle \phi_0| \rho^+_k \rho_k |\phi_0\rangle = N S(k) \delta(k). \] (32)

Where \( S(k) \) is the static structure factor and, in neutral superfluids, is equal to \( k/(2mc) \) in the long wavelength limit (\( c \) is the speed of sound). We are left with

\[
\langle k| \partial_X |\psi\rangle / \langle k|k\rangle^{1/2} = -4\pi^2 \frac{\rho}{m} \delta(k_z) k_y \left( \frac{S(k)}{N} \right)^{1/2}. \] (33)

Given that \( \epsilon(k) \approx k^2/(2mS(k)) \), we can write

\[
M^{xy}_0 = M^{yx}_0 = 0, \] (34)

\[
M^{xx}_0 = M^{yy}_0 = 4\hbar^2 \rho L_z \int d^2k k_y^2 S(k)^2 = \frac{\pi\hbar^2 \rho L_z}{m^2 c^2} \int_{k_{min}}^{k_{max}} \frac{dk}{k} = \frac{\pi\hbar^2 \rho L_z}{m^2 c^2} \log \left( \frac{R}{a} \right), \] (35)

Where \( R \) and \( a \) are large and small distances cutoffs. In the static limit one obtains the size dependent logarithmic divergence of the mass as predicted by Duan from the hydrodynamic equations of the superfluid. A finite frequency calculation will cut \( R \), by the phonon wavelength \( R \to \lambda = 2\pi c/\omega \).

A very illustrative fact is that the additional vortex mass is related to the energy of a static vortex by the Einstein relation \( E_{\text{static}} = M_0 c^2 \). Where

\[
E_{\text{static}} = \int dr \frac{1}{2} \rho v^2 = \frac{\pi\hbar^2 \rho}{m^2} \log \left( \frac{R}{a} \right). \] (36)

This is not surprising, and should be expected for any wave-equation like dynamics like the ones involved for phonons.

V. CONCLUSIONS

Starting from an adiabatic expansion for the effective action of a vortex we have obtained, in a closed and coherent way, the vortex velocity dependent part of the transverse force acting on a rectilinear vortex in a neutral and homogeneous superfluid.

In this circumstances, the force that depends on the vortex velocity can be completely determined by conditions far away from the vortex core and is proportional to the superfluid density, which is correct in light of Vinen’s experiment.

Within the formalism used, the renormalization of the vortex mass depends only on the form of the static structure factor for the fluid. The obtained result coincides with similar calculations derived from hydrodynamic equations.

More research is currently under way to completely determine normal and superfluid velocities dependent part of the transverse force, and to elucidate the effects caused by charge and the loss of translation invariance, as well as the roles of phonon radiative damping and the longitudinal part of the Iordanskii force.

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APPENDIX A: WAVE EQUATION FOR PHONONS IN PRESENCE OF A VORTEX

Starting from the Gross-Pitaevskii nonlinear Schrödinger equation\[ \text{eq}\]
\[
i\hbar \frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m} \nabla^2 \Psi + \lambda (|\Psi|^2 - \rho_0) \Psi,
\]
(A1)
a Madelung transformation\[ \text{eq}\] \[ \Psi = \sqrt{\rho} e^{iS} \] leaves us with the continuity equation and the Euler-Bernoulli equation\[ \text{eq}\] :
\[
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0,
\]
(A2)
\[
\frac{\hbar}{m} \ddot{S} + \frac{v^2}{2} + c^2 \frac{\rho - \rho_0}{\rho_0} = 0.
\]
(A3)

Where \( \mathbf{v} = \frac{\hbar}{m} \nabla S \) and we have left out the “quantum pressure” term, which is irrelevant for our purposes. After some algebra, separating the static phase due to the vortex and linearizing, one obtains the following equation of motion for the phonon part of the phase:
\[
c^2 \nabla^2 \psi - \ddot{\psi} - \frac{\kappa_s}{\pi \rho} \dot{\phi} \cdot \nabla \psi = 0.
\]
(A4)

This expression is not valid in the immediate vicinity of the vortex, where the assumptions made are not valid. In general this will only be important when either dealing with short-wavelength excitations or the s-wave modes, that can reach these regions.

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