Abstract
The paper is devoted to function theory on symplectic manifolds. We study a natural class of functionals involving the double Poisson brackets from the viewpoint of their robustness properties with respect to small perturbations in the uniform norm. We observe a hierarchy of such robustness properties. The methods involve Hofer’s geometry on the symplectic side and Landau-Hadamard-Kolmogorov inequalities on the function-theoretic side.
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1 Introduction and main results

A mainstream topic of modern symplectic topology is the study of rigidity properties of subsets and Hamiltonian diffeomorphisms of symplectic manifolds. A number of recent developments (see [3, 5, 21, 8, 6, 2]) show that there is another manifestation of symplectic rigidity which takes place on function spaces associated to a symplectic manifold. This circle of problems, which we call function theory on symplectic manifolds, lies in the focus of the present paper.

1.1 A dichotomy

Let \((M, \omega)\) be a connected symplectic manifold (open or closed). Denote by \(C_c^\infty(M)\) the space of smooth compactly supported functions on \(M\) equipped with the Poisson bracket \(\{F, G\}\). Write \(\| \cdot \|\) for the standard uniform norm (also called the \(C^0\)-norm) on it: \(\|F\| := \max_{x \in M} |F(x)|\). As it was shown in [6] the functionals \((F, G) \mapsto \max\{F, G\}\) and \((F, G) \mapsto -\min\{F, G\}\) are lower semicontinuous on \(C_c^\infty(M) \times C_c^\infty(M)\) with respect to the uniform norm. Thus, even though the Poisson bracket of a pair of functions is defined via their first derivatives, it exhibits a robust behavior under \(C^0\)-small perturbations.

In the present paper we explore the double Poisson bracket. We focus on non-negative\(^1\) functionals \(\Phi^v(F, G)\) of the form

\[
\Phi^v(F, G) = v_1 \cdot \max\{\{F, G\}, F\} - v_2 \cdot \min\{\{F, G\}, F\} \\
+ v_3 \cdot \max\{\{F, G\}, G\} - v_4 \cdot \min\{\{F, G\}, G\},
\]

where \(v = (v_1, v_2, v_3, v_4) \in \mathbb{R}^4\) is a non-zero vector with non-negative entries. Interestingly enough, these functionals exhibit different patterns of behavior depending on \(v\). To highlight the phenomenon, given a non-negative functional \(\Phi(F, G)\), we form a new functional

\[
\Phi(F, G) := \lim inf_{F', G' \overset{C^0}{\longrightarrow} F, G} \Phi(F', G')
\]

\(^1\)In the case of a closed \(M\) the mean value of a Poisson bracket of two functions on \(M\) is zero, hence its minimum is non-positive and its maximum is non-negative. In the open case the latter properties are obvious since we work with compactly supported functions.
and introduce the following terminology: A functional $\Phi$ is called \textit{weakly robust} if $\overline{\Phi}(F, G) > 0$ whenever $\Phi(F, G) > 0$. With this language $\Phi$ is \textit{lower semicontinuous} if $\overline{\Phi} = \Phi$.

**Theorem 1.1** (Dichotomy).

(i) If either $v_3 = v_4 = 0$ or $v_1 = v_2 = 0$, $\Phi^v$ is weakly robust but not lower semicontinuous;

(ii) If at least one of $v_1, v_2$ is positive and at least one of $v_3, v_4$ is positive, $\Phi^v$ is lower semicontinuous.

In order to verify the failure of lower semicontinuity in (i) we proceed as follows. First we consider the case when $(M, \omega) = (\mathbb{R}^2(p, q), dp \wedge dq)$. We put $F = F(p), G = G(q)$, and

$$F_N(p, q) = F(p) + \frac{1}{N} a(q) \sin NF(p).$$

Note that $F_N \overset{C^0}{\to} F$ as $N \to \infty$. We construct a rather explicit example of the functions $F, G, a$ so that for sufficiently large $N$

$$\max(\pm \{\{F_N, G_N\}, F_N\}) < 0.99 \cdot \max(\pm \{\{F, G\}, F\}).$$

This example can be implanted into arbitrary symplectic manifolds which eventually yields the desired result (see Section 4 and 5 for the details). The remaining statements of the Dichotomy Theorem deserve a more detailed discussion.

### 1.2 Landau-Hadamard inequality for the Poisson bracket

Let $(M, \omega)$ be a connected symplectic manifold of dimension $2n$. Put $\text{osc} F := \max F - \min F$. Note that, given $F, G \in C_c^\infty(M)$, one has $\int_M \{F, G\} \omega^n = 0$, and thus $\text{osc} \{F, G\} \geq ||\{F, G\}||$.

The next inequality is a variant of the classical Landau-Hadamard inequality [14, 7], cf. [18], in the context of the Poisson brackets:

**Proposition 1.2.** Assume $F, G \in C_c^\infty(M)$ and $G \not\equiv 0$. Then

$$\max\{\{F, G\}, F\} \geq \frac{||\{F, G\}||^2}{2 \cdot \text{osc} G},$$

(2)
The proof is given in Section 2 below. Since, as we mentioned at the beginning of the paper, the functional \( \| \{ F, G \} \| \) is lower semicontinuous [6], we readily get that
\[
\liminf_{F', G' \to F, G} \max\{ \{ F', G' \}, F' \} \geq \frac{\| \{ F, G \} \|^2}{2 \cdot \text{osc} \, G} > 0 ,
\]
which implies that the functional \( \max\{ \{ F, G \}, F \} \) is weakly robust. After elementary algebraic manipulations this yields weak robustness in Theorem 1.1(i), see Section 5.

1.3 The convergence rate

In the case when a functional \( \Phi \) is lower semicontinuous we investigate the “convergence rate” in the limit (1) (cf. [5]). For every \( \epsilon > 0 \) put
\[
\overline{\Phi}_\epsilon(F, G) = \inf_{\|F' - F\| \leq \epsilon, \|G' - G\| \leq \epsilon} \Phi(F', G') .
\]
We are interested in upper bounds for the difference
\[
\Phi(F, G) - \overline{\Phi}_\epsilon(F, G)
\]
in terms of \( F, G \) and \( \epsilon \) as \( \epsilon \to 0 \). L.Buhovsky [2] discovered such a (sharp!) upper bound for the functional \( \Phi(F, G) = \max\{ F, G \} \). To state this result, we put
\[
\Psi(F, G) := \| \{ \{ \{ F, G \}, F \}, F \} + \{ \{ F, G \}, G \}, G \} | | .
\]
One can show that \( \Psi(F, G) > 0 \) provided \( \{ F, G \} \neq 0 \) (see Corollary 2.2 below). With this notation, Buhovsky derived the following 2/3-law: There exists \( \epsilon_0(F, G) > 0 \) such that for any \( 0 < \epsilon < \epsilon_0(F, G) \)
\[
\Phi(F, G) - \overline{\Phi}_\epsilon(F, G) \leq C \cdot \Psi(F, G)^{\frac{2}{3}} \epsilon^{\frac{2}{3}} ,
\]
where \( C > 0 \) is a numerical constant. Furthermore, Buhovsky showed that his estimate captures sharp asymptotics in \( \epsilon \). Buhovsky’s proof of (5) is based on an ingenious application of the energy-capacity inequality. In Section 3.2 we reprove (5) by our methods.

For the case of the double bracket we have the following estimate of the convergence rate:
Theorem 1.3 (Convergence rate). Let
\[ \Phi(F, G) = \max\{\{F, G\}, F\} + \max\{\{F, G\}, G\} . \]

Then for every \( \epsilon > 0 \)
\[ \Phi(F, G) - \Phi_{\epsilon}(F, G) \leq C(F, G) \cdot \epsilon^{\frac{1}{3}} , \quad (6) \]
where \( C(F, G) \) is a positive constant depending on \( F \) and \( G \).

The proof is given in Section 3.3 below. It is unclear to us whether the theorem above can be improved:

Question 1.4. Is the power law \( \epsilon^{\frac{1}{3}} \) in inequality (6) asymptotically sharp as \( \epsilon \to 0 \)?

We return to this question in Remark 3.4 below.

As an immediate consequence of inequality (6) we get that the functional \( \Phi \) is lower semicontinuous. With a little extra work, we deduce from this Theorem 1.1(ii), see Section 5.

For the proof of Theorem 1.3 we use the approach initiated in [6] which is based on the following ingredient from “hard” symplectic topology: Denote by \( \text{Ham}^c(M) \) the group of Hamiltonian diffeomorphisms of \( M \) generated by Hamiltonian flows with compact support. Then sufficiently small segments of one-parameter subgroups of the group \( \text{Ham}^c(M) \) of Hamiltonian diffeomorphisms of \( M \) minimize the “positive part of the Hofer length” among all paths on the group in their homotopy class with fixed end points. This was proved by D.McDuff in [16, Proposition 1.5] for closed manifolds and in [17, Proposition 1.7] for open ones; see also [1], [13], [4], [15], [9], [19] for related results in this direction. In fact, our method readily generalizes to any (in general, “infinite-dimensional”) Lie group equipped with a bi-invariant (Finsler) semi-norm, provided sufficiently short segments of 1-parameter subgroups are minimal geodesics. It would be interesting to formalize this remark and to find new significant examples.

The results discussed above can be viewed as a symplectic counter-part of the following classical problem of approximation theory: find the best uniform approximation of a given function (say, of a periodic function of one real variable) by functions with given bounds on derivatives. This problem
was solved in the 1960s, see Sections 6.2.1 and 7.2.3 of Korneichuk's book [12] and the references therein. For instance, one can extract from Korneichuk's results that the functional taking a smooth periodic function $u$ on $\mathbb{R}$ to the uniform norm of its derivative is lower semicontinuous in the uniform norm and obeys the $2/3$-law (see [2] for a direct proof and some generalizations). It would be interesting to explore further the connection between approximation theory and function theory on symplectic manifolds.

**Organization of the paper.** In Section 2 we prove a version of the Landau-Hadamard inequality for the double Poisson bracket and thus complete the proof of weak robustness of the functional $\max\{\{F,G\}, F\}$. In Section 3, after recalling some preliminaries on Hofer's geometry, we give a new proof of Buhovsky’s $2/3$-law (5) for the ordinary Poisson bracket and prove the Convergence Rate Theorem for the double bracket. In Section 4 we construct an example which in particular shows that the functional $\text{osc} \{\{F,G\}, F\}$ is not lower semicontinuous. The proof of the Dichotomy Theorem is completed in Section 5. Finally, in Section 6 we present some generalizations of our results on weak robustness to higher iterated Poisson brackets and formulate open problems.

**Convention on the Poisson bracket:** Our convention concerning the Poisson bracket is as follows: Given a Hamiltonian function $G \in C^\infty_c(M)$, its Hamiltonian vector field, denoted by $sgrad G$, is defined by the condition $dG(\cdot) := \omega(\cdot, sgrad G)$. The Poisson bracket of a pair of functions $F, G \in C^\infty_c(M)$ is then defined by
\[
\{F, G\} := \omega(sgrad G, sgrad F) = dF(sgrad G) = \frac{d}{dt} \big|_{t=0} F \circ g_t ,
\]
where $g_t$ is the Hamiltonian flow generated by $G$.

## 2 The Landau-Hadamard inequality

We shall need the following version of the classical Landau-Hadamard inequality.

**Proposition 2.1.** Let $u$ be a non-constant twice differentiable function on $\mathbb{R}$ which is bounded with its two derivatives. Assume that $|u'|$ attains its maximal value. Then
\[
\sup u'' \geq \frac{||u'||^2}{2 \cdot \text{osc} \ u} .
\]
Proof. Assume without loss of generality that $\|u\| = |u'(0)|$. Denote $A = \sup u''$. For every $t$

$$\text{osc } u \geq u(0) - u(t) = -tu'(0) - \int_0^t \int_0^s u''(z) \, dz \geq -tu'(0) - At^2/2. \tag{8}$$

Note that $A > 0$, otherwise $u$ is either constant or unbounded. Substituting $t = -u'(0)/A$ into (8) we get inequality (7).

**Proof of Proposition 1.2**: Take a point $x \in M$ so that $\|\{F,G\}\| = \{F,G\}(x)$. Denote by $f_t$ the Hamiltonian flow of $F$ and put $u(t) := -G(f_t x)$. Then $u'(t) = \{F,G\}(f_t x)$ and $u''(t) = \{\{F,G\}, F\}(f_t x)$. Note that

$$\sup u'' \leq \max\{\{F,G\}, F\}, \|u'\| = \|\{F,G\}\|, \text{ osc } u \leq \text{ osc } G.$$

Applying inequality (7) to $u$ we get that

$$\max\{\{F,G\}, F\} \geq \frac{\|\{F,G\}\|^2}{2 \cdot \text{ osc } G},$$

as required.

**Corollary 2.2.** Let $F, G \in C^\infty_c(M)$ be a pair of functions with $\{F,G\} \neq 0$. Then

$$I(F,G) := \{\{F,G\}, F\} + \{\{F,G\}, G\} \neq 0.$$  

**Proof.** We shall use the following classical identity which readily follows from the definition of the Poisson bracket and the Stokes formula:

$$\int_M \{P, Q\} R \omega^n = \int_M \{R, P\} Q \omega^n$$

for all functions $P, Q, R \in C^\infty_c(M)$. This implies that

$$\int_M I(F,G) \cdot \{F,G\} \omega^n = -\int_M \{\{F,G\}, F\}^2 + \{\{F,G\}, G\}^2 \omega^n.$$

If $I(F,G) \equiv 0$ we have that $\{\{F,G\}, F\} \equiv 0$. By Proposition 1.2 this yields $\{F,G\} = 0$. □
3 Poisson bracket via Hofer’s geometry

3.1 Preliminaries

Let \((M, \omega)\) be a connected symplectic manifold of dimension \(2n\). Write \(\tilde{\text{Ham}}(M)\) for the universal cover of the group \(\text{Ham}(M)\) of Hamiltonian diffeomorphisms of \((M, \omega)\), where the base point is chosen to be the identity map \(\text{Id}\). Denote by \(\phi_H^t \in \tilde{\text{Ham}}(M)\), \(t \in \mathbb{R}\), the lift of the Hamiltonian flow generated by a (time-dependent) Hamiltonian \(H\). We set \(\phi_H := \phi_H^1\) and say that \(\phi_H\) is generated by \(H\). If \(H\) is time-independent, we often abbreviate \(\phi_H^t = h_t\).

If \(F\) is a function on \(M\) and \(\phi \in \tilde{\text{Ham}}(M)\), we (by a slight abuse of notation) write \(F \circ \phi\) for the composition of \(F\) with the projection of \(\phi\) to \(\text{Ham}(M)\).

For a time-dependent Hamiltonian \(H(x, t)\) we set \(H_t = H(\cdot, t)\). Denote by \(\mathcal{F}\) the set of all the Hamiltonians \(H\) on \(M\) such that

- if \(M\) is open, the union of supports of \(H_t, t \in [0, 1]\), is compact;
- if \(M\) is closed, \(H_t\) has zero mean for all \(t\): \(\int_M H_t \omega^n = 0\).

The group \(\tilde{\text{Ham}}(M)\) carries a conjugation-invariant functional \(\rho\) (called the “positive part” of the Hofer’s norm) defined by

\[
\rho(\phi) := \inf_H \int_0^1 \max_{x \in M} H(x, t) \, dt ,
\]

where the infimum is taken over all (time-dependent) Hamiltonians \(H \in \mathcal{F}\) generating \(\phi\).

We shall often use the following well known properties of the functional \(\rho\) which readily follow from the definition.

**Proposition 3.1.**

(i) (conjugation invariance) \(\rho(\psi \phi \psi^{-1}) = \rho(\phi)\) for all \(\phi, \psi \in \tilde{\text{Ham}}(M)\).

(ii) (triangle inequality) \(\rho(\phi \psi) \leq \rho(\phi) + \rho(\psi)\) for all \(\phi, \psi \in \tilde{\text{Ham}}(M)\).

(iii) \(\rho(\phi_F^1 \phi_G) \leq \int_0^1 \max(G_t - F_t) \, dt\) for all \(F, G \in \mathcal{F}\).
Combining items (ii) and (iii) of the proposition, we get that
\[
|\rho(\phi_F) - \rho(\phi_G)| \leq \max(\rho(\phi_F^{-1}\phi_G), \rho(\phi_G^{-1}\phi_F)) \leq \int_0^1 ||F_t - G_t|| \, dt .
\] (9)

Let us illustrate the conjugation invariance of \(\rho\) and the triangle inequality by proving the following lemma which will be useful in the sequel.

**Convention on commutators:** We write \([\phi, \psi]\) for the commutator \(\phi\psi\phi^{-1}\psi^{-1}\) of elements \(\phi, \psi \in \widehat{\text{Ham}}(M)\).

**Lemma 3.2.** For every elements \(a, b, c, d \in \widehat{\text{Ham}}(M)\)
\[
|\rho([a, b]) - \rho([c, d])| \leq \rho(a^{-1}c) + \rho(c^{-1}a) + \rho(b^{-1}d) + \rho(d^{-1}b) .
\] (10)

**Proof.** Set \(e := [a, b]^{-1}[c, d]\). Note that \(e\) is conjugate to \(fg\) with \(f = b^{-1}a^{-1}cd, g = c^{-1}d^{-1}ba\). In turn, \(f\) is conjugate to \((a^{-1}c)(db^{-1})\) and \(g\) is conjugate \((d^{-1}b)(ac^{-1})\). Using the conjugation invariance of \(\rho\) and the triangle inequality we conclude that \(\rho(e)\) does not exceed the right-hand side of (10). A similar analysis proves the same bound for \(\rho(e^{-1})\). But the left inequality in (9) shows that the left-hand side of (10) does not exceed \(\max(\rho(e), \rho(e^{-1}))\), and thus inequality (10) follows. \(\square\)

The key Hofer-geometric ingredient used in the proofs below is as follows (McDuff, [16, Proposition1.5]): for every time-independent function \(H \in \mathcal{F}\) there exists \(\delta > 0\) so that
\[
\rho(\phi_t H) = t \cdot \max H \quad \forall t \in (0, \delta) .
\] (11)

### 3.2 Buhovsky’s 2/3 law

As a warm up we prove formula (5). Put
\[
P := \{F, G\}, \quad I := \{\{P, F\}, F\} + \{\{P, G\}, G\} .
\]

Note that \(\Psi = ||I||\). Recall from Corollary 2.2 that \(\Psi > 0\). Consider the Hamiltonian flow
\[
v_\tau := \phi_{(F+G)/2} F - g \cdot f g \cdot \phi_{(F+G)/2} .
\]
A lengthy but straightforward calculation (which we checked by the slightly modified Maple-based Lie Tools Package software [20]) shows that the corresponding Hamiltonian $V(\tau) := V(x, \tau)$ has expansion

$$V(\tau) = 2\tau P + \frac{\tau^3}{6} I + O(\tau^4). \quad (12)$$

Let $t$ be a small parameter, and $\tau \in [0, 1]$ be the time variable. Consider the flow $v_t \sqrt{\tau}$ whose time one map equals $v_t$. By (12) this flow is generated by the Hamiltonian

$$(2\sqrt{\tau})^{-1} tV(t\sqrt{\tau}) = t^2 P + R,$$

where

$$R = \frac{t^4 \tau}{12} I + O(t^5).$$

Fix $\delta > 0$. By (9) there exists $t_0 > 0$ so that for every $0 < t < t_0$

$$|\rho(v_t) - \rho(\phi_{t^2P})| \leq \frac{1 + \delta}{24} \Psi \cdot t^4. \quad (13)$$

Decreasing if necessary $t_0$ we have from (11) that $\rho(\phi_{t^2P}) = t^2 \max P$ for $0 < t < t_0$. Furthermore, $v_t$ is conjugate to $[f_t, g_t]$ and hence $\rho(v_t) = \rho([f_t, g_t])$. Thus (13) yields

$$\rho([f_t, g_t]) \geq t^2 \max\{F, G\} - \frac{1 + \delta}{24} \Psi \cdot t^4, \quad (14)$$

for $0 < t < t_0$.

Now put $K_\delta = \frac{1 + \delta}{24} \Psi$. Choose a small positive $\epsilon$ so that $(4\epsilon \cdot K_\delta^{-1})^{1/3} \leq t_0$. Let $F', G'$ be functions with $||F - F'|| \leq \epsilon, ||G' - G|| \leq \epsilon$, and let $f'_t, g'_t$ be the corresponding Hamiltonian flows. By an elementary ODE Lemma 2.2 of [6]

$$\rho([f'_t, g'_t]) \leq t^2 \max\{F', G'\}, \quad \text{for all } t. \quad (15)$$

Next we claim that

$$|\rho([f'_t, g'_t]) - \rho([f_t, g_t])| \leq 8\epsilon t \quad \text{for all } t. \quad (16)$$

When $M$ is an open manifold, this readily follows from Lemma 3.2 and Proposition 3.1(iii), and actually we get the numerical constant 4 instead of 8 in (16). When $M$ is a closed manifold, we have to be a bit more careful since
the inequality in Proposition 3.1(iii) holds only for normalized Hamiltonians. For any function \( H \in C^\infty(M) \) define its normalization by

\[ H_{\text{norm}} := H - \frac{1}{\text{Volume}(M)} \cdot \int_M H \omega^n. \]

Clearly, for any two functions \( H, H' \) we have

\[ ||H_{\text{norm}} - H'_{\text{norm}}|| \leq 2||H - H'||. \]

With this in mind, we again use Lemma 3.2 and apply Proposition 3.1(iii) to the normalizations of the functions \( tF, tF', tG, tG' \). This readily yields inequality (16).

Combining inequalities (14),(15) and (16) one gets that for all \( t, 0 < t < t_0 \),

\[ \max\{F', G'\} \geq \max\{F, G\} - (8\epsilon t^{-1} + K_\delta t^2), \]

and hence

\[ \max\{F, G\} - \max\{F', G'\} \leq 8\epsilon t^{-1} + K_\delta t^2. \]

As a function of \( t, t > 0 \) (for fixed \( \epsilon \) and \( K_\delta \)), the right-hand side reaches its minimum at \( t = (4\epsilon \cdot K_\delta^{-1})^{1/3} \) which, by our choice of \( \epsilon \), belongs to the interval \((0, t_0)\). Hence, we may substitute this \( t \) in the right-hand side which yields

\[ \max\{F, G\} - \max\{F', G'\} \leq C(1 + \delta)^{1/3} \Psi^{1/3} \epsilon^{2/3}, \]

where \( C \) is a numerical constant. This immediately yields the desired formula (5).

\[ \square \]

### 3.3 Convergence rate for the double bracket

In this section we prove Theorem 1.3. Throughout the proof we use notation

\[ \theta(F, G) = [\phi_{-F} \phi_{-G}, \phi_{F+G}]. \]

**Lemma 3.3.** For all \( F, G \in C^\infty_c(M) \)

\[ \rho(\theta(F, G)) \leq (\max\{\{F, G\}, F\} + \max\{\{F, G\}, G\})/2 \]
Proof. The diffeomorphism $\theta(F,G)$ can be generated by a Hamiltonian flow

$$[\phi_{-F}\phi_{-G}, \phi_{(F+G)}].$$

The corresponding Hamiltonian is given by

$$H(x, \tau) = (F + G)\phi_G\phi_F - (F + G)\phi_{-F}\phi_{-G}\phi_{(F+G)}\phi_G\phi_F.$$

Clearly,

$$\max_x H(x, \tau) = \max_x (F + G - (F + G)\phi_{-F}\phi_{-G}) = \max_x (G + F\phi_G - G\phi_{-F} - F).$$

Denote

$$Y := G + F\phi_G - G\phi_{-F} - F.$$

Then

$$\rho(\theta(F,G)) \leq \max_x H = \max_x Y. \quad (17)$$

On the other hand, it is easy to see that

$$Y(x) = \int_1^x \left( \{F,G\}\phi_{uG} + \{G,F\}\phi_{-uF} \right) du =$$

$$= \int_0^1 \int_0^u \left( \{F,G\}G\phi_{uG} + \{G,F\}(-F)\phi_{-uF} \right) dw du \leq$$

$$\leq (\max_x \{F,G\}) \int_0^1 \int_0^x dw du =$$

$$= (\max_x \{F,G\})/2.$$

Hence

$$\max_x Y \leq (\max_x \{F,G\})/2.$$

By (17) this implies

$$\rho(\theta(F,G)) \leq (\max_x \{F,G\})/2,$$

as needed.
Denote \( A := \frac{1}{2}\{\{F, G\}, F\} \) and \( B := \frac{1}{2}\{\{F, G\}, G\} \). Consider the flow
\[
v_\tau = \phi_{\tau(F-G)/6} \circ \theta(\tau F, \tau G) \circ \phi_{\tau(F-G)/6}^{-1}.
\]
A lengthy but straightforward calculation (which we checked by the slightly modified Maple-based Lie Tools Package software [20]) shows that the corresponding Hamiltonian \( V(\tau) := V(x, \tau) \) has expansion
\[
V(\tau) = 3\tau^2(A + B) + \tau^4Q + O(\tau^5),
\]
where \( Q \) is a Lie polynomial of \( F \) and \( G \) whose monomials are 4-times-iterated Poisson brackets. Let \( s, t \) be small parameters. Replacing \( F \to sF, G \to tG \) and making the change of time \( \tau \to \tau^4 \) we get that the element
\[
u_{s,t} := \phi(sF-tG)/6 \circ \theta(sF, tG) \circ \phi_{(sF-tG)/6}^{-1}
\]
is generated by Hamiltonian \( s^2tA + st^2B + R \) with
\[
||R|| \leq E \cdot \sum_{i=1}^{4} s^i t^{5-i},
\]
where \( E \) is a constant depending on \( F \) and \( G \). Applying (9) and (11) (here hard symplectic topology enters the play) and taking into account that \( u_{s,t} \) is conjugate to \( \theta(sF, tG) \) we get that for sufficiently small \( s, t > 0 \)
\[
\rho(\theta(sF, tG)) = \rho(u_{s,t}) \geq \max(s^2tA + st^2B) - E \cdot \sum_{i=1}^{4} s^i t^{5-i}.
\]
Take any \( F', G' \) with \( ||F - F'|| \leq \epsilon, ||G' - G|| \leq \epsilon \). Put
\[
A' := \frac{1}{2}\{\{F', G'\}, F'\}, \quad B' := \frac{1}{2}\{\{F', G'\}, G'\}.
\]
By Lemma 3.3,
\[
\rho(\theta(sF', tG')) \leq s^2t \max A' + st^2 \max B'.
\]
Furthermore,
\[
|\rho(\theta(sF, tG)) - \rho(\theta(sF', tG'))| \leq 16\epsilon(s + t).
\]
The proof of this inequality is similar to the proof of inequality (16) above: it readily follows from Lemma 3.2 and Proposition 3.1(iii). We omit the details.

Combining inequalities (19), (20) and (21) we get that for all sufficiently small $s, t > 0$

$$s^2t \max A' + st^2 \max B' \geq \max(s^2tA + st^2B) - E \cdot \sum_{i=1}^{4} s^i t^{5-i} - 16\epsilon(s + t). \quad (22)$$

Put

$$2\Delta := \max A + \max B - \max A' - \max B'. \quad (23)$$

We have to find an upper bound on $\Delta$ assuming that $\Delta > 0$. Without loss of generality assume that $\max A - \max A' \geq \Delta$.

Since $\Delta > 0$, (23) yields

$$\max B' \leq \||A|| + \||B||. \quad (24)$$

Further,

$$\max(s^2tA + st^2B) \geq s^2t \max A - st^2 \||B||.$$

Substituting these inequalities into (22) we get that

$$s^2t \max A' + st^2(\||A|| + \||B||) \geq s^2t \max A - st^2 \||B|| - E \cdot \sum_{i=1}^{4} s^i t^{5-i} - 16\epsilon(s + t) .$$

Since $\Delta \leq \max A - \max A'$, we conclude that for sufficiently small $s$ and $t$

$$\Delta \leq Rs^{-2}t^{-1},$$

where $R = (\||A|| + 2||B||)st^2 + E \cdot \sum_{i=1}^{4} s^i t^{5-i} + 16\epsilon(s + t)$. Let us balance this inequality: we choose $s = \epsilon^{\frac{4}{5}}$, $t = \epsilon^{\frac{2}{5}}$ (we assume that $\epsilon$ was chosen sufficiently small so that the last inequality is valid for these $s$ and $t$) and get that

$$\Delta \leq \text{const}(F, G) \cdot \epsilon^{\frac{4}{5}},$$

as required.

Remark 3.4. Let us make the following manipulation with inequality (22): put $s = t$, divide by $t^3$ and rewrite the inequality as follows:

$$\max(A + B) - \max A' - \max B' \leq 4Et^2 + 16\frac{\epsilon}{t^2}. \quad (25)$$
At first glance this is not too encouraging since we have to estimate from above the quantity $\max A + \max B - \max A' - \max B'$, while in general $\max A + \max B$ is greater than $\max(A + B)$. As we have seen in the proof above, we bypassed this difficulty by letting $s$ and $t$ to have different asymptotical behavior in $\epsilon$. It might well happen that at this point we lost sharpness in our bound $\sim \epsilon^{1/3}$ for the convergence rate given in Theorem 1.3, cf. Question 1.4 above. Here is some evidence in favor of this possibility: assume for a moment that $\max A + \max B = \max(A + B)$. Putting $t = \epsilon^{1/4}$ in inequality (25) we get that the right hand side is of the order $\sim \epsilon^{1/2}$. At the same time it is easy to exhibit examples of functions $F, G$ on the 2-sphere with, in notations of Theorem 1.3,

$$\Phi(F, G) - \overline{\Phi}_t(F, G) \leq C \cdot \epsilon^{1/2}.$$ 

Thus $\sim \epsilon^{1/2}$ could be considered as another candidate for the sharp power law in the convergence rate.

### 4 Decreasing $\max\{\{F, G\}, F\}, -\min\{\{F, G\}, F\}$

In this section we prove the following result.

**Theorem 4.1.** On every symplectic manifold there exists a collection of functions $F_N, F, G$ so that $F_N \overset{C^0}{\to} F$ and

$$\liminf_{N \to \infty} \max\{\{F_N, G\}, F_N\} < 0.99 \cdot \max\{\{F, G\}, F\},$$

$$\liminf_{N \to \infty} (-\min\{\{F_N, G\}, F_N\}) < -0.99 \cdot \min\{\{F, G\}, F\}.$$ 

**Beginning the construction:** To make the example more transparent we first describe it in the two-dimensional case. Consider $M = \mathbb{R}^2$ with the standard symplectic form $\omega = dp \wedge dq$. Recall that the Poisson bracket is defined by $\{F, G\} = dF(\text{grad } G)$, so $\{p, q\} = -1$. We first look for $F, G, F_N : \mathbb{R}^2 \to \mathbb{R}$ in the form

$$F = u(p), \quad G = -v(q), \quad F_N = u(p) + \frac{1}{N}a(q) \sin Nu(p).$$

Here $u, v, a$ are compactly supported in $\mathbb{R}$, thus at this stage $F, G, F_N$ do not have compact supports in $\mathbb{R}^2$ yet. This will be corrected later. The choice of
\( u(p) \) is essentially arbitrary, while \( v(q), a(q) \) will be chosen in a special way below.

Observe that
\[
\{ \{ F, G \}, F \} = u'(p)^2 v''(q) ,
\]
so
\[
\max \{ \{ F, G \}, F \} = \max u'(p)^2 \max v''(q) .
\]
Furthermore, a straightforward but lengthy calculation shows that
\[
\{ \{ F'_N, G \}, F'_N \} = u'(p)^2 R(p, q) + O\left( \frac{1}{N} \right),
\]
where
\[
R(p, q) = v''(q)(a(q) \cos Nu(p) + 1)^2 + a'(q)v'(q)(a(q) + \cos Nu(p)) .
\]
Write \( v'(q) = w(q) \) so that
\[
R(p, q) = w'(q)(a(q) \cos Nu(p) + 1)^2 + a'(q)w(q)(a(q) + \cos Nu(p)) . \quad (26)
\]
Introduce the function
\[
r(\alpha, \gamma, z) := (\alpha z + 1)^2 - \gamma(\alpha + z) .
\]
With this notation
\[
R(p, q) = w'(q)r\left( a(q), -a'(q)w(q)/w'(q), \cos Nu(p) \right)
\]
whenever \( w'(q) \neq 0 \).

**Lemma 4.2.** For the specific fixed values \( \alpha = 1.1 \) and \( \gamma = 1.63 \) the function \( z \mapsto r_{\alpha, \gamma}(z) := r(\alpha, \gamma, z) \) satisfies the following inequality:
\[
-0.99 < r_{\alpha, \gamma}(z) < 0.99 \quad \forall z \in [-1, 1] .
\]

**Proof.** Write
\[
r(z) := r_{1.1, 1.63}(z) = 1.21z^2 + 0.57z - 0.793 .
\]
We have to check the values of \( r \) at the endpoints \( \pm 1 \) and at the critical point. We have \( r(-1) = -0.153, r(1) = 0.987 \). The critical value equals \(-0.57^2/(4 \cdot 1.21) - 0.793 \approx -0.86 \). We conclude that \( |r(z)| < 0.99 \) for \( z \in [-1, 1] \).
Now we are ready to describe our example. Fix real numbers $c_1 < c_2 < c_3 < c_4$ so that $c_{i+1} - c_i = \delta$, where $\delta > 0$ will play the role of a small parameter in our construction. Fix $\kappa > 0$ so that the inequality in Lemma 4.2 holds for the function $r_{\alpha, 1.63}(z)$ (over $[-1, 1]$) for every $\alpha \in [1.1 - \kappa, 1.1 + \kappa]$. The compactly supported functions $w$ and $a$ are chosen as follows:

**CONDITIONS ON $w(q)$:**

(i) $\max_{[c_2, c_3]} w' = 1$, $\min_{[c_2, c_3]} w' = -1$, $w'(c_2) = 0.001$, $w'(c_3) = -0.001$.

(ii) $1 \leq w(q) \leq 2$ for $q \in [c_1, c_4]$.

(iii) $|w'(q)| \leq 0.01$ for $q \in \mathbb{R}_+ \setminus [c_2, c_3]$.

(iv) $|w(q)| \leq 3$ for $q \in \mathbb{R}_+ \setminus [c_1, c_4]$.

(v) $\max w = -\min w = 1$.

(vi) $\int_{-\infty}^{+\infty} w(q) \, dq = 0$.

**CONDITIONS ON $a(q)$:**

(i) $a'(q) = -1.63 w'(q)/w(q)$ for $q \in [c_1, c_4]$.

(ii) $a(q) \in [1.1 - \kappa, 1.1 + \kappa]$ for $q \in [c_1, c_4]$.

(iii) $|a'(q)| \leq 0.03$ and $|a(q)| \leq 2$ for $q \in \mathbb{R}_+ \setminus [c_1, c_4]$.

Assumption (iii) on $a$ is compatible with (i) since on $[c_1, c_2] \cup [c_3, c_4]$ we have $|a'(q)| \leq 2 \cdot 0.01 : 1 = 0.02$. Assumption (ii) is achieved by taking $\delta$ as small as needed.

**THE BOUND ON $R(p, q)$:** Let us check that

$$|R(p, q)| \leq 0.99 \cdot \max w' = -0.99 \cdot \min w' = 0.99 .$$

**(27)**

**CASE 1:** For $q \in [c_1, c_4]$ we have

$$|R(p, q)| = |w'(q) \cdot |r(\alpha, 1.63, z)||,$n

with $\alpha \in [1.1 - \kappa, 1.1 + \kappa]$ and $z \in [-1, 1]$. By Lemma 4.2 and due to the choice of $\kappa$, we have $|R(p, q)| \leq 0.99 \cdot |w'(q)| \leq 0.99$.

**CASE 2:** For $q \in \mathbb{R}_+ \setminus [c_1, c_4]$ we have

$$-0.99 < -0.36 = -0.01 \cdot (2 + 1)^2 - 0.03 \cdot 3 \cdot (2 + 1) \leq$$
\[ \leq R(p, q) \leq 0.01 \cdot (2 + 1)^2 + 0.03 \cdot 3 \cdot (2 + 1) = 0.36 < 0.99. \]

This completes the proof of (27).

**Summary:** Since \( \{\{F, G\}, F\} = u'(p)^2w'(q) \) and
\[
\{\{F_N, G\}, F_N\} = u'(p)^2R(p, q) + O\left(\frac{1}{N}\right),
\]
we conclude that the functions \( F, F_N, G \) satisfy inequalities in Theorem 4.1.

**Making the functions compactly supported in \( \mathbb{R}^2 \):** According to our construction, the support of the function \( u = u(p) \), as a function on \( \mathbb{R} \), is contained in some closed interval \( I \) and the supports of \( v = v(q) \) and \( a = a(q) \), as functions on \( \mathbb{R} \), are both contained in some closed interval \( J \). Hence the supports of \( F \) and \( F_N \) in \( \mathbb{R}^2 \) are contained in \( I \times \mathbb{R} \) and the support of \( G \) is contained in \( \mathbb{R} \times J \).

Let us choose a cut-off function \( \phi : \mathbb{R}^2 \to [0, 1] \) which is equal to 1 on \( I \times J \). Then
\[
\{\phi F, \phi G\} = \phi^2\{F, G\},
\]
since \( \phi \) is constant on \( \text{supp} F \cap \text{supp} G \). For the same reason, since \( \text{supp} \{F, G\} \subset I \times J \) and \( \phi \) is constant on \( \text{supp} F \cap \text{supp} \{F, G\} \), we have
\[
\{\phi F, \{\phi F, \phi G\}\} = \phi^3\{F, \{F, G\}\}.
\]
Since \( \text{supp} \{F, \{F, G\}\} \subset \text{supp} F \cap \text{supp} \{F, G\} \subset I \times J \) we get that
\[
\max\{\phi F, \{\phi F, \phi G\}\} = \max\{F, \{F, G\}\}.
\]

Replacing \( F \) by \( F_N \) and noticing that the previous considerations depend only on the supports of \( F \) and \( G \), we get that
\[
\{\phi F_N, \{\phi F_N, \phi G\}\} = \phi^3\{F_N, \{F_N, G\}\}
\]
and
\[
\max\{\phi F_N, \{\phi F_N, \phi G\}\} = \max\{F_N, \{F_N, G\}\}
\]
for the same reason as above.

The same equalities holds for the minima. Thus the functions \( \tilde{F} := \phi F, \tilde{F}_N := \phi F_N, \tilde{G} := \phi G \) are compactly supported in \( \mathbb{R}^2 \) and satisfy the inequalities in Theorem 4.1.
Implanting into a symplectic manifold: Now the example can be easily generalized to a higher-dimensional Darboux chart (and hence implanted into any symplectic manifold), cf. [6], proof of Theorem 1.6.

Namely, assume \( \dim M = 2n > 2 \). In a local Darboux chart with coordinates \( p_1, q_1, \ldots, p_n, q_n \) on \( M \) choose an open cube \( P = K^{2n-2} \times K^2 \), where \( K^{2n-2} \) is an open cube in the \( (p_1, q_1, \ldots, p_{n-1}, q_{n-1}) \)-coordinate plane and \( K^2 \) is a open square in the \( (p_n, q_n) \)-coordinate plane. Fix a smooth compactly supported function \( \chi : K^{2n-2} \to [0, 1] \) which reaches the value 1 at some point. Given a smooth compactly supported function \( L \) on \( K^2 \), define the function \( \chi L \in C_c^\infty(M) \) as

\[
\chi L(p_1, q_1, \ldots, p_n, q_n) := \chi(p_1, q_1, \ldots, p_{n-1}, q_{n-1})L(p_n, q_n)
\]
on \( P \) and as zero outside \( P \).

Now pick functions \( \tilde{F}, \tilde{G}, \tilde{F}_N \in C_c^\infty(K^2) \) as above. Set

\[
F' := \chi \tilde{F}, \quad G' := \chi \tilde{G}, \quad F'_N := \chi \tilde{F}_N \in C_c^\infty(P) \subset C_c^\infty(M).
\]

Clearly \( F'_N \) converges uniformly to \( F' \). It is also clear that

\[
\{\{F', G'\}, F'\} = \{\{\chi \tilde{F}, \chi \tilde{G}\}, \chi \tilde{F}\} = \chi^3 \{\{\tilde{F}, \tilde{G}\}, \tilde{F}\},
\]

\[
\{\{F'_N, G'\}, F'_N\} = \{\{\chi \tilde{F}_N, \chi \tilde{G}\}, \chi \tilde{F}_N\} = \chi^3 \{\{\tilde{F}_N, \tilde{G}\}, \tilde{F}_N\},
\]

because the Poisson bracket of \( \chi \) and any function of \( p_n, q_n \) vanishes identically. Recalling how \( \chi \) was chosen we see that

\[
\max \{\{F', G'\}, F'\} = \max \{\{\tilde{F}, \tilde{G}\}, \tilde{F}\},
\]

\[
\max \{\{F'_N, G'\}, F'_N\} = \max \{\{\tilde{F}_N, \tilde{G}\}, \tilde{F}_N\}.
\]

Hence \( F', G', F'_N \) satisfy the required properties (since \( \tilde{F}, \tilde{G}, \tilde{F}_N \) do). This completes the proof of the theorem. \( \Box \)

**Remark 4.3.** It would be interesting to find out how small can the ratio

\[
\liminf_{F', G' \xrightarrow{C^0} F, G} \frac{\osc \{\{F', G'\}, F'\}}{\osc \{\{F, G\}, F\}}
\]

be.
be made by varying $F$ and $G$ so that $\text{osc}\{\{F,G\},F\} \neq 0$. Clearly, this ratio always belongs to $(0,1]$. In the example constructed above it is no bigger than 0.99. In fact, one can slightly modify that example to show that the ratio

$$\lim \inf_{F,G; \underset{C_0\rightarrow F,G}{\overset{\overset{C_0\rightarrow F,G}{\rightarrow}}} F,G'} \max \ \{\{F',G'\},F'\}$$

max $\{\{F,G\},F\}$

can be made arbitrarily small by an appropriate choice of $F$ and $G$.

5 The proof of the Dichotomy Theorem: conclusion

We shall write $\mathbb{R}_+^4$ for the non-negative orthant of $\mathbb{R}^4$. The functionals $\Phi^v(F,G)$ respect the natural actions of the dihedral group $D_4$ on vectors $v \in \mathbb{R}_+^4$ and the variables $(F,G)$. In particular, define linear transformations $A, B, C$ of $\mathbb{R}^4$ by

$$A(v_1, v_2, v_3, v_4) = (v_2, v_1, v_3, v_4) ,$$

$$B(v_1, v_2, v_3, v_4) = (v_1, v_2, v_4, v_3) ,$$

$$C(v_1, v_2, v_3, v_4) = (v_3, v_4, v_1, v_2)$$

which generate the $D_4$-action on $\mathbb{R}_+^4$. Then

$$\Phi^v(F, -G) = \Phi^{Au}(F, G) ,$$

$$\Phi^v(-F, G) = \Phi^{Bu}(F, G) ,$$

$$\Phi^v(-G, -F) = \Phi^{Cu}(F, G) .$$

Next, $\Phi^v$ obeys the following scaling laws: given $\alpha, \beta > 0$, we have $\Phi^v(\alpha F, \beta G) = \Phi^w(F, G)$ with

$$w = (\alpha^2 \beta v_1, \alpha \beta^2 v_2, \alpha^2 \beta v_3, \alpha \beta^2 v_4) .$$

Note now that all the properties of functionals $\Phi^v$ appearing in the Dichotomy Theorem are invariant under the action of the dihedral group and rescaling.
Proof of Theorem 1.1(i) Assume that either \( v_3 = v_4 = 0 \) or \( v_1 = v_2 = 0 \). Applying if necessary the dihedral group we can assume without loss of generality that

\[
\Phi^v(F, G) = v_1 \max\{\{F, G\}, F\} - v_2 \min\{\{F, G\}, F\}, \quad v_1 > 0, v_2 \geq 0.
\]

By Theorem 4.1 \( \Phi^v \) is not lower semicontinuous. Further,

\[
\Phi^v(F, G) \geq v_1 \max\{\{F, G\}, F\},
\]

and hence \( \Phi^v \) is weakly robust by inequality (3).

Proof of Theorem 1.1(ii) Put

\[
\mu_+(F, G) = \max\{\{F, G\}, F\}, \quad \mu_-(F, G) = -\min\{\{F, G\}, F\},
\]

\[
\nu_+(F, G) = \max\{\{F, G\}, G\}, \quad \nu_-(F, G) = -\min\{\{F, G\}, G\}.
\]

With this notation \( \mu_+(F, G) + \nu_+(F, G) \) is lower semicontinuous by Theorem 1.3. We shall need the following auxiliary result.

Lemma 5.1. Let \((F_i, G_i) \xrightarrow{C^0} (F, G)\). If \( \mu_+(F_i, G_i) \leq K < \infty \) for all \( i \)

\[
\liminf_{i \to \infty} \nu_-(F_i, G_i) \geq \nu_-(F, G) .
\]

Similarly, if \( \nu_+(F_i, G_i) \leq K < \infty \) for all \( i \)

\[
\liminf_{i \to \infty} \mu_-(F_i, G_i) \geq \mu_-(F, G) .
\]

Proof. Assume that \( \mu_+(F_i, G_i) \leq K < \infty \) for all \( i \). First we prove that

\[
\liminf_{i \to \infty} \nu_+(F_i, G_i) \geq \nu_+(F, G) .
\]

Assume on the contrary that along a subsequence

\[
\lim_{i \to \infty} \nu_+(F_i, G_i) \leq E < \nu_+(F, G) .
\]

Pick \( 0 < \alpha < (\nu_+(F, G) - E)/K \). Then for \( i \) sufficiently large

\[
\mu_+(\alpha F_i, G_i) + \nu_+(\alpha F_i, G_i) = \alpha^2 \mu_+(F_i, G_i) + \alpha \nu_+(F_i, G_i) \leq \alpha (\alpha K + E)
\]

\[
< \nu_+(\alpha F, G) < \mu_+(\alpha F, G) + \nu_+(\alpha F, G) ,
\]

which contradicts lower semicontinuity of \( \mu_+ + \nu_+ \) at \( (\alpha F, G) \). This proves (30). Replacing \( F \) by \(-F\), we obtain the desired inequality (28) from (30). The proof of (29) is analogous.
Now we are ready to complete the proof of Theorem 1.1(ii). Let \( v \in \mathbb{R}_4^+ \) be such that at least one of \( v_1, v_2 \) is positive and at least one of \( v_3, v_4 \) is positive. Applying the action of the dihedral group and rescaling, we can achieve that 
\[ v_1 = v_3 = 1, \]
and thus, without loss of generality, 
\[ \Phi^v = \mu_+ + \nu_+ + v_2 \cdot \mu_- + v_4 \cdot \nu_. \]
Assume that \( (F_i, G_i) \xrightarrow{C_0} (F, G) \) so that \( \Phi^v(F_i, G_i) \) is bounded. The lower semicontinuity of \( \mu_+ + \nu_+ \) and Lemma 5.1 yield
\[ \liminf_{i \to \infty} \Phi^v(F_i, G_i) \geq \Phi^v(F, G). \]
This finishes off the proof.

6 Higher iterated brackets: discussion

Denote by \( \mathcal{P}_N \), \( N \in \mathbb{N} \), the set of all Lie monomials in two variables involving \( N \)-times-iterated Poisson brackets. Given monomials \( p_1, \ldots, p_d \in \mathcal{P}_N \) and non-negative numbers \( \alpha_j, \beta_j \geq 0 \), consider a functional \( \Phi(F, G) \), given by
\[ \sum_{j=1}^d \alpha_j \cdot \max p_j(F, G) - \beta_j \cdot \min p_j(F, G). \]
In the case \( N \geq 3 \) the problem of detecting whether \( \Phi \) is weakly robust or lower semicontinuous is at the moment almost completely out of reach. The simplest case where the answer is unknown to us is \( N = 3, d = 1, p(F, G) = \{ \{F, G\}, F\} \).

To emphasize the main difficulty, let us recall that our strategy of proving the lower semi-continuity in the case of the ordinary bracket and the double bracket is as follows: We design an expression of the form \( u_t = \prod_j \phi_{a_j F + b_j G} \) so that the Hofer’s (semi)norm \( \rho(u_t) \) admits “tight” lower and upper bounds in terms of the maxima/minima of Lie polynomials involving monomials \( p_j \) entering \( \Phi \). For instance, for \( \Phi(F, G) = \{F, G\} \) we use the flow \( u_t = [\phi_{F^t}, \phi_{G^t}] \) and for \( \Phi(F, G) = \max\{\{F, G\}, F\} + \max\{\{F, G\}, G\} \) we use the flow \( u_t = [\phi_{-F} \phi_{-G}, \phi_{F+G}] \). Combining the above-mentioned lower and upper bounds, we obtain an inequality involving iterated Poisson brackets, which eventually yields the desired semicontinuity.
For a general functional of the form (31) it is unclear how to design expressions \( u_t \) as above leading to “tight” lower and upper bounds for \( \rho(u_t) \), and it is even unclear whether such the expressions do exist at all. Thus new ideas are needed.

**Question 6.1. Is the functional**

\[
\Phi_N(F,G) = \sum_{p \in P_N} \text{osc } p(F,G)
\]

**lower semicontinuous?**

Note that the answer is affirmative for \( N = 1, 2 \).

As far as the weak robustness is concerned, we are able to settle a particular case which is a direct generalization of Proposition 1.2. Denote by \( \text{ad}_F : C_c^\infty(M) \to C_c^\infty(M) \) the operator \( G \mapsto \{G,F\} \).

**Proposition 6.2.** For every \( N \in \mathbb{N} \) the functional \( \Phi(F,G) = \text{osc } (\text{ad}_F)^N G \) is weakly robust.

*Proof. Indeed, by Kolmogorov’s generalization of the Landau-Hadamard inequality [10, 11], cf. [18], there exists a constant \( C_N > 0 \) so that for every smooth function \( v \in C^\infty(\mathbb{R}) \), which is bounded with its \( N \) derivatives and does not vanish identically, one has the following lower bound on the uniform norm of the \( N \)-th derivative \( v^{(N)} \) of \( v \) in terms of its first derivative \( v' \):

\[
\text{osc } v^{(N)} \geq C_N \cdot \frac{\|v'\|^N}{\|v\|^{N-1}}.
\]

Choose a point \( x \) in the symplectic manifold \( M \) where the uniform norm of \( \{G,F\} \) is attained. Applying the Kolmogorov inequality to the function \( v(t) = G(\phi^t_F(x)) \) we get that

\[
\Phi(F,G) \geq C_N \frac{\|\{F,G\}\|^N}{\|G\|^{N-1}}.
\]

Since the functional \( \{F,G\} \) is lower semicontinuous, the above inequality readily yields the weak robustness of \( \Phi \). \( \square \)
Inequality (32) can be iterated as follows. For integers \( m \geq 1, k \geq 0 \) introduce the functional

\[
\Phi_{k,m}(F, G) = \text{osc} \left( (\text{ad}_H)^m G \right), \text{where } H = (\text{ad}_G)^k F.
\]

**Proposition 6.3.** There exist constants \( C_{k,m} > 0 \) so that

\[
\Phi_{k,m}(F, G) \geq C_{k,m} \cdot \frac{\|\{F, G\}\|^{(k+1)m}}{\|F\|^{km} \|G\|^{m-1}}.
\]

In particular, \( \Phi_{k,m} \) is weakly robust.

**Proof.** Applying (32) twice we get

\[
\Phi_{k,m}(F, G) \geq \text{const} \cdot \frac{\|\{H, G\}\|^m}{\|G\|^{m-1}} \geq \text{const} \cdot \frac{\|(\text{ad}_G)^{k+1} F\|^m}{\|G\|^{m-1}}
\]

\[
\geq \text{const} \cdot \frac{\|\{F, G\}\|^{(k+1)m}}{\|F\|^{km} \|G\|^{m-1}}.
\]

\( \square \)

**Problem 6.4.** Prove that for \( N \geq 3 \) the functional \( \text{osc} \left( (\text{ad}_F)^N G \right) \) is not lower semicontinuous.

In Section 4 we proved this for \( N = 2 \). It is not clear whether our method extends to \( N \geq 3 \). A natural generalization of this problem would be to detect the failure of lower semicontinuity for the functionals \( \Phi_{k,m} \) from Proposition 6.3 (except the trivial case \( k = 0, m = 1 \)).

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