Synchronization of Bernoulli sequences on shared letters

Samy Abbes
Université Paris Diderot/IRIF CNRS UMR 8243
samy.abbes@univ-paris-diderot.fr
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Abstract

The topic of this paper is the distributed and incremental generation of long executions of concurrent systems, uniformly or more generally with weights associated to elementary actions.

Synchronizing sequences of letters on alphabets sharing letters are known to produce a trace in the concurrency theoretic sense, \( i.e. \), a labeled partially ordered set. We study the probabilistic aspects by considering the synchronization of Bernoulli sequences of letters, under the light of Bernoulli and uniform measures recently introduced for trace monoids.

We introduce two algorithms that produce random traces, using only local random primitives. We thoroughly study some specific examples, the path model and the ring model, both of arbitrary size. For these models, we show how to generate any Bernoulli distributed random traces, which includes the case of uniform generation.

1—Introduction

The developments of concurrency theory and of model checking theory have urged the development of a theory of probabilistic concurrent systems. A central issue is the problem of the uniform generation of long executions of concurrent systems. Executions of a concurrent system can be represented as partial orders of events. Each partial order has several sequentializations, the combinatorics of which is non trivial. Therefore, the uniform generation of executions of a concurrent system is much different from the uniform generation of their sequentializations. The later can be done using uniform generation techniques for runs of transition systems, at the expense of an increasing amount of complexity due to the combinatorics of sequentializations \cite{18,9,20}. Yet, it still misses the overall goal of uniform generation among executions of the system, since it focuses on their sequentializations.

We consider the framework of trace monoids \cite{10}, also called monoids with partial commutations \cite{6}. Trace monoids have been studied as basic models of concurrent systems since several decades \cite{17,11}. One uses the algebraic commutation of generators of the monoid to render the concurrency of elementary actions. The elements of the trace monoid, called traces, represent the finite executions of the concurrent system \cite{10}. More sophisticated concurrency models build on trace monoids, for instance executions of 1-safe Petri nets correspond to regular languages of trace monoids \cite{19}.

The topic of this paper is the effective uniform generation of large traces, \( i.e. \), large elements in a trace monoid. It has several potential applications in the model checking and in the simulation of concurrent systems.
In a recent work co-authored with J. Mairesse [2], we have shown that the notion of Bernoulli measure for trace monoids provides an analogous, in a framework with concurrency, of classical Bernoulli sequences—i.e., the mathematical model of memoryless coin tossing. In particular, Bernoulli measures encompass the uniform measure, an analogous for trace monoids of the maximal entropy measure. Therefore Bernoulli measures are an adequate theoretical ground to work with for the random generation of traces, in particular for the uniform generation.

Bernoulli sequences are highly efficiently approximated by random generators. An obvious, but nevertheless crucial feature of their generation is that it is incremental. For the random generation of large traces, we shall also insist that the generation procedure is incremental. Furthermore, another desirable feature is that it is distributed, in a sense that we explain now.

We consider trace monoids attached to networks of alphabets sharing common letters. The synchronization of several sequences of letters on different local alphabets sharing common letters is known to be entirely encoded by a unique element of a trace monoid. If \( \Sigma \) denotes the union of all local alphabets, then the \emph{synchronization trace monoid} is the monoid with the presentation by generators and relations \( \mathcal{M} = \langle \Sigma \mid ab = ba \rangle \), where \((a, b)\) ranges over pairs of letters that do not occur in any common local alphabet. Hence, seeing local alphabets as “resources”, two distinct letters \( a \) and \( b \) commute in \( \mathcal{M} \) if and only if they do not share any common resource—a standard paradigm in concurrency theory.

In this framework, our problem rephrases as follows: given a synchronization trace monoid, design a probabilistic protocol to reconstruct a global random trace, uniformly among traces, and in a distributed and incremental way. By “distributed”, we mean that the random primitives should only deal with the local alphabets. The expression “uniformly among traces” deserves also an explanation, since traces of a monoid are countably many. One interpretation is to fix a size \( k \) for target traces, and to retrieve a trace uniformly distributed among those of size \( k \). Another interpretation is to consider infinite traces, i.e., endless executions of the concurrent system. It amounts in an idealization of the case with large size traces. We then rely on the notion of uniform measure for infinite traces, which happens to have nicer properties than the uniform distribution on traces of fixed size. As explained above, the uniform measure belongs to the largest class of Bernoulli measures for trace monoids. Hence, a slightly more general problem is the distributed and incremental reconstruction of any Bernoulli measure attached to a synchronization trace monoid.

We introduce two algorithms that partly solve this problem, the Probabilistic Synchronization Algorithm (PSA) and the Probabilistic Full Synchronization Algorithm (PFSA), the latter building on the former. According to the topology of the network, one algorithm or the other shall be applied. We show that the problem of generating traces according to a Bernoulli measure is entirely solved for some specific topologies of the network, namely for the path topology and for the ring topology—it could also be applied successfully to a tree topology. It is only partly solved for a general topology. Yet, even in the case of a general topology, our procedure outputs large random traces according to a Bernoulli scheme, although it is unclear how to tune the probabilistic parameters in order to obtain uniformity. Furthermore, the amount of time needed to obtain a trace of size \( k \) is linear with \( k \) in average.

Several works analyzing the exchange of information in concurrent systems restrict their studies to tree topologies, see for instance [13], and very few has been said on the probabilistic aspects. In particular, designing an incremental and distributed procedure to uniformly randomize a system with a ring topology has been an unsolved problem in the literature so far.

How does our method compare with standard generation methods based on tools from Analytic Combinatorics, such as Boltzmann samplers techniques? There exists
a normal form for traces, the so-called Cartier-Foata normal form, from which one
derives a bijection between traces of a given trace monoid and words of a regular
language. This seems to draw a direct connection with Boltzmann sampling of words
from regular languages ([12, p.590], [4]). This approach however suffers from some
drawbacks. First, the rejection mechanism which is at the very heart of the Boltzmann
sampling approach prevents the construction to be incremental, as we seek. This
could be avoided by considering instead the generation of the Markov chain of the
elements of the normal form associated to the uniform measure on infinite traces,
as investigated partly in [3]. But this leads to a second problem, namely that the
randomness now concentrates on the set of cliques of the trace monoid, a set which
size grows exponentially fast with the number of generators of the monoid in general.
It henceforth misses the point of being a distributed generation.

Outline. Section 2 illustrates on small examples the two algorithms for generat-
ing random traces from a network of alphabets, the PSA and the PFSA, to be fully
analyzed in forthcoming sections. Sections 3 and Section 4 are two preliminary sec-
tions, gathering material on the combinatorics of trace monoids for the first one, and
material on Bernoulli measures for trace monoids for the second one. The non-random
algorithms for the synchronization of sequences, possibly infinite, that we present in
Section 3 seem to be new, although the fundamental ideas on which they rest are not
new. A new result on the theory of Möbius valuations is given at the end of Section 4.

Our main contributions are organized in Sections 5 and 6. Section 5 is devoted
to the Probabilistic Synchronization Algorithm (PSA). The analysis of the algorithm
itself is quite simple and short. The most striking contributions are in the examples.
In particular, for the surprising case of the path model, the PSA is shown to work the
best one could expect.

Section 6 is devoted to the description and to the analysis of the Probabilistic
Full Synchronization Algorithm (PFSA), both from the probabilistic point of view
and from the complexity point of view. Examples are examined; the ring model of
arbitrary size is precisely studied, and we obtain the satisfying result of simulating
any Bernoulli scheme on infinite traces by means of the PFSA.

Finally, Section 7 discusses some additional complexity issues and suggests pers-
pectives.

2—Illustrating the PSA and PFSA algorithms

In this section, we illustrate our two generation algorithms on small examples, as well
as the tuning of their probabilistic parameters, at an informal and descriptive level.

2.1 — Illustrating the Probabilistic Synchronization Algorithm (1)

Let \( a_0, a_1, a_2, a_3, a_4 \) be five distinct symbols, and consider the four alphabets:

\[
\Sigma_1 = \{a_0, a_1\}, \quad \Sigma_2 = \{a_1, a_2\}, \quad \Sigma_3 = \{a_2, a_3\}, \quad \Sigma_4 = \{a_3, a_4\}.
\]

To each alphabet \( \Sigma_i \) is attached a device able to produce a random sequence
\( Y_i = (Y_{i,1}, Y_{i,2}, \ldots) \) of letters \( Y_{i,j} \in \Sigma_i \). The random letters are independent and
identically distributed, according to a distribution \( p_i = \left( p_i(a_{i-1}), p_i(a_i) \right) \) to be
determined later, but with positive coefficients. Furthermore, we assume that the
devices themselves are probabilistically independent with respect to each other. For
instance, the beginnings of the four sequences might be:

\[
Y_1 = (a_1 a_0 a_1 a_0 \ldots) \quad Y_2 = (a_1 a_2 a_2 a_2 \ldots) \\
Y_3 = (a_2 a_3 a_3 a_2 \ldots) \quad Y_4 = (a_4 a_3 a_3 a_4 \ldots)
\]
Figure 1: Heap of pieces corresponding to the vector $Y$

\[
Y = \begin{pmatrix}
a_1 & a_0 & a_0 & \ldots \\
a_1 & a_2 & a_2 & a_2 & \ldots \\
a_2 & a_2 & a_3 & a_3 & a_2 & \ldots \\
a_4 & a_3 & a_3 & \ldots \\
\end{pmatrix}
\]

Figure 2: Synchronization graph of the path model with five generators

Then we stack these four sequences in a unique vector $Y$ with four coordinates, and we join the matching letters from one coordinate to its neighbor coordinates, in their order of appearance. The border elements $a_0$ and $a_4$ do not match with any other element; some elements are not yet matched (for example, the second occurrence of $a_1$ in $Y_1$), and we do not picture them. This yields:

Finally, we identify each matching pair with a single piece, labeled with the matching letter and we impose a rotation of the whole picture by a quarter turn counterclockwise. We obtain the heap of pieces depicted in Figure 1. It is apparent on this picture that pieces $a_i$ and $a_{i+1}$ share a sort of common resource. Therefore we depict the topology of the system resulting from the synchronization of the network of alphabets $(\Sigma_1, \Sigma_2, \Sigma_3, \Sigma_4)$ in Figure 2 by drawing an undirected graph with pieces as vertices, and an edge between two pieces whenever they share a common resource, or equivalently, whenever they both appear in the same alphabet.

The PSA algorithm consists, for each device attached to each alphabet $\Sigma_i$, to do the following: 1) Generate its own local random sequence; 2) Communicate with the neighbors in order to tag the matching of each of the occurring letters. Since the coefficients $p_i(a_{i-1})$ and $p_i(a_i)$ are positive, each coordinate $Y_i$ has infinitely many occurrences of both letters $a_{i-1}$ and $a_i$. Furthermore, each occurrence of a letter $a_j$ with $j \neq 0, 4$ will eventually match a corresponding occurrence in a neighbor coordinate. Therefore the random heap that we obtain, say $\xi$, is infinite if the algorithm keeps running forever. The PSA produces incremental finite approximations of $\xi$. The probabilistic analysis of the PSA consists in determining the probability distribution of the theoretical infinite random heap $\xi$ that the PSA would produce if it was to run indefinitely.

In order to characterize the probability distribution of $\xi$, we shall denote by $\uparrow x$, for each finite possible heap $x$, the probabilistic event that $\xi$ starts with $x$. This means
that the finite heap $x$ can be seen at the bottom of $\xi$. Equivalently: $\xi \in \uparrow x$ if, after waiting long enough, the finite heap $x$ can be seen at the bottom of the current heap produced by the PSA.

We aim at evaluating the probability $\mathbb{P}(\uparrow x)$ for every finite heap $x$. For this, consider an arbitrary possible sequentialization of $x$, seen as a successive piling of different occurrences of the elementary pieces, which we write symbolically as $x = x_1 \cdots x_k$ with $x_j \in \{a_0, a_1, a_2, a_3, a_4\}$. Then $\mathbb{P}(\uparrow x)$ is obtained as the following product:

$$\mathbb{P}(\uparrow x) = t_{x_1} \cdots t_{x_k},$$

where the coefficient $t_{x_j}$ corresponds to the probability of having the letter $x_j$ appearing either on its single coordinate if $x_j = a_0$ or $x_j = a_4$, or on both coordinates to which it belongs otherwise. Hence:

$$t_{a_0} = p_1(a_0), \quad t_{a_1} = p_1(a_1) \cdot p_2(a_1), \quad t_{a_2} = p_2(a_2) \cdot p_3(a_2), \quad t_{a_3} = p_3(a_3) \cdot p_4(a_3), \quad t_{a_4} = p_4(a_4).$$

For instance, if the local probabilities $p_i$ were uniform, $p_i = (1/2 \quad 1/2)$, we would have:

$$t_{a_0} = \frac{1}{2}, \quad t_{a_1} = \frac{1}{4}, \quad t_{a_2} = \frac{1}{4}, \quad t_{a_3} = \frac{1}{4}, \quad t_{a_4} = \frac{1}{2}.$$  

This choice, which may seem natural at first, would actually produce a random heap with a bias, namely it would favor the appearance of pieces $a_0$ and $a_4$. Since $t_{a_0} = 1/2$ and $t_{a_1} = 1/4$ in this case, it would produce on average twice more occurrences of $a_0$ than occurrences of $a_1$.

However, for uniform generation purposes, it is desirable to obtain all coefficients $t_{a_i}$ equal. Can we tune the initial probability distributions $p_i$ in order to achieve this result? Introduce the Möbius polynomial $\mu(z) = 1 - 5z + 6z^2 - 2z^3$, and consider its root of smallest modulus, namely $q_0 \approx 0.308$. It turns out that the only way to have all coefficients $t_{a_i}$ equal is to make them precisely equal to $q_0$. In turn, this imposes conditions on the local probability distributions $p_1, p_2, p_3, p_4$ with only one solution, yielding for this example:

$$p_1(a_0) = q_0 \approx 0.308, \quad p_1(a_1) = 1 - q_0 \approx 0.692$$

$$p_2(a_1) = \frac{q_0}{1 - q_0} \approx 0.445, \quad p_2(a_2) = 1 - \frac{q_0}{1 - q_0} \approx 0.555$$

$$p_3(a_2) = \frac{1 - q_0}{1 - q_0} \approx 0.555, \quad p_3(a_3) = \frac{q_0}{1 - q_0} \approx 0.445$$

$$p_4(a_3) = 1 - q_0 \approx 0.692, \quad p_4(a_4) = q_0 \approx 0.308$$

This array of positive numbers has the sought property that the product of any two numbers along the depicted diagonals equals $q_0$, and the sum of every line equals 1; and $q_0$ is the only real allowing this property for an array of this size. Running the PSA with these values for the local probability distributions produces a growing random heap which is uniform, in a precise meaning that will be formalized later in the paper.

### 2.2 Illustrating the Probabilistic Synchronization Algorithm (2)

Consider the following network of alphabets with the four letters $a_0, a_1, a_2, a_3$:

$$\Sigma_1 = \{a_0, a_1\}, \quad \Sigma_2 = \{a_1, a_2\}, \quad \Sigma_3 = \{a_2, a_3\}, \quad \Sigma_4 = \{a_3, a_0\}.$$
Figure 3: Synchronization graph of the ring model with four generators

Figure 4: Output heap resulting from an instance of the PSA for the ring model with four generators

Analogously to the previous example, the synchronization graph for this network is depicted in Figure 3. Let us try to apply the same generation technique as in the previous example. For each alphabet $\Sigma_i$, we consider a random sequence $Y_i$ of letters of this alphabet. Given the symmetry of the network of alphabets, in order to obtain a uniform distribution we need to consider this time uniform local distributions $p_i = (1/2, 1/2)$ for $i = 1, 2, 3, 4$. This yields for instance:

$Y_1 = a_0 a_1 a_1 a_0 \ldots$  $Y_2 = a_1 a_2 a_2 a_1 \ldots$  $Y_3 = a_3 a_2 a_3 a_2 \ldots$  $Y_4 = a_0 a_3 a_0 a_3 \ldots$

The construction of the stacking vector $Y$ with the matching of letters yields:

$$Y = \begin{pmatrix}
\begin{array}{cccc}
1 & a_0 & a_1 & \ldots \\
1 & a_1 & a_2 & \ldots \\
1 & a_3 & a_2 & \ldots \\
1 & a_0 & a_3 & \ldots \\
\end{array}
\end{pmatrix}$$

the two occurrences of $a_0$ being connected.

The corresponding heap is depicted on Figure 4. It is then impossible to extend this heap while still respecting the sequences $Y_1, Y_2, Y_3, Y_4$; not because some letter is waiting for its matching, but because the different letters already present are blocking each other, creating a cycle that prevents to interpret the rest of the vector as an extension of the heap of Figure 4. Such a cycle will appear with probability 1 at some point, whatever the sequences $Y_1, Y_2, Y_3, Y_4$. Henceforth the PSA only outputs finite heaps for this model.

2.3 — Illustrating the Probabilistic Full Synchronization Algorithm

In the previous example, we have seen that the PSA outputs a finite heap with probability 1, whereas we would like to obtain heaps arbitrary large. The Probabilistic Full Synchronization Algorithm (PFSA) is designed to solve this issue. We will now describe how to tune it, for the ring model with four generators introduced above and illustrated in Figure 3, in order to produce infinite heaps uniformly distributed.
First, the theory of Bernoulli measures for trace monoids (see below in Section 4) tells us that each piece of the monoid must be given the probabilistic weight \( q_1 \), root of smallest modulus of the Möbius polynomial \( \mu(z) = 1 - 4z + 2z^2 \), hence \( q_1 = 1 - \sqrt{2} / 2 \).

Second, we arbitrarily choose a piece, say \( a_3 \), to be removed. We are left with the following network of alphabets: \( \Sigma'_1 = \{a_0, a_1\} \), \( \Sigma'_2 = \{a_1, a_2\} \), \( \Sigma'_3 = \{a_2\} \), \( \Sigma'_4 = \{a_0\} \). From the heaps point of view, this is equivalent to the network with only two alphabets \( (\Sigma'_1, \Sigma'_2) \), and sharing thus the only letter \( a_1 \). We will now design a variant of the \textsf{PSA} algorithm, to be applied to the network \( (\Sigma'_1, \Sigma'_2) \), outputting \textit{finite} heaps with probability 1, and attributing the probabilistic parameter \( q_1 \) to each of the three pieces \( a_0, a_1, a_2 \). This is possible by considering \textit{sub-probability} distributions \( p'_1, p'_2 \) associated to \( \Sigma'_1, \Sigma'_2 \) instead of \textit{probability} distributions for the local generations. A possible choice, although not the unique one, is the following:

\[
p'_1(a_0) = q_1 = 1 - \sqrt{2} / 2 \quad p'_1(a_1) = 1 - q_1 = \sqrt{2} / 2
\]

\[
p'_2(a_1) = \frac{q_1}{1 - q_1} = \sqrt{2} - 1 \quad p'_2(a_2) = q_1 = 1 - \sqrt{2} / 2
\]

Note that the first line sums up to 1 whereas the second line sums up to less than 1, and this guarantees that the \textsf{PSA} executed with these values will output a finite heap and stop with probability 1.

Let \( \xi'_1 \) be the output of the \textsf{PSA} on \( (\Sigma'_1, \Sigma'_2) \) with the above parameters. It is a finite heap built with occurrences of \( a_0, a_1 \) and \( a_2 \) only. Then, we add the piece \( a_3 \) on top of \( \xi'_1 \), and we check whether the obtained heap is \textit{pyramidal}, which means that no piece can be removed from it without moving the last piece \( a_3 \) (see Figure 5). If it is not pyramidal, we reject it, and re-run the \textsf{PSA}, producing another instance of \( \xi'_1 \), until \( \xi'_1 \cdot a_3 \) is pyramidal. At the end of this process, we obtain a pyramidal heap \( \xi_1 = \xi'_1 \cdot a_3 \) built with the four generators \( a_0, a_1, a_2, a_3 \), and a unique occurrence of \( a_3 \).

The process of computing this heap \( \xi_1 \) is the first loop of the \textsf{PFSA}. We initialize what will be the final output of the algorithm by setting \( y_1 = \xi_1 \). We then reproduce the above random procedure, yielding a pyramidal heap \( \xi_2 \) and then we form the heap \( y_2 = y_1 \cdot \xi_2 \), obtained by simply piling up \( \xi_2 \) on top of \( y_1 \). We iterate this procedure: outputting \( \xi_3 \) pyramidal, we put \( y_3 = y_2 \cdot \xi_3 \), and so on. Then the increasing heaps \( y_1, y_2, \ldots \) approximate an infinite heap on the four generators \( a_0, a_1, a_2, a_3 \) which we claim to be uniformly distributed.
3—Preliminaries on trace monoids

An alphabet is a finite set, usually denoted by $\Sigma$, the elements of which are called letters. The free monoid generated by $\Sigma$ is denoted by $\Sigma^*$.

3.1 — Combinatorics of trace monoids

3.1.1 Definitions — An independence pair on the alphabet $\Sigma$ is a binary, irreflexive and symmetric relation on $\Sigma$, denoted $I$. The trace monoid $M = M(\Sigma, I)$ is the presented monoid $M = \langle \Sigma \mid ab = ba \text{ for all } (a,b) \in I \rangle$. Hence, if $R$ is the smallest congruence on $\Sigma^*$ that contains all pairs $(ab, ba)$ for $(a, b)$ ranging over $I$, then $M$ is the quotient monoid $M = \Sigma^*/R$. Elements of $M$ are called traces [10]. The unit element is denoted by $e$, and the concatenation in $M$ is denoted by $\cdot$.

The length $|x|$ of a trace $x \in M$ is the length of any word in the equivalence class $x$. The left divisibility relation on $M$ is denoted by $\leq$, it is defined by $x \leq y \iff \exists z \in M : y = x \cdot z$.

3.1.2 Cliques — A clique of $M$ is any element $x \in M$ of the form $x = a_1 \cdot \ldots \cdot a_n$ with $a_i \in \Sigma$ and such that $(a_i, a_j) \in I$ for all $i \neq j$. Cliques thus defined are in bijection with the cliques, in the graph-theoretic sense, of the pair $(\Sigma, I)$ seen as an undirected graph, that is to say, with the set of complete sub-graphs of $(\Sigma, I)$.

The set of cliques is denoted by $C$. The unit element is a clique, called the empty clique. The set of non empty cliques is denoted by $C^*$.

For instance, for $M = \langle a_0, a_1, a_2, a_3, a_4 \mid a_i a_j = a_j a_i \text{ for } |i - j| > 1 \rangle$, we have $C = \{ e, a_0, a_1, a_2, a_3, a_4, a_0 a_2, a_0 a_3, a_0 a_4, a_1 a_3, a_1 a_4, a_2 a_4, a_0 a_2 a_4 \}$. We call this trace monoid the path model with five generators, it corresponds to the example introduced in Section 2.1. Note that the synchronization graph depicted in Figure 2 is not $(\Sigma, I)$ but its complementary.

3.1.3 Growth series and Möbius polynomial — The growth series of $M$ is the formal series

$$Z_M(t) = \sum_{x \in M} t^{|x|}.$$ 

The Möbius polynomial [6] is the polynomial $\mu_M(t)$ defined by

$$\mu_M(t) = \sum_{c \in C} (-1)^{|c|} t^{|c|}.$$ 

For the path model with five generators introduced above, we have $\mu_M(t) = 1 - 5t + 6t^2 - t^3$.

The Möbius polynomial is the formal inverse of the growth series [6] [22]:

$$Z_M(t) = 1/\mu_M(t).$$

The Möbius polynomial has a unique root of smallest modulus, say $p_0$. This root is real and lies in $(0, 1]$, and coincides with the radius of convergence of the power series $Z_M(t)$ [6] [14].

3.1.4 Multivariate Möbius polynomial — The Möbius polynomial has a multivariate version, $\mu_M(t_1, \ldots, t_N)$ where $t_1, \ldots, t_N$ are formal variables associated with the generators $a_1, \ldots, a_N$ of $M$. It is defined by:

$$\mu_M(t_1, \ldots, t_N) = \sum_{c \in C} (-1)^{|c|} t_{j_1} \cdot \ldots \cdot t_{j_n},$$

where the variables $t_{j_1}, \ldots, t_{j_n}$ correspond to the letters such that $c = a_{j_1} \cdot \ldots \cdot a_{j_n}$. 

8
3.1.5 Irreducibility — The dependence relation associated with $\mathcal{M} = \mathcal{M}(\Sigma, I)$ is the binary, symmetric and reflexive relation $D = (\Sigma \times \Sigma) \setminus I$. The trace monoid $\mathcal{M}$ is irreducible if the pair $(\Sigma, D)$ is connected as a non oriented graph.

If $\mathcal{M}$ is irreducible, then the root $p_0$ of $\mathcal{M}$ is simple [16].

3.2 — The heap of pieces interpretation of traces

3.2.1 The picture — Viennot’s theory provides a visualization of traces as heaps of pieces [22]. Picture each trace as the piling of dominoes labeled by the letters of the alphabet, and such that dominoes associated with two letters $a$ and $b$ fall to the ground according to parallel lanes, and disjoint if and only if $(a,b) \in I$. See an illustration on Figure 6 (i)–(iii), for $\mathcal{M} = \langle a, b, c, d \mid ac = ca, ad = da, bd = db \rangle$.

3.2.2 Traces as labeled ordered sets — The formalization of this picture is done by interpreting each trace $x$ of a trace monoid as an equivalence class, up to isomorphism, of a $\Sigma$-labeled partial order, which we describe now.

Let $x$ be a trace, written as a product of letters $x = a_1 \cdot \ldots \cdot a_n$. Let $\pi = \{1, \ldots, n\}$, and define a labeling $\phi : \pi \to \Sigma$ by $\phi(i) = a_i$. Equip $\pi$ with the natural ordering on integers $\leq$. Then remove the pair $(i, j)$ from the strict ordering relation $<$ on $\pi$ whenever, by a finite number of adjacent commutations of distinct letters, the sequence $(a_1, \ldots, a_n)$ can be transformed into a sequence $(a_{\sigma(1)}, \ldots, a_{\sigma(n)})$ such that $\sigma(i) > \sigma(j)$.

The set of remaining ordered pairs $(i, j)$ is a partial ordering on $\pi$, which only depends on $x$, and which defines the heap associated with $x$. See Figure 6 (iv).

3.3 — Completing trace monoids with infinite traces

3.3.1 Infinite traces — Let $(x_n)_{n \geq 1}$ and $(y_n)_{n \geq 1}$ be two nondecreasing sequences of traces: $x_n \leq x_{n+1}$ and $y_n \leq y_{n+1}$ for all integers $n \geq 1$. We identify $(x_n)_{n \geq 1}$ and $(y_n)_{n \geq 1}$ whenever they satisfy:

$$(\forall n \geq 1 \ \exists m \geq 1 \ x_n \leq y_m) \land (\forall n \geq 1 \ \exists m \geq 1 \ y_n \leq x_m).$$

This identification is an equivalence relation between nondecreasing sequences. The quotient set is denoted by $\overline{\mathcal{M}}$. It is naturally equipped with a partial ordering, such that the mapping $\mathcal{M} \to \overline{\mathcal{M}}$ associating the (equivalence class of the) constant sequence $x_n = x$ to any trace $x \in \mathcal{M}$, is an embedding of partial orders.
We identify thus $\mathcal{M}$ with its image in $\overline{\mathcal{M}}$ through the embedding $\mathcal{M} \rightarrow \overline{\mathcal{M}}$. The set $\partial \mathcal{M}$ of infinite traces is defined by:

$$\partial \mathcal{M} = \overline{\mathcal{M}} \setminus \mathcal{M}.$$ 

The set $\partial \mathcal{M}$ is called the boundary of $\mathcal{M}$ [2]. Visually, elements of $\partial \mathcal{M}$ correspond to infinite countable heaps, that is to say, limiting heaps obtained by piling up countable many pieces.

3.3.2 Properties of $(\mathcal{M}, \leq)$ — The partially ordered set $(\mathcal{M}, \leq)$ is complete with respect to least upper bound of nondecreasing sequences. And any element $\xi \in \mathcal{M}$ is the least upper bound of a nondecreasing sequence of elements of $\mathcal{M}$ [2].

3.4 — Synchronization of sequences

In this section we introduce an alternative way of looking at trace monoids, by means of vectors of words. This emphasizes the distributed point of view on trace monoids.

3.4.1 Network of alphabets and synchronizing trace monoid. — A family $(\Sigma_1, \ldots, \Sigma_N)$ of $N$ alphabets, not necessarily disjoint, is called a network of alphabets. Let $\Sigma = \Sigma_1 \cup \ldots \cup \Sigma_N$. Define $R = \{1, \ldots, N\}$ as the set of resources, and consider the product monoid

$$H = (\Sigma_1)^* \times \ldots \times (\Sigma_N)^*.$$  

(1)

Identify each $a \in \Sigma$ with the element $H(a) = (a_i)_{i \in R}$ of $H$ defined by:

$$\forall i \in R \quad a_i = \begin{cases} 
\epsilon \text{ (the empty word)}, & \text{if } a \not\in \Sigma_i, \\
a, & \text{if } a \in \Sigma_i. 
\end{cases}$$

For each letter $a \in \Sigma$, the set of resources associated with $a$ is the subset $R(a)$ defined by

$$R(a) = \{ i \in R : a \in \Sigma_i \}.$$  

Let $\mathcal{G}$ be the sub-monoid of $H$ generated by the collection $\{H(a) : a \in \Sigma\}$. Then $\mathcal{G}$ is isomorphic with the trace monoid $\mathcal{M} = \mathcal{M}(\Sigma, I)$, where the independence relation $I$ on $\Sigma$ is defined by

$$(a, b) \in I \iff R(a) \cap R(b) = \emptyset.$$  

(2)

The monoid $\mathcal{M}$ is called the synchronization trace monoid, or simply the synchronization monoid, of the network $(\Sigma_1, \ldots, \Sigma_N)$. Examples have been given in Section 2. It can be proved that every trace monoid is isomorphic to a synchronization trace monoid [7].

3.4.2 Synchronization of sequences. — Let $(\Sigma_1, \ldots, \Sigma_N)$ be a network of alphabets. Each monoid $(\Sigma_i)^*$ being equipped with the prefix ordering, we equip the product monoid $H$ defined in (1) with the product order, and the sub-monoid $\mathcal{G}$ with the induced order.

Assume given a vector of sequences $Y = (Y_1, \ldots, Y_N) \in H$. The synchronization trace of $Y$ is defined as the following least upper bound in $\mathcal{G}$:

$$X = \bigvee \{ Z \in \mathcal{G} : Z \leq Y \}.$$  

(3)

This least upper bound does indeed exist in $\mathcal{G}$: it is obtained by taking least upper bounds component-wise, which are all well defined.
Identifying $\mathcal{G}$ with the trace monoid $\mathcal{M}$ defined in (2) and above, the element $X$ is thus the largest trace among those tuples below $Y$.

For example, consider the ring model with four generators introduced in Section 2.2, with $\Sigma_1 = \{a_0, a_1\}$, $\Sigma_2 = \{a_1, a_2\}$, $\Sigma_3 = \{a_2, a_3\}$ and $\Sigma_4 = \{a_3, a_0\}$, and the vector:

$$Y = \begin{pmatrix} a_0 a_1 a_0 \\ a_1 a_2 a_2 a_1 \\ a_3 a_2 a_3 a_2 \\ a_0 a_3 a_0 a_3 \end{pmatrix}$$

One easily convinces oneself that the synchronization of $Y$ is the following:

$$X = \begin{pmatrix} a_0 a_1 \\ a_1 a_2 \\ a_3 a_2 \\ a_0 a_3 \end{pmatrix}$$

corresponding to $a_0 \cdot a_1 \cdot a_3 \cdot a_2$ in the trace monoid.

We shall see below an algorithmic way to determine the synchronization of any given vector $Y \in \mathcal{H}$.

3.4.3 On-line computation features. — The algorithmic computation of the synchronization trace of a given vector of sequences might belong to the folklore of concurrency theory; see for instance [7]. In what follows, an special emphasis is given to the distributed and on-line nature of this computation.

The remaining of this section is devoted to provide an algorithm taking a vector of sequences $Y \in \mathcal{H}$ as input, and outputting the synchronization trace of $Y$. Furthermore, the input trace $Y'$ might not be entirely known at the time of computation. Instead, we assume that only a sub-trace $Y' \leq Y$ feeds the algorithm, together with the knowledge whether some further input is about to come or not. Therefore, we need our algorithm to produce the two following outputs:

1. The best approximation $X'$ of the synchronization trace $X$, given only the input $Y'$, which is merely the synchronization trace of $Y'$.

2. And one of the following tags:
   - DL for “Deadlock” if some more input is about to come, and yet the algorithm has already reached the synchronization trace of $Y$, i.e., if $X' = X$ whatever the continuation of $Y'$.
   - WFI for “Waiting for Input” if some more input is about to come, and the trace $X'$ might be extended.
   - EOF for “End of file” if there is no more input to come.

3.4.4 Detecting deadlocks — Consider, as in Sections 3.4.1–3.4.2, a network $(\Sigma_1, \ldots, \Sigma_N)$ of alphabets. Given a finite word $Y_i \in (\Sigma_i)^*$, we denote by $\overline{Y}_i$ the word $\overline{Y}_i = Y_i \cdot \dagger_i$, where $\dagger_i$ is an additional symbol which can be either EOF of WFI, to be interpreted as “no more input will ever come” if $\dagger_i = \text{EOF}$, or as “some more input might arrive” if $\dagger_i = \text{WFI}$. Finally, let $\overline{Y} = (\overline{Y}_1, \ldots, \overline{Y}_N)$.

We first introduce a basic routine described in pseudo-code below (Algorithm 1). The input of the routine is the vector $\overline{Y}$. Its output is as follows:

1. If the synchronization trace $X$ of $Y$ is non empty, then the routine outputs a minimal piece of $X$.

2. If the synchronization trace $X$ of $Y$ is empty, then the routine outputs a flag $\ddagger$ explaining why $X$ is empty:
a) If $\mathbf{Y} = (\text{eof}, \ldots, \text{eof})$, then $\ddagger = \text{eof}$. Note that this necessarily entails that $\mathbf{Y}$ is empty.

b) In case at least one input component of $\mathbf{Y}$ does not carry the symbol $\ddagger_i = \text{eof}$, then:
   i. If adding some letters to some of the components of $\mathbf{Y}$ carrying $\ddagger_i = \text{wfi}$ could yield a non empty synchronization trace, then $\ddagger = \text{wfi}$.
   ii. If no letter can be added to the components of $\mathbf{Y}$ carrying $\ddagger_i = \text{wfi}$ to yield a non empty synchronization trace, then $\ddagger = \text{dl}$.

To detect if a piece $u$ is minimal in a heap, when the heap is given by its representation in $\mathcal{G}$, it is necessary and sufficient to check that it is minimal in all the coordinates where it occurs, that is to say, in all the components belonging to the set of resources $\mathcal{R}(u)$. This justifies that Algorithm 1 is sound. Observe that the algorithm is non deterministic, as there may be several minimal pieces.

Algorithm 1 executes in constant time, the constant growing linearly with the number of edges in the dependence graph $(\Sigma, D)$ of the associated trace monoid. If communicating processes are devised, one per each alphabet, and able to write to a common register, they will be able to perform Algorithm 1 in a distributed way. We shall not work out the details of the distributed implementation, since it would be both outside the scope of the paper and out of the range of expertise of the author. People from the distributed algorithms community will probably find it routine.

3.4.5 Computation of the synchronization trace. — With Algorithm 1 at hand, we can now give an algorithm to compute the synchronization trace of a given vector of sequences $\mathbf{Y} \in \mathcal{H}$. This is the topic of the Synchronization Algorithm (Algorithm 2), of which we give the pseudo-code below, and which takes as input a vector $\mathbf{Y} = (\mathbf{Y}_1, \ldots, \mathbf{Y}_N)$ of the same kind as Algorithm 1.

The Synchronization Algorithm iteratively executes Algorithm 1 and collects the minimal pieces thus obtained to form its own output. If $\mathbf{Y} = (\mathbf{Y}_1, \ldots, \mathbf{Y}_N) \in \mathcal{H}$ and if $M \leq \mathbf{Y}$, then we denote by $Z = M \setminus \mathbf{Y}$ the unique vector $Z \in \mathcal{H}$ such that $M \cdot Z = \mathbf{Y}$.

As in the requirements stated above in Section 3.4.3, the Synchronization Algorithm outputs both the synchronization trace of its input and a tag advertising if the computation is over, either because a deadlock has been reached or because the input feed is over, or if it is still waiting for some input that might cause the synchronization trace computed so far to be extended. This feature will be crucial when considering its execution on sequences of letters which are possibly infinite, see below in Section 3.4.6.

The Synchronization Algorithm can be executed in a distributed way by $N$ communicating processes, one for each coordinate, and it runs in time linear with the size of the synchronization trace $\mathbf{X}$.

3.4.6 Feeding the Synchronization Algorithm with a possibly infinite input — We have defined the synchronization trace of a vector $\mathbf{Y} \in \mathcal{H}$ in Section 3.4.2. The same definition applies if one or several components of $\mathbf{Y}$ are infinite, by putting:

$$\mathbf{X} = \bigvee \{ Z \in \mathcal{G} : Z \leq \mathbf{Y} \},$$

and this least upper bound is always well defined in $\overline{\mathcal{M}}$ since least upper bounds of nondecreasing sequences always exist in $(\mathcal{M}, \leq)$ as recalled in Section 3.3.2. We call the element $\mathbf{X}$ of $\overline{\mathcal{M}}$ thus defined the generalized synchronization trace of $\mathbf{Y}$.

Adapting the algorithmic point of view to an infinite input introduces obviously some issues, which can be addressed by observing that the output of the Synchronization Algorithm is nondecreasing with its input. We will idealize the situation where the Synchronization Algorithm is repeatedly fed with a nondecreasing input by saying
Algorithm 1 Determines a minimal piece of the synchronizing trace $X$ of $Y = (Y_1, \ldots, Y_N)$

**Require:** $\overline{Y}_1, \ldots, \overline{Y}_N$  \quad \text{▷ Recall that } \overline{Y}_i = Y_i \cdot \uparrow_i

1: for all $i \in \{1, \ldots, N\}$ do
2: \quad $u_i \leftarrow$ first letter of $\overline{Y}_i$  \quad \text{▷ $u_i$ is either a real letter or } \uparrow_i
3: end for
4: $H \leftarrow \{i : u_i \neq \text{EOF}\}$  \quad \text{▷ The set of indices of interest}
5: $\overline{H} \leftarrow \{i : u_i = \text{EOF}\}$  \quad \text{▷ The complementary set of } H
6: if $H = \emptyset$ then
7: \quad return EOF
8: end if
9: $K \leftarrow \{i \in H : u_i \neq \text{wfi}\}$  \quad \text{▷ The set of indices with real letters}
10: if $K = \emptyset$ then
11: \quad return wfi
12: end if
13: for all $i \in K$ do
14: \quad if $\mathcal{R}(u_i) \cap \overline{H} \neq \emptyset$ then
15: \quad \quad $M_i \leftarrow \text{DL}$
16: \quad else
17: \quad \quad if $u_j = u_i$ for all $j \in \mathcal{R}(u_i)$ then  \quad \text{▷ Case where } u_i \text{ is minimal in all the expected components of } Y$
18: \quad \quad \quad $M_i \leftarrow u_i$
19: \quad else
20: \quad \quad \quad if $u_j \in \{u_i, \text{wfi}\}$ for all $j \in \mathcal{R}(u_i)$ then  \quad \text{▷ Synchronization for } u_i \text{ is possible in the future}$
21: \quad \quad \quad \quad $M_i \leftarrow \text{wfi}$
22: \quad \quad \quad else
23: \quad \quad \quad \quad $M_i \leftarrow \text{DL}$  \quad \text{▷ The piece } u_i \text{ is not minimal in } X$
24: \quad \quad end if
25: \quad end if
26: end if
27: end for
28: if $M_i = \text{DL}$ for all $i \in K$ then
29: \quad return DL  \quad \text{▷ Case 2(b)i the synchronization trace } X \text{ is empty since it has no minimal piece}$
30: end if
31: else
32: \quad if $M_i = \text{wfi}$ for all $i \in K$ then  \quad \text{▷ Case 2(b)i (again)}$
33: \quad \quad return wfi$
34: \quad else
35: \quad \quad return one $M_i$ with $M_i \notin \{\text{DL, wfi}\}$  \quad \text{▷ Any } M_i \notin \{\text{DL, wfi}\} \text{ is minimal in the synchronization trace } X$
36: \quad end if
37: end if
38: end if
39: end if
Algorithm 2 Synchronization Algorithm: computes the synchronization trace of $Y$

**Require:** $Y = (Y_1, \ldots, Y_N)$

1. $X \leftarrow \emptyset$ \quad \triangleright \text{Initialize the variable } X \text{ with the empty heap}
2. call Algorithm 1 with input $Y$
3. $M \leftarrow \text{output of Algorithm 1}$
4. while $M \notin \{dl, wfi, eof\}$ do
5. \quad $X \leftarrow X \cdot M$
6. \quad $Y \leftarrow M \setminus Y$
7. \quad call Algorithm 1 with input $Y$
8. \quad $M \leftarrow \text{output of Algorithm 1}$
9. end while
10. return $(X, M)$

that it is fed with a vector with possibly infinite components. The output, possibly infinite, is defined as the least upper bound in $\mathcal{M}$ of the nondecreasing sequence of finite output heaps. It coincides of course with the generalized synchronization trace $X$ defined in (4).

To describe more precisely this algorithmic procedure, consider a vector $Y$ of sequences, some of which may be infinite. We assume that $Y$ is effectively given through some sampling, i.e., as an infinite concatenation of finite vectors $Z_k \in \mathcal{H}$:

$$Y = Z_1 \cdot Z_2 \cdot Z_3 \cdots$$

If a component of $Y$ is finite, the corresponding component of the vector $Z_k$ will be the empty word for $k$ large enough.

Furthermore, we assume that a primitive is able to produce, for each integer $k \geq 1$, a vector $\overline{Z}_k = Z_k \cdot \overline{t}_k$, where $\overline{t}_k$ is itself a vector $\overline{t}_k = (\overline{t}_{k,i})$, with $\overline{t}_{k,i} \in \{\text{eof}, \text{wfi}\}$, in such a way that an occurrence of eof marks the finiteness of the corresponding component of $Y$. Formally, we assume that the following two properties hold, for all components:

$$(Y_i \text{ is a finite sequence}) \iff (\exists k \geq 1 \quad \overline{t}_{k,i} = \text{eof})$$

and

$$\forall k \geq 1 \quad \overline{t}_{k,i} = \text{eof} \quad \iff (\forall k' \geq k \quad \overline{t}_{k',i} = \text{eof})$$

The Generalized Synchronization Algorithm, the pseudo-code of which is given below in Algorithm 3, is then recursively fed with $\overline{Z}_1, \overline{Z}_2, \ldots$, writing out to its output register $X$. The output of Algorithm 3 is the least upper bound, in $\mathcal{M}$, of the sequence of heaps that recursively appear in the register $X$.

The Generalized Synchronization Algorithm exits its while loop if and only if the generalized synchronization trace is finite. In all cases, its output (as defined above) is the generalized synchronization trace of the vector $Y$, regardless of the decomposition $(Z_k)_{k \geq 1}$ that feeds its input.

## 4—Preliminaries on probabilistic trace monoids

We denote by $\mathbb{R}^*_+$ the set of positive reals.

### 4.1 — Bernoulli and finite Bernoulli sequences

In this section we collect classical material found in many textbooks [5]. We pay a special attention to presenting this material so as to prepare for its generalization to trace monoids.
Algorithm 3 Generalized Synchronization Algorithm

Require: \((Z_k)_{k \geq 1}\) \(\triangleright\) The \(Z_k\)s recursively feed the input
1: \(X \leftarrow e\) \(\triangleright\) Initialize the variable \(X\) with the empty heap
2: \(k \leftarrow 1\)
3: call Algorithm 2 with input \(Z_1\)
4: \((U, \dagger) \leftarrow \text{output of Algorithm 2}\) \(\triangleright\) \(\dagger \in \{\text{EOF, DL, WFI}\}\)
5: while \(\dagger = \text{WFI}\) do
6: \(X \leftarrow X \cdot U\)
7: \(k \leftarrow k + 1\)
8: call Algorithm 2 with input \(Z_k\)
9: \((U, \dagger) \leftarrow \text{output of Algorithm 2}\)
10: end while

4.1.1 Bernoulli sequences — Classically, a Bernoulli sequence on an alphabet \(\Sigma\) is an
infinite sequence \((X_n)_{n \geq 1}\) of independent and identically distributed (i.i.d.) random
variables, where each \(X_i\) takes its values in \(\Sigma\). In order to eliminate degenerated
cases, we assume that the common probability distribution, say \(\rho\), over \(\Sigma\) of all
\(X_i\) is positive on \(\Sigma\); hence \(\rho\) is bound to satisfy:
\[
\forall a \in \Sigma \quad \rho_a > 0,
\sum_{a \in \Sigma} \rho_a = 1.
\]

4.1.2 Bernoulli measures — The canonical probability space associated with the
Bernoulli sequence \((X_n)_{n \geq 1}\) is the triple \((\partial \Sigma^*, \mathcal{F}, P)\) defined as follows: the set \(\partial \Sigma^*\) is the set of infinite sequences with values in \(\Sigma\). The \(\sigma\)-algebra \(\mathcal{F}\) is the \(\sigma\)-algebra
generated by the countable collection of elementary cylinders \(\uparrow x\), for \(x\) ranging over
the free monoid \(\Sigma^*\), and defined by
\[
\uparrow x = \{\omega \in \partial \Sigma^* : x \leq \omega\},
\]
where \(x \leq \omega\) means that the infinite sequence \(\omega\) starts with the finite word \(x\). Finally, \(P\) is the unique probability measure on \((\partial \Sigma^*, \mathcal{F})\) which takes the following values on elementary cylinders:
\[
\forall x \in \Sigma^* \quad P(\uparrow x) = f(x);
\]
here \(f : \Sigma^* \to \mathbb{R}_+^*\) is the unique positive function satisfying:
\[
\forall a \in \Sigma \quad f(a) = \rho_a, \quad \forall x, y \in \Sigma^* \quad f(xy) = f(x)f(y), \quad (5)
\]
where \(xy\) denotes the concatenation of words \(x\) and \(y\). Bernoulli sequences correspond
exactly to probability measures \(P\) on \((\partial \Sigma^*, \mathcal{F})\) with the following property:
\[
\forall x \in \Sigma^* \quad P(\uparrow x) > 0,
\forall x, y \in \Sigma^* \quad P(\uparrow xy) = P(\uparrow x)P(\uparrow y).
\]
Such probability measures on \((\partial \Sigma^*, \mathcal{F})\) are called Bernoulli measures.

4.1.3 Uniform Bernoulli measure — Among Bernoulli measures associated with the
alphabet \(\Sigma\), one and only one is uniform, in the following sense:
\[
\forall x, y \in \Sigma^* \quad |x| = |y| \implies P(\uparrow x) = P(\uparrow y),
\]
where \(|x|\) denotes the length of the word \(x\). It is characterized by \(P(\uparrow x) = p_0^{|x|}\),
where \(p_0 = 1/|\Sigma|\).
4.1.4 Finite Bernoulli sequences — Let \( \rho = (\rho_a)_{a \in \Sigma} \) be a sub-probability distribution over \( \Sigma \), hence bound to satisfy:

\[
\forall a \in \Sigma \quad \rho_a > 0, \quad \sum_{a \in \Sigma} \rho_a < 1.
\]

To each sub-probability distribution \( \rho \) we associate the function \( f : \Sigma^* \rightarrow \mathbb{R} \) defined as in (5), and also the following quantities:

\[
\varepsilon = 1 - \sum_{a \in \Sigma} \rho_a, \quad Z = \sum_{x \in \Sigma^*} f(x) = \frac{1}{\varepsilon} < \infty.
\]

Finally we define the sub-Bernoulli measure \( \nu_\rho \) as the probability distribution over the countable set \( \Sigma^* \), equipped with the discrete \( \sigma \)-algebra, and defined by:

\[
\forall x \in \Sigma^* \quad \nu_\rho(\{x\}) = \frac{1}{Z} f(x).
\]

A finite Bernoulli sequence is the random sequence of letters that compose a word \( x \in \Sigma^* \), drawn at random according to the probability measure \( \nu_\rho \) on \( \Sigma^* \).

An effective way to produce a finite Bernoulli sequence according to a sub-probability distribution \( p = (p_i) \) is the following. Consider a stopping symbol \( \text{EOF} \), and extend \( p \) to a probability distribution by setting \( p(\text{EOF}) = 1 - \sum_i p_i \). Then output a Bernoulli sequence according to \( p \), until the symbol EOF first occurs. The letters before the first occurrence of EOF form the sought finite sequence.

4.1.5 Full elementary cylinders — It is convenient to consider the following completion of \( \Sigma^* \):

\[
\Sigma** = \Sigma^* \cup \partial \Sigma^*.
\]

Hence both Bernoulli measures and sub-Bernoulli measures are now defined on the same space \( \Sigma** \). For each word \( x \in \Sigma^* \), we define the full elementary cylinder \( \uparrow x \) as follows:

\[
\uparrow x = \{ \xi \in \Sigma** : x \preceq \xi \}.
\]

Here, \( x \preceq \xi \) has the same meaning as above if \( \xi \) is an infinite sequence; and it means that \( x \) is a prefix of \( \xi \) if \( \xi \) is a finite word. We gather the description of both Bernoulli and sub-Bernoulli measures in the following result.

• **Theorem 4.1** — Let \( P \) be a probability measure on \( \Sigma** = \Sigma^* \cup \partial \Sigma^* \). Assume that the function \( f : \Sigma^* \rightarrow \mathbb{R} \) defined by \( f(x) = P(\uparrow x) \) is positive multiplicative, that is to say, satisfies:

\[
\forall x \in \Sigma^* \quad f(x) > 0 \\
\forall x, y \in \Sigma^* \quad f(xy) = f(x)f(y).
\]

Define \( \rho = (\rho_a)_{a \in \Sigma} \) and \( \varepsilon \) by:

\[
\forall a \in \Sigma \quad \rho_a = f(a), \quad \varepsilon = 1 - \sum_{a \in \Sigma} \rho_a.
\]

Then one and only one of the two following possibilities occurs:

1. \( \varepsilon = 0 \). In this case, \( P \) is concentrated on \( \partial \Sigma^* \), and characterized by:

\[
\forall x \in \Sigma^* \quad P(\uparrow x) = P(\uparrow x) = f(x).
\]

The series \( \sum_{x \in \Sigma^*} f(x) \) is divergent, and \( P \) is a Bernoulli measure.
2. \( \varepsilon > 0 \). In this case, \( P \) is concentrated on \( \Sigma^* \). The series \( Z = \sum_{x \in \Sigma^*} f(x) \) is convergent, and satisfies:
\[
\varepsilon = \frac{1}{Z}, \quad \forall x \in \Sigma^* \quad P(\{x\}) = \varepsilon f(x).
\]
The measure \( P \) is a sub-Bernoulli measure.

4.2 — Bernoulli and sub-Bernoulli measures on trace monoids

The notions of Bernoulli measure and of sub-Bernoulli measure extend to trace monoids the notions of Bernoulli sequences and of finite Bernoulli sequences. They provide a theoretical ground for concurrency probabilistic models, in the framework of trace monoids.

4.2.1 Valuations — Let \( \mathbb{R}_+^* \) be equipped with the monoid structure \((\mathbb{R}_+^*, \times, 1)\). A valuation on a trace monoid \( M = M(\Sigma, I) \) is a morphism of monoids \( f : M \to \mathbb{R}_+^* \).

It is thus a function \( f : M \to \mathbb{R}_+^* \) satisfying:
\[
f(1) = 1, \quad \forall x, y \in M \quad f(x \cdot y) = f(x)f(y).
\]

4.2.2 Möbius transform; Möbius and sub-Möbius valuations — Let \( f : M \to \mathbb{R} \) be a valuation. The Möbius transform of \( f \) is the function \( h : C \to \mathbb{R} \) defined by:
\[
\forall c \in C \quad h(c) = \sum_{c' \in C : c' \geq c} (-1)^{|c'| - |c|} f(c'), \quad \text{only defined on cliques.}
\]

An alternative expression for the Möbius transform is the following. For each clique \( c \in C \), let \( M(c) \) be the sub-trace monoid generated by those letters \( a \in \Sigma \) such that \( a \parallel c \). Here, \( a \parallel c \) reads as “\( a \) parallel to \( c \)”, and means that \( (a, b) \in I \) for all letters \( b \) that occur in the clique \( c \). Then:
\[
h(c) = f(c) \cdot \mu_{M(c)}(t_1, \ldots, t_N), \quad \text{with } t_i = f(a_i), \quad (7)
\]
where \( \mu_{M(c)} \) denotes the multivariate Möbius polynomial (see Section 3.1.4) of the trace monoid \( M(c) \). By convention, the expression \( \mu_{M(c)}(t_1, \ldots, t_N) \) actually involves only the variables \( t_i \) associated with those generators belonging to \( M(c) \).

In particular, \( h(1) = \mu_M(t_1, \ldots, t_N) \)—that was already observable on (6).

A valuation \( f \) is said to be [2]:

1. A Möbius valuation if:
\[
h(1) = 0, \quad \forall c \in C \quad h(c) > 0. \quad (8)
\]

2. A sub-Möbius valuation if:
\[
h(1) > 0, \quad \forall c \in C \quad h(c) > 0. \quad (9)
\]

Equivalently, as seen from [7], if \( \Sigma = \{a_1, \ldots, a_N\} \) and if we put \( t_i = f(a_i) \) for \( i \in \{1, \ldots, N\} \), then \( f \) is:

1. A Möbius valuation if:
\[
\mu_M(t_1, \ldots, t_N) = 0, \quad \forall c \in C \quad \mu_{M(c)}(t_1, \ldots, t_N) > 0. \quad (10)
\]

2. A sub-Möbius valuation if:
\[
\mu_M(t_1, \ldots, t_N) > 0, \quad \forall c \in C \quad \mu_{M(c)}(t_1, \ldots, t_N) > 0. \quad (11)
\]
4.2.3 Cylinders and $\sigma$-algebras on $M$ and on $\partial M$ — Recall that we have introduced infinite traces in Section 3.3, yielding the completion $\overline{M} = M \cup \partial M$.

To each trace $x \in M$, we associate the elementary cylinder $\uparrow x \subseteq \partial M$ and the full elementary cylinder $\uparrow\uparrow x \subseteq M$, defined as follows:

\[ \uparrow x = \{ \xi \in M : x \leq \xi \}, \quad \uparrow\uparrow x = \{ \xi \in \partial M : x \leq \xi \} = \uparrow x \cap \partial M. \]

The set $M$ is equipped with the $\sigma$-algebra $F$ generated by the collection of full elementary cylinders, and the set $\partial M$ is equipped with the $\sigma$-algebra $\mathcal{F}$ induced by $F$ on $\partial M$. Both $\sigma$-algebras are Borel $\sigma$-algebras for compact and metrisable topologies.

The restriction of $F$ to $\partial M$ is the discrete $\sigma$-algebra.

4.2.4 Multiplicative probability measures — In order to state an analogous result to Theorem 4.1 for trace monoids, we introduce the following definition, borrowed from [2].

**Definition 4.2**— A probability measure $P$ on $(M, F)$ is said to be **multiplicative** whenever it satisfies the following property:

\[ \forall x \in M \quad P(\uparrow\uparrow x) > 0, \quad \forall x, y \in M \quad P(\uparrow (x \cdot y)) = P(\uparrow x) \cdot P(\uparrow y). \]

We define the valuation $f : M \to \mathbb{R}$ associated with $P$ and the number $\varepsilon$ by:

\[ \forall x \in M \quad f(x) = P(\uparrow x), \quad \varepsilon = h(e), \]

where $h : \mathcal{F} \to \mathbb{R}$ is the M"obius transform of $f$.

The relationship between multiplicative measures and M"obius and sub-M"obius valuations is as follows [2, 3].

**Theorem 4.3**— There is a bijective correspondence between multiplicative measures $P$ and valuations which are either M"obius or sub-M"obius. The alternative is the following:

1. $\varepsilon = 0$. The valuation is M"obius. In this case, $P$ is concentrated on $\partial M$ and characterized by:

\[ \forall x \in M \quad P(\uparrow x) = f(x). \]

The series $\sum_{x \in M} f(x)$ is divergent. We say that $P$ is a Bernoulli measure.

2. $\varepsilon > 0$. The valuation is sub-M"obius. In this case, $P$ is concentrated on $M$ and satisfies:

\[ \forall x \in M \quad P(\{x\}) = \varepsilon f(x). \]

The series $Z = \sum_{x \in M} f(x)$ is convergent and satisfies $Z = 1/\varepsilon$. We say that $P$ is a sub-Bernoulli measure.

4.2.5 Uniform multiplicative measures — A valuation $f : M \to \mathbb{R}$ is said to be **uniform** if $f(a)$ is constant, for $a$ ranging over $\Sigma$. This is equivalent to saying that $f(x)$ only depends on the length of $x$, and also equivalent to saying that $f(x) = p^{|x|}$ for some real $p$.

A multiplicative measure is uniform if the associated valuation is uniform. The following result describes uniform multiplicative measures [2, 3]. They are related to the root of smallest modulus of the M"obius polynomial of the trace monoid (see Section 3.1.3).
• Theorem 4.4—Uniform multiplicative measures \( \mathbb{P} \) on a trace monoid \( \mathcal{M} \) are in bijection with the half closed interval \((0,p_0]\), where \( p_0 \) is the root of smallest modulus of \( \mu_\mathcal{M} \). The correspondence associates with \( \mathbb{P} \) the unique real \( p \) such that \( \mathbb{P}([x,\infty)) = p^{|x|} \) for all \( x \in \mathcal{M} \).

The alternative is the following:

1. \( p = p_0 \). In this case, \( \mathbb{P} \) is Bernoulli (concentrated on the boundary).

2. \( p < p_0 \). In this case, \( \mathbb{P} \) is sub-Bernoulli (concentrated on the monoid).

4.2.6 Extension and restriction of valuations — We shall need the following result for the construction of the PFSA in Section 6.3.

• Theorem 4.5—Let \( \mathcal{M} = \mathcal{M}(\Sigma, I) \) be an irreducible trace monoid.

1. Let \( f : \mathcal{M} \to \mathbb{R}_+^\ast \) be a Möbius valuation, let \( \Sigma' \) be any proper subset of \( \Sigma \) and let \( \mathcal{M}' \) be the submonoid of \( \mathcal{M} \) generated by \( \Sigma' \). Then the restriction \( f' : \mathcal{M}' \to \mathbb{R}_+^\ast \) of \( f \) to \( \mathcal{M}' \) is a sub-Möbius valuation.

2. Let \( \Sigma' = \Sigma \setminus \{a\} \), where \( a \) is any element of \( \Sigma \), let \( \mathcal{M}' \) be the submonoid of \( \mathcal{M} \) generated by \( \Sigma' \), and let \( f' : \mathcal{M}' \to \mathbb{R}_+^\ast \) be a sub-Möbius valuation. Then there exists a unique Möbius valuation \( f : \mathcal{M} \to \mathbb{R}_+^\ast \) that extends \( f' \) on \( \mathcal{M}' \).

Proof. Proof of point 1 Let \( f, f' \) and \( \Sigma' \) be as in the statement. Let also \( h \) and \( h' \) denote the Möbius transforms of \( f \) and of \( f' \). Let \( S \) be the series with nonnegative terms:

\[
S = \sum_{x \in \mathcal{M}'} f(x).
\]

We claim that this series is convergent. To prove it, recall from [6] that a pair \((\gamma, \gamma')\) of cliques is said to be in normal form, denoted by \( \gamma \to \gamma' \), if \( \forall b \in \gamma' \exists \alpha \in \gamma (a,b) \notin I \). Any trace \( x \in \mathcal{M} \) can be uniquely written as a product \( x = \gamma_1 \cdots \gamma_k \) of cliques such that \( \gamma_i \to \gamma_{i+1} \) holds for all \( i = 1, \ldots, k - 1 \).

Let the nonnegative matrices \( A = (A_{\gamma,\gamma'})_{(\gamma,\gamma') \in \mathcal{C} \times \mathcal{C}} \) and \( B = (B_{\gamma,\gamma'})_{(\gamma,\gamma') \in \mathcal{C} \times \mathcal{C}} \), where \( \mathcal{C} \) denotes the set of nonempty cliques of \( \mathcal{M} \), be defined by:

\[
A_{\gamma,\gamma'} = 1(\gamma \in \mathcal{M}') \cdot 1(\gamma' \in \mathcal{M}') \cdot 1(\gamma \to \gamma') \cdot f(\gamma'), \quad B_{\gamma,\gamma'} = 1(\gamma \to \gamma') \cdot f(\gamma').
\]

It follows from the existence and uniqueness of the normal form for traces, decomposing the traces \( x \in \mathcal{M} \) according to the number of terms of their normal form, that the series \( S \) writes as:

\[
S = 1 + \sum_{k \geq 1} I \cdot A^k \cdot J = 1 + I \cdot \left( \sum_{k \geq 1} A^k \right) \cdot J,
\]

where \( I \) and \( J \) are row and column vectors filled with 1s. We know by [1] Lemma 6.4 that the matrix \( B \) has spectral radius 1, since \( f \) is assumed to be Möbius, and that it is a primitive matrix since \( \mathcal{M} \) is irreducible. But \( A \preceq B \) with \( A \neq B \). Therefore, it follows from the Perron-Frobenius Theorem [2] that \( A \) has spectral radius \( < 1 \) and thus that the series \( S \) is convergent.

By the Möbius inversion formula [22], it entails that the relation \( (\sum_{x \in \mathcal{M}'} f(x)) \cdot h'(e) = 1 \) holds in the fields of reals. Hence \( h'(e) > 0 \). It remains to prove that \( h'(\delta) > 0 \) also holds for any non empty clique \( \delta \) of \( \mathcal{M}' \). This is a bit easier to prove. For any non empty clique \( \delta \) of \( \mathcal{M}' \), let \( \mathcal{M}(\delta) \) and \( \mathcal{M}(\delta) \) be the submonoids of \( \mathcal{M}' \).
and of $\mathcal{M}$ respectively, be defined as in Section 4.2.2. Then $\mathcal{M}(\delta) \subseteq \mathcal{M}(\delta)$, hence the following inequalities between series with nonnegative terms hold:

$$\sum_{x \in \mathcal{M}(\delta)} f(x) \leq \sum_{x \in \mathcal{M}(\delta)} f(x) = \frac{1}{h(\delta)} < \infty.$$ 

Therefore, as above, the equality $h'(\delta) \cdot (\sum_{x \in \mathcal{M}(\delta)} f(x)) = 1$ holds in the field of reals, proving that $h'(\delta) > 0$, which completes the proof that $f'$ is a sub-M"obius valuation.

**Proof of point 2.** Let $a, \Sigma'$ and $f' : \mathcal{M}' \to \mathbb{R}_+^*$ be as in the statement. Let also $h' : \mathcal{C}' \to \mathbb{R}$ the M"obius transform of $f'$, where $\mathcal{C}'$ is the set of cliques of $\mathcal{M}'$. Assuming that a M"obius extension $f : \mathcal{M} \to \mathbb{R}_+^*$ of $f'$ exists, we first prove its uniqueness. For each positive real $t$, let $f_t : \mathcal{M} \to \mathbb{R}$ be the valuation defined by $f_t(a) = t$ and $f_t(\alpha) = f'(\alpha)$ for $\alpha \in \Sigma'$. Then any valuation on $\mathcal{M}$ extending $f'$ is of the form $f_t$ for some $t$. Let $h_t : \mathcal{C} \to \mathbb{R}$ denote the M"obius transform of $f_t$. We evaluate $h_t(e)$ as follows:

$$h_t(e) = \sum_{\gamma \in \mathcal{C}' : a \in \gamma} (-1)^{|\gamma|} f_t(\gamma) + \sum_{\gamma \in \mathcal{C}' : a \notin \gamma} (-1)^{|\gamma|} f_t(\gamma).$$

On the one hand, observing the equality of sets $\mathcal{C}' = \{ \gamma \in \mathcal{C} : a \notin \gamma \}$, we recognize $h'(e)$ in the second sum. On the other hand, using the notation already introduced $a \parallel \gamma$, for $\gamma \in \mathcal{C}$, to denote that $a \notin \gamma$ and $a \cdot \gamma \in \mathcal{C}'$, the range of the first sum above is in bijection with the set of cliques $\delta \in \mathcal{C}'$ such that $a \parallel \delta$, the bijection associating $\delta$ with $\gamma = a \cdot \delta$. We then have $(-1)^{|\gamma|} f_t(\gamma) = (-t) \cdot (-1)^{|\delta|} f'(\delta)$.

Henceforth:

$$h_t(e) = (-t) \cdot K + h'(e), \quad \text{with } K = 1 - K' \quad \text{and} \quad K' = \sum_{\delta \in \mathcal{C}' : a \| \delta \neq e} (-1)^{|\delta|+1} f'(\delta).$$

Let $X$ be a random trace associated with the sub-M"obius valuation $f'$. We evaluate the probability of the event $U = \{ \exists b \in \Sigma' : a \parallel b \land a \leq X \}$. The inclusion-exclusion principle yields:

$$\mathbb{P}(U) = \sum_{\delta \in \mathcal{C}' : a \| \delta \neq e} (-1)^{|\delta|+1} \mathbb{P}(X \geq \delta) = K',$n

the later equality since $\mathbb{P}(X \geq \delta) = f'(\delta)$ by definition of $X$. The event $U$ has probability less than 1, otherwise all pieces of $\Sigma'$ would be parallel to $a$, contradicting that $\mathcal{M}$ is irreducible. We deduce that $K' \in [0, 1)$ and thus $K \in [0, 1]$. In particular, if $f_t$ is M"obius, it entails that $h_t(e) = 0$, and since we have just seen that $K \neq 0$, it implies that $t = h'(e)/K$, proving the sought uniqueness.

Let us now prove the existence of a M"obius extension $f$. Let $\mathcal{M}(\delta)$ denote as above, for any clique $\delta \in \mathcal{C}$, the sub-monoid of $\mathcal{M}$ generated by those letters $b \in \Sigma$ such that $b \parallel \delta$. For each $\delta \in \mathcal{C}$, we introduce the formal series:

$$G_\delta(t) = \sum_{x \in \mathcal{M}(\delta)} f_t(x).$$

Now, let $t_0$ be the radius of convergence of the power series $G_\delta(t) = \sum_{x \in \mathcal{M}} f_t(x)$. Since all the power series $G_\delta(t)$ have non negative coefficients, they satisfy $G_\delta(t) \leq G_\delta(t) < \infty$ for all $t \in [0, t_0)$. In particular, the radius of convergence $r_\delta$ of $G_\delta$ satisfies $r_\delta \geq t_0$. Actually, reasoning as in the proof of point 1 and invoking the Perron-Frobenius Theorem, we see that the strict inequality $r_\delta > t_0$ holds for all $\delta \neq e$. Therefore, for all $\delta \neq e$, the series $G_\delta(t_0)$ is convergent, and thus the M"obius inversion formula yields the following equality in the field of reals: $G_\delta(t_0) \cdot h_{t_0}(\delta) = 1$. We conclude that $h_{t_0}(\delta) > 0$ for all cliques $\delta \neq e$. Since $h_{t_0}(e) = 0$, we conclude that $f_{t_0}$ is the sought M"obius valuation extending $f'$. \qed
5—The Probabilistic Synchronization Algorithm

In this section we consider a network of alphabets sharing some common letters. We then wish to generate random traces of the synchronization monoid, in a distributed and incremental way. The first idea that comes in mind is to generate local Bernoulli sequences, and to see what is the synchronization trace of these sequences. This constitutes the Probabilistic Synchronization Algorithm, which is thus a probabilistic variant of the Generalized Synchronization Algorithm which was described in Algorithm 3.

The random traces thus obtained can be either finite or infinite. In all cases their probability distribution is multiplicative, hence the theory of Bernoulli and sub-Bernoulli measures for trace monoids is the adequate tool for their study. After having established this rather easy result, we turn to specific examples. We obtain the non trivial result that for path models, any Bernoulli or sub-Bernoulli measure can be generated by this simple technique.

5.1 — Description of the PSA

Let \((\Sigma_1, \ldots, \Sigma_N)\) be a network of \(N\) alphabets with \(N \geq 2\), let \(\Sigma = \bigcup_{i=1}^{N} \Sigma_i\), and let \(\mathcal{M} = \mathcal{M}(\Sigma, I)\) be the synchronization trace monoid, as described in Section 3.4.1. Recall that we identify \(\mathcal{M}\) with the sub-monoid \(\mathcal{G} \subseteq \mathcal{H}\) defined in Section 3.4.1 and that indices in \(1, \ldots, N\) are seen as resources.

Assume that, to each resource \(i \in \{1, \ldots, N\}\), is attached a device able to produce a \(\Sigma_i\)-Bernoulli sequences \(Y_i\), either finite or infinite, with a specified probability or sub-probability distribution \(p_i\) on \(\Sigma_i\). As we have seen in Section 4.1.4, the generation of such Bernoulli sequences is effective, together with the information that the sequence is over in case where \(p_i\) is a sub-probability distribution.

Henceforth, it is straightforward for each device to produce a sampling of \(Y_i\) under the form \(Z_{i,1}, Z_{i,2}, \ldots\), and moreover to tag each sub-sequence \(Z_{i,k}\) with an additional symbol \(\hat{t}_{i,k} \in \{\text{eof}, \text{wfi}\}\) in order to deliver a sequence \((Z_{i,k})_{k \geq 1}\) with \(Z_{i,k} = Z_{i,k} \cdot \hat{t}_{i,k}\) as specified in Section 3.4.6. The symbol \(\hat{t}_{i,k}\) is given the value wfi until the device decides the sequence is over (if it ever does), after which the symbol \(\hat{t}_{i,k}\) is given the value eof.

The Probabilistic Synchronization Algorithm (PSA), the pseudo-code of which is given below in Algorithm 4, consists in executing in parallel both the local generation of the sequences \(Y_1, \ldots, Y_N\), and the Generalized Synchronization Algorithm described previously in Algorithm 3.

Algorithm 4 Probabilistic Synchronization Algorithm

```
Require: \(p_1, \ldots, p_N\) \(\Rightarrow\) Probability or sub-probability distributions
1: while Algorithm 3 does not exit do
2:   for all \(i \in \{1, \ldots, N\}\) do
3:     for all \(k = 1, 2, \ldots\) do
4:       generate the \(k^{th}\) sampling \(Z_{i,k}\) of a Bernoulli sequence according to \(p_i\)
5:       feed Algorithm 3 with \(Z_{i,k}\)
6:     end for
7:   end for
8: end while
```

5.2 — Analysis of the algorithm
5.2.1 Distribution of PSA random traces — The trace $y \in \mathcal{M}$, output of the execution of the PSA (Algorithm 4), is random. What is its distribution?

- **Theorem 5.1**— The probability distribution $\mathbb{P}$ of the random trace produced by the PSA is a multiplicative probability measure on $\mathcal{M}$. The valuation $f : \mathcal{M} \to \mathbb{R}$ associated with this measure by $f(x) = \mathbb{P}(\uparrow x)$ is such that:

$$\forall a \in \Sigma \quad f(a) = \prod_{i \in R(a)} p_i(a),$$

(12)

where $p_i$ is the probability or sub-probability distribution on $\Sigma_i$, and $R(a)$ is the set of resources associated with $a$.

**Proof.** For each letter $a \in \Sigma$, let $q_a$ be the real number defined by:

$$q_a = \prod_{i \in R(a)} p_i(a).$$

Let $y \in \mathcal{M}$ be the random trace produced by the PSA. Fix $z \in \mathcal{M}$ a trace, and let $(z_1, \ldots, z_N)$ be the representation of $z$ in $\mathcal{G}$. Then $y \geq z$ holds if and only if, for every index $i \in \{1, \ldots, N\}$, the corresponding sequence $x_i$ starts with $z_i$.

Let $z = a_1 \cdot \ldots \cdot a_j$ be any decomposition of $z$ as a product of generators $a_k \in \Sigma$. Since each sequence $x_i$ produced by the device number $i$ is Bernoulli or sub-Bernoulli with distribution or sub-distribution $p_i$, and since the sequences are mutually independent, we have thus:

$$\mathbb{P}(y \geq z) = \prod_{k=1}^{j} q_{a_k}.$$  

In other words, if $f : \mathcal{M} \to \mathbb{R}$ is the valuation defined by $f(x) = \mathbb{P}(\uparrow x)$, one has $f(z) = \mathbb{P}(\uparrow z)$. This shows that $\mathbb{P}$ is multiplicative. 

5.2.2 Small values — We shall see that not all multiplicative probabilities on $\mathcal{M}$ can be reached by this technique. However, as long as only “small values” are concerned, the PSA can reach any target multiplicative measure. Let us first introduce the following convention: if $f : \mathcal{M} \to \mathbb{R}$ is a valuation, and if $\varepsilon$ is a real number, we denote by $\varepsilon f$ the valuation which values on generators are given by:

$$\forall a \in \Sigma \quad (\varepsilon f)(a) = \varepsilon f(a).$$

Hence the value of $\varepsilon f$ on arbitrary traces are given by:

$$\forall x \in \mathcal{M} \quad (\varepsilon f)(x) = \varepsilon |x| f(x).$$

The following result states that, at the expense of having small synchronizing traces, the relative frequency of letters in traces produced by the PSA suffers from no constraint.

- **Proposition 5.2**— Let $(\Sigma_1, \ldots, \Sigma_N)$ be $N$ alphabets, let $\mathcal{M} = \mathcal{M}(\Sigma, I)$ be the synchronization trace monoid, and let $t : \mathcal{M} \to \mathbb{R}$ be a valuation on $\mathcal{M}$. Then there exists $\varepsilon > 0$ and sub-probability distributions $(p_i)_{1 \leq i \leq N}$, with $p_i$ a sub-probability distribution on $\Sigma_i$, such that the valuation $f$ characterizing the random trace $y \in \mathcal{M}$ produced by the PSA with respect to $(p_i)_{1 \leq i \leq N}$, satisfies:

$$f = \varepsilon t.$$
Proof. For $\varepsilon > 0$ to be specified later, let $(p_i)_{1 \leq i \leq N}$ be the family of real valued functions, $p_i : \Sigma_i \to \mathbb{R}$, defined by:

$$\forall a \in \Sigma_i \quad p_i(a) = (\varepsilon t(a))^{1/|R(a)|}.$$ 

For $\varepsilon > 0$ small enough, all $p_i$ are sub-probability distributions over $\Sigma_i$. According to Th. 5.1, the valuation $f$ describing the distribution of the random trace produced by the PSA satisfies:

$$f(a) = \prod_{i \in R(a)} p_i(a) = \varepsilon t(a).$$

The proof is complete.

5.2.3 Reduction to irreducible trace monoids — Assume that the trace monoid is not irreducible. Hence the dependence relation $(\Sigma, D)$ has several connected components, let us say that it has two components $(S_1, D_1)$ and $(S_2, D_2)$ to simplify the discussion.

Let $I_1$ and $I_2$ be the dependence relations on $S_1$ and $S_2$ defined by:

$$I_1 = (S_1 \times S_1) \cap I = (S_1 \times S_1) \setminus D_1, \quad I_2 = (S_2 \times S_2) \cap I = (S_2 \times S_2) \setminus D_2.$$ 

Putting $\mathcal{M}_1 = \mathcal{M}(S_1, I_1)$ and $\mathcal{M}_2 = \mathcal{M}(S_2, I_2)$, we have:

$$\mathcal{M} = \mathcal{M}_1 \times \mathcal{M}_2, \quad \overline{\mathcal{M}} = \overline{\mathcal{M}_1} \times \overline{\mathcal{M}_2}.$$ 

It follows from Theorem 5.1 that the distribution $P$ of the random trace $y$ produced by the PSA is a tensor product $P_1 \otimes P_2$; probabilistically speaking, $y$ is obtained as the concatenation of two independent traces $y_1 \in \mathcal{M}_1$ and $y_2 \in \mathcal{M}_2$. The probability measures $P_1$ and $P_2$, of $y_1$ and $y_2$ respectively, are identical as those deriving from PSA algorithms restricted to the alphabets concerning $S_1$ and $S_2$ respectively.

In conclusion, the PSA algorithm decomposes as a product of sub-PSA algorithms on the irreducible components of the synchronization trace monoid. Hence there is no loss of generality in assuming, for the sake of analysis, that the synchronization trace monoid is irreducible.

5.3 — Example: the path model

The path model is close to the dimer model, a topic of numerous studies in Combinatorics and in Statistical Physics [23, 15]. Here we shall see that the path model is a framework where the PSA works at its best, in the sense that any multiplicative measure on $\overline{\mathcal{M}}$ can be obtained through the PSA.

The path model is defined as the trace monoid $\mathcal{M} = \mathcal{M}(\Sigma, I)$ on $N+1$ generators: $\Sigma = \{a_0, \ldots, a_N\}$, where $N \geq 0$ is a fixed integer. The independence relation is defined by:

$$I = \{(a_i, a_j) : |i - j| > 1\}.$$ 

Then $\mathcal{M}$ is the synchronization monoid of the $N$-tuple of alphabets $(\Sigma_1, \ldots, \Sigma_N)$ with $\Sigma_i = \{a_{i-1}, a_i\}$ for $i \in \{1, \ldots, N\}$. The set of resources of $a_i$ is:

$$R(a_i) = \begin{cases} 
\{1\}, & \text{if } i = 0, \\
\{N\}, & \text{if } i = N, \\
\{i, i + 1\}, & \text{if } 0 < i < N.
\end{cases}$$  

(13)

Assume that each of the $N$ alphabets $\Sigma_1, \ldots, \Sigma_N$ is equipped with a positive probability distribution, say $p_1, \ldots, p_N$. Then, with probability 1, all tuples $(x_1, \ldots, x_N)$,
where \( x_i \) is a Bernoulli infinite sequence distributed according to \( p_i \), are synchronizing. Indeed, each \( x_i \) contains infinitely many occurrences of \( a_{i-1} \) and of \( a_i \), and this is enough to ensure the synchronization. Therefore, the PSA yields a multiplicative measure entirely supported by the set of infinite traces of \( M \), hence a Bernoulli measure on \( \partial M \).

Actually, the following result shows that every Bernoulli measure on the path model can be obtained through the execution of the PSA.

**Theorem 5.3**—Let \( \mathcal{P} \) be a Bernoulli measure on \( \partial M \), where \( M \) is the synchronization monoid associated with the path model. For \( i \in \{0, \ldots, N\} \), put:

\[
t_i = \mathcal{P}(\uparrow a_i).
\]

Then there exists a unique tuple \( (p_1, \ldots, p_N) \), with \( p_i \) a positive probability distribution over \( \Sigma_i = \{a_{i-1}, a_i\} \), such that the random infinite trace \( \xi \in \partial M \) produced by the PSA based on \( (p_1, \ldots, p_N) \) has the distribution probability \( \mathcal{P} \).

The probability distributions \( p_1, \ldots, p_N \) are computed recursively by:

\[
\begin{align*}
p_1(a_0) &= t_0, & p_1(a_1) &= 1 - t_0 \quad (14) \\
p_i(a_{i-1}) &= \frac{t_{i-1}}{p_{i-1}(a_{i-1})}, & p_i(a_i) &= 1 - p_i(a_{i-1}). \quad (15)
\end{align*}
\]

**Remark.** The relations \((14)–(15)\) are necessary conditions, almost immediate to establish from the result of Theorem 5.1. What is not obvious is that the numbers thus defined stay within \((0,1)\), yielding indeed probability distributions \( (p_i(a_{i-1}) \ p_i(a_i)) \) over \( \Sigma_i \), and that \( p_N(a_N) = t_N \).

**Proof.** Let \( f : M \rightarrow \mathbb{R} \) be the valuation associated with \( \mathcal{P} \). According to Theorem 4.3 \( f \) is a Möbius valuation—we will use this fact in a moment.

**Proof of uniqueness of \((p_1, \ldots, p_N)\) and proof of \((14)–(15)\).** With respect to a tuple \((p_1, \ldots, p_N)\) of positive probability distributions over \( (\Sigma_1, \ldots, \Sigma_N) \), the PSA produces a random infinite trace \( \xi \in \partial M \). Let \( g : M \rightarrow \mathbb{R} \) be the valuation associated with the probability distribution of \( \xi \). Then, according to Theorem 5.1 one has:

\[
\forall i \in \{0, \ldots, N\} \quad g(a_i) = \prod_{r \in R(a_i)} p_r(a_i).
\]

Referring to \((13)\), \( f = g \) is equivalent to:

\[
\begin{align*}
t_0 &= p_1(a_0) \quad (16) \\
t_N &= p_N(a_N) \quad (17)
\end{align*}
\]

\[
0 < i < N \quad t_i = p_i(a_i)p_{i+1}(a_i) \quad (18)
\]

It follows at once that the tuple \((p_1, \ldots, p_N)\) inducing \( \mathcal{P} \) through the PSA, if it exists, is unique and satisfies necessarily the recurrence relations \((14)–(15)\).

**Proof of existence of \((p_1, \ldots, p_N)\).** Instead of starting from the recurrence relations \((14)–(15)\), we use a different formulation. For \( i \in \{-1, \ldots, N\} \), let \( M_{0,i} \) be the sub-trace monoid of \( M \) generated by \( \{a_0, \ldots, a_i\} \). Let also \( \mu_{0,i} \) be the evaluation on
(t_0, \ldots, t_i) of the multivariate Möbius polynomial of M_{0,i} (see Section 3.1.4). Hence:

\[
\begin{align*}
\mu_{0,-1} &= 1, \\
\mu_{0,0} &= 1 - t_0, \\
\mu_{0,1} &= 1 - t_0 - t_1, \\
\mu_{0,2} &= 1 - t_0 - t_1 - t_2 + t_0t_2, \\
\mu_{0,3} &= 1 - t_0 - t_1 - t_2 - t_3 + t_0t_2 + t_0t_3 + t_1t_3, \\
\mu_{0,4} &= 1 - t_0 - t_1 - t_2 - t_3 - t_4 + t_0t_2 + t_0t_3 + t_0t_4 + t_1t_3 + t_1t_4 + t_2t_4 - t_0t_2t_4 \\
&\ldots
\end{align*}
\]

For any \( i \in \{-1, \ldots, N - 2\} \), the monoid \( M_{0,i} \) coincides with the sub-monoid \( M^{(\gamma)} \) as defined in Section 4.2.2, where \( c_i \) is the following clique of \( M \):

\[
c_i = \begin{cases} 
 a_{i+2} \cdot a_{i+4} \cdot \ldots \cdot a_N, & \text{if } N - i \equiv 0 \pmod{2}, \\
 a_{i+2} \cdot a_{i+4} \cdot \ldots \cdot a_{N-1}, & \text{if } N - i \equiv 1 \pmod{2}.
\end{cases}
\]

The clique \( c_i \) is non empty as long as \( i < N - 1 \). Therefore, according to (10), the Möbius conditions on \( f \) ensure the positivity of all numbers \( \mu_{0,i} \) for \( i < N - 1 \).

Let \( i \in \{0, \ldots, N - 2\} \). Any clique \( \gamma \) of \( M_{0,i+1} \) either belongs to \( M_{0,i} \), or contains an occurrence of \( a_{i+1} \). In the later case, this clique \( \gamma \) is of the form \( \gamma = a_{i+1} \cdot \gamma' \), with \( \gamma' \) ranging over cliques of \( M_{0,i-1} \). It follows at once that the following recurrence relation holds:

\[
0 \leq i < N - 1 \quad \mu_{0,i+1} = \mu_{0,i} - t_{i+1}\mu_{0,i-1} \quad (19)
\]

Similarly, any clique \( \gamma \) of \( M_{0,N} \) is either contained in \( M_{0,N-2} \), or it contains an occurrence of \( a_{N-1} \), in which case it writes as \( \gamma = a_{N-1} \cdot \gamma' \) with \( \gamma' \) ranging over cliques of \( M_{0,N-3} \), or it contains occurrence of \( a_N \), in which case it writes as \( \gamma = a_N \cdot \gamma' \) with \( \gamma' \) ranging over cliques of \( M_{0,N-2} \). We deduce: \( \mu_{0,N} = \mu_{0,N-2} - t_{N-1}\mu_{0,N-3} - t_N\mu_{0,N-2} \). But \( \mu_{0,N} \) is the evaluation on \( (t_0, \ldots, t_N) \) of the multivariate Möbius polynomial of \( M = M_{0,N} \), hence \( \mu_{0,N} = 0 \) since \( f \) is a Möbius valuation. We obtain:

\[
(1 - t_N)\mu_{0,N-2} - t_{N-1}\mu_{0,N-3} = 0. \quad (20)
\]

Now, since all the numbers \( \mu_{0,i} \) for \( i \in \{-1, \ldots, N - 2\} \) are positive, we define the family \((p_i)_{-1 \leq i \leq N}\) as follows:

\[
\begin{align*}
p_1(a_0) &= t_0, & p_1(a_1) &= 1 - t_0, \quad (21) \\
1 < i < N & \quad p_i(a_{i-1}) = t_{i-1} \frac{\mu_{0,i-3}}{\mu_{0,i-2}}, & p_i(a_i) = \frac{\mu_{0,i-1}}{\mu_{0,i-2}}, \quad (22) \\
p_N(a_{N-1}) &= 1 - t_N, & p_N(a_N) &= t_N. \quad (23)
\end{align*}
\]

All numbers appearing in (21), (22) and (23) are positive, and equations (16) and (17) are satisfied. As for (18), we write, for \( 1 < i < N - 1 \):

\[
p_i(a_i)p_{i+1}(a_i) = \frac{\mu_{0,i-1}}{\mu_{0,i-2}} \frac{\mu_{0,i-2}}{\mu_{0,i-1}} = t_i
\]

For \( i = N - 1 \), we have:

\[
p_{N-1}(a_{N-1})p_N(a_{N-1}) = \frac{\mu_{0,N-2}}{\mu_{0,N-3}} (1 - t_N) = t_{N-1},
\]

25
the later equality by (20). We have shown so far that (16), (17) and (18) are satisfied.

It remains to see that all \( p_i(a_{i-1}) \) for \( i \) ranging over \( \{2, \ldots, N - 1\} \), are positive probability vectors, since this is trivially true for \( i = 1 \) and for \( i = N \).

We have already observed that they are all positive vectors. For \( 1 < i < N \), one has:

\[
p_i(a_i) + p_i(a_{i-1}) = \frac{\mu_{0,i-1} + t_{i-1}\mu_{0,i-3}}{\mu_{0,i-2}} = 1
\]

by virtue of (19). Hence each \( p_i \) is indeed a positive probability distribution on \( \Sigma_i \), which completes the proof. \( \square \)

Theorem 5.3 can be adapted to the case of sub-Bernoulli measures instead of Bernoulli measures, still for the path model, as follows. In a nutshell: every sub-Bernoulli measure can be simulated by synchronization of sub-Bernoulli sequences, but there is no uniqueness in the choice of the local sub-probability distributions.

**Theorem 5.4**—Let \( \mathbb{P} \) be a sub-Bernoulli measure on \( \mathcal{M} \), where \( \mathcal{M} \) is the synchronization monoid associated with the path model. For \( i \in \{0, \ldots, N\} \), put:

\[
t_i = \mathbb{P}(\uparrow a_i).
\]

Then there exists a tuple \( (p_1, \ldots, p_N) \), with \( p_i \) either a probability or a sub-probability distribution over \( \Sigma_i = \{a_{i-1}, a_i\} \), such that the random trace \( \xi \in \mathcal{M} \) produced by the PSA based on \( (p_1, \ldots, p_N) \) is finite with probability 1 and has the distribution \( \mathbb{P} \).

**Proof.** Using the same notations as in the proof of Theorem 5.3, we define \( (p_i)_{1 \leq i \leq N} \) as follows:

\[
\begin{align*}
p_1(a_0) &= t_0 & p_1(a_1) &= 1 - t_0 \\
1 < i < N & \quad p_i(a_{i-1}) = t_{i-1}\frac{\mu_{0,i-3}}{\mu_{0,i-2}} & p_i(a_i) &= \frac{\mu_{0,i-1}}{\mu_{0,i-2}} \\
p_N(a_{N-1}) &= \frac{t_{N-1}}{p_{N-1}(a_{N-1})} & p_N(a_N) &= t_N
\end{align*}
\]

The only difference with the definitions introduced in the proof of Theorem 5.3 lies in (26). As a consequence, all \( p_i(a_{i-1}) \) and \( p_i(a_i) \) are positive. They satisfy \( p_i(a_i)p_{i+1}(a_i) = t_i \) for all \( i \in \{2, \ldots, N - 1\} \), and obviously \( p_1(a_0) = t_0 \) and \( p_N(a_N) = t_N \). What remains to be proved is that the sums \( p_i(a_{i-1}) + p_i(a_i) \) stay within \((0, 1]\) for all \( i \in \{1, \ldots, N\} \).

For \( i \in \{1, \ldots, N - 1\} \), the sum is 1, just as in the proof of Theorem 5.3. And for \( i = N \), we have:

\[
p_N(a_{N-1}) + p_N(a_N) \leq 1 \iff t_N + t_{N-1}\frac{\mu_{0,N-3}}{\mu_{0,N-2}} \leq 1 \\
\iff \mu_{0,N-2} - t_N\mu_{0,N-2} - t_{N-1}\mu_{0,N-3} \geq 0 \\
\iff \mu_M(t_0, \ldots, t_N) \geq 0
\]

The last condition is satisfied since \( \mathbb{P} \) is a sub-Bernoulli measure. The proof is complete. \( \square \)

### 5.4 — Finitary cases

#### 5.4.1 Example of a ring model —

Consider the ring model already introduced in Section 2.2. The trace monoid is:

\[
\mathcal{M} = \langle a_0, a_1, a_2, a_3 \mid a_0a_2 = a_2a_0, \ a_1a_3 = a_3a_1 \rangle.
\]
This is the synchronization monoid associated with the network of alphabets \((\Sigma_0, \Sigma_1, \Sigma_2, \Sigma_3)\) given by \(\Sigma_0 = \{a_0, a_0\}\), \(\Sigma_1 = \{a_1, a_1\}\), \(\Sigma_2 = \{a_2, a_2\}\) and \(\Sigma_3 = \{a_3, a_3\}\). Contrasting with the path model, we shall see on an example that the PSA for this ring model with four generators produces finite traces with probability 1.

Let \(p_0, \ldots, p_3\) be uniform distributions on \(\Sigma_0, \ldots, \Sigma_3\). According to Theorem 5.1 the PSA yields a multiplicative and uniform probability measure \(\Pr\) on \(\mathcal{M}\) with parameter \(p = (1/2)(1/2) = 1/4:\)

\[
\forall x \in \mathcal{M}, \quad \Pr(x) = \left(\frac{1}{4}\right)^{|x|}.
\] (27)

The Möbius polynomial of \(\mathcal{M}\) is \(\mu_\mathcal{M}(t) = 1 - 4t + 2t^2\), which root of smallest modulus is \(p_0 = 1 - 1/\sqrt{2}\). Since \(1/4 < p_0\), it follows from Theorem 4.4 that the PSA produces a finite trace \(y \in \mathcal{M}\) with probability 1.

The average length of \(y\) is easily computed. Following the notations introduced in Section 4.2 we put \(G(z) = \sum_{x \in \mathcal{M}} |x|z^{|x|}\), \(p = 1/4\) and \(\varepsilon = \mu_\mathcal{M}(p)\). Using that \(\Pr(y = x) = \varepsilon p^{|x|}\) and \(G(z) = 1/\mu_\mathcal{M}(z)\), we have:

\[
\mathbb{E}|y| = \sum_{x \in \mathcal{M}} |x|\Pr(x) = \varepsilon p \frac{dG}{dz} |_{z = p} = -p \frac{\mu_\mathcal{M}(p)}{\mu_\mathcal{M}(p)} = 6.
\]

### 5.4.2 Generalization: trees and cycles

— Since the probability distribution of the random element produced by the PSA is a multiplicative probability measure, we know by Theorem 4.3 that the trace is either finite with probability 1 or infinite with probability 1. The examples studied above show that both cases may occur indeed.

It turns out that the dichotomy is solved by a simple criterion, as stated below.

**Proposition 5.5**—Assume that the synchronization monoid \(\mathcal{M} = \mathcal{M}(\Sigma, I)\) associated with a tuple of alphabets \((\Sigma_1, \ldots, \Sigma_N)\) is irreducible, and let \(D\) be the dependence relation of \(\mathcal{M}\). Let \(p_1, \ldots, p_N\) be positive probability distributions on \(\Sigma_1, \ldots, \Sigma_N\) respectively. Then

1. The PSA based on \((p_1, \ldots, p_N)\) produces an infinite trace with probability 1 if and only if:
   
   a) \(i \neq j \implies |\Sigma_i \cap \Sigma_j| \leq 1\); and
   
   b) the non oriented graph \((\Sigma, D)\) has no cycle.

2. If the output \(X\) of the PSA is finite with probability 1, then the length \(|X|\) has a finite average.

**Proof.** Assume first that both conditions \([1a]\) and \([1b]\) are met. Let \(Y = (Y_i)_{i \in \Sigma}\) be a tuple of infinite sequences \(Y_i \in (\Sigma_i)^\omega\). With probability 1, for any distinct \(i\) and \(j\) such that \(\Sigma_i \cap \Sigma_j \neq \emptyset\), the sequence \(Y_i\) has infinitely many occurrences of the unique element belonging to \(\Sigma_i \cap \Sigma_j\). Just as in the case of the path model, this is enough to guarantee that the synchronization trace of \(Y\) is infinite.

Conversely, assume that one of conditions \([1a]\) and \([1b]\) is not met, for instance condition \([1a]\). Let \(i\) and \(j\) be distinct indices such that \(\Sigma_i \cap \Sigma_j\) has at least two distinct elements \(a\) and \(b\). Let \(Y = (Y_i)_{i \in \Sigma}\) be a random vector of sequences such that the synchronization trace of \(Y\) is infinite. Since \(\mathcal{M}\) is assumed to be irreducible, we observe that, with probability 1, if the synchronization trace of \(Y\) is infinite then all coordinates of \(Y\) are infinite. Hence there is no loss of generality in assuming that both coordinate \(Y_i\) and \(Y_j\) are infinite. Then the order of occurrences of \(a\) and \(b\) in both coordinates must be the same, which has probability 0 to occur.
Finally, assume that condition 1b is not met, hence the presence of a cycle \((r_1, r_2, \ldots, r_k)\) in the dependence relation. We assume without loss of generality that \(r_1, \ldots, r_{k-1}\) are pairwise disjoint. Let \(a_1 \in \Sigma_{r_1} \cap \Sigma_{r_2}, a_2 \in \Sigma_{r_2} \cap \Sigma_{r_3}, \ldots, a_k \in \sigma_{r_k} \cap \Sigma_{r_1}\). Focusing on the coordinates \(r_1, \ldots, r_k\) of \(Y\) only, a pattern of the form \((a_1 a_k, a_2 a_1, a_3 a_2, \ldots, a_{k-1} a_{k-2}, a_k a_{k-1})\) shall occur with probability 1. Since such a pattern is blocking the synchronization, it follows that the synchronization trace is finite with probability 1.

We have proved so far the equivalence stated in point 1. We now come to the proof of point 2 and assume that \(|X| < \infty\) with probability 1. The average of \(|X|\) is computed as the following mathematical expectation:

\[
\mathbb{E}[|X|] = \sum_{x \in M} |x| P(X = x) = \varepsilon \sum_{x \in M \setminus \{e\}} |x| f(x),
\]

where \(f\) is the sub-Möbius valuation associated with \(P\), and \(\varepsilon\) is the constant given by the Möbius transform of \(f\) evaluated at the empty heap (see Section 4.2.4). Let \(G(\lambda)\) be the power series:

\[
G(\lambda) = \sum_{x \in M} \lambda^{|x|} f(x)
\]

Then \(G(1) = \varepsilon^{-1} < \infty\) according to Theorem 4.1 point 2 and since \(G\) has non-negative coefficients, it implies by the Pringsheim Theorem that the radius of convergence of \(G\) is greater than 1. Hence so does its derivative, and thus \(G'(1) < \infty\). Since \(\mathbb{E}[|X|] = \varepsilon G'(1)\), the result of point 2 follows.

### 6—The Probabilistic Full Synchronization Algorithm

The PSA produces random traces, either finite or infinite. Proposition 5.5 shows that the ability of the PSA to produce finite or infinite traces does not depend on the probabilistic parameters one chooses to equip the local alphabets with. It rather depends on the structure of the synchronization monoid.

To be sound, testing procedures and statistical averaging techniques require arbitrary large traces, which binds us to the mathematical model of infinite traces. In case where the PSA fails to produce infinite traces, we are thus left with an unsolved problem. Yet, we can produce finite traces... and the most natural thing to try from there, is to start the PSA over and over, and to concatenate the finite random traces obtained at each execution of the PSA. The limiting trace is infinite and random, couldn’t it just be the one we were looking for?

It is quite surprising to realize that this strategy fails in general. Of course the unlimited concatenation of finite traces, with a positive average length, necessarily produces an infinite trace. But the failure comes from the distribution of this random infinite trace. We will show on an example below that it is not a Bernoulli measure in general. In particular, the uniform measure is thus unreachable by this technique.

Nevertheless, we introduce an algorithm based on recursive concatenation of finite random traces and that outputs, if executed indefinitely, an infinite trace which is always distributed according to a Bernoulli measure. The intermediate, finite random traces, are obtained by a trial-and-reject procedure based on the PSA. The whole procedure constitutes the Probabilistic Full Synchronization Algorithm (PFSA). In Section 6.4 we show that any Bernoulli measure can be simulated by the output of the PFSA for the ring model.

### 6.1 — Convolution of probability distributions and random walks

28
6.1.1 Convolution and random walks — We recall the general definition of convolution for a countable monoid \( \mathcal{M} \). Let \( \nu \) and \( \theta \) be two probability distributions on \( \mathcal{M} \). Assume that \( X \) and \( Y \) are two independent random variables with values in \( \mathcal{M} \), distributed according to \( \nu \) and to \( \theta \) respectively. Then the convolution \( \nu \star \theta \) is the distribution of the random variable \( X \cdot Y \), and it is given by the Cauchy product formula:
\[
\forall x \in \mathcal{M} \quad \nu \star \theta(x) = \sum_{(y,z) \in \mathcal{M} : y \cdot z = x} \nu(y) \cdot \theta(z).
\]
The convolution product is associative.

Given a probability distribution \( \nu \) over \( \mathcal{M} \), let \( (X_n)_{n \geq 1} \) be a sequence of independent random variables with values in \( \mathcal{M} \), and identically distributed according to \( \nu \). The random walk associated with \( \nu \) is the sequence of random variables \( (Y_n)_{n \geq 0} \) defined by \( Y_0 = e \), the unit element of the monoid, and inductively: \( Y_{n+1} = Y_n \cdot X_{n+1} \) for all integers \( n \). If \( \nu_n \) denotes the distribution of \( Y_n \), we have \( \nu_n = \nu \ast (\nu^{*n}) \) for all integers \( n \), the \( n \)th convolution power of \( \nu \) with itself.

Each trajectory \( (Y_n)_{n \geq 0} \) of the random walk is nondecreasing for the divisibility relation in the monoid. Hence, if we assume now that \( \mathcal{M} \) is a trace monoid, the nondecreasing sequence \( (Y_n)_{n \geq 1} \) has a least upper bound in the completion \( \overline{\mathcal{M}} \), say \( Y_\infty = \bigvee_{n \geq 1} Y_n \). We introduce the notation \( \nu^{*\infty} \) for the probability distribution of \( Y_\infty \), which we call the limit distribution of the random walk (it is also called the harmonic measure of the random walk).

6.1.2 An example where the concatenation of PSA traces does not yield a limit multiplicative measure — We consider the ring synchronization monoid on four generators \( \mathcal{M} = \langle a_0, a_1, a_2, a_3 \mid a_0a_2 = a_2a_0, a_1a_3 = a_3a_1 \rangle \), corresponding to the network of alphabets \( \{\Sigma_0, \Sigma_1, \Sigma_2, \Sigma_3\} \) with \( \Sigma_0 = \{a_0, a_2\} \), \( \Sigma_1 = \{a_0, a_1\} \), \( \Sigma_2 = \{a_1, a_2\} \) and \( \Sigma_3 = \{a_2, a_3\} \). Each alphabet \( \Sigma_i \), for \( i = 0, \ldots, 3 \), is equipped with the uniform probability distribution. Let \( \nu^{*\infty} \) be the distribution on \( \partial \mathcal{M} \) of the infinite trace obtained by concatenating infinitely many independent copies of a finite trace generated by the PSA. Then we claim: the limit distribution \( \nu^{*\infty} \) is not Bernoulli.

Seeking a contradiction, assume that it is. Clearly, \( \nu^{*\infty} \) is concentrated on the boundary \( \partial \mathcal{M} \). And for symmetry reasons, it is necessarily the uniform distribution, and thus given by \( \nu^{*\infty}(\uparrow x) = p_0^{|z|} \) where \( p_0 \) is the root of smallest modulus of the Möbius polynomial \( \mu(z) = 1 - 4z + 2z^2 \).

We extend the concatenation of traces \( x \cdot y \) with \( (x, y) \in \mathcal{M} \times \mathcal{M} \) to the case where \( y \) is an infinite trace by putting \( x \cdot y = \bigvee \{x \cdot y_n : n \geq 1\} \) for \( y = \bigvee \{y_n : n \geq 1\} \), and this definition does not depend on the choice of the sequence \( (y_n)_{n \geq 1} \). This yields also an extension of the notion of convolution \( \nu \star \theta \) to the case where \( \nu \) is concentrated on \( \mathcal{M} \), but \( \theta \) might be a probability distribution on \( \overline{\mathcal{M}} \). The construction of \( \nu^{*\infty} \) implies the fix point property \( \nu_p \ast \nu^{*\infty} = \nu^{*\infty} \), where \( \nu_p(\uparrow x) = p^{|x|} \) is the distribution of the PSA, here given by \( p = 1/4 \). In particular for the cylinder \( \uparrow a_0 \), this yields:
\[
p_0 = \sum_{k \geq 0} \nu_p(\{a_0^k\}) \nu^{*\infty}(\uparrow a_0) + \nu_p(\uparrow a_0) = p_0(1 - 4p + 2p^2) \frac{1}{1 - p} + p
\]
Simplifying by \( p \neq 0 \), we obtain: \( p = (3p_0 - 1)/(2p_0 - 1) \), and since \( 1 - 4p_0 + 2p_0^2 = 0 \), it yields \( p = p_0 \), a contradiction. Actually, we have shown the strongest result that no random walk based on the distributions \( \nu_p(\uparrow x) = p^{|x|} \) with \( p \in (0, p_0) \) has a Bernoulli measure as limit distribution \( \nu^{*\infty} \).

6.2 — First hitting times and pyramidal heaps
6.2.1 First hitting times — First hitting times for random heaps formalize the idea of the first time of occurrence of a given piece—yet, without an explicit notion of time at hand. It generalizes to random heaps the analogous notion, for a Bernoulli sequence, of first time of reaching a given letter.

Let $M = M(\Sigma, I)$ be a trace monoid, and let $a \in \Sigma$ be a given letter. The number of occurrences of $a$ in the congruent words defining a trace $x$ is constant, and depends thus only on the trace $x$. We denote it $|x|_a$. For any infinite trace $\xi \in \partial M$, let $L_a(\xi) = \{x \in M : x \leq \xi \land |x|_a > 0\}$. If non-empty, the set $L_a(\xi)$ has a minimum which we denote by $V_a(\xi)$, and it satisfies $|V_a(\xi)|_a = 1$. Intuitively, $V_a(\xi)$ represents the smallest sub-trace of $\xi$ with at least an occurrence of $a$.

If $\partial M$ is equipped with a Bernoulli measure $\mathbb{P}$, then $L_a(\xi) \neq \emptyset$ with probability 1. Hence, neglecting a set of zero probability, we may assume that $V_a : \partial M \to M$ is well defined. The mapping $V_a$ is called the first hitting time of $a$. The distribution of the first hitting time of $a$ is the probability distribution of the random variable $V_a$.

It is a discrete probability distribution on $M$, which we denote by $\mathbb{P}_a$, and which is defined by $\mathbb{P}_a(x) = \mathbb{P}(V_a = x)$ for all $x \in M$.

We will base our random generation of infinite heaps on the following result.

• Theorem 6.1— Let $\mathbb{P}$ be a Bernoulli measure equipping the boundary $\partial M$ of an irreducible trace monoid $M = M(\Sigma, I)$. Let $a \in \Sigma$, and let $\mathbb{P}_a$ be the distribution of the first hitting time of $a$. Then $\mathbb{P}_a^\infty = \mathbb{P}$, where $\mathbb{P}_a^\infty$ is the limit distribution of the random walk on $M$ associated with $\mathbb{P}_a$.

Sketch of proof. Let $(V^n)_{n \geq 0}$ be the sequence of iterated stopping times associated with the first hitting time $V_a$, as defined in [1] Def. 5.2. Under the probability $\mathbb{P}$, it follows from [1] Prop. 5.3 that, for each integer $n$, $V^n$ has the same distribution as the $n^{th}$ step of the random walk associated with $\mathbb{P}_a$. Since $M$ is assumed to be irreducible, the sequence $(V^n)_{n \geq 0}$ is exhaustive as defined in [1] Def. 5.5, from which the result derives.\hfill \Box

As a consequence, if we can simulate the distribution $\mathbb{P}_a$ of the first hitting time of some piece $a$, we will be able to simulate a $\mathbb{P}$-distributed infinite heap. The improvement lies in the fact that first hitting times are finite heaps. Our next task consists thus in studying more closely the distribution of the first hitting time, after which we shall see how to simulate it.

6.2.2 Pyramidal heaps and the distribution of the first hitting time — Recall that any trace has an interpretation as a labeled partial order of pieces (see Section 6.2.2). A trace $x$ is pyramidal if, as labeled partially ordered set, it has a unique maximal element (a notion introduced by Viennot [22]). Any trace of the form $x = V_a(\xi)$ for some $\xi \in \partial M$ is pyramidal, with its unique occurrence of $a$ as its unique maximal piece.

Then the set $V_a$ of traces $x \in M$ in the image of the mapping $V_a : \partial M \to M$ can be described as follows: $V_a$ is the set of pyramidal traces $x \in M$ such that the piece $a$ only occurs as the unique maximal piece; see Figure 7. Furthermore, we observe that, if $x \in V_a$, then $\{V_a = x\} = \uparrow x$ (an intuitive property, also proved in [1] Prop. 4.2). It follows that the distribution $\mathbb{P}_a$ of the first hitting time has the following simple expression:

$$\forall x \in M \quad \mathbb{P}_a(x) = \begin{cases} 0, & \text{if } x \notin V_a \\ \mathbb{P}(\uparrow x), & \text{if } x \in V_a \end{cases} \tag{28}$$

6.2.3 Generating pyramidal heaps — We consider a network of alphabets $(\Sigma_1, \ldots, \Sigma_N)$ with $\Sigma = \Sigma_1 \cup \ldots \cup \Sigma_N$, such that the synchronization trace monoid $M = M(\Sigma, I)$ is irreducible. We pick an arbitrary letter $a \in \Sigma$. For each $i \in \{1, \ldots, N\}$, let

30
\[ x = c \cdot b \cdot d \cdot c \cdot b \cdot d \cdot a \]

\[ y = c \cdot b \cdot d \cdot c \cdot b \cdot a \]

Figure 7: (i): For \( \mathcal{M} = (a, b, c, d : ac = ca, ad = da, bd = db) \): a trace \( x \notin \mathcal{V}_a \) (it is not pyramidal) and its associated labeled poset. (ii): a trace \( y \in \mathcal{V}_a \) and its associated labeled poset on the right.

\[ \Sigma'_i = \Sigma_i \setminus \{a\} \] if \( a \in \Sigma_i \), and \( \Sigma'_i = \Sigma_i \) otherwise. Then the synchronization monoid \( \mathcal{M}' \) of \( (\Sigma'_1, \ldots, \Sigma'_N) \) coincides with the sub-monoid of \( \mathcal{M} \) generated by \( \Sigma \setminus \{a\} \).

We assume, for each \( i \in \{1, \ldots, N\} \), to be given \( p'_i \), either a probability distribution or a sub-probability distribution over \( \Sigma'_i \), such that the output \( X \) of the PSA executed on \( (\Sigma'_1, \ldots, \Sigma'_N) \) with those parameters, is finite with probability 1.

Given the parameters \( (p'_1, \ldots, p'_N) \), we consider the execution of the trial-and-reject procedure described in pseudo-code in Algorithm 5 below. At each run of the loop, the algorithm needs to decide whether some trace \( V \) is pyramidal in \( \mathcal{M} \) or not. It is clear that a—far from being optimal—scanning procedure, examining all elements of \( V \) starting from the right, will successfully complete this job.

**Algorithm 5** Outputting a pyramidal trace in \( \mathcal{V}_a \)

**Require:**
1: \( X \leftarrow e \quad \triangleright \text{Initialization} \\
2: \textbf{repeat} \\
3: \quad \textbf{call} Algorithm 4 \quad \triangleright \text{Calling the PSA} \\
4: \quad X \leftarrow \text{output of Algorithm 4} \\
5: \quad V \leftarrow X \cdot a \\
6: \textbf{until} \ V \text{ is pyramidal in } \mathcal{M} \\
7: \textbf{return} V \\

By assumption, the PSA which is repeatedly called in Algorithm 5 always terminates and outputs a finite trace. Since the probability of success if positive, since for instance \( X = e \) yields a success and has positive probability to be output by the PSA, Algorithm 5 always terminates. The probability distribution of the output is given in the following lemma, stated with the notations introduced above.

- **Lemma 6.2**—Let \( f' : \mathcal{M}' \rightarrow \mathbb{R}^*_+ \) be the sub-Möbius valuation associated with the output of the PSA running on \( \mathcal{M}' \). Then the distribution of the output \( V \) of Algorithm 5 is concentrated on \( \mathcal{V}_a \) and given by:

\[ \forall v \in \mathcal{V}_a \quad P(V = v) = K \cdot f'(v/a), \quad (29) \]
where \( v/a \) denotes the heap obtained by removing from \( v \) its unique maximal piece \( a \), and \( K \) is a normalization constant.

Proof. Let \( Q \) denote the probability distribution of the output \( X \) of the PSA running on \( M' \) with the specified parameters \((p_1', \ldots, p_N')\). Then \( f'(x) = Q(\uparrow x) \) for \( x \in M' \). According to Theorem 4.3, for some constant \( \varepsilon > 0 \), we have \( Q(X = x) = \varepsilon f'(x) \) for all \( x \in M' \).

The rejection procedure amounts to considering the distribution of \( V = X \cdot a \) conditioned on \( X \cdot a \in V_a \). Hence the probability for Algorithm 5 to issuing an element \( v \in V_a \) is:

\[
P(V = v) = \frac{Q(X \cdot a = v)}{Q(X \cdot a \in V_a)} = \frac{Q(X = v/a)}{Q(X \cdot a \in V_a)} = K \cdot f'(v/a),
\]

where \( K \) is the constant \( K = \varepsilon / Q(X \cdot a \in V_a) \).

Note that the form (29) is almost that of a valuation evaluated at \( v \). The contribution of the last piece \( a \) is missing, but the constant \( K \) is adequately placed to play the role of this missing contribution. This will be used in the proof of Theorem 6.3.

6.3 — The Probabilistic Full Synchronization Algorithm

We are now ready for constructing a probabilistic algorithm generating Bernoulli-distributed infinite traces. The framework consists of a network \((\Sigma_1, \ldots, \Sigma_N)\) of alphabets, such that the synchronization trace monoid \( M = M(\Sigma, I) \) is irreducible, with \( \Sigma = \Sigma_1 \cup \cdots \cup \Sigma_N \).

6.3.1 Description of the algorithm — Having chosen an arbitrary piece \( a \in \Sigma \), we consider a family \((p_1', \ldots, p_N')\) of probabilistic parameters as above, i.e., with the constraint that the PSA executed on the sub-monoid \( M' \) generated by \( \Sigma \setminus \{a\} \) and with these parameters, outputs a finite trace with probability 1.

The Probabilistic Full Synchronization Algorithm (PFSA) is described in pseudocode in Algorithm 6 below. The PFSA is an endless loop, incrementally writing to its output register \( X \). It simulates thus the random walk on \( M \) with increments distributed according to the distribution established in Lemma 6.2.

---

**Algorithm 6** Probabilistic Full Synchronization Algorithm

**Require:**
1. \( X \leftarrow e \) \hfill \triangleright \text{ Initialization}
2. repeat
3. \( V \leftarrow \text{output of Algorithm 5} \) \hfill \triangleright \text{Random pyramidal trace } V \in V_a
4. \( X \leftarrow X \cdot V \) \hfill \triangleright \text{Increments the random walk}
5. until false

The analysis of Algorithm 6 is twofold: a probabilistic analysis carried on below and a complexity analysis carried on in Section 6.3.3.

6.3.2 Probabilistic analysis of the PFSA — Recall that, by convention, the output of the PFSA is the random infinite heap, least upper bound in \( \partial M \) of the finite heaps recursively written in its output register. The probability distribution of this infinite heap is as follows.

- **Theorem 6.3** — We consider the execution of the PFSA in the framework described in Section 6.3.1 and we adopt the same notations. Let \( f' : M' \rightarrow \mathbb{Z}_+ \) be the sub-Möbius valuation associated with the PSA executed with the chosen parameters \((p_1', \ldots, p_N')\), and let \( X_\infty \) be the output (with the convention recalled above) of the PFSA.
Then $X_\infty$ is distributed according to a Bernoulli measure on $\partial \mathcal{M}$. The associated valuation $f : \mathcal{M} \to \mathbb{R}$ is the Möbius valuation on $\mathcal{M}$ that extends $f' : \mathcal{M}' \to \mathbb{R}$ (the existence and uniqueness of which are stated in Theorem 4.5).

**Proof.** Let $f : \mathcal{M} \to \mathbb{R}^*_+$ be as in the statement. Let $\mathbb{P}$ be the Bernoulli measure on $\partial \mathcal{M}$ defined by $\mathbb{P}(\uparrow x) = f(x)$, which is well defined according to Theorem 1.3.

Letting $\mathbb{Q}$ be the distribution of the increment $V$, we have to prove that $\mathbb{P} = \mathbb{Q}$. Let $g : \Sigma \to \mathbb{R}^*_+$ be the probability distribution of the increment $V$ in the PFSA. Since $\mathbb{Q}$ is the limit distribution of the random walk on $\mathcal{M}$ with increments distributed identically to $V$, it follows from Theorem 6.1 that we only need to show that $g(x) = \mathbb{P}(V_a = x)$ for all $x \in \Sigma_a$.

According to (28), we have:

\[ \forall x \in \Sigma_a \quad \mathbb{P}(V_a = x) = \mathbb{P}(\uparrow x) = f(x). \quad (30) \]

Whereas, according to Lemma 6.2 we have for some constant $K$:

\[ \forall x \in \Sigma_a \quad g(x) = K f'(x/a) = \frac{K}{f(a)} f(x). \quad (31) \]

Summing up over $\Sigma_a$ in (30) yields $\sum_{x \in \Sigma_a} f(x) = 1$, whereas summing up over $\Sigma_a$ in (31) yields $1 = (K/f(a)) \cdot (\sum_{x \in \Sigma_a} f(x)) = K/f(a)$. Therefore $K = f(a)$, which yields after re-injecting in (31): $g(x) = f(x)$ and thus $g(x) = \mathbb{P}(V_a = x)$ for all $x \in \Sigma_a$. Since $\mathbb{P}$ has the desired properties, the proof is complete. \(\square\)

### 6.3.3 Complexity analysis of the PFSA

We will limit our analysis to the following observation: **the size of the output register of the PFSA grows linearly with time in average.**

Indeed, each PSA call in Algorithm 5 takes a finite amount of time in average according to Proposition 5.5 point 2. Since the probability of success in the trial-and-reject procedure of Algorithm 5 is positive, it will thus execute in average in a fixed amount of time, whence the average linear growth of the output register of the PFSA.

Another question is to compute adequately the probabilistic parameters. We will discuss it briefly in Section 7 after having examined some examples.

### 6.4 Example: ring models

#### 6.4.1 A general result

For the ring models, the following result shows that any Bernoulli measure can be simulated by executions of the PFSA.

**Theorem 6.4**—Let $(a_0, \ldots, a_{N-1})$ be $N$ distinct symbols, and let $(\Sigma_0, \ldots, \Sigma_{N-1})$ be the network of alphabets defined by:

\[ 0 < i \leq N - 1 \quad \Sigma_i = \{a_{i-1}, a_i\}, \quad \Sigma_0 = \{a_{N-1}, a_0\}. \]

The synchronization monoid $\mathcal{M} = \mathcal{M}(\Sigma, I)$ is described as follows:

\[ \Sigma = \{a_0, \ldots, a_{N-1}\}, \quad I = \{(a_i, a_j) : (i-j \mod N \geq 2) \land (j-i \mod N \geq 2)\}. \]

Then any Bernoulli measure on $\partial \mathcal{M}$ can be simulated by the endless execution of the PFSA, and in particular the uniform measure on $\partial \mathcal{M}$.

**Proof.** Let $\mathbb{P}$ be a target Bernoulli measure on $\partial \mathcal{M}$, and let $f : \mathcal{M} \to \mathbb{R}^*_+$ be the associated Möbius valuation. We pick $a_0$ as the piece to be removed. Let $\mathcal{M}'$ be the submonoid of $\mathcal{M}$ generated by $a_1, \ldots, a_N$. Then $\mathcal{M}'$ is a path model. Furthermore,
since $\mathcal{M}$ is irreducible, it follows from Theorem 4.5 point 1 that the restriction of $f$ to $\mathcal{M}'$ is sub-Möbius. According to Theorem 5.4, the associated probability distribution can be obtained by running the PSA with suitable parameters. Running the PFSA based on this instance of the PSA, we generate a Bernoulli measure which Möbius valuation, say $g$, extends to $\mathcal{M}$ the restriction of $f$ to $\mathcal{M}'$. By the uniqueness of the extension of sub-Möbius valuations to Möbius valuations (Theorem 4.5 point 2), it follows that $g = f$. 

6.4.2 Example — We consider the example of the ring model $\mathcal{M}$ on five generators, the synchronization graph of which is depicted in Figure 8 (i). We aim at generating the uniform measure on $\partial \mathcal{M}$, say $\nu$, characterized by $\nu(\uparrow x) = p_{|x|}$, where $p_0 = 1 - \frac{\sqrt{5}}{10}$ is the root of smallest modulus of $\mu_{\mathcal{M}}(t) = 1 - 5t + 5t^2$.

We pick $a_0$ as our distinguished piece, as depicted in Figure 8 (ii). Then, we wish to run on the sub-monoid

$$\mathcal{M}' = \langle a_1, a_2, a_3, a_4 \mid a_1a_3 = a_3a_1, a_1a_4 = a_4a_1, a_2a_4 = a_4a_2 \rangle$$

a PSA with associated valuation $f' : \mathcal{M}' \to \mathbb{R}$ such that $f'(x) = p_{|x|}$ for all $x \in \mathcal{M}'$. Note that $p_0$ is not the root of $\mu_{\mathcal{M}'}$!

Referring to the computations performed in the proof of Theorem 5.4, adequate solutions for $p_2$, $p_3$ and $p_4$, respectively on $\langle a_1, a_2 \rangle$, on $\langle a_2, a_3 \rangle$ and on $\langle a_3, a_4 \rangle$, are obtained as follows:

$$p_2(a_1) = p_0 = \frac{1}{2} - \frac{\sqrt{5}}{10} \approx 0.276 \quad p_2(a_2) = 1 - p_0 = \frac{1}{2} + \frac{\sqrt{5}}{10} \approx 0.724$$

$$p_3(a_2) = \frac{p_0}{1 - p_0} = \frac{3}{2} - \frac{\sqrt{5}}{2} \approx 0.382 \quad p_3(a_3) = \frac{1 - 2p_0}{1 - p_0} = -\frac{1}{2} + \frac{\sqrt{5}}{2} \approx 0.618$$

$$p_4(a_3) = \frac{p_0(1 - p_0)}{1 - 2p_0} = \frac{1}{\sqrt{5}} \approx 0.447 \quad p_4(a_4) = p_0 = \frac{1}{2} - \frac{\sqrt{5}}{10} \approx 0.276$$

7—Computational issues and perspectives

When trying to use the PFSA in practice for simulation and testing, one might be concerned by the fact that it incrementally outputs heaps with a particular shape, namely they are all pyramidal. Furthermore, the tip of these pyramidal heaps is always labeled with the same letter, corresponding to an arbitrary choice made before executing the algorithm. Actually, there are good reasons not to worry about that. Indeed, a large class of statistics on heaps can safely be computed on these particular
heaps, and they will asymptotically be indistinguishable from statistics computed on arbitrary large heaps. For instance, an asymptotics of the speedup, i.e. the ratio height over number of pieces in large heaps, can be estimated in this way. Precise results on this topic are found in [1], under the name of cut-invariance.

Another concern is the following. Even if the PFSA were proved to be able to simulate any Bernoulli measure for any topology, not only for the ring topology, there is no doubt that its execution would still need the precomputation of adequate probabilistic parameters, and in particular the root of smallest modulus of the Möbius polynomial of the synchronization monoid. Given that the determination of this polynomial on a general synchronization graph is an NP-complete problem (since the independence set decision problem is NP-complete [8]), this precomputation method appears unrealistic to be used in practice.

However, what we need is not the Möbius polynomial itself, but only its root of smallest modulus. It is not theoretically forbidden to think that this root might be approximated in polynomial time, even though the Möbius polynomial is hard to find. Actually, one can even think of a feedback procedure based on our generation algorithms to find an approximation of this root. Indeed, we could execute the generation algorithm with arbitrary parameters, then adjust the probabilistic parameters in order to increase uniformity, then re-run the generation algorithm and re-adjust the parameters, and so on. It is reasonable to expect that such a procedure would lead the parameters to converge toward the critical value entailing uniformity, which is the root of smallest modulus of the Möbius polynomial. This interesting question may deserve a dedicated work.

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