Dessins d’Enfants, Their Deformations and Algebraic the Sixth Painlevé and Gauss Hypergeometric Functions

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Abstract

We consider an application of Grothendieck’s dessins d’enfants to the theory of the sixth Painlevé and Gauss hypergeometric functions: two classical special functions of the isomonodromy type. It is shown that, higher order transformations and the Schwarz table for the Gauss hypergeometric function are closely related with some particular Belyi functions. Moreover, we introduce a notion of deformation of the dessins d’enfants and show that one dimensional deformations are a useful tool for construction of algebraic the sixth Painlevé functions.

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1 Introduction

In this paper we report a further development of the method of RS-transformations recently introduced [11] in the theory of Special Functions of the Isomonodromy Type (SFITs) [10]. It was already discussed [11] that RS-transformations are a useful tool for solution of many problems in the theory of SFITs, in particular, for construction of higher order transformations and special values of these generally transcendental functions, say, algebraic values at algebraic points.

Technically the most complicated problem in the method of RS-transformations is construction of their R-parts, i.e., rational functions with some special properties. In many cases it is not a priori clear whether such rational functions actually exist.

The progress achieved in this work is based on the observation that the Belyi functions [3], which are important in many questions of algebraic geometry, play also an important role in construction of the RS-transformations. It is well known that in connection with the theory of the Belyi functions Grothendieck [7] suggested the theory of “Dessins d’Enfants”. We show that for the theory of SFITs we need not only the theory of dessins d’enfants but also a theory of their deformations. This theory would help to establish existence of the R-parts of RS-transformations. As long as existence of the desired rational function is proved it immediately imply existence of the S-part of the corresponding RS-transformation. Since construction of the S part is performed pure algorithmically in terms of the coefficients of the R-part and the original linear ODE, which is subjected by the RS-transformation. Of course, the existence does not tell how to construct the R-part explicitly; it is a separate problem. However, many theoretical conclusions, e.g., differential equations that obey the corresponding SFITs, can be found without their explicit constructions. One can write, say, an ansatz for the RS-transformation with unknown coefficients. The latters can be calculated numerically, or studied in some other way. Exact calculation of the monodromy parameters of the SFITs constructed via the method of RS-transformations also does not require explicit expressions, it is enough to know the above mentioned ansatz with the numerical values of the coefficients. In fact, the existence is helpful even in finding the explicit formulae too: since the knowledge, that the function $R$ actually exists, stimulates to continuing efforts in obtaining these formulae even though a few first attempts failed. It is the last comment that was important for the author in doing this work. It is important to stress in the very beginning, that possibly the most interesting part of this work concerning a relation between deformations of the dessins d’enfants and existence of the $R$-parts is only conjectured rather than proved.

In this paper, instead of dealing with the general theory of SFITs, we continue to consider application of the method of RS-transformations to the theory of two classical one variable SFITs, namely, the sixth Painlevé and Gauss hypergeometric functions. More precisely, we are interested in explicit constructions of algebraic solutions of the sixth Painlevé equations [2, 11], a topic that have been attracted recently a considerable attention, and also discuss related questions for its linear analog, the Gauss hypergeometric function [11].

The main new observation concerning algebraic solutions of the sixth Painlevé equation which is made in this paper is that a wide class of these solutions (possibly all?) can be constructed via a pure algebraic procedure without any “touch” of differential equations at all! The procedure reads as follows:
1. Take a proper dessin d’enfant (a bicolour graph);
2. Consider its one dimensional deformations (tricolour graphs);
3. For each tricolour graph there exists a rational function the $R$-part of some $RS$-transformation;
4. One of the critical points of $R$ is a solution of the sixth Painlevé equation.

As is mentioned above we do not have a general proof of the existence in item 3. In all our examples this existence is obtained via an explicit construction, which is, of course, technically the most complicated part of this scheme. At the same time there is a substantial simplification comparing to the work [2], as to get explicit formulae for the algebraic solutions we do not have to find explicitly $S$-parts of the corresponding $RS$-transformations. Of course, without the $S$-parts we cannot obtain explicit solutions for the associated monodromy problems. But the latters are additional problems which are not directly related with the original one of finding algebraic solutions for the sixth Painlevé equation.

In [2] we began a classification of $RS$-transformations generating the algebraic sixth Painlevé functions. As is explained in the beginning of this Introduction the principle problem here is a classification of $R$-parts of these transformations. It is important to notice that these $R$-parts can be used in many other $RS$-transformations, say, for SFITs related with isomonodromy deformations of matrix ODEs with the matrix dimension higher than 2. Therefore, the problem of classification of the $R$-parts goes beyond a particular problem related with the algebraic sixth Painlevé functions. The scheme of classification of the $R$-parts that rise out from the deformation point of view is that it is enough to classify one dimensional deformations of the dessins d’enfants.

Our main new result for the Gauss hypergeometric function is an explicit formulae for the special Belyi function that allows one to construct three octic transformations which act on finite sets, we call them clusters, of transcendental the Gauss hypergeometric functions. Before, the only known higher order transformations, that act beyond the Schwarz cluster of algebraic the Gauss hypergeometric functions, were quadratic and cubic transformations, and their compositions.

Now we briefly overview the content of the paper.

In Section 2 we introduce a notion of deformations of the dessins. More precisely, we begin with a presentation of the dessins as bicolour graphs on the Riemann sphere. Then one dimensional deformations of the dessins are defined as special tricolour graphs which can be obtained by some simple rules from the bicolour ones. After that the main statements concerning a relation between the tricolour graphs and a class of rational functions are formulated as conjectures. Then we establish a proposition allowing one to calculate algebraic solutions of the sixth Painlevé equation directly from the latter rational functions. In the remaining part of the section we consider some special examples and discuss questions important for the theory of the sixth Painlevé equation.

In Section 3 deformations of the dessins for the Platonic solids are studied. Here and in Section 2 we obtained many different algebraic solutions of the sixth Painlevé equations, some of them are, new, the other related with the known solutions not in a straightforward way. However, it is only a secondary goal of this paper, as well as of the papers [11, 2], to enrich a “zoology” of algebraic solutions of the sixth Painlevé equation. The main purpose is to study different features of the method of $RS$-transformations and better understand its place in the theory of SFITs [10]. Another goal we achieve in Section 3 is to show that all genus zero algebraic solutions of the sixth Painlevé equation in the special case classified by Dubrovin and Mazzocco [6] can be constructed with the help of $RS$-transformations. This is aimed towards a check of my conjecture [11], that all algebraic solutions for the sixth Painlevé equation can be generated via the method of $RS$-transformations and, so-called, the Okamoto transformation (see [14]). It seems that all algebraic solutions of zero
genus that so far appeared in the literature have been now reconstructed by the method of $RS$-transformations or related with the ones that are constructed by this method via the certain transformations. However, Dubrovin and Mazzocco [6] have shown that there exists one more, genus one, algebraic solution of the sixth Painlevé equation. At this stage I cannot confirm that this genus one algebraic solution can be produced in accordance with the above conjecture. On the other hand, there are still a few complicated dessins to be examined to confirm or disprove the conjecture.

In connection with the conjecture mentioned in the previous paragraph it is interesting and instructive to check a closely related, though much simpler case, of the Gauss hypergeometric function, especially taking into account that a complete classification of the cases when the general solution of the Euler equation for the Gauss hypergeometric function is algebraic is known due to H. A. Schwarz [19]. Actually, in Section 4 we show that the whole Schwarz list can be generated via the method of $RS$-transformations, whose $R$ parts are the Belyi functions starting with the simplest Fuchsian ODE with two singular points. This possibly, a new constructive point of view allows one to find in a straightforward, though in some cases tedious way, explicit formulae for all algebraic Gauss hypergeometric functions. We call the set of these functions the Schwarz cluster.

Section 5 is a continuation of the previous work [1] devoted to higher order transformations for the Gauss hypergeometric function. In [1] we have found few new higher order algebraic transformations for the Gauss hypergeometric functions, however, all these transformations except quadratic and cubic ones, act within the Schwarz cluster. Thus it was an interesting question to understand whether there exist transformations of the order higher than 3 and are not compositions of quadratic and cubic transformations, which act on transcendental the Gauss hypergeometric functions. In [1] we found a numeric construction of an octic transformation which has this property. Although we were able to find a numerical solution with much more digits than that indicated in [1] and thus it was no any doubt that this transformation actually exists, we didn’t have a mathematical proof of the existence. In this Section an identification of this transformation with one of the Belyi functions, immediately gives the desired proof. Moreover, by using a better computer the corresponding Belyi function is calculated explicitly. This makes straightforward an explicit construction of, actually, three different octic $RS$-transformations. These transformations together with the quadratic and cubic transformations, and their inverses define three different clusters of the transcendental Gauss hypergeometric functions which are related via algebraic higher order transformations, i.e., have the same type of transcendency. We call them Octic Clusters and present them explicitly at the end of Section 5 in the corresponding tables.

Discussing in Sections 2 and 3 different questions concerning particular algebraic solutions we make references to the quadratic transformations for the sixth Painlevé equation. For the convenience of the reader in Appendix we give an overview of these transformations in the soul of the present work from the “Belyi functions” angle of view. I hope that even a specialist may find this outlook interesting.

Acknowledgement and Comments After the work was finished and put into the web archive I got a letter from P. Boalch, who informed me about two very interesting works [4, 5] that are closely related and substantially overlap with this work. I would like to thank him for this very important information and a subsequent informal discussion of the related issues. Below we make necessary comments concerning these papers.

The solution presented in item 3 (Cross) of Subsection 3.4 was explicitly constructed in the recent work by Boalch [4]. He used the method suggested in [6], which is very
different from the one considered here. Moreover, in his work a relation of this solution with the famous Klein’s quartic algebraic curve in $\mathbb{P}^2$ of genus 3 that has a maximum possible number of holomorphic automorphisms is established.

Theorem 2.1 formulated in terms of the $R$-parts rather than tricolour graphs and modulo explicit formulae for the coefficients of the sixth Painlevé equation was first established in the work by Ch. F. Doran [5] (Section 4, Theorem 4.5). Doran’s work is based essentially on the scalar second order Fuchsian equation and gives a deep and general theoretical insight based on many remarkable results known for the Belyi functions, Hurwitz spaces, arithmetic Fuchsian groups, etc.

At the same time Doran does not introduce a new concept of the deformation of the dessins like we do, also he does not perform any explicit constructions of the algebraic solutions leaving an opportunity for the interested reader to apply the method by J.-M. Couveignes.

Our deformation technique, assuming the validity of Conjectures 2.1 and 2.2 allows one to immediately reproduce the classification results of [5] formulated as Corollaries 4.6–4.8 and continue in a systematic way a production of further “solvable” types of the suitable $R$-parts. Thus, our examples are not just an illustration of the classification [5]: many of them, say, the deformations studied in Section 2 or Subsection 3.1 go beyond Corollaries 4.6–4.8. I also call attention of the reader to the discussion of the renormalization aspect. It is clear that if Conjectures 2.1 and 2.2 are valid they give an answer on the “inverse problem”, i.e., classification of the types of $R$-parts that generate algebraic sixth Painlevé’s functions (see [2] and Remark 12 of [5]). Of course, this answer is not absolutely explicit, but in principle for any given type a finite number of operations is required to check whether there exists the corresponding tricolour graph or not and thus give an answer on the inverse problem. I hope that further studies will pour more light on these conjectures, so that more explicit statements will be available. It is also worth to mention that all $R$-parts presented in this work are found by a straightforward though complicated method explained in Remark 2.1.

The work by Doran rise a priority question. This question can be separated into two ones: 1. The idea of using of $RS$-transformations in the context of SFTTs in particular the sixth Painlevé equation, and 2. Appearance of the Belyi functions and more generally the Hurwitz curves as their $R$-parts. The method of $RS$-transformations were used by the author [13] (1991) for a construction of the quadratic transformations for the sixth Painlevé equation. Application of the $RS$-transformations for construction of the higher order transformations for SFTTs and algebraic SFTTs was reported by the author at Workshop on Isomonodromic Deformations and Applications in Physics, Montreal (Canada), May 1 – 6, 2000 and published in [11]. The works [11] and [2] were reported at Workshops in Strasbourg (February, 2001) and Ōtsu (August, 2001). We also note that the original scheme of [11] always uses both $R$- and $S$- part of the $RS$-transformation. As is clear from this work it is a more general procedure, than that based on Theorem 2.1 (Theorem 4.5 of [5]), some of the known solutions we were not able to reproduce without the $S$-parts. Strictly speaking this fact does not mean that such solutions cannot be obtained via Theorem 2.1 by finding some other suitable $R$-parts, however, construction of the $R$-parts is a much more complicated enterprise than construction of the $S$-parts. Note that this is a more general content than that reported in [5].

At the same time, the fact of the relation of the $R$-parts with the Belyi functions and their deformations was noticed by me only in this work, after I decided to reproduce via the method of $RS$-transformations the results reported in [6] and also explicitly construct
the octic transformation for the Gauss hypergeometric function from \[1\]: this requires a more detailed study of the corresponding rational functions. So that the second fact was first found by Doran and only rediscovered here.

2 Deformations of Dessins d’Enfants and Algebraic Solutions of the Sixth Painlevé Equation

Proposition 2.1 Let \( R : \mathbb{CP}^1 \rightarrow \mathbb{CP}^1 \) be a rational function of degree \( n \) with \( k \geq 3 \) critical values. Denote by \( k_i \) the number of critical points corresponding to the \( i \)-th critical value and put

\[
m = \sum_{i=1}^{3} k_i - n - 2.
\]

(2.1)

Then \( m \geq 0 \). Moreover, \( R \) is the Belyi function with 3 critical values iff \( m = 0 \).

Proof. The Riemann–Hurwitz formula gives

\[
\sum_{i=1}^{k} k_i = (k - 2)n + 2.
\]

Summing up it with Equation (2.1) we arrive at

\[
m + \sum_{i=4}^{k} k_i = (k - 3)n,
\]

(2.2)

where we assume that the sum is equal to 0 if \( k < 4 \). Notice that \( \sum_{i=4}^{k} k_i \leq (k - 3)(n - 1) \), therefore, Equation (2.2) implies \( m \geq k - 3 \geq 0 \). Moreover, if \( k = 3 \), then \( m = 0 \) again by virtue of Equation (2.2).

Remark 2.1 The condition \( m \geq 0 \) has a very lucid sense and intuitively evident: We can assume that the first three critical values are located at 0, 1, and \( \infty \) and define the function \( R \) and \( R - 1 \) by two rational expressions with indeterminate preimages of 0, 1 and \( \infty \) with the prescribed multiplicities. Writing then the consistency condition we arrive at a system of algebraic equations for the indeterminate preimages. In this setting the condition \( m \geq 0 \) says that the number of equations in the system, \( n \), should not be greater than the number of unknown parameters, \( \sum_{i=1}^{3} k_i - 2 \). This necessary condition however is not sufficient for existence of \( R \), see an example in Remark 4.2. If \( m > 0 \) and the corresponding function \( R \) exists, it may depend on \( m \) parameters. We call, sometimes, such functions \( m \)-dimensional deformation of the Belyi functions.

I learned from the recent paper \[8\] that this Proposition essentially coincides with Silverman’s proof of the abc-theorem for polynomials. A minor difference occurs since in our setting it is natural to count all critical points including the point at \( \infty \), whilst in the abc-setting the \( \infty \) point is excluded. Below in Remark 2.2 we give also a “graphical” insight on this Proposition.

For a description of rational functions, \( z = R(z_1) \), we use a symbol which is called their type and denoted \( R(\ldots|\ldots|\ldots) \). In the space between two neighbouring vertical lines or the line and one of the parentheses, which we call the box, we write a partition of \( \deg R \) into the sum of multiplicities of critical points of the function \( R(z_1) \) corresponding to one of its critical values in the descending order. The total number of boxes is normally supposed to coincide with the number of critical values of the function \( R \). However, in the case \( m = 1 \) which is studied in this and the next Sections we do not indicate the
fourth “evident” box: $|2 + 1 + \ldots + 1|$. At the same time, where convenient we include the boxes for non-critical values: $|1 + \ldots + 1|$.

Consider on the Riemann sphere bicolour connected graphs; with black and white vertices, and faces homeomorphic to a circle. The valencies of the black and white vertices are defined in the usual manner. These valencies can be equal to any natural number. The vertices of the same colour are not connected by edges. We introduce a notion of the black edge, i.e., the path in the graph connecting two black vertices: it contains only two black vertices, its end points, and one white vertex. Any two black vertices can belong to a few different black edges or cycles. Any cycle should contain at least one black vertex. The loops are not allowed. We also define the black order of a face as a number of the black edges in its boundary.

With each Belyi function we can associate now a bicolour connected graph in the following way. The black vertices are preimages of $\infty$ and the white vertices are preimages of 1. Their valencies equal to their respective multiplicities. Each face corresponds to a preimage of 0, the black orders of faces equal to the multiplicities of the corresponding preimages. Conversely, for each bicolour graph with the properties stated above, there exists the unique, modulo fractional linear transformations of the independent variable, Belyi function whose associated bicolour graph is isomorphic to the given one. In case if all valencies of the white vertices equal 2, we do not indicate them at all and, instead of a bicolour graph, get a usual planar graph with the black edges and black orders of faces coinciding with usual notion of edges and orders of faces. Note that after we dismiss the white vertices the loops may appear on the latter planar graph.

Pictures of such bicolour graphs on the Riemann sphere we call dessins d’enfants or just the dessins. These dessins define on the Riemann sphere, so called bipartite maps [20].

We define now tricolour graphs. It is also connected graphs on the Riemann sphere with black and white vertices that obey the conditions for the bicolour graphs, but also with one blue vertex. Any cycle should contain at least one black or blue vertex, any edge connecting vertices of different colours. The loops again are not allowed. The boundary of every face should contain at least one black vertex. Valency of the blue vertex equals 4. Again in case when all the valencies of the white vertices equal 2 we are not indicating them and instead of a tricolour graph get a bicolour graph with all black vertices and only one blue vertex. The latter bicolour graph of course, has nothing to do with the black-white bicolour graphs introduced in the previous paragraph. In particular, the black-blue bicolour graph may contain loops. For the tricolour graphs we keep the same notions of the black edge and black order of face, the blue point is not counted in both cases.

Graphically the tricolour graphs can be viewed as obtained from the bicolour ones as a result of simple “deformations” see examples below and in the next Section. Therefore we call them the deformation dessins or very often, where there is no cause for a confusion, just the dessins.

**Remark 2.2** At this stage introduction of tricolour graphs with only one blue vertex looks somewhat artificial; but it is what we need for applications to the theory of the sixth Painlevé equation. If we would think of a more general rational functions needed for classification of $RS$-transformations for multivariable SFITs we have to introduce multicolour graphs. Say, via a recurrence procedure by considering one dimensional deformations of $n$-colour graphs we arrive to $n$- or $n + 1$-colour graphs. At each stage we can either add a vertex with a new or one of the “old” colours. We have to admit a
coalescence of the vertices of the same colours, so that the valencies of the colour vertices would be even and $\geq 4$. The vertices of the same colour correspond to critical points for the same critical value of $R$. The multiplicities of these critical points coincides with a half of valencies of the corresponding colour vertices. Thus the colour vertices correspond to multiple critical points of $R$. The dimension $m$ of the deformation equals to one half of the sum of valencies of the colour vertices minus their total number.

**Conjecture 2.1** In conditions of Proposition 2.1 for any rational function $z = R(z_1)$ with $m = 1$ whose first three critical values are 0, 1 and $\infty$, there exists a tricolour graph such that:
1. There is a one-to-one correspondence between its faces, white, and black vertices and critical points of the function $R$ for the critical values 0, 1, and $\infty$, respectively.
2. Black orders of the faces and valencies of the vertices coincide with multiplicities of the corresponding critical points.

**Remark 2.3** As the reader will see in Section 3, with the same rational function $z = R(z_1)$ with $m = 1$ can be associated a few seemingly different tricolour graphs. The reason is that $R$ actually is the function of two variables, $R(z_1, y)$, where $y \in \mathbb{C}P^1$ is a parameter. As the function of $y$ $R$ has different branches. The different tricolour graphs should be related with these different branches. These also should lead to some equivalence relation on the set of the tricolour graphs. Examples, considered in Items 4 and 5 of Subsection 3.3 of Section 3 show that we cannot just announce all tricolour graphs whose associated rational functions have the same type equivalent; since these rational functions can be different as the functions of $y$. Of course, until this equivalence relation is established in cases when there are a few different functions $R(z_1, y)$ of $y$ having the same type as the rational functions of $z_1$ it is complicated to relate them with the proper tricolour graphs. In the examples of Subsection 3.3 referenced above to make such a distinction we were motivated by a symmetry that exists in one of the examples. This remark explains why uniqueness for the function-graph correspondence in Conjecture 2.1 and uniqueness modulo fractional linear transformations for the inverse correspondence in the following Conjecture 2.2 are not stated.

**Conjecture 2.2** For any tricolour graph there exists a function $z = R(z_1, y)$, which is a rational function of $z_1 \in \mathbb{C}P^1$ with four critical values. Three of them: 0, 1, and $\infty$, and the corresponding critical points are related with the tricolour graph as it is stated in Conjecture 2.1. The variable $y$ denotes the unique second order critical point corresponding to the fourth critical value of $z = R(z_1, y)$. $R$ is an algebraic function of $y$ of the zero genus.

**Corollary 2.1** In the conditions of Conjecture 2.2 There is a representation of the function $R$ as the ratio of coprime polynomials of $z_1$ such that its coefficients and $y$ allow a simultaneous rational parametrization.

Consider a rational parametrization of $y = y(s)$ and $z = R(z_1, y) \equiv R_1(z_1, s)$ with some parameter $s$ as is stated in Corollary 2.1. We can change the role of the variables $z_1$ and $s$, i.e., consider $R_1(z_1, s)$ as the rational function of $s$ and treat $z_1$ as an auxiliary parameter. In this case we call $R_1(z_1, s)$ the conjugate function with respect to $R(z_1)$. Making fractional linear transformations of $R$ interchanging its critical points we get new rational functions of equivalent types. However, their conjugate functions have,
generically, not equivalent types (see examples below). The reason is that the conjugate functions have dimensions \( m \geq 1 \), however, instead of \( m \) parameters they have only one, \( z_1 \). Critical points of the conjugate functions depend generically on \( z_1 \). However, each function has \( m \) critical points independent of \( z_1 \) we call them *additional critical points*.

The set of the additional critical points of all conjugate functions coincides with the set of values of the parameter \( s \) such that the function \( R \) changes its type and therefore coincides with one of the Belyi functions.

Recall the canonical form of the sixth Painlevé equation,

\[
\frac{d^2 y}{dt^2} = \frac{1}{2} \left( \frac{1}{y} \left( \frac{1}{y-1} + \frac{1}{y-t} \right) + \left( \frac{1}{t} + \frac{1}{t-1} + \frac{1}{y-t} \right) \frac{dy}{dt} \right)^2 - \left( \frac{1}{t} + \frac{1}{t-1} + \frac{1}{y-t} \right) \frac{dy}{dt}
+ \frac{y(y-1)(y-t)}{t^2(t-1)^2} \left( \alpha_6 + \beta_6 \frac{t}{y^2} + \gamma_6 \frac{t-1}{(y-1)^2} + \delta_6 \frac{t(t-1)}{(y-t)^2} \right),
\]

the conjugate functions where \( \alpha_6, \beta_6, \gamma_6, \delta_6 \in \mathbb{C} \) are parameters. For a convenience of comparison of the results obtained here with the ones from the other works we will use also parametrization of the coefficients in terms of the formal monodromies \( \hat{\theta}_k \):

\[
\alpha_6 = \frac{(\hat{\theta}_{\infty} - 1)^2}{2}, \quad \beta_6 = -\frac{\hat{\theta}_0^2}{2}, \quad \gamma_6 = \frac{\hat{\theta}_1^2}{2}, \quad \delta_6 = \frac{1 - \hat{\theta}_0^2}{2}.
\]

To formulate our main result, showing a relation between the tricolour graphs and algebraic solutions of Equation (2.3), we recall a notion of the *RS-symbol* for RS-transformations and introduce some necessary notation.

The *RS-symbol* is designed for a description of RS-transformations for arbitrary SFITs related with the Fuchsian ODEs. Below instead of a general definition we refer to a more specific situation considered in this paper. The brief notation for the RS-symbols needed here is \( RS^k_i(3) \) for \( k = 2, 3, 4 \). This notation means that we map a \( 2 \times 2 \) matrix Fuchsian ODE with 3 singular point into analogous ODE but with \( k \) singular points. The extended notation for the RS-symbol, instead of the number 3 in the parentheses, uses three boxes, each one contains two rows of numbers. In the first raw of each box there is just a rational number, in the second raw the sum of integers representing multiplicities of the preimages of critical values of the rational function \( R \), the \( R \)-part of the RS-transformation. It is supposed that three critical values of the function \( R \) are located at 0, 1, and \( \infty \). The boxes in the notation of the RS-symbol are ordered accordingly. For transformations studied in this work we do not have a need to indicate other critical values (if any). The whole set of the preimages of the critical values: 0, 1, and \( \infty \) is divided (non-uniquely!) on two sets of apparent and non-apparent points: we call them *apparent and non-apparent sets*, respectively.

The apparent set is the union of apparent sets of the boxes. The apparent set of the \( i \)-th box, where \( i = 0, 1, \infty \), consists of all points whose multiplicities are divisible by some natural number \( \geq 2 \). We denote by \( n_i \geq 2 \) the *greatest common divisor* of the apparent set of the \( i \)-th box. In particular, this set can be empty. In the last case we formally put \( n_i = 1/\theta_i \), where \( \theta_i \) is a parameter. Let a number of the apparent points in the \( i \)-th box be \( N_i \geq 0 \). If \( N_i \geq 1 \), then multiplicity of the \( j \)-th apparent point in the \( i \)-th box can be written as \( k^j_i n_i \), where \( j = 1, \ldots, N_i \) and \( k^j_i \in \mathbb{N} \).

A critical point of \( i \)-th critical value is non-apparent iff its multiplicity is not divisible by \( n_i \). The union of such point is the non-apparent set of \( R \). Non-apparent sets for the transformations \( RS^2_1(3) \), \( RS^2_3(3) \) or \( RS^2_5(3) \) consist of 4, 3, or 2 points, respectively.
In general, we call the $R$-part of some $RS$-transformation with three or more critical values normalized iff the set $\{0, 1, \infty\}$ is a subset of the set of critical values of $R$ and also a subset of its non-apparent set (if the non-apparent set consists of only two preimages it should coincide with the set $\{0, \infty\}$). The rational function $R$ with two critical values is normalized, iff the set of its critical values is a subset of $\{0, 1, \infty\}$ and the non-apparent set obey the same condition as in the previous sentence.

Further in this section we consider only $RS_2^2(3)$-transformations. Suppose that their $R$-part are normalized, then the non-apparent set consists of four points: $0, 1, \infty$, and $t$. Denote their multiplicities as $m_0, m_1, m_\infty$, and $m_t \in \mathbb{N}$.

Definition 2.1 $RS_4^2(3)$-symbol is called special if for $i = 0, 1, \infty$ the rational number in the first raw of the $i$-th box is $1/n_i$.

Remark 2.4 The numbers $1/n_i$ in first rows of the boxes of special $RS$-symbols equal to formal monodromy of the $i$-th singular point of the original Fuchsian ODEs.

Theorem 2.1 Let the normalized rational function $z = z(z_1)$ corresponding to a tricolour dessin be the $R$-part of some $RS_2^2(3)$-transformation with a special $RS$-symbol. Put $\varepsilon = 0$ or $1$ depending on whether $\sum_{i=1}^{3} \sum_{j=1}^{N_i} k_{ij}$ is even or odd, respectively. Then the double critical point $y$ of the function $z(z_1)$, corresponding to its fourth critical value, considered as the function of the fourth non-apparent critical point $t$, is an algebraic solution of the sixth Painlevé equation (2.3) for the following $\hat{\theta}$-tuple:

$$
\hat{\theta}_0 = \frac{m_0}{n z(0)}, \quad \hat{\theta}_1 = \frac{m_1}{n z(1)}, \quad \hat{\theta}_t = \frac{m_t}{n z(t)}, \quad \hat{\theta}_\infty = \varepsilon + (-1)^\varepsilon \frac{m_\infty}{n z(\infty)}.
$$

Proof. We assume that the reader is acquainted with that how the method of $RS$-transformations is working to produce algebraic solutions of the sixth Painlevé equation (see [2]). A suitable $RS$-transformation can be viewed as a composition of some $R$-transformation with a finite number of elementary Schlesinger transformations which are successively applied to the associated linear $2 \times 2$ matrix ODE stating with a matrix form of the Gauss hypergeometric equation. The key observation is that for special $RS$-symbols all the elementary Schlesinger transformations can be chosen to have the same upper-triangular structure. Thus $y$, the root of the equation $R'(z_1) = 0$, is also a root of $\{21\}$-element of the successively transformed coefficient matrix of the associated linear ODE at each step of application of the elementary Schlesinger transformations.

Remark 2.5 In formulation of Theorem 2.1 we can, of course, instead of mentioning of the special $RS$-symbol, formulate all necessary conditions to be imposed on the rational function $z(z_1)$ pure graphically, in terms of the valencies of the tricolour graph.

For $\varepsilon = 1$ the sign minus in the above formula for $\hat{\theta}_\infty$ is not essential as it does not change the coefficients of Equation (2.3). We keep it to get smaller absolute values for $\hat{\theta}_\infty$.

Now we consider some special deformations of the dessins and corresponding constructions for algebraic solutions of the sixth Painlevé equation which follows from Theorem 2.1.

Remark 2.6 On all figures throughout the paper we follow the convention that blue vertices are indicated similarly to the white ones but with a larger diameter than the latters. In case the picture contains only one “white vertex” it is actually the blue one.
It is worth to notice that applying to this solution the Okamoto to transformation together with Proposition 2.2 (see Equation (1.3)). Hereafter, we call these and similar deformations of the other dessins Twist, Cross, and Join, respectively. Note that two face distributions, namely, 2+2+2+2 and 3+3+1+1 which pass through the necessary condition of Proposition 2.1 cannot be realized as the deformations of the first dessin on Figure 1. However, the latter face distribution can be obtained as a face deformation of the dessin for the Belyi function of the type $R(3+3+2|2+2+2+2|3+3+2)$. The first one cannot be realized as a deformation of any dessin and thus does not define any rational function. There are several other seemingly different deformations that lead to the same face distributions. In this case, they can be interpreted as different branches of the same algebraic solution. Below we give exact formulae for the deformations of the Belyi function for all three deformed dessins presented on Figure 1 together with the corresponding solutions, $y(t)$, of the sixth Painlevé equation calculated via Theorem 2.1.

1. Twist. $RS_1^2\left(\begin{array}{c}1/5 \\ 5+1+1+1 \end{array}\right)\begin{array}{c}1/2 \\ 2+\ldots+2 \end{array}\left(\begin{array}{c}1/3 \\ 3+3+2 \end{array}\right)$:

$$z = \frac{2^{214}s^3(5s^2 + 118s + 5)(s + 1)^{10}}{(3s^3 + 95s^2 + 25s + 5)^3(5s^3 + 25s^2 + 95s + 3)^3},$$

$$a = \frac{(s - 1)(s^4 + 12s^3 - 410s^2 + 12s + 1)}{128s(s + 1)\sqrt{s(5s^2 + 118s + 5)}},$$

$$c_1 \equiv c_1(s) = \frac{(5s^6 + 462s^5 + 8535s^4 + 3060s^3 + 195s^2 + 30s + 1)}{16s(3s^3 + 95s^2 + 25s + 5)\sqrt{s(5s^2 + 118s + 5)}}, c_2 = -c_1(1/s),$$

$$t = \frac{1}{2} - \frac{(s - 1)(25(s^8 + 1) + 760(s^7 + s) + 4924(s^6 + s^2) + 75464(s^5 + s^3) + 329174s^4)}{2^{11}s(s + 1)\sqrt{s(5s^2 + 118s + 5)}},$$

$$y = \frac{1}{2} - \frac{(s - 1)(5(s^6 + 1) + 58(s^5 + s) + 1771(s^4 + s^2) + 8620s^3)\sqrt{s(5s^2 + 118s + 5)}}{8s(s + 1)(5s^3 + 25s^2 + 95s + 3)(3s^3 + 95s^2 + 25s + 5)},$$

$$\hat{\theta}_0 = \frac{1}{5}, \quad \hat{\theta}_1 = \frac{1}{5}, \quad \hat{\theta}_t = \frac{1}{5}, \quad \hat{\theta}_\infty = \frac{1}{3}.$$

It is worth to notice that applying to this solution the Okamoto transformation together with

$^1$The usual convention $\sum^0_{t=0} = 0$ is assumed.

$^2$The one different from 0, 1, and $\infty$.

\[\text{Figure 1: One parameter "face" deformations of the dessin for the Belyi function } R(5 + 2 + 1|2 + 2 + 2 + 2|3 + 3 + 2). \text{ Distributions of the black orders of faces are indicated under the dessins.}\]
with the so-called Bäcklund transformations we can obtain algebraic solutions of Equation \[2.3\] for the following \(\theta\)-tuples:

\[
\left( \frac{1}{15}, \frac{1}{15}, \frac{1}{15}, \frac{7}{15} \right) \quad \text{and} \quad \left( \frac{4}{15}, \frac{4}{15}, \frac{4}{15}, \frac{2}{15} \right).
\]

It is readily seen that by introducing a new variable \(z_2 = (z_1 - 1/2)\sqrt{s(5s^2 + 118s + 5)}\) we get a new function \(z(z_2)\) which has the same type as \(z(z_1)\) but parametrized by \(s\) rationally. However, to relate our rational function with a solution of the sixth Painlevé equation \[2.3\] we have, as it is explained earlier, to normalize it by placing, via a fractional linear transformation, three of its non-apparent zeroes or poles at 0, 1, and \(\infty\). In this example the normalization procedure leads to the appearance of a genus 1 parametrization. The following Remark \[2.7\] helps in many cases to simplify parametrizations of the \(R\)-parts of \(RS\)-transformations regardless whether we use them for construction of solutions via Theorem \[2.1\] or with a help of the complete construction of the corresponding \(RS\)-transformations including their \(S\)-parts. As the reader will see below with this function \(z(z_1)\) one can associate another \(RS\)-transformation, such that Theorem \[2.1\] is not applicable. Thus, one actually have to build up the \(S\)-part of the latter \(RS\)-transformation to find the corresponding solution of the sixth Painlevé equation.

**Remark 2.7** \(S\)-parts of the \(RS\)-transformations are symmetric functions of the apparent critical points of their \(R\)-parts.

Thus, we need to know a parametrization of the symmetric functions, \(c_1 + c_2\) and \(c_1 c_2\), of the apparent critical points \(c_1\) and \(c_2\) rather than their individual parametrization. Of course, the latter leads to a simplification of the parametrization. Moreover, it is clear that such simplification can theoretically reduce the genus of the parametrization. In particular, in item 5 of Subsection 3.3 we have an example where it actually happens. Below we show that the simplified parametrization can be obtained with the help of the Zhukovski type transformation, \(s + 1/s = 2^5 s_1 + 2\). The latter reduces a degree of the parametrization by two, however in this case the genus of parametrization remains unchanged.

\[
z(z_1) = \frac{(5s_1 + 4)(8s_1 + 1)^5}{8(30s_1^3 + 40s_1^2 + 10s_1 + 1)^3} \frac{(z_1 - 1/2 - a)^5 z_1(z_1 - 1)(z_1 - t)}{(z_1^2 - (1 + c_1 + c_2)z_1 + 1/4 + c_1 c_2 + (c_1 + c_2)/2)^3},
\]

\[
a = \frac{(8s_1^2 + 4s_1 - 3)s_1}{2\sqrt{s_1(8s_1 + 1)/(5s_1 + 4)}} \frac{1}{4} + c_1 c_2 = -\frac{320s_1^6 + 1344s_1^5 + 1560s_1^4 + 480s_1^3 - 60s_1 + 1}{8(30s_1^3 + 40s_1^2 + 10s_1 + 1)(5s_1 + 4)},
\]

\[
c_1 + c_2 = -\frac{(6s_1^2 + 4s_1 - 1)(4s_1 + 5)s_1^2}{(30s_1^3 + 40s_1^2 + 10s_1 + 1)} \frac{(8s_1 + 1)}{\sqrt{s_1(8s_1 + 1)/(5s_1 + 4)}},
\]

\[
t = \frac{1}{2} - \frac{(800s_1^4 + 960s_1^3 + 312s_1^2 + 100s_1 + 15)s_1}{2(8s_1 + 1)^2 \sqrt{s_1(8s_1 + 1)/(5s_1 + 4)}}, \quad y = \frac{1}{2} - \frac{(40s_1^2 + 22s_1^2 + 16s_1 + 3)s_1}{2(30s_1^4 + 40s_1^3 + 10s_1 + 1)(8s_1 + 1)}.
\]

It is also worth to note that although we have a simpler parametrization of the function \(z(z_1)\), the first parametrization is still needed in the theory of SFITs with several variables, where the critical points \(c_1\) and \(c_2\) are included into the non-apparent set, so that the second parametrization just cannot be used for a purpose of construction of \(RS\)-transformations.
Consider now the associated conjugation functions. It is clear that in this case some of them are rational functions on the torus. We consider here only conjugate functions rational on the Riemann sphere. To get them we have to normalize the function \(z(z_2)\) mentioned in the paragraph right after the first parametrization of \(z(z_1)\). The only way to do it, if we wish to keep a rational parametrization, is to use for the normalization apparent critical points: the zero of order 5 and poles of the orders 3, 3, and 2, together with one non-apparent zero. We call the normalization degenerate if at least one of the critical points 0, 1, or \(\infty\) is apparent. All in all we can arrange 33 different degenerate normalizations of \(z(z_2)\) and thus to get 33 different rational conjugation functions.

For example, the function \(z(z_2)\) normalized such that it has a zero of the fifth order at 0, and poles of the third order at 0, 1, and \(\infty\) reads,

\[
\tilde{z} = \frac{\tilde{z}^5(z_1 - 1 + s_0^2)(5s_0 + 1)^2\tilde{z}_1^2 + (5s_0 + 1)(5s_0^2 + 49s_0^2 + 115s_0 + 23)\tilde{z}_1 - 24(s_0^2 + 6s_0 + 1)(s_0^5 + 4s_0 + 1))}{64(z_1 - 1)^2((3s_0^2 + 15s_0^2 + 25s_0 + 5)z_1 - 8(s_0 + 1)(s_0^5 + 4s_0 + 1))}, \tag{2.4}
\]

This function differs from the one in the beginning of this item, by a fractional linear transformation of \(z_1\) related with the normalization discussed above:

\[
\tilde{z}(\tilde{z}_1) = z(z_1), \quad z_1 \equiv M(\tilde{z}_1) = \frac{\tilde{t}_1 - c_0}{\tilde{t}_1 - \tilde{t}_+} \frac{\tilde{z}_1 - \tilde{t}_+}{\tilde{z}_1 - c_0},
\]

where

\[
\tilde{t}_1 = 1 - s_0^2, \quad \tilde{t}_+ = -24s_0^2 + 23 + s_0^2 \pm \frac{(s_0 + 5)}{2} \sqrt{\frac{(5s_0^2 + 22s_0 + 5)(s_0 + 5)}{5s_0 + 1}},
\]

\[
c_0 = \frac{8(s_0 + 1)(s_0^2 + 4s_0 + 1)}{(3s_0^2 + 15s_0^2 + 25s_0 + 5)}, \quad s_0 = \frac{5 - s}{5s - 1},
\]

and \(s\) is exactly the same as in the formulae in the beginning of this item. The functions \(\tilde{t}_\pm\) are the conjugate roots of the quadratic polynomial of \(z_1\) in the numerator of function (2.4) and \(c_0\) is its second order pole.

The function \(\tilde{z}(\tilde{z}_1)\) has a degenerate normalization. Thus Theorem 2.1 is not applicable to it. However, in this particular case the first order zeroes of \(\tilde{z}(\tilde{z}_1)\) unexpectedly have an “apparent behaviour” so that Theorem 2.1 actually works. Therefore, by putting: \(m_0 = 5, m_1 = 3, m_t = 1, m_\infty = 3, \epsilon = 1, n_0 = 1, \) and \(n_\infty = 2,\) and formally applying Theorem 2.1 to the function \(\tilde{z}(\tilde{z}_1)\), we arrive at the conclusion that

\[
\tilde{y} = \frac{5(1 - s_0^2)(s_0^2 + 6s_0 + 1)(s_0^5 + 4s_0 + 1)}{(5s_0 + 1)(3s_0^3 + 15s_0^2 + 25s_0 + 5)},
\]

being considered as the function of either argument \(\tilde{t}\) or \(\tilde{t}_\pm\), solves the sixth Painlevé equation (2.8) for

\[
\hat{\theta}_0 = 5, \quad \hat{\theta}_1 = \frac{3}{2}, \quad \hat{\theta}_t = 1, \quad \hat{\theta}_\infty = -\frac{1}{2} \tag{2.5}
\]

Actually, \(y(\tilde{t})\) and \(y(\tilde{t}_\pm)\) are just different branches of the same genus 0 algebraic function, so that \(\tilde{t} = \tilde{t}(s_0)\) and \(y = y(s_0)\) is its rational parametrization.

The functions \(y(t)\) and \(\tilde{y}(\tilde{t})\) are related via the fractional linear transformation:

\[
y = M(\tilde{y}), \quad t = M(\tilde{t}).
\]
Of course, the latter transformation can be called “fractional linear” only conventionally as it parametrized via the same parameter \( s_0 \) (or \( s \)) as \( \tilde{t} \) and \( \tilde{y} \). At this point it is worth to notice that solutions \( y(t) \) and \( \tilde{y}(\tilde{t}) \) are not related by any of transformations that act on the set of general solutions of the sixth Painlevé equation.

Moreover, we can treat \( \tilde{y} \) as the function of \( \tilde{t}_1 = c_0 \) as such it is a rational function, \[
\tilde{y} = \frac{5\tilde{t}_1(\tilde{t}_1 - 2)}{3\tilde{t}_1 - 8},
\]
which solves the sixth Painlevé equation for the same \( \hat{\theta} \)-tuple \([2,5]\).

Now we consider function \([2,3]\) as the rational function of \( s_0 \) treating \( z_1 \) as a parameter, i.e., as the conjugate function. Its type is \( R(1 + \ldots + 1|2 + 2|2 + 2 + 2) \) and therefore this function has four additional critical points. These points are located at \( s_0 = 0, \infty, \) and \(-7/5 \pm 2\sqrt{6}/5 \). We write the corresponding Belyi functions in terms of the variables \( z \) and \( z_1 \), introduced in the very beginning of this item, as the values of the function \( z(z_1) = z(z_1, s) \) at \( s = 5, 1/5, \) and \(-11/5 \pm 4\sqrt{6}/5 \), respectively. The first two Belyi functions have the same type \( R(5 + 1|2 + 2 + 2|3 + 2 + 1) \), they are related with a simple change of argument, \( z_1 \to 1 - z_1 \); whilst the last two functions coincide and their type is \( R(5 + 2 + 1|2 + \ldots + 3 + 3 + 2) \):

\[
s = 5 : \quad z(z_1) = \frac{20(z_1 - 16/25)^5(z_1 - 1)}{27z_1(z_1 - 128/125)^3}, \quad s = \frac{1}{5} : \quad z(1 - z_1),
\]

\[
s = -11/5 \pm 4\sqrt{6}/5 : \quad z = -\frac{5(z_1 - 27/25)^5z_1^2(z_1 - 1)}{8(z_1^2 + 216/125z_1 + 729/1000)^3}.
\]

For completeness we mention that with this twist one can associate another \( RS \)-transformation, namely, \( RS_1^2 \left( \begin{array}{c} 2/5 \\ 5 + 1 + 1 + 1 \\ 4 \\ 3 + 3 + 2 \end{array} \right) \). However the corresponding solution of Equation \([2,3]\) cannot be calculated via Theorem \([2,1]\). To find it explicitly one has to construct the \( S \) part of the transformation like in the work \([2]\). Here we only notice that this solution also gain an elliptic parametrization and corresponds to the following \( \hat{\theta} \)-tuple:

\[
\hat{\theta}_0 = \frac{2}{5}, \quad \hat{\theta}_1 = \frac{2}{5}, \quad \hat{\theta}_t = \frac{2}{5}, \quad \hat{\theta}_\infty = \frac{2}{3}.
\]

Again the Okamoto transformation together with the Bäcklund ones allow one to get algebraic solutions for the following \( \hat{\theta} \)-tuples:

\[
\left( \begin{array}{c} 2 \\ 15 \\
2 \\ 15 \\
2 \\ 15 \\
14 \\ 15 \end{array} \right) \quad \& \quad \left( \begin{array}{c} 8 \\ 15 \\
8 \\ 15 \\
8 \\ 15 \\
4 \\ 15 \end{array} \right) \sim \left( \begin{array}{c} 7 \\ 15 \\
7 \\ 15 \\
7 \\ 15 \\
11 \\ 15 \end{array} \right)
\]
2. Cross. \( RS_4^2 \left( \begin{array}{ccc} 1/4 & 1/2 & 1/3 \\ 4 + 2 + 1 + 1 & 2 + \ldots + 2 & 3 + 3 + 2 \end{array} \right) : \)

\[
z = -\frac{256s^6}{(3s^2 - 2)^3(s^2 - 6)^3} \frac{(z_1 - a)^4(z_1 - t)^2z_1(z_1 - 1)}{(z_1 - c_1)^3(z_1 - c_2)^3}, \quad c_1 = \frac{(s + 1)^2(s^2 + 2s - 2)^2}{4s^2(3s^2 - 2)},
\]

\[
a = -\frac{1}{32s^3}(s^2 - 4s - 2)(s^2 + 2s - 2)^2, \quad c_2 = \frac{(s - 2)^2(s^2 + 2s - 2)^2}{16s(s^2 - 6)},
\]

\[
t = \frac{1}{4s^4}(s - 2)^2(s + 1)^2(s^2 + 2s - 2), \quad y(t) = -\frac{(s + 1)(s^2 + 2s - 2)(s^2 - 4s - 2)}{2s(s - 2)(3s^2 - 2)(s^2 - 6)}.
\]

\[
\hat{\theta}_0 = \frac{1}{4}, \quad \hat{\theta}_1 = \frac{1}{4}, \quad \hat{\theta}_1 = \frac{1}{2}, \quad \hat{\theta}_\infty = \frac{1}{3}.
\]

The certain transformations as in the previous cases imply now algebraic solutions for the following \( \hat{\theta} \)-tuples:

\[
\left( \frac{1}{12}, \frac{1}{12}, \frac{2}{6}, \frac{3}{3} \right) \quad \& \quad \left( \frac{5}{12}, \frac{5}{12}, \frac{1}{6}, \frac{1}{3} \right).
\]

One can associate with this deformation dessin one more \( RS \)-transformation with the following symbol \( RS_4^2 \left( \begin{array}{ccc} 1/2 & 1/2 & 1/2 \\ 4 + 2 + 1 + 1 & 2 + \ldots + 2 & 3 + 3 + 2 \end{array} \right) \). The corresponding explicit solution for \( \hat{\theta} \)-tuple \((1/2, 1/2, 3/2, -3/2)\) is easy to find by a renormalization of the function \( z(z_1) \) via a fractional linear transformation of \( z_1 \) analogously to that as it is done in the previous item.

The conjugate function is of the type \( R(4 + \ldots + 4 + 2 + \ldots + 2 | 2 + \ldots + 2 | 3 + \ldots + 3) \).

The general function of this type depends on 4 arbitrary parameters. The conjugate function has four additional critical points: \( \pm \sqrt{2} \) and \( \pm i \sqrt{2} \). The original function at this values of the parameter \( s \) coincides with the Belyi functions of the types \( R(6 + 1 + 1^2 + \ldots + 2 | 3 + 3 + 2) \) and \( R(4 + 2 + 1 + 1 | 2 + \ldots + 2 | 6 + 6 + 2 + 2) \), respectively:

\[
s = \pm \sqrt{2} : \quad z = \frac{(z_1 - 1/2)^6z_1(z_1 - 1)}{2(z_1^2 - z_1^1 - 1/32)^3},
\]

\[
s = \sqrt{-2} : \quad z = \frac{(z_1 - 1/2 - 5\sqrt{-2}/4)^4(z_1 - 1/2 + 11\sqrt{-2}/2)^2z_1(z_1 - 1)}{128(z_1^1 - 1/2 + 7\sqrt{-2}/16)^6}.
\]

By fractional linear transformations of \( z_1 \) one can get a few other non-equivalent conjugate functions, e.g., the one of the type \( R(2 + \ldots + 2 + 1 + \ldots + 1 | 2 + \ldots + 2 | 6 + 6 + 2 + 2) \).

The latter function has 6 additional critical points located at 0, \( \infty \), and \( \pm 1 \pm \sqrt{3} \). To all of them corresponds the same Belyi function of the type \( R(1 + 1 | 2, 2) \), \( z = -4z_1(z_1 - 1) \).

3. Join. \( RS_4^2 \left( \begin{array}{ccc} 1/3 & 1/2 & 1/3 \\ 3 + 2 + 2 + 1 | 2 + \ldots + 2 | 3 + 3 + 2 \end{array} \right) : \)

\[
z = -\frac{(s^2 + 3)^6}{(s^2 + 1)^3(s^2 + 9)^3} \frac{(z_1 - a)^3z_1^2(z_1 - 1)^2(z_1 - t)}{(z_1 - c_1)^3(z_1 - c_2)^3}, \quad c_1 = \frac{(s + 1)^2}{2(s^2 + 3)}, \quad c_2 = \frac{(s + 1)^2}{2(s^2 + 1)},
\]

\[
a = \frac{(s^2 + 6s + 3)}{2(s^2 + 3)}, \quad c_1 = \frac{(s + 3)^2}{2(s^2 + 9)}, \quad c_2 = \frac{(s + 1)^2}{2(s^2 + 1)}.
\]
For completeness we mention that this dessin generates one more RS-transformation, $RS^2_4 \left( \frac{1}{2} \left| \frac{2}{3} + 2 + 1 \right| \frac{2}{3} + 2 \right| 3 + 3 + 2 \). The corresponding solution solves Equation (2.3) for $\hat{t}$-tuple (3/2, 1/2, 3/2, -1/2). Its explicit construction can be made in a standard way by a renormalization of the function $z(z_1)$.

By using the Okamoto, Bäcklund, Quadratic (see Appendix) transformations and their compositions one can construct from the solutions presented in this item many different algebraic solutions. We are not writing here the corresponding $\hat{t}$-tuples as care must be taken when considering action of the transformations on $\hat{t}$-tuples: transformations that work on general solutions can degenerate on some particular ones and lead to “solutions” like $y(t) = 0$, $y(t) = 1$, $y(t) = \infty$. Whether it happens or not on some intermediate step of a chain of transformations is not clear until the actual calculation for the particular solutions is done.

The conjugate function is of the type $R(3 + 3 + 1 \ldots + 1 | 2 + \ldots + 2 | 3 + \ldots + 3)$. It has four additional critical points at 0, $\infty$, and $\pm \sqrt{3}$, and the original function at these values of the parameter $s$ coincides with the Belyi functions of the types $R(2 + 2 | 2 + 2 | 2 + 2)$ and $R(3 + 2 + 2 + 1 | 2 + \ldots + 2 | 6 + 2)$, respectively:

\[
\begin{align*}
\hat{s} = 0, \infty : & \quad z = -\frac{4\hat{s}^2(z_1 - 1)^2}{(2\hat{s}z_1 - 1)^2}, \\
\hat{s} = \pm \sqrt{3} : & \quad z = -\frac{2\hat{s}(z_1 - 1/2 \mp \sqrt{3}/2)^3 \hat{s}^2(z_1 - 1/2 \pm 5\sqrt{3}/18)}{64(z_1 - 1/2 \mp \sqrt{3}/4)^6},
\end{align*}
\]

By making a fractional linear transformation one can get a few other nonequivalent conjugate functions; one of them is of the type $R(2 + \ldots + 2 + 1 \ldots + 1 | 2 + \ldots + 2 | 6 + 2 + 2 + \ldots + 2)$. It has six additional critical points at 0, $\infty$, $1 \pm i\sqrt{2}$, and $-1 \pm i\sqrt{2}$.

The Belyi functions corresponding to the last four critical points, $s = \pm 1 + \sqrt{2}$, have the same type $R(3 + 3 + 2 | 2 + \ldots + 2 | 3 + 3 + 2)$ and by the quadratic transformation, $\hat{s} = (z_1 - (1 \pm 1)/2)^2$, can be reduced to the function of the type $R(3 + 1 | 2 + 2 | 3 + 1)$, namely, $z = -64\hat{s}(\hat{s} - 1)^3/(8\hat{s} + 1)^3$. 

\[
t = \frac{(s + 1)^2(s + 3)^2(s^2 - 2s + 3)}{2(s^2 + 3)^3}, \quad y(t) = \frac{(s + 1)(s + 3)(s^2 - 2s + 3)(s^2 + 6s + 3)}{2(s^2 + 1)(s^2 + 3)(s^2 + 9)},
\]

$\hat{\theta}_0 = \frac{2}{3}, \quad \hat{\theta}_1 = \frac{2}{3}, \quad \hat{\theta}_t = \frac{1}{3}, \quad \hat{\theta}_\infty = \frac{1}{3}$. 

By making the fractional linear transformation of $z_1$, $\tilde{z}_1 = (1 - t)z_1/(z_1 - t)$, mapping the points 0, 1, $\infty$, $t$ to 0, 1, 1 - $t$, $\infty$, respectively, we obtain another algebraic solution, $\tilde{y}(\tilde{t}) = (1 - t)y(t)/(y(t) - 1)$ and $\tilde{t} = 1 - t$. Changing the notation $\tilde{y}$ and $\tilde{t}$ back to $y$ and $t$ this solution can be written as follows,

\[
t = \frac{(s - 1)^2(s - 3)^2(s^2 + 2s + 3)}{2(s^2 + 3)^3}, \quad y(t) = \frac{(s - 1)(s - 3)(s^2 + 6s + 3)}{4s(s^2 + 3)},
\]

$\hat{\theta}_0 = \frac{2}{3}, \quad \hat{\theta}_1 = \frac{2}{3}, \quad \hat{\theta}_t = \frac{2}{3}, \quad \hat{\theta}_\infty = \frac{2}{3}$. 

The conjugate function is of the type $R(3 + 3 + 1 \ldots + 1 | 2 + \ldots + 2 | 3 + \ldots + 3)$. It has four additional critical points at 0, $\infty$, and $\pm \sqrt{3}$, and the original function at these values of the parameter $s$ coincides with the Belyi functions of the types $R(2 + 2 | 2 + 2 | 2 + 2)$ and $R(3 + 2 + 2 + 1 | 2 + \ldots + 2 | 6 + 2)$, respectively:
3 Deformations of Dessins for the Platonic Solids and Algebraic Solutions of the Sixth Painlevé Equation

The Platonic solids have reach symmetry groups, therefore their dessins can be obtained as actions of the certain finite rotation groups on simpler dessins called the truncated dessins. In terms of the corresponding Belyi functions it means that these functions are compositions of the Belyi functions for the truncated dessins and a monomial $z = z_1^n$ with some integer $n$. We refer to the interesting paper [15] for details. As is shown in this section and the following one these irreducible truncated dessins are very important in the theory of algebraic the Sixth Painlevé and Gauss hypergeometric functions.

In the case when the $\hat{\theta}$-tuple is of the form $(0,0,0,\hat{\theta}_\infty)$, where $\hat{\theta}_\infty \in \mathbb{C}$ is a parameter, algebraic solutions for the sixth Painlevé equations were classified by Dubrovin and Mazocco [6]. They found five cases, namely: $\theta_\infty = -1/2$, so-called, Tetrahedron solution, $\theta_\infty = -2/3$ - Cube solution, another solution for $\theta_\infty = -2/3$ - Great Dodecahedron solution, $\theta_\infty = -4/5$ - Icosahedron solution, and $\theta_\infty = -2/5$ - Great Icosahedron solution. They also gave a rational parametrization for all solutions, except the Great Dodecahedron one. They mention that the last one is an algebraic function of genus 1 and produced via computer simulations incredibly long algebraic equation for this solution, which can be found in the preprint version of their work. Here we show that all algebraic solutions of zero genus mentioned above can be constructed via the method of $RS$-transformations whose $R$-parts are obtained as deformations of the dessins for the Platonic solids. Here we introduce also different kinds of vertex deformations of the dessins which we call Splits. Instead of the splits one can always consider the face deformations of the dual dessins.

3.1 Deformations of Folded Truncated Cube (Tetrahedron Solution)

Below we define, the folded truncated cube dessin. This a very simple dessin has four one dimensional deformations: twist, join, B- and W-splits. However, the first three of them generate equivalent $RS$-transformations: the B-split is dual to the twist and the latter defines the same face distribution as the join. The last two dessins can be continuously transformed one into another, when the blue vertex passes through the white one. Thus they correspond to two different branches of the same function $z(z_1)$ (as the function of the deformation parameter $s$) and define the same algebraic solution of the sixth Painlevé equation. So there are two nonequivalent tricolor dessins. With each one can be associated one (seed) algebraic solution. Both solutions were constructed via the method of $RS$-transformations in Section 3 of [2], where, of course, their deformation nature was not discussed.

The Belyi function corresponding to the truncated cube is of the type $R(4+1+1|2+2+2|3+3)$ (see [15]). Interchanging the colours of white and black vertices we go to the dual dessin. The folding procedure for the latter dessin is the transition to the depicted dessin based on the following decomposition of the types of the Belyi functions, $R(4 + 1 + 1|3 + 3|2 + 2 + 2) = R(2 + 1|3|2 + 2 + 2) \circ R(2|2)$.

\footnote{http://arXiv.org/abs/math/9806056}
The symbol of the $RS$-transformation associated with the $W$-split reads $\mathcal{R}\mathcal{S}_4^2 \left( \begin{array}{c|cc} \theta_0 \\ \begin{array}{c} 2+1 \\ 2+1 \\ 2+1 \end{array} \end{array} \right)$. The complete list of formulae related with this $RS$-transformation including the corresponding solution can be found in [2] Subsection 3.1.1. Below we consider only the twist, which generates Tetrahedron Solution of [6], treating it with the help of Theorem 2.1. Corresponding $RS$-symbol is $\mathcal{R}\mathcal{S}_4^2 \left( \begin{array}{c|cc} \theta_0 \\ \begin{array}{c} 1+1+1 \\ 1/2 \\ 2+1 \end{array} \end{array} \right)$.

$$z = (1 - s^2)^3 \frac{z_1(1 - z_1 - t)}{(1 - s^3) z_1 - 1}^2,$$

$$t = \frac{(2s + 1)}{(1 - s)(s + 1)^3},$$

$$y(t) = \frac{(2s + 1)}{(s + 1)(s^2 + s + 1)};$$

$$\hat{\theta}_0 = \hat{\theta}_1 = \hat{\theta}_t \in \mathbb{C}, \quad \hat{\theta}_{\infty} = 1/2.$$

Since $\hat{\theta}_0 = \hat{\theta}_1 = \hat{\theta}_t$ arbitrary and $y(t)$ does not depend on them we arrive at the following algebraic equation for $y(t)$:

$$\frac{t}{y^2} - \frac{(t - 1)}{(y - 1)^2} + \frac{t(t - 1)}{(y - t)^2} = 0;$$

Actually, the solution found and called in [6] Tetrahedron solution is more complicated and corresponds to a more special $\hat{\theta}$-tuple, $\hat{\theta}_0 = \hat{\theta}_1 = \hat{\theta}_t = 0, \quad \hat{\theta}_{\infty} = -1/2$. The latter solution is related with the one presented here via a Bäcklund transformation and the associated Fuchsian linear ODE has the same monodromy group. See details in Remark 2 of Subsection 3.1.2 of [2].

3.2 Deformations of Truncated Tetrahedron (Cube Solution)

There are two deformations of Truncated Tetrahedron with each one we can associate one solution. Both solutions were obtained via the method of $RS$-transformations in [10][2]. However here we provide the first one with a simpler parametrization than that given in the cited works and discuss application of the quadratic transformations to the second solution. The quadratic transformations are interesting in the context of finding algebraic solutions with nontrivial genus, as their application often leads to the solutions with elliptic parametrization.

The Belyi function for the Dual Truncated Tetrahedron reads,

$$z = \frac{64z_1(1 - z_1)^3}{(8z_1^2 + 20z_1 - 1)^2} \quad \text{and} \quad z - 1 = \frac{(8z_1 + 1)^3}{(8z_1^2 + 20z_1 - 1)^2}.$$  

$^4$There is a misprint in formula for $\rho$ given in this Subsection, denominator of this formula should be squared.
\[ t = \frac{1}{4}(2 - s)(s + 1)^2, \quad y(t) = \frac{1}{2s}(s + 1)(2 - s), \]
\[ \hat{\theta}_0 = \hat{\theta}_1 = \frac{1}{2} \hat{\theta}_t \in \mathbb{C}, \quad \hat{\theta}_\infty = \frac{2}{3}. \]

Similar to the Subsection 3.1 no efforts required to derive an algebraic equation for the function \( y(t) \)\(^5\),
\[ \frac{t}{y^2} - \frac{(t - 1)}{(y - 1)^2} + \frac{4t(t - 1)}{(y - t)^2} = 0; \]
Concerning the relation of this solution to Cube solution of \( \text{[a]} \) one can make analogous remark as at the end of Subsection 3.1: the latter Cube solution satisfies the sixth Painlevé equation only for \( \hat{\theta}_0 = \hat{\theta}_1 = \frac{1}{2} \hat{\theta}_t = 0 \) \( \theta = -2/3 \) and related to the one constructed here via a Bäcklund transformation.

2. \( B \)-Split (Cube Solution?): \( RS^2_2 \left( \begin{array}{c|c|c|c} 1/3 & 1/3 & 1/2 & 2+1+1 \\ \hline 3+1 & 3+1 & 2+1+1 & \end{array} \right) \).

\[ z = \frac{\rho(z_1 - a)^3z_1}{(z_1 - c)^2(z_1 - 1)}, \quad z - 1 = \frac{\rho(z_1 - b)^3(z_1 - t)}{(z_1 - c)^2(z_1 - 1)}, \quad \rho = \frac{(2s + 1)^3}{(3s + 1)^2(1 - 3s^2)^2}, \]
\[ a = \frac{(1 - 3s^2)}{(2s + 1)}(3s^2 + 2s + 1), \quad b = 1 - 3s^2, \quad c = \frac{(1 - 3s^2)}{(3s + 1)}(3s^2 + 3s + 1), \]
\[ t = \frac{(1 - 3s^2)}{(2s + 1)^3}(3s^2 + 3s + 1)^2, \quad y(t) = \frac{(3s^2 + 2s + 1)(3s^2 + 3s + 1)}{(2s + 1)(3s + 1)}, \]
\[ \hat{\theta}_0 = \frac{1}{3}, \quad \hat{\theta}_1 = \frac{1}{2}, \quad \hat{\theta}_t = \frac{1}{3}, \quad \hat{\theta}_\infty = \frac{1}{2}. \]

We can further apply the quadratic transformation given in Example 3 of Appendix \( \text{[a]} \) Put in this Example \( \hat{\theta}_t^0 = 2/3 \) and \( \hat{\theta}_\infty^0 = 0 \). Then we see that the \( \theta \)-tuple of Example 3 coincides with the \( \hat{\theta} \)-tuple for the solution obtained above, so that we can make inverse quadratic transformation of the above solution. Redenoting variables \( t_0 \) and \( y_0(t_0) \) back as \( t \ y(t) \) we arrive at the following solution:
\[ t = \frac{1}{4} + \frac{(3s+1)(s+1)(3s^3+3s^2+2s+1)^2}{4\sqrt{(s+1)(3s+2)}}, \quad y(t) = \frac{1}{2} - \frac{9(s+1)^4-2(3s+2)^2}{6(s+1)(3s+1)\sqrt{(s+1)(3s+2)}}, \]
\[ \hat{\theta}_0 = 0, \quad \hat{\theta}_1 = 0, \quad \hat{\theta}_t = \frac{2}{3}, \quad \hat{\theta}_\infty = 0. \]

We see that the resulted solution has an elliptic parametrization. However, the study of its Puiseux expansions at the singular points \( t = 0, 1, \infty \) shows that they looks very similar to the ones for Cube solution. So, most probably, this solution can be rationally parametrized and via a Bäcklund transformation interchanging \( (0, 0, \theta_t, 0) \mapsto (0, 0, 0, \theta_t) \), mapped to Cube solution.

So here we have described a mechanism of appearance of elliptic parametrization of algebraic solutions via an application of the quadratic transformations. This mechanism is seemingly different from the normalization one explained in Section \( \text{[b]} \). At the moment it is not clear whether such mechanisms could really produce an elliptic algebraic solution. As it is explained in Appendix \( \text{[a]} \) the quadratic transformations are also generated by \( RS \)-transformations. However, to get the complete list of the quadratic transformations by applying the method of \( RS \)-transformations to the \( 2 \times 2 \) matrix Fuchsian system

\(^5\)In \( \text{[a]} \) there is a misprint in the sign before the last term in this equation.
with four singularities in Jimbo-Miwa parametrization, we need the Okamoto transformation, which can be treated as a reparametrization of the Fuchsian system. So that we can use the method of $RS$-transformations to the Fuchsian system in the Okamoto parametrization also for production of the algebraic solutions. In the latter setting we do not need to separately apply any quadratic transformations: we just apply the same $RS$-transformations to two copies of the Fuchsian system in the different parametrizations. Therefore, if one can actually produce via the method of $RS$-transformations an algebraic solution with a nontrivial genus, then its appearance can be “explained” as a special case of the first normalization mechanism of Section 2.

Since throughout the paper we often reference the Okamoto transformation, it is probably worth to notice, that this transformation does not change $t$ and is rational in $y(t), y'(t),$ and $t$. Thus it cannot change a genus of our algebraic solutions.

### 3.3 Deformations of Truncated Cube

On the pictures below all nonequivalent one dimensional deformations of Truncated Cube are presented. On the first four dessins the white vertices are not indicated. It is interesting that here we have two nonequivalent $W$-Splits which define the same face distributions.

**Truncated Cube**

**CC-Join**

**LC-Join**

**B-Split**

**LW-Split**

**CW-Split**

1. **CC-Join.** The function $z(z_1)$, as the function of $y$, has two branches. The dessin corresponding to the second branch has the edge going from the upper B-vertex around the inner circle, joining with itself, and finally connecting with the B-vertex on the inner circle. We can associate two $RS$-transformations with this dessin: $RS_4^2 \left( \frac{1}{2} \left| \begin{array}{c} 1/2 \\ 2+2+1+1 \\ 3+3 \end{array} \right| \theta \right)$ and $RS_4^2 \left( \frac{\theta}{2} \left| \begin{array}{c} 1/2 \\ 2+2+2 \\ 3+3 \end{array} \right| \right)$. Notice that the $R$-type of these transformations can be presented as a composition of $R$-types of the degrees 3 and 2, $R(2+2+1+1|2+2+2|3+3) = R(2+1|2+1|3) \circ R(2)$. This results in that both transformations generate exactly the same solution as $RS_4^2 \left( \frac{\theta_0}{1+1} \left| \begin{array}{c} 1/2 \\ 2 \end{array} \right| \frac{\theta_{\infty}}{1+1} \right)$, but formally they give different and more restricted $\hat{\theta}$-tuple than the latter. The reader can find this solution in [2] Section 2.

2. **LC-Join.** The function $z(z_1)$ as the function of $y$ has three branches. Two other branches correspond to two Crosses: one of the inner circle another of the external. Cor-
responding RS-transformation is $RS^2_4 \left( \begin{array}{c} \theta_0 \\ 3+1+1+1 \end{array} \right), \left( \begin{array}{c} 1/2 \\ 2+2+2 \\ 1/3 \\ 3+3 \end{array} \right)$.

$$z = -\frac{4(p_1 - p_3)p_1^4p_3^4}{3^6(4s+1)^9(10s+1)^9} (z_1(z_1 - 1)(z_1 - t) - 1 - \frac{(z_1^3 + b_2z_1^2 + b_1z_1 + b_0)^2}{(z_1 - c_1)3(z_1 - c_2)^3},$$

$$b_0 = \frac{s^3(11s + 2)^3p_3^3}{(4s + 1)^6(10s + 1)^6}, \quad b_1 = \frac{p_3^2(p_1^4 + p_2^4)}{3^4(4s + 1)^6(10s + 1)^6}, \quad b_2 = -\frac{p_3(p_1^2 + p_2^2)}{3^4(4s + 1)^6(10s + 1)^6},$$

$$c_1 = \frac{(11s + 2)^2p_3}{3^2(4s + 1)^3(10s + 1)}; \quad c_2 = \frac{3^2s^2p_3}{(4s + 1)(10s + 1)^3},$$

$$t = \frac{3^3s^3(11s + 2)^3p_3}{(p_1 - p_3)(4s + 1)^3(10s + 1)^3}, \quad \hat{y}(t) = \frac{3^2s^2(11s + 2)^2}{(p_1 - p_3)(4s + 1)(10s + 1)}, \quad \hat{\theta}_0 = \hat{\theta}_1 = \hat{\theta}_t = \theta_0 \in \mathbb{C}, \quad \hat{\theta}_\infty = 3\theta_0 + 1.$$
produces a solution for \((0,0,0,2/3)\). This construction should be examined in view of Great Dodecahedron solution.

The solution associated with the first two RS-transformation can be found with the help of Theorem 2.4

\[
\begin{align*}
    z &= \rho \frac{(z_1-a)^4 z_1(z_1-1)}{(z_1-c)^3(z_1-t)}, \\
    \rho &= \frac{2^6 s^6}{(3s^2-1)(s^2+1)^3}, \\
    a &= \frac{(3s-1)(s+1)^3}{24^3 s^3}, \\
    b_0 &= -\frac{(2s-1)(s+1)^8}{2^9 s^6}, \\
    b_1 &= \frac{(15s^4+36s^3-22s^2+4s-1)(s+1)^4}{2^8 s^6}, \\
    b_2 &= -\frac{(3s^4+12s^3+6s^2-1)}{2^3 s^3}, \\
    c &= \frac{(s+1)^4}{2^4 s(s^2+1)}, \\
    t &= \frac{(s+1)^2(2s-1)^2}{8s^3(3s^2-1)}, \\
    y(t) &= \frac{(3s-1)(2s-1)(s+1)^3}{4s(3s^2-1)(s^2+1)}, \\
    \hat{\theta}_0 &= \frac{1}{4}, \\
    \hat{\theta}_1 &= \frac{1}{4}, \\
    \hat{\theta}_2 &= \frac{1}{3}, \\
    \hat{\theta}_\infty &= \frac{1}{3}.
\end{align*}
\]

Starting from this solution and using the quadratic transformations and Okamoto transformation (see Appendix A) one can obtain algebraic solutions for the following \(\hat{\theta}\)-tuples:

\[
\begin{align*}
    &\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{4}\right), \\
    &\left(0, 0, \frac{1}{12}, \frac{7}{12}\right), \\
    &\left(\frac{1}{24}, \frac{1}{24}, \frac{5}{24}, \frac{19}{24}\right), \\
    &\left(\frac{7}{24}, \frac{7}{24}, \frac{11}{24}, \frac{13}{24}\right).
\end{align*}
\]

The solution corresponding to the second RS-transformation is a renormalization of the function \(z(z_1)\):

\[
\bar{z} = z(M^{-1}(\bar{z}_1)), \quad \bar{z}_1 = M(z_1) = \frac{(1-c)z_1}{z_1-c}; \quad \bar{t} = M(t), \quad \bar{y} = M(y).
\]

Redenoting back \(\bar{t} \to t\) and \(\bar{y} \to y\) we get:

\[
\begin{align*}
    t &= -\frac{(2s-1)^2}{8s}, \\
    y(t) &= \frac{(s-1)(3s-1)(2s-1)}{12s(s^2+1)}, \\
    \hat{\theta}_0 &= \frac{1}{4}, \\
    \hat{\theta}_1 &= \frac{1}{4}, \\
    \hat{\theta}_2 &= \frac{1}{2}, \\
    \hat{\theta}_\infty &= -\frac{1}{2}.
\end{align*}
\]

To the general solution for the latter \(\hat{\theta}\)-tuple all kinds of the known transformations are applicable and generate solutions for the following \(\hat{\theta}\)-tuples:

\[
\begin{align*}
    &\left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right), \\
    &\left(0, 0, \frac{1}{2}, 0\right), \\
    &\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right), \\
    &\left(0, 0, \frac{1}{4}, \frac{1}{4}\right), \\
    &\left(\frac{1}{8}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8}\right), \\
    &\left(\frac{3}{8}, \frac{3}{8}, \frac{3}{8}, \frac{3}{8}\right).
\end{align*}
\]

Care should be taken to check whether application of some transformations will not degenerate this particular solution, i.e., do not map them to 0, 1, or \(\infty\).

4. LW-Split. The function \(z(z_1)\) as the function of \(y\) has two branches. Another branch corresponds to the twist of the dessin which is a multiplication of Folded Truncated Cube and the segment corresponding to the following composition of the types for the associated Belyi functions, \(R(4 + 2|2+2+1+1|3+3) = R(2+1|2+1|3) \circ R(2|2)\) (see the picture). The type of the resulted dessin is also a composition of the types of degrees 3 and 2, \(R(4+1+1|2+2+1+1|3+3) = R(2+1|2+1|3) \circ R(2)\). Therefore LW-Split dessin generate exactly the same solution as the one in item 1 (CC-Join).
and we refer again to Section 2 of \cite{2}. The RS-transformation associated with the LW-split is \( RS_3^2 \left( \begin{array}{ccc} 1/4 & 1/2 & 1/3 \\ 4+1+1 & 2+2+1+1 & 3+3 \end{array} \right) \). Therefore, this solution corresponds to the following \( \hat{\theta} \)-tuple \((1/4,1/4,1/2,1/2)\). However, since the same solution can be produced via a simpler RS-transformation, it solves Equation \((2.3)\) for a more general \( \hat{\theta} \)-tuple. It is actually interesting to compare this solution with the one for CW-Split in the next item as the latter has exactly the same RS-symbol. Therefore, we provide below the details of this RS-transformation. Our solution differs from the one cited in \cite{2} by fractional linear transformations of \( y \) and \( t \).

\[
\begin{align*}
z &= \frac{(z_1 - a)^4 z_1 (z_1 - t)}{(z_1 - c_1)^3 (z_1 - c_2)^3}, \\
z - 1 &= - \frac{3(s^2 - 4s + 1)^2}{s^2(s + 2)^2} \frac{(z_1^2 + b_1 z_1 + b_0)(z_1 - 1)}{(z_1 - c_1)^3 (z_1 - c_2)^3}, \\
a &= \frac{(s^2 + 4s + 1)}{(s + 2)s}, \\
c_1 &= \frac{(s^2 + 4s + 1)}{(s + 2)^2}, \\
b_0 &= \frac{(s^2 + 4s + 1)^2}{9s^2(s + 2)^2}, \\
b_1 &= -\frac{2(5s^2 + 2s - 4)(s^2 + 4s + 1)}{9s^2(s + 2)^2}, \\
t &= \frac{(s^2 - 1)(s^2 + 4s + 1)}{s^2(s + 2)^2}, \\
y(t) &= \frac{(s^2 - 1)}{(s + 2)s}, \\
\hat{\theta}_0 = \hat{\theta}_t \in \mathbb{C}, \\
\hat{\theta}_1 = \hat{\theta}_\infty - 1 \in \mathbb{C}.
\end{align*}
\]

The algebraic equation for \( y(t) \) is very simple, \( (y - 1)^2 = 1 - t \).

5. **CW-Split.** Exactly the same deformation as CW-Split, as well as a few others which are related with different branches of the same function \( z(z_1) \) can be also obtained as a cross and joins of the dessin shown on the picture. RS-symbol of the transformation associated with CW-Split reads exactly the same as for LW-Split, \( RS_3^2 \left( \begin{array}{ccc} 1/4 & 1/2 & 1/3 \\ 4+1+1 & 2+2+1+1 & 3+3 \end{array} \right) \). However its R-part the function \( z(z_1) \) as the function of \( y \) is different! It is important to mention that now we keep the denominator of function \( z(z_1) \) in the non-factorized form. If we factorize it, like in the previous item, then only an elliptic parametrization of the function \( z(z_1) \) is possible.

\[
\begin{align*}
z &= \frac{(z_1 - a)^4 z_1 (z_1 - t)}{(z_1^2 + c_1 z_1 + c_0)^3}, \\
z - 1 &= - \frac{3(s^2 - 2^2)(s^2 - 4s - 2)^2}{16(s + 1)^3(s - 1)^3} \frac{(z_1^2 + b_1 z_1 + b_0)(z_1 - 1)}{(z_1^2 + c_1 z_1 + c_0)^3}, \\
a &= \frac{(s^2 + 2s + 2)(s^2 - 2)}{4(s - 1)^3}, \\
c_1 &= -\frac{(s^2 + 2)(s^2 - 2)}{12(s - 1)^3(s + 1)}, \\
b_0 &= -\frac{(s^2 - 2)^2 (s^2 + 2)^3}{144(s^2 - 4s - 2)(s - 1)^6}, \\
b_1 &= -\frac{(s^2 - 2)(s^5 - 6s^4 + 10s^3 - 32s^2 + 12s - 20)}{18(s^2 - 4s - 2)(s - 1)^3}, \\
t &= \frac{(s^2 - 2)(s^2 + 2)^3}{16(s + 1)^3(s - 1)^3}, \\
y(t) &= -\frac{(s^2 - 2s + 2)(s^2 + 2)^2}{4(s + 1)(s^2 - 4s - 2)(s - 1)^2}, \\
\hat{\theta}_0 = \hat{\theta}_t = \frac{1}{4}, \\
\hat{\theta}_1 = \hat{\theta}_\infty = \frac{1}{7}.
\end{align*}
\]
We can further apply to this solution a quadratic transformation, the inverse to the one in Example 3 of Appendix A. As the result we get a new solution again denoted as $y(t)$:

$$
t = \frac{1}{2} + \frac{(s^8 - 28s^6 + 96s^4 - 112s^2 + 16)}{16s\sqrt{(1-s^2)(s^2-4)}}, \quad y(t) = \frac{1}{2} + \frac{s(7s^4 - 44s^2 + 28)}{2((s^2-2)\sqrt{(1-s^2)(s^2-4)})},
$$

$$
\hat{\theta}_0 = 0, \quad \hat{\theta}_1 = 0, \quad \hat{\theta}_t = \frac{1}{2}, \quad \hat{\theta}_\infty = 0.
$$

Due to the obvious symmetry we can use the Zhukovski transformation to parametrize this solution rationally:

$$
\frac{s^2}{2} + \frac{2}{s^2} = \frac{5 - s_1^2}{2}; \quad t = \frac{(s_1 - 1)(s_1 + 3)^3}{16s_1^3}, \quad y(t) = \frac{(s_1 + 1)(s_1 + 3)^2}{2s_1(15 + s_1^2)}.
$$

This solution can be mapped further to produce Tetrahedron solution considered in Subsection 3.1. The $\hat{\theta}$-tuple for LW-Split solution of the previous item are such that one can also try to apply to it the same quadratic transformation, however it is exactly one of those exceptional solutions for which this transformation fails: it maps this solution into the critical values of the sixth Painlevé equation \cite{23}, depending on the choice of the branches to 0 or 1.

### 3.4 Deformations of Truncated Dodecahedron (Icosahedron Solutions)

Truncated Icosahedron and its dual solid Truncate Dodecahedron and their Belyi functions were introduced by Magot and Zvonkin \cite{15} in connection with their study of the Belyi functions of the Archimedian solids. Here we consider only face deformations of the corresponding dessins. They are indicated on Figure 2.

![Truncated Dodecahedron](image)

Figure 2: Some face deformations of the dessin for the Belyi function of Truncated Dodecahedron $R(5 + 5 + 1 + 1 | 2 + 2 + 2 + 2 + 2 + 2 | 3 + 3 + 3 + 3)$.

Actually the last dessin on Figure 2 is a join of the dessin for the Belyi function of the type $R(8 + 2 + 1 + 1 | 2 + \ldots + 2 | 3 + \ldots + 3)$ (see the picture), rather than the cross of Truncated Dodecahedron. However, we consider it here since it is interesting to get a solution for the $\hat{\theta}$-tuple proportional to 1/7.

1. **Twist.** This deformation produces both, Icosahedron and Great Icosahedron solutions
of $\mathbb{E}$ and also one more solution which is related with Tetrahedron solution of $\mathbb{E}$.
\[
z = \frac{3^3(s^2 + 3)^5(s^2 - 5)^5(s^2 + 4s - 1)^5(s^2 - 4s - 1)}{2^{10}(s + 3)^{12}(s - 1)^{20}} (z_1 - a)^5 z_1(z_1 - 1)(z_1 - t) \\
a = \frac{2^4(s^2 - 5)}{(s - 1)(s^2 + 3)(s + 3)^3}, \quad c_3 = -\frac{(s^2 - 5)(s^6 + 4s^5 - 3s^4 - 8s^3 + 115s^2 - 60s + 15)}{(s - 1)^5(s + 3)^3},
\]
\[
c_2 = \frac{(s^2 - 5)^2}{2^4(s - 1)^{10}(s + 3)^6} (s^{12} + 8s^{11} + 10s^{10} - 40s^9 + 1135s^8 + 3408s^7 - 10036s^6 - 14160s^5 + 71055s^4 - 78040s^3 + 39050s^2 - 9480s + 1185), \quad c_0 = \frac{2^8s^2(s^2 - 5)^4}{(s + 3)^{10}(s - 1)^{10}},
\]
\[
c_1 = \frac{8(s^2 - 5)^3(15 - 105s + 525s^2 - 705s^3 + 107s^4 + 461s^5 + 183s^6 + 29s^7 + 2s^8)}{(s - 1)^{10}(s + 3)^9}
\]

Consider the following RS-symbol: $RS_4^2 \left( \begin{array}{ccc} 1/5 \\ 5 + 4 + 1 + 1 + 1 \\ 1/2 \\ 2 + \ldots + 2 \\ 3 + \ldots + 3 \\ 1/3 \\ 6 \\ 4 \end{array} \right)$, the corresponding solution found via Theorem 2.1 is as follows:
\[
t = -\frac{2^8s^3(s^2 - 5)}{(s - 1)^5(s + 3)^3(s^2 - 4s - 1)}, \quad y(t) = -\frac{2^6s^2}{(s - 1)(s + 3)(s^2 + 3)(s^2 - 4s - 1)},
\]
\[
\theta_0 = \theta_1 = \theta_t = \theta_\infty = \frac{1}{5}.
\]

This solution can be mapped by the Okamoto transformation (Appendix A) to the one for the $\theta$-tuple $(0, 0, 0, 2/5)$ and then, by a Bäcklund transformation, to Great Icosahedron solution of $\mathbb{E}$ corresponding to the following $\theta$-tuple, $(0, 0, 0, -2/5)$.

Another $RS$-transformation that is associated with this deformation is $RS_4^2 \left( \begin{array}{ccc} 2/5 \\ 5 + 4 + 1 + 1 + 1 \\ 1/2 \\ 2 + \ldots + 2 \\ 3 + \ldots + 3 \\ 1/3 \\ 6 \\ 4 \end{array} \right)$. To find explicit formulae for the corresponding solution of the sixth Painlevé equation one has to find the $S$-part of this $RS$-transformation similar to the examples of $\mathbb{E}$. The corresponding $\theta$-tuple reads $(2/5, 2/5, 2/5, 2/5)$. By means of the same transformations as for the previous solution, this one can be mapped to Icosahedron Solution of $\mathbb{E}$ corresponding to the following $\theta$-tuple, $(0, 0, 0, -4/5)$.

We can construct one more $RS$-transformation by rearranging the normalization of the function $z(z_1)$, i.e., with the help of the function $\tilde{z}(\tilde{z}_1) = z(M^{-1}(\tilde{z}_1))$, where the fractional linear transformation $M$ is defined as follows, $\tilde{z}_1 \equiv M(z_1) = \frac{(1-a)z_1}{z_1 - a}$. Then the function $\tilde{y}(\tilde{t})$, where $\tilde{t} = M(t)$ and $\tilde{y} = M(y)$ is a new solution of the sixth Painlevé equation $\mathbb{E}$. Omitting the sign $\sim$ in the notation of the new solution we get:
\[
t = \frac{2^4s^3}{(s + 3)^3(s - 1)}, \quad y(t) = \frac{4(s^2 + 4s - 1)s^2}{5(s + 3)(s^2 + 3)(s - 1)},
\]
\[
\theta_0 = \theta_1 = \theta_t = \frac{1}{4}, \quad \theta_\infty = -\frac{1}{4}.
\]

This solution by the Okamoto transformation can be mapped to the one for the $\hat{\theta}$-tuple $(0, 0, 0, 1/2)$ and the latter, according to $\mathbb{E}$, has to coincide with Tetrahedron Solution.
The first transformation is generated by the symbol
\[ RS_1^2 \begin{pmatrix} 1/5 \\ 5 + 3 + 2 + 1 + 1 \\ 2 + \ldots + 2 \\ 3 + \ldots + 3 \end{pmatrix} \]
whose \( R \)-part reads as follows,
\[
\begin{align*}
    z &= 2^5 3^3 (s - 2)^5 \left( (z_1 - a)^5 (z_1 - 1)^2 z_1 (z_1 - t) \right) / \left( (s + 3)^9 (s - 2)^6 (z_1^4 + c_3 z_1^3 + c_2 z_1^2 + c_1 z_1 + c_0)^3 \right), \\
    a &= (s - 1) (s^2 - 5) / 2^3 (s - 2)^2, \\
    c_0 &= s^2 (s^2 - 5)^4 / 2^4 (s + 3)^4 (s - 2)^6, \\
    c_1 &= -8 s^4 - 35 s^3 + 65 s^2 - 75 s + 45 (s^2 - 5)^3 / 4 (s + 3)^4 (s - 2)^6, \\
    c_2 &= 5(s - 1)(2s^3 + 2s^2 - 3s - 9)(s^2 - 5)^2 / 2(s - 2)^4 (s + 3)^4, \\
    c_3 &= -2(s^2 - 5) (2s^3 + 5 s^2 - 15) / (s + 3)^3 (s - 2)^2.
\end{align*}
\]
Then Theorem 2.1 provides the corresponding solution,
\[
\begin{align*}
    t &= 2 s^3 / (s - 2)^2 (s + 3)^3, \\
    y(t) &= s^2 - 1 / 3(s - 2) (s + 3), \\
    \hat{\theta}_0 &= 1 / 5, \\
    \hat{\theta}_1 &= 2 / 5, \\
    \hat{\theta}_L &= 1 / 3, \\
    \hat{\theta}_\infty &= 2 / 3.
\end{align*}
\]
With this \( R \)-part one can associate another \( RS \)-transformation by choosing \( 2/5 \) instead of \( 1/5 \) in the first box of the \( RS \)-symbol written above. However, to find the corresponding solution of the sixth Painlevé equation, one has to construct explicitly the \( S \)-part of this transformation. The corresponding \( \hat{\theta} \)-tuple of the latter solution is \( (6/5, 4/5, 2/5, 2/5) \). By means of the certain transformations this solution can be mapped to the one corresponding to exactly the same \( \hat{\theta} \)-tuple as the the first solution \( y(t) \). A natural question, whether these solutions are different, requires a further investigation. It is worth to notice that by the Okamoto transformation solution (3.3) one can associate another \( RS \)-part as the the first solution \( y(t) \). The latter solution by the quadratic transformation given in item 3 of Appendix A can be further transformed to solutions for the following \( \hat{\theta} \)-tuples:
\[
\left( \frac{1}{10}, \frac{1}{5}, \frac{1}{5}, \frac{4}{5} \right) \quad \text{and} \quad \left( \frac{3}{10}, \frac{2}{5}, \frac{3}{2}, \frac{3}{5} \right).
\]
Another transformation associated with this join is defined by the symbol
\[ RS_1^2 \begin{pmatrix} 1/3 \\ 5 + 3 + 2 + 1 + 1 \\ 2 + \ldots + 2 \\ 3 + \ldots + 3 \end{pmatrix} \] Its \( R \)-part, \( \hat{z}(\tilde{z}_1) \), is obtained from the \( R \)-part of the first \( RS \)-transformation by the following renormalizing fractional linear transformation,
\[
\hat{z}(\tilde{z}_1) = z \left( M^{-1}(\tilde{z}_1) \right), \quad \hat{z}_1 = M(z_1) = \frac{(1 - a) z_1}{z_1 - a}.
\]
The solution is \( \hat{y}(\hat{t}) \), where \( \hat{t} = M(t) \) and \( \hat{y} = M(y(t)) \), with \( t \) and \( y(t) \) are defined by Equations (3.3). As usual omitting the sign \( \sim \), we have the following explicit parametric form for \( \hat{y}(\hat{t}) \):
\[
\begin{align*}
    t &= 2 s^3 / (s + 1)(s - 2)^2, \\
    y(t) &= -\frac{(s - 3) s^2}{5(s + 1)(s - 2)}, \\
    \hat{\theta}_0 &= 1 / 3, \\
    \hat{\theta}_1 &= 2 / 3, \\
    \hat{\theta}_L &= 1 / 3, \\
    \hat{\theta}_\infty &= 2 / 3.
\end{align*}
\]
Finally, there is one more normalization of the function \( z(z_1) \) which produces one more RS-transformation, \( RS_4^1 \left( \begin{array}{ccc} 1/2 & 1/2 & 1/3 \\ 5 + 3 + 2 + 1 + 1 & 2 + \ldots + 2 & 3 + \ldots + 3 \\ 6 \\ 4 \end{array} \right) \). Normalizing transformation reads,

\[
\tilde{z}(\tilde{z}_1) = z \left( M^{-1}(\tilde{z}_1) \right), \quad \tilde{z}_1 = M(z_1) \equiv \frac{z_1}{z_1 - a}.
\]

The corresponding solution of the sixth Painlevé equation is given in terms of the latter fractional linear transformation \( M \) and the solution \( y(t) \) (see Equations \( 3.3 \)) exactly by the same formulae as the ones for \( \tilde{y}(\tilde{t}) \) from the previous paragraph,

\[
t = -\frac{16s^3}{(s + 1)(s - 3)^3}, \quad y = \frac{8s^2(s - 2)}{5(s + 1)(s - 3)^2},
\]

\[
\hat{\theta}_0 = \frac{1}{2}, \quad \hat{\theta}_1 = \frac{3}{2}, \quad \hat{\theta}_t = \frac{1}{2}, \quad \hat{\theta}_\infty = -\frac{3}{2}.
\]

3. Cross. With this dessin we can associate four RS-transformations. The first one is \( RS_4^2 \left( \begin{array}{ccc} 1/7 & 1/2 & 1/3 \\ 7 + 2 + 1 + 1 + 1 & 2 + \ldots + 2 & 3 + \ldots + 3 \\ 6 \\ 4 \end{array} \right) \). Its \( R \)-part and the corresponding algebraic solution are as follows:

\[
z = -\frac{3^3(3s^2 + 1)^7(z_1 - a)^7z_1(z_1 - 1)(z_1 - t)}{(7s^2 + 1)^4 z_1^4 + c_3 z_1^3 c_2 z_1^2 + c_1 z_1 + c_0)^3},
\]

\[
a = -\frac{(s - 1)(2s^2 + s + 1)^2}{2(3s^2 + 1)}, \quad c_3 = \frac{2(s - 1)(57s^6 + 57s^5 + 71s^4 + 22s^3 + 15s^2 + s + 1)}{(7s^2 + 1)^2},
\]

\[
c_2 = -\frac{42s^6 - 42s^5 + 16s^4 - 44s^3 - 16s^2 - 10s + 5)(2s^2 + s + 1)^2}{2(7s^2 + 1)^2},
\]

\[
c_1 = -\frac{7(s - 1)(3s^4 - 3s^3 + 6s^2 - 3s + 1)(2s^2 + s + 1)^4}{2(7s^2 + 1)^2}, \quad c_0 = \frac{(3s - 1)^2(2s^2 + s + 1)^6}{16(7s^2 + 1)^2},
\]

\[
t = -\frac{(2s^2 + s + 1)^2(3s - 1)^3}{2(7s^2 + 1)^2}, \quad y(t) = -\frac{(s - 1)(2s^2 + s + 1)(3s - 1)^2}{2(7s^2 + 1)(3s^2 + 1)}, \quad (3.4)
\]

\[
\hat{\theta}_0 = \hat{\theta}_1 = \hat{\theta}_t = \frac{1}{7}, \quad \hat{\theta}_\infty = \frac{5}{7}.
\]

Two other RS-transformations are based on the same \( R \)-part but the first box of their RS-symbol contains \( 2/7 \) and \( 3/7 \) instead of \( 1/7 \). To get explicit formulae for the corresponding solutions one has to construct the \( S \)-part of these transformations. Here we only mention that these solutions corresponds to the following \( \theta \)-tuples:

\[
\left( \begin{array}{cccc} 2 & 2 & 2 & 4 \\ 7 & 7 & 7 & 7 \end{array} \right) \quad \text{and} \quad \left( \begin{array}{cccc} 3 & 3 & 3 & 1 \\ 7 & 7 & 7 & 7 \end{array} \right).
\]

Remark 3.1 It is interesting to note that if we apply the Okamoto and certain other transformations to solution \( \tilde{y}(\tilde{t}) \) we can get algebraic solutions exactly for both \( \hat{\theta} \)-tuples written above! It should be however checked whether solutions constructed via the Okamoto transformation and RS-transformations coincide?
The fourth RS-transformation is 
\[ RS^2_4 \begin{pmatrix} \frac{1}{2} & 1/2 & 1/3 \\ 7 + 2 + 1 + 1 & 2 + \ldots + 2 & 3 + \ldots + 3 \end{pmatrix}. \]

Its R-part is the following renormalization \( \tilde{z}(\tilde{z}_1) \) of the function \( z(z_1) \):
\[ \tilde{z}(\tilde{z}_1) = z(M^{-1}(\tilde{z}_1)), \quad \tilde{z}_1 = M(z_1) = \frac{(1-a)z_1}{z_1 - a}. \]

The new solution \( \tilde{y}(\tilde{t}) \) is given by the usual formulae: 
\[ \tilde{t} = M(t), \quad \tilde{y} = M(y(t)), \]
where \( t \) and \( y(t) \) are given by Equations (3.4). Explicit form for \( \tilde{y}(\tilde{t}) \) (with the omitted sign \( \sim \)) is:
\[ t = \frac{(3s - 1)^3(s + 1)}{16s}, \quad y(t) = -\frac{(2s^2 - s + 1)(3s - 1)^2}{14s(3s^2 + 1)}, \]
\[ \hat{\theta}_0 = \hat{\theta}_1 = \hat{\theta}_t = \frac{1}{2}, \quad \hat{\theta}_\infty = -\frac{5}{2}. \]

### 4 The Schwarz Cluster

Recall the Euler equation for the Gauss hypergeometric function,
\[ z(1-z) \frac{d^2 u}{dz^2} + (c - (a + b + 1)z) \frac{d u}{d z} - abu = 0. \quad (4.1) \]

Define the parameters,
\[ \lambda = 1 - c, \quad \mu = b - a, \quad \nu = c - a - b, \]

and introduce in the set of triples \((\lambda, \mu, \nu)\) an equivalence relation: two triples are called equivalent iff one can be transformed into another by a permutation and transformation of the form
\[ \lambda \to l \pm \lambda, \quad \mu \to m \pm \mu, \quad \nu \to n \pm \nu, \]
where the integers \(l, m, n\) are such that \(l + m + n\) is an even number. If a triple is equivalent to the one that obey the following relation \(\lambda + \mu + \nu = 1\), than it is called degenerate. In the degenerate case one independent solution is an elementary function, the other can be expressed in terms of elementary and the incomplete Beta functions via an application of a finite sequence of simple transformations.

H. A. Schwarz [19] proved that in the non-degenerate case the general solution of Equation (4.1) is an algebraic function iff the corresponding parameters \(\lambda, \mu, \nu\) can be reduced to one of the 15 cases listed in the following table.

| \(N\) | \(\lambda\) | \(\mu\) | \(\nu\) | \(N\) | \(\lambda\) | \(\mu\) | \(\nu\) | \(N\) | \(\lambda\) | \(\mu\) | \(\nu\) | \(N\) | \(\lambda\) | \(\mu\) | \(\nu\) |
|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|
| 1     | 1/3   | 1/3   | 1/3   | 4     | 1/3   | 1/3   | 1/3   | 7     | 2/3   | 1/3   | 1/3   | 10    | 3/5   | 3/5   | 1/5   |
| 2     | 1/3   | 1/3   | 1/3   | 5     | 2/3   | 1/3   | 1/3   | 8     | 3/5   | 3/5   | 1/3   | 11    | 2/3   | 1/3   | 1/3   |
| 3     | 1/3   | 1/3   | 1/3   | 6     | 1/2   | 1/3   | 1/5   | 9     | 1/2   | 1/3   | 1/5   | 12    | 2/3   | 1/3   | 1/5   |
|       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |
|       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |
|       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |
|       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |

Table 1: The Schwarz List

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In Table 1 integers $p$ and $n$ are such that $2p \leq n$. Further in the article instead of the scalar form of the hypergeometric equation (4.1) we refer to its matrix form

$$
\frac{d}{dz} \Psi = \left( \frac{A_0}{z} + \frac{A_1}{z-1} \right) \Psi,
$$

where $A_0$ and $A_1 \in sl_2(\mathbb{C})$. We also assume that $A_0 + A_1 = -\frac{\theta_\infty}{2}\sigma_3$, with $\theta_\infty \in \mathbb{C}$ and $\sigma_3 = \text{diag}\{1, -1\}$. Due to these conditions and the freedom in normalization, $\Psi - > \exp\{c\sigma_3\} \Psi$, $c \in \mathbb{C}$, the matrices $A_0$ and $A_1$, can be parametrized by the corresponding formal monodromies: $\theta_0$, $\theta_1$, and $\theta_\infty$, where $\pm \theta_k/2$ are the eigenvalues of the matrices $A_k$. Here we assume that $\theta_\infty \neq 0$ and $A_0$ is not a diagonal matrix. Under these assumptions a fundamental solution $\Psi$ can be presented in terms of independent solutions of Equation (4.1), for the parameters:

$$
\lambda = \theta_\infty - 1, \quad \mu = \theta_0, \quad \nu = \theta_1.
$$

This parametrization is not used here and therefore omitted; it can be found, say, in [1]. In the following instead of the triples $(\lambda, \mu, \nu)$ we always use the $\theta$-triples, $(\theta_0, \theta_1, \theta_\infty)$, note $-1$ in Equations (4.3). On the set of the $\theta$-triples we assume the same equivalence relation as for $(\lambda, \mu, \nu)$ triples and make no difference between the triples belonging to the same equivalence class.

The degenerate case of Equation (4.1) in the matrix framework is described in the following way. Let the matrix $A_0$, and hence $A_1$, be triangular, but not diagonal. The lower triangular case corresponds to $a = 0$ and the upper to $b = 0$ in the parametrization considered in [4]. The triangular structure implies the following relation for the $\theta$-triple: $\theta_0 + \theta_1 + \theta_\infty = 0$. The general degenerate case is obtained from the one of the triangular cases mentioned above by an application of a finite number of Schlesinger transformations. These transformations change the $\theta$-triple such that they satisfy the equation, $\theta_0 + \theta_1 + \theta_\infty = 2k$ with some $k \in \mathbb{Z}$. If $k \neq 0$ the corresponding Equation (4.2) does not have a triangular structure, however its monodromy group remains isomorphic to the group of the triangular equation and has in particular the same triangular representation in a proper normalization. In the degenerate case the general solution can be also algebraic in infinite number of cases they can be deduced from the certain cases when the incomplete Beta function is algebraic.

Under our conditions Equation (4.2) has three singular points at 0, 1, and $\infty$.

**Theorem 4.1** All Equations (4.2) corresponding to the parameters from the Schwarz List (Table 1) can be obtained as inverse RS-transformations or compositions of RS-transformations and inverse RS-transformations of the $2 \times 2$ matrix Fuchsian ODE with two singular points,

$$
\frac{d}{dz} \Phi = \frac{A}{z} \Phi, \quad \text{where} \quad A \in sl_2(\mathbb{C}),
$$

has rational eigenvalues. Moreover, R-parts of the RS-transformations are the Belyi functions.

**Proof of Theorem 4.1** We divide the 15 cases of Table 1 into two sets; the first one contains all cases with $\lambda = 1/2$ except the case 9 and the second all the other cases. In Proposition 4.1 we show how to construct RS-transformations from Equation (4.2) to Equation (4.4). This proves that every Equation (4.2) corresponding to the first set is RS-inverse to Equation (4.4). In Proposition 4.2 we prove that there exist RS-transformations
from Equations (4.2) corresponding to the first set into Equations (4.2) for the second one.

Remark 4.1 The fundamental solution of Equation (4.4) is \( \Phi = Gz^{r_{\sigma}}C \), where \( G \) and \( C \in SL(2, \mathbb{C}) \), and \( r \) is a rational number. All RS-transformations that appear in the following Propositions 4.1–4.3 can be constructed explicitly, therefore, in fact, our proof provides an explicit construction of all general algebraic solutions of Equation (4.1).

Proposition 4.1 If a \( \theta \)-triple, defining Equation (4.2), coincides with one of \( \theta \)-triples corresponding to the rows of Table 1 with \( \lambda = 1/2 \) except the row 9, i.e., the rows 1, 2, 4, 6, and 14, then there exists an RS-transformation, whose \( R \)-part is the Belyi function, which maps Equation (4.2) to Fuchsian Equation (4.4).

Proof. Below we consider each of the 5 cases of Equation (4.2) in the corresponding item.

1. Case 1. Actually, the general solution of Equation (4.1) for the triple \((1/2, \theta_1, 1/2)\) with an arbitrary \( \theta_1 \in \mathbb{C} \) is an elementary function. This function is algebraic for rational values of \( \theta_1 \). These facts can be observed by application of transformation

\[
RS_2^2 \begin{pmatrix} 1/2 \\ 2 \\ 1 + 1 \\ 2 \end{pmatrix} \theta_1 \begin{pmatrix} 1/2 \\ 3 + 1 \\ 2 + 2 \\ 2 + 3 \end{pmatrix} \]

with the \( R \)-part \( z = z_1^2 \), which maps Equation (4.2) to Equation (4.4).

2. Case 2. Here we use the transformation

\[
RS_2^2 \begin{pmatrix} 1/3 \\ 3 + 1 \\ 2 + 2 \\ 2 + 3 \end{pmatrix} \frac{1}{3 + 1} \frac{1/2}{2 + 2} \frac{1/3}{2 + 3}, \]

with the Belyi function corresponding to the truncated tetrahedron [15], \( z = -\frac{64(z_1^2+1)^3}{(z_1(z_1-8))^3} \).

3. Case 4. The required transformation is

\[
RS_2^2 \begin{pmatrix} 1/4 \\ 4 + 1 + 1 \\ 2 + 2 + 2 \\ 2 + 3 \\ 3 + 3 \\ 3 \end{pmatrix} \frac{1}{4 + 1 + 1} \frac{1/2}{2 + 2 + 2} \frac{1/3}{2 + 3}, \]

with the Belyi function corresponding to the truncated cube [15], \( z = -\frac{108(z_1^2+1)^4z_1}{(z_1^2-12z_1+1)^3} \).

4. Cases 6 and 14. Transformations

\[
RS_2^2 \begin{pmatrix} 1/3 \\ 3 + \ldots + 3 \\ 2 + \ldots + 2 \\ 2 + 5 + 5 + 1 + 1 \end{pmatrix} \frac{1/3}{3 + \ldots + 3} \frac{1/2}{2 + \ldots + 2} \frac{1/5}{2 + 5 + 5 + 1 + 1}, \]

and

\[
RS_2^2 \begin{pmatrix} 1/3 \\ 3 + \ldots + 3 \\ 2 + \ldots + 2 \\ 2/5 \\ 5 + 5 + 1 + 1 \end{pmatrix} \frac{1/3}{3 + \ldots + 3} \frac{1/2}{2 + \ldots + 2} \frac{1/5}{5 + 5 + 1 + 1}, \]

with the Belyi function of the truncated icosahedron [15], \( z = \frac{(z_1^4+228z_1^3+494z_1^2-228z_1+1)^3}{1728z_1(z_1^4-11z_1-1)^3} \).

Proposition 4.2 Fundamental solutions of Equation (4.2) corresponding to \( \theta \)-triples for the rows: 3, 5, 7, 8, 9, 10, 11, 12, 13, and 15 of Table 1 can be constructed as RS-transformations of the fundamental solutions of Equation (4.2) corresponding to \( \theta \)-triples of the remaining rows of Table 1 i.e., all the rows with \( \lambda = 1/2 \) different from the 9-th one. Moreover, \( R \) parts of these transformations are the Belyi functions.

Proof. At the end of item 6.2 of [1] it is shown that there are quadratic, cubic, and sextic RS-transformations that map Equation (4.2) corresponding to the row 6 of Table 1 into the ones for the rows: 7, 8, 9, 11, 12, and 13. At the end of item 6.3 of the same work it is explained that using quadratic, cubic, and sextic RS-transformations one can also map
the “case” 14 of Table 1 to the cases: 7, 9, 11, 12, 13, and 15. To complete the proof we have to show how to obtain the cases 3, 5, and 10.

To get the case 3 consider the quartic transformation, \( RS_3^2 \left( \begin{array}{c|c|c} \theta_0 \\ 2 + 1 + 1 \\ 1/2 \\ 2 + 2 \\ 1/4 \\ 4 \end{array} \right) \), where \( \theta_0 \) is arbitrary. An explicit form of the \( R \) part of this transformation, which is the Belyi function, can be found in item 4.2.1.A of [2]. This \( RS \) transformation can be also obtained as a composition of two quadratic transformations (cf. [1]). The \( \theta \)-triple of the resulting Equation (4.2) is \((\theta_0, \theta_0, 2\theta_0 - 1)\). Thus choosing \( \theta_0 = 1/3 \) we get the case 3 from the fourth one.

The Belyi function that allows one to build the \( RS \)-transformation from the case 6 of Table 1 to the case 10 is of the type \( RS_3^2 \left( \begin{array}{c|c|c} 1/3 \\ 3 + 3 + 2 \\ 1/2 \\ 2 + \ldots + 2 \\ 4 \\ 5 + 2 + 1 \end{array} \right) \), its corresponding Belyi function reads

\[
z = y - \frac{(100z^2 - 1728z + 729)^3}{64(z_1^2 - 27z_2 - 27)}, \quad z - 1 = y - \frac{(25000z^4 - 80000z^3 + 105300z^2 - 69984z + 19683)^2}{64(z_1 - 1)(25z_2 - 27)^3}.
\]

(4.5)

It the dual function for the first dessin on Figure 11.

The transformation \( RS_3^2 \left( \begin{array}{c|c|c} 1/3 \\ 4 + 3 + 3 \\ 1/2 \\ 2 + \ldots + 2 \\ 4 + 4 + 1 + 1 \\ 5 \\ 4 \end{array} \right) \) maps the case 4 to the case 5. Its \( R \) part is the following Belyi function,

\[
z = \frac{(320z^3 - 320z - 1)^3}{4z_1(z_1 - 1)(128z^2 - 128z + 5)^4}, \quad z - 1 = \frac{(2z_1 - 1)^2(16384z^4 - 32768z^3 + 15616z^2 + 768z - 1)^2}{4z_1(z_1 - 1)(128z^2 - 128z + 5)^4}.
\]

(4.6)

**Remark 4.2** The Belyi function (4.5) is easy to find with the help of Maple 8. Although at first glance Function (4.6) is only a little more complicated than Function (4.5), I was not able to get it by analysing the corresponding system of algebraic equations (see Remark 2.1) like it was done to find Function (4.5); note that the factor \((2z_1 - 1)\) in the second Equation (4.6) is not a priori evident, therefore the ansatz for its numerator is a square of the general polynomial of the fifth order with indeterminate coefficients.

Moreover, there is a simpler ansatz for the function of degree 8 with a proper number of the parameters that seems to produce a simpler Belyi function. It is Grothendieck’s theory of “Dessin’s d’Enfants” that helped to find a correct degree and symmetry of the Belyi function (see explanation on Figure 8).

**Remark 4.3** Of course, there are some other transformations that map Equations (4.2) corresponding to different cases of Table 1 to each other including some autotransformations. In Proposition 4.2 we give an example of the transformation of the case 2 to the
case 3 of Table 1. Due to the appearance of an arbitrary parameter \( s \), this transformation allows one to get a fundamental solution of Equation (1.2) for the case 3 in terms of the corresponding Schlesinger transformation without usage of the explicit form of the initial fundamental solution of Equation (1.2) for the case 2.

**Proposition 4.3** The transformation \( RS_3^2 \left( \begin{array}{c} 1/3 \\ 3+1+1+1 \\ 1/2 \\ 2+2+2 \\ 1/3 \\ 3+3 \end{array} \right) \) maps Equation (1.2) corresponding to the third row of Table 1 to Equation (1.2) for the second row.

**Proof.** For the proof it is enough to present the rational function of the type \( R(3+1+1|2+2+2|3+3) \), which is given by Equation (5.1).

### 5 One Irreducible Octic Transformation

It is indicated in [1] that Belyi function of the following type \( R(7+1|2+2+2|3+3+1+1) \), i.e.,

\[
z = \frac{\rho z_1(z_1-a)^7}{(z_1-1)(z_1-c_1)^3(z_1-c_2)^3}, \quad z-1 = \frac{\rho(z_1-b_1)^2(z_1-b_2)^2(z_1-b_3)^2(z_1-b_4)^2}{(z_1-1)(z_1-c_1)^3(z_1-c_2)^3},
\]

(5.1)
defines three irreducible octic transformations of the hypergeometric function, which in terms of \( \theta \)-triples read,

\[
\begin{align*}
\left( \frac{1}{2}, \frac{1}{3}, \frac{1}{7} \right) & \rightarrow \left( \frac{1}{2}, \frac{2}{3}, \frac{2}{7} \right), \\
\left( \frac{1}{2}, \frac{2}{3}, \frac{2}{7} \right) & \rightarrow \left( \frac{1}{3}, \frac{3}{3}, \frac{3}{7} \right), \\
\left( \frac{1}{3}, \frac{3}{3}, \frac{3}{7} \right) & \rightarrow \left( \frac{1}{3}, \frac{2}{3}, \frac{2}{7} \right).
\end{align*}
\]

(5.2)

In [1] Function (5.1) was not identified as the Belyi function, therefore its existence was not strictly speaking established. However, we were able there to calculate with a very high accuracy its coefficients numerically. Now, the existence of this transformation follows from the dessin presented on the picture below.

Moreover, Maple 8 and a substantially better computer than that used in calculations for [1] allow to find them now explicitly. To present the result we rewrite function (5.1) in the following form:

\[
z = \frac{\rho z_1(z_1-a)^7}{(z_1-1)(z_1^3+c_1z_1+c_0)^3}, \quad z-1 = \frac{\rho(z_1^4+b_3z_1^3+b_2z_1^2+b_1z_1+b_0)^2}{(z_1-1)(z_1^3+c_1z_1+c_0)^3},
\]

(5.3)

where

\[
\rho = \frac{1-i3\sqrt{3}}{112}, \quad a = \frac{27-139\sqrt{3}}{98}, \quad c_0 = \frac{-5697+i2349\sqrt{3}}{208912}, \quad c_1 = \frac{-513+i435\sqrt{3}}{784},
\]

(5.4)

\[
\hat{b}_0 = \frac{60507+i142803\sqrt{3}}{1317668}, \quad \hat{b}_1 = \frac{249399+i38313\sqrt{3}}{124456}, \quad \hat{b}_2 = \frac{-4293+i28251\sqrt{3}}{5488}, \quad \hat{b}_3 = \frac{-83+i129\sqrt{3}}{28}.
\]

(5.5)

It is, of course, straightforward to find explicit formulae for the roots: \( b_1, b_2, b_3, b_4 \) and \( c_1, c_2 \). We omit them as the ones for \( b_k, k = 1, 2, 3, 4 \) are very cumbersome. Moreover, corresponding RS-transformations are actually symmetric functions of these roots, i.e., they can be expressed in terms of the coefficients (5.4) and (5.5).

Each of the octic transformations (5.2) together with the quadratic, cubic, and their inverse ones (cf. [10]) generates a cluster of the hypergeometric functions which have the same type of transcendency. We have presented these clusters in terms of the corresponding \( \theta \)-triples in Table 2.

Note that the first cluster contains the hypergeometric functions corresponding to the triples \( \left( \frac{5}{7}, \frac{5}{7}, \frac{5}{7} \right) \) \& \( \left( \frac{5}{7}, \frac{5}{7}, \frac{5}{7} \right) \), the second – \( \left( \frac{5}{7}, \frac{5}{7}, \frac{5}{7} \right) \) \& \( \left( \frac{5}{7}, \frac{5}{7}, \frac{5}{7} \right) \), and the third – \( \left( \frac{1}{7}, \frac{1}{7}, \frac{1}{7} \right) \) \& \( \left( \frac{1}{7}, \frac{1}{7}, \frac{1}{7} \right) \).
A Appendix. On Quadratic Transformations for the Sixth Painlevé Equation

Existence of the quadratic transformations for the sixth Painlevé equation were discovered in [12] via an artificial transformation found for the similarity reductions of the so-called three-wave resonant system. In the subsequent work [13] one quadratic transformation was derived explicitly via the method of RS transformations. The latter transformation acts on the θ-tuples as,

\[
\left( \frac{1}{2}, \dot{\theta}_i^0, \ddot{\theta}_i^0, \pm \frac{1}{2} \right) \quad \mapsto \quad \left( \dot{\theta}_i^0, \ddot{\theta}_i^0, \dddot{\theta}_i^0, \dot{\theta}_i^0 \right).
\]

(A.1)

For the solution \( y(t) \) corresponding to the right \( \dot{\theta} \)-tuple and depending on \( t = 4\sqrt{t_0}/(1 + \sqrt{t_0})^2 \), where \( t_0 \) is the independent variable for the original solution \( y_0(t_0) \) corresponding to the left \( \dot{\theta} \)-tuple, it was obtained an explicit though complicated formula in terms of \( y_0(t_0) \). Recently K. Okamoto and his collaborators found that this formula can be actually substantially simplified. Some time ago Manin [16] rediscovered the so-called elliptic form of the sixth Painlevé equation given first by Painlevé. He applied then the Landen transformation for the elliptic functions and found another transformation in terms of the “elliptic variables” for the sixth Painlevé equation. It was later mentioned by Okamoto that it is also a quadratic transformation however, it is not equivalent to the one given above. Actually, Manin’s transformation cannot be obtained via the method of RS-transformation if we apply it to the associated linear Fuchsian \( 2 \times 2 \) matrix ODE in the Jimbo-Miwa parametrization [9]; in this parametrization we get only quadratic transformation (A.1) and its equivalent forms. The Jimbo-Miwa parametrization, which is proved to be a very helpful one for studies of almost all questions related with the theory of the sixth Painlevé equation, has one drawback: the complete group of symmetries

\[
\begin{array}{|c|c|c|c|}
\hline
N & \theta_0 & \theta_1 & \theta_\infty \\
\hline
1 & 1 & 1 & 1 \\
2 & 1 & 2 & 1 \\
3 & 1 & 3 & 1 \\
\hline
\end{array}
\]
for this equation cannot be realized as the group action on the solutions of the Fuchsian ODE. Therefore if we want to build the whole theory for the sixth Painlevé equation based on the isomonodromy problem for the \((2 \times 2)\) matrix linear Fuchsian ODE, then for studies of some questions, especially the ones related with the symmetries and transformations, we have to consider alternatively another parametrization of this Fuchsian ODE by employing one special transformation found by Okamoto. We call the last transformation the Okamoto transformation and related parametrization the Okamoto parametrization (see Appendix \[14\] Equation (A.5)\(^6\)). Recently, in the work \[17\] a representation of the complete group of Okamoto’s transformations as the symmetries for solutions of a linear ODE in matrix dimension greater than 2 has been obtained.

If we apply the method of RS-transformation to the \(2 \times 2\) Fuchsian ODE with four singular points in the Okamoto parametrization, then we can obtain the Manin transformation for the canonical form of the sixth Painlevé equation. Here I will not give an explicit form of the Okamoto parametrization and the “RS-derivation” of the Manin transformation. Instead I present the final answer in the soul close to the presentation in the main part of this paper.

Now we describe an algorithm how to construct quadratic transformations of the Manin type. Consider the quadratic Belyi functions, \(z(z_1)\) mapping \(\mathbb{C}P^1\) into itself:

\[
\begin{align*}
z &= z_1^2, \\
z &= -4z_1(z_1 - 1), \\
z &= \frac{z_1^2}{4(z_1 - 1)}.
\end{align*}
\]

**Remark A.1** These functions, of course, have only two critical points on the Riemann sphere. However, Proposition \[2.4\] is applicable for them too: if we formally put the multiplicity of the absent third critical point equal zero, then it gives \(m = 0\), as it should be for the Belyi functions.

Here we study transformations of the type: \(RS^2_4(4)\), therefore we incorporate in our notation one more point \(t \in \mathbb{C}P^1 \setminus \{0, 1, \infty\}\). Thus the types of these functions read, respectively, as follows: \(R(2|1 + 1|1 + 1|2)\), \(R((1 + 1|2|1 + 1|2)\), and \(R(2|2|1 + 1|1 + 1)\), where the third box is for \(t\) and the last one for the point at \(\infty\).

**Remark A.2** In this case it is also convenient to treat all four points 0, 1, \(t\), \(\infty\) on “equal footing” rather than normalize the critical values of the Belyi functions necessarily at 0, 1, and \(\infty\). Finally the latter ambiguity is absorbed by the corresponding transformations for the sixth Painlevé equation. However, instead of making these transformations afterwards it is easier to prepare the desired quadratic transformation from the very beginning as it does not require any special efforts. For example, we consider also the function \(z = 1 + (1 - t)(z_1 - 1)^2/(4z_1)\), which is of the type \(R(1 + 1|2|1 + 1)\). Clearly, we have 6 different quadratic Belyi functions.

For a given type of the Belyi function consider the \(\hat{\theta}\)-tuple whose members are denoted as \(\hat{\theta}^0_k\), \(k = 0, 1, t_0, \infty\). The parameters \(\hat{\theta}^0_k\) corresponding to the boxes of the type symbol with 2 vanish if \(k = 0, 1, t_0\), or \(\hat{\theta}^0_k = 1\) for \(k = \infty\), the other two members of the \(\theta\)-tuple are arbitrary. Now, denote \(y_0(t_0)\) any solution of the sixth Painlevé equation \[2.8\] with \(t = t_0\) and the \(\hat{\theta}\)-tuple defined above. The set of the preimages \(\{z^{-1}(0), z^{-1}(1), z^{-1}(t_0), z^{-1}(\infty)\}\) contains four non-apparent points, namely, these are preimages of the points with \(1 + 1\)

\[6\]There is a misprint in Equation (A.6) of \[14\]; instead of \(y^2_0\) in the numerator there should be \(y_0\).
The function transformations interchanging 0 counting their inverses. However, of course, if we consider them modulo fractional linear formation. All in all this construction gives $6 \cdot 24 = 144$ quadratic transformations without counting their inverses. However, of course, if we consider them modulo fractional linear transformations interchanging 0, 1, $\infty$ for both the initial equation and for final one, then we have actually only two seed not equivalent transformations, since there is a difference between two cases, namely, depending on whether $\infty$ is set as a critical value of the Belyi function or not. If we further factorize them modulo Bäcklund transformations, then we left with only one seed transformation. This seed transformation can be derived from the quadratic transformation found by the author \cite{A1} via application of the Okamoto transformation. Therefore, all in all there is only one seed quadratic transformation for the sixth Painlevé equation. However, in those cases when the explicit formula is needed it is convenient to use directly Proposition \cite{A.1} as it allows, in many situations, to avoid compositions of a cumbersome transformations.

Below we consider examples of quadratic transformations constructed by means of Proposition \cite{A.1} for the Belyi functions \cite{A.2}. Example 3 is the algebraic form of the transformation obtained by Manin.

1. \begin{align*}
z(z_1) &= z_1^2, \\
t &= \frac{4\sqrt{t_0}}{(1 + \sqrt{t_0})^2}, \\
\tilde{z}(z_1) &= \frac{2}{1 + \sqrt{t_0}} \frac{z_1 + \sqrt{t_0}}{z_1 + 1}, \\
y(t) &= \frac{2}{1 + \sqrt{t_0}} \frac{\sqrt{y_0(t_0) + \sqrt{t_0}}}{\sqrt{y_0(t_0) + 1}}, \\
(0, \theta_0, 0, 1) &\mapsto \left( \frac{1}{2} \theta_0, 0, 1, \frac{1}{2} \theta_0, 1 + \frac{1}{2} \theta_0 \right).
\end{align*}

2. \begin{align*}
z(z_1) &= -4z_1(z_1 - 1), \\
t &= -\frac{(1 - \sqrt{1 - t_0})^2}{4\sqrt{1 - t_0}}, \\
\tilde{z}(z_1) &= \frac{(1 - \sqrt{1 - t_0}) z_1}{2z_1 - 1 - \sqrt{1 - t_0}}, \\
y(t) &= -\frac{(1 - \sqrt{1 - t_0}) (1 - \sqrt{1 - y_0(t_0)})}{2 \left( \sqrt{1 - t_0} + \sqrt{1 - y_0(t_0)} \right)}, \\
(\theta_0, 0, 0, 1) &\mapsto \left( \frac{1}{2} \theta_0, 0, 1, \frac{1}{2} \theta_0, 1 + \frac{1}{2} \theta_0 \right).
\end{align*}
3. \( z(z_1) = \frac{z_1^2}{4(z_1 - 1)} \), \( \ddot{z}(z_1) = \frac{(z_1 - 2t_0 + 2\sqrt{t_0^2 - t_0})}{(1 - 2t_0 + 2\sqrt{t_0^2 - t_0})} \), \( \tau = 2 \left( t_0 + \sqrt{t_0^2 - t_0} \right) \),

\[ t = \frac{4\sqrt{t_0^2 - t_0}}{(1 - 2t_0 + 2\sqrt{t_0^2 - t_0})}, \quad y(t) = \frac{2\left( y_0(t_0) + \sqrt{y_0(t_0)^2 - y_0(t_0) - t_0 + \sqrt{t_0^2 - t_0}} \right)}{(1 - 2t_0 + 2\sqrt{t_0^2 - t_0})}, \quad \left( 0, 0, \hat{\theta}_0^0, \hat{\theta}_0^\infty \right) \to \left( \frac{1}{2} \hat{\theta}_0^0, \frac{1}{2} (\hat{\theta}_0^0 - 1), \frac{1}{2} \hat{\theta}_0^0, 1 + \frac{1}{2} (\hat{\theta}_0^0 - 1) \right). \]

Remark A.4 In all formulae 1–3 above the branches of the square roots with \( y_0(t_0) \) can be taken independently of the branches of square roots with \( t_0 \); it does not change the mapping between the \( \theta \)-tuples. All transformations are invertible.

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