Generalized common index jump theorem with applications to closed characteristics on star-shaped hypersurfaces and beyond

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Abstract

In this paper, we first generalize the common index jump theorem of Long-Zhu in 2002 and Duan-Long-Wang in 2016 to the case where the mean indices of symplectic paths are not required to be all positive. As applications, we study closed characteristics on compact star-shaped hypersurfaces in $\mathbb{R}^{2n}$, when both positive and negative mean indices may appear simultaneously. Specially we establish the existence of at least $n$ geometrically distinct closed characteristics on every compact non-degenerate perfect star-shaped hypersurface $\Sigma$ in $\mathbb{R}^{2n}$ provided that every prime closed characteristic possesses nonzero mean index. Furthermore, in the case of $\mathbb{R}^6$ we remove the nonzero mean index condition by showing that the existence of only finitely many geometrically distinct closed characteristics implies that each of them must possess nonzero mean index. We also generalize the above results about closed characteristics on non-degenerate star-shaped hypersurfaces to closed Reeb orbits of non-degenerate contact forms on a broad class of prequantization bundles.

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1 Introduction and main results

There are three goals in this paper. The first one is to generalize the common index jump theorem (CIJT for short below). This theorem is the first result which exhibits certain common index property of iterates of more than one but finitely many symplectic matrix paths. This theorem was discovered and proved by Y. Long and C. Zhu in 2002 (Theorems 4.1 and 4.3 of [LoZ], cf. also Theorems 11.1.1 and 11.2.1 of [Lon4]), and yields a breakthrough in the study of the multiplicity and stability of closed characteristics on compact convex hypersurfaces in $\mathbb{R}^{2n}$. In the last twenty years, this theorem has been applied to study various problems, provides an important method in the studies of the multiplicity and stability of closed characteristics in symplectic and contact dynamics. Specially it is one of the few methods which work effectively for such studies when the dimension of the symplectic manifold is not only 4. In [Wan2] and [Wan3] of 2016, W. Wang found certain useful symmetric property in CIJT. In 2016 also, an important extension of this CIJT was established by H. Duan, Y. Long and W. Wang (Theorem 3.5 of [DLW]), which gives an enhanced version of the CIJT by giving precise indices for iterates of related symplectic paths near the carefully chosen iterates in the study. This enhanced CIJT has been used to establish sharp estimates on the multiplicity and stability of prime closed geodesics on compact simply-connected bumpy Finsler manifolds whose loop spaces possess bounded Betti number sequences, provided the number of prime closed geodesics is finite and each of them possesses non-zero Morse index, which implies the positivity of their mean indices in [DLW]. This enhanced CIJT has also been applied to the studies of closed characteristics on star-shaped hypersurfaces in $\mathbb{R}^{2n}$ by H. Duan, H. Liu, Y. Long and W. Wang in [DLLW] and on other contact manifolds by V. Ginzburg, B. Z. Gürel and L. Macarini in [GGM] for example, under the assumption that all the prime closed characteristics possess positive (or negative) mean indices. Note that in Theorems 5.1 and 5.2 as well as Corollaries 5.3 and 5.4 of [GGu], V. Ginzburg and B. Z. Gürel extended the enhanced CIJT of [DLW] to the case admitting the mean indices of all symplectic paths being nonzero via a new index recurrence arguments. Subsequently, based on [GGu], V. Ginzburg, B. Z. Gürel and L. Macarini in Theorem 4.1 of [GGM] further studied the enhanced CIJT of [DLW] for strongly non-degenerate symplectic paths with positive mean indices. Note that based on the theorems these authors gave interesting results about Reeb orbits on contact manifolds. But their extensions of the enhanced CIJT missed the precise values of indices of some crucial iterates as those listed in Theorem 3.6 below, and missed the symmetric property of the CIJT produced by the vertices in the
cube $[0, 1]^d$ in the proof of CIJT discovered first in [LoZ] and then in [DLW] as those in the Step 2 of the proof of Theorem 1.2 below where the two opposite vertices are used. Note that such missing might be due to the index recurrence arguments. Note also that these missing properties are very crucial in our study in the current paper and for the future study, more precisely in order to get sharp estimates on multiplicities of periodic orbits, we do need to compute Morse type number quantities accurately and to apply the mentioned symmetric property of CIJM. Thus the first goal of this paper is to further extend this enhanced CIJT of [DLW] to the case that there exist prime symplectic paths possessing positive as well as negative mean indices simultaneously and the above mentioned information can be reserved at the same time by generalizing the method of [DLW].

The main idea in the proof of the CIJT in [LoZ] is to show that there exist large suitable iterate of each path among the given finitely many prime symplectic paths such that the corresponding enlarged index intervals at their these iterates possess a non-empty common intersection interval which is sufficiently large to contain certain integers, and the number of these integers yields a lower bound estimate for the number of these prime symplectic paths, provided all of these paths possess positive mean indices. Note that when prime closed characteristics on a compact star-shaped hypersurface in $\mathbb{R}^{2n}$ are considered, in general some of them may possess positive mean indices and the others possess negative mean indices. Then the iterated index sequences of them may tend to positive infinity as well as negative infinity simultaneously, and consequently it seems impossible to apply the CIJT to get a common intersection interval of the iterated enlarged index intervals of all the prime symplectic paths. To overcome this difficulty, suggested by the resonance identity for a star-shaped hypersurface in $\mathbb{R}^{2n}$ with only finitely many prime closed characteristics which was proved by H. Liu, Y. Long and W. Wang in [LLW] of 2014 (cf. Theorem 2.3 below), we can consider iterates of all these prime symplectic paths together by adding some $-1$ parameter to the negative mean indices, and we have improved the common index jump theorem (Theorems 4.1 and 4.3 of [LoZ] as well as Theorems 11.1.1 and 11.2.1 of [Lon4]) to the new enhanced common index jump Theorems 3.4 and 3.6 below to deal with mixed mean indices. This improvement allows our Theorem 1.2 below to deal with closed characteristics on non-degenerate star-shaped hypersurfaces when their mean indices are non-zero.

The second goal of this paper is to apply this generalized enhanced common index jump Theorem to study the multiplicity and stability of closed characteristics on compact star-shaped hypersurfaces in $\mathbb{R}^{2n}$. In this paper, we let $\Sigma$ be always a $C^3$ compact hypersurface in $\mathbb{R}^{2n}$ strictly star-shaped with respect to the origin, i.e., the tangent hyperplane at any $x \in \Sigma$ does not intersect the origin. We denote the set of all such hypersurfaces by $\mathcal{H}_{st}(2n)$, and denote by $\mathcal{H}_{con}(2n)$ the subset of $\mathcal{H}_{st}(2n)$ which consists of all strictly convex hypersurfaces. We consider closed characteristics $(\tau, y)$
on $\Sigma$, which are solutions of the following problem

$$
\begin{aligned}
\dot{y} &= JN_\Sigma(y), \\
y(\tau) &= y(0),
\end{aligned}
$$

(1.1)

where $J = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}$, $I_n$ is the identity matrix in $\mathbb{R}^n$, $\tau > 0$, $N_\Sigma(y)$ is the outward normal vector of $\Sigma$ at $y$ normalized by the condition $N_\Sigma(y) \cdot y = 1$. Here $a \cdot b$ denotes the standard inner product of $a, b \in \mathbb{R}^{2n}$. A closed characteristic $(\tau, y)$ is prime, if $\tau$ is the minimal period of $y$. Two closed characteristics $(\tau, y)$ and $(\sigma, z)$ are geometrically distinct, if $y(R) \neq z(R)$. We denote by $\mathcal{T}(\Sigma)$ the set of geometrically distinct closed characteristics $(\tau, y)$ on $\Sigma \in \mathcal{H}_{st}(2n)$. A closed characteristic $(\tau, y)$ is non-degenerate if $1$ is a Floquet multiplier of $y$ of precisely algebraic multiplicity $2$; hyperbolic if $1$ is a double Floquet multiplier of it and all the other Floquet multipliers are not on $U = \{ z \in \mathbb{C} \mid |z| = 1 \}$, i.e., the unit circle in the complex plane; elliptic if all the Floquet multipliers of $y$ are on $U$. We call a $\Sigma \in \mathcal{H}_{st}(2n)$ non-degenerate if all the closed characteristics on $\Sigma$, together with all of their iterations, are non-degenerate.

The study on closed characteristics in the global sense started in 1978, when the existence of at least one closed characteristic was first established on any $\Sigma \in \mathcal{H}_{st}(2n)$ by P. Rabinowitz in [Rab] and on any $\Sigma \in \mathcal{H}_{con}(2n)$ by A. Weinstein in [Wei] independently. Since then the existence of multiple closed characteristics on $\Sigma \in \mathcal{H}_{con}(2n)$ has been deeply studied by many mathematicians, for example, studies in [EkL], [EkH], [HWZ1], [Szu], [LoZ], [LLZ], [Wan2], [Wan3], [WHL] as well as [Lon4] and references therein.

For the star-shaped hypersurfaces, We are only aware of a few papers about the multiplicity of closed characteristics. In [Gir] of 1984 and [BLMR] of 1985, $\# \mathcal{T}(\Sigma) \geq n$ for $\Sigma \in \mathcal{H}_{st}(2n)$ was proved under some pinching conditions. In [Vit2] of 1989, C. Viterbo proved a generic existence result for infinitely many closed characteristics on star-shaped hypersurfaces. In [HuL] of 2002, X. Hu and Y. Long proved that $\# \mathcal{T}(\Sigma) \geq 2$ for non-degenerate $\Sigma \in \mathcal{H}_{st}(2n)$. In [HWZ2] of 2003, H. Hofer, K. Wysocki, and E. Zehnder proved that $\# \mathcal{T}(\Sigma) = 2$ or $\infty$ holds for every non-degenerate $\Sigma \in \mathcal{H}_{st}(4)$ provided that all stable and unstable manifolds of the hyperbolic closed characteristics on $\Sigma$ intersect transversally. Furthermore, recently this alternative result was proved to be true for every non-denernerate $\Sigma \in \mathcal{H}_{st}(4)$ without the above transversal condition by D. Cristofaro-Gardiner, M. Hutchings and D. Pomerleano in [CGHP]. In [CGH] of 2016, D. Cristofaro-Gardiner and M. Hutchings proved that $\# \mathcal{T}(\Sigma) \geq 2$ for every contact manifold $\Sigma$ of dimension three. Later various proofs of this result for star-shaped hypersurfaces have been given in [GHHM], [LLo1] and [GiG]. There are also some multiplicity results for closed orbits of dynamically convex Reeb flows, cf., [GuK] and [AbM].

On the stability problem, we refer the readers to [Eke], [DDE], [Lon1]-[Lon3], [LoZ], [WHL], [Wan1] for convex hypersurfaces and [LiL], [LLo2], [CGHHL] for star-shaped hypersurfaces. The
following index perfect condition was first introduced by H. Duan, H. Liu, Y. Long and W. Wang in [DLLW] of 2018, for the star-shaped hypersurfaces in $\mathbb{R}^{2n}$ which is much weaker than the dynamically convexity condition introduced by H. Hofer, K. Wysocki, and E. Zehnder in [HWZ1] of 1998 (cf. also [HWZ2] in 2003).

**Definition 1.1.** We call a compact star-shaped hypersurface $\Sigma$ in $\mathbb{R}^{2n}$ perfect, if for every prime closed characteristic $(\tau, y)$ on $\Sigma$, the Maslov-type index of each good $m$-th iterate $(m\tau, y)$ of $(\tau, y)$ with some $m \in \mathbb{N}$ satisfies $i(y, m) \neq -1$ if $n$ is even, or $i(y, m) \notin \{-2, -1, 0\}$ if $n$ is odd.

Here an iterate $(m\tau, y)$ of a prime closed characteristic $(\tau, y)$ on $\Sigma$ with $m \in \mathbb{N}$ is called *good*, if its Maslov-type index has the same parity as that of $(\tau, y)$, otherwise it is called *bad*. Note that for a bad closed characteristic $(m\tau, y)$, the element $x^m$ corresponding to $(m\tau, y)$ satisfies $\beta(x^m) = -1$ in Lemma 2.2 below, and consequently the equivariant critical module of the functional $F_{a,K}$ at $S^1\cdot x^m$ must be trivial, i.e., $x^m$ is homologically invisible and thus can be ignored in the Morse theory study. This property was used first in Definition 4.8 and Remark 4.9 via Proposition 4.2 of [LLW] to compute the Euler characteristic $\hat{\chi}(y)$. Then this property was used in Section 3 of [GGM] to compute the local equivariant symplectic homology. (cf. also the condition (A) below Theorem 1.2). Here in the current paper, we shall use this property in the computations of the Morse type numbers in Section 4 below.

In [DLLW], the authors proved specially the existence of at least $n$ closed characteristics on every non-degenerate perfect star-shaped hypersurfaces in $\mathbb{R}^{2n}$ when every closed characteristic possesses positive mean index. Then V. Ginzburg, B.Z. G"urel and L. Macarini in [GGM] obtained the same multiplicity result of closed Reeb orbits on contact manifolds under non-degenerate condition and the index perfect condition introduced in [DLLW] (i.e., the perfect condition given in Definition 1.1), provided the contact form $\alpha$ is index-positive (or index-negative), i.e., all contractible periodic orbit $\gamma$ of $\alpha$ possess positive (or negative) mean index. Most recently V. Ginzburg and L. Macarini in [GM] obtained some optimal multiplicity results of closed Reeb orbits on symmetric contact spheres under the so called strong dynamical convexity which extended results of C. Liu, Y. Long and C. Zhu in [LLZ] of 2002.

When we consider prime closed characteristics on a compact star-shaped hypersurface in $\mathbb{R}^{2n}$, a priori the mean indices of some of them can be non-positive. By the resonance identity for a star-shaped hypersurface in $\mathbb{R}^{2n}$ with only finitely many prime closed characteristics proved by H. Liu, Y. Long and W. Wang in [LLW] of 2014 (cf. Theorem 2.3 below), at least one of these prime closed characteristics must possess positive mean index, but some of the others may possess zero or negative mean indices. Thus even if we assume that every prime closed characteristic possesses nonzero mean index, their iterated index sequences may still tend to positive infinity as well as negative infinity simultaneously as we mentioned before. To overcome this difficulty, by Theorem 2.3 below, we notice that the existence of some prime closed characteristics possessing positive mean
index can be crucially used to construct actually the mentioned common index intersection interval, and that the prime closed characteristics with negative mean indices make no contributions to it and thus can be ignored in some sense. This understanding is rigorously realized by applying our new generalized enhanced common index jump Theorem 3.6 together with the mentioned existence of prime closed characteristics possessing positive mean indices so that we can deal with positive and negative mean indices simultaneously, and establish the following more general existence result, provided the mean indices of all the prime closed characteristics are non-zero. Note that another key observation in our proof is that the Morse inequalities still hold under non-zero mean index condition when the degrees of Morse-type numbers and Betti numbers are carefully chosen (cf. (2.16) and (2.17) below).

**Theorem 1.2.** Let $\Sigma$ be a compact non-degenerate perfect star-shaped hypersurface in $\mathbb{R}^{2n}$. If every prime closed characteristic on $\Sigma$ possesses nonzero mean index, then there exist at least $n$ geometrically distinct closed characteristics. Furthermore, if the total number of prime closed characteristics on $\Sigma$ is finite, then $\Sigma$ carries at least $n$ non-hyperbolic closed characteristics with even Maslov-type indices when $n$ is even, and at least $n$ closed characteristics with odd Maslov-type indices and at least $(n-1)$ of them are non-hyperbolic when $n$ is odd.

Based on Theorem 3.6 below, we can generalize Theorem 1.2 about closed characteristics on star-shaped hypersurfaces to closed Reeb orbits of contact forms on a broad class of prequantization bundles, which improves Theorem 2.1 and Theorem 2.10 of [GGM]. To clarify it, we review some terminologies from contact geometry, following Section 2 of [GGM].

Let $(M^{2n+1}, \xi)$ be a closed contact manifold satisfying $c_1(\xi)|_{\pi_2(M)} = 0$ and $\alpha$ be the contact form supporting the contact structure $\xi$. A non-degenerate periodic orbit $\gamma$ is called *good* if its Conley-Zehnder index $\mu(\gamma)$ has the same parity as that of the underlying simple closed orbit. Note that the Maslov-type index of $\gamma$ equals to $\mu(\gamma) - 1$. In the following, let $(M^{2n+1}, \xi)$ be a prequantization circle bundle over closed integral symplectic manifolds $(B^{2n}, \omega)$, i.e., the first Chern class of the principle bundle $M \to B$ is $-|\omega|$. Denote by $\chi(B)$ the Euler characteristic of $B$ and by

$$c_B := \inf\{k \in \mathbb{N} \mid \exists S \in \pi_2(B) \text{ with } \langle c_1(TB), S \rangle = k\}$$

its minimal Chern number. We impose the following condition which is weaker than the condition (F) in Section 2 of [GGM]:

(A) (i) The manifold $(M^{2n+1}, \xi)$ admits a strong symplectic filling $(W, \Omega)$ which is symplectically aspherical, i.e., $\Omega|_{\pi_2(W)} = 0$ and $c_1(TW)|_{\pi_2(W)} = 0$, and the map $\pi_1(M) \to \pi_1(W)$ induced by the inclusion is injective.

(ii) The contact form $\alpha$ is non-degenerate, the mean index $\hat{\mu}(\gamma)$ is nonzero for every contractible periodic orbit $\gamma$ of $\alpha$ and has no contractible good periodic orbits $\gamma$ such that $\mu(\gamma) = 0$ if $n$ is odd or $\mu(\gamma) \in \{0, \pm 1\}$ if $n$ is even.
**Theorem 1.3.** Let $(M^{2n+1}, \xi)$ be a prequantization $S^1$-bundle of a closed symplectic manifold $(B, \omega)$ such that $\omega|_{\pi_2(B)} = 0$ and $c_B > n/2$ and, furthermore, $H_k(B; \mathbb{Q}) = 0$ for every odd $k$ or $c_B > n$. Let $\alpha$ be a contact form supporting $\xi$ and assume that $M$ and $\alpha$ satisfy condition (A). Then $\alpha$ carries at least $r_B$ geometrically distinct contractible periodic orbits. Furthermore, if there exist finitely many geometrically distinct contractible closed orbits, then $\alpha$ carries at least $r_B^{non-hyp}$ geometrically distinct contractible non-hyperbolic periodic orbits, where $r_B^{non-hyp} := r_B - \dim H_n(B; \mathbb{Q})$ and

$$r_B := \begin{cases} 
\chi(B) + 2 \dim H_n(B; \mathbb{Q}), & \text{if } n \text{ is odd,} \\
\chi(B) + 4 \dim H_{n-1}(B; \mathbb{Q}), & \text{if } n \text{ is even.}
\end{cases}$$

(1.2)

**Remark 1.4.** (1) The proof of Theorem 1.3 follows the proofs of Theorems 2.1 and 2.10 of [GGM] via replacing the enhanced common jump theorem of H. Duan, Y. Long and W. Wang by our Theorem 3.6 and the proof of Theorem 1.2 below. We should emphasize that under nonzero mean index condition, there are similar Morse inequalities in the setting of equivariant symplectic homology to (2.16)-(2.17) below, cf., (68) in p.221 of [HM].

(2) Note that we are unable to weaken the condition (NF) in Section 2 of [GGM] as (A), since the proof in [GGM] relies on the machinery of positive equivariant symplectic homology. A remarkable observation by F. Bourgeois and A. Oancea in Section 4.1.2 of [BO] is that under suitable additional assumptions that all closed contractible Reeb orbits on $M$ are non-degenerate and have Conley-Zehnder index strictly greater than $3 - n$, the positive equivariant symplectic homology is defined and well-defined even when $M$ does not have a symplectic filling. But the existence of closed contractible Reeb orbit with negative mean index will destroy this assumption and then the positive equivariant symplectic homology is not well defined, thus we are unable to weaken the index-positive condition in (NF) of [GGM] to the nonzero mean index condition.

An important result of V. Bangert and W. Klingenberg in [BaK] implies that if $c$ is a closed geodesic on a compact Riemannian (or Finsler) manifold $M$ such that it possesses zero mean index and $c$ is neither homologically invisible nor an absolute minimum of the energy functional, then there exist infinitely many closed geodesics on $M$. In fact, we tend to believe that when the number of prime closed geodesics on a compact Finsler manifold or prime closed characteristics on a compact star-shaped hypersurface in $\mathbb{R}^{2n}$ is finite, then each one of them must be homologically visible as well as variationally visible (cf. [BaK] and [Lon4] for definitions). Motivated by the result in [BaK] and the well-known weakly non-resonant ellipsoids, we tend to believe that the following conjecture for closed characteristics should hold:

**Conjecture 1.5.** If there exist only finitely many geometrically distinct closed characteristics on a compact star-shaped hypersurface $\Sigma$ in $\mathbb{R}^{2n}$, then no one of them possesses zero mean index.
Our third goal of this paper is to give a positive answer to this conjecture below in the case of \( n = 3 \) for non-degenerate star-shaped hypersurfaces. But up to our knowledge, this conjecture seems to be rather challenging in its full generality.

**Theorem 1.6.** If there exist only finitely many geometrically distinct closed characteristics on a compact non-degenerate star-shaped hypersurface \( \Sigma \) in \( \mathbb{R}^6 \), then every prime closed characteristic possesses nonzero mean index.

Using Theorem 1.6, we can remove the nonzero mean index condition in Theorem 1.2 in the case of \( n = 3 \):

**Corollary 1.7.** If \( \Sigma \) is a compact non-degenerate perfect star-shaped hypersurface in \( \mathbb{R}^6 \), then there exist at least three geometrically distinct closed characteristics. Furthermore, if the total number of prime closed characteristics on \( \Sigma \) is finite, then \( \Sigma \) carries at least three geometrically distinct closed characteristics with odd Maslov-type indices and at least two of them are non-hyperbolic.

Note that one may generalize our Theorem 1.6 to contact manifolds by the idea of our proof of Theorem 1.6, especially the key observation in our Lemma 5.1 that the Viterbo index \( i(y^m) \) always equals to \(-4\) for every closed characteristic \((\tau, y)\) with zero mean index and all \( m \in \mathbb{N} \).

In this paper, let \( \mathbb{N}, \mathbb{N}_0, \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C} \) and \( \mathbb{R}^+ \) denote the sets of natural integers, non-negative integers, integers, rational numbers, real numbers, complex numbers and positive real numbers respectively. We define the functions

\[
[a] = \max\{k \in \mathbb{Z} \mid k \leq a\}, \quad \{a\} = a - [a], \quad E(a) = \min\{k \in \mathbb{Z} \mid k \geq a\}, \quad \varphi(a) = E(a) - [a]. \tag{1.3}
\]

Denote by \( a \cdot b \) and \( |a| \) the standard inner product and norm in \( \mathbb{R}^{2n} \). Denote by \( \langle \cdot, \cdot \rangle \) and \( \| \cdot \| \) the standard \( L^2 \) inner product and \( L^2 \) norm. For an \( S^1 \)-space \( X \), we denote by \( X_{S^1} \) the homotopy quotient of \( X \) by \( S^1 \), i.e., \( X_{S^1} = S^\infty \times S^1 X \), where \( S^\infty \) is the unit sphere in an infinite dimensional complex Hilbert space. In this paper we use \( \mathbb{Q} \) coefficients for all homological and cohomological modules. By \( t \to a^{+} \), we mean \( t > a \) and \( t \to a \).

2 Mean index identities for closed characteristics on compact star-shaped hypersurfaces in \( \mathbb{R}^{2n} \)

In this section, we briefly review the mean index identities for closed characteristics on \( \Sigma \in \mathcal{H}_{sd}(2n) \) developed in [LLW] which will be needed in Section 4. All the details of proofs can be found in [LLW]. Now we fix a \( \Sigma \in \mathcal{H}_{sd}(2n) \) and assume the following condition on \( T(\Sigma) \):

(F) There exist only finitely many geometrically distinct prime closed characteristics \( \{(\tau_j, y_j)\}_{1 \leq j \leq k} \) on \( \Sigma \).
Let $j : \mathbb{R}^{2n} \to \mathbb{R}$ be the gauge function of $\Sigma$, i.e., $j(\lambda x) = \lambda$ for $x \in \Sigma$ and $\lambda \geq 0$, then $j \in C^3(\mathbb{R}^{2n} \setminus \{0\}, \mathbb{R}) \cap C^0(\mathbb{R}^{2n}, \mathbb{R})$ and $\Sigma = j^{-1}(1)$. Let $\hat{\tau} = \inf_{1 \leq j \leq k} \tau_j$ and $T$ be a fixed positive constant. Then following [Vit1] and Section 2 of [LLW], for any $a > \frac{\hat{\tau}}{\tau}$, we can construct a function $\varphi_a \in C^\infty(\mathbb{R}, \mathbb{R}^+)$ which has 0 as its unique critical point in $[0, +\infty)$. Moreover, $\frac{\varphi_a''(t)}{a}$ is strictly decreasing for $t > 0$ together with $\varphi_a(0) = \varphi'_a(0)$ and $\varphi''_a(0) = 1 = \lim_{t \to 0^+} \frac{\varphi'_a(t)}{a}$. The precise definition of $\varphi_a$ and the dependence of $\varphi_a$ on $a$ are given in Lemma 2.2 and Remark 2.3 of [LLW] respectively. As in [LLW], we define a Hamiltonian function $H_a \in C^3(\mathbb{R}^{2n} \setminus \{0\}, \mathbb{R}) \cap C^1(\mathbb{R}^{2n}, \mathbb{R})$ satisfying $H_a(x) = a\varphi_a(j(x))$ on $U_A = \{ x \mid a\varphi_a(j(x)) \leq A \}$ for some large $A$, and $H_a(x) = \frac{1}{2}a|x|^2$ outside some even larger ball with $\epsilon_a > 0$ small enough such that outside $U_A$ both $\nabla H_a(x) \neq 0$ and $H''_a(x) < \epsilon_a$ hold as in Lemmas 2.2, 2.4 and Proposition 2.5 of [LLW].

We consider the following fixed period problem

$$\dot{x}(t) = JH'_a(x(t)), \quad x(0) = x(T). \quad (2.1)$$

Then solutions of (2.1) are $x \equiv 0$ and $x = ry(\tau t/T)$ with $\frac{\varphi'_a(y)}{\rho} = \frac{\tau}{\alpha}$, where $(\tau, y)$ is a solution of (1.1). In particular, non-zero solutions of (2.1) are in one to one correspondence with solutions of (1.1) with period $\tau < aT$.

For any $a > \frac{\hat{\tau}}{\tau}$, we can choose some large constant $K = K(a)$ such that

$$H_{a,K}(x) = H_a(x) + \frac{1}{2}K|x|^2 \quad (2.2)$$

is a strictly convex function, that is,

$$(\nabla H_{a,K}(x) - \nabla H_{a,K}(y), x - y) \geq \frac{\epsilon}{2}|x - y|^2, \quad (2.3)$$

for all $x, y \in \mathbb{R}^{2n}$, and some positive $\epsilon$. Let $H^*_{a,K}$ be the Fenchel dual of $H_{a,K}$ defined by

$$H^*_{a,K}(y) = \sup\{ x \cdot y - H_{a,K}(x) \mid x \in \mathbb{R}^{2n} \}. \quad \text{The dual action functional on} \ X = W^{1,2}(\mathbb{R}/T\mathbb{Z}, \mathbb{R}^{2n}) \text{is defined by}$$

$$F_{a,K}(x) = \int_0^T \left[ \frac{1}{2}(J\dot{x} - Kx, x) + H^*_{a,K}(-J\dot{x} + Kx) \right] dt. \quad (2.4)$$

Then $F_{a,K} \in C^{1,1}(X, \mathbb{R})$ and for $KT \not\in 2\pi\mathbb{Z}$, $F_{a,K}$ satisfies the Palais-Smale condition and $x$ is a critical point of $F_{a,K}$ if and only if it is a solution of (2.1). Moreover, $F_{a,K}(x_a) < 0$ and it is independent of $K$ for every critical point $x_a \neq 0$ of $F_{a,K}$.

When $KT \not\in 2\pi\mathbb{Z}$, the map $x \mapsto -J\dot{x} + Kx$ is a Hilbert space isomorphism between $X = W^{1,2}(\mathbb{R}/(T\mathbb{Z}); \mathbb{R}^{2n})$ and $E = L^2(\mathbb{R}/(T\mathbb{Z}), \mathbb{R}^{2n})$. We denote its inverse by $M_K$ and the functional

$$\Psi_{a,K}(u) = \int_0^T \left[ -\frac{1}{2}(M_Ku, u) + H^*_{a,K}(u) \right] dt, \quad \forall u \in E. \quad (2.5)$$
Then $x \in X$ is a critical point of $F_{a,K}$ if and only if $u = -J\dot{x} + Kx$ is a critical point of $\Psi_{a,K}$.

Suppose $u$ is a nonzero critical point of $\Psi_{a,K}$. Then the formal Hessian of $\Psi_{a,K}$ at $u$ is defined by

$$Q_{a,K}(v) = \int_0^T (-M_K v \cdot v + H''_{a,K}(u)v \cdot v)dt,$$

which defines an orthogonal splitting $E = E_- \oplus E_0 \oplus E_+$ of $E$ into negative, zero and positive subspaces. The index and nullity of $u$ are defined by $i_K(u) = \dim E_-$ and $\nu_K(u) = \dim E_0$ respectively. Similarly, we define the index and nullity of $x = M_Ku$ for $F_{a,K}$, we denote them by $i_K(x)$ and $\nu_K(x)$. Then we have

$$i_K(u) = i_K(x), \quad \nu_K(u) = \nu_K(x),$$

which follow from the definitions (2.4) and (2.5). The following important formula was proved in Lemma 6.4 of [Vit2]:

$$i_K(x) = 2n([KT/2\pi] + 1) + i^v(x) \equiv d(K) + i^v(x),$$

where the Viterbo index $i^v(x)$ does not depend on $K$, but only on $H_a$.

By the proof of Proposition 2 of [Vit1], we have that $v \in E$ belongs to the null space of $Q_{a,K}$ if and only if $z = M_Kv$ is a solution of the linearized system

$$\dot{z}(t) = JH''_a(x(t))z(t).$$

Thus the nullity in (2.7) is independent of $K$, which we denote by $\nu^v(x) \equiv \nu_K(u) = \nu_K(x)$.

By Proposition 2.11 of [LLW], the index $i^v(x)$ and nullity $\nu^v(x)$ coincide with those defined for the Hamiltonian $H(x) = \bar{j}(x)^\alpha$ for all $x \in \mathbb{R}^{2n}$ and some $\alpha \in (1,2)$. Especially $1 \leq \nu^v(x) \leq 2n - 1$ always holds.

For every closed characteristic $(\tau, y)$ on $\Sigma$, let $aT > \tau$ and choose $\varphi_a$ as above. Determine $\rho$ uniquely by $\frac{\varphi_a'(\rho)}{\rho} = \frac{\tau}{aT}$. Let $x = \rho y(T_{\tau/\rho})$. Then we define the index $i(\tau, y)$ and nullity $\nu(\tau, y)$ of $(\tau, y)$ by

$$i(\tau, y) = i^v(x), \quad \nu(\tau, y) = \nu^v(x).$$

Then the mean index of $(\tau, y)$ is defined by

$$\hat{i}(\tau, y) = \lim_{m \to \infty} \frac{i(m\tau, y)}{m}.$$
Clearly both of $F_{a,K}$ and $\Psi_{a,K}$ are $S^1$-invariant. For any $\kappa \in \mathbf{R}$, we denote by

$$
\Lambda_{a,K}^\kappa = \{ u \in L^2(\mathbf{R}/(T\mathbf{Z}); \mathbf{R}^{2n}) \mid \Psi_{a,K}(u) \leq \kappa \},
$$

$$
X_{a,K}^\kappa = \{ x \in W^{1,2}(\mathbf{R}/(T\mathbf{Z}), \mathbf{R}^{2n}) \mid F_{a,K}(x) \leq \kappa \}.
$$

For a critical point $u$ of $\Psi_{a,K}$ and the corresponding $x = M_K u$ of $F_{a,K}$, let

$$
\Lambda_{a,K}(u) = \Lambda_{a,K}^{\Psi_{a,K}(u)} = \{ w \in L^2(\mathbf{R}/(T\mathbf{Z}), \mathbf{R}^{2n}) \mid \Psi_{a,K}(w) \leq \Psi_{a,K}(u) \},
$$

$$
X_{a,K}(x) = X_{a,K}^{F_{a,K}(x)} = \{ y \in W^{1,2}(\mathbf{R}/(T\mathbf{Z}), \mathbf{R}^{2n}) \mid F_{a,K}(y) \leq F_{a,K}(x) \}.
$$

Clearly, both sets are $S^1$-invariant. Denote by $\text{crit}(\Psi_{a,K})$ the set of critical points of $\Psi_{a,K}$. Because $\Psi_{a,K}$ is $S^1$-invariant, $S^1 \cdot u$ becomes a critical orbit if $u \in \text{crit}(\Psi_{a,K})$. Note that by the condition (F), the number of critical orbits of $\Psi_{a,K}$ is finite. Hence as usual we can make the following definition.

**Definition 2.1.** Suppose $u$ is a nonzero critical point of $\Psi_{a,K}$, and $N$ is an $S^1$-invariant open neighborhood of $S^1 \cdot u$ such that $\text{crit}(\Psi_{a,K}) \cap (\Lambda_{a,K}(u) \cap N) = S^1 \cdot u$. Then the $S^1$-critical module of $S^1 \cdot u$ is defined by

$$
C_{S^1, q}(\Psi_{a,K}, S^1 \cdot u) = H_q((\Lambda_{a,K}(u) \cap N)_{S^1}, ((\Lambda_{a,K}(u) \setminus S^1 \cdot u) \cap N)_{S^1}).
$$

Similarly, we define the $S^1$-critical module $C_{S^1, q}(F_{a,K}, S^1 \cdot x)$ of $S^1 \cdot x$ for $F_{a,K}$.

We fix $a$ and let $u_K \neq 0$ be a critical point of $\Psi_{a,K}$ with multiplicity $\text{mul}(u_K) = m$, that is, $u_K$ corresponds to a closed characteristic $(\tau, y) \subset \Sigma$ with $(\tau, y)$ being $m$-iteration of some prime closed characteristic. Precisely, we have $u_K = -J\dot{x} + Kx$ with $x$ being a solution of (2.1) and $x = \rho y(T^t)$ with $\dot{\rho}(T) = \frac{\tau}{a}$. Moreover, $(\tau, y)$ is a closed characteristic on $\Sigma$ with minimal period $\tau_m$. For any $p \in \mathbf{N}$ satisfying $p\tau < aT$, we choose $K$ such that $pK \notin \frac{\mathbf{Z}}{T\mathbf{Z}}$, then the $p$th iteration $u_{pK}^p$ of $u_K$ is given by $-J\dot{x}^p + pKx^p$, where $x^p$ is the unique solution of (2.1) corresponding to $(p\tau, y)$ and is a critical point of $F_{a,pK}$, that is, $u_{pK}^p$ is the critical point of $\Psi_{a,pK}$ corresponding to $x^p$.

**Lemma 2.2.** (cf. Proposition 4.2 and Remark 4.4 of [LLW]) If $u_{pK}^p$ is non-degenerate, i.e., $\nu_{pK}(u_{pK}^p) = 1$, let $\beta(x^p) = (-1)^{i_{pK}(u_{pK}^p) - i_{K}(u_K)} = (-1)^{i^w(x^p) - i^w(x)}$, then

$$
C_{S^1, q-d(pK)+d(K)}(F_{a,K}, S^1 \cdot x^p) = C_{S^1, q}(F_{a,pK}, S^1 \cdot x^p)
$$

$$
= C_{S^1, q}(\Psi_{a,pK}, S^1 \cdot u_{pK}^p)
$$

$$
= \begin{cases} 
\mathbf{Q}, & \text{if } q = i_{pK}(u_{pK}^p) \text{ and } \beta(x^p) = 1, \\
0, & \text{otherwise}. 
\end{cases}
$$

**Theorem 2.3.** (cf. Theorem 1.1 of [LLW] and Theorem 1.2 of [Vit2]) Suppose that $\Sigma \in \mathcal{H}_{at}(2n)$ satisfying $\#\mathcal{T}(\Sigma) < +\infty$. Denote by $\{(\tau_j, y_j)\}_{1 \leq j \leq k}$ all the geometrically distinct prime closed
characteristics. Then the following identities hold

\[
\sum_{1 \leq j \leq k \atop i(y_j) > 0} \frac{\dot{\chi}(y_j)}{i(y_j)} = \frac{1}{2}, \quad \sum_{1 \leq j \leq k \atop i(y_j) < 0} \frac{\dot{\chi}(y_j)}{i(y_j)} = 0,
\]  

(2.11)

where \( \dot{\chi}(y) \in \mathbb{Q} \) is the average Euler characteristic given by Definition 4.8 and Remark 4.9 of [LLW].

In particular, if all \( y^m \)'s are non-degenerate for \( m \geq 1 \), then

\[
\dot{\chi}(y) = \begin{cases} 
(-1)^{i(y)}, & \text{if } i(y^2) - i(y) \in 2\mathbb{Z}, \\
-\frac{1}{2} & \text{otherwise}
\end{cases}
\]

(2.12)

Let \( F_{a,K} \) be a functional defined by (2.4) for some \( a, K \in \mathbb{R} \) large enough and let \( \epsilon > 0 \) be small enough such that \( \lbrack -\epsilon, 0 \rbrack \) contains no critical values of \( F_{a,K} \). For \( b \) large enough, The normalized Morse series of \( F_{a,K} \) in \( X^{-\epsilon} \setminus X^{-b} \) is defined, as usual, by

\[
M_a(t) = \sum_{q \geq 0, 1 \leq j \leq q} \dim C_{S^1, q}(F_{a,K}, S^1 \cdot v_j)t^{q-d(K)},
\]

(2.13)

where we denote by \( \{S^1 \cdot v_1, \ldots, S^1 \cdot v_p\} \) the critical orbits of \( F_{a,K} \) with critical values less than \(-\epsilon\). The Poincaré series of \( H_{S^1 \ast}(X, X^{-\epsilon}) \) is \( t^{d(K)}Q_a(t) \), according to Theorem 5.1 of [LLW], if we set \( Q_a(t) = \sum_{k \in \mathbb{Z}} q_k t^k \), then

\[
q_k = 0 \quad \forall k \in \mathbb{I},
\]

where \( I \) is an interval of \( \mathbb{Z} \) such that \( I \cap [i(\tau, y), i(\tau, y) + \nu(\tau, y) - 1] = \emptyset \) for all closed characteristics \( (\tau, y) \) on \( \Sigma \) with \( \tau \geq aT \). Then by Section 6 of [LLW], we have

\[
M_a(t) - \frac{1}{1 - t^2} + Q_a(t) = (1 + t)U_a(t),
\]

where \( U_a(t) = \sum_{i \in \mathbb{Z}} u_i t^i \) is a Laurent series with nonnegative coefficients. If there is no closed characteristic with \( i = 0 \), then

\[
M(t) - \frac{1}{1 - t^2} = (1 + t)U(t),
\]

(2.14)

where \( M(t) = \sum_{p \in \mathbb{Z}} M_p t^p \) denotes \( M_a(t) \) as \( a \) tends to infinity. In addition, we also denote by \( b_p \) the coefficient of \( t^p \) of \( \frac{1}{1 - t^2} = \sum_{p \in \mathbb{Z}} b_p t^p \), i.e. there holds \( b_p = 1 \), for all \( p \in 2\mathbb{N}_0 \), and \( b_p = 0 \) for all \( p \notin 2\mathbb{N}_0 \).

For any two positive integers \( n_1 \) and \( n_2 \), it follows from (2.14) that

\[
\sum_{p = -2n_1 + 1}^{2n_2 + 1} M_p t^p - \sum_{p = -2n_1 + 1}^{2n_2 + 1} b_p t^p = (1 + t) \sum_{p = -2n_1}^{2n_2 + 1} u_p t^p - u_{2n_2 + 1} t^{2n_2 + 2} - u_{-2n_1} t^{-2n_1},
\]

(2.15)
which, through letting $t = -1$, yields the following Morse inequality

\[
\sum_{p=-2n_1+1}^{2n_2+1} (-1)^p M_p \leq \sum_{p=-2n_1+1}^{2n_2+1} (-1)^p b_p.
\]

Similarly we have

\[
\sum_{p=-2n_1}^{2n_2} (-1)^p M_p \geq \sum_{p=-2n_1}^{2n_2} (-1)^p b_p.
\]

### 3 The generalized common index jump theorem for symplectic paths

In [Lon2] of 1999, Y. Long established the basic normal form decomposition of symplectic matrices. Based on this result he further established the precise iteration formulae of indices of symplectic paths in [Lon3] of 2000.

As in [Lon3], denote by

\[
N_1(\lambda, b) = \begin{pmatrix} \lambda & b \\ 0 & \lambda \end{pmatrix}, \quad \text{for } \lambda = \pm 1, \ b \in \mathbb{R},
\]

\[
D(\lambda) = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}, \quad \text{for } \lambda \in \mathbb{R} \setminus \{0, \pm 1\},
\]

\[
R(\theta) = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}, \quad \text{for } \theta \in (0, \pi) \cup (\pi, 2\pi),
\]

\[
N_2(e^{\sqrt{-1} \theta}, B) = \begin{pmatrix} R(\theta) & B \\ 0 & R(\theta) \end{pmatrix}, \quad \text{for } \theta \in (0, \pi) \cup (\pi, 2\pi) \text{ and}
\]

\[
B = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix} \text{ with } b_j \in \mathbb{R}, \text{ and } b_2 \neq b_3.
\]

Here $N_2(e^{\sqrt{-1} \theta}, B)$ is non-trivial if $(b_2 - b_3) \sin \theta < 0$, and trivial if $(b_2 - b_3) \sin \theta > 0$.

As in [Lon3], the $\circ$-sum (direct sum) of any two real matrices is defined by

\[
\begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix}_{2i \times 2i} \circ \begin{pmatrix} A_2 & B_2 \\ C_2 & D_2 \end{pmatrix}_{2j \times 2j} = \begin{pmatrix} A_1 & 0 & B_1 & 0 \\ 0 & A_2 & 0 & B_2 \\ C_1 & 0 & D_1 & 0 \\ 0 & C_2 & 0 & D_2 \end{pmatrix}.
\]

For every $M \in \text{Sp}(2n)$, the homotopy set $\Omega(M)$ of $M$ in $\text{Sp}(2n)$ is defined by

\[
\Omega(M) = \{ N \in \text{Sp}(2n) | \sigma(N) \cap U = \sigma(M) \cap U \equiv \Gamma \text{ and } \nu_\omega(N) = \nu_\omega(M) \forall \omega \in \Gamma \},
\]

where $\sigma(M)$ denotes the spectrum of $M$, $\nu_\omega(M) \equiv \dim \ker \left( M - \omega I \right)$ for $\omega \in U$. The component $\Omega^0(M)$ of $P$ in $\text{Sp}(2n)$ is defined by the path connected component of $\Omega(M)$ containing $M$. 

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Lemma 3.1. (cf. [Lon3], Lemma 9.1.5 and List 9.1.12 of [Lon4]) For \( M \in \text{Sp}(2n) \) and \( \omega \in \mathbb{U} \), the splitting number \( S^\pm_M(\omega) \) (cf. Definition 9.1.4 of [Lon4]) satisfies
\[
S^\pm_M(\omega) = 0, \quad \text{if} \quad \omega \notin \sigma(M), \quad \text{(3.5)}
\]
\[
S^\pm_{M_i(1,a)}(1) = \begin{cases} 1, & \text{if} \quad a \geq 0, \\ 0, & \text{if} \quad a < 0. \end{cases} \quad \text{(3.6)}
\]
For any \( M_i \in \text{Sp}(2n_i) \) with \( i = 0 \) and \( 1 \), there holds
\[
S^\pm_{M_0 \circ M_1}(\omega) = S^\pm_{M_0}(\omega) + S^\pm_{M_1}(\omega), \quad \forall \omega \in \mathbb{U}. \quad \text{(3.7)}
\]

We have the following decomposition theorem

Theorem 3.2. (cf. [Lon3] and Theorem 1.8.10 of [Lon4]) For any \( M \in \text{Sp}(2n) \), there is a path \( f : [0, 1] \to \Omega^0(M) \) such that \( f(0) = M \) and
\[
f(1) = M_1 \circ \cdots \circ M_k, \quad \text{(3.8)}
\]
where each \( M_i \) is a basic normal form listed in (3.1)-(3.4) for \( 1 \leq i \leq k \).

For every \( \gamma \in \mathcal{P}_\tau(2n) \equiv \{ \gamma \in C([0, \tau], \text{Sp}(2n)) \mid \gamma(0) = I_{2n} \} \), we extend \( \gamma(t) \) to \( t \in [0, m\tau] \) for every \( m \in \mathbb{N} \) by
\[
\gamma^m(t) = \gamma(t - j\tau)\gamma(\tau)^j \quad \forall \quad j\tau \leq t \leq (j + 1)\tau \quad \text{and} \quad j = 0, 1, \ldots, m-1, \quad \text{(3.9)}
\]
as in p.114 of [Lon2]. As in [LoZ] and [Lon4], we denote the Maslov-type indices of \( \gamma^m \) by \( (i(\gamma, m), \nu(\gamma, m)) \).

Then the following iteration formula from [LoZ] and [Lon4] can be obtained.

Theorem 3.3. (cf. Theorem 9.3.1 of [Lon4]) For any path \( \gamma \in \mathcal{P}_\tau(2n) \), let \( M = \gamma(\tau) \) and \( C(M) = \sum_{0 < \theta < 2\pi} S^-_M(e^{\sqrt{-1} \theta}) \). We extend \( \gamma \) to \( [0, +\infty) \) by its iterates. Then for any \( m \in \mathbb{N} \) we have
\[
i(\gamma, m) = m(i(\gamma, 1) + S^+_M(1) - C(M)) + 2 \sum_{\theta \in (0, 2\pi)} E \left( \frac{m\theta}{2\pi} \right) S^-_M(e^{\sqrt{-1} \theta}) - (S^+_M(1) + C(M)), \quad \text{(3.10)}
\]
and
\[
\hat{i}(\gamma, 1) = i(\gamma, 1) + S^+_M(1) - C(M) + \sum_{\theta \in (0, 2\pi)} \frac{\theta}{\pi} S^-_M(e^{\sqrt{-1} \theta}). \quad \text{(3.11)}
\]

Theorem 3.4. Fix an integer \( q > 0 \). Let \( \mu_i \geq 0 \) and \( \beta_i \) be integers for all \( i = 1, \ldots, q \). Let \( \alpha_{i,j} \) be positive numbers for \( j = 1, \ldots, \mu_i \) and \( i = 1, \ldots, q \). Let \( \delta \in (0, \frac{1}{2}) \) satisfying \( \delta \max_{1 \leq i \leq q} \mu_i < \frac{1}{2} \). Suppose
\[ D_i \equiv \beta_i + \sum_{j=1}^{\mu_i} \alpha_{i,j} \neq 0 \text{ for } i = 1, \cdots, q. \] Then there exist infinitely many \((N, m_1, \cdots, m_q) \in \mathbb{N}^{q+1}\) such that

\[
m_i \beta_i + \sum_{j=1}^{\mu_i} E(m_i \alpha_{i,j}) = g_i N + \Delta_i, \quad \forall \ 1 \leq i \leq q.
\] (3.12)

\[
\min \{m_i \alpha_{i,j}, 1 - \{m_i \alpha_{i,j}\}\} < \delta, \quad \forall \ j = 1, \cdots, \mu_i, \quad 1 \leq i \leq q,
\] (3.13)

\[
m_i \alpha_{i,j} \in \mathbb{N}, \quad \text{if } \alpha_{i,j} \in \mathbb{Q},
\] (3.14)

where

\[
g_i = \begin{cases} 1, & \text{if } D_i > 0, \\ -1, & \text{if } D_i < 0, \end{cases} \quad \Delta_i = \sum_{0 < m_i \alpha_{i,j} < \delta} 1, \quad \forall \ 1 \leq i \leq q.
\] (3.15)

**Remark 3.5.** When \(D_i > 0\) for all \(1 \leq i \leq q\), this is precisely the Theorem 4.1 of [LoZ] (also cf. Theorem 11.1.1 of [Lon4]).

**Proof of Theorem 3.4.** By assumption \(D_i \neq 0, \forall 1 \leq i \leq q\), we further assume that there exists some integer \(0 \leq q_0 \leq q\) with \(D_i < 0\) for \(0 \leq i \leq q_0\) and \(D_i > 0\) for \(q_0 + 1 \leq i \leq q\). Next we will do with both of these two cases simultaneously. In fact we only need to use \(g_i N\) and \(m_i = \left(\frac{g_i N}{MD_i}\right) + \chi_i \) to replace the corresponding \(N\) and \(m_i\)s in the proof of Theorem 4.1 of [LoZ] (cf. Theorem 11.1.1 of [Lon4]). For reader’s convenience and because the proof is almost self-contained, in the following we only give some different points and details.

In order to get (3.12), we consider

\[
m_i D_i = \frac{g_i N}{MD_i} MD_i - \left\{ \frac{g_i N}{MD_i} \right\} MD_i + \chi_i MD_i
\]

\[
= g_i N + \left( \chi_i - \left\{ \frac{g_i N}{MD_i} \right\} \right) MD_i, \quad \forall \ 1 \leq i \leq q,
\] (3.16)

where, to get (3.14), we require \(M \in \mathbb{N}\) to satisfy \(M \alpha_{i,j} \in \mathbb{N}\) when \(\alpha_{i,j} \in \mathbb{Q}\) for \(j = 1, \cdots, \mu_i\), and \(\chi_i \in \{0, 1\}\) will be determined later.

Set

\[
m_i = \left(\frac{g_i N}{MD_i}\right) + \chi_i \ M.
\] (3.17)

Then by (1.3) and (3.16), following the proofs from (4.11) and (4.13) of [LoZ] (or (11.1.11) to (11.1.13) of [Lon4]) and using \(\Delta_i\) and \(\delta\) defined there, it yields

\[
m_i \beta_i + \sum_{j=1}^{\mu_i} E(m_i \alpha_{i,j}) = m_i D_i + \sum_{j=1}^{\mu_i} (\varphi(m_i \alpha_{i,j}) - \{m_i \alpha_{i,j}\})
\]

\[
= g_i N + \left( \chi_i - \left\{ \frac{g_i N}{MD_i} \right\} \right) MD_i + \sum_{j=1}^{\mu_i} (\varphi(m_i \alpha_{i,j}) - \{m_i \alpha_{i,j}\})
\]
\[
\begin{align*}
&= q_i N + \left( \chi_i - \left\{ \frac{q_i N}{MD_i} \right\} \right) MD_i + \Delta_i \\
&- \sum_{0 < \{m_i \alpha_{i,j} \} < \delta} \{ m_i \alpha_{i,j} \} + \sum_{0 < 1 - \{m_i \alpha_{i,j} \} < \delta} (1 - \{m_i \alpha_{i,j} \}),
\end{align*}
\]

which, together with requiring (3.16) and (3.18) simultaneously, implies that

\[
\left| m_i \beta_i + \sum_{j=1}^{\mu_i} E(m_i \alpha_{i,j}) - q_i N - \Delta_i \right| \leq \left| \left\{ \frac{q_i N}{MD_k} \right\} - \chi_i \right| M |D_i| + \mu_i \delta, \quad 1 \leq i \leq q.
\]

Notice that \( \delta \max_{1 \leq i \leq q} \mu_i < \frac{1}{2} \) holds by assumption. So by (3.19), in order to obtain (3.12) we need to choose \( M, N \in \mathbb{N} \) and \( \chi_i \) such that the following estimate holds

\[
\left| \left\{ \frac{q_i N}{MD_i} \right\} - \chi_i \right| M |D_i| < \frac{1}{2}.
\]

On the other hand, by the choice (3.17) of \( m_i \), we have

\[
\{ m_i \alpha_{i,j} \} = \left\{ \left( \left\{ \frac{q_i N}{MD_i} \right\} + \chi_i \right) M \alpha_{i,j} \right\} = \left\{ \frac{q_i N \alpha_{i,j}}{D_i} + \left( \chi_i - \left\{ \frac{q_i N}{MD_i} \right\} \right) \right\} M \alpha_{i,j} = \{ A_{i,j}(q_i N) + B_{i,j}(q_i N) \}, \quad j = 1, \ldots, \mu_i, \quad 1 \leq i \leq q,
\]

where

\[
A_{i,j}(q_i N) = \left\{ \frac{q_i N \alpha_{i,j}}{D_i} \right\} - \chi_{i,j}, \quad B_{i,j}(q_i N) = \left( \chi_i - \left\{ \frac{q_i N}{MD_i} \right\} \right) M \alpha_{i,j},
\]

and \( \chi_{i,j} \in \{0, 1\} \) will be determined later.

Following the arguments between (4.18) and (4.20) of [LoZ], it can be easily shown that \( \{ m_i \alpha_{i,j} \} \) must be close enough to 0 or 1, i.e., satisfying (3.13), if

\[
\max \{|A_{i,j}(q_i N)|, |B_{i,j}(q_i N)|\} < \frac{\delta_1}{3}, \quad \text{for } 0 < \delta_1 < \delta.
\]

By (3.20) and (3.23), in order to get (3.12)-(3.14) we only need to choose integers \( \chi_i, \chi_{i,j} \in \{0, 1\} \) and infinitely many integers \( N \in \mathbb{N} \) such that all the quantities

\[
\left| \left\{ \frac{q_i N \alpha_{i,j}}{D_i} \right\} - \chi_{i,j} \right|, \quad \left| \left\{ \frac{q_i N}{MD_i} \right\} - \chi_i \right|
\]

can be made simultaneously to be small enough, which can be reduced to a dynamical problem on a torus (cf. pages 233-234 of [Lon4]). Here we omit rest of details in [Lon4].

In 2002, Y. Long and C. Zhu [LoZ] has established the common index jump theorem for symplectic paths, which has become one of the main tools to study the periodic orbit problem in Hamiltonian and symplectic dynamics. In [DLW] of 2016, H. Duan, Y. Long and W. Wang further improved this theorem to an enhanced version which gives more precise index properties of \( \gamma_k^{2m_k} \).
and $\gamma_k^{2m_k \pm m}$ with $1 \leq m \leq \bar{m}$ for any fixed $\bar{m}$. Under the help of Theorem 3.4, following the proofs of Theorem 3.5 in [DLW], next we further generalize this theorem to the case of admitting the existence of symplectic paths with negative mean indices.

**Theorem 3.6.** (Generalized common index jump theorem for symplectic paths) Let $\gamma_i \in \mathcal{P}_n(2n)$ for $i = 1, \ldots, q$ be a finite collection of symplectic paths with nonzero mean indices $\hat{i}(\gamma_i, 1)$. Let $M_i = \gamma_i(\tau_i)$. We extend $\gamma_i$ to $[0, +\infty)$ by (3.9) inductively.

Then for any fixed $\bar{m} \in \mathbb{N}$, there exist infinitely many $(q + 1)$-tuples $(N, m_1, \ldots, m_q) \in \mathbb{N}^{q+1}$ such that the following hold for all $1 \leq i \leq q$ and $1 \leq m \leq \bar{m}$,

\begin{align*}
\nu(\gamma_i, 2m_i - m) &= \nu(\gamma_i, 2m_i + m) = \nu(\gamma_i, m), \\
i(\gamma_i, 2m_i + m) &= 2\theta_i N + i(\gamma_i, m), \\
i(\gamma_i, 2m_i - m) &= 2\theta_i N - i(\gamma_i, m) - 2(S_{M_i}^-(1) + Q_i(m)), \\
i(\gamma_i, 2m_i) &= 2\theta_i N - (S_{M_i}^+(1) + C(M_i) - 2\Delta_i),
\end{align*}

where

$$
\theta_i = \begin{cases} 
1, & \text{if } \hat{i}(\gamma_i, 1) > 0, \\
-1, & \text{if } \hat{i}(\gamma_i, 1) < 0, 
\end{cases}
$$

$$
\Delta_i = \sum_{0 < \{m_i \theta_i \pi \} < \delta} S_{M_i}^-(e^{\sqrt{-1} \theta}), 
Q_i(m) = \sum_{e^{\sqrt{-1} \theta} \in \sigma(M_i), \{m_i \theta_i \pi \} = \phi(\frac{m_i \theta_i \pi}{2\pi}) = 0} S_{M_i}^-(e^{\sqrt{-1} \theta}).
$$

More precisely, by (3.17) and (4.40), (4.41) in [LoZ], we have

$$
m_i = \left(\frac{N}{M|\hat{i}(\gamma_i, 1)|}\right) + \chi_i M, \quad \forall 1 \leq i \leq q,
$$

where $\chi_i = 0$ or $1$ for $1 \leq i \leq q$ and $\frac{M\theta_i}{\pi} \in \mathbb{Z}$ whenever $e^{\sqrt{-1} \theta} \in \sigma(M_i)$ and $\frac{\theta_i}{\pi} \in \mathbb{Q}$ for some $1 \leq i \leq q$. Furthermore, by (3.24), for any $\epsilon > 0$, we can choose $N$ and $\{\chi_i\}_{1 \leq i \leq q}$ such that

$$
\left|\left\{\frac{N}{M|\hat{i}(\gamma_i, 1)|}\right\} - \chi_i\right| < \epsilon, \quad \forall 1 \leq i \leq q.
$$

**Proof.** For $1 \leq i \leq q$, let $\mu_i = \sum_{0 < \theta < 2\pi} S_{M_i}^-(e^{\sqrt{-1} \theta}), \alpha_{i,j} = \frac{\theta_j}{\pi}$ where $e^{\sqrt{-1} \theta_j} \in \sigma(M_i)$ for $1 \leq j \leq \mu_i$, and $D_i = i(\gamma_i, 1) + S_{M_i}^+(1) - C(M_i) + \sum_{\theta \in (0, 2\pi)} \frac{\theta}{\pi} S_{M_i}^-(e^{\sqrt{-1} \theta})$. Then Theorem 3.6 can be proved by Theorem 3.4 and using $\theta_i N$ and $m_i = \left(\frac{N}{M|\hat{i}(\gamma_i, 1)|}\right) M$ to replace the corresponding $N$ and $m_i$s in the proof of Theorem 3.5 of [DLW]. Here we omit all details.

**Remark 3.7.** Let $l = q + \sum_{k=1}^{q} \mu_k$, and

$$
v = \left(\frac{1}{M|\hat{i}(\gamma_1, 1)|}, \ldots, \frac{1}{M|\hat{i}(\gamma_1, 1)|}, \frac{\alpha_{1,1}}{|\hat{i}(\gamma_1, 1)|}, \ldots, \frac{\alpha_{1,\mu_1}}{|\hat{i}(\gamma_1, 1)|}, \ldots, \frac{\alpha_{q,1}}{|\hat{i}(\gamma_q, 1)|}, \ldots, \frac{\alpha_{q,\mu_q}}{|\hat{i}(\gamma_q, 1)|}\right) \in \mathbb{R}^l.
$$
Theorem 3.6 also shows that for any given small \( \epsilon > 0 \) one can find a vertex
\[
\chi = (\chi_1, \ldots, \chi_q, \chi_{11}, \ldots, \chi_{1, \mu_1}, \ldots, \chi_{q, 1}, \ldots, \chi_{q, \mu_q})
\]
of the cube \([0, 1]^q\) and infinitely many \( N \in \mathbb{N} \) such that \( |\{Nv\} - \chi| < \epsilon \).

**Theorem 3.8.** (cf. Theorem 2.1 of [HuL] and Theorem 6.1 of [LLo2]) Suppose \( \Sigma \in \mathcal{H}_{st}(2n) \) and \((\tau, y) \in \mathcal{T}(\Sigma)\). Then we have
\[
i(m, \tau, y) = i(y, m) - n, \quad \nu(m, \tau, y) = \nu(y, m), \quad \forall m \in \mathbb{N}, \tag{3.32}
\]
where \( i(m, \tau, y) \) and \( \nu(m, \tau, y) \) are the index and nullity of \((m, \tau, y)\) defined in Section 2, \( i(y, m) \) and \( \nu(y, m) \) are the Maslov-type index and nullity of \((m, \tau, y)\) (cf. Section 5.4 of [Lon3]). In particular, we have \( \hat{i}(\tau, y) = \hat{i}(y, 1) \), where \( \hat{i}(\tau, y) \) is given in Section 2, \( \hat{i}(y, 1) \) is the mean Maslov-type index (cf. Definition 8.1 of [Lon4]). Hence we denote it simply by \( \hat{i}(y) \).

## 4 Proof of Theorem 1.2

In order to prove Theorem 1.2, let \( \Sigma \in \mathcal{H}_{st}(2n) \) be a non-degenerate perfect star-shaped hypersurface which possesses only finitely many prime closed characteristics \( \{(\tau_k, y_k)\}_{k=1}^q \) with \( \hat{i}(y_k, 1) \neq 0 \). Note that there exist at least one closed characteristic on \( \Sigma \) with positive mean index by the first identity of (2.11) in Theorem 2.3. So without loss of generality, the following mixed mean index condition holds:

**(MMI)** There exists an integer \( q_0 \in [1, q] \) such that \( \hat{i}(y_k, 1) > 0 \) for \( 1 \leq k < q_0 \) and \( \hat{i}(y_k, 1) < 0 \) for \( q_0 + 1 \leq k \leq q \).

Denote by \( \gamma_k \equiv \gamma_{y_k} \) the associated symplectic path of \((\tau_k, y_k)\) for \( 1 \leq k \leq q \). Then by Lemma 3.3 of [HuL] and Lemma 3.2 of [Lon1], there exists \( P_k \in Sp(2n) \) and \( U_k \in Sp(2n - 2) \) such that
\[
M_k \equiv \gamma_k(\tau_k) = P_k^{-1}(N_1(1, 1) \circ U_k)P_k, \quad \forall 1 \leq k \leq q, \tag{4.1}
\]
where every \( U_k \) has the following form:
\[
R(\theta_1) \circ \cdots \circ R(\theta_r) \circ D(\pm 2)^{\circ s} \\
\circ N_2(e^{\alpha_j \sqrt{-1}}, A_{1j}) \cdots \circ N_2(e^{\alpha_j \sqrt{-1}}, A_{kj}) \circ N_2(e^{\beta_j \sqrt{-1}}, B_{1j}) \cdots \circ N_2(e^{\beta_{r_j} \sqrt{-1}}, B_{r_j}),
\]
where \( \frac{\theta_j}{2\pi} \notin \mathbb{Q} \) for \( 1 \leq j \leq r; \frac{\alpha_j}{2\pi} \notin \mathbb{Q} \) for \( 1 \leq j \leq r_s; \frac{\beta_j}{2\pi} \notin \mathbb{Q} \) for \( 1 \leq j \leq r_0 \) and
\[
r + s + 2r_s + 2r_o = n - 1. \tag{4.2}
\]

**Proof of Theorem 1.2.**
We prove Theorem 1.2 in two cases:

**Case 1.** $n$ is even.

We continue the proof in three steps.

**Step 1.** The first set of iterates for the choice of the vertex $\chi$ in the cube $[0,1]^l$.

By (MMI), we have $\hat{i}(y_k) = \hat{i}(y_k,1) > 0$ for $1 \leq k \leq q_0$ and $\hat{i}(y_k) = \hat{i}(y_k,1) < 0$ for $q_0+1 \leq k \leq q$, which implies that $i(y_k,m) \to +\infty$ for $1 \leq k \leq q_0$ and $i(y_k,m) \to -\infty$ for $q_0+1 \leq k \leq q$ as $m \to +\infty$. So the positive integer $\bar{m}$ defined by

\[
\bar{m} = \max_{1 \leq k \leq q_0} \{ \min_{m_0 \in \mathbb{N}} \ | \ i(y_k, m + l) \geq i(y_k, l) + n + 1, \ \forall l \geq 1, m \geq m_0 \} 
\]

is well-defined and finite.

For the integer $\bar{m}$ defined in (4.3), it follows from Theorem 3.6 and Remark 3.7 that there exist a vertex $\chi$ of $[0,1]^l$ and infinitely many $(q + 1)$-tuples $(N, m_1, \ldots, m_q) \in \mathbb{N}^{q+1}$ such that for any $1 \leq k \leq q$, there holds

\[
\bar{m} + 2 \leq \min \{ 2m_k, 1 \leq k \leq q \}, \quad (4.4)
\]

\[
i(y_k, 2m_k - m) = 2q_k N - 2 - i(y_k, m), \quad \forall 1 \leq m \leq \bar{m}, \quad (4.5)
\]

\[
i(y_k, 2m_k) = 2q_k N - 1 - C(M_k) + 2\Delta_k, \quad (4.6)
\]

\[
i(y_k, 2m_k + m) = 2q_k N + i(y_k, m), \quad \forall 1 \leq m \leq \bar{m}, \quad (4.7)
\]

where note that $S_{M_k}^+(1) = 1, Q_k(m) = 0, \forall m \geq 1$ by (4.1)-(4.2).

By the definition (4.3) of $\bar{m}$ and (4.6), for any $m \geq \bar{m} + 1$, we obtain

\[
i(y_k, 2m_k - m) \leq i(y_k, 2m_k) - n - 1
\]

\[
= 2N - n - 2 + 2\Delta_k - C(M_k)
\]

\[
\leq 2N - 3, \quad 1 \leq k \leq q_0, \quad (4.8)
\]

\[
i(y_k, 2m_k - m) \geq i(y_k, 2m_k) + n + 1
\]

\[
= -2N + n + 2\Delta_k - C(M_k)
\]

\[
\geq -2N + 1, \quad q_0 + 1 \leq k \leq q, \quad (4.9)
\]

\[
i(y_k, 2m_k + m) \geq i(y_k, 2m_k) + n + 1
\]

\[
= 2N + n - C(M_k) + 2\Delta_k
\]

\[
\geq 2N + 1, \quad 1 \leq k \leq q_0, \quad (4.10)
\]

\[
i(y_k, 2m_k + m) \leq i(y_k, 2m_k) - n - 1
\]
where we use the facts $2\Delta_k - C(M_k) \leq n - 1$ and $C(M_k) \leq n - 1$.

Then by (4.5)-(4.11) and Theorem 3.8, we obtain

$$
i(y_k^{2m_k - m}) \leq 2N - n - 3, \quad \forall \ m + 1 \leq m \leq 2m_k - 1, \quad 1 \leq k \leq q_0, \quad (4.12)$$
$$
i(y_k^{2m_k - m}) \geq -2N - n + 1, \quad \forall \ m + 1 \leq m \leq 2m_k - 1, \quad q_0 + 1 \leq k \leq q, \quad (4.13)$$
$$
i(y_k^{2m_k - m}) = 2g_k N - 2n - 2 - i(y_k^m), \quad \forall \ 1 \leq m \leq \bar{m}, \quad (4.14)$$
$$
i(y_k^{2m_k}) = 2g_k N - C(M_k) + 2\Delta_k - n - 1, \quad (4.15)$$
$$
i(y_k^{2m_k + m}) = 2g_k N + i(y_k^m), \quad \forall \ 1 \leq m \leq \bar{m}, \quad (4.16)$$
$$
i(y_k^{2m_k + m}) \geq 2N - n + 1, \quad \forall \ m \geq \bar{m} + 1, \quad 1 \leq k \leq q_0, \quad (4.17)$$
$$
i(y_k^{2m_k + m}) \leq -2N - n - 3, \quad \forall \ m \geq \bar{m} + 1, \quad q_0 + 1 \leq k \leq q. \quad (4.18)$$

**Claim 1:** For $N \in \mathbb{N}$ in Theorem 3.6 satisfying (4.12)-(4.18), we have

$$\sum_{k=1}^{q} 2m_k \tilde{\chi}(y_k) = N. \quad (4.19)$$

In fact, let $\epsilon < \frac{1}{1 + 2M \sum_{1 \leq k \leq q} |\tilde{\chi}(y_k)|}$, by Theorem 2.3 and (MMI) we have

$$\sum_{k=1}^{q} \tilde{\chi}(y_k) \cdot |\tilde{i}(y_k)| = \sum_{\tilde{i}(y_k) > 0} \tilde{\chi}(y_k) - \sum_{\tilde{i}(y_k) < 0} \tilde{\chi}(y_k) = \frac{1}{2},$$

which, together with (3.30)-(3.31), yields

$$\left| N - \sum_{k=1}^{q} 2m_k \tilde{\chi}(y_k) \right| = \left| \sum_{k=1}^{q} \frac{2N \tilde{\chi}(y_k)}{|\tilde{i}(y_k)|} - \sum_{k=1}^{q} 2\tilde{\chi}(y_k) \left( \left\lfloor \frac{N}{M |\tilde{i}(y_k)|} \right\rfloor + \chi_k \right) M \right| \leq 2M \sum_{k=1}^{q} |\tilde{\chi}(y_k)| \left\lfloor \frac{N}{M |\tilde{i}(y_k)|} \right\rfloor - \chi_k \right| \leq 2M \epsilon \sum_{k=1}^{q} |\tilde{\chi}(y_k)| < 1. \quad (4.20)$$

It proves Claim 1 since each $2m_k \tilde{\chi}(y_k)$ is an integer.

Now by Lemma 2.2, it yields

$$\sum_{m=1}^{2m_k} (-1)^{d(K)+i(y_k^m)} \dim C_{S^1, d(K)+i(y_k^m)}(F_{a,K}, S^1 \cdot x_k^m)$$
\[ \sum_{m=1}^{2m_k} (-1)^i(y_k^m) \dim C_{S^1, d(K)+i(y_k^m)}(F_{a,K}, S^1 \cdot x_k^m) \]
\[ = \sum_{i=0}^{m_k-1} \sum_{m=2i+1}^{2i+2} (-1)^i(y_k^m) \dim C_{S^1, d(K)+i(y_k^m)}(F_{a,K}, S^1 \cdot x_k^m) \]
\[ = \sum_{i=0}^{m_k-1} \sum_{m=1}^{2} (-1)^i(y_k^m) \dim C_{S^1, d(K)+i(y_k^m)}(F_{a,K}, S^1 \cdot x_k^m) \]
\[ = m_k \sum_{m=1}^{2} (-1)^i(y_k^m) \dim C_{S^1, d(K)+i(y_k^m)}(F_{a,K}, S^1 \cdot x_k^m) \]
\[ = 2m_k \chi(y_k), \quad \forall 1 \leq k \leq q, \quad (4.21) \]

where \( x_k \) is the critical point of \( F_{a,K} \) corresponding to \( y_k \), and we choose large enough \( K \) such that \( d(K) = 2n([K/2\pi] + 1) \geq -i(y_k^m) \) for \( 1 \leq m \leq 2m_k \) and \( 1 \leq k \leq q \).

For \( 1 \leq k \leq q \), by (4.17)-(4.18) and Lemma 2.2, we know that all \( y_k^{2m_k+m} \)'s with \( m \geq \bar{m} + 1 \) have no contribution to the alternative sum \( \sum_{p=-2N-n-1}^{2N-n-1}(-1)^p M_p \), where the Morse-type number \( M_p \) is defined in (2.14). Similarly again by Lemma 2.2 and (4.12)-(4.13), all \( y_k^{2m_k-m} \)'s with \( \bar{m} + 1 \leq m \leq 2m_k - 1 \) only have contribution to \( \sum_{p=-2N-n-1}^{2N-n-1}(-1)^p M_p \).

For \( 1 \leq m \leq \bar{m} \), by (4.16) and Lemma 2.2, we know that all \( y_k^{2m_k+m} \)'s with \( -n \leq i(y_k^m) \) for \( 1 \leq k \leq q_0 \), or \( i(y_k^m) \leq -n - 2 \) for \( q_0 + 1 \leq k \leq q \), have no contribution to the alternative sum \( \sum_{p=-2N-n-1}^{2N-n-1}(-1)^p M_p \). Similarly again by Lemma 2.2 and (4.14), for \( 1 \leq m \leq \bar{m} \), all \( y_k^{2m_k-m} \)'s with \( -n \leq i(y_k^m) \) and \( 1 \leq k \leq q_0 \), or \( i(y_k^m) \leq -n - 2 \) for \( q_0 + 1 \leq k \leq q \), only have contribution to \( \sum_{p=-2N-n-1}^{2N-n-1}(-1)^p M_p \).

Since \( i(y_k^m) \neq -n - 1 \) when \((m\tau_k, y_k)\) is good, by (MMI), Definition 1.1 and Theorem 3.8, we set

\[
M^\circ_+ (k) = \begin{cases} 
\# \{1 \leq m \leq \bar{m} \mid i(y_k^m) \leq -n - 2, \ i(y_k^{2m_k+m}) \in 2Z, \ i(y_k) \in 2Z \}, & \text{if } 1 \leq k \leq q_0, \\
\# \{1 \leq m \leq \bar{m} \mid i(y_k^m) \leq -n, \ i(y_k^{2m_k+m}) \in 2Z, \ i(y_k) \in 2Z \}, & \text{if } q_0 + 1 \leq k \leq q,
\end{cases}
\]

\[
M^\circ_+ (k) = \begin{cases} 
\# \{1 \leq m \leq \bar{m} \mid i(y_k^m) \leq -n - 2, \ i(y_k^{2m_k+m}) \in 2Z - 1, \ i(y_k) \in 2Z - 1 \}, & \text{if } 1 \leq k \leq q_0, \\
\# \{1 \leq m \leq \bar{m} \mid i(y_k^m) \leq -n, \ i(y_k^{2m_k+m}) \in 2Z - 1, \ i(y_k) \in 2Z - 1 \}, & \text{if } q_0 + 1 \leq k \leq q,
\end{cases}
\]

\[
M_+^\circ (k) = \begin{cases} 
\# \{1 \leq m \leq \bar{m} \mid i(y_k^m) \leq -n - 2, \ i(y_k^{2m_k-m}) \in 2Z, \ i(y_k) \in 2Z \}, & \text{if } 1 \leq k \leq q_0, \\
\# \{1 \leq m \leq \bar{m} \mid i(y_k^m) \leq -n, \ i(y_k^{2m_k-m}) \in 2Z, \ i(y_k) \in 2Z \}, & \text{if } q_0 + 1 \leq k \leq q,
\end{cases}
\]

\[
M_+^\circ (k) = \begin{cases} 
\# \{1 \leq m \leq \bar{m} \mid i(y_k^m) \leq -n - 2, \ i(y_k^{2m_k-m}) \in 2Z - 1, \ i(y_k) \in 2Z - 1 \}, & \text{if } 1 \leq k \leq q_0, \\
\# \{1 \leq m \leq \bar{m} \mid i(y_k^m) \leq -n, \ i(y_k^{2m_k-m}) \in 2Z - 1, \ i(y_k) \in 2Z - 1 \}, & \text{if } q_0 + 1 \leq k \leq q,
\end{cases}
\]

which, together with \( i(y_k^{2m_k+m}) - i(y_k^{2m_k-m}) \in 2Z \) by (4.14) and (4.16), yields

\[
M_+^\circ (k) = M^-_+ (k), \quad M_+^\circ (k) = M^\circ_+ (k), \quad \forall 1 \leq k \leq q. \quad (4.22)
\]
For $1 \leq k \leq q_0$ and $1 \leq m \leq \tilde{m}$ satisfying $i(y_k^m) \leq -n - 2$, and for $q_0 + 1 \leq k \leq q$ and $1 \leq m \leq \tilde{m}$ satisfying $i(y_k^m) \geq -n$, by (4.14) and (4.16) it yields

\begin{align*}
i(y_k^{2m_k-m}) & \geq 2N - n, \quad i(y_k^{2m_k+m}) \leq 2N - n - 2, \quad \forall \ 1 \leq k \leq q_0, \quad (4.23) \\
i(y_k^{2m_k-m}) & \leq -2N - n - 2, \quad i(y_k^{2m_k+m}) \geq -2N - n, \quad \forall \ q_0 + 1 \leq k \leq q. \quad (4.24)
\end{align*}

So, for $1 \leq m \leq \tilde{m}$, by (4.23)-(4.24) and Lemma 2.2, we know that all $y_k^{2m_k+m}$s with $i(y_k^m) \leq -n - 2$ and $1 \leq k \leq q_0$, or $i(y_k^m) \geq -n$ and $q_0 + 1 \leq k \leq q$, only have contribution to the alternative sum $\sum_{p=-2N-n-1}^{2N-n-1} (-1)^p M_p$, and all $y_k^{2m_k-m}$s with $i(y_k^m) \leq -n - 2$ and $1 \leq k \leq q_0$, or $i(y_k^m) \geq -n$ and $q_0 + 1 \leq k \leq q$, have no contribution to $\sum_{p=-2N-n-1}^{2N-n-1} (-1)^p M_p$.

Thus for the Morse-type numbers $M_p$’s in (2.14), by (4.21)-(4.24) we have

\begin{align*}
\sum_{p=-2N-n-1}^{2N-n-1} (-1)^p M_p & = \sum_{k=1}^{q} \sum_{1 \leq m \leq \tilde{m} + \bar{m}} (-1)^{d(K) + i(y_k^m)} \dim C_{S^1,d(K)+i(y_k^m)}(F_{a,K}, S^1 \cdot x_k^m) \\
& = \sum_{k=1}^{q} \sum_{n=1}^{2m_k} (-1)^{d(K) + i(y_k^m)} \dim C_{S^1,d(K)+i(y_k^m)}(F_{a,K}, S^1 \cdot x_k^m) \\
& \quad + \sum_{k=1}^{q} \left[ M^e_k(k) - M^o_k(k) \right] - \sum_{k=1}^{q} \left[ M^e_k(k) - M^o_k(k) \right] \\
& \quad - \sum_{1 \leq k \leq q_0} (-1)^{i(y_k^{2m_k})} \dim C_{S^1,d(K)+i(y_k^{2m_k})}(F_{a,K}, S^1 \cdot x_k^{2m_k}) \\
& \quad - \sum_{q_0 + 1 \leq k \leq q} (-1)^{i(y_k^{2m_k})} \dim C_{S^1,d(K)+i(y_k^{2m_k})}(F_{a,K}, S^1 \cdot x_k^{2m_k}) \\
& = \sum_{k=1}^{q} 2m_k \tilde{x}(y_k) \\
& \quad - \sum_{1 \leq k \leq q_0} (-1)^{i(y_k^{2m_k})} \dim C_{S^1,d(K)+i(y_k^{2m_k})}(F_{a,K}, S^1 \cdot x_k^{2m_k}) \\
& \quad - \sum_{q_0 + 1 \leq k \leq q} (-1)^{i(y_k^{2m_k})} \dim C_{S^1,d(K)+i(y_k^{2m_k})}(F_{a,K}, S^1 \cdot x_k^{2m_k}). \quad (4.25)
\end{align*}

In order to exactly know whether the iterate $y_k^{2m_k}$ of $y_k$ has contribution to the alternative sum $\sum_{p=-2N-n-1}^{2N-n-1} (-1)^p M_p$, $1 \leq k \leq q$, we set

\begin{align*}
N^e_+ & = \# \{ q_0 + 1 \leq k \leq q \mid i(y_k^{2m_k}) \leq -2N - n - 2, \ i(y_k^{2m_k}) \in 2Z, \ i(y_k) \in 2Z \} \\
& \quad + \# \{ 1 \leq k \leq q_0 \mid i(y_k^{2m_k}) \geq 2N - n, \ i(y_k^{2m_k}) \in 2Z, \ i(y_k) \in 2Z \}, \quad (4.26) \\
N^o_+ & = \# \{ q_0 + 1 \leq k \leq q \mid i(y_k^{2m_k}) \leq -2N - n - 2, \ i(y_k^{2m_k}) \in 2Z - 1, \ i(y_k) \in 2Z - 1 \}
\end{align*}
where, furthermore, $\Delta \leq q [\text{LoZ})$ and Remark 3.7 that there exist also infinitely many ($N$)

where the first equality holds by Claim 1, the second equality follows from (4.25) and the definitions

Similar to (4.12)-(4.18), for $\hat{\chi}$ Step 2. So (4.30) gives the following estimate

Thus by Claim 1, (4.25), the definitions of $N_+^e$ and $N_+^o$ and (2.16), we have

where the first equality holds by Claim 1, the second equality follows from (4.25) and the definitions
of $N_+^e$ and $N_+^o$, and the last equality follows from $b_{2j} = 1$ and $b_{2j-1} = 0$ for $0 \leq j \leq N - \frac{n-2}{2}$ by (2.16) where $n$ is even.

So (4.30) gives the following estimate

Step 2. The second set of iterates for the choice of the dual vertex $\hat{\chi} = 1 - \chi$ in the cube $[0,1]^l$.

Similar to (4.12)-(4.18), for $\hat{\chi} = \hat{\chi} - \chi$ of the cube $[0,1]^l$ with $\chi$ chosen below (4.3) where $\hat{\chi} = (1, \cdots, 1)$, it follows from Theorem 3.6 (also cf. Theorem 2.8 of [HaW] and Theorem 4.2 of [LoZ]) and Remark 3.7 that there exist also infinitely many ($q+1$)-tuples $(N', m'_1, \cdots, m'_q) \in \mathbb{N}^{q+1}$ such that for any $1 \leq k \leq q$, there holds

where, furthermore, $\Delta_k$ and $\Delta_k'$ satisfy the following relationship

$$\Delta_k + \Delta_k' = C(M_k), \quad \forall \ 1 \leq k \leq q,$$ (4.39)
by the fact \( \hat{\chi} = 1 - \chi \) and the proof of Claim 4 in the proof of Theorem 1.1 of [DLW] or (42) in Theorem 2.8 of [HaW].

Similarly, we define

\[
N^e_+ = \# \{ q_0 + 1 \leq k \leq q \mid i(y^{2m'_k}_k) \leq -2N' - n - 2, \ i(y^{2m'_k}_k) \in 2\mathbb{Z}, \ i(y_k) \in 2\mathbb{Z} \} \\
+ \# \{ 1 \leq k \leq q_0 \mid i(y^{2m'_k}_k) \geq 2N' - n, \ i(y^{2m'_k}_k) \in 2\mathbb{Z}, \ i(y_k) \in 2\mathbb{Z} \} \\
N^o_+ = \# \{ q_0 + 1 \leq k \leq q \mid i(y^{2m'_k}_k) \leq -2N' - n - 2, \ i(y^{2m'_k}_k) \in 2\mathbb{Z} - 1, \ i(y_k) \in 2\mathbb{Z} - 1 \} \\
+ \# \{ 1 \leq k \leq q_0 \mid i(y^{2m'_k}_k) \geq 2N' - n, \ i(y^{2m'_k}_k) \in 2\mathbb{Z} - 1, \ i(y_k) \in 2\mathbb{Z} - 1 \} \\
N^e_- = \# \{ q_0 + 1 \leq k \leq q \mid i(y^{2m'_k}_k) \geq -2N' - n, \ i(y^{2m'_k}_k) \in 2\mathbb{Z}, \ i(y_k) \in 2\mathbb{Z} \} \\
+ \# \{ 1 \leq k \leq q_0 \mid i(y^{2m'_k}_k) \leq 2N' - n - 2, \ i(y^{2m'_k}_k) \in 2\mathbb{Z} - 1, \ i(y_k) \in 2\mathbb{Z} - 1 \}. 
\]

So by (4.35) and (4.39) it yields

\[
i(y^{2m'_k}_k) = 2\theta_k N' - C(M_k) + 2(C(M_k) - \Delta_k) - n - 1 = 2\theta_k N' + C(M_k) - 2\Delta_k - n - 1. 
\]

So by definitions (4.26)-(4.29) and (4.40)-(4.43) we have

\[
N^e_\pm = N^e_+, \quad N^o_\pm = N^o_+. 
\]

Thus, carrying out the arguments similar to (4.30)-(4.31), by Claim 1, the definitions of \( N^e_+ \) and \( N^o_+ \) and (2.16), we have

\[
N' + N^o_+ - N^e_+ = \sum_{k=1}^{q} 2m'_k \hat{\chi}(y_k) + N^o_+ - N^e_+ \\
= \sum_{p=-2N'-n-1}^{2N'-n-1} (-1)^p M_p \\
\leq \sum_{p=-2N'-n-1}^{2N'-n-1} (-1)^p b_p = \sum_{p=0}^{2N'-n-2} b_p \\
= N' - \frac{n}{2},
\]

which, together with (4.45), implies

\[
N^e_- = N^e_+ \geq \frac{n}{2}. 
\]

**Step 3. The summary.**

So by (4.31) and (4.47) it yields

\[
q \geq N^e_+ + N^e_- \geq n. 
\]
In addition, any hyperbolic closed characteristic \( y_k \) must have \( i(y_k^{2m_k}) = 2q_kN - n - 1 \) since there holds \( C(M_k) = 0 \) in the hyperbolic case. However, by (4.26) and (4.28), there exist at least \((N_e^+ + N_e^-)\) closed characteristics with even indices \( i(y_k^{2m_k}) \). So all these \((N_e^+ + N_e^-)\) closed characteristics are non-hyperbolic. Then (4.48) shows that there exist at least \( n \) distinct non-hyperbolic closed characteristics. Now (4.26), (4.28) and (4.48) show that all these non-hyperbolic closed characteristics and their iterations have even Maslov-type indices. This completes the proof of Case 1.

**Case 2.** \( n \) is odd.

In this case, (MMI) still holds. Here the arguments are similar to those in the proof of Case 1. So we only give some different parts in the proof and omit other details.

**Claim 2:** There exist at least \((n - 1)\) geometrically distinct non-hyperbolic closed characteristics denoted by \( \{y_k\}_{k=1}^{n-1} \) with odd Maslov-type indices on such hypersurface \( \Sigma \).

Here one crucial and different point from the proof of Case 1 is that we need to consider the alternative sum \( \sum_{p=-2N-n}^{2N-n}(-1)^pM_p \) (cf. (4.52)) instead of \( \sum_{p=-2N-n-1}^{2N-n-1}(-1)^pM_p \) (cf. (4.25)). This difference is mainly due to the different parity of \( n \). Since the method is similar to that in proof of Case 1, we only list some necessary parts.

At first, there holds \( i(y_k^m) \notin \{-n-2,-n-1,-n\} \) when \((m\tau_k, y_k)\) is good, by (MMI), Definition 1.1 and Theorem 3.8, we set

\[
\begin{align*}
\bar{M}_+^e(k) &= \left\{ \begin{array}{l}
\# \{1 \leq m \leq \bar{m} \mid i(y_k^m) \leq -n - 3, \ i(y_k^{2m_k+m}) \in 2\mathbb{Z}, \ i(y_k) \in 2\mathbb{Z} \}, \text{ if } 1 \leq k \leq q_0, \\
\# \{1 \leq m \leq \bar{m} \mid i(y_k^m) \geq -n + 1, \ i(y_k^{2m_k+m}) \in 2\mathbb{Z}, \ i(y_k) \in 2\mathbb{Z} \}, \text{ if } q_0 + 1 \leq k \leq q,
\end{array} \right.
\end{align*}
\]

\[
\begin{align*}
\bar{M}_+^o(k) &= \left\{ \begin{array}{l}
\# \{1 \leq m \leq \bar{m} \mid i(y_k^m) \leq -n - 3, \ i(y_k^{2m_k+m}) \in 2\mathbb{Z} - 1, \ i(y_k) \in 2\mathbb{Z} - 1 \}, \text{ if } 1 \leq k \leq q_0, \\
\# \{1 \leq m \leq \bar{m} \mid i(y_k^m) \geq -n + 1, \ i(y_k^{2m_k+m}) \in 2\mathbb{Z} - 1, \ i(y_k) \in 2\mathbb{Z} - 1 \}, \text{ if } q_0 + 1 \leq k \leq q,
\end{array} \right.
\end{align*}
\]

\[
\begin{align*}
\bar{M}_-^o(k) &= \left\{ \begin{array}{l}
\# \{1 \leq m \leq \bar{m} \mid i(y_k^m) \leq -n - 3, \ i(y_k^{2m_k-m}) \in 2\mathbb{Z}, \ i(y_k) \in 2\mathbb{Z} \}, \text{ if } 1 \leq k \leq q_0, \\
\# \{1 \leq m \leq \bar{m} \mid i(y_k^m) \geq -n + 1, \ i(y_k^{2m_k-m}) \in 2\mathbb{Z}, \ i(y_k) \in 2\mathbb{Z} \}, \text{ if } q_0 + 1 \leq k \leq q,
\end{array} \right.
\end{align*}
\]

which, together with \( i(y_k^{2m_k+m}) - i(y_k^{2m_k-m}) \in 2\mathbb{Z} \) by (4.14) and (4.16), yields

\[
\bar{M}_+^e(k) = \bar{M}_+^o(k), \quad \bar{M}_-^o(k) = \bar{M}_-^o(k), \quad \forall 1 \leq k \leq q. \tag{4.49}
\]

For \( 1 \leq k \leq q_0 \) and \( 1 \leq m \leq \bar{m} \) satisfying \( i(y_k^m) \leq -n - 3 \), and for \( q_0 + 1 \leq k \leq q \) and \( 1 \leq m \leq \bar{m} \) satisfying \( i(y_k^m) \geq -n + 1 \), by (4.14) and (4.16) it yields

\[
\begin{align*}
\begin{aligned}
i(y_k^{2m_k-m}) &\geq 2N - n + 1, & i(y_k^{2m_k+m}) &\leq 2N - n - 3, & \forall 1 \leq k \leq q_0, \\
i(y_k^{2m_k-m}) &\leq -2N - n - 3, & i(y_k^{2m_k+m}) &\geq -2N - n + 1, & \forall q_0 + 1 \leq k \leq q,
\end{aligned}
\end{align*}
\]
Then, similarly to the equation (4.25), we have
\[
\sum_{p=2N-n}^{2N-n} (-1)^p M_p = \sum_{k=1}^{q} 2m_k \chi(y_k) \\
- \sum_{1 \leq i \leq q_0, i(y_k + 2m_k) \geq 2N-n+1} (-1)^{(y_k + 2m_k)} \dim C_{S^1, d(K)}(F_{a,K}, S^1 \cdot x_k^{2m_k}) \\
- \sum_{q_0+1 \leq i \leq q, i(y_k + 2m_k) \leq 2N-n-3} (-1)^{(y_k + 2m_k)} \dim C_{S^1, d(K)}(F_{a,K}, S^1 \cdot x_k^{2m_k}). \quad (4.52)
\]

Denote by \( H^e_+, H^o_+, H^e_-, H^o_- \) the numbers similarly defined by (4.26)-(4.29) where \( \pm 2N-n \) and \( \pm 2N-n-2 \) are replaced by \( \pm 2N-n+1 \) and \( \pm 2N-n-3 \), respectively.

Then by Claim 1, (4.52), the definitions of \( H^e_+ \) and \( H^o_+ \), and (2.16), we have
\[
N + H^o_+ - H^e_+ = \sum_{k=1}^{q} 2m_k \chi(y_k) + H^o_+ - H^e_+ \\
= \sum_{p=-2N-n}^{2N-n} (-1)^p M_p \\
\leq \sum_{p=-2N-n}^{2N-n} (-1)^p b_p \\
= \sum_{p=0}^{2N-n-1} b_p = \frac{2N-n-1}{2} + 1 \\
= N - \frac{n-1}{2}, \quad (4.53)
\]

which yields
\[
H^e_+ \geq H^e_+ - H^o_+ \geq \frac{n-1}{2}. \quad (4.54)
\]

Similarly, denote by \( H^e_+, H^o_+, H^e_- H^o_- \) the numbers similarly defined by (4.40)-(4.43) where \( \pm 2N'-n \) and \( \pm 2N'-n-2 \) are replaced by \( \pm 2N'-n+1 \) and \( \pm 2N'-n-3 \), respectively, and these numbers satisfy the following relationship
\[
H^e_\pm = H^e_\pm, \quad H^o_\pm = H^o_\pm. \quad (4.55)
\]

Similarly to the inequality (4.46), by the same arguments above and (4.55) we can obtain
\[
H^e_+ = H^e_+ \geq H^e_+ - H^o_+ \geq \frac{n-1}{2}. \quad (4.56)
\]

Therefore it follows from (4.54) and (4.56) that
\[
q \geq H^e_+ + H^e_- \geq \frac{n-1}{2} + \frac{n-1}{2} = n-1. \quad (4.57)
\]
By the same arguments in the proof of Case 1, it follows from the definitions of \( H^+ \) and \( H^- \) that these \((n-1)\) distinct closed geodesics are non-hyperbolic, and the Viterbo indices of them and their iterations are even, and thus the Maslov-type indices of them and their iterations are odd. This completes the proof of Claim 2.

**Claim 3:** There exist at least another geometrically distinct closed characteristic different from those found in Claim 2 with odd Maslov-type indices on such hypersurface \( \Sigma \).

In fact, for those \((n-1)\) distinct closed characteristics \( \{y_k\}_{k=1}^{n-1} \) found in Claim 2, there holds \( i(y_k^{2m}) \neq \pm 2N - n - 1 \) by the definitions of \( H^+ \) and \( H^- \), which, together with (4.12)-(4.18) and (MMI), yields

\[
i(y_k^m) \neq 2N - n - 1, \quad \forall \ m \geq 1, \quad k = 1, \cdots, n - 1. \quad (4.58)
\]

Then by Lemma 2.2 it yields

\[
\sum_{1 \leq k \leq n-1} \dim C_{S^1, d(K)+2N-n-1}(F_{a,K}, S^1 \cdot x_k^m) = 0. \quad (4.59)
\]

By (4.12)-(4.14), (4.16)-(4.17) and (MMI), it yields \( i(y_k^m) \neq 2N - n - 1 \) for any \( m \neq 2m_k \) and \( 1 \leq k \leq q \). Therefore, by (4.59) and (2.14) we obtain

\[
\sum_{n \leq k \leq q} \dim C_{S^1, d(K)+2N-n-1}(F_{a,K}, S^1 \cdot x_k^{2m_k}) = \sum_{n \leq k \leq q, \ m \geq 1} \dim C_{S^1, d(K)+2N-n-1}(F_{a,K}, S^1 \cdot x_k^m) = \sum_{1 \leq k \leq q, \ m \geq 1} \dim C_{S^1, d(K)+2N-n-1}(F_{a,K}, S^1 \cdot x_k^m) = M_{2N-n-1} \geq b_{2N-n-1} = 1. \quad (4.60)
\]

Now by (4.60) and Lemma 2.2, it yields that there exist at least another closed characteristic \( y_n \) with \( i(y_n^{2m_n}) = 2N - n - 1 \) and \( i(y_n^{2m_n}) - i(y_n) \in 2\mathbb{Z} \). Thus \( y_n \) and its iterations have odd Maslov-type indices. This completes the proof of Claim 3.

Now for Case 2, Theorem 1.2 follows from Claim 2 and Claim 3. The proof of Theorem 1.2 is finished.

\[\square\]

**5 Proof of Theorem 1.6**

In this section, we prove Conjecture 1.5 for the case of \( n = 3 \), i.e., Theorem 1.6, by contradiction. We assume first the following condition (C):

\[(C) \text{ There are finitely many prime closed characteristics } \{(\tau_k, y_k)\}_{1 \leq k \leq q} \text{ on } \Sigma \in \mathcal{H}_{st}(6), \text{ and } \hat{i}(y_k) = 0 \text{ for } 1 \leq k \leq q_0 \text{ with some integers } q_0 \in [1, q] \text{ and } q \in \mathbb{N}.\]
Let $P_{\Sigma} = \{m\tau_k \mid 1 \leq k \leq q, m \in \mathbb{N}\}$ be the period set of all closed characteristics on $\Sigma$.

Denote by $\gamma_k \equiv \gamma_{y_k}$ the associated symplectic path of $(\tau_k, y_k)$ for $1 \leq k \leq q$. Then by Lemma 3.3 of [HuL] and Lemma 3.2 of [Lon1], there exists $P_k \in Sp(6)$ and $U_k \in Sp(4)$ such that

$$M_k \equiv \gamma_k(\tau_k) = P_k^{-1}(N_1(1,1)\circ U_k)P_k, \quad \forall 1 \leq k \leq q,$$

where every $U_k$ has the following form by Theorem 3.2:

$$R(\theta_1) \circ \cdots \circ R(\theta_r) \circ D(\pm 2)^\circ s$$
$$\circ N_2(e^{\alpha_1\sqrt{-1}}, A_1) \circ \cdots \circ N_2(e^{\alpha_r\sqrt{-1}}, A_r) \circ N_2(e^{\beta_1\sqrt{-1}}, B_1) \circ \cdots \circ N_2(e^{\beta_r\sqrt{-1}}, B_{r_0}),$$

where $\frac{\theta_j}{2\pi} \notin Q$ for $1 \leq j \leq r$; $\frac{\alpha_j}{2\pi} \notin Q$ for $1 \leq j \leq r_s$; $\frac{\beta_j}{2\pi} \notin Q$ for $1 \leq j \leq r_0$ and

$$r + s + 2r_s + 2r_0 = 2.$$

Hence by (5.1), Theorem 3.8 and the precise index iteration formulae for symplectic paths due to Y. Long (cf. [Lon3] or Chapter 8 of [Lon4]), we have

$$i(y_k^m) = m(i(y_k) + n + 1 - r) + 2 \sum_{j=1}^r \left\lfloor \frac{m\theta_j}{2\pi} \right\rfloor + r - 1 - n$$
$$= m(i(y_k) + 4 - r) + 2 \sum_{j=1}^r \left\lfloor \frac{m\theta_j}{2\pi} \right\rfloor + r - 4, \quad \forall m \geq 1, 1 \leq k \leq q,$$

where in (5.3), we have used $E(a) = [a] + 1$ for $a \in \mathbb{R} \setminus \mathbb{Z}$. Thus

$$\tilde{i}(y_k) = i(y_k) + 4 - r + \sum_{j=1}^r \frac{\theta_j}{\pi}, \quad \forall 1 \leq k \leq q.$$

**Lemma 5.1.** When $\tilde{i}(y_k) = 0$ with $1 \leq k \leq q_0$, we have $i(y_k^m) = -4$ for any $m \in \mathbb{N}$.

**Proof.** By (5.4) we have

$$i(y_k) + 4 - r + \sum_{j=1}^r \frac{\theta_j}{\pi} = 0, \quad \forall 1 \leq k \leq q_0.$$

Note that $\frac{\theta_j}{\pi} \notin Q$ and $0 \leq r \leq 2$ by (5.2), and then it yields $r = 0$ or 2.

If $r = 0$, then $i(y_k) = -4$ by (5.5). Together with (5.3), it yields $i(y_k^m) = -4$ for all $m \in \mathbb{N}$.

If $r = 2$, note that $\sum_{j=1}^2 \frac{\theta_j}{\pi} \in (0, 4)$ and $i(y_k) \in 2\mathbb{Z}$ by (8.1.8) of Theorem 8.1.4 and (8.1.29) of Theorem 8.1.7 of [Lon4], then by (5.5) we have $\sum_{j=1}^2 \frac{\theta_j}{\pi} = 2$ and $i(y_k) = -4$. Since $\sum_{j=1}^2 \frac{m\theta_j}{2\pi} = m$ implies that $\sum_{j=1}^2 \left\lfloor \frac{m\theta_j}{2\pi} \right\rfloor = m - 1$, so by (5.3) we have

$$i(y_k^m) = -2m + 2 \sum_{j=1}^r \left\lfloor \frac{m\theta_j}{2\pi} \right\rfloor - 2 = -2m + 2(m - 1) - 2 = -4, \quad \forall m \geq 1.$$
The proof of Lemma 5.1 is finished.

Remark 5.2. In our proof of Theorem 1.6 below, we shall apply results in Section 7 of [Vit2] frequently. Note that in the Theorem 7.1 of [Vit2] and its proof, the two end points of the index interval \( I \) were carefully avoided (cf. Theorem 7.1 on p.637, (7.24) and (7.25) in p.647, and the top part on p.648 of [Vit2]), which are due to the effect of the \( S^1 \)-action on the homologies with two adjacent dimensions (cf. Corollary in Appendix 1 on p.653 of [Vit2]). In our proof of Theorem 1.6 we are dealing with only non-degenerate critical orbits. Note that because in our case the critical orbit \( S^1 \cdot y_k^m \) studied below is always orientable on the star-shaped hypersurface \( \Sigma \), by the homological correspondence of the Theorem I.7.5 and the comments below it on pp.78-79 of [Cha], only the homology with dimension to be precisely equal to the Morse index of the critical orbit survives. Therefore in our case, all the results in (7.24) and (7.25) of [Vit2] work for all \( k \in I \), not only for \( k \in I^0 \). Specially, by (5.7) below, whenever the dimension of the related homology is \( d(K) - 3 \) or \( d(K) - 5 \), i.e., the Viterbo index of the corresponding closed characteristic is \(-3\) or \(-5\), results in (7.24) and (7.25) of [Vit2] can be applied. Such arguments have no requirement on those Viterbo index near and is not \(-3\) and \(-5\) of the related closed characteristics, and thus can be applied to the case in the current paper. This understanding is applied below, whenever we apply Theorem 7.1 of [Vit2]. We refer readers also to Theorem 5.1 and its proof in [LLW] which generalized Theorem 7.1 of [Vit2] to the degenerate case.

Proof of Theorem 1.6.

Based on Lemma 5.1, we carry out the proof of Theorem 1.6 in several steps below.

Step 1. On one hand, by Lemma 5.1, there always holds \( i(y_k^m) = -4 \) for any \( 1 \leq k \leq q_0 \) and \( m \in \mathbb{N} \). On the other hand, note that \( \hat{i}(y_k) > 0 \) (respectively \( < 0 \)) implies \( i(y_k^m) \to +\infty \) (respectively \( -\infty \)) as \( m \to +\infty \). Thus iterates \( y_k^m \) of every \( y_k \) for \( q_0 + 1 \leq k \leq q \) have indices satisfying \( i(y_k^m) \neq -3 \) and \(-5\) for any large enough \( m \in \mathbb{N} \). Therefore for large enough \( a \), all the closed characteristics \( y_k^m \) for \( 1 \leq k \leq q \) with period larger than \( aT \), which implies that the iterate number \( m \) is very large, will have their Viterbo indices:

\[
\begin{align*}
\{ \\
eq 4, & \quad \text{when } \hat{i}(y_k) = 0, \\
\text{or (ii) different from } -3, -4 \text{ and } -5, & \quad \text{when } \hat{i}(y_k) \neq 0. 
\end{align*}
\]

(5.7)

Step 2. For \( a \in \mathbb{R} \), let \( X^-(a,K) = \{ x \in X \mid F_{a,K}(x) < 0 \} \) with \( K = K(a) \) as defined in the above (2.2) as well as in Section 7 of [Vit2]. Note also that the origin 0 of \( X \) is not contained in \( X^-(a,K) \) by definition. Because the Hamiltonian function \( H_{a,K} \) is quadratic homogeneous as assumed at the beginning of Section 7 of [Vit2] due to the study there being near the origin, the functional \( F_{a,K} \) is homogeneous too.
For any large enough positive \( a < a' \), we fix the same constant \( K > 0 \) in (2.2) to be sufficiently large such that the Hamiltonian function \( H_{t,K}(x) \) is strictly convex for every \( t \in [a,a'] \). Now let \( A = X^-(a,K) \) and \( A' = X^-(a',K) \). Because the period set \( P_2 \) defined at the beginning of this section is discrete, we choose the above constants \( a \) and \( a' \) carefully such that \( aT \) and \( a'T \) do not belong to \( P_2 \). Note that for \( t \in [a,a'] \) because every critical orbit \( S^1 \cdot x \) of the functional \( F_{t,K} \) always possesses the critical value \( F_{t,K}(S^1 \cdot x) = 0 \) as mentioned in p.639 of [Vit2] and by (2.7) of [LLW], the boundary sets of \( A \) and \( A' \), i.e., \( \{ x \mid F_{t,K}(x) = 0 \} \) with \( t = a \) or \( a' \), possess no critical orbits, and specially the origin \( 0 \) of \( X \) is not contained in \( A \) and \( A' \). Therefore by the homogeneity mentioned above we have

\[
H_{S^1,a}(A',A) = H_{S^1,a}(A' \cap S(X), A \cap S(X)), \quad \forall i \in \mathbb{Z}, \tag{5.8}
\]

where \( S(X) \) is the unit sphere of \( X \).

Because each critical orbit \( S^1 \cdot x \) of the functional \( F_{t,K} \) for some \( t \in [a,a'] \) corresponds to an iterate \( y_k^m \) for some \( 1 \leq k \leq q \) and \( m \in \mathbb{N} \), we denote this critical orbit by \( S^1 \cdot x_{t,k,m} \). Thus by the definition of \( F_{t,K} \) we have the period \( T_{t,k,m} \) of \( x_{t,k,m} \) satisfies \( T_{t,k,m} = tT \). Consequently every critical orbit \( S^1 \cdot x_{t,k,m} \) of \( F_{t,K} \) for some \( t \in [a,a'] \) contained in \( (A' \setminus A) \cap S(X) \) must satisfy

\[
aT \leq T_{t,k,m} \leq a'T. \tag{5.9}
\]

Thus the period of the corresponding \( y_k^m \) is also \( T_{t,k,m} \) and satisfies (5.9) too. Consequently the total number of such critical orbits contained in \( (A' \setminus A) \cap S(X) \) is finite, which is denoted by \( \hat{j} \). That is, there exist precisely \( \hat{j} \) times of \( t \in (a,a') \) which we denote by \( t_j \) with \( 1 \leq j \leq \hat{j} \) satisfying \( a < t_1 < \ldots < t_j < a' \), such that \( F_{t_j,K} \) with \( 1 \leq j \leq \hat{j} \) possesses critical orbits in \( (A' \setminus A) \cap S(X) \), and any other \( F_{t,K} \) with \( t \in [a,a'] \) \( \{ t_j \mid 1 \leq j \leq \hat{j} \} \) possesses no any critical orbit in \( (A' \setminus A) \cap S(X) \).

In order to compute the homology in (5.8), we introduce below a new functional \( \tilde{t} \), which is motivated by the proof of Proposition 3 in Appendix 1 of [Vit2].

**Claim 4.** The partial derivative \( \frac{\partial}{\partial t} F_{t,K}(x) \) of \( F_{t,K}(x) \) with respect to \( t \in [a,a'] \) satisfies \( \frac{\partial}{\partial t} F_{t,K}(x) < 0 \) for all \( (x,t) \in S(X) \times [a,a'] \).

In fact, by the definition of \( H_t(x) \) in Section 2, it is strictly increasing in \( t \) when \( x \neq 0 \), and then so is \( H_{t,K}(x) \). Then the Fenchel dual function \( H^*_{t,K}(y) \) is strictly decreasing in \( t \) when \( y \neq 0 \). Consequently \( F_{t,K}(x) \) is strictly decreasing in \( t \) too. Thus Claim 4 is proved.

Now based on Claim 4 and the well-known implicity function theorem, there exists a unique smooth function \( \tilde{t} : S(X) \to \mathbb{R} \) given by the equation

\[
F_{\tilde{t}(x),K}(x) = 0, \quad \forall x \in S(X). \tag{5.10}
\]

It further implies

\[
\frac{\partial}{\partial t} F_{\tilde{t}(x),K}(x) \tilde{t}'(x) + F'_{\tilde{t}(x),K}(x) = 0, \quad \forall x \in S(X). \tag{5.11}
\]
Then for \((x_0, t_0) \in S(X) \times \mathbb{R}\), we have that \(x_0\) is a critical point of \(\hat{t}\) with critical value \(t_0\) if and only if \(F'_{t_0,K}(x_0) = 0\) by (5.11). Note that \(\hat{t}\) is \(S^1\)-invariant since so is \(F_{t,K}\).

**Claim 5.** At any critical point \(x_0\) of \(\hat{t}\) with critical value \(t_0\), we have

\[
C_{S^1, \ast}(\hat{t}, x_0) \cong C_{S^1, \ast}(F_{t_0,K \mid S(X)}, x_0).
\]

In fact, let \(U\) be a small enough \(S^1\)-invariant open neighborhood of \(S^1 \cdot x_0\) in \(S(X)\). Since \(\partial F_{t,K}(x_0)/\partial t < 0\) by Claim 4, we obtain that \(\hat{t}(x) \leq t_0\) for \(x \in U\) if and only if \(F_{t_0,K}(x) \leq 0\) by (5.10). Thus we obtain

\[
\begin{align*}
\{ x \in U \mid \hat{t}(x) \leq t_0 \} &= \{ x \in U \mid F_{t_0,K}(x) \leq 0 \}, \\
\{ x \in U \mid \hat{t}(x) \leq t_0 \} \setminus \{ S^1 \cdot x_0 \} &= \{ x \in U \mid F_{t_0,K}(x) \leq 0 \} \setminus \{ S^1 \cdot x_0 \}. 
\end{align*}
\]

Then by the definition of \(S^1\)-critical module in Sections I.4 and I.7 of [Cha], the two \(S^1\)-critical modules in (5.12) are isomorphic to each other at every dimension. Thus Claim 5 holds.

**Remark 5.3.** Note that the isomorphic identity (5.12) holds without further showing that functional \(\hat{t}(x)\) is \(C^2\) and its Morse index and nullity at its critical point \(x_0\) with critical value \(t_0\) are the same as those of the functional \(F_{t_0,K}(x_0)\) at its critical point \(x_0\), although these can be proved by using the implicity function theorem and (5.11). Here the Hessian matrices of these two functionals differ by only a positive constant which can be obtained by differentiating both sides of (5.11) with respect to \(x\), and then evaluating at the critical points respectively.

By Claim 4 and (5.10), we then obtain

\[
\begin{align*}
A' \cap S(X) &= \{ x \in S(X) \mid F_{a',K}(x) < 0 \} = \{ x \in S(X) \mid \hat{t}(x) < a' \}, \\
A \cap S(X) &= \{ x \in S(X) \mid F_{a,K}(x) < 0 \} = \{ x \in S(X) \mid \hat{t}(x) < a \}.
\end{align*}
\]

Note that both \(a\) and \(a'\) are regular values of \(\hat{t}\) since \(aT\) and \(a'T\) do not belong to \(P_\Sigma\). Then for small enough \(\epsilon > 0\), \(A' \cap S(X)\) and \(A \cap S(X)\) are \(S^1\)-homotopy equivalent with \(\hat{t}^{a'-\epsilon}\) and \(\hat{t}^{a+\epsilon}\) respectively, where \(\hat{t}^\kappa\) denotes the level set \(\hat{t}^\kappa = \{ x \in S(X) \mid \hat{t}(x) \leq \kappa \}\). Thus by the homotopy invariance of the homology, we obtain

\[
H_{S^1,d(K)+i}(A' \cap S(X), A \cap S(X)) = H_{S^1,d(K)+i}(\hat{t}^{a'-\epsilon}, \hat{t}^{a+\epsilon}), \quad \forall i \in \mathbb{Z}.
\]

**Step 3.** Now for the chosen large enough \(a\) and \(a'\) with \(a < a'\), by (5.7) there exists no any closed characteristic whose period locates between \(aT\) and \(a'T\) possessing Viterbo index \(-3\) or \(-5\). Therefore by the discussion in pp.78-79 of [Cha], we obtain

\[
C_{S^1,d(K) - 3}(F_{t_j,K \mid S(X)}, S^1 \cdot x_{t_j,k,m}) = C_{S^1,d(K) - 5}(F_{t_j,K \mid S(X)}, S^1 \cdot x_{t_j,k,m}) = 0.
\]
which together with (5.12) yields
\[ C_{S^1,d(K)-3}(\hat{t}, S^1, x_{t_j,k,m}) = C_{S^1,d(K)-5}(\hat{t}, S^1, x_{t_j,k,m}) = 0. \] (5.15)

Combining (5.15) with an equivariant version of Theorem I.4.3 of [Cha], i.e., the Morse inequality, we obtain
\[ H_{S^1,d(K)-3}(\hat{t}^{a'}, \hat{t}^{a'+\epsilon}) = H_{S^1,d(K)-5}(\hat{t}^{a' - \epsilon}, \hat{t}^{a'+\epsilon}) = 0. \] (5.16)

Therefore, combining (5.16) with (5.8) and (5.14), we obtain
\[ H_{S^1,d(K)-3}(A', A) = H_{S^1,d(K)-5}(A', A) = 0. \] (5.17)

**Step 4.** Now we consider the following exact sequence of the triple \((X, A', A)\)
\[ \cdots \longrightarrow H_{S^1,d(K)-3}(A', A) \xrightarrow{i_{3*}} H_{S^1,d(K)-3}(X, A) \xrightarrow{j_{3*}} H_{S^1,d(K)-3}(X, A') \]
\[ \xrightarrow{\partial_{3*}} H_{S^1,d(K)-4}(A', A) \xrightarrow{i_{4*}} H_{S^1,d(K)-4}(X, A) \xrightarrow{j_{4*}} H_{S^1,d(K)-4}(X, A') \]
\[ \xrightarrow{\partial_{4*}} H_{S^1,d(K)-5}(A', A) \longrightarrow \cdots. \] (5.18)

It follows from (7.4) on p.639 of [Vit2] that the homomorphisms \(j_{3*}\) in (5.18) is a zero map. Thus (5.17) and (5.18) yield
\[ H_{S^1,d(K)-3}(X, A) = \text{Ker}(j_{3*}) = \text{Im}(i_{3*}) = H_{S^1,d(K)-3}(A', A) = 0. \] (5.19)

Now we fix the above chosen \(a' > 0\) and choose another large enough \(a'' > a'\), and enlarge the constant \(K\) in (2.2) chosen above (5.8) so that the conclusions between (5.7) and (5.8) hold when we replace \((a, a')\) by \((a', a'')\). Then repeating the above proof with the long exact sequence of the triple \((X, A'', A')\) instead of \((X, A', A)\) in the above arguments with \(A'' = X^{-1}(a'', K)\), similarly we obtain
\[ H_{S^1,d(K)-3}(X, A') = 0. \] (5.20)

Together with (5.19) and (5.20), (5.18) yields
\[ 0 \xrightarrow{\partial_{3*}} H_{S^1,d(K)-4}(A', A) \xrightarrow{i_{4*}} H_{S^1,d(K)-4}(X, A) \xrightarrow{j_{4*}} H_{S^1,d(K)-4}(X, A') \xrightarrow{\partial_{4*}} 0. \] (5.21)

**Step 5.** When \(a\) increases, we always meet infinitely many closed characteristics with Viterbo index \(-4\) due to the existence of \(y_k\) with \(\hat{t}(y_k) = 0\) for \(1 \leq k \leq q_0\) by Lemma 5.1. For the above chosen large enough \(a < a'\), there exist only finitely many closed characteristics among \(\{y_k^m \mid 1 \leq k \leq q_0, m \geq 1\}\) such that their periods locate between \(aT\) and \(a'T\). Therefore for the corresponding critical orbits \(S^1 \cdot x_{t_j,k,m}\) of \(F_{t_j,K}\), all of them possess Morse index \(d(K) - 4\). Then
by the equivariant version of Theorem I.4.2 as well as the discussions there on pp. 78-79 of [Cha],
when the condition (C) holds, i.e., \( q_0 \geq 1 \) here, we obtain

\[
H_{S^1, d(K)-4}(A', A) = H_{S^1, d(K)-4}(A' \cap S(X), A \cap S(X))
= H_{S^1, d(K)-4}(\hat{t}_{a'-\epsilon}, \hat{t}_{a+\epsilon})
= \bigoplus_{aT \leq T_t, k, m \leq a'T}
C_{S^1, d(K)-4}(\hat{t}, S^1 \cdot x_{t, k, m})
= \bigoplus_{aT \leq T_t, k, m \leq a'T}
C_{S^1, d(K)-4}(F_{t, K} | S(X), S^1 \cdot x_{t, k, m})
= \bigoplus_{aT \leq T_t, k, m \leq a'T}
Q \neq 0,
\]

(5.22)

where the first equality follows from (5.8), the second equality follows from (5.14), the fourth
equality follows from (5.12), and for the third equality we give more explanations as follows:

Denote by

\[
M_q(a, a') = \bigoplus_{aT \leq T_t, k, m \leq a'T}
\text{rank } C_{S^1, q}(\hat{t}, S^1 \cdot x_{t, k, m}),
\]

\[
\beta_q(a, a') = \text{rank } H_{S^1, q}(\hat{t}_{a'-\epsilon}, \hat{t}_{a+\epsilon}).
\]

Then \( M_{d(K)-3}(a, a') = M_{d(K)-5}(a, a') = 0 \) and \( \beta_{d(K)-3}(a, a') = \beta_{d(K)-5}(a, a') = 0 \) hold by (5.15)
and (5.16) respectively, which together with an equivariant version of Theorem I.4.3 of [Cha] yield
\( M_{d(K)-4}(a, a') = \beta_{d(K)-4}(a, a') \). Then the third equality in (5.22) holds.

**Step 6.** By the exactness of the sequence (5.21) and (5.22), we obtain

\[
H_{S^1, d(K)-4}(X, A) = H_{S^1, d(K)-4}(A', A) \bigoplus H_{S^1, d(K)-4}(X, A') \neq 0.
\]

Then, by our choice of \( a, a' \) and \( a'' \), and replacing \( (X, A', A) \) by \( (X, A'', A') \) in the above arguments,
similarly we obtain

\[
H_{S^1, d(K)-4}(X, A') \neq 0. \tag{5.23}
\]

Now on one hand, if \( j_{4*} \) in (5.21) is a trivial homomorphism, then by the exactness of the
sequence (5.21) it yields

\[
H_{S^1, d(K)-4}(X, A') = \text{Ker}(\partial_{4*}) = \text{Im}(j_{4*}) = 0,
\]

which contradicts to (5.23). Therefore \( j_{4*} \) in (5.21) is a non-trivial homomorphism.

However, on the other hand, by (7.4) of [Vit2], \( j_{4*} \) in (5.21) is a zero homomorphism. This
contradiction completes the proof of Theorem 1.6.
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