On the way towards a generalized entropy maximization procedure

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Abstract

We propose a generalized entropy maximization procedure, which takes into account the generalized averaging procedures and information gain definitions underlying the generalized entropies. This novel generalized procedure is then applied to Rényi and Tsallis entropies. The generalized entropy maximization procedure for Rényi entropies results in the exponential stationary distribution asymptotically for $q \in [0, 1]$ in contrast to the stationary distribution of the inverse power law obtained through the ordinary entropy maximization procedure. Another result of the generalized entropy maximization procedure is that one can naturally obtain all the possible stationary distributions associated with the Tsallis entropies by employing either ordinary or $q$-generalized Fourier transforms in the averaging procedure.

Key words: entropy maximization procedure, generalized entropies

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1 Introduction

The inverse power law distributions are ubiquitous in nature, emerging in such diverse fields as subregion laser cooling [1], the heartbeat histograms of healthy patients [2], plasmas [3], conservative motion in 2-D periodic potentials [4], controlled decoherence [5], rheology of steady-state draining foams [6], DNA slippage step-length distributions [7], econophysics [8], earthquake models [9], to name but a few.

On the other hand, it is well-known that the stationary distribution obtained from the maximization of the Boltzmann-Gibbs (BG) entropy is of exponential form. In this sense, it cannot always be used in modeling phenomena

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exhibiting inverse power law behavior. Therefore, there has been an increasing interest in the generalized entropies such as Tsallis [10], Rényi [11] and Sharma-Mittal (SM) [12] entropies whose stationary solutions are of inverse power law form. Although these different entropy measures yield to inverse power law stationary distributions, they differ from one another in many aspects. For example, Rényi entropy is additive whereas Tsallis entropy is not. Both Tsallis and SM entropies are nonadditive, but the former is one parameter generalization whereas the latter is a two-parameter generalization of BG entropy. However, one common structure underlying these generalized entropies is that all of them are obtained through the joint generalization of the averaging procedures and the concept of information gain. For example, Rényi entropy preserves the same definition of information gain as BG measure, but makes use of a different averaging procedure than the one used for BG entropy, namely, exponential averaging procedure. Although this averaging procedure seems to stem from information-theoretic approaches, it has wide range of applications in nonequilibrium statistical physics. For example, the free energy difference for arbitrary transformations is nothing but the exponential average of total work done during the process as given by Jarzynski equality [13]. This equality is the cornerstone of experimental investigations in several diverse fields [14].

On the other hand, Tsallis measure preserves the same averaging procedure as BG measure i.e., linear averaging, but generalizes the concept of information gain by deforming the logarithmic function. The SM entropy benefits generalizations of both kind in its structure [15].

Although the mathematical structure of these generalized entropies is well understood, the entropy maximization procedure (EMP) applied to these generalized measures does not take the aforementioned structure into account, in general. In fact, the choice of constraints mostly relies on trial and error or the arguments of suitability in order to obtain a stationary distribution of inverse power law. For example, it is emphasized in the literature that Tsallis and Rényi entropies are monotonic functions of one another and therefore yield the same stationary distribution under the same constraints. However, there is no reason to apply the same set of constraints to both, since their mathematical structure is completely different. At this point, it is worth remark that one can obtain a stationary distribution of inverse power law form even by using BG entropy with suitably chosen constraints [16]. However, we discard this possibility, since one cannot justify the form of constraint necessary for this maximization in a reasonable manner [16, 17]. In short, it has been an open problem how the constraints for the generalized entropies must be chosen (see Ref. [18] for the choice of constraints and the nonadditive formalism).

In this paper, we propose a new EMP, which takes into account the general mathematical structure of the generalized entropy measures in terms of
the underlying definition of the information gain and generalized averaging procedure. This novel procedure will be hereafter called generalized entropy maximization procedure (GEMP).

The outline of the paper is as follows. In section II, we review the mathematical structure of the Rényi entropy and generalized averaging procedures. The application of GEMP to Rényi entropy is presented in Section III. Section IV is the application of GEMP to Tsallis entropy. Conclusions are presented in Section V.

2 Generalized averaging procedures and Rényi entropy

It is well-known that BG entropy is the linear average of the elementary information gain \( \log_b(1/p_i) \) associated with an event of probability \( p_i \) i.e.,

\[
S_{BG}(p) = \left\langle \log_b \left( \frac{1}{p_i} \right) \right\rangle_{\text{lin}}
\]

(1)

where the linear average is defined as

\[
\langle x_i \rangle_{\text{lin}} \equiv \sum_i^W p_i x_i
\]

(2)

\( W \) being the total number of configurations of the system. It should be noted that from here on we will use the natural base i.e., \( b = e \) and denote \( \log_e(1/p_i) \) as \( \ln(1/p_i) \) without loss of generality. In order to generalize BG entropy, A. Rényi considered whether other forms of averaging procedures are possible or not. He then adopted the generalized averaging procedure developed by Kolmogorov and Nagumo [19, 20]. Kolmogorov and Nagumo independently showed that the averaging procedure must be extended to quasi-linear mean defined as

\[
S = f^{-1} \left[ \sum_i^W p_i f \left( \ln \frac{1}{p_i} \right) \right]
\]

(3)

where \( f \) is a strictly monotone continuous and invertible function called Kolmogorov-Nagumo function (K-N function). The importance of this extension of the averaging procedure is understood, since it succeeds the generalization by preserving conformity to Kolmogorov axioms of probability as shown by Kolmogorov and Nagumo. Rényi then showed that only two possible K-N functions exist.
if one is restricted to additive measures i.e., for two systems described by two independent probability distributions $A$ and $B$, the entropy measure satisfies $S(A \cup B) = S(A) + S(B \mid A)$, where the conditional probability $S(B \mid A)$ is defined as $S(B \mid A) = \sum_i p_i(A)S(B \mid A = A_i)$. The first possible K-N function is the linear mean defined in Eq. (2) and reads $f(x) = x$. The linear mean of the information gain results in the BG entropy i.e., Eq. (1). The second possibility is the exponential averaging defined by

$$f(x) = c_1e^{(1-q)x} + c_2$$

where $q$ is a real parameter, and $c_1$, $c_2$ are two arbitrary constants [11, 15]. The arbitrary constants $c_1$ and $c_2$ can be chosen, without loss of generality, as $\frac{1}{1-q}$ and $\frac{1}{q-1}$, respectively. The general expression for the exponential average of a quantity, using Eqs. (3) and (4), can then be written as

$$\langle x_i \rangle_{\exp} = \frac{1}{1-q} \ln \left( 1 + \sum_i p_i e^{(1-q)x_i} - \langle 1 \rangle_{\lin} \right).$$

The expression $\langle 1 \rangle_{\lin}$ is equal to one (more on this later) so that

$$\langle x_i \rangle_{\exp} = \frac{1}{1-q} \ln \left( \sum_i p_i e^{(1-q)x_i} \right).$$

Now, it is not difficult to see that the exponential average of the ordinary information gain results in the Rényi entropy [11, 15] i.e.,

$$S_R(p) = \langle \ln \left( \frac{1}{p_i} \right) \rangle_{\exp} = \frac{1}{1-q} \ln \left( \sum_i p_i^q \right)$$

where $\langle \cdot \rangle_{\exp}$ stands for the exponential averaging procedure defined by Eq. (6).

Since the exponential average becomes the linear average in the $q \to 1$ limit, the Rényi entropy becomes the BG entropy in the same limit, i.e., as $q \to 1$. In other words, the only difference between the Rényi and BG entropies is due to the different averaging procedure used although the same definition of the information gain $\ln(1/p_i)$ is preserved in both measures.

Note that, through Eq. (5), the exponential average of 1 can be written as

$$\langle 1 \rangle_{\exp} = \frac{1}{1-q} \ln \left[ 1 + e^{(1-q)\langle 1 \rangle_{\lin} - \langle 1 \rangle_{\lin}} \right].$$
The above equality shows us that once the normalization through linear averaging procedure is carried out i.e., $\langle 1 \rangle_{\text{lin}} = 1$, the normalization through exponential averaging procedure i.e., $\langle 1 \rangle_{\text{exp}} = 1$ is ensured, and vice versa.

At this point, it is worth mentioning that if one sets the arbitrary constants in Eq. (4) as $c_1 = 1$, $c_2 = 0$ and $1 - q = -\beta$, then the Jarzynski equality $\Delta F = -\beta \ln \langle \exp(-\beta W) \rangle_{\text{lin}}$ [13] can simply be written as $\Delta F = \langle W \rangle_{\text{exp}}$.

### 3 \textbf{GEMP and Rényi entropy}

Since the seminal work of Jaynes, the entropy maximization procedure played an important role in obtaining the stationary distribution associated with a particular entropy measure. For example, the maximization of the BG entropy has been carried out by using the following functional

$$\Phi_{BG} = \left\langle \ln \frac{1}{p_i} \right\rangle_{\text{lin}} - \alpha \langle 1 \rangle_{\text{lin}} - \beta \langle \varepsilon_i \rangle_{\text{lin}}$$

(9)

to obtain the concomitant stationary distribution

$$p_i = \exp(-S_{BG} + \beta U - \beta \varepsilon_i),$$

(10)
denoting $\langle \varepsilon_i \rangle_{\text{lin}}$ by $U$, as usual. The stationary distribution in Eq. (10) can be cast into the form $p_i = \frac{e^{-\beta \varepsilon_i}}{Z}$, where the partition function is given by $Z = e^{S_{BG} - \beta U}$. It should be noted that the linear averages in Eq. (9) are carried out using ordinary probability distribution (e.g., not escort distribution), since ordinary information gain written in terms of ordinary logarithmic function is used in the definition of BG entropy. This fact can be better understood by remembering that a function $f(x)$ is completely and uniquely determined by its moments when the moments are calculated in terms of the ordinary probability distribution and with the help of ordinary Fourier transform. Although the Rényi entropy preserves the same definition of ordinary information gain (and must therefore be maximized using ordinary probability distribution), the averaging procedure is exponential rather than linear. However, the maximization of the Rényi entropy too is done exactly in the same way as in BG entropy in the literature [21-29] i.e.,

$$\Phi_R = \left\langle \ln \frac{1}{p_i} \right\rangle_{\text{exp}} - \alpha \langle 1 \rangle_{\text{lin}} - \beta \langle \varepsilon_i \rangle_{\text{lin}},$$

(11)
which is \textit{completely inconsistent}, since the same averaging procedure is not used throughout the functional to be maximized. Instead, a consistent maximization would require the use of the following functional

\[ \Phi_R = \left\langle \ln \frac{1}{p_i} \right\rangle_{\exp} - \alpha \left\langle 1 \right\rangle_{\exp} - \beta \left\langle \varepsilon_i \right\rangle_{\exp}. \]  \hspace{1cm} (12)

The maximization of this consistent functional yields

\[ \frac{\delta \Phi_R}{\delta p_i} = \frac{q p_i^{q-1}}{\sum_j p_j^q} - \alpha \left( 1 - e^{q-1} \right) - \beta \left[ e^{(1-q)(\varepsilon_i - U)} - e^{(q-1)U} \right] = 0 \]  \hspace{1cm} (13)

where \( U \) is consistently calculated through \( \left\langle \varepsilon_i \right\rangle_{\exp} \). After a little algebra, one obtains the stationary distribution

\[ p_i = \left[ \frac{\sum_j p_j^q}{q} (1 - e^{q-1}) + \frac{\sum_j p_j^q}{q} \left( e^{(1-q)(\varepsilon_i - U)} - e^{(q-1)U} \right) \right]^{\frac{1}{q-1}}. \]  \hspace{1cm} (14)

In order to eliminate the Lagrange multiplier \( \alpha \), we multiply Eq. (13) by \( p_i \) and sum over the index \( i \) to obtain

\[ \alpha = q + \beta \left[ e^{(q-1)U} - 1 \right]. \]  \hspace{1cm} (15)

By substituting \( \alpha \) given by Eq. (15) into Eq. (14), we obtain the stationary distribution associated with the Rényi entropy measure as

\[ p_i = \frac{\left[ 1 - \beta^* (1 - e^{(1-q)(\varepsilon_i - U)}) \right]^{\frac{1}{q-1}}}{Z_q} \]  \hspace{1cm} (16)

with the notation \( [a]_+ = \max\{0, a\} \). The parameter \( \beta^* \) is equal to \( \frac{q}{q} \), and the partition function \( Z_q \) is given by \( Z_q = (\sum_j p_j^q)^{\frac{1}{1-q}} \). This stationary distribution asymptotically decays as an exponential i.e., \( p_i \propto e^{-\xi_i} \) for \( q \in [0, 1] \). This result is in complete agreement with the one obtained from the method of multinomial coefficients used by Oikonomou [30], and is different than the one obtained through ordinary EMP, since the latter obtains a stationary distribution of inverse power law form [21-29]. We emphasize again that the ordinary maximization of the Rényi entropy is based on the functional in Eq. (11) and therefore inconsistent.
The analysis of the stationary distribution given by Eq. (16) shows that it asymptotically attains a constant value, independent of the microstate energy $\varepsilon_i$, for $q > 1$. Therefore, the Rényi measure cannot be used as an entropy in a thermostatical sense for the region $q > 1$. This fact has previously been agreed upon due to the fact that the Rényi entropy is neither concave nor convex in the aforementioned interval. However, a consistent maximization of the Rényi measure simply shows the physical reason at the level of the stationary distribution (i.e., Eq. (16)), why this measure cannot be used for $q > 1$. The stationary solution obtained through ordinary EMP of the Rényi measure, on the other hand, does not show such an inconsistency for the interval $q > 1$. This result also verifies the recent findings of Oikonomou and Tirnakli given in [31], where they mentioned that the inverse power law distributions obtained from the ordinary EMP for Rényi entropies do not make these entropies extensive. Thus, these probability distribution functions do not maximize Rényi entropies. Finally, we note that the stationary distribution given by Eq. (16) becomes the one given by BG entropy i.e., Eq. (10) in the $q \to 1$ limit.

4 GEMP and Tsallis entropy

We will now work in the continuous domain so that the summation of the previous section will be replaced by integral. As we have noted before, generalized entropies is based on the joint generalization of the concept of information gain and the averaging procedure. The Rényi entropy (which is additive like BG entropy) preserves the ordinary definition of the information gain, but generalizes the averaging procedure. Tsallis entropy (which is nonadditive) is another generalized entropy measure, and is based on the generalization of the information gain only, since it preserves the linear averaging procedure of the BG entropy. Tsallis entropy reads

$$S_q = \left\langle \ln_q \left( \frac{1}{p} \right) \right\rangle_{\text{lin}}$$

(17)

where $\ln_q(x)$ is $q$-logarithm given by

$$\ln_q(x) = \frac{x^{1-q} - 1}{1 - q}.$$  

(18)

In other words, Tsallis entropy preserves the linear averaging procedure, but generalizes the definition of the information gain $\ln(1/p)$ to the $q$-information
gain defined as \( \ln_q(1/p) \). Then, it is evident that, according to GEMP, the functional to be maximized must be of the form

\[
\Phi_q = \langle \ln_q 1/p \rangle_{\text{lin}} - \alpha \langle . \rangle_{\text{lin}} - \beta \langle . . \rangle_{\text{lin}} \tag{19}
\]

where \( \alpha \) and \( \beta \) are as usual Lagrange multipliers associated with the linear average of 1 and the microstate energy \( \varepsilon \), respectively. Although we know that the constraints must be written as linear averaged quantities in accordance with the fundamental structure of the Tsallis entropy, the form of the probability distribution to be used in the functional is not trivial due to the generalization of the ordinary definition of the information gain. This is denoted by the brackets with single and double dots. The number of dots in the brackets are different, since the same probability distribution may not be used in the linear averaging procedure.

Our goal is now to determine the form of the function \( f(x) \) to be used in the linear averaging procedure (since Tsallis entropy preserves the linear averaging procedure as it is), in Eq. (19). One might here consider three choices: the first choice is to employ ordinary definition of the probability function \( f(x) \), which simply corresponds to ordinary Fourier transform implying all finite moments to be calculated through \( \langle x^n \rangle_{\text{lin}} = \int dx f(x)x^n, (n = 0, 1, \ldots) \). This is called first choice of constraints in the literature [10,18]. However, the second and third choices of constraints are related to \( q \)-Fourier transform [32] instead of the ordinary one, since a function \( f(x) \) can alternatively be determined by its \( q \)-moments as a result of rewriting the definition of the information gain through \( q \)-logarithm. Within this scheme, the third choice can easily be obtained as follows: the \( q \)-Fourier transform [32] of probability density \( f(x) \) is given by

\[
F_q[f](\xi) = \int_{-\infty}^{+\infty} dx e_q(i\xi x)^{q-1})f(x), q \geq 1 \tag{20}
\]

where the \( q \)-exponential i.e., \( \exp_q(x) \) is

\[
\exp_q(x) = \left[ 1 + (1 - q)x \right]_+^{1/q} \tag{21}
\]

Then, it is possible to show that

\[
\frac{1}{v_{q_n}} \left[ \frac{d^n}{d\xi^n} F_q[f](\xi) \right]_{\xi=0} = i^n \left\{ \prod_{m=0}^{n-1} \left[ 1 + m(q - 1) \right] \right\} \langle x^n \rangle_{q_n}, n = 1, 2, \ldots \tag{22}
\]
where $q_n = 1 + n(q - 1)$ and $q_n$-mean moments $\langle x^n \rangle_{q_n}$ are given by

$$\langle x^n \rangle_{q_n} = \frac{\int dx x^n f_{q_n}}{\int dx f_{q_n}}$$  \hspace{1cm} (23)

with

$$v_{q_n} = \int_{-\infty}^{+\infty} dx [f(x)]^{q_n}.$$  \hspace{1cm} (24)

This result obtained by Tsallis et al. [32] is very important from our point of view, since, as emphasized by Tsallis et al. too, it shows that one must use the following generalized escort distributions

$$f_{q_n}(x) = \left[ f(x) \right]^{q_n} \frac{\int dx f(x)}{\int dx [f(x)]^{q_n}}, \hspace{0.5cm} n = 0, 1, 2, ...$$  \hspace{1cm} (25)

whenever one needs to calculate an averaged quantity in $q$-space. The results of Tsallis et al. [32] can be summarized as follows: the probability density that must be used in Eq. (19) is the generalized escort distributions given by Eq. (25). If we want to obtain the linear average of a constant, we substitute $n = 0$ in Eq. (25). For any first moment, we substitute $n = 1$ and so on. Therefore, we write, for the linear average of 1 in the $q$-space, as

$$\langle 1 \rangle_{\text{lin}} = \int dx f_{q_n}(x) \times 1 = \frac{\int dx f(x)}{\int dx [f(x)]^{q_n}} = 1.$$  \hspace{1cm} (26)

The inspection of the above equation shows us that the normalization has to be carried out in terms of the ordinary probability density function $f(x)$. Next, we consider the linear average of the microstate energy $\varepsilon(x)$ in $q$-space

$$\langle \varepsilon \rangle_{\text{lin}} = \int dx f_{q_1}(x) \times \varepsilon = \frac{\int dx \varepsilon f(x)}{\int dx [f(x)]^{q_1}} = \frac{\int dx \varepsilon f(x)}{\int dx [f(x)]^{q}}.$$  \hspace{1cm} (27)

At this point, it should be emphasized that the linear average above is taken in terms of the generalized distribution given by Eq. (25) as a result of employing $q$-moments. The maximization of the functional in Eq. (19), subject to constraints Eqs. (26) and (27), yields the well-known stationary distribution

$$p = \frac{[1 - (1 - q)\beta(\varepsilon - U_q)]/ \int dx p(x)^q}{Z_q^{1/(1-q)}}$$  \hspace{1cm} (28)
where $U_q = \langle \varepsilon \rangle_{\text{lin}}$, and the partition function $Z_q$ is given by

$$Z_q = \int dx \left[ 1 - (1 - q) \beta \frac{(\varepsilon - U_q)}{\int dx p(x)^q} \right]^{1/(1-q)}.$$  (29)

Eq. (28) is the well-known probability distribution associated with the third choice of constraints [18]. Moreover, the probability distribution associated with the second choice of constraints can be obtained from the formalism above solely by using unnormalized $q$-moments.

Summing up, the algebra underlying the first choice is ordinary Fourier transform with $q'$, whereas the second and third choices require the use of $q$-Fourier transform with $q$, satisfying $q' = 2 - q$ [31, 33]. Therefore, all choices used so far in nonextensive statistical mechanics [18] emerge naturally within GEMP.

### 5 Conclusions

The emergence of the generalized entropy measures rendered the generalization of the maximization procedure necessary. These generalized entropy measures generally stem from the interplay of the generalization of the information gain and/or averaging procedure. For instance, the Rényi entropy is obtained as a generalization of BG entropy through the generalized averaging procedure i.e., the exponential average, whereas Tsallis entropy is a generalization through a novel definition of information gain. Despite all these generalizations of the entropy measures in terms of information gain and/or averaging procedures, the maximization procedure has in general been applied without taking these changes into account properly. We therefore proposed a generalized entropy maximization procedure, which takes into account the averaging procedure and information gain underlying the generalized entropies. This novel procedure was then applied to the Rényi and Tsallis entropies.

Since the Rényi entropy is a generalization of BG entropy in terms of exponential averages, the generalized maximization procedure requires the consistent use of the exponentially averaged constraints. This novel procedure applied to the Rényi entropy then yields a stationary distribution, which asymptotically decays as an exponential for $q \in [0, 1]$, instead of inverse power law distributions obtained from the ordinary maximization procedure. It should be noted that this result is in complete agreement with the one obtained from the method of multinomial coefficients [30]. Moreover, the inspection of the stationary distribution obtained from the maximization of the Rényi entropy shows that it asymptotically attains a constant value, independent of the microstate energy $\varepsilon_i$, for $q > 1$. Therefore, the Rényi measure cannot be used as
a thermodynamical entropy in the region $q > 1$. This interval was generally excluded by recourse to the fact that the Rényi measure is neither concave nor convex in the aforementioned interval. The consistent maximization of the Rényi measure in this work simply shows the underlying reason why this measure cannot be used for the region $q > 1$ at the level of the stationary distribution. The stationary solution obtained through ordinary maximization of the Rényi measure, on the other hand, seems to be valid for all $q$ values.

The nonadditive Tsallis entropy preserves the linear averaging procedure in its definition, but deforms the definition of the information gain. As a result, there exist two possibilities: the first one is to carry out the averaged constraints in terms of moments determined by the ordinary Fourier transform, whereas the second possibility is to make use of $q$-moments, stemming from the $q$-Fourier transform, in the averaging procedure. The former corresponds to the probability distribution associated with the first choice of constraints. On the other hand, the probability distribution associated with the third choice of constraints is obtained from the latter. Finally, the use of unnormalized $q$-moments results in the probability distribution related to the second choice of constraints. In other words, in our unifying scheme, all choices of constraints naturally emerge depending on the adoption of either ordinary or $q$-deformed Fourier transforms. The resulting stationary distributions, in all cases, are genuine inverse power laws.

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References

[1] F. Bardou, J. P. Bouchaud, O. Emile, A Aspect, and C. Cohen-Tannoudji, Phys. Rev. Lett. 72, 203 (1994).
[2] C. -K. Peng, J. Mietus, J. M. Hausdorff, S. Havlin, H. E. Stanley, and A. L. Goldberger, Phys. Rev. Lett. 70, 1343 (1993).
[3] B. M. Boghosian, Phys. Rev. E 53, 4754 (1996).
[4] J. Klafter and G. Zumofen, Phys. Rev. E 49, 4873 (1994).
[5] H. Schomerus and E. Lutz, Phys. Rev. A 77, 062113 (2008).
[6] R. Soller and S. A. Koehler, Phys. Rev. Lett. 100, 208301 (2008).
[7] B. Borstnik and D. Pumpernik, Phys. Rev. E 71, 031913 (2005).
[8] S. Picozzi and B. J. West, Phys. Rev. E 66, 046118 (2002).
[9] D. Sornette, C. Vanneste, and L. Knopoff, Phys. Rev. A 45, 8351 (1992).
[10] C. Tsallis, J. Stat. Phys. 52, 479 (1988).
[11] A. Rényi, On measures of entropy and information, in: Proceedings of the Fourth Berkeley Symposium on Mathematics, Statistics and Probability, vol. 1, University California Press, Berkeley, 1961, pp. 547-561.
[12] B. D. Sharma and D. P. Mittal, J. Math. Sci. 10, 28 (1975).
[13] C. Jarzynski, Phys. Rev. Lett. 78, 2690 (1997).
[14] S. Park and K. Schulten, J. Chem. Phys. 120, 5946 (2004); J. Liphardt et al., Science 296, 1832 (2002).
[15] M. Masi, Phys. Lett. A 338, 217 (2005).
[16] E. W. Montroll and M. F. Shlesinger, J. Stat. Phys. 32, 209 (1983).
[17] C. Tsallis, S. V. F. Levy, A. M. C. Souza, and R. Maynard, Phys. Rev. Lett. 75, 3589 (1995).
[18] C. Tsallis, R. S. Mendes, and A. R. Plastino, Physica A 261, 534 (1998).
[19] A. N. Kolmogorov, Atti Accad. Naz. Lincei Mem. Cl. Sci. Fis. Mat. Natur. Sez. 12, 388 (1930).
[20] M. Nagumo, Jpn. J. Math. 7, 71 (1930).
[21] E. K. Lenzi, R. S. Mendes, L. R. da Silva, Physica A 280, 337 (2000).
[22] A. G. Bashkirov, Physica A 340, 153 (2004).
[23] A. G. Bashkirov, Phys. Rev. Lett. 93, 130601 (2004).
[24] A. G. Bashkirov, Theor. and Math. Phys. 149, 1559 (2006).
[25] A. R. Vasconcellos, E. Laureto, E. A. Meneses, and R. Luzzi, Chaos, Solitons & Fractals 28, 8 (2006).
[26] A. R. Plastino and A. Plastino, Phys. Lett. A 226, 257 (1997).
[27] S. Martínez, F. Nicholás, F. Pennini, and A. Plastino, Physica A 286, 489 (2000).
[28] A. Figueiredo, M. A. Amato, and T. M. da Rocha Filho, Physica A 367, 191 (2006).
[29] P. Jizba and T. Arimitsu, Physica A 365, 76 (2006).
[30] Th. Oikonomou, Physica A 386, 119 (2007).
[31] Th. Oikonomou and U. Tirnakli, arXiv:0808.1673v2 (2008).
[32] C. Tsallis, A. R. Plastino, and R. F. Alvarez-Estrada, arXiv:0802.1698v2 (2008).
[33] G. B. Bağcı, Int. J. Mod. Phys. B 22, 3381 (2008).