Dispersion relation for anisotropic media

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I. INTRODUCTION

Development of the modern microscopic technology
(nano-technology) provides a possibility to manufacture
materials of a rather non-ordinary electromagnetic pa-
rameters. This situation recall the theoretical investiga-
tion of wave propagation in a media with a generic
constitutive law.

A wide class of media is characterized by a linear con-
stitutive law and in general can be described by four
$3 \times 3$ matrices. Two of these matrices, \( \varepsilon^{ij} \) and \( \mu^{ij} \) (permittivity and permeability matrices), describe the pure electric and the magnetic properties of the matter. Two
additional matrices describe relativity smaller electric-
magnetic cross-term effects.

The ordinary textbook’s description of a media with
two anisotropic matrices \( \varepsilon^{ij} \) and \( \mu^{ij} \) is based on a dia-
ogonalization of one of them \([1],[2],[3]\). This algebraic pro-
cedure is always possible for a symmetric matrix. More-
over, if both matrices are symmetric and one of them is
positive definite, both of them can be diagonalized. Even
with this simplification, the corresponded dispersion re-
lation is given in a rather complicated form. Moreover,
it is clear that the diagonalization technique is not ap-
plicable in a general case when both matrices \( \varepsilon^{ij} \) and \( \mu^{ij} \)
are not symmetric nor positive definite.

In the current paper, we study the wave propagation
in a generic media in the framework of premetric electrodynamics approach \([4],[5],[9]\). Our final result is a
compact tensorial dispersion relation is derive-
d. The resulted formula are useful for a
theoretical study of electromagnetic wave propagation in a classical media and in a modern type of
media with a generic linear constitutive relation including metamaterials.

II. ANISOTROPIC MEDIA IN THE METRIC-FREE DESCRIPTION

Let us start with a metric-free four dimensional system of Maxwell equations

\[
\varepsilon^{\alpha\beta\gamma\delta} F_{\beta\gamma,\delta} = 0 , \quad \mathcal{H}^{\alpha\beta} = 4\pi J^\alpha . \quad (2.1)
\]

It includes two antisymmetric tensors — the field strength \( F_{\alpha\beta} \) and the field excitation \( \mathcal{H}^{\alpha\beta} \). The Greek indices change in the range \( \alpha,\beta,\cdots = 0,1,2,3, \) the comma
denotes the partial derivatives relative to the coordi-
nates \( \{x^0,x^1,x^2,x^3\} = \{ct,x,y,z\} \). In sequel, the Ro-
man indices will be used for the spatial coordinates,
\( i,j,k \cdots = 1,2,3 \).

The \( (1+3) \)-decomposition of the field tensors reads

\[
E_i = F_{0i} , \quad B^i = -\frac{1}{c}\varepsilon^{ijk} F_{jk} , \quad (2.2)
\]
\[
D^i = \mathcal{H}^{0i} , \quad H_i = \frac{1}{c}\varepsilon^{ijk} \mathcal{H}_{jk} . \quad (2.3)
\]

The electric current is given by

\[
J^0 = \rho , \quad J^i = \frac{1}{c} j^i . \quad (2.4)
\]

In this notation, the system (2.1) is rewritten in the or-

}
The constitutive tensor $\chi^{\alpha\beta\gamma\delta}$ is antisymmetric in two pairs of indices, so it has, in general, 36 independent components. Such generic constitutive tensor can be represented by four 3-dimensional matrices of 9 independent components. We will use a representation of the form

$$\chi^{\alpha\beta\gamma\delta} = \left( \begin{array}{c} \varepsilon^{ij} \gamma^j_i \\ \gamma^j_i \pi_{ij} \end{array} \right).$$

In this paper, we restrict to an electromagnetic media which describes by two tensors $\varepsilon^{ij}$ and $\pi_{ij}$. Two additional tensors $\gamma^j_i$ and $\pi^j_i$ represent the electromagnetic cross-terms, which are relatively small for most types of the dielectric materials. We will consider, however, a some type of a generalized anisotropic media. In particular, we will not require the matrices $\varepsilon^{ij}$ and $\pi_{ij}$ to be symmetric, positive definite, nor even invertible. Consequently, we will use a constitutive tensor of 18 independent components

$$\chi^{00ij} = \varepsilon^{ij}, \quad \chi^{ijkl} = -\varepsilon^{ijm} \varepsilon^{kln} \pi_{mn}.$$

In three-dimensional form, the corresponded constitutive relation is given by

$$D^i = \varepsilon^{ij} E_j, \quad H_i = \pi_{ij} B^j.$$  

For a regular matrix $\pi_{ij}$, an inverse permeability matrix

$$(\pi^{-1})^{ij} = \mu^{ij}$$

is defined. With this notation, the constitutive relation takes the ordinary form

$$D^i = \varepsilon^{ij} E_j, \quad B^i = \mu^{ij} H_j.$$  

III. A GENERAL DISPERSION RELATION

A covariant dispersion relation for a generic constitutive tensor $\chi^{\alpha\beta\gamma\delta}$ recently accept a considerable interest [5], [6]. Here we briefly recall the necessary notations and the main stages of the derivation as it given in [9].

Our aim is to establish the necessary conditions for existence of physically non-trivial solutions of the source-free system

$$\epsilon^{\alpha\beta\gamma\delta} F_{\beta\gamma,\delta} = 0, \quad \chi^{\alpha\beta\gamma\delta} F_{\beta\gamma,\delta} = 0.$$  

Here the ordinary condition of the geometric optics approximation is accepted. In particular, we consider the media parameters encoded in $\chi^{\alpha\beta\gamma\delta}$ as varied slowly relative to the change of the electromagnetic field.

The first equation of (3.1) has a standard solution in term of the vector potential $A_{\alpha}$

$$F_{\alpha\beta} = \frac{1}{2} (A_{\alpha,\beta} - A_{\beta,\alpha}).$$

Consequently, the second equation of (3.1) takes the form

$$\chi^{\alpha\beta\gamma\delta} A_{\gamma,\beta\delta} = 0.$$  

Let us look for a solution of this equation in the form of a monochromatic wave ansatz

$$A_{\alpha} = a_\alpha e^{iq_\beta x^\beta}.$$  

We substitute this ansatz into (3.3) and treat the amplitude of the field $a_{\alpha}$ and the wave covector $q_\beta$ as slow functions of a spacetime point. Consequently, we come to an algebraic system

$$M^{\alpha\delta} q_\delta = 0$$

with a characteristic matrix

$$M^{\alpha\delta} = \chi^{\alpha\beta\gamma\delta} q_\beta q_\gamma.$$  

This matrix evidently satisfies the relations

$$M^{\alpha\delta} q_\alpha = 0, \quad M^{\alpha\delta} q_\delta = 0.$$  

These relations have a clear physical meaning. The first equation represents the charge conservation law, while the second one means that an ansatz (3.4) with $q_\alpha \sim a_\alpha$ is a solution of (3.5). Certainly this solution is not physically meaningful, because it corresponds to a zero value of the field $F_{\alpha\beta}$, i.e., it is related to the gauge invariance of the field equations.

Thus we are looking for a solutions of the system (3.5) constrained by the relations (3.6). On the matrix language, these relations mean that the columns and the rows of the matrix $M^{\alpha\delta}$ are linearly dependent, i.e., the matrix is singular. Consequently, a system always has a non-zero solution. However we need more of that, in fact, we are looking for an additional linear independent solution. Only this one will be of a physical meaning.

It is an algebraic fact, that a linear system has two independent solution only if the adjoint of the characteristic matrix equal to zero. So we come to an equation

$$(\text{adj} M)_{\alpha\beta} = 0.$$  

On a first view, it seems that we require here 16 conditions for 16 components of the matrix $M^{\alpha\beta}$. In fact, the situation is much different. It can be proved [9] that, for a matrix which satisfies the conditions (3.7), the adjoint matrix is of the form

$$(\text{adj} M)_{\alpha\beta} = \lambda(q) q_\alpha q_\beta.$$  

Consequently, a necessary condition for existence of a physically meaningful solution for a wave propogational system is expressed by a scalar equation

$$\lambda(q) = 0.$$  

The function $\lambda(q)$ is a homogeneous 4-th order polynomial in the wave covector $q_\alpha$. It's explicit forms are given in [5], [9].
IV. AN ANISOTROPIC DISPERSION RELATION

For a generalized anisotropic media with a constitutive tensor (2.9), we apply an ordinary (1+3)-decomposition of the wave vector

\[ q^\alpha = (w, k^i) . \]

The characteristic matrix has now the entries

\[ M^{00} = \varepsilon^{ij} k_i k_j, \quad M^{0i} = -\varepsilon^{ij} k_j w, \quad M^{i0} = -\varepsilon^{ij} k_i w, \]

and

\[ M^{ij} = -\varepsilon^{ij} w^2 + \varepsilon^{mnp} \pi_{mp} k_n k_q . \]

We write the latter equation in a short form

\[ M^{ij} = \varepsilon^{ij} w^2 - \tau^{ij} . \]

where a matrix \( \tau^{ij} \) is defined as

\[ \tau^{ij} = \varepsilon^{mnp} \pi_{mp} k_n k_q . \]

Due to the relations \( \tau^{ij} k_i = \tau^{ij} k_j = 0 \) it is singular.

In correspondence with (3.9), it is enough to calculate only one component of the adjoint matrix. Write

\[(adj \ M)_{00} = \frac{1}{3!} \varepsilon_{i_1 i_2 i_3} \varepsilon_{j_1 j_2 j_3} M^{i_1 j_1} M^{i_2 j_2} M^{i_3 j_3} . \]

Substituting (4.4) we derive for \( (adj \ M)_{00} = \lambda w^2 \)

\[ \lambda = \frac{1}{3!} w^4 \left( \varepsilon_{i_1 i_2 i_3} \varepsilon_{j_1 j_2 j_3} \varepsilon^{i_1 j_1} \varepsilon^{i_2 j_2} \varepsilon^{i_3 j_3} \right) - \frac{1}{2!} w^2 \left( \varepsilon_{i_1 i_2 i_3} \varepsilon_{j_1 j_2 j_3} \varepsilon^{i_1 j_1} \varepsilon^{i_2 j_2} \tau^{i_3 j_3} \right) + \frac{1}{2!} \left( \varepsilon_{i_1 i_2 i_3} \varepsilon_{j_1 j_2 j_3} \varepsilon^{i_1 j_1} \tau^{i_2 j_2} \tau^{i_3 j_3} \right) . \]

In the first term, we recognize the determinant of the matrix \( \varepsilon \). In the second and the third terms, the adjoint of the matrices \( \varepsilon \) and \( \tau \) are extracted. Consequently, the desired dispersion relation obtains a compact matrix form (the central dot denotes the matrix multiplication)

\[ w^4 (det \varepsilon) - w^2 tr (\varepsilon^T \cdot adj \varepsilon) + tr (\varepsilon^T \cdot adj \tau) = 0 \]

(4.8)

This equation can be rewritten in the explicit tensorial form. The adjoint of the matrix \( \tau \) is calculated by substituting (4.5) into the definition

\[ (adj \tau)_{ij} = \frac{1}{2!} \varepsilon_{i_1 i_2 i_3} \varepsilon_{j_1 j_2 j_3} \varepsilon^{i_1 j_1} \varepsilon^{i_2 j_2} \varepsilon^{i_3 j_3} . \]

(4.9)

The result is a scalar function multiplied by \( k_i k_j \)

\[ (adj \tau)_{ij} = (adj \pi)^{mnp} k_m k_n k_i k_j . \]

(4.10)

For an invertible matrices \( \varepsilon, \mu \), we use (4.10) to rewrite the expression (4.8) in a form

\[ w^4 - 2 (\psi^{ij} k_i k_j) w^2 + \frac{\varepsilon_{i_1 i_2 i_3} \varepsilon_{j_1 j_2 j_3} \mu^{mnp} k_m k_n}{det \varepsilon} = 0 , \]

(4.11)

where

\[ \psi^{ij} = \frac{1}{2} \varepsilon^{mnp} \varepsilon^{pq} \varepsilon^{i_1 j_1} \mu^{mnp} . \]

(4.12)

Note some straightforward facts resulted from this expression:

(1) The dispersion relation (4.11) is symmetric under interchange between \( \varepsilon \) and \( \mu \).

(2) A criterion for the absence of zero-frequency modes with \( w = 0 \) for some \((k_1, k_2, k_3) \neq (0, 0, 0)\) takes the form: The symmetric parts of the matrices \( \varepsilon \) and \( \mu \) are definite (positive or negative).

(3) The necessary and sufficient condition for hyperbolicity (four real \( w \) for any real \( k \)) is expressed as a system of inequalities

\[ \psi_{ij} k_i k_j > 0 \quad \text{positive definite} \]

(4.13)

and

\[ (\psi_{ij} k_i k_j)^2 \geq \frac{\varepsilon_{i_1 j_1} \mu_{mnp} \varepsilon_{i_2 j_2}}{det \varepsilon} \frac{\varepsilon_{i_3 j_3}}{det \mu} > 0 . \]

(4.14)

(4) In the non-birefringence case \([11]\) with a unique optical metric, the dispersion relation (4.11) takes the form

\[ w^2 - \psi^{ij} k_i k_j = 0 . \]

(4.15)

Thus the optical metric is of Minkowski signature if and only if the matrix \( \psi^{ij} \) is positive definite.

V. EXAMPLES

A. Isotropic case

It easy to check that in the isotropic case with \( \varepsilon_{ij} = \varepsilon \delta_{ij}, \mu_{ij} = \mu \delta_{ij} \) (4.11) yields the ordinary dispersion relation \((\varepsilon \mu) w^2 - k^2 = 0 \).

B. Diagonal case

Let us consider a more involved example of two diagonal matrices

\[ \varepsilon = diag(\varepsilon_1, \varepsilon_2, \varepsilon_3), \quad \mu = diag(\mu_1, \mu_2, \mu_3) . \]

(5.1)

The last term of (4.11) takes the form

\[ \frac{\varepsilon_1 k_1^2 + \varepsilon_2 k_2^2 + \varepsilon_3 k_3^2}{\varepsilon_1 \varepsilon_2 \varepsilon_3} \frac{\mu_1 k_1^2 + \mu_2 k_2^2 + \mu_3 k_3^2}{\mu_1 \mu_2 \mu_3} . \]

(5.2)

The coefficient of the second term of (4.11) also easily calculated

\[ k_1^2 \left( \frac{1}{\varepsilon_2 \mu_3} + \frac{1}{\varepsilon_3 \mu_2} \right) + k_2^2 \left( \frac{1}{\varepsilon_1 \mu_3} + \frac{1}{\varepsilon_3 \mu_1} \right) + k_3^2 \left( \frac{1}{\varepsilon_1 \mu_2} + \frac{1}{\varepsilon_2 \mu_1} \right) . \]

(5.3)
Consequently, the dispersion relation in this case takes the form

\[
\left(\varepsilon_1\varepsilon_2\varepsilon_3\mu_1\mu_2\mu_3\right)w^4 + \left(\varepsilon_1\mu_1k_1^2(\varepsilon_2\mu_3 + \varepsilon_3\mu_2) + \varepsilon_2\mu_2k_2^2(\varepsilon_1\mu_3 + \varepsilon_3\mu_1) + \varepsilon_3\mu_3k_3^2(\varepsilon_1\mu_2 + \varepsilon_2\mu_1)\right)w^2 + (\varepsilon_1k_1^2 + \varepsilon_2k_2^2 + \varepsilon_3k_3^2)(\mu_1k_1^2 + \mu_2k_2^2 + \mu_3k_3^2) = 0 \quad (5.4)
\]

In a special case \(\mu_1 = 1\), this formula coincides with the one given in [2],[3].

C. Magnetized ferrite

Magnetized ferrite materials are described by an isotropic dielectric constant

\[
\varepsilon = \text{diag}(\varepsilon, \varepsilon, \varepsilon).
\quad (5.5)
\]

Under influence of the magnetic field the initially isotropic magnetic matrix obtains an anisotropic modification. For a magnetic field directed as the z-axis [12],

\[
\mu = \begin{pmatrix}
\mu & iq & 0 \\
-iq & \mu & 0 \\
0 & 0 & \mu_0
\end{pmatrix}.
\quad (5.6)
\]

The last term of (4.11) takes now the form

\[
\frac{k^2}{\varepsilon^2} \cdot \frac{\mu(k_1^2 + k_2^2) + \mu_0k_3^2}{\mu_0(\mu^2 - q^2)}.
\quad (5.7)
\]

The coefficient of the second term of (4.11) is easily calculated. The inverse matrix

\[
\mu^{-1} = \frac{1}{r\mu_0}\begin{pmatrix}
\mu & iq & 0 \\
-iq & \mu & 0 \\
0 & 0 & \mu_0
\end{pmatrix} \quad \text{where} \quad r = \mu^2 - q^2, \quad \frac{\mu^2}{\mu_0}.
\quad (5.8)
\]

Consequently the second term of (4.11) takes the form

\[
\frac{w^2}{\mu_0r} \left[ \frac{\mu(k_1^2 + k_2^2) + \mu_0k_3^2}{\mu_0(\mu^2 - q^2)} \right].
\quad (5.9)
\]

The resulting dispersion relation is

\[
r\mu_0^2\varepsilon^2w^4 - w^2\varepsilon_0\left[2k^2 + (r - 1)(k^2 - k_3^2)\right] + k^2 \left[ k^2 + \left(\frac{\mu_0}{\mu} - 1\right)k_3^2 \right] = 0.
\quad (5.10)
\]

In absent of an exterior magnetic field, \(q = 0\) and \(\mu = \mu_0\). Consequently, \(r = 1\) and the isotropic dispersion relation reinstated.

VI. CONCLUSION

A short dispersion relation is derived for a generalized anisotropic media. The corresponded matrices are not required to be symmetric, positive definite, and even invertible. This compact form is much simpler for calculation and for theoretical analysis than the ones represented in the electromagnetic literature [13], [14]. The algebraic consideration presented here can be also useful for a compact algebraic representation of a general bianisotropic tensorial dispersion relation represented recently in [15].

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