AdS/BCFT Correspondence for Higher Curvature Gravity: An Example

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Abstract

We consider the effects of higher curvature terms on a holographic dual description of boundary conformal field theory. Specifically, we consider three-dimensional gravity with a specific combination of Ricci tensor square and curvature scalar square, so called, new massive gravity. We show that a boundary entropy and an entanglement entropy are given by similar expression with those of the Einstein gravity case when we introduce an effective Newton’s constant and an effective cosmological constant. We also show that the holographic g-theorem still holds in this extension, and we give some comments about the central charge dependence of boundary entropy in the holographic construction. In the same way, we consider new type black holes and comment on the boundary profile. Moreover, we reproduce these results through auxiliary field formalism in this specific higher curvature gravity.

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1 Introduction

Anti-de-Sitter/Conformal Field Theory (AdS/CFT) correspondence has opened new and promising research directions, after it has been proposed [1, 2] as a concrete realization of holographic principle [3, 4]. Succinctly speaking, these directions are related on one side to studies about strongly coupled quantum field theories, and, on the other side, to those about gravity or string theories on AdS space. While the original conjecture was about supersymmetric Yang-Mills theory and supergravity (or superstrings) on AdS space, this original form of the correspondence has been extended in various ways. One of the interesting instances of AdS/CFT correspondence is the case of $AdS_3/CFT_2$ correspondence, which is a prime example of how the black hole entropy may be obtained from field theory computations [5] and which has been deeply related to understanding black hole entropy microscopically [6].

Recently, there have been some attempts to extend AdS/CFT correspondence to the case with boundaries in the CFT side [7]. Specifically, the simplest form of proposals is that CFT on a domain $M$ with conformal boundary conditions on a boundary $\partial M$, which is called boundary conformal field theory (BCFT), corresponds to gravity on AdS space with a boundary $Q$. The bulk boundary $Q$ is taken on bulk AdS space by demanding that its intersection with asymptotic AdS boundary $M$ be given by $\partial M$ and it preserves the same amount of symmetry with the conformal boundary $\partial M$. To construct dual of BCFT, the Neumann boundary conditions are taken on $Q$ with the introduction of matter stress tensors, while the usual Dirichlet boundary conditions are done on $M$.

Through this correspondence between two dimensional BCFT and three dimensional gravity on AdS space with boundary, many interesting results in BCFT$_2$ are obtained by holographic computations. The so-called boundary entropy [8] is one of the interesting physical quantities obtained in this way. The boundary entropy in BCFT$_2$ is defined by logarithm of $g$-function which is a disk amplitude of boundary state $|B_\alpha\rangle$ with a boundary condition $\alpha$. It counts the degeneracy of boundary ground states and plays a role of boundary central charge. Moreover, it satisfies g-theorem [9] which says that it decreases along renormalization group flow and so is regarded as the boundary analogue of c-theorem [10]. It has been known that the boundary entropy has interesting connection with the entanglement entropy which has also a holographic dual description. Consequently, the boundary entropy was shown to be matched by independent computations from holographic dual of disk amplitude and dual of entanglement entropy. This agreement gives us some consistency checks of various holographic constructions.

On the other hand, there are other interesting directions for the extension of $AdS_3/CFT_2$ correspondence by considering higher derivative terms on three-dimensional gravity. One of them is specific on three dimensions since it contains three-dimensional Chern-Simons term [11, 12]. This extension with some subtleties leads to the chiral gravity conjecture and/or log-gravity conjecture [13, 14]. The others are related to the introduction of higher curvature terms with specific combination of coefficients which can be extended to higher dimensions [15, 16, 17]. Since all these higher derivative gravities allow AdS space as a solution for some parameter range, one may study the effects of higher derivative terms on the dual CFT side. One lesson obtained from these studies on higher curvature theories is that asymptotic fall-off boundary conditions of bulk fields are important to characterize theories.

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According to these developments in the extensions of AdS/CFT correspondence, one of natural questions is what are the effects of higher curvature terms on the boundary entropy and entanglement entropy. As a rule, these constructions of BCFT quantities or AdS/BCFT correspondence are expected to hold even with higher curvature theories. Specifically, higher curvature effects on the entanglement entropy has been studied recently in several works [20]∼[22]. In this paper we focus on three-dimensional gravity with higher curvature terms to see their effects on the boundary entropy. Our main interest is the so-called new massive gravity (NMG) [15]∼[29], which has a specific combination of Ricci tensor square and Ricci scalar one. Since NMG is unique in the sense that its Lagrangian is determined by holographic c-theorem, it is a natural playground for AdS/BCFT correspondence. Interestingly, it turns out that there is an effective description of NMG which reproduces most of the direct computations.

This paper is organized as follows. In section 2, we review some background materials to fix our conventions. We present our holographic computation of the boundary entropy in section 3 with some indications to its extension to even higher curvature terms. In this section holographic g-theorem for NMG is also derived. In section 4, we give some results about black holes and their thermal properties. We rederive various results by the effective description of NMG in section 5. In section 6 we summarize our results with some comments. Various formulae relevant to the main part are relegated to Appendixes.

2 A Brief review : boundary entropy in dual BCFT and new massive gravity

In this section we review the holographic construction dual to BCFT with a boundary ∂M following [7] and also summarize NMG on three dimensions to clarify some points and to fix our conventions.

2.1 Holographic boundary entropy

A boundary entropy is an interesting physical quantity in two dimensional CFT with boundary, or BCFT (see for a brief review [30]). It is considered as the boundary analogue of central charge and has an interesting connection with an entanglement entropy. To introduce a boundary entropy, let us consider the disk amplitude of boundary state |B_α⟩ of a certain BCFT with a boundary condition α, which is denoted as g_α. Then, the boundary entropy associated with a boundary is defined by [8]

\[ S_{bd} \equiv \ln g_\alpha = \ln \langle 0 | B_\alpha \rangle , \tag{1} \]

where |0⟩ denotes the vacuum state. Interestingly, this boundary entropy is related to the so-called entanglement entropy.

The entanglement entropy of the subsystem A is given by von Neumann entropy \( S_A = -\text{Tr} \rho_A \ln \rho_A \) for the reduced density matrix \( \rho_A \) which is the partial trace of the total density matrix \( \rho \) with respect to the complement of A. In a two dimensional CFT defined on a half line \( x < 0 \), it has been
known that the entanglement entropy of the finite interval $A : -\mu \leq x < 0$ is given by

$$S_A = \frac{c}{6} \ln \frac{2\mu}{\epsilon} + \ln g + \frac{c'_1}{2},$$

(2)

where $\epsilon$ is UV cut-off, $c$ is the central charge of $CFT_2$, $\ln g$ is boundary entropy and $c'_1$ is a non-universal constant. Note that this result in the two-dimensional CFT is obtained by the so-called replica method while the geometric approach to entanglement entropy (EE) is preferred in generic spacetime dimension. In the geometric approach it is shown that coefficient of the trace anomaly formula is related to EE with logarithmic behavior of EE in even dimensions.

Recently, the above two formulae for boundary entropy are realized holographically as $AdS/BCFT$ correspondence [7, 32]. To construct these holographic duals to BCFT quantities, it seems natural to consider Wick-rotated three-dimensional bulk gravity. The relevant Euclidean bulk action dual to BCFT with one boundary $\partial M$ for pure Einstein gravity is given by

$$S_E = S_N + S_M + S_Q + S_Q^{\text{mat}},$$

(3)

where

$$S_N = \frac{1}{16\pi G} \int_N d^3x \sqrt{g} \left[ R + \frac{2}{L^2} \right], \quad S_M = -\frac{1}{8\pi G} \int_M d^2x \gamma K,$$

$$S_Q = -\frac{1}{8\pi G} \int_Q d^2x \sqrt{\gamma} K, \quad S_Q^{\text{mat}} = \frac{1}{8\pi G} \int_Q d^2x \sqrt{\gamma} T.$$  

(4)

In the recent proposal on $AdS/BCFT$ correspondence, the boundary conditions on $AdS$ boundary $M$ are taken as the usual Dirichlet ones while those on bulk boundary $Q$ which corresponds to the CFT boundary $\partial M$ are done as the Neumann ones. To be a holographic dual of BCFT, the boundary $Q$ in the bulk $AdS$ space has isometry dual to conformal symmetry preserved by the CFT boundary conditions. Note that there are two terms on the boundary $Q$, one of which is the usual Gibbons-Hawking(GH) term and the other represents the matter contribution localized on $Q$. For a matter contribution written by $T$ the boundary condition on the bulk boundary $Q$ is taken by

$$8\pi GT_Q^{ij} = T^{ij}, \quad 8\pi G T_Q^{ij} \equiv K^{ij} - K^{ij},$$

(5)

where $T_Q^{ij}$ is the so-called Brown-York stress tensor [33, 34] on $Q$ and $K_{ij}$ denotes extrinsic curvature for $Q$ (see below for our convention). Though the scalar function $T$ does not need to be a constant in general, the conservation of Brown-York stress tensor requires the constancy of $T$. We will give some comments about this requirement in the next section.

As the simplest construction, one may try to preserve $SO(2,1)$ symmetry by the boundary conditions among the full $SO(2,2)$ symmetry of bulk $AdS_3$ space. As holographic construction this symmetry needs to be preserved by the bulk boundary $Q$ as isometries. Explicitly, $AdS_3$ space in Poincare coordinates $(z, \tau, x)$ can be represented as $AdS_2$ fibration over a line in coordinates $(\rho, \tau, y)$ as

$$ds^2 = \frac{L^2}{z^2} \left[ dz^2 + d\tau^2 + dx^2 \right] = d\rho^2 + \cosh^2 \left( \frac{\rho}{L} \right) \left[ \frac{L^2}{y^2} (d\tau^2 + dy^2) \right],$$

(6)

where two coordinates are related by coordinate transformations

$$z \equiv \frac{y}{\cosh(\rho/L)}, \quad x \equiv y \tanh \frac{\rho}{L}, \quad y \geq 0.$$
Then, the relevant boundary $Q$ can be taken as the AdS$_2$ hypersurface given by $\rho = \rho_*$ with an induced metric $ds^2 = (L^2/y^2)(dy^2 + d\tau^2)$ and the bulk domain $N$ is specified by the range $-\infty < \rho < \rho_*$. This bulk boundary $Q$ can also be represented by $(\tau, z, x(z))$ in original Poincare coordinates with
\[
x(z) = z \sinh \left( \frac{\rho_*}{L} \right) = z \frac{TL}{\sqrt{1 - T^2L^2}},
\]
where we have used the boundary condition on $Q$ given in eq. (5) to replace $\rho_*$ by $T$ in the second equality. One can see that the domain of BCFT is given by $x < 0$ from this equation, since the asymptotic boundary $M$ is specified by $z = 0$ while the intersection of $Q$ and $M$ is given by $x(0) = 0$ and $(z, x) \to (0, -y_0)$ for $(\rho, y) \to (-\infty, y_0)$. As usual, we take $z = \epsilon$ as the geometric cut-off corresponding to field theory UV cut-off.

After an appropriate conformal transformation from the half space $x < 0$ to a round disk $\tau^2 + x^2 \leq r_D^2$, it turns out that the holographic dual of this round disk in the BCFT is given by the domain $N$ in the bulk AdS$_3$ as
\[
\tau^2 + x^2 + \left( z - r_D \sinh \frac{\rho_*}{L} \right)^2 - r_D^2 \cosh^2 \frac{\rho_*}{L} \leq 0,
\]
which is a suitable form for obtaining boundary entropy holographically as a disk amplitude.

### 2.2 New Massive Gravity

New massive gravity (NMG) is a three-dimensional gravity with higher curvature terms, which is regarded as the covariant completion of Pauli-Fierz massive graviton theory [15]. Furthermore, it is shown that NMG is the unique extension of Einstein gravity consistent with holographic c-theorem [16, 35]. The Lagrangian of NMG we will consider in the following is given by
\[
S = \frac{1}{16\pi G} \int d^3x \sqrt{-g} \left[ \sigma R + \frac{2}{\ell^2} + \frac{1}{m^2} K \right],
\]
where $\sigma$ takes 1 or $-1$ and $K$ is a specific combination of scalar curvature square and Ricci tensor square defined by
\[
K = R_{\mu\nu}R^{\mu\nu} - \frac{3}{8} R^2.
\]
Our convention is such that the parameter $m^2$ and cosmological constant $\ell^2$ can take positive or negative values. The equations of motion of NMG are given by
\[
\mathcal{E}_{\mu\nu} = \sigma G_{\mu\nu} - \frac{1}{\ell^2} g_{\mu\nu} + \frac{1}{2m^2} K_{\mu\nu} = 0,
\]
where $K_{\mu\nu}$ is
\[
K_{\mu\nu} = g_{\mu\nu} \left( 3R_{\alpha\beta}R^{\alpha\beta} - \frac{13}{8} R^2 \right) + \frac{9}{2} RR_{\mu\nu} - 8R_{\mu\alpha}R^\alpha_{\nu} + \frac{1}{2} \left( 4D^2 R_{\mu\nu} - D_\mu D_\nu R - g_{\mu\nu} D^2 R \right).
\]
Now, let us consider the generalized Gibbons-Hawking (GH) boundary term in NMG. To obtain the generalized GH term in NMG, it is useful to introduce an auxiliary field. In summary, the above NMG action can be rewritten in terms of auxiliary field $f_{\mu\nu}$ as

$$S = \frac{1}{16\pi G} \int d^3x \sqrt{-g} \left[ \sigma R + \frac{2}{\ell^2} + f^{\mu\nu} G_{\mu\nu} - \frac{m^2}{4} \left( f^{\mu\nu} f_{\mu\nu} - f^2 \right) \right].$$

(13)

In this representation the EOMs of this action are given by

$$\sigma G_{\mu\nu} - \frac{1}{\ell^2} g_{\mu\nu} = T^B_{\mu\nu}, \quad f_{\mu\nu} = \frac{2}{m^2} \left( R_{\mu\nu} - \frac{1}{4} R g_{\mu\nu} \right),$$

(14)

where

$$T^B_{\mu\nu} = \frac{m^2}{2} \left[ f_{\mu\alpha} f^\alpha_{\nu} - f f_{\mu\nu} - \frac{1}{4} \left( f^{\alpha\beta} f_{\alpha\beta} - f^2 \right) g_{\mu\nu} \right] + \frac{1}{2} f R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} - 2 f_{(\mu} G^\alpha_{\nu)} + \frac{1}{2} f^{\alpha\beta} G_{\alpha\beta} g_{\mu\nu}$$

$$- \frac{1}{2} \left[ D^2 f_{\mu\nu} + D_{\mu} D_{\nu} f - 2 D^\alpha D_{(\mu} f_{\nu)\alpha} + \left( D^\alpha D_{\beta} f^{\alpha\beta} - D^2 f \right) g_{\mu\nu} \right].$$

This form of NMG is useful to obtain the generalized GH term. To specify various quantities in GH term, we take the relevant metric in the ADM-decomposed form as

$$ds^2 = N^2 d\eta^2 + \gamma_{ij} \left( dx^i + N^i d\eta \right) \left( dx^j + N^j d\eta \right).$$

(15)

According to this metric decomposition, the auxiliary field $f^{\mu\nu}$ can be decomposed as

$$f^{\mu\nu} = \begin{pmatrix} s & h^i \\ h_i & f^{ij} \end{pmatrix}.$$  

With this decomposition the generalized GH term was obtained in the form of

$$S_{GH} = \frac{1}{16\pi G} \int d^2x \sqrt{-\gamma} \left[ 2 \sigma K + \hat{f}^{ij} K_{ij} - \hat{f} K \right],$$

(16)

where $\hat{f}$ and $\hat{f}^{ij}$ are defined by

$$\hat{f}^{ij} \equiv f^{ij} + 2 h^{(i} N^{j)} + s N^i N^j, \quad \hat{f} \equiv \gamma_{ij} \hat{f}^{ij}.$$

Note that the first term proportional to $\sigma$ is the GH term in pure Einstein gravity case and our convention for the extrinsic curvature is

$$K_{ij} = \frac{1}{2N} \left( \partial_\eta \gamma_{ij} - \nabla_i N_j - \nabla_j N_i \right),$$

(17)

where $\nabla_i$ denotes a covariant derivative for the metric $\gamma_{ij}$.

### 3 Boundary entropy and holographic g-theorem

#### 3.1 Boundary Entropy as a disk amplitude

As in the Einstein gravity case we consider $AdS$ space with its asymptotic boundary $M$ and bulk boundary $Q$ to realize boundary entropy of BCFT dual to higher curvature gravity. By omitting
the $AdS$ boundary $M$ which leads to counter terms for the divergence cancelation, one may denote the Euclidean action as

$$S_E = S + S_Q + S_{Q}^{\text{mat}},$$

where

$$S = -\frac{1}{16\pi G} \int_{N} \sqrt{g} \left[ \sigma R + \frac{2}{l^2} + f^{\mu\nu} G_{\mu\nu} - \frac{m^2}{4} \left( f^{\mu\nu} f_{\mu\nu} - f^2 \right) \right],$$

$$S_Q = -\frac{1}{16\pi G} \int_{Q} \sqrt{\gamma} \left[ 2\sigma K + \hat{f}^{ij} K_{ij} - \hat{f} K \right],$$

$$S_{Q}^{\text{mat}} = \frac{1}{8\pi G} \int_{Q} \sqrt{\gamma} T,$$

and $T$ denotes the contribution from some matter fields localized on the bulk boundary surface $Q$. Though the function $T$ denotes just the Lagrangian for matters on $Q$ in the generic case, we confine ourselves to a simple case such as the scalar function $T$ without metric dependence, which might be realized as ‘infinitely massive’ scalar fields with potential $\hat{S}$. This means that a certain matter might be distributed on the surface $Q$, which is specified by the consistency with equations of motion or boundary condition on $Q$. The simplest case is given by the constant matter density $T$, which corresponds to the boundary cosmological constant or constant scalar potential localized in $Q$ and is the our main focus.

The Brown-York stress tensor for the above action with the convention $T_{ij} = -\frac{2}{\sqrt{\gamma}} \frac{\delta S}{\delta \gamma_{ij}}$ in Euclidean signature is given by

$$8\pi GT_{Q}^{ij} = \left( \sigma + \frac{1}{2} \hat{s} - \frac{1}{2} \hat{f} \right) (K^{ij} - K_{ij}) - \nabla^{i} \hat{h}_{j}^{i} + \frac{1}{2} D_{\eta} \hat{f}^{ij} + K_{k}^{(i} \hat{f}^{j)k} + \gamma^{ij} \left( \nabla_{k} \hat{h}^{k} - \frac{1}{2} D_{\eta} \hat{f} \right),$$

where hatted quantities are defined by

$$\hat{s} = N^{2} s, \quad \hat{h}^{i} = N(h^{i} + s N^{i}), \quad \hat{f}^{ij} = f^{ij} + 2h^{(i} N^{j)} + s N^{i} N^{j}, \quad \hat{f} = \gamma_{ij} \hat{f}^{ij},$$

and the covariant derivatives along $\eta$ are

$$D_{\eta} \hat{f}^{ij} = \frac{1}{N} \left( \partial_{\eta} \hat{f}^{ij} - N^{k} \partial_{k} \hat{f}^{ij} + \hat{f}^{kj} \partial_{k} N^{i} + \hat{f}^{ik} \partial_{k} N^{j} \right),$$

$$D_{\eta} \hat{f} = \frac{1}{N} \left( \partial_{\eta} \hat{f} - N^{i} \partial_{j} \hat{f} \right).$$

By imposing the Neumann boundary condition on the bulk boundary $Q$, one obtains the boundary condition on the surface $Q$ as

$$8\pi GT_{Q}^{ij} = T \gamma^{ij}. \quad (24)$$

Noting a useful relation

$$D_{\eta} \hat{f} = \gamma_{ij} D_{\eta} \hat{f}^{ij} + 2K_{ij} \hat{f}^{ij},$$

one can see that the tension of boundary $Q$ is given by

$$2T = 8\pi GT_{Q}^{ij} \gamma_{ij} = \left( \sigma + \frac{1}{2} \hat{s} - \frac{1}{2} \hat{f} \right) K - \frac{1}{2} D_{\eta} \hat{f}. \quad (25)$$

\footnote{The kinetic term of scalar fields are effectively dropped by ‘infinite mass’. One may also recall that the nature of higher derivative terms might be different between gravity and matters in three dimensions. As was realized in three-dimensional TMG or NMG, higher derivative terms in these gravity theories can be made harmless and have the same weight as the Einstein-Hilbert term, which might be problematic in matter fields by the existence of ghosts. This constrasts with the standard string theory approach, in which all higher derivative terms are treated as small corrections and then those in gravity and matters can be treated on the same footing.}
It is also interesting to see that the Brown-York stress tensor of NMG already indicates the possibility of its non-conservation because of various hatted quantities enters in its expression. In the case of Einstein gravity the so-called second Gauss-Codacci equation implies the conservation of Brown-York tensor in some cases, for instance BTZ black holes [37]. Explicitly, the second Gauss-Codacci equation is given in our convention by

\[ \nabla^j (K_{ij} - K_{ij}) = -NR_i^j, \]  

where \( R_i^j \) is \( \eta i \) component of the bulk Ricci tensor. The conservation of Brown-York tensor charges at the asymptotic infinity is a usual assumption in AdS/CFT correspondence, which is related to the conformal invariance in the CFT side. The conservation of Brown-York tensor charges at the bulk boundary \( Q \) might also be related to conformal symmetry, which implies the constancy of the tension of \( Q \) by the boundary condition on \( Q \). Even for BTZ black holes in NMG, the conservation of Brown-York tensor charges can be seen in the same way with Einstein case in spite of the hatted quantities. However, that is not the case for the so-called new type black holes which are allowed at special value of parameters in NMG. As was shown in [26, 27] these black holes are stable and compatible with holographic renormalization, but they are mysterious in the view point of the AdS/CFT correspondence and their relevance in dual CFT is still unclear as was point out in [27]. Therefore, we study the possibility of the AdS/BCFT correspondence without the conservation of Brown-York tensor charges keeping these new type black holes in mind.

Since all the formulae in NMG as well as GH terms and the above tension \( T \) are written in terms of the ADM-decomposed metric, it is useful to rewrite AdS metric in the ADM-decomposed form. Specifically, the AdS\(_3\) metric in coordinates \((\rho, \tau, y)\) in eq. (6) is already in the ADM-decomposed form.

Explicitly, the on-shell value of auxiliary field \( f_{\mu\nu} \) on AdS space is given by

\[ f_{\mu\nu} = -\frac{1}{m^2 L^2} \tilde{g}_{\mu\nu}, \]

where \( \tilde{g}_{\mu\nu} \) denotes the background AdS metric. Then, the hatted quantities for the above ADM-decomposed metric are given by

\[ \hat{s} = -\frac{1}{m^2 L^2}, \quad \hat{h}^i = 0, \quad \hat{f}^{ij} = -\frac{1}{m^2 L^2} \hat{g}^{ij}, \quad \hat{f} = -\frac{2}{m^2 L^2}. \]  

One may note that \( L \) is not identical with the cosmological constant \( \ell \) in NMG.

Let us take the same boundary \( A : \rho = \rho_* \) with the Einstein case for the computation of boundary entropy in the holographic setup. By a straightforward computation, one obtains the total action value or the disk amplitude in NMG with a boundary \( A : \rho = \rho_* \) as

\[ S_E = \frac{L}{4G} \left( \sigma + \frac{1}{2m^2 L^2} \right) \left[ \frac{\rho_*^2}{2 \epsilon^2} + \frac{\rho_*}{\epsilon} \sinh \left( \frac{\rho_*}{L} \right) + \log \left( \frac{\epsilon}{r_D} \right) - \frac{1}{2} - \frac{\rho_*}{L} \right]. \]  

Now, one can see that the (regularized) boundary entropy for the boundary \( A \) is given by

\[ S_{bd} = S_E(0) - S_E(\rho_*) = \frac{\rho_*}{4G} \left( \sigma + \frac{1}{2m^2 L^2} \right) = \frac{\rho_*}{4G_{eff}}, \quad G_{eff} \equiv \frac{1}{(\sigma + \frac{1}{2m^2 L^2})G}. \]
Since the central charge of CFT dual to NMG is given by\(^{2}\)
\[
c = \frac{3L}{2G} \left( \sigma + \frac{1}{2m^2L^2} \right),
\]  
(30)

this boundary entropy can be written as\(^{3}\)
\[
S_{bd} = \frac{c \rho^*}{6L},
\]  
(31)

which is the same form with Einstein gravity except the adaptation of the central charge to NMG. The behavior of boundary entropy proportional to central charge is expected to be a universal one in the holographic construction of boundary entropy. (see [40] for another example in six dimensional supergravity setup.)

3.2 Boundary entropy from holographic entanglement entropy

The holographic realization of the entanglement entropy suggests that higher curvature terms would require some modifications of the original proposal on holographic entanglement entropy(HEE) in the pure Einstein gravity case. Recalling that the original proposal for HEE is given by the minimized area of relevant (hyper)surface \(S_A\) in the Euclideanized bulk space, one naturally expects the minimization of some functional defined on \(S_A\) would be a correct way to make modifications. There are some suggestions to identify this functional at least for the so-called Lovelock gravity [20, 21]. One of the results in these studies is that the Wald formula type of entropy is not a correct functional for generic higher curvature gravity.

One may note that the Wald type of entropy is shown to be inadequate because of the discrepancy of HEE and Wald entropy in the dependence of various central charges in the trace anomaly. However, in the \(AdS_3/CFT_2\) case of our interest, there is just single central charge \(c\), which enters at the trace anomaly formula as
\[
\langle T^i_i \rangle = \frac{c}{24\pi} R.
\]

Therefore, we propose the Wald type of entropy as a correct HEE functional in our case. More concretely, we propose that Wald entropy is the correct way to compute HEE for any configuration in NMG with Brown-Heannaux fall-off boundary conditions. The necessity of these boundary conditions and the generality of proposal will be explained in section 5. In this section, we adopt this prescription for HEE and test its validity for a specific configuration. A convenient tool for the computation of Wald type of entropy is a specific central charge \(a^*_d\) introduced in [35, 41] as
\[
a^*_d = \left. \frac{\pi^{d/2}L^{d+1}}{d \Gamma(d/2)} \right|_{AdS}.
\]  
(32)

One can see that \(a^*_2 = c/12\) in the case of \(d = 2\). Then, Wald type of entropy as HEE is already computed in [35] and reduces in our case to
\[
S_{HEE} = S_W = \frac{2}{L} a^*_2 \int_{S_A} dx \sqrt{\gamma} = \frac{c}{6L} \int_{S_A} dx \sqrt{\gamma}.
\]  
(33)

\(^{2}\)See, for example [23, 38, 39] for this expression of central charge of dual CFT for NMG.

\(^{3}\)One may note that \(\rho^*\) has dimension of length.
Now, let us use this formula to compute HEE in our setup and verify that the relation between HEE and boundary entropy is not modified even with higher curvature terms. One may regard this computation as the verification of our prescription for HEE in our configuration, since in three dimensions the relation between entanglement entropy and boundary entropy is completely determined by conformal symmetry up to some constant. For the zero time slice \( \tau = 0 \), the bulk integration region is given by the same way with the Einstein case as \(-\infty < \rho \leq \rho_*\). Therefore, with some cutoff \( \rho_\infty = L \ln(2\mu/\epsilon) \), one obtains

\[
S_{\text{HEE}} = \frac{c}{6L} \int_{-\infty}^{\rho_*} d\rho = \frac{c}{6L} (\rho_* + \rho_\infty),
\]

which is the holographic construction for EE in Eq. (2) and leads to the boundary entropy as

\[
S_{\text{bd}} = S_{\text{HEE}}(\rho_\infty) - S_{\text{HEE}}(0) = \frac{c}{6L} \frac{\rho_*}{4G'}.
\]

One can see that the complete match of this result with Eq. (29) which is obtained in a different way.

3.3 Holographic g-theorem for higher curvature gravities

One interesting quantity in BCFT is the so-called \( g \)-function which is a monotonic function characterizing the boundary renormalization group flow and coincides with the boundary entropy at the conformal points of the boundary. Since boundary degrees of freedom localized at the boundary is independent of those in the bulk in a generic situation, the boundary renormalization group flow is independent of bulk renormalization group flow and so the boundary \( g \)-function is defined for a non-conformal boundary condition while the bulk is at the conformal point. In this subsection we derive holographic g-theorem in NMG from the null energy condition on matters localized on the bulk boundary \( Q \).

Since the bulk in BCFT is at conformal point, it is natural to take \( AdS_3 \) space as the background geometry for gravity. One may take the bulk boundary \( Q \) in \( AdS_3 \) space as \( (r,t,x(r)) \) in the Fefferman-Graham (or \( AdS \)-kink) coordinates which are related to Poincare coordinates as \( z \equiv e^{-r} \). In these coordinates \( AdS_3 \) space is written as

\[
ds^2 = L^2 \left[ dr^2 + e^{2A(r)} \left( -dt^2 + dx^2 \right) \right], \quad A(r) \equiv r.
\]

Since one is interested in the renormalization group flow of the \( g \)-function, the bulk boundary \( Q \) needs not to be \( AdS_2 \) space along the flow except at the conformal points. Therefore, the profile \( x(r) \) of the bulk boundary is taken as a generic function in this subsection. To address holographic g-theorem in NMG, it is convenient to use the ADM-decomposed form of \( AdS_3 \) metric in Lorentzian signature\(^4\). The above metric of \( AdS_3 \) space can be rewritten as the ADM-decomposed form with respect to the bulk hypersurface \( Q \) like the following

\[
ds^2 = L^2 \left[ \frac{1}{e^{-2A(r)} + x'(r)^2} d\eta^2 + e^{2A(r)} \left\{ -dt^2 + \left( e^{-2A(r)} + x'(r)^2 \right) \left( dr + \frac{x'(r)}{e^{-2A(r)} + x'(r)^2} d\eta \right)^2 \right\} \right], \quad (35)
\]

\(^4\)The overall sign of stress tensor should be changed in this case.
where we have introduced new coordinates $\eta$ as
\[ \eta \equiv x - x(r) . \]
The lapse and shift functions in this ADM decomposition are read as
\[
N(r) = \frac{L}{\sqrt{e^{-2A(r)} + x'(r)^2}}, \quad N^i = (N^t, N^r) = \left( 0, \frac{x'(r)}{e^{-2A(r)} + x'(r)^2} \right).
\]

Since the above metric form is ADM decomposed one, the indices of the extrinsic curvature, $K_{ij}$ take only those for surfaces. Those do not need to take all bulk indices in our setup. Note that the extrinsic curvature of the surface $Q$ in these coordinates is given by
\[
K_{tt} = L e^{2A(r)} \frac{x'(r)}{\sqrt{e^{-2A(r)} + x'(r)^2}}, \quad K_{tr} = 0, \quad K_{rr} = -L \frac{e^{2A(r)} x'(r)^2 + 2A'(r)x'(r) + x''(r)}{\sqrt{e^{-2A(r)} + x'(r)^2}}.
\] (36)

Null vectors on the surface $Q$ are given by
\[
\xi^i = \frac{e^{-A(r)}}{L} \left( \pm 1, \frac{1}{\sqrt{e^{-2A(r)} + x'(r)^2}} \right).
\] (37)

Let us recall that the null energy condition for matters localized on $Q$ with respect to any null vector, $\xi^i \pm$, is given by
\[
T_{ij}^{\text{mat}} \xi^i \pm \xi^j \pm \geq 0. \quad (38)
\]

Since $T_{ij}^{\text{mat}} = T_{ij}^Q$ by boundary condition on $Q$, this null energy condition can be written as $T_{ij}^Q \xi^i \pm \xi^j \pm \geq 0$. Now, it is straightforward to apply this condition to NMG with boundary terms. Using values of hatted auxiliary fields $\hat{f}_j$, $\hat{h}^i$, $\hat{s}$ and $\hat{f}$ given in eq. (27), through a straightforward calculation, one can see that null energy condition for null vectors in eq. (37) implies
\[
-\frac{1}{8\pi GL} \left[ \sigma + \frac{1}{2m^2 L^2} \right] \frac{e^{-3A(r)} \left( e^{A(r)} x'(r) \right)'}{\left( e^{-2A(r)} + x'(r)^2 \right)^{3/2}} \geq 0.
\] (39)

Note that this condition reduces to the pure Einstein gravity case by taking $\sigma = 1$ and $m^2 \to \infty$. In this expression one can see that the central charge of dual CFT appears as an overall coefficient.

Since we have chosen the Lagrangian parameter range such that $\sigma + 1/2m^2 L^2 > 0$ or equivalently the positive central charge of dual CFT, one obtains the constraint on the profile $x(r)$ as the consequence of null energy condition
\[
\left( e^{A(r)} x'(r) \right)' \leq 0.
\] (40)

This is the essential ingredient for holographic g-theorem and shows that the holographic g-theorem still holds in NMG. According to the previous result, it is natural to take the holographic g-function with respect to the boundary $Q$ specified by $(r, t, x(r))$ as
\[
\ln g(r) = \frac{c}{6} \text{arcsinh} \left[ -e^{A(r)} x'(r) \right],
\] (41)

\footnote{This choice of $g$-function is slightly different from \cite{7}.}
where \( c \) denotes the central charge of CFT dual to NMG given in eq. (30). This gives us the correct value of boundary entropy at \( x(\infty) = 0 \) through eq. (40) and the coordinate relation. Now, one can see that the constraint on \( x(r) \) by null energy condition implies
\[
\frac{\partial \ln g(r)}{\partial r} \geq 0.
\]
This is a holographic version of g-theorem in NMG and reduces to the ordinary one in the Einstein gravity case.

Though we have considered only the case of \( A(r) = r \), which represents \( AdS_3 \) space, one may try a more generic function \( A(r) \) which represents a \( AdS_3 \) kink connecting two \( AdS_3 \) spaces at the end points. This gravity background corresponds to the simultaneous renormalization group flow in the bulk and the boundary. Even in this situation one may ask the boundary \( g \)-function can be introduced and interpreted as the description of boundary degrees of freedom. One can show that our holographic \( g \)-function does the job at least Einstein gravity by elevating the central charge \( c \) to the central charge function \( c(r) \sim 1/A'(r) \) in this case.

4 Boundary entropy and thermal property

4.1 Boundary entropy from BTZ black holes

As in the previous section the bulk boundary \( Q \) in \( AdS_3 \) space is taken as \((\tau, z, x(z))\) in the Poincare coordinates. The basic reason for this choice of \( Q \) comes from the fact that \( Q \) should preserve \( SO(2,1) \) isometry which is manifest in the representation of \( AdS_3 \) space by the \( AdS_2 \) fibration over a line. Since BTZ black holes are locally isomorphic to \( AdS_3 \), one may still take \( Q \) in the same way as the case of \( AdS_3 \) space, which turns out to be the correct choice. In the following, we consider BTZ black holes in NMG and obtain boundary entropy from these black holes through the relation between thermal partition function and BTZ black holes. Here, we follow closely the Einstein gravity case.

Let us take the interval of its length \( \Delta x \) on the domain of the BCFT \((x < 0)\) for BTZ black holes. Using the boundary conditions on \( Q \), it is straightforward to obtain a profile of the boundary \( Q \) in the geometry of BTZ black holes. Euclideanized BTZ black holes in Schwarzschild coordinates are represented by
\[
\begin{align*}
\text{ds}^2 &= \frac{L^2}{z^2} \left[ f(z) d\tau^2 + \frac{dz^2}{f(z)} + dx^2 \right],
\end{align*}
\]
where \( f(z) = 1 - z^2/z_H^2 \) and \( z_H \) is a location of the event horizon. Euclidean time \( \tau \) compactified on a circle satisfies the periodicity \( \tau \sim \tau + 2\pi z_H \) and the temperature of the dual BCFT is given by \( T_{\text{BCFT}} = 1/2\pi z_H \). Through the coordinate transformation from \((x, z)\) to \((\eta, z)\) with the relation \( d\eta = dx - x'(z)dz \), the above BTZ black hole metric can be rewritten in the form of the ADM-decomposed metric
\[
\begin{align*}
\text{ds}^2 &= \frac{L^2}{z^2} \left[ \frac{1}{1 + f(z)x'^2(z)} d\eta^2 + f(z) d\tau^2 + \frac{1 + f(z)x'^2(z)}{f(z)} \left( dz + \frac{f(z)x'(z)}{1 + f(z)x'^2(z)} d\eta \right)^2 \right],
\end{align*}
\]
which is more suitable for our purpose.

By introducing an effective (constant) tension on the bulk boundary $Q$ as

$$T_{\text{eff}} \equiv \frac{1}{\left(\sigma + \frac{1}{2m^2L^2}\right)} T,$$

one can see that the boundary condition becomes (see Appendix A.2)

$$T_{\text{eff}} = \frac{x'(z)}{L\sqrt{1 + f(z)x'(z)^2}},$$

and $x = x(z)$ satisfies

$$\frac{dx}{dz} = \frac{T_{\text{eff}}L}{\sqrt{1 - T_{\text{eff}}^2L^2f(z)}}.$$  

This result is simply borrowed from the pure Einstein gravity case except the replacement of $T$ by $T_{\text{eff}}$, since the metric and boundary conditions take the same form with the Einstein case except the replacement of $T$ by $T_{\text{eff}}$. As a result, the profile of the boundary $Q$ can be described by

$$x(z) = z_H \arcsinh \left( \frac{T_{\text{eff}}L}{\sqrt{1 - T_{\text{eff}}^2L^2z_H}} \right).$$

One can see that this profile inserted in the above ADM-decomposed metric for BTZ black holes gives us the representation of those black holes as $AdS_2$ fibration over a line. One can explicitly verify that $\eta = \text{const.}$ hypersurface is the metric of $AdS_2$, indeed. Therefore, the bulk boundary $Q$ has $SO(2,1)$ isometry group as the case of $AdS_3$ space.

On the contrary to the form of the profile $x = x(z)$ which can be borrowed from Einstein gravity case, the on-shell value of total action needs to be computed separately in NMG in order to obtain HEE of the interval $A$ with length $\Delta x$ at the asymptotic boundary. Through a straightforward computation the on-shell value of the total action is given by (see Appendix A.2)

$$S_E = S_{\text{bk}} + 2S_{\text{bd}} = -\frac{\pi c}{3} \Delta x \cdot T_{BCFT} - \frac{Lz_H}{2G} \left( \sigma + \frac{1}{2m^2L^2} \right) \left[ \frac{x(z)}{z^2} \right] z_H^2 + c \frac{z_H \Delta x}{6} \frac{1}{\epsilon^2}.$$

To obtain the correct physical result, one needs to be careful by considering the physical radius at the position $z = \epsilon$. The asymptotic two dimensional geometry for BCFT at $z = \epsilon$ in Poincare coordinates is taken by

$$ds_{BCFT}^2 = f(\epsilon)d\tau^2 + dx^2.$$  

Accordingly, the physical radius of $\tau$ direction on this two dimensional surface should be taken as

$$z_H = \sqrt{f(\epsilon)z_H}.$$

Finally, using $x(z_H) = z_H \arctanh(T_{\text{eff}} L)$ and discarding the divergent parts by appropriate counter terms, one can see that the action value for the entanglement entropy is given by

$$S_E = -\frac{\pi c}{6} \Delta x \cdot T_{BCFT} - \frac{c}{3} \arctanh(T_{\text{eff}} L).$$
where $T_{BCFT} = 1/2\pi z_H$. Note that the final form of total Euclidean action value in this case is identical with the one in Einstein gravity except the adaptation of central charge relevant to NMG.

In the holographic construction standard thermal partition function is given at the leading order by the on-shell value of Euclidean gravity action with temporal length $\beta$

$$Z_{th}^{BCFT}(\beta) = e^{-S_E(\beta)}.$$  \hspace{1cm} (51)

This thermal partition function also takes the same form with the Einstein gravity case though the central charge is different. Therefore, one can see that the various properties inherited from $S_E$ are same with the Einstein gravity case. At a high temperature the BCFT partition function, which is defined on cylinder of the length $\Delta x$ with two boundaries $\alpha, \beta$, is given by

$$Z_{\alpha\beta} \approx g_\alpha g_\beta e^{-E_0 \Delta x}.$$  \hspace{1cm} (52)

In our case $S_E$ is given by $S_E = S_{bk} + 2 S_{bd}$. Since $S_{bk} \sim 1/\beta$ and $S_{bd}$ is independent of $\beta$, one can see that $S_{th} = -2S_{bk} - 2S_{bd}$. As a result, thermal entropy for BTZ black holes for the system of length $\Delta x$ is given by

$$S_{th} = \frac{\pi c}{3} \Delta x \cdot T_{BCFT} + \frac{c}{3} \text{arctanh}(T_{eff} L).$$  \hspace{1cm} (53)

### 4.2 Holographic thermal property of AdS black holes

The metrics of thermal solitons and non-rotating new type black holes [42] are given by

$$ds^2 = \frac{L^2}{z^2} \left[ dr^2 + \frac{1}{h(z)} dz^2 + h(z) dx^2 \right], \hspace{1cm} h(z) = 1 + Bz + Cz^2,$$  \hspace{1cm} (54)

$$ds^2 = \frac{L^2}{z^2} \left[ f(z) d\tau^2 + \frac{1}{f(z)} dz^2 + dx^2 \right], \hspace{1cm} f(z) = 1 + bz + cz^2,$$  \hspace{1cm} (55)

Let us focus on $B = b = 0$ case which corresponds to the case of non-rotating BTZ black holes and the corresponding thermal solitons. Through the coordinate transformation from $(x, z)$ to $(\eta, z)$ with the relation $d\eta = dx - x'(z) dz$, the thermal AdS metric can be rewritten in the form of the ADM-decomposed metric

$$ds^2 = \frac{L^2}{z^2} \left[ \frac{h(z)}{1 + h(z) x'(z)^2} d\eta^2 + d\tau^2 + \frac{1 + h(z)^2 x'(z)^2}{h(z)} \left( dz + \frac{h(z)^2 x'(z)}{1 + h(z)^2 x'(z)^2} d\eta \right)^2 \right].$$  \hspace{1cm} (56)

From the boundary condition, the profile of $Q$ can be found (see Appendix.A.3). By introducing an effective (constant) tension on the bulk boundary $Q$ as

$$T_{eff} \equiv \frac{1}{(\sigma + \frac{1}{2m^2 L^2})} T,$$  \hspace{1cm} (57)
one can see that the boundary condition becomes
\[ T_{eff} = \frac{h(z)^2 x'(z)}{L \sqrt{h(z)(1 + h(z)^2 x'(z)^2)}} , \]
and \( x = x(z) \) satisfies
\[ \frac{dx(z)}{dz} = \frac{L T_{eff}}{h(z) \sqrt{h(z) - L^2 T_{eff}^2}} . \] (58)

This result is the same as the pure Einstein gravity case except the replacement of \( T \) by \( T_{eff} \).

Therefore, the profile of the boundary \( Q \) can be described by
\[ x(z) = z_0 \arctan \left( \frac{LT_{eff} z}{z_0 \sqrt{h(z) - L^2 T_{eff}^2}} \right) . \] (59)

The on-shell value of the total action is given by (see Appendix A.3)
\[ S_E = \frac{L z_H}{2G} \left( \sigma + \frac{1}{2m^2 L^2} \right) \left[ -\frac{\pi}{2z_0} + \frac{1}{e^2} \left( x(\epsilon) + \frac{\pi z_0}{2} \right) - \frac{x(\epsilon)}{z_0^2} \right] . \] (60)

The smoothness of the geometry near \( z = z_0 \) gives us the periodicity of \( x \) as \( x \sim x + 2\pi z_0 \), which means that the radius of \( x \) direction is \( z_0 \). As in the case of BTZ black holes, the asymptotic two-dimensional geometry for BCFT at \( z = \epsilon \) is taken as
\[ ds^2 = d\tau^2 + h(\epsilon) dx^2 . \] (61)

By matching the physical radius of \( x \) as
\[ \tilde{z}_0 = \sqrt{h(\epsilon)} z_0 , \]
and discarding the divergent part by appropriate counter terms, one can see that the quantity relevant to the entanglement entropy is given by
\[ S_E = -\left( \sigma + \frac{1}{2m^2 L^2} \right) \pi L z_H \frac{z_0}{S G z_0} = -\frac{\pi}{24 \Delta x} \frac{c}{T_{BCFT}} , \] (62)
where \( T_{BCFT} = 1/(2\pi z_H) \). This also takes the same form as the Einstein case and it leads to the same boundary entropy in NMG as the one in Einstein gravity except scaling of Newton’s constant and cosmological constant through central charge. Since all the expression is identical with the Einstein gravity case, thermal properties are same with the Einstein gravity case. For instance, thermal solitons are preferred at low temperatures and BTZ black holes at higher temperatures, of which phase transition point is given by \( z_0 = z_H \).

Next, let us consider new type black hole case [23] in NMG, which exists only at the special parameters given by \( \sigma = 1 \), and \( 2m^2 L^2 = 1 \). Euclideanized new type black holes in Schwarzschild coordinates can be represented by (43) with \( f(z) = (1 - z/z_H)(1 - z/z_m) \) and \( z_H \) is a location of the event horizon and \( z_m \) is a certain parameter in new type black holes [42]. Euclidean time \( \tau \) compactified on a circle satisfies the periodicity \( \tau \sim \tau + 4\pi \left( \frac{1}{z_H} - \frac{1}{z_m} \right)^{-1} \) and the temperature of
the dual BCFT is given by \( T_{BCFT} = \frac{1}{4\pi} \left( \frac{1}{z_H} - \frac{1}{z_H} \right) \). Through the coordinate transformation from \((x, z)\) to \((\eta, z)\) with the relation \(d\eta = dx - x'(z)dz\), the metric (43) can be rewritten in the form of the ADM-decomposed metric (44).

Even though the metric takes the same form with the BTZ black hole case, there are some definite differences between them, taking into account boundary entropy. In fact, some issues described in the following should be resolved in new type black holes. Specifically, as mentioned before, \( R^n_i \) vanishes for BTZ black holes and the charge conservation of Brown-York tensor holds from the second Gauss-Codacci equation. However, for new type black holes one can see that \( R^n_i \) does not vanish as follows:

\[
R^n_z = \frac{(2 - 2f(z) + zf'(z))}{2L^2} x'(z) .
\] (63)

Note that BTZ black holes satisfy the relation \( 2 - 2f(z) + zf'(z) = 0 \). Moreover, charges of Brown-York tensor for new type black holes are not conserved in general (see Appendix.B). This phenomenon is attributed to the existence of a new parameter in the metric of (static) new type black holes which makes their geometrical and physical properties different from those of the BTZ black holes. The charge conservation of Brown-York tensor is a usual assumption in AdS/CFT correspondence, which is related to the conformal invariance in the CFT side. However, it is not so clear in NMG that the conservation of Brown-York tensors is requirement or relevant assumption.

For new type black holes there is no profile of \( Q \) to satisfy the boundary condition (24) with a constant tension \( T \). One may consider a more generic matter Lagrangian to find the profile of \( Q \) from the boundary condition. But in these cases the situation would be different from the simplest case we considered in this paper. It is expected that the charge conservation for Brown-York tensor would not be satisfied and the surface \( Q \) would not be \( AdS_2 \), in contrast to BTZ black hole case. Therefore \( AdS/BCFT \) for new type black holes should be separately investigated further. This is because of the property of new type black holes not locally isomorphic to \( AdS_3 \), contrary to BTZ black hole case. So, for new type black holes, one might need to take a different ansatz for \( Q \) to satisfy charge conservation and boundary condition.

Even with the problem of specifying the surface \( Q \), however, it is expected that the new type black holes with bulk boundary have a similar thermal property to BTZ black holes with bulk boundary.

5 A simple derivation

In this section we rederive our results presented in the previous sections through a few simple steps. First, let us recall that the radius \( L \) of AdS space in NMG is related to the parameters \( \sigma, m \) and \( l \) in the Lagrangian (13) by

\[
\frac{1}{L^2} = \frac{1}{L^2} \left( \sigma - \frac{1}{4m^2L^2} \right).
\] (64)

Second, one may notice that the auxiliary field formalism in some kinds of higher derivative theories is particularly appropriate for \( AdS/CFT \) correspondence. After the construction of three-dimensional NMG action, there are some studies on critical and non-critical gravities with higher derivatives in various dimensions. One of the interesting observations in these theories on AdS...
space is the possibility of the consistent truncation for unwanted ghost-like modes by appropriate boundary conditions. It was argued that this consistent truncation in four-dimensional case may be elevated to the full non-linear level [43] and indeed shown that is the case through symmetric auxiliary fields [44] (see also [45]). In the following we propose how to extend this effective description of specific higher curvature gravity to NMG even with a bulk boundary.

By taking the same procedure in the case of four dimensional non-critical Einstein-Weyl gravity [44], one may lift the value in AdS space to the full metric field for the auxiliary tensor field $f_{\mu\nu}$. Explicitly, this means that one can take the auxiliary field $f_{\mu\nu}$ as

$$f_{\mu\nu} = -\frac{1}{m^2 L^2} g_{\mu\nu}, \quad f = -\frac{3}{m^2 L^2},$$

where $g_{\mu\nu}$ denotes the fluctuating metric field not the metric value in AdS space. By inserting this ansatz to two tensor form of NMG bulk action given in (13), one obtains

$$\mathcal{L}_{NMG} = \frac{1}{16\pi G} \left[ \sigma R + \frac{2}{L^2} + \frac{1}{2m^2 L^2} R + \frac{3}{2m^2 L^2} \right] = \frac{1}{16\pi G} \left( \sigma + \frac{1}{2m^2 L^2} \right) \left[ R + \frac{2}{L^2} \right].$$

Note that this is the just action for the pure Einstein gravity with rescaled Newton’s constant $G_{eff} \equiv G/(\sigma + 1/2m^2 L^2)$ and with the substituted cosmological constant, $L$.

This simple computation explains various results about NMG in the setup of AdS/CFT correspondence. For example, the central charge of dual CFT for NMG is given simply by the rescaling of Newton’s constant and the substitution of cosmological constant as

$$c = \frac{3L}{2G} \left( \sigma + \frac{1}{2m^2 L^2} \right).$$

Moreover, one can see that the GH term in NMG is also reduced to the one in the pure Einstein gravity by the same lift of auxiliary field in the boundary. To see this, note that the relevant auxiliary fields in NMG GH term are taken by

$$\hat{f}_{ij} = -\frac{1}{m^2 L^2} \gamma_{ij}, \quad \hat{f} = -\frac{2}{m^2 L^2}.$$

Then, the GH term (16) becomes

$$S_{GH}^{NMG} = \frac{1}{16\pi G} \int d^2 x \sqrt{-\gamma} \left[ 2\sigma K - \frac{1}{m^2 L^2} K + \frac{2}{m^2 L^2} K \right] = \frac{1}{8\pi G_{eff}} \int d^2 x \sqrt{-\gamma} K,$$

which shows us that NMG GH term also reduced to the one in the pure Einstein gravity with the same rescaling of Newton’s constant. Another point to address in the boundary entropy is the existence of matter sector on the boundary $Q$, which is described, in the simplest case, by

$$S_{mat}^{Q} = -\frac{1}{8\pi G} \int_{Q} \sqrt{\gamma} T.$$
Hence, the boundary action on $Q$ becomes
\[ S_Q + S_Q^{\text{mat}} = \frac{1}{8\pi G_{\text{eff}}} \int_Q d^2x \sqrt{\gamma} \left[ K - T_{\text{eff}} \right]. \] (70)

This is our result for the effective description of NMG with a bulk boundary in the context of the AdS/BCFT correspondence.

The above short computation reveals that the effective description of NMG in the setup of AdS/CFT correspondence is given simply by the pure Einstein gravity with some rescaling and substitution. Even with boundary, the above effective description becomes unchanged. As a result, most of our computation in previous sections are understood as simple rescaling of Newton’s constant and the substitution of the cosmological constant. This simple derivation holds for BTZ black holes and other cases except new type black holes.

One can consider this simple derivation as another check of our results given in the previous sections. Some comments may clarify the meaning of this derivation since the role of higher derivatives may be somewhat different according to the spacetime dimensions. Note that the above derivation is based on the possibility of consistent truncation. In the case of four-dimensional conformal or noncritical Einstein-Weyl gravity, this truncation is achieved by appropriate boundary conditions at the asymptotic infinity since the unwanted ghost-like massive modes fall off more slowly than log or massless modes. On the contrary, three-dimensional NMG is regarded as the covariant completion of Pauli-Fierz massive graviton theory, and so propagating physical massive graviton modes are allowed with an appropriate choice of Lagrangian parameters. Accordingly, one may worry about the inappropriate truncation of physical modes in the NMG case. However, one may recall that the unitarity of graviton modes and dual CFT is incompatible in the case of the three-dimensional Einstein gravity and even in the NMG case [23]. All our results in the previous sections belong to the parameter regions $c > 0$ where dual CFT is unitary or BTZ black holes have positive mass while the gravitons are ghost-like modes and can be truncated for $m^2 > 0$. Therefore, gravitons may be truncated consistently in our setup of AdS/CFT correspondence which requires dual CFT be unitary. More explicitly, our boundary conditions taken at the asymptotic infinity are those of Brown-Henneaux [46] which excludes new type black holes. This explains the above derivation does not apply to those black holes. See Appendix B for some attempts of AdS/BCFT correspondence in new type black holes.

## 6 Conclusion

It is natural to ask higher curvature effects on holographic description of BCFT after its construction for Einstein gravity. As in HEE, AdS/BCFT correspondence has no generic prescription for higher curvature effects, and so we have explored one simple extension of the original proposal of AdS/BCFT correspondence on Einstein gravity by considering the correspondence for the so-called NMG on three dimensions. NMG is most natural setting to see higher curvature effects in the sense that it is unique curvature square gravity consistent with holographic c-theorem.

In this paper we have followed closely the construction in the Einstein gravity case and computed in three different ways the boundary entropy for BCFT defined on a half space or its conformally-
related disk. Concretely, boundary entropy is obtained through its relation with disk amplitude, HEE and black holes. We showed that all approaches give us the same result for boundary entropy and it is simply given by the same form with the Einstein gravity case except the fact that the relevant central charge is taken for NMG case not for the pure Einstein gravity one. By using auxiliary field formulation of NMG, we have also showed that the holographic g-theorem still holds in NMG. Thermal properties of $BCFT$ are also studied holographically in NMG. Furthermore, we have showed that these results can be understood by the effective description of NMG by the auxiliary field which is recently analyzed and advocated in the case of non-critical Einstein gravity in four dimensions.

Our study is just a small step toward understanding of higher curvature effects on $AdS/BCFT$ correspondence. One immediate question one may ask is about effects by even higher curvature terms. There are several extensions of NMG consistent with holographic c-theorem with even more higher curvature terms and it is interesting to see their effects on the correspondence. Most natural expectation is that only the central charge will be modified and all the results will be same with Einstein gravity except the substitution in various expression by the modified central charge. The counter terms for the divergence cancelation should also be addressed properly in the case of BCFT since the counter terms are more subtle with higher curvature terms as shown in [26]. Another interesting question is the higher curvature effects on higher dimensional spacetime. In this case Lovelock gravity may be more adequate for the analysis as in HEE studies.

It needs to be mentioned that there may be log-modes in higher derivative theories at a specific value of higher derivative couplings, which is beyond the scope of this paper. As can be seen from the fact that the simple derivation through auxiliary field formalism depends on the truncation of slow fall-off modes including log ones, the original proposal for $AdS/BCFT$ correspondence in Einstein gravity should be modified when these slow fall-off modes in higher derivative gravity are included. Specifically in the case of $AdS_3/BCFT_2$, this means that modified construction needs to be considered for the more general fall-off boundary conditions than the Brown-Henneaux ones. At the specific coupling in NMG required for the existence of log-modes, the central charge vanishes and the theory is argued to be null under the strict Brown-Henneaux boundary conditions, which is consistent with our results about the zero g-function value at that coupling. However, the boundary entropy of dual CFT may be non zero with log-modes, since log modes are related to non-unitary log-CFT according to log-gravity conjecture and it is plausible to envisage some non-vanishing boundary entropy in this case.

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A Some formulae for on-shell action value of NMG

In this appendix, we present some computational details to obtain boundary entropy, \( \text{i.e.} \) \( \ln g \), associated with a boundary \( \partial M \). In our convention, Euclidean NMG actions with boundary terms on \( Q \), while omitting asymptotic boundary term on \( M \), are taken as Eq. (4) which can be written as

\[
S_{bk} \equiv S = -\frac{1}{16\pi G} \int_N \sqrt{g} L_{bk}, \quad S_{bd} \equiv S_Q + S_Q^{\text{mat}} = -\frac{1}{8\pi G} \int_Q \sqrt{\gamma} L_{bd}. \tag{A.1}
\]

A.1 Disk amplitude

The following are some steps to obtain the on-shell value of Euclidean action given in eq. (28) in order for the boundary entropy as the disk amplitude: Let us first consider the coordinates transformation from \((\tau, x, z)\) to \((r, \theta, \phi)\) in accordance with

\[
\tau = r \sin \theta \cos \phi, \quad x = r \sin \theta \sin \phi, \quad z = r_D \sinh \frac{\rho_*}{L} + r \cos \theta.
\]

Then, the metric (6) becomes

\[
ds^2 = L^2 dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 \tag{A.2}
\]

and surface integration does

\[
\int_Q \sqrt{\gamma} = \int d\theta d\phi \frac{L^2 r^2 \sin \theta}{(r \cos \theta + r_D \sinh \frac{\rho_*}{L})^2} \bigg|_{r=r_D \cosh \frac{\rho_*}{L}}.
\]

Using the above metric (A.2) and applying the boundary condition (24), one can see that integrands for the bulk and boundary parts become

\[
L_{bk} = -\frac{4}{L^2} \left( \sigma + \frac{1}{2m^2 L^2} \right), \quad L_{bd} = r_D \frac{\sinh \frac{\rho_*}{L}}{L} \left( \sigma + \frac{1}{2m^2 L^2} \right) \frac{1}{r},
\]

where we used the tension given by

\[
T = r_D \frac{\sinh \frac{\rho_*}{L}}{L} \left( \sigma + \frac{1}{2m^2 L^2} \right) \frac{1}{r}. \tag{A.3}
\]

Some values related to auxiliary fields \( f^{\mu\nu} \) are as follows;

\[
\hat{s} = \frac{1}{m^2 L^2}, \quad \hat{f} = -\frac{2}{m^2 L^2}, \quad D_r \hat{f} = 0, \quad \frac{1}{2} \gamma_{ij} D_r \hat{f}^{ij} = \frac{r_D \sinh \frac{\rho_*}{L}}{m^2 L^2} \cdot \frac{2}{r}, \quad K_{ij} \hat{f}^{ij} = -\frac{r_D \sinh \frac{\rho_*}{L}}{m^2 L^2} \cdot \frac{2}{r}.
\]
As a result, the on-shell value of total Euclidean action for the disk are given by

\[ S_E = S_{bk} + S_{bd} \]

\[ = \frac{1}{4\pi GL^2} \int_{\gamma} L^3 \left( \sigma + \frac{1}{2m^2L^2} \right) dz \int_{r_0}^{r_\ast} 2\pi r dr \]

\[ - \frac{1}{2\pi G} \int_{\gamma} \frac{L^2 r^2 \sin \theta}{(r \cos \theta + r_D \sinh \frac{\rho_s}{L})^2} \cdot \frac{r_D \sinh \frac{\rho_s}{L}}{L} (\sigma + \frac{1}{2m^2L^2}) \left|_{r=r_D \cosh \frac{\rho_s}{L}} \right. \]

where \( \epsilon \) is introduced for the UV cutoff. Parameters \( r_0, r_\ast \) and \( \theta_0 \) for the integration ranges are given by

\[ r_0 = r_D e^{\frac{\rho_s}{L}}, \quad r_\ast = r_D \cosh^2 \frac{\rho_s}{L} - (z - r_D \sinh \frac{\rho_s}{L}), \quad \theta_0 = -\arccos \left( \frac{\tan \frac{\rho_s}{L}}{L} \right). \]

**A.2 BTZ black hole**

Using the ADM decomposition, the BTZ black hole metric can be represented by eq. (44) with \( f(z) = 1 - z^2/\mu L \). The lapse function \( N(z) \) and shift vector \( N^i \) are given by

\[ N(z) = \frac{L}{z \sqrt{1 + f(z)x'(z)^2}}, \quad N^i = (N^\tau, N^z) = \left( 0, \frac{f(z)x'(z)}{1 + f(z)x'(z)^2} \right). \]

For a space-like unit vector \( n^\mu \) normal to the hypersurface \( Q \), the induced metric \( \gamma_{\mu\nu} \) are introduced as \( \gamma_{\mu\nu} = g_{\mu\nu} - n_\mu n_\nu \). In our case \( n^\mu \) is given by

\[ n^\mu = \frac{z}{L \sqrt{1 + f(z)x'(z)^2}} \left( 0, -x'(z)f(z), 1 \right). \]  

(A.4)

The induced metric on the surface \( Q \) in a matrix form is given by

\[ \gamma_{\tau\tau} = \frac{L^2 f(z)}{z^2}, \quad \gamma_{zz} = \frac{L^2 (1 + f(z)x'(z)^2)}{z^2 f(z)}, \quad \gamma_{\tau z} = 0. \]  

(A.5)

Then, the extrinsic curvature defined by (17) is given by

\[ K_{\tau\tau} = \frac{L f(z)x'(z)}{z^2 \sqrt{1 + f(z)x'(z)^2}}, \quad K_{zz} = \frac{L [x'(z) + f(z)f(z)x'(z)^2 - z x''(z)]}{z^2 f(z) \sqrt{1 + f(z)x'(z)^2}}, \quad K_{\tau z} = 0. \]  

(A.6)

After all, the boundary condition gives us the relation for the profile of the surface \( Q \) as

\[ T = \left( \sigma + \frac{1}{2m^2L^2} \right) \frac{x'(z)}{L \sqrt{1 + f(z)x'(z)^2}}. \]  

(A.7)

Some values of hatted quantities relevant to the computation are

\[ \hat{s} = -\frac{1}{m^2L^2}, \quad \hat{h}^i = 0, \quad \hat{f} = -\frac{2}{m^2L^2}, \]

\[ \hat{f}^{\tau\tau} = -\frac{z^2}{m^2L^4 f(z)}, \quad \hat{f}^{zz} = -\frac{z^2}{m^2L^4 (1 + f(z)x'(z)^2)}. \]
The integrands for the bulk action, the boundary one and the extrinsic curvature scalar in the BTZ black hole case are given by

\[ L_{bk} = -\frac{4}{L^2} \left( \sigma + \frac{1}{2m^2L^2} \right), \quad L_{bd} = \left( \sigma + \frac{1}{2m^2L^2} \right) K - T, \quad K = \frac{2x'^{(z)}}{L\sqrt{1 + f(z)x''^{(z)^2}}} \quad \text{(A.8)} \]

Using above quantities, integrands and extrinsic curvature scalar, the on-shell values of the action can be obtained. Firstly, the value of bulk action is given by

\[ S_{bk} = -\frac{1}{16\pi G} \int d\tau dx \frac{L^3}{z} \left[ -\frac{4}{L^2} \left( \sigma + \frac{1}{2m^2L^2} \right) \right] = -c \frac{\Delta x}{6z_H} + \frac{Lz_H}{G} \left( \sigma + \frac{1}{2m^2L^2} \right) \int_{\epsilon}^{z_H} \frac{dz}{z^3} \cdot x(z) + \frac{c}{6} \frac{z_H \Delta x}{\epsilon^2}. \quad \text{(A.9)} \]

And the action value for the boundary part becomes

\[ S_{bd} = -\frac{Lz_H}{4G} \left( \sigma + \frac{1}{2m^2L^2} \right) \int_{\epsilon}^{z_H} \frac{dz}{z^2} \cdot x'(z) = -\frac{Lz_H}{4G} \left( \sigma + \frac{1}{2m^2L^2} \right) \left[ \frac{x(z)}{z^2} \right]_{\epsilon}^{z_H} - \frac{Lz_H}{2G} \left( \sigma + \frac{1}{2m^2L^2} \right) \int_{\epsilon}^{z_H} \frac{dz}{z^3} \cdot x(z). \quad \text{(A.10)} \]

Finally, the on-shell value of total Euclidean action for the BTZ black hole case is given by

\[ S_E = S_{bk} + 2S_{bd} = -\frac{c}{6} \frac{\Delta x}{z_H} - \frac{c}{3} \frac{x(z_H)}{z_H} + \frac{c}{3} \frac{z_H x(\epsilon)}{\epsilon^2} + \frac{c}{6} \frac{z_H \Delta x}{\epsilon^2}. \quad \text{(A.11)} \]

### A.3 Thermal AdS$_3$ or thermal solitons

The metric of thermal solitons can be written in the form of the ADM-decomposed one for the bulk boundary $Q$ as eq.(56) with $h(z) = 1 - z^2/z_0^2$. Lapse function and shift vector can be seen as

\[ N = \frac{Lh(z)}{z\sqrt{h(z)(1 + h(z)^2x'(z)^2)}}, \quad N^i = (N^\tau, N^z) = \left( 0, \frac{h(z)^2x'(z)}{1 + h(z)^2x'(z)^2} \right). \quad \text{(A.12)} \]

According to the same procedure for the BTZ black hole case, we can obtain the induced metric for the thermal solitons. Choosing a space-like unit vector normal to $Q$

\[ n^\mu = \frac{z}{L\sqrt{h(z)(1 + h(z)^2x'(z)^2)}} \left( 0, -h(z)^2x'(z), 1 \right). \quad \text{(A.13)} \]

Then, induced metric $\gamma_{ij}$ on $Q$ can be given in the form of

\[ \gamma_{\tau\tau} = \frac{L^2}{z^2}, \quad \gamma_{zz} = \frac{L^2(1 + h(z)^2x'(z)^2)}{z^2h(z)}, \quad \gamma_{\tau z} = 0. \quad \text{(A.14)} \]

Now, using eq.(17) with the above induced metric, the extrinsic curvature can be read as

\[ K_{\tau\tau} = \frac{Lh(z)^2x'(z)}{z^2\sqrt{h(z)(1 + h(z)^2x'(z)^2)}}, \quad K_{\tau z} = 0 \quad \text{(A.15)} \]

\[ K_{zz} = \frac{L[x'(z)(1 + h(z)^2x'(z)^2) - z(h(z)x''(z) + h'(z)x'(z))]}{z^2\sqrt{h(z)(1 + h(z)^2x'(z)^2)}}. \quad \text{(A.16)} \]
Adopting the boundary condition on the bulk boundary $Q$, one finally obtains

$$ T = \left( \sigma + \frac{1}{2m^2L^2} \right) \frac{h(z)^2x'(z)}{L \sqrt{h(z)(1 + h(z)^2x'(z)^2)}}. \quad (A.17) $$

Some values of hatted quantities in this case are given by

$$ \hat{s} = - \frac{1}{m^2L^2}, \quad \hat{h}^i = 0, \quad \hat{f} = \frac{2}{m^2L^2}, $$

$$ \hat{f}^{\tau\tau} = - \frac{z^2}{m^2L^2}, \quad \hat{f}^{zz} = \frac{2h(z)}{m^2L^4(1 + h(z)^2x'(z)^2)}, $$

and extrinsic curvature scalar is

$$ K = \frac{2h(z)^2x'(z)}{L \sqrt{h(z)(1 + h(z)^2x'(z)^2)}}. \quad (A.18) $$

The Euclidean action is given by

$$ S_{bk} = \frac{L}{4\pi G} \left( \sigma + \frac{1}{2m^2L^2} \right) \int d\tau dz dx \frac{1}{z^3} $$

$$ = \frac{Lz_H}{G} \left( \sigma + \frac{1}{2m^2L^2} \right) \left[ \int_{z_*}^{z_0} dz \frac{1}{z^3} \left( x(z) + \frac{\pi z_0}{2} \right) + \int_{z_*}^{z_0} dz \frac{\pi z_0}{z^3} \right], \quad (A.19) $$

$$ S_{bd} = \frac{-Lz_H}{2G} \left( \sigma + \frac{1}{2m^2L^2} \right) \int_{z_*}^{z_0} dz \frac{1}{z^2} h(z)x'(z). \quad (A.20) $$

The on-shell value of the total action is given by

$$ S_E = S_{bk} + S_{bd} = \frac{Lz_H}{2G} \left( \sigma + \frac{1}{2m^2L^2} \right) \left[ - \frac{\pi}{2z_0} + \frac{1}{\epsilon^2} \left( x(\epsilon) + \frac{\pi z_0}{2} \right) - \frac{x(\epsilon)}{z_0} \right]. \quad (A.21) $$

### B New type black holes and HEE

In this section, let us consider new type black holes in NMG. Euclideanized new type black holes can be recast in the form of the ADM-decomposed metric (44) with $f(z) = (1 - z/z_H)(1 - z/z_0)$. First, consider action values (A.1) for new type black holes. Note that both BTZ black hole and new type black hole have the following relation,

$$ f^n(z) = 0 \quad (n \geq 3), $$

$$ z^2 f''(z) + 2(f(z) - zf'(z) - 1) = 0. \quad (B.1) $$

Using these relation, the integrands of bulk action and boundary action become

$$ \mathcal{L}_{bk} = \left( \sigma + \frac{1}{2m^2L^2} \right) \frac{2zf'(z) - 4f(z)}{L^2}, \quad (B.2) $$

$$ \mathcal{L}_{bd} = \left( \sigma + \frac{1}{2m^2L^2} \right) K + \frac{(2f(z) - zf'(z) - 2)(2f(z) - zf'(z))f(z)x'(z)^3}{4m^2L^3(1 + f(z)x'(z)^2)^{3/2}} - T, \quad (B.3) $$
where

\[ K = \frac{[f(z)(4f(z) - zf'(z))x'(z)^3 + (4f(z) - 2zf'(z))x'(z) - 2zf(z)x''(z)]}{2L(1 + f(z)x'(z)^2)^{3/2}}. \]

For BTZ black holes and AdS space, there is another relation which is given by

\[ zf'(z) - 2f(z) + 2 = 0. \]  \hspace{1cm} (B.4)

Using this relation, it can be seen that AdS space and BTZ black holes have the same value of the bulk part, and same structure of the boundary part. Using the profile of the boundary \( Q, x'(z) \) obtained from (A.7), boundary parts for AdS space and BTZ black holes have the same value, \( T \), i.e., (A.8). However, the results of the new type black holes are definitely different from these cases.

Now, in order to find the profile of \( Q \) in new type black holes, let us consider the boundary condition given by (21). Using the relations (B.1), we have two equations for \( \tau \tau \) and \( zz \) components, which are separately given by

\[
0 = \frac{z^2}{L^3 f(z) (1 + f(z)x'(z)^2)^{3/2}} \left[ \left( \sigma + \frac{1}{2m^2L^2} \right) f(z)^2 x'(z)^3 + \sigma \left( f(z) - \frac{1}{2} zf'(z) \right) x'(z) \right.
\]
\[
+ \frac{1}{2m^2L^2} \left( f(z) \left( 7 - 6f(z) + 2zf'(z) \right) + \frac{1}{2} zf'(z) \left( zf'(z) + 1 \right) \right) x'(z),
\]
\[
- zf(z) \left( \sigma - \frac{1}{2} zf'(z) \right) x''(z) - LT \left( 1 + f(z)x'(z)^2 \right)^{3/2}, \] \hspace{1cm} (B.5)

\[
0 = \frac{z^2 f(z)}{2L^3 (1 + f(z)x'(z)^2)^{3/2}} \left[ \sigma \left( 2f(z) - zf'(z) \right) x'(z) - 2LT \sqrt{1 + f(z)x'(z)^2} \right.
\]
\[
+ \frac{1}{2m^2L^2} \left( 2f(z) - zf'(z) - 1 \right) \left( 2f(z) - zf'(z) \right) x'(z) \]. \hspace{1cm} (B.6)

Therefore from \( zz \) component, we obtain

\[ T = \frac{2f(z) - zf'(z)}{2L \sqrt{1 + f(z)x'(z)^2}} \left( \sigma + \frac{2f(z) - zf'(z) - 1}{2m^2L^2} \right) x'(z). \] \hspace{1cm} (B.7)

This equation gives us the profile of \( Q \). For BTZ black holes, with Eq. (B.4), it is easy to see that Eq. (B.7) reduces to Eq. (46). However, we should check if this is a correct solution of boundary condition by checking \( \tau \tau \) component. For BTZ black holes, with \( x'(z) \) obtained from (A.7), it is easy to check that Eq. (B.5) is satisfied. For new type black holes at the special parameters, however, the \( \tau \tau \) component with \( x'(z) \) from Eq. (B.7) gives us the following condition.

\[
\frac{1}{L^2 (zf''(z) - 2f(z))^6} \left[ 4Tz^2 \left( 2f(z) - zf'(z) - 2 \right) \left( (zf'(z) - 2f(z))^4 - 6L^2 T^2 f(z) \right) \right] = 0, \] \hspace{1cm} (B.8)

where we put \( \sigma = 1 \) and \( 2m^2L^2 = 1 \) for simplicity. The first term of the numerator vanishes for BTZ black holes, because of the relation (B.4). But new type black holes do not satisfy the relation. Therefore for the new type black hole there is no solution to satisfy the boundary condition with constant \( T \).
Next, let us consider the energy conservation for the new type black holes.

\[ \nabla_j T^{\mathcal{Q}}_{ij} = \frac{1}{8\pi G} \left[ \left( \sigma + \frac{1}{2} \hat{s} - \frac{1}{2} \hat{f} \right) \left( -N R^\eta_{ij} \right) + (K \gamma_{ij} - K_{ij}) \nabla^j \left( \sigma + \frac{1}{2} \hat{s} - \frac{1}{2} \hat{f} \right) + \nabla^j H_{ij} \right], \tag{B.9} \]

where we used the second Gauss-Codacci equation \( (26) \) and

\[ H^{ij} \equiv -\nabla^{(i} \hat{h}^{j)} + \frac{1}{2} D_\eta \hat{f}^{ij} + K^{(i} \hat{f}^{j)k} + \gamma^{ij} \left( \nabla_k \hat{h}^k - \frac{1}{2} D_\eta \hat{f} \right). \tag{B.10} \]

Using Eq. (B.1), we find that

\[ \nabla_j T^{\mathcal{Q}}_{\tau j} = 0, \tag{B.11} \]

\[ \nabla_j T^{\mathcal{Q}}_{z j} = \frac{1}{8\pi G} \left[ \left( \sigma + \frac{1}{2} \hat{s} - \frac{1}{2} \hat{f} \right) \left( -N R^\eta_{ij} \right) - \frac{(2f(z) - zf'(z) - 2) (2f(z) - zf'(z) + 4) x'(z)}{4m^2 L^3 z (1 + f(z)x'(z)^2)^{3/2}} \right] 
\]

\[ = \frac{(2f(z) - zf'(z) - 2) x'(z)}{16GL\pi z (1 + f(z)x'(z)^2)^{3/2}} \left[ \sigma(1 + f(z)x'(z)^2) \right. \]

\[ + \left. \frac{1}{2m^2 L^2} \left( (2f(z) - zf'(z) - 1) f(z)x'(z)^2 - 4f(z) + 2zf'(z) - 1 \right) \right], \tag{B.12} \]

where

\[ N = \frac{L}{z\sqrt{1 + f(z)x'(z)^2}}, \quad R^\eta_{ij} = \frac{(2 - 2f(z) + zf'(z))}{2L^2} x'(z). \tag{B.13} \]

For BTZ black hole case, it is easy to check that eq. (B.12) is zero from eq. (B.4).
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