Vacuum Polarization Using Quantum Mechanical Path Integrals

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Abstract

We compute the quantum vacuum polarization for a pure neutral scalar field theory within the context of single-particle quantum mechanics. The loop diagram is computed without ever encountering loop-momentum integrals. Our approach is based on standard Feynman path integrals. Contact is made to scalar QED.

1 Introduction

This paper was initiated by several recent articles on so-called string-inspired calculations in scalar and spinor quantum electrodynamics (QED). World-line techniques and standard quantum mechanical path integrals were used to calculate effective actions and loop processes without employing any loop-momentum integrals. Our intention here is to demonstrate in a didactic and rather explicit manner how these methods can be put to work in a simplified model which avoids charge, spin or polarization degrees of freedom, but brings out all the necessary physics that one would encounter in more realistic models.

String-inspired methods in quantum field theory were first used in the works of Bern and Kosower\textsuperscript{1}. These authors and Strassler\textsuperscript{2} then realized that some of the well-known vacuum processes in QED and QCD can be computed rather easily with the aid of one-dimensional path integrals for relativistic point particles. Similar techniques and results can also be found in the monograph by Polyakov\textsuperscript{3}. String-inspired methods, particularly in QED, were then extensively studied in a series of papers by Schmidt, Schubert and Reuter; cf., e.g., Ref. 4, where the state of the art is reviewed extensively. There are also contributions by McKeon\textsuperscript{5} and various co-authors who have proved that world-line methods are extremely useful.
2 Vacuum Polarization in a Model Field Theory

\[ \mathcal{L}' = \frac{g}{2} \psi^2 \phi \]

To have a comparatively simple model let us consider an interaction Lagrangian

\[ \mathcal{L}' = \frac{g}{2} \psi^2 (x) \phi(x), \quad (1) \]

where \( g \) is the coupling constant. In the following we will be mainly interested in the one-loop vacuum graph (Fig. 1).

\[ i \Gamma_N(x_1, \ldots, x_N) \equiv N = 1, 2, \ldots, \infty \]

Figure 1: One-loop vacuum graph in a \( \psi^2 \phi \)-theory.

In QED the particle circulating in the loop would be the electron which is tied to an arbitrary number of off-shell photons. As is well known (see, e.g., Ref. 6), loop graphs belong to a subclass of Feynman diagrams called one-particle-irreducible diagrams. Their associated one-particle-irreducible amplitudes \( \Gamma_N(x_1, x_2, \ldots, x_N) \) can be obtained with the aid of a generating functional:

\[ \Gamma[\phi] = \sum_{N=0}^{\infty} \frac{1}{N!} \int d^4x_1 d^4x_2 \ldots d^4x_N \Gamma_N(x_1, \ldots, x_N) \phi(x_1) \ldots \phi(x_N). \quad (2) \]

Our interest lies with \( N = 2 \). But for the time being we will let \( N = 1, 2, \ldots, \infty \). Now recall from potential theory that a particle of mass \( m \) travelling to all orders in an external field \( \phi(x) \) is given by
\[ \Delta_+[\phi] = \Delta_+ + \Delta_+ g\phi \Delta_+ + \Delta_+ g\phi \Delta_+ g\phi \Delta_+ + \ldots \]

where \( \Delta_+ \) is the Green’s function of the freely propagating \( \psi \)-particle (\( \Delta_+ \equiv \Delta_+[\phi = 0] \)) which satisfies the Green’s function equation \((-\partial^2 + m^2)\Delta_+(x - y) = \delta^4(x - y)\), or in momentum space, \((p^2 + m^2)\Delta_+(p) = 1\). Our metric signature is \((-+,+,+,+)\). Summing up the terms in the geometric (Born-)series \(\text{Eq. (3)}\) we obtain

\[ \Delta_+[\phi] = \Delta_+ (1 + g\phi \Delta_+ + g\phi \Delta_+ g\phi \Delta_+ + \ldots) = \Delta_+ (1 - g\phi \Delta_+)^{-1}. \]

It is rather trivial to rewrite Eq. \(\text{Eq. (4)}\) in the form

\[ (p^2 + m^2 - g\phi) \Delta_+[\phi] = 1, \]

or in \(x\)-representation

\[ (-\partial^2 + m^2 - g\phi(x)) \Delta_+(x, y|\phi) = \delta^4(x - y). \]

Given these simple facts we can give an analytical expression for our graph Fig. \(\text{Fig. 1}\), namely,

\[ i\Gamma_N(x_1, \ldots, x_N) = (N - 1)! \frac{1}{2} (g\Delta_+(x_1, x_2))(g\Delta_+(x_2, x_3)) \ldots (g\Delta_+(x_N, x_1)). \]

The individual factors have the following origin:

1. The factor \((N - 1)!\) takes into account that, after fixing one of the \(\phi\)-lines, a total of \((N - 1)!\) topological inequivalent graphs can be created by permutation of the remaining \(\phi\)-lines.

2. One factor \(g\) is assigned to each vertex.

3. A free propagator \(\Delta_+\) is assigned to each \(\psi\)-line.

4. The factor \(\frac{1}{2}\) is indicative of a neutral scalar field theory.
\( \Gamma_N \) does not, by definition, contain external propagators. Substituting Eq. (7) into (2) we obtain \( \Gamma[\phi] \) in one-loop approximation

\[
i\Gamma[\phi] = \frac{1}{2} \sum_{N=1}^{\infty} \frac{(N-1)!}{N!} \int d^4x_1 \ldots d^4x_N \left( g \right)^N \Delta_+(x_1, x_2) \ldots \Delta_+(x_N, x_1) \phi(x_1) \ldots \phi(x_N)
\]

\[
= \frac{1}{2} \sum_{N=1}^{\infty} \frac{1}{N} \int d^4x_1 \ldots d^4x_N \left( g \phi(x_1) \Delta_+(x_1, x_2) \right) \ldots \left( g \phi(x_N) \Delta_+(x_N, x_1) \right)
\]

\[
= \frac{1}{2} \int d^4x_1 \sum_{N=1}^{\infty} \frac{1}{N} \int d^4x_2 \ldots d^4x_N \left( \phi \Delta_+ | x_2 \right) \ldots \left( \phi \Delta_+ | x_1 \right)
\]

\[
= \frac{1}{2} \int d^4x_1 \sum_{N=1}^{\infty} \frac{1}{N} \left( \phi \Delta_+ \right)^N | x_1 \rangle = \frac{1}{2} \int d^4x \left( \int \sum_{N=1}^{\infty} \frac{1}{N} \left( \phi \Delta_+ \right)^N \right) | x \rangle
\]

\[
= -\frac{1}{2} \int d^4x \langle x | \ln(1 - g \phi \Delta_+) | x \rangle = -\frac{1}{2} \text{Tr} \ln(1 - g \phi \Delta_+).
\]

Here we used the series \( \ln(1 - x) = -\sum_{n=1}^{\infty} \frac{x^n}{n}, x \in (-1, +1) \). We also made use of the completeness relation \( \int d^4 \langle y | y \rangle = 1 \) and wrote \( \langle x | g \phi \Delta_+ | y \rangle = g \phi(x) \langle x | \Delta_+ | y \rangle = g \phi(x) \Delta_+(x, y) \). So our final result reads

\[
i\Gamma[\phi] = \frac{1}{2} \text{Tr} \ln(1 - g \phi \Delta_+)^{-1}, \quad (8)
\]

or with the aid of (4):

\[
i \Gamma[\phi] = \frac{1}{2} \text{Tr} \ln \left[ \frac{\Delta_+[\phi]}{\Delta_+[0]} \right] \quad (9)
\]

and since

\[
\Delta_+[0] \equiv \Delta_+ = \frac{1}{p^2 + m^2 - i\epsilon} \quad \text{and} \quad \Delta_+[\phi] = \frac{1}{p^2 + m^2 - g \phi - i\epsilon},
\]

we have

\[
i \Gamma[\phi] = -\frac{1}{2} \text{Tr} \ln \frac{p^2 + m^2 - g \phi - i\epsilon}{p^2 + m^2 - i\epsilon} \cdot (10)
\]

Here we employ the formula

\[
\ln \frac{a}{b} = \int_{0}^{\infty} \frac{ds}{s} e^{-is(b-i\epsilon)} - \int_{0}^{\infty} \frac{ds}{s} e^{-is(a-i\epsilon)} \quad (11)
\]
and obtain
\[ i \Gamma[\phi] = \frac{1}{2} \int_0^\infty \frac{ds}{s} \text{Tr} e^{-is(p^2-g\phi+m^2-i\epsilon)} + \frac{1}{2} \int_0^\infty \frac{ds}{s} \text{Tr} e^{-is(p^2+m^2-i\epsilon)}. \tag{12} \]

Since the last term is \( \phi \)-independent it is usually dropped.

Now we turn to the computation of
\[ i \Gamma[\phi] = \frac{1}{2} \int_0^\infty \frac{ds}{s} \text{Tr} e^{-is(p^2-g\phi+m^2)} \]
\[ = \frac{1}{2} \int_0^\infty \frac{ds}{s} e^{-ism^2} \int d^4x \langle x|e^{-i(p^2-g\phi)s}|x \rangle. \tag{13} \]

Introducing the “Hamiltonian” \( H = p^2 - g\phi \), \( p_\mu = \frac{i}{2} \partial_\mu \), we need to calculate the trace of the quantum mechanical transition amplitude
\[ \langle x, s|y \rangle = \langle x|e^{-iHs}|y \rangle. \tag{14} \]

Instead of working with the Hamiltonian we now switch over to the Lagrangian description of our system so that we can make use of Feynman’s path integral representation of Eq. (14).

Since \( L = p\dot{x} - H = \dot{x} = \frac{\partial H}{\partial p} = 2p \); \( p = \frac{\dot{x}}{2} \), we find
\[ L = \frac{\dot{x}^2}{2} - \left( \frac{\dot{x}^2}{4} - g\phi \right) = \frac{\dot{x}^2}{4} + g\phi. \tag{15} \]

Now, Feynman’s path integral representation of the transition amplitude is given by
\[ \langle x, s|y \rangle = \mathcal{N} \int_{x(0)=y, x(s)=x} Dx(\tau) e^{iS_\text{cl}[x(\tau)]}, \tag{16} \]
where
\[ S_\text{cl}[x(\tau)] = \int_0^s d\tau L(x(\tau), \dot{x}(\tau)), \quad L = \frac{\dot{x}^2}{4} + g\phi. \tag{17} \]

This is all we need from Feynman’s book \(^7\) or any other monograph on Feynman path integrals in single-particle quantum mechanics\(^8\). Already at this stage we want to emphasize
that nowhere in the sequel do we have to compute a loop-momentum integral as is usually required in other field theoretic approaches.

The normalization factor $\mathcal{N}$ in (16) is determined from the free theory, $g = 0$, i.e., $H_0 = p^2$. In this case we have

$$\langle x | e^{-isp^2} | y \rangle = \mathcal{N} \int_{x(0)=y}^{{x(s)=x}} \mathcal{D}x \, e^{i \int_0^s \frac{dx^2}{4}}.$$  \hspace{1cm} (18)

Taking the trace on the left-hand side gives us ($(dx) \equiv d^4x$)

$$\int (dx) \langle x | e^{-isp^2} | x \rangle = \int (dx) \langle x | e^{-isp^2} \rangle \cdot 1 | x \rangle, \quad 1 = \int (dp) \langle p | p \rangle$$

$$= \int (dx) \int (dp) e^{-isp^2} \langle x | p \rangle \langle p | x \rangle, \quad \langle x | p \rangle = \frac{e^{ipx}}{(2\pi)^2} \cdot \int (dx) = V_4$$

$$= V_4 \int \frac{(dp)}{(2\pi)^4} e^{-isp^2} = V_4 \left( \frac{-i}{(4\pi)^2} \right) \frac{1}{s^2}. \hspace{1cm} (19)$$

Here the four-dimensional integral over the (3+1)-dimensional momentum space is computed as

$$\int \frac{(dp)}{(2\pi)^4} e^{-isp^2} = \left( \int_{-\infty}^{\infty} \frac{dp_1}{2\pi} e^{-isp_1} \right)^3 \left( \int_{-\infty}^{\infty} \frac{dp_0}{2\pi} e^{-isp_0} \right) = \left( \frac{1}{2\pi} \left( \frac{\pi}{is} \right)^{1/2} \right)^3 \frac{1}{2\pi} \left( \frac{\pi}{s} \right)^{1/2}$$

$$= \frac{1}{(4\pi)^2} \frac{1}{is^2}.$$  

The trace on the right-hand side of Eq. (18) is written as

$$\mathcal{N} \int (dx) \int \mathcal{D}x e^{i \int_0^s \frac{dx^2}{4}} \equiv \mathcal{N} \int \mathcal{D}x e^{i \int_0^s \frac{dx^2}{4}},$$

so that we end up with the useful relation

$$\mathcal{N} \int \mathcal{D}x e^{i \int_0^s \frac{dx^2}{4}} = V_4 \frac{1}{(4\pi)^2} \frac{1}{s^2}. \hspace{1cm} (20)$$

Here then is our path integral representation for the one-loop process with an arbitrary
number of external off-shell $\phi$-particle lines (s. Eq. (13)):

$$
\Gamma[\phi] = \frac{1}{2} \int_0^\infty ds \ N \int_{x(0)=x(s)} D_x e^{i \int_0^s d\tau \left[ \frac{\partial^2}{\partial t^2} + g\phi \right]}.
$$

(21)

Using a perturbative expansion of the right-hand side of Eq. (21) we can write

$$
\Gamma[\phi] = -\frac{i}{2} \int_0^\infty ds \ e^{-im^2 s} N \int_{x(0)=x(s)} D_x e^{i \int_0^s d\tau \left[ \frac{\partial^2}{\partial t^2} + \sum_{i=1}^N \frac{(ig)^N}{N!} \int_0^s d\tau_i \phi(x(\tau_i)) \right]}.
$$

(22)

with

$$
\Gamma_N[\phi] = -\frac{i}{2} \frac{(ig)^N}{N!} \int_0^\infty ds \ e^{-im^2 s} N \int_{x(0)=x(s)} D_x e^{i \int_0^s d\tau \left[ \frac{\partial^2}{\partial t^2} \prod_{i=1}^N + \sum_{i=1}^N \frac{(ig)^N}{N!} \int_0^s d\tau_i \phi(x(\tau_i)) \right]}.
$$

(23)

At this point we specialize the $\phi$-field to a sum of plane waves,

$$
\phi(x) = \sum_{i=1}^N e^{ik_i x}.
$$

(24)

For $N = 2$ the $\phi$-term on the right-hand side of Eq. (23) would read

$$
\int_0^s d\tau_1 d\tau_2 \phi(x(\tau_1)) \phi(x(\tau_2)) = \int_0^s d\tau_1 d\tau_2 \left( e^{ik_1 x(\tau_1)} + e^{ik_2 x(\tau_1)} \right) \left( e^{ik_1 x(\tau_2)} + e^{ik_2 x(\tau_2)} \right)
$$

$$
= \int_0^s d\tau_1 d\tau_2 \left( \ldots + e^{ik_1 x(\tau_1)} e^{ik_2 x(\tau_2)} + e^{ik_2 x(\tau_1)} e^{ik_1 x(\tau_2)} + \ldots \right)
$$

$$
= 2! \int_0^s d\tau_1 d\tau_2 e^{ik_1 x(\tau_1)} e^{ik_2 x(\tau_2)} + \ldots.
$$

(25)
Here we kept only mixed terms in $k_1$ and $k_2$, i.e., each $\phi$-mode occurs only once.

Generalizing to $N$ we would find $N!$ instead of $2!$ in Eq. (23). This factor $N!$ then cancels the $N!$ that stands in the denominator of Eq. (23). So far we have

$$\Gamma_N[k_1,k_2,\ldots,k_N] = (-i)^{1/2}(ig)^N \int_0^{\infty} \frac{ds}{s} N \int Dx \ e^{i \int_0^s d\tau \left[ \frac{1}{2} m^2 - \frac{x^4}{4} \right]} \left( \prod_{i=1}^N \int_0^s d\tau_i e^{ik_i x(\tau_i)} \right). \quad (26)$$

Introducing the “current” for $x$,

$$j(\tau) = i \sum_{j=1}^N k_j \delta(\tau - \tau_j), \quad (27)$$

we can rewrite Eq. (26) in the form

$$\Gamma_N[k_1,k_2,\ldots,k_N] = (-i)^{1/2}(ig)^N \int_0^{\infty} \frac{ds}{s} e^{-im^2s} \left( \prod_{i=1}^N \int_0^s d\tau_i \right) N \int Dx \ e^{i \int_0^s d\tau \left[ \frac{1}{2} m^2 - \frac{x^4}{4} \right]} \ e^{i \int_0^s d\tau j(\tau)x(\tau)} \ \left( \prod_{i=1}^N \int_0^s d\tau_i e^{ik_i x(\tau_i)} \right). \quad (28)$$

The operator $\frac{d^2}{d\tau^2}$, acting on $x(\tau)$ with periodical boundary condition $x(s) = x(0)$, has zero-modes $x_0$:

$$\frac{d^2}{d\tau^2} x(\tau) = \lambda_n x(\tau), \quad x(s) = x(0),$$

zero mode $x \equiv x_0 \equiv \text{const.}: \frac{d^2}{dx^2} x_0 = 0, \ \lambda_0 = 0$.

These zero modes will be separated from their orthogonal non-zero mode partners $\xi(\tau)$ by writing $x(\tau) = x_0 + \xi$ with $\int_0^s d\tau \xi(\tau) = 0$, i.e., $\int_0^s d\tau x(\tau) = x_0$ and

$$\int Dx = \int d^4 x_0 \ \int D\xi. \quad (29)$$

Since $x(\tau)$ (and therefore $\xi(\tau)$) is periodic we can write

$$x(\tau) = x_0 + \sum_{n \neq 0} c_n e^{2\pi in \tau}. \quad (30)$$
The zero-mode contribution in the source term of Eq. (28) yields
\[ \int_0^s j x d\tau = x_0 i \sum_{j=1}^N k_j \delta(\tau - \tau_j) = e^{ix_0 \sum_{j=1}^N k_j} \]
and therefore
\[ \int d^4 x_0 e^{ix_0 \sum_{j=1}^N k_j} = (2\pi)^4 \delta^4(k_1 + k_2 + \cdots + k_N). \tag{31} \]

The result of combining all this information with Eq. (28) produces the result
\[ \Gamma_N[k_1, \ldots, k_N] = -\frac{1}{2}(ig)^N(2\pi)^4 \delta \left( \sum_{j=1}^N k_j \right) \int_0^\infty \frac{ds}{(4\pi)^2 s^3} \left. e^{-im^2 s} \right| \prod_{i=1}^N \int_0^s d\tau_i \]
\[ \times \int \mathcal{D}\xi \left. e^{-\frac{i}{4} \int_0^s d\tau \frac{d^2 \xi}{d\tau^2} \xi} \right. \]
\[ \cdot \left. \int \mathcal{D}\xi \left. e^{-\frac{i}{4} \int_0^s d\tau \frac{d^2 \xi}{d\tau^2} \xi} \right. \right| \mathcal{D}\xi e^{i S[\xi]}, \tag{32} \]

Use has also been made of formula (20):
\[ \mathcal{N} \int d^4 x_0 \int_{\xi(s) = \xi(0)} \mathcal{D}\xi e^{\frac{i}{4} \int_0^s d\tau (\xi^2 - ij\xi)} = -\frac{i}{(4\pi)^2} \frac{1}{s^2} V, \]

where \( \dot{x}_0 = 0 \) and the four-volume \( \int d^4 x_0 = V \) is cancelled on both sides.

In Eq. (32) we meet the path integral
\[ \int_{\xi(0) = \xi(s)} \mathcal{D}\xi e^{\frac{i}{4} \int_0^s d\tau \left( \frac{\xi^2}{4} - ij\xi \right)} = \int_{\xi(0) = \xi(s)} \mathcal{D}\xi e^{i S[\xi]}, \]

where
\[ S[\xi] = \int_0^s d\tau L(\xi, \dot{\xi}), \quad \text{and} \quad L(\xi, \dot{\xi}) = \frac{\dot{\xi}^2}{4} - ij\xi. \tag{33} \]

Using Eq. (33) we find from \( \frac{d}{d\tau} \frac{\partial L}{\partial \dot{\xi}} - \frac{\partial L}{\partial \xi} = 0 \) the equation of motion,
\[ \frac{1}{2} \frac{d^2}{d\tau^2} \xi(\tau) = -ij(\tau), \tag{34} \]
which can be solved with the ansatz

$$\xi(\tau) = -i \int_0^s d\tau' G(\tau, \tau') j(\tau'),$$

where the Green’s function equation is given by (s. later)

$$\frac{1}{2} \frac{d^2}{d\tau^2} G(\tau, \tau') = \delta(\tau - \tau') - \frac{1}{s}. \quad (35)$$

Indeed we can write

$$\frac{1}{2} \frac{d^2}{d\tau^2} \xi(\tau) = -i \int_0^s d\tau' \left[ \delta(\tau - \tau') - \frac{1}{s} \right] j(\tau') = -ij(\tau) + \frac{i}{s} \int_0^s d\tau' j(\tau')$$

$$= -ij(\tau) + \frac{i}{s} \int_0^s d\tau' \sum_{j=1}^N k_j \delta(\tau' - \tau_j) = -ij(\tau) - \frac{1}{s} \sum_{j=1}^N k_j$$

$$= -ij(\tau)$$

following from the \(\delta\)-function in Eq. (32): \(\sum_{j=1}^N k_j = 0\).

We now must compute the path integrals occurring in Eq. (32) which are of Gaussian type. So in an intermediate step we just consider a single Gaussian integral:

$$\int dx \frac{e^{i(mx^2 + vx)}}{\sqrt{-i\pi}} = \int dx \frac{e^{im(x + \frac{v}{2m})^2} e^{-i\frac{v^2}{4m}}}{\sqrt{-i\pi}} = \int dx \frac{e^{imx^2} e^{-i\frac{v^2}{4m}}}{\sqrt{-i\pi} \sqrt{m}} = \int ds \frac{e^{-s^2}}{\sqrt{\pi}} \sqrt{\frac{1}{m}} = 1 \cdot \frac{e^{-\frac{v^2}{4m}}}{\sqrt{m}}, \quad x = \sqrt{-\frac{1}{m}}.$$

The result for a product of coupled Gaussian integrals is therefore the generalization

$$\int \frac{dx_1}{\sqrt{-i\pi}} \cdots \int \frac{dx_n}{\sqrt{-i\pi}} \frac{e^{i(\sum_{l,m,x_l M_{lm} x_m + \sum_l x_lv_l)}}}{\sqrt{\det M}} = \frac{e^{-i\sum_{l,m,v_l(M^{-1})_{lm} v_m}}}{\sqrt{\det M}}, \quad (36)$$

which can be proved by making a rotation on the \(x\)’s and \(v\)’s which diagonalizes \(M\) and so reduces to the one-variable case where now \(\sqrt{m}\) is replaced by \(\sqrt{\det M} = \sqrt{\prod_{j=1}^n M_j}\), where the \(M_j\) are the eigenvalues of \(M\). So when we first compute the discrete version of the path integrals in Eq. (32) and then go to the continuous limit, we evidently obtain
(substituting $M = -\frac{1}{4} d^2/d\tau^2$, $v = \frac{1}{4} j$ in Eq. (36))

$$
\int \mathcal{D}\xi e^{i \int_0^1 d\tau \xi \left( -\frac{1}{4} \frac{d^2}{d\tau^2} \right) \xi} \left( e^{i \sum_{i,j} v_i (\frac{d}{d\tau})_i v_j} \right) \frac{1}{\sqrt{\det M}} - \sqrt{\det M}
$$

$$
\int \mathcal{D}\xi e^{i \int_0^1 d\tau \xi \left( -\frac{1}{4} \frac{d^2}{d\tau^2} \right) \xi} \left( e^{i \sum_{i,j} v_i (\frac{d}{d\tau})_i v_j} \right) \frac{1}{\sqrt{\det M}} - \sqrt{\det M}
$$

where

$$
G_{\mu\nu}(\tau, \tau') = \eta_{\mu\nu} G(\tau, \tau'), \quad \text{and} \quad G(\tau, \tau') = \langle \tau | 2 \left( \frac{d^2}{d\tau^2} \right)^{-1} | \tau' \rangle.
$$

Now we can write Eq. (32) in the form

$$
\Gamma_N[k_1, \ldots, k_N] = -\frac{1}{2} (ig)^N (2\pi)^4 \delta (k_1 + \cdots + k_N)
$$

$$
\int_0^\infty \frac{ds}{(4\pi)^2 s^3} e^{-im^2 s} \left( \prod_{i=1}^N \int_0^s d\tau_i \right) e^{-\frac{1}{2} \int_0^s \int_0^{\tau'} d\tau'' j^\mu(\tau) G_{\mu\nu}(\tau, \tau'') j^\nu(\tau'')}
$$

Note the structure expressed in Eq. (38), where loop-particle, mass $m$, and off-shell $\phi$-particles are factorized in such a way that the scalar particle circulating in the loop becomes multiplied by the exponential term which is solely due to the $\phi$-particles tied to the loop.

With $j(\tau)$ given in Eq. (27) and

$$
\int_0^\infty \frac{ds}{(4\pi)^2 s^3} e^{-im^2 s} \left( \prod_{i=1}^N \int_0^s d\tau_i \right) e^{-\frac{1}{2} \int_0^s \int_0^{\tau'} d\tau'' j^\mu(\tau) G_{\mu\nu}(\tau, \tau'') j^\nu(\tau'')}
$$

we finally obtain

$$
\Gamma_N[k_1, \ldots, k_N] = -\frac{1}{2} (ig)^N (2\pi)^4 \delta (k_1 + \cdots + k_N)
$$

$$
\int_0^\infty \frac{ds}{(4\pi)^2 s^3} e^{-im^2 s} \left( \prod_{i=1}^N \int_0^s d\tau_i \right) e^{\frac{1}{2} \sum_{i,j} k_i k_j G(\tau_i, \tau_j)}
$$

As is shown below, $G(\tau, \tau) = 0 = \hat{G}(\tau, \tau)$, so that there are no terms with $k_i^2$ present, i.e., without the use of on-shell conditions.
Now we must devote a few lines to the Green’s function of the problem. First note that the spectrum and the eigenmodes of the operator $\frac{\partial}{\partial \tau}$ are given by

$$\text{Spectrum}(\frac{\partial}{\partial \tau}) = i \frac{2\pi}{s} n, \quad \langle \tau | f_n \rangle = f_n(\tau) = \frac{1}{\sqrt{s}} e^{i(\frac{2\pi}{s})n\tau}, \quad n \in \mathbb{Z}, \quad f_n(\tau + s) = f_n(\tau).$$

One can check orthogonality,

$$\langle f_n | f_m \rangle = \int_0^s d\tau f_n^*(\tau)f_m(\tau) = \delta_{nm},$$

and completeness,

$$\sum_{n=-\infty}^{\infty} f_n(\tau_2)f_n^*(\tau_1) = \frac{1}{s} \sum_{n=-\infty}^{\infty} e^{\frac{2\pi}{s}(\tau_2-\tau_1)n} = \sum_{m=-\infty}^{\infty} \delta(\tau_2 - \tau_1 - ms),$$

by Poisson’s formula. Note that for $0 \leq \tau_1, \tau_2 \leq s$ we obtain

$$\sum_{n=-\infty}^{\infty} f_n(\tau_2)f_n^*(\tau_1) = \delta(\tau_2 - \tau_1) \quad \text{in} \quad \int_0^s d\tau \ldots .$$

We are interested in the spectrum of $\partial^2_\tau$ which is given by $\text{Spectrum}(\partial^2_\tau) = \left(\frac{i\frac{2\pi}{s} n}{s} \right)^2 = -\frac{4\pi^2}{s^2} n^2$.

Earlier we defined the Green’s function

$$G(\tau_2, \tau_1) \equiv 2\langle \tau_2 | (\partial^2_\tau)^{-1} | \tau_1 \rangle,$$

which we write as

$$2\langle \tau_2 | \frac{1}{\partial^2_\tau} | \tau_1 \rangle = 2 \sum_{n=-\infty}^{\infty} \langle \tau_2 | f_n \rangle \langle f_n | \frac{1}{\partial^2_\tau} | f_m \rangle \langle f_m | \tau_1 \rangle$$

$$= 2 \sum_{n=-\infty}^{\infty} f_n(\tau_2) \frac{1}{-\frac{4\pi^2}{s^2} n^2} f_n^*(\tau_1) = 2s \sum_{n=-\infty, n \neq 0}^{\infty} \frac{e^{2\pi in\frac{\tau_2 - \tau_1}{s}}}{(2\pi in)^2}$$

$$= |\tau_2 - \tau_1| - \frac{(\tau_2 - \tau_1)^2}{s} - \frac{s}{6} = G(\tau_2, \tau_1).$$

It can be easily seen that the constant $-\frac{s}{6}$ drops out in scattering amplitude calculations, so that we can omit it at the beginning.
Now we are able to prove the Green’s function equation mentioned earlier in Eq. (35):

\[
\frac{1}{2} \partial_{\tau}^2 G(\tau, \tau') \overset{(40,41)}{=} \frac{1}{s} \sum_{n=-\infty}^{\infty} e^{2\pi i n (\tau - \tau')} = \frac{1}{s} \sum_n e^{2\pi i n (\tau - \tau')} - \frac{1}{s} = \sum_m \delta(\tau - \tau' - ms) - \frac{1}{s} \\
= \sum_{m=0}^{\infty} \delta(\tau - \tau') - \frac{1}{s} \quad \text{for} \quad 0 \leq \tau, \tau' \leq s.
\]

(\frac{1}{s} \text{ comes from the zero-mode, } n = 0, \text{ which is subtracted.}) With the above definition of the Green’s function Eq. (42), we can easily prove the following relations:

\[
\begin{align*}
G(\tau, \tau') &= G(\tau' - \tau) = G(\tau', \tau), \quad \text{symmetry} \\
\partial_{\tau} G(\tau, \tau') &\equiv \dot{G}(\tau, \tau') = \text{sign}(\tau - \tau') - \frac{2(\tau - \tau')}{s} \\
\dot{G}(\tau, \tau') &= -\dot{G}(\tau', \tau), \quad \cdot = \frac{d}{d\tau}.
\end{align*}
\]

Since we wanted to calculate the vacuum polarization diagram with two \(\phi\)-lines we now specialize to \(N = 2\) in Eq. (39):

\[
\begin{align*}
\Gamma_2[k_1, k_2] &= \frac{1}{2} g^2 (2\pi)^4 \delta^4(k_1 + k_2) \int_0^{\infty} \frac{ds}{(4\pi)^2 s^3} e^{-im^2s} \int_0^{s} d\tau_1 \int_0^{s} \int_0^{s} d\tau_2 e^{2i(k_1 \cdot k_2 G(\tau_1, \tau_2) + k_2 \cdot k_1 G(\tau_2, \tau_1))} \\
&= \frac{1}{2} g^2 (2\pi)^4 \delta^4(k_1 + k_2) \int_0^{\infty} \frac{ds}{(4\pi)^2 s^3} e^{-im^2s} \int_0^{s} d\tau_1 \int_0^{s} d\tau_2 e^{ik_1 \cdot k_2 G(\tau_1, \tau_2)},
\end{align*}
\]

where we used the symmetry of \(G\): \(G(\tau_1, \tau_2) = G(\tau_2, \tau_1)\). Since \(G(\tau_1, \tau_2)\) is periodic and is only dependent on the difference \((\tau_1 - \tau_2)\), the integration over \(\tau_2\) is trivial; after the \(\tau_1\)-integration there is no dependence on \(\tau_2\) left and hence the integration over \(\tau_2\) gives just \(s\). We can then choose \(\tau_2 = 0\):

\[
\Gamma_2[k_1, k_2] = \frac{1}{2} g^2 (2\pi)^4 \delta^4(k_1 + k_2) \int_0^{\infty} \frac{ds}{(4\pi)^2 s^3} e^{-im^2s} \int_0^{s} d\tau_1 e^{ik_1 \cdot k_2 G(\tau_1)}.
\]

Now we use

\[
G(\tau_1) \equiv G(\tau_1, 0) = -\frac{\tau_1^2}{s} + \tau_1 \\
v = \frac{2\tau_1}{s} - 1, \quad d\tau_1 = \frac{s}{2} dv, \quad \tau_1 = (v + 1) \frac{s}{2} \\
0 \leq \tau_1 \leq s, \quad -1 \leq v \leq 1, \quad G = \frac{s}{4}(1 - v^2)
\]
and obtain

\[ \Gamma_2[k_1, k_2] = \frac{1}{2} g^2 2(2\pi)^4 \delta^4(k_1 + k_2) \int_0^\infty \frac{ds}{2(4\pi)^2 s} e^{-im^2 s} \int_{-1}^1 dv e^{-i k^2 \frac{4}{m}(1-v^2)} \]

\[ = -(2\pi)^4 \delta^4(k_1 + k_2) \Pi(k^2), \tag{46} \]

with

\[ \Pi(k^2) = -\frac{g^2}{2} \int_0^\infty \frac{ds}{(4\pi)^2 s} e^{-im^2 s} \frac{1}{2} \int_{-1}^1 dv e^{-i k^2 \frac{4}{m}(1-v^2)}. \tag{47} \]

An integration by parts on the variable \( v \) then produces

\[ \Pi(k^2) = -\frac{g^2}{2(4\pi)^2} \int_0^\infty \frac{ds}{s} e^{-im^2 s} + \frac{g^2}{2(4\pi)^2} \frac{k^2}{2} \int_0^1 dv v^2 \left[ m^2 + \frac{k^2}{4}(1-v^2) \right]. \tag{48} \]

Now let us assume that the \( \phi \)-particle is massless; then we can regularize \( \Pi(k^2) \) at \( k^2 = 0 \) and construct \( \Pi^R(k^2) = \Pi(k^2) - \Pi(0) \), which is equivalent to writing Eq. (48) in the simple form (we drop \( R \) again):

\[ \Pi(k^2) = \frac{g^2}{2(4\pi)^2} \frac{k^2}{2} \int_0^1 dv v^2 \left[ m^2 + \frac{k^2}{4}(1-v^2) \right]. \tag{49} \]

The same procedure can be carried through in Eq. (47). There we would need

\[ \int_0^\infty \frac{ds}{s} \int_0^1 dv e^{-im^2 s} \left( e^{-is \frac{k^2}{m}(1-v^2)} - 1 \right) = \int_0^1 dv \int_0^\infty \frac{ds}{s} \left( e^{-is \frac{m^2+k^2}{4m^2}(1-v^2)} - e^{-im^2 s} \right) \]

\[ \overset{(11)}{=} -\int_0^1 dv \ln \frac{m^2+k^2}{m^2(1-v^2)} - \int_0^1 dv \ln \left( 1 + \frac{k^2}{4m^2(1-v^2)} \right) \]

and so obtain another version of \( \Pi(k^2) \), namely,

\[ \Pi(k^2) = \frac{g^2}{(4\pi)^2} \frac{1}{2} \int_0^1 dv \ln \left( 1 + \frac{k^2}{4m^2(1-v^2)} \right). \tag{50} \]
Substituting \( x = \frac{1+v}{2} \) into Eq. (50) we finally arrive at

\[
\Pi(k^2) = \frac{g^2}{2(4\pi)^2} \int_{-1}^{1} dv \ln \left( 1 + \frac{k^2}{4m^2} (1-v^2) \right) \\
= \frac{g^2}{2(4\pi)^2} \int_{0}^{1} dx \ln \left( 1 + \frac{k^2}{m^2} x(1-x) \right).
\]

(51)

It is also instructive to derive the spectral representation of our vacuum polarization diagram. To do this we start from Eq. (49) and substitute \( v = \left( 1 - \frac{4m^2}{M^2} \right)^{1/2} \). Our end result is then presented as

\[
\Pi(k^2) = k^2 \int_{(2m)^2}^{\infty} dM^2 \frac{\sigma(M^2)}{k^2 + M^2 - i\epsilon},
\]

(52)

where the so-called spectral measure is given by

\[
\sigma(M^2) = \frac{1}{2} \left( \frac{g}{4\pi} \right)^2 \left( 1 - \frac{4m^2}{M^2} \right)^{1/2} M^2.
\]

(53)

Finally we write for the modified massless \( \phi \)-particle propagator \( \tilde{\Delta}_+^\phi \):

\[
k^2 \left[ 1 - k^2 \int_{(2m)^2}^{\infty} dM^2 \frac{\sigma(M^2)}{k^2 + M^2 - i\epsilon} \right] \tilde{\Delta}_+^\phi(k) = 1
\]

(54)

or

\[
\tilde{\Delta}_+^\phi(k) = \frac{1}{k^2 - i\epsilon} \frac{1}{1 - k^2 \int_{(2m)^2}^{\infty} dM^2 \frac{\sigma(M^2)}{k^2 + M^2 - i\epsilon}}
\]

(55)

\[
= \frac{1}{k^2 - i\epsilon} + \int_{(2m)^2}^{\infty} dM^2 \frac{\sigma(M^2)}{k^2 + M^2 - i\epsilon},
\]

(56)

where we expanded the second factor of Eq. (53). We have hereby reproduced the
Lehmann-Källen spectral representation:

\[
\bar{\Delta}_+^\sigma(k) = \frac{1}{k^2 - i\epsilon} + \frac{1}{2} \left( \frac{g}{4\pi} \right)^2 \int_{(2m)^2}^\infty \frac{dM^2}{M^2} \left( 1 - \frac{4m^2}{M^2} \right)^{1/2} \frac{1}{k^2 + M^2 - i\epsilon}. \tag{57}
\]

It is interesting to compare expressions Eq. (53) and Eq. (57) with those occurring in scalar QED. There we would find

\[
\bar{\Delta}_+^{\gamma_{\mu\nu}} = \left( g_{\mu\nu} - \frac{k_{\mu}k_{\nu}}{k^2} \right) \bar{\Delta}_+^{\gamma}(k^2), \quad \sigma(M^2) = \frac{1}{3} \left( \frac{e^2}{4\pi} \right)^2 \frac{\left( 1 - \frac{4m^2}{M^2} \right)^{3/2}}{M^2}, \tag{58}
\]

\[
\bar{\Delta}_+^{\gamma}(k^2) = \frac{1}{k^2 - i\epsilon} + \frac{1}{3} \left( \frac{e^2}{4\pi} \right)^2 \int_{(2m)^2}^\infty \frac{dM^2}{m^2} \left( 1 - \frac{4m^2}{M^2} \right)^{3/2} \frac{1}{k^2 + M^2 - i\epsilon}. \tag{59}
\]

Incidentally, Eq. (51) corresponds to the scalar QED-case:

\[
\Pi(k^2) = \left( \frac{e^2}{4\pi} \right)^2 \int_0^1 dx \left( 2x - 1 \right)^2 \ln \left[ 1 + \frac{k^2}{m^2} x (1 - x) \right]. \tag{60}
\]

### 3 Conclusion

We have presented a one-loop calculation for a simplified model field theory which is based on standard quantum mechanical path integrals. We found that loop-momentum integrals can be avoided and be replaced with simple Feynman path integrals. All the results known from ordinary field theory can thus be obtained with much less labor. Since our calculations have great similarity with those occurring in QED, the reader should now be able to pursue his or her own studies in more realistic models.

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