Higher Rank Symmetry Defects: Defect Bound States and Hierarchical Proliferation

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Symmetry defects, e.g., vortices in conventional superfluids, play a critical role in a complete description of symmetry-breaking phases. In this Letter, we develop the theory of symmetry defects in spontaneously higher-rank symmetry (HRS) breaking phases. By Noether’s theorem, HRS is associated with the conservation law of higher moments, e.g. dipoles, quadrupoles, and angular moments. We establish finite-temperature phase diagrams by identifying a series of topological phase transitions via the renormalization group flow equations and Debye-Hückel approximation. Accordingly, a series of Kosterlitz-Thouless topological transitions are found to occur successively at different temperatures, which are triggered by proliferation of defects, defect bound states, and so on. Such a hierarchical proliferation brings rich phase structures. Meanwhile, a screening effect from sufficiently high density of defect bound states leads to instability and collapse of the intermediate temperature phases, which further enriches the phase diagrams. For concreteness, we consider an example in which “angular moments” are conserved. We then present the general theory, in which other types of HRS can be analyzed in a similar manner. Further directions are present at the end of the paper.

Introduction.— The concept of symmetry and its spontaneous breaking has influenced quantum physics, from phases and phase transitions to the Standard Model of fundamental forces. A symmetry breaking phase can be characterized by an order parameter that varies with respect to symmetry operation and fluctuates in the vicinity of mean-field configurations. Distinct from smooth fluctuations, such as phonons and rotons, a singularity can be induced by symmetry defects, which tends to disorder the phase and restore symmetry. In a conventional 2D superfluid where U(1) global symmetry related to particle number conservation gets spontaneously broken, for example, the symmetry defects “superfluid vortex” at sufficiently low temperatures are confined into dipole-like bound states such that the superfluid phase is sustained against thermal fluctuations. At a critical value $T_c$, vortices begin to be released from bound states, which eventually results in a vortex proliferation that destroys the superfluid phase and restores symmetry at temperatures higher than $T_c$. This is one of great achievements that lead to 2016 Nobel Prize in Physics, namely, celebrated Kosterlitz-Thouless (KT) physics [1–3], which provides one of most influential prototypes of topological phase transitions driven by symmetry defects.

As symmetry is an indispensable ingredient in the above KT physics, the goal of this Letter is to explore KT-like physics in many-body systems where higher-rank symmetry (HRS) gets spontaneously broken. Let us first give a brief introduction to HRS. Electromagnetism textbook tells us, by starting with the particle density $\rho$, we can express a series of multipole moments, such as $\rho x^a$, $\rho x^a x^b$, etc. While the conservation of charge, i.e., the integration of $\rho$ over the whole space, is a very familiar fact in, e.g., metals and insulators, it is also meaningful for considering systems with conserved higher moments. Furthermore, Noether’s theorem indicates that each of conserved quantities here must be associated with an underlying continuous symmetry, which is nothing but aforementioned HRS. In the literature, the condensed matter physics of such type of symmetry has been initiated and paid great attentions to in the field of fracton physics [4, 5] where diverse quantum phenomena have been discovered, including system-size dependent noise-immune ground state degeneracy, exotic quantum dynamics, see, e.g., Refs. [6–17]. As a side, while there are alternative terminologies in the literature, here we call these symmetries “higher rank symmetry” due to the resultant higher-rank tensor gauge field after gauging it.

By noting that there have been great triumphs in the conventional spontaneously symmetry-breaking phase, we are motivated to ask if there is novel many-body physics by instead considering HRS. For example, in the framework of HRS, what can we expect for the superfluid-like phase, off-diagonal long-range order, Mermin-Wagner-like theorem, symmetry defects, and KT-like physics? Along this line of thinking, Refs. [18–21] construct concrete minimal models for realizing unconventional ordered phases dubbed “fractonic superfluids”, in which higher-rank $U(1)$ symmetry gets spontaneously broken in various fashions. Many exotic phenomena beyond the conventional symmetry-breaking phases have been identified in various channels [18–24]. In this Letter, we will focus on HRS defects and HRS-generalization of KT physics.

In this Letter, we establish a general framework of finite-temperature phase diagrams and a series of KT topological phase transitions driven by proliferation of either HRS symmetry defects or defect bound states. We discover that, thermally activated HRS defects generally constitute a hierarchy of bound states, such as dipole with neutral charge [Terminology clarification: Hereafter, unless otherwise specified, charge means the vorticity/winding number of HRS defect, while dipole, quadrupole, etc. really are formed by defects rather than original condensed bosons.], quadrupole [Fig. 1(a)] with both neutral charge and dipole moment, and so on. They bear a series of self-energy divergence, e.g., $\ln L L^2 \cdots$ with $L$ being the system size. Upon increasing temperature, various bound states succes-
sively get dissolved, from which new bound states of defects are successively released and then proliferate. After a series of KT transitions, HRS defects are finally released from all bound states and become mobile at high temperatures.

We take the conservation of so-called angular moments as a concrete example [19, 20] and then go to general theory. Through renormalization-group (RG) analysis and Debye-Hückel approximation, we identify two characteristic KT transitions and three distinct phases as shown in Fig. 1(b).

In Phase-I, defects are confined into quadrupole bound states such that superfluidity is sustained. If further increasing temperature with $\rho_0^{-1}$ being in the domain (A, B), Phase-I is transited into Phase-II at $T_c$, which is the critical temperature of the first KT transition where quadrupole bound states get dissolved and dipole bound states get proliferated. In Phase-II, superfluidity is destroyed. Then, if temperature is further increased, dipole bound states form a plasma with high density which drastically renormalizes the bare interaction between defects.

As a result, the second KT transition occurs at $T_2$, which sends the system into Phase-III. As the defects are no more bound in the form of BS, defects are wildly mobile in Phase-III, which is similar to the episode in the movie Pirates of the Caribbean: the enraged goddess Calypso, once being released from human form, generates a massive maelstrom. On the other hand, if $\rho_0^{-1}$ is either too small or too large, Phase-I is directly transited to Phase-III in Fig. 1.

**Defect solutions.**— We start with the following minimal Hamiltonian that conserves angular moments [19, 20],

$$H = |\hat{\partial}_1 \hat{\Phi}_1|^2 + |\hat{\partial}_2 \hat{\Phi}_2|^2 + \hat{V}_0 + \hat{V}_1,$$

(1)

where $\hat{V}_0 = \sum_{\alpha=1}^2 -\mu \hat{\rho}_\alpha + \frac{\alpha}{2} \hat{\rho}^2_\alpha$ and $\hat{\rho}_\alpha = \hat{\Phi}_\alpha \hat{\Phi}_\alpha$. $\hat{\Phi}_{1,2}$ denote two components of complex bosonic fields. $\hat{V}_0$ includes a repulsive interaction and a chemical potential term. The term $\hat{V}_1 = \hat{K} |\hat{\Phi}_1 \hat{\partial}_2 \hat{\Phi}_2 + \hat{\Phi}_2 \hat{\partial}_1 \hat{\Phi}_1|^2$ represents a two-particle correlated hopping process. Apart from the particle number conservation for each component, the model has an extra $U(1)$ to conserve the total angular moments that are a vectorial generalization of dipoles:

$$Q_{12} = \int d^2r (\hat{\rho}_1 y - \hat{\rho}_2 x),$$

(2)

where $r = (x, y)$. Indeed, one may check that

$$[Q_{12}, H] = 0$$

(3)

upon applying canonical quantization and regarding $Q_{12}$ as a generator of HRS. Thus, a group element of HRS can be expressed as $e^{i\theta Q_{12}}$. Here we discard symmetry-respecting terms with more complicated momentum dependence as all these terms are irrelevant at low energies. Once $\hat{V}_0$ forms a Mexican hat, we obtain a superfluid phase with off-diagonal long-range order and non-vanishing order parameters $\langle \hat{\Phi}_1 \rangle = \langle \hat{\Phi}_2 \rangle \neq 0$, whose low-energy effective Hamiltonian reads

$$\hat{H}[\hat{\theta}_1, \hat{\theta}_2] = \frac{1}{2} \rho_0 \left[ (\hat{\partial}_1 \hat{\theta}_1)^2 + (\hat{\partial}_2 \hat{\theta}_2)^2 + (\hat{\partial}_1 \hat{\theta}_2 + \hat{\partial}_2 \hat{\theta}_1)^2 \right],$$

where $\hat{\theta}_{1,2}$ are Goldstone modes. Without loss of generality, we set $K \rho_0 = 1$ with $\rho_0$ being the superfluid stiffness, which simplifies the formulas. Symmetry defects denoted as $\Theta$ can arise due to the compactness of $\theta_{1,2}$, which have the following general solutions:

$$\Theta \equiv (\theta_1, \theta_2) = \ell \left( \frac{y}{a} \varphi(r) - \frac{x}{a} \ln \frac{r}{a} - \frac{x}{a} \varphi(r) - \frac{y}{a} \ln \frac{r}{a} \right).$$

(4)

The general rules of constructing symmetry defects have been established in Ref. [19]. Here $a$ is the size of the defect core and a multi-valued function $\varphi(r)$ is the angle of $r$ relative to the core. The defect charge $\ell$ is quantized as a momentum that is well-understood on a lattice.

**Phase-I: defects are confined in quadrupoles.**— We consider $N$ defects $\{\Theta_i\}$ carrying winding number (i.e., “charge”) $\ell_i$ with the core coordinates $r_i$ ($i = 1, \ldots, N$), and then the total energy can be expressed in the momentum space:

$$E'[\{\Theta_i\}] = \frac{\rho_0}{2} \sum_{i,j=1}^N \ell_i \ell_j \int d^2q \mathcal{Z}_q e^{i q(r_i - r_j)}$$

(5)

where $\mathcal{Z}_q = \frac{2}{q^2}$ and the momentum $q = |q|$. In this expression, a series of severely infrared divergent terms can be identified. By observing

$$\int d^2q \mathcal{Z}_q e^{i q(r_i - r_j)} = 2\pi \left( \frac{L}{a} \right)^2 - \pi \frac{|r_i - r_j|^2}{a^2} \ln \frac{L}{a} + f,$$

(6)
we can single out two types of divergence, i.e., $L^2$ and $\ln L$, while the letter $f$ incorporates all finite terms. In the low temperature region, we have to impose the constraints on the defect configuration for energetic consideration. Suppression of the $L^2$ divergence requires the net charges vanish, i.e.,

$$\sum_{i=1}^{N} \ell_i = 0. \tag{7}$$

Under the charge neutral condition, suppression of $\ln L$ divergence further requires the net dipole moments vanish, i.e.,

$$\sum_{i=1}^{N} \ell_i \mathbf{r}_i = 0. \tag{8}$$

Once both conditions are satisfied, all divergence in Eq. (6) disappears, yielding the screened interacting energy:

$$T[\Theta_i, \Theta_j] = \pi \rho_0 \sum_{i,j} \ell_i \ell_j \frac{|\mathbf{r}_i - \mathbf{r}_j|^2}{a^2} \ln \frac{|\mathbf{r}_i - \mathbf{r}_j|}{a}. \tag{9}$$

Despite being canceled out, the divergence in Eq. (6) physically allows us to introduce self-energy formulas for three distinct objects, namely, a single defect, a dipole bound state (BS), and a quadrupole BS:

1. **A single defect.** The self-energy ($E_{\text{self}}^\delta$) of a single defect is polynomially divergent ($\sim L^2$) as $L \to \infty$:

$$E_{\text{self}}^\delta = \pi \rho_0 L^2 \frac{a^2}{L} \to \infty \tag{10}$$

which is apparently different from the logarithmic divergent self-energy of vortices (i.e., defects) in the conventional KT physics.

2. **A dipole bound state.** The self-energy ($E_{\text{self}}^d$) of a dipole BS formed by two defects with opposite charges $\pm \ell$ is logarithmically divergent ($\sim \ln L$) as $L \to \infty$:

$$E_{\text{self}}^d = \pi \rho_0 \frac{p^2}{a} \ln \frac{L}{a} \to \infty \tag{11}$$

with $p = |\mathbf{p}|$ and $\mathbf{p} = \ell \xi \equiv \ell(\mathbf{r}_1 - \mathbf{r}_2)$ being the dipole moment. Thus, from the expression of self-energy, a dipole BS here resembles a vortex in the conventional KT physics.

3. **A quadrupole bound state.** A quadrupole BS is formed by four defects with vanishing charges and vanishing dipole moments, as shown in Fig. 1(a). Its self-energy ($E_{\text{self}}^q$) is given by ($\Delta_{\pm} = |\xi \pm \zeta|$, $\xi = |\xi|$, $\zeta = |\zeta|$):

$$E_{\text{self}}^q = \pi \rho_0 \sum_{i=\pm} \frac{\Delta_i^2}{a^2} \ln \frac{\Delta_i}{a} - 2 \sum_{\tau=\xi,\zeta} \frac{\tau^2}{a} \ln \frac{\tau}{a} < \infty. \tag{12}$$

The finiteness of this self-energy is expected since the definition of quadrupole BSs exactly meets the aforementioned two conditions.

Bearing the above analysis in mind, we conclude that, at the thermodynamic limit, HRS defects are energetically confined into quadrupole BSs at low temperatures (Phase-I). Phase-I shows the algebraic long-range order since the correlation function $\langle \Phi(\mathbf{r}_1)\Phi(\mathbf{r}_2) \rangle$ behaves as a power-law function at long distances due to the gapless phonon mode excitations with a stiffness renormalized by defects.

**Phase-II: defects are confined in dipoles.**— When the temperature is increased to some critical value $T_{c1}$, dipole BSs are completely released from quadrupole BSs and proliferate wildly, which leads to a topological phase transition (named as “the first KT transition”) from Phase-I to Phase-II.

To analytically understand the phase transition, we introduce the dipole BS fugacity $y_d \equiv \exp(-\beta E_d)$ with $E_d$ being the dipole BS core energy and $\beta$ being the inverse temperature. In general, by definition, a dipole BS is a stable BS object when $E_d < 2\varepsilon$, with $\varepsilon$ being the core energy of a single defect $\Theta$. The thermally activated dipole BSs can weaken the superfluid stiffness $\rho$, which can be justified via the perturbative RG analysis. It should be noted that, in the RG calculation of the first KT transition, it is enough to keep track of the renormalization effect of released dipole BSs on superfluidity since an isolated defect $\Theta$ suffers from much severe energy divergence than a dipole BS. The $\beta$-equations for the superfluid stiffness $\rho$ and the fugacity $y_d$ are given by [see SM]:

$$\frac{dy_d^{-1}}{dl} = 2\beta y_d^2 + O(y_d^4), \tag{13}$$

$$\frac{dy_d}{dl} = (2 - \pi \beta \rho) y_d + O(y_d^2), \tag{14}$$

where $l$ is the energy scale. As the right-hand side of Eq. (13) is positive-definite, we can find the critical temperature

$$T_{c1} = \frac{\pi \rho_0}{2}. \tag{15}$$

at the sign change point of the factor “$(2 - \pi \beta \rho)$” in Eq. (14). Alternatively, $T_{c1}$ can be simply obtained by a free energy argument in which sign change of the free-energy of a test object (i.e., an externally inserted dipole BS) occurs at $T_{c1}$. The derivation in the argument is almost the same as that in the conventional KT physics except that the test object here is not a vortex (defect) but a dipole BS.

In Phase-II, the superfluidity is fully destroyed, and thus we do not need to consider the gapless phonon excitations. Instead, the system consists of mobile dipole BSs, while a few defects $\Theta$ can be thermally activated. Since a dipole BS is by definition formed by two $\Theta$’s with opposite charges, we can deduce the Hamiltonian of the interacting dipole BSs directly from Eq. (9),

$$H_D = \int d^2 \mathbf{q} \sum_{ij} \left( \frac{\rho_0 q_i q_j}{q^4} + \delta_{ij} \varepsilon_d \right) D_i(\mathbf{q})D_j(-\mathbf{q}), \tag{16}$$

where $D(\mathbf{q}) \equiv (D_1(\mathbf{q}), D_2(\mathbf{q}))$ is the dipole BS density field [See SM]. Physically, the first term in Eq. (16) accounts for the self-energy divergence shown in Eq. (11) while the second
term is the core energy. When temperature increases, dipole BSs reach high density, which will substantially renormalize the interaction (9) between two bare defects $\Theta$. Below we will discuss this renormalization.

**Phase-III: defects are deconfined.**—When the temperature is close to the topological phase transition (named as “the second KT transition”) from Phase-II to Phase-III, dipole BSs form a plasma due to sufficiently high density and equivalently we have a condensate of dipole BSs. Therefore, the interaction between two defects $\Theta$, i.e., $I$ in Eq. (9) gets strongly renormalized. This dynamical renormalization can be quantitatively treated via the standard Debye–Hückel approximation by approximating the dipole BSs as a continuous independent field. At low energies and small $q$, we keep the leading orders, yielding an effective Hamiltonian for defects [see SM]:

$$H_{\text{eff}} = \int d^2 q \left[ \frac{E_d}{q^2} + (E_s - \rho_0 \lambda_d^4) s(q) s(-q) \right],$$  \hspace{1cm} (17)

where $s(q)$ is the Fourier transformation of a defect density $s(r) \equiv \sum_i \delta(r-r_i)$ and $\lambda_d = \sqrt{E_d/\rho_0}$ denotes the Debye–Hückel screening length. Physically, the dipole BS plasma brings two significant effects: 1) to renormalize the interaction from Eq. (9) to logarithmic divergence, we have the bound state that can be thermally activated from the ground states with conservation of the first $m$-th moments

$$Q_{\alpha_1\alpha_2\cdots\alpha_n} \equiv \sum_i \ell_i r_{i\alpha_1} r_{i\alpha_2} \cdots r_{i\alpha_n} = 0, \quad n \leq m.$$  \hspace{1cm} (21)

where $Q_{\alpha}$ describes defect interaction and generally we assume $Q_{\alpha} \propto 1/q^{2n}$ ($n \in \mathbb{Z}$) at low energies, i.e., the IR limit with small momentum $q$, which leads to energy divergence of a single defect. We can expose the divergent features:

$$\int d^2 q \mathcal{U}_q e^{iq(r_i-r_j)} \propto \int d^2 q \frac{P_{2n-2}}{q^{2n}} + \text{finite}$$  \hspace{1cm} (20)

with $P_n(r) = \sum_{k=0}^{n} \frac{(iqr)^k}{k!}$ and finite denoting the remaining non-divergent terms. By inspecting the power-law and logarithmic divergence, we have the bound state that can be thermally activated from the ground states with conservation of the first $m$-th moments.

We fix our attention on defects carrying topological charges $\pm 1$, and at low energies, the minimal bound state with the first $2n$ moments vanishing, contain $2^n$ defects, with half $\ell = +1$ and half $\ell = -1$ and the spatial alignment of these vortices is strongly constrained.

Accordingly, we can establish a series of KT topological transitions that successively occur at different temperatures due to proliferation of defects and defect BSs. The effect of hierarchical proliferation can be theoretically characterized by means of RG flow equations and the Debye–Hückel approximation. More interestingly, a screening effect from sufficiently high density of defect bound states can lead to instability and collapse of the intermediate temperature phases, which further enriches the phase diagrams. In particular, the first KT point is determined by the stiffness, while the other ones are related to the bound state core energies.

**Discussions.**—In this Letter, we have investigated the KT-like physics when the conventional $U(1)$ symmetry is replaced by higher rank symmetry and the conventional superfluid vortex is replaced by higher symmetry defect. The model in Eq. (1) can be promisingly realizability in the cold atomic gas subjected to an optical lattice by tuning a two-particle state [25]. Eq. (18) indicates that $T_{c2}$ is determined by $E_d$, i.e., the core energy of dipole-like bound states of defects. Naively, a core energy manifests as the increase in free energy due to destruction of the superfluidity, i.e., $E_d \sim \alpha^2 f_{\text{cond}}$ with $f_{\text{cond}}$ being the superfluid energy density. Along with the Josephson scaling relation $\rho(T) \sim T_{c2} \xi$ and $f_{\text{cond}} \sim T_{c2} \xi^{-2}$, we can estimate the core energy $E_d \sim \pi \rho_0/2$ at the optimal value of $a$ that minimizes the total energy $\delta_{\text{self}}^e + E_d$ of a single
dipole BS with a unit charge. Simultaneously, a numerical simulation on a proper lattice regularized model is desired to benchmark the phase diagram in Fig. 1 by incorporating the defect core physics. One may also decorate these defects [26] and then defect condensation [27–30] may generate new classes of Symmetry Protected Topological (SPT) phases beyond group cohomology [31]. The non-equilibrium properties are also of fundamental importance, since the high rank symmetry enforces dynamics with constraint kinetics. Example is the Kibble-Zurek mechanism of the forming of high-rank defects that is triggered through a continuous phase transition at finite rate. Recently, HRS has attracted much attention from fields other than condensed matter physics, see, e.g., Refs. [24, 32–35]. It is interesting to study effect of defects.

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SUPPLEMENTAL MATERIALS

OUTLINE

This supplemental material (SM) is devoted to work out the technical details for the main text of renormalization procedure for the Kosterlitz-Thouless (KT) transitions and the screened interaction of the higher rank symmetry defects. In SM-1, we unify the defect Hamiltonian by introducing defect densities. In SM-2, we give the details on derivations of KT transition. In SM-3, we consider the high temperature plasma phase in Phase-III.

SM-1: DEFECT DENSITY AND UNIFIED HAMILTONIAN

The defects $\Theta$ are the building blocks for dipole and quadrupole bound states. Hence, we can unify the description of defect interaction. Given a defect system, in general, it contains isolated defects $\Theta$, dipole bound states and the quadrupole bound states. Here we emphasize the isolated defects $\Theta$ in order to distinguish ones that are confined in bound states. We introduce a density field $s(r)$ of total defects $\Theta$,

$$ s(r) = \sum_i \ell_i \delta(r - r_i), \quad (22) $$

where $\ell_i$ is the defect charge and $r_i$ is the position of the defect core.

A dipole BS is formed by two neighboring defects $\Theta$ with opposite charges $\pm \ell$ that are separated by $\xi$,

$$ \ell \Theta(r) - \ell \Theta(r - \xi) = \ell \xi \cdot \nabla \Theta(r). \quad (23) $$

Here we use the notation $\Theta$ to denote a defect with charge $\ell = 1$. We accordingly introduce the density of dipole bound states

$$ D(r) = \sum_i \ell_i \frac{(\xi_{ix} \xi_{iy})}{a^2} \delta(r - r_i) \equiv \sum_i I_i \delta(r - r_i), \quad (24) $$

while in a lattice, $\xi_{ix}/a$ and $\xi_{iy}/a$ take integer values. From the relation in Eq. (23), we reach a condition

$$ s(r) = -a \nabla \cdot D(r), \quad (25) $$

where $s(r)$ represents the density of confined defects $\Theta$ in a dipole bound state.

Similarly, a quadrupole bound state consists of four defects $\Theta$,

$$ \ell \Theta(r) - \ell \Theta(r - \xi) - \ell \Theta(r - \zeta) + \ell \Theta(r - \xi - \zeta) = \sum_{ij} \ell_i \ell_j \xi_{ij} \partial_i \partial_j \Theta(r), \quad (26) $$

where $\partial_i \partial_j \Theta(r)$ represents the configuration of a quadrupole bound state. Thus, we can introduce a quadrupole bound state density,

$$ Q_{ab}(r) = \sum_i \ell_i \frac{\xi_{ia} \xi_{ib}}{a^2} \delta(r - r_i), \quad (27) $$

which can be related to the density of defect $\Theta$ via

$$ s(r) = a^2 \sum_{ij} \partial_i \partial_j Q_{ij}(r). \quad (28) $$

In summary, for a defect system, if we apply the notations $\bar{s}(r)$, $D(r)$ and $Q$ respectively for isolated defect density, density of bound states and density of quadrupole bound states, along with $s(r)$ as the density of all defects $\Theta$, we have the condition,

$$ s(r) = \bar{s}(r) - a \nabla \cdot D(r) + a^2 \sum_{ij} \partial_i \partial_j Q_{ij}(r). \quad (29) $$
From the concrete configuration of defects $\Theta$, we can evaluate the velocity field $v_{ij}$

$$v_{ij}(r) = -\sum_i \delta_{ij} \left( -1 - \ln \frac{|r - r'|}{a} \right) = -\int d^2 r' \delta_{ij} \left( -1 - \ln \frac{|r - r'|}{a} \right) s(r')$$

$$= \int d^2 q \frac{s(q)}{q^2} e^{iqr},$$

(30)

which simply is indicative of the relation,

$$v_{ij}(q) = \delta_{ij} \frac{s(q)}{q^2}.$$  

(31)

Consequently, the defect system can be characterized by the Hamiltonian

$$H = \frac{\rho_0}{2} \int d^2 q \frac{2}{q^2} s(q) s(-q).$$

(32)

where $s(q)$ represents a density of total defects $\Theta$. From Eq. (32), one can recover the interaction formula in the main text. For example, when $s(r) = -a \nabla \cdot \nabla D(r)$, we obtain the Hamiltonian for interacting dipole bound states.

SM-2: RG EQUATION AND KT TRANSITIONS

2.1: Superfluid stiffness

In 2d conventional superfluids $^{25}F^2$, the topological defects will destroy the superfluidity at high temperatures. The superfluid stiffness will jump to zero above critical temperature. The same thing happens in superfluid phases of higher-rank symmetry. In this section, we will introduce a general theory on the renormalization on the superfluid stiffness by higher-rank defects.

We start with the model introduced in the main text with conserved angular moment. The low-energy effective Hamiltonian for the superfluid phase reads can be formulated in the form as:

$$H = \frac{\rho_0}{2} \int d^2 r [ (\partial_1 \theta_1)^2 + (\partial_2 \theta_2)^2 + (\partial_1 \theta_2 + \partial_2 \theta_1)^2],$$

(33)

where $\theta_{1,2}$ are gapless Golstone modes. The bare superfluid stiffness $\rho_0$ is a hallmark of the superfluid phase, which can be reduced by thermal defects. By introducing the velocity fields $v_{ij}(i, j = 1, 2)$,

$$v_{ij} = \delta_{ij} \partial_i \theta_i + (1 - \delta_{ij}) (\partial_i \theta_j + \partial_j \theta_i),$$

(34)

the Hamiltonian in Eq. (33) takes a more compacted form

$$H = \frac{\rho_0}{2} \int d^2 r \sum_{ij} v_{ij}^2.$$

(35)

In general, due to activated thermal topological defects, we can decompose the fields $\theta_i$ into a smooth part $\hat{\theta}_i$ and singular part $\theta^s_i$, i.e.

$$\theta_i = \hat{\theta}_i + \theta^s_i.$$  

(36)

And the velocity field follows a similar decomposition, the smooth part $\hat{v}_{ij}$ describing the superfluidity and the singularity part $v_{\text{s},ij}$ describing topological defects,

$$v_{ij} = \hat{v}_{ij} + v_{\text{s},ij},$$

(37)

where $\hat{v}_{ij} = \delta_{ij} \partial_i \hat{\theta}_i + (1 - \delta_{ij}) (\partial_i \hat{\theta}_j + \partial_j \hat{\theta}_i)$ and $v_{\text{s},ij} = \delta_{ij} \partial_i \theta^s_i + (1 - \delta_{ij}) (\partial_i \theta^s_j + \partial_j \theta^s_i)$. Therefore, we have a free energy by means of the velocity fields

$$F(\hat{v}_{ij}) = -T \ln \int Dv_{\text{s},ij} \exp (-\beta H),$$

(38)
where
\[ H = \frac{\rho_0}{2} \int d^2r \sum_{ij} \left( \tilde{v}_{ij}^2 + 2\tilde{v}_{ij}v_{s,ij} + v_{s,ij}^2 \right). \] (39)

The renormalization effect can be measured by the change of a free energy in response to a small constant velocity shift to the smooth component \( \tilde{v}_{ij} \rightarrow \tilde{v}_{ij} + u_{ij} \),
\[ F(\tilde{v}_{ij} + u_{ij}) - F(\tilde{v}_{ij}) = \frac{1}{2} \Omega \rho_R \sum_{ij} u_{ij}^2 + O(u_{ij}^3), \] (40)
where \( \rho_R \) is the renormalized superfluid stiffness
\[ \rho_R = \frac{1}{\Omega} \frac{\delta^2 F(\tilde{v}_{ij} + u_{ij})}{\delta u_{ij} \delta u_{ij}} \bigg|_{u=0}, \] (41)
and \( \Omega \) is the system area. In the dilute limit, after averaging over defect configuration, we can extract \( \rho_R \),
\[ \rho_R = \rho_0 - \beta \rho_0^2 \int d^2r \sum_{ij} \langle v_{s,ij}(r)v_{s,ij}(0) \rangle \]
\[ = \rho_0 - \beta \rho_0^2 \lim_{q \to 0} \frac{2}{q^4} \langle s(q)s(-q) \rangle, \] (42)
where \( \langle \cdot \rangle \) averages over all defect configurations and a relation \( v_{s,ij}(q) = \delta_{ij}s(q) \) is adopted. The formula in Eq. (42) is our starting point to derive the RG equation of KT transition. The remaining task is to compute the correlator \( \langle s(q)s(-q) \rangle \).

2.2: First transition \( T_{c1} \): from Phase-I to Phase-II

Topological defects \( \Theta \) are confined into quadrupoles in the low temperature. As the temperature increases, the quadrupole bound states begin to disassociate into dipole bounds states at the first transition and then dipole bound states fully split into freely movable defects \( \Theta \) at the second transition. Physically, we need to introduce the core energy \( E_c \) for a defect \( \Theta \) while \( E_d \) is the core energy for a dipole bound state. In general, we have to require \( E_d < 2E_c \) in order for the terminology bound state to matter.

Around the first transition, dipole bound states can be thermally activated to reduce the superfluid stiffness. Thus, we need to evaluate the dipole bound state contribution to the correlators. In the grand ensemble \( \Xi = \sum_{D(r)} e^{-\beta H} \), the Hamiltonian \( H \) can approximately only involve dipole bound states,
\[ H = \frac{1}{2} \rho_0 \int_{|r-r'|>a} d^2r d^2r' D(r) \cdot D(r')(2\pi \log \frac{|r-r'|}{a} - 2\pi \frac{r_i r'_j}{r^2} + \int d^2q E_a D(q) \cdot D(-q). \] (43)
As discussed in the main text, the system should meet the dipole charge neutral condition \( \sum_i I_i = 0 \). In the low energy limit, since the core energy is large as compared with the temperature, and the dipole bound states are dilute, we can calculate physical quantity in a power series of the fugacity \( y_d = e^{-\beta E_d} \). With the dipole charge neutral condition, we can explicitly conduct the calculation,
\[ \lim_{q \to 0} \frac{\langle s(q)s(-q) \rangle}{q^4} = \frac{1}{4\Omega} \int_{|r-r'|>a} d^2r d^2r' \langle D(r') \cdot D(r') \rangle e^{-iq \cdot (r-r')} \]
\[ = -\frac{1}{4\Omega} \int_{|r-r'|>a} d^2r d^2r' \langle D(r') \cdot D(r') \rangle (r - r')^2 \]
\[ = -\frac{2\pi}{4} \sum_{i \neq j} \frac{dr_{ij}}{a^2} (I_i \cdot 1 J_{ij} I_j^3), \] (44)
where \( r_{ij} = r_i - r_j \). In the dilute limit, we can only consider a pair of dipole bound states. Regarding the energy in Eq. (43), there are 5 cases in the leading order, \( \mathbf{I}_i = (1,0), \mathbf{I}_j = (-1,0) \) or \( \mathbf{I}_i = (-1,0), \mathbf{I}_j = (1,0) \) or \( \mathbf{I}_i = (0,1), \mathbf{I}_j = (0,-1) \) or \( \mathbf{I}_i = (0,-1), \mathbf{I}_j = (0,1) \) or \( \mathbf{I}_i = \mathbf{I}_j = (0,0) \). With a fixed separation \( r_{ij} \), the first four cases have equal contributions to grand ensemble. We hence obtain

\[
\langle \mathbf{l}_i \cdot \mathbf{l}_j r_{ij}^3 \rangle = -4y_d^2 \left( \frac{r_{ij}}{a} \right)^3 2\pi \beta \rho_0 ,
\]

which yields,

\[
\lim_{q \to 0} \frac{s(q)s(-q)}{q^4} = -8\pi^2 y_d^4 \int_{r > a} \frac{dr}{r^2} (\frac{r}{a})^3 2\pi \beta \rho_0 + O(y_d^4).
\]

We are ready to send Eq. (46) back to Eq. (42). After keeping the lowest order in \( \rho_R^{-1} \), we have

\[
\rho_R^{-1} = \rho_0^{-1} - \frac{8\pi y_d^2}{4} \int_{r > a} \frac{dr}{r^2} (\frac{r}{a})^3 2\pi \beta \rho_0 + O(y_d^4).
\]

The RG equation arises when we enforce the standard regularization scheme by dividing the integral into two parts,

\[
\int_{r > a} = \int_{a} + \int_{a}^{\infty},
\]

which leads to two equations,

\[
\rho_R^{-1} = \rho_0^{-1} (\delta l) + \frac{8\pi y_d^2}{4} \int_{a}^{\infty} \frac{dr}{a^2} (\frac{r}{a})^3 2\pi \beta \rho_0 + O(y^4)
\]

and

\[
\rho_0^{-1} (\delta l) = \rho_0^{-1} + \frac{8\pi y_d^2}{4} \int_{a} \frac{dr}{a^2} (\frac{r}{a})^3 2\pi \beta \rho_0 + O(y^4),
\]

where \( y(\delta l) = ye^{(2-\beta \rho_0)\delta l} \). Therefore, we reach the RG equations for the first transition

\[
\frac{d\rho}{dl} = \frac{8\pi}{4} \beta y^2 + O(y^4),
\]

\[
\frac{dy}{dl} = \gamma(2 - \pi \beta \rho) + O(y^2).
\]

Obviously, the critical temperature is given by

\[
T_c = \frac{\pi}{2} \rho_0.
\]

Above \( T_c \), the superfluid stiffness gets renormalized into zero, indicating that the superfluid phase is broken down. We then goes to a phase with the dipole bound states as freely mobile charge carriers.

Towards the RG equations, we have introduced a lattice regularization on which the dipole charges can only take integer values. Also, one can repeat the above procedures in a continuum space, in which \( \mathbf{l}_i \cdot \mathbf{l}_j = l_i l_j \xi_i \cdot \xi_j \) includes a part from the relative angular \( \xi_i \cdot \xi_j = \xi_i \xi_j \cos(\theta_{ij}) \). It will alter the coefficient before \( y^2 \) in Eq. (51), but the physics will not be changed.

### 2.3: Second transition \( T_{c2} \): from Phase-II to Phase-III

Around the second transition point, the density of quadrupole bound states vanishes, \( Q_{ij} = 0 \). The dipole bound states have sufficiently high density to form a plasma, which allows us to apply the Debye-Hückel approximation. The Debye-Hückel approximation takes the density \( D(r) \) as a continuous function. At the stage, the Hamiltonian reads

\[
H = \frac{\rho_0}{2} \int d^2 q \frac{2}{q^4} s(q)s(-q) + \int d^2 r \left[ E_s s^2(r) + E_q D^2(r) \right],
\]

\[
\lim_{q \to 0} \frac{s(q)s(-q)}{q^4} = 8\pi^2 y_d^4 \int_{r > a} \frac{dr}{r^2} (\frac{r}{a})^3 2\pi \beta \rho_0 + O(y_d^4).
\]

We are ready to send Eq. (46) back to Eq. (42). After keeping the lowest order in \( \rho_R^{-1} \), we have

\[
\rho_R^{-1} = \rho_0^{-1} - \frac{8\pi y_d^2}{4} \int_{r > a} \frac{dr}{r^2} (\frac{r}{a})^3 2\pi \beta \rho_0 + O(y_d^4).
\]

The RG equation arises when we enforce the standard regularization scheme by dividing the integral into two parts,

\[
\int_{r > a} = \int_{a} + \int_{a}^{\infty},
\]

which leads to two equations,

\[
\rho_R^{-1} = \rho_0^{-1} (\delta l) + \frac{8\pi y_d^2}{4} \int_{a}^{\infty} \frac{dr}{a^2} (\frac{r}{a})^3 2\pi \beta \rho_0 + O(y^4)
\]

and

\[
\rho_0^{-1} (\delta l) = \rho_0^{-1} + \frac{8\pi y_d^2}{4} \int_{a} \frac{dr}{a^2} (\frac{r}{a})^3 2\pi \beta \rho_0 + O(y^4),
\]

where \( y(\delta l) = ye^{(2-\beta \rho_0)\delta l} \). Therefore, we reach the RG equations for the first transition

\[
\frac{d\rho}{dl} = \frac{8\pi}{4} \beta y^2 + O(y^4),
\]

\[
\frac{dy}{dl} = \gamma(2 - \pi \beta \rho) + O(y^2).
\]

Obviously, the critical temperature is given by

\[
T_c = \frac{\pi}{2} \rho_0.
\]

Above \( T_c \), the superfluid stiffness gets renormalized into zero, indicating that the superfluid phase is broken down. We then goes to a phase with the dipole bound states as freely mobile charge carriers.

Towards the RG equations, we have introduced a lattice regularization on which the dipole charges can only take integer values. Also, one can repeat the above procedures in a continuum space, in which \( \mathbf{l}_i \cdot \mathbf{l}_j = l_i l_j \xi_i \cdot \xi_j \) includes a part from the relative angular \( \xi_i \cdot \xi_j = \xi_i \xi_j \cos(\theta_{ij}) \). It will alter the coefficient before \( y^2 \) in Eq. (51), but the physics will not be changed.
where \( s(r) = \bar{s}(r) - \nabla \cdot D(r) \). The plasma of dipole bound states can screen the interaction between defects \( \Theta \). We first evaluate the correlator of dipole bound state. Note that a partition function for the dipole bound states is \( \Xi_d = \int D D \exp(-\beta H_d) \), where

\[
H_d = \int d^2 q \left( \rho_0 q_i q_j + \delta_{ij} E_d \right) D_i(q) D_j(-q),
\]

(55)

The correlator can be calculated straightforward

\[
\langle D_i(q) D_j(-q) \rangle = \frac{1}{\Xi_d} \int D D \exp(-\beta H_d) D_i(q) D_j(-q)
= \frac{1}{\beta \Xi_d} \left( \delta_{ij} - \frac{q_i q_j}{q^2} \right) + \frac{q_i q_j}{\beta \Xi_d q^2} \frac{q^2}{\lambda_d^2 + q^2}.
\]

(56)

where \( \lambda_d = \sqrt{E_d/\rho_0} \) is the Debye-Hückel screening length of dipole bound states. With the correlator in Eq. (56), we can integrate out the dipole bound states to give the screened interaction of defects \( \Theta \),

\[
H = \int d^2 q \frac{E_s}{q^2} \lambda_d^{-2} + q^2 \bar{s}(q)\bar{s}(-q) + \Xi_c \int d^2 r \bar{s}(q)\bar{s}(-q)
= \int d^2 q \frac{E_s}{q^2} \bar{s}(q)\bar{s}(-q) + \Xi_c (1 - \lambda_d^4) \int d^2 r \bar{s}(q)\bar{s}(-q),
\]

(57)

where in the second line we make the expansion in the low energy. There are two main effects from the plasma screening effect. Firstly, it renormalizes the form of the defect interaction to a logarithmic one. Second, it reduces the core energy from \( E_s \) to \( E_s (1 - 2\lambda_d^4) \). When the effective core energy is positive \( 1 - \lambda_d^{-4} > 0 \), the model in Eq. (57) coincides with a Coulomb gas, and the KT transition occurs at

\[
T_{c2} = \pi \Xi_d,
\]

(58)

which only depends on the core energy of defects \( \Theta \). We also should require \( T_{c2} > T_{c1} \), giving rise to a condition \( \Xi_d < \rho_0 < 2\Xi_c \). However, the resultant Hamiltonian in Eq. (57) is no longer stable when \( 1 - 2\lambda_d^4 < 0 \). The fourth-order terms work and the ground state favors a finite density of defects \( \Theta \),

\[
\langle \bar{s}(q) \rangle \neq 0.
\]

(59)

In this case, there will be no intermediate phase-II. That is, the quadrupole bound states split into isolated defects \( \Theta \) directly.

**SM-3: HIGH TEMPERATURE PLASMA IN PHASE-III**

At high temperature \( T \gg T_{c2} \) region, both dipole and quadrupole bound states disassociate into free isolated defects. This is a plasma phase of defects \( \Theta \), and then the interactions between \( \Theta \) will get screened. Here we present the details on the screening effect in Phase-III.

At high temperature range, the Hamiltonian only involves density of isolated defects, which reads

\[
H = \frac{\rho_0}{2} \int d^2 q \frac{2}{q^2} s(q) s(-q) + \Xi_s \int d^2 r s^2(r).
\]

(60)

We regularize the system with the constant by a lattice with a lattice constant \( a \). Then a plasma phase is referred to as one defect per site,

\[
\sum_{\ell \neq 0} n_{\ell}(r) = 1,
\]

(61)

where \( n_{\ell}(r) \) is the number of a defect with charge \( \ell \) at site \( r \), \( n_{\ell}(r) = \sum_{\delta} \delta_{\ell,\ell',\delta} (r - r_i) \) with a relation \( s(r) = \sum_{\ell} \ell n_{\ell}(r) \). Then, we reformulate the Hamiltonian in (60)

\[
H = \frac{\rho_0}{2} \int d^2 r d^2 r' \sum_{\ell, \ell'} \ell' n_{\ell}(r) U(r - r') n_{\ell'}(r') + \Xi_s \int d^2 r \sum_{\ell} \ell^2 n_{\ell}(r),
\]

(62)
where $U(r)$ is the Fourier transformation of $2/q^4$. $n_\ell(r)$ is a function of the individual position $r_i$ of the defects. In order to obtain the interaction screened by the plasma, we decompose all the defects $n_\ell(r)$ into the continuous plasma field $\langle n_\ell(r) \rangle$ and the additional defects placed in the plasma $\Delta n_\ell(r)$,

$$n_\ell(r) = \langle n_\ell(r) \rangle + \delta n_\ell(r).$$  \hspace{1cm} (63)

Keeping the terms containing $\langle n_\ell(r) \rangle$, we have the free energy at the mean-field level [7]

$$F = \frac{\rho_0}{2} \int d^2r d^2r' \sum_{\ell,\ell'} \ell \ell' \langle n_\ell(r) \rangle U(r-r') \langle n_{\ell'}(r') \rangle - \int d^2r \sum_{\ell} \ell \langle n_\ell(r) \rangle \phi^\text{ext}(r)$$

$$+ E_s \int d^2r \sum_{\ell} \ell^2 \langle n_\ell(r) \rangle + \frac{1}{\beta} \sum_{\ell} \int d^2r \langle n_\ell(r) \rangle \ln \langle n_\ell(r) \rangle,$$  \hspace{1cm} (64)

with an external field $\phi^\text{ext}(r) = \sum_\ell \ell \delta n_\ell(r') U(r-r')$. In equilibrium, minimize $F$ under the condition in (61), and we obtain the mean field solution

$$\langle n_\ell(r) \rangle = \frac{\exp \left[ \beta \ell \phi^\text{ext}(r) - \beta \ell^2 E_s - \beta \ell \rho_0 \int d^2r' \sum_{\ell' \neq 0} \ell' \langle n_{\ell'}(r') \rangle U(r-r') \right]}{\sum_{\ell \neq 0} \exp \left[ \beta \ell \phi^\text{ext}(r) - \beta \ell^2 E_s - \beta \ell \rho_0 \int d^2r' \sum_{\ell' \neq 0} \ell' \langle n_{\ell'}(r') \rangle U(r-r') \right]}.$$  \hspace{1cm} (65)

The effective interaction for $\delta n_\ell(r)$ can be yielded by the linear response function

$$\chi_{ss}(r,r') = \sum_{\ell} \frac{\delta \langle n_\ell(r) \rangle}{\delta \phi^\text{ext}(r') \bigg|_{\phi^\text{ext}=0}}.$$  \hspace{1cm} (66)

At $\phi^\text{ext}(r) = 0$, $\langle n_\ell(r) \rangle$ is spatially uniform, and we have a condition,

$$\sum_{\ell} \ell \langle n_\ell(r') \rangle = 0.$$  \hspace{1cm} (67)

In this case, the equilibrium density distribution is

$$\langle n_\ell(r) \rangle = \frac{e^{-\beta E_s \ell^2}}{\sum_{\ell \neq 0} e^{-\beta E_s \ell^2}}.$$  \hspace{1cm} (68)

In response to fluctuation field $\phi^\text{ext}(r)$, the $\langle n_\ell(r) \rangle$ is changed according to Eq. (65). We can evaluate the susceptibility $\chi_{ss}(r,r')$ as

$$\chi_{ss}(r,r') = \frac{\lambda'}{E_s} \left[ \delta (r-r') - \rho_0 \int d^2r'' \chi_{ss}(r'',r') U(r-r'') \right],$$  \hspace{1cm} (69)

and in the momentum space,

$$\chi_{ss}(q) = \left( \frac{2\rho_0}{q^4} + \frac{E_s}{\lambda'} \right)^{-1},$$  \hspace{1cm} (70)

with a coefficient $\lambda'$

$$\lambda' = \frac{\sum_{\ell \neq 0} \beta E_s \ell^2 e^{-\beta E_s \ell^2}}{\sum_{\ell \neq 0} e^{-\beta E_s \ell^2}}.$$  \hspace{1cm} (71)

In the high temperature limit, we have $\beta E_s \rightarrow 0$ and $\lambda' = 1/2$. Therefore, we derive the mean-field Hamiltonian for defects $\Theta$ in the plasma phase in Phase-III,

$$H = \frac{\rho_0}{2} \int d^2q \left( \frac{2}{q^4} - \chi_{ss}(q) \right) \delta \bar{s}(q) \delta \bar{s}(-q)$$

$$= \frac{1}{2} \int d^2q \frac{2\rho_0}{q^4 + \lambda_s^{-4}} \delta \bar{s}(q) \delta \bar{s}(-q),$$  \hspace{1cm} (72)

where $\lambda_s$ is the Debye-Hücke screening length with $\lambda_s^{-4} = 2\lambda'/\rho_0/E_s$. We can find that the screening effect leads to a short ranged interaction.