TWO-PARAMETER DEFORMATIONS OF LOOP ALGEBRAS
AND SUPERALGEBRAS

Valeriy N. Tolstoy
Institute of Nuclear Physics, Moscow State University
119899 Moscow, Russia (e-mail: tolstoy@anna19.npi.msu.su)

Abstract
We discuss two-parameter deformations of an universal enveloping algebra $U(g[u])$ of a polynomial loop algebra $g[u]$, where $g$ is a finite-dimensional complex simple Lie algebra (or superalgebra). These deformations are Hopf algebras. One deformation called Drinfeldian is a quantization of $U(g[u])$ in the direction of a classical r-matrix which is a sum of the simplest rational and trigonometric r-matrices. Another deformation (discussed only for the case $g = sl_2$) is a twisting of the usual Yangian $Y_\eta(sl_2)$.

1 Introduction
As it is well known, an universal enveloping algebra $U(g[u])$ of a polynomial loop (current) Lie algebra $g[u]$, where $g$ is a finite-dimensional complex simple Lie algebra, admits two type deformations: a trigonometric deformation $U_q(g[u])$ and a rational deformation or Yangian $Y_\eta(g)$ [1]. (In the case $g = sl_n$ there also exists an elliptic quantum deformation of $U(sl_n[u])$, which is not discussed in this report). The algebras $U_q(g[u])$, and $Y_\eta(g)$ are quantizations of $U(g[u])$ in the direction of the simplest trigonometric and rational solutions of the classical Yang-Baxter equation over $g$, respectively. These deformations are one-parameter. It turns out that $U(g[u])$ also admits two-parameter deformations. Here we discuss two type of such deformations which are Hopf algebras.

A Hopf algebra of the first type called the rational-trigonometric quantum algebra or the Drinfeldian $D_{q\eta}(g)$ [2] is a quantization of $U(g[u])$ in the direction of a classical r-matrix which is a sum of the simplest rational and trigonometric r-matrices. The Drinfeldian $D_{q\eta}(g)$ contains $U_q(g)$ as a Hopf subalgebra, and $U_q(g[u])$ and $Y_\eta(g)$ are its limit quantum algebras when the deformation parameters of $D_{q\eta}(g)$ $\eta$ goes to 0 and $q$ goes to 1, respectively. These results are easy generalized to a supercase, i.e. when $g$ is a finite-dimensional contragredient simple superalgebra.

A Hopf algebra of the second type discussed only for the case $g = sl_2$ is obtained by twisting of the usual Yangian $Y_\eta(sl_2)$. The twisted Yangian $Y_{\eta\zeta}(sl_2)$ is a quantization $U(sl_2[u])$ in the direction of a classical r-matrix $r(u, v) = \eta c_2/(u - v) + \zeta h_\alpha \wedge e_{-\alpha}$, where $c_2$ is the $sl_2$ Casimir element. Detailed description of $Y_{\eta\zeta}(sl_2)$ is given in [3].

2 Drinfeldian $D_{q\eta}(g)$
Let $g$ be a finite-dimensional complex simple Lie algebra of a rank $r$ with a standard Cartan matrix $A = (a_{ij})^r_{i,j=1}$, with a system of simple roots $\Pi := \{\alpha_1, \ldots, \alpha_r\}$, and
with a maximal positive root \( \theta \). Let \( U_q(g) \) be a standard \( q \)-deformation of the universal enveloping algebra \( U(g) \) with Chevalley generators \( k_{\alpha_i}^\pm, e_{\pm \alpha_i} \) \((i = 1, 2, \ldots, r)\) and with the defining relations

\[
[k_{\alpha_i}, k_{\alpha_j}] = 0, \quad k_{\alpha_i} e_{\pm \alpha_j} k_{\alpha_i}^{-1} = q^{\pm (\alpha_i, \alpha_j)} e_{\pm \alpha_j},
\]

\[
[e_{\alpha_i}, e_{-\alpha_i}] = \frac{k_{\alpha_i} - k_{\alpha_i}^{-1}}{q - q^{-1}}, \quad (\text{ad}_q e_{\pm \alpha_i})^{1-a_{ij}} e_{\pm \alpha_j} = 0 \quad \text{for } i \neq j,
\]

\[ (2.1) \]

\[ (2.2) \]

**Definition 2.1** The Drinfeldian \( D_q(g) \) is generated as an associative algebra over \( \mathbb{C}[\eta] \) by the algebra \( U_q(g) \) and the elements \( \xi_{\delta-\theta}, k_{\delta}^\pm \) with the relations:

\[
[k_{\delta}^\pm, \text{everything}] = 0 \quad k_{\alpha_i} \xi_{\delta-\theta} k_{\alpha_i}^{-1} = q^{-(\alpha_i, \theta)} \xi_{\delta-\theta},
\]

\[
[e_{-\alpha_i}, \xi_{\delta-\theta}] = a \left[ e_{-\alpha_i}, \bar{e}_{-\theta} \right], \quad (\text{ad}_q e_{\alpha_i})^{n_{\alpha}} \xi_{\delta-\theta} = a (\text{ad}_q e_{\alpha_i})^{n_{\alpha}} \bar{e}_{-\theta}
\]

\[ (2.3) \]

\[ (2.4) \]

for \( n_{i} = 1 + 2(\alpha_i, \theta)/(\alpha_i, \alpha_i) \), and

\[
[[e_{\alpha_i}, \xi_{\delta-\theta}]_q, \xi_{\delta-\theta}]_q = -a^2 \left[ [e_{\alpha_i}, \bar{e}_{-\theta}]_q, \bar{e}_{-\theta} \right]_q + a \left[ [e_{\alpha_i}, \bar{e}_{-\theta}]_q, \xi_{\delta-\theta} \right]_q + a \left[ [e_{\alpha_i}, \xi_{\delta-\theta}]_q, \bar{e}_{-\theta} \right]_q
\]

\[ (2.5) \]

for \( g \neq sl_2 \) and \((\alpha_i, \theta) \neq 0\),

\[
[[[e_{\alpha_i}, \xi_{\delta-\theta}]_q, \xi_{\delta-\theta}]_q, \xi_{\delta-\theta}]_q = a^3 \left[ [ [e_{\alpha_i}, \bar{e}_{-\theta}]_q, \bar{e}_{-\theta} \right]_q, \bar{e}_{-\theta} \right]_q
\]

\[ (2.6) \]

for \( g = sl_2 \). The Hopf structure of \( D_q(g) \) is defined by the formulas \( \Delta_{D_q}(x) = \Delta_q(x) \), \( S_{D_q}(x) = S_q(x) \) \((x \in U_q(g))\) and it is the same for the elements \( k_{\delta}^\pm \) and \( k_{\alpha_i} \). The comultiplication and the antipode of \( \xi_{\delta-\theta} \) are given by

\[
\Delta_{D_q}(\xi_{\delta-\theta}) = \xi_{\delta-\theta} \otimes 1 + 1 \otimes \xi_{\delta-\theta}
\]

\[ (2.7) \]

\[
S_{D_q}(\xi_{\delta-\theta}) = -k_{\delta-\theta} \xi_{\delta-\theta} + a (S_q(\bar{e}_{-\theta}) + k_{\delta-\theta} \bar{e}_{-\theta})
\]

\[ (2.8) \]

where \( a := \eta/(q - q^{-1}) \), \((\text{ad}_q e_\beta) e_\gamma = [e_\beta, e_\gamma]_q \), and the vector \( \bar{e}_{-\theta} \) is any \( U_q(g) \) element of the weight \(-\theta\), such that \( g \geq \lim_{q \to 1} \bar{e}_{-\theta} \neq 0 \).

The right-hand sides of the relations \( (2.4), (2.8) \) are nonsingular at \( q = 1 \).

**Theorem 2.1** (i) The Drinfeldian \( D_{q\eta}(g) \) is a two-parameter quantization of \( U(g[\eta]) \), where \( g[\eta] \) is a central extension of \( g[\eta] \), in the direction of a classical \( r \)-matrix which is a sum of the simplest rational and trigonometric \( r \)-matrices.

(ii) The Hopf algebra \( D_{q=1,\eta}(g) \) is isomorphic to the Yangian \( Y_\eta'(g) \) (with a central element). Moreover, \( D_{q=0}(g) = U_q(g[\eta]) \).
In the supercase, i.e. when $g$ is a simple finite-dimensional contragredient Lie superalgebra all the commutators and the q-commutators are replaced by the supercommutators and the q-supercommutators. Moreover we have to add some additional Serre relations if they exist.

3 Twisted Yangian $Y_{\eta\zeta}(sl_2)$

In the case $sl_2$ from (2.3)-(2.8) at $q = 1$ we can obtain that the Yangian $Y_{\eta}(sl_2)$ is generated by the $sl_2$ elements $h_\alpha, e_{\pm\alpha}$ and the element $\xi_{\delta-\alpha}$ with the relations:

\[
[h_\alpha, \xi_{\delta-\alpha}] = -2\xi_{\delta-\alpha}, \quad [e_{-\alpha}, \xi_{\delta-\alpha}] = \eta e_{-\alpha},
\]

(3.9)

\[
[e_{\alpha}, [e_{\alpha}, [e_{\alpha}, \xi_{\delta-\alpha}]]] = 6\eta e_{\alpha}, \quad [[[e_{\alpha}, \xi_{\delta-\alpha}], \xi_{\delta-\alpha}], \xi_{\delta-\alpha}] = 6\eta\xi_{\delta-\alpha}^2.
\]

(3.10)

\[
\Delta(\xi_{\delta-\alpha}) = \xi_{\delta-\alpha} \otimes 1 + 1 \otimes \xi_{\delta-\alpha} + \eta e_{-\alpha} \otimes h_\alpha, \quad S(\xi_{\delta-\alpha}) = -\xi_{\delta-\alpha} + \eta e_{-\alpha} h_\alpha,
\]

(3.11)

where we put $(\alpha, \alpha) = 2$.

Using the twisting element $F = \sum_{k\geq0}(\zeta^k/k!)(\prod_{i=0}^{k-1}(h_\alpha+2i))\otimes e_{-\alpha}$ one can calculate the new coproduct $\Delta^{(F)}(x) := F\Delta(x)F^{-1}$ and antipode $S^{(F)} := uS(x)u^{-1}$ ($x \in Y_{\eta}(sl_2)$) where $u := \sum_{k\geq0}(-\zeta^k/k!)(\prod_{i=0}^{k-1}(h_\alpha+2i))e_{-\alpha}$. The result is the twisted Yangian $Y_{\eta\zeta}(sl_2)$. It is not difficult to show that $Y_{\eta\zeta}(sl_2)$ is a quantization of $U(sl_2[u])$ with the classical r-matrix $r(u,v) = \eta c_2/(u-v) + \zeta h_\alpha \wedge e_{-\alpha}$, where $c_2$ is the $sl_2$ Casimir element. See [3] for details.

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