Long-wavelength Cosmological Perturbation in the Universe with Multiple Perfect Fluids

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Abstract

We investigate the large scale cosmological perturbation in the Universe with multiple perfect fluids. Using the long-wavelength approximation with Hamilton-Jacobi method, we derive the formula for the gauge invariant comoving curvature perturbation. As an application of our approach, we examine the large scale perturbation in a brane cosmology.

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I. INTRODUCTION

The analysis of the large-scale cosmological perturbation is important issue to obtain the information on the initial density fluctuation which was generated during the era of the inflationary expansion of the Universe. The scales of cosmological interest such as large-angle cosmic microwave background (CMB) have spent most of their time far outside the Hubble radius and have re-entered only relatively recently in the history of the Universe.

If we concentrate on only the evolution of the large-scale fluctuation, the long-wavelength approximation (gradient expansion) of Einstein’s equation with the Hamilton-Jacobi (HJ) method[1, 2, 3, 4] is convenient to obtain the solution of the fluctuation. Following this method, the large-scale fluctuation is given by the lowest order of the spatial gradient expansion of the Einstein equation and can be recognized as the fluctuation of the spatially dependent integration constants contained in the spatially inhomogeneous “background” solution. The large-scale fluctuation of the metric and the matter field can be obtained by differentiating the solution of the background Einstein equation with respect to the constant of integration contained in it. This approach provides an easier method to give the solution of the long-wavelength perturbation and is useful especially for the Universe with multiple scalar fields or multiple fluids.

In this paper, we consider the behavior of the long-wavelength curvature perturbation. We aim to derive the formula for the gauge invariant curvature perturbation for the Einstein gravity with multiple fluids using the long-wavelength approximation with the HJ method. Contrary to the case of multiple scalar fields[1, 2], we can obtain explicit scale factor dependence of the curvature perturbation. As an application of our formalism, we consider the large-scale fluctuation in the brane cosmology[5, 6, 7]. The analysis of the perturbation in the brane cosmology is complicated because the perturbation equations do not form a closed system on the brane in general. But for the large-scale scalar type fluctuation, the perturbation equations contain a closed form on the brane[5, 7], which may be solved without solving for the bulk perturbations. Bulk effects produce a non-adiabatic mode of fluctuation caused by the degrees of freedom of the Weyl curvature. The Universe can be treated as the system with multi component perfect fluids.

The plan of this paper is as follows. In Sec. II, we introduce the long-wavelength approximation of Einstein’s equation with perfect fluid. In Sec. III, we derive the expression for the gauge invariant curvature perturbation using the HJ method. We apply this method to the brane cosmology in Sec. IV. Sec. V is devoted to summary. We use units in which $c = \hbar = 8\pi G = 1$ throughout the paper.

II. LONG-WAVELENGTH APPROXIMATION OF THE EINSTEIN GRAVITY WITH PERFECT FLUIDS

We consider the Universe with two component perfect fluids of which equation of state is given by $p^{(i)} = (\Gamma_i - 1)\rho^{(i)}$ $(i = 1, 2)$ where $\Gamma_i$ is assumed to be constant. The form of the line element is

$$ds^2 = -(N^2 - N^i N_i)dt^2 + 2N_i dtdx^i + e^{2\alpha} dx^2,$$

where $N$ is the lapse function, $N^i$ is the shift vector and $e^{\alpha}$ is the scale factor of the Universe. We assume that each component of the metric can have spatial dependence. By introducing
the velocity potential $\chi_i$, the Hamiltonian which derives the long-wavelength dynamics of the Universe is:

$$H_T = \int d^3x (N\mathcal{H} + N_i\mathcal{H}_i),$$

$$\mathcal{H} = -\frac{e^{-3\alpha}}{12} P^2 + e^{3\alpha} (\rho^{(1)} + \rho^{(2)}), \quad \rho^{(1)} = (e^{-3\alpha} P_{\chi_1})^{\Gamma_1}, \quad \rho^{(2)} = (e^{-3\alpha} P_{\chi_2})^{\Gamma_2}$$

$$\mathcal{H}_i = -\frac{1}{3} P_{\alpha,i} - P_{\alpha}\alpha_{,i} - P_{\chi_1}\chi_{1,i} - P_{\chi_2}\chi_{2,i}.$$

The three velocity of the fluid is given by

$$v^{(i)}_{j} = -\frac{1}{\epsilon^{(i)}}\chi_{i,j}.$$

$\epsilon^{(i)}$ is the specific enthalpy of the fluid which has the following relation to the number density $n^{(i)}$:

$$\epsilon^{(i)} = \frac{\rho^{(i)} + p^{(i)}}{n^{(i)}}, \quad n^{(i)} = \left(\frac{\partial p^{(i)}}{\partial \epsilon^{(i)}}\right)_s.$$  

Using these relations, the specific enthalpy can be written as

$$\epsilon^{(i)} = G_i \left(\rho^{(i)} \right)^{\frac{\Gamma_i - 1}{\Gamma_i}},$$

and the three velocity is

$$v^{(i)}_{j} = -e^{-3\alpha} \frac{P_{\chi_i}}{\Gamma_i \rho^{(i)}} \chi_{i,j}.$$

By variating this Hamiltonian with respect to $\alpha, P_{\alpha}, \chi_i, P_{\chi_i}, N, N^i$, we have the following equations

$$\dot{\alpha} = \frac{\delta H_T}{\delta P_{\alpha}} = -\frac{N}{6} e^{-3\alpha} P_{\alpha} - \frac{1}{3} N^i \alpha_{,i} - N^j \alpha_{,i},$$

$$\dot{\chi}_i = \frac{\delta H_T}{\delta P_{\chi_i}} = N \Gamma_i e^{-3\Gamma_i \alpha} (P_{\chi_i})^{\Gamma_i - 1} - N^j \chi_{i,j},$$

$$-\dot{P}_{\alpha} = \frac{\delta H_T}{\delta \alpha} = N^j e^{-3\alpha} P_{\alpha}^2 + 3e^{3\alpha} \left((\Gamma_1 - 1)\rho^{(1)} + (\Gamma_2 - 1)\rho^{(2)} + (N^j P_{\alpha})_{,j}\right),$$

$$-\dot{P}_{\chi_i} = \frac{\delta H_T}{\delta \chi_i} = (N^j \chi_{i,j}),$$

$$\mathcal{H} = 0, \quad \mathcal{H}_i = 0.$$

Equations (11) are the Hamiltonian constraint and the momentum constraint. If we assume that the shift vector is zero, we have

$$3 \left(\frac{\dot{\alpha}}{N}\right)^2 = \rho^{(1)} + \rho^{(2)},$$

$$-\frac{2}{N} \left(\frac{\dot{\alpha}}{N}\right) - 3 \left(\frac{\dot{\alpha}}{N}\right)^2 = (\Gamma_1 - 1)\rho^{(1)} + (\Gamma_2 - 1)\rho^{(2)},$$

$$\dot{\rho}^{(1)} + 3\alpha \Gamma_1 \rho^{(1)} = 0, \quad \dot{\rho}^{(2)} + 3\alpha \Gamma_2 \rho^{(2)} = 0.$$
Although these equations have the same form as for a flat Friedman-Robertson-Walker (FRW) universe with perfect fluids, variables $\alpha, \rho^{(1)}, \rho^{(2)}$ can have the spatial dependence and this system represents the evolution of the Universe with the long-wavelength inhomogeneity. The spatial dependence of each variable is determined by the momentum constraint.

We treat this system using HJ method. By introducing the generating functional $S[\alpha, \chi_1, \chi_2]$, the conjugate momenta of the dynamical variables are replaced by

$$P_\alpha = \frac{\delta S}{\delta \alpha}, \quad P_{\chi_1} = \frac{\delta S}{\delta \chi_1}, \quad P_{\chi_2} = \frac{\delta S}{\delta \chi_2}.$$ (15)

The Hamiltonian constraint and the momentum constraint become

$$\mathcal{H} = e^{-3\alpha} \left[ -\frac{1}{12} \left( \frac{\delta S}{\delta \alpha} \right)^2 + e^{6\alpha} \left( e^{-3\alpha} \frac{\delta S}{\delta \chi_1} \right)^{r_1} + e^{6\alpha} \left( e^{-3\alpha} \frac{\delta S}{\delta \chi_2} \right)^{r_2} \right] = 0,$$ (16)

$$\mathcal{H}_i = \frac{1}{3} \left( \frac{\delta S}{\delta \alpha} \right)_{,i} - \left( \frac{\delta S}{\delta \alpha} \right) \alpha_{,i} - \left( \frac{\delta S}{\delta \chi_1} \right) \chi_{1,i} - \left( \frac{\delta S}{\delta \chi_2} \right) \chi_{2,i} = 0.$$ (17)

For $N^i = 0$, the evolution equations are

$$\dot{\alpha} = -\frac{N}{6} e^{-6\alpha} \left( \frac{\delta S}{\delta \alpha} \right),$$ (18)

$$\dot{\chi}_1 = N \Gamma_1 \left( e^{-3\alpha} \frac{\delta S}{\delta \chi_1} \right)^{r_1-1}, \quad \dot{\chi}_2 = N \Gamma_2 \left( e^{-3\alpha} \frac{\delta S}{\delta \chi_2} \right)^{r_2-1}.$$ (19)

By assuming the following form of the generating functional

$$S = -2 \int d^3 x e^{3\alpha} H(\chi_1, \chi_2),$$ (20)

the Hamiltonian constraint becomes

$$-3H^2 + (-2H_{\chi_1})^{r_1} + (-2H_{\chi_2})^{r_2} = 0.$$ (21)

As this equation does not contain $\chi_1$ and $\chi_2$ explicitly, the form of its solution can be written as

$$H = H(\chi_1 + d_1, \chi_2 + d_2),$$ (22)

where $d_1(\mathbf{x})$ and $d_2(\mathbf{x})$ are spatially dependent constants of integration. The momentum constraint becomes

$$\mathcal{H}_i = 2e^{3\alpha} \left[ -H_{,i} + H_{\chi_1,\chi_1,i} + H_{\chi_2,\chi_2,i} \right] = -2e^{3\alpha} (H_{d_1,1,i} + H_{d_2,2,i}) = 0.$$

For $N^i = 0$, the conjugate momentum $P_{\chi_1}$ and $P_{\chi_2}$ become constant in time and we have

$$P_{\chi_1} = -2e^{3\alpha} H_{\chi_1} = -2c_1, \quad P_{\chi_2} = -2e^{3\alpha} H_{\chi_2} = -2c_2,$$ (23)

where $c_1(\mathbf{x})$ and $c_2(\mathbf{x})$ are spatially dependent constants. Hence the momentum constraint gives the following relation between spatially dependent constants :

$$c_1 d_{1,i} + c_2 d_{2,i} = 0.$$ (24)
The evolution equations are
\[ \dot{\alpha} = NH, \]
\[ \dot{\chi}_1 = N \Gamma_1 (-2H \chi_1)^{\Gamma_1-1}, \]
\[ \dot{\chi}_2 = N \Gamma_2 (-2H \chi_2)^{\Gamma_2-1}. \]  
\((21),(23), (24)\) and \((25)\) are basic equations to analyze the long-wavelength dynamics of the Universe. The spatial inhomogeneity of the dynamical variables \(\alpha, \chi_1, \chi_2\) comes from the spatially dependent constants \(c_1(x), c_2(x), d_1(x), d_2(x)\). Solving HJ equation is formally equivalent to obtaining the homogeneous background solution. Once the background solution with constants of integration is obtained, the spatial dependence of the long-wavelength fluctuation is determined through the momentum constraint.

**III. LINEAR PERTURBATION ABOUT A FLAT FRW UNIVERSE**

To obtain the curvature perturbation, we consider the linear perturbation about the homogeneous background. This means that variables in the equations \((21),(23), (24)\) and \((25)\) are linearized as follows:
\[ \alpha \to \alpha(t) + \delta \alpha(t,x), \quad N \to 1 + n(t,x), \quad \chi_i \to \chi_i(t) + \delta \chi_i(t,x), \]
\[ c_i \to c_i + \delta c_i(x), \quad d_i \to d_i + \delta d_i(x). \]

The background equations are given by
\[ -3H^2 + (-2H \chi_1)^{\Gamma_1} + (-2H \chi_2)^{\Gamma_2} = 0, \]
\[ \dot{\alpha} = H, \quad \dot{\chi}_1 = \Gamma_1 (-2H \chi_1)^{\Gamma_1-1}, \quad \dot{\chi}_2 = \Gamma_2 (-2H \chi_2)^{\Gamma_2-1}. \]  
\((26)\) These equations describes the evolution of spatially flat FRW universe. The constants contained in these equations do not have spatial dependence. By using equation \((23)\), the background solution of the velocity potential can be written as
\[ \chi_1 = \chi_1(\alpha, c_1, c_2, d_1, d_2), \quad \chi_2 = \chi_2(\alpha, c_1, c_2, d_1, d_2). \]  
\((27)\)

The evolution equations for linear perturbation are
\[ \delta \dot{\alpha} = nH + H \chi_1 \delta \chi_1 + H \chi_2 \delta \chi_2, \]  
\[(28)\]
\[ \delta \dot{\chi}_1 = n \Gamma_1 (-2H \chi_1)^{\Gamma_1-1}, \]
\[ -2 \Gamma_1 (\Gamma_1 - 1) (-2H \chi_1)^{\Gamma_1-2} (H \chi_1 \chi_1 \delta \chi_1 + H \chi_1 \chi_2 \delta \chi_2 + H \chi_1 d_1 \delta d_1 + H \chi_1 d_2 \delta d_2), \]
\[(29)\]
\[ \delta \dot{\chi}_2 = n \Gamma_2 (-2H \chi_2)^{\Gamma_2-1}, \]
\[ -2 \Gamma_2 (\Gamma_2 - 1) (-2H \chi_2)^{\Gamma_2-2} (H \chi_1 \chi_2 \delta \chi_1 + H \chi_2 \chi_2 \delta \chi_2 + H \chi_2 d_1 \delta d_1 + H \chi_2 d_2 \delta d_2). \]
\[(30)\]

In equation \((28)\), we do not have terms which contain \(\delta d_i\) because this equation give the relation between \(\delta \alpha\) and \(\delta \chi_i\), and is not the evolution equation. Indeed this equation has the same form as the momentum constraint in the conventional cosmological perturbation theory\[8, 9\].

We solve the perturbation equation in \(\delta \alpha = 0\) gauge. This gauge condition corresponds to the zero curvature gauge\[10\]. In this gauge, by eliminating the lapse function \(n\) and using
the background solution (27), we can rewrite the evolution equation for $\delta \chi_i$ as the following form:

\[
\begin{pmatrix}
\delta \chi_1 + \delta d_1 \\
\delta \chi_2 + \delta d_2
\end{pmatrix}
\bigg|_{i,\alpha} =
X^{-1}
\begin{pmatrix}
\delta \chi_1 + \delta d_1 \\
\delta \chi_2 + \delta d_2
\end{pmatrix}
+ 
\frac{H_{\chi_1} \delta d_1 + H_{\chi_2} \delta d_2}{H^2}
\left(
\frac{\Gamma_1(-2H_{\chi_1})}{\Gamma_1} \right)
\left(
\frac{\Gamma_2(-2H_{\chi_2})}{\Gamma_2}
\right),
\] (31)

where the matrix $X$ is defined by using the background solution (27) as follows

\[
X \equiv
\begin{pmatrix}
\chi_{1,c_1} & \chi_{1,c_2} \\
\chi_{2,c_1} & \chi_{2,c_2}
\end{pmatrix}.
\] (32)

If we substitute the following form of the solution to the evolution equation (31),

\[
\begin{pmatrix}
\delta \chi_1 + \delta d_1 \\
\delta \chi_2 + \delta d_2
\end{pmatrix}
= X
\begin{pmatrix}
C_1(t) \\
C_2(t)
\end{pmatrix},
\] (33)

we have

\[
X
\begin{pmatrix}
C_1 \\
C_2
\end{pmatrix}
|_{\alpha}
= c_0 \frac{e^{-3\alpha}}{H}
\begin{pmatrix}
\chi_{1,\alpha} \\
\chi_{2,\alpha}
\end{pmatrix},
\] (34)

where $c_0 = c_1 \delta d_1 + c_2 \delta d_2$ is a constant which has no spatial dependence. This comes from equation (24). Therefore, $\delta \chi_1$ and $\delta \chi_2$ are given by

\[
\begin{pmatrix}
\delta \chi_1 \\
\delta \chi_2
\end{pmatrix}
= X
\left[
\begin{pmatrix}
\delta c_1 \\
\delta c_2
\end{pmatrix}
+ c_0 \int d\alpha \frac{e^{-3\alpha}}{H} X^{-1}
\left(
\begin{pmatrix}
\chi_{1,\alpha} \\
\chi_{2,\alpha}
\end{pmatrix}
\right)
\right]
- \begin{pmatrix}
\delta d_1 \\
\delta d_2
\end{pmatrix},
\] (35)

where $\delta c_1(x)$ and $\delta c_2(x)$ are spatially dependent constants.

Now we evaluate the gauge invariant variable corresponding to the curvature perturbation on comoving slice

\[
\mathcal{R} \equiv -\delta \alpha - H \frac{(\rho^{(1)} + p^{(1)})v^{(1)} + (\rho^{(2)} + p^{(2)})v^{(2)}}{(\rho^{(1)} + p^{(1)}) + (\rho^{(2)} + p^{(2)})}
= -\delta \alpha + \frac{H}{H_{\chi_1} \delta \chi_1 + H_{\chi_2} \delta \chi_2},
\] (36)

where we have used equation (6) to express the velocity perturbation by $\delta \chi$. This quantity gives the spatial curvature perturbation on the comoving slice and equivalent to Bardeen’s parameter $\zeta$ in the long-wavelength limit. To proceed our calculation, we prepare formulas which comes from the conservation law equation (23). By differentiating equation (23) with respect to $\alpha$, we have

\[
M
\begin{pmatrix}
\chi_{1,\alpha} \\
\chi_{2,\alpha}
\end{pmatrix}
= -3
\begin{pmatrix}
H_{\chi_1} \\
H_{\chi_2}
\end{pmatrix},
M \equiv
\begin{pmatrix}
H_{\chi_1 \chi_1} & H_{\chi_1 \chi_2} \\
H_{\chi_2 \chi_1} & H_{\chi_2 \chi_2}
\end{pmatrix},
\] (37)

By differentiating equation (23) with respect to $c_1, c_2$, we have

\[
MX = e^{-3\alpha}
\begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix}.
\]
Using these relations, the gauge invariant curvature perturbation $\mathbf{R}$ in the $\delta \alpha = 0$ gauge becomes

$$
\mathbf{R} = \frac{H}{\dot{H}} (H_{\chi_1} \delta \chi_1 + H_{\chi_2} \delta \chi_2)
= -\frac{1}{3} \frac{\delta c_1 \Gamma_1 \rho^{(1)} + \delta c_2 \Gamma_2 \rho^{(2)}}{\Gamma_1 \rho^{(1)} + \Gamma_2 \rho^{(2)}} + c_0 \left( \int dt e^{-3\alpha} - \frac{H}{\dot{H}} e^{-3\alpha} \right)
$$

(38)

where energy density of each fluids is given by $\rho^{(1)} = (-2c_1) \Gamma_1 e^{-3\Gamma_1 \alpha}, \rho^{(2)} = (-2c_2) \Gamma_2 e^{-3\Gamma_2 \alpha}$. This is the main result of this paper. The first term of this expression corresponds to the contribution of the growing mode of adiabatic and iso-curvature perturbation. This term is same as the one derived by using the local conservation of energy momentum tensor[11]. The second term corresponds to decaying mode. In our analysis, the decaying mode cannot have spatial dependence, but it gives the correct time dependence. For $\Gamma_1 = \Gamma_2$, the system reduces to a single component fluid case and $\mathbf{R}$ obeys the following evolution equation

$$
\dot{\mathbf{R}} = 0 \quad \text{for} \quad \Gamma_1 = 1,
$$

(39)

$$
\ddot{\mathbf{R}} + 3\dot{\mathbf{R}} = 0 \quad \text{for} \quad \Gamma_1 \neq 1
$$

(40)

In the situation when the energy density of one fluid dominates $\rho^{(1)} \gg \rho^{(2)}$, the time dependence of the scale factor is $e^{\alpha} \approx (t/t_0)^\frac{2}{3 \Gamma_1}$ and the behavior of the curvature perturbation is given by

$$
\mathbf{R} \approx -\frac{\delta c_1}{c_1} - \frac{1}{3} (\frac{\delta c_2}{c_2} - \frac{\delta c_1}{c_1}) \frac{\Gamma_2 \rho_2}{\Gamma_1 \rho_1} + c_0 \left( \int dt e^{-3\alpha} - \frac{H}{\dot{H}} e^{-3\alpha} \right)
= -\frac{\delta c_1}{c_1} - \frac{1}{3} (\frac{\delta c_2}{c_2} - \frac{\delta c_1}{c_1}) \frac{\Gamma_2 (-2c_2) \Gamma_1}{\Gamma_1 (-2c_1) \Gamma_1} \times e^{-3(\Gamma_2 - \Gamma_1) \alpha} - c_0 t_0 \times \frac{2(\Gamma_1 - 1)}{2 - \Gamma_1} e^{\frac{2}{3}(\Gamma_2 - \Gamma_1) \alpha}.
$$

(41)

The first term corresponds to the adiabatic growing mode which is constant in time. The second term corresponds to the iso-curvature mode and the third term is the decaying mode. The decaying mode vanishes in the $\Gamma_1 = 1$ case.

**IV. APPLICATION TO THE BRANE COSMOLOGY**

We apply the method developed in Sec. III to evaluate the large scale curvature perturbation in the brane cosmology[5, 6, 7]. By assuming the Randall-Sundrum type brane model, the four dimensional Einstein equation on the spatially homogeneous brane universe is

$$
3H^2 = \rho + \frac{l^2}{12} \rho^2 + \rho_\varepsilon,
$$

$$
-2\dot{H} - 3H^2 = p + \frac{l^2}{12} (\rho^2 + 2pp) + \frac{\rho_\varepsilon}{3},
$$

$$
\dot{\rho} + 3H(\rho + p) = 0, \quad p = (\Gamma - 1)\rho, \quad \dot{\rho}_\varepsilon + 4H\rho_\varepsilon = 0,
$$

(42)

where $l^2$ is the inverse of brane tension and $\rho_\varepsilon$ is the energy density of the Weyl matter. The unconventional feature of the brane cosmology is the appearance of $\rho^2$ and $\rho_\varepsilon$ terms.
on the right hand side of the Einstein equation. The Hamiltonian for the long-wavelength dynamics which derives equation (42) in the homogeneous limit is
\begin{equation}
H_T = \int d^3x (N\mathcal{H} + N^i\mathcal{H}_i),
\end{equation}
\begin{equation}
\mathcal{H} = -\frac{e^{-3\alpha}}{12} P_\alpha^2 + e^{3\alpha} \left( \rho + \frac{l^2}{12}\rho^2 + \rho_\varepsilon \right), \quad \rho = (e^{-3\alpha} P_{\chi_1})^\Gamma, \quad \rho_\varepsilon = (e^{-3\alpha} P_{\chi_2})^\frac{2}{3},
\end{equation}
\begin{equation}
\mathcal{H}_i = \frac{P_{\alpha,i}}{3} - P_{\alpha,i} - P_{\chi_1,\chi_1,i} - P_{\chi_2,\chi_2,i}.
\end{equation}
The effective energy density and the effective pressure for the perfect fluid are defined by
\begin{equation}
\rho = \rho + \frac{l^2}{12}\rho^2,
\end{equation}
\begin{equation}
p = (\Gamma - 1)\rho + \frac{l^2}{12}(2\Gamma - 1)\rho^2.
\end{equation}
The specific enthalpy for the effective matter field is given by
\begin{equation}
\varepsilon = \Gamma \rho \frac{\Gamma - 1}{\Gamma} \left( 1 + \frac{l^2}{6}\rho^2 \right).
\end{equation}
For the Weyl matter,
\begin{equation}
\varepsilon = \frac{4}{3}\rho_{\varepsilon}^{1/4}.
\end{equation}
Hence the three velocity of the effective matter field and the Weyl matter becomes
\begin{align*}
v_M &= -\frac{\delta\chi_1}{\varepsilon} = -\frac{\delta\chi_1}{\Gamma \rho \frac{\Gamma - 1}{\Gamma} \left( 1 + \frac{l^2}{6}\rho^2 \right)},
\end{align*}
\begin{align*}
v_{\varepsilon} &= -\frac{\delta\chi_2}{\varepsilon} = -\frac{\delta\chi_2}{\frac{4}{3}\rho_{\varepsilon}^{1/4}}.
\end{align*}
The gauge invariant variable corresponding to the curvature perturbation on comoving slice is
\begin{align*}
\mathcal{R} &= -\delta\alpha - H (\rho \rho_M + \rho_p M) V_M + (\rho \rho_\varepsilon + \rho_\varepsilon p) V_{\varepsilon} \\
&= -\delta\alpha + \frac{H}{\rho} \frac{\Gamma}{\rho} \frac{\delta\chi_1 + \rho_{\varepsilon}^{3/4} \delta\chi_2}{\rho_{\varepsilon}^{1/4}} \\
&= -\delta\alpha + \frac{H}{\rho} (H_1 \delta\chi_1 + H_2 \delta\chi_2).
\end{align*}
This expression is same as the result of standard cosmology (39). The explicit form obtained in the \(\delta\alpha = 0\) gauge is
\begin{equation}
\mathcal{R} = \frac{1}{3} \frac{\Gamma}{H} \left( \rho + \frac{l^2}{6}\rho^2 \right) \frac{\delta\chi_1}{\delta e_1} + \frac{4}{3}\rho_{\varepsilon} \frac{\delta\chi_2}{\delta e_2} + c_0 \left( \int dt e^{-3\alpha} - \frac{H}{H} e^{-3\alpha} \right).
\end{equation}
The first term of this expression reproduces the result of Ref. [3] which was derived by using the local conservation law of the fluids. Our method include the contribution of the decaying mode. The time dependence of the curvature perturbation can be obtained in the following three cases:
a. $l^2 \rho^2$ dominates The time dependence of the scale factor is $e^\alpha \approx (t/t_0)^{\frac{3}{4\Gamma}}$ and

$$
\mathcal{R} \approx -\frac{\delta c_1}{3c_1} - \frac{8}{3\Gamma} \left( \frac{\delta c_2}{c_2} - \frac{\delta c_1}{c_1} \right) \frac{\rho \varepsilon}{l^2 \rho^2} + c_0 \left( \int dt e^{-3\alpha} - \frac{H}{H} e^{-3\alpha} \right)
= -\frac{\delta c_1}{3c_1} - \frac{8}{3\Gamma} \left( \frac{\delta c_2}{c_2} - \frac{\delta c_1}{c_1} \right) \left( -\frac{2c_2}{l^2(-2c_1)^{2r}} \right)^{4/3} \times e^{2(3r-2)\alpha} - c_0 t_0 \times \frac{2\Gamma - 1}{1 - \Gamma} e^{3(r-1)\alpha}. \quad (50)
$$

b. $\rho_\varepsilon$ dominates The time dependence of the scale factor is $e^\alpha \approx (t/t_0)^{1/2}$ and

$$
\mathcal{R} \approx -\frac{\delta c_2}{3c_2} - \frac{\Gamma}{4} \left( \frac{\delta c_2}{c_2} - \frac{\delta c_1}{c_1} \right) \left( \frac{\rho}{\rho_\varepsilon} + \frac{l^2 \rho^2}{6 \rho_\varepsilon} \right) + c_0 \left( \int dt e^{-3\alpha} - \frac{H}{H} e^{-3\alpha} \right)
= -\frac{\delta c_2}{3c_2} - \frac{\Gamma}{4} \left( \frac{\delta c_2}{c_2} - \frac{\delta c_1}{c_1} \right) \left( -\frac{2c_1}{l^2(-2c_2)^{2r}} \right)^{4/3} \times e^{(4-3r)\alpha} + \frac{l^2}{6} \left( -\frac{2c_2}{(-2c_1)^{2r}} \right)^{4/3} \times e^{-2(3r-2)\alpha}
- c_0 t_0 \times e^{-\alpha}. \quad (51)
$$

c. $\rho$ dominates The time dependence of the scale factor is $e^\alpha \approx (t/t_0)^{\frac{2}{3\Gamma}}$ and

$$
\mathcal{R} \approx -\frac{\delta c_1}{3c_1} - \frac{4}{9\Gamma} \left( \frac{\delta c_2}{c_2} - \frac{\delta c_1}{c_1} \right) \frac{\rho}{\rho_\varepsilon} + c_0 \left( \int dt e^{-3\alpha} - \frac{H}{H} e^{-3\alpha} \right)
= -\frac{\delta c_1}{3c_1} - \frac{4}{9\Gamma} \left( \frac{\delta c_2}{c_2} - \frac{\delta c_1}{c_1} \right) \left( -\frac{2c_1}{(-2c_1)^{2r}} \right)^{4/3} \times e^{-(4-3r)\alpha} - c_0 t_0 \times \frac{2(\Gamma - 1)}{2 - \Gamma} e^{-\frac{3}{2}(3r-2)\alpha}. \quad (52)
$$

V. SUMMARY

We have derived a formula of long-wavelength curvature perturbation for the Universe with multi-component fluids using HJ method. The non-decaying mode of the curvature perturbation is explicitly expressed as the function of scale factor of the Universe. This mode is the mixture of the adiabatic growing mode which is constant in time and the time varying iso-curvature mode. In the case of the multiple-scalar field[1], the curvature perturbation cannot be expressed explicitly as the function of the scale factor (see Appendix) because we cannot obtain the solution of the background scalar fields as the function of the scale factor in general. For the perfect fluids case, each component of fluids evolves separately and this makes it possible to obtain the background and the perturbation solution.

Although the decaying mode obtained in this paper reproduces the same time dependence as the result of the conventional cosmological perturbation theory, it cannot have the spatial dependence because of the momentum constraint. The homogeneous decaying mode obtained here can be absorbed to the background solution by re-defining the background constant of integration. This is related to the feature of decaying mode in zero curvature gauge and co-moving gauge; the decaying mode under these gauge conditions becomes higher order in long-wavelength expansion compared with that of other gauge conditions[2].

As the application of the HJ method, the large scale curvature perturbation in the brane cosmology was derived. Our method reproduced the result obtained by using standard cosmological perturbation theory with the local conservation of the energy-momentum tensor[3]. It is interesting to investigate the brane model with the scalar field. But we could not obtain the Hamiltonian which derives the long-wavelength dynamics of the brane cosmology with the scalar field. This is left as our future problem.
**APPENDIX A: CURVATURE PERTURBATION IN THE MULTIPLE SCALAR FIELDS SYSTEM**

We review the derivation of the long-wavelength curvature perturbation in the Einstein gravity with multiple scalar fields. HJ equation and the evolution equation are

\[-3H^2 + 2H_{\phi_1}^2 + 2H_{\phi_2}^2 + V(\phi_1, \phi_2) = 0, \quad (A1)\]
\[\dot{\alpha} = NH, \quad (A2)\]
\[\dot{\phi}_1 = -2NH\phi_1, \quad \dot{\phi}_2 = -2NH\phi_2 \quad (A3)\]

The solution of the HJ equation is given by

\[H = H(\phi_1, \phi_2, d_1, d_2) \quad \text{where} \quad d_1(\mathbf{x}) \quad \text{and} \quad d_2(\mathbf{x}) \quad \text{are spatially dependent constants.} \]

From the momentum constraint, the spatial dependence of these constants are determined:

\[H_{d_1,d_1,i} + H_{d_2,d_2,i} = 0.\]

Other constants of motion in the system are given by

\[e^{3\alpha} H_{d_1} = c_1(\mathbf{x}) \equiv e^{-3\alpha_0(\mathbf{x})}, \quad e^{3\alpha} H_{d_2} = c_2(\mathbf{x}) \equiv e^{-3\alpha_0(\mathbf{x})} f(\mathbf{x}), \quad (A4)\]

From this, we can express \(\phi_1, \phi_2\) as the function of the scale factor:

\[\phi_1 = \phi_1(\alpha + \alpha_0, f, d_1, d_2), \quad \phi_2 = \phi_2(\alpha + \alpha_0, f, d_1, d_2). \quad (A5)\]

The evolution equation for the perturbation is

\[\delta \dot{\alpha} = nH + H_{\phi_1} \delta \phi_1 + H_{\phi_2} \delta \phi_2, \quad (A6)\]
\[\delta \dot{\phi}_1 = -2nH\phi_1 - 2(H_{\phi_1}\delta \phi_1 + H_{\phi_1,\phi_2}\delta \phi_2 + H_{\phi_1,d_1}\delta d_1 + H_{\phi_1,d_2}\delta d_2), \quad (A7)\]
\[\delta \dot{\phi}_2 = -2nH\phi_2 - 2(H_{\phi_2}\delta \phi_1 + H_{\phi_2,\phi_2}\delta \phi_2 + H_{\phi_2,d_1}\delta d_1 + H_{\phi_2,d_2}\delta d_2). \quad (A8)\]

The gauge invariant variable corresponding to the curvature perturbation on comoving slice is given by

\[\mathcal{R} = -\delta \alpha - \frac{H^3}{2H\alpha} (\phi_{1,\alpha} \delta \phi_1 + \phi_{2,\alpha} \delta \phi_2). \quad (A9)\]

We evaluate this expression in the \(\delta \alpha = 0\) gauge. For this purpose, we prepare the relation derived form equation \((A4)\). By differentiating equation \((A4)\) with respect to \(c_1, c_2\), we have

\[\begin{bmatrix} H_{\phi_1,d_1} & H_{\phi_2,d_1} \\ H_{\phi_1,d_2} & H_{\phi_2,d_2} \end{bmatrix}, \quad \begin{bmatrix} \phi_{1,c_1} \\ \phi_{2,c_1} \end{bmatrix}\]

\[MX = e^{-3\alpha} I, \quad M = \begin{bmatrix} H_{\phi_1,d_1} & H_{\phi_2,d_1} \\ H_{\phi_1,d_2} & H_{\phi_2,d_2} \end{bmatrix}, \quad X = \begin{bmatrix} \phi_{1,c_1} & \phi_{1,c_2} \\ \phi_{2,c_1} & \phi_{2,c_2} \end{bmatrix} \quad (A10)\]

The evolution equation in the \(\delta \alpha = 0\) gauge can be written as

\[\begin{bmatrix} \delta \phi_1 \\ \delta \phi_2 \end{bmatrix} = X^\alpha X^{-1} \begin{bmatrix} \delta \phi_1 \\ \delta \phi_2 \end{bmatrix} - \frac{2}{H} \begin{bmatrix} H_{\phi_1,d_1} & H_{\phi_1,d_2} \\ H_{\phi_2,d_1} & H_{\phi_2,d_2} \end{bmatrix} \begin{bmatrix} \delta d_1 \\ \delta d_2 \end{bmatrix}. \quad (A11)\]

By substituting

\[\begin{bmatrix} \delta \phi_1 \\ \delta \phi_2 \end{bmatrix} = X \begin{bmatrix} C_1(t) \\ C_2(t) \end{bmatrix}\]
to the evolution equation, we have
\[ X \begin{pmatrix} C_1 \\ C_2 \end{pmatrix},_\alpha = -\frac{2}{H} M^T \begin{pmatrix} \delta d_1 \\ \delta d_2 \end{pmatrix} = -\frac{2e^{-3\alpha}}{H} (X^{-1})^T \begin{pmatrix} \delta d_1 \\ \delta d_2 \end{pmatrix} \]
and the solution of equation (A11) is given by
\[ \begin{pmatrix} \delta \phi_1 \\ \delta \phi_2 \end{pmatrix} = X \left[ \begin{pmatrix} \delta c_1 \\ \delta c_2 \end{pmatrix} + \int d\alpha \frac{e^{-3\alpha}}{H} (X^{-1})(X^{-1})^T \begin{pmatrix} \delta d_1 \\ \delta d_2 \end{pmatrix} \right], \quad (A12) \]
where \( \delta c_1, \delta c_2 \) are spatially dependent constants. Using this solution, the curvature perturbation becomes
\[ R = -\frac{H^3}{2H_\alpha} (\phi_{1,\alpha} \phi_{2,\alpha}) X \left[ \begin{pmatrix} \delta c_1 \\ \delta c_2 \end{pmatrix} + \int d\alpha \frac{e^{-3\alpha}}{H} (X^{-1})(X^{-1})^T \begin{pmatrix} \delta d_1 \\ \delta d_2 \end{pmatrix} \right]. \quad (A13) \]
For the growing mode \( \delta d_1 = \delta d_2 = 0 \), we have
\[ R = \delta \alpha_0 + \frac{\phi_{1,\alpha} \phi_{1,\alpha} + \phi_{2,\alpha} \phi_{2,\alpha} \delta f}{(\phi_{1,\alpha})^2 + (\phi_{2,\alpha})^2} \delta f. \quad (A14) \]
The first term corresponds to the adiabatic growing mode which is constant in time, and the second term corresponds to the iso-curvature mode. The explicit scale factor dependence can be obtained with the assumption of the slow rolling approximation and the specific form of the scalar field potential[1].

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