A Complete Lattice Technicolor Model

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ABSTRACT: We construct a lattice gauge theory using reduced staggered fermions and gauge fields which provides a non-perturbative realization of a complete technicolor model; one which treats both strong and weakly coupled gauge symmetries on an equal footing. We show that the model is capable of developing a Higgs phase at non zero lattice spacing via the formation of fermion condensates. We further show that while the broken symmetry associated with this phase has a vector character in the lattice theory it is realized as an axial symmetry in the continuum limit in agreement with the Vafa Witten theorem.

KEYWORDS: Lattice gauge theory, chiral gauge theories, spontaneous symmetry breaking.
1. Introduction

The idea that the Higgs mechanism can occur through the formation of fermionic condensates is an attractive one when constructing many theories of Beyond Standard Model (BSM) physics and finds application in technicolor, composite Higgs models, tumbling and grand unification schemes [1, 2, 3, 4]. Lattice realizations of these scenarios thus potentially give a rigorous setting for understanding how non-perturbative dynamics in models without elementary scalars can spontaneously break gauge symmetries and potentially can give us new tools to analyze such theories.

In a lattice theory Elitzur’s theorem [5] guarantees that any condensate which is not invariant under the gauge symmetry must necessarily have vanishing expectation value. Instead, the gauge invariant way to understand spontaneous gauge symmetry breaking and the operation of the Higgs mechanism in such theories is that it proceeds via the condensation of a \( \phi \dagger \phi \) where \( \phi \) is a fermion bilinear which carries a non-trivial representation of the gauge group. One can think of this bilinear as a composite Higgs field.

In this paper we construct a lattice realization of these ideas using a set of two massless staggered lattice fermions. Each of these fields can be reduced by simply restricting one field to even parity sites and the other to sites of odd parity. The key observation is that in the absence of single site mass terms the kinetic terms for each of these reduced staggered fermions may be gauged independently. We will use this observation to construct a class of lattice technicolor theories in which the gauge forces factorize into strong and weakly interacting sectors with both reduced fields coupling identically to the strong interaction but differing in their weak interactions [6]. In the limit in which the weak gauge couplings are set to zero the strong interactions generate the usual single site condensate characteristic
of staggered fermions. This condensate, which couples the two reduced fields, is a strong interaction singlet but will spontaneously break a subgroup of the weak symmetries. Once the weak symmetries are gauged these bilinear condensates must remain zero but a non-trivial four fermion condensate can form instead corresponding to the onset of a dynamical Higgs mechanism.

Furthermore, we show that while the broken symmetries have a vector character at non-zero lattice spacing, they are to be interpreted as axial symmetries in the continuum limit. This result is then compatible with the Vafa Witten theorem which forbids the spontaneous breaking of vector symmetries \[7\]. This is a necessary condition for this Higgs phase to survive the continuum limit.

2. Lattice model

We start with two staggered fields \( \chi \) and \( \xi \). After restricting them to odd/even sites, we can define new fields \( \psi \) and \( \lambda \) as:

\[
\begin{align*}
\bar{\psi}_+(x) &= \frac{1}{2} (1 + \epsilon(x)) \chi(x), \\
\bar{\lambda}_+(x) &= \frac{1}{2} (1 + \epsilon(x)) \xi(x), \\
\psi_-(x) &= \frac{1}{2} (1 - \epsilon(x)) \chi(x), \\
\lambda_-(x) &= \frac{1}{2} (1 - \epsilon(x)) \xi(x)
\end{align*}
\]

where the parity of a lattice site is given by \( \epsilon(x) = (-1)^{\sum_{\mu=1}^{4} x_\mu} \). The fields \( \psi \) and \( \lambda \) are termed reduced staggered fermions since each contains half the number of degrees of freedom of the usual staggered fermion and corresponds to two rather than four Dirac fermions in the continuum limit \[8, 9\]. The resultant lattice action can then be written as,

\[
S = \sum_{x, \mu} \eta_\mu(x) (\bar{\psi}_+(x) (\psi_-(x + \mu) - \psi_-(x - \mu)) + \sum_{x, \mu} \eta_\mu(x) \bar{\lambda}_-(x) (\lambda_+(x + \mu) - \lambda_+(x - \mu))
\]

where the phase \( \eta_\mu(x) \) is the usual staggered quark phase given by

\[
\eta_\mu(x) = (-1)^{\sum_{i=1}^{x_i} x_i}.
\]

Since the fields \( \lambda \) and \( \psi \) in the action are uncoupled we can take them to transform in different representations of one or more internal symmetry groups. In this paper we will assume a gauge symmetry of the form \( SU(N) \times SU(M) \times SU(M) \) with the reduced staggered fields \( \psi \) and \( \lambda \) transforming in the \((\Box, \Box, 1)\) and \((\Box, 1, \Box)\) representations (a similar construction was used in an earlier work by Banks et. al. \[13\]). We will take the \( SU(N) \) gauge coupling to be large while the two \( SU(M) \)'s are assumed weakly coupled and act on the fields in the following way:

\[
\bar{\psi}_+ \rightarrow \bar{\psi}_+ G^\dagger, \quad \psi_- \rightarrow G \psi_-, \quad \bar{\lambda}_- \rightarrow \bar{\lambda}_- H^\dagger, \quad \lambda_+ \rightarrow H \lambda_+
\]

where \( G \) and \( H \) denote the corresponding weak symmetry transformations and we have suppressed the variation of the fields under the strong symmetries. Notice that it is impossible to write down a single site mass term that preserves all the weak symmetries. The
usual staggered mass term
\[ \bar{\psi}_+(x) \lambda_+(x) + \bar{\lambda}_-(x) \psi_-(x) \] (2.5)
is *not* invariant but instead transforms as a bifundamental under the weak groups. In the absence of such a mass term all the gauge symmetries can be made local by inserting appropriate gauge links between the \( \psi \) and \( \lambda \) fields on neighboring sites. The gauging of the lattice kinetic term is given explicitly as
\[
S_K = \sum_{x,\mu} \bar{\psi}_+(x) \left( U_\mu(x) V_\mu(x) \psi_-(x + \mu) - U_\mu^\dagger(x - \mu) V_\mu^\dagger(x + \mu) \phi_-(x) \right) + \sum_{x,\mu} \bar{\lambda}_-(x) \left( W_\mu(x) V_\mu(x) \lambda_+(x + \mu) - W_\mu^\dagger(x - \mu) V_\mu^\dagger(x + \mu) \phi_-(x) \right),
\] (2.6)
where \( V_\mu(x) \) is the lattice gauge field for the strong interactions and \( U_\mu(x) \) and \( W_\mu(x) \) correspond to the weak gauge groups.

While there are no single site fermion bilinears that are gauge invariant it is nonetheless possible to write down a gauge invariant lattice four fermion term:
\[
\sum_\mu \phi_+^\dagger(x) U_\mu(x) W_\mu^\dagger(x) \phi_-(x),
\] (2.7)
where the composite Higgs field \( \phi(x) \) (a strong interaction singlet) is given by
\[
\phi_+^\dagger(x) = \bar{\psi}_+(x) \lambda_+(x), \quad \phi_-(x) = \bar{\lambda}_-(x) \psi_-(x)
\] (2.8)
If a condensate of \( \phi^\dagger \phi \) develops it will signal the appearance of a lattice phase in which a subset of the weak interactions are Higgsed. This subset can be determined in the limit where the weak coupling is set to zero; in this case one expects the strong interaction to break the weak symmetries down to their diagonal subgroup
\[
SU_D(M) = \text{diag}(SU(M) \times SU(M))
\] (2.9)
The Higgsed gauge fields will thus correspond to the broken generators
\[
\tau_a = \tau^1_a - \tau^2_a
\] (2.10)
with \( \tau^1 \) and \( \tau^2 \) the generators of the two weak groups respectively.

In order to see such a Higgs phase, we need to add a small perturbation to the above action via an auxiliary field \( \phi \).
\[
\delta S = g \sum_x \left( \phi(x) \bar{\psi}_+(x) \lambda_+(x) + \phi^\dagger(x) \bar{\lambda}_-(x) \psi_-(x) \right)
\] (2.11)
This field \( \phi(x) \) is a local field which is needed to couple the two reduced fermions together and must transform under the gauge symmetry as a bifundamental in the weak groups in order that the perturbation is gauge invariant. To render the path integral well defined after integration over \( \phi(x) \) one must then also add a suitable action for \( \phi(x) \). We choose an additional simple term \( \sum_x \phi^\dagger(x) \phi(x) \). The effect of these Yukawa terms is to add a small gauge invariant four fermion interaction to the action that favors the conjectured
symmetry breaking pattern. Spontaneous breaking of the global weak symmetries occurs iff the fermion bilinear takes a vacuum expectation value in the thermodynamic limit as the auxiliary coupling is sent to zero.

Finally to build the entire model requires the addition of Wilson plaquette terms for the three independent lattice gauge fields.

3. Strong dynamics and chiral symmetry breaking

In the rest of the paper will consider the case where the weak symmetry groups are $SU(2)$. Let us first examine the case where the weak gauge coupling is set to zero and the weak symmetry is purely global. Since each reduced staggered field contributes 2 Dirac fermions in the continuum limit the theory will contain eight Dirac fields with global symmetry $SU_V(8) \times SU_A(8)$. As a result of strong interactions we expect that this symmetry will break according to the pattern

$$SU_V(8) \times SU_A(8) \rightarrow SU_V(8).$$

(3.1)

The $SU(2) \times SU(2)$ weak symmetries of the staggered lattice theory constitute a subgroup of this continuum global symmetry. The precise embedding of these weak symmetries in the continuum limit can be obtained by the following argument. We start by assuming that each staggered field yields 2 Dirac fermions in the continuum eg. $\psi_1 \rightarrow \{\Psi_1^1, \Psi_1^2\}$ with the upper indices reflecting this extra factor of two. We can then arrange these continuum fields into an eight component vector

$$\left( \Psi_1^1 \Lambda_1^1 \Psi_1^1 \Lambda_2^1 \Psi_1^2 \Lambda_1^2 \Psi_1^2 \Lambda_2^2 \right)$$

(3.2)

In this representation the broken generators $\tau_a = \tau_a^1 - \tau_a^2$, $a = 1 \ldots 3$ take the explicit form

$$\tau_1 = \begin{pmatrix} 0 & \sigma_3 \\ \sigma_3 & 0 \end{pmatrix} \times I_2 \quad \tau_2 = \begin{pmatrix} 0 & i\sigma_3 \\ -i\sigma_3 & 0 \end{pmatrix} \times I_2 \quad \tau_3 = \begin{pmatrix} \sigma_3 & 0 \\ 0 & -\sigma_3 \end{pmatrix} \times I_2$$

(3.3)

where the two dimensional unit matrix $I_2$ represents this extra factor of two degeneracy associated to the upper indices of $\Psi, \Lambda$. We will neglect the $I_2$ factor in what follows since it enters trivially in our analysis. On the lattice the usual single site fermion condensate takes the form

$$\sum_{a=1}^2 \bar{\psi}_+^a \Lambda_+^a + \bar{\Lambda}_-^a \psi_-^a$$

(3.4)

corresponding to two independent staggered fermion condensates. This clearly breaks the lattice $SU(2) \times SU(2)$ symmetry down to its diagonal subgroup. We then expect that this breaking pattern corresponds to a continuum condensate of the form

$$\Sigma = \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_1 \end{pmatrix} \times I_2$$

(3.5)
Standard universal arguments tell us that it should be possible to change basis for our fermion fields to force this condensate to take the canonical flavor symmetric form $\Sigma = I_8$. To accomplish this first diagonalize $M \to P^\dagger MP$ using the unitary transformation

$$P = \begin{pmatrix} \frac{1}{\sqrt{2}} (\sigma_3 + \sigma_1) & 0 \\ 0 & \frac{1}{\sqrt{2}} (\sigma_3 - \sigma_1) \end{pmatrix}$$

(3.6)

This results in a condensate of the form

$$\Sigma' = \begin{pmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix}$$

(3.7)

To transform this to the unit matrix we employ the non-anomalous chiral transformation $M' \to QM'Q$ with

$$Q = \begin{pmatrix} D & 0 \\ 0 & D^\dagger \end{pmatrix}$$

(3.8)

where the $2 \times 2$ matrix

$$D = \begin{pmatrix} 1 & 0 \\ 0 & i\gamma_5 \end{pmatrix}$$

(3.9)

To find the explicit form of the broken generators in this new basis we transform them according to the rule $Q^\dagger P^\dagger \tau_a PQ$. It is straightforward to show that the broken generators acquire an axial character in the new basis (we have reinserted the $I_2$ at this point)

$$\tau'_1 = \gamma_5 \begin{pmatrix} 0 & -i\sigma_1 \\ i\sigma_1 & 0 \end{pmatrix} \times I_2 \quad \tau'_2 = \gamma_5 \begin{pmatrix} 0 & \sigma_3 \\ \sigma_3 & 0 \end{pmatrix} \times I_2 \quad \tau'_3 = \gamma_5 \begin{pmatrix} \sigma_2 & 0 \\ 0 & -\sigma_2 \end{pmatrix} \times I_2$$

(3.10)

This confirms our earlier arguments that the broken generators do indeed correspond to axial symmetries in the continuum limit. We see that the staggered fermion action we use picks out a particular breaking direction corresponding to a specific embedding of the weak symmetries into the global symmetry group. The fact that the broken symmetries are axial in the continuum limit is a necessary condition, according to the Vafa-Witten theorem, for this broken phase of the lattice theory to survive the continuum limit.

One should contrast this with the situation in the continuum where the canonical chiral symmetry breaking pattern given in eqn. 3.1 gives rise to an infinite number of degenerate vacua related by an $SU(8)$ transformation. Depending on the vacuum that is picked any given $SU(2)$ subgroup may be broken or left intact by the corresponding vacuum condensate.

4. Vacuum alignment and the Higgs phase

We now examine the situation when the weak gauge coupling is switched on. In the continuum this breaks the degeneracy of the strong interaction vacua and, in principle, the system adopts a unique vacuum state determined by the weak interactions. The question of which strong interaction vacuum now yields the true lowest energy state in the presence
of the weak interactions is termed the vacuum alignment problem. We thank Maarten Golterman and Yigal Shamir for pointing this out to us and for sharing their notes [11]. The usual folklore is that in a vector-like continuum theory the system will adopt a vacuum in which the weak gauge interactions are left unbroken [12, 13].

In our lattice model the situation is less clear; the vacuum state is unique at non-zero lattice spacing and corresponds to a broken weak $SU(2)$ symmetry. We expect for non zero lattice spacing that this vacuum is not disturbed for sufficiently weak $SU(2)$ gauge coupling and hence the lattice theory should be found in a Higgs phase. The question of what happens in the continuum limit is then somewhat subtle; the weak interactions presumably favor a vacuum in which $SU(2)$ is left unbroken but this will cost an energy arising from strong interactions. For example, a condensate consisting of four 1-link condensates can be constructed in the lattice theory which will not break $SU(2)$. In the continuum limit universality arguments would suggest that it has the same energy as the single site condensate. However to convert the single site condensate favored by the strong interactions at non zero lattice spacing into a single link condensate requires a microscopic rearrangement of lattice degrees of freedom which will require that the system pass through intermediate configurations whose energy is higher. This may render the $SU(2)$ broken vacuum stable after the weak coupling constant is switched on.

5. Numerical results

To cement these conclusions we have simulated the model for the case of an $SU(3) \times SU(2) \times SU(2)$ gauge symmetry using the RHMC algorithm and working in the phase quenched approximation. We use an auxiliary scalar field which is a bi-fundamental in the two weak groups.

As a first step, we analyzed the behavior of the gauge invariant four fermion term given by

$$\sum_{\mu} g <\bar{\psi}_+(x)\lambda_+(x) > U_\mu(x)W_\mu^+(x) <\bar{\lambda}_-(x + \mu)\psi_-(x + \mu) >.$$ (5.1)

This four fermion condensate, while strictly not an order parameter, is expected to be enhanced in the Higgs phase and this is indeed what we see in fig. 1. Clearly for $\beta_S > 6.0$ the condensate is small and approaches zero while for $\beta_S < 6.0$ it grows and approaches a plateau within errors (we fix the weak coupling $\beta_W = 10.0$ throughout the numerical work). Notice that the plot in fig. 1 was constructed with an auxiliary coupling $g = 0.1$. It is also very important to see how the condensate behaves for fixed strong coupling $\beta_S$ as we vary $g$. Fig. 2 shows a plot at $\beta_S = 5.5$ as we scan in $g$. Clearly the value of the condensate is insensitive to the auxiliary coupling until $g$ becomes very small. We conclude that the magnitude of the four fermion condensate is determined primarily by the value of the strong coupling constant as expected. Further evidence that the lattice is indeed in a Higgs phase can be derived by plotting both the strong and weak Polyakov lines as a function of the strong coupling constant (again we fix the auxiliary Yukawa coupling $g = 0.1$). The

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1. The reduced staggered theory has a sign problem for non zero lattice spacing. This allows a condensate that breaks vector symmetries to exist away from the continuum limit.
results are shown in fig. 3. The behavior of the strong Polyakov line corresponds to what is expected for a system with a confining, chirally broken phase for small $\beta_S$ (strong coupling) and a deconfined phase at large $\beta_S$. The weak Polyakov line is more interesting; for large $\beta_S$ it is large consistent with a deconfined phase as would be expected for such a large value of the bare inverse coupling $\beta_W = 10.0$ on a small volume. However, at the point that the strong interactions confine, this behavior changes dramatically and the weak Polyakov line falls to much smaller values. Since the Polyakov line measures the energy of an isolated fermion in the fundamental representation of the group a change in this value indicates a change in the energy of such a weakly charged fermion. This we interpret as evidence of a Higgs phase.

Clearly these numerical results have only been obtained for very small lattice volumes but are certainly consistent with our earlier theoretical arguments for the appearance of a Higgs phase. To understand whether this Higgs phase survives the continuum limit will require a careful analysis on much larger lattices than we have attempted in this paper.

6. Discussion

We have shown how a model based on reduced staggered fermions can be constructed that spontaneously breaks an exact global $SU(2)$ symmetry as a result of strong dynamics. Remarkably, while the $SU(2)$ enters the theory as a vector symmetry acting on the
staggered lattice fields, we show that it should be interpreted in the continuum limit as an axial symmetry of Dirac fermions. According to the Vafa Witten theorem this is a necessary condition for this broken phase to survive the continuum limit. Furthermore since the continuum theory is vector-like this weak SU(2) symmetry may then be gauged without generating anomalies [14, 15] and one would expect the broken symmetry phase to reappear as a Higgs phase. In support of these theoretical arguments we see numerical evidence of an enhancement in the four fermion condensate and a dramatic change in the weak Polyakov line once the strong sector confines and breaks chiral symmetry.

This model constitutes an example of a lattice theory which implements a full technicolor dynamics; it includes both a strong sector that drives chiral symmetry breaking and a complete non-perturbative treatment of the weak symmetries. We have argued that the resultant Higgs phase may survive the continuum limit. We stress that the correct gauge invariant description of the dynamical Higgs mechanism in this context is that it proceeds via condensation of an entirely gauge invariant four fermion operator built from strong interaction singlets. To the best of our knowledge, this is the first lattice theory that has been constructed with these properties.

The observation that lattice vector symmetries can yield broken continuum axial symmetries in the presence of a non-perturbative condensate is intriguing and deserves further study. It can be seen in a more familiar example; staggered quark formulations of QCD. The

\[^2\text{see [16] for a similar conclusion in a different model}\]
Figure 3: Absolute value of the strong (circles) and weak (crosses) Polyakov lines vs $\beta_S$ with $g = 0.1$

massless staggered theory possesses a $U(1) \times U(1)$ symmetry corresponding to independent vector $U(1)$ rotations of the two reduced fields that make up the complete staggered field. The usual single site condensate that forms will break this to the diagonal $U(1)$. Following the same arguments as used in section 3 one can show that this broken $U(1)$ generator acquires a factor of $\gamma_5$ in the continuum limit where it can be interpreted as giving rise to an axial symmetry. Indeed this broken $U(1)$ is nothing more than the usual $U(1)_A$ of staggered fermions. In this case however the lattice fermion measure is not invariant under a local $U(1)$ transformation and so this symmetry cannot be gauged. This is consistent with the existence of the continuum axial anomaly.

Clearly it is important to study this model further on larger lattices so that the continuum limit can be monitored more carefully. We hope to report on such a study in the future.

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