Non-equilibrium dynamics in Ising like models with biased initial condition

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We investigate the dynamical fixed points of the zero temperature Glauber dynamics in Ising-like models. The stability analysis of the fixed points in the mean field calculation shows the existence of an exponent that depends on the coordination number \( z \) in the Ising model. For the generalised voter model, a phase diagram is obtained based on this study. Numerical results for the Ising model for both the mean field case and short ranged models on lattices with different values of \( z \) are also obtained. A related study is the behaviour of the exit probability \( E(x_0) \), defined as the probability that a configuration ends up with all spins up starting with \( x_0 \) fraction of up spins. An interesting result is \( E(x_0) = x_0 \) in the mean field approximation when \( z = 2 \), which is consistent with the conserved magnetisation in the system. For larger values of \( z \), \( E(x_0) \) shows the usual finite size dependent non linear behaviour both in the mean field model and in Ising model with nearest neighbour interaction on different two dimensional lattices. For such a behaviour, a data collapse of \( E(x_0) \) is obtained using \( y = \frac{1-x_0}{L^1} \) as the scaling variable and \( f(y) = \frac{1}{2/(1+y)} \) appears as the scaling function. The universality of the exponent and the scaling factor is investigated.

\section{I. INTRODUCTION}

Non-equilibrium dynamics associated with spin systems quenched from a high temperature have been extensively studied in the past. Various features associated with the ordering dynamics have been explored for the Ising model defined by the Hamiltonian \( H = -J \sum_i \sigma_i \sigma_j \) (\( \sigma_i = \pm 1 \)). Classical spin models have no intrinsic dynamics, however, one can study the stochastic time evolution using certain dynamical algorithms that maintain the detailed balance. Glauber dynamics is one of the popular choices that reduces to a simple energy lowering scheme at zero temperature. To study the ordering process, the system is taken to be completely disordered (i.e., at a high temperature) initially and suddenly cooled to a lower temperature \( T \); we consider \( T = 0 \) specifically in this paper. In finite systems, the one dimensional Ising-Glauber model, following such a zero temperature quench, always ends up with all spins up or down irrespective of the initial fraction of up spin \( x_0 \). In higher dimensions, striped and blinkers state can also be reached when the initial state is completely disordered, i.e., \( x_0 = 0.5 \). On the other hand, there are a fairly large number of models which use Ising spins but without any energy function associated with it, for example the Voter model. In such models, the system evolves by a given dynamical rule.

Various features in the ordering process, for example domain growth, persistence, aging, time evolution of the order parameter and other relevant quantities have been studied for quite some time, particularly in the spin models. Exit probability is another feature associated with the non-equilibrium dynamics that has received a fair amount of attention more recently. The exit probability \( E(x_0) \) is defined as the probability that a all-up configuration is reached starting from \( x_0 \) fraction of up spins. \( E(x_0) \) is linear in the one dimensional Ising model and the voter model (in all dimensions): \( E(x_0) = x_0 \); this occurs due to the conservation of the order parameter. In contrast, in the two dimensional Ising system \( E(x_0) \) is non-linear and shows strong finite size effects. The exit probability as well as the dynamics have also been studied in the recent past for binary opinion dynamics models using mean field and several other analyses on hypergraphs, networks and simple square lattices.

In this paper, we have considered the dynamics of Ising and Ising like models where the evolution of the fraction of up spins (\( x \)) is studied. A mean field approach leads to the identification of the fixed points. We note that a non-trivial fixed point is \( x = 0.5 \) which corresponds to a disordered state. The stability of this fixed point is studied by starting from a biased but uncorrelated initial condition where the initial fraction \( x_0 \) deviates from 0.5. The results for the mean field Ising model, obtained for different values of \( z \), the coordination number, are compared with the short range model in finite dimensions. The fixed points for the generalised voter model (GVM) are also obtained parametrically and the stability analysis leads to obtaining the mean field phase diagram in the two parameter plane.

The evolution of \( x \) in time helps in understanding the behaviour of the exit probability. The exit probability is computed numerically for the Ising-Glauber model in square, triangular lattices and the mean field Ising model. The results for finite sizes show the existence of a scaling function with which two parameters can be associated, as noted in some earlier studies.

\section{II. MEAN FIELD CALCULATION IN THE ISING MODEL}

\subsection{A. Master equation approach}

We first consider the Ising model in the mean field approximation. The master equation for the variable \( x(t) \), the fraction of up spins at time \( t \) is set up after calculating the spin flip probabilities. The system evolves...
under the zero temperature Glauber dynamics, i.e., spins are flipped when energy decreases by it and flipped with probability $1/2$ when energy does not change by flipping. For a particular spin, a neighbouring spin here is simply another spin with which it interacts and the number of such neighbours or the coordination number $z$ is taken as a variable.

1. $z=2$

We first consider the case $z=2$. Suppressing the argument $t$ for $x$, an up spin flips with probability (i) $(1-x)^2$, when it has two neighbouring down spin (ii)$2x(1-x)/2$ when it has 1 down neighbour and 1 up neighbour. This can happen in two ways and for each of the cases the spin flips with probability $1/2$.

Denoting $P_+$ ($P_-$) as the total probability that a up (down) spin flips, one can therefore write,

$$P_+ = (1-x)^2 + x(1-x)$$
$$P_- = x^2 + x(1-x)$$

The evolution equation for $x(t)$ can be expressed in general as

$$\frac{dx}{dt} = (1-x)P_+ - xP_-,$$

which reduces to $\frac{dx}{dt} = 0$ using eq. (1). This implies $x(t) = x_0$, i.e., the dynamics conserve the magnetisation $m(t) = 2x(t) - 1$ such that $E(x_0) = x_0$ in this case obviously.

2. $z=4$

We next consider the case for $z=4$. In this case, an up spin flips with probability (i) $(1-x)^3$ if it has 4 neighbouring down spins, (ii) $4x(1-x)^3$ when it has 3 down neighbours and 1 up neighbour which can happen in 4 ways, (iii) $6x^2(1-x)^2/2$ in case of 2 up and 2 down neighbours (possible in 6 ways and the spin flips with probability 1/2 in each case).

Therefore,

$$P_+ = (1-x)^3 + 4x(1-x)^3 + 3x^2(1-x)^2$$
$$P_- = x^4 + 4x^3(1-x) + 3x^2(1-x)^2.$$  

At the steady state, we obtain from the master equation, 

$$\frac{dx}{dt} = -2x^3 + 3x^2 - x = 0,$$

with the solutions $x^* = 0, 0.5, 1$.

To check the stability of the solutions, we consider $x = x^* + \delta(t)$ where $\delta(t)$ is the deviation from the fixed point.

For both $x^* = 0$ and 1, considering only up to linear order terms in $\delta$, we get, $\frac{d\delta}{dt} = -\delta$, the solution of which is

$$\delta(t) = \delta_0 \exp(-t).$$

where $\delta_0 = \delta(0)$. The negative exponent implies that $x^* = 0$ and 1 are stable fixed points.

For $x^* = 0.5$, one gets

$$\delta(t) = \delta_0 \exp\left(\frac{1}{2}t\right).$$

The positive exponent here implies that $x^* = 0.5$ is an unstable fixed point. Of course, $\delta$ cannot increase indefinitely and its extreme values are $\pm 0.5$. The stability analysis thus shows that the system ends up with all spins up/down (for $\delta_0$ positive/negative). The magnetisation $m(t) = 2\delta(t)$ here.

![Flow diagram for the mean field Ising model for $z > 2$.](image)

FIG. 1: (a) Flow diagram for the mean field Ising model for $z > 2$. $x = 0$ and 1 are the stable fixed points and $x = 0.5$ is unstable. (b) shows the flow diagram for the generalised voter model. $x_1 = 0.5$ is the unstable fixed point and $x_2 = \frac{1}{2} + \frac{1}{2} \sqrt{\frac{2x_1 + x_2 - 2}{2x_1 - 2}}$ and $x_3 = \frac{1}{2} - \frac{1}{2} \sqrt{\frac{2x_1 + x_2 - 2}{2x_1 - 2}}$ are stable fixed points provided that the exponent is positive.

3. $z=6$

A similar analysis is done for $z = 6$. Here, $P_+$ and $P_-$ can be expressed as

$$P_+ = (1-x)^6 + 6x(1-x)^5 + 15x^2(1-x)^4 + 10x^3(1-x)^3$$
$$P_- = x^6 + 6x^5(1-x) + 15x^4(1-x)^2 + 10x^3(1-x)^3$$

and therefore the master equation can be written as

$$\frac{dx}{dt} = 6x^5 - 15x^4 + 10x^3 - x.$$  

Solving the steady state equation $\frac{dx}{dt} = 0$, one gets, $x^* = 0, \frac{3 + \sqrt{7}}{2}, \frac{3 - \sqrt{7}}{2}, 1$. The third and fourth solution being unphysical, $x^* = 0, 0.5$ and 1 are the only physical solutions. Considering $x = x^* + \delta(t)$ in Eq. (8), the solution becomes $\delta(t) = \delta_0 \exp(-t)$ for $x^* = 0$ and 1 which are stable fixed points. For $x^* = 0.5$, one obtains

$$\delta(t) = \delta_0 \exp\left(\frac{7}{8}t\right).$$
which shows that \( x^* = 0.5 \) is again an unstable fixed point. The larger value of the exponent in the \( z = 6 \) case indicates that the dynamics are faster for \( z = 6 \) compared to that in the \( z = 4 \) case. Fig. 1 shows the flow diagram for the mean field Ising model for \( z = 4 \) and \( z = 6 \).

From the above studies, we conclude that in the mean field approximation, in general \( \delta(t) = \delta_0 \exp(\gamma t) \) where \( \gamma \) increases with \( z \). This behaviour is a short time one as the system reaches the stable fixed points at long times, confirmed by the simulation results discussed in the next subsection. Hence \( \gamma \) can be interpreted as an inverse time scale over which the exponential growth of \( \delta(t) \) can be observed.

### B. Simulation results

To check the mean field results we have conducted simulations where a spin interacts with randomly chosen \( z \) neighbours. The system consists of \( N \) spins and the choice of the random neighbour is made in an annealed manner which implies the interaction can take place with any different spins at each step in general.

We defer the discussion on the \( z = 2 \) case to section [IV] and consider the cases \( z = 4 \) and 6 where we expect an unstable point at \( x = 0.5 \). We have started from a fixed initial fraction of up spin \( x_0 = 0.5 + \delta_0 \) with \( \delta_0 > 0 \) and studied how the fraction \( \delta(t) = x(t) - 0.5 \), evolves in time. \( N \) updates constitute one single Monte Carlo step. Here, we have considered only those configurations for which positive consensus is attained to obtain the exponent \( \gamma \) and compare with the result found in the analytical calculation.

\( \delta(t) \) shows an exponential growth with time which shows consistency with the results of section [IIA] as \( N \) is increased. The results for \( z = 4 \) shown in Fig. 2 indicate the exponential growth becomes more noticeable as \( N \) increases and that the associated exponent \( \approx 0.5 \) is independent of \( \delta_0 \) for all practical purposes. A data collapse for different values of \( \delta_0 \) is obtained by scaling \( \delta(t) \) by \( \delta_0 \), shown in Fig. 3 which is also consistent with the analytical results.

A similar estimation has been done for \( z = 6 \) by considering the interaction of the selected spin with randomly chosen 6 neighbours. \( \delta(t) \) shows an exponential behaviour with time again and the exponent is \( \approx 0.86 \). Fig. 4 shows the results. The exponent \( 7/8 \) obtained in section [IIA] for \( z = 6 \) agrees fairly well with the simulation results.

It should be mentioned here that the saturation is obtained very rapidly, the exponential fitting is therefore valid only for a few initial time steps. The saturation is enhanced for larger values of \( \delta_0 \) and \( z \).

### FIG. 2: Variation of \( \delta(t) \) are shown with time for \( z = 4 \) for different \( \delta_0 \) where mean field approach is used. This plot also shows the data for several system sizes. Data are fitted to the exponential function, mentioned in the key. Maximum number of configuration was 5000. \( \delta(t) \) attains the saturation value faster for larger \( \delta_0 \) and the process is slower for smaller value of \( \delta_0 \).

### FIG. 3: Data collapse of \( \delta(t) \) are shown with time for several \( \delta_0 \) where \( z = 4 \) and \( z = 6 \) and data are fitted to an exponential form as mentioned in the key. These data are for a system of 2\(^{16} \) spins averaged over a maximum of 5000 realisations. Inset shows the unscaled data. In this simulation mean field approach is used.

### III. SIMULATIONS FOR SHORT RANGE MODELS ON LATTICES

The simulations for the Ising model are repeated on two dimensional lattices where the spins have short range interactions. We consider the vicinity of the unstable fixed point again, such that \( x(0) = 0.5 + \delta(0) \) and study the evolution of \( \delta(t) \) where \( x(t) = 0.5 + \delta(t) \).

In order to check the dependence on \( z \), we have considered square lattices with nearest and nearest plus next nearest neighbours and triangular lattices with nearest neighbours such that \( z = 4, 8 \) and 6 respectively.

It is well known that the absolute value of the magnetisation grows as \( t^{\beta} \) in the ordering process of the Ising model in all finite dimensions when the initial configura-
The inset of Fig. 5 for $z = 6$. Data for several values of $\delta_0$ are also shown. Inset shows the data collapse for different $\delta_0$ for a system of $2^{16}$ spins. As $\delta_0$ increases, $\delta(t)$ rapidly saturates. Minimum Number of different initial configuration was 2000 and mean field approximation is used for this simulation.

It is observed that for any value of $z$ and $\delta_0$, $\delta(t)$ shows a power law behaviour with time before reaching the saturation value for all values of $\delta_0$:

$$\delta(t) \sim t^\beta.$$  

The results for $z = 4$ are shown in Fig. 5. The value of $\beta$ depends on $\delta_0$, as $\delta_0$ increases (which means the system is more ordered to begin with), it decreases as shown in the inset of Fig. 5 for $z = 4$. This is understandable, in the limit $\delta_0 \to 0.5$, the system is almost static such that the time dependence is weak reflected by a smaller value of $\beta$.

In the triangular lattice, where $z = 6$, $\delta(t)$ is also found to show a power law variation with time according to Eq. 10. As $\delta_0$ increases, $\beta$ decreases as indicated by the data presented in Fig. 6. The values of $\beta$ are reasonably close to those obtained in the square lattice.

We also consider the the Ising model with a Moore neighbourhood where next nearest neighbour interactions are included and $z = 8$. The Hamiltonian of this system is given by

$$H = -J_1 \sum_{<i,j>} S_i S_j - J_2 \sum_{<i,j>'} S_i S_j,$$  

where $J_1$ and $J_2$ are the strengths of interaction for nearest neighbour and next nearest neighbour respectively. We have considered the interactions to be equal in strength, $J_1 = J_2$. Here, $z = 8$ and once again we find a behaviour similar to $z = 4, 6$ in two dimensions (see Fig. 7).

It is also interesting to check whether for the same value of $z$ but in a different dimension, the value of $\beta$ remains the same. For this, simulations have been conducted on cubic lattice Ising system where $z = 6$ as in
the triangular lattice. $\delta(t)$ shows a power law variation in this case also; however, the exponent $\beta$ is larger compared to the two dimensional case (see Fig. 8).

The above results show that the exponent $\beta$ is independent of $z$ in two dimensions while for three dimensions, with the same $z$ we find a different value of $\beta$ when $\delta_0 \neq 0$.

IV. EXIT PROBABILITY

In this section, we present the results for $E(x_0)$, the probability that the system ends up in a state with all up spins, starting from an initial state with $x_0$ fraction of up spins. Since some results are already known for the short range Ising models, we first discuss that and then continue to report the results for the mean field case.

A. Results for nearest neighbour interactions

Next, we have studied the exit probability for the two dimensional nearest neighbour Ising models. Exit probability $E(x_0)$ is known to have a linear behaviour $E(x_0) = x_0$ for one dimensional Ising Glauber model. In two dimensional model $E(x_0)$ is non linear and shows strong finite size effects [18, 20]. As the system size increases the curves become steeper and the data suggest that $E(x_0)$ approaches a step function in the thermodynamic limit. Finite size scaling can be done using the form

$$E(x_0, L) = f\left[\frac{x_0 - x_c}{x_c} L^{\frac{2}{\nu}}\right]$$

as observed in [12], where $f(y) \to 0$ for $y \ll 0$ and is equal to 1 for $y \gg 0$. Therefore, a data collapse for different system size can be obtained when $E(x_0)$ is plotted against $\frac{x_0 - x_c}{x_c} L^{\frac{2}{\nu}}$ where $x_c = 0.5$. On square lattices, an Ising system freezes into a striped configuration for $x_0 = 0.5$ in 33.9 percent cases (an exact result [3]) in the thermodynamic limit. Numerical simulations show that the freezing probability has strong system size dependence [3, 4]. However, the dynamical scaling behaviour remains intact in spite of the freezing. Very close to $x_0 = 0.5$, such frozen striped states may occur with a nonzero probability in finite systems as shown in [4]. While calculating $E(x_0)$, such configurations have been discarded.

The data collapse is obtained using eye estimation for square lattice Ising model when $\nu \approx 1.3$ agreeing with the result of [14, 20]. The collapsed data can be fitted to the form

$$f(y) = \frac{1 + \tanh(\lambda y)}{2}$$

as in [13]. The value of $\lambda$ turns out to be 1.10 using GNUFIT.

To get a more accurate value of $\nu$ required for obtaining best data collapse, we have employed another method used previously in [20]. We have calculated $y = x_0 - x_c L^{1/\nu}$ for the different values of $L$. As the data collapse are supposed to fit to the form of Eq. (13) we have chosen the range of $\nu$ and $\lambda$ for which the collapse and fitting seem good. We have varied the values of $\nu$ and $\lambda$ in steps of 0.001 and for every pair we have calculate the error $\epsilon$ given by

$$\epsilon = \frac{1}{n} \sqrt{\frac{1}{n} \sum_{n} [f(y) - E(x_0)]^2}$$

The pair of $\nu$ and $\lambda$ for which the minimum value of $\epsilon$ is obtained are the optimal values required for best data collapse and scaling function. The values of $\nu$ and $\lambda$ are 1.307 and 1.111 using this method and the results are shown in Fig. 9.
These results were already available from previous studies, although for smaller system sizes. We repeat these simulations as our aim is to determine if any universality in the scaling behaviour exists in two dimensional Ising systems. Hence we have studied the exit probability in a triangular lattice (number of nearest neighbours $z = 6$). To obtain the best data collapse the least square method has been employed in this case also and is graphically illustrated in Fig. 10. The data collapse of $E(x_0)$ for different system sizes is obtained with $\nu = 1.204$ using the above method and the scaled data is fitted according to Eq. (13). Fig. 11 shows the results. The value of $\nu$ is close but not exactly equal to the value obtained for square lattice. $\lambda = 0.857$ is definitely different.

Here we present the results for the exit probability $E(x_0)$ using mean field approach where the $z$ neighbours are chosen randomly.

For $z = 2$, the exit probability shows a linear behaviour $E(x_0) = x_0$ (see Fig. 12). This is in consistency with the conservation we noted for $x$ in section II A. It may seem surprising that the mean field result with $z = 2$ gives the exact result known for the one dimensional Ising model. We attempt to justify why this happens in the following way. We note that for the voter model, the $i$th spin $\sigma_i$ flips with a probability

$$w(\sigma_i) = \frac{1}{2} [1 - \sigma_i \sum_j \sigma_j / z]$$

(15)

where $j$ is a neighbour of $i$. This probability is valid in any dimension. Thus the above dynamics in the voter model conserve the total spin in any dimension. It is well known that in one dimension, the voter model dynamics coincide with the Ising dynamics where $z = 2$. In the mean field calculations for $z = 2$ it is evident that the voter model dynamics are being used precisely and since in the latter, conservation is valid always, we get a result which is the exact one for the one dimensional Ising model too. It is interesting to note that hence for the $z = 2$ case, it does not matter whether one picks up randomly any two neighbours or strictly the nearest neighbours as far as conservation is concerned. We have also checked that if the choice of neighbors is done randomly in a quenched manner, the results remain the same.

For other values of $z$, $E(x_0)$ becomes steeper in the mean field case than that was obtained using nearest neighbour interactions. Here, a data collapse is obtained by plotting $E(x_0)$ against $\frac{x_0 - x_c}{L^{1/\nu'}}$ (where $N$ is the total number of spins) using $\nu' = 2$ for both $z = 4$ and $z = 6$. The scaled data are fitted to the form of Eq.
where $y = \frac{x_0 - x}{z}$, and the value of $\lambda$ obtained for $z = 4$ is approximately 0.48 and $\lambda \approx 0.58$ when $z = 6$. Data collapse of $E(x_0)$ is shown in Fig. 13 for $z = 4$ and $z = 6$.

Table I shows the values of $\nu$, $\nu'$ and $\lambda$ obtained numerically for the short ranged and mean field Ising models. Note that for the short ranged systems, $\nu' = 2\nu$. These results are discussed in the last section.

![Fig. 12: $E(x_0)$ is shown against $x_0$ for $z = 2$ where mean field approach is used. $E(x_0)$ shows a linear variation with $x_0$. Number of different initial configuration was 5000 for these simulations.](image)

![Fig. 13: (a) Data collapse of $E(x_0)$ is shown for different system sizes using $\nu' = 2$ where $z = 4$. (b) shows the data collapse of $E(x_0)$ for different system sizes with $\nu' = 2$ for $z = 6$. Mean field approximation is used in these simulations where number of different initial configuration was 5000. Data are fitted to the functions as mentioned in the key.](image)

### V. GENERALISED VOTER MODEL (GVM)

We have considered next the generalised voter model. We first describe the model on a square lattice where a spin variable $\sigma_i = \pm 1$ is associated with every site of the lattice. The time evolution is governed by single spin flip stochastic dynamics; the spin flip probability $w_i(\sigma)$ for the $i$-th spin is given by

\[
    w_i(\sigma) = \frac{1}{2}[1 - \sigma_i f_i(\sigma)], \tag{16}
\]

where $f_i(\sigma) = f(\sum_\delta \sigma_{i+\delta})$, a function of the sum of the nearest neighbor spin variables. The model is defined taking $f(0) = 0$, $f(2) = -f(-2) = z_1$ and $f(4) = -f(-4) = z_2$, where $z_1$ and $z_2$ are restricted to $z_1 \leq 1$ and $z_2 \leq 1$. The original voter model corresponds to $z_1 = 0.5$ and $z_2 = 1$ and Ising model is recovered for $z_1 = 1$, $z_2 = 1$.

In the mean field approximation to obtain the master equation, the above dynamical rule is followed which means that $z = 4$ is taken and the parameters defined as above. For an up spin, flipping probabilities $P_i$ ($i = 0$ to 4), when there are $i$ neighboring spins in the up state are given by

\[
    P_0 = \frac{1}{2}(1 + z_2)(1 - x)^4, \\
    P_1 = \frac{1}{2}(1 - z_2)x^4, \\
    P_2 = 2(1 - z_1)x^3(1 - x), \\
    P_3 = 3x^2(1 - x)^2, \\
    P_4 = 2(1 + z_1)x(1 - x)^3, \\
    P_5 = \frac{1}{2}(1 + z_2)(1 - x)^4. \tag{17}
\]

The total probability $P_+$ that an up spin flips is $P_+ = \sum_{i=0}^{4} P_i$ such that

\[
    P_+ = \frac{1}{2}(1 - z_2)x^4 + 2(1 - z_1)x^3(1 - x) + 3x^2(1 - x)^2 + 2(1 + z_1)x(1 - x)^3 + \frac{1}{2}(1 + z_2)(1 - x)^4. \tag{18}
\]

Similarly the probability $P_-$ that a down spin flips is

\[
    P_- = \frac{1}{2}(1 - z_2)(1 - x)^4 + 2(1 - z_1)x(1 - x)^3 + 3x^2(1 - x)^2 + 2(1 + z_1)x^3(1 - x) + \frac{1}{2}(1 + z_2)x^4. \tag{18}
\]

Therefore, the master equation $\frac{dx}{dt} = (1 - x)P_- - xP_+$ reduces to

\[
    \frac{dx}{dt} = x^3(-4z_1 + 2z_2) + x^2(6z_1 - 3z_2) + x(-2z_1 + 2z_2 - 1) + \frac{1}{2} - \frac{z_2}{2}. \tag{19}
\]

**TABLE I: $\nu$, $\nu'$ and $\lambda$ obtained for Ising model using numerical simulations.**

| Quantity | Nearest neighbour (NN) interaction | Mean field |
|----------|-----------------------------------|------------|
| $\nu$, $\nu'$ | $\nu = 1.307(1)$, $\nu' = 1.204(1)$ (NN models) | $\nu' \sim 2$, $\nu' \sim 2$ |
| $\lambda$ | $1.111(1)$ | $0.857(1)$ | $\sim 0.48 \sim 0.58$ |
Putting the values $z_1 = 0.5$ and $z_2 = 1$ in eq. [19] one gets $\frac{dx}{dt} = 0$, consistent with the voter model result that there is conservation in any dimension. On the other hand, by taking $z_1 = 1$ and $z_2 = 1$ in Eq. [19] Eq. [3] is recovered for the Ising model with $z = 4$.

For general values of $z_1$ and $z_2$, the steady state condition leads to three fixed points $x_1, x_2, x_3$ where $x_1 = 0.5, x_2 = 1 + \frac{1}{2} \sqrt{\frac{2z_1 + z_2 - 2}{2z_1 - z_2}}$, and $x_3 = \frac{1}{2} - \frac{1}{2} \sqrt{\frac{2z_1 + z_2 - 2}{2z_1 - z_2}}$. Now, let us take $x(t) = x_1 + \delta(t)$, i.e., the behaviour close to the fixed point $x_1 = 0.5$. Considering up to linear term in $\delta$ only, $\delta(t)$ is found to be

$$\delta(t) = \delta_0 \exp (z_1 + \frac{1}{2} z_2 - 1) t,$$  \hspace{1cm} (20)

where $\delta_0 = \delta(t = 0)$. The exponent is thus $z_1 + \frac{1}{2} z_2 - 1$.

We will now analyse the sign of the exponent and thus the stability of the fixed point $x_1 = 0.5$ which corresponds to a completely disordered state. Since the magnetisation $m$ is given by $2r - 1$, $|m| = (\frac{2z_1 + z_2 - 2}{2z_1 - z_2})^\frac{1}{2}$ for $x_2$ and $x_3$, $m$ can have non-zero values for $x_2$ and $x_3$, provided $\frac{2z_1 + z_2 - 2}{2z_1 - z_2} \geq 0$ and also we require $|m| \leq 1$. The first criterion is satisfied ($|m| > 0$) when either (i) $2z_1 + z_2 - 2 \geq 0$ and $2z_1 - z_2 > 0$ or when (ii) $2z_1 + z_2 - 2 \leq 0$ and $2z_1 - z_2 < 0$. We note in the first case the first condition implies the second and in the next case the second condition implies the first one. Hence, for $|m| > 0$, $2z_1 + z_2 - 2$ and $2z_1 - z_2$ can in principle be either both positive or both negative. However, the condition that $|m| \leq 1$ is violated for case (ii) since $z_1, z_2 \leq 2$ and hence $m = 0$ is the only possible solution when $2z_1 + z_2 - 2 < 0$. Thus the only condition for an ordered region to exist is that the quantity $2z_1 + z_2 - 2$ must be positive. This is consistent with the fact that the exponent (which is an identical expression in $z_1, z_2$), has to be positive to make the $x_1 = 0.5$ (i.e. $m = 0$) fixed point unstable. On the other hand, when it is stable, i.e., $2z_1 + z_2 - 2 < 0$, $m = 0$ is the only solution. Fig. 14 shows the flow diagram of the generalised voter model.

Hence the phase boundary between the ordered and disordered phases is given by $2z_1 + z_2 - 2 = 0$. We have plotted the phase diagram in Fig. 14 where the magnitude of the magnetisation is also indicated. Obviously, the mean field phase diagram shows a larger region that is ordered compared to the two dimensional case.

VI. SUMMARY AND CONCLUSIONS

We have studied the dynamics in zero temperature Ising-like systems with up/down symmetry with different coordination number $z$. Using mean field approximation, it is observed that the dynamics always lead to one unstable fixed point which corresponds to the disordered state for $z > 2$. This fixed point is precisely $x = 0.5$ where $x$ is the fraction of up spins. The stability of this fixed point has been considered by introducing a small deviation $\delta_0$ from 0.5 in $x$. This essentially means we have a biased initial condition in the system with unequal fractions of up and down spins. The initial bias is generally considered to be small such that the system does not have any appreciable correlation.

For the unstable fixed points we obtain an initial exponential growth of $\delta(t)$ with time which strongly depends on the coordination number $z$. The growth is characterised by an exponent $\gamma$ that increases with $z$. These results have been checked by numerical simulations for the mean field Ising model.

The simulations of the short ranged Ising model in two dimensions on the other hand showed that the behaviour of $\delta(t)$ is a power law with time. The power law exponent is non-universal and depends on $\delta_0$. The exit probability has also been calculated which for the two dimensional Ising model shows the expected nonlinear behaviour. The exponent $\nu$ and the scaling factor $\lambda$ related to the finite size behaviour have been calculated. It appears that $\nu$ shows a weak dependence on the lattice structure (i.e., $z$) while for $\lambda$ the values are appreciably different (Table II). The exit probability study for the mean field model on the other hand shows $\nu'$ is independent of $z$ while $\lambda$ again shows strong dependence.

We have also conducted a similar study for the two parameter generalised voter model. In this case, we find that the stability of the disordered fixed point depends on the parameter values and it is possible to obtain a phase diagram based on this analysis.

Our studies show that the behaviour of $\delta(t)$ which is related to magnetisation for the fixed point $x = 0.5$ is different in the mean field case and the short range model. However, when the number of neighbours $z = 2$, the mean field result that the dynamics conserve the ensemble magnetisation is the same as that of the one dimensional Ising model or the voter model. We have justified this result on the basis of the voter model dynamics. Hence an important conclusion is that for $z = 2$, the results are independent of the range of the interaction.

The instability of the $x = 0.5$ fixed point for the higher values of $z$ indicates the exit probability should be a step function in the mean field case in the thermodynamic limit. This behaviour is found to be true for the short range models as well in which the exit probability for
larger lattice sizes show the tendency to approach a step function. However, the exponents associated with the finite size scaling analysis are quite different quantitatively. In particular, $\nu'$ is independent of $z$ in the mean field case and less compared to the value obtained for the model on two dimensional lattices.

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