From deep to Shallow: Equivalent Forms of Deep Networks in Reproducing Kernel Kreĭn Space and Indefinite Support Vector Machines

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Abstract

In this paper we explore a connection between deep networks and learning in reproducing kernel Kreĭn space. Our approach is based on the concept of push-forward - that is, taking a fixed non-linear transform on a linear projection and converting it to a linear projection on the output of a fixed non-linear transform, aka pushing the weights forward through the non-linearity. Applying this repeatedly from the input to the output of a deep network, the weights can be progressively “pushed” to the output layer, resulting in a flat network that has the form of a fixed non-linear map (whose form is determined by the structure of the deep network) followed by a linear projection determined by the weight matrices - that is, we take a deep network and convert it to an equivalent (indefinite) support vector machine. We then investigate the implications of this transformation for capacity control and generalisation, and provide a bound on generalisation error in the deep network in terms of generalisation error in reproducing kernel Kreĭn space.

1 Introduction

In machine learning, a clear distinction is often drawn between kernel methods such as support vector machines, which were overwhelmingly popular in the early-mid 2000s, and deep networks that have come to dominate the field since Hinton et al’s groundbreaking paper [37]. Kernel methods are often characterised as elegant but limited - founded on beautiful mathematical theory (reproducing kernel Hilbert space etc), and intuitive (max-margin in feature space, geometric interpretation of support vectors etc), but inflexible and incapable of scaling to the needs of big-data - while deep networks are characterised as utilitarian but superior in terms of performance, scalability, and flexibility. Thus deep networks have come to dominate in many areas, while kernel methods have to retreated to niche applications.

An argument often made to explain the superior expressive power and performance of deep networks is the apparent complexity (and hence capacity) of such networks. Kernel methods learn a linear relation in a feature space, where all nonlinearity is contained in the fixed map from input space to feature space, and deep networks are built from many layers of non-linearity interspersed with linear maps (weight matrices). Thus it is natural to conclude that (a) there is little or no crossover between the two methods, and (b) that deep networks are naturally more expressive.

In this paper we show that the distinction is not so clear-cut. We show that a large family of deep networks can, in fact, be precisely represented as single-layer networks where, as is the case with kernel methods, non-linearity is confined to a fixed feature map from input space to feature space - as encoded by a Kreĭn kernel - and learning is restricted to constructing a linear map from feature space to target space - or, equivalently, learning in reproducing kernel Kreĭn space. To be precise we
show that, with some minor restrictions, deep networks can be “collapsed” to single-layer networks in the form of a fixed non-linear layer (feature map) followed by a learned linear layer (weight vector). Moreover we show that, given the structure of the deep network (number of layers, width of each layer, and activation functions), the set of all possible trained machines (as elucidated by the weight matrices) is actually smaller than the set of possible trained machines for the corresponding single-layer network.

We note that the connection between kernel methods and deep networks has a long history. For example [51] showed (in 1996) that, as the width of a neural network goes to infinity, a single layer neural network with iid random parameters converges to a function drawn from a GP. This can be extended to multi-layered nets [38] by assuming random weights up to (but not including) the output layer. Indeed, the use of random weights to derive approximate kernels has become a popular means of linking deep networks and kernel methods [60][9][8][21][50][20]. More recently, neural tangent kernels [35][6] built on this by considering the path of weights during gradient descent training of a deep network, deriving a kernel that approximate the network around some solution (or average set of solutions). However such approaches do not provide a 1-1 equivalence between kernel methods and deep networks, which is our goal here. An alternative approach appears in the construction of arc-cosine kernels [15]. For activation functions of the form $\sigma(\xi) = (\xi)^n$, $n = 0, 1, 2, \ldots$, where $n = 1$ corresponds to ReLU, letting the width of the network go to infinity, arc-cosine kernels capture the feature map of the network (depth is achieved by composition of kernels). However this method is restricted to networks of infinite width, which limits the insight gained for deep networks, whereas our approach works for networks of arbitrary width and depth.

In recent years a significant amount of literature has been generated investigating the generalisation properties of deep networks under various assumptions [52][54][55][31][11][27][5][24][40][49][50][58][50]. In this paper we approach the problem of generalisation bounds indirectly, which both simplifies the derivation and generalises the results. To be precise, by constructing an equivalence between deep networks and kernel methods, we are able to analyse the capacity of the deep network in question by bounding it by the capacity of the corresponding indefinite SVM. Assuming the deep network is regularised using an elementwise norm on the weight matrices, we give an equivalent regularisation scheme for the “flattened” deep network representation. We then show that the resulting (effective) regularisation term imposed by the deep network weight regularisation places an upper bound on the corresponding (naive) regularisation term for an SVM-type approach. This allows us show that the set of reachable functions in the deep network with bounded (norm) weight matrices is a subset of the corresponding set of reachable functions in the SVM approach. Thus we can bound for example the Rademacher complexity of deep networks in terms of the Rademacher complexity of a corresponding indefinite (Krein) SVM, allowing a set of results to be directly transferred from the SVM context to the deep network context.

The remainder of the paper is organised as follows. In section 2 we define our assumptions regarding deep network structure; and in section 3 we present a review of indefinite (Krein) support vector machines, with a focus on the primal formulation. In section 4 we show a network satisfying our assumptions can be flattened to obtain an equivalent, feature-space representation consisting of a fixed non-linear map followed by a projection onto $\mathbb{R}$, which we show is equivalent to an indefinite SVM in section 5. Finally, in section 6 we show how regularisation in deep networks and regularisation in indefinite SVMs are linked, and hence how we may transfer generalisation bounds that apply to the indefinite SVM directly to the deep network.

1.1 Notation

We use $\mathbb{N} = \{0, 1, \ldots\}$, $\mathbb{N}_+ = \{1, 2, \ldots\}$, $\bar{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$, $\mathbb{N}_n = \{0, 1, \ldots, n - 1\}$, $\mathbb{R}_+ = \{x \in \mathbb{R} | x > 0\}$. Hilbert spaces are denoted $\mathcal{H}$ and Krein spaces $\mathcal{K}$. We define $(x)_+ = \max\{x, 0\}$ to be the positive part. For (countable) vectors $a, b, a_i$ denotes the $i$th element of $a$, $a \circ b$ is the element-wise product, $a^{oc}$ the element-wise power (scalar $c$), $|a|$ the element-wise absolute, $\text{sgn}(a)$ the elementwise sign, $(a)_+$ the elementwise positive part, and $\text{sum}(a) = \sum a_i$. We use a number of variations of inner product, so for clarity we try to use visually distinct notations for each
We are interested in the case of regularised training, where regularisation is applied via an elementwise norm on the weights. Specifically, given a training dataset \( \{(x_i, y_i) \in \mathbb{R}^D \times \mathbb{R} | i \in \mathbb{N}_N \} \) drawn from some distribution, weight matrices are selected to solve:

\[
\min_{w[q] \in \mathbb{R}^{H_q \times H_{q-1}}} \frac{1}{N} \sum_{i=1}^{N} L(y_i, o(x_i)) + \lambda \frac{1}{d} \sum_{q} \|W[q]\|_{2}^{2}
\]

where the first term is the empirical risk \( L \) is the loss function, which will vary depending on the purpose of the network (classification, regression etc.), the second term is a regularisation term, and

![Figure 1: Machine learning architectures. (a) shows the physical deep network architecture, and (b) the representation of the same deep network architecture in “flat” (feature space) form of non-linear feature map \( \varphi : \mathbb{R}^D \rightarrow \mathcal{F} \) followed by linear projection onto \( \mathbb{R} \), where the weight vector \( w \in \mathcal{F} \) is the “push-forward” of the weight matrices in the deep network.](image)
we assume that the matrix norms $\| \cdot \|_{q}$ are elementwise norms. We use min (minimise) in the loose sense that a local minimiser suffices (stat stationarise) is used in the same context in [42], but this does not appear to be standard notation in deep networks.

We aim to show that the deep network (1) can be rewritten in feature-space form:

$$o(x) = (w, \varphi(x))_g = \sum_i g_i w_i \varphi_i(x)$$

which allows us to build a connection between deep networks and support vector machines. Before proceeding, however, we first present some background on the theory of indefinite (Kreǐn) support vector machines.

2.1 Non-Entire Activation Functions

While the “entire function” requirement on the activation functions $\sigma_{[q]}$ is necessary, we note that more general functions $\sigma_{[q]}$ can be approximated to arbitrary precision using an entire surrogate. For example, if $\sigma_{[q]}$ is continuous then it may always be approximated to arbitrary precision by a finite sum $\hat{\sigma}_{[q]}(\cdot) = \sum_i \beta_i \kappa(\cdot, \zeta_i)$, where $\beta_i, \zeta_i \in \mathbb{R}$ and $\kappa$ is an entire universal kernel [47] (for example, $\kappa(\zeta) = \exp(-\zeta^2)$). In this way we may construct arbitrarily close entire approximations to e.g. the tanh activation function. Thus, though our analysis is restricted to entire activation functions, this should not be seen as a serious limiting factor.

2.2 A Note on the ReLU Activation Function

The ReLU (Rectified Linear Unit) activation $\sigma_+(\xi) = (\xi)_+$ function is popular in deep networks, so it is worth considering it in more detail. It is not entire, but can be approximated to arbitrary accuracy by $\sigma_{+\xi}(\xi) = \lim_{c \to 0_+} \frac{1}{2} (1 + \text{erf}(\frac{\xi}{c}))$, which is an entire function. When discussing ReLU networks we implicitly mean the limit of some sequence $\sigma_{c_0+}, \sigma_{c_1+}, \ldots$, where $c_0 \geq c_1 \geq \ldots \to 0$.

3 Preliminaries II: Indefinite Support Vector Machines

Indefinite (or Kreǐn) support vector machines (SVMs) [41, 43, 30, 76, 63] are an extension of support vector machines [17, 12, 14, 70, 19, 32, 73, 74] that relax the usual positive definiteness requirement on the kernel, based on the observation that indefinite kernels outperform positive definite kernels in some cases. They may be interpreted [59, 57, 56] as a form of regularised learning in reproducing kernel Kreǐn space theory [13, 7] RKKS. Typically, indefinite SVMs are introduced without reference to the primal formulation often found in standard SVM theory (for example [17]), but as we require the primal formulation here we now give a brief introduction from this perspective using the Kreǐn-kernel trick. Our approach is loosely based on [17], extended to the indefinite case. A more conventional presentation from reproducing kernel Kreǐn space theory is presented in the supplementary.

We consider a function of the simple, linear form:

$$o(x) = (w, \varphi(x))_g = \sum_i g_i w_i \varphi_i(x)$$

where the feature map $\varphi : \mathbb{R}^D \to \mathcal{F}$ and the metric $g \in \mathcal{F}$ are defined a-priori (implicitly, as we will see, by a Kreǐn kernel). We note that this is the same as the primal form of the trained machine in SVM theory, excepting that it involves a weighted indefinite inner product rather than the usual inner product; that is, it is an indefinite SVM primal. In SVM learning, as in deep networks, the goal is to mimic the input/output relation embodied by the training set $(x_i, y_i) \in \mathbb{R}^D \times \mathbb{R} | i \in \mathbb{N}$). In an indefinite SVM this is done by minimising the stabilised risk minimisation problem (56 equation (1), [42]):

$$\min_{w \in \mathcal{G}} \frac{1}{N} \sum_i L(y_i, (w, \varphi(x_i))_g) + \lambda \| (w, w)_g \|$$

where once again we use min in the loose sense, as local minima are allowed (see [42] for discussion, as well as an alternative notation). Distinct from [59, 57, 56], we regularise (stabilise [42]) via $\| (w, w)_g \|$, which will allow us to place a tighter bound on the Rademacher complexity. We next provide a representer theory:

**Theorem 1 (Representer Theory)** Any solution $w^*$ to (4) can be represented as $w^* = \sum_i \alpha_i \varphi(x_i)$, where $\alpha \in \mathbb{R}^N$. Defining $K(x, x') = (\varphi(x), \varphi(x'))_g$, the optimal $o^* : \mathbb{R}^D \to \mathbb{R}$ is $o^*(x) = \sum_i \alpha_i K(x, x_i)$. 


We aim to show that the deep network (1) can be rewritten in feature-space form: written:

\[ \prod_{i=1}^{N} \langle \varphi_i(x_i), \varphi_j(x_j) \rangle_{\mathcal{G}} \]

and so \( w = \sum_{i} \alpha_i \varphi_i(x_i) \) for some \( \alpha \in \mathbb{R}^N \). Substituting into (3) we have \( o(x) = \sum_i \alpha_i K(x, x_i) \) for \( K \) defined.

Note that, for \( K \) as per theorem [1] the stabilised risk minimisation problem (4) can be rewritten in terms of \( \alpha \) as:

\[
\min_{\alpha \in \mathbb{R}^N} \frac{1}{N} \sum_i L \left( y_i, \sum_j \alpha_j K(x_i, x_j) \right) + \lambda \left| \sum_{i,j} \alpha_i \alpha_j K(x_i, x_j) \right|
\]

(5)

In this formulation \( K \) is a Kreın kernel, which is a function \( K : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R} \) that can be written as a difference \( K = K_+ - K_- \) between positive definite kernels \( K_\pm \) [59] Proposition 7]. Note that \( K \) in theorem [1] can be split like this using (where \( (a)_+ = \max\{0, a\} \), \( (a)_- \) elementwise):

\[
K_{\pm}(x, x') = \langle \varphi_{\pm}(x), \varphi_{\pm}(x') \rangle_{(\pm)G^+}
\]

(6)

Trivially, \( K_\pm \) are positive definite, so \( K \) in theorem [1] is a Kreın kernel. Conversely, given a Kreın kernel \( K \), by definition there exist positive definite \( K_\pm \) (non-uniquely) such that \( K = K_+ - K_- \). Hence there exists implicit, finite or countably infinite dimensional (by Mercer’s theorem) expansions \( K_{\pm}(x, x') = \langle \varphi_{\pm}(x), \varphi_{\pm}(x') \rangle_1 \), so \( K(x, x') = \langle \varphi(x), \varphi(x) \rangle_{\mathcal{G}} \) where \( \varphi(x) = [\varphi_+, \varphi_-](x) \) and \( \mathcal{G} = [+1, -1] \). So, as for standard SVMs, we don’t actually need to know the feature map and metric; rather, we just need a Kreın kernel to implicitly define a feature map and metric. We call this the Kreın kernel trick by analogy.

Formally, as discussed in the supplementary, \( o \in \mathcal{K}_K \) lies is a reproducing kernel Kreın space (RKKS) \( \mathcal{K}_K \) defined (non-uniquely) by Kreın kernel \( K \) [59]. Recalling that \( K(x, x') = \langle \varphi(x), \varphi(x') \rangle_{\mathcal{G}} \) then:

\[
\mathcal{K}_K = \{ f(\cdot) = (w, \varphi(\cdot))_{\mathcal{G}} \mid w \in \mathcal{F} \}
\]

is a vector space of functions equipped with an indefinite inner product \( \langle (w, \varphi(\cdot))_{\mathcal{G}}, (w', \varphi(\cdot))_{\mathcal{G}} \rangle_{\mathcal{K}_K} = \langle w, w' \rangle_{\mathcal{G}} \). Hence the risk minimisation problem (4) can be written:

\[
\min_{o \in \mathcal{K}_K} \frac{1}{N} \sum_i L \left( y_i, o(x_i) \right) + \lambda \| (o, o)_{\mathcal{K}_K} \|
\]

(7)

4 Flat Representations for Deep Network

We aim to show that the deep network [1] can be rewritten in feature-space form:

\[
o(x) = (w, \varphi(x))_{\mathcal{G}} = \sum_{i} g_i w_i \varphi_i(x)
\]

(8)

where \( \langle \cdot, \cdot \rangle_{\mathcal{G}} \) is an indefinite-inner-product, \( \varphi : \mathbb{R}^d \rightarrow \mathcal{F} \) is a non-linear feature map and \( \mathcal{G} \) a metric, such \( \varphi \) and \( \mathcal{G} \) depend only on the network structure; and \( w \in \mathcal{F} \) is a weight vector that solves a regularised risk minimisation problem:

\[
\min_{w \in \mathcal{F}, \mathcal{G}} \frac{1}{N} \sum_i L \left( y_i, (w, \varphi(x_i))_{\mathcal{G}} \right) + \lambda \rho_{\text{final}}(w)
\]

(9)

for appropriate \( \mathcal{F}_{\text{fin}} \) and \( \rho_{\text{fin}} : \mathcal{F} \rightarrow \mathbb{R} \) such that the trained networks [8] and [1] are functionally equivalent. This representation is analogous to the trained SVM primal [1], which will allow us to analyse deep networks from the same perspective as indefinite SVMs.

Central to our approach is the push-forward operation, converting a nonlinear function of a multilinear product of vectors to a multilinear product of the non-linear images of the original vectors - that is\footnote{This is essentially an indefinite m-kernel (moment function, tensor kernel) expansion \cite{68,61,62,22}.}:

\[
\sigma \left( (x, \ldots, x^m)_{m, \mu} \right) \rightarrow \left( \phi(x), \ldots, \phi(x^m) \right)_{m, \gamma \circ \phi(\mu)} = \sum_{i} \gamma_i \phi_i(\mu) \phi_i(x), \ldots, \phi_i(x^m)
\]

We call the image \( \phi(x) \) of \( x \) the push-forward of \( x \), as it heuristically represents the result of pushing \( x \) (forwards) through \( \sigma \). This is enabled by the following lemma:
Lemma 2 Let \( \sigma \) be an entire function with Taylor expansion \( \sigma(\xi) = \sum a_i \xi^i \), and let \( \langle \cdot , \cdot \rangle \), be an \( m \)-indefinite-inner-product defined by metric \( \mu \in \mathbb{R}^n \) (section 1.1). Then:

\[
\sigma \left( \langle \mathbf{x}, \ldots, \mathbf{x}'''' \rangle \right)_{m, \mu} = \langle \phi(\mathbf{x}), \ldots, \phi(\mathbf{x}''') \rangle_{m, \gamma_{[q]} \circ \phi(\mu)}
\]

(10)

where \( \phi : \mathbb{R}^n \to \mathcal{F} \) is a feature map and \( \gamma \in \mathcal{F} \), both independent of \( m \) and \( \mu \). Using multi-index notation, \( \phi(\mathbf{x}) = \left[ \phi_t(\mathbf{x}) \right]_{i \in \mathbb{N}^n} \) and \( \gamma = [\gamma_t]_{i \in \mathbb{N}^n} \), where:

\[
\phi_t(\mathbf{x}) = \prod_j x_j^{t_j}, \quad \gamma_t = \left( \sum_{i \leq t} a_{\text{sum}(i)} \right)
\]

(11)

Moreover, \( \forall t \in \mathbb{R}^D, \forall u_{(0)}, \ldots, u_{(n-1)}, v, v', \ldots, \mu \in \mathbb{R}^P : 

\[
\sigma \left( \sum_j t_j \langle u_{(j)}, v, v', \ldots \rangle_{m, \mu} \right) = \langle \phi(t_*), \phi(u_*), \phi(v_*), \phi(v'_*), \ldots \rangle_{m+1, \gamma \circ \phi(\mu_*)}
\]

(12)

where \( t_* = t \otimes 1_p \) and \( u_* = [u^{(0)}_{1*} u^{(1)}_{1*} \ldots]^T, v_* = 1_n \otimes v, v'_* = 1_n \otimes v', \ldots, \mu_* = 1_n \otimes \mu \).

Proof: Equations (10) and (11) follows from the multinomial expansion of \( \sigma \) and subsequent collection of terms, while the sequel (12) also follows with some index shuffling. See supplementary for details. \( \square \)

Recalling that we are considering only entire activation functions, we have by Lemma 2 (equation (11)) that:

\[
\sigma_{[q]} \left( \langle \mathbf{x}, \ldots, \mathbf{x}'''' \rangle \right)_{m, \mu} = \langle \phi(\mathbf{x}), \ldots, \phi(\mathbf{x}''') \rangle_{m, \gamma_{[q]} \circ \phi(\mu)}
\]

(13)

where the indices \( [q] \) again denotes layer \( q \). Note that we do not place a subscript on \( \phi \) as, by Lemma 2 (equation (11)), this depends on the input dimension and not the activation function \( \sigma_{[q]} \). The next step is to apply this push-forward repeatedly: starting with \( \mathbf{x} \) and \( \mathbf{W}_{[0]} \) at layer 0, obtain a push-forward representation of \( \mathbf{x} \) and \( \mathbf{W}_{[0]} \) at layer 1, then a push-forward representation of \( \mathbf{x}, \mathbf{W}_{[0]} \) and \( \mathbf{W}_{[1]} \) at layer 2, and so on to the output layer. The result of this procedure is summarised by the following theorem:

Theorem 3 The deep network \( \mathbf{f} \) has equivalent form:

\[
o(\mathbf{x}) = (\mathbf{w}, \varphi(\mathbf{x})) = \sum_i g_i w_i \varphi_i(\mathbf{x})
\]

(14)

where the feature map, weight vector and metric are:

\[
\varphi(\mathbf{x}) = \phi(1_{H_{d-2}} \otimes \phi(\ldots \cdot 1_{H_2} \otimes \phi(1_{H_0} \otimes \phi(\mathbf{x}))))
\]

(15)

\[
\mathbf{g} = [\gamma_{d-1} \otimes \phi(1_{H_{d-2}} \otimes (\gamma_{d-2} \circ \phi(\ldots \cdot 1_{H_0} \otimes (\gamma_{[0]} \circ \phi(\mathrm{D}) \ldots))))]
\]

where \( \mathbf{W}_{k,i} \) is row \( i \) of matrix \( \mathbf{W} \) (Matlab style notation).

Proof: The proof follows by repeated application of Lemma 2. See supplementary for details. \( \square \)

4.1 Regularisation in Flat Representations

Theorem 3 provides the desired flat equivalent form (14) of the deep network (1). To show that \( \mathbf{w} \) is equivalently the result of solving a regularised risk minimisation problem of form (9) we must (a) define a restricted feature space \( \mathcal{F}_{\text{fin}} \) consisting of only those weight vectors \( \mathbf{w} \) that are in-principle realisable by weight matrices \( \mathbf{W}_{[q]} \) by (15), and (b) construct a regularisation term \( r_{\\text{fin}} \) such that it evaluates as the regularisation term of the equivalent (weight matrix based) deep network form regularisation term - that is:

\[
\min_{\mathbf{w} \in \mathcal{F}_{\text{fin}}} \frac{1}{N} \sum_i L(y_i, (\mathbf{w}, \varphi(\mathbf{x}_i))) + \lambda r_{\\text{fin}}(\mathbf{w})
\]

(16)

For the curious we provide an explicit form of the feature map, metric and weight expansion in the supplementary material.
where once again we use \( \min \) in the loose sense, as local minima are allowed. The set of realisable weights is:

\[
\mathcal{F}_{\text{ini}} = \{ \mathbf{w} = \bigotimes_q \phi_{[q,d-1]}(\mathbf{W}_{[q]}) : \mathbf{W}_{[q]} \in \mathbb{R}^{H_q \times H_q - 1} \} \tag{17}
\]

and the regularisation function is:

\[
\mathcal{R}_{\text{ini}}(\mathbf{w}) = \text{argmin}_{\mathbf{w}_{[q]} \in \mathbb{R}^{H_q \times H_q - 1}} \left\{ \frac{1}{d} \sum_q \| \mathbf{W}_{[q]} \|^2_{[q]} \right\}
\]

\[
= \text{argmin}_{\mathbf{w}_{[q]} \in \mathbb{R}^{H_q \times H_q - 1}} \left\{ \frac{1}{d} \sum_q \| \mathbf{W}_{[q]} \|^2_{[q]} \right\}
\tag{18}
\]

In this expression we use the word “sel” to mean select - that is, a process of selecting between the multiple regularisation terms corresponding to the different weight matrices that correspond to the same (flat) weight vector \( \mathbf{w} \), and hence implementing the same function \( o : \mathbb{R}^D \to \mathbb{R} \). So:

**Observation 1** Weight-matrix based regularisation in deep networks is ill-defined, insofar as the same network (in terms of function) can have multiple instantiations (different weight matrices), and hence multiple distinct regularisation penalties (depending on implementation).

**Example 1** Consider a ReLU network with weight matrices \( \mathbf{W}_{[q]} \); and let \( a_0, a_1, \ldots, a_{d-1} \in \mathbb{R}^+ \) : \( \prod_q a_q = 1 \). Replacing the weight matrices with \( \mathbf{W}_{[q]} = a_q \mathbf{W}_{[q]} \) will not change the network function, as \( \sigma_+ (a \xi) = a \sigma_+ (\xi) \forall a \in \mathbb{R}^+ \), but will change the regularisation penalty, as \( \frac{1}{d} \sum_q \| \mathbf{W}_{[q]} \|^2_{[q]} \neq \frac{1}{d} \sum_q \| \mathbf{W}_{[q]} \|^2_{[q]} \).

As the weight matrices act multiplicatively (in series from input to output, precisely so in ReLU networks), it is perhaps more natural to consider the geometric mean of their norms rather than the arithmetic mean when regularising (this has been noted before - for example [22]). This motivates us to restrict ourselves to the following variant of the regularisation function [18], which selects compatible weight matrices to calculate regularisation based on the alignment of the arithmetic and geometric means of their norms:

\[
\tilde{\mathcal{R}}_{\text{ini}}(\mathbf{w}) = \frac{1}{d} \sum_q \| \mathbf{W}_{[q]} \|^2_{[q]}
\]

where: \( \mathbf{W}_{[q]} = \text{argmin} \left\{ \frac{1}{d} \sum_q \| \mathbf{W}_{[q]} \|^2_{[q]} - \left( \prod_q \| \mathbf{W}_{[q]} \|^2_{[q]} \right) \right\} \tag{19}

such that: \( \mathbf{w} = \bigotimes_q \phi_{[q,d-1]}(\mathbf{W}_{[q]}) \)

Finally for this section we note that, for a ReLU network, using the standard regularisor of form \( \{19\} \), \( \mathbf{w} = \bigotimes_q \phi_{[q,d-1]}(\mathbf{W}_{[q]}) \) such that:

\[
\tilde{\mathcal{R}}_{\text{ini}}(\mathbf{w}) = \frac{1}{d} \sum_q \| \mathbf{W}_{[q]} \|^2_{[q]} = \left( \prod_q \| \mathbf{W}_{[q]} \|^2_{[q]} \right) \frac{1}{d}
\tag{20}
\]

**Proof:** We aim to show that, for a ReLU network, all weight matrices \( \mathbf{W}_{[q]} \) satisfying the conditions of \( \{19\} \) must therefore satisfy \( \{20\} \). We will do this by contradiction. Let \( \mathbf{W}_{[q]} \) be a set of weight matrices that satisfy the conditions of \( \{19\} \) but not \( \{20\} \). As noted in example [1] for any \( a_0, a_1, \ldots, a_{d-1} \) for which \( \prod_q a_q = 1 \), the weight matrices \( \mathbf{W}_{[q]} = a_q \mathbf{W}_{[q]} \) will define the same network (functionally speaking) as the weight matrices \( \mathbf{W}_{[q]} \) - that is, \( \mathbf{w} = \bigotimes_q \phi_{[q,d-1]}(\mathbf{W}_{[q]}) = \bigotimes_q \phi_{[q,d-1]}(\mathbf{W}_{[q]}) \). Setting \( a_q = \prod_q \| \mathbf{W}_{[q]} \|^2_{[q]} \), we see that \( \prod_q a_q = 1 \), and moreover \( \| \mathbf{W}_{[q]} \|^2_{[q]} = \prod_q \| \mathbf{W}_{[q]} \|^2_{[q]} \) \( \{19\} \) and \( \{20\} \), but this does not hold for the weight matrices \( \mathbf{W}_{[q]} \). Hence \( \| \mathbf{W}_{[q]} \|^2_{[q]} - \left( \prod_q \| \mathbf{W}_{[q]} \|^2_{[q]} \right) \frac{1}{d} = 0 < \frac{1}{d} \sum_q \| \mathbf{W}_{[q]} \|^2_{[q]} - (\prod_q \| \mathbf{W}_{[q]} \|^2_{[q]} \frac{1}{d}) \). But this contradicts the assertion that \( \mathbf{W}_{[q]} \) satisfies the conditions of \( \{19\} \) (it is not the minimiser), which completes the proof. \( \square \)

5 **Equivalent SVMs for Deep Networks**

As shown in section [4] deep networks can be flattened to an equivalent (flat) form [8]:

\[
\mathbf{o}(\mathbf{x}) = (\mathbf{w}, \varphi(\mathbf{x}))_\mathbf{r} = \sum_i g_i w_i \varphi_i(\mathbf{x})
\]
We have shown that both a deep network and its equivalent indefinite SVM are trained networks of the form (equivalent flat formulation for deep networks, primal form for indefinite SVMs):

$$
\min_{w \in \mathcal{F}_{NN}} \frac{1}{N} \sum_i L(y_i, (w, \varphi(x_i))_{\mathcal{F}}) + \lambda r_{NN}(w)
$$

where \( \mathcal{F}_{NN} \) and \( r_{NN} \) are defined by (17) and (19), respectively. We define an equivalent (indefinite) SVM (for the deep network) to be an indefinite SVM using the same feature map \( \varphi: \mathbb{R}^D \to \mathcal{F} \) and metric \( g \) as the deep network (in flat form) that solves the regularised risk minimisation problem:

$$
\min_{w \in \mathcal{F}} \frac{1}{N} \sum_i L(y_i, (w, \varphi(x_i))_{\mathcal{F}}) + \lambda |(w, w)_{\mathcal{F}}|
$$

Clearly the feature map is countably infinite dimensional, so the primal form of the equivalent SVM is not useful; however we may use the Krein kernel trick to encapsulate the feature map in a Krein kernel and then solve (5) to get \( o(x) = \sum_i \alpha_i K_{NN}(x, x_i) \). Specifically:

**Theorem 4** Let the feature map \( \varphi: \mathbb{R}^D \to \mathcal{F} \) and metric \( g \) be defined by the deep network (1) as per theorem [3] Then:

$$
K_{NN}(x, x') = \langle \varphi(x), \varphi(x') \rangle_{\mathcal{F}} = \sigma_{[d-1]}(H_{d-2}\sigma_{[d-2]}(H_{d-3} \ldots H_1\sigma_{[1]}(\langle x, x' \rangle_1))))
$$

(21)

is the corresponding Krein kernel.

**Proof:** The proof follows by direct application of the definitions (theorem [3] and lemma [2]). See supplementary. □

An indefinite SVM using Krein kernel \( K_{NN} \) trained on a particular dataset will learn a relation \( o: \mathbb{R}^D \to \mathbb{R} \) of the same form (in primal representation) (but different weights) as that learned by the deep network (flattened representation) from whose structure (21) was derived and that has been trained on the same dataset.

### 5.1 Implications

We have shown that both a deep network and its equivalent indefinite SVM are trained networks of the form (equivalent flat formulation for deep networks, primal form for indefinite SVMs):

$$
o(x) = (w, \varphi(x))_{\mathcal{F}} = \sum_i g_i w_i \varphi_i(x)
$$

where the weight \( w \) minimises the relevant stabilised risk minimisation problem:

$$
\min_{w \in \mathcal{F}_{NN}} \frac{1}{N} \sum_i L(y_i, (w, \varphi(x))_{\mathcal{F}}) + \lambda r_{NN}(w)
$$

and \( M \in \{\text{NN, SVM}\} \), and we have defined \( r_{NN}(w) = |(w, w)_{\mathcal{F}}| \) and \( \mathcal{F}_{NN} = \mathcal{F} \). The distinction between the approaches lies in the definition of the restricted feature space \( \mathcal{F}_d \) and the form of the regulariser \( r_d \). We note that:

1. The space of \( \mathcal{F}_{NN} \) of realisable weights of the deep network is significantly smaller than the space of realisable weights for the equivalent SVM. The former is the projection of the weight matrices by (17), whereas the latter is the entire feature space \( \mathcal{F}_{NN} = \mathcal{F} \).

2. The regularisation term \( r_{NN} \) of the flattened deep network (18) is structured and depends on the structure of the deep network, whereas the regularisation term \( r_{NN}(w) = |(w, w)_{\mathcal{F}}| \) of the equivalent SVM is trivial.

We may therefore expect that the capacity of the equivalent SVM will be larger than the capacity of the deep network from which it was derived - a fact that we demonstrate shortly - which may help to explain the surprisingly good performance of deep networks even on relatively small datasets. The precise effect of the structured regularisation term in flattened deep networks is unclear, but we note that regularising with non-Euclidean products generally leads to reproducing kernel Banach space learning ([22]) most clearly, but also ([79],[77],[78],[25],[71],[75]), with some early precursors e.g. ([44]) in the case of non-Euclidean norm-based regularisation, or reproducing kernel Krein space learning (as discussed above) in the case of indefinite inner product-based regularisation, so we speculate that this structured regularisation may be connected to one (or possibly some amalgam of both) of these.
It is also interesting to consider the effect of network width, particularly in the limit of an infinitely wide network. In the flat form of the deep network, the width appears only indirectly in the expressions (17) for the space of realisable weight vectors and the regularisation term (18). As the regularised risk (16) is a weighted sum of empirical risk and regularisation, we would expect that the weights $\mathbf{W}_{\mathbf{g}}$ would decrease (in magnitude, as measured by the norms) as the widths increase (the regularisation term grows proportional to width, which is balanced against the empirical risk, which does not grow). In terms of the equivalent indefinite SVM, the width $H_0$ of the network appears in the expressions for the Kreîn kernel (21) in theorem 4. Assuming bounded concave activation functions (eg tanh), $K$ approaches a step function as $H \to \infty$, and for ReLU, $K$ becomes undefined in the limit (though we would expect $\alpha \to 0$ based on [5] in this case).

6 Bounding the Capacity of Deep Networks through Equivalent SVMs

In the previous section we speculated that the space of realisable weight vectors for the flattened deep network may be significantly smaller than the space of realisable weight vectors for the equivalent SVM, which would imply that the capacity of the deep network is lower than the capacity of the equivalent SVM. In this section we prove this using Rademacher complexity analysis.

We consider a deep network of the form defined by (1) in section 2, and an equivalent indefinite SVM would decrease (in magnitude, as measured by the norms) as the widths increase (the regularisation term grows proportional to width, which is balanced against the empirical risk, which does not grow). In terms of the equivalent indefinite SVM, the width $H_0$ of the network appears in the expressions for the Kreîn kernel (21) in theorem 4. Assuming bounded concave activation functions (eg tanh), $K$ approaches a step function as $H \to \infty$, and for ReLU, $K$ becomes undefined in the limit (though we would expect $\alpha \to 0$ based on [5] in this case).

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We consider a deep network of the form defined by (1) in section 2, and an equivalent indefinite SVM (as per section 3) with Kreîn kernel $K_{\mathbf{w}}$ defined by the structure of the deep network as per theorem 4, with corresponding RKKS $\mathcal{K}_{K_{\mathbf{w}}}$ as per section 5. Given the preliminary, the deep network and its equivalent (indefinite) SVM have the form:

$$ o(x) = (w, \varphi(x)) = \sum g_i w_i \varphi_i(x) \in \mathcal{K}_{K_{\mathbf{w}}} $$

where $w$ solves a stabilised risk minimisation problem (we use subscript $\mathcal{M}$ to indicate, respectively, the deep network and its equivalent (indefinite) SVM):

$$ \min_{w \in \mathcal{F} \subseteq \mathcal{H}} \frac{1}{N} \sum_{i} L(y_i, (w, \varphi(x_i))_{\mathbf{g}}) + \lambda r_{\mathcal{M}}(w) $$

which, if we interpret $\lambda \geq 0$ as a Lagrange multiplier, is equivalent to the constrained optimisation problem:

$$ \min_{w \in \mathcal{F} \subseteq \mathcal{H}} \frac{1}{N} \sum_{i} L(y_i, (w, \varphi(x_i))_{\mathbf{g}}) $$

where: $r_{\mathcal{M}}(w) \leq R_{\mathcal{M}}$ for appropriate $R_{\mathcal{M}}$ (as we are using min in the loose sense that local minima suffice we may apply Lagrange multiplier theory, which in this case guarantees only that local optima are found for non-convex $r_{\mathcal{M}}$). Hence we define our hypothesis space as:

$$ \mathcal{F}_{\mathcal{M}} = \{ o(\cdot) = (w, \varphi(\cdot))_{\mathbf{g}} | w \in \mathcal{F} \mathcal{M} \land r_{\mathcal{M}}(w) \leq R_{\mathcal{M}} \} $$

or, explicitly for a deep network or its equivalent SVM:

$$ \mathcal{F}_{\mathcal{NN}} = \{ o(\cdot) = (w, \varphi(\cdot))_{\mathbf{g}} | w \in \mathcal{F} \mathcal{NN} \land r_{\mathcal{NN}}(w) \leq R_{\mathcal{NN}} \} $$

$$ \mathcal{F}_{\mathcal{SVM}} = \{ o(\cdot) = (w, \varphi(\cdot))_{\mathbf{g}} | \|w\|_{G_{\mathcal{NN}}} \leq R_{\mathcal{SVM}} \} $$

(22)

where $\mathcal{F}_{\mathcal{SVM}}$ is a ball of radius $R_{\mathcal{SVM}}$ in $\mathcal{K}_{K_{\mathbf{w}}}$.

The Rademacher complexity $\mathcal{R}_{\mathcal{N}}(\mathcal{F}_{\mathcal{M}})$ of a set of functions $\mathcal{F}_{\mathcal{M}}$ is a measure of the capacity (expressive power) of the set. Letting $x_0, x_1, \ldots$ be sampled according to distribution $\nu$, and $\sigma_0, \sigma_1, \ldots \in \{-1, 1\}$ be Rademacher random variables, the Rademacher complexity is defined as [46]:

$$ \mathcal{R}_{\mathcal{N}}(\mathcal{F}_{\mathcal{M}}) = \mathbb{E}_{\nu, \sigma} \left[ \sup_{o \in \mathcal{F}_{\mathcal{M}}} \left\| \frac{1}{\sqrt{N}} \sum_{i} \sigma_i o(x_i) \right\| \right] $$

(23)

$\mathcal{R}_{\mathcal{N}}(\mathcal{F}_{\mathcal{SVM}})$ can be bounded in terms of the associated kernel (that is, the positive definite kernel $K = K_{\mathcal{NN}} + K_{\mathcal{SVM}}$, where $K = K_{\mathcal{NN}} - K_{\mathcal{SVM}}$ [59]. However this results in a very loose bound - for example the associated kernel for $K(x, x') = \text{erf}(\langle x, x' \rangle_1)$ is $\tilde{K}(x, x') = \text{erfi}(\langle x, x' \rangle_1)$ (see supplementary), which is convex and unbounded as $\|x\| \to \infty$. To avoid this we have used the alternative regulariser $r_{\mathcal{SVM}}(w) = \|w\|_{G_{\mathcal{SVM}}}$ in [4], which enables:
We finish by considering two special cases where the bound (26) can be simplified or tightened: Moreover if the ReLU activation function is used everywhere then functions),

$$M$$

A network with additional factors

$$L$$

complexity (specifically

$$f$$

it follows from (25) and (22) that

$$0 \leq \sigma_{[d-1]}(c^2_{[d-1]} \| W_{[d-1]} \|_{2}^2 \sigma_{[d-2]}(\ldots \sigma_{[0]}(c^2_{[0]} \| W_{[0]} \|_{2}^2)) \|) \leq (\text{w}, \text{w})_g$$

$$\leq (LM)^d \prod_q \| W_{[q]} \|_{F}^2 \leq \left( \frac{LM}{d} \sum_q \| W_{[q]} \|_{F}^2 \right)^d$$

where

$$L = \left( \prod_q L_q \right)^{1/d}$$

and

$$M = \left( \prod_q c^2_q \right)^{1/d}$$

are the geometric means of $$L_q$$ and $$c^2_q$$, respectively. Moreover if the ReLU activation function is used everywhere then $$L = 1$$ and the bounds simplify to:

$$0 \leq (\text{w}, \text{w})_g \leq M^d \prod_q \| W_{[q]} \|_{F}^2 = \left( \frac{M}{d} \sum_q \| W_{[q]} \|_{F}^2 \right)^d$$

for $$r_{\text{SVN}}$$ as per (19).

Proof: See supplementary.

It follows from this and (18), (19) that:

$$0 \leq r_{\text{SVN}}(\text{w}) = |(\text{w}, \text{w})_g| \leq (LMr_{\text{SVN}}(\text{w}))^d$$

and we see that, as speculated previously, the hypothesis space of the deep network is smaller than the hypothesis space of the equivalent indefinite SVM in the sense that, if we set

$$R_{\text{SVN}} = (LMr_{\text{SVN}})^d$$

it follows from (25) and (22) that $$R_{\text{SVN}} \subseteq \psi_{\text{SVN}}$$. Using the well-known properties of Rademacher complexity (specifically $$A \subseteq B \Rightarrow R_N(A) \leq R_N(B)$$, eg [10] Theorem 12) and the bound (23):

$$R_N(\psi_{\text{SVN}}) \leq R_N(\psi_{\text{SVN}}) \leq \frac{1}{\sqrt{N}} \left( LM R_{\text{SVN}} \right)^d \sqrt{\int K_{\text{SVN}}(x, x) d\nu(x)}$$

That is, we can transfer the capacity/complexity bound on the equivalent indefinite SVM to the deep network with additional factors $$L$$ (the geometric mean of the Lipschitz constants for the activation functions), $$M$$ (a constant dependent on the matrix norms) and the depth $$d$$ of the network.

We finish by considering two special cases where the bound (26) can be simplified or tightened:

- **Tanh-type Networks**: Let $$\sigma_{[q]}(\xi) \in [-1, +1]$$ be concave non-decreasing with $$\sigma_{[q]}(0) = 0$$ for all $$q$$. Then $$K_{\text{SVN}}(x, x) = k_{[q]}(\| x \|_2^2)$$, where $$k_{[q]}$$ shares the same properties as $$\sigma_{[q]}$$, so $$0 \leq K_{\text{SVN}}(x, x) \leq 1$$, and hence:

$$R_N(\psi_{\text{SVN}}) \leq R_N(\psi_{\text{SVN}}) \leq \frac{1}{\sqrt{N}} \left( LM R_{\text{SVN}} \right)^d \sqrt{\int K_{\text{SVN}}(x, x) d\nu(x)}$$

$$R_N(\psi_{\text{SVN}}) \leq R_N(\psi_{\text{SVN}}) \leq \frac{1}{\sqrt{N}} \left( LM R_{\text{SVN}} \right)^d \sqrt{\int \sigma_{[q]}(0)}$$

(27)
**ReLU Networks:** Let \( \sigma_q(\xi) = (\xi)_+ \forall q \). Then \( K_{\text{nn}}(x,x) = H^d\|x\|_2^2 \), where \( H = (\prod_q H_q)^{1/d} \) (the geometric mean width), and hence:

\[
R_N(f_{\text{nn}}) \leq R_N(f_{\text{svm}}) \leq \left( \frac{M\sqrt{\prod H_q}}{\sqrt{\nu}} \right)^d \sqrt{\nu \|X\|_2^4} \tag{28}
\]

We note that the network width and all distribution dependence disappears entirely in complexity bound (27) for tanh-type networks. The ReLU bound is looser in this sense, as it depends on both the network width and the (raw) second moment of the data distribution. Unfortunately we see that the bound for Rademacher complexity in the ReLU network fails for the infinite-width case, though the tanh-type networks the bound remains entirely unaffected.

We finish by analysing the dependency of our bound on the various characteristics of the deep network:

- **Depth:** All of our bounds grow exponentially with depth. This is in line with other results (for example [11, 54] that show similar dependence.

- **Width:** For tanh-type networks our capacity bound is independent of network width, while for ReLU networks our capacity bound grows as \( O(H^{d/2}) \). The latter result is in line with existing results (for example [52, 27]), while the former appears novel.

- **Dataset size:** Our capacity bounds shrink as \( O(\sqrt[4]{N}) \). This is typical of Rademacher type bounds.

- **Data distribution:** For tanh-type networks our bound is independent of the data distribution (in the sense that (26) is bounded by (27)), while for ReLU type networks the bound is proportional to \( \sqrt{\nu \|X\|_2^2} \).

- **Regularisation Type:** Our approach requires regularisation based on an elementwise norms of the weight matrices. The impact of this is captured by the constant \( M \), which depends on the “closeness” of the norm to the Frobenius norm. Our bound grows as \( O(M^d) \), so the tightest bound (\( M = 1 \)) is obtained using Frobenius norm regularisation.

- **Activation Function:** Our generalisation bound depends on the geometric mean \( L \) of the Lipschitz constants of the activaiton functions and grows as \( O(L^d) \), where we note that in the ReLU network \( L = 1 \).

### 7 Conclusions

We have explored a novel connection between deep networks and learning in reproducing kernel Kreın space. We have shown how a deep network can be converted to an equivalent (flat) form consisting of a fixed non-linear feature map followed by a learned linear projection onto \( \mathbb{R} \), which is functionally identical to an indefinite SVM. Using this as a base, we have explored the issue of capacity and generalisation in deep networks by bounding them in terms of capacity in regularised learning in reproducing kernel Kreın space.
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1 Supplementary: Reproducing Kernel Kreın Space - Revision, Primal Form and Feature Space

Reproducing kernel Hilbert space theory [41, 73, 67] is ubiquitous in machine learning [17, 16, 19, 26, 28, 33, 39, 48, 64, 66, 65, 67, 59]. Motivated by the observation that indefinite kernels outperform RKHS kernels in some cases [41, 43, 50, 76], reproducing kernel Kreın spaces (RKKSs) have been studied in [59, 57, 56, 42, 63]. In this section we present a quick overview of reproducing kernel Kreın space theory (see [13, 7, 59, 57, 56] for alternatives). As per [13, 7], define:

**Definition 1** A Kreın space \( \mathcal{K} \) is a vector space equipped with an indefinite inner product \((\cdot, \cdot)_\mathcal{K}\) that may be decomposed into a direct difference \( \mathcal{K} = \mathcal{H}_+ \oplus \mathcal{H}_- \) of \((\cdot, \cdot)_\mathcal{K}\)-orthogonal Hilbert spaces \( \mathcal{H}_\pm \) (that is, \((f_+, g_-)_\mathcal{K} = 0 \) \( \forall f_+ \in \mathcal{H}_+, g_- \in \mathcal{H}_- \)) such that:

\[
(f, g)_\mathcal{K} = (f_+, g_+)_\mathcal{H}_+ - (f_-, g_-)_\mathcal{H}_-
\]

where \( f = f_+ \| f_-, g = g_+ \| g_-\) and \( f_\pm, g_\pm \in \mathcal{H}_\pm \) (here \( \| \) denotes the orthogonal sum). The associated Hilbert space \( \mathcal{K} = \mathcal{H}_+ \oplus \mathcal{H}_- \) over \( \mathbb{R} \) such that:

\[
(f, g)_\mathcal{K} = (f_+, g_+)_\mathcal{H}_+ + (f_-, g_-)_\mathcal{H}_-
\]

The strong topology on \( \mathcal{K} \) is induced by the metric

\[
d^2(f, g) = \| f - g \|_\mathcal{K}^2 = (f - g, g - g)_\mathcal{K}.
\]

Note that the decomposition of \( \mathcal{K} \) into \( \mathcal{H}_\pm \) is not unique in general. However the strong topology induced by the associated Hilbert space norm is independent of the decomposition [56]. Reproducing kernel Kreın space is defined as [3, 59]:

**Definition 2 (Reproducing Kernel Kreın Space)** A reproducing kernel Kreın space (RKKS) \( \mathcal{K} \) is a Kreın space of functions \( f : \mathbb{X} \to \mathbb{R} \) such that \( \forall x \in \mathbb{X} \) the point evaluation functional \( L_x : H \to \mathbb{R}, L_x(f) = f(x) \), is continuous with respect to the strong topology.

([59, Proposition 61]) For every RKKS \( \mathcal{K} \) there exists a symmetric reproducing (Kreın) kernel \( K : \mathbb{X} \times \mathbb{X} \to \mathbb{R} \), where \( f(x) = (K(x, \cdot), f)_{\mathcal{K}} \forall f \in \mathcal{K} \) and \( K(x, x') = (K(x, x'), K'(x', x'))_{\mathcal{K}} \), and \( K \) can be decomposed as \( K = K_+ - K_- \) where \( K_\pm \) are positive-definite reproducing kernels for \( \mathcal{H}_\pm \). The associated Hilbert space \( \mathcal{K} \) is a RKHS with reproducing kernel (associated kernel) \( K = K_+ + K_- \).

Any Kreın kernel \( K \) that can be decomposed as \( K = K_+ - K_- \) defines a reproducing kernel Kreın space \( \mathcal{K}_K \), and it can be shown that any symmetric, jointly analytic \( K : \mathbb{X} \times \mathbb{X} \to \mathbb{R} \) is a Kreın kernel [3]. In this paper we are primarily concerned with Kreın kernels of the form:

\[
K(x, x') = k((x, x')_g)
\]

where we use the notation \( (x, x')_g = \sum_i g_i x_i x'_i \) to represent the weighted indefinite inner product (in the special case \( g > 0 \) we instead write \( (x, x')_g \) to emphasise that this is an inner-product in this case). Clearly if \( k \) is analytic then \( K \) must be a Kreın-kernel. Indeed, if \( k \) is entire then we can construct the Taylor expansion \( k(\chi) = \sum a_i \chi^i \), and it follows that:

\[
\begin{align*}
K(x, x') &= k((x, x')_g) = k_+((x, x')_g) - k_-((x, x')_g) = (\varphi(x), \varphi(x'))_{\mathcal{K}} \\
\hat{K}(x, x') &= \hat{k}((x, x')_g) = \hat{k}_+((x, x')_g) + \hat{k}_-((x, x')_g) = (\varphi(x), \varphi(x'))_{\mathcal{K}}
\end{align*}
\]

where, using multi-index notation \( \varphi(x) = [\varphi_1(x)]_{i \in \mathbb{N}_{\text{dim}(x)}} \) and \( \gamma = [\gamma_i]_{i \in \mathbb{N}_{\text{dim}(x)}} \), where \( \varphi_1(x) = \prod_j x_j^\gamma \) and \( \gamma = (1_{\text{ord}(j)} \gamma_j)_j a_{\text{ord}(i)} \). We may further note that:

\[
K_\pm((x, x')_g) = \langle \varphi(x), \varphi(x') \rangle_{(\pm \gamma)_{\mathcal{K}}}
\]

where \( (a)_+ = \max\{a, 0\} \) and \( (a)_+ = [(a_0)_+, (a_1)_+, \ldots] \). See table [1] for examples of functions \( k \) defining Kreın kernels, along with the functions \( \hat{k} \) defining the associated kernel. Importantly in our context, we note that this expansion applies to more general entire functions of \( m \)-indefinite-inner-products, specifically:

\[
\begin{align*}
K((x, x', \ldots, x''')_m, g) &= k_+((x, x', \ldots, x''')_m, g) - k_-((x, x', \ldots, x''')_m, g) \\
&= (\varphi(x), \varphi(x'), \ldots, \varphi(x''))_{\mathcal{K}} \\
\hat{K}((x, x', \ldots, x''')_m, g) &= \hat{k}_+((x, x', \ldots, x''')_m, g) + \hat{k}_-((x, x', \ldots, x''')_m, g) \\
&= (\varphi(x), \varphi(x'), \ldots, \varphi(x''))_{\mathcal{K}}
\end{align*}
\]
Table 1: Expansion series for Kreinin kernels. In each case $K(x,x') = \langle (\varphi(x), \varphi(x')) \rangle_m$ is a Kreinin kernel with associated kernel $\tilde{K}(x,x') = \langle (\varphi(x), \varphi(x')) \rangle_m$. The result is non-convex, but nevertheless representor theory applies to all saddle points. Alternatively, \cite{[56]} apply regularisation via the associated RKHS norm - that is, a regularisation term of the form $\lambda (f,f)_{\mathcal{K}_F}$. Once again the problem is non-convex, but superior results are reported. Following \cite{[59]}, consider the following (equivalent) regularised risk minimisation problems:

$$\begin{align*}
\min_{w \in F} L(w,x_i) + \lambda h(w) & \quad \text{(weight-centric form (primal))} \\
\min_{f \in \mathcal{K}_F} L(f,x_i) + \lambda h(f) & \quad \text{(function-centric form (dual))}
\end{align*}
$$

$w^* = \min_{w \in F} \sum_i L((w,\varphi(x_i))) + \lambda h((w,w))$

$\lambda (f,f)_{\mathcal{K}_F}$
where \( \{ (x_i, y_i) : i \in \mathbb{N} \} \) is some training set, \( L \) is a (differentiable) loss function, and \( h \) is differentiable. As per [59], it is not difficult to see that this has a solution of the form \( f^*(\cdot) = \sum \alpha_i K(\cdot, x_i) \) (or, equivalently in weight-centric notation, \( w^* = \sum \alpha_i \phi(x_i) \)), where \( \alpha \in \mathbb{R}^n \).

Note that, while (30) appears directly analogous to a typical regularised risk minimisation problem in reproducing kernel Hilbert space, the non-convexity of this form makes finding \( \alpha \) somewhat complicated [59], which may explain why it does not appear to have been widely adopted despite promising performance.

## 2 Supplementary: Details of Proofs

**Lemma 2:** Let \( \sigma \) be an entire function with Taylor expansion \( \sigma(\xi) = \sum_i a_i \xi^i \), and let \((\cdot, \ldots)_{m, \mu}\) be an \( m \)-indefinite-inner-product defined by metric \( \mu \in \mathbb{R}^n \) (section 1.7). Then:

\[
\sigma((\cdot, \ldots, x^{(m+1)} ))_{m, \mu} = ((\phi(\cdot), \ldots, \phi(x^{(m+1)} )))_{m, \gamma \circ \phi(\mu)}
\]

where \( \phi : \mathbb{R}^n \to \mathcal{F} \) is a feature map and \( \gamma \in \mathcal{F} \), both independent of \( m \) and \( \mu \). Using multi-index notation, \( \phi(x) = [\phi_i(x)]_{i \in \mathbb{N}} \) and \( \gamma = [\gamma_i]_{i \in \mathbb{N}} \), where:

\[
\phi_i(x) = \prod_j x_j^{i_j}, \quad \gamma_i = \left( \frac{\text{sum}(i)}{i_1! \cdots i_n!} \right) a_{\text{sum}(i)}
\]

Moreover, \( \forall t \in \mathbb{R}^D, \forall u_0, \ldots, u_{(n-1)}, v, v', \ldots, \mu \in \mathbb{R}^p \):

\[
\sigma \left( \sum_j t_j (u_{(j)}(v, v', \ldots)_{m, \mu}) = \ldots \right)
\]

where \( t_\bullet = t \otimes 1_p \) and \( u_\bullet = [u_0^T, u_{(1)}^T, \ldots]_T \).

**Proof:** Equation (10) follows directly by substituting the \( m \)-indefinite-inner-product into the Taylor expansion of \( k \) and applying the multinomial expansion. For (12) we expand, noting that:

\[
t_\bullet = t \otimes 1_p \quad \eta_{(0)} = u_i(1, \ldots, t_\bullet \otimes 1_p), \quad t_{\bullet i} = t_{i-p} \otimes 1_p,
\]

where \( \cdot | \cdot \) is floor:

\[
\sum_j \in \mathbb{N} t_j (u_{(j)}(v, v', \ldots)_{m, \mu}) = \sum_{j \in \mathbb{N} \cap \mathbb{N}_p} t_j u_{(j)} v_{(j)} \ldots \mu_k
\]

Substituting and apply (10):

\[
k \left( \sum_j \in \mathbb{N} t_j (u_{(j)}(v, v', \ldots)_{m, \mu}) \right) = ((\phi(\bullet), \phi(u_\bullet), \phi(v_\bullet), \phi(v'_\bullet), \ldots)_{m+1, \gamma \circ \phi(\bullet)_{\mu}}
\]

which completes the proof.

**Theorem 3:** The deep network (7) has equivalent form:

\[
\phi(x) = \phi(1_{H_{d-2}} \otimes \phi(\ldots, \phi(1_{H_1} \otimes \phi(1_{H_0} \otimes \phi(x)))))
\]

where the feature map, weight vector and metric are [9]

\[
\phi(x) = \phi(1_{H_{d-2}} \otimes \phi(\ldots, \phi(1_{H_1} \otimes \phi(1_{H_0} \otimes \phi(x))))))
\]

\[
W = \bigotimes \phi_{[d-1]}(W_{[i]})
\]

\[
g = \gamma_{[d-1]} \otimes \phi(1_{H_{d-2}} \otimes \phi(\ldots, \phi(1_{H_1} \otimes \phi(1_{H_0} \otimes \phi(x))))))
\]

\[
(W_{i}, \text{is row } i \text{ of matrix } W \text{ (Matlab style notation).})
\]

\[
\text{(which depend only on the deep network structure and):}
\]

\[
\phi_{[i,j]}(W) = 1_{H_j} \otimes \phi(\ldots 1_{H_{i+1}} \otimes \phi(W_{0,i} \otimes \phi(W_{1,i}) \otimes \ldots \otimes (1_{H_i} \otimes \phi(1_{H_0} \otimes \phi(1_{H_1})))
\]

\[
\text{where } W_{i}, \text{ is row } i \text{ of matrix } W \text{ (Matlab style notation).}
\]

\footnote{For the curious we provide an explicit form of the feature map, metric and weight expansion in the supplementary material.}
Proof: Let \( m_0 = 1 \) and \( m_q = \dim(x_{[q-1]}) \) (see below). Let \( w_{[q],i} = W_{[q],i} \), and let \( o_{[q]}(x) \) denote the output of layer \( q \). We proceed as follows:

Layer 0: As per figure 1 and equation (10), the output of layer 0 is:

\[
\begin{align*}
o_{[0]}(x) &= \begin{bmatrix} \sigma_0((w_{[0,0],0},x)) \\ \sigma_0((w_{[0,1],0},x)) \\ \vdots \\ \sigma_0((w_{[0,i],0},x)) \\ \vdots \end{bmatrix} \\
&= \begin{bmatrix} \sigma_0((w_{[0,0],0},x,1))_{2} \\ \sigma_0((w_{[0,1],0},x,1))_{2} \\ \vdots \\ \sigma_0((w_{[0,i],0},x,1))_{2} \\ \vdots \end{bmatrix}
\end{align*}
\]  

where \( x \) has been propagated through layer 0 to obtain \( x_{[1]} \), and likewise \( W_{[0]} \) has been propagated through layer 0 to obtain \( w_{[0,1]} \). Specifically:

\[
\begin{align*}
x_{[1]} & = \phi(x) \\
g_{[1]} & = \gamma_{[0]} \circ \phi(1_d) \\
w_{[0,1],i} & = \phi(w_{[0,i]} \times 1_m)
\end{align*}
\]

Layer 1: As per figure 1 and equation (12), the output of layer 1 is:

\[
\begin{align*}
o_{[1]}(x) &= \begin{bmatrix} \sigma_1((w_{[1,0]} \times 1_m,x_{[1]})) \\ \sigma_1((w_{[1,1]} \times 1_m,x_{[1]})) \\ \vdots \\ \sigma_1((w_{[1,i]} \times 1_m,x_{[1]})) \\ \vdots \end{bmatrix} \\
&= \begin{bmatrix} \sigma_1((w_{[1,0]} \times 1_m,x_{[1]}))_{3} \\ \sigma_1((w_{[1,1]} \times 1_m,x_{[1]}))_{3} \\ \vdots \\ \sigma_1((w_{[1,i]} \times 1_m,x_{[1]}))_{3} \\ \vdots \end{bmatrix}
\end{align*}
\]

where \( x_{[1]} \) has been propagated through layer 1 to obtain \( x_{[2]} \), and likewise for the weights \( W_{[1]} \) is the result of propagating weights \( W_{[1]} \) through layers \( i, i+1, \ldots, j-1 \). So:

\[
\begin{align*}
x_{[2]} & = \phi(x_{[1]}) \\
g_{[2]} & = \gamma_{[1]} \circ \phi(g_{[1]}) \\
w_{[0,2]} & = \phi(w_{[0,1]}) \\
w_{[1,2],i} & = \phi(w_{[1,2],i} \times 1_m)
\end{align*}
\]

Layer \( q \): Repeating the same approach, at layer \( q \):

\[
\begin{align*}
o_{[q]}(x) &= \begin{bmatrix} \sigma_q((w_{[q,0]} \times 1_m,x_{[q-1]})) \\ \sigma_q((w_{[q,1]} \times 1_m,x_{[q-1]})) \\ \vdots \\ \sigma_q((w_{[q,i]} \times 1_m,x_{[q-1]})) \\ \vdots \end{bmatrix} \\
&= \begin{bmatrix} \sigma_q((w_{[q,0]} \times 1_m,x_{[q-1]}))_{q} \\ \sigma_q((w_{[q,1]} \times 1_m,x_{[q-1]}))_{q} \\ \vdots \\ \sigma_q((w_{[q,i]} \times 1_m,x_{[q-1]}))_{q} \\ \vdots \end{bmatrix}
\end{align*}
\]

where propagation through layer \( q \) gives:

\[
\begin{align*}
x_{[q+1]} & = \phi(x_{[q]}) \\
g_{[q+1]} & = \gamma_{[q]} \circ \phi(g_{[q]}) \\
w_{[0,q+1]} & = \phi(w_{[0,q]}) \\
w_{[1,q+1],i} & = \phi(w_{[1,q+1],i} \times 1_m)
\end{align*}
\]

\[
\begin{align*}
x_{[q+1]} & = 1_{H_q} \times x_{[q+1]} \\
g_{[q+1]} & = 1_{H_q} \times g_{[q+1]} \\
w_{[0,q+1]} & = 1_{H_q} \times w_{[0,q+1]} \\
w_{[1,q+1],i} & = 1_{H_q} \times w_{[1,q+1],i}
\end{align*}
\]

Output layer: Propagation through the output layer \( d-1 \) follows the same formula, noting that \( n_{d-1} = 1 \) and so \( x_{[d]} = x_{[d]} \), \( w_{[d-1,d]} = w_{[d-1,d]} \), etc. Hence:

\[
o(x) = o_{[d-1]}(x) = ((w_{[d-1,d],0} \times x_{[d-1]}), \ldots, w_{[1,d],w_{[0,d]}}, x_{[d]})_{d+1:G(d)}
\]

where (32) applies with \( q = d-1 \).
To simplify our notation we define the functions:

\[ \phi_{[i,j]}(z_0, z_1, \ldots) = 1_{H_j} \otimes \phi(\ldots 1_{H_{i+1}} \otimes \phi((\phi^T(z_0 \otimes 1_{m_i}), \phi^T(z_1 \otimes 1_{m_i}), \ldots)^T)) \]

\[ \varphi(x) = \phi_{[0,d-1]}(x, x, n_0 \quad \text{terms}, x) = \phi(1_{H_{d-2}} \otimes \phi(\ldots 1_{H_1} \otimes \phi(1_{H_0} \otimes \phi(x)))) \]

where \( \phi_{[i,j]} \) takes (vectorised) weight vectors from the input of layer \( i \) and propagates them to the output of layer \( j \), and \( \varphi \) does the same for input vectors. Using this notation, it is not difficult to see that \( \forall q \in \mathbb{N}_d; \)

\[ w_{[q,d]} = \phi_{[q,d]}(w_{[0,0]}, w_{[q,1]}, \ldots, w_{[q,n_q-1]}) \]

and hence, defining:

\[ w = w_{[d-1,d]} \odot \ldots \odot w_{[1,d]} \odot w_{[0,d]} \]

\[ g = g_{[d]} = \gamma_{[d-1]} \otimes \phi(1_{H_{d-2}} \otimes (\gamma_{[d-2]} \otimes \phi(\ldots 1_{H_0} \otimes (\gamma_{[0]} \otimes \phi(1_D)) \ldots))) \]

the overall network may be written in the simple form:

\[ y(x) = ((w, \varphi(x)))_{2,g} \]

Finally, using the form of \( \varphi \) (a monomial map with terms of the form \( x_j^i \)) we have that:

\[ 1_{m_1} = \phi(\ldots 1_{H_0} \otimes \phi(1_{H_0} \otimes \phi(1_D))) \]

and also \( \phi(a \otimes b) = \phi(a) \otimes \phi(a) \) and \( \phi(a \otimes b) = \phi(a) \otimes \phi(a) \). It follows that:

\[ \phi_{[i,j]}(W) = 1_{H_j} \otimes \phi(\ldots 1_{H_{i+1}} \otimes \phi(\begin{bmatrix} \phi(W_{0,:} \otimes 1_{m_i}) \\ \phi(W_{1,:} \otimes 1_{m_i}) \\ \vdots \end{bmatrix} \circ (1_{H_1} \otimes \phi(\ldots 1_{H_0} \otimes \phi(1_D))))) \]

which completes the proof. \( \Box \)

**Theorem 4** Let the feature map \( \varphi : \mathbb{R}^D \rightarrow \mathcal{F} \) and metric \( g \) be defined by the deep network \( \Pi \) as per theorem 3. Then:

\[ K_{\Pi}(x, x') = ((\varphi(x), \varphi(x')))_{g} = \sigma_{[d-1]}(\ldots H_{d-2}\sigma_{[d-2]}(H_{d-3}\ldots H_1\sigma_{[1]}(H_0\sigma_{[0]}((\langle x, x' \rangle_1))))) \]

is the corresponding Krein kernel.

**Proof:** Using the notation and definitions in the proof of theorem 3 and applying the definitions (theorem 3 and lemma 2):

\[ K_{\Pi}(x, x') = ((\varphi(x), \varphi(x')))_{g} = ((\phi_{[0,d-1]}(x), \phi_{[0,d-1]}(x'))_{2,g_{[d]}} = (\phi(\phi_{[0,d-2]}(x), \phi(\phi_{[0,d-2]}(x')))_{\gamma_{[d-1]} \otimes \phi(\sigma_{[d-1]})} = \sigma_{[d-1]}(H_{d-2}\sigma_{[d-2]}(\phi_{[0,d-3]}(x), \phi_{[0,d-3]}(x'))_{\gamma_{[d-3]} \otimes \phi(\sigma_{[d-3]})}) = \ldots = \sigma_{[d-1]}(H_{d-2}\sigma_{[d-2]}(\ldots H_1\sigma_{[1]}(H_0\sigma_{[0]}((\langle x, x' \rangle_1))))) \]

and likewise, the associated kernel \( \tilde{K}_{\Pi} \) is:

\[ \tilde{K}_{\Pi}(x, x') = ((\varphi(x), \varphi(x')))_{\tilde{g}} = ((\phi(\phi_{[0,d-2]}(x), \phi(\phi_{[0,d-2]}(x')))_{\gamma_{[d-1]} \otimes \phi(\sigma_{[d-1]})]) = \tilde{\sigma}_{[d-1]}(H_{d-2}\tilde{\sigma}_{[d-2]}(\phi_{[0,d-3]}(x), \phi_{[0,d-3]}(x'))_{\gamma_{[d-3]} \otimes \phi(\sigma_{[d-3]})}) = \ldots = \tilde{\sigma}_{[d-1]}(H_{d-2}\tilde{\sigma}_{[d-2]}(\ldots H_1\tilde{\sigma}_{[1]}(H_0\tilde{\sigma}_{[0]}((\langle x, x' \rangle_1))))) \]
where, if $\sigma_q(x) = \sum_i a_i x^i$ (recall that $\sigma_q$ is entire, so this Taylor series exists) then $\sigma_q(x) = \sum_i |a_i| x^i$. See Table 1 in the supplementary for examples.

**Theorem 5.** Let $K_{\text{lin}} : \mathbb{R}^D \times \mathbb{R}^D \rightarrow \mathbb{R}$ be a kernel such that $x \rightarrow K_{\text{lin}}(x, x) \in L_1(\mathbb{R}^D, \nu)$ and $K_{\text{lin}}(x, x) \geq 0 \ \forall x \in \mathbb{R}^D$. Then:

$$
\mathcal{R}_N(\mathbf{f}_{g_{\nu d}}) \leq \frac{R_{g_{\nu d}}}{\sqrt{N}} \left( \int_x K_{\text{lin}}(x, x) \, d\nu(x) \right)^{1/2} \quad (24)
$$

where $\mathbf{f}_{g_{\nu d}}$ is as per (22).

**Proof:** Following [46], we first prove the following, where we use the reproducing kernel property at steps labelled *, the Cauchy-Schwarz inequality at the step labelled #, and the fact that $K(x, x) \geq 0$ at the step labelled †:

$$
\mathbb{E}_{\sigma} \left[ \sup_{f \in \mathcal{K}_{\text{lin}}} \left| \sum_i \epsilon_i f(x_i) \right|^2 \right] = \mathbb{E}_{\sigma} \left[ \sup_{f \in \mathcal{K}_{\text{lin}}} \left| \sum_i (\sigma_i K_{\text{lin}}(x_i, \cdot), f(\cdot))_{\mathcal{K}_{\text{lin}}} \right|^2 \right] \\
\leq \mathbb{E}_{\sigma} \left[ \sup_{f \in \mathcal{K}_{\text{lin}}} \left| \sum_i (\sigma_i K_{\text{lin}}(x_i, \cdot), f(\cdot))_{\mathcal{K}_{\text{lin}}} \right|^2 \right] \\
\leq \mathbb{E}_{\sigma} \left[ \sup_{f \in \mathcal{K}_{\text{lin}}} \left| \sum_i (\sigma_i K_{\text{lin}}(x_i, \cdot), f(\cdot))_{\mathcal{K}_{\text{lin}}} \right|^2 \right]
$$

Then, using the properties of Rademacher complexity (again following [46]), we have that:

$$
\mathbb{E}_{\mu, \sigma} \left[ \sup_{f \in \mathcal{K}_{\text{lin}}} \left| \frac{1}{\sqrt{N}} \sum_i \epsilon_i f(x_i) \right| \right] \leq \frac{1}{\sqrt{N}} \left( \mathbb{E}_{\mu, \sigma} \left[ \sup_{f \in \mathcal{K}_{\text{lin}}} \left| \sum_i \epsilon_i f(x_i) \right|^2 \right] \right)^{1/2} \\
\leq \frac{1}{\sqrt{N}} \mathbb{E}_{\mu} \left[ \sum_i K_{\text{lin}}(x_i, x_i) \right]^{1/2} \\
\leq \frac{R_{g_{\nu d}}}{\sqrt{N}} \left( \int_x K_{\text{lin}}(x, x) \, d\mu(x) \right)^{1/2}
$$

which completes the proof. □

**Lemma 6.** For the deep network described:

$$(w, w)_k = \sum_{d=1}^D \sigma_{[d-1]} \left( \|W_{[d-1]}\|_2^2 \sum_{d'=1}^{d-2} \sigma_{[d-2]} \left( \|W_{[d-2]}\|_2^2 \cdots \sum_{d''} \sigma_{[0]} \left( \|W_{[0]}\|_2^2 \right) \right) \right)$$

If $\sigma_q$ is non-decreasing Lipschitz with constant $L_q$, concave on $\mathbb{R}_+$, and $\sigma_q(0) = 0$; and $\| \cdot \|_F$ (the Frobenius norm) are topologically equivalent with $e_q' \| W \|_{[q]} \leq \| W \|_F \leq e_q \| W \|_{[q]}$ for all $W$ (which is true for elementwise norms, which we assume throughout); then:

$$
0 \leq \sigma_{[d-1]} \left( e_{[d-1]}^2 \|W_{[d-1]}\|_{[d-1]}^2 \cdots \sigma_{[0]} \left( e_0^2 \|W_{[0]}\|_{[0]}^2 \right) \right) \leq (w, w)_k \\
\leq M \prod_q \| W_{[q]} \|_2^2 \leq M \left( \frac{1}{2} \sum_q \| W_{[q]} \|_{[q]}^2 \right)^d
$$

where $M = \prod_q L_q e_q^2$ is a positive constant. Moreover if the ReLU activation function is used everywhere and $\| \cdot \|_{[q]} = \| \cdot \|_F$ then $M = 1$ and the bounds simplify to:

$$
0 \leq (w, w)_k = \prod_q \| W_{[q]} \|_{[q]}^2 = \left( \frac{1}{2} \sum_q \| W_{[q]} \|_F^2 \right)^d
$$

for $r_{\text{lin}}$ as per [17].
Proof: We have from theorem that $w = \bigotimes q \phi_{[i,d-1]}(W[q])$, so:

$$
\langle w, w \rangle_{2,g} = \langle \phi \left( 1_{H_{d-2}} \otimes \phi \left( \phi_{[d-3,0]}(W_{[d-1]}) \right) \right),
\phi \left( 1_{H_{d-2}} \otimes \phi \left( \phi_{[d-3,0]}(W_{[d-2]}) \right) \right) \rangle \ldots
$$

Using Lemma we see that:

$$
\langle w, w \rangle_{2,g} = \sum_i \sigma_{[d-1]} \left( \left\| W_{[d-1] i, :} \right\|_F^2 \right) \ldots
$$

and the first expression follows by recursion, noting that a single weight matrix is “peeled off” with each successive layer.

It follows from the first expansion and the fact that $\sigma_{[q]}$ is non-decreasing and $\sigma_{[q]}(0) = 0$ that $\langle w, w \rangle_{2,g} \geq 0$. For the lower bound we use that $\sigma_{[q]}(0) = 0$ and $\sigma_{[q]}$ is concave, so $\forall a, b \geq 0, \sigma_{[q]}(a + b) \leq \sigma_{[q]}(a) + \sigma_{[q]}(b)$, and hence:

$$
\sigma_{[q]} \left( \left\| W_{[q]} \right\|_F^2 \right) = \sigma_{[q]} \left( \sum_i \left\| W_{[q] i, :} \right\|_F^2 \right) \leq \sum_i \sigma_{[q]} \left( \left\| W_{[q] i, :} \right\|_F^2 \right)
$$

which, applied repeatedly to the first equation in the theorem, and using the topological equivalence of norms $\left\| \cdot \right\|_F$ and $\left\| \cdot \right\|_F^2$ gives the desired lower bound on $\langle w, w \rangle_{2,g}$.

For the upper bounds we use that $\sigma_{[q]}$ is non-decreasing, Lipschitz (or appropriately bounded), and $\sigma_{[q]}(0) = 0$, and that $\left\| \cdot \right\|_F$ and $\left\| \cdot \right\|_F^2$ are topologically equivalent, to obtain the first bound; and the AM-GM inequality to obtain the second.

In the case of ReLU and Frobenius normalisation we immediately have that $L_q = e_q^2 = 1$, so $M = 1$; and $\sigma_{[q]}(\xi) = (\xi)_+$ everywhere in the bounds as all arguments are non-negative. It is then straightforward to see that:

$$
0 \leq \langle w, w \rangle_g = \prod_q \left\| W_{[q]} \right\|_F^2 \leq \left( \frac{1}{d} \sum_q \left\| W_{[q]} \right\|_F^2 \right)^d
$$

Finally we use proof of Claim 1) and the ambiguity in the definition of $r_{\text{lin}}$ to obtain the final equality.

$\square$
3 Supplementary: An Explicit Form of the Feature Map, Metric and Weight Vector

To obtain an explicit form for the feature map φ, metric g and weights w for the flat representation of a deep network, let \( \tilde{\phi}_{[q], j[q], j[q]} (W_{[q]}) = \prod_{k_{[q]}} W_{[q]_{[q]_{[q]_{[q]}}} \), \( \tilde{\phi}_{[q], i[q], j[q]} (x) = \prod_{k_{[q]}} x_{[q]_{[q]_{[q]_{[q]}}} \), \( j[q] \in \mathbb{N}_{H_q}, i[0] \in \mathbb{N}^D, k[0] \in \mathbb{N}^D, k[q] = (i[q-1], j[q-1]) \). The feature map is:

\[
\varphi (x) = \tilde{\phi}_{[d-1]} \left( \prod_{i[d-2]} \tilde{\phi}_{[d-2]} \left( \prod_{i[d-1]} \left( \tilde{\phi}_{[d]} (x) \right) \right) \right). 
\]

Noting that:

\[
\tilde{\phi}_{[0, d-1]} (W_{[0]}) = \prod_{k_{[d-2]}} \prod_{i[d-1]} \prod_{j[d-1]} \prod_{i[0]} \prod_{j[0]} x_{[0]_{[d-1]}} \prod_{k_{[0]}} w_{[0]_{[d-1]}}^{(i[0-1], i[d-2], j[d-2], i[d-1], j[d-1], i[0], j[0], k[0])}, \quad i[0-1]
\]

and:

\[
\tilde{\phi}_{[q, d-1]} (W_{[q]}) = \prod_{k_{[d-2]}} \prod_{i[d-1]} \prod_{j[d-1]} \prod_{i[0]} \prod_{j[0]} w_{[q]_{[d-1]}}^{(i[q-1], i[d-2], j[d-2], i[d-1], j[d-1], i[0], j[0], k[0])}, \quad i[d-1]
\]

we see that the weight vector is:

\[
w = \bigotimes_q \tilde{\phi}_{[q, d-1]} (W_{[q]}). 
\]

and finally, by inspection, the metric is:

\[
g = \gamma_{[d-1]} (i_{[d-1], 0}) \prod_{i[d-2]} \gamma_{[d-2]} (i_{[d-2], j_{[d-2]}}) \prod_{i[0]} \gamma_{[0]} (i_{[0], j_{[0]}}).
\]