Oscillatory behavior of nonlinear Hilfer fractional difference equations

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Abstract
In this paper, we study the oscillation behavior for higher order nonlinear Hilfer fractional difference equations of the type

\[ \Delta_{\alpha}^{\beta} y(x) + f_{1}(x, y(x + \alpha)) = \omega(x) + f_{2}(x, y(x + \alpha)), \quad x \in \mathbb{N}_{\beta+n-\alpha}, \]

\[ \Delta_{\alpha}^{k-(\beta-\gamma)} y(x)|_{x=\beta+n-\gamma} = y_{k}, \quad k = 0, 1, \ldots, n, \]

where \([\alpha] = n, n \in \mathbb{N}, 0 \leq \beta \leq 1\). We introduce some sufficient conditions for all solutions and give an illustrative example for our results.

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1 Introduction
In recent years, fractional differential equations and fractional difference equations have been attractive areas for researchers. This is because using in modeling real problems fractional order equations gives highly accurate results rather than integer order equations [1, 2]. Studying the behavior of solutions is very important for analyzing equations, so the existence and uniqueness, stability, and oscillation of the solutions are the areas where researchers have worked most, recently. Many studies have been done on the oscillation of fractional differential equations [3–11], functional differential equations [12–15], and dynamic equations on time scales [16, 17]. However, few researchers addressed the oscillation of fractional difference equations [18–28].

In [29], Haider et al. introduced a new definition of a fractional difference operator which is a generalization of Riemann–Liouville and Caputo type difference operator. This operator interpolates the Riemann–Liouville like fractional difference (\(\beta = 0\)) and the Caputo like fractional difference (\(\beta = 1\)). The type-parameter produces more types of stationary states and provides an extra degree of freedom on the initial condition. No one has studied, to the best of our knowledge, the oscillation of equations involving the Hilfer difference operator in the literature.
In [5], Grace et al. initiated the oscillation theory for fractional differential equations of the form
\begin{align*}
D_0^\alpha y(x) + f_1(x, y) &= ν(x) + f_2(x, y), \\
\lim_{t \to a^+} J^{1-\alpha}_a y(x) &= b_1,
\end{align*}
where $D_0^\alpha$ is the Riemann–Liouville differential operator of order $\alpha$, $0 < \alpha \leq 1$ and the functions $f_1, f_2, ν$ are continuous. The results are also stated when the Riemann–Liouville differential operator is replaced by Caputo’s differential operator.

In [21], Marian et al. gave similar conclusions for the oscillation behavior of the nonlinear fractional difference equations of the form
\begin{align*}
\Delta_1^\alpha y(x) + f_1(x, y(x + α)) &= ν(x) + f_2(x, y(x + α)), \\
\Delta_1^{α-1} y(x) |_{x=0} &= x_0,
\end{align*}
where $\Delta_1^\alpha$ denotes the Riemann–Liouville like discrete fractional difference operator of order $\alpha$, $0 < \alpha \leq 1$. In [22], Marian et al. obtained some new results for the initial value problem (1).

In [20], Kısalar et al. considered higher order fractional nonlinear difference equation of the form
\begin{align*}
\Delta_1^\alpha y(x) + f_1(x, y(x + α)) &= ν(x) + f_2(x, y(x + α)), \\
\Delta_1^{α-1} y(x) |_{x=0} &= x_0,
\end{align*}
where $\Delta_1^\alpha$ denotes the Riemann–Liouville like discrete fractional difference operator of order $\alpha$ and $m \geq 1$.

This paper aims to state some oscillation criteria for a class of higher order nonlinear Hilfer fractional difference equations. Some sufficient conditions will be given for the oscillation of the solution of Hilfer fractional difference equations. The results also contain new conditions for the oscillation of the solutions of the Riemann–Liouville and Caputo difference equations.

2 Preliminaries

Definition 1 ([30]) Suppose $f$ is a real valued function defined on $\mathbb{N}_a$ and $α > 0$. Then the $α$th fractional sum of $f$ is defined by
\begin{align}
\Delta_a^{-α} f(x) := \sum_{t=a}^{x-α} h_{α-1}(x, σ(t)) f(t)
\end{align}
for $x \in \mathbb{N}_{a+α}$, where $t^α$ is the generalized falling function and $h_α(t, τ) = \frac{(t-τ)^α}{Γ[α+1]}$ is the $α$th fractional Taylor monomial.

Definition 2 ([30]) Let $f$ be a real valued function defined on $\mathbb{N}_a$ and $[α] = n$. Then the $α$th Riemann–Liouville fractional difference of $f$, defined by
\begin{align}
\Delta_a^α f(x) := Δ^n Δ_a^{-(n-α)} f(x), \quad x \in \mathbb{N}_{a+n-α}.
\end{align}
Lemma 1 ([30]) Let \( f : \mathbb{N}_a \to \mathbb{R}, k \in \mathbb{N}_0, m - 1 < \alpha < n \) and \( n - 1 < \beta \leq n \). Then
\[ \Delta_{x+\alpha}^{-\gamma} \Delta_{x+\alpha}^\beta f(x) = \Delta_{x+\alpha}^{-\gamma} f(x) - \sum_{i=0}^{n-1} h_{\alpha+\gamma+i}(x, a + n - \alpha \Delta_{x+\gamma}^i f(a + n - \beta), \) for \( x \in \mathbb{N}_{x+\gamma+n-\alpha}. \)

Theorem 1 (Fractional sum power rule [30]) Let \( \mu \geq 0 \) and \( \nu > 0 \). Then
\[ \Delta_{x+\alpha}^{-\nu} (t-a)^\mu = \frac{\Gamma(\mu+1)}{\Gamma(\mu+\nu+1)} (t-a)^{\mu+\nu} \] for \( t \in \mathbb{N}_a + \mu + \nu. \)

In [29], Haider et al. introduced a Hilfer like fractional difference operator.

Definition 3 Assume \( f : \mathbb{N}_a \to \mathbb{R} \). Then the fractional difference of order \( n - 1 < \alpha < n \) and type \( 0 \leq \beta \leq 1 \) is defined by
\[ \Delta_{x+\alpha}^\beta f(x) = \Delta_{x+\alpha}^{-\beta}(n-\alpha) \Delta_{x+\alpha}^{1-\beta}(n-\alpha) f(x) \]
for \( x \in \mathbb{N}_{x+n-\alpha}. \)

Lemma 2 The Hilfer fractional difference can be written as follows:
\[ \Delta_{x+\alpha}^\beta f(x) = \Delta_{x+\alpha-n+\gamma}^{-\gamma} \Delta_{x+\alpha}^n \Delta_{x+\alpha}^{-n+\gamma} f(x), \]
where \( \gamma = \alpha + \beta(n-\alpha). \)

Lemma 3 Let \( f \) be a real valued function defined on \( \mathbb{N}_a, n - 1 < \alpha < n \) and \( 0 \leq \beta \leq 1 \). Then
\( i \) \( \Delta_{x+\alpha}^\alpha \Delta_{x+\alpha}^\beta f(x) = \Delta_{x+\alpha}^{-\alpha}(n-\alpha) \Delta_{x+\alpha}^\beta f(x), \)
\( ii \) \( \Delta_{x+\alpha}^\beta \Delta_{x+\alpha}^\alpha f(x) = \Delta_{x+\alpha}^{-\beta}(n-\alpha) \Delta_{x+\alpha}^\alpha f(x), \)
for \( x \in \mathbb{N}_{x+1}. \)

Proof (i) We have
\[ \Delta_{x+\alpha-n-\gamma}^\alpha \Delta_{x+\alpha}^\beta f(x) = \Delta_{x+\alpha-n-\gamma}^\alpha \Delta_{x+\alpha-n+\gamma} \Delta_{x+\alpha}^{-\gamma} \Delta_{x+\alpha}^n \Delta_{x+\alpha}^{-1+\gamma} f(x) \]
\[ = \Delta_{x+\alpha-n+\gamma} \Delta_{x+\alpha}^{-n} \Delta_{x+\alpha}^{-1+\gamma} f(x) \]
\[ = \Delta_{x+\alpha-n+\gamma} \Delta_{x+\alpha}^{-n} \Delta_{x+\alpha}^\beta f(x). \]

(ii) We have
\[ \Delta_{x+\alpha}^\beta \Delta_{x+\alpha}^\alpha f(x) = \Delta_{x+\alpha}^{\beta}(n-\alpha) \Delta_{x+\alpha-n+\gamma}^\gamma f(x) \]
\[ = \Delta_{x+\alpha-n+\gamma} \Delta_{x+\alpha}^{\beta}(n-\alpha) f(x) \]
\[ = \Delta_{x+\alpha}^{\beta}(n-\alpha) \Delta_{x+\alpha}(n-\alpha) f(x). \]
In this paper, we denote the oscillation criterion of the nonlinear Hilfer like fractional difference equation

\[
\Delta^\alpha_{a+1} y(x) + f_1(x, y(x + \alpha)) = \omega(x) + f_2(x, y(x + \alpha)), \quad x \in \mathbb{N}_{a+1},
\]

where \( n - 1 < \alpha \leq n \) (\( n \in \mathbb{N}_0 \)) and \( 0 \leq \beta \leq 1 \), \( \omega \) and \( f_k : [0, +\infty) \times \mathbb{R} \to \mathbb{R}, k = 1, 2 \) are continuous.

**Lemma 4** ([31]; Young’s inequality)

(i) Assume \( \chi, \xi \geq 0, u > 1 \) and \( \frac{1}{u} + \frac{1}{v} = 1 \). Then the following inequality holds if and only if \( \xi = \chi^{u-1} \):

\[
\chi \xi \leq \frac{1}{u} \chi^u + \frac{1}{v} \xi^v. \tag{7}
\]

(ii) Assume \( \chi \geq 0, \xi > 0, 0 < u < 1 \) and \( \frac{1}{u} + \frac{1}{v} = 1 \). Then the following inequality holds if and only if \( \xi = \chi^{u-1} \):

\[
\chi \xi \geq \frac{1}{u} \chi^u + \frac{1}{v} \xi^v. \tag{8}
\]

**Lemma 5** The unique solution of the initial value problem (6) is

\[
y(x) = \sum_{k=0}^{n-1} h_{\gamma-n-\xi k}(x, a + n - \gamma) y_k
\]

\[
+ \sum_{t=a+1}^{x-a} h_{a-1}(x, \sigma(t)) \left[ \omega(t) + f_2(t, y(t + \alpha)) - f_1(t, y(t + \alpha)) \right]
\]

for all \( x \in \mathbb{N}_{a+1} \).

**Proof** Applying the \( \Delta^{-\alpha}_{a+1} \alpha \) operator to both sides of (6), we get

\[
\Delta^{-\alpha}_{a+1} \Delta^\alpha_{a} y(x) = \Delta^{-\alpha}_{a+1} \left[ \omega(x) + f_2(x, y(x + \alpha)) - f_1(x, y(x + \alpha)) \right]. \tag{10}
\]

Using equation (i) in Lemma 3 for the left-hand side of (10), we have

\[
\Delta^{-\alpha}_{a+1} \Delta^\alpha_{a} y(x) = \Delta^{-\alpha(\alpha+\beta(n-\alpha))}_{a+1} \Delta^\beta_{a} \Delta^{-\alpha}_{a} y(x)
\]

\[
= y(x) - \sum_{k=0}^{n-1} h_{a+1}(x, a + n - \beta(n-\alpha)) \Delta^\beta_{a} \Delta^{-\alpha}_{a} y(x)
\]

\[
\times \sum_{k=0}^{n-1} h_{\gamma-n-\xi k}(x, a + n - \gamma) \Delta^\beta_{a} \Delta^{-\alpha}_{a} y(x)
\]

\[
= y(x) - \sum_{k=0}^{n-1} h_{\gamma-n-\xi k}(x, a + n - \gamma) \Delta^\beta_{a} \Delta^{-\alpha}_{a} y(x).
\]
where \( \gamma = \alpha + \beta(n - \alpha) \). Hence,

\[
y(x) = \sum_{k=0}^{n-1} h_{y_{n+k}}(x, a + n - \gamma) \Delta_{a}^{k}(n-\gamma)y(a + n - \gamma) + \Delta_{a+1}^{-\alpha} \left[ \omega(x) + f_{2}(x, y(x + \alpha)) - f_{1}(x, y(x + \alpha)) \right] \\
= \sum_{k=0}^{n-1} h_{y_{n+k}}(x, a + n - \gamma) y_{k} + \sum_{t=a+1}^{x-\alpha} h_{\alpha}(x, \sigma(t)) \left[ \omega(t) + f_{2}(t, y(t + \alpha)) - f_{1}(t, y(t + \alpha)) \right].
\]

This completes the proof. \( \square \)

### 3 Main results

In this section, we will contemplate the following conditions:

\[
f_{k}(x, y) \frac{y}{y} > 0, \quad (k = 1, 2), y \neq 0, x \geq x_{0}, \tag{11}
\]

and

\[
|f_{1}(x, y)| \geq |q_{1}(x)||y|^\mu \quad \text{and} \quad |f_{2}(x, y)| \leq |q_{2}(x)||y|^\nu, \quad y \neq 0, x \geq x_{0}, \tag{12}
\]

where \( q_{k} : [x_{0}, \infty) \to \mathbb{R}^{+}, \quad k = 1, 2 \) are continuous functions and \( \mu, \nu > 0 \) are real numbers. Also, we obtain another oscillation criterion using the following condition:

\[
|f_{1}(x, y)| \leq |q_{1}(x)||y|^\mu \quad \text{and} \quad |f_{2}(x, y)| \geq |q_{2}(x)||y|^\nu, \quad y \neq 0, x \geq x_{0}, \tag{13}
\]

where \( q_{k} : [x_{0}, \infty) \to \mathbb{R}^{+}, \quad k = 1, 2 \), are continuous functions and \( \mu, \nu > 0 \) are real numbers.

**Theorem 2** Assume the conditions (11) and (12) hold for \( \mu > \nu \). If

\[
\liminf_{x \to \infty} x^{1-\gamma} \sum_{t=T}^{x-a} h_{\alpha-1}(x, \sigma(t)) \left[ \omega(t) + K_{\mu, \nu}(t) \right] = -\infty \tag{14}
\]

and

\[
\limsup_{x \to \infty} x^{1-\gamma} \sum_{t=T}^{x-a} h_{\alpha-1}(x, \sigma(t)) \left[ \omega(t) - K_{\mu, \nu}(t) \right] = \infty, \tag{15}
\]

where \( K_{\mu, \nu}(t) = (\mu / \nu - 1)[v \eta(t) / \mu]^{\mu / (\mu - \nu)} q_{1}(v(t))^\nu / (\nu - \mu) \), then every solution of (6) is oscillatory for every sufficiently large \( T \).

**Proof** Suppose \( y(x) \) is a non-oscillatory solution of Eq. (6). In this case, assume that \( T > a \) is sufficiently large such that \( y(x) > 0 \) for \( x \geq T \).
Let $F(x) = \omega(x) + f_2(x, y(x + \alpha)) - f_1(x, y(x + \alpha))$. Then we have

$$
y(x) = \sum_{k=0}^{n-1} h_{y-n+k}(x, a + n - \gamma)y_k + \sum_{t=x-1}^{x-\alpha} h_{a-1}(x, \sigma(t))F(t)
$$

$$
\leq \sum_{k=0}^{n-1} h_{y-n+k}(x, a + n - \gamma)|y_k| + \sum_{t=x+1-a}^{x-\alpha} h_{a-1}(x, \sigma(t))|F(t)|
$$

$$
+ \sum_{t=x+1-a}^{x-\alpha} h_{a-1}(x, \sigma(t))\left[\omega(t) + f_2(t, y(t + \alpha)) - f_1(t, y(t + \alpha))\right]
$$

$$
\leq \sum_{k=0}^{n-1} h_{y-n+k}(x, a + n - \gamma)|y_k| + \sum_{t=x+1-a}^{T-1} h_{a-1}(x, \sigma(t))|F(t)|
$$

$$
+ \sum_{t=x+1-a}^{T-1} h_{a-1}(x, \sigma(t))\left[\omega(t) + q_2(t)y'(t + \alpha) - q_1(t)y''(t + \alpha)\right].
$$

Define

$$
\Phi(x) = \sum_{k=0}^{n-1} h_{y-n+k}(x, a + n - \gamma)|y_k|
$$

and

$$
\Psi(x, T) = \sum_{t=x+1-a}^{T-1} h_{a-1}(x, \sigma(t))|F(t)|;
$$

hence

$$
y(x) \leq \Phi(x) + \Psi(x, T) + \sum_{t=x+1-a}^{T-1} h_{a-1}(x, \sigma(t))\left[\omega(t) + q_2(t)y'(t + \alpha) - q_1(t)y''(t + \alpha)\right],
$$

for $x > T$. Let $t \geq T$ and take $\chi = |y|^v$, $\xi = vq_2(t)/\mu q_1(t)$, $u = \mu/v$ and $v = \mu/(\mu - v)$. Then we have

$$
q_2(t)|y(t + \alpha)|^v - q_1(t)|y(t + \alpha)|^u = \frac{\mu q_1(t)}{v} \left[|y(t + \alpha)|^v - \frac{vq_2(t)}{\mu q_1(t)}\frac{|y(t + \alpha)|^v}{\mu/v} \right]
$$

$$
= \frac{\mu q_1(t)}{v} \left[\chi^v - \frac{1}{u} \chi^u\right]
$$

$$
\leq \frac{\mu q_1(t)}{v} \frac{1}{v} \chi^v = K_{\mu,v}(t).
$$

Using (18) in inequality (17) we obtain

$$
y(x) \leq \Phi(x) + \Psi(x, T) + \sum_{t=x+1-a}^{T-1} h_{a-1}(x, \sigma(t))\left[\omega(t) + K_{\mu,v}(t)\right], \quad x > T.
$$
Multiplying both sides of (19) with $\Gamma(\gamma)x^{1-\gamma}$, we get

$$0 < \Gamma(\gamma)x^{1-\gamma}y(x) \leq \Gamma(\gamma)x^{1-\gamma}\Phi(x) + \Gamma(\gamma)x^{1-\gamma}\Psi(x, T)$$

$$+ \Gamma(\gamma)x^{1-\gamma}\sum_{t>T} h_{a-1}(x, \sigma(t))[\omega(t) + K_{\mu,v}(t)]$$  \hspace{1cm} (20)

for $t \geq T$. We consider two cases.

Case (i). Assume $0 < \alpha \leq 1$. Then $n = 1$ and $0 < \gamma \leq 1$. Also we have $\Phi(x) = |y_0|h_{\gamma-1}(x, a + 1 - \gamma)$ for $x \geq T$, and

$$\Gamma(\gamma)x^{1-\gamma}\Phi(x) = \Gamma(\gamma)x^{1-\gamma}|y_0|h_{\gamma-1}(x, a + 1 - \gamma)$$

$$= |y_0|x^{1-\gamma}(x -(a + 1 - \gamma)\gamma^{-1}$$

$$= |y_0|x^{1-\gamma}\frac{\Gamma(x-(a + 1 - \gamma)+1)}{\Gamma(x-(a + 1 - \gamma)-(\gamma-1)+1)}$$

$$= |y_0|x^{1-\gamma}\frac{\Gamma(x-a+\gamma)}{\Gamma(x-a+\gamma+(1-\gamma))}$$

and

$$\Gamma(\gamma)x^{1-\gamma}\Psi(x, T) = \Gamma(\gamma)x^{1-\gamma}\sum_{t>a+1-\alpha} h_{a-1}(x, \sigma(t))|F(t)|$$

$$= \frac{\Gamma(\gamma)}{\Gamma(\alpha)} \sum_{t=a+1-\alpha}^{T-1} x^{1-\gamma}(x-t-1)^{a-1}|F(t)|$$

$$= \frac{\Gamma(\gamma)}{\Gamma(\alpha)} \sum_{t=a+1-\alpha}^{T-1} x^{1-\gamma} \frac{\Gamma(x-t)}{\Gamma(x-t+1-\alpha)}|F(t)|$$

$$= \frac{\Gamma(\gamma)}{\Gamma(\alpha)} \sum_{t=a+1-\alpha}^{T-1} (x^{1-\alpha})^{1-\beta} \frac{\Gamma(x-t)}{\Gamma(x-t+1-\alpha)}|F(t)|$$

$$= \frac{\Gamma(\gamma)}{\Gamma(\alpha)} \sum_{t=a+1-\alpha}^{T-1} \frac{1}{(x^{1-\alpha})^{\beta}} x^{1-\alpha} \frac{\Gamma(x-t)}{\Gamma(x-t+1-\alpha)}|F(t)|$$

and using the asymptotic expansion formula

$$\lim_{x \to \infty} \frac{\Gamma(x)x^\epsilon}{\Gamma(x+\epsilon)} = 1, \quad \epsilon > 0,$$

we have

$$\lim_{x \to \infty} \left[ \Gamma(\gamma)x^{1-\gamma}\Phi(x) + \Gamma(\gamma)x^{1-\gamma}\Psi(x, T) \right] = M < \infty, \quad x > T. \hspace{1cm} (21)$$

Taking the limit inferior of inequality (20) as $x \to \infty$,

$$\liminf_{x \to \infty} x^{1-\gamma}\sum_{t>T} h_{a-1}(x, \sigma(t))[\omega(t) + K_{\mu,v}(t)] > -M > -\infty$$

and we have a contradiction to (14).
Case (ii). Assume \( n - 1 < \alpha < n, n \geq 2 \). Then \( n - 1 < \gamma < n \) and \( \gamma > \alpha \) with \( \gamma = \alpha + \beta(n - \alpha) \);

\[
\Gamma(\gamma)x^{1-\gamma}\Phi(x) = \Gamma(\gamma)x^{1-\gamma}\sum_{k=0}^{n-1} h_{\gamma-n+k}(x, a + n - \gamma)|y_k|
\]

\[
= \Gamma(\gamma)x^{1-\gamma}\sum_{k=0}^{n-1} h_{\gamma-n+k}(x, a + n - \gamma)|y_k|
\]

\[
= \Gamma(\gamma)x^{1-\gamma}\sum_{k=0}^{n-1} (x - a - n + \gamma)^{2-n+k} \Gamma(\gamma-n+k+1) |y_k|
\]

\[
= \Gamma(\gamma)x^{1-\gamma}\sum_{k=0}^{n-1} \frac{\Gamma(x - a - n + \gamma + 1)}{\Gamma(x - a - k + \gamma + 1)} |y_k|
\]

\[
= \Gamma(\gamma)x^{1-\gamma}\sum_{k=0}^{n-1} \frac{\Gamma(x - a - n + \gamma + 1)}{\Gamma(x - a - k + \gamma + 1)} \times \frac{\Gamma(x - a - k + \gamma)}{\Gamma(x - a - k + 1)} |y_k|
\]

\[
= \Gamma(\gamma)x^{1-\gamma}\sum_{k=0}^{n-1} \frac{1}{\Gamma(x - a + \gamma - (k - 1)) \cdots (x - a + \gamma - (n - 1))}
\]

\[
\times \frac{\Gamma(x - a - k + \gamma)}{\Gamma(x - a - k + 1)} \times \frac{1}{\Gamma(\gamma - n + k + 1)} |y_k|
\]

and

\[
\Gamma(\gamma)x^{1-\gamma}\Psi(x, T) = \Gamma(\gamma)x^{1-\gamma}\sum_{t=\alpha+1}^{T-1} h_{\alpha-1}(x, \sigma(t))|F(t)|
\]

\[
= \frac{\Gamma(\gamma)}{\Gamma(\alpha)} \sum_{t=\alpha+1}^{T-1} x^{1-\gamma}(x - t - 1)^{\alpha-1} |F(t)|
\]

\[
= \frac{\Gamma(\gamma)}{\Gamma(\alpha)} \sum_{t=\alpha+1}^{T-1} \frac{\Gamma(x - t)}{\Gamma(x - t + 1 - \alpha)} |F(t)|
\]

\[
= \frac{\Gamma(\gamma)}{\Gamma(\alpha)} \sum_{t=\alpha+1}^{T-1} \frac{\Gamma(x - t)}{\Gamma(x - t + 1 - \alpha)} \times \frac{1}{\Gamma(x - t + 1 - \alpha)} |F(t)|
\]

Then using the asymptotic expansion formula, we obtain

\[
\lim_{x \to \infty} \left[ \Gamma(\gamma)x^{1-\gamma}\Phi(x) + \Gamma(\gamma)x^{1-\gamma}\Psi(x, T) \right] = 0, \quad x \geq T.
\]

Hence, taking the limit inferior of inequality (20) as \( x \to \infty \), we get

\[
\liminf_{x \to \infty} \frac{x^{\sigma-\alpha}}{x^{\sigma-\alpha}} \sum_{t=T}^{\infty} h_{\alpha-1}(x, \sigma(t)) \left[ \omega(t) + K_{\mu, \nu}(t) \right] > 0,
\]

which is a contradiction to condition (14).

Thus we complete the proof of the theorem. \( \square \)
Theorem 3 Suppose $\alpha \geq 1$ and assume that (11) and (13) valid for $\mu < \nu$. If

$$
\liminf_{x \to \infty} x^{1-\gamma} \sum_{t=T}^{x-a} h_{a-1}(x, \sigma(t)) [\omega(t) - K \mu, \nu(t)] = -\infty
$$

(22)

and

$$
\limsup_{x \to \infty} x^{1-\gamma} \sum_{t=T}^{x-a} h_{a-1}(x, \sigma(t)) [\omega(t) + K \mu, \nu(t)] = \infty,
$$

(23)

where $K_{\mu,\nu}(t)$ is defined as in Theorem 2, then for every sufficiently large $T$ every bounded solution of (6) is oscillatory.

Proof Assume $y(x)$ is a non-oscillatory and bounded solution of (6). Then for $M_1, M_2 \in \mathbb{R}$

$$
M_1 \leq y(x) \leq M_2, \quad x \geq a.
$$

(24)

Suppose that $y(x) > 0$ for $x \geq T > a$. Using inequality (8) and condition (13), we get

$$
q_2(t)|y(t + \alpha)|^\nu - q_1(t)|y(t + \alpha)|^\mu \geq K_{\mu,\nu}(t), \quad t \geq T,
$$

(25)

similarly to Theorem 2. Define

$$
\Phi(x) = \sum_{k=0}^{n-1} h_{n-\alpha+k}(x, a + n - \gamma)|y_k|
$$

and

$$
\Psi(x, T) = \sum_{t=a+1-a}^{T-1} h_{a-1}(x, \sigma(t)) |F(t)|.
$$

Then we obtain for $x \geq T$

$$
\Gamma(\gamma)x^{1-\gamma}y(x) \geq \Gamma(\gamma)x^{1-\gamma} \Phi(x) + \Gamma(\gamma)x^{1-\gamma} \Psi(x, T)
$$

$$
+ \Gamma(\gamma)x^{1-\gamma} \sum_{t=T}^{x-a} h_{a-1}(x, \sigma(t)) [\omega(t) + K \mu, \nu(t)], \quad x > T,
$$

(26)

and also using (24)

$$
\Gamma(\gamma)x^{1-\gamma}M_2 \geq \Gamma(\gamma)x^{1-\gamma} \Phi(x) + \Gamma(\gamma)x^{1-\gamma} \Psi(x, T)
$$

$$
+ \Gamma(\gamma)x^{1-\gamma} \sum_{t=T}^{x-a} h_{a-1}(x, \sigma(t)) [\omega(t) + K \mu, \nu(t)], \quad x > T.
$$

(27)

We consider two cases for the proof.
Case (i) Assume $\alpha = 1$. Then $\gamma = 1$ and $\Phi(x) = h_{\gamma-1}(x, a + 1 - \gamma)|y_0| = |y_0|$, $\Psi(x, T) = \sum_{t=a}^{T-1}|F(t)|$. Hence, we see from (27)

$$
M_2 - |y_0| - \sum_{t=a}^{T-1}|F(t)| \geq \sum_{t=a}^{x-a} h_{\alpha-1}(x, \sigma(t))[\omega(t) + K_{\mu, \nu}(t)], \quad x > T,
$$

and

$$
\limsup_{x \to \infty} x^{1-\gamma} \sum_{t=a}^{x-a} h_{\alpha-1}(x, \sigma(t))[\omega(t) + K_{\mu, \nu}(t)] \leq \left[ M_2 - |y_0| - \sum_{t=a}^{T-1}|F(t)| \right] < \infty,
$$

which is a contradiction to (23).

Case (ii) Assume $\alpha > 1$. Then $\gamma > 1$. As in the proof of Theorem 2, using the asymptotic expansion formula we have

$$
\lim_{x \to \infty} \left[ \Gamma(\gamma) x^{1-\gamma} \Phi(x) + \Gamma(\gamma) x^{1-\gamma} \Psi(x, T) \right] = 0, \quad x \geq T.
$$

Since $\lim_{x \to \infty} x^{1-\gamma} = 0$, from (27)

$$
\limsup_{x \to \infty} x^{1-\gamma} \sum_{t=a}^{x-a} h_{\alpha-1}(x, \sigma(t))[\omega(t) + K_{\mu, \nu}(t)] \leq 0 < \infty,
$$

which is a contradiction to (23).

Example 1 Consider the following initial value problem:

$$
\Delta^{\frac{1-\gamma}{2}} y(x) + y^2(x + \frac{1}{3}) e^{x^{\frac{1}{2}}} + y^\frac{v}{2}(x + \frac{1}{3}) e^{x^{\frac{1}{2}}} + (\bigg((x + \frac{1}{3})^2 - x \bigg)^\frac{v}{2} e^{x^{\frac{1}{2}}}
$$

$$
\Delta^{-(1-\frac{2}{3})} y \left( \frac{1}{3} \right) = 0,
$$

(28)

where $\alpha = 1/3$, $\beta = 1/2$ and $\gamma = 2/3$, $y(x) = x^2$ is a non-oscillatory solution of (28). Here, $\mu = 2$, $\nu = 1/5$, $q_1(x) = q_2(x) = e^{x^{\frac{1}{2}}}$ and $\omega(x) = \frac{3}{2(1-x^2)} (\frac{2}{5} x^\frac{2}{5} - x^\frac{3}{5}) + (x + \frac{1}{3})^2 - (x + \frac{1}{3})^2 e^{x^{\frac{1}{2}}}$. However, condition (14) is not fulfilled because of $\omega(x) \geq 0$ and $\liminf_{x \to \infty} x^{1-\gamma} \sum_{t=a}^{x-a} h_{\alpha-1}(x, \sigma(t))[\omega(t) + K_{\mu, \nu}(t)] \geq 0$.

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