Approximation of Fourier and its conjugate series by triple Euler product summability

Smita Sonker¹ and Paramjeet Sangwan²

Department of Mathematics, National Institute of Technology Kurukshetra (India)-136119

smita.sonker@gmail.com

Abstract. The paper is related to the concept of degree of approximation of Fourier and its conjugate series with the help of triple Euler product means. New theorems based on triple Euler product transform have been established and proved under general conditions.

1. Introduction

Fourier series has a great value in applied and theoretical mathematics. In the mid-eighteenth century, various physical problems like heat conduction patterns, vibrations and oscillations gave emphasis towards the study of Fourier series. Degree of approximation by single or double product summability methods have been studied by many researchers like Mohanty and Nanda [3], Mittal and Singh [10] established a result on T.C.1–summability of the Fourier series. Sahney and Goel [5], Qureshi [6], Chandra [7], Lal and Kushwaha [11], Sonker and Singh [12], Shukla, Srivastav and Rathore [17] worked on degree of approximation with the help of various conditions and different classes. Mittal and Prasad [8] determined the theorem on sequence of Fourier coefficients. Lal and Nigam [9] carried out his work on (N, p, q) summability. Nigam [14] worked on product summability of (C, 2) (E, 1). Saxena and Prabhakar [16] worked on double Euler summability. After doing literature survey it seems that nothing has been done still towards in triple Euler product means.

2. Definitions and Notations

Let a function (or signal) which is periodic with time period of $2\pi$ is denoted at point $x$ by ‘$g$’ and integrable as Lebesgue for the limit (-π to π).

Fourier series given by

$$g(x) \sim \frac{a_0}{2} + \sum_{m=1}^{\infty} (a_m \cos mx) + \sum_{m=1}^{\infty} (b_m \sin mx) = \sum_{m=1}^{\infty} A_m(x) \quad (2.1)$$

The imaginary part of eqn (2.1) is the conjugate series and is given by

$$\bar{g}(x) \sim \sum_{m=1}^{\infty} (b_m \cos mx) - \sum_{m=1}^{\infty} (a_m \sin mx) = \sum_{m=1}^{\infty} B_m(x) \quad (2.2)$$
\[ \phi(t) = g(x + t) + g(x - t) - 2g(t) \]
\[ \psi(t) = \frac{1}{2} [g(x + t) - g(x - t)] \]
\[ Z_m(t) = \frac{1}{\pi(2)^{m+1}} \sum_{h=0}^{m} \left( \frac{m}{h} \right) \frac{1}{(2)^h} \sum_{i=0}^{h} \left( \frac{1}{2^i} \right) \left( \sum_{j=0}^{i} \sin \left( \frac{j+\frac{1}{2}}{2^h} \right) t \right) \]
\[ \overline{Z}_m(t) = \frac{1}{\pi(2)^{m+1}} \sum_{h=0}^{m} \left( \frac{m}{h} \right) \frac{1}{(2)^h} \sum_{i=0}^{h} \left( \frac{1}{2^i} \right) \left( \sum_{j=0}^{i} \cos \left( \frac{j+\frac{1}{2}}{2^h} \right) t \right) \]

Let \( \sum_{m=0}^{\infty} u_m \) is given infinite series and \( \{s_m\} \) is \( m \)th partial sum of series.

(I) If \( E^{1}_{m} = m E^{1} = \frac{1}{(2)^m} \sum_{h=0}^{m} \left( \frac{m}{h} \right) s_h \rightarrow s \) as \( m \rightarrow \infty \),
then \( \sum_{m=0}^{\infty} u_m \) known as summable using single Euler (E, 1) means to ‘s’ (throughout ‘s’ denotes a definite no).

(II) If \( E^{1}_{m} E^{1}_{h} = m E^{1,1} = \frac{1}{(2)^m} \sum_{h=0}^{m} \left( \frac{m}{h} \right) E^{1}_{h} \rightarrow s \) as \( m \rightarrow \infty \),
then \( \sum_{m=0}^{\infty} u_m \) known as summable using double Euler (E, 1) product means to ‘s’.

(III) If \( E^{1}_{m} E^{1}_{h} E^{1}_{i} = m E^{1,1,1} = \frac{1}{(2)^m} \sum_{h=0}^{m} \left( \frac{m}{h} \right) \frac{1}{(2)^h} \sum_{i=0}^{h} \left( \frac{h}{i} \right) E^{1}_{i} \rightarrow s \) as \( m \rightarrow \infty \)
\[ = \frac{1}{(2)^m} \sum_{h=0}^{m} \left( \frac{m}{h} \right) \frac{1}{(2)^h} \sum_{i=0}^{h} \left( \frac{h}{i} \right) \frac{1}{(2)^i} \sum_{j=0}^{i} \left( \frac{j}{i} \right) \rightarrow s \) as \( m \rightarrow \infty \),
then \( \sum_{m=0}^{\infty} u_m \) known as summable using triple Euler (E, 1) product means.

3. Main Theorems

The aim of this study is to generalize the theorems of Saxena and Prabhakar [16] for triple Euler product means.

**Theorem 3.1.** Let \( \{p_{m}\} \) is +ive sequence which is monotonic and non-increasing with real constants
\[ P_{m} = \sum_{w=0}^{m} p_{w} \rightarrow \infty, \; \text{as} \; m \rightarrow \infty. \]
If \( \phi \) satisfy the conditions as below
\[ \phi(t) = \int_{0}^{t} \phi(u) du = o \left( \frac{t}{\beta(t)} P_{t} \right) \; \text{as} \; t \rightarrow +0 \]  \hspace{1cm} (3.1)
provided \( \beta \) is +ive, non-increasing and monotonic function of \( t \).
\[ \log m = O([\beta(m)].P_{m}), \; \text{as} \; m \rightarrow \infty \]  \hspace{1cm} (3.2)
then approximation of the function at \( x = t \) using triple Euler product means is given by
\[ |t_{m} E^{1,1,1} - g(x)| = O(1) \; \text{as} \; m \rightarrow \infty \]
where \( t_{m} E^{1,1,1} \) denotes (E, 1)(E, 1)(E, 1) transform of partial sums of the series (2.1).
**Theorem 3.2:** Let \( \{p_m\} \) be a sequence which is monotonic and non-increasing with real constants

\[
P_m = \sum_{w=0}^{m} p_w \to \infty \quad \text{as} \quad m \to \infty.
\]

If \( \psi \) satisfies the conditions as below

\[
\psi(t) = \int_0^t |\psi(u)| du = o\left(\frac{t}{\beta\left(\frac{t}{1}\right)} P_t\right) \quad \text{as} \quad t \to +0
\]

(3.3)

provided \( \beta \) is +ive, non-increasing and monotonic function

\[
\log m = O[\beta(m)]P_m \quad \text{as} \quad m \to \infty
\]

then eqn. (2.2) is summable to

\[
\tilde{g}(x) = -\frac{1}{2\pi} \int_0^{2\pi} \psi(t) \cot \left(\frac{t}{2}\right) dt
\]

at each point of the existence of this integral, then approximation of the function at \( x = t \) using triple Euler product means is given by

\[
|t_m^{E^1E^1E^1} - \tilde{g}(x)| = O(1) \quad \text{as} \quad m \to \infty,
\]

where \( t_m^{E^1E^1E^1} \) denotes \((E, 1)(E, 1)(E, 1)\) transform of partial sums of the series (2.2).

4. **Lemmas**

To prove the main theorems, the following lemmas are as given below:

**Lemma 4.1:** \( |Z_m(t)| = O(m) \), for \( 0 \leq t \leq \frac{1}{m} \); \( \sin mt \leq m \sin mt \).

**Proof:**

\[
|Z_m(t)| \leq \frac{1}{\pi(2)^{m+1}} \left( \sum_{h=0}^{m} \sum_{l=0}^{h} \frac{(m)}{(2^h)^l j} \left( \sin \left(\frac{j+\frac{1}{2}}{2}\right) t \right) \right)
\]

\[
\leq \frac{1}{\pi(2)^{m+1}} \left( \sum_{h=0}^{m} \sum_{l=0}^{h} \frac{(m)}{(2^h)^l j} \left( \sin \left(\frac{2i+1}{2}\right) \right) \right)
\]

\[
= \frac{1}{\pi(2)^{m+1}} \left( \sum_{h=0}^{m} \sum_{l=0}^{h} \frac{(m)}{(2^h)^l j} \left( 2h+1 \right) \right)
\]

\[
= \frac{1}{\pi(2)^{m+1}} \left( \sum_{h=0}^{m} \frac{(m)}{(2^h)^l j} \left( 2h+1 \right) \right)
\]

\[
= \frac{1}{\pi(2)^{m+1}} \left( \sum_{h=0}^{m} \frac{(m)}{(2^h)^l j} \right) = \frac{1}{2\pi(2)^m} (2m+1) \sum_{h=0}^{m} \frac{(m)}{(h)} = O(m).
\]

**Lemma 4.2:** \( |Z_m(t)| = O\left(\frac{1}{t}\right) \), for \( \frac{1}{m} \leq t \leq \pi \), \( t \leq \pi \sin \left(\frac{t}{2}\right) \), and \( \sin(mt) \leq 1 \).

**Proof:**
\[ |Z_m(t)| \leq \frac{1}{\pi(2)^{m+1}} \left| \sum_{h=0}^{m} \left( \begin{array}{c} m \\ h \\ \end{array} \right) \frac{1}{(2)^h} \sum_{i=0}^{h} \left( \begin{array}{c} h \\ i \\ \end{array} \right) \left( \frac{1}{(2)^i} \sum_{j=0}^{i} \frac{(j)}{\sin \left( \frac{t}{2} \right)} \right) \right| \]

\[ \leq \frac{1}{\pi(2)^{m+1}} \left| \sum_{h=0}^{m} \left( \begin{array}{c} m \\ h \\ \end{array} \right) \frac{1}{(2)^h} \sum_{i=0}^{h} \left( \begin{array}{c} h \\ i \\ \end{array} \right) \left( \frac{1}{(2)^i} \sum_{j=0}^{i} \frac{(j)}{\pi} \right) \right| \]

\[ = \frac{1}{t(2)^{m+1}} \left| \sum_{h=0}^{m} \left( \begin{array}{c} m \\ h \\ \end{array} \right) \frac{1}{(2)^h} \sum_{i=0}^{h} \left( \begin{array}{c} h \\ i \\ \end{array} \right) \left( \frac{1}{(2)^i} \sum_{j=0}^{i} (j) \right) \right| \]

\[ \leq \frac{1}{t(2)^{m+1}} \left| \sum_{h=0}^{m} \left( \begin{array}{c} m \\ h \\ \end{array} \right) \frac{1}{(2)^h} \sum_{i=0}^{h} (i) \right| = \frac{1}{2t(2)^m} \sum_{h=0}^{m} \left( \begin{array}{c} m \\ h \\ \end{array} \right) = O \left( \frac{1}{t} \right). \]

**Lemma 4.3:** \( |Z_m(t)| = O \left( \frac{1}{t} \right), \) for \( 0 \leq t \leq \frac{1}{m} \), \( t \leq \pi \sin \left( \frac{1}{2} \right) \) and \( |\cos(mt)| \leq 1. \)

**Proof:**

\[ |Z_m(t)| \leq \frac{1}{\pi(2)^{m+1}} \left| \sum_{h=0}^{m} \left( \begin{array}{c} m \\ h \\ \end{array} \right) \frac{1}{(2)^h} \sum_{i=0}^{h} \left( \begin{array}{c} h \\ i \\ \end{array} \right) \left( \frac{1}{(2)^i} \sum_{j=0}^{i} \frac{(j)}{\sin \left( \frac{t}{2} \right)} \right) \right| \]

\[ \leq \frac{1}{\pi(2)^{m+1}} \left| \sum_{h=0}^{m} \left( \begin{array}{c} m \\ h \\ \end{array} \right) \frac{1}{(2)^h} \sum_{i=0}^{h} \left( \begin{array}{c} h \\ i \\ \end{array} \right) \left( \frac{1}{(2)^i} \sum_{j=0}^{i} \frac{(j)}{\pi} \right) \right| \]

\[ \leq \frac{1}{t(2)^{m+1}} \left| \sum_{h=0}^{m} \left( \begin{array}{c} m \\ h \\ \end{array} \right) \frac{1}{(2)^h} \sum_{i=0}^{h} (i) \right| = O \left( \frac{1}{t} \right). \]

**Lemma 4.4:** \( |\overline{Z_m}(t)| = O \left( \frac{1}{t} \right), \) for \( \frac{1}{m} \leq t \leq \pi, \) \( t \leq \pi \sin \left( \frac{1}{2} \right). \)

**Proof:**

\[ |\overline{Z_m}(t)| \leq \frac{1}{\pi(2)^{m+1}} \left| \sum_{h=0}^{m} \left( \begin{array}{c} m \\ h \\ \end{array} \right) \frac{1}{(2)^h} \sum_{i=0}^{h} \left( \begin{array}{c} h \\ i \\ \end{array} \right) \left( \frac{1}{(2)^i} \sum_{j=0}^{i} \frac{(j)}{\sin \left( \frac{t}{2} \right)} \right) \right| \]

\[ \leq \frac{1}{\pi(2)^{m+1}} \left| \sum_{h=0}^{m} \left( \begin{array}{c} m \\ h \\ \end{array} \right) \frac{1}{(2)^h} \sum_{i=0}^{h} \left( \begin{array}{c} h \\ i \\ \end{array} \right) \left( \frac{1}{(2)^i} \sum_{j=0}^{i} \frac{(j)}{\pi} \right) \right| \]

\[ \leq \frac{1}{t(2)^{m+1}} \left| \sum_{h=0}^{m} \left( \begin{array}{c} m \\ h \\ \end{array} \right) \frac{1}{(2)^h} \sum_{i=0}^{h} (i) \right| = \frac{1}{2t(2)^m} \sum_{h=0}^{m} \left( \begin{array}{c} m \\ h \\ \end{array} \right) \left| e^{i(j+\frac{1}{2})t} \right| \right| = O \left( \frac{1}{t} \right). \]
\[ \leq \frac{1}{t(2)^{m+1}} \left| \sum_{h=0}^{t-1} \left( \sum_{i=0}^{m} \left( \frac{h}{2^h} \sum_{i=0}^{m} \left( \frac{h}{2} \sum_{i=0}^{m} \left( \frac{i}{2} \sum_{i=0}^{m} \left( \frac{j}{2} \sum_{i=0}^{m} \phi \right) \right) \right) \right) \right| \]

Now,

\[ |J_1| \leq \frac{1}{t(2)^{m+1}} \left| \sum_{h=0}^{t-1} \left( \sum_{i=0}^{m} \left( \frac{h}{2^h} \sum_{i=0}^{m} \left( \frac{i}{2} \sum_{i=0}^{m} \left( \frac{j}{2} \sum_{i=0}^{m} \phi \right) \right) \right) \right| \]

\[ \leq \frac{1}{t(2)^{m+1}} \left| \sum_{h=0}^{t-1} \left( \sum_{i=0}^{m} \left( \frac{h}{2^h} \sum_{i=0}^{m} \left( \frac{i}{2} \sum_{i=0}^{m} \left( \frac{j}{2} \sum_{i=0}^{m} \phi \right) \right) \right) \right| \]

\[ \leq \frac{1}{t(2)^{m+1}} \left| \sum_{h=0}^{t-1} \left( \sum_{i=0}^{m} \left( \frac{h}{2^h} \sum_{i=0}^{m} \left( \frac{i}{2} \sum_{i=0}^{m} \left( \frac{j}{2} \sum_{i=0}^{m} \phi \right) \right) \right) \right) \right| \]

Now,

\[ |J_2| \leq \frac{1}{t(2)^{m+1}} \left| \sum_{h=0}^{t-1} \left( \sum_{i=0}^{m} \left( \frac{h}{2^h} \sum_{i=0}^{m} \left( \frac{i}{2} \sum_{i=0}^{m} \left( \frac{j}{2} \sum_{i=0}^{m} \phi \right) \right) \right) \right) \right| \]

\[ \leq \frac{1}{t(2)^{m+1}} \left| \sum_{h=0}^{t-1} \left( \sum_{i=0}^{m} \left( \frac{h}{2^h} \sum_{i=0}^{m} \left( \frac{i}{2} \sum_{i=0}^{m} \left( \frac{j}{2} \sum_{i=0}^{m} \phi \right) \right) \right) \right) \right| \]

\[ \leq \frac{1}{t(2)^{m+1}} \left| \sum_{h=0}^{t-1} \left( \sum_{i=0}^{m} \left( \frac{h}{2^h} \sum_{i=0}^{m} \left( \frac{i}{2} \sum_{i=0}^{m} \left( \frac{j}{2} \sum_{i=0}^{m} \phi \right) \right) \right) \right) \right| \]

5. Proof of Theorems

**Proof of Theorem 3.1:** According to Titchmarsh [1], let \( s_m(g;x) \) is partial sum of Fourier series (2.1).

\[ |s_m(g;x) - g(x)| = \frac{1}{2\pi} \int_0^\pi \phi(t) \frac{\sin \left( \frac{m+1}{2} t \right)}{\sin \frac{t}{2}} \, dt. \]

The (E, 1) (E, 1) transform of \( s_m(g;x) \) is given by

\[ |t_m^{E_1,E_1} - g(x)| = \frac{1}{\pi(2)^{m+1}} \left| \sum_{h=0}^{m} \left( \frac{h}{2^h} \sum_{i=0}^{m} \left( \frac{i}{2} \sum_{i=0}^{m} \phi \right) \right) \right| \]

\[ \leq \frac{1}{\pi(2)^{m+1}} \left| \sum_{h=0}^{m} \left( \frac{h}{2^h} \sum_{i=0}^{m} \left( \frac{i}{2} \sum_{i=0}^{m} \phi \right) \right) \right| \]

By using the assumptions of theorem, it is to be shown that

\[ \int_0^\pi |\phi(t)| |Z_m(t)| \, dt = O(1) \quad \text{as} \quad m \to \infty. \]

We set the limit of \( \gamma \) from 0 to \( \pi \).
Collecting equations (5.3), (5.4) and (5.5), we get

\[
|t_m^E x^1 \cdot E^1 - g(x)| = \int_0^\pi |\phi(t)| |Z_m(t)| dt = \left[ \int_0^\pi |\phi(t)| + \int_1^m |\phi(t)| + \int_m^\pi |\phi(t)| \right] |Z_m(t)| dt
\]

\[= I_1 + I_2 + I_3 \quad \text{(say)}. \tag{5.2}\]

Throughout the paper, second mean value theorem is using for the second term integral. Using Lemma 4.1, equations (3.1) and (3.2), we have

\[
|I_1| \leq \int_0^\pi |\phi(t)||Z_m(t)| dt = O(m) \left[ \int_0^m |\phi(t)| dt \right] = O(m) \left[ o\left( \frac{1}{P_m \cdot m\beta(m)} \right) \right]
\]

\[= O\left( \frac{1}{P_m \cdot \beta(m)} \right) = O\left( \frac{1}{\log m} \right) = O(1) \quad \text{as } m \to \infty. \tag{5.3}\]

Using Lemma 4.2, equations (3.1) and (3.2), we have

\[
|I_2| \leq \int_0^\pi |\phi(t)||Z_m(t)| dt = O\left( \int_0^\gamma |\phi(t)| \left( \frac{1}{t} \right) dt \right) = O\left( \int_0^\gamma \frac{1}{t} |\phi(t)| dt \right) + \int_0^\gamma \frac{1}{t^2} |\phi(t)| dt
\]

\[= O\left( \frac{1}{P_m \cdot \beta(m)} \right) + \int_0^\gamma \frac{1}{P_m \cdot \beta(m)} + o\left( \frac{1}{P_m \cdot \beta(m)} \right) \int_\gamma^m 1. du
\]

\[= O\left( \frac{1}{\log m} \right) + O\left( \frac{1}{\log m} \right) = O(1) \quad \text{as } m \to \infty. \tag{5.4}\]

By considering summability regularity condition and taking Riemann-Lebesgue theorem

\[
|I_3| \leq \int_0^\pi |\phi(t)||Z_m(t)| dt = O(1) \quad \text{as } m \to \infty. \tag{5.5}\]

Collecting equations (5.3), (5.4) and (5.5), we get

\[
|t_m^E x^1 \cdot E^1 - g(x)| = O(1) \quad \text{as } m \to \infty.
\]

**Proof of Theorem 3.2:** On using Riemann-Lebesgue theorem and according to Lal [9], let \( \bar{s}_m (g; x) \) be partial sum of the series (2.2).

\[
|\bar{s}_m (g; x) - \bar{g}(x)| = \frac{1}{2\pi} \int_0^\pi \psi(t) \frac{\cos \left( \frac{m + \frac{1}{2}}{2} t \right)}{\sin \frac{t}{2}} dt.
\]

The \((E, 1)(E, 1)\) transform of \( \bar{s}_m (g; x) \) is given by

\[
|t_m^E x^1 \cdot E^1 - \bar{g}(x)| = \frac{1}{\pi(2)^{m+1}} \sum_{h=0}^{m} \left( \begin{array}{c} m \\ h \end{array} \right) \frac{1}{2} \sum_{i=0}^{h} \left( \begin{array}{c} h \\ i \end{array} \right) \frac{1}{2^i} \int_0^\pi \psi(t) \left[ \sum_{j=0}^{i} \left( \cos \left( \frac{j}{2} + \frac{1}{2} \right) \right) \right] dt
\]
\[ = \int_0^\pi |\psi(t)| |Z_m(t)| dt. \]

According to assumptions of theorem, it is to be shown that
\[ \int_0^\pi |\psi(t)| |Z_m(t)| dt = O(1) \quad \text{as} \quad m \to \infty. \]

Using limit \( 0 < \gamma < \pi \) and on taking \( \psi(t) \) as \( \psi \), we have
\[
|t_m^\gamma \hat{f}_1 \hat{f}_1 - \hat{g}(x)| = \int_0^\pi |\psi(t)| |Z_m(t)| dt = \left[ \int_0^{\frac{\pi}{m}} |\psi(t)| + \int_{\frac{\pi}{m}}^\pi |\psi(t)| \right] |Z_m(t)| dt
\]
\[ = L_i + L_2 + L_3 \quad \text{(say)}. \quad (5.6) \]

Consider Lemma 4.3, equations (3.2) and (3.3)
\[
|L_1| \leq \int_0^{\frac{\pi}{m}} |\psi(t)| |Z_m(t)| dt = O(m) \left[ \int_0^{\frac{\pi}{m}} |t|^\gamma |\psi| dt \right] = O(m) \left[ \int_0^{\frac{\pi}{m}} |\psi| dt \right]
\]
\[ = O \left( \frac{1}{\log m} \right) = O(1) \quad \text{as} \quad m \to \infty. \quad (5.7) \]

Considering Lemma 4.4, equations (3.2) and (3.3)
\[
|L_2| \leq \int_0^\pi |\psi(t)| |Z_m(t)| dt = O \left[ \int_0^\pi \frac{1}{|t|^\gamma} |\psi| dt \right] = O \left[ \int_0^\pi \frac{1}{|t|} |\psi| dt \right]
\]
\[ = O \left[ \int_0^\pi \frac{1}{|t|^\gamma} \left( \frac{1}{|t|^\gamma} \right) dt \right] = O \left[ \int_0^\pi \frac{1}{|t|^\gamma} \right] dt
\]
\[ = O \left( \int_0^\frac{1}{\log m} \right) + O \left( \frac{1}{\log m} \right) = O(1) \quad \text{as} \quad m \to \infty. \quad (5.8) \]

Using regularity condition in method of summability and Riemann-Lebesgue theorem, we have
\[
|L_3| \leq \int_0^\pi |\psi(t)| |Z_m(t)| dt = O(1) \quad \text{as} \quad m \to \infty. \quad (5.9)
\]

Collecting equations (5.7), (5.8) and (5.9), we have
\[
|t_m^\gamma \hat{f}_1 \hat{f}_1 - \hat{g}(x)| = O(1) \quad \text{as} \quad m \to \infty.
\]
6. Corollary

**Corollary 6.1.** If \( \{p_n\} \) be positive sequence which is monotonic and non-increasing and conditions (3.1) and (3.2) are satisfied, then the series (2.1) reduces to \( E^1 \cdot E^1 \) product means [16] if we take one \( E^1_i = 1 \) in triple product and approximation of the function at \( x = t \) using \( E^1 \cdot E^1 \) product means is given by

\[
\left| t_m^E \cdot E^1 \cdot E^1 - g(x) \right| = O(1) \quad \text{as } m \to \infty.
\]

**Corollary 6.2.** If \( \{p_n\} \) be positive sequence which is monotonic and non-increasing and conditions (3.2) and (3.3) are satisfied, then the series (2.2) reduces to \( \overline{E^1} \cdot \overline{E^1} \) product means [16] if we take \( \overline{E^1_i} = 1 \) in triple product and approximation of the function at \( x = t \) using \( \overline{E^1} \cdot \overline{E^1} \) product means is given by

\[
\left| t_m^\overline{E^1} \cdot \overline{E^1} - \overline{g}(x) \right| = O(1) \quad \text{as } m \to \infty.
\]

7. Conclusion

This paper focuses on the approximation of functions with the help of \( E^1 \cdot E^1 \cdot E^1 \) product means of series (2.1) and (2.2). Through this research work, it has been concluded that the main theorem is a generalized form which can be reduced to familiar results. Various results related to \( (E^1)^2 \cdot X \) and \( X \cdot (E^1)^2 \) product means of series (2.1) and (2.2) have been reviewed.

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