Integral equation models for thermoacoustic imaging of acoustic dissipative tissue

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Abstract

The main task of thermoacoustic imaging is the estimation of a function, denoted by \( \phi \), which depends on the electromagnetic absorption function and the optical scattering properties of the tissue. In the absence of acoustic dissipation, the parameter function \( \phi \) can be estimated from one of the three types of projections (spherical, circular or planar). In the case of acoustic dissipative wave propagation in tissue, it is no longer possible to explicitly calculate the projection of \( \phi \) from the respective pressure data (measured by point, planar or line detectors). The goal of this paper is to derive for each of the three types of pressure detectors, an integral equation that allows estimation of the respective projections of \( \phi \). The advantage of this approach is that known reconstruction formulas for \( \phi \) using the respective projection data can then be exploited.

1. Introduction

The basic setup of thermoacoustic imaging can be described as follows (cf [14, 16, 19, 22, 29, 35]). First the tissue, e.g. human breast, is illuminated by a short pulse of radio-frequency wave. Then the absorbed electromagnetic energy heats the tissue up and causes deformations which generate a pressure wave within the tissue. This pressure wave is measured on a surface surrounding the tissue by point, line or planar detectors. The inverse problem consists in estimating a parameter function \( \phi \) which depends on the electromagnetic absorption function and the optical scattering properties of the tissue from this pressure data. Since the actual cause of the pressure wave is an electromagnetic wave, thermoacoustic tomography (TAT) is a hybrid imaging method. If the electromagnetic wave is generated by a laser, then the imaging method is called photoacoustic tomography (PAT). The advantage of these hybrid imaging methods is on the one hand the good contrast between the electromagnetic absorption of e.g. cancerous cells and healthy tissue (due to electromagnetic illumination) and on the other hand...
the good spatial resolution (due to ultrasound measurements). Another advantage of TAT (and PAT) is that the technological advance allows measuring different geometric setups with point, line or planar detectors (cf [1, 7, 26, 35]) and therefore various explicit reconstruction formulas can be applied (cf [5, 10, 15, 16, 30–32, 34]).

In this paper we derive integral equation models that take dissipation into account (cf [2, 20, 21]) and allow the application of known reconstruction formulas. There are two reasons for this goal. First, it is desirable to increase the resolution of thermoacoustic imaging and second, there will be a resolution limit if the inverse problem is ill-posed. Yet the degree of ill-posedness of this problem is not known. This paper provides the means to tackle these two problems which are of considerable interest in TAT (and PAT). In the following we explain this more precisely in the case of point detectors.

First we briefly discuss the direct problem. The direct problem models the propagation of a pressure wave $p_{\text{att}} = p_{\text{att}}(x,t)$ in acoustic dissipative tissue generated by very short heating of the tissue due to electromagnetic waves. In the case of homogeneous and isotropic acoustic dissipation, the direct problem can be modeled as follows (cf theorem 4.2.1 in [8]):

$$p_{\text{att}} = G * f$$  \hspace{1cm} (*_{x,t} \text{ spacetime convolution},)  \hspace{1cm} (1)$$

where

$$G(x,t) = \mathcal{F}^{-1} \left\{ \frac{\exp[i k(\omega)|x|]}{4\pi |x|} \right\} \hspace{1cm} (\mathcal{F} \text{ Fourier transform})$$  \hspace{1cm} (2)$$

corresponds to an attenuated spherical wave with a complex-valued wave number $k(\omega)$ and origin in $(x,t) = (0,0)$ and $f$ corresponds to the source term of the pressure function $p_{\text{att}}$. For example, the wave number of an attenuated spherical wave that obeys the frequency power law with exponent $\gamma \in (0,1)$ satisfies (cf [25, 27])

$$i k(\omega) = -\alpha_0(-i \omega)^\gamma + \frac{i \omega}{c_0} = |\omega|^\gamma \exp(-i \text{sgn}(\omega) \pi \gamma / 2).$$

The negative of the real part of the first right-hand side term, i.e. $\alpha(\omega) := \alpha_0 |\omega|^\gamma$, is the attenuation law and the phase speed $c(\omega)$ is determined by

$$\frac{\omega}{c(\omega)} := -k(\omega) + i \alpha(\omega) = \alpha_0 |\omega|^\gamma \text{sgn}(\omega) \tan(\pi \gamma / 2) + \frac{i \omega}{c_0}.$$  \hspace{1cm} (3)$$

$k(\omega)$ and the value $c_0$ determine $\alpha(\omega)$ and $c(\omega)$, and vice versa. In the absence of the attenuation ($\alpha_0 = 0$), $G$ is the retarded Green’s function of the standard wave equation with phase speed $c_0$ and source term $f$. If $G$ is modeled appropriately, then $f$ is the same source term as in the absence of acoustic dissipation (cf section 2 and [22]), i.e.

$$f(x,t) = \frac{\partial I(t)}{\partial t} \phi(x) \hspace{1cm} (I \text{ short time impulse}),$$  \hspace{1cm} (4)$$

where $\phi$ depends on the electromagnetic absorption function and the optical scattering properties of the tissue. The essential requirement for $G$ is causality, i.e. the speed of the wave front (cf [13])

$$\{ (x, T(x)) \in \mathbb{R}^4 | T(x) := \text{sup} \{ t > 0 | G(x, \tau) = 0 \text{ for all } \tau \leq t \} \}$$

has to be bounded from above. Here $T(|x|)$ denotes the travel time of the wave front from 0 to $x$. Below we will see that causality plays an important role in thermoacoustic imaging.

The respective inverse problem can be formulated as follows. Let $\Gamma$ denote a surface covered with point detectors surrounding the tissue. From (1) and (3), we obtain the integral equation

$$\int_{\mathbb{R}^3} \tilde{G}(x-y,t) \phi(y) \, dy = p_{\text{att}}(x,t) \hspace{1cm} \text{for } x \in \Gamma, \ t \in [0, T].$$  \hspace{1cm} (4)$$
where $G := G \ast \frac{\partial I(t)}{\partial t}$ and $T$ is sufficiently large. The inverse problem corresponds to the estimation of $\phi$ in integral equation (4) and the subsequent inversion of $\phi = F(\alpha, t)$.

In this paper we show that integral equation (4) can be reformulated to

$$
\int_0^\infty N(t, t') R_{sp}(\phi)(x, t') \, dt' = p_{un}(x, t) \quad (N \in C^\infty(\mathbb{R}^2 \setminus \{(0, 0)\}),
$$

(5)

where $R_{sp}(\phi)$ denotes the spherical projection of $\phi$, i.e.

$$
R_{sp}(\phi)(x, t) := \int_{S^2} \phi(x') \, d\lambda^2(x') \quad x \in \mathbb{R}^3, \quad t \geq 0.
$$

(6)

Here $d\lambda^2$ denotes the Lebesgue measure on $\mathbb{R}^2$. The causality of $G$ ensures that $N(t, t') = 0$ if $t' > t$, i.e. the upper integration limit in (5) can be replaced by $t$. This means that the pressure function $p = p(x, t)$ does not depend on information from the future. For some surfaces $\Gamma$, $\phi$ can be reconstructed from $R_{sp}(\phi)$ via exact or approximate explicit reconstruction formulas. For example, if $\Gamma$ is a sphere, then (cf e.g. [15, 16])

$$
\phi(x) = -\frac{1}{2\pi} \text{div} \int_{S^2} \left[ \frac{\partial}{\partial t} \left( t R_{sp}(\phi)(x', t) \right) \right]_{t' = |x - x'|} \, d\lambda^2(x'),
$$

(7)

where $\text{div}$ denotes the exterior normal of $S^2$. In the absence of attenuation if $I(t) = \delta(t)$, then the projection $R_{sp}(\phi)$ can be calculated explicitly from (cf [11])

$$
\frac{\partial}{\partial t} \left( \frac{R_{sp}(\phi)(x, t)}{4\pi t} \right) = p(x, t) \quad (p \text{ unattenuated data}).
$$

(8)

However, in the presence of attenuation, a set of 1D-integral equations such as (5) has to be solved to obtain the projection $R_{sp}(\phi)$. Moreover, we see that if an efficient and accurate explicit reconstruction formula for $\phi$ from data $R_{sp}(\phi)$ exists, then the iterative estimation of $\phi$ via integral equation (4) is unfavorable.

Under special circumstances integral equation (5) can be written as a convolution equation which allows estimation of the effect of attenuation on the data. Assume that the illumination is instant, i.e. $I(t) = \delta(t)$, and that $\phi$ has one sharp peak at $x = x_0$, i.e. $\phi(x) = \delta(x - x_0)$. Let $c_0 := 1$ and $x_M$ be a measurement position on the boundary of the tissue. Then $p(x, t) = \delta'(t - |x - x_0|)/(4\pi |x - x_0|)$ and (8) with $f(t) \delta'(t) = -f'(0) \delta(t)$ imply $R_{sp}(\phi)(x, t) = \delta(t - |x - x_0|)$. From this, (5) and the representation of $N(t, |x|)$ (cf (35))

$$
N(t, |x|) = -\frac{1}{4\pi |x|} \left| \frac{\partial K(|x|, t - t')}{\partial t'} \right|_{t' = |x|},
$$

we get

$$
p_{un}(x_M, t) = N(t, |x|) = K(|x_M - x_0|, t) \ast p(x_M, t) \quad (x_0 \text{ fixed}).
$$

We see that the attenuated data can be interpreted as blurred unattenuated pressure data where the blur depends on the distance between the measurement point and the peak of $\phi$. This fact explains why current methods in TAT work well if blurring is assumed to occur. If in addition $|x_M - x_0|$ is constant for each detector position $x_M$, then the attenuated data $p_{un}(x_M, t)$ can be interpreted as unattenuated pressure with non-instant illumination $I(t) = K(|x_M - x_0|, t)$.

The goal of this paper is to derive 1D-integral equations that relate spherical, circular and planar projections to attenuated pressure data measured by point, line and planar detectors, respectively. As described above, acoustic dissipation can be included in TAT with these models. We note that this is a theoretical paper that is not concerned with numerical simulations.
The outline of this paper is as follows. The direct problem of thermoacoustic imaging of dissipative tissue is modeled in section 2. In section 3 an integral equation model is derived that allows the estimation of the unattenuated pressure data in case \( I(t) = \delta(t) \). Then, from this model, the basic theorem of thermoacoustic imaging of dissipative tissue is derived in case \( I(t) \neq \delta(t) \). Subsequently, in section 4, integral equation models for the three types of projections are derived that allow applying the explicit reconstruction formulas of TAT. In the appendix it is proven that the wave equation modeled in section 2 has a unique Green’s function satisfying the ‘initial condition’ \( G|_{t=0} = 0 \) and causality, defined as in section 2.

2. Modeling of the direct problem

In this section we model the direct problem of thermoacoustic imaging of dissipative tissue. First we summarize essential facts about wave attenuation and causality and then model the stress tensor and the source term for thermoacoustic imaging of dissipative tissue. For more details we refer to [13, 18, 20, 23–25].

Causal wave attenuation in tissue

Consider a viscous medium that is homogeneous with respect to density, compressibility and attenuation. It is well known that the Fourier transform \( \hat{G}(\mathbf{x}, \omega) \) of the attenuated spherical pressure wave (2) satisfies the following Helmholtz equation:

\[
\nabla^2 \hat{G}(\mathbf{x}, \omega) + k^2(\omega) \hat{G}(\mathbf{x}, \omega) = -\frac{\delta(\mathbf{x})}{\sqrt{2\pi}}.
\]

In the presence of attenuation the wave number \( k \) is of the form (cf e.g. [24, 25, 27])

\[
i k(\omega) = -\alpha_*(\omega) + i \frac{\omega}{c_0} \quad (c_0 \text{ constant})
\]

and in the absence of attenuation the complex-valued function \( \alpha_* \) vanishes. The real part of \( \alpha_* \) models the attenuation law. If the imaginary part of \( \alpha_* \) vanishes, then causality of \( G \) cannot hold (cf e.g. [24, 25, 27]). In order to formulate the Helmholtz equation in the spacetime domain for a general source term \( f \), we define \( D_* \) as the time convolution operator with kernel defined by

\[
\tilde{K}_*(\omega) := \frac{\alpha_*(\omega)}{\sqrt{2\pi}} \quad (\hat{\cdot} \text{ denotes Fourier transform}).
\]

From

\[
\begin{align*}
    k(\omega)^2 &= -(-ik(\omega))^2 \\
    &= -\left(\alpha_*(\omega) + \frac{-i\omega}{c_0}\right)^2,
\end{align*}
\]

it follows that

\[
\mathcal{F}^{-1}[k(\omega)^2](t) \ast f = -\left(D_* + \frac{1}{c_0} \frac{\partial}{\partial t}\right)^2
\]

and hence an attenuated pressure wave \( p_{\text{att}} := G \ast_{x,t} f \) satisfies the wave equation

\[
\nabla^2 p_{\text{att}}(\mathbf{x}, t) - \left(D_* + \frac{1}{c_0} \frac{\partial}{\partial t}\right)^2 p_{\text{att}}(\mathbf{x}, t) = -f(\mathbf{x}, t).
\]

The function \( \alpha_* : \mathbb{R} \to \mathbb{C} \) is called complex attenuation law and \( \alpha := \text{Re}(\alpha_*) \) is called (real) attenuation law. In order that \( p_{\text{att}} \) is real valued \( \alpha \) must be even and \( \text{Im}(\alpha_*) \) must be odd. Attenuation occurs only if \( \alpha \) is positive. Moreover, the function \( \alpha_* \) is restrained by the
requirement of a bounded wave front speed of the Green’s function $G_\gamma$ of (10) \((\text{causality})\). If the wave front speed this is bounded from above by $c_1 \in (0, \infty)$, then this is equivalent to $G_\gamma (x, t + T(|x|)) = 0$ if $t < 0$ for all $x \neq 0$, where $T(|x|) \in (|x|/c_1, \infty)$ is the travel time of the wave front from point $0$ to point $x$. In the case of a constant wave front speed $c_0$, causality is equivalent to

$$K(|x|, t) := 4\pi |x| G_\gamma \left( x, t + \frac{|x|}{c_0} \right) = 0 \quad \text{if} \quad t < 0 \quad \text{for all} \quad x \neq 0. \quad (11)$$

since $T(|x|) = |x|/c_0$, we note that $\hat{K}(|x|, \omega) = e^{-a_\gamma(\omega)|x|/\sqrt{2\pi}}$ \((\text{cf theorem 9 in the appendix})\).

\textbf{Remark 1.} In the literature often a less strong definition of causality is used that demands the existence of a (retarded) Green’s function that vanishes for $t < 0$. However, this requirement is not related to the speed of the wave front. We note that in [18] an equation for acoustic propagation in inhomogeneous media with relaxation losses was derived which satisfies our definition of causality and a frequency power law with exponent 2 for small frequencies. However, the tissue obeys an approximate frequency power law with exponent in (1), (2) and therefore an attenuated wave equation model for thermoacoustic imaging is still required.

According to experiments the real attenuation law of a variety of viscous media similar to the tissue satisfy (at least approximately) a frequency power law \((\text{cf [25, 28]})\):

$$\alpha(\omega) = \alpha_0(\omega)^\gamma \quad \text{for} \quad \gamma \in (1, 2], \quad \alpha_0 = \text{const.}$$

This led to the complex attenuation laws \((\text{cf [23–25, 27]})\)

$$\alpha_s(\omega) = \tilde{\alpha}_0 (-i \omega)^\gamma \quad \text{with} \quad \tilde{\alpha}_0 := \frac{\alpha_0}{\cos(\pi \gamma/2)}, \quad (12)$$

which violate causality for the range $\gamma \in (1, 2]$ relevant for thermoacoustic imaging (cf appendix). For thermoacoustic imaging we propose the following complex attenuation laws:

$$\alpha_s(\omega) = \frac{\alpha_0 (-i \omega)}{c_0 \sqrt{1 + (-i \tau_0 \omega)^{\gamma-1}}} \quad \text{(} \gamma \in (1, 2], \tau_0 > 0\text{)}, \quad (13)$$

where the square root of $1 + (-i \tau_0 \omega)^{\gamma-1}$ is understood as the root that guarantees a nonnegative real part of $\alpha_s$.

\textbf{Remark 2.} We note that the above complex power function is defined by

$$w^\gamma = e^{\gamma(\log w + i \phi)} \quad \text{for} \quad w = r e^{i\phi} \in C^{-}, \quad (14)$$

where $C^- := \mathbb{C} \setminus \{z \in \mathbb{C} | \text{Re}(z) \leq 0, \text{Im}(z) = 0\}$.

\textbf{Remark 3.} In the appendix we prove that the Green’s function $G_\gamma$ of (10) with $\alpha_s$ defined as in (12) does not satisfy the causality requirement (11). It is also proved in the appendix that if $\alpha_s$ is defined as in (13), then (11) is satisfied. Models (13) are appropriate for thermoacoustic imaging, since causality is satisfied and $\text{Re}(\alpha_s)$ is approximately a frequency power law with exponent $\gamma \in (1, 2]$ for small frequencies (cf figure 1). Since the wave number in the thermo-viscous case defined as in [24] and [12] satisfies $k^2 = (\omega/c_0)^2 (1 - i \tau_0 \omega)^{-1}$ \((\text{cf equation (9) in [24]})\), $\text{Re}(\alpha_s)$ for $\gamma = 2$ corresponds to the thermo-viscous attenuation law. More precisely, the complex attenuation law $\alpha_s^{\text{tv}}(\omega)$ determined by the thermo-viscous wave equation and $\alpha_s(\omega)$ defined as in (13) with $\gamma = 2$ are related by

$$\alpha_s^{\text{tv}}(\omega) = \alpha_s(\omega) + \frac{i\omega}{c_0} \quad \text{for} \quad \gamma = 2$$

such that $\text{Re}(\alpha_s^{\text{tv}}(\omega)) = \text{Re}(\alpha_s(\omega))$. 

Figure 1. Comparison of the real part of (13) with $\alpha_0 = \beta_0 \frac{\rho_0}{c_0 \tau_0}$ and the power law $\alpha(\omega) = |\tau_0 \omega|^{\gamma}$ for $\gamma = 1.5$. For liquids: $\tau_0 = 10^{-6}$ ms (left picture) and for gases: $\tau_0 = 10^{-4}$ s (right picture) (cf [12]) if the frequency is in MHz. Experimental demonstrations of the power law are performed for the range 0-60 MHz (cf e.g. [25]).

Remark 4. All mathematical results in this paper are valid if $\alpha_*$ defined as in (13) is replaced by a complex attenuation law that satisfies (11), $\alpha_*(z) \neq i/z/c_0$ for each $z \in \mathbb{C}$ with $\text{Im}(z) > 0$ and has a positive monotonic increasing real part on $\mathbb{R}$. For example if $\alpha_*(\omega) = i\omega/c_0$ with $\omega \in \mathbb{R}$, then $4\pi |x| G(x, t) = \delta(t)$, i.e. $G$ is not a wave.

Modeling of the stress tensor and the source term

Since equation (10) is an integro-differential equation, it is not self-evident that the source term $f$ has the same structure as in the absence of wave attenuation. In order to model the source term we first model the temperature-dependent stress tensor.

According to [17] the stress tensor of an inviscid fluid can be modeled by

$$\sigma_{i,j} = -c_0^2 \rho_0 \alpha_{th}(T - T_0) \delta_{i,j} - p_1 \delta_{i,j}$$

where $p_1 = p + c_0^2 (\rho - \rho_0)$.

Here $p$, $\rho_0$, $c_0$, $T_0$ are reference values of the pressure, the density, the sound speed and the temperature, respectively, and $\alpha_{th}$ is the thermal expansion coefficient. Wave equation (10) shows that a dissipative medium has a long memory (cf [3]) and therefore we claim that attenuation is accounted for if $\sigma_{i,j}$ is replaced by $K_1 \ast \sigma_{i,j}$, where $\ast$ denotes time convolution and $K_1$ is an appropriate kernel. We show that this ansatz implies wave equation (10) with the same thermoacoustic source term as in the absence of wave attenuation. (This calculation can be considered as another justification of model equation (10)). Since $\sigma_{1,1} = \sigma_{2,2} = \sigma_{3,3}$ correspond to the negative total pressure in the absence of acoustic dissipation, our claim implies

$$p_{\text{att}} = \tilde{p} + c_0^2 K_1 \ast \rho (\rho - \rho_0) + c_0^2 \rho_0 \alpha_{th} K_1 \ast (T - T_0)$$

with $\tilde{p} := p \int_{\mathbb{R}} K_1(t) \, dt = \sqrt{2\pi} p \hat{K}_1(0+)$.  

If the activation time of the illumination device is very short, say $\Delta t$, then the absorbed heat in the small volume $d\Omega$ in point $x$ at $t = \Delta t$ can be modeled by (cf [6])

$$\Delta Q(x, \Delta t) = \alpha_{el} c_0 e_0(\mathbf{x}) \int e^{-\alpha_{el}|x-x_s|} \, dx_s I(t) \Delta t \, d\Omega$$

$$=: \alpha_{el} I_0(\alpha_{el}, x) I(t) \Delta t \, d\Omega,$$
where $\alpha_{el}$, $\epsilon_0$ and $x_s$ denote the electromagnetic absorption coefficient, the absorbed energy flow density and the origin of an electromagnetic point source, respectively. Moreover, $I = I(t)$ denotes a short time pulse. On the other hand the heat in the small volume $d\Omega$ in point $x \in \Omega$ at $t = \Delta t$ due to the rise of temperature is

$$\Delta Q(x, \Delta t) = \rho_\Omega c_p (T(x, \Delta t) - T_0) \, d\Omega,$$

where $\rho_\Omega$ denotes the density of the tissue and $c_p$ denotes the specific heat capacity of the tissue at constant pressure. In summary we get

$$\frac{dT}{dt} = \frac{\alpha_{el} \, I_0(\alpha_{el}, \cdot)}{\rho_\Omega \, c_p} \, I. \quad (16)$$

Since a state equation refers always to the local comoving system, the total time derivative $\frac{d}{dt}$ appears. From (15) we infer the state equation

$$\frac{1}{c_0^2} \frac{dp_{att}}{dt} = K_1(\alpha_{th}, \cdot) \frac{d\rho}{dt} + \rho_0 \alpha_{th} \, K_1(\alpha_{th}, \cdot) \frac{dT}{dt}. \quad (17)$$

If the sound speed is not too large, then $\frac{d}{dt}$ can be replaced by $\frac{\partial}{\partial t}$. In the following we assume that this simplification is appropriate. The linearized equation of motion and the linearized continuity equation

$$\rho_0 \frac{\partial v}{\partial t} = -\nabla p_{att} \quad \text{and} \quad \frac{\partial \rho}{\partial t} + \rho_0 \nabla \cdot v = 0$$

imply

$$\nabla^2 p_{att} - \frac{\partial^2 \rho}{\partial t^2} = 0.$$  

From this, $\rho_\Omega \approx \rho_0$, (17) and (16), we infer

$$K_1(t) \star \nabla^2 p_{att}(x, t) - \frac{1}{c_0^2} \frac{\partial^2 p_{att}(x, t)}{\partial t^2} = -K_1(t) \star f(x, t), \quad (18)$$

with

$$f = \frac{\alpha_{th} \alpha_{el} I_0(\alpha_{el}, \cdot)}{c_p} \frac{\partial I}{\partial t}.$$  

Comparison of (10) and (18) in Fourier space shows that both equations are equivalent if

$$\hat{K}_1(\omega) := \frac{1}{\sqrt{2 \pi}} \left( \frac{\omega}{c_0 k(\omega)} \right)^2, \quad k(\omega) = i \alpha_\omega(\omega) + \frac{\omega}{c_0}.$$  

(19)

Now we can formulate the direct problem.

**The direct problem**

Let $T, T_1 > 0$ with $T_1 \ll T$ and e.g. $\phi \in L^2(\Omega)$ and $I \in L^2(\mathbb{R})$ with $\text{supp}(I) \subset [0, T_1]$. The direct problem of TAT is to solve wave equation (18) on $\mathbb{R}^3 \times (0, T)$ with

$$f(x, t) := \phi(x) \, I'(t) \quad \text{where} \quad \phi := \frac{\alpha_{th} \alpha_{el} I_0}{c_p}$$

such that

$$p_{att}|_{t=0} = 0. \quad (20)$$

Let $\delta(t)$ denote the delta distribution. Frequently, it is assumed that $I(t) = \delta(t)$, since the speed of light is much larger than the speed of sound. More precisely, thermal and stress confinement is assumed. Then we have

$$f(x, t) = \phi(x) \, \delta'(t) \quad (22)$$

and $\phi$ corresponds to an initial value function.
Remark 5. In the appendix we show that wave equation (18) has a unique Green’s function \( G_\gamma (f(x, t) = \delta(x) \delta(t)) \) satisfying \( G_\gamma \big|_{t<0} = 0. \)

3. Derivation of the basic theorem

Before we explain the purpose of this section we introduce some important notions and assumptions.

(A1) Without loss of generality we assume \( c_0 = 1. \) \( \alpha_\ast \) is defined as in (13) if \( \gamma \in (1, 2) \) and \( \alpha_\ast := 0 \) if \( \gamma = 0. \) Moreover, \( k \) is defined as in (19).

(A2) \( p_{\text{at}} \) is the unique solution of (18) with (21) and \( f(x, t) = \phi(x) I'(t) \) and \( \tilde{p} \) is the unique solution of (18) with \( \alpha_0 = 0 \) (no attenuation), (21) and \( f(x, t) = \phi(x) \delta'(t). \)

If \( I(t) \neq \delta(t) \), then we assume that the signal satisfies

(A3) \( I \in C(\mathbb{R}) \) and \( \text{supp}(I) \subset [0, T_1] \) for some \( T_1 > 0. \)

If \( I(t) = \delta(t) \), then the unattenuated wave \( p \) is equal to \( \tilde{p} \). The uniqueness of the Green’s function of (18) is proven in theorem 9 in the appendix.

The goal of this section is to derive an integral equation that allows the estimation of \( \tilde{p} \) from the data \( p_{\text{at}} \). This will be the basic theorem of thermoacoustic imaging of acoustic dissipative tissue.

The following two lemmas provide important properties of a distribution that plays an important role in deriving the basic theorem.

Lemma 1. Let (A1) be satisfied and

\[
\hat{M}_\gamma (\omega, t') := \frac{\omega}{\sqrt{2\pi}} \frac{e^{i k(\omega)t'}}{k(\omega)} \quad (\omega, t' \in \mathbb{R}).
\]  

(a) \( M_\gamma \) satisfies for \( t' \geq 0 \)

\[
K_1 * t \frac{\partial^2 M_\gamma}{\partial t'^2} = -\frac{\partial^2 M_\gamma}{\partial t'^2} = 0 \quad \text{with} \quad \frac{\partial M_\gamma}{\partial t'} \bigg|_{t'=0} = -\delta'(t).
\]  

(b) If \( \alpha_\ast \neq 0 \), then \( M_\gamma \in C^\infty(\mathbb{R}^2 \setminus \{(0, 0)\}). \)

(c) If \( \alpha_\ast = 0 \), then \( M_\gamma (t, t') = \delta(t - |t'|). \)

Proof.

(a) In Fourier space, equation (24) and definition (23) imply

\[
\frac{(-i\omega)^2}{(ik(\omega))^2} \frac{\partial^2 \hat{M}_\gamma (\omega, t')}{\partial t'^2} - \frac{(-i\omega)^2}{(ik(\omega))^2} \hat{M}_\gamma (\omega, t') = 0,
\]

\[
\frac{\partial^2 \hat{M}_\gamma (\omega, t')}{\partial t'^2} = (ik(\omega))^2 \hat{M}_\gamma (\omega, t') \quad t' > 0.
\]

From both identities, it follows that \( M_\gamma (t, t') \) satisfies the first identity in (24). The second identity in (24) follows from (23):

\[
\frac{\partial M_\gamma (t, t')}{\partial t'} \bigg|_{t'=0} = \mathcal{F}^{-1} \left\{ \frac{i\omega}{\sqrt{2\pi}} \right\} (t) = -\delta'(t).
\]
Let $\alpha* \neq 0$. For fixed $|t'| = T > 0$, $M_y$ can be considered as an oscillatory integral (cf section 7.8 in [8])

$$I_{\phi,a}(u) = \int e^{i\phi(t, \omega)} a(t, \omega) u(t) \, dt \, d\omega \quad u \in C^\infty_0(\mathbb{R})$$

with the phase function $\phi(t, \omega) = -t \omega$ in $\Gamma = \mathbb{R} \times (\mathbb{R} \setminus \{0\})$ and $a(t, \omega) = \frac{1}{\sqrt{2\pi}} \omega k(\omega) e^{i k(\omega) T}$, since $a$ is $C^\infty$ and rapidly decreasing with respect to $\omega$. From theorem 7.8.3 in [8] and $d\phi/d\omega(t, \omega) = 0$ if $t = 0$,

it follows that $\text{sing supp}(I_{\phi,a}) \subseteq \{0\}$. That is to say $M_y(t, T)$ is $C^\infty$ for $t > 0$. Similarly if $t = T$ is fixed, then it follows that $M_y(T, t')$ is $C^\infty$ for $|t'| > 0$. This shows that $M_y$ is $C^\infty$ on $\mathbb{R}^2 \setminus \{(0, 0)\}$.

(c) The claim follows from $k(\omega) = \omega (\alpha* = 0)$ and the property $F\{\delta(t - t')\}(\omega) = e^{i\omega t'/\sqrt{2\pi}}$. This concludes the proof. □

**Lemma 2.** Let (A1) be satisfied with $\alpha* \neq 0$ and $M_y$ be defined as in (23).

(a) If $t' = 0$, then

$$M_y(t, 0) = 0 \quad \text{for} \quad t < 0.$$ 

(b) If $t' \neq 0$, then $t \mapsto M_y(t, t')$ is a rapidly decreasing $C^\infty$-function on $\mathbb{R}$ and satisfies

$$M_y(t, t') = 0 \quad \text{for} \quad t < |t'|. \quad (25)$$

**Proof.**

(a) From (23) with $t' = 0$ and (13) we get

$$\sqrt{2\pi} \hat{M}_y(\omega, 0) = \frac{\omega}{k(\omega)} = \frac{\sqrt{1 + (-i\tau_0 z)^{y-1}}}{\alpha_0 + \sqrt{1 + (-i\tau_0 z)^{y-1}}} \quad (\alpha_0 > 0).$$

Here the power functions are defined on $\mathbb{C}\setminus(-\infty, 0]$. Since $\alpha_0 + \sqrt{1 + (-i\tau_0 z)^{y-1}}$ maps

$$\{z \in \mathbb{C} \mid \text{Im}(z) \geq 0\} \quad \text{to} \quad \{w \in \mathbb{C} \mid \text{Re}(w) \geq \alpha_0\},$$

we see that $\hat{M}_y(z, 0)$ is holomorphic for all $\text{Im}(z) \geq 0$. Hence conditions (C1) and (C2) of theorem 6 in the appendix are satisfied. Moreover, we see that there exists a polynomial $P$ such that

$$|\hat{M}_y(z, 0)| \leq P(|z|) \quad \text{for} \quad \text{Im}(z) \geq 0.$$ 

According to theorem 6 in the appendix, $M_y(t, 0)$ vanishes for $t < 0$.

(b) Let $\alpha* \neq 0$ and $t' \neq 0$. Then $\hat{M}_y(\omega, t')$ is rapidly decreasing and $C^\infty$- and thus $M_y(t, t')$ is a rapidly decreasing $C^\infty$-function on $\mathbb{R}$. Hence property (25) is satisfied if

$$M_y(t + |t'|, t') = 0 \quad \text{for} \quad t < 0. \quad (26)$$

Because of (23) and the second definition in (19) we have

$$M_y(t + |t'|, t') = F^{-1}\{\hat{M}_y(\omega, t') e^{-i\omega|t'|}\}(t) = F^{-1}\left\{\frac{\omega}{k(\omega)} e^{-\alpha*(\omega)|t'|} \sqrt{2\pi}\right\}(t).$$
According to part (a) of this proof and theorem 8 in the appendix both functions

\[
\mathcal{F}^{-1}\left\{ \frac{e^{-\alpha t}}{\sqrt{2\pi}} \right\} (t)
\]

vanish for \( t < 0 \) and therefore their convolution vanishes for \( t < 0 \), too. This proves property (26) and concludes the proof. \( \square \)

Let \( \tau_1, \tau_2 \in \mathbb{R} \). We recall that if \( f_1 \) and \( f_2 \) are the two distributions with support in \([\tau_1, \infty)\) and \([\tau_2, \infty)\), respectively, then \( f_1 \ast f_2 \) is well defined and (cf [8])

\[
\text{supp}(f_1 \ast f_2) \subseteq \text{supp}(f_1) + \text{supp}(f_2) \subseteq [\tau_1 + \tau_2, \infty). \tag{27}
\]

Remark 6. For completeness, we sketch the proof of part (a) of lemma 2 for the general case of a complex attenuation law \( \alpha \) that satisfies (11), \( \alpha(z) \neq i z/c_0 \) for each \( z \in \mathbb{H} \) and has a positive monotonic increasing real part on \( \mathbb{R} \). We have to prove properties (C1)–(C3) of theorem 6 for \( z_k(z) \). Firstly, causality condition (11) implies \( \mathcal{F}^{-1}\{\alpha(t)\} = 0 \) if \( t < 0 \), which is equivalent to the Kramers–Kronig relation in [27]. Then \( \alpha \) is analytic on \( \mathbb{H} \) by theorem 6, and since \( k \) has no zeros (\( \alpha(z) \neq i z/c_0 \)), \( z/k(z) \) is analytic on \( \mathbb{H} \). Hence (C1) is satisfied. Secondly, (C2) holds because of the remark after theorem 7.4.3 in [8]. Thirdly, (11) also implies for fixed \( t' > 0 \) that \( M_{\gamma}(t, t') = 0 \) if \( t < 0 \) and thus by theorem 6

\[
\frac{|z|}{k(z)} \leq P(|z|) \quad z \in \mathbb{H} \quad (P \text{ polynomial}).
\]

The maximum modulus principle implies that the second left-hand side term grows for \( |z| \to \infty \), which implies that there exists a constant \( C_1 > 0 \) such that

\[
\frac{|z|}{k(z)} C_1 \leq \frac{|z|}{k(z)} \leq P(|z|) \quad \text{on} \quad z \in \mathbb{H}.
\]

But this is (C3) and thus the claim is proven.

Now we derive the first integral relation that relates the attenuated pressure data with the unattenuated pressure data in case \( I(t) = \delta(t) \). Note that in this case \( p \equiv \tilde{p} \) (cf (A2)).

Theorem 1. Let (A1), (A2) with \( I(t) = \delta(t) \) be satisfied and \( M_{\gamma} \) be defined as in (23). Then \( p \) solves the integral equation

\[
p_{\text{att}}(x, t) = \int_0^\infty M_{\gamma}(t, t') p(x, t') \, dt'. \tag{28}
\]

Here the upper limit of integration can be replaced by \( t \).

Proof. Let \( t' \geq 0 \) and \( n \in \{0, 1\} \). Lemma 2 implies that

\[
\frac{\partial^n}{\partial t'^n} M_{\gamma}(t, t') = 0 \quad \text{for} \quad t < t'
\]

and that \( K_1 \) defined as in (19) is causal. From this and (27), it follows that

\[
K_1(t) \ast \frac{\partial^n}{\partial t'^n} M_{\gamma}(t, t') = 0 \quad \text{if} \quad t < t' \quad \text{and} \quad n \in \{0, 1\}. \tag{29}
\]

Moreover, \( p \) satisfies (\( \gamma = 0, K_1(0) = \delta(t) \))

\[
\nabla^2 p(x, t') = \frac{\partial^2 p(x, t')}{\partial t'^2} \quad \text{for} \quad t' > 0,
\]

\[
p(\cdot, 0+) = \phi \quad \text{and} \quad \frac{\partial p}{\partial t}(\cdot, 0+) = 0. \tag{30}
\]
For convenience we introduce the notions
\[ \Box p := K_1(t) \ast (\nabla^2 p) - \frac{\partial^2 p}{\partial t^2} \] (31)
and
\[ q(x, t) := \int_0^\infty M_{\gamma}(t, t')p(x, t') \, dt'. \]

Kernel property (25) implies that the upper limit can be replaced by \( t \). From (24) we get
\[ \Box q(x, t) = \int_0^\infty K_1(t) \ast \left[ M_{\gamma}(t, t')\nabla^2 p(x, t') - \frac{\partial^2 M_{\gamma}(t, t')}{\partial t^2} p(x, t') \right] \, dt', \]
which simplifies with integration by parts together with (29) and (30) to
\[ \Box q(x, t) = \phi(x) \left[ K_1(t) \ast \frac{\partial M_{\gamma}(t, t')}{\partial t'} \right]_{t'=0} = -\phi(x) \delta(t) \ast \tau_1(t). \] (32)

Because \( p_{\text{att}} \) is the unique solution of (18) satisfying (21) (cf theorem 9 in the appendix), it follows that \( p_{\text{att}} = q \). This concludes the proof. \( \Box \)

From theorem 1, assumption (A3) and (27), we get:

**Theorem 2.** Let (A1)–(A3) be satisfied and \( M_{\gamma} \) be defined as in (23). Then \( \tilde{p} \) solves the integral equation
\[ p_{\text{att}}(x, t) = \int_0^\infty [I(t) \ast M_{\gamma}(t, t') \phi](x, t') \, dt' \] (33)
The upper limit of integration can be replaced by \( t \).

4. Integral equation models for projections of \( \phi \)

In this section we derive integral equation models for the three types of projections in TAT. For various situations, the function \( \phi \) can be calculated via explicit reconstruction formulas from projections of \( \phi \) (cf [5, 10, 15, 16, 30–32, 34]). In the case of reconstructions with limited view, we refer to [33] and the literature cited there.

In this section we use the following notions and assumptions.

(1) The region of interest (tissue) \( \Omega \subset \mathbb{R}^3 \) is a subset of the open ball \( B_{R_0}(0) \) with radius \( R_0 > 0 \).

(2) \( d\lambda^m \) denotes the Lebesgue measure on \( \mathbb{R}^m \) for \( m \in \{1, 2, 3\} \).

**Case of point detectors**

The simplest setup of point detector, which guarantees stable reconstruction, is the sphere \( \Gamma = \partial B_{R_0}(0) \) that encloses the region of interest. Then the set of data is
\[ \{ p_{\text{att}}(x, t) \mid x \in \Gamma, t \in [0, T] \} \quad (T > 0 \text{ sufficiently large}). \]

Inserting the spherical mean representation (8) into integral equation (33) and performing integration by parts yield
\[ p_{\text{att}}(x, t) = -\int_0^t \frac{\partial}{\partial t'} [I(t) \ast M_{\gamma}(t, t') \phi](x, t') \frac{R_{\gamma}(\phi)(x, t')}{4\pi t'} \, dt' \quad \text{for} \quad t \geq 0, \]
since (27) holds and
\[
\left[ \frac{R_{sp}(\phi)(x, t')}{4\pi t'} \right]_{t'=0} = \left[ \int_0^{t'} \tilde{p}(x, \tau) \, d\tau \right]_{t'=0} = 0.
\]
This implies

**Theorem 3.** Let (A1)–(A3) be satisfied and \( M_\gamma \) be defined as in (23). The spherical projection \( R_{sp}(\phi) \) of \( \phi \) satisfies
\[
N_{\gamma}(t, t') = -\frac{1}{\sqrt{2\pi}} \frac{\partial M_\gamma(t, t')}{\partial t'} = -\frac{\partial K(t', t - t')}{\partial t},
\]
which together with theorem 3 imply the representation
\[
N_{\gamma}(t, |x|) = \frac{1}{4\pi |x|} \frac{\partial K(|x|, t - |x|)}{\partial t} = -\frac{1}{4\pi |x|} \frac{\partial K(|x|, t - t')}{|x|} \bigg|_{t'=|x|} \tag{35}
\]
if \( I(t) = \delta(t) \).

**Case of planar detectors**

In order to define a data setup of planar detectors we need

**Definition 1.** Let \( n \in S^1 \) and \( E(n, t) \) denote a plane normal to \( n \) with normal distance \( t \) to the origin. Let \( \phi \) be integrable. We define the planar projection of \( \phi \) by
\[
R_{pl}(\phi)(n, t) := \int_{E(n, t)} \phi(x') \, d\lambda_2(x') \quad n \in S^1, \quad t \geq 0.
\]

Let \( \Gamma = S^2 \). The simplest setup of planar detectors, which guarantees stable reconstruction, is given by
\[
\{ R_{pl}(p_{att}(\cdot, t))(n, R_0)|n \in \Gamma, t \in [0, T]\} \quad (T > 0 \text{ sufficiently large}).
\]

According to [7, 35] the identity
\[
R_{pl}(\tilde{p}(\cdot, s))(n, R_0) = 2R_{pl}(\phi)(n, R_0 - s) \quad n \in S^1, \quad s \geq 0
\]
holds. This identity and integral equation (33) imply
\[
R_{pl}(p_{att}(\cdot, t))(n, R_0) = \int_0^{R_0} 2[I(t) *_{s} M_{\gamma}(t, s)]R_{pl}(\phi)(n, R_0 - s) \, ds
\]
\[
= \int_{R_0-t}^{R_0} 2[I(t) *_{t'} M_{\gamma}(t, R_0 - t')]R_{pl}(\phi)(n, t') \, dt'.
\]
This leads to
Theorem 4. Let (A1)–(A3) be satisfied and $M_\gamma$ be defined as in (23). The planar projection $R_{pl}$ of $\Phi$ satisfies

$$R_{pl}(\rho_{pl}(\cdot, t))(\mathbf{n}, R_0) = \int_{R_0 - t}^{R_0} N_\gamma(t, t') R_{pl}(\Phi)(\mathbf{n}, t') \, dt' \quad \mathbf{n} \in S^1, \; t > 0$$

where

$$N_\gamma(t, t') = 2 \{ I(t) \ast_I M_\gamma(t, R_0 - t') \}. \quad (37)$$

Case of line detectors

In this case the data setup is more complicated. Again we start with a definition.

Definition 2. Let $\mathbf{n} \in S^1$ and $E(\mathbf{n}, 0)$ be defined as in definition 1. For each $\mathbf{x} \in E$, $l_\mathbf{n}(\mathbf{x})$ denotes the line passing through $\mathbf{x}$ normal to $E$. For integrable $\Phi$ we define the line integral operator by

$$\Phi_\mathbf{n} := L_\mathbf{n}(\Phi)(\mathbf{x}) := \int_{l_\mathbf{n}(\mathbf{x})} \Phi(\mathbf{x}') \, d\lambda^1(\mathbf{x}') \quad \mathbf{x} \in E.$$  

We define the circular projection of $\Phi_\mathbf{n}$ by

$$R_{circ}(\Phi_\mathbf{n})(\mathbf{x}, t) := \int_{\partial B_t(\mathbf{x}) \cap E} \Phi_\mathbf{n}(\mathbf{x}') \, d\lambda^1(\mathbf{x}') \quad \mathbf{x} \in E, \; t \geq 0.$$  

Actually we are concerned with three inverse problems. The first is concerned with the estimation of the circular projections $R_{circ}(\Phi_\mathbf{n})$ for each $\mathbf{n} \in S^1$, the second is concerned with the estimation of $\Phi_\mathbf{n}$ for each $\mathbf{n} \in S^1$ and the third is concerned with the estimation of $\Phi$ from the set $\{\Phi_\mathbf{n} \mid \mathbf{n} \in S^1\}$. The latter problem corresponds to the inversion of the linear Radon transform and will not be discussed in this paper. For each $\mathbf{n} \in S^1$ let e.g. $\Gamma = \partial B_{R_0}(0) \cap E(\mathbf{n})$, which guarantees a stable reconstruction. Consider the sets of data

$$M(\mathbf{n}) = \{ L_\mathbf{n}(\rho_{pl}(\cdot, t)) (\mathbf{x}) | \mathbf{x} \in \Gamma, t \in [0, T] \} \quad (T > 0 \text{ sufficiently large}).$$

For each fixed $\mathbf{n}$ and data set $M(\mathbf{n})$ we derive an integral equation for $R_{circ}(\Phi_\mathbf{n})$. In [1] it was shown that

$$L_\mathbf{n}(\tilde{\rho}(\cdot, s))(\mathbf{x}) = \frac{1}{2\pi} \frac{\partial}{\partial s} \int_{0}^{t} \frac{R_{circ}(\Phi_\mathbf{n})(\mathbf{x}, t')}{\sqrt{s^2 - t'^2}} \, dt'.$$  

Let $\tilde{M}_\gamma(\cdot, s) := I(\cdot) \ast_I M_\gamma(\cdot, s)$. We note that the geometric property

$$\{ (t', s) \in [0, t]^2 | s \in [0, t], t' \in [0, s] \} = \{ (t', s) \in [0, t]^2 | t' \in [0, t], s \in [t', t] \}$$

implies

$$\int_{0}^{t'} \int_{0}^{t} dt' \, ds = \int_{0}^{t} \int_{t'}^{t} ds \, dt'.$$

From this, (33), (38), and integration by parts, we get

$$L_\mathbf{n}(\rho_{pl}(\cdot, t))(\mathbf{x}) = - \frac{1}{2\pi} \int_{0}^{t} \int_{0}^{t} \frac{\partial \tilde{M}_\gamma(t, s)}{\partial s} \frac{R_{circ}(\Phi_\mathbf{n})(\mathbf{x}, t')}{\sqrt{s^2 - t'^2}} \, dt' \, ds$$

$$= - \frac{1}{2\pi} \int_{0}^{t} \int_{t'}^{t} \left[ \frac{1}{\sqrt{s^2 - t'^2}} \frac{\partial \tilde{M}_\gamma(t, s)}{\partial s} \right] ds R_{circ}(\Phi_\mathbf{n})(\mathbf{x}, t') \, dt',$$
since $M_\nu(t, t) = 0$ for $t > 0$ and
\[
\frac{1}{2\pi} \left[ \int_0^t \frac{R_{\text{circ}}(\Phi_{\nu})(x, t')}{\sqrt{s^2 - t'^2}} \, dt' \right]_{s=0} = \left[ \int_0^t L_n(p, \tau)(x) \, d\tau \right]_{s=0} = 0.
\]
We infer

**Theorem 5.** Let (A1)–(A3) be satisfied and $M_\nu$ be defined as in (23). For $n \in S$ let $L_n$ and $R_{\text{circ}}$ be defined as in definition 2. Then
\[
L_n(p_{\text{att}}(\cdot, t))(x) = \int_0^t N_\nu(t, t') R_{\text{circ}}(\Phi_{\nu})(x, t') \, dt' \quad x \in E, \quad t > 0 \tag{39}
\]
with kernel
\[
N_\nu(t, t') = -\frac{1}{2\pi} \int_t^t \left[ \frac{1}{\sqrt{s^2 - t'^2}} I(t) \ast \frac{\partial M_\nu(t, s)}{\partial s} \right] \, ds. \tag{40}
\]

5. Conclusions

One main requirement for the model presented in this paper is causality. Apart from the pure mathematical interest there are two main reasons for this objective. First, a stable numerical solution of the direct problem requires a finite wave front speed. For example, the CFL condition requires this property and can be satisfied by the model (10) with (13). Second, due to the sensitivity (ill-posedness) of many inverse problems, an accurate modeling of the direct problem is crucial. Indeed, the integral equation models derived in section 4 reveal that the inverse problem is ill-posed and hence small modeling errors may cause large estimation errors if real measurement data are used. More precisely, that the kernel function $(t, t') \in [0, \infty)^2 \mapsto M_\nu(t, t') \in \mathbb{R}$ in integral equation (28) is $C^\infty$ and satisfies
\[
M_\nu(t, t) = 0 \quad \text{for} \quad t > 0 \quad \text{(by continuity)},
\]
indicates that the estimation of the unattenuated data is severely ill-posed (cf Satz 3.6 in [9]). Indeed, the singular values of a discretization of $\hat{M}_\nu(\omega, t')$ decrease exponentially. This paper discloses a potentially strategy for solving the main TAT problem in the presence of acoustic dissipation, but due to the possibly severely ill-posedness of the problem, only a limited increase of resolution of standard TAT methods can be expected.

Appendix

In the appendix we prove that the frequency power law (12) for $\gamma \in (1, 2]$ fails causality, defined as in section 2, and that the complex attenuation law proposed in (13) permits causality. Moreover, we prove that the attenuated wave equation has a unique Green’s function that vanishes for $t < 0$. For this purpose we need theorem 4 in [4], which is stated below (cf theorem 7.4.3 and the following remark in [8]).

Let $\mathbb{H} := \{ z \in \mathbb{C} \mid \text{Im}(z) > 0 \}$ denote the upper open complex half plane and $S'(\mathbb{R})$ denote the space of tempered distributions on $\mathbb{R}$ with range in $\mathbb{C}$.

**Theorem 6.** A distribution $f \in S'(\mathbb{R})$ is causal, i.e. $\text{supp}(f) \subseteq [0, \infty)$, if and only if

(C1) $\hat{f} : \mathbb{R} \rightarrow \mathbb{C}$ can be extended to a function $F : \mathbb{H} \rightarrow \mathbb{C}$ that is holomorphic.

(C2) For all fixed $\eta > 0$ and $\xi \in \mathbb{R}$, $F(\xi + i \eta)$ considered as a distribution with respect to the variable $\xi$ is tempered, and $F(\xi + i \eta)$ converges (in the sense of $S'$) when $\eta \rightarrow 0$. 

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(C3) There exists a polynomial \( P \) such that
\[
|F(z)| \leq P(|z|) \quad \text{for} \quad \text{Im}(z) \geq \epsilon > 0.
\]

If all three conditions are satisfied, then \( F \) is the Fourier–Laplace transform of \( f \).

**Remark 7.** The definition of the Fourier transform in this paper has a different sign as in [4] and thus \( \mathbb{H} \) is the upper half plane and not the lower half plane.

**Theorem 7.** For \( \alpha_0 > 0 \) and \( \gamma \in (1, 2) \) let \( \alpha_\ast \) be defined as in (12). Then \( \mathcal{F}^{-1}\{e^{-\alpha_\ast(\omega)|x|}\}(t) \) is not a causal function.

**Proof.** Let \( x \in \mathbb{R}^3 \) be arbitrary but fixed.

1. Since \( e^{-\alpha_\ast(\omega)|x|} \) is a rapidly decreasing function, its inverse Fourier transform is also a rapidly decreasing function and hence theorem 6 is applicable. The holomorphic power function \( (-iz)^\gamma \) (cf remark 2) defined on \( \mathbb{C}^- \) is the unique holomorphic extension of the function \( \omega \in \mathbb{R} \mapsto (-i \omega)^\gamma \in \mathbb{C} \) and hence
\[
F(z) = \exp[-\alpha_\ast(z)|x|] \quad \text{for} \quad z \in \mathbb{H}
\]
is the unique holomorphic extension of \( \omega \in \mathbb{R} \mapsto \exp[-\alpha_\ast(\omega)|x|] \in \mathbb{C} \).

2. We show that condition (C3) cannot be satisfied for the sequence \( (z_n)_{n \in \mathbb{N}} \) defined by
\[
z_n := in \quad \text{for} \quad n \in \mathbb{N}.
\]
From \(-iz_n = n \) and \( \cos \left( \frac{x}{2} \gamma \right) < 0 \) for \( \gamma \in (1, 2] \), it follows
\[
|F(z_n)| = \left| \exp \left\{ -\alpha_\ast \left( \frac{-inz_n}{\cos \left( \frac{\gamma}{2} \right)} \right)x \right\} \right| = \exp \left\{ \alpha_0 \frac{n^\gamma}{\cos \left( \frac{\gamma}{2} \right)} \right\} ,
\]
which cannot be bounded by a polynomial \( P(n) \). \( \square \)

**Remark 8.**

1. We note that the following modification of the power law
\[
\alpha_\ast(\omega) = \tilde{\alpha}_0(\gamma)(-i\omega)^\gamma + \alpha_1(-i\omega) \quad (\gamma \in (1, 2]),
\]
which also appears in the literature, does not satisfy causality, since
\[
|F(z_\ast)| = \exp \left\{ \alpha_0 \frac{n^\gamma}{\cos \left( \frac{\gamma}{2} \right)} \right\} - \alpha_1 n
\]
cannot be bounded by a polynomial for \( \gamma > 1 \).

2. Let \( \gamma \in (0, 1) \). It is easy to see that \( |F(z)| \) is bounded by a constant, since \( \text{Re} (\alpha_\ast(z)) \) is always positive and thus \( |F(z)| \) decreases exponentially for \( |z| \to \infty \).

**Theorem 8.** For \( \alpha_0, \tau_0 > 0 \) and \( \gamma \in (1, 2] \), let \( \alpha_\ast \) be defined as in (13). Then \( \mathcal{F}^{-1}\{e^{-\alpha_\ast(\omega)|x|}\}(t) \) is a causal function.

**Proof.** Let \( x \in \mathbb{R}^3 \) be arbitrary but fixed.

1. Since \( e^{-\alpha_\ast(\omega)|x|} \) is a rapidly decreasing function, its inverse Fourier transform is also a rapidly decreasing function and thus theorem 6 is applicable. Let \( M_\ast := \{ z \in \mathbb{C} | \text{Im}(z) > -\epsilon \} \). Since \( \alpha_\ast(z) \) is holomorphic on \( M_\ast \) for sufficiently small \( \epsilon \), this function is the unique holomorphic extension of \( \omega \in \mathbb{R} \mapsto \alpha_\ast(\omega) \in \mathbb{C} \). Thus
\[
F(z) = \exp[-\alpha_\ast(z)|x|] \quad \text{for} \quad z \in \mathbb{H}
\]
is the unique extension of \( \omega \in \mathbb{R} \mapsto \exp[-\alpha_\ast(\omega)|x|] \in \mathbb{C} \). This proves conditions (C1) and (C2) of theorem 6.
(2) Since the inequality in (C3) is equivalent to
\[
\exp[-\text{Re}(\alpha_*(z))] \leq P(|z|) \quad \text{for } \text{Im}(z) \geq \epsilon > 0,
\]
(C3) is satisfied if
\[
\text{Re}(\alpha_*(\mathbb{H})) \subseteq [0, \infty).
\]
Since
\[
\text{Re}(\alpha_*) = \frac{\alpha_0}{c_0}(w_1z_2 + w_2z_1) \quad (z_2 > 0)
\]
with
\[
f(z) := \frac{1}{\sqrt{1 + (-ir_0z)^\gamma - 1}} =: w_1 + iw_2 \quad \text{for } z \in \mathbb{H},
\]
we have to show that
\[
w_1 > 0 \quad \text{and} \quad w_2 z_1 \geq 0.
\]
Let \( f_1 = -i r_0 z, f_2(z) := \sqrt{1 + z^{r_0}} \) and \( f_3(z) := \frac{1}{z} \), so that \( f = f_3 \circ f_2 \circ f_1 \). Moreover, let
\[
M_1 := \{z_1 + iz_2 \in \mathbb{C}|z_1 < 0, z_2 > 0\},
M_2 := \{z_1 + iz_2 \in \mathbb{C}|z_1 > 0, z_2 > 0\},
M_3 := \{z_1 + iz_2 \in \mathbb{C}|z_1 > 0, z_2 < 0\},
N_1 := \{iz_2 \in \mathbb{C}|z_2 > 0\} \quad \text{and} \quad N_2 := \{z_1 \in \mathbb{C}|z_1 > 0\}.
\]
We see that
\[
f_1(M_1) \subseteq M_2, \quad f_1(N_1) \subseteq N_2, \quad f_1(M_2) \subseteq M_3,
\]
\[
f_2(M_2) \subseteq M_2, \quad f_2(N_2) \subseteq N_2, \quad f_2(M_3) \subseteq M_3,
\]
From
\[
\frac{1}{z_1 + iz_2} = \frac{z_1 - iz_2}{z_1^2 + z_2^2} \quad z_1^2 + z_2^2 \neq 0,
\]
we get
\[
f_3(M_2) \subseteq M_3, \quad f_3(N_2) \subseteq N_2, \quad f_3(M_3) \subseteq M_2.
\]
Therefore we end up with
\[
f(M_1) \subseteq M_3, \quad f(N_1) \subseteq N_2, \quad f(M_2) \subseteq M_2,
\]
since \( f = f_3 \circ f_2 \circ f_1 \). In other words condition (A.1) is satisfied and hence condition (C3) holds. This concludes the proof. \( \square \)

**Theorem 9.** For \( \alpha_0, \tau_0 > 0 \) and \( \gamma \in (1, 2] \), let \( \alpha_* \) be defined as in (13). Then equation (18) with \( f(x, t) = \delta(x) \delta(t) \) and
\[
p_{att} \mid_{t<0} = 0 \quad \text{and} \quad p_{att} = 0 \quad \text{at} \quad t = 0,
\]
has a unique solution
\[
p_{att}(x, t) = F^{-1} \left\{ \frac{\exp[i k(\omega)|x|]}{4\pi |x|} \right\} \quad \text{with} \quad k(\omega) := i \alpha_*(\omega) + \frac{\omega}{c_0}.
\]

**Proof.** Since \( p_{att} \) is the Green’s function, we write \( G_\gamma \) instead of \( p_{att} \). The Helmholtz equation of (18) reads as follows
\[
\nabla^2 G_\gamma + k^2 G_\gamma = -\delta(x) / \sqrt{2\pi}
\]
and has the only two solutions
\[ \hat{G}_\gamma(x, \omega) = \frac{1}{\sqrt{2\pi}} \frac{\exp[\text{i}k(\omega)|x|]}{4\pi|x|} \quad s = \pm 1. \]

If condition (A.2) holds, then according to theorem 6, we have for \( z = z_1 + iz_2 \in \mathbb{H} \) and \( x \neq 0 \):
\[ e^{-s \text{Re}(\alpha^*(z)) |x|} \leq P(|z|) \quad z_1 \in \mathbb{R}, \quad z_2 > 0 \]
(A.3)
for some polynomial \( P \). From theorems 8 and 6, we get
\[ e^{-Re(\alpha^*(z)) |x|} \leq Q(|z|) \quad z_1 \in \mathbb{R}, \quad z_2 \geq \epsilon > 0 \]
for some polynomial \( Q \), i.e. (A.3) holds with \( P := Q \) for \( s = 1 \). If \( s = -1 \), then the left-hand side of (A.3) grows exponentially, since \( \text{Re}(\alpha^*(\omega)) > 0 \) is increasing, i.e. condition (A.2) cannot be satisfied for \( s = -1 \). Hence there exists a unique solution. □

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