Real Valued Functions for the BFKL Eigenvalue

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Abstract: We consider known expressions for the eigenvalue of the Balitsky-Fadin-Kuraev-Lipatov (BFKL) equation in $N = 4$ super Yang-Mills theory as a real valued function of two variables. We define new real valued functions of two complex conjugate variables that have a definite complexity analogous to the weight of the nested harmonic sums. We argue that those functions span a general space of functions for the BFKL eigenvalue at any order of the perturbation theory.

Keywords: BFKL equation; real valued functions; maximal transcendentality

1. Introduction

The analytic solution of the Balitsky-Fadin-Kuraev-Lipatov (BFKL) [1–4] equation is traditionally represented as the eigenvalue problem in the space of real Mellin coordinates of conformal spin $n$ and anomalous dimension $\nu$. Those two variables are real, but the known leading order (LO) and the next-to-leading order (NLO) BFKL [5] eigenvalues are written in terms of polygamma functions and their generalizations as functions of complex variables $z = -\frac{1}{2} + i\nu + \frac{|n|}{2}$ and $\bar{z} = -\frac{1}{2} - i\nu + \frac{|n|}{2}$. The overall expression is real due to the cancellation of the imaginary part and it would be more natural to use real valued functions to describe the real valued BFKL eigenvalue. As a starting point we use the NLO BFKL eigenvalue in $N = 4$ SYM derived from the corresponding QCD expression by Kotikov and Lipatov [6,7] due to the uniform complexity of the functions that build it. The uniform complexity of NLO BFKL eigenvalue in $N = 4$ SYM is related to the principle of the maximal transcendentality formulated by Kotikov and Lipatov [8], which can be used as a practical tool in loop calculations.

The recent results for higher order corrections to the BFKL eigenvalue stem from integrability techniques use nested harmonic sums as a convenient space of functions. The nested harmonic sums are functions of one variable and the currently available results for higher order corrections to the BFKL eigenvalue are given in the analytic form as functions of either $\nu$ or $n$, but not both of them. The nested harmonic sums represent a convenient choice of possible space of functions mostly due to their pole structure that coincides with that of polygamma functions and their generalizations, as well as, because of the fact that they are properly labeled with so-called weights that allow to identify the complexity of expression (A few words to be said about the definition of transcendentality and complexity we use in our analysis. The transcendental constants are well known in mathematics (\pi, \log 2 etc.) and defined as irrational numbers, which are not roots of any non-trivial polynomial with rational coefficients. We extend the notion of transcendentality to functions that give a transcendental constant in the limit of the infinite value of the argument. We use the analytically continued nested harmonic sums as a convenient space of functions for that purpose. The weight and depth of nested harmonic sums determine their complexity and allow to prescribe a definite “transcendentality” to a certain function. It is worth emphasizing that a notion of “transcendentality” is still to be defined in a rigorous mathematical way, but for all practical purposes its definition using the limit of well defined nested harmonic sums works exceptionally well). The maximal complexity

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of functions in the BFKL eigenvalue depends on the loop order and gives a connection between the corresponding expressions in QCD and $N = 4$ SYM due to the principle of maximal transcendentality. In this paper we continue to investigate the analytic properties of the BFKL eigenvalue and related special functions [9–16]. We put an emphasis on special functions that span the space of functions relevant for the BFKL eigenvalue and are universal at any order of the perturbation theory. In this paper we propose real valued functions of two complex conjugate variables as a candidate to define the proper space of functions for this purpose. The new real valued function combine the useful features of the nested harmonic sums and generalized polygamma functions such as a well defined complexity (weight) and the correct pole structure.

The paper is organized as follows. In the next section we analyze the functions that appear in LO and NLO expressions of the BFKL eigenvalue in $N = 4$ SYM. Then we show that the special real valued combinations of those functions possess a common functional form of a well defined complexity (i.e., weight). In the last section we list all possible real valued functions at weight one and two. In the Appendix A we provide a full list of the new real valued functions at weight three.

2. BFKL Eigenvalue

In this section we consider the BFKL eigenvalue in $N = 4$ Super Yang-Mills (SYM) theory and the leading order (LO) and the next-to-leading (NLO) order that was calculated directly from Feynman diagrams. Then we extend our analysis to the known higher order corrections to the BFKL eigenvalue obtained using integrability techniques. This LO and NLO expressions can be written as follows (see paper by Kotikov and Lipatov [6,7]),

$$\omega = 4\bar{a}[\chi(n, \gamma) + \delta(n, \gamma)]$$

where $\gamma = \frac{1}{2} - i\nu$ and

$$\chi(n, \gamma) = 2\psi(1) - \psi(M) - \psi(1 - \bar{M}),$$

$$\delta(n, \gamma) = \phi(M) + \phi(1 - \bar{M}) - \frac{\alpha_0}{2\pi}(\rho(M) + \rho(1 - \bar{M}))$$

in terms of

$$M = \gamma + \frac{|n|}{2}, \quad \bar{M} = \gamma - \frac{|n|}{2}.$$ 

Here $\psi(z)$ is the Euler $\psi$-function defined in through the logarithmic derivative of the Gamma function $\psi(z) = \frac{d\ln \Gamma(z)}{dz}$. The coupling $\bar{a} = \frac{g^2 N_c}{16\pi^2}$ is given in terms of the coupling constant $g$ in the DREG scheme (Dimensional Regularization (DREG) scheme is the prescription for regularizing loop integrals by changing the dimension of the integrated momenta by a small negative value. For example, in the case of the loop integration over four dimensional $d = 4$ momenta one replaces the dimension of the integration by $d = 4 - \epsilon$, namely writing $d^{4-\epsilon}k$ instead of $d^4k$. The dimensional regularization is used for infra-red and ultra-violet divergent integrals, where the divergences appear as powers of $1/\epsilon$). The functions $\rho(M)$ and $\phi(M)$ are give by

$$\rho(M) = \beta'(M) + \frac{1}{2}\zeta(2)$$

and

$$\phi(M) = 3\zeta(3) + \psi''(M) - 2\Phi_2(M) + 2\beta'(M)(\psi(1) - \psi(M))$$
where
\[
\beta'(M) = \frac{1}{4} \left[ \psi' \left( \frac{M + \frac{1}{2}}{2} \right) - \psi' \left( \frac{M}{2} \right) \right] = - \sum_{r=0}^{\infty} \frac{(-1)^r}{(M+r)^2}
\]  
(7)

and
\[
\Phi_2(M) = \sum_{k=0}^{\infty} \frac{\beta'(k+1)}{k+M} + \sum_{k=0}^{\infty} \frac{(-1)^k \psi'(k+1)}{k+M} - \sum_{k=0}^{\infty} \frac{(-1)^k (\psi(k+1) - \psi(1))}{(k+M)^2}.
\]  
(8)

The expressions of \( \chi(n, \gamma) \) and \( \delta(n, \gamma) \) are real functions of \( \gamma = \frac{1}{2} - iv \) and for real \( v \) and \( n \). They are built of polygamma functions and their generalizations, which are complex functions of one complex variable. They appear in \( \chi(n, \gamma) \) and \( \delta(n, \gamma) \) in particular combinations that cancel the imaginary part making the overall expression being real for real values of \( v \) and \( n \). In the next section we analyze those special combinations and define a real-valued functions of \( v \) and \( n \).

3. Definitions

In this section we analyze the real valued combinations of the complex functions of one variable that build the BFKL eigenvalue. Based on this analysis we define the real-valued functions of two real variables with a definite complexity that matches the weight of the corresponding nested harmonic sums.

The nested harmonic sums are defined \([17–21]\) in terms of nested summation for \( n \in \mathbb{N} \)
\[
S_{a_1, a_2, \ldots, a_k}(n) = \sum_{n \geq i_1 \geq i_2 \geq \cdots \geq i_k \geq 1} \frac{\text{sign}(a_1)^{i_1} \cdots \text{sign}(a_k)^{i_k}}{i_1^{a_1} \cdots i_k^{a_k}}.
\]  
(9)

The harmonic sums are defined for real integer values of \( a_i \) (excluding zero), which build the alphabet of the possible negative and positive indices uniquely labeling \( S_{a_1, a_2, \ldots, a_k}(n) \). In the definition of nested harmonic sums \( S_{a_1, a_2, \ldots, a_k}(n) \) in \( (9) \) \( k \) is the depth and \( w = \sum_{i=1}^{k} |a_i| \) is the weight. The nested harmonic sums are defined for positive integer values of the argument and require analytic continuation to the complex plane done using their integral representation.

In our analysis of the functions in the BFKL eigenvalue we start with the digamma functions defined in terms of the logarithmic derivative of the Gamma function as follows
\[
\psi(z) \equiv \frac{d \ln \Gamma(z)}{dz}.
\]  
(10)

For the purpose of the present discussion we use the series representation of the digamma function
\[
\psi(z) - \psi(1) = \sum_{k=1}^{\infty} \frac{z-1}{k(k+z-1)},
\]  
(11)

where \( z \) is a complex variable
\[
z = x + iy
\]  
(12)

for real \( x \) and \( y \). The digamma function appears in the BFKL eigenvalue in \( (1) \) as a linear combination of arguments \( z \) and \( \bar{z} \)
\[
\psi(z) + \psi(z) - 2\psi(1) = \sum_{k=0}^{\infty} \left( \frac{2}{k+1} - \frac{1}{k+z} - \frac{1}{k+\bar{z}} \right)
\]  
(13)

\[
= \sum_{k=1}^{\infty} \left( \frac{2}{k+1} - \frac{2(k+x)}{(k+x)^2 + y^2} \right).
\]  
(14)
In a similar way we can reduce the the first derivative of digamma function (called trigamma function) defined by

$$\psi'(z) = \sum_{k=1}^{\infty} \frac{1}{(k+z)^2}. \quad (15)$$

The real-valued combination of trigamma function reads

$$\psi'(z) + \psi'(z) = \sum_{k=0}^{\infty} \left( \frac{1}{(k+z)^2} + \frac{1}{(k+z)^2} \right) = 2 \sum_{k=0}^{\infty} \frac{(k+x)^2 - y^2}{((k+x)^2 + y^2)^2}. \quad (16)$$

We proceed with the second derivative of the digamma function and get

$$\psi''(z) + \psi''(z) = -2 \sum_{k=0}^{\infty} \left( \frac{1}{(k+z)^3} + \frac{1}{(k+z)^3} \right) = -4 \sum_{k=0}^{\infty} \frac{(k+x)^3 - 3y^2(k+x)}{((k+x)^2 + y^2)^3}. \quad (17)$$

From the three expressions in (11), (15) and (17) we can extract a general structure of their terms

$$\sum_{k=0}^{\infty} \frac{(k+x)^{\alpha-2}\beta y^{2\beta}}{((k+x)^2 + y^2)^{\gamma}} \quad (18)$$

where $\alpha$ and $\beta$ are positive integers subject to the following relation $\alpha - 2\beta \geq 0$. The expression in (18) is a real-valued function of $z$ and $\bar{z}$ with a definite complexity depending on the choice of the parameter $\alpha$ that corresponds to the weight of nested harmonic sums. It is important to emphasize that the complexity of the functions in (18) is fully determined by $\alpha$ and does not depend on the parameter $\beta$, i.e., is the same for any positive integer $\beta$ that satisfies the relation $\alpha - 2\beta \geq 0$.

Next we analyze more complicated functions $F_i(z)$ that are present in the NLO BFKL eigenvalue

$$F_1(z) = \sum_{k=0}^{\infty} \frac{(-1)^k \psi(k+1)}{k+z}, \quad (19)$$

$$F_2(z) = \sum_{k=0}^{\infty} \frac{(-1)^k (\psi(k+1) - \psi(1))}{(k+z)^2}, \quad (20)$$

$$F_3(z) = \sum_{k=0}^{\infty} \frac{\beta' (k+1)}{k+z}, \quad (21)$$

where

$$\beta'(z) = -\sum_{r=0}^{\infty} \frac{(-1)^r}{(r+z)^2}. \quad (22)$$

The functions $F_i(z)$ always appear in the NNLO eigenvalue (3) in the real valued combinations $F(z) + F(z)$. We start with $F_1(z)$ for $z = x + iy$ and write

$$F_1(z) + F_1(z) = \sum_{k=0}^{\infty} (-1)^k \psi(k+1) \left( \frac{1}{k+z} + \frac{1}{k+z} \right)$$

$$= 2 \sum_{k=0}^{\infty} \frac{(-1)^k (k+x)}{(k+x)^2 + y^2} \psi(k+1)$$

$$= 2 \sum_{k=0}^{\infty} \frac{(-1)^k (k+x)}{(k+x)^2 + y^2} \left( -S_2(k) + \frac{\pi^2}{6} \right). \quad (23)$$
where we used the analytic continuation of the harmonic sum
\[ \psi'(k+1) = -S_2(k) + \frac{\pi^2}{6}. \] (24)

Next we consider \( F_2(z) \) and its real valued combination
\[
F_2(z) + F_2(\bar{z}) = \sum_{k=0}^{\infty} (-1)^k \left( \psi(k+1) - \psi(1) \right) \left( \frac{1}{(k+z)^2} + \frac{1}{(k+\bar{z})^2} \right)
= 2 \sum_{k=0}^{\infty} (-1)^k \frac{(k+x)^2 - y^2}{((k+x)^2 + y^2)^2} S_1(k),
\] (25)
where the analytic continuation for the harmonic sum is given by
\[
S_1(k) = \psi(k+1) - \psi(1).
\] (26)

Finally we repeat the analysis for \( F_3(z) \).
\[
F_3(z) + F_3(\bar{z}) = \sum_{k=0}^{\infty} \left( \frac{1}{k+z} + \frac{1}{k+\bar{z}} \right) \beta'(k+1)
= \sum_{k=0}^{\infty} \frac{2(k+x)}{(k+x)^2 + y^2} \left( -S_{-2}(k) + \frac{\pi^2}{12} \right)
\] (27)
where we used
\[
\beta'(z) = -\sum_{r=0}^{\infty} \frac{(-1)^r}{(r+2)^2}
\] (28)
and the relation
\[
\beta'(k+1) = -S_{-2}(k) + \frac{\pi^2}{12}
\] (29)
for the analytic the analytic continuation for the harmonic sum from even \( k \) to natural numbers
\[
S_{-2}^{+}(k) = (-1)^k S_{-2}(k) + \left( 1 + (-1)^k \right) \frac{\pi^2}{12}.
\] (30)

The \( \beta'(z) \) also appears in the NNLO BFKL eigenvalue in the real valued combination
\[
\beta'(z) + \beta'(\bar{z}) = -2 \sum_{k=0}^{\infty} (-1)^k \frac{(k+x)^2 - y^2}{((k+x)^2 + y^2)^2}
\] (31)
where \( x = \text{Re}(z) \) and \( y = \text{Im}(z) \).

From the real valued expression in (11), (15), (17), (23), (25), (27) and (31) one can read out the general form of the real-valued functions
\[
\sum_{k=0}^{\infty} \frac{(-1)^k(k+x)^2 - y^2}{((k+x)^2 + y^2)^2} S_{\{a_i\}}^+(k)
\] (32)
where \( a \) and \( \beta \) are natural numbers and \( \eta = 0, 1 \). Here \( S_{\{a_i\}}^+(k) \) denotes nested harmonic sum analytically continued from even values of \( k \) to natural numbers. The details of this analytic continuation can be found in the paper by Kotikov and Velizhanin [22]. In order to have a function with a well defined weight we set \( \gamma + 2\beta = |a| \). The weight of the function in this case is given by \( w = \alpha + \sum_{i=0}^{d} |a_i| \), where \( d \) is the depth of the harmonic sum \( S_{\{a_i\}}(k) \).
For sake of clarity of presentation we can eliminate $\eta$ and set $a$ to be integer number not including zero. Thus for a given weight we define the most general form of functions that can appear in the BFKL eigenvalue

$$D^\beta_{\alpha,(a_1)}(z,\bar z) \equiv \sum_{k=0}^{\infty} \frac{\text{sign}(a)^k (k+x) |\alpha| - 2\beta y^{2\beta}}{((k+x)^2 + y^2)^{|\alpha|}} \mathcal{S}(a_i)(k),$$  

(33)

where $\alpha$ is a non-zero integer number and $\beta$ is a natural number (including zero) subject to the condition $|\alpha| - 2\beta \geq 0$. Then the weight of the $D^\beta_{\alpha,(a_1)}(z,\bar z)$ function in (33) reads

$$w = |\alpha| + \sum_{i=1}^{d} |a_i|.$$  

(34)

The weight in (34) is equivalent to the weight of the nested harmonic sums of the same complexity.

The variables $x = \text{Re}(z)$ and $y = \text{Im}(z)$ are $x = \frac{1}{2} + \frac{\eta}{2}$ and $y = \nu$ in the traditional notation of the BFKL approach. Because of the similarity to the dispersive integrals we choose to call the new functions in (33) the **dispersive functions**. The dispersive functions defined in (33) are real-valued functions of two complex conjugate variables $z$ and $\bar z$ and properly capture the analytic structure of the BFKL eigenvalue, which has isolated poles at $\nu = \pm i m / 2$ for $m \in \mathbb{Z}$. One of the main features of the dispersive functions in (33) is that they are defined at definite complexity, which is useful for building a functional basis and then fitting the free coefficients based on the known results. This approach is parallel to the one originally used by Gromov, Levkovich-Maslyuk and Sizov [23] in calculating the functional form of the NNLO BFKL eigenvalue in $N = 4$ SYM for $n = 0$ in terms of the nested harmonic sums (see also a parallel calculation by Velizhanin [24]). The expressions for other values of the conformal spin in other related calculations can be found in [25–32].

The real valued dispersive functions can be used in other approaches to the BFKL physics such as probabilistic approach [33] or Reggeon effective action approach [34]. The nested harmonic sums defined in (9) were chosen for constructing the functional basis primarily because of the convenience of their labeling, which also defines their complexity in terms of the so-called weight. The weight $w$ of nested harmonic sum $S_{a_1,a_2,...}(n)$ is the sum of the absolute values of their indices, namely $w = \sum |a_i|$. In the case of the dispersive functions $D^\beta_{\alpha,(a_1)}(z,\bar z)$ defined in (33) the complexity is given by $|\alpha| + \sum |a_i|$ and it is equivalent to the weight $w$ of the nested harmonic sums. The important difference between the dispersive functions $D^\beta_{\alpha,(a_1)}(z,\bar z)$ and the nested harmonic sums is that the latter are complex functions of one complex variable (after the analytic continuation), while the dispersive functions $D^\beta_{\alpha,(a_1)}(z,\bar z)$ are real valued functions of a complex variable $z$ and its complex conjugate $\bar z$. There is a limited number of $D^\beta_{\alpha,(a_1)}(z,\bar z)$ functions at a given weight as listed below at weight one and two. The list of all possible $D^\beta_{\alpha,(a_1)}(z,\bar z)$ can be found in the Appendix A.

At weight $w = 1$ the functions defined in (33) read

$$D^1_0(z,\bar z) = \sum_{k=0}^{\infty} \frac{(x+k)}{(x+k)^2 + y^2}$$  

(35)

$$D^\infty_1(z,\bar z) = \sum_{k=0}^{\infty} \frac{(-1)^k (x+k)}{(x+k)^2 + y^2}$$  

(36)
and at the weight two \( w = 2 \) they are given by

\[
D^0_0(z, \bar{z}) = \sum_{k=0}^{\infty} \frac{(x+k)^2}{(x+k)^2 + y^2}
\]

(37)

\[
D^0_2(z, \bar{z}) = \sum_{k=0}^{\infty} \frac{(-1)^k (x+k)^2}{(x+k)^2 + y^2}
\]

(38)

\[
D^0_{1,1}(z, \bar{z}) = \sum_{k=0}^{\infty} \frac{(x+k)}{(x+k)^2 + y^2} S_1(k)
\]

(39)

\[
D^0_{1,-1}(z, \bar{z}) = \sum_{k=0}^{\infty} \frac{(-1)^k (x+k)}{(x+k)^2 + y^2} S_{-1}(k)
\]

(40)

\[
D^0_{1,-1}(z, \bar{z}) = \sum_{k=0}^{\infty} \frac{(-1)^k (x+k)}{(x+k)^2 + y^2} S_{-1}(k)
\]

(41)

\[
D^1_2(z, \bar{z}) = \sum_{k=0}^{\infty} \frac{y^2}{(x+k)^2 + y^2}
\]

(42)

\[
D^1_{2,-1}(z, \bar{z}) = \sum_{k=0}^{\infty} \frac{(-1)^k y^2}{(x+k)^2 + y^2}
\]

(43)

\[
D^1_{2,-1}(z, \bar{z}) = \sum_{k=0}^{\infty} \frac{(-1)^k y^2}{(x+k)^2 + y^2}
\]

(44)

At weight \( w = 3 \) there are 24 irreducible real valued dispersive functions \( D^\beta_{\alpha,\{a_i\}}(z, \bar{z}) \)

\[
\{ D^0_{1,2}(z, \bar{z}), D^0_{1,-2}(z, \bar{z}), D^0_{-1,2}(z, \bar{z}), D^0_{-1,-2}(z, \bar{z}), D^0_{1,1,1}(z, \bar{z}), D^0_{1,-1,1}(z, \bar{z}), D^0_{-1,1,1}(z, \bar{z}), D^0_{-1,-1,1}(z, \bar{z}), D^0_{1,1,-1}(z, \bar{z}), D^0_{1,-1,-1}(z, \bar{z}), D^0_{-1,1,-1}(z, \bar{z}), D^0_{-1,-1,-1}(z, \bar{z}), D^0_{1,1,1}(z, \bar{z}), D^0_{1,-1,1}(z, \bar{z}), D^0_{-1,1,1}(z, \bar{z}), D^0_{-1,-1,1}(z, \bar{z}), \}
\]

(45)

and they are given in the Appendix A.

The number of irreducible \( D^\beta_{\alpha,\{a_i\}}(z, \bar{z}) \) functions at a given weight should be compared to the number nested harmonic sums of one variable at the same weight. At weight \( w = 1 \) there are only two \( D^\beta_{\alpha,\{a_i\}}(z, \bar{z}) \) functions as well as the harmonic sums \( S_1(z) \) and \( S_{-1}(z) \).

At weight \( w = 2 \) there are eight \( D^\beta_{\alpha,\{a_i\}}(z, \bar{z}) \) functions compared to six independent nested harmonic sums of the linear basis

\[
\{ S_2(z), S_{-2}(z), S_{1,1}(z), S_{1,-1}(z), S_{-1,1}(z), S_{-1,-1}(z) \}.
\]

(46)

At weight \( w = 3 \) there are 24 linearly independent irreducible \( D^\beta_{\alpha,\{a_i\}}(z, \bar{z}) \) functions compared to 18 independent nested harmonic sums of the linear basis

\[
\{ S_3(z), S_{-3}(z), S_{1,2}(z), S_{1,-2}(z), S_{-1,2}(z), S_{-1,-2}(z), S_{2,1}(z), S_{2,-1}(z), S_{-2,1}(z), S_{-2,-1}(z), S_{1,1,1}(z), S_{1,1,-1}(z), S_{1,-1,1}(z), S_{1,-1,-1}(z), S_{-1,1,1}(z), S_{-1,1,-1}(z), S_{-1,-1,1}(z), S_{-1,-1,-1}(z) \}.
\]

(47)

It is worth emphasizing that for pure real \( z \), i.e., for \( y = 0 \) the number of the irreducible \( D^\beta_{\alpha,\{a_i\}}(z, \bar{z}) \) functions and the nested harmonic sums coincide at weight \( w = 2 \) and \( w = 3 \). The number of \( D^\beta_{\alpha,\{a_i\}}(z, \bar{z}) \) functions is larger than the number of the linearly independent
nested harmonic sums at a given weight that were used to define $D^\beta_{a_i} (z, \bar{z})$ functions through real valued pairwise combinations $S_{a_i} (z - 1) + S_{a_i} (\bar{z} - 1)$. This fact implies that the space of $D^\beta_{a_i} (z, \bar{z})$ function is larger than the space of harmonic sums and includes functions that cannot be expressed in terms of real valued combinations $S_{a_i} (z - 1) + S_{a_i} (\bar{z} - 1)$. Same holds for the non-linear basis of nested harmonic sums, where one makes use of the quasi-shuffle identities for building a full set functions for the irreducible basis.

4. Summary and Discussion

In this paper we define new real valued functions $D^\beta_{a_i} (z, \bar{z})$ of a complex variable $z$ and its complex conjugate $\bar{z}$. The definition stems from the real valued combinations of generalized polygamma functions in the known expressions of the BFKL eigenvalue in $N = 4$ super Yang-Mills theory. The functions $D^\beta_{a_i} (z, \bar{z})$ have an analytic structure similar to that of the polygamma functions and its generalizations as well as to the nested harmonic sums analytically continued to the complex plane. We named $D^\beta_{a_i} (z, \bar{z})$ functions the dispersive functions because of similarity in their definitions to the dispersive integrals. The important feature of $D^\beta_{a_i} (z, \bar{z})$ functions is that they are of definite complexity, which corresponds to the weight of nested harmonic sums. The complexity or transcendentality of the functions is important for building a functional basis for perturbative calculations, where the complexity is determined by loop order. In the case of the multi-Regge kinematics of the BFKL approach each perturbative order increases the maximal transcendentality of the final expression by two units. For example, the expression of the BFKL eigenvalue at the leading order has maximal transcendentality one, at the next-to-leading order its maximal transcendentality is three, for the next-to-next-to-leading order the maximal transcendentality is five etc. The definite transcendentality of $D^\beta_{a_i} (z, \bar{z})$ functions is very convenient for building the finite functional basis, which counts about four hundred terms at weight five, needed for the next-to-next-to-leading order of the BFKL eigenvalue.

The number of $D^\beta_{a_i} (z, \bar{z})$ functions is larger than a number of possible real valued combinations of generalized polygamma functions building the LO and NLO BFKL eigenvalue. This means that $D^\beta_{a_i} (z, \bar{z})$ functions cannot be expressed in terms real valued combinations of generalized polygamma functions of one variable in the form of $S_{a_i} (z - 1) + S_{a_i} (\bar{z} - 1)$. In this paper we provide a complete list of $D^\beta_{a_i} (z, \bar{z})$ up to weight three.

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Appendix A

Here we make an explicit list of the dispersive functions defined in (33) at weight \( w = 3 \). Their general expression is given by

\[
D_{\alpha,\beta}(z,\bar{z}) = \sum_{k=0}^{\infty} \frac{\text{sign}(\alpha)^k (k + x)^{|\alpha| - 2\beta} y^2}{((k + x)^2 + y^2)^{|\alpha|}} S_{\alpha}(k).
\] (A1)

At weight \( w = 3 \) they read

\[
D^0_{1,2}(z,\bar{z}) = \sum_{k=0}^{\infty} \frac{(k + x)}{(x + k)^2 + y^2} S_2(k)
\] (A2)

\[
D^0_{1,-2}(z,\bar{z}) = \sum_{k=0}^{\infty} \frac{(k + x)}{(x + k)^2 + y^2} S_{-2}(k)
\] (A3)

\[
D^0_{0,2}(z,\bar{z}) = \sum_{k=0}^{\infty} \frac{(-1)^k(k + x)}{(x + k)^2 + y^2} S_2(k)
\] (A4)

\[
D^0_{0,-2}(z,\bar{z}) = \sum_{k=0}^{\infty} \frac{(-1)^k(k + x)}{(x + k)^2 + y^2} S_{-2}(k)
\] (A5)

\[
D^0_{1,1,1}(z,\bar{z}) = \sum_{k=0}^{\infty} \frac{(k + x)}{(x + k)^2 + y^2} S_{1,1}(k)
\] (A6)

\[
D^0_{0,1,1}(z,\bar{z}) = \sum_{k=0}^{\infty} \frac{(-1)^k(k + x)}{(x + k)^2 + y^2} S_{1,1}(k)
\] (A7)

\[
D^0_{1,1,-1}(z,\bar{z}) = \sum_{k=0}^{\infty} \frac{(k + x)}{(x + k)^2 + y^2} S_{1,-1}(k)
\] (A8)

\[
D^0_{1,-1,-1}(z,\bar{z}) = \sum_{k=0}^{\infty} \frac{(k + x)}{(x + k)^2 + y^2} S_{-1,-1}(k)
\] (A9)

\[
D^0_{-1,1,1}(z,\bar{z}) = \sum_{k=0}^{\infty} \frac{(-1)^k(k + x)}{(x + k)^2 + y^2} S_{1,1}(k)
\] (A10)

\[
D^0_{-1,-1,1}(z,\bar{z}) = \sum_{k=0}^{\infty} \frac{(-1)^k(k + x)}{(x + k)^2 + y^2} S_{1,1}(k)
\] (A11)

\[
D^0_{-1,1,-1}(z,\bar{z}) = \sum_{k=0}^{\infty} \frac{(-1)^k(k + x)}{(x + k)^2 + y^2} S_{1,-1}(k)
\] (A12)

\[
D^0_{-1,-1,-1}(z,\bar{z}) = \sum_{k=0}^{\infty} \frac{(-1)^k(k + x)}{(x + k)^2 + y^2} S_{-1,-1}(k)
\] (A13)
\[ D_0^0(z, z) = \sum_{k=0}^{\infty} \frac{(k+x)^3}{((x+k)^2 + y^2)^3} \quad (A14) \]
\[ D_{-3}^0(z, z) = \sum_{k=0}^{\infty} \frac{(-1)^k(k+x)^3}{((x+k)^2 + y^2)^3} \quad (A15) \]
\[ D_{2,1}^0(z, z) = \sum_{k=0}^{\infty} \frac{(k+x)^2}{((x+k)^2 + y^2)^2} S_1(k) \quad (A16) \]
\[ D_{2,-1}^0(z, z) = \sum_{k=0}^{\infty} \frac{(k+x)^2}{((x+k)^2 + y^2)^2} S_{-1}(k) \quad (A17) \]
\[ D_{-2,1}^0(z, z) = \sum_{k=0}^{\infty} \frac{(-1)^k(k+x)^2}{((x+k)^2 + y^2)^2} S_1(k) \quad (A18) \]
\[ D_{-2,-1}^0(z, z) = \sum_{k=0}^{\infty} \frac{(-1)^k(k+x)^2}{((x+k)^2 + y^2)^2} S_{-1}(k) \quad (A19) \]
\[ D_1^1(z, z) = \sum_{k=0}^{\infty} \frac{(k+x)y^2}{((x+k)^2 + y^2)^3} \quad (A20) \]
\[ D_{-3}^1(z, z) = \sum_{k=0}^{\infty} \frac{(-1)^k(k+x)y^2}{((x+k)^2 + y^2)^3} \quad (A21) \]
\[ D_{2,1}^1(z, z) = \sum_{k=0}^{\infty} \frac{y^2}{((x+k)^2 + y^2)^2} S_1(k) \quad (A22) \]
\[ D_{2,-1}^1(z, z) = \sum_{k=0}^{\infty} \frac{y^2}{((x+k)^2 + y^2)^2} S_{-1}(k) \quad (A23) \]
\[ D_{-2,1}^1(z, z) = \sum_{k=0}^{\infty} \frac{(-1)^ky^2}{((x+k)^2 + y^2)^2} S_1(k) \quad (A24) \]
\[ D_{-2,-1}^1(z, z) = \sum_{k=0}^{\infty} \frac{(-1)^ky^2}{((x+k)^2 + y^2)^2} S_{-1}(k) \quad (A25) \]

There are 24 functions \( D_{a,(a)}^b(z, z) \) at weight \( w = 3 \). Here we use a compact notation for the nested harmonic sums \( S_{(a)}(k) \) that denote the corresponding sums \( \overline{\mathbb{S}}_{(a)}(k) \) that are analytically continued from even positive values of the argument to natural numbers using the prescription of Kotikov and Velizhanin [22].

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