BiHom-Lie colour algebras structures

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January 4, 2022

Abstract

BiHom-Lie Colour algebra is a generalized Hom-Lie Colour algebra endowed with two commuting multiplicative linear maps. The main purpose of this paper is to define representations and a cohomology of BiHom-Lie colour algebras and to study some key constructions and properties. Moreover, we discuss $\alpha^k\beta^l$-generalized derivations, $\alpha^k\beta^l$-quasi-derivations and $\alpha^k\beta^l$-quasi-centroid. We provide some properties and their relationships with BiHom-Jordan colour algebra.

Keywords: Cohomology, Representation, BiHom-Lie colour algebra, $\alpha^k\beta^l$-generalized derivation, BiHom-Jordan colour algebra.

MSC(2010): 17B75, 17B55, 17B40.

Introduction

As generalizations of Lie algebras, Hom-Lie algebras were introduced motivated by applications in Physics and to deformations of Lie algebras, especially Lie algebras of vector fields. Hom-Lie colour algebras are the natural generalizations of Hom-Lie algebras and Hom-Lie superalgebras. In recent years, they have become an interesting subject of mathematics and physics. A cohomology of Lie colour algebras were introduced and investigated in [16, 17] and representations of Lie colour algebras were explicitly described in [8]. Hom-Lie colour algebras were studied first in [19], while in the particular case of Hom-Lie superalgebras, a cohomology theory was provided in [3]. Notice that for Hom-Lie algebras, cohomology was described in [1, 15, 18] and representations also in [5].

A BiHom-algebra is an algebra in such a way that the identities defining the structure are twisted by two homomorphisms $\alpha, \beta$. This class of algebras was introduced from a categorical approach in [10] as an extension of the class of Hom-algebras. More applications of BiHom-Lie algebras, BiHom-algebras, BiHom-Lie superalgebras and BiHom-Lie admissible superalgebras can be found in [7, 20].

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In the present article, we introduce and study the BiHom-Lie colour algebras, which can be viewed as an extension of BiHom-Lie (super)algebras to $\Gamma$-graded algebras, where $\Gamma$ is any abelian group.

The paper is organized as follows. In Section 1, we recall definitions and some key constructions of BiHom-Lie colour algebras and provide a list of twists of BiHom-Lie colour algebras. In Section 2 we introduce a multiplier $\sigma$ on the abelian group $\Gamma$ and provide constructions of new BiHom-Lie colour algebras using the twisting action of the multiplier $\sigma$. We show that the $\sigma$-twist of any BiHom-Lie colour algebra is still a BiHom-Lie colour algebra.

In Section 3, we extend the classical concept of Lie admissible algebras to BiHom-Lie colour settings. Hence, we obtain a more generalized algebra class called BiHom-Lie colour admissible algebras. We also explore some other general class of algebras: $G$ BiHom-associative colour algebras, where $G$ is any subgroup of the symmetric group $S_3$, using which we classify all the BiHom-Lie colour admissible algebras.

In Section 4, we construct a family of cohomologies of BiHom-Lie colour algebras, discuss representation theory in connection with cohomology. In the last section, we discuss homogeneous $\alpha\beta$-generalized derivations and the $\alpha\beta$-centroid of BiHom-Lie colour algebras. We generalize to Hom-setting the results obtained in [6]. Moreover, We show that $\alpha^{-1}\beta^2$-derivations of BiHom-Lie colour algebras give rise to BiHom-Jordan colour algebras.

1 Definitions, proprieties and Examples

In the following we summarize definitions of BiHom-Lie and BiHom-associative colour algebraic structures generalizing the well known Hom-Lie and Hom-associative colour algebras. Throughout the article we let $K$ be an algebraically closed field of characteristic 0 and $K^* = K \setminus \{0\}$ be the group of units of $K$.

Let $\Gamma$ be an abelian group. A vector space $V$ is said to be $\Gamma$-graded, if there is a family $(V_\gamma)_{\gamma \in \Gamma}$ of vector subspace of $V$ such that

$$V = \bigoplus_{\gamma \in \Gamma} V_\gamma.$$ 

An element $x \in V$ is said to be homogeneous of degree $\gamma \in \Gamma$ if $x \in V_\gamma$, $\gamma \in \Gamma$, and in this case, $\gamma$ is called the degree of $x$. As usual, we denote by $\overline{\gamma}$ the degree of an element $x \in V$. Thus each homogeneous element $x \in V$ determines a unique group of element $\overline{\gamma} \in \Gamma$ by $x \in V_{\overline{\gamma}}$. Fortunately, We can drop the symbol ” − “, since confusion rarely occurs. In the sequel, we will denote by $\mathcal{H}(V)$ the set of all the homogeneous elements of $V$.

Let $V = \bigoplus_{\gamma \in \Gamma} V_\gamma$ and $V' = \bigoplus_{\gamma \in \Gamma} V'_{\gamma}$ be two $\Gamma$-graded vector spaces. A linear mapping $f : V \rightarrow V'$ is said to be homogeneous of degree $\gamma \in \Gamma$ if $f(V_\gamma) \subseteq V'_{\gamma + \nu}$, $\forall \gamma \in \Gamma$. If in addition $f$ is homogeneous of degree zero, i.e. $f(V_\gamma) \subseteq V'_{\gamma}$ holds for any $\gamma \in \Gamma$, then $f$ is said to be even.

An algebra $A$ is said to be $\Gamma$-graded if its underlying vector space is $\Gamma$-graded, i.e. $A = \bigoplus_{\gamma \in \Gamma} A_\gamma$, and if, furthermore $A_\gamma A_{\gamma'} \subseteq A_{\gamma + \gamma'}$, for all $\gamma, \gamma' \in \Gamma$. It is easy to see that if $A$ has a unit element $e$, it follows that $e \in A_0$. A subalgebra of $A$ is said to be graded if it is graded as a subspace of $A$.

Let $A'$ be another $\Gamma$-graded algebra. A homomorphism $f : A \rightarrow A'$ of $\Gamma$-graded algebras is by definition a homomorphism of the algebra $A$ into the algebra $A'$, which is, in addition an even
Definition 1.1. Let $\mathbb{K}$ be a field and $\Gamma$ be an abelian group. A map $\varepsilon : \Gamma \times \Gamma \to \mathbb{K}^\ast$ is called a skewsymmetric bicharacter on $\Gamma$ if the following identities hold, for all $a, b, c$ in $\Gamma$.

1. $\varepsilon(a, b) \varepsilon(b, a) = 1,$
2. $\varepsilon(a, b + c) = \varepsilon(a, b) \varepsilon(a, c),$
3. $\varepsilon(a + b, c) = \varepsilon(a, c) \varepsilon(b, c).$

The definition above implies, in particular, the following relations:

$\varepsilon(a, 0) = \varepsilon(0, a) = 1$, $\varepsilon(a, a) = \pm 1$, for all $a \in \Gamma$.

If $x$ and $x'$ are two homogeneous elements of degree $\gamma$ and $\gamma'$ respectively and $\varepsilon$ is a skewsymmetric bicharacter, then we shorten the notation by writing $\varepsilon(x, x')$ instead of $\varepsilon(\gamma, \gamma')$.

Definition 1.2. A BiHom-Lie colour algebra is a 5-uple $(A, \lbrack \cdot, \cdot \rbrack, \varepsilon, \alpha, \beta)$ consisting of a $\Gamma$-graded vector space $A$, an even bilinear mapping $\lbrack \cdot, \cdot \rbrack : A \times A \to A$ (i.e. $[A_a, A_b] \subseteq A_{a+b}$ for all $a, b \in \Gamma$), a bicharacter $\varepsilon : \Gamma \times \Gamma \to \mathbb{K}^\ast$ and two even homomorphism $\alpha, \beta : A \to A$ such that for homogeneous elements $x, y, z$ we have

$$\alpha \circ \beta = \beta \circ \alpha,$$
$$\alpha([x, y]) = [\alpha(x), \alpha(y)], \quad \beta([x, y]) = [\beta(x), \beta(y)],$$
$$[\beta(x), \alpha(y)] = -\varepsilon(x, y)[\beta(y), \alpha(x)],$$
$$\bigcirc_{x, y, z} \varepsilon(z, x)[\beta^2(x), [\beta(y), \alpha(z)]] = 0 \quad (\varepsilon\text{-BiHom-Jacobi condition})$$

where $\bigcirc_{x, y, z}$ denotes summation over the cyclic permutation on $x, y, z$.

Obviously, a Hom-Lie colour algebra $(A, [\cdot, \cdot], \varepsilon, \alpha)$ is a particular case of BiHom-Lie colour algebra. Conversely a BiHom-Lie colour algebra $(A, [\cdot, \cdot], \varepsilon, \alpha, \alpha)$ with bijective $\alpha$ is a Hom-Lie colour algebra $(A, [\cdot, \cdot], \varepsilon, \alpha)$.

Remark 1.3. A Lie colour algebra $(A, [\cdot, \cdot], \varepsilon)$ is a Hom-Lie colour algebra with $\alpha = \text{Id}$, since the $\varepsilon\text{-Hom-Jacobi condition}$ reduces to the $\varepsilon\text{-Jacobi condition}$ when $\alpha = \text{Id}$. If, in addition, $\varepsilon(x, y) = 1$ or $\varepsilon(x, y) = (-1)^{|x||y|}$, then the BiHom-Lie colour algebra is a classical Hom-Lie algebra or Hom-Lie superalgebra. Using definitions of [10, 20], BiHom-Lie algebras and BiHom-Lie superalgebras are also obtained when $\varepsilon(x, y) = 1$ and $\varepsilon(x, y) = (-1)^{|x||y|}$ respectively.

Definition 1.4.

1. A BiHom-Lie colour algebra $(A, [\cdot, \cdot], \varepsilon, \alpha, \beta)$ is multiplicative if $\alpha$ and $\beta$ are even algebra morphisms, i.e., for any homogenous elements $x, y \in A$, we have

$$\alpha([x, y]) = [\alpha(x), \alpha(y)] \quad \text{and} \quad \beta([x, y]) = [\beta(x), \beta(y)].$$
2. A BiHom-Lie colour algebra \((\mathcal{A}, [\cdot, \cdot], \varepsilon, \alpha, \beta)\) is regular if \(\alpha\) and \(\beta\) are even algebra automorphisms.

We recall in the following the definition of BiHom-associative colour algebra which provide a different way for constructing BiHom-Lie colour algebra by extending the fundamental construction of Lie colour algebras from associative colour algebra via commutator bracket multiplication.

**Definition 1.5.** A BiHom-associative colour algebra is a 5-tuple \((\mathcal{A}, \mu, \varepsilon, \alpha, \beta)\) consisting of a \(\Gamma\)-graded vector space \(\mathcal{A}\), an even bilinear map \(\mu : \mathcal{A} \times \mathcal{A} \to \mathcal{A}\) (i.e. \(\mu(\mathcal{A}_a, \mathcal{A}_b) \subset \mathcal{A}_{a+b}\)) and two even homomorphisms \(\alpha, \beta : \mathcal{A} \to \mathcal{A}\) such that \(\alpha \circ \beta = \beta \circ \alpha\)

\[
\alpha(x)(yz) = (xy)\beta(z).
\]

In the case where \(xy = \varepsilon(x,y)yx\), the BiHom-associative colour algebra \((\mathcal{A}, \mu, \alpha, \beta)\) is called commutative BiHom-associative colour algebra.

Obviously, a Hom-associative colour algebra \((\mathcal{A}, \mu, \alpha)\) is a particular case of a BiHom-associative colour algebra, namely \((\mathcal{A}, \mu, \alpha, \alpha)\). Conversely, a BiHom-associative colour algebra \((\mathcal{A}, \mu, \alpha, \alpha)\) with bijective \(\alpha\) is the Hom-associative colour algebra \((\mathcal{A}, \mu, \alpha)\).

**Proposition 1.6.** Let \((\mathcal{A}, \mu, \varepsilon, \alpha, \beta)\) be a BiHom-associative colour algebra defined on the vector space \(\mathcal{A}\) by the multiplication \(\mu\) and two bijective homomorphisms \(\alpha\) and \(\beta\). Then the quadruple \((\mathcal{A}, [\cdot, \cdot], \varepsilon, \alpha, \beta)\), where the bracket is defined for \(x, y \in \mathcal{H}(\mathcal{A})\) by

\[
[x, y] = xy - \varepsilon(x, y)\alpha^{-1}(\beta(y))(\alpha\beta^{-1}(x)),
\]

is a BiHom-Lie colour algebra.

**Proof.** First we check that the bracket product \([\cdot, \cdot]\) is compatible with the structure maps \(\alpha\) and \(\beta\). For any homogeneous elements \(x, y \in \mathcal{A}\), we have

\[
[\alpha(x), \alpha(y)] = \alpha(x)\alpha(y) - \varepsilon(x, y)\alpha^{-1}(\beta(\alpha(y)))(\alpha\beta^{-1}(\alpha(x)))
= \alpha(x)\alpha(y) - \varepsilon(x, y)\beta(y)(\alpha\beta^{-1}(x))
= \alpha([x, y]).
\]

The second equality holds since \(\alpha\) is even and \(\alpha \circ \beta = \beta \circ \alpha\). Similarly, one can prove that \(\beta([x, y]) = [\beta(x), \beta(y)]\).

It is easy to verify that \([\beta(x), \alpha(y)] = -\varepsilon(x, y)[\beta(y), \alpha(x)]\).

Now we prove the \(\varepsilon\)-BiHom-Jacobi condition. For any homogeneous elements \(x, y, z \in \mathcal{A}\), we have

\[
\varepsilon(z, x)[\beta^2(x), [\beta(y), \alpha(z)]] = \varepsilon(z, x)[\beta^2(x), \beta(y)\alpha(z) - \varepsilon(y, z)\alpha^{-1}(\beta(\alpha(z))\alpha\beta^{-1}(\beta(y)))]
= \varepsilon(z, x)[\beta^2(x), \beta(y)\alpha(z)] - \varepsilon(z, x)\varepsilon(y, z)[\beta^2(x), \alpha^{-1}(\beta(\alpha(z))\alpha\beta^{-1}(\beta(y)))]
= \varepsilon(z, x)\left(\beta^2(x)\beta(y)\alpha(z)\right) - \varepsilon(z, x)\varepsilon(y, z)\left(\alpha^{-1}(\beta^2(y))\beta(z)\alpha(z)\right)
- \varepsilon(z, x)\varepsilon(y, z)\left(\beta^2(x)\beta(z)\alpha(y)\right) - \varepsilon(z, x)\varepsilon(y, z)\left(\alpha^{-1}(\beta^2(z))\beta(y)\alpha(z)\right).
\]
Similarly, we have
\[
\varepsilon(x, y)[\beta^2(y), [\beta(z), \alpha(x)]] = 
\varepsilon(x, y) \left( \beta^2(y)(\beta(z)\alpha(x)) - \varepsilon(y, z + x)(\alpha^{-1}(\beta^2(z))\beta(x))\alpha(\beta(y)) \right) \\
- \varepsilon(x, y)\varepsilon(z, x) \left( \beta^2(y)(\beta(x)\alpha(z)) - \varepsilon(y, z + x)(\alpha^{-1}(\beta^2(x))\beta(z))\alpha(\beta(y)) \right).
\]

\[
\varepsilon(y, z)[\beta^2(z), [\beta(x), \alpha(y)]] = 
\varepsilon(y, z) \left( \beta^2(z)(\beta(x)\alpha(y)) - \varepsilon(z, x + y)(\alpha^{-1}(\beta^2(x))\beta(y))\alpha(\beta(z)) \right) \\
- \varepsilon(y, z)\varepsilon(x, y) \left( \beta^2(z)(\beta(y)\alpha(x)) - \varepsilon(z, x + y)(\alpha^{-1}(\beta^2(y))\beta(x))\alpha(\beta(z)) \right).
\]

By the associativity, we have
\[
\circ_{x, y, z} \varepsilon(z, x)[\beta^2(x), [\beta(y), \alpha(z)]] = 0.
\]
And this finishes the proof. \(\square\)

The following theorem generalizes the result of [19]. In the following, starting from a BiHom-Lie colour algebra and an even Lie colour algebra endomorphism, we construct a new BiHom-Lie colour algebra. We say that it is obtained by twisting principle or composition method.

**Theorem 1.7.** Let \( (\mathcal{A}, [, .], \varepsilon) \) be an ordinary Lie colour algebra and let \( \alpha, \beta : \mathcal{A} \rightarrow \mathcal{A} \) two commuting even linear maps such that \( \alpha([x, y]) = [\alpha(x), \alpha(y)] \) and \( \beta([x, y]) = [\beta(x), \beta(y)] \), for all \( x, y \in \mathcal{H}(\mathcal{A}) \). Define the even linear map \( \{, , \} : \mathcal{A} \times \mathcal{A} \rightarrow \mathcal{A} \)
\[
\{x, y\} = [\alpha(x), \beta(y)], \quad \forall \ x, y \in \mathcal{H}(\mathcal{A}).
\]
Then \( \mathcal{A}(\alpha, \beta) = (\mathcal{A}, \{, , \}, \varepsilon, \alpha, \beta) \) is a BiHom-Lie colour algebra.

**Proof.** Obviously \( \{, , \} \) is BiHom-\( \varepsilon \)-skewsymmetric. Furthermore \( (\mathcal{A}, \{, , \}, \varepsilon, \alpha, \beta) \) satisfies the \( \varepsilon \)-BiHom-Jacobi condition. Indeed
\[
\circ_{x, y, z} \varepsilon(z, x)[\beta^2(x), \{\beta(y), \alpha(z)\}] = 
\varepsilon(z, x) \{[\beta^2(x), \alpha(\beta(y))], \beta(z)\}
= \varepsilon(z, x) \{[\alpha(\beta^2(x)), \beta(z)], \alpha(\beta(y))\}
= 0.
\]
\(\square\)

**Claim:** More generally, let \( (\mathcal{A}, [, .], \varepsilon, \alpha, \beta) \) be a BiHom-Lie colour algebra and \( \alpha', \beta' : \mathcal{A} \rightarrow \mathcal{A} \) even linear maps such that \( \alpha'([x, y]) = [\alpha'(x), \alpha'(y)] \) and \( \beta'([x, y]) = [\beta'(x), \beta'(y)] \), for all \( x, y \in \mathcal{H}(\mathcal{A}) \), and any two of the maps \( \alpha, \beta, \alpha', \beta' \) commute. Then \( (\mathcal{A}, [, .], \varepsilon, \alpha, \beta) = \{, , \} \circ (\alpha' \otimes \beta') \) is a BiHom-Lie colour algebra.

**Remark 1.8.** Let \( (\mathcal{A}, [, .], \varepsilon) \) be a Lie colour algebra and \( \alpha \) be a Lie colour algebra morphism, then \( (\mathcal{A}, [, .], \varepsilon, \alpha) \) is a multiplicative Hom-Lie colour algebra.

**Example 1.9.** We construct an example of a BiHom-Lie colour algebra, which is not a Lie colour algebra starting from the orthosymplectic Lie superalgebra. We consider in the sequel the
matrix realization of this Lie superalgebra.  
Let \( \text{osp}(1,2) = A_0 \oplus A_1 \) be the Lie superalgebra where \( A_0 \) is spanned by 
\[
H = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad X = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.
\]
and \( A_1 \) is spanned by 
\[
F = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad G = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}.
\]
The defining relations (we give only the ones with non-zero values in the right-hand side) are 
\[
[H, X] = 2X, \quad [H, Y] = -2Y, \quad [X, Y] = H, \\
[Y, G] = F, \quad [X, F] = G, \quad [H, F] = -F, \quad [H, G] = G, \\
[G, F] = H, \quad [G, G] = -2X, \quad [F, F] = 2Y.
\]
Let \( \lambda, \kappa \in \mathbb{R}^* \), we consider the linear maps \( \alpha_{\lambda} : \text{osp}(1,2) \to \text{osp}(1,2) \) and \( \beta_{\kappa} : \text{osp}(1,2) \to \text{osp}(1,2) \) defined by 
\[
\alpha_{\lambda}(X) = \lambda^2 X, \quad \alpha_{\lambda}(Y) = \frac{1}{\lambda^2} Y, \quad \alpha_{\lambda}(H) = H, \quad \alpha_{\lambda}(F) = \frac{1}{\lambda} F, \quad \alpha_{\lambda}(G) = \lambda G, \\
\beta_{\kappa}(X) = \kappa^2 X, \quad \beta_{\kappa}(Y) = \frac{1}{\kappa^2} Y, \quad \beta_{\kappa}(H) = H, \quad \beta_{\kappa}(F) = \frac{1}{\kappa} F, \quad \beta_{\kappa}(G) = \kappa G.
\]
Obviously, we have \( \alpha_{\lambda} \circ \beta_{\kappa} = \beta_{\kappa} \circ \alpha_{\lambda} \). For all \( H, X, Y, F \) and \( G \) in \( \text{osp}(1,2) \), we have 
\[
\alpha_{\lambda}([H, X]) = \alpha_{\lambda}(2X) = 2\lambda^2 X, \quad \alpha_{\lambda}([H, Y]) = \alpha_{\lambda}(-2Y) = -2\frac{1}{\lambda^2} Y, \\
\alpha_{\lambda}([X, Y]) = \alpha_{\lambda}(H) = H, \quad \alpha_{\lambda}([Y, G]) = \alpha_{\lambda}(F) = \frac{1}{\lambda} F, \\
\alpha_{\lambda}([X, F]) = \alpha_{\lambda}(G) = \lambda G, \quad \alpha_{\lambda}([H, F]) = \alpha_{\lambda}(-F) = -\frac{1}{\lambda} F, \\
\alpha_{\lambda}([H, G]) = \alpha_{\lambda}(G) = \lambda G, \quad \alpha_{\lambda}([G, F]) = \alpha_{\lambda}(H) = H, \\
\alpha_{\lambda}([G, G]) = \alpha_{\lambda}(-2X) = -2\lambda^2 X, \quad \alpha_{\lambda}([F, F]) = \alpha_{\lambda}(2Y) = 2\frac{1}{\lambda^2} Y.
\]
On the other hand, we have 
\[
\alpha_{\lambda}(H), \alpha_{\lambda}(X) = [H, \lambda^2 X] = 2\lambda^2 X, \quad \alpha_{\lambda}(H), \alpha_{\lambda}(Y) = [H, \frac{1}{\lambda^2} Y] = -2\frac{1}{\lambda^2} Y, \\
\alpha_{\lambda}(X), \alpha_{\lambda}(Y) = [\lambda^2 X, \frac{1}{\lambda^2} Y] = H, \quad \alpha_{\lambda}(Y), \alpha_{\lambda}(G) = \frac{1}{\lambda^2} Y, \lambda G = \lambda G, \\
\alpha_{\lambda}(X), \alpha_{\lambda}(F) = [\lambda^2 X, \frac{1}{\lambda} F] = \lambda G, \quad \alpha_{\lambda}(Y), \alpha_{\lambda}(F) = \frac{1}{\lambda} F, \\
\alpha_{\lambda}(X), \alpha_{\lambda}(F) = [\lambda^2 X, \frac{1}{\lambda} F] = \lambda G, \quad \alpha_{\lambda}(Y), \alpha_{\lambda}(F) = \frac{1}{\lambda} F, \\
\alpha_{\lambda}(H), \alpha_{\lambda}(G) = [H, \lambda G] = \lambda G, \quad \alpha_{\lambda}(G), \alpha_{\lambda}(F) = \lambda G, \frac{1}{\lambda} F = H, \\
\alpha_{\lambda}(G), \alpha_{\lambda}(G) = [\lambda G, \lambda G] = -2\lambda^2 X, \quad \alpha_{\lambda}(F), \alpha_{\lambda}(F) = \frac{1}{\lambda} F, \frac{1}{\lambda} F = 2\frac{1}{\lambda^2} Y.
\]
Therefore, for $a, a' \in \mathfrak{osp}(1,2)$, we have
\[
\alpha_\lambda([a, a']) = [\alpha_\lambda(a), \alpha_\lambda(a')].
\]
Similarly, we have
\[
\beta_\kappa([a, a']) = [\beta_\kappa(a), \beta_\kappa(a')].
\]
Applying Theorem 1.7, we obtain a family of BiHom-Lie colour algebras $\mathfrak{osp}(1,2)_{\alpha_\lambda, \beta_\kappa} = (\mathfrak{osp}(1,2), \{\cdot, \cdot\} = \{\cdot, \cdot\} \circ (\alpha_\lambda \otimes \beta_\kappa), \alpha_\lambda, \beta_\kappa)$ where the BiHom-Lie colour algebra bracket $\{\cdot, \cdot\}$ on the basis elements is given, for $\lambda, \kappa \neq 0$, by
\[
\begin{align*}
\{H, X\} &= 2\kappa^2 X, & \{H, Y\} &= -\frac{2}{\kappa^2} Y, & \{X, Y\} &= \left(\frac{\lambda}{\kappa}\right)^2 H, \\
\{Y, G\} &= \frac{\kappa}{\lambda^2} F, & \{X, F\} &= \frac{\lambda^2}{\kappa} G, & \{H, F\} &= -\frac{1}{\kappa} F, & \{H, G\} &= \kappa G, \\
\{G, F\} &= \frac{\lambda}{\kappa} H, & \{G, G\} &= -2\kappa X, & \{F, F\} &= 2\frac{\lambda}{\kappa} Y.
\end{align*}
\]
These BiHom-Lie colour algebras are not Hom-Lie colour algebras for $\lambda \neq 1$. Indeed, the left-hand side of the $\varepsilon$-Hom-Jacobi identity, for $\beta_\kappa = \text{id}$, leads to
\[
\{\alpha_\lambda(X), \{Y, H\}\} - \{\alpha_\lambda(H), \{X, Y\}\} + \{\alpha_\lambda(Y), \{H, X\}\} = 2\frac{\lambda^6 - 1}{\lambda^4} H,
\]
and also
\[
\{\alpha_\lambda(H), \{F, F\}\} - \{\alpha_\lambda(F), \{H, F\}\} + \{\alpha_\lambda(F), \{F, H\}\} = 4\frac{1 - \lambda}{\lambda^2} Y.
\]
Then, they do not vanish for $\lambda \neq 1$.

**Example 1.10.** Let $(A, [\cdot, \cdot]_A, \varepsilon, \alpha, \beta)$ be a BiHom-Lie colour algebra. Then the vector space $A' := A \otimes \mathbb{K}[t, t^{-1}]$ can be considered as the algebra of Laurent polynomials with coefficients in the BiHom-Lie colour algebra $A$. Note that $A'$ can be endowed with a natural $\Gamma$-grading as follows: an element $x \in A'$ is said to be homogeneous of degree $a \in \Gamma$, if there exist an element $x_a \in A$ with degree $a$ and $f(t) \in \mathbb{K}[t, t^{-1}]$, such that $x = x_a \otimes f(t)$. Put $\alpha' = \alpha \otimes \text{Id}_A$, $\beta' = \beta \otimes \text{Id}_A$ and define an even bilinear multiplication $[\cdot, \cdot]_{A'}$ on $A'$ by
\[
[x \otimes f(t), y \otimes g(t)] = [x, y] \otimes f(t)g(t)
\]
for all $x, y \in H(A)$ and $f(t), g(t) \in \mathbb{K}[t, t^{-1}]$. Then $(A', [\cdot, \cdot]_{A'}, \varepsilon, \alpha', \beta')$ is a BiHom-Lie colour algebra. Indeed, For any homogeneous elements $x, y, z \in A$, and $f(t), g(t), h(t) \in \mathbb{K}[t, t^{-1}]$, we have
\[
[\beta'(x \otimes f(t)), \alpha'(y \otimes g(t))] = [\beta(x) \otimes f(t), \alpha(y) \otimes g(t)]
\]
\[
= [\beta(x), \alpha(y)] \otimes f(t)g(t)
\]
\[
= -\varepsilon(x, y)[\beta(y), \alpha(x)] \otimes f(t)g(t) = -[\beta(x), \alpha(x)] \otimes f(t)g(t)
\]
\[
= [\beta'(y \otimes g(t)), \alpha'(x \otimes f(t))]\]
and
\[
\circ_{x, y, z} \varepsilon(z, x)[\beta'^2(x \otimes f(t)), [\beta'(y \otimes g(t)), \alpha'(z \otimes h(t))]]
\]
\[
= \circ_{x, y, z} \varepsilon(z, x)[\beta'^2(x \otimes f(t)), [\beta'(y \otimes g(t)), \alpha(z) \otimes h(t)]]
\]
\[
= \circ_{x, y, z} \varepsilon(z, x)[\beta'^2(x), [\beta(y), \alpha(z)] \otimes f(t)g(t)h(t)] = 0,
\]
since $(A, [\cdot, \cdot]_A, \varepsilon, \alpha, \beta)$ is a BiHom-Lie colour algebra.
2 \(\sigma\)-Twists of BiHom-Lie colour algebras

In this section, we shall give a close relationship between BiHom-Lie colour algebras corresponding to different form \(\sigma\) on \(\Gamma\).

Let \((A, [~, ~], \varepsilon, \alpha, \beta)\) be a BiHom-Lie colour algebra. Given any mapping \(\sigma : \Gamma \times \Gamma \to \mathbb{K}^*\), we define on the \(\Gamma\)-graded vector space \(A\) a new multiplication \([~, ~]^{\sigma}\) by the requirement that

\[
[x, y]^{\sigma} = \sigma(x, y)[x, y],
\]

for all the homogenous elements \(x, y\) in \(A\). The \(\Gamma\)-graded vector space \(A\), endowed with the multiplication \([~, ~]^{\sigma}\), is a \(\Gamma\)-graded algebra that will be called a \(\sigma\)-twist and will be denoted by \(A^{\sigma}\). We will looking for conditions on \(\sigma\), which ensure that \((A^{\sigma}, [~, ~]^{\sigma}, \varepsilon, \alpha, \beta)\) is also a BiHom-Lie colour algebra.

The bilinear mapping \([~, ~]^{\sigma}\) is a \(\varepsilon\)-skewsymmetric if and only if

1. \(\sigma\) is symmetric, i.e. \(\sigma(x, y) = \sigma(y, x)\), for any \(x, y \in \Gamma\).
2. Furthermore, the product \([~, ~]^{\sigma}\) satisfies the \(\varepsilon\)-BiHom-Jacobi condition if and only if \(\sigma(x, y)\sigma(z, x + y)\) is invariant under cyclic permutations of \(x, y, z \in \Gamma\).

We call such a mapping \(\sigma : \Gamma \times \Gamma \to \mathbb{K}^*\) satisfying both (1) et (2) a symmetric multiplier on \(\Gamma\).

Then we have:

**Proposition 2.1.** With the above notations. Let \((A, [~, ~], \varepsilon, \alpha, \beta)\) be a BiHom-Lie colour algebra and suppose that \(\sigma\) is a symmetric multiplier on \(\Gamma\). Then the \(\sigma\)-twist \((A^{\sigma}, [~, ~]^{\sigma}, \varepsilon, \alpha, \beta)\) is also a BiHom-Lie colour algebra under the same twisting \(\varepsilon\).

**Proof.** For any homogeneous elements \(x, y \in A\), we have

\[
[\alpha(x), \alpha(y)]^{\sigma} = \sigma(\alpha(x), \alpha(y))[\alpha(x), \alpha(y)] = \sigma(x, y)\alpha([x, y]) = \alpha([x, y]^{\sigma}).
\]

Similarly, one can prove that \(\beta([x, y]^{\sigma}) = [\beta(x), \beta(y)]^{\sigma}\).

Since \(\sigma\) is symmetric, we have

\[
[\beta(x), \alpha(y)]^{\sigma} = \sigma(x, y)[\beta(x), \alpha(y)] = -\sigma(x, y)\varepsilon(x, y)[\beta(y), \alpha(x)] = -\varepsilon(x, y)[\beta(y), \alpha(x)]^{\sigma}.
\]

Now we prove the \(\varepsilon\)-BiHom-Jacobi condition. For any homogeneous elements \(x, y \in A\), we have

\[
\varepsilon(z, x)[\beta^2(x), [\beta(y), \alpha(z)]^{\sigma}] = \varepsilon(z, x)[\beta^2(x), \sigma(y, z)[\beta(y), \alpha(z)]^{\sigma}] = \varepsilon(z, x)\sigma(y, z)\sigma(x, y + z)[\beta^2(x), [\beta(y), \alpha(z)].
\]

Using (2), we have \(\varepsilon(z, x)[\beta^2(x), [\beta(y), \alpha(z)]^{\sigma}]^{\sigma} = 0. \square

**Corollary 2.2.** Let \((A', [~, ~]', \varepsilon', \alpha, \beta)\) be a second BiHom-Lie colour algebra and \(\sigma\) be a symmetric multiplier on \(\Gamma\). Let \(f : A \to A'\) be a homomorphism of BiHom-Lie colour algebra, so \(f\) is also a homomorphism of BiHom-Lie colour algebras \((A^{\sigma}, [~, ~]^{\sigma}, \varepsilon, \alpha, \beta)\) into \((A'^{\sigma}, [~, ~]'^{\sigma}, \varepsilon', \alpha, \beta)\).
is said to be associated with $\sigma$. Let Proposition 2.4.

\[ f([x, y]) = f(\sigma(x, y)[x, y]) = \sigma(x, y)f([x, y]) = \sigma(x, y)[f(x), f(y)]' = [f(x), f(y)]'\sigma. \]

\[ \square \]

**Remark 2.3.** It is easy to construct a large class of symmetric multipliers on $\Gamma$ as follows. Let $\omega$ be an arbitrary mapping of $\Gamma$. Then the map $\tau : \Gamma \times \Gamma \to \mathbb{K}^*$ defined by

\[ \tau(x, y) = \omega(x + y)\omega(x)^{-1}\omega(y)^{-1}, \forall x, y \in \Gamma, \]

is a symmetric multiplier on $\Gamma$.

Let $\sigma : \Gamma \times \Gamma \to \mathbb{K}^*$ be a map endowing $A$ with a new multiplication defined by (2), we define a mapping $\delta : \Gamma \times \Gamma \to \mathbb{K}^*$ by

\[ \delta(x, y) = \sigma(x, y)\sigma(y, x)^{-1}, \forall x, y \in \Gamma. \]

Then it follows that

\[ [x, y] = -\varepsilon(x, y)\delta(x, y)[y, x] \]

for any homogenous $x, y \in A$.

In [77], M. Scheunert provided the necessary and sufficient condition on $\sigma$ which ensure that $\varepsilon\delta$ is a bicharacter on $\Gamma$ where $\varepsilon\delta(x, y) = \varepsilon(x, y)\delta(x, y)$ for all $x, y \in \Gamma$. It turns out that $A^\sigma$ is a $\Gamma$-graded $\varepsilon\delta$-Lie algebra if and only if

\[ \sigma(x, y + z)\sigma(y, z) = \sigma(x, y)\sigma(x + y, z), \forall x, y, z \in \Gamma. \]

Proof. Using Equation (3) and the fact that $\sigma$ is symmetric, it follows that

\[ [\beta(x), \alpha(y)]\sigma = -\varepsilon(x, y)\sigma(x, y)[\beta(y), \alpha(x)]\sigma. \]

Now for any homogenous elements $x, y, z \in A$, one has

$\cup_{x, y, z} \varepsilon\delta(z, x)[\beta^2(x), [\beta(y), \alpha(z)]\sigma = \sigma(x, y)\sigma(y, z)\sigma(z, x) \cup_{x, y, z} \varepsilon(z, x)[\beta^2(x), [\beta(y), \alpha(z)] = 0.$

Then $(A^\sigma, [\cdot, \cdot], \varepsilon, \delta, \alpha, \beta)$ is a BiHom-Lie colour algebra. \[ \square \]

**Corollary 2.5.** Let $(A', [\cdot, \cdot], \varepsilon, \delta, \alpha, \beta)$ be a second BiHom-Lie colour algebra. Suppose we are given a multiplier $\sigma$ on $\Gamma$; let $\delta$ be the bicharacter on $\Gamma$ associated with it. If $f : A \to A'$ is a homomorphism of BiHom-Lie colour algebras, then $f$ is also a homomorphism of BiHom-Lie colour algebra $(A^\sigma, [\cdot, \cdot], \varepsilon, \delta, \alpha, \beta)$ into $(A'^\sigma, [\cdot, \cdot], \varepsilon, \delta, \alpha, \beta)$. 9
3 BiHom-Lie colour admissible algebras

In this section, we aim to extend the notions and results about Lie admissible (colour) algebras to more generalized cases: BiHom-Lie colour admissible algebras and flexible BiHom-Lie colour admissible algebras. We will also explore some other general classes of such kind of algebras: G-BiHom-associative colour algebra, which we use to classify BiHom-Lie colour admissible algebras. In this section, we always assume that the structure maps α and β are bijective.

Definition 3.1. Let \((A, \mu, \varepsilon, \alpha, \beta)\) be a BiHom-Lie colour algebra on the \(\Gamma\)-graded vector space \(A\) defined by an even multiplication \(\mu\) and an algebra endomorphism \(\alpha\) and \(\beta\). Let \(\varepsilon\) be a bicharacter on \(\Gamma\). Then \((A, [, , ], \varepsilon, \alpha, \beta)\) is said to be a BiHom-Lie colour admissible algebra if the bracket defined by

\[
[x, y] = xy - \varepsilon(x, y)(\alpha^{-1}\beta(y))(\alpha\beta^{-1}(x))
\]

satisfies the BiHom-Jacobi identity for all homogeneous elements \(x, y, z \in A\).

Let \((A, [, , ], \varepsilon, \alpha, \beta)\) be a BiHom-Lie colour algebra. Define a new commutator product \([, , ]'\) by

\[
[x, y]' = [x, y] - \varepsilon(x, y)[\alpha^{-1}\beta(y), \alpha\beta^{-1}(x)], \quad \forall \ x, y \in \mathcal{H}(A).
\]

It is easy to see that \([\beta(x), \alpha(y)]' = -\varepsilon(x, y)[\beta(y), \alpha(x)]'\). Moreover, we have

\[
\bigcirc_{x, y, z} \varepsilon(z, x)[\beta^2(x), [\beta(y), \alpha(z)]]' = \bigcirc_{x, y, z} \varepsilon(z, x)[\beta^2(x), [\beta(y), \alpha(z)]] - \varepsilon(y, z)[\alpha^{-1}(\beta^2(z)), \beta^{-1}(\alpha^2(y))]'
\]

Our discussion above now shows:

Proposition 3.2. Any BiHom-Lie colour algebra is BiHom-Lie colour admissible.

Let \((A, \mu, \varepsilon, \beta, \alpha)\) be a colour BiHom-algebra, that is respectively a vector space \(A\), a multiplication \(\mu\), a bicharacter \(\varepsilon\) on the abelian group \(\Gamma\) and two linear maps \(\alpha\) and \(\beta\). Notice that there is no conditions required on the given data. Let

\[
x, y = xy - \varepsilon(x, y)(\alpha^{-1}\beta(y))(\beta^{-1}\alpha(x))
\]

be the associated colour commutative. A BiHom associator \(as_{\alpha, \beta}\) of \(\mu\) is defined by

\[
as_{\alpha, \beta}(x, y, z) = \alpha(x)(yz) - (xy)\beta(z), \quad \forall \ x, y, z \in \mathcal{H}(A). \tag{6}
\]

A colour BiHom-algebra is said to be flexible if \(as_{\alpha, \beta}(x, y, x) = 0\), for all \(x, y \in \mathcal{H}(A)\).

Now let us introduce the notation:

\[
S(x, y, z) = \bigcirc_{x, y, z} \varepsilon(z, x)as_{\alpha, \beta}(\alpha^{-1}\beta^2(x), \beta(y), \alpha(z)).
\]

Then we have the following properties:
Lemma 3.3. 
\[ S(x, y, z) = \circlearrowleft_{x,y,z} \varepsilon(z, x)[\beta^2(x), \beta(y)\alpha(z)]. \]

Proof. The assertion follows by expanding the commutators on the right hand side:
\[ \varepsilon(z, x)[\beta^2(x), \beta(y)\alpha(z)] + \varepsilon(x, y)[\beta^2(y), \beta(z)\alpha(x)] + \varepsilon(y, z)[\beta^2(z), \beta(x)\alpha(y)] \]
\[ = \varepsilon(z, x)\beta^2(x)(\beta(y)\alpha(z)) - \varepsilon(x, y)(\alpha^{-1}\beta^2(y)\beta(z))\alpha(\beta(x)) + \varepsilon(x, y)\beta^2(y)(\beta(z)\alpha(x)) \]
\[ - \varepsilon(y, z)(\alpha^{-1}\beta^2(z)\beta(x))\alpha(\beta(y)) + \varepsilon(y, z)\beta^2(z)(\beta(x)\alpha(y)) - \varepsilon(z, x)(\alpha^{-1}\beta^2(x)\beta(y))\alpha(\beta(z)) \]
\[ = \circlearrowleft_{x,y,z} \varepsilon(z, x)\alpha_\beta(\alpha^{-1}\beta^2(x), \beta(y), \alpha(z)) = S(x, y, z). \]

Proposition 3.4. A colour BiHom-algebra \((A, \mu, \varepsilon, \beta, \alpha)\) is BiHom-Lie colour admissible if and only if it satisfies
\[ S(x, y, z) = \varepsilon(x, y)\varepsilon(y, z)\varepsilon(z, x)S(x, z, y), \ \forall x, y, z \in \mathcal{H}(A). \]

Proof. From Lemma 3.3 for all \(x, y, z \in \mathcal{H}(A)\) we have
\[ S(x, y, z) - \varepsilon(x, y)\varepsilon(y, z)\varepsilon(z, x)S(x, z, y) \]
\[ = \circlearrowleft_{x,y,z} \varepsilon(z, x)[\beta^2(x), \beta(y)\alpha(z)] - \varepsilon(x, y)\varepsilon(y, z)\varepsilon(y, x) \circlearrowleft_{x,y,z} \varepsilon(y, x)[\beta^2(x), \beta(z)\alpha(y)] \]
\[ = \circlearrowleft_{x,y,z} \varepsilon(z, x)[\beta^2(x), \beta(y)\alpha(z)] - \varepsilon(y, z)\beta(z)\alpha(y) \]
\[ = \circlearrowleft_{x,y,z} \varepsilon(z, x)[\beta^2(x), \beta(y)\alpha(z)] = 0, \]
which proves the result.
for any homogeneous elements $x_1, x_2, x_3$ in $A$. Let $\text{sgn}(\sigma)$ denote the signature of $\sigma \in S_3$. We have the following useful lemma:

**Lemma 3.5.** A colour BiHom-algebra $(A, \mu, \varepsilon, \beta, \alpha)$ is a BiHom-Lie admissible colour algebra if and only if the following condition holds:

$$
\sum_{\sigma \in S_3} (-1)^{\text{sgn}(\sigma)} (-1)^{|\sigma(x_1, x_2, x_3)|} a_{\alpha, \beta} \circ \sigma(\alpha^{-1} \beta^2(x_1), \beta(x_2), \alpha(x_3)) = 0.
$$

**Proof.** One only needs to verify the BiHom-$\varepsilon$-Jacobi identity. By straightforward calculation, the associated color commutator satisfies

$$
\bigotimes_{x_1, x_2, x_3} \varepsilon(x_3, x_1)[\beta^2(x_1), [\beta(x_2), \alpha(x_3)]]

= \sum_{\sigma \in S_3} (-1)^{\text{sgn}(\sigma)} (-1)^{|\sigma(x_1, x_2, x_3)|} a_{\alpha, \beta} \circ \sigma(\alpha^{-1} \beta^2(x_1), \beta(x_2), \alpha(x_3)).
$$

Let $G$ be a subgroup of $S_3$, any colour BiHom-algebra $(A, \mu, \varepsilon, \beta, \alpha)$ is said to be $G$-BiHom-associative if the following equation holds:

$$
\sum_{\sigma \in G} (-1)^{\text{sgn}(\sigma)} (-1)^{|\sigma(x_1, x_2, x_3)|} a_{\alpha, \beta} \circ \sigma(\alpha^{-1} \beta^2(x_1), \beta(x_2), \alpha(x_3)) = 0, \forall \ x_1, x_2, x_3 \in H(A).
$$

**Proposition 3.6.** Let $G$ be a subgroup of the symmetric group $S_3$. Then any $G$-BiHom-associative colour algebra $(A, \mu, \varepsilon, \beta, \alpha)$ is BiHom-Lie admissible.

**Proof.** The $\varepsilon$-skew symmetry follows straightaway from the definition. Assume that $G$ is a subgroup of $S_3$. Then $S_3$ can be written as the disjoint union of the left cosets of $G$. Say $S_3 = \bigcup_{\sigma \in I} \sigma G$, with $I \subseteq S_3$ and for any $\sigma, \sigma' \in I$,

$$
\sigma \neq \sigma' \implies \sigma G \bigcap \sigma' G = \emptyset.
$$

Then one has

$$
\sum_{\sigma \in S_3} (-1)^{\text{sgn}(\sigma)} (-1)^{|\sigma(x_1, x_2, x_3)|} a_{\alpha, \beta} \circ \sigma(\alpha^{-1} \beta^2(x_1), \beta(x_2), \alpha(x_3))

= \sum_{\tau \in I} \sum_{\sigma \in \tau G} (-1)^{\text{sgn}(\sigma)} (-1)^{|\sigma(x_1, x_2, x_3)|} a_{\alpha, \beta} \circ \sigma(\alpha^{-1} \beta^2(x_1), \beta(x_2), \alpha(x_3)) = 0.
$$
Now we provide a classification of BiHom-Lie colour admissible algebras via $G$-BiHom-associative colour algebras. The subgroups of $S_3$ are:

$$G_1 = \{Id\}, \ G_2 = \{Id, \sigma_1\}, \ G_3 = \{Id, \sigma_2\}, \ G_4 = \{Id, \sigma_1 \sigma_2 = (1 \ 3)\}, \ G_5 = A_3, \ G_6 = S_3,$$

where $A_3$ is the alternating subgroup of $S_3$.
We obtain the following types of BiHom-Lie admissible colour algebras.

- The $G_1$-BiHom-associative colour algebras are the colour BiHom-algebras defined in Definition 1.5.

- The $G_2$-BiHom-associative colour algebras satisfy the condition:
  $$\beta^2(x)(\beta(y)\alpha(z)) - (\alpha^{-1}\beta^2(x)\beta(y))\alpha(\beta(z)) = \varepsilon(x, y)\left(\alpha(\beta(y))(\alpha^{-1}\beta^2(x)\alpha(z)) - (\beta(y)\alpha^{-1}\beta^2(x))\alpha(\beta(z))\right).$$

- The $G_3$-BiHom-associative colour algebras satisfy the condition:
  $$\beta^2(x)(\beta(y)\alpha(z)) - (\alpha^{-1}\beta^2(x)\beta(y))\alpha\beta(z) = \varepsilon(y, z)\left(\beta^2(x)(\beta(z)\alpha(y)) - (\alpha^{-1}\beta^2(x)\alpha(z))\beta^2(y)\right).$$

- The $G_4$-BiHom-associative colour algebras satisfy the condition:
  $$\beta^2(x)(\beta(y)\alpha(z)) - (\alpha^{-1}\beta^2(x)\beta(y))\alpha\beta(z) = \varepsilon(x, y)\varepsilon(y, z)\varepsilon(x, z)\left(\alpha^2(z)(\beta(y)\alpha^{-1}\beta^2(x)) - (\alpha(z)\beta(y))\alpha^{-1}\beta^2(x)\right).$$

- The $G_5$-BiHom-associative colour algebras satisfy the condition:
  $$(\alpha^{-1}\beta^2(x)\beta(y))\alpha\beta(z) - \varepsilon(x, y + z)\alpha\beta(y)(\alpha(z)\alpha^{-1}\beta^2(x)) - \varepsilon(x + y, z)\alpha^2(z)(\alpha^{-1}\beta^2(x)\beta(y))$$
  $$= (\alpha^{-1}\beta^2(x)\beta(y))\alpha\beta(z) - \varepsilon(x, y + z)(\beta(y)\alpha(z))\alpha^{-1}\beta^3(x) - \varepsilon(x + y, z)(\alpha(z)\alpha^{-1}\beta^2(x))\beta^2(y).$$

- The $G_6$-BiHom-associative colour algebras are the BiHom-Lie colour admissible algebras.

**Remark 3.7.** Moreover, if in the previous identities we consider $\beta = \alpha$, then we obtain a classification of Hom-Lie admissible colour algebras [17].

### 4 Cohomology and Representations of BiHom-Lie colour algebras

#### 4.1 Cohomology of BiHom-Lie colour algebras

We extend first to BiHom-Lie colour algebras, the concept of $A$-module introduced in [3, 5, 18], and then define a family of cohomology complexes for BiHom-Lie colour algebras.
In the sequel we assume that the BiHom-Lie colour algebra is a regular BiHom-Lie colour algebra $(V,\rho,\alpha_V,\beta_V)$, where $V$ is a $\Gamma$-graded vector space, $\alpha_V,\beta_V : V \to V$ are two even commuting linear maps and $\rho : A \to End(V)$ is an even linear map such that, for all $x,y \in \mathcal{H}(A)$ and $v \in V$, we have

$$\rho(\alpha(x)) \circ \alpha_V = \alpha_V \circ \rho(x),$$

$$\rho(\beta(x)) \circ \beta_V = \beta_V \circ \rho(x),$$

$$\rho(\beta(x)) \circ \beta_V(v) = \rho(\alpha(x)) \circ \rho(y) - \varepsilon(x,y)\rho(\beta(y)) \circ \rho(\alpha(x))(v). \quad (7)$$

The cohomology of Lie colour algebras was introduced in [17]. In the following, we define cohomology complexes of BiHom-Lie colour algebras.

Let $(A,\cdot,\varepsilon,\alpha,\beta)$ be a regular BiHom-Lie colour algebra and $(V,\rho,\alpha_V,\beta_V)$ be a representation of $A$. In the sequel, we denote $\rho(x)(v)$ by a bracket $[x,v]_V$.

The set of $n$-cochains on $A$ with values in $V$, which we denote by $C^n(A,V)$, is the set of skewsymmetric $n$-linear maps $f : A^n \to V$, that is

$$f(x_1,\ldots,x_i,x_{i+1},\ldots,x_n) = -\varepsilon(x_i,x_{i+1})f(x_1,\ldots,x_{i+1},x_i,\ldots,x_n), \quad 1 \leq i \leq n - 1.$$

For $n = 0$, we have $C^0(A,V) = V$.

We set

$$C^n_{\alpha,\beta}(A,V) = \{ f : A^n \to V : f \in C^n(A,V) \text{ and } f \circ \alpha = \alpha_V \circ f, \ f \circ \beta = \beta_V \circ f \}.$$

We extend this definition to the case of integers $n < 0$ and set

$$C^n_{\alpha,\beta}(A,V) = \{0\} \text{ if } n < -1 \quad \text{and} \quad C^0_{\alpha,\beta}(A,V) = V.$$

A map $f \in C^n(A,V)$ is called even (resp. of degree $\gamma$) when $f(x_1,\ldots,x_i,\ldots,x_n) \in V_{\gamma_1+\gamma_2+\ldots+\gamma_n}$ for all elements $x_i \in A_{\gamma_i}$ (resp. $f(x_1,\ldots,x_i,\ldots,x_n) \in V_{\gamma_1+\gamma_2+\ldots+\gamma_n}$).

A homogeneous element $f \in C^n_{\alpha,\beta}(A,V)$ is called $n$-cochain or sometimes in the literature $n$-Hom-cochain.

Next, for a given integer $r$, we define the coboundary operator $\delta^r_{\alpha,\beta}$

**Definition 4.1.** Let $(A,\cdot,\varepsilon,\alpha,\beta)$ be a regular BiHom-Lie colour algebra. A representation of $A$ is a 4-tuple $(V,\rho,\alpha_V,\beta_V)$, where $V$ is a $\Gamma$-graded vector space, $\alpha_V,\beta_V : V \to V$ are two even commuting linear maps and $\rho : A \to End(V)$ is an even linear map such that, for all $x,y \in \mathcal{H}(A)$ and $v \in V$, we have

$$\rho(\alpha(x)) \circ \alpha_V = \alpha_V \circ \rho(x),$$

$$\rho(\beta(x)) \circ \beta_V = \beta_V \circ \rho(x),$$

$$\rho(\beta(x)) \circ \beta_V(v) = \rho(\alpha(x)) \circ \rho(y) - \varepsilon(x,y)\rho(\beta(y)) \circ \rho(\alpha(x))(v). \quad (7)$$

The cohomology of Lie colour algebras was introduced in [17]. In the following, we define cohomology complexes of BiHom-Lie colour algebras.

Let $(A,\cdot,\varepsilon,\alpha,\beta)$ be a regular BiHom-Lie colour algebra and $(V,\rho,\alpha_V,\beta_V)$ be a representation of $A$. In the sequel, we denote $\rho(x)(v)$ by a bracket $[x,v]_V$.

The set of $n$-cochains on $A$ with values in $V$, which we denote by $C^n(A,V)$, is the set of skewsymmetric $n$-linear maps $f : A^n \to V$, that is

$$f(x_1,\ldots,x_i,x_{i+1},\ldots,x_n) = -\varepsilon(x_i,x_{i+1})f(x_1,\ldots,x_{i+1},x_i,\ldots,x_n), \quad 1 \leq i \leq n - 1.$$

For $n = 0$, we have $C^0(A,V) = V$.

We set

$$C^n_{\alpha,\beta}(A,V) = \{ f : A^n \to V : f \in C^n(A,V) \text{ and } f \circ \alpha = \alpha_V \circ f, \ f \circ \beta = \beta_V \circ f \}.$$

We extend this definition to the case of integers $n < 0$ and set

$$C^n_{\alpha,\beta}(A,V) = \{0\} \text{ if } n < -1 \quad \text{and} \quad C^0_{\alpha,\beta}(A,V) = V.$$

A map $f \in C^n(A,V)$ is called even (resp. of degree $\gamma$) when $f(x_1,\ldots,x_i,\ldots,x_n) \in V_{\gamma_1+\gamma_2+\ldots+\gamma_n}$ for all elements $x_i \in A_{\gamma_i}$ (resp. $f(x_1,\ldots,x_i,\ldots,x_n) \in V_{\gamma_1+\gamma_2+\ldots+\gamma_n}$).

A homogeneous element $f \in C^n_{\alpha,\beta}(A,V)$ is called $n$-cochain or sometimes in the literature $n$-Hom-cochain.

Next, for a given integer $r$, we define the coboundary operator $\delta^r_{\alpha,\beta}$

**Definition 4.2.** We call, for $n \geq 1$ and for any integer $r$, a $n$-coboundary operator of the BiHom-Lie colour algebra $(A,\cdot,\varepsilon,\alpha,\beta)$ the linear map $\delta^n_{\alpha,\beta} : C^n_{\alpha,\beta}(A,V) \to C^{n+1}_{\alpha,\beta}(A,V)$ defined by

$$\delta^n_{\alpha,\beta}(f)(x_0,\ldots,x_n) = \sum_{0 \leq s < t \leq n} (-1)^t \varepsilon(x_{s+1} + \ldots + x_{t-1},x_t)f(\beta(x_0),\ldots,\beta(x_{s-1}),[\alpha^{-1}\beta(x_s),x_1,\beta(x_{s+1}),\ldots,\hat{x_1},\ldots,\beta(x_n)])$$

$$+ \sum_{s=0}^n (-1)^s \varepsilon(\gamma + x_0 + \ldots + x_{s-1},x_s)[\alpha\beta^{r+n-1}(x_s),f(x_0,\ldots,\hat{x_s},\ldots,x_n)]_V,$$

where $f \in C^n_{\alpha,\beta}(A,V)$, $\gamma$ is the degree of $f$, $(x_0,\ldots,x_n) \in \mathcal{H}(A)^{\otimes n+1}$ and $\hat{x}$ indicates that the element $x$ is omitted.

In the sequel we assume that the BiHom-Lie colour algebra $(A,\cdot,\varepsilon,\alpha,\beta)$ is multiplicative.
For $n = 1$, we have
\[
\delta^1_r : C^1(A, V) \rightarrow C^2(A, V)
\]
\[
f \mapsto \delta^1_r(f)
\]
such that for two homogeneous elements $x, y$ in $A$
\[
\delta^1_r(f)(x, y) = \varepsilon(\gamma, x)[\alpha \beta(x), f(y)]_V - \varepsilon(\gamma + x, y)[\alpha \beta(y), f(x)]_V - f([\alpha^{-1} \beta(x), y])
\]
and for $n = 2$, we have
\[
\delta^2_r : C^2(A, V) \rightarrow C^3(A, V)
\]
\[
f \mapsto \delta^2_r(f)
\]
such that, for three homogeneous elements $x, y, z$ in $A$, we have
\[
\delta^2_r(f)(x, y, z) = \varepsilon(\gamma, x)[\alpha \beta^2(x), f(y, z)]_V - \varepsilon(\gamma + x, y)[\alpha \beta^2(y), f(x, z)]_V + \varepsilon(\gamma + x + y, z)[\alpha \beta^2(z), f(x, y)]_V - f([\alpha^{-1} \beta(x), y, z])
\]
(9)
\[
\delta^2_r(f)(x, y, z) = \varepsilon(\gamma, x)[\alpha \beta^2(x), f(y, z)]_V - \varepsilon(\gamma + x, y)[\alpha \beta^2(y), f(x, z)]_V + \varepsilon(\gamma + x + y, z)[\alpha \beta^2(z), f(x, y)]_V - f([\alpha^{-1} \beta(x), y, z])
\]
(10)

**Lemma 4.3.** With the above notations, for any $f \in C^n_{\alpha, \beta}(A, V)$, we have
\[
\delta^n_r(f) \circ \alpha = \alpha V \circ \delta^n_r(f),
\]
\[
\delta^n_r(f) \circ \beta = \beta V \circ \delta^n_r(f), \quad \forall \ n \geq 2
\]

Thus, we obtain a well-defined map $\delta^n_r : C^n_{\alpha, \beta}(A, V) \rightarrow C^{n+1}_{\alpha, \beta}(A, V)$.

**Proof.** Let $f \in C^n_{\alpha, \beta}(A, V)$ and $(x_0, \ldots, x_n) \in H(A)^{\otimes n+1}$, we have
\[
\delta^n_r(f) \circ \alpha(x_0, \ldots, x_n)
\]
\[
= \delta^n_r(f)(\alpha(x_0), \ldots, \alpha(x_n))
\]
\[
= \sum_{0 \leq s \leq t \leq n} (-1)^t \varepsilon(x_0 + \ldots + x_{t-1}, x_t) f(\alpha \beta(x_0), \ldots, \alpha \beta(x_{t-1}), [\alpha \beta^2(x_t), \alpha(x_t)], \alpha \beta(x_{t+1}), \ldots, \hat{x}_t, \ldots, \alpha \beta(x_n))
\]
\[
+ \sum_{n \geq s \geq t \geq 0} (-1)^s \varepsilon(\gamma + x_0 + \ldots + x_{s-1}, x_s) \alpha \beta^{n-1} \beta(x_s), f(\alpha(x_0), \ldots, \hat{x}_s, \ldots, \alpha(x_n))]_V
\]
\[
= \sum_{0 \leq s \leq t \leq n} (-1)^t \varepsilon(x_0 + \ldots + x_{t-1}, x_t) \alpha V \circ f(\beta(x_0), \ldots, \beta(x_{t-1}), [\alpha^{-1} \beta(x_t), x_t], \beta(x_{t+1}), \ldots, \hat{x}_t, \ldots, \beta(x_n))
\]
\[
+ \sum_{n \geq s \geq t \geq 0} (-1)^s \varepsilon(\gamma + x_0 + \ldots + x_{s-1}, x_s) \alpha V [\alpha \beta^{n-1} \beta(x_s), f(x_0, \ldots, \hat{x}_s, \ldots, x_n)]_V
\]
\[
= \alpha V \circ \delta^n_r(f)(x_0, \ldots, x_n).
\]

Then $\delta^n_r(f) \circ \alpha = \alpha V \circ \delta^n_r(f)$.

Similarly, we have
\[
\delta^n_r(f) \circ \beta(x_0, \ldots, x_n) = \beta V \circ \delta^n_r(f)(x_0, \ldots, x_n),
\]
which completes the proof. \(\square\)

**Theorem 4.4.** Let $(A, [\ldots], \varepsilon, \alpha, \beta)$ be a multiplicative BiHom-Lie colour algebra and $(V, \alpha_V, \beta_V)$ be an $A$-module. Then the pair $(\bigoplus_{n \geq 0} C^n_{\alpha, \beta}, \delta^n_r)$ is a cohomology complex. That is the maps $\delta^n_r$
satisfy $\delta^n_r \circ \delta^{n-1}_r = 0, \quad \forall \ n \geq 2, \forall \ r \geq 1.$
Proof. For any \( f \in C^{n-1}(A, V) \), we have

\[
\delta^n \circ \delta^{n-1}_p(f(x_0, \ldots, x_n)) = \sum_{s < t} (-1)^{s} \varepsilon(x_0 + \cdots + x_{t-1}, x_s) \delta^{n-1}_p(f(x_0, \ldots, x_{s-1}), [\alpha^{-1} \beta(x_s), x_t], \beta(x_{s+1}), \ldots, \beta(x_n))
\]

\[+ \sum_{s=0}^{n} (-1)^{s} \varepsilon(f + x_0 + \cdots + x_{s-1}, x_s) \alpha \beta^{n+n-1}(x_s), \delta^{n-1}_p(f(x_0, \ldots, \tilde{x}_s, \ldots, x_n)) \varepsilon.
\]

From (11) we have

\[
\delta^{n-1}_p(f)(\beta(x_0), \ldots, \beta(x_{s+1}), [\alpha^{-1} \beta(x_s), x_t], \beta(x_{s+1}), \ldots, \beta(x_n)) = \sum_{s' < t'} (-1)^{s'} \varepsilon(x_{s'+1} + \cdots + x_{t'-1}, x_{s'}) f\left(\beta^2(x_0), \ldots, \beta^2(x_{s'-1}), [\alpha^{-2} \beta^2(x_s), \alpha^{-1} \beta(x_t)], \beta^2(x_{s'+1}), \ldots, \tilde{x}_{t'}, \beta^2(x_n)\right)
\]

\[+ \sum_{s' < s < t'} (-1)^{s'} \varepsilon(x_{s'+1} + \cdots + [x_s, x_t] + \cdots + x_{t'-1}, x_{t'}) f\left(\beta^2(x_0), \ldots, \beta^2(x_{s'-1}), [\alpha^{-1} \beta(x_s), \beta(x_{s'}), \beta^2(x_{s'+1}), \beta([\alpha^{-1} \beta(x_s), x_t]), \beta^2(x_{s'+1}), \ldots, \tilde{x}_{t'}, \beta^2(x_n)\right)
\]

\[+ \sum_{s' < s < t'} (-1)^{s'} \varepsilon(x_{s'+1} + \cdots + x_{s'-1} + [x_s, x_t] + x_{s'+1} + \cdots + \tilde{x}_t + \cdots + x_{t'-1}, x_{t'}) f\left(\beta^2(x_0), \ldots, \beta^2(x_{s'-1}), [\alpha^{-1} \beta(x_s), \beta(x_{s'}), \beta^2(x_{s'+1}), \beta([\alpha^{-1} \beta(x_s), x_t]), \beta^2(x_{s'+1}), \ldots, \tilde{x}_{t'}, \beta^2(x_n)\right)
\]

\[+ \sum_{s < t' < t} (-1)^{s'} \varepsilon(x_{s'+1} + \cdots + x_{t'-1}, x_{t'}) f\left(\beta^2(x_0), \ldots, [\alpha^{-2} \beta^2(x_s), \alpha^{-1} \beta(x_t)], \beta(x_{s'}), \beta^2(x_{s'+1}), \ldots, \tilde{x}_{t'}, \beta^2(x_n)\right)
\]

\[+ \sum_{s < t' < t} (-1)^{s'} \varepsilon(x_{s'+1} + \cdots + x_{t'-1}, x_{t'}) f\left(\beta^2(x_0), \ldots, \beta^2(x_{s'-1}), [\alpha^{-2} \beta^2(x_s), \alpha^{-1} \beta(x_t)], \beta(x_{s'}), \beta^2(x_{s'+1}), \ldots, \tilde{x}_{t'}, \beta^2(x_n)\right)
\]
\[ [\alpha^{-1}\beta(x_{s'}), \beta(x_t), \ldots, \beta^2(x_n)] \] (19)

\[ + \sum_{s < s' < t} (-1)^r \varepsilon(x_{s'+1} + \ldots + \hat{x}_t + \ldots + x_{t-1}, x_t) \] (20)

\[ f\left(\beta^2(x_0), \ldots, \beta^2(x_{s-1}), \beta([\alpha^{-1}\beta(x_s), x_t]), \beta^2(x_{s+1}), \ldots, [\alpha^{-1}\beta((x_s'), \beta(x_t)), \ldots, \tilde{x}_t, \ldots, \tilde{x}_{t'}, \ldots, \beta^2(x_n)\right) \]

\[ + \sum_{t < t' < s} (-1)^r \varepsilon(x_{t'+1} + \ldots + \hat{x}_{t'_{t'}} + \ldots + x_{t'-1}, x_{t'}) \] (21)

\[ f\left(\beta^2(x_0), \ldots, \beta^2(x_{s-1}), \beta([\alpha^{-1}\beta(x_s), x_t]), \beta^2(x_{s+1}), \ldots, \tilde{x}_t, \ldots, [\alpha^{-1}\beta((x_s'), \beta(x_t)), \ldots, \tilde{x}_{t'}, \ldots, \beta^2(x_n)\right) \]

\[ + \sum_{0 < s' < s} (-1)^r \varepsilon(\gamma + x_0 + \ldots + x_{s'-1}, x_{s'}) [\alpha^{r+n-1}(x_{s'}), f(\beta(x_0), \ldots, \tilde{x}_{s'}, [x_s, x_t], \ldots, \beta(x_n))] V \] (22)

\[ + (-1)^s \varepsilon(\gamma + x_0 + \ldots + x_{s-1}, [x_s, x_t]) [\alpha^{r+n-3}([\alpha^{-1}\beta(x_s), x_t]), f(\beta(x_0), \ldots, [x_s, x_t], \beta(x_{s+1}), \ldots, \tilde{x}_t, \ldots, \beta(x_n))] V \] (23)

\[ + \sum_{s < s' < t} (-1)^r \varepsilon(\gamma + x_0 + \ldots + [x_s, x_t] + \ldots + x_{s'-1}, x_{s'}) \] (24)

\[ [\alpha^{r+n-2}(x_{s'}), f(\beta(x_0), \ldots, [\alpha^{-1}\beta(x_s), x_t], \ldots, \tilde{x}_{s'+1}, \ldots, \beta(x_n))] V \]

\[ + \sum_{t < s'} (-1)^r \varepsilon(\gamma + x_0 + [x_s, x_t] + \ldots + \hat{x}_t + \ldots + x_{t'-1}, x_{s'}) \] (25)

\[ [\alpha^{r+n-2}(x_{s'}), f(\beta(x_0), \ldots, [\alpha^{-1}\beta(x_s), x_t], \ldots, \tilde{x}_{s'+1}, \ldots, \beta(x_n))] V . \]

The identity (12) implies that

\[ [\alpha^{r+n-1}(x_s), \delta^{n-1}_p(f)(x_0, \ldots, \tilde{x}_s, \ldots, x_n)] V = [\alpha^{r+n-1}(x_s), \sum_{s' < t' < s} (-1)^r \varepsilon(x_{s'+1} + \ldots + x_{t'-1}, x_{t'}) \]

\[ f\left(\beta(x_0), \ldots, [\alpha^{-1}\beta(x_{s'}), x_{t'}], \beta(x_{s'+1}), \ldots, \tilde{x}_{s'+1}, \beta(x_{s+1}), \ldots, \beta(x_n)\right)] V \]

\[ + [\alpha^{r+n-1}(x_s), \sum_{s' < s < t} (-1)^{r-1} \varepsilon(x_{s'+1} + \ldots + \hat{x}_t + \ldots + x_{t'-1}, x_{t'}) \]

\[ f(\beta(x_0), \ldots, [\alpha^{-1}\beta(x_{s'}), x_{t'}], [\alpha^{-1}\beta(x_{s'}), x_{t'}], \beta(x_{s'+1}), \ldots, \tilde{x}_{s'+1}, \ldots, \beta(x_n))] V \] (26)
Thus by the $\varepsilon$-bihom-Jacobi condition, we obtain
\[\sum_{s<t}(1)^{s-1}\varepsilon(x_{s+1} + \ldots + x_{t-1}, x_t) = 0.\]

By the $\varepsilon$-bihom-Jacobi condition, we obtain
\[\sum_{s<t}(1)^{s}(x_{s+1} + \ldots + x_{t-1}, x_t) = 0.\]

Also, we have
\[\sum_{s<t}(1)^{s}(x_{s+1} + \ldots + x_{t-1}, x_t) + \sum_{s<t}(-1)^{s}(x_{s+1} + \ldots + x_{t-1}, x_t) = 0.\]

By a simple calculation, we get
\[\sum_{s<t}(1)^{s}(x_{s+1} + \ldots + x_{t-1}, x_t) + \sum_{s<t}(-1)^{s}(x_{s+1} + \ldots + x_{t-1}, x_t) = 0,\]
and
\[\sum_{s<t}(1)^{s}(x_{s+1} + \ldots + x_{t-1}, x_t) + \sum_{s<t}(-1)^{s}(x_{s+1} + \ldots + x_{t-1}, x_t) = 0.\]
Similarly, we have
\[
\sum_{s<t}(-1)^s\varepsilon(x_{s+1} + \ldots + x_{t-1}, x_t)(13 + 21) = 0
\]
and
\[
\sum_{s<t}(-1)^s\varepsilon(x_{s+1} + \ldots + x_{t-1}, x_t)(16 + 19) = 0.
\]
Therefore \(\delta^n_x \circ \delta^{n-1}_x = 0\), which completes the proof. \(\square\)

Let \(Z^n_r(A, V)\) (resp. \(B^n_r(A, V)\)) denote the kernel of \(\delta^n_x\) (resp. the image of \(\delta^{n-1}_x\)). The spaces \(Z^n_r(A, V)\) and \(B^n_r(A, V)\) are graded submodules of \(C^n_{\alpha, \beta}(A, V)\) and according to Proposition 4.4 we have
\[
B^n_r(A, V) \subseteq Z^n_r(A, V).
\]
(30)
The elements of \(Z^n_r(A, V)\) are called \(n\)-cocycles, and the elements of \(B^n_r(A, V)\) are called the \(n\)-coboundaries. Thus, we define a so-called cohomology groups
\[
H^n_r(A, V) = \frac{Z^n_r(A, V)}{B^n_r(A, V)}.
\]
We denote by \(H^n_r(A, V) = \bigoplus_{\gamma \in \Gamma}(H^n_r(A, V))_\gamma\) the space of all \(r\)-cohomology group of degree \(\gamma\) of the BiHom-Lie colour algebra \(A\) with values in \(V\).
Two elements of \(Z^n_r(A, V)\) are said to be cohomologous if their residue classes modulo \(B^n_r(A, V)\) coincide, that is if their difference lies in \(B^n_r(A, V)\).

### 4.2 Adjoint representations of BiHom-Lie colour algebras

In this section, we generalize to BiHom-Lie colour algebras some results from \([3]\) and \([18]\). Let \((A, [ , ], \varepsilon, \alpha, \beta)\) be a regular BiHom-Lie colour algebra. We consider that \(A\) represents on itself via the bracket with respect to the morphisms \(\alpha, \beta\).

Now, we discuss adjoint representations of a BiHom-Lie colour algebra.

The adjoint representations are generalized in the following way.

**Definition 4.5.** An \(\alpha^*\beta^\dagger\)-adjoint representation, denoted by \(ad_{s,l}\), of a BiHom-Lie colour algebra \((A, [ , ], \varepsilon, \alpha, \beta)\) is defined as
\[
ad_{s,l}(a)(x) = [\alpha^*\beta^\dagger(a), x], \quad \forall \ a, x \in \mathcal{H}(A).
\]

**Lemma 4.6.** With the above notations, we have \((A, ad_{s,l}(\cdot)(\cdot), \alpha, \beta)\) is a representation of the BiHom-Lie colour algebra \((A, [ , ], \varepsilon, \alpha, \beta)\). It satisfies
\[
\begin{align*}
ad_{s,l}(\alpha(x)) \circ \alpha &= \alpha \circ ad_{s,l}(x), \\
ad_{s,l}(\beta(x)) \circ \beta &= \beta \circ ad_{s,l}(x), \\
ad_{s,l}([\beta(x), y]) \circ \beta &= ad_{s,l}(\alpha\beta(x)) \circ ad_{s,l}(\beta(y)) - \varepsilon(x, y)ad_{s,l}(\beta(y)) \circ ad_{s,l}(x).
\end{align*}
\]

**Proof.** First, the result follows from
\[
ad_{s,l}(\alpha(x))(\alpha(y)) = [\alpha^*\beta^\dagger(\alpha(x)), \alpha(y)] = \alpha([\alpha^*\beta^\dagger(x), y]) = \alpha \circ ad_{s,l}(x)(y).
\]
Similarly, we have

$$ad_{s,t}(\beta(x))(\beta(y)) = \beta \circ ad_{s,t}(x)(y).$$

Note that the $\varepsilon$-BiHom skew symmetry condition implies

$$ad_{s,t}(x)(y) = -\varepsilon(x, y)[\alpha^{-1}\beta(y), \alpha^{s+1}\beta^{l-1}(x)].$$

On one hand, we have

$$ad_{s,t}([\beta(x), y])(\beta(z)) = -\varepsilon(x + y, z)[\alpha^{-1}\beta^2(z), \alpha^{s+1}\beta^{l-1}([\beta(x), y])] = -\varepsilon(x + y, z)[\alpha^{-1}\beta^2(z), [\alpha^{s+1}\beta^l(x), \alpha^{s+1}\beta^{l-1}(y)]]$$

On the other hand, we have

$$\left(\frac{ad_{s,t}(\alpha)(x) \circ ad_{s,t}(y)}{ad_{s,t}(\alpha)(y) \circ ad_{s,t}(x)}\right)(z) = ad_{s,t}([\alpha\beta](x))(y) = \varepsilon(x, y)[\alpha^{-1}\beta(\alpha^{-1}\beta(\varepsilon)), \alpha^{s+2}\beta^{l-1}(y)]) - \varepsilon(x, y)[\alpha^{-1}\beta(\alpha^{-1}\beta(y)), \alpha^{s+2}\beta^{l-1}(x)]$$

$$= -\varepsilon(x + y, z)[\beta^{l-1}(\alpha^{-2}\beta(y)), \alpha^{s+2}\beta(\varepsilon)] - \varepsilon(x + y, z)[\beta^{l-1}(\alpha^{-2}\beta(z), \alpha^{s+1}\beta^{l-1}(x)), \alpha^{s+2}\beta^l(\alpha^{-1}(y))]$$

$$= \varepsilon(x, y)[\beta^{l-1}(\alpha^{-2}\beta(y), \alpha^{s+1}\beta^{l-1}(y))) + \varepsilon(x, y + z)[\beta^{l-1}(\alpha^{-2}\beta(y, \alpha^{s+1}\beta^{l-1}(x)]$$

$$= -\varepsilon(x + y, z)[\beta^{l-1}(\alpha^{-2}\beta(y), \alpha^{s+1}\beta^{l-1}(y))) + \varepsilon(x, y + z)[\beta^{l-1}(\alpha^{-2}\beta(\alpha^{-1}(y)), \alpha^{s+1}\beta^{l-1}(x)]$$

$$= -\varepsilon(x + y, z)[\beta^{l-1}(\alpha^{-2}\beta(y), \alpha^{s+1}\beta^{l-1}(y))) + \varepsilon(x, y + z)[\beta^{l-1}(\alpha^{-2}\beta(\alpha^{-1}(y)), \alpha^{s+1}\beta^{l-1}(x)]$$

Thus, the definition of $\alpha^s\beta^l$-adjoint representation is well defined. The proof is completed. 

The set of $n$-cochains on $A$ with coefficients in $\mathcal{A}$, which we denote by $C^n_{\alpha,\beta}(\mathcal{A}, \mathcal{A})$, is given by

$$C^n_{\alpha,\beta}(\mathcal{A}, \mathcal{A}) = \{f \in C^n(\mathcal{A}, \mathcal{A}) : f \circ \alpha^{\otimes n} = \alpha \circ f, \ f \circ \beta^{\otimes n} = \beta \circ f\}.$$ 

In particular, the set of $0$-cochains is given by

$$C^0_{\alpha,\beta}(\mathcal{A}, \mathcal{A}) = \{x \in \mathcal{H}(\mathcal{A}) : \alpha(x) = x, \ \beta(x) = x\}.$$ 

Now, we aim to study cochain complexes associated to $\alpha^s\beta^l$-adjoint representations of a BiHom-Lie colour algebra $\mathcal{A}[\alpha, \beta, \varepsilon, \alpha, \beta]$. 

**Proposition 4.7.** Associated to the $\alpha^s\beta^l$-adjoint representation $ad_{s,t}$ of the BiHom-Lie colour algebra $\mathcal{A}[\alpha, \beta, \varepsilon, \alpha, \beta], \ D \in C^1_{\alpha,\beta}(\mathcal{A}, \mathcal{A})$ is 1-cocycle of degree $\gamma$ if and only if $D$ is an $\alpha^{s+2}\beta^{l-1}$-derivation of degree $\gamma$ (i.e. $D \in (Der_{\alpha^{s+2}\beta^{l-1}}(\mathcal{A})).$)

**Proof.** The conclusion follows directly from the definition of the coboundary $\delta$. $D$ is closed if and only if

$$\delta(D)(x, y) = -D([\alpha^{-1}\beta(x), y] + \varepsilon(\gamma, x)[\alpha^{s+1}\beta^l(x), y]) - \varepsilon(\gamma + x, y)[\alpha^{s+1}\beta^l(y), D(x)] = 0.$$ 

So

$$D([\alpha^{-1}\beta(x), y]) = [D(x), \alpha^{s+1}\beta^l(y)] + \varepsilon(\gamma, x)[\alpha^{s+1}\beta^l(x), D(y)]$$ 

which implies that $D$ is an $\alpha^{s+2}\beta^{l-1}$-derivation of $(\mathcal{A}[\alpha, \beta, \varepsilon, \alpha, \beta])$ of degree $\gamma$. 

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Proposition 4.8. Associated to the $\alpha^s\beta^l$-adjoint representation $ad_{\alpha^s\beta^l}$, we have

$$H^0(A, A) = \{ x \in H(A) : \alpha(x) = x, \beta(x) = x, \ [x, y] = 0 \}.$$  

$$H^1(A, A) = \frac{Der_{\alpha^{s+2}\beta^{-1}}(A)}{Inn_{\alpha^{s+2}\beta^{-1}}(A)}.$$  

Proof. For any 0-BiHom cochain $x \in C^0_{\alpha, \beta}(A, A)$, we have $d_{\alpha^s\beta^l}(x)(y) = -[\alpha^{s+1}\beta^{l-1}(y), x] = [\alpha^{-1}\beta(x), \alpha^{s+2}\beta^{l-2}(y)].$

Therefore, $x$ is a closed 0-BiHom-cochain if and only if

$$[\alpha^{-1}\beta(x), \alpha^{s+2}\beta^{l-2}(y)] = 0,$$

which is equivalent to

$$\alpha^{-s-2}\beta^{-l+2}\left([\alpha^{-1}\beta(x), \alpha^{s+2}\beta^{l-2}(y)]\right) = [x, y] = 0.$$  

Therefore, the set of 0-BiHom cocycle $Z^0(A, A)$ is given by

$$Z^0(A, A) = \{ x \in C^0_{\alpha, \beta}(A, A) : [x, y] = 0, \ \forall \ y \in H(A) \}.$$  

As, $B^0(A, A) = \{0\}$, we deduce that

$$H^0(A, A) = \{ x \in H(A) : \alpha(x) = x, \beta(x) = x, \ [x, y] = 0, \ \forall \ y \in H(A) \}.$$  

By Proposition 4.7 we have $Z^1(A, A) = Der_{\alpha^{s+2}\beta^{-1}}(A)$. Furthermore, it is obvious that any exact 1-BiHom-cochain is of the form $-[\alpha^{s+1}\beta^{l-1}(\cdot), x]$ for some $x \in C^0_{\alpha, \beta}(A, A)$. Therefore, we have $B^1(A, A) = Inn_{\alpha^{s+1}\beta^{-1}}(A)$, which implies that

$$H^1(A, A) = \frac{Der_{\alpha^{s+2}\beta^{-1}}(A)}{Inn_{\alpha^{s+2}\beta^{-1}}(A)}.$$

\[\square\]

4.3 The coadjoint representation $\widetilde{ad}$

In this subsection, we explore the dual representations and coadjoint representations of BiHom-Lie colour algebras. Let $(A, [\cdot, \cdot], \varepsilon, \alpha, \beta)$ be a BiHom-Lie colour algebra and $(V, \rho, \alpha V, \beta V)$ be a representation of $A$. Let $V^*$ be the dual vector space of $V$. We define a linear map $\tilde{\rho} : A \to End(V^*)$ by $\tilde{\rho}(x) = -^t\rho(x)$. Let $f \in V^*, x, y \in H(A)$ and $v \in V$. We compute the right hand side of the identity (7).

$$\tilde{\rho}(\alpha x) \circ \tilde{\rho}(y) - \varepsilon(x, y) \tilde{\rho}(\beta(y)) \circ \tilde{\rho}(\alpha x))(f)(v)$$

$$= \tilde{\rho}(\alpha x) \left( -\varepsilon(x, y)f \circ \rho(\alpha)(y) \right) - \varepsilon(x, y) \tilde{\rho}(\beta(y)) \left( -\varepsilon(x, f) \circ \rho(\alpha)(y) \right)$$

$$= -\varepsilon(x + y, f) \left( \rho(\alpha x) \circ \rho(\beta(y))(v) - \varepsilon(x, y)\rho(y) \circ \rho(\alpha x)(v) \right).$$

On the other hand, we set that the twisted map for $\tilde{\rho}$ is $\tilde{\beta} = -^t\beta$, the left hand side of (7) writes

$$\tilde{\rho}([\beta x], y) \circ \tilde{\beta}_M(f)(v) = \tilde{\rho}([\beta x], y)(f \circ \beta)(v)$$

$$= -\varepsilon(x + y, f) \circ \beta([\beta x], y)(v).$$

Therefore, we have the following Proposition:
Proposition 4.9. Let \((A, [., .], \varepsilon, \alpha)\) be a BiHom-Lie colour algebra and \((M, \rho, \beta)\) be a representation of \(A\). Let \(M^*\) be the dual vector space of \(V\). The triple \((V^*, \tilde{\rho}, \tilde{\beta})\), where \(\tilde{\rho} : A \rightarrow \text{End}(V^*)\) is given by \(\tilde{\rho}(x) = -\rho(x)\), defines a representation of BiHom-Lie colour algebra \((A, [., .], \varepsilon, \alpha)\) if and only if
\[
\beta \circ \rho([\beta(x), y]) = \rho(\alpha(x)) \circ \rho(\beta(y)) - \varepsilon(x, y)\rho(y) \circ \rho(\alpha(x)).
\] (31)

We obtain the following characterization in the case of adjoint representation.

Corollary 4.10. Let \((A, [., .], \varepsilon, \alpha)\) be a BiHom-Lie colour algebra and \((A, \text{ad}, \alpha)\) be the adjoint representation of \(A\), where \(\text{ad} : A \rightarrow \text{End}(A)\). We set \(\text{ad} : A \rightarrow \text{End}(A^*)\) and \(\text{ad}(x)(f) = -f \circ \text{ad}(x)\). Then \((A^*, \tilde{\text{ad}}, \tilde{\alpha})\) is a representation of \(A\) if and only if
\[
\alpha \circ \text{ad}([x, y]) = \text{ad}(x) \circ \text{ad}(\alpha(y)) - \varepsilon(x, y)\text{ad}(y) \circ \text{ad}(\alpha(x)), \quad \forall \ x, y \in \mathcal{H}(A).
\]

5 Generalized \(\alpha^k\beta^l\)-Derivations of BiHom-Lie colour algebras

The purpose of this section is to study the homogeneous generalized \(\alpha^k\beta^l\)-derivations and homogeneous \(\alpha^k\beta^l\)-centroid of BiHom-Lie colour algebras, as well as \(\alpha^k\beta^l\)-quasi-derivations and \(\alpha^k\beta^l\)-quasi-centroid. The homogeneous generalized derivations were discussed first in \([6]\).

Let \((A, [., .], \varepsilon, \alpha, \beta)\) be a BiHom-Lie colour algebra. We set \(P\gamma(A) = \{D \in \text{End}(A) : D(A_{\gamma}) \subset A_{\gamma+\gamma} \text{ for all } \gamma \in \Gamma\}\).

It turns out that \((P\gamma(A) = \bigoplus_{\gamma \in \Gamma} P\gamma(A), [., .], \alpha, \beta)\) is a BiHom-Lie colour algebra with the Lie colour bracket
\[
[D, \gamma] = \gamma \circ D - \varepsilon(\gamma, \tau)D \circ \gamma
\]
for all \(D, \gamma, \tau \in \mathcal{H}(P\gamma(A))\) and with \(\alpha : A \rightarrow A\) is an even homomorphism.

A homogeneous \(\alpha^k\beta^l\)-derivation of degree \(\gamma\) of \(A\) is an endomorphism \(D \in P\gamma(A)\) such that
\[
[D, \alpha] = 0, \quad [D, \beta] = 0,
\]
\[
D([x, y]) = [D(x), \alpha^k\beta^l(y)] + \varepsilon(\gamma, x)\alpha^k\beta^l(x), D(y)], \quad \forall x, y \in \mathcal{H}(A).
\]

We denote the set of all homogeneous \(\alpha^k\beta^l\)-derivations of degree \(\gamma\) of \(A\) by \(\text{Der}_{\gamma^k\beta^l}(A)\). We set
\[
\text{Der}_{\gamma^k\beta^l}(A) = \bigoplus_{\gamma \in \Gamma} \text{Der}_{\gamma^k\beta^l}(A), \quad \text{Der}(A) = \bigoplus_{k, l \geq 0} \text{Der}_{\gamma^k\beta^l}(A).
\]

The set \(\text{Der}(A)\) provided with the colour-commutator is a Lie colour algebra. Indeed, the fact that \(\text{Der}_{\gamma^k\beta^l}(A)\) is \(\Gamma\)-graded implies that \(\text{Der}(A)\) is \(\Gamma\)-graded
\[
(\text{Der}(A))_{\gamma} = \bigoplus_{k, l \geq 0} (\text{Der}_{\gamma^k\beta^l}(A))_{\gamma}, \quad \forall \ \gamma \in \Gamma.
\]

Definition 5.1.

1. A linear mapping \(D \in \text{End}(A)\) is said to be an \(\alpha^k\beta^l\)-generalized derivation of degree \(\gamma\) of \(A\) if there exist linear mappings \(D', D'' \in \text{End}(A)\) of degree \(\gamma\) such that
\[
[D, \alpha] = 0, \quad [D', \alpha] = 0, \quad [D'', \alpha] = 0, \quad [D, \beta] = 0, \quad [D', \beta] = 0, \quad [D'', \beta] = 0,
\]
\[
D'([x, y]) = [D(x), \alpha^k\beta^l(y)] + \varepsilon(\gamma, x)\alpha^k\beta^l(x), D'(y)], \quad \forall x, y \in \mathcal{H}(A)
\]
2. A linear mapping \( D \in \text{End}(A) \) is said to be an \( \alpha^k\beta^l \)-quasi-derivation of degree \( \gamma \) of \( A \) if there exist linear mappings \( D' \in \text{End}(A) \) of degree \( \gamma \) such that

\[
[D, \alpha] = 0, \quad [D', \alpha] = 0, \quad [D, \beta] = 0, \quad [D', \beta] = 0,
\]

\[
D'([x, y]) = [D(x), \alpha^k\beta^l(y)] + \varepsilon(\gamma, x)[\alpha^k\beta^l(x), D(y)], \quad \forall \; x, y \in \mathcal{H}(A).
\]

The sets of generalized derivations and quasi-derivations will be denoted by \( G\text{Der}(A) \) and \( Q\text{Der}(A) \), respectively.

**Definition 5.2.**

1. The set \( C(A) \) consisting of linear mappings \( D \) with the property

\[
[D, \alpha] = 0,
\]

\[
D([x, y]) = [D(x), \alpha^k\beta^l(y)] = \varepsilon(\gamma, x)[\alpha^k\beta^l(x), D(y)], \quad \forall \; x, y \in \mathcal{H}(A).
\]

is called the \( \alpha^k\beta^l \)-centroid of degree \( \gamma \) of \( A \).

2. The set \( Q\text{C}(A) \) consisting of linear mappings \( D \) with the property

\[
[D, \alpha] = 0,
\]

\[
[D(x), \alpha^k\beta^l(y)] = \varepsilon(\gamma, x)[\alpha^k\beta^l(x), D(y)], \quad \forall \; x, y \in \mathcal{H}(A).
\]

is called the \( \alpha^k\beta^l \)-quasi-centroid of degree \( \gamma \) of \( A \).

**Proposition 5.3.** Let \( (A, [\cdot, \cdot], \varepsilon, \alpha, \beta) \) be a multiplicative BiHom-Lie colour algebra. Then

\[
[\text{Der}(A), C(A)] \subseteq C(A).
\]

**Proof.** Assume that \( D_\gamma \in \text{Der}_{\alpha^k\beta^l}(A), \quad D_\eta \in C_{\alpha^r\beta^s}(A) \). For arbitrary \( x, y \in \mathcal{H}(A) \), we have

\[
[D_\gamma D_\eta(x), \alpha^{k+s}\beta^{l+t}(y)] = D_\gamma([D_\eta(x), \alpha^s\beta^l(y)]) - \varepsilon(\gamma, \eta + x)[\alpha^k\beta^l(D_\eta(x)), D_\gamma(\alpha^s\beta^l(y))]
\]

\[
= D_\gamma([D_\eta(x), \alpha^s\beta^l(y)]) - \varepsilon(\gamma, \eta + x)[D_\eta(\alpha^k\beta^l(x)), D_\gamma(\alpha^s\beta^l(y))]
\]

\[
= D_\gamma D_\eta([x, y]) - \varepsilon(\gamma, \eta + x)[\alpha^{k+s}\beta^{l+t}(x), D_\eta D_\gamma(y)].
\]

and

\[
[D_\eta D_\gamma(x), \alpha^{k+s}\beta^{l+t}(y)] = D_\eta([D_\gamma(x), \alpha^k\beta^l(y)])
\]

\[
= D_\eta D_\gamma([x, y]) - \varepsilon(\gamma, x)[D_\eta(\alpha^k\beta^l(x)), D_\gamma(y)]
\]

\[
= D_\eta D_\gamma([x, y]) - \varepsilon(\gamma, x)D_\eta([\alpha^k\beta^l(x), D_\gamma(y)]).
\]

Now, let \( \Delta'_{\gamma'} \in C_{\alpha^r\beta^s}(A) \) then we have:

\[
\Delta'_{\gamma'}([x, y]) = \Delta'_{\gamma'}([D_\gamma(x), \alpha^k\beta^l(y)] + \varepsilon(\gamma, x)[\alpha^k\beta^l(x), D'_\gamma(y)])
\]

\[
= \Delta'_{\gamma'} D_\gamma(x), \alpha^{k+s}\beta^{l+t}(y)] + \varepsilon(\gamma + \gamma', x)[\alpha^{k+s}\beta^{l+t}(x), \Delta'_{\gamma'} D_\gamma(y)].
\]

Then \( \Delta'_{\gamma'} D_\gamma \in G\text{Der}_{\alpha^{k+s}\beta^{l+t}}(A) \) and is of degree \( (\gamma + \gamma') \).
Proposition 5.4. \( C(A) \subseteq QDer(A) \).

**Proof.** Let \( D_\gamma \in C_{\alpha^s \beta^t}(A) \) and \( x, y \in \mathcal{H}(A) \), then we have
\[
[D_\gamma(x), \alpha^k \beta^l(y)] + \varepsilon(\gamma, x)[\alpha^k \beta^l(x), D_\gamma(y)] = [D_\gamma(x), \alpha^k \beta^l(y)] + [D_\gamma(x), \alpha^k \beta^l(y)] \\
= 2D_\gamma([x, y]) \\
= D_\gamma([x, y]).
\]
Then \( D_\gamma \in QDer_{\alpha^k \beta^l}(A). \)

**Proposition 5.5.** \( QC(A), QC(A) \subseteq QDer(A) \).

**Proof.** Assume that \( D_\gamma \in \mathcal{H}(QC_{\alpha^s \beta^t}(A)) \) and \( D_\tau \in \mathcal{H}(QC_{\alpha^s \beta^t}(A)) \). Then for all \( x, y \in \mathcal{H}(A) \), we have
\[
[D_\gamma(x), \alpha^k \beta^l(y)] = \varepsilon(\gamma, x)[\alpha^k \beta^l(x), D_\gamma(y)]
\]
and
\[
[D_\tau(x), \alpha^s \beta^t(y)] = \varepsilon(\tau, x)[\alpha^s \beta^t(x), D_\tau(y)].
\]
Hence, on the other hand, we have
\[
[[D_\gamma, D_\tau](x), \alpha^{k+s} \beta^{l+t}(y)] = [(D_\gamma \circ D_\tau - \varepsilon(\gamma, \tau)D_\gamma \circ D_\tau)(x), \alpha^{k+s} \beta^{l+t}(y)] \\
= [D_\gamma \circ D_\tau(x), \alpha^{k+s} \beta^{l+t}(y)] - \varepsilon(\gamma, \tau)[D_\gamma \circ D_\tau(x), \alpha^{k+s} \beta^{l+t}(y)] \\
= \varepsilon(\gamma + \tau, x)[\alpha^{k+s} \beta^{l+t}(x), D_\gamma \circ D_\tau(y)] \\
- \varepsilon(\gamma, \tau)[\varepsilon(\gamma + \tau, x)[\alpha^{k+s} \beta^{l+t}(x), D_\gamma \circ D_\tau(y)] \\
= \varepsilon(\gamma + \tau, x)[\alpha^{k+s} \beta^{l+t}(x), [D_\gamma, D_\tau](y)] + [[D_\gamma, D_\tau](x), \alpha^{k+s} \beta^{l+t}(y)],
\]
which implies that \( [[D_\gamma, D_\tau](x), \alpha^{k+s} \beta^{l+t}(y)] + [[D_\gamma, D_\tau](x), \alpha^{k+s} \beta^{l+t}(y)] = 0. \)
Then \( [D_\gamma, D_\tau] \in GDer_{\alpha^{k+s} \beta^{l+t}}(A) \) and is of degree \((\gamma + \tau)\).

5.1 BiHom-Jordan colour algebras and Derivations

We show in the following that \( \alpha^{-1} \beta^2 \)-derivations of BiHom-Lie colour algebras give rise to BiHom-Jordan colour algebras. First we introduce a definition of colour BiHom-Jordan algebra.

**Definition 5.6.** A colour BiHom-algebra \((A, \mu, \varepsilon, \alpha, \beta)\) is a BiHom-Jordan colour algebra if hold the identities
\[
\alpha \circ \beta = \beta \circ \alpha, \quad (32)
\]
\[
\mu(\beta(x), \alpha(y)) = \varepsilon(x, y)\mu(\beta(y), \alpha(x)), \quad (33)
\]
\[
\circ_{x,y,w} \varepsilon(w, x + z)a_{\alpha,\beta}\left(\mu(\beta^2(x), \alpha(\beta(y))), \alpha^2 \beta(z), \alpha^3(w)\right) = 0. \quad (34)
\]
for all \( x, y, z \) and \( w \) in \( \mathcal{H}(A) \) and where \( a_{\alpha,\beta} \) is the BiHom-associator defined in \( \ref{eq:31} \).
The identity \( \ref{eq:34} \) is called BiHom-Jordan colour identity.

Observe that when \( \beta = \alpha \), the BiHom-Jordan colour identity reduces to the Hom-Jordan colour identity.

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Proposition 5.7. Let \( (A,[.,.],\varepsilon,\alpha,\beta) \) be a multiplicative BiHom-Lie colour algebra. Consider the operation \( D_1 \circ D_2 = D_1 \circ D_2 + \varepsilon(d_1, d_2) \cdot D_2 \circ D_1 \) for all \( \alpha^{-1} \beta^2 \)-derivations \( D_1, D_2 \in \mathcal{H}(Pl(A)) \). Then the 5-tuple \( (Pl(A), \bullet, \varepsilon, \alpha, \beta) \) is a BiHom-Jordan colour algebra.

Proof. Assume that \( D_1, D_2, D_3, D_4 \in \mathcal{H}(Pl(A)) \), we have

\[
\beta(D_1) \bullet \alpha(D_2) = \beta(D_1) \circ \alpha(D_2) + \varepsilon(d_1, d_2) \beta(D_2) \circ \alpha(D_1)
\]

\[
= \varepsilon(d_1, d_2)(\beta(D_2) \circ \alpha(D_1) + \varepsilon(d_2, d_1)) \beta(D_1) \circ \alpha(D_2)
\]

Since

\[
\left((\beta^2(D_1) \bullet \alpha \beta(D_2)) \bullet \alpha^2 \beta(D_3)\right) \bullet \beta \alpha^3(D_4)
\]

\[
= \left((\beta^2(D_1) \alpha \beta(D_2)) \alpha^2 \beta(D_3)\right) \beta \alpha^3(D_4) + \varepsilon(d_1 + d_2 + d_3, d_4) \beta \alpha^3(D_4)
\]

\[
+ \varepsilon(d_1 + d_2, d_3) \left(\alpha^2 \beta(D_3)(\beta^2(D_1) \alpha \beta(D_2))\right) \beta \alpha^3(D_4)
\]

\[
+ \varepsilon(d_1 + d_2, d_3) \varepsilon(d_1 + d_2 + d_3, d_4) \beta \alpha^3(D_4) \left(\alpha^2 \beta(D_3)(\beta^2(D_1) \alpha \beta(D_2))\right)
\]

\[
+ \varepsilon(d_1, d_2) \varepsilon(d_1 + d_2 + d_3, d_4) \beta \alpha^3(D_4) \left(\alpha^2 \beta(D_3)(\alpha \beta(D_3) \beta^2(D_1))\right) \beta \alpha^3(D_4)
\]

\[
+ \varepsilon(d_1, d_2) \varepsilon(d_1 + d_2, d_3) \varepsilon(d_1 + d_2 + d_3, d_4) \beta \alpha^3(D_4) \left(\alpha^2 \beta(D_3)(\alpha \beta(D_2) \beta^2(D_1))\right)
\]

and

\[
\alpha(\beta^2(D_1) \bullet \alpha \beta(D_2)) \bullet (\beta^2(D_3) \bullet \alpha^3(D_4))
\]

\[
= (\alpha \beta^2(D_1) \alpha^2 \beta(D_2))(\alpha \beta^2(D_3) \alpha^3(D_4)) + \varepsilon(d_1 + d_2 + d_3 + d_4) \alpha \beta^2(D_3) \alpha^3(D_4)
\]

\[
+ \varepsilon(d_1, d_2) \alpha^2 \beta(D_3)(\alpha \beta^2(D_1) \alpha \beta^2(D_2))
\]

\[
+ \varepsilon(d_3, d_4) \varepsilon(d_1 + d_2 + d_3 + d_4) \alpha^2 \beta(D_3)(\alpha \beta^2(D_1) \alpha \beta^2(D_2))
\]

\[
+ \varepsilon(d_1, d_2) \varepsilon(d_1 + d_2 + d_3 + d_4) \alpha^2 \beta(D_3)(\alpha \beta^2(D_1) \alpha \beta^2(D_2))
\]

Then we have

\[
\varepsilon(d_1, d_2 + d_3) \alpha \beta^3(D_1) \bullet \alpha \beta(D_2), \alpha \beta(D_3), \alpha^2 \beta(D_4)
\]

\[
= \varepsilon(d_1, d_2 + d_3) \varepsilon(d_1 + d_2 + d_3, d_4) \alpha \beta^3(D_1) \bullet \alpha \beta(D_2), \alpha \beta(D_3), \alpha^2 \beta(D_4)
\]

\[
+ \varepsilon(d_1, d_2) \alpha \beta^3(D_1) \bullet \alpha \beta(D_2), \alpha \beta(D_3), \alpha^2 \beta(D_4)
\]

\[
+ \varepsilon(d_3, d_4) \varepsilon(d_1 + d_2 + d_3 + d_4) \alpha \beta^3(D_1) \bullet \alpha \beta(D_2), \alpha \beta(D_3), \alpha^2 \beta(D_4)
\]

\[
+ \varepsilon(d_1, d_2) \varepsilon(d_1 + d_2 + d_3 + d_4) \alpha \beta^3(D_1) \bullet \alpha \beta(D_2), \alpha \beta(D_3), \alpha^2 \beta(D_4)
\]

\[
- \varepsilon(d_2, d_4) \beta \alpha^3(D_4) \left(\beta^2(D_1) \alpha \beta(D_2) \alpha \beta^2(D_1)\right)
\]

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and so the statement holds.

Therefore, we get

$$\varepsilon(d_1, d_1 + d_3)\varepsilon(d_1 + d_2, d_3)\left(\alpha^2\beta(D_3)\beta^2(D_1)\alpha\beta(D_2)\right)\beta\alpha^3(D_4)$$

$$-\varepsilon(d_1, d_2)\varepsilon(d_2, d_4)\varepsilon(D_4)\left((\alpha\beta(D_2)\beta^2(D_1))\alpha^2\beta(D_3)\right)$$

$$-\varepsilon(d_2, d_4)\varepsilon(d_4, d_3)\varepsilon(D_3)\left((\alpha\beta(D_3)\beta^2(D_1))\alpha^2\beta(D_2)\right)\beta\alpha^3(D_4).$$

Therefore, we get

$$\bigcirc_{D_1, D_2, D_3} \varepsilon(d_1, d_1 + d_3)\varepsilon(d_1 + d_2, d_3)\varepsilon(d_1 + d_2, d_3)\left(\beta^2(D_1) \circ \alpha\beta(D_2), \alpha\beta(D_3), \alpha^3(D_4)\right) = 0,$$

and so the statement holds. □

**Corollary 5.8.** Let $$(\mathcal{A}, [\cdot, \cdot], \varepsilon, \alpha, \beta)$$ be a multiplicative BiHom-Lie colour algebra. Consider the operation

$$D_1 \bullet D_2 = D_1 \circ D_2 - \varepsilon(d_1, d_2) D_2 \circ D_1$$

for all $D_1, D_2 \in \mathcal{H}(QC(\mathcal{A}))$. Then the 5-tuple $$(QC(\mathcal{A}), \bullet, \varepsilon, \alpha, \beta)$$ is a BiHom-Jordan colour algebra.

**Proof.** We need only to show that $D_1 \bullet D_2 \in QC(\mathcal{A})$, for all $D_1, D_2 \in \mathcal{H}(QC(\mathcal{A}))$.

Assume that $x, y \in \mathcal{H}(\mathcal{A})$, we have

$$[D_1 \bullet D_2(x), \alpha^{k+s} \beta^{l+t}(y)]$$

$$= \varepsilon(d_1, d_2)[D_2(x), \alpha^{k+s} \beta^{l+t}(y)]$$

$$+ \varepsilon(d_1, d_2)[D_1(x), \alpha^{k+s} \beta^{l+t}(y)]$$

$$-\varepsilon(d_1, d_2)[D_1(x), D_2(x), \alpha^{k+s} \beta^{l+t}(y)]$$

$$= \varepsilon(d_1, d_2)[D_1(x), \alpha^{k+s} \beta^{l+t}(x), D_2(y)]$$

$$+ \varepsilon(d_1, d_2)[D_2(x), \alpha^{k+s} \beta^{l+t}(x), D_1(y)]$$

$$-\varepsilon(d_1, d_2)[D_1(x), D_2(x), \alpha^{k+s} \beta^{l+t}(x), D_1(y)]$$

Hence $D_1 \bullet D_2 \in QC(\mathcal{A})$. □

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