Approximate Counting for Complex-Weighted Boolean Constraint Satisfaction Problems

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Abstract: Constraint satisfaction problems (or CSPs) have been extensively studied in, for instance, artificial intelligence, database theory, graph theory, and statistical physics. From a practical viewpoint, it is beneficial to approximately solve CSPs. When one tries to approximate the total number of truth assignments that satisfy all Boolean constraints for (unweighted) Boolean CSPs, there is a known trichotomy theorem; namely, all such counting problems are neatly classified into three categories under polynomial-time approximation-preserving reductions. In contrast, we obtain a dichotomy theorem of approximate counting for complex-weighted Boolean CSPs, provided that all complex-valued unary constraints are freely available to use. The expressive power of those unary constraints enables us to prove a stronger, complete classification theorem. This makes a step forward in the quest for the approximation-complexity classification of all counting CSPs. To deal with complex weights, we employ proof techniques of factorization and arity reduction along the line of solving Holant problems. We introduce a novel notion of T-constructibility that naturally induces approximation-preserving reducibility. Our result also gives an approximation analogue of the dichotomy theorem on the complexity of exact counting for complex-weighted Boolean CSPs.

Keywords: constraint satisfaction problem, constraint, T-constructibility, Holant problem, signature, approximation-preserving reduction, dichotomy theorem

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1 Background, New Challenges, and Achievement

Constraint satisfaction problems (or CSPs) have appeared in many different contexts, such as graph theory, database theory, type inferences, scheduling, and notably artificial intelligence, from which the notion of CSPs was originated. The importance of CSPs comes partly from the fact that the framework of the CSP is broad enough to capture numerous natural problems that arise in real applications. Generally, an input instance of a CSP is a set of “variables” (over a specified domain) and a set of “constraints” (such a set of constraints is sometimes called a constraint language) among these variables. We are particularly interested in the case of Boolean variables throughout this paper.

As a decision problem, a CSP asks whether there exists an appropriate variable assignment, which satisfies all the given constraints. In particular, Boolean constraints can be expressed by Boolean functions or equivalently propositional logical formulas; hence, the CSPs with Boolean constraints are all NP problems. Typical examples of CSP are the satisfiability problem (or SAT), the vertex cover problem, and the colorability problem, all of which are known to be NP-complete. On the contrary, other CSPs, such as the perfect matching problem on planar graphs, fall into P. One naturally asks what kind of constraints make them solvable efficiently or NP-complete. To be more precise, we first restrict our attention on CSP instances that depend only on constraints chosen from a given set $\mathcal{F}$ of constraints. Such a restricted CSP is conventionally denoted CSP($\mathcal{F}$). A classic dichotomy theorem of Schaefer [15] states that if $\mathcal{F}$ is included in one of six clearly specified classes, CSP($\mathcal{F}$) belongs to P; otherwise, it is indeed NP-complete. To see the significance of this theorem, let us recall a result of Ladner [17], who demonstrated under the P $\neq$ NP assumption that all NP problems fill infinitely many disjoint categories located between the class P and the class of NP-complete problems. Schaefer’s claim implies that there are no intermediate categories for Boolean CSPs.

Another challenging question is to count the number of satisfying assignments for a given CSP instance. The counting satisfiability problem, #SAT, is a typical counting CSP (or succinctly, #CSP), which is known
to be complete for Valiant’s counting class \#P \cite{Valiant}. Restricted to a set \( \mathcal{F} \) of Boolean constraints, Creignou and Hermann \cite{CreignouHermann} gave a dichotomy theorem, concerning the computational complexity of the restricted counting problem \#CSP(\( \mathcal{F} \)).

If all constraints in \( \mathcal{F} \) are affine\(^3\) then \#CSP(\( \mathcal{F} \)) is solvable in polynomial time. Otherwise, \#CSP(\( \mathcal{F} \)) is \#P-complete.

In real applications, constraints often take real numbers, and this fact leads us to concentrate on “weighted” \#CSPs (namely, \#CSPs with arbitrary-valued constraints). In this direction, Dyer, Goldberg, and Jerrum \cite{DyerGoldbergJerrum} extended the above result to nonnegative-weighted Boolean \#CSPs. Eventually, Cai, Lu, and Xia \cite{CaiLuXia} further pushed the scope of Boolean \#CSPs to complex-weighted Boolean \#CSPs, and thus all Boolean \#CSPs have been completely classified.

However, when we turn our attention from exact counting to (randomized) approximate counting, a situation looks quite different. Instead of the aforementioned dichotomy theorems, Dyer, Goldberg, and Jerrum \cite{DyerGoldbergJerrum} presented a trichotomy theorem for the complexity of approximately counting the number of satisfying assignments for each Boolean CSP instance. What they actually proved is that, depending on the choice of a set \( \mathcal{F} \) of Boolean constraints, the complexity of approximately solving \#CSP(\( \mathcal{F} \)) can be classified into three categories.

If all constraints in \( \mathcal{F} \) are affine, then \#CSP(\( \mathcal{F} \)) is polynomial-time solvable. Otherwise, if all constraints in \( \mathcal{F} \) are in a well-defined class, known as \( IM_2 \), then \#CSP(\( \mathcal{F} \)) is equivalent in complexity to \#BIS. Otherwise, \#CSP(\( \mathcal{F} \)) is equivalent to \#SAT. The equivalence is defined via polynomial-time approximation-preserving reductions (or AP-reductions, in short).

Here, \#BIS is the problem of counting the number of independent sets in a given bipartite graph.

There still remains a nagging question on the approximation complexity of a “weighted” version of Boolean \#CSPs: what happens if we expand the scope of Boolean \#CSPs from unweighted ones to complex-weighted ones? Unfortunately, there have been few results showing the hardness of approximately solving \#CSPs with real/complex-valued constraints, except for, e.g., \cite{DyerGoldbergJerrum}. Unlike unweighted constraints, when we deal with complex-valued constraints, a significant complication occurs as a result of massive cancellations of weights in the process of summing all weights given by constraints. This situation demands a quite different approach toward the complex-weighted \#CSPs. Do we still have a classification theorem similar to the theorem of Dyer et al. or something quite different? In this paper, we answer this question under a reasonable assumption that all unary (i.e., arity 1) constraints are freely available to use. Meanwhile, let the notation \#CSP\( ^* (\mathcal{F}) \) denote the complex-weighted counting problem \#CSP(\( \mathcal{F} \)) that satisfies this extra assumption. A free use of unary constraints appeared in the past literature for Holant problems \cite{GeffnerHaywardXia} \cite{GeffnerHaywardXia}. Even in case of bounded-degree Boolean \#CSPs, Dyer et al. \cite{DyerGoldbergJerrum} assumed free unary Boolean-valued constraints. Although it is reasonable, this extra assumption makes the approximation complexity of \#CSP\( ^* (\mathcal{F}) \) look quite different from the approximation complexity of \#CSP(\( \mathcal{F} \)), except for the case of Boolean-valued constraints. If we restrict our interest on Boolean constraints, then the only nontrivial unary constraints are \( \Delta_0 \) and \( \Delta_1 \) (which are called “constant constraints” and will be explained in Section 2) and thus, as shown in \cite{DyerGoldbergJerrum}, we can completely eliminate them from the definition of \#CSP\( ^* (\mathcal{F}) \) using polynomial-time randomized approximation algorithms. The elimination of those constant constraints is also possible in our general setting of complex-weighted \#CSPs when all values are limited to algebraic complex numbers.

For the approximation complexity of \#CSP\( ^* (\mathcal{F}) \)'s, the expressive power of unary complex-valued constraints leads us to a dichotomy theorem—Theorem 1.1—which depicts a picture that looks quite different from that of Dyer et al.

**Theorem 1.1** Let \( \mathcal{F} \) be any set of complex-valued constraints. If \( \mathcal{F} \subseteq \mathcal{E} \mathcal{D} \), then \#CSP\( ^* (\mathcal{F}) \) is solvable in polynomial time. Otherwise, \#CSP\( ^* (\mathcal{F}) \) is as hard as \#SAT\_\( \mathcal{C} \) under AP-reductions.

Here, the counting problem \#SAT\_\( \mathcal{C} \) is a complex-weighted analogue of \#SAT and the set \( \mathcal{E} \mathcal{D} \) is composed of products of the equality/disequality constraints (which will be explained in Section 5) together with unary constraints.

This theorem marks a significant progress in the quest for determining the approximation complexity of all counting problems \#CSP(\( \mathcal{F} \)) in the most general form. Our proof heavily relies on the previous work of Dyer et al. \cite{DyerGoldbergJerrum} \cite{DyerGoldbergJerrum} and, particularly, the work of Cai et al. \cite{CaiLuXia} \cite{CaiLuXia}, which is based on a theory of signature (see, e.g., \cite{Hladik} \cite{Hladik}) that formulate underlying concepts of holographic algorithms (which are Valiant’s \cite{Valiant} \cite{Valiant} \cite{Valiant} \cite{Valiant}).

\(^3\)An affine relation is a set of solutions of a certain set of linear equations over GF(2).
manifestation of a new algorithmic design method of solving seemingly-intractable counting problems in polynomial time). A challenging issue for this paper is that core arguments of Dyer et al. [15] exploited Boolean natures of Boolean-valued constraints and they are not designed to lead to a dichotomy theorem for complex-valued constraints. Cai’s theory of signature, on the contrary, deals with complex-valued constraints (which are formally called signatures); however, the theory has been developed over polynomial-time Turing reductions but it is not meant to be valid under AP-reductions. For instance, a useful technical tool known as polynomial interpolation, which is frequently used in an analysis of exact-counting of Holant problems, is no longer applicable in general. Therefore, our first task is to re-examine the well-known results in this theory and salvage its key arguments that are still valid for our AP-reductions. From that point on, we need to find our own way to establish an approximation theory.

Toward forming a solid approximation theory, a notable technical tool developed in this paper is a notion of T-constructibility in tandem with the aforementioned AP-reducibility. Earlier, Dyer et al. [14] utilized an existing notion of (faithful and perfect) implementation for Boolean-valued constraints in order to induce their desired AP-reductions. The T-constructibility similarly maintains the validity of the AP-reducibility; in addition, it is more suitable to handle complex-valued constraints in a more systematic fashion. Other proof techniques involved in proving our main theorem include (1) factorization (of Boolean parts) of complex-valued constraints and (2) arity reduction of constraints. Factoring complex-valued constraints helps us conduct crucial analyses on fundamental properties of those constraints, and reducing the arities of constraints helps construct, from constraints of higher arity, binary constraints, which we can handle directly by a case-by-case analysis. In addition, a particular binary constraint—Implies—plays a pivotal role in the proof of Theorem 1.1. This situation is quite different from [6, 7], which instead utilized the affine property.

To prove our dichotomy theorem, we will organize the subsequent sections in the following fashion. Section 2 gives the detailed descriptions of our key terminology: constraints, Holant problems, counting CSPs, and AP-reductions. In particular, an extension of the notion of randomized approximation scheme over non-negative integers to arbitrary complex numbers is described in Section 2. Briefly explained in Section 5 is the concept of T-constructibility, a technical tool developed exclusively in this paper. For readability, a basic property of T-constructibility is proven in Section 10. Section 3 introduces several crucial sets of constraints, which are bases of our key results. Toward our main theorem, we will develop solid foundations in Sections 4 and 5. Notably, a free use of “arbitrary” unary constraint is heavily required in Section 6 to prove approximation-complexity bounds of #CSP*(f). As an important ingredient of the proof of the dichotomy theorem, we will present in Section 5 the approximation hardness of #CSP*(f) for two types of constraints f. The dichotomy theorem is finally proven in Section 9 achieving the goal of this paper.

Given a constraint, if its outcomes are limited to algebraic complex numbers, we succinctly call the constraint an algebraic constraint. When all input instances are only algebraic constraints, as we noted earlier, we can further eliminate the constant constraints and thus strengthen the main theorem. To describe our next result, we introduce a special notation #CSP∗,κ(F) to indicate #CSP*(F) in which (i) all input instances are limited to algebraic constraints and (ii) free unary constraints take neither of the forms c · ∆0 nor c · ∆1 for any constant c. Similarly, #SATκ is induced from #SATC by limiting node-weights within algebraic complex numbers. The power of AP-reducibility leads us to establish the following corollary of the main theorem.

**Corollary 1.2** Let F be any set of complex-valued constraints. If F ⊆ ED, then #CSP∗,κ(F) is solvable in polynomial time; otherwise, #CSP∗,κ(F) is at least as hard as #SATκ under AP-reductions.

This corollary will be proven in Section 9. A key to the proof of the corollary is an AP-equivalence between #CSP∗,κ(F) and #CSP∗,κ(F) for any constraint set F, where the subscript “κ” in #CSP∗,κ(F) emphasizes the restriction on input instances within algebraic constraints. This AP-equivalence is a direct consequence of the elimination of ∆0 and ∆1 from #CSP∗,κ(F) and this elimination will be demonstrated in Section 10.

**Outline of the Proof of the Main Theorem:** The proof of the dichotomy theorem is outlined as follows. First, we will establish in Section 4 the equivalence between #SATC and #CSP*(OR), where OR represents the logical “or” on two Boolean variables. This makes it possible to work solely with #CSP*(OR), instead of #SATC in the subsequent sections. When a constraint set F is completely included in ED, we will show in Lemma 6.4 that #CSP*(F) is polynomial-time solvable. On the contrary, when F is not included in ED, we choose a constraint f not in ED. Such a constraint will be treated by Proposition 9.1 in which we will AP-reduce #CSP*(OR) to #CSP*(f). The proof of this proposition will be split into two cases, depending on whether or not f has “imp support,” which is a property associated with the constraint Implies. When f has such a property, Proposition 8.1 helps establish the hardness of #CSP*(f); namely, an AP-reduction
of \#CSP(\(f\)) from \#CSP(\(OR\)) and thus from \#SAT\(_C\). In contrast, if \(f\) lacks the property, then we will examine two subcases. If \(f\) is a non-zero constraint, then Lemma 7.5 together with Proposition 6.8 leads to the hardness of \#CSP(\(f\)). Otherwise, Proposition 7.7 establishes the desired AP-reduction. Therefore, the proof of the theorem is completed.

Now, we begin with an explanation of basic definitions.

## 2 Basic Definitions

This section briefly presents fundamental notions and notations, which will be used in later sections. For any finite set \(A\), the notation \(|A|\) denotes the cardinality of \(A\). A string over an alphabet \(\Sigma\) is a finite sequence of symbols from \(\Sigma\) and \(|x|\) denotes the length of a string \(x\), where an alphabet is a non-empty finite set of "symbols." Let \(\mathbb{N}\) denote the set of all natural numbers (i.e., non-negative integers). For convenience, \(\mathbb{N}^+\) denotes \(\mathbb{N} - \{0\}\). Moreover, \(\mathbb{R}\) and \(\mathbb{C}\) denote respectively the sets of all real numbers and of all complex numbers. Given a complex number \(\alpha\), let \(|\alpha|\) and \(\arg(\alpha)\) respectively denote the absolute value and the argument of \(\alpha\), where we always assume that \(-\pi < \arg(\alpha) \leq \pi\). The special notation \(A\) represents the set of all algebraic complex numbers. For each number \(n \in \mathbb{N}\), \([n]\) expresses the integer set \([1, 2, \ldots, n]\). The notation \(A^T\) for any matrix \(A\) indicates the transposed matrix of \(A\). We always treat "vectors" as row vectors, unless stated otherwise.

For any undirected graph \(G = (V, E)\) (where \(V\) is a node set and \(E\) is an edge set) and a node \(v \in V\), an incident set \(E(v)\) of \(v\) is the set of all edges incident on \(v\), and \(\text{deg}(v) = |E(v)|\) is the degree of \(v\). When we refer to nodes in a given undirected graph, unless there is any ambiguity, we call such nodes by their labels instead of their original node names. For example, if a node \(v\) has a label of Boolean variable \(x\), then we often call it "node \(x\)," although there are many other nodes labeled \(x\), as far as it is clear from the context which node is referred to. Moreover, when \(x\) is a Boolean variable, as in this example, we succinctly call any node labeled \(x\) a "variable node."

### 2.1 Constraints, Signatures, Holant Problems, and \#CSP

The most fundamental concept in this paper is "constraint" on the Boolean domain. A function \(f\) is called a (complex-valued) constraint of arity \(k\) if it is a function from \(\{0, 1\}^k\) to \(\mathbb{C}\). Assuming the standard lexicographic order on \(\{0, 1\}^k\), we express \(f\) as a series of its output values, which is identified with an element in the complex space \(\mathbb{C}^{2^k}\). For instance, if \(k = 1\), then \(f\) equals \((f(0), f(1))\), and if \(k = 2\), then \(f\) is expressed as \((f(00), f(01), f(10), f(11))\). A constraint \(f\) is symmetric if the values of \(f\) depend only on the Hamming weights of inputs; otherwise, \(f\) is called asymmetric. When \(f\) is a symmetric constraint of arity \(k\), we use another notation \(f = [f_0, f_1, \ldots, f_k]\), where each \(f_i\) is the value of \(f\) on inputs of Hamming weight \(i\). As a concrete example, when \(f\) is the equality function \((EQ_k)\) of arity \(k\), it is expressed as \([1, 0, \ldots, 0, 1]\) (including \(k-1\) zeros). We denote by \(\mathcal{U}\) the set of all unary constraints and we use the following special unary constraints: \(\Delta_0 = [1, 0]\) and \(\Delta_1 = [0, 1]\). These constraints are often referred to as "constant constraints."

Before introducing \#CSPs, we will give a brief description of Holant problem; however, we focus our attention only on "bipartite Holant problems" whose input instances are "signature grids" containing bipartite graphs \(G\), in which all nodes on the left-hand side of \(G\) are labeled by signatures in \(\mathcal{F}_1\) and all nodes on the right-hand side of \(G\) are labeled by signatures in \(\mathcal{F}_2\), where "signature" is another name for complex-valued constraint, and \(\mathcal{F}_1\) and \(\mathcal{F}_2\) are two sets of signatures. Formally, a bipartite Holant problem, denoted Holant(\(\mathcal{F}_1|\mathcal{F}_2\)), (on a Boolean domain) is a counting problem defined as follows. The problem takes an input instance, called a signature grid \(\Omega = (G, \mathcal{F}_1|\mathcal{F}_2, \pi)\), that consists of a finite undirected bipartite graph \(G = (V_1|V_2, E)\) (where all nodes in \(V_1\) appear on the left-hand side and all nodes in \(V_2\) appear on the right-hand side), two finite subsets \(\mathcal{F}_1 \subseteq \mathcal{F}_1\) and \(\mathcal{F}_2 \subseteq \mathcal{F}_2\), and a labeling function \(\pi: V_1 \cup V_2 \rightarrow \mathcal{F}_1 \cup \mathcal{F}_2\) such that \(\pi(V_1) \subseteq \mathcal{F}_1\), \(\pi(V_2) \subseteq \mathcal{F}_2\), and each node \(v \in V_1 \cup V_2\) is labeled \(\pi(v)\), which is a function mapping \([0, 1]^{\text{deg}(v)}\) to \(\mathbb{C}\). For convenience, we often write \(f_v\) for this \(\pi(v)\). Let \(\text{Asn}(E)\) denote the set of all edge assignments \(\sigma: E \rightarrow \{0, 1\}\). The bipartite Holant problem is meant to compute the complex value Holant\(\Omega\):

\[
\text{Holant}_\Omega = \sum_{\sigma \in \text{Asn}(E)} \prod_{v \in V_1 \cup V_2} f_v(\sigma|E(v)),
\]

where \(\sigma|E(v)\) denotes the binary string \((\sigma(w_1), \sigma(w_2), \cdots, \sigma(w_k))\) if \(E(v) = \{w_1, w_2, \ldots, w_k\}\), sorted in a certain pre-fixed order associated with \(f_v\).
Let us define complex-weighted Boolean \( \#CSP \) problems associated with a set \( F \) of constraints. Conventionally, a complex-weighted Boolean \( \#CSP \) problem, denoted \( \#CSP(F) \), takes a finite set \( \Omega \) of “elements” of the form \( (h, (x_1, x_2, \ldots, x_k)) \) on Boolean variables \( x_1, x_2, \ldots, x_n \), where \( h \in F \) and \( i_1, \ldots, i_k \in [n] \). The problem outputs the value \( \text{csp}_\Omega \):

\[
\text{csp}_\Omega = \sum_{(h, x') \in \Omega} \prod_{\sigma} h(\sigma(x_{i_1}), \sigma(x_{i_2}), \ldots, \sigma(x_{i_k})),
\]

where \( x' = (x_{i_1}, x_{i_2}, \ldots, x_{i_k}) \) and \( \sigma : \{x_1, x_2, \ldots, x_n\} \to \{0, 1\} \) ranges over the set of all variable assignments.

Exploiting a close resemblance to Holant problems, we intend to adopt the Holant framework and redefine \( \#CSP(F) \) in a form of “bipartite graphs” as follows: an input instance to \( \#CSP(F) \) is a triplet \( \Omega = (G, X | F, \pi) \), which we call a “constraint frame” (to distinguish it from the aforementioned conventional framework), where \( G \) is an undirected bipartite graph whose left-hand side contains nodes labeled by Boolean variables and the right-hand side contains nodes labeled by constraints in \( F \). Throughout this paper, we take this constraint-frame formalism to treat complex-weighted Boolean \( \#CSP \); that is, we always assume that an input instance to \( \#CSP(F) \) is a certain constraint frame \( \Omega \) and an output of \( \#CSP(F) \) is the value \( \text{csp}_\Omega \).

The above concept of constraint frame is actually inspired by the fact that \( \#CSP(F) \) can be viewed as a special case of bipartite Holant problem \( \text{Holant}(\{EQ_{k}\}_{k \geq 1} | F) \) by the following translation: any constraint frame \( \Omega \) given to \( \#CSP(F) \) is viewed as a signature grid \( \Omega' = (G, \{EQ_{k}\}_{k \geq 1} | F', \pi) \) in which each Boolean variable appearing in the original constraint frame \( \Omega \) corresponds to all edges incident on a node labeled \( EQ_{k} \) in \( G \), and thus each variable assignment for \( \Omega \) matches the corresponding 0-1 edge assignment for \( \Omega' \).

To improve readability, we often omit the set notation and express, e.g., \( \#CSP(f, g) \) and \( \#CSP(f, F, G) \) to mean \( \#CSP(\{f, g\}) \) and \( \#CSP(\{f\} \cup F \cup G) \), respectively. When we allow unary constraints to appear in any instance freely, we succinctly write \( \#CSP^{*}(F) \) instead of \( \#CSP(F, U) \). In the rest of this paper, we will target the counting problems \( \#CSP^{*}(F) \).

**Our Treatment of Complex Numbers.** Here, we need to address a technical issue concerning how to treat complex numbers as well as complex-valued functions. Recall that each input instance to a \( \#CSP \) involves a finite set of constraints, which are actually complex-valued functions. How can we compute or manipulate those functions? More importantly, how can we “express” them as part of input instances even before starting to compute their values?

The past literature has exhibited numerous ways to treat complex numbers in an existing framework of theory of string-based computation. There are several reasonable definitions of “polynomial-time computable” complex numbers. They vary depending on which viewpoint we take. To state our results independent of the definitions of computable complex numbers, however, we rather prefer to treat complex numbers as basic “objects.” Whenever complex numbers are given as part of input instances, we implicitly assume that we have a clear and concrete means of specifying those numbers within a standard framework of computation. Occasionally, however, we will limit our interest within a scope of algebraic numbers, as in Lemma 5.2.

To manipulate such complex numbers algorithmically, we are limited to perform only “primitive” operations, such as, multiplications, addition, division, etc., on the given numbers in a very plausible fashion. The execution time of an algorithm that handles those complex numbers is generally measured by the number of those primitive operations. To given complex numbers, we apply such primitive operations only; therefore, our assumption on the execution time of the operations causes no harm in a later discussion on the computability of \( \#CSP(F) \). (See [3,4] for further justification.)

By way of our treatment of complex numbers, we naturally define the function class \( FP_{C} \) as the set of all complex-valued functions that can be computed deterministically on input strings in time polynomial in the sizes of the inputs.

### 2.2 Randomized Approximation Schemes

We will lay out a notion of randomized approximation scheme, particularly, working on complex numbers. Let \( F \) be any counting function mapping from \( \Sigma^{*} \) (over an appropriate alphabet \( \Sigma \)) to \( \mathbb{C} \). Our goal is to approximate each value \( F(x) \) when \( x \) is given as an input instance to \( F \). A standard approximation theory (see, e.g., [1]) deals mostly with natural numbers; however, treating complex numbers in the subsequent sections requires an appropriate modification of the standard definition of computability and approximation. In what follows, we will make a specific form of complex-number approximation.
A fundamental idea behind “relative approximation error” is that a maximal ratio between an approximate solution \( w \) and a true solution \( F(x) \) should be close to 1. Intuitively, a complex number \( w \) is an “approximate solution” for \( F(x) \) if a performance ratio \( z = w/F(x) \) (as well as \( z = F(x)/w \)) is close enough to 1. In case when our interest is limited to “real-valued” functions, we can expand a standard notion of relative approximation of functions producing non-negative integers (e.g., [1]) and we demand 2-\( \varepsilon \)-approximating both the real parts and the imaginary parts of the complex numbers.

\[ 2^{-\varepsilon} \leq \left| \frac{w}{F(x)} \right| \leq 2^\varepsilon \quad \text{and} \quad \left| \arg \left( \frac{w}{F(x)} \right) \right| \leq \varepsilon, \]

provided that we apply the following exceptional rule: when \( F(x) = 0 \), we instead require \( w = 0 \). Notice that this way of approximating both the real parts and the imaginary parts of the complex numbers.

A randomized approximation scheme for (complex-valued) \( F \) is a randomized algorithm that takes a standard input \( x \in \Sigma^* \) together with an error tolerance parameter \( \varepsilon \in (0,1) \), and outputs a 2\( \varepsilon \)-approximate solution (which is a random variable) for \( F(x) \) with probability at least 3/4. A fully polynomial-time randomized approximation scheme (or simply, \( FPRAS \)) for \( F \) is a randomized approximation scheme for \( F \) that runs in time polynomial in \(|x|,1/\varepsilon\).

Next, we will describe our notion of approximation-preserving reducibility among counting problems. Of numerous existing notions of approximation-preserving reductions (see, e.g., [1]), we choose a notion introduced by Dyer et al. [12], which can be viewed as a randomized variant of Turing reducibility, described by a mechanism of oracle Turing machine. Given two counting functions \( F \) and \( G \), a polynomial-time (randomized) approximation-preserving (Turing) reduction (or AP-reduction, in short) from \( F \) to \( G \) is a randomized algorithm \( N \) that takes a pair \( (x,\varepsilon) \in \Sigma^* \times (0,1) \) as input, uses an arbitrary randomized approximation scheme (not necessarily polynomial time-bounded) \( M \) for \( G \) as oracle, and satisfies the following three conditions: (i) \( N \) is a randomized approximation scheme for \( F \); (ii) every oracle call made by \( N \) is of the form \( (w,\delta) \in \Sigma^* \times (0,1) \) satisfying \( 1/\delta \leq p(|x|,1/\varepsilon) \), where \( p \) is a certain absolute polynomial, and an oracle answer is an outcome of \( M \) on the input \( (w,\delta) \); and (iii) the running time of \( N \) is bounded from above by a certain polynomial in \(|x|,1/\varepsilon\), not depending on the choice of the oracle \( M \). In this case, we write \( F \leq_{AP} G \) and we also say that \( F \) is AP-reducible (or AP-reduced) to \( G \). If \( F \leq_{AP} G \) and \( G \leq_{AP} F \), then \( F \) and \( G \) are AP-equivalent\(^6\) and we write \( F \equiv_{AP} G \). The following lemma is straightforward.

\[ \text{Lemma 2.1} \quad \text{If } F \subseteq G, \text{ then } #\text{CSP}^*(F) \leq_{AP} #\text{CSP}^*(G). \]

3 Underlying Relations and Constraint Sets

A relation of arity \( k \) is a subset of \( \{0,1\}^k \). Such a relation can be viewed as a “function” mapping Boolean variables to \( \{0,1\} \) (by setting \( R(x) = 0 \) and \( R(x) = 1 \) whenever \( x \notin R \) and \( x \in R \), respectively, for every \( x \in \{0,1\}^k \)) and it can be treated as a Boolean constraint. For instance, logical relations \( OR, NAND, XOR, \) and \( Implies \) are all expressed as Boolean constraints in the following manner: \( OR = [0,1,1], NAND = [1,1,0], XOR = [0,1,0], \) and \( Implies = (1,1,0) \). The negation of \( XOR \) is \([1,0,1] \) and it is simply denoted \( EQ \) for convenience. Notice that \( EQ \) coincides with \( EQ_2 \).

For each \( k \)-ary constraint \( f \), its underlying relation is the relation \( R_f = \{ x \in \{0,1\}^k \mid f(x) \neq 0 \} \), which characterizes the non-zero part of \( f \). A relation \( R \) belongs to the set \( \text{IMP} \) (slightly different from \( IM \) in [13]) if it is logically equivalent to a conjunction of a certain “positive” number of relations of the form \( \Delta_0(x) \), \( \Delta_1(x) \), and \( Implies(x,y) \). It is worth mentioning that \( EQ_2 \in IM \) but \( EQ_1 \notin IM \). Moreover, the empty relation \( \emptyset \) also belongs to \( IM \).

The purpose of this paper is to extend the scope of the approximation complexity of \#CSPs from Boolean constraints of Dyer et al. [17], stated in Section 4 to complex-valued constraints. To simplify later descriptions, it is better for us to introduce the following six special sets of constraints, the first of which has been

\(^6\)This concept was called “AP-interreducible” by Dyer et al. [12] but we prefer this term, which is originated from “Turing equivalent” in computational complexity theory.
already introduced in Section 2.1. The notation \( f \equiv 0 \) below means that \( f(x_1, \ldots, x_k) = 0 \) for all \( k \) vectors \((x_1, \ldots, x_k) \in \{0,1\}^k\), where \( k \) is the arity of \( f \).

1. Denote by \( \mathcal{U} \) the set of all unary constraints.

2. Let \( \mathcal{NZ} \) be the set of all constraints \( f \) of arity \( k \geq 1 \) such that \( f(x_1, x_2, \ldots, x_k) \neq 0 \) for all \((x_1, x_2, \ldots, x_k) \in \{0,1\}^k\). We succinctly call such functions non-zero functions. Notice that this case is different from the case where \( f \equiv 0 \). Obviously, \( \Delta_0, \Delta_1 \not\in \mathcal{NZ} \).

3. Let \( \mathcal{DG} \) denote the set of all constraints \( f \) of arity \( k \geq 1 \) such that \( f(x_1, x_2, \ldots, x_k) = \prod_{i=1}^k g_i(x_i) \) for certain unary constraints \( g_1, g_2, \ldots, g_k \). A constraint in \( \mathcal{DG} \) is called degenerate. Obviously, \( \mathcal{DG} \) includes \( \mathcal{U} \) as a proper subset.

4. Define \( \mathcal{ED} \) to be the set of functions \( f \) of arity \( k \geq 1 \) such that \( f(x_1, x_2, \ldots, x_k) = \left( \prod_{i=1}^h h_i(x_{j_i}) \right) \left( \prod_{i=1}^m g_i(x_{m_i}, x_{n_i}) \right) \) with \( \ell_1 \leq \ell_2 \geq 0, \ell_1 + \ell_2 \geq 1 \), and \( 1 \leq j_1, m_1, n_1 \leq k \), where each \( h_i \) is a unary constraint and each \( g_i \) is either the binary equality \( EQ \) or the disequality \( XOR \). Clearly, \( \mathcal{DG} \subseteq \mathcal{ED} \) holds. The name “ED” refers to its key components, “equality” and “disequality.” See [8] for its basic property.

5. Let \( \mathcal{IM} \) be the set of all constraints \( f \not\in \mathcal{NZ} \) of arity \( k \geq 1 \) such that \( f(x_1, x_2, \ldots, x_k) = \left( \prod_{i=1}^h h_i(x_{j_i}) \right) \left( \prod_{i=1}^m f\text{implies}(x_{m_i}, x_{n_i}) \right) \) with \( \ell_0, \ell_1 \geq 0, \ell_1 + \ell_2 \geq 1 \), and \( 1 \leq j_1, m_1, n_1 \leq k \), where each \( h_i \) is a unary constraint.

We will present four simple properties of the above-mentioned sets of constraints. The first property concerns the set \( \mathcal{NZ} \) of non-zero constraints. Notice that non-zero constraints will play a quite essential role in Lemma 4.3 and Proposition 8.4. In what follows, we claim that two sets \( \mathcal{DG} \) and \( \mathcal{ED} \) coincide with each other, when they are particularly restricted to non-zero constraints.

**Lemma 3.1** Let \( f \) be any constraint of arity \( k \geq 1 \) in \( \mathcal{NZ} \). It holds that \( f \in \mathcal{DG} \) iff \( f \in \mathcal{ED} \).

**Proof.** Let \( f \) be any non-zero constraint of arity \( k \). Note that \( f \in \mathcal{NZ} \) iff \( |R_f| = 2^k \), where \( |R_f| \) is the cardinality of the set \( R_f \). Since \( \mathcal{DG} \subseteq \mathcal{ED} \), it is enough to show that \( f \in \mathcal{ED} \) implies \( f \in \mathcal{DG} \). Assume that \( f \) is in \( \mathcal{ED} \). Since \( f \) is a product of certain constraints of the forms: \( EQ, XOR, \) and unary constraints. Since \( |R_f| = 2^k \), \( f \) cannot be made of \( EQ \) as well as \( XOR \) as its “factors,” and thus it should be of the form \( \prod_{i=1}^k U_i(x_i) \), where each \( U_i \) is a non-zero unary constraint. We therefore conclude that \( f \) is degenerate and it belongs to \( \mathcal{DG} \).

**Lemma 3.2** Let \( f \) be any constraint. If \( f \not\in \mathcal{DG} \), then it holds that \( |R_f| \geq 2 \).

**Proof.** We prove the lemma by contrapositive. Take a constraint \( f(x_1, x_2, \ldots, x_k) \) of arity \( k \geq 1 \) and assume that \( |R_f| \leq 1 \). When \( |R_f| = 0 \), since \( f \)’s output is always zero, \( f(x_1, \ldots, x_k) \) can be expressed as, for instance, \( \Delta_0(x_1) \Delta_1(x_1) \). On the contrary, when \( |R_f| = 1 \), we assume that \( R_f = \{(a_1, a_2, \ldots, a_k)\} \) for a certain vector \((a_1, a_2, \ldots, a_k) \in \{0,1\}^k\). Let us define \( b \) as \( b = f(a_1, a_2, \ldots, a_k) \). It is not difficult to show that \( f(x_1, x_2, \ldots, x_k) \) equals \( b \cdot \prod_{i=1}^k \Delta_{a_i}(x_i) \). Thus, \( f \) belongs to \( \mathcal{DG} \), as required.

Several sets in the aforementioned list satisfy the closure property under multiplication. For any two constraints \( f \) and \( g \) of arities \( c \) and \( d \), respectively, the notation \( f \cdot g \) denotes the function defined as follows. For any Boolean vector \((x_1, \ldots, x_k) \in \{0,1\}^k\), let \((f \cdot g)(x_{m_1}, \ldots, x_{m_k}) = f(x_{j_1}, \ldots, x_{j_d})g(x_{j_1}, \ldots, x_{j_d}) \) if \( \{m_1, \ldots, m_k\} = \{i_1, \ldots, i_c\} \cup \{j_1, \ldots, j_d\} \), where the order of the indices in \( \{m_1, \ldots, m_k\} \) should be predetermined from \((i_1, \ldots, i_c)\) and \((j_1, \ldots, j_d)\) before multiplication. For instance, we obtain \((f \cdot g)(x_1, x_2, x_3, x_4)\) from \((f(x_1, x_2, x_3, x_4) \cdot g(x_2, x_4, x_1))\).

**Lemma 3.3** For any two constraints \( f \) and \( g \) in \( \mathcal{ED} \), the constraint \( f \cdot g \) is also in \( \mathcal{ED} \). A similar result holds for \( \mathcal{DG}, \mathcal{NZ}, \mathcal{IM}, \) and \( \mathcal{IMP} \).

**Proof.** Assume that \( f, g \in \mathcal{ED} \). Note that \( f \) and \( g \) are both products of constraints, each of which has one of the following forms: \( EQ, XOR, \) unary constraints. Clearly, the multiplied constraint \( f \cdot g \) is a product of those factors, and hence it is in \( \mathcal{ED} \). The other cases are similarly proven.

Exponentiation can be considered as a special case of multiplication. To express an exponentiation, we introduce the following notation: for any number \( r \in \mathbb{R} - \{0\} \) and any constraint \( f \), let \( f^r \) denote the function defined as \( f^r(x_1, \ldots, x_n) \equiv (f(x_1, \ldots, x_n))^r \) for any \( k \)-tuple \((x_1, \ldots, x_k) \in \{0,1\}^k \).
Lemma 3.4 For any number $m \in \mathbb{N}^+$ and any constraint $f$, $f \in \mathcal{E}D$ iff $f^m \in \mathcal{E}D$. A similar result holds for $\mathcal{D}G, \mathcal{N}Z, \mathcal{I}M, \text{ and } \mathcal{I}M P$.

Proof. Let $m \geq 1$. Since $f^m$ is the $m$-fold function of $f$, by Lemma 3.3, $f \in \mathcal{E}D$ implies $f^m \in \mathcal{E}D$. Next, we intend to show that $f^m \in \mathcal{E}D$ implies $f \in \mathcal{E}D$. Let us assume that $f^m \in \mathcal{E}D$. By setting $g = f^m$, it holds that $f(x_1, \ldots, x_n) = (g(x_1, \ldots, x_n))^{1/m}$ for any vector $(x_1, \ldots, x_n) \in \{0, 1\}^n$. Now, assume that $g = g_1 \cdots g_k$, where each $g_i$ is one of $\text{EQ}, \text{XOR}$, or unary constraints. If $g_i$ is either $\text{EQ}$ or $\text{XOR}$, then we define $h_i = g_i$. If $g_i$ is a unary function, define $h_i = (g_i)^{1/m}$, which is also a unary constraint. Obviously, all $h_i$’s are well-defined and also belong to $\mathcal{E}D$, because $\mathcal{E}D$ contains all unary constraints. Since $f = h_1 \cdots h_k$, by the definition of $\mathcal{E}D$, we conclude that $f$ is in $\mathcal{E}D$.

The second part of the lemma can be similarly proven. \hfill \Box

4 Typical Counting Problems

We will discuss the approximation complexity of special counting problems that has arisen naturally in the past literature. When we use complex numbers in the subsequent discussion, we always assume our special way of handling those numbers, as discussed in Section 2.

The counting satisfiability problem, $\#\text{SAT}$, is a problem of counting the number of truth assignments that make each given propositional formula true. This problem was proven to be complete for $\#P$ under AP-reduction [12]. Dyer et al. [15] further showed that $\#\text{SAT}$ possesses the computational power equivalent to $\#\text{CSP(OR)}$ under AP-reduction, namely, $\#\text{CSP(OR)} \equiv_{AP} \#\text{SAT}$.

Nevertheless, to deal particularly with complex-weighted counting problems, it is desirable to introduce a complex-weighted version of $\#\text{SAT}$. In the following straightforward way, we define $\#\text{SAT}_C$, a complex-weighted version of $\#\text{SAT}$. Let $\phi$ be any propositional formula (with three logical connectives, $\neg$ (not), $\lor$ (or), and $\land$ (and)) and let $V(\phi)$ be the set of all variables appearing in $\phi$. Let $\{w_x\}_{x \in V(\phi)}$ be any series of node-weight functions $w_x : \{0, 1\} \rightarrow C - \{0\}$. Given such a pair $(\phi, \{w_x\}_{x \in V(\phi)})$, $\#\text{SAT}_C$ asks to compute the sum of all weights $w(\sigma)$ for every truth assignment $\sigma$ satisfying $\phi$, where $w(\sigma)$ denotes the product of all $w_x(\sigma(x))$ for any $x \in V(\phi)$. If $w_x(\sigma(x))$ always equals 1 for every pair of $\sigma$ and $x \in V(\phi)$, then we immediately obtain $\#\text{SAT}$. This indicates that $\#\text{SAT}_C$ naturally extends $\#\text{SAT}$. The notation $\#\text{CSP}^+(\mathcal{F})$ denotes the counting problem $\#\text{CSP}(\mathcal{F}, U \cap \mathcal{N}Z)$.

Proof of Lemma 4.1. We can modify the construction of an AP-reduction from $\#\text{SAT}$ to $\#\text{IS}$, given in [12], by adding a node-weight function to each variable node. Hence, we instantly obtain $\#\text{SAT}_C \leq_{AP} \#\text{IS}_C$. We leave the details of the proof to the avid reader. Next, we claim that $\#\text{IS}_C$ and $\#\text{CSP}^+(\text{NAND})$ are AP-equivalent. Because this claim is a concrete example of how to relate $\#\text{CSPs}$ to more popular counting problems, here we include the proof of the claim.

Claim 1 $\#\text{IS}_C \equiv_{AP} \#\text{CSP}^+(\text{NAND})$.

Proof. We want to show that $\#\text{IS}_C$ is AP-reducible to $\#\text{CSP}^+(\text{NAND})$. Let $G = (V, E)$ and $\{w_x\}_{x \in V}$ be any instance pair to $\#\text{IS}_C$. In the way described below, we will construct a constraint frame $\Omega = (G', X, F', \pi)$ that becomes an input instance to $\#\text{CSP}^+(\text{NAND})$, where $G' = (V', E')$ is an undirected bipartite graph whose $V'$ and $E'$ (\(\subseteq V \times V\)) are defined by the following procedure. Choose any edge $(x, y) \in E$, prepare three new nodes $v_1, v_2, v_3$ labeled $\text{NAND}, w_x, w_y$, respectively, and place four edges $(x, v_1), (y, v_1), (x, v_2), (y, v_3)$ into $E'$. At the same time, place these new nodes into $V'$. In case where variable $x$ (resp.) has been already used to insert a new node $v_2$ ($v_3$, resp.), we no longer need to add the node $v_2$ ($v_3$, resp.). We define $X$ to

\[ \text{Proof.} \quad \text{We want to show that } \#\text{IS}_C \text{ is AP-reducible to } \#\text{CSP}^+(\text{NAND}). \quad \text{Let } G = (V, E) \text{ and } \{w_x\}_{x \in V} \text{ be any instance pair to } \#\text{IS}_C. \quad \text{In the way described below, we will construct a constraint frame } \Omega = (G', X, F', \pi) \text{ that becomes an input instance to } \#\text{CSP}^+(\text{NAND}), \quad \text{where } G' = (V', E') \text{ is an undirected bipartite graph whose } V' \text{ and } E' (\subseteq V \times V') \text{ are defined by the following procedure. Choose any edge } (x, y) \in E, \quad \text{prepare three new nodes } v_1, v_2, v_3 \text{ labeled } \text{NAND}, w_x, w_y, \text{ respectively, and place four edges } (x, v_1), (y, v_1), (x, v_2), (y, v_3) \text{ into } E'. \quad \text{At the same time, place these new nodes into } V'. \quad \text{In case where variable } x \text{ (resp.) has been already used to insert a new node } v_2 \text{ (resp.), we no longer need to add the node } v_2 \text{ (resp.). We define } X \text{ to} \]
be the set of all labels of the nodes in $V$ and define $F'$ to be $\{w_x\}_{x \in V} \cup \{\text{NAND}\}$. A labeling function $\pi$ is naturally induced from $G'$, $X$, and $F'$ ans we omit its formal description.

Now, we want to use variable assignment to compute $\text{csp}_\Omega$. Given any independent set $S$ for $G$, we define its corresponding variable assignment $\sigma_S$ as follows: for each variable node $x \in V$, let $\sigma_S(x) = S(x)$. Note that, for every edge $(x,y)$ in $E$, $x,y \in S$ iff $\text{NAND}(\sigma_S(x),\sigma_S(y)) = 0$. Let $\tilde{V}$ denote a subset of $V'$ whose elements have the label $\text{NAND}$. Since all unary constraints appearing as node labels in $V'$ are $w$'s, $w(S)$ coincides with $\prod_{x \in \tilde{V}} \prod_{x \in \tilde{V}} f_x(\sigma_S(x),\sigma_S(y)) \cdot \prod_{x \in \tilde{V}} w_x(\sigma_S(x))$, where each label $f_x$ of node $v$ is $\text{NAND}$. Using this equality, it is not difficult to show that $\text{csp}_\Omega$ equals the outcome of $\text{#IS}_C$ on the instance $(G,\{w_x\}_{x \in V})$. Therefore, $\text{#IS}_C$ is AP-reducible to $\text{#CSP}^+(\text{NAND})$.

Next, we will construct an AP-reduction from $\text{#CSP}^+(\text{NAND})$ to $\text{#IS}_C$. Given any input instance $\Omega = (G,X,F',\pi)$ with $G = (V_1 \cup V_2, E)$ to $\text{#CSP}^+(\text{NAND})$, we first simplify $G$ as follows. Notice that $F'$ is a finite subset of $\{\text{NAND}\} \cup \mathcal{U}$. If any two distinct nodes $v_1, v_2 \in V_2$ labeled $u_1, u_2 \in \mathcal{U}$, respectively, satisfy $E(v_1) = E(v_2)$, then we merge the two nodes into one node with new label $u'$, where $u'(x) = u_1(x)u_2(x)$. Similarly, if $v_1, v_2 \in V_2$ with the same label $\text{NAND}$ satisfy $E(v_1) = E(v_2)$, then we delete the node $v_1$ and all its incident edges. By abusing the notation, we denote the obtained graph by $G$.

From the graph $G$, we define another graph $G' = (V_1,E')$ with $E' = \{(x,y) \in V_1 \times V_1 \mid \exists v \in V_2 \text{ s.t. } v \text{ has label } \text{NAND} \text{ and } x,y \in E(v)\}$. Let $x$ be any variable that appears in $G'$. For each node $w$ in $V_1$ labeled $x$, if $w$ is adjacent to a certain node whose label is a unary constraint, say, $u$, then define $w_{u'}$ to be $u$; otherwise, define $w_{u'}(x) = 1$ for any $x \in \{0,1\}$. Let $\tilde{V}$ be the set of all nodes in $V_2$ whose labels are $\text{NAND}$. Fix a variable assignment $\sigma$ arbitrarily and define $S_\sigma = \{x \in V_1 \mid \sigma(x) = 1\}$. It follows that $w(S_\sigma) = \prod_{x \in V_2} f_x(\sigma(x_1),\ldots,\sigma(x_k))$, where each $k$-tuple $(x_1,\ldots,x_k)$ depends on the choice of $f_x$. Thus, $\sum_{\sigma} w(S_\sigma)$ equals $\text{csp}_\Omega$. Moreover, it holds that $\prod_{x \in V_2} f_x(\sigma(x_1),\ldots,\sigma(x_k)) = \prod_{x \in V_2} f_x(\sigma(x),\sigma(y)) \cdot \prod_{x \in V_2} w_x(\sigma(x))$. Hence, $\prod_{x \in V_2} f_x(\sigma(x_1),\ldots,\sigma(x_k)) \neq 0$ iff $S_\sigma$ is an independent set. These conditions give the desired AP-reduction from $\text{#CSP}^+(\text{NAND})$ to $\text{#IS}_C$. This completes the proof of Claim $\blacksquare$

Naturally, $\text{#CSP}^+(\text{NAND})$ is AP-reducible to $\text{#CSP}^+(\text{NAND})$. To complete the proof, we want to show that $\text{#CSP}^+(\text{NAND}) \leq_{\text{AP}} \text{#CSP}^+(\text{OR})$. This is easily shown by, roughly speaking, exchanging the roles of 0 and 1 in variable assignments. More precisely, given an instance $\Omega = (G,X,F',\pi)$ to $\text{#CSP}^+(\text{NAND})$, we build another instance $\Omega'$ by replacing any unary constraint $u$ by $\pi$, where $\pi = [b,a]$ if $u = [a,b]$, and by replacing $\text{NAND}$ by $\text{OR}$. It clearly holds that $\text{csp}_\Omega = \text{csp}_\Omega'$, and thus $\text{#CSP}^+(\text{NAND}) \leq_{\text{AP}} \text{#CSP}^+(\text{OR})$. $\blacksquare$

We remark that, by carefully checking the above proof, we can AP-decrease $\text{#SAT}_C$ to $\text{#CSP}^+(\text{OR})$ instead of $\text{#CSP}^+(\text{OR})$. For another remark, we need two new notations. The first notation $\text{#CSP}^+_\lambda(F)$ indicates the counting problem obtained from $\text{#CSP}^+(F)$ under the restriction that input instances are limited to algebraic constraints. When the outcomes of all node-weight functions of $\text{#SAT}_C$ are limited to algebraic complex numbers, we briefly write $\text{#SAT}_\lambda$. Similar to the first remark, we can prove that $\text{#SAT}_\lambda$ is AP-reducible to $\text{#CSP}^+_\lambda(\text{OR})$. This fact will be used in Section $\text{6}$. 

5 T-Constructibility

One of key technical tools of Dyer et al. [14] in manipulating Boolean constraints is a notion of “implementation,” which is used to help establish certain AP-reductions among $\text{#CSP}$s with Boolean constraints. In light of our AP-reducibility, we prefer a more “operational” or “mechanical” approach toward the manipulation of constraints in a rather systematic fashion. Here, we will present our key technical tool, called T-constructibility, of constructing target constraints from a given set of presumably simpler constraints by applying repeatedly such mechanical operations, while maintaining the AP-reducibility. This key tool will be frequently used in Section $\text{6}$ to establish several AP-reductions among $\text{#CSP}$s with constraints.

In an exact counting case of, e.g., Cai et al. [6,7,8], numerous “gadget” constructions were used to obtain required properties of constraints. Our systematic approach with the T-constructibility naturally supports most gadget constructions and the results obtained by them can be re-proven by appropriate applications of T-constructibility. The minimal set of constraints that are T-constructed from a fixed set $G$ of “basis” constraints, denoted $\text{CL}_T(G)$, together with arbitrary free unary constraints is certainly an interesting research object in promoting our understanding of the AP-reducibility. An advantage of taking such a systematic approach can be exemplified, for instance, by Lemma [72] in which we are able to argue the closure property under
AHF-reducibility (without the projection operation). This property is a key to the subsequent lemmas and propositions. This line of study was lately explored in [2].

To pursue notational succinctness, we use the following notations in the rest of this paper. For any index $i \in [k]$ and any bit $c \in \{0, 1\}$, let the notation $f^{x_i = c}$ denote the function $g$ satisfying that $g(x_1, \ldots, x_i-1, x_i, x_{i+1}, \ldots, x_k) = f(x_1, \ldots, x_i-1, c, x_{i+1}, \ldots, x_k)$ for any vector $(x_1, \ldots, x_i-1, x_{i+1}, \ldots, x_k) \in \{0, 1\}^{k-1}$. Similarly, for any two distinct indices $i, j \in [k]$, we denote by $f^{x_i = x_j}$ the function $g$ defined as $g(x_1, \ldots, x_i-1, x_i, x_{i+1}, \ldots, x_k) = f(x_1, \ldots, x_i-1, x_i, x_{i+1}, \ldots, x_k)$. Moreover, let $f^{x_i = *}$ be the function $g$ defined as $g(x_1, \ldots, x_i-1, x_i, x_{i+1}, \ldots, x_k) = \sum_{x_i \in \{0, 1\}} f(x_1, \ldots, x_i-1, x_i, x_{i+1}, \ldots, x_k)$, where $x_i$ is no longer a free variable. By extending these notations naturally, we can write, e.g., $f^{x_i = 0, x_m = *}$ as the shorthand for $(f^{x_i = 0})^{x_m = *}$ and $f^{x_i = 1, x_m = 0}$ for $(f^{x_i = 1})^{x_m = 0}$.

We say that a constraint $f$ of arity $k$ is T-constructible (or T-constructed) from a constraint set $\mathcal{G}$ if $f$ can be obtained, initially from constraints in $\mathcal{G}$, by applying recursively a finite number (possibly zero) of functional operations described below.

1. Permutation: for two indices $i, j \in [k]$ with $i < j$, by exchanging two columns $x_i$ and $x_j$ in $(x_1, \ldots, x_i, \ldots, x_j, \ldots, x_k)$, transform $g$ into $f$ that is defined as $g(x_1, \ldots, x_i, \ldots, x_j, \ldots, x_k) = g(x_1, \ldots, x_j, \ldots, x_i, \ldots, x_k)$.
2. Pinning: for an index $i \in [k]$ and a bit $c \in \{0, 1\}$, build $g^{x_i = c}$ from $g$.
3. Projection: for an index $i \in [k]$, build $g^{x_i = *}$ from $g$.
4. Linking: for two distinct indices $i, j \in [k]$, build $g^{x_i = x_j}$ from $g$.
5. Expansion: for an index $i \in [k]$, introduce a new “free” variable, say, $y$ and transform $g$ into $f$ that is defined by $g'(x_1, \ldots, x_i, y, x_{i+1}, \ldots, x_k) = g(x_1, \ldots, x_i, x_{i+1}, \ldots, x_k)$.
6. Multiplication: from two constraints $g_1$ and $g_2$ of arity $k$ sharing the same input variable series $(x_1, \ldots, x_k)$, build $g_1 \cdot g_2$, that is, $(g_1 \cdot g_2)(x_1, \ldots, x_k) = g_1(x_1, \ldots, x_k)g_2(x_1, \ldots, x_k)$.
7. Normalization: for a constant $\lambda \in \mathbb{C} \setminus \{0\}$, build $\lambda g$ from $g$, where $\lambda g$ is defined as $(\lambda g)(x_1, \ldots, x_k)$.

When $f$ is T-constructible from $\mathcal{G}$, we write $f \leq_{\text{con}} \mathcal{G}$. In particular, when $\mathcal{G}$ is a singleton, say, $\{g\}$, we also write $f \leq_{\text{con}} g$ instead of $f \leq_{\text{con}} \{g\}$ for succinctness. With this notation $\leq_{\text{con}}$, an earlier notation $CL^*_T(\mathcal{G})$ can be formally defined as $CL^*_T(\mathcal{G}) = \{f \mid f \leq_{\text{con}} \mathcal{G} \cup U\}$.

As is shown below, T-constructibility induces a partial order among all constraints. The proof of the following lemma is rather straightforward, and thus we omit it entirely and leave it to the avid reader.

**Lemma 5.1** For any three constraints $f, g$, and $h$, it holds that (i) $f \leq_{\text{con}} f$ and (ii) $f \leq_{\text{con}} g$ and $g \leq_{\text{con}} h$ imply $f \leq_{\text{con}} h$.

The usefulness of T-constructibility comes from the following lemma, which indicates the invariance of T-constructibility under AP-reductions. For readability, we place the proof of the lemma in Section 10.

**Lemma 5.2** If $f \leq_{\text{con}} g$, then $\text{CSP}^*(f, F) \leq_{\text{AP}} \text{CSP}^*(g, F)$ for any set $F$ of constraints.

### 6 Expressive Power of Unary Constraints

In the rest of this paper, we aim at proving our dichotomy theorem (Theorem 11). Its proof, which will appear in Section 14, is comprised of several crucial ingredients. A starting point of the proof of the dichotomy theorem is a tractability result of #CSP’s, which states that #CSP($F$) is solvable in polynomial-time if $F \subseteq \mathcal{E}D$. A free use of arbitrary unary constraint plays an essential role in this section.

For the proof of our dichotomy theorem, we need to consider a “factorization” of a given constraint $g$. Recall that the definition of $\mathcal{E}D$. When $g$ is in $\mathcal{E}D$, $g$ should be expressed as $g = g_1 \cdot g_2 \cdots g_n$, where each $g_i$ is one of $\text{EQ}$, $\text{XOR}$, and unary constraints. For convenience, we call the list $L = \{g_1, g_2, \ldots, g_n\}$ of all those factors a factor list for $g$.

**Lemma 6.1** For any constraint set $F$, if $F \subseteq \mathcal{E}D$, then $\text{CSP}^*(F)$ is in $\text{FP}_C$.

**Proof.** Consider any constraint frame $\Omega = (G, X | F', \pi)$ given as an input instance to #CSP($F$), where $G = (V_1 \cup V_2, E)$ is an bipartite undirected graph and $F'$ is a finite subset of $F \cup U$. Here, we examine the situation where $\mathcal{F} \subseteq \mathcal{E}D$. To simplify our later algorithm, we first modify $\Omega$ as follows. Since $F' \subseteq \mathcal{E}D \cup U \subseteq \mathcal{E}D$, it is possible to replace all occurrences of each constraint in $F'$ by its “factors” without changing the
outcome of \( \text{csp}_\Omega \). In what follows, we assume that \( G \) is composed of nodes whose labels are limited to \( \text{EQ}, \text{XOR} \), and unary constraints.

For each node \( v \) labeled \( \text{EQ} \) that is adjacent to two nodes having variable labels, say, \( x_1 \) and \( x_2 \), we merge these two nodes into a single node with the label \( x_1 \), and then we delete from the graph the node \( v \) and all edges that have been incident to the node \( x_2 \). After this deletion, we hereafter assume that there is no node with the label \( \text{EQ} \). Moreover, if two nodes \( v_1 \) and \( v_2 \) both labeled \( \text{XOR} \) are adjacent to the same nodes in \( V_i \), then we delete the node \( v_2 \) and its incident edges since the node \( v_2 \) is redundant for the calculation of \( \text{csp}_\Omega \). Henceforth, let us assume that no such node pair of \( v_1 \) and \( v_2 \) exists.

Consider all connected components of the obtained graph. Choose such a connected component \( G' = (V'_i, V'_2, E') \), which forms a bipartite subgraph of \( G \). Note that \( G' \) consists only of nodes whose labels are \( \text{XOR} \) or unary constraints. Let \( \Omega' \) be a constraint frame obtained from \( \Omega \) by restricting its scope within \( G' \). We select a special node, say, \( v \) in \( V'_i \) as follows. If there exists a cycle that contains all nodes in \( V'_i \), then we choose any node in \( V'_i \) as \( v \). Otherwise, we choose a node \( v \) in \( V'_i \) that is not adjacent to two nodes having the label \( \text{XOR} \). Let \( x \) be the label of this node \( v \). To compute the value \( \text{csp}_\Omega \), it suffices to consider a Boolean value of this particular variable \( x \). It is important to note that the Boolean values of the remaining variables are automatically induced by the choice of the value of \( x \). Hence, \( \text{csp}_\Omega \) is easily computed by assigning only two values (0 or 1) to \( x \). This implies that we can calculate the value \( \text{csp}_\Omega \) in time associated with the number of connected components times the maximal computation time for \( \text{csp}_{\Omega'} \). Therefore, \( \#\text{CSP}(\mathcal{F}) \) belongs to FP\(_C \).

Henceforth, we will focus our attention on the remaining case where \( \mathcal{F} \not\subseteq \mathcal{CD} \). As a basis to the subsequent analysis, the rest of this section is devoted to explore fundamental properties of binary constraints \( f \) and it shows numerous complexity bounds of \( \#\text{CSP}(\mathcal{F})'s \). A key to our study is an expressive power of free unary constraints. We begin with a quick reminder that, since all unary constraints are free to use, it obviously holds that \( \#\text{CSP}(\Delta_0, \Delta_1, \mathcal{F}) = \#\text{AP} \#\text{CSP}(\mathcal{F}) \) for any set \( \mathcal{F} \) of constraints.

Earlier, in the proof of Lemma \ref{lem:ap}, we have demonstrated the AP-equivalence between \( \#\text{CSP}(\text{OR}) \) and \( \#\text{CSP}(\text{NAND}) \) by a simple technique of swapping the roles of 0 and 1. However, this technique is not sufficient to prove that \( \#\text{CSP}(\text{OR}, \mathcal{F}) = \#\text{AP} \#\text{CSP}(\text{NAND}, \mathcal{F}) \) for an arbitrary constraint set \( \mathcal{F} \). A use of unary constraint, on the contrary, helps us establish this stronger AP-equivalence.

**Proposition 6.2** For any constraint set \( \mathcal{F} \), \( \#\text{CSP}(\text{OR}, \mathcal{F}) = \#\text{CSP}(\text{NAND}, \mathcal{F}) \).

**Proof.** We will show only one direction of \( \#\text{CSP}(\text{OR}, \mathcal{F}) \leq \#\text{AP} \#\text{CSP}(\text{NAND}, \mathcal{F}) \), since the opposite direction is similarly proven. For brevity, let \( f = \text{NAND} \) and set \( u = [1, -1] \). Now, we claim that \( \text{OR} \leq \text{con} \{ f, u \} \). For this purpose, let us define \( g(x_1, x_2) = \sum_{x_2 \in [0, 1]} f(x_1, x_2) u(x_3) \). It is not difficult to show that \( g \) equals \( \text{OR} \in [0, 1] \). Hence, \( \text{OR} \) is \( \text{T} \)-constructed from \( \{ f, u \} \), as requested. From this \( \text{T} \)-constructibility, by Lemma \ref{lem:tconstruct} we obtain an AP-reduction from \( \#\text{CSP}(\text{OR}, \mathcal{F}) \) to \( \#\text{CSP}(f, \mathcal{F}) \). The last term obviously equals \( \#\text{CSP}(f, \mathcal{F}) \) because \( u \) is a unary constraint. Therefore, we conclude that \( \#\text{CSP}(\text{OR}, \mathcal{F}) \leq \#\text{CSP}(f, \mathcal{F}) \). \( \square \)

Now, let us consider other binary constraints. Of them, our next target is constraints having the forms: (0, \( a, b, 1 \)) or (1, \( a, b, 0 \)) with \( \{a, b\} \neq \emptyset \).

**Lemma 6.3** Let \( a, b \in \mathbb{C} \) with \( ab \neq 0 \) and let \( f \) be any constraint of the form: either (0, \( a, b, 1 \)) or (1, \( a, b, 0 \)). For any constraint set \( \mathcal{F} \), the following statement holds: \( \#\text{CSP}(\mathcal{F}) \leq \#\text{CSP}(f) \).

**Proof.** Consider the case where \( f = (0, a, b, 1) \). Let \( u = [1, ab] \) for brevity. We want to claim that \( \text{OR} \) is \( \text{T} \)-constructed from the constraint set \( \{ f, u \} \). To show this claim, define \( g(x_1, x_2) = f(x_1, x_2) u(x_1) u(x_2) \). A simple calculation leads us to the conclusion that \( g = [0, a^2 b^2, a^2 b^2] \). By normalizing \( g \) appropriately, we immediately obtain another constraint \( g' = [0, 1, 1] \), which clearly equals \( \text{OR} \). By the definition of \( g \), it thus follows that \( \text{OR} \leq \text{con} \{ f, u \} \). Lemma \ref{lem:tconstruct} then implies that \( \#\text{CSP}(\mathcal{F}) \) is AP-reducible to \( \#\text{CSP}(f, \mathcal{F}) \). Since \( u \) is unary, the last term coincides with \( \#\text{CSP}(f, \mathcal{F}) \), yielding the desired consequence of the lemma. For the case of \( f = (1, a, b, 0) \), a similar argument shows that \( \#\text{CSP}(\text{NAND}, \mathcal{F}) \leq \#\text{CSP}(f, \mathcal{F}) \). By Proposition \ref{prop:ap} it is possible to replace \( \text{NAND} \) by \( \text{OR} \), and therefore the desired consequence follows. \( \square \)

We will examine other binary constraints of the form (0, \( a, b, 0 \)) with \( ab \neq 0 \).

**Lemma 6.4** Let \( a, b \in \mathbb{C} \) with \( ab \neq 0 \). For any set \( \mathcal{F} \) of constraints, \( \#\text{CSP}(\mathcal{F}) \leq \#\text{CSP}(((0, a, b, 0), \mathcal{F}) \).

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Proof. Let $f = (0, a, b, 0)$ with $ab \neq 0$. Now, we “symmetrize” $f$ by setting $g(x_1, x_2) = f(x_1, x_2)f(x_2, x_1)$, which yields the equation $g = [0, ab, 0]$. We normalize $g$ and then obtain $[0, 1, 0]$, which is exactly XOR. We thus conclude that $\text{XOR} \leq_{\text{con}} f$, implying that $\#\text{CSP}^*(\text{XOR}, F)$ is AP-reducible to $\#\text{CSP}^*(f, F)$ by Lemma 5.3.

Next, we move our interest to the special relation $\text{Implies}$. Unlike the case of unweighted Boolean $\#\text{CSP}$s [11], where it remains open whether $\#\text{CSP}(\text{Implies})$ is AP-equivalent to $\#\text{CSP}(\text{OR})$, a heavy use of non-zero unary constraints leads to a surprising AP-equivalence between $\#\text{CSP}^*(\text{Implies}, F)$ and $\#\text{CSP}^*(\text{OR}, F)$ for any constraint set $F$.

**Proposition 6.5** For any constraint set $F$, it holds that $\#\text{CSP}^*(\text{Implies}, F) \equiv_{\text{AP}} \#\text{CSP}^*(\text{OR}, F)$.

This proposition directly follows from the lemma stated below together with Lemma 5.2, which translates T-constructibility into AP-reducibility.

**Lemma 6.6**

1. There exists a finite set $G \subseteq U$ such that $\text{Implies} \leq_{\text{con}} G \cup \{\text{OR}\}$.

2. There exists a finite set $G \subseteq U$ such that $\text{OR} \leq_{\text{con}} G \cup \{\text{Implies}\}$.

**Proof.** For ease of the description that follows, we set $f = \text{Implies}$.

(1) Here, we intend to claim the T-constructibility of $f$ from the set $\{\text{OR}, u_1, u_2, u_3\}$, where $u_1 = [1, -1/2]$, $u_2 = [2, -2/3]$, and $u_3 = [-1, -1/8]$. We will prove this claim by building a series of T-constructible constraints. First, we define $g(x, y) = \sum_{z \in \{0, 1\}} \text{OR}(x, z)\text{OR}(y, z)$. This implies that $g = [1, 1, 2]$ and $f \leq_{\text{con}} \text{OR}$. Next, let $h(x, y) = \sum_{z \in \{0, 1\}} g(x, z)g(y, z)u_1(z)$, which equals $(1/2, -1, 0, -3)$. Clearly, it holds that $h \leq_{\text{con}} \{g, u_1\}$. Moreover, let $h'(x, y) = h(x, y)u_2(x)$, implying $h'(1, -2, 0, -2)$. Finally, we set $p(x, y) = \sum_{z \in \{0, 1\}} h'(x, z)h'(y, z)u_3(z)$. A simple calculation shows that $p = (1, 1, 0, 1)$. Since $p$ is T-constructible from $\{h', u_3\}$, we then obtain $f \leq_{\text{con}} \{\text{OR}, u_1, u_2, u_3\}$, as requested.

(2) We will show that $\text{OR}$ is T-constructed from the set $\{f, \Delta_0, u_1, u_2\}$, where $u_1 = [-1, -8]$ and $u_2 = [49, 24]$. To prove this claim, we introduce the following two useful constraints: $h_2 = [2, 1, 1]$ and $h_3 = [2, 1, 1, 1]$. In what follows, we will prove (i) $\text{OR} \leq_{\text{con}} \{h_2, u_1, u_2\}$ and (ii) $h_2 \leq_{\text{con}} \{f, \Delta_0\}$. From Statements (i) and (ii), it immediately follows by Lemma 5.1 that $\text{OR} \leq_{\text{con}} \{f, \Delta_0, u_1, u_2\}$, as requested.

(i) We start with defining $g(x, y) = \sum_{z \in \{0, 1\}} h_2(x, z)h_2(y, z)u_1(z)$. It is easy to check that $g = (0, -6, -4, -7)$. With this $g$, we define $s(x, y) = g(x, y)g(y, x)u_2(x)u_2(y)$, which equals $(0, a, a, a)$, where $a = 28224$. By normalizing $s$ properly, we immediately obtain the constraint $\text{OR}$. Therefore, it holds that $\text{OR} \leq_{\text{con}} \{h_2, u_1, u_2\}$.

(ii) We note that $E_{\Delta}$ is T-constructed from $f$ because $E_{\Delta}(x, y, z)$ equals $f(x, y)f(y, z)f(z, x)$. Using $E_{\Delta}$, we define

$$p(x_1, y_1, z_1) = \sum_{x_2, y_2, z_2 \in \{0, 1\}} E_{\Delta}(x_2, y_2, z_2)f(x_1, x_2)f(y_1, y_2)f(z_1, z_2).$$

This definition implies that $p = [2, 1, 1, 1]$, and thus $p$ equals $h_3$. This means that $h_3 \leq_{\text{con}} \{E_{\Delta}, \text{Implies}\}$. Since $h_2 = h_3^0$, $h_2$ is T-constructed from $\{h_3, \Delta_0\}$. From all the obtained results, we easily conclude that $h_2 \leq_{\text{con}} \{f, \Delta_0\}$.

Next, we target constraints of the forms $(1, a, 0, b)$ and $(1, 0, a, b)$ with $ab \neq 0$.

**Lemma 6.7** Let $f = (1, a, 0, b)$ with $a, b \in \mathbb{C}$. If $ab \neq 0$, then $\#\text{CSP}^*(\text{OR}, F) \leq_{\text{AP}} \#\text{CSP}^*(f, F)$ holds for any constraint set $F$. By permutation as well as normalization, $(1, 0, a, b)$ also yields the same consequence.

**Proof.** Let $f = (1, a, 0, b)$ with $ab \neq 0$. In this proof, we use two unary constraints: $u = [1, a/b]$ and $v = [1, 1/a^2]$. With a help of Proposition 5.3, our goal is now set to show the T-constructibility of $\text{Implies}$ from $\{f, u, v\}$. Firstly, by defining $g(x_1, x_2) = f(x_1, x_2)u(x_1)$, we obtain $g = (1, a, 0, a)$. This implies that $g \leq_{\text{con}} \{f, u\}$. Secondly, we define $h(x_1, x_2) = \sum_{x_3 \in \{0, 1\}} g(x_1, x_3)g(x_3, x_2)v(x_3)$. A simple calculation shows that $h = (1, 1, 0, 1)$. This concludes that $\text{Implies}$ is T-constructed from $\{g, v\}$. By combining those two results, we obtain $\text{Implies} \leq_{\text{con}} \{f, u, v\}$ by Lemma 5.1. The desired result then follows immediately because $u$ and $v$ are unary constraints.

Now, we consider the case of constraints $f$ having the form $(1, x, y, z)$ and demonstrate the hardness of $\#\text{CSP}^*(f)$. For complex-valued constraints $f$, this case is quite special because, if they are Boolean, they all become $[1, 1, 1]$ and fall into $\mathcal{DG}$. 

Proposition 6.8 Let $x, y, z \in \mathbb{C}$. If both $xyz \neq 0$ and $xy \neq z$ hold, then $\#CSP^\ast(OR, \mathcal{F}) \leq_{AP} \#CSP^\ast((1, x, y, z), \mathcal{F})$ for any set $\mathcal{F}$ of constraints.

When $xy = z$, on the contrary, the constraint $f = (1, x, y, z)$ becomes degenerate, since $f(x_1, x_2)$ equals $[1, y|x_1| - [1, x|x_2])$. Proposition (6.8) is a direct consequence of two lemmas—Lemmas 6.3 and 6.10—each of which handles a different case. Let us begin with the case where $xyz \neq 0$ and $xy \neq \pm z$.

Lemma 6.9 Let $x, y, z \in \mathbb{C}$. If $xyz \neq 0$ and $xy \neq \pm z$, then $\#CSP^\ast(OR, \mathcal{F}) \leq_{AP} \#CSP^\ast((1, x, y, z), \mathcal{F})$ for any set $\mathcal{F}$ of constraints.

Proof. Let $f = (1, x, y, z)$ with $xyz \neq 0$. Assuming that $xy \neq \pm z$, we first want to show that $\#CSP^\ast(\text{Implies}, \mathcal{F})$ is AP-reducible to $\#CSP^\ast(f, \mathcal{F})$. With an unknown variable $a$, set $u = [1, a]$. Now, we define $g(x_1, x_2) = \sum_{x_3 \in [0, 1]} f(x_1, x_2)i(x_3, x_2)j(x_3, x_1)u(x_3)$. A simple calculation provides an equation $g = (1 + aax_2y, xy + a2y), y(1 + axz), xy^2 + ax^2)$. By setting $a$ to be $-1/xx$, we obtain $g = (1 - xy/z, x(y - z/x), 0, xy^2 - z^2/x)$, which implies $g \leq_{con} \{f, u\}$. It thus follows that $\#CSP^\ast(g, \mathcal{F}) \leq_{AP} \#CSP^\ast(f, \mathcal{F})$ by Lemma 6.2. Note that three entries in $g$ are non-zero, since $xy \neq z$ and $x^2y^2 \neq z^2$. Apply Lemma 6.3 to a normalized $g$. As an immediate consequence, we obtain an AP-reduction from $\#CSP^\ast(OR, \mathcal{F})$ to $\#CSP^\ast(g, \mathcal{F})$. The final result is obtained by combining the two AP-reductions.

Finally, we consider the remaining case where $xy = -z$; that is, constraints of the form $(1, x, y, -xy)$, which is excluded in the previous lemma.

Lemma 6.10 Let $x, y \in \mathbb{C}$. If $xy \neq 0$, then $\#CSP^\ast(OR, \mathcal{F}) \leq_{AP} \#CSP^\ast((1, x, y, -xy), \mathcal{F})$ for any constraint set $\mathcal{F}$.

Proof. Our proof strategy is to reduce this case to Lemma 6.3. Let $f = (1, x, y, -xy)$ and assume that $xy \neq 0$. Define $u = [1, a]$ and consider the constraint $g$ defined by $g(x_1, x_2) = \sum_{x_3 \in [0, 1]} f(x_1, x_2)i(x_3, x_2)j(x_3, x_1)u(x_3)$. This $g$ satisfies $g = (1 + axy, x(1 - axy), y(1 - axy), xy(1 + axy))$. If we choose $a = 2/xy$, then we have $g = (3, -x, -y, 3y)$, which equals $3 \cdot (1, -x/3, -y/3, y/3)$. For simplicity, set $x' = -x/3, y' = -y/3, z' = y$. Note that $x', y', z'$ are all non-zero. Now, we set $h = (1, x', y', z')$ that is obtained by normalizing $g$. Since $x'y' \neq \pm z'$, we can apply Lemma 6.3 to this $h$ and the desired consequence then follows.

7 Useful Properties of Specific Constraints

We have shown in the previous section numerous complexity bounds of $\#CSP^\ast(f)$’s when $f$’s are of arity 2. Our next step is to show similar bounds of $\#CSP^\ast(f)$’s for constraints $f$ of higher arities. To achieve our goal, we first explore fundamental properties of constraints related to $\mathcal{E}D, \mathcal{I}M$, and $\mathcal{N}Z$ so that those properties will contribute to proving the desired hardness results in Section 8.

Underlying relations of $f$ play a distinguishing role in our analysis of the behaviors of the counting problems $\#CSP^\ast(f)$. In particular, basic properties of relations in $\text{IMP}$ become a crucial part of the proof of our dichotomy theorem. Let us recall that a relation $R$ in $\text{IMP}$ is expressed as a product of the constant constraints as well as $\text{Implication}$. To handle relations in $\text{IMP}$, it is convenient to introduce a notion of “imp. support.” A constraint $f$ is said to have $\text{imp. support}$ if $R_f$ is in $\text{IMP}$. It is not difficult to show that all constraints in $\text{IM}$ have imp. support. The converse also holds for any binary constraint.

Lemma 7.1 For any binary constraint $f$, it holds that $f \in \text{IM}$ iff $R_f \in \text{IMP}$.

Proof. Since the underlying relation of any constraint $f$ in $\text{IM}$ belongs to $\text{IMP}$, it suffices to show that if $R_f \in \text{IMP}$ then $f \in \text{IM}$. Assume that $R_f$ is in $\text{IMP}$. Depending on the form of $R_f$, we consider two cases separately.

(i) Consider the case where $f$ has the form $(x, y, 0, z)$ with $x, y, z \in \mathbb{C}$ and $y \neq 0$. It is easy to check that $f(x_1, x_2)$ always equals $[y, z|x_1| - [y, l|x_2|\text{Implies}(x_1, x_2)]$. Thus, $f$ should belong to $\text{IM}$.

(ii) Next, we consider the case where $f$ has the form $(x, 0, 0, z)$ with $x, z \in \mathbb{C}$. Obviously, $f(x_1, x_2)$ always coincides with $[x, z|x_1|\text{Implies}(x_1, x_2)]\text{Implies}(x_2, x_1)$. This shows that $f$ is in $\mathcal{I}M$.

The first useful property is a closure property under a certain restricted case of T-constructibility. In what follows, we will show that the T-constructibility without the projection operation preserves the membership
Finally, if there exists a factor $\Delta^i_1$, complication, we are focused on specific indices in the following argument.

In Section 4, we have defined the concept of “factor list” for a given constraint in $\mathcal{ED}$. Similarly, for a relation $R$ in $\mathcal{IMP}$, we can define its “factor list” using its factors of the forms, $\Delta_0, \Delta_1, \text{ and } \text{Implies}$.

**Proof of Lemma 7.2.** Let $f$ be any $k$-ary constraint and let $G$ be any constraint set. Assume that $f$ is $T$-constructible from $G$ using no projection operation.

1. If all constraints in $G$ have imp support, then $f$ also has imp support.
2. If all constraints in $G$ are in $\mathcal{ED}$, then $f$ is also in $\mathcal{ED}$.

To $\mathcal{ED}$ and the property of imp support; in other words, the set $\mathcal{ED}$ as well as the set of all constraints that have imp support is closed under $T$-constructibility with no projection operation.

**Lemma 7.2** Let $f$ be any constraint and let $G$ be any constraint set. Assume that $f$ is $T$-constructible from $G$ using no projection operation.

1. If all constraints in $G$ have imp support, then $f$ also has imp support.
2. If all constraints in $G$ are in $\mathcal{ED}$, then $f$ is also in $\mathcal{ED}$.

Proof of Lemma 7.2 Let $f$ be any $k$-ary constraint and let $G$ be any constraint set. Assume that $f$ is $T$-constructed from $g$ (or $\{g_1, g_2\}$ in the case of the multiplication operation) in $G$ by a single application of one of the operations described in Section 4 except for the projection operation. The proof proceeds by induction on the number of operations that are applied to $T$-construct $f$ from $G$. Clearly, the basis case (i.e., $f \in G$) is trivial.

1. Assume that $g$ has imp support and let $L$ be a factor list for $R_g$. We aim at proving that $R_f$ is in $\mathcal{IMP}$ by modifying this factor list $L$ step by step. Because the cases for the operations of normalizing, permutation, and expansion are trivial, we will concentrate on the remaining operations. For ease of notational complication, we focus on specific indices in the following argument.

**Pinning** Let us consider the case $f = g^{i_1=i_0}$. To keep our proof clean, we set $i = 1$ without loss of generality. Notice that $R_f = R_g^{i_1=i_0}$. Now, we need to eliminate all occurrences of $x_1$ from $L$. For any index $j \in [k]$, if there is a factor Implies($x_1, x_j$) in $L$, then we delete it from the list. If a factor Implies($x_j, x_1$) exists in $L$, then we replace it by $\Delta_0(x_j)$. If $L$ contains a factor $\Delta_0(x_1)$, then we simply delete it from $L$.

2. The proof for $\mathcal{ED}$ is in essence similar to (1); in particular, the multiplication and the linking operations are treated almost identically. Here, we note only a major difference. In the case of the pinning operation, say, $f = g^{i_1=i_0}$, if there exists a factor of the form EQ($x_1, x_j$) (XOR($x_1, x_j$), resp.) in a factor list $L$ for $R_g$, then we replace it with $\Delta_0(x_j)$ ($\Delta_1(x_j)$, resp.) in a factor list $L$ for $R_f$. The newly obtained list becomes a factor list for $R_f$, and thus $R_f$ belongs to $\mathcal{IMP}$ since $R_g$ is in $\mathcal{IMP}$.

3. Finally, assume that $f = g_1 \cdot g_2$. We denote by $L_1$ and $L_2$ two factor lists for $R_{g_1}$ and $R_{g_2}$, respectively. We combine these two lists into the union $L_1 \cup L_2$, which becomes a factor list for $R_f$. Therefore, $f$ has imp support.

4. The proof for $\mathcal{ED}$ is in essence similar to (1); in particular, the multiplication and the linking operations are treated almost identically. Here, we note only a major difference. In the case of the pinning operation, say, $f = g^{i_1=i_0}$, if there exists a factor of the form EQ($x_1, x_j$) (XOR($x_1, x_j$), resp.) in a factor list $L$ for $R_g$, then we replace it with $\Delta_0(x_j)$ ($\Delta_1(x_j)$, resp.). This manipulation eliminates the variable $x_1$ from the list $L$, and thus the resulting list becomes a factor list for $R_f$. \hfill \Box

For any constraint $f$ having imp support, by its definition, its underlying relation $R_f$ can be factorized as $R_f = g_1 \cdot g_2 \cdots g_m$, where each factor $g_i$ is one of the following forms: $\Delta_0(x), \Delta_1(x)$, and $\text{Implies}(x, y)$ ($x$ and $y$ may be the same). The factor list $L = \{g_1, g_2, \ldots, g_m\}$ for $R_f$ is said to be imp-distinctive if (i) no single variable appears both in $\Delta_c$ and $\text{Implies}$ in $L$, where $c \in \{0, 1\}$, and (ii) no factor of the form $\text{Implies}(x, x)$ belongs to $L$. In Lemma 7.3 we will show that such an imp-distinctive list always exists for an arbitrary constraint $f$ with imp support although such a list may not be unique in general.

**Lemma 7.3** For each constraint $f$ having imp support, there exists an imp-distinctive factor list for $R_f$.

**Proof.** This proof is similar to the proof of [13, Lemma 4]. Let $f$ be any constraint that has imp support. Let $L$ be any factor list for $R_f$, composed of relations of the forms $\Delta_0(x), \Delta_1(x)$, and $\text{Implies}(x, y)$, where $x$ and $y$ are appropriate variables. Since this list $L$ may not be imp-distinctive in general, we need to run the following five processes repeatedly to make $L$ imp-distinctive.

(i) From the factor list $L$, delete all factors of the form $\text{Implies}(x, x)$. After this process, we assume that $L$ contains no such factor. (ii) If $\{\Delta_0(x), \text{Implies}(x, y)\} \subseteq L$, then delete $\text{Implies}(x, y)$. (iii) If

$\text{(This notion is called “normalized” in [13]; however, we have already used the term “normalization” in a different context.)}$
\( \{ \Delta_0(y), \text{Implies}(x, y) \} \subseteq L \), then replace \( \text{Implies}(x, y) \) in \( L \) by \( \Delta_0(x) \). (iv) If \( \{ \Delta_1(x), \text{Implies}(x, y) \} \subseteq L \), then replace the factor \( \text{Implies}(x, y) \) in \( L \) by \( \Delta_1(y) \). (v) If \( \{ \Delta_1(y), \text{Implies}(x, y) \} \subseteq L \), then delete \( \text{Implies}(x, y) \) from \( L \).

Assume that no process is further applicable to the obtained factor list, say, \( L' \). We want to make a claim that \( L' \) is indeed imp-distinctive. Toward a contradiction, let us assume otherwise. Suppose that there exists a variable \( x \) appearing in both \( \Delta_c \) (where \( c \in \{0, 1\} \) and \( \text{Implies} \) in \( L' \). When \( c = 0 \), we can further apply either Process (ii) or Process (iii) to \( L' \). This is a contradiction against the definition of \( L' \). The case of \( c = 1 \) is similar. Next, suppose that a variable \( x \) appears in \( \text{Implies}(x, x) \) in \( L' \). In this case, Process (i) can be applied to \( L' \), a contradiction. Therefore, it follows that \( L' \) is imp-distinctive.

We have utilized a certain form of “factorization” of constraints. In fact, most constraints \( f \) can be expressed as products of a finite number of certain types of “factors,” which are usually “simpler” than the original constraints. Here, we look for particular factorization that is obtained by factors of the following forms: \( \Delta_0(x) \), \( \Delta_1(x) \), and \( \text{EQ}(x, y) \). After dividing \( f \) by those factors, the remaining portion of the constraint can be described by a notion of “simple form.” To explain this notion, we need to introduce new terminology. For each constraint \( f \) of arity \( k \), its representing Boolean matrix \( M_f \) is composed of rows indexed by all instances \( a = (a_1, a_2, \ldots, a_k) \) in \( R_f \) (in the standard lexicographical order) and columns indexed by numbers in \([k]\), and each \((a, i)\)-entry of \( M_f \) is a Boolean value \( a_i \). We say that a constraint is in simple form if its Boolean matrix does not contain all-0 columns, all-1 columns, or any pair of identical columns. Clearly, any all-0 constraint \( f \) (i.e., \( f \equiv 0 \)) cannot be in simple form.

As shown in Lemma 7.3, it is always possible to factorize any given constraint \( f \) into two factors, at least one of which must be in simple form. For the proof of this lemma, we will deal with a representing Boolean matrix \( M_f \) of the constraint \( f \) and we will execute a sweeping procedure that eliminates, one by one, unwanted columns of \( M_f \) until the remaining matrix becomes a simple form. The lemma will become useful in the proof of Proposition 5.11. Recall that \( \text{EQ}_1 \) represents \([1, 1]\).

**Lemma 7.4** Let \( f \) be any constraint of arity \( k \geq 1 \). If \( f \) never belongs to \((\text{IMP} \cap \text{ED}) \cup \{\text{EQ}_1\}\) after any normalization, then there exist two indices \( m \) and \( m' \) with \( 1 \leq m \leq m' \leq k \), an arity-\(m\) relation \( R \in (\text{IMP} \cap \text{ED}) \cup \{\text{EQ}_1\} \), and a constraint \( g \) of arity \( k - m \) such that (after properly permuting variable indices) \( f(x_1, \ldots, x_k) = R(x_1, \ldots, x_m)g(x_{m+1}, \ldots, x_k) \), \( m \neq k \), \( g \leq_{\text{con}} f \), and \( g \) is in simple form. Moreover, \( f \) has imp support iff \( g \) has imp support, and \( f \in \text{ED} \) iff \( g \in \text{ED} \).

**Proof.** Let \( f \) be any constraint of arity \( k \geq 1 \). Assume that \( f \) cannot equal \( c \cdot R' \) for a certain constant \( c \in \mathbb{C} \) and a certain relation \( R' \in (\text{IMP} \cap \text{ED}) \cup \{\text{EQ}_1\} \). To generate a constraint of simple form, we run the following algorithm, called a sweeping procedure. The algorithm uses two parameters \( g \) and \( R \), and it updates them at each step until \( g \) becomes the desired simple form. Initially, we set \( g \) to \( f \) and set \( R \) to be \( \text{EQ}_1 \) over a single variable, say, \( x_1 \). Suppose below that, after an appropriate re-ordering of variable indices, \( g \) and \( R \) have the forms \( g(x_c, x_{c+1}, \ldots, x_k) \) and \( R(x_1, x_2, \ldots, x_d) \) for two numbers \( d \) and \( c \) satisfying \( 1 \leq c \leq d \leq k \). Let \( M_g \) denote the representing Boolean matrix of \( g \).

(i) Assume that there exists an all-0 column indexed, say, \( i \) in \( M_g \). We then delete this column \( i \) from \( M_g \). When this situation happens, \( g(x_c, \ldots, x_k) \) must be factorized into \( \Delta_0(x_i)g^{x_i=0}(x_c, \ldots, x_{i-1}, x_{i+1}, \ldots, x_k) \). After the deletion of the column \( i \), we update \( g \) to be \( g^{x_i=0} \) and set \( R \) to be \( \Delta_0 \cdot R \) that is defined as \( (\Delta_0 \cdot R)(x_1, \ldots, x_d, x_i) = \Delta_0(x_i)R(x_1, \ldots, x_d) \) if \( d < i \), and \( (\Delta_0 \cdot R)(x_1, \ldots, x_d) = \Delta_0(x_i)R(x_1, \ldots, x_d) \) if \( i \leq d \). (ii) If an all-1 column, say, \( i \) exists in \( M_g \), then we delete the column \( i \). Since \( g(x_c, \ldots, x_k) = \Delta_1(x_i)g^{x_i=1}(x_c, \ldots, x_{i-1}, x_{i+1}, \ldots, x_k) \), we can update \( g \) and \( R \) to \( g^{x_i=1} \) and \( \Delta_1 \cdot R \), respectively. (iii) Assuming that there are no all-0 and all-1 columns, if there is a pair of identical columns, say, \( i \) and \( j \) \((i < j)\), then we delete the column \( i \). Note that \( g(x_c, \ldots, x_k) \) equals \( \text{EQ}(x_1, x_j)g^{x_1=x_j}(x_c, \ldots, x_{i-1}, x_{i+1}, \ldots, x_k) \). After the deletion, we update \( g \) and \( R \) respectively to \( g^{x_1=x_j} \) and \( \text{EQ} \cdot R \), where \( (\text{EQ} \cdot R)(x_1, \ldots, x_d, x_i, x_j) = \text{EQ}(x_1, x_j)R(x_1, \ldots, x_d, x_i, x_j) \) if \( d < i \), \( (\text{EQ} \cdot R)(x_1, \ldots, x_i, x_j) = \text{EQ}(x_i, x_j)R(x_1, \ldots, x_{i-1}, x_j, x_{i+1}, \ldots, x_d) \) if \( i \leq d < j \), and \( (\text{EQ} \cdot R)(x_1, \ldots, x_d) = \text{EQ}(x_1, x_j)R(x_1, \ldots, x_d) \) if \( j \leq d \).

After an execution of the above sweeping procedure, we obtain a relation \( R \) and a constraint \( g \) satisfying the equation \( f(x_1, \ldots, x_k) = R(x_1, \ldots, x_m)g(x_{m+1}, \ldots, x_k) \) (after an appropriate permutation of variable indices). In particular, when \( m = 1 \), none of the cases (i)–(iii) occurs, and thus \( R \) equals \( \text{EQ}_1 \) and \( g \) coincides with \( f \). When \( m \neq 1 \), the procedure guarantees that \( R \) is a product of some of \( \Delta_0 \), \( \Delta_1 \), and \( \text{EQ} \). In either case, \( R \) belongs to \((\text{IMP} \cap \text{ED}) \cup \{\text{EQ}_1\}\). Now, we will show that \( m \neq k \). If \( m = k \), then \( g \) has no input variable. Thus, \( g \) becomes a constant, say, \( c \in \mathbb{C} \), which implies that \( f \) equals \( c \cdot R \). Obviously, \( f \) belongs to \((\text{IMP} \cap \text{ED}) \cup \{\text{EQ}_1\}\) after appropriate normalization. This is a clear contradiction against our assumption,
and therefore we conclude that \( m \neq k \). Moreover, \( g \) should be in simple form. The procedure clearly ensures that \( g \leq_{\text{con}} f \).

The second part of the lemma is shown as follows. Assume that \( f \) has imp support. By the behavior of the sweeping procedure, \( g \) should be T-constructible from \( f \) without the projection operation. Lemma \([7, 2] \) then ensures that \( g \) has imp support as well. Finally, we will show that if \( g \) has imp support then \( f \) has imp support. As a starting point, assume that \( g \) has imp support. Notice that, since \( f = R \cdot g \), \( R_f \) coincides with \( R \cdot R_g \). Since both \( R_g \) and \( R \) are in \( \text{IMP} \), Lemma \([6] \) shows that \( R \cdot R_g \) belongs to \( \text{IMP} \). Therefore, \( f \) has imp support. In a similar fashion, it is easy to show, using Lemma \([7, 2] \), that \( f \in \mathcal{ED} \) iff \( g \in \mathcal{ED} \).

Non-zero constraints require a special attention for weighted \#CSPs because their underlying relations are all equal to \( \{0, 1\}^k \) (where \( k \) is the arities of the constraints) and they cannot be dealt with simply by a factorization technique. However, we can show that every non-degenerate constraint in \( \mathcal{NZ} \) T-constructs a quite useful binary constraint residing in \( \mathcal{NZ} \).

**Lemma 7.5** Let \( f \in \mathcal{NZ} \) be any constraint of arity \( k \geq 2 \) and let \( F \) be any set of constraints. If \( f \notin \mathcal{DG} \), then there exists a constraint \( h = (1, x, y, z) \) for which \( xyz \neq 0 \), \( z \neq xy \), \( h \leq_{\text{con}} f \). In particular, \( h \) (before normalizing) has the form \( f_{x=a_3, \ldots, x_k=a_k} \) for certain constants \((a_3, \ldots, a_k) \in \{0, 1\}^{k-2} \), after an appropriate permutation of variable indices.

**Proof.** This proof is part of the proof of \([8, \text{Lemma 4.4}] \) meant for exact counting of weighted \#CSPs with complex-valued constraints. A similar argument for non-negative constraints is found in the proof of \([14, \text{Lemma 14}] \). Since the proof of this lemma is not difficult, for completeness, we include the proof.

Assume that \( f \) is a non-zero constraint of arity \( k \geq 2 \). For each index \( i \in [k] \), using the assumption \( f \in \mathcal{NZ} \), we define a constraint \( g_i \) as \( g_i(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_k) = f_{x_1=1}(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_k) / f_{x_1=0}(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_k) \). Let us show by contradiction the existence of an index \( i \in [k] \) such that \( g_i \) is not a constant function. Suppose otherwise. For each index \( i \in [k] \), we define \( u_i = [1, b_i] \), where \( b_i \in \mathbb{C} \) is a unique constant satisfying \( g_i(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_k) = b_i \) for any vector \((x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_k) \in \{0, 1\}^{k-1} \). Such \( b_i \) actually exists because \( g_i \) is a constraint function. Since \( f_{x_1=1}(x_2, \ldots, x_k) = b_i f_{x_1=0}(x_2, \ldots, x_k) \), it follows that \( f(x_1, \ldots, x_k) = u_i(x_1) f_{x_1=0}(x_2, \ldots, x_k) \). By a similar argument, we also have \( f(x_1, \ldots, x_k) = u_i(x_1) u_2(x_2) f_{x_1=0,x_2=0}(x_3, \ldots, x_k) \). After repeating this argument, in the end, we obtain \( f(x_1, \ldots, x_k) = f_{x_1=0,\ldots,x_k=0} \prod_{i=1}^k u_i(x_i) \), where \( f_{x_1=0,\ldots,x_k=0} \) is a certain complex number. This indicates that \( f \) is degenerate, and thus it belongs to \( \mathcal{DG} \), a contradiction. Therefore, for a certain index \( i \in [k] \), \( g_i \) is not a constant function. Set \( i = 1 \) for simplicity.

Let us choose a sequence \((a_3, \ldots, a_k) \in \{0, 1\}^{k-2} \) for which \( g_1(0, a_3, \ldots, a_k) \neq g_1(1, a_3, \ldots, a_k) \). Define \( h = f_{x_3=a_3, \ldots, x_k=a_k} \), which must have the form \((w, x, y, z) \). From \( h \in \mathcal{NZ} \), \( x y z \neq 0 \) follows immediately. By normalizing \( h \) appropriately, we can assume that \( h \) takes the form of \((1, x, y, z) \) with \( x y z \neq 0 \). Moreover, from \( h(1, 0)/h(0, 0) \neq h(1, 1)/h(0, 1) \), we obtain the inequality \( x y \neq z \).

## 8 Imp Support and the Hardness of \#CSPs

Based on various properties given in Sections 5–7, we will present, in Propositions 8.1 and 8.7, two hardness results on the approximation complexity of certain counting problems \#\text{CSP}(f) \). We will show these results by building appropriate AP-reductions from \#\text{CSP}(\text{OR}) \), indicating the \#P-hardness of the \#\text{CSP}(f)’s by Lemma 14. Moreover, those results look “complementarily:” that is, Proposition 8.1 deals with constraints having imp support whereas Proposition 8.7 targets constraints lacking imp support. They will become a core of the proof of our main theorem in Section 9.

Our first focal point is to discuss constraints that have imp support. Particularly, we are interested in the case where the constraints are not in \( \mathcal{ED} \).

**Proposition 8.1** Let \( f \) be any constraint having imp support and let \( F \) be any constraint set. If \( f \notin \mathcal{ED} \), then \( \#\text{CSP}(\text{OR}, F) \leq_{\text{AP}} \#\text{CSP}(f, F) \).

The proof of this proposition requires five claims concerning the relation \( \text{Implies} \). We begin with a useful result, shown in \([13] \), on relations residing outside of \( \text{IMP} \cup \mathcal{NZ} \). For its description, we introduce two additional notations. For any two vectors \( a = (a_1, \ldots, a_k) \) and \( b = (b_1, \ldots, b_k) \) in \( \{0, 1\}^k \), the notation \( a \land b \) denotes the vector \((a_1 \land b_1, \ldots, a_k \land b_k) \) and \( a \lor b \) denotes \((a_1 \lor b_1, \ldots, a_k \lor b_k) \), where \( a_i \land b_i = \min\{a_i, b_i\} \) and \( a_i \lor b_i = \max\{a_i, b_i\} \).
Lemma 8.2 [15 Corollary 18] For any nonempty relation \( R \notin \text{IMP} \cup \text{N} \), there are two distinct instances \( a \) and \( b \) in \( R \) such that either \( a \wedge b \notin R \) or \( a \vee b \notin R \) holds.

Secondly, we present a simple characterization of binary constraints not in \( \text{IM} \cup \text{N} \).

**Lemma 8.3** For any constraint \( f \) of arity \( 2 \) with \( f \neq 0 \), \( f \notin \text{IM} \cup \text{N} \) iff \( f \) is of the form \((a, b, c, d)\) with \( ad = 0 \) and \( bc \neq 0 \).

**Proof.** Let \( f \) be any binary constraint satisfying that \( f \neq 0 \). Since \( f \) is binary, by Lemma 8.2, it holds that \( f \notin \text{IM} \) iff \( R_f \notin \text{IMP} \).

(Only If-part) Assume that \( f \) does not belong to \( \text{IM} \cup \text{N} \). Since \( R_f \notin \text{IMP} \), Lemma 8.2 guarantees the existence of two distinct elements \( a' = (a_1, a_2) \) and \( b' = (b_1, b_2) \) in \( R_f \) satisfying that either \( a' \wedge b' \notin R_f \) or \( a' \vee b' \notin R_f \). Let us consider the first case where \( a_1 = b_1 = 0 \). Since \( a' \neq b' \), we obtain \( a_2 \neq b_2 \) and thus \( \{a' \wedge b', a' \vee b'\} = \{a', b'\} \subseteq R_f \), a contradiction. The second case \( a_1 = b_2 = 0 \) is similar. Consider the third case \( a_1 \neq b_1 \). Without loss of generality, we set \( a_1 = 0 \) and \( b_1 = 1 \). When \( a' = (0, 0) \), we obtain both \( a' = a' \wedge b' \) and \( b' = a' \vee b' \), leading to a contradiction. When \( a' = (0, 1) \), \( b' \) should equal \((1, 0)\) because, otherwise, \( a' = a' \wedge b' \) and \( b' = a' \vee b' \) follow. Since either \( a' \wedge b' \notin R_f \) or \( a' \vee b' \notin R_f \), \( R_f \)'s outcome should be one of the following three forms: \((0, 1, 1, 1)\), \((1, 1, 1, 0)\), and \((0, 1, 1, 0)\). In other words, \( R_f \) (seen as a function) equals \( \text{OR}, \text{NAND}, \text{or} \text{XOR} \). From this consequence, the lemma immediately follows.

(If-part) Let \( f = (a, b, c, d) \) with \( a, b, c, d \in \mathbb{C} \) and assume that \( ad = 0 \) and \( bc \neq 0 \). This instantly implies \( f \notin \text{N} \). Next, we wish to show that \( f \notin \text{IM} \). Toward a contradiction, assume that \( f \) is in \( \text{IM} \), implying \( R_f \in \text{IMP} \). Lemma 8.2 then yields an imp-distinctive factor list for \( R_f \). Such a list should be a subset of \( \{\text{Imp}(x_1, x_2), \text{Imp}(x_2, x_1), \Delta_0(x_j), \Delta_1(x_j) \mid j = 1, 2\} \). Let us consider all possible imp-distinctive lists for \( R_f \). By checking them carefully, we can find that all the lists define only 13 binary relations, excluding \( \text{OR}, \text{NAND}, \text{and} \text{XOR} \). In addition, it is not difficult to show that, for any binary relation \( R \notin \{\text{OR}, \text{NAND}, \text{XOR}\} \) if \( R = R_f \) then either \( ad \neq 0 \) or \( bc = 0 \) holds. This clearly contradicts our assumption on \( f \). Therefore, we reach a conclusion that \( f \notin \text{IM} \).

As an immediate corollary of Lemma 8.3, we obtain a characterization of binary constraints in \( \text{IM} \). Recall that \( \text{IM} \cap \text{N} = \emptyset \).

**Corollary 8.4** For any constraint \( f = (a, b, c, d) \) with \( a, b, c, d \in \mathbb{C} \) and \( f \neq 0 \), \( f \in \text{IM} \) iff \( bc = 0 \).

**Proof.** Let \( f = (a, b, c, d) \) with \( a, b, c, d \in \mathbb{C} \) and \( f \neq 0 \). Lemma 8.3 states that \( f \notin \text{IM} \cup \text{N} \) iff either \( ad \neq 0 \) or \( bc = 0 \) holds. Obviously, \( f \notin \text{IM} \) implies \( f \notin \text{N} \). Moreover, it holds that \( f \notin \text{N} \) iff \( abcd = 0 \). Because if \( ad \neq 0 \) and \( abcd = 0 \) then at least one of \( b \) and \( c \) should be 0, the corollary follows immediately.

The third claim is more technical. To explain it, we need to introduce a directed graph \( G_{f,L} \) induced from a factor list \( L \) for \( R_f \). The graph \( G_{f,L} \) consists of nodes whose names are variables \( x_1, x_2, \ldots, x_k \) appearing in \( R_f(x_1, x_2, \ldots, x_k) \) and of edges \((x,y)\) whenever a factor \( \text{Imp}(x,y) \) is in \( L \). We call this \( G_{f,L} \) an imp graph of \( R_f \) and \( L \). We say that a factor list \( L \) for \( R_f \) is good if (i) \( L \) consists only of \( \text{Imp}(x,y) \)'s, (ii) every node in \( G_{f,L} \) is adjacent to at least one node in \( G_{f,L} \), and (iii) there is no cycle in \( G_{f,L} \). Note that, whenever \( f \) has a good factor list, Condition (iii) prohibits \( f \) from belonging to \( \mathcal{E}D \).

The notation \( \text{COMP}_1(f) \) for a constraint \( f \) of arity \( k \) means the set \( \{f_{x=c} \mid i \in [k], c \in \{0, 1\}\} \). Furthermore, let \( \text{COMP}_2(f) = \{f_{x=c} : j-d \mid i, j \in [k], i \neq j, c, d \in \{1, 0\}\} \). Every constraint in \( \text{COMP}_1(f) \cup \text{COMP}_2(f) \) is obviously \( T \)-constructible from \( f \) by applications of the pinning operation.

**Lemma 8.5** For any constraint \( f \) of arity \( k \geq 3 \), if \( R_f \) has a good factor list, then there exists a constraint \( h \in \text{COMP}_1(f) \cup \text{COMP}_2(f) \) such that \( R_h \) has a good factor list.

**Proof.** Let \( R_f \) be the underlying relation of an arity-\( k \) constraint \( f \) defined on \( k \) Boolean variables \( \{x_1, \ldots, x_k\} \). Let \( L \) be any good factor list for \( R_f \) and let \( G_{f,L} = (V,E) \) be an imp graph of \( R_f \) and \( L \) with \( V = \{x_1, \ldots, x_k\} \). There are two cases to handle differently.

(1) Suppose that there exists an index \( i \in [n] \) for which (i) \( (x_i, x_j) \notin E \) holds for all indices \( j \in [k] - \{i\} \) and (ii) the incident set \( E(x_i) \) of the node \( x_i \) is a singleton. By the property of the imp graph, a certain index \( j \in [k] - \{i\} \) must satisfy that \( (x_i, x_j) \in E \). Since \( |E(x_i)| = 1 \), this node \( x_j \) should be unique. Now, we are focused on this particular node \( x_j \).

(a) Assume that \( |E(x_i)| > 1 \). For the desired \( h \) stated in the lemma, we set \( h = f_{x_i=1} \), which belongs to \( \text{COMP}_1(f) \). Next, we want to show that \( R_h \) has a good factor list. Let us define \( L' = L - \{\text{Imp}(x_j, x_i)\} \).
It is easy to show that \( L' \) is a factor list for \( R_h \). With this list \( L' \), define \( G_h \) to be an imp graph of \( R_h \) and \( L' \).

Note that \( G_{h, L'} \) contains no node named \( x_i \). Obviously, every node in \( G_{h, L'} \) is adjacent to at least one node in \( G_{h, L'} \). Moreover, there is no cycle in \( G_{h, L'} \) because any cycle in \( G_{h, L'} \) becomes a cycle in \( G_{f, L} \). Therefore, \( L' \) is a good factor list for \( R_h \).

(b) Assume that \( |E(x_j)| = 1 \).
This means \( E(x_j) = \{ x_i \} \), and the graph \( H = \{ \{ x_i, x_j \}, \{ (x_j, x_i) \} \} \) forms a connected component of \( G_{f, L} \).
Here, we set \( h = f^{x_i=1, x_j=1} \) so that \( h \) belongs to \( \text{COMP}_2(f) \).
We define \( L' = L - \{ \text{Implies}(x_j, x_i) \} \), which becomes a factor list for \( R_h \).
Note that \( L' \) cannot be empty because, otherwise, \( L \) consists only of \( \text{Implies}(x_j, x_i) \) and thus \( k = 2 \) follows, a contradiction.
Now, we claim that \( L' \) is good.
Let \( G_{h, L'} \) be an imp graph of \( R_h \), which has neither the node \( x_i \) nor the node \( x_j \).
Note that every node in \( G_{h, L'} \) is adjacent to at least one node because deleting the subgraph \( H \) does not affect the adjacency property of the other nodes in \( G_{f, L} \). Thus, \( L' \) is a good factor list for \( R_h \).

(2) Assume that Case (1) does not happen.
Choose a variable \( x_i \) so that \( (x_j, x_i) \notin E \) for any \( j \in [k] - \{ i \} \).
Such a variable should exist because there is no cycle in \( G_{f, L} \). The desired \( h \) is now defined as \( h = f^{x_i=0} \), which clearly falls into \( \text{COMP}_1(f) \).
Let \( L' = L - \{ \text{Implies}(x_i, x_j) \mid j \in [k] - \{ i \} \} \). This \( L' \) becomes a factor list for \( R_h \).
If any node \( x_j \) with \( j \neq i \) is deleted from \( G_{f, L} \), then \( |E(x_j)| = 1 \) follows and this \( x_j \) satisfies Case (1).
This is a contradiction; hence, an imp graph of \( R_h \) and \( L' \) lacks only the node \( x_i \). This ensures that the properties of \( L \) are naturally inherited to \( L' \); therefore, \( L' \) is good.

The notion of good factor list is closely related to that of simple form. Exploring this relationship, we can prove the following corollary, in which we decrease the arity of a given constraint while maintaining the imp-support property and the non-membership property to \( \mathcal{E} \mathcal{D} \).

**Corollary 8.6** Let \( f \) be any constraint of arity \( k \geq 3 \). Assume that \( f \) is in simple form. If \( f \) has imp support and \( f \notin \mathcal{E} \mathcal{D} \), then there exists a constraint \( h \) of arity less than \( k \) such that \( h \) has imp support, \( h \notin \mathcal{E} \mathcal{D} \), and \( h \leq_{\text{con}} f \).

**Proof.** Let \( k \geq 3 \) and let \( f \) be any arity-\( k \) constraint having imp support. Assume that \( f \) is in simple form but it is not in \( \mathcal{E} \mathcal{D} \).
Since \( f \) has imp support, by Lemma \([62] \), every constraint \( h \) that is \( T \)-constructed from \( f \) by applications of the pinning operation has imp support.
Since \( R_f \in \text{IMP} \), we choose an imp-distinctive factor list \( L \) for \( R_f \).
Note that every factor in \( L \) is of the form \( \text{Implies} \) because \( L \) contains a factor \( \Delta_i \), where \( c \in \{0,1\} \), then \( M_f \) must contain an all-\( c \) column, a contradiction against the simple-form property of \( f \).

To appeal to Lemma \([53] \), we need to show that \( L \) is a good factor list for \( R_f \).
Let \( G_{f, L} \) denote an imp graph of \( R_f \) and \( L \).
Firstly, we deal with a situation where there exists a variable that appears in no factor in \( L \).
We choose such a variable, say, \( x_i \), and define \( h = f^{x_i=0} \), which clearly belongs to \( \text{COMP}_1(f) \). Moreover, the value of \( x_i \) does not affect the computation of \( f \); thus, it follows that \( f^{x_i=0}(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_k) = f^{x_i=1}(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_k) \).
Therefore, we obtain \( f(x_1, \ldots, x_k) = h(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_k) \), which implies that \( h \) has imp support.
Since \( f \notin \mathcal{E} \mathcal{D} \), we conclude that \( h \notin \mathcal{E} \mathcal{D} \). Hereafter, we assume that every variable appears in at least one factor in \( L \).

Secondly, we will show that \( G_{f, L} \) has no cycle. Suppose otherwise; namely, for a certain series \((x_{i_1}, x_{i_2}, \ldots, x_{i_m})\) of variables, the set \( \{ \text{Implies}(x_{i_j}, x_{i_{j+1}}), \text{Implies}(x_{i_m}, x_{i_1}) \mid j \in [m-1] \} \) is included in \( L \).
Clearly, \( M_f \) includes two identical columns \( t_1 \) and \( t_m \), and thus \( f \) cannot be in simple form, a contradiction.
Therefore, no cycle exists in \( G_{f, L} \). Overall, we conclude that \( L \) is a good factor list for \( R_f \). Lemma \([53] \) then gives a constraint \( h \) such that \( h \) is \( T \)-constructed from \( f \) by one or more applications of the pinning operation and \( R_h \) has a good factor list. Therefore, \( h \) should have imp support. Moreover, the definition of good factor list implies that \( h \) should not belong to \( \mathcal{E} \mathcal{D} \). The use of the pinning operation guarantees that the arity of \( h \) should be less than \( k \).

Finally, we will give the proof of Proposition \([81] \).

**Proof of Proposition 8.1.** Let \( f \) be any constraint of arity \( k \geq 1 \) and let \( F \) be any constraint set. We will show by induction on \( k \) that if \( f \) has imp support but it is not in \( \mathcal{E} \mathcal{D} \) then \#\text{CSP}^+(\text{OR}, F) \) is AP-reduced to \#\text{CSP}^+(f, F).
Let us assume that \( f \) has imp support and \( f \notin \mathcal{E} \mathcal{D} \).
Note that \( f \neq 0 \) because, otherwise, \( f \) belongs to \( \mathcal{E} \mathcal{D} \).

[Basis Case: \( k = 1 \)] In this case, the proposition is trivially true, because all unary functions are already in \( \mathcal{E} \mathcal{D} \).

[Next Case: \( k = 2 \)] Assume that \( f = (a, b, c, d) \) with \( a, b, c, d \in \mathbb{C} \). Since \( f \) is a binary constraint, the imp-support property of \( f \) makes \( f \) belong to \( \mathcal{I} \mathcal{M} \). Since \( f \neq 0 \), Corollary \([81] \) yields \( bc = 0 \). Now, we examine
the following three possible cases.

(1) The first case is that \( b = 0 \) but \( c \neq 0 \). Let us examine all four possible values of \( f \). Write \( u \) for the constraint \([c, d]\). (i) If \( f = (0, 0, c, 0) \), then \( f \) is clearly in \( \mathcal{ED} \). (ii) Let \( f = (0, 0, c, d) \) with \( d \neq 0 \). The value \( f(x_1, x_2) \) actually equals \( \Delta_1(x_1)u(x_2) \), and thus \( f \) belongs to \( \mathcal{ED} \). (iii) If \( f = (a, 0, c, 0) \) with \( a \neq 0 \), then \( f \) has the form \( u(x_1)\Delta_0(x_2) \), implying \( f \in \mathcal{ED} \). These three cases immediately lead to a contradiction against the assumption \( f \notin \mathcal{ED} \). (iv) The remaining case is that \( f = (a, 0, c, d) \) with \( ad \neq 0 \). By normalizing \( f \) appropriately, we may assume that \( f \) has the form \((1, 0, c, d)\). Now, we apply Lemma 6.7 and then obtain the desired AP-reduction from \( \#\text{CSP}^*(\mathcal{OR}, \mathcal{F}) \) to \( \#\text{CSP}^*(f, \mathcal{F}) \).

(2) The second case where \( b \neq 0 \) and \( c = 0 \) is symmetric to Case (1) and is omitted.

(3) Let us consider the third case where \( b = c = 0 \). There are four possible choices for \( f \): (i') \( f = (a, 0, 0, 0) \) with \( a \neq 0 \), (ii') \( f = (0, 0, 0, d) \) with \( d \neq 0 \), (iii') \( f = (a, 0, 0, d) \) with \( ad \neq 0 \), and (iv') \( f = (0, 0, 0, 0) \). In all those four cases, clearly \( f \) belongs to \( \mathcal{ED} \), a contradiction. This completes the case of \( k = 2 \).

[Induction Case: \( k \geq 3 \)] As the induction hypothesis, we assume that the proposition is true for any constraint of arity less than \( k \).

(1) Assume that \( f \) falls into \((\text{IMP} \cap \mathcal{ED}) \cup \{\text{EQ}_1\}\) after appropriate normalization; in other words, \( f \) equals \( c \cdot \mathcal{R} \), where \( c \in \mathcal{C} \) and \( \mathcal{R} \in (\text{IMP} \cap \mathcal{ED}) \cup \{\text{EQ}_1\} \). There are two cases, \( \mathcal{R} \in \text{IMP} \cap \mathcal{ED} \) or \( \mathcal{R} = \text{EQ}_1 \), to consider. In either case, however, \( f \) belongs to \( \mathcal{ED} \). This contradicts our assumption.

(2) Assume that Case (1) does not occur. Lemma 7.3 then provides a relation \( \mathcal{R} \) in \((\text{IMP} \cap \mathcal{ED}) \cup \{\text{EQ}_1\}\) and a constraint \( g \) in simple form that satisfies \( g \leq_{\text{con}} f \) and \( f = \mathcal{R} \cdot g \). Moreover, the second part of Lemma 7.3 implies that \( g \) has imp support and \( g \) does not belong to \( \mathcal{ED} \). Firstly, we consider the case where \( \mathcal{R} \neq \text{EQ}_1 \). Since \( f \neq g \), an execution of the sweeping procedure given in the proof of Lemma 7.3 makes the arity of \( g \) smaller than that of \( f \). The induction hypothesis therefore implies that \( \#\text{CSP}^*(\mathcal{OR}, \mathcal{F}) \leq_{\text{AP}} \#\text{CSP}^*(g, \mathcal{F}) \). Since \( g \leq_{\text{con}} f \), by Lemmas 5.1 and 5.2 we obtain \( \#\text{CSP}^*(\mathcal{OR}, \mathcal{F}) \leq_{\text{AP}} \#\text{CSP}^*(f, \mathcal{F}) \). Secondly, we consider the case of \( \mathcal{R} = \text{EQ}_1 \). This case implies \( f = g \), and thus \( f \) should be in simple form. Appealing to Corollary 5.6 we obtain a constraint \( h \) of arity smaller than \( k \) satisfying that \( h \leq_{\text{con}} f \), \( h \notin \mathcal{ED} \), and \( h \) has imp support. Our induction hypothesis then ensures that \( \#\text{CSP}^*(\mathcal{OR}, \mathcal{F}) \leq_{\text{AP}} \#\text{CSP}^*(h, \mathcal{F}) \). Moreover, since \( h \leq_{\text{con}} f \), \( \#\text{CSP}^*(h, \mathcal{F}) \) is AP-reduced to \( \#\text{CSP}^*(f, \mathcal{F}) \) by Lemma 5.2. The desired conclusion of the proposition follows by combining those two AP-reductions.

In Proposition 8.7, we have discussed constraints with imp support. Our second focal point is to discuss constraints that lack imp support, provided that they are chosen from the outside of \( \mathcal{ED} \cup \mathcal{NZ} \).

**Proposition 8.7** Let \( f \) be any constraint not in \( \mathcal{ED} \cup \mathcal{NZ} \). If \( f \) has no imp support, then \( \#\text{CSP}^*(\mathcal{OR}, \mathcal{F}) \leq_{\text{AP}} \#\text{CSP}^*(f, \mathcal{F}) \) for any constraint set \( \mathcal{F} \).

The proof of this proposition relies on Lemma 8.3 which gives a complete characterization of binary constraints inside \( \mathcal{IM} \cup \mathcal{NZ} \). The proposition is proven easily by an assist of Lemma 7.2 as well.

**Proof of Proposition 8.7** Let \( f \) be any constraint of arity \( k \geq 1 \) and assume that \( f \) has no imp support and \( f \notin \mathcal{ED} \cup \mathcal{NZ} \). Our proof proceeds by induction on \( k \). The base case \( k = 1 \) is trivial since all unary constraints belong to \( \mathcal{ED} \). Next, assume that \( k = 2 \). Notice that \( f \) cannot be in \( \mathcal{IM} \) since \( R_f \notin \text{IMP} \) by Lemma 7.1. We then apply Lemma 8.3 to \( f \). It then follows that \( f \) must have one of the following forms: \((0, b, c, 0), (0, b, c, d), \) and \((a, b, c, 0) \). Since \( f \notin \mathcal{ED} \), \( f \) cannot be of the form \((0, b, c, 0) \). In all the other cases, Lemma 6.3 establishes an AP-reduction from \( \#\text{CSP}^*(\mathcal{OR}, \mathcal{F}) \) to \( \#\text{CSP}^*(f, \mathcal{F}) \) for any constraint set \( \mathcal{F} \).

Finally, assume that \( k \geq 3 \). Now, we want to build a constraint \( g \notin \mathcal{ED} \cup \mathcal{NZ} \) of arity two such that \( g \leq_{\text{con}} f \) and \( g \) has no imp support. Since \( R_f \notin \text{IMP} \cup \mathcal{NZ} \), Lemma 8.2 supplies two vectors \( a = (a_1, \ldots, a_k) \) and \( b = (b_1, \ldots, b_i) \) in \( R_f \) satisfying either \( a \land b \notin R_f \) or \( a \lor b \notin R_f \) (or both). First, we will claim that (*) there are indices \( i, j \in [k] \) such that \((a_i, b_j) = (0, 1) \) and \((a_j, b_i) = (1, 0) \). Assume otherwise; namely, either \((a_i, b_i) \in \{1, 0\} \times \{0, 1\} \) for all \( i \in [k] \) or \((a_i, b_i) \in \{1, 0\} \times \{0, 1\} \) for all \( i \in [k] \). Let us consider the case where \( a \land b \notin R_f \). It easily follows that either \( a = a \lor b = b \lor b \) holds. This is a contradiction against the assumption \( a \lor b \notin R_f \). The other case where \( a \land b \notin R_f \) is similarly treated. Therefore, the claim (*) should hold.

Hereafter, we assume that \( a \lor b \notin R_f \) since the other case (i.e., \( a \land b \notin R_f \)) is similarly handled. For simplicity, let \((a_1, b_1) = (0, 1) \) and \((a_2, b_2) = (1, 0) \). Now, we recursively define a new constraint \( g \). Initially, we set \( f_2 = f \). If \( f_{i-1} \) (\( 3 \leq i \leq k \)) has been already defined, then we define \( f_i \) as follows. For each bit \( c \in \{0, 1\} \), if \((a_i, b_i) = (c, c) \), then set \( f_i = f_{i-1}^= \). If \((a_i, b_i) = (a_1, b_1) \), then let \( f_i = f_{i-1}^> \). If \((a_i, b_i) = (a_2, b_2) \), then let \( f_i = f_{i-1}^< \). Finally, we define \( g \) to be \( f_k \). By this construction of \( g, (0, 1) \) and \((1, 0) \) are in \( R_g \); however, \((1, 1) \)
is not in $R_f$ because $a \lor b = (1,1, c_3, c_4, \ldots, c_k) \notin R_f$ (for certain bits $c_i$'s) implies $g(1,1) = 0$.  In summary, it holds that $g(0,1)g(1,0) \neq 0$ and $g(0,0)g(1,1) = 0$.  Lemma \ref{lem:imp support} then concludes that $g$ is not in $IM \cup NZ$.  In particular, since $g$ is of arity two, $g$ has no imp support by Lemma \ref{lem:imp support}.  Moreover, the above construction is actually $T$-construction, and thus this fact ensures that $g \subseteq_{con} f$.  Because this $T$-construction obviously uses no projection operation, by Lemma \ref{lem:transitivity}, $f \notin ED$ implies $g \notin ED$.  To end our proof, we will claim that $g(0,0) \neq 0$.  Assume otherwise; namely, $g$ has the form $(0, x, y, 0)$ with $xy \neq 0$.  Obviously, $g$ belongs to $ED$, a contradiction.  Hence, $g(0,0) \neq 0$ holds.  We then conclude that $g$ equals $(w, x, y, 0)$ for certain non-zero constants $x, y, w$.  By Lemma \ref{lem:imp support} it follows that $#CSP^*(OR, F) \leq_{AP} #CSP^*(g,F)$.  Since $g \subseteq_{con} f$, we obtain the desired consequence.

\section{Dichotomy Theorem}

Our dichotomy theorem states that all counting problems of the form $#CSP^*(F)$ can be classified into exactly two categories, one of which consists of polynomial-time solvable problems and the other consists of $#P$-hard problems, assuming that $#SAT \notin FP$.  This theorem steps forward in a direction toward a complete analysis of a more general form of constraint than Boolean constraints (e.g., \cite{dyer}).  Theorem also gives an approximation version of the dichotomy theorem of Cai et al. \cite{cai} for exact counting problems.  Here, we rephrase the theorem given in Section \ref{sec:main result} as follows.

\begin{theorem}[rephrased] Let $F$ be any set of constraints.  If $F \subseteq ED$, then $#CSP^*(F)$ is in $FP$.  Otherwise, $#SAT \leq_{AP} #CSP^*(F)$ holds.
\end{theorem}

Through Sections\ref{sec:hardness} to \ref{sec:hardness}, we have developed necessary foundations to the proof of this dichotomy theorem.  Now, we are ready to apply them properly to prove the theorem.  The next proposition is a center point of the proof of the theorem.  To simplify a later discussion, however, the proposition targets only a single constraint, instead of a set of constraints as in the theorem.

\begin{proposition}
Let $f$ be any constraint.  If $f$ is not in $ED$, then $#CSP^*(OR, F) \leq_{AP} #CSP^*(f,F)$ holds for any set $F$ of constraints.
\end{proposition}

\begin{proof}
Let $F$ be any constraint set.  We want to establish an AP-reduction from $#CSP^*(OR, F)$ to $#CSP^*(f,F)$.  First, suppose that $f$ has imp support.  Since $f \notin ED$, we apply Proposition \ref{prop:imp support} and instantly obtain the desired AP-reduction from $#CSP^*(OR, F)$ to $#CSP^*(f,F)$, as requested.  Next, suppose that $f$ has no imp support.  To finish the proof, we hereafter consider two independent cases.

[Case: $f \notin \mathcal{N}Z$] Since $f \notin ED \cup \mathcal{N}Z$, Proposition \ref{prop:ED} leads to an AP-reduction from $#CSP^*(OR, F)$ to $#CSP^*(f,F)$.

[Case: $f \in \mathcal{N}Z$] Notice that $f \notin \mathcal{D}G$ since $\mathcal{D}G \subseteq ED$.  Lemma \ref{lem:hardness} provides a constraint $h = (1, x, y, z)$ satisfying that $xyz \neq 0$, $z \neq xy$, and $h \subseteq_{con} f$.  To this $h$, we apply Proposition \ref{prop:hardness} from which it follows that $#CSP^*(h, F) \leq_{AP} #CSP^*(f,F)$.  Since $h \subseteq_{con} f$, Lemma \ref{lem:transitivity} implies $#CSP^*(h, F) \leq_{AP} #CSP^*(f,F)$.  Combining those two AP-reductions, we obtain the desired AP-reduction from $#CSP^*(OR, F)$ to $#CSP^*(f,F)$.

Therefore, we have completed the proof.
\end{proof}

Finally, we give the long-awaited proof of Theorem \ref{thm:main result} and accomplish the main task of this paper.

\begin{proof}
Let $F$ be any constraint set.  Then \ref{lem:hardness} implies that $#CSP^*(F)$ belongs to $FP$.  Henceforth, we assume that $F \subseteq ED$.  From this assumption, we choose a constraint $f \in F$ for which $f \notin ED$.  Proposition \ref{prop:imp support} then yields the AP-reduction: $#CSP^*(OR) \leq_{AP} #CSP^*(f)$.  By the transitivity of AP-reducibility, $#CSP^*(OR) \leq_{AP} #CSP^*(f)$ follows.  Note that, by Lemma \ref{lem:hardness} we obtain $#SAT \leq_{AP} #CSP^*(OR)$.  Therefore, we conclude that $#SAT$ is AP-reducible to $#CSP^*(f)$.
\end{proof}

As demonstrated in Theorem \ref{thm:main result} a free use of unary constraint helps us obtain a truly stronger claim—dichotomy theorem—than a trichotomy theorem of Dyer et al. \cite{dyer} on unweighted Boolean $#CSP$s.  Is this phenomenon an indication that we could eventually prove a similar type of dichotomy theorem for all weighted Boolean $#CSP$s?  In our dichotomy theorem, we have shown that all seemingly “intractable” $#CSP$s are at least as hard as $#SAT$.  Are those problems are all AP-equivalent to $#SAT$?  Those questions demonstrate
that we will have a long way to acquire a full understanding of the approximation complexity of the weighted 
#CSPs.

Next, we wish to prove Corollary \ref{cor1} which strengthens Theorem \ref{thm1} when limiting the free use of “arbitrary” constraints within “algebraic” constraints. Recall from Section \ref{sec1} that an algebraic constraint outputs only algebraic complex numbers. Moreover, we recall the notations \#CSP\(^+\)(F) and \#SAT\(_\lambda\) from Section \ref{sec1}. Similarly, we write \#CSP\(_\lambda\)(F) to denote \#CSP\(_\lambda\)(F) whose instances are only algebraic constraints. Now, let us re-state the corollary given in Section \ref{sec1}

**Corollary 1.2** (rephrased) Let \( \mathcal{F} \) be any set of constraints. If \( \mathcal{F} \subseteq \mathcal{E}\mathcal{D} \), then \#CSP\(_\lambda\)(F) is in FP\(_\lambda\); otherwise, \#SAT\(_\lambda\) \(\leq_{AP}\) \#CSP\(_\lambda\)(F) holds.

Earlier, Dyer et al. \cite{Dyer} demonstrated how to eliminate two constant constraints—\( \Delta_0 \) and \( \Delta_1 \)—using randomized approximation schemes for unweighted Boolean \#CSPs. Similarly, we can eliminate those two constraints and thus prove the corollary by approximating them by any two non-zero constraints of the following forms: \([1, \lambda] \) and \([\lambda, 1]\) with |\(\lambda\)| < 1. In Lemma \ref{lem9.2}, we will demonstrate how to eliminate from \#CSP\(_\lambda\)(F) all unary constraints whose output values contain zeros. This elimination is possible by a use of AP-reductions and this exemplifies a significance of the AP-reducibility.

**Lemma 9.2** For any constraint set \( \mathcal{F} \), it holds that \#CSP\(_\lambda\)(F) \(\equiv_{AP}\) \#CSP\(_\lambda\)(F).

An argument of Dyer et al. \cite{Dyer} for their claim of eliminating both \( \Delta_0 \) and \( \Delta_1 \) exploits their use of non-negative integers. However, since our target is arbitrary (algebraic) complex numbers, the proof of Lemma \ref{lem9.2} demands a quite different argument. To make the paper readable, we postpone the proof until the last section. Finally, we will give the proof of Corollary \ref{cor1} using Lemma \ref{lem9.2}.

**Proof of Corollary \ref{cor1}** Using Lemma \ref{lem9.2} all results in this paper on \#CSP\(^+\)(F)'s can be restated in terms of \#CSP\(_\lambda\)(F)'s. Therefore, we obtain from Theorem \ref{thm1} that \#CSP\(_\lambda\)(F) \(\in\) FP\(_\lambda\) if \( \mathcal{F} \subseteq \mathcal{E}\mathcal{D} \), and \#CSP\(_\lambda\)(OR) \(\leq_{AP}\) \#CSP\(_\lambda\)(F) otherwise. It thus remains to show that \#SAT\(_\lambda\) \(\leq_{AP}\) \#CSP\(_\lambda\)(OR).

As remarked in the end of Section \ref{sec1} following a similar argument given in the proof of Lemma \ref{thm1}, it is possible to prove that \#SAT\(_\lambda\) is AP-reducible to \#CSP\(_\lambda\)(OR). By Lemma \ref{lem9.2} it follows that \#SAT\(_\lambda\) \(\leq_{AP}\) \#CSP\(_\lambda\)(OR). Therefore, the corollary holds.

**10 Proofs of Lemmas 5.2 and 9.2**

This last section will fill the missing proofs of Sections \ref{sec5} and \ref{sec10} to complete the proofs of our main theorem and its corollary. First, we will give the proof of Lemma \ref{lem9.2}. A use of algebraic numbers in the lemma ensures the correctness of a randomized approximation scheme used in the proof of the lemma. Underlying ideas of the scheme come from the proofs of \cite{Dyer} Lemma 10 and \cite{Williams} Theorem 3(2). Particularly, the latter relies on the following well-known lower bound of the absolute values of polynomials in algebraic numbers.

**Lemma 10.1** \cite{Dyer} Let \( \alpha_1, \ldots, \alpha_m \in \mathbb{A} \) and let \( c \) be the degree of \( \mathbb{Q}(\alpha_1, \ldots, \alpha_m) / \mathbb{Q} \). There exists a constant \( e > 0 \) that satisfies the following statement for any complex number \( \alpha \) of the form \( \sum_k a_k \left( \prod_{i=1}^m \alpha_i^{k_i} \right) \), where \( k = (k_1, \ldots, k_m) \) ranges over \( [N_1] \times \cdots \times [N_m] \), \( (N_1, \ldots, N_m) \in \mathbb{N}^m \), and \( a_k \in \mathbb{Z} \). If \( \alpha \neq 0 \), then \( |\alpha| \geq (\sum_k |a_k|)^{1-c} \prod_{i=1}^m e^{-cN_i} \).

Now, we start the proof of Lemma \ref{lem9.2}.

**Proof of Lemma \ref{lem9.2}** Since any input instance to \#CSP\(_\lambda\)(F) is obviously an instance to \#CSP\(_\lambda\)(F), it easily follows that \#CSP\(_\lambda\)(F) is AP-reduced to \#CSP\(_\lambda\)(F). Hereafter, we wish to prove the other direction, namely, \#CSP\(_\lambda\)(F) \(\leq_{AP}\) \#CSP\(_\lambda\)(F). Since \#CSP\(_\lambda\)(F) coincides with \#CSP\(_\lambda\)(F, \(\Delta_0, \Delta_1\)), we wish to demonstrate how to eliminate \( \Delta_0 \) from \#CSP\(_\lambda\)(F, \(\Delta_0, \Delta_1\)). Without loss of generality, we aim at proving that \#CSP\(_\lambda\)(F, \(\Delta_0\)) is AP-reducible to \#CSP\(_\lambda\)(F).

Let \( \Omega = (G, X|F', \pi) \) be any constraint frame given as an input instance to \#CSP\(_\lambda\)(F, \(\Delta_0\)), where \( G = (V_1|V_2, E), X = \{x_1, x_2, \ldots, x_n\} \), and \( F' \subseteq F \cup \{\Delta_0\} \). Let \( n \) be the number of distinct variables used in \( G \). If \( F \) contains \( \Delta_0 \), the lemma is trivially true. Henceforth, we assume that \( \Delta_0 \not\subseteq F \). Choose any complex number \( \lambda \) satisfying \( 0 < |\lambda| < 1 \) and define \( u(x) = [1, \lambda] \), which is clearly in \( U \cap \mathcal{N}\mathcal{Z} \). For later use, let \( |\Omega| \) denote \( \prod_{v \in V_2} \max\{1, |f_v|\} \), where \( |f_v| = \max\{|f_v(x)| \mid x \in \{0, 1\}^k \} \) and \( k \) is the arity of \( f_v \).
First, we modify the graph $G$ as follows. Let us select all nodes in $V_1$ that are adjacent to certain nodes in $V_2$ having the label $\Delta_0$. We first merge all selected nodes into a single node, say, $v$ with a “new” label, say, $x$, and then delete all the nodes labeled $\Delta_0$ and their incident edges. Finally, we attach a “new” node labeled $\Delta_0$ to the node $v$ by an additional single edge. It is not difficult to show that this modified graph produces the same output value as its original one. In what follows, we assume that the constraint $\Delta_0$ appears exactly once as a node label in the graph $G$ and it depends only on the variable $x$.

Let $G_0$ be the graph obtained from $G$ by removing the unique node $\Delta_0$. Its associated constraint frame is briefly denoted $\Omega_0$. Note that $\text{csp}_{\Omega_0}$ can be expressed as $\sum_{x \in \{0,1\}} h(x)$ using an appropriate complex-valued function $h$ depending on the value of $x$. With this $h$, $\text{csp}_{\Omega_0}$ is calculated as $\sum_{x \in \{0,1\}} h(x)\Delta_0(x)$, which obviously equals $h(0)$. Moreover, let $u_a = u^a$ for any fixed number $m \in \mathbb{N}^+$. Denote by $G_a$ the graph obtained from $G$ by replacing $\Delta_0$ by $u_a$ and let $\Omega_a$ be its associated constraint frame. Since $u_a = [1, \lambda^m]$, it holds that $\text{csp}_{\Omega_a} = \sum_a h(x)u_a(x) = h(0) + \lambda^m h(1)$. Letting $K = h(1)$, we obtain $\text{csp}_\Omega = \text{csp}_{\Omega_m} - \lambda^m K$. Note that, for each fixed variable assignment $\sigma : X \to \{0,1\}$, the product of the outcomes of all constraints is at most $|\Omega|$. Since there are $2^n$ distinct variable assignments, $|\Omega|$ is thus upper-bounded by $2^n|\Omega|$. Meanwhile, we assume that $\text{csp}_\Omega \neq 0$. Since all entries of any constraint in $\mathcal{F}'$ are taken from $\Lambda$, we want to apply Lemma 10.3. For a use of this lemma, however, we need to express the value $|\text{csp}_\Omega|$ using three series $\{a_k\}_k$, $\{a_k\}_k$, and $\{k_i\}_k$, given in the lemma. Let us define them as follows. Let $I = \{(v,w) | v \in V_2, w \in \{0,1\}\}$, where $v$ is the arity of $f_v$. Here, we assume a fixed enumeration of all elements in $I$. For each variable assignment $\sigma : X \to \{0,1\}$, we define a vector $k(\sigma) = (k_i(\sigma))_{i \in I}$ as follows: for each $i = (v,w) \in I$, let $k_i(\sigma) = 1$ if $f_v$ depends on a certain variable series $(x_i, \ldots, x_k)$ and $w$ equals $\sigma(x_i), \ldots, \sigma(x_k)$; otherwise, let $k_i(\sigma) = 0$. Moreover, let $N_i (i \in I)$ equal $1$ and set $N = \prod_{i \in I} N_i$. For any vector $k \in N$, let $a_k = 1$ if there exists a valid assignment $\sigma$ satisfying $k = k(\sigma)$; otherwise, let $a_k = 0$. Finally, let $\alpha(v,w) = f_v(w)$, where $(v,w) \in I$. By these definitions, the value $\text{csp}_\Omega = \sum_{I \subseteq V_2} \prod_{i \in I} f_v(\sigma(x_i), \ldots, \sigma(x_k)) \sum_{k \in N} a_k \prod_{i \in I} k_i(\alpha)$. Now, Lemma 10.3 provides two constants $c, e > 0$ for which $|\text{csp}_\Omega|$ is lower-bounded by the value $\sum_{k \in N} a_k \prod_{i \in I} e^{c N_i}$, from which we obtain $|\text{csp}_\Omega| > d$. For convenience, whenever $d \geq 1$, we automatically set $d = 1/2$ so that we can always assume that $0 < d < 1$.

Now, we will describe an AP-reduction $N$ from $\# \text{CSP}_\lambda^\delta(\mathcal{F}, \Delta_0)$ to $\# \text{CSP}_\lambda^\delta(\mathcal{F})$. Let $(\Omega, 1/\epsilon)$ be any input to the algorithm $N$, where $\epsilon \in (0, 1)$ is an error tolerance parameter. Without loss of generality, we assume that $0 < \epsilon < 1/4$. Let $M$ be an arbitrary oracle that is a randomized approximation scheme designed to solve $\# \text{CSP}_\lambda^\delta(\mathcal{F})$. To compute the value $\text{csp}_\Omega$ on the given input $(\Omega, 1/\epsilon)$, $N$ works as follows.

Let $\delta = \epsilon/2$ and choose a positive integer $m$ for which (i) $2^n + \delta |\Omega| \lambda^m \leq (1 - 2^{-\delta})d$ and (ii) $2^n |\Omega| \lambda^m \leq \delta d$, where $n = |X|$ and $\delta^\prime = \frac{\delta}{\epsilon}$ Next, $N$ constructs the constraint frame $\Omega_m$ from $\Omega$. To the oracle $M$, $N$ makes a query with a query word $(\Omega_m, 1/\delta)$. Let $z$ denote an oracle answer from $M$. Notice that $z$ is a random variable. If $|z| < d$, then $N$ outputs $0$; otherwise, it outputs $z$.

The correctness of the above algorithm $N$ is shown as follows. Let us focus on the case where $\text{csp}_\Omega = 0$; in other words, $\text{csp}_{\Omega_m} - \lambda^m K = 0$, which is equivalent to $\text{csp}_{\Omega_m} = \lambda^m K$. Since $z$ is a $2^\delta$-approximate solution for $\text{csp}_{\Omega_m}$, the value $z$ must satisfy that $2^{-\delta} |\text{csp}_{\Omega_m}| \leq |z| \leq 2^\delta |\text{csp}_{\Omega_m}|$. It thus follows by Condition (i) that

$$|z| \leq 2^\delta |\text{csp}_{\Omega_m}| \leq 2^\delta |\lambda^m K| \leq 2^{n+\delta} |\Omega| \lambda^m \leq (1 - 2^{-\delta})d < d.$$

In this case, the algorithm $N$ outputs $0$, that is, $N(\Omega, 1/\epsilon) = 0$. This means that $N$ correctly computes $\text{csp}_\Omega$ with high probability.

Let us consider the other case where $\text{csp}_\Omega \neq 0$. Due to the choice of $\delta$, $|\text{csp}_\Omega| > d$ holds. Since the oracle returns a $2^\delta$-approximate solution $z$ for $\text{csp}_{\Omega_m}$, it follows that $2^{-\delta} |\text{csp}_{\Omega_m}| \leq |z| \leq 2^\delta |\text{csp}_{\Omega_m}|$. Now, we want to show that (iii) $2^{-\epsilon} |\text{csp}_\Omega| \leq 2^{-\delta} (|\text{csp}_\Omega| - |\lambda^m K|)$ and (iv) $2^\delta (|\text{csp}_\Omega| + |\lambda^m K|)$, because these bounds together imply

$$2^{-\epsilon} |\text{csp}_\Omega| \leq 2^{-\delta} (|\text{csp}_\Omega| - |\lambda^m K|) \leq |z| \leq 2^\delta (|\text{csp}_\Omega| + |\lambda^m K|) \leq 2^\epsilon |\text{csp}_\Omega|.$$

In other words, $2^{-\epsilon} \leq |z|/|\text{csp}_\Omega| \leq 2^\epsilon$ holds. Our next task is to prove Conditions (iii) and (iv). Condition (iii) is equivalent to $|\lambda^m K| \leq (1 - 2^{-\delta}) |\text{csp}_\Omega|$, whereas Condition (iv) is equivalent to $|\lambda^m K| \leq (2^\delta - 1) |\text{csp}_\Omega|$. Since $2^{\delta - 1} \geq 1 - 2^{-\delta}$ holds for our choice of $\delta$, Condition (iv) follows instantly from Condition (iii). By Condition (i), we obtain $|\lambda^m K| \leq 2^n |\Omega| \lambda^m \leq (1 - 2^{-\delta})d$. This immediately implies Condition (iii) since $|\text{csp}_\Omega| > d$.
To complete the proof, we still need to show that \(|\arg(z / \text{csp}_\Omega)| \leq \varepsilon\) whenever \(\text{csp}_\Omega \neq 0\). Let us assume that \(\text{csp}_\Omega \neq 0\). Since \(z\) is an output of \(M\) on the input \((\Omega_m, 1/\delta)\), it holds that \(|\arg(z) - \arg(\text{csp}_\Omega_m)| \leq \delta\). Now, we set \(\theta = |\arg(\text{csp}_\Omega_m) - \arg(\text{csp}_\Omega)|\). Notice that the value \(\theta\) represents an angle in the complex plane between two vectors \(\text{csp}_\Omega_m\) and \(\text{csp}_\Omega\). Since \(\text{csp}_\Omega_m = \text{csp}_\Omega + \lambda^m K\), the value \(\theta\) is maximized when the vector \(\lambda^m K\) is perpendicular to the line extending the vector \(\text{csp}_\Omega_m\). This implies that \(|\text{csp}_\Omega \sin \theta| \leq |\lambda^m K|\). Condition (ii) implies that \(|\lambda^m K| \leq 2m|\Omega||\lambda|^m \leq \delta' d\). Because \(|\text{csp}_\Omega| > d\), we also obtain \(\sin \theta \leq \frac{|\lambda^m K|}{|\text{csp}_\Omega|} \leq \frac{\delta'}{2}\). Since \(\delta' = \frac{\varepsilon^2}{\delta} = \frac{\varepsilon}{\theta} \leq \frac{\delta}{2}\), we may assume that \(0 \leq \theta \leq \frac{\delta}{2}\). Within this range, it always holds that \(\frac{\varepsilon}{\theta} \leq \sin \theta\). Therefore, we conclude that \(\theta \leq \frac{\varepsilon}{\theta} \delta = \delta + \delta = \varepsilon\).

By following the above argument closely, it is also possible to prove that \(#\text{CSP}^*_k(F, \Delta_t)\) is AP-reducible to \(#\text{CSP}^*_k(F)\). Therefore, we have completed the proof of the lemma.

In our argument toward the dichotomy theorem, we have omitted the proof of Lemma 5.2, which shows a fundamental property of T-constructibility. Now, we will give the proof of the lemma. The proof will proceed by induction on the number of operations applied to construct a target constraint.

**Proof of Lemma 5.2** Let \(F\) be any set of constraints. For simplicity, assume that \(f\) is obtained from \(g\) (or \(\{g_1, g_2\}\) in the case of the multiplication operation) by applying exactly one of the seven operations described in Section 5. Our purpose is to show that \(#\text{CSP}^*(f, F)\) is AP-reducible to \(#\text{CSP}^*(g, F)\). Notationally, we set \(\Omega = (G, X, |H|, \pi)\) and \(\Omega = (G', X', |H'|, \pi')\) to be any constraint frames associated with \(g\) and \(f\), respectively. Note that \(H\) and \(H'\) are finite subsets of \((g) \cup F \cup U\) and \((f) \cup F \cup U\), and \(X\) and \(X'\) are both finite sets of Boolean variables. To improve the readability, we assume that \(X = X' = \{x_1, x_2, \ldots, x_n\}\). Let \(\varepsilon\) be any error tolerance parameter. For each operation, we want to explain how to produce \(G\) and \(\pi\) from \(G'\) and \(\pi'\) in polynomial time so that, after making a query \((\Omega, 1/\varepsilon)\) to any oracle (which is a randomized approximation scheme solving \(#\text{CSP}^*(g, F)\)), from its oracle answer \(\text{csp}_\Omega\), we can compute a 2\(^\varepsilon\)-approximate solution for \(\text{csp}_\Omega\) with high probability. This procedure indicates that \(#\text{CSP}^*(f, F) \leq_{\text{AP}} #\text{CSP}^*(g, F)\). For simplicity, we will omit the mentioning of \(\varepsilon\) in the following description.

- **Permutation** Assume that \(f\) is obtained from \(g\) by exchanging two indices \(i\) and \(j\) of variables \(\{x_i, x_j\}\). From \(G'\), we build \(G\) by swapping only the labels \(x_i\) and \(x_j\) of the corresponding nodes (without changing any edge incident on them). The labeling function \(\pi\) is also naturally induced from \(\pi'\). Clearly, this step requires linear time. Our underlying randomized approximation scheme \(N\) works as follows: it first constructs \(\Omega\) from \(\Omega'\), makes a single query to obtain a 2\(^\varepsilon\)-approximate solution \(z\) for \(\text{csp}_\Omega\) from the oracle \(#\text{CSP}^*(g, F)\), and outputs \(z\) instantly. Since \(\text{csp}_\Omega = \text{csp}_\Omega\), the output of \(N\) is also a 2\(^\varepsilon\)-approximate solution for \(\text{csp}_\Omega\).

- **Pinning** Let \(f = g^{x_i = c}\) for \(i \in [k]\) and \(c \in \{0, 1\}\). From \(G'\), we construct \(G\) in polynomial time as follows: append a new node whose label is \(\Delta_x\) and connect it to the node labeled \(x_i\) by a new edge. Because \(\text{csp}_\Omega = \text{csp}_\Omega\) holds, an algorithm similar to the one in the previous case can approximate \(\text{csp}_\Omega\).

- **Projection** Let \(f = g^{x_i = x_j}\) with index \(i \in [k]\). Notice that \(f\) does not have the variable \(x_i\). For simplicity, assume that \(G'\) has no node with the label \(x_i\). Let \(V'\) denote the set of all nodes having the label \(f\) in \(G'\). Now, we construct \(G\) from \(G'\) by adding a new node labeled \(x_i\) to \(V_2\), by replacing the label \(f\) by \(g\), and by connecting the node \(x_i\) to all nodes in \(V'\) by new single edges. This transformation implies that \(\text{csp}_\Omega = \text{csp}_\Omega\).

The rest is similar to the previous cases.

- **Linking** Let \(f = g^{x_i = x_j}\) and assume that \(i < j\). In this case, we obtain \(G\) from \(G'\) by replacing the label \(f\) by \(g\) and adding an extra edge between the node \(x_j\) and the node \(g\). Notice that there are now two different edges between the node \(x_j\) and the node \(g\). Using this new graph \(G\), we obtain \(\text{csp}_\Omega = \text{csp}_\Omega\).

- **Expansion** Let \(f(x_1, \ldots, x_i, y, x_{i+1}, \ldots, x_k) = g(x_1, \ldots, x_i, x_{i+1}, \ldots, x_k)\), where \(y\) is a free variable. To define \(G\), we delete from \(G'\) any edge between the node \(y\) and any node labeled \(f\). Note that, in general, we cannot remove the node \(y\) from \(G'\) because it may be connected to other nodes although \(y\) does not have the variable \(y\). Since any node labeled \(f\) has not initially depended on the node \(y\), it follows that \(\text{csp}_\Omega = \text{csp}_\Omega\). Since a 2\(^\varepsilon\)-approximate solution \(z\) for \(\text{csp}_\Omega\) can be obtained as an answer to an oracle query regarding \(\Omega\), \(z\) approximates \(\text{csp}_\Omega\), as well.

- **Multiplication** Assume that \(f = g_1 \cdot g_2\), where \(g_1\) and \(g_2\) share the same input variable series. We intend to define \(G\) in the following way. Since \(\{g_1, g_2\}\) is a factor list for \(f\), we first replace every node labeled \(f\) in \(G'\) by two new nodes having the labels \(g_1\) and \(g_2\), each of which has the same incident set as \(f\) does. Using \(#\text{CSP}^*(g_1, g_2, F)\) as an oracle, we obtain a 2\(^\varepsilon\)-approximate solution \(z\) for \(\text{csp}_\Omega\) as an oracle answer. Since \(\text{csp}_\Omega = \text{csp}_\Omega\), \(z\) approximates \(\text{csp}_\Omega\).
Normalization] Let $f = \lambda \cdot g$ for a constant $\lambda \in \mathbb{C} - \{0\}$. Define $G$ to be $G'$ except that every occurrence of $f$ is replaced by $g$. Let $n$ be the number of nodes in $G'$ that have the label $f$. Since $csp_{\Omega} = \lambda^n \cdot csp_{\Omega}$, we first obtain a $2^\varepsilon$-approximate solution $z$ for $csp_{\Omega}$ by making a query to the oracle $\#CSP^*(g, \mathcal{F})$. We then multiply $z$ by $\lambda^m$ and output the resulting value. Clearly, this value approximates $csp_{\Omega}^*$. From all seven cases discussed above, we conclude that $\#CSP^*(f, \mathcal{F})$ is AP-reducible to $\#CSP^*(g, \mathcal{F})$. This finishes the proof of Lemma 5.2.

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