Abstract: The pivotal aim of the present work is to find the solution for fractional Caudrey-Dodd-Gibbon (CDG) equation using \( q \)-homotopy analysis transform method (\( q \)-HATM). The considered technique is graceful amalgamations of Laplace transform technique with \( q \)-homotopy analysis scheme, and fractional derivative defined with Atangana-Baleanu (AB) operator. The fixed point hypothesis considered in order to demonstrate the existence and uniqueness of the obtained solution for the projected fractional-order model. In order to illustrate and validate the efficiency of the future technique, we analysed the projected model in terms of fractional order. Moreover, the physical behaviour of \( q \)-HATM solutions have been captured in terms of plots for diverse fractional order and the numerical simulation is also demonstrated. The obtained results elucidate that, the considered algorithm is easy to implement, highly methodical as well as accurate and very effective to examine the nature of nonlinear differential equations of arbitrary order arisen in the connected areas of science and engineering.

Keywords: Laplace transform; Atangana-Baleanu derivative; Caudrey-Dodd-Gibbon equation; \( q \)-homotopy analysis method; fixed point theorem

1 Introduction

Fractional calculus (FC) was originated in Newton’s time, but lately, it fascinated the attention of many scholars. From the last thirty years, the most intriguing leaps in scientific and engineering applications have been found within the framework of FC. The concept of the fractional derivative has been industrialized due to the complexities associated with a heterogeneities phenomenon. The fractional differential operators are capable to capture the behaviour of multifaceted media having diffusion process. It has been a very essential tool and many problems can be illustrated more conveniently and more accurately with differential equations having arbitrary order. Due to the swift development of mathematical techniques with computer software’s, many researchers started to work on generalised calculus to present their viewpoints while analysing many complex phenomena.

Numerous pioneering directions are prescribed for the diverse definitions of fractional calculus by many senior researchers, and which prearranged the foundation [1–6]. Calculus with fractional order is associated to practical ventures and it extensively employed to nanotechnology [7], human diseases [8, 9], chaos theory [10], and other areas [11–34]. The numerical and analytical solution for these equations illustrating these models have an impartment role in portraying nature of nonlinear problems ascends in connected areas of science.

In order to demonstrate the efficiency of the future scheme, we consider fifth-order nonlinear CDG equation of the form [35, 36]

\[
 u_t + u_{xxxxx} + 30uu_{xxx} + 30u_xu_{xx} + 180u^2u_x = 0 .
\]  

(1)

The above equation is a class of KdV equation and further, it possesses distinct and diverse properties. The CGD equation is also familiar as Sawada-Kotera equation [37]. Due to the importance of the considered problem, it has been magnetized the attention of many researchers from diverse areas. In 1984, Weiss illustrated the Painlevé’ property for the Eq. (1) [38]. It has been proved that it has a strong physical background in fluid [39] and also has N-soliton solutions [40].

In the present scenario, many important and nonlinear models are methodically and effectively analysed with the help of fractional calculus. There have been diverse definitions are suggested by many senior research scholars, for instance, Riemann, Liouville, Caputo and Fabrizio.
However, these definitions have their own limitations. The Riemann–Liouville derivative is unable to explain the importance of the initial conditions; the Caputo derivative has overcome this shortcoming but is impotent to explain the singular kernel of the phenomena. Later, in 2015 Caputo and Fabrizio defeated the above obliges [41], and many researchers are considered this derivative in order to analyse and find the solution for diverse classes of nonlinear complex problems. But some issues were pointed out in CF derivative, like non-singular kernel and non-local, these properties are very essential in describing the physical behaviour and nature of the nonlinear problems. In 2016, Atangana and Baleanu introduced and natured the novel fractional derivative, namely AB derivative. This novel derivative defined with the aid of Mittag–Leffler functions [42]. This fractional derivative buried all the above-cited issues and helps us to understand the natural phenomena systematically and effectively.

In the present framework, we consider the fractional Caudrey-Dodd-Gibbon (FCDG) equation of the form

\[ \begin{align*}
  &A^{BC}D^{\alpha}_{a}u(x, t) + u_{xxxx} + 30u_{xxx} + 30u_{xx} + 180u^2u_x = 0, \\
  &0 < \alpha \leq 1,
\end{align*} \tag{2} \]

where \( \alpha \) is fractional-order and defined with AB fractional operator. The fractional-order is introduced in order to incorporate the memory effects and hereditary consequence in the phenomenon and these properties aid us to capture essential physical properties of the nonlinear problems.

Recently, many mathematicians and physicists developed very effective and more accurate methods in order to find and analyse the solution for complex and nonlinear problems arisen in science and engineering. In connection with this, the homotopy analysis method (HAM) proposed by Chinese mathematician Liao Shijun [43]. HAM has been profitably and effectively applied to study the behaviour of nonlinear problems without perturbation or linearization. But, for computational work, HAM requires huge time and computer memory. To overcome this, there is an essence of the amalgamation of a considered method with well-known transform techniques.

In the present investigation, we put an effort to find and analysed the behaviour of the solution obtained for the FCDG equation by applying \( q \)-HAM. The future algorithm is the combination of \( q \)-HAM with LT [44]. Since \( q \)-HAM is an improved scheme of HAM; it does not require discretization, perturbation or linearization. Recently, due to its reliability and efficacy, the considered method is exceptionally applied by many researchers to understand physical behaviour diverse classes of complex problems [45–53]. The projected method offers us more freedom to consider the diverse class of initial guess and the equation type complex as well as nonlinear problems; because of this, the complex NDEs can be directly solved. The novelty of the future method is it aids a modest algorithm to evaluate the solution and it natured by the homotopy and auxillary parameters, which provides the rapid convergence in the obtained solution for a nonlinear portion of the given problem. Meanwhile, it has prodigious generality because it plausibly contains the results obtained by many algorithms like \( q \)-HAM, HPM, ADM and some other traditional techniques. The considered method can preserve great accuracy while decreasing the computational time and work in comparison with other methods. The considered nonlinear problem recently fascinated the attention of researchers from different areas of science. Since FCDG equation plays a significant role in portraying several nonlinear phenomena and also which are the generalizations of diverse complex phenomena, many authors find and analysed the solution using analytical as well as numerical schemes [54–61].

### 2 Preliminaries

Recently, many authors considered these derivatives to analyse a diverse class of models in comparison with classical order as well as other fractional derivatives, and they prove that AB derivative is more effective while analysing the nature and physical behaviour of the models [62–65]. Here, we define the basic notion of Atangana-Baleanu derivatives and integrals [42].

**Definition 1.** The fractional Atangana-Baleanu-Caputo derivative for a function \( f \in H^1(a, b) \) \((b > a, \ a \in [0, 1])\) is presented as follows

\[ \begin{align*}
  &A^{BC}D^{\alpha}_{a} f(t) = \frac{B[a]}{\Gamma(1 - \alpha)} \int_{a}^{t} f(\vartheta) E_{\alpha} \left[ \frac{(t - \vartheta)^{\alpha}}{\alpha - 1} \right] d\vartheta. \tag{3} \end{align*} \]

where \( B[a] \) is a normalization function such that \( B(0) = B(1) = 1 \).

**Definition 2.** The AB derivative of fractional order for a function \( f \in H^1(a, b), \ b > a, \ a \in [0, 1] \) in Riemann-Liouville sense presented as follows

\[ \begin{align*}
  &A^{BR}D^{\alpha}_{a} f(t) = \frac{B[a]}{\Gamma(1 - \alpha)} \frac{d}{dt} \int_{a}^{t} f(\vartheta) E_{\alpha} \left[ \frac{(t - \vartheta)^{\alpha}}{\alpha - 1} \right] d\vartheta. \tag{4} \end{align*} \]
Definition 3. The fractional AB integral related to the non-local kernel is defined by

\[ a^\alpha I^\alpha_a f(t) = \frac{1 - \alpha}{B[\alpha]}f(t) + \frac{\alpha}{B[a]T(a)} \int_a^t f(\xi)(t - \xi)^{\alpha - 1} d\xi \] (5)

Definition 4. The Laplace transform (LT) of AB derivative is defined by

\[ L \left[ a^\alpha D^\alpha_a f(t) \right] = \frac{B[\alpha]}{s^\alpha} \left[ f(t) - \frac{\alpha}{s^\alpha + (a/1-a)} \right] \] (6)

Theorem 1. The following Lipschitz conditions respectively hold true for both Riemann-Liouville and AB derivatives defined in Eqs. (3) and (4) [42],

\[ \|a^\alpha D^\alpha_a f_1(t) - a^\alpha D^\alpha_a f_2(t)\| < K_1 \|f_1(x) - f_2(x)\| \] (7)

and

\[ \|a^\alpha D^\alpha_a f_1(t) - a^\alpha D^\alpha_a f_2(t)\| < K_2 \|f_1(x) - f_2(x)\| \] (8)

Theorem 2. The time-fractional differential equation

\[ a^\alpha D^\alpha_a f(t) = s(t) \]

has a unique solution and which is defined as [42]

\[ f(t) = \frac{1 - \alpha}{B[a]}s(t) + \frac{\alpha}{B[a]T(a)} \int_0^t s(\xi)(t - \xi)^{\alpha - 1} d\xi \] (9)

3 Fundamental idea of the considered scheme

In this segment, we consider the arbitrary order differential equation in order to demonstrate the fundamental solution procedure of the projected algorithm

\[ a^\alpha D^\alpha_a v(x, t) + R v(x, t) + N v(x, t) = f(x, t) \]

with the initial condition

\[ v(x, 0) = g(x) \] (10)

where \( a^\alpha D^\alpha_a v(x, t) \) symbolise the AB derivative of \( v(x, t) \). \( f(x, t) \) signifies the source term, \( R \) and \( N \) respectively denotes the linear and nonlinear differential operator. On using the LT on Eq. (10), we have after simplification

\[ L [v(x, t)] - \frac{g(x)}{s} + \frac{1}{B[a]} \left( 1 - \alpha + \frac{\alpha}{s^\alpha} \right) \{ L [Rv(x, t)] + L [Nv(x, t)] \} + L [Nv(x, t)] - L [f(x, t)] = 0. \] (12)

The non-linear operator is defined as follows

\[ N [\varphi(x, t; q)] = L [\varphi(x, t; q)] - \frac{g(x)}{s} + \frac{1}{B[a]} \left( 1 - \alpha + \frac{\alpha}{s^\alpha} \right) \{ L [R \varphi(x, t; q)] + L [N \varphi(x, t; q)] \} - L [f(x, t)]. \] (13)

Here, \( \varphi(x, t; q) \) is the real-valued function with respect to \( x, t \) and \( \{ q \in [0, \frac{1}{n}] \} \). Now, we define a homotopy as follows

\[ (1 - n q) L [\varphi(x, t; q) - \varphi_0(x, t)] = h q N [\varphi(x, t; q)], \] (14)

where \( L \) is signifying \( LT \), \( q \in [0, \frac{1}{n}] \) \((n \geq 1)\) is the embedding parameter and \( h = 0 \) is an auxiliary parameter. For \( q = 0 \) and \( q = \frac{1}{n} \), the results are given below hold true

\[ \varphi(x, t; 0) = \varphi_0(x, t), \quad \varphi(x, t; \frac{1}{n}) = v(x, t). \] (15)

Thus, by intensifying \( q \) from 0 to \( \frac{1}{n} \), the solution \( \varphi(x, t; q) \) varies from \( \varphi_0(x, t) \) to \( v(x, t) \). By using the Taylor theorem near to \( q \), we defining \( \varphi(x, t; q) \) in series form and then we get

\[ \varphi(x, t; q) = \varphi_0(x, t) + \sum_{m=1}^{\infty} v_m(x, t) q^m, \] (16)

where

\[ v_m(x, t) = \frac{1}{m!} \frac{\partial^m \varphi(x, t; q)}{\partial q^m} \bigg|_{q=0}. \] (17)

The series (14) converges at \( q = \frac{1}{n} \) for the proper chase of \( \varphi_0(x, t), n \) and \( h \). Then

\[ v(x, t) = \varphi_0(x, t) + \sum_{m=1}^{\infty} v_m(x, t) \left( \frac{1}{n} \right)^m. \] (18)

Now, \( m \)-times differentiating Eq. (15) with \( q \) and later dividing by \( m! \) and then putting \( q = 0 \), we obtain

\[ L [v_m(x, t) - k_m v_{m-1}(x, t)] = h R_m (\vec{v}_{m-1}), \] (19)

where the vectors are defined as

\[ \vec{v}_m = \{ v_0(x, t), v_1(x, t), \ldots, v_m(x, t) \}. \] (20)

On applying inverse LT on Eq. (19), one can get

\[ v_m(x, t) = k_m v_{m-1}(x, t) + h L^{-1} [R_m (\vec{v}_{m-1})], \] (21)

where

\[ R_m (\vec{v}_{m-1}) = L [v_{m-1}(x, t)] - \left( 1 - \frac{k_m}{n} \right). \]
\[
\begin{align*}
&\left(\frac{g(x)}{s} + \frac{1}{B[a]} \left(1 - \alpha + \frac{a}{s^a}\right) L[f(x, t)]\right) \\
&+ \frac{1}{B[a]} \left(1 - \alpha + \frac{a}{s^a}\right) L[Rv_{m-1} + H_{m-1}],
\end{align*}
\]

and
\[
k_m = \begin{cases} 
0, & m \leq 1, \\
\infty, & m > 1.
\end{cases}
\]

In Eq. (22), \( H_m \) signifies homotopy polynomial and presented as follows
\[
H_m = \frac{1}{m!} \left[ \theta^m \varphi(x, t; q) \right]_{q=0}.
\]

and \( \varphi(x, t; q) = \varphi_0 + q\varphi_1 + q^2\varphi_2 + \ldots \).

By the aid of Eqs. (21) and (22), one can get
\[
v_m(x, t) = (k_m + h)v_{m-1}(x, t) - \left(1 - \frac{k_m}{n}\right)
\]
\[
L^{-1}\left(\frac{g(x)}{s} + \frac{1}{B[a]} \left(1 - \alpha + \frac{a}{s^a}\right) L[f(x, t)]\right)
\]
\[
+ hL^{-1} \left\{\frac{1}{B[a]} \left(1 - \alpha + \frac{a}{s^a}\right) L[Rv_{m-1} + H_{m-1}]\right\}.
\]

Using the Eq. (25), one can get the series of \( v_m(x, t) \).
Lastly, the series \( q \)-HATM solution is defined as
\[
v(x, t) = v_0(x, t) + \sum_{m=1}^{\infty} v_m(x, t) \left(\frac{1}{n}\right)^m.
\]

4 Solution for FCDG equation

In order to present the solution procedure and efficiency of the future scheme, in this segment, we consider FCDG equation of fractional order. Further by the help of obtained results, we made an attempt to capture the behaviour of \( q \)-HATM solution for different fractional order.

By the help of Eq. (2), we have
\[
A^{B/a} D_t^\alpha u(x, t) + u_{xxxx} + 30u u_{xx} + 30u_x u_{xx} + 180u^2 u_x = 0,
\]
\[
0 < \alpha \leq 1,
\]
with initial condition
\[
u(x, 0) = \mu^2 sech^2(\mu x).
\]

Taking LT on Eq. (27) and then using the Eq. (28), we get
\[
L[u(x, t)] = \frac{1}{s} \left(\mu^2 sech^2(\mu x)\right) + \frac{1}{B[a]} \left(1 - \alpha + \frac{a}{s^a}\right)
\]
\[
L\left(\frac{\partial^5 u}{\partial x^5} + 30\frac{\partial^3 u}{\partial x^3} + 30\frac{\partial u}{\partial x} \frac{\partial^2 u}{\partial x^2} + 180u^2 \frac{\partial u}{\partial x}\right).
\]

The non-linear operator \( N \) is presented with the help of future algorithm as below
\[
N[\varphi(x, t; q)] = L[\varphi(x, t; q)] - \frac{1}{s} \left(\mu^2 sech^2(\mu x)\right)
\]
\[
+ \frac{1}{B[a]} \left(1 - \alpha + \frac{a}{s^a}\right) L\left(\frac{\partial^5 \varphi}{\partial x^5} + 30\frac{\partial^3 \varphi}{\partial x^3} + 30\frac{\partial \varphi}{\partial x} \frac{\partial^2 \varphi}{\partial x^2} + 180\varphi^2 \frac{\partial \varphi}{\partial x}\right).
\]

The deformation equation of \( m \)-th order by the help of \( q \)-HATM at \( H(x, t) = 1 \), is given as follows
\[
L[u_m(x, t) - k_m u_{m-1}(x, t)] = hR_{1, m} \left[u_{m-1}, v_{m-1}\right],
\]

where
\[
R_m [u_{m-1}] = L[u_{m-1}(x, t)] - \left(1 - \frac{k_m}{n}\right)
\]
\[
\left\{\frac{1}{s} \left(\mu^2 sech^2(\mu x)\right) + \frac{1}{B[a]} \left(1 - \alpha + \frac{a}{s^a}\right)
\]
\[
L\left(\frac{\partial^5 u_{m-1}}{\partial x^5} + 30\sum_{i=0}^{m-1} \frac{\partial^3 u_{m-i-1}}{\partial x^3} + 30\sum_{i=0}^{m-1} \frac{\partial u_i}{\partial x} \frac{\partial^2 u_{m-i-1}}{\partial x^2} + \sum_{i=0}^{m-1} \frac{\partial u_i}{\partial x} \frac{\partial u_{m-i-1}}{\partial x}\right).
\]

On applying inverse LT on Eq. (31), it reduces to
\[
u_m(x, t) = k_m u_{m-1}(x, t) + hL^{-1} \left[R_m \left[u_{m-1}\right]\right].
\]

On simplifying the above equation systematically by using \( u_0(x, t) = \frac{1}{s} \left(\mu^2 sech^2(\mu x)\right) \) we can evaluate the terms of the series solution
\[
u(x, t) = u_0(x, t) + \sum_{m=1}^{\infty} u_m(x, t) \left(\frac{1}{n}\right)^m.
\]

5 Existence of solutions for the future model

Here, we considered the fixed-point theorem in order to demonstrate the existence of the solution for the considered model. Since the considered model cited in the system (27) is non-local as well as complex; there are no particular algorithms or methods exist to evaluate the exact solutions. However, under some particular conditions the existence of the solution assurances. Now, the system (27) is considered as follows:
\[
A^{B/a} D_t^\alpha [u(x, t)] = G(x, t, u).
\]

The foregoing system is transformed to the Volterra integral equation using Theorem 2, and which as follows
\[
u(x, t) - u(x, 0) = \left(1 - \frac{1}{B(a)}\right) G(x, t, u)
\]
\[
\int_{0}^{t} G(x, t, u) \, dt.
\]
The contraction. The recursive form of Eq. (36) defined as follows

\[ G(x, t, u) = \sum_{i=1}^{n} \phi_{ni}(x, t). \]

Notice that

By using Eq. (38) after applying the norm on the Eq. (41), one can get

\[ \|\phi_n(x, t)\| \leq \left( \left( \frac{1 - a}{B(a)} \right) \eta + \frac{a}{B(a) \Gamma(a)} \eta \right)^n. \]

Theorem 3. The kernel \( G \) satisfies the Lipschitz condition and contraction if the condition

\[ 0 \leq \left( \delta^5 + 30 \delta \left( 2 \left( a^2 + b^2 + ab \right) + \delta \right) \right) < 1 \]

Proof. In order to prove the required result, we consider the two functions \( u \) and \( u_1 \), then

\[ \|G(x, t, u) - G(x, t, u_1)\| \leq \delta \|u(x, t) - u(x, t_1)\|. \]

This gives, the Lipschitz condition is obtained for \( G_1 \). Further, we can see that if \( 0 \leq \left( \delta^5 + 30 \delta \left( 2 \left( a^2 + b^2 + ab \right) + \delta \right) \right) < 1 \), then it implies the contraction. The recursive form of Eq. (36) defined as follows

\[ u_n(x, t) = G(x, t, u_{n-1}) \]

The associated initial condition is

\[ u(x, 0) = u_0(x, t). \]

The successive difference between the terms is presented as follows

\[ \phi_n(x, t) = u_n(x, t) - u_{n-1}(x, t) \]

Similarly, at \( t_0 \) we can obtain

\[ \|K_{n+1}(x, t)\| \leq \left( \left( \frac{1 - a}{B(a)} \right) + \frac{a t_0}{B(a) \Gamma(a)} \right)^{n+1} \eta^{n+1} M. \]

As \( n \) approaches to \( \infty \), we can see that form Eq. (50), \( \|K_n(x, t)\| \) tends to 0.
Next, it is a necessity to demonstrate uniqueness for the solution of the considered model. Suppose \( u^*(x, t) \) be the other solution, then we have

\[
\begin{align*}
& \quad u(x, t) - u^*(x, t) = \left( 1 - \frac{a}{B(a)} \right) \left( G(x, t, u) - G \left( x, t, u^* \right) \right) \\
& \quad + \frac{a}{B(a) \Gamma(a)} \int_0^t \left( G \left( x, \zeta, u \right) - G \left( x, \zeta, u^* \right) \right) d\zeta. \quad \text{(48)}
\end{align*}
\]

On applying norm, the Eq. (48) simplifies to

\[
\begin{align*}
& \quad \|u(x, t) - u^*(x, t)\| = \left\| \left( 1 - \frac{a}{B(a)} \right) \left( G(x, t, u) - G \left( x, t, u^* \right) \right) \\
& \quad + \frac{a}{B(a) \Gamma(a)} \int_0^t \left( G \left( x, \zeta, u \right) - G \left( x, \zeta, u^* \right) \right) d\zeta \right\|
\end{align*}
\]

\[
\leq \left( 1 - \frac{a}{B(a)} \right) \eta \|u(x, t) - u^*(x, t)\| + \frac{a}{B(a) \Gamma(a)} \eta t \|u(x, t) - u^*(x, t)\|. \quad \text{(49)}
\]

On simplification

\[
\|u(x, t) - u^*(x, t)\| \left( 1 - \frac{a}{B(a)} \eta - \frac{a}{B(a) \Gamma(a)} \eta t \right) \leq 0.
\]

(50)

From the above condition, it is clear that \( u(x, t) - u^*(x, t) \), if

\[
\left( 1 - \frac{a}{B(a)} \eta - \frac{a}{B(a) \Gamma(a)} \eta t \right) \geq 0.
\]

(51)

Hence, Eq. (51) evidences our essential result.

**Theorem 5.** Suppose \( u_n(x, t) \) and \( u(x, t) \) define in the Banach space \( (B[0, T], \| \cdot \|) \). Then series solution defined in Eq. (26) converges to the solution of the Eq. (10), if \( 0 < \lambda_1 < 1 \).

**Proof:** Let consider the sequence \( \{S_n\} \) which is the partial sum of the Eq. (26), we have to prove \( \{S_n\} \) is Cauchy sequence in \( (B[0, T], \| \cdot \|) \). Now consider

\[
\|S_{n+1}(x, t) - S_n(x, t)\| = \|u_{n+1}(x, t)\|
\]

\[
\leq \lambda_1 \|u_n(x, t)\|
\]

\[
\leq \lambda_1^2 \|u_{n-1}(x, t)\|
\]

\[
\leq \ldots
\]

\[
\leq \lambda_1^{n+1} \|u_0(x, t)\|
\]

Now, we have for every \( n, m \in N \) \((m \leq n)\)

\[
\|S_n - S_m\| = \|S_n - S_{n-1}\| + \|S_{n-1} - S_{n-2}\| + \ldots
\]

\[
+ \|S_{m+1} - S_m\| \leq \|S_n - S_{n-1}\| + \|S_{n-1} - S_{n-2}\| + \ldots
\]

\[
+ \|S_{m+1} - S_m\| \leq \left( \lambda_1^n + \lambda_1^{n-1} + \ldots + \lambda_1^{m+1} \right) \|u_0\|
\]

\[
\leq \lambda_1^{m+1} \left( \lambda_1^{n-m-1} + \lambda_1^{n-m-2} + \ldots + \lambda_1 + 1 \right) \|u_0\|
\]

\[
\leq \lambda_1^{m+1} \left( \frac{1 - \lambda_1^{n-m}}{1 - \lambda_1} \right) \|u_0\|. \quad \text{(52)}
\]

But \( 0 < \lambda_1 < 1 \), therefore \( \lim_{n,m \to \infty} \|S_n - S_m\| = 0 \). Hence, \( \{S_n\} \) is the Cauchy sequence. Similarly, we can demonstrate for the second case. This proves the required result.

**Theorem 6.** For the series solution (26) of the Eq. (10), the maximum absolute error is presented as

\[
\|u(x, t) - \sum_{n=0}^M u_n(x, t)\| \leq \lambda_1^{M+1} \left( \frac{1 - \lambda_1^{n-m}}{1 - \lambda_1} \right) \|u_0(x, t)\|.
\]

**Proof:** By the help of Eq. (56), we get

\[
\|u(x, t) - S_n\| = \lambda_1^{n+1} \left( \frac{1 - \lambda_1^{n-m}}{1 - \lambda_1} \right) \|u_0(x, t)\|.
\]

But \( 0 < \lambda_1 < 0 \Rightarrow 1 - \lambda_1^{n-m} < 1 \). Hence, we have

\[
\|u(x, t) - \sum_{n=0}^M u_n(x, t)\| \leq \lambda_1^{M+1} \|u_0(x, t)\|.
\]

This ends the proof.

## 6 Results and discussion

In this manuscript, we find the solution for CDG equation having arbitrary order using a novel scheme namely, q-HATM with the help of Mittag-Leffler law. In the present segment, we demonstrate the effect of fractional order in the obtained solution with distinct parameters offered by the future method. In Figures 1 to 3, the nature of q-HATM solution for different arbitrary order is presented in terms of 2D plots. From these plots, we can see that considered problem conspicuously depends on fractional order. In order to analyse the behaviour of obtained solution with respect to homotopy parameter \( h \), the \( h \)-curves are drown for diverse \( \mu \) and presented in Figure 4. In the plots, the horizontal line represents the convergence region of the q-HATM solution and these curves aid us to adjust and handle the convergence province of the solution. For an appropriate value of \( h \), the achieved solution quickly converges to the exact solution. Further, the small variation in the physical behaviour of the complex models stimulates the enormous new results to analyse and understand nature in a better and systematic manner. Moreover, from all the plots we can see that the considered method is more accurate and very effective to analyse the considered complex coupled fractional order equations.
7 Conclusion

In this study, the $q$-HATM is applied lucratively to find the solution for arbitrary order CDG equations. Since AB derivatives and integrals having fractional order are defined with the help of generalized Mittag-Leffler function as the non-singular and non-local kernel, the present investigation illuminates the effeteness of the considered derivative. The existence and uniqueness of the obtained solution are demonstrated with the fixed point hypothesis. The results obtained by the future scheme are more stimulating as compared to results available in the literature. Further, the projected algorithm finds the solution for the nonlinear problem without considering any discretization, perturbation or transformations. The present investigation illuminates, the considered nonlinear phenomena noticeably depend on the time history and the time instant and which can be proficiently analysed by applying the concept of calculus with fractional order. The present investigation helps the researchers to study the behaviour nonlinear problems gives very interesting and useful consequences. Lastly, we can conclude the projected method is extremely methodical, more effective and very accurate, and which can be applied to analyse the diverse classes of nonlinear problems arising in science and technology.

References

[1] Liouville J., Memoire sur quelques questions de geometrie et de mecanique, et sur un nouveau genre de calcul pour resoudre questions, J. Ecole. Polytech., 1832, 13, 1-69.

[2] Riemann G.F.B., Versuch Einer Allgemeinen Auffassung der Integration und Differentiation, Gesammelte Mathematische Werke, Leipzig, 1896.

[3] Caputo M., Elasticita e Dissipazione, Zanichelli, Bologna, 1969.

[4] Miller K.S., Ross B., An introduction to fractional calculus and fractional differential equations, A Wiley, New York, 1993.
Figure 3: (a) Surface of $u(x, t)$, (b) 2D plot of $u(x, t)$ at $t = 10$ at $\mu = 0.5$, $\hbar = -1$, $n = 1$ and $\alpha = 1$.

Figure 4: $\hbar$-curves for q-HATM solution with distinct $\alpha$ at $x = 1$ and $t = 0.01$ for (a) $n = 1$ and (b) $n = 2$.

[5] Podlubny I., Fractional Differential Equations, Academic Press, New York, 1999.

[6] Kilbas A.A., Srivastava H.M., Trujillo J.J., Theory and applications of fractional differential equations, Elsevier, Amsterdam, North-Holland, 2006.

[7] Baleanu D., Guvenc Z.B., Machado J.A.T., New trends in nanotechnology and fractional calculus applications, Springer Dordrecht Heidelberg, London New York, 2010.

[8] Gao W., Veeresha P., Prakasha D.G., Novel dynamic structures of 2019-nCoV with nonlocal operator via powerful computational technique, Biology, 2020, 9(5), DOI: 10.3390/biology9050107

[9] Veeresha P., Prakasha D.G., Baskonus H.M., Solving smoking epidemic model of fractional order using a modified homotopy analysis transform method, Math. Sci., 2019, 13(2), 115–128

[10] Baleanu D., Wu G.C., Zeng S.D., Chaos analysis and asymptotic stability of generalized Caputo fractional differential equations, Chaos Solit. Fractals, 2017, 102, 99-105

[11] Veeresha P., Prakasha D.G., Baskonus H.M., New numerical surfaces to the mathematical model of cancer chemotherapy effect in Caputo fractional derivatives, Chaos, 2019, 29(013119), DOI: 10.1063/1.5074099

[12] Baskonus H.M., Sulaiman T.A., Bulut H., On the new wave behavior to the Klein-Gordon-Zakharov equations in plasma physics, Indian J. Phys., 2019, 93 (3), 393-399

[13] Veeresha P., Prakasha D.G., Kumar D., Baleanu D., Singh J., An efficient computational technique for fractional model of generalized Hirota-Satsuma coupled Korteweg–de Vries and coupled modified Korteweg–de Vries equations, J. Comput. Nonlinear Dynam., 2020, 15(7), 071003.

[14] Veeresha P., Prakasha D.G., Baleanu D., An efficient numerical technique for the nonlinear fractional Kolmogorov-Petrovskii-Piskunov equation, Mathematics, 2019, 7(3), DOI: 10.3390/math7030265

[15] Goufo F.D., Kumar S., Mugisha S.B., Similarities in a fifth-order evolution equation with and with no singular kernel, Chaos Solit. Fract., 2020, 130.

[16] Gao W., Ghanbari B., Baskonus H.M., New numerical simulations for some real world problems with Atangana–Baleanu fractional derivative, Chaos Solit. Fract., 2020, 128, 34-43.

[17] Ghanbari B., Kumar S., Kumar R., A study of behaviour for immune and tumor cells in immunogenetic tumour model with non-singular fractional derivative, Chaos Solit. Fractals, 2020, 133, DOI: 10.1016/j.chaos.2020.109619

[18] Gao W., Veeresha P., Prakasha D.G., Baskonus H.M., Yel G., New approach for the model describing the deathly disease in
pregnant women using Mittag-Leffler function, Chaos Solitons Fractals, 2020, 134, DOI: 10.1016/j.chaos.2020.109696

[19] Kumar S., Nisar K.S., Kumar R., Cattani C., Samet B., A new Rabotnov fractional-exponential function based fractional derivative for diffusion equation under external force, Math. Methods Appl. Sci., 2020, 43(7), 4460-4471.

[20] Veeresha P., Prakasha D.G., An efficient technique for two-dimensional fractional order biological population model, International Journal of Modeling, Simulation, and Scientific Computing, 2020 (in press), DOI: 10.1142/S1793962320500051

[21] Sulaiman T.A., Bulut H., Baskonus H.M., Optical solitons to the fractional perturbed NLS in nano-fibers, Discrete Contin. Dyn. Syst. Ser. S, 2020, 13(3), 925-936

[22] Jleli M., Kumar S., Kumar R., Samet B., Analytical approach for time fractional wave equations in the sense of Yang-Abdel-Aty-Cattani via the homotopy perturbation transform method, Alexandria Eng. J., 2020 (in press), DOI: 10.1016/j.aej.2019.12.022

[23] Gao W., Silambarasan R., Baskonus H.M., Anand R.V., Reza-Akhand S., Jleli M., Kumar S., Kumar R., Samet B., A modified analytical approach with existence and uniqueness for fractional Cauchy reaction-diffusion equations, Adv. Differ. Equ., 2020, 28, DOI: 10.1186/s13662-019-2488-3

[24] Kumar S., Kumar R., Agarwal R.P., Samet B., A modified numerical scheme for fractional Caudrey-Dodd-Gibbon equation using Haar wavelet and Adams-Bashforth-Moulton methods, Math. Methods Appl. Sci., 2020, 43(8), 5564-5578

[25] Veeresha P., Baskonus H.M., Prakasha D.G., Gao W., Yel G., Regaining new numerical solution of fractional Schistosomiasis disease arising in biological phenomena, Chaos Solit. Fract., 2020, 133, DOI: 10.1016/j.chaos.2020.109661

[26] Kumar S., Kumar R., Singh J., Nisar K.S., Kumar D., An efficient numerical scheme for fractional model of HIV-1 infection of CD4+ T-Cells with the effect of antiviral drug therapy, Alexandria Eng. J., 2020, (in press), DOI: 10.1016/j.aej.2019.12.046

[27] Kumar S., Ghosh S., Samet B., Goufo E.F.D., An analysis for heat equations arises in diffusion process using new Yang-Abdel-Aty Cattani fractional operator, Math. Methods Appl. Sci., 2020, (in press), DOI: 10.1002/mma.6347

[28] Alishabanan A., Jleli M., Kumar S., Samet B., Generalization of Caputo-Fabrizio fractional derivative and applications to electrical circuits, Front. Phys., 2020, (in press) DOI: 10.3389/fphy.2020.00064

[29] Prakasha D.G., Veeresha P., Analysis of Lakes pollution model with Mittag-Leffler kernel, J. Ocean Eng. Sci., 2020, (in press), DOI: 10.1016/j.joes.2020.01.004

[30] Ravichandran C., Logeswari K., Jarad F., New results on existence in the framework of Atangana–Baleanu derivative for fractional integro-differential equations, Chaos Solit. Fract., 2019, 125, 194-200.

[31] Ravichandran C., Jothimani K., Baskonus H.M., Villammal N., New results on nondensely characterized integrodifferential equations with fractional order, Eur. Phys. J. Plus, 2018, 133(3), DOI: 10.1140/epjp/i2018-11966-3

[32] Gao W., Veeresha P., Prakasha D.G., Baskonus H.M., Yel G., New numerical results for the time-fractional Phi-four equation using a novel analytical approach, Symmetry2020, 12(3), DOI: 10.3390/sym12030478

[33] Veeresha P., Prakasha D.G., Solution for fractional generalized Zakharov equations with Mittag-Leffler function, Results Eng., 2020, 5, (in press) DOI: 10.1016/j.rineng.2019.100085

[34] Nissar A., Vahidi J., Ghomi M.J., Mighani M., Reconstruction of variational iterative method for solving fifth order Caudrey-Dodd-Gibbon (CDG) equation, Int. J. Sci. Eng. Invest., 2012, 1(6), 38-41.

[35] Bibi S., Ahmed N., Faisal I., Mohyud-Din S.T., Rafiq M., Khan U., Some new solutions of the Caudrey-Dodd-Gibbon (CDG) equation using the conformable derivative, Adv. Diff. Equat. 2019, 89.

[36] Salas A., Some solutions for a type of generalized Sawada-Kotera equation, Appl. Math. Comp., 2008, 196(2), 812-817.

[37] Weiss J., On Classes of Integrable Systems and the Painlevé Property, J. Math. Phys., 1984, 25(1), 13-24.

[38] Dai Z.D., Jiang M.R., Wang S.H., Homoclinic orbits and periodic solitons for Boussinesq equation with even constraint, Chaos Solit. Fract., 2005, 26(4), 1189-1194.

[39] Caputo M., Fabricio M., A new definition of fractional derivative without singular kernel, Prog. Fract. Differ. Appl., 2015, 1(2), 73-85.

[40] Atangana A., Baleanu D., New fractional derivatives with nonlocal and non-singular kernel theory and application to heat transfer model, Therm. Sci., 2016, 20, 763-769.

[41] Liao S.J., Homotopy analysis method: a new analytic method for nonlinear problems, Appl. Math. Mech., 1998, 19, 957-962.

[42] Singh J., Kumar D., Swroop R., Numerical solution of time- and space-fractional coupled Burgers’ equations via homotopy algorithm, Alexandria Eng. J., 2016, 55(2), 1753-1763.

[43] Srivastava H.M., Kumar D., Singh J., An efficient analytical technique for fractional model of vibration equation, Appl. Math. Model., 2017, 45, 192-204.

[44] Gao W., Veeresha P., Prakasha D.G., Senel B., Baskonus H.M., Iterative method applied to the fractional nonlinear systems arising in thermoelectricity with Mittag-Leffler kernel, Fractals, 2020, (in press), DOI: 10.1142/S0218348X2040040X

[45] Prakasha D.G.,Veeresha P., Singh J., Fractional approach for potential Kadomtsev-Petviashvili equation, Commum. Nonlinear Sci. Numer. Simulat., 2010, 15(9), 2331-2336.

[46] Dai Z.D., Huang J., Jiang M.R., Wang S.H., Homoclinic orbits and periodic solitons for Boussinesq equation with even constraint, Chaos Solit. Fract., 2005, 26(4), 1189-1194.

[47] Caputo M., Prakasha D.G., Solution for fractional Zakharov-Kuznetsov equation using two reliable techniques, Chin. J. Phys., 2019, 60, 313-330.

[48] Kumar D., Agarwal R.P., Singh J., A modified numerical scheme and convergence analysis for fractional model of Lienard’s equation, J. Comput. Appl. Math. 2018,399, 405-413.

[49] Veeresha P., Prakasha D.G., Baskonus H.M., An efficient technique for coupled fractional Whitham-Broer-Kaup equations describing the propagation of shallow water waves, Adv. Intell. Syst. Comput. 2020, 49-75.

[50] Veeresha P., Prakasha D.G., Baskonus H.M., Novel simulations to the time-fractional Fisher’s equation, Math. Sci., 2019, 13(1), 33-42.

[51] Prakash A., Veeresha P., Prakasha D.G., An efficient numerical technique for fractional Caudrey-Dodd-Gibbon equation, Int. J. Bifurcation Chaos, 2019, 29(12), 2050126.

[52] Prakash A., Veeresha P., Prakasha D.G., Goyal M., A homotopy approach for fractional order multi-dimensional telegraph
equation via Laplace transform, Eur. Phys. J. Plus, 2019, 134, 19.
[53] Veeresha P., Baskonus H.M., Prakasha D.G., Gao W., Yel G.,
Regarding new numerical solution of fractional Schistosomiasis disease arising in biological phenomena, Chaos Solitons Fractals, 2020, 133.
[54] Xu Y.-G., Zhou X.-W., Yao L., Solving the fifth order Caudrey–Dodd–Gibbon (CDG) equation using the exp-function method, Appl. Math. Comput., 2008, 206, 70–73.
[55] Chan W.L., Zheng Y., Bäcklund transformations for the Caudrey–Dodd–Gibbon–Sawada–Kotera equation and its $\lambda$-modified equation, J. Math. Phys., 1989, 30(9), 2065-2068.
[56] Chen H., Xu Z., Dai Z., Breather soliton and cross two-soliton solutions for the fifth order Caudrey-Dodd-Gibbon (CDG) equation, Internat. J. Numer. Methods Heat Fluid Flow, 2015, 25(3), 651-655.
[57] Karaagac B., A numerical approach to Caudrey Dodd Gibbon equation via collocation method using quintic B-spline basis, TWMS J. App. Eng. Math., 2019, 9(1), 1-8.
[58] Abdollahzadeh M., Hosseini M., Ghanarpour M., Shirvani H.,
Exact travelling solutions for fifth order Caudrey-Dodd-Gibbon equation, Int. J. Appl. Math. Comput., 2010, 2(4), 81–90.
[59] Jiang B., Bi Q., A study on the bilinear Caudrey-Dodd-Gibbon equation, Nonlinear Anal., 2010, 72, 4530-4533.
[60] Salas A.H., Hurtado O.G., Castillo J.E., Computing multi-soliton solutions to Caudrey-Dodd-Gibbon equation by Hirota's method, Int. J. Phys. Sci., 2011, 6(34), 7729–7737.
[61] Geng X., He G., Wu L., Riemann theta function solutions of the Caudrey-Dodd-Gibbon-Sawada-Kotera hierarchy, J. Geom. Phys., 2019, 140, 85-103.
[62] Singh J., Kumarv, Hammouch Z., Atangana A., A fractional epidemiological model for computer viruses pertaining to a new fractional derivative, Appl. Math. Comput., 2018, 316, 504-515.
[63] Prakasha D.G., Veeresha P., Baskonus H.M., Analysis of the dynamics of hepatitis E virus using the Atangana-Baleanu fractional derivative, Eur. Phys. J. Plus 2019, 134, 241.
[64] Atangana A., Alkahtani B.T., Analysis of the Keller-Segel model with a fractional derivative without singular kernel, Entropy, 2015, 17, 4439-4453.
[65] Atangana A., Alkahtani B.T., Analysis of non-homogenous heat model with new trend of derivative with fractional order, Chaos Solit. Fract., 2016, 89, 566-571.