The Number of Point-Splitting Circles

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Abstract

Let $S$ be a set of $2n+1$ points in the plane such that no three are collinear and no four are concyclic. A circle will be called point-splitting if it has 3 points of $S$ on its circumference, $n-1$ points in its interior and $n-1$ in its exterior. We show the surprising property that $S$ always has exactly $n^2$ point-splitting circles, and prove a more general result.

1 Introduction

Our starting point is the following problem, which first appeared in the 1962 Chinese Mathematical Olympiad [4].

Problem 1.1. Let $S$ be a set of $2n+1$ points in the plane such that no three are collinear and no four are concyclic. Prove that there exists a circle which has 3 points of $S$ on its circumference, $n-1$ points in its interior, and $n-1$ in its exterior.

Following [6, p.48], we call such a circle point-splitting for the given set of points. For the rest of sections 1 and 2, $S$ denotes an arbitrary set of $2n+1$ points in general position in the plane, where $n$ is a fixed integer.

There are several solutions to problem 1.1. Perhaps the easiest one is the following. Let $A$ and $B$ be two consecutive vertices of the convex hull of $S$. We claim that some circle going through $A$ and $B$ is point-splitting. All circles through $A$ and $B$ have their centers on the perpendicular bisector $\ell$ of the segment $AB$. Pick a point $O$ on $\ell$ which lies on the same side of $AB$
as $S$, and is so far away from $AB$ that the circle $\Gamma$ with center $O$ and going through $A$ and $B$ completely contains $S$. This can clearly be done. Now slowly “push” $O$ along $\ell$, moving it towards $AB$. The circle $\Gamma$ will change continuously with $O$. As we do this, $\Gamma$ will stop containing some points of $S$. In fact, it will lose the points of $S$ one at a time: if it lost $P$ and $Q$ simultaneously, then points $P, Q, A$ and $B$ would be concyclic. We can move $O$ so far away past $AB$ that, in the end, the circle will not contain any points of $S$.

Originally, $\Gamma$ contained all the points of $S$. Now, as it loses one point of $S$ at a time, we can decide how many points we want it to contain. In particular, if we stop moving $O$ when the circle is about to lose the $n$-th point $P$ of $S$, then the resulting $\Gamma$ will be point-splitting: it will have $A$, $B$ and $P$ on its circumference, $n - 1$ points inside it, and $n - 1$ outside it, as shown in Figure 1.

The above proof hints that any set $S$ has several different point-splitting circles. We can certainly construct one for each pair of consecutive vertices of the convex hull of $S$. In fact, the argument above can be modified to show that, for any two points of $S$ we can find a point-splitting circle going through them. The reader might find it instructive to work out a proof.

This suggests that we ask the following question. What can we say about the number $N_S$ of point-splitting circles of $S$? At first sight, it seems that we really cannot say very much about this number. Point-splitting circles seem hard to “control”, and harder to count.

We should be able to find upper and lower bounds for $N_S$ in terms of $n$. Right away we know that $N_S \geq n(2n + 1)/3$, since we can find a point-
splitting circle for each pair of points of $S$, and each such circle is counted by three different pairs. Computing an upper bound seems more difficult. If we fix points $A$ and $B$ of $S$, it is indeed possible that all $2n - 1$ circles through $A$, $B$ and another point of $S$ are point-splitting. The reader is invited to check this. This is not likely to happen very often in a set $S$, and we can get some upper bound out of this. However, it is hard to make this precise and get a non-trivial upper bound.

When $S$ consists of 5 points, the situation is simple enough that we can actually show that $N_S = 4$ always. This was done in [2]. It was also proposed, but not chosen, as a problem for the 1999 International Mathematical Olympiad. Notice that our lower bound above gives $N_S \geq 4$.

In a different direction, problem 5 of the 1998 Asian-Pacific Mathematical Olympiad, proposed by the author, stated the following.

**Proposition 1.2.** $N_S$ has the same parity as $n$.

This result follows easily from the nontrivial observation that, for any $A$ and $B$ in $S$, the number of point-splitting circles that go through $A$ and $B$ is odd.

The following result brings together the above considerations.

**Theorem 1.3.** Any set $S$ of $2n+1$ points in the plane in general position has exactly $n^2$ point-splitting circles.

Theorem 1.3 is the main result of this paper. In section 2 we prove that every set of $2n+1$ points in the plane in general position has the same number of point-splitting circles. In section 3 we prove that this number is exactly $n^2$. In section 4 we present some questions that arise from our work.

2 $N_S$ is Constant

At this point, we could go ahead and prove the very counterintuitive Theorem 1.3, suppressing the motivation behind its discovery. With the risk of making the argument seem longer, we believe that it is worthwhile to present a natural way of realizing and proving that the number of point-splitting circles of $S$ depends only on $n$. Therefore, we ask the reader to forget momentarily the punchline of this article.

Suppose that we are trying to find out whatever we can about the number $N_S$. As mentioned in Section 1, this number does not seem very tractable
and it is not clear how much we can say about it. Being optimistic, we can
hope to be able to answer the following two questions.

**Question 2.1.** What are the sharp lower and upper bounds $m = m_{2n+1}$
and $M = M_{2n+1}$ for $N_S$?

**Question 2.2.** What are all the values that $N_S$ takes in the interval
$[m, M]$?

Question 2.1 seems considerably difficult. To answer it completely, we
would first need to prove an inequality $m \leq N_S \leq M$, and then construct
suitable sets $S_{\text{min}}$ and $S_{\text{max}}$ which achieve these bounds. To see how difficult
this is, the reader is invited to try to construct *any* set $S$ of $2n+1$ points for
which the number $N_S$ can be easily computed.

At this point question 2.1 seems very hard, so let us focus on Question 2.2 instead. Here is a first approach.

Intuitively, since the set $S$ can be transformed continuously, we should
expect the value of $N_S$ to change “continuously” with it. Suppose we start
with the set $S_{\text{min}}$ (with $N_S = m$) and move its points continuously so that
we end up with $S_{\text{max}}$ (with $N_S = M$). The value of $N_S$ should change
“continuously” as $S$ changes continuously. By “continuity” we would guess
that $N_S$ sweeps all the integers between $m$ and $M$ as $S$ changes from $S_{\text{min}}$
to $S_{\text{max}}$.

Right away, we know that this is not entirely true. By Proposition 1.2
we know that the parity of $N_S$ is determined by $n$, so $N_S$ will not sweep all
the integers between $m$ and $M$. This is not too surprising, since we haven’t
made precise the meaning of the statement that the value of $N_S$ should change “continuously” as $S$ changes continuously. The above guess assumed
that the value of $N_S$ can only jump by 1 as $S$ is transformed continuously.
(That is, if we have a set with $k - 1$ point-splitting circles and we deform it
continuously into a set with $k + 1$ point-splitting circles, then somewhere in
the middle we must have had a set with $k$ point-splitting circles.) We have
no reason to assume that.

We can still hope that, as $S$ changes, $N_S$ sweeps all the integers of the
right parity between $m$ and $M$. To show this, we would have to show that
the value of $N_S$ can only jump by 2 as $S$ is transformed continuously. This
is a reasonable statement which we can try to prove.

In any case, the natural question to ask is what kind of “continuity” the
value of $N_S$ satisfies as $S$ changes continuously. We certainly expect that if
two sets $S$ and $T$ look very very much alike, then the difference $N_S - N_T$ should be small. We have to find a way to make this statement precise.

Suppose we have sets $S_{\text{min}} = \{P_1, \ldots, P_{2n+1}\}$ and $S_{\text{max}} = \{Q_1, \ldots, Q_{2n+1}\}$ that achieve the upper and lower bounds for $N_S$, respectively. Now slowly transform $S_{\text{min}}$ into $S_{\text{max}}$: first send $P_1$ to $Q_1$ continuously along some path, then send $P_2$ to $Q_2$ continuously along some other path, and so on. We can think of our set $S$ as changing with time. At the initial time $t = 0$, our set is $S(0) = S_{\text{min}}$. At the final time $t = T$, our set is $S(T) = S_{\text{max}}$. In between, $S(t)$ varies continuously with respect to $t$. How does $N_{S(t)}$ vary “continuously” with time? How small can we make $N_{S(t+\Delta t)} - N_{S(t)}$ for small enough $\Delta t$? This is the question we need to ask.

**Technical Remark.** As we move from $S(0)$ to $S(T)$ continuously, it is likely that several intermediate sets $S(t)$, with $0 < t < T$, are not in general position. Strictly speaking, we should only consider those times $t$ when $S(t)$ is in general position; when $S(t)$ is not in general position, we should decree that $S(t)$ is undefined, and have a discontinuity at $t$.

We shall see that we can go from $S(0)$ to $S(T)$ with only finitely many such discontinuous points. At such a discontinuity $t$, we still need to know how small we can make $N_{S(t+\Delta t)} - N_{S(t-\Delta t)}$ for small $\Delta t$.

For small enough $\Delta t$, the set $S(t + \Delta t)$ is a very slight deformation of $S(t)$. What is missing is an understanding of what can make $N_S$ change as the set $S$ changes very slightly from $S(t)$ to $S(t + \Delta t)$, and how small this change is. Let us answer this question.

Notice that, in the way we defined the deformation from $S_{\text{min}}$ to $S_{\text{max}}$, the points of $S$ moved only one at a time. Let us focus for now on the interval of time where $P_1$ moves towards $Q_1$.

Suppose that the number $N_S$ changes between time $t$ and time $t + \Delta t$. Then it must be the case that for some $i, j, k$ and $l$ the circle $P_iP_jP_k$ contained (or did not contain) point $P_l$ at time $t$, but at time $t + \Delta t$ it does not (or does) contain it. For this to be true, it must have happened that somewhere between times $t$ and $t + \Delta t$, these four points must have been concyclic, or three of them must have been collinear. Since $P_1$ is the only point that has moved, we can conclude that $P_1$ must have crossed a circle or a line determined by the other points; this is what caused $N_S$ to change. We will call the circles and lines determined by the points $P_2, P_3, \ldots, P_{2n+1}$ the boundaries.
We can choose the path along which $P_1$ is going to move towards $Q_1$. To make things easier, we may assume that $P_1$ never crosses two of the boundaries at the same time. This can clearly be guaranteed: we know that these boundaries intersect pairwise in finitely many points, and all we have to do is avoid these intersection points in the path from $P_1$ to $Q_1$. We can also assume that $\Delta t$ is small enough that $P_1$ crosses exactly one boundary between times $t$ and $t + \Delta t$. Let us see how $N_S$ changes in this time interval.

It will be convenient to call a circle $P_i P_j P_k$ $(a, b)$-splitting (where $a + b = 2n - 2$) if it has $a$ points of $S$ inside it and the remaining $b$ outside it. For example, an $(n - 1, n - 1)$-splitting circle is just a point-splitting circle.

First assume that $P_1$ crosses line $P_i P_j$, going from position $P_1(t) = A$ to position $P_1(t + \Delta t) = B$. From the remarks made above, we know that only circle $P_1 P_i P_j$ can change the value of $N_S$ by becoming or ceasing to be point-splitting. Assume that circle $AP_1 P_j$ was $(a, b)$-splitting. Since $P_1$ only crossed the boundary $P_i P_j$ when going from $A$ to $B$, the region common to circles $AP_1 P_j$ and $BP_1 P_j$ cannot contain any points of $S$, as indicated in Figure 2.

The region outside of both circles cannot contain points of $S$ either. For circle $AP_1 P_j$ to be $(a, b)$-splitting, the other two regions must then contain $a$ and $b$ points respectively, as shown. Therefore, circle $BP_1 P_j$ is $(b, a)$-splitting. It follows that $AP_1 P_j$ was point-splitting if and only if $BP_1 P_j$ is point-splitting (if and only if $a = b = n - 1$). We conclude that the value of $N_S$ doesn't change when $P_1$ crosses a line determined by the other points; it can only change when $P_1$ crosses a circle.
Now assume that $P_1$ crosses circle $P_iP_jP_k$, going from position $P_1(t) = A$ inside the circle to position $P_1(t + \Delta t) = B$ outside it. (The other case, when $P_1$ moves inside the circle, is analogous.) We can assume that $P_1$ crossed the arc $P_iP_j$ of the circle that doesn’t contain point $P_k$. Notice that $A$ must be outside triangle $P_iP_jP_k$ if we want $P_1$ to cross only one boundary in the time interval considered. Assume that circle $AP_iP_j$ was $(a, b)$-splitting. As before, we know that the only regions of Figure 3 containing points of $S$ are the one common to circles $AP_iP_j$ and $BP_iP_j$, and the one outside both of them. They must contain $a - 1$ and $b$ points respectively, for circle $AP_iP_j$ to be $(a, b)$-splitting. In this case, the value of $N_S$ can change only by circles $P_iP_jP_k$, $P_iP_kP_j$, $P_kP_jP_i$ and $P_1P_iP_j$ becoming or ceasing to be point-splitting. It is clear that circle $P_iP_jP_k$ went from being $(a, b)$-splitting to being $(a - 1, b + 1)$-splitting. The same is true of circle $P_1P_iP_j$.

It is also not hard to see, by a similar argument, that circles $P_1P_jP_k$ and $P_1P_kP_i$ both went from being $(a - 1, b + 1)$-splitting to being $(a, b)$-splitting. Again, the key assumption is that $P_1$ only crossed the boundary $P_iP_jP_k$ in this time interval.

So, by having $P_1$ cross circle $P_iP_jP_k$, we have traded two $(a, b)$-splitting and two $(a - 1, b + 1)$-splitting circles for two $(a - 1, b + 1)$-splitting and two $(a, b)$-splitting circles, respectively. It follows that the number $N_S$ of
point-splitting circles remains constant when $P_1$ crosses a circle $P_iP_jP_k$ also.

We had shown that, as we moved $P_1$ to $Q_1$, $N_S$ could only possibly change in a time interval when $P_1$ crossed a boundary determined by the other points. But now we see that, even in such a time interval, $N_S$ does not change! Therefore moving $P_1$ to $Q_1$ doesn’t change the value of $N_S$. Similarly, moving $P_i$ to $Q_i$ doesn’t change $N_S$ either, for any $1 \leq i \leq 2n + 1$. It follows that $N_S$ is the same for $S_{min}$ and $S_{max}$. In fact, $N_S$ is the same for any set $S$ of $2n + 1$ points in general position!

$$3 \quad N_S = n^2$$

Now that we know that the number $N_S$ depends only on the number of points in $S$, define $N_{2n+1}$ to be the number of point-splitting circles for a set of $2n+1$ points in general position. We compute $N_{2n+1}$ recursively.

Construct a set $S$ of $2n+1$ points as follows. First consider the vertices of a regular $2n-1$-gon with center $O$. Now move them very slightly to positions $P_1, \ldots, P_{2n-1}$ so that they are in general position. The difference will be so slight that all the lines $OP_i$ still split the remaining points into two sets of equal size, and all the circles $P_iP_jP_k$ still contain $O$. Also consider a point $Q$ which is so far away from the others that it lies outside of all the circles formed by the points considered so far. Of course, we need $Q$ to be in general position with respect to the remaining points. Let us count the number of point-splitting circles of $S = \{O, P_1, \ldots, P_{2n-1}, Q\}$.

First consider the circles of the form $P_iP_jP_k$. These circles contain $O$ and don’t contain $Q$; so they are point-splitting for $S$ if and only if they are point-splitting for $\{P_1, \ldots, P_{2n-1}\}$. Thus there are $N_{2n-1}$ such circles.

Next consider the circles $OP_iP_j$. It is clear that these circles contain at most $n-2$ other $P_k$’s. They do not contain $Q$, so they contain at most $n-2$ points, and they are not point-splitting.

Finally consider the circles that go through $Q$ and two other points $X$ and $Y$ of $S$. Circle $QXY$ splits the remaining points in the same way that line $XY$ does. More specifically, circle $QXY$ contains a point $P$ of $S$ if and only if $P$ is on the same side of line $XY$ that $Q$ is. This follows easily from the fact that $Q$ lies outside circle $PXY$. Therefore we have to determine how many lines determined by two of the points of $S - \{Q\}$ split the remaining points of this set into two sets of $n-1$ points each. This question is much
easier; it is clear from our construction that the lines $OP_i$ do this and the lines $P_i P_j$ do not. It follows that the $2n - 1$ circles $OP_i Q$ are point-splitting, and the circles $P_i P_j Q$ are not.

Summarizing, the point-splitting circles of $S$ are the $N_{2n-1}$ point-splitting circles of $S - \{O, Q\}$ and the $2n - 1$ circles $OP_i Q$. Therefore $N_{2n+1} = N_{2n-1} + 2n - 1$. Since $N_3 = 1$, it follows inductively that $N_{2n+1} = n^2$. This completes the proof of Theorem 1.3.

**Theorem 3.1.** Consider a set of $2n + 1$ points in general position in the plane, and two non-negative integers $a < b$ such that $a + b = 2n - 2$. There are exactly $2(a + 1)(b + 1)$ circles which are either $(a, b)$-splitting or $(b, a)$-splitting for the set of points.

**Sketch of Proof.** The argument of Section 2 carries directly to this situation, to show that the number of circles in consideration, which we denote $N(a, b)$, only depends on $a$ and $b$. Therefore it suffices to compute it recursively, using the set $S$ above. It is essential in the proof that $a < n - 1$.

Just as above, there are $N(a - 1, b - 1)$ such circles among the circles $P_i P_j P_k$. Among the $OP_i P_j$ there are exactly $2n - 1$ such circles, namely the circles $OP_i P_{i+a+1}$ (taking subscripts modulo $2n - 1$). There are also $2n - 1$ such circles among the $QP_i P_j$, namely the circles $QP_i P_{i+a+1}$. Finally, there are no such circles among the $OP_i Q$. Therefore $N(a, b) = N(a - 1, b - 1) + 4n - 2 = N(a - 1, b - 1) + 2a + 2b + 2$.

Repeating the above argument for $a = 0$, we get that $N(0, b) = 2b + 2$. If we combine this and the recursive relation obtained, Theorem 3.1 follows by induction.

It is worth mentioning at this point that Theorems 1.3 and 3.1 are closely related to a beautiful result of D.T. Lee, which gives a sharp bound for the number of vertices of an order $j$ Voronoi diagram. The connection is obtained if we embed our set $S$ of points on the surface of a sphere. Then the point-splitting circles of $S$ are put in correspondence with the “point-splitting planes” of a three-dimensional convex polytope with $2n + 1$ vertices. These are known to be related to Voronoi diagrams. See [1, p. 397] for more details on this, and a proof of a result essentially equivalent to Theorems 1.3 and 3.1.
4 Questions

Our work completely determines the number of point-splitting circles, as well as the total number of \((a, b)\)-splitting and \((b, a)\)-splitting circles for a set of points in general position in the plane. However, we know very little about these numbers for sets of points that are not in general position.

The situation here is much more subtle. For example, the number of point-splitting circles of a set \(S\) is not uniquely determined by the subsets of \(S\) which are concyclic. Consider the following example. Let \(S_1\) and \(S_2\) be the two sets of seven points shown in Figure 4. Both of them are almost in general position; the only exception is that, for each of the two sets, there is a circle going through four points of the set. In \(S_1\), this circle \(\Gamma_1\) contains exactly one point of \(S_1\) inside it. In \(S_2\), this circle \(\Gamma_2\) contains no points of \(S_2\) inside it. In analogy with Theorem 1.3, where it did not matter which points were inside which circles, we might hope that \(S_1\) and \(S_2\) have the same number of point-splitting circles.

Unfortunately this is not the case. If we move \(A, B, C\) or \(D\) very slightly to put \(S_1\) in general position, the resulting set will have 9 point-splitting circles by Theorem 1.3. It is easy to see that exactly two of \(ABC, BCD, CDA\) and \(DAB\) are among these circles. When we deform the set back to \(S_1\), these 9 circles will still be point-splitting, but two of them will deform into \(\Gamma_1\). So \(S_1\) has 8 point-splitting circles.

Similarly, if we move \(a, b, c\) or \(d\) very slightly to put \(S_2\) in general position,
the resulting set will have 9 point-splitting circles. But now we can see that when we deform the set back to $S_2$, none of these circles will deform into $\Gamma_2$, because $\Gamma_2$ contains no points of $S_2$. Therefore $S_2$ has 9 point-splitting circles.

Even if the number of point-splitting circles is not constant, we might be able to say something about it. As a small example, consider all sets of seven points which are almost in general position, except that four of them are concyclic. It is possible to show, by an argument similar to the above, that such a set can only have 8 or 9 point-splitting circles. It seems reasonable that, in general, one might be able to define some measure of how far a set $S$ is from being in general position, and to obtain bounds for $N_S$ in terms of that measure.

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