Sobolev embeddings with weights in complete riemannian manifolds.

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February 25, 2019

Abstract
We prove Sobolev embedding Theorems with weights for vector bundles in a complete riemannian manifold. As a consequence, under geometric hypotheses, we improve some "classical" Sobolev embedding Theorems for vector bundles in a complete riemannian manifold.

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1 Introduction.

Let \((M, g)\) be a complete riemann manifold. The Sobolev inequalities in \(M\) for functions play a major role in the study of differential operators and nonlinear functional analysis. They are valid in \(\mathbb{R}^n\) or if \(M\) is compact.

If \(M\) has a Ricci curvature bounded from below then we have a characterisation of the riemannian manifolds where they are valid.

**Theorem 1.1.** Let \((M, g)\) be a complete riemannian manifold of dimension \(n\) with Ricci curvature bounded from below. The Sobolev embeddings for functions are valid for \(M\) if and only if there is a uniform lower bound for the volume of balls which is independent of their center, namely if and only if \(\inf_{x \in M} \text{Vol}(B(x, 1)) > 0\).

The necessity is a well known fact, see p.18 in [Hebey, 1996] and was generalised by Carron [Carron, 1994]. It can be used to get that, in general, the Sobolev embeddings are not valid in a complete riemannian manifold, even if the Ricci curvature is bounded from below. (See [Hebey, 1996, Proposition 3.13, p. 30]).

The sufficiency of Theorem 1.1 was done by Varopoulos [Varopoulos, 1989], see Theorem 3.18, p. 37 in [Hebey, 1996], based on the work of Coulhon and Saloff-Coste [Coulhon and Saloff-Coste, 1993]. We have also using Theorem 1.1: ([Hebey, 1996, Corollary 3.19, p. 38])

**Corollary 1.2.** The Sobolev embeddings for functions are valid for complete manifolds with Ricci curvature bounded from below and positive injectivity radius.

The case of vector bundles was also studied long time ago. For instance by Cantor [Cantor, 1974] and many other authors. In particular N. Lohouë [Lohouë, 1985] studied Sobolev inequalities for the bundle of \(p\)-forms. On complete non-compact Riemannian manifolds, he got inequalities of the form:

\[
\|\omega\|_{L^s(M)} \leq C(\|d(\omega)\|_{L^t(M)} + \|d^*(\omega)\|_{L^r(M)} + \|\omega\|_{L^r(M)}),
\]

where \(d\) is the exterior derivative and \(d^*\) its formal adjoint and with \(\frac{1}{s} = \frac{1}{r} - \frac{1}{n}\).

In order to get vanishing theorems on the \(L^{p,q}\)-cohomology and to study the \(L^{p,q}\)-torsion on complete non-compact Riemannian manifolds, X-D. Li [Li, 2010] extends strongly N. Lohouë’s result. In particular he studied this kind of questions in the case of complete manifolds with weight "à la Witten". The introduction in [Li, 2010] is quite complete and there is a rich set of references on these questions in it.

The aim of this work is to introduce weights, given by the geometry, on a complete riemannian manifold \((M, g)\) in order to have Sobolev embeddings, on vector bundles, always valid with these weights, without any curvature conditions nor volume control.

Then, in the Applications Section, we show that geometric conditions on \((M, g)\) allow us to forget these weights and retrieve "classical" Sobolev embeddings.

Let \((M, g)\) be a complete riemannian manifold and let \(G := (H, \pi, M)\) be a complex \(C^m\) vector bundle over \(M\) of rank \(N\) with fiber \(H\) with a smooth scalar product \((\ , \ )\). We make the hypothesis that we have a metric connection \(\nabla^G\) on \(G\), i.e. for any vector field \(X\) on \(M\) we have \(X(u, v) = (\nabla_X u, v) + (u, \nabla_X v)\) for any smooth sections \(u, v\) of \(G\). We shall call a vector bundle with these properties an adapted vector bundle.
This work is presented as follow:
• In the next Section we define the $\epsilon$-admissible balls and their main properties. We shall first prove local results for bundles on these balls.
• Then in order to get global results we group these balls via a Vitali type covering in SubSection 2.1.
• In Section 3 we define the vector bundle $G$ we are interested in and precise the metric connexion $\nabla^G$ on it we shall use.
• In Section 4 we define the Sobolev spaces of smooth sections of $G$, with weights. We prove in this Section a generalisation of a nice result of T. Aubin [Aubin, 1982] which says that, in order to prove Sobolev embeddings for smooth sections of $G$ with weights, we have just to prove them at the first level.
• In Section 5 we prove the local estimates, i.e. for smooth sections of $G$ in the $\epsilon$-admissible balls.
• In SubSection 5.3 we also prove a local Gaffney type inequality, using a result by C. Scott [Scott, 1995].
• In Section 6 we prove the global estimates for functions, smooth sections of $G$ and Gaffney type in $L^r$.
• In Section 7 we get "classical" Sobolev embeddings by use of a Theorem of Hebey-Herzlich [Hebey and Herzlich, 1997, Corollary, p. 7]. First in the case of riemannian manifold with bounded geometry. This implies the validity of Sobolev embeddings for vector bundles in compact riemannian manifold without boundary.
• In SubSection 7.3 we deduce from the compact case without boundary the validity of Sobolev embeddings for vector bundles in compact riemannian manifold with smooth boundary. We use here the method of the "double" manifold.
• Finally in SubSection 7.4 we study the case of hyperbolic manifolds.

2 Admissible balls.

Definition 2.1. Let $(M, g)$ be a riemannian manifold and $x \in M$. We shall say that the geodesic ball $B(x, R)$ is $(0, \epsilon)$-admissible if there is a chart $(B(x, R), \varphi)$ such that:

(*) $(1 - \epsilon)\delta_{ij} \leq g_{ij} \leq (1 + \epsilon)\delta_{ij}$ in $B(x, R)$ as bilinear form.

We shall say that the geodesic ball $B(x, R)$ is $(1, \epsilon)$-admissible if moreover

(**) $\sum_{|\beta|=1} R \sup_{i,j=1,\ldots,n, y \in B_x(R)} |\partial^\beta g_{ij}(y)| \leq \epsilon$.

We shall denote $\mathcal{A}(0, \epsilon)$ the set of $(0, \epsilon)$-admissible balls on $M$ and $\mathcal{A}(1, \epsilon)$ the set of $(1, \epsilon)$-admissible balls on $M$.

The $(0, \epsilon)$-admissible balls will be adapted to functions and the $(1, \epsilon)$-admissible balls will be adapted to smooth sections of $G$.

Definition 2.2. Let $x \in M$, we set $R'(x) = \sup \{R > 0 :: B(x, R) \in \mathcal{A}(\epsilon)\}$. We shall say that $R_\epsilon(x) := \min (1, R'(x)/2)$ is the $\epsilon$-admissible radius at $x$.

Here $\mathcal{A}(\epsilon)$ will be either $\mathcal{A}(0, \epsilon)$ or $\mathcal{A}(1, \epsilon)$ depending on the context, functions or smooth sections of $G$. The notation $R_\epsilon(x)$ will means either $R_{0,\epsilon}(x)$ or $R_{1,\epsilon}(x)$ depending on the choice of $\mathcal{A}(\epsilon)$.

Clearly if $B(x, R) \in \mathcal{A}(\epsilon)$, i.e. is $\epsilon$-admissible, then so is $B(x, S)$ if $S \leq R$. 

3
Let \( x, y \in M \). Suppose that \( R'(x) > d_g(x, y) \), with \( R'(x) \) as in Definition 2.2, and consider the ball \( B(y, \rho) \) of center \( y \) and radius \( \rho := R'(x) - d_g(x, y) \). This ball is contained in \( B(x, R'(x)) \) hence, by definition of \( R'(x) \), we have that all the points in \( B(y, \rho) \) verify the condition (*) and eventually (**). So, by definition of \( R'(y) \), we have that \( R'(y) \geq R'(x) - d_g(x, y) \). If \( R'(x) \leq d_g(x, y) \) this is also true because \( R'(y) > 0 \). Exchanging \( x \) and \( y \) we get that \( |R'(y) - R'(x)| \leq d_g(x, y) \).

Hence \( R'(x) \) is 1-lipschitzian so it is continuous. So the \( \epsilon \)-admissible radius \( R_\epsilon(x) := \min(1, R'(x)/2) \) is also continuous hence measurable.

The harmonic radius \( r_H(1 + \epsilon, k, \alpha) \) is defined in the book [Hebey, 1996, p. 4], by adding a second condition to (*), for \( \alpha \in (0, 1) \):
\[
\sum_{1 \leq |\beta| \leq k} R^{[|\beta|]} \sup_{i,j=1,...,n, y \in B_x(R)} |\partial^{\beta} g_{ij}(y)| +
+ R^{k+\alpha}(x) \sup_{i,j=1,...,n, y \neq z \in B(x, R), |\beta| = k} \left| \frac{\partial^{\beta} g_{ij}(y) - \partial^{\beta} g_{ij}(z)}{d_g(y, z)^\alpha} \right| \leq \epsilon
\]
and asking moreover that the chart \( (B(x, R), \varphi) \) has harmonic coordinates.

So our admissible radius \( R_\epsilon \) is bigger than the harmonic radius \( r_H(1 + \epsilon, 0, \alpha) \) because we do not require the coordinates to be harmonic and we keep only the condition (*).

The same our admissible radius \( R_{1,\epsilon} \) is bigger than the harmonic radius \( r_H(1 + \epsilon, 1, \alpha) \).

The interest for these balls is that they provide estimates independent of the point in the manifold \( M \).

Of course, without any extra hypotheses on the riemannian manifold \( M \), we have \( \forall \epsilon > 0, \forall x \in M \), taking \( g_{ij}(x) = \delta_{ij} \) in a chart on \( B(x, R) \) and the radius \( R \) small enough, the ball \( B(x, R) \) is \( \epsilon \)-admissible.

**Remark 2.3.** Because on our admissible ball \( B(x, R_\epsilon(x)) \) there is a diffeomorphism from \( B(x, R) \) to \( \varphi(B(x, R)) \subset \mathbb{R}^n \), i.e. on an open set in the tangent space \( T_x M \), we get that the injectivity radius \( r_{inj}(x) \) always verifies \( r_{inj}(x) \geq R_\epsilon(x) \).

**Lemma 2.4.** (Slow variation of the admissible radius) Let \( (M, g) \) be a riemannian manifold then with \( R(x) = R_\epsilon(x) = \epsilon \)-admissible radius at \( x \in M \). We get:
\[
\forall y \in B(x, R(x)) \text{ we have } R(x)/2 \leq R(y) \leq 2R(x).
\]

**Proof.**
Let \( x, y \in M \) and \( d(x, y) \) the riemannian distance on \( (M, g) \). Let \( y \in B(x, R(x)) \) then \( d(x, y) \leq R(x) \) and suppose first that \( R(x) \geq R(y) \). Then, because \( R(x) = R'(x)/2 \), we get \( y \in B(x, R'(x)/2) \) hence we have \( B(y, R'(x)/2) \subset B(x, R'(x)) \). But by the definition of \( R'(x) \), the ball \( B(x, R'(x)) \) is admissible and this implies that the ball \( B(y, R'(x)/2) \) is also admissible for exactly the same constants and the same chart; this implies that \( R'(y) \geq R'(x)/2 \) hence \( R(y) \geq R(x)/2 \), so \( R(x) \geq R(y) \geq R(x)/2 \).

If \( R(x) \leq R(y) \) then \( d(x, y) \leq R(x) \Rightarrow d(x, y) \leq R(y) \Rightarrow x \in B(y, R'(y)/2) \Rightarrow B(x, R'(y)/2) \subset B(y, R'(y)). \) Hence the same way as above we get \( R(y) \geq R(x) \geq R(y)/2 \Rightarrow R(y) \leq 2R(x) \). So in any case we proved that
\[
\forall y \in B(x, R(x)) \text{ we have } R(x)/2 \leq R(y) \leq 2R(x).
\]
2.1 Vitali covering.

**Lemma 2.5.** Let $\mathcal{F}$ be a collection of balls $\{B(x,r(x))\}$ in a metric space, with $\forall B(x,r(x)) \in \mathcal{F}$, $0 < r(x) \leq R$. There exists a disjoint subcollection $\mathcal{G}$ of $\mathcal{F}$ with the following properties: every ball $B$ in $\mathcal{F}$ intersects a ball $C$ in $\mathcal{G}$ and $B \subset 5C$.

This is a well known lemma, see for instance [Evans and Gariepy, 1992], section 1.5.1.

Fix $\epsilon > 0$ and let $\forall x \in M$, $r(x) := R_\epsilon(x)/10$, where $R_\epsilon(x)$ is the admissible radius at $x$, we built a Vitali covering with the collection $\mathcal{F} := \{B(x,r(x))\}_{x \in M}$. The previous lemma gives a disjoint subcollection $\mathcal{G}$ such that every ball $B$ in $\mathcal{F}$ intersects a ball $C$ in $\mathcal{G}$ and we have $B \subset 5C$. We set $\mathcal{D}(\epsilon) := \{x \in M :: B(x,r(x)) \in \mathcal{G}\}$ and $\mathcal{C}_\epsilon := \{B(x,5r(x)), x \in \mathcal{D}(\epsilon)\}$: we shall call $\mathcal{C}_\epsilon$ a $\epsilon$-admissible covering of $(M,g)$. We notice that $B(x,5r(x)) = B(x, R_\epsilon(x)/2)$.

Then we have [Amar, 2018b, Proposition 6.12],

**Proposition 2.6.** Let $(M,g)$ be a riemannian manifold, then the overlap of a $\epsilon$-admissible covering $\mathcal{C}(\epsilon)$ is less than $T = \frac{(1+\epsilon)^n/2}{(1-\epsilon)^n/2}(100)^n$, i.e.

$\forall x \in M$, $x \in B(y,5r(y))$, where $B(y, r(y)) \in \mathcal{G}(\epsilon)$, for at most $T$ such balls.

Moreover we have

$\forall f \in L^1(M), \sum_{j \in \mathbb{N}} \int_{B(x_j, r(x_j))} |f(x)| dv_g(x) \leq T\|f\|_{L^1(M)}$.

And its Corollary [Amar, 2018b, Corollary 6.13],

**Corollary 2.7.** Let $(M,g)$ be a complete riemannian manifold. Consider the covering by the balls $\{B(x, R_\epsilon(x)), x \in \mathcal{D}(\epsilon)\}$. Then the overlap of the associated covering verifies:

$T_1 \leq \frac{(1+\epsilon)^n/2}{(1-\epsilon)^n/2}(100)^n \times 2^n$.

3 Vector bundle.

Let $(M,g)$ be a complete riemannian manifold and let $G := (H, \pi, M)$ be an adapted complex $C^m$ vector bundle over $M$ of rank $N$ with fiber $H$. Recall that this means that $G$ has a smooth scalar product $(\ , \ )$ and a metric connection $\nabla^G : C^\infty(M, G) \to C^\infty(M, G \otimes T^*M)$, i.e. verifying $d(u,v) = (\nabla^G u, v) + (u, \nabla^G v)$, where $d$ is the exterior derivative on $M$. See [Taylor, 2000, Section 13].

**Lemma 3.1.** The $\epsilon$-admissible balls $B(x, R_\epsilon(x))$ trivialise the bundle $G$.

**Proof.** Because if $B(x,R)$ is an $\epsilon$-admissible ball, we have by Remark 2.3 that $R \leq r_{\text{inj}}(x)$. Then, one can choose a local frame field for $G$ on $B(x,R)$ by radial parallel translation, as done in [Taylor, 2000, Section 13, p.86-87], see also [Mazzucato and Nistor, 2006, p. 4, eq. (1.3)]]. This means that the $\epsilon$-admissible ball also trivialises the bundle $G$. ■
If $\partial_j := \partial/\partial x_j$ in a coordinate system on, say $B(x_0, R)$, and with a local frame $\{e_\alpha\}_{\alpha=1,\ldots,N}$, we have, for a smooth sections of $G$, $u = u^\alpha e_\alpha$ with the Einstein summation convention. We set:

$$\nabla_{\partial_j} u = (\partial_j u^\alpha + u^\beta \Gamma_{\beta j}^{G, \alpha}) e_\alpha,$$

the Christoffel coefficients $\Gamma_{\beta j}^{G, \alpha}$ being defined by $\nabla_{\partial_j} e_\beta = \Gamma_{\beta j}^{G, \alpha} e_\alpha$.

We shall make the following hypothesis, for $B(x_0, R) \in \mathcal{A}(1, \epsilon)$:

$$(\text{FDG}) \forall x \in B(x_0, R), \quad \left| \Gamma_{\beta j}^{G, \alpha}(x) \right| \leq C(n, G, \epsilon) \sum_{|\beta| = 1} \sup_{i, j, \alpha} \left| \partial^\beta g_{ij}(x) \right|,$$

the constant $C$ depending only on $n, \epsilon$ and $G$.

This hypothesis is natural in the sense that it is true for tensor bundles over $M$.

**Lemma 3.2.** Let $F$ be a tensor bundle over $M$. Then the hypothesis $(\text{FDG})$ is true.

**Proof.**
Let $\Gamma_{ij}^k$ be the Christoffel coefficients of the Levi-Civita connexion on the tangent bundle $TM$. We have

$$\Gamma_{ij}^k = \frac{1}{2} g^{kl} \left( \frac{\partial g_{kl}}{\partial x^i} + \frac{\partial g_{ij}}{\partial x^k} - \frac{\partial g_{jk}}{\partial x^i} \right). \tag{3.1}$$

Now on $B(x_0, R) \in \mathcal{A}(\epsilon)$, we have $(1 - \epsilon) \delta_{ij} \leq g_{ij} \leq (1 + \epsilon) \delta_{ij}$ as bilinear forms. Hence

$$\forall x \in B(x_0, R), \quad \left| \Gamma_{ij}^k(x) \right| \leq \frac{C}{2} (1 + \epsilon) \sum_{|\beta| = 1} \sup_{i, j, \alpha} \left| \partial^\beta g_{ij}(x) \right|$$

in a coordinates chart on $B(x_0, R)$.

Let $\omega := \omega_k dx^k$ be a 1-form in this chart. Then $\nabla_{\partial_j} \omega = (\partial_j \omega_k - \Gamma_{kj}^l \omega_l) dx^k$. In particular we have $\nabla_{\partial_j} dx^m = (-\Gamma_{kj}^m) dx^k$. Hence $(\text{FDG})$ is true. The same if $X$ is a vector field, $X := X^k \partial_k$ we have $\nabla_{\partial_j} X = (\partial_j X^k + \Gamma_{kj}^l X^l) \partial_k$. So $\nabla_{\partial_j} \partial_i = (\Gamma_{ij}^k) \partial_k$ and $(\text{FDG})$ is still true.

Now on $(p, q)$ tensors, using the fact that $\nabla$ is a derivation, we have that $\nabla_{\partial_j}(dx^{k_1} \otimes \cdots \otimes dx^{k_q} \otimes \partial_{i_1} \otimes \cdots \otimes \partial_{i_p}) =$$\nabla_{\partial_j}(dx^{k_1} \otimes \cdots \otimes dx^{k_q} \otimes \partial_{i_1} \otimes \cdots \otimes \partial_{i_p}) =$$\nabla_{\partial_j}(dx^{k_1} \otimes \cdots \otimes dx^{k_q} \otimes \partial_{i_1} \otimes \cdots \otimes \partial_{i_p}) =$

is a linear combination of $\Gamma_{ij}^k$ with constant coefficients, hence the $(\text{FDG})$ is also true, with a constant depending only on the dimension $n$ of $M$ and $p, q, \epsilon$.

A special mention on the bundle $\Lambda^p(M)$ of $p$-forms on $M$, which is a sub bundle of the $(0, p)$ tensors bundle. It also has the $(\text{FDG})$ property.

The proof is complete.

**4 Sobolev spaces for smooth sections of $G$ with weight.**

We have seen that $\nabla^G : C^\infty(M, G) \to C^\infty(M, G \otimes T^*M)$. On the tensor product of two Hilbert spaces we put the canonical scalar product $(u \otimes \omega, v \otimes \mu) := (u, v)(\omega, \mu)$, with $u \otimes \omega \in G \otimes T^*M$, and completed by linearity to all elements of the tensor product. On $T^*M$ we have the Levi-Civita connexion $\nabla^M$, which is of course a metric one, and on $G$ we have the metric connexion $\nabla^G$ so we define a connexion on the tensor product $G \otimes T^*M$:

$$\nabla^{G \otimes T^*M}(u \otimes \omega) = (\nabla^G u) \otimes \omega + u \otimes (\nabla^{T^*M} \omega)$$

by asking that this connexion be a derivation. We get easily that $\nabla^{G \otimes T^*M} : C^\infty(M, G \otimes T^*M) \to C^\infty(M, G \otimes (T^*M)^{\otimes 2})$
is still a metric connexion, i.e.

\[ d(u \otimes \omega, v \otimes \mu) = (\nabla^{G \otimes T^*M}(u \otimes \omega), \ v \otimes \mu) + (u \otimes \omega, \ \nabla^{G \otimes T^*M}(v \otimes \mu)) \]

We define by iteration \( \nabla^j u := \nabla(\nabla^{j-1} u) \) on the section \( u \) of \( G \) and the associated pointwise scalar product \((\nabla^j u(x), \nabla^j v(x))\) which is defined on \( G \otimes (T^*M)^{\otimes j} \), with again the metric connection

\[ d(\nabla^j u, \nabla^j v)(x) = (\nabla^{j+1} u, \nabla^j v)(x) + (\nabla^j u, \nabla^{j+1} v)(x) \]

Let \( w \) be a weight on \( M \), i.e. a positive measurable function on \( M \). If \( k \in \mathbb{N} \) and \( r \geq 1 \) are given, we denote by \( C_{G}^{k,r}(M, w) \) the space of smooth sections of \( G \), \( \omega \in C^\infty(M) \) such that \( |\nabla^j \omega| \in L^r(M, w) \) for \( j = 0, \ldots, k \) with the pointwise modulus associated to the pointwise scalar product. Hence

\[ C_{G}^{k,r}(M, w) := \{ \omega \in C^\infty(M), \ \forall j = 0, \ldots, k, \int_M |\nabla^j \omega|^r(x) w(x) dv(x) < \infty \} \]

with \( dv \) the volume measure on \( (M,g) \).

Now we have, see M. Cantor [Cantor, 1974, Definition 1 & 2, p. 240] for the case without weight:

**Definition 4.1.** The Sobolev space \( W_{G}^{k,r}(M, w) \) is the completion of \( C_{G}^{k,r}(M, w) \) with respect to the norm:

\[ \|\omega\|_{W_{G}^{k,r}(M,w)} = \sum_{j=0}^{k} \left( \int_M |\nabla^j \omega(x)|^r w(x) dv(x) \right)^{1/r}. \]

The usual case is when \( w \equiv 1 \). Then we write simply \( W_{G}^{k,r}(M) \).

We shall apply these well known facts to generalise a nice result of T. Aubin.

Let \( w(x), \ w'(x) \) be weights on the complete riemannian manifold \( (M,g) \). We have:

**Proposition 4.2.** Let \( (M,g) \) be a complete riemannian manifold. If \( W_{G}^{1,r_0}(M, w) \) is embedded in \( L_{G}^{s_0}(M, w') \), with \( \frac{1}{s_0} = \frac{1}{r_0} - \frac{1}{n} \) \( (1 \leq r_0 < n) \), then \( W_{G}^{k,r}(M, w) \) is embedded in \( W_{G}^{l,s}(M, w') \), with \( \frac{1}{s_0} = \frac{1}{r} - \frac{(k-l)}{n} > 0 \).

**Proof.**

We shall copy the proof of Proposition 2.11, p. 36 in [Aubin, 1982], replacing \( L^s(M) \) by \( L^s_G(M, w') \) and \( W^{k,r}(M) \) by \( W_{G}^{k,r}(M, w) \) and simplifying a little bit the argument.

Let \( m \) be an integer and let \( \omega \in C_{G}^{m+1} \). We have pointwise:

\[ |\nabla |\nabla^m \omega|| \leq |\nabla^{m+1} \omega|. \quad (4.2) \]

To see this, by \( |\nabla^m \omega|^2(x) = (\nabla^m \omega, \nabla^m \omega)(x) \), we have

\[ \nabla |\nabla^m \omega|^2 = \nabla(\nabla^m \omega, \nabla^m \omega)(x) = 2(\nabla^{m+1} \omega, \nabla^m \omega), \]

the last equality because \( \nabla \) is a metric connection.

By the Cauchy-Schwartz inequality, we get

\[ |\nabla |\nabla^m \omega|^2|(x) \leq 2|\nabla^{m+1} \omega| |\nabla^m \omega|(x). \]

Setting \( F(x) := |\nabla^m \psi|(x) \), because \( \nabla \) is a derivation, we also have \( \nabla F^2(x) = 2F(x) \nabla F(x) \), hence:

\[ |\nabla |\nabla^m \psi|^2| = 2|F||\nabla |F|| \leq 2|\nabla^{m+1} \omega| |\nabla^m \omega|, \]

so \( |\nabla |\nabla^m \omega|| \leq |\nabla^{m+1} \omega|. \)
Since \( W^{1,r_0}_G(M,w) \) is embedded in \( L^q_G(M,w') \), there exists a constant \( A \), such that for all \( \varphi \in W^{1,r_0}_G(M,w) \): (for now on, we do not indicate the subscript to ease the notation.)

\[
\| \varphi \|_{L^{r_0}(M,w')} \leq A(\| \nabla \varphi \|_{L^{r_0}(M,w)} + \| \varphi \|_{L^{r_0}(M,w)}).
\]

Let us apply this inequality with \( \varphi = |\nabla^m \omega| \), assuming \( \varphi \), which is a function now, belongs to \( W^{1,r_0}_G(M,w) \):

\[
\| \nabla^m \omega \|_{L^{r_0}(M,w')} \leq A(\| \nabla |\nabla^m \omega| \|_{L^{r_0}(M,w)} + \| \nabla^m \omega \|_{L^{r_0}(M,w)}),
\]

hence, using (4.2) and integrating,

\[
\| \nabla^m \omega \|_{L^{r_0}(M,w')} \leq A(\| \nabla^{m+1} \omega \|_{L^{r_0}(M,w)} + \| \nabla^m \omega \|_{L^{r_0}(M,w)}). \tag{4.3}
\]

Now let \( \omega \in W^{k,r}_G(M,w) \cap C^\infty(G) \). Applying inequalities (4.2) and (4.3) with \( r = r_0 \) and \( m = k - 1 \), \( k - 2, \ldots, \) we find, for \( \nabla^j \omega \),

\[
\| \nabla^{k-1} \omega \|_{L^{s_{k-1}}(M,w')} \leq A(\| \nabla^k \omega \|_{L^r(M,w)} + \| \nabla^{k-1} \omega \|_{L^r(M,w)}),
\]

\[
\| \nabla \omega \|_{L^{s_{k-1}}(M,w')} \leq A(\| \nabla^k \omega \|_{L^r(M,w)} + \| \nabla^{k-1} \omega \|_{L^r(M,w)}),
\]

\[
\| \omega \|_{L^{s_{k-1}}(M,w')} \leq A(\| \nabla \omega \|_{L^r(M,w)} + \| \nabla^{k-1} \omega \|_{L^r(M,w)});
\]

Thus

\[
\| \omega \|_{W^{k-1,s_{k-1}}(M,w')} \leq 2A \| \omega \|_{W^{k,r}_G(M,w)}.
\]

Therefore a Cauchy sequence in \( W^{k,r}_G(M,w) \) of \( C^\infty \) sections of \( G \) is a Cauchy sequence in \( W^{k-1,s_{k-1}}_G(M,w') \), and the preceding inequality holds for all \( \psi \in W^{k,r}_G(M,w) \) and we get:

\[
W^{k,r}_G(M,w) \subset W^{k-1,s_{k-1}}_G(M,w').
\]

Now with \( w = w' \) we prove similarly the following embeddings:

\[
W^{k-1,s_{k-1}}_G(M,w) \subset W^{k-2,s_{k-2}}_G(M,w') \subset \cdots \subset W^{L,s_L}_G(M,w').
\]

Hence

\[
W^{k,r}_G(M,w) \subset W^{L,s_L}_G(M,w').
\]

Proposition 4.2 says that, in order to prove Sobolev embeddings with weights, we have just to prove that \( W^{1,r}_G(M,w) \) is embedded in \( L^q_G(M,w') \), with \( \frac{1}{r} = \frac{1}{r_0} - \frac{1}{n} \) \((1 \leq r < n)\).

This is very important here because we shall have just to deal with first order Sobolev spaces. Hence we have to work only with \( \nabla^G u \) which, by our assumption (FDG), implies at most the first order derivatives of the metric tensor.

The aim now is to prove that \( W^{1,r_0}_G(M,w) \) is embedded in \( L^{s_0}_G(M,w') \), with \( \frac{1}{s_0} = \frac{1}{r_0} - \frac{1}{n} \) \((1 \leq r_0 < n)\) then we shall be able to apply Proposition 4.2.

5 Local estimates.

5.1 Sobolev comparison estimates for functions.

Lemma 5.1. We have the Sobolev comparison estimates where \( B(x,R) \) is a \((0, \varepsilon)\)-admissible ball in \( M \) and \( \varphi : B(x,R) \to \mathbb{R}^n \) is the admissible chart relative to \( B(x,R) \),

\[
\forall u \in W^{1,r}(B(x,R)), \quad \| u \|_{W^{1,r}(B(x,R))} \leq C \| u \circ \varphi^{-1} \|_{W^{1,r}(\varphi(B(x,R)))},
\]

and, with \( B_\varepsilon(0,t) \) the euclidean ball in \( \mathbb{R}^n \) centered at 0 and of radius \( t \),

\[
\| u \|_{W^{1,r}(B_\varepsilon(0,(1-\varepsilon)R))} \leq C \| u \|_{W^{1,r}(B(x,R))},
\]
The constants $C, C'$ depending only on $\epsilon, n$ and not on $B \in A(\epsilon)$.

Proof.
We have to compare the norms of $u, \nabla u$, with the corresponding ones for $v := u \circ \varphi^{-1}$ in $\mathbb{R}^n$.
First we have because $(1 - \epsilon)\delta_{ij} \leq g_{ij} \leq (1 + \epsilon)\delta_{ij}$ in $B(x, R)$:

$$B_\epsilon(0, (1 - \epsilon)R) \subset \varphi(B(x, R)) \subset B_\epsilon(0, (1 + \epsilon)R).$$

For functions, in a coordinates chart, we have $(\nabla u)_j := \partial_j u$ hence with $\forall y \in B(x, R), z := \varphi(y)$, we get:

$$\forall y \in B(x, R), |u(y)| = |v(z)|, |\nabla u(y)| \leq |\partial v(z)|.$$ Integrating this we get:

$$\|\nabla u\|_{L^r(B(x, R))} \leq \|\partial v\|_{L^r(B(\epsilon(0, (1 + \epsilon)R))}.$$ So, using that

$$\|u\|_{W^{1,r}(B(x, R))} = \|\nabla u\|_{L^r(B(x, R))} + \|u\|_{L^r(B(x, R))},$$

we get

$$\|u\|_{W^{1,r}(B(x, R))} \leq C(\|\partial v\|_{L^r(B(\epsilon(0, (1 + \epsilon)R))} + \|v\|_{L^r(B(\epsilon(0, (1 + \epsilon)R))}),$$

hence

$$\|u\|_{W^{1,r}(B(x, R))} \leq C\|v\|_{W^{1,r}(\varphi(B(x, R))}.$$ Of course all these estimates can be reversed so we also have

$$\|v\|_{W^{1,r}(B(\epsilon, (1 - \epsilon)R))} \leq C\|u\|_{W^{1,r}(B(x, R)).}$$

This ends the proof of the lemma.

\[\square\]

We have to study the behavior of the Sobolev embeddings w.r.t. the radius. Set $B_R := B_\epsilon(0, R)$ an euclidean ball in $\mathbb{R}^n$. For this purpose we have by [Amar, 2018a, Lemma 7.7] in the special case $m = 1$.

**Lemma 5.2.** We have, with $t := \frac{1}{t} = \frac{1}{r} - \frac{1}{n}$,

$$\forall R, 0 < R \leq 1, \forall u \in W^{1,r}(B_R), \|u\|_{L^r(B_R)} \leq CR^{-1} \|u\|_{W^{1,r}(B_R)}$$

the constant $C$ depending only on $n, r$.

Proof.
Start with $R = 1$, then we have, by the Sobolev embeddings which are valid on balls in $\mathbb{R}^n$, with $t := \frac{1}{t} = \frac{1}{r} - \frac{1}{n}$,

$$\forall v \in W^{1,r}(B_1), \|v\|_{L^r(B_1)} \leq C\|v\|_{W^{1,r}(B_1)}$$

(5.4)

where $C$ depends only on $n$ and $r$. For $u \in W^{1,r}(B_R)$ we set

$$\forall x \in B_1, y := Rx \in B_R, v(x) := u(y).$$

Then we have

$$\partial v(x) = \partial u(y) \times \frac{\partial y}{\partial x} = R\partial u(y);$$

So we get, because the jacobian for this change of variables is $R^{-n}$,

$$\|\partial v\|_{L^r(B_1)}^r = \int_{B_1} |\partial v(x)|^r dm(x) = \int_{B_R} |\partial u(y)|^r \frac{R^r}{R^n} dm(x) = R^{-n} \|\partial u\|_{L^r(B_R)}^r.$$ So
\[ \|\partial u\|_{L^r(B_R)} = R^{-1 + n/r} \|\partial v\|_{L^r(B_1)}. \tag{5.5} \]

And of course \( \|u\|_{L^r(B_R)} = R^{n/r} \|v\|_{L^r(B_1)} \).

So with 5.4 we get

\[ \|u\|_{L^t(B_R)} = R^{n/t} \|v\|_{L^t(B_1)} \leq CR^{n/t} \|v\|_{W^{1,r}(B_1)}. \tag{5.6} \]

But

\[ \|u\|_{W^{1,r}(B_R)} = \|u\|_{L^r(B_R)} + \|\partial u\|_{L^r(B_R)}, \]

and

\[ \|v\|_{W^{1,r}(B_1)} = \|v\|_{L^r(B_1)} + \|\partial v\|_{L^r(B_1)}, \]

so

\[ \|v\|_{W^{1,r}(B_1)} := R^{-n/r} \|u\|_{L^r(B_R)} + R^{1-n/r} \|\partial u\|_{L^r(B_R)}. \]

Because we have \( R \leq 1 \), we get

\[ \|v\|_{W^{1,r}(B_1)} \leq R^{-n/r} \|u\|_{L^r(B_R)} + \|\partial u\|_{L^r(B_R)} = R^{-n/r} \|u\|_{W^{1,r}(B_R)}. \]

Putting it in (5.6) we get

\[ \|u\|_{L^t(B_R)} \leq CR^{n/t} \|v\|_{W^{1,r}(B_1)} \leq CR^{-n(\frac{1}{t} - \frac{1}{r})} \|u\|_{W^{1,r}(B_R)}. \]

But, because \( t := \frac{1}{t} = \frac{r}{n} - \frac{1}{n} \), we get \( (\frac{1}{t} - \frac{1}{n}) = \frac{1}{n} \) and

\[ \|u\|_{L^t(B_R)} \leq CR^{1} \|u\|_{W^{1,r}(B_R)}. \]

The constant \( C \) depends only on \( n, r \). The proof is complete. \[ \blacksquare \]

**Lemma 5.3.** Let \( x \in M \) and \( B(x,R) \) be a \((0,\epsilon)\)-admissible ball in the complete riemannian manifold \((M, g)\); we have, with \( 1/s = 1/r - 1/n \),

\[ \forall u \in W^{1,r}(B(x,R)), \|u\|_{L^s(B(x,R))} \leq CR^{-2} \|u\|_{W^{1,r}(B(x,R))}, \]

the constant \( C \) depending only on \( n, r \) and \( \epsilon \).

**Proof.**

By Lemma 5.2, we get in \( \mathbb{R}^n \):

\[ \|u\|_{L^s(B_R)} \leq CR^{-1} \|u\|_{W^{1,r}(B_R)} \]

so we can apply the comparison Lemma 5.1:

\[ \|u\|_{L^s(B(x,R))} \leq C \|u \circ \varphi^{-1}\|_{L^s(B(R))}, \]

which gives:

\[ \|u\|_{L^s(B(x,R))} \leq CR^{-1} \|u\|_{W^{1,r}(B_R)}. \]

Again with the reverse inequalities in the comparison Lemma 5.1:

\[ \|u\|_{W^{1,r}(B_R)} \leq CR^{-1} \|u\|_{W^{1,r}(B(x,R))}. \]

So we get

\[ \|u\|_{L^t(B(x,R))} \leq CR^{-2} \|u\|_{W^{1,r}(B(x,R))}, \]

The constant \( C \) being independent of \( x \in M \) and of \( R \). The proof is complete. \[ \blacksquare \]
5.2 Sobolev comparison estimates for sections of $G$.

**Lemma 5.4.** We have the Sobolev comparison estimates where $B(x, R)$ is a $(1, \epsilon)$-admissible ball in $M$ and $\varphi : B(x, R) \to \mathbb{R}^n$ is the admissible chart relative to $B(x, R)$. Set $v := \varphi^* \omega$, then:
\[ \forall \omega \in W^{1,r}_G(B(x, R)), \quad \| \omega \|_{W^{1,r}_G(B(x, R))} \leq (1 + C\epsilon)R^{-1}\| v \|_{W^{1,r}(\varphi(B(x, R)))}, \]
and, with $B_\epsilon(0, t)$ the euclidean ball in $\mathbb{R}^n$ centered at $0$ and of radius $t$,
\[ \| v \|_{W^{1,r}(B_\epsilon(0, (1-\epsilon)R))} \leq (1 + C\epsilon)R^{-1}\| \omega \|_{W^{1,r}(B(x, R))}. \]

We also have:
\[ \forall \omega \in L^r_G(B(x, R)), \quad \| \omega \|_{L^r_G(B(x, R))} \leq (1 + C\epsilon)\| v \|_{L^r(\varphi(B(x, R)))}, \]

and
\[ \| v \|_{L^r(B_\epsilon(0, (1-\epsilon)R))} \leq (1 + C\epsilon)\| \omega \|_{L^r_G(B(x, R))}. \]

**Proof.**

We have to compare the norms of $\omega$, $\nabla \omega$, with the corresponding ones for $v := \varphi^* \omega$ in $\mathbb{R}^n$. By Lemma 3.1 the $\epsilon$-admissible ball $B(x, R)$ trivialises the bundle $G$, hence the image of a sections of $G$ in $\mathbb{R}^n$, is just vectors of functions. Precisely $v := \varphi^* \omega \in \varphi(B(x, R)) \times \mathbb{R}^n$.

We have because $(1 - \epsilon)\delta_{ij} \leq g_{ij} \leq (1 + \epsilon)\delta_{ij}$ in $B(x, R)$:
\[ B_\epsilon(0, (1 - \epsilon)R) \subset \varphi(B(x, R)) \subset B_\epsilon(0, (1 + \epsilon)R). \]

Let $\omega$ be a section of $G$ in $M$. By our assumption (FDG) we have that $\nabla \omega$ depends on the first order derivatives of the metric tensor $g$.

Because of (3.1) we get, with the fact that $B(x, R)$ is $(1, \epsilon)$-admissible, with $\eta := \frac{\epsilon}{R}$,
\[ \sum_{|\beta|=1} R \sup_{i,j=1,\ldots,n, \ y \in B_\epsilon(R)} \left| \partial^{\beta} g_{ij}(y) \right| \leq \epsilon \Rightarrow \left| \Gamma^k_{ij} \right| \leq C\eta. \]

Hence
\[ \forall y \in B(x, R), \ |\omega(y)| = |v(z)|, \quad |\nabla \omega(y)| \leq |\partial \omega| + |\Phi|, \]
where $\Phi$ depends on the coefficients of $\omega$ and on the first order derivatives of the metric tensor $g$.

So
\[ |\nabla \omega(y)| \leq |\partial v(z)| + C\eta |v(z)|. \]  \hspace{1cm} (5.7)

Integrating this we get
\[ \| \nabla \omega(y) \|_{L^r(B(x, R))} \leq \| \partial v \|_{L^r(B_\epsilon(0, (1+\epsilon)R))} + C\eta \| v \|_{L^r(B_\epsilon(0, (1+\epsilon)R))}. \]

We also have the reverse estimates
\[ \| \partial v \|_{L^r(B_\epsilon(0, (1-\epsilon)R))} \leq \| \nabla \omega \|_{L^r(B(x, R))} + C\eta \| \omega \|_{L^r(B(x, R))}. \]

So, using that
\[ \| \omega \|_{W^{1,r}_G(B(x, R))} \leq \| \nabla \omega \|_{L^r_G(B(x, R))} + \| \omega \|_{L^r_G(B(x, R))}, \]
we get
\[ \| \omega \|_{W^{1,r}_G(B(x, R))} \leq \| \partial v \|_{L^r(B_\epsilon(0, (1+\epsilon)R))} + C\eta \| v \|_{L^r(B_\epsilon(0, (1+\epsilon)R))} \leq \left( 1 + C\epsilon \right) \| v \|_{W^{1,r}(B_\epsilon(0, (1+\epsilon)R))}. \]

Now for $R \leq 1$, we get:
\[ 1 + C\eta = 1 + C\frac{\epsilon}{R} = R^{-1}(R + C\epsilon) \leq R^{-1}(1 + C\epsilon). \]

The second part follows the same lines but is easier because there is no derivations.

The proof of the lemma is complete. \hfill \blacksquare
Lemma 5.5. Let $x \in M$ and $B(x, R)$ be a $(1, \epsilon)$-admissible ball in the complete riemannian manifold $(M, g)$; we have, with $1/s = 1/r - 1/n$,
\[ \forall \omega \in W^{1,r}_G(B(x, R)), \|\omega\|_{L^s_G(B(x, R))} \leq CR^{-2}\|\omega\|_{W^{1,r}_G(B(x, R))}, \]
the constant $C$ depending only on $n$, $r$ and $\epsilon$.

Proof.
Because the image of a section of $G$ in $\mathbb{R}^n$ is just vectors of functions by Lemma 3.1, Lemma 5.2 is also true for $\varphi^*\omega$:
\[ \|\varphi^*\omega\|_{L^s(B_R)} \leq CR^{-1}\|\varphi^*\omega\|_{W^{1,r}(B_R)}, \]
so we can apply the second part in the comparison Lemma 5.4:
\[ \|\omega\|_{L^s_G(B(x, R))} \leq C\|\varphi^*\omega\|_{L^s(B_R)}, \]
which gives:
\[ \|\omega\|_{L^s_G(B(x, R))} \leq CR^{-1}\|\varphi^*\omega\|_{W^{1,r}(B_R)}. \]
Again with the reverse inequalities in the comparison Lemma 5.4:
\[ \|\varphi^*\omega\|_{W^{1,r}(B_R)} \leq CR^{-1}\|\omega\|_{W^{1,r}_G(B(x, R))}, \]
So we get
\[ \|\omega\|_{L^s_G(B(x, R))} \leq CR^{-2}\|\omega\|_{W^{1,r}_G(B(x, R))}, \]
The constant $C$ being independent of $x \in M$ and of $R$. The proof is complete.

5.3 Local Gaffney type inequality in $L^r$.

We shall restrict here to the case of the bundle of the bundle $\Lambda^p(M)$ of $p$-forms on $M$.

Of course the operator $d$ on $p$-forms is local and so is $d^*$ as a first order differential operator on $M$.

Let $B := B(x_0, R)$ be a $(1, \epsilon)$-admissible ball in the complete riemannian manifold $(M, g)$ and $(B, \varphi)$ be a coordinates chart. Let $\omega$ be a $p$-form in $M$. Let $\chi$ be a smooth cut-off function, $\chi \in C^1_0(B)$, $0 \leq \chi \leq 1$, $\chi \equiv 1$ in $B^1 := B(x_0, R/2)$. We consider the $p$-form $\chi\omega$.

Read in the chart $(B, \varphi)$ with the local coordinates $x$, we get $\omega = a_J dx^J$ with $J = (j_1, \ldots, j_p)$ is a multi-index of length $p$ and the functions $a_J$ are in $W^{1,r}(B)$. We get then that $d(\chi\omega) = \chi da_J \wedge dx^J + a_J d\chi \wedge dx^J$, hence, with $da_J = \frac{\partial a_J}{\partial x^j} dx^j$, we deduce $d\omega = \frac{\partial a_J}{\partial x^j} dx^j \wedge dx^J$ and $d(\chi\omega) = \chi \frac{\partial a_J}{\partial x^j} dx^j \wedge dx^J + a_J d\chi \wedge dx^J$.

We shall take the following notation from the book by C. Voisin [Voisin, 2002].

With a local Hodge $*$ operator $\Lambda^p \to \Lambda^{n-p}$ (locally $M$ is always orientable), i.e. in a coordinates chart $U$ with $\omega = a_J dx^J \in \Lambda^p$, it is defined as:
\[ *\omega := (-1)^p\sigma(J) a_J \cdot dx^J \in \Lambda^{n-p} \]
with $J^c$ is the complement of $J$ in $(1, 2, \ldots, n)$ and $\sigma(J)$ is 0 or 1. We have: $\int_U \omega \wedge \omega^* = \int_U \sum_{|J|=p} |a_J|^2 dv$. Using the link between the $*$ Hodge operator and the adjoint $d^*$ of $d$, (see [Voisin, 2002, Section 5.1.2, p. 118]), we get:
\[ d^* = (-1)^p *^{-1} d* \] on $\Lambda^p$.

Hence here we have:
\[ d^*(\chi\omega) = (-1)^p(n-p)-n+p-1(*d(*\chi\omega))) = L_K \frac{\partial a_J}{\partial x^j} dx^J + F_K(a_J, \frac{\partial \chi}{\partial x^j}) dx^J, \]
where $L_K$ is linear in the $\frac{\partial a_I}{\partial x^j}$ and $F_K$ is linear in the $a_I$ and in the $\frac{\partial \chi}{\partial x^j}$, and $|K| = p - 1$. Moreover $L_K$, $F_K$ have compact support in $B$.

Now for the covariant derivative $\nabla_M$ on $M$, we have, in our chart $(B, \varphi)$, using (5.7)

$$|\nabla_M(\chi \omega)| \leq |\partial(\chi \omega)| + c \frac{\epsilon}{R} |\chi \omega|.$$  \hspace{1cm} (5.8)

On the other hand, in $\mathbb{R}^n$ we have $|\nabla_{\mathbb{R}^n}(\chi \omega)| = |\partial(\chi \omega)|$ just because $\nabla_{\mathbb{R}^n}(\chi \omega) = \partial(\chi \omega)$.

Now we are in position to apply the following Proposition 4.3 in [Scott, 1995]:

**Proposition 5.6. (Gaffney type inequality for $L^r$)**

$$\|\nabla_{\mathbb{R}^n} \omega\|_{L^r} \leq C D_r(\omega) \text{ for } \omega \in C_0^\infty(\Lambda^p(\mathbb{R}^n))$$

where $D_r(\omega) := \|d\omega\|_{L^r(\mathbb{R}^n)} + \|d^*\omega\|_{L^r(\mathbb{R}^n)}$ and $C = C(n, r)$.

From it, we get in $\mathbb{R}^n$,

$$\|\partial(\chi \omega)\|_{L^r(\varphi(B))} \leq C(\|d(\chi \omega)\|_{L^r(\varphi(B))} + \|d^*\omega\|_{L^r(\varphi(B))}).$$

So, by (5.8), we get, because $\chi \equiv 1$ in $B$:

$$\|\nabla_M \omega\|_{L^r(B)} \leq C(\|d(\chi \omega)\|_{L^r(B)} + \|d^*\omega\|_{L^r(B)}) + c \epsilon R^{-1} \|\omega\|_{L^r(B)},$$

because, by condition (*) in the definition of the $\epsilon$-admissible ball $B$, we have that the Lebesgue measure in $\varphi(B)$ and the volume measure on $B$ are equivalent.

Now, because $\|d\chi\|_{\infty} \leq c R^{-1}$ in $B$:

$$\|d(\chi \omega)\|_{L^r(B)} \leq \|d\chi\|_{\infty} \|\omega\|_{L^r(B)} + \|d\omega\|_{L^r(B)} \leq c R^{-1} \|\omega\|_{L^r(B)} + \|d\omega\|_{L^r(B)},$$

and the same

$$\|d^*\omega\|_{L^r(B)} \leq c R^{-1} \|\omega\|_{L^r(B)} + \|d^\ast \omega\|_{L^r(B)}.$$

Hence, with new constants $c, C$ depending only on $n, r, p, \epsilon$,

$$\|\nabla_M \omega\|_{L^r(B)} \leq C(\|d(\omega)\|_{L^r(B)} + \|d^*\omega\|_{L^r(B)}) + c R^{-1} \|\omega\|_{L^r(B)},$$

So we proved the corollary of Scott’s Proposition 5.6:

**Corollary 5.7.** Let $B := B(x_0, R)$ be a $(1, \epsilon)$-admissible ball in the complete riemannian manifold $(M, g)$ and set $B^1 := B(x_0, R/2)$. Let $\omega$ be a $p$-form in $M$. We have the local $L^r$ Gaffney’s inequality:

$$\|\nabla_M \omega\|_{L^r(B)} \leq C(\|d(\omega)\|_{L^r(B)} + \|d^*\omega\|_{L^r(B)}) + c R^{-1} \|\omega\|_{L^r(B)},$$

the constants $c, C$ depending only on $n, r, p, \epsilon$.

## 6 Global results.

### 6.1 Global estimates for sections of $G$.

**Lemma 6.1.** We have, for any section $f$ of $G$ with $w(x) := R(x)^\mu$ and $B(x) := B(x, R(x))$, with $x \in \mathcal{D}(\epsilon)$ and $R(x) := R_\epsilon(x)$, that:

$$\forall \tau \geq 1, \|f\|_{L^\tau(M, w)} \simeq \sum_{x \in \mathcal{D}(\epsilon)} R(x)^\mu \|f\|_{L^\tau(B(x))}. $$

**Proof.**

We consider the function $|f|$. We set $w(x) := R(x)^\mu$ to get an adapted weight. Let $x \in \mathcal{D}(\epsilon)$, then $B(x) := B(x, R(x)) \in \mathcal{C}_\epsilon$. 

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We have, because \( C_e \) is a covering of \( M \),
\[
\|f\|_{L^r(M,w)}^r := \int_M |f(x)|^r w(x)dv(x) \leq \sum_{x \in D(e)} \int_{B(x)} |f(y)|^r R(y)^\mu dv(y).
\]
Because we have, by Lemma 2.4, \( \forall y \in B, R(y) \leq 2R(x) \), we get
\[
\sum_{x \in D(e)} \int_{B(x)} |f(y)|^r R(y)^\mu dv(y) \leq \sum_{x \in D(e)} 2^\mu R(x)^\mu \int_{B(x)} |f(y)|^r dv(y) \leq 2^\mu \sum_{x \in D(e)} R(x)^\mu \|f\|_{L^r(B(x))}^r.
\]
Hence we get
\[
\|f\|_{L^r(M,w)}^r \leq 2^\mu \sum_{x \in D(e)} R(x)^\mu \|f\|_{L^r(B)}^r.
\]
To get the reverse inequality we still use Lemma 2.4, to have \( \forall y \in B, R(x) \leq 2R(y) \) so we get:
\[
\sum_{x \in D(e)} R(x)^\mu \int_{B(x)} |f(y)|^r dv(y) \leq 2^\mu \sum_{x \in D(e)} R(y)^\mu |f(y)|^r dv(y).
\]
Now we use the fact that the overlap of \( C_e \) is bounded by \( T \),
\[
\sum_{x \in D(e)} \int_{B(x)} R(y)^\mu |f(y)|^r dv(y) \leq 2^\mu T \int_M R(y)^\mu |f(y)|^r dv(y) = 2^\mu T \|f\|_{L^r(M,w)}^r.
\]
So we get
\[
\sum_{x \in D(e)} R(x)^\mu \|f\|_{L^r(B)}^r \leq 2^\mu T \|f\|_{L^r(M,w)}^r.
\]
The proof is complete.

**Lemma 6.2.** Let \( u \in W^{1,r}_G(M,w) \) where \( w(x) = R(x)^\gamma \). We have:
\[
\sum_{B \in C_e, j=0,1} R(x)^\gamma \|\nabla^j u\|_{L^r_G(B)}^r \simeq \|u\|_{W^{1,r}_G(M,w)}^r
\]
and
\[
\sum_{B \in C_e} R(x)^\gamma \|u\|_{W^{1,r}_G(B)}^r \simeq \|u\|_{W^{1,r}_G(M,w)}^r.
\]

**Proof.**
We start with \( u \in W^{1,r}_G(M,w) \) where \( w(x) = R(x)^\gamma \). This means:
\[
\forall j \in \mathbb{N}, j \leq 1, \int_M \left| \nabla^j u(y) \right|^r R(y)^\gamma dv(y) \leq C \|u\|_{W^{1,r}_G(M,w)}^r.
\]
Take a Vitali \( \epsilon \)-covering \( C_e \) of \( M \). Let \( B := B(x,R(x)) \in C_e \); because \( u \in W^{1,r}_G(M,w) \), we have \( u \in W^{1,r}_G(B,w) \).
By Lemma 6.1 we have, with \( f := |\nabla^j u| \) and \( r = r \),
\[
\int_M \left| \nabla^j u(y) \right|^r R(y)^\gamma dv(y) \simeq \sum_{x \in D(e)} R(x)^\gamma \|\nabla^j u\|_{L^r_G(B(x))}^r.
\]
Hence, adding on \( j, j \leq 1, \)
\[
\sum_{x \in D(e), j \leq 1} R(x)^\gamma \|\nabla^j u\|_{L^r_G(B(x))}^r \leq C \|u\|_{W^{1,r}_G(M,w)}^r.
\] (6.9)
Using that \( \|u\|_{W^{1,r}(B(x,R))} = \|\nabla u\|_{L^r(B(x,R))} + \|u\|_{L^r(B(x,R))} \),
we deduce from (6.9):
\[
\sum_{B \in C_\epsilon} R(x)^\gamma \|u\|_{W^{1,r}_G(B)}^r \leq C \|u\|_{W^{1,r}_G(M,w)}^r.
\]

For the converse, because:
\[
\|u\|_{W^{1,r}_G(M,w)}^r \leq \sum_{r=0}^1 \int \|\nabla^j u(y)\|^r R(y)^\gamma dv(y)
\]
we get again with Lemma 6.1,
\[
\int \|\nabla^j u(y)\|^r R(y)^\gamma dv(y) \simeq \sum_{x \in D(\epsilon)} R(x)^\gamma \|\nabla^j u\|_{L^r_G(B(x))}^r.
\]
Hence
\[
\|u\|_{W^{1,r}_G(M,w)}^r \leq \sum_{x \in D(\epsilon), j=0,1} \sum_{B \in C_\epsilon} R(x)^\gamma \|\nabla^j u\|_{L^r_G(B(x))}^r \simeq \sum_{x \in D(\epsilon)} \sum_{j=0,1} R(x)^\gamma \|u\|_{W^{1,r}_G(B(x))}^r.
\]
Now from
\[
\sum_{B \in C_\epsilon, j=0,1} R(x)^\gamma \|\nabla^j u\|_{L^r_G(B(x))}^r \simeq \|u\|_{W^{1,r}_G(M,w)}^r \tag{6.10}
\]
the constants underlying \( \simeq \) being independent of \( B \), and with:
\[
\sum_{j=0,1} \|\nabla^j u\|_{L^r_G(B(x))}^r = \|u\|_{W^{1,r}_G(B)}^r,
\]
we get
\[
\sum_{j=0,1} R(x)^\gamma \|\nabla^j u\|_{L^r_G(B(x))}^r \simeq R(x)^\gamma \|u\|_{W^{1,r}_G(B)}^r.
\]
So, porting in (6.10), we get
\[
\sum_{B \in C_\epsilon} R(x)^\gamma \|u\|_{W^{1,r}_G(B(x))}^r \simeq \|u\|_{W^{1,r}_G(M,w)}^r.
\]
The proof is complete. \( \blacksquare \)

**Lemma 6.3.** We have:
\[
r \geq r' \Rightarrow \sum_{j \in \mathbb{N}} a_j^r \leq \left( \sum_{j \in \mathbb{N}} a_j^{r'} \right)^{r/r'} ; \quad r \leq r' \Rightarrow \sum_{j \in \mathbb{N}} a_j^r \geq \left( \sum_{j \in \mathbb{N}} a_j^{r'} \right)^{r/r'}.
\]

Proof.
The comparison of the norms of \( \ell^r(N) \) and \( \ell^{r'}(N) \) gives the result. \( \blacksquare \)

**Lemma 6.4.** Let \( B = B(x,R) \) be an \( \epsilon \)-admissible ball in \( M \). We have, for \( \omega \in L^s_G(B) \) with \( s \geq r \):
\[
\|\omega\|_{L^s_G(B)} \leq c(n, \epsilon) R^{n-\frac{r}{2}} \|\omega\|_{L^s_G(B)}^s,
\]
with \( c \) depending only on \( n \), \( \epsilon \) and \( G \).

Proof.
First suppose that \( B \) is in \( \mathbb{R}^n \) with the Lebesgue measure. Let \( \omega \in L^s(B) \). Because \( \frac{d\nu}{|B|} \) is a probability measure on \( B \), where \( |B| \) is the volume of the ball \( B \) and \( s \geq r \), we get
\[
\left( \int_B |\omega(y)|^r \frac{d\nu(y)}{|B|} \right)^{1/r} \leq \left( \int_B |\omega(y)|^s \frac{d\nu(y)}{|B|} \right)^{1/s},
\]
hence
\[ \| \omega \|_{L^r(B)} \leq |B|^{\frac{1}{2} - \frac{1}{r}} \| \omega \|_{L^r(B)}. \]

Now back to the manifold, for \( B_x := B(x, R) \) a \( \epsilon \)-admissible ball we get
\[ \forall y \in B_x, \, (1 - \epsilon)^n \leq |\det g(y)| \leq (1 + \epsilon)^n, \]
hence we have, comparing the Lebesgue measure in \( \mathbb{R}^n \) with the volume measure in \( M \),
\[ \forall x \in M, \, (1 - \epsilon)^{n/2} \nu_n R^n \leq \text{Vol}(B(x, R(x))) \leq (1 + \epsilon)^{n/2} \nu_n R^n, \]
where \( \nu_n \) is the volume of the unit ball in \( \mathbb{R}^n \). So, because the ball \( B \) trivialises the bundle \( G \) by Lemma 3.1, on the manifold \( M \), we have
\[ \| \omega \|_{L^r_{\nu}(B)} \leq c(n, \epsilon, G) R^{\frac{n}{2} - \frac{n}{r}} \| \omega \|_{L^r_{\nu}(B)} \]
with \( c \) depending only on \( n, \epsilon \) and \( G \).

Now we fix \( x \in \mathcal{D}(\epsilon) \) hence the ball \( B := B(x, R(x)) \) is fixed. We have, with \( 1/s = 1/r - 1/n \),
that \( W^{1,r}(B) \subset L^s(B) \), by Lemma 5.3 for functions or by Lemma 5.5 for sections of \( G \) with:
\[ \forall u \in W^{1,r}_G(B(x, R)), \, \| u \|_{L^s_G(B(x, R))} \leq C R(x)^{-2} \| u \|_{W^{1,r}_G(B(x, R))}. \]
So
\[ R(x)^{2 + \gamma/r} \| u \|_{L^s_G(B(x, R))} \leq C R(x)^{\gamma/r} \| u \|_{W^{1,r}_G(B(x, R))}. \tag{6.11} \]

Now the aim is to get the global form of the previous result.

Let \( w'(x) := R(x)^\nu \) with \( \nu := s(2 + \gamma/r) \). Using (6.11), raising to the power \( s \) and adding, we get:
\[ \sum_{B \in \mathcal{C}_x} R(x)^{\gamma/r} \| u \|_{L^s_G(B)}^s \leq C \sum_{B \in \mathcal{C}_x} R(x)^{\gamma/r} \| u \|_{W^{1,r}_G(B(x, R))}^s. \tag{6.12} \]

Because \( s \geq r \), the Lemma 6.3 gives:
\[ \sum_{B \in \mathcal{C}_x} R(x)^{\gamma/r} \| u \|_{W^{1,r}_G(B(x, R))} \leq \left( \sum_{B \in \mathcal{C}_x} R(x)^{\gamma} \| u \|_{W^{1,r}_G(B(x, R))} \right)^{s/r}. \]

Now we use Lemma 6.2 with \( w(x) := R(x)^\gamma \),
\[ \sum_{B \in \mathcal{C}_x} R(x)^{\gamma} \| u \|_{W^{1,r}_G(B(x, R))} \simeq \| u \|_{W^{1,r}_G(M, w)} \]
to get, with Lemma 6.1, with \( \mu = \nu := 2s + s\gamma/r \),
\[ \| u \|_{L^s_G(M, w')} \simeq \sum_{B \in \mathcal{C}_x} R(x)^{\nu} \| u \|_{L^s_G(B(x, R))} \]
hence, by (6.12),
\[ \| u \|_{L^s_G(M, w')} \leq C \left( \sum_{B \in \mathcal{C}_x} R(x)^{\gamma} \| u \|_{W^{1,r}_G(B(x, R))} \right)^{s/r} \simeq C \| u \|_{W^{1,r}_G(M, w)}. \]

So we proved the following Sobolev embedding Theorem with weights:

**Theorem 6.5.** Let \( 1/s = 1/r - 1/n > 0 \). Let \( u \in W^{1,r}_G(M, w) \) with \( w(x) := R(x)^\gamma \). Then we have \( u \in L^s_G(M, w') \), with \( \nu := s(2 + \gamma/r) \) and \( w' := R(x)^\nu \), with the control:
\[ \| u \|_{L^s_G(M, w')} \leq C \| u \|_{W^{1,r}_G(M, w)}. \]

Theorem 6.5 with Proposition 4.2 give:
Theorem 6.6. Let \((M, g)\) be a complete riemannian manifold. Let \(w(x) := R(x)^{\gamma} \text{ and } w'(x) := R(x)^{\gamma'}\) with \(\nu := s(2 + \gamma/r)\). Then \(W_{G}^{m,r}(M, w)\) is embedded in \(W_{G}^{k,s}(M, w')\), with \(\frac{1}{s} = \frac{1}{r} - \frac{(m - k)}{n} > 0\) and:

\[
\forall u \in W_{G}^{m,r}(M, w), \quad \|u\|_{W_{G}^{k,s}(M, w')} \leq C\|u\|_{W_{G}^{m,r}(M, w)}.
\]

Remark 6.7. The weights, as function of the \(\epsilon\)-admissible radius \(R_{\epsilon}(x)\), do not depend on the fact that we work with functions or sections of \(G\) but the radius itself depends on that fact. The \((1, \epsilon)\) admissible radius for the sections of \(G\) is smaller than the \((0, \epsilon)\) one for functions.

In [Amar, 2018b, Theorem 6.23, p. 21], we proved, with the bundle \(\Lambda^{p}(M)\) of \(p\)-forms, the following:

Theorem 6.8. Let \(M\) be a complete non compact riemannian manifold of class \(C^{2}\) without boundary. Let \(\alpha > 0\), \(r \geq 2\) and \(k\) the smallest integer such that, with \(\frac{1}{r_{k}} = \frac{1}{2} - \frac{2k}{n}\), we have \(r_{k} \geq r\). For any \(\omega \in L^{r}([0, T + \alpha], L^{r}_{p}(M, w_{1})) \cap L^{r}([0, T + \alpha], L_{p}^{2}(M))\) there is a \(u \in L^{r}([0, T], W^{2,r}_{p}(M, w_{2}))\) such that \(\partial t u + \Delta u = \omega\) and:

\[
\|\partial_{t} u\|_{L^{r}([0, T], L^{r}_{p}(M, w_{2}))} + \|u\|_{L^{r}([0, T], W^{2,r}_{p}(M, w_{2}))} \leq c_{1}\|\omega\|_{L^{r}([0, T + \alpha], L^{r}_{p}(M, w_{1}))} + c_{2}\sqrt{T + \alpha}\|\omega\|_{L^{r}([0, T + \alpha], L^{2}_{p}(M))}
\]

where the weights functions are: \(w_{1}(x) = R(x)^{(\frac{r}{2} - \frac{2k}{n} + 2)}\) and \(w_{2}(x) = R(x)^{(3 + 8k)}\) if we work only with functions and \(w_{2}(x) = R(x)^{(3 + 12k)}\) for any \(p\)-forms, \(p \geq 1\).

As a corollary we get, if we are interested in estimates \(L^{r} - L^{s}\),

Corollary 6.9. Let \(M\) be a complete non compact riemannian manifold of class \(C^{2}\) without boundary. Let \(\alpha > 0\), \(r \geq 2\) and \(k\) the smallest integer such that, with \(\frac{1}{r_{k}} = \frac{1}{2} - \frac{2k}{n}\), we have \(r_{k} \geq r\). For any \(\omega \in L^{r}([0, T + \alpha], L^{r}_{p}(M, w_{1})) \cap L^{r}([0, T + \alpha], L_{p}^{2}(M))\) there is a \(u \in L^{r}([0, T], L^{s}_{p}(M, w_{2}))\) such that \(\partial_{t} u + \Delta u = \omega\) and:

\[
\|\partial_{t} u\|_{L^{r}([0, T], L^{r}_{p}(M, w_{2}))} + \|u\|_{L^{r}([0, T], L^{s}_{p}(M, w_{2}))} \leq c_{1}\|\omega\|_{L^{r}([0, T + \alpha], L^{r}_{p}(M, w_{1}))} + c_{2}\sqrt{T + \alpha}\|\omega\|_{L^{r}([0, T + \alpha], L_{p}^{2}(M))}
\]

with \(\frac{1}{s} = \frac{1}{r} - \frac{2}{n} > 0\) and where the weights functions are: \(w_{1}(x) = R(x)^{(\frac{r}{2} - \frac{2k}{n} + 2)}\) and \(w_{2}(x) = R(x)^{(2 + (3 + 12k)/r)}\) if we work only with functions and \(w_{2}(x) = R(x)^{(s(2 + (3 + 12k)/r)}\) for any \(p\)-forms, \(p \geq 1\).

Proof.

By Theorem 6.6 we have \(W_{G}^{m,r}(M, w) \subset W_{G}^{k,s}(M, w')\), with \(\frac{1}{s} = \frac{1}{r} - \frac{(m - k)}{n} > 0\). So we choose \(m = 2\), \(k = 0\) and \(w = R(x)^{(3 + 12k)}\) which gives \(w'(x) = R(x)^{\nu}\) with \(\nu = s(2 + (3 + 12k)/r)\). We get then, again with \(G = \Lambda^{p}(M)\):

\[
\forall u \in W^{2,r}_{p}(M, w), \quad \|u\|_{L^{p}_{r}(M, w')} \leq C\|u\|_{W^{2,r}_{p}(M, w)}.
\]

If we work with functions, we have \(\nu = s(2 + (3 + 8k)/r)\).

Putting this in Theorem 6.8, this finishes the proof of the corollary. ■
6.2 Global Gaffney type inequality in $L^r$.

Let $B := B(x, R)$ be a $(1, \epsilon)$-admissible ball in the complete riemannian manifold $(M, g)$ and set $B^1 := B(x, R/2)$. Let $\omega$ be a $p$-form in $M$. We have the local $L^r$ Gaffney’s inequality by Corollary 5.7: 
\[
\|\nabla_M \omega\|_{L^r(B)} \leq C\|d(\omega)\|_{L^r(B)} + \|d^s(\omega)\|_{L^r(B)} + cR^{-1}\|\omega\|_{L^r(B)},
\]
the constants $c, C$ depending only on $n, r, p, \epsilon$. Because $R \leq 1$, we get, with another constant $c$: 
\[
\|\omega\|_{W^{1,r}(B)} \leq \|\nabla_M \omega\|_{L^r(B)} + \|\omega\|_{L^r(B)} \leq C\|d(\omega)\|_{L^r(B)} + \|d^s(\omega)\|_{L^r(B)} + cR^{-1}\|\omega\|_{L^r(B)}.
\]

To globalise this, we proceed as above by summing over the balls $B$ and $B^1$. Raising to the power $r$ and adding, we get 
\[
\sum_{x \in \mathcal{D}(\epsilon)} \|\omega\|_{W^{1,r}(B)}^r \leq C \sum_{x \in \mathcal{D}(\epsilon)} \|d(\omega)\|_{L^r(B)}^r + \|d^s(\omega)\|_{L^r(B)}^r + cR^{-r}\|\omega\|_{L^r(B)}^r.
\]
As above we have 
\[
\|\omega\|_{W^{1,r}(M)}^r \leq \sum_{x \in \mathcal{D}(\epsilon)} \|\omega\|_{W^{1,r}(B)}^r
\]
and, because the overlap of the covering $\{B(x, R_\epsilon(x)) : x \in \mathcal{D}(\epsilon)\}$ is bounded by $T_1$ by Corollary 2.7, we get 
\[
\sum_{x \in \mathcal{D}(\epsilon)} (\|d(\omega)\|_{L^r(B)}^r + \|d^s(\omega)\|_{L^r(B)}^r) \leq T_1 (\|d(\omega)\|_{L^r(M)}^r + \|d^s(\omega)\|_{L^r(M)}^r)
\]
and 
\[
\sum_{x \in \mathcal{D}(\epsilon)} R_\epsilon^{-r}(x) \|\omega\|_{L^r(B(x, R_\epsilon(x)))}^r \leq T_1 \|\omega\|_{L^r(M, w)}^r,
\]
where $w(x)$ is the weight $w(x) := R_\epsilon^{-r}(x)$. So we proved the global Gaffney’s type inequality with weight.

**Theorem 6.10.** Let $(M, g)$ be a complete riemannian manifold. Let $r \geq 1$ and $w(x) := R(x)^{-r}$. Let $\omega$ be a $p$-form in $M$. We have: 
\[
\|\omega\|_{W_p Statement 5.6} \leq C\|d(\omega)\|_{L^r_p(M)} + \|d^s(\omega)\|_{L^r_{p-1}(M)} + \|\omega\|_{L^r_p(M, w)}.
\]

So using Theorem 6.6 with $\gamma = 0$, $m = 1$, $k = 0$, $\frac{1}{s} = \frac{1}{r} - \frac{1}{n} > 0$, $w'(x) = R_\epsilon(x)^{2s}$, plus Theorem 6.10, we get

**Corollary 6.11.** Let $(M, g)$ be a complete riemannian manifold. Let $r \geq 1$ and $w(x) := R(x)^{-r}$. Let $\omega$ be a $p$-form in $M$. We have:
\[
\|\omega\|_{L^s_p(M, w')} \leq C\|d(\omega)\|_{L^r_p(M)} + \|d^s(\omega)\|_{L^r_{p-1}(M)} + \|\omega\|_{L^r_p(M, w')}
\]
with $\frac{1}{s} = \frac{1}{r} - \frac{1}{n} > 0$, $w'(x) = R_\epsilon(x)^{2s}$.

This corollary is a kind of N. Lohoué’s result [Lohoué, 1985] with weights and without any geometric conditions on the riemannian manifold $(M, g)$.

7 Applications.

We shall give some examples where we have classical estimates using that $\forall x \in M$, $R_\epsilon(x) \geq \delta$, via [Hebey and Herzlich, 1997, Corollary, p. 7] (see also Theorem 1.3 in the book by Hebey [Hebey, 1996]). We have:
Corollary 7.1. Let \((M,g)\) be a complete riemannian manifold. If the injectivity radius verifies \(r_{\text{inj}}(x) \geq i > 0\) and the Ricci curvature verifies \(Rc(M,g)(x) \geq \lambda g_x\) for some \(\lambda \in \mathbb{R}\) and all \(x \in M\), then there exists a positive constant \(\delta > 0\), depending only on \(n, \epsilon, \lambda, \alpha, i\) such that for any \(x \in M\), \(r_H(1 + \epsilon, 0, \alpha)(x) \geq \delta\).

Hence \(R_{0, \epsilon}(x) \geq \delta\) all \(x \in M\).

If we have \(|Rc(M,g)(x)| \leq C\) for all \(x \in M\), then there exists a positive constant \(\delta > 0\), depending only on \(n, \epsilon, C, \alpha, i\) and \(c\), such that for any \(x \in M\), \(r_H(1 + \epsilon, 1, \alpha)(x) \geq \delta\).

Hence \(R_{1, \epsilon}(x) \geq \delta\) all \(x \in M\).

Proof.
The Theorem of Hebey and Herzlich gives that, under these hypotheses, for any \(\alpha \in (0, 1)\) that \(\forall x \in M\), \(r_H(1 + \epsilon, 0, \alpha)(x) \geq \delta\).

Now recall that \(r_H(1 + \epsilon, 0, \alpha)(x)\) is the sup of \(S\) such that, in an harmonic coordinates patch,

1) \((1 - \epsilon)\delta_{ij} \leq g_{ij} \leq (1 + \epsilon)\delta_{ij}\) in \(B(x,S)\) as bilinear forms,

2) \(S^\alpha(x) \sup_{i,j=1,\ldots,n, y \neq z \in B(x,S)} \left| \frac{g_{ij}(y) - g_{ij}(z)}{d(y, z)^\alpha} \right| \leq \epsilon.\)

So when we take the sup for \(R_{0, \epsilon}(x)\) on any smooth coordinates patch we get \(r_H(1 + \epsilon, 0, \alpha)(x) \leq R_{0, \epsilon}(x)\).

The same way we get \(r_H(1 + \epsilon, 1, \alpha)(x) \leq R_{1, \epsilon}(x)\).

The proof is complete. ■

As a corollary we retrieve Corollary 3.19, p. 38, in [Hebey, 1996]:

Corollary 7.2. Let \((M,g)\) be a complete riemannian manifold. If we have the injectivity radius bounded below and the Ricci curvature verifying \(Rc(x) \geq \lambda g_x\) for some \(\lambda \in \mathbb{R}\) and all \(x \in M\) then the Sobolev embeddings for functions are valid in \((M,g)\).

Remark 7.3. Because the proof of the Theorem of Hebey and Herzlich does not use the Theorem of Varopoulos, we get here a different proof of Corollary 7.2.

We get also a Sobolev embedding for sections of \(G\).

Corollary 7.4. Let \((M,g)\) be a complete riemannian manifold. Let \(G := (H, \pi, M)\) be a complex \(C^m\) vector bundle over \(M\) as above. If the injectivity radius is bounded below and the Ricci curvature is bounded, then the "classical" Sobolev embeddings for sections of \(G\) are valid in \((M,g)\).

M. Cantor [Cantor, 1974] proved that, for \(E\) a tensor bundle over the complete riemannian manifold \((M,g)\), we have the Sobolev embedding theorems provided that:

(C1) the injectivity radius is bounded away from zero.

(C2) There is a \(\delta\) such that for each \(x \in M\) and \(V, W \in T_xM\), the sectional curvature \(|K_x(V, W)| < \delta\).

Because the condition (C2) implies that the Ricci curvature is bounded by Lemma 7.11, our result concerning classical Sobolev embeddings is better. But, under the same hypotheses (C1) and (C2), M. Cantor proved also a Theorem of Gagliardo-Nirenberg type which we have not. Also our proofs are completely different.

We also have a global Gaffney's type inequality:
Corollary 7.5. Let \((M,g)\) be a complete riemannian manifold. If the injectivity radius is bounded below and the Ricci curvature is bounded, then the global Gaffney’s type inequality in \(L^r\) is valid:

\[
\|\omega\|_{W^{1,r}_p(M)} \leq C \left( \|d(\omega)\|_{L^{p+1}_r(M)} + \|d^*(\omega)\|_{L^{r-1}_{p-1}(M)} + \|\omega\|_{L^r_p(M,w)} \right).
\]

Also Corollary 6.11 take the form:

Corollary 7.6. Let \((M,g)\) be a complete riemannian manifold with bounded Ricci curvature and positive injectivity radius. Let \(r \geq 1\) and let \(\omega\) be a \(p\)-form in \(M\). We have:

\[
\|\omega\|_{L^{s}_p(M)} \leq C \left( \|d(\omega)\|_{L^{p+1}_r(M)} + \|d^*(\omega)\|_{L^{r-1}_{p-1}(M)} + \|\omega\|_{L^r_p(M)} \right)
\]

with \(\frac{1}{s} = \frac{1}{r} - \frac{1}{n} > 0\).

This result has to be linked to N. Lohoué’s one [Lohoué, 1985]: he proved the same result under the stronger hypothesis that \((M,g)\) has a 2-order bounded geometry plus some other hypotheses on the laplacian and the range of \(r\). Here already 0-order bounded geometry is enough. In fact we just need that the Ricci curvature be bounded, not the complete curvature tensor.

### 7.1 Bounded geometry

Now we shall see a family of riemannian manifolds verifying a stronger condition than the one asks in Corollary 7.1: namely we ask that the complete curvature \(R\) is bounded instead of solely the Ricci curvature.

**Definition 7.7.** A riemannian manifold \(M\) has \(k\)-order bounded geometry if:

- the injectivity radius \(r_{\text{inj}}(x)\) at \(x \in M\) is bounded below by some constant \(\delta > 0\) for any \(x \in M\)
- for \(0 \leq j \leq k\), the covariant derivatives \(\nabla^j R\) of the curvature tensor are bounded in \(L^\infty(M)\) norm.

The Corollary 7.1 gives, in particular, that if \((M,g)\) has a 0-order bounded geometry, then we get that \(\forall x \in M, R_{(1,\cdot)}(x) \geq \delta\). Again we need only to control the Ricci curvature instead of the complete curvature tensor. Hence we get the classical Sobolev embedding Theorems:

**Theorem 7.8.** Let \((M,g)\) be a complete riemannian manifold. Let \(G := (H,\pi,M)\) be a complex smooth vector bundle over \(M\) with a smooth metric \((\ , \ )\) and a metric connection \(\nabla^G\). Suppose \((M,g)\) has a 0-order bounded geometry. Let \(0 \leq k < m\) and \(1/s = 1/r - (m-k)/n\). Let \(u \in W^{m,r}_G(M)\). Then we have \(u \in W^{k,s}_G(M)\) with the control:

\[
\|u\|_{W^{k,s}_G(M)} \leq C \|u\|_{W^{m,r}_G(M)}.
\]

We shall give some examples of such a situation.

### 7.2 Examples of manifolds of bounded geometry.

These examples are precisely taken from [Eldering, 2012, p. 40], where there is more explanations.

- Euclidean space with the standard metric trivially has bounded geometry.
• A smooth, compact Riemannian manifold $M$ has bounded geometry as well; both the injectivity radius and the curvature including derivatives are continuous functions, so these attain their finite minima and maxima, respectively, on $M$. If $M \in C^{m+2}$, then it has bounded geometry of order $m$.

• Non compact, smooth Riemannian manifolds that possess a transitive group of isomorphisms (such as the hyperbolic spaces $\mathbb{H}^n$) have $m$-order bounded geometry since the finite injectivity radius and curvature estimates at any single point translate to a uniform estimate for all points under isomorphisms.

More manifolds of bounded geometry can be constructed with these basic building blocks in the following ways.

• The product of a finite number of manifolds of bounded geometry again has bounded geometry, since the direct sum structure of the metric is inherited by the exponential map and curvature.

• If we take a finite connected sum of manifolds with bounded geometry such that the gluing modifications are smooth and contained in a compact set, then the resulting manifold has bounded geometry again.

7.3 Compact riemannian manifold with smooth boundary.

We have seen that a compact riemannian manifold of class $C^2$ without boundary has a 0-order bounded geometry, so the Sobolev embeddings are valid on it. We shall deduce that they are also valid in the case of a compact riemannian manifold with a $C^\infty$ smooth boundary. But first we have, using that a compact riemannian manifold has 0-order bounded geometry:

**Corollary 7.9.** Let $(M,g)$ be a compact riemannian manifold without boundary. Let $G := (H,\pi,M)$ be an adapted complex smooth vector bundle over $M$. Then the Sobolev embeddings for sections of $G$ are valid in $(M,g)$. Precisely we have: $W^{m,r}_G(M)$ is embedded in $W^{k,s}_G(M)$, with $\frac{1}{s} = \frac{1}{r} - \frac{(m-k)}{n} > 0$ and:

$$\forall u \in W^{m,r}_G(M), \|u\|_{W^{k,s}_G(M)} \leq C\|u\|_{W^{m,r}_G(M)}.$$  

Now let $M$ be a $C^\infty$ smooth connected compact riemannian manifold with a $C^\infty$ smooth boundary $\partial M$. We want to show how the results in case of a compact boundary-less manifold apply to this case.

A classical way to get rid of an "annoying boundary" of a manifold is to use its "double". For instance: Duff [Duff, 1952], Hörmander [Hörmander, 1994, p. 257]. Here we copy the following construction from [Guneysu and Pigola, 2015, Appendix B].

The "Riemannian double" $\Gamma := \Gamma(M)$ of $M$, obtained by gluing two copies, $M$ and $M_2$, of $M$ along $\partial M$, is a compact Riemannian manifold without boundary. Moreover, by its very construction, it is always possible to assume that $\Gamma$ contains an isometric copy of the original manifold $M$. We shall also write $M$ for this isometric copy to ease notation.

We take $u \in W^{m,r}_G(M)$ and we want to show that $u \in W^{k,s}_G(M)$, with $\frac{1}{s} = \frac{1}{r} - \frac{(m-k)}{n} > 0$.

We shall suppose that $G$ extends smoothly to $\Gamma$, i.e. the connexion is smooth and still is a metric connexion on $\Gamma$, and the scalar product also is smooth in $\Gamma$. For instance this is the case if $G = \Lambda^p(M)$, the bundle of $p$-forms in $M$.  

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The Seeley Theorem [Seeley, 1964] in the version of Lions [Lions, 1964], tells us that any function \( f \in W^{m,r}(M) \) can be extended to \( \Gamma \) as \( \tilde{f} \in W^{m,r}(\Gamma) \). By use of a finite covering of \( M \) by balls \( B(x, R(x)) \) with center \( x \in M \) and trivializing the bundle \( G \), and an associated partition of unity, this result of Seeley can be made valid to a section \( u \in W^{m,r}_G(M) \). So we have an extension \( \tilde{u} \in W^{m,r}_G(\Gamma) \).

Using Corollary 7.9 we get that:

\[
\tilde{u} \in W^{k,s}_G(\Gamma), \text{ with } \frac{1}{s} = \frac{1}{r} - \frac{(m-k)}{n} > 0 \text{ and:}
\]

\[
\forall \tilde{u} \in W^{m,r}_G(\Gamma), \|\tilde{u}\|_{W^{k,s}_G(\Gamma)} \leq C\|\tilde{u}\|_{W^{m,r}_G(\Gamma)}.
\]

Hence, restricting \( \tilde{u} \) to \( M \) we get, a fortiori, \( u \in W^{k,s}_G(M) \), and:

\[
\forall u \in W^{m,r}_G(M), \|u\|_{W^{k,s}_G(M)} \leq C\|u\|_{W^{m,r}_G(M)}.
\]

So we proved:

**Theorem 7.10.** Let \((M, g)\) be a compact riemannian manifold with a smooth boundary. Let \( G := (H, \pi, M) \) be a complex vector bundle over \( M \), which admit a smooth extension to a "double" manifold \( \Gamma \). Then the Sobolev embeddings for sections of \( G \) are valid in \((M, g)\). Precisely we have:

\[
W^{m,r}_G(M) \text{ is embedded in } W^{k,s}_G(M), \text{ with } \frac{1}{s} = \frac{1}{r} - \frac{(m-k)}{n} > 0 \text{ and:}
\]

\[
\forall u \in W^{m,r}_G(M), \|u\|_{W^{k,s}_G(M)} \leq C\|u\|_{W^{m,r}_G(M)}.
\]

### 7.4 Hyperbolic manifolds.

These are manifolds such that the sectional curvature \( K_M \) is constantly \(-1\). For them we have first that the Ricci curvature is bounded.

**Lemma 7.11.** Let \((M, g)\) be a complete Riemannian manifold such that \( H \leq K_M \leq K \) for constants \( H, K \in \mathbb{R} \). Then we have that \( \|Rc\|_{\infty} \leq \max(|H|, |K|) \).

**Proof.**

Take a tangent vector \( v \in T_x M \), with \(|v| = 1\), we obtain the Ricci curvature \( Rc \) of \( v \) at \( x \) by extending \( v = v_n \) to an orthonormal basis \( v_1, \ldots, v_n \). Then we compute the Ricci curvature along \( v \):

\[
Rc_x(v) = \frac{1}{n-1} \sum_{j=1}^{n-1} \langle R_x(v, v_j)v, v_j \rangle.
\]

where \( R \) denotes the Riemannian curvature tensor. On the other hand, the sectional curvature \( K_{M,x}(v, v_j) \) for \( j < n \) is given by (remember the \( v_j \) are orthonormal):

\[
K_{M,x}(v, v_j) = \langle R_x(v, v_j)v, v_j \rangle
\]

So we get

\[
Rc_x(v) = \frac{1}{n-1} \sum_{j=1}^{n-1} \langle R_x(v, v_j)v, v_j \rangle = \frac{1}{n-1} \sum_{j=1}^{n-1} K_{M,x}(v, v_j),
\]

hence \( H \leq Rc_x(v) \leq K \Rightarrow |Rc_x| \leq \max(|H|, |K|) \).

To get estimates on the Ricci tensor \( Rc_{(M,g)}(x)(u, v) \), we notice that \( Rc_x(v) = Rc_{(M,g)}(x)(v, v) \) and we get the estimates by polarisation.

To get that the injectivity radius \( r_{inj}(x) \) is bounded below we shall use a Theorem by Cheeger, Gromov and Taylor [Cheeger et al., 1982]:

\[
\text{...}
\]
**Theorem 7.12.** Let \((M, g)\) be a complete Riemannian manifold such that \(K_M \leq K\) for constants \(K \in \mathbb{R}\). Let \(0 < r < \frac{\pi}{4\sqrt{K}}\) if \(K > 0\) and \(r \in (0, \infty)\) if \(K \leq 0\). Then the injectivity radius \(r_{inj}(x)\) at \(x\) satisfies 
\[
r_{inj}(x) \geq r\frac{\text{Vol}(B_M(x, r))}{\text{Vol}(B_M(x, r)) + \text{Vol}(B_{T_xM}(0, 2r))},
\]
where \(B_{T_xM}(0, 2r)\) denotes the volume of the ball of radius \(2r\) in \(T_xM\), where both the volume and the distance function are defined using the metric \(g^* := \exp_p^* g\) i.e. the pull-back of the metric \(g\) to \(T_xM\) via the exponential map.

This Theorem leads to the definition:

**Definition 7.13.** Let \((M, g)\) be a Riemannian manifold. We shall say that it has the **lifted doubling property** if we have:

\[
(LDP) \quad \exists \beta, \gamma > 0 :: \forall x \in M, \exists r \geq \beta, \text{Vol}(B_{T_xM}(0, 2r)) \leq \gamma \text{Vol}(B_M(x, r)),
\]
where \(B_{T_xM}(0, 2r)\) denotes the volume of the ball of radius \(2r\) in \(T_xM\), and both the volume and the distance function are defined on \(T_xM\) using the metric \(g^* := \exp_p^* g\) i.e. the pull-back of the metric \(g\) to \(T_xM\) via the exponential map.

Hence we get:

**Corollary 7.14.** Let \((M, g)\) be a complete Riemannian manifold such that \(K_M \leq K\) for a constant \(K \in \mathbb{R}\). For instance an hyperbolic manifold. Suppose moreover that \((M, g)\), has the lifted doubling property. Then:
\[
\forall x \in M, \ r_{inj}(x) \geq \frac{\beta}{1 + \gamma}.
\]

**Proof.**

By the (LDP) we get, for a \(r \geq \beta\),
\[
\text{Vol}(B_{T_xM}(0, 2r)) \leq \gamma \text{Vol}(B_M(x, r)).
\]

We apply Theorem 7.12 of Cheeger, Gromov and Taylor to get
\[
r_{inj}(x) \geq \frac{r}{\text{Vol}(B_M(x, r)) + \text{Vol}(B_{T_xM}(0, 2r))},
\]
So
\[
\frac{\text{Vol}(B_M(x, r))}{\text{Vol}(B_M(x, r)) + \text{Vol}(B_{T_xM}(0, 2r))} \geq \frac{1}{1 + \gamma}
\]
hence, because \(r \geq \beta\), we get the result.

So finally we get

**Corollary 7.15.** Let \((M, g)\) be a complete Riemannian manifold such that \(H \leq K_M \leq K\) for constants \(H, K \in \mathbb{R}\), where \(K_M\) is the sectional curvature of \(M\). Suppose moreover that \((M, g)\) has the lifted doubling property. Then \(\exists \delta > 0, \forall x \in M, R_{1,\epsilon}(x) \geq \delta\).

This implies that the Sobolev embeddings are valid for sections of \(G\) in that case.

**Proof.**

Because the Lemma 7.11 gives that the Ricci curvature is bounded. The Corollary 7.14 gives \(\forall x \in M, r_{inj}(x) \geq \frac{\beta}{1 + \gamma}\). So the Corollary 7.1 completes the proof.
Remark 7.16. In the case the hyperbolic manifold \((M, g)\) is simply connected, then by the Hadamard Theorem [do Carmo, 1993, Theorem 3.1, p. 149], we get that the injectivity radius is \(\infty\), so we have also the classical embedding Theorems in this case, even for sections of \(G\).

References

[Amar, 2018a] Amar, E. (2018a). The LIR method. \(L^r\) solutions of elliptic equation in a complete riemannian manifold. \textit{J. Geometric Analysis}. To appear. DOI: 10.1007/s12220-018-0086-3. 9

[Amar, 2018b] Amar, E. (2018b). Sobolev solutions of parabolic equation in a complete riemannian manifold. \textit{Arxiv}. arXiv:1812.04411. 5, 17

[Aubin, 1982] Aubin, T. (1982). \textit{Nonlinear Analysis on Manifolds. Monge-Ampère Equations}, volume 252. Springer-Verlag, New York. 3, 7

[Cantor, 1974] Cantor, M. (1974). Sobolev inequalities for riemannian bundles. \textit{Bull Am. Math. Soc.}, 80:239–243. 2, 7, 19

[Carron, 1994] Carron, G. (1994). Inégalités isopérimétriques sur les variétés riemanniennes. Master’s thesis, Université Joseph Fourier, Grenoble. These de Doctorat. 2

[Cheeger et al., 1982] Cheeger, J., Gromov, M., and Taylor, M. (1982). Finite propagation speed, kernel estimates for functions of the Laplace operator, and the geometry of complete riemannian manifolds,, \textit{J. Differential Geom.}, 17:15–53. 22

[Coulhon and Saloff-Coste, 1993] Coulhon, T. and Saloff-Coste, L. (1993). Isopérimétrie pour les groupes et les variétés. \textit{Revista Mathematica Iberoamericana}, 9:293–314. 2

[do Carmo, 1993] do Carmo, M. P. (1993). \textit{Riemannian geometry}. Mathematics. Birkhäuser Boston. 24

[Duff, 1952] Duff, G. (1952). Differential forms in manifolds with boundary. \textit{Ann. of Math.}, 56:115–127. 21

[Eldering, 2012] Eldering, J. (2012). Persistence of noncompact normally hyperbolic invariant manifolds in bounded geometry. \textit{Ph. d. Thesis}. Ph. d. Thesis Universiteit Utrecht. 20

[Evans and Gariepy, 1992] Evans, L. C. and Gariepy, R. F. (1992). \textit{Measure theory and fine properties of functions}. Studies in Advanced Mathematics. CRC Press, Boca Raton. 5

[Guneysu and Pigola, 2015] Guneysu, B. and Pigola, S. (2015). Calderon-Zygmund inequality and Sobolev spaces on noncompact riemannian manifolds. \textit{Advances in Mathematics}, 281:353–393. 21

[Hebey, 1996] Hebey, E. (1996). \textit{Sobolev spaces on Riemannian manifolds.}, volume 1635 of \textit{Lecture Notes in Mathematics}. Springer-Verlag, Berlin. 2, 4, 18, 19
[Hebey and Herzlich, 1997] Hebey, E. and Herzlich, M. (1997). Harmonic coordinates, harmonic radius and convergence of riemannian manifolds. *Rend. Mat. Appl.* (7) 17 (1997), no. 4, 569-605 (1998), 17(4):569–605. 3, 18

[Hörmander, 1994] Hörmander, L. (1994). *The Analysis of Linear Partial Differential Operators III*, volume 274 of *Grundlehren der mathematischen Wissenschften*. Springer. 21

[Li, 2010] Li, X.-D. (2010). Sobolev inequalities on forms and $L_{p,q}$-cohomology on complete riemannian manifolds. *J. Geom. Anal.*, 20:354–387. 2

[Lions, 1964] Lions, J. L. (1964). Extension of $C^\infty$ functions defined in a half space. *Math. Reviews*, MR0165392 (29 #2676). 22

[Lohoué, 1985] Lohoué, N. (1985). Inégalités de Sobolev pour les formes différentielles sur une variété riemannienne. *C. R. Acad. Sci. Paris*, 301(6):277–280. 2, 18, 20

[Mazzucato and Nistor, 2006] Mazzucato, A.-L. and Nistor, V. (2006). Mapping properties of heat kernels, maximal regularity, and semi-linear parabolic equations on noncompact manifolds. *Journal of Hyperbolic Differential Equations*, 3(4):599–629. 5

[Scott, 1995] Scott, C. (1995). $L^p$ theory of differential forms on manifolds. *Transactions of the American Mathematical Society*, 347(6):2075–2096. 3, 13

[Seeley, 1964] Seeley, R. (1964). Extension of $C^\infty$ functions defined in a half space. *Proc. Amer. Math. Soc.*, 15:625–626. 22

[Taylor, 2000] Taylor, M. E. (2000). *Differential Geometry*. Course of the University of North Carolina. University of North Carolina. www.unc.edu/math/Faculty/met/diffg.html. 5

[Varopoulos, 1989] Varopoulos, N. (1989). Small time gaussian estimates of heat diffusion kernels. *Bulletin des Sciences Mathématiques.*, 113:253–277. 2

[Voisin, 2002] Voisin, C. (2002). *Théorie de Hodge et géométrie algébrique complexe.*, volume 10 of *Cours spécialisé*. S.M.F. 12