STRICHARTZ ESTIMATES IN SPHERICAL COORDINATES

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Abstract. In this paper we study Strichartz estimates for dispersive equations which are defined by radially symmetric pseudo-differential operators, and of which initial data belongs to spaces of Sobolev type defined in spherical coordinates. We obtain the space time estimates on the best possible range including the endpoint cases.

1. Introduction

In this paper we consider the Cauchy problem of linear dispersive equations:

\[ iu_t - \omega(|\nabla|)u = 0 \quad \text{in} \ R^{1+n}, \quad u(0) = \varphi \quad \text{in} \ R^n, \quad n \geq 2 \quad (1.1) \]

where \( \omega(|\nabla|) \) is the pseudo-differential operator of which multiplier is \( \omega(|\xi|) \). Typical examples of \( \omega \) are \( \rho^a (0 < a \neq 1) \), \( \sqrt{1 + \rho^2} \), \( \rho \sqrt{1 + \rho^2} \), and \( \frac{\rho}{\sqrt{1 + \rho^2}} \) which describe Schrödinger type equation [22], Klein-Gordon or semirelativistic equation [15], iBq, and imBq (see [10] and references therein).

The solution can formally be given by

\[ u(t, x) = \frac{1}{(2\pi)^n} \int_{R^n} e^{ix \cdot \xi - t\omega(|\xi|)} \hat{\varphi}(\xi) \, d\xi. \]

Here \( \hat{\varphi} \) is the Fourier transform of \( \varphi \) defined by \( \int_{R^n} e^{-ix \cdot \xi} \varphi(x) \, dx \). There have been a lot of works on the space time estimates for the solution \( u \) which play important roles in recent studies on nonlinear dispersive equations. (See Cazenave [5], Sogge [27] and Tao [33] and references therein.) Especially, when \( \omega(\rho) = \rho^a, \ a \neq 0, \) the solution satisfies

\[ \|u\|_{L^q_t L^p_x} \leq C\|\varphi\|_{H^s} \quad (1.2) \]

with \( s = \frac{n}{2} - \frac{a+1}{q} \), which is known as Strichartz estimates. These estimates were first established by Strichartz [31] for \( q = p \) and were generalized to mixed norm
(q ≠ p) spaces by Ginibre and Velo [16, 17] except the endpoint cases, which were later proven by Keel and Tao [20].

It is well known that the estimate (1.2) is possible only if $n/p + 2/q ≤ n/2$, $q ≥ 2$ when $a > 0$ and $n/p + 2/q ≤ n/2$, $q ≥ 2$ when $a = 1$ as it can be easily seen by Knapp’s example. In actual applications of (1.2) to various problems, depending on the problems being considered, the existence of proper $(p, q)$ for which (1.2) holds is crucial. Hence, there have been attempts to extend the range $p, q$ by suitable generalizations [32, 30]. As it was observed in [30, 25], the estimates have wider ranges of admissible $p, q$ when $\phi$ is a radial function. It is due to the fact that Knapp’s examples are non-radial. However, to make these estimates on the extended range hold for general functions which are no longer radial, additional regularity in angular direction is necessary.

For precise description we now define a function spaces of Sobolev type in spherical coordinates. Let $\Delta_\sigma = \sum_{1≤i<j≤n} \Omega_{ij}^2$, $\Omega_{ij} = x_i \partial_j - x_j \partial_i$, be the Laplace-Beltrami operator defined on the unit sphere in $\mathbb{R}^n$ and set $D_\sigma = \sqrt{1-\Delta_\sigma}$. For $|s| < n/2$, $\alpha \in \mathbb{R}$, we denote by $\dot{H}^s_{\sigma}H^\alpha$ the space $\{ f ∈ S' : \| f \|_{\dot{H}^s_{\sigma}H^\alpha} = \| |\nabla|^s D_\sigma f \|_{L^2} < \infty \}$.

It should be noted that $C^\infty_c$ is dense in $\dot{H}^s_{\sigma}H^\alpha$ since $|s| < n/2$. So a natural generalization of (1.2) might be

$$\| u \|_{L^q_t L^r_x} ≤ C\| \phi \|_{\dot{H}^s_{\sigma}H^\alpha}.$$  (1.3)

In fact, for the wave equation ($\omega(\rho) = \rho$) Strebenz [30] obtained almost optimal range of $q, r$ and almost sharp required regularity (see also Section 4.6). In [21, 1.3] was shown for $\omega(\rho) = \rho^a$, $\frac{1}{q} < (n-1)(\frac{1}{2} - \frac{1}{p})$, $q ≥ 2$ and $\alpha ≥ \frac{1}{q}$ by utilizing Rodnianski’s argument of [30] and weighted Strichartz estimates (see [11, 14, 7]). Recently, Guo and Wang [18] considered the estimate (1.3) with $\omega(\rho) = \rho^a$ and radially symmetric initial data, and found the optimal range of $p, q$ except some endpoint cases.

In a different direction one may try to extend (1.2) to include more general $\omega$. Let us consider $\omega \in C^\infty(0, \infty)$ which satisfies the following properties:

(i) $\omega'(\rho) > 0$, and either $\omega''(\rho) > 0$ or $\omega''(\rho) < 0$,

(ii) $|\omega^{(k)}(\rho_1)| \sim |\omega^{(k)}(\rho_2)|$ for $0 < \rho_1 < \rho_2 < 2\rho_1$,

(iii) $\rho|\omega^{(k+1)}(\rho)| \lesssim |\omega^{(k)}(\rho)|$. 

...
We also define a pseudo-differential operator \( D_{\omega}^{s_1, s_2} \) by setting
\[
\mathcal{F}(D_{\omega}^{s_1, s_2} f)(\xi) = \left( \frac{\omega'(|\xi|)}{|\xi|} \right)^{s_1} |\omega''(|\xi|)|^{s_2} \hat{f}(\xi).
\]
Here \( \mathcal{F} \) denotes the Fourier transform. In \cite{12} (also see \cite{19} for earlier result), the authors proved the following: If \( \omega \) satisfies the conditions (i), (ii) for \( k = 1, 2 \), and (iii) for \( 1 \leq k \leq \left\lceil \frac{n}{2} \right\rceil + 1 \), then
(1.4) \[
\|u\|_{L^p_t L^q_x} \lesssim \|D_{\omega}^{s_1, s_2} \varphi\|_{\dot{H}^s}
\]
holds for \( 2 \leq p, q \leq \infty \), \( \frac{2}{q} + \frac{n}{p} \leq \frac{n}{2} \) and \( (n, p, q) \neq (2, \infty, 0) \) with
(1.5) \[
s_1 = \left( \frac{1}{4} - \frac{1}{2p} \right) - \frac{1}{q}, \quad s_2 = \frac{1}{2p} - \frac{1}{4}, \quad s = n \left( \frac{1}{2} - \frac{1}{p} \right) - \frac{2}{q}.
\]
It is obvious that the range and the exponents are sharp \((1.4)\).

In this note we try to unify the estimates \((1.2)\) and \((1.3)\) in a single framework. More precisely, we consider the estimates
(1.6) \[
\|u\|_{L^p_t L^q_x} \lesssim \|D_{\omega}^{s_1, s_2} \varphi\|_{\dot{H}^s}
\]
which have a wider range \( p, q \) of boundedness than \((1.4)\). Allowing some regularity loss in spherical variables, we want to find the best possible range of \( p, q \) for \((1.6)\). In fact, using a Knapp’s example which is adapted to radial function one can see that \((1.6)\) is possible only if
(1.7) \[
\frac{1}{q} \leq \frac{2n - 1}{2} \left( \frac{1}{2} - \frac{1}{p} \right).
\]
(See Section 4.1.) Since we already have the usual Strichartz estimates \((1.4)\) on the range \( \frac{2}{q} + \frac{n}{p} \leq \frac{n}{2} \), we are mainly interested in the estimates for \((p, q)\) which is contained in the region \( \frac{2}{q} + \frac{n}{p} > \frac{n}{2} \).

The following is our first result which establishes \((1.6)\) in the best possible range of \( p, q \) except an endpoint.

**Theorem 1.1.** Let \( n \geq 2, 2 \leq p, q \leq \infty \) and \( s_1, s_2, s \) given by \((1.5)\). Suppose that \( \omega \in C^\infty(0, \infty) \) satisfies the conditions (i), (ii) for \( k = 1, 2 \), and (iii) for \( 1 \leq k \leq \max(4, \left\lceil \frac{n}{2} \right\rceil + 1) \). If \( \frac{n}{2} \left( \frac{1}{2} - \frac{1}{p} \right) \leq \frac{1}{q} \leq \frac{2n - 1}{2} \left( \frac{1}{2} - \frac{1}{p} \right) \), \( (n, p, q) \neq (2, \infty, 2) \), and \((p, q) \neq (\frac{2(2n-1)}{2n-3}, 2)\), the solution \( u \) to \((1.1)\) satisfies \((1.6)\) for \( \alpha > \frac{5n - 1}{3n - 5} \left( \frac{n}{p} + \frac{2}{q} - \frac{n}{2} \right) \).

\(^1\)In \cite{12} the condition (iii) was assumed for \( k \geq 1 \) to get \((1.4)\) but (iii) for \( 1 \leq k \leq \left\lceil \frac{n}{2} \right\rceil + 1 \) is enough as it is obvious from the proof in \cite{12}.
Theorem 1.1 generalizes Shao’s results in [25] where \( \omega(\rho) = \rho^2 \) and radial data were considered. In [18], some estimates for \((p, q)\) on the sharp line \( \frac{1}{q} = \frac{2n-1}{2n-3} \left( \frac{1}{2} - \frac{1}{p} \right) \) were obtained when \( p \leq q, \ \omega(\rho) = \rho^2 \) and the initial datum \( \phi \) is radial. But our results include all the estimates on the sharp line except for \((p, q) \neq \left( \frac{2(2n-1)}{2n-3}, 2 \right)\), which is left open and seems to be beyond the method of this paper. But it is possible to obtain the following weak type inequality when the initial datum is radial:

\[
\|u\|_{L^2_t L^\infty_x} \lesssim \|D^{s_1, s_2}_\omega \varphi\|_{\dot{H}^s}
\]

where \( s_1, s_2, \) and \( s \) are given by (1.5) with \((p, q) = \left( \frac{2(2n-1)}{2n-3}, 2 \right)\). We provide a brief proof of this in Section 4.4.

It should be noted that if \( \omega \) satisfies (iv) below instead of (iii), then \( D^{s_1, s_2}_\omega \) simplifies so that \( D^{s_1, s_2}_\omega \sim |\omega''(\rho)|^{-\frac{1}{q}} \). Although Theorem 1.1 gives a sharp estimate in \((q, p)\) pairs, there is no reason to believe that the angular regularity is sharp. Substantial improvement should be possible by obtaining refined estimates for Bessel function.

When \( \omega \) satisfies

\[
(iv) \quad \rho |\omega''(\rho)| \sim |\omega'(\rho)| \quad \text{for } \rho > 0,
\]

the angular regularity can be improved further. (See Section 3.3). The following is our second result which also improves the angular regularity result in [21].

**Theorem 1.2.** Let \( n \geq 2, 2 \leq p, q \leq \infty \) and \( s_1, s_2, s \) given by (1.5). Suppose that \( \omega \in C^\infty(0, \infty) \) satisfies the conditions (i), (ii) for \( k = 1, 2 \), (ii) for \( 1 \leq k \leq \max(4, \left\lceil \frac{n}{2} \right\rceil) + 1 \), and (iv). If \( \frac{1}{2} - \frac{1}{p} \leq \frac{1}{q} \leq \frac{2n-1}{2n-3} \left( \frac{1}{2} - \frac{1}{p} \right) \), \((n, p, q) \neq (2, \infty, 2)\), and \((p, q) \neq \left( \frac{2(2n-1)}{2n-3}, 2 \right)\), the solution \( u \) to (1.1) satisfies (1.6) for \( \alpha > \frac{1}{2} \frac{2n-1}{n-1} \left( \frac{p}{2} + \frac{q}{2} - \frac{n}{2} \right) \).

Compared to previous works, our approach here is simpler and more systematic so that we can provide a simplified proof of the result in [30] (see Section 4.6). By spherical harmonic expansion (1.6) the matters basically reduce to one dimensional situation but it involves with a family operators which are given by Bessel functions of different orders. To get the desired estimate, the growth of bounds depending on the orders needs to be controlled in a uniform way. It will be done by comparing spatial scale and the orders of Bessel functions. Our novelty here is the use of a temporal localization (see Lemma 3.2) which is available only after frequency and spatial localizations. This enables us to reduce the estimate in time to that of the same scale in space so that we suffices to work on local estimates, and it also plays a role in obtaining precise estimates for general \( \omega \). (See Section 3)
This paper is organized as follows: In Section 2 we consider the asymptotic behavior of Bessel function. In Section 3 we obtain various preliminary estimates via space-frequency-time localization which are expected to be useful for related problems and in Section 4 the proofs of Theorems (1.1), (1.2) are given.

If not specified, $A \lesssim B$, $A \sim B$ mean $A \leq CB$, $C^{-1}A \leq B \leq CB$, respectively, for some generic constant $C$.

2. Estimates for Bessel functions

For the proofs of theorems we need estimates for Bessel functions $J_\nu$, which depend on $\nu$. When $\nu$ is bounded, estimates are easy to obtain. We start by recalling some basic properties of Bessel functions.

Let $\nu_0 > 1$ be a fixed number. If $0 \leq \nu \leq \nu_0$, then

\begin{equation}
|J_\nu(r)| \lesssim 1, \text{ if } r \lesssim 1,
\end{equation}

\begin{equation}
J_\nu(r) = r^{-\frac{\nu}{2}}(b_+e^{ir} + b_-e^{-ir}) + \Psi(r), \text{ if } r \gg 1,
\end{equation}

where $|\Psi(r)| \lesssim r^{-\frac{3}{2}}$. For instance see [28, 36]. If $\nu > \nu_0$, then we have

\begin{equation}
|J_\nu(r)| \lesssim \exp(-C\nu), \text{ if } r \ll \nu.
\end{equation}

It is easy to show by making use of the Poisson representation ([28, 36])

\begin{equation}
J_\nu(r) = \frac{(\frac{r}{\nu})^\nu}{\Gamma(\nu + \frac{1}{2})\Gamma(\frac{1}{2})} \int_{-1}^{1} e^{irs}(1 - s^2)^{\nu - \frac{1}{2}} ds
\end{equation}

and Stirling’s formula ([23]) $\Gamma(t) \sim \sqrt{2\pi t^{t-\frac{1}{2}}}e^{-t}$ for large $t$.

For simplicity we denote $z + \overline{z}$ by

$z + \overline{z}$

so that $\overline{z} + \overline{z}$ stands for the complex conjugate of terms appearing before $+\overline{z}$. We now make use of the following representation of Bessel function (see [1] or Lemma 3 of [2]);

\begin{equation}
J_\nu(r) = 2(r^2 - \nu^2)^{-1/4} (c_\nu e^{i\theta(r)} + \overline{c_\nu}) + h_\nu(r),
\end{equation}

where

\[
\theta(r) = r \left[ \left(1 - \frac{\nu^2}{r^2}\right)^{\frac{1}{2}} - \left(\frac{\pi}{2} - \cos^{-1}\frac{\nu}{r}\right) \right], \quad |h_\nu(r)| \lesssim r^{-1}.
\]
It can be obtained by the stationary phase method and Schl"afli’s integral representation (see p.176, [36]) which is given by

\begin{equation}
J_\nu(r) = \frac{1}{\pi} \int_0^\pi e^{i(r \sin \theta - \nu \theta)} d\theta - \frac{\sin(\nu \pi)}{\pi} \int_0^\infty e^{-\nu \tau - r \sin \tau} d\tau.
\end{equation}

The following lemma gives asymptotic bounds for $J_\nu$ when $\nu$ is large.

**Lemma 2.1.** Let $\nu \geq \frac{1}{2}$. Then the following holds:

\begin{align}
|J_\nu(r)| \lesssim r^{-\frac{1}{2}}, \quad & \text{if } r \geq 2\nu, \\
J_\nu(r) = (c_\nu r^{-1/2} + \tilde{c}_\nu \nu^2 r^{-\frac{3}{2}}) e^{i\nu} + \mathcal{C} + \Psi_\nu(r), \quad & \text{if } r \geq 4\nu^{\frac{3}{2}},
\end{align}

where $c_\nu = \frac{e^{-i(\frac{\nu}{2} + \frac{\nu^2}{2})}}{2\sqrt{2\pi}}$, $\tilde{c}_\nu = -2ic_\nu$, and $|\Psi_\nu(r)| \lesssim r^{-1}$.

**Proof of Lemma 2.1.** We rewrite $e^{i\theta(r)}$ as

\[ e^{i\theta(r)} = e^{ir}(1 + i(\theta(r) - r)) + e^{ir}(e^{i\theta(r)-r} - 1 - i(\theta(r) - r)). \]

Substituting this into (2.5), we obtain

\[ J_\nu(r) = r^{-\frac{1}{2}} \left( 1 - \frac{\nu^2}{r^2} \right)^{-1/4} \left( c_\nu e^{ir}(1 + i(\theta(r) - r)) + \mathcal{C} \right) + \tilde{h}_\nu(r), \]

where

\[ \tilde{h}_\nu(r) = h_\nu(r) + \left( c_\nu \frac{e^{ir}(e^{i\theta(r)-r} - 1 - i(\theta(r) - r))}{(r^2 - \nu^2)^{1/4}} + \mathcal{C} \right). \]

Let $\theta_1(r) = \left( 1 - \frac{\nu^2}{r^2} \right)^{-1/4} - 1$ and $\theta_2(r) = i(\frac{3\nu^2}{2r} + \theta(r) - r)$. Then

\[ J_\nu(r) = r^{-\frac{1}{2}}(1 + \theta_1(r)) \left( c_\nu e^{ir}(1 - i\frac{2\nu^2}{r} + \theta_2(r)) + \mathcal{C} \right) + \tilde{h}_\nu(r) + \Psi_\nu(r), \]

where

\[ \Psi_\nu(r) = \tilde{h}_\nu(r) + r^{-1/2}\theta_1(r) \left( c_\nu e^{ir} \left( 1 - i\frac{3\nu^2}{2r} + \theta_2(r) \right) + \mathcal{C} \right) \]

\[ + r^{-\frac{1}{2}}(1 + \theta_1(r)) \left( c_\nu e^{ir} \theta_2(r) + \mathcal{C} \right). \]

Taylor’s theorem gives that $|\tilde{h}_\nu(r)| \lesssim r^{-1}$, $|\theta_1(r)| \lesssim \frac{\nu^2}{r^2}$ and $|\theta_2(r)| \lesssim \frac{\nu^2}{r}$. Hence $|\Psi_\nu(r)| \lesssim r^{-1}$ for $r \geq 4\nu^{\frac{3}{2}}$. This completes the proof of Lemma 2.1 \hfill \square
3. Estimates via space-frequency localization

In this section we obtain estimates via localization on both space and frequency sides. Let \( 0 < \lambda_0 \lesssim 1 \) and \( \omega \in C(1/2, 2) \cap C^4(1/2, 2) \) satisfy that
\[
|\omega'(\rho)| \sim 1, \quad \lambda_0 \leq |\omega''(\rho)| \lesssim 1, \quad |\omega^{(3)}(\rho)| + |\omega^{(4)}(\rho)| \lesssim 1
\]
if \( 1/2 \leq \rho \leq 2 \). For \( R > 0 \), let us set \( \chi_R = \chi_{\{|x| \leq 2R\}} \) and define
\[
\mathcal{T}_R^\nu h(t, r) = \chi_R(r)r^{-\frac{\nu-2}{2}} \int e^{-it\omega(\rho)}J_\nu(r\rho)\beta(\rho)h(\rho)d\rho,
\]
where \( \beta \in C_c^\infty(1/2, 2) \). In what follows \( \beta \) may be different at each occurrence but we keep the same notation as long as it is contained uniformly in \( C_c^\infty(1/2, 2) \).

We denote by \( \mathfrak{L}_p^n \) the space \( L^p(r^{n-1} dr) \).

**Proposition 3.1.** Let \( R \gtrsim 1 \) and \( \mathcal{T}_R^\nu \) be defined by (3.2). If \( 2 \leq p, q \leq \infty, \nu \geq 0, \) and \( 2/q \geq 1/2 - 1/p \), then there is a constant \( C = C(n, p, q) > 0 \), independent of \( \lambda_0, \nu, R \), such that
\[
\|\mathcal{T}_R^\nu h\|_{L^p_xL^q_t} \leq C\lambda_0^{-\frac{\nu}{2}(\frac{1}{2} - \frac{1}{p})}(1 + \nu)^{\frac{\nu}{2}(\frac{1}{2} - \frac{1}{p})}R^\frac{\nu}{2} - \frac{2n-1}{2}(\frac{1}{2} - \frac{1}{p})}\|h\|_2.
\]

The exponent on \( R \) is sharp as it can be shown by the example for the necessary condition (see Section 4.1).

For the proof, we need to show the cases \( (p, q) = (2, \infty), (2, 2), (\infty, 2), \) and \( (\infty, 4) \) since the other estimates follow from interpolation. The first two cases are straightforward from the square function estimates for Bessel functions such that
\[
\int_{r \sim R} |J_\nu(r)|^2 dr \lesssim 1
\]
for all \( \nu \geq 0 \) and \( R > 0 \). This can be shown by using (2.1), (2.2) when \( 0 \leq \nu \leq \nu_0 \), \( \int_0^\infty |J_\nu(r)|^2dr/r = 1/(2\nu) \) when \( \nu \geq \nu_0 \) and \( R \lesssim \nu \) (see p. 405 of [36]), and (2.7) when \( R \gg \nu > \nu_0 \).

Now, by Schwarz inequality and (3.4) we have
\[
|\mathcal{T}_R^\nu h(t, r)| \lesssim R^{-\frac{n-2}{2}} \sup_{r \sim R} \left( \int_{\rho \sim 1} |J_\nu(r\rho)|^2 \, d\rho \right)^{\frac{1}{2}} \|h\|_2 \lesssim R^{-\frac{n-1}{2}} \|h\|_2.
\]
Hence, this shows (3.3) for \( (p, q) = (2, \infty) \). Similarly, since \( |\omega'(\rho)| \sim 1 \), by the change of variables \( \omega(\rho) \mapsto \rho \), Plancherel’s theorem in \( t \) and Schwarz’s inequality we have
\[
\|\mathcal{T}_R^\nu h\|_{L^2_tL^2_x}^2 \lesssim \int_{\rho \sim 1} |h(\omega^{-1}(\rho))|^2 \int_{r \sim R} |J_\nu(r\rho)|^2 r \, dr \, d\rho.
\]
By (3.4) and reversing change of variables we obtain (3.3) for \( (p, q) = (2, 2) \).
It remains to show (3.3) for $(\infty, 2)$, and $(\infty, 4)$. For this purpose we use a localization property of $T_R^\nu$ in $t$, which is possible because supp $\beta \subset (1/2, 2)$.

### 3.1. Temporal localization

Let $\chi_R$ be a measurable function supported $[0, 2R]$ with $\|\chi_R\|_\infty \leq 1$. Let $\tilde{\omega} \in C^4(1/2, 2)$ and let us define

\[(3.7) \quad R^\nu R g(t, r) = \tilde{\chi}_R(r) \int e^{-it\tilde{\omega}(\rho)}J_\nu(r\rho)\beta(\rho)g(\rho)d\rho.\]

Making use of the localization of scale $R$ in $r$ and the fact that supp $\beta \subset (1/2, 2)$, it is possible to localize the estimate for $R^\nu R$ obtaining sharp estimates for $\mathcal{T}_{1, \nu}^R$ for $\nu \leq \nu^* = \nu_R$. Note that $\|\mathcal{T}_{1, \nu}^R\|_{L^q_x L^p_t} \leq \frac{4\nu^*+1}{2q-2}$.

#### Lemma 3.2

Let $R \geq 1$, $\nu = \nu(k) = \frac{n-2+2k}{2}$, $k = 0, 1, 2, \ldots$, and let $I$ be an interval of length $R$. Suppose that $|\tilde{\omega}'(\nu)| \sim 1$, $|\tilde{\omega}^{(k)}(\nu)| \lesssim 1$, $k = 2, 3, 4$, on the support of $\beta$. And suppose that

\[(3.8) \quad \|\chi_I(t)R^\nu R g\|_{L^q_t L^p_x} \leq CR^b \|g\|_2\]

for $2 \leq p, q \leq \infty$, $B \geq 1$, and $b \geq \frac{n}{p} + \frac{1}{q} - 1$. Then $\|R^\nu R g\|_{L^q_t L^p_x} \leq CBR^b \|g\|_2$.

Note that $\mathcal{T}_{1, \nu}^R h(r, t) = r^{-\frac{n-2+2k}{2}}R^\nu R h(r, t)$ with a proper $\tilde{\chi}_R$ if $\varpi = \tilde{\omega}$. From Section 4.1 we see that a lower bound for $\mathcal{T}_{1, \nu}^R : L^2 \rightarrow L^q_t L^p_x$ is $CR^{b+\frac{n-1}{2}} + \frac{1}{2}$ which is bigger than or equal to $CR^{\frac{n}{p}+\frac{1}{q}-1}$ if $p \geq 2$ and $R$ is large enough. So we can use this localization for $\mathcal{T}_{1, \nu}^R$ for any loss in bound.

**Proof of Lemma 3.2.** It should be noted that $\nu = 0$ or $\nu \geq 1/2$ from the definition of $\nu(k)$. By Plancherel’s theorem it suffices to show that

\[(3.9) \quad \|\chi_I(t)R^\nu R \hat{g}\|_{L^q_t L^p_x} \leq CR^b \|g\|_2\]

for $B \geq 1$, $b \geq \frac{n}{p} + \frac{1}{q} - 1$ implies

\[\|R^\nu R \hat{g}\|_{L^q_t L^p_x} \leq CBR^b \|g\|_2.\]

Using the integral representation (2.6) of Bessel function we write

\[(3.10) \quad R^\nu R \hat{g} = R_1 g + R_2 g,\]

where

\[R_1 g = \frac{\tilde{\chi}_R(r)}{\pi} \int_0^\pi \int_0^\infty e^{-it\tilde{\omega}(\rho)} e^{i(r\rho \sin \theta - \nu \theta)} \beta(\rho)\hat{g}(\rho) d\rho d\theta,\]

\[R_2 g = \frac{-\sin(\nu \pi)}{\pi} \tilde{\chi}_R(r) \int_0^\infty \int e^{-it\tilde{\omega}(\rho)} e^{-\nu \tau - r \rho \sin \tau} \beta(\rho)\hat{g}(\rho) d\rho d\tau.\]
Note that if \( \nu = 0 \), then \( \mathcal{R}_2 = 0 \).

If \( \nu \geq 1/2 \), then we claim that for \( 2 \leq p, q \leq \infty \)
\[
\| \mathcal{R}_2 g \|_{L^p_t L^q_x} \leq CR^{\frac{2}{\nu} + \frac{1}{q} - 1} \| g \|_2.
\]
In view of interpolation, it is sufficient to consider the cases \( (p, q) = (2, 2), (2, \infty), (\infty, 2) \),
and \( (\infty, \infty) \). By Hölder’s inequality we only need to show (3.11) for \( (p, q) = (\infty, 2), (\infty, \infty) \).
Using the fundamental theorem of calculus and Hölder’s inequality,
we have
\[
\sup_{0 \leq r \leq 2R} |\mathcal{R}_2 g(t, r)| \leq |\mathcal{R}_2 h(t, 3R/2)| + \int_0^{2R} |\partial_r \mathcal{R}_2 g(t, r)| dr
\leq |\mathcal{R}_2 h(t, 3R/2)| + CR^{\frac{2}{\nu}} \|\partial_r \mathcal{R}_2 g(t, \cdot)\|_{L^2(0, 2R)}.
\]
Since \( |\tilde{\omega}'(\rho)| \sim 1 \), by the change of variables \( \tilde{\omega}(\rho) \mapsto \rho \) and Plancherel’s theorem
\[
\| \mathcal{R}_2 g(t, 3R/2) \|_{L^2_t} \leq C \int_0^\infty e^{-\nu \tau} e^{-\frac{2}{3} R \sinh \tau} d\tau \| g \|_2 \leq C \frac{\| g \|_2}{\nu + R}.
\]
To handle \( \| \partial_r \mathcal{R}_2 g(t, \cdot) \|_{L^2_r(2R)} \), observe that
\[
\partial_r \mathcal{R}_2 g(t, r) = \frac{\sin(\nu \pi)}{\pi} \hat{\chi}_R(r) \int e^{-it\rho} \left( \int_0^\infty e^{-\nu \tau - r \phi(\rho) \sinh \tau} \sinh \tau d\tau \right)
\times \phi(\rho) \beta(\phi(\rho)) \hat{g}(\phi(\rho)) \phi'(\rho) d\rho,
\]
where \( \phi = \tilde{\omega}^{-1} \). Since \( \phi'(\rho) \sim 1 \), by Plancherel’s theorem we have
\[
\| \partial_r \mathcal{R}_2 g \|_{L^2_t L^2_r(2R)} \leq C \int_0^\infty e^{-\nu \tau} \| e^{-r \sinh \tau} \|_{L^2_r(2R)} \sinh \tau d\tau \| g \|_2 \leq C \int_0^\infty e^{-\nu \tau} e^{-R \sinh \tau (\sinh \tau)^{\frac{1}{2}}} d\tau \| g \|_2 \leq \frac{C}{\nu - 1/2 + R} \| g \|_2.
\]
Since \( \nu \geq 1/2 \), we get the desired estimate (3.11) for \( (p, q) = (2, \infty) \). The estimate for \( (p, q) = (\infty, \infty) \) is straightforward because
\[
|\mathcal{R}_2 g| \leq C \int_0^\infty e^{-\nu \tau - \frac{R}{2} \sinh \tau} d\tau \| g \|_2.
\]
By (3.10), (3.11), and (3.9) it is now enough to show that \( \| \chi(t) \mathcal{R}_1 g \|_{L^p_t L^q_x} \leq BR^b \| g \|_2 \) implies \( \| \mathcal{R}_1 g \|_{L^p_t L^q_x} \leq CBR^b \| g \|_2 \). Since \( |\tilde{\omega}'(\rho)| \sim 1 \), by the change of variables \( \rho \to \phi(\rho) = \tilde{\omega}^{-1}(\rho) \) and Plancherel’s theorem, we may replace \( \mathcal{R}_1 \) by \( \tilde{\mathcal{R}}_1 \) which is given by
\[
\tilde{\mathcal{R}}_1 g = \chi_R(r) \int_0^\pi \int e^{-it\rho} e^{i(r\phi(\rho) \sin \theta - \nu \theta)} \tilde{\beta}(\rho) \hat{g}(\rho) d\rho d\theta
\]
for some \( \tilde{\beta} \in C^\infty_c(1/2, 2) \). Matters are reduced to showing that

\[
\| \chi_I(t) \tilde{R}_1 g \|_{L^q_t L^p_r} \leq BR^b \| g \|_2
\]

for \( b \geq -1 \) implies

\[
\| \tilde{R}_1 g \|_{L^q_t L^p_r} \leq CBR^b \| g \|_2.
\]

We note that

\[
\tilde{R}_1 g(t, r) = \tilde{\chi}_R(r)(K_r * g)(t)
\]

with

\[
K_r(t) = \frac{1}{2\pi} \int_0^\pi \int_0^{\pi} e^{-i(t\rho - r\phi(\rho)\sin \theta)} \tilde{\beta}(\rho) d\rho \ e^{-iv\theta} d\theta.
\]

Since \( 0 \leq r \leq 2R \) and \( |\frac{d}{d\rho}(t\tilde{\omega}(\rho) - r\rho \sin \theta)| \geq C|t| \) for some \( C > 0 \) if \( |t| \geq MR \) for some large \( M > 0 \). From the condition on \( \tilde{\omega} \) and integration by parts (three times) it follows that for \( 0 \leq a \leq 3 \),

\[
|K_r(t)| \leq C(1 + |t|)^{-3} \leq CR^{-a}(1 + |t|)^{-3+a}
\]

if \( r \sim R \) and \( |t| \geq MR \). Now the argument is rather standard. Indeed, let \( \{I\} \) be a collection of disjoint intervals with sidelength \( \sim R \) which partition \( \mathbb{R} \). Let us denote by \( \tilde{I} \) the interval \( \{ t : \text{dist}(t, I) \leq 5MR \} \). Breaking \( g = \chi_{\tilde{I}} g + \chi_{\tilde{I}^c} g \), by (3.14) we see that for \( t \in I \)

\[
|\tilde{R}_1 g(t, r)| \leq |\tilde{R}_1 (\chi_{\tilde{I}} g)(t, r)| + CR^{-a}\mathcal{E} \ast |g|(t),
\]

where \( \mathcal{E}(t) = (1 + |t|)^{-3+a} \). Thus, when \( q \neq \infty \), taking \( a = 1 \), we see that

\[
\int_\mathbb{R} \| \tilde{R}_1 g(t, \cdot) \|_{L^q_r}^q dt \leq \sum_I \int_I \| \tilde{R}_1 g(t, \cdot) \|_{L^q_r}^q dt \leq C \sum_I \int_I \| \tilde{R}_1 (\chi_{\tilde{I}} g) \|_{L^q_r}^q dt + CR^{-q} \int_\mathbb{R} (\mathcal{E} \ast |g|)^q(t) dt \leq CB^a R^b \| \chi_{\tilde{I}} g \|_2^q + CR^{-q} \| g \|_2^q \leq CB^a R^b \| g \|_2^q.
\]

For the third inequality we use the hypothesis (3.12) and the fact that \( \mathcal{E} \in L^1 \cap L^\infty \) and for the Fourth inequality we use the fact that \( b \geq \frac{n}{p} + \frac{1}{q} - 1 \). Hence summation along \( I \) gives the desired estimate (3.13). When \( q = \infty \), the argument is even simpler.

We omit the detail. \( \Box \)
Now let us set
\begin{equation}
R_\Omega g(t, r) = \tilde{\chi}_R(r) \int e^{-it\bar{\omega}(\rho)} \Omega(r\rho)\beta(\rho)g(\rho)d\rho.
\end{equation}

From the proof of Lemma 3.2 it is obvious that the same statement remains valid even if we replace $R_\nu^r$ by $R_\nu^\Psi g$ provided that $R \geq 4\nu^\frac{8}{5}$. Here $\Psi_\nu(r) = J_\nu(r) - [(c_\nu r^{-1/2} + \tilde{c}_\nu \nu^2 r^{-\frac{3}{2}}) e^{ir} + C.C]$ which is given in (2.8). In fact, since we already have (3.11), one needs to check that
\begin{equation}
K_r(t) = \frac{1}{2\pi} \int e^{-it\rho} \left( \int_0^\pi e^{i(r\phi(\rho)\sin\theta - \nu\theta)} d\theta 
- \left[ \frac{c_\nu}{(\phi(\rho)r)^{\frac{1}{2}}} + \frac{\tilde{c}_\nu \nu^2}{(\phi(\rho)r)^{\frac{3}{2}}} \right] e^{ir\phi(\rho)} + C.C \right) \beta(\rho)d\rho
\end{equation}
satisfies (3.14) if $r \sim R$, $|t| \geq MR$. It is easy to see using the fact that $R \geq 4\nu^\frac{8}{5}$. One can handle each term separately. Then the rest of the argument is straightforward. The similar implication is also valid for $R_\Omega$ with $\Omega = \rho^{-\frac{1}{4}} e^{\pm i\rho} \nu^2 \rho^{-\frac{1}{2}} e^{\pm i\rho}$. For future use we summarize it in the following.

**Lemma 3.3.** Let $R \geq 1$, $\nu$, $I$ and $\bar{\omega}$ be the same as in Lemma 3.2 and let $R_\Omega$ be defined by (3.15). Suppose that $R \geq 4\nu^\frac{8}{5}$ and (3.8) holds for $B \geq 1$ and $2 \leq p, q \leq \infty$. If $\Omega = \rho^{-\frac{1}{4}} e^{\pm i\rho} \nu^2 \rho^{-\frac{1}{2}} e^{\pm i\rho}$, $\Psi_\nu$, then $\| R_\Omega g \|_{L^p_t L^q_x} \leq CBR^h \| g \|_2$.

We now return to the proof of Proposition 3.1.

3.2. **Proof of (3.3) for $(p, q) = (\infty, 2)$, $(\infty, 4)$.** We show that for $q = 2, 4$,
\begin{equation}
\| T_\nu^R h \|_{L^q_t L^\infty_x} \leq C\lambda_0^{\frac{1}{4}} (1 + \nu)^{\frac{3}{2}} R^{\frac{4n-1}{2}} \| h \|_2.
\end{equation}
The estimate for $q = 2$ follows from the case $q = 4$. In fact, by Lemma 3.2 it is sufficient to show (3.16) for $q = 2$ when $I$ is an interval of side length $\sim R$ but this follows from Hölder’s inequality and the estimate (3.16) with $q = 4$. Hence we are reduced to showing that
\begin{equation}
\| T_\nu^R h \|_{L^4_t L^\infty_x} \leq C\lambda_0^{\frac{1}{4}} (1 + \nu)^{\frac{3}{2}} R^{-\frac{n-1}{2}} \| h \|_2.
\end{equation}
For this we consider the following three cases, separately:
\begin{equation}
(1) : R \ll \nu, \quad (2) : \nu \lesssim R \lesssim \nu^\frac{8}{5}, \quad (3) : \nu^\frac{8}{5} \ll R.
\end{equation}

**Case (1).** From (2.3) we have $\| T_\nu^R h \|_{L^4_t L^\infty_x} \lesssim e^{-C\nu R^{\frac{n-2}{2}}} \| h \|_2$. By Lemma 3.2 we get
\begin{equation}
\| T_\nu^R h \|_{L^4_t L^\infty_x} \lesssim e^{-C\nu R^{\frac{1}{4} - \frac{n-2}{2}}} \| h \|_2 \lesssim e^{-C\nu^\frac{3}{4} R^{-\frac{n-1}{2}}} \| h \|_2 \lesssim R^{-\frac{n-1}{2}} \| h \|_2,
\end{equation}
which is acceptable.

Case (2). By (3.5) we have \( \| T_R^\nu h \|_{L^4_{t,x}} \lesssim R^{-\frac{n+1}{2}} \| h \|_2 \). Then, by Lemma 3.2 and Hölder’s inequality \( \| T_R^\nu h \|_{L^4_{t,x}} \leq CR^{\frac{1}{4} - \frac{n+1}{2}} \| h \|_2 \). So, if \( R \sim \nu \), then \( \| T_R^\nu h \|_{L^4_{t,x}} \lesssim \nu^\frac{4}{5} R^{-\frac{n+1}{2}} \| h \|_2 \). If \( \nu \ll R \lesssim \nu^\frac{5}{8} \), then \( \| T_R^\nu h \|_{L^4_{t,x}} \lesssim \nu^\frac{5}{8} R^{-\frac{n+1}{2}} \| h \|_2 \). So we have (3.17).

Case (3). For simplicity let us set

\[
T_\Omega h(t,r) = r^{-\frac{n-2}{2}} \int e^{-it\varpi(\rho)} \Omega(\rho) \beta(\rho) h(\rho) d\rho,
\]

and

\[
T_{\Omega,R} h(t,r) = \chi_R(r) T_\Omega h(t,r).
\]

Since \( r \sim R \), using (2.8), we need to consider \( R_\Omega \) with

\[
\Omega(\rho) = \rho^{-\frac{1}{2}} e^{i\varphi}, \quad \nu^2 \rho^{-\frac{1}{2}} e^{i\varphi}, \quad \Psi(\rho) = O(1/\rho)
\]

and for (3.17) it is sufficient to show that

\[
\| T_\Omega h \|_{L^4_{t,x}} \lesssim C.\lambda_0^{-\frac{n}{4}} \| h \|_2.
\]

For \( \Omega(\rho) = O(1/\rho) \), by Schwarz’s inequality \( | T_{\Omega,R} h(t,r) | \leq CR^{-\frac{n}{2}} \| h \|_2 \). So we get the required bound from Hölder’s inequality. Hence we only need to consider the cases \( \Omega(\rho) = \rho^{-\frac{1}{2}} e^{i\varphi}, \nu^2 \rho^{-\frac{1}{2}} e^{i\varphi} \). These two cases can be handled similarly. In fact, since \( \nu^\frac{5}{8} \ll R \), we get the desired bound (3.17) if we show that

\[
\| T h \|_{L^4_{t,x}} \lesssim C.\lambda_0^{-\frac{n}{4}} \| h \|_2,
\]

where

\[
T h(r,t) = \tilde{\chi}_R(r) \int e^{i(-t\varpi(\rho)\pm \varphi)} \beta(\rho) h(\rho) d\rho.
\]

By duality it is equivalent to \( \| T^* H \|_{L^2} \leq C.\lambda_0^{-\frac{n}{4}} \| H \|_{L^4_{t,x}} \), where \( T^* \) is the adjoint operator of \( T \). It again follows from

\[
\| T T^* H \|_{L^4_{t,x}} \leq C.\lambda_0^{-\frac{n}{4}} \| H \|_{L^4_{t,x}}.
\]

Now note that

\[
TT^* H(t,r) = \int \int K(t-s,r,r') |r'(n-1)| H(s,r') dsdr',
\]

where

\[
K(t,r) = \tilde{\chi}_R(r) \tilde{\chi}_R(r') e^{i(-t\varpi(\rho)\pm \varphi)} \beta(\rho) d\rho.
\]

\[\text{The bound } \nu^\frac{5}{8} \text{ is actually decided by the term } \nu^2 \rho^{-\frac{1}{2}} e^{i\varphi}.\]
Since \( \lambda_0 \leq |\omega''(r)| \lesssim 1 \), by van der Corput (see for instance page 334 of \[28\]), it follows that \( |K(t, r)| \leq C \lambda_0^{-\frac{3}{2}} |t|^{-1/2} \). So we get

\[
\|TT^*H\|_{L^6_tL^\infty} \leq C \lambda_0^{-\frac{1}{2}} \left\| \int |t-s|^{-\frac{1}{2}} \|H(\cdot,s)\|_{L^6_s} ds \right\|_{L^4_t}.
\]

Then by Hardy-Littlewood-Sobolev inequality we get the desired bound. This completes the proof of (3.3), and hence Proposition (3.1). □

**Remark 1.** From the above proof (Case (3)) it is obvious that if \( \Omega = \rho^{-\frac{3}{2}} e^{\pm ip}, \Psi_\nu(\rho) = O(1/\rho) \), then for \( R \gtrsim 1, 2 \leq p, q \leq \infty \) and \( 2/q \geq 1/2 - 1/p \),

\[
(3.20) \quad \|T_{\Omega,R}h\|_{L^q_t L^p} \lesssim \lambda_0^{-\frac{1}{2}(\frac{3}{2} - \frac{1}{p})} R^{\frac{1}{2} - \frac{2n-1}{2} + \frac{1}{2}}\frac{1}{(\frac{3}{2} - \frac{1}{p})} \|h\|_2
\]

and if \( \Omega = \nu^2 \rho^{-\frac{3}{2}} e^{\pm ip} \), for \( 2 \leq q \leq 4 \)

\[
(3.21) \quad \|T_{\Omega,R}h\|_{L^q_t L^p} \lesssim \lambda_0^{-\frac{1}{2}} \nu^2 R^{-1} R^{\frac{1}{2} - \frac{2n-1}{2}} \|h\|_2
\]

By (3.3), (2.8) and (3.20) for \( \Omega = \rho^{-\frac{3}{2}} e^{\pm ip}, \Psi_\nu(\rho) = O(1/\rho) \), we also have for \( \Omega = \nu^2 \rho^{-\frac{3}{2}} e^{\pm ip} \),

\[
(3.22) \quad \|T_{\Omega,R}h\|_{L^q_t L^p} \lesssim R^{\frac{1}{2}} \|h\|_2
\]

Hence, by interpolation between (3.21) and (3.22) we see that if \( \Omega = \nu^2 \rho^{-\frac{3}{2}} e^{\pm ip} \),

\[
(3.23) \quad \|T_{\Omega,R}h\|_{L^q_t L^p} \lesssim \lambda_0^{-\frac{1}{2}(\frac{3}{2} - \frac{1}{p})} (\nu^2 R^{-1})^{1 - \frac{2}{p}} R^{\frac{1}{2} - \frac{2n-1}{2} + \frac{1}{2}} \|h\|_2
\]

provided that \( 2 \leq p, q \leq \infty \) and \( 2/q \geq 1/2 - 1/p \). Hence, when \( R \geq 2\nu^2 \), one gets uniform bounds so that if \( \Omega = \nu^2 \rho^{-\frac{3}{2}} e^{\pm ip}, \rho^{-\frac{3}{2}} e^{\pm ip}, \Psi_\nu(\rho) = O(1/\rho) \),

\[
(3.24) \quad \|T_{\Omega,R}h\|_{L^q_t L^p} \lesssim \lambda_0^{-\frac{1}{2}(\frac{3}{2} - \frac{1}{p})} R^{\frac{1}{2} - \frac{2n-1}{2} + \frac{1}{2}} \|h\|_2,
\]

whenever \( 2 \leq p, q \leq \infty \) and \( 2/q \geq 1/2 - 1/p \).

### 3.3. An improvement on angular regularity.

In what follows we improve the bound in \( \nu \) but at the expense of the bounds in \( \lambda_0 \). This is why we need an extra condition on \( \omega \) in Theorem 1.2.

**Proposition 3.4.** Let \( R \gtrsim 1 \) and \( T^\nu_R \) be defined by (3.2). If \( 2 \leq p, q \leq \infty, \nu \geq 0, \) and \( 1/q \geq 1/2 - 1/p \), then there is a constant \( C = C(n,p,q) > 0 \), independent of \( \lambda_0, \nu, R \), such that

\[
(3.25) \quad \|T^\nu_R h\|_{L^q_t L^p} \leq C \lambda_0^{-\frac{1}{2}(\frac{3}{2} - \frac{1}{p})} (1 + \nu)^{\frac{1}{2}(\frac{3}{2} - \frac{1}{p})} R^{\frac{1}{2} + \frac{2n-1}{2} - \frac{1}{p}} \|h\|_2.
\]
Proof. From the proof Proposition 3.1 (see (3.5) and (3.6)) and Remark 1 (see (3.24)) we recall that the estimates (3.25) for \((p, q) = (2, 2), (2, \infty)\) are already obtained. Hence, for the proof of Proposition 3.4 it is sufficient to show (3.25) for \((p, q) = (\infty, 2)\). By Lemma 3.2 this follows from

\[
\|\chi_I T_\nu h\|_{L^2_t L^\infty_r} \leq C\lambda_0^{-\frac{1}{2}} (1 + \nu)^{\frac{1}{2}} R^{\frac{3}{4} - \frac{3}{4} n} \|h\|_2.
\]

Here \(I\) is an interval of length \(\sim R\). For the case \(R \ll \nu\) it is easy to check (3.26) as before and the case \(R \gg \nu^2\) is already handled (see (3.24) in Remark 1). Hence to show (3.26) we may assume \(\nu \ll R \lesssim \nu^2\).

To treat this case we use (2.5). The contribution from \(h_\nu\) in (2.5) is \(O(R^{\frac{3}{2}} \|h\|_2)\). So, it is acceptable. Hence it is enough to show that

\[
\|T_\pm h\|_{L^2_t L^\infty_r} \leq C\lambda_0^{-\frac{1}{2}} \nu^{\frac{1}{2}} R^{\frac{1}{2}} \|h\|_2,
\]

where

\[
T_\pm g(t, r) = \chi_I(t) \tilde{\chi}_{R}(r) \int e^{-it\tilde{\omega}(\rho) \pm i\theta(r)} \beta_\nu(\rho, r) g(\rho) d\rho
\]

and \(\beta_\nu(\rho, r) = \beta(\rho) \left( 1 - \nu^2 \frac{\rho^2}{r^2} \right)^{-\frac{1}{4}}\). We only show the estimate for \(T_+\). The other can be handle similarly. Following the previous argument we need to show that

\[
\|T_+ T_+^* H\|_{L^2_t L^\infty_r} \leq C\lambda_0^{-1} (1 + \nu)^{\frac{1}{2}} R^{\frac{3}{2}} \|H\|_{L^2_t L^\infty_r}.
\]

Since

\[
T_+ T_+^* H = \int \int \chi_I(t) \chi_I(s) K(t - s, r, r') [r^{m-1} H(s, r')] dr' ds,
\]

and

\[
K(t, r, r') = \int e^{-it\tilde{\omega}(\rho) + i\theta(r) - \theta(r')} \beta_\nu(\rho, r) \beta_\nu(\rho, r') d\rho.
\]

Now let us observe that for \(\nu \ll r, r \sim 1\)

\[
\left| \frac{d^2}{d\rho^2} \theta(r\rho) \right| \lesssim \frac{\nu^2}{r} \lesssim \nu.
\]

So, if \(|t| \geq C\lambda_0^{-1} \nu\) for some large \(C\),

\[
\left| \frac{d^2}{d\rho^2} \left( -t\tilde{\omega}(\rho) + \theta(r\rho) - \theta(r'\rho) \right) \right| \geq C\lambda_0 |t|.
\]

Hence, from van der Corput lemma we get

\[
|K(t, r, r')| \leq C\lambda_0^{-\frac{1}{2}} |t|^{-\frac{1}{2}}.
\]
if $|t| \geq C\lambda_0^{-1}\nu$. Hence using trivial bounds $|K(t, r, r')| = O(1)$ for $|t| \leq C\lambda_0^{-1}\nu$ we see that
\[
\int \chi_I(t)\chi_I(s) \sup_{r,r'} |K(s-t, r, r')| dt, \int \chi_I(t)\chi_I(s) \sup_{r,r'} |K(s-t, r, r')| ds
\]
are bounded by
\[
C \int_0^{\lambda_0^{-1}\nu} dt + C\lambda_0^{-\frac{1}{2}} \int_0^R t^{-\frac{1}{2}} dt \leq C\lambda_0^{-1}\nu + C\lambda_0^{-\frac{1}{2}} R^{\frac{1}{2}} \leq C\lambda_0^{-1}\nu R^{\frac{1}{2}}
\]
because $\nu \ll R$. Then by Schur’s test we get the desired bound. $\square$

**Remark 2 (The wave equation).** For the wave equation $\omega(\rho) = \pm \rho$, the estimates are much easier to show. Let us consider the operator
\[
W^\nu_R h(t, r) = \chi_R(r) r^{-\frac{n-2}{2}} \int e^{-it\omega(\rho)} J_{\nu}(r\rho) \beta(\rho) h(\rho) d\rho.
\]
Then we have for $2 \leq p, q \leq \infty$
\begin{equation}
||W^\nu_R h||_{L^q_t L^p_r} \leq CR^{\frac{1}{2} + \frac{n-1}{p} - \frac{n-1}{q}} ||h||_2.
\end{equation}
We only need to show the estimates for $(p, q) = (2, \infty), (2, 2), (\infty, \infty), (2, \infty)$. In fact, the case $(p, q) = (2, \infty)$ is a consequence of Plancherel’s theorem. So, we can apply Hölder’s inequality and Lemma 3.2 to the estimate (3.28) with $(p, q) = (2, \infty)$ to get (3.28) for $(p, q) = (2, 2)$. When $(p, q) = (\infty, \infty)$, the desired estimate can be obtained by Schwarz’s inequality and (3.4) (cf. (3.5)). So similarly the case $(p, q) = (\infty, 2)$ also follows by Hölder’s inequality and Lemma 3.2.

### 4. Proofs of Theorem 1.1, 1.2

In this section we prove Theorem 1.1 and 1.2 making use of the estimates in the previous section. The estimates other than those on the sharp line ($\frac{1}{q} = \frac{2n-1}{2}(\frac{1}{2} - \frac{1}{p})$) are relatively easy to show once we have Propositions 3.1. However, to get the estimate along the sharp line we need additional works which actually correspond to showing inhomogeneous estimates.

We start with proving the necessity of the condition $\frac{1}{q} \leq \frac{2n-1}{2}(\frac{1}{2} - \frac{1}{p})$ for (1.6).

#### 4.1. The necessary condition (1.7)

Let $\varphi$ be radially symmetric function such that $\hat{\varphi}$ is supported in $\{||\xi|| \sim 1\}$. Since $\hat{\varphi}$ is also radial, we can write the solution $u$ to (1.1) as
\[
u(t, x) = C|x|^{-\frac{n-2}{2}} \int e^{-it\omega(\rho)} \rho^{\frac{\nu}{2}} J_{\frac{n-2}{2}}(|x|\rho) \hat{\varphi}(\rho) d\rho.
\]
Fix $\rho_0 \in [1, 2]$. For $R \gg 1$, let us choose $\varphi$ such that

$$\hat{\varphi}(\xi) = \rho^{-\frac{n+1}{2}} \phi(R\frac{1}{2}(\rho - \rho_0)), \quad \rho = |\xi|,$$

where $\phi \in C^\infty_c(1, 1)$. By the asymptotic of Bessel function (2.2) we have

$$\left|\frac{d}{d\rho}(-t\omega(R^{-\frac{1}{2}}\rho + \rho_0) - R^{-\frac{1}{2}}|x|)\right| \geq CR\frac{1}{2}$$

for any $R \geq 1$. By integration by parts, we see that the second integral in (4.1) is $O(R\frac{1}{2})$ if $t, |x| \sim R$. By integration by parts, we see that the second integral in (4.1) is $O(RM)$ for any $M$ if $t, |x| \sim R$. Hence, for $t, |x| \sim R$ and $||x| - \omega(\rho_0)t| \lesssim R^\frac{1}{2}$

$$|u(t, x)| \gtrsim R^{-\frac{n}{2}}.$$

Therefore it follows that

$$\|u\|_{L^p_tL^q_xL^\infty_\nu} \gtrsim R^{-\frac{n+1}{2} + \frac{1}{q} + \frac{2n-1}{2p}}.$$

On the other hand $\|\varphi\|_{L^2_x} \sim R^{-\frac{1}{4}}$. Since $\varphi$ is a radial function, $\|\varphi\|_{L^2_x} = \|\varphi\|_{L^2_xH^1_\sigma}$. So the estimate (1.6) implies that $R^{-\frac{n}{2} + \frac{1}{2} + \frac{2n-1}{2p}} \lesssim R^{-\frac{1}{4}}$. Letting $R \to \infty$, we get the condition (1.7).

### 4.2 Frequency localization

By Littlewood-Paley theory, scaling and orthogonality the estimate (1.6) can be obtained from the estimates for the simpler operator $T^\nu$ which is defined by

$$T^\nu h(t, r) = r^{-\frac{n-2}{2}} \int e^{-it\varpi(\rho)} J_\nu(r\rho) \rho^n \beta(\rho) h(\rho) d\rho.$$

**Lemma 4.1.** Let $2 \leq p, q \leq \infty$, $\gamma \geq 0$, and $\varpi \in C(1/2, 2) \cap C^4(1/2, 2)$ which satisfies (3.1). Suppose that for $\nu = \nu(k) = \frac{n-2+2k}{2}$, $k \geq 0$,

$$\| T^\nu h \|_{L^p_tL^q_x} \leq C(1 + \nu)^\gamma \lambda_0^{\frac{1}{2p} - \frac{1}{4}} \| h \|_{2}.$$
Then the solution \( u \) to (1.1) satisfies (1.6) with \( s_1, s_2 \) and \( s \) satisfying (1.5) provided that \( \alpha = \gamma + (n - 1) \left( \frac{1}{2} - \frac{1}{p} \right) \) when \( p > \infty \) and \( \alpha > \gamma + (n - 1) \left( \frac{1}{2} - \frac{1}{p} \right) \) when \( p = \infty \).

**Proof.** Let \( N > 0 \) denote dyadic numbers and let \( \beta \in C^\infty_c(1/2, 2) \) such that \( \sum N \beta(|\xi|/N) = 1, |\xi| \neq 0 \). Then we define \( P_N \) to be the projection operator given by

\[
\tilde{P}_N f(\xi) = \beta \left( \frac{|\xi|}{N} \right) \tilde{f}(\xi).
\]

Since \( 2 \leq p \leq \infty \) and \( q \geq 2 \), by Littlewood-Paley theory, Minkowski’s inequality and Sobolev embedding on the unit sphere \( S^{n-1} \) it follows that

\[
\|e^{-it\omega(|\xi|)}P_N\varphi\|_{L^q_tL^r_x} \lesssim \left\| \left| \sum N \right| P_N e^{-it\omega(|\xi|)}|\varphi|^2 \right\|_{L^p_tL^2_x} \leq \left( \sum N \|e^{-it\omega(|\xi|)}P_N\varphi\|_{L^q_tL^r_x}^2 \right)^{\frac{1}{2}} \leq \left( \sum N \|e^{-it\omega(|\xi|)}P_N D_{\sigma}^{(\alpha-\gamma)}\varphi\|_{L^q_tL^r_x}^2 \right)^{\frac{1}{2}}.
\]

(Note that \( \alpha - \gamma = (n - 1) \left( \frac{1}{2} - \frac{1}{p} \right) \) if \( p > \infty \) and \( \alpha - \gamma > \frac{n-1}{2} \) when \( p = \infty \).) Then, by orthogonality it is sufficient for (1.6) to show that

\[
\|e^{-it\omega(|\xi|)}P_N\varphi\|_{L^q_tL^r_x} \leq C\|D_{\omega}^{s_1,s_2}P_N\varphi\|_{\dot{H}^{s_1}_x\dot{H}^{s_2}_x}
\]

with \( C \), independent of \( N \). By the property (ii) of \( \omega \) it reduces to

\[
\|e^{-it\omega(|\xi|)}P_N\varphi\|_{L^q_tL^r_x} \leq W_N N^n \left( \frac{1}{2} - \frac{1}{p} \right) - \frac{1}{2} \|\varphi\|_{\dot{H}^{s_2}_x},
\]

where \( W_N = C(\omega'(N)N^{-1})^{\frac{1}{2} - \frac{1}{p}} |\omega''(N)|^{\frac{1}{2} - \frac{1}{p}} \). By rescaling

\[
\xi \rightarrow N\xi, \ x \rightarrow N^{-1}x, \ t \rightarrow (N\omega'(N))^{-1}t,
\]

and (1.5) this is equivalent with

\[
\|e^{-it\varphi(|\xi|)}P_1\varphi\|_{L^q_tL^r_x} \leq C\lambda_0^{\frac{1}{p} - \frac{1}{2}} \|\varphi\|_{\dot{H}^{s_2}_x},
\]

where

\[
\varrho(\rho) = \frac{\omega(N\rho)}{N\omega'(N)}, \ \lambda_0 = \left| \frac{N\omega''(N)}{\omega'(N)} \right|.
\]

Since \( \|\varphi\|_{\dot{H}^{s_2}_x} = \|\widehat{\varphi}\|_{\dot{H}^{s_2}_x} \) by Plancherel’s theorem and orthogonality of spherical harmonics, we are reduced to showing that

\[
\|Tf\|_{L^q_tL^r_x} \leq C\lambda_0^{\frac{1}{p} - \frac{1}{2}} \|f\|_{\dot{H}^{s_2}_x},
\]
for \( f \) supported in \([\frac{1}{2}, 2]\), where

\[
(4.6) \quad Tf(t, x) = \int e^{i(x-t\omega(|\xi|))} \beta(|\xi|) f(\xi) \, d\xi.
\]

We now expand \( f \) by the orthonormal basis \( \{Y_k^l\}, k \geq 0, 1 \leq l \leq d(k) \) of spherical harmonics (here \( d(k) \) is the dimension of spherical harmonics of order \( k \)) such that

\[
f(\xi) = f(\rho \sigma) = \sum_{k \geq 0} \sum_{1 \leq l \leq d(k)} a_k^l(\rho) Y_k^l(\sigma).
\]

We use the identities \( \hat{Y}_k^l(\rho \sigma) = c_{n,k} d^{n-2} \nu Y_k^l(-\sigma) \) (see [29]) to get

\[
(4.7) \quad Tf(t, x) = \sum_{k,l} c_{n,k} T^{\nu}(a_k^l)(t, r) Y_k^l(x/|x|), \quad r = |x|,
\]

where \( |c_{n,k}| = (2\pi)^{\frac{n}{2}}, k \geq 0 \) for some positive constant \( C \) which is not depending on \( k \). By orthogonality among \( \{Y_k^l\} \) and Minkowski’s inequality

\[
\|Tf\|_{L_p^q; L_3^2} \leq C \left( \sum_{k,l} \|T^{\nu(k)}(a_k^l)\|_{L_p^q; L_3^2}^2 \right)^{\frac{1}{2}}.
\]

Since \( \omega \) satisfies the conditions \((i) - (iii)\), it is easy to check that \( \omega, \lambda_0 \) in \((4.5)\) verifies the condition \((3.1)\). Hence by the estimate \((4.2)\) and the identity \( \|f\|_{L_p^q H_3^2} = \left\| \left( \sum_{k,l}(1+k+n-2)\alpha|a_k^l|^2 \right)^{\frac{1}{2}} \right\|_{L_p^q L_3^2} \) which follows from the fact that \( -\Delta_{\omega} Y_k^l = k(k+n-2)Y_k^l \), we get

\[
\|Tf\|_{L_p^q L_3^2} \leq C \lambda_0^{\frac{1}{2p} - \frac{1}{4}} \left( \sum_{k,l} \left( 1 + \nu(k) \right)^{2\gamma} \|a_k^l\|_{L_2^p L_2^q}^2 \right)^{\frac{1}{2}} \leq C \lambda_0^{\frac{1}{2p} - \frac{1}{4}} \|f\|_{L_p^q H_3^2}.
\]

This completes the proof. \( \square \)

### 4.3 Proof of Theorem 1.1

From the results [12], we already have estimates \((1.6)\) for \( \frac{n}{2} \left( \frac{1}{2} - \frac{1}{r} \right) = \frac{1}{q} \) with \( \alpha = 0 \). So, by interpolation it is enough to consider estimates near or on the sharp line \( \left( \frac{1}{q} = \frac{2n-1}{2} \left( \frac{1}{2} - \frac{1}{p} \right) \right) \). Hence by Lemma 4.1 we only need to show \((4.2)\) with \( \gamma = \frac{\delta}{2n} \left( \frac{1}{2} - \frac{1}{p} \right) + \epsilon \) for any \( \epsilon > 0 \) if \( \frac{n}{2} \left( \frac{1}{2} - \frac{1}{r} \right) < \frac{1}{q} \leq \frac{2n-1}{2} \left( \frac{1}{2} - \frac{1}{p} \right) \) and \( q \neq 2 \). In fact, note that \( (n-1 + \frac{1}{2}) \left( \frac{1}{2} - \frac{1}{p} \right) = \frac{5n-1}{2(n-1)} \) as \((p, q)\) to \((\frac{4n-2}{2n-3}, 2)\). So, we interpolate \((1.6)\) with \((p, q)\) arbitrarily close to \((\frac{4n-2}{2n-3}, 2)\) and \((1.4)\) to get the desired estimate.

\[^3c_{n,k} = (2\pi)^{\frac{n}{2}} i^{-k}\]
4.3.1. Estimates away from the sharp line. We show (1.6) for \( \frac{n}{2} \left( \frac{1}{2} - \frac{1}{q} \right) < \frac{1}{q} < \frac{2n-1}{2} \left( \frac{1}{2} - \frac{1}{p} \right) \). We break the operator in space radially. Fix a dyadic number \( R_0 \geq 1 \). We write

\[
\mathcal{T}_\nu h = \chi_{\{r < R_0\}} \mathcal{T}_\nu h + \chi_{\{r \geq R_0\}} \mathcal{T}_\nu h.
\]

The first is easy to handle. In fact, we show that for \( 2 \leq p, q \leq \infty \),

\[
\| \chi_{\{r < R_0\}} \mathcal{T}_\nu h \|_{L^q_t L^p_x} \leq C \|h\|_2.
\]  

From (4.7) we note that

\[
c_{n,k} \mathcal{T}_\nu h(t,|x|) Y^l_k \left( -\frac{x}{|x|} \right) = \int e^{i(x \cdot \xi - t\omega(|\xi|))} h(|\xi|) Y^l_k (\frac{\xi}{|\xi|}) \beta(|\xi|) \, d\xi.
\]

Then the estimate (4.8) for \( (p, q) = (2, \infty) \) follows from Plancherel’s theorem. Also, taking \( L^2 \) norm in angular variables (on \( S^{n-1} \)) and Schwarz’s inequality we get

\[
|\mathcal{T}_\nu h(t, r)| \leq C \|h\|_2.
\]

Interpolation establishes (4.8) for \( 2 \leq p \leq \infty, q = \infty \). Now, by Lemma 3.2 it is sufficient for (4.8) to show

\[
\| \chi_{\{r \geq R_0\}} \mathcal{T}_\nu h (t, r) \|_{L^q_t L^p_x} \leq C \|h\|_2
\]

for \( 2 \leq p, q \leq \infty \). It follows by Hölder’s inequality. Hence we get the desired estimate (4.8).

Recalling (3.2), we further break \( \chi_{\{r \geq R_0\}} \mathcal{T}_\nu h \) to get

\[
\chi_{\{r \geq R_0\}} \mathcal{T}_\nu h = \sum_{R \text{dyadic}, R \geq R_0} \mathcal{T}_R^\nu h,
\]

After triangle inequality we apply Propositions 3.1 to get

\[
\| \chi_{\{r \geq R_0\}} \mathcal{T}_\nu h \|_{L^q_t L^p_x} \leq C \lambda_0 \left( \frac{1}{2} - \frac{1}{p} \right)^{\frac{n}{4} \left( \frac{1}{2} - \frac{1}{p} \right)} \left( 1 + \nu \right)^{\frac{3}{4} - \frac{1}{p}} \|h\|_2.
\]

provided that \( 2 \leq p, q \leq \infty, 2/q \geq 1/2 - 1/p \), and \( \frac{1}{q} < \frac{2n-1}{2} \left( \frac{1}{2} - \frac{1}{p} \right) \). From this and (4.8) we get the estimate (4.2) and hence (1.6).

4.3.2. Estimate along the sharp line \( \frac{1}{q} + \frac{2n-1}{2p} - \frac{2n-1}{4} = 0 \). We now fill up the remaining estimates along the sharp line by showing estimate (4.2) for \( \frac{1}{q} = \frac{2n-1}{2} \left( \frac{1}{2} - \frac{1}{p} \right), q \neq 2 \). Again by Lemma 4.1 it is enough to show (4.2). It will be done by making use of \( TT^* \) argument. Since the estimate for \( (p, q) = (2, \infty) \) is trivial, we may assume \( p \neq 2 \).

By (4.8) it is sufficient to consider \( \chi_{\{r \geq R_0\}} \mathcal{T}_\nu h \). We further break it (up to the cases in (3.18)) to get

\[
\chi_{\{r \geq R_0\}} \mathcal{T}_\nu h = \left( \sum_{R_0 \leq R < 5R_0} + \sum_{R \geq 5R_0} \right) \mathcal{T}_R^\nu h.
\]
The first sum is easy to handle. From Proposition 3.1 we have (3.3). So, by direct summation we see that for \( \frac{1}{q} + \frac{2n-1}{2p} - \frac{2n-1}{4} = 0 \)

\[
\| \sum_{R_0 \leq R \leq 5 \nu \frac{\delta}{2}} T'_R h \|_{L_t^q L_x^p} \leq C (\log \nu) \nu^\frac{1}{2} (\frac{1}{2} - \frac{1}{p}) \| h \|_2.
\]

To obtain the desired estimate for \( \sum_{R \geq 5 \nu \frac{\delta}{2}} T'_R h \), it is sufficient to show that

\[
\| \chi_{(r \geq 5 \nu \frac{\delta}{2})} T_{\Omega} h \|_{L_t^q L_x^p} \leq C (1 + \nu)^\frac{1}{4} (\frac{1}{2} - \frac{1}{p}) \lambda_0^{\frac{1}{2} - \frac{4}{q}} \| h \|_2
\]

with \( \Omega = J_\nu \). (See (3.19).) For the proof of (4.2), using (2.8) in Lemma 2.1 we only need to show this with

\[
\Omega(\rho) = \rho^{-1/2} e^{\pm i \rho}, \quad \nu^2 \rho^{-\frac{3}{2}} e^{\pm i \rho}, \quad \Psi_\nu(\rho) = O(\rho^{-1}).
\]

First we handle the case \( \Omega = \Psi_\nu \). We break the operator dyadically so that

\[
\| \sum_{R \geq 5 \nu \frac{\delta}{2}} \chi_R(r) T_{\Psi_\nu} h \|_{L_t^q L_x^p} \leq \sum_{R \geq 5 \nu \frac{\delta}{2}} \| \chi_R(r) T_{\Psi_\nu} h \|_{L_t^q L_x^p}.
\]

Since \( \Psi_\nu(\rho) = O(\rho^{-1}) \), \( \| T_{\Psi_\nu} h \|_{L_t^q L_x^p} \leq CR^{-\frac{a}{4}} \). From Lemma 3.3 and Hölder’s inequality, it follows that

\[
\| \chi_R(r) T_{\Psi_\nu} h \|_{L_t^q L_x^p} \leq CR^{-\frac{a}{2} + \frac{1}{q} + \frac{n}{q}} \| h \|_2
\]

for \( p, q \geq 2 \). Since \( \frac{1}{q} = \frac{2n-1}{2} - \frac{1}{p} \), \( p \neq 2 \), we get for some \( \epsilon > 0 \)

\[
\| \sum_{R \geq 5 \nu \frac{\delta}{2}} \chi_R(r) T_{\Psi_\nu} h \|_{L_t^q L_x^p} \leq C \sum_{R \geq 5 \nu \frac{\delta}{2}} R^{-\epsilon} \| h \|_2 \leq C \| h \|_2.
\]

When \( \Omega(\rho) = \nu^2 \rho^{-\frac{3}{2}} e^{\pm i \rho} \), using (3.23) we obtain the desired bound by direct summation. Indeed, if \( \Omega(\rho) = \nu^2 \rho^{-\frac{3}{2}} e^{\pm i \rho} \), by (3.23)

\[
\| \sum_{R \geq 5 \nu \frac{\delta}{2}} \chi_R(r) T_{\Omega} h \|_{L_t^q L_x^p} \leq C \lambda_0^{-\frac{1}{2} (\frac{1}{2} - \frac{1}{p})} \sum_{R \geq 5 \nu \frac{\delta}{2}} (\nu^2 R^{-1})^{\frac{1}{4} (\frac{1}{2} - \frac{1}{p})} R^{\frac{1}{2} - \frac{2n}{q} (\frac{1}{2} - \frac{1}{p})} \| h \|_2
\]

\[
\leq C (1 + \nu)^\frac{1}{4} (\frac{1}{2} - \frac{1}{p}) \lambda_0^{\frac{1}{2} - \frac{4}{q}} \| h \|_2.
\]

We now handle the case \( \Omega(\rho) = \rho^{-1/2} e^{\pm i \rho} \) which is the main term. By discarding some irrelevant factors it is sufficient to show that

\[
\| \sum_{R \geq 5 \nu \frac{\delta}{2}} S_R h \|_{L_t^q L_x^p} \leq C \lambda_0^{\frac{1}{2} - \frac{1}{4}} \| h \|_2,
\]

where

\[
S_R h(t, r) = R^{-\frac{n-1}{2}} \chi_R(r) \int e^{i(-tw(\rho) \pm r \rho)} \beta(\rho) h(\rho) d\rho
\]
and \( \varpi \) satisfies \((3.1)\). By duality it is equivalent with

\[
\| \sum_{R \geq 5^{\frac{n-1}{2}}} S_R^* H \|_2 \leq C \lambda_0^{\frac{1}{2p} - \frac{1}{q}} \| H \|_{L_t^q \mathcal{C}^p}.
\]

Here \( S_R^* \) is the adjoint of \( S_R \). Hence it suffices to show that

\[
\| \sum_{R, R' \geq 5^{\frac{n-1}{2}}} S_RS_{R'}^* H \|_{L_t^q \mathcal{C}^p} \leq C \lambda_0^{\frac{1}{2p} - \frac{1}{q}} \| H \|_{L_t^q \mathcal{C}^p}
\]

provided that \( \frac{1}{q} = \frac{2n-1}{2}(\frac{1}{2} - \frac{1}{p}), \ q \neq 2, \ p \neq 2 \).

From \((3.20)\) in Remark \(1\) we have

\[
\|S_R h\|_{L_t^q \mathcal{C}^p} \leq C \lambda_0^{\frac{1}{2p} - \frac{1}{q}} R^{\frac{1}{2} - \frac{2n-1}{2}(\frac{1}{2} - \frac{1}{p})} \|h\|_2
\]

provided that \( 2 \leq p, q \leq \infty, \nu \geq 0, \) and \( 2/q \geq 1/2 - 1/p \). By duality we have for \( 2 \leq p, q \leq \infty \) and \( 2/q \geq 1/2 - 1/p \)

\[
\|S_R^* H\|_2 \leq \lambda_0^{\frac{1}{2p} - \frac{1}{q}} R^{\frac{1}{2} - \frac{2n-1}{2}(\frac{1}{2} - \frac{1}{p})}\|H\|_{L_t^q \mathcal{C}^p}.
\]

Hence it follows that

\[
\|S_RS_{R'}^* H\|_{L_t^q \mathcal{C}^p} \leq C \lambda_0^{\frac{1}{2p} - \frac{1}{q}} R^{\frac{1}{2} - \frac{2n-1}{2}(\frac{1}{2} - \frac{1}{p})} R'^{\frac{1}{2} - \frac{2n-1}{2}(\frac{1}{2} - \frac{1}{p})}\|H\|_{L_t^q \mathcal{C}^p}
\]

provided that \( 2 \leq p, q \leq \infty \) and \( 2/q \geq 1/2 - 1/p \) and \( 2 \leq \tilde{p}, \tilde{q} \leq \infty \) and \( 2/\tilde{q} \geq 1/2 - 1/\tilde{p} \). However to get estimates at the critical line the estimates are still not enough. To get over it, we make an observation which is stated in the following lemma.

**Lemma 4.2.** Let us denote \( \max(R, R') \) by \( R^* \) and \( \min(R, R') \) by \( R_* \). If \( 1 \leq q, \tilde{q} \leq \infty \),

\[
\|S_RS_{R'}^* H\|_{L_t^q \mathcal{C}^p} \leq C \lambda_0^{\frac{1}{2p} - \frac{1}{q}} R^{\frac{1}{2} - \frac{2n-1}{2}(\frac{1}{2} - \frac{1}{p})} R'^{\frac{1}{2} - \frac{2n-1}{2}(\frac{1}{2} - \frac{1}{p})}\|H\|_{L_t^q \mathcal{C}^p}.
\]

**Proof.** By \((4.12)\) we may assume that \( R^* \geq 8R_* \). Note that

\[
S_RS_{R'}^* H(t, r) = \int \int K_{R,R'}(t-s, r, r')\nu^{m-1}H(s, r')dsdr',
\]

where

\[
K_{R,R'}(t, r, r') = (RR')^{-\frac{2n-1}{4}}C_R(r)\tilde{C}_{R'}(r') \int e^{i(2\pi \nu (\rho \pm (r-r')\rho))} \beta^2(\rho) d\rho.
\]

We first break the kernel \( K_{R,R'} \) so that

\[
K_{R,R'}(t, r, r') = K_1(t, r, r') + K_2(t, r, r'),
\]

where
where
\[ K_1(t, r, r') = \chi_{|R^*|/8 \leq |t| \leq 8R^*} K_{R^*}(t, r, r'). \]

Since \(|\varpi'(\rho)| \lesssim 1, |\frac{d}{dr}(-t\varpi(\rho) \pm (r-r')\rho)| \geq C \max(|t|, R^*) \) if \(|t| \leq R^*/8 \) or \(|t| \geq 8R^* \).

Hence by integration by parts (three times) we see that
\[ |K_2(t, r, r')| \leq (R^*)^{-a}(1 + t)^{-(3-a)} \]
for any \(0 \leq a \leq 3\). Hence, the contribution from \(K_2\) is negligible. Therefore for \((4.13)\)

it is sufficient to consider the operator
\[ S_{R,R^*}H(t, r) = \int \int K_1(t - s, r, r')[r^{n-1}H(s, r')]dsdr' \]
instead of \(S_RS_{R^*}^\ast\). Since \(|\varpi''| \gtrsim \lambda_0\) and \(K_1(\cdot, r, r')\) is supported in \([R/8, 8R]\), by the van der Corput lemma it follows that
\[ |K_1(t, r, r')| \leq C(R^*\lambda_0)^{-\frac{1}{2}}. \]

By the standard argument, obviously we may assume that the temporal supports of \(S_{R,R^*}H\), \(H\) are contained in an interval of length \(\sim R^*\), by Hölder’s inequality and the above kernel estimate we have for \(1 \leq q \leq 2 \leq \tilde{q} \leq \infty \)

\[
\|S_{R,R^*}H\|_{L^q_t L^\infty_x} \leq C(R^*)^{\frac{1}{2}+\frac{1}{q}}(RR')^{-\frac{2n-2}{q}}(R^*\lambda_0)^{-\frac{1}{2}}\|H\|_{L^\tilde{q}_t L^1_x}.
\]

Hence we get the desired estimate \((4.13)\). \(\square\)

Now we interpolate \((4.12)\) and \((4.13)\) to improve the estimate \((4.12)\). In particular, taking \(p = \tilde{p} = 2\) in \((4.12)\), we have

\[
\|S_RS_{R^*}^\ast H\|_{L^2_t L^\infty_x} \lesssim CR^{\frac{1}{2}}R^{\frac{1}{2}}\|H\|_{L^\tilde{q}_t L^1_x}
\]

provided that \(2 \leq q, \tilde{q} \leq \infty\). Then we interpolate it with \((4.13)\) to get
\[
(4.14) \quad \|S_RS_{R^*}^\ast H\|_{L^p_t L^\infty_x} \leq C\lambda_0^{\frac{1}{2p'\cdot\frac{1}{p}}} \min\left(\frac{R}{R'}, \frac{R'}{R}\right)^{\epsilon} \times R^{\frac{1}{2}}R^{\frac{1}{2}}R^{\frac{1}{2} - \frac{2n-2}{2p'}}(\frac{1}{2} - \frac{1}{p}) \|H\|_{L^\tilde{q}_t L^1_x}
\]

for some \(\epsilon = \epsilon(p, q, \tilde{q}) > 0\) provided that \(2 < p \leq \infty\) and \(0 \leq \frac{1}{q}, \frac{1}{\tilde{q}} < \frac{1}{2} + \frac{1}{2p}\). Clearly we may assume that \(\epsilon\) continuously depends on \(\frac{1}{p}, \frac{1}{q}, \frac{1}{\tilde{q}}\). So, if \(\Delta\) is a compact subset of \(\{(\frac{1}{p'}, \frac{1}{q'}, \frac{1}{\tilde{q}}'): \frac{1}{2} > \frac{1}{p} \geq 0, \ 0 \leq \frac{1}{q}, \frac{1}{\tilde{q}} < \frac{1}{2} + \frac{1}{2p}\}\), there is a uniform lower bound \(\epsilon_0 = \epsilon_0(\Delta)\) such that \(\epsilon(1/p, 1/q, \tilde{q}) \geq \epsilon_0 > 0\) if \((1/p, 1/q, \tilde{q}) \in \Delta\).
4.3.3. Endpoint estimates for $2 < p < \frac{2n}{n-1}$. We firstly show (4.10) for $p < \frac{2n}{n-1}$. The remaining case will be handled differently.

Fix $p, q$ such that $\frac{1}{q} = \frac{2n-1}{2}(\frac{1}{2} - \frac{1}{p})$, $2 < p < \frac{2n}{n-1}$. We write

$$\sum_{R, R' \geq 50^k} S_RS^*_R H = \sum_{k=\infty}^{\infty} \left( \sum_{R, R' \geq 50^k : \frac{R'}{R} = 2^k} S_RS^*_R H \right).$$

Then by (4.14) each of summand in the inner summation satisfies

$$\|S_R S^*_R H\|_{L^q_t L^p_x} \leq C \lambda_0^{-\frac{1}{2p} - \frac{1}{2q}} (R'2^k/2)^{(\frac{1}{2} + \frac{1}{2} - \frac{2n-1}{2} - \frac{2}{p})} 2^{-\epsilon[k]} \|H\|_{L^2_t L^2_x}$$

for some $\epsilon > 0$ if $0 \leq \frac{1}{s}, \frac{1}{s'} < \frac{1}{4} + \frac{1}{2p}$.

We now use a summation argument due to Bourgain [4] (also see [6] for a generalization.) For reader’s convenience we state a version which we need here (see [24] for a simple proof.)

**Lemma 4.3.** Let $\varepsilon_1, \varepsilon_2 > 0$. Let $A, B$ be Banach spaces and $1 \leq r_1, r_2, s_1, s_2 < \infty$. Suppose that $\{T_j\}_{j=\infty}^{-\infty}$ be a collection of operators satisfying that $\|T_j F\|_{L^{r_1}(B)} \leq C M_1 2^{\varepsilon_1 j} F\|_{L^{r_1}(A)}$ and $\|T_j F\|_{L^{r_2}(B)} \leq C M_2 2^{-\varepsilon_2 j} F\|_{L^{r_2}(A)}$. Then

$$\|\sum T_j F\|_{L^{r, \infty}(B)} \leq C M_1^\theta M_2^{1-\theta} F\|_{L^{r, \infty}(A)},$$

where $\theta = \varepsilon_2/(\varepsilon_1 + \varepsilon_2)$, $1/r = \theta/r_1 + (1-\theta)/r_2$ and $1/s = \theta/s_1 + (1-\theta)/s_2$. Here $L^{r, \alpha}$ denotes the Lorentz space.

Let us denote

$$I_p = \left\{ \left( \frac{1}{s}, \frac{1}{s'} \right) : \frac{1}{s} + \frac{1}{s'} = \frac{2n-1}{2}(1 - \frac{2}{p}), \ 0 \leq \frac{1}{s}, \frac{1}{s'} < \frac{1}{4} + \frac{1}{2p} \right\}.$$

The open line segment $I_p$ is not empty as long as $\frac{2n-1}{2}(1 - \frac{2}{p}) < \frac{1}{2} + \frac{1}{p}$ (equivalently $p < \frac{2n}{n-1}$). By Lemma 4.3 we get for $2 < p < \frac{2n}{n-1}$

$$\|\sum_{\frac{R}{R'} = 2^k} S_RS^*_R H\|_{L^2_t L^2_x} \leq C 2^{-\epsilon[k]} \|H\|_{L^{r, \infty}_t L^{s, \infty}_x}$$

provided $(\frac{1}{s}, \frac{1}{s'}) \in I_p$. Since $\frac{1}{s'} < 2 < s$, by real interpolation among the estimates (4.15) for $(\frac{1}{s}, \frac{1}{s'}) \in I_p$ they can be strengthened to strong type. Hence we have that if $2 < p < \frac{2n}{n-1}$ and $(\frac{1}{s}, \frac{1}{s'}) \in I_p$, then

$$\|\sum_{\frac{R}{R'} = 2^k} S_RS^*_R H\|_{L^q_t L^p_x} \leq C \lambda_0^{-\frac{1}{2p} - \frac{1}{2q}} 2^{-\epsilon[k]} \|H\|_{L^2_t L^2_x}.$$
So for $2 < p < \frac{2n}{n-1}$ and $(\frac{1}{s}, \frac{1}{q}) \in I_p$ we get
\[
\left\| \sum_{k=-\infty}^{\infty} \left( \sum_{R, R' \geq 5^{\frac{k}{\theta}}; \frac{R}{R'} = 2^k} S_R S_{R'}^* H \right) \right\|_{L_t^p L_{x}^q} \leq C \lambda_0^{\frac{1}{2p} \frac{1}{2q}} \sum_{k=-\infty}^{\infty} 2^{-|k|} \|H\|_{L_t^p L_{x}^q} \leq C \lambda_0^{\frac{1}{2p} \frac{1}{2q}} \|H\|_{L_t^p L_{x}^q}.
\]

In particular if we take $s = \tilde{s} (= q)$, we get the desired estimate (4.10) for $p < \frac{2n}{n-1}$.

4.3.4. **Endpoint estimates for $\frac{2n}{n-1} \leq p < \frac{2(2n-1)}{2n-3}$**. After squaring the left hand side of (4.9), we rearrange it so that
\[
\sum_{R, R' \geq 5^{\frac{k}{\theta}}; \frac{R}{R'} = 2^k} \langle S_R^* H, S_{R'}^* H \rangle = \sum_{k=-\infty}^{\infty} \left( \sum_{R, R' \geq 5^{\frac{k}{\theta}}; \frac{R}{R'} = 2^k} \langle S_R^* H, S_{R'}^* H \rangle \right).
\]

Hence the desired estimate (4.9) follows if we show that for $2 < p < \frac{2(2n-1)}{2n-3}$
\begin{equation}
(4.17) \quad \left| \sum_{R, R' \geq 5^{\frac{k}{\theta}}; \frac{R}{R'} = 2^k} \langle S_R^* H, S_{R'}^* G \rangle \right| \leq C 2^{-|k|} \lambda_0^{\frac{1}{2p} \frac{1}{2q}} \|H\|_{L_t^p L_{x}^q} \|G\|_{L_t^q L_{x}^q}.
\end{equation}

From (4.16) we already established this inequality for $2 < p < \frac{2n}{n-1}$. (Also the estimate without extra $2^{-|k|}$ factor is not difficult to obtain by repeating the above argument if $p < \frac{4n-2}{2n-3}$.) To get this for $\frac{2n}{n-1} \leq p < \frac{2(2n-1)}{2n-3}$, it is sufficient to show that
\begin{equation}
(4.18) \quad \left| \sum_{R, R' \geq 5^{\frac{k}{\theta}}; \frac{R}{R'} = 2^k} \langle S_R^* H, S_{R'}^* G \rangle \right| \leq C \lambda_0^{\frac{2n-2}{2n-3} \frac{1}{2p}} \|H\|_{L_t^p L_{x}^{\frac{4n+2}{2n+4}}} \|G\|_{L_t^q L_{x}^{\frac{4n+2}{2n+4}}}.
\end{equation}

Then interpolating this with (4.17) for $2 < p < \frac{2n}{n-1}$ we get (4.17) for $2 < p < \frac{4n-2}{2n-3}$.

To show (4.18) we adopt bilinear interpolation argument which was used to show the endpoint Strichartz estimate [20]. Let us denote by $\ell^*_r$ the pace of sequences $\{Z_R\}_{R \text{ dyadic}}$ with norm
\[
\|\{Z_R\}\|_{\ell^*_r} = \left\{ \begin{array}{ll}
\left( \sum_{R \text{ dyadic}} |R^s Z_R|^r \right)^{\frac{1}{r}} & \text{if } r \neq \infty, \\
\sup_{R \text{ dyadic}} |R^s Z_R| & \text{if } r = \infty.
\end{array} \right.
\]

We will use the fact (see Theorem 5.6.2 in [3]) that if $0 < q_0, q_1 \leq \infty$ and $s_0 \neq s_1$, then for $q \leq \infty$,
\begin{equation}
(4.19) \quad \left( \ell^s_{q_0}, \ell^s_{q_1} \right)_{\theta, q} = \ell^s_q;
\end{equation}
where \( s = (1-\theta)s_0 + \theta s_1 \). And we also recall the following fact from real interpolation (see [34], section 1.18.4). Let \( A_0, A_1 \) be Banach spaces. If \( 1 \leq p_0, p_1 < \infty, 0 < \theta < 1 \), and \( \frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1} \), then

\[
\left( L^{p_0}(A_0), L^{p_1}(A_1) \right)_{\theta, p} = L^p((A_0, A_1)_{\theta, p}).
\]

Here \((A_0, A_1)_{\theta, p}\) denotes the real interpolation space.

We consider the bilinear operator which is defined by

\[
B_k(H, G)_{R'} = \begin{cases} 
\langle S_{r}^* H, S_{r}^* G \rangle & \text{if } R = 2^k R', R, R' \geq 5^\nu R, \\
0 & \text{otherwise.}
\end{cases}
\]

By (4.12) and duality, taking \( q = \tilde{q} = 2 \) particularly, we have for \( 2 \leq p, \tilde{p} \leq \infty \)

\[
|\langle S_{r}^* H, S_{r}^* G \rangle| \lesssim C\lambda_0^{\frac{1}{Rp} - \frac{1}{\tilde{q}p}} R^{\frac{2n-1}{p} - \frac{2n-1}{\tilde{q}} - \frac{2n-3}{p} - \frac{2n-3}{\tilde{q}}} \|H\|_{L^2_{r} L^p} \|G\|_{L^2_{r} L^\tilde{p}}.
\]

We rewrite it as

\[
\langle S_{2^k R}^* H, S_{2^k R}^* G \rangle \lesssim C\lambda_0^{\frac{1}{Rp} - \frac{1}{\tilde{q}p}} 2^{k(\frac{2n-1}{p} - \frac{2n-1}{\tilde{q}}) - \frac{2n-3}{p} - \frac{2n-3}{\tilde{q}}} \|H\|_{L^2_{r} L^p} \|G\|_{L^2_{r} L^\tilde{p}}.
\]

which is valid for \( 2 \leq p, \tilde{p} \leq \infty \). Let us set

\[
\beta(p, \tilde{p}) = \frac{2n-3}{2} - \frac{2n-1}{2} \left( \frac{1}{p} + \frac{1}{\tilde{p}} \right).
\]

Then (4.21) implies that for \( 2 \leq p, \tilde{p} \leq \infty \)

\[
B_k : L^2_{r} L^p \times L^2_{r} L^{\tilde{p}} \to \ell^\beta(p, \tilde{p})
\]

is bounded with bound \( C\lambda_0^{\frac{1}{Rp} - \frac{1}{\tilde{q}p}} 2^{k(\frac{2n-1}{p} - \frac{2n-1}{\tilde{q}} - \frac{2n-3}{p} - \frac{2n-3}{\tilde{q}})} \).

Now we apply the following interpolation lemma. See [3] (exercise 5(a) in section 3.13) or [34] (section 1.19.5).

**Lemma 4.4.** Let \( A_0, A_1, B_0, B_1, C_0, \) and \( C_1 \) be Banach spaces, and \( T \) be a bilinear operator such that \( T : A_i \times B_i \to C_i \) with norm \( M_i \) for \( i = 0, 1 \). If \( 0 < \theta < 1 \), \( 1 \leq r \leq \infty \), and \( \frac{1}{a} = \frac{1}{b} + \frac{1}{c} - 1 \), then

\[
T : (A_0, A_1)_{\theta, c} \times (B_0, B_1)_{\theta, b} \to (C_0, C_1)_{\theta, a}
\]

is bounded with norm \( M_0^{1-\theta} M_1^\theta \).

Choosing \((p_0, \tilde{p}_0), (p_1, \tilde{p}_1)\) \in [2, \infty]^2\) such that \( \beta(p_0, \tilde{p}_0) \neq \beta(p_1, \tilde{p}_1) \), we apply Lemma 4.4 to the estimates (4.22) with \((p, \tilde{p}) = (p_0, \tilde{p}_0), (p_1, \tilde{p}_1)\). Then it follows that if \( 0 < \theta < 1 \)

\[
B_k : (L^2_{r} L^p, L^2_{r} L^{\tilde{p}})_{\theta, 1} \times (L^2_{r} L^p, L^2_{r} L^{\tilde{p}})_{\theta, 1} \to \ell^\beta(p_0, \tilde{p}_0), \ell^\beta(p_1, \tilde{p}_1)_{\theta, 1}
\]
with bound $C\lambda_0^{\frac{1}{2}\left(\frac{n-2}{2p}+\frac{n}{p_0}\right)-\frac{1}{2}\left(\frac{n-2}{2p}+\frac{n}{p_1}\right)}2^{k\left(\frac{2n-1}{2p}-\frac{2n-3}{2}\right)}$. Here $\mathcal{L}_{r}^{p,r}$ is the Lorentz space defined with measure $r^{-n-1}dr$. Since there are plenty of choices of $(p_0, \tilde{p}_0)$ and $(p_1, \tilde{p}_1)$, by a proper choice $(p_0, \tilde{p}_0)$, and by (4.19), (4.20), we have for $2 < p, \tilde{p} < \infty$

$$B_k : L^2_1 \mathcal{Q}_{r}^{p,1} \times L^2_1 \mathcal{Q}_{r}^{\tilde{p},1} \to \ell_1$$

$C\lambda_0^{\frac{1}{2}\left(\frac{n-2}{2p}+\frac{n}{p_0}\right)-\frac{1}{2}\left(\frac{n-2}{2p}+\frac{n}{p_1}\right)}2^{k\left(\frac{2n-1}{2p}-\frac{2n-3}{2}\right)}$. In particular, for $2 < p, \tilde{p} < \infty$ and $\frac{2n-3}{2} - \frac{2n-1}{2}(\frac{1}{p} + \frac{1}{\tilde{p}}) = 0$ ($\beta(p, \tilde{p}) = 0$) we have

$$| \sum_{R, R' \geq 5^{n+1}2^{k}} \langle S_R^* H, S_{R'}^* G \rangle | \leq C\lambda_0^{\frac{1}{2p}}2^{k\left(\frac{2n-1}{2p}-\frac{2n-3}{2}\right)}\| H \|_{L^2_1 \mathcal{Q}_{r}^{p,2}}\| G \|_{L^2_1 \mathcal{Q}_{r}^{\tilde{p},2}}.$$  

Now we can interpolate these estimate by applying Lemma 4.4 again to the bilinear operator

$$(H, G) \to \sum_{R, R' \geq 5^{n+1}2^{k}} \langle S_R^* H, S_{R'}^* G \rangle.$$  

This time we choose $a = \infty$ and $b = c = 2$ and use (4.20) to get

$$| \sum_{R, R' \geq 5^{n+1}2^{k}} \langle S_R^* H, S_{R'}^* G \rangle | \leq C\lambda_0^{\frac{1}{2p}}2^{k\left(\frac{2n-1}{2p}-\frac{2n-3}{2}\right)}\| H \|_{L^2_1 \mathcal{Q}_{r}^{p,2}}\| G \|_{L^2_1 \mathcal{Q}_{r}^{\tilde{p},2}}.$$  

provided that $2 < p, \tilde{p} < \infty$ and $\frac{2n-3}{2} - \frac{2n-1}{2}(\frac{1}{p} + \frac{1}{\tilde{p}}) = 0$. Taking $p = \tilde{p} = \frac{2(2n-1)}{2n-3}$, we get the desired (1.18) since $\frac{2(2n-1)}{2n-3} \geq 2$.

### 4.4. Weak type bounds for radial functions for \((p, q) = \left(\frac{2(2n-1)}{2n-3}, 2\right)\): Proof of (1.18)

From (4.23) we see that the bilinear operator

$$(H, G) \to \left\{ \sum_{R, R' \geq 5^{n+1}2^{k}} \langle S_R^* H, S_{R'}^* G \rangle \right\}_{k=-\infty}^{\infty}$$

is bounded from $L^2_1 \mathcal{Q}_{r}^{p,2} \times L^2_1 \mathcal{Q}_{r}^{\tilde{p},2}$ to $\ell_1^{\infty}$ with bounds $\lambda_0^{\frac{1}{2p}} - \frac{2n-1}{2p}$ provided that $2 < p, \tilde{p} < \infty$ and $\frac{2n-3}{2} - \frac{2n-1}{2}(\frac{1}{p} + \frac{1}{\tilde{p}}) = 0$. We can interpolate these estimates using Lemma 4.4. Then by (4.19), and (4.20), in particular we get

$$\sum_{k=-\infty}^{\infty} \left| \left( \sum_{R, R' \geq 5^{n+1}2^{k}} \langle S_R^* H, S_{R'}^* G \rangle \right) \right| \leq C\lambda_0^{\frac{2n-3}{2\left(2n-1\right)}}\| H \|_{L^2_1 \mathcal{Q}_{r}^{2n+1,1}}\| G \|_{L^2_1 \mathcal{Q}_{r}^{2n+1,1}}.$$  

By duality it gives the weak type bound

$$\| \sum_{R \geq 5^{n+1}} S_R h \|_{L^2_1 \mathcal{Q}_{r}^{2n-3,3}} \leq C\lambda_0^{\frac{2n-3}{2\left(2n-1\right)}}\| h \|_2.$$
We now proceed to prove (1.8) for radial $\varphi$. From Littlewood-Paley theory and real interpolation we have $\| \sum_N P_N f \|_{L^p_x L^q_t} \leq C \| \sum_N |P_N f|^2 \|_{L^{p'}_x L^{q'}_t}$ for $1 < p < \infty$. If $p > 2$ and $q \geq 2$, using this and Minkowski’s inequality, we see that

$$\| e^{-it\omega(|\nabla|)} \varphi \|_{L_x^q L^2_t} \lesssim \left( \sum_N \| P_N e^{-it\omega(|\nabla|)} \varphi \|_{L^2_x} \right)^{\frac{1}{2}} \lesssim \left( \sum_N \| e^{-it\omega(|\nabla|)} P_N \varphi \|_{L^2_x L^p_t} \right)^{\frac{1}{2}}.$$

Then by scaling (4.3), for (1.8) it is enough to show that

$$\| e^{-it\varphi(\nabla)} P_1 \varphi \|_{L^{\frac{4n-2}{n-1}}_x L^\infty_t} \leq C \lambda_0^\frac{2n-3}{2(n-2)} \| \varphi \|_{L^2}$$

where $\varphi$ and $\lambda_0$ are given by (4.5). Note that $\| f \|_{L^p_x} = \| g \|_{L^q_r}$ when $f(x) = g(|x|)$. Hence by Plancherel’s theorem and Fourier transform of radial function we are reduced to showing that

$$\| T^{\frac{n-2}{2}} h \|_{L^{4n-2}_{x,t} L^\infty} \leq C \lambda_0^\frac{2n-3}{2(n-2)} \| h \|_{L^2}.$$

(cf. Proof of Lemma 4.1). This follows from (4.8) and (4.24). Therefore we get (1.8).

4.5. **Proof of Theorem 1.2**. Theorem 1.2 can be proven similarly as Theorem 1.1. Once we have Lemma 3.4, we can routinely follow the arguments for the proof of Theorem 1.1. The only difference comes from the additional assumption (iv) (see (4.23) and (4.35)) by which we have $\lambda_0 \sim 1$ at (4.5). Hence we do not lose anything when applying Lemma 3.4 even though its bound is not sharp in $\lambda_0$. Then the remaining is almost identical with the proof of Theorem 1.1. We omit the details.

4.6. **Remark for wave equation**. In [30], the estimate (1.3) was proven for $\omega(\rho) = \rho$, $s = n\left(\frac{1}{2} - \frac{1}{p}\right) - \frac{1}{q}$, $\alpha > \frac{2}{q} - (n-1)\left(\frac{1}{2} - \frac{1}{p}\right)$ provided that

$$\frac{n-1}{2} - \frac{1}{2} - \frac{1}{p} \leq \frac{1}{q} < (n-1)\left(\frac{1}{2} - \frac{1}{p}\right).$$

It was shown by Knapp’s example that the estimate fails if $\frac{1}{q} > (n-1)\left(\frac{1}{2} - \frac{1}{p}\right)$. The example in [30] also shows that $\alpha \geq \frac{2}{q} - (n-1)\left(\frac{1}{2} - \frac{1}{p}\right)$ is necessary for (1.3).

Let us set

$$W^\nu h(t, r) = r^{-\frac{n-2}{2}} \int e^{-it\rho} J_\nu(r\rho) \rho^\frac{\alpha}{2} \beta(\rho) h(\rho) d\rho.$$
Similarly as before, it is easy to see that
\[
\|\chi_{\{r<R_0\}} \mathcal{W}^\nu h\|_{L^q_t L^p_x} \leq C\|h\|_2. \quad (4.26)
\]
Hence, by (3.28) it follows that for \(2 \leq p, q \leq \infty\) and \(\frac{1}{q} < (n-1)(\frac{1}{2} - \frac{1}{p})\)

Using Lemma 4.3 it is also possible to get some weak type estimates for \(\mathcal{W}^\nu\) along the sharp line \(\frac{1}{q} = (n-1)(\frac{1}{2} - \frac{1}{p})\) but the strong endpoint estimates are not possible by the method in Section 4.3 because \(\omega'' = 0\). By the argument for the proof of Lemma 4.1 the following is easy to see.

**Lemma 4.5.** Let \(\omega(\rho) = \rho\) and \(2 \leq p, q \leq \infty, \gamma \geq 0\). Suppose that (4.26) holds for \(\nu = \nu(k) = \frac{n-2+2k}{2}, k \geq 0\). Then the solution \(u\) to (1.1) satisfies (1.3) with \(s = n(\frac{1}{2} - \frac{1}{p}) - \frac{1}{q}\) provided that \(\alpha = (n-1)(\frac{1}{2} - \frac{1}{p})\) when \(p > \infty\) and \(\alpha > (n-1)(\frac{1}{2} - \frac{1}{p})\) when \(p = \infty\).

Hence we get (1.3) for \(2 \leq p, q \leq \infty\) and \(\frac{1}{q} < (n-1)(\frac{1}{2} - \frac{1}{p})\) provided \(\sigma \geq (n-1)(\frac{1}{2} - \frac{1}{p})\). Now note \((n-1)(\frac{1}{2} - \frac{1}{p}) = \frac{2}{q} - (n-1)(\frac{1}{2} - \frac{1}{p})\) if \(\frac{1}{q} = (n-1)(\frac{1}{2} - \frac{1}{p})\). Hence interpolating these estimates with the usual Strichartz estimates for the wave equation (along the sharp line \(\frac{1}{q} = \frac{n-2+2k}{2}(\frac{1}{2} - \frac{1}{p})\)) recovers the aforementioned results in [30]. This also shows that if one obtains (4.20) on the sharp line \(\frac{1}{q} = (n-1)(\frac{1}{2} - \frac{1}{p})\), then the optimal angular regularity \(\alpha = \frac{2}{q} - (n-1)(\frac{1}{2} - \frac{1}{p})\) for (1.3) also follows.

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