Non-equilibrium Ising Model on a 2D Additive Small-World Network

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In this work, we have studied the Ising model with one- and two-spin flip competiting dynamics on a two-dimensional additive small-world network (A-SWN). The system model consists of a $L \times L$ square lattice where each site of the lattice is occupied by a spin variable that interacts with the nearest neighbor spins and it has a certain probability $p$ of being additionally connected at random to one of its farther neighbors. The dynamics present in the system can be defined by the probability $q$ of being in contact with a heat bath at a given temperature $T$ and, at the same time, with a probability of $1-q$ the system is subjected to an external flux of energy into the system. The contact with the heat bath is simulated by one-spin flip according to the Metropolis prescription, while the input of energy is mimicked by the two-spin flip process, involving a simultaneous flipping of a pair of neighboring spins. We have employed Monte Carlo simulations to obtain the thermodynamic quantities of the system, such as, the total $m^g_T$ and staggered $m^A_T$ magnetizations per spin, the susceptibility $\chi_L$, and the reduced fourth-order Binder cumulant $U_4$. We have built the phase diagram for the stationary states of the model in the plane $T$ versus $q$, showing the existence of two continuous transition lines for each value of $p$: one line between the ferromagnetic $F$ and paramagnetic $P$ phases, and the other line between the $P$ and antiferromagnetic $AF$ phases. Therefore, we have shown that the phase diagram topology changes when $p$ increases. Using the finite-size scaling analysis, we also obtained the critical exponents for the system, where varying the parameter $p$, we have observed a different universality class from the Ising model in the regular square lattice to the A-SWN.

A. Introduction

In the 1960s, the dynamic behavior of the Ising model was successfully described by Glauber [1] and Kawasaki [2] mechanisms. This instigated interest in the competition between the Glauber and Kawasaki stochastic processes, one-spin flip and two-spin exchange, respectively, in this model. This competition can be simulated by the Glauber process with probability $q$ simulating the system in contact with a heat bath at a temperature $T$, and at the same time, with probability $1-q$, the Kawasaki process mimics an input of energy into the system. Each of these dynamical processes singly satisfies the detailed balance condition, which drives the system toward equilibrium. However, when both act simultaneously, the detailed balance is no longer satisfied and the system is forced out of equilibrium.

The Ising model on a regular square lattice has the critical temperature and universality class are given by the critical exponents well-known exactly at the equilibrium state [3]. Therefore, the stationary non-equilibrium states were obtained by the two competing dynamic processes described above, and a self-organization is observed by the disappearance of the ordered ferromagnetic $F$ phase in the transition to the paramagnetic $P$ phase, and identification of the ordered antiferromagnetic $AF$ phase, as we increase the flow of energy into the system [3]. However, through the Monte Carlo simulations (MC) the critical exponents of the system have been obtained, and because it is a system with the same symmetry, spatial dimension, and range interactions, the exponents are the same as at the equilibrium state model, and known exactly [3]. In the same way, Godoy and Figueredo investigated the mixed-spin Ising model, which does not admit spin exchanges between the spin sublattices, consequently do not admit to utilizing the Kawasaki dynamic. Thus, the competing dynamic was made by the one- and two-spin flip mechanisms, and even with that, they have also obtained the self-organization phenomena [7], and based on the critical behavior of the system, the universality class of the system is the same that the Ising model with only spin-$1/2$ [6]. Therefore, in the non-equilibrium models, the universality class of the stationary critical behavior is the same as in the equilibrium models. All of these works were studied on regular square lattices.

By using graph theory, Watts and Strogatz quantify the properties of Small-World phenomena as demonstrated in Milgram’s 1967 study [6]. As an underlying assumption of the Watts-Strogatz model (WS-model) [10], vertices of graphs are sites of networks, and edges are connections between sites of the networks. By introducing a disorder parameter $p$, as the probability of randomly rewriting each one of the connections in a regular lattice, we can obtain the SWN in specific regions in the interval $0 < p \leq 1$. The SWN regime is identified in regions of $p$ where the network possesses local clustering, $C(p)$, of a regular lattice, but at the same time has an average distance between any two sites, $l(p)$, characteristic of a random lattice. In addition to the WS-model, some variants of this model were also developed to describe the properties of an SWN. One of these variants [11] uses a regular square lattice, and we can add a long-range interaction to each site with a certain probability $p$. This leads to a small typical separation, preserving the clustering property of a regular lattice. While we have described the rewiring SWN (R-SWN) in the WS-model [10], this last form is known as additive SWN (A-SWN) [11].

These networks have been used in numerous physical models since the initial SWN model was put forth [12–14], including the Ising model in 1D, 2D, and 3D for...
Figure 1. Schematic representation of the system and the A-SWN. The blue square dots indicate the sites on one of the sublattices, the white square dots are the sites on the other sublattice, the solid black lines are the nearest-neighbor interactions $J$ between pairs of spins, and the blue-dashed lines are long-range interaction $J_{ik}$ added to the network with a certain probability $p$. (a) for $p = 0$, (b) for $p = 0.5$ and (c) for $p = 1$.

the investigation of the critical phenomena at equilibrium system $^{13,24}$. According to these findings for the Ising model, an order to the disorder phase transition is established for $T \neq 0$ with $0 < p \leq 1$, and it is seen that the addition of long-range interactions changes the critical behavior of the system.

The interesting behavior of the Ising model at the equilibrium SWN, its investigation was also carried out about the non-equilibrium phase transitions by the competing dynamics: analytically in 1D $^{25}$, by MC simulations in 2D $^{26}$, and by the Gaussian model in 3D $^{27}$. In all of these works they have been using the competition between the Glauber and Kawasaki dynamics, and have no conclusions about the mean-field critical behavior observed at the equilibrium Ising model on an SWN $^{18,20–22,24,28}$. However, in 2D and 3D systems, is obtained the $AF - P$ and $F - P$ phase transitions, characteristic of the self-organization phenomena, and observed in all of the other systems at the non-equilibrium state by the competing dynamics.

In the present work, we have investigated the Ising model in a two-dimensional A-SWN, where each site of the network is occupied by a spin variable spin-1/2 that can assume values $\pm 1$. We limit by one the number of long-range interactions that each site can receive with probability $p$, and divide the network into two sublattices, each new interaction created should connect these sublattices. The system is in a non-equilibrium regime by competing between two dynamic processes that do not conserve the order parameter: with competition probability $q$, the one-spin flip process simulates the system in contact with a heat bath at temperature $T$, and with competition probability $1 - q$ the two-spin flip process mimics the system subjected to an external energy flux into it. Therefore, the system is studied at the non-equilibrium regime due to competing dynamics. We verified the phase transition between the $AF$ and $F$ ordered phases to the $P$ disordered phase, and if the system is in this A-SWN regime, it exhibit the same mean-field critical behavior observed at equilibrium systems with long-range interactions by the A-SWN, see Ref. $^{28}$. The behavior of the phase transitions, phases diagrams and critical exponents by FSS analysis also are described and compared with those of Ref. $^{28}$.

This work is organized as follows: In Section $B$, we describe the model, the network, and the motion equations for the non-equilibrium Ising model. In Section $C$, we present the MC simulation method used. The behavior of the phase transitions, phase diagrams, and critical exponents by FSS analysis is described in Section $D$. Finally, in Section $E$ we present our conclusions.

B. Model

The Ising model with $N = L^2$ spins $\sigma_i = \pm 1$ on a regular square lattice $L \times L$, periodic boundary conditions, and a nearest-neighbor ferromagnetic interaction of strength $J$ has been studied in this work (see Fig. $\Pi a$)). On the other hand, with a certain probability $p$, we can add one long-range interaction $J_{ik}$ to each site of that regular square lattice. We divided the system into two sublattices to add the long-range interactions $J_{ik}$, in which one sublattice plays the role of central spins, while the other sublattice contains the spins in which the central spins can connect, to beyond their nearest neighbors. Thus, to choose a long-range interaction $J_{ik}$ for a site $i$, the sublattice of $i$ will be the sublattice of the central spins, then, we choose randomly a site $k$ from another sublattice. If the site $k$ does not be one of its nearest neighbors already naturally coupled with $i$, we picked a random number $0 < r < 1$, and if $r \leq p$ (with $p$ predefined), then we couple the site $k$ to the neighbors of site $i$, and for the site $k$ we couple the site $i$ to its neighbors. The attempt to add a long-range interaction $J_{ik}$ is made once to each site that does not have a long-range interaction $J_{ik}$ in the network, and as result, we have a network with an average coordination number $z = 4 + p$. Therefore, we can think as an example of some situations: i) for $p = 0$, i.e., the probability of adding a long-range interaction $J_{ik}$ to any site on the lattice is zero, therefore, we have a regular square lattice, see Fig. $\Pi a$); ii) for $p = 0.5$, we are in the A-SWN regime because in addition to the conservation of $C(p)$, and we also have an average short
path length between network sites, through the shortcuts created by the long-range interaction $J_{ik}$ added between the sublattices, see Fig. 1(b); finally, for $p = 1$, all sites on the network have a long-range interaction $J_{ik}$ connecting the two sublattices, and consequently, it is the network with the shortest typical separation between the sites on the network, see Fig. 1(c).

Thus, as the regular structure in $p = 0$ keeps unaltered (Fig. 1(a)), we have a high local clustering for any value of $p$, and conform we increase $p$, the long-range interaction $J_{ik}$ is added to the network, creating shortcuts between the sites that before in the simple regular lattice would be more distant, consequently decreasing the typical distance $l(p)$ of the network. The $l(p)$ scales linearly $l(p \rightarrow 0) \sim L/2$ and logarithmically $l(p \rightarrow 1) \sim \ln(L^{1.77})$, these being regimes referred to as the “large-world” and “small-world” respectively. The cross-over between these regimes occurs when the average number of shortcuts is about one, or in other words, we can say in the SWN regime when $p \geq 2L^{-2}$. Versed on this, our study is based on $p \geq 0.25$ values, where the A-SWN is found and the decay of $l$ as a function of $p$ undergoes less, i.e., having approximately the same value of $l$.

The ferromagnetic Ising spin energy is described by the Hamiltonian of the form:

$$
H = -J \sum_{(i,j)} \sigma_i \sigma_j - \sum_{(i,k)} J_{ik} \sigma_i \sigma_k,
$$

where $J$ is the nearest-neighbor ferromagnetic interaction, and $J_{ik}$ is the long-range interaction on the A-SWN. The first sum is over all the pair of nearest-neighbor spins on the regular square lattice and the second sum is made over all the pairs of spins $(i,k)$ connected through long-range interaction on the A-SWN. Here, we always are considering $J_{ik} = J = 1$.

We are dealing with the non-equilibrium Ising model and in an SWN, being the time evolution of the states of the system governed by two competing dynamical processes: one simulating the contact of the system with a heat bath at temperature $T$, with the one-spin flip process and probability $q$ to occur, and at the same time but with probability $(1-q)$ to occur, the system is subjected to an external flux of energy into the system with the competition it is possible to find stationary states for the order parameter in the $AF$, $F$, and $P$ phases. It is worth noting that to reach the stationary state in the AF phase was of fundamental importance to use the $J_{ik}$ between the sublattices, because of the antiparallel ordering in which this phase is characterized.

C. Monte Carlo simulations

Let $(k,l)$ and $(k',l')$ be the coordinates of a site in our two-dimensional SWN and one of your neighbors respectively. The periodic boundary conditions were used in all our simulations. Starting the initial state of the system with all spins aligned in the same direction, a new configuration is generated by the following the Markov process: for a given temperature $T$, competition probability $q$, and additive probability $p$, we choose a random spin from the lattice, i.e., we choose a coordinate $k$ and $l$ at random. Then we generate a random number $\xi$ between zero and one, and if $\xi \leq q$ we choose the one-spin flip process.

In this process, the flipping probability is dependent on $W(\sigma_{kl} \rightarrow \sigma'_{kl})$, which is given by the Metropolis prescription as follows:

$$
W(\sigma_{kl} \rightarrow \sigma'_{kl}) = \left\{ \begin{array}{ll}
\exp(-\Delta E_{kl}/k_B T) & \text{if } \Delta E_{kl} > 0 \\
1 & \text{if } \Delta E_{kl} \leq 0
\end{array} \right.,
$$

where $\Delta E_{kl}$ is the change in the energy after flipping the spin $\sigma_{kl} \rightarrow \sigma'_{kl}$, $k_B$ is the Boltzmann constant, and $T$ is the absolute temperature, thus, the new state is accepted if $\Delta E_{kl} \leq 0$, and in the case of $\Delta E_{kl} > 0$ we choose another random number $1 < \xi_1 < 0$ and if $\xi_1 \leq \exp(-\Delta E_{kl}/k_B T)$ the new state is also accepted, but if none of the conditions are satisfied, we do not change the state of the system. On the other hand, if

the process independent of the temperature, where the energy of the system increases by one external flow of energy into it. $G$ and $V$ are described by

$$
G = \sum_{i} \{W(\sigma_i \rightarrow \sigma'_i)p(\{\sigma\}, t) + W(\sigma'_i \rightarrow \sigma_i)p(\{\sigma'\}, t)\},
$$

$$
V = \sum_{i,j} \{W(\sigma_i \sigma_j \rightarrow \sigma'_i \sigma'_j)p(\{\sigma\}, t) + W(\sigma'_i \sigma'_j \rightarrow \sigma_i \sigma_j)p(\{\sigma'\}, t)\},
$$

where $\{\sigma'\}$ denotes the spin configurations after the spin flipping, $W(\sigma_i \rightarrow \sigma'_i)$ is the transition rate between states in the one-spin flip process, and $W(\sigma_i \sigma_j \rightarrow \sigma'_i \sigma'_j)$ the transition rate between the states in the two-spin flip process, with the order parameter being conserved in none of the dynamic processes.

If $0 < q < 1$, we have two dynamics processes acting simultaneously, the detailed balance is not satisfied and the system is forced out of equilibrium. As these processes favor the states of higher and lower energy of the system, with the competition it is possible to find stationary states for the order parameter in the AF, $F$, and $P$ phases. It is worth noting that to reach the stationary state in the AF phase was of fundamental importance to use the $J_{ik}$ between the sublattices, because of the antiparallel ordering in which this phase is characterized.
ξ > q, the two-spin flip process is chosen. In this case, in addition to the spin σ_{kl}, we also randomly choose one of its neighbors σ_{k′l′}, which can be either the nearest neighbor or the farthest neighbor coming from a j_{kl}. In this process, the two spins chosen are flipping simultaneously, and for that, the two-spin flip probability is dependent on \( W(σ_{kl}σ_{k′l′} → σ′_{kl}σ′_{k′l′}) \), which is given by

\[
W(σ_{kl}σ_{k′l′} → σ′_{kl}σ′_{k′l′}) = \begin{cases} 0 & \text{if } ΔE_{kl,k′l′} ≤ 0 \\ 1 & \text{if } ΔE_{kl,k′l′} > 0 \end{cases},
\]

where \( ΔE_{kl,k′l′} \) is the change in the energy after flipping the spins \( σ_{kl} \) and \( σ_{k′l′} \). Thus, in this process, the new state is just accepted if \( ΔE_{kl,k′l′} > 0 \).

Repeating the Markov process \( N \) times, we have one Monte Carlo Step (MCS). In our simulations, for \( p ≠ 0 \), we have waited for \( 2 × 10^4 \) MCS for the system to reach the stationary state, for all the lattice sizes. We used \( 5 × 10^3 \) MCS to calculate the thermal averages of the quantities of interest. The average over the samples was done using 25 independent samples for any lattice. On the other hand, for the case \( p = 0 \), we needed to wait for \( 5 × 10^3 \) MCS to reach the equilibrium state, and after \( 3 × 10^5 \) MCS to calculate the thermal average, only over one sample.

The measured thermodynamic quantities in our simulations are: the total magnetization per spin \( m^F \), the staggered magnetization per spin \( m^F_{AF} \), the magnetic susceptibility \( χ_L \) and the reduced fourth-order Binder cumulant \( U_L \):

\[
m^F_L = \frac{1}{N}\left[\langle \sum_{kl} σ_{kl} \rangle\right], \tag{7}
\]

\[
m^F_{AF} = \frac{1}{N}\left[\langle \sum_{kl} (-1)^{(k+l)} σ_{kl} \rangle\right], \tag{8}
\]

\[
χ_L = \frac{N}{k_BT}\left[⟨m^2⟩ - ⟨m⟩^2\right], \tag{9}
\]

\[
U_L = 1 - \frac{[⟨m^4⟩]}{3[⟨m^2⟩]^2}, \tag{10}
\]

where \([\ldots]\) denotes the average over the samples, \( ⟨\ldots⟩ \) is the thermal average over the MCS in the stationary state, and \( m \) can be \( m^F \) or \( m^F_{AF} \) in Eq. \( \text{9} \) and \( \text{10} \), respectively. The lattice sizes from \( L = 24 \) to \( L = 256 \) are simulated and the data are analyzed via finite-size scaling theory (FSS). These Eqs. \( \text{7}, \text{8}, \text{9} \) and \( \text{10} \) obey the following FSS relations in the neighborhood of the stationary critical point \( \lambda_C \):

\[
m = L^{-β/ν}m_0(L^{1/ν}ε), \tag{11}
\]

\[
χ_L = L^{γ/ν}χ_0(L^{1/ν}ε), \tag{12}
\]

\[
U_L = U_0(L^{1/ν}ε), \tag{13}
\]

where \( ε = (λ_0 − λ_C)/λ_C \), \( λ \) can be \( T \) or \( q \). Here \( m_0, \chi_0 \) and \( U_0 \) are scaling functions, where \( β, γ, \) and \( ν \) are the critical exponents related to magnetization, susceptibility, and the length correlation, respectively. The derivative of Eq. \( \text{13} \) with respect to the parameter \( λ \) gives us the following scaling relation:

\[
U'_{L} = \frac{L^{1/ν}}{λ_C}U'_0(L^{1/ν}ε). \tag{14}
\]

We have determined the critical exponent relations \( β/ν, γ/ν \) and \( ν \) from slope of a log-log plot of \( m^F_0(λ_C), χ_L(λ_C) \) or \( U'_0(λ_C) \) versus lattice size \( L \) respectively. We also have used another alternative method to estimate the values of the critical exponents, the data collapse from the scaling functions.

D. Results and Discussions

In this section, we illustrate and discuss the results of the magnetic properties of the Ising model on a 2D AF-SWN at the non-equilibrium regime by the two competing dynamics. For the study about the critical behavior and phase transitions at the non-equilibrium system, it was convenient to fix the temperature \( T \), additive probability \( p \), and to use the competition parameter \( q \) as a variable to transit between the ordered to disordered phases in the regions of \( T \) and \( p \) of the phase diagram. It is convenient because the two-spin flip mechanism is independent of the temperature \( T \), and in the present work we do not have used \( p \) as a variable to identify the phase transitions.

Before studying the thermal phase diagrams, we will present the best results for the behavior of thermodynamic quantities and critical point values. These results were obtained where most sites have the same coordination number \( z = 5 \). Therefore, in Fig. 2 we have shown one of the best results for the thermodynamic quantities obtained in the stationary state as a function of \( q \), for fixed \( p = 0.75 \) and \( T = 1 \). We can see the self-organization in the system, by finding an AF phase, being represented in the staggered magnetization \( m^F_{AF} \). These because in high values of \( q \) we have the transition between the \( F \) to \( P \) phase (see Fig. 2(d)) and from this \( P \) phase to the ordered AF phase (see Fig. 2(a)) as we increase the flow of energy into the system (\( q → 0 \)). For these magnetizations, we also have their respective reduced fourth-order Binder cumulants, \( U^F_{AF} \) (Fig. 2(b)) and \( U^F_L \) (Fig. 2(e)) beyond the magnetic susceptibilities \( χ^F_L \) (Figs. 2(c)) and \( χ^F_L \) (Fig. 2(f)).
Figure 2. Thermodynamic quantities in the phase transitions of the non-equilibrium system, with \( p = 0.75 \), \( T = 1 \), and lattice sizes \( L \) shown in the figures. (a) Staggered magnetization \( m_{AF}^p \) in (b) and (c) we have respectively the Binder cumulant \( U_{AF}^L \) and the staggered susceptibility \( \chi_{AF}^p \). (d) Total magnetization \( m_p^L \) of the system is represented, the Binder cumulant \( U_p^L \) in (e), and the total susceptibility \( \chi_p^L \) in (f). The error bars in the magnetization are smaller than the size of the symbols, so for a better interpretation of the results, these were omitted.

The thermodynamic quantities for the other \( p \) values in the A-SWN regime, such as \( p = 0.25, p = 0.5, p = 0.75, \) and \( p = 1 \), have also been computed. In the order to compare the behavior during phase transitions, we also exhibited the same thermodynamic quantities in the conventional square lattice Ising model, \( p = 0 \), in Fig. 3. These result can see in details for the \( m_{AF}^L \) and \( m_p^L \) in Figs. 3(a) and 3(d), respectively, \( U_{AF}^L \) in Fig. 3(b) and \( U_p^L \) in Fig. 3(e), in addition to \( \chi_{AF}^L \) in Fig. 3(c) and \( \chi_p^L \) in Fig. 3(f). We have presented only the smaller \( (L = 24) \) and the larger \( (L = 256) \) linear lattice size and they are enough so that we can observe the finite-size behavior and the critical point change \( q_c \) as we increase \( p \). On the other hand, for the calculation of \( q_c \), we have used all six lattice sizes of the system.

To evaluate the \( q_c \), we have employed two methods. Firstly, we obtained by extrapolating the susceptibility discontinuity to when \( L \to \infty \), which returns \( q_c(\infty) \), using finite lattice sizes \( 24 \leq L \leq 256 \), in the plot of maximum susceptibility as a function of \( 1/L \). Secondly, we obtained by the crossing of the Binder cumulant curves for the different lattice sizes \( L \). In Fig. 3 the values of \( q \) are displayed where the susceptibility has its maximum value, \( \chi_{AF}^{max} \), as a function of \( 1/L \) for the values of \( p \) selected. We also have the best fit of the points, which is a linear fit, and for the extrapolation, when \( L \to \infty \), we have the estimated of \( q_c \) by using the linear coefficient, i.e., we have made the infinite-size extrapolation in according to \( q(\chi_{AF}^{max}) = q_c(\infty) = \alpha L^{-1} \). By extrapolation, the critical points \( q_c(\infty) \) in the transition between the \( AF - P \) phases are represented in Fig. 3(a) and the transition between the \( P - P \) phases are represented in Fig. 3(b).

The critical point values using magnetic susceptibility data, \( q_c^{\chi} \), and their respective errors are exhibited in Table 4 for the transition between the \( AF - P \) phases. In this transition, we also have used the crossing of the Binder cumulant curves in the selected lattice sizes \( 24 \leq L \leq 256 \), to obtain another estimate for the critical points, \( q_{cU}^\chi \), which are shown in Table 4 and the characterization of the second-order phase transition in the system [28-31]. For the transition between \( F - P \)
Figure 4. Extrapolation of the critical transition probability $q_c$ obtained for linear lattice sizes $24 \leq L \leq 256$, and for different values of $p$ as indicated in the figures. (a) The points in the transition between $AF - P$ phases are represented, and (b) the points in the transition between the $P - F$ phases. The values of critical transition probabilities $q_c(L \to \infty)$ can be seen in Table I and Table II respectively.

Table I. Critical competition probability $q_c$, based on the extrapolating of the susceptibility discontinuity and in the $AF - P$ phase transition, for $T = 1$

| $p$  | $q_c^\chi$ | $\beta/\nu$ | $\gamma/\nu$ | $\nu$  |
|------|------------|--------------|--------------|--------|
| 0    | 0.00149 ± 0.0001 | 0.16 ± 0.05  | 1.77 ± 0.03  | 1.04 ± 0.09 |
| 0.25 | 0.03966 ± 0.002  | 0.46 ± 0.07  | 1.07 ± 0.06  | 1.07 ± 0.03 |
| 0.5  | 0.07176 ± 0.003  | 0.45 ± 0.04  | 1.06 ± 0.04  | 0.96 ± 0.09 |
| 0.75 | 0.09062 ± 0.002  | 0.46 ± 0.03  | 1.04 ± 0.02  | 1.02 ± 0.05 |
| 1.0  | 0.09774 ± 0.002  | 0.48 ± 0.02  | 1.05 ± 0.02  | 1.06 ± 0.08 |

With the critical point values, we built the phase diagram which shows the regions on the plane of $T$ versus $q$, where the $F$, $P$, and $AF$ phases are found. The phase diagrams are presented in Fig. 5 for different values of $p$, where we can see the greater the probability of adding $J_{ik}$, the greater the region where we find the ordered phases.

Now, in order to better understand the behavior of these phases (see Fig. 5), we can relate these ordered phases to dynamics used in the competition. The $AF$
phase, observing the q-axis, is found when q → 0 and the order parameter \( m_{\text{L}}^{\text{AF}} \) → 1 (see the figures on the left side in Fig. 5), i.e., when the two-spin flip dynamic prevails in the competition. This is because, in the dynamics that simulates the system with an external energy flow into it, the change in the spin states is only accepted if it increases the energy of the system. Considering the Hamiltonian model, Eq. (1), the state of the highest energy to which the dynamics lead the system is the one where the spins are aligned antiparallel. The antiparallel order also can be achieved through the A-SWN, because if we analyze locally, the antiferromagnetic phase occurs when a central spin is in the up (down) state, and its neighbors, to whom it is connected, are in the down (up) state. Extending this analysis to the entire network, an ordering of this type only occurs when we have well-defined what are the central spins and what sites they can connect to, otherwise, completely random long-range interactions can connect two distant sites in the network that the highest local energy configuration of one of these is unfavorable to the local antiparallel ordering of the other site, thus, making it impossible to obtain the stationary state with an AF phase in the system. In this context, the \( F \) phase is found in the limit that q → 1 and the order parameter \( m_{\text{L}}^{\text{F}} \) → 1 (see the figures of the right side in Fig. 5), i.e., when the one-spin flip dynamic prevails. This dynamic is responsible to simulate the system in contact with the heat bath at temperature \( T \), and favors the lowest energy state through the thermal equilibrium, in which all spins have the same state following the Hamiltonian system, so, if we wanted to, we could treat them without the sublattices in the A-SWN regime. On the other hand, when none of the dynamics prevails, i.e., between the extremes of the probability q-value, no one of the expected order phase types is found in the system. Thus, we have most of the values of q, the \( P \) phase in the system is found, where both \( m_{\text{L}}^{\text{AF}} \) → 0 and \( m_{\text{L}}^{\text{F}} \) → 0. Another important observation is that the phase diagram topology changes when p increases, but the phases do not disappear.

| p | q | \( \beta/\nu \) | \( \gamma/\nu \) | \( \nu \) |
|---|---|---|---|---|
| 0 | 0.964 ± 0.002 | 0.11 ± 0.02 | 1.67 ± 0.09 | 1.06 ± 0.09 |
| 0.25 | 0.917 ± 0.002 | 0.47 ± 0.06 | 1.01 ± 0.06 | 0.96 ± 0.09 |
| 0.5 | 0.895 ± 0.001 | 0.46 ± 0.02 | 1.04 ± 0.02 | 0.96 ± 0.09 |
| 0.75 | 0.883 ± 0.002 | 0.46 ± 0.01 | 1.06 ± 0.01 | 0.99 ± 0.06 |
| 1.0 | 0.880 ± 0.001 | 0.49 ± 0.01 | 1.0 ± 0.02 | 1.03 ± 0.04 |

After the presentation of the phase diagrams by exploiting the thermal variations of the order parameters, the Binder cumulant and the magnetic susceptibility, we can now study the critical behavior of these quantities in the vicinity of the phase transitions using the FSS method to evaluate some critical exponents of the model. Therefore, to obtain the critical exponents, we also used two methods, both referring to the FSS method, using the scale relations of Eqs. (11), (12), and (14). One of the methods refers to the value of the thermodynamic quantities at the critical point, in which when we make a log-log plot of the value of these quantities as a function of \( L \). Using the scale relations, we obtain ratios between the critical exponents through the slope of the line of best fit of those points. In Fig. 6(a) and (d) we were able to find the ratio \( -\beta/\nu \) in the \( AF - P \) and \( F - P \) phase transitions, respectively, using the scaling.
relation of the Eq. (11), through the slope in the linear fit of the points for each selected value of $p$, as indicated in the figures. The same can be done using the scaling relation of the Eq. (12), however, the critical exponent ratio is $\gamma/\nu$ and obtained by the slope of the linear fits of Figs. 6(b) and 6(e), for the different values of $p$ and in the $AF-P$ and $F-P$ phase transitions, respectively. Finally, the ratios between the values of the critical exponents obtained previously, it is useful to use the scaling relation of Eq. (14), which we have used the data of the Binder cumulant derivative to obtain information related to the critical exponent of correlation length, $\nu$. Here, they were obtained from the linear fit of the curves of Figs. 6(c) and 6(f) for the different $p$ values and $AF-P$ and $F-P$ phase transitions, respectively. It is worth noting that as our interest is in the slope of the log-log plot, we changed the linear coefficients of the straight lines to separate the lines and thus making it easier for the reader to see the fits. All the ratios between the values of the critical exponents obtained by the log-log plot of the scaling relations can be seen in Table I for the $AF-P$ phase transitions, and in Table II for the $F-P$ phase transitions.

Another method used to obtain the critical exponents is through the scaling functions in Eqs. (11), (12) and (14), in the around of the critical point. For this, we isolate the scale function and plot it in a log-log plot through the curves of $m^F L^{\beta/\nu}$ and $m^{AF} L^{\beta/\nu}$ as a function of $|x| L^{1/\nu}$, resulting in a single curve for all lattice sizes $L$ if we have the correct critical exponents and critical points adjusted in the scaling relations. In this method, the data collapse can also be obtained in a plot that does not have the axes on the logarithmic scale, but the asymptotic behavior that relates to the critical exponents are not present. We can obtain the critical exponents because the data collapse in the vicinity of the critical point, depends on the correct critical exponents of the system to occur, in this way, we adjust them to obtain the best data collapse in the criticality, and consequently, the exponents involved in this data collapses are the critical exponents of the system. All values of the critical exponents obtained by data collapse of the scaling

| $p$ | $q_c^L$ | $\beta$ | $\nu$ ($m^F$) | $\gamma$ | $\nu$ ($\chi^F$) | $d_{eff}$ |
|-----|--------|--------|--------------|--------|---------------|--------|
| 0   | 0.00145 ± 0.00004 | 0.125 ± 0.03 | 1.0 ± 0.05 | 1.75 ± 0.04 | 1.0 ± 0.05 | 2.00 ± 0.21 |
| 0.25 | 0.0390 ± 0.0009 | 0.46 ± 0.06 | 0.96 ± 0.06 | 1.06 ± 0.05 | 1.0 ± 0.05 | 2.02 ± 0.29 |
| 0.5  | 0.0715 ± 0.0006 | 0.44 ± 0.05 | 1.0 ± 0.05 | 1.05 ± 0.03 | 1.0 ± 0.04 | 1.93 ± 0.24 |
| 0.75 | 0.0921 ± 0.0007 | 0.48 ± 0.04 | 0.98 ± 0.06 | 1.05 ± 0.06 | 1.0 ± 0.04 | 2.03 ± 0.31 |
| 1.0  | 0.0978 ± 0.0004 | 0.48 ± 0.03 | 1.02 ± 0.06 | 1.06 ± 0.06 | 0.98 ± 0.04 | 2.02 ± 0.23 |

Table III. Critical competition probability $q_c$, based on the crossing of the fourth-order Binder cumulant curves and in the $AF-P$ phase transition, for $T = 1$. The effective dimension is given by hyperscaling relation $d_{eff} = 2/\nu + \gamma/\nu$.

| $p$ | $q_c^L$ | $\beta$ | $\nu$ ($m^{AF}$) | $\gamma$ | $\nu$ ($\chi^{AF}$) | $d_{eff}$ |
|-----|--------|--------|-----------------|--------|-------------------|--------|
| 0   | 0.965 ± 0.0004 | 0.125 ± 0.02 | 1.0 ± 0.05 | 1.70 ± 0.07 | 1.01 ± 0.03 | 1.95 ± 0.20 |
| 0.25 | 0.917 ± 0.0009 | 0.45 ± 0.05 | 1.0 ± 0.05 | 1.05 ± 0.05 | 1.0 ± 0.04 | 1.95 ± 0.26 |
| 0.5  | 0.895 ± 0.0008 | 0.46 ± 0.04 | 0.95 ± 0.07 | 1.05 ± 0.04 | 1.0 ± 0.04 | 2.02 ± 0.12 |
| 0.75 | 0.883 ± 0.0005 | 0.47 ± 0.03 | 1.0 ± 0.06 | 1.04 ± 0.06 | 1.0 ± 0.05 | 1.98 ± 0.23 |
| 1.0  | 0.879 ± 0.0006 | 0.49 ± 0.04 | 1.0 ± 0.07 | 1.02 ± 0.05 | 0.99 ± 0.06 | 2.00 ± 0.26 |

Table IV. Critical competition probability $q_c$, based on the crossing of the fourth-order Binder cumulant curves and in the $F-P$ phase transition, for $T = 1$. The effective dimension is given by hyperscaling relation $d_{eff} = 2/\nu + \gamma/\nu$. 

Figure 7. Data collapse by FSS analysis for (a) staggered magnetization $m^F_{1}$ and (c) susceptibility $\chi^{AF}$ for different values of $L$ as indicated in the figure, and fixed $p = 0.75$. In (b) and (d) we have the data collapse for $m^F_{1}$ and $\chi^{AF}$, respectively, but for all values of $p$, in which the critical exponents were obtained using the best data collapse for all lattice sizes, and here we only display the lattice sizes $L = 24$ (+) and $L = 256$ (□). The $\varepsilon$ parameter is set to $\varepsilon = (q - q_c)/q_c$. The dashed lines represent the asymptotic behavior of the scale functions. The values of the critical exponents $\beta$, $\gamma$, $\nu$ ($m^{AF}$), and $\nu$ ($\chi^{AF}$) of the best data collapse can be seen in Table III.
The critical exponents are not independent one each other, but related by simple scaling laws, as is the case with the hyperscaling law $d_{\text{eff}} = 2\beta/\nu + \gamma/\nu$, in which we have as a result the effective dimension $d_{\text{eff}}$ of the system. With this law, we see that the system has approximately the same critical exponents in the A-SWN regime, as we are returned that $d_{\text{eff}} \approx 4.0$ with the mean field critical exponents, and $d_{\text{eff}} \approx 2.0$ following the data in the tables for $p = 0$, obtained with the scale relationships of Eqs. (11), (12) and (14).

The universality class can be defined by the set of exponents in the phase transition, as in the case of the second-order phase transitions, in which systems very different from each other can share the same set of critical exponents. In general, these systems share the same spatial dimension, symmetries, and range of interactions. Here, following the set of critical exponents obtained at $p = 0$, we have the same universality class of the equilibrium Ising model in the regular square lattice. However, in the A-SWN regime $(0 < p \leq 1)$, we have long-range interactions in the system, and, due to its consequent set of critical exponents, the system belongs mean-field universality class. By comparing with the results obtained for the Ising model in the two-dimensional A-SWN at the thermodynamic equilibrium regime [24, 28], we see that both the non-equilibrium model and the equilibrium model have the same universality class, mean-field universality class, in stationary critical behavior.

### E. Conclusions

In this work, we have developed MC simulations to study the thermodynamic quantities and the critical behavior of the non-equilibrium Ising model on a 2D A-
Figure 9. (a) Representation of the critical exponents presented in Tables III and I for the critical behavior of the system in the AF − P phase transition, taking into account the mean-field scale relationships for the A-SWN regime \((0 < p \leq 1)\). (b) Representation of the critical exponents presented in Tables IV and II for the system in the F − P phase transition, also using the mean-field scale relations in the A-SWN regime \((0 < p \leq 1)\). In both figures, we have the comparison with the mean-field critical exponents (MF) by the dotted lines, \(\gamma = 1.0\) and \(\beta = \nu = 0.5\).
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