WEYL GROUP INVARIANTS

MASAKI KAMEKO AND MAMORU MIMURA

Abstract. For any odd prime $p$, we prove that the induced homomorphism from the mod $p$ cohomology of the classifying space of a compact simply-connected simple connected Lie group to the Weyl group invariants of the mod $p$ cohomology of the classifying space of its maximal torus is an epimorphism except for the case $p = 3$, $G = E_8$.

1. Introduction

Let $p$ be an odd prime. Let $G$ be a compact connected Lie group. Let $T$ be a maximal torus of $G$. We denote by $W$ the Weyl group $N_G(T)/T$ of $G$. We write $H^*(X)$ for the mod $p$ cohomology of a space $X$. Then, the Weyl group $W$ acts on $G$, $T$, $G/T$, $BG$, $BT$ and their cohomologies through the inner automorphism. The mod $p$ cohomology of $BT$ is a polynomial algebra $\mathbb{Z}/p[t_1, \ldots, t_n]$. We denote by $H^*(BT)^W$ the ring of invariants of the Weyl group $W$. Since $G$ is path connected, the action of the Weyl group on $BG$ is homotopically trivial and so the action of the Weyl group on the mod $p$ cohomology $H^*(BG)$ is trivial. Therefore, we have the induced homomorphism

$$\eta^* : H^*(BG) \to H^*(BT)^W.$$ 

If $H_*(G; \mathbb{Z})$ has no $p$-torsion, the induced homomorphism is an isomorphism. Even if $H_*(G; \mathbb{Z})$ has $p$-torsion, the fact that the induced homomorphism $\eta^*$ is an epimorphism was proved by Toda in [10] for $(G, p) = (F_4, 3)$ and announced in [11] for $(G, p) = (E_6, 3)$, respectively. However the results depend on the computation of the Weyl group invariants. The purpose of this paper is not only to show the following theorem but also to give a proof without explicit computation of the Weyl group invariants.

We denote by $y_2$ a generator of $H^2(BG)$ for $(G, p) = (PU(p), p)$ and by $y_4$ a generator of $H^4(BG)$ for $(G, p) = (F_4, 3), (E_6, 3), (E_7, 3), (E_8, 5)$. Let $Q_i$ be the Milnor operation defined by $Q_0 = \beta$, $Q_1 = \psi^1\beta - \beta\psi^1$, $Q_2 = \psi^pQ_1 - Q_1\psi^p$, ..., where $\psi^i$ is the $i$-th Steenrod reduced power operation. Let $e_2 = Q_0Q_1y_2$, $e_3 = Q_1Q_2y_4$ for the above $H^*(BG)$’s. For a graded vector space $M$, we denote by $M^{even}$, $M^{odd}$ for graded subspaces of $M$ spanned by even degree elements and odd degree elements, respectively.

**Theorem 1.1.** For $(G, p) = (PU(p), p), (F_4, 3), (E_6, 3), (E_7, 3), (E_8, 5)$, the induced homomorphism $\eta^*$ above is an epimorphism. Moreover, we have

$$H^*(BT)^W = H^{even}(BG)/(e_k),$$
where $k = 2$ for $(G, p) = (PU(p), p)$ and $k = 3$ for $(G, p) = (F_4, 3), (E_6, 3), (E_7, 3), (E_8, 5)$. If $G$ is a simply-connected, simple, compact connected Lie group, then $G$ is one of the classical groups $SU(n)$, $Sp(n)$ and $Spin(n)$ or one of the exceptional groups $G_2$, $F_4$, $E_6$, $E_7$, $E_8$. Since $H_*(G; \mathbb{Z})$ has no $p$-torsion except for the cases $(G, p) = (F_4, 3), (E_6, 3), (E_7, 3), (E_8, 3)$ and $(E_8, 5)$, the above theorem provides a supporting evidence for the following conjecture.

Conjecture 1.2. Let $p$ be an odd prime. Let $G$ be a simply-connected, simple, compact connected Lie group. Then, the induced homomorphism $\eta^*$ above is an epimorphism.

To prove this conjecture, it remains to prove the case $(G, p) = (E_8, 3)$. However, the mod 3 cohomology of $BE_8$ seems to be rather different from the other cases. For instance, the Rothenberg-Steenrod spectral sequence for the mod $p$ cohomology for $(G, p)$’s in Theorem 1.1 collapses at the $E_2$-level but the one for the mod 3 cohomology of $BE_8$ is known not to collapse at the $E_2$-level and its computation is still an open problem. See [4].

In this paper, for $(G, p)$’s in Theorem 1.1, we examine the Leray-Serre spectral sequence associated with the fibre bundle $BT \to BG$ and show that at the level of the spectral sequence, $E^r_0 \neq (E^r_0)^W$ for some $r$ but we have $E^\infty_0 = (E^\infty_0)^W$ at the end. In the case $(G, p) = (E_8, 3)$, the cohomology of the base space $BE_8$ is not yet known; since we need the cohomology of the base space in order to examine the Leray-Serre spectral sequence, we do not deal with the case $(G, p) = (E_8, 3)$ in this paper.

The paper is organized as follows. In §2, we recall the Leray-Serre spectral sequence and the action of the Weyl group on it. In §3, we recall the invariant theory of the Weyl group of non-toral elementary abelian $p$-subgroup of $G$ in order to describe the cohomology of the base space $BG$. In §4, we recall the cohomology of $BG$ and prove Proposition 4.3. In §5, using Proposition 4.3, we compute the Leray-Serre spectral sequence. As a consequence of the computation of the Leray-Serre spectral sequence, we obtain Theorem 1.1.

The result in this paper is announced in [6], where we deal with the case $(G, p) = (PU(p), p)$ in detail.

2. The Weyl group and the spectral sequence

As in §1, let $G$ be a compact connected Lie group. We consider the Leray-Serre spectral sequence associated with the fibre bundle

$$G/T \buildrel \eta \over \to BT \buildrel \eta \over \to BG.$$ 

Since $BG$ is simply connected, the $E_2$-term is given by

$$H^*(BG) \otimes H^*(G/T).$$

It converges to gr $H^*(BT)$. Moreover, the Weyl group acts on this spectral sequence and its action is given by

$$r^*(y \otimes x) = y \otimes r^*x,$$

where $r$ is an element in $W$. We fix a set of generators $\{r_j\}$ and denote by $\sigma_j$ the induced homomorphism $1 - r_j^*$. It is clear that

$$H^*(G/T)^W = \bigcap_j \text{Ker} \sigma_j,$$
and $\sigma_j(x \otimes y) = x \otimes \sigma_j(y)$. Moreover, we have

$$(E^*,*)^W = \bigcap_j \text{Ker } \sigma_j.$$ 

To relate the Weyl group invariants of $H^*(BT)$ with the one of $E_\infty$-term, that is $\text{gr } H^*(BT)$, of the spectral sequence, we use the following lemma.

**Lemma 2.1.** Suppose that $f : M \to N$ is a filtration preserving homomorphism of finite dimensional vector spaces with filtration. Denote by $\text{gr } f : \text{gr } M \to \text{gr } N$ the induced homomorphism between associated graded vector spaces. Then, we have

$$\dim \text{Ker } \text{gr } f \geq \dim \text{Ker } f.$$ 

It is clear that $E^*_\infty = \text{Im } \eta^* : H^*(BG) \to H^*(BT)^W$, so that $\dim E^*_\infty \leq \dim H^*(BT)^W$. By Lemma 2.1 above, we have

$$\sum_{s'} \dim (E^*_{s-s',s'})^W \geq \dim H^*(BT)^W.$$ 

Hence, if we have

$$(E^*_{s-s'})^W = E^*_{s-0},$$ 

we obtain

$$\dim H^*(BT)^W \leq \dim E^*_\infty$$

and the desired result $E^*_\infty = H^*(BT)^W$.

In [2], Kac mentioned the following theorem and Kitchloo gave the detailed account of it in §5 of [7].

**Theorem 2.2 (Kac, Kitchloo).** Let $p$ be an odd prime. Let $G$ be a compact connected Lie group. Let $T$ be a maximal torus of $G$ and $W$ the Weyl group of $G$. Then, we have $H^*(G/T)^W = H^0(G/T) = Z/p$.

Theorem 2.2 is the starting point of this paper. By Theorem 2.2, we have

$$(E^*,*)^W = (H^*(BG) \otimes H^*(G/T))^W = (H^*(BG) \otimes \mathbb{Z}/p) = E^*_2.$$ 

Recall that the cohomology $H^*(G/T)$ has no odd degree generators. So, if $H_*(G;Z)$ has no $p$-torsion, then the $E_2$-term has no odd degree generators. Hence, it collapses at the $E_2$-level. Thus, we have that

$$(E^*_{s-s'})^W = E^*_{s-0} = H^*(BG).$$

Therefore, it is clear that the induced homomorphism $\eta^* : H^*(BG) \to H^*(BT)^W$ is an isomorphism if $H_*(G;Z)$ has no $p$-torsion.

However, for $(G, p)$ in Theorem 2.1, $H_*(G;Z)$ has $p$-torsion and we have odd degree generators in the $E_2$-level. These odd degree generators do not survive to the $E_\infty$-level. So, the spectral sequence does not collapse at the $E_2$-level. We deal with the spectral sequence for $(G, p)$ in Theorem 1.3 in §4 and we will see that $(E^*_{r-s'})^W \neq E^*_{r,0}$ for some $r$ but still $(E^*_{s-s'})^W = E^*_{s-0}$ holds.

We end this section by recalling from [2] the description of the mod $p$ cohomology of $G/T$ for $(G, p)$'s in Theorem 1.3.
We denote by $p$ the image of the induced homomorphism $i^{*} : H^{*}(BT) \rightarrow H^{*}(G/T)$, $m = 2$ for $(G,p) = (PU(p),p)$ and $m = 2p + 2$ for $(G,p) = (F_{4},3), (E_{6},3), (E_{7},3)$ and $(E_{8},5)$.

3. Invariant Theory

In order to describe the odd degree generators of $H^{*}(BG)$ for $(G,p)$ in Theorem 1.1 we consider non-toral elementary abelian $p$-subgroups of $G$.

Non-toral elementary abelian $p$-subgroups of a compact connected Lie group $G$ and their Weyl groups are described in [1] not only for $(G,p)$ in Theorem 1.1 but also for $(G,p) = (E_{6},3), (PU(p),p)$. For $(G,p)$'s in Theorem 1.1 there exists a unique, up to conjugacy, maximal non-toral elementary abelian $p$-subgroup $A$. Their Weyl groups $W(A) = N_{G}(A)/C_{G}(A)$ are also determined in [1]. We refer the reader to [1] for the details. From now on, we consider the case $(G,p)$ in Theorem 1.1 only.

We denote by $\xi : A \rightarrow G$ the inclusion of $A$ into $G$ and the induced map $BA \rightarrow BG$ by the same symbol $\xi : BA \rightarrow BG$. Although the Weyl group invariants $H^{*}(BT)^{W}$ is not yet known, we determined in [3] the ring of invariants $H^{*}(BA)^{W(A)}$. It is rather easy to describe it in terms of Dickson-Mui invariants because the Weyl groups $W(A)$ are $SL_{2}(\mathbb{Z}/p)$ for $(G,p) = (PU(p),p), SL_{3}(\mathbb{Z}/p)$ for $(G,p) = (F_{4},3), (E_{6},5)$ and

$$\left\{ \begin{pmatrix} 1 & \ast & \ast & \ast \\ 0 & \ast & \ast & \ast \\ 0 & 0 & g \\ 0 & 0 & 0 & g \end{pmatrix} \right\}, \left\{ \begin{pmatrix} \ast & \ast & \ast & \ast \\ 0 & \ast & \ast & \ast \\ 0 & 0 & g \\ 0 & 0 & 0 & g \end{pmatrix} \right\}$$

for $(G,p) = (E_{6},3), (E_{7},3)$, respectively, where $g$ ranges over $SL_{3}(\mathbb{Z}/3)$ and $\varepsilon$ ranges over $(\mathbb{Z}/3)^{\times}$.

In order to describe the image of $\xi^{*} : H^{*}(BG) \rightarrow H^{*}(BA)^{W(A)}$ for the above $(G,p)$, firstly, we recall from [3] the invariant theory of special linear groups and related groups. Let $p$ be an odd prime and $A_{n}$ the elementary abelian $p$-group of rank $n$. We need the cases $n = 2, 3, 4, p = 3, 5$ only. However, since some arguments seem to be comprehensive when we put them in the general setting, we recall the invariant theory without any restriction on $p$ and $n$ for the time being. We have

$$H^{*}(BA_{n}) = \mathbb{Z}/p[t_{1}, \ldots, t_{n}] \otimes \Lambda(dt_{1}, \ldots, dt_{n}),$$

where $dt_{i}$'s are generators of $H^{1}(BA_{n})$, $t_{i} = \beta dt_{i}$, and $\beta$ is the Bockstein homomorphism. Denote by $G_{n}, G'_{n}$ subgroups of $GL_{n}(\mathbb{Z}/p)$ consisting of the following matrices:

$$\begin{pmatrix} 1 & \ast & \ldots & \ast \\ 0 & \ast & \ldots & \ast \\ \vdots & g & \ast & \ldots \\ 0 & \ast & \ldots & \ast \end{pmatrix}, \begin{pmatrix} \alpha & \ast & \ldots & \ast \\ 0 & \ast & \ldots & \ast \\ \vdots & g & \ast & \ldots \\ 0 & \ast & \ldots & \ast \end{pmatrix},$$

respectively, where $g$ ranges over $SL_{n-1}(\mathbb{Z}/p)$ and $\alpha$ ranges over $(\mathbb{Z}/p)^{\times}$. We use the letter $H$ to denote one of $SL_{n}(\mathbb{Z}/p), G_{n}, G'_{n}$.

Now, we define some elements in the ring of invariants. Let $V_{n}$ be the vector space over $\mathbb{Z}/p$ spanned by $t_{1}, \ldots, t_{n}$ and $V_{n-1}$ the subspace spanned by $t_{2}, \ldots, t_{n}$.
We denote the element $dt_1 \cdots dt_n$ by $u_n$ and let $e_n = Q_0 \cdots Q_{n-1} u_n$. We also denote $dt_2 \cdots dt_n$ by $u_{n-1}$ and let $e_{n-1} = Q_0 \cdots Q_{n-2} u_{n-1}$. The Dickson invariants $c_{n,i}$’s are defined by

$$\prod_{x \in V_n} (X - x) = \sum_{j=0}^{n} (-1)^j c_{n,n-j} X^{p^{n-j}}.$$ 

We also consider the Dickson invariants $d_{n-1,i}$ given by

$$\prod_{x \in V_{n-1}} (X - x) = \sum_{j=0}^{n-1} (-1)^j d_{n-1,n-1-j} X^{p^{n-1-j}}.$$ 

We define $O_{n-1}$ to be

$$Q_{n-1} - c_{n-1,n-2} Q_{n-2} - \cdots + (-1)^{n-1} c_{n-1,0} Q_0.$$ 

Let $f_n = O_{n-1}(dt_1)$. Then, it is clear that we have

$$u_{n-1} = f_n^{-1} O_{n-1} u_n$$

and

$$e_{n-1} = f_n^{-1} Q_0 \cdots Q_{n-2} O_{n-1} u_n.$$ 

For the sake of notational simplicity, we denote $\{0, \ldots, n-1\}, \{0, \ldots, n-2\}$ by $\Delta_n$, $\Delta_{n-1}$, respectively. For a subset $I = \{i_1, \ldots, i_r\}$, where $0 \leq i_1 < \cdots < i_r \leq n-1$, of $\Delta_n$, let

$$Q_I u_n = Q_{i_1} \cdots Q_{i_r} u_n$$

and $Q_I u_n = u_n$. In order to deal with $H = SL_n(\mathbb{Z}/p), G_n, G'_n$ in the same manner, we need the following definition. We define $\overline{Q}_i$ by $\overline{Q}_i = Q_i$ for $i = 0, \ldots, n-2$. Let $\overline{\pi}_n = u_n$ for $H = SL_n(\mathbb{Z}/p), G_n$ and $\overline{\pi}_n = f_n^{-p-2} u_n$ for $H = G'_n$. We define $\overline{Q}_{n-1}$ by

$$\overline{Q}_{n-1} = Q_{n-1}, \overline{Q}_n = f_n^{-1} O_{n-1}, \overline{Q}_n^{-p-1} O_{n-1}$$

for $H = SL_n(\mathbb{Z}/p), G_n, G'_n$, respectively. We define $\overline{Q}_I \overline{\pi}_n$ to be $\overline{Q}_{i_{1}} \cdots \overline{Q}_{i_{r}} \overline{\pi}_n$ and $\overline{Q}_0 \overline{\pi}_n = \overline{\pi}_n$.

We denote the ring of invariants

$$\mathbb{Z}/p[t_1, \ldots, t_n]^H$$

by $R$. We have

$$R = \mathbb{Z}/p[c_{n,1}, \ldots, c_{n,n-1}, e_n]$$

for $H = SL_n(\mathbb{Z}/p)$,

$$R = \mathbb{Z}/p[e_{n-1,1}, \ldots, e_{n-1,n-2}, e_{n-1}, f_n]$$

for $H = G_n$,

$$R = \mathbb{Z}/p[e_{n-1,1}, \ldots, e_{n-1,n-2}, e_{n-1}, f_n^{p-1}]$$

for $H = G'_n$.

Moreover, we have

$$H^*(BA_n)^H = R\{1, Q_I u_n\}$$

for $H = SL_n(\mathbb{Z}/p)$,

$$H^*(BA_n)^H = R\{1, Q_I u_{n-1}, Q_K u_n\}$$

for $H = G_n$,

$$H^*(BA_n)^H = R\{1, Q_I u_{n-1}, f_n^{p-2} Q_K u_n\}$$

for $H = G'_n$.

where $I$ ranges over all the proper subsets of $\Delta_n$, $J$ ranges over all the proper subsets of $\Delta_{n-1}$ and $K$ ranges over all the subsets of $\Delta_{n-1}$. With the above definitions of $\overline{Q}_I$’s and $\overline{\pi}_n$, we may write

$$H^*(BA_n)^H = R\{1, \overline{Q}_I \overline{\pi}_n\},$$

where $I$ ranges over all the proper subsets of $\Delta_n$. 


Secondly, we consider a subspace $F_i$ of $H^*(BA_n)^H$. Since there holds
\[
H^*(BA_n)^{G_n} \subset H^*(BA_n)^{G_n},
\]
we introduce a filtration on $H^*(BA_n)^{SL_n(Z/p)}$ and $H^*(BA_n)^{G_n}$. Let $w(\overline{Q}I)$ be the number of elements in $I$. For monomials in $H^*(BA_n)^{SL_n(Z/p)}$, we define $w(-)$ by
\[
w(e_n) = n,
w(e_{n,j}) = 0 \quad (j = 1, \ldots, n - 1),
\]
and
\[
w(xy) = w(x) + w(y),
\]
where $x, y$ are monomials in $H^*(BA_n)^{SL_n(Z/p)}$. For monomials in $H^*(BA_n)^{G_n}$, we define $w(-)$ by
\[
w(e_{n-1}) = n,
w(e_{n-1,j}) = 0 \quad (j = 1, \ldots, n - 2),
w(f_n) = 0,
\]
and
\[
w(xy) = w(x) + w(y),
\]
where $x, y$ are monomials in $H^*(BA_n)^{G_n}$. We denote by $F_{j+i,i}$ the subspace spanned by elements $x$ such that $j + i \geq w(x) \geq i$, that is, $F_{j+i,i} = F_i \oplus F_{i+1} \oplus \cdots \oplus F_{i+j}$. Since
\[
H^*(BA_n)^{G_n} \subset H^*(BA_n)^{G_n},
\]
by abuse of notation, we denote
\[
H^*(BA_n)^{G_n} \cap F_i
\]
by $F_i$. It is clear that
\[
F_{\infty,i} = \mathbb{Z}/p[e_k] \otimes F_{n-1+i,i},
\]
where $k = n$ for $H = SL_n(Z/p)$ and $k = n - 1$ for $H = G_n, G'$, $G$. It is also clear that
\[
H^*(BA_n)^H = R \oplus F_{\infty,0}.
\]

We will see in Theorem 4.2 that $R \oplus F_{\infty,0}$ is the image of $\xi^*$ for $(G, p) = (PU(p), p)$, that $R \oplus F_{\infty,1}$ is the image of $\xi^*$ for $(G, p) = (F_4, 3)$ and $(E_8, 5)$, that $R \oplus F_{\infty,2}$ is the image of $\xi^*$ for $(G, p) = (E_6, 3)$ and that $R \oplus (F_{\infty,2} \cap (\overline{Q}3u_4))$ is the image of $\xi^*$ for $(G, p) = (E_7, 3)$ where $(\overline{Q}3u_4)$ is the $R$-submodule generated by $Qfu_3 = Qfu_4$.

Next, we consider the multiplication on $R \oplus F_{\infty,i}$. For $I = \{i_1, \ldots, i_r\}, J = \{j_1, \ldots, j_s\} \subset \Delta_n$ such that $I \cap J = \emptyset$, we denote the sign of the permutation
\[
\begin{pmatrix}
1 & \cdots & r & \vdots & r + 1 & \cdots & r + s \\
i_1 & \cdots & i_r & j_1 & \cdots & j_s
\end{pmatrix}
\]
simply by $\text{sgn}(I, J)$. If one of $I, J$ is empty, we set $\text{sgn}(I, J) = 1$.

**Lemma 3.1.** If $I \cup J \neq \Delta_n$, then $Qfu_n \cdot Qfu_n = 0$. If $I \cup J = \Delta_n$, then
\[
Qfu_n \cdot Qfu_n = (-1)^{nr + r^2} \text{sgn}(K, I \setminus K) \text{sgn}(I \setminus K, J) e_n Qfu_n,
\]
where $K = I \cap J$ and $r$ is the number of elements in $I \setminus K$. 
Proof. If $I \cup J \neq \Delta_n$, then $w(Q_{I \setminus K}u_n) + w(Q_J u_n) < n$. So, it is clear that
$$Q_{I \setminus K}u_n \cdot Q_J u_n = 0.$$ On the other hand, we have
$$Q_K(Q_{I \setminus K}u_n \cdot Q_J u_n) = sgn(K, I \setminus K)Q_I u_n \cdot Q_J u_n,$$ since $Q_k Q_j = 0$ for $k \in K \subset J$. Hence, we have $Q_I u_n \cdot Q_J u_n = 0$ as desired.

Next, we deal with the case $I \cup J = \Delta_n$ and $I \cap J = \emptyset$. There holds
$$Q_{I \setminus \{i_1\}}u_n \cdot Q_J u_n = 0,$$ since $w(Q_{I \setminus \{i_1\}}u_n) + w(Q_J u_n) < n$. On the other hand, we have
$$Q_{i_1}(Q_{I \setminus \{i_1\}}u_n \cdot Q_J u_n) = Q_I u_n \cdot Q_J u_n + (-1)^{n+r-1}Q_{I \setminus \{i_1\}}u_n \cdot Q_{i_1}Q_J u_n.$$ Hence, we have
$$Q_I u_n \cdot Q_J u_n = (-1)^{n+r}Q_{I \setminus \{i_1\}}u_n \cdot Q_{i_1}Q_J u_n.$$ Therefore, we obtain
$$Q_I u_n \cdot Q_J u_n = (-1)^{n+r}(-1)^{n+r-1} \cdots (-1)^{n+1} u_n \cdot Q_{i_1} \cdots Q_{i_z}Q_{i_1}Q_J u_n$$
$$= (-1)^{nr+r(r+1)/2}(-1)^{r(r-1)/2}u_n \cdot Q_I Q_J u_n$$
$$= (-1)^{rn+r^2}sgn(I, J)u_n \cdot e_n,$$ which is the desired result.

Now, we deal with the case $I \cup J = \Delta_n$ and $K = I \cap J$. Then, we have
$$Q_{I \setminus K}u_n \cdot Q_J u_n = (-1)^{rn+r^2}sgn(I \setminus K) e_n u_n.$$ Hence, we have
$$Q_K(Q_{I \setminus K}u_n \cdot Q_J u_n) = (-1)^{rn+r^2}sgn(I \setminus K, J) e_n Q_K u_n.$$ On the other hand, we have
$$Q_K(Q_{I \setminus K}u_n \cdot Q_J u_n) = sgn(K, I \setminus K)Q_I u_n \cdot Q_J u_n,$$ since $Q_k Q_j u_n = 0$ for $k \in K \subset J$. This completes the proof. \qed

Remark 3.2. For $H = G_n, G'_n$, it is easy to see that we have the same formula
$$\overline{Q_J \overline{u}_n} \cdot \overline{Q_J \overline{u}_n} = (-1)^{nr+r^2}sgn(K, I \setminus K)sgn(I \setminus K, J) e_n \overline{Q_K \overline{u}_n}.$$ By Lemma 3.1 and by Remark 3.2, we see that $F_i : F_j \subset F_{i+j}$. Therefore, $R \oplus F_{\infty, i}$ is closed under the multiplication.

Finally, we end this section by considering the direct sum decomposition
$$R \oplus F_{\infty, i} = N_0 \oplus N_1.$$ We use this decomposition in order to deal with differentials in the spectral sequence in §5. For $0 \leq \ell \leq n-2$, let $E(\ell)$ be the subspace of $F_{\infty, 0}$ spanned by $\{x \overline{Q} \overline{u}_n \mid \ell \in I, x \in R\}$ and $\hat{E}(\ell)$ the subspace of $F_{\infty, 0}$ spanned by $\{x \overline{Q} \overline{u}_n \mid \ell \not\in I, x \in R\}$. Let
$$N_0 = R \oplus (F_{\infty, i} \cap E(\ell))$$ and
$$N_1 = F_{\infty, i} \cap \hat{E}(\ell).$$ Let
$$z_\ell = Q_0 \cdots Q_{\ell} \cdots Q_{n-2} \overline{Q}_{n-1} \overline{u}_n.$$
From now on, we assume that $i \leq n - 1$, so that $z_i \in N_i$. Let $F_{j,i} = F_{j,i}/(z_i)$ if $z_i \in F_{i,j}$. Then, it is easy to see that the following proposition holds.

**Proposition 3.3.** There hold the following:

1. Suppose that $i < n$. If $n - i$ is even, then $F_{i}^{\text{even}} = F_{i}$. If not, $F_{i}^{\text{even}} = \{0\}$.
2. If $n$ is even, then $F_{n}^{\text{even}} = F_{n}$. If not, $F_{n}^{\text{even}} = \{0\}$.
3. $F_{n-1} \cap \hat{E}(\ell) = \{0\}$.
4. $F_{n} \cap \hat{E}(\ell) = F_{n}$.

The Milnor operation $Q_{\ell}$ induces a short exact sequence

$$0 \to N_{1} \xrightarrow{Q_{\ell}} N_{0} \to R/(e_{k}) \oplus F_{i} \cap E(\ell) \to 0.$$ 

The multiplication by $z_i$ induces an isomorphism

$$0 \to N_{0} \xrightarrow{z_{i}} N_{1} \to F_{n-2+i,i} \cap \hat{E}(\ell) \to 0.$$ 

We have the following proposition:

**Proposition 3.4.** For $(n, i) = (2, 0), (3, 1), (4, 2)$, there exist short exact sequences

1. $0 \to N_{0} \xrightarrow{z_{i}} N_{1} \to N_{1}^{\text{even}}/(e_{k}) \to 0$.
2. $0 \to N_{1} \xrightarrow{Q_{\ell}} N_{0} \to N_{0}^{\text{even}}/(e_{k}) \to 0$.

**Proof.** From the observation above, it suffices to show that

$$R/(e_{k}) \oplus F_{i} \cap E(\ell) = N_{1}^{\text{even}}/(e_{k})$$

and

$$F_{n-2+i,i} \cap \hat{E}(\ell) = N_{0}^{\text{even}}/(e_{k}).$$

Since $N_{0} \oplus N_{1}^{\text{even}}/(e_{k}) = R/(e_{k}) \oplus F_{n-1+i,i}^{\text{even}}$, it suffices to show that

$$F_{n-1+i,i}^{\text{even}} = F_{i} \cap E(\ell) \oplus F_{n-2+i,i} \cap \hat{E}(\ell).$$

By Proposition 3.3, we have the following table.

| $(n, i)$ | $(2, 0)$ | $(3, 1)$ | $(4, 2)$ |
|---------|---------|---------|---------|
| $F_{i} \cap E(\ell)$ | $F_{i}^{\text{even}} \cap E(\ell)$ | $F_{i}^{\text{even}} \cap E(\ell)$ | $F_{i}^{\text{even}} \cap E(\ell)$ |
| $\overline{F_{i}} \cap \hat{E}(\ell)$ | $\overline{F_{0}}^{\text{even}} \cap \hat{E}(\ell)$ | $\overline{F_{1}}^{\text{even}} \cap \hat{E}(\ell)$ | $\overline{F_{2}}^{\text{even}} \cap \hat{E}(\ell)$ |
| $\overline{F}_{i+1} \cap \hat{E}(\ell)$ | $\overline{F}_{0}^{\text{even}} \cap \hat{E}(\ell)$ | $\overline{F}_{1}^{\text{even}} \cap \hat{E}(\ell)$ | $\overline{F}_{2}^{\text{even}} \cap \hat{E}(\ell)$ |
| $\overline{F}_{i+2} \cap \hat{E}(\ell)$ | $\overline{F}_{0}^{\text{even}} \cap \hat{E}(\ell)$ | $\overline{F}_{1}^{\text{even}} \cap \hat{E}(\ell)$ | $\overline{F}_{2}^{\text{even}} \cap \hat{E}(\ell)$ |

This table completes the proof. \qed

4. Cohomology of classifying spaces

Now, we investigate $H^{*}(BG)$ and $H^{*}(BA)^{W(A)}$ for $(G, p)$ in Theorem 1.1. Throughout the rest of this section, we put
where $y$ is the generator of $H^2(BG)$ such that $\xi^*(y_2) = u_2$ and $y_3 = Q_0 y_2$, $y_{2p+1} = Q_1 y_2$. Then, we have

$$\xi^* M_0 = R \{ 1, Q_1 u_2 \} = R \oplus (F_{0,0} \cap E(1)),$$

$$\xi^* M_1 = R \{ Q_0 u_2, u_2 \} = F_{0,0} \cap \tilde{E}(1).$$
For \((G, p) = (F_4, 3)\) and \((E_8, 5)\), the rank of the elementary abelian \(p\)-subgroup \(A\) is 3. Since \(F_{3,1} = R/(e_3)\{Q_0u_3, Q_1u_3, Q_2u_3, Q_0Q_1u_3, Q_0Q_2u_3, Q_1Q_2u_3, e_3u_3\}\, we put
\[
M_0 = R\{y_2p^2+2, y_2p^2+3, y_2p^2+2p+1\},
M_1 = R\{y_2p+3, y_2p+3y_2p^2+2, y_4, y_2p+2\},
\]
where \(y_4\) is the generator of \(H^4(BG)\) such that \(\xi^*(y_4) = Q_0u_3, y_2p+2 = \varphi^1y_4, y_2p+3 = -Q_1y_4, y_2p^2+2 = \varphi p^2\varphi^1y_4, y_2p^2+3 = -Q_2y_4, y_2p^2+2p+1 = -Q_2y_2p+2\). Then, we have
\[
\begin{align*}
\xi^* M_0 &= R\{1, Q_2u_3, Q_0Q_2u_3, Q_1Q_2u_3\} = R \oplus (F_{\infty,1} \cap E(2)), \\
\xi^* M_1 &= R\{Q_0Q_1u_3, e_3u_3, Q_0u_3, Q_1u_3\} = F_{\infty,1} \cap \overline{E}(2).
\end{align*}
\]
For \((G, p) = (E_6, 3)\) and \((E_7, 3)\), the rank of the elementary abelian \(p\)-subgroup \(A\) is 4. First, we consider the case \((G, p) = (E_6, 3)\). Let \(\overline{E}_3 = f_4^{-1}O_3\). Since
\[
F_{5,2} = R/(e_3)\{Q_1u_4, e_3Q_1u_4\},
\]
where \(w(Q_1u_4) = 2\) and \(w(Q_1u_4) = 0, 1\), we denote by \(y_4, y_10\) the generators of \(H^4(BG)\), \(H^{10}(BG)\) such that \(\xi^*(y_4) = Q_0\overline{E}_3u_4 = Q_0u_3, \xi^*(y_{10}) = Q_0Q_1u_4\), respectively. Let \(y_9 = -Q_1y_4, y_8 = \varphi^1y_4, y_20 = \varphi^3\varphi^1y_4\) and let \(y_{22} = \varphi^3y_{10}, y_{26} = \varphi^1y_{22}\). Let \(y_{30} = y_{10}y_{20}\). Let
\[
M_0 = R\{1, y_{22}, y_{26}, y_{20}, Q_2y_{10}, Q_2y_4, Q_2y_8, Q_2y_{30}\}, \\
M_1 = R\{y_9, y_9y_{22}, y_9y_{26}, y_9y_{20}, y_{10}, y_4, y_8, y_{30}\}.
\]
Then, we have
\[
\begin{align*}
\xi^* M_0 &= R\{1, Q_2Q_2u_4, Q_2\overline{E}_3u_4, Q_2\overline{E}_3u_4, Q_0Q_1Q_2u_4, Q_0Q_2Q_2u_4, Q_1Q_2\overline{E}_3u_4, e_3Q_2u_4\} \\
&= (R \oplus F_{\infty,2}) \cap E(2), \\
\xi^* M_1 &= R\{Q_0Q_1Q_3u_4, e_3\overline{E}_3u_4, Q_0Q_1Q_3u_4, Q_0Q_1Q_3u_4, e_3Q_1u_4\} \\
&= F_{\infty,2} \cap \overline{E}(2),
\end{align*}
\]
where \(i\) ranges over \(\{0, 1\}\).
As for \((G, p) = (E_7, 3)\), let
\[
M_0 = R\{1, y_{20}, Q_2y_4, Q_2y_8\}, \\
M_1 = R\{y_9, y_9y_{20}, y_4, y_8\}.
\]
Then, we have
\[
\begin{align*}
\xi^* M_0 &= R\{1, Q_2\overline{E}_3u_4, Q_2Q_2\overline{E}_3u_4\} = R \oplus (F_{\infty,2} \cap (\overline{E}_3u_4) \cap E(2)), \\
\xi^* M_1 &= R\{Q_0Q_1Q_3u_4, e_3\overline{E}_3u_4, Q_0Q_1Q_3u_4\} = F_{\infty,2} \cap (\overline{E}_3u_4) \cap \overline{E}(2),
\end{align*}
\]
where \(i\) ranges over \(\{0, 1\}\).
One can verify the following proposition by direct computations but it also follows immediately from Proposition 3.4.

**Proposition 4.3.** For \((G, p)\) in Theorem 1.1, there holds

1. \(\xi^* M_0 \oplus \xi^* M_1 = \text{Im} \xi^*\).

Moreover, there exist the following short exact sequences:

2. \(0 \to M_0 \xrightarrow{\text{Im} \xi^*} M_1 \to M_0^{\text{even}}/(e_k) \to 0\),
3. \(0 \to M_1 \xrightarrow{Q_k^{-1}} M_0 \to M_0^{\text{even}}/(e_k) \to 0\),
where \(m = 2, k = 2\) for \((G, p) = (PU(p), p)\) and \(m = 2p + 2, k = 3\) for \((G, p) = (F_4, 3), (E_6, 3), (E_7, 3), (E_8, 5)\).

5. The spectral sequence

In this section, we complete the proof of Theorem 1.1 by computing the Leray-Serre spectral sequence associated with

\[
G/T \xrightarrow{\iota} BT \xrightarrow{\eta} BG
\]

for \((G, p)\) in Theorem 1.1. We put \(m = 2, k = 2\) for \((G, p) = (PU(p), p)\) and \(m = 2p + 2, k = 3\) for \((G, p) = (F_4, 3), (E_6, 3), (E_7, 3), (E_8, 5)\).

The \(E_2\)-term of the spectral sequence is given by

\[
E_2 = H^*(BG) \otimes H^* (G/T)
\]
as an \(H^*(BG) \otimes S\)-algebra. The algebra generator is \(1 \otimes x_m\). So, the first nontrivial differential is determined by \(d_r(1 \otimes x_m)\) for some \(r \geq 2\).

**Proposition 5.1.** For \(r < m + 1, d_r = 0\). The first nontrivial differential is \(d_{m+1}\) and there holds

\[
d_{m+1}(1 \otimes x_m) = \alpha(y_{m+1} \otimes 1)
\]

for some \(\alpha \neq 0 \in \mathbb{Z}/p\).

**Proof.** Suppose that \(d_{r_0}(1 \otimes x_{m}) \neq 0\) for some \(r_0 < m + 1\). Then, up to degree \(\leq m\), \(E_{r_0+1}\)-term is generated by \(1 \otimes 1\) as an \(H^*(BG) \otimes S\)-module. So, for \(r_1 \geq r_0\), \(\operatorname{Im} d_{r_1}\) does not contain any element of degree less than or equal to \(m + 1\). Hence, \(y_{m+1} \otimes 1\) survives to the \(E_{\infty}\)-term. Then, \(\eta^* (y_{m+1}) \neq 0\). This contradicts the fact \(E_{\infty}^{\text{odd}} = \{0\}\), since \(\deg y_{m+1} = m + 1\) is odd. Therefore, we have \(d_r(1 \otimes x_m) = 0\) for \(r < m + 1\).

Next, we verify that \(d_{m+1}(1 \otimes x_m) = \alpha(y_{m+1} \otimes 1)\) for some \(\alpha \neq 0 \in \mathbb{Z}/p\). If \(\operatorname{Im} d_{m+1}\) does not contain \(y_{m+1} \otimes 1\), then up to degree \(\leq m + 1\), the spectral sequence collapses at the \(E_{m+2}\)-level and \(y_{m+1} \otimes 1\) survives to the \(E_{\infty}\)-term. As in the above, it is a contradiction. Hence, the proposition holds.

To consider the next nontrivial differential, first we show the following lemmas.

**Lemma 5.2.** Both

1. the multiplication by \(y_{m+1}\) and
2. the multiplication by \(e_k\)

are trivial on \(\ker \xi^*\).

**Proof.** Suppose that \(z \in \ker \xi^*\). Then, \(\xi^*(z \cdot y_{m+1}) = 0\) and \(\deg (z \cdot y_{m+1})\) is odd. Hence, we have \(z \cdot y_{m+1} = 0\) in \(H^*(BG)\).

We also get \(Q_{k-1}(z \cdot y_{m+1}) = 0\). On the other hand, since \(\xi^* (Q_{k-1}z) = 0\) and since \(\deg (Q_{k-1}z)\) is odd, we have \(Q_{k-1}z = 0\) in \(H^*(BG)\). Hence, we get

\[
Q_{k-1}(z \cdot y_{m+1}) = Q_{k-1} z \cdot y_{m+1} + (-1)^{k-1} z \cdot e_k = (-1)^{k-1} z \cdot e_k = 0.
\]

So, we obtain \(z \cdot e_k = 0\). Thus, we have the desired results.

By choosing suitable elements in \(H^*(BG)\) as generators, we may consider a subalgebra of \(H^*(BG)\), which maps to \(R\) in \(H^*(BA)^{(A)}\). By abuse of notation, we call it \(R\). Then, we may consider

\[
E_{m+1} = \cdots = E_2 = (M_0 \oplus M_1 \oplus \ker \xi^*) \otimes H^*(G/T),
\]
as an $R \otimes S$-module. By Propositions 5.1 and 4.3 (2) and Lemma 5.2 (1), we have the $E_{m+2}$-term:
\[E_{m+2} = (M_1 \otimes N_{p-1}) \oplus (M^{even}_1/(e_k) \otimes N_{2p-2}) \oplus (M_0 \otimes N_0) \oplus (\text{Ker} \xi^* \otimes H^*(G/T)),\]
where $N_{\leq i}$ is the $S$-submodule of $H^*(G/T)$ generated by $x_m^k$ ($k \leq i$) and $N_i$ is the $S$-submodule generated by a single element $x_m^i$ in $H^*(G/T)$. Observe that the above direct sum decomposition is in the category of $R \otimes S$-modules.

Now, we investigate the action of the Weyl group on the spectral sequence in terms of $\sigma_j$. Recall that we define $\{r_j\}$ to be the generators of the Weyl group $W$ and that $\sigma_j = 1 - r_j^*$ where $\sigma_j$ acts on the spectral sequence by $\sigma_j(y \otimes x) = y \otimes \sigma_j(x)$ and so it commutes with the differential $d_r$ for $r \geq 2$.

**Lemma 5.3.** There holds $\sigma_j(x_m^i) \in N_{\leq i - 1}$ for all $\sigma_j$.

**Proof.** Since $d_{m+1}$ commutes with $\sigma_j$ and since $\sigma_j(y_{m+1} \otimes 1) = 0$, we have
\[d_{m+1}(\sigma_j(1 \otimes x_m)) = 0.\]
Suppose that $\sigma_j(x_m) = \beta x_m + s$ for some $\beta \in \mathbb{Z}/p$ and $s$ in $S$. Then, we have
\[d_{m+1}(\beta(1 \otimes x_m) + 1 \otimes s) = \alpha \beta(y_{m+1} \otimes 1) = 0.\]
Therefore, we have $\beta = 0$ and $\sigma_j(x_m) \in N_0 = S$. In general, we have
\[\sigma_j(xy) = \sigma_j(x)y + x\sigma_j(y) - \sigma_j(x)\sigma_j(y).\]
Hence, we have
\[\sigma_j(x_m^i) = \sigma_j(x_m)x_m^{i-1} + x_m\sigma_j(x_m^{i-1}) - \sigma_j(x_m)\sigma_j(x_m^{i-1}) \in N_{\leq i-1},\]
as desired. \qed

**Remark 5.4.** By Lemma 5.3, $\sigma_j$ acts trivially on $N_i = N_{\leq i}/N_{\leq i-1}$. Hence, it is easy to see that
\[(E_{m+2}^{r,s})^W = (M_1^{odd} \oplus e_kM^{even}_1) \otimes N_{p-1} \oplus (M^{even}_1/(e_k) \otimes M_0 \oplus \text{Ker} \xi^*) \otimes \mathbb{Z}/p \neq E_{m+2}^{r,0}.\]

Now, we begin to compute the next nontrivial differential.

**Proposition 5.5.** For $r \geq m + 2$ such that $E_r = E_{m+2}$, we have
\[d_r(M_0 \otimes N_0) = d_r(\text{Ker} \xi^* \otimes H^*(G/T)) = d_r(M_1^{even}/(e_k) \otimes N_{\leq p-2}) = \{0\}.\]

**Proof.** Since $M_0 \otimes N_0$ is generated by $M_0 \otimes \mathbb{Z}/p$ as an $R \otimes S$-module, $d_r(M_0 \otimes N_0) = \{0\}$ holds for $r \geq m + 2$. For $M^{even}_1/(e_k) \otimes N_{\leq p-2}$, there exist no odd degree generators. Hence, we have $d_r(M^{even}_1/(e_k) \otimes N_{\leq p-2}) \subset E_{m+2}^{odd} = M^{odd}_1 \otimes N_{p-1} \oplus M^{odd}_0 \otimes N_0$. On the one hand, the multiplication by $e_k \otimes 1$ is zero on $M^{even}_1/(e_k) \otimes N_{\leq p-2}$. On the other hand, the multiplication by $e_k \otimes 1$ is a monomorphism on $M^{odd}_1 \otimes N_{p-1} \oplus M^{odd}_0 \otimes N_0$. Hence, we have
\[d_r(M^{even}_1/(e_k) \otimes N_{\leq p-2}) = \{0\}.\]
Finally, by Lemma 5.2, the same holds for $\text{Ker} \xi^* \otimes H^*(G/T)$ and so we obtain
\[d_r(\text{Ker} \xi^* \otimes H^*(G/T)) = \{0\}.\]

Let $n = m(p - 1)$ for the sake of notational simplicity. Next, we show the following proposition.

**Proposition 5.6.** If $r \geq m + 2$ and if $d_r$ is nontrivial, then $r \geq n + 1$. 

Proof. Suppose that we have a nontrivial differential $d_r$ for some $r < n + 1$, say,

$$d_r(z \otimes x_m^{p-1}) = z_{i_1} \otimes x'_{i_1} + \cdots + z_{i_t} \otimes x'_{i_t},$$

where $z \in M_1$, $1 \leq i_1 < \cdots < i_t \leq L$, $\{z_1, \ldots, z_L\}$ is a basis for $M^1$,

and $x'_{i_1}, \ldots, x'_{i_t} \in H^{n-r+1}(G/T)$, $x'_{i_1}, \ldots, x'_{i_t} \neq 0$. Since $H^*(G/T)^W = \mathbb{Z}/p$, for $x'_{i} \neq 0$ in $H^{n-r+1}(G/T)$, there exists $\sigma_j$ such that $\sigma_j(x'_{i}) \neq 0$. Therefore, we have

$$\sigma_j d_r(z \otimes x_m^{p-1}) \neq 0.$$

On the other hand, by Lemma 5.3, we have $\sigma_j(x_m^{p-1}) \in N_{\leq p-2}$. Hence, by Proposition 5.3 above, we have

$$\sigma_j d_r(z \otimes x_m^{p-1}) \in d_r(M^1 \otimes N_{\leq p-2}) = \{0\}.$$

This is a contradiction. Hence, we have $r \geq n + 1$. \qed

Finally, we complete the computation of the spectral sequence.

**Proposition 5.7.** There holds $d_{n+1}(M_1 \otimes N_{p-1}) = (M_0^{\text{odd}} \oplus e_k M_0^{\text{even}}) \otimes N_0$.

Proof. The $E_{n+1}$-term is equal to

$$M_1 \otimes N_{p-1} \oplus M^1 \otimes (e_k) \otimes N_{\leq p-2} \oplus M_0 \otimes N_0 \oplus (\text{Ker} \xi^*) \otimes H^*(G/T)$$

and

$$d_{n+1}(M^1 \otimes (e_k) \otimes N_{\leq p-2} \oplus M_0 \otimes N_0 \oplus (\text{Ker} \xi^*) \otimes H^*(G/T)) = \{0\}.$$

Since $M^1 \otimes (e_k) \otimes N_{\leq p-2} \oplus M_0 \otimes N_0 \oplus (\text{Ker} \xi^*) \otimes H^*(G/T)$ is generated by the elements of the second degree less than $n$, that is, the elements in $E^*_{n,n'}$ ($n' < n$), it is clear that

$$d_r(M^1 \otimes (e_k) \otimes N_{\leq p-2} \oplus M_0 \otimes N_0 \oplus (\text{Ker} \xi^*) \otimes H^*(G/T)) = \{0\}$$

for all $r \geq n + 1$.

On the other hand, since all the elements in $(M_0^{\text{odd}} \oplus e_k M_0^{\text{even}}) \otimes \mathbb{Z}/p$ do not survive to the $E_\infty$-term and since $d_r(M_0 \otimes N_0) = \{0\}$ for all $r \geq 2$, all the elements in $(M_0^{\text{odd}} \oplus e_k M_0^{\text{even}}) \otimes \mathbb{Z}/p$ must be hit by nontrivial differentials.

Suppose that there exists an element in $(M_0^{\text{odd}} \oplus e_k M_0^{\text{even}}) \otimes \mathbb{Z}/p$ which is not hit by $d_{n+1}$. Let $z \otimes 1$ be such an element with the lowest degree $s$. Up to degree $s$, by Proposition 5.3, we see that

$$d_{n+1} : M_1 \otimes N_{p-1} \to (M_0^{\text{odd}} \oplus e_k M_0^{\text{even}})^{i+n+1} \otimes N_0$$

is an isomorphism for $i < s$.

Then, Ker $d_{n+1}$ is equal to $M_1 \otimes (e_k) \otimes N_{\leq p-2} \oplus M_0 \otimes (\text{Ker} \xi^*) \otimes H^*(G/T)$ up to degree $s$. Therefore, for $r \geq n + 2$, Im $d_r = \{0\}$ up to degree $\leq s$. Hence the element $z \otimes 1$ survives to the $E_\infty$-term. This is a contradiction. So, the proposition holds. \qed

Thus, by Propositions 5.3 and 5.7, we have

$$E_{n+2} = (M^1 \otimes (e_k) \otimes N_{p-2}) \oplus (M^1 \otimes (e_k) \otimes N_0) \oplus (\text{Ker} \xi^* \otimes H^*(G/T)).$$

Since there are no odd degree elements in the $E_{n+2}$-term, the spectral sequence collapses at the $E_{n+2}$-level and we obtain $E_\infty = E_{n+2}$ and

$$(E^*_{n,n'})^W = E_{n,n'}^0 = (M^1 \otimes (e_k) \otimes M^0 \otimes (e_k) \otimes (\text{Ker} \xi^*) \otimes \mathbb{Z}/p.$$
This completes the proof of Theorem 1.1.

References

[1] K. Andersen, J. Grodal, J. Møller, and A. Viruel, The classification of p-compact groups for p odd, Ann. of Math. (2) 167 (2008), no. 1, 95–210. MR2373153 (2009a:55012)

[2] V. G. Kac, Torsion in cohomology of compact Lie groups and Chow rings of reductive algebraic groups, Invent. Math. 80 (1985), no. 1, 69–79. MR0784529 (86m:57041)

[3] M. Kameko, Cohomology of the cyclic group \( \mathbb{Z}/p \). Surikaisekikenkyusho Kokyuroku No.1679 (2010), 98–112.

[4] M. Kameko and M. Mimura, On the Rothenberg-Steenrod spectral sequence for the mod 3 cohomology of the classifying space of the exceptional Lie group \( E_8 \), in Proceedings of the Nishida Fest (Kinosaki 2003), 213–226, Geom. Topol. Monogr., 10 Geom. Topol. Publ., Coventry. MR2402786 (2009g:55019)

[5] M. Kameko and M. Mimura, M"ui invariants and Milnor operations, in Proceedings of the School and Conference in Algebraic Topology, 107–140, Geom. Topol. Monogr., 11 Geom. Topol. Publ., Coventry. MR2402803 (2009g:55020)

[6] M. Kameko and M. Mimura, The Weyl group invariants – the case of projective unitary group \( PU(p) \) –, Surikaisekikenkyusho Kokyuroku, to appear.

[7] N. Kitchloo, On the topology of Kac-Moody groups. [arXiv:0810.0851]

[8] M. Mimura and Y. Sambe, On the cohomology mod \( p \) of the classifying spaces of the exceptional Lie groups. I, J. Math. Kyoto Univ. 19 (1979), no. 3, 553–581. MR0553232 (81c:55025)

[9] M. Mimura and Y. Sambe, Collapsing of the Eilenberg-Moore spectral sequence mod 5 of the compact exceptional group \( E_8 \), J. Math. Kyoto Univ. 21 (1981), no. 2, 203–230. MR0625605 (82j:57040)

[10] H. Toda, Cohomology mod 3 of the classifying space \( BF_4 \) of the exceptional group \( F_4 \), J. Math. Kyoto Univ. 13 (1973), 97–115. MR0321086 (47 #9619)

[11] H. Toda, Cohomology of the classifying space of exceptional Lie groups, in Manifolds-Tokyo 1973 (Proc. Internat. Conf., Tokyo, 1973), 265–271, Univ. Tokyo Press, Tokyo. MR0368059 (51 #4301)