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Partitioned Finite Element Method for port-Hamiltonian systems with Boundary Damping: Anisotropic Heterogeneous 2D wave equations

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Abstract A 2D wave equation with boundary damping of impedance type can be recast into an infinite-dimensional port-Hamiltonian system (pHs) with an appropriate feedback law, where the structure operator \( \mathcal{J} \) is formally skew-symmetric. It is known that the underlying semigroup proves dissipative, even though no dissipation operator \( \mathcal{R} \) is to be found in the pHs model. The Partitioned Finite Element Method (PFEM) introduced in Cardoso-Ribeiro et al. (2018), is structure-preserving and provides a natural way to discretize such systems. It gives rise to a non null symmetric matrix \( \mathcal{R} \). Moreover, since this matrix accounts for boundary damping, its rank is very low: only the basis functions at the boundary have an influence. Lastly, this matrix can be factorized out when considering the boundary condition as a feedback law for the pHs, involving the impedance parameter. Note that pHs – as open system – is used here as a tool to accurately discretize the wave equation with boundary damping as a closed system.

In the worked-out numerical examples in 2D, the isotropic and homogeneous case is presented and the influence of the impedance is assessed; then, an anisotropic and heterogeneous wave equation with space-varying impedance at the boundary is investigated.

Keywords: Port-Hamiltonian systems (pHs), distributed-parameter system (DPS), structure preserving discretization, partitioned finite element method (PFEM), boundary damping.

1. INTRODUCTION

The analysis and modeling of multi-physical system as port-Hamiltonian systems (pHs) originates from Maschke and van der Schaft (1992), where the authors proposed a Hamiltonian formulation of lumped-parameter systems by introducing ports which account for energy exchange with the external environment. In van der Schaft and Maschke (2002), the authors provided a generalization of pHs formulation to distributed-parameter systems in a differential-geometrical setting. For an introductory overview, we refer the reader to e.g. van der Schaft and Jeltsema (2014); Duindam et al. (2009). Subsequently, several works have been carried out on the construction of a mathematical framework for linear infinite-dimensional pHs with the purpose of proving existence, uniqueness, and stability of the solutions, see e.g. Jacob and Zwart (2012).

In an attempt to perform numerical simulation or to apply some dedicated control laws, a growing body of structuring-preserving (SP) methods has been investigated in the last decade. By SP discretization, we mean a mimetic method that preserves the structure and all the properties of the infinite-dimensional pHs, when passing to the finite-dimensional system. In this direction, the authors in Golo et al. (2004) proposed a "mixed finite element method", also a pseudo-spectral method is provided for 1D pHs in in Moulla et al. (2012); a generalization of this method to arbitrary spatial dimension using mixed Galerkin discretization was recently proposed in Kotyczka et al. (2018).

A recent SP method, first introduced in Cardoso-Ribeiro et al. (2018), named Partitioned Finite Element Method (PFEM), is a kind of mixed finite element methods, based on the weak formulation of the port-Hamiltonian system, where an integration by parts is performed on a subset of the variables only. In this method, boundary control and observation appear naturally, which allows simulation and control of infinite-dimensional pHs. A major advantage of PFEM is that it can be easily implemented in a Finite Element numerical software, and does not require further specific treatment.

In this paper, we study the effect of boundary damping on a wave equation in 2D: the main parameter of interest is the impedance \( Z \). The first systematic study and simulation of such a wave equation with an absorbing boundary condition was conducted in Engquist and Majda (1977), while in Cowsar et al. (1996), a mixed finite element method with different time integration schemes was fully studied. Otherwise, this paper presents an analysis and discretization of the dissipative pHs of anisotropic hetero-

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geneous wave equation with boundary damping. A class of dissipative PHSs with structural and fluid damping was studied in Matignon and Hélie (2013) and Le Gorrec and Matignon (2013), while in Villegas et al. (2009) a boundary damping was used to get the exponential stability of a 1-D system. The theoretical analysis of our dissipative PHSs heavily relies on the results of Kurula and Zwart (2015). Then, the discretization procedure is carried out in two steps: first, the PHS system with boundary control and observation is considered alone, and PFEM is applied to it, leading to collocated boundary control and observation of finite dimension; second, an output feedback control law is applied, at the discrete level, which mimicks the impedance boundary condition; moreover, in the last stage, a dissipation matrix $R$ appears in a straightforward way. The motivation for such an approach relies on avoiding the explicit discretization of $D(J)$: indeed, since the boundary condition is natural (i.e. included in the domain of the unbounded operator), and not essential (i.e. included in the weak formulation of interest), one might think about adding some Lagrange multipliers (DAE) to overcome this difficulty, but this strategy proves irrelevant for such a problem. Finally, another advantage of this two-step approach is to provide a factorization of the dissipation matrix $R$ giving meaningful information about its low rank, as shown in § 3.4.

The paper is organized as follows, Section 2 reformulates the mathematical analysis of our wave equation based on results in Kurula and Zwart (2015). In Section 3, the discretization of the PHS formulation of the boundary controlled and observed wave equation is presented first, then the output feedback control law related to the impedance boundary condition is applied, the resulting matrices and their properties are given. Section 4 is dedicated to the simulation results of two worked-out examples of the wave equation, as a closed system.

2. INFINITE-DIMENSIONAL PORT-HAMILTONIAN SYSTEM

2.1 Physical model at stake

We consider the 2-dimensional heterogeneous anisotropic wave equation defined for all $t \geq 0$ as

$$\rho(x) \partial_t^2 w(t,x) = \text{div} \left( \nabla w(t,x), x \in \Omega, \right)$$

$$Z(x) \left( \nabla w(t,x), \right) \cdot \n + \partial_t w(t,x) = 0, x \in \partial \Omega,$n(0,x) = w_0(x), x \in \Omega, t = 0,$n(0,x) = w_1(x), x \in \Omega, t = 0,$

where $\Omega \subset \mathbb{R}^N$ is an open bounded spatial domain with Lipschitz-continuous boundary $\partial \Omega$, $w(t,x)$ is the deflection from the equilibrium position at point $x \in \Omega$ and time $t \geq 0$, and vector $n$ denotes the outward normal at the boundary. The physical parameters in the domain are $\rho \in L^\infty(\Omega)$ (positive and bounded from below) and $\nabla$ is the usual Hilbert spaces associated to the scalar products $(\cdot, \cdot)_\Omega$ and $(\cdot, \cdot)_{\Omega}$. Then $H^1(\Omega)$ and $H^2(\Omega)$ are the usual Sobolev spaces, where $H^3(\Omega) := \{ \psi \in L^2(\Omega)^N, \text{div} \psi \in L^2(\Omega) \}$. 

In 1D, the critical value of impedance providing the balance between Dirichlet trace of the velocity and normal trace of the stress in (3) is called characteristic impedance $Z_c$. When $Z < Z_c$ we get a dominant Dirichlet boundary condition, while for $Z > Z_c$ the normal trace of the stress is dominant. From acoustical literature, if $\nabla = T_{\Omega}/\rho_0$, and $\rho = \rho_0$ are constant then the characteristic impedance is given by $Z_c \equiv \sqrt{T_{\Omega}/\rho_0}$. 

2.2 Formulation as an infinite-dimensional Hamiltonian system

In order to formulate the wave equation with impedance boundary condition as a (closed) Hamiltonian system, we first introduce the energy variables $\alpha := [\alpha_q \alpha_p]^T$ and $(\cdot, \cdot)_{\Omega}$. Then $H^1(\Omega)$ and $H^2(\Omega)$ are the usual Sobolev spaces, where $H^2(\Omega) := \{ v_q \in L^2(\Omega)^N, \text{div} v_q \in L^2(\Omega) \}$.

The power balance of this system is computed by the time derivation of the Hamiltonian along the trajectories, it reads thanks to Green's formula

$$\frac{d}{dt} \mathcal{H}(t) = - ||\sqrt{Z} e_q \cdot n||_{L^2(\partial \Omega)}^2 \leq 0,$$

where $|| \cdot ||_{L^2(\partial \Omega)}$ is the norm on $L^2(\partial \Omega)$.

More compactly, (2) can be rewritten as

$$\partial_t \alpha = J e \quad \text{or} \quad \partial_t \alpha = J Q \alpha,$$

where $J := \begin{bmatrix} 0 & \nabla \cr \text{div} & 0 \end{bmatrix}$ is the unbounded interconnection operator, and $Q := \begin{bmatrix} \nabla & 0 \cr 0 & 1/\rho \end{bmatrix}$ is the bounded operator containing the physical parameters of the system, it is symmetric and positive definite. Finally, (3) also writes with the energy variables as: $Z (\nabla \cdot \alpha_q \cdot n + \alpha_p) = 0$.
The dynamical part of pHs (6) in weak form reads
\[
\langle \partial_t \alpha_q, v_q \rangle_{\Omega} = \langle - (e_p, \text{div } v_q), v_q \rangle_{\Omega} + \langle u_0, v_q \rangle_{\partial \Omega},
\]
where \(v_q\) and \(e_p\) are sufficiently smooth test functions.

Applying Green’s formula, it appears explicitly in the new system: in the sequel, since the control term \(u_0\) is chosen, then the corresponding subsystem to be integrated is (7). Thus
\[
\begin{align*}
\langle \partial_t \alpha_q, v_q \rangle_{\Omega} &= \langle - (e_p, \text{div } v_q), v_q \rangle_{\Omega} + \langle u_0, v_q \rangle_{\partial \Omega}, \\
\langle \partial_t \alpha_p, v_p \rangle_{\Omega} &= \langle \text{div } e_q, v_p \rangle_{\Omega},
\end{align*}
\]

Note that the physical meaning of this control is not addressed here, as the aim is to provide an accurate discretization of (3).

3.3 PFEM

The weak form of (6), with \(v_q\), \(v_p\) and \(v_\theta\) sufficiently smooth scalar or vector-valued functions, is then
\[
\begin{align*}
\langle \partial_t \alpha_q, v_q \rangle_{\Omega} &= -(e_p, \text{div } v_q)_{\Omega} + \langle u_0, v_q \rangle_{\partial \Omega}, \\
\langle \partial_t \alpha_p, v_p \rangle_{\Omega} &= \langle \text{div } e_q, v_p \rangle_{\Omega}, \\
\langle v_\theta, y_\theta \rangle_{\Omega} &= \langle \psi_\theta, e_\theta \cdot n \rangle_{\partial \Omega}.
\end{align*}
\]

A suitable approximation of (8), starts by choosing families of finite-dimensional bases (in general polynomials, conforming with the mathematical framework). Then the proposed families of approximation are
\[
\begin{align*}
V_q &= \text{span}(\varphi_q^1)_{1 \leq m \leq N_q}, \\
V_p &= \text{span}(\varphi_p^1)_{1 \leq m \leq N_p}, \\
V_\theta &= \text{span}(\psi_\theta^m)_{1 \leq m \leq N_\theta}.
\end{align*}
\]

Henceforward, we define the notation
\[
\begin{align*}
\alpha_q(t, x) &\approx \alpha_q^0(t, x) = \sum_{k=1}^{N_q} \alpha_q^k(t) \varphi_q^k(x) = \Phi_q^T \cdot \alpha_q^k, \\
e_q(t, x) &\approx e_q^0(t, x) = \sum_{k=1}^{N_q} e_q^k(t) \varphi_q^k(x) = \Phi_q^T \cdot e_q^k, \\
\alpha_p(t, x) &\approx \alpha_p^0(t, x) = \sum_{k=1}^{N_p} \alpha_p^k(t) \varphi_p^k(x) = \Phi_p^T \cdot \alpha_p^k.
\end{align*}
\]

Similarly for the boundary variables
\[
\begin{align*}
u_\theta(t, x) &\approx u_\theta^0(t, x) = \sum_{m=1}^{N_\theta} u_\theta^m(t) \psi_\theta^m(x) = \Psi_\theta^T \cdot u_\theta^m, \\
y_\theta(t, x) &\approx y_\theta^0(t, x) = \sum_{m=1}^{N_\theta} y_\theta^m(t) \psi_\theta^m(x) = \Psi_\theta^T \cdot y_\theta^m.
\end{align*}
\]

Plugging the approximation (9)–(10)–(11) into the weak form (8), and choosing any test functions \(v_q\), \(v_p\) and \(v_\theta\) in \(V_q\), \(V_p\) and \(V_\theta\) leads to
\[
\begin{align*}
\Sigma_{i=1}^{N_q} \langle \varphi_q^i, \varphi_q^i \rangle_{\Omega} \frac{d}{dt} \alpha_q^i &= -\sum_{k=1}^{N_q} \langle \varphi_q^k, \text{div } \varphi_q^i \rangle_{\Omega} e_q^k, \\
+ \langle \sum_{m=1}^{N_\theta} \psi_\theta^m \cdot \varphi_q^i, \psi_\theta^m \rangle_{\partial \Omega} u_\theta^k.
\end{align*}
\]

In matrix form, we obtain
\[
\begin{align*}
M_q \frac{d}{dt} \alpha_q &= D \cdot \alpha_p + B \cdot \psi_\theta, \\
M_p \frac{d}{dt} \alpha_p &= -D^T \cdot \alpha_q, \\
M_\theta \frac{d}{dt} \psi_\theta &= B^T \cdot \alpha_q,
\end{align*}
\]
\[(M_q)_{ij} = (\varphi_q^j, \varphi_q^i)_\Omega, \quad M_q = \int_\Omega \Phi_q . \Phi_q^T \in \mathbb{R}^{N_q \times N_q},\]
\[(M_p)_{k\ell} = (\varphi_p^\ell, \varphi_p^k)_\Omega, \quad M_p = \int_\Omega \Phi_p . \Phi_p^T \in \mathbb{R}^{N_p \times N_p},\]
\[(D)_{jk} = -(\varphi_p^j, \text{div } \varphi_q^k)_\Omega, \quad D = -\int_\Omega \text{div } \Phi_q . \Phi_q^T \in \mathbb{R}^{N_p \times N_p},\]
\[(B)_{jm} = (\varphi^m_\rho, \varphi_q^j)_\Omega, \quad B = \int_\Omega \Phi_q . n . \Phi_q^\rho \in \mathbb{R}^{N_q \times N_\rho},\]
\[(D_{m\rho})_{n\rho} = (\varphi^m_\rho, \varphi^\rho_n)_\Omega, \quad D_{m\rho} = \int_\Omega \Phi_q . \Phi_q^\rho \in \mathbb{R}^{N_q \times N_\rho}.
\]

System (12) is a finite-dimensional pHs, as defined in Egger et al. (2018), i.e. including mass matrices.

For simulation and control purposes, it is very interesting to get the discrete counterpart of \( Q \) and substitute the discrete part of the variable \( e \) by \( \alpha \). To do so, we rewrite the constitutive relation (1) in a weak sense

\[ (e_q, v_q)_\Omega = (\mathbf{T} \alpha_q, v_q)_\Omega \]

by writing and testing \( e_q \) and \( \alpha_q \) over the family of basis functions \( \{\varphi_q^i\}_{1 \leq i \leq N_q} \), we have

\[ \left[ \int_\Omega \Phi_q . \Phi_q^T \right] e_q = \left[ \int_\Omega \Phi_q . \mathbf{T} \Phi_q^T \right] \alpha_q, \]

then we can define \( Q_T \) as the spatially averaged value of \( \mathbf{T} \) on the basis functions \( \{\varphi_q^i\}_{1 \leq i \leq N_q} \) as

\[ e_q = Q_T \alpha_q, \quad Q_T \in \mathbb{R}^{N_q \times N_q}. \]

For example in a homogeneous isotropic problem, i.e. \( \mathbf{T} = T_0 l_2 \) with \( T_0 \equiv \text{const} \), then \( Q_T = T_0 l_2 \), i.e. \( e_q = Q_T \).

Likewise for \( e_p = \frac{1}{\rho} \alpha_p \), we define \( Q_p \) as

\[ e_p = \left[ \int_\Omega \Phi_p . \Phi_p^T \right]^{-1} \left[ \int_\Omega \Phi_p . \mathbf{T} \Phi_p^T \right] \alpha_p \]

\[ \Rightarrow e_p = Q_p \alpha_p, \quad Q_p \in \mathbb{R}^{N_p \times N_p}. \]

Hence, we can write (12) as

\[ \begin{pmatrix} M_q & 0 \\ 0 & M_p \end{pmatrix} \frac{d}{dt} \begin{pmatrix} \alpha_q \\ \alpha_p \end{pmatrix} = \begin{pmatrix} 0 & -D^T \\ -D & 0 \end{pmatrix} \begin{pmatrix} Q_T & 0 \\ 0 & Q_p \end{pmatrix} \begin{pmatrix} \alpha_q \\ \alpha_p \end{pmatrix} + \begin{pmatrix} B & 0 \end{pmatrix} y_\theta. \]

\[ \Rightarrow \frac{d}{dt} y_\theta = M_{\rho0}^{-1} B^T Q_T \alpha_q. \]

Discrete Hamiltonian Then we define the discrete Hamiltonian as

\[ \mathcal{H}_d(t) := \frac{1}{2} \int_\Omega \alpha_q^T \mathbf{T} \cdot \alpha_q + \alpha_p^T \mathbf{T}^\rho \cdot \mathbf{T} \alpha_p \]

\[ = \frac{1}{2} \left( \alpha_q^T \left( \int_\Omega \Phi_q . \mathbf{T} \Phi_q^T \right) \alpha_q + \alpha_p^T \left( \int_\Omega \Phi_p . \mathbf{T} \Phi_p^T \right) \alpha_p \right) \]

\[ = \frac{1}{2} \left( \alpha_q^T M_q Q_T \alpha_q + \alpha_p^T M_p Q_p \alpha_p \right) \]

Moreover, the discrete power balance is

\[ \frac{d}{dt} \mathcal{H}_d(t) = y_\theta^T M_\theta y_\theta := (u_\theta, y_\theta)_{\theta}. \]

which perfectly fits with the pHs framework of Egger et al. (2018).

3.4 Closed-loop system

The boundary condition (3), \( Z e_q . n + e_p = 0 \), corresponds to an output feedback law \(-Z\), hence system (6) can be rewritten with \( u_\theta = -Z y_\theta \), and once again, the closure relation is to be approximated in a weak sense

\[ (u_\theta, \psi_\theta)_\Omega = (Z y_\theta, \psi_\theta)_\Omega. \]

\[ \Rightarrow \psi_\theta = Z \mathcal{D}_y \psi_\theta, \quad \mathcal{D}_y = M_{\rho0}^{-1} B^T Q_T \alpha_q. \]

Thus we can plug the latter expression in the discrete system (13)

\[ \begin{cases} M_q \frac{d}{dt} \alpha_q = D \alpha_q + Z \mathcal{D}_y \psi_\theta \\ M_p \frac{d}{dt} \alpha_p = -D^T Q_T \alpha_q \end{cases} \]

Let us set

\[ R_Z := B \mathcal{D}_y M_{\rho0}^{-1} B^T, \]

then the finite-dimensional dissipative Hamiltonian system derived from system (2)-(3) is

\[ \begin{pmatrix} M_q & 0 \\ 0 & M_p \end{pmatrix} \frac{d}{dt} \begin{pmatrix} \alpha_q \\ \alpha_p \end{pmatrix} = \begin{pmatrix} -R_Z D & 0 \\ 0 & Q_p \end{pmatrix} \begin{pmatrix} \alpha_q \\ \alpha_p \end{pmatrix} \]

More compactly

\[ M_{d0} \frac{d}{dt} \alpha = (J_d - R) Q_d \alpha, \]

where \( \alpha = \begin{pmatrix} \alpha_q \\ \alpha_p \end{pmatrix} \), \( M_d = \begin{pmatrix} M_q & 0 \\ 0 & M_p \end{pmatrix} \), \( J_d = \begin{pmatrix} 0 & D \\ -D^T & 0 \end{pmatrix} \), \( Q_d = \begin{pmatrix} Q_T & 0 \\ 0 & Q_p \end{pmatrix} \) and \( R \geq 0 \), furthermore \( R_Z \) is of maximum rank \( N_\theta \), since \( M_{Z0} \) is.
Figure 1. The spectrum for \( L_x = 2, T = 1 \) and \( \rho = \frac{1}{6} \) for: (left): \( Z = 0 \); (center): \( Z = 0.1 \); (right): \( Z = 10^8 \approx \infty \).

Discrete Hamiltonian We get the discrete power balance, counterpart of (4), from (14)

\[
\frac{d}{dt} H_d(t) = \left\langle u_0, y_0 \right\rangle_\partial - \left\langle Z_d y_0, y_0 \right\rangle_\partial \leq 0.
\]

Indeed, \( Z_d \) is a positive semi-definite matrix with respect to \( \cdot, \cdot \)\(_\partial\), since \( \left\langle Z_d y_0, y_0 \right\rangle_\partial := y_0^T M_0 Z_d y_0 = y_0^T M_Z y_0 \) and \( M_Z \) is positive semi-definite by construction.

4. NUMERICAL SIMULATION: TWO CASE STUDIES

In § 4.1, the classical test case of a uniform impedance is investigated, and an optimal value of this impedance is discussed. Then, in § 4.2, our method is applied to a more complex test case, providing coherent results.

The simulation and computation below are performed using FEniCS software and Python 3. We consider a rectangle domain \( \Omega = (0, L_x) \times (0, 1) \) and the chosen finite elements families are \( RT_0 \) (lowest order Raviart-Thomas element), \( P_1 \) (Lagrange element) and \( P_1 \) respectively for \( q, p \) and \( \vartheta \) variables. The associated numbers of degrees of freedom are \( N_q = 369, N_p = 140 \) and \( N_\vartheta = 48 \).

4.1 A simple test case: isotropic, homogeneous medium with uniform impedance at the boundary

Following Kergomard et al. (2006) in the 1D case, where both the spectrum and the eigenfunctions are analytically computed (the Riesz basis property of the eigenfunctions is also fully proved), we can infer some spectral properties of the 2D case when \( \Omega \) is rectangle.

First, we look at the spectrum for three different values of \( Z = 0, 0.1, \infty \). As said in § 2.1, \( Z = 0 \) and \( Z = \infty \) mean that the system is conservative, implying that their spectrums must lie on \( \mathbb{i} \mathbb{R} \), which is confirmed in the left and right plot in Figure 1 (numerically we chose \( \infty \approx 10^8 \)). For \( Z = 0.1 \), we see that the eigenvalues of \( Z = 0 \) moved from \( \mathbb{i} \mathbb{R} \) to the left half-plane.

Remark 1. Since zero is always an eigenvalue, one may show that the proposed method preserves also the stationary fields.

Optimal value of the impedance It is interesting to see the behavior of the spectrum in term of \( Z \). Then, Figure 2 shows the largest non-null real part of the eigenvalues of the system (16) (i.e. \( -\max_{\lambda \neq 0} \Re(\lambda) \)) as a function of \( Z \). The computations are done for \( T_0 = 5 \) and \( \rho_0 = 3 \), then the peak in Figure 2 allows to find the numerical optimal impedance \( Z_{\text{opt}} = 1.16 \) in order to obtain the optimal decay rate in (17).

Another important point already emphasized on Figure 1 is remarkable on Figure 2: if \( Z = 0, \infty \), we indeed have a conservative equation, i.e. \( JQ \) is skew-adjoint in the appropriate functional space.

4.2 A more involved test case: anisotropic, heterogeneous medium with space-varying impedance at the boundary

For an anisotropic heterogeneous case, we compute the discrete Hamiltonian (Figure 3), with physical parameters \( \rho, \mathbf{T} \) and \( Z \) taking different values in the domain and at the boundary. Let \( L_x = 3, \Omega = \bigcup_{i=1}^4 \Omega_i \) and \( \partial \Omega = \bigcup_{i=1}^5 \Gamma_i \), we set \( \mathbf{T}|_{\Omega_1 \cup \Omega_3} = \begin{bmatrix} 1 & 0.5 \\ 0.5 & 1 \end{bmatrix}, \mathbf{T}|_{\Omega_2 \cup \Omega_4} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}, \rho|_{\Omega_1 \cup \Omega_3} = 0.5, \rho|_{\Omega_2 \cup \Omega_4} = 1 \). Figure 3 shows the evolution of the discrete Hamiltonian over time. In \( t \in [0, \frac{1}{3} t_f] \), we have a conservative system: we do not apply boundary control, indeed we see the conservation of the Hamiltonian. Just after, for \( t > \frac{1}{3} t_f \), we apply the output feedback control law, mimicking the impedance boundary condition, then the system becomes dissipative. Note however that it does not decay to 0 due to the existence of stationary solutions (\( \lambda = 0 \) belongs to the spectrum).

The time integration is performed thanks to Crank-Nicholson scheme (2nd order accurate, unconditionally stable) with \( t_f = 20 \) and \( dt = 10^{-3} \). The initial data are a trigonometric vector function for \( \alpha_q(0) \) and zero for \( \alpha_p(0) \), which prove compatible with the boundary conditions.
5. CONCLUSION AND PERSPECTIVES

To sum up, a structure-preserving discretization of the anisotropic heterogeneous wave equation with boundary damping has been presented. From the proposed strategy, the dissipation matrix $R$ that was hidden in $D(JQ)$ in the continuous model has been derived, and, thanks to an equivalent formulation in terms of output feedback control law, a meaningful factorization of $R$ that makes its low rank obvious has been obtained. New scalar products associated to the mass matrices fit perfectly into the finite element framework, furthermore they allow for recovering a discrete Hamiltonian system from the discrete Hamiltonian $H_d$. The overall theoretical and numerical work has been illustrated by simulation results, both on the well-known constant coefficient cases and on some more general situation, either anisotropic, heterogeneous or with space-varying impedance.

As interesting perspectives in the near future, let us mention first the optimal order of convergence of PFEM depending on the choice of the finite elements, see Serhani et al. (2019), and second the impact of the refinement of the mesh at the boundary on the decay rate of the damped solutions.

Then, in the spirit of Egger et al. (2018), a structure-preserving reduction procedure could eventually be applied to such a dissipative system in order to reduce it to a relatively small dimension for efficient simulation and control purposes.

Finally, in many applications, e.g. in acoustics, the problem involves a time-varying impedance and gives rise to dynamic boundary damping: the associated filters are either of low-pass type $Z(s) = \frac{Z_0}{s+\tau}$, or of high-pass type $Z(s) = \frac{Z_\infty}{s+\tau}$, and even non-rational, see e.g. Monteghetti et al. (2018): applying PFEM to these systems would be a major breakthrough for aeronautical applications.

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