Superconducting tetrahedral quantum bits

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We propose a new design for a quantum bit with four superconducting islands in the topology of a symmetric tetrahedron, uniformly frustrated with one-half flux-quantum per loop and one-half Cooper-pair per island. This structure emulates a noise-resistant spin-1/2 system in a vanishing magnetic field. The tetrahedral quantum bit combines a number of advances such as a doubly degeneracy ground state minimizing decoherence via phonon radiation, a weak quadratic sensitivity to electric and magnetic noise, relieved constraints on the junction fabrication, a large freedom in manipulation, and attractive measurement schemes. The simultaneous appearance of a degenerate ground state and a weak noise sensitivity are consequences of the tetrahedral symmetry, while enhanced quantum fluctuations derive from the special magnetic frustration. We determine the spectral properties of the tetrahedral structure within a semiclassical analysis and confirm the results numerically. We show how proper tuning of the charge-frustration selects a doubly degenerate ground state and discuss the qubit’s manipulation through capacitive and inductive coupling to external bias sources. The complete readout of the spin-components $\sigma_i$, $i = x, y, z$, is achieved through coupling of the internal qubit currents to external junctions driven close to criticality during the measurement.

I. INTRODUCTION

Superconducting solid-state qubits (short for quantum bits) are promising candidates for the future construction of quantum information processors. They appear in a variety of designs: in the charge-version the quantum information is stored in the number of excess Cooper pairs residing on a small superconducting island — this design requires fabrication of ultra-small structures and is susceptible to charge noise. In the flux4 and phase5,6 versions the information is encoded in the current state of the device — this is a macroscopic variable susceptible to flux noise. The new design by Vion et al.7 is ‘in between’, with the energy scales for the charge- ($E_C = C^2 / 2C$) and the phase- ($E_J = \Phi_0 I_c / 2\pi C$) degrees of freedom roughly balancing one another (here, $C$ and $I_c$ are the capacitance and the maximal current of the device, $\Phi_0 = hc / 2e$ is the flux quantum); its ground state is non-degenerate and the limitations are close to those of the charge device.

The novel tetrahedral qubit design we propose below operates in the phase-dominated regime and exhibits two remarkable physical properties: first, its non-Abelian symmetry group (the tetrahedral group $T_d$) leads to the natural appearance of degenerate states and appropriate tuning of parameters provides us with a doubly degenerate groundstate. Our tetrahedral qubit then emulates a spin-1/2 system in a vanishing magnetic field, the ideal starting point for the construction of a qubit. Manipulation of the tetrahedral qubit through external bias signals translates into application of magnetic fields on the spin; the application of the bias to different elements of the tetrahedral qubit corresponds to rotated operations in spin space. Furthermore, geometric quantum computation via Berry phases8,9 might be implemented through adiabatic change of external variables. Going one step further, one may hope to make use of this type of systems in the future physical realization of non-Abelian anyons, thereby aiming at a new generation of topological devices10,11,12,13 which keep their protection even during operation.14

![FIG. 1: (a) Tetrahedral superconducting qubit involving four islands and six junctions (with Josephson coupling $E_J$ and charging energy $E_C$); all islands and junctions are assumed to be equal and arranged in a symmetric way. The islands are attributed phases $\phi_i$, $i = 0, \ldots, 3$. The qubit is manipulated via bias voltages $v_i$ and bias currents $i_i$. In order to measure the qubit’s state it is convenient to invert the tetrahedron as shown in (b) — we refer to this version as the ‘connected’ tetrahedron with the inner dark-grey island in (a) transformed into the outer ring in (b). The measurement involves additional measurement junctions with couplings $E_m \gg E_J$ on the outer ring which are driven by external currents $I_m$ (schematic, see Fig. 6 for details); the large coupling $E_m$ effectively binds the ring segments into one island.](cond-mat/supr-con/1410.5637v1)

The second property we wish to exploit is geometric frustration: In our tetrahedral qubit discussed below it appears in an extreme way by rendering the classical minimal states continuously degenerate along a line in parameter space. Semi-classical states then appear
only through a fluctuation-induced potential, reminiscent of the Casimir effect, and the concept of inducing 'order from disorder'. The quantum-tunneling between these semi-classical states defines the operational energy scale of the qubit, which turns out to be unusually large due to the weakness of the fluctuation-induced potential. Hence the geometric frustration present in our tetrahedral qubit provides a natural boost for the quantum fluctuations without the stringent requirements on the smallness of the junction capacitances, thus avoiding the disadvantages of both the charge- and the phase-device: The larger junctions reduce the demands on the fabrication process and the susceptibility to charge noise and mesoscopic effects, while the large operational energy scale due to the soft fluctuation-induced potential reduces the effects of flux noise. Both types of electromagnetic noise, charge- and flux noise, appear only in second order (cf. also Ref. [1]).

In the following (Sec. 2), we first introduce the structure of our tetrahedral qubit and then find the low-energy part of its spectrum. We proceed in three steps, beginning with the (highly degenerate) classical solution; subsequently, we demonstrate how the fluctuation induced potential reduces this degeneracy to three semi-classical states and finally, we analyze the tunneling between them in order to arrive at the final answer for the phase dominated regime. We confirm this solution with the help of numerical calculations and extend it to the charge dominated regime. The inclusion of external fields breaking the (tetrahedral) symmetry of the device prepares the discussion of the qubit’s manipulation schemes (section 3). In section 4, we discuss various measurement schemes and end with the conclusions in Sec. 5. A first account on part of this work has been given in Ref. [18].

II. TETRAHEDRON

A. Device structure

Consider the planar structure made from four superconducting islands interconnected via six (conventional) Josephson junctions with equal couplings $E_J$; connecting each island with all the others produces the topology of a tetrahedron, see Fig. 1(a). The islands are numbered through ‘0’ to ‘3’, with the island ‘0’ residing in the center, and are assigned phases $\phi_i$, $i = 0, \ldots, 3$. The three triangular loops are small (i.e., $E_J \ll (\Phi_0/2\pi)^2/L_\Delta$ with $L_\Delta$ the loop inductance), allowing us to neglect fluxes induced by currents flowing in the structure. In the absence of external magnetic fields, the classical energy of this arrangement is given by the sum $V_0 = \sum_{i<j} E_J [1 - \cos(\phi_{ij})]$ with the difference variables $\phi_{ij} = \phi_j - \phi_i$. A slightly modified version of this device with the inner island converted into an enclosing ring is shown in Fig. 1(b) — choosing appropriate parameters, this variant exhibits the same physical properties as the original isolated tetrahedral qubit. In addition, this second design ideally lends itself for measurement of the qubit state. Below, we treat the ring as one connected island; the strong junctions with $E_m \gg E_J$ used in the measurement process will be discussed later.

Next, we bias the structure through an external magnetic field, frustrating each sub-loop with a flux $\Phi_0/2$ (these are three triangular sub-loops in the isolated version of Fig. 1(a) and 4 sub-loops in the connected version of Fig. 1(b)). We include the effect of this flux in a symmetric gauge by adding the phase $\pi$ to each of the difference variables $\phi_{ij}$; the energy (up to a trivial constant; we measure phases with respect to the phase $\phi_0$ of the central/ring island)

$$V_x = E_J [\cos(\phi_1 + \cos(\phi_2 + \cos(\phi_3 + \cos(\phi_1 - \phi_3) + \cos(\phi_2 - \phi_1))]$$

then is minimized (with $V_x = -2E_J$) along the lines

$$\phi_3 = \pm\pi; \phi_2 - \phi_1 = \pm\pi; \psi_3 = \phi_1 + \phi_2 \in [-\pi, \pi]$$

and their analogs obtained by cyclic permutation $3 \to 1 \to 2 \to 3$. These minimal-energy lines run in the planes of the cube $[-\pi, \pi]^3$ defined in phase space $\{\phi_1, \phi_2, \phi_3\}$, see Fig. 2(a). The huge (linear) classical degeneracy can be easily understood via a reformulation of the potential $V_1$ in terms of the complex variables $z_k = \exp(i\phi_k)$,

$$V_x = V_1 \sum_{k=0}^3 |z_k|^2 - 4;$$

this expression is minimal for $\sum_k z_k = 0$. The two conditions defined by this (complex) equation imply that one out of the three variables $\phi_k$ can be freely chosen, thus defining lines of minimal potential energy. Of particular relevance are the minimal energy configurations $O_i$, $i = 1, 2, 3$ on the cube edges where two minimal-energy lines join. These configurations involve two opposite junctions with a phase difference $\phi_{ij} = 0$, while the remaining junctions involve maximal phase differences $\pi$; following three consecutive segments on the cube, these minimal states rotate through $2\pi$, see Fig. 2(a) (here, we have included the bias phases $\pi$ on each link; in the original variables these states involve 2 strained junctions with phase difference $\pi$ and 4 unstrained ones with phase difference 0; note the absence of currents in these configurations).

We account for the quantum dynamics in the array via the capacitive term

$$T = \frac{\hbar^2}{4e^2} \sum_{i<j} C_{ij} \left(\dot{\phi}_{ij}\right)^2 + \sum_{i=1}^3 C_g \left(\dot{\phi}_i\right)^2 + C_0 \left(\dot{\phi}_0\right)^2$$

in the Lagrangian $\mathcal{L} = T - V_x$; here, $C_{ij}$ denotes the capacitance of the junctions, while $C_g$ and $C_0$ are the capacitances to the ground of the islands $i = 1, 2, 3$ and of the center/ring island. An additional term $\hbar \sum_i \dot{\phi}_i$ appears when charges $2e\phi_i$ are induced on the islands; we will discuss the effect of this topological term later. For the isolated tetrahedron of Fig. 1(a) we have $C_0 = C_g$; going over to center of mass $(4\Phi = \sum_i \phi_i)$ and relative coordinates $(\dot{\phi}_i = \phi_i - \Phi)$ the difference variables $\dot{\phi}_{ij} = \phi_{ij}$
pick an additional capacitance $C_0/4$ and the corresponding part of the kinetic energy can be written in the form $T_{rel} = (\hbar^2/16E_C) \sum_{i<j} \phi^2_{ij}$, with the capacitive energy $E_C = e^2/2C$ and $C = C_I + C_0/4$. The identical expression for the kinetic energy is obtained for the inverted tetrahedron in Fig. 1(b) if we choose a large self-capacitance $C_0/C_I \rightarrow \infty$ for the ring and a small one $C_0/C_I \rightarrow 0$ for the other islands, then $E_C = e^2/2C_I$; this limit describes the inverted tetrahedron connected to the outside world via (large) superconducting wires fixing the phase $\phi_0 = 0$. The following discussion applies to both designs of Fig. 1; we first assume that $E_C \ll E_J$, placing the array into the phase-dominated regime.

**B. Semi-classical Analysis**

Next, we account for quantum fluctuations associated with the degenerate line-minima. It is convenient to introduce the (non-orthogonal) variables $\chi_k = (\phi_i + \phi_j - \phi_k)/2$ with $i,j,k \in \{1,2,3\}$ and mutually different, where both the kinetic and potential energy terms acquire a simpler form, $T_{rel} = (\hbar^2/4E_C)[\chi_1^2 + \chi_2^2 + \chi_3^2]$ and $V_\gamma = 2E_J[\cos\chi_1 \cos\chi_2 + \cos\chi_1 \cos\chi_3 + \cos\chi_2 \cos\chi_3]$. The new coordinates are directed along the potential minima which are parametrized by fixing two coordinates to 0 and $\pm \pi$ and have the third run through the interval $[0, \pi[, e.g., the reference line (denoted by $\gamma$) connecting $O_2$ with $O_1$ in Fig. 2(a) is parametrized by $\chi_1 = 0$, $\chi_2 = -\pi$, $\chi_3 \in [0, \pi]$. In the vicinity of $\gamma$ the potential energy takes the form $V_\gamma \approx E_J(-2 + \delta\chi_1^2(1 - \cos\chi_3) + \delta\chi_3^2(1 + \cos\chi_3))$; the two fast oscillatory modes $\chi_1$ and $\chi_2$ appear with a curvature which depends on the adiabatic coordinate $\chi = \chi_3$, see Fig. 2(a). Their zero-point fluctuations produce an induced potential

$$V_\gamma(\chi) = \frac{1}{2} [\hbar \omega_1(\chi) + \hbar \omega_2(\chi)],$$

with the frequencies of the fast modes

$$\omega_{1,2}(\chi) = \omega \sqrt{[1 \pm \cos(\chi)]/2}$$

and the frequency scale $\omega = \sqrt{8E_JE_C}/\hbar$. Near the edges $\chi = 0, \pi$ one of the modes goes to zero, as is clear from the potential shape shown in Fig. 2(a), and we have to refine our analysis. We then remain with only one fast and two slow modes. Expanding $V_\gamma$ around the point $O_2$ = (0, $-\pi$, 0) (in $\chi_1$ coordinates) we arrive at the potential $V_\gamma = (E_J/2)[-4 + 4\delta\chi_1^2 + \delta\chi_3^2(1 + \delta\chi_1^2 + \delta\chi_2^2)]$; integration over the fast mode $\chi_2$ and transformation to momenta provides us with the Hamiltonian

$$H_t \approx -E_C \left[ \frac{d^2}{d\chi_1^2} + \frac{d^2}{d\chi_3^2} \right] + E_J \delta\chi_1^2 \delta\chi_3^2 - \kappa(\delta\chi_1^2 + \delta\chi_3^2),$$

where $\kappa = \langle \delta\chi_3^2 \rangle = (E_C/8E_J)^{1/2}$. Dimensional analysis tells that the low lying levels of this quartic anisotropic oscillator are of the order of

$$\Omega \equiv \omega \left( \frac{E_C}{E_J} \right)^{1/6} \ll \omega_1;$$

the numerical factors determining the exact positions of the groundstate and of the non-equidistant higher levels have to be determined numerically and the results are summarized in Table I; the groundstate energy is accurately described by the expression $\Omega_0/\Omega \approx 0.311 - 0.129(E_C/E_J)^{1/6}$, where the scale dependence of the correction easily follows from first order perturbation theory in the term $-E_J\kappa(\delta\chi_1^2 + \delta\chi_2^2)/2$.

Repeating this analysis for the other classical line-minima, we arrive at three distinct quantum states $|O_i\rangle$ (at energies $-2E_J + \hbar\omega_i/2 + \hbar\Omega_0$) associated with the
three classical zero-current states $O_i$ described above. These isolated quantum states are generated through a fluctuation-induced potential reminding about the Casimir force between metallic plates or the van der Waals interaction between neutral atoms. This is just the mechanism producing order from disorder originally proposed by Villain where the huge classical ground state degeneracy (which does not follow from the symmetry properties of the system) is removed by quantum fluctuations, here, selecting the three points $|O_i\rangle$ as new ground states.

The low-energy spectrum near the points $O_i$ exhibits non-equidistant levels $\Omega_i$ even deep in the semiclassical regime, allowing for the use of the tetrahedral structure as a simple Josephson junction qubit of the type introduced in Refs. 20,21. Moreover, both symmetry arguments and the numerical data tell that the first excited level is twofold degenerate, such that we effectively deal with a spin 1 system with an easy-plane anisotropy $H_{S=1} = (\Omega_1 - \Omega_2)S^z_2$.

Before going to the full quantum description, let us discuss the above semiclassical version of the device, as it exhibits a number of interesting features by itself. First, the potential defines a doubly-periodic junction with two distinct minima. The potential $V_1(\chi)$ can be mapped out experimentally through the measurement of the Josephson current $I_J(\chi)$ that can be pushed through the structure. E.g., fixing the phase $\phi_2$ between the central island and the island '2' via a flux-biased external loop defines the two classical minimal solutions $(\phi_2 - \pi, 0, 0)$ and $(0, -\pi, \phi_2)$ (in $\chi_i$-coordinates). The running coordinate $\chi = \chi_1$ or $\chi = \chi_3$ is equal to $\phi_2$ (up to a trivial shift) and thus related to the external bias flux $\Phi$ via $\chi = -2\pi\Phi/\Phi_0$. The current $I = -|e|\Phi/E$ then is given by the expression $I_J(\chi) = (2e/h)\partial_\chi V_1(\chi)$ and is double periodic in the interval $0 \leq \Phi < \Phi_0$ (we define the charge of the electron as $-e$ and $e > 0$). Alternatively, one may measure the frequencies $\omega_{1,2}(\chi)$, cf. Eq. 5, directly via the resonant absorption of an ac-signal.

At nonzero temperatures (but still $T \ll E_J$) the induced potential is driven thermally and involves the free energy of the two fast oscillating modes,

$$F_t(\chi, T) = \sum_{i=1}^{2} \left[ \frac{\hbar \omega_i(\chi)}{2} + T \log \left( 1 - e^{-\hbar \omega_i(\chi)/T} \right) \right].$$

Thermal fluctuations become relevant for temperatures $T > \hbar \omega_1$ and lead to an increase in the barrier $\delta F_t(T) \equiv F_t(\pi/2, T) - F_t(0, T)$,

$$\delta F_t(T) = \left( \hbar \omega_t/2 \right) \left[ \sqrt{2} - (1 + 2\Omega_0/\omega_t) \right] + T \log[\omega_t/2\Omega_0].$$

As a result, rather then decreasing, the phase stiffness in the tetrahedron increases with temperature and hence the Josephson current $I_J(T) \propto \partial_\chi F_t(\Phi, T)$ increases with temperature until the fluctuation-induced potential disappears due to level broadening: thermally induced quasiparticles produce a level broadening $\hbar/RC \sim (E_J/E_C/\hbar \Delta \exp(-\Delta/T))$, where we have used the Ambegaokar-Baratoff relation $I_R \sim \Delta/e$. This broadening should remain small on the scale of the level separation $\hbar \omega_t$, from which we obtain the condition that $T < \Delta/\ln(\hbar \omega_t/\Delta)$. Beyond this temperature, the Josephson current is expected to decrease again, resulting in a non-monotonic behavior of $I_J(T)$.

In order to arrive at a fully quantum mechanical description of the tetrahedron, we have to account for the tunneling processes between the points $O_i$, cf. Fig. 2. It is important to note that each set of 4 mid-edge points residing in one plane $\phi_i = 0$ has to be identified with one quantum mechanical state $|O_i\rangle$; on the other hand, the pair of classically degenerate lines connecting two states $|O_i\rangle$ and $|O_j\rangle$ in one of the faces describe different tunneling trajectories, which have to be added coherently in order to arrive at the tunneling matrix element between the two states (the other two trajectories on the opposite face are equivalent). Let us concentrate on the tunneling process between $|O_1\rangle$ and $|O_2\rangle$; the two tunneling trajectories follow the lines parametrized by $\chi_1, \chi_2 \in \{0, -\pi\}$, $\chi_3 \in [0, \pi]$, see Fig. 2(a). The tunneling processes described by these two trajectories flip the phase across the four junctions ‘11’, ‘01’, ‘02’, and ‘23’ by $\pm \pi$, which corresponds to a fluxon $\Phi_0/2$ traversing the tetrahedron as shown in Fig. 2(b) (the same arguments apply to the connected tetrahedron). The phase difference between the two trajectories then corresponds to taking a full fluxon $\Phi_0$ around the two islands ‘1’ and ‘2’, which translates into the Aharonov-Bohm-Casher phase $\exp(2\pi i(q_1 + q_2))$, with $q_i$ the charge on the island ‘i’ measured in units of $2e$ (see Ref. 22 for a detailed discussion of charge-induced interference effects in small superconducting structures; these phases are generated by the topological term $\hbar \sum_i q_i \phi_i$ in the Lagrangian $\mathcal{L}$ in the presence of charges $2eq_i$, cf. the note below 30). Combining this phase factor with the modulus $|a|$ we arrive at the tunneling amplitude

$$t_{12} = -2|a| \cos[\pi(q_1 + q_2)]$$
between the states $|O_1\rangle$ and $|O_2\rangle$; a similar analysis provides the amplitudes for all the other pairs. In the absence of any charge frustration (i.e., for integer charge $q$, on each island) the system gains energy from hopping and hence $t < 0$, thus defining the sign in $\mathcal{S}$. The modulus $|a|$ of the tunnelling amplitude follows from the semi-classical description of the one-dimensional motion under the barrier $V_t(\chi)$ as given by Eq. [4] and takes the form $|a| \approx \hbar/t(\Omega_0)exp[-S_t(\Omega_0)]$, with $T(\Omega_0)$ the classical period of motion and $S_t(\Omega_0)$ the dimensionless action, both evaluated at the ground state energy $\hbar\Omega_0$.

$$S_t = \left(\frac{32E_J}{E_C}\right)^{1/4} \int_0^{\chi_0} d\chi \sqrt{2\cos(\chi/2) - 1 - 2\Omega_0/\omega_t}^{1/2}$$

$$\sim 1.88 \left(\frac{E_J}{E_C}\right)^{1/4}, \quad (E_J/E_C)^{1/4} \gg 1,$$  \hspace{1cm} (11)

$$T = \frac{8\hbar}{E_J} \left(\frac{\sqrt{2}\Omega_0}{\Omega}\right)^{1/2} \left(\frac{E_J}{E_C}\right)^{2/3},$$  \hspace{1cm} (12)

where $\chi_0 = 2 \arccos([1+2\Omega_0/\omega_t]/\sqrt{2})$. The semi-classical analysis describing tunneling at the correct ground state energy $\hbar\Omega_0$ gives very accurate results, see below; the simple asymptotic form describes tunneling from the bottom of the well and is valid only deep in the quasi-classical regime. The result obtained here for the tetrahedron is smaller than the usual tunneling action $S \propto (E_J/E_C)^{1/2}$ of a Josephson junction device involving only the square root of the parameter $E_J/E_C$; the unconventional dependence on the ratio $E_J/E_C$ is a consequence of the fluctuation-induced (rather than classical) barrier and puts less stringent requirements (e.g., with respect to the smallness of the junction) on the fabrication process of this new type of qubits.

The above tunneling action can be probed via a measurement of the current-voltage characteristic of the device: applying a fixed bias current across two islands, a finite voltage appears due to quantum and thermally induced phase slips. At low temperatures the appearance of quantum phase slips involves the finite action $S_{ps} = 2\hbar S_t$ and results in an exponentially small resistivity $R \propto \exp(-2S_t)$. At higher temperatures $T > U_{ps}\hbar/S_{ps}$ the phase slips are created thermally via activation over the barrier $U_{ps} = \delta F_t$. The phase-slip induced resistivity then exhibits an unconventional saturation at high temperatures: increasing $T$ beyond $\hbar\omega_t(E_C/E_J)^{1/4}$, the resistivity first increases with temperature. As $T > \hbar\omega_t$ the barrier $\delta F_t$ itself increases linearly in $T$ and the further rise of $R$ saturates at a value $R \propto \Omega_0/\omega_t$; hence care has to be taken not to confuse the saturation in $R$ at high temperatures with the (exponentially small) quantum resistance surviving as $T \to 0$.

Let us return to the quantum description of our tetrahedron and study its low-energy spectrum as determined by the quantum coherent oscillations between semi-classical states $|O_i\rangle$: the appearance of the island charges $2e\, q_i$ in the tunneling amplitudes $t_{ij}$ manifests itself in this level structure. Assuming a uniform distribution of the total charge $2e\, q$ on the isolated tetrahedron, the matrix elements take the form $t_{ij} = t = -2|a|\cos(q\pi/2)$ and we have to diagonalize the matrix

$$H_O = \begin{pmatrix} 0 & t & t \\ t & 0 & t \\ t & t & 0 \end{pmatrix}.$$  \hspace{1cm} (13)

Depending on the value of $q$, the low-energy spectrum of the isolated tetrahedron splits into singlets and doublets involving the energies $E_s = 2t$ and $E_d = -t$, or remains tri-fold degenerate with $E_s(t = 0) = 0$ (here, energies are measured with respect to the unperturbed value $-2E_J + \hbar(\omega_t/2 + \Omega_0)$); the ground state is a

- singlet if $q = 4k$, even,
- doublet if $q = 4k + 2$, even,
- triplet if $q = 2k + 1$, odd.

In the connected tetrahedron of Fig. 1(b) the charge is not quantized on the inner islands; however, the dependence of the tunneling amplitude on the island charges $q_i$ remains valid and the above spectrum is recovered under appropriate biasing of the inner islands with either zero, one-quarter, or one-half Cooper-pair. Biasing the tetrahedron into the charge state $q_i = 1/2$ then establishes a ground state doublet with eigenstates

$$|+\rangle = \frac{1}{\sqrt{3}}(|O_1\rangle + \zeta|O_2\rangle + \zeta^*|O_3\rangle),$$

$$|-\rangle = \frac{1}{\sqrt{3}}(|O_1\rangle + \zeta^*|O_2\rangle + \zeta|O_3\rangle),$$  \hspace{1cm} (15)

suitable for the implementation of a quantum bit; here, $\zeta = \exp(2\pi i/3)$. These degenerate states involve bonds resonating with opposite chirality in the device, see Fig. 2(a), and are reminiscent of the resonating dimer bonds in the topologically protected qubits discussed in Refs. [10, 11]. Also, such chiral states appear as degenerate spin-singlet ground states in the vanadium tetrahedron of the pyrochlore system. The doublet $|\pm\rangle$ is protected by the gap $\Delta_1 = 3t$, separating it from the next excited (singlet) state

$$|0\rangle = |O_1\rangle + |O_2\rangle + |O_3\rangle/\sqrt{3}.$$  \hspace{1cm} (16)

Combining the results of Eqs. [10, 11, and 12] we obtain the protective and operational energy scale $t$ of the qubit

$$\frac{t}{E_J} = \frac{1}{4} \left(\frac{\Omega}{\sqrt{2}\Omega_0}\right)^{1/2} \left(\frac{E_C}{E_J}\right)^{2/3} \exp[-S_t(\Omega_0)].$$  \hspace{1cm} (17)

### C. Charge limit

We briefly extend our analysis to the charge-dominated regime with $E_C \gg E_J$. It is convenient to go over to a Hamiltonian description; starting from the above Lagrangian (cf. Eqs. [11] and [13]) and eliminating the variable $\phi_0$ one obtains the expression

$$H = E_C\{Q_1^2 + Q_2^2 + Q_3^2 + (Q_1 + Q_2 + Q_3)^2\} + \sum_{\pi} V_{\pi}(\phi_1, \phi_2, \phi_3),$$  \hspace{1cm} (18)
where \( \bar{Q}_i = -i \partial_{q_i} - q_i \) and \( q_i \) is the induced charge on the \( i \)-th island (in units of \( 2e \)), \( q_i = \sum_j C_{ij} V_j \) for the connected tetrahedron, while an additional term \( |q - (C/4) \sum_j V_j|/4 \) with \( C = \sum_{ij} C_{ij} \) the total capacitance, has to be added for the isolated tetrahedron (here, \( C_{ij} \) denotes the capacitance matrix, see \( \text{[4]} \), and \( V_j \) are the bias potentials applied to the islands). For the isolated tetrahedron the total charge \( q = \sum_i q_i \) is integer, while the total induced charge \( q = \sum_j q_j \) can take any value in the connected device. The Hamiltonian \( \text{[6]} \) describing both devices becomes identical under symmetric bias and for specific values \( q_i = k/4 \) with \( k \) an integer; under these conditions the maximal symmetry \( S_4 \), i.e., the tetrahedral symmetry \( T_4 \) is established. Note that a symmetric bias with equal charges \( q_i = q/3 \) on the three inner islands of the connected tetrahedron in general guarantees only for a \( S_3 \) point symmetry.

We determine the spectrum for the uniformly charged isolated tetrahedron with \( q = 4k + 2 \). In the limit \( E_C \gg E_J \) the operators \( Q_i \) take on integer values and the charging term is minimized by distributing two bosons onto the four sites avoiding double occupancy (in this limit, the term \( E_C (\bar{Q}_1 + \bar{Q}_2 + \bar{Q}_3)^2 \) in \( \text{[6]} \) describes the charging energy \( E_C Q_0^2 \) of the middle island). The resulting six states ‘01’, ‘02’, ‘03’, ‘23’, ‘31’, and ‘12’ are degenerate with an energy \( E_0 = 2E_C \). The hopping term \( V_r \) lifts this degeneracy through the mixing via the Josephson coupling \( E_J \): each state ‘ij’ hosting Cooper-pairs on the islands ‘i’ and ‘j’ exchanges particles with all other states except for ‘kl’, where \( k, l \neq i, j \). The Hamiltonian describing the mixing of the six two-Boson states may be written as a matrix product

\[
H_{2B} = \begin{pmatrix}
0 & i & i \\
i & 0 & i \\
i & i & 0
\end{pmatrix} \otimes \begin{pmatrix}
1 & 1 \\
1 & 1
\end{pmatrix},
\]

(19)

where the sign of the tunneling amplitude \( \tilde{t} = E_J/2 > 0 \) is positive for our frustrated tetrahedron (again, we choose a symmetric gauge with all Josephson couplings reversed in sign). The eigenvalues of the direct matrix product in \( \text{[6]} \) are given by the product of the eigenvalues \( (2\tilde{t}, -\tilde{t}, -\tilde{t}) \) of its first factor and those of the second factor, \( (0, 2) \); correspondingly we find the first 6 levels at the energies

\[
\begin{align*}
E_d &= -E_J \quad \text{doublet}, \\
E_t &= 0 \quad \text{triplet}, \\
E_s &= 2E_J \quad \text{singlet}.
\end{align*}
\]

(20)

The first excitation now is a triplet rather than the singlet state found in the opposite limit \( E_J \gg E_C \): hence, decreasing \( E_J/E_C \) from large values, the singlet and triplet energies cross each other and the first excitation gap changes over from \( 3\tilde{t} \) (at \( E_J \gg E_C \)) to \( 2\tilde{t} = E_J \) (for \( E_C \gg E_J \)). The precise location where this crossing appears can be found from the numerical analysis described below. The same analysis can be repeated for the connected tetrahedron; the six states degenerate under the capacitive term then involve either one or two charges on the three inner islands and the mixing term \( V_r \) describes charges hopping between the three islands as well as hopping of one charge to and from the ring.

D. Numerical Results

The above results can be verified numerically via diagonalization of the Hamiltonian \( \text{[6]} \) with the help of a Lanczos algorithm; in the charge basis the mixing term \( V_r \) then describes the hopping of charges between the islands. Going to a phase representation, the bias charges \( q_j \) are conveniently accounted for via the boundary condition for the wave function, \( \Psi_n(\phi_1 + 2\pi \delta_{1k}, \phi_2 + 2\pi \delta_{2k}, \phi_3 + 2\pi \delta_{3k}) = \exp(-2\pi i q_j) \Psi_n(\phi_1, \phi_2, \phi_3) \), \( k = 1, 2, 3 \), after a suitable gauge transformation. The results of such an analysis for the charge state \( q_1 = 1/2 \) is shown in Fig. 3, where the excitation gap \( \Delta_4 \) protecting the qubit states against higher excitations is shown as a function of \( E_J/E_C \). The crossover from the charge to the phase dominated regime, where the singlet and triplet excited states cross, takes place at \( E_J/E_C \approx 5 \). The analytic results \( \text{[17]} \) and \( \text{[20]} \) describe well the data away from the crossover regime. One expects the quasi-classic result to become exact in the limit of large \( E_J/E_C \); however, the result \( \text{[24]} \) has been calculated using the one-dimensional approximation \( V_I \) for the potential and one cannot expect perfect agreement with the numerical data. Still, the quasi-classic approximation turns out accurate over a very large regime extending down to parameters \( E_J/E_C \) of order 10: scaling the dashed line in Fig. 3 by 0.8 the quasi-classic result cannot be distinguished from the numerical data for \( E_J/E_C > 20 \).

The numerical results show that the suppression of the tunneling amplitude \( t \) is indeed small in the tetrahedron. E.g., choosing a value \( E_J/E_C \) of order 100 the energy scale \( t \) of the qubit is suppressed by a factor \( \sim 2/1000 \) with respect to the energy scale \( E_J \) of the junctions. For a conventional device this suppression involves an action \( S \approx \epsilon \sqrt{E_J/E_C} \), with the numerical \( \epsilon \) depending on the specific setup. E.g., for the 4-junction loops studied in Ref. \( \text{[24]} \) the numerical \( \epsilon \approx 1.6 \) and choosing the same value of \( E_J/E_C \) this implies a suppression of quantum fluctuations by a factor \( \sim \exp(-16) \sim 10^{-7} \); such a device then resides deep in the semi-classical regime and quantum effects are heavily suppressed.

In summary, we can tune our tetrahedral structure such as to realize a doubly-degenerate groundstate corresponding to a spin-1/2 system in zero magnetic field: the device can be realized using moderately large junctions with \( E_J/E_C \) of order 100 while keeping an appreciable operational energy scale \( t \), a consequence of the particular frustration in the device. Below, we will show that this ground state remains robust to quadratic order in the external noise and hence provides a suitable starting
m3', and 'm3-m1', cf. Fig. 1(b). The bias angles \( \delta \) symmetric setup, this flux will induce the phase shifts in a bias angle \( \delta_3 \). We denote the corresponding deviations by \( \delta_\Phi \) and \( \delta_\chi \). We denote \( \delta_3 \equiv \delta (\delta \chi_3 - \delta \chi_1) \).

The bias angles \( \delta_i \) along the links ‘j-k’ then have to account for this flux via the modified form \( \delta_i^e = 2\pi (\Phi - \Phi/3)/\Phi_0 \), where \( \Phi \) denotes the flux through the loop ‘m_j-m_k-k_j’-m_j’. Note that we have assumed that the currents circulating in the outer ring are still sufficiently small such as not to produce significant self-fields. While the charge bias \( \delta_i^Q \) only affects the tunneling matrix elements \( t_{ij} \), i.e., the kinetic energy term, the flux bias \( \delta_i^F \) modifies both the potential and the kinetic energy terms in the Lagrangian.

We first determine the modification of the potential energy \( V_\chi \), cf. (1), due to applied fluxes \( \delta_i^F \),

\[
\frac{V_\chi}{2E_J} = \cos(\chi_1 + \frac{\delta_i^F}{2}) \cos(\chi_2 - \frac{\delta_i^F}{2}) + \cos(\chi_3 + \frac{\delta_i^F}{2})
\times \cos(\chi_1 - \frac{\delta_i^F}{2}) + \cos(\chi_2 + \frac{\delta_i^F}{2}) \cos(\chi_3 - \frac{\delta_i^F}{2}).
\]

Assuming small perturbations, we expand (21) in \( \delta_i^F \ll \omega_1/E_J \); we concentrate on the point \( Q_3 = (0, -\pi, 0) \) and combine the result with the kinetic term to arrive at the Hamiltonian (6) with the additional term

\[
\frac{\delta H}{E_J} = - \left[ \delta_i^F (\delta \chi_3 - \delta \chi_1) + \delta_i^F (\delta \chi_3 - \delta \chi_1) \right]
+ \frac{\delta_i^F (\delta \chi_2 - \delta \chi_1)}{2} \left[ \delta_i^F + \frac{\delta_i^F}{2} \right].
\]

Classically, the force term in (22) lowers the energy indefinitely as the system runs away along the degenerate classical minimal lines; e.g., a perturbation \( \delta_i^F > 0 \) produces a runaway either along the \((0, 0, \delta \chi_3)\) direction or along the \((- \delta \chi_1, 0, 0)\) direction. However, quantum fluctuations generate a finite potential along these lines, resulting in a linear response in the coordinates \( \delta \chi_i \) and a quadratic change in energy \( v_\chi \). Indeed, second-order perturbation theory in the force term of (21) produces the result

\[
\frac{v_\chi}{E_J} = -\nu \left[ (\delta_i^F + \delta_j^F)^2 + (\delta_i^F + \delta_j^F)^2 \right]
+ (\epsilon_{02} - \epsilon_{03} - \epsilon_{01}) + (\epsilon_{31} - \epsilon_{12} - \epsilon_{23}),
\]

with

\[
\nu \approx 1.0 \left( E_J/E_C \right)^{1/3}
\]

obtained from a numerical solution of the perturbed eigenvalue problem combining (6) and (22). In (23) we have dropped the term \((E_J/2)(\delta_i^F + \delta_j^F - \delta_k^F)^2\) as it is small by a factor \((E_J/E_C)^{1/3}\) compared to the leading term; also, we have added a term due to deviations \( \epsilon_{ij} E_J \) in the junction couplings. Equivalent expressions for the other minima follow from cyclic permutation of the indices. As a result, we obtain the relative shifts \( v_{ij} \equiv v_i - v_j \) \((i, j, k = 1, 2, 3, k \neq i, j)\)

\[
\frac{v_{ij}}{E_J} = \nu (\delta_i^F - \delta_j^F)(\delta_i^F + \delta_j^F + 2\delta_k^F)
+ (\epsilon_{0j} - \epsilon_{0i} + \epsilon_{ki} - \epsilon_{jk})
\]

point for the construction of a qubit; we will concentrate on this doubly-degenerate case and its use for quantum computing in the following.

E. External fields

Fabrication errors and external bias induce splittings and shifts in the levels. With respect to the qubit’s functionality, random external signals produce decoherence, while prescribed bias signals are used for its manipulation. Fabrication errors mainly affect the coupling \( E_J \) of the junctions — we denote the corresponding deviations by \( \epsilon_{ij} \). External bias signals appear randomly through fluctuations in the local magnetic field and through stray charges; artificially generated signals can be applied through properly placed current loops biasing the sub-loops of the tetrahedron (currents \( i_i \) in Fig. 1) and through capacitive charging of the islands (via voltages \( v_i \)). We denote the corresponding bias fields by \( \delta_i^Q \equiv \epsilon_{ij} \Phi_0 \) and \( \delta_i^\Phi \equiv \epsilon_{ij} \Phi_0 \), \( i, j = 1, 2, 3 \). E.g., for the isolated tetrahedron the flux \( \Phi_3 \) penetrating the sub-loop ‘0-1-2’ is encoded in a bias angle \( \delta_3 \equiv \delta (\phi_2 - \phi_1) = -2\pi \Phi_3/\Phi_0 \) along the link ‘1-2’; equivalent definitions apply to the other sub-loops with the cyclic replacement \( 3 \rightarrow 1 \rightarrow 2 \rightarrow 3 \).

For the connected tetrahedron we have to account for the total flux \( \Phi \) threading the outer ring; assuming a symmetric setup, this flux will induce the phase shifts \( \delta = -2\pi \Phi/3\Phi_0 \) in each of the segments ‘m1-m2’, ‘m2-m3’, and ‘m3-m1’, cf. Fig. 1(b). The bias angles \( \delta_i^Q \) along the links ‘j-k’ then have to account for this flux via the modified form \( \delta_i^F = 2\pi (\Phi - \Phi/3)/\Phi_0 \), where \( \Phi \) denotes the flux through the loop ‘m_j-m_k-k_j’-m_j’. Note that we have assumed that the currents circulating in the outer ring are still sufficiently small such as not to produce significant self-fields. While the charge bias \( \delta_i^Q \) only affects the tunneling matrix elements \( t_{ij} \), i.e., the kinetic energy term, the flux bias \( \delta_i^F \) modifies both the potential and the kinetic energy terms in the Lagrangian.

We first determine the modification of the potential energy \( V_\chi \), cf. (1), due to applied fluxes \( \delta_i^F \),

\[
\frac{V_\chi}{2E_J} = \cos(\chi_1 + \frac{\delta_i^F}{2}) \cos(\chi_2 - \frac{\delta_i^F}{2}) + \cos(\chi_3 + \frac{\delta_i^F}{2})
\times \cos(\chi_1 - \frac{\delta_i^F}{2}) + \cos(\chi_2 + \frac{\delta_i^F}{2}) \cos(\chi_3 - \frac{\delta_i^F}{2}).
\]

Assuming small perturbations, we expand (21) in \( \delta_i^F \ll \omega_1/E_J \); we concentrate on the point \( Q_3 = (0, -\pi, 0) \) and combine the result with the kinetic term to arrive at the Hamiltonian (6) with the additional term

\[
\frac{\delta H}{E_J} = - \left[ \delta_i^F (\delta \chi_3 - \delta \chi_1) + \delta_i^F (\delta \chi_3 - \delta \chi_1) \right]
+ \frac{\delta_i^F (\delta \chi_2 - \delta \chi_1)}{2} \left[ \delta_i^F + \frac{\delta_i^F}{2} \right].
\]

Classically, the force term in (22) lowers the energy indefinitely as the system runs away along the degenerate classical minimal lines; e.g., a perturbation \( \delta_i^F > 0 \) produces a runaway either along the \((0, 0, \delta \chi_3)\) direction or along the \((- \delta \chi_1, 0, 0)\) direction. However, quantum fluctuations generate a finite potential along these lines, resulting in a linear response in the coordinates \( \delta \chi_i \) and a quadratic change in energy \( v_\chi \). Indeed, second-order perturbation theory in the force term of (21) produces the result

\[
\frac{v_\chi}{E_J} = -\nu \left[ (\delta_i^F + \delta_j^F)^2 + (\delta_i^F + \delta_j^F)^2 \right]
+ (\epsilon_{02} - \epsilon_{03} - \epsilon_{01}) + (\epsilon_{31} - \epsilon_{12} - \epsilon_{23}),
\]

with

\[
\nu \approx 1.0 \left( E_J/E_C \right)^{1/3}
\]

obtained from a numerical solution of the perturbed eigenvalue problem combining (6) and (22). In (23) we have dropped the term \((E_J/2)(\delta_i^F + \delta_j^F - \delta_k^F)^2\) as it is small by a factor \((E_J/E_C)^{1/3}\) compared to the leading term; also, we have added a term due to deviations \( \epsilon_{ij} E_J \) in the junction couplings. Equivalent expressions for the other minima follow from cyclic permutation of the indices. As a result, we obtain the relative shifts \( v_{ij} \equiv v_i - v_j \) \((i, j, k = 1, 2, 3, k \neq i, j)\)
in the minima. First, we note that small random fluxes do not affect these positions in the first order of these fluctuations; the corrections appearing in quadratic order then are small. Second, we note that fabrication errors $\epsilon_{ij}$ in the junction couplings can be compensated by appropriate choices of bias fluxes $\delta_i^*$. In the determination of the perturbed tunnelling matrix elements $t_{ij}$, we ignore the modifications arising due to fabrication errors and concentrate on the effects of flux and charge signals, random or externally applied. By way of example, we calculate the tunneling matrix elements $t_{12}$ and $t_{21}$ connecting the states $|O_1\rangle$ and $|O_2\rangle$. The presence of perturbing fluxes $\delta_1^*$ and $\delta_2^*$ shifts the potential $V_\nu$ by $v$ along the line $\gamma$, to lowest order in $\delta_1^*$, $v(\chi) = -E_J \sin \chi (\delta_1^* + \delta_2^*)$. This shift produces the changes $\delta S_{12}^1 = \mp s(\delta_1^* + \delta_2^*)$ (the correction $\delta S_{12}^1$ applies to the trajectory $\gamma$ in Fig. 2(a)) with

$$s \approx 1.5 \left(E_J/E_C\right)^{3/4}$$

in the action $S_1$ determining the modulus $|a|$ of the tunnelling amplitude $v_{12}$; as before, the expression $|v_{12}|$ is valid deep in the semi-classical regime.

The presence of perturbing charges $\delta_q^1$ and $\delta_q^2$ modifies the Aharonov-Bohm-Casher phase associated with the two trajectories (a charge $Q$ encircling a flux $\Phi$ counter clockwise produces the phase $\exp(2\pi i(Q/\Phi))$); they pick up the additional phases $\exp(\pm i\pi(\delta_q^1 + \delta_q^2))$, with the plus sign belonging to the trajectory $\gamma$, cf. Fig. 2(b). Combining the perturbations in the fluxes and charges, the change in the tunneling amplitudes $\delta t_{12} = t_{12}^* + it_{12}^*$ takes the form

$$\frac{\delta t_{12}}{t} = e^{-i\delta S_{12}^1 + i\pi(\delta_q^1 + \delta_q^2)} + e^{-i\delta S_{12}^2 + i\pi(\delta_q^1 + \delta_q^2)} - \frac{2}{2}$$

expanding the exponentials, the symmetric and antisymmetric parts are given by the expressions

$$t_{12}^* = t \left[ e^{2\pi i / 2} (\delta_{1}^2 + \delta_{2}^2) / 2 + \pi \sqrt{2} \Phi_2 \right],$$

$$t_{12}^* = \pi s t (\delta_{1}^2 + \delta_{2}^2) (\delta_{1}^2 + \delta_{2}^2);$$

further terms quadratic in $\delta^*$ (e.g., arising from the next term in the expansion of $S_1$) are small by the factor $(E_C/E_J)^{1/4}$. With respect to the qubit’s stability, we note that all perturbations appear in second order of the small quantities $\delta^*$ and $\delta^*$. In the further analysis below we will drop the flux bias term $\alpha \delta^2$ in $t_{ij}$ against the (parametrically large) shifts $v_i$ in the potential energy.

We determine the new energy levels perturbatively: the perturbation $\delta H$, written in the space of semi-classical ground states $|O_i\rangle$, takes the form

$$\delta H_0 = \begin{pmatrix} v_1 & t_{12}^* + it_{12}^* & t_{34}^* + it_{34}^* & v_3 \\ t_{12}^* - it_{12}^* & v_2 & t_{23}^* + it_{23}^* & v_3 \\ t_{34}^* + it_{34}^* & v_2 & t_{23}^* + it_{23}^* & v_3 \\ t_{12}^* + it_{12}^* & v_2 & t_{23}^* + it_{23}^* & v_3 \end{pmatrix}.$$ (29)

Next, we find the corresponding matrix elements $\langle \pm | \delta H | \pm \rangle$ in the projected space spanned by the doublet $|\pm\rangle$, cf. (15). It is convenient to cast the result of this calculation into the form

$$H_{\text{qubit}} = e_{1d} 1 + h \cdot \sigma$$

with $\sigma = (\sigma_x, \sigma_y, \sigma_z)$ the Pauli matrices; switching of the ‘magnetic’ fields $h_x$ and $h_z$ then produces the standard amplitude- and phase-shift operations required for the manipulation of the individual qubit. The shift $e_{1d}$ and the effective ‘magnetic’ field $h$ are given by the expressions

$$e_{1d} = \frac{1}{3} (v_1 + v_2 + v_3 - (t_{12}^* + t_{23}^* + t_{34}^*)),
$$

$$h_x = \frac{1}{3} \left[ v_1 - \frac{1}{2} (v_2 + v_3) + (2t_{23}^* - t_{12}^* - t_{34}^*) \right],
$$

$$h_y = \frac{1}{2} \left[ v_3 - v_2 + \frac{1}{\sqrt{3}} (t_{12}^* - t_{34}^*) \right],
$$

$$h_z = -\frac{1}{\sqrt{3}} \left( t_{12}^* + t_{23}^* + t_{34}^* \right);$$ (31)

note that the perturbations $v_i$ in the potential come with the large amplitude $E_J$, while those in the kinetic energy $(t_{ij}^{*a})$ involve the smaller energy scale $t$ of the tunneling matrix element; in (31) we keep both terms as we might choose to manipulate the qubit via changes in the charges $\delta^a$ alone. Expressing the perturbations in terms of the flux- and charge-parameters $\delta_1^*$ and $\delta_2^*$ we obtain the results

$$e_{1d} = -\frac{4\nu E_J}{3} (\delta_{1}^2 + \delta_{2}^2 + \delta_{3}^2 + \delta_{1}^2 \delta_{2}^2 + \delta_{1}^2 \delta_{3}^2 + \delta_{2}^2 \delta_{3}^2)
$$

$$+ \frac{t \pi^2}{6} \left[ (\delta_{1}^2 + \delta_{2}^2)^2 + (\delta_{3}^2 + \delta_{4}^2)^2 + (\delta_{5}^2 + \delta_{6}^2)^2 \right],
$$

$$h_x = \frac{\nu E_J}{6} \left[ 2(\delta_{1}^2 + \delta_{3}^2)^2 - (\delta_{1}^2 + \delta_{2}^2)^2 - (\delta_{6}^2 + \delta_{7}^2)^2 \right]
$$

$$+ \frac{t \pi^2}{6} \left[ 2(\delta_{3}^2 - \delta_{2}^2 - \delta_{3}^2)^2 - 2(\delta_{1}^2 - \delta_{2}^2)^2 - 2(\delta_{4}^2 - \delta_{5}^2)^2 \right]
$$

$$- 2(\delta_{5}^2 - \delta_{6}^2)^2,$$

$$h_y = \frac{\nu E_J}{2\sqrt{3}} \left[ 2(\delta_{1}^2 + \delta_{3}^2)^2 + (\delta_{5}^2 - \delta_{6}^2)^2 \right]
$$

$$+ \frac{t \pi^2}{2\sqrt{3}} \left[ 2(\delta_{3}^2 - \delta_{2}^2 + 2(\delta_{1}^2 - \delta_{2}^2)^2 \right]
$$

$$+ \frac{t \pi^2}{2\sqrt{3}} \left[ 2(\delta_{1}^2 + \delta_{3}^2)^2 + (\delta_{5}^2 + \delta_{6}^2)^2 \right]
$$

$$+ (\delta_{5}^2 + \delta_{6}^2)(\delta_{1}^2 + \delta_{2}^2) + (\delta_{5}^2 + \delta_{6}^2)(\delta_{3}^2 + \delta_{4}^2) \right].$$ (32)

For convenience, we also cite the energy shift $e_{s}$ of the singlet state,$

$$e_s = \frac{1}{3} (v_1 + v_2 + v_3 + 2(t_{12}^* + t_{23}^* + t_{34}^*)),$$

$$= -\frac{4\nu E_J}{3} (\delta_{1}^2 + \delta_{2}^2 + \delta_{3}^2 + \delta_{1}^2 \delta_{2}^2 + \delta_{1}^2 \delta_{3}^2 + \delta_{2}^2 \delta_{3}^2)
$$

$$- \frac{t \pi^2}{3} \left[ (\delta_{1}^2 + \delta_{2}^2)^2 + (\delta_{3}^2 + \delta_{4}^2)^2 + (\delta_{5}^2 + \delta_{6}^2)^2 \right].$$

While the bias variables $\delta_q$ and $\delta_q^*$ as well as the fields $e_{1d}$ and $h$ do respect the symmetry of the tetrahedron, the
bias fields $h_x$ and $h_y$ do not. The tetrahedral symmetry may be made explicit by reexpressing the planar field $h_\perp$ in the hexagonal basis $e_1 = (1,0)$, $e_2 = (-1/2, -\sqrt{3}/2)$, $e_3 = (-1/2, \sqrt{3}/2)$, $h_\perp = \sum_i h_i e_i$ with

$$h_i = \frac{\mu E_i}{3} (\delta_i^x + \delta_i^y)^2 + \frac{t_\pi^2}{3}[\delta_i^x \delta_i^z + 2\delta_i^y (\delta_i^x + \delta_i^z)^2];$$ (34)

the relation between the tetrahedral symmetries and the rotations in spin space then becomes obvious. The three symmetric fields $h_i$ relate to the two cartesian fields via $h_x = h_1 - (h_2 + h_3)/2$ and $h_y = \sqrt{3}(h_2 - h_3)/2$; there is no inverse transformation.

III. NOISE SENSITIVITY AND MANIPULATION

The above results exhibit two remarkable features of our tetrahedral qubit:

i) Both, charge and flux noise appear only to quadratic order, a feature which is easily traced back to the absence of polarization charges and currents in the qubit’s ground state. This guarantees for a long decoherence time of the tetrahedral qubit, similar to the ‘quaternionium’ discussed by Vion et al. Numerical analysis confirms the weak susceptibility to charge noise: a uniform random bias $V \in [-V_0, V_0]$ acting on the islands produces a small doublet splitting $\delta \approx 0.2 \Delta_0(2eV_0/E_C)^2$, roughly independent of $E_I/E_C$. At the same time, the quadratic dependence on bias allows for a qubit manipulation via ac-fields and the dangerous low-frequency noise can be blocked with appropriate filters.

ii) The tetrahedral qubit admits a large variety of manipulation schemes using either magnetic or electric bias. The ‘planar fields’ $h_x$ and $h_y$ can be manipulated via changes in flux alone, e.g., setting $\delta_2^x = \delta_3^x = \delta_1^y = \delta^z/a_x$ with $a_x = -(1 \pm 2)$, we can direct the field along the $x$-axis, while the flux state $\delta_1^z = \delta^y((1 + a_y)/2)^{1/2}$, $\delta_2^y/a_y = \delta_3^y = \delta^x$; with $a_y = -(2 + \sqrt{3})$ produces a field pointing in the $y$-direction (here, we assume no charge bias, $\delta_1^y = 0$; with these choices of parameters the subleading term $E_I(\delta_1^z - \delta_2^y + 2\delta_3^y)/2$ in $h_\perp$ contributing to $h_x$ with $(E_I/6)[\delta_1^z + \delta_2^y - 2\delta_3^y]$ and to $h_y$ with $(E_I/2\sqrt{3})[\delta_1^z - \delta_2^y - \delta_3^y]$), vanish as well). Proper choice of amplitudes and phases allows for the generation of a rotating planar field. The axial field $h_z$, however, involves a modification of fluxes and charges, cf. [22]: choosing a uniform flux- $\delta_1^z = \delta^z$ and a uniform charge-bias $\delta_1^y = \delta^y$ produces a pure axial field

$$h_z = -4\pi \sqrt{3} st\delta^x \delta^y.$$ (35)

The free manipulation of the ‘magnetic field’ $h(t)$ allows for the implementation of Berry-phase type phenomena [26,27,28]. E.g., the following provides a simple realization of the NOT-operator in the basis $|s,a\rangle = |+\rangle \pm |\rangle)/\sqrt{2}$: adiabatically rotating the transverse components $h_{x,y} = h_\perp \exp(i\omega_{h_\perp} t)$, while keeping the $z$-component $h_z$ fixed, defines the operator $U_{\text{Berry}} = \exp i\sigma_z t/2$ after one period of rotation. Here, $\Omega = 2\pi(1 - h_z/\sqrt{h_0^2 + h_0^2})$ is the solid angle spanned by the rotating field-cone. Selecting a field vector $h$ pointing at an angle of $60^\circ$ with respect to the $x$-axis, the resulting operator $\exp(i\sigma_z t/2)$ changes the relative sign of the components along $|\rangle$, i.e., the eigenstates $|s,a\rangle$ of the operator $\sigma_x$ transform into one another upon each period of field rotation. Note that operator $U_{\text{Berry}}$ does not depend upon the rotating field frequency $\omega_{h_\perp}$ as long as $\omega_{h_\perp}$ is much shorter than the qubit’s decoherence time $t_{\text{dec}}$.

IV. MEASUREMENTS

Finally, we discuss several potential procedures for the measurement of our qubit’s state. We assume an ideal symmetric device. As the qubit reacts to external bias only in second order, the measurement of its state involves a two-step process: In a first step, the qubit is pushed away from the symmetric point of operation and to internal currents and polarization charges. In a second step, these signals have to be measured by appropriate devices and the meter reading will tell about the qubit’s state.

As a simple illustration we consider the measurement of the qubit’s state in the basis $|\rangle, |\rangle, |0\rangle\rangle$, i.e., the operator $\sigma_z$, using a charge bias $\delta^y \ll 1$. This charge bias generates a current flow within the qubit and the associated flux can be measured by an external SQUID loop, cf. Fig. 4(a). We express the matrix $\delta HO$ in the basis $\{|+, |-, |0\rangle\rangle\}$

$$\delta HO = \begin{pmatrix} e_0 + i h_y & h_x - i h_y & e_0^+ + i h_y^+ \\ h_x + i h_y & e_0 - i h_y & e_0^+ - i h_y^+ \\ e_0^+ - i h_y^+ & e_0 - i h_y & e_s \end{pmatrix},$$ (36)

with $e_{d,s}, h_x, h_y, h_z$ given in [22] and [31] above and

$$e_{0}^\pm = \frac{1}{6}[2v_1 - v_2 - v_3 - (2t_{23}^x - t_{31}^x - t_{12}^x)],$$

$$h_{0y}^\pm = \frac{1}{2\sqrt{3}}[v_3 - v_2 - (t_{12}^y - t_{31}^y)] \pm \frac{1}{2}(t_{12}^y - t_{31}^y).$$ (37)

A finite ‘magnetic field’ $h = (0,0,h_z)$ induces the shifts $\delta E^\pm = \pm h_z$ in the two states $|\rangle, |\rangle$. Charge-biasing the device induces the current (we assume $\delta^y \ll 1$)

$$I_y = \frac{2e}{\hbar} \delta E^\pm \frac{\partial E^\pm}{\partial \delta^y} + \frac{2\pi e s t}{\sqrt{3}\hbar}(\delta^y + \delta_j^y + \delta_k^y)$$ (38)
in the corresponding sub-loop ‘0 − j − k’ (the above signs apply to the isolated tetrahedron and have to be reversed for the connected device; a positive current runs counter clockwise around the loop). Alternatively, applying a (small) external flux-bias induces the voltage (see Fig. 4(b))

\[
V_i^\pm = \frac{1}{2} \frac{2 \pi}{\sqrt{3}} \frac{\delta E_i^\pm}{\delta \phi_i^\pm} \approx \frac{\pi}{3\sqrt{6}} \frac{2 \delta_i^\pm + \delta_j^\pm + \delta_k^\pm}{2e} \tag{39}
\]

on the island ‘i’ which can be measured via a single electron transistor (SET) device\(^{2m}\), here i, j, k ∈ \{1, 2, 3\} and pairwise different. Choosing the bias in the ratio \(\delta_i^{\pm} : \delta_j^{\pm} : \delta_k^{\pm} = 3 : -1 : -1\) limits the current/voltage to the loop/island ‘i’.

![Figure 4: Measurement setups allowing the identification of the qubit state.](image)

**FIG. 4:** Measurement setups allowing the identification of the qubit state. (a) An external charge bias induces currents in the qubit structure with a circularity depending on the qubit state. The flux associated with these currents is inductively coupled to a SQUID which is driven close to criticality during the measurement. Depending on the qubit flux, the SQUID is driven overcritical and the associated voltage is measured. (b) An external current bias induces polarization charges in the qubit structure with a polarity depending on the qubit state. The associated charge is capacitively coupled to a SET which is driven close to (charge) frustration during the measurement. Depending on the qubit’s polarization charge the SET is driven into the conducting state and the associated current is measured. (c) Similar to (a), but with qubit currents directly channelled through the measurement junctions with large couplings \(E_m \gg E_j\); again, the presence of a voltage \(V_m\) in the external loop carries the information on the qubit state. (d) States of the measurement junctions (with phase \(\gamma\)) at fixed classical driving current \(I_m\) with a slowly decaying state \(|0\rangle\) (decay rate \(\Gamma_0\)) and a fast decaying state \(|1\rangle\) (decay rate \(\Gamma_1 \gg \Gamma_0\)) depending on the qubit state. We assume a slow energy relaxation for the qubit, \(\Gamma_E \ll \Gamma_1\).

Several issues have to be considered in the measurement process, cf. the discussion in Ref.\(^{51}\). Relevant parameters are the dephasing- and mixing rates \(\Gamma_\varphi\) and \(\Gamma_E\) induced by the measurement apparatus and their relation to the separation \(E_{01}\) between the qubit eigenstates; in a weak measurement scheme we have \(\Gamma_\varphi \ll E_{01}\), while a projective measurement is characterized by a strong coupling with \(E_{01} \ll \Gamma_\varphi\) such that the quantum evolution of the system is quenched rapidly. While dephasing transforms a coherent superposition of states into a classical mixture through elimination of the off-diagonal elements in the qubit’s density matrix, the mixing induces transitions between the qubit’s states and thus spoils the measurement. Usually, a good measurement setup makes use of decoherence in transferring classical information to the measurement device but avoids mixing, hence \(\Gamma_\varphi \gg \Gamma_E\).

The simplest situation is realized when the measured observable commutes with the qubit Hamiltonian — in this case the measurement preserves the qubit’s eigenstates. On the other hand, if the measured observable does not commute with the qubit Hamiltonian, the measurement has to be completed before mixing sets in; hence \(\Gamma_E < \Gamma_{\text{meas}} < \Gamma_\varphi \ll E_{01}/\hbar\) for a weak measurement, while the sequence \(\Gamma_E < \Gamma_{\text{meas}} < E_{01}/\hbar \ll \Gamma_\varphi\) applies to the projective measurement (here, \(\Gamma_{\text{meas}} = 1/t_{\text{meas}}\) denotes the inverse measuring time).

![Figure 5: Current-voltage characteristics of meter devices.](image)

**FIG. 5:** Current-voltage characteristics of meter devices. The solid/dashed lines refer to the different quantum states of the qubit shifting the characteristic of the meter device. (a) In an overdamped SQUID the current-voltage characteristic is single valued, with a voltage due to a finite rate of individual phase-slips. \(I_m\) is the imposed measuring current. (b) The current in the SET results from a continuous flow of individual electrons traversing the island. \(V_m\) is the imposed gate voltage during measurement. (c) An underdamped SQUID/Josephson junction exploits an instability where a single phase-slip triggers a transition to the dissipative branch.

The measurement schemes described in Ref.\(^{31}\) involve dissipative meter devices, e.g., an overdamped SQUID with a well defined (single-valued) current-voltage characteristic as depicted in Fig. 5(a) or a SET with the characteristic shown in Fig. 5(b). In both cases the measurement involves numerous dissipative events, either many phase-slips producing the voltage across the SQUID device or many electrons traversing the island of the SET. The fluctuations due to the phase slips/electrons act back on the qubit, enforcing its loss of phase coherence. Such measurement schemes can be implemented in terms of weak or strong (projective) measurements.

This type of measurement has to be contrasted with a meter characterized by an instability, such as an un-
derdamped SQUID or Josephson junction with a characteristic as shown in Fig. 6(c). Such a meter does not couple dissipatively to the qubit but switches to a dissipative state only after the measurement, e.g., after the occurrence of one phase slip, hence $\Gamma_\varphi \ll \Gamma_{01}$; the measurement is weak and generates a small imaginary part to the energy of the qubit eigenstate which then may be determined in a decay process. The qubit states are identified through their (exponentially) different decay rates. Such a measurement scheme has been used recently by Vion et al. In their setup, an additional measurement junction is introduced into the qubit loop. The current generated by the qubit is superimposed on an external measurement current and drives the measurement junction towards criticality, see Fig. 6(c). Here, we make use of the symmetric setup shown in Fig. 6(a) involving six classical measurement junctions with equal couplings $E_m \gg E_J$. The measuring currents $I_{ni}$ are fed into the device through the points ‘n1’, ‘n2’, and ‘n3’ and removed at the points ‘m1’, ‘m2’, and ‘m3’. The measuring current $I_{m}$ flowing in the segments ‘n3-m1’ and ‘n3-m2’ (and in the other through the yx feeding the measurement junctions, requiring to feed a current $I_{ext} \approx 2I_{m,c}$ into the tetrahedron via the input lines ‘n1’, ‘n2’, ‘n3’ and retraining them via the output lines ‘m1’, ‘m2’, ‘m3’. A convenient measuring setup is the symmetric one with equal currents crossing the junctions ‘n3-m1’ and ‘n3-m2’ (and other pairs obtained through cyclic permutation). Driving the qubit with external bias fields $\delta_i^S$ and $\delta_i^D$ induces additional currents $I_{q,i}$ in the loops ‘mj-mk-k-j-mj’ which are characteristic for the quantum state of the qubit. Depending on the relative flow direction of the qubit currents and the measuring current, the six measuring junctions are either driven towards $(I_m + I_{q,i} \sim I_{m,c})$ or away $(I_m + I_{q,i} < I_{m,c})$ from criticality. The voltage pattern appearing on the six junctions then allows for the identification of the qubit state. Within this scheme, care has to be taken not to spoil the symmetry of the measurement setup by a circular current flowing in the outer ring, hence the system should be flux biased such that the total flux $\Phi$ through the ring remains zero. This can be easily achieved by compensating the fluxes $\Phi_i$ in the loops ‘mj-mk-k-j-mj’ through an equal and opposite flux $\Phi_{\Delta} = -\sum_{i=1}^{3} \Phi_i$ through the central triangular loop.

Below, we study two schemes, a weak measurement in the Hamiltonian basis with $\Gamma_\varphi < \Gamma_{01}$ and a projective measurement onto the current basis with $\Gamma_{01} < \Gamma_\varphi$. In both cases we discuss the measurement of two operators, the measurement of $\sigma_z$ and of $\sigma_x$.

A. Measurements in the qubit’s eigenbasis

A proper measurement in the energy eigenbasis imposes a number of constraints on the measuring device. First of all, the phase decoherence rate has to be small, $\Gamma_\varphi < \Gamma_{01}$. The decay rate $\Gamma_1$ via tunneling of the high energy state (low barrier state, cf. Fig. 6(d)) should be large, $\Gamma_1 \gg \Gamma_{E}$ such that the system tunnels before decaying to the (metastable) ground state. Finally, the measurement time $t_{meas}$ should be larger than the inverse tunneling rate, $t_{meas} > \Gamma_{1}^{-1}$. Note, that phase decoherence before tunneling, i.e., $\Gamma_\varphi > \Gamma_1$, is not a necessary requirement in this type of measurement; the decay may as well proceed out of the coherent state, i.e., a superposition of the qubit states. In this case, the first phase slip triggers the projection, while the subsequent phase-slips produce the large voltage signal.

In order to measure the operator $\sigma_n$, i.e., the spin projection onto the axis $n$, we first apply a ‘magnetic field’ $\mathbf{h} = h \mathbf{n}$ directed along $\mathbf{n}$. This is achieved via proper
charge- and flux-biasing as described by the equations (42). The doublet space \{\ket{+}, \ket{-}\} is then split with new qubit eigenstates \ket{\pm} and \ket{\mp} separated by the ‘Zeeman’ energy \(\Delta E = 2\hbar\). The currents \(I^{(n-n_\text{m})}\) flowing in the segments ‘mj-nk’ across the measuring junctions are equal to the loop currents \(I_{q,i}\) in ‘mj-nk-k-j-mj’ and thus are determined by the derivatives

\[
I^{(n_n-m)} = I_{q,i} = -\frac{2e}{h} \frac{\partial h}{\partial \delta^q_i}.
\]

In order to avoid the flow of a circular current in the outer ring, the driving flux bias \(\delta^q_i\) has to be properly compensated as described above.

We proceed with the evaluation of the currents (40) associated with the measurement of \(\sigma_z\) and \(\sigma_y\) and generated by the application of magnetic fields \(h_z\) and \(h_x\) along the z- and x-axis, respectively. Starting with the measurement of \(\sigma_z\), the projection along the z-axis, we choose a uniform charge and flux bias \((-\to h = (0,0,h_z))\), cf. (22) and obtain the qubit loop currents

\[
I_{q,i}^z = \pm \frac{8\pi \text{est}}{\sqrt{3}h} \delta^q_i, \quad (41)
\]

\(i = 1,2,3\), in the qubit state \(\ket{\pm}\). Hence all loops are equally driven and voltage signals on the triple ‘n1-m3’, ‘n2-m1’, ‘n3-m2’ identify the qubit state \(\ket{+}\), while finite voltages on the junctions ‘n1-m2’, ‘n2-m3’, ‘n3-m1’ are associated with the \(\ket{-}\) state, provided that \(\delta^q_i > 0\).

Second, the projection along the x-axis can be measured by applying a ‘field’ along \(h_x\) using a flux bias \(\delta^x_i = \delta^q_i = -\delta^q_i \equiv \delta^x\). This bias generates a field \(h_x = 4\nu E_j/3\) and imprints qubit currents of magnitude

\[
I_{q,i}^x = \pm \frac{8\pi \text{est} E_j}{3h} \delta^x_i \quad (42)
\]

for \(i = 2,3\) in the qubit state \(\ket{\pm_x}\), while \(I_{q,1}^x = 0\). Hence the pairs ‘n2-m1’, ‘n3-m2’, and ‘n2-m3’, ‘n3-m1’ are equal and oppositely driven, while the junctions ‘n1-m2’ and ‘n1-m3’ do not experience any additional drive due to the qubit structure.

### B. Projective measurement in the current basis

A projective measurement in the current basis requires to switch on a strong decoherence \(\Gamma \approx \text{est}\) during the measurement. This decoherence then projects the state of the qubit at onset of dissipation onto the current basis and keeps it there via the Zeno (watchdog) effect, cf. Ref. 31. A suitable circuit allowing to turn on decoherence is shown in Fig. (22b). The admittance between the points ‘d1’ and ‘d2’ is given by the expression \(Z^{-1}(\omega) = 1/2R - E_m/2i\hbar\omega R_Q\), with \(R_Q = h/4e^2\) the quantum resistance. Choosing parameters \(R \lesssim R_Q\) (this guarantees a sufficient decoherence in the ‘on’ state, see below), \(E_m \sim (10-100)E_j\), and an operating frequency \(\omega \lesssim 0.1E_j\), we find that \(Z \approx -2iR_Q(h\omega/E_m)\). Hence, at zero applied current \(I\) the conductance between the points ‘d1’ and ‘d2’ is dominated by the Josephson junctions and is mostly imaginary at low frequencies, allowing us to ignore dissipation (‘off’ state). However, when the system is biased by a current \(I\) larger than critical, the Josephson current disappears and the resistance \(R\) provides a significant source of dissipation as quantified through the dimensionless parameter \(\alpha = R_Q/R\).

In addition, while the Josephson junction itself involves a large quasi-particle resistance at low temperatures, when driven with a large current the junction switches to the resistive state with the resistance \(R_m\); the latter is related to \(E_m\) via the Ambegaokar-Baratoff relation \(R_m/R_Q = \Delta/E_m\). The projective measurement in the current basis then starts with switching on a strong dissipation (such that \(\Gamma \gg \text{est}\)) which is diagonal in the eigenbasis of the qubit’s current operator \(I_q\).

Before continuing with the discussion of an appropriate projective measurement in the current basis, we discuss one more issue related to the quality of such a measurement. The above measurement schemes (via coupling to a SQUID or a SET) are diagonal in the qubit’s energy subspace spanned by \(\ket{+}\) and \(\ket{-}\) and can be implemented in terms of weak and strong (projective) measurements, cf. Ref. 31. However, extending the analysis to the low-energy subspace \{\ket{+}, \ket{-}, \ket{0}\}, we find for the current operator \(\hat{I}_1 = (2e/h)\partial \delta H_Q / \partial \delta^q_1 = 0\) the expression (cf. 31a)

\[
2\pi \text{est}/h\sqrt{3} \left( \begin{array}{ccc} -\delta^q_{12} - \delta^q_{31} & 0 & \delta^q_{12} \zeta - \delta^q_{31} \zeta^* \\ 0 & \delta^q_{12} + \delta^q_{31} & -\delta^q_{12} \zeta - \delta^q_{31} \zeta^* \\ \delta^q_{12} \zeta^* + \delta^q_{31} \zeta & -\delta^q_{12} \zeta^* - \delta^q_{31} \zeta^* & 0 \end{array} \right),
\]

where \(\delta^q_{ij} = \delta^q_i \zeta \delta^q_j \zeta^*\): choosing a charge-bias \(\delta^q_i/3 = \delta^q = -\delta^q_i = -\delta^q_3\), this reduces to

\[
\hat{I}_1 = -\frac{4\pi \text{est}}{h\sqrt{3}} \left( \begin{array}{ccc} 0 & 2 & 0 \\ -2 & 0 & -1 \\ 1 & -1 & 0 \end{array} \right).
\]

Similar expressions apply to the other current operators \(\hat{I}_2\) and \(\hat{I}_3\) and, replacing \(2\pi \text{est}/h\sqrt{3} \to \pi \text{est}/2e\sqrt{3}\) in the prefactor and substituting \(\delta^q_{ij} \to \delta^q_i + \delta^q_j\), to the voltage operators \(\hat{V}_i = (1/2e)\partial \delta H_Q / \partial \delta^q_i\).

Hence a projective measurement onto the current basis is in fact non-ideal as the measured observable, the current, does not commute with the Hamiltonian when going beyond the qubit sector. In a ‘high quality’ measurement of our qubit state one would request that the off-diagonal matrix elements are much smaller than the matrix elements in the doublet subspace. Otherwise, the measurement has to be repeated many times in order to arrive at a proper readout of the qubit state. We then need to identify special measurement configurations where the off-diagonal matrix elements remain small or even vanish altogether.
We will now give a specific example how this goal can be achieved for a projective measurement of the $\sigma_z$ operator. We will show that a charge-bias with

$$\delta_2^Q = -\delta_3^Q = \delta^z, \quad \delta_1^Q = 0$$

results in a high-quality measurement of $\sigma_z$ and $\sigma_y$ if we choose a large bias $\delta^Q = 1/2$. This bias induces a current flow of equal magnitude but different chirality in the loops (cf. Fig. 6(a)) 'm1-1-3-m3-m1' (current $I_2$) and 'm2-2-1-m1-m2' (current $I_3$; loop currents circulating counter clockwise are positive); no current is induced in the loop 'm3-3-2-m2-3'. The total current through the link 'm1-1' is given by $I_{m1} = I_2 - I_3$ and will be used to drive the measurement current $I_m$ fed symmetrically into the tetrahedron at the points 'm' overcritical. We first describe the setup projecting the link current $I_{m1}$ and subsequently derive an explicit expression for the current operator. In a third step, we then describe the actual measurement process in detail.

We will show below that the link current $I_{m1}$ will not mix to the third state $|0\rangle$. However, this property is not shared by the individual loop currents $I_2$ and $I_3$. It is then important to devise a setup projecting the state onto the link current $I_{m1}$ rather than the loop currents $I_2$ or $I_3$. This is achieved by the symmetric coupling of three 'dissipators' in between the points 'm1-m2', 'm2-m3', and 'm3-m1'. Turning on one of these dissipators renders the current in the corresponding link classical. In our setup we wish both loop current, $I_2$ and $I_3$, to be classical and hence switch on the dissipators 'm1-m2' and 'm1-m3'. In this situation it is the link current $I_{m1}$ which is coupled to the dissipative reservoir (the segment 'm2-m3' remains quantum and can be contracted to a point, cf. Fig. 7) and hence the quantum state is projected to a state with fixed current $I_{m1}$. In an ideally symmetric setup this current will be symmetrically split into the two arms 'm1-m2' and 'm1-m3' and, depending on the qubit state, will drive a specific set of junctions overcritical.

Let us then turn to the calculation of the link current $I_{m1}$. In a first step, we have to generalize the tunneling matrix elements (28) to allow for a large charge bias,

$$t + t_{ij}^Q = t \cos(\pi \delta_{ij}^Q),$$

$$t_{ij}^Q = s t \delta_{ij}^{\sigma} \sin(\pi \delta_{ij}^Q).$$

Assuming the above form of charge bias, the Hamiltonian in the basis of semi-classical states $\{|O_1\rangle, |O_2\rangle, |O_3\rangle\}$ takes the form $H_O = \theta O_{1r} + ist O_{1i}$ with

$$h_O = \begin{pmatrix}
0 & \cos(\pi \delta^Q) & \cos(\pi \delta^Q) \\
\cos(\pi \delta^Q) & 0 & 1 \\
\cos(\pi \delta^Q) & 1 & 0
\end{pmatrix}$$

and

$$h_{Oi} = \begin{pmatrix}
0 & \delta_{12}^* \sin(\pi \delta^Q) & \delta_{31}^* \sin(\pi \delta^Q) \\
-\delta_{12}^* \sin(\pi \delta^Q) & 0 & 0 \\
-\delta_{31}^* \sin(\pi \delta^Q) & 0 & 0
\end{pmatrix}.$$

where $\delta_{ij}^* = \delta_{ij}^Q + \delta_{ij}^Q$. The link current operator then takes the form (after setting $\delta_{ij}^Q = 0$; note the sign change for the connected tetrahedron, $\dot{I}_i = -(2e/h)\partial H_O/\partial \delta_{ij}^Q$)

$$\dot{I}_{m1} = \frac{2e^2 i}{h} \sin(\pi \delta^Q) \begin{pmatrix}
0 & -1 & 1 \\
1 & 0 & 0 \\
-1 & 0 & 0
\end{pmatrix}.$$

In measuring the qubit’s state, we first drive the tetrahedron adiabatically towards the measuring point $\delta^Q = 1/2$. At small values of $\delta = (\pi \delta^Q)^2/2$ the doublet splits and we find the new eigenvalues $E_3$ and associated eigenvectors $|e_3\rangle$ (up to normalization),

$$E_3/t = \begin{pmatrix}
1 & |e_3\rangle \\
1 & 1
\end{pmatrix}$$

$$-1 + \frac{\delta}{4} |s_3\rangle \approx \frac{2(1+\delta/3)|O_1\rangle - (O_2) - (O_3)}{\sqrt{1+8\delta/3}}.$$

$$2 - \frac{\delta}{4} |0_3\rangle \approx \frac{(1-\delta/3)|O_2\rangle + (O_2) + (O_3)}{\sqrt{3-2\delta/3}}.$$

Note that at $\delta = 0$ the eigenvectors $|a_0\rangle$ and $|s_0\rangle$ defining the low-energy qubit subspace correspond to the (anti-)symmetric combinations $|a_0\rangle = [(|+) + (|+) / \sqrt{2} and |s_0\rangle = [(|+\rangle + |\rangle) / \sqrt{2}$ and starting with a qubit state $|\Psi\rangle = (|\Psi_+\rangle + (|\Psi_-\rangle / \sqrt{2} and $\psi_{a0} = (|\psi_+\rangle - (|\psi_-\rangle / \sqrt{2}$ or, in matrix form,

$$\begin{pmatrix}
\psi_{a0} \\
\psi_{s0}
\end{pmatrix} = \hat{T} \begin{pmatrix}
\psi_+ \\
\psi_-
\end{pmatrix} = \frac{1 + \delta/3}{\sqrt{2}} \begin{pmatrix}
1 & -1 & 0 \\
1 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix} \begin{pmatrix}
\psi_+ \\
\psi_- \\
\psi_0
\end{pmatrix}.$$

Driving the bias up to $\delta^Q = 1/2$ the spectrum and eigen-
vectors deform into
\[ E_{1/2}/t = |\epsilon_1/2\rangle, \]
\[ -1 |a_{1/2}/rangle = i(|O_2\rangle - |O_3\rangle)/\sqrt{2}, \]
\[ 0 |s_{1/2}/rangle = |O_1\rangle, \]
\[ 1 |0_{1/2}/rangle = ||O_2\rangle + |O_3\rangle)/\sqrt{2}. \]

In this new basis \{a_{1/2}/rangle, |s_{1/2}/rangle, |0_{1/2}/rangle\} the current operator \( I_{m} \) takes the form
\[ \hat{I}_{m-1} = 2\sqrt{2V_{est}}/\hbar \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}; \]

hence after the adiabatic evolution, the current measures the \( \sigma_x \) operator in the basis \( |\rangle \) and has no matrix elements with the third state \( |0_{1/2}/rangle \). At the same time, the qubit wave function takes the form \(|\Psi\rangle = e^{-i/2}\psi_{n_0}|a_{1/2}/rangle + e^{i/2}\psi_{n_0}|s_{1/2}/rangle\), where \( \theta = \int dt(E_a - E_x)/\hbar \) denotes the additional phase picked up in the adiabatic evolution of the qubit amplitudes \( \psi_{n_0} \) and \( \psi_{n_0} \) until the measurement is performed. In order to reexpress the current through the original qubit amplitudes \( \Psi_\pm \) we define the matrix
\[ \hat{T}_\theta = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{-i/2} & -e^{-i/2} \\ e^{i/2} & e^{i/2} \\ 0 & 0 \end{pmatrix} \]
and find the result
\[ \hat{T}_\theta^* \hat{I}_{m-1} \hat{T}_\theta = \frac{2\sqrt{2V_{est}}}{\hbar} \begin{pmatrix} \cos \theta & i \sin \theta & 0 \\ -i \sin \theta & -\cos \theta & 0 \\ 0 & 0 & 1 \end{pmatrix}. \]

describing a high quality measurement of the qubit operators \( \pm \sigma_x \) at \( \theta = n\pi \) and \( \pm \sigma_y \) at \( \theta = (n + 1/2)\pi \) (or any component residing in the y-z plane for angles \( \theta \) in between). Applying the charge bias to cyclically permuted islands \( 1 \rightarrow 2 \rightarrow 3 \rightarrow 1 \) produces measurements of spin-projections in planes rotated by the corresponding angles \( \pm 2\pi/3 \).

In the following, we restrict the discussion to the measurement of \( \sigma_z \), i.e., we assume that \( \theta = n\pi \) with \( n \) even in the specific discussion below. Following the result \( I_{m-1} \) has opposite signs for the two qubit-states \( |+\rangle \) and \(-\rangle \) and its measurement allows for the determination of the qubit’s final wave function.

In the actual measurement, the system is driven symmetrically with equal external currents \( I_{m} = I_{m} \) entering the system at the points ‘1’, ‘2’, ‘3’ (cf. Fig. 6(a)) and leaving symmetrically through the points ‘1’, ‘2’, and ‘3’. The external current \( I_{m} \) is chosen close to, but below twice the critical current of the large junctions in the ring, \( I_{m} < 2I_{m} = 4eE_{m}/h \). Accounting for the additional currents induced in the tetrahedron under the charge bias \( \delta^2 = 1/2 \), the currents \( I^{(n_2-m_1)} \) and \( I^{(n_3-m_1)} \) through the large measurement junctions are equal to \( (I_{m} \pm \sqrt{2V_{est}})/2 \) for the \( |\pm\rangle \) states of the doublet, while the currents \( I^{(n_2-m_2)} \) and \( I^{(n_3-m_2)} \) assume the values \( (I_{m} \mp \sqrt{2V_{est}})/2 \) for the same eigenstates. Note that in the singlet state the bare measurement current \( I_{m}/2 \) flows through the junctions as no current is induced in the singlet state \( |0\rangle \) under the above charge bias. As demonstrated in Ref. 11, the measurement current \( I_{m} \) and the induced current \( 2\sqrt{2V_{est}}/t \) can be chosen such that the switching probabilities \( P_\pm \) into the transient voltage-state of the large measurement junctions are strongly different for the \( |\pm\rangle \) states. Then the location of voltage pulses on the junctions ‘2-n1’ and ‘3-n1’ or on the complementary junctions ‘2-n3’ and ‘3-n3’ tells us whether the qubit was in the \( |+\rangle \) or \(-\rangle \) state just before the measurement. Furthermore, the absence of any voltage pulse is the signature of the singlet state. Finally, we discuss the projective measurement of the operator \( \sigma_x \). Following 14, a flux bias produces finite ‘magnetic fields’ \( h_{xy} \) which are bilinear in \( \delta^x \); one then may expect that an appropriate flux bias will produce loop currents which are diagonal in the basis \( \{a, s\} \), where \(|a\rangle = ((|+\rangle - |\rangle))/\sqrt{2}, \quad |s\rangle = ((|+\rangle + |\rangle))/\sqrt{2}. \)

Here, we concentrate on the main contribution (proportional to \( \nu \)) originating from the modification \( v_j \) in the energies of the semi-classical minima \( |O_j\rangle \). The current operators \( \hat{I}_j = -(2e/\hbar)\partial\delta H_j/\partial\delta^x \) evaluated in the basis \( \{+\rangle, |\rangle, |0\rangle \} \) take the form
\[ \hat{I}_j = \frac{2eV_j}{3h} \begin{pmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ \lambda_2 & \lambda_3 & \lambda_1 \\ \lambda_3 & \lambda_1 & \lambda_2 \end{pmatrix} \]
with \( \lambda_i = 4(2\delta^x + \delta^y + \delta^z) \) and
\[ \lambda_1 = (2\delta^x + \delta^y + \delta^z) + i\sqrt{3}(\delta^y - \delta^z), \]
\[ \lambda_2 = -(2\delta^x + \delta^y - \delta^z) + i\sqrt{3}(\delta^z - \delta^x), \]
\[ \lambda_3 = -(2\delta^x - \delta^y - \delta^z) - i\sqrt{3}(\delta^x - \delta^z). \]

Applying a specific bias \( \delta^x = \delta^y = \delta^z = \delta^x \) produces a field along \( h_{xy} \) (cf. 14) and induces the current \( I^{(n_3-3)} = I_1 - I_2 = -I^{(n_2-2)} \) in the loop ‘1-n1-3-2-m1’,
\[ \hat{I}_{m-3} = -\frac{8eV_j\delta^x}{3h} \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix} \]
the current operator \( \hat{I}_{m-1} \) vanishes. Transforming to the basis states \( \{a, s\} \), this takes the form
\[ \hat{I}_{m-3} = -\frac{8eV_j\delta^z}{3h} \begin{pmatrix} 3 & 0 & 0 \\ 0 & 1 & -\sqrt{2} \\ 0 & -\sqrt{2} & 2 \end{pmatrix} \]
obviously, the anti-symmetric state \( |a\rangle \) is already a good eigenstate of the current operator, while the states \( |s\rangle \)
Assume that just before the measurement the wave function to identify the qubit state: Let us assume that the above constellation allows for a well defined statistical procedure to identify the qubit state. The actual measurement of the current is carried out with the same (symmetric) measurement current configuration as discussed in the previous section. The only difference is that in the present measurement the current eigenvalues are proportional to \( j = 1 \) and \( j = 0 \) instead of \( \pm 1 \), i.e., the switching of (one of the two) junctions ‘n2-m1’, ‘n3-m2’ (or, depending on the sign of flux bias, ‘n2-m3’, ‘n3-m1’) is characteristic for a \( j = 1 \) eigenstate, whereas the absence of any switching corresponds to the \( j = 0 \) state.

**V. CONCLUSIONS**

The CH\(_4\) molecule can be viewed as a molecular analogue to the tetrahedral superconducting structure with the same tetrahedral symmetry group. The non-Abelian character of this symmetry group is responsible for the natural appearance of degenerate states. However, contrary to the situation in atomic and molecular physics, where such degenerate levels usually correspond to excited states, the macroscopic device discussed here can be tuned such that the non-Abelian character of its symmetry group manifests itself in the appearance of a degenerate ground state. In order to do so, we have to bias the device, both electrically and magnetically, to the maximally frustrated point with half-flux \( \Phi_0/2 \) threading each sub-loop and half-Cooper-pair charge \( e \) induced on each island. The ground state is a doublet equivalent to a spin-1/2 system in a vanishing magnetic field.

Given the above analogy to molecular physics, one may ask whether a splitting of the doublet ground state due to a Jahn-Teller type instability may appear in our system as well. In the superconducting tetrahedron this would correspond to a paramagnetic instability with a spontaneous breaking of time reversal symmetry: paramagnetic currents would lower the system energy due to their interaction with the self-generated magnetic field and the ground state doublet would split — the magnitude of this effect is determined by the magnetic inductance of the device which we have set to zero in our analysis above. However, in order to realize such an instability, the energy gain should be linear in the spontaneously generated flux; since our device exhibits only a quadratic dependence on flux, cf. [92], this type of instability is absent.

On a technical level, the superconducting tetrahedral qubit comes with a number of practically useful features: i) The weak quadratic sensitivity to electric and magnetic noise implies long coherence times for this device, similar to the ‘quantonium’ discussed by Vion et al. [2]. At the same time, the quadratic dependence on bias allows for a manipulation via ac-fields rather than the usual dc-bias, hence the most dangerous low-frequency part of the noise spectrum can be blocked from the qubit via appropriate filters. ii) The degenerate ground state allows to avoid the appearance of phonon radiation during idle time [2]. Assume a qubit with states \( |0\rangle \) and \( |1\rangle \) residing at different energies \( E_0 \neq E_1 \). A superposition state \( |\psi(t)\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \) will induce voltage oscillations \( V = \frac{\hbar\phi}{2e} \propto (E_1 - E_0)/2e \) across the Josephson junction which couple to the underlying lattice via the piezo-electric effect. Hence, the junction acts as an antenna emitting phonons which contributes to the energy relaxation rate of the qubit. In our tetrahedral qubit, \( E_1 = E_0 \) and this loss channel is avoided during idle time. iii) The tetrahedral qubit can be fabricated with junctions of relatively large size. This is the consequence of a weak \( \propto \exp[-\text{const}.(E_J/E_C)^{1/4}] \) rather than the usual \( \propto \exp[-\text{const}.(E_J/E_C)^{1/2}] \) suppression of the qubit’s operational energy scale and entails two important advantages: first, less stringent requirements on the fabrication process and better junction uniformity, and second, an improved robustness of the qubit with respect to charge noise originating from fluctuating stray charges. The physical origin of this benevolent behavior is found in the huge classical ground state degeneracy originating from junctions with a simple \( \propto \sin \varphi \) current-phase relation combined with a maximal magnetic frustration; this degeneracy then is lifted only due to quantum fluctuations [22,16]. iv) The tetrahedral qubit admits a large variety of manipulation schemes — arbitrary manipulations of the effective ‘spin 1/2’ ground state can be implemented through either magnetic or electric bias fields. v) The quantum measurement can be performed with respect to different basis states and us-
ing either charge or flux bias. We have described schemes operating in the qubit eigenbasis and projective measurements onto the current basis. In the former setup, we have described detailed procedures for the detection of both ‘spin’ projections $\sigma_x$ and $\sigma_y$. In the latter scheme, we have identified a high-quality measurement scheme for the operators $\sigma_x$ and $\sigma_z$ through appropriate charge bias; the flux bias scheme produces a non-ideal but acceptable setup for the measurement of $\sigma_x$. Proper rotation of the bias scheme by $\pm 2\pi/3$ provides a measurement of the corresponding rotated spin-components.

The above advantages seem worth the additional complexity of the device. Still, one may pose the question whether the same benefits can be implemented with a simpler device. E.g., the $C_{3v}$ symmetry group of the symmetric three-junction loop also contains a two-dimensional representation and appropriate charging with $q_i = 1/3$-charge per island produces a doublet ground state suitable for quantum computation. However, this simpler design does not exhibit the quadratic stability under charge noise — charge biasing one island reduces the symmetry to $Z_2$ with only one-dimensional representations and the doublet splits in linear order in $\delta^2$. On the contrary, charge biasing the tetrahedron reduces the symmetry from $T_d$ to $C_{3v}$, which still contains a two-dimensional representation. Indeed, the two complex conjugated states $|\pm\rangle$ react the same way to a charge bias $\delta^2$ and hence the doublet splits only in quadratic order. This behavior, the indifferent response of the two ground state wave functions to a local perturbation is strongly reminiscent of the idea of topological protection, where the fault tolerance of the device is implemented on the hardware rather than the software level. The above symmetry arguments then show, that in order to benefit from a protected degenerate ground state doublet, the qubit design requires a certain minimal complexity; it seems to us that the tetrahedron exhibits the minimal symmetry requirements necessary for this type of protection and thus the minimal complexity necessary for its implementation.

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