SINGULARITY FORMATION FOR THE TWO-DIMENSIONAL HARMONIC MAP FLOW INTO $S^2$

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Abstract. We construct finite time blow-up solutions to the 2-dimensional harmonic map flow into the sphere $S^2$, $u_t = \Delta u + |\nabla u|^2 u$ in $\Omega \times (0,T)$,

$u = \varphi$ on $\partial \Omega \times (0,T)$

$u(\cdot,0) = u_0$ in $\Omega$,

where $\Omega$ is a bounded, smooth domain in $\mathbb{R}^2$ and $u : \Omega \times (0,T) \to S^2$, $u_0 : \bar{\Omega} \to S^2$, smooth, $\varphi = u_0|_{\partial \Omega}$. Given any points $q_1, \ldots, q_k$ in the domain, we find initial and boundary data so that the solution blows-up precisely at those points. The profile around each point is close to an asymptotically singular scaling of a 1-corotational harmonic map. We build a continuation after blow-up as a $H^1$-weak solution with a finite number of discontinuities in space-time by “reverse bubbling”, which preserves the homotopy class of the solution after blow-up.

1. Introduction and main result

Let $\Omega$ be a bounded domain in $\mathbb{R}^2$ with smooth boundary $\partial \Omega$. We denote by $S^2$ the standard 2-sphere. We consider the harmonic map flow for maps from $\Omega$ into $S^2$, given by the semilinear parabolic equation

$u_t = \Delta u + |\nabla u|^2 u$ in $\Omega \times (0,T)$

$u = \varphi$ on $\partial \Omega \times (0,T)$

$u(\cdot,0) = u_0$ in $\Omega$,

for a function $u : \Omega \times [0,T) \to S^2$. Here $u_0 : \bar{\Omega} \to S^2$ is a given smooth map and $\varphi = u_0|_{\partial \Omega}$. Local existence and uniqueness of a classical solution follows from the works [5,14,31]. Equation (1.1) formally corresponds to the negative $L^2$-gradient flow for the Dirichlet energy $\int_\Omega |\nabla u|^2 dx$. This energy is decreasing along smooth solutions $u(x,t)$:

$\frac{\partial}{\partial t} \int_\Omega |\nabla u(\cdot,t)|^2 = - \int_\Omega |u_t(\cdot,t)|^2$.

Struwe [31] established the existence of an $H^1$-weak solution, where just for a finite number of points in space-time loss of regularity occurs. This solution is unique within the class of weak solutions with degreasing energy, see Freire [15].

If $T > 0$ designates the first instant at which smoothness is lost, we must have

$\|\nabla u(\cdot,t)\|_\infty \to +\infty$ as $t \uparrow T$. 

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Several works have clarified the possible blow-up profiles as \( t \uparrow T \). The following fact follows from results by Ding-Tian \[13\], Lin-Wang \[18\], Qing \[23\], Qing-Tian \[25\], Struwe \[31\], Topping \[33\] and Wang \[36\]:

Along a sequence \( t_n \to T \) and points \( q_1, \ldots, q_k \in \Omega \), not necessarily distinct, \( u(x, t_n) \) blows-up occurs at exactly those \( k \) points in the form of bubbling. Precisely, we have

\[
u(x, t_n) - u_\ast(x) - \sum_{i=1}^{k} \left[ U_i \left( \frac{x - q_i^n}{\lambda_i^n} \right) - U_i(\infty) \right] \to 0 \quad \text{in } H^1(\Omega) \quad (1.4)\]

where \( u_\ast \in H^1(\Omega) \), \( q_i^n \to q_i \), \( 0 < \lambda_i^n \to 0 \), satisfy for \( i \neq j \),

\[
\frac{\lambda_i^n}{\lambda_j^n} + \frac{\lambda_j^n}{\lambda_i^n} + \frac{|q_i^n - q_j^n|^2}{\lambda_i^n \lambda_j^n} \to +\infty.
\]

The \( U_i \)'s are entire, finite energy harmonic maps, namely solutions \( U : \mathbb{R}^2 \to S^2 \) of the equation

\[
\Delta U + |\nabla U|^2 U = 0 \quad \text{in } \mathbb{R}^2, \quad \int_{\mathbb{R}^2} |\nabla U|^2 < +\infty.
\]

After stereographic projection, \( U \) lifts to a smooth map in \( S^2 \), so that its value \( U(\infty) \) is well-defined. It is known that \( U \) is in correspondence with a complex rational function or its conjugate. Its energy corresponds to the absolute value of the degree of that map times the area of the unit sphere, and hence

\[
\int_{\mathbb{R}^2} |\nabla U|^2 = 4\pi m, \quad m \in \mathbb{N}, \quad (1.5)
\]

see Topping \[33\].

In particular, \( u(\cdot, t_n) \rightharpoonup u_\ast \) in \( H^1(\Omega) \) and for some positive integers \( m_i \), we have

\[
|\nabla u(\cdot, t_n)|^2 \rightharpoonup |\nabla u_\ast|^2 + \sum_{i=1}^{k} 4\pi m_i \delta_{q_i} \quad (1.6)
\]

in the measures sense, were \( \delta_q \) denotes the unit Dirac mass at \( q \).

Topping \[34\] estimated the blow-up rates as \( \lambda_i^n = o(T - t_n)^{\frac{1}{2}} \) (also valid for more general targets), a fact that tells that the blow-up is of “type II”, namely it does not occur at a self-similar rate.

A decomposition similar to \((1.4)\) holds if blow-up occurs in infinite time, \( T = +\infty \). In such a case one has the additional information that \( u_\ast \) is a harmonic map, and the convergence in \((1.4)\) also holds uniformly in \( \Omega \) (the latter is called the “no-neck property”), see Qing and Tian \[25\]. Finer properties of the bubble-decomposition have been found by Topping \[33\].

A least energy entire, non-trivial harmonic map is given by

\[
\omega(x) = \frac{1}{1 + |x|^2} \left( \frac{2x}{|x|^2 - 1} \right), \quad x \in \mathbb{R}^2,
\]

which satisfies

\[
\int_{\mathbb{R}^2} |\nabla \omega|^2 = 4\pi, \quad \omega(\infty) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.
\]
Very few examples are known of solutions which exhibit the singularity formation phenomenon (1.6), and all of them concern single-point blow-up in radially symmetric corotational classes. When $\Omega$ is a disk or the entire space, a 1-corotational solution of (1.1) is one of the form
\[ u(x, t) = \left( e^{i\theta} \sin v(r, t), \cos v(r, t) \right), \quad x = re^{i\theta}. \]
Within this class, (1.1) reduces to the scalar, radially symmetric problem
\[ v_t = v_{rr} + \frac{v_r}{r} - \frac{\sin v \cos v}{r^2}. \] (1.8)
We observe that the function
\[ w(r) = \pi - 2 \arctan(r) \]
is a steady state of (1.8) which corresponds precisely to the harmonic map $\omega$ in (1.7). Indeed,
\[ \omega(x) = \left( e^{i\theta} \sin w(r), \cos w(r) \right). \]
Chang, Ding and Ye [6] found the first example of a blow-up solution of Problem (1.1)-(1.3) (which was previously conjectured not to exist). It is a 1-corotational solution in a disk with the blow-up profile
\[ u(x, t) = \omega \left( \frac{x}{\lambda(t)} \right) + O(1). \] (1.9)
with $O(1)$ bounded in $H^1$-norm and $0 < \lambda(t) \to 0$ as $t \to T$. No information on the blow-up rate $\lambda(t)$ is obtained. Angenent, Hulshof and Matano [1] estimated the blow-up rate of 1-corotational maps as $\lambda(t) = o(T-t)$. Using matched asymptotics formal analysis for Problem (1.8), van den Berg, Hulshof and King [3] demonstrated that this rate for 1-corotational maps should generically be given by
\[ \lambda(t) \sim \kappa \frac{T - t}{|\log(T - t)|} \]
for some $\kappa > 0$. Raphael and Schweyer [28] succeeded to rigorously construct an entire 1-corotational solution with this blow-up rate.

In this paper we deal with the general, nonsymmetric case in (1.1)-(1.3). Our first result asserts that for any given finite set of points of $\Omega$ and suitable initial and boundary values, then a solution with a simultaneous blow-up at those points exists, with a profile resembling a translation and rotation of that in (1.9) around each bubbling point.

To state our result, we observe that the functions
\[ U_{\lambda, q, Q}(x) := Q\omega \left( \frac{x - q}{\lambda} \right) \]
with $\lambda > 0$, $q \in \mathbb{R}^2$ and $Q$ an orthogonal matrix in $\mathbb{R}^3$ do solve Problem (1.5), and all share the least energy property:
\[ \int_{\mathbb{R}^2} |\nabla U_{\lambda, q, Q}|^2 = 4\pi. \]
We shall denote, for \( \alpha \in \mathbb{R} \) by \( Q_\alpha \), the \( \alpha \)-rotation around the third axis,

\[
Q_\alpha \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} y_1 \cos \alpha - y_2 \sin \alpha \\ y_1 \sin \alpha + y_2 \cos \alpha \\ y_3 \end{bmatrix}.
\]

**Theorem 1.** Given \( T > 0 \) and points \( q = (q_1, \ldots, q_k) \in \Omega^k \), there exist \( u_0 \) and \( \varphi \) such the solution \( u_q(x,t) \) of Problem \((1.1)-(1.3)\) blows-up at exactly those \( k \) points as \( t \uparrow T \). More precisely, there exist numbers \( \kappa_i^* > 0 \), \( \alpha_i^* \) and a function \( u_* \in H^1(\Omega) \cap C(\bar{\Omega}) \) such that

\[
u \equiv u_q(x,t) - u_*(x) - \sum_{j=1}^{k} Q_{\alpha_j^*} \left[ \omega \left( \frac{x - q_j}{\lambda_i} \right) - \omega(\infty) \right] \rightarrow 0 \quad \text{as} \quad t \uparrow T, \quad (1.10)
\]

in the \( H^1 \) and uniform senses in \( \Omega \), where

\[
\lambda_i(t) = \kappa_i^* \frac{T - t}{|\log(T - t)|^2}. \quad (1.11)
\]

In particular, we have

\[
|\nabla u(\cdot,t)|^2 \rightarrow |\nabla u_*|^2 + 4\pi \sum_{j=1}^{k} \delta_{q_j} \quad \text{as} \quad t \uparrow T.
\]

Raphael and Schweyer \([28]\) proved the stability of their solution within the 1-corrortational class, namely perturbing slightly its initial condition in its equivalent form in the radial equation \((1.8)\), the associated radial solution exhibits a similar blow-up at a slightly different time. On the other hand, Merle, Raphaël and Rodnianski \([22]\), based on the instability they observed in equivariant blow up solutions of the critical Schrödinger map problem, expected that the stability of blow up for the harmonic map flow could be affected by the presence of rotations. Formal and numerical evidence of instability was later found by van den Berg and Williams \([4]\), and led them to conjecture that radial bubbling loses its stability if special perturbations off the radially symmetric class are made to the initial data. The proof of Theorem 1 provides strong evidence that this is the case, and in fact, a slight strengthening of the estimates we already have should allow us to conclude that there exists a manifold in \( C^1(\bar{\Omega},S^2) \) with codimension \( k \), that contains \( u_q(x,0) \), such that if \( u_0 \) lies in that manifold and it is sufficiently close to \( u_q(x,0) \), then the solution \( u(x,t) \) of problem \((1.1)-(1.3)\) blows-up at exactly \( k \) points \( \tilde{q} \) close to \( q \), at a time \( \tilde{T} \) close to \( T \), with an expansion of the form \((1.10)\).

The solutions in Theorems 1 are classical in \([0,T)\). Our next result concerns the continuation of the solution after blow-up. As we have mentioned Struwe \([31]\) defined a global \( H^1 \)-weak solution of \((1.1)-(1.3)\). Struwe’s solution is obtained by just dropping the bubbles appearing at the blow-up time and then restarting the flow. The energy has jumps at each blow-up time generated by this procedure and it is decreasing. Decreasing energy suffices for uniqueness of the weak solution, as proven in \([15]\). On the other hand the bubble-dropping procedure modifies in time the topology of the image of the solution map. Topping \([34]\) showed a different way to construct a continuation after blow up in the symmetric 1-corrortational class. The solution in \([6]\) is continued after blow-up by attaching a bubble with opposite orientation, which unfolds continuously the energy. The solution referred
to is a reverse bubbling solution. As emphasized in [34], this continuation has the advantage that, unlike Struwe’s solution, it preserves the homotopy class of the map after blow-up. Formal asymptotic rates for 1-corrotational reverse bubbling was found in [3]. In [2] other forms of continuation of radial solutions were found.

We establish that Topping’s continuation can be made without symmetry assumptions, with exact asymptotics, for the solution in Theorem 1. We define bubble with reverse orientation to that of \( \omega \) as

\[
\bar{\omega}(x) = \frac{1}{1 + |x|^2} \begin{pmatrix}
-2x \\
|x|^2 - 1
\end{pmatrix} = \begin{pmatrix}
-e^{i\theta} \sin w(r) \\
\cos w(r)
\end{pmatrix}.
\]

**Theorem 2.** Let \( u_q(x, t) \) be the solution in Theorem 1. Then \( u_q \) can be continued as an \( H^1 \)-weak solution in \( \Omega \times (0, T + \delta) \), which is continuous except at the points \( (q_i, T) \), with the property that, besides expansion (1.10), we have \( u_q(x, T) = u_*(x) \)

\[
u_q(x, t) - u_*(x) - \sum_{j=1}^{k} Q_{\alpha_j^*} \left[ \bar{\omega} \left( \frac{x - q_i}{\lambda_i} \right) - \bar{\omega}(\infty) \right] \rightarrow 0 \text{ as } t \downarrow T,
\]
in the \( H^1 \) and uniform senses in \( \Omega \), where

\[
\lambda_i(t) = \kappa_i^* \frac{t - T}{|\log(t - T)|^2}.
\]

We observe that the energy in this continuation fails to be decreasing: it has a jump exactly at time \( T \) and it goes back to its previous level immediately after.

Before proceeding into the proof we make some further comments. It is plausible that the solutions of the form described in Theorem 1 represent a form of “generic” bubbling phenomena for the two-dimensional harmonic map flow, still too general in the form (1.4). For instance, it is reasonable to think, yet unknown, that the limits along any sequence should have the same elements in the bubble decomposition. On the other hand, is it possible to have bubbles other than those induced by \( \omega \) or \( \bar{\omega} \), and or decomposition in several bubbles at the same point? Some evidence is already present in the literature. It is known that in the more general symmetry class of the \( d \)-corrotational ones, \( d \geq 1 \),

\[
u(x, t) = \begin{pmatrix}
e^{i\theta} \sin v(r, t) \\
\cos v(r, t)
\end{pmatrix}, \quad x = re^{i\theta}
\]

are steady states \( v = w_d(r) = \pi - 2 \arctan(v^d) \) which do not lead to blow-up, at least for \( d \geq 4 \) (conjectured for \( d = 2, 3 \)). See Guan-Gustafson-Tsai [16]. On the other hand, no bubble trees in finite time exist in the 1-corrotational class. See Van der Hout [35]. Infinite time multiple bubbling was found by Topping [33] in a target different from \( S^2 \). On the other hand, bubbling rates faster than (1.11) do exist in the 1-corrotational case, but they are not stable, see Raphaël and Schweyer [29]. Many other results on bubbling phenomena, and regularity for harmonic maps and the harmonic map flow are available in the literature, we refer the reader to the book the book by Lin and Wang [19].

In bubbling phenomena in this and related problems very little is known in nonradial situations. The method in [28,29], was successfully applied to very related blow-up phenomena in dispersive equations in symmetric classes. See for instance Rodnianski-Sterbenz [30] Merle-Raphaël-Rodnianski [22], Raphaël [26], Raphaël-Rodnianski [27]. Our results share a flavor with finite time multiple blow-up in
the subcritical semilinear heat equation, as in the results by Merle and Zaag [21].
Bubbling associated to the critical exponent has been recently studied in [9, 10].
Our approach is parabolic in nature. It is based on the construction of a good
approximation and then linearizing inner and outer problems. An appropriate
inverse for the inner equation is then found (which works well if the parameters
of the problems are suitably adjusted) which makes it possible the application of
fixed point arguments. The general approach, which we call inner-outer gluing,
has already been applied to various singular perturbation elliptic problems, see for
instance [11, 12]. A major difficulty we have to overcome is the coupled nonlocal
ODE satisfied by the scaling and rotation parameter. (See equation (2.23).)

2. Construction of a first approximation

2.1. Setting up the problem. As we have said, a central step in the proof of
Theorem 1 is based on the construction of a good approximation to an actual
solution of (1.1)-(1.3). In order to keep notation to a minimum, we shall do this
in the case \( k = 1 \) of a single bubbling point. We will later indicate the necessary
changes in the case of a general \( k \). Given a \( T > 0 \) and a point \( q \in \Omega \), we are looking
for a solution \( u(x, t) \) of the equation

\[
S(u) := -u_t + \Delta u + |\nabla u|^2 u = 0, \quad |u| = 1 \quad \text{in } \Omega \times (0, T)
\]  

(2.1)

which at main order looks like

\[
U(x, t) := U_{\lambda, \xi, \alpha}(x) = Q_\alpha \omega \left( \frac{x - \xi}{\lambda} \right)
\]

where \( \omega \) is the canonical least energy harmonic map (1.7). We want to find functions
\( \xi(t), \lambda(t) \) and \( \alpha(t) \) of class \( C^1[0, T] \) such that

\[
\xi(T) = q, \quad \lambda(T) = 0,
\]

in such a way that a solution \( u(x, t) \) which blows-up at time \( T \) and the point \( q \)
with a profile given at main order by \( U \) exists. We shall a priori make the following
assumptions on the coefficients:

\[
|\dot{\lambda}(t)| \leq \frac{C|\log T|}{\log^2(T-t)}, \quad |\dot{\xi}(t)| \leq C(T-t)^\sigma, \quad |\lambda\dot{\alpha}| \leq \frac{C|\log T|}{\log^2(T-t)},
\]

where \( T > 0 \) is a fixed number which we will reduce as many times as needed in
what follows, and \( C \) is a constant independent of \( T \) which we shall fix later on.

More precisely we will find a suitable choice of these parameter functions and a
well-chosen small perturbation of \( U \) such that the solution \( u(x, t) \) of (2.1) with the
initial condition of this perturbation has the desired property.

For each fixed \( t \), \( U \) is a harmonic map:

\[
\Delta U + |\nabla U|^2 U = 0
\]

and hence \( S(U) = -U_t \). We want to find a solution \( u(x, t) \) of (2.1) such that
for suitable choices of \( \alpha, \lambda \) and \( \xi \) looks like a small perturbation of \( U, U + p \) and
\( |U + p| \equiv 1 \). It is convenient to parametrize the admissible perturbations \( p(x, t) \) in
terms of free small functions \( \varphi : \Omega \times [0, T) \rightarrow \mathbb{R}^3 \) in the form

\[
p(\varphi) := \Pi_U \varphi + a(\Pi_U \varphi)U,
\]
where
\[ \Pi_{U \perp} \varphi := \varphi - (\varphi \cdot U) U, \quad a(\Pi_{U \perp} \varphi) := \sqrt{1 + (\varphi \cdot U)^2} - |\varphi|^2 - 1 = \sqrt{1 - |\Pi_{U \perp} \varphi|^2} - 1, \]
so that
\[ |U + p(\varphi)|^2 = 1 \]
indeed holds. Thus, we want to find a small function \( \varphi \) with values in \( \mathbb{R}^3 \) such that
\[ u = U + \Pi_{U \perp} \varphi + a(\Pi_{U \perp} \varphi) U, \]
solves (2.1). We compute
\[ S(U + \Pi_{U \perp} \varphi + aU) = -U_t - \partial_\alpha \Pi_{U \perp} \varphi + L_U(\Pi_{U \perp} \varphi) + N_U(\varphi) + b(\varphi) U \]
where
\[ L_U(\Pi_{U \perp} \varphi) = \Delta \Pi_{U \perp} \varphi + |\nabla U|^2 \Pi_{U \perp} \varphi + 2(\nabla U \cdot \nabla \Pi_{U \perp} \varphi) U \]
\[ N_U(\Pi_{U \perp} \varphi) = \left[ 2\nabla(au) \cdot \nabla(U + \Pi_{U \perp} \varphi) + 2\nabla U \cdot \nabla \Pi_{U \perp} \varphi + |\nabla \Pi_{U \perp} \varphi|^2 \right. \]
\[ + |\nabla(aU)|^2 \left. \right] \Pi_{U \perp} \varphi - aU_t + 2\nabla a\nabla U, \]
\[ b(\Pi_{U \perp} \varphi) = \Delta a - a_t + (|\nabla(U + \Pi_{U \perp} \varphi + aU)|^2 - |\nabla U|^2)(1 + a) - 2\nabla U \cdot \nabla \Pi \varphi \]
A useful observation we make is that under this condition if \( \varphi \) just solves an equation of the form
\[ -U_t - \partial_\alpha \Pi_{U \perp} \varphi + L_U(\Pi_{U \perp} \varphi) + N_U(\Pi_{U \perp} \varphi) + b(x, t) U = 0 \quad (2.2) \]
for some scalar function \( \tilde{b}(x, t) \) and \( |\varphi| \leq \frac{1}{2} \), then \( u = U + \Pi_{U \perp} \varphi + a(\varphi) U \) solves equation (2.1), namely \( S(u) = 0 \). Indeed, \( u \) satisfies
\[ S(u) + b_0 U = 0 \]
where \( b_0 = \tilde{b} - b(\Pi_{U \perp} \varphi) \). We claim that necessarily \( b_0 \equiv 0 \). Indeed for this \( u \) we have that
\[ -b_0(x, t) U \cdot u = S(u) \cdot u = -\frac{1}{2} \frac{d}{dt} |u|^2 + \frac{1}{2} \Delta(|u|^2) = 0. \]
On the other hand, by definition \( U \cdot u = 1 + a(\Pi_{U \perp} \varphi) \) and since \( |\varphi| \leq \frac{1}{2} \), we check that \( |a(\Pi_{U \perp} \varphi)| \leq \frac{1}{4} \). Thus \( U \cdot u > 0 \) and hence \( b_0 \equiv 0 \).
We begin by computing the error of approximation by just inserting \( U \) into equation (2.1). Since \( U(\cdot, t) \) defines a harmonic map, we have that
\[ S(U) = -U_t = -\lambda \partial_\lambda U_{\lambda, \xi, \alpha} - \lambda \partial_\alpha U_{\lambda, \xi, \alpha} - \partial_\xi U_{\lambda, \xi, \alpha} \cdot \hat{\xi} \]
\[ = \frac{\lambda}{\lambda} Q_\alpha \nabla \omega(y) \cdot y - \lambda \partial_\alpha Q_\alpha \omega(y) + Q_\alpha \nabla \omega(y) \cdot \hat{\xi}, \quad y = \frac{x - \xi}{\lambda}. \]
Each of the terms above is orthogonal to \( U \) since \( |U_{\lambda, \xi, \alpha}|^2 = 1 \), in other words they are tangent vectors to \( S^2 \) at the point \( \bar{U}(x, t) \). If we represent \( \omega(y) \) in polar coordinates,
\[ \omega(y) = \begin{pmatrix} e^{i\theta} \sin w(\rho) \\ \cos w(\rho) \end{pmatrix}, \quad w(\rho) = \pi - 2 \arctan(\rho), \quad y = \rho e^{i\theta}, \]
we notice that
\[ w_{\rho} = -\frac{2}{1 + \rho^2}, \quad \sin w = -\rho w_{\rho} = \frac{2\rho}{1 + \rho^2}, \quad \cos w = \frac{\rho^2 - 1}{1 + \rho^2}. \]
The vectors

\[ E_1(y) = \begin{pmatrix} e^{i\theta} \cos w(\rho) \\ -\sin w(\rho) \end{pmatrix}, \quad E_2(y) = \begin{pmatrix} i e^{i\theta} \\ 0 \end{pmatrix}, \]

constitute an orthonormal basis of the tangent space to \( S^2 \) at the point \( \omega(y) \). The error becomes in this language

\[ S(U) = -U_t = \mathcal{E}_0 + \mathcal{E}_1, \tag{2.3} \]

where

\[ \mathcal{E}_0(x,t) = Q_\alpha \left[ \frac{\dot{\lambda}}{\lambda} \rho w(\rho) E_1(y) + \dot{\alpha} \rho w(\rho) E_2(y) \right] \tag{2.4} \]

\[ \mathcal{E}_1(x,t) = \frac{\dot{\xi}_1}{\lambda} w(\rho) Q_\alpha \left[ \cos \theta E_1(y) + \sin \theta E_2(y) \right] \]
\[ + \frac{\dot{\xi}_2}{\lambda} w(\rho) Q_\alpha \left[ \sin \theta E_1(y) - \cos \theta E_2(y) \right], \tag{2.5} \]

with

\[ y = x - \xi = \rho e^{i\theta}, \quad \rho = |y|. \]

To solve Equation (2.2), we shall find a first correction \( \Phi^* \) and recast the problem into the form

\[ 0 = -U_t - \partial_t \Pi_{U_\perp} \Phi^* + L_U[\Pi_{U_\perp} \Phi^*] \]
\[ - \partial_t \Pi_{U_\perp} \Phi + L_U[\Pi_{U_\perp} \Phi + N_U[\Pi_{U_\perp} \Phi^* + \Phi]] + \tilde{b}(x,t)U. \tag{2.6} \]

The correction \( \Phi^* \) will be such that the new error given by the first row in the above equation will have the term \( \mathcal{E}_0 \) in (2.4) canceled at main order away from the point \( \xi \). In order to compute a useful expansion of the new error and for later purposes we need some expression for the term \( L_U[\Pi_{U_\perp} \Phi] \) at a general function \( \Phi(x) \).

### 2.2. Some computations for \( L_U \)

We consider a generic \( C^1 \) function \( \Phi : \Omega \to \mathbb{C} \times \mathbb{R} \), that we express in the form

\[ \Phi(x) = \begin{pmatrix} \varphi_1(x) + i \varphi_2(x) \\ \varphi_3(x) \end{pmatrix}. \]

We also denote

\[ \varphi = \varphi_1 + i \varphi_2, \quad \bar{\varphi} = \varphi_1 - i \varphi_2 \]

and define the operators

\[ \text{div} \varphi = \partial_{x_1} \varphi_1 + \partial_{x_2} \varphi_2, \quad \text{curl} \varphi = \partial_{x_1} \varphi_2 - \partial_{x_2} \varphi_1. \]

**Lemma 2.1.** We have that

\[ L_U[\Pi_{U_\perp} \Phi] = \Pi_{U_\perp} \Delta \Phi + \tilde{L}_U[\Phi] \tag{2.7} \]

where

\[ \tilde{L}_U[\Phi] := |\nabla U|^2 \Pi_{U_\perp} \Phi - 2 \nabla(\Phi \cdot U) \nabla U, \tag{2.8} \]

with

\[ \nabla(\Phi \cdot U) \nabla U = \sum_{i=1}^2 \partial_{x_i} (\Phi \cdot U) \partial_{x_i} U. \]
Proof. We have that
\[ \Delta(\Phi \cdot U) = (\Delta \Phi) \cdot U + 2\nabla \Phi \cdot \nabla U - (\Phi \cdot U) |\nabla U|^2 \]
so that
\[ \Delta \Pi_{U \perp} \Phi = \Pi_{U \perp} \Delta \Phi - 2(\nabla \Phi \cdot \nabla U)U + 2\Phi \cdot U |\nabla U|^2 U + 2\nabla(\Phi \cdot U) \nabla U \]
Now,
\[ \nabla [(\Phi \cdot U) U] \cdot \nabla U = (\Phi \cdot U) |\nabla U|^2 \]
hence
\[ \nabla \Pi_{U \perp} \Phi \cdot \nabla U = \nabla \Phi \cdot \nabla U - (\Phi \cdot U) |\nabla U|^2. \]
It follows that
\[ L_U[\Pi_{U \perp} \Phi] = \Pi_{U \perp} \Delta \Phi + |\nabla U|^2 \Pi_{U \perp} \Phi - 2\nabla(\Phi \cdot U) \nabla U \]
as desired. \[\square\]

The following lemma provides a useful formula for computing the operator \( \hat{L}_U \) in (2.8) in polar coordinates.

**Lemma 2.2.** If we write in polar coordinates
\[ \Phi(x) = \Phi(r, \theta), \quad x = \xi + re^{i \theta}, \quad \rho = \frac{r}{\lambda} \]
then we have
\[ \hat{L}_U[\Phi] = -\frac{2}{\lambda} w_\rho(\rho) \left[ (\Phi_r \cdot U) Q_1 E_1 - \frac{1}{r} (\Phi_\theta \cdot U) Q_2 E_2 \right]. \] (2.9)

Proof. We have that
\[ \nabla(\Phi \cdot U) \nabla U = \partial_r (\Phi \cdot U) \partial_r U + \frac{1}{r^2} \partial_\theta (\Phi \cdot U) \partial_\theta U \]
\[ = (\Phi_r \cdot U) \partial_r U + \frac{1}{r^2} (\Phi_\theta \cdot U) \partial_\theta U \]
\[ + (\Phi \cdot \partial_r U) \partial_r U + \frac{1}{r^2} (\Phi \cdot \partial_\theta U) \partial_\theta U. \]
We see that
\[ \partial_r U = \frac{1}{\lambda} w_\rho(\rho) E_1, \quad \frac{1}{r} \partial_\theta U = \frac{1}{\lambda} \frac{\sin w(\rho)}{\rho} E_2 = -\frac{1}{\lambda} w_\rho(\rho) E_2. \]
Hence
\[ 2\nabla(\Phi \cdot U) \nabla U = \frac{2}{\lambda} w_\rho(\rho) \left[ (\Phi_r \cdot U) E_1 - \frac{1}{r} (\Phi_\theta \cdot U) E_2 \right] \]
\[ + \frac{2}{\lambda} w_\rho(\rho)^2 \left[ (\Phi \cdot E_1) E_1 + (\Phi \cdot E_2) E_2 \right]. \]
On the other hand, \(|\nabla U|^2 = 2w_\rho^2\) and
\[ \Pi_{U \perp} \Phi = (\Phi \cdot E_1) Q_1 E_1 + (\Phi \cdot E_2) Q_2 E_2 \]
hence
\[ \hat{L}_U[\Phi] = |\nabla U|^2 \Pi_{U \perp} \Phi - 2\nabla(\Phi \cdot U) \nabla U = \frac{2}{\lambda} w_\rho(\rho) \left[ (\Phi_r \cdot U) E_1 - \frac{1}{r} (\Phi_\theta \cdot U) E_2 \right] \]
and the proof is concluded. \[\square\]

Next we single out two consequences of Formula (2.9) which will be crucial for the error computations.
Corollary 2.1. We have the validity of the formula
\[
\hat{L}_U[\Phi] = \hat{L}_U[\Phi]_0 + \hat{L}_U[\Phi]_1 + \hat{L}_U[\Phi]_2,
\] (2.10)
where
\[
\begin{align*}
\hat{L}_U[\Phi]_0 &= \lambda^{-1} \rho w_p^2 \left[ \operatorname{div}(e^{-i\alpha} \varphi) Q_\alpha E_1 + \operatorname{curl}(e^{-i\alpha} \varphi) Q_\alpha E_2 \right], \\
\hat{L}_U[\Phi]_1 &= -2\lambda^{-1} w_p \cos \omega \left[ (\partial_{x_1} \varphi_3) \cos \theta + (\partial_{x_2} \varphi_1) \sin \theta \right] Q_\alpha E_1 \\
&\quad - 2\lambda^{-1} w_p \cos \omega \left[ (\partial_{x_1} \varphi_3) \sin \theta - (\partial_{x_2} \varphi_1) \cos \theta \right] Q_\alpha E_2, \\
\hat{L}_U[\Phi]_2 &=\lambda^{-1} \rho w_p^2 \left[ \operatorname{div}(e^{i\alpha} \tilde{\varphi}) e^{2\theta} - \operatorname{curl}(e^{i\alpha} \tilde{\varphi}) \sin 2\theta \right] Q_\alpha E_1 \\
&\quad +\lambda^{-1} \rho w_p^2 \left[ \operatorname{div}(e^{i\alpha} \tilde{\varphi}) e^{2\theta} + \operatorname{curl}(e^{i\alpha} \tilde{\varphi}) \cos 2\theta \right] Q_\alpha E_2.
\end{align*}
\]

Proof. Let us assume first $\alpha = 0$. We notice that
\[
\hat{\Phi}_r = \cos \theta \partial_{x_1} \Phi + \sin \theta \partial_{x_2} \Phi \\
\frac{1}{r} \hat{\Phi}_\theta = -\sin \theta \partial_{x_1} \Phi + \cos \theta \partial_{x_2} \Phi.
\]
Then
\[
\begin{align*}
\varphi_r \cdot U &= \sin w \left[ \partial_{x_1} \varphi_1 \cos^2 \theta + \partial_{x_2} \varphi_1 \cos \theta \sin \theta + \partial_{x_1} \varphi_2 \sin \theta \cos \theta + \partial_{x_2} \varphi_2 \sin \theta \sin \theta \right] \\
&\quad + \cos w \left[ \partial_{x_1} \varphi_3 \cos \theta + \partial_{x_2} \varphi_3 \sin \theta \right] \\
&= \frac{1}{2} \sin w \left[ (\partial_{x_1} \varphi_1 + \partial_{x_2} \varphi_2) + \cos 2\theta (\partial_{x_1} \varphi_1 - \partial_{x_2} \varphi_2) + (\partial_{x_2} \varphi_1 + \partial_{x_1} \varphi_2) \sin 2\theta \right] \\
&\quad + \cos w \left[ \partial_{x_1} \varphi_3 \cos \theta + \partial_{x_2} \varphi_3 \sin \theta \right],
\end{align*}
\]
while
\[
\begin{align*}
\frac{1}{r} \varphi_\theta \cdot U &= \sin w \left[ -\partial_{x_1} \varphi_1 \cos \theta \sin \theta + \partial_{x_2} \varphi_1 \cos^2 \theta - \partial_{x_1} \varphi_2 \sin^2 \theta + \partial_{x_2} \varphi_2 \cos \theta \sin \theta \right] \\
&\quad + \cos w \left[ -\partial_{x_1} \varphi_3 \sin \theta + \partial_{x_2} \varphi_3 \cos \theta \right] \\
&= \frac{1}{2} \sin w \left[ (\partial_{x_2} \varphi_1 - \partial_{x_1} \varphi_2) + \cos 2\theta (\partial_{x_2} \varphi_1 + \partial_{x_1} \varphi_2) + (\partial_{x_2} \varphi_2 - \partial_{x_1} \varphi_1) \sin 2\theta \right] \\
&\quad + \cos w \left[ -\partial_{x_1} \varphi_3 \sin \theta + \partial_{x_2} \varphi_3 \cos \theta \right].
\end{align*}
\]
Since $\sin w = -\rho w_p$, we obtain from Formula (2.9)
\[
\begin{align*}
\hat{L}_U[\Phi] &= \lambda^{-1} \rho w_p^2 \left[ \operatorname{div} \varphi E_1 + \operatorname{curl} \varphi E_2 \right] \\
&\quad + \lambda^{-1} \rho w_p^2 \left[ \operatorname{div} \tilde{\varphi} \cos 2\theta - \operatorname{curl} \tilde{\varphi} \sin 2\theta \right] E_1 \\
&\quad + \lambda^{-1} \rho w_p^2 \left[ \operatorname{div} \tilde{\varphi} \sin 2\theta + \operatorname{curl} \tilde{\varphi} \cos 2\theta \right] E_2 \\
&\quad - 2\lambda^{-1} w_p \cos \omega \left[ \partial_{x_1} \varphi_3 \cos \theta + \partial_{x_2} \varphi_3 \sin \theta \right] E_1 \\
&\quad - 2\lambda^{-1} w_p \cos \omega \left[ \partial_{x_1} \varphi_3 \sin \theta - \partial_{x_2} \varphi_3 \cos \theta \right] E_2.
\end{align*}
\]
For the case of a general $\alpha$, we observe that we have the identity
\[
\hat{L}_U[\Phi] = Q_\alpha \hat{L}_{Q^{-\alpha} U}[Q^{-\alpha} \Phi]
\]
hence we obtain the desired result by just substituting in the above formula $\varphi$ by $e^{-i\alpha} \varphi$. The proof is complete. \qed
Corollary 2.2. Assume that \( \Phi(x) \) has the form (in polar coordinates)

\[
\Phi(x) = \left( \phi(r)e^{i\theta} \right), \quad x = \xi + re^{i\theta}, \quad \rho = \frac{r}{\lambda}
\]

where \( \phi(r) \) is complex valued. Then

\[
\hat{L}_U[\Phi] = \frac{2}{\lambda} w_\rho(\rho)^2 \left[ \Re (e^{-i\alpha}\phi(r))Q_\alpha E_1 + \frac{1}{r} \Re (e^{-i\alpha}\phi(r))Q_\alpha E_2 \right].
\]

(2.11)

Proof. We have

\[
\Phi_r \cdot U = \begin{bmatrix} \phi_r e^{i\theta} & 0 \\ \theta \end{bmatrix} \cdot \begin{bmatrix} e^{i(\theta+\alpha)} \sin w \\ \cos w \end{bmatrix} = \Re (\phi_r e^{-i\alpha}) \sin w
\]

\[
\frac{1}{r} \Phi_\theta \cdot U = \frac{1}{r} \begin{bmatrix} i\phi e^{i\theta} \\ 0 \end{bmatrix} \cdot \begin{bmatrix} e^{i(\theta+\alpha)} \sin w \\ \cos w \end{bmatrix} = \frac{1}{r} \Re (i\phi e^{-i\alpha}) \sin w.
\]

Since \( \sin w = -\rho w_\rho \), formula (2.9) then yields the validity of (2.11). \( \square \)

We notice that Lemma 2.1 has a consequence that the operator \( L_U \) transforms functions orthogonal to \( U \) into functions orthogonal to \( U \). It is then natural to seek for a formula to compute the coordinates of \( L_U[\Phi] \) along \( Q_\alpha E_1 \) and \( Q_\alpha E_2 \) for a function of the form

\[
\Phi = \varphi_1 Q_\alpha E_1 + \varphi_2 Q_\alpha E_2,
\]

in terms of differential operators in \( \varphi_1 \) and \( \varphi_2 \). The following result provides such a computation in polar coordinates.

Lemma 2.3. If \( \Phi(x) \) has the form

\[
\Phi(x) = \varphi_1(\rho, \theta)Q_\alpha E_1 + \varphi_2(\rho, \theta)Q_\alpha E_2, \quad x = \xi + \lambda \rho e^{i\theta}
\]

then

\[
L_U[\Phi] = \lambda^{-2} \left( \partial_\rho^2 \varphi_1 + \frac{\partial_\rho \varphi_1}{\rho} + \frac{\partial_\theta^2 \varphi_1}{\rho^2} + (2w_\rho - \frac{1}{\rho^2}) \varphi_1 - \frac{2}{\rho^2} \partial_\theta \varphi_2 \cos w \right) Q_\alpha E_1
\]

\[
+ \lambda^{-2} \left( \partial_\rho^2 \varphi_2 + \frac{\partial_\rho \varphi_2}{\rho} + \frac{\partial_\theta^2 \varphi_2}{\rho^2} + (2w_\rho - \frac{1}{\rho^2}) \varphi_2 + \frac{2}{\rho^2} \partial_\theta \varphi_1 \cos w \right) Q_\alpha E_2.
\]

Proof. Let us assume that

\[
\Phi(\rho, \theta) = \varphi_1(\rho, \theta)Q_\alpha E_1 + \varphi_2(\rho, \theta)Q_\alpha E_2.
\]

We notice that

\[
\Delta_x \Phi = \lambda^{-2} \left( \partial_\rho^2 \Phi + \frac{1}{\rho} \partial_\rho \Phi + \frac{1}{\rho^2} \partial_\theta^2 \Phi \right).
\]

Since \( \Phi \cdot U = 0 \) we get

\[
L_U[\Phi] = \Pi_{U \perp} \Delta_x \Phi + |\nabla U|^2 \Phi.
\]

Then

\[
\Delta_x (\varphi_1 Q_\alpha E_1) = (\Delta_x \varphi_1) Q_\alpha E_1 + 2\lambda^{-2} \partial_\rho \varphi_1 \partial_\rho Q_\alpha E_1 + \varphi_1 Q_\alpha \Delta_x E_1
\]

We have that

\[
Q_\alpha E_{1 \rho} = -U w_\rho,
\]

\[
Q_\alpha E_{1 \rho \rho} = -w_{\rho \rho} U - Q_\alpha E_1 w_\rho^2,
\]

\[
Q_\alpha E_{1 \theta \theta} = -\cos w (\sin w U + \cos w Q_\alpha E_1).
\]
Thus
\[ \lambda^2 \Delta_x (Q_\alpha E_1) = -Q_\alpha E_1 \left( \frac{w^2}{\rho^2} + \frac{\cos^2 w}{\rho^2} \right) + U \left( \frac{w_{\rho \rho}}{\rho} + \frac{w_\rho}{\rho} + \frac{\sin w \cos w}{\rho^2} \right). \]

By definition of \( w(\rho) \) we have
\[ w_{\rho \rho} + \frac{w_\rho}{\rho} - \frac{\sin w \cos w}{\rho^2} = 0. \]

Hence
\[ \lambda^2 \Delta_x (Q_\alpha E_1) = -Q_\alpha E_1 \left( \frac{w^2}{\rho^2} + \frac{\cos^2 w}{\rho^2} \right) = - \frac{1}{\rho^2} Q_\alpha E_1 - 2 \frac{\sin w \cos w}{\rho^2} U. \]

Thus we have
\[ \lambda^2 \Delta_x (\phi_1 Q_\alpha E_1) = \lambda^2 (\Delta_x \phi_1) Q_\alpha E_1 - 2 \phi_1 w_{\rho} U + \frac{2}{\rho^2} \phi_1 \cos w Q_\alpha E_2 - \frac{\phi_1}{\rho^2} E_1 - 2 \frac{\sin w \cos w}{\rho^2} U. \]

Using this and \((2.2)\) we find after a direct computation
\[ L U [\phi_1 Q_\alpha E_1] = \left( \Delta_x \phi_1 + (2 w^2 - \frac{1}{\rho^2}) \phi_1 \right) Q_\alpha E_1 + \frac{2}{\rho^2} \phi_1 \cos w Q_\alpha E_2. \]

On the other hand, we find similarly
\[ \lambda^2 \Delta_x (\phi_2 Q_\alpha E_2) = \lambda^2 (\Delta_x \phi_2) Q_\alpha E_2 - \frac{2}{\rho^2} \phi_2 (\sin w U + \cos w Q_\alpha E_1) \]

and hence
\[ L U [\phi_2 E_2] = \left( \Delta_x \phi_2 + \lambda^{-2} (2 w^2 - \frac{1}{\rho^2}) \phi_2 \right) Q_\alpha E_2 - \lambda^{-2} \frac{2}{\rho^2} \phi_2 \cos w Q_\alpha E_1. \]

The proof is concluded. \( \square \)

2.3. Defining \( \Phi^* \). Equation \((2.6)\) can be approximated by the linear problem
\[ -\partial_t \Pi_{U, \perp} \Phi + L U [\Pi_{U, \perp} \Phi] + \mathcal{E}^* + b(x,t)U = 0 \quad (2.12) \]

where
\[ \mathcal{E}^* = -\partial_t \Pi_{U, \perp} \Phi^* + L U [\Pi_{U, \perp} \Phi^*] - U_t. \quad (2.13) \]

Our purpose is to build a function \( \Phi^* \) which reduces the error \emph{far away} from the blow-up point \( q_i \), in the sense that \( \mathcal{E}^* \) above is smaller than the largest term \( \mathcal{E}_0 \) of the initial error \(-U_t\), given by \((2.4)\),
\[ \mathcal{E}_0(x,t) = -\frac{2r}{r^2 + \lambda^2} \left[ \lambda Q_\alpha E_1 + \lambda \Phi Q_\alpha E_2 \right], \quad x = \xi + r e^{i\theta}. \]

The modified error \( \mathcal{E}^* \) can also be written as
\[ \mathcal{E}^* = \Pi_{U, \perp} [-\partial_t \Phi^* + \Delta \Phi^*] - U_t + \tilde{L}_U [\Phi^*] - (U \cdot \Phi^*) U_t \]

Since \( \tilde{L}_U \) has a “fast decay in \( \rho^* \)” according to the expressions in Corollaries 2.1 and 2.2, to achieve the required error reduction we consider a function \( \Phi^*(x,t) \) that satisfies
\[ \mathcal{E}_0 - \partial_t \Phi^* + \Delta \Phi^* \approx 0. \quad (2.14) \]

We assume that \( \Phi^* \) decomposes into the form
\[ \Phi^* := \Phi^0[\lambda, \alpha, \xi] + Z^*(x,t) \]
where
\[
Z^*(x, t) = \begin{bmatrix} z^*(x, t) \\ \bar{z}_1^*(x, t) \end{bmatrix}, \quad z^*(x, t) = z_1^*(x, t) + i\bar{z}_2^*(x, t)
\]
is a solution of the heat equation
\[
-\partial_t Z^* + \Delta Z^* = 0 \quad \text{in } \Omega \times (0, \infty)
\]
independent of the parameter functions, on whose initial and boundary we will later make specific assumptions. \(\Phi_0^0\) is an explicit function that satisfies (2.14) and defines an operator on \((\lambda, \alpha, \xi)\).

To define \(\Phi_0^0\), we first observe that away from the point \(\xi\) we have the validity of the approximation
\[
E_0 \approx -\frac{2r}{r^2 + \lambda^2} \left[ (\dot{\lambda} + i\lambda\dot{\alpha})e^{i(\theta + \alpha)} \right].
\]
Thus
\[
\Phi_0^0(x, t) = \begin{bmatrix} \phi_0^0(x, t) \\ 0 \end{bmatrix}
\]
will satisfy (2.14) if \(\phi_0^0\) is an approximate solution of
\[
-\phi_t^0 + \Delta_x \phi^0 - \frac{2r}{r^2 + \lambda^2}(\dot{\lambda} + i\lambda\dot{\alpha})e^{i(\alpha + \theta)} = 0 \quad \text{in } \mathbb{R}^2 \times (0, T).
\]
We let
\[
\phi_0^0(x, t) := re^{i\theta} \psi^0(z(r), t), \quad z(r) = \sqrt{r^2 + \lambda^2}
\]
where \(\psi(z, t)\) satisfies the equation
\[
\psi_t = \psi_{zzz} + \frac{3\psi_z}{z} + \frac{p(t)}{z^2}, \quad p(t) = -2(\dot{\lambda} + i\lambda\dot{\alpha})e^{i\alpha},
\]
which corresponds to the radially symmetric form of an inhomogeneous heat equation in \(\mathbb{R}^4\). Duhamel’s formula allows us quickly to get to the following expression for a weak solution to this problem as
\[
\psi(z, t) = \int_{-T}^t p(s) k(z, t - s) \, ds, \quad k(z, t) = \frac{1 - e^{-\frac{z^2}{4}}}{z^2},
\]
where \(p(t)\) is assumed to also be defined for negative values of \(t\). The reason for taking this formula integrated from a negative time \(-T\) rather than from 0 is rather subtle and will be made clear later on. In fact it will be convenient to consider \(p(t)\) defined as
\[
p(t) = \begin{cases} 
-2(\dot{\lambda} + i\lambda\dot{\alpha})e^{i\alpha} & \text{if } t \geq 0, \\
-2\dot{\lambda}(0) & \text{if } t < 0.
\end{cases}
\]
We need the parameter functions \(\lambda(t)\) and \(\alpha(t)\) to be defined for \(t \in [-T, T]\). The precise choice for these functions will be made later on. It will be convenient to use the notation
\[
p(t) = \lambda(t)e^{i\alpha(t)},
\]
which is then considered to be defined in \([-T, T]\).

The function \(\Phi_0^0\) defines an operator
\[
\Phi_0^0[\alpha, \lambda, \xi] = \begin{pmatrix} \varphi^0(r, t)e^{i\theta} \\ 0 \end{pmatrix}
\]
where
\[ \varphi^0(r,t) = \int_{-T}^{t} p(s) r k(z(r), t-s) \, ds, \quad z(r) = \sqrt{r^2 + \lambda^2}, \]
so that we find for \( t > 0, \)
\[ -\Phi_t^0 + \Delta_x \Phi^0 = \tilde{R}_0 + \tilde{R}_1 \quad \tilde{R}_0 = \begin{pmatrix} \mathcal{R}_0 \\ 0 \end{pmatrix}, \quad \tilde{R}_1 = \begin{pmatrix} \mathcal{R}_1 \\ 0 \end{pmatrix} \]
where
\[
\mathcal{R}_0 := -re^{i\theta} \frac{p(t)}{z^2} + re^{i\theta} \frac{\lambda^2}{z^4} \int_{-T}^{t} p(s) (zk_z - z^2 k_{zz})(z(r), t-s) \, ds
\]
and
\[
\mathcal{R}_1 := -\xi(t) \int_{-T}^{t} p(s) k(z(r), t-s) \, ds
+ \frac{r}{z^2} e^{i\theta}(\lambda \dot{\lambda}(t) + \text{Re}(re^{i\theta} \xi(t))) \int_{-T}^{t} p(s) zk_z(z(r), t-s) \, ds.
\]

2.4. Estimating the error \( \mathcal{E}^* \). In accordance with formula (2.13) we compute
\[
\mathcal{E}^* = \mathcal{E}_0^* + \mathcal{E}_1^* + \mathcal{E}_2^* + \mathcal{E}_3^*
\]
where, with the notation in formula (2.10),
\[
\begin{align*}
\mathcal{E}_0^* &= \Pi_{U^\perp} \mathcal{R}_0 + \tilde{L}_U[\Phi^0] + \tilde{L}_U[Z^*]_0 + \mathcal{E}_0 \\
\mathcal{E}_1^* &= \tilde{L}_U[Z^*]_1 + \mathcal{E}_1 \\
\mathcal{E}_2^* &= \tilde{L}_U[Z^*]_2 \\
\mathcal{E}_3^* &= \Pi_{U^\perp} \mathcal{R}_1 + (Z^* + \Phi^0) \cdot U. \end{align*}
\]
We observe that actually all terms in \( \tilde{R}_1 \) are smaller than those in \( \mathcal{R}_0 \). To compute \( \Pi_{U^\perp} \mathcal{R}_0 \) we use the following general fact: for a complex valued function \( f(r) \) we have
\[
\Pi_{U^\perp} \begin{pmatrix} f(r)e^{i\theta} \\ 0 \end{pmatrix} = \cos w(\rho) \text{Re}(f(r)e^{-i\alpha})Q_\alpha E_1 + \text{Im}(f(r)e^{-i\alpha})Q_\alpha E_2.
\]
Using this formula, the facts
\[
\frac{\lambda^2r}{z^4} = \frac{1}{4\lambda} \rho w^2, \quad \frac{r}{z^2}(1 - \cos w) = \frac{1}{2\lambda} \rho w^2
\]
and Corollaries 2.1, 2.2 we find
\[
\begin{align*}
\mathcal{E}_0^* &= \frac{1}{\lambda} \rho w^2 \left[ \lambda + \int_{-T}^{t} \text{Re}(p(s)e^{-i\alpha(t)})rk_z(z, t-s) \, ds \right] Q_\alpha E_1 \\
&\quad + \frac{1}{4\lambda} \rho w^2 \cos w \left[ \int_{-T}^{t} \text{Re}(p(s)e^{-i\alpha(t)}) (zk_z - z^2 k_{zz})(z, t-s) \, ds \right] Q_\alpha E_1 \\
&\quad + \frac{1}{4\lambda} \rho w^2 \left[ \int_{-T}^{t} \text{Im}(p(s)e^{-i\alpha(t)}) (zk_z - z^2 k_{zz})(z, t-s) \, ds \right] Q_\alpha E_2 \\
&\quad + \frac{1}{\lambda} \rho w^2 \left[ \text{div}(z^* e^{-i\alpha(t)}) + \int_{-T}^{t} \text{Re}(p(s)e^{-i\alpha(t)})k(z, t-s) \, ds \right] Q_\alpha E_1 \\
&\quad + \frac{1}{\lambda} \rho w^2 \left[ \text{curl}(z^* e^{-i\alpha(t)}) + \int_{-T}^{t} \text{Im}(p(s)e^{-i\alpha(t)})k(z, t-s) \, ds \right] Q_\alpha E_2 
\end{align*}
\]
\[ E_1^* = \frac{1}{\lambda} w_\rho \left[ \text{Re} \left( (\xi_1 - i\xi_2)e^{i\theta} \right) Q_\alpha E_1 + \text{Im} \left( (\xi_1 - i\xi_2)e^{i\theta} \right) Q_\alpha E_2 \right] \] (2.17)

\[ + \frac{2}{\lambda} w_\rho \cos \omega \text{Re} \left( (\partial_{x_1} z_3^* - i\partial_{x_2} z_3^*)e^{i\theta} \right) Q_\alpha E_1 \]

\[- \frac{2}{\lambda} w_\rho \cos \omega \text{Im} \left( (\partial_{x_1} z_3^* - i\partial_{x_2} z_3^*)e^{i\theta} \right) Q_\alpha E_2 \]

\[ E_2^* = \frac{1}{\lambda} \rho w_\rho^2 \text{Re} \left( (\text{div}(e^{i\alpha} z^*) + i \text{curl}(e^{i\alpha} z^*)e^{2i\theta}) \right) Q_\alpha E_1 \]

\[ + \frac{1}{\lambda} \rho w_\rho^2 \text{Im} \left( (\text{div}(e^{i\alpha} z^*) + i \text{curl}(e^{i\alpha} z^*)e^{2i\theta}) \right) Q_\alpha E_2 \]

\[ E_3^* = \Pi_{U^2} [\tilde{R}_1] - (\Phi^* \cdot U) U_t. \]

We remark that in these expressions, \( E_1^* \) and \( \text{div} z^* \), \( \text{curl} z^* \) are evaluated at \((x,t)\) with \( x \in \Omega \), and we are using \( y = \frac{x - \xi(t)}{\rho(t)} \) and polar coordinates for \( y = pe^{i\theta} \).

2.5. **Improving the inner error: the choice at main order of \( \lambda, \alpha, \xi \).** We come back now to our original linearized problem (2.12) which we can rewrite as that of finding \( \varphi \) such that

\[ E(\varphi) := -\varphi_t + L_U[\varphi] + E^* + b(x,t)U = 0, \quad \varphi \cdot U = 0. \] (2.18)

Examining the terms in \( E^* \) we see that the introduction of the first correction \( \Phi^* \) was precisely that of reducing the outer size of the initial error \( U_t \), namely away from the point \( x = \xi \). What we discuss next is how to adjust the parameter functions in such a way that a correction \( \varphi \) can be found so that \( E(\varphi) \) is globally smaller than the first error \( U_t \). As we will see, such a choice of parameters is possible under suitable assumptions on the initial and boundary data of \( Z^*(x,t) \) which we state next. We let \( Z^*(x,t) \) solve

\[ Z_t^* = \Delta Z^* \quad \text{in } \Omega \times (0,\infty), \]

\[ Z^*(\cdot,t) = Z_0^* \quad \text{in } \partial \Omega \times (0,\infty), \]

\[ Z_1^*(\cdot,0) = Z_0^* + Z_1^* \quad \text{in } \Omega. \]

Here \( Z_0^*(x) \) is defined as follows. Let us consider a point \( q_0 \in \Omega \) and a smooth function

\[ \tilde{Z}_0(x) = \begin{bmatrix} \tilde{Z}_01(x) + i\tilde{Z}_02(x) \\ \tilde{Z}_03(x) \end{bmatrix} \begin{bmatrix} \tilde{Z}_0(x) \\ \tilde{Z}_03(x) \end{bmatrix} \]

that satisfies the following assumptions

\[ \tilde{Z}_0(q_0) = 0, \]

\[ \text{div } \tilde{Z}_0(q_0) < 0. \] (2.19)

In addition, we assume that the map \( D\tilde{Z}_0(q_0) \) is nonsingular. Then we choose

\[ Z_0^*(x) := \delta \tilde{Z}_0(x) \]

where \( \delta > 0 \) is a fixed small number, to be reduced if necessary. The reason for assumptions (2.19) will be made clear below. The main point is that they will imply that a “right choice” of the parameter functions can indeed be made.

If we write

\[ \Phi(x,t) = Q_\alpha \phi(y,t), \quad y = \frac{x - \xi}{\rho}, \quad \rho = |y|, \]
then Equation (2.18) becomes
\[ -\lambda^2 Q_{-\alpha} \partial_t \Phi + L_\omega[\phi] + \lambda^2 Q_{-\alpha} \mathcal{E}^* + b \phi = 0, \quad \phi \cdot U = 0. \]

An improvement of the approximation is obtained if we solve the elliptic equation
\[ L_\omega[\phi] + \lambda^2 Q_{-\alpha} \mathcal{E}^* = 0, \quad \phi \cdot \omega(y) = 0, \quad \lim_{|y| \to \infty} \phi(y, t) = 0 \quad \text{in } \mathbb{R}^2 \quad (2.20) \]
where \( \mathcal{E}^* \) is extended as zero outside of \( \Omega \). The decay condition is needed in order to essentially not modify the size of the full error far away. We consider necessary conditions for (2.20) to be solvable. Let us consider the functions
\[
Z_{01}(y) = \rho w_\rho(\rho) E_1(y) \\
Z_{02}(y) = \rho w_\rho(\rho) E_2(y) \\
Z_{11}(y) = w_\rho(\rho) [\cos \theta E_1(y) + \sin \theta E_2(y)] \\
Z_{12}(y) = w_\rho(\rho) [-\sin \theta E_1(y) + \cos \theta E_2(y)].
\]

These four decaying functions annihilate \( L_\omega \), namely \( L_\omega[Z_{lj}] = 0 \). Indeed, they correspond to generators of invariance under \( \lambda \)-dilations, \( \alpha \)-rotations and respective translations in \( y_1 \) an \( y_2 \) directions at \( \omega \), for the equation \( \Delta U + |\nabla U|^2 U = 0 \). Testing equation (2.20) and integrating by parts we obtain, thanks to the involved decays,
\[ \int_{\mathbb{R}^2} L_\omega[\phi] \cdot Z_{lj} \, dy = \int_{\mathbb{R}^2} L_\omega[Z_{lj}] \cdot \phi = 0, \]
and hence
\[ \int_{\mathbb{R}^2} Q_{-\alpha} \mathcal{E}^* \cdot Z_{lj} \, dy = 0 \quad \text{for all } \ l, j. \]

If we write
\[ \mathcal{E}^*(x, t) = \mathcal{E}^{*1} Q_\alpha E_1 + \mathcal{E}^{*2} Q_\alpha E_2 \]
and we consider these relations for \( i = 0 \), they read as
\[ 0 = \int_{\mathbb{R}^2} Q_{-\alpha} \mathcal{E}^* \cdot Z_{0j} \, dy = \int_0^{2\pi} d\theta \int_0^{\infty} \mathcal{E}^{*j}(\xi + \lambda \rho e^{i\theta}, t) \rho^2 w_\rho \, d\rho \]
Examining the components in the expansion (2.15) of \( \mathcal{E}^* \), we see that the contributions to the above integrals of \( \mathcal{E}^{*1} \) and \( \mathcal{E}^{*2} \) are of smaller order due to the \( \theta \)-integration. The main contribution arises from \( \mathcal{E}^{*0}_0 \). Using Formula (2.16) we have as an approximation of main order with error \( \mathcal{O}(1) \to 0 \) as \( t \to T \),
\[ \lambda \int_{\mathbb{R}^2} Q_{-\alpha} \mathcal{E}^* \cdot Z_{0j} \, dy = -2\pi \mathcal{A}_j[\lambda, \alpha, \xi] + \mathcal{O}(1), \quad j = 1, 2, \]
\[
\mathcal{A}_1[\lambda, \alpha, \xi] = 2 \text{Re}(ae^{-ia}) + \mathcal{B}_1[\lambda, \alpha] \\
\mathcal{A}_2[\lambda, \alpha, \xi] = 2 \text{Im}(ae^{-ia}) + \mathcal{B}_2[\lambda, \alpha]
\]
where
\[ a(t) = e^{-ia(t)}(\text{div } z^*(\xi(t), t) + i \text{curl } z^*(\xi(t), t)), \]
and the operators \( \mathcal{B}_1, \mathcal{B}_2 \) are defined as
\[
\mathcal{B}_1[\lambda, \alpha](t) = \int_{-T}^t \text{Re}(p(s)e^{-ia(t)}) \Gamma_1 \left( \frac{\lambda(t)^2}{t-s} \right) \frac{ds}{t-s} + 2 \lambda(t) \quad (2.21) \\
\mathcal{B}_2[\lambda, \alpha](t) = \int_{-T}^t \text{Im}(p(s)e^{-ia(t)}) \Gamma_2 \left( \frac{\lambda(t)^2}{t-s} \right) \frac{ds}{t-s} \quad (2.22)
\]
where $\Gamma_j(\tau)$, $j = 1, 2$ are the smooth functions defined as follows:

$$
\Gamma_1(\tau) = \int_0^\infty \rho^3 w_\rho^3 \left[ K(\zeta) + \zeta K_\zeta(\zeta) - \frac{\rho^2}{1 + \rho^2} \right]_{\zeta = \tau(1 + \rho^2)} d\rho
$$

$$
\Gamma_2(\tau) = \int_0^\infty \rho^3 w_\rho^3 \left[ K(\zeta) + \frac{1}{4} \zeta K_\zeta(\zeta) - \frac{1}{4} \zeta^2 K_\zeta^2(\zeta) \right]_{\zeta = \tau(1 + \rho^2)} d\rho
$$

where

$$
K(\zeta) = \frac{1 - e^{-\frac{\zeta}{2}}}{\zeta}.
$$

These functions satisfy

$$
\Gamma_j(0) = \frac{1}{2}, \quad \Gamma_j(\tau) = O(\tau^{-1}) \quad \text{as} \quad \tau \to +\infty.
$$

In the above computation we have used the facts

$$
\text{div}(z^* e^{-i\alpha}) + i \text{curl}(z^* e^{-i\alpha}) = e^{-i\alpha}(\text{div} z^* + i \text{curl} z^*)
$$

and

$$
\int_0^\infty \rho^3 w_\rho^3 d\rho = -2.
$$

We also recall that

$$
p(t) = \begin{cases} -2(\dot{\lambda}(t) + i\lambda\dot{\alpha}(t))e^{i\alpha(t)} & \text{if } t \geq 0, \\
-2\lambda(0) & \text{if } t < 0. \end{cases}
$$

It is convenient to rewrite, with the convention $\alpha = 0$ and $\dot{\lambda} \equiv \dot{\lambda}(0)$ in $(-T, 0)$,

$$
\mathcal{B}_1[\lambda, \alpha](t) = 2 \int_{-T}^t \dot{\lambda}(s) \Gamma_1 \left( \frac{\lambda(t)^2}{t - s} \right) ds + \theta_1(t)
$$

where

$$
\theta_1(t) = 2 \int_{-T}^t \left[ \dot{\lambda}(s) \frac{\sin(\alpha(t) - \alpha(s))}{t - s} + \dot{\lambda}(s) \frac{\cos(\alpha(t) - \alpha(s)) - 1}{t - s} \right] \Gamma_1 \left( \frac{\lambda(t)^2}{t - s} \right) ds \quad + 2\lambda,
$$

and

$$
\mathcal{B}_2[\lambda, \alpha](t) = 2 \int_{-T}^t \dot{\lambda}(s) \Gamma_2 \left( \frac{\lambda(t)^2}{t - s} \right) ds + \theta_2(t)
$$

with

$$
\theta_2(t) = 2 \int_{-T}^t \left[ \dot{\lambda}(s) \frac{\cos(\alpha(t) - \alpha(s)) - 1}{t - s} + \dot{\lambda}(s) \frac{\sin(\alpha(t) - \alpha(s))}{t - s} \right] \Gamma_2 \left( \frac{\lambda(t)^2}{t - s} \right) ds.
$$

We would like to find functions $\alpha(t)$ and $\lambda(t)$ that solve the equation $\mathcal{A}(\lambda, \alpha) = 0$. Doing so in exact form seems not possible. However, for our purposes it will suffice to do so in approximate form. We begin by identifying functions $\lambda(t), \alpha(t)$ which satisfy

$$
\mathcal{A}_j(\lambda, \alpha)(t) = o(1), \quad j = 1, 2,
$$

where $o(1)$ vanishes at $t = T$ and is uniformly small with $T$. We will be able to achieve this by means of the simple ansatz

$$
\dot{\lambda}(t) = -\frac{\kappa}{\log^2(T - t)} |\log T|, \quad \alpha(t) = \gamma,
$$

(2.24)
where the constants $\kappa > 0$ and $\gamma$ are at main order independent of $T$. Observe that such a $\kappa$ must indeed be positive because $\dot{\lambda}$ decreases to zero. We observe that for this choice of $\lambda$ and $\alpha$ we automatically have
\[
\mathcal{B}_2(\alpha, \lambda) \equiv 0, \quad \theta_1 \equiv 0.
\]
We claim that the following holds: for $\lambda$ and $\alpha$ of the form (2.24) we have that
\[
|\mathcal{B}_1(\lambda, \alpha)(t) - k| \lesssim \kappa \frac{\log(|\log T|)}{|\log T|} \quad (2.25)
\]
In fact, let us assume that $\lambda(t)$ and $\alpha(t)$ have the form (2.24). We have that
\[
\mathcal{B}_1(\lambda, \alpha) = 2 \int_{-T}^{t} \frac{\dot{\lambda}(s)}{t - s} \Gamma_1 \left( \frac{\lambda(t)^2}{t - s} \right) ds.
\]
Since
\[
\Gamma_1(\tau) = \begin{cases} \frac{1}{\tau} + O(\tau) & \text{for } 0 \leq \tau < 1, \\ O(\tau^{-1}) & \text{for } \tau \geq 1, \end{cases}
\]
then
\[
2 \int_{-T}^{t} \frac{\dot{\lambda}(s)}{t - s} \Gamma_1 \left( \frac{\lambda(t)^2}{t - s} \right) ds = \int_{-T}^{t} \frac{\dot{\lambda}(s)}{t - s} ds + \lambda^2 \int_{-T}^{t} \frac{\dot{\lambda}(s)}{(t - s)^2} ds + O(1) ds
\]
\[
+ \frac{1}{\lambda^2} \int_{-T}^{t} \dot{\lambda}(s) O(1) ds
\]
\[
= \int_{-T}^{t} \frac{\dot{\lambda}(s)}{t - s} ds + O(\dot{\lambda}(t)).
\]
Now,
\[
\int_{-T}^{t} \frac{\dot{\lambda}(s)}{t - s} ds = \int_{-T}^{t} \frac{\dot{\lambda}(s)}{T - s} ds + \int_{t}^{-T} \frac{\dot{\lambda}(s)}{T - s} ds + O(\dot{\lambda}(t))
\]
\[
= \int_{-T}^{t} \frac{\dot{\lambda}(s)}{T - s} ds + \dot{\lambda}(t) \left[ \log|\frac{1}{2}(T - t)| - 2 \log(\lambda(t)) \right] + O(\dot{\lambda}(t))
\]
\[
= \int_{-T}^{t} \frac{\dot{\lambda}(s)}{T - s} ds - \dot{\lambda}(t) \log(T - t) + O(\dot{\lambda}(t) \log |\log(T - t)|).
\]
Letting
\[
\psi(t) = \int_{-T}^{t} \frac{\dot{\lambda}(s)}{T - s} ds - \dot{\lambda}(t) \log(T - t)
\]
we see that,
\[
\log(T - t) \frac{d\psi}{dt}(t) = \frac{d}{dt} (\log^2(T - t) \dot{\lambda}(t)) = 0
\]
from the explicit form of $\dot{\lambda}(t)$. Hence $\psi(t) = \psi(T)$ for all $t$. As a conclusion we find
\[
\mathcal{B}_1[\lambda, \alpha](t) = \int_{-T}^{T} \frac{\dot{\lambda}(s)}{T - s} ds + O(\dot{\lambda} \log |\log(T - t)|),
\]
Finally, we explicitly compute
\[
\int_{-T}^{T} \frac{\dot{\lambda}(s)}{T - s} = -\kappa + O \left( \frac{\kappa}{|\log T|} \right)
\]
Combining the above estimates readily yields (2.25). We are ready to choose \( \gamma \) and \( \kappa \) that satisfy the approximate system (2.23). We have that

\[
\mathcal{A}_1(\lambda, \alpha) = \kappa(1 + o(1)) + \cos \gamma \text{ div } z^*(\xi, t) + \sin \gamma \text{ curl } z^*(\xi, t)
\]

\[
\mathcal{A}_2(\lambda, \alpha) = -\sin \gamma \text{ div } z^*(\xi, t) + \cos \gamma \text{ curl } z^*(\xi, t).
\]

The requirement that the above expression vanishes at \( t = T \) yields

\[
\kappa(1 + o(1)) + e^{-i\gamma}(\text{div } z^*_0(q, T) + i \text{ curl } z^*_0(q, T)) = 0.
\]

We assume that \( \text{div } z^*_0(q_0) < 0 \). The above system is solvable as a perturbation of the limiting equation as \( T \to 0 \),

\[
\kappa_0 = -(\text{div } z^*_0(q_0) + i \text{ curl } z^*_0(q_0))e^{-i\gamma_0}
\]

which is solvable with

\[
\kappa_0 = \sqrt{\text{div } z^*_0(q_0)^2 + \text{curl } z^*_0(q_0)^2}, \quad \gamma_0 = \arctan \left( \frac{\text{curl } z^*_0(q_0)}{\text{div } z^*_0(q_0)} \right).
\]

Therefore, our first approximation (valid as \( T \to 0 \)) of the functions \( \lambda(t) \) and \( \alpha(t) \) is given by

\[
\dot{\lambda}_0(t) := -\frac{\kappa_0|\log T|}{\log^2(T - t)}, \quad \alpha_0(t) := \gamma_0.
\] (2.26)

Next we will justify the choice at main order of \( \xi(t) \). To do so we argue in a similar way as before, now testing the error against the functions \( Z^*_1, j = 1, 2 \).

In this case we need

\[
\lambda \int_{\mathbb{R}^2} Q_{-\alpha} E_1^* \cdot Z^*_1 dy = 0, \quad j = 1, 2,
\]

namely orthogonality with respect to the generators of space translations. These relations are well approximated by

\[
\lambda \int_{\mathbb{R}^2} Q_{-\alpha} E_1^* \cdot Z^*_1 dy = 0, \quad j = 1, 2
\]

which, from the expression for \( E_1^* \) becomes approximately the ODE for \( \xi(t) \)

\[
\dot{\xi}(t) \int_0^\infty w^2 \rho d\rho + c \int_0^\infty \rho w^2 \cos w \, d\rho = 0, \quad c = \partial_{x_1} z^*_3(\xi, t) + i \partial_{x_1} z^*_3(\xi, t).
\]

But we directly check that

\[
\int_0^\infty \rho w^2 \cos w \, d\rho = 0.
\]

Hence \( \dot{\xi}(t) = 0 \). It is natural in our setting that \( \xi(0) = q_0 \). Then we take as a first order approximation the function

\[
\xi_0(t) := q_0.
\] (2.27)
2.6. The final ansatz. Let us fix $\lambda_0(t), \alpha_0(t), \xi_0(t)$ as in (2.26) and (2.27). We write
\[ \lambda(t) = \lambda_0(t) + \lambda_1(t), \quad \alpha(t) = \gamma_0 + \alpha_1(t), \quad \xi(t) = q + \xi_1(t). \]
Recalling (2.6), what we look for is a small solution $\varphi$ of
\[ \mathcal{E}^* - \partial_t \Pi_{U^+} \varphi + L_U(\Pi_{U^+} \varphi) + N_U(\Pi_{U^+} [\varphi^* + \varphi]) + b(x,t)U = 0 \tag{2.28} \]
where
\[ \varphi^* = \Phi[\lambda, \alpha, \xi] + Z^*, \]
and $\mathcal{E}^*$ is expanded in (2.15). We recall that
\[ N_U(\Pi_{U^+} \varphi) = [2\nabla (aU) \cdot \nabla (U + \Pi_{U^+} \varphi) + 2\nabla U \cdot \nabla \Pi_{U^+} \varphi + |\nabla \Pi_{U^+} \varphi|^2] \Pi_{U^+} \varphi, \]
\[ a(\Pi_{U^+} \varphi) = \sqrt{1 - |\Pi_{U^+} \varphi|^2} - 1. \]
In terms of the original problem (1.1)–(1.3), we want that
\[ u = U + \Pi_{U^+} [\varphi^* + \varphi] + a(\Pi_{U^+} [\varphi^* + \varphi])U \]
solves the boundary value problem
\[ u_t = \Delta u + |\nabla u|^2 u \quad \text{in } \Omega \times (0,T) \]
\[ u = u_{\partial \Omega} \quad \text{on } \partial \Omega \times (0,T) \]
\[ u(\cdot,0) = u_0 \quad \text{in } \Omega \]
Let us define
\[ u_{\partial \Omega}(x) := \frac{Z_0^*(x) + e_3}{|Z_0^*(x) + e_3|}, \quad e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \]
Thus, on $\Pi_{U^+} [\varphi]$ we impose the nonlinear boundary condition
\[ \Pi_{U^+} [\varphi] = u_{\partial \Omega} - U + a(\Pi_{U^+} [\varphi^*] + \Pi_{U^+} [\varphi])U - \Pi_{U^+} [\varphi^*] \quad \text{on } \partial \Omega, \]
a relation that can be uniquely solved for a small $\Pi_{U^+} [\varphi]$ in the form
\[ \Pi_{U^+} [\varphi] = (u_{\partial \Omega} - U) - \Pi_{U^+} [\varphi^*] + a(u_{\partial \Omega} - U, \Pi_{U^+} [\varphi^*]) \quad \text{on } \partial \Omega, \tag{2.29} \]
\[ |\tilde{a}(V,W)| \lesssim |V|^2 + |W|^2 \]
provided that $\varphi^*$ and $u_{\partial \Omega} - U$ are sufficiently small on $\partial \Omega$. In what follows, we will select specific values of the parameter functions and an initial condition $\varphi_0$ (and correspondingly $u_0$) so that the desired blow-up phenomenon takes place.

3. The strategy of proof: the outer-inner gluing scheme

Using formula (2.7), and possibly modifying $b(x,t)$, we can rewrite equation (2.28) as
\[ 0 = \mathcal{E}^* - \partial_t \varphi + \Delta \varphi + |\nabla U|^2 \varphi - 2\varphi \cdot U \nabla U \]
\[ - (\varphi \cdot U) U_t + N_U(\Pi_{U^+} \varphi) + b(x,t)U \tag{3.1} \]
Let us consider a smooth cut-off function $\eta_0(s)$ with $\eta_0(s) = 1$ for $s < 1$ and $= 0$ for $s > \frac{1}{2}$. We also consider an increasing function
\[ R(t) > 0, \quad R(t) \to \infty \quad \text{as } t \uparrow T. \]
We define
\[ \eta(x,t) := \eta_0 \left( \frac{x - \xi(t)}{R(t)\lambda(t)} \right), \quad \tau_\lambda(t) = \tau_0 + \int_0^t \frac{ds}{\lambda(s)^2}. \]
so that
\[ \tau_\lambda \sim \tau_0 + \frac{1}{\lambda_0} \frac{\log^2(T-t)}{\log T} \]
and hence
\[ \tau_\lambda \sim \tau_0 + \frac{1}{\lambda_0} \frac{\log^2 \lambda_0}{\log T} \]
and consider a function \( \varphi(x,t) \) of the special form
\[ \varphi(x,t) = \eta Q_\alpha \phi \left( \frac{x - \xi(t)}{\lambda(t)}, \tau_\lambda(t) \right) + \psi(x,t), \quad (3.2) \]
for a function \( \phi(y,\tau) \) with
\[ \phi(y,\tau_0) = 0, \quad \phi(y,\tau) \cdot \omega(y) \equiv 0. \]
Noticing that
\[ \partial_t \phi = \frac{1}{\lambda^2} \partial_x \phi - \lambda y \cdot \nabla_y \phi - \frac{\lambda}{\lambda} y \cdot \nabla_y \phi, \]
it is straightforward to check that \( \varphi \) given by (3.2) solves (3.1) if the pair \( (\phi,\psi) \) satisfies the following system of equations
\begin{align*}
\partial_t \phi &= L_\omega[\phi] + \lambda^2 Q_{-\alpha} \Pi_U^+ E^* \\
&\quad + |\nabla \omega(y)|^2 \Pi_{U}^+ Q_{-\alpha} \psi - 2 \nabla_y (Q_{-\alpha} \psi \cdot \omega) \nabla \omega(y) \quad \text{in } D_{2R}, \quad (3.3)
\end{align*}
with
\begin{align*}
\phi &= 0 \quad \text{in } B_{2R(0)}(0) \times \{\tau_0\}, \\
\partial_t \psi &= \Delta_x \psi + (1 - \eta) \left[ |\nabla_x U|^2 \psi - 2 \nabla_x (\psi \cdot U) \nabla_x U \right] \\
&\quad + Q_\alpha \left[ \Delta_x \eta \phi + 2 \lambda^{-1} \nabla_x \eta \nabla_y \phi \right] - \lambda \eta \phi + (\partial_t Q_\alpha) \phi \\
&\quad - \lambda \eta \nabla_y \phi - \lambda \eta \xi \cdot \nabla_y \phi \\
&\quad - (\psi \cdot U) U_t + N(\varphi^* + \psi + \eta Q_\alpha \phi) + (1 - \eta) \Pi_{U}^+ E^* \quad \text{in } \Omega \times (0,T), \quad (3.4)
\end{align*}
\[ \psi = \psi_{0\Omega} \quad \text{on } \partial \Omega \times (0,T), \]
\[ \psi = \psi_0 \quad \text{in } \Omega \times \{0\}, \]
where \( \psi_{0}(x) \) is a small function that will be later determined, and in agreement with (2.29) we take the boundary condition
\[ \psi_{0\Omega}(\lambda,\alpha,\xi) = (u_{\partial \Omega} - U) - \Pi_{U}^+ [\varphi^*] + \tilde{a}(u_{\partial \Omega} - U, \Pi_{U}^+ [\varphi^*]) \quad \text{on } \partial \Omega. \]
Here we have set for \( \gamma > 0 \),
\[ D_{\gamma R} = \{ (x,\tau) / \tau \in (\tau_0,\infty), \ |y| \leq \gamma R(\tau) \}, \quad R(\tau) = R(t_\lambda(\tau)), \quad t_\lambda = \tau_\lambda^{-1} \]
so that \( R(\tau) \sim |\log \tau|^m \) and we recall
\[ L_\omega[\phi] = \Delta_y \phi + |\nabla \omega(y)|^2 \phi - 2(\nabla \omega(y) \cdot \nabla_y \phi) \omega(y). \]
We call equation (3.3) the inner problem and (3.4) the outer problem. The strategy we will use to solve this system is the following: we will first fix a function \( \phi(y,t) \) in a suitable class together with parameter functions \( \xi, \alpha, \lambda \) and solve for
ψ equation (3.4) as an operator \( \psi = \Psi[\lambda, \alpha, \xi, \phi] \). Then we substitute this \( \Psi \) in equation (3.3) and get
\[
\begin{align*}
\partial_t \psi &= L_\omega[\phi] + G(\lambda, \alpha, \xi, \phi) \quad \text{in } D_{2R} \\
\psi(\cdot, 0) &= 0 \quad \text{in } B_{2R(0)}
\end{align*}
\] (3.5)
where
\[
G(\lambda, \alpha, \xi, \phi) = \lambda^2 Q_{-\alpha} \Pi_{U^*} \mathcal{E}^* + \left| \nabla \omega(y) \right|^2 Q_{-\alpha} \Psi - 2 \nabla_y (Q_{-\alpha} \Psi \cdot \omega) \nabla_y \omega
\]
We will subsequently solve (3.5) recasting it as a system which we solve for \( \phi \) and the parameters \( \lambda, \alpha, \xi \). This resolution is carried out by a fixed point argument involving an inverse for the linear problem
\[
\begin{align*}
\partial_\tau \phi &= L_\omega[\phi] + h(y, \tau) \quad \text{in } D_{2R} \\
\phi(\cdot, 0) &= 0 \quad \text{in } B_{2R(0)}
\end{align*}
\] (3.6)
We shall find a solution \( \phi \) of (3.6) that defines a linear operator of the functions \( h \) which satisfies good \( L^\infty \)-weighted estimates provided that certain further orthogonality conditions hold.

The class of functions \( h \) we will consider is well-represented by the time-space asymptotic behavior of the error term \( \lambda^2 Q_{-\alpha} \Pi_{U^*} \mathcal{E}^* \) regarded as a function of the scaled variables \( (y, \tau) \). From the decomposition (2.10), the assumptions on the parameter functions \( \lambda, \alpha, \xi \), and the estimates we have worked out for the part of the error involving the operator \( \Phi[\lambda, \alpha, \xi] \), we see that
\[
\left| \lambda^2 Q_{-\alpha} \Pi_{U^*} \mathcal{E}^*(\xi + \lambda y, t) \right| \lesssim \frac{\lambda(t)}{1 + \rho^2}, \quad \rho = |y|.
\] (3.7)
We observe that
\[
\tau_{\lambda}(t) \sim \frac{1}{\log^2 T} \frac{\log^4 (T - t)}{T - t} \sim \frac{1}{\lambda(t)} \frac{\log^2 (T - t)}{\log T},
\]
and hence
\[
\lambda(\tau) := \lambda(t_{\lambda}(\tau)) \sim \frac{1}{\tau + T} \frac{\log^2 (\tau + T)}{\log T} \sim \lambda_0(\tau) := \frac{1}{\log T} \frac{\log^2 \tau}{\tau}.
\]

4. Linear theory for the inner problem

At the very heart of capturing the bubbling structure is the construction of an inverse for the linearized heat operator around the basic harmonic map. We consider the linear equation
\[
\begin{align*}
\partial_\tau \phi &= L_\omega[\phi] + h(y, \tau) \quad \text{in } D_{2R} \\
\phi(\cdot, \tau_0) &= 0 \quad \text{in } B_{2R(\tau_0)} \\
\phi \cdot \omega &= 0 \quad \text{in } D_{2R}
\end{align*}
\] (4.1)
where
\[
D_{2R} = \{(y, \tau) / \tau \in (\tau_0, \infty), y \in B_{2R(\tau)}(0)\}
\]
and we assume \( h \cdot \omega = 0 \) in \( D_{2R} \). We observe that a priori we are not imposing boundary conditions. Our purpose is to construct a solution \( \phi \) of the above problem that defines a linear operator of \( h \) and satisfies good bounds in terms of suitable norms.
The type of functions $h$ we would like to consider have separate multiplicative decays in space and time in the way of estimate (3.7). For given numbers $a, \nu > 0$, we assume that $h$ satisfies for some $K > 0$,\[ |h(y, \tau)| \leq K \frac{\tau^{-\nu}}{1 + |y|^a} \text{ in } D_{2R}. \]

We define the norm $\| h \|_{a, \nu}$ as the least number $K$ for which this inequality holds, that is,
\[ \| h \|_{a, \nu} = \sup_{D_{2R}} \tau^\nu (1 + |y|^a) |h(y, \tau)|. \]

In the best of the worlds we would like to find a solution $\phi$ to (4.1) that satisfies $\| \phi \|_{a-2, \nu} \leq C \| h \|_{a, \nu}$. Under certain orthogonality conditions on $h$, we will be able to find a solution with an estimate like that, however only close to $\partial B_{2R}(0)$, being deteriorated in the interior. These orthogonality conditions are similar to those used in Section §2 to derive the main order equations for the parameters $\lambda, \alpha, \xi$.

We recall that these conditions are referred to the generators of invariances under dilation, rotation and translations given by
\[
\begin{align*}
Z_{0j}(y) &= Z_0(\rho)E_j(y), \quad j = 1, 2, \\
Z_{11}(y) &= Z_1(y)(\cos \theta E_1(y) + \sin \theta E_2(y)), \\
Z_{12}(y) &= Z_1(y)(\sin \theta E_1(y) - \cos \theta E_2(y)) \quad (4.2)
\end{align*}
\]

where
\[
Z_0(\rho) = \frac{\rho}{\rho^2 + 1} = -\frac{1}{2} \rho \omega_\rho, \quad Z_1(\rho) = \frac{1}{\rho^2 + 1} = \frac{1}{2} \omega_\rho,
\]
so that $L_\omega[Z_{kj}] = 0$. We consider the following $L^2$-weighted projections onto $Z_{kj}(y)$, $j = 1, 2$ of a function $h(y, \tau)$,
\[
\tilde{h}_k(y, \tau) := \sum_{j=1}^{2} \frac{\chi_{Z_{kj}}(y)}{\rho \chi_{Z_{kj}}} \int_{B_{2R}} h(x, \tau) \cdot Z_{kj}(z) \, dz, \quad \chi(y) = \frac{1}{1 + |y|^a}.
\]

All functions $h(y, \tau)$ with $h(y, \tau) \cdot \omega(y) \equiv 0$ can be expressed in polar form as
\[ h(y, \tau) = h^1(\rho, \theta, \tau)E_1(y) + h^2(\rho, \theta, \tau)E_2(y), \quad y = \rho e^{i\theta}. \]

We can also expand in Fourier series
\[ \tilde{h}(\rho, \theta, \tau) := h^1 + \rho h^2 = \sum_{k=-\infty}^{\infty} \tilde{h}_k(\rho, \tau) e^{ik\theta}, \quad \tilde{h}_k = \tilde{h}_{k1} + i\tilde{h}_{k2} \]
so that
\[ h(y, \tau) = \sum_{k=-\infty}^{\infty} h_k(y, \tau) = h_0(y, \tau) + h_1(y, \tau) + h_\perp(y, \tau), \]
where
\[ h_k(y, \tau) = \text{Re} \left( \tilde{h}_k(\rho, \tau) e^{ik\theta} \right) E_1 + \text{Im} \left( \tilde{h}_k(\rho, \tau) e^{ik\theta} \right) E_2. \]

Our main result in this section is the following.
Proposition 4.1. Let $1 < a < 2$, $\nu > 0$. For each $h$ with $\|h\|_{a,\nu} < +\infty$ there exists a solution $\phi = \phi[h]$ of Problem (4.1), which defines a linear operator of $h$ and satisfies the following estimate in $\mathcal{D}_2^R$:

\[
\begin{align*}
(1 + |y|) |\nabla_y \phi(y, \tau)| + |\phi(y, \tau)| \lesssim & \tau^{-\nu} R^{\frac{2-a}{2}} \|h^1\|_{a,\nu} \\
& + \frac{\tau^{-\nu} R^{\frac{2-a+\sigma_1}{2}}}{1 + |y|^{\sigma_1}} \|\tilde{h}_1\|_{a,\nu} + \frac{\tau^{-\nu} R^4}{1 + |y|^2} \|\bar{h}_1\|_{a,\nu} \\
& + \frac{\tau^{-\nu} R^2}{1 + |y|} \min\{1, R^\frac{2-a}{2} |y|^{-2}\} \|h_0 - \bar{h}_0\|_{a,\nu} \\
& + \frac{\tau^{-\nu} R^2}{1 + |y|} \|\bar{h}_1\|_{a,\nu},
\end{align*}
\]

where $\sigma_1 \in (0, 1)$.

The construction of the operator $\phi[h]$ as stated in the proposition will be carried out mode by mode in the Fourier series expansion. We shall use the convention that $h(y, \tau) = 0$ for $|y| > 2R(\tau)$. Let us write

\[
\phi = \sum_{k=-\infty}^{\infty} \phi_k, \quad \phi_k(y, \tau) = \text{Re} (\varphi_k(\rho, \tau)e^{ik\theta}) E_1 + \text{Im} (\varphi_k(\rho, \tau)e^{ik\theta}) E_2.
\]

We shall build a solution of (4.1) by solving separately each of the equations

\[
\begin{align*}
\partial_\tau \phi_k &= \mathcal{L}_k[\varphi_k] + \tilde{h}_k(y, \tau) = 0 \quad \text{in } \mathcal{D}_4^R, \quad (4.3) \\
\phi_k(y, \tau_0) &= 0 \quad \text{in } B_{4R_0}(0), \quad R_0 = R(\tau_0)
\end{align*}
\]

which, are equivalent to the problems

\[
\begin{align*}
\partial_\tau \varphi_k &= \mathcal{L}_k[\varphi_k] + \tilde{h}_k(\rho, \tau) \quad \text{in } \tilde{D}_4^R, \\
\varphi_k(\rho, 0) &= 0 \quad \text{in } (0, 4R_0)
\end{align*}
\]

with

\[
\tilde{D}_4^R = \{(\rho, \tau) / \tau \in (\tau_0, \infty), \rho \in (0, 4R(\tau))\}
\]

and we recall

\[
\mathcal{L}_k[\varphi_k] := \partial_\rho^2 \varphi_k + \partial_\rho \frac{\varphi_k}{\rho} - (k^2 + 2k \cos w + \cos(2w)) \frac{\varphi_k}{\rho^2}
\]

We have the validity of the following result.

Lemma 4.1. Let $\nu > 0$ and $-1 < a < 2$, $a \neq 0, 1$. Assume that

\[
\|h_k(y, \tau)\|_{a,\nu} < +\infty.
\]

Then problem (4.3) has a unique bounded solution $\phi_k(y, \tau)$ of the form

\[
\phi_k(y, \tau) = \text{Re} (\varphi_k(\rho, \tau)e^{ik\theta}) E_1 + \text{Im} (\varphi_k(\rho, \tau)e^{ik\theta}) E_2
\]

which in addition satisfies the boundary condition

\[
\phi_k(y, \tau) = 0 \quad \text{for all } \tau \in (0, \infty), \quad y \in \partial B_{R(\tau)}(0).
\]

These solutions satisfy the estimates

\[
\begin{align*}
|\phi_k(y, \tau)| &\leq C \|h\|_{a,\nu} \tau^{-\nu} k^{-2} R^{2-a} \quad \text{if } k \neq 0, 1, \\
|\phi_0(y, \tau)| &\leq C \|h\|_{a,\nu} \lambda_0(\tau)^\nu (1 + \rho)^{-1} \left\{ \begin{array}{ll}
R^2 & \text{if } a > 1, \\
R^{3-a} & \text{if } a < 1,
\end{array} \right.
\end{align*}
\]
Let us now consider \( k \) \( \phi \) parts of \( k \) with \( \phi \) \( g(\tau) = 0 \) for all \( \tau \in (\tau_0, +\infty) \)
\( \phi_k(0, \rho) = 0 \) in \((0, 4R_0)\),
\[
\mathcal{L}_k[\phi_k] = \partial_{\rho}\phi_k + \frac{\partial_{\rho}\phi_k}{\rho} - (k^2 + 2k\cos w + \cos(2w))\frac{\phi_k}{\rho^2}
\]
dominates both, real and imaginary parts of \( \varphi_{-1}(\rho, \tau) \). As a conclusion, we find
\[
\varphi(\rho) := Z_{-1}(\rho) \int_0^{4R} \frac{dr}{\rho Z_{-1}(r)^2} \int_0^r g(s)Z_{-1}(s)\, ds,
\]
Here we have used that \( \mathcal{L}_1[Z_{-1}] = 0 \). Let us call \( \varphi_0(\rho) \) the function in (4.6) with \( g(\rho) := 2(1 + \rho)^{-a} \). Assuming that \( a < 2 \), we readily estimate \( |\varphi(\rho)| \leq CR^2-a \).

Let us call \( \varphi(x, \tau) = \tau^{-\nu}\varphi(\rho) \). Then we see that
\[
-\varphi_{\tau}(\rho, \tau) + \mathcal{L}_{-1}[\varphi(\rho, \tau)] + \tau^{-\nu} = -\nu\tau^{-\nu-1}\varphi(\rho) - \frac{\tau^{-\nu}}{(1+\rho)^a} \leq -\tau^{-\nu}(1+\rho)^{-a} [1 - C\tau^{-1}R^{2-a}(1+\rho)^a] < 0
\]
in \( \tilde{D}_{4R} \). Indeed, since \( R(\tau) \sim \tau^n \), the inequality globally holds provided that \( t_0 \) was chosen sufficiently large.

Then for \( k = -1 \) the barrier \( h_{x,\nu}\varphi(\rho, \tau) \) dominates both, real and imaginary parts of \( \varphi_{-1}(\rho, \tau) \). As a conclusion, we find
\[
|\varphi_{-1}(y, \tau)| \leq C \| h_{x,\nu}\varphi(\rho, \tau) \|
\]
in \( \mathcal{D}_{4R} \). Let us now consider \( \phi_k \) with \( \phi_k \geq 2 \) and the function \( \varphi(\rho, \tau) \) as above. Now we find
\[
-\varphi_{\tau}(\rho, \tau) + \mathcal{L}_k[\varphi(\rho, \tau)] \leq (\mathcal{L}_k - \mathcal{L}_{-1})[\varphi(\rho, \tau)]
\]
\[
\leq -C\lambda_0^2(k^2 - 1 + 2(k-1)) \frac{1}{\rho^2}(1+\rho)^{2-a}
\]
\[
< -C(k^2 - 1 + 2(k-1)) \frac{\tau^{-\nu}}{(1+\rho)^a}
\]
in \( \tilde{D}_{4R} \). The latter quantity is negative provided that \( |k| \geq 2 \) and \( k \neq -2 \). Indeed, there is a \( c > 0 \) such that for all such \( k \)'s we have
\[
-\varphi_{\tau}(\rho, \tau) + \mathcal{L}_k[\varphi(\rho, \tau)] + \frac{ck^2\tau^{-\nu}}{(1+\rho)^a} < 0
\]
in \( \tilde{D}_{4R} \).
and hence we get the estimate
\[ |φ_k(y, τ)| ≤ \frac{C}{k^2} \|h\|_{a, ν} τ^{-ν} R^{2-a} \text{ in } D_{4R} \]
for all \( k \neq 0, 1, -2 \). Now, we observe that the function
\[ Z_{-2}(ρ) = \frac{ρ^3}{ρ^2 + 1} \]
satisfies \( L_{-2}[Z_{-2}] = 0 \) and that the formula
\[ φ(ρ) = Z_{-2}(ρ) \int_ρ^R \frac{dr}{r Z_{-2}(r)^2} \int_0^r g(s) Z_{-2}(s) s ds, \]
yields a solution of \( L_{-2}[φ] = g \). Choosing again \( g = (1 + ρ)^{-a} \) we find that
\[ |φ(ρ)| ≤ CR^{2-a}. \]
Arguing as before, we get
\[ |φ_{-2}(y, τ)| ≤ C \|h\|_{a, ν} τ^{-ν} R^{2-a} \text{ in } D_{4R}. \]
The cases \( k = 0 \) and \( k = 1 \) produce poorer estimates. The functions
\[ Z_0(ρ) = \frac{ρ}{ρ^2 + 1}, \quad Z_1(ρ) = \frac{1}{ρ^2 + 1} \]
satisfy \( L_0[Z_0] = 0, \) \( L_1[Z_1] = 0 \). We have that for \( k = 0, 1 \) the formulas
\[ φ^k(ρ) = Z_0(ρ) \int_ρ^{4R} \frac{dr}{ρ Z_k(r)^2} \int_0^r g(s) Z_k(s) s ds, \]
produce for \( g = (1 + ρ)^a \) solutions of \( L_k[φ^k] + g = 0 \) that satisfy
\[ |φ_0(ρ)| ≤ C(1 + ρ)^{-1} \begin{cases} R^2 & \text{if } a > 1, \\ R^{3-a} & \text{if } a < 1, \end{cases} \]
and
\[ |φ_1(ρ)| ≤ C(1 + ρ)^{-2} \begin{cases} R^4 & \text{if } a > 0, \\ R^{4-a} & \text{if } a < 0. \end{cases} \]
From here, the corresponding estimates for \( φ_k(y, τ) \) follow as stated. The proof is concluded. \( \Box \)

We can get gradient estimates for the solutions built in the above lemma by means of the following.

**Lemma 4.2.** Let \( φ \) be a solution of the equation
\[ \partial_τ φ = L_ω[φ] + h(y, τ) \text{ in } D_{4γR} \]
\[ φ(·, τ_0) = 0 \text{ in } B_{2R(0)}. \]
Given numbers \( a, b, ν, γ \), there exists a \( C \) such that for some \( M > 0 \) we have
\[ |φ(y, τ)| + (1 + |y|)^2 |h(y, τ)| ≤ M R^b τ^{-ν} (1 + |y|)^{-a} \text{ in } D_{4γR}, \]
then
\[ (1 + |y|) |∇_y φ(y, τ)| ≤ C M R^b τ^{-ν} (1 + |y|)^{-a} \text{ in } D_{3γR} \]
and we recall
\[ D_{γR} = \{(y, τ) / |y| < γ R(τ), \quad τ ∈ (τ_0, ∞)\}. \]
Proof. To prove the remaining gradient estimates, we fix a large positive number \( \rho \), a vector \( e \) with \( |e| = 1 \), and a number \( \tau_1 \geq 2\tau_0 \). We assume that \( \rho \leq \frac{7}{2} \gamma R(\tau_1) \) so that the ball \( B_{\rho/2}(\rho e) \) is contained in \( B(0, 4\gamma R(\tau_1)) \). Let us define

\[
\tilde{\phi}(z, t) := \phi(\rho e + \rho z, \tau_1 + \rho^2 t).
\]

Then \( \tilde{\phi}(z, t) \) satisfies an equation of the form

\[
\tilde{\phi}_t = \Delta_z \tilde{\phi} + A\nabla_z \tilde{\phi} + B\tilde{\phi} + \tilde{h}(z, t) \quad \text{in} \quad B_1(0) \times (0, 2)
\]

with coefficients \( A(z, t) \) and \( B(z, t) \) uniformly bounded by \( O(\rho^{-2}) \) in \( B_1(0) \times (0, 2) \) and

\[
\tilde{h}(z, t) = \rho^2 h(\rho e + \rho z, \tau_1 + \rho^2 t).
\]

We observe that

\[
\lambda_0(\tau)^\nu \lesssim \lambda_0(\tau_1)^\nu, \quad R(\tau)^b \lesssim R(\tau_1)^b, \quad \text{for all} \quad \tau \in (\tau_1, \tau_1 + R(\tau_1)^2).
\]

Standard parabolic estimates and assumption (4.8) yield

\[
\| \nabla \tilde{\phi} \|_{L^\infty(B_{\frac{1}{2}}(0) \times (1, 2))} \lesssim \| \tilde{\phi} \|_{L^\infty(B_{\frac{1}{2}}(0) \times (0, 2))} + \| \tilde{h} \|_{L^\infty(B_{\frac{1}{2}}(0) \times (0, 2))} \lesssim M R(\tau_1)^b \tau_1^{-\nu} \rho^{2-a},
\]

so that in particular

\[
\rho |\nabla \tilde{\phi}(\rho e, \tau_1 + \rho^2)| = |\nabla \tilde{\phi}(0, 1)| \lesssim M R(\tau_1)^b \tau_1^{-\nu} \rho^{2-a}
\]

provided that \( \rho \leq \frac{7}{2} \gamma R(\tau_1) \). Let us now fix a \( \tau > 2\tau_0 \) and consider a \( \rho \) with \( \rho < 3\gamma R(\tau) \). By fixing \( \tau_0 \) sufficiently large we get that

\[
\tau - 9\gamma^2 R^2(\tau) > \frac{T}{2} \quad \text{for all} \quad \tau > \tau_0.
\]

Let us set \( \tau_1 = \tau - \rho^2 \). Then \( \tau_1 > 2\tau_0 \) if \( \tau > 4\tau_0 \) and

\[
\rho \leq 3\gamma R(\tau) \leq \frac{7}{2} \gamma R\left(\frac{T}{2}\right) \leq \frac{7}{2} \gamma R(\tau - 9\gamma^2 R(\tau)^2) \leq \frac{7}{2} \gamma R(\tau_1).
\]

On the other hand, \( R(\tau_1) \leq R(\tau) \), and

\[
\lambda_0(\tau_1) \leq \lambda_0(\tau - R(\tau)^2) \leq \lambda_0\left(\frac{T}{2}\right) \leq 4\lambda_0(\tau).
\]

Hence, for \( \tau > 4\tau_0 \) and \( \rho \leq 3\gamma R(\tau) \) we have

\[
\rho |\nabla \phi(\rho e, \tau)| \lesssim M R(\tau)^b \tau^{-\nu} \rho^{2-a}.
\]

Thus if we fix \( \tau_0 \) sufficiently large we get that for any \( \tau > 4\tau_0 \) and \( |y| \leq 3\gamma R(\tau) \)

\[
(1 + |y|) |\nabla \phi(y, \tau)| \lesssim M R(\tau)^b \tau^{-\nu} (1 + |y|)^{2-a}.
\]

We obtain that these bounds are as well valid for \( \tau < 4\tau_0 \) by the use of the similar parabolic estimates up to the initial time (with condition 0). Local estimates on bounded sets near the origin give the remaining region of \( D_{3\gamma R} \). The proof is concluded.

Our next goal is to construct an inverse for modes \( k = 0, 1 \) with better control but subject to a certain solvability condition. Let us consider again equation (4.3) for \( k = 0 \) and the functions \( Z_{0j}(y) \) defined in (4.2) . We have the following result.
Lemma 4.3. Let assume that \( 1 < a < 2, \ k = 0 \) and
\[
\int_{B_{2R}} h_0(y) \cdot Z_0^j(y) \, dy = 0 \quad \text{for all} \quad \tau \in (\tau_0, +\infty) \tag{4.9}
\]
for \( j = 1, 2 \). Then there exist a solution \( \phi_0 \) to Equation (4.3) for \( k = 0 \) that defines a linear operator of \( h_0 \) and satisfies the estimate in \( D_{3R} \).
\[
|\phi_0(y, \tau)| \lesssim \|h_0\|_{a, \nu} R^{\frac{5-a}{2}} \lambda_0^k(1 + |y|)^{-1} \min\{1, R^{\frac{5-a}{2}} |y|^{-2}\} \tag{4.10}
\]
A central feature of estimate (4.10) is that it matches the size of the solutions obtained in Lemma 4.1 for \( k \neq 0, 1 \) when \( |y| \sim R \).

Proof. We observe that conditions (4.9) can be written as
\[
\int_{0}^{2R} h_0(\rho, \tau) Z_0(\rho) \, d\rho = 0 \quad \text{for all} \quad \tau \in (\tau_0, +\infty). \tag{4.11}
\]
Let us consider the complex valued functions
\[
\tilde{H}_0(\rho, \tau) := -Z_0(\rho) \int_{\rho}^{\infty} \frac{1}{sZ_0(\zeta)^2} \int_{s}^{\infty} \tilde{h}_0(\zeta, \tau) Z_0(\zeta) \, d\zeta, \quad k = 0, 1,
\]
where we have extended \( \tilde{h}(\rho, \tau) = 0 \) for \( \rho > 2R(\tau) \). They are well-defined thanks to (4.11). Then the function
\[
H_0(y, \tau) := \text{Re}(\tilde{H}_0(\rho, \tau)) E_1(y) + \text{Re}(\tilde{H}_k(\rho, \tau)) E_2(y)
\]
solves
\[
L_\omega[H_0(y, \tau)] = h_0(y, \tau) \quad \text{in} \quad D_{4R}
\]
and satisfies
\[
|H_0(y, \tau)| \lesssim \tau^{-\nu}(1 + |y|)^{2-a}\|h_0\|_{a, \nu} \quad \text{in} \quad D_{4R}.
\]
Moreover, elliptic gradient estimates yield
\[
|\nabla_y H_0(y, \tau)| \lesssim \tau^{-\nu}(1 + |y|)^{1-a}\|h_0\|_{a, \nu} \quad \text{in} \quad D_{3R}.
\]
Let us consider the problem
\[
\Phi_\tau = L_\omega[\Phi] + H_0(y, \tau) \quad \text{in} \quad D_{4R},
\]
\[
\Phi(y, 0) = 0 \quad \text{in} \quad B_{4R_0}(0)
\]
\[
\Phi(y, \tau) = 0 \quad \text{for all} \quad \tau \in (\tau_0, \infty), \quad y \in \partial B_{4R_0}(0)
\]
According to Lemma 4.1, this problem has unique solution \( \Phi = \Phi_k \) that satisfies the estimates
\[
|\Phi_0(y, \tau)| \lesssim C\|H_0\|_{a-2, \nu} \lambda_0(\tau)^\nu (1 + |y|)^{-1} R^{5-a} \quad \text{in} \quad D_{4R}.
\]
Applying Lemma 4.2 we deduce that, also,
\[
|\nabla_y \Phi_0(y, \tau)| \lesssim \|H_0\|_{a-2, \nu} \lambda_0(\tau)^\nu (1 + |y|)^{-2} R^{5-a} \quad \text{in} \quad D_{3R}
\]
Let us write
\[
\Phi_{0j} := \partial_{y_j} \Phi_0, \quad H_{0j} := \partial_{y_j} H_0
\]
Then we have
\[
\partial_\tau \Phi_{0j} = L_\omega[\Phi_{0j}] + \partial_{y_j} |\nabla_\omega|^2 \Phi_0 + 2\nabla \partial_{y_j} \omega \nabla \Phi_0 + H_{0j}(y, \tau) \quad \text{in} \quad D_{3R},
\]
\[
\Phi_{0j}(y, \tau) = 0 \quad \text{for all} \quad \tau \in (\tau_0, \infty), \quad y \in \partial B_{3R_0}(0)
\]
According to Lemma 4.2 and the above estimates we obtain that
\[
(1 + |y|)|\nabla \Phi_0(y, \tau)| \lesssim \|h_0\|_{a, \nu} \lambda_0(\tau)^{n} (1 + |y|)^{-2} R^{3-a} + \|h_0\|_{a, \nu} \lambda_0(\tau)^{n} (1 + |y|)^{4-a} \text{ in } D_{3R}.
\]
Then we define
\[
\phi_0 := L_\omega[\Phi_0]
\]
so that \( \phi = \phi_0 \) solves
\[
\begin{align*}
\phi_r &= L_\omega[\phi] + h_0(y, \tau) \text{ in } D_{3R}, \\
\phi(y, \tau) &= 0 \text{ for all } \tau \in (\tau_0, \infty), \quad y \in B_{3R_0}(0)
\end{align*}
\]
and defines a linear operator of the function \( h_0 \). Moreover, observing that
\[
|L_\omega[\Phi_0]| \lesssim |D_y^2\Phi_0| + O(\rho^{-a}) |\Phi_0| + O(\rho^{-2}) |D_y\Phi_0|
\]
we then get the estimate
\[
|\phi_0(y, \tau)| \lesssim \|h_0\|_{a, \nu} R^{3-a} \tau^{-\nu} (1 + |y|)^{-3}.
\] (4.12)

To complete the proof of estimate (4.10), we let \( \varphi_0 \) be the complex valued function defined as
\[
\phi_0(y, \tau) = \text{Re} (\varphi_0(\rho, \tau)) E_1 + \text{Im} (\varphi_0(\rho, \tau)) E_2
\]
so that letting \( R' = R^{\frac{\nu}{\nu-a}} \ll R \), using the notation in (4.5), \( \varphi_0 \) satisfies the equation
\[
\partial_\tau \varphi_0 = \mathcal{L}_0[\varphi_0] + h_0(\rho, \tau) \text{ in } \hat{D}_{R'},
\]
\[
\varphi_0(0, \rho) = 0 \text{ in } (0, R'),
\]
and from (4.12), in addition we have
\[
|\varphi_0(R', \tau)| \lesssim \|h_0\|_{a, \nu} \tau^{-\nu} R'.
\]
Similarly as in (4.7) we let
\[
\varphi^0(\rho, \tau) := C \|h_0\|_{a, \nu} \tau^{-\nu} Z_0(\rho) \int_0^{10R'} \frac{dr}{r^2} \int_0^r (1 + s)^{-a} Z_0(s) s \, ds,
\]
which for a suitably large choice of the constant \( C \) becomes a supersolution for the real and imaginary parts of equation (4.13), which also dominates their boundary values at \( R' \). This then yields
\[
|\varphi_0(y, \tau)| \lesssim \|h_0\|_{a, \nu} \tau^{-\nu} |R'|^2 (1 + |y|)^{-1}, \quad |y| < R'.
\]
Combining this estimate and (4.12) yields the validity of (4.10). The proof is concluded.

Next we deal with the mode \( k = 1 \). The following result holds

**Lemma 4.4.** Let assume that \( 1 < a < 2, k = 1 \) and
\[
\int_{B_{2R}} h_1(y) \cdot Z_1^j(y) \, dy = 0 \text{ for all } \tau \in (\tau_0, +\infty)
\]
for \( j = 1, 2 \). Then there exist a solution \( \phi_1 \) to Equation (4.3) for \( k = 1 \) that defines a linear operator of \( h_1 \) and for any \( \sigma > 0 \) satisfies the estimate in \( D_{3R} \),
\[
|\phi_1(y, \tau)| \lesssim \|h_1\|_{a, \nu} R^{2-a+\sigma} \lambda_0^{-\nu}(1 + |y|)^{-\sigma}.
\]
We will establish this result by solving the equation in entire $\mathbb{R}^2$ after extending $h$ as zero outside the considered region. Thus, we consider a function $h(y, \tau)$ defined in entire $\mathbb{R}^2 \times (1, +\infty)$ of the form

$$ h = \text{Re} (\tilde{h}e^{i\theta}) E_1 + \text{Im} (\tilde{h}e^{i\theta}) E_2, \quad (4.14) $$

that satisfies the orthogonality conditions for $j = 1, 2$

$$ \int_{\mathbb{R}^2} h(\cdot, \tau) \cdot Z_{1j} = 0 \quad \text{for all } \tau \in (1, \infty). \quad (4.15) $$

We consider the problem

$$ \partial_\tau \phi \ = \ L_\omega[\phi] + h \quad \text{in } \mathbb{R}^2 \times (1, \infty), \quad (4.16) $$

$$ \phi(\cdot, 0) = 0 \quad \text{in } \mathbb{R}^2. $$

By standard parabolic theory, this problem has a unique solution, which is therefore of the form

$$ \phi = \text{Re} (\varphi e^{i\theta}) E_1 + \text{Im} (\varphi e^{i\theta}) E_2, \quad (4.17) $$

where the complex valued function $\varphi(\rho, \tau)$ solves the initial value problem

$$ \partial_\tau \varphi = \mathcal{L}_1[\varphi] + \tilde{h}(\rho, \tau) \quad \text{in } (0, \infty) \times (1, \infty), \quad (4.18) $$

$$ \varphi(0, \rho) = 0 \quad \text{in } (0, \infty), $$

$$ \mathcal{L}_1[\varphi] = \partial_\rho^2 \varphi + \frac{\partial_\rho \varphi}{\rho} - (1 + 2 \cos w + \cos(2w)) \frac{\varphi}{\rho^2}. $$

Let us write

$$ \|h\|_{b, \tau_1} := \sup_{\tau \in (0, \tau_1)} \tau^\nu \|(1 + |y|^b)h\|_{L^\infty(\mathbb{R}^2)} . $$

We have the validity of the following result.

**Lemma 4.5.** Let $0 < \sigma < 1$, $\nu > 0$. There exists a constant $C > 0$ such that the following holds. Assume that $\|h\|_{2+\sigma, \tau_1} < +\infty$ and that $h$ has the form (4.14) and satisfies the orthogonality conditions (4.15). Then for any sufficiently large $\tau_1 > 0$ the solution $\phi$ of Problem (4.16) satisfies the estimate

$$ \|\phi\|_{\sigma, \tau_1} \leq C \|h\|_{2+\sigma, \tau_1}. \quad (4.19) $$

**Proof.** Let $h$ be a function with $\|h\|_{2+\sigma, \infty} < +\infty$ of the form (4.14) and $\phi$ of the form (4.17) the solution of the initial value problem (4.16). We claim that given $\tau_1 > 0$ we have that $\|\phi\|_{2+\sigma, \tau_1} < +\infty$. Indeed, by standard linear parabolic theory $\phi(y, \tau)$ is locally bounded in time and space. More precisely, given $R > 0$ there is a $K = K(R, \tau_1)$ such that

$$ |\phi(y, \tau)| \leq K \quad \text{in } B_R(0) \times (0, \tau_1]. $$

If we fix $R$ large and take $K_1$ sufficiently large, we see that $K_1 \rho^{-\sigma}$ is a supersolution for the real and imaginary parts of the equivalent complex valued equation (4.18) and $\rho > R$. As a conclusion, we find that $|\phi| \leq 2K_1 \rho^{-\sigma}$, and therefore $\|\phi\|_{\sigma, \tau_1} < +\infty$ for any $\tau_1 > 0$. We claim that

$$ \int_{\mathbb{R}^2} \phi(\cdot, \tau) \cdot Z_{1j} = 0 \quad \text{for all } \tau \in (1, \tau_1), \quad j = 1, 2. \quad (4.20) $$

Indeed, let us test the equation against

$$ Z_{1j}\eta, \quad \eta(y) = \eta_0(R^{-1}|y|). $$
\[ \eta_0 \text{ is a smooth cut-off function with } \eta_0(r) = 1 \text{ for } r < 1 \text{ and } = 0 \text{ for } r > 2 \]

and \( R \) is an arbitrary large constant. We find that

\[
\int_{\mathbb{R}^2} \phi(\cdot, \tau) \cdot Z_{1j} \eta = \int_0^\tau ds \int_{\mathbb{R}^2} \phi(\cdot, s) \cdot (L_\omega[\eta Z_{1j}] + h \cdot Z_{1j} \eta).
\]

(4.21)

On the other hand,

\[
\int_{\mathbb{R}^2} \phi(\cdot, \tau) \cdot (L_\omega[\eta Z_{1j}] + h \cdot Z_{1j} \eta)
\]

\[
= \int_{\mathbb{R}^2} \phi(\cdot, \tau) \cdot (Z_{1j} \Delta \eta + 2 \nabla \eta \cdot \nabla Z_{1j}) - h \cdot Z_{1j}(1 - \eta R)
\]

\[
= O(R^{-2-\sigma})
\]

uniformly on \( \tau \in (0, \tau_1) \). Letting \( R \to +\infty \) in (4.21) we get that (4.20) holds.

Now we claim that there exists a constant \( C \) such that for all \( \tau_1 > 0 \) sufficiently large, any \( h \) and \( \phi \) of the form (4.14) with and \( \|\phi\|_{\sigma, \tau_1} < +\infty \), which solves (4.16) and satisfies relations (4.20) we have the validity of the estimate

\[
\|\phi\|_{\sigma, \tau_1} \leq C \|h\|_{2+\sigma, \tau_1}.
\]

(4.22)

so that in particular estimate (4.19) holds.

To prove (4.22) we assume by contradiction the existence of sequences \( \tau_1^n \to +\infty \) and \( \phi_n, h_n \) of the form (4.14), (4.17) satisfying

\[
\partial_\tau \phi_n = L_\omega[\phi_n] + h_n \quad \text{in } \mathbb{R}^2 \times (1, \tau_1^n),
\]

\[
\int_{\mathbb{R}^2} \phi_n(\cdot, \tau) \cdot Z_{1j} = 0 \quad \text{for all } \tau \in (1, \tau_1^n),
\]

\[
\phi_n(\cdot, 0) = 0 \quad \text{in } \mathbb{R}^2,
\]

for which

\[
\|\phi_n\|_{\sigma, \tau_1^n} = 1, \quad \|h_n\|_{2+\sigma, \tau_1^n} \to 0.
\]

(4.23)

We claim first that

\[
\sup_{1 < \tau < \tau_1^n} \tau^n |\phi_n(y, \tau)| \to 0
\]

uniformly on compact subsets of \( y \in \mathbb{R}^2 \). If not we have that for some \( |y_n| \leq M \) and \( 1 < \tau_2^n < \tau_1^n \)

\[
(\tau_2^n)^\nu (1 + |y_n|^\sigma) |\phi(y_n, \tau_2^n)| \geq \frac{1}{2}.
\]

Clearly we must have \( \tau_2^n \to +\infty \). Let us define

\[
\tilde{\phi}_n(y, \tau) = (\tau_2^n)^\nu \phi_n(y, \tau_2^n + \tau).
\]

Then

\[
\partial_\tau \tilde{\phi}_n = L_\omega[\tilde{\phi}_n] + \tilde{h}_n \quad \text{in } \mathbb{R}^2 \times (1, \tau_2^n, 0)
\]

where \( \tilde{h}_n \to 0 \) uniformly on compact subsets of \( \mathbb{R}^2 \times (-\infty, 0] \) and

\[
|\tilde{\phi}_n(y, \tau)| \leq \frac{1}{1 + |y|^\sigma} \quad \text{in } \mathbb{R}^2 \times (1 - \tau_2^n, 0].
\]
From standard parabolic estimates, we find that passing to a subsequence, $\tilde{\phi}_n \to \tilde{\phi}$ uniformly on compact subsets of $\mathbb{R}^2 \times (-\infty, 0]$ where $\tilde{\phi} \neq 0$ and

$$\tilde{\phi}_\tau = L_\omega[\tilde{\phi}] \quad \text{in } \mathbb{R}^2 \times (-\infty, 0],$$

$$\int_{\mathbb{R}^2} \tilde{\phi}(\cdot, \tau) \cdot Z_{1j} = 0 \quad \text{for all } \tau \in (-\infty, 0],$$

$$|\tilde{\phi}(y, \tau)| \leq \frac{1}{1 + |y|^\sigma} \quad \text{in } \mathbb{R}^2 \times (-\infty, 0], \quad j = 1, 2,$$

$$\tilde{\phi}(y, \tau) = \text{Re} (\tilde{\varphi}(\rho, \tau)e^{i\theta}) E_1 + \text{Im} (\tilde{\varphi}(\rho, \tau)e^{i\theta}) E_2.$$

We claim that necessarily $\tilde{\phi} = 0$ which is a contradiction. By standard parabolic regularity $\tilde{\phi}(y, \tau)$ is a smooth function. A scaling argument shows that

$$(1 + |y|)^{-1}|D_y \tilde{\phi}| + |\tilde{\phi}_\tau| + |D_y^2 \tilde{\phi}| \leq C(1 + |y|)^{-2-\sigma}.$$  

Differentiating the equation in $\tau$, we also get $\partial_\tau \phi_\tau = L_\omega[\phi_\tau]$ and we find the estimates

$$(1 + |y|)^{-1}|D_y \tilde{\phi}_\tau| + |\tilde{\phi}_{\tau\tau}| + |D_y^2 \tilde{\phi}_\tau| \leq C(1 + |y|)^{-3-\sigma}.$$  

Testing suitably the equations (taking into account the asymptotic behaviors in $y$ in integrations by parts) we find the relations

$$\frac{1}{2} \partial_\tau \int_{\mathbb{R}^2} |\tilde{\phi}_\tau|^2 + B(\tilde{\phi}, \tilde{\phi}_\tau) = 0$$

where

$$B(\tilde{\phi}, \tilde{\phi}) = -\int_{\mathbb{R}^2} L_\omega[\tilde{\phi}] \cdot \tilde{\phi} - \int_{\mathbb{R}^2} |\nabla \tilde{\phi}|^2 - |\nabla \omega|^2 |\tilde{\phi}|^2.$$  

It is useful to observe the following: since

$$\tilde{\phi}(y, \tau) = \text{Re} (\tilde{\varphi}(\rho, \tau)e^{i\theta}) E_1 + \text{Im} (\tilde{\varphi}(\rho, \tau)e^{i\theta}) E_2$$

then we compute, using that $L_1[w_\rho] = 0$,

$$B(\tilde{\phi}, \tilde{\phi}) = -\int_{-\infty}^{\infty} L_1[\varphi] \tilde{\varphi} \rho d\rho = \int_{-\infty}^{\infty} |(w_\rho^{-1}\tilde{\varphi})| w_\rho^2 \rho d\rho \geq 0.$$  

We also get

$$\int_{\mathbb{R}^2} |\tilde{\phi}_\tau|^2 = -\frac{1}{2} \partial_\tau B(\tilde{\phi}, \tilde{\phi}).$$  

From these relations we find

$$\partial_\tau \int_{\mathbb{R}^2} |\tilde{\phi}_\tau|^2 \leq 0, \quad \int_{-\infty}^{0} d\tau \int_{\mathbb{R}^2} |\tilde{\phi}_\tau|^2 < +\infty$$

and hence $\tilde{\phi}_\tau = 0$. Thus $\tilde{\phi}$ is independent of $\tau$ and therefore $L_\omega[\tilde{\phi}] = 0$. Since $\tilde{\phi}$ is at mode 1, this implies that $\tilde{\phi}$ is a linear combination of $Z_{1j}$, $j = 1, 2$. Since $\int_{\mathbb{R}^2} \tilde{\phi} \cdot Z_{1j} = 0$, $j = 1, 2$ we conclude that $\tilde{\phi} = 0$, a contradiction. We conclude that (4.24) indeed holds. From (4.23), we have that for a certain $y_n$ with $|y_n| \to \infty$

$$(\tau_2^n)^\nu |y_n|^\sigma |\tilde{\phi}_n(y_n, \tau_2^n)| \geq \frac{1}{2}.$$  

Now we let

$$\tilde{\phi}_n(z, \tau) := (\tau_2^n)^\nu |y_n|^\sigma \phi_n(y_n + |y_n|^{-1}z, |y_n|^{-2}\tau + \tau_2^n)$$
so that
\[ \partial_\tau \hat{\phi}_n = \Delta_z \hat{\phi}_n + a_n \cdot \nabla_z \hat{\phi}_n + b_n \hat{\phi}_n + \hat{h}_n(z, \tau) \]
where
\[ \hat{h}_n(z, \tau) = (\tau_n^2)^\nu |y_n|^{2+\sigma} h_n(y_n + |y_n|^{-1} z, |y_n|^{-2} \tau + \tau_n^2). \]
By assumption on \( \hat{h}_n \) we find that
\[ |\hat{h}_n(z, \tau)| \leq o(1) |\hat{y}_n + z|^{-2-\sigma} (|\tau_n^2|^{-1}|y_n|^{-2} \tau + 1)^{-\nu} \]
with
\[ \hat{y}_n = \frac{y_n}{|y_n|} \rightarrow -\hat{e} \]
with \(|\hat{e}| = 1\). Thus \( \hat{h}_n(z, \tau) \rightarrow 0 \) uniformly on compact subsets of \( \mathbb{R}^2 \setminus \{\hat{e}\} \times (-\infty, 0] \) and the same property holds for \( a_n \) and \( b_n \). Besides \( |\hat{\phi}_n(0, 0)| \geq \frac{1}{2} \) and
\[ |\hat{\phi}_n(z, \tau)| \leq |\hat{y}_n + z|^{-\sigma} (|\tau_n^2|^{-1}|y_n|^{-2} \tau + 1)^{-\nu}. \]
As a conclusion, we may assume that \( \hat{\phi}_n \rightarrow \hat{\phi} \neq 0 \) uniformly over compact subsets of \( \mathbb{R}^2 \setminus \{\hat{e}\} \times (-\infty, 0] \) where
\[ \hat{\phi}_\tau = \Delta_z \hat{\phi} \quad \text{in} \quad \mathbb{R}^2 \setminus \{\hat{e}\} \times (-\infty, 0], \]
and
\[ |\hat{\phi}(z, \tau)| \leq |z - \hat{e}|^{-\sigma} \quad \text{in} \quad \mathbb{R}^2 \setminus \{\hat{e}\} \times (-\infty, 0]. \]
Moreover, the mode 1 assumption for \( \phi_n \) translates for \( \hat{\phi} \) into
\[ \hat{\phi}(z, \tau) = \begin{bmatrix} \varphi(\rho, \tau) e^{i\theta} \\ 0 \end{bmatrix}, \quad z = \hat{e} + \rho e^{i\theta} \]
for a complex valued function \( \varphi \) that solves
\[ \varphi_\tau = \varphi_{\rho\rho} + \frac{\varphi_\rho}{\rho} - \frac{4\varphi}{\rho^2} \quad \text{in} \quad (0, \infty) \times (-\infty, 0], \quad (4.25) \]
\[ |\varphi(\rho, \tau)| \leq \rho^{-\sigma} \quad \text{in} \quad (0, \infty) \times (-\infty, 0]. \]
Let us set
\[ u(\rho, t) = (\rho^2 + t)^{-\sigma/2} + \frac{\varepsilon}{\rho^2} \]
Then
\[ -u_t + \Delta u - \frac{4u}{\rho^2} < (\rho^2 + t)^{-\sigma/2 - 1}[\sigma(\sigma + 2) - 4 + \frac{\sigma}{2}] < 0. \]
It follows that the function \( u(x, \tau + M) \) is a positive supersolution for the real and imaginary parts of equation \( (4.25) \) in \( (0, \infty) \times [-M, 0] \). We find then that \( |\varphi(\rho, \tau)| \leq 2u(\rho, \tau + M) \). Letting \( M \rightarrow +\infty \) we find
\[ |\varphi(\rho, \tau)| \leq \frac{2\varepsilon}{\rho^2} \]
and since \( \varepsilon \) is arbitrary we conclude \( \varphi = 0 \). Hence \( \hat{\phi} = 0 \), a contradiction that concludes the proof of the lemma. \( \square \)
Proof of Lemma 4.4. We take $h$ to be the extension as zero of the function $h_1$ as in the statement of the lemma. Then we let $\phi$ be the unique solution of the initial value problem (4.16), which clearly defines a linear operator of $h_1$. From Lemma 4.5 we have that for any $\tau_1 > 0$

$$|\phi(y, \tau)| \leq C \tau^{-\nu}(1 + |y|)^{-\alpha}h_{2+\sigma,\tau_1}$$

for all $\tau \in (0, \tau_1)$, $y \in \mathbb{R}^2$.

By assumption we have that $\|h_1\|_{a,\nu} < +\infty$, and hence

$$|h(y, \tau)| \leq C \tau^{-\nu}(1 + |y|)^{-\alpha}h_{1,\tau_1}$$

and we find

$$\|h\|_{2+\sigma,\tau_1} \leq R(\tau_1)^{2+\sigma-a}h_{1,\tau_1}$$

for an arbitrary $\tau_1$. It follows that

$$|\phi(y, \tau)| \leq C \tau^{-\nu}(1 + |y|)^{-\alpha}R(\tau)^{2+\sigma-a}h_{1,\tau_1}$$

for all $(\tau, y) \in (1, \infty) \times \mathbb{R}^2$.

Then simply letting $\phi_1 := \phi|_{D_{2R}}$, the result follows. □

Proof of Proposition 4.1. We let $h$ be defined in $D_{2R}$ with $\|h\|_{a,\nu} < +\infty$, with $1 < a < 2$. We extend $h(y, \tau)$ as zero for $|y| > 2R(\tau)$ and consider the problem

$$\partial_\tau \phi = L_\omega[\phi] + h \quad \text{in} \quad D_{4R}$$

$$\phi(\cdot, \tau_0) = 0 \quad \text{in} \quad B_{4R_0}.$$

Let $\phi_k$ be the solution estimated in Lemma 4.1 of

$$\partial_\tau \phi_k = L_\omega[\phi_k] + h_k \quad \text{in} \quad D_{4R}$$

$$\phi_k(\cdot, \tau_0) = 0 \quad \text{on} \quad \partial B_{4R} \quad \text{for all} \quad \tau \in (\tau_0, \infty),$$

$$\phi_k(\cdot, \tau_0) = 0 \quad \text{in} \quad B_{4R_0}.$$

In addition we let $\phi_{01}, \phi_{11}$ solve

$$\partial_\tau \phi_{k1} = L_\omega[\phi_{k1}] + \tilde{h}_k \quad \text{in} \quad D_{4R}$$

$$\phi_{k1}(\cdot, \tau) = 0 \quad \text{on} \quad \partial B_{4R} \quad \text{for all} \quad \tau \in (\tau_0, \infty),$$

$$\phi_{k1}(\cdot, \tau_0) = 0 \quad \text{in} \quad B_{4R_0}.$$

for $k = 0, 1$. Let us consider the functions $\phi_{k2}$ constructed in Lemma 4.3 that solve for $k = 0, 1$

$$\partial_\tau \phi_{k2} = L_\omega[\phi_{k2}] + h_k - \tilde{h}_k \quad \text{in} \quad D_{4R}$$

$$\phi_{k2}(\cdot, \tau_0) = 0 \quad \text{in} \quad B_{4R_0}.$$

We define

$$\phi := \sum_{k=0,1} (\phi_{k1} + \phi_{k2}) + \sum_{k \neq 0,1} \phi_k$$

which is a bounded solution of the equation

$$\phi_\tau = L_\omega[\phi] + h(y, \tau) \quad \text{in} \quad D_{4R}$$
that defines a linear operator of \( h \). Applying the estimates for the components in Lemmas 4.1, 4.3 and 4.4 we obtain
\[
|\phi(y, \tau)| \lesssim \tau^{-\nu} R^{2-a} \|h_1\|_{a, \nu} + \frac{\tau^{-\nu} R^{2-a+\sigma}}{1 + |y|^\sigma} \|h_1 - \bar{h}_1\|_{a, \nu} + \frac{\tau^{-\nu} R^4}{1 + |y|^2} \|\bar{h}_1\|_{\nu, a}
\]
\[
+ \frac{R^{a+\nu}}{1 + |y|} \max\{1, R^{a+\nu}|y|^{-2}\} \|h_0 - \bar{h}_0\|_{a, \nu} + \frac{\tau^{-\nu} R^2}{1 + |y|} \|\bar{h}_0\|_{a, \nu},
\]
in \( D_{3R} \). Finally, Lemma 4.2 yields that the same bound is valid for
\[
(1 + |y|) |\nabla_y \phi(y, \tau)| \quad \text{in} \quad D_{2R}.
\]
The function \( \phi \big|_{D_{2R}} \) solves (4.1), it defines a linear operator of \( h \) and satisfies the required estimates. The proof is concluded.

\[\square\]

5. Estimates for the heat equation

The purpose of this section is to construct and provide estimates for a solution of the heat equation in \( \Omega \times (0, T) \) that will be used to solve the outer problem (3.4) via a fixed point argument. More precisely, given \( q \in \Omega \) and sufficiently a small positive numbers \( T \) we consider the problem
\[
\begin{cases}
\psi_t = \Delta_x \psi + f(x, t) & \text{in } \Omega \times (0, T) \\
\psi = 0 & \text{on } \partial\Omega \times (0, T) \\
\psi(q, t) = 0 \\
\psi(x, 0) = (c_1 e_1 + c_2 e_2 + c_3 e_3) \eta_1 & \text{in } \Omega
\end{cases}
\]
for suitable constants \( c_1, c_2, c_3 \), where
\[
e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.
\]
and \( \eta_1 \) is a smooth cut-off function with compact support, such that \( \eta_1 \equiv 1 \) in a neighborhood of \( q \).

Our purpose is to construct a solution of this problem that defines a linear operator of \( f \) and that satisfies good estimates in norms suitably adapted to the terms appearing in equation (3.4). Let us examine the terms appearing in (3.4), in particular
\[
f_1 := Q_\alpha \left[ \Delta_x \eta \phi + 2\lambda^{-1} \nabla_x \eta \nabla_y \phi - \partial_t \eta \phi \right], \quad f_2 := (1 - \eta) \Pi_{U^\perp} \mathcal{E}^*.
\]
The model \( \phi \) we would like to consider satisfies, in agreement with the estimate in Proposition 4.1,
\[
R(t)|\nabla_y \phi(y, t)| + |\phi(y, t)| \lesssim \lambda_0(t) R(t)^{-2-a}, \quad |y| \sim R(t), \quad y = \frac{x - \xi(t)}{\lambda(t)},
\]
where \( 1 < a < 2 \). Hence
\[
|f_1(x, t)| \lesssim \frac{1}{\lambda_0} \chi_{\{R \lambda_0 < r < 2R \lambda_0\}} R^{-a}, \quad |f_2(x, t)| \lesssim \frac{\lambda_0}{\lambda_0^2 + r^2} \chi_{\{r > R \lambda_0\}},
\]
where \( r = |x - \xi(t)| \).
Here and in what follows we set
\[ R(t) = \lambda_0(t)^{-\beta}, \quad \beta = \frac{1}{4} + \sigma \] (5.2)
and
\[ a = 2 - \sigma, \]
for a sufficiently small fixed number \( \sigma > 0 \).

We introduce the following weights that are adequate to uniformly estimate the above terms and others appearing in (3.4).

\[
\begin{align*}
\varrho_1 &:= \lambda_0^{\nu - 2} R^{-a} \lambda_{\{r < 2R\lambda_0\}} \\
\varrho_2 &:= T^{-a_0}(1 - \eta) \frac{\lambda_0}{r^2 + \lambda_0^2} \\
\varrho_3 &:= \frac{\lambda_0^{1/2 + \sigma_2}}{r + \lambda_0} \\
\varrho_4 &:= 1,
\end{align*}
\]
(5.3)
where \( \sigma_0, \sigma_2 > 0 \) are small and \( \nu \in (0, 1) \) is close to 1. For a function \( f(x, t) \) we consider the \( L^\infty \)-weighted norm
\[
\| f \|_* := \sup_{\Omega \times (0, T)} \left( 1 + \sum_{i=1}^4 \varrho_i(x, t) \right)^{-1} |f(x, t)|.
\]
The factor \( T^\sigma \) in front of \( \varrho_2 \) is there as a simple trick to have the error \( \mathcal{E}^* \) small in the outer problem.

Let
\[
\| \psi \|_* = \lambda_0(0)^{-\nu} R(0)^{2-a} |\log T| \sup_{\Omega \times (0, T)} |\psi(x, t)|
\]
(5.4)
\[
+ \sup_{\Omega \times (0, T)} \lambda_0(t)^{-\nu} R(t)^{a-2} |\log(T - t)||\psi(x, t) - \psi(x, T)|
\]
\[
+ \sup_{\Omega \times (0, T)} \lambda_0(t)^{-\nu} R(t)^{a-1} |\nabla \psi(x, t)|
\]
\[
+ \sup_{\Omega \times (0, T)} \lambda_0(t)^{-\nu} R(t)^{a-1} |\nabla \psi(x, t) - \nabla \psi(x, T)|
\]
\[
+ \sup_{\Omega \times (0, T)} \lambda_0(t_2)^{1-\nu+2\gamma} R(t_2)^{a-1+2\gamma} \frac{|\nabla \psi(x, t) - \nabla \psi(x', t)|}{|x - x'|^{2\gamma}}
\]
\[
+ \sup_{\Omega \times (0, T)} \lambda_0(t_2)^{1-\nu+2\gamma} R(t_2)^{a-1+2\gamma} \frac{|\nabla \psi(x, t_2) - \nabla \psi(x, t_1)|}{(t_2 - t_1)^\gamma},
\]
where the last sup is taken over \( x \in \Omega \) and \( 0 \leq t_1 < t_2 \leq T \) such that \( t_2 - t_1 \leq \frac{1}{10}(T - t_2) \).

Our main result in this section is the following.

**Proposition 5.1.** For \( T, \varepsilon > 0 \) there is a linear operator that maps a function \( f : \Omega \times (0, T) \to \mathbb{R}^3 \) with \( \| f \|_{**} < \infty \) into \( \psi, c_1, c_2, c_3 \) so that (5.1) is satisfied. Moreover the following estimates hold
\[
\| \psi \|_* \leq C\| f \|_{**}.
\]
(5.5)
To prove Proposition 5.1 we consider
\[
\begin{aligned}
\psi_t &= \Delta \psi + f \quad \text{in } \Omega \times (0, T) \\
\psi(x, 0) &= 0, \quad x \in \Omega \\
\psi(x, t) &= 0, \quad x \in \partial \Omega, t \in (0, T),
\end{aligned}
\tag{5.6}
\]
and let \( q \) be a point in \( \Omega \).

**Lemma 5.1.** Let \( \psi \) solve (5.6) with \( f \) such that
\[
|f(x, t)| \leq \lambda_0(t)^{\nu - 2} R(t)^{-a} \chi_{\{|x-q| \leq 2\lambda_0(t)R(t)\}},
\]
where \( a \in (1, 2), \nu \in (0, 1), \) and assume \( R \) is given by (5.2) and
\[
\nu - \frac{1}{2} > \beta(2 - a).
\]
Then
\[
|\psi(x, t)| \leq C\lambda_0(0)^\nu R(0)^{2-a} |\log T|,
\tag{5.7}
\]
\[
|\psi(x, t) - \psi(x, T)| \leq C\lambda_0(t)^\nu R(t)^{2-a} |\log(T - t)|,
\tag{5.8}
\]
\[
|\nabla \psi(x, t)| \leq C\lambda_0(t)^{\nu - 1} R(t)^{1-a},
\tag{5.9}
\]
\[
|\nabla \psi(x, t) - \psi(x, T)| \leq C\lambda_0(t)^{\nu - 1} R(t)^{1-a},
\tag{5.10}
\]
and for any \( \mu \in (0, 1) \),
\[
|\nabla \psi(x, t) - \nabla \psi(x', t)| \leq C\lambda_0(t)^{\nu + \mu - 2} R(t)^{\mu-a} |x - x'|^{1-\mu}
\tag{5.11}
\]
for any \( 0 \leq t_1 \leq t_2 \leq T \) such that \( t_2 - t_1 \leq \frac{1}{10}(T - t_2) \).

The proof is in appendix B.1.

**Lemma 5.2.** Let \( \psi \) solve (5.6) with \( f \) such that
\[
|f(x, t)| \leq \frac{\lambda_0(t)^b}{|x - q| + \lambda_0(t)},
\]
where \( 0 < b < 1 \). Then
\[
|\psi(x, t)| \leq C\lambda_0(0)^b T^{1/2}, \quad x \in \mathbb{R}^2, \quad 0 \leq t \leq T.
\]
\[
|\psi(x, t) - \psi(x, T)| \leq \begin{cases} 
\lambda_0(t)^b(T - t)^{1/2}, & \text{if } b < 1/2 \\
|\log T|^b(T - t), & \text{if } b > 1/2 
\end{cases}
\]
\[
|\nabla \psi(x, t)| \leq C\lambda_0(0)^b |\log \lambda_0(0)|, \quad x \in \mathbb{R}^2, \quad 0 \leq t \leq T,
\tag{5.12}
\]
\[
|\nabla \psi(x, t_2) - \nabla \psi(x, t_1)| \leq C\lambda_0(t_2)^{b-1}(t_2 - t_1)^{1/2}
\tag{5.13}
\]
for any \( 0 \leq t_1 \leq t_2 \leq T \) such that \( t_2 - t_1 \leq \frac{1}{10}(T - t_2) \).

**Lemma 5.3.** Let \( \psi \) solve (5.6) with \( f \) such that
\[
|f(x, t)| \leq \frac{(T - t)|\log(T - t)|^b}{|x - q|^2 + \lambda^2}
\]
where \( b \in \mathbb{R} \). Then
\[
|\psi(x, t)| \leq CT |\log(T)|^{b+2},
\tag{5.14}
\]
\[ |\psi(x,t) - \psi(x,T)| \leq C(T-t) \begin{cases} |\log(T)|^{b+2} & \text{if } b < -2 \\ |\log(T-t)|^{b+2} & \text{if } b > -2 \end{cases} \]  

(5.15)

**Lemma 5.4.** Let \( \psi \) solve (5.6) with \( f \) such that \[ |f(x,t)| \leq 1, \]

Then
\[ |\psi(x,t)| \leq Ct. \]
\[ |\psi(x,t) - \psi(x,T)| \leq C(T-t)|\log(T-t)|. \]
\[ |\nabla \psi(x,t)| \leq T^{1/2} \]
\[ |\nabla \psi(x,t_2) - \nabla \psi(x,t_1)| \leq C|t_2 - t_1|^{1/2}. \]

The proofs of Lemmas 5.2, 5.3 and 5.4 are similar to the one of Lemma 5.1 and we will omit them.

**Proof of Proposition 5.1.** We first show that the solution \( \psi \) of (5.1) with \( c_1 = c_2 = c_3 = 0 \) satisfies \( \|\psi\|_* \leq C\|f\|_* \). Given \( f \) with \( \|f\|_* < \infty \) we decompose \( f = \sum f_i \) with \( |f_i| \leq C\|f\|_* \theta_i \). By linearity it is sufficient to prove that when \( f \) is each of the \( \theta_i \), the corresponding \( \psi \) has finite \( \|\|_* \) norm.

The case \( f = \theta_1 \) is direct from Lemma 5.1.

Suppose now \( f = \theta_2 \) and let \( \psi \) denote the corresponding solution. Note that
\[ \varrho_2 = T^{-\sigma_0} (1 - \eta) \frac{\lambda_0}{r^2 + \lambda_0} \leq T^{-\sigma_0} \frac{\lambda_0^\beta}{r + \lambda_0}. \]  

(5.16)

By (5.12) we get
\[ R(0)^{a-1} \sup_{T \times (0,T)} |\nabla \psi| \leq CT^{-\sigma_0} \lambda_0(0) \beta(2-a) \log \lambda_0(0). \]  

(5.17)

Thanks to (5.16) again, and using (5.13) we find
\[ |\nabla \psi(x,t_2) - \nabla \psi(x,t_1)| \leq CT^{-\sigma_0} \lambda_0(t_2) \beta(t_2 - t_1)^{1/2}. \]

for any \( 0 \leq t_1 \leq t_2 \leq T \) such that \( t_2 - t_1 \leq \frac{1}{10}(T - t_2) \). Note that
\[ \lambda_0(t_2)^{\beta-1}(t_2 - t_1)^{1/2} \leq C\lambda_0(t_2)^{\beta-1}(T-t_2)^{1-\gamma}(t_2-t_1)^{\gamma} \]

and assuming \( \gamma < \frac{a\beta}{3-2a} \) (\( \gamma < \frac{1}{8} \) for \( a \approx 2, \beta \approx \frac{1}{4} \)) we obtain
\[ |\nabla \psi(x,t_2) - \nabla \psi(x,t_1)| \leq CT^{-\sigma_0} \lambda_0(t_2)^\sigma \lambda_0(t_2)^{2\gamma} R(t_2)^{a-1+2\gamma}(t_2-t_1)^\gamma \]  

(5.18)

for some \( \sigma > 0 \).

Since
\[ \varrho_2 \leq T^{-\sigma_0} \frac{|\log(T)| (T-t)}{|\log(T-t)|^2} \leq T^{-\sigma_0} \frac{(T-t)}{|\log(T-t)|^b} \leq T^{-\sigma_0} \frac{(T-t)}{|\log(T-t)|^b} \]  

(5.19)

with \( b = -2 + \sigma, \sigma > 0 \) small, we can use (5.15) and we get
\[ |\psi(x,t) - \psi(x,T)| \leq C|\log T|(T-t)|\log(T-t)|^\sigma \]
\[ \leq C\lambda_0(t) R(t)^{2-\sigma} \log(T-t). \]  

(5.20)

Using (5.19) and (5.14), with \( b = -2 + \sigma, \sigma > 0 \) small, we see that
\[ |\psi(x,t)| \leq CT^{1-\sigma_0} |\log T|^\sigma. \]  

(5.21)
From (5.17), (5.18), (5.20), (5.21) we see that
\[ \|\psi\|_* < +\infty. \]

The proof in the cases that \( f \) is given by the other \( \varrho_i \) is similar, using Lemmas 5.2, 5.3, and 5.4.

Finally, from the above estimates it is clear that we can choose \( c_i \) such that \( \psi(q,T) = 0 \). To do this we let \( \psi_0 \) denote the solution with all \( c_i = 0 \) and let \( \psi_i \) the solution with \( f = 0 \) and initial condition \( e_i \). Let
\[ \psi = \psi_0 + \sum c_i \psi_i \]
Then for \( T > 0 \) small there is unique choice of \( c_i \) such that \( \psi(q,T) = 0 \). Moreover \( |c_i| \leq C \lambda_0(0) R(0)^{2-a} |\log T| \|f\|_{**} \) and hence \( \psi \) satisfies (5.5). □

6. Estimates for the \( \lambda - \alpha \) system

Let \( a : [0,T] \to \mathbb{C} \) be a continuous function. In this section we want to solve approximately the system of equations

\[
2 \int_{t-s}^t \frac{1}{t-s} \left( \lambda(s) \cos(\alpha(t) - \alpha(s)) + \lambda \dot{\alpha}(s) \sin(\alpha(t) - \alpha(s)) \right) \Gamma_1 \left( \frac{\lambda(t)^2}{t-s} \right) ds
- 2\lambda = 2 \text{Re}(e^{-i\alpha(t)} a(t)) \tag{6.1}
\]

\[
2 \int_{t-s}^t \frac{1}{t-s} \left( -\lambda(s) \sin(\alpha(t) - \alpha(s)) + \lambda \dot{\alpha}(s) \cos(\alpha(t) - \alpha(s)) \right) \Gamma_2 \left( \frac{\lambda(t)^2}{t-s} \right) ds
= 2 \text{Im}(e^{-i\alpha(t)} a(t)) \tag{6.2}
\]

for \( t \in [0,T] \).

We recall that \( \Gamma_i \) are smooth on \([0,\infty)\), with the behavior
\[ \Gamma_i(0) = \frac{1}{2}, \quad \Gamma_i(x) = O(x^{-1}) \quad \text{as} \quad x \to \infty. \]

We will assume that \( a \) satisfies
\[
|a(t) - a(T)| \leq CR(t)^{1-a}, \tag{6.3}
\]
\[
|a(t) - a(s)| \leq C \frac{(t-s)^\gamma}{\lambda_0(t)^{2\gamma} R(t)^{a+2\gamma-1}} \tag{6.4}
\]

for \( s \leq t \leq T \), such that \( t - s \leq \frac{1}{10}(T-t) \), where \( \gamma \in (0,\frac{1}{2}) \) and \( a \in (1,2) \). We recall that \( R(t) = \lambda_0(t)^{-\beta} \), where
\[
\lambda_0(t) = |\log T| \frac{(T-t)}{|\log(T-t)|^2}, \quad t \leq T
\]

and \( \beta > \frac{1}{4}, \beta \approx \frac{1}{4} \). We also assume
\[
a(T) = \kappa_0 e^{i\gamma_0}, \quad \text{with} \quad \kappa_0 < 0, \quad |\gamma_0| \leq \frac{1}{2}. \tag{6.5}
\]

We will change the unknowns \( \lambda \) and \( \alpha \) a single complex valued function as follows:
\[
p(t) = \lambda(t) e^{i\alpha(t)}. \]
We rewrite (6.1) as
\[ 2 \Re \left( e^{-i \alpha(t)} \int_{-T}^{t} \frac{\dot{p}(s)}{t-s} \Gamma_1 \left( \frac{|p(t)|^2}{t-s} \right) ds \right) - 2 \frac{d}{dt} |p(t)| = 2 \Re(e^{-i \alpha} a(t)) \]
and (6.2) as
\[ 2 \Im \left( e^{-i \alpha(t)} \int_{-T}^{t} \frac{\dot{p}(s)}{t-s} \Gamma_2 \left( \frac{|p(t)|^2}{t-s} \right) ds \right) = 2 \Im(e^{-i \alpha} a(t)) \]

Let
\[ 2 \int_{-T}^{t} \frac{\dot{p}(s)}{t-s} \Gamma_i \left( \frac{|p(t)|^2}{t-s} \right) ds = \mathcal{B}_0[p](t) + \tilde{\mathcal{B}}_i[p](t) \]
where
\[ \mathcal{B}_0[g](t) = \int_{-T}^{t-\lambda_0(t)^2} \frac{g(s)}{t-s} ds \]
\[ \tilde{\mathcal{B}}_i[p](t) = \int_{-T}^{t-\lambda_0(t)^2} \frac{\dot{p}(s)}{t-s} \left( 2 \Gamma_i \left( \frac{|p(t)|^2}{t-s} \right) - 1 \right) ds + 2 \int_{t-\lambda_0(t)^2}^{t} \frac{\dot{p}(s)}{t-s} \Gamma_i \left( \frac{|p(t)|^2}{t-s} \right) ds. \]

Let
\[ \tilde{\mathcal{B}}[p](t) = \frac{\dot{p}(t)}{|p(t)|} \Re \left( \frac{\ddot{p}(t)}{|p(t)|} \tilde{\mathcal{B}}_1[p](t) \right) + i \frac{p(t)}{|p(t)|} \Im \left( \frac{\ddot{p}(t)}{|p(t)|} \tilde{\mathcal{B}}_2[p](t) \right). \]

Then the system (6.1), (6.2) is equivalent to
\[ \int_{-T}^{t-\lambda_0(t)^2} \frac{\dot{p}(s)}{t-s} ds - 2 \frac{d}{dt} |p(t)| + \tilde{\mathcal{B}}[p](t) = 2a(t). \]

Let us introduce some notation. We work with $\kappa \in \mathbb{C}$ and let $p_{0, \kappa}$ be the function
\[ p_{0, \kappa}(t) = \kappa \log T \int_{t}^{T} \frac{1}{|\log(T-s)|} ds, \quad t \leq T, \quad (6.7) \]
so that
\[ p_{0, \kappa}(t) = -\frac{\kappa |\log T|}{|\log(T-t)|^2}. \]

We will always consider
\[ \frac{1}{C_1} \leq |\kappa| \leq C_1 \quad (6.8) \]
where $C_1 > 0$ is a large fixed constant and therefore we have
\[ \frac{1}{C_1} \lambda_0 \leq |p_{0, \kappa}| \leq \tilde{C}_1 \lambda_0, \]
with $\tilde{C}_1 > 0$.

The main result of the section is the following.
Proposition 6.1. Assume $\beta > \frac{1}{2}$. Let $a(t)$ satisfy (6.3), (6.4) and (6.5). Then, for $T > 0$ small enough there is $p : [-T, T] \to \mathbb{C}$ of class $C^1$ such that

$$\left| \int_{-T}^{t-\lambda_0(t)^2} \frac{\tilde{p}(s)}{s} ds - 2 \frac{d}{dt} |p(t)| + \tilde{B}[p](t) - 2a(t) \right| \leq C(T-t)^\sigma_2 R(t)^{1-a} \left( \|a(\cdot) - a(T)\|_{\mu,t-1} + \|a(\cdot) - a(T)\|_{\gamma,m,t-1} \right),$$  

(6.9)

for some fixed $\sigma_2 > 0$. Moreover $p$ satisfies

$$|\dot{p}(t)| \leq C|\dot{\lambda}_0(t)|.$$

The proof of this proposition is given later in this section.

The norms in the RHS of (6.9) are defined as

$$\|g\|_{\mu,t} = \sup_{t \in [-T,T]} (T-t)^{-\mu} |\log(T-t)|^l |g(t)|,$$

(6.10)

where $g \in C([-T,T]; \mathbb{C})$ with $g(T) = 0$ and $\mu \in (0,1)$, $l \in \mathbb{R}$, and

$$\|f\|_{\gamma,m,t} = \sup(T-t)^{-m} |\log(T-t)|^l |f(t) - f(s)| > (t-s)^\gamma,$$

(6.11)

where the supremum is taken over $-T \leq s \leq t \leq T$ such that $t - s \leq \frac{1}{10}(T-t)$, $f \in C([-T,T]; \mathbb{C})$ such that $f(T) = 0$, and $m \in (0,1)$, $\mu \in (0,\infty)$, and $l \in \mathbb{R}$. In the RHS of (6.9) we use the following specific values

$$\mu = \beta(a-1), \quad m = -2\gamma + \beta(a+2\gamma-1),$$

while $\gamma$, $l$ have to satisfy

$$l \leq 2\beta(a-1) + 1 - 4\gamma(1-\beta), \quad 0 < \gamma < \frac{\beta(a-1)}{2(1-\beta)}.$$

(6.12)

The solution $p = p_\kappa + p_1$ in Proposition 6.1 is fully described by $\kappa$ and $p_1$. Moreover this proposition defines $(\kappa, p_1)$ as a nonlinear map of $a$.

We construct the function $p$ in this proposition by linearization, and the first approximation is a function $p_\kappa$ that deals with the case of constant $a$.

Lemma 6.1. Given $\kappa \in \mathbb{C}$, there is a function $p_\kappa$ such that for some constant $c(\kappa)$

$$\left| B_0[p_\kappa](t) - 2 \frac{d}{dt} |p_\kappa(t)| + \tilde{B}[p_\kappa](t) - c(\kappa) \right| \leq C(T-t)^{\delta_1},$$

(6.13)

for some $\delta_1 > \frac{1}{2}$ and a constant $c(\kappa)$, and

$$|\dot{p}_\kappa(t) - \dot{p}_0(\kappa(t))| \leq C_0 \frac{\log \left( T \right)^{k-1} \log(T)^2}{\log(T-t)^{k+1}},$$

where $k < 2$ can be taken close to 2. $C_0$ is independent of $T$, $t$ once $k$ is fixed.

The proof of this lemma is in appendix A.

To prove Proposition 6.1, we take $p$ of the form

$$p = p_\kappa + p_1,$$

where $p_\kappa$ is the function constructed in Lemma 6.1, for some $\kappa \in \mathbb{C}$ to be determined. The function $p_1(t)$ will have the property

$$p_1(t) = o(p_\kappa(t)),$$

as $t \to T$. 

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We would like that
\[ B_0[p_\kappa] + B_0[p_1] - \frac{2}{dt} |p_\kappa + p_1| + 1 \tilde{B}[p_\kappa + p_1](t) = 2a(t) + O((T - t)^\sigma R(t)^{1-\sigma}). \] (6.14)

Given \( \delta > 0 \), let us decompose
\[ B_0 = S_\delta + R_\delta \]
where \( S_\delta, R_\delta \) are defined by
\[
S_\delta[g](t) = g(t)(1 + \delta) \log(T - t) - 2 \log(\lambda_0(t)) + \int_{t-}\frac{g(s)}{t-s} ds
\]
\[
R_\delta[g](t) = -\int_{t-}^{t-}\frac{g(t) - g(s)}{t-s} ds.
\]
The idea is to replace \( B_0[p_1] \) by \( S_\delta[p_1] \) in (6.14), that is consider
\[ B_0[p_\kappa] + S_\delta[p_1] - \frac{2}{dt} |p_\kappa + p_1| + 1 \tilde{B}[p_\kappa + p_1](t) = 2a(t) \] (6.15)
in \( [0, T] \). If we find a solution to this problem, then we have obtained
\[ B_0[p_\kappa] + B_0[p_1] - \frac{2}{dt} |p_\kappa + p_1| + 1 \tilde{B}[p_\kappa + p_1] - 2a(t) = R_\delta[p_1] \]
in \( [0, T] \). We will later show that
\[ |R_\delta[p_1]| \leq C(T - t)^\sigma R(t)^{1-\sigma}. \] (6.16)

Let us split
\[ S_\delta[g] = L_0[g] + L_1[g] \]
where
\[
L_0[g] = (1 - \delta) \log(T - t)|g(t)
\]
\[
L_1[g] = 4 \log(|\log(T - t)|) - 2 \log(\kappa) - 2 \log(|\log(T)|)|g(t)
\]
\[
+ \int_{t-}^{t-}\frac{g(s)}{t-s} ds.
\]
Let \( \eta \) be a smooth cut-off function such that
\[ \eta(s) = 1 \quad \text{for} \quad s \geq 0, \quad \eta(s) = 0 \quad \text{for} \quad s \leq -\frac{1}{4}. \] (6.18)

We consider then the modified linear equation
\[ L_0[g] + \eta\left(\frac{t}{T}\right)L_1[g] = f + c \quad \text{in} \quad [-T, T]. \] (6.19)

Here \( c = L_1[g](T) \) and has to be there because all other terms in the equation vanish at \( T \). Thanks to this cut-off function, we need only to consider the values of \( L_1[g](t) \) for \( t \geq -\frac{T}{4} \). Then in the definition of \( L_1[g], t - (T - t)^{1+\delta} \geq t - \frac{1}{2}(T - t) \geq -T \) of \( T > 0 \) is small.

Given a continuous function \( f \) in \([-T, T]\) with a certain modulus of continuity at \( T \), we would like to solve (6.19). We do this in the following weighted spaces. Let \( \mu \in (0, 1), \: \nu \in \mathbb{R} \). We consider \( g \) in the space \( C([-T, T]; \mathbb{C}) \cap C^1([-T, T]; \mathbb{C}) \) with \( g(T) = 0 \) and the norm \( \|g\|_{\mu, l} \) defined in (6.10). For the right hand side of (6.19)
we take the space $C([-T, T]; \mathbb{C})$ with $f(T) = 0$ and the norm $\|f\|_{\mu, l-1}$. The next lemma asserts the solvability of (6.19) in the weighted spaces introduced above.

**Lemma 6.2.** Let $\delta \in (0, \frac{1}{2})$ and $T > 0$ be sufficiently small. Assume $\|f\|_{\mu, l-1} < \infty$ where $\mu \in (0, 1)$, $l \in \mathbb{R}$. Then for $T > 0$ small there is a solution $g$ of (6.19) that defines a linear operator of $f$ and such that

$$\|g\|_{\mu, l} \leq C\|f\|_{\mu, l-1}.$$  \hfill (6.20)

**Proof.** We invert $L_0$ directly by the formula

$$L_0^{-1}[f](t) = \frac{f(t)}{(1 - \delta) \log(T - t)},$$  \hfill (6.21)

and construct a solution of (6.19) as a fixed point of

$$g = L_0^{-1}\left[f - \eta(T) (L_1[g](t) - L_1[g](T))\right].$$  \hfill (6.22)

It is clear that

$$\|L_0^{-1}[f]\|_{\mu, l} \leq \frac{1}{1 - \delta} \|f\|_{\mu, l-1}.$$  \hfill (6.23)

We claim that

$$\|L_1[g](\cdot) - L_1[g](T)\|_{\mu, l-1} \leq (\delta + \frac{C\log |\log T|}{|\log T|})\|g\|_{\mu, l}.$$  \hfill (6.24)

Indeed consider the term

$$|(4 \log(|\log(T - t)|) - 2 \log(\kappa) - 2 \log(|\log(T)|))g(t)| \leq C \log |\log(T - t)| \frac{(T - t)^\mu}{|\log(T - t)|}\|g\|_{\mu, l},$$

and this gives

$$\|(4 \log(|\log(T - t)|) - 2 \log(\kappa) - 2 \log(|\log(T)|))g(t)\|_{\mu, l-1} \leq \frac{C\log |\log T|}{|\log T|}\|g\|_{\mu, l}.$$  

To estimate the integral term we decompose

$$\int_{-T}^{T-(T-t)^{1+\delta}} \frac{g(s)}{t-s} ds - \int_{-T}^{T} \frac{g(s)}{T-s} ds = I_1 + I_2 + I_3$$

where

$$I_1 = \int_{-T-(T-t)^{1/2}}^{T-(T-t)^{1/2}} \frac{g(s)}{t-s} ds$$

$$I_2 = \int_{-T}^{T-(T-t)/2} \frac{g(s)}{t-s} \left(\frac{1}{t-s} - \frac{1}{T-s}\right) ds$$

$$I_3 = \int_{T-(T-t)/2}^{T} \frac{g(s)}{T-s} ds.$$
Then
\[ |I_1| \leq \|g\|_{\mu,t} \int_{t-(T-t)^{\frac{1}{2}}}^{t-(T-t)^{\frac{1}{2}}} \frac{(T-s)^{\mu}}{|\log(T-s)|^{\frac{3}{2}}(t-s)} \, ds \]
\[ = \|g\|_{\mu,t} \int_{(T-t)^{1+\delta}}^{(T-t)^{1+\delta}} \frac{(T-t+r)^{\mu}}{|\log(T-t+r)|^{\frac{3}{2}}r} \, dr \]
\[ = \|g\|_{\mu,t} \frac{(T-t)^{\mu}}{|\log(T-t)|^{\frac{3}{2}}} \int_{(T-t)^{1+\delta}}^{1+O(\frac{r}{T-T})} \frac{1}{r} \, dr \]
\[ \leq \|g\|_{\mu,t} \frac{(T-t)^{\mu}}{|\log(T-t)|^{\frac{3}{2}}} (\delta |\log(T-t)| + C). \]

\[ |I_2| \leq \|g\|_{\mu,t} \frac{(T-t)^{\mu}}{|\log(T-t)|^{\frac{3}{2}}} \int_{T-t}^{t-(T-t)^{\frac{1}{2}}} \frac{(T-s)^{\mu}}{|\log(T-s)|^{\frac{3}{2}}(T-s-t)} \, ds \]
\[ \leq C\|g\|_{\mu,t} \frac{(T-t)^{\mu}}{|\log(T-t)|^{\frac{3}{2}}}. \]

and
\[ |I_3| \leq \|g\|_{\mu,t} \int_{t-(T-t)^{\frac{1}{2}}}^{T} \frac{(T-s)^{\mu-1}}{|\log(T_s)|^{\frac{3}{2}}} \, ds \]
\[ \leq C\|g\|_{\mu,t} \frac{(T-t)^{\mu}}{|\log(T-t)|^{\frac{3}{2}}}. \]

These estimates imply (6.24). Then this inequality combined with (6.23) shows that
\[ \left\| L_0^{-1} \left[ \frac{t}{T} \left( L_1[g](t) - L_1[g](T) \right) \right] \right\|_{\mu,t} \leq \frac{1}{1 - \delta (\delta + C \frac{\log|\log T|}{|\log T|})} \left\| g \right\|_{\mu,t}. \]

Then for \( \delta \in (0, \frac{1}{2}) \) and \( T > 0 \) sufficiently small this operator is a contraction and we obtain the conclusion of the lemma.

The next step is to use the linear operator just constructed to find a solution to a small modification of (6.15). For this modification it is convenient that \( a \) is defined in \([-T,T]\). So Consider the given function \( a : [-T,T] \rightarrow \mathbb{C} \) satisfying (6.3), (6.4) and extend it continuously by constant for \( t \leq 0 \).

**Lemma 6.3.** Let \( \delta \in (0, \frac{1}{2}) \) and \( T > 0 \) be sufficiently small. Then for any \( \kappa \in \mathbb{C} \) satisfying (6.8) there is a solution \( p_1 \) to
\[ B_0[p_\kappa] + L_0[p_1] + \eta \left( \frac{t}{T} \right) L_1[p_1] - 2 \frac{d}{dt} [p_\kappa + p_1] + \tilde{B} [p_\kappa + p_1](t) = 2a(t) + c \tag{6.25} \]
in \([-T,T]\) for some \( c \in \mathbb{C} \). Moreover this solution satisfies
\[ \|p_1\|_{\mu,t} \leq C |\log T|^{(\frac{3}{2} - \beta(a-1))} \]
for some constant \( C \), where
\[ \mu = \beta(a - 1), \quad l \leq 2\beta(a - 1) + 1. \tag{6.26} \]

**Proof.** Let \( S \) denote the linear operator constructed in Lemma 6.2. Let
\[ E_\kappa(t) = B_0[p_\kappa](t) - 2 \frac{d}{dt} [p_\kappa(t)] + \tilde{B}[p](t) - c(t). \]

Then
\[ |E_\kappa(t)| \leq C |\log T| \int_{T-t}^{t-(T-t)^{\frac{1}{2}}} \frac{(T-s)^{\mu}}{|\log(T-s)|^{\frac{3}{2}}(t-s)} \, ds \]
\[ \leq C \|g\|_{\mu,t} \frac{(T-t)^{\mu}}{|\log(T-t)|^{\frac{3}{2}}} \int_{(T-t)^{1+\delta}}^{1+O(\frac{r}{T-T})} \frac{1}{r} \, dr \]
\[ \leq C \|g\|_{\mu,t} \frac{(T-t)^{\mu}}{|\log(T-t)|^{\frac{3}{2}}} \delta |\log(T-t)| + C \]

This implies that
\[ \left\| L_0^{-1} \left[ \frac{t}{T} \left( L_1[g](t) - L_1[g](T) \right) \right] \right\|_{\mu,t} \leq \frac{1}{1 - \delta (\delta + C \frac{\log|\log T|}{|\log T|})} \left\| g \right\|_{\mu,t}. \]
Then to find a solution to (6.25) it is sufficient to find a solution \( p_1 \) of the fixed point problem

\[
\dot{p}_1 = A[p_1]
\]

where \( \dot{p} = A[p_1] \) is defined by \( \dot{p}(T) = 0 \) and

\[
\frac{d}{dt} \dot{p} = S \left[ -E_\kappa + 2 \frac{d}{dt}(|p_\kappa + p_1| - |p_\kappa|) - \tilde{B}[p_\kappa + p_1](t) + \tilde{B}[p_\kappa](t) + 2(a(t) - a(T)) \right].
\]

The solution \( p_1 \) we look for is much smaller than \( p_\kappa \) and then it makes sense to solve this problem by the contraction mapping theorem.

We note that assumption (6.3) implies that choosing \( \mu \) and \( l \) as in (6.26) we have

\[
\|a(\cdot) - a(T)\|_{\mu,l^{-1}} \leq C|\log T|^{\beta(a-1)}, \quad (6.27)
\]

and Lemma 6.1 gives

\[
\|E_\kappa\|_{\mu,l^{-1}} \leq C T^{\delta_1 - \mu} |\log T|^{\delta_1 - \mu}. \quad (6.28)
\]

Let \( M_1 = C_0|\log T|^{\beta(a-1)} \) where \( C_0 \) is a sufficiently large fixed constant. We claim that \( A \) is a contraction in ball \( B_{M_1} \) of the space of complex valued functions \( p_1 \in C^1([-T,T]) \) with \( p_1(T) = 0 \) and with the norm \( \|p_1\|_{\mu,l} \). Note that with this norm

\[
|p_1(t)| \leq C\|\dot{p}_1\|_{\mu,l} |(T-t)^{a+1}| \log(T-t)|^l.
\]

Let us estimate, for \( p_1, p_2 \in B_{M_1} \), the quantity

\[
\left\| \frac{d}{dt} |p_\kappa + p_1| - \frac{d}{dt} |p_\kappa + p_2| \right\|_{\mu,l^{-1}}.
\]

For these estimates it is useful to notice that with the choice of \( M_1 \), if \( \|p_1\|_{\mu,l} \leq M_1 \) we have

\[
\left| \frac{1}{\lambda_0} \right| + \left| \frac{\dot{p}_1}{\lambda_0} \right| << 1
\]

for \( T > 0 \) small.

Then we compute

\[
\frac{d}{dt} |p_\kappa + p_1| - \frac{d}{dt} |p_\kappa + p_2| = \left( \frac{(p_\kappa + p_1) \cdot (\dot{p}_\kappa + \dot{p}_1)}{|p_\kappa + p_1|} - \frac{(p_\kappa + p_2) \cdot (\dot{p}_\kappa + \dot{p}_2)}{|p_\kappa + p_2|} \right)
\]

\[
= (p_1 - p_2) \cdot \int_0^1 \frac{\dot{p}_\kappa + \dot{p}_\zeta}{|p_\kappa + p_\zeta|} d\zeta + (\dot{p}_1 - \dot{p}_2) \cdot \int_0^1 \frac{p_\kappa + p_\zeta}{|p_\kappa + p_\zeta|} d\zeta
\]

\[
- (p_1 - p_2) \cdot \int_0^1 (p_\kappa + p_\zeta) \frac{(p_\kappa + p_\zeta) \cdot (\dot{p}_\kappa + \dot{p}_\zeta)}{|p_\kappa + p_\zeta|^3} d\zeta
\]
where \( \mathbf{p}_\zeta = \zeta \mathbf{p}_1 + (1 - \zeta) \mathbf{p}_2 \). We then have

\[
\left| (\mathbf{p}_1 - \mathbf{p}_2) \cdot \frac{\delta \mathbf{p}_\zeta + \hat{D}_\zeta}{|\mathbf{p}_\zeta + \mathbf{p}_\zeta|} \right| \leq C \frac{\lambda_0(t) (T - t)^{1+\mu}}{\lambda_0(t) |\log(T - t)|^\mu} \|\mathbf{p}_1 - \mathbf{p}_2\|_{\mu,t} \\
\leq C \frac{(T - t)^{\mu}}{\log(T - t)^t} \|\mathbf{p}_1 - \mathbf{p}_2\|_{\mu,t}.
\]

With similar estimates for the other terms we obtain

\[
\left\| \frac{d}{dt} \mathbf{p}_\zeta + \mathbf{p}_1 - \frac{d}{dt} \mathbf{p}_\zeta + \mathbf{p}_2 \right\|_{\mu,t-1} \leq C \frac{1}{|\log T|} \|\mathbf{p}_1 - \mathbf{p}_2\|_{\mu,t}. \tag{6.29}
\]

Next we estimate

\[
\|\mathbf{B}[\mathbf{p}_\zeta + \mathbf{p}_1] - \mathbf{B}[\mathbf{p}_\zeta + \mathbf{p}_2]\|_{\mu,t-1}
\]

for \( \mathbf{p}_1, \mathbf{p}_2 \in \mathbb{T}_{M_1} \). The computation here is analogous to that of Lemma A.7. Let

\[
D_{i,a} = \frac{(\mathbf{p}_\zeta + \mathbf{p}_1)(t)}{|(\mathbf{p}_\zeta + \mathbf{p}_1)(t)|} \Re \left( \frac{(\mathbf{p}_\zeta + \mathbf{p}_1)(t)}{|(\mathbf{p}_\zeta + \mathbf{p}_1)(t)|} \mathbf{B}_{i,a}[\mathbf{p}_\zeta + \mathbf{p}_1](t) \right)
\]

\[
- \frac{(\mathbf{p}_\zeta + \mathbf{p}_2)(t)}{|(\mathbf{p}_\zeta + \mathbf{p}_2)(t)|} \Re \left( \frac{(\mathbf{p}_\zeta + \mathbf{p}_2)(t)}{|(\mathbf{p}_\zeta + \mathbf{p}_2)(t)|} \mathbf{B}_{i,a}[\mathbf{p}_\zeta + \mathbf{p}_2](t) \right),
\]

where \( \mathbf{B}_{i,a} \) is defined in (A.16). We write

\[
D_{i,a} = \int_0^1 \frac{d}{d\zeta} \left[ \frac{(\mathbf{p}_\zeta + \mathbf{p}_\zeta)(t)}{|(\mathbf{p}_\zeta + \mathbf{p}_\zeta)(t)|} \Re \left( \frac{(\mathbf{p}_\zeta + \mathbf{p}_\zeta)(t)}{|(\mathbf{p}_\zeta + \mathbf{p}_\zeta)(t)|} \mathbf{B}_{i,a}[\mathbf{p}_\zeta + \mathbf{p}_\zeta](t) \right) \right] d\zeta
\]

where \( \mathbf{p}_\zeta = \zeta \mathbf{p}_1 + (1 - \zeta) \mathbf{p}_2 \). Let us analyze the terms in this expression. For this we note that

\[
\left| \frac{d}{d\zeta} \left[ \frac{(\mathbf{p}_\zeta + \mathbf{p}_\zeta)(t)}{|(\mathbf{p}_\zeta + \mathbf{p}_\zeta)(t)|} \right] \right| = \left| \frac{(\mathbf{p}_1(t) - \mathbf{p}_2(t))}{|(\mathbf{p}_\zeta + \mathbf{p}_\zeta)(t)|} \frac{(\mathbf{p}_\zeta + \mathbf{p}_\zeta)(t) \cdot (\mathbf{p}_1(t) - \mathbf{p}_2(t))}{|(|\mathbf{p}_\zeta + \mathbf{p}_\zeta)(t)|^3}} \right| \leq 2 \|\mathbf{p}_1(t) - \mathbf{p}_2(t)| \|_{|\mathbf{p}_\zeta + \mathbf{p}_\zeta)(t)|.}
\]

Using \( A.18 \) we get

\[
\left| \Re \left( \frac{(\mathbf{p}_\zeta + \mathbf{p}_\zeta)(t)}{|(\mathbf{p}_\zeta + \mathbf{p}_\zeta)(t)|} \mathbf{B}_{i,a}[\mathbf{p}_\zeta + \mathbf{p}_\zeta](t) \right) \right| \leq C \left| \frac{(\mathbf{p}_1(t) - \mathbf{p}_2(t))}{|(\mathbf{p}_\zeta + \mathbf{p}_\zeta)(t)|} \frac{|\log T|}{|\log(T - t)|^2} \right| \leq C \frac{(T - t)^\mu}{|\log(T - t)|^t} \|\mathbf{p}_1 - \mathbf{p}_2\|_{\mu,t}.
\]
Let us consider

\[
\frac{d}{dt} \tilde{B}_{i,a} [p_\kappa + p_\zeta] (t) = \int_{-T}^{t-\lambda_0(t)^2} \frac{[p_\kappa - \hat{p}_2(s)]^2}{t-s} \left( 2 \Gamma_i \left( \frac{[p_\kappa + p_\zeta(t)]^2}{t-s} \right) - 1 \right) ds + 4 \langle p_\kappa(t) + p_\zeta(t) \rangle \cdot (p_1(t) - p_2(t))
\]

\[. \int_{-T}^{t-\lambda_0(t)^2} \frac{[\hat{p}_\kappa + \hat{p}_\zeta(s)]^2}{(t-s)^2} \Gamma_i \left( \frac{[p_\kappa + p_\zeta(t)]^2}{t-s} \right) ds. \]

We estimate the first term above

\[\left| \int_{-T}^{t-\lambda_0(t)^2} \frac{[p_\kappa - \hat{p}_2(s)]^2}{t-s} \left( 2 \Gamma_i \left( \frac{[p_\kappa + p_\zeta(t)]^2}{t-s} \right) - 1 \right) ds \right| \leq C \int_{-T}^{t-\lambda_0(t)^2} \frac{[p_\kappa - \hat{p}_2(s)]^2}{(t-s)^2} \left( \frac{[p_\kappa + p_\zeta(t)]^2}{t-s} \right) ds \]

\[\leq C \| p_1 - p_2 \|_{\mu,t} \lambda_0(t)^2 \int_{-T}^{t-\lambda_0(t)^2} \frac{(T-s)^\mu}{(t-s)^2 |\log(T-s)|} ds\]

and by Lemma A.5

\[\int_{-T}^{t-\lambda_0(t)^2} \frac{(T-s)^\mu}{(t-s)^2 |\log(T-s)|} ds \leq \frac{C (T-t)^\mu}{\lambda_0(t)^2 |\log(T-t)|}.\]

Therefore

\[\left| \int_{-T}^{t-\lambda_0(t)^2} \frac{[p_\kappa - \hat{p}_2(s)]^2}{t-s} \left( 2 \Gamma_i \left( \frac{[p_\kappa + p_\zeta(t)]^2}{t-s} \right) - 1 \right) ds \right| \leq C \frac{C (T-t)^\mu}{|\log(T-t)|} \| p_1 - p_2 \|_{\mu,t}.\]

For the second term in (6.30) we compute

\[\left| (p_\kappa(t) + p_\zeta(t)) \cdot (p_1(t) - p_2(t)) \right| \int_{-T}^{t-\lambda_0(t)^2} \frac{[p_\kappa + p_\zeta(s)]^2}{(t-s)^2} \Gamma_i \left( \frac{[p_\kappa + p_\zeta(t)]^2}{t-s} \right) ds \]

\[\leq C \lambda_0(t) \| p_1 - p_2 \|_{\mu,t} \int_{-T}^{t-\lambda_0(t)^2} \frac{(T-t)^{1+\mu}}{|\log(T-t)|} \frac{\lambda_0(s)}{(t-s)^2} ds \]

\[\leq C \frac{(T-t)^\mu}{|\log(T-t)|} \| p_1 - p_2 \|_{\mu,t}.\]

Thus we have proved that

\[| D_{i,a} | \leq C \frac{(T-t)^\mu}{|\log(T-t)|} \| p_1 \|_{\mu,t}. \]

The estimate of \( D_{i,b} \) defined by

\[D_{i,b} = \frac{(p_\kappa + p_2)(t)}{|p_\kappa + p_2|(t)} \text{Re} \left( \frac{(p_\kappa + p_2)(t)}{|p_\kappa + p_2|(t)} \tilde{B}_{i,b} [p_\kappa + p_2] (t) \right) \]

\[\quad - \frac{(p_\kappa + p_2)(t)}{|p_\kappa + p_2|(t)} \text{Re} \left( \frac{(p_\kappa + p_2)(t)}{|p_\kappa + p_2|(t)} \tilde{B}_{i,b} [p_\kappa + p_2] (t) \right),\]
is very similar, the only difference appears in
\[ \frac{d}{dt} B_{i,b}^T \left( |p_\kappa + p_\zeta| \right) \]
\[ = 2 \int_{t-\lambda_0(t)^2}^{t} \frac{p_1(s) - p_2(s)}{t-s} \Gamma_i \left( \frac{|p_\kappa + p_\zeta(t)|^2}{t-s} \right) ds \]
\[ + 4(p_\kappa(t) + p_\zeta(t)) \cdot (p_1(t) - p_2(t)) \int_{t-\lambda_0(t)^2}^{t} \frac{\dot{p}_1 + \dot{p}_2(s)}{(t-s)^2} \Gamma_i \left( \frac{|p_\kappa + p_\zeta(t)|^2}{t-s} \right) ds. \]

We estimate the first term above
\[ \left| \int_{t-\lambda_0(t)^2}^{t} \frac{\dot{p}_1(s) - \dot{p}_2(s)}{t-s} \Gamma_i \left( \frac{|p_\kappa + p_\zeta(t)|^2}{t-s} \right) ds \right| \]
\[ \leq \frac{C}{|p_\kappa + p_\zeta(t)^2|} \int_{t-\lambda_0(t)^2}^{t} |\dot{p}_1(s) - \dot{p}_2(s)| ds \]
\[ \leq C \frac{(T-t)^\mu}{|\log(T-t)|^l} \|p_1 - p_2\|_{\mu,l}. \]

The second term is estimated by
\[ \left| (p_\kappa(t) + p_\zeta(t)) \cdot (p_1(t) - p_2(t)) \int_{t-\lambda_0(t)^2}^{t} \frac{\dot{p}_1 + \dot{p}_2(s)}{(t-s)^2} \Gamma_i \left( \frac{|p_\kappa + p_\zeta(t)|^2}{t-s} \right) ds \right| \]
\[ \leq C\lambda_0(t) \frac{(T-t)^{1+\mu}}{|\log(T-t)|^l} \|p_1 - p_2\|_{\mu,l} \frac{1}{\lambda_0(t)^4} \int_{t-\lambda_0(t)^2}^{t} \frac{|\log T|}{|\log(T-s)|^2} ds \]
\[ \leq C \frac{(T-t)^\mu}{|\log(T-t)|^l} \|p_1 - p_2\|_{\mu,l}. \]

We conclude that
\[ \|D_{i,b}\| \leq C \frac{(T-t)^\mu}{|\log(T-t)|^l} \|p_1 - p_2\|_{\mu,l}, \]
and this combined with (6.31) gives
\[ \|\tilde{B}[p_\kappa + p_1] - \tilde{B}[p_\kappa + p_2]\|_{\mu,l-1} \leq C \frac{1}{|\log T|} \|p_1 - p_2\|_{\mu,l}. \]

Let us verify that \( A \) maps \( \overline{B}_{M_1} \) into itself. Let \( p_1 \in \overline{B}_{M_1} \). By (6.20), (6.27), (6.28), (6.29), and (6.32)
\[ \|A[p_1]\|_{\mu,l} \leq C(\|E_\kappa\|_{\mu,l-1} + \frac{d}{dt}(\|p_\kappa + p_1| - |p_\kappa|))_{\mu,l-1} \]
\[ + \|\tilde{B}[p_\kappa + p_1] - \tilde{B}[p_\kappa]\|_{\mu,l-1} + \|a(\cdot) - a(T)\|_{\mu,l-1} \]
\[ \leq C(\|\log T\|^{\beta(a-1)} + T^{\delta_1 - \mu}|\log T|^l + \frac{M_1}{|\log T|}) \]
\[ \leq M_1. \]

Also thanks to (6.20), (6.29), and (6.32) we see that \( A \) is a contraction in \( \overline{B}_{M_1} \). This finishes the proof. \qed

To be able to obtain estimate (6.16), we need to show that the solution constructed in Lemma 6.3 has some Hölder regularity inherited from the one of \( a \), see (6.4). We then have the following result.
Lemma 6.4. Let $\delta \in (0, \frac{1}{2})$. Let $a$ satisfy (6.3), (6.4). Let $\mu, l$ satisfy (6.26).
Assume $\gamma$ in (6.4) is such that $0 < \gamma < \frac{\beta(a-1)}{2(1-\beta)}$.

Then the solution $\mathbf{p}_1$ constructed in Lemma 6.3 satisfies
$$\|\mathbf{p}_1\|_{\gamma,m,l} \leq C(\log T)^{\beta(a-1)} + |\log T|^{l-1-\beta(a-1)-2\beta\gamma}).$$
where $m = -2\gamma + \beta(a + 2\gamma - 1)$ and $l < 1 + 2\beta(a - 1) - 4\gamma(1 - \beta)$.

For the proof we need an estimate for the operator $S$ constructed in Lemma 6.2.

Lemma 6.5. Let $S$ denote the linear operator constructed in Lemma 6.2. Assume
$\mu, m, \gamma \in (0, 1)$, $m \leq \mu - \gamma$, $l \in \mathbb{R}$. Then $S$ satisfies
$$\|S(f)\|_{\gamma,m,l} \leq C(\|f\|_{\mu,l-1} + \|f\|_{\gamma,m,l-1}).$$

Proof. The proof uses the fixed point characterization (6.22) of the operator and the mapping properties of $L_0$ and $L_1$ with respect to the Hölder norms $\|\cdot\|_{\gamma,m,l}$.

The operator $L_0^{-1}$ defined by (6.21) satisfies: if $m < \mu - \gamma$ and any $l, l'$, or if $m = \mu - \gamma$ and $l' \geq l - 1$, then
$$\|L_0^{-1}[f]\|_{\gamma,m,l+1} \leq \frac{1}{1 - \delta} \|f\|_{\gamma,m,l} + C\|f\|_{\mu,l'}.$$ (6.33)

Indeed, writing $g(t) = \frac{f(t)}{\log(T-t)}$ we have, for $-T \leq s \leq t \leq T$ such that $t - s \leq \frac{1}{10}(T - t)$:
$$|g(t) - g(s)|$$
$$\leq \left| \frac{f(t) - f(s)}{\log(T-t)} \right| + \left| f(s) \right| \left| \frac{1}{\log(T-t)} \right| - \left| \frac{1}{\log(T-s)} \right|$$
$$\leq \|f\|_{\gamma,m,l}(t - s)^\gamma \left( \frac{1}{\log(T-t)} \right)^m + C\|f\|_{\mu,l'} \left( \frac{1}{\log(T-t)} \right)^{m + 1}$$
$$\leq \|f\|_{\gamma,m,l}(t - s)^\gamma \left( \frac{1}{\log(T-t)} \right)^m + C\|f\|_{\mu,l'} \left( \frac{1}{\log(T-t)} \right)^m$$
$$\leq \|f\|_{\gamma,m,l}(t - s)^\gamma \left( \frac{1}{\log(T-t)} \right)^m + C\|f\|_{\mu,l'} \left( \frac{1}{\log(T-t)} \right)^m$$
$$\leq \|f\|_{\gamma,m,l}(t - s)^\gamma \left( \frac{1}{\log(T-t)} \right)^m + C\|f\|_{\mu,l'} \left( \frac{1}{\log(T-t)} \right)^m$$
$$\leq \|f\|_{\gamma,m,l}(t - s)^\gamma \left( \frac{1}{\log(T-t)} \right)^m + C\|f\|_{\mu,l'} \left( \frac{1}{\log(T-t)} \right)^m$$

This proves (6.33).

Let $L_1$ be defined defined in (6.17) and
$$\tilde{L}_1[g](t) = \eta(\frac{t}{T})(L_1[g](t) - L_1[g](T)),$$
where $\eta$ is defined in (6.18). Then we claim that if $m \leq \mu - \gamma$ then
$$\|\tilde{L}_1[g]\|_{\gamma,m,l-1} \leq \left( \delta + C \frac{\log |\log T|}{|\log T|} \right) (\|g\|_{\gamma,m,l} + \|g\|_{\mu,l}).$$ (6.34)

Let
$$-T \leq t_1 < t_2 < T, \quad t_2 - t_1 \leq \frac{T - t_2}{10}$$ (6.35)
and then note that
$$\tilde{L}_1[g](t_2) - \tilde{L}_1[g](t_1) = h_1 + h_2$$
We estimate

\[ h_1 = \left( \eta \left( \frac{t_2}{T} \right) - \eta \left( \frac{t_1}{T} \right) \right) \left( L_1[g](t_2) - L_1[g](T) \right) \]

\[ h_2 = \eta \left( \frac{t_1}{T} \right) (L_1[g](t_2) - L_1[g](t_1)) \]

Then

\[ |h_1| \leq C \frac{t_2 - t_1}{T} |L_1[g](t_2) - L_1[g](T)| \]

and by (6.24)

\[ |h_1| \leq C \frac{t_2 - t_1}{T} \frac{(T - t_2)^\mu}{|\log(T - t_2)|^{\alpha - 1}} \left( \delta + C \frac{\log |\log T|}{|\log T|} \right) \|g\|_{\mu, l} \]

\[ \leq C (t_2 - t_1)^\gamma \frac{(T - t_2)^{\mu - \gamma}}{|\log(T - t_2)|^{\alpha - 1}} \left( \delta + C \frac{\log |\log T|}{|\log T|} \right) \|g\|_{\mu, l} \]

To estimate \( h_2 \) we only need to consider \( t_1 \geq -\frac{T}{4} \) because of the cut-off function. It is convenient to split

\[ L_1 = L_{11} + L_{12} \]

where

\[ L_{11}[g](t) = (4 \log(|\log(T - t)|) - 2 \log(\kappa) - 2 \log(|\log(T)|))g(t) \]

\[ L_{12}[g](t) = \int_{-T}^{t} \frac{g(s)}{t - s} ds. \]

Then

\[ L_{12}[g](t_2) - L_{12}[g](t_1) = \int_{(T-t_2)^{1+\delta}}^{(T-t_1)^{1+\delta}} \frac{g(t_2 - r)}{r} dr \]

\[ + \int_{(T-t_1)^{1+\delta}}^{(T-t_2)/2} \frac{g(t_2 - r) - g(t_1 - r)}{r} dr \]

\[ + \int_{(T-t_2)/2}^{(T-t_1)^{1/2}} \frac{g(t_1 - r)}{r} dr. \]

Note that assuming \( T > 0 \) small and (6.35) we have that \( (T - t_1)^{1+\delta} \leq (T - t_2)/2 \). We estimate

\[ \left| \int_{(T-t_2)^{1+\delta}}^{(T-t_1)^{1+\delta}} \frac{g(t_2 - r)}{r} dr \right| \]

\[ \leq \|g\|_{\mu, l} \int_{(T-t_2)^{1+\delta}}^{(T-t_1)^{1+\delta}} \frac{(T - t_2 + r)^\mu}{|\log(T - t_2 + r)|^{\alpha - 1}} \frac{1}{r} dr. \]

But

\[ \int_{(T-t_2)^{1+\delta}}^{(T-t_1)^{1+\delta}} \frac{(T - t_2 + r)^\mu}{|\log(T - t_2 + r)|^{\alpha - 1}} \frac{1}{r} dr \leq C \frac{(T - t_2)^\mu}{|\log(T - t_2)|^{\alpha - 1}} \frac{t_2 - t_1}{T - t_2} \]

\[ \leq C (t_2 - t_1)^\gamma \frac{(T - t_2)^{\mu - \gamma}}{|\log(T - t_2)|^{\alpha - 1}} \]

\[ \leq C \frac{(t_2 - t_1)^\gamma}{|\log T|} \frac{(T - t_2)^{\mu - \gamma}}{|\log(T - t_2)|^{\alpha - 1}}. \]
We estimate the second term:
\[ \left| \int_{(T-t_1)^{1+\delta}}^{(T-t_2)^{1/2}} \frac{g(t_2 - r) - g(t_1 - r)}{r} \, dr \right| \]
\[ \leq \| g \|_{\gamma,m,l} \int_{(T-t_1)^{1+\delta}}^{(T-t_2)^{1/2}} \frac{(T - t_2 + r)^\mu}{\log(T - t_2 + r)} \, \frac{(t_2 - t_1)^\gamma}{r} \, dr \]
\[ = \| g \|_{\gamma,m,l} (t_2 - t_1)^\gamma \int_{(T-t_1)^{1+\delta}}^{(T-t_2)^{1/2}} \frac{(T - t_2)^\mu}{\log(T - t_2)} \] \[ \cdot \int_{(T-t_1)^{1+\delta}}^{(T-t_2)^{1/2}} \left| 1 + \frac{\log(1 + \frac{r}{T - t_2})}{\log(T - t_2)} \right|^{-l} \left( 1 + \frac{r}{T - t_2} \right)^\mu \frac{1}{r} \, dr, \]
and we estimate
\[ \int_{(T-t_1)^{1+\delta}}^{(T-t_2)^{1/2}} \left| 1 + \frac{\log(1 + \frac{r}{T - t_2})}{\log(T - t_2)} \right|^{-l} \left( 1 + \frac{r}{T - t_2} \right)^\mu \frac{1}{r} \, dr \]
\[ = \int_{(T-t_1)^{1+\delta}}^{(T-t_2)^{1/2}} \frac{1}{r} \left( 1 + O(\frac{r}{T - t_2}) \right) \, dr \]
\[ \leq \delta |\log(T - t_2)| + C. \]

With this we deduce
\[ \left| \int_{(T-t_1)^{1+\delta}}^{(T-t_2)^{1/2}} \frac{g(t_2 - r) - g(t_1 - r)}{r} \, dr \right| \]
\[ \leq \| g \|_{\gamma,m,l} (t_2 - t_1)^\gamma \int_{(T-t_1)^{1+\delta}}^{(T-t_2)^{1/2}} \frac{(T - t_2)^\mu}{\log(T - t_2)} \left( \delta |\log(T - t_2)| + C \right) \]
\[ \leq \| g \|_{\gamma,m,l} (t_2 - t_1)^\gamma \int_{(T-t_1)^{1+\delta}}^{(T-t_2)^{1/2}} \frac{(T - t_2)^\mu}{\log(T - t_2)} \] \[ \left( \delta + \frac{C}{|\log T|} \right). \]

For the third term in (6.36) we compute
\[ \left| \int_{(T-t_1)^{1/2}}^{(T-t_2)^{1/2}} \frac{g(t_1 - r)}{r} \, dr \right| \leq \| g \|_{\mu,l} \int_{(T-t_2)^{1/2}}^{(T-t_1)^{1/2}} \frac{(T - t_1 + r)^\mu}{\log(T - t_1 + r)} \frac{1}{r} \, dr \]
and we estimate the integral
\[ \int_{(T-t_2)^{1/2}}^{(T-t_1)^{1/2}} \frac{(T - t_1 + r)^\mu}{\log(T - t_1 + r)} \frac{1}{r} \, dr \leq C \frac{(T - t_1)^\mu}{\log(T - t_1)} \frac{t_2 - t_1}{T - t_2} \]
\[ \leq C(t_2 - t_1)^\gamma \frac{(T - t_1)^{\mu - \gamma}}{|\log(T - t_1)|}. \]

Since \( m \leq \mu - \gamma \), we obtain the desired estimate for \( L_{12} \).

For \( L_{11} \), the largest term in \( L_{11}[g](t_2) - L_{11}[g](t_1) \) is
\[ \log(|\log(T - t_2)|)g(t_2) - \log(|\log(T - t_1)|)g(t_1) = l_1 + l_2 \]
where
\[ l_1 = |\log(|\log(T - t_2)|) - \log(|\log(T - t_1)|)|g(t_2) \]
\[ l_2 = \log(|\log(T - t_2)|)(g(t_2) - g(t_1)). \]
Then
\[ |l_1| \leq C \frac{t_2 - t_1}{(T - t_2) \log(T - t_2)} \frac{(T - t_2)^\mu}{|\log(T - t_2)|^{m+1}} \|\gamma\|_{\mu,l} \]

and
\[ |l_2| \leq \log(|\log(T - t_1)|) (t_2 - t_1) \frac{(T - t_2)^m}{|\log(T - t_2)|^{m+1}} \|\gamma,m,l\| \]

This concludes the proof of (6.34). Then the proofs is obtained from the contraction mapping theorem.

**Proof of Lemma 6.4.** In Lemma 6.3 \(p_1\) is constructed as the solution of the fixed point problem (6.25). The following estimates hold
\[
\left\| \frac{d}{dt} [p_\kappa + p_1] - \frac{d}{dt} [p_\kappa + p_2] \right\|_{\gamma,m,l-1} \leq C \frac{1}{|\log T|} \left( \|p_1 - p_2\|_{\gamma,m,l} + \|p_1 - p_2\|_{\mu,l} \right)
\]
and
\[
\left\| \tilde{B} [p_\kappa + p_1] - \tilde{B} [p_\kappa + p_2] \right\|_{\gamma,m,l-1} \leq C \frac{1}{|\log T|} \left( \|p_1 - p_2\|_{\gamma,m,l} + \|p_1 - p_2\|_{\mu,l} \right),
\]
for \(p_1, p_2\) in \(\overline{B}_M\) (defined in Lemma 6.3). The proof is a lengthy computation that we omit. Then the fixed point characterization of the solution \(p_1\) yields
\[
\|\tilde{p}_1\|_{\gamma,m,l} \leq C (|a(\cdot) - a(T)|_{\mu,l-1} + |a(\cdot) - a(T)|_{\gamma,m,l-1}).
\]

We have already computed
\[
|a(\cdot) - a(T)|_{\mu,l-1} \leq C |\log T|^{\beta(a-1)}.
\]

Next we choose
\[
m = -2\gamma + \beta(a + 2\gamma - 1)
\]
and note that \(m < \mu - \gamma\) is equivalent to \(\beta < \frac{1}{2}\), which is true, and \(m > 0\) holds provided we take \(0 < \gamma < \frac{2\beta(a-1)}{4(1-\beta)}\) (if \(\beta \approx \frac{1}{4}\), \(a \approx 2\), \(\gamma < \frac{1}{4}\) suffices). In order for \(|a(\cdot) - a(T)|_{\gamma,m,l-1}\) to be finite we need \(l < 1 + 2\beta(a-1) - 4\gamma(1-\beta)\) and in this case we get
\[
|a(\cdot) - a(T)|_{\gamma,m,l-1} \leq C |\log T|^{l-1-\beta(a-1)-2\beta\gamma}.
\]

**Proof of Proposition 6.1.** We apply Lemma 6.3 with \(\delta \in (0, \frac{1}{2})\) such that
\[
2\beta - 1 + \delta > 0,
\]
which is possible because \(\beta > \frac{1}{4}\), and with \(\mu, l\) as in (6.26). Recall that \(a \approx 2\), \(\beta \approx \frac{1}{4}\) and then \(\mu \approx \frac{1}{4}\).

By Lemma 6.3 there is \(p_1\) satisfying (6.25) and
\[
\|\tilde{p}_1\|_{\mu,l} \leq C |\log T|^{\beta(a-1)}.
\]

(6.38)
In equation (6.25) the constant \( c \) depends on \( \kappa \) and we claim that it is possible to choose \( \kappa \) satisfying (6.8) such that \( c = 0 \). Evaluating (6.25) at \( t = T \) we find
\[
E_0[\dot{p}_\kappa](T) + E_0[\dot{p}_1](T) = 2a(T) + c. \tag{6.39}
\]
But evaluating (6.13) at \( t = T \) we get
\[
c(\kappa) = E_0[\dot{p}_\kappa](T) = \int_{-T}^{T} \frac{\dot{p}_\kappa(s)}{T - s} ds.
\]
and this implies
\[
c(\kappa) = \kappa(1 + O\left(\frac{1}{|\log T|}\right)) \tag{6.40}
\]
as \( T \to 0 \). On the other hand, using (6.38)
\[
|E_0[\dot{p}_1](T)| \leq C \frac{T^\mu}{|\log T|^{\beta(a-1)+1}} \tag{6.41}
\]
Since the left hand side of (6.39) is a continuous function of \( \kappa \), from (6.40) and (6.41) we see that there exists \( \kappa \) such that \( c = 0 \) in (6.39). Moreover \( \kappa = |a(T)|(1 + O\left(\frac{1}{|\log T|}\right)) \).

Now let us prove the estimate (6.9). For \( t \in [0, T] \) we have
\[
E_0[\dot{p}_\kappa] + E_0[\dot{p}_1] - 2 \frac{d}{dt} |p_\kappa + p_1| + E[|p_\kappa + p_1|] - c = R_\delta[\dot{p}_1],
\]
and therefore we have to estimate \( R_\delta[\dot{p}_1] \). By Lemma 6.4
\[
\|\dot{p}_1\|_{\gamma, m, l} \leq C(|\log T|^{\beta(a-1)} + |\log T|^{d-1-\beta(a-1)-2\beta\gamma}).
\]
Then
\[
|R_\delta[\dot{p}_1]| \leq \int_{t-(T-t)^{1+\delta}}^{t-\lambda_0(t)^2} \left|\frac{\dot{p}_1(t) - \dot{p}_1(s)}{t - s}\right| ds
\]
\[
\leq C\|\dot{p}_1\|_{\gamma, m, l} \int_{t-(T-t)^{1+\delta}}^{t-\lambda_0(t)^2} |T - s|^{-\alpha} |\log(T - s)|^{-\gamma} \left|\frac{t - s}{t - s}\right| ds
\]
\[
\leq C\|\dot{p}_1\|_{\gamma, m, l} (T - t)^{m + (1+\delta)\gamma} |\log(T - t)|^{-\gamma}.
\]
Note that
\[
m + (1 + \delta)\gamma = -2\gamma + \beta(a + 2\gamma - 1) + (1 + \delta)\gamma
\]
\[
= \beta(a - 1) - 2\gamma + (1 + \delta)\gamma + 2\gamma\beta
\]
\[
= \beta(a - 1) + \gamma(2\beta + \delta - 1).
\]
By (6.37) we obtain the estimate
\[
|R_\delta[\dot{p}_1]| \leq C(T - t)^\sigma R(t)^{1-a}\|\dot{p}_1\|_{\gamma, m, l},
\]
for some \( \sigma > 0 \). \( \square \)
7. Solving the outer-inner gluing system

In this section we use the operators built in the previous sections to solve System (3.3)-(3.4). It is convenient to separate the inner problem (3.3) into two equations. We write

\[ G(\lambda, \alpha, \xi, \psi) = \chi_{B_{2R}} \left[ |\nabla^2 Q_{-\alpha} \cdot \xi \eta \cdot \psi \cdot \omega| \right] , \]

where \( \chi_{B_{2R}} \) is 1 in \( B_{2R} \) and 0 outside, and let for \( l = 0, 1, j = 1, 2 \),

\[ c_{lj}(\lambda, \alpha, \xi, \psi) = c_{lj}(\tau) := \frac{1}{\int_{B_{2R}} \chi_{Z_{lj}}} \int_{B_{2R}} G(\lambda, \alpha, \xi, \psi) Z_{lj} \, dy \]

where \( \chi(y) = \frac{1}{1+|y|} \), so that for

\[ \tilde{G}(\lambda, \alpha, \xi, \psi) := \sum_{l,j} c_{lj}(\tau) \chi_{Z_{lj}}(y), \]

we have

\[ \int_{B_{2R}} (G - \tilde{G}) Z_{lj} \, dy = 0 \quad \text{for all} \quad \tau \in (\tau_0, \infty), \]

and for all \( l = 0, 1, j = 1, 2 \). Let us write \( \phi = \phi_1 + \phi_2 \). Then \( \phi \) solves equation (3.3) if \( \phi_1 \) and \( \phi_2 \) respectively solve (using summation convention)

\[ \begin{align*}
\partial_t \phi_1 &= L_\omega[\phi_1] + (G - \tilde{G})(\lambda, \alpha, \xi, \psi) \quad \text{in} \quad D_{2R}, \\
\partial_t \phi_2 &= L_\omega[\phi_2] + c_{lj}(\lambda, \alpha, \xi, \psi) \chi_{Z_{lj}} \quad \text{in} \quad D_{2R}, \\
\phi_1 &= \phi_2 = 0 \quad \text{in} \quad B_{2R(\tau_0)}(0) \times \{\tau_0\}. 
\end{align*} \]

Let us rewrite equation (3.4) as

\[ \begin{align*}
\partial_t \psi &= \Delta_x \psi + H(\psi, \phi, \lambda, \alpha, \xi) \quad \text{in} \quad \Omega \times (0, T), \\
\psi &= \psi_{\partial \Omega}(\psi, \phi, \lambda, \alpha, \xi) \quad \text{on} \quad \partial \Omega \times (0, T),
\end{align*} \]

where

\[ H(\psi, \phi, \lambda, \alpha, \xi) = (1 - \eta) \left[ |\nabla_x U|^2 \psi - 2 \nabla_x (\psi \cdot U) \nabla_x U \right] + Q_\alpha \left[ \Delta_x \eta \phi + 2 \lambda^{-1} \nabla_x \eta \nabla_y \phi - \partial_t \eta \phi \right] + (\partial_t Q_\alpha) \phi - \lambda^{-1} \lambda y \cdot \nabla_y \phi - \lambda^{-1} \xi \cdot \nabla_y \phi - (\psi \cdot U) U_t + N(\phi^* \phi + \psi + \eta Q_\alpha \phi) + (1 - \eta) \Pi_{U \cdot E^*} \psi. \]

We let \( \phi =: T[h] \) be the linear operator built in Proposition 4.1 and \( \psi = S[g] \) that in Proposition 5.1.

To complete the system of equations we will specify a choice of the parameters \( \lambda, \alpha \) and \( \xi \) to make \( c_{lj} \) as small as possible, that is, to make the function \( G(\lambda, \alpha, \xi, \psi) \) as orthogonal as possible to the elements of the kernel of \( L_\omega \) in modes 0 and 1. The components in these modes are the ones that deteriorate the estimate of the solution of the inner linear problem given in Proposition 4.1.

It is convenient to combine the parameters \( \lambda, \alpha \) into the single complex valued function

\[ p(t) = \lambda(t) e^{i\alpha(t)}. \quad (7.1) \]

We will be using a norm for \( p \) than in particular implies

\[ |\dot{p}(t)| \leq C \frac{|\log T|}{|\log(T - t)|^2}. \]
We describe first this choice in mode 0, that is, \( c_0 \). Recalling the expression \( \mathcal{E}^* = \mathcal{E}_0^* + \mathcal{E}_1^* + \mathcal{E}_2^* + \mathcal{E}_3^* \) and using (2.16), we have that

\[
\lambda \int_{B_{2R}} Q_{-a} \Pi_{U^+} \mathcal{E}_0^* \cdot Z_{01} = -2\pi \left[ B_1[\lambda, \alpha] + 2 \text{Re}(a_*(t)e^{-i\alpha}) \right] + O\left( \frac{1}{R} \right)
\]

\[
\lambda \int_{B_{2R}} Q_{-a} \Pi_{U^+} \mathcal{E}_0^* \cdot Z_{02} = -2\pi \left[ B_2[\lambda, \alpha] + 2 \text{Im}(a_*(t)e^{-i\alpha}) \right] + O\left( \frac{1}{R} \right)
\]

where the operators \( B_i \) are defined in (2.21), (2.22) and

\[
a_*(t) = \text{div} z^*(\xi(t), t) + i \text{curl} z^*(\xi(t), t).
\]

We let

\[
\tilde{a}_j(t; \lambda, \alpha, \xi, \psi) = -\frac{1}{2\pi} \int_{B_{2R}} \left[ Q_{-a} \Pi_{U^+} (\mathcal{E}_1^* + \mathcal{E}_2^* + \mathcal{E}_3^*) + \tilde{L}_\omega (Q_{-a} \psi) \right] \cdot Z_{0j}
\]

where

\[
\tilde{L}_\omega (Q_{-a} \psi) = |\nabla \omega(y)|^2 \Pi_{U^+} Q_{-a} \psi - 2 \nabla_y (Q_{-a} \psi \cdot \omega) \nabla \omega(y),
\]

and

\[
\tilde{a}(t; \lambda, \alpha, \xi, \psi) = \tilde{a}_1(t; \lambda, \alpha, \xi, \psi) + i \tilde{a}_2(t; \lambda, \alpha, \xi, \psi).
\]

We would like to solve for \( \lambda \) and \( \alpha \) in the equations

\[
B_1[\lambda, \alpha] + 2 \text{Re}(\tilde{a}_1(t) + \tilde{a})e^{-i\alpha(t)}) = 0
\]

\[
B_2[\lambda, \alpha] + 2 \text{Im}(\tilde{a}_1(t) + \tilde{a})e^{-i\alpha(t)}) = 0.
\]

Proposition 6.1 gives us a map \( A \) that given a function \( a : [0, T] \to \mathbb{C} \) with certain properties produces \( p : [-T, T] \to \mathbb{C} \) such that if we write \( p = \lambda e^{i\alpha} \) with \( \lambda > 0 \) and \( \alpha \in \cdots \) then

\[
B_1[\lambda, \alpha] + 2 \text{Re}(\tilde{a}_1(t) + \tilde{a})e^{-i\alpha(t)}) = o(R(t)^{1-a})
\]

\[
B_2[\lambda, \alpha] + 2 \text{Im}(\tilde{a}_1(t) + \tilde{a})e^{-i\alpha(t)}) = o(R(t)^{1-a}),
\]

with estimates to be made more precise later on. We define the first two components of \( \Theta[\lambda, \alpha, \xi, \psi] \) as

\[
\Theta_1[\lambda, \alpha, \xi, \psi](t) = |A(a_* + \tilde{a}; \lambda, \alpha, \xi, \psi)|
\]

\[
\Theta_2[\lambda, \alpha, \xi, \psi](t) = \arg A(a_* + \tilde{a}; \lambda, \alpha, \xi, \psi).
\]

where \( \arg \) is the unique argument in \([-\frac{\pi}{2}, \frac{\pi}{2}]\).

We define the third component of \( \Theta \) as the new \( \xi \) to get orthogonality of \( G - \overline{C} \) with respect to the functions \( Z_{1j} \). To do this it is convenient to decompose \( \mathcal{E}_1^* \) in (2.17) as

\[
\mathcal{E}_1^* = \mathcal{E}_{1a}^* + \mathcal{E}_{1b}^*
\]

where

\[
\mathcal{E}_{1a}^* = \frac{1}{\lambda} w_\rho \left[ \text{Re} \left( (\dot{\xi}_1 - i\dot{\xi}_2)e^{i\theta} \right) Q_\alpha E_1 + \text{Im} \left( (\dot{\xi}_1 - i\dot{\xi}_2)e^{i\theta} \right) Q_\alpha E_2 \right]
\]

\[
\mathcal{E}_{1b}^* = \frac{2}{\lambda} w_\rho \cos \omega \left[ \text{Re} \left( (\partial \xi_1 z_3^* - i\partial \xi_2 z_4^*)e^{i\theta} \right) Q_\alpha E_1 \right]
\]

\[
- \frac{2}{\lambda} w_\rho \cos \omega \left[ \text{Im} \left( (\partial \xi_1 z_3^* - i\partial \xi_2 z_4^*)e^{i\theta} \right) Q_\alpha E_2 \right].
\]

We define

\[
\Theta_3[\lambda, \alpha, \xi, \psi] = \dot{\xi}
\]
where $\tilde{\xi}(t) \in \mathbb{R}^2$ is given by
\[\begin{aligned}
\frac{d}{dt}\tilde{\xi}_j &= \frac{1+R^2}{2\pi}\int_{B_R} \left(\xi_{1b}^* + \xi_{33}^* + \tilde{L}_\omega[Q-\alpha \psi]\right) \cdot Z_{13} \, dy.
\end{aligned}\]

(7.4)

With this definition we have
\[c_{1j}(\Theta(\lambda, \alpha, \xi, \psi)) = 0, \quad j = 1, 2.\]

In order to find a solution of system (3.3), (3.4) it suffices to find a solution to
\[\begin{aligned}
\phi_1 &= T[(G-G)(\lambda, \alpha, \xi, \psi)],
\phi_2 &= T[c_{1j}(\Theta(\lambda, \alpha, \xi, \psi))\chi_{zj}],
\psi &= S[H, \psi_{[\Omega]}](\psi, \phi_1 + \phi_2, \lambda, \alpha, \xi).
\end{aligned}\]

(7.5)

Problem (7.5) is a fixed point equation in the unknown functions $\phi, \psi, \lambda, \alpha, \xi$ that we will solve using Shauder’s fixed point. Let us describe the spaces where we set up this fixed point problem. We will take $\phi_1, \phi_2$ in the space
\[X(\alpha, \nu) = \{ \phi \in C(\overline{D}_{2R}) : \nabla_y \phi \in C(\overline{D}_{2R}), \| \phi \|_{X(\alpha, \nu)} < \infty \},\]

where
\[\| \phi \|_{X(\alpha, \nu)} = \sup_{(y, \tau) \in \overline{D}_{2R}} \frac{\lambda_0}{1+|y|} \min\{1, R^{2-\alpha}|y|^{-2}\} \left[ (1+|y|)|\nabla_y \phi(y, \tau)| + |\phi(y, \tau)| \right]\]

and $a \in (1, 2)$ is close to 2, $\nu \in (0, 1)$ is close to 1. We take $\psi$ in the space
\[Y(a', \nu, \gamma') = \{ \psi \in C(\overline{\Omega} \times [0, T]) : \nabla_x \psi \in C(\overline{\Omega} \times [0, T]), \| \psi \|_{Y(a', \nu, \gamma')} < \infty, \}\]

where
\[\| \psi \|_{Y(a', \nu, \gamma')} = \lambda_0(0)^{-\nu} R(0)^{2-a'} |\log T| \sup_{\Omega \times (0, T)} |\psi(x, t)|\]
\[+ \sup_{\Omega \times (0, T)} \lambda_0(t)^{-\nu} R(t)^{a'-1} |\log(T-t)| |\psi(x, t) - \psi(x, T)|\]
\[+ \lambda_0(0)^{1-\nu} R(0)^{a'-1} \sup_{\Omega \times (0, T)} |\nabla \psi(x, t)|\]
\[+ \sup_{\Omega \times (0, T)} \lambda_0(t)^{1-\nu} R(t)^{a'-1} |\nabla \psi(x, t) - \nabla \psi(x, T)|\]
\[+ \sup_{\Omega \times (0, T)} \lambda_0(t_1)^{1-\nu+2\gamma'} R(t_2)^{a'-1+2\gamma'} |\nabla \psi(x, t_2) - \nabla \psi(x, t_1)|\]
\[+ \lambda_0(t_2)^{1-\nu+2\gamma'} R(t_2)^{a'-1+2\gamma'} \frac{1}{(t_2-t_1)^{2\gamma'}} |x-x'|^{2\gamma'}\]

The last supremum is taken over $0 \leq t_1 < t_2 < T$, $t_2 - t_1 \leq \frac{1}{10}(T-t_2)$ and with parameters $a' < a$ and $\gamma' < \gamma$. Note that compared to the norm $\| \|_*$ defined in (5.4), this norm is weaker and actually the inclusion $Y(a, \nu, \gamma)$ into $Y(a', \nu, \gamma')$ is compact.

**Proof of Theorem 1.** Let $R_1 > 0$ be small and fixed. We set up the fixed point problem (7.5) in the following sets. We take $\phi_1, \phi_2 \in B_{R_1}(0)$ of the space $X(\alpha, \nu)$ and we take $\psi$ in $B_{R_1}(0)$ of the space $Y(a', \nu, \gamma')$. For the parameters $\lambda$, $\alpha$, we
consider these two together as in (7.1) and take $p = p_\kappa + p_1$ as in Proposition 6.1, with $\kappa$ satisfying

$$\frac{1}{C_1} \leq |\kappa| \leq C_1$$  \hspace{1cm} (7.6)

with $C_1 > 0$ a large fixed constant, and $p_1$ is in the closed ball of radius $R_1$ of the space of $C^1([-T, T])$ functions with $p_1(T) = 0$ and with the norm

$$\|p_1\|_{\mu_0} = \sup_{t \in [-T, T]} (T-t)^{-\mu_0}|p_1(t)|.$$  

We take $0 < \mu_0 < \beta(a' - 1)$.

As for $\xi$ we consider the closed ball of radius $R_1$ of the space $C^1([0, T])$ with $\xi(T) = q$ and norm

$$\|\xi\|_{\sigma_3} = \sup_{t \in [0, T]} (T-t)^{-\sigma_3}|\xi(t)|,$$

for some $\sigma_3 > 0$ fixed. In the rest of the proof we assume

$$\begin{aligned}
\|\phi_1\|_{X_{(a, \nu)}} + \|\phi_2\|_{X_{(a, \nu)}} + \|\psi\|_{Y_{(a', \nu', \gamma')}} + \|p_1\|_{\mu'} + \|\xi\|_{\sigma_3} & \leq R_1, \\
\text{and } \kappa & \text{ satisfies } (7.6).
\end{aligned}$$  \hspace{1cm} (7.7)

The function $\phi$ will always denote $\phi_1 + \phi_2$. We then consider the map

$$(\phi_1, \phi_2, \psi, p, \xi) \mapsto F(\phi_1, \phi_2, \psi, \hat{\psi}, \hat{p}, \hat{\xi})$$

with

$$\begin{aligned}
\hat{\phi}_1 & = T[(G - \tilde{G})(\lambda, \alpha, \xi, \psi)] \\
\hat{\phi}_2 & = T[c_j(\Theta(\lambda, \alpha, \xi, \psi))\chi_{lj}], \\
\hat{\psi} & = S[H, \psi_{\partial a'}](\psi, \phi_1 + \phi_2, \lambda, \alpha, \xi), \\
\hat{p} & = A(a_0 + \delta(\cdot; \lambda, \alpha, \xi, \psi))
\end{aligned}$$  \hspace{1cm} (7.8)

and with the function $\hat{\xi}$ given by (7.4). We will show that the map $F$ takes the set defined by (7.7) into itself and is compact.

We first claim that if $R_1 > 0$ is fixed sufficiently small, then

$$\|H(\psi, \phi, \lambda, \alpha, \xi)\|_{*\sigma} \leq C(T^{\sigma_0} + R_1).$$  \hspace{1cm} (7.9)

To prove this, we will be using constantly that the inclusion

$$B_{2\lambda R}(\xi) \subset B_{C\lambda R}(q),$$

holds. For this it is sufficient to have

$$|\xi - q| \leq C\lambda R,$$

which in turn holds because

$$|\xi - q| \leq (T-t)^{1+\sigma_3}, \quad \text{and} \quad \lambda R = \lambda^{1-\beta} \sim \left(\frac{|\log T|(T-t)}{|\log(T-t)|^2}\right)^{1-\beta}. $$

Let us consider $(1 - \eta)|\nabla_x U|^2\psi$. Because $\psi(q, T) = 0$ and by definition of $\|Y_{(a', \nu', \gamma')}\$ we have

$$\begin{aligned}
|\psi(x, t)| & \leq (|\psi(x, t) - \psi(x, T)| + |\psi(x, T) - \psi(q, T)|) \\
& \leq (r + \lambda_0(t)(T-t)^{2-a'}|\log(T-t)|)\|\psi\|_{Y_{(a', \nu', \gamma')}}.
\end{aligned}$$  \hspace{1cm} (7.10)
Similarly
\[
|(1 - \eta) \nabla_x U \cdot \nabla_x \psi| \leq (1 - \eta) \frac{\lambda_0^2}{(r + \lambda_0)^2} (|\psi(x, t) - \psi(z, t)| + |\psi(x, T) - \psi(q, T)|)
\]
\[
\leq C(1 - \eta) \frac{\lambda_0^2}{(r + \lambda_0)^2} (r + \lambda_0 R(t)^{2-a'} |\log(T - t)|) \|\psi\|_{Y(a', \nu, \gamma')}
\]
\[
\leq C(1 - \eta) \frac{\lambda_0}{(r + \lambda_0)^2} \|\psi\|_{Y(a', \nu, \gamma')}
\]
\[
\leq C g_2 \|\psi\|_{Y(a', \nu, \gamma')},
\]
with \( g_2 \) defined in (5.3). Hence
\[
\|(1 - \eta) \nabla U \cdot \nabla \psi\|_{\ast \ast} \leq C \|\psi\|_{Y(a', \nu, \gamma')}.
\]

Similarly
\[
|(1 - \eta) \nabla_x (\psi \cdot U) \nabla_x \psi| \leq (1 - \eta) \|\nabla_x U\|^2 |\psi| + (1 - \eta) \|\nabla_x U\| |\nabla \psi|.
\]
The first term above was already estimated. The second term is
\[
(1 - \eta) \|\nabla_x U\| |\nabla \psi| \leq (1 - \eta) \frac{\lambda_0}{r^2 + \lambda_0} \|\nabla \psi\|_{L^\infty(\Omega \times (0, T))}
\]
\[
\leq C g_2 \|\psi\|_{Y(a', \nu, \gamma')},
\]
and so
\[
\|(1 - \eta) \nabla_x U\| |\nabla \psi|\|_{\ast \ast} \leq C \|\psi\|_{Y(a', \nu, \gamma')}.
\]

Let us analyze now \((\psi \cdot U) U_t\). Thanks to (7.10), (2.3), (2.4), and (2.5) we find
\[
|((\psi \cdot U) U_t| \leq |\psi| |U_t|
\]
\[
\leq C \left( r + \lambda_0 R^{2-a'} |\log(T - t)| \right) \left[ \frac{1}{\lambda} \left| \frac{\dot{\lambda}}{\lambda} \right| \frac{\lambda}{r + \lambda_0} + \frac{1}{r^2 + \lambda_0} \right] \|\psi\|_{Y(a', \nu, \gamma')}
\]
\[
\leq C \left( r + \lambda_0 R^{2-a'} |\log(T - t)| \right) \left[ \frac{\lambda_0}{r + \lambda_0} + \frac{\lambda_0}{r^2 + \lambda_0} \right] \|\psi\|_{Y(a', \nu, \gamma')}.
\]

Using that \( |\dot{\lambda}| + \lambda |\dot{\lambda}| \leq C |\dot{\lambda}_0| \leq C \) and \( |\xi| \leq C \) we get
\[
|((\psi \cdot U) U_t| \leq C \left( r + \lambda_0 R^{2-a'} |\log(T - t)| \right) \frac{1}{r + \lambda_0} \|\psi\|_{Y(a', \nu, \gamma')}
\]
\[
\leq C \left[ 1 + \frac{\lambda_0 R^{2-a'}}{r + \lambda_0} \right] \|\psi\|_{Y(a', \nu, \gamma')}.
\]

Taking \( a' \) close to 2 (\( a' > \frac{3}{2} \) suffices) we see that
\[
|((\psi \cdot U) U_t| \leq C (g_3 + g_4) \|\psi\|_{Y(a', \nu, \gamma')},
\]
with \( g_3, g_4 \) as in (5.3). Therefore
\[
\|(\psi \cdot U) U_t\|_{\ast \ast} \leq C \|\psi\|_{Y(a', \nu, \gamma')}.
\]

Next consider \( Q_\alpha \Delta_x \eta \phi \). By the definition of \( \| \|_{X(a, \nu')} \) we have
\[
|\phi(y, \tau)| + (1 + |y|) |\nabla \phi(y, \tau)| \leq \|\phi\|_{X(a, \nu')} \lambda_0^\nu R^{2-a}, \quad \text{for } R \leq |y| \leq 2R.
\]
Hence
\[ |Q_\alpha \Delta_x \eta \phi| \leq \frac{1}{\lambda^0 R^2} \chi_{|[x-\xi| \leq 2\lambda R]} |\phi(y, \tau)| \]
\[ \leq C \lambda_0^{\nu-2} R^{-\alpha} \chi_{|[x-q| \leq C \lambda_0 R]} \| \phi \|_{X(a, \nu)} \]
\[ \leq C \rho_1 \| \phi \|_{X(a, \nu)} \]
so that
\[ \| Q_\alpha \Delta_x \eta \phi \|_{**} \leq C \| \phi \|_{X(a, \nu)}. \]
Similarly
\[ \| (\partial_t \eta) Q_\alpha \phi \|_{**} + \| Q_\alpha \lambda^{-1} \nabla_x \eta \nabla_y \phi \|_{**} \leq C \| \phi \|_{X(a, \nu)}. \]

Let us analyze \((\delta_t Q_\alpha) \eta \phi\). We have
\[ |(\delta_t Q_\alpha) \eta \phi| \leq |\hat{\delta} \phi| \chi_{|[x-\xi| \leq 2\lambda_0(t) R(t)]} \]
\[ \leq C \lambda_0^{\nu-1} R^2 \chi_{|[x-q| \leq C \lambda_0(t) R(t)]} \| \phi \|_{X(a, \nu)}. \]

We remark that we can achieve \(a \in (1, 2), \ a \approx 2, \ \beta > \frac{1}{4}, \ \beta \approx \frac{1}{4} \) such that \((a+2) \beta < 1\) (take for example \(a = 2 - 4\sigma, \ \beta = \frac{1+2\sigma}{4} \) with \(\sigma > 0\) small). Then
\[ \nu - 1 - 2\beta > \nu - 2 + a\beta \]
and we deduce
\[ \| (\partial_t Q_\alpha) \eta \phi \|_{**} \leq C \| \phi \|_{X(a, \nu)}. \]

Let us analyze \(-\eta \lambda^{-1} \hat{\lambda} y \cdot \nabla_y \phi - \eta \lambda^{-1} \hat{\xi} \cdot \nabla_y \phi\). Using that \(|\hat{\lambda}| \leq C\)
\[ |\lambda^{-1} \eta \lambda y \cdot \nabla_y \phi| \leq C \lambda_0^{\nu-1} R^2 \chi_{|[x-q| \leq 2\lambda_0(t) R(t)]} \| \phi \|_{X(a, \nu)}; \]
and similarly, because \(|\hat{\xi}| \leq C\),
\[ |\eta \lambda^{-1} \hat{\xi} \cdot \nabla_y \phi| \leq C \lambda_0^{\nu-1} R^2 \chi_{|[x-q| \leq 2\lambda_0(t) R(t)]} \| \phi \|_{X(a, \nu)}; \]

It follows that
\[ \| \lambda^{-1} \eta \lambda y \cdot \nabla_y \phi \|_{**} + \| \eta \lambda^{-1} \hat{\xi} \cdot \nabla_y \phi \|_{**} \leq C \| \phi \|_{X(a, \nu)}. \]

Let us analyze \((1 - \eta) \Pi_{U'; E^*}\). Using (3.7) we have
\[ |(1 - \eta) \Pi_{U'; E^*}| \leq (1 - \eta) \frac{\lambda_0}{r^2 + \lambda_0^2} \]
and hence
\[ \| (1 - \eta) \Pi_{U'; E^*} \|_{**} \leq CT^\sigma_0. \]
Finally, the proof of the estimate
\[ \| N(\psi^* + \psi + \eta Q_\alpha \phi) \|_{**} \leq C (\| \psi \|_{Y(a', \nu, \gamma')} + \| \phi \|_{X(a, \nu)}); \]
is analogous as the one of the previous terms. We omit the details. Collecting the above estimates we obtain (7.9).

With the notation from (7.8), it follows from (7.9) and Proposition 5.1
\[ \| \tilde{\psi} \|_{**} = \| S[H, \psi]\phi_1 + \phi_2, \lambda, \alpha, \xi \|_{**} \leq C R_1. \]

(7.11)

Let us consider the first equation in (7.5). Thanks to (2.9) we see that
\[ |\tilde{L}_w[Q_{-\alpha} \psi]| \leq C \frac{\lambda_0}{1 + \rho^2} \| \nabla \psi \|_{L^\infty} \]
\[ \leq C \frac{\lambda_0}{1 + \rho^2} T^{\nu-1 + \beta(a'-1)} \| \psi \|_{Y(a', \nu, \gamma')}. \]
Let us fix $a_1 \in (a, 2)$ and $\nu_1 \in (\nu, 1)$. This combined with (3.7) implies
\[
\|(G - \tilde{G})(\lambda, \alpha, \xi, \psi)\|_{a_1, \nu_1} \leq CT^{1-\nu_1} + CT^{\nu-1+\beta(a'-1)}R_1.
\]
Using now Proposition 4.1 and the notation (7.8) we get
\[
\|\tilde{\phi}_1\|_{X(a_1, \nu_1)} = \|T[(G - \tilde{G})(\lambda, \alpha, \xi, \psi)]\|_{X(a_1, \nu_1)} \leq CT^{1-\nu_1} + CT^{\nu-1+\beta(a'-1)}R_1. \tag{7.12}
\]

Let us consider $\phi_2$. Because of the choice of $\tilde{\xi}$, we have $c_1 j = 0$ and because of the choice of $\tilde{p}$ we have, thanks to Proposition 6.1,
\[
|c_0 j| \leq C(T - t)^{p_{2}} R(t)^{1-a'}(||a(\cdot) - a(T)||_{\mu, l-1} + ||a(\cdot) - a(T)||_{\gamma', m, l-1}), \tag{7.13}
\]
where
\[
a = a_\ast + \tilde{a},
\]
with $a_\ast$ is defined in (7.2) and $\tilde{a}$ is defined in (7.3), which depends on $\lambda, \alpha, \xi, \psi$.

The norms in (7.13) are defined in (6.10), (6.11) with $\mu = \beta(a' - 1)$, $m = -2\gamma' + \beta(a' + 2\gamma' - 1)$, and any fixed $l$ satisfying the restriction in (6.12). It follows from the definition of the norm $\|Y(a', \nu, \gamma')$, the formula for $\tilde{a}$ (7.3) and the expression for the operator $\tilde{L}_\omega$ (analogous to (2.9)), that
\[
\|a(\cdot) - a(T)||_{\gamma', m, l-1} \leq C\|\psi\|_{Y(a', \nu, \gamma')} \leq CR_1.
\]
Using Proposition 4.1 and with the notation (7.8) we get
\[
\|\phi_2\|_{X(a_1, \nu_1)} = T[c_{ij}(\Theta(\lambda, \alpha, \xi, \psi))\chi Z_{ij}] \leq CR_1. \tag{7.14}
\]
This holds because the spatial decay of $\chi Z_{0,j}$ is $\frac{1}{4\rho^2}$ and we have chosen $\nu_1$ sufficiently close to $\nu$, depending on $\sigma$.

Let us write $\tilde{p} = \tilde{p}_\kappa + \tilde{p}_1$. Then from Proposition 6.1 we get that $\tilde{\kappa}$ satisfies (7.6) and $\tilde{p}_1$ satisfies
\[
\|\tilde{p}_1\|_{\mu, l} \leq C|\log T|^\beta(a'-1). \tag{7.15}
\]
Finally, let us note that from definition (7.4) and the assumption $\|\psi\|_{Y(a', \nu, \gamma')}$ imply that
\[
\|\tilde{\xi}\|_{\mu_1} \leq C\bar{R}_1, \tag{7.16}
\]
where $\mu_1 = \nu - 1 + \beta(a - 1)$.

We note that the estimates (7.11), (7.12), (7.14), (7.15), (7.16) imply that $(\tilde{\phi}_1, \tilde{\phi}_2, \tilde{\psi}, \tilde{p}, \tilde{\xi})$ satisfy (7.7). Moreover, using standard parabolic estimates we see that $\mathcal{F}$ is compact from the set (7.7) into itself. The existence of a solution follows then from Schauder’s fixed point theorem. \hfill \Box

\section*{Appendix A. Proof of Lemma 6.1}
Let us first sketch the proof of Lemma 6.1. We look for $p_\kappa$ of the form
\[
p_\kappa = p_{0, \kappa} + p_1,
\]
where $p_{0, \kappa}$ is defined in (6.7), and we would like
\[
\mathcal{B}_0[p_{0, \kappa}] + \mathcal{B}_0[p_1] - \frac{d}{dt}(p_{0, \kappa} + p_1) + \mathcal{B}(p_{0, \kappa} + p_1)(t) - c(\kappa) = O((T - t)^{\delta_1}). \tag{A.1}
\]
The idea is to replace \( B_0[p_1] \) by \( S_\delta[p_1] \) in (A.1) and try to solve the corresponding equation. We claim that if \( \delta_0 > 0 \) is small, then we can find \( p_1 \) such that

\[
B_0[p_{0,\kappa}] + S_\delta[p_1] - 2 \frac{d}{dt} |p_{0,\kappa} + p_1| + \tilde{B}[p_{0,\kappa} + p_1](t) - c = 0 \tag{A.2}
\]

in \([0, T]\) for some \( c \). This means that instead of (A.1) we have obtained

\[
B_0[p_{0,\kappa}] + B_0[p_1] - 2 \frac{d}{dt} |p_{0,\kappa} + p_1| + \tilde{B}[p_{0,\kappa} + p_1] - c = R_\delta[p_1]
\]

in \([0, T]\). We will later show that

\[
|R_\delta[p_1]| \leq C(T - t)^{\delta_0},
\]

but since the only information we have on \( \delta_0 \) is that it is small, this estimate is not sufficient to obtain the conclusion of Lemma 6.1. Therefore we look for an improvement of \( p \) of the form \( p = p_{0,\kappa} + p_1 + p_2 \) with a new unknown \( p_2 \) and we look for it solving

\[
S_\delta[p_2] - 2 \frac{d}{dt} |p_{0,\kappa} + p_1 + p_2| + 2 \frac{d}{dt} |p_{0,\kappa} + p_1| + \tilde{B}[p_{0,\kappa} + p_1 + p_2] - \tilde{B}[p_{0,\kappa} + p_1] = c + R_\delta[p_1], \quad \text{in } [0, T],
\]

some constant \( c \), where now \( \delta \in (0, \frac{1}{2}) \) is arbitrary.

We claim it is possible to find \( p_2 \) satisfying this equation. Then \( p_\kappa = p_{0,\kappa} + p_1 + p_2 \) satisfies

\[
B_0[p_\kappa] - 2 \frac{d}{dt} |p_\kappa| + \tilde{B}[p_\kappa] = c + R_\delta[p_2]
\]

in \([0, T]\) with \( c \) a constant. Finally we show that

\[
|R_\delta[p_2](t)| \leq C(T - t)^{\delta + \delta_0}.
\]

As explained previously, we first construct a function \( p \) of the form \( p = p_{0,\kappa} + p_1 \), with \( p_{0,\kappa} \) defined in (6.7), satisfying (A.2) in \([0, T]\), where \( \delta_0 > 0 \) is fixed sufficiently small. This function \( p_1 \) will be obtained by linearization. The first computation we do is an estimate for how well \( p_{0,\kappa} \) is an approximate solution to (A.2). Let

\[
E := B_0 |p_{0,\kappa}|,
\]

where \( B_0 \) is given by (6.6).

**Lemma A.1.** Let \( p_{0,\kappa} \) be given by (6.7) and assume \( \kappa \in \mathbb{C} \) satisfies (6.8). Then

\[
|E(t) - E(T)| \leq C \frac{\log T \log |\log(T - t)|}{|\log(T - t)|^2}, \quad -T \leq t \leq T. \tag{A.3}
\]

**Proof.** Let us write

\[
E(t) = \int_{t-(T-t)}^{t-(T-t)} \frac{p_{0,\kappa}(s)}{t-s} ds + \int_{t-(T-t)}^{t-(T-t)} \frac{p_{0,\kappa}(s)}{t-s} ds
\]

\[
= \int_{t-(T-t)}^{t} \frac{p_{0,\kappa}(s)}{T-s} ds - \int_{t-(T-t)}^{t} \frac{p_{0,\kappa}(s)}{T-s} ds
\]

\[
+ \int_{t-(T-t)}^{t-(T-t)} \frac{p_{0,\kappa}(s)}{t-s} \left( \frac{1}{t-s} - \frac{1}{T-s} \right) ds + \int_{t-(T-t)}^{(T-t) \lambda_0(t)^2} \frac{p_{0,\kappa}(s)}{t-s} ds
\]
We estimate
\[ \left| \int_{t-(T-t)}^{t} \frac{\dot{p}_{0,\kappa}(s)}{T-s} \, ds \right| \leq C\kappa \int_{t-(T-t)}^{t} \frac{1}{(T-s)\log(T-s)^2} \, ds \]
\[ \leq \frac{C\kappa|\log(T)|}{(T-t)\log(T-t)^2} \int_{t-(T-t)}^{t} ds \]
\[ \leq \frac{C\kappa|\log(T)|}{|\log(T-t)|^2} \]

and
\[ \left| \int_{-T}^{t-(T-t)} \frac{\dot{p}_{0,\kappa}(s)}{T-s} \left( \frac{1}{1 - \frac{1}{T-s}} \right) \, ds \right| \]
\[ \leq C\kappa|\log(T)|(T-t) \int_{-T}^{t-(T-t)} \frac{1}{|\log(T-s)|^2(t-s)(T-s)} \, ds \]
\[ \leq C\kappa|\log(T)|(T-t) \int_{-T}^{t-(T-t)} \frac{1}{|\log(T-s)|^2(T-s)^2} \, ds \]
\[ \leq \frac{C\kappa|\log(T)|}{|\log(T-t)|^2}. \]

With the fourth term in \( E \) we proceed as follows
\[ \int_{t-(T-t)}^{t-(T-t)} \frac{\dot{p}_{0,\kappa}(s)}{T-s} \, ds \]
\[ = p_{0,\kappa}(t) \int_{t-(T-t)}^{t-(T-t)} \frac{1}{t-s} \, ds - \int_{t-(T-t)}^{t-(T-t)} \frac{\dot{p}_{0,\kappa}(t) - \dot{p}_{0,\kappa}(s)}{t-s} \, ds \]
\[ = p_{0,\kappa}(t)(\log(T-t) - 2\log(\lambda_0)) - \int_{t-(T-t)}^{t-(T-t)} \frac{\ddot{p}_{0,\kappa}(t) - \ddot{p}_{0,\kappa}(s)}{t-s} \, ds. \]

But
\[ \left| \int_{t-(T-t)}^{t-(T-t)} \frac{\ddot{p}_{0,\kappa}(t) - \ddot{p}_{0,\kappa}(s)}{t-s} \, ds \right| \leq \sup_{s \leq t} |p_{0,\kappa}(s)|(T-t) \]
\[ \leq \frac{C\kappa|\log(T)|}{|\log(T-t)|^2}. \]

Therefore we have obtained
\[ E = \int_{-T}^{t} \frac{\dot{p}_{0,\kappa}(s)}{T-s} \, ds + p_{0,\kappa}(t)(\log(T-t) - 2\log(\lambda_0)) + O\left( \frac{\kappa|\log(T)|}{|\log(T-t)|^2} \right). \]

But
\[ p_{0,\kappa}(t)|\log(T-t)| + \int_{0}^{t} \frac{\dot{p}_{0,\kappa}(s)}{T-s} \, ds = c \]
for some constant $c$. Indeed, differentiating
\[
\frac{d}{dt} \left( \mathbf{p}_{0, \kappa}(t) \right| \log(T - t) \right) + \int_0^t \frac{\dot{\mathbf{p}}_{0, \kappa}(s) \cdot \kappa(s)}{T - s} \, ds = \frac{d}{dt} \left( \mathbf{p}_{0, \kappa}(t) \right| \log(T - t) \right) + 2 \frac{\dot{\mathbf{p}}_{0, \kappa}(t) \cdot \kappa(t)}{T - t}.
\]
This shows that
\[
E(t) = E(T) + O\left( \frac{\log(T) \log(|\log(T)|) + \log(|\log(T - t)|)}{|\log(T - t)|^2} \right),
\]
which implies the estimate (A.3).

Next we construct an almost right inverse for the operator $S_{\delta_0}$. It will be useful to decompose
\[
S_{\delta_0}[g] = \tilde{L}_0[g] + \tilde{L}_1[g]
\]
where
\[
\tilde{L}_0[g](t) = (1 - \delta_0) \left| \log(T - t) \right| g(t) + \int_{-T}^t \frac{g(s)}{T - s} \, ds
\]
and $\tilde{L}_1$ contains all other terms, that is,
\[
\tilde{L}_1[g](t) = \int_{-(T - t)}^{t - (T - t) + \delta_0} \frac{g(s)}{t - s} \, ds - \int_{t-(T-t)}^t \frac{g(s)}{T - s} \, ds
\]
\[
+ \int_{-T}^{t-(T-t)} g(s) \left( \frac{1}{t - s} - \frac{1}{T - s} \right) \, ds
\]
\[
+ \left( 4 \log(|\log(T - t)|) - 2 \log(|\log(T - t)|) \right) g(t).
\]

Given a continuous function $f$ in $[-T, T]$ with a certain modulus of continuity at $T$, we would like to find $g$ such that
\[
S_{\delta_0}[g] = f \quad \text{in } [-T, T].
\]
We will not quite obtain this, but we will solve a modified version of this equation. Let $\eta$ be a smooth cut-off function such that
\[
\eta(s) = 1 \quad \text{for } s \geq 0, \quad \eta(s) = 0 \quad \text{for } s \leq -\frac{1}{4} \quad \text{(A.4)}
\]
We will be able to find a function $g$ such that
\[
\tilde{L}_0[g] + \eta\left( \frac{t}{T} \right) \tilde{L}_1[g] = f + c \quad \text{in } [-T, T]. \quad \text{(A.5)}
\]
We do this in suitable weighted spaces. Let $k > 1$. We consider $g$ in the space $C([-T, T]; \mathbb{C}) \cap C^1([-T, T]; \mathbb{C})$ with
\[
g(T) = 0
\]
and the norm
\[
\|g\|_{*, k} = \sup_{t \in [-T, T]} |\log(T - t)|^k |\dot{g}(t)|. \quad \text{(A.6)}
\]
This norm controls
\[
\sup_{t \in [-T, T]} (T - t)^{-1} |\log(T - t)|^k |g(t)|
\]
For the right hand side of (A.5) we take the space \( C([-T, T]; \mathbb{C}) \) with \( f(T) = 0 \) and the norm
\[
\|f\|_{**} = \sup_{t \in [-T, T]} |\log(T - t)|^k |f(t)|. \tag{A.7}
\]

Note that in (A.5) the expression \( \eta(t)^{\frac{1}{T}} \tilde{L}_1 \hat{g}(t) \) is well defined for \( g \) of class \( C^1 \) in \([-T, T]\). Indeed, because of the cut-off function, \( \tilde{L}_1 \hat{g}(t) \) needs to be computed only for \( t \geq -\frac{T}{4} \), and for \( t \geq -\frac{T}{4} \) the integrals appearing in \( L_1 \hat{g} \) are well defined, since they start at either \(-T\) or \( t - \frac{1}{4}(T - t) = \frac{3}{4}t - \frac{1}{4}T \geq -T\).

The next lemma gives the solvability of (A.5) in the weighted spaces introduced above.

**Lemma A.2.** Let \( k > \beta - 1 \) and assume \( \kappa = O(1) \) as \( T \to 0 \). Then, there is \( \delta_0 > 0 \), so that for \( 0 < \delta_0 \leq \delta_0 \), and \( T > 0 \) small, there is a linear operator \( T_1 \) such that \( g = T_1[f] \) satisfies (A.5) for some constant \( c \) and
\[
\|g\|_{**} \leq C \|f\|_{**} + |c|. \tag{A.8}
\]

The constant \( C \) is independent of \( T, \delta_0 \).

First we want to find an inverse for \( \tilde{L}_0 \), namely given \( f \) find \( g \) such that \( \tilde{L}_0 \hat{g} = f \).

To do this, we differentiate this equation and we get
\[
\frac{\hat{g}(t)}{1 - \delta_0} + \frac{\hat{g}(t)}{\log(T - t)} = \frac{1}{1 - \delta_0 \log(T - t)} \frac{\dot{f}(t)}{1 - \delta_0}. \tag{A.9}
\]

Then we can write a particular solution for \( \hat{g} \) to (A.9) as
\[
\hat{g}(t) = \frac{f(t)}{(1 - \delta_0) \log(T - t)} + \beta - 1 \frac{\hat{g}(t)}{\log(T - t)} - \beta \int_t^T \frac{|\log(T - s)|^{\beta - 2}}{T - s} f(s) ds, \tag{A.10}
\]
where \( \beta = \frac{2 - \delta_0}{\delta_0} \) and where we have assumed that \( |\log(T - s)|^{\beta - 2} f(s) \) is integrable near \( T \) (for example \( f(s) = O(|\log(T - s)|^{-k}) \) with \( k > \beta - 1 \) suffices).

Define the operator
\[
T_0[f] = g, \tag{A.11}
\]
where \( g \) is such that \( \hat{g} \) is given by (A.10) and \( g(T) = 0 \). Note that \( g = T_0[f] \) solves (A.9) and therefore \( \tilde{L}_0 \hat{g} = f + c \) for some constant \( c \).

**Lemma A.3.** Assume \( k > \beta - 1 \). Then for \( f \in C([-T, T]; \mathbb{C}) \) with \( f(T) = 0 \)
\[
\|T_0[f]\|_{**} \leq C \|f\|_{**}. \tag{A.12}
\]

The constant is independent of \( T, \kappa \).

**Proof.** This is direct from (A.10). \( \Box \)

**Proof of Lemma A.2.** We construct \( g \) as a solution of the fixed point problem
\[
g = T_0 \left[ f - \eta(t)^{\frac{1}{T}} \tilde{L}_1 \hat{g} \right].
\]
where \( T_0 \) is the operator constructed in (A.11) and \( \eta \) is the cut-off function (A.4). By Lemma A.3

\[
\|T_0[\tilde{L}_1[g]]\|_{*,k+1} \leq \frac{C}{k+1-\beta} \|\tilde{L}_1[g]\|_{**,k}.
\]

Let us analyze the different terms in \( \tilde{L}_1 \), which we denote by

\[
\tilde{L}_1 = \sum_{j=1}^{4} \tilde{L}_{1j}
\]

where

\[
\tilde{L}_{11}[g](t) = \int_{t-(T-t)^{1+\delta_0}}^{t-(T-t)^{1+\delta_0}} \frac{\dot{g}(s)}{t-s} ds
\]

\[
\tilde{L}_{12}[g](t) = \int_{t-(T-t)^{1+\delta_0}}^{t} \frac{\dot{g}(s)}{T-s} ds
\]

\[
\tilde{L}_{13}[g](t) = \int_{-T}^{t-(T-t)^{1+\delta_0}} \dot{g}(s) \left( \frac{1}{t-s} - \frac{1}{T-s} \right) ds
\]

\[
\tilde{L}_{14}[g](t) = (4 \log(|\log(T-t)|) - 2 \log(|\log(T)|)) \dot{g}(t).
\]

Then we have

\[
|\tilde{L}_{11}[g](t)| \leq \|g\|_{*,k+1} \int_{t-(T-t)^{1+\delta_0}}^{t-(T-t)^{1+\delta_0}} \frac{1}{(t-s) \log(T-s)^{k+1}} ds
\]

\[
\leq \|g\|_{*,k+1} \int_{t-(T-t)^{1+\delta_0}}^{t-(T-t)^{1+\delta_0}} \frac{1}{\log(T-t)^{k+1}} ds
\]

\[
\leq \|g\|_{*,k+1} \frac{\delta_0}{\log(T-t)^{k}}
\]

and therefore

\[
\|\tilde{L}_{11}[g]\|_{**,k} \leq \delta_0 \|g\|_{*,k+1}.
\]

We also find that

\[
|\tilde{L}_{12}[g](t)| \leq \|g\|_{*,k+1} \int_{t-(T-t)^{1+\delta_0}}^{t} \frac{1}{(T-s) \log(T-s)^{k+1}} ds
\]

\[
\leq \|g\|_{*,k+1} \int_{t-(T-t)^{1+\delta_0}}^{t} \frac{1}{T-s} ds
\]

\[
\leq \|g\|_{*,k+1} \int_{t-(T-t)^{1+\delta_0}}^{t} \frac{1}{T-s} ds
\]

\[
\leq \|g\|_{*,k+1} \frac{1}{\log(T-t)^{k+1}} \log(2),
\]

which implies

\[
\|\tilde{L}_{12}[g]\|_{**,k} \leq \frac{\log(2)}{|\log(T)|} \|g\|_{*,k+1}.
\]
Concerning \( \tilde{L}_{13} \) we have
\[
|\tilde{L}_{13}[g](t)| \leq \|g\|_{*,k+1} \int_{-T}^{t-(T-t)} \frac{1}{\log(T-s)^{k+1}} \left( \frac{1}{t-s} - \frac{1}{T-s} \right) ds
\]
\[
\leq C\|g\|_{*,k+1}(T-t) \int_{-T}^{t-(T-t)} \frac{1}{(T-s)^2 \log(T-s)^{k+1}} ds
\]
\[
\leq C\|g\|_{*,k+1} \frac{1}{|\log(T-t)|^{k+1}}
\]
and this gives
\[
\|\tilde{L}_{13}[g]\|_{*,k} \leq \frac{C}{|\log(T)|} \|g\|_{*,k+1}.
\]
Finally
\[
|\tilde{L}_{14}[g](t)| \leq C\|g\|_{*,k+1} \frac{\log(|\log(T-t)|) + \log(|\log(T)|)}{|\log(T-t)|^{k+1}}
\]
and hence using that \( \kappa = O(1) \) we get
\[
\|\tilde{L}_{14}[g]\|_{*,k} \leq \frac{C \log(|\log(T)|)}{|\log T|} \|g\|_{*,k+1}.
\]

Therefore
\[
\|T_0[\tilde{L}_1[g]]\|_{*,k+1} \leq \frac{C}{k+1-\beta} \|\tilde{L}_1[g]\|_{*,k}
\]
\[
\leq \frac{C}{k+1-\beta} \left( \delta_0 + \frac{1}{|\log T|} + \frac{\log |\log T|}{|\log T|} \right) \|g\|_{*,k+1}.
\]
we get a contraction if \( \delta_0 > 0 \) is fixed small and then \( T > 0 \) is sufficiently small.

Next we are going to show that there is \( p \) satisfying (A.2). Let \( T_1 \) be the operator constructed in Lemma A.2 for \( T > 0, \delta_0 > 0 \) small. We will apply inequality (A.8) with \( k < 2 \) close to 2. The constant in this inequality remains bounded as \( \delta_0 \to 0^+ \), because \( \beta = \frac{2-k}{1-\delta_0} \to 2 \) as \( \delta_0 \to 0^+ \).

To obtain a function \( p \) satisfying (A.2) it is sufficient to solve the fixed point problem
\[
p_1 = A[p_1]
\]}
where
\[
A[p_1] = T_1 \left[ -\tilde{E} + 2 \frac{d}{dt} |p| - (\tilde{E}[p](t) - \tilde{E}[p](T)) \right]
\]
\( p = p_{0,\kappa} + p_1 \) and \( \tilde{E}(t) = E(t) - E(T) \).

**Lemma A.4.** Let \( k > 0, k < 2 \) close to 2 and \( \delta_0 > 0 \) small. Then for \( T > 0 \) small there is a function \( p_1 \) satisfying (A.13) and moreover
\[
\|p_1\|_{*,k+1} \leq M
\]
where
\[
M = C_0 |\log(T)|^{k-1} \log(|\log(T)|)^2,
\]}
with \( C_0 \) a fixed large constant.
For the proof we use the norm (A.7) with $k < 2$, $k$ close to 2 so $k + 1 < 3$ is close to 3. We work with $\mathbf{p}_1$ in the space $X = C([-T, T]; C) \cap C^1([-T, T]; C)$ with the norm $\| \cdot \|_{*, k+1}$ defined in (A.6). By Lemma A.2
\[
\| A[p_1] \|_{*, k+1} \leq C (\| E - E(T) \|_{*, k} + \| \tilde{B}[p](t) - \tilde{B}[p](T) \|_{*, k})
\]
and by Lemma A.1
\[
\| E - E(T) \|_{*, k} \leq C |\log T|^{k-1} \log(|\log T|)^2.
\]
We take in $X$ the closed ball $\overline{B}_M(0)$ of center 0 and radius $M$. The proof of Lemma A.4 consists in showing that $A : \overline{B}_M(0) \to \overline{B}_M(0)$ is a contraction. The estimates required for this are contained in the following lemmas.

**Lemma A.5.**

a) If $a > 1$, $b > 0$ then
\[
\int_{-T}^{t - \lambda_0(t)^2} \frac{1}{(t-s)^a \log(T-s)^b} ds \leq C \frac{\lambda_0(t)^{(2(1-a))}}{|\log(T-t)|^b}, \quad t \in [0, T].
\]

b) If $\mu \in (0, 1)$, $l \in \mathbb{R}$ then
\[
\int_{-T}^{t - \lambda_0(t)^2} \frac{(T-s)^\mu}{(t-s)^a \log(T-s)^b} ds \leq C \frac{(T-t)^\mu}{\lambda_0(t)^2 |\log(T-t)|^l}.
\]

**Proof.** Let us start with property a). Consider first $t \in [0, T]$. Then we can write
\[
\int_{-T}^{t - \lambda_0(t)^2} \frac{1}{(t-s)^a \log(T-s)^b} ds = \int_{-T}^{t - (T-t)} \frac{1}{(t-s)^a \log(T-s)^b} ds + \int_{t - (T-t)}^{t - \lambda_0(t)^2} \frac{1}{(t-s)^a \log(T-s)^b} ds.
\]

Then
\[
\int_{-T}^{t - (T-t)} \frac{1}{(t-s)^a \log(T-s)^b} ds \leq C \int_{-T}^{t - (T-t)} \frac{1}{(T-s)^a \log(T-s)^b} ds \leq C \frac{|(T-t)|^{1-a}}{|\log(T-t)|^b} ds \leq C \frac{\lambda_0(t)^{(2(1-a))}}{|\log(T-t)|^b}.
\]

The other integral is
\[
\int_{-(T-t)}^{t - \lambda_0(t)^2} \frac{1}{(t-s)^a \log(T-s)^b} ds \leq \frac{1}{|\log(T-t)|^b} \int_{-(T-t)}^{t - \lambda_0(t)^2} \frac{1}{(t-s)^a} ds \leq C \frac{\lambda_0(t)^{(2(1-a))}}{|\log(T-t)|^b}.
\]

Now consider $t \in [-T, 0]$:
\[
\int_{-T}^{t - \lambda_0(t)^2} \frac{1}{(t-s)^a \log(T-s)^b} ds \leq \frac{1}{|\log(T-t)|^b} \int_{-T}^{t - \lambda_0(t)^2} \frac{1}{(t-s)^a} ds \leq C \frac{\lambda_0(t)^{(2(1-a))}}{|\log(T-t)|^b}.
\]
As for property b). Again consider first $t \in [0,T]$. Then

\[
\int_{-T}^{t-\lambda_0(t)^2} \frac{(T-s)^\mu}{(t-s)^2} \log|T-s| t^t \, ds = \int_{-T}^{t-(T-t)} \ldots + \int_{t-(T-t)}^{t-\lambda_0(t)^2}.
\]

Then

\[
\int_{-T}^{t-(T-t)} \frac{(T-s)^\mu}{(t-s)^2} \log|T-s| t^t \, ds \leq C \int_{-T}^{t-(T-t)} \frac{(T-s)^{\mu-2}}{|\log(T-s)|} \, ds \leq C \frac{(T-t)^{\mu-1}}{|\log(T-t)|} t^t.
\]

and

\[
\int_{t-(T-t)}^{t-\lambda_0(t)^2} \frac{(T-s)^\mu}{(t-s)^2} \log|T-s| t^t \, ds \leq C \frac{(T-t)^\mu}{|\log(T-t)|} \int_{t-(T-t)}^{t-\lambda_0(t)^2} \frac{1}{(t-s)^2} \, ds \leq C \frac{(T-t)^\mu}{\lambda_0(t)^2 |\log(T-t)|} t^t.
\]

The case $t \in [-T,0]$ is handled similarly as in part a).

Lemma A.6. Let $M$ be given by (A.15). For $\|p_1\|_{*,k+1} \leq M$ we have

\[
\|\tilde{E}[p_{0,\kappa} + p_1] (\cdot) - \tilde{E}[p_{0,\kappa}] (T)\|_{**,k} \leq C |\log(T)|^{k-1}.
\]

Proof. For these estimates it is useful to notice that with the choice of $M = C |\log(T)|^{k-1} \log(|\log(T)|)^2$, if $\|p_1\|_{*,k+1} \leq M$ we have

\[
\left| \frac{p_1}{p_{0,\kappa}} \right| \leq \frac{M}{\kappa|\log(T)|^k} < 1
\]

for $T > 0$ small.

Let us write

\[
\tilde{E}_{i,a}[p](t) = \int_{-T}^{t-\lambda_0(t)^2} \frac{p(s)}{t-s} \left( 2 \Gamma_i \left( \frac{|p(t)|^2}{t-s} \right) - 1 \right) \, ds \quad \text{(A.16)}
\]

\[
\tilde{E}_{i,b}[p](t) = 2 \int_{t-\lambda_0(t)^2}^{t} \frac{p(s)}{t-s} \Gamma_i \left( \frac{|p(t)|^2}{t-s} \right) \, ds.
\]

so that

\[
\tilde{B}[p](t) = \frac{p(t)}{|p(t)|} \text{Re} \left( \frac{\bar{p}(t)}{|p(t)|} \tilde{E}_{1,a}[p](t) \right) + i \frac{p(t)}{|p(t)|} \text{Im} \left( \frac{\bar{p}(t)}{|p(t)|} \tilde{E}_{2,a}[p](t) \right)
\]

\[
+ \frac{p(t)}{|p(t)|} \text{Re} \left( \frac{\bar{p}(t)}{|p(t)|} \tilde{E}_{1,b}[p](t) \right) + i \frac{p(t)}{|p(t)|} \text{Im} \left( \frac{\bar{p}(t)}{|p(t)|} \tilde{E}_{2,b}[p](t) \right).
\]

Then to prove the statement of the lemma it is sufficient to show that

\[
|\tilde{E}_{i,a}[p](t)| + |\tilde{E}_{i,b}[p](t)| \leq C \frac{|\log T|}{|\log(T-t)|^2}.
\]

(A.18)
Using Lemma A.5 we find

\[ |\tilde{B}_{i,a}[p](t)| \leq C\lambda_0(t)^2 \int_{-\tau}^{t-\lambda_0(t)^2} \frac{|\tilde{p}(s)|}{(t-s)^2} \, ds \]

\[ \leq C\lambda_0(t)^2 |\log T| \int_{-\tau}^{t-\lambda_0(t)^2} \frac{1}{(t-s)^2 |\log(T-s)|^2} \, ds \]

\[ \leq C \frac{|\log T|}{|\log(t-T)|^2}. \]

Similarly

\[ |\tilde{B}_{i,b}[p](t)| \leq C \frac{1}{\lambda_0(t)^2} \int_{t-\lambda_0(t)^2}^{t} |\tilde{p}(s)| \, ds \]

\[ \leq C \frac{|\log T|}{|\log(t-T)|^2}. \]

This proves (A.18). \( \square \)

**Lemma A.7.** Let \( M \) be given by (A.15). For \( \| p_i \|_{*,k+1} \leq M \), \( i = 1, 2 \) we have

\[ \| \tilde{B} [p_{0,0} + p_1] - \tilde{B} [p_{0,0} + p_2] \|_{*,k} \leq C \frac{1}{|\log T|^{k+1}} \| p_1 - p_2 \|_{*,k+1}. \]

**Proof.** By (A.17) it is sufficient to obtain the estimate

\[ |D_{i,a}| \leq C \frac{1}{|\log(T-t)|^{k+1}} \| p_1 - p_2 \|_{*,k+1}, \]  \hspace{1cm} (A.19)

where

\[ D_{i,a} = \frac{(p_{0,0} + p_1)(t)}{|(p_{0,0} + p_1)(t)|} \text{Re} \left( \frac{(p_{0,0} + \tilde{p}_1)(t)}{|(p_{0,0} + \tilde{p}_1)(t)|} \tilde{B}_{i,a}[p_{0,0} + p_1](t) \right) \]

\[ - \frac{(p_{0,0} + p_2)(t)}{|(p_{0,0} + p_2)(t)|} \text{Re} \left( \frac{(p_{0,0} + \tilde{p}_2)(t)}{|(p_{0,0} + \tilde{p}_2)(t)|} \tilde{B}_{i,a}[p_{0,0} + p_2](t) \right), \]

and similar differences for the operators \( \tilde{B}_{i,b} \). We write

\[ D_{i,a} = \int_0^1 \frac{d}{d\zeta} \left[ \frac{(p_{0,0} + c)(t)}{|(p_{0,0} + c)(t)|} \text{Re} \left( \frac{(p_{0,0} + \tilde{c})(t)}{|(p_{0,0} + \tilde{c})(t)|} \tilde{B}_{i,a}[p_{0,0} + c](t) \right) \right] \, d\zeta \]

where \( p_c = \zeta p_1 + (1 - \zeta)p_2 \), and note that

\[ \frac{d}{d\zeta} \left[ \frac{(p_{0,0} + c)(t)}{|(p_{0,0} + c)(t)|} \text{Re} \left( \frac{(p_{0,0} + \tilde{c})(t)}{|(p_{0,0} + \tilde{c})(t)|} \tilde{B}_{i,a}[p_{0,0} + c](t) \right) \right] \]

\[ = \text{Re} \left( \frac{(p_{0,0} + c)(t)}{|(p_{0,0} + c)(t)|} \tilde{B}_{i,a}[p_{0,0} + c](t) \right) \frac{d}{d\zeta} \left( \frac{(p_{0,0} + \tilde{c})(t)}{|(p_{0,0} + \tilde{c})(t)|} \tilde{B}_{i,a}[p_{0,0} + \tilde{c}](t) \right) \]

\[ + \frac{(p_{0,0} + c)(t)}{|(p_{0,0} + c)(t)|} \text{Re} \left( \tilde{B}_{i,a}[p_{0,0} + c](t) \frac{d}{d\zeta} \left( \frac{(p_{0,0} + \tilde{c})(t)}{|(p_{0,0} + \tilde{c})(t)|} \tilde{B}_{i,a}[p_{0,0} + \tilde{c}](t) \right) \right) \]

\[ + \frac{(p_{0,0} + c)(t)}{|(p_{0,0} + c)(t)|} \text{Re} \left( \frac{(p_{0,0} + \tilde{c})(t)}{|(p_{0,0} + \tilde{c})(t)|} \frac{d}{d\zeta} \tilde{B}_{i,a}[p_{0,0} + \tilde{c}](t) \right). \]
We estimate the first term above

\[
\begin{aligned}
& \left| \frac{d}{d \zeta} \left( \frac{p_{0, \kappa} + p_\zeta(t)}{|p_{0, \kappa} + p_\zeta(t)|} \right) \right| \\
& = \left| \frac{(p_1 - p_2)(t)}{|(p_{0, \kappa} + p_\zeta(t))|} \cdot \frac{(p_{0, \kappa} + p_\zeta(t))(p_{0, \kappa} + p_\zeta(t)) \cdot (p_1 - p_2)(t)}{|(p_{0, \kappa} + p_\zeta(t))^3|} \right| \\
& \leq 2 \left| (p_1 - p_2)(t) \right| \left| (p_{0, \kappa} + p_\zeta(t)) \right|
\end{aligned}
\]

Using (A.18)

\[
\begin{aligned}
& \left| \Re \left( \frac{(p_{0, \kappa} + p_\zeta(t))}{|p_{0, \kappa} + p_\zeta(t)|} \right) B_{i, a}[p_{0, \kappa} + p_\zeta](t) \right| \frac{d}{d \zeta} \left( \frac{(p_{0, \kappa} + p_\zeta(t))}{|p_{0, \kappa} + p_\zeta(t)|} \right) \\
& \leq C \left| (p_1 - p_2)(t) \right| \frac{|\log T|}{|(p_{0, \kappa} + p_\zeta(t))| |\log(T - t)|^2} \\
& \leq \frac{C}{|\log(T - t)|^{k+1}} \|p_1 - p_2\|_{*, k+1}.
\end{aligned}
\]

The second term in (A.20) is estimated analogously.

Let us consider

\[
\begin{aligned}
d & \frac{d}{d \zeta} B_{i, a}[p_{0, \kappa} + p_\zeta](t) \\
& = \int_{-T}^{t-\lambda_0(t)^2} \frac{\dot{p}_1(s) - \dot{p}_2(s)}{t-s} \left( 2 \Gamma_i \left( \frac{|(p_{0, \kappa} + p_\zeta(t))|^2}{t-s} \right) - 1 \right) \, ds \\
& + 4 (p_{0, \kappa}(t) + p_\zeta(t)) \cdot (p_1(t) - p_2(t)) \\
& \quad \cdot \int_{-T}^{t-\lambda_0(t)^2} \frac{(p_{0, \kappa} + p_\zeta(s))}{(t-s)^2} \left( 2 \Gamma_i \left( \frac{|(p_{0, \kappa} + p_\zeta(t))|^2}{t-s} \right) - 1 \right) \, ds.
\end{aligned}
\]

We estimate the first term above

\[
\begin{aligned}
& \left| \int_{-T}^{t-\lambda_0(t)^2} \frac{\dot{p}_1(s) - \dot{p}_2(s)}{t-s} \left( 2 \Gamma_i \left( \frac{|(p_{0, \kappa} + p_\zeta(t))|^2}{t-s} \right) - 1 \right) \, ds \right| \\
& \leq C \int_{-T}^{t-\lambda_0(t)^2} \frac{|\dot{p}_1(s) - \dot{p}_2(s)| |(p_{0, \kappa} + p_\zeta(t))|^2}{t-s} \, ds \\
& \leq C\|p_1 - p_2\|_{*, k+1} \lambda_0(t)^2 \int_{-T}^{t-\lambda_0(t)^2} \frac{1}{(t-s)^2 |\log(T - s)|^{k+1}} \, ds
\end{aligned}
\]

and by Lemma A.5

\[
\int_{-T}^{t-\lambda_0(t)^2} \frac{1}{(t-s)^2 |\log(T - s)|^{k+1}} \, ds \leq \frac{C}{\lambda_0(t)^2 |\log(T - t)|^{k+1}}.
\]

Therefore

\[
\begin{aligned}
& \left| \int_{-T}^{t-\lambda_0(t)^2} \frac{\dot{p}_1(s) - \dot{p}_2(s)}{t-s} \left( 2 \Gamma_i \left( \frac{|(p_{0, \kappa} + p_\zeta(t))|^2}{t-s} \right) - 1 \right) \, ds \right| \\
& \leq \frac{C}{|\log(T - t)|^{k+1}} \|p_1 - p_2\|_{*, k+1}.
\end{aligned}
\]
For the second term in (A.21) we compute
\[
\left| (p_{0,\kappa}(t) + p_{\zeta}(t)) \cdot (p_1(t) - p_2(t)) \right| \int_{-\lambda_0(t)^2}^{t - \lambda_0(t)^2} \frac{(p_{0,\kappa} + \hat{p}_{\zeta})(s)}{(t - s)^2} \frac{\Delta_i \left( \frac{|(p_{0,\kappa} + p_{\zeta})(s)|^2}{(t - s)^2} \right)}{t - s} ds
\]
\[
\leq C \lambda_0(t) \|p_1 - p_2\|_{*,k+1} \int_{-\lambda_0(t)^2}^{t - \lambda_0(t)^2} \frac{\lambda_0(s)}{(t - s)^2} ds
\]
\[
\leq \frac{C}{\|\log(T - t)\|^{k+1}} \|p_1 - p_2\|_{*,k+1}.
\]
Thus we have obtained the estimate (A.19).

The estimate of \(D_{i,b} \) defined by
\[
D_{i,b} = \frac{(p_{0,\kappa} + p_1(t))}{(p_{0,\kappa} + p_2(t))} \text{Re} \left( \frac{(p_{0,\kappa} + \hat{p}_{\zeta})(t)}{|(p_{0,\kappa} + p_1(t))|} \tilde{B}_{i,b}[p_{0,\kappa} + p_1(t)] \right)
\]

is very similar, the only difference appears in
\[
\frac{d}{d\zeta} \tilde{B}_{i,b}[p_{0,\kappa} + p_1(t)]
\]
\[
= 2 \int_{-\lambda_0(t)^2}^{t} \frac{\hat{p}_1(s) - \hat{p}_2(s)}{t - s} \Gamma_i \left( \frac{|(p_{0,\kappa} + p_{\zeta})(t)|^2}{t - s} \right) ds
\]
\[
+ 4(p_{0,\kappa}(t) + p_{\zeta}(t)) \cdot (p_1(t) - p_2(t)) \int_{-\lambda_0(t)^2}^{t} \frac{(p_{0,\kappa} + \hat{p}_{\zeta})(s)}{(t - s)^2} \frac{\Gamma_i \left( \frac{|(p_{0,\kappa} + p_{\zeta})(t)|^2}{t - s} \right)}{t - s} ds.
\]
We estimate the first term above
\[
\left| \int_{-\lambda_0(t)^2}^{t} \frac{\hat{p}_1(s) - \hat{p}_2(s)}{t - s} \Gamma_i \left( \frac{|(p_{0,\kappa} + p_{\zeta})(t)|^2}{t - s} \right) ds \right|
\]
\[
\leq \frac{C}{\|\log(T - t)\|^{k+1}} \int_{-\lambda_0(t)^2}^{t} |\hat{p}_1(s) - \hat{p}_2(s)| ds
\]
\[
\leq \frac{C}{\|\log(T - t)\|^{k+1}} \|p_1 - p_2\|_{*,k+1}.
\]
The second term is estimated by
\[
\left| (p_{0,\kappa}(t) + p_{\zeta}(t)) \cdot (p_1(t) - p_2(t)) \int_{-\lambda_0(t)^2}^{t} \frac{(p_{0,\kappa} + \hat{p}_{\zeta})(s)}{(t - s)^2} \frac{\Gamma_i \left( \frac{|(p_{0,\kappa} + p_{\zeta})(t)|^2}{t - s} \right)}{t - s} ds \right|
\]
\[
\leq C \lambda_0(t) \|p_1 - p_2\|_{*,k+1} \frac{T - t}{\|\log(T - t)\|^{k+1}} \frac{1}{\lambda_0(t)^2} \int_{-\lambda_0(t)^2}^{t} \frac{\log T}{\|\log(T - s)\|^2} ds
\]
\[
\leq \frac{C}{\|\log(T - t)\|^{k+1}} \|p_1 - p_2\|_{*,k+1}.
\]
We conclude that
\[
|D_{i,b}| \leq \frac{C}{\|\log(T - t)\|^{k+1}} \|p_1 - p_2\|_{*,k+1},
\]
and this combined with (A.19) gives the result of the lemma.
Proof of Lemma A.4. By Lemmas A.6 and A.7 and direct estimates for the term \( \frac{d}{dt}|p| \) appearing in (A.2) we see that \( A \) is a contraction in the ball \( \mathcal{B}_M \) of \( X \) (defined with the norm \( \| \cdot \|_{s,k+1} \)).

Next we want to show that the solution \( p_1 \) constructed in Lemma A.4 has a good Hölder behavior, which will allow us to estimate \( R_{\delta_0}[p_1] \).

**Lemma A.8.** Let \( p_1 \) be the solution constructed in Lemma A.4. Then

\[
|\dot{p}_1(t)| \leq C \frac{\log T}{|\log(T - t)|^3(T - t)} \\
|\frac{d^3}{dt^3}p_1(t)| \leq C \frac{\log T}{|\log(T - t)|^3(T - t)^2}.
\]

Proof. Note that the fixed point problem (A.13) is equivalent to

\[
\tilde{L}_0[p_1] + \eta\left(\frac{t}{T}\right)\tilde{L}_1[p_1] + E - 2\frac{d|p|}{dt} + \tilde{B}[p](t) = c, \quad t \in [-T, T],
\]

for some constant \( c \).

We differentiate equation (A.24):

\[
(1 - \delta_0)|\log(T - t)|\dot{p}_1 + (1 - \delta_0)\frac{\ddot{p}_1}{T - t} + \frac{1}{T}\eta'(\frac{t}{T})\tilde{L}_1[p_1] + \eta(\frac{t}{T})\frac{d\tilde{L}_1[p_1]}{dt} \\
+ \frac{d}{dt}\tilde{E} + \frac{d}{dt}\tilde{B}[p](t) - 2\frac{d^2}{dt^2}|p| = 0,
\]

where \( p = p_{0,k} + p_1 \). Using (A.25), using (A.14), we see that

\[
(1 - \delta_0)|\log(T - t)|\dot{p}_1 + \eta\left(\frac{t}{T}\right)\tilde{L}_1[p_1] + D\tilde{B}[p; p_1](t) - 2\frac{p \cdot \ddot{p}_1}{|p|^3} = h,
\]

with a function \( h \) satisfying

\[
|h(t)| \leq C \frac{\log T}{|\log(T - t)|^2(T - t)},
\]

and where \( D\tilde{B}[p; v](t) \) is the derivative of \( \tilde{B} \) at \( p \) in the direction \( v \), and is given by

\[
D\tilde{B}[p; v](t) = \left( \frac{v(t)}{|p(t)|} - \frac{p(p \cdot v(t))}{|p(t)|^3} \right) \text{Re} \left( \frac{\ddot{p}(t)}{|p(t)|} \tilde{B}_1[p](t) \right) \\
+ \frac{p(t)}{|p(t)|} \text{Re} \left( \frac{\ddot{v}(t)}{|p(t)|} \tilde{B}_1[p](t) \right) \\
+ \frac{p(t)}{|p(t)|} \text{Re} \left( \frac{\ddot{p}(t)}{|p(t)|} D\tilde{B}_1[p; v](t) \right) \\
+ i \left( \frac{v(t)}{|p(t)|} - \frac{p(p \cdot v(t))}{|p(t)|^3} \right) \text{Im} \left( \frac{\dot{p}(t)}{|p(t)|} \tilde{B}_2[p](t) \right) \\
+ i \frac{p(t)}{|p(t)|} \text{Im} \left( \frac{\ddot{v}(t)}{|p(t)|} \tilde{B}_2[p](t) \right) \\
+ i \frac{p(t)}{|p(t)|} \text{Im} \left( \frac{\dot{p}(t)}{|p(t)|} D\tilde{B}_2[p; v](t) \right).
\]
Here $D\tilde{B}_{i}[p;v](t)$ represent the derivatives of the operators $\tilde{B}_{i}$ at $p$ and direction $v$ and they are explicitly given by

$$
D\tilde{B}_{i}[p;v](t) = \int_{-T}^{t-\lambda_{0}(t)^{2}} \frac{\hat{v}(s)}{t-s} \left(2\Gamma_{i} \left(\frac{|p(t)|}{t-s} \right)^{2} - 1 \right) ds \\
+ 4p(t) \cdot v(t) \int_{-T}^{t-\lambda_{0}(t)^{2}} \frac{\tilde{p}(s)}{(t-s)^{2}} \Gamma'_{i} \left(\frac{|p(t)|}{t-s} \right) ds \\
+ 2\int_{t-\lambda_{0}(t)^{2}}^{t} \frac{\hat{v}(s)}{t-s} \Gamma_{i} \left(\frac{|p(t)|}{t-s} \right) ds \\
+ 4p(t) \cdot v(t) \int_{t-\lambda_{0}(t)^{2}}^{t} \frac{\tilde{p}(s)}{(t-s)^{2}} \Gamma'_{i} \left(\frac{|p(t)|}{t-s} \right) ds.
$$

To prove estimate (A.27), first observe that with a calculation similar to Lemma A.1 we get

$$
\left| \frac{d}{dt} \tilde{E}(t) \right| \leq C \frac{|\log T|}{|\log(T-t)|^{2}(T-t)}.
$$

The other terms in $h$ are of this size or smaller. For example, for $\tilde{L}_{11}$ defined in (A.12), we find that

$$
\frac{d}{dt} \tilde{L}_{11}[\tilde{p}_{1}] = \frac{d}{dt} \int_{(T-t)^{1+\delta_{0}}}^{T-t} \frac{\tilde{p}_{1}(r)}{r} dr \\
= \int_{(T-t)^{1+\delta_{0}}}^{T-t} \frac{\tilde{p}_{1}(r)}{r} dr - \frac{\tilde{p}_{1}(T-t)}{T-t} + (1+\delta_{0}) \frac{\tilde{p}_{1}(T-t)^{1+\delta_{0}}}{T-t}
$$

and we see that

$$
\left| \frac{\dot{\lambda}_{1}(T-t)}{T-t} \right| \leq C \frac{|\log T|^{k-1} \log(|\log T|)^{2}}{|\log(T-t)|^{k+1}(T-t)} \leq C \frac{|\log T|}{|\log(T-t)|^{2}(T-t)}
$$

To deduce estimate (A.22) we use again mapping properties of $\tilde{L}_{1}$ and $D\tilde{B}[p;v](t)$, this time with the norm

$$
\|f\|_{*,j,m} = \sup_{t \in (-T,T)} (T-t)^{j} |\log(T-t)|^{m} |f(t)|.
$$

We note that (A.27) gives

$$
\|h\|_{*,1,2} \leq C |\log T|.
$$

We claim that

$$
\|\tilde{L}_{1}\tilde{p}\|_{*,j,m} \leq \left( \delta_{0} + \frac{C}{|\log T|} \right) \|\tilde{p}\|_{*,j,m+1}.
$$
We have

\[ |L_1[p]\phi(t)| \leq \|\phi\|_{*,j,m+1} \int_{t-(T-t)^{1+\delta_0}}^{t-(T-t)^{1+\delta_0}/2} \frac{1}{(T-s)^{3}|\log(T-s)|^{m+1}(t-s)} ds \]

\[ \leq \|\phi\|_{*,j,m+1} \int_{t-(T-t)^{1+\delta_0}/2}^{t-(T-t)^{1+\delta_0}/2} \frac{1}{(T-s)^{3}(t-s)} ds, \]

but

\[ \int_{t-(T-t)^{1+\delta_0}/2}^{t-(T-t)^{1+\delta_0}/2} \frac{1}{(T-s)^{3}(t-s)} ds = \int_{t-(T-t)^{1+\delta_0}/2}^{t-(T-t)^{1+\delta_0}/2} \frac{(T-t)^{\frac{1}{t}}}{r} dr \]

\[ = (T-t)^{-\frac{1}{t}} \int_{t-(T-t)^{1+\delta_0}/2}^{t-(T-t)^{1+\delta_0}/2} 1 + O\left(\frac{r}{T-t}\right) dr \]

\[ \leq (T-t)^{-\frac{1}{t}} (\delta_0 |\log(T-t)| + C). \]

Thus

\[ |L_1[g](t)| \leq \left( \delta_0 + \frac{C}{\log(T-t)^{-1}} \right) \frac{1}{(T-t)^{3}|\log(T-t)|^{m+1}} \|\phi\|_{*,j,m+1}, \]

and this proves (A.29).

A similar estimate holds for \( D\tilde{B}[p;v] \) and from (A.26) and (A.28) we deduce (A.22). The proof of (A.23) is analogous. \( \square \)

Now we can estimate the remainder \( R_{\delta_0}[p_1] \). Let us recall that up to here \( \delta_0 > 0 \) is a small constant so that the contraction mapping principle can be applied to \( A \) in Lemma A.4.

**Lemma A.9.** Let \( p_1 \) be the solution constructed in Lemma A.4. Then

\[ |R_{\delta_0}[p_1](t)| \leq C \frac{|\log T|}{|\log(T-t)|^{3}|T-t|^{\delta_0}}, \quad \text{(A.30)} \]

\[ \frac{d}{dt} R_{\delta_0}[p_1](t) \leq C \frac{|\log T|}{|\log(T-t)|^{3}|T-t|^{{\delta_0}-1}}. \quad \text{(A.31)} \]

**Proof.** We have, thanks to (A.22)

\[ |R_{\delta_0}[p_1](t)| \leq \int_{t-(T-t)^{1+\delta_0}}^{t-(T-t)^{1+\delta_0}/2} \frac{|p_1(t) - p_1(s)|}{t-s} ds \]

\[ \leq \sup_{s \in \{t-(T-t)^{1+\delta_0}, t-(T-t)^{1+\delta_0}/2\}} |p_1(s)|(T-t)^{1+\delta_0} \]

\[ \leq C \frac{|\log T|}{|\log(T-t)|^{3}|T-t|^{\delta_0}}. \]

This proves (A.30).
We compute
\[
\frac{d}{dt} R_{\delta_0}[\hat{p}_1](t) = -\frac{d}{dt} \int_{t-(T-t)^{1+\delta_0}}^{t-\lambda_0(t)\delta_0} \frac{\hat{p}_1(t) - \hat{p}_1(s)}{t-s} ds \\
= -\frac{d}{dt} \int_{\lambda_0(t)^2}^{(T-t)^{1+\delta_0}} \frac{\hat{p}_1(t) - \hat{p}_1(t-r)}{r} dr \\
= -(1 + \delta_0) \frac{\hat{p}_1(t - (T-t)^{1+\delta_0})}{T-t} - 2(1 + \delta_0) \frac{\hat{p}_1(t - \lambda_0(t)^2) \lambda_0(t)}{\lambda_0(t)} \\
- \int_{\lambda_0(t)^2}^{(T-t)^{1+\delta_0}} \frac{\hat{p}_1(t) - \hat{p}_1(t-r)}{r} dr.
\]

By (A.22) and (A.23) we get (A.31).

The final step to prove Lemma 6.1 is to correct once more the function \( p_{0,\kappa} + p_1 \) to achieve a smaller error. To do this, we consider \( p = p_{0,\kappa} + p_1 + p_2 \) and solve the following equation for \( p_2 \):

\[
S_0[p_2] - 2 \frac{d}{dt} |p_{0,\kappa} + p_1| + 2 \frac{d}{dt} |p_{0,\kappa} + p_1| + \tilde{E}|p_{0,\kappa} + p_1| + p_2 - \tilde{E}|p_{0,\kappa} + p_1| \\
= c - R_{\delta_0}[\hat{p}_1], \quad \text{in } [0, T],
\]

some constant \( c \), where now \( \delta \in (0, \frac{1}{2}) \) is arbitrary.

**Lemma A.10.** For any \( \delta \in (0, \frac{1}{2}) \) there is \( p_2 \) satisfying (A.32). Moreover
\[
|\hat{p}_2(t)| \leq C(T-t)^{\delta_0} \\
|\hat{p}_2(t)| \leq C(T-t)^{\delta_0-1}
\]

We omit the details of the proof of this lemma, since it is to that of Lemma A.4.

**Proof of Lemma 6.1.** Let \( p = p_{0,\kappa} + p_1 + p_2 \) with \( p_2 \) the solution constructed in Lemma A.10. Then for some constant \( c \)
\[
2 \int_{-T}^{T} \frac{\hat{p}_s(s)}{t-s} \Gamma_1 \left( \frac{\hat{p}_s(t)^2}{t-s} \right) ds + 2\hat{p}_s(t) - c = -R_{\delta}[\hat{p}_2],
\]
and using (A.33) we find
\[
|R_{\delta}[\hat{p}_2]| \leq \int_{t-(T-t)^{1+\delta_0}}^{t-\lambda_0(t)^2} |\hat{p}_2(t) - \hat{p}_2(s)| \frac{ds}{t-s} \\
\leq C(T-t)^{\delta_0}.
\]

Since \( \delta_0 > 0 \) and \( \delta \in (0, \frac{1}{2}) \) is arbitrary, we have \( \delta + \delta_0 > \frac{1}{2} \) in (6.13).

**Appendix B. Estimates for the solution of the heat equation**

**B.1. Proof of Lemma 5.1.** The proof of the estimates is done by analyzing the bounded solution \( \psi \) of
\[
\begin{aligned}
\partial_t \psi_0 &= \Delta \psi_0 + f \quad \text{in } \mathbb{R}^2 \times (0, T), \\
\psi_0(x, 0) &= 0 \quad x \in \mathbb{R}^2,
\end{aligned}
\]
with
\[
f(x, t) = \chi_{\{|y| \leq 2 \lambda_0(s)R(s)\}} \lambda_0(t)^{r-2} R(t)^{-a}.
\]
The solution to (5.6) is then given by $\psi = \psi_0 + \psi_1$ where $\psi_1$ solves the homogeneous heat equation in $\Omega \times (0, T)$ with boundary condition given by $-\psi_0$. In the sequel we prove that the estimates (5.7)–(5.8) are valid for $\psi_0$. Then the conclusion for $\psi_1$ follows from standard parabolic estimates. In what follows we denote by $\psi$ the bounded solution to (B.1).

**Proof of (5.7).** We have, using the heat kernel,

$$
\psi(x, t) = C \int_0^t \int \lambda_0(s)^{\nu-2} R(s)^{-a} \int_{|y| \leq 2\lambda_0(s)R(s)} e^{-\frac{|x-y|^2}{4(t-s)}} \, dy \, ds
$$

which is the desired estimate (5.7).

To show (5.8), we consider the integrals $\int_{t-\lambda_0(t)^2}^t$ and $\int_{t-\lambda_0(t)^2}^t$. We have

$$
\int_{t-\lambda_0(t)^2}^t \lambda_0(s)^{\nu-2} R(s)^{-a} \int_{|z| \leq 2\lambda_0(s)R(s)(t-s)^{-1/2}} e^{-|\tilde{x} - z|^2} \, dz \, ds
$$

which is the desired estimate (5.8).

**Proof of (5.8).** Using the heat kernel we have

$$
|\psi(x, t) - \psi(x, T)| \leq I_1 + I_2 + I_3,
$$
where

\[ I_1 = \int_0^{t-(T-t)} \int_{\mathbb{R}^2} |G(x-y, t-s) - G(x-y, T-s)| |f(y, s)| \, dy \, ds \]

\[ I_2 = \int_t^{t-(T-t)} \int_{\mathbb{R}^2} |G(x-y, t-s) - G(x-y, T-s)| |f(y, s)| \, dy \, ds \]

\[ I_3 = \int_t^T \int_{\mathbb{R}^2} |G(x-y, t-s) - G(x-y, T-s)| |f(y, s)| \, dy \, ds. \]

We estimate the first integral

\[ I_1 \leq (T-t) \int_0^1 \int_0^{t-(T-t)} \lambda_0(s) v^2 R(s)^{-a} \int_{|y| \leq 2\lambda_0(s)R(s)} |\partial_t G(x-y, t_v-s)| \, dy \, ds \, dv, \]

where \( t_v = vT + (1-v)(T-t) \). We have

\[ \int_{|y| \leq 2\lambda_0(s)R(s)} |\partial_t G(x-y, t_v-s)| \, dy \]

\[ \leq C \frac{1}{(t_v-s)^2} \int_{|y| \leq 2\lambda_0(s)R(s)} e^{-\frac{|x-y|^2}{t_v-s}} \left( 1 + \frac{|x-y|^2}{t_v-s} \right) \, dy \]

\[ \leq C \frac{1}{(t_v-s)^2} \int_{|z| \leq 2\lambda_0(s)R(s)(t_v-s)^{-1/2}} e^{-|z|^2} \left( 1 + |\tilde{x} - z|^2 \right) \, dz. \]

We then get

\[ \int_0^{t-(T-t)} \lambda_0(s) v^2 R(s)^{-a} \int_{|y| \leq 2\lambda_0(s)R(s)} |\partial_t G(x-y, t_v-s)| \, dy \, ds \]

\[ \leq C \int_0^{t-(T-t)} \lambda_0(s) v R(s)^2 - a \frac{ds}{(t_v-s)^2} \]

\[ \leq C \int_0^{t-(T-t)} \lambda_0(s) v R(s)^2 - a \frac{ds}{(T-s)^2} \]

\[ \leq C \frac{\log T^{|v-\beta(2-a)|/(T-t)^{v-\beta(2-a)}}}{|\log(T-t)|^{2v-2\beta(2-a)}}, \]

where we have used \( R(t) = \lambda_0(t)^{-\beta} \). Therefore

\[ I_1 \leq C \lambda_0(t) v^2 R(t)^{2-a}. \]

Next we estimate \( I_2 \):

\[ I_2 \leq \int_{t-(T-t)}^t \int_{|y| \leq 2\lambda_0(s)R(s)} |G(x-y, t-s)| \lambda_0(s) v^2 R(s)^{-a} \, dy \, ds \]

\[ + \int_{t-(T-t)}^t \int_{|y| \leq 2\lambda_0(s)R(s)} |G(x-y, T-s)| \lambda_0(s) v^2 R(s)^{-a} \, dy \, ds. \]
These two integrals are very similar. Let us compute the first one
\[
\int_{t-(T-t)}^{t} \int_{|y| \leq 2\lambda_0(s)R(s)} |G(x - y, t - s)| \lambda_0(s)^{\nu - 2} R(s)^{-a} \, dy \, ds
\]
\[
\leq C \int_{t-(T-t)}^{t} \frac{\lambda_0(s)^{\nu - 2} R(s)^{-a}}{t - s} \int_{|y| \leq 2\lambda_0(s)R(s)} e^{-\frac{|x - y|^2}{4(t-s)}} \, dy \, ds
\]
\[
= C \int_{t-(T-t)}^{t} \frac{\lambda_0(s)^{\nu - 2} R(s)^{-a}}{t - s} \int_{|z| \leq 2\lambda_0(s)R(s)(t-s)^{-1/2}} e^{-|\bar{z} - z|^2/4} \, dz \, ds.
\]
We split this integral in \( \int_{t-\lambda_0(t)^2}^{t-\lambda_0(t)^2} \) ... and estimate
\[
\int_{t-(T-t)}^{t} \lambda_0(s)^{\nu - 2} R(s)^{-a} \int_{|z| \leq 2\lambda_0(s)R(s)(t-s)^{-1/2}} e^{-|\bar{z} - z|^2/4} \, dz \, ds
\]
\[
\leq C \int_{t-(T-t)}^{t} \frac{\lambda_0(s)^{\nu} R(s)^{2-a}}{t - s} ds
\]
\[
\leq C\lambda_0(t)^\nu R(t)^{2-a} |\log(T-t)|.
\]
For the second part we have
\[
\int_{t-\lambda_0(t)^2}^{t} \lambda_0(s)^{\nu - 2} R(s)^{-a} \int_{|z| \leq 2\lambda_0(s)R(s)(t-s)^{-1/2}} e^{-|\bar{z} - z|^2/4} \, dz \, ds
\]
\[
\leq C\lambda_0(t)^\nu R(t)^{-a},
\]
and therefore, summarizing,
\[
\int_{t-(T-t)}^{t} \int_{|y| \leq 2\lambda_0(s)R(s)} |G(x - y, t - s)| \lambda_0(s)^{\nu - 2} R(s)^{-a} \, dy \, ds
\]
\[
\leq C\lambda_0(t)^\nu R(t)^{2-a} |\log(T-t)|.
\]
Similar computations show that
\[
\int_{t-(T-t)}^{t} \int_{|y| \leq 2\lambda_0(s)R(s)} |G(x - y, T - s)| \lambda_0(s)^{\nu - 2} R(s)^{-a} \, dy \, ds
\]
\[
\leq C\lambda_0(t)^\nu R(t)^{2-a} |\log(T-t)|
\]
and we obtain
\[
I_2 \leq C\lambda_0(t)^\nu R(t)^{2-a} |\log(T-t)|.
\]
Finally
\[
I_3 \leq C \int_{t}^{T} \int_{|y| \leq 2\lambda_0(s)R(s)} e^{-\frac{|x - y|^2}{4(t-s)}} \lambda_0(s)^{\nu - 2} R(s)^{-a} \, dy \, ds
\]
\[
\leq C \int_{t}^{T} \frac{\lambda_0(s)^{\nu - 2} R(s)^{-a}}{T - s} \int_{|z| \leq 2\lambda_0(s)R(s)(T-s)^{-1/2}} e^{-|\bar{z} - z|^2/4} \, dz \, ds
\]
\[
\leq C \int_{t}^{T} \frac{\lambda_0(s)^{\nu} R(s)^{2-a}}{(T - s)} ds
\]
\[
\leq C\lambda_0(t)^\nu R(t)^{2-a}.
\]
This finishes the proof of (5.8).
Proof of (5.9). Using the heat kernel we have

\[
|\nabla \psi(x, t)| \leq C \int_0^t \int_{|y| \leq 2\lambda_0(s)R(s)} e^{-\frac{|x-y|^2}{2(t-s)^2}}dyds
\]

\[
= C \int_0^t \int_{|z| \leq 2\lambda_0(s)R(s)} e^{-|\bar{z}-z|^2/4|\bar{z} - z|}dzds
\]

\[
\leq C \int_0^t \int_{|z| \leq 2\lambda_0(s)R(s)} e^{-\frac{|z|^2}{4(1 + |z|)}}dzds
\]

where \( \bar{x} = (t - s)^{-1/2} x \). Then

\[
\int_0^{t-1/2(T-t)} \int_{|z| \leq 2\lambda_0(s)R(s)} e^{-\frac{|z|^2}{4(1 + |z|)}}dzds
\]

\[
\leq C \int_0^{t-1/2(T-t)} \int_{|z| \leq 2\lambda_0(s)R(s)} e^{-\frac{|z|^2}{4}}e^{-\beta(2-a)ds}
\]

\[
\leq C \int_0^{t-1/2(T-t)} \lambda_0(s)^2 \int_{|z| \leq 2\lambda_0(s)R(s)} e^{-\frac{|z|^2}{4}}(1 + |z|)dzds
\]

\[
= C|\log T|^{\nu-\beta(2-a)} \int_0^{t-1/2(T-t)} \int_{|z| \leq 2\lambda_0(s)R(s)} e^{-\frac{|z|^2}{4}}(1 + |z|)dzds
\]

\[
\leq C\lambda_0(0)^{-1/2} R(0)^{2-a}T^{-1/2}
\]

\[
\leq C\lambda_0(0)^{-1} R(0)^{1-a}.
\]

Here we need \( \nu - 1/2 - \beta(2-a) > 0 \).

To estimate the integral

\[
\int_{t-1/2(T-t)} \lambda_0(s)^{\nu-2} R(s)^{-a} \int_{|z| \leq 2\lambda_0(s)R(s)(t-s)^{-1/2}} e^{-\frac{|z|^2}{4}(1 + |z|)}dzds
\]

let us define

\[
g(v) = \int_0^{2v} e^{-\rho^2/4}(1 + \rho)\rho d\rho
\]

so that

\[
\int_{t-1/2(T-t)} \lambda_0(s)^{\nu-2} R(s)^{-a}(t-s)^{-1/2} \int_{|z| \leq 2\lambda_0(s)R(s)(t-s)^{-1/2}} e^{-\frac{|z|^2}{4}(1 + |z|)}dzds
\]

\[
= \int_{t-1/2(T-t)} \lambda_0(s)^{\nu-2} R(s)^{-a} \int_{t-s}^{t} g \left( \frac{\lambda_0(s)R(s)}{(t-s)^{1/2}} \right) ds
\]

We change variables

\[
\frac{t-s}{\lambda_0(s)^2 R(s)^2} = u
\]

and note that

\[
\frac{du}{ds} = \frac{1}{\lambda_0(s)^2 R(s)^2} \left( 1 + \frac{2\lambda_0(s)}{\lambda_0(s)(t-s)} + \frac{2R(s)}{R(s)(t-s)} \right).
\]
Then
\[
\int_{t-\frac{1}{10}(T-t)}^{t} \frac{\lambda_0(s)^{\nu-2} R(s)^{-a}}{(t-s)^{1/2}} \int_{|z| \leq 2\lambda_0(s)R(s)(t-s)^{-1/2}} e^{-|z|^2/4(1+|z|)} \, dz \, ds \\
= \int_{0}^{\nu \lambda_0(t-\frac{1}{10}(T-t)-2)R(t-\frac{1}{10}(T-t))} \lambda_0(s)^{\nu-1} R(s)^{1-a} u^{-1/2} g(u^{-1/2}) \lambda_0(s) (t-s)^{1/2} \right| du \\
\leq C \lambda_0(t)^{\nu-1} R(t)^{1-a}.
\]
This establishes (5.9). \(\square\)

**Proof of (5.10).** Using the heat kernel we have
\[
\partial_{x_v} \psi(x, t) - \partial_{x_v} \psi(x, T) \\
= \int_{0}^{t} \int_{\mathbb{R}^2} (\partial_{x_v} G(x-y, t-s) - \partial_{x_v} G(x-y, T-s)) f(y, s) \, dy \, ds \\
- \int_{0}^{T} \int_{\mathbb{R}^2} \partial_{x_v} G(x-y, T-s) f(y, s) \, dy \, ds,
\]
and so
\[
|\partial_{x_v} \psi(x, t) - \partial_{x_v} \psi(x, T)| \leq I_1 + I_2 + I_3,
\]
where
\[
I_1 = \int_{0}^{t-(T-t)} \int_{\mathbb{R}^2} |\partial_{x_v} G(x-y, t-s) - \partial_{x_v} G(x-y, T-s)| f(y, s) \, dy \, ds \\
I_2 = \int_{t-(T-t)}^{T} \int_{\mathbb{R}^2} |\partial_{x_v} G(x-y, T-s) - \partial_{x_v} G(x-y, T-s)| f(y, s) \, dy \, ds \\
I_3 = \int_{t}^{T} \int_{\mathbb{R}^2} |\partial_{x_v} G(x-y, T-s)| f(y, s) \, dy \, ds.
\]
For the first integral, we have
\[
I_1 \leq C(T-t) \int_{0}^{1} \int_{0}^{t-(T-t)} \frac{\lambda_0(s)^{\nu-2} R(s)^{-a}}{(t_v-s)^{5/2}} \left\{ \int_{|y| \leq 2\lambda_0(s)R(s)} e^{-\frac{|x-y|^2}{4(t_v-s)^3}} \left( \frac{|x-y|}{(t_v-s)^{1/2}} + \frac{|x-y|^3}{(t_v-s)^{3/2}} \right) \, dy \right\} ds \, dv
\]
where \(t_v = vT + (1-v)(T-t)\). Changing variables
\[
\int_{|y| \leq 2\lambda_0(s)R(s)} e^{-\frac{|x-y|^2}{4(t_v-s)^3}} \left( \frac{|x-y|}{(t_v-s)^{1/2}} + \frac{|x-y|^3}{(t_v-s)^{3/2}} \right) \, dy \\
= (t_v-s) \int_{|z| \leq 2\lambda_0(s)R(s)(t-s)^{-1/2}} e^{-|\tilde{x}_v-z|^2/4(|\tilde{x}_v-z| + |\tilde{x}_v-z|^3)} \, dz
\]
where \( \tilde{x}_v = x(t_v - s)^{-1/2} \). We then need to estimate
\[
\int_0^{t-(T-t)} \frac{\lambda_0(s)\nu-2 R(s)^{-a}}{(t_v - s)^{3/2}} \int_{|z| \leq 2\lambda_0(s)R(s)(t-s)^{-1/2}} e^{-|\tilde{x}_v - z|^2/4} (|\tilde{x}_v - z| + |\tilde{x}_v - z|^3) \, dz \, ds
\]
\[
\leq C \int_0^{t-(T-t)} \frac{\lambda_0(s)\nu-2 R(s)^{-a}}{(T-s)^{3/2}} \int_0^{2\lambda_0(s)R(s)(t-s)^{-1/2}} e^{-\rho^2/4 (1 + \rho^3)} \rho \, d\rho \, ds.
\]
We note that for \( 0 \leq s \leq t - (T-t) \). Therefore
\[
\int_0^{t-(T-t)} \frac{\lambda_0(s)\nu-2 R(s)^{-a}}{(T-s)^{3/2}} e^{-\rho^2/4 (1 + \rho^3)} \rho \, d\rho \, ds
\]
\[
\leq C \int_0^{t-(T-t)} \frac{\lambda_0(s)\nu-2 R(s)^{-a}}{(T-s)^{3/2}} \lambda_0(s)R_0(s)(t-s)^{-1/2} \, ds
\]
\[
\leq C \int_0^{t-(T-t)} \frac{\lambda_0(s)\nu-1 R(s)^{-a}}{(T-s)^2} \, ds
\]
\[
\leq C(T-t)^{-1} \lambda_0(t)^{-1} R(t)^{1-a}.
\]
Therefore
\[
I_1 \leq C \lambda_0(t)^{-1} R(t)^{1-a}.
\]
To estimate \( I_2 \) it is sufficient to bound the terms
\[
\int_{t-(T-t)}^{t} \int_{|y| \leq 2\lambda_0(s)R(s)} |\nabla_x G(x - y, t - s)| \lambda_0(s)^{\nu-2} R^{-a} \, dy \, ds,
\]
\[
\int_{t-(T-t)}^{t} \int_{|y| \leq 2\lambda_0(s)R(s)} |\nabla_x G(x - y, T - s)| \lambda_0(s)^{\nu-2} R^{-a} \, dy \, ds.
\]
Let us start with:
\[
\int_{t-(T-t)}^{t} \int_{|y| \leq 2\lambda_0(s)R(s)} |\nabla_x G(x - y, t - s)| \lambda_0(s)^{\nu-2} R(s)^{-a} \, dy \, ds
\]
\[
\leq C \int_{t-(T-t)}^{t} \frac{\lambda_0(s)^{\nu-2} R(s)^{-a}}{(t-s)^{1/2}} \int_{|z| \leq 2\lambda_0(s)R(s)(t-s)^{-1/2}} e^{-|\tilde{x} - z|^2/4} |\tilde{x} - z| \, dz \, ds,
\]
\[
\leq C \int_{t-(T-t)}^{t} \frac{\lambda_0(s)^{\nu-2} R(s)^{-a}}{(t-s)^{1/2}} \int_0^{2\lambda_0(s)R(s)(t-s)^{-1/2}} e^{-\rho^2/4} (1 + \rho) \rho \, d\rho \, ds,
\]
where \( \tilde{x} = (t-s)^{-1/2} x \). We note that for \( s \in [t - (T-t), t] \) the inequality
\[
\frac{\lambda_0(s)R(s)}{(t-s)^{1/2}} \leq 1
\]
is equivalent to \( s \leq s^* \) for some \( s^* \in (t - (T-t), t) \), and that for \( s \leq s^* \)
\[
\int_0^{2\lambda_0(s)R(s)(t-s)^{-1/2}} e^{-\rho^2/4} (1 + \rho) \rho \, d\rho \leq C \lambda_0(s)^2 R(s)^2 (t-s)^{-1}.
\]
and therefore a similar proof gives the result. Let $I_3$ be given by \((5.11)\).

\[
\int_{t-(T-t)}^{s^*} \frac{\lambda_0(s)^{\nu-2} R(s)^{-a}}{(t-s)^{1/2}} \int_0^{2\lambda_0(s)R(s)(t-s)^{-1/2}} e^{-\rho^2 (1 + \rho) \rho \, dp \, ds}.
\]

Then

\[
\int_{t-(T-t)}^{s^*} \frac{\lambda_0(s)^{\nu-2} R(s)^{-a}}{(t-s)^{1/2}} \int_0^{2\lambda_0(s)R(s)(t-s)^{-1/2}} e^{-\rho^2 (1 + \rho) \rho \, dp \, ds} \leq C \lambda_0(t)^{\nu-2} R(t)^{2-a} \int_{t-(T-t)}^{s^*} \frac{1}{(t-s)^{3/2}} \, ds \leq C \lambda_0(t)^{\nu-1} R(t)^{1-a}.
\]

The integral on $[s^*, t]$ is estimated

\[
\int_{s^*}^{t} \frac{\lambda_0(s)^{\nu-2} R(s)^{-a}}{(t-s)^{1/2}} \int_0^{2\lambda_0(s)R(s)(t-s)^{-1/2}} e^{-\rho^2 (1 + \rho) \rho \, dp \, ds} \leq C \lambda_0(t)^{\nu-2} R(t)^{-a} \int_{s^*}^{t} \frac{1}{(t-s)^{1/2}} \, ds \leq C \lambda_0(t)^{\nu-1} R(t)^{1-a}.
\]

In the same way we get

\[
\int_{t-(T-t)}^{t} \int_{|y| \leq 2\lambda_0(s)R(s)} |\nabla_x G(x - y, T - s)| \lambda_0(s)^{-1} R^{-a} \, dyds \leq C \lambda_0(t)^{\nu-1} R(t)^{1-a},
\]

and therefore

\[
I_2 \leq C \lambda_0(t)^{\nu-1} R(t)^{1-a}.
\]

We deal now with $I_3$:

\[
I_3 \leq C \int_t^{T} \frac{\lambda_0(s)^{\nu-2} R(s)^{-a}}{(T-s)^2} \int_{|y| \leq 2\lambda_0(s)R(s)} e^{-\frac{|x-y|^2}{4(T-s)}} |x-y| \, dyds
\]

\[
\leq C \int_t^{T} \frac{\lambda_0(s)^{\nu-2} R(s)^{-a}}{(T-s)^{1/2}} \int_{|z|(T-s)^{1/2} \leq 2\lambda_0(s)R(s)} e^{-|\tilde{x}-z|^{2/4}} |\tilde{x}-z| \, dzds
\]

\[
\leq C \int_t^{T} \frac{\lambda_0(s)^{\nu-\beta(2-a)}}{(T-s)^{3/2}} ds
\]

\[
= C |\log T|^{\nu-\beta(2-a)} \int_t^{T} \frac{(T-s)^{\nu-\beta(2-a)-3/2}}{|\log(T-s)|^{2\nu-2\beta(2-a)}} ds
\]

\[
\leq |\log T|^{\nu-\beta(2-a)} (T-t)^{\nu-1/2-\beta(2-a)} |\log(T-t)|^{2-2\beta(2-a)}
\]

\[
\leq C \lambda_0(t)^{\nu-1} R(t)^{1-a}.
\]

\[\square\]

Proof of (5.11). Let $0 < t_1 < t_2 < T$. We assume that $t_2 < 2t_1$. In the other case a similar proof gives the result. Let $f$ be given by (B.2). Using the heat kernel we have

\[
|\partial_x \psi(x, t) - \partial_x \psi(x, T)| \leq I_1 + I_2 + I_3,
\]
where
\[ I_1 = \int_0^{t_1-(t_2-t_1)} \int_{\mathbb{R}^2} |\partial_x, G(x - y, t_1 - s) - \partial_x, G(x - y, t_2 - s)| f(y, s) \, dy \, ds \]
\[ I_2 = \int_{t_1-(t_2-t_1)}^{t_1} \int_{\mathbb{R}^2} |\partial_x, G(x - y, t_1 - s) - \partial_x, G(x - y, t_2 - s)| f(y, s) \, dy \, ds \]
\[ I_3 = \int_{t_1}^{t_2} \int_{\mathbb{R}^2} |\partial_x, G(x - y, t_2 - s)| f(y, s) \, dy \, ds. \]

For the first integral, we have
\[ I_1 \leq (t_2 - t_1) \int_0^1 \int_0^{t_1-(t_2-t_1)} \int_{\mathbb{R}^2} |\partial_t \partial_x, G(x - y, t_v - s)| f(y, s) \, dy \, ds \, dv \]
\[ \leq (t_2 - t_1) \int_0^1 \int_0^{t_1-(t_2-t_1)} \lambda_0(s)^{\nu-2} R(s)^{-a} \left\{ \frac{1}{(t_v - s)^{5/2}} \right\} \]
\[ \int_{|y| \leq 2\lambda_0(s)R(s)} e^{-\frac{(x-y)^2}{4(t_v - s)}} \left( \frac{|x - y|}{(t_v - s)^{1/2}} + \frac{|x - y|^3}{(t_v - s)^{3/2}} \right) \, dy \, ds \, dv \]
where \( t_v = vt_2 + (1-v)(t_2 - t_1) \). Changing variables
\[ \int_{|y| \leq 2\lambda_0(s)R(s)(t_v - s)^{-1/2}} e^{-|\tilde{x}_v - z|^2/4} \left( |\tilde{x}_v - z| + |\tilde{x}_v - z|^3 \right) \, dz \]
where \( \tilde{x}_v = x(t_v - s)^{-1/2} \). We then need to estimate
\[ \int_0^{t_1-(t_2-t_1)} \lambda_0(s)^{\nu-2} R(s)^{-a} \int_{|z| \leq 2\lambda_0(s)R(s)(t_v - s)^{-1/2}} e^{-|\tilde{x}_v - z|^2/4} (|\tilde{x}_v - z| + |\tilde{x}_v - z|^3) \, dz \, ds \]
\[ \leq C \int_0^{t_1-(t_2-t_1)} \lambda_0(s)^{\nu-2} R(s)^{-a} \int_0^{2\lambda_0(s)R(s)(t_v - s)^{-1/2}} e^{-\rho^2/4(1 + \beta^3) \rho} \, d\rho \, ds. \]
For any \( 0 < \mu < 1 \)
\[ \int_0^{2\lambda_0(s)R(s)(t_v - s)^{-1/2}} e^{-\rho^2/4(1 + \beta^3) \rho} \, d\rho \leq C(\lambda_0(s)R(s)(t_v - s)^{-1/2})^{\mu}. \]
Therefore
\[ I_1 \leq C(t_2 - t_1) \int_0^{t_1-(t_2-t_1)} \lambda_0(s)^{\nu-2} R(s)^{-a} \int_0^{2\lambda_0(s)R(s)(t_v - s)^{-1/2}} e^{-\rho^2/4(1 + \beta^3) \rho} \, d\rho \, ds \]
\[ \leq C(t_2 - t_1) \int_0^{t_1-(t_2-t_1)} \lambda_0(s)^{\mu+\nu-2} R(s)^{\mu-a} \int_0^{(t_2 - s)^{3/2+\mu/2}} \frac{1}{(t_2 - s)^{3/2+\mu/2}} \, ds. \]
Recall that \( R(t) = \lambda_0(t)^{-\beta} \). If \( \mu + \nu - 2 - \beta(\mu - a) \leq 0 \) we have
\[ \int_0^{t_1-(t_2-t_1)} \lambda_0(s)^{\mu+\nu-2} R(s)^{\mu-a} \, ds \leq \lambda_0(t_1)^{\mu+\nu-2} R(t_1)^{\mu-a} \int_0^{(t_2-t_1)} \frac{1}{(t_2 - s)^{3/2+\mu/2}} \, ds \]
\[ \leq C\lambda_0(t_1)^{\mu+\nu-2} R(t_1)^{\mu-a}(t_2 - t_1)^{-1/2-\mu/2}. \]
If $b := \mu + \nu - 2 - \beta(\mu - a) > 0$

$$\int_0^{t_1 - (t_2 - t_1)} \frac{\lambda_0(s)^b}{(t_2 - s)^{3/2 + \mu/2}} \, ds = \int_0^{t_1 - (T - t_1)} \frac{\lambda_0(s)^b}{(T - s)^{3/2 + \mu/2}} \, ds + \int_{t_1 - (T - t_1)}^{t_1 - (t_2 - t_1)} \, ds,$$

and, assuming $b - 1/2 - \mu/2 < 0$ (which we have),

$$\int_0^{t_1 - (T - t_1)} \frac{\lambda_0(s)^b}{(t_2 - s)^{3/2 + \mu/2}} \, ds \leq C \int_0^{t_1 - (T - t_1)} \frac{\lambda_0(s)^b}{(T - s)^{3/2 + \mu/2}} \, ds$$

$$= \int_0^{t_1 - (T - t_1)} \frac{\lambda_0(s)^b}{(T - s)^{3/2 + \mu/2}} \, ds \leq C \int_0^{t_1 - (T - t_1)} \frac{|\log T|^b (T - s)^b - 3/2 - \mu/2}{|\log(T - t_1)|^b} \, ds$$

$$\leq C \lambda_0(t_2)^b (t_2 - t_1)^{-1/2 - \mu/2}.$$

while

$$\int_0^{t_1 - (T - t_1)} \frac{\lambda_0(s)^b}{(t_2 - s)^{3/2 + \mu/2}} \, ds \leq C \lambda_0(t_2)^b (t_2 - t_1)^{-1/2 - \mu/2}.$$

In any case we obtain

$$I_1 \leq C \lambda_0(t_2)^{\mu + \nu - 2} R(t_2)^{\mu - a} (t_2 - t_1)^{1/2 - \mu/2}.$$

To estimate $I_2$ it is sufficient to bound the terms

$$\int_0^{t_1} \int_{|y| \leq 2 \lambda_0(s) R(s)} |\nabla_x G(x - y, t_1 - s)| \lambda_0(s)^{\nu - 2} R^{-a}(s) \, dy \, ds,$$

$$\int_0^{t_1} \int_{|y| \leq 2 \lambda_0(s) R(s)} |\nabla_x G(x - y, t_2 - s)| \lambda_0(s)^{\nu - 2} R^{-a}(s) \, dy \, ds.$$

Let us start with:

$$\int_0^{t_1} \int_{|y| \leq 2 \lambda_0(s) R(s)} |\nabla_x G(x - y, t_1 - s)| \lambda_0(s)^{\nu - 2} R^{-a}(s) \, dy \, ds$$

$$\leq C \int_{t_1 - (t_2 - t_1)}^{t_1} \frac{\lambda_0(s)^{\nu - 2} R(s)^{-a}}{(t_1 - s)^{1/2}} \int_{|z| (t_1 - s)^{1/2} \leq 2 \lambda(s) R(s)} e^{-|\tilde{x} - z|^2} |\tilde{x} - z| \, dz \, ds,$$

$$\leq C \int_{t_1 - (t_2 - t_1)}^{t_1} \frac{\lambda_0(s)^{\nu - 2} R(s)^{-a}}{(t_1 - s)^{1/2}} \int_{0}^{2 \lambda_0(s) R(s) (t_1 - s)^{1/2}} e^{-\rho^2 (1 + \rho) \rho} \, d\rho \, ds,$$

where $\tilde{x} = (t - s)^{-1/2} x$. But then, for $0 < \mu < 1:

$$\int_0^{t_1} \frac{\lambda_0(s)^{\nu - 2} R(s)^{-a}}{(t_1 - s)^{1/2}} \int_{0}^{2 \lambda_0(s) R(s) (t_1 - s)^{1/2}} e^{-\rho^2 (1 + \rho) \rho} \, d\rho \, ds$$

$$\leq C \int_{t_1 - (t_2 - t_1)}^{t_1} \frac{\lambda_0(s)^{\nu - 2 + \mu} R(s)^{-a + \mu}}{(t_1 - s)^{1/2 + \mu/2}} \, ds$$

$$\leq C \lambda_0(t_2)^{\nu - 2 + \mu} R(t_2)^{-a + \mu} (t_2 - t_1)^{1/2 - \mu/2}.$$
The estimate of the other integral is similar:
\[
\int_{t_1}^{t_2} \left( \int_{|y| \leq 2\lambda_0(s)R(s)} |\nabla_y G(x - y, t_2 - s)\lambda_0(s)\nu^{-2} R^{-\alpha}(s) dy ds \right)
\leq C\lambda_0(t_2)^{\nu-2+\mu} R(t_2)^{-\alpha+\mu}(t_2 - t_1)^{1/2-\mu/2}.
\]

We deal now with \( I_3 \):
\[
I_3 \leq C \int_{t_1}^{t_2} \frac{\lambda_0(s)^{\nu-2} R(s)^{-\alpha}}{(t_2 - s)^{1/2}} \left( \int_{|y| \leq 2\lambda_0(s)R(s)} e^{-\frac{|x - y|^2}{4(t_2 - s)}} |x - y| dy ds \right)
\leq C \int_{t_1}^{t_2} \frac{\lambda_0(s)^{\nu-2} R(s)^{-\alpha}}{(t_2 - s)^{1/2}} \left( \int_{|z| \leq 2\lambda_0(s)R(s)(t_2 - s)^{-1/2}} e^{-|z|^2/4(1 + |z|)} dz ds \right).
\]

For \( \mu \in (0, 1) \) we have the inequality
\[
\int_{|z| \leq A} e^{-|z|^2/4(1 + |z|)} dz \leq CA^{\mu}.
\]

Therefore
\[
\int_{t_1}^{t_2} \frac{\lambda_0(s)^{\nu-2} R(s)^{-\alpha}}{(t_2 - s)^{1/2}} \left( \int_{|z| \leq 2\lambda_0(s)R(s)(t_2 - s)^{-1/2}} e^{-|z|^2/4(1 + |z|)} dz ds \right)
\leq C \int_{t_1}^{t_2} \frac{\lambda_0(s)^{\mu+\nu-2} R(s)^{\mu-\alpha}}{(t_2 - s)^{1/2+\mu/2}} ds
\leq C\lambda_0(t_2)^{\mu+\nu-2} R(t_2)^{\mu-\alpha}(t_2 - t_1)^{1/2-\mu/2}.
\]

This proves (5.11).

\[\square\]

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