CHIP-FIRING MAY BE MUCH FASTER THAN YOU THINK

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Abstract. A new bound (Theorem 4.4) for the duration of the chip-firing game with \( N \) chips on a \( n \)-vertex graph is obtained, by a careful analysis of the pseudo-inverse of the discrete Laplacian matrix of the graph. This new bound is expressed in terms of the entries of the pseudo-inverse.

It is shown (Section 5) to be always better than the classic bound due to Björner, Lovász and Shor. In some cases the improvement is dramatic.

For instance: for strongly regular graphs the classic and the new bounds reduce to \( O(nN) \) and \( O(n + N) \), respectively. For dense regular graphs - \( d = (\frac{1}{2} + \epsilon)n \) - the classic and the new bounds reduce to \( O(N) \) and \( O(n) \), respectively.

This is a snapshot of a work in progress, so further results in this vein are in the works.

1. Introduction

Consider the following solitaire game: some chips are placed on the vertices 1, 2, \ldots, \( n \) of a graph \( G \) so that vertex \( i \) has degree \( d_i \) and receives \( a_i \) chips. Then the player performs a series of moves which are called “firings”. In each such move she selects a vertex for which \( a_i \geq d_i \) and moves one chip from \( i \) to each of \( i \)’s neighbours. If there are no possible moves, the game ends. It is also possible for the game to enter an infinite loop, when a quondam position comes up again.

This rather innocuous-sounding game is actually laden with a staggering amount of very deep properties. Papers on it may be found under different names in journals devoted to mathematics, physics and theoretical computer science. We shall now very briefly mention some of these connections and then proceed to outline our new contribution.

1.1. A portal to bibliography. The chip-firing game, in the form described above, was introduced by Björner, Lovász and Shor [7] in 1991. A physicist would recognize this as a fixed-energy variant of the “sandpile model” [2, 15] which has been suggested as a possible simple model for the emergence of power laws in various of natural phenomena, for instance earthquakes.

The chip firing game is related to potential theory on graphs and to arithmetic geometry. We refer the reader to [3, 6, 26, 29] for more on this. A close relation to random walks is described in [27]. For a description of the relation between the chip
firing game to lattice theory we refer to [16] and for the chip firing game in the role of a universal computer to [17].

Finally, we suggest [1, 3, 14, 16, 22, 29] as possible entry points to the quite vast literature on the subject.

2. A remarkable property

Theorem 2.1. [7] Given a connected graph and an initial distribution of chips, either every legal game can be continued indefinitely, or every legal game terminates after the same number of moves with the same final position. The number of times a given vertex is fired is the same in every legal game.

Theorem 2.1 may feel rather surprising at first glance. While the inherent determinism of the game might not be that surprising, all things considered, the fact that the game always takes the same exact number of moves is harder to stomach.

However, the derivation in [7] elucidates all: the set of legal gameplays, considered as sequences, forms an ‘antimatroid with repetitions’ and the terminating games correspond to the bases. Thus we see that the surprising claim about game duration is just good old equicardinality of bases in heavy disguise.

2.1. Goal of the paper. In this paper we shall concentrate on the problem of obtaining structural bounds on the possible duration of a game. This problem has received some attention and a number of bounds are known.

However, computer simulations (cf. Section 3.1) show that the extant bounds are often far too pessimistic and that the game tends to end much faster than predicted by them. Therefore, there is a need to develop new bounds which will be closer to the actual values of the game duration.

3. The problem of game duration

Let us pause to fix notation for the rest of the paper. We will be dealing with a connected graph $G$ with $n$ vertices and $m$ edges. The number of chips in play will be denoted $N$. Let us assume that the game terminates in $s$ moves and let $x_i$ be the number of times that vertex $i$ has been fired during the game.

A result from [7] says that in a terminating game we must necessarily have $N \leq 2m - n$. Furthermore, if $N < m$ then every initial distribution with $N$ chips will terminate.

The first bound on game duration was given by Tardos [33]:

Theorem 3.1. [33] Suppose that the diameter of the graph $G$ is $D$. Then

$$s \leq nND.$$ 

This result is sometimes quoted in a less precise from as $s = O(n^4)$ which holds since clearly $D = O(n)$ and $N < 2m = O(n^2)$. Tardos [33] also shows an example of a game which does take $O(n^4)$ moves to terminate. Note that for directed graphs it was proved by Eriksson [18] that no such polynomial bound is possible.
Recall that the Laplacian matrix $L$ (cf. [30, 31]) of the graph $G = (V, E)$ whose vertices are labelled \{1, 2, \ldots, n\} is:

$$L_{ij} = \begin{cases} 
-1, & \text{if } (i, j) \in E \\
0, & \text{if } (i, j) \notin E \text{ and } i \neq j \\
d_i, & \text{if } i = j.
\end{cases}$$

Denote by $a \in \mathbb{R}^n$ the vector representing the chip distribution at some moment and by $a' \in \mathbb{R}^n$ the vector representing the distribution of the chips after firing vertex $i$. We see that $a'$ is obtained from $a$ by subtracting the $i$th column of $L$. Therefore we have the following important fact, observed first in [7]:

**Theorem 3.2.** [7] Suppose that a terminating game is played on $G$. Let $a \in \mathbb{R}^n$ represent the initial chip distribution and $b \in \mathbb{R}^n$ the final chip distribution. Then

$$Lx = a - b.$$ 

Björner, Lovász and Shor [7] derived from (1) the following bound:

**Theorem 3.3.** [7] Let the eigenvalues of $L$ be $0 < \lambda_2 \leq \ldots \leq \lambda_n$. Then

$$s \leq \frac{2nN}{\lambda_2}.$$ 

The smallest non-trivial Laplacian eigenvalue $\lambda_2$ is often called the algebraic connectivity of $G$ (cf. [19, 28]).

The bounds of Theorems 3.1 and 3.3 are elegant and crisp. However, as we will now show by examples, they tend to severely overestimate the number of moves required for the game.

### 3.1. A few exemplary games.

The chip-firing game can be easily implemented on a computer and because of Theorem 2.1 we need not worry about the choice of vertices to fire at each stage - we can just choose any vertex we like.

Therefore, we can examine what happens when we play the game on three graphs: the (in)famous Petersen graph, the Schlafli graph [1] and the Paley graph with 109 vertices. In all three cases all $N$ chips were initially placed on a single vertex.

The table below compares the actual number of moves expended in the games with the foregoing upper bounds.

| Name       | $n$ | $N$   | Theorem 3.1 | Theorem 3.3 | Actual duration |
|------------|-----|-------|-------------|-------------|-----------------|
| Petersen   | 10  | 14    | 280         | 140         | 8               |
| Schlafli   | 27  | 215   | 11610       | 967         | 13              |
| Paley(109) | 109 | 2900  | 632200      | 12828       | 53              |

Clearly, there is some serious overestimation here. Our main result will be to provide a new bound for the class of strongly regular graphs, to which these three graphs belong.

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1The complement of the collinearity graph of $GQ(2, 4)$
Roughly speaking, instead of a $O(nN)$ bound we will be able to obtain a $O(n + N)$ bound.

4. Main Implicit Bound

The quantity that we are interested in is

$$s = \sum_{i=1}^{n} x_i.$$ 

If the matrix $L$ were invertible, we could have solved for $x$ as

$$x = L^{-1}(a - b)$$

and then derived estimates on $s$. However, as is well-known, the rank of a Laplacian matrix of a graph with $c$ connected components is $n - c$ and thus our $L$ is singular.

Nevertheless, in [7] it was observed that it is possible to use the so-called Moore-Penrose pseudo-inverse of $L$ in a similar way. We shall only give here the briefest of introductions to pseudo-inversion, referring the interested reader to the books [5,12].

4.1. A crash course on generalized inversion. Let $A$ be a $m \times n$ complex matrix. Then there exists a unique $n \times m$ matrix $X$ such that:

(i) $AXA = A$.

(ii) $XAX = X$.

(iii) $AX$ and $XA$ are Hermitian.

Clearly, if $A$ is square and invertible, then $X = A^{-1}$. The matrix $X$ is denoted $A^\dagger$ and called the Moore-Penrose pseudo-inverse of $A$.

The foregoing axiomatic definition does not tell us how to compute $A^\dagger$ but fortunately there is another characterization which does. Let

$$A = U\Sigma V^*$$

be the singular value decomposition of $A$, where $U$ and $V$ are orthogonal matrices of appropriate orders and $\Sigma$ is a $m \times n$ matrix which is “generalized diagonal”, in the sense that $\Sigma_{ij} = 0$ for $i \neq j$.

It is easy to see that $\Sigma^\dagger$ is obtained by replacing every non-zero entry in $\Sigma$ by its inverse. Then we have

$$A^\dagger = V\Sigma^\dagger U^*.$$ 

(2)

If the matrix $A$ has a spectral decomposition we can deduce from (2) a very convenient representation of the pseudo-inverse:

$$A = \sum_{i=1}^{k} \lambda_i E_i, \quad A^\dagger = \sum_{i=1}^{k} \lambda_i^{-1} E_i,$$

(3)

where the $\lambda_i$ are the distinct nonzero eigenvalues of $A$ and the $E_i$ are the corresponding orthogonal projections.

There is quite a large number of papers devoted to describing $L^\dagger$ combinatorially. We may point out [4] for trees and [13] for weighted multigraphs as good entrance points to this subject.
4.2. An implicit bound for any graph. Before we state the new bound, we need to collate two very useful observations made by the pioneers of the subject.

Lemma 4.1. Suppose that a terminating game was played on $G$. Then at least one vertex $k$ has not fired during the game, that is $x_k = 0$.

Let us denote the standard basis vectors of $\mathbb{R}^n$ as $e_1, e_2, \ldots, e_n$.

Lemma 4.2. Suppose that a terminating game was played on $G$. Let $k$ be a vertex such that $x_k = 0$. Then

$$s = -n e_k L^\dagger (a - b).$$

Theorem 3.3 was in fact derived in [7] from (4) by what in effect amounts to “bounding” the spectral decomposition of $L^\dagger = \sum_{i=1}^{k} \lambda_i^{-1} E_i$ by $\frac{1}{\lambda} \sum_{i=1}^{k} E_i$. However, it is possible to get stronger bounds on $s$ if we delve more deeply into the actual entries of $L^\dagger$. Our next theorem, which is the first new main result of the paper, provides a bound for $s$ in terms of the entries of $L^\dagger$.

To state our theorem it will be convenient to introduce the following notation:

$$f = \max_{i=1}^{n} \{L^\dagger_{ii}\}, \quad o = \max_{i \neq j} \{|L^\dagger_{ij}|\}.$$

Lemma 4.3. $f \geq o$.

Proof. Suppose that $o = |L^\dagger_{ij}|$. Consider the $2 \times 2$ submatrix $H$ based on the $i$th and $j$th lines of $L^\dagger$. Since $H$ is positive semidefinite we have $0 \leq |H| = L^\dagger_{ii} L^\dagger_{jj} - o^2 \leq f^2 - o^2$. \hfill $\square$

Theorem 4.4 (Main Implicit Bound). Suppose that a terminating game was played on $G$. Suppose that the maximum degree of $G$ is $\Delta$. Let $k$ be a vertex such that $x_k = 0$. Then

$$s \leq n \left( f(\Delta - 1) + o(2N - \Delta + 1) \right).$$

Proof. Denote $d = a - b$. We have from (4) that $s$ equals $-n$ times the scalar product of the $k$th row of $L^\dagger$ with the vector $d$. Now, we observe that as the vertex $k$ had not been fired, we must have $a_k \leq \Delta - 1$ or else $k$ would have fired at some stage. On the other hand, $b_k \leq \Delta - 1$ as the game terminated. Therefore $|d_k| \leq \Delta - 1$. On the other hand $\sum_{i \neq k} |d_i| \leq 2N - |d_k|$. We can combine our observations to write:

$$s = -n(L^\dagger_{kk} d_k + \sum_{i \neq k} L^\dagger_{ki} d_k) \leq n(f(|d_k|) + o(2N - |d_k|)) \leq n(f(\Delta - 1) + o(2N - \Delta + 1)).$$

\hfill $\square$

5. Improving the Björner-Lovász-Shor bound

Since $f \geq o$, the Main Implicit Bound has the following corollary:

Corollary 5.1.

$$s \leq 2nNf.$$
We now observe that even this corollary is strong enough to imply Theorem 3.3.

**Corollary 5.2** (Björner-Lovász-Shor).

\[ s \leq \frac{2nN}{\lambda_2}. \]

*Proof.* Schur’s majorization theorem (cf. [23, Theorem 4.3.26]) tells us that the largest diagonal entry \( f \) of \( L^\dagger \) is bounded from above by the largest eigenvalue \( \frac{1}{\lambda_2} \) of \( L^\dagger \).

As a warm-up let us now obtain a modest improvement upon Theorem 3.3 for vertex-transitive graphs:

**Theorem 5.3.** Let \( G \) be a vertex-transitive graph. Then

\[ s \leq \frac{2(n - 1)N}{\lambda_2}. \]

*Proof.* All the diagonal entries of \( L^\dagger \) are equal in this case to \( f \). Thus we have:

\[ f = \frac{\text{Tr}(L^\dagger)}{n} = \frac{\sum_{i=2}^{n} \frac{1}{\lambda_i}}{n} \leq \frac{n - 1}{n} \cdot \frac{1}{\lambda_2}. \]

\[ \square \]

Now let us proceed to improve upon Theorem 3.3 for all graphs, using a more complicated estimate for \( f \). This will require some setting-up. First we recall a well-known formula for \( L^\dagger \) which can be deduced from (3). \( J \) denotes the all-ones matrix and \( c \neq 0 \):

\[ L^\dagger = (L + cJ)^{-1} - \frac{1}{cn^2} J. \]

**Theorem 5.4.** [21, Theorem 5.1] Let \( A \) be a positive definite matrix whose eigenvalues are contained in the interval \([a, b], a > 0\). Then

\[ A_{ii}^{-1} \leq \frac{a + b - a_{ii}}{ab}. \]

Now we can prove our result:

**Theorem 5.5.** For any graph \( G \) we have:

\[ s \leq 2nN \cdot \frac{\lambda_2 + \frac{\lambda_n(n-1)}{n} - \delta}{\lambda_2 \lambda_n}. \]

*Proof.* Let \( T = L + cJ \) with \( c = \frac{\delta}{n} \). The spectrum of \( T \) is identical to that of \( L \), except for the zero which becomes \( cn = \lambda_n \). Therefore, the smallest and largest eigenvalues of \( T \) are \( \lambda_2 \) and \( \lambda_n \), respectively. We use (3) and (7) to write:

\[ L_{ii}^\dagger \leq T_{ii}^{-1} \leq \frac{\lambda_2 + \lambda_n - d_i - c}{\lambda_2 \lambda_n} = \frac{\lambda_2 + \frac{\lambda_n(n-1)}{n} - d_i}{\lambda_2 \lambda_n}. \]

Therefore

\[ f \leq \frac{\lambda_2 + \frac{\lambda_n(n-1)}{n} - \delta}{\lambda_2 \lambda_n} \]

and the conclusion follows immediately from Corollary 5.1.

\[ \square \]
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Theorem 5.5 is always stronger than Theorem 2.1 because $\lambda_2 \leq \delta$ by a classic result of Fiedler [19].

6. BEATING $o \leq f$ FOR DENSE REGULAR GRAPHS

The explicit results of the previous section were obtained by substituting certain estimates for $f$ into the Main Implicit Bound. We will obtain even better explicit results by estimating $o$ in its own right, improving upon the simple $o \leq f$. This is rather difficult to do, but nevertheless we will now show a way of deriving an estimate for $o$ in the case of dense regular graphs.

Let $A$ be a $n \times n$ matrix. Denote $R_i(A) = \sum_{j=1, j \neq i}^n |a_{ij}|$ and $\sigma_i(A) = \frac{R_i(A)}{|a_{ii}|}$. If $\sigma_i(A) \leq 1$ for all $i = 1, 2, \ldots, n$ we say that the matrix $A$ is diagonally dominant and if $\sigma_i(A) < 1$ for all $i$, then $A$ is said to be strictly diagonally dominant or SDD. Clearly, the Laplacian matrix $L$ is diagonally dominant but not SDD.

The inverse of an SDD matrix is not necessarily SDD. However, Ostrowski [32] observed that a weak form of diagonal dominance does carry over to the inverse. Namely: if $A$ is SDD and $B = A^{-1}$, then $|b_{ji}| \leq \sigma_j(A)|b_{ii}|$.

Ostrowski’s theorem was later rediscovered by Yong and Wang [35] whose work has rekindled interest in such results, in the context of the so-called Fiedler-Markham conjecture. However, they are embedded as technical lemmata in various papers and are not readily available to the casual peruser. We are going to use one such result which is particularly simple to apply while being rather effective:

**Theorem 6.1.** [25, Theorem 2.4] Let $A$ be SDD. Let $B = A^{-1} = (b_{ij})$. Then it holds that:

$$|b_{ji}| \leq \max_{i \neq j} \left\{ \frac{|a_{ii}| - |a_{il}|}{|a_{ii}| - \sum_{k \neq i} |a_{ik}|} |b_{ii}| \right\}, \text{ for all } j \neq i.$$

We are now in a position to prove:

**Theorem 6.2.** Let $G$ be a connected $d$-regular graph on $n$ vertices. If $d > \frac{n}{2} - 1$, then

$$o \leq \frac{f}{2d - n + 3} + \frac{2}{n^2}.$$

**Proof.** Let $T = L + J$. We claim that $T$ is an SDD matrix. Indeed, $t_{ii} = d + 1$ and $R_i(T) = n - d - 1$. Denote

$$f_T = \max_{i=1}^n \{|T_i^{-1}\}, \quad o_T = \max_{i \neq j} \{|T_{ij}^{-1}\}.$$

As a consequence of (3) we have

$$f = f_T - \frac{1}{n^2}, \quad o \leq o_T + \frac{1}{n^2}.$$

Since the only off-diagonal entries of $T$ are 0 and 1 we can apply Theorem 6.1 with $t_{ii} = 1$, $t_{il} = d + 1$, and $\sum_{k \neq i} |t_{ik}| = n - d - 2$ to obtain:

$$o_T \leq \frac{f_T}{2d - n + 3}.$$
Finally,
\[ o \leq o_T + \frac{1}{n^2} \leq \frac{f_T}{2d - n + 3} + \frac{1}{n^2} = \frac{f + \frac{1}{n^2}}{2d - n + 3} + \frac{1}{n^2} \leq \frac{f}{2d - n + 3} + \frac{2}{n^2}. \]

\[ \square \]

Theorem 6.2 is a relative estimate for \( o \) in the sense that it depends on \( f \). We shall see in the next section a situation in which extra combinatorial structure allows us to provide absolute estimates for \( o \).

In the presentation of the next result we aim rather more for elegance of expression than for the utmost optimization of lower-order terms and so we use \( 2N \) instead of \( 2N - d + 1 \) and estimate \( f \) via \( \frac{1}{\lambda^2} \) rather than via the sharper but more cumbersome expression in (8).

**Theorem 6.3.** Let \( G \) be a connected \( d \)-regular graph on \( n \) vertices and suppose that \( d = (\frac{1}{2} + \epsilon)n, \ 0 < \epsilon < \frac{1}{2} \). Then

\[ s \leq \frac{n(d - 1)}{\lambda_2} + \frac{N}{\lambda_2^{ce}} + \frac{4N}{n}. \]

**Proof.** Apply the Main Implicit Bound in the simplified form \( s \leq n(f(d - 1) + 2No) \), together with Theorem 6.2. Note that \( 2d - n + 3 = 2en + 3 \geq 2en \).

We can offer a probabilistic comparison of Theorem 6.3 with Theorem 3.3 based on a result by Juhász [24]:

**Theorem 6.4.** [24, Theorem 2] Let \( G(n, p) \) be a random graph. Then the algebraic connectivity \( \lambda_2 \) of \( G \) satisfies for any \( \epsilon > 0 \):

\[ \lambda_2(G) = pn + o(n^{\frac{1}{2}+\epsilon}) \text{ in probability.} \]

Therefore we are justified in writing \( \frac{1}{\lambda_2} = O(n) \), in the probabilistic sense. Also \( d = \Theta(n) \) and \( N = O(n^2) \) and we see that the BLS bound reduces to \( O(N) \) while our new bound reduces to \( O(n) \) (probabilistically). Since for the game to even start we must have \( N \geq d = \Omega(n) \), our bound is always at least as good as the BLS one.

7. STRONGLY REGULAR GRAPHS

**Definition 7.1.** [8] A strongly regular graph with parameters \((n, k, a, c)\) is a \( k \)-regular graph on \( n \) vertices such that any two adjacent vertices have a common neighbour and any two non-adjacent vertices have \( c \) common neighbours.

If \( G \) is strongly regular with parameters \((n, k, a, c)\) we shall also write compactly that \( G \) is \( SRG(n, k, a, c) \). Nice expositions of the theory of strongly regular graphs may be found in e.g., [10, 11, 20, 34]. The graphs discussed in Section 3.1 are all strongly regular: Petersen is \( SRG(10, 3, 0, 1) \), the Schl"{a}fli graph is \( SRG(27, 16, 10, 8) \) and Paley(109) is \( SRG(109, 54, 25, 26) \).

Recall that the adjacency matrix \( A \) of a strongly regular graph has exactly three distinct eigenvalues: \( k, \theta, \tau \), with \( \theta > 0 \) and \( \tau < 0 \). It is well known (cf. [20, p. 220])
that the eigenvalues $\theta, \tau$ are given by:

$$\theta = \frac{(a - c) + \sqrt{\Delta}}{2}, \quad \tau = \frac{(a - c) - \sqrt{\Delta}}{2},$$

where $\Delta = \sqrt{(a - c)^2 + 4(k - c)}$.

For the duration of this section $G$ will refer to a connected $SRG(n, k, a, c)$. We will also denote $d = a - c$ and since $c = 0$ would have implied a disconnected $G$ (cf. [20, p. 218]) we freely assume that $c \geq 1$ and thus $d \leq a - 1$.

7.1. **Auxiliary results.** We now collate a number of facts that will be used in the proof of our new result, Theorem 7.9.

**Lemma 7.2.**

$$\theta + \tau = a - c = d.$$  

**Lemma 7.3.**

$$(k - \theta)(k - \tau) = k(k - d - 1) + c.$$  

**Proof.**

$$(k - \theta)(k - \tau) = \left(k - \frac{d + \sqrt{\Delta}}{2}\right)\left(k - \frac{d - \sqrt{\Delta}}{2}\right) = \frac{(2k - d)^2 - \Delta}{4} = \frac{4k^2 - 4kd - 4(k - c)}{4}. \tag*{\Box}$$

**Lemma 7.4.** [20, p. 244]

$$(k - \theta)(k - \tau) = nc.$$  

**Lemma 7.5** (Taylor, Levingstone). [9, p. 7] *Suppose that $G$ is connected and $G \neq K_n$. Then*

$$k \geq 2a - c + 3$$

*and equality holds if and only if $G$ is a pentagon.*

**Lemma 7.6.** Suppose that $G$ is connected and $G \neq K_n, C_5$. Then

$$d \leq \frac{k - 5}{2}.$$  

**Proof.** By Lemma 7.5 it follows that $k \geq 2a - c + 4$ under our assumptions. Therefore:

$$k \geq 2a - c + 4 = d + 4 + a \geq d + 4 + d + 1 = 2d + 5. \tag*{\Box}$$

**Lemma 7.7.** [20, p. 219]

$$k(k - a - 1) = (n - k - 1)c.$$  

**Lemma 7.8.**

$$2(n - k) \geq -d.$$  

**Proof.** From Lemma 7.7 we have that $(n - k) = \frac{k(k - a - 1)}{c} + 1$. Therefore our claim is equivalent to

$$2k(k - a - 1) \geq c(c - a - 2)$$

which always holds by elementary algebra. \tag*{\Box}
7.2. Chip-firing on SRG is fast. Theorems 3.1 and 3.3 imply a $O(nN)$ bound on game duration for a strongly regular graph, but by using our method it can be improved considerably to $O(n + N)$. Specifically we have:

**Theorem 7.9.** Let $G$ be a connected SRG($n, k, a, c$). Then

$$s \leq n \frac{k - 1}{k - 2} + \frac{2(2N - k + 1)}{c}.$$

**Proof.** The claim will follow from Theorem 4.4 and the following two estimates:

(10) $$f \leq \frac{1}{k - 2};$$

(11) $$o \leq \frac{2}{nc}.$$

The three orthogonal eigenprojections of $A$ are:

(12) $$E_0 = \frac{J}{n}, \quad E_1 = \frac{1}{\theta - \tau}(A - \tau I - \frac{k - 1}{n}J), \quad E_2 = \frac{1}{\tau - \theta}(A - \theta I - \frac{k - \theta}{n}J)$$

and we have

$$A = kE_0 + \theta E_1 + \tau E_2.$$

Since $E_0 + E_1 + E_2 = I$ we infer the following expression for $L = kI - A$:

$$L = (k - \theta)E_1 + (k - \tau)E_2.$$

Applying (3) we have

(13) $$L^1 = \frac{1}{k - \theta}E_1 + \frac{1}{k - \tau}E_2.$$

Now a tedious but straightforward calculation from (12) and (13) yields:

$$L^\dagger_{ii} = \frac{k(n - 2) - (n - 1)(\theta + \tau)}{n(k - \theta)(k - \tau)}.$$

Let us use this expression in conjunction with Lemmata 7.2 and 7.3 to estimate $f$:

$$f = \frac{k(n - 2) - (n - 1)(\theta + \tau)}{n(k - \theta)(k - \tau)} \leq \frac{k - (\theta + \tau)}{(k - \theta)(k - \tau)} = \frac{k - d}{k(k - d - 1) + c} \leq \frac{k - d}{k(k - d - 1)}.$$

Furthermore, Lemma 7.6 enables us to complete the estimate:

$$f \leq \frac{k - d}{k(k - d - 1)} = \frac{1}{k} \left(1 + \frac{1}{k - d - 1}\right) \leq \frac{1}{k} \left(1 + \frac{1}{k - \frac{k-5}{2} - 1}\right) = \frac{k + 5}{k(k + 3)} \leq \frac{1}{k - 2}.$$

This proves estimate (10).

Now we consider the off-diagonal entry $L^\dagger_{ij}$. Once again, we calculate it from (12) and (13) but now there are two possible cases: when $i$ and $j$ are adjacent and when they are not. In the former case we have

$$L^\dagger_{ij} = \frac{n - 2k + \theta + \tau}{n(k - \theta)(k - \tau)} = \frac{n - 2k + d}{n(k - \theta)(k - \tau)},$$
and in the latter case
\[ L_{ij}^\dagger = \frac{\theta + \tau - 2k}{n(k - \theta)(k - \tau)} = \frac{d - 2k}{n(k - \theta)(k - \tau)}. \]

The denominator of both expressions is equal to \( n^2c \) by Lemma 7.4. We now claim that the numerators are bounded in absolute value by \( 2n \), which will prove estimate (11). In fact, for the first expression we can even show the slightly stronger fact that the numerator’s modulus is bounded by \( n \).

Indeed, \( n - 2k + d \leq n \) is obvious and \( n - 2k + d \geq -n \) is exactly the claim of Lemma 7.8. For the second expression, \( d - 2k \leq 2n \) is obvious and \( d - 2k \geq -2n \) is once again Lemma 7.8. \( \square \)

Let us now reproduce Table 1 with two additional column indicating the new bounds of Theorems 4.4 and 7.9.

**Table 2. Upper bounds on the number of moves**

| Name        | \( n \) | \( N \) | Theorem 3.1 | Theorem 3.3 | Theorem 4.4 | Theorem 7.9 | Actual \( s \) |
|-------------|--------|--------|-------------|-------------|-------------|-------------|----------------|
| Petersen    | 10     | 14     | 280         | 140         | 24          | 72          | 8              |
| Schläfi    | 27     | 215    | 11610       | 967         | 81          | 132         | 13             |
| Paley(109) | 109    | 2900   | 632200      | 12828       | 318         | 536         | 53             |

The improvement is palpable.

**Example 7.10.** For the Paley graph on \( q \) vertices the Tardos bound is \( \approx \frac{q^3}{2} \) and the Björner-Lovász-Shor bound is \( \approx q^2 \) while Theorem 7.9 yields a bound of \( \approx 5q \) which is quite close to \( \frac{q^2}{2} \) which is (according to numerical evidence) probably the right answer.

**Remark 7.11.** Our estimates are rather sharp since:

- The Schläfi graph is SRG(27, 16, 10, 8) and has \( o = \frac{5}{972} > \frac{1}{216} = \frac{1}{nc} \).
- The triangular graph \( T(21) \) (that is, the line graph of \( K_{21} \)) is SRG(210, 38, 19, 4) and has \( f = \frac{4769}{176400} > \frac{1}{37} = \frac{1}{\kappa - 1} \).

8. Notes

Some time after obtaining the main result of the paper (Theorem 4.4) I have had the pleasure of reading the paper [3] by Baker and Shokrieh and realized, with some surprise and not a little gratification, that they had - among other considerations in a remarkable and far-ranging work - carried out an analysis of the problem of game duration along similar lines, stressing the importance of generalized inverses (Moore-Penrose or others). Cf. especially their Section 3.2 and Remark 5.4.

The crucial difference between the analysis in [3] and the present paper is the observation that the off-diagonal entries of \( L^\dagger \) have different dynamics from those of the diagonal entries and that this fact can be exploited to sharply reduce the bounds.
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References

[1] S. Backman. Combinatorial divisor theory for graphs. http://www.aco.gatech.edu/doc/Backman_thesis.pdf, 2014. Ph.D. Thesis, Georgia Institute of Technology.
[2] P. Bak. How Nature Works: The Science of Self-Organized Criticality. Oxford University Press, 1997.
[3] M. Baker and F. Shokrieh. Chip-firing games, potential theory on graphs, and spanning trees. J. Combin. Theory Ser. A, 120(1):164–182, 2013.
[4] R. B. Bapat. Moore-Penrose inverse of the incidence matrix of a tree. Linear Multilinear Algebra, 42(2):159–167, 1997.
[5] A. Ben-Israel and T. N. E. Greville. Generalized inverses: theory and applications. John Wiley & Sons, 1974.
[6] N. Biggs. Algebraic potential theory on graphs. Bull. Lond. Math. Soc., 29(6):641–682, 1997.
[7] A. Björner, L. Lovász, and P. W. Shor. Chip-firing games on graphs. Eur. J. Comb., 12(4):283–291, 1991.
[8] R. Bose. Strongly regular graphs, partial geometries and partially balanced designs. Pacific J. Math., 13:389–419, 1963.
[9] A. E. Brouwer, A. M. Cohen, and A. Neumaier. Distance-regular graphs, volume 3. Folge, 18 of Ergebnisse der Mathematik und ihrer Grenzgebiete. Springer-Verlag, 1989.
[10] A. E. Brouwer and W. H. Haemers. Spectra of Graphs, volume 223 of Universitext. Springer, 2012.
[11] P. J. Cameron and J. H. van Lint. Designs, graphs, codes and their links, volume 22 of London Mathematical Society Student Texts. Cambridge University Press, 1991.
[12] S. L. Campbell and C. D. Meyer. Generalized inverses of linear transformations. Number 4 in Surveys and References Works in Mathematics. Pitman, 1979.
[13] P. Y. Chebotarev and E. V. Shamis. On proximity measures for graph vertices. Autom. Remote Control, 59(10, Part 2):1443–1459, 1998.
[14] F. Chung and R. B. Ellis. A chip-firing game and Dirichlet eigenvalues. Discrete Math., 257(2–3):341–355, 2002.
[15] D. Dhar. Self-organized critical state of sandpile automaton models. Phys. Rev. Lett., 64(14):1613–1616, 1990.
[16] Éric Goles, M. Latapy, C. Magnien, M. Morvan, and H. D. Phan. Sandpile models and lattices: a comprehensive survey. Theor. Comput. Sci., 322(2):383–407, 2004.
[17] Éric Goles and M. Margenstern. Universality of the chip-firing game. Theor. Comput. Sci., 172(1–2):121–134, 1997.
[18] K. Eriksson. No polynomial bound for the chip firing game on directed graphs. Proc. Am. Math. Soc., 112(4):1203–1205, 1991.
[19] M. Fiedler. Algebraic connectivity of graphs. Czechoslovak Math. J., 23(98):298–305, 1973.
[20] C. Godsil and G. Royle. Algebraic Graph Theory, volume 207 of Graduate Texts in Mathematics. Springer, 2001.
[21] G. H. Golub and C. Brezinski. Matrices, moments and quadrature. In Numerical analysis 1993 (Dundee, 1993), volume 303 of Pitman Res. Notes Math. Ser., pages 105–156. Longman Sci. Tech., Harlow, 1994.
[22] A. E. Holroyd, L. Levine, K. Mészáros, Y. Peres, J. Propp, and D. B. Wilson. Chip-firing and rotor-routing on directed graphs. Sidoravicius, Vladas (ed.) et al., In and out of equilibrium 2. Papers celebrating the 10th edition of the Brazilian school of probability. Birkhäuser. Progress in Probability 60, pp. 331-364., 2008.
[23] R. A. Horn and C. R. Johnson. Matrix Analysis. Cambridge University Press, 1985.
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[24] F. Juhász. The asymptotic behaviour of Fiedler’s algebraic connectivity for random graphs. *Discrete Math.*, 96(1):59–63, 1991.

[25] H.-B. Li, T.-Z. Huang, S.-Q. Shen, and H. Li. Lower bounds for the minimum eigenvalue of Hadamard product of an $M$-matrix and its inverse. *Linear Algebra Appl.*, 420(1):235–247, 2007.

[26] D. Lorenzini. Smith normal form and Laplacians. *J. Comb. Theory, Ser. B*, 98(6):1271–1300, 2008.

[27] L. Lovász and P. Winkler. Mixing of random walks and other diffusions on a graph. Rowlinson, Peter (ed.), Surveys in combinatorics, 1995. Proceedings of the 15th British combinatorial conference held in July 1995 at the University of Stirling, UK. Cambridge: Cambridge University Press. Lond. Math. Soc. Lect. Note Ser. 218, pp. 119–154., 1995.

[28] N. M. Maia de Abreu. Old and new results on algebraic connectivity of graphs. *Linear Algebra Appl.*, 423(1):53–73, 2007.

[29] C. Merino. The chip-firing game. *Discrete Math.*, 302(1–3):188–210, 2005.

[30] R. Merris. Laplacian matrices of graphs: A survey. *Linear Algebra Appl.*, 197–198:143–176, 1994.

[31] B. Mohar. Some applications of Laplace eigenvalues of graphs. In G. Hahn and G. Sabidussi, editors, *Graph symmetry: algebraic methods and applications*, volume 497 of *NATO ASI Ser. C*, pages 225–275. Kluwer Academic Publishers, 1997.

[32] A. M. Ostrowski. Note on bounds for determinants with dominant principal diagonal. *Proc. Am. Math. Soc.*, 3:26–30, 1952.

[33] G. Tardos. Polynomial bound for a chip firing game on graphs. *SIAM J. Discrete Math.*, 1(3):397–398, 1988.

[34] J. H. van Lint and R. M. Wilson. *A course in combinatorics*. Cambridge University Press, 1992.

[35] X. Yong and Z. Wang. On a conjecture of Fiedler and Markham. *Linear Algebra Appl.*, 288:259–267, 1999.

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