UNIFYING DAGS AND UGS

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Abstract. We introduce a new class of graphical models that generalizes Lauritzen-Wermuth-Frydenberg chain graphs by relaxing the semi-directed acyclicity constraint so that only directed cycles are forbidden. Moreover, up to two edges are allowed between any pair of nodes. Specifically, we present local, pairwise and global Markov properties for the new graphical models and prove their equivalence. We also present an equivalent factorization property.

1. INTRODUCTION

Lauritzen-Wermuth-Frydenberg chain graphs (LWF CGs) are usually described as unifying directed acyclic graphs (DAGs) and undirected graphs (UGs) (Lauritzen, 1996, p. 53). However, this is arguable because the only constraint that DAGs and UGs jointly impose is the absence of directed cycles, whereas LWF CGs forbid semi-directed cycles which is a stronger constraint. Moreover, LWF CGs do not allow more than one edge between any pair of nodes. In this work, we consider graphs with directed and undirected edges but without directed cycles. The graphs can have up to two different edges between any pair of nodes. Therefore, our graphs truly unify DAGs and UGs. Hence, we call them UDAGs.

As we will see, UDAGs generalize LWF CGs. Two other such generalizations that can be found in the literature are reciprocal graphs (RGs) (Koster, 1996) and acyclic graphs (AGs) (Lauritzen and Sadeghi, 2017). The main differences between UDAGs and these two classes of graphical models are the following. UDAGs are not a subclass of RGs because, unlike RGs, they can have semi-directed cycles. UDAGs are a subclass of AGs. However, Lauritzen and Sadeghi define a global Markov property for AGs but no local or pairwise Markov property. We define the three properties for UDAGs. Lauritzen and Sadeghi do define though a pairwise Markov property for a subclass of AGs called chain mixed graphs (CMGs), but no local Markov property. Moreover, UDAGs are not a subclass of CMGs because, unlike CMGs, they can have semi-directed cycles. In addition to the local, pairwise and global...
Markov properties, we also define a factorization property for UDAGs. Such a property exists for RGs but not yet for AGs or CMGs.

Finally, it is worth mentioning that our work complements that by Richardson (2003), where DAGs and covariance (bidirected) graphs are unified.

The rest of the paper is organized as follows. Section 2 introduces some notation and definitions. Sections 3 and 4 present the global, local and pairwise Markov properties for UDAGs and prove their equivalence. Section 5 does the same for the factorization property. Section 6 closes the paper with some discussion.

2. PRELIMINARIES

In this section, we introduce some concepts about graphical models. Unless otherwise stated, all the graphs and probability distributions in this paper are defined over a finite set of random variables $V$. The elements of $V$ are not distinguished from singletons. An UDAG $G$ is a graph with possibly directed and undirected edges but without directed cycles, i.e. $A \rightarrow \cdots \rightarrow A$ is forbidden. There may be up to two different edges between any pair of nodes. Edges between a node and itself are not allowed. We denote by $A \rightarrow B$ or $A \leftarrow B$ or both are in $G$.

Given an UDAG $G$, the parents of a set $X \subseteq V$ are $\text{pa}(X) = \{ B | B \rightarrow A \text{ in } G \text{ with } A \in X \}$. The children of $X$ are $\text{ch}(X) = \{ B | A \rightarrow B \text{ is in } G \text{ with } A \in X \}$. The neighbors of $X$ are $\text{ne}(X) = \{ B | A - B \text{ is in } G \text{ with } A \in X \}$. The ancestors of $X$ are $\text{an}(X) = \{ B | B \rightarrow \cdots \rightarrow A \text{ is in } G \text{ with } A \in X \}$. Moreover, $X$ is called ancestral set if $X = \text{an}(X)$. The sets just defined are defined with respect to $G$. When they are defined with respect to another UDAG, this is indicated with a subscript.

Given an UDAG $G$, the moral graph of $G$ is the UG $G^m$ such that $A - B$ is in $G^m$ if and only if $A \rightarrow B$, $A \leftarrow B$, or $A \rightarrow C \leftarrow D \leftarrow B$ is in $G$. Given a set $W \subseteq V$, we let $G_W$ denote the subgraph of $G$ induced by $W$. Given an UG $H$, we let $H^W$ denote the marginal subgraph of $H$ over $W$, i.e. the edge $A - B$ is in $H^W$ if and only if $A - B$ is in $H$ or $A - V_1 - \cdots - V_n - B$ is $H$ with $V_1, \ldots, V_n \notin W$. A set of nodes of $H$ is complete if there exists an undirected edge between every pair of nodes in the set. A clique of $H$ is a maximal complete set of nodes. The cliques of $H$ are denoted as $\text{Cl}(H)$.

A route between two nodes $V_1$ and $V_n$ of an UDAG $G$ is a sequence of (not necessarily distinct) edges $E_1, \ldots, E_{n-1}$ in $G$ such that $E_i$ links the nodes $V_i$ and $V_{i+1}$. A route is called a path if the nodes in the route are all different. An undirected route is a route whose edges are all undirected. A section of a route $\rho$ is a maximal undirected subroute of $\rho$. A section $V_2 - \cdots - V_{n-1}$ of $\rho$ is called collider section if
$V_1 \rightarrow V_2 - \ldots - V_{n-1} \leftarrow V_n$ is a subroute of $\rho$. Given a set $Z \subseteq V$, $\rho$ is said to be $Z$-active if (i) every collider section of $\rho$ has a node in $Z$, and (ii) every non-collider section of $\rho$ has no node in $Z$.

Let $X$, $Y$, $W$ and $Z$ be disjoint subsets of $V$. We represent by $X \perp_p Y|Z$ that $X$ and $Y$ are conditionally independent given $Z$ in a probability distribution $p$. Every probability distribution $p$ satisfies the following four properties: Symmetry $X \perp_p Y|Z \Rightarrow Y \perp_p X|Z$, decomposition $X \perp_p Y \cup W|Z \Rightarrow X \perp_p Y|Z \cup W$, and contraction $X \perp_p Y|Z \cup W \land X \perp_p W|Z \Rightarrow X \perp_p Y \cup W|Z$.

If $p$ is strictly positive, then it also satisfies the intersection property $X \perp_p Y|Z \cup W \land X \perp_p W|Z \cup Y \Rightarrow X \perp_p Y \cup W|Z$.

3. GLOBAL MARKOV PROPERTY

Given three disjoint sets $X, Y, Z \subseteq V$, we say that $X$ is separated from $Y$ given $Z$ in an UDAG $G$, denoted as $X \perp Y|Z$, if every path in $(G_{an(X \cup Y \cup Z)})^m$ between a node in $X$ and a node in $Y$ has a node in $Z$. As the theorem below proves, this is equivalent to saying that there is no route in $G$ between a node of $X$ and a node of $Y$ that is $Z$-active. Note that these separation criteria generalize those developed by Lauritzen (1996) and Studený (1998) for LWF CGs.

**Theorem 1.** The two separation criteria for UDAGs in the paragraph above are equivalent.

**Proof.** Assume that there is a $Z$-active route $\rho$ in $G$ between $A \in X$ and $B \in Y$. Clearly, every node in a collider section is in $an(Z)$. Moreover, every node in a non-collider section is ancestor of $A$, $B$ or a node in a collider section, which implies that it is in $an(A \cup B \cup Z)$. Therefore, there is a route between $A$ and $B$ in $(G_{an(X \cup Y \cup Z)})^m$. Moreover, the route can be modified into a route $\varrho$ that circumvents $Z$ by noting that there is an edge $V_1 \rightarrow V_n$ in $(G_{an(X \cup Y \cup Z)})^m$ whenever $V_1 \rightarrow V_2 - \ldots - V_{n-1} \leftarrow V_n$ is a subroute of $\rho$. The route $\varrho$ can be converted into a path by removing loops.

Conversely, assume that there is a path $\rho$ in $(G_{an(X \cup Y \cup Z)})^m$ between $A \in X$ and $B \in Y$ that circumvents $Z$. Note that $\rho$ can be converted into a route $\varrho$ in $G$ as follows: If the edge $V_1 \rightarrow V_n$ in $\rho$ was added to $(G_{an(X \cup Y \cup Z)})^m$ because $V_1 \rightarrow V_n$, $V_1 \leftarrow V_n$ or $V_1 \rightarrow V_2 - \ldots - V_{n-1} \leftarrow V_n$ was in $G_{an(X \cup Y \cup Z)}$, then replace $V_1 \rightarrow V_n$ with $V_1 \rightarrow V_n$, $V_1 \leftarrow V_n$, or $V_1 \rightarrow V_2 - \ldots - V_{n-1} \leftarrow V_n$, respectively. Note that the non-collider sections of $\varrho$ have no node in $Z$ for $\rho$ to circumvent $Z$, whereas the collider sections of $\varrho$ have all their nodes in $an(X \cup Y \cup Z)$ by definition of $(G_{an(X \cup Y \cup Z)})^m$.

Note that we can assume without loss of generality that all the collider sections of $\varrho$ have some node in $an(Z)$ because, otherwise, if there is a collider section with no node in $an(Z)$ but with some node $C$ in $an(X)$ then there is a route $A' \leftarrow \cdots \leftarrow C$ with $A' \in X$ which can
replace the subroute of \( \varrho \) between \( A \) and \( C \). Likewise for \( an(Y) \) and some \( B' \in Y \).

Finally, note that every collider section \( V_1 \to V_2 \to \cdots \to V_{n-1} \Leftarrow V_n \) of \( \varrho \) that has no node in \( Z \) must have a node \( V_i \) in \( an(Z) \setminus Z \) with \( 2 \leq i \leq n-1 \), which implies that there is a route \( V_i \to \cdots \to C \) where \( C \) is the only node of the route that is in \( Z \). Therefore, we can replace the collider section with \( V_1 \to V_2 \to \cdots \to V_i \to \cdots \to C \Leftarrow \cdots \Leftarrow V_i \to \cdots \to V_{n-1} \Leftarrow V_n \). Repeating this step results in a \( Z \)-active route between a node in \( X \) and a node in \( Y \).

We say that a probability distribution \( p \) satisfies the global Markov property with respect to an UDAG \( G \) if \( X \perp_p Y \mid Z \) for all disjoint sets \( X, Y, Z \subseteq V \) such that \( X \perp Y \mid Z \) in \( G \). Note that two non-adjacent nodes in \( G \) are not necessarily separated. For example, \( C \perp D \mid Z \) does not hold for any \( Z \subseteq \{A, B, E, F, H\} \) in the UDAG in Figure 11. This drawback is shared by AGs. Although this problem cannot be solved for general AGs (Lauritzen and Sadeghi, 2017, Figure 6), it can be solved for the subclass of CMGs by adding edges without altering the separations represented (Lauritzen and Sadeghi, 2017, Corollary 3.1). Unfortunately, a similar solution does not exist for UDAGs. For example, adding the edge \( C \to D \) to the UDAG in Figure 1 makes \( A \perp B \mid D \) cease holding, whereas adding the edge \( C \leftarrow D \) makes \( A \perp B \mid C \cup F \) cease holding. Adding two edges between \( C \) and \( D \) does not help either, since one of them must be \( C - D \). The following lemma characterizes the problematic pairs of nodes.

**Lemma 1.** Given two non-adjacent nodes \( V_1 \) and \( V_n \) in an UDAG \( G \), \( V_1 \perp V_n \mid Z \) does not hold for any \( Z \subseteq V \setminus (V_1 \cup V_n) \) if and only if \( V_1 \to V_2 \to \cdots \to V_{n-1} \Leftarrow V_n \) is in \( G \), and \( V_i \in an(V_1 \cup V_n) \) for some \( 1 \leq i \leq n \).\(^3\)

**Proof.** To prove the if part, assume without loss of generality that \( V_i \in an(V_1) \). This together with the route in the lemma imply that \( G \) has a route \( \rho \) of the form \( V_i \Leftarrow \cdots \Leftarrow V_i \to \cdots \to V_{n-1} \Leftarrow V_n \). If no node in \( Z \) is in \( \rho \), then \( V_1 \perp V_n \mid Z \) does not hold due to \( \rho \). If \( C \in Z \) is in the subroute \( V_i \to \cdots \to V_{n-1} \Leftarrow V_n \) of \( \rho \), then \( V_1 \perp V_n \mid Z \) does not hold due to the route in the lemma. Finally, if \( C \in Z \) is in the subroute \( V_1 \Leftarrow \cdots \Leftarrow V_i \) of \( \rho \), then \( V_1 \perp V_n \mid Z \) does not hold due to the route \( V_1 \to V_2 \to \cdots \to V_i \to \cdots \to C \Leftarrow \cdots \Leftarrow V_i \to \cdots \to V_{n-1} \Leftarrow V_n \).

To prove the only if part, simply consider \( Z = \emptyset \) and note that \( V_1 \) and \( V_n \) are adjacent in \( (G_{an(V_1 \cup V_n)})^m \) only if \( G \) has a subgraph of the form described in the lemma. \( \square \)

\(^3\)In the terminology of Lauritzen and Sadeghi (2017), this route is a primitive inducing walk.
3.1. Algorithm. Since there may be infinite many routes in an UDAG $G$, one may wonder if the separation criterion based on ruling out $Z$-active routes that we have presented above is of any use in practice. The algorithm below shows how to implement it to check in a finite number of steps whether $X \perp Y \mid Z$ holds. The algorithm is a generalization of the one developed by Studená (1998) for LWF CGs, which was later slightly improved by Sonntag et al. (2015). The algorithm basically consists in repeatedly executing some rules to build the sets $U_1, U_2, U_3 \subseteq V$, which can be described as follows.

- $B \in U_1$ if and only if there exists a $Z$-active route between $A \in X$ and $B$ in $G$ which ends with the subroute $V_i \rightarrow V_{i+1} - \cdots - V_{i+k} = B$ with $k \geq 1$.
- $B \in U_2$ if and only if there exists a $Z$-active route between $A \in X$ and $B$ in $G$ which does not end with the subroute $V_i \rightarrow V_{i+1} - \cdots - V_{i+k} = B$ with $k \geq 1$.
- $B \in U_3$ if and only if there exists a node $C \in U_1 \cup U_2$ and a route $C = V_1 \rightarrow V_2 - \cdots - V_k = B$ in $G$ with $k \geq 2$ such that $\{V_2, \ldots, V_k\} \cap Z \neq \emptyset$.

The algorithm starts with $U_1 = X$ and $U_2 = U_3 = \emptyset$. The algorithm executes the following rules until $U_1$, $U_2$ and $U_3$ cannot be further enlarged.

- $C \in U_2$, $C \leftarrow D$ is in $G$, and $D \notin Z \Rightarrow D \in U_2$.
- $C \in U_1 \cup U_2$, $C \rightarrow D$ is in $G$, and $D \notin Z \Rightarrow D \in U_1$.
- $C \in U_1$, $C - D$ is in $G$, and $D \notin Z \Rightarrow D \in U_1$.
- $C \in U_1 \cup U_2$, $C \rightarrow D$ is in $G$, and $D \in Z \Rightarrow D \in U_3$.
- $C \in U_1$, $C - D$ is in $G$, and $D \in Z \Rightarrow D \in U_3$.
- $C \in U_3$, and $C - D$ is in $G \Rightarrow D \in U_3$.
- $C \in U_3$, $C \leftarrow D$ is in $G$, and $D \notin Z \Rightarrow D \in U_2$.

One can prove that, when the algorithm halts, there is a $Z$-active route between each node in $U_1 \cup U_2$ and some node in $X$. The proof is identical to the one for LWF CGs by Studená (1998, Lemma 5.2) and Sonntag et al. (2015, Proposition 1). Therefore, $X \perp Y \mid Z$ if and only if $Y \subseteq V \setminus (U_1 \cup U_2)$. Moreover, the algorithm above makes it straightforward to adapt to UDAGs the algorithm for learning LWF CGs via answer set programming developed by Sonntag et al. (2015).
A \leftarrow B \rightarrow C \quad D \rightarrow E
\quad F \rightarrow H \quad I \rightarrow J \rightarrow K

Figure 2. Example where the local Markov property can be improved by considering only maximal ancestral sets.

4. LOCAL AND PAIRWISE MARKOV PROPERTIES

We say that a probability distribution $p$ satisfies the local Markov property with respect to an UDAG $G$ if for any ancestral set $W$,

$$A \perp p W \setminus (A \cup ne(G_W)_m(A)) | ne(G_W)_m(A)$$

for any $A \in W$. Similarly, we say that a probability distribution $p$ satisfies the pairwise Markov property with respect to $G$ if for any ancestral set $W$,

$$A \perp p B | W \setminus (A \cup B)$$

for any $A, B \in W$ such that $B \notin ne(G_W)_m(A)$.

Theorem 2. Given a probability distribution $p$ satisfying the intersection property, $p$ satisfies the local Markov property with respect to an UDAG $G$ if and only if it satisfies the pairwise Markov property with respect to $G$.

Proof. The if part follows by repeated application of the intersection property. The only if part follows by the weak union property. □

Theorem 3. Given a probability distribution $p$ satisfying the intersection property, $p$ satisfies the pairwise Markov property with respect to an UDAG $G$ if and only if it satisfies the global Markov property with respect to $G$.

Proof. The if part is trivial. To prove the only if part, let $W = an(X \cup Y \cup Z)$ and note that the pairwise and global Markov properties are equivalent for UGs (Lauritzen, 1996, Theorem 3.7). □

Note that the local Markov property for LWF CGs specifies a single independence for each node (Lauritzen, 1996, p. 55). However, the local Markov property for UDAGs specifies many more independences, specifically an independence for any node and ancestral set containing the node. All in all, our local Markov property serves its purpose, namely to identify a subset of the independences specified by the global Markov property that implies the rest. In the next section, we show how to reduce this subset.
4.1. Reduction. The number of independences specified by the local Markov property for UDAGs can be reduced by considering only maximal ancestral sets for any node $A$, i.e., those ancestral sets $W'$ such that $A \in W'$ and $ne_{(G_{W'})}(A) \subset ne_{(G_{W''})}(A)$ for any ancestral set $W''$ such that $W' \subset W''$. Note that there may be several maximal ancestral sets $W'$ for $A$, each for a different set $ne_{(G_{W'})}(A)$ as will be shown. The independences for the non-maximal ancestral sets follow from the independences for the maximal ancestral sets by the decomposition property. In other words, for any non-maximal ancestral set $W$ and $A \in W$,

$$A \perp_{p} W \setminus (A \cup ne_{(G_{W'})}(A)) | ne_{(G_{W'})}(A)$$

follows from

$$A \perp_{p} W' \setminus (A \cup ne_{(G_{W'})}(A)) | ne_{(G_{W'})}(A)$$

where $W'$ is the maximal ancestral set for $A$ such that $ne_{(G_{W'})}(A) = ne_{(G_{W'})}(A)$. In the UDAG in Figure 2, for instance, $W_1 = \{A, B, C, D\}$, $W_2 = \{A, B, C, D, E, I, J, K\}$, and $W_3 = \{A, B, C, D, E, F, H, I, J, K\}$ are three ancestral sets that contain the node $B$. However, only $W_2$ and $W_3$ are maximal for $B$: $W_1$ is not maximal because $W_1 \subset W_2$ but $ne_{(G_{W_1})}(B) = ne_{(G_{W_2})}(B)$, and $W_2$ is maximal because $W_2 \subset W_3$ and $ne_{(G_{W_2})}(B) \subset ne_{(G_{W_3})}(B)$. Note that $W_1$ specifies $B \perp_{p} D | \{A, C\}$, and $W_2$ specifies $B \perp_{p} D, E, I, J, K | \{A, C\}$. Clearly, the latter independence implies the former by the decomposition property. Therefore, there is no need to specify both independences, as the local Markov property does. It suffices to specify just the second.

A more convenient characterization of maximal ancestral sets is the following. An ancestral set $W'$ is maximal for $A \in W'$ if and only if $W' = V \setminus \{(ch(A) \cup de(ch(A))) \setminus W'\}$. To see it, note that $B \in ne_{(G_{W'})}(A)$ if and only if $A \leftarrow B$, $A \rightarrow B$, or $A \rightarrow C \rightarrow \cdots \rightarrow D \leftarrow B$ is in $G_{W'}$. Note that all the parents and neighbors of $A$ are in $W'$, because $W'$ is ancestral. However, if there is some child $B$ of $A$ that is not in $W'$, then any ancestral set $W''$ that contains $W'$ and $B$ or any node that is a descendant of $B$ will be such that $ne_{(G_{W'})} = ne_{(G_{W''})}(A)$.

The number of independences specified by the pairwise Markov property can also be reduced by considering only maximal ancestral sets. This can be proven in the same way as Theorem 2.

5. FACTORIZATION PROPERTY

**Theorem 4.** Given a probability distribution $p$ satisfying the intersection property, $p$ satisfies the pairwise Markov property with respect to an UDAG $G$ if and only if for any ancestral set $W$,

$$p(W) = \prod_{K \in Cl((G_{W'})^{m})} \varphi(K)$$

where $\varphi(K)$ is a non-negative function.
Figure 3. Example where the factorization property can be improved by considering only maximal ancestral sets.

Proof. It suffices to recall the equivalence between the pairwise Markov property and the factorization property for UGs (Lauritzen, 1996, Theorem 3.9). □

5.1. Reduction. The number of factorizations specified by the factorization property for UDAGs can be reduced by considering only maximal ancestral sets, i.e. those ancestral sets $W'$ such that $(G_{W'})^m$ is a proper subgraph of $((G_{W''})^m)^{W'}$ for any ancestral set $W''$ such that $W' \subset W''$. These maximal ancestral sets do not necessarily coincide with the ones defined in Section 4.1. The factorizations for the non-maximal ancestral sets follow from the factorizations for the maximal ancestral sets. To see it, note that for any non-maximal ancestral set $W$, the probability distribution $p(W)$ can be computed by marginalization from $p(W')$ where $W'$ is any maximal ancestral set such that $((G_{W'})^m)^W = (G_{W'})^m$. Note also that $p(W)$ factorizes according to $((G_{W'})^m)^W$ and thus according to $(G_{W'})^m$, by Studeny (1997, Lemma 3.1) and Lauritzen (1996, Theorems 3.7 and 3.9). In the UDAG in Figure 3, for instance, $W_1 = \{A, B\}$, $W_2 = \{A, B, C, D, E\}$, and $W_3 = \{A, B, C, D, E, F, H, I\}$ are three ancestral sets. However, only $W_1$ and $W_3$ are maximal: $W_2$ is not maximal because $W_2 \subset W_3$ but $((G_{W_3})^m)^{W_2} = (G_{W_2})^m$, and $W_1$ is maximal because $W_1 \subset W_3$ and $(G_{W_1})^m$ is a proper subgraph of $((G_{W_3})^m)^{W_1}$. Note that $W_3$ specifies

$$p(W_3) = \varphi(A, B)\varphi(A, C)\varphi(B, E)\varphi(C, D)\varphi(D, E) \cdot \varphi(A, F)\varphi(B, I)\varphi(F, H)\varphi(H, I)$$

and $W_2$ specifies

$$p(W_2) = \varphi'(A, B)\varphi'(A, C)\varphi'(B, E)\varphi'(C, D)\varphi'(D, E).$$
Clearly, the former factorization implies the latter by taking
\[ \varphi'(A, B) = \varphi(A, B) \sum_{F, H, I} \varphi(A, F) \varphi(B, I) \varphi(F, H) \varphi(H, I) \]
\[ \varphi'(A, C) = \varphi(A, C) \]
\[ \varphi'(B, E) = \varphi(B, E) \]
\[ \varphi'(C, D) = \varphi(C, D) \]
\[ \varphi'(D, E) = \varphi(D, E). \]
Therefore, there is no need to specify both factorizations, as the factorization property does. It suffices to specify just the first.

A more convenient characterization of maximal ancestral sets is the following. An ancestral set \( W' \) is maximal if and only if \( \text{pa}(A \cup \text{an}(A) \setminus W') \cap W' \) is not a complete set in \((G_{W'})^m\) for any node \( A \in V \setminus W' \).

To see it, note that any ancestral set \( W'' \) that contains \( W' \cup \text{an}(A) \setminus W' \) will also contain \( \text{an}(A) \setminus W' \). Note also that no node \( B \in A \cup \text{an}(A) \setminus W' \) has a neighbor or child in \( W' \) because, otherwise, \( B \in W' \) which is a contradiction. So, any such node \( B \) can only have parents in \( W' \). Moreover, since \( \text{pa}(A \cup \text{an}(A) \setminus W') \cap W' \) is not a complete set in \((G_{W'})^m\), there must be two nodes in \( \text{pa}(A \cup \text{an}(A) \setminus W') \cap W' \) that are not adjacent in \((G_{W'})^m\). However, there is a path between these two nodes in \((G_{W''})^m\), which implies that \((G_{W'})^m\) is a proper subset of \(((G_{W''})^m)^{W''}\).

6. DISCUSSION

Some natural questions to tackle in the future are the characterization of Markov equivalent UDAGs, and the parameterization of the factorization for UDAGs proposed before. We are also interested in giving a causal interpretation to UDAGs. For instance, the DAG subgraph of a UDAG may represent the causal relations in the domain, and the UG subgraph may represent the compatibility relations in the domain. An example of compatibility relations is soft-constraints. A (hard) constraint specifies which value combinations are feasible, e.g. Boyle’s law relates the pressure and volume of a gas as \( \text{pressure} \cdot \text{volume} = \text{constant} \) if the temperature and amount of gas remain unchanged within a closed system (Dawid, 2010, p. 77). Since we only consider strictly positive probability distributions in this paper, no value combination is unfeasible. Therefore, the previous constraint may be approximated by the soft-constraint \( \text{pressure} \cdot \text{volume} \sim \mathcal{N}(\text{constant}, \sigma) \) where \( \sigma \) is a small value. Many scientific laws are symmetric and, thus, one should represent them as soft-constraints rather than as causal relations. In other words, one should prefer \( \text{pressure} \rightarrow \text{volume} \) or \( \text{pressure} \leftarrow \text{volume} \). This is something one can

\[2\]In the terminology of Frydenberg (1990), \( A \cup \text{an}(A) \setminus W' \) is a non-simplicial set in \((G_{W'})^m\).
readily do with UDAGs. It would also be interesting to extend UDAGs to allow feed-back loops, since some may argue that the preferred representation should be \textit{pressure} \textRightarrow \textit{volume}.

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