Hopf Algebras Consisting of Finite Sets

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Abstract

In this article we generalise Connes-Kreimer Hopf algebra to case of abstract finite sets by introducing the notations parallel to ones in quantum field theory. Firstly, we introduce the notations of collapsing and quotient motivated by same notations of Feynman diagrams. With help of those notations we construct two coalgebra structures for finite sets, such that the power-set of every finite set carries on coalgebra structures. Secondly, we exhibit the notation of forest for the case of finite set which appeared in quantum field theory originally, and then we show that every forest induces a Hopf algebra consisting of finite sets.

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1 Introduction

In the present article we generalise the structure of Connes-Kreimer Hopf algebra (see A. Connes an D. Kreimer [1], [2] for Feynman diagrams to the situation of abstract finite sets. The heartening suggestion in H. Figueroa and J.M. Gracia-Bondia [3] (see J. M. Gracia-Bondia, J. C. Varilly and H. Figueroa [4] also) shows that Connes-Kreimer’s coproduct of Feynman diagrams can be admitted to subgraphs. A subgraph of a connected

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Feynman diagram is a subdiagram determined by its vertices completely. Somehow we can centre on vertices for the structure of Hopf algebra of Feynman diagrams. This is our motivation to generalise the structure of Hopf algebra about Feynman diagrams to the case of abstract finite sets. Roughly speaking, a abstract finite set can be viewed as set of "vertices”.

In order to generalise the structures concerning Feynman diagrams we need to ”translate” some notations of Feynman diagrams into language of set theory. We show that in the following table. Subgraphs are translated as subsets or sequences of disjoint union subsets, factorisations of Feynman amplitudes are translated to be partitions of some subset, collapsing or quotient and forest of Feynman diagrams have been translated also. We establish those notations in a formal way from pure algebraic viewpoint, actually, in our case there is not sub-divergence to be considered. By generalising the collapsing or quotient to the case of abstract finite sets we can construct two types of coalgebra structure for any finite set, as consequence, we show that the power-set of a finite set carries two natural coalgebra structures. Moreover, we emphasize the notion of forest plays key role in our construction. With the help of forest in the sense of abstract finite set we can construct Hopf algebra for every finite set.

The article is organized as following. In section 2 we discuss some notation we will need and quotient or collapsing for subsets are discussed. In section 3 we construct two types of coproducts in finite sets. Finally, in section 4 we discuss the notations of forest and factorisations. We prove every forest induce a Hopf algebra consisting set of factorisations.

## 2 Quotient and collapsing

In this article every issue can be reduced to the case of finite sets, therefore, we restrict our consideration in the case of finite sets only.

**Partitions:** For an abstract finite set $A$, $\#A = d$ ($d$ is a positive integer), let $(I_1, \cdots, I_l)$ be a sequence of disjoint non-trivial subsets in $A$, it is denoted by $(I_i)$ also for short, we can always regard $(I_i)$ as a partition of some set. Actually, let $I = \bigcup_{i=1}^l I_i$, then $(I_i)$ is a partition of $I$, i.e. $(I_i) \in \text{part}(I)$. Thus, for simplicity, we call the sequence of disjoint subsets in forest

| set theory          | quantum field theory                   |
|---------------------|----------------------------------------|
| subset              | subgraph                               |
| collapsing and quotient | collapsing and quotient of Feynman diagrams |
| partition           | factorisation of Feynman amplitude     |
| forest              | forest of Feynman diagrams             |
| subsets in forest   | connected diagrams                     |

Table 1: default
non-trivial subsets of $A$ the partition in $A$ below. For two partitions $(I_1, \cdots, I_l)$ and $(J_1, \cdots, J_k)$ in $A$, let $I = \bigcup_{i=1}^l I_i$ and $J = \bigcup_{j=1}^k J_j$, we need the following notations:

- **Joint:** $(I_i \cap J_j)$ is a partition of $I \cap J$ denoted by $(I_i) \cap (J_j)$. If $I \cap J = \emptyset$ we say $(I_i)$ and $(J_j)$ are disjoint denoted by $(I_i) \cap (J_j) = \emptyset$.

- **Union:** If $(I_i) \cap (J_j) = \emptyset$, $(I_1, \cdots, I_l; J_1, \cdots, J_k)$ is a partition of $I \cup J$ denoted by $(I_i) \cup (J_j)$.

- **Inclusion:** If for $\forall I_i$, $\exists J_j$, such that $I_i \subset J_j$, we say $(J_j)$ includes $(I_i)$ denoted by $(I_i) \subset (J_j)$. In this case we call $(I_i)$ is a sub-partition of $(J_j)$.

**Quotient and collapsing:** For a given finite set $A(\#A = d > 0)$, let $U, I$ be subsets of $A$, we want to construct an operation called quotient or collapsing which can be regarded as a map:

$$
\text{collapsing} \quad \mathcal{P}(A) \times \mathcal{P}(A) \rightarrow \mathcal{P}(A) \cup \mathcal{P}^2(A),
$$

where $\mathcal{P}(A)$ denotes the power-set of $A$, and $\mathcal{P}^2(A) = \mathcal{P}(\mathcal{P}(A))$. We define the quotient of $U$ by $I$ denoted by $U / I$ in the following way.

**Definition 2.1.**

- If $I \cap U = \emptyset$, we have
  $$
  U / I = U.
  $$

- If $U \subset I$, we have
  $$
  U / I = \emptyset.
  $$

- If $I, U$ satisfy the following condition:
  $$
  I \cap U \neq \emptyset, U \setminus I \neq \emptyset,
  $$
  (2.1)
  we have

  - When $\#(I \cap U) = 1$, which means $I \cap U = \{a\} \ (a \in A)$, we define
    $$
    U / I := U \setminus I;
    $$
  - When $\#(I \cap U) > 1$, we define
    $$
    U / I := (U \setminus I) \cup \{U \cap I\}.
    $$
  (2.2)

If condition (2.1) is available we call the quotient is non-trivial.

**Remark 2.1.**
We call procedure from pair \((U, I)\) to quotient \(U/I\) the collapsing. Roughly speaking, a single element collapses to \(\emptyset\), and a subset \(I \cap U\) with \(#(I \cap U) > 1\) collapses to a new “ideal element”. The notations of quotient and collapsing are motivated by the quotient and collapsing of Feynman diagrams.

Particularly, we have
\[ U/U = \emptyset, U/\emptyset = U. \]

The definition 2.1 shows that the quotient \(U/I\) determined by \(U \cap I\) solely. Actually, we can take \(I' = I \cap U\) instead of \(I\), where we identify the \(I\) with \(I'\) as same “ideal element”. The key idea is that the part dug from \(U\) collapses to a “ideal element”. Generally, for two sets \(I_1, I_2\), if \(I_1 \cap U = I_2 \cap U\) we have \(U/I_1 = U/I_2\).

Moreover, if \(J, U/I \subset A \cup \mathcal{P}(A)\) satisfy the condition (2.1), we can make collapsing again. By definition 2.1 we have
\[ (U/I)/J = (((U \setminus I) \cup \{I\}) \setminus J) \cup \{J\}. \]

Where we are interested in non-trivial quotient. Thus, noting remark 2.1, we can assume \(J \subset U/I\) without loss of generality. Let \(J' = J \cap (U \setminus I)\), we discuss the different cases as follows:

\(\{I\} \subset J: J = J' \cup \{I\}\), we have
\[ (U/I)/J = ((U \setminus I) \setminus J') \cup \{J\} = ((U \setminus I) \setminus J) \cup \{J\}, \]

thus
\[ (U/I)/J = (U \setminus I)/J. \]

Where we work on \(\mathcal{P}(A) \cup \mathcal{P}^2(A)\) at all, or, collapsing is the following map:
\[ \mathcal{P}(A) \cup \mathcal{P}^2(A) \times \mathcal{P}(A) \cup \mathcal{P}^2(A) \longrightarrow \mathcal{P}(A) \cup \mathcal{P}^2(A). \] \hspace{1cm} (2.3)

\(\{I\} \notin J: \) We have
\[ (U/I)/J = (U \setminus (I \cup J)) \cup \{I, J\}. \]

For three finite sets \(U, V, I\), if condition (2.1) is valid for them such that both of \(U/I\) and \(V/I\) are non-trivial, it is obvious that we have
\[ (U/I) \cap (V/I) = (U \cap V)/I, \] \hspace{1cm} (2.4)
and
\[ (U/I) \cup (V/I) = (U \cup V)/I. \] \hspace{1cm} (2.5)

Particularly, if \(U \subset V\) we have
\[ U/I \subset V/I. \]
Additionally, if \( U \cap V = \emptyset \), and \( I \cap J = \emptyset \), and \( U/I, V/J \) are non-trivial, we have

\[
(U/I) \cup (V/J) = ((U \cup V)/I)/J.
\]

(2.6)

Now we consider the situation of partitions. For a given partition \((I_1, \cdots, I_l)\) and subset \( U \) in \( A \), we define the quotient \( U/(I_i) \) inductively.

\[
U/(I_i) := (\cdots((U/I_1)/I_2 \cdots)/I_l.
\]

(2.7)

By definition 2.1 we have

\[
U/(I_i) = (U \setminus I) \cup \{I_i| \#I_i > 1, 1 \leq i \leq l\},
\]

where \( I = \bigcup_{i=1}^{l} I_i \). There are formulas similar to (2.4), (2.5) and (2.6) in the case of partitions.

**Remark 2.2.**

- In summary, collapsing or quotient concerning the partitions in \( A \) is the map (2.3).
- For two partitions \((I_1)_{1 \leq i \leq k}\) and \((J_1)_{1 \leq i \leq k}\), if \( U \cap I_i = U \cap J_i \) \((1 \leq i \leq k)\), we will identify two sets \( U/(I_i) \) and \( U/(J_i) \) due to similar reason in remark 2.1.

We now discuss the quotient of partitions by partitions. Let \((I_1, \cdots, I_l)\) and \((J_1, \cdots, J_k)\) are two partitions in \( A \). We assume \( I \cap J \neq \emptyset \), where \( I = \bigcup_{i=1}^{l} I_i \) and \( J = \bigcup_{j=1}^{k} J_j \). The indices of \( J_j \) can be divided into two classes, for example, \( \{1, \cdots, p\} \) and \( \{p+1, \cdots, k\} \), such that \( J_j \cap I = \emptyset \) \((1 \leq j \leq p)\) and \( J_j \cap I \neq \emptyset \) \((p < j \leq k)\),

For a \( J_j \), when \( J_j \cap I \neq \emptyset \), there are two possible cases:

- There some \( I_i \) such that \( I_i \supset J_j \).
- There is a sub-sequence \((I_{i_1}, \cdots, I_{i_j})\) such that \( J_j \cap I_{i_1}, \cdots, J_j \cap I_{i_j} \neq \emptyset \).

Let \( J_j(p < j \leq p+q) \) belong to the second case mentioned above, and \( J_j(p+q < j \leq k) \) belong the first case. Now we define the quotient of \((J_j)\) by \((I_i)\) denoted by \((J_j)/(I_i)\) as follows:

- For \( 1 \leq j \leq p \), \( \tilde{J}_j = J_j \).
- For \( p+1 \leq j \leq p+q \), \( \tilde{J}_j = (J_j \setminus I) \cup \{I_i| I_i \cap J_j \neq \emptyset\} \).

For \( j > p+q \) we have \( J_j/(I_i) = \emptyset \). It is obvious that \((\tilde{J}_j) \in \text{part}(J/(I_i))\). Actually, we can identify \((J_j)/(I_i)\) with \((J_j/(I_i))\).

We are really interested in the case of \((U/(I_i))/(J_j)\), which will be very useful in discussion below, where \( U \subset A \) and \((J_1, \cdots, J_k)\) is a partition in \( A/(I_i) \). Now we have
Proposition 2.1. Let \((I_1, \ldots, I_l)\) be a partition in \(A, U \subset A,(J_1, \ldots, J_k)\) be a partition in \(A/(I_l)\), then there is a partition \((K_\lambda)\) in \(A, (I_i) \subset (K_\lambda), \) such that
\[
(U/(I_l))/(J_j) = U/(K_\lambda),
\]
and
\[
(K_\lambda)/(I_i) = (J_j).
\]
Where \(U\) satisfies the conditions such that \(U/(I_l)\) and \((U/(I_l))/(J_j)\) are non-trivial.

Proof. Without loss of generality, we can assume \(#I_i, #J_j > 1 (1 \leq i \leq l, 1 \leq j \leq k)\).
Actually, the partition \((I_i)\) can be divided into two parts \((I_i) = (I_i') \cup (I_i'')\), such that \(#I_i' \geq 1\) and \(#I_i'' \geq 1\). Then \((I_i'')\) is a partition in \(A \setminus \bigcup'_i I_i''\). It is easy to check that
\[
A/(I_i) = (A \setminus \bigcup'_i I_i')/(I_i''),
\]

Let \(J = \bigcup_{i=1}^k J_i, J' = J \cap I^c, J'' = J \setminus \bigcup \{I_i\}, J_j' = J_j \cap I^c, J_j'' = J_j \setminus \bigcup \{I_i\} (j = 1, \ldots, k)\), where \(\{I_1, \ldots, I_l\}\) is the set of "ideal elements" in \(A/(I_l)\). We discuss the problem for three cases respectively.

Case of \(J'' = \emptyset\): In this case \(J \subset I^c\), thus \((J_j)\) is a partition in \(I^c\). If we take \((K_\lambda) = (I_i) \cup (J_j)\), then \(K \subset A, K = \bigcup_{\lambda} K_\lambda, \) which means \((K_\lambda)\) is a partition in \(A\). From (2.6) we know that the formula (2.7) is valid. Noting
\[
(K_\lambda)/(I_i) = (I_1/I_1, \ldots, I_l/I_l; J_1/I_l \setminus /\emptyset, \ldots, J_k/I_l \setminus \emptyset) = (\emptyset, \ldots, \emptyset; J_1, \ldots, J_k),
\]
thus we can identify \((K_\lambda)/(I_i)\) with \((J_j)\) naturally.

Case of \(J'' = \{I_i\}\): In this case we take \(K_j = J'_j \cup (\bigcup_{i \in J''} I_i)\) for \(J'' \neq \emptyset\) and \(K_j = J_j\) for \(J'' = \emptyset\). Then we have
\[
(U/(I_l)) \setminus \bigcup_{j=1}^k J_j = U \setminus \bigcup_{j=1}^k K_j.
\]
If we identify \(K_j\) with \(J_j\) as "ideal elements", we get formula (2.7). We note that for \(J'' \neq \emptyset\) we have
\[
I_i \subset K_j \iff I_i \in J''_j,
\]
thus \(K_j/(I_i || I_i \subset K_j) = J_j\), so the formula (2.8) is valid.

Case of \(J'' \neq \emptyset\) and \(J'' \neq \{I_i\}\): Let \(\{I_i || I_i \notin J\} = \{I_{i_1}, \ldots, I_{i_p}\}\), we take the partition \((K_1, \ldots, K_k, K_{k+1}, \ldots, K_{k+p})\) as follows: For \(1 \leq j \leq k\),
\[
K_j = J'_j \cup (\bigcup_{i \in J''_j} I_i), \hspace{1cm} J''_j \neq \emptyset
\]
\[
K_j = J_j, \hspace{1cm} J''_j = \emptyset,
\]
for \(j > k, K_{k+q} = I_{i_q}\). Then both of formulas (2.7) and (2.8) are valid.
3 Two coalgebra structures

Now we want to construct the coproduct for finite sets.

3.1 The first oproduct

Let $A$ be a finite set. We will construct a coproduct

$$\Delta_1 : P(A) \cup P^2(A)_{dis} \rightarrow (P(A) \cup P^2(A)_{dis}) \otimes (P(A) \cup P^2(A)_{dis}),$$

where

$$P^2(A)_{dis} = \{\{I_1, \ldots, I_k\}|I_i \subset A, I_i \cap I_j = \emptyset, i \neq j\}.$$ 

In other word $P^2(A)_{dis}$ can be regarded as the set of partitions in $A$.

**Definition 3.1.** We define the coproduct $\Delta_1$ as follows:

- $\Delta_1 \emptyset = \emptyset \otimes \emptyset$. (3.1)
- If $U \in P(A) \cup P^2(A)_{dis}$, $\#U = 1$

$$\Delta_1 U = U \otimes \emptyset + \emptyset \otimes U.$$ (3.2)

- If $U \in P(A) \cup P^2(A)_{dis}$, $\#U = d, d > 1$, and $U$ adapts to a partition $(U_l) \in \text{part}(U)$, then the coproduct admitting to $(U_l)$ is defined to be

$$\Delta_1(U; (U_l)) = U \otimes \emptyset + \emptyset \otimes U$$

$$\sum_{(I_i) \in \text{part}(I_l), I_l \subset (U_l), I \subset U} (I_i; (I_i)) \otimes (U/(I_i), (U_l)/(I_i)).$$ (3.3)

Particularly

$$\Delta_1 U = U \otimes \emptyset + \emptyset \otimes U + \sum_{(I_i) \in \text{part}(I_l), I \subset U} (I_i; (I_i)) \otimes (U/(I_i)).$$ (3.4)

In the formula (3.4) we recognize $U$ admits to a trivial partition $(U)$, and in the sum (3.4), if $I = U$ the partition $(I_i)$ consists of two non-trivial subsets at least.

About coproduct defined in definition 3.1 we have

**Theorem 3.1.** For a subset $U \in P(A) \cup P^2(A)_{dis}$ we have

$$(\Delta_1 \otimes id)\Delta_1 U = (id \otimes \Delta_1)\Delta_1 U.$$ (3.5)

and

$$(\Delta_1 \otimes id)\Delta_1(U, (U_l)) = (id \otimes \Delta_1)\Delta_1(U, (U_l)).$$ (3.6)
Proof. For simplicity we give the proof of (3.5) only. The formula (3.6) can be proved by similar way. It is enough for us to consider the reduced coproduct $\Delta'_1$, where

$$
\Delta'_1 U = \Delta_1 U - (U \otimes \emptyset + \emptyset \otimes U).
$$

According to the formula (3.4) we have

$$(\Delta'_1 \otimes id)\Delta'_1 U = \sum_{(I_i) \in \text{part}(I), I \subset U} \Delta'_1(I; (I_i)) \otimes (U/(I_i)),$$

and

$$(id \otimes \Delta'_1)\Delta'_1 U = \sum_{(I_i) \in \text{part}(I), I \subset U} (I; (I_i)) \otimes \Delta'_1(U/(I_i)).$$

Noting that

$$\Delta'_1(I; (I_i)) = \sum_{(J_j) \subset (I_i)} (J; (J_j)) \otimes (I/(J_j), (I_i)/(J_j)).$$

In the previous sum $J = \bigcup_j J_j$. On the other hand, due to the formula (3.4) and proposition 2.1 we have

$$\Delta'_1(U/(I_i)) = \sum_{(J_j) \in \text{part}(J), J \subset U/(I_i)} (J; (J_j)) \otimes (U/(I_i)/(J_j))$$

$$= \sum_{(I_i) \subset (K \lambda)} (K/(I_i), (K \lambda)/(I_i)) \otimes (U/(K \lambda)),$$

where $K = \bigcup \lambda K \lambda$. Comparing the expression of $(\Delta'_1 \otimes id)\Delta'_1 U$ and one of $(id \otimes \Delta'_1)\Delta'_1 U$, we know that the formula (3.5) is valid. 

The following conclusion can be proved by induction.

**Proposition 3.1.** Let $m = \max_i \{\#U_i\}$, then we have

$$\Delta'_1^{m-1}(U, (U_i)) = 0. \quad (3.7)$$

### 3.2 The second coproduct

For a finite set $A$, we consider a non-trivial subset $I, J \subset A$. We now define the quotient of $J$ by subset $I$ denoted by $J/I$ in a simple way:

$$J/I := J \setminus I, \quad (3.8)$$

**Remark 3.1.**

We define

$$J/J = \emptyset, J/\emptyset = J. \quad (3.9)$$
It is obvious that
\[ \#(J/I) = \#J - \#I. \quad (3.10) \]

For \( I_2 \subset J/I_1 \), we have
\[ (J/I_1)/I_2 = J/(I_1 \cup I_2). \]

We can now define the coproduct
\[ \Delta_2 : \mathcal{P}(A) \to \mathcal{P}(A) \otimes \mathcal{P}(A). \]

**Definition 3.2.** Let \( A \) be a finite set, we define the coproduct \( \Delta_2 \) as following:

- \( \Delta_2 \emptyset = \emptyset \otimes \emptyset. \quad (3.11) \)
- For \( J \subset A \),
  - If \( \#J = 1 \)
    \[ \Delta_2 J = J \otimes \emptyset + \emptyset \otimes J. \quad (3.12) \]
  - If \( \#J = d, \ d > 1 \),
    \[ \Delta_2 J = J \otimes \emptyset + \emptyset \otimes J + \sum_{I \subset J, 1 < \#I < d} I \otimes (J/I). \quad (3.13) \]

About the coproduct defined in above definition we have

**Theorem 3.2.** Let \( J \subset A \), we have
\[ (\Delta_2 \otimes \text{id})\Delta_2 J = (\text{id} \otimes \Delta_2)\Delta_2 J, \ \forall J \in \mathcal{P}(A). \quad (3.14) \]

If \( \#J = d, \ d > 1 \), then
\[ (\Delta_2')^{d-1} J = 0, \quad (3.15) \]
where \( \Delta_2' \) is reduced coproduct
\[ \Delta_2' J = \Delta_2 J - (J \otimes \emptyset + \emptyset \otimes J). \]

**Proof.** Let \( J \in \mathcal{P}(A), \ \#J = d > 1 \), then
\[ (\Delta_2' \otimes \text{id})\Delta_2 J = \sum_{I \subset J, 0 < \#I < d} \Delta_2 I \otimes (J/I), \]
and
\[ (\text{id} \otimes \Delta_2')\Delta_2 J = + \sum_{I \subset J, 0 < \#I < d} I \otimes \Delta_2'(J/I). \]

Then we have
\[ \Delta_2 I = \sum_{K \subset I, 0 < \#K < \#I} K \otimes (I/K). \]
On the other hand we have
\[
\begin{align*}
&= \sum_{K \subseteq I^c, 0 < \#K < \#I^c} K \otimes (J/I) \\
&= \sum_{K \subseteq I^c, 0 < \#K < \#I^c} ((I \cup K) \subseteq J/(I \cup K))
\end{align*}
\]

Thus the formula (3.17) is valid. The formula (3.18) can easily be proved by induction for \#J. □

**Proposition 3.2.** For two subsets \(I, J\), if \(I \cap J = \emptyset\), we have
\[
\Delta_2(I \cup J) = \Delta_2 I \cup \Delta_2 J. \tag{3.16}
\]

**Proof.** Let \(#I = p\), \(#J = q\), \(K = I \cup J\), by definition of \(\Delta_2\) we have
\[
\begin{align*}
\Delta_2 I &= \sum_{k=1}^{p-1} \sum_{I_k \subseteq I} I_k \otimes (I \setminus I_k), \\
\Delta_2 J &= \sum_{l=1}^{q-1} \sum_{J_l \subseteq J} J_l \otimes (J \setminus J_l),
\end{align*}
\]

in above sums \(#I_k = k\), \(#J_l = l\). Therefore,
\[
\begin{align*}
\Delta_2 I \cup \Delta_2 J &= \sum_{m=2}^{p+q-2} \sum_{k+l=m} (I_k \cup J_l) \otimes ((I \setminus I_k) \cup (J \setminus J_l)) \\
&= \sum_{m=2}^{p+q-2} \sum_{K_m \subseteq K} K_m \otimes (K \setminus K_m),
\end{align*}
\]

in above sum \(#K_m = m\) and \(K_m \cap I = I_k, K_m \cap J = J_l\). Finally, we have
\[
\begin{align*}
&= K \otimes \emptyset + \emptyset \otimes K + I \otimes J + J \otimes I \\
&+ \sum_{m=2}^{p+q-2} \sum_{K_m \subseteq K} K_m \otimes (K \setminus K_m) \\
&+ \sum_{k=1}^{p-1} \sum_{I_k \subseteq I} (I_k \cup J) \otimes ((K \setminus I_k) \\
&+ \sum_{l=1}^{q-1} \sum_{J_l \subseteq J} (I \cup J_l) \otimes (K \setminus J_l) \\
&= K \otimes \emptyset + \emptyset \otimes K + \sum_{m=1}^{p+q-1} \sum_{K_m \subseteq K} K_m \otimes (K \setminus K_m).
\end{align*}
\]

Thus the formula (3.16) is valid. □
Remark 3.2.

- If $I \cap J \neq \emptyset$, the formula (3.16) is not valid. For example, let $I = \{a, b\}$ and $J = \{b, c\}$, it is to check that
  \[
  \Delta_2 I \cup \Delta_2 J \neq \Delta_2 (I \cup J).
  \]

- For the case of $\Delta_1$, even though $I \cap J = \emptyset$ the formula similar to (3.16) may not be valid. For example, let $I = \{a, b\}, J = \{c, d\}$, in $\Delta_1 (I \cup J)$ the "ideal element" will appear.

4 Hopf algebra

In this section we construct an example of Hopf algebra consists of finite sets which can basically be recognised to be a translation of Connes-Kreimer Hopf algebra consisting of Feynman diagrams by language of set theory.

4.1 Factorisations and forest:

We will start at a set $A$ which may be infinite. For given finite subset $I \subset A$ and a partition $(I_i) \in \text{part}(I)$, we call the pair $(I, (I_i))$ is a factorisation in $A$. The factorisations what we will discuss concern finite subset of $A$ only. Let $(I, (I_i))$ and $(J, (J_j))$ be two factorisations, if $I \cap J = \emptyset$, then $(I \cup J, (I_i) \cup (j_j))$ is a factorisation which is called the union of $(I, (I_i))$ and $(J, (J_j))$ denoted by $(I, (I_i)) \cup (J, (J_j))$. We call the pair $(I, (I))$ is a trivial factorisation. Each factorisation can be expressed as disjoint union of some trivial factorisations obviously, actually we have $(I, (I_i)) = \bigcup_i (I_i, (I_i))$. To construct Hopf algebra we introduce the notation of forest as following:

Definition 4.1. A family of finite subsets of $A$ is called a forest in $A$ denoted by $F$ if

- $\emptyset \in F$;
- For any two subsets $I, J \in F$ we have $I \cap J = \emptyset$, or $I \subset J$, or $J \subset I$.
- If $A$ is a finite set, then $A \in F$.

Remark 4.1.

- The terminology "forest" was appeared in Zimmermann’s forest formula concerning renormalisation in quantum field theory originally. The forest in original sense is a set of Feynman diagrams satisfying some conditions. In present article we translate "forest" into the language of set theory with same conditions.

- If $A$ is finite set, we can assume $A$ is not the disjoint union of other subsets in $F$. In fact, if $A = A_1 \cup A_2$, $A_1 \cap A_2 = \emptyset$, $A_i \in F$ ($i = 1, 2$), we can divid $F$ into two disjoint parts such that they are forests in $A_i$ ($i = 1, 2$) respectively.
If a forest $F$ satisfies the condition: any set in $F$ cannot be the disjoint union of some other sets in $F$, we call this forest $F$ a primary forest. Each forest $F$ can induce a primary forest by dropping the subsets which can be expressed as disjoint union of subsets in $F$, we denote it by $F_{pri}$.

**Lemma 4.1.** Let $F$ be a primary forest in $A$, $I_1, \cdots, I_k \in F$, $J = \bigcup_{i=1}^{k} I_i$. Then there is an unique factorisation $(J, (J_1, \ldots, J_l))$ where $J_1, \ldots, J_l \in F$.

**Proof.** We can prove existence by induction. Let $I_i \in F$ ($i = 1, \cdots, k$), and $I_i \cap I_j = \emptyset$ ($1 \leq i < j \leq k - 1$) by assumption of induction. There are two possible cases for $I_k$, $I_k \cap (\bigcup_{1 \leq i \leq k-1} I_i) = \emptyset$, or $I_k \cap (\bigcup_{1 \leq i \leq k-1} I_i) \neq \emptyset$. In the first case we take $J_i = I_i$ ($i = 1, \cdots, k$); in the second case we have $I_k \subset I_l$ for some $l$, therefore, we can take $J_i = I_i$ ($i = 1, \cdots, k-1$).

Now we turn to uniqueness, let

$$
\bigcup_{i=1}^{l} J_i = \bigcup_{\lambda=1}^{p} V_\lambda,
$$

where $J_i, V_\lambda \in F$, $J_i \cap J_j = \emptyset$ ($i \neq j$), $V_\lambda \cap V_\mu = \emptyset$ ($\lambda \neq \mu$). For each $J_i$ there some $V_{\lambda_1}, \cdots, V_{\lambda_p}$ such that $J_i \cap V_{\lambda_j} \neq \emptyset$, $j = 1, \cdots, p$, and $J_i \cap V_\lambda = \emptyset$ for other $\lambda$. Thus $l \geq k$. Similarly, $l \leq k$, so we have $l = k$. Furthermore, from the definition of primary forest we know that for each $J_i$ there is an unique $V_\lambda$ such that $J_i \cap V_\lambda \neq \emptyset$, and vice versa. Thus, we have $J_i = V_\lambda$, which means that there is one-one correspondence between $J_i$ and $V_\lambda$. Therefore the uniqueness is valid. \[\square\]

Let

$$
A_F = \{I_1 \cup \cdots \cup I_k | U_1, \cdots, I_k \in F, I_i \cap I_j \neq \emptyset, i \neq j\}.
$$

(4.1)

$A_F$ is a set of factorisations generated by $F$. It is obvious that $A_F = A_{F_{pri}}$. Therefore, we can always assume the forest $F$ is primary one. It is obvious that for two factorisations $(I, (I_i)), (J, (J_j)) \in A_F$, $I \cap J = \emptyset$, we have $(I, (I_i)) \cup (J, (J_j)) \in A_F$. If $F$ is a primary forest, $U \in F$, then $U$ adapts to a unique factorisation $(U, (U))$ which is trivial one in $A_F$. In this case we call $(U, (U))$ is connected. Each factorisation in $A_F$ is the disjoint union of some connected ones. By lemma 4.1 the decomposition $(I, (I_i)) = \bigcup_i (I_i, (I_i))$ is unique, where $(I_i, (I_i))$ are connected, or, each $I_i \in F$.

**Proposition 4.1.** Let $\Omega$ be a set of factorisations in $A$, there is a forest $F$ such that $\Omega = A_F$, if the following conditions are available:

- If $(I, (I_i))_{1 \leq i \leq l} \in \Omega$, then $(I_i, (I_i)) \in \Omega$, $i = 1, \cdots, l$.
- If $(I, (I_i)), (J, (J_j)) \in \Omega$ and $I \cap J = \emptyset$, then $(I \cup J, (I_i) \cup (J_j)) \in \Omega$.
- Every factorisation $(I, (I_i)) \in \Omega$ has an unique decomposition

$$
(I, (I_i)) = \bigcup_i (I_i, (I_i)).
$$
For two factorisations \((I, (I_i)), (J, (J_j))\) \(\in \Omega\), if \(I \cap J \neq \emptyset\), we have \((I, (I_i)) \subset (I, (I \setminus J, I \cap J))\) or \((J, (J_j)) \subset (J, (J \setminus I, I \cap J))\).

**Proof.** To prove the proposition we need to construct a forest from \(\Omega\). In the case of \(\mathcal{A}\) being infinite set we take \(\mathcal{F}\) in the following way:

\[
\mathcal{F} = \{V \subset \mathcal{A}|(V, (V)) \text{ appears in some decomposition of } (I, (I_i)) \in \Omega\} \cup \{\emptyset\}.
\]

Let \(V, W \in \mathcal{F}, V \cap W \neq \emptyset\). By the definition of \(\mathcal{F}\), there are two factorisations \((I, (I_i)), (J, (J_j))\) \(\in \Omega\), such that \(V = I_i\) and \(W = J_j\) for some \(i\) and \(j\). We can assume \(i = j = 1\) without loss of generality. Noting \(I \cap J \neq \emptyset\), there are a few possible cases as following:

- **If \(I = J\):** By the uniqueness of decomposition of the factorisations we have \(V = W\).
- **If \((I, (I_i)) \subset (J, (J_j))\)** but \(I \neq J\): In this case there is a \(J_j\), for example, let it be \(J_k\), such that \(J_k \cap I = \emptyset\). Now we take two new factorisations in \(\Omega\), \((V, (V))\) and \((W \cup J_k, (W, J_k))\), from the final condition of \(\Omega\) we have \((V, (V)) \subset (W \cup J_k, (W, J_k))\), thus, \(V \subset W\).
- **If \(I \setminus J \neq \emptyset\) and \(J \setminus I \neq \emptyset\):** There is some \(I_i\), for example \(I_1\), such that \(I_1 \cap J = \emptyset\). Similarly, \(J_k \cap I = \emptyset\), we construct two new factorisations \((V \cup I_1, (V, I_1))\) and \((W \cup J_k, (W, J_k))\), noting \(V \cap W \neq \emptyset\), thus we have

\[
(V \cup I_1, (V, I_1)) \subset (V \cup I_1, (W \cap V, (V \setminus W) \cup I_1)),
\]

or,

\[
(W \cup J_k, (W, J_k)) \subset (W \cup J_k, (W \cap V, (W \setminus V) \cup J_k)).
\]

Which means \(V \subset W\) or \(W \subset V\).

If \(\mathcal{A}\) is finite we put \(\mathcal{A}\) into \(\mathcal{F}\) addtionally. From the definition of \(\mathcal{F}\), it is obvious that \(\Omega = \mathcal{A}_F\).

For a non-trivial subset \(\mathcal{B}\) of \(\mathcal{A}\), from forest \(\mathcal{F}\) we can define the restriction of \(\mathcal{F}\) on \(\mathcal{B}\) denoted by \(\mathcal{F}|_{\mathcal{B}}\),

\[
\mathcal{F}|_{\mathcal{B}} = \{U \cap \mathcal{B}|U \in \mathcal{F}\} \cup \{\mathcal{B}\}.
\]  

\(\mathcal{F}|_{\mathcal{B}}\) is a forest obviously, but, it may not be primary one even though \(\mathcal{F}\) is primary forest. However, if we consider a factorisation \((I, (I_i)) \in \mathcal{A}_F\), we can check that \(\mathcal{F}|_I\) is a primary forest. The forest \(\mathcal{F}|_I\) generates a set of factorisations denoted by \(I_{\mathcal{F}|_I}\). On the other hand we can discuss the restriction of \(\mathcal{A}_F\) on \(I\) denoted by \(\mathcal{A}_{F|_I}\),

\[
\mathcal{A}_{F|_I} = \{(J, (J_j)) \in \mathcal{A}_F|J \subset I\}.
\]  

It is obvious that we have

\[
I_{\mathcal{F}|_I} = \mathcal{A}_{F|_I}.
\]
With the help of restrictions of forest and set of factorisations we can always reduce the discussion to the case of finite set.

For a given forest \( F \) and a set of factorisations \( \mathcal{A}_F \) generated by \( F \), we will discuss quotient of them. Let \((I, (I_1, \cdots, I_k)) \in \mathcal{A}_F\), we know that \( I_i \in F (1 \leq i \leq k) \) and \( I_i \cap I_j = \emptyset \ (i \neq j) \). We consider the quotient of factorisations

\[
\mathcal{A}_F/(I_i) = \{(K/(I_i), (K_\lambda)/(I_i)) | (K, (K_\lambda)) \in \mathcal{A}_F\}. \tag{4.5}
\]

Let \((K, (K_\lambda)) \in \mathcal{A}_F\), we want to calculate \((K/(I_i), (K_\lambda)/(I_i))\). Noting

\[
K/(I_i) = \bigcup_\lambda (K_\lambda/(I_i)), \quad (K_\lambda/(I_i)) \cap (K_{\lambda_2}/(I_i)) = \emptyset, \ (\lambda_1 \neq \lambda_2),
\]

it is obvious that

\[
(K/(I_i), (K_\lambda)/(I_i)) = (K, (K_\lambda)), \text{ for } K \cap I = \emptyset,
\]

and

\[
(K/(I_i), (K_\lambda)/(I_i)) = (\emptyset, \emptyset), \text{ for } (K, (K_\lambda)) \subset (I, (I_i)).
\]

The real interesting case is that \( K \cap I \neq \emptyset \), but there is not inclusion relation \((K, (K_\lambda)) \subset (I, (I_i))\), which is the situation of non-trivial quotient actually. Because of \( K_\lambda, I_i \in F \), for each \( K_\lambda \), there are the following possible cases:

- **Case of** \( K_\lambda \cap I = \emptyset \): We have \( K_\lambda/(I_i) = K_\lambda \).

- **Case of** \( K_\lambda \subset I_i \) for **some** \( i \): We have \( K_\lambda/(I_i) = \emptyset \).

- **Some** \( I_i \) is **non-trivial subset** of \( K_\lambda \): We have

\[
K_\lambda/(I_i) = K_\lambda/(I_{i_1}, \cdots, I_{i_p}),
\]

where \( I_{i_1}, \cdots, I_{i_p} \subset K_\lambda \), \( I_i \cap K_\lambda = \emptyset \) for other indices \( i \).

Furthermore, we can take quotient for forest \( F \) to result in a forest in \( \mathcal{A}_F/(I_i) \) denoted by \( F/(I_i) \),

\[
F/(I_i) = \{B/(I_i) | B \in F\}. \tag{4.6}
\]

Similar to above discussion, for each \( B \in F \), there are three cases under our consideration. If \( B \cap I = \emptyset \), we have \( B/(I_i) = B \). If there is some \( I_i \) such that \( B \subset I_i \) then \( B/(I_i) = \emptyset \). For final case we have \( B/(I_i) = B/(I_{j_1}, \cdots, I_{j_y}) \), where \( I_{j_1}, \cdots, I_{j_y} \subset B \) and \( I_i \cap B = \emptyset \) for other \( i \). Thus \( F/(I_i) \) is a forest indeed. If \( F \) is primary, so is \( F/(I_i) \).

Combining the discussion about the factorisations we know that \( \mathcal{A}_F/(I_i) \) is generated by \( F/(I_i) \) really. Actually we have

\[
\mathcal{A}_F/(I_i) = (\mathcal{A}/(I_i))_{F/(I_i)}. \tag{4.7}
\]

Additionally, we can discuss the quotient of \( \mathcal{A}_F|I \). Now we have conclusion similar to proposition 2.1.
Proposition 4.2. Let \((A, (A_λ)) ∈ \mathcal{A}_F\), \((I, (I_i)) ∈ \mathcal{A}_F|_A\), \((J, (J_j)) ∈ (\mathcal{A}_F|_A)/(I_i)\), there is a factorisation \((K,(K_λ)) ∈ \mathcal{A}_F|_A\) such that

\[
(A/\langle I_i \rangle)/(J_j) = A/(K_λ),
\]
and

\[
(I, \langle I_i \rangle) ⊂ (K, (K_λ)), \quad (J, \langle J_j \rangle) = (K/\langle I_i \rangle, (K_λ)/\langle I_i \rangle).
\]

Proof. According to the definition of \(A/\langle I_i \rangle\) we know that for each \((J, \langle J_j \rangle) ∈ (\mathcal{A}_F|_A)/(I_i)\), there is a \((K, (K_λ)) ∈ \mathcal{A}_F|_A\) such that

\[
(J, \langle J_j \rangle) = (K/\langle I_i \rangle, (K_λ)/\langle I_i \rangle).
\]

However, in general we have not

\[
(A/\langle I_i \rangle)/(J_j) = A/(K_λ)
\]

for \((K, (K_λ))\) mentioned above. We need to modify \((K, (K_λ))\) to get a new factorisation \((K', (K'_λ)) \in \mathcal{A}_F|_A\), such that both of two formulas in proposition 2.4 are valid. Without loss of generality we assume that

- When \(1 ≤ λ ≤ p\), \(K_λ \cap I = \emptyset\).
- When \(p < λ ≤ p+q\), there is some \(I_i\) being a non-trivial subset of \(K_λ\).
- When \(λ > p+q\), \(K_λ \subset I_i\) for some \(i\).

Then \(J = (K \setminus I) \cup \{I_i|I_i\ \text{belongs to the second case mentioned above}\}\), and

\[
(J_j) = (K_1, \cdots, K_p, K_{p+1}/\langle I_i \rangle_{i \subset K_{p+1}}, \cdots, K_{p+q}/\langle I_i \rangle_{i \subset K_{p+q}}).
\]

We change \((K, (K_λ))\) in the following way, let \(K'_λ = K_λ (1 ≤ λ ≤ p+q); K'_{p+q+l} = I_{i_l}\), where \(l = 1, \cdots, m\) such that \(\{I_{i_l}\}_{1 ≤ l ≤ m}\) satisfies \(I_{i_l} \cap K = \emptyset\), or, \(I_{i_l} \supset K_λ\) for some \(λ > p+q\). Then \((K', (K'_λ)) \in \mathcal{A}_F\) and satisfies the formulas in the proposition 2.4.

Up to now we have discussed the forest and its quotient and restriction. Continuously, we can take quotient of quotient, or, restriction for quotient, or, quotient of restriction, and so on. All of forests yielded by above procedure can generate sets of factorisations. Let \(\mathcal{F}_A\) denote the set of the factorisations yielded by above procedure and their disjoint union. Then each factorisation in \(\mathcal{F}_A\) can be decomposed into the disjoint union of connected factorisations. Let \(\mathcal{H}_F\) denote the algebra over \(\mathbb{C}\) generated by \(\mathcal{F}_A\), the multiplication in \(\mathcal{H}_F\) is disjoint union denoted by \(\bullet\) which is commutative obviously. The addition in \(\mathcal{H}_F\) is formal one. It is obvious that \((\emptyset, (\emptyset))\) plays the role of unit denoted by \(1\) simply. We want to prove \(\mathcal{H}_F\) is a Hopf algebra for suitable coproduct.
4.2 Hopf algebra

Coproduct:

**Definition 4.2.** For an element $h \in \mathcal{H}_F$ with form $h = (A, (A_a))$ we define coproduct $\triangle_F$ to be

$$\triangle_F h = \triangle_F(A_1, (A_{1})) \cdot \cdots \cdot \triangle_F(A_k, (A_k)).$$

(4.9)

For a connected factorisation $(A, (A))$, we define the coproduct to be

$$\triangle_F(A, (A)) = (A, (A)) \otimes 1 + 1 \otimes (A, (A)) + \sum_{(I, (I)) \in A|A} (I, (I)) \otimes A/(I).$$

(4.10)

**Remark 4.2.** The coproduct $\triangle_F$ is very similar to $\triangle_1$ discussed in previous section. The difference between them is $\triangle_F$ depends on the forest $F$.

**Theorem 4.1.** For coproduct $\triangle_F$ we have

$$(\triangle_F \otimes \text{id}) \triangle_F = (\text{id} \otimes \triangle_F) \triangle_F.$$  

(4.11)

**Proof.** It is sufficient for us to consider the cases of connected factorisations and reduced coproduct. We will take $A$ instead of $(A, (A))$ for short. Additionally, we can think of the forest $F$ relates to $A$, thus we will work for $A_F$ and $F_A$. We recall reduced coproduct is

$$\triangle'_F A = \triangle_F A - (A \otimes 1 + 1 \otimes A).$$

From formula (2.31) we have

$$(\triangle'_F \otimes \text{id}) \triangle'_F A = \sum_{(I, (I)) \in A_F} \triangle'_F(I, (I)) \otimes A/(I),$$

and

$$(\text{id} \otimes \triangle'_F) \triangle'_F A = \sum_{(I, (I)) \in A_F} (I, (I)) \otimes \triangle'_F(A/(I)).$$

We will discuss

$$\triangle'_F(I, (I)) = \sum_{(J, (J)) \in A_F/(I)} (J, (J)) \otimes (I, (I))/(J),$$

and

$$\triangle'_F(A/(I)) = \sum_{(J, (J)) \in A_F/(I)} (J, (J)) \otimes (A/(I))/(J).$$

in details respectively. From proposition 4.1 we know that there is a $(K, (K_\lambda)) \in A_F$ such that

$$(A/(I))/(J) = A/(K_\lambda),$$

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and

\((I, (I_i)) \subset (K, (K_{\lambda})), (J, (J_j)) = (K \backslash (I_i), (K_{\lambda}) \backslash (I_i))\).

Therefore, we have

\[
\sum_{(I, (I_i)) \in A_F, (J, (J_j)) \in A_H} (J, (J_j)) \otimes (I, (I_i)) / (J_j) \otimes A / (I_i),
\]

and

\[
\sum_{(K, (K_{\lambda})) \in A_F, (I, (I_i)) \subset (K, (K_{\lambda}))} (I, (I_i)) \otimes (K, (K_{\lambda})) / (I_i) \otimes A / (K_{\lambda}).
\]

Previous two formulas show that the formula (2.32) is valid.

Let \(A\) be a finite set equipped with a primary forest \(F\), a set \(U \in F\) is called a minimal set of \(F\), if there is not another set \(V \in F\) such that \(V \subset U\). Because \(A\) is a finite set, the minimal set of \(F\) exists obviously. Another obvious fact is that each \(U \in F\) must include a minimal set of \(F\). From the formula (2.31) in definition 2.5 we know that for a minimal set \(U\) of \(F\) we have

\[
\Delta_F U = U \otimes 1 + 1 \otimes U,
\]

or, \(\Delta_F^U = 0\). Thus, by induction we can prove the following proposition:

**Proposition 4.3.** Let \((I, (I_i)) \in A_F, m = \max\{\#I_i\}\), then

\[
(\Delta_F^m)^{m-1} (I, (I_i)) = 0.
\]  \hspace{1cm} (4.12)

**Antipode:** In \(H_F\) we define the unit and counit as following:

\[
u : C \longrightarrow H_F, \quad \nu : c \mapsto c \emptyset.
\]  \hspace{1cm} (4.13)

\[
\eta : H_F \longrightarrow C, \quad \eta : \emptyset \mapsto 1; \quad \eta : (I, (I_i)) \mapsto 0.
\]  \hspace{1cm} (4.14)

Then \(H_F\) becomes a bi-algebra. Proposition 2.5 shows that \(H_F\) is co-nilpotent bialgebra, thus, there must be antipode on \(H_F\) such that it becomes a Hopf algebra. The antipode is given by

\[
S(h) = -h + \sum_{n \geq 1} (-1)^{n+1} m^n \circ (\Delta_F^n)^{n-1}(h).
\]  \hspace{1cm} (4.15)

The explicit formula of antipode is same as one in H.Figueroa, Jose M. Gracia-Bondia?. We recall some content in H.F, J.M.GB? now. Of cause, here translation and some modification will be carried out.
Definition 4.3. A chain is a sequence of factorisations in $A$, denoted by $C$:

\[ C : (U(1), (U_{i_1}^{(1)})) \subset (U(2), (U_{i_2}^{(2)})) \subset \cdots \subset (U^k, (U_{i_k}^{(k)})), \]

such that, if $U_{i_1}^{(j)} \subset U_{i_1+1}^{(j+1)}$, $U_{i_1}^{(j)}$ is non-trivial subset of $U_{i_1+j+1}$, ($1 \leq j \leq k - 1$). Each $U_{i_k}^k$ is non-trivial subset of $A$. The set of chains in $A$ is denoted by $\mathcal{F}_C$. Where $k$ is called the length of chain $C$ denoted by $l(C)$. For each chain $C$, let

\[ \Omega(C) = (U(1), (U_{i_1}^{(1)})) \bullet (U(2), (U_{i_2}^{(2)})) / (U_{i_2}^{(1)}) \bullet \cdots \bullet (U^k, (U_{i_k}^{(k)})) / (U_{i_{k-1}}^{(k-1)}) \bullet A / (U_{i_k}^{(k)}). \]  

(4.16)

Now we have

Theorem 4.2.

\[ S(A) = -A + \sum_{C \in \mathcal{F}_C} (-1)^{l(C)-1} \Omega(C). \]  

(4.17)

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