ON ISOPERIMETRIC INEQUALITIES WITH RESPECT TO INFINITE MEASURES

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Abstract. We study isoperimetric problems with respect to infinite measures on $\mathbb{R}^n$. In the case of the measure $\mu$ defined by $d\mu = e^{c|x|^2} \, dx$, $c \geq 0$, we prove that, among all sets with given $\mu$–measure, the ball centered at the origin has the smallest (weighted) $\mu$–perimeter. Our results are then applied to obtain Polya-Szegö-type inequalities, Sobolev embeddings theorems and a comparison result for elliptic boundary value problems.

Key words: Isoperimetric inequalities, infinite measures, Steiner symmetrization, Schwarz symmetrization, comparison result, Polya-Szegö inequality.

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1. Introduction

Consider an elliptic boundary value problem of the following type,

$$\begin{cases}
-\text{div}(\varphi(x)|\nabla u|^{p-2}\nabla u) = f(x) & \text{in } \Omega \\
u = 0 & \text{on } \partial\Omega,
\end{cases}$$

where $\Omega$ is an open subset of $\mathbb{R}^N$, possibly unbounded, and $f$ belongs to suitable weighted Lebesgue spaces. We are interested in sharp explicit a-priori bounds for the weak solution to (1.1). Such type of problem can be examined by symmetrization methods. However, the presence of the weight function in the operator in (1.1) does not allow us to use the classical approach via Schwarz symmetrization given e.g. in [23], and [3]. This leads us to introduce an appropriate symmetrization based on a weighted isoperimetric inequality which is related to the structure of the operator. A similar approach which is based on the isoperimetric inequality for Gauss measure has been carried out by the authors in [5], (see also [14], [9]).

In this paper we study isoperimetric inequalities for infinite measures, together with properties of corresponding weighted symmetrizations.

To be more precise, let $\mu$ be a measure on $\mathbb{R}^n$ defined by

$$d\mu = \varphi(x) \, dx,$$

where $\varphi$ is a positive continuous function. For any smooth set $\Omega$, we denote by

$$P_\mu(\Omega) = \int_{\partial\Omega} \varphi(x) \, d\mathcal{H}_{n-1}(x)$$

the weighted perimeter of $\Omega$ w.r.t. $\mu$ and for any fixed number $m > 0$, we denote by $I_\mu(m)$ the isoperimetric function, that is

$$I_\mu(m) := \inf\{P_\mu(\Omega) : \Omega \text{ smooth}, \mu(\Omega) = m\}.$$
We are interested in finding isoperimetric sets, that is smooth sets which realize the infimum in (1.2).

Such a problem has been treated in various settings. For example, if \( \mu \) is the Lebesgue measure on \( \mathbb{R}^n \), then the isoperimetric sets are the balls, i.e. if \( \varphi(x) \equiv 1 \), then \( I_\mu(m) = P_\mu(B) \), for any ball \( B \) in \( \mathbb{R}^n \) with \( \mu(B) = m \) (see for instance [27], [30]).

Moreover if \( \mu \) is the Gauss measure, then the isoperimetric sets are the half-spaces of \( \mathbb{R}^n \), i.e. if \( \varphi(x) = \exp(-c|x|^2) \) for some \( c > 0 \) and \( m \in (0, \mu(\mathbb{R}^n)) \), then \( I_\mu(m) = P_\mu(H) \), where \( H \) is any Euclidean half-space with \( \mu(H) = m \) (see for instance [7], [15], [20]).

Isoperimetric inequalities and their connections with rearrangements have received considerable interest in the last decades (see e.g. [19], [10], [29], [24], [25] and the references cited therein). In the paper [6], the authors recently analyzed symmetrizations w.r.t. finite measures on \( \mathbb{R}^n \).

Here we investigate infinite measures on \( \mathbb{R}^n \) together with Steiner and Schwarz symmetrizations. One of our results is the following: If \( \mu \) is a measure defined by
\[
d\mu = \exp(c|x|^2)dx, \quad \text{for some } c > 0,
\]
then the only isoperimetric sets are Euclidean balls which are centered at the origin, i.e. one has
\[
I_\mu(m) = P_\mu(B_R),
\]
where \( R \) is chosen in such a way that \( \mu(B_R) = m \). Alternatively one can express this isoperimetric inequality by using the notion of weighted Schwarz symmetrization \( U^* \) of a set \( U \), which is the Euclidean ball centered at the origin such that \( \mu(U) = \mu(U^*) \). With this notation, (1.4) is equivalent to
\[
P_\mu(U) \geq P_\mu(U^*), \quad \text{for any smooth set } U,
\]
with equality if and only if \( U = U^* \) (see Theorem 5.1).

We note that a proof of inequality (1.5), but without the equality case, was given by Borell already in 1986, in an unpublished preprint (see [8]). Theorem 4.1 was the subject of an earlier preprint of the authors, and it has been presented at several conferences since 2005. Recently Benguria and Linde in [4] used this result to obtain eigenvalue bounds for the Dirichlet Schrödinger operator.

We emphasize that after having finished our paper we learned that Theorem 4.1 has been also proved independently of us by Rosales, Canete, Bayle and Morgan ([26]).

All mentioned proofs are based on the observation that Steiner symmetrization does not increase the perimeter. Note, our proof differs from the one given in [26] in that it does not make use of the smoothness of a minimizing set. Therefore we include it here, for the convenience of the readers. It is performed in several steps.

First we study the one-dimensional case. More precisely, we consider a measure on \( \mathbb{R} \) given by
\[
d\mu_1 = \psi(x_1) dx_1,
\]
where \( \psi \) is an even, positive and continuous function on \( \mathbb{R} \) and we prove that (1.4) holds iff \( \psi \) is log-convex. Then we consider a more general measure \( \mu \) whose density is the product of two continuous functions \( \psi : \mathbb{R} \to \mathbb{R}^+ \), and \( \rho : \mathbb{R}^{n-1} \to \mathbb{R}^+ \) depending on \( x_1 \) and
we prove that Steiner symmetrization with respect to this measure decreases the perimeter (see Section 3.2 for the definition of Steiner symmetrization). The last step consists in approximating the ball \( U^\star \) by an appropriate sequence of consecutive Steiner symmetrizations. In view of the product structure of the density and the invariance w.r.t. rotations of the measure \((1.3)\), this leads to \((1.5)\).

Isoperimetric inequality \((1.5)\) has various consequences. For example, by Talenti’s result \((29)\), inequality \((1.5)\) implies a Polya-Szegő type inequality (see Theorem 5.2) and the equality case is also studied. Moreover, we prove a Sobolev type imbedding theorem in a weighted space w.r.t. the measure \(\mu\) defined in \((1.3)\) (see Theorem 5.3). The best constant in such an inequality is obtained in a special case (see Corollary 5.3).

We now outline the content of the paper. In Section 2 we introduce some notation and provide basic information about the weighted perimeter. In Section 3 we study the Steiner symmetrization w.r.t. general measures. In particular we prove that Steiner symmetrization decreases the weighted perimeter of a set having given measure. We also show a general form of Polya-Szegő type inequalities principle (see Theorem 4.3). The proof of inequality \((1.5)\) is given in Section 4. Finally, Section 5 contains the comparison result.

2. Notation and preliminaries

In the whole paper \(\mu\) will denote a measure on \(\mathbb{R}^n\) defined by
\[
d\mu = \varphi(x) \, dx,
\]
where \(\varphi \in C(\mathbb{R}^n)\) and \(\varphi(x) > 0\) for any \(x \in \mathbb{R}^n\). Moreover \(\mu_1\) will denote a measure on \(\mathbb{R}\) defined by
\[
d\mu_1 = \psi(x) \, dx,
\]
where \(\psi \in C(\mathbb{R})\) and \(\psi(x) > 0\) for any \(x \in \mathbb{R}\).

We will always assume that the measures are infinite, that is
\[
\mu(\mathbb{R}^n) = +\infty, \quad \mu_1(\mathbb{R}) = +\infty.
\]

By \(\mathcal{G}M^n\) we denote the set of \(\mu\)-measurable sets with finite Lebesgue measure and by \(\mathcal{F}^n\) the set of all \(\mu\)-measurable functions on \(\mathbb{R}^n\) such that \(\{x : u(x) > t\} \in \mathcal{M}^n\) for every \(t > \inf u\).

Let \(\Omega\) be a domain in \(\mathbb{R}^n\) and \(p \in [1, +\infty)\). We denote by \(L^p(\varphi, \Omega)\) the space of \(\mu\)-measurable functions \(u\) such that
\[
\|u\|_{p,\Omega} \equiv \left( \int_\Omega |u|^p \, d\mu \right)^{1/p} < +\infty,
\]
endowed with the norm \((2.4)\). Furthermore, let \(W^{1,p}(\varphi, \Omega)\) denote the weighted Sobolev space containing all functions \(u \in L^p(\varphi, \Omega)\) with weak derivatives \(u_{x_i} \in L^p(\varphi, \Omega)\), \(i = 1, \ldots, n\), and let
\[
\|\|u\|\|_{p,\Omega} := \|u\|_{p,\Omega} + \|\nabla u\|_{p,\Omega}
\]
be a norm in this space. Finally, let \(W^{1,p}_0(\varphi, \Omega)\) be the closure of \(C_0^\infty(\Omega)\) under the norm \((2.5)\). If \(\Omega = \mathbb{R}^n\) in one of the above spaces, then we will omit the subindex \(\Omega\) in the norms.
For subsets $A, B, M$ of $\mathbb{R}^n$, let $A + B = \{x + y : x \in A, y \in B\}$ denote the Minkowski sum of $A$ and $B$, and
\[
M_r : = \{x \in \mathbb{R}^n : \text{dist} \{x; M\} < r\} = M + B_r, \quad r > 0, 
\]
the exterior parallel sets of $M$, where $B_r$ is the ball of radius $r$ with the center at the origin. We will call a set $M \subset \mathbb{R}^n$ smooth, if $M$ is bounded, open, and if there is a number $\varepsilon > 0$ such that for every $x_0 \in \partial M$, $\partial M \cap B_{\varepsilon}(x_0)$ is a Lipschitz graph, and $M \cap B_{\varepsilon}(x_0)$ lies on one side of $\partial M \cap B_{\varepsilon}(x_0)$ only. Observe that, this definition includes polyhedra and excludes the presence of “veils” and inner “slices” of $M$.

If $M$ is a Borel set then we denote by $\mu^+(M)$ the (lower outer) Minkowski $\mu$-content of the boundary of $M$ (see, for instance, [10], p.69) which is defined by
\[
\mu^+(M) = \liminf_{r \to 0^+} \frac{\mu(M_r) - \mu(M)}{r} .
\]
If $\varphi \in W^{1,1}_{loc}(\mathbb{R}^n)$, then we define the $\mu$-perimeter (in the sense of De Giorgi) by
\[
P_\mu(M) = \sup \left\{ \int_M \text{div} (\varphi(x)h(x)) \, dx : h \in C^1_0(\mathbb{R}^n, \mathbb{R}^n), |h| \leq 1 \right\}.
\]
The following properties are well-known for the Lebesgue measure (see, for instance, [10]) and their proofs carry over to general measures $\mu$ defined in (2.1) .

1) Both $\mu^+(M)$ and $P_\mu(M)$ can be seen as a “weighted surface measure” of $M$, that is, if $M$ is a smooth set then
\[
\mu^+(M) = P_\mu(M) = \int_{\partial M} \varphi(x) \mathcal{H}_{n-1}(dx) < +\infty.
\]

2) Lower semicontinuity: If $\{M_k\} \subset \mathcal{M}^n$, $M \in \mathcal{M}^n$, and $\mu(M \Delta M_k) \to 0$ as $k \to \infty$, then
\[
\liminf_{k \to \infty} P_\mu(M_k) \geq P_\mu(M).
\]

3) If $M \in \mathcal{M}^n$, and $P_\mu(M) < \infty$, then there is a sequence of smooth sets $\{M_k\}$ such that $\mu(M \Delta M_k) \to 0$ as $k \to \infty$, and such that
\[
\lim_{k \to \infty} P_\mu(M_k) = P_\mu(M).
\]

4) If $M$ is a Borel set in $\mathbb{R}^n$, then
\[
P_\mu(M) \leq \mu^+(M).
\]

We mention that the theory of sets with finite $\mu$-perimeter is imbedded in the framework of BV-functions space, $BV(\varphi, \mathbb{R}^n)$, defined as the set of all functions $u \in L^1(\varphi, \mathbb{R}^n)$ such that
\[
\|Du\|_{BV} := \sup \left\{ \int_{\mathbb{R}^n} u(x) \, \text{div} \left(\varphi(x)h(x)\right) \, dx, h \in C^1_0(\mathbb{R}^n, \mathbb{R}^n), |h| \leq 1 \right\} < +\infty.
\]
Notice that if $M$ has finite $\mu$-perimeter then the characteristic function of $M$, $\chi_M$ belongs to $BV(\varphi, \mathbb{R}^n)$ and moreover $\|D\chi_M\|_{BV} = P_\mu(M)$. Furthermore, if $u \in W^{1,1}(\varphi, \mathbb{R}^n)$ then $\|Du\|_{BV} = \|\nabla u\|_1$. 
Finally we recall some well-known derivation formulas (cf. [1], [18] and [11]). We set $\mathbb{R}_0^+ = [0, +\infty)$ and we denote a point $x \in \mathbb{R}^n$ by $x = (x_1, x')$ where $x_1 \in \mathbb{R}$ and $x' \in \mathbb{R}^{n-1}$ ($n \geq 2$).

Let $u \in \mathcal{F}^n$, and let $m_u$ denote its $\mu$-distribution function w.r.t. the variable $x_1$, that is

$$m_u(t, x') := \mu_1(\{u(\cdot, x') > t\}), \quad t \in \mathbb{R}, \ x' \in \mathbb{R}^{n-1}. \quad (2.13)$$

We also set

$$u_- := \text{ess inf } u,$$

and

$$V_u := \{(t, x') : t > u_-, x' \in \mathbb{R}^{n-1}\}. \quad (2.14)$$

Let $u \in W^{1,p}(\varphi, \mathbb{R}^n)$ for some $p \in [1, +\infty)$. We define

$$D_u(x') := \{x_1 \in \mathbb{R} : u_{x_1}(x_1, x') = 0\}, \ x' \in \mathbb{R}^{n-1}, \quad (2.15)$$

and the superlevel sets

$$E_u(t, x') := \{u(x_1, x') > t\} \equiv \{u(\cdot, x') > t\}, \quad (t, x') \in V_u. \quad (2.16)$$

Notice that, since $u(\cdot, x') \in W^{1,p}(\varphi, \mathbb{R})$ for a.e. $x' \in \mathbb{R}^{n-1}$, the Sobolev imbedding theorem tells us that $u(\cdot, x')$ is continuous for a.e. $x' \in \mathbb{R}^{n-1}$. Hence $E_u(t, x')$ is open and $\partial E_u(t, x')$ is countable for a.e. $(t, x') \in V_u$.

Let us consider $g \in L^1(\mathbb{R}^n)$. Defining

$$F(t, x') := \int_{E_u(t, x')} g(x_1, x')\, dx_1, \quad (t, x') \in \mathbb{R} \times \mathbb{R}^{n-1}, \quad (2.17)$$

we then have that $F(\cdot, x') \in BV(\mathbb{R})$ for a.e. $x' \in \mathbb{R}^{n-1}$, and the Fleming-Rishel formula (see, for instance, [17]) tells us that

$$\frac{\partial}{\partial t} F(t, x') = \int_{\partial E_u(t, x')} \frac{g(\cdot, x')}{|u_{x_1}(\cdot, x')|}\, d\mathcal{H}_0 \quad \text{for a.e. } (t, x') \in V_u. \quad (2.18)$$

Moreover, we have by the co-area formula (see, e.g., [17])

$$\int_{\mathbb{R}} F(t, x')\, dt = \int_{\mathbb{R}\setminus D_u(x')} g(x_1, x')\, dx_1 = \int_{u_-(x')}^{u_+(x')} \int_{\partial E_u(t, x')} g(\cdot, x')\, d\mathcal{H}_0\, dt, \quad (2.19)$$

for a.e. $x' \in \mathbb{R}^{n-1}$.

Now let us assume that $u$ is a smooth function and satisfies

$$\mathcal{L}^1(D_u(x')) = 0 \quad \text{for a.e. } x' \in \mathbb{R}^{n-1}. \quad (2.20)$$

Here $\mathcal{L}^1$ denotes the 1-dimensional Lebesgue measure. Then the following derivation formula holds true (cf [1]; see also [18] and [11]).

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1Here and in the following we will write $\{u(\cdot, x') > t\}$ for $\{x_1 : u(x_1, x') > t\}$. 
Lemma 2.1. If $g \in C^1$, then we have

$$\frac{\partial}{\partial x_i} \int_{E_u(t,x')} g(x_1, x') \, dx_1 = \int_{E_u(t,x')} \frac{\partial g}{\partial x_i}(x_1, x') \, dx_1 + \int_{\partial E_u(t,x')} g(\cdot, x') \frac{u_{x_i}(\cdot, x')}{|u_{x_1}(\cdot, x')|} \, dH_0.$$  

3. Measure on the real line

Let $\psi$ a positive, even and continuous function on $\mathbb{R}$, and such that

$$\int_{\mathbb{R}} \psi(x) \, dx = +\infty.$$  

We then define a measure $\mu_1$ on $\mathbb{R}$ by

$$d\mu_1 = \psi(x) \, dx.$$  

Hence the primitive of $\psi$,

$$\Psi(x) := \int_0^x \psi(t) \, dt, \quad x \in \mathbb{R},$$

is strictly increasing and odd. We introduce a continuous, positive function $J$ by

$$J(y) = \psi(\Psi^{-1}(y)), \quad y \in \mathbb{R},$$

where $\Psi^{-1} : \mathbb{R} \to \mathbb{R}$ is the inverse of $\Psi$. Notice that $J$ is even, and (3.1) implies that

$$\Psi^{-1}(y) = \int_0^y \frac{dt}{J(t)}, \quad y \in \mathbb{R}.$$  

If $M \in \mathcal{M}^1$, and if $\mu_1(M) < +\infty$, then there exists a unique number $c \geq 0$ such that

$$\mu_1((-c, c)) = \mu_1(M),$$

and we set

$$M^* = (-c, c).$$

If $M \in \mathcal{M}^1$, and if $\mu_1(M) = +\infty$ then we set $M^* = \mathbb{R}$. We call the set $M^*$ the $\mu_1$-symmetrization of $M$.

Remark 3.1. By the above definition, $M^*$ is some centered interval $(-c, c)$, $(c \in \mathbb{R}^+_0 \cup \{+\infty\})$. Notice that since we deal with measurable sets we usually do not distinguish between two sets $M, N$ which are equivalent, that is, which satisfy $\mu_1(M \Delta N) = 0$. On the other hand, the above definition can be easily precised if $M$ is open or closed:

If $M$ is open then the above settings have to be understood in pointwise sense, so that $M^*$ is then open, too. If $M$ is closed then we replace the open intervals in the above definition by closed intervals, so that $M^*$ is closed.
Remark 3.2. Obviously, by definition

\( \mu_1(M) = \mu_1(M^*) \quad \forall M \in \mathcal{M}^1. \)

It is also easy to confirm the following monotonicity properties, \((M, N \in \mathcal{M}^1)\):

\( M \subset N \implies M^* \subset N^*, \)

\( M^* \cup N^* \subset (M \cup N)^*, \quad M^* \cap N^* \supset (M \cap N)^*, \)

\( \mu_1(M \setminus N) \geq \mu_1(M^* \setminus N^*), \quad \mu_1(M \Delta N) \geq \mu_1(M^* \Delta N^*). \)

We now ask for a condition on the measure \( \mu_1 \) such that the \( \mu_1 \)-rearrangement decreases the perimeter, that is we ask for a condition such that the following isoperimetric inequality holds

\( P_{\mu_1}(M) \geq P_{\mu_1}(M^*) \equiv J(\mu_1(M)), \quad \forall M \in \mathcal{M}^1. \)

Such a condition is given by Theorem 3.1 below

Theorem 3.1. Inequality \((3.9)\) holds iff \( J \) is convex. Furthermore, let us assume that equality holds in \((3.9)\); if \( J \) is convex, then \( M \) is equivalent to an interval, while if \( J \) is strictly convex, then \( M = M^*. \)

Remark 3.3. \( J \) is convex iff \( \log \psi \) is convex (or equivalently, if \( \psi \) is log-convex). A typical case is

\( \psi(x) = e^{\frac{|x|^2}{c}}, \)

where \( c \in \mathbb{R}_0^+ \). Obviously if \( c > 0 \) then \( J \) is strictly convex.

Remark 3.4. Isoperimetric inequality \((3.9)\) and property \((2.11)\) imply an isoperimetric inequality for the Minkowsky \( \mu_1 \)-content, i.e.

\( \mu_1^+(M) \geq \mu_1^+(M^*) \quad \text{for every Borel set } M. \)

Moreover it is easy to see that this also implies

\( \mu_1(M_r) \geq \mu_1((M^*)_r) \quad \forall r > 0, \)

which is equivalent to

\( (M_r)^* \supset (M^*)_r \quad \forall r > 0. \)

Proof of Theorem 3.1: First let us assume that \( \mu_1 \) satisfies \((3.9)\), and let \( I \) be any finite interval \((a, b), \ a < b\). Setting \( \alpha = \Psi(a), \ \beta = \Psi(b) \), we have that \( I^* = (-c, c) \), where \( \Psi(c) = (\beta - \alpha)/2 \), then \((3.9)\) reads as

\[ J(\alpha) + J(\beta) \geq 2J\left(\frac{\beta - \alpha}{2}\right). \]

Since \( J(\alpha) = J(-\alpha) \), this implies that

\[ J(s) + J(t) \geq 2J\left(\frac{s + t}{2}\right) \quad \forall s, t \in \mathbb{R}, \]

which means that \( J \) is convex.

Now let us assume that \( J \) is convex, and let \( M \) be a smooth set. Then \( M = \bigcup_{i=1}^{n}(a_i, b_i) \)
where \( m \in \mathbb{N}, a_i < b_i \), and the intervals \([a_i, b_i]\) are mutually disjoint, \( i = 1, 2, \ldots, m \). Setting \( \alpha_i = \Psi(a_i), \beta_i = \Psi(b_i), i = 1, \ldots, m \), we find, using the convexity and evenness of \( J \),

\[
P_{\mu_1}(M) = \sum_{i=1}^{m} (J(\alpha_i) + J(\beta_i)) \geq \sum_{i=1}^{m} 2J\left(\frac{\beta_i - \alpha_i}{2}\right)
\]

\[
\geq 2mJ\left(\sum_{i=1}^{m} \frac{\beta_i - \alpha_i}{2m}\right) \geq 2J\left(\sum_{i=1}^{m} \frac{\beta_i - \alpha_i}{2}\right) = \mu_1^+(M^*)
\]

By property (2.10) this also implies (3.9) for sets \( M \in \mathcal{M}^1 \), proving the first statement of the Theorem.

Now we assume that equality holds in (3.9) and that \( J \) is convex. W.l.o.g. we may assume that also \( \mu_1(M) > 0 \). For \( r > 0 \), let \( \theta_r(M, x) := \mu_1(M \cap B_r(x))[\mu_1(B_r(x))]^{-1} \) and define the upper density of the set \( M \) at \( x \) by \( \theta(M, x) := \limsup_{r \to 0} \theta_r(M, x) \). Suppose that \( M' \) is the set of upper density points of \( M \), i.e. \( M' = \{ x \in \mathbb{R} : \theta(M, x) = 1 \} \). Since \( \mu_1(M \Delta M') = 0 \), it is sufficient to study \( M' \) instead of \( M \). We first claim that \( M' \) is convex.

Assume that \( M' \) is not convex. Then there are points \( x_i \in \mathbb{R}, i = 1, 2, 3, x_1 < x_3 < x_2 \), such that \( \theta(M, x_1) = \theta(M, x_2) = 1 \) and \( \theta(M, x_3) = 0 \). Then for \( r < (1/4) \min\{x_3 - x_1; x_2 - x_3\} \), and small enough, we have that \( \theta_r(M, x_i) \geq (7/8), i = 1, 2, \) and \( \theta_r(M, x_3) \leq (1/8) \). Let \( \{M_k\} \) a sequence of smooth sets such that \( \mu_1(M_k \Delta M) \to 0 \) and \( P_{\mu_k}(M_k) \to P_{\mu_1}(M) \), as \( k \to \infty \). For \( k \) large enough - say \( k \geq k_0 \) - we still have \( \theta_r(M_k, x_i) \geq (3/4), i = 1, 2, \) and \( \theta_r(M_k, x_3) \leq (1/4) \). This implies that the sets \( B_r(x_i) \cap M_k, i = 1, 2, \) and \( B_r(x_3) \setminus M_k \) are nonempty for these \( k \). In other words, if \( k \geq k_0 \), then there is a nonempty interval \( I_k = (y_k^1, y_k^2) \) such that \( I_k \subset (x_1 - r, x_2 + r), I_k \cap M_k = \emptyset \) and \( y_k^1, y_k^2 \in \partial M_k \). Setting \( N_k := I_k \cup M_k \) we then have in view of the isoperimetric property (3.9),

\[
P_{\mu_k}(M_k) - J(\mu_k(M_k)) \geq P_{\mu_k}(N_k) - J(\mu_k(N_k)) + \psi(y_k^1) + \psi(y_k^2)
\]

\[
\geq \psi(y_k^1) + \psi(y_k^2) \geq \delta, \forall k \geq k_0,
\]

for some \( \delta > 0 \) independent on \( k \). Passing to the limit for \( k \to \infty \), this also implies \( P_{\mu_1}(M) - J(\mu_1(M)) \geq \delta > 0 \), that is a contradiction. Hence \( M' \) is convex.

Now we assume that equality holds in (3.9) and that \( J \) is strictly convex. Hence \( M' = (a, b) \), where \( a, b \in \mathbb{R}, a < b \). Setting \( \alpha = \Psi(a), \beta = \Psi(b) \), we have that \( M^* = (-c, c) \), where \( \Psi(c) = (\beta - \alpha)/2 \), so that

\[
J(\alpha) + J(\beta) = 2J\left(\frac{\beta - \alpha}{2}\right).
\]

The strict convexity of \( J \) then implies that \( |\alpha| = |\beta| \), that is \( b = -a \). The Theorem is proved.

\[\square\]

### 4. Product Measures on \( \mathbb{R}^n \)

In this section we prove isoperimetric inequalities with respect to product measures on \( \mathbb{R}^n \) and we apply them to obtain integral inequalities in Sobolev spaces.

We deal with a product measure \( \mu \) on \( \mathbb{R}^n \) defined by

\[
d\mu = \psi(x_1)\rho(x') \, dx,
\]

where \( x = (x_1, x_2, \ldots, x_n) = (x_1, x') \) is a point in \( \mathbb{R}^n \), \( (n \geq 2) \), \( \psi \) is a function as in the previous section and \( \rho \) is a positive, continuous function on \( \mathbb{R}^{n-1} \).
4.1. **Isoperimetric inequalities.** If \( M \subset \mathbb{R}^n \) we introduce \( x' \)-slices, \( x' \in \mathbb{R}^{n-1} \), by
\[
M(x') = \{ x_1 : (x_1, x') \in M \}.
\]
Notice that if \( M \in \mathcal{M}^n \), and if \( \mu(M) < \infty \) then \( \mu_1(M(x')) < \infty \) for almost every \( x' \in \mathbb{R}^{n-1} \).
For \( M \in \mathcal{M}^n \), we define its Steiner \( \mu \)-symmetrization \( M^\ast \) by
\[
M^\ast := \left\{ x = (x_1, x') : x_1 \in \left( M(x') \right)^\ast, \ x' \in \mathbb{R}^{n-1} \right\}.
\]
Then
\[
\mu(M) = \mu(M^\ast),
\]
and it is easy to see that the monotonicity properties (3.6)-(3.8) carry over to Steiner \( \mu \)-symmetrization.

As in the one-dimensional case, the above definitions will be read pointwise for open and closed sets.
If \( M \) is open (respectively closed) then the function \( \varphi(x') := \mu(M(x')) \), \( x' \in \mathbb{R}^{n-1} \), is lower (respectively upper) semicontinuous. Since the sets \( (M(x'))^\ast \), \( x' \in \mathbb{R}^{n-1} \), are open (respectively closed) it then follows that \( M^\ast \) is open (respectively closed), too.
Steiner-like rearrangements are characterized by the fact that the isoperimetric property (3.11) on slices carries over to sets on \( \mathbb{R}^n \) (see [10]). By adapting the proof of Theorem 3.1 in [6], we can prove the following result

**Lemma 4.1.** The property (3.11) holds for Borel sets \( M \) on \( \mathbb{R}^n \).

By the definition of \( \mu^+ \) and \( P_\mu \), and by property (2.10) we then also have

**Corollary 4.1.** The following inequalities hold
\[
\mu^+(M) \geq \mu^+(M^\ast) \quad \text{for every Borel set } M, \text{ and}
\]
\[
P_\mu(M) \geq P_\mu(M^\ast) \quad \text{for every } M \in \mathcal{M}^n.
\]

The next property follows easily from Lemma 4.1

**Corollary 4.2.** Let \( M, N \) opens sets in \( \mathbb{R}^n \) with \( M \subset N \). Then
\[
\text{dist } \{ M ; \partial N \} \leq \text{dist } \{ M^\ast ; \partial N^\ast \}.
\]

Finally we analyze the equality case in (1.5). The following result holds

**Theorem 4.1.** Assume that equality holds in (1.5) for some \( M \in \mathcal{M}^n \). Then \( M(x') \) is either empty or equivalent to an interval for almost every \( x' \in \mathbb{R}^{n-1} \). Moreover, if \( J \) is strictly convex, then \( M = M^\ast \).

The proof of Theorem 4.1 depends on a precise estimate for the deficit of the perimeter under Steiner symmetrization for polyhedra. This approach is well-known in the case of the uniform Lebesgue measure \( \varphi \equiv 1 \) (see [10], chapter 14). Let us first introduce some notation. Let \( \nabla' \) denote the vector of derivatives \( (\partial/\partial x_2, \ldots, \partial/\partial x_n) \). If \( x' \in \mathbb{R}^{n-1} \) then let \( l_{x'} \) denote the line \( \{ (t, x') : t \in \mathbb{R} \} \). Let \( \mathcal{P} \) denote the set of polyhedra \( \Pi \) in \( \mathbb{R}^n \) such that \( l_{x'} \cap \Pi \) is either empty or consists of a finite number of points for every \( x' \in \mathbb{R}^{n-1} \). The map \( p : \partial \Pi \rightarrow \mathbb{R}^{n-1} \) will be called a projection. If \( \Pi \in \mathcal{P} \) then \( \mathbb{R}^{n-1} \) is splitted into a finite numbers of domains \( Q \) such that the part of \( \partial \Pi \) which is projected into \( Q \) consists of a
finite number $2m$ of components $\Gamma_j$ whose projections onto $Q$ are one-to-one. (The number $m$ depends on $Q$, and those for which $m = 0$ will not be considered further on.) Each $\Gamma_j$ permits an (affine) representation $x_1 = z_j(x')$, $x' \in Q$. Then it follows that

$$P(\Pi) = \sum^* \int_Q \sum_{j=1}^{2m} \sqrt{1 + |\nabla' z_j|^2} \psi(z_j) \rho(x') \, dx',$$

where the sum $\sum^*$ is taken over all the $Q$ for which $m \geq 1$. After the Steiner symmetrization, the area of the boundary will be

$$P(\Pi^*) = \sum^* \int_Q \sum_{j=1}^{2m} \sqrt{1 + |\nabla' z_j|^2} \psi(z_j) \rho(x') \, dx',$$

where the function $z : Q \to \mathbb{R}^n_+$ is given by

$$\sum_{j=1}^{2m} (-1)^j \Psi(z_j) = 2\Psi(z).$$

**Lemma 4.2.** Let $\Pi \in \mathcal{P}$. Then, with the above notations,

$$P(\Pi) - P(\Pi^*) \geq \left( P(\Pi^*) \right)^{-1} \sum^* \int_Q \sqrt{\sum_{j=1}^{2m} \psi(z_j) - 2\psi(z)} \rho(x') \, dx'.$$

**Proof:** For convenience, we set $y_j := \Psi(z_j)$, $j = 1, \ldots, 2m$, $y = \Psi(z)$, and $J := \psi(\Psi^{-1})$. Recall that $J$ is convex by our assumptions. Then we find

$$\sum_{j=1}^{2m} \sqrt{1 + |\nabla' z_j|^2} \psi(z_j) - 2\sqrt{1 + |\nabla' z|^2} \psi(z)$$

$$= \sum_{j=1}^{2m} \sqrt{J(y_j)^2 + |\nabla' y_j|^2} - 2\sqrt{J(y)^2 + |\nabla' y|^2}$$

$$\geq \frac{J(y)}{\sqrt{J(y)^2 + |\nabla' y|^2}} \left( \sum_{j=1}^{2m} J(y_j) - 2J(y) \right)$$

$$= \sqrt{1 + |\nabla' z|^2}.$$  \hfill (4.11)

Integrating (4.11) and applying Cauchy-Schwarz inequality we have (4.10).

**Proof of Theorem 4.1** For $x \in \mathbb{R}^n$ and $r > 0$ let $\theta_r(M, x) := \mu(M \cap B_r(x)) \mu(B_r(x))^{-1}$, and define $\theta(M, x')$ and the set $M'$ of upper density points of $M$ as in the proof of Theorem 1. As before, we may restrict ourselves to the set $M'$ instead of $M$. Choose a sequence of polyhedra $\{\Pi_k\}$ such that $\mu(\Pi_k \Delta M) \to 0$ and $P(\Pi_k) \to P(\mu(M)$ as
\[ k \to \infty. \] Without loss of generality, we may also assume that \( \Pi_k \in \mathcal{P}, k = 1, 2, \ldots. \) Since \( \mu(\Pi_k \Delta M^*) = 0 \) as \( k \to \infty, \) we have by the lower semicontinuity of the perimeter,

\[ \lim_{k \to \infty} (P_\mu(\Pi_k) - P_\mu(\Pi^*_k)) = 0. \tag{4.12} \]

Set

\[ R_k := \{ x' \in \mathbb{R}^{n-1} : l_{x'} \cap \Pi_k \text{ has at least two components} \}, \]

and introduce a measure \( \nu \) on \( \mathbb{R}^{n-1} \) by

\[ d\nu = \rho(x')\,dx'. \]

Since the function \( \psi \) is bounded away from 0, the previous Lemma 2 together with (4.12) tells us that

\[ \lim_{k \to \infty} \nu(R_k) = 0. \tag{4.13} \]

We claim that \( M(x') \) is convex for almost every \( x' \in \mathbb{R}^{n-1}. \) Assume that this is not the case. Then there are points \( x^i = (z_i, x^i_0), i = 1, 2, 3, \) with \( x^i_0 \in \mathbb{R}^{n-1}, \) and \( z_1 < z_2 < z_3, \) such that \( \theta(M, x^1) = \theta(M, x^2) = 1, \) and \( \theta(M, x^3) = 0. \) Let \( \varepsilon > 0 \) and small (the exact choice of \( \varepsilon \) being specified later). Choose \( r = r(\varepsilon) > 0, \) and small enough such that \( \theta_r(M, x^i) \geq 1 - \varepsilon, i = 1, 2, \) and \( \theta_r(M, x^3) \leq \varepsilon. \) For \( k \) large enough - say \( k \geq k_\varepsilon \) - we then still have \( \theta_r(\Pi_k, x^1) \geq 1 - 2\varepsilon, i = 1, 2, \) and \( \theta_r(\Pi_k, x^3) \leq 2\varepsilon. \) Let

\[ H_k := \{ x = (x_1, x') \in B_r(x^3) : \nu(l_{x'} \cap \Pi_k \cap B_r(x^1)) > 0, \text{ and } \nu(l_{x'} \cap \Pi_k \cap B_r(x^2)) > 0 \} \]

By choosing \( \varepsilon \) small enough we can achieve that \( \mu(H_k) > (1/2)\mu(B_r(x^3)) \), and in view of \( \theta_r(\Pi_k, x^3) \leq 2\varepsilon, \) also that \( \mu(H_k \setminus \Pi_k) > (1/4)\mu(B_r(x^3)) \). Hence there is a number \( c_0 > 0 \) which depends only on \( \varepsilon, \) but not on \( k, \) such that \( \nu(R_k) \geq c_0. \) But this contradicts (4.13). This proves the claim.

Hence there is a nullset \( N \subset \mathbb{R}^n, \) a measurable set \( G \subset \mathbb{R}^{n-1}, \) and measurable functions \( z_i, i = 1, 2, \) such that

\[ \quad \quad \quad \quad \quad \quad \quad M = N \cup \{(x_1, x') : z_1(x') < x_1 < z_2(x'), x' \in G\}. \]

Using Lemma 2 and the limit property (4.12) we have

\[ 0 \geq \left( P_\mu(M) \right)^{-1} \int_G \sqrt{\psi(z)} \left( \sum_{j=1}^{2} \psi(z_j) - 2\psi(z) \right) \rho(x')\,dx', \tag{4.14} \]

where \( z \) is given by \( 2\Psi(z) = \Psi(z_2) - \Psi(z_1). \) Using the strict convexity of \( J \) this implies that \( z_2 = -z_1 = z \) on \( G \) and the Theorem is proved.

\section*{4.2. Steiner \( \mu \)-symmetrization of functions.} If \( u \in \mathcal{F}^n \) we define its \textit{Steiner \( \mu \)-symmetrization} (w.r.t. \( x_1 \)) \( u^* \) by

\[ u^*(x) := \sup \left\{ t \in \mathbb{R} : x \in \{ u > t \}^* \right\}, \quad x \in \mathbb{R}^n. \tag{4.15} \]

By its definition, the function \( u^* \) is nonincreasing and right-continuous w.r.t. the variable \( x_1. \) Moreover \( u \) and \( u^* \) are \textit{equimeasurable} functions, that is

\[ \{ u > t \}^* = \{ u^* > t \} \quad \text{and} \quad \{ u \geq t \}^* = \{ u^* \geq t \} \quad \forall t > \inf u, \tag{4.16} \]
which implies that \( \forall t > \inf u \) and for a.e. \( x' \in \mathbb{R}^{n-1} \),
\[
\mu\bigl(\{u > t\}\bigr) = \mu\bigl(\{u^* > t\}\bigr), \quad \mu\bigl(\{u \geq t\}\bigr) = \mu\bigl(\{u^* \geq t\}\bigr),
\]
and
\[
\mu\bigl(\{u(\cdot, x') > t\}\bigr) = \mu\bigl(\{u^*(\cdot, x') > t\}\bigr),
\]
(4.17)
\[
\mu\bigl(\{u(\cdot, x') \geq t\}\bigr) = \mu\bigl(\{u^*(\cdot, x') \geq t\}\bigr).
\]
Furthermore, the monotonicity (3.6) implies
(4.18)
\[
u, v \in F^n, \quad u \leq v \implies u^* \leq v^*.
\]

**Remark 4.1.** We will generally not distinguish between \( u \) and its equivalence class given by all measurable functions which differ from \( u \) on a nullset. But if \( u \) is continuous, then the sets \( \{u > t\} \), \( t \in \mathbb{R} \) are open, and we will agree that the above definition of \( u^* \) has to be understood in pointwise sense. Furthermore it is easy to see that the sets \( \{u^* > t\} \), (respectively \( \{u^* \geq t\} \)) are open (respectively closed), \( t \in \mathbb{R} \), so that \( u^* \) is continuous too.

**Remark 4.2.** An equivalent definition of \( u^* \) can be given by using the \( \mu \)-distribution function of \( u \) (w.r.t. \( x_1 \)), \( m_u \), defined by
\[
m_u(x', t) := \mu_1\bigl(\{u(\cdot, x') > t\}\bigr), \quad x' \in \mathbb{R}^{n-1}, \quad t > \inf u.
\]
The function \( m_u \) is nonnegative, nonincreasing, right-continuous w.r.t. the variable \( t \), and
(4.19)
\[
u(x_1, x') = \sup\{t \in \mathbb{R} : m_u(x', t) > \mu_1((-x_1, x_1))\}, \quad x = (x_1, x') \in \mathbb{R}^n.
\]

Proceeding analogously as we did in [6] for a certain class of Steiner-like rearrangements w.r.t. a finite measure, we can prove the following properties.

**Theorem 4.2.**

1) If \( u \in L^1_+ (\mu, \mathbb{R}^N) \), then
(4.20)
\[
\int_{\mathbb{R}^n} u \ d\mu = \int_{\mathbb{R}^n} u^* \ d\mu.
\]

2) If \( u \in F^n \) and if \( \varphi : \mathbb{R} \to \mathbb{R} \) is a nondecreasing function, then
(4.21)
\[
\varphi(u^*) = \left(\varphi(u)\right)^*.
\]

3) (Cavalieri’s principle)

If \( f : \mathbb{R} \to \mathbb{R} \) is continuous or nondecreasing, \( u \in F^n \) and if \( f(u) \in L^1(\mu, \mathbb{R}^n) \), then \( f(u^*) \in L^1(\mu, \mathbb{R}^n) \) and
(4.22)
\[
\int_{\mathbb{R}^n} f(u) \ d\mu = \int_{\mathbb{R}^n} f(u^*) \ d\mu.
\]

4) Let \( F \in C\left((\mathbb{R}^+_0)^2\right) \), \( F(0, 0) = 0 \), and
(4.23)
\[
F(A, B) - F(a, B) - F(A, b) + F(a, b) \geq 0
\]
for all \(a, b, A, B \in \mathbb{R}_0^+\) with \(a \leq A\), \(b \leq B\).
Furthermore, let \(u, v \in F^n_+\) such that \(F(u, 0), F(0, v), F(u, v) \in L^1(\mu, \mathbb{R}^n)\). Then

\[
\left(4.24\right) \quad \int_{\mathbb{R}^n} F(u, v) \, d\mu \leq \int_{\mathbb{R}^n} F(u^*, v^*) \, d\mu.
\]

5) (Nonexpansivity of the rearrangement)
Let \(G \in C(\mathbb{R}_0^+)\) continuous, nondecreasing and convex with \(G(0) = 0\), and let \(u, v \in F^n_+\) such that \(G(|u|), G(|v|), G(|u - v|) \in L^1(\mu, \mathbb{R}^n)\). Then

\[
\left(4.25\right) \quad \int_{\mathbb{R}^n} G(|u^* - v^*|) \, d\mu \leq \int_{\mathbb{R}^n} G(|u - v|) \, d\mu.
\]

6) Let \(u, v \in L^2_+(\mu, \mathbb{R}^n)\). Then

\[
\left(4.26\right) \quad \int_{\mathbb{R}^n} uv \, d\mu \leq \int_{\mathbb{R}^n} u^*v^* \, d\mu.
\]

7) Let \(u, v \in L^\infty(\mathbb{R}^n) \cap F^n\). Then

\[
\left(4.27\right) \quad \|u^* - v^*\|_\infty \leq \|u - v\|_\infty.
\]

Remark 4.3. If \(F \in C^2\) then \(\left(4.28\right)\) is equivalent to

\[
\left(4.28\right) \quad \frac{\partial^2 F(\sigma, \tau)}{\partial \sigma \partial \tau} \geq 0, \quad \forall \sigma, \tau \in \mathbb{R}_0^+.
\]

Proof of Theorem 4.2: The proofs of the properties 1)-3), and of 5)-7) mimic the proofs of analogous properties in [6].

4) We proceed similarly as in [12]. In view of \(\left(4.23\right)\), there exists a nonnegative measure, denoted by \(dF_{\sigma\tau}\), such that

\[
F(s, t) - F(s, 0) - F(0, t) = \int_{0}^{t} \int_{0}^{s} dF_{\sigma\tau}
\]

\[
\left(4.29\right) \quad = \int \int_{(\mathbb{R}_0^+)^2} \chi(0, s)(\sigma) \chi(0, t)(\tau) dF_{\sigma\tau} \quad \forall s, t \in \mathbb{R}_0^+.
\]

Notice that in the case \(F \in C^2\) we have

\[
dF_{\sigma\tau} = \frac{\partial^2 F(\sigma, \tau)}{\partial \sigma \partial \tau} \, d\sigma \, d\tau,
\]

which is obviously nonnegative by \(\left(4.23\right)\).
Choosing \(s = u(x)\) and \(t = v(x)\) in \(\left(4.29\right)\) and then integrating we find

\[
\int_{\mathbb{R}^n} F(u, v) \, d\mu = \int_{\mathbb{R}^n} F(u, 0) \, d\mu + \int_{\mathbb{R}^n} F(0, v) \, d\mu
\]

\[
\quad + \iint \mu\left(\{u > \sigma\} \cap \{v > \tau\}\right) \, dF_{\sigma\tau}.
\]
An analogous expression for \( \int_{\mathbb{R}^n} F(u^*, v^*) \, d\mu \) holds. Since from (3.7), we have
\[
(4.31) \quad \mu \left( \{ u > \sigma \} \cap \{ v > \tau \} \right) \leq \mu \left( \{ u^* > \sigma \} \cap \{ v^* > \tau \} \right) \quad \forall \sigma, \tau > 0.
\]
Then (4.24) follows from (4.30) and (4.31).

**Remark 4.4.** Let \( M = \mathbb{R} \times M' \), where \( M \in \mathcal{M}^{n-1} \), and let \( u, v \in \mathcal{F}^n \). Then the properties (1), and 3)–6) of Theorem 4.2 hold with the range of integration restricted to \( M \). Indeed the proof of this result can be easily obtained by Theorem 4.2 setting
\[
U(x) := \begin{cases} u(x) & \text{if } x \in M \\ \inf u & \text{if } x \in \mathbb{R}^n \setminus M \end{cases}
\]
and therefore
\[
U^*(x) := \begin{cases} u^*(x) & \text{if } x \in M \\ \inf u & \text{if } x \in \mathbb{R}^n \setminus M. \end{cases}
\]

We conclude this subsection with the following property, which is easy to prove but it is crucial for the next section.

Let \( u \) be a function belonging to \( C(\mathbb{R}^n) \). Denote the modulus of continuity of \( u \), by
\[
(4.32) \quad \omega_u(t) := \sup \{ |u(x) - u(y)| : |x - y| < t \}, \quad t > 0.
\]
Notice that \( u \) is uniformly continuous iff \( \lim_{t \searrow 0} \omega_u(t) = 0 \).

**Proposition 4.1.** Let \( u \in \mathcal{C}(\mathbb{R}^n) \cap \mathcal{F}^n \). Then \( u^* \in \mathcal{C}(\mathbb{R}^n) \) and
\[
(4.33) \quad \omega_u(t) \geq \omega_{u^*}(t) \quad \forall t > 0.
\]
In particular, if \( u \) is Lipschitz continuous with constant \( L \), then also \( u^* \) is Lipschitz continuous with a Lipschitz constant \( L^* \) such that \( L^* \leq L \).

**Proof:** By Remark 4.1, and by Corollary 4.2 (4.6), since \( u \) is continuous we have that
\[
(4.34) \quad \text{dist} \left\{ \{ u > t \}; \partial \{ u > s \} \right\} \leq \text{dist} \left\{ \{ u^* > t \}; \partial \{ u^* > s \} \right\} \quad \forall s, t \in \mathbb{R} \text{ with } s < t,
\]
which implies (4.33). Since \( L = \sup \{ \omega_u(t)/t : t > 0 \} \), and similarly for \( u^* \), the second assertion follows, too.

### 4.3. Integral inequalities in Sobolev spaces

In this subsection we state integral inequalities which involve derivatives of a function and its rearrangement. Variants of them are well-known for different types of rearrangements, including Steiner symmetrization (see, for instance, [19, 11, 16]), and they are usually referred as Polya-Szegő type inequalities. Theorem 4.3 below can be shown as Theorem 6.1 and Corollary 6.1 in [6], and its proof is therefore omitted.

**Theorem 4.3.** (Polya-Szegő’s principle)
Let \( G = G(y, v, x') \) be a function belonging to \( L^\infty(\mathbb{R}^n \times \mathbb{R}_+^n \times \mathbb{R}^{n-1}) \) where \( y = (y_1, \ldots, y_n) \in \mathbb{R}^n \). Assume also that \( G \) is continuous in \( v \), convex in \( y \), even in \( y_1 \) and nondecreasing in \( y_i \) with \( y_i > 0 \). Moreover let \( u \in L^\infty(\mathbb{R}^n) \) a nonnegative Lipschitz continuous function with compact support. Then
\[
(4.35) \quad \int_{\mathbb{R}^n} G(\nabla u, u, x') \, d\mu \geq \int_{\mathbb{R}^n} G(\nabla u^*, u^*, x') \, d\mu.
\]
Moreover, if \( u \in W^{1,p}_+(\mu, \mathbb{R}^n) \), for some \( p \in [1, \infty) \), then \( u^* \in W^{1,p}_+(\mu, \mathbb{R}^n) \), and inequality (4.35) holds if
\[
|G(y, v, x')| \leq C|y|^p \quad \text{for some } C > 0,
\]
for any \((y, v, x') \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^{n-1}\). Finally, if \( \Omega \) is a bounded domain in \( \mathbb{R}^n \) and \( u \) is a nonnegative function belonging to \( W^{1,p}_0(\mu, \Omega) \) for some \( p \in [1, \infty) \), then \( u^* \in W^{1,p}_0(\mu, \Omega^*) \).

Using Theorem 4.4 and proceeding analogously as in [6], proof of Theorem 5.4, one can obtain a criterion for the equality case in the inequality (4.35). We omit the proof.

**Theorem 4.4.** Let \( u \in W^{1,p}(\varphi, \mathbb{R}^n) \) for some \( p \in [1, \infty) \). Furthermore, let the function \( J := \psi(\Psi^{-1}) \) be strictly convex, let \( G \in C(\mathbb{R}^n) \), \( G = G(y) \), \( y = (y_1, \ldots, y_n) \), \( G \) convex and strictly increasing in \( y_1 \) for \( y_1 > 0 \), and such that
\[
|G(y)| \leq C(1 + |y|^p), \quad \text{for some } C > 0.
\]

Finally, assume that
\[
\int_{\mathbb{R}^n} G(\nabla u) \, d\mu = \int_{\mathbb{R}^n} G(\nabla u^*) \, d\mu.
\]

Then \( u = u^* \).

## 5. Radial Measures

In this section we consider measures \( \mu \) whose density is a radially symmetric function i.e.
\[
d\mu = \varphi(|x|) \, dx,
\]
where \( \varphi \in C(\mathbb{R}^+_n) \) is positive.

We prove isoperimetric inequalities with respect two special measures whose densities are
\[
\varphi_1(|x|) = \exp \{c|x|^2\}, \quad c > 0, \quad \varphi_2(|x|) = |x|^{1-n} \exp \{a(|x|)\},
\]
where \( a \in C(\mathbb{R}_+^n) \) is convex.

If \( M \in \mathcal{M}^n \), and if \( \mu(M) < +\infty \), then let \( M^* \) denote the ball \( B_R \) such that \( \mu(M) = \mu(B_R) \). If \( \mu(M) = +\infty \) then let \( M^* = \mathbb{R}^n \). We call \( M^* \) the Schwarz \( \mu \)-symmetrization of \( M \). As in the previous sections, we replace this definition by pointwise ones for open and closed sets. Thus, if \( M \) is open/closed with finite \( \mu \)-measure, then let \( M^* \) the open/closed ball centered at zero, having the same measure as \( M \).

We ask for additional conditions on the measure \( \mu \) such that the following isoperimetric inequality holds
\[
P_\mu(M) \geq P_\mu(M^*), \quad \forall M \in \mathcal{M}^n,
\]
with equality iff \( M = M^* \). Although we are not able to give a necessary and sufficient condition for (5.2) to hold, we show below that the above isoperimetric property holds if
\[
d\mu = \exp \{c|x|^2\} \, dx, \quad c > 0
\]
or if
\[
d\mu = |x|^{1-n} \exp \{a(|x|)\} \, dx, \quad \text{where } a \in C(\mathbb{R}_+^n) \text{ is convex.}
\]
Theorem 5.1. Let $\mu$ be the measure defined by (5.1) with $\varphi$ defined by (5.3). Then

\[ M^* + B_r \subset \left(M + B_r\right)^*, \quad \forall \text{ Borel sets and } \forall r > 0, \]

and (5.2) holds. Furthermore, if $M, N$ are open sets with $M \subset N$, then

\[ \text{dist} \{M; \partial N\} \leq \text{dist} \{M^*; \partial N^*\}. \]

Finally, if $P_\mu(M) = P_\mu(M^*)$ for some $M \in \mathcal{M}^n$, then $M = M^*$.

Proof: Let $M$ compact, $M \subset \overline{B}_R$ for some $R > 0$, and set

\[ A(M) := \{N \subset \mathbb{R}^n: N \text{ compact, } N \subset \overline{B}_R, \mu(N) = \mu(M), \mu(N + B_r) \leq \mu(M + B_r) \quad \forall r > 0\}. \]

Letting

\[ \delta := \inf\{\mu(N\Delta M^*): N \in A(M)\}, \]

there exists a sequence $\{N_k\} \subset A(M)$ with

\[ \lim_{k \to \infty} \mu(N_k \Delta M^*) = \delta. \]

Since the $N_k$’s are equibounded, there is a subsequence $\{N_{k'}\}$ which converges in Haussdorf distance to some set $N$, which also implies that $N \in A(M)$ and $\mu(N\Delta M^*) = \delta$. Assume that $\delta > 0$. Then we find two density points $\eta, \zeta$ of $N$ and $M^*$ such that $\eta \in M^* \setminus N$ and $\zeta \in N \setminus M^*$. After some rotation of the coordinate system

\[ x = \rho(\xi), \quad (x, \xi \in \mathbb{R}^n), \]

we achieve

\[ \rho(\eta) = y = (y_1, y'), \quad \rho(\zeta) = z = (z_1, y'), \]

for some $y' \in \mathbb{R}^{n-1}, y_1, z_1 \in \mathbb{R}$. Defining $N'$ by

\[ N' := \rho(N) \equiv \{x = \rho(\xi): \xi \in N\} \]

let $(N')^*$ denote its Steiner $\mu$-symmetrization w.r.t. the variable $x_1$. Notice that $\mu(N'\Delta M^*) = \delta$, and, due to the product structure of $\varphi$, we have that $(N')^* = ((N')^*)^* = N^*$, and $N', (N')^* \in A(M)$. Since the slices $(N'(y'))^*$ and $M^*(y')$ are intervals centered at zero, it is easy to see that

\[ \mu_1\left((N'(y'))^* \cap M^*(y')\right) > \mu_1\left(N'(y') \cap M^*(y')\right). \]

This also implies

\[ \mu((N')^* \cap M^*) > \mu((N') \cap M^*), \]

and thus

\[ \mu((N')^* \Delta M^*) < \mu((N') \Delta M^*), \]

contradicting the minimality of $\delta$. Hence $N = M^*$, and (5.2) is proved for compact sets. It is easy to see that this also implies property (5.3) for compact sets, and by a simple approximation argument, as well for Borel sets. It is well-known that (5.5) also implies (5.6), and the lower semicontinuity of the perimeter $P_\mu$ yields (5.2).

Assume finally that $P_\mu(M) = P_\mu(M^*)$ for some $M \in \mathcal{M}^n$. Let $\rho$ any rotation about the origin, and let $^*$ denote $\mu$-Steiner symmetrization in direction $x_1$. Then $[(\rho M)^*]^* = M^*$, which means that $P_\mu(\rho M) = P_\mu((\rho M)^*)$. By Theorem 4.1 this implies that $\rho M = (\rho M)^*$. 
Since $\rho$ was arbitrary we just have proved that $M$ is symmetric w.r.t. every $(n-1)$-hyperplane through the origin. Hence $M = M^\star$.

Next we rewrite the isoperimetric inequality in terms of $\mu(M)$. Let $I_\mu(m)$ be the isoperimetric function defined in (1.2),

\begin{align}
 h(r) &:= n\omega_n e^{\sigma^2} r^{n-1} \quad \text{and} \\
 H(r) &= \int_0^r h(t) \, dt.
\end{align}

Then

\[ P_\mu(M^\star) = h(H^{-1}(\mu(M))) = I(\mu(M^\star)). \]

Therefore (5.2) reads as follows:

**Corollary 5.1.** If $\mu(M) < +\infty$, then

\[ P_\mu(M) \geq h(H^{-1}(\mu(M))) = I(\mu(M^\star)). \] (5.9)

Now let us define the $\mu$-Schwarz symmetrization of functions with respect to the measure $\mu$ defined in (5.1) with $\varphi$ defined by (5.3). First we introduce a function $\tilde{u}: [0, +\infty[ \to \mathbb{R}$ defined by

\[ \tilde{u}(s) = \inf \{ t \in \mathbb{R} : m_u(t) \leq s \}. \]

Notice that $\tilde{u}$ is a nonincreasing and right-continuous function. Observe also that $\tilde{u}(s)$ is the inverse function of $m_u(t)$, if $\tilde{u}(s)$ is not constant on intervals. In this case, the following equality holds,

\[ \frac{\partial \tilde{u}(s)}{\partial s} = \left[ \frac{\partial m_u(t)}{\partial t} \right]^{-1}, \]

where $s = m_u(t)$.

If $u \in \mathcal{F}^n$ we define the *Schwarz $\mu$-symmetrization of $u$* by

\[ u^\star(x) := \sup \{ t \in \mathbb{R} : x \in \{ u > t \}^\star \}, \quad x \in \mathbb{R}^n. \] (5.11)

Observe that, by definition of $\tilde{u}$ and $u^\star$, one has

\[ u^\star(x) = \tilde{u}(H(|x|)), \quad \text{for a.e. } x \in \Omega^\star. \]

By definition $u^\star$ is radially symmetric and radially decreasing. Moreover $u$ and $u^\star$ are equimeasurable functions. If $u$ is continuous then we will understand this definition in pointwise sense, so that $u^\star$ is continuous, too. The assertions of Theorem 4.2 hold as well for the Schwarz $\mu$-symmetrization.

As in case of the Steiner $\mu$-symmetrization, the isoperimetric property (5.6) implies the following estimate for the modulus of continuity.

**Proposition 5.1.** Let $\mu$ be given by (5.1) and (5.3). Then

\[ \omega_u \geq \omega_{u^\star} \quad \forall u \in C(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n) \cap \mathcal{F}^n. \] (5.12)
Isoperimetric property (5.2) and a result due to Talenti [31], imply some Polya-Szegö type inequalities, that is, integrals involving gradients of a nonnegative Lipschitz function having compact support decrease under weighted Schwarz symmetrization. The fact that the equality case in these inequalities occurs only in symmetric situations can be shown analogously as in [8], and by using Theorem 4.4 above. Using arguments as in the proofs of Theorem 5.2 and Corollary 5.2 this leads to norm inequalities in $W^{1,p}(\mathbb{R}^n)$. We omit the proofs.

**Theorem 5.2.** Let $u$ a nonnegative Lipschitz continuous function on $\mathbb{R}^n$ with compact support, and let $G \in C(\mathbb{R}_+^n)$ nonnegative and convex with $G(0) = 0$. Then

\begin{equation}
(5.13) \quad \int_{\mathbb{R}^n} G(|\nabla u|) \, d\mu \geq \int_{\mathbb{R}^n} G(|\nabla u^*|) \, d\mu.
\end{equation}

Moreover, if $G$ is strictly convex then (5.13) holds with equality sign only if $u = u^*$. Furthermore, if $u \in W^{1,p}(\mu, \mathbb{R}^n)$ is nonnegative, for some $p \in [1, \infty)$, then $u^* \in W^{1,p}(\mu, \mathbb{R}^n)$, and (5.13) holds with $G(t) = t^p$. Finally, if $\Omega$ is a domain in $\mathbb{R}^n$ and $u \in W^{1,p}_0(\mu, \Omega)$ is nonnegative, then $u^* \in W^{1,p}_0(\mu, \Omega^*)$.

The isoperimetric inequality leads to several integral inequalities which compare an $L^p$-weighted norm of the gradient of a function with an $L^q$-weighted norm of the same function when the measure $\mu$ is given by (5.1) and (5.3). This type of results are also proved in a different way in [23].

**Theorem 5.3.** There are constants $C = C(p, q) > 0$ such that for every $u \in C_0^\infty(\mathbb{R}^n)$,

\begin{equation}
(5.14) \quad \|\nabla u\|_p \geq C(p, q)\|u\|_q,
\end{equation}

where $q \in [p, np/(n-p)]$ for $p \in [1, n)$, $q \in [n, +\infty)$ for $p = n$, and $q \in [p, +\infty)$ for $p \in (n, +\infty)$. Moreover, there are constants $C(p) > 0$ such that for every $u \in C_0^\infty(\mathbb{R}^n)$,

\begin{equation}
(5.15) \quad \|\nabla u\|_p \geq C(p)\|u\|_{C^{0,1-(n/p)}(\mathbb{R}^n)},
\end{equation}

if $p \in (n, +\infty)$.

**Proof :**

1) Let $h, H$ be given by (5.7) and (5.8). It is then easy to see that

$$h(r)^q \geq CH(r) \quad \forall r \in [0, +\infty), \quad \text{for some } C > 0,$$

if $q \in [1, n/(n-1)]$. Applying Theorem 2.1.1 of [22], this implies (5.14) with $p = 1$ and $q \in [1, n/(n-1)]$.

2) Let $p > 1$, $t > 0$, $q \in [1, n/(n-1)]$, $q < p/(p-1)$, and $u \in C_0^\infty(\mathbb{R}^n)$. Applying 1) and Hölder’s inequality we have that

\begin{equation}
(5.16) \quad \|u|u|^{t-1}\|_q \leq C(1, q)\|t|u|^{t-1}|\nabla u|_1 \leq C_0\|\nabla u\|_p\|u|^{t-1}\|_{p'},
\end{equation}

for some $C_0 > 0$, where $p' = p/(p-1)$. Choosing $t = p'/p' - q$, we obtain (5.14) for $p \in (1, n)$ with $q \in [p, np/(n-p)]$, and for $p \geq n$ with $q \in [n, +\infty)$. 
3) Since $\varphi(t) = \exp \{ct^2\} \geq 1$, it results
\[
\|\nabla u\|_p \geq \left( \int_{\mathbb{R}^n} |\nabla u|^p \, dx \right)^{1/p},
\]
we obtain (5.15) from Morrey’s Imbedding Theorem. From this we also obtain (5.14) for $p \in [n, +\infty)$ with $q = +\infty$.

Let $X^{p,q}$ denote the closure of $C^\infty_0(\mathbb{R}^n)$ with respect to the norm
\[
\|u\|_{p,q} := \|u\|_p + \|\nabla u\|_p, \quad p, q \in [1, +\infty).
\]
From (5.14) one immediately obtains the following results.

**Corollary 5.2.**
1) Let $p \in [1, +\infty)$, $q \in [p, np/(n-p)]$ for $p < n$, and $q \in [p, +\infty)$ for $p \geq n$. Then
\[
X^{p,q} = W^{1,p}(\mathbb{R}^n, \varphi),
\]
2) Let $\Omega$ any domain in $\mathbb{R}^n$ and $p \in [1, +\infty)$. Then $W^{1,p}_0(\Omega, \varphi) \subset W^{1,p}(\mathbb{R}^n, \varphi)$, and
\[
\|\nabla u\|_{p,\Omega} \geq C(p, p)\|u\|_{p,\Omega} \quad \forall u \in W^{1,p}_0(\Omega, \varphi).
\]

To our knowledge, the problem of finding the best constants in the inequalities (5.14) is still open. Here we solve such a problem in the special case $p = q = 2$.

**Corollary 5.3.** Let $c > 0$. Then there holds
\[
\inf \left\{ \frac{\|\nabla u\|^2}{\|u\|^2} : u \in C^\infty_0(\mathbb{R}^n) \right\} = 2cn = \frac{\|\nabla (e^{-c|x|^2})\|^2}{\|e^{-c|x|^2}\|^2}.
\]

**Proof:** Consider the following eigenvalue problem for the harmonic oscillator,
\[
-\Delta v + c^2|x|^2v = \lambda v \quad \text{in} \quad \mathbb{R}^n.
\]
The spectrum and the eigenfunctions are explicitly known (see [32], p.104 ff.). In particular, the spectrum is given by \{ $\lambda = \lambda_k : (2k-2+n)c, k = 1, 2, \ldots$ \}, the eigenvalue $\lambda_1 = cn$ is simple and a corresponding eigenfunction is $v_1 = \exp \{ -c|x|^2/2 \}$. This implies
\[
\int_{\mathbb{R}^n} \left( \nabla (\exp(-c|x|^2/2)) \right)^2 + c^2|x|^2 \exp(-c|x|^2) - cn \exp(-c|x|^2) \, dx = 0.
\]
Now let $u \in C^\infty_0(\mathbb{R}^n)$. Setting $v := u \exp \{ c|x|^2/2 \}$ we then find by partial integration
\[
\|\nabla u\|^2 + 2cn\|u\|^2 = \int_{\mathbb{R}^n} \left( \nabla v^2 + c^2|x|^2|v|^2 - cn|v|^2 \right) \, dx,
\]
and the assertion follows from (5.22).

Finally we prove an isoperimetric inequality w.r.t. a measure $\mu$ that is given by
\[
d\mu = |x|^{1-n}\exp \{ a(|x|) \} \, dx, \quad \text{where} \ a \in C(\mathbb{R}^n_+) \text{ is convex}.
\]
Notice that the measure $\mu$ above is singular at the origin. This implies in particular that the outer Minkowski content of the set $\{0\}$ is positive, namely

$$\mu^+\{0\} = n\omega_n e^{a(0)}, \quad (\omega_n: \text{measure of the } n\text{-dimensional unit ball}).$$

**Lemma 5.1.** Let $\Omega$ be a smooth open set in $\mathbb{R}^n$ which contains an open neighborhood of the origin, and such that $\mu(\Omega) < \infty$, where the measure $\mu$ is given by (5.23), and let $B_R$ the ball with $\mu(B_R) = \mu(\Omega)$, $(R > 0)$. Then

$$\mu^+(\Omega) = \int\limits_{\partial \Omega} |x|^{1-n} \exp \{a(|x|)\} \mathcal{H}^{n-1}(dx) \geq n\omega_n e^{a(R)} = \mu^+(B_R).$$

**Proof:** Consider

$$f(x) := e^{-a(|x|)} \text{ div } \left(|x|^{-n} e^{a(|x|)} \right)$$

and denote by $\nu$ the exterior normal to $\Omega$. By Green’s Theorem, we have

$$\mu^+(\Omega) = \int\limits_{\partial \Omega} |x|^{1-n} \exp \{a(|x|)\} \mathcal{H}^{n-1}(dx) \geq \int\limits_{\partial \Omega} |x|^{-n} \exp \{a(|x|)\} (x \cdot \nu) \mathcal{H}^{n-1}(dx) = n\omega_n e^{a(0)} + \int\limits_{\Omega} \text{div} \left(|x|^{-n} \exp \{a(|x|)\} \right) dx \geq n\omega_n e^{a(0)} + \int\limits_{B_R} f(x) d\mu = \int\limits_{\partial B_R} |x|^{-n} \exp \{a(|x|)\} (x \cdot \nu) \mathcal{H}^{n-1}(dx) = \mu^+(B_R).$$

Analogously to Theorem 5.2, the following result holds

**Theorem 5.4.** Let $u$ a nonnegative Lipschitz continuous function with compact support, and suppose that

$$(5.25) \quad u(0) = \text{ess sup } u.$$ 

Then (5.13) holds.

**Proof:** The proof can be carried out analogously as the proof of Theorem 6.1 in [6], taking into account that the superlevel sets $\{x: u(x) > t\}$, $(\sup u > t > 0)$, contain an open neighborhood of the origin. We leave the details to the reader.

6. **Comparison results**

In this Section we assume that $\mu$ is the measure defined by (5.1) with $\varphi$ given by (5.3).

We will prove a comparison result for weak solutions $u$ to nonlinear elliptic problems. This implies an estimate of the Schwarz $\mu$-symmetrization in terms of the solution of a related
radially symmetric problem. We mention that similar results for the classical Schwarz symmetrization are well-known (see, for instance, [2], [28], [3]). We also mention that a related comparison theorem holds for the symmetrization in Gauss space (see [5], [?]).

Consider the following nonlinear elliptic problem

\[
\begin{aligned}
-\text{div}(a(x,u,\nabla u)) &= f\varphi & \text{in } \Omega \\
u &= 0 & \text{on } \partial \Omega.
\end{aligned}
\]

Here \(\Omega\) is an open subset of \(\mathbb{R}^N, N \geq 2\), \(p\) is a real number with \(1 < p < N\), and \(a: \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N\) is a Carathéodory function satisfying

\[
\begin{aligned}
a(x,s,\xi)\xi &\geq \varphi(|x|)|\xi|^p, \\
|a(x,s,\xi)| &\leq \varphi(|x|) \left[|\xi|^{p-1} + |s|^{p-1} + a_0(x)\right], & a_0(x) &\in L^p(\mu, \Omega), \ a_0 \geq 0, \\
(a(x,s,\xi) - a(x,s,\eta), \xi - \eta) &> 0, & \xi &\neq \eta,
\end{aligned}
\]

for almost every \(x \in \Omega\) and for every \(s \in \mathbb{R}, \xi \in \mathbb{R}^N, \eta \in \mathbb{R}^N\). Moreover we assume that

\[
\begin{aligned}
f &\in L^{p^*}(\varphi, \Omega).
\end{aligned}
\]

We will say that \(u \in W^{1,p}_0(\Omega, \varphi)\) is a weak solution to the problem (6.1) if it satisfies

\[
\begin{aligned}
\int_{\Omega} a(x,u,\nabla u)\nabla \psi \, dx = \int_{\Omega} f\varphi \psi \, dx, & \quad \forall \psi \in W^{1,p}_0(\Omega, \varphi).
\end{aligned}
\]

The existence of a weak solution is a consequence of the Sobolev type inequality given by Theorem 5.3 and an adaptation of classical results due to J. Leray and J.-L. Lions (cf. [21]).

The main result of this Section is

**Theorem 6.1.** Let \(u\) be a weak solution to the problem (6.1). Denote by \(v = v^*(|x|) \in W^{1,p}_0(\Omega^*, \varphi)\) the function defined by

\[
v(|x|) = \int_{H(|x|)} \left| \frac{1}{I(r)|^p \left( \int_0^r \tilde{f}(\sigma) \, d\sigma \right)^{\frac{1}{p-1}} \right| \, dr,
\]

which is a weak solution to the problem

\[
\begin{aligned}
-\text{div}(\varphi(x)|\nabla u|^{p-2}\nabla u) &= f^*(x)\varphi(x) & \text{in } \Omega^*, \\
v &= 0 & \text{on } \partial \Omega^*.
\end{aligned}
\]

Then

\[
u^*(x) \leq v^*(x),
\]

for a.e. \(x \in \Omega^*\). Moreover, for any \(1 \leq q < p\), there holds

\[
\|\nabla u\|_q \leq \|\nabla v\|_q.
\]
Proof: Denote by $\psi_h = \begin{cases} \text{sign } u & \text{if } |u| > t + h \\ \frac{u - t \text{sign } u}{h} & \text{if } t < |u| \leq t + h \\ 0 & \text{otherwise.} \end{cases}$

Since $u$ belongs to $W^{1,p}_0(\Omega, \varphi)$ the function $\psi_h$ can be considered as a test function in (6.6) and we have

$$\int_{\Omega} a(x, u, \nabla u) \nabla \psi_h \, dx = \int_{\Omega} f \psi_h \varphi \, dx,$$

or, equivalently,

$$\frac{1}{h} \int_{t < |u| \leq t + h} a(x, u, \nabla u) \nabla u \, dx = \int_{|u| > t} f \text{sign } u \varphi \, dx + \frac{1}{h} \int_{t < |u| \leq t + h} f(u - t \text{sign } u) \varphi \, dx.$$

By using the ellipticity condition (6.2), Hardy’s inequality, and by letting $h$ go to zero, we have that

$$- \frac{d}{dt} \int_{|u| > t} \varphi(|x|) |\nabla u|^p \, dx \leq \int_{0} \tilde{f}(\sigma) \, d\sigma.$$

Moreover, by Hölder inequality, we find

$$- \frac{d}{dt} \int_{|u| > t} \varphi(|x|) |\nabla u| \, dx \leq \left( - \frac{d}{dt} \int_{|u| > t} \varphi(|x|) |\nabla u|^p \, dx \right)^{1/p} (-m_u'(t))^{1/p'}.$$

On the other hand, from Federer’s coarea formula (cf. [17]), we obtain

$$- \frac{d}{dt} \int_{|u| > t} \varphi(|x|) |\nabla u| \, dx = \int_{|u| = t} \varphi(|x|) \mathcal{H}_{n-1}(dx).$$

Combining (6.10), (6.11) and (6.12), we deduce

$$\left( \int_{|u| = t} \varphi(|x|) \mathcal{H}_{n-1}(dx) \right)^p \leq \left( -m_u'(t) \right)^{p/p'} \int_{0} \tilde{f}(\sigma) \, d\sigma.$$

Now we apply the isoperimetric inequality given by Corollary 5.1 that is

$$\int_{u = t} \varphi(|x|) \mathcal{H}_{n-1}(dx) \geq h(H^{-1}(m_u(t))) = I(m_u(t)).$$

Combining (6.13) and (6.14), we get

$$- \left[ I(m_u(t)) \right]^p \left( -m_u'(t) \right)^{p/p'} \leq \int_{0} \tilde{f}(\sigma) \, d\sigma, \quad t \geq \text{ess inf } u.$$
This implies
\[-\frac{d\tilde{u}}{ds} \leq \frac{1}{[I(s)]^{\mu'}} \left( \int_0^s \tilde{f}(r) \, dr \right)^{\frac{1}{p-1}}.\]

Now an integration between \(s > 0\) and \(\mu(\Omega^*)\) gives
\[ \tilde{u}(s) \leq \int_s^{\mu(\Omega^*)} \frac{1}{[I(r)]^{\mu'}} \left( \int_0^r \tilde{f}(\sigma) \, d\sigma \right)^{\frac{1}{p-1}} \, dr. \]

Choosing \(s = H(|x|) > 0\) this concludes the proof of (6.8).

By Hölder’s inequality, we have for any \(1 \leq q < p\),
\[(6.16) \quad -\frac{d}{dt} \int_{|u| > t} \varphi(|x|) |\nabla u|^q \, dx \leq \left( \frac{d}{dt} \int_{|u| > t} \varphi(|x|) |\nabla u|^p \, dx \right)^{q/p} (-\mu'_u(t))^{1-q/p}.\]

Using (6.10) this leads to
\[(6.17) \quad -\frac{d}{dt} \int_{|u| > t} \varphi(|x|) |\nabla u|^q \, dx \leq \left( \int_0^s \tilde{f}(s) \, ds \right)^{q/p} (-\mu'_u(t))^{1-q/p}.\]

Integrating between 0 and \(+\infty\) then gives
\[(6.18) \quad \int_{\Omega} \varphi(|x|) |\nabla u|^q \, dx \leq \int_0^{+\infty} \frac{1}{[-\mu'(t)]^{q/p}} \left( \int_0^s \tilde{f}(s) \, ds \right)^{q/p} (-d\mu(t)),\]
from which one has, by (6.15),
\[(6.19) \quad \int_{\Omega} \varphi(|x|) |\nabla u|^q \, dx \leq \int_0^{+\infty} \left( \frac{1}{I(s)} \right)^{q/(p-1)} \left( \int_0^s f^*(r) \, dr \right)^{q/(p-1)} \, ds.\]

This is (6.9).

**Remark 6.1.** We emphasize that the proof of the comparison result carries over to domains with infinite \(\mu\)-measure. Indeed, since the solutions \(u\) and \(v\) belong to weighted Sobolev spaces, their level sets \(\{x \in \Omega : |u| > t\}\) and \(\{x \in \Omega : |v| > t\}\) have finite measure and therefore we can apply the isoperimetric inequality (5.9) to such sets. Notice that in this case one has to replace \(\Omega^*\) in the symmetrized problem (6.7) by \(\mathbb{R}^n\).
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