Regularizing the quark-level linear $\sigma$ model.

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Abstract

We show that the finite difference, $-i\pi m^2$, between quadratic and logarithmic divergent integrals $\int d^4p \left[ m^2(p^2 - m^2)^{-2} - (p^2 - m^2)^{-1} \right]$, as encountered in the linear $\sigma$ model, is in fact regularization independent.

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Given that the scalar meson $\sigma$ (now called the $f_0(400 - 900)$ in [1]) will shortly be reinstated in the 1996 particle data group tables, but perhaps with a slightly extended mass range, we believe it important for theorists to reconsider seriously the original $SU(2)$ chiral linear $\sigma$ model field theory [2].

In a recent letter [3] it was shown that a dimensionally regularized quark-level linear $\sigma$ model (L$\sigma$M) not only dynamically generates the spontaneously broken Gell-Mann-Lévy interacting Lagrangian density [2]

$$L_{\sigma M}^{\text{int}} = g\bar{\psi} (\sigma’ + i\gamma_5 \vec{\tau} \cdot \vec{\pi}) \psi + g’ \sigma’ (\sigma’^2 + \vec{\pi}^2) - \lambda (\sigma’^2 + \vec{\pi}^2)^2 / 4 \quad (1)$$

with the chiral-limiting meson-quark and meson-meson couplings

$$g = \frac{m_q}{f_\pi}, \quad g’ = \frac{m_\sigma^2}{2f_\pi} = \lambda f_\pi, \quad (2)$$

but that the L$\sigma$M parameters satisfy the one-loop order relations

$$m_\sigma = 2m_q, \quad m_q = 2\frac{f_\pi}{\sqrt{3}} \approx 650 \text{ MeV}, \quad g = \frac{2\pi}{\sqrt{N_c}}. \quad (3)$$

Then for color number $N_c = 3$ and chiral limiting $f_\pi \approx 90 \text{ MeV}$, (2) and (3) require

$$m_q = f_\pi \frac{2\pi}{\sqrt{3}} \approx 325 \text{ MeV}, \quad m_\sigma = 2m_q \approx 650 \text{ MeV}, \quad (4)$$

$$g = \frac{2\pi}{\sqrt{3}} = 3.6276, \quad g_{\pi NN} = 3g_{A\pi} \approx 13.68, \quad (5)$$

for the measured value [4] $g_A \approx 1.2573$. Since the predicted $\pi NN$ coupling in (5) is near the phenomenologically determined [5] $g_{\pi NN} = 13.40 \pm 0.08$ and [3] predicts a reasonable constituent quark mass $m_q$ near $M_N/3$, along with a Nambu-Jona-Lasinio [6] scalar $\sigma$ mass which is the average [7] of the non-strange empirical $\sigma$ mass extracted in [1], the dynamically generated [3] L$\sigma$M appears to reflect reality.

In this note we show that the crucial new relations of (3) are in fact regularization scheme independent. Specifically they hold not only for dimensional regularization, but also analytic regularization [8], a symmetrical approach to generalized functions [9], and for Pauli-Villars regularization.

We begin by quickly reviewing the dynamical generation of the L$\sigma$M Lagrangian [3] and chiral couplings [2] starting from the more basic chiral quark model (CQM) massless Lagrangian

$$L_{CQM} = \bar{\psi} [i\vec{\gamma} \cdot \vec{D} + g (\sigma + i\gamma_5 \vec{\tau} \cdot \vec{\pi})] \psi + \left[(\partial \sigma)^2 + (\partial \vec{\pi})^2\right] / 2. \quad (6)$$
Here the bare quark, $\sigma$ and pion masses are zero in (6); the quark and $\sigma$ masses must be dynamically generated from the meson-quark chiral driving term

$$g \bar{\psi} (\sigma + i\gamma_5 \vec{\tau} \cdot \vec{\pi}) \psi,$$

while the pion remains massless due to the underlying conservation of the axial current $\partial^\mu \vec{A}_\mu = 0$, which in turn ensures the quark-level Goldberger-Treiman relation (GTR) $f_\pi g = m_q$ in (2).

The pion decay constant $f_\pi$ and quark mass $m_q$ are simultaneously non-perturbatively generated in the spirit of NJL gap equations $\delta f_\pi = f_\pi$ and $\delta m_q = m_q$ via the quark loops in Figure 1. Since there are no mass terms in the CQM Lagrangian (3), the physical masses $m_q$ and $m_\sigma$ in Fig. 1 equal the counterterm masses. Due to the GTR $m_q = f_\pi g$, Fig. 1a generates the chiral-limiting log-divergent gap equation (with $\frac{d^4 p}{d^4 p_1}$)

$$1 = -4iN_c g^2 \int \frac{d^4 p}{(p^2 - m_q^2)^2},$$

which is a compositeness condition in the context of (10). On the other hand, Fig. 1b with two quark flavors generates (11) the counterterm $m_q$ (quadratically divergent) mass gap

$$m_q = \frac{-8iN_c g^2}{m_\sigma^2} \int \frac{d^4 p m_q}{p^2 - m_q^2}. \hspace{1cm} (8)$$

The blending of the logarithmic and quadratic divergent loop integrals in (7) and (8) is where (3) specialized to the dimensional regularization scheme (dim. reg.), leading to the lemma in the limit of $2l \rightarrow 4$ dimensions,

$$\int d^4 p \left[ \frac{m^2}{(p^2 - m^2)^2} - \frac{1}{(p^2 - m^2)^2} \right] = \lim_{l \rightarrow 2} \frac{im^{2l-2}}{(4\pi)^l} \left[ \Gamma(2 - l) + \Gamma(1 - l) \right] = \frac{-im^2}{(4\pi)^2}. \hspace{1cm} (9)$$

Since this lemma (9) was derived from dimensional continuation, one might suspect that it is specific to that regularization and is therefore subject to criticism. We now demonstrate that the result (9) holds more generally.

First we reformulate the left hand side (l.h.s) of (9) with the aid of the
identity

\[
I = \int \! \bar{d}^{4}p \left[ \frac{m^2}{(p^2 - m^2)^2} - \frac{1}{p^2 - m^2} \right] = \left( m^2 \frac{d}{dm^2} - 1 \right) \int \! \bar{d}^{4}p \frac{p^2}{p^2 - m^2}.
\]

The r.h.s. of (10) is more amenable to alternative regularization schemes than is the l.h.s. of (9) or (10). Specifically the analytic regularization scheme uses (eq. 3.30 of [8]) non-integral \( \Sigma \),

\[
- i(4\pi)^2 \int \frac{\bar{d}^{4}p}{(p^2 - m^2)^{\Sigma}} = \frac{(-1)^{-\Sigma} \Gamma(\Sigma - 2)}{\Gamma(\Sigma)(m^2)^{\Sigma-2}} = \frac{(-m^2)^{2-\Sigma}}{(\Sigma - 1)(\Sigma - 2)}.
\]

Inserting (11) into the r.h.s. of (10) in the limit \( \Sigma \rightarrow 1 \) then yields

\[
- i(4\pi)^2 \left( m^2 \frac{d}{dm^2} - 1 \right) \int \frac{\bar{d}^{4}p}{(p^2 - m^2)^{\Sigma}} = \left[ \frac{2 - \Sigma}{(\Sigma - 1)(\Sigma - 2)} - \frac{1}{(\Sigma - 1)(\Sigma - 2)} \right] (-m^2)^{2-\Sigma} \rightarrow -m^2.
\]

Substituting (12) back into (10) then gives for the analytic regularization

\[
I = \int \! \bar{d}^{4}p \left[ \frac{m^2}{(p^2 - m^2)^2} - \frac{1}{p^2 - m^2} \right] \rightarrow -im^2 (4\pi)^2,
\]

which is precisely the result of the dim. reg. lemma (8); \( \zeta \)-function regularization is essentially equivalent to that.

To verify that the scheme-independent relation (10) also reproduces the original dimensional regularization lemma (3), one replaces the analytic regularization scheme integral (11) by [8]

\[
- i(4\pi)^l \int \frac{\bar{d}^{2l}p}{p^2 - m^2} = -\Gamma(1 - l)(m^2)^{l-1}
\]

in \( 2l \) dimensions. Using the identity \( \Gamma(1 - l) = \Gamma(2 - l)/(1 - l) \) in (14) and substituting the latter into the r.h.s. of (11), one finds as \( l \rightarrow 2 \),

\[
- i(4\pi)^l \left( m^2 \frac{d}{dm^2} - 1 \right) \int \frac{\bar{d}^{2l}p}{p^2 - m^2} = -\Gamma(2 - l) \frac{[(l - 1) - 1]}{1 - l} (m^2)^{l-1} \rightarrow -m^2.
\]
Since the r.h.s. of (15) is the same as the r.h.s. of (12), the dim. reg. lemma (9) is clearly recovered, which is hardly surprising.

Because both the dimensional and analytic regularization schemes involve $\Gamma$–functions, we should also consider alternative regularization schemes not containing them. Specifically we study the symmetric generalized function scheme advocated by Lodder [9] involving natural logarithmic functions:

$$-i(4\pi)^2 \int \frac{d^4p}{p^2 - m^2} = -m^2 \left( \ln \left( \frac{m^2}{M^2} \right) + C \right).$$  \hspace{1cm} (16)$$

Here $C$ is an indeterminate constant and $M$ is some mass scale. Then the analogue of (12) and (15) is

$$-i(4\pi)^2 \left( m^2 \frac{d}{dm^2} - 1 \right) \int \frac{d^4p}{(p^2 - m^2)}$$

$$= -m^2 \left( \ln \left( \frac{m^2}{M^2} \right) + C \right) - m^2 + m^2 \left( \ln \left( \frac{m^2}{M^2} \right) + C \right) = -m^2, \hspace{1cm} (17)$$

gain giving (13).

Finally, turning to Pauli-Villars regularization, the l.h.s. of (9) or (10) can be alternatively written in the form

$$I = \int \frac{d^4p}{p^2} \left[ \frac{m^4}{(p^2 - m^2)^2} - 1 \right].$$  \hspace{1cm} (18)$$

Next introduce an ultraviolet cut-off $\Lambda$ and sum over massive fermions (masses $M_j$) with probabilities $c_j$. The integral $I$ thereby reduces to

$$I = \sum_j ic_j(\Lambda^2 - M_j^2)/(4\pi)^2. \hspace{1cm} (19)$$

Applying the Pauli-Villars sum rules, \cite{8, 12} \sum $c_j = 0$, \sum $c_j M_j^2 = m^2$ (which eliminates the quadratic divergence or massless tadpole), we remain once again with $I = -im^2/(4\pi)^2$, namely the r.h.s. of (9). We contend that any other reasonable regularization scheme will lead to the same final result (13).

Returning to the Fig. 1b counterterm mass gap equation (8), we cancel out the quark mass $m_q$ from both sides of (8) to write

$$m_{\sigma}^2 = -8iN_c g^2 \int \frac{d^4p}{p^2 - m_q^2}. \hspace{1cm} (20)$$

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Subtracting the quadratic divergent integral in (20) from the log-divergent mass gap integral of (7) weighted by $2m_q^2$, the regularization independent lemma (9) then leads to

$$m_\sigma^2 = 2m_q^2 \left[ 1 + \frac{g^2 N_c}{4\pi^2} \right]. \quad (21)$$

Moreover, the quark bubble plus tadpole graphs of Fig. 2 generate the counterterm $\sigma$ mass (squared) from the CQM Lagrangian (6) giving

$$m_\sigma^2 = \frac{16i N_c g^2}{\pi^2} \left[ \int \frac{m_q^2 \, d^4p}{(p^2 - m_q^2)^2} - \int \frac{\bar{d}^4p}{(p^2 - m_q^2)} \right] = \frac{N_c g^2 m_q^2}{\pi^2}, \quad (22)$$

by virtue of the same lemma (9). Solving the two equations (21) and (22) in terms of the two parameters $m_q^2$ and $N_c g^2$, one obtains the two key LoSM relations in (3), with resulting physical scales given in (4) and (5).

To reaffirm the NJL relation $m_\sigma = 2m_q$ in this LoSM context and to test the consistency of the counterterm mass relations (8) or (21) and (22), we express the integral version of (22) as

$$m_\sigma^2 = -4m_q^2 + 2m_\sigma^2. \quad (23)$$

Here the first log-divergent integral in (22) is replaced by $-4m_q^2$ in (23) using the log-divergent gap equation (9), and the second quadratic-divergent integral in (22) is replaced by the counterterm $2m_\sigma^2$ in (23) using (8) or (21). The solution of (23) is trivially $m_\sigma = 2m_q$, as anticipated.

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[7] While the $\sigma$ has mass range $400 - 900$ MeV in [4], the LσM width is $\Gamma(\sigma \pi \pi) = \frac{3}{2} q |m_{\sigma}^2/f_{\pi}|^2 / 8\pi m_{\sigma}^2 \approx 700$ MeV for the average $\sigma$ mass $m_{\sigma} = 650$ MeV and is as wide as the mass. This is the main reason why the $\sigma$ took so long to establish.

[8] For analytic, dimensional and Pauli-Villars regularization, see the review by R. Delbourgo, Rep. Prog. Phys. 39, 345 (1976). For a review of zeta-function regularization, see E. Elizalde et al. “Zeta Regularization”, World Scientific, Singapore 1994.

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[11] Disregarding the propagator $\sigma$ mass when evaluating the dominant log-divergent behaviour, the quark self-energy amplitudes for the sum of $\vec{\pi}$– and $\sigma$– mediated loops sum to

$$\Sigma(p) = ig^2 \int \frac{d^4k}{k^2 [(p+k)^2 - m_{\sigma}^2]} \left[ -3\gamma_5 (\not{p} + \not{k} + m_q) \gamma_5 + (\not{p} + \not{k}) + m_q \right].$$
Introducing Feynman parameter $x$ and shifting the loop 4–momentum as $k \rightarrow k - px$, the above loop integral vanishes on the quark mass-shell

$$\Sigma \left( \not{p} = m_q, p^2 = m_q^2 \right) \propto \int_0^1 dx \left[ -2 \left( 1 - x \right) + 1 \right] = 0.$$ 

For this reason we have dropped the standard $\vec{\pi}, \sigma$ contributions to the quark self-energy; they cancel to leading order by chiral symmetry.

[12] See eg C. Itzykson and J. Zuber, “Quantum Field Theory”, McGraw-Hill, NY (1980), page 411.
Figure Captions

Fig. 1(a) Quark loop determination of the decay constant $f_\pi$.
Fig. 1(b) Mass gap quark tadpole.
Fig. 2 Quark loop induced $(\sigma \text{ mass})^2$. 

This figure "fig1-1.png" is available in "png" format from:

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