Noether identities in Einstein–Dirac theory 
and the Lie derivative of spinor fields

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Abstract

We characterize the Lie derivative of spinor fields from a variational point of view by resorting to the theory of the Lie derivative of sections of gauge-natural bundles. Noether identities from the gauge-natural invariance of the first variational derivative of the Einstein (–Cartan)–Dirac Lagrangian provide restrictions on the Lie derivative of fields.

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1 Introduction

It is nowadays widely recognized the preminent rôle played by the gauge-natural functorial approach to the geometric description of (classical) field theories [1, 2, 3, 4]. Physical fields are assumed to be geometrically represented by sections of fiber bundles functorially associated with some jets prolongation of the relevant principal bundle by means of left actions of Lie groups on manifolds, usually tensor spaces. Such an approach enables to functorially define the Lie derivative of physical fields with respect to gauge-natural lifts of (prolongations of) infinitesimal principal automorphisms of the underlying principal bundle [5].

This concept generalizes that of the natural lift of an infinitesimal automorphism of the basis manifold to the bundle of higher order frames [3]. The structure group of the total space is generalized to be the semidirect product of a differential group on a Lie group $G$ — the gauge group — which is not in general a subgroup of a differential group. In particular, in the Einstein(-Cartan)-Dirac theory, the coupling between gravitational and fermionic fields requires the use of the concept of a spin-tetrad, which turns out to be a gauge-natural object (see e.g. Refs. [3] and references therein).

However, since this construction involves the enlargement of the class of morphisms of the category, such an approach yields an indeterminacy in the concept of conserved quantities. In fact the vertical components of an infinitesimal principal automorphism are completely independent from the components of its projection on the tangent bundle to the basis manifold. This implies that...
there is a priori no natural way of relating infinitesimal gauge transformations with infinitesimal transformations of the basis manifold, e.g. of space-time (see e.g. in particular Ref. [8]). It is generally believed that such an indeterminacy is somehow unavoidable, and that ad hoc restrictions [10, 11] on the allowed automorphisms of the gauge-natural bundle must be performed, in order to coherently and uniquely define a geometric concept of the Lie derivative of sections of gauge-natural bundles. For a quite exhaustive review see e.g. Refs. [12, 6, 7, 8].

For reasons which will be clear later, due to its invariance with respect to contact structure induced by jets, we consider the geometric framework finite order variational sequences [13] as the most suitable for the definition of the class of infinitesimal automorphisms of the gauge-natural bundle with respect to which the Lie derivative of fields can be defined unambiguously. In particular, in Refs. [15, 18, 17, 19, 20] a variational sequence on gauge-natural bundles was considered, pointing out some important properties of the Lie derivative of sections of bundles [11, 5] and relative consequences on the content of Noether Theorems [21].

We stress the very important fact — although underestimated — that in the category of gauge-natural bundles it is possible to relate the Lie derivative of sections of bundles with the vertical part — *not* the vertical component — of jet prolongations of gauge-natural lifts of infinitesimal principal automorphisms (they coincide up to a sign). The concept of the vertical part of a projectable vector field, together with other important related decompositions induced by the contact structure on jet bundles will be defined in the next Section.

In Ref. [18] for the first time the fact has been pointed out that — when taken as variation vector fields, in order to derive covariantly and canonically conserved Noether currents — such vertical parts are constrained by generalized Jacobi equations. A restriction on the Lie derivative of fields is then immediately derived by the simple request of the covariance of conserved quantities generated by gauge-natural symmetries. We notice that necessary and sufficient conditions (Bergmann-Bianchi identities [22]) for the existence of canonical covariant conserved currents and associated superpotentials can be suitably interpreted as Noether identities [23]. By representing the Noether Theorems [21] in terms of the generalized gauge-natural Jacobi morphism, the Lie derivative of spinor fields is then accordingly characterized as the above mentioned indeterminacy disappears along the kernel of the generalized gauge-natural Jacobi morphism.

### 2 Variational sequences on jets of gauge-natural bundles

Consider a fibered manifold \( \pi : Y \to X \), with \( \dim X = n \) and \( \dim Y = n + m \). For \( s \geq q \geq 0 \) integers we are concerned with the \( s \)-jet space \( J_s Y \) of \( s \)-jet prolongations of (local) sections of \( \pi \); in particular, we set \( J_0 Y \equiv Y \). We recall the natural fiberings \( \pi_s^q : J_s Y \to J_q Y \), \( s \geq q \), \( \pi^s : J_s Y \to X \), and, among these, the affine fiberings \( \pi^s_{s-1} \) [24].
By adopting a multi-index notation, the charts induced on \( J_sY \) are denoted by \((x^\alpha, y^\alpha_1, \ldots, y^\alpha_s)\), with \(0 \leq |\alpha| \leq s\); in particular, we set \( y^\alpha_0 \equiv y^\alpha \). The local vector fields and forms of \( J_sY \) induced by the above coordinates are denoted by \((\partial^\alpha_i)\) and \((d_{\partial^\alpha_i})\), respectively.

Let \( C^s_{s-1}Y \simeq J_sY \times_{J_{s-1}Y} V^*J_{s-1}Y \). For \( s \geq 1 \), we have a natural splitting:

\[
J_sY \times_{J_{s-1}Y} T^*J_{s-1}Y = \left( J_sY \times_{J_{s-1}Y} T^*X \right) \oplus C^s_{s-1}Y.
\]

Given a vector field \( \Xi : J_sY \to TJ_sY \), the splitting (1) yields \( \Xi \circ \pi^{s+1} = \Xi_H + \Xi_V \) where, if \( \Xi = \Xi^\gamma \partial_\gamma + \Xi^\alpha_1 \partial^{\alpha_1} \), then we have \( \Xi_H = \Xi^\gamma D_\gamma \) and the invariant expression \( \Xi_V = (\Xi^\gamma_1 - y^\alpha_1 \Xi^\gamma) \partial^{\alpha_1} \). We shall call \( \Xi_H \) and \( \Xi_V \) the horizontal and the vertical part of \( \Xi \), respectively. The splitting (1) induces also a decomposition of the exterior differential on \( Y \), \((\pi^{s-1}_s)^* \circ d = d_H + d_V \), where \( d_H \) and \( d_V \), the horizontal and vertical differential [14, 24, 25].

Let \( P \to X \) be a principal bundle with structure group \( G \). Let \( r \leq k \) be integers and \( W^{(r,k)}P \) be the gauge-natural prolongation of \( P \) with structure group \( W^{(r,k)}_nG \) [1, 2]. Let \( F \) be any manifold and \( \zeta : W^{(r,k)}_nG \times F \to F \) be a left action of \( W^{(r,k)}_nG \) on \( F \). There is a naturally defined right action of \( W^{(r,k)}_nG \) on \( W^{(r,k)}_nP \times F \) so that we can associate in a standard way to \( W^{(r,k)}_nP \) the gauge-natural bundle of order \((r,k)\) \( Y_\zeta \simeq W^{(r,k)}_nP \times F \) [1, 2].

Let \( A^{(r,k)} = TW^{(r,k)}_nP / W^{(r,k)}_nG \) \((r \leq k)\) be the vector bundle over \( X \) of right invariant infinitesimal automorphisms of \( W^{(r,k)}_nP \). The gauge-natural lift \( \Theta \) functorially associates with any right-invariant local automorphism \((\Phi, \phi)\) of the principal bundle \( W^{(r,k)}_nP \) a unique local automorphism \((\Phi_\zeta, \phi)\) of the associated bundle \( Y_\zeta \). An infinitesimal version can be defined:

\[
\Theta : Y_\zeta \times X \to TY_\zeta : (y, \Xi) \mapsto \tilde{\Xi}(y),
\]

where, for any \( y \in Y_\zeta \), one sets: \( \tilde{\Xi}(y) = \frac{d}{dt}[(\Phi_\zeta \circ t)(y)]_{t=0} \), and \( \Phi_\zeta \circ t \) denotes the (local) flow corresponding to the gauge-natural lift of \( \Phi_t \). This mapping fulfills important linearity properties [4].

Although the jet prolongation up to a given order of a gauge-natural bundle is again a gauge-natural bundle associated with some gauge natural prolongation of the underlying principal bundle [5], in general — as it can be easily seen also from the corresponding local (invariant) expression — the generalized vector field \( j_* \Xi = (j_* \Xi)_V \) is not the gauge-natural lift of some infinitesimal principal automorphism.

Following Ref. [5] we give the definition of the Lie derivative of a section of a gauge-natural bundle. Notice that this object is uniquely and functorially defined by the right invariant vector field \( \Xi \).

**Definition 1** Let \( \gamma \) be a (local) section of \( Y_\zeta \), \( \tilde{\Xi} \in A^{(r,k)} \) and \( \hat{\Xi} \) its gauge-natural lift. We define the **generalized Lie derivative** of \( \gamma \) along the vector field \( \Xi \) to be the (local) section \( \hat{\Xi}_\gamma : X \to VY_\zeta \), given by \( \hat{\Xi}_\gamma = T\gamma \circ \xi - \hat{\Xi} \circ \gamma \).
Remark 1 The Lie derivative of sections is an homomorphism of Lie algebras; furthermore for any local section \( \gamma \) of \( Y_\zeta \), the mapping \( \Xi : \mathcal{L}_{\Xi} \gamma \) is a linear differential operator. As a consequence, for any gauge-natural lift \( \hat{\Xi} \), the fundamental relation hold true:

\[
\hat{\Xi}_V = -\mathcal{L}_{\hat{\Xi}}.
\]

(3)

2.1 Variational derivatives and Noether identities

Let us now introduce variational sequences on gauge-natural bundles. For \( s \geq 0 \) (resp. \( 0 \leq q \leq s \)) we consider the sheaves \( \Lambda^p_s \) of \( p \)-forms on \( J_s Y_\zeta \) (resp. \( \mathcal{H}^p_{(s,q)} \) and \( \mathcal{H}^q_s \) of horizontal forms with respect to the projections \( \pi^p_q \) and \( \pi^0_0 \)). Furthermore, for \( 0 \leq q < s \), let \( \mathcal{C}^p_{(s,q)} \subset \mathcal{H}^p_{(s,q)} \) and \( \mathcal{C}^p_s \subset \mathcal{C}^p_{(s+1,s)} \) be the sheaves of contact forms, i.e. horizontal forms valued into \( C^*_s[Y_\zeta] \). The fibered splitting \( \mathfrak{H} \) yields the sheaf splitting \( \mathcal{H}^p_{(s+1,s)} = \bigoplus_{t=0}^s \mathcal{C}^t_{(s+1,s)} \wedge \mathcal{H}^t_{s+1} \). Let the surjective map \( h \) be the restriction to \( \Lambda^p_s \) of the projection of the above splitting onto the non-trivial summand with the highest value of \( t \). Set \( \Theta^*_s := \ker h + d \ker h \).

The following \( s \)-th order variational sequence associated with the fibered manifold \( Y_\zeta \to X \), where the integer \( I \) depends on the dimension of the fibers of \( Y_\zeta \), is an exact resolution of the constant sheaf \( \mathbb{R} Y_\zeta \) over \( Y_\zeta \):

\[
\begin{array}{cccccccc}
0 & \mathbb{R} Y_\zeta & \to \Lambda^0_s & \mathbin{\xrightarrow{\mathcal{E}_0}} \Lambda^1_s/\Theta^1_s & \mathbin{\xrightarrow{\mathcal{E}_1}} \cdots & \mathbin{\xrightarrow{\mathcal{E}_{I-1}}} \Lambda^I_s/\Theta^I_s & \mathbin{\xrightarrow{\mathcal{E}_I}} \Lambda^{I+1}_s & \mathbin{\xrightarrow{d}} 0 .
\end{array}
\]

For our purposes we refer to the representation of a truncated variational sequence due to Vitolo \cite{25} where \( \Lambda^p_s/\Theta^p_s = C^{p-n}_s \wedge \mathcal{H}^{n-h}_{s+1}/h(d \ker h) \) with \( 0 \leq p \leq n + 2 \). Further developments can be found e.g. in Refs. \cite{14 15 16 18 17 19 20 21}.\cite{25}

Let now \( \eta \in C^1_s \wedge C^1_{(s,0)} \wedge \mathcal{H}^{n-h}_{s+1} \); then there is a unique morphism \( k_\eta \in C^1_{(2s,s)} \odot C^1_{(2s,0)} \wedge \mathcal{H}^{n-h}_{2s+1} \) such that, for all \( \Xi : Y_\zeta \to V Y_\zeta, C^1_s[j_2 \Xi \otimes K_\eta] = E_{j_2 \Xi}, \) where \( E \) is the the generalized Euler–Lagrange form \cite{13}; \( C^1_s \) stands for tensor contraction on the first factor and \( \odot \) denotes inner product.

Let \( \mathcal{L}_{\mathcal{J},\Xi} \) be the variational Lie derivative \cite{14}. The First and the Second Noether Theorem \cite{21} then read as follows (compare with Ref. \cite{22}).

**Theorem 1** Let \( [\alpha] = h(\alpha) \in \mathcal{V}^n_s \). Then we have locally

\[
\mathcal{L}_{\mathcal{J},\Xi} h(\alpha) = \Xi \mathcal{E}_n [h(\alpha)] + d_H (j_2 \Xi \mathcal{V} \mathcal{P}_{dH}[h(\alpha)] + \mathcal{E}[h(\alpha)]),
\]

where \( \mathcal{P}_{dH}[h(\alpha)] \) is the generalized momentum associated with \( h(\alpha) \).

**Theorem 2** Let \( \alpha \in \Lambda^{s+1}_s \). Then we have globally

\[
\mathcal{L}_{\mathcal{J},\Xi} [\alpha] = \mathcal{E}(j_{s+1} \Xi \mathcal{V}) [h(\alpha)] + C^1_s (j_2 \Xi \mathcal{V} \otimes K_{hda}).
\]
Definition 2 We say \( \lambda \) to be a gauge-natural invariant Lagrangian if the gauge-natural lift \((\tilde{\Xi}, \xi)\) of any vector field \( \tilde{\Xi} \in \mathcal{A}^{(r,k)} \) is a symmetry for \( \lambda \), i.e. if \( \mathcal{L}_{j+1}\tilde{\Xi} \lambda = 0 \). In this case the projectable vector field \( \tilde{\Xi} \) is called a gauge-natural symmetry of \( \lambda \).

Let \( \lambda \) be a Lagrangian and let \( \tilde{\Xi}_V \) be considered a variation vector field. Let us set \( \chi(\lambda, \tilde{\Xi}_V) = C_1(\tilde{\Xi}_V \otimes K_{hd}\mathcal{E}_{j+1,\Xi_V} \lambda) = E_{j,\tilde{\Xi}}hd\mathcal{E}_{j+1,\Xi_V} \lambda \). Because of linearity properties of \( K_{hd}\mathcal{E}_{j+1,\Xi_V} \lambda \), by using a global decomposition formula due to Kolář \[29\], we can decompose the morphism defined above as \( \chi(\lambda, \tilde{\Xi}_V) = E_{\chi(\lambda, \tilde{\Xi}_V)} + F_{\chi(\lambda, \tilde{\Xi}_V)} \), where \( F_{\chi(\lambda, \tilde{\Xi}_V)} \) is a local horizontal differential which can be globalized by means of the fixing of a connection; however we will not fix any connection a priori in the present paper. Such a decomposition is a kind of integration by parts, which provides us with a globally defined gauge-natural morphism playing a relevant rôle \[18\].

Definition 3 We call the morphism \( J(\lambda, \tilde{\Xi}_V) = F_{\chi(\lambda, \tilde{\Xi}_V)} \) the gauge-natural generalized Jacobi morphism associated with the Lagrangian \( \lambda \) and the variation vector field \( \tilde{\Xi}_V \).

Such a morphisms has been also represented on finite order variational sequence modulo horizontal differentials \[15\] and thereby proved to be self-adjoint along solutions of the Euler–Lagrange equations, a result already well known for first order field theories \[30\]. The same property has been also proved in finite order variational sequences on gauge-natural bundles \[20\] without quotienting out horizontal differentials.

Because of linearity properties of the Lie derivative of sections of gauge-natural bundles, we can consider the form \( \omega(\lambda, \tilde{\Xi}_V) = -\mathcal{L}_{\tilde{\Xi}}|\mathcal{E}_\nu(\lambda) \) as a new Lagrangian defined on an extended space \( J_{2s}(\mathcal{A}^{(r,k)} \times \mathcal{Y}) \). This Lagrangian plays a very important rôle in the study of conserved quantities. In fact, it is for example remarkable that when \( \omega(\lambda, \tilde{\Xi}_V) \) is an horizontal differential (i.e. a null Lagrangian) from the First Noether Theorem \[1\] we get a conservation law which holds true along any section of the gauge natural bundle (not only along solutions of the Euler–Lagrange equations).

It is also remarkable that the new Lagrangian \( \omega \), in principle, is not gauge-natural invariant. In fact from the gauge-natural invariance of \( \lambda \) we only infer that, for any \( \tilde{\Xi} \), \( \mathcal{L}_{j+1}\tilde{\Xi}[\mathcal{L}_{j+1}\tilde{\Xi}_V \lambda] = \mathcal{L}_{j+1}[\tilde{\Xi}_{\mathcal{E}} \lambda] + \mathcal{L}_{j+1}\tilde{\Xi}_V \mathcal{L}_{j+1}\tilde{\Xi}_V \lambda = \mathcal{L}_{j+1}[\tilde{\Xi}_{\mathcal{E}} \lambda] \) and a priori neither \( [\tilde{\Xi}_{\mathcal{E}}, \tilde{\Xi}_V] = 0 \), nor it is the gauge-natural lift of some infinitesimal principal automorphism. Nevertheless it is still possible to derive some invariance properties of the new Lagrangian \( \omega(\lambda, \tilde{\Xi}_V) \) restricted along the kernel of the gauge-natural generalized gauge-natural Jacobi morphism as well as corresponding Noether conservation laws and Noether identities \[21\].

Let then \( \delta^2 \lambda = \mathcal{L}_{j+1}\tilde{\Xi}_V \mathcal{L}_{j+1}\tilde{\Xi}_V \lambda \) be the gauge-natural second variation of \( \lambda \) taken with respect to vertical parts of gauge-natural lifts of infinitesimal principal automorphisms. First we generalize a classical result \[30\].

Proposition 1 Let \( \tilde{\Xi}_V \in \mathcal{R} \), where \( \mathcal{R} \) is the kernel of the gauge-natural Jacobi
morphism. We have:
\[ L_{j+1} \xi_{[L_{j+1} \xi_V \lambda]} \equiv 0. \]

**Proof.**
\begin{align*}
L_{j+1} \xi_{[L_{j+1} \xi_V \lambda]} &= L_{j+1} \xi_{[L_{j+1} \xi_V \lambda]} + \\
&+ L_{j+1} \xi_V [L_{j+1} \xi_V \lambda] = L_{j+1} \xi_H [L_{j+1} \xi_V \lambda] + \delta^2 G_{\lambda}.
\end{align*}

Theorem 2 implies \( \delta^2 G_{\lambda} = J(\lambda, \hat{\xi}_V) \), thus we get the assertion. \[ \text{QED} \]

**Theorem 3** The existence of canonical covariant conserved currents and superpotentials associated with a gauge-natural invariant Lagrangian is equivalent to the existence of Noether identities associated with the invariance properties of the first variational derivative of the Lagrangian \( \lambda \) taken with respect to vertical parts of gauge-natural lifts lying in \( \mathcal{R} \).

**Proof.** From the above Proposition we see that \( L_{j+1} \xi_{[L_{j+1} \xi_V \lambda]} \equiv 0 \). This last condition means that \( \hat{\xi} \) is a gauge-natural symmetry of the new Lagrangian \( L_{j+1} \xi_V \lambda = \omega(\lambda, \hat{\xi}_V) \). This invariance implies also the existence of Noether identities from the gauge-natural invariance of the Euler–Lagrange morphism \( \mathcal{E}_n(\hat{\xi}_V | \mathcal{E}_n(\lambda)) \) and ultimately from the corresponding invariance properties of the first variational derivative of the Lagrangian \( \lambda \). It is easy to verify that Bergmann-Bianchi identities for the existence of canonical covariant conserved currents and superpotentials associated with the invariance of \( \lambda \) coincide with the condition \( \hat{\xi}_V | \mathcal{E}_n(\hat{\xi}_V | \mathcal{E}_n(\lambda)) = 0 \) (see e.g. Refs. [18, 20] and [15, 30]) equivalent to \( L_{j+1} \xi_{[\mathcal{E}_n(\hat{\xi}_V | \mathcal{E}_n(\lambda))] = 0} \).

**3 Einstein(–Cartan)–Dirac theory**

Let \( X \) be a 4-dimensional manifold admitting Lorentzian structures \((SO(1, 3)^c\)-reductions) and let \( \Lambda \) be the epimorphism which exhibits \( SPIN(1, 3)^c \) as the twofold covering of \( SO(1, 3)^c \) [2, 3].

We recall that a free spin structure on \( X \) is a \( SPIN(1, 3)^c \)-principal bundle \( \pi : \Sigma \to X \) and a bundle map inducing a spin-frame on \( \Sigma \) given by \( \Lambda : \Sigma \to L(X) \) defining a metric \( g \) via the reduced subbundle \( SO(X, g) = \Lambda(\Sigma) \) of \( L(X) \) [3, 7, 8].

Now, let \( \rho \) be the left action of the group \( W^{(1, 0)} SPIN(1, 3)^c \) on the manifold \( GL(4, \mathbb{R}) \) given by \( \rho : ((A, S), \theta) \to \Lambda(S) \circ \theta \circ A^{-1} \) and consider the associated bundle (a gauge-natural bundle of order \((1, 0))
\[ \Sigma_{\rho} \equiv W^{(1, 0) \Sigma} \times_{\rho} GL(4, \mathbb{R}) : \]
the bundle of spin-tetrads \( \theta \) [9].
The induced metric is $g_{\mu\nu} = \theta^a_{\mu} \theta^b_{\nu} \eta_{ab}$, where $\theta^a_{\mu}$ are local components of a spin tetrad $\theta$ and $\eta_{ab}$ the Minkowski metric.

Let $so(1,3) \simeq spin(1,3)$ be the Lie algebra of $SO(1,3)$. One can consider the left action of $W^{(1,1)}$ $spin(1,3)^c$ on the vector space $(\mathbb{R}^4)^* \otimes so(1,3)$.

The associated bundle is a gauge-natural bundle of order $(1,1)$:

$$\Sigma_l \cong W^{(1,1)} \Sigma \times_\Sigma ((\mathbb{R}^4)^* \otimes so(1,3)) \simeq J_1(\Sigma/Z_2)/SO(1,3)^c,$$

the bundle of spin-connections $\omega$.

If $\tilde{\gamma}$ is the linear representation of $spin(1,3)^c$ on the vector space $\mathbb{C}^4$ induced by the choice of matrices $\gamma$ we get a $(0,0)$-gauge-natural bundle

$$\Sigma_\gamma \cong \Sigma \times_{\tilde{\gamma}} \mathbb{C}^4,$$

the bundle of spinors. A spinor connection $\tilde{\omega} = (id \otimes (T\varLambda)^{-1})(\omega)$ - where $T\varLambda$ defines the isomorphism of Lie algebras $spin(1,3) \simeq so(1,3)$ - is locally given by $\tilde{\omega}_a = -\frac{1}{4} \epsilon_{ab} \gamma_{ab}$.

In the following the Einstein–Cartan Lagrangian will be the base preserving morphism $\lambda_{EC} : \Sigma_\mu \times J_1 \Sigma \rightarrow \Lambda^4 T^* \mathbb{X}$, locally given by $\lambda_{EC}(\theta, J^1 \omega) \cong -\frac{1}{2} \Omega_{ab} \wedge e^{ab}$, where $\Omega_{ab}$ is the curvature of $\omega$, $k = \frac{8\pi G}{c^4}$, $e^{ab} = e_a |(e_b)| \epsilon$ and $\epsilon$ is the standard volume form on $\mathbb{X}$. Locally $\epsilon = \det||\theta|| d_0 \wedge \ldots d_3$, $e_a = \epsilon_b \partial_{\mu}$, where $||e^a_{\mu}||$ is the inverse of $||\theta||$.

The Dirac Lagrangian is the base preserving morphism

$\lambda_D : \Sigma_\mu \times \Sigma \times J_1 \Sigma_\gamma \rightarrow \Lambda^4 T^* \mathbb{X}$, locally given by $\lambda_D(\theta, \omega, J^1 \psi) \cong \left( \Lambda^2 (\tilde{\psi} \gamma^a \nabla_a \psi - \nabla_a \tilde{\psi} \gamma^a \psi) - m \tilde{\psi} \psi \right) \epsilon$, wher $\alpha = hc$, $\nabla$ the covariant derivative with respect to a connection on $\Sigma_\mu$, and $\tilde{\psi}$ the adjoint with respect to the standard $spin(1,3)^c$-invariant product on $\mathbb{C}^4$. Under the assumption of a minimal coupling the total Lagrangian of a gravitational field interacting with spinor matter is $\lambda = \lambda_{EC} + \lambda_D$.

Let now $\Xi$ be a $spin(1,3)^c$-invariant vector field on $\Sigma$. The lagrangian $\lambda$ is invariant with respect to any lift $\overline{\Xi}$ of $\Xi$ to the total space of the theory. By the First Noether Theorem the following conserved Noether current has been found in Ref. [31]:

$$\epsilon(\lambda, \Xi) = \xi^b \left( \frac{1}{k} \partial_a T^a_b + T^a_b \epsilon_a + \Xi^a_b \left( \frac{1}{2k} \partial_\nu \epsilon_{ab} - u^c_{ab} \epsilon_c \right) + d_H \left( \frac{1}{2k} \Xi^{ab}_c \epsilon_{ab} \right) \right),$$

where $L_{\Xi} \omega^a_b = \xi |\Omega^a_b + \nabla^{a}_{b} \Xi^b_{c} | \in \mathbb{D} \Xi^a_b, \mathbb{D}$ is the covariant exterior derivative with respect to the connection $\omega$, and $\Xi^a_b = \Xi^{a}_b - \omega^a_{b\mu} \xi^\mu$ is the vertical part of $\Xi$ with respect to $\omega$. The corresponding superpotential is $\nu(\lambda, \Xi) \cong -\frac{1}{2k} \Xi^a_b \epsilon_{ab}$.

### 3.1 Lie derivative of spinor fields

The natural splitting induced by the contact structure provides us with the vertical part $\Xi^a_v = \Xi^a - u^a_{\mu} \xi^\mu$, where $(x^a, u^a)$ are local coordinates on the total gauge-natural bundle and $A$ is a multindex. On the other hand, since
the vertical part with respect to the spinor-connection \( \tilde{\omega} \) is given by

\[
\hat{\Xi}^A_v = \hat{\Xi}^A_v + \tilde{\omega}_A^\mu \xi^\mu.
\]

In the above Section we saw that, without the fixing of a connection a priori, the existence of canonical global conserved quantities in field theory is related with gauge-natural invariant properties of \( -\mathcal{L}_{\Xi} \mid \mathcal{E}_n(\lambda) \) and corresponding Noether identities:

\[
(-1)^{|\sigma|} D_\sigma( D_\mu(-\mathcal{L}_{\Xi}^\mu)^{ab} \partial_{cd}(\partial_{ab}(-\frac{1}{2k} \Omega_{ab} \wedge e^{ab} + 
\left(\frac{i\alpha}{2}(\bar{\psi} \gamma^a \nabla_a \psi - \nabla_a \bar{\psi} \gamma^a \psi) - m \bar{\psi} \psi) \epsilon\right)) - \sum_{|\alpha| = 0}^{s-|\sigma|} (-1)^{|\mu + \alpha|!} \frac{(\mu + \alpha)!}{\mu!}! D_\alpha \partial_{cd}(\partial_{ab}(-\frac{1}{2k} \Omega_{ab} \wedge e^{ab} + \n\left(\frac{i\alpha}{2}(\bar{\psi} \gamma^a \nabla_a \psi - \nabla_a \bar{\psi} \gamma^a \psi) - m \bar{\psi} \psi) \epsilon\right)) = 0,
\]

with \( 0 \leq |\sigma| \leq 1, 0 \leq |\mu| \leq 1 \) and the fibered local coordinates in the total bundle we denote by \((x^\mu, \theta^a_\mu, \omega^{ab}_\mu, \omega^{ab}_{\mu+1}, \psi)\). Such identities imply, after some manipulations, that \( \Xi^a_{\nu} = -\tilde{\nabla}^{[a} \xi^{b]} \) (the so-called Kosmann lift \([10]\)), where \( \tilde{\nabla} \) is the covariant derivative with respect to the standard transposed connection on \( \Sigma_\rho \). On the other hand, the Lie derivative of spinor fields can be expressed in terms of \( \Xi^a_{\nu} \) as follows:

\[
\mathcal{L}_{\Xi}^a \psi = \xi^a e_\alpha \psi + \frac{1}{4} \tilde{\Xi}^{ab}_{\nu} \gamma^a \gamma^b \psi = \xi^a \nabla_a \psi - \frac{1}{4} \nabla_{(a} \xi^{b]} \gamma^a \gamma^b \psi = \\
= \xi^a \partial_{a} \psi + \frac{1}{4} \tilde{\Xi}^{ab}_{K} \gamma^a \gamma^b \psi - \frac{1}{4} \nabla_{(a} \xi^{b]} \gamma^a \gamma^b \psi,
\]

where \( \tilde{\Xi}^a_{K} \) is the horizontal part of \( \Xi \) with respect to the spinor-connection. We see that, because of relation (3), once the expression of \( \Xi^a_{\nu} \) derived by Eq. (5) has been substituted in Eq. (4), we obtain a condition involving the spinor-connection \( \tilde{\omega} \).

This result has been pointed out in Ref. [23]. It agrees with an analogous one obtained in Ref. [12] within a different approach to conservation laws for the Einstein–Cartan theory of gravitation.

In Ref. [6, 7] a geometric interpretation of the Kosmann lift as a reductive lift has been recovered for the definition of a \( SO(1,3)^e \)-reductive Lie derivative of spinor fields. We justify the “naturality” of the Kosmann lift from a variational point of view: it characterizes the only gauge-natural lift which ensures the naturality condition \( \mathcal{L}_{j+1}^a \Xi^a_{\nu} \mid \mathcal{L}_{j+1} \Xi^a_{\nu}, \lambda \equiv 0 \) holds true. Along such a lift not only the initial Lagrangian \( \lambda \) is by assumption invariant, but also its first variational derivative, taken with respect to vertical parts of gauge-natural lifts lying in the kernel of the generalized Jacobi morphism, \( \omega(\lambda, \mathcal{R}) \) is it, thus implying
that either $[\hat{\Xi}_H, \hat{\Xi}_V] = 0$ or it is a gauge-natural lift. The Hamiltonian content of such a naturality condition and implications on conserved quantities have been investigated in Refs. \[32, 19, 23\]. The two approaches are strictly related: in a forthcoming paper \[33\] we study how the kernel of the Jacobi morphism induces a split structure which is also reductive.

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References

[1] D.J. Eck: Gauge-natural bundles and generalized gauge theories, *Mem. Amer. Math. Soc.* 247 (1981) 1–48.

[2] L. Fatibene, M. Ferraris, M. Francaviglia, M. Godina: Gauge formalism for general relativity and fermionic matter, *Gen. Rel. Grav.*, 30(9) (1998) 1371–89.

[3] L. Fatibene, M. Francaviglia: *Natural and gauge natural formalism for classical field theories. A geometric perspective including spinors and gauge theories*; Kluwer Academic Publishers, Dordrecht, 2003.

[4] L. Fatibene, M. Francaviglia, M. Palese: Conservation laws and variational sequences in gauge-natural theories, *Math. Proc. Camb. Phil. Soc.* 130 (2001) 555–569.

[5] I. Kolář, P.W. Michor, J. Slovák: *Natural Operations in Differential Geometry*, (Springer–Verlag, N.Y., 1993).

[6] M. Godina, P. Matteucci: Reductive $G$-structures and Lie derivatives, *J. Geom. Phys.* 47 (1) (2003) 66–86.

[7] M. Godina, P. Matteucci: The Lie derivative of spinor fields: theory and applications, *Int. J. Geom. Methods Mod. Phys.* 2 (2) (2005) 159–188.

[8] P. Matteucci: Einstein-Dirac theory on gauge-natural bundles, *Rep. Math. Phys.* 52 (1) (2003) 115–139.

[9] S. Weinberg: *Gravitation and cosmology: principles and applications of the general theory of relativity*, Wiley, New York, 1972.

[10] Y. Kosmann: Dérivée de Lie de spineurs, *C. R. Acad. Sci. Paris Sér. A-B* 262 (1966) A289–A292; – Dérivée de Lie de spineurs. Applications, *C. R. Acad. Sci. Paris Sér. A-B* 262 (1966) A394–A397.
[11] A. Lichnerowicz: Spineurs harmoniques, *C.R.Acad.Sci.Paris* **257** (1963) 7–9.

[12] M. Ferraris, M. Francaviglia, M. Raiteri: Conserved Quantities from the Equations of Motion (with applications to natural and gauge natural theories of gravitation) *Class. Quant. Grav.* **20** (2003) 4043–4066.

[13] D. Krupka: Variational Sequences on Finite Order Jet Spaces, *Proc. Diff. Geom. and its Appl.* (Brno, 1989); J. Janyška, D. Krupka eds.; World Scientific (Singapore, 1990) 236–254.

[14] M. Francaviglia, M. Palese, R. Vitolo: Symmetries in Finite Order Variational Sequences, *Czech. Math. J.* **52(127)** (2002) 197–213.

[15] M. Francaviglia, M. Palese, R. Vitolo: The Hessian and Jacobi Morphisms for Higher Order Calculus of Variations, *Diff. Geom. Appl.* **22** (1) (2005) 105–120.

[16] M. Kríbek, J. Musilová: Representation of the variational sequence by differential forms, *Rep. Math. Phys.* **51** (2003) 251–258.

[17] M. Palese, E. Winterroth: Covariant gauge-natural conservation laws, *Rep. Math. Phys.* **54** (3) (2004) 349–364.

[18] M. Palese, E. Winterroth: Global Generalized Bianchi Identities for Invariant Variational Problems on Gauge-natural Bundles, *Arch. Math. (Brno)* **41** (3) (2005) 289–310.

[19] M. Palese, E. Winterroth: Gauge-natural field theories and Noether theorems: canonical covariant conserved currents, *Rend. Circ. Mat. Palermo* (2) **Suppl. 79** (2006), 161–174.

[20] M. Palese, E. Winterroth: The relation between the Jacobi morphism and the Hessian in gauge-natural field theories, *Theoret. Math. Phys.* **152**(2) (2007), 1191–1200.

[21] E. Noether: Invariante Variationsprobleme, *Nachr. Ges. Wiss. Göttingen*, *Math. Phys. Kl. II* (1918) 235–257.

[22] J.L. Anderson, P.G. Bergmann: Constraints in Covariant Field Theories, *Phys. Rev.* **83** (5) (1951) 1018–1025.

[23] E.H.K. Winterroth: *Variational derivatives of gauge-natural invariant Lagrangians and conservation laws*, PhD thesis University of Torino, 2007.

[24] D.J. Saunders: *The Geometry of Jet Bundles*, Cambridge Univ. Press (Cambridge, 1989).

[25] R. Vitolo: Finite Order Lagrangian Bicomplexes, *Math. Proc. Camb. Phil. Soc.* **125** (1) (1999) 321–333.
[26] J.F. Pommaret: Spencer Sequence and Variational Sequence, *Acta Appl. Math.* 41 (1995), 285–296.

[27] I. Kolář, R. Vitolo: On the Helmholtz operator for Euler morphisms, *Math. Proc. Cambridge Phil. Soc.*, 135 (2) (2003) 277–290.

[28] A. Trautman: Noether equations and conservation laws, *Comm. Math. Phys.* 6 (1967) 248–261.

[29] I. Kolář: A Geometrical Version of the Higher Order Hamilton Formalism in Fibred Manifolds, *J. Geom. Phys.*, 1 (2) (1984) 127–137.

[30] H. Goldschmidt, S. Sternberg: The Hamilton–Cartan Formalism in the Calculus of Variations, *Ann. Inst. Fourier, Grenoble* 23 (1) (1973) 203–267.

[31] M. Godina, P. Matteucci, J.A. Vickers: Metric-affine gravity and the Nester-Witten 2-form, *J. Geom. Phys.*, 39 (4) (2001) 265–275.

[32] M. Francaviglia, M. Palese, E. Winterroth: Second variational derivative of gauge-natural invariant Lagrangians and conservation laws, *Proc. IX Int. Conf. Diff. Geom. Appl., Prague 2004*, J. Bures et al. eds., (Charles University, 2005) 591–604.

[33] M. Palese, E. Winterroth: Lagrangian reductive structures on gauge-natural bundles, preprint (2007).