On Transiso Graphs of Groups of Order Less Than 32

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(Received March 18, 2016)

Abstract: For a finite group $G$ and a divisor $d$ of $|G|$, the transiso graph $\Gamma_d(G)$ is a graph whose vertices are subgroups of $G$ of order $d$ and two distinct vertices $H_1$ and $H_2$ are adjacent if and only if there exist normalized right transversals $S_1$ and $S_2$ of $H_1$ and $H_2$ respectively in $G$ such that $S_1 \equiv S_2$ with respect to the right loop structure induced on them. In the present paper, we have determined some finite groups $G$ for which the graphs $\Gamma_d(G)$ are complete for each divisor $d$ of $|G|$. We have also discussed the completeness of transiso graphs for groups of order less than 32.

Keywords: Right loop, Normalized right transversal, Transiso graph, t-group.

2010 AMS Classification No.: 05C25, 20N05.

1. Introduction

Let $G$ be a finite group and $H$ be a subgroup of $G$. A normalized right transversal (NRT) $S$ of $H$ in $G$ is a subset of $G$ obtained by selecting one and only one element from each right coset of $H$ in $G$ and $1 \in S$. An NRT $S$ has an induced binary operation $\circ$ given by \( \{x \circ y\} = S \cap H_{xy} \), with respect to which $S$ is a right loop with identity $1$ (Smith¹, p.42, Lal²). Conversely, every right loop can be embedded as an NRT in a group with some universal property (Lal², p.76). Let $\langle S \rangle$ be the subgroup of $G$ generated by $S$ and $H_S$ be the subgroup $H \cap \langle S \rangle$. Then, $H_S = \{xy(x \circ y)^{-1} | x, y \in S\}$ and $H_S S = \langle S \rangle$. Identifying $S$ with the set $H \setminus G$ of all right cosets of $H$ in $G$, we get a

*Author is supported by University Grants Commission, India
transitive permutation representation \( \chi_S : G \rightarrow \text{Sym}(S) \) defined by \( \{ \chi_S(g)(x) = S \cap Hxg : g \in G, x \in S \} \). The kernel \( \ker \chi_S \) of this action is \( \text{Core}_G(H) \), the core of \( H \) in \( G \). The group \( G_S = \chi_S(H_S) \) is known as the group torsion of the right loop \( S \) (Lal\(^2\), p. 75) which depends only on the right loop structure \( \circ \) on \( S \) and not on the subgroup \( H \). Since \( \chi_S \) is injective on \( S \) and if we identify \( S \) with \( \chi(S) \), then \( \chi(S_S \circ S_H) \) is known as the group torsion of the right loop structure on \( S \) and \( S \) is an NRT of \( G_S \) in \( G_S S \). One can also verify that the identity map on \( S \) is \( \chi_S \) and \( \chi_S \) is \( \chi(S_S \circ S_H) \) which also depends only on the right loop structure on \( S \) and \( S \) is an NRT of \( G_S \) in \( G_S S \). Also \( (S, \circ) \) is a group if and only if \( G_S \) is trivial.

Let \( \mathcal{T}(G,H) \) denote the set of all normalized right transversals (NRTs) of \( H \) in \( G \). Two NRTs \( S, T \in \mathcal{T}(G,H) \) are said to be isomorphic (denoted by \( S \cong T \)), if their induced right loop structures are isomorphic. A subgroup \( H \) is normal in \( G \) if and only if all NRTs of \( H \) in \( G \) are isomorphic to the quotient group \( G/H \) (Cameron\(^3\)).

Throughout the paper, we will assume that \( G \) is a finite group and \( d \) is a divisor of the order \(|G|\) of the group \( G \). Let \( V_d(G) \) be the set of all subgroups of \( G \) of order \( d \). We define a graph \( \Gamma_d(G) = (V_d(G), E_d(G)) \) with \( \{H_1, H_2\} \in E_d(G) \) if and only if there exists \( S_i \in \mathcal{T}(G,H_i) \) \((i=1,2)\) such that \( S_1 \cong S_2 \) with respect to the right loop structure induced on \( S_i \). We will call this graph a transiso graph (Kakkar and Mishra\(^4\)). If \( G \) has no subgroup of order \( d \), then \( \Gamma_d(G) \) is a null graph (a graph having empty vertex set and empty edge set). If \( G \) has unique subgroup of order \( d \), then \( \Gamma_d(G) \) is an empty graph (a graph having empty edge set). We will denote transiso graph \( \Gamma_d(G) \) by \( \Gamma_d \) if there is no confusion about \( G \). A group \( G \) is called a t-group if \( \Gamma_d(G) \) is a complete graph for each divisor \( d \) of \(|G|\).

In this paper, we have determined all t-groups of the order less than 32. In the Section 2, we have recalled some preliminary results related to transiso graph from Kakkar and Mishra\(^4\). We have also discussed about the relation of adjacency and proved that the direct product of two t-groups of co-prime order is a t-group. In the Section 3, we have discussed about the
transiso graphs of some non-abelian groups like dicyclic groups, quasidihedral groups and the groups of the order $pq, 4p, 2pq$ and $2p^2$ for distinct odd prime $p$ and $q$. We have classified all the t-groups of order less than 32 in the Section 4.

2. Preliminaries

We first recall the following results of Kakkar and Mishra⁴ and prove some elementary results which will be used in the present paper.

Proposition 1: A subgroup of a group $G$ is always adjacent with its automorphic images in $\Gamma_d(G)$ for any divisor $d$ of $|G|$. 

Proposition 2: Let $H_1$ and $H_2$ be corefree subgroups of $G$. Let $S_i \in \mathcal{I}(G, H_i) \ (i=1,2)$ such that $S_1 \cong S_2$ and $\langle S_i \rangle = G$. Then, an isomorphism between $S_1$ and $S_2$ can be extended to an automorphism of $G$ which sends $H_1$ onto $H_2$.

Proposition 3: A finite abelian group $G$ is a t-group if and only if each Sylow subgroup of $G$ is either elementary abelian or cyclic.

Corollary 1: An elementary abelian group is a t-group.

Proposition 4: The dihedral group $D_{2n}$ of order $2n$ is a t-group.

One can easily observe that the number of vertices in the graph $\Gamma_d(G)$ is equal to the number of subgroups of order $d$ and is given by

$$|V_d(D_{2n})| = \begin{cases} 
1 & \text{if } d \text{ is odd.} \\
\frac{2n}{d} & \text{if } d \text{ is even and does not divide } n. \\
\frac{2n}{d} + 1 & \text{if } d \text{ is even and divides } n.
\end{cases}$$

Proposition 5: Let $G$ be a non $p$-central finite $p$-group. Then, $\Gamma_d(G)$ is complete if and only if whenever $H$ is a non-normal subgroup of $G$ of order $p$, $G \cong H \times K$ for some subgroup $K$ of $G$ with $G/L \cong K$ for any normal subgroup $L$ of $G$ of order $p$. 


**Proposition 6:** Let \( p \) be an odd prime and \( G \) be a non-abelian group. Then,

1. If the group \( G \) is a t-group and \( |G| = p^3 \), then \( G \) is of exponent \( P \) (and hence \( G \cong C_{p^2} \times C_p \)).

2. If \( |G| = p^4 \), then \( \Gamma_p(G) \) is not a complete graph.

3. If \( |G| = p^5 \), then \( \Gamma_p(G) \) is not complete unless \( \Phi(G) = Z(G) = G' \cong C_p^2 \).

Let \( G \) be a finite group and \( d \) be a divisor of \( |G| \). Let us define a relation \( \sim_d \) on the set \( V_d(G) \) of all subgroups of the group \( G \) of order \( d \) such that two subgroups \( H_1 \) and \( H_2 \) are related by the relation \( \sim_d \) if either \( H_1 = H_2 \) or \( H_1 \) and \( H_2 \) are adjacent in the graph \( \Gamma_p(G) \). We call this relation \( \sim_d \) the relation of adjacency in the graph \( \Gamma_p(G) \). It is trivial that the relation \( \sim_d \) is reflexive and symmetric on \( V_d(G) \).

**Proposition 2.1:** If the relation \( \sim_d \) defined above is a transitive relation on \( V_d(G) \), then \( \Gamma_p(G) \) is either a complete graph or a disjoint union of complete graphs.

**Proof:** Assume that the relation \( \sim_d \) is a transitive relation on \( V_d(G) \). Then, it is an equivalence relation on \( V_d(G) \) and hence it gives a partition of \( V_d(G) \) and each component of this partition corresponds to a complete graph.

**Lemma 2.1:** Let \( H_i \) and \( K_i \) (\( i = 1, 2 \)) be subgroups of the groups \( G_i \) such that there exist NRTs \( S_i \in \mathcal{T}(G, H_i) \) and \( T_i \in \mathcal{T}(G, H_i) \) with \( S_i \cong T_i \). Then, \( S_1 \times S_2 \cong T_1 \times T_2 \).

**Proof:** One can easily observe that \( S_1 \times S_2 \in \mathcal{T}(G_1 \times G_2, H_1 \times H_2) \), for an element \( (g_1, g_2) \in G_1 \times G_2 \) can be expressed as \( (h_1, h_2)(s_1, s_2) \), where \( h_i \in H_i \) and \( s_i \in S_i \) (\( i = 1, 2 \)). Similarly, \( T_1 \times T_2 \in \mathcal{T}(G_1 \times G_2, K_1 \times K_2) \). Then, the map \( f \times g : S_1 \times S_2 \to T_1 \times T_2 \) given
by \((s_1,s_2)\in(f(s_1),g(s_2))\), is a right loop isomorphism where \(f : S_1 \rightarrow T_1\)
and \(g : S_2 \rightarrow T_2\) are right loop isomorphisms.

**Proposition 2.2:** The direct product of two t-groups of co-prime order is a t-group.

**Proof:** Let \(G_1\) and \(G_2\) be two t-groups of co-prime order. Let \(G = G_1 \times G_2\) and \(H, K\) be subgroups of \(G\) of same order. Then by [Suzuki, p. 141], \(H = H_1 \times H_2\) and \(K = K_1 \times K_2\) for some subgroups \(H_1, K_1 \in G_1\) and \(H_2, K_2 \in G_2\) such that \(|H_1| = |K_1| = d_1\) and \(|H_2| = |K_2| = d_2\). Since \(G_1\) and \(G_2\) are t-groups, \(H_1 \sim_{d_1} K_1\) and \(H_2 \sim_{d_2} K_2\). Therefore by Lemma 2.1, the subgroups \(H\) and \(K\) are adjacent in the corresponding transiso graph. Hence the group \(G\) is also a t-group.

**Lemma 2.2:** Let \(G\) be a finite group and \(H\) be a non-normal subgroup of prime order. Then, an NRT \(S\) of \(H\) in \(G\) is either a subgroup of \(G\) or \(S \cong H\).

**Proof:** Let \(S\) be an NRT of \(H\) in \(G\). Then, either \(H_S = \{1\}\) or \(H_S = H\). If \(H_S = \{1\}\), then \(S\) is a subgroup of \(G\). Now, assume that \(H_S = H\). Since \(H\) is core-free, \(G_S \cong H_S\). We also observe that \(S\) is not a group in this case.

### 3. Transiso Graphs for Some Non-Abelian Groups

In this section, we have determined transiso graphs for some non-abelian groups like dicyclic groups, quasidihedral groups and the groups of the order \(pq, 4p, 2pq\) and \(2p^2\) for distinct odd primes \(p\) and \(q\). The dicyclic group (or binary dihedral group) \(Q_{4n} = \langle a, b | a^{2n}, a^n b^2, abab^{-1} \rangle\) is a group of order \(4n\) for \(n \geq 1\) (Roman, p. 347). It is a non-abelian group for \(n > 1\) and it is a cyclic group for \(n = 1\) (that is, \(Q_4 \cong C_4\)). A generalized quaternion group is a special case of the dicyclic group \(Q_{4n}\) when \(n = 2^k\) for some positive integer \(k\).

In order to prove the Proposition 3.1, we need the following elementary lemma.
Lemma 3.1: A subgroup of the dicyclic group $Q_{4n}$ is either cyclic or dicyclic. Moreover, if $d$ is a divisor of $4n$, then

1. there is unique subgroup (namely $\left\langle a^{\frac{2n}{d}} \right\rangle$) of $Q_{4n}$ of order $d$ if $4$ does not divide $d$,

2. there are $i$ subgroups $\left\langle a^i b \right\rangle, 0 \leq j < i$ of order $d$ conjugate to each other if $4$ divides $d$ and $i = \frac{4n}{d}$ is odd,

3. a subgroup of order $d$ is either $\left\langle a^i \right\rangle$ or conjugate to one of $\left\langle a^i, b \right\rangle$ or $\left\langle a^i, ab \right\rangle$ if $4$ divides $d$ and $i = \frac{4n}{d}$ is even.

Proof: Let $H$ be a nontrivial proper subgroup of $Q_{4n}$ of order $d$. Clearly $\langle a \rangle$ is maximal cyclic subgroup of $Q_{4n}$ of index 2. The composite homomorphism $H \rightarrow Q_{4n} \rightarrow Q_{4n}/\langle a \rangle$ is either trivial or onto with the kernel $H \cap \langle a \rangle = \left\langle a^i \right\rangle$ for unique divisor $i$ of $2n$. If the homomorphism is trivial, then $H \cap \langle a \rangle = \left\langle a^i \right\rangle$ for unique divisor $i = \frac{4n}{d}$ of $2n$. Therefore the subgroup $H$ is cyclic in this case.

Now, if the homomorphism is onto, then $H / \left\langle a^i \right\rangle \cong Q_{4n} / \langle a \rangle \cong C_2$. Since $H \not\subseteq \langle a \rangle$, $H$ has an element $a^i b$ and $a^n \subseteq \left\langle a^i \right\rangle$ for $\left( a^i b \right)^2 = a^n \in H$.

Therefore $H \cap \langle a \rangle = \left\langle a^i \right\rangle$ for unique divisor $i = \frac{4n}{d}$ of $n$. Now, we have an appropriate element $a^i b \in H \setminus \langle a \rangle$ where $0 \leq j < i$, such that $H = \left\langle a^i, a^j b \right\rangle$.

Clearly $H$ is a dicyclic group \[ \text{precisely } H \cong Q_{\frac{4n}{d}} \] for \( \left( a^i \right)^{\frac{4n}{d}} = 1, \left( a^i \right)^{\frac{4n}{d}} = \left( a^j b \right)^2 \) and \( \left( a^j b \right) a^i \left( a^j b \right)^{-1} = \left( a^i \right)^{-1} \).

Now, we prove the next part of the lemma.

Let $H$ be a subgroup of $Q_{4n}$ of order $d$ and $i = \frac{4n}{d}$. If $d$ is not a multiple of 4, then there is no subgroup of $Q_{4n}$ of order $d$ which is dicyclic.
and so $H = \langle a^i \rangle$ is a cyclic subgroup. If $d$ is a multiple of 4, then there are two cases.

If $d \nmid 2n$ i.e. $i$ is odd, then $H$ cannot be contained in $\langle a \rangle$ so $H$ is dicyclic subgroup of the form $\langle a', a'b \rangle$. If $i \leq j$, then we can find $l$ such that $0 \leq l < i$ and $H = \langle a', a'b \rangle$. Thus we conclude that $0 \leq j < i$ and hence there are $i$ subgroups of order $d$ which are conjugates.

If $d \mid 2n$ i.e., $i$ is even, then $H$ is either $\langle a^i \rangle$ or of the form $\langle a', a'b \rangle$.

Using above arguments, we see that there are $\frac{i}{2}$ subgroups conjugate to $\langle a', b \rangle$ and $\frac{i}{2}$ subgroups conjugate to $\langle a', ab \rangle$.

One can easily observe that an abelian normal subgroup of the group $Q_{4n}$ is cyclic subgroup contained in the maximal cyclic subgroup and a non-abelian normal subgroup of $Q_{4n}$ has index less than or equal to 2.

**Proposition 3.1:** The dicyclic group $Q_{4n} = \langle a, b \mid a^{2n}, a^i b^2, abab^{-1} \rangle$ of order $4n$ is a t-group.

**Proof:** Let $d$ be a divisor of $4n$ and $i = \frac{4n}{d}$.

First assume that $4 \nmid d$. Then by Lemma 3.1, there is unique subgroup of $Q_{4n}$ of order $d$ and so $\Gamma_d(Q_{4n})$ is trivially a complete graph.

Now assume that $4 \mid d$ and $i$ is odd. Then by Lemma 3.1, there are $i$ subgroups of order $d$ conjugate to $\langle a', b \rangle$ and so $\Gamma_d(Q_{4n})$ is a complete graph.

Finally assume that $4 \mid d$ and $i$ is even. Then, a subgroup of order $d$ is either $H_1 = \langle a^i \rangle$ or conjugate to exactly one of $H_2 = \langle a', b \rangle$ or $H_3 = \langle a', ab \rangle$. Note that $H_1$ is a normal subgroup of $Q_{4n}$ and so its all NRTs are isomorphic to $Q_{4n}/H_1 \cong D_{\frac{4n}{2}}$. 


Now, choose \( S_2 = \left\{ a^{2j+k}b^k \mid 0 \leq j < \frac{i}{2}, k = 0,1 \right\} \) in \( \mathcal{T}(Q_{4n}, H_2) \) and \( S_3 = \left\{ a^{2j}b^k \mid 0 \leq j < \frac{i}{2}, k = 0,1 \right\} \) in \( \mathcal{T}(Q_{4n}, H_3) \). Note that \( \langle S_2 \rangle = \langle a^2, ab \rangle \) and \( \langle S_3 \rangle = \langle a^2, b \rangle \). Then, \( H_{S_2} = \langle S_2 \rangle \cap H_2 = \langle d' \rangle \leq \langle S_2 \rangle \) and \( H_{S_3} = \langle S_3 \rangle \cap H_3 = \langle d' \rangle \leq \langle S_3 \rangle \). Therefore \( G_{S_2} = G_{S_3} = \{1\} \) and hence \( S_2 \) and \( S_3 \) are groups.

Let \( \circ_2 \) denote the induced binary operation on \( S_2 \) as described in the Section 1. One can observe that, \( (a^2)^{\frac{i}{2}} = (ab)^2 = (ab \circ_2 a^2)^2 = 1 \). This implies that \( S_2 \cong D_{2^n} \). One can similarly observe that \( S_3 \cong D_{2^n} \). This shows that the graph \( \Gamma_{d}(Q_{4n}) \) is complete.

It follows from the Lemma 3.1 that the number of vertices in the graph \( \Gamma_{d}(Q_{4n}) \) is given by

\[
|V_d(Q_{4n})| = \begin{cases} 
1 & \text{if 4 does not divide } d. \\
\frac{4n}{d} & \text{if 4 divides } d \text{ and } \frac{4n}{d} \text{ is odd.} \\
\frac{4n}{d} + 1 & \text{if 4 divides } d \text{ and } \frac{4n}{d} \text{ is even.}
\end{cases}
\]

The quasidihedral (or semidihedral) group \( QD_{2^n} = \langle a,b \mid a^{2^{n-1}}, b^2, bab^{-1}a^{2^{n-2}+1} \rangle \) is a non-abelian group of order \( 2^n \) where \( n > 4 \) (Gorenstein\(^7\), p. 191). Its subgroup structure can be given by the following lemma.

**Lemma 3.2:** A proper nontrivial subgroup of the quasidihedral group \( QD_{2^n} \) is either cyclic or dihedral or generalized quaternion.

**Proof:** The proof is similar to that of the Lemma 3.1. From theorem 4.10 of Gorenstein\(^7\) (p. 199), it follows that an abelian normal subgroup of the quasidihedral group \( QD_{2^n} \) of order \( d = 2^m \) is cyclic (precisely \( \left\langle a^{2^{n-m-1}} \right\rangle \)) and a non-abelian normal subgroup of \( QD_{2^n} \) has index less than or equal to 2.
Now, we have the following proposition from which it follows that the quasidihedral group $QD_{2^n}$ is not a t-group.

**Proposition 3.2**: Let $G$ be the quasidihedral group $QD_{2^n}$ and $d = 2^m$ be a divisor of $2^n$. Then, the graph $\Gamma_d(G)$ is complete if and only if $d \neq 2$.

**Proof**: First assume that $d \neq 2$. Then by Lemma 3.2, a subgroup of $G$ of order $d = 2^m$ is either $H_1 = \langle a^{2^{n-m-1}} \rangle \cong C_{2^m}$ or conjugate to exactly one of $H_2 = \langle a^{2^{n-m}}, b \rangle$ or $H_3 = \langle a^{2^{n-m}}, ab \rangle$. Note that $H_1$ is a normal subgroup of $QD_{2^n}$ and so its all NRTs are isomorphic to $QD_{2^n}/H_1 \cong D_{2^{n-m}}$.

Now choose $S_2 = \{a^{2^{j+k}}b^k \mid 0 \leq j < 2^{n-m-1}, k = 0, 1\}$ in $\mathcal{T}(QD_{2^n}, H_2)$ and $S_3 = \{a^{2^{j+k}}b^k \mid 0 \leq j < 2^{n-m-1}, k = 0, 1\}$ in $\mathcal{T}(QD_{2^n}, H_2)$. Note that $\langle S_2 \rangle = \langle a^2, ab \rangle$ and $\langle S_3 \rangle = \langle a^2, b \rangle$. Then, $H_{S_2} = \langle S_2 \rangle \cap H_2 = \langle a^{2^{n-m}} \rangle \leq \langle S_2 \rangle$ and $H_{S_3} = \langle S_3 \rangle \cap H_3 = \langle a^{2^{n-m}} \rangle \leq \langle S_3 \rangle$. Therefore $G_{S_2} = G_{S_3} = \{1\}$ and hence $S_2$ and $S_3$ are groups.

Let $\circ_2$ denote the induced binary operation on $S_2$ as described in the Section 1. One can observe that, $(a^2)^{2^{n-m-1}} = (ab)^2 = (ab \circ_2 a^2)^2 = 1$. This implies that $S_2 \cong D_{2^{n-m}}$. One can similarly observe that $S_3 \cong D_{2^{n-m}}$. This shows that the graph $\Gamma_d(QD_{2^n})$ is complete.

Finally assume that $d = 2$. Then, a subgroup of $G$ of order 2 is either $H_1 = \langle a^{2^{n-2}} \rangle$ or a conjugate to $H_2 = \langle b \rangle$. Since $H_1 \not\leq G$, every NRT of $H_1$ in $G$ is isomorphic to $G/H_1 \cong D_{2^{n-1}}$.

Let $H$ be a non-normal subgroup of $QD_{2^n}$ of order 2. Then, $H$ is contained in $\langle a^2, b \rangle \cong D_{2^{n-1}}$ and $H$ is a conjugate to the subgroup $\langle b \rangle$. Clearly the core $Core_G(H)$ of $H$ in $QD_{2^n}$ is trivial. Now let $S$ be an NRT of $H$ in $QD_{2^n}$. Then, the order of $H_S = H \cap \langle S \rangle$ is less than or equal to 2.
If $|H_S|=1$, then $S=\langle S \rangle$ is a subgroup of $QD_{2^n}$. Therefore $S$ is equal to either $<a>$ or $\langle a^2, ab \rangle \cong Q_{2^{n-1}}$.

Finally if $|H_S|=2$, then $H_S=H$ and $\langle S \rangle=G$. Therefore, $G_S \cong H_S/\text{Core}_{H_S}(H_S) = H / \text{Core}_G(H) \cong H$. Since $G_S$ is nontrivial, $S$ is not a group. Hence $S \not\cong D_{2^{n-1}}$.

It can be trivially observed that the number of vertices in the graph $\Gamma_d(QD_{2^n})$ is equal to the number of subgroups of $QD_{2^n}$ of order $d$ and is given by

$$V_d(QD_{2^n}) = \begin{cases} 
1 & \text{if } d=1 \text{ or } d=2^n. \\
2^{n-2}+1 & \text{if } d=2. \\
2^{n-m}+1 & \text{if } d=2^m \text{ with } 0 < m < n.
\end{cases}$$

**Proposition 3.3:** Let $p$ and $q$ be distinct odd primes. Then, a group of order either $pq$ or $4p$ or $2pq$ is t-group.

**Proof:** Observe that a nontrivial proper subgroup of a group of order $pq$ is a Sylow subgroup. Hence any two subgroups of same order are adjacent in corresponding transiso graph.

By classification of groups of order $4p$ (Burnside\(^8\), p.132-137), a non-abelian group of order $4p$ is isomorphic to exactly one of $D_{4n}$, $Q_{4n}$, the alternating group $Alt(4)$ (for $p=3$), $C_p \times C_4$ (for $p \equiv 1 \pmod{4}$). The groups $D_{4n}$ and $Q_{4n}$ are t-groups from the propositions 4 and 3.1. Since any two subgroups of the group $Alt(4)$ of equal order are conjugate therefore the group $Alt(4)$ is also a t-group.

Let $H_1$ and $H_2$ be two distinct subgroups of $C_p \times C_4$ of order 2. Then, there exist unique Sylow 2-subgroup $K_i$ of $C_p \times C_4$ containing $H_i$ where $i=1,2$. Since $K_1$ and $K_2$ are conjugate, the subgroups $H_1$ and $H_2$ are conjugate. So $H_1$ and $H_2$ are adjacent in $\Gamma_2(C_p \times C_4)$.

A non-abelian group of order $2pq$ is isomorphic to exactly one of the groups $D_{2pq}$, $D_q \times C_p$, $D_p \times C_q$ and $C_2 \times (C_q \times C_p)$, $(C_q \times C_p) \times C_2$ (when
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\( p \) divides \( q - 1 \) (Ghorbani and Larki\(^8\), p. 50). \( D_{2q} \times C_p, D_{2p} \times C_q \) and \( C_2 \times (C_q \times C_p) \) are t-groups due to the Proposition 2.2. Order of the normalizer \( N_G(H) \) of a Sylow \( p \) -subgroup \( H \) of \( (C_q \times C_p) \times C_2 \) is \( 2p \) and \( H \) is unique Sylow \( p \) -subgroup of \( N_G(H) \). Since all Sylow \( p \) -subgroups are conjugate; therefore their normalizers are also conjugate.

**Proposition 3.4:** Let \( G \) be a non-abelian group of order \( 2p^2 \) for some odd prime \( p \). Then, the group \( G \) is t-group if and only if \( G \) is isomorphic to either the dihedral group \( D_{2p^2} \) or \((C_p)^2 \times C_2\).

**Proof:** It is well known that a non-abelian group of order \( 2p^2 \) is isomorphic to exactly one of the groups \( D_{2p^2}, (C_p)^2 \times C_2 \) and \( C_p \times D_{2p} \) (Burnside\(^8\), p.132-137).

Let \( G = \langle a, b, c | a^p, b^p, c^2, [a, b], (ac)^2, (bc)^2 \rangle \cong (C_p)^2 \times C_2 \). Then, all subgroups of \( \langle a, b \rangle \cong (C_p)^2 \) are normal in \( G \) and their quotients are dihedral groups \( D_{2p} \). Hence \( \Gamma_p(G) \) is a complete graph. Now \( \Gamma_{2p}(G) \) is also complete as there are several NRTs of a subgroup \( H \) of \( G \) order \( 2p \) which are isomorphic to the cyclic group of order \( p \). So \( G \) is a t-group.

Now, let \( G \cong C_p \times D_{2p} = \langle a, b, c | a^p, b^p, c^2, [a, b], [a, c], (bc)^2 \rangle \). Then, it is obvious that \( < a > \) and \( < b > \) are normal subgroups of \( G \) of order \( p \) such that \( G/ < a > \cong D_{2p} \) and \( G/ < b > \cong C_{2p} \). Hence \( \Gamma_p(G) \) is not a complete graph.

4. Classification of T-Groups of Order Less Than 32

Abelian t-groups are already determined by Proposition 3 which tells that a finite abelian group \( G \) is a t-group if and only if it is isomorphic to the direct sum of a cyclic group \( C \) and a direct sum \( A \) of some elementary abelian groups, where \(|A|\) and \(|C|\) are co-prime.

Non-abelian groups of the order 12, 20, 21, 28 and 30 are t-groups by Proposition 3.3 and a non-abelian t-group of the order 18 can be determined by Proposition 3.4. By Propositions 3.1 and 4, it is clear that the non-abelian
groups of order 8 and $2p$ (for odd prime $p \leq 13$) are t-groups. In Propositions 4.1 and 4.2, we have determined non-abelian t-groups of the order 16 and 24 respectively. We recall that a finite $p$-group $P$ is $p$-central if each subgroup of $P$ of order $p$ is contained in the center $Z(P)$.

**Proposition 4.1:** Let $G$ be a non-abelian group of order 16. Then, the group $G$ is a t-group if and only if $G$ is isomorphic to either dihedral group $D_{16}$ or dicyclic group $Q_{16}$.

**Proof:** If $G$ is a 2-central group, then it is isomorphic to one of the groups $Q_8, C_4 \rtimes C_4$ and $C_2 \times Q_8$ (Wild). By Proposition 3.1, $Q_{16}$ is a t-group. The group $C_4 \rtimes C_4 = \langle a, b \mid a^4, b^4, abab^{-1} \rangle$ has three normal subgroups $\langle a^2 \rangle, \langle b^2 \rangle$ and $\langle a^2 b^2 \rangle$ of order 2 with quotient groups isomorphic to the groups $C_4 \times C_2, D_8$ and $Q_8$ respectively. Therefore the graph $\Gamma_2(C_4 \rtimes C_4)$ is not complete and hence $C_4 \rtimes C_4$ is not a t-group. The group $C_2 \times Q_8 = \langle a, b, c \mid a^2, b^4, b^2 c^2, [a, b], [a, c], bcbc^{-1} \rangle$ is not a t-group, for it has three normal subgroups $\langle a \rangle, \langle b^2 \rangle$ and $\langle ab \rangle$ of order 2 with quotient groups isomorphic to the groups $Q_8, (C_2)^3$ and $Q_8$ respectively. Therefore $C_4 \rtimes C_4$ is not a t-group.

If $G$ is a non 2-central group which is also a t-group, then $\Gamma_2(G)$ is a complete graph and hence by Proposition 5, $G$ should be isomorphic to a nontrivial semidirect product $H \rtimes K$ of a non-normal subgroup $H$ of $G$ of order 2 and a normal subgroup $K$ of $G$ of order 8 such that for any normal subgroup $L$ of $G$ of order 2, $K$ is isomorphic to $G/L$. From a result of Wild we observe that there are five groups $(C_4 \times C_2) \rtimes_1 C_2, C_8 \rtimes C_2, QD_{16} = C_8 \rtimes_1 C_2, D_{16} = D_8 \rtimes C_2$ and $(C_4 \times C_2) \rtimes_2 C_2$ of required semidirect product type. Proposition 4 asserts that the group $D_{16}$ is a t-group and the group $QD_{16}$ is not a t-group by Proposition 3.2. The groups $(C_4 \times C_2) \rtimes_1 C_2, C_8 \rtimes C_2$ and $(C_4 \times C_2) \rtimes_2 C_2$ have normal subgroups of order 2 such that corresponding quotient groups are isomorphic to $D_8, C_4 \times C_2$ and $(C_2)^3$ respectively. Therefore these groups are not t-groups.
Lemma 4.1: Let \( G \) be the group \( C_2 \times \text{Alt}(4) \). Then, the graph \( \Gamma_2(G) \) is not a complete graph.

Proof: First note that \( N = C_2 \times \{1\} \) is a normal subgroup of \( G = C_2 \times \text{Alt}(4) \) of order 2, where \( I \) is the identity element of \( \text{Alt}(4) \) and every NRT of \( N \) in \( G \) is isomorphic to \( G/N \cong \text{Alt}(4) \).

Now, choose a non-normal subgroup \( H \) of \( G \) of order 2 which is contained the subgroup \( C_2 \times \text{Alt}(4) \) of \( G \).

Let \( S \) be an NRT of \( H \) in \( G \). Note that \( S' = S \cap (C_2 \times \text{Alt}(4)) \) is an NRT of \( H \) in \( C_2 \times \text{Alt}(4) \) and \( \langle S' \rangle = C_2 \times \text{Alt}(4) \). Hence by Lemma 2.2, \( S \) can not be a group. Thus, the subgroups \( H \) and \( N \) are not adjacent in the graph \( \Gamma_2(G) \), that is, the graph \( \Gamma_2(G) \) is not complete.

Proposition 4.2: Let \( G \) be a non-abelian group of order 24. Then, the group \( G \) is a t-group if and only if \( G \) is isomorphic to a semidirect product of two t-groups of co-prime order except the groups \( C_2 \times \text{Alt}(4) \) and \( (C_2 \times C_6) \times C_2 \).

Proof: We know that there are 12 non isomorphic non-abelian groups of order 24 (Burnside\(^8\), p.101-104) and 9 of them are semidirect product of two t-groups of co-prime order.

It is obvious that the groups \( C_3 \times C_8 \) and \( SL(2,3) \) are t-groups, for any two subgroups of respective groups of equal order are conjugate. The groups \( Q_{24} \) and \( D_{24} \) are also t-groups by Propositions 3.1 and 4 respectively. By Proposition 2.2, we see that the groups \( C_3 \times D_8, \ C_3 \times Q_8 \) and \( C_2 \times D_{12} \cong (C_2)^2 \times D_6 \) are t-groups. It is clear from example 2.2 of Kakkar and Mishra\(^4\) that the symmetric group \( \text{Sym}(4) \) is not a t-group. One can observe that \( \langle a^2 \rangle \times \text{Alt}(3) \cong C_6 \) and \( \{1\} \times \text{Sym}(3) \) are normal subgroups of the group \( <a> \times \text{Sym}(3) \cong C_4 \times D_6 \) such that their quotient groups are \( (C_2)^2 \) and \( C_4 \) respectively. So \( \Gamma_6(C_4 \times D_6) \) is not a complete graph and hence the group \( C_4 \times D_6 \) is not a t-group. Similarly \( C_2 \times D_{12} \) is not a t-group since there are two normal subgroups \( C_2 \times \{1\} \) and \( \{1\} \times Z(D_{12}) \) of order 2 such that their quotient groups are \( D_{12} \) and \( Q_{12} \).
Now, consider $G = (C_2 \times C_6) \times C_2$. It has a normal subgroup $H$ of order 2 such that $G / H \cong D_{12}$. Let $K$ be a subgroup of $G$ of order 2 contained in the subgroup isomorphic to $D_{12}$. Then, there is no NRT $S \in \mathcal{T}(G,H)$ such that $S = D_{12}$, for otherwise $S = \langle S \rangle$ and $S \cap H = H$ which contradicts the fact that $S$ is an NRT. Therefore the group $(C_2 \times C_6) \times C_2$ is not a t-group. Finally by Lemma 2.2, the group $C_2 \times \text{Alt}(4)$ is not a t-group.

Acknowledgement

Authors are thankful to Prof. R. P. Shukla, Department of Mathematics, University of Allahabad, India and Dr. Vipul Kakkar, School of Mathematics, Harish-Chandra Research Institute, Allahabad, India for suggesting this problem and their valuable discussions.

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