The equivalence problem for KDV-type equations

Mostafa Hesamiarshad and Mehdi Nadjafikhah
Department of Mathematics, College of Basic Sciences, Karaj Branch, Islamic Azad
University, Alborz, Iran
E-mail: m.hesami@tuyiau.ac.ir and m–nadjafikhah@iust.ac.ir

Abstract. The moving coframe method has been applied to solve the local equivalence problem for KDV-type equations in two independent variables under the action of a pseudo-group of contact transformations. The structure equations, the complete sets of differential invariants for symmetry groups and equivalent conditions of these equations are found.

Keywords: Contact transformations, Equivalence problem, KDV-type equations, Moving coframe.
Introduction

In the beginning of twentieth century, Elie Cartan developed a uniform method for analyzing the differential invariants of many geometric structures, nowadays called the ‘Cartan equivalence method’. Also, the method of equivalence is a systematic procedure that allows one to decide whether two systems of differential equations can be mapped one to another by a transformation taken in a given pseudo-group. Later, C. Ehresmann and S.S. Chern introduced two important concepts to the method of equivalence: jets spaces and G-structures. In recent years, with the help of mathematical software, many authors have successfully applied the method of equivalence to many interesting problems: classifications of differential equations [2, 10, 11], holonomy groups [8], inverse variational problems [9] and general relativity [12, 13].

In this paper, we consider a local equivalence problem for the class of equations

\[ u_{xxx} = u_t + Q(u, u_x) \]  

under the contact transformation of a pseudo-group. Equation (1) is the standard Korteweg-de Vries (KDV) equation if \( Q = uu_x \), it is the Modified KDV-equation if \( Q = u^2u_x \), and it is generalized KDV equation if \( Q = h(u)u_x \). Two equations are said to be equivalent if there exists a contact transformation maps one equation to another. We use Elie Cartan’s method of equivalence [1, 6, 7] which in form developed by Fels and Olver [2, 3] and as stated by Morozov [4] to compute the Maurer-Cartan forms, structure equations and basic invariants for symmetry groups of equations. Cartan’s solution to the equivalence problem states that two equations are (locally) equivalent if and only if Cartan’s test satisfied and essential torsion coefficients in the structure equations are constant or their classifying manifolds (locally) overlap.

The symmetry classification problem for classes of differential equations is closely related to the problem of local equivalence: symmetry groups of two equations are necessarily isomorphic if these equations are equivalent while, in general the converse of this issue is not true. For the symmetry analysis of (1) the reader is referred to [5].

1. Pseudo-group of contact transformations of differential equations

In this section we describe the local equivalence problem for differential equations under the action of the pseudo group of contact transformations. Two equations are said to be equivalent if there exists a contact transformation which maps the equations to each other. We apply Elie Cartan’s structure theory of Lie pseudo-groups to obtain necessary and sufficient conditions under which equivalence mappings can be found. This theory describes a Lie pseudo-group in terms of a set of invariant differential 1-forms called Maurer-Cartan forms, which contain all information about the pseudo-group. In particular, they give basic invariants and operators of invariant differential, which in terms allow us to solve equivalence problem for submanifolds under the action of the pseudo-group. Recall that expansions of exterior differentials of Maurer-Cartan forms in terms of the form themselves, yields the Cartan structure equation for the prescribed
pseudogroup.
Suppose \( \pi : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n \) is a trivial bundle with the local base coordinates \((x_1, \ldots, x_n)\) and the local fibre coordinates \((u_1, \ldots, u_m)\); then \( J^1(\pi) \) is denoted by the bundle of the first-order jets of sections of \( \pi \), with the local coordinates \((x_i, u_\alpha, p_\alpha^i)\), \( i \in \{1, \ldots, n\}, \alpha \in \{1, \ldots, m\} \), where \( p_\alpha^i = \frac{\partial u_\alpha}{\partial x_i} \). For every local section \((x_i, f_\alpha(x))\) of \( \pi \), the corresponding 1-jet \((x_i, f_\alpha(x), \frac{\partial f_\alpha(x)}{\partial x_i})\) is denoted by \( j_1(f) \). A differential 1-form \( \nu \) on \( J^1(\pi) \) is called a contact form, if it is annihilated by all 1-jets of local sections: \( j_1(f)^*\nu = 0 \). In the local coordinates every contact 1-form is a linear combination of the forms \( \nu^\alpha = du_\alpha - p_\alpha^i dx_i, \alpha \in \{1, \ldots, m\} \). A local diffeomorphism:

\[
\Delta : J^1(\pi) \rightarrow J^1(\pi), \quad \Delta : (x_i, u_\alpha, p_\alpha^i) \rightarrow (\xi_i, \eta_\alpha, \zeta'^i) \tag{2}
\]

is called a contact transformation, if for every contact 1-form \( \nu \), the form \( \Delta^*\tau \) is also contact. Suppose \( \mathcal{R} \) is a first-order differential equation in \( m \) dependent and \( n \) independent variables. We consider \( \mathcal{R} \) as a subbundle in \( J^1(\pi) \). Suppose \( Cont(\mathcal{R}) \) is the group of contact symmetries for \( \mathcal{R} \). It consists of all the contact transformations on \( J^1(\pi) \) mapping \( \mathcal{R} \) to itself.

It was shown in [4] that the following differential 1-forms:

\[
\Theta^\alpha = a^\alpha_\beta (du_\beta - p_\beta^j dx_j),
\Xi^i = b_j^i dx_j + c_\beta^i \Theta^\beta,
\Sigma^i_\alpha = a^\alpha_\beta B^j_\beta dp_\beta^j + f_\beta^i \Theta^\beta + g^i_\alpha \Xi^j.
\]

are the Maurer-Cartan forms of \( Cont(J^1(\pi)) \). They are defined on \( J^1(\pi) \times \mathcal{H} \), where \( \mathcal{H} = (a^\alpha_\beta, b_j^i, c_\beta^i, f_\beta^i, g^i_\alpha) | \alpha, \beta \in \{1, \ldots, m\}, i, j \in \{1, \ldots, n\} \), \( \det(a^\alpha_\beta), \det(b_j^i) \neq 0 \), \( g^i_\alpha = g^i_\alpha \) and \( (B^j_\beta) \) is the inverse matrix for \( (b_j^i) \). They satisfy the structure equations

\[
d\Theta^\alpha = \Phi^\alpha_\beta \wedge \Theta^\beta + \Xi^k \wedge \Sigma^\alpha_k,
d\Xi^i = \Psi^i_\gamma \wedge \Xi^\gamma + \Pi^i_\gamma \wedge \Theta^\gamma,
d\Sigma^i_\alpha = \Phi^\alpha_\beta \wedge \Sigma^\beta_k - \Psi^k_\beta \wedge \Sigma^\alpha_k + \Lambda^\alpha_\beta \wedge \Theta^\beta + \Omega^\alpha_{ij} \wedge \Xi^j,
\]

where the forms \( \Phi^\alpha_\beta, \Psi^i_\gamma, \Pi^i_\gamma, \Lambda^\alpha_\beta \) and \( \Omega^\alpha_{ij} \) depend on differentials of the coordinates of \( \mathcal{H} \).

Differential equations defines a submanifold \( \mathcal{R} \subset J^1(\pi) \). The Maurer-Cartan forms for its symmetry pseudo-group \( Cont(\mathcal{R}) \) can be found from restrictions \( \theta^\alpha = i^* \Theta^\alpha, \xi^i = i^* \Xi^i \) and \( \sigma^\alpha_i = i^* \Sigma^\alpha_i \), where \( i = i_0 \times id : \mathcal{R} \times \mathcal{H} \rightarrow J^1(\pi) \times \mathcal{H} \) with \( i_0 : \mathcal{R} \rightarrow J^1(\pi) \) is defined by our differential equations. In order to compute the Maurer-Cartan forms for the symmetry pseudo-group, we implement Cartan’s equivalence method. Firstly, the forms \( \theta^\alpha, \xi^i, \sigma^\alpha_i \) are linearly dependent, i.e. there exists a nontrivial set of functions \( U_\alpha, V_i, W^i_\alpha \) on \( \mathcal{R} \times \mathcal{H} \) such that \( U_\alpha \theta^\alpha + V_i \xi^i + W^i_\alpha \sigma^\alpha_i \equiv 0 \). Setting these functions equal to some appropriate constants allows us to introduce a part of the coordinates of \( \mathcal{H} \) as functions of the other coordinates of \( \mathcal{R} \times \mathcal{H} \). Secondly, we substitute the obtained values into the forms \( \phi^\alpha_\beta = i^* \Phi^\alpha_\beta \) and \( \psi^i_k = i^* \Psi^i_k \) coefficients of semi-basic forms \( \phi^\alpha_\beta \) at \( \sigma^\alpha_i, \xi^i \), and the coefficients of semi-basic forms \( \psi^i_k \) at \( \sigma^\alpha_i \) are lifted invariants of \( Cont(\mathcal{R}) \).
We set them equal to appropriate constants and get expressions for the next part of the coordinates of $\mathcal{H}$, as functions of the other coordinates of $\mathcal{R} \times \mathcal{H}$. Thirdly, we analyze the reduced structure equations

\[
\begin{align*}
\frac{d\theta^\alpha}{\alpha} &= \phi^\alpha_{\beta} \wedge \theta^\beta + \xi^k \wedge \sigma^\alpha_k, \\
\frac{d\xi^i}{\beta} &= \psi^i_{\beta} \wedge \xi^k + \pi^i_\gamma \wedge \theta^\gamma, \\
\frac{d\sigma^\alpha_i}{\gamma} &= \phi^\alpha_{\gamma} \wedge \sigma^\gamma_i - \psi^i_{\gamma} \wedge \sigma^\alpha_k + \lambda^\alpha_{\beta\gamma} \wedge \theta^\beta + \omega^\alpha_{\beta\gamma} \wedge \xi^\gamma.
\end{align*}
\]

If the essential torsion coefficients are dependent on the group parameters, then we may normalize them to constants and find some new part of the group parameters, which, upon being substituted into the reduced modified Maurer-Cartan forms, allows us to repeat the procedure of normalization. This process has two results. First, when the reduced lifted coframe appears to be involutive, this coframe is the desired set of defining forms for $\text{Cont}(\mathcal{R})$. Second, when the coframe is not involutive we should apply the procedure of prolongation described in [6].

### 2. Structure and invariants of symmetry groups for KDV-type equations

Consider the following system equivalent to (1) of first order:

\[
\begin{align*}
u_x &= v, & \quad v_x &= w, & \quad w_x &= u_t + Q(u, v).
\end{align*}
\]

We apply the method described in the previous section to the class of equations (3). We denote that $t = x_1, x = x_2, u = u_1, v = u_2, w = u_3, u_t = p^1_1, u_x = p^2_1, v_t = p^2_2, w_x = p^2_3$. We consider this system as a sub-bundle of the bundle $J^1(\pi), \pi : \mathbb{R}^2 \times \mathbb{R}^3 \rightarrow \mathbb{R}^3$, with local coordinates $\{x_1, x_2, u_1, u_2, u_3, p^1_1, p^2_1, p^3_1\}$, where the embedding $\iota$ is defined by the following equalities:

\[
\begin{align*}
p^1_2 &= u_2, & p^2_2 &= u_3, & p^3_2 &= p^1_1 + Q(u_1, u_2).
\end{align*}
\]

The forms $\theta^\alpha = \iota^* \Theta^\alpha, \alpha \in \{1, 2, 3\}, \xi^i = \iota^\# \Xi^i, i \in \{1, 2\}$, are linearly dependent. The group parameters $a^\alpha \beta, b^i_j$ should satisfy the simultaneous conditions $\det(a^\alpha \beta) \neq 0, \det(b^i_j) \neq 0$. Linear dependence between the forms $\sigma^\alpha_i$ are

\[
\begin{align*}
\sigma^1_2 &= 0, & \quad \sigma^2_2 &= 0, & \quad \sigma^3_2 &= \sigma^1_3.
\end{align*}
\]

Computing the linear dependence conditions (5) gives the group parameters $a^1_1, a^2_1, a^3_1, a^1_2, a^2_2, a^3_2, b^i_1, f^i_1, f^i_2, f^i_3, f^i_4, f^i_5, f^i_6, g^i_1, g^i_2, q^i_1$ as functions of other group parameters and the local coordinates $\{x_1, x_2, u_1, u_2, u_3, p^1_1, p^2_1, p^3_1\}$ of $\mathcal{R}_1$. In particular,

\[
\begin{align*}
f^i_1 &= -\frac{a^2_1 a^3_2 a_2^2 - (a^2_1)^2 a^2_3 - a^2_1 (a^2_2)^2 + g^2_2 c^2_1 (a^3_2)^2 b^1_1 a_2^2 + g^2_1 c^1_1 (a^3_2)^2 b^1_1 a^2_2}{(a^3_2)^2 b^1_1 a^2_2}, \\
g^i_1 &= \frac{-p^1_2 a^3_2 b^2_2 - p^1_2 a^2_1 b^2_2 + Q^1_2 a^2_2 + p^1_2 b^1_2 a^2_2 + u_3 b^2_1 a^2_1 + u_3 b^2_1 a_2^2}{b^1_1 (b^2_2)^2}, \quad b^1_2 = 0, \\
f^i_2 &= -\frac{a^2_3 a_2^2 + a^2_1 a^2_3 + g^2_2 c^2_1 a^2_2 b^2_2 a^2_3 + g^2_1 c^1_1 a^2_2 b^2_2 a^2_3}{a^3_2 b^2_1 a^2_2}, \quad a^2_3 = 0,
\end{align*}
\]
Thus, we can assume $c = 1$ completely.

The expressions for $f_{11}$, $f_{22}$, $f_{23}$, $g_{12}$ and $g_{22}$ are too long to be written out here completely.

The analysis of the semi-basic modified Maurer-Cartan forms $\phi_\beta$, $\psi_k$ at the obtained values of the group parameters gives the following normalization.

The form $\psi_1$ is semi-basic, and $\psi_1 = -c_3^1 \sigma_1$. So we take $c_3 = 0$. For the semi-basic form $\phi_2$ we have

$$\phi_2^1 \equiv c_2^1 \sigma_1^1 \pmod{\theta^1, \theta^2, \theta^3, \xi^1, \xi^2},$$

thus, we can assume $c_1 = 0$. And for the semi-basic form $\phi_3$ we have

$$\phi_3^1 \equiv -c_3^2 a_3^3 b_3^2 u_3 - c_3^2 a_3^3 b_3^2 p_1^2 + f_{13}^1 (b_2^2)^3 \xi^1 \pmod{\theta^1, \theta^2, \theta^3},$$

so we set the coefficient at $\xi^1$ equal to 0 and find

$$f_{13}^1 = c_3^2 a_3^3 (b_2^2 p_1^2 - b_1^2 u_3) / (b_2^2)^3.$$

By doing the analysis of the modified semi-basic Maurer-Cartan forms in the same way, we can normalize the following group parameters:

$$a_1^2 = a_1^3 = a_2^3 = 0,$$
$$b_1^1 = (b_2^2)^3,$$
$$c_1^1 = c_2^1 = c_2^3 = c_3^3 = 0,$$
$$f_{12}^1 = f_{21}^3 = f_{13}^3 = f_{12}^1 = f_{12}^2 = f_{11}^3 = 0,$$

We denote by $\mathcal{S}_1$ the subclass of equations (3) such that: $Q_{uv} = 0$, $Q_{uw} = 0$ and $Q_{vu} = 0$. In this case, the structure equations of the symmetry group for system (3) have the form:

$$d\theta^1 = -\theta^1 \wedge (\eta_4 + 2\eta_5) - \theta^2 \wedge \xi^2 + \xi^1 \wedge \sigma_1^1,$$
$$d\theta^2 = -\theta^2 \wedge (\eta_4 + \eta_5) - \theta^3 \wedge \xi^2 + \xi^1 \wedge \sigma_1^2,$$
$$d\theta^3 = -\theta^3 \wedge \eta_4 + \xi^1 \wedge \sigma_1^3 + \xi^2 \wedge \sigma_1^1,$$
$$d\xi^1 = -3\xi^1 \wedge \eta_5,$$
$$d\xi^2 = -\xi^2 \wedge \eta_5,$$
In the analysis of structure equations we can’t absorb any group parameters more than before. Besides, the Cartan character is 5 and the indetermination degree is 3, thus the involution test fails. So we adopt the procedures of prolongation to compute the new structure equations:

\[
\begin{align*}
    d\sigma^1 &= -\xi^1 \wedge \eta_3 + \xi^2 \wedge \sigma^2_1 - \sigma^1_1 \wedge (\eta_4 - \eta_5), \\
    d\sigma^2 &= -\xi^1 \wedge \eta_1 + \xi^2 \wedge \sigma^2_1 - \sigma^2_2 \wedge (\eta_4 - 2\eta_5), \\
    d\sigma^3 &= -\xi^1 \wedge \eta_2 - \xi^2 \wedge \eta_3 + \sigma^2_1 \wedge (\eta_4 - 3\eta_5).
\end{align*}
\]

In structure equations (7), the forms \(\eta_1, \ldots, \eta_5\) on \(J^2(\pi) \times H\) depend on differentials of the parameters of \(H\), while the forms \(\beta_1, \beta_2, \beta_3\) depend on differentials of the prolongation variables. In the structure equations (7) the degree of indetermination is 3 and the Cartan characters are \(s_1 = 3, s_2 = \ldots = s_{13} = 0\). Consequently, Cartan’s test for the lifted coframe \(\{\theta^1, \theta^2, \theta^3, \xi^1, \xi^2, \sigma^2_1, \sigma^2_2, \sigma^2_3, \eta_1, \eta_2, \eta_3, \eta_4, \eta_5\}\) is satisfied. Therefore, the coframe is involutive. All the essential torsion coefficients in the structure equations (7) are constant. By applying Theorem 11.8 of [6], we have:

**Theorem 2.1 All systems from \(S_1\) are (locally) equivalent under contact transformations.**

We denote by \(S_2\) the subclass of equations (3) such that \(Q_{u^2} = 0, Q_{u^3} = 0\) and \(Q_{uv} \neq 0\). \(S_2\) is completely described by equations of the form \(u_{xxx} = u_t + Au + Bu_x + Cu_{xx} + D\) with \(A, B, C, D \in \mathbb{R}\) and \(C \neq 0\). KDV equation belongs to \(S_2\). The analysis of the structure equations gives the following essential torsion coefficients and the corresponding normalization:

\[
\begin{align*}
    d\sigma^2 &= \omega^2_{11} \wedge \xi^1 + \phi^3 \wedge \sigma^2_1 - 2\psi^2 \wedge \sigma^2_1 + \frac{C}{(b_2^2)^4} \theta^1 \wedge \theta^3 + \cdots
\end{align*}
\]
thus, we can assume \( a_3^3 = \frac{C}{(b_2^2)^3} \); and
\[
\begin{align*}
d\sigma_1^2 &= \omega_{11}^3 \wedge \xi^2 - 6\omega_{11}^1 \wedge \sigma_1^1 + \theta^1 \wedge \theta^2 - \frac{Cv}{(b_2^2)^3} \theta^3 \wedge \xi^2 + \ldots
\end{align*}
\]
thus, we can assume \( b_2^2 = \sqrt{Cv} \).

After this normalization, the structure equations of coframe \( \{ \theta^1, \theta^2, \theta^3, \xi^1, \xi^2, \sigma_1, \sigma_1^1, \sigma_1^2 \} \) is,
\[
\begin{align*}
d\theta^1 &= \frac{2}{3} \theta^1 \wedge \theta^2 - \left( \frac{2}{3} I_2 - I_3 - 1 \right) \theta^1 \wedge \xi^1 + \frac{2}{3} I_1 \theta^1 \wedge \xi^2 - \theta^2 \wedge \xi^2 + \xi^1 \wedge \sigma_1^1, \\
d\theta^2 &= (I_3 - I_2 + 1) \theta^2 \wedge \xi^1 + I_1 \theta^2 \wedge \xi^2 - \theta^3 \wedge \xi^2 + \xi^1 \wedge \sigma_1^2, \\
d\theta^3 &= -\frac{4}{3} \theta^2 \wedge \theta^3 - \left( \frac{4}{3} I_2 - I_3 - 1 \right) \theta^3 \wedge \xi^1 + \frac{4}{3} I_1 \theta^3 \wedge \xi^2 + \xi^2 \wedge \sigma_1^1 + \xi^1 \wedge \sigma_1^2, \\
d\xi^1 &= \theta^2 \wedge \xi^1 - I_1 \xi^1 \wedge \xi^2, \\d\xi^2 &= -\theta^1 \wedge \xi^1 + \frac{1}{3} \theta^2 \wedge \xi^2 - \left( \frac{1}{3} I_2 - 1 \right) \xi^1 \wedge \xi^2, \\
d\sigma_1^1 &= \pi_3 \wedge \xi^1 - I_1 \theta^1 \wedge \xi^2 - \theta^2 \wedge \xi^2 - \frac{5}{3} \theta^2 \wedge \sigma_1^1 + \left( \frac{5}{3} I_2 - I_3 - 2 \right) \xi^1 \wedge \sigma_1^1 \\
&\quad - \frac{5}{3} I_1 \xi^2 \wedge \sigma_1^1 + \xi^2 \wedge \sigma_1^2, \\
d\sigma_2^1 &= \pi_1 \wedge \xi^1 + \theta^1 \wedge \theta^3 - I_1 \theta^1 \wedge \xi^2 - \theta^3 \wedge \xi^2 - 2\theta^2 \wedge \sigma_1^2 - 2I_1 \xi^2 \wedge \sigma_1^2 \\
&\quad + \xi^2 \wedge \sigma_1^3, \\
d\sigma_3^1 &= \pi_2 \wedge \xi^1 + \pi_3 \wedge \xi^2 + \theta^1 \wedge \sigma_1^1 + \theta^2 \wedge \theta^3 - I_1 \theta^2 \wedge \xi^2 - \frac{7}{3} \theta^2 \wedge \sigma_1^3 \\
&\quad - \frac{7}{3} I_1 \xi^2 \wedge \sigma_1^3,
\end{align*}
\]
where
\[
\begin{align*}
I_1 &= \frac{w}{\sqrt{Cv^2}}, \\
I_2 &= -\frac{Bw + Cuw + v_i}{Cv^2}, \\
I_3 &= \frac{A}{Cv}.
\end{align*}
\]
are invariants of the symmetry group of an equation of \( \mathcal{S}_2 \).

Consider the subclass of equations \( \mathcal{S}_3 \) such that \( Q_{uvv} \neq 0 \) and \( Q_{uv} \neq 0 \). We denote this subclass by \( \mathcal{S}_3 \). For an equation from \( \mathcal{S}_3 \) we normalize \( a_3^3 = \frac{\left( Q_{uvv} \right)^4}{(Q_{uv})^2} \) and \( b_2^2 = \frac{Q_{uv}}{Q_{uvv}} \). After the absorption of torsion we have the coframe \( \theta = \{ \theta_1, \theta_2, \theta_3, \xi^1, \xi^2, \sigma_1, \sigma_1^1, \sigma_1^2, \sigma_1^3 \} \), with the structure equations
\[
\begin{align*}
d\theta^1 &= (L_1 - 2L_2) \theta^1 \wedge \theta^2 + (L_3 - L_1 + 2L_4) \theta^1 \wedge \xi^1 - \theta^2 \wedge \xi^2 \\
&\quad + (L_6 - 2L_7) \theta^1 \wedge \xi^2 + \xi^1 \wedge \sigma_1^1, \\
d\theta^2 &= (3L_1 - 2L_6) \theta^1 \wedge \theta^2 + (2L_4 + L_3 - 3L_5) \theta^2 \wedge \xi^1 - \theta_3 \wedge \xi^2 \\
&\quad + (2L_6 - 3L_7) \theta^2 \wedge \xi^2 + \xi^1 \wedge \sigma_1^2, \\
d\theta^3 &= (4L_1 - 3L_6) \theta^1 \wedge \theta^3 + (4L_2 - 3L_1) \theta^2 \wedge \theta^3 + (3L_6 - 4L_7) \theta^3 \wedge \xi^2
\end{align*}
\]
+ (3L_4 + L_3 - 4L_5)\theta^3 \wedge \xi^1 + \xi^1 \wedge \sigma_1^3 + \xi^2 \wedge \sigma_1^1,
\]
\[d\xi^1 = 3(L_8 - L_1)\theta^1 \wedge \xi^1 + 3(L_1 - L_2)\theta^2 \wedge \xi^1 + 3(L_7 - L_6)\xi^1 \wedge \xi^2,
\]
\[d\xi^2 = \xi^1 \wedge \theta^1 + (L_8 - L_1)\theta^1 \wedge \xi^2 - \theta^2 \wedge \xi^1 + (L_1 - L_2)\theta^2 \wedge \xi^2 + (L_4 - L_5 + L_9)\xi^1 \wedge \xi^2,
\]
\[d\tau^1 = \pi_3 \wedge \xi^1 - L_{10} \theta^1 \wedge \xi^2 + (5L_1 - 4L_8)\theta^1 \wedge \sigma_1^1 - L_9 \theta^2 \wedge \xi^2 + (5L_2 - 4L_1)\theta^2 \wedge \sigma_1^1 + (5L_5 - L_3 - 4L_4 - L_9)\xi^1 \wedge \sigma_1^1 + \xi^2 \wedge \sigma_1^1 + (5L_7 - 4L_6)\xi^2 \wedge \sigma_1^1,
\]
\[d\tau^2 = \pi_1 \wedge \xi^1 + L_{11} \theta^1 \wedge \theta^2 + \theta^1 \wedge \theta^3 + (6L_1 - 5L_8)\theta^1 \wedge \sigma_1^2 + \theta^2 \wedge \theta^3 - L_{10} \theta^2 \wedge \xi^2 + (6L_2 - 5L_1)\theta^2 \wedge \sigma_1^2 - L_9 \theta^3 \wedge \xi^2 + (6L_7 - 5L_6)\xi^2 \wedge \sigma_1^2 + \xi^2 \wedge \sigma_1^3,
\]
\[d\tau^3 = \pi_2 \wedge \xi^1 + \pi_3 \wedge \xi^2 + L_{11} \theta^1 \wedge \theta^3 + (7L_1 - 6I_8)\theta^1 \wedge \sigma_1^3 + \theta^1 \wedge \sigma_1^1 + \theta^2 \wedge \theta^3 + (7L_2 - 6I_1)\theta^2 \wedge \sigma_1^3 + \theta^2 \wedge \sigma_1^1 - L_{10} \theta^3 \wedge \xi^2 + (7L_7 - 6I_6)\xi^2 \wedge \sigma_1^3.
\]

Where the following functions
\[L_1 = \frac{Q_{uv^2}Q_{uv}}{(Q_{uv})^3}, \quad L_7 = \frac{(wQ_{v^3} + vQ_{uv^2})}{Q_{uv}},
\]
\[L_2 = \frac{Q_{v^3}(Q_{uv})^2}{(Q_{uv})^4}, \quad L_{10} = \frac{(Q_{uv})^4(wQ_{uv} + vQ_{uv})}{(Q_{uv})^4},
\]
\[L_3 = \frac{Q_{uv}(Q_{uv})^2}{(Q_{uv})^3}, \quad L_6 = \frac{Q_{v^2}(uQ_{uv^2} + vQ_{uv})}{(Q_{uv})^2},
\]
\[L_8 = \frac{Q_{u^2v}}{(Q_{uv})^2}, \quad L_9 = \frac{(Q_{uv})^3(wQ_{uv^2} + vQ_{uv})}{(Q_{uv})^3},
\]
\[L_{11} = \frac{Q_{uv^2}Q_{uv}}{(Q_{uv})^2}, \quad L_4 = \frac{Q_{v^2}(uQ_{vQ_{uv^2}} + uQ_{u^2v} + wQ_{vQ_{uv^2}} + vQ_{uv^2})}{(Q_{uv})^4},
\]
\[L_5 = \frac{(Q_{uv})^2(vQ_{vQ_{uv^2}} + uQ_{uv^2} + wQ_{vQ_{uv^2}} + vQ_{uv^2})}{(Q_{uv})^3}.
\]

are invariants of the symmetry group of an equations of $S_3$.

Finally, we denote by $S_4$ the subclass of equations \([3]\) such that $Q_{u^2} \neq 0$, $Q_{uv} \neq 0$ and $Q_{vu} = 0$, modified and generalized KDV equations belong to this subclass. For an equation from $S_4$ we normalize $a_3^2 = (\frac{Q_{uv}}{Q_{uv^2}})^3$ and $b_3^2 = \frac{uQ_{uv^2}}{Q_{u^2}}$. After absorption of torsion, we have the coframe $\theta = \{\theta_1, \theta_2, \theta_3, \xi^1, \xi^2, \sigma_1^1, \sigma_1^2, \sigma_1^3\}$, with the following structure equations
\[d\theta^1 = 2M_1 \theta_1 \wedge \theta_2 + (M_4 - 3M_2 + 2M_3)\theta_1 \wedge \xi_1 + (2M_5 - 3M_6)\theta_1 \wedge \xi_2 - \theta_2 \wedge \xi_2 + \xi_1 \wedge \sigma_1^1,
\]
\[d\theta^2 = (4M_1 - 3M_7)\theta_1 \wedge \theta_2 + (M_4 - 4M_2 + 3M_3)\theta_2 \wedge \xi_1 - \theta_3 \wedge \xi_2 + (3M_5 - 4M_6)\theta_2 \wedge \xi_2 + \xi_1 \wedge \sigma_1^2,
\]
\[d\theta^3 = (5M_1 - 4M_7)\theta_1 \wedge \theta_3 - 4M_1 \theta_2 \wedge \theta_3 + (M_4 - 5M_2 + 4M_3)\theta_3 \wedge \xi_1 - \xi_1 \wedge \sigma_1^3 + (4M_5 - 5M_6)\theta_3 \wedge \xi_2 + \xi_2 \wedge \sigma_1^1,
\]
\[ d\xi^1 = 3(M_7 - M_1)\theta_1 \wedge \xi_1 + 3M_1\theta_2 \wedge \xi_1 + 3(M_6 - M_5)\xi_1 \wedge \xi_2, \]
\[ d\xi^2 = \xi_1 \wedge \theta_1 + (M_7 - M_1)\theta_1 \wedge \xi_2 + (M_8 - M_2 + M_3)\xi_1 \wedge \xi_2, \]
\[ + M_1\theta_2 \wedge \xi_2, \]
\[ d\sigma^1 = \pi_3 \xi_2 \wedge \xi_1 - (M_8 + M_9)\theta_1 \wedge \xi_2 + (6M_1 - 5M_7)\theta_1 \wedge \sigma_1^1 + \xi_2 \wedge \sigma_1^2 \]
\[ - 5M_1\sigma_2 \wedge \sigma_1^2 + (6M_2 - 5M_3 - M_4 - M_8)\xi_1 \wedge \sigma_1^1 \]
\[ - M_9\theta_2 \wedge \xi_2 - (6M_6 - 5M_5)\xi_2 \wedge \sigma_1^2, \]
\[ d\sigma^2 = \pi_1 \xi_1 \wedge \theta_1 - \theta_1 \wedge \theta_1 + \theta_1 \wedge \theta_3 + (7M_1 - 6M_7)\theta_1 \wedge \sigma_1^2 + \xi_2 \wedge \sigma_1^3 \]
\[ - (M_8 + M_9)\theta_2 \wedge \xi_2 - 6M_1\theta_2 \wedge \sigma_1^2 - M_8\theta_3 \wedge \xi_2 \]
\[ + (7M_6 - 6M_5)\xi_2 \wedge \sigma_1^2, \]
\[ d\sigma^3 = \pi_2 \xi_1 + \pi_3 \xi_2 + \theta_1 \wedge \theta_3 + \theta_1 \wedge \sigma_1^1 + (8M_1 - 7M_7)\theta_1 \wedge \sigma_1^3 \]
\[ + \theta_2 \wedge \theta_3 - 7M_1\theta_2 \wedge \sigma_1^3 - (M_8 + M_9)\xi_2 \wedge \sigma_1^3 \]
\[ + (8M_6 - 7M_5)\xi_2 \wedge \sigma_1^3. \]

Where the functions
\[ M_1 = \frac{Q_{u^2v}(Q_{u^2})^2}{(Q_{u^2})^4}, \quad M_2 = \frac{Q_{u^2v}(Q_{uv})^2(vQ_v + u_t)}{(Q_{u^2})^3}, \]
\[ M_3 = \frac{(Q_{uv})^3(v_tQ_{u^2v} + vQ_vQ_{u^3} + wQ_vQ_{u^2v} + u_tQ_{u^3})}{(Q_{u^2})^4}, \]
\[ M_4 = \frac{Q_u(Q_{uv})^3}{(Q_{u^2})^3}, \quad M_5 = \frac{Q_{uv}(vQ_{u^3} + wQ_{u^2v})}{(Q_{u^2})^2}, \]
\[ M_6 = \frac{vQ_{u^2v}}{Q_{u^2}^2}, \quad M_7 = \frac{vQ_{u^2}Q_{u^3}}{(Q_{uv})^3}, \quad M_8 = \frac{v(Q_{uv})^4}{(Q_{u^2})^3}, \]
\[ M_9 = \frac{w(Q_{uv})^5}{(Q_{u^2})^4}, \]

are invariants of the symmetry group of an equation from \( S_1 \).
The structure equations (8), (10) and (12) do not contain any torsion coefficient depending on the group parameters. Their degree of indeterminacy \( r^{(1)} \) is 3, whereas the reduced characters are \( s_1 = 3, s_2 = \ldots = s_8 = 0 \). So, Cartan’s test for each of them is satisfied and the coframes are involutive. By applying Theorem 15.12 of [6] to above calculations we have following statement:

**Theorem 2.2** The class of equation (1) is divided into four subclasses \( S_1 \) to \( S_4 \) invariant under an action of the pseudo-group of contact transformations:

- **\( S_1 \)** consists of all systems (1) such that \( Q_{u^2} = 0 \), \( Q_{u^2} = 0 \) and \( Q_{uvu} = 0 \);
- **\( S_2 \)** consists of all systems (1) such that \( Q_{u^2} = 0, Q_{u^2} = 0 \) and \( Q_{uvu} \neq 0 \);
- **\( S_3 \)** consists of all systems (1) such that \( Q_{u^2} \neq 0 \) and \( Q_{uvu} \neq 0 \);
- **\( S_4 \)** consists of all systems (1) such that \( Q_{u^2} \neq 0, Q_{uvu} \neq 0 \) and \( Q_{u^2} = 0 \).

All equation from \( S_1 \) is equivalent to \( u_{xxx} = u_t \).
The basic differential invariants for equations from the subclass \( S_2 \) are the functions \( I_1, I_2 \) and \( I_3 \) defined by (9). Two equations from \( S_2 \) are equivalent with regard to the pseudo-group of contact transformations whenever they have the same functional dependence.
among the invariants $I_1, I_2$ and $I_3$.

The basic differential invariants for equations from the subclass $S_3$ are the functions $L_1, \ldots, L_{11}$ defined by (11). Two systems from $S_3$ are equivalent with respect to the pseudo-group of contact transformations whenever they have the same functional dependence among the invariants $L_1, \ldots, L_{11}$.

The basic differential invariants for equations from the subclass $S_4$ are the functions $M_1, \ldots, M_9$ defined by (13). Two systems from $S_4$ are equivalent with respect to the pseudo-group of contact transformations whenever they have the same functional dependence among the invariants $M_1, \ldots, M_9$.

**Conclusion.** In this paper, the moving coframe method of [2, 4] is applied to the local equivalence problem for a class of systems of KDV-type equations under the action of a pseudo-group of contact transformations. We found four subclasses and showed that every system of KDV-type equations can be transformed to a system from one of these subclasses. The structure equations and the invariants for all subclasses were also found.

---

[1] E.Cartan, Les Problemes d’equivalence, Oeuvres Completes Vol. 2, Gauthiers-Villars, Paris, 1953.

[2] M.Fels, P.J.Olver, Moving coframes, I. A practical algorithm, Acta. Appl. Math 51(1998) 161-213.

[3] M.Fels, P.J.Olver, Moving coframes. II. Regularization and theoretical foundations, Acta Appl. Math 55(1999)127-208.

[4] O.Morozov, Moving coframes and symmetries of differential equations, J. Phys. A: Math. Gen. 35 (2002) 2965-2977.

[5] F. Gungor, V.I.Lahno and R.Z.Zhdanov, Symmetry classification of KdV-type nonlinear evolution equations, journal of mathematical phisycs, volume 45, number 6, June 2004.

[6] P.J.Olver, Equivalence, Invariants, and Symmetry, Cambridge University Press, Cambridge, 1995.

[7] R.B.Gardner, The Method of Equivalence and Its Applications, SIAM, Philadelphia, 1989.

[8] Bryant, R.: Two exotic holonomies in dimension four, path geometries, and twistor theory, Proc. Symp. in Pure Math. 53, 3385-3507(1991)

[9] Fels, M. E.: The inverse problem of the calculus of variations for scalar fourth-order ordinary differential equations. Trans. Amer. Math. Soc. 348, 50075029(1996)

[10] Kamran, N., Milson, R., Olver, P.J.: Invariant modules and the reduction of nonlinear partial differential equations to dynamical systems, Adv. in Math., 156, 286-319(2000)

[11] Kamran, N., Tenenblat, K.: On differential equations describing pseudospherical surfaces, J. Diff. Eq., 115, 75-98(1995)

[12] Brans, C. H.: Invariant Approach to the Geometry of Spaces in General Relativity, J. Math. Phys. 6, 94(1965)

[13] Karlhede, A.: A review of the geometrical equivalence of metrics in general relativity, General Relativity and Gravitation 12, 693(1980)