Finite Subgroups of $\text{PGL}_2(K)$

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To Ramanan on his 70th birthday

Abstract. We classify, up to conjugacy, the finite subgroups of $\text{PGL}_2(K)$ of order prime to $\text{char}(K)$.

Introduction

The aim of this note is to describe, up to conjugacy, the finite subgroups of $\text{PGL}_2(K)$, for an arbitrary field $K$. Throughout the paper, we consider only subgroups whose order is prime to the characteristic of $K$.

When $K = \mathbb{C}$, or more generally when $K$ is algebraically closed, the answer is well known: any such group is isomorphic to $\mathbb{Z}/r$, $D_r$ (the dihedral group), $\mathfrak{S}_4$, $\mathfrak{S}_5$, and there is only one conjugacy class for each of these groups. If $K$ is arbitrary, the group $\text{PGL}_2(K)$ is contained in $\text{PGL}_2(\overline{K})$, so the subgroups of $\text{PGL}_2(K)$ are among the previous list; it is not difficult to decide which subgroups occur for a given field $K$, see §1.

So the only question left is to describe the conjugacy classes in $\text{PGL}_2(K)$ of the subgroups in the list. In §2 we give a general answer for subgroups of $G(K)$, for an algebraic group $G$, in terms of (non-abelian) Galois cohomology. We illustrate the method on one example in §3, and apply it to the case $G = \text{PGL}_2$ in §4.

The motivation for looking at this question was to understand the appearance of the Brauer group in the case of $\left(\mathbb{Z}/2\right)^2$ considered in [B]. The result is somewhat disappointing, as it turns out that this case (which could be treated directly, as in [B]) is the only one where some second Galois cohomology group plays a role. At least our method explains this role, and hopefully may be useful in other situations.

1. The possible subgroups

We repeat that whenever we mention a finite group, we always assume that its order is prime to the characteristic of $K$. The following is classical (see [S2], 2.5).

**Proposition 1.1.** 1) $\text{PGL}_2(K)$ contains $\mathbb{Z}/r$ and $D_r$ if and only if $K$ contains $\zeta + \zeta^{-1}$ for some primitive $r$-th root of unity $\zeta$.

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\footnote{We denote by $D_r$ the dihedral group with $2r$ elements.}
2) \( \text{PGL}_2(K) \) contains \( \mathfrak{A}_4 \) and \( \mathfrak{S}_4 \) if and only if \(-1\) is the sum of two squares in 

\[ K. \]

3) \( \text{PGL}_2(K) \) contains \( \mathfrak{A}_5 \) if and only if \(-1\) is the sum of two squares and \( 5 \) is a square in \( K. \)

Proof. One way to prove this is to use the isomorphism \( \text{PGL}_2(K) \cong \text{SO}(\mathcal{K}, q) \), where \( q \) is the quadratic form \( q(x, y, z) = x^2 + yz \) on \( \mathcal{K}^3 \) ([D], II.9). If a group \( H \) embeds into \( \text{SO}(\mathcal{K}, q) \), we have a faithful representation \( \rho \) of \( H \) in \( \mathcal{K}^3 \), which preserves an indefinite quadratic form.

- Case \( H = \mathbb{Z}/r \) : let \( q \) be a generator; the existence of \( q \) forces the eigenvalues of \( \rho(g) \) in \( \mathcal{K} \) to be of the form \( (\zeta, \zeta^{-1}, 1) \), with \( \zeta \) a primitive \( r \)-th root of 1. This implies \( \zeta + \zeta^{-1} \in \mathcal{K} \). Conversely, if \( \lambda := \zeta + \zeta^{-1} \) is in \( \mathcal{K} \), the homography \( z \mapsto \frac{(\lambda + 1)z - 1}{z + 1} \) is an element of order \( r \) of \( \text{PGL}_2(K) \).

- Case \( H = \text{D}_r \) : by the previous case, if \( \text{D}_r \subset \text{PGL}_2(K) \), \( \lambda := \zeta + \zeta^{-1} \) is in \( \mathcal{K} \). Conversely if \( \lambda \in \mathcal{K} \), the homographies \( z \mapsto 1/z \) and \( z \mapsto \frac{(\lambda + 1)z - 1}{z + 1} \) generate a subgroup of \( \text{PGL}_2(K) \) isomorphic to \( \text{D}_r \).

- Cases \( H = \mathfrak{A}_4, \mathfrak{S}_4 \) or \( \mathfrak{A}_5 \). The representation \( \rho \) must be irreducible. Each of the groups \( \mathfrak{A}_4 \) and \( \mathfrak{S}_4 \) has exactly one irreducible 3-dimensional representation with trivial determinant, which is defined over the prime field; the only invariant quadratic form (up to a scalar) is the standard form \( q_0(x, y, z) = x^2 + y^2 + z^2 \). Thus \( \mathfrak{A}_4 \) and \( \mathfrak{S}_4 \) are contained in \( \text{PGL}_2(K) \) if and only if \( q_0 \) is equivalent to \( \lambda q \) for some \( \lambda \in \mathcal{K}^* \), which means that \( q_0 \) represents 0.

Since \( \mathfrak{A}_5 \) contains elements of order 5, the condition \( \sqrt{5} \in \mathcal{K} \) is necessary. Suppose this is the case, and put \( \varphi = \frac{1}{2}(1 + \sqrt{5}) \); the subgroup of \( \text{SO}(\mathcal{K}, q_0) \) preserving the icosahedron with vertices

\[ \{ (\pm 1, 0, \pm \varphi), (\pm \varphi, \pm 1, 0), (0, \pm \varphi, \pm 1) \} \]

is isomorphic to \( \mathfrak{A}_5 \). It follows as above that \( \mathfrak{A}_5 \) embeds in \( \text{SO}(\mathcal{K}, q) \) if and only if \( q_0 \) represents 0. \( \square \)

2. Some Galois cohomology

2.1. In this section we consider an algebraic group \( G \) over \( K \), and a subgroup \( H \subset G(K) \). We choose a separable closure \( K_s \) of \( K \), and put \( g := \text{Gal}(K_s/K) \). We are interested in the set of embeddings \( H \hookrightarrow G(K) \) which are conjugate in \( G(K_s) \) to the natural inclusion \( i : H \hookrightarrow G(K) \), modulo conjugacy by an element of \( G(K) \). We denote this (pointed) set by \( \text{Emb}_i(H, G(K)) \).

We will use the standard conventions for non-abelian cohomology, as explained for instance in [S3], ch. I, §5. We will also use the notation of [S3] for Galois cohomology: if \( G \) is an algebraic group over \( K \), we put \( H^i(K, G) := H^i(g, G(K_s)) \).

Proposition 2.2. Let \( Z \) be the centralizer of \( H \) in \( G(K_s) \). The pointed set \( \text{Emb}_i(H, G(K)) \) is canonically isomorphic to the kernel of the natural map \( H^1(K, Z) \to H^1(K, G) \).

Proof. Let \( X \subset G(K_s) \) be the subset of elements \( g \) such that \( g^{-1} \sigma g \in Z \) for all \( \sigma \in g \). The group \( G(K) \) (resp. \( Z \)) acts on \( X \) by left (resp. right) multiplication. By [S3], ch. I, 5.4, cor. 1, the kernel of \( H^1(K, Z) \to H^1(K, G) \) is identified with the (left) quotient by \( G(K) \) of the subset of \( g \)-invariant elements in \( G(K_s)/Z \); but this
subset is by definition $X/Z$, so we can identify our kernel to the double quotient $G(K)\backslash X/Z$.

For every $g \in X$, the conjugate embedding $gig^{-1}$ belongs to $\text{Emb}_i(H, G(K))$. Any element $j \in \text{Emb}_i(H, G(K))$ is of the form $gig^{-1}$ for some $g \in G(K_s)$; for $\sigma \in g$, the element $\sigma g$ again conjugates $i$ to $j$, hence $g^{-1}\sigma g \in Z$ and $g \in X$. Thus the map $g \mapsto gig^{-1}$ from $X$ to $\text{Emb}_i(H, G(K))$ is surjective. Two elements $g$ and $g'$ of $X$ give the same element in $\text{Emb}_i(H, G(K))$ if and only if $g'$ belongs to the double coset $G(K)gZ$. Therefore the above map induces a canonical bijection $G(K)\backslash X/Z \rightarrow \text{Emb}_i(H, G(K))$. 

2.3. Let us write down the correspondence explicitly: a class in our kernel is represented by a 1-cocycle $g \rightarrow Z$ which becomes a coboundary in $G$, hence is of the form $\sigma \mapsto g^{-1}\sigma g$ for some $g \in X$; we associate to this class the embedding $gig^{-1}$.

2.4. We are actually more interested in the set $\text{Conj}(H, G(K))$ of subgroups of $G(K)$ which are conjugate to $H$ in $G(K_s)$, modulo conjugacy by $G(K)$. Associating to an embedding its image defines a surjective map $im : \text{Emb}_i(H, G(K)) \rightarrow \text{Conj}(H, G(K))$. The normalizer $N$ of $H$ in $G(K_s)$ acts on $H$ by automorphisms, hence also on $\text{Emb}_i(H, G(K))$. Two embeddings with the same image differ by an automorphism of $H$, which must be induced by an element of $N$ if the embeddings are conjugate under $G(K_s)$. It follows that $im$ induces an isomorphism $\text{Emb}_i(H, G(K))/N \rightarrow \text{Conj}(H, G(K))$.

2.5. Let us translate this in cohomological terms. Let $H^1(K, Z)_0$ denote the kernel of the map $H^1(K, Z) \rightarrow H^1(K, G)$. An element $n$ of $N$ acts on $\text{Emb}_i(H, G(K))$ by $j \mapsto j \circ \text{int}(n^{-1})$; if $j = gig^{-1}$, this amounts to replace $g$ by $gn$, hence the 1-cocycle $\varphi : \sigma \mapsto g^{-1}\sigma g$ by $n^{-1}\varphi n$. This formula defines an action of $N$ on $H^1(K, Z)$ which preserves $H^1(K, Z)_0$; the map $g \mapsto gHg^{-1}$ induces an isomorphism of pointed sets $H^1(K, Z)_0/N \rightarrow \text{Conj}(H, G(K))$.

3. An example

3.1. In this section we fix an integer $r \geq 2$, prime to $\text{char}(K)$, and we assume that $K$ contains a primitive $r$-th root of unity $\zeta$. We consider the matrices $A, B \in M_r(K)$ defined on the canonical basis $(e_1, \ldots, e_r)$ of $K^r$ by

$$A \cdot e_i = e_{i+1}, \quad B \cdot e_i = \zeta^i e_i$$

for $1 \leq i \leq r$, with the convention $e_{r+1} = e_1$.

The matrices $A$ and $B$ generate the $K$-algebra $M_r(K)$, with the relations

$$A^r = B^r = I, \quad BA = \zeta AB.$$  

Their classes $A, B$ in $\text{PGL}_r(K)$ commute; we consider the embedding $i : (\mathbb{Z}/r)^2 \hookrightarrow \text{PGL}_r(K)$ which maps the two basis vectors to $A$ and $B$. The image $H$ of $i$ is its own centralizer; in particular, $H$ is a maximal commutative subgroup of $\text{PGL}_r(K)$.

By the Kummer exact sequence (and the choice of $\zeta$), the group $H^1(K, \mathbb{Z}/r)$ is identified with $K^*/K^{*r}$; the pointed set $H^1(K, \text{PGL}_r)$ can be viewed as the set of isomorphism classes of central simple $K$-algebras of dimension $r^2$ ([S1], X.5). 

**Lemma 3.2.** Let $\alpha, \beta \in K^*$, and let $\tilde{\alpha}, \tilde{\beta}$ be their images in $K^*/K^{*r}$. The map $H^1(i) : H^1(K, \mathbb{Z}/r)^2 \rightarrow H^1(K, \text{PGL}_r)$ associates to $(\tilde{\alpha}, \tilde{\beta})$ the class of the cyclic
$K$-algebra $A_{\alpha,\beta}$ generated by two variables $x,y$ with the relations $x^r = \alpha$, $y^r = \beta$, $yx = \zeta xy$.

**Proof.** We choose $\alpha', \beta'$ in $K^*$ with $\alpha'^r = \alpha$ and $\beta'^r = \beta$. The Kummer isomorphism associates to $(\alpha, \beta)$ the homomorphism $(a,b) : g \rightarrow (\mathbb{Z}/r)^2$ defined by

$$\sigma \alpha' = \zeta^{a(\sigma)} \alpha', \quad \sigma \beta' = \zeta^{b(\sigma)} \beta'$$

for each $\sigma \in g$.

Its image in $H^1(K, \text{PGL}_r(K_\alpha))$ is the class of the 1-cocycle $\sigma \mapsto A^a(\sigma) B^b(\sigma)$.

Now let us recall how we associate to the algebra $A_{\alpha,\beta}$ a cohomology class $[A_{\alpha,\beta}]$ in $H^1(K, \text{PGL}_r)$ (loc. cit.). We choose an isomorphism of $K_\alpha$-algebras $u : M_r(K_\alpha) \rightarrow A_{\alpha,\beta} \otimes_K K_\alpha$. For each $\sigma \in g$, $u^{-1} \sigma u$ is an automorphism of $M_r(K_\alpha)$, hence of the form $\text{int}(g_\sigma)$ for some $g_\sigma$ in $\text{PGL}_r(K_\alpha)$. Then $[A_{\alpha,\beta}]$ is the class of the 1-cocycle $\sigma \mapsto g_\sigma$.

In our case we define $u$ on the generators $A, B$ by $u(A) = \beta' y^{-1}$, $u(B) = \alpha'^{-1} x$. Then the automorphism $u^{-1} \sigma u$ multiplies $A$ by $\zeta^{b(\sigma)}$ and $B$ by $\zeta^{-a(\sigma)}$, which gives $g_\sigma = A^a(\sigma) B^b(\sigma)$ as above.

**3.3.** The exact sequence

$$1 \rightarrow G_m \rightarrow \text{GL}_r \rightarrow \text{PGL}_r \rightarrow 1$$

gives rise to a coboundary homomorphism $\partial_r : H^1(K, \text{PGL}_r) \rightarrow H^2(K, G_m) = \text{Br}(K)$ which is injective (loc. cit.). The class $[A_{\alpha,\beta}] \in \text{Br}(K)$ is the symbol $(\alpha, \beta)$; it depends only on the classes of $\alpha$ and $\beta$ (mod. $K^*$). The map $(\alpha, \beta)_r : (K^*/K^{*r})^2 \rightarrow \text{Br}(K)$ is bilinear and alternating. Since $\partial_r$ is injective, we find:

**Proposition 3.4.** The set $\text{Emb}_r((\mathbb{Z}/r)^2, \text{PGL}_r(K))$ is isomorphic to the set of couples $(\alpha, \beta)$ in $(K^*/K^{*r})^2$ such that $(\alpha, \beta)_r = 0$. 

We will describe the correspondence more explicily in the case $r = 2$ in the next section.

**4. Conjugacy classes in PGL_{2}(K)**

**Proposition 4.1.** Assume that $K$ is separably closed. Two finite subgroups of $\text{PGL}_{2}(K)$ which are isomorphic (and of order prime to char($K$)) are conjugate.

**Proof.** Again this is certainly well-known; we give a quick proof for completeness. The possible subgroups are those which appear in Proposition 1.1.

An element of order $r$ of $\text{PGL}_{2}(K)$ comes from a diagonalizable element of $\text{GL}_{2}(K)$, hence is conjugate to the homothety $z \mapsto \zeta z$ for some $\zeta \in \mu_{r}(K)^{2}$; thus a cyclic subgroup of order $r$ of $\text{PGL}_{2}(K)$ is conjugate to the group $H_{r}$ of homotheties $z \mapsto \lambda z$, $\lambda \in \mu_{r}(K)$.

There is only one group $D_{r}$ containing $H_{r}$, namely the subgroup generated by $H_{r}$ and the involution $z \mapsto 1/z$; it follows that all dihedral subgroups of order $2r$ are conjugate to this subgroup.

For the three remaining groups, we use again the isomorphism $\text{PGL}_{2}(K) \cong \text{SO}_{3}(K)$. The groups $\mathfrak{A}_{4}$ and $\mathcal{S}_{4}$ have exactly one irreducible representation of dimension 3 with trivial determinant, while $\mathfrak{A}_{5}$ has two such representations which differ by an outer automorphism: this is elementary in characteristic 0, and the general case follows by [I], ch. 15. Therefore two isomorphic subgroups $H$ and $H'$ of $\text{SO}_{3}(K)$ of this type are conjugate in $\text{GL}_{3}(K)$. The only quadratic forms

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2As usual we denote by $\mu_{r}(K)$ the group of $r$-th roots of unity in $K$. 

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preserved by $H$ or $H'$ are the multiple of the standard form; thus the element $g$ of $GL_3(K)$ which conjugates $H$ to $H'$ must satisfy $^tgg = \lambda I$ for some $\lambda \in K$. Replacing $g$ by $\pm \mu g$, with $\mu^2 = \lambda^{-1}$, we have $g \in SO_3(K)$, hence our assertion. □

Recall that the determinant induces a homomorphism $\overline{\det} : PGL_2(K) \to K^*/K^{*2}$.

**Theorem 4.2.** 1) $PGL_2(K)$ contains only one conjugacy class of subgroups isomorphic to $\mathbb{Z}/r \ (r > 2)$, $\mathfrak{A}_4$, $\mathfrak{S}_4$ or $\mathfrak{A}_5$.

2) The conjugacy classes of cyclic subgroups of order 2 of $PGL_2(K)$ are parametrized by $K^*/K^{*2}$: to $\alpha \in K^*$ (mod. $K^{*2}$) corresponds the involution $z \mapsto \alpha/z$.

3) The homomorphism $\overline{\det} : PGL_2(K) \to K^*/K^{*2}$ induces a bijective correspondence between:

- conjugacy classes of subgroups of $PGL_2(K)$ isomorphic to $(\mathbb{Z}/2)^2$;
- subgroups $G \subset K^*/K^{*2}$ of order $\leq 4$, such that $(-\alpha, -\beta_2) = 0$ for all $\alpha, \beta$ in $G$ (see (3.3)).

4) Assume that $\mu_r(K)$ has order $r$. The conjugacy classes of subgroups $D_r$ of $PGL_2(K)$ are parametrized by $K^*/K^{*2} \mu_r(K)$. The subgroup corresponding to $\alpha \in K^*$ (mod. $K^{*2} \mu_r(K)$) consists of the homographies $z \mapsto \zeta z$ and $z \mapsto \alpha \eta/z$, for $\zeta, \eta \in \mu_r(K)$.

**Proof.** Using Proposition 4.1 we can apply the method of §3. We give the list of the subgroups of $PGL_2(K, \mathbb{K})$ and their centralizers:

| $H$          | $\mathbb{Z}/2$ | $\mathbb{Z}/r \ (r > 2)$ | $\mathbb{Z}/2 \times \mathbb{Z}/2$ | $D_r \ (r > 2)$ | $\mathfrak{A}_4$ | $\mathfrak{S}_4$ | $\mathfrak{A}_5$ |
|-------------|----------------|---------------------------|-----------------------------------|----------------|----------------|----------------|----------------|
| $\mathbb{Z}$ | $\mathbb{G}_m \times \mathbb{Z}/2$ | $\mathbb{G}_m$ | $\mathbb{Z}/2 \times \mathbb{Z}/2$ | $\mathbb{Z}/2$ | 1              | 1              | 1              |

In case 1), we have $H^1(K, \mathbb{Z}) = \{1\}$ (using $H^1(K, \mathbb{G}_m) = \{1\}$). The result follows from (2.5).

Case 2): This is the case where a direct approach is definitely simpler than our method, so we follow the former and leave the latter to the reader. Let $s$ be an involution of $PGL_2(K)$, and let $\alpha \in K^*$ such that $\alpha \equiv -\overline{\det}(s)$ (mod. $K^{*2}$). Then $s$ is represented by a matrix $A \in GL_2(K)$ with $A^2 = \alpha I$. In a basis $(v, Av)$ of $K^2$, we have $A = \begin{pmatrix} 0 & \alpha \\ 1 & 0 \end{pmatrix}$, hence $s$ is conjugate to the involution $z \mapsto \alpha/z$. This implies 2).

Case 3): Let $\iota : (\mathbb{Z}/2)^2 \to PGL_2(K)$ be the embedding which maps the basis vectors $e_1$ and $e_2$ to the involutions $z \mapsto 1/z$ and $z \mapsto -z$. By Proposition 3.4 the set $\text{Emb}_i((\mathbb{Z}/2)^2, PGL_2(K))$ is canonically isomorphic to the set of couples $(\alpha, \beta)$ in $(K^*/K^{*2})^2$ with $(\alpha, \beta)_2 = 0$.

We make the correspondence explicit following (2.3). Let $\alpha, \beta \in K^*$ with $(\alpha, \beta)_2 = 0$. This means that the conic $x^2 - \alpha y^2 - \beta z^2 = 0$ is isomorphic to $\mathbb{P}^1_K$, thus there exists $\lambda, \mu \in K$ with $\lambda^2 - \alpha - \beta \mu^2 = 0$. We choose $\alpha'$ and $\beta'$ in $K_s$ such that $\alpha'^2 = \alpha$ and $\beta'^2 = \beta$; as above we define the homomorphisms $a$ and $b : g \mapsto \mathbb{Z}/2$ by

$^\sigma \alpha' = (-1)^{a(\sigma)} \alpha'$ and $^\sigma \beta' = (-1)^{b(\sigma)} \beta'$ for each $\sigma \in g$.
Put $\theta := \frac{\beta' \mu}{\lambda + \alpha'} = \frac{\lambda - \alpha'}{\beta' \mu}$; let $g \in \text{PGL}_2(K_s)$ be the homography $z \mapsto \frac{\alpha z - \theta}{z + \theta}$. An easy computation gives

$$g^{-1} \sigma g = i(a(\sigma), b(\sigma)).$$

Thus the embedding of $(\mathbb{Z}/2)^2$ associated to $(\alpha, \beta)$ is $g ig^{-1}$; it maps $e_1$ to the homography $h_1 : z \mapsto \frac{\lambda u - \alpha}{z - \lambda}$, and $e_2$ to $h_2 : z \mapsto \alpha / z$. Note that $\det(h_1) = -\beta$ and $\det(h_2) = -\alpha$.

Now we have to take into account the action of the normalizer $N$ of $H$ in $\text{PGL}_2(K_s)$. This is the subgroup $S_3$ generated by $H$ and the homographies

$$n_1 : z \mapsto \frac{z + 1}{z - 1}, \quad n_2 : z \mapsto \iota z,$$

where $\iota$ is a square root of $-1$. We apply the recipe of (2.5). Since $n_1 \in \text{PGL}_2(K)$, it acts on $H_1(K, H)$ through its action on $H$, which permutes $e_1$ and $e_2$; thus it maps $(\alpha, \beta) \in (K^*/K^{*2}) \times (K^*/K^{*2})$ to $(\beta, \alpha)$. The action of $n_2$ on $H$ fixes $e_2$ and exchanges $e_1$ with $e_1 + e_2$; to get the action on $H_1(K, H)$ we have to multiply by the class of the cocycle $\sigma \mapsto n_2^{-1} \sigma n_2$, that is, $\sigma \mapsto i(\sigma(i) / i) e_2$. Hence $n_2$ acts on $H_1(K, H)$ by

$$n_2 \cdot (\alpha, \beta) = (\alpha, -\alpha \beta).$$

Let $G_{\alpha, \beta}$ be the subgroup of $K^*/K^{*2}$ generated by $-\alpha$ and $-\beta$; it is the image of $H$ by the homomorphism $\det : \text{PGL}_2(K) \to K^*/K^{*2}$. If $G_{\alpha, \beta} \cong (\mathbb{Z}/2)^2$, the orbit $N \cdot (\alpha, \beta)$ in $(K^*/K^{*2}) \times (K^*/K^{*2})$ has 6 elements, which are the couples $(-x, -y)$ with $x, y \in G_{\alpha, \beta}, x \neq y$. If $G_{\alpha, \beta} \cong (\mathbb{Z}/2)$, the orbit has 3 elements, which are the couples $(-x, y)$ with $x, y \in G_{\alpha, \beta}, (x, y) \neq (1, 1)$. Finally if $G_{\alpha, \beta}$ is trivial the orbit consists only of $(-1, -1)$. Thus the conjugacy classes of subgroups $(\mathbb{Z}/2)^2$ in $\text{PGL}_2(K)$ are parametrized by the subgroups $G \subset K^*/K^{*2}$ of order $\leq 4$, with the property $(-\alpha, -\beta)_2 = 0$ for each $\alpha, \beta$ in $G$.

Case 4): The group $D_r$ is generated by two elements $s, t$ with the relations $s^2 = t^r = 1$ and $sts = t^{-1}$. We choose a primitive $r$-th root of unity $\zeta$ and consider the embedding $i : D_r \hookrightarrow \text{PGL}_2(K)$ such that $i(s)$ is the involution $z \mapsto 1 / z$ and $i(t)$ the homothety $z \mapsto \zeta z$. The centralizer is $\mathbb{Z}/2$, generated by the involution $z \mapsto -z$. As in case 2) it follows that $\text{Emb}(D_r, \text{PGL}_2(K))$ is isomorphic to $H_1(K, \mathbb{Z}/2)$. Also the previous argument shows that the embedding corresponding to $\alpha \in K^*$ is the conjugate of $i$ by the homography $z \mapsto \alpha' z$, with $\alpha^2 = \alpha$, so it maps $s$ to $z \mapsto \alpha / z$ and $t$ to $z \mapsto \zeta z$.

To complete the picture we have to take into account the action of the normalizer $N$ of $i(D_r)$ in $\text{PGL}_2(K_s)$. This is the subgroup $D_{2r}$, generated by $i(s) : z \mapsto 1 / z$ and the homothety $n : z \mapsto \eta z$, where $\eta \in K_s$ is a primitive $2r$-th root of unity. The action of $i(s)$ is trivial, and $n$ acts by multiplication by the cocycle $\sigma \mapsto n^{-1} \sigma n$, which corresponds to the class of $\eta^2$ in $K^*/K^{*2}$. Since $\eta^2$ generates $\mu_r(K)$, the assertion 4) follows.

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FINITE SUBGROUPS OF $\text{PGL}_2(K)$

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