Distributed Linear Quadratic Optimal Control:
Compute Locally and Act Globally

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Abstract—In this paper we consider the distributed linear quadratic control problem for networks of agents with single integrator dynamics. We first establish a general formulation of the distributed LQ problem and show that the optimal control gain depends on global information on the network. Thus, the optimal protocol can only be computed in a centralized fashion. In order to overcome this drawback, we aim at designing protocols that can be computed in a decentralized way. We will, instead of minimizing a given global cost functional, consider cost functionals that can be represented as a sum of local cost functionals, each associated with one of the agents. In order to achieve ‘good’ performance of the controlled network, each agent will compute its own local gain, using sampled information of its neighboring agents. This decentralized computation will obviously not result in optimality of the global network behavior. However, we will show that the resulting network will reach consensus. A simulation example is provided to illustrate the performance of the proposed protocol.

I. MOTIVATION AND PROBLEM FORMULATION

This paper deals with distributed linear quadratic (LQ) optimal control. The problem is to interconnect a finite number of identical agents according to a given network graph so that consensus is achieved in an optimal way. Each agent receives information only from its neighbors, in the form of a linear feedback involving the relative states amplified by a certain constant gain. Such control law is called a distributed diffusive control law. In order to determine an optimal gain, a quadratic cost functional is introduced. The problem of minimizing this cost functional over all distributed diffusive control laws that achieve consensus is then called the distributed LQ problem corresponding to the given cost functional.

The distributed LQ control problem has attracted much attention in the past, see e.g. [1]–[4]. In [5], for a single integrator network a particular global cost functional was considered, and it was shown that computation of the optimal control gain requires exact knowledge of the Laplacian matrix of the graph and all initial states of the agents. More recently, an inverse distributed optimal control problem was addressed in [6]. It was shown that a given distributed control protocol can be made optimal by choosing a suitable cost criterion. Later on, in [7], for a given cost functional conditions were established under which certain distributed control laws are suboptimal. The common feature of all work referred to above is that the computation of the control gains needs global information on the network. In the present paper we will address this drawback and present a decentralized design method for distributed controllers. For related work, we also refer [8]–[12], to name a few.

In this paper we will show that, for a general class of distributed LQ problems, the computation of the optimal gain requires exact knowledge of the network graph and all initial states of the agents. Thus, an optimal distributed diffusive control law can be computed only by a (virtual) supervisor that has global information on the network. In other words, although the resulting optimal control law would operate in a distributed fashion, the actual computation can only be performed in a centralized way.

Thus, we will argue that formulating a distributed LQ problem as a problem of minimizing a global cost functional is highly unpractical. Indeed, the centralized computation requires that the local optimal gain needs to be re-designed in case that changes in the network occur. For example, by adding or removing agents from the network, its graph will change, and new initial states will occur while existing ones will disappear. Therefore, the computation of the interconnection gain should, instead, be done in a decentralized way. This will then enable ‘plug-and-play’ operations on the network, since each agent will be able to automatically recompute its local gain whenever a new agent is added or removed.

In order to decentralize the computation we will, instead of minimizing a given global cost functional, consider cost functionals that can be written as a sum of local cost functionals, each associated with one of the agents. In order to achieve ‘good’ performance of the controlled network, each agent will compute its own local gain. This decentralized computation will obviously not result in optimality of the global network behavior. However, we will show that the resulting network will still reach consensus.

The outline of this paper is as follows. In Section II, we will introduce a general formulation of the distributed LQ problem. We will argue that distributed LQ problems are characterized by two properties, both depending on the given network graph: (i) the state weighting matrix in the cost functional has a particular structure, and (ii) the minimization takes place over a set of distributed diffusive control laws. In Section III, we will show that computation of the optimal control laws requires complete knowledge of the network graph and the initial state of the entire network. As argued above, this is highly unpractical, and therefore, in Section IV, we will propose a decentralized method to compute optimal (local) control laws. In order to do this, we need to apply ideas from linear quadratic tracking, and these will be reviewed in Section V. Then, in Section VI, we will compute
the proposed local optimal control laws, and show that the network reaches consensus if all agents apply their own local gain. To illustrate the designed control protocol, a simulation example is provided in Section VII. Finally, in Section VIII, we will give some concluding remarks.

**Notation**

We denote by \( \mathbb{R} \) the field of real numbers. The space of \( n \)-dimensional real vectors is denoted by \( \mathbb{R}^n \). The vector in \( \mathbb{R}^N \) with all components equal to 1 is denoted by \( 1_N \).

For a symmetric matrix \( P \), we write \( P > 0 \) (\( P \geq 0 \)) if \( P \) is positive (semi-)definite. We use \( \text{diag}(a_1, a_2, \ldots, a_n) \) to denote the \( n \times n \) diagonal matrix with \( a_1, a_2, \ldots, a_n \) on its diagonal. For a linear map \( A : X \to Y \), the kernel and image of \( A \) are denoted by \( \ker(A) := \{ x \in X \mid Ax = 0 \} \) and \( \text{im}(A) := \{ Ax \mid x \in X \} \), respectively.

In this paper, a graph is denoted by \( G = (V, E) \) with \( V = \{ 1, 2, \ldots, N \} \) the node set and \( E \subset V \times V \) the edge set. For \( i, j \in V \), an edge from node \( i \) to \( j \) is represented by \( (i, j) \in E \). The degree of agent \( i \) is defined as \( N_i := \{ j \in V \mid (i, j) \in E \} \). The adjacency matrix of \( G \) is equal to \( A = [a_{ij}] \in \mathbb{R}^{N \times N} \), where \( a_{ij} = 1 \) if \( (i, j) \in E \) and \( a_{ij} = 0 \) otherwise.

For a symmetric matrix \( R \), we write \( A \) diagonal for some symmetric matrix \( \bar{W} \) and \( Q \) weighting matrix. Thus, \( LWL \) is positive (semi-)definite. We use \( \text{diag}(a_1, a_2, \ldots, a_n) \) to denote the \( n \times n \) diagonal matrix with \( a_1, a_2, \ldots, a_n \) on its diagonal.

To show this, note that as admissible control laws we only allow control laws that operate in a distributed diffusive way and achieve consensus, i.e. the controlled trajectories converge to \( \text{im}(1_N) \), the span of the vector of ones. Thus the class of control laws over which we want to minimize (3) consists of all control laws of the form \( u = -gLx \), with \( g > 0 \).

Write \( Q = C^TC \) for some \( C \). Now, let \( \bar{x}(t) \) denote any nonzero state trajectory generated by the control law \( u = -Lx \), and let \( \bar{u}(t) = -L\bar{x}(t) \). It is well known that we then have \( \bar{x}(t) \to c1_N \) for some nonzero constant \( c \).

Clearly, in order for the cost functional (3) to make sense, the control law \( u = -Lx \) should give finite cost, which means that we should have \( J(x_0, \bar{u}) < \infty \). Since this implies \( \int_0^\infty \bar{x}(t)C^T C \bar{x}(t)dt < \infty \), we find that \( C \bar{x}(t) \to 0 \). Thus we obtain \( 1_N \in \ker(C) \), equivalently, \( \ker(L) \subset \ker(C) \).

We thus conclude that there exists a matrix \( V \) such that \( C = VL \) so the state weighting matrix \( Q \) must be of the form \( Q = LV^T VL \) for some matrix \( V \). This proves our claim.

We have thus shown that, for a general LQ cost functional to make sense in the context of distributed diffusive control for multi-agent systems, it must necessarily be of the form

\[
J(u, x_0) = \int_0^\infty x(t)^T L W L x(t) + u(t)^T R u(t) dt, \quad \text{for some } W \geq 0 \text{ and } R > 0.
\]

The corresponding distributed LQ problem is to minimize, for the system (2) with initial state \( x_0 \), the cost functional (4) over all control laws of the form \( u = -gLx \) with \( g > 0 \).

As an illustration, we will now provide two important special cases of LQ cost functionals. The first one was studied before in [7] and [5]:

\[
J(u, x_0) = \sum_{i=1}^N \int_0^\infty \sum_{j \in N_i} q(x_i(t) - x_j(t))^2 + ru_i^2(t) dt, \quad \text{(5)}
\]

where \( q \) and \( r \) are positive real numbers. Clearly, (5) is equal to \( J(x_0, u) = \int_0^\infty x(t)^T 2qL x(t) + ru_i^2(t) dt \). Note that \( 2qL = L(2qL)^L \) with \( L^L \) the Moore-Penrose inverse of \( L \) (which is indeed positive semi-definite). Thus this cost functional is of the form (4) with \( W = 2qL^L \) and \( R = rI \).

As a second example, consider

\[
J(x_0, u) = \sum_{i=1}^N \int_0^\infty q(x_i(t) - a_i(t))^2 + ru_i^2(t) dt, \quad \text{(6)}
\]

with

\[
a_i(t) := \frac{1}{d_i} \left( x_i(t) + \sum_{j \in N_i} x_j(t) \right). \quad \text{(7)}
\]

Here, \( q \) and \( r \) are positive weights, \( d_i \) denotes the node degree of agent \( i \) and \( N_i \) its set of neighbors. The idea of the cost functional (6) is to minimize the sum of the deviations between the state \( x_i(t) \) and the average \( a_i(t) \) of the states of its neighbors (including itself) and the control energy. In order to put this in the form (4), define

\[
G := (D + I)^{-1}(A + I),
\]

where \( D \) is the degree matrix and \( A \) the adjacency matrix. Then clearly \( a(t) = G x(t) \), where \( x = (x_1, x_2, \ldots, x_N)^T \) and \( a = (a_1, a_2, \ldots, a_N)^T \). It is then easily seen that

\[
J(x_0, u) = \int_0^\infty q x(t)^T (I - G)^T (I - G) x(t) + ru_i^2(t) u(t) dt.
\]
Since \((I-G)^T(I-G) = L(D+I)^{-2}L\), we conclude that \((6)\) is a special case of \((4)\) with \(W = q(D+I)^{-2}\) and \(R = rI\).

### III. CENTRALIZED OPTIMAL GAIN

In this section we will briefly give a solution to the general distributed LQ problem introduced in Section II, showing that, indeed, the computation of the optimal protocol requires global information on the network graph and the initial state of the entire network.

Consider the general cost functional \((4)\) together with the dynamics \((2)\) with given initial state \(x_0\). Substituting the general admissible control law \(u = -gLx\) into the cost functional yields

\[
J(g) := x_0^T \left( \int_0^\infty e^{-gLt} (LWL + g^2LRL) e^{-gLt} dt \right) x_0
\]

(9)

Clearly, we need to minimize \(J(g)\) over \(g > 0\). Substituting \(gt = \tau\), we find

\[
J(g) := x_0^T \int_0^{\infty} e^{-\tau L} \left( \frac{1}{g} LWL + gLRL \right) e^{-\tau L} d\tau x_0.
\]

Define \(X_0 := \int_0^{\infty} e^{-\tau L} LWLe^{-\tau L} d\tau\) and \(Y_0 := \int_0^{\infty} e^{-\tau L} LRL e^{-\tau L} d\tau\). It turns out that both integrals indeed exist, and that they can be computed as particular solutions of the Lyapunov equations

\[
-LX - XL + LWL = 0 \quad (10a)
\]

\[
-LY -YL + LRL = 0 \quad (10b)
\]

Indeed, although \(L\) is not Hurwitz, these equations do have positive semi-definite solutions \(X\) and \(Y\) and, in fact, \(X_0\) is the unique positive semi-definite solution \(X\) to \((10a)\) with the property that \(\text{im}(1_N) \subset \ker(X)\). Likewise \(Y_0\) is the unique positive semi-definite solution \(Y\) of \((10b)\) with the property that \(\text{im}(1_N) \subset \ker(Y)\) (see Proposition 1 in [13]). It follows from \((10b)\) that, in fact, \(\ker(Y_0) = \text{im}(1_N)\). Thus we see that \(J(g) = \frac{1}{2} x_0^T X_0 x_0 + g x_0^T Y_0 x_0\).

In order to minimize \(J(g)\) we distinguish three cases. (i) If \(x_0 \in \ker(Y_0) = \text{im}(1_N)\) then we must have \(x_0 \in \ker(X_0)\) as well, so \(J(g) = 0\) for all \(g > 0\) and every \(g > 0\) is optimal. (ii) If \(x_0^T Y_0 x_0 > 0\) and \(x_0^T X_0 x_0 = 0\) then no optimal \(g > 0\) exists. (iii) If \(x_0^T Y_0 x_0 > 0\) and \(x_0^T X_0 x_0 > 0\) then an optimal \(g > 0\) exists and can be shown to be equal to \(g^* = \left( x_0^T X_0 x_0 \right)^{\frac{1}{2}} \left( x_0^T Y_0 x_0 \right)^{-\frac{1}{2}}\).

As announced in Section I, it should now be clear that the computation of the optimal gain \(g\) requires exact knowledge of the network graph in the form of the Laplacian \(L\). Also, the optimal gain highly depends on the global initial state of the network.

### IV. TOWARDS DECENTRALIZED COMPUTATION

In this section we will propose a new approach to compute 'good' local gains that can be computed in a decentralized way. Instead of doing this for the general LQ cost functional \((4)\), we will zoom in on the particular case given by \((6)-(7)\).

In order to decentralize the computation, instead of minimizing the global cost functional \((6)\) for the multi-agent system \((2)\), we write it as a sum of local cost functionals and assign to each agent one of these local cost functionals. More specifically, the associated local cost functional for agent \(i\) is given by

\[
J_i(u_i) = \int_0^{\infty} q \left( x_i(t) - a_i(t) \right)^2 + ru_i^2(t) \ dt,
\]

(11)

where \(a_i(t)\) is defined in \((7)\), for \(i = 1, 2, \ldots, N\). The idea of this local cost functional is to minimize the quadratic difference between the state of the \(i\)-th agent and the average state of its neighboring agents (including itself) and to penalize the local control energy. By doing this, agent \(i\) would already try to make the difference between its own state and the average of the states of its neighbors (including itself) small. Together, these local control laws could be hoped to form a global protocol that reaches consensus for the network. Note, however, that the complete future trajectory \(a_i(t)\) associated with the neighboring agents appears in the local cost functional \((11)\). This future trajectory is not known, so also not available to the \(i\)-th agent. It is therefore impossible for agent \(i\) to compute the optimal local control law.

Because direct minimization of the local cost functional \((11)\) is impossible, as an alternative we will replace each of these local optimal control problems by a sequence of linear quadratic tracking problems that do turn out to be tractable.

More specifically, we choose a sampling period \(T > 0\), and introduce the following sampling procedure. For each nonnegative integer \(k\), at time \(t = kT\) the \(i\)-th agent receives the sampled state value \(x_i(kT)\) of its neighboring agents and takes the average of these, which is given by

\[
a_i(kT) = \frac{1}{d_i} + 1 \left( x_i(kT) + \sum_{j \in N_i} x_j(kT) \right).
\]

(12)

Then, the \(i\)-th agent minimizes the cost functional

\[
J_{i,k}(u) = \int_0^{\infty} e^{-2\alpha t} \left( q \left( x_i(t) - a_i(kT) \right)^2 + ru_i^2(t) \right) dt.
\]

(13)

In fact, this is a discounted linear quadratic tracking problem with constant reference signal \(a_i(kT)\) and discount factor \(\alpha > 0\). By solving this linear quadratic tracking problem, agent \(i\) obtains an optimal control law over an infinite time interval. However, agent \(i\) applies this control law only on the time interval \([kT, (k+1)T]\).

Then, at time \(t = (k+1)T\) the above procedure is repeated, i.e. agent \(i\) receives the updated average \(a_i((k+1)T)\), and subsequently solves the discounted tracking problem with cost functional \(J_{i,k+1}(u)\) which involves the constant updated reference signal \(a_i((k + 1)T)\). By performing this control design procedure sequentially at each sampling time \(kT\), we then obtain a single control law for agent \(i\) over the entire interval \([0, \infty)\).

Based on this control design procedure for the individual agents, we will obtain a distributed control protocol for the entire multi-agent system, simply by letting all agents compute their own control law. In the sequel we will analyze this protocol and show that it achieves consensus for the network:

**Definition 1:** A distributed control protocol is said to achieve consensus for the network if \(x_i(t) - x_j(t) \to 0\)
as \( t \to \infty \) for all initial states of agents \( i \) and \( j \), for all \( i, j = 1, 2, \ldots, N \).

In order to obtain an explicit expression for the control protocol proposed above, we will study the linear quadratic tracking problem for a single linear system. This will be done in the next section.

V. THE DISCOUNTED LQ TRACKING PROBLEM

In this section, we will deal with the discounted linear quadratic tracking problem for a given linear system. The linear quadratic tracking problem has been studied before, see e.g. [14]. Here, however, we will solve it by transforming it into a standard linear quadratic control problem.

Consider the continuous-time linear time-invariant system

\[
\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0,
\]

with \( A \in \mathbb{R}^{n \times n} \) and \( B \in \mathbb{R}^{n \times m} \), and where \( x(t) \in \mathbb{R}^n \), \( u(t) \in \mathbb{R}^m \) denote the state and the input, respectively. We assume that the pair \((A, B)\) is stabilizable. Given is also a constant reference signal \( r \). Since the pair \((A, B)\) is stabilizable, the linear quadratic tracking problem has been studied before, and we will introduce a discounted quadratic cost functional given by

\[
J(u) = \int_0^\infty e^{-\alpha t} \left( \langle x(t) - r \rangle^\top Q (x(t) - r) + u(t)^\top R u(t) \right) dt
\]

where \( Q > 0 \) and \( R > 0 \) are given weight matrices and \( \alpha > 0 \) is a discount factor [14]. The linear quadratic tracking problem is to determine for every initial state \( x_0 \) a piecewise continuous input function \( u(t) \) that minimizes the cost functional (15).

To solve this problem, we introduce the variables

\[
z(t) = e^{-\alpha t} x(t), \quad z_r(t) = e^{-\alpha t} r, \quad v(t) = e^{-\alpha t} u(t),
\]

and denote \( \xi(t) = (z^\top(t), z_r^\top(t))^\top \). Then we obtain an auxiliary system in terms of \( \xi \) and \( v \), given by

\[
\dot{\xi}(t) = A \xi(t) + B \nu(t), \quad \xi_0 = (x_0^\top, r^\top)^\top,
\]

where \( \xi_0 \in \mathbb{R}^{2n} \) is the initial state and

\[
A_e = \begin{pmatrix} A - \alpha I & 0 \\ 0 & -\alpha I \end{pmatrix}, \quad B_e = \begin{pmatrix} B \\ 0 \end{pmatrix}.
\]

In terms of the new variables \( \xi \) and \( v \), the cost functional (15) can be written as

\[
J(v) = \int_0^\infty \langle \xi(t) \rangle^\top Q_e \xi(t) + v^\top(t) R v(t) dt,
\]

where \( Q_e = \begin{pmatrix} Q & -Q \\ -Q & Q \end{pmatrix} \). The problem is now to find, for every initial state \( \xi_0 \), a piecewise continuous input function \( v(t) \) that minimizes this cost functional. This is a so-called a free endpoint standard LQ control problem, see [15, pp. 218].

Since the pair \((A, B)\) is stabilizable, the pair \((A_e, B_e)\) is also stabilizable and hence the input function \( v(t) \) that minimizes the cost functional \( J(v) \) is generated by the feedback law

\[
v(t) = -R^{-1} B_e^\top P_e \xi(t),
\]

where \( P_e^- \) is the smallest positive semi-definite solution of the Riccati equation

\[
A_e^\top P_e^- + P_e^- A_e - P_e^- B_e R^{-1} B_e^\top P_e^- + Q_e = 0.
\]

Now, partition \( P_e^- \) as

\[
\begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix},
\]

where all blocks have dimension \( n \times n \). Recalling (16) and (17), we then immediately find an expression for the input function \( u(t) \)

that minimizes the cost functional (15) for the system (14) and reference signal \( r = r(t) \).

Theorem 1: The input function \( u(t) \) that minimizes the cost functional (15) is generated by the control law

\[
u(t) = K_1 x(t) + K_2 r,
\]

where \( K_1 = -R^{-1} B_1^\top P_1 \) and \( K_2 = -R^{-1} B_2^\top P_2 \).

Remark 2: Let \( e(t) := x(t) - r \) denote the tracking error. Because \( Q > 0 \), the control law (19) only guarantees that \( e(t) = e^{-\alpha t} e(t) \) tends to zero as \( t \) goes to infinity. Thus, the feedback law that minimizes the LQ tracking cost functional (15) only guarantees the actual tracking error \( e(t) \) to be exponentially bounded with growth rate \( \alpha > 0 \). Note that \( \alpha > 0 \) can be taken arbitrarily small.

It will be shown however that, for the multi-agent system case, the control design method established in this section will, nevertheless, lead to a protocol that achieves consensus.

VI. CONSENSUS ANALYSIS

In this section, we will show that, by adopting the control design method for the multi-agent system (2) as proposed in Section IV, the resulting distributed control protocol achieves consensus for the entire network.

As already explained in Section IV, we choose a sampling period \( T > 0 \) and introduce a sampling procedure. For each nonnegative integer \( k \), at time \( t = kT \) the \( i \)-th agent receives the sampled state value of its neighboring agents (including itself) and minimizes the cost functional (13), which is a discounted linear quadratic tracking problem with constant reference signal \( r(t) = a_t(kT) \) and discount factor \( \alpha > 0 \).

According to the theory on the discounted LQ tracking problem described in Section V, the local optimal control law for agent \( i \) at time \( t = kT \) over the whole time horizon \([0, \infty)\) is therefore of the form

\[
u_{i,(k)}(t) = u_{i,k}(x(t)) + g_{i,k} a_t(kT),
\]

in which the control gains \( g_{i,k} \) and \( g_{i,k}' \) can be computed explicitly by solving the Riccati equation (18) associated with the LQ tracking problem for agent \( i \).

Lemma 3: Consider, at time \( t = kT \), the \( i \)-th agent of the multi-agent system (1) with associated local cost functional (13). Denote

\[
\bar{A} = \begin{pmatrix} -\alpha & 0 \\ 0 & -\alpha \end{pmatrix}, \quad \bar{B} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \bar{Q} = \begin{pmatrix} q & -q \\ -q & q \end{pmatrix}.
\]

Let \( \bar{P} := \begin{pmatrix} p_1 & p_{12} \\ p_{12} & p_2 \end{pmatrix} \) be the smallest positive semi-definite solution of the Riccati equation

\[
\bar{A}^\top \bar{P} + \bar{P} \bar{A} - \bar{R} \bar{P} \bar{B}^\top \bar{P} + \bar{Q} = 0.
\]

Then the local control law (20) with \( g_{i,k} := -r^{-1} p_1 \) and \( g_{i,k}' := -r^{-1} p_{12} \) minimizes the cost (13) for agent \( i \).

Proof: This follows immediately from Theorem 1.

Next, agent \( i \) applies the control law (20) only on the time interval \([kT, (k+1)T)\). Then, at time \( t = (k+1)T \) the above procedure is repeated.

Since, for all \( i = 1, 2, \ldots, N \) and \( k = 0, 1, \ldots \), the matrices \( \bar{A}, \bar{B} \) and \( \bar{Q} \) are independent of \( i \) and \( k \), the same
holds for the gains $g_{i,k}$ and $g'_{i,k}$. In the sequel, we will therefore drop the subscripts in the control gains $g_{i,k}$ and $g'_{i,k}$ and denote them by $g$ and $g'$, respectively. Moreover, using (21), we compute $g = r^{-1}(\alpha - \sqrt{\alpha^2 + 4r})$ and $g' = -g$.

By performing this procedure sequentially at each sampling time $kT$, we then obtain a single control law for agent $i$ over the entire interval $[0, \infty)$ as

$$u_i(k) = gx_i(t) - ga_i(kT), \quad t \in [kT, (k+1)T),$$

where $g = r^{-1}(\alpha - \sqrt{\alpha^2 + 4r}) < 0$.

Recall that $a(t) = Gx(t)$, with $G$ given by (8), and that $a(kT) = Gx(kT)$. Therefore, the local control laws for the individual agents lead to a distributed control protocol

$$u_k(t) = gx(t) - gGx(kT), \quad t \in [kT, (k+1)T).$$

Now, by applying the protocol (23) to the multi-agent system (1), we find that the controlled network is represented by

$$\dot{x}(t) = gx(t) - gGx(kT), \quad t \in [kT, (k+1)T).$$

In the remainder of this section, we will analyze this representation, and show that consensus is achieved, i.e. for each initial state $x(0) = x_0$ we have $x_i(t) - x_j(t) \to 0$ as $t$ tends to infinity.

In order to do this, note that the solution of (24) with initial state $x(0) = x_0$ is given by

$$x(t) = e^{gT}x(kT) - \int_k^t e^{g(T-\tau)}gGx(kT) d\tau,$$

for $t \in [kT, (k+1)T)$, $k = 0, 1, 2, \ldots$. Obviously, for each initial state $x_0$, the corresponding solution $x(t)$ is continuous.

From (25) we see that the sequence of network states $x(kT)$ evaluated at the discrete time instances $kT$, $k = 0, 1, \ldots$ satisfies the difference equation

$$x((k+1)T) = \Gamma x(kT),$$

where $\Gamma = e^{gT}I - (e^{gT} - 1)G$.

Clearly, the network reaches consensus if and only if for each $x_0$, $x_i(kT) - x_j(kT) \to 0$ as $t$ tends to infinity.

We proceed with analyzing the eigenvalues of $\Gamma$.

**Lemma 4:** The matrix $G$ has an eigenvalue 1 with algebraic multiplicity equal to one and associated eigenvector $1_N$. The remaining eigenvalues of $G$ are all real and have absolute value strictly less than 1.

**Proof:** Since $L = D - A$, we have $G = I - (D + I)^{-1}L$. Hence we have $D^2G = I = D^{-\frac{1}{2}}LD^{-\frac{1}{2}}$ where $D = D + I$. Note that the right hand side is symmetric and hence has only real eigenvalues. Thus, by matrix similarity, $G$ also has only real eigenvalues.

Next, we show that $G$ has a simple eigenvalue 1 with associated eigenvector $1_N$. First note that

$$G1_N = (I - (D + I)^{-1}L)1_N = 1_N.$$  

Hence, indeed, 1 is an eigenvalue of $G$ with eigenvector $1_N$. Since $G$ is similar to a symmetric matrix, it is diagonalizable, so the algebraic multiplicity of its eigenvalue 1 must be equal to its geometric multiplicity. Suppose now that 1 is not a simple eigenvalue. Then there must exist a second eigenvector, say $\psi$, which is linearly independent of $1_N$. This implies $G\psi = \lambda\psi$. Then $L\psi = 0$, so $\psi$ must be a multiple of $1_N$. This is a contradiction. We conclude that the eigenvalue 1 is indeed simple.

Finally, it follows from Gershgorin’s Theorem [16] that every eigenvalue $\lambda$ of $G$ satisfies $-1 < \lambda \leq 1$.

Before we give the main result of this paper, we first review the following proposition.

**Proposition 5:** Consider the discrete-time system

$$x(k+1) = Ax(k), \quad x(0) = x(0), \quad y(k) = Cx(k),$$

with $A \in \mathbb{R}^{n \times n}$ and $C \in \mathbb{R}^{p \times n}$, where $x(k) \in \mathbb{R}^n$ is the state, $x_0$ is the initial state and $y(k) \in \mathbb{R}^p$ is the output. Then, $y(k) \to 0$ as $k \to \infty$ for all initial states $x_0$ if and only if $X_+(A) \subset \ker(C)$. Here, $X_+(A)$ is the unstable subspace, i.e., the sum of the generalized eigenspaces of $A$ associated with its eigenvalues in $\{ \lambda \in \mathbb{C} \mid |\lambda| \geq 1 \}$.

**Proof:** A proof can be given by generalizing the results [15, pp. 99] to the discrete time case.

We are now ready to present the main result of this paper.

**Theorem 6:** Consider the multi-agent system (1). Let $T > 0$ be a sampling period, $\alpha > 0$ a discount factor, and let $q, r > 0$ be given weights. Let $P$ be the smallest positive semi-definite solution of the Riccati equation (21) and partition $\tilde{P} := \begin{pmatrix} p_1 & p_12 \\ p_12 & p_2 \end{pmatrix}$. Then the distributed control protocol (23) with $g = -r^{-1}p_1$ and $g' = -r^{-1}p_12$ achieves consensus for the controlled network (24).

**Proof:** The network reaches consensus if and only if $Lx(kT) \to 0$ as $k \to \infty$. Since $\ker(L) = \text{im}(1_N)$, it then follows from Proposition 5 that consensus is achieved if and only if $X_+(\Gamma) \subset \ker(L)$; equivalently, the sum of the generalized eigenspaces of $\Gamma$ corresponding to the eigenvalues $\lambda$ with $|\lambda| \geq 1$ is equal to $\text{im}(1_N)$.

Indeed, we will show that all eigenvalues $\lambda$ of $\Gamma$ are real and satisfy $-1 < \lambda \leq 1$, and $\lambda = 1$ is a simple eigenvalue with associated eigenvector $1_N$.

Recall that $\Gamma = e^{gT}I - (e^{gT} - 1)G$. Hence, $\mu$ is an eigenvalue of $\Gamma$ if and only if $\mu = e^{gT} - \lambda(e^{gT} - 1)$ where $\lambda$ is an eigenvalue of $G$. It was shown in Lemma 4 that all eigenvalues $\lambda$ of $G$ are real and satisfy $-1 < \lambda \leq 1$ and, moreover, $\lambda = 1$ is a simple eigenvalue. Using the fact that $g < 0$ we thus obtain that the eigenvalues $\mu$ of $\Gamma$ satisfy $-1 < \mu \leq 1$ and $\mu = 1$ is a simple eigenvalue of $\Gamma$.

Finally, we will show $\mu = 1$ has eigenvector $1_N$. Indeed, this follows from $\Gamma 1_N = (e^{gT}I - (e^{gT} - 1)G)1_N = 1_N$. This completes the proof.

**Remark 7:** By analyzing the eigenvalues $\mu$ of $\Gamma$ satisfying $-1 < \mu < 1$, it can be seen that, for given $\alpha$, the convergence rate of the difference equation (26) increases with increasing sampling period $T$. The total time it takes to reach a disagreement smaller than a given tolerance is then the product of the number of iterations in (26) and this sampling period. It might therefore be more advantageous to use a smaller sampling period with a larger number of required iterations, but yet leading to a smaller total time. In other words, the choice of sampling period is a trade-off between the total time required to obtain an acceptable disagreement, and the number of iterations in (26).
VII. SIMULATION

Consider a network of six agents with single integrator dynamics \( \dot{x}_i(t) = u_i(t), i = 1, 2, \ldots, 6 \), where the initial states are \( x_{10} = 1, x_{20} = 2, x_{30} = -1, x_{40} = -2, x_{50} = 1 \) and \( x_{60} = 3 \). We assume that the communication among these agents is represented by an undirected circle graph with six nodes. First, we take the sampling period to be equal to \( T = 10 \). On the time interval \( t \in [kT, (k + 1)T) \), \( k = 0, 1, \ldots \), we consider the local cost functional (13) for agent \( i \). We choose the weights to be \( q = 2, r = 1 \) and the discount factor \( \alpha = 0.01 \). We adopt the control design proposed in Theorem 6 and compute the smallest positive semi-definite of the Riccati equation \( A^\top P + PA - r^{-1}PBB^\top P + Q = 0 \) with

\[
A = \begin{pmatrix}
-0.01 & 0 \\
0 & -0.01
\end{pmatrix}, \quad B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad Q = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}.
\]

This Riccati equation has a unique positive semi-definite solution which is given by

\[
P = \begin{pmatrix}
1.4042 & -1.4042 \\
-1.4042 & 1.4042
\end{pmatrix}.
\]

Thus we find the control gains \( g = -1.4042 \) and \( g' = 1.4042 \). Subsequently, the local control law for agent \( i \) is given by \( u_{i,k}(t) = -1.4042x_i(t) + 1.4042a_i(kT) \) for \( t \in [kT, (k + 1)T) \) and \( i = 1, 2, 3 \) and \( k = 0, 1, \ldots \).

In Figure 1 we have plotted the controlled trajectories of the individual agents. It can be seen that the protocol resulting from the local control laws indeed achieves consensus. The results of a second simulation, this time with sampling period \( T = 0.1 \), are plotted in Figure 2.

By comparing Figure 1 and 2, it can be seen that the network reaches consensus faster by taking a smaller sampling period.

VIII. CONCLUSION

We have studied the distributed linear quadratic control problem for a network of agents with single integrator dynamics. We have shown that the computation of control gains that minimize global cost functionals need global information, in particular the initial states of all agents and the Laplacian matrix. We have also shown that this drawback can be overcome by transforming the global cost functional into discounted local cost functionals and assigning each of these to an associated agent. In such a way, each agent computes its own control gain, using sampled information of its neighboring agents. Finally, we have shown that the resulting control protocol achieves consensus for the network.

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