In this Letter, we consider fluctuation-induced Casimir forces or pressures in liquid layers caused by mode-coupling between hydrodynamic modes in the presence of shear flow. Using planar Couette flow as an example, we show how the shear-induced pressure enhancement depends on the Reynolds number. Explicit expressions for these shear-induced Casimir pressures are presented which complete and correct expressions currently available in the literature. These nonequilibrium Casimir pressures are considerably smaller than those in liquid layers in the presence of a temperature gradient. Furthermore, computer simulations of model fluids in shear observe effects from molecular correlations at nanoscales that have a different physical origin. The idea that such computer simulations probe shear-induced Casimir pressures resulting from coupling of long-wavelength hydrodynamic modes is erroneous.

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When large and long-range fluctuations are present, they will induce forces in confined fluids [1]. They are commonly referred to as Casimir-like forces in analogy to forces induced by vacuum fluctuations between two conducting plates [2]. Well-known examples are Casimir forces due to critical fluctuations [3] or due to long-range correlations in condensed systems with Goldstone modes [1]. It has now been well established that even longer-range thermal fluctuations exist in fluids in nonequilibrium states [4]. The physical reason is that the presence of a gradient breaks the symmetry and causes a coupling between long-wavelength hydrodynamic modes [5].

In this Letter we consider Casimir forces due to long-range velocity fluctuations in liquids in shear [4,5]. For the case of a liquid subjected to planar laminar Couette flow, we have obtained explicit expressions for the shear-induced pressure enhancement \( \delta p \), which complete and correct results obtained by previous authors [6-12]. We provide quantitative estimates for the magnitude of these shear-induced Casimir pressures. We clarify an essential difference between the Casimir pressures resulting from macroscopic long-range fluctuations and forces resulting from the effect of fluctuations at nanoscales which are the ones observed in computer simulations [13-15].

To maintain consistency with a previous analysis of velocity fluctuations in planar Couette flow by two of us [20,21], we continue to use a coordinate system which the \( x \) coordinate is in the stream-wise direction, the \( y \) coordinate in the span-wise direction, and the \( z \) coordinate in the wall-normal direction. The liquid layer is located between two horizontal plates located at \( z = \pm L \) moving with constant velocities \( \pm U \) in the \( z \) direction. We assume no-slip boundary conditions for the velocities at \( z = \pm L \) [22]. The local fluid velocity can be decomposed as \( \mathbf{v} = \mathbf{v}_0(z) + \delta \mathbf{v} \), where \( \mathbf{v}_0 = \mathbf{v}_0(\gamma z, 0, 0) \) is the average velocity dependent on the shear rate \( \gamma = U/L \) with a component only in the stream-wise direction \( x \), and where \( \delta \mathbf{v}(\mathbf{r}, t) \) is a fluctuating-velocity contribution dependent on location \( \mathbf{r} = (x, y, z) \) and on time \( t \). Just as in our previous work on a pressure enhancement induced by a temperature gradient [23], we use nonlinear fluctuating hydrodynamics. Since density fluctuations have been shown to decay faster than velocity fluctuations [2], the leading fluctuation renormalization to the pressure tensor is, see Supplemental Material:

\[
\delta P(\mathbf{r}) = \rho \langle \delta \mathbf{v}(\mathbf{r}) \delta \mathbf{v}(\mathbf{r}) \rangle_{\text{NE}}.
\]  

(1)

Here \( \rho \) is the mass density and the average is taken over the stationary nonequilibrium state which is independent of the time \( t \). The diagonal elements \( \delta p_{ii} = \rho \langle \delta v_i(0) \delta v_i(t) \rangle \) contribute to the shear-induced pressure enhancement, such that \( \delta p \equiv \frac{1}{2} (\delta p_{xx} + \delta p_{yy} + \delta p_{zz}) \). The off-diagonal elements all vanish except for \( \delta p_{xz} = \rho \langle \delta v_x(t) \delta v_z(t) \rangle \), yielding a fluctuation-induced contribution to the shear viscosity \( \eta \) [11]. We find that all diagonal elements depend on the shear rate \( \gamma \) and the Reynolds number \( \text{Re} = \gamma L^2/\nu \) as

\[
\delta p_{ii} = V_{ii}^* k_B T \left( \frac{\nu}{\zeta} \right)^{3/2} \varphi_{ii}^* (\text{Re}) \, .
\]

(2)

where \( k_B \) is Boltzmann’s constant, \( T \) the temperature, and \( \nu \) the kinematic viscosity. Here \( \varphi (\text{Re}) \) is a crossover function such that \( \lim_{\text{Re} \to \infty} \varphi_{ii}^*(\text{Re}) = 1 \) and \( \lim_{\text{Re} \to 0} \varphi_{ii}^*(\text{Re}) = (V^0_{ii}/V_{ii}^*)(\text{Re})^{1/2} \). Specifically, the two limiting cases are

\[
\delta p_{ii}^\infty \equiv \lim_{\text{Re} \to \infty} \delta p_{ii} = V_{ii}^* k_B T \left( \frac{\nu}{\zeta} \right)^{3/2},
\]

\[
\delta p_{ii}^0 \equiv \lim_{\text{Re} \to 0} \delta p_{ii} = V_{ii}^0 k_B T \left( \frac{\nu}{\zeta} \right)^2,
\]

(3)

(4)

We emphasize that the anomalous dependence of \( \delta p \) on \( \gamma^{3/2} \) and on \( L^2 \) is a consequence of a coupling between macroscopic viscous modes and not from static or dynamic correlations at molecular scales.
In previous publications we have shown how the correlation functions for the wall-normal velocity fluctuations and for the wall-normal vorticity fluctuations can be derived by solving the fluctuating Orr-Sommerfeld and Squire equations [20, 21]. For large $L$ and, hence, for large Re, we have obtained an exact solution of these fluctuating hydrodynamics equations, since in this limit the solution becomes independent of the boundary conditions. For small $L$ and, hence, for small Re, the solution is strongly affected by the no-slip boundary condition for the velocity. In this limit it is difficult to get an exact solution [24, 22] and we have settled for an approximate solution in a Galerkin approximation [20, 21]. Explicit expressions for the diagonal elements of the shear-induced pressure tensor can be directly related to the solutions previously obtained for the wall-normal velocity and vorticity fluctuations as shown in the Supplemental Material and which does not involve any new physics [20]. The only additional step required is integration of the correlation functions over wave numbers so as to get the intensity of the velocity fluctuations in real space. The resulting values are $V_{\infty}^{xx} = +0.0847$, $V_{\infty}^{yy} = +0.0173$, $V_{\infty}^{zz} = +0.0106$, and $V_{\infty}^{xy} = +0.001498$, $V_{\infty}^{yz} = +0.000480$, $V_{\infty}^{zx} = +0.000392$. Actually, the solution from the fluctuating hydrodynamics equations yield expressions for $V_{\infty}^{ii}$ that depend on the $z$ coordinate as a consequence of the boundary conditions at $z = \pm L$. However, just as in the case of the Casimir pressures induced by a temperature gradient [23], mechanical equilibrium, combined with conservation of mass, causes a uniform pressure enhancement equal to the height-averaged value obtained from the fluctuating hydrodynamics equations. Hence, we only quote here the height-averaged values for $V_{\infty}^{ii}$. We thus conclude:

$$\delta p_{\infty}^{\infty} = \frac{1}{3} \sum_i \delta p_{\infty i}^{\infty} = +0.0375 \ k_B T (\frac{c}{c_0})^{3/2},$$

$$\delta p_{\infty}^{0} = \frac{1}{3} \sum_i \delta p_{\infty i}^{0} = +0.000770 \ k_B T L (\frac{c}{c_0})^2.$$

We do not consider contributions from the sound modes here, since they are smaller by a factor of $(U/c)^{1/2}$, where $c$ is the speed of sound. Terms of higher order in the shear-rate have also been neglected as discussed in Section III of the Supplemental Material [26].

Attempts to determine the shear-induced pressure tensor in the absence of boundary conditions have been made by Kawasaki and Gunton [9] and by Yamada and Kawasaki [10]. While they did find that the shear-induced pressure varies with the shear rate as $\gamma^{3/2}$, the numerical coefficients are substantially different from the values found by us as shown in Table I.

Ernst et al. [11] determined the traceless part of the shear-induced pressure tensor using a kinetic-theory approach. Our results for the traceless part of the shear-induced pressure tensor are in perfect agreement with those obtained by Ernst et al. as shown in Table II. In this table we also see perfect agreement with their off-diagonal stress, $V_{xz}$, which gives a generalized viscosity. Hence, we are confident that we have obtained correct expressions for the shear-induced pressure enhancement as given by Eqs. (5) and (6). Wada and Sasa [12] have only determined the wall-normal component of the shear-induced pressure tensor. They find $V_{xz}^{\infty} = +0.0106$ in agreement with our result, but their value $V_{xz}^{0} = 0.000553$ slightly differs from $V_{xz}^{0} = 0.000392$ found by us. The reason is that Wada and Sasa used periodic boundary conditions which are mathematically convenient, but physically unrealistic.

To estimate the magnitude of the shear-induced pressure enhancement we consider water, which is the liquid commonly used in Couette-flow experiments [27–34]. The smallest gap width thus far employed is about 1.5 mm [31]. The possible experimental plate velocities $U$ may be up to 0.5 m s$^{-1}$ [37]. A gap width of 1 mm ($L = 0.5$ mm) and plate velocities $U = \pm 0.5$ m s$^{-1}$ imply Re $\approx 280$, which is still well below the critical Reynolds number for the onset of turbulence [27]. Substituting $\nu = 8.93 \times 10^{-7}$ m$^2$s$^{-1}$ for the kinematic viscosity of water at 273.15 K [36] into Eqs. (5) and (6) we obtain the estimates

$$\delta p_{\infty}^{\infty} = 6 \times 10^{-9} \text{ Pa} \quad \text{and} \quad \delta p_{\infty}^{0} = 2 \times 10^{-9} \text{ Pa},$$

i.e., the shear-induced pressure enhancement is somewhere between $10^{-9}$ and $10^{-8}$ Pa. It is interesting to compare this shear-induced pressure enhancement with those in a liquid layer with the same gap width either from critical fluctuations $\delta p = -2 \times 10^{-11}$ Pa (from Ref. [37], corrected for a sign error) or from nonequilibrium temperature fluctuations caused by the presence of a temperature gradient (25 K /mm) $\delta p = 5 \times 10^{-4}$ Pa [23].

We see that the shear-induced pressure enhancement is many orders of magnitude smaller than the Casimir pressures induced by the presence of a temperature gradient. One reason is that temperature fluctuations decay more slowly than velocity fluctuations and, hence, are more strongly impacted by the presence of a temperature gradient. Another reason is that the shear-induced pressure enhancement has a kinetic origin, while the pressure enhancement from a temperature gradient has a potential origin that in liquids is several orders of magnitude larger.

An important consequence is that, unlike the case of a temperature gradient [23], effects from short-range corre-

### Table I: Comparison with literature

|               | $V_{xz}^{\infty}$ | $V_{xy}^{\infty}$ | $V_{yy}^{\infty}$ |
|---------------|-------------------|-------------------|-------------------|
| Kawasaki and Gunton [9] | +0.0050 | -0.0046 | -0.0017 |
| Yamada and Kawasaki [10] | +0.0428 | +0.0173 | +0.0106 |
| This work | +0.0847 | +0.0046 | +0.0106 |
lations to the shear-induced pressure enhancement may not be negligible. To elucidate a possible contribution from short-range correlations, we note from nonequilibrium statistical mechanics that \( \delta p = \kappa \gamma^2 \), where \( \kappa \) is a nonlinear Burnett coefficient. These nonlinear Burnett coefficients are known to diverge as \( L \to \infty \) [38]. We may decompose this Burnett coefficient as the sum of a finite short-range contribution \( \kappa^{(0)} \) and a long-range contribution \( L \kappa \) [23], yielding a short-range (SR) and a long-range (LR) contribution the shear-induced pressure enhancement:

\[
\delta p = \delta p_{\text{SR}} + \delta p_{\text{LR}},
\]

where \( \delta p_{\text{SR}} = \kappa^{(0)} \gamma^2 \) and \( \delta p_{\text{LR}} = L \kappa \gamma^2 \). Comparing with Eq. (10), we note that the shear-induced Casimir pressure arises from the same long-wavelength hydrodynamic modes that cause the nonlinear Burnett coefficient \( \kappa \) to diverge. A complete kinetic theory for the nonlinear Burnett coefficients of real fluids is not available, but it is possible to get an order-of-magnitude estimate by extending the theory of Enskog for the transport properties of a dense gas of hard spheres to the quadratic level [39]. Starting from an expression for the pressure tensor of a gas of hard spheres provided by Dufty [10] and retaining only the collisional transfer contribution, which is the dominant one at high densities, we obtain

\[
\delta p_{\text{SR}} \approx \rho \sigma^2 n \sigma^3 \frac{17 \pi}{45} \chi^2,
\]

where \( \sigma \) is the hard-sphere diameter, \( n \) the number density, and \( \chi \) the value of the radial distribution function at contact between the spheres. For liquid water \( \rho = mm = 10^3 \text{ kg m}^{-3}, m = 3 \times 10^{-26} \text{ kg}, \sigma = 3 \times 10^{-10} \text{ m} \). Estimating \( \chi \approx 5 \) for a dense liquid, we then conclude from Eq. (8) that for water with \( L = 0.5 \text{ mm} \) and \( U = 0.5 \text{ m s}^{-1} \)

\[
\delta p_{\text{SR}} \approx 2 \times 10^{-10} \text{ Pa}.
\]

On comparing Eq. (10) with Eq. (7) we see that the SR contribution to the induced-pressure enhancement is indeed smaller than the LR contribution to the shear-induced pressure enhancement, but may not be entirely negligible, even at \( L = 0.5 \text{ mm} \). The SR contribution becomes even more important at smaller values of \( L \). From Eq. (9) it follows that, for a fixed velocity \( U \), \( \delta p_{\text{SR}} \) will increase as \( L^{-2} \), while \( \delta p_{\text{LR}} \), due to the long-range velocity fluctuations, will only increase either as \( L^{-3/2} \) for large values of \( \text{Re} \) or even less as \( L^{-1} \) for small values of \( \text{Re} \).

Another important consequence is that most computer simulations of model fluids under shear have generally been misinterpreted [14–19]. Investigators have either claimed to have found agreement [14–16] or disagreement [17–19] with the predictions of Eqs. (3) and (4). However, these computer simulations probe effects of fluctuations at nanoscales, which have a completely different physical origin and need to be distinguished from the long-range macroscopic fluctuations responsible for the shear-induced Casimir pressure described by Eqs. (2) – (4).

The first molecular dynamics (MD) simulations on a 3-dimensional sheared fluid consisting of a small number of Lennard-Jones (LJ) particles were performed by Evans [13]. He found results that seemed, especially near the triple point, to indicated a nonequilibrium (NE) pressure enhancement that was proportional to \( \gamma^{3/2} \), but with a coefficient that was orders of magnitude larger than the coefficient to be expected from Eq. (3). He noted a similarity with the so-called molasses tail observed in MD simulations of the equilibrium stress-tensor time correlation functions that determines the shear viscosity [41]. In turns out that in this time-dependent correlation function, again near the triple point of LJ particles or near freezing of hard-sphere particles, an apparent long-time tail proportional to \( \gamma^{1/2} \) appears, but with a coefficient, again, several orders of magnitude larger than the theoretically expected long-time tail coefficient. It was subsequently realized that this molasses tail was not due to long-wave length MC effects, but was due to molecular scale MC effects related to structural relaxation in dense fluids [42–45]. This theory explains the magnitude of observed molasses tails and predicts that this \( 1/\gamma^{1/2} \) behavior will crossover to an exponential decay on a structural time scale \( \tau_s = S(k_0)/D k_0^2 \), where \( D \) is the self-diffusion coefficient and \( k_0 \) the wave number where the static structure factor \( S(k) \) has its maximum [43]. For a review of these molecular scale MC effects, the reader is referred to a forthcoming book of Dorfman et al. [46]. Another complication is that the computer simulations use extremely large shear rates \( \gamma \approx 10^{11} - 10^{12} \text{ s}^{-1} \). At such large shear rates, where \( \gamma > \gamma_s^{-1} \), the NE pressure is also determined by molecular-scale MC effects. The molecular-scale effects will not only depend on the intermolecular potential adopted, but, at a given density, also on the number of free paths sampled, and, hence, on the number of particles used in the simulations.

Generally, it never makes sense, except in some asym-
totic limit, to fit the shear-induced pressure enhancement in terms of a simple power law. For example, Eq. (8) suggests for sufficiently large $L$ a fit to

$$\delta p = A_{SR} \gamma^2 + \delta p_{LR}$$

with $\delta p_{LR} \propto L^2$ for $Re < 1$ or $\delta p_{LR} \propto \gamma^{3/2}$ for $Re > 1$, and with $A_{SR}$ independent of $L$. In MD simulations that probe structural relaxation effects the appropriate fit should have an additional term $\delta p_s \propto \gamma^{3/2}$ for $\gamma > \tau_s^{-1}$ and $\delta p_s \propto \gamma^2$ for $\gamma < \tau_s^{-1}$. That is, for $\gamma > \tau_s^{-1}$, $\delta p_s$ will renormalize $\delta p_{LR}$ in Eq. (11) and for $\gamma < \tau_s^{-1}$, $\delta p_s$ will renormalize $A_{SR}$ in this equation.

Almost all discussions of computer-simulation studies currently available [14–19] have ignored the effects of the molecular-scale correlations that dominate at nanoscales. Lee and Cumming [14, 15] found an enhancement $\propto \gamma^{3/2}$, but without checking the coefficient, they assumed to have found agreement with both the results of Evans [13] and with Eq. (3), which is impossible as explained above. Sadus and coworkers have found effective exponents for the shear-rate dependence ranging from 1.5 to 2 without any theoretical analysis of the results.

The theoretical expression, Eq. (2), for the shear-induced pressure enhancement follows from a solution of the fluctuating hydrodynamics equations for the long-range velocity fluctuations. Recently, Varghese et al. [17] have tried to obtain a numerical solution of the fluctuating hydrodynamics equations for a multiparticle collision dynamics fluid, a model with an ideal-gas equation of state. They conclude that the shear-induced pressure enhancement obtained over about one decade of the shear-rate appears to scale as $\gamma^2$ and therefore does not agree with Eq. (5). However, the magnitude of the enhancement seems to be indeed of the order given by our Eqs. (5) and (6). It would be of interest to pursue such calculations for a larger ranges of $L$ and $Re$ numbers, so as to probe a possible crossover from a behavior $\propto L^2 \gamma^2$ for small $Re$ to $\propto \gamma^{3/2}$ for large $Re$. Varghese et al. [17] conclude their paper with the following comment: “It therefore remains for further theoretical and simulation studies to establish a unified picture of the exponent associated with the hydrodynamic pressure under shear”. This Physical Review Letter attempts to address this issue.

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[1] M. Kardar and R. Golestanian, Rev. Mod. Phys. 71, 1233 (1999).
[2] G. L. Klimchitskaya, U. Mohideen, and V. M. Mostepanenko, Rev. Mod. Phys. 81, 1827 (2009).
[3] M. Krech, The Casimir Effect in Critical Systems (World Scientific, Singapore, 1994).
[4] J. R. Dorfman, T. R. Kirkpatrick, and J. V. Sengers, Annu. Rev. Phys. Chem. 45, 213 (1994).
[5] J. M. Ortiz de Zárate and J. V. Sengers, Hydrodynamic Fluctuations in Fluids and Fluid Mixtures (Elsevier, Amsterdam, 2006).
[6] J. F. Lutsko and J. W. Dufty, Phys. Rev. A 32, 3040 (1985).
[7] J. F. Lutsko and J. W. Dufty, Phys. Rev. E 66, 041206 (2002).
[8] J. V. Sengers and J. M. Ortiz de Zárate, J. Non-Newtonian Fluid Mech. 165, 925 (2010).
[9] K. Kawasaki and J. D. Gunton, Phys. Rev. A 8, 2048 (1973).
[10] T. Yamada and K. Kawasaki, Progr. Theor. Phys. (Japan) 53, 111 (1975).
[11] M. H. Ernst, B. Cichocki, J. R. Dorfman, J. Sharma, and H. van Beijeren, J. Stat. Phys. 18, 237 (1978).
[12] H. Wada and S. I. Sasa, Phys. Rev. E 67, 065302(R) (2003).
[13] D. J. Evans, Phys. Rev. A 23, 1988 (1981).
[14] S. H. Lee and P. T. Cummings, J. Chem. Phys. 99, 3919 (1993).
[15] S. H. Lee and P. T. Cummings, J. Chem. Phys. 101, 6206 (1994).
[16] G. Marcelli, B. D. Todd, and R. J. Sadus, Phys. Rev. E 63, 021204 (2001).
[17] G. Marcelli, B. D. Todd, and R. J. Sadus, Phys. Rev. E 64, 021201 (2001).
[18] J. Ge, B. D. Todd, G. Wu, and R. J. Sadus, Phys. Rev. E 67, 061201 (2003).
[19] A. Ahmed, P. Mausbach, and R. J. Sadus, Phys. Rev. E 82, 011201 (2010).
[20] J. M. Ortiz de Zárate and J. V. Sengers, Phys. Rev. E 77, 026306 (2008).
[21] J. M. Ortiz de Zárate and J. V. Sengers, Phys. Rev. E 79, 046308 (2009).
[22] S. Chandrasekhar, Hydrodynamic and Hydromagnetic Stability (Oxford Univ. Press, Oxford, 1961), Dover edition, 1981.
[23] T. R. Kirkpatrick, J. M. Ortiz de Zárate, and J. V. Sengers, Phys. Rev. Lett. 110, 235902 (2013).
[24] J. M. Ortiz de Zárate and J. V. Sengers, J. Stat. Phys. 144, 774 (2011).
[25] J. M. Ortiz de Zárate and J. V. Sengers, J. Stat. Phys. 150, 540 (2013).
[26] See supplemental material.
[27] N. Tillmark and P. Alfredson, J. Fluid Mech. 235, 89 (1992).
[28] F. Daviaud, J. Hegseth, and P. Bergé, Phys. Rev. Lett. 69, 2511 (1992).
[29] O. Dauchot and E. Daviaud, Phys. Fluids 7, 335 (1995).
[30] S. Bottin, F. Daviaud, P. Manneville, and O. Dauchot,
[31] A. Prigent, G. Grégoire, H. Chaté, and O. Dauchot, Physica D 174, 100 (2003).
[32] M. Couliou and R. Monchaux, Phys. Fluids 27, 034101 (2015).
[33] L. Klotz, G. Lemoult, I. Frontczak, L. S. Tuckerman, and J. E. Wesfried, Phys. Rev. Fluids 2, 043904 (2017).
[34] L. Klotz and J. Wesfreid, J. Fluid Mech. 829, R4 (2017).
[35] R. Monchaux, private communication.
[36] Revised Supplementary Release on Properties of Liquid Water at 0.1 MPa, IAPWS SR6-08 (2011), available at www.iapws.org.
[37] T. R. Kirkpatrick, J. M. Ortiz de Zárate, and J. V. Sengers, Phys. Rev. E 93, 012148 (2016).
[38] J. J. Brey, J. Chem. Phys. 79, 4585 (1983).
[39] H. van Beijeren and J. R. Dorfman, Physica 68, 437 (1973).
[40] J. W. Dufty, Mol. Phys. 100, 2331 (2002).
[41] J. J. Erpenbeck and W. Wood, J. Stat. Phys. 24, 455 (1981).
[42] H. van Beijeren, Phys. Lett. A 105, 191 (1984).
[43] T. R. Kirkpatrick, Phys. Rev. Lett. 53, 1735 (1984).
[44] T. R. Kirkpatrick, Phys. Rev. A 32, 3130 (1985).
[45] T. R. Kirkpatrick and J. C. Nieuwoudt, Phys. Rev. A 33, 2651 (1986).
[46] J. R. Dorfman, H. van Beijeren, and T. R. Kirkpatrick, Contemporary Kinetic Theory of Matter (Cambridge University Press, to be published).
[47] A. Varghese, G. Gompper, and R. G. Winkler, Phys. Rev. E 96, 062617 (2017).
In this supplemental material we show how the expressions for the diagonal elements $\langle \delta v \delta v \rangle_{NE}$ are related to the correlation functions previously obtained for the wall-normal velocity fluctuations $\delta v_z$ and for the wall-normal vorticity fluctuations $\delta \omega_z = \partial_x \delta v_x - \partial_y \delta v_y$ in shear [1, 2]. We continue to use here dimensionless variables with position $r$ in terms of $L$, wave vector $q$ in terms of $L^{-1}$, and velocity $v$ in terms of $L \gamma$. In addition, we continue to define a dimensionless strength of the thermal noise as [1]:

$$\tilde{S} = \frac{k_B T}{\rho L^2 \gamma^2 L^2 \Re}.$$  \hfill (S.1)

We specifically consider the limits of large $L$ and small $L$ at a fixed velocity $U$ which correspond to large Re and small Re, respectively.

\section{I. Calculation for Large Re}

For large $L$ we can neglect the boundary conditions and solve the fluctuating Orr-Sommerfeld and Squire equations by applying a 3-dimensional Fourier transformation [1,2]. We then obtain for the equal time correlation functions in momentum space:

$$\langle \delta v_z (q) \delta v_z (q') \rangle_{NE} = C_{zz}^{NE} (q) (2\pi)^3 \delta (q - q'),$$  \hfill (S.2)

$$\langle \delta \omega_z (q) \delta \omega_z (q') \rangle_{NE} = W_{zz}^{NE} (q) (2\pi)^3 \delta (q - q'),$$  \hfill (S.3)

$$\langle \delta v_z (q) \delta \omega_z (q') \rangle_{NE} = B_{zz}^{NE} (q) (2\pi)^3 \delta (q - q').$$  \hfill (S.4)

The functions $C_{zz}^{NE} (q)$, $W_{zz}^{NE} (q)$, and $B_{zz}^{NE} (q)$ are given by

$$C_{zz}^{NE} (q) = 2 \tilde{S} \Re \frac{q_x q_y^2}{q^4} \int_0^\infty d\beta (q_z + q_x \beta) e^{-\Gamma (\beta, q)},$$  \hfill (S.5)

$$W_{zz}^{NE} (q) = 2 \tilde{S} \frac{q_x^2}{q^2} \int_0^\infty d\beta \left[ \beta^2 + (q_z + \beta q_x)^2 \right] \left[ \text{atan} \left( \frac{q_z + q_x \beta}{q_x} \right) - \text{atan} \left( \frac{q_z}{q_x} \right) \right] e^{-\Gamma (\beta, q)},$$  \hfill (S.6)

$$B_{zz}^{NE} (q) = 2 \tilde{S} \frac{q_x q_y}{q^2 q_z} \int_0^\infty d\beta \left[ \beta^2 + (q_z + \beta q_x)^2 \right] \left[ \text{atan} \left( \frac{q_z + \beta q_x}{q_x} \right) - \text{atan} \left( \frac{q_z}{q_x} \right) \right] e^{-\Gamma (\beta, q)}.$$  \hfill (S.7)

with

$$\Gamma (\beta, q) = \frac{2\beta}{\Re} \left( q_x^2 \beta^2 + 3 q_x^2 q_z + 3 \beta^2 \right),$$  \hfill (S.8)

as given by Eq. (39) and Eq. (43b) in [2], which are exactly the same as Eqs. (S.5) and (S.6) here, while Eq. (S.7) for the cross-correlation can be obtained following the same techniques and is first presented here. In these equations $q_x$ is the magnitude of the component $q_x$ of the wave vector in the $x$-$y$ plane, i.e., parallel to the plates. From these equations we can obtain the correlation functions for $\delta v_x$ and $\delta v_y$ by noting that

$$\delta v_x = -\frac{1}{q_x} (q_x q_y \delta v_y - i q_y q_x \delta \omega_z),$$  \hfill (S.9)

$$\delta v_y = -\frac{1}{q_x} (q_y q_x \delta v_x + i q_x q_y \delta \omega_z).$$  \hfill (S.10)
so that
\[
\langle \delta v_x^*(q) \delta v_x(q') \rangle_{\text{NE}} = C_{xx}^{\text{NE}}(q) (2\pi)^3 \delta(q - q'),
\]
\[
\langle \delta v_y^*(q) \delta v_y(q') \rangle_{\text{NE}} = C_{yy}^{\text{NE}}(q) (2\pi)^3 \delta(q - q'),
\]
with
\[
C_{xx}^{\text{NE}}(q) = \frac{q_x^2}{q_\parallel^2} C_{zz}^{\text{NE}}(q) + \frac{q_y^2}{q_\parallel^2} W_{zz}^{\text{NE}}(q) + 2 \frac{q_x q_y q_z}{q_\parallel^2} B_{zz}^{\text{NE}}(q),
\]
\[
C_{yy}^{\text{NE}}(q) = \frac{q_y^2}{q_\parallel^2} C_{zz}^{\text{NE}}(q) - \frac{q_x q_y q_z}{q_\parallel^2} W_{zz}^{\text{NE}}(q) - 2 \frac{q_x q_y q_z}{q_\parallel^2} B_{zz}^{\text{NE}}(q).
\]
Integration of Eqs. (S.5), (S.13), and (S.14) yield the diagonal elements of \( \langle \delta v \delta v \rangle_{\text{NE}} \) in real space for large Re. As an example, we consider the computation of \( V_{\infty}^{\text{NE}} \). In the dimensionless units used in this Supplemental Material we have:
\[
\tilde{S}\text{Re} V_{\infty}^{\text{NE}}(\text{Re})^{3/2} = \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} C_{zz}^{\text{NE}}(q) \, dq.
\]
To evaluate the coefficient \( V_{\infty}^{\text{NE}} \), after substitution of Eq. (S.5) into Eq. (S.15), we adopt spherical coordinates for the integration over \( q \). We first integrate over the magnitude \( q \) of the vector \( q \), which can be done analytically and yields the prefactor \((\text{Re})^{3/2}\). A second integration over the polar angle can also be performed analytically taking advantage of the symmetry properties of the integral. The final double integral, over the azimuthal angle and over the parameter \( \beta \), can be simplified but not performed analytically and has been evaluated numerically:
\[
V_{\infty} = \frac{\sqrt{3}}{32\pi^3} \Gamma\left(\frac{1}{4}\right)^2 \int_0^\infty \frac{d\beta}{\beta^2} \int_0^{\pi} \frac{(\beta + \cos \theta)(\sin \theta)^2}{(\beta^2 + 3\beta \cos \theta + 3)^2} \, d\theta \approx 0.0106,
\]
which is the value quoted in the main text of the Letter and in Table I. The other coefficients, \( V_{\infty}^{\text{NE}} \) and \( V_{\infty}^{\text{NE}} \), are evaluated in a similar fashion.

**II. CALCULATION FOR SMALL Re**

Small Re means narrow layers. Hence, we need to take the boundary conditions for the velocity fluctuations at \( z = \pm L \) into account explicitly. As shown in [1, 2], this is accomplished by applying a Fourier transformation only in the stream-wise and span-wise directions, while retaining the dependence of the wall-normal coordinate \( z \). In that case, for the equal-time correlation functions we have:
\[
\langle \delta v_x^*(q_\parallel, z) \delta v_z(q_\parallel, z') \rangle_{\text{NE}} = C_{zz}^{\text{NE}}(q_\parallel, z, z') (2\pi)^2 \delta(q_\parallel - q_\parallel),
\]
\[
\langle \delta v_y^*(q_\parallel, z) \delta v_z(q_\parallel, z') \rangle_{\text{NE}} = W_{zz}^{\text{NE}}(q_\parallel, z, z') (2\pi)^2 \delta(q_\parallel - q_\parallel),
\]
\[
\langle \delta v_z^*(q_\parallel, z) \delta v_z(q_\parallel, z') \rangle_{\text{NE}} = B_{zz}^{\text{NE}}(q_\parallel, z, z') (2\pi)^2 \delta(q_\parallel - q_\parallel),
\]
where the presence of boundaries breaks the translational invariance in the wall-normal direction. We can obtain the correlation functions for \( \delta v_x \) and \( \delta v_y \) by noting that
\[
\delta v_x(q_\parallel, z) = \frac{1}{q_\parallel} \left[ q_x \partial_z \delta v_z(q_\parallel, z) + q_y \delta v_z(q_\parallel, z) \right],
\]
\[
\delta v_y(q_\parallel, z) = \frac{1}{q_\parallel} \left[ q_y \partial_z \delta v_z(q_\parallel, z) - q_x \delta v_z(q_\parallel, z) \right],
\]
so that
\[
\langle \delta v_x^*(q_\parallel, z) \delta v_x(q_\parallel, z') \rangle_{\text{NE}} = C_{xx}^{\text{NE}}(q_\parallel, z, z') (2\pi)^2 \delta(q_\parallel - q_\parallel),
\]
\[
\langle \delta v_y^*(q_\parallel, z) \delta v_y(q_\parallel, z') \rangle_{\text{NE}} = C_{yy}^{\text{NE}}(q_\parallel, z, z') (2\pi)^2 \delta(q_\parallel - q_\parallel),
\]
with

\[ C_{x\parallel}(q_{\parallel}, z, z') = \frac{q_{\parallel}^2}{q_{\parallel}^2} \partial_z \partial_{z'} C_{x\parallel}^{\text{NE}}(q_{\parallel}, z, z') + \frac{q_{\parallel}^2}{q_{\parallel}^2} W_{x\parallel}^{\text{NE}}(q_{\parallel}, z, z') + \frac{q_{\parallel} q_{\parallel}}{q_{\parallel}^4} \left[ \partial_z B_{x\parallel}^{\text{NE}}(q_{\parallel}, z, z') + \partial_{z'} B_{x\parallel}^{\text{NE}}(q_{\parallel}, z, z') \right], \tag{S.24} \]

\[ C_{y\parallel}(q_{\parallel}, z, z') = \frac{q_{\parallel}^2}{q_{\parallel}^2} \partial_z \partial_{z'} C_{y\parallel}^{\text{NE}}(q_{\parallel}, z, z') + \frac{q_{\parallel}^2}{q_{\parallel}^2} W_{y\parallel}^{\text{NE}}(q_{\parallel}, z, z') - \frac{q_{\parallel} q_{\parallel}}{q_{\parallel}^4} \left[ \partial_z B_{y\parallel}^{\text{NE}}(q_{\parallel}, z, z') + \partial_{z'} B_{y\parallel}^{\text{NE}}(q_{\parallel}, z, z') \right]. \]

As stated in the main text, to solve the fluctuating hydrodynamics equations and following \cite{1, 2}, we adopt a Galerkin approximation. Specifically, to satisfy the boundary conditions, we assume that

\[ \delta v_z(\omega, q_{\parallel}, z) = (z^2 - 1)^2 \left[ A_0(q_{\parallel}) + A_1(q_{\parallel}) z + A_2(q_{\parallel}) z^2 + \cdots \right], \]

\[ \delta w_z(\omega, q_{\parallel}, z) = (z^2 - 1) \left[ B_0(q_{\parallel}) + B_1(q_{\parallel}) z + B_2(q_{\parallel}) z^2 + \cdots \right], \tag{S.25} \]

where the coefficients \( A_N(\omega, q_{\parallel}) \) and \( B_N(\omega, q_{\parallel}) \) are determined by projection of the equations onto the basis used for the expansion \cite{S.23} itself, i.e., \((z^2 - 1)^2 z^N\) and \((z^2 - 1) z^N\), and solving the resulting algebraic equations. In practice, this Galerkin approach is only useful when the expansion \cite{S.25} is truncated at some low order, and we have truncated at \( N = 1 \).

As before and for illustrative purposes, we consider in detail here only \( V_{x\parallel}^0 \). For the nonequilibrium contribution \( C_{x\parallel}^{\text{NE}}(q_{\parallel}, z) \) in Eq. \((S.17)\) the first-order Galerkin method described above, after substitution of \( z = z' \), explicitly gives:

\[ C_{x\parallel}^{\text{NE}}(q_{\parallel}, z) = \tilde{S}Re \left( 1 - z^2 \right)^4 \left[ C_1(Re, q_{\parallel}) - C_2(Re, q_{\parallel}) z^2 \right] Re^2 \tag{S.26} \]

with

\[ C_1(Re, q_{\parallel}) = \frac{3465}{128} \frac{q_{\parallel}^2 (11 + q_{\parallel}^2) \left( 1089 + 411 q_{\parallel}^2 + 42 q_{\parallel}^4 + 2 q_{\parallel}^6 \right)}{11(63 + 12 q_{\parallel}^2 + 2 q_{\parallel}^4) + 4 q_{\parallel}^2 (11 + q_{\parallel}^2) q_{\parallel}^2 Re^2 \left( 495 + 44 q_{\parallel}^2 + 2 q_{\parallel}^4 \right)}, \tag{S.27} \]

and

\[ C_2(Re, q_{\parallel}) = \frac{3465}{128} \frac{11 q_{\parallel}^4 \left( 1089 + 411 q_{\parallel}^2 + 42 q_{\parallel}^4 + 2 q_{\parallel}^6 \right)}{(63 + 12 q_{\parallel}^2 + 2 q_{\parallel}^4) + 4 q_{\parallel}^2 (11 + q_{\parallel}^2) q_{\parallel}^2 Re^2}. \tag{S.28} \]

Next, we can define a \( z \)-dependent component \( V_{x\parallel}^0(z) \) which, in terms of the dimensionless units adopted here, will be given by the expression:

\[ V_{x\parallel}^0(z) = \lim_{Re \to 0} \frac{1}{SRe^3 \left( 2\pi \right)^2} \int_{R^2} C_{x\parallel}^{\text{NE}}(q_{\parallel}, z) \, dq_{\parallel}. \tag{S.29} \]

Upon substitution of Eq. \((S.26)\) into Eq. \((S.29)\) after taking the \( Re \to 0 \) limit, we obtain:

\[ V_{x\parallel}^0(z) = \frac{1}{2 \pi^2} \frac{1}{(2 - 1)^4} \left[ \int_{R^2} C_1(0, q_{\parallel}) \, dq_{\parallel} - z^2 \int_{R^2} C_2(0, q_{\parallel}) \, dq_{\parallel} \right]. \] \tag{S.30}

The two integrals are convergent and can be performed analytically, but the result is long and not particularly informative. We prefer to display the result numerically, namely

\[ V_{x\parallel}^0(z) = \left( 1 - z^2 \right)^4 (1.584 - 6.812 z^2) \times 10^{-3}. \tag{S.31} \]

As explained in the Letter, the quantity relevant for the estimation of fluctuation-induced pressures is obtained upon averaging over the wall-normal coordinate, or

\[ V_{x\parallel}^0 = \frac{1}{2} \int_{-1}^{1} V_{x\parallel}^0(z) \, dz = 3.92 \times 10^{-4}, \tag{S.32} \]

which is the value quoted in the main text. The other coefficients, \( V_{x\parallel}^0 \) and \( V_{y\parallel}^0 \), are evaluated in a similar fashion.
III. OTHER CONTRIBUTIONS TO THE SHEAR-INDUCED PRESSURE ENHANCEMENT

First, there are short-ranged corrections to Eq. (9) that are of $O(\gamma^4)$. In dimensionless variables, these terms are of relative $O(\gamma \sigma / v_{th}^2)$, with $v_{th} \propto \sqrt{k_B T/m}$ a thermal velocity. For realistic laboratory shear rates these are very small corrections.

Second, the long wavelength nonlinear terms in the nonlinear fluctuating hydrodynamic equations that renormalize the pressure are,

$$\langle \delta P_{ij} \rangle = \langle \rho v_i v_j + p(\rho, \epsilon) \delta_{ij} - \rho \overline{v_i v_j} - p(\overline{\rho}, \overline{\epsilon}) \delta_{ij} \rangle. \quad (S.33)$$

Here the over-line denotes average values and $\epsilon$ is the internal energy density. In principle all of these nonlinearities will lead to long-ranged renormalizations of the pressure. If we neglect density and internal energy density fluctuations, then we obtain Eq. (1). Taking into account density fluctuations leads to additional renormalizations of the pressure. However, Lutsko and Dufty [3] have shown that in general density fluctuations decay faster in space than velocity fluctuations. Their work suggest that density nonlinearities will lead to a $\gamma^{11/6}$ contribution to the pressure, with a relatively large coefficient. Compared to the $\gamma^{3/2}$ terms, this term is of relative $[\gamma \sigma / v_{th}]^{1/3}$, which is again quite small. In any case, it would be difficult to distinguish this term from all of the analytic $\gamma^2$ terms.

We also note that we have considered isothermal flow, that is, possible viscous-heating effects have been neglected [4]. This condition is commonly satisfied in computer simulations by special dynamical rules keeping the temperature constant. However, in real experiments a pressure increase resulting from viscous heating may not be negligible.

[1] J. M. Ortiz de Zárate and J. V. Sengers, Phys. Rev. E 77, 026306 (2008).
[2] J. M. Ortiz de Zárate and J. V. Sengers, Phys. Rev. E 79, 046308 (2009).
[3] J. F. Lutsko and J. W. Dufty, Phys. Rev. E 66, 041206 (2002).
[4] J. W. Dufty and J. Lutsko, in Recent Developments in Nonequilibrium Thermodynamics: Fluids and Related Topics, edited by J. Casas-Vázquez, D. Jou, and J. M. Rubí (Springer, Berlin, 1986), vol. 253 of Lecture Notes in Physics, pp. 47–84.