Mixed Schur-Weyl Duality Between General Linear Lie Algebras and Cyclotomic Walled Brauer Algebras

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Abstract. Motivated by Brundan-Kleshchev’s work on higher Schur-Weyl duality, we establish mixed Schur-Weyl duality between general linear Lie algebras and cyclotomic walled Brauer algebras in an arbitrary level. Using weakly cellular bases of cyclotomic walled Brauer algebras, we classify highest weight vectors of certain mixed tensor modules of general linear Lie algebras. This leads to an efficient way to compute decomposition matrices of cyclotomic walled Brauer algebras arising from mixed Schur-Weyl duality, which generalizes early results on level two walled Brauer algebras.

1. Introduction

The classical Schur-Weyl duality sets up a closed relationship between polynomial representations of general linear groups and representations of symmetric groups [13]. In [8], Brundan and Stroppel established higher super Schur-Weyl duality between general linear Lie superalgebras $\mathfrak{gl}_{m|n}$ and level two degenerate Hecke algebras. In order to generalize their results in mixed cases, affine walled Brauer algebras and their cyclotomic quotients were introduced in [21]. See also [4, 23]. Moreover, using weakly cellular bases of level two walled Brauer algebras, highest weight vectors of some mixed tensor modules have been classified in [22]. This leads to an efficient way to compute decomposition matrices of such level two walled Brauer algebras via the structures of indecomposable tilting modules in the category of finite dimensional rational representations for $\mathfrak{gl}_{m|n}$. In particular, such level two walled Brauer algebras are multiplicity free in the sense that their decomposition numbers are either 1 or 0.

Motivated by Brundan-Kleshchev’s remarkable work on higher Schur-Weyl duality between general linear Lie algebras and degenerate cyclotomic Hecke algebras [6], we will extend the mixed Schur-Weyl duality in [3, 15, 21–23] to an arbitrary level as follows.

Let $R$ be a commutative ring contains 1, $\omega_a, \overline{\omega}_a$, $a \in \mathbb{N}$ such that $\overline{\omega}_a$ are determined by $\omega_b$’s via [21, Corollary 4.3]. The affine walled algebra $B_{r, t}^{\text{aff}}$ can be realized as the free $R$-module $R[x_r] \otimes B_{r, t}(\omega_0) \otimes R[x_t]$, the tensor product of the walled Brauer algebra $B_{r, t}(\omega_0)$ with two polynomial algebras $R[x_r] := R[x_1, x_2, \cdots, x_r]$ and $R[x_t] := R[t_1, t_2, \cdots, t_t]$, such that $R[x_r] \otimes R\mathfrak{S}_r$ and $R\overline{\mathfrak{S}}_t \otimes R[x_t]$ are isomorphic to the degenerate affine Hecke algebras $H_r^{\text{aff}}$ and $H_t^{\text{aff}}$ respectively (where $\mathfrak{S}_r$ and $\overline{\mathfrak{S}}_t$ are symmetric groups contained in $B_{r, t}(\omega_0)$)

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generated by $s_i$'s and $\overline{s}_j$'s, respectively), and further, the following relations are satisfied (cf. Definition 2.1)

ea) $e_1(x_1 + \overline{x}_1) = (x_1 + \overline{x}_1)e_1 = 0$, $s_1e_1s_1x_1 = x_1s_1e_1s_1$, $\overline{s}_1e_1\overline{s}_1\overline{x}_1 = \overline{x}_1\overline{s}_1e_1\overline{s}_1$,

b) $s_i\overline{x}_1 = \overline{x}_1s_i$, $\overline{s}_ix_1 = x_1\overline{s}_i$, $x_1(e_1 + \overline{x}_1) = (e_1 + \overline{x}_1)x_1$,

c) $e_1x_i^a = \omega_a e_1$, $e_1\overline{x}_i^a = \overline{\omega}_a e_1$, $\forall a \in \mathbb{N}$.

Throughout, unless otherwise stated, we will work over the ground field $\mathbb{C}$. Let $\mathfrak{g}$ be the general linear Lie algebra $\mathfrak{gl}_n$. Let $W$ be the linear dual of the natural $\mathfrak{g}$-module $V$. We consider the mixed tensor product $M^{r,t} := M \otimes V^{\otimes r} \otimes W^{\otimes t}$ for various positive integers $r$ and $t$, where $M$ is any highest weight $\mathfrak{g}$-module. Let $\Omega = \sum_{i,j=1}^n e_{i,j} \otimes e_{i,j}$, where $e_{i,j} \in \mathfrak{g}$ is a matrix unit. Via $\Omega$, there is a well-defined right action of $B^{\text{aff}}_{k,r,t}$ on $M^{r,t}$, commuting with the left action of $\mathfrak{g}$ such that $x_1, \overline{x}_1, e_1, s_i$ and $\overline{s}_j$ act as certain endomorphisms of $M^{r,t}$ in Definition 3.4. See also [23] for some special $\mathfrak{g}$-module $M$. In order to get an action of a cyclotomic walled Brauer algebra on $M^{r,t}$ in arbitrary level, we need to pick $M$ as a highest weight $\mathfrak{g}$-module as follows. This is motivated by Brundan and Kleshchev’s work in [6].

Suppose that $(q_1 \geq q_2 \geq \cdots \geq q_k)$ is a partition of $n$. Let $\mathfrak{p}$ be the parabolic subalgebra of $\mathfrak{g}$ such that the corresponding Levi subalgebra $\mathfrak{l} = \mathfrak{gl}_{q_1} \oplus \mathfrak{gl}_{q_2} \oplus \cdots \oplus \mathfrak{gl}_{q_k}$. Let $\mathcal{O}^\mathfrak{p}$ be the parabolic category $\mathcal{O}$ with respect to $\mathfrak{p}$. Then $\mathcal{O}^\mathfrak{p}$ is a full subcategory of $\mathcal{O}$ such that $M$ is semisimple as an $U(\mathfrak{l})$-module and locally $\mathfrak{u}$-finite, where $\mathfrak{u}$ is the nilradical of $\mathfrak{p}$ and $U(\mathfrak{l})$ is the universal enveloping algebra of $\mathfrak{l}$. For each $c = (c_1,c_2,\cdots,c_k) \in \mathbb{C}^k$ consider $\mathfrak{p}$-dominant weight $\delta_c := \sum_{i=1}^k c_i (\varepsilon_{p_{i-1}+1} + \varepsilon_{p_{i-1}+2} + \cdots + \varepsilon_{p_i})$, where $p_i = \sum_{j=1}^i q_j$, $1 \leq i \leq k$ and $p_0 = 0$. As a left $\mathfrak{g}$-module, the corresponding parabolar Verma module $M^\mathfrak{p}(\delta_c)$ is irreducible, projective and injective. If $M^{r,t}_c = M^\mathfrak{p}(\delta_c) \otimes V^{\otimes r} \otimes W^{\otimes t}$, then $M^{r,t}_c$ turns out to be a $(U, B_{k,r,t})$-bimodule where $U$ is the universal enveloping algebra of $\mathfrak{g}$ and $B_{k,r,t}$ is the cyclotomic walled Brauer algebra whose parameters are determined by Definition 3.11 and 3.11. One of the main results of this paper is that there is an algebra epimorphism

$$\varphi : B_{k,r,t} \to \text{End}_{\mathcal{O}}(M^{r,t}_c)^{\text{op}}.$$  \hspace{1cm} (1.1)

If $r + t \leq q_k$, then $\varphi$ is an algebra isomorphism.

We remark that affine and cyclotomic walled Brauer algebras were defined in [4] via affine oriented Brauer category and cyclotomic oriented Brauer category. See also [23]. In [4] §5.5] Brundan et. al show that affine and cyclotomic walled Brauer algebras in [4] are isomorphic to those considered in [21]. When (3.11) holds, the homomorphism $\varphi$ in (1.1) has been observed in [4] §4] and a proof of surjectivity is sketched in [4] Remark 4.14].

If we allow $t = 0$, then $B_{k,r,t}$ turns out to be the level $k$ degenerate Hecke algebra $\mathcal{H}_{k,r}$ and the surjectivity in (1.1) has been proved in [6]. Motivated by [19][22], we use weakly cellular bases of $B_{k,r,t}$ to classify highest weight vectors of $M^{r,t}_c$ under the assumption $r + t \leq q_k$. Such a result is enough for us to establish an explicit relationship between multiplicities of
parabolic Verma modules in any indecomposable direct summand of \( M_{c,t} \), (which is in fact an indecomposable tilting module in \( \mathcal{O}^\mathfrak{p} \)) and multiplicities of simple modules in any cell module of \( \mathcal{B}_{k,r,t} \). This determines decomposition matrices of \( \mathcal{B}_{k,r,t} \) which arise from mixed Schur-Weyl duality (see Theorem 7.19 and Remark 6.12). Motivated by our works on Birman-Murakami-Wenzl algebras in [20], we conjecture that decomposition numbers of cyclotomic walled Brauer algebras over \( \mathbb{C} \) can be determined by those in Theorem 7.19 together with some results on Morita equivalences. We hope to settle this problem in the future.

We organize the paper as follows. In Section 2, we recall some of results on affine walled Brauer algebras and their cyclotomic quotients in [21][22]. In Section 3, we prove the surjectivity in (1.1) under the assumption that \( r + t < q_k \). In order to deal with the general case, we need to state some of results on the duality between finite \( W \) algebras (resp., its associated graded algebras) and cyclotomic walled Brauer algebras (resp., its associated graded algebras) in sections 4–5. Such observations heavily depend on Brundan and Kleshchev’s influential work [6]. In section 6, following Brundan-Kleshchev’s idea in [6], we prove the surjectivity of \( \varphi \) in (1.1) in general cases. Finally, we classify highest weight vectors of \( M_{c,t} \) under the assumption \( r + t < q_k \) and hence to compute decomposition matrices of cyclotomic walled Brauer algebras arising from mixed Schur-Weyl duality.

2. Affine and cyclotomic walled Brauer algebras

Throughout this section, \( R \) is a commutative ring containing 1, \( \omega_a \) and \( \overline{a} \) for all \( a \in \mathbb{N} \) such that \( \omega_a \)'s are determined by \( \omega_y \)'s via [21] Corollary 4.3.

Definition 2.1. [21] Fix \( r, t \in \mathbb{Z}^{>0} \). The affine walled Brauer algebra \( \mathcal{B}_{r,t}^{\text{aff}} \) is the associative \( R \)-algebra generated by \( e_1, x_1, \overline{1}, s_i \) (1 \( \leq i \leq r-1 \)), \( \overline{s}_j \) (1 \( \leq j \leq t-1 \)), subject to the following relations

\begin{align*}
(1) & \quad s_i^2 = 1, \ 1 \leq i < r, \quad (14) & \quad \overline{s}_i^2 = 1, \ 1 \leq i < t, \\
(2) & \quad s_is_j = s_js_i, \ |i-j| > 1, \quad (15) & \quad \overline{s}_i\overline{s}_j = \overline{s}_j\overline{s}_i, \ |i-j| > 1, \\
(3) & \quad s_is_{i+1}s_i = s_{i+1}s_is_{i+1}, \ 1 \leq i < r-1, \quad (16) & \quad \overline{s}_i\overline{s}_{i+1}\overline{s}_i = \overline{s}_{i+1}\overline{s}_i\overline{s}_{i+1}, \ 1 \leq i < t-1, \\
(4) & \quad s_i e_1 = e_1 s_i, \quad 2 \leq i < r, \quad (17) & \quad \overline{s}_i e_1 = e_1 \overline{s}_i, \quad 2 \leq i < t, \\
(5) & \quad e_1 s_1 e_1 = e_1, \quad (18) & \quad e_1 \overline{s}_1 e_1 = e_1, \\
(6) & \quad e_1^2 = \omega_0 e_1, \quad (19) & \quad e_1 s_1 \overline{s}_1 e_1 s_1 = e_1 s_1 \overline{s}_1 e_1 \overline{s}_1, \\
(7) & \quad s_i \overline{s}_j = \overline{s}_j s_i, \quad (20) & \quad s_1 e_1 s_1 \overline{s}_1 e_1 = \overline{s}_1 e_1 s_1 \overline{s}_1 e_1, \\
(8) & \quad e_1(x_1 + \overline{1}) = (x_1 + \overline{1})e_1 = 0, \quad (21) & \quad x_1(e_1 + \overline{1}) = (e_1 + \overline{1})x_1, \\
(9) & \quad e_1 s_1 x_1 s_1 = s_1 x_1 s_1 e_1, \quad (22) & \quad e_1 \overline{s}_1 \overline{x}_1 \overline{s}_1 = \overline{x}_1 \overline{s}_1 e_1, \\
(10) & \quad s_i x_1 = x_1 s_i, \quad 2 \leq i < r, \quad (23) & \quad \overline{s}_i \overline{x}_1 = \overline{x}_1 \overline{s}_i, \quad 2 \leq i < t, \\
(11) & \quad s_i \overline{x}_1 = \overline{x}_1 s_i, \quad 1 \leq i < r, \quad (24) & \quad \overline{s}_i x_1 = x_1 \overline{s}_i, \quad 1 \leq i < t, \\
(12) & \quad e_1 x_1^k e_1 = \omega_k e_1, \quad \forall k \in \mathbb{Z}^{>0}, \quad (25) & \quad e_1 \overline{x}_1^k e_1 = \overline{x}_1 \overline{e}_1, \quad \forall k \in \mathbb{Z}^{>0}, \\
(13) & \quad x_1(s_1 x_1 s_1 - s_1) = (s_1 x_1 s_1 - s_1)x_1. \quad (26) & \quad \overline{x}_1(\overline{s}_1 \overline{x}_1 \overline{s}_1 - \overline{s}_1) = (\overline{s}_1 \overline{x}_1 \overline{s}_1 - \overline{s}_1)\overline{x}_1.
\end{align*}
In \[23\], Sartori defined affine walled Brauer algebras over \( \mathbb{C} \). See also \[4\]. For convenience, write \( \emptyset = \emptyset \) and \( n = \{1, 2, \cdots, n\} \) for any positive integer \( n \). The following result follows from Definition \[2.1\] immediately.

**Lemma 2.2.** There is an \( R \)-linear anti-involution \( \sigma \) on \( D_{r,t} \) which fixes all generators \( x_1, \overline{x}_1, e_1, s_i, \overline{s}_j, i \in r-1 \) and \( j \in t-1 \).

The affine walled Brauer algebra \( D_{r,t}^{\text{aff}} \) contains two subalgebras generated by \( \{x_1, s_i \mid i \in r-1\} \) and \( \{\overline{x}_1, \overline{s}_j \mid j \in t-1\} \), which are isomorphic to the degenerate affine Hecke algebras \( D_{e}^{\text{aff}} \) and \( D_{t}^{\text{aff}} \), respectively. Also, the subalgebra of \( D_{r,t}^{\text{aff}} \) generated by \( \{e_1, s_i, \overline{s}_j \mid i \in r-1, j \in t-1\} \) is isomorphic to the walled Brauer algebra \( D_{r,t}(\omega_0) \) with respect to the parameter \( \omega_0 \) in \[15, 25\]. Later on, we will need another definition of \( D_{r,t}(\omega_0) \) via walled Brauer diagrams so as to describe the actions of \( D_{r,t}^{\text{aff}} \) and its cyclotomic quotients on mixed tensor product of certain modules in parabolic category \( O \) for general linear Lie algebras.

Recall that \( r \) and \( t \) are two positive integers. A \textit{walled \((r, t)\)-Brauer diagram} \( D \) is a diagram with \((r+t)\) vertices on the top and bottom rows, and vertices on both rows are labeled from left to right by \( r, \cdots, 2, 1, \Omega, \overline{\Omega}, \cdots, \tau \). Each vertex \( i \in \{1, 2, \cdots, r\} \) (resp., \( \tau \in \{\overline{\Omega}, \overline{\Omega}, \cdots, \tau\} \)) on a row has to be connected to a unique vertex, say \( \overline{j} \) (resp., \( j \)) on the same row or a unique vertex \( j \) (resp., \( \overline{j} \)) on the other row. There are four types of pairs \([i, j], [i, \overline{j}], [\overline{i}, j] \) and \([\overline{i}, \overline{j}]\) such that the pairs \([i, j] \) and \([\overline{i}, \overline{j}]\) are called \textit{vertical edges}, and the pairs \([i, j] \) and \([\overline{i}, \overline{j}]\) are called \textit{horizontal edges}. If we imagine that there is a wall which separates the vertices \( 1, \Omega \) on both top and bottom rows, then a \textit{walled \((r, t)\)-Brauer diagram} is a diagram with \((r+t)\) vertices on both rows such that each vertical edge can not cross the wall and each horizontal edge has to cross the wall. For convenience, we call a \textit{walled \((r, t)\)-Brauer diagram} a \textit{walled Brauer diagram} if there is no confusion.

Let \( D_1 \circ D_2 \) be the \textit{composition} \( D_1 \circ D_2 \) of two walled Brauer diagrams \( D_1 \) and \( D_2 \). Then \( D_1 \circ D_2 \) can be obtained by putting \( D_1 \) above \( D_2 \) and connecting each vertex on the bottom row of \( D_1 \) to the corresponding vertex on the top row of \( D_2 \). Removing all circles of \( D_1 \circ D_2 \) yields a walled Brauer diagram, say \( D_3 \). Let \( n(D_1, D_2) \) be the number of circles appearing in \( D_1 \circ D_2 \). Then the \textit{product} \( D_1 D_2 \) of \( D_1 \) and \( D_2 \) is defined to be \( \omega_0^{n(D_1, D_2)} D_3 \), where \( \omega_0 \in R \). The \textit{walled Brauer algebra} \( D_{r,t}(\omega_0) \) with respect to the parameter \( \omega_0 \) is the associative algebra over \( R \) spanned by all \textit{walled \((r, t)\)-Brauer diagrams} with product defined as above.

Nikitin \[17\] proved that two previous definitions of walled Brauer algebras are isomorphic. The corresponding isomorphism sends \( e_1 \) (resp., \( s_i \) resp., \( \overline{s}_j \)) to the walled Brauer diagram whose edges are of form \([k, k] \) or \([\overline{k}, \overline{k}] \) except two horizontal edges \([1, \Omega] \) on both top and bottom rows (resp., two vertical edges \([i, i+1] \) and \([i+1, i] \) resp., \([\overline{j}, \overline{j}+1] \) and \([\overline{j}+1, \overline{j}] \)). The walled Brauer algebra defined via walled Brauer diagrams in \[9\] is isomorphic to the opposite of that defined as above.
Suppose \( u_i, \overline{u}_i \in R, i \in k \). Let \( I \) be the two-sided ideal of \( \mathcal{B}_{r,t}^{aff} \) generated by \( f(x_1) \) and \( g(\overline{x}_1) \), where \( f(x_1) = \prod_{i=1}^{k} (x_1 - u_i) \) and \( g(\overline{x}_1) = \prod_{i=1}^{k} (\overline{x}_1 - \overline{u}_i) \) such that \( e_1 f(x_1) = (-1)^k e_1 g(\overline{x}_1) \).

The cyclotomic (or level) \( k \)-walled Brauer algebra \( \mathcal{B}_{k,r,t} \) is the quotient algebra \( \mathcal{B}_{r,t}^{aff}/I \).

Rewrite \( f(x_1) = 0 \) as \( x_1^k + \sum_{t=0}^{k-1} a_{k-t} x^t = 0 \). If

\[
\omega_\ell = -(a_1 \omega_{\ell-1} + \cdots + a_k \omega_{\ell-k}), \text{ for all } \ell \geq k,
\]

then \( \mathcal{B}_{k,r,t} \) is called admissible. Let \( \mathbb{N}_k = \{0, 1, \ldots, k - 1\} \). If \( (\alpha, \beta) \in \mathbb{N}_k \times \mathbb{N}_k \), write \( x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_r^{\alpha_r}, \) and \( \overline{x}^\beta = \overline{x}_1^{\beta_1} \overline{x}_2^{\beta_2} \cdots \overline{x}_r^{\beta_r} \), where \( x_{i+1} = s_i x_i s_i - s_i, \) \( i \in r-1 \), and \( \overline{x}_{j+1} = \overline{s}_j \overline{x}_j \overline{s}_j - \overline{s}_j, \) \( j \in t-1 \). We call \( x^\alpha D \overline{x}^\beta \) a regular monomial of \( \mathcal{B}_{k,r,t} \), where \( D \) is a walled Brauer diagram.

**Theorem 2.3.** [22 Theorem 2.12] Let \( \mathcal{B}_{k,r,t} \) be defined over \( R \).

a) As an \( R \)-module, \( \mathcal{B}_{k,r,t} \) is spanned by all regular monomials.

b) \( \mathcal{B}_{k,r,t} \) is free over \( R \) with rank \( k^{r+t}(r+t)! \) if and only if \( \mathcal{B}_{k,r,t} \) is admissible. In this case, all regular monomials of \( \mathcal{B}_{k,r,t} \) consist of an \( R \)-basis of \( \mathcal{B}_{k,r,t} \).

3. CYCLOTOMIC WALLED BRAUER ALGEBRAS AND PARABOLIC CATEGORY \( \mathcal{O} \)

Throughout, let \( \mathfrak{g} \) be the general linear Lie algebra \( \mathfrak{gl}_n \) over \( \mathbb{C} \). Then \( \mathfrak{g} = \mathfrak{n}^+ \oplus \mathfrak{h} \oplus \mathfrak{n}^- \) such that the Cartan subalgebra \( \mathfrak{h} \) consists of all diagonal \( n \times n \) matrices, and \( \mathfrak{n}^+ \) (resp., \( \mathfrak{n}^- \)) consists of all strictly upper (resp., lower) triangle \( n \times n \) matrices. For any \( i, j \in n \), let \( e_{i,j} \) be the usual matrix unit. Then \( \{e_{i,j} \mid i \in n\} \) consists of a basis of \( \mathfrak{h} \). Let \( \{e_{i,i} \mid i \in n\} \) be the dual basis of \( \{e_{i,i} \mid i \in n\} \) in the sense that \( e_i(e_{j,j}) = \delta_{i,j} \). Then any \( \lambda \in \mathfrak{h}^* \), called a weight of \( \mathfrak{g} \), can be written as

\[
\lambda = \lambda_1 e_1 + \lambda_2 e_2 + \cdots + \lambda_n e_n, \quad \lambda_i \in \mathbb{C}.
\]

In this paper, a \( \mathfrak{g} \)-module \( M \) is always a left \( \mathfrak{g} \)-module. A non-zero vector \( m \in M \) is of weight \( \lambda \) if \( e_i m = \lambda_i m \), for any \( i \in n \). If \( \mathfrak{n}^+ m = 0 \), then \( m \) is called a highest weight vector of \( M \) with highest weight \( \lambda \). A highest weight module is a \( \mathfrak{g} \)-module generated by a highest weight vector.

Throughout, \( V \) is the natural module of \( \mathfrak{g} \) with a basis \( \{v_i \mid i \in n\} \). Let \( W = \text{Hom}_\mathbb{C}(V, \mathbb{C}) \) be its linear dual with a basis \( \{v_i^* \mid i \in n\} \) such that \( v_i^*(v_j) = \delta_{i,j} \). Then

\[
e_{i,j} v_k = \delta_{j,k} v_i, \quad \text{and} \quad e_{i,j} v_k^* = -\delta_{i,k} v_j^*.
\]

So, \( V \) is a highest weight \( \mathfrak{g} \)-module with the highest weight \( \epsilon_1 \) and \( W \) is a highest weight \( \mathfrak{g} \)-module with the highest weight \( -\epsilon_n \).

**Definition 3.1.** For two positive integers \( r \) and \( t \) and any highest weight \( \mathfrak{g} \)-module \( M \), define

\[
M^{r,t} = M \otimes V^{\otimes r} \otimes W^{\otimes t}.
\]
We order the positions of tensor factors of $M^{r,t}$ according to the total ordered set $(J, \prec)$ such that $J = \{0\} \cup J_1 \cup J_2$ with $J_1 = \{r, ..., 2, 1\}$ and $J_2 = \{\overline{1}, \overline{2}, ..., \overline{t}\}$ and

$$0 < r < r - 1 < \cdots < 1 < \overline{1} < \cdots < \overline{t}. \quad (3.2)$$

**Definition 3.2.** According to the total ordered set $J$, we define $I(n, r + t)$ to be the set of all maps $J_1 \cup J_2 \to \mathbb{N}$. So, each $i \in I(n, r + t)$ is of form $(i_r, i_{r-1}, \cdots, i_1, i_{\overline{1}}, i_{\overline{2}}, \cdots, i_{\overline{t}})$. Define $i^L = (i_r, i_{r-1}, \cdots, i_1)$ and $i^R = (i_{\overline{1}}, i_{\overline{2}}, \cdots, i_{\overline{t}})$.

**Lemma 3.3.** For $i \in I(n, r + t)$, define $v_i = v'_1 \otimes \ldots \otimes v'_{r,t}$ where $v'_1 = v_i \otimes v_{i_{r-1}} \otimes \cdots \otimes v_{i_1}$ and $v'_{r,t} = v'_{r} \otimes v'_{\overline{t}} \otimes \cdots \otimes v'_{\overline{1}}$. Then $\{v \otimes v_i \mid v \in S, i \in I(n, r + t)\}$ is a basis of $M^{r,t}$, where $S$ is a basis of $M$.

Let $C$ be the quadratic Casimir element of the universal enveloping algebra $U$ with respect to $\mathfrak{g}$. Then $C = \sum_{i,j \in \mathbb{N}} e_{i,j} e_{j,i}$. Let

$$\Omega = \frac{1}{2}(\Delta(C) - C \otimes 1 - 1 \otimes C) = \sum_{i,j \in \mathbb{N}} e_{i,j} \otimes e_{j,i}, \quad (3.3)$$

where $\Delta$ is the co-multiplication of $U$. For any $a, b \in J$ with $a \prec b$, define $\pi_{a,b} : U^\otimes 2 \to U^\otimes (r+t+1)$ by

$$\pi_{a,b}(x \otimes y) = 1 \otimes \cdots \otimes 1 \otimes ^{\text{th}} x \otimes 1 \otimes \cdots \otimes 1 \otimes ^{\text{th}} y \otimes 1 \otimes \cdots \otimes 1. \quad (3.4)$$

Since $C$ is a central element of $U$, $\pi_{a,b}(\Omega)|_{M^{r,t}} \in \text{End}_U(M^{r,t})$.

**Definition 3.4.** We define some elements of $\text{End}_U(M^{r,t})$ as follows:

$$s_i = \pi_{i+1,i}(\Omega)|_{M^{r,t}} \quad (i \in r - 1), \quad \overline{s}_j = \pi_{j+\overline{1},j}(\Omega)|_{M^{r,t}} \quad (j \in t - 1),$$

$$x_1 = -\pi_{0,1}(\Omega)|_{M^{r,t}}, \quad \overline{x}_1 = -\pi_{0,\overline{1}}(\Omega)|_{M^{r,t}}, \quad e_1 = -\pi_{1,1}(\Omega)|_{M^{r,t}}. \quad (3.5)$$

We always assume that $\text{End}_U(M^{r,t})$ acts on the left of $M^{r,t}$.

**Proposition 3.5.** Suppose that $M$ is a highest weight module for $\mathfrak{g}$. There is an affine walled Brauer algebra $\mathcal{B}_{r,t}^{\text{aff}}$ with some special parameters $\omega_0 = n$ and $\omega_i, i \in \mathbb{Z}^{>0}$, such that there is a well-defined right action of $\mathcal{B}_{r,t}^{\text{aff}}$ on $M^{r,t}$, which gives an algebra homomorphism $\varphi : \mathcal{B}_{r,t}^{\text{aff}} \to \text{End}_U(M^{r,t})^{\text{op}}$ sending $e_1, x_1, \overline{x}_1, s_i, \overline{s}_j$ to the same symbols in Definition 3.4 for all $i \in r - 1$ and $j \in t - 1$.

**Proof.** It follows from [3] that $e_1, s_i, \overline{s}_j$’s satisfy the relations for $\mathcal{B}_{r,t}(n)$. So, we need only verify (8)-(13) and (21)-(26) in Definition 2.1. One can verify them by arguments similar to those in [2]. \qed

**Assumption 3.6.** Fix positive integers $q_1, q_2, \cdots, q_k$ such that $\sum_{i=1}^k q_i = n$. Following [6], we consider any $d = (d_1, d_2, \cdots, d_k) \in \mathbb{C}^k$ such that $d_i - d_j \in \mathbb{Z}$ if and only if $d_i = d_j$. \hspace{1cm}
Let \( c_i = d_i + p_i - q_1 \), for all \( i \in \mathbb{Z} \), where \( p_i = \sum_{j=1}^{i} q_j \), \( 1 \leq i \leq k \). Define \( p_0 = 0 \) and 
\[
\delta_c = \sum_{i=1}^{k} c_i (\varepsilon_{p_{i-1} + 1} + \varepsilon_{p_{i-1} + 2} + \cdots + \varepsilon_{p_i}).
\]

Let \( \mathfrak{p} \) be the parabolic subalgebra of \( \mathfrak{g} \) such that the corresponding Levi subalgebra \( \mathfrak{l} \) is \( \mathfrak{gl}_{q_1} \oplus \mathfrak{gl}_{q_2} \oplus \cdots \oplus \mathfrak{gl}_{q_k} \). Let \( \Phi_1 \) be the root system of \( \mathfrak{l} \) and denote the corresponding set of simple roots by \( \Delta_1 \). Recall that the category \( \mathcal{O} \) is the category of finitely generated \( \mathfrak{g} \)-modules which are locally finite over \( \mathfrak{n}^+ \) and semi-simple over \( \mathfrak{h} \). Let \( \mathcal{U}(\mathfrak{l}) \) be the universal enveloping algebra of \( \mathfrak{l} \). Then \( \mathcal{O}^p \) is a full subcategory of \( \mathcal{O} \) such that for each object \( M \) in \( \mathcal{O}^p \), \( M \) is both semisimple as a \( \mathcal{U}(\mathfrak{l}) \)-module and locally \( u \)-finite, where \( u \) is the nilradical of \( \mathfrak{p} \). Let \( \Lambda^p \) be the subset of \( \mathfrak{h}^* \) consisting of all \( \lambda \) such that \( (\lambda, \alpha) \in \mathbb{N} \) for any \( \alpha \in \Delta_1 \). Each \( \lambda \in \Lambda^p \) is called a \( \mathfrak{p} \)-dominant integral weight. For any \( \lambda \in \mathfrak{h}^* \), let \( M^p(\lambda) \) be the usual parabolic Verma module with respect to a highest weight \( \lambda \). Then \( M^p(\lambda) \) is the maximal quotient of the ordinary Verma module \( M(\lambda) \) which is locally \( \mathfrak{p} \)-finite. So, \( M^p(\lambda) = 0 \) if \( \lambda \) is not \( \mathfrak{p} \)-dominant.

Obviously, \( \delta_c \in \Lambda^p \), where \( \delta_c \) is given in the Assumption 3.6. Let \( M_c := M^p(\delta_c) \). It is well known that \( M_c \) is irreducible, projective, injective in \( \mathcal{O}^p \) (see e.g. [6]). In the remaining part of this paper, we always assume that 
\[
M^{\mathfrak{c}^\prime} := M_c \otimes V^\otimes r \otimes W^\otimes t.
\]

So, \( M^{\mathfrak{c}^\prime} \in \mathcal{O}^p \). Recall that \( \{v_i \mid i \in \mathbb{N}\} \) is a basis of \( V \) and \( \{v^*_i \mid i \in \mathbb{N}\} \) is its dual basis. Let \( p_i \)’s be in Assumption 3.6.

**Lemma 3.7.** Let \( m \) be the highest weight vector of \( M_c \), which is unique up to non-zero multiple. If \( i \in \mathbb{N} \) with \( p_{\ell - 1} < i \leq p_\ell \) for some \( \ell \in \mathbb{Z} \), then

a) \( m \otimes v_i x_1 = -c_i m \otimes v_i - \sum_{1 \leq j \leq p_{\ell - 1}} e_{i,j} m \otimes v_j \),

b) \( m \otimes v^*_i x_1 = c_i m \otimes v^*_i + \sum_{p_{\ell - 1} < j \leq n} e_{j,i} m \otimes v^*_j \).

**Definition 3.8.** Let \( B_q = \bigcup_{i=1}^{k} \bigcup_{h=1}^{d-1} \mathbb{N}_i \times \mathbb{N}_h \), where \( \mathbb{N}_i = \{p_i - 1, p_i - 2, \cdots, p_i\} \) for any \( i \in \mathbb{Z} \). Let \( \preceq \) be the lexicographic order on \( B_q \) in the sense that \( (i_1, j_1) \preceq (i_2, j_2) \) if either \( i_1 \leq i_2 \) or \( i_1 = i_2 \) and \( j_1 \leq j_2 \). If \( (i_1, j_1) \preceq (i_2, j_2) \) and \( (i_1, j_1) \neq (i_2, j_2) \), we write \( (i_1, j_1) \prec (i_2, j_2) \).

If \( (i_\ell, j_\ell) \in B_q \), for all \( \ell \in a \), and \( a \in \mathbb{Z}^{a_0} \), and if \( \alpha \in \mathbb{N}^a \), we write 
\[
e{a_j}^{\alpha} = e_{i_{a_0}, j_{a_0}} e_{i_{a_0-1}, j_{a_0-1}} \cdots e_{i_{1}, j_{1}}.\]

Abusing of notation, we identify \( i \) (resp., \( j \)) with \( (i_a, i_{a-1}, \cdots, i_1) \) (resp., \( (j_a, j_{a-1}, \cdots, j_1) \)).

In this case, we say that both \( i \) and \( j \) are of lengths \( a \). If \( a = 0 \), we set \( e_{i_j}^{\alpha} = 1 \).

**Lemma 3.9.** Let \( S \) be the set of all elements \( e_{k_j}^{\alpha} m \otimes v_1 \in M^{\mathfrak{c}^\prime} \), where

a) \( m \) is the highest weight vector of \( M_c \), and \( i \in I(n, r + t) \), and \( a \in \mathbb{N}^a \),
Lemma 3.10. Let $m$ be the highest weight vector of $M_e$. Suppose $h \leq k-1$.

a) If $j \in \bigcup_{i=1}^{h} p_i$, then $m \otimes v_j x_1^h = 0$ up to some terms in $M_e \otimes V$ with degrees $\leq h-1$.

b) If $j \in \bigcup_{i=1}^{h} p_{k-i+1}$, then $m \otimes v_j x_1^h = 0$ up to some terms in $M_e \otimes W$ with degrees $\leq h-1$.

c) If $j \in p_i$, $h+1 \leq i \leq k$, then
\[ m \otimes v_j x_1^h = (-1)^h \sum_{l=1}^{h} \sum_{j_i \in p_i} e_{j_{h-1},j_h} \cdots e_{j_1,j_2}e_{j_{i+1},j_i}m \otimes v_{j_{i+1}} \]
up to some terms in $M_e \otimes V$ with degrees $\leq h-1$, where $1 \leq i_h < i_{h-1} < \cdots < i_1 \leq i-1$.

d) If $j \in p_i$, $1 \leq i \leq k-h$, then
\[ m \otimes v_j x_1^h = \sum_{l=1}^{h} \sum_{j_i \in p_i} e_{j_{h},j_{h-1}} \cdots e_{j_{i+1},j_i}e_{j_{i+1},j_i}m \otimes v_{j_{i+1}} \]
up to some terms in $M_e \otimes W$ with degrees $\leq h-1$, where $i+1 \leq i_1 < i_2 < \cdots < i_h \leq k$.

Proof. We prove (a) and (c) by induction on $h$. One can check (b)-(d), similarly. The case $h = 1$ for both (a) and (c) follows from Lemma 3.7. In general, (a) follows from inductive assumption on $h-1$ for both (a) and (c). If $j \in p_i$ and $h+1 \leq i \leq k$, then by inductive assumption on $h-1$, up to some terms with degree $< h-1$, there are some integers $i_1, i_2, \ldots, i_{h-1}$ such that $1 \leq i_{h-1} < \cdots < i_2 < i_1 \leq i-1$ and
\[ v_j \otimes mx_1^h = (-1)^{h-1} \sum_{l=1}^{h-1} \sum_{j_i \in p_i} e_{j_{h-2},j_{h-1}} \cdots e_{j_1,j_2}e_{j_{i+1},j_i}m \otimes v_{j_{i+1}}x_1 \]
up to some terms with degree $\leq h-1$. Note that $m$ is the highest weight vector of $M_e$. If $j_h \in p_{i_h}$ and $i_h \geq i_{h-1}, e_{j_{h-1},j_h}e_{j_{h-2},j_{h-1}} \cdots e_{j_{i+1},j_i}m$ is a linear combinations of basis elements of $M_e$ with degrees $\leq h-1$, proving (c). \qed

Definition 3.11. Recall that $p_i$’s and $e_j$’s are in Assumption 3.6. Define

a) $u_i = -c_i + p_{i-1}$, and $\overline{u}_i = c_i + n - p_i$, $i \in k$.

b) $f(x) = \prod_{i=1}^{k}(x - u_i)$ and $g(x) = \prod_{i=1}^{k}(x - \overline{u}_i)$. 

Lemma 3.12. Let $M_c = M^p(\delta_c)$ where $\delta_c$ is in the Assumption 3.6.

a) $M_c \otimes V$ has a parabolic Verma flag

\[ 0 = M_0 \subset M_1 \subset M_2 \subset \cdots \subset M_k = M_c \otimes V \quad (3.8) \]

such that $M_i/M_{i-1} \cong M^p(\delta_c + \epsilon_{p_{i-1}+1})$, where $M_i$ is generated by $\{m \otimes v_{p_0+1}, m \otimes v_{p_1+1}, \cdots, m \otimes v_{p_{i-1}+1}\}$. Moreover, $\prod_{j=1}^{i}(x_1 - u_j)$ acts on $M_i$ trivially.

b) $M_c \otimes W$ has a parabolic Verma flag

\[ 0 = N_{k+1} \subset N_k \subset \cdots \subset N_1 = M_c \otimes W \quad (3.9) \]

such that $N_i/N_{i+1} \cong M^p(\delta_c - \epsilon_{p_i})$, where $N_i$ is generated by $\{m \otimes v_{p_k}^*, m \otimes v_{p_{k-1}}^*, \cdots, m \otimes v_{p_1}^*\}$. Moreover, $\prod_{j=1}^{k}(x_1 - \varpi_j)$ acts on $N_i$ trivially.

Proof. By [14, Theorem 3.6], both $M_c \otimes V$ and $M_c \otimes W$ have parabolic Verma flags as required. It is well known that $C$ acts on $M^p(\lambda)$ as the scalar $\langle \lambda, \lambda + 2\rho \rangle$, where

\[ \rho = -\varepsilon_2 - 2\varepsilon_3 - \cdots - (n-1)\varepsilon_n. \quad (3.10) \]

By (3.3), $\Omega$ acts on $M^p(\delta_c + \varepsilon_i)$ as the scalar $\langle \delta_c, \varepsilon_i \rangle - (i-1)$. Similarly, it acts on $M^p(\delta_c - \varepsilon_i)$ as the scalar $-(\delta_c, \varepsilon_i) - (n-i)$. Therefore, $\prod_{j=1}^{i}(x_1 - u_j)$ (resp., $\prod_{j=1}^{k}(x_1 - \varpi_j)$) acts on $M_i$ (resp., $N_i$) trivially. \qed

Lemma 3.13. The generating function of parameters $\omega_a$’s in Proposition 3.5 satisfies

\[ 1 + \sum_{a=0}^{\infty} \frac{\omega_a}{a!} x^a = \prod_{i=1}^{k} \frac{u + n - u_i}{u + u_i}, \quad (3.11) \]

if we use $M_c$ to replace $M$ in Proposition 3.5, where $u_i$ and $\varpi_j$ are defined in Definition 3.7.

Proof. Let $E$ be the $n \times n$ matrix such that the $(i,j)$th entry is the matrix unit $e_{i,j}$. It is well known that the Gelfand invariant $\text{tr}(E^a)$ is central in $U$ for any $a \in \mathbb{N}$ (see, e.g., [16, Corollary 7.1.4]). On the other hand, for any $g$-module $M$, $e_1 x_1^a e_1$ acts on $M \otimes V \otimes W$ as $(-1)^a \text{tr}(E^a) \otimes e_1$, $a \in \mathbb{N}$. Let $\chi : Z(U) \to \mathbb{C}[\ell_1, \ell_2, \cdots, \ell_n]^{GL_n}$ be the Harish-Chandra isomorphism, where $Z(U)$ is the center of $U$ and $\ell_i = e_{i,i} - i + 1$, $i \in \mathbb{N}$. It follows from [16, Corollary 7.1.4] that

\[ 1 + \sum_{a=0}^{\infty} \frac{(-1)^a \chi(\text{tr}(E^a))}{(u - n + 1)^{a+1}} = \prod_{i=1}^{k} \frac{u + \ell_i + 1}{u + \ell_i}. \]

If $M = M_c$, then $\omega_a = (-1)^a \chi(\text{tr}(E^a))(\delta_c)$ (see Assumption 3.6). Using $u$ instead of $u - n + 1$ yields (3.11). \qed

Lemma 3.14. Let $f(x_1)$ and $g(\varpi_1)$ be defined in Definition 3.7.

a) The set $\{e_1, e_1 x_1, \cdots, e_1 x_1^{k-1}\}$ is $\mathbb{C}$-linear independent if we consider it as a subset of $\text{End}_U(M_c^{\text{aff}})$.

b) $e_1 f(x_1) = (-1)^k e_1 g(\varpi_1)$ in $\mathbb{R}_{\text{aff}}$. 

Remark 3.16. Define $x'_i = x_1$ and $x'_i = s_{i-1}x'_i s_{i-1}$ for $1 < i \leq r$. Similarly, define $\pi'_i$ for $1 \leq i \leq t$. Since $x'_i$ (resp., $\pi'_i$) acts on $M_{c,t}$ as $-\pi_{0,i}(\Omega)$ (resp., $-\pi_{0,i}(\Omega)$), and $x_i = x'_i$ (resp., $\pi_i = \pi'_i$) in $\mathcal{B}_{k,r,t}$, up to a linear combination of some basis elements of $M_{c,t}$ with lower degrees, we have formulae for $m \otimes v_i x_i$ (resp., $m \otimes v_i \pi_i$) similar to those in Lemma 3.10, where $i \in I(n,r+t)$.

Theorem 3.17. If $r + t \leq \min\{q_1, q_2, \ldots, q_k\}$, then the algebra homomorphism $\varphi : \mathcal{B}_{k,r,t} \to \text{End}_\mathcal{O}(M_{c,t})^{\text{op}}$ in Proposition 3.15 is an algebra isomorphism.
Proof. We claim that the images of all regular monomials of $\mathcal{B}_{k,r,t}$ are linear independent in $\text{End}_\mathcal{O}(M_{c,r,t})^\text{op}$. If so, by Theorem 2.31(a), $\varphi$ is injective. On the other hand, by adjoint associativity, there is a $\mathbb{C}$-linear isomorphism

$$\text{End}_\mathcal{O}(M_{c,r,t}) \cong \text{End}_\mathcal{O}(M_c \otimes V^{\otimes r+t}). \quad (3.16)$$

If $r + t \leq \min\{q_1, q_2, \cdots, q_k\}$, then $\dim \text{End}_\mathcal{O}(M_c \otimes V^{\otimes r+t}) = k^{r+t}(r + t)!$. By Proposition 3.15, $\mathcal{B}_{k,r,t}$ is admissible and hence the dimension of $\mathcal{B}_{k,r,t}$ is $k^{r+t}(r + t)!$ (see Theorem 2.3), forcing $\varphi$ to be an isomorphism.

It remains to prove our claim. Recall that a regular monomial of $\mathcal{B}_{k,r,t}$ is of form $x^\alpha D^\beta$ where $D$ is a walled Brauer diagram and $(\alpha, \beta) \in \mathbb{N}_k \times \mathbb{N}_k$. For each $x^\alpha D^\beta$, we assume that $x^\alpha D^\beta$ acts on the left of $M_{c,r,t}$. In other words, when we consider the right action of $\mathcal{B}_{k,r,t}$, $x^\alpha D^\beta$ should be replaced by $\sigma(x^\alpha D^\beta)$, where $\sigma$ is the $R$-linear anti-involution in Lemma 2.2. Such elements consist of an $\mathbb{C}$-basis of $\mathcal{B}_{k,r,t}$.

Motivated by Brundan-Stroppel’s work in [9] and Lemma 3.10 and Remark 3.16 we define a labeled walled Brauer diagram for any $x^\alpha D^\beta$ as follows. In this case, we identify an edge of $D$ as an arrow and call the starting point as a source and the endpoint as a head.

- The vertices $\{r, r - 1, \cdots, 1\}$ (resp., $\{\overline{1}, \overline{2}, \cdots, \overline{t}\}$) on the bottom (resp., top) row of $D$ are called sources of corresponding arrows of $D$. The other vertices of $D$ will be called heads of corresponding arrows of $D$.
- For any $i \in \mathbb{Z}$, there are $\alpha_i$ beads at the $i$-th vertex on the top row of $D$.
- For any $i \in \mathbb{Z}$, there are $\beta_i$ beads at the $i$-th vertex on the bottom row of $D$.
- For the $i$-th vertex on the bottom row of $D$, we label it as $p_{k-1} + (r - i + 1)$, where
  - $p_{k-1}$ is given in Assumption 3.6.
- For the $i$-th vertex on the top row of $D$, we label it as $p_{k-1} + r + i$.
- If there is no bead at the head of an arrow of $D$, then we label the head the same labeling of the corresponding source.
- If there are $h$ beads at the head of an arrow, and if the labeling of the source is $p$, we label the head with positive integer $p - \sum_{i=1}^h q_{k-i}$ where $q_i$’s are given in Assumption 3.6.

Since $r + t \leq \min\{q_1, q_2, \cdots, q_k\}$, the above setting is well-defined. Moreover, for each $x^\alpha D^\beta$, we obtain two sequences of positive integers $(\alpha, D, \beta)^b$ and $(\alpha, D, \beta)^t$, which are obtained by reading labeling according to the vertices $r, r - 1, \cdots, 1, \overline{1}, \overline{2}, \cdots, \overline{t}$ on the bottom (resp., top) row of the labeled walled Brauer diagram. The key point is that we always fix the labeling of the sources of $D$ as above and hence both $(\alpha, D, \beta)^b$ and $(\alpha, D, \beta)^t$ are uniquely determined by the triple $(\alpha, D, \beta)$ (see Example 3.18).

Recall that $p_i$’s are positive integers in Assumption 3.6. For $i \in \{r, r - 1, \cdots, 1\}$ (resp., $\overline{i} \in \{\overline{1}, \overline{2}, \cdots, \overline{t}\}$), if there is no bead on the edge which contains $i$ (resp. $\overline{i}$), define $\mathcal{Y}_i = 1$. ...
(resp. \( \mathcal{Y}_T = 1 \)); otherwise, there are \( h \) beads on the edge which contains \( i \) (resp., \( j \)) at the bottom (resp., top) row, define

\[
\begin{align*}
(1) \quad & \mathcal{Y}_i = e_{pk-i+1}e_{pk-i+2} \cdots e_{pk-h+i}e_{pk-h-1+i}, \\
(2) \quad & \mathcal{Y}_j = e_{pk-i-r+i}e_{pk-i-r+1} \cdots e_{pk-h+i+r}e_{pk-h-1+i}, \\
(3) \quad & \mathcal{Y} = \mathcal{Y}_1 \mathcal{Y}_2 \cdots \mathcal{Y}_i \mathcal{Y}_j \mathcal{Y}_j \cdots \mathcal{Y}_T.
\end{align*}
\]

Now, we assume that \( \sum_{\alpha,D,\beta} a_{\alpha,D,\beta} \sigma(x^\alpha D^\beta) = 0 \), where \( D \) ranges over all walled Brauer diagrams and \( (\alpha, \beta) \in \mathbb{N}_k^r \times \mathbb{N}_k^s \). If there is an \( a_{\alpha,D,\beta} \neq 0 \) for some \( (\alpha, \beta) \in \mathbb{N}_k^r \times \mathbb{N}_k^s \), we consider \( x^\gamma D^\delta \) among such regular monomials such that \( \sum_i \gamma_i + \sum_j \delta_j \) is maximal. If \( \sum_i \gamma_i + \sum_j \delta_j > 0 \), we write

\[
\begin{align*}
a) \quad & b = (\gamma, D, \delta)^b = (b_r, b_{r-1}, \cdots, b_1; b_T, b_{T-1}, \cdots, b_T) \in I(n, r + t), \\
b) \quad & w = (\gamma, D, \delta)^t = (w_r, w_{r-1}, \cdots, w_1; w_T, w_{T-1}, \cdots, w_T) \in I(n, r + t).
\end{align*}
\]

By Lemma 3.10 and Remark 3.16, the coefficient of \( \mathcal{Y}_m \otimes u_w \) in \( (m \otimes v_b) \sum_{\alpha,D,\beta} a_{\alpha,D,\beta} \sigma(x^\alpha D^\beta) \) is \( a_{\gamma,D,\delta} \) up to a sign, forcing \( a_{\gamma,D,\delta} = 0 \), a contradiction. The key point is that there is a basis element \( \mathcal{Y}_m \) of \( M_e \) such that the coefficient of the basis element \( \mathcal{Y}_m \otimes u_w \) in \( (m \otimes v_b) \sum_{\alpha,D,\beta} a_{\alpha,D,\beta} \sigma(x^\alpha D^\beta) \) is \( a_{\gamma,D,\delta} \) up to a sign, where \( \mathcal{Y}_m \) is of the highest degree \( \sum_i \gamma_i + \sum_j \delta_j \) and \( \mathcal{Y} \) is determined uniquely by both \( b \) and \( w \). Finally, we consider regular monomials \( x^\alpha D^\beta \) with degree 0. In this case, we consider all walled Brauer diagrams as elements in \( \text{End}_O(V^{\otimes r} \otimes W^t) \). When \( r + t \leq n \), it is well known that \( \mathcal{B}_{r,t}(n) \) acts faithfully on \( V^{r,t} \). So, \( a_{\alpha,D,\beta} = 0 \) for all regular monomials \( x^\alpha D^\beta \) with degree 0. 

\[ \square \]

**Example 3.18.** We give an example to illustrate that \( (\alpha, D, \beta)^b \) and \( (\alpha, D, \beta)^t \) are uniquely determined by \( x^\alpha D^\beta \in \mathcal{B}_{k,r,t} \) and the labeling of the sources of \( D \). We assume \( k = 2 \) and \( r = t = 3 \). Fix \( q_1 \) and \( q_2 \) such that \( q_1 + q_2 = n \). If \( \alpha = (1, 0, 1) \) and \( \beta = (0, 1, 1) \), and \( D = e_1s_1\overline{s}_2 \), then \( (\alpha, D, \beta)^b = (q_1 + 1, q_1 + 2, q_1 + 3; q_1 + 2, 6, 5) \) and \( (\alpha, D, \beta)^t = (1, q_1 + 3, 4; q_1 + 4, q_1 + 5, q_1 + 6) \). In this case, \( \mathcal{Y} = e_{q_1+1,1}e_{q_1+4,4}e_{q_1+5,5}e_{q_1+6,6} \).

4. **Graded cyclotomic walled Brauer algebras**

In this section, we assume that \( q = (q_1, q_2, \cdots, q_k) \) is a partition of \( n \). Consider the tableau \( t \) with respect to \( q \) such that there are \( q_i \) boxes in the \( i \)th column and moreover, the numbers
1, 2, \cdots, n are inserted into the boxes along the columns from left to right. For example,
\[
t = \begin{array}{ccc}
1 & 2 & 5 \\
3 & 6 & 8 \\
4 & 7 & 9
\end{array}
\quad \text{if } (q_1, q_2, q_3) = (4, 3, 2). \tag{4.1}
\]
For any \( i \in \mathbb{N} \), following \cite{6}, let row\((i) = \ell \) (resp., col\((i) = m \)) if the box containing \( i \) is in the \( \ell \)th row and \( m \)th column of the \( t \) (see e.g. \((4.1)\)). Define the nilpotent matrix
\[
e = \sum_{(i,j) \in K} e_{i,j} \in \mathfrak{g}, \tag{4.2}
\]
where
\[
K = \{(i, j) \mid 1 \leq i, j \leq n, \text{row}(i) = \text{row}(j), \text{col}(i) = \text{col}(j) - 1\}. \tag{4.3}
\]
It is known that there is a \( \mathbb{Z} \)-grading \( \mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}_i \) on \( \mathfrak{g} \) by declaring that \( e_{i,j} \) is of degree \( \text{col}(j) - \text{col}(i) \). The parabolic subalgebra \( \mathfrak{p} \) is \( \mathfrak{f} \bigoplus_{i > 0} \mathfrak{g}_i \), where \( \mathfrak{f} \), which is the corresponding Levi subalgebra, is \( \mathfrak{g}_0 \). Let \( \mathfrak{m} = \bigoplus_{i < 0} \mathfrak{g}_i \) and define \( \mathfrak{g}_e \) to be the centralizer of \( e \) in \( \mathfrak{g} \). Then the universal enveloping algebra \( U(\mathfrak{g}_e) \) is a graded subalgebra of \( U(\mathfrak{p}) \). In this section, we assume that \( B_{r,t} \) is an affine walled Brauer algebra with arbitrary parameters \( \omega_a \)'s such that \( \omega_0 = n \) and moreover, \( B_{k,r,t} \) is admissible. However, when we use graded cyclotomic walled Brauer algebras \( gr(B_{k,r,t}) \) next section, we will show that the parameters for \( B_{k,r,t} \) come from Lemma 3.12 and (3.11). By Proposition 3.15 \( B_{k,r,t} \) is admissible.

**Lemma 4.1.** As a \( \mathbb{Z} \)-graded algebra, \( gr(B_{k,r,t}) \) is generated by \( e_1, x_1, \overline{x}_1, s_i, \overline{s}_j, 1 \leq i \leq r - 1 \) and \( 1 \leq j \leq t - 1 \) such that all relations in Definition 2.7 hold except (12)-(13), (25)-(26) and (21) which are replaced by the following relations:

\begin{enumerate}
\item[a)] \( s_1 x_1 s_1 x_1 = x_1 s_1 x_1 s_1 \), \quad \item[d)] \( x_1^k = \overline{x}_1^k = 0 \), \quad \item[b)] \( \overline{x}_1 x_1 \overline{x}_1 = \overline{x}_1 \overline{x}_1 \overline{x}_1 \), \quad \item[e)] \( e_1 x_1^h e_1 = e_1 \overline{x}_1^h e_1 = 0 \), for \( h \geq 1 \). \quad \item[c)] \( x_1 \overline{x}_1 = \overline{x}_1 x_1 \),
\end{enumerate}

For any positive integers \( r \) and \( t \), define
\[
V^{r,t} = V^{\otimes r} \otimes W^{\otimes t}, \tag{4.4}
\]
where \( V \) is the natural module for \( \mathfrak{g} \) and \( W \) is the linear dual of \( V \). We order the positions of tensor factors of \( V^{r,t} \) according to the total ordered set \( (J_1 \cup J_2, <) \) where \( < \) is given in \((3.2)\). So,
\[
r < r - 1 < \cdots < 1 < \overline{T} < \cdots < \overline{t}.
\]
It is easy to see that \( V^{r,t} \) has a basis which consists of all elements \( v_i \), \( i \in (n, r + t) \). Make \( V \) and \( W \) into a graded \( U(\mathfrak{g}_e) \)-module by declaring that \( \text{deg} \, v_i = -\text{deg} \, v_i^* = k - \text{col}(i) \), it leads to gradings on \( V^{r,t} \) and \( \text{End}(V^{r,t}) \). Recall that \( \pi_{a,b} \) in \((3.4)\) and \( \Omega \) in \((3.3)\). An graded algebra homomorphism between two graded algebras is a graded homomorphism with degree zero.
Proposition 4.2. Let $e \in \mathfrak{g}$ be given in (4.2). There is a graded algebra homomorphism $\varphi : \text{gr}(\mathcal{B}_{k,r,t}) \rightarrow \text{End}_{U(\mathfrak{g}_e)}(V^{r,t})^{op}$ such that

(1) $\varphi(e_1) = -\pi_{1,1}(\Omega)$,
(2) $\varphi(\pi_j) = \pi_{j-j,1}(\Omega)$, $1 \leq j \leq t - 1$,
(3) $\varphi(s_i) = \pi_{i+1,i}(\Omega)$, $1 \leq i \leq r - 1$,
(4) $\varphi(x_1) = -1^{\otimes r-1} \otimes e \otimes 1^t$,
(5) $\varphi(\pi_1) = -1^t \otimes e \otimes 1^{\otimes t-1}$.

Proof. It follows from the conditions (1)-(3) and Proposition 3.5 that $\varphi(e_1)$, $\varphi(s_i)$’s and $\varphi(\pi_j)$’s satisfy relations for the walled Brauer algebra $\mathcal{B}_{r,t}(n)$, a subalgebra of $\mathcal{B}_{k,r,t}$. Moreover, since $e$ is the nilpotent matrix in (4.2) and $e^k = 0$, $\varphi(x_1^k) = \varphi(\pi_1^k) = 0$. The conditions in Lemma 4.1(a)-(c)(e) immediately follow from the definitions. One can verify other relations by straightforward computation. We show that

$$\varphi(e_1)(\varphi(x_1) + \varphi(\pi_1)) = (\varphi(x_1) + \varphi(\pi_1))\varphi(e_1) = 0$$

as an example and leave the others to the reader. We have $(v_i \otimes v_j^*) e_1 = 0$ if $i \neq j$. Otherwise,

$$(v_i \otimes v_j^*)(e_1(x_1 + \pi_1)) = \sum_{j=1}^n v_j \otimes v_j^*(x_1 + \pi_1)$$

$$= -\sum_{(i,j) \in K} v_i \otimes v_j^* - \sum_{(i,j) \in K} v_i \otimes (-v_j^*) = 0.$$

If $(v_i \otimes v_j^*)(x_1 + \pi_1) e_1 \neq 0$, then $(j, i) \in K$. So, $(v_i \otimes v_j^*)(x_1 + \pi_1) e_1 = -(v_j \otimes v_j^* - v_i \otimes v_i^*) e_1 = 0$, and (4.5) follows.

For the simplification of notation, we denote by $\text{End}(M)$ the set of all linear endomorphisms for any $\mathbb{C}$-space $M$. Since $V^{r,t}$ is a graded $(U(\mathfrak{g}_e), \text{gr}(\mathcal{B}_{k,r,t}))$-bimodule, it leads to the graded algebra homomorphism

$$\psi : U(\mathfrak{g}_e) \rightarrow \text{End}_{\text{gr}(\mathcal{B}_{k,r,t})}(V^{r,t}).$$

(4.6)

Define the flip map

$$\text{flip} : \text{End}(V^{\otimes r}) \otimes \text{End}(V^{\otimes t}) \rightarrow \text{End}(V^{\otimes r}) \otimes \text{End}(W^{\otimes t})$$

(4.7)

such that $\text{flip}(f \otimes g) = f \otimes g^*$, for any $f \in \text{End}(V^{\otimes r})$ and $g \in \text{End}(V^{\otimes t})$, where $g^* \in \text{End}(W^{\otimes t})$ such that

$$g^*(v^*(w)) = v^*(g(w)), \forall w \in V^{\otimes t}.$$  

(4.8)

In this paper, we identify $I(n,r)$ with $I(n,r + 0)$. Similarly, we identify $I(n,t)$ with $I(n,0 + t)$. For each $i \in I(n,r)$ (resp., $j \in I(n,t)$), following Lemma 3.3 define $v_i = v_{i_1} \otimes v_{i_{r-1}} \otimes \cdots \otimes v_{i_1}$ (resp., $v_j = v_{j_1}^* \otimes v_{j_2}^* \otimes \cdots \otimes v_{j_r}^*$). If there is no confusion, we also write $v_i^* = v_{j_1}^* \otimes v_{j_2}^* \otimes \cdots \otimes v_{j_r}^*$.

Definition 4.3.  

a) For $i, j \in I(n,r)$, define $e_{ij} \in \text{End}(V^{\otimes r})$ such that $e_{ij}(v_k) = \delta_{j,k}v_i$ for any $k \in I(n,r)$.
b) For $i,j \in I(n,t)$, define $f_{i,j} \in \text{End}(W^{\otimes t})$ such that $f_{i,j}(v_k^*) = \delta_{j,k} v_i^*$.

Lemma 4.4. [9] Lemma 7.6] Let $\phi$ be the linear map defined by the following commutative diagram

$$
\begin{array}{ccc}
\text{End}(V^{\otimes r+t}) & \xrightarrow{\phi} & \text{End}(V^{r,t}) \\
\downarrow b & & \downarrow c \\
\text{End}(V^{\otimes r}) \otimes \text{End}(V^{\otimes t}) & \xrightarrow{\text{flip}} & \text{End}(V^{\otimes r}) \otimes \text{End}(W^{\otimes t})
\end{array}
$$

(4.9)

where $b,c$ are canonical linear isomorphisms. Then $\phi$ is a $\mathfrak{g}_k$-module isomorphism.

In fact, [9] Lemma 7.6] is for general linear Lie superalgebra $\mathfrak{g}_{m|n}$ and further, $\phi$ is a $\mathfrak{g}_{m|n}$-module isomorphism. We need its special case $n = 0$. We also call $\phi$ the flip map and will denote it by flip, too. Recall that $e$ is the nilpotent matrix defined in (4.2). The following result can be verified easily. Note that $(g_1 g_2)^* = g_2^* g_1^*$ for any $g_1, g_2 \in \text{End}(V^{\otimes t})$ (see (4.8)).

Lemma 4.5. Let $K$ be given in (4.3). For any $i,j \in I(n,t)$, and $\beta \in \mathbb{N}^t$, define

$$e_i e_j^\beta = e_{i_1,j_1} e_{i_2,j_2}^\beta \otimes \cdots \otimes e_{i_t,j_t} e_{i_1,j_1}^\beta,$$

$$e_i (e_j^\beta f_{j,i}) = (e_j^\beta f_{j,i}) (e_i e_j^\beta).$$

Then $e^* = \sum_{(i,j) \in K} f_{j,i}$ and $(e_i e_j^\beta)^* = (e_j^\beta f_{j,i})$.

In this paper, we always denote by $S_{r+t}$ the symmetric group on $r + t$ letters \{r, $\cdots$, 2, 1, $\overline{r}$, $\overline{r-1}$, $\cdots$, $\overline{1}$\}. We can identify each permutation $w \in S_{r+t}$ with its permutation diagram such that the vertices at the both rows are indexed by $r, r-1, \cdots, 1, \overline{r}, \overline{r-1}, \cdots, \overline{1}$ from left to right. There is a linear isomorphism

$$\text{flip} : C S_{r+t} \rightarrow \mathcal{B}_{r,t}(n),$$

(4.10)

sending a permutation diagram to the corresponding walled Brauer diagram obtained by adding an imaginary wall between the 1th and $\overline{1}$th vertices, and flipping the part of the diagram which is at the right hand of the wall. Let $\mathcal{H}_{k,r+t}$ be a level $k$ degenerate Hecke algebra. The current definition of a level $k$ degenerate Hecke algebra is different from the usual one. Our $x_r, x_{r-1}, \cdots, x_1, \overline{x}_1, \overline{x}_2, \cdots, \overline{x}_t$‘s are the same as $-x_1, -x_2, \cdots, -x_{r+t}$ in usual sense. Moreover, $s_{r-i}$ is the usual $s_i$, $i \in r-1$ and $\overline{s}_j$ is the usual $s_{r+j}$, $j \in t-1$. The special one which switches 1, $\overline{1}$ is the usual $s_r$. We keep this setting so as to be compatible with $\mathcal{B}_{k,r,t}$. The associated graded algebra $\text{gr}\mathcal{H}_{k,r+t}$ has a basis

$$\{x^\alpha \overline{x}^\beta w \mid (\alpha, \beta) \in N_r^t \times N^t_k, w \in S_{r+t}\}.$$  

(4.11)

It follows from (4.10) and Theorem 2.3 that $\text{gr} \mathcal{B}_{k,r,t}$ has a basis

$$\{x^\alpha \overline{\text{flip}}(w) \overline{x}^\beta \mid (\alpha, \beta) \in N^t_k \times N^r_k, w \in S_{r+t}\},$$

(4.12)

where $\overline{\text{flip}}$ is given in (4.10). This leads to a linear isomorphism

$$\overline{\text{flip}} : \text{gr}\mathcal{H}_{k,r+t} \rightarrow \text{gr} \mathcal{B}_{k,r,t}.$$  

(4.13)
Motivated by [9], for any \( w \in \mathcal{S}_{r,t} \) and any \( i, j \in I(n, r + t) \), there is a labeled diagram \( w_{ij} \) obtained by labeling the vertices at the bottom (resp., top) row of \( w \) according to the sequence \( i_r, i_{r-1}, \ldots, i_1, i_T, i_{T-1}, \ldots, i_T \) (resp., \( j_r, j_{r-1}, \ldots, j_1, j_T, j_{T-1}, \ldots, j_T \) ) from left to right. Similarly, for any walled Brauer diagram \( D \) and any \( i, j \in I(n, r + t) \), there is a labeled diagram, say, \( D_{ij} \), obtained by labeling the vertices at the bottom (resp., top) row of \( D \) according to the sequence \( i_r, i_{r-1}, \ldots, i_1, i_T, i_{T-1}, \ldots, i_T \) (resp., \( j_r, j_{r-1}, \ldots, j_1, j_T, j_{T-1}, \ldots, j_T \) ). Following [9], we call a labeled diagram a consistently labeled diagram if the vertices at the ends of each edge are labeled with the same number. For \( x \in \{w_{ij}, D_{ij}\} \), define

\[
\text{wt}(x) = \begin{cases} 
1, & \text{if } x \text{ is consistently;} \\
0, & \text{otherwise.} 
\end{cases}
\] (4.14)

The following result is the special case of [9] Lemma 7.3–7.4 for \( \mathfrak{gl}_{n|0} \).

**Lemma 4.6.** Suppose \( i \in I(n, r + t) \).

a) Each \( w \in \mathcal{S}_{r,t} \) acts on the right of \( V^\otimes(r+t) \) via \( \sum_{i,j} \text{wt}(w_{ij})e_{iL}e_{jL} \otimes e_{iR}e_{jR} \), for all \( i, j \in I(n, r + t) \), where \( e_{iL}e_{jL} \) and \( e_{iR}e_{jR} \) are defined in Definition 4.3.

b) Each walled Brauer diagram \( D \) acts on the right of \( V^{r,t} \) as \( \sum_{i,j} \text{wt}(D_{ij})e_{iL}e_{jL} \otimes f_{iR}f_{jR} \), for all \( i, j \in I(n, r + t) \).

**Lemma 4.7.** There is a graded algebra homomorphism \( \phi : \text{gr}\mathcal{H}_{k,r+t} \rightarrow \text{End}_{U(\mathfrak{gl})}(V^\otimes(r+t))^{\text{op}} \) such that

\[
\begin{align*}
\phi(x_1) &= -1^{r-1} \otimes e \otimes 1^\otimes t, & d) \phi(s_i) &= \pi_{i+1}^1(\Omega), & 1 \leq i \leq r - 1, \\
\phi(\pi_1) &= -1^{r} \otimes e \otimes 1^{\otimes(t-1)}, & e) \phi(\pi_j) &= \pi_{j+1}^{t-j}(\Omega), & 1 \leq j \leq t - 1.
\end{align*}
\]

Moreover, we have the commutative diagram as follows,

\[
\begin{array}{ccc}
\text{gr}\mathcal{H}_{k,r+t} & \xrightarrow{\text{flip}} & \text{gr}\mathcal{B}_{k,r,t} \\
\phi \downarrow & & \phi \downarrow \\
\text{End}(V^{\otimes(r+t)}) & \xrightarrow{\text{flip}} & \text{End}(V^{r,t}),
\end{array}
\] (4.15)

where flip (resp., \( \text{flip} \)) is given in (4.9) (resp., (4.13)) and \( \phi \) is in Proposition 4.2.

**Proof.** For any \( w \in \mathcal{S}_{r,t} \), it follows from [9] Lemma 7.7 that

\[
\text{flip}(\phi(w)) = \sum_{i,j \in I(n, r + t)} \text{wt}(w_{ij})e_{iL}e_{jL} \otimes f_{iR}f_{jR} = \varphi(\text{flip}(w))
\] (4.16)

where \( w \) acts on \( V^{r,t} \) as in Lemma 4.6a). So,

\[
\text{flip} \circ \phi(x_1^{\alpha_1}x_2^{\alpha_2} \cdots x_r^{\alpha_r}x_1^{\beta_1}x_2^{\beta_2} \cdots x_t^{\beta_t}w) = \text{flip}(\sum_{i,j \in I(n, r + t)} \text{wt}(w_{ij})e_{iL}e_{jL}e^{\alpha} \otimes e_{iR}e_{jR}e^{\beta})
\]

\[
= \sum_{i,j \in I(n, r + t)} \text{wt}(w_{ij})e_{iL}e^{\alpha} \otimes e_{iR}e_{jR}e^{\beta})^* = \sum_{i,j \in I(n, r + t)} \text{wt}(w_{ij})e_{iL}e^{\alpha} \otimes (e^{\beta})^*f_{iR}f_{jR}
\]

\[
= \varphi(x_1^{\alpha_1}x_2^{\alpha_2} \cdots x_r^{\alpha_r} \text{flip}(w)x_1^{\beta_1}x_2^{\beta_2} \cdots x_t^{\beta_t}w) = \varphi(\text{flip}(x_1^{\alpha_1}x_2^{\alpha_2} \cdots x_r^{\alpha_r}x_1^{\beta_1}x_2^{\beta_2} \cdots x_t^{\beta_t}w)
\]
where the third (resp., forth) equality follows from Lemma [4, 5](resp., [4, 16]). This completes the proof. □

**Theorem 4.8.** Let \( \varphi \) be the graded algebra homomorphism in Proposition [4, 2]. Then \( \varphi \) is surjective, and it is injective if \( r + t \leq q_k \).

Proof. By Lemma [4, 3], the flip map in [4, 15] is \( \mathfrak{g}_n \)-homomorphism. So, there is a \( \mathbb{C} \)-linear isomorphism \( \operatorname{End}_U(\mathfrak{g}_n)(V^{\otimes(r+t)}) \cong \operatorname{End}_U(\mathfrak{g}_n)(V^{r,t}) \) and hence the results follow from Lemma [4, 7] and [6, Theorem 2.4] which says that \( \phi \) in (4.15) is an epimorphism and moreover, \( \phi \) is injective if \( r + t \leq q_k \). □

5. **Finite W-algebras and cyclotomic walled Brauer algebras**

Throughout this section, we go on assuming that \( q = (q_1, q_2, \ldots, q_k) \) is a partition of \( n \) and \( d = (d_1, d_2, \ldots, d_k) \in \mathbb{C}^k \) in Assumption [3, 6]. So, \( d_i - d_j \in \mathbb{Z} \) if and only if \( d_i = d_j \). Recall that \( \mathfrak{p} \) is the parabolic subalgebra of \( \mathfrak{g} \) whose Levi subalgebra is \( \mathfrak{g}_{l_{q_1}} \oplus \mathfrak{g}_{l_{q_2}} \oplus \cdots \oplus \mathfrak{g}_{l_{q_k}} \). It follows from [6] that there are two algebra automorphisms of \( U(\mathfrak{p}) \), say \( \eta_d \) and \( \eta \) such that, for each \( e_{i,j} \in \mathfrak{p} \),

\[
\begin{align*}
\eta_d(e_{i,j}) &= e_{i,j} + \delta_{i,j}d_{\operatorname{col}(i)}, \\
\eta(e_{i,j}) &= e_{i,j} + \delta_{i,j}(q_1 - q_{\operatorname{col}(j)} - q_{\operatorname{col}(j)+1} - \cdots - q_k).
\end{align*}
\]

(5.1)

Recall that \( \mathfrak{m} \subseteq \mathfrak{g} \) such that \( \mathfrak{g} = \mathfrak{p} \oplus \mathfrak{m} \). Let \( \chi : U(\mathfrak{m}) \to \mathbb{C} \) be the homomorphism sending \( x \) to \((x, e)\) for all \( x \in \mathfrak{m} \), where \((a, b) = \operatorname{tr}(ab)\) for \( a, b \in \mathfrak{g} \), and \( \operatorname{tr}(\ ) \) is the usual trace function defined on \( \mathfrak{g} \). Following [6], define \( I_\chi \) to be the kernel of the homomorphism \( \chi \). Then the finite W-algebra

\[
U(\mathfrak{g}, e) = \{ u \in U(\mathfrak{p}) | [x, \eta(u)] \in U(\mathfrak{g})I_\chi \text{ for all } x \in \mathfrak{m} \}.
\]

(5.2)

It is a subalgebra of \( U(\mathfrak{p}) \). The grading on \( U(\mathfrak{p}) \) induces a filtration on \( U(\mathfrak{g}, e) \) as follows:

\[
0 \subseteq U(\mathfrak{g}, e)^{(0)} \subseteq U(\mathfrak{g}, e)^{(1)} \subseteq \cdots
\]

with \( U(\mathfrak{g}, e)^{(i)} = U(\mathfrak{g}, e) \cap \bigoplus_{j=0}^i U(\mathfrak{p})^{(j)} \) and \( U(\mathfrak{p})^{(j)} \) consists of all elements of \( U(\mathfrak{p}) \) with degree \( j \). Let \( \operatorname{gr}(U(\mathfrak{g}, e)) \) be the associated graded algebra. By [6, Lemma 3.1], there is a graded algebra isomorphism

\[
\operatorname{gr}(U(\mathfrak{g}, e)) \cong U(\mathfrak{g}_e).
\]

(5.3)

The \( V^{r,t} \) in [4, 4] can be considered as a left \( U(\mathfrak{p}) \)-module and the action of \( U(\mathfrak{p}) \) is defined as

\[
u \cdot v = \eta_d(u)v, \forall u \in U(\mathfrak{p}), \text{ and } v \in V^{r,t},
\]

(5.4)

where \( \eta_d \) is defined in (5.1). In order to emphasis the \( \eta_d \), the left \( U(\mathfrak{p}) \)-module \( V^{r,t} \) is denoted as \( V^{r,t}_d \). Since \( V^{r,t} \) is a graded vector space, \( V^{r,t}_d \) can be considered as a graded \( U(\mathfrak{p}) \)-module.
Via restriction, it can be considered as a left graded $U(\mathfrak{g})$-module. The corresponding representation $\psi_d$ is

$$\psi_d = \psi \circ \eta_d \quad (5.5)$$

where $\psi$ is defined in (4.6). We remark that all results in this section can be found in [6] when $t = 0$. In general, the proofs still depend on Brundan-Kleshchev's idea in [6].

Let $C_d = C_1d$ be the 1-dimensional $p$-module such that each $e_{i,j} \in p$ acts on $1_d$ via the scalar $\delta_{i,j} \text{col}(i)$. Since $V^{r,t}$ can be considered as a $U(\mathfrak{g})$-module with respect to $\psi$, it leads to the left $U(\mathfrak{g})$ structure on $C_d \otimes V^{r,t}$. The following result can be verified easily.

**Lemma 5.1.** As $U(\mathfrak{g})$-modules, $C_d \otimes V^{r,t} \cong V^{r,t}_d$, and the corresponding isomorphism sends $1_d \otimes v$ to $v$, for each $v \in V^{r,t}$.

Via restriction, both $C_d$ and $V^{r,t}_d$ are $U(\mathfrak{g},e)$-modules. Let $\Psi_d : U(\mathfrak{g},e) \to \text{End}(V^{r,t}_d)$ be the corresponding algebra homomorphism. Via (5.3), the associated graded homomorphism

$$\text{gr}\Psi_d : \text{gr}(U(\mathfrak{g},e)) \to \text{End}(V^{r,t}_d) \quad (5.7)$$

coincides with $\psi_d$ in (5.5) (see [6, Lemma 3.1] for the explicit description on the isomorphism in (5.3)). Let $\mathcal{C}$ be the category of all $\mathfrak{g}$-modules on which $x - \chi(x)$ acts locally nilpotently for all $x \in \mathfrak{m}$. Following [6], let

$$Q_\chi = U/\mathcal{U}I_\chi. \quad (5.8)$$

Then $Q_\chi$ is a $(U, U(\mathfrak{g},e))$-bimodule. By Skryabin’s theorem in [24], the functor

$$Q_\chi \otimes \text{U}(\mathfrak{g},e) ? : U(\mathfrak{g},e)-\text{mod} \to \mathcal{C} \quad (5.9)$$

is an equivalence of categories. It follows from [5, §8.1] that the inverse of $Q_\chi \otimes \text{U}(\mathfrak{g},e) ?$ is $\text{Wh}(?) : \mathcal{C} \to U(\mathfrak{g},e)$-mod such that, for each object $M \in \mathcal{C}$,

$$\text{Wh}(M) = \{ v \in M \mid xv = \chi(x)v \text{ for all } x \in \mathfrak{m} \},$$

and the action of $U(\mathfrak{g},e)$ on $\text{Wh}(M)$ is defined by

$$u \cdot v = \eta(u)v, \text{ for } u \in U(\mathfrak{g},e), v \in \text{Wh}(M). \quad (5.10)$$

Suppose that $X$ is a finite dimensional $U$-module. It is well known that $M \otimes X \in \mathcal{C}$ for any $M \in \mathcal{C}$ and the functor $\otimes X : \mathcal{C} \to \mathcal{C}$ is exact. The endomorphisms $x_1, s_i, i \in \mathbb{Z} - 1$ in Definitions 5.2–5.3 have been defined in [6].

**Definition 5.2.** Define the endomorphisms $s_i, s_j, e_1, x_1$ and $\overline{x}_1$ of functors $\otimes V^{r,t}$ such that for any $U$-module $M$, $s_i(M), s_j(M), e_1(M), x_1(M), \overline{x}_1(M)$ are exactly the same as $s_i, s_j, e_1, x_1$ and $\overline{x}_1$ in Definition 3.3.
Recall that $X$ is a finite dimensional $U$-module. By [5] §8.2, there is an exact functor

\[ \otimes X : \mathbf{U}(g,e)\text{-mod} \to \mathbf{U}(g,e)\text{-mod}, \quad M \mapsto \text{Wh}(\langle Q \otimes \mathbf{U}(g,e) M \rangle \otimes X). \quad (5.11) \]

Given two finite dimensional $g$-modules $X, Y$ and a $\mathbf{U}(g,e)$-module $M$, there is a natural associativity isomorphism (see [5] (8.8)):

\[ a_{M,X,Y} : (M \otimes X) \otimes Y \cong M \otimes (X \otimes Y). \quad (5.12) \]

**Definition 5.3.** There are endomorphisms of functors $\otimes V^g \otimes W^t$, say $s_i$ and $\pi_i$, $\eta_1, x_1$ and $\pi_1$ such that for any $\mathbf{U}(g,e)$-module $M$, $s_i(M), \pi_i(M), ev_1(M), \pi_1(M)$ are defined through the Skryabin’s equivalence and (5.11)–(5.12) and Definition 5.2.

**Lemma 5.4.** [6] Suppose $M$ is a $\mathbf{U}(p)$-module $M$ and $X$ is a finite dimensional $U$-module. As $\mathbf{U}(g,e)$-modules, $M \otimes X \cong M \otimes X$ and the corresponding isomorphism $\mu_{M,X}$ sends $(\eta(u)1_\chi \otimes m) \otimes x \mapsto um \otimes x$ for all $u \in \mathbf{U}(p), m \in M$ and $x \in X$, where $1_\chi = 1 + U \chi_\chi \in U/UL_X$.

Using Lemma 5.4 and (5.12) repeatedly yields the following results.

**Corollary 5.5.** As $\mathbf{U}(g,e)$-modules, $C_d \otimes V^g \otimes W^t \cong C_d \otimes V^r \otimes W^t \cong C_d \otimes V^r \otimes C_d \otimes V^r \cong V^r$.

The corresponding isomorphism $C_d \otimes V^g \otimes W^t \cong V^r$ will be denoted by $\mu_{r,t}$. We are going to determine the action of $\mathcal{B}_{r,t}^{\text{aff}}$ on $C_d \otimes V^g \otimes W^t$. Via $\mu_{r,t}$, we lift the action of $\mathcal{B}_{r,t}^{\text{aff}}$ on $C_d \otimes V^g \otimes W^t$ to $V^r$. The actions of generators $x_1, \tau_1, s_i, \pi_i, e_1, i \in r - 1$, $j \in t - 1$ of $\mathcal{B}_{r,t}^{\text{aff}}$ on $C_d \otimes V^r \subseteq (Q \otimes \mathbf{U}(g,e) C_d) \otimes V^r$ are defined as $-\pi_0, -\pi_{i+1}, \pi_{i+1}, \pi_{i+1} = (-\pi_{i+1}, \pi_{i+1})$, respectively (cf. Definition 3.3). Note that the positions of tensor factors of $(Q \otimes \mathbf{U}(g,e) C_d) \otimes V^r$ are ordered as $0, r, r - 1, \ldots, 1, \tau, \pi, \pi, \tau, \pi$, and the tensor factor at the 0-position is $Q \otimes \mathbf{U}(g,e) C_d$. So the actions of $e_1, s_i, \pi_i$ are the same as those in Definition 3.3. However, the actions of $x_1$ and $\pi_1$ are different. They will be described in Lemma 5.8. The following result follows from Theorem 8.1 and Corollary 8.2 in [7]. See [8] Lemma 3.2 for $t = 0$. Recall that $i^L$ and $i^R$ in Definition 3.2 for any $i \in I(n, r + t)$.

**Lemma 5.6.** For all $i, j \in I(n, r + t)$, there exist elements $x_{ij} \in \mathbf{U}(p)$ such that

a) $[e_{i,j}, \eta(x_{ij})] + \sum_k \eta(x_{k,j}) - \sum_k \eta(x_{k,j}) \in \text{UI}_X$ for each $e_{i,j} \in m$, where $k \in I(n, r + t)$ such that $k^R = i^R$, and $k^L$ is obtained by using $j$ instead of an $i$ in $i^L$, and $h \in I(n, r + t)$ such that $h^L = i^L$, and each $h^R$ is obtained by using $i$ instead of an $j$ in $i^R$;

b) $x_{ij}$ acts on $C_d$ as $\delta_{i,j}$;

c) $\mu_{C_d, V^r}(v_j) = \sum_{i \in I(n, r + t)} (\eta(x_{ij}) \chi_\chi \otimes 1_d) \otimes v_i \in C_d \otimes V^r$.

**Definition 5.7.** Suppose $i \in I(n, r + t)$. 

a) If there is a $k \in \mathbb{N}$ such that row$(k) = row(i_1)$ and col$(k) = col(i_1) - 1$, then $k$ is unique. In this case, define $i_1^k = (i_r, \ldots, i_2, k)$. If there is no such a $k$, define $i_1^k = \emptyset$ and $v_i^0 = 0$.

b) If there is a $k \in \mathbb{N}$ such that row$(k) = row(i_\tau)$ and col$(k) = col(i_\tau) + 1$, then $k$ is unique. In this case, define $i_\tau^k = (k, i_\tau, \ldots, i_\tau)$. If there is no such a $k$, define $i_\tau^k = \emptyset$ and $v_i^\emptyset = 0$.

Assume $i \in I(n, r + t)$. Recall that $\mathcal{G}_r$ (resp., $\mathcal{G}_c$) acts on $i^L$ (resp., $i^R$) via place permutations. If we assume $t = 0$, then the following result has been given in \cite[Lemma 3.3]{Rui}. 

**Lemma 5.8.** Suppose $i \in I(n, r + t)$.

a) $v_i x_1 = -v_i^1 \otimes v_i^R - (d_{\text{col}(i_1)} + q_{\text{col}(i_1)} - q_1)v_i + \sum_h v_i^h \otimes v_i^R - \sum_k v_k$, where $h \in \mathbb{N}$ with $\text{col}(i_h) < \text{col}(i_1)$ and $k \in I(n, r + t)$ such that, if there is an $i_1$, which appears in $i^R$, then $k^R$ is obtained by using $j$ instead of an $i_1$ in $i^R$ such that $j \in \mathbb{N}$ and $\text{col}(i_1) > \text{col}(j)$ and moreover, $k^L = (i_r, \cdot, i_3, i_2, j)$.

b) $v_i x_1 = v_i^L \otimes v_i^R + (d_{\text{col}(i_\tau)} + n - q_1)v_i + \sum_h v_i^h \otimes v_i^R(\Omega) - \sum_k v_k$, where $h \in \mathbb{L}$ with $\text{col}(i_h) > \text{col}(i_\tau)$ and $k \in I(n, r + t)$ such that, if there is an $i_\tau$, which appears in $i^L$, then $k^L$ is obtained by using some $j$ instead of an $i_\tau$ in $i^L$ such that $j \in \mathbb{N}$ and $\text{col}(i_\tau) < \text{col}(j)$, and moreover, $k^R = (j, i_\tau, i_\tau, \cdot, j)$.

**Proof.** For the simplification of notation, write $\mu = \mu_{\mathcal{C}, V_{r, t}}$. By Lemma 5.6(c),

$$v_i x_1 = -\mu(\pi_{i, \emptyset}(\Omega) \sum_{s \in I(n, r + t)} (\eta(x_{s, 1})L_{1} \otimes 1_d) \otimes v_j) = -\sum_{j, k} \mu((e_{s}, s, \eta(x_{s, 1})L_{1} \otimes 1_d) \otimes v_k)$$

where $j, k \in I(n, r + t)$ with $j^R = k^R$, and $j_i = k_i$, $2 \leq i \leq r$. Recall that we always write $i = (i_r, i_{r-1}, \cdots, i_3, i_2, i_1, i_\tau, i_\tau, \cdots)$ for any $i \in I(n, r + t)$.

If $\text{col}(j_1) \leq \text{col}(k_1)$, then $e_{j_1, k_1} \in \mathcal{P}$. By Lemmas 5.4 and 5.6(b), and (5.1), $\mu((e_{j_1, k_1} \eta(x_{s, 1})L_{1} \otimes 1_d) \otimes v_k) = 0$ unless $i = j = k$. In the later case,

$$\mu((e_{s}, s, \eta(x_{s, 1})L_{1} \otimes 1_d) \otimes v_k) = (d_{\text{col}(i_1)} + q_{\text{col}(i_1)} + \cdots + q_k - q_1)v_i.$$  \hfill (5.13)

If $\text{col}(j_1) > \text{col}(k_1)$. By Lemma 5.6(a),

$$e_{j_1, k_1} \eta(x_{s, 1})L_{1} = \eta(x_{s, 1})e_{j_1, k_1}L_{1} - \sum_h \eta(x_{h, 1})L_{1} + \sum_s \eta(x_{s, 1})L_{1}$$ \hfill (5.14)

where each $h$ is obtained from $j$ by using $k_1$ instead of some $j_1$ in $j^L$, and each $s$ is obtained from $j$ by using $j_1$ instead of some $k_1$ in $j^R$. Note that $\chi(e_{j_1, k_1}) = 0$ unless $k_1$ is equal to the entry in the $1$th position of $j_1^R$. In the later case, $\chi(e_{j_1, k_1}) = 1$. So, (a) follows from (5.13)-(5.14) and Lemma 5.6(b), immediately. Finally, (b) can be verified similarly. \hfill \square

**Proposition 5.9.** There is an algebra homomorphism $\Phi : \mathcal{B}_{r, t} \to \operatorname{End}U(\mathcal{C}, V)^{\text{op}}$ for some affine walled Brauer algebra $\mathcal{B}_{r, t}$, sending generators $e_1, x_1, \Phi_1, s_i, \Phi_j$ to $e_1(\mathcal{C}), x_1(\mathcal{C}), \Phi_1(\mathcal{C}), s_i(\mathcal{C}), \Phi_j(\mathcal{C})$ in Definition 5.3 for all $i \in r - 1$ and $j \in t - 1$. 


Proof. It follows from Skryabin equivalence and Proposition 3.5 that all relations in Definition 2.1 hold except

a) \( e_1(x_1 + \overline{x}_1) = (x_1 + \overline{x}_1)e_1 = 0, \)

b) \( e_1x_1^ie_1 = \omega_0e_1, e_1\overline{x}_1e_1 = \overline{\omega}_0e_1, \) for any \( a \in \mathbb{N}, \) where \( \omega_0, \overline{\omega}_0 \) are some scalars in \( \mathbb{C}. \)

Since \( Q_\chi \otimes u(\mathfrak{g}, e) \mathbb{C}_d \) is cyclic module, (a) follows from Skryabin equivalence and arguments similar to those for \( e_1(x_1 + \overline{x}_1) = (x_1 + \overline{x}_1)e_1 = 0 \) in the proof of Proposition 3.5. Finally, (b) follows from the formulae on both \( v_i x_1 \) and \( v_i \overline{x}_1 \) in Lemma 5.8 together with the fact that \( e_1 \) acts on \( V_{\beta,t} \) via \( -\pi_{1,\beta}(\Omega). \) \( \square \)

The following result can be proved by arguments similar to those in the proof of [6, Lemma 3.4]. The only difference is that we need to use Lemma 5.8 instead of [6, Lemma 3.3].

**Lemma 5.10.** The minimal polynomial of the endomorphism of \( V_{\beta,t} \) defined by \( x_1 \) (resp., \( x_1^t \)) is \( f(x) \) (resp., \( g(x) \)) in Definition 3.11.

**Theorem 5.11.** Let \( B_{k,r,t} = B^\text{aff}_{r,t} / J \) where \( J \) is the two-sided ideal generated by \( f(x_1) \) and \( g(\overline{x}_1). \) Then \( B_{k,r,t} \) is admissible. The algebra homomorphism \( \Phi \) in Proposition 5.9 factors through \( B_{k,r,t}. \) The corresponding algebra homomorphism, which will be denoted by \( \Phi \) again, is always surjective. It is injective if \( r + t \leq q_k. \)

Proof. By Lemma 5.8 and the description of the action of \( e_1, \) it is easy to see that \( \{e_1, e_1\overline{x}_1, \cdots, e_1\overline{x}_1^{k-1}\} \) is \( \mathbb{C} \)-linear independent. So, the first assertion follows from arguments similar to those in the proof of Lemma 3.14. Recall that two graded algebra homomorphisms \( \psi_d \) in (5.5) and \( \varphi \) in (4.15). By Proposition 4.2 \( \text{gr}(\Phi)(x) = \varphi(x) \) if \( x \in \{e_1, s_i, \overline{s}_j\}. \) Using Lemma 5.8 yields \( \text{gr}(\Phi)(x) = \varphi(x) \) for \( x \in \{x_1, \overline{x}_1\}. \) So, \( \text{gr}(\Phi) = \varphi. \) On the other hand, it follows from [6, Lemma 3.1] that \( \psi_d = \text{gr}(\Psi_d), \) where \( \Psi_d \) is given in (5.6). Now, the result follows from [6, Lemma 3.6] and Theorem 4.8. \( \square \)

6. Epimorphisms in (1.1)

Throughout this section, we go on assume that \( (q_1 \geq q_2 \geq \cdots \geq q_k) \) is a partition of \( n. \)

We also keep Assumption 3.6. Following [6, Section 4], let

\[
\Lambda_d = \{\mu_1 \varepsilon_1 + \cdots + \mu_n \varepsilon_n \in \Lambda^p \mid \mu_i - d_j \in \mathbb{Z}, \text{ for any } i \in \mathfrak{p}_j, j \in \mathfrak{k}\}, \tag{6.1}
\]

where \( \Lambda^p \) is the set of \( \mathfrak{p} \)-dominant weights with respect to the parabolic subalgebra \( \mathfrak{p} \) of \( \mathfrak{g} \) and \( \mathfrak{p}_j \)'s are defined in Definition 3.8. Following [6], an element \( \mu \in \Lambda_d \) is called standard if the entries in each row of \( s \) increase weakly from left to right, where \( s \) is obtained from \( t \) in (4.1) by using \( \mu_i - i + 1 \) instead of \( i \) for all \( i \in n. \) Let

\[
\overline{\Lambda}_d = \{\mu \in \Lambda_d \mid \mu \text{ is standard}\}. \tag{6.2}
\]
Lemma 6.2. If \( M \) and \( \nu \) are objects in \( \mathcal{O}_d \) such that each \( \mu \in \Lambda_d^{r,t} \) satisfies \( \sum_{i \in \mathbb{Z}_+} \nu_i = r - t \) if \( \mu - \delta_c \sum_{i \in \mathbb{Z}_+} \nu_i \epsilon_i \). Let \( \Lambda_d^{r,t} = \{ \mu \in \Lambda_d^{r,t} \mid \sum_{\nu_i > 0} \nu_i \leq r \} \). \( \overline{\Lambda}_d^{r,t} = \Lambda_d^{r,t} \cap \overline{\Lambda}_d^{r,t} \).

Let \( \mathcal{O}_d \) be the Serre subcategory of \( \mathcal{C}^s \) generated by the irreducible modules \( \{ \mathcal{L}(\mu) \mid \mu \in \Lambda_d \} \). Let \( \mathcal{O}_d^{r,t} \) be the Serre subcategory of \( \mathcal{O}_d \) generated by the irreducible modules \( \{ \mathcal{L}(\mu) \mid \mu \in \Lambda_d^{r,t} \} \). Recall that \( M_{c,t}^{r,t} = M_c \otimes V_{\otimes r} \otimes W_{\otimes t} \). Then \( M_{c,t}^{r,t} \) is an object in \( \mathcal{O}_d \).

Lemma 6.2. If \( \mu \in \Lambda_c \), then \( [M_{c,t}^{r,t} : \mathcal{L}(\mu)] \neq 0 \) only if \( \mu \in \Lambda_d^{r,t} \). In particular, \( M_{c,t}^{r,t} \) is an object in \( \mathcal{O}_d^{r,t} \).

Proof. It follows from [14] Theorem 3.6] that \( M_{c,t}^{r,t} \) has a parabolic Verma flag such that each section is of form \( \mathcal{M}^r(\mu) \) for some \( \mu \in \Lambda_d^{r,t} \). On the other hand, if \( [M_{c,t}^{r,t} : \mathcal{L}(\mu)] \neq 0 \), then \( [\mathcal{M}^r(\nu) : \mathcal{L}(\mu)] \neq 0 \) for some \( \nu \in \Lambda_d^{r,t} \). This implies that \( \nu \) and \( \mu \) are in the same block and hence \( \nu = w \cdot \mu \) for some \( w \in \mathcal{S}_n \) and “.” is the usual dot action. So, \( \mu \in \Lambda_d^{r,t} \).

Lemma 6.3. Let \( \mathcal{P}(\mu) \) be the projective cover of simple \( \mathfrak{g} \)-module \( \mathcal{L}(\mu) \in \mathcal{C}^s \). Then \( \mathcal{P}(\mu) \) is a direct summand of \( M_{c,t}^{r,t} \) if and only if \( \mu \in \overline{\Lambda}_d^{r,t} \).

Proof. It is well known that \( \mathcal{M}(\delta_c) \) projective and injective and hence tilting, where \( \delta_c \) is in Assumption 3.6. So is \( M_{c,t}^{r,t} \) and hence each indecomposable direct summand of \( M_{c,t}^{r,t} \) is tilting. If \( \mathcal{P}(\mu) \) is a direct summand of \( M_{c,t}^{r,t} \), then \( \mathcal{P}(\mu) \) is tilting and hence self-dual. By [6] Theorem 4.6, \( \mu \) is standard. Since \( \mathcal{P}(\mu) \) has a parabolic Verma flag with top section \( \mathcal{M}(\mu) \), \( \mathcal{M}(\mu) \) appears in a parabolic Verma flag of \( M_{c,t}^{r,t} \), forcing \( \mu \in \overline{\Lambda}_d^{r,t} \). Conversely, for every \( \mu \in \overline{\Lambda}_d^{r,t} \), by [5] Theorem 4.5, there is a \( \nu \) either in \( \overline{\Lambda}_d^{r-1,t} \) or \( \overline{\Lambda}_d^{r,t-1} \) such that \( f_i \nu = \mu \) or \( e_i \nu = \mu \) for some \( i \in \mathbb{Z} \) where \( \overline{\Lambda}_d^{r,t} \) are known as Kashiwara operators. So, the simple module \( \mathcal{L}(\mu) \) is a quotient of either \( \mathcal{L}(\nu) \otimes V \) or \( \mathcal{L}(\nu) \otimes W \). This implies that \( \mathcal{P}(\mu) \) is a direct summand of \( \mathcal{P}(\nu) \otimes V \) or \( \mathcal{P}(\nu) \otimes W \). By induction assumption, \( \mathcal{P}(\nu) \) is an indecomposable direct summand of either \( M_{c,t}^{r-1,t} \) or \( M_{c,t}^{r,t-1} \). Therefore, \( \mathcal{P}(\mu) \) is an indecomposable direct summand of \( M_{c,t}^{r,t} \).

Lemma 6.4. [7] Lemma 8.18] Let \( \nabla : \mathcal{O}_d \to \mathcal{R}_d(e) \) be the Whittaker functor defined in [7] Lemma 8.20]. For any \( M \in \mathcal{O}_d \) and any finite dimensional \( \mathfrak{g} \)-module \( X \), there is a natural isomorphism

\[
\nu_{M,X} : \nabla(M \otimes X) \to \nabla(M) \oplus V
\]  
(6.3)
Lemma 6.8. (cf. [6] from (b), Proposition 6.6 and Corollary 5.5. (a) was proved in [6] and (b) follows from Lemma 6.4 and (a). Finally, (c) follows from the naturality of (1) and (3) are (8.44)–(8.45) in the proof of [7, Lemma 8.19]. One can verify (2)

\[ \text{Lemma 6.7.} \quad \text{Recall that \( s_i(M), \tilde{s}_j(M), e_1(M), x_1(M) \) and \( \tilde{\tau}_i(M) \) for any \( g \)-module \( M \) and any \( U(g,e) \)-module \( M \) in Definitions 5.2, 5.3. The following results are motivated by [6].}

**Proposition 6.6.** Suppose \( M \in \mathcal{O}_d \). We have:

1. \( x_1(V(M)) \circ \nu_{M,V} = \nu_{M,V} \circ V(x_1(M)) \),
2. \( \tau_1(V(M)) \circ \nu_{M,W} = \nu_{M,W} \circ V(\tau_1(M)) \),
3. \( (\nu_{M,V} \circ 1_V) \circ \nu_{M\otimes V,V} \circ V(s_1(M)) = s_1(V(M)) \circ (\nu_{M,V} \circ 1_V) \circ \nu_{M\otimes V,V} \),
4. \( (\nu_{M,W} \circ 1_W) \circ \nu_{M\otimes W,W} \circ V(\tau_1(M)) = \tau_1(V(M)) \circ (\nu_{M,W} \circ 1_W) \circ \nu_{M\otimes W,W} \),
5. \( (\nu_{M,V} \circ 1_W) \circ \nu_{M\otimes V,W} \circ V(e_1(M)) = e_1(V(M)) \circ (\nu_{M,V} \circ 1_W) \circ \nu_{M\otimes V,W} \).

Similarly, we have the equalities for \( s_i(M) \) and \( \tilde{s}_j(M) \). In particular, when \( M = M^p(\delta_c) \), \( V(M^p,\xi) \) is an \((U(g,e), \mathcal{B}^{aff})\)-bimodule, where \( M^p,\xi \) is defined in (3.6) and \( \omega_{\xi} \)'s satisfy (3.11).

**Proof.** (1) and (3) are (8.44)–(8.45) in the proof of [7, Lemma 8.19]. One can verify (2) and (4) similarly. Finally, (5) follows from the naturality of \( \nu_{M,V\otimes W} \) and (6.4) and the definitions of \( e_1(M) \) and \( e_1(V(M)) \). The last assertion follows from relations in (1)–(5) and Proposition 3.15. \( \square \)

**Lemma 6.7.** Recall that \( C_d = C1_d \) is the 1-dimensional \( p \)-module such that \( e_{1,d} = \delta_{i,j}d_{col(i)}1_d, \forall e_{ij} \in p \).

1. The \( U(g,e) \)-module \( C_d \) is projective in \( R_d(e) \) and moreover, \( C_d \cong V(M_c) \), where \( M_c \) is the parabolic Verma module with respect to \( \delta_c \) in Assumption 3.6.
2. As \( U(g,e) \)-modules, \( V(M_c^{r,t}) \cong C_d \circledast V^{\otimes r} \circledast W^{\otimes t} \).
3. As \( (U(g,e), \mathcal{B}^{aff}) \)-bimodules, \( V(M_c^{r,t}) \cong V_d^{r,t} \).

**Proof.** (a) was proved in [6] and (b) follows from Lemma 5.4 and (a). Finally, (c) follows from (b), Proposition 6.6 and Corollary 5.5. \( \square \)

**Lemma 6.8.** (cf. [6] Lemma 5.7.) For any \( \mu \in \lambda_d \), let \( Q(\mu) = V(P(\mu)) \). If \( \mu \in \lambda_d^{\otimes} \), then \( Q(\mu) \) is the projective cover of the simple object \( D(\mu) \) in \( R_d(e) \).
Proof. If \( t = 0 \), this is \([6\) Lemma 5.7]. Suppose \( t > 0 \) and \( \mu \in \Lambda_d^{r,t} \). By \([5\) Theorem 4.5], \( P(\mu) \) is a direct summand of \( P(\nu) \otimes W \) for some \( \nu \in \Lambda_d^{r,t-1} \). Since \( \mathcal{V}(?) \) is exact, \( Q(\mu) \) is a direct summand of \( \mathcal{V}(P(\nu) \otimes W) \cong Q(\nu) \otimes W \). It follows from the proof of \([6\) Lemma 5.7] that the functor \( \otimes W \) sends projective objects in \( R_d(e) \) to projective objects in \( R_d(e) \). By inductive assumption, \( Q(\nu) \) is projective and so is \( Q(\mu) \). Write

\[
P(\nu) \otimes W = \oplus \mu P(\mu)^{m_\mu}
\]

Then \( \dim \text{Hom}_{U(\mu, e)}(Q(\nu) \otimes W, D(\mu)) = m_\mu \) (see the display at the end of the proof of \([6\) Lemma 5.7] where we switch the role between \( V \) and \( W \)). So, \( Q(\mu) \) is indecomposable. By Lemma \( 6.5 \) and the exactness of \( \mathcal{V}(?) \), \( Q(\mu) \) has to be the projective cover of \( D(\mu) \).

For convenience, let \( i_{r,t} \) be the isomorphism in Lemma \( 6.7(b) \), which is obtained by composing isomorphisms in Lemma \( 6.7(a) \) and Lemma \( 6.4 \). Recall that \( \mu_{r,t} \) is the isomorphism in Corollary \( 5.5 \). Let \( j_{r,t} = \mu_{r,t} \circ i_{r,t} \). Then \( j_{r,t} \) is the isomorphism in Lemma \( 6.7(c) \).

Lemma 6.9. For any object \( M \in \mathcal{O}_{d}^{r-t} \), let

\[
\gamma_{M}^{r,t} : \text{Hom}_{\mathcal{O}}(M_{c}^{r,t}, M) \to \text{Hom}_{U(\mu, e)}(V_{d}^{r,t}, \mathcal{V}(M))
\]

be the map sending \( f \) to \( \mathcal{V}(f) \circ j_{r,t}^{-1} \). Then \( \gamma_{M}^{r,t} \) is a \( \mathcal{B}_{r,t}^{\text{aff}} \)-module isomorphism.

Proof. By Proposition \( 6.6 \) and Lemma \( 6.7(c) \), \( \gamma_{M}^{r,t} \) is a \( \mathcal{B}_{r,t}^{\text{aff}} \)-homomorphism. So, it suffices to prove that \( \gamma_{M}^{r,t} \) is a linear isomorphism. Note that \( M_{c}^{r,t} \) is projective, and the exact functor \( \mathcal{V} \) sends a projective module to a projective module (see Lemma \( 6.8 \)). So, both \( \text{Hom}_{\mathcal{O}}(M_{c}^{r,t}, ?) \) and \( \text{Hom}_{U(\mu, e)}(V_{d}^{r,t}, ?) \) are exact functors. It suffices to check that \( \gamma_{M}^{r,t} \) a linear isomorphism for any \( \mu \in \Lambda_d^{r-t} \). By Lemma \( 6.7(c) \), we need to show

\[
\text{Hom}_{\mathcal{O}}(M_{c}^{r,t}, L(\mu)) \to \text{Hom}_{U(\mu, e)}(\mathcal{V}(M_{c}^{r,t}), \mathcal{V}(L(\mu))), f \mapsto \mathcal{V}(f)
\]

is a linear isomorphism. By Lemma \( 6.3 \), each indecomposable summands of \( M_{c}^{r,t} \) is of form \( P(\nu) \) for some \( \nu \in \Lambda_d^{r-t} \). Therefore it is enough to show

\[
\text{Hom}_{\mathcal{O}}(P(\nu), L(\mu)) \to \text{Hom}_{U(\mu, e)}(\mathcal{V}(P(\nu)), \mathcal{V}(L(\mu))), f \mapsto \mathcal{V}(f)
\]

(6.5)

is a linear isomorphism for each \( \mu \in \Lambda_d^{r-t}, \nu \in \Lambda_d^{r-t} \). By Lemmas \( 6.5 \) and \( 6.8 \) RHS (resp., LHS) of (6.5) is of dimension \( \delta_{\mu, \nu} \). So, it is enough to prove that the linear map in (6.5) is a linear isomorphism if \( \mu = \nu \). Since \( \mathcal{V} \) is exact, by Lemma \( 6.5 \) \( \mathcal{V}(f) \neq 0 \) for any \( 0 \neq f \in \text{Hom}_{\mathcal{O}}(P(\nu), L(\nu)) \). This implies that the map from (6.5) is a linear isomorphism.

Corollary 6.10. There is a \( \mathcal{B}_{r,t}^{\text{aff}} \)-isomorphism \( k_{r,t} : \text{End}_{\mathcal{O}}(M_{c}^{r,t}) \to \text{End}_{U(\mu, e)}(V_{d}^{r,t}) \) sending \( f \in \text{End}_{\mathcal{O}}(M_{c}^{r,t}) \) to \( j_{r,t} \circ \mathcal{V}(f) \circ j_{r,t}^{-1} \).

Theorem 6.11. Let \( \varphi : \mathcal{B}_{k,r,t} \to \text{End}_{\mathcal{O}}(M_{c}^{r,t})^{\mathcal{O}} \) be the algebra homomorphism in Proposition \( 6.13 \). Then \( \Phi = k_{r,t} \circ \varphi \), where \( \Phi \) (resp., \( k_{r,t} \)) is given in Theorem \( 5.17 \) (resp., Corollary \( 6.10 \)). So, \( \varphi \) is always surjective, and it is injective if \( r + t \leq q_k \).
Proof. By Corollary 6.10, $\Phi = k_{r,t} \circ \varphi$. The second result follows from Theorem 5.14 and the first assertion. \hfill \Box

Remark 6.12. Let $\mathcal{B}_{k,r,t}$ be the cyclotomic walled Brauer algebras $\mathcal{B}_{r,t}^{\text{aff}}/I$ where $I$ is the two-sided ideal generated by $f(x_1) = \prod_{i=1}^k (x_1 - u_i)$ and $g(x_1) = \prod_{i=1}^k (x_1 - \overline{u}_i)$ satisfying $e_1 f(x_1) = (-1)^k e_1 g(x_1)$, and $\omega_0$’s are determined by (3.11). By Brundan-Kleshchev’s arguments in [6], one can choose a partition $(q_1, q_2, \ldots, q_k)$ of $n$ such that $\omega_0 = n$ and $u_i$’s are determined by Definition 3.11. By Lemma 3.12, there are some $\overline{v}_i, 1 \leq i \leq k$ such that $g_1(\overline{x}_1) = \prod_{i=1}^k (\overline{x}_1 - v_i)$ acts trivially on $M_c \otimes V \otimes W$. By arguments in the proof of Lemma 3.14, we have $g(\overline{x}_1) = g_1(\overline{x}_1)$. So, it is enough for us to assume that $(q_1, q_2, \ldots, q_k)$ is a partition of $n$ when we study the representations of $\mathcal{B}_{k,r,t}$ whose parameters are arisen from mixed Schur-Weyl duality.

7. DECOMPOSITION NUMBERS OF $\mathcal{B}_{k,r,t}$ ARISING FROM MIXED SCHUR-WEYL DUALITY

In this section, we work over the ground field $\mathbb{C}$. The aim of this section is to classify highest weight vectors of $M_{c,r,t}$ under the assumption $r + t \leq \min\{q_1, q_2, \ldots, q_k\}$, where $q = (q_1, q_2, \ldots, q_k)$ is given in Assumption 3.6. This in turn gives an efficient way to compute decomposition numbers of $\mathcal{B}_{k,r,t}$ arising from mixed Schur-Weyl duality. Since we use Theorem 3.17, we do not assume that $(q_1, q_2, \ldots, q_k)$ is a partition. First, we consider the case $t = 0$.

Recall that the degenerate affine Hecke algebra $\mathcal{H}_{r,t}^{\text{aff}}$ generated by $x_1$ and $s_i, i \in \mathbb{Z}$ in section 2 and $x_{i+1} = s_i x_i s_i - s_i, i \in \mathbb{Z}$. The current $x_i$’s are the usual $-x_i$’s in [6] since we use $-\pi_{1,0}(\Omega)|_{M_{r,t}}$ instead of $\pi_{1,0}(\Omega)|_{M_{r,t}}$ in [6]. Thus, our current eigenvalues of $x_1$ are the same as those in [6] by multiplying $-1$. The cyclotomic (or level $k$) degenerate Hecke algebra $\mathcal{H}_{k,r} := \mathcal{H}_{r,t}^{\text{aff}}/I$, where $I$ is the two-sided ideal generated by $f(x_1) = \prod_{i=1}^k (x_1 - u_i)$ in Definition 3.11.

For each composition $\lambda = (\lambda_1, \lambda_2, \ldots)$, let $|\lambda| = \sum \lambda_i$. A $k$-partition (resp., composition) $\lambda$ of $r$ is of form $(\lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(k)})$ where each $\lambda^{(i)}$ is a partition (resp., composition) such that $|\lambda| = \sum_{i=1}^k |\lambda^{(i)}| = r$. Let $\Lambda_k^+(r)$ be the set of all $k$-partitions of $r$. For each $\lambda \in \Lambda_k^+(r)$, the Young diagram $Y(\lambda)$ is a collection of boxes arranged in left-justified rows with $\lambda_i$ boxes in the $i$th row of $Y(\lambda)$. A $\lambda$-tableau $\bullet$ is obtained by inserting elements $i, 1 \leq i \leq r$ into $Y(\lambda)$ without repetition. A $\lambda$-tableau $\bullet$ is said to be standard if the entries in $\bullet$ increase both from left to right in each row and from top to bottom in each column. Let $\mathcal{T}^{\text{std}}(\lambda)$ be the set of all standard $\lambda$-tableaux. Let $t^\lambda \in \mathcal{T}^{\text{std}}(\lambda)$ be obtained from $Y(\lambda)$ by adding $1, 2, \ldots, r$ from left to right along the rows of $[\lambda]$. Let $t^\lambda \in \mathcal{T}^{\text{std}}(\lambda)$ be obtained from $Y(\lambda)$ by adding $1, 2, \ldots, r$ from top to bottom along the columns of $Y(\lambda)$. For example, if $\lambda = (3, 2)$, then

$$t^\lambda = \begin{array}{ccc} 1 & 2 & 3 \\ 4 & 5 \end{array} \quad \text{and} \quad t_\lambda = \begin{array}{c} 1 \ 3 \ 5 \\ 2 \ 4 \end{array}. \quad (7.1)$$
If $\lambda \in \Lambda_k^+(r)$, then the corresponding Young diagram $Y(\lambda)$ is $(Y(\lambda^{(1)}), Y(\lambda^{(2)}), \ldots, Y(\lambda^{(k)}))$. In this case, a $\lambda$-tableau $s = (s_1, s_2, \ldots, s_k)$ is obtained by inserting elements $i \in \mathbb{Z}$ into $Y(\lambda)$ without repetition. A $\lambda$-tableau $s$ is said to be standard if the entries in each $s_i$, $i \in \mathbb{Z}$, increase both from left to right in each row and from top to bottom in each column. Let $\mathcal{F}^{std}(\lambda)$ be the set of all standard $\lambda$-tableaux. Let $t^\lambda \in \mathcal{F}^{std}(\lambda)$ be obtained from $Y(\lambda)$ by adding $1, 2, \ldots, r$ from left to right along the rows of $Y(\lambda^{(1)})$ and then $Y(\lambda^{(2)})$ and so on. Let $t_\lambda \in \mathcal{F}^{std}(\lambda)$ be obtained from $\lambda$ by adding $1, 2, \ldots, r$ from top to bottom along the columns of $\lambda^{(k)}$ and then $\lambda^{(k-1)}$, and so on. For example, if $\lambda = ((3, 2), (3, 1)) \in \Lambda_k^+(9)$, then

$$t^\lambda = \begin{pmatrix} 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & 2 & 3 & 6 & 7 & 8 \end{pmatrix} \quad \text{and} \quad t_\lambda = \begin{pmatrix} 5 & 7 & 9 \\ 6 & 8 & 2 \\ 1 & 3 & 4 \end{pmatrix}. \quad (7.2)$$

Recall that $\mathcal{S}_r$ acts on the right of the set $\{1, 2, \ldots, r\}$ (i.e., the right action). Then $\mathcal{S}_r$ acts on the right of a $\lambda$-tableau $s$ by permuting its entries. For example, if $\lambda = ((3, 2), (3, 1)) \in \Lambda_k^+(9)$, and $w = s_1 s_2$, then $t^\lambda w = \begin{pmatrix} 3 & 1 & 2 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$. Write $d(s) = w$ for $w \in \mathcal{S}_r$ if $t^\lambda w = s$. Then $d(s)$ is uniquely determined by $s$. Let $w_\lambda = d(t_\lambda)$. Following [10], we define $[\lambda] = [a_0, a_1, \ldots, a_k]$ for any $\lambda \in \Lambda_k^+(r)$ such that $a_0 = 0$ and $a_i = \sum_{j=1}^i \lambda(j)$. Denote $[\lambda] \leq [\mu]$ if $a_i \leq b_i$ for $1 \leq i \leq k$, provided that $[\mu] = [b_0, b_1, \ldots, b_k]$. Let $w_{[\lambda]} \in \mathcal{S}_r$ be defined by

$$(a_{i-1} + l)w_{[\lambda]} = r - a_i + l, \quad \text{for all } i \text{ with } a_{i-1} < a_i, \quad 1 \leq l \leq a_i - a_{i-1}. \quad (7.4)$$

For example,

$$w_{[\lambda]} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 6 & 7 & 8 & 9 & 2 & 3 & 4 & 5 & 1 \end{pmatrix} \quad \text{if } [\lambda] = [0, 4, 8, 9].$$

Let $t^i$ (resp., $t_i$) be the $i$th subtableau of $t^\lambda$ (resp., $t^\lambda w_{[\lambda]}^{-1}$), and define $w(i)$ by $t^i w(i) = t_i$. Likewise, if we define $\tilde{t}^i$ (resp., $\tilde{t}_i$) the $i$th subtableau of $t^\lambda w_{[\lambda]}$ (resp., $t_\lambda$), and $\tilde{w}(i)$ with $\tilde{t}^i \tilde{w}(i) = \tilde{t}_i$, then

$$w_\lambda = w(1) w(2) \cdots w(k) w_{[\lambda]} = w_{[\lambda]} \tilde{w}(k) \tilde{w}(k-1) \cdots \tilde{w}(1), \quad w_{[\lambda]}^{-1} w(i) w_{[\lambda]} = \tilde{w}_{k-i+1}. \quad (7.5)$$

The row stabilizer $\mathcal{S}_\lambda$ of $t^\lambda$ for $\lambda \in \Lambda_k^+(r)$ is known as the Young subgroup of $\mathcal{S}_r$ with respect to $\lambda$. It is the same as the Young subgroup $\mathcal{S}_{\lambda}$ with respect to the composition $\lambda$, which is obtained from $\lambda$ by concatenation. For example, if $\lambda = ((3, 2), (3, 1))$ then $\lambda = (3, 2, 3, 1)$. The following definition follows from [2].

**Definition 7.1.** Suppose $\lambda \in \Lambda_k^+(r)$ and $u_1, u_2, \ldots, u_k \in \mathbb{C}$. Let $x_\lambda = \pi_{[\lambda]} \pi_{\lambda} x_{\lambda}$, $y_\lambda = \pi_{[\lambda]} \pi_{\lambda} y_{\lambda}$, where

a) $\pi_{[\lambda]} = \prod_{i=1}^k \pi_{a_i(u_{i+1})}$, and $\tilde{\pi}_{[\lambda]} = \prod_{i=1}^k \tilde{\pi}_{a_i(u_{k-i})}$, and $\pi_{a}(u) = \prod_{i=1}^a (x_i - u)$ for $u \in \mathbb{C}$ and $a \in \mathbb{Z}^+$ and $\pi_{0}(u) = 1$,

b) $x_{\lambda} = \sum_{w \in \mathcal{S}_{\lambda}} w$ and $y_{\lambda} = \sum_{w \in \mathcal{S}_{\lambda}} (-1)^{\ell(w)} w$, where $\ell(w)$ is the length of $w$. 

For any $\lambda \in \Lambda_k^+(r)$, the conjugate $\lambda'$ of $\lambda$ is of form $(\mu^{(k)}, \mu^{(k-1)}, \ldots, \mu^{(1)})$ where $\mu^{(i)}$ is the conjugate of $\lambda^{(i)}$. For $s, t \in \mathcal{F}^s(\lambda)$, let $x_{st} = d(s)^{-1}x_{\lambda}d(t)$ and $y_{st} = d(s)^{-1}y_{\lambda}d(t)$.

**Theorem 7.2.** $\mathcal{H}_{k,r}$ is a cellular algebra in the sense of [12] with both $S_1$ and $S_2$ being its cellular bases, where $S_1 = \{x_{st} \mid s, t \in \mathcal{F}^{std}(\lambda), \lambda \in \Lambda_k^+(r)\}$ and $S_2 = \{y_{st} \mid s, t \in \mathcal{F}^{std}(\lambda), \lambda \in \Lambda_k^+(r)\}$. The required anti-involution is the $\mathbb{C}$-linear anti-involution fixing generators $x_1$ and $s_i, i \in r-1$.

For each $\lambda \in \Lambda_k^+(r)$, following [12], define $C(\lambda)$ to be the cell module with respect to the cellular basis $S_2$ in Theorem 7.2. The classical Specht module $S^\lambda = x_{\lambda}w_{\lambda\lambda'}\mathcal{H}_{k,r}$. It is well-known that

$$C(\lambda') \cong S^\lambda, \forall \lambda \in \Lambda_k^+(r).$$

(7.6)

**Definition 7.3.** For any $\lambda \in \Lambda_k^+(r)$, define $\lambda = \sum_{1 \leq i \leq k} \sum_{p_{i-1} < j \leq p_i} \lambda^{(i)}_{j-p_{i-1}} \varepsilon_j$ and $\lambda = \delta_c + \hat{\lambda}$, where $p_i$’s and $\delta_c$ are in Assumption 3.6.

Recall that $V$ is the natural $\mathfrak{g}$-module with a basis $\{v_1, v_2, \ldots, v_n\}$. Then its linear dual $W$ has a basis $\{v_1^*, v_2^*, \ldots, v_n^*\}$ such that $(v_i, v_j^*) = \delta_{i,j}$. Recall that any element in $I(n, r)$ is of form $\mathbf{i} = (i_r, i_{r-1}, \ldots, i_1)$ and $v_1 = v_{i_r} \otimes v_{i_{r-1}} \otimes \cdots \otimes v_{i_1}$.

**Definition 7.4.** Suppose $\lambda \in \Lambda_k^+(r)$. Define

1) $\mathbf{i}_\lambda = (i_{\lambda^{(k)}}, i_{\lambda^{(k-1)}}, \ldots, i_{\lambda^{(1)}}) \in I(n, r)$, where $i_{\lambda^{(j)}} = ((p_j)^{\lambda^{(j)}}, \ldots, (p_j-1+1)^{\lambda^{(j)}})$, and
2) $v_{\mathbf{i}} = m \otimes v_{\lambda}w_{\lambda\lambda'}d(t)$, for any $t \in \mathcal{F}^{std}(\lambda')$, where $m$ is the highest weight vector of $M_c$.

Recall that $\mathfrak{p}$ is the parabolic subalgebra of $\mathfrak{g}$ whose Levi subalgebra $\mathfrak{l} = \mathfrak{gl}_{q_1} \oplus \mathfrak{gl}_{q_2} \oplus \cdots \oplus \mathfrak{gl}_{q_n}$. Let $M_c \in \mathcal{O}_\mathfrak{p}$ be the parabolic Verma module with respect to the highest weight $\delta_c$ in Assumption 3.6. The following result, which will be used to classify highest weight vectors of $M_c^r$, may be well-known for experts. We leave the proof to the reader.

**Lemma 7.5.** Suppose that $N$ is a finite dimensional $\mathfrak{g}$-module. For any $\mathfrak{g}$-highest weight vector $v_\mu \in M_c \otimes N$, there is a unique $\mathfrak{l}$-highest weight vector $w \in N$ with weight $\mu - \delta_c$ such that $v_\mu - m \otimes w \in M_c^\perp \otimes N$, where $m$ is the highest weight vector of $M_c$ and $M_c^\perp$ is the direct sum of weight spaces $(M_c)_\nu$ such that $\nu < \delta_c$.

We need the following well-known results (see, e.g. [22]).

**Lemma 7.6.** Suppose $\lambda$ and $\mu$ are two compositions of $r$ and $\mu'$ is the conjugate of $\mu$. Then $x_{\lambda\mu}C_{\mathcal{S}_\tau}y_{\mu'} = 0$ unless $\lambda \leq \mu$.

**Lemma 7.7.** Suppose $n \geq r$. There is a bijection between the set of dominant weights of $V^\otimes r$ and $\Lambda^+(r, n)$, the set of partitions of $r$ with at most $n$ parts. Further, the $\mathbb{C}$-space of $\mathfrak{g}$-highest weight vectors with highest weight $\lambda$ has a basis $\{v_{\lambda}w_{\lambda\lambda'}d(t) \mid t \in \mathcal{F}^{std}(\lambda')\}$. 

Theorem 7.8. Suppose \( r \leq \min\{q_1, q_2, \cdots, q_k\} \).

a) There is a bijection between \( \Lambda_c^\epsilon (r) \) and the set of \( \mathfrak{p} \)-dominant weights \( \mu \) such that \( M_c^{r,0} \) contains at least a highest weight vector with highest weight \( \mu \).

b) Let \( V_\lambda \) be the \( \mathbb{C} \)-space which consists of all \( \mathfrak{g} \)-highest weight vectors of \( M_c^{r,0} \) with highest weight \( \lambda = \delta_c + \lambda \). Then \( \{ v_t \mid t \in \mathcal{P}^{std}(\lambda') \} \) is a basis of \( V_\lambda \).

Proof. (a) follows from Lemma 7.6 and (b). We claim that \( v_t \) is a \( \mathfrak{g} \)-highest weight vector of \( M_c^{r,0} \). Since \( d(t) \) is invertible, it is enough for us to consider the case \( d(t) = 1 \).

By Lemma 7.7, \( e_{i,j+1}v_1 = 0 \) for any \( e_{i,j+1} \in \mathfrak{n}^+ \cap \mathfrak{h} \), where \( \mathfrak{n}^+ \) is the positive part of \( \mathfrak{g} \).

It remains to show that \( e_{\pi_i \pi_j}v_1 = 0 \) for any \( 1 \leq i \leq k \). If \( v_{\pi_i} \) does not occur in \( v_1 \), then \( e_{\pi_i \pi_j}v_1 = 0 \). Otherwise, \( v_{\pi_i} \) occurs in \( v_1 \), forcing \( \lambda^{(i+1)} \neq \emptyset \). Recall that \( [\lambda] = [a_0, a_1, \cdots, a_{k-1}, a_k] \).

\[
eq \sum_{1 \leq a \leq \lambda^{(i+1)}} m \otimes v_{\lambda^{(i+1)}} \otimes \cdots \otimes v_{\lambda^{(i+2)}} \otimes v_{\lambda^{(i+1)}} \otimes \cdots \otimes v_{\lambda^{(i+1)}} w_{\lambda^{(i+1)}},
\]

where \( i_a \) is obtained from \( i_{\lambda^{(i+1)}} \) by using \( p_i \) instead of \( p_i \) at \( (a_i + a)th \) position. Let \( j = (i_{\lambda^{(k)}}, i_{\lambda^{(i+2)}}, i_1, i_{\lambda^{(i)}}, \cdots, i_{\lambda^{(1)}}) \in (n, r) \). Then

\[
eq m \otimes v_j h w\lambda |\tilde{\pi}[\lambda]. \quad (7.7)
\]

where \( h = \sum_{1 \leq a \leq \lambda^{(i+1)}} (a_i + 1, a_i + a) \), and \( (i, j) \) is the permutation which switches \( i \) and \( j \) and fixes others. So, \( h w\lambda | = w\lambda | h_1 \) for some \( h_1 \) in the group algebra of the Young subgroup \( S_{\lambda'} \) of \( \mathfrak{S}_r \) with respect to the composition \( (r - a_k - 1, \cdots, a_2 - a_1, a_1 - a_0) \), and hence \( h_1 \tilde{\pi}[\lambda] = \tilde{\pi}[\lambda] h_1 \), and

\[
eq m \otimes v_j h w\lambda | \tilde{\pi}[\lambda] = m \otimes v_{\lambda^{(1)}} \otimes \cdots \otimes v_{\lambda^{(k)}} \otimes v_1 \otimes v_{\lambda^{(i+1)}} \otimes \cdots \otimes v_{\lambda^{(i+1)}} \tilde{\pi}[\lambda] h_1.
\]

For the simplification of notation, write \( b_i = r - a_{k-i}, 1 \leq i \leq k \). Then the tensor factor of \( m \otimes v_{\lambda^{(1)}} \otimes \cdots \otimes v_{\lambda^{(i+1)}} \otimes v_1 \otimes v_{\lambda^{(i+2)}} \otimes \cdots \otimes v_{\lambda^{(i+2)}} \) at \( (b_{k-i+1} - 1)th \) position is \( v_{\pi_i} \). So, it suffices to verify

\[
eq m \otimes v_{\lambda^{(1)}} \otimes \cdots \otimes v_{\lambda^{(i)}} \otimes v_1 \otimes v_{\lambda^{(i+2)}} \otimes \cdots \otimes v_{\lambda^{(i+1)}} \otimes v_{\lambda^{(i+2)}} \otimes \cdots \otimes v_{\lambda^{(i+2)}} (1, b_{k-i-1} + 1)^2 \tilde{\pi}[\lambda] = 0. \quad (7.8)
\]

However, the tensor factor of \( m \otimes v_{\lambda^{(1)}} \otimes \cdots \otimes v_{\lambda^{(i)}} \otimes v_1 \otimes v_{\lambda^{(i+1)}} \otimes \cdots \otimes v_{\lambda^{(i)}} \otimes v_{\lambda^{(i+1)}} \otimes \cdots \otimes v_{\lambda^{(i)}} \) at \( 1st \) position is \( v_{\pi_i} \). Since we are assuming that \( \lambda^{(i+1)} \neq \emptyset \), \( b_{k-i} > b_{k-i+1} \). Note that \( \pi_a(x) \) commutes with \( s_j \) for any \( j \neq a \). We have

\[ (1, b_{k-i+1} + 1)^2 \tilde{\pi}[\lambda] = \prod_{j=1}^i (x_{1} - u_j) h, \text{ for some } h \in \mathcal{H}_{k,r}. \]

Since \( m \otimes v_{\pi_i} \in M_i \), where \( M_i \) is given in Lemma 3.12 (7.8) follows from Lemma 3.12 a).

Next we verify \( v_1 \neq 0 \). Write

\[
v = v_1 w_{\lambda y \lambda'} = \sum_{i \in I(n, r)} b_i v_1. \quad (7.9)
\]
Since $w_{[\lambda]}$ is invertible, by Lemma 7.6 $v \neq 0$. Let $V_\lambda$ be the set of all $v_1$ in (7.9) such that $b_1 \neq 0$. Obviously, for any $v_1 \in V_\lambda$,

$$\{i_{b_j+1}, i_{b_j+2}, \ldots, i_{b_k}\} \subseteq p_{k-j+1}, 1 \leq j \leq k,$$

(7.10)

where $p_{k-j+1}$'s are defined in Definition 3.8. On the other hand, $\tilde{\pi}_{[\lambda]}$ contains a unique term $x_1^{\alpha_1}x_2^{\alpha_2} \cdots x_r^{\alpha_r}$ such that $\alpha \in N^r_k$ and $\sum \alpha_i$ is maximal. By Lemma 3.10 and Remark 3.16, $m \otimes v_1 x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_r^{\alpha_r}$ contains a unique tensor with highest degree $\sum \alpha_i$. Also, it is the unique term of $m \otimes v_1 \tilde{\pi}_{[\lambda]}$ with highest degree $\sum \alpha_i$. On the other hand, via arguments similar to those in the proof of Theorem 3.17, one can easily see that $\{m \otimes v_1 x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_r^{\alpha_r} \mid v_1 \in V_\lambda\}$ is linearly independent. So $v_1 \neq 0$. By Lemma 7.5 the $m$-component of $v_1$ is $a_i v(t)$ for some $a_i \in \mathbb{C}$, where $v$ is given in (7.9). By Lemma 7.4 $\{vd(t) \mid t \in \mathcal{S}^{std}(\lambda')\}$ is linear independent. Using Lemma 7.5 shows that $\{v_1 \mid t \in \mathcal{S}^{std}(\lambda')\}$ is linear independent, too. This completes the proof of (b).

Theorem 7.9. For any $\lambda \in \Lambda^+_k(r)$, let $\tilde{\lambda}$ be defined in Theorem 7.8. As right $\mathcal{H}_{k,r}$-modules, $\text{Hom}_\Omega(M^\ast(\tilde{\lambda}), M^r_{c,0}) \cong C(\lambda')$.

Proof. For any $t \in \mathcal{S}^{std}(\lambda')$, by the universal property of parabolic Verma modules, we define $f_t \in \text{Hom}_{\mathcal{O}}(M^\ast(\tilde{\lambda}), M^r_{c,0})$ such that $f_t(m_{\tilde{\lambda}}) = v_1$, where $m_{\tilde{\lambda}}$ is the highest weight vector of $M^\ast(\tilde{\lambda})$. By Theorem 7.8 $\{f_t \mid t \in \mathcal{S}^{std}(\lambda')\}$ is a basis of $\text{Hom}_\Omega(M^\ast(\tilde{\lambda}), M^r_{c,0})$.

Let $\phi : V_{\tilde{\lambda}} \rightarrow S^\lambda$ be the linear isomorphism sending $v_1$ to $x_{\lambda}w_{\lambda} y_{\lambda} d(t)$, where $S^\lambda$ is the classical Specht module for $\mathcal{H}_{k,r}$ (see (7.6)). We claim that $\phi$ is an $\mathcal{H}_{k,r}$-homomorphism. In fact, by Theorem 7.2

$$y_{\lambda} d(t) h = \sum_{s \in \mathcal{S}^{std}(\lambda')} a_s y_{\lambda} d(s) + \sum_{s_1, s_2 \in \mathcal{S}^{std}(\nu), \nu \triangleright \lambda'} a_{s_2} a_{s_1} d(s_2)^{-1} y_{\nu} d(s_1).$$

(7.11)

for any $h \in \mathcal{H}_{k,r}$ and some $a_s, a_{s_2} \in \mathbb{C}$. It is well known that $x_{\lambda} \mathcal{H}_{k,r} y_{\nu} = 0$ if $\lambda, \nu \in \Lambda^+_k(r)$ and $\lambda \triangleright \nu$. Since $\lambda \triangleright \nu$ if and only if $\lambda' \triangleright \nu'$, we have

$$\phi(v_1) h = \sum_{s \in \mathcal{S}^{std}(\lambda')} a_s x_{\lambda}w_{\lambda} y_{\lambda} d(s) = \sum_{s \in \mathcal{S}^{std}(\lambda')} a_s \phi(v_1).$$

In order to complete the proof of our claim, we need to verify

$$m \otimes v_1 x_{\lambda} w_{\lambda} d(s_2)^{-1} y_{\nu} = 0$$

(7.12)

Since we are assuming $\nu \triangleright \lambda'$, either $[\nu] = [\lambda']$ or $[\lambda'] \prec [\nu]$. In the first case, (7.12) follows from Lemma 7.6. Write $[\nu] = [0, b_1, b_2, \cdots, b_k]$ and $[\lambda'] = [0, a_1, a_2, \cdots, a_k]$. In the second case, there is an $i$ such that $a_j = b_j$ for $j < i$ and $a_i = b_i$. So,

$$m \otimes v_1 x_{\lambda} w_{\lambda} d(s_2)^{-1} \pi_{[\nu]} = m \otimes v_1 \pi_{b_i}(u_{k-i}) \cdots \pi_{b_{k-i}}(u_1)(a_i + 1, 1) \pi_{b_i}(u_{k-i}) \cdots \pi_{b_{k-i}}(u_{k-i+1})$$

where $v_1 = v_1 x_{\lambda} w_{\lambda} d(s_2)^{-1}(a_i + 1, 1)$ and $j_1 \in p_{k-i}$. Since $(x_1 - u_1) \cdots (x_1 - u_{k-i})$ is a factor of $\pi_{b_i}(u_{k-i}) \cdots \pi_{b_{k-i}}(u_1)$, by Lemma 3.12(a), $m \otimes v_1 \pi_{b_i}(u_{k-i}) \cdots \pi_{b_{k-i}}(u_1) = 0$. So
\(v_h = \sum_{s \in \mathcal{F}_{\text{std}}(\lambda)} a_s v_s\) and \(\phi(v_h) = \phi(v_h)\). This proves our claim and hence \(V_\lambda^+ \cong \Lambda^\lambda\) as right \(\mathcal{H}_{k,r}\)-modules. Via it, it is routine to check that there is an \(\mathcal{H}_{k,r}\)-isomorphism \(\text{Hom}_\Sigma(M_f(\lambda), M^{r,0}_c) \cong \Lambda^\lambda\). By (7.6), the result follows. \(\square\)

By similar arguments as Theorem [7.8] we can also give a classification of highest weight vectors of \(M^{r,t}_c\). In this case, the parameters \(u_i\) of cyclotomic Hecke algebra \(\mathcal{H}_{k,t}\) should be replaced by \(\pi_{k-i+1}\)'s in Definition 3.11 for any \(i \in \underline{k}\). In this case, we define

\[
\lambda^* = \sum_{1 \leq i \leq k} \sum_{p_i-1 < j < p_i} -\lambda^{(i)}_{p_j-1} \varepsilon_j, \quad \text{and} \quad \hat{\lambda}^* = \delta_c + \lambda^* \text{ for any } \lambda \in \Lambda^+_k(t). \tag{7.13}
\]

**Definition 7.10.** Suppose \(\lambda \in \Lambda^+_k(t)\). Define

- a) \(\lambda'^o = (\lambda^{(k)}, \lambda^{(k-1)}, \ldots, \lambda^{(1)})\),
- b) \(i_{\lambda'} = (i_{\lambda(k)}, \ldots, i_{\lambda(2)}, i_{\lambda(1)}) \in I(n, t)\), where \(i_{\lambda(\cdot)} = ((p_j)_{\lambda(1)}, \ldots, (p_j - 1 = 1)_{\lambda(j)})\),
- c) \(v^*_t = m \otimes v^*_{i_{\lambda'}} w_{\lambda'^o y_{\lambda'^o} d}(t)\) for any \(t \in \mathcal{F}_{\text{std}}((\lambda'^o))\).

**Corollary 7.11.** Suppose that \(t \leq \min\{q_1, q_2, \ldots, q_k\}\).

- a) There is a bijection between \(\Lambda^+_k(t)\) and the set of \(\mathfrak{p}\)-dominant weights \(\lambda\) such that \(M^{r,t}_c\) contains at least a highest weight vector with highest weight \(\lambda\).

- b) The \(\mathbb{C}\)-space \(V_{\lambda^*}\) of \(\mathfrak{g}\)-highest weight vectors of \(M^{r,t}_c\) with highest weight \(\hat{\lambda}^*\) has a basis \(\{v^*_t \mid t \in \mathcal{F}_{\text{std}}((\lambda'^o))\}\).

We are going to classify highest weight vectors of \(M^{r,t}_c\) under the assumption \(r+t \leq \min\{q_1, q_2, \ldots, q_k\}\). We need some of results on a cellular basis of a cyclotomic walled Brauer algebra as follows. Fix \(r, t, f \in \mathbb{Z}^{>0}\) with \(f \leq \min\{r, t\}\). Define

\[
\mathcal{D}^{r,f}_{r,t} = \{s_{r-f+1, i_1, \ldots, i_{r-f+1}, j_1, \ldots, j_{k-t-1}} \mid r \geq i_r > \cdots > i_{r-f+1}, j_k \geq k + f - t\}. \tag{7.14}
\]

For each \(c \in \mathcal{D}^{r,f}_{r,t}\), as in (7.14), let \(\kappa_c\) be the \(r\)-tuple

\[
\kappa_c = (k_1, \ldots, k_r) \in \mathbb{N}^r_k \text{ and } k_i = 0 \text{ unless } i = i_r, i_{r-1}, \ldots, i_{r-f+1}. \tag{7.15}
\]

Note that \(\kappa_c\) may have more than one choice for a fixed \(c\), and it may be equal to \(\kappa_d\) although \(c \neq d\) for \(c, d \in \mathcal{D}^{r,f}_{r,t}\). Let \(\mathbf{N}_f = \{\kappa_c \mid c \in \mathcal{D}^{r,f}_{r,t}\}\). If \(\kappa_c \in \mathbf{N}_f\), define \(x^\kappa = \prod_{i=1}^r x_i^{k_i}\). In [21], we consider poset \((\Lambda_{k,r,t}, \succeq)\), where

\[
\Lambda_{k,r,t} = \left\{ (f, \lambda, \mu) \mid (\lambda, \mu) \in \Lambda^+_k(r-f) \times \Lambda^+_k(t-f), 0 \leq f \leq \min\{r, t\} \right\}, \tag{7.16}
\]

such that \((f, \lambda, \mu) \succeq (\ell, \alpha, \beta)\) for \((f, \lambda, \mu), (\ell, \alpha, \beta) \in \Lambda_{k,r,t}\) if either \(f > \ell\) or \(f = \ell\) and \(\lambda \succeq_1 \alpha\), and \(\mu \succeq_2 \beta\), and in case \(f = \ell\), the orders \(\succeq_1\) and \(\succeq_2\) are dominant orders on \(\Lambda^+_k(r-f)\) and \(\Lambda^+_k(t-f)\) respectively. Define

\[
e_{i,j} = s_{j, i, s_i} \delta_{i, s_i} \delta_{s_i, i} \text{ for } i, j \text{ with } 1 \leq i \leq r \text{ and } 1 \leq j \leq t. \tag{7.17}
\]
Definition 7.12. For \((f, \mu, \nu) \in \Lambda_{k,r,t}\), define
\[
\delta(f, \mu, \nu) = (\mathcal{F}^{\text{std}}(\mu) \times \mathcal{F}^{\text{std}}(\nu)) \times D_{r,t}^f \times N_f.
\]
For any \((s, d, \kappa_d), (t, c, \kappa_c) \in \delta(f, \mu, \nu)\), define
\[
C(s, d, \kappa_d), (t, c, \kappa_c) = x^{kd} d^{-1} e^f y_{st} c x^{kc},
\]
where \(e^f = e_{r,t} e_{r-1,t-1} \cdots e_{r-f+1,t-f+1}\) if \(f \geq 1\) and \(e^0 = 1\).

Theorem 7.13. The set
\[
\mathcal{C} = \{C(s, c, \kappa_c), (t, d, \kappa_d) | (s, c, \kappa_c), (t, d, \kappa_d) \in \delta(f, \lambda), \forall (f, \lambda) \in \Lambda_{k,r,t}\},
\]
is a weakly cellular basis of \(\mathcal{B}_{k,r,t}\) over \(\mathbb{C}\) in the sense of [11]. The required anti-involution is \(\sigma\) in Lemma 2.2.

Proof. This result has been proved in [21] for \(k = 2\). In general, see Remark 3.8 in [22]. \(\square\)

For each \((f, \mu, \nu) \in \Lambda_{k,r,t}\), let \(C(f, \mu, \nu)\) be the cell module with respect to the weakly cellular basis of \(\mathcal{B}_{k,r,t}\) in Theorem 7.13. The following result can be proved by arguments similar to those in the proof of Proposition 3.9 in [21].

Lemma 7.14. Let \(\tilde{C}(f, \mu, \nu) = e^f x^f y_{mc} w_{d, \lambda} y_{mc} y_{d, \lambda} \mathcal{B}_{k,r,t} \mathcal{B}_{k,r,t} (mod \mathcal{B}_{k,r,t}^{f+1})\), where \(\mathcal{B}_{k,r,t}^{f+1}\) is the two-sided ideal generated by \(e^f+1\).

a) The set \(\{e^f x^f y_{mc} w_{d, \lambda} y_{mc} y_{d, \lambda} d(t) dx^a (mod \mathcal{B}_{k,r,t}^{f+1}) | (t, d, \kappa_d) \in \delta(f, \mu', \nu')\}\) is a basis of \(\tilde{C}(f, \mu, \nu)\).
b) As right \(\mathcal{B}_{k,r,t}\)-modules, \(\tilde{C}(f, \mu, \nu) \cong C(f, \mu', \nu')\).

Definition 7.15. Assume \(r + t \leq \min\{q_1, q_2, \ldots, q_k\}\) and \((f, \mu, \nu) \in \Lambda_{k,r,t}\). Define

1. \(i = (1, \cdots, 1, i_\mu) \in I(n, r)\), where \(i_\mu\) is defined in Definition 7.4.
2. \(j = (i_\mu, 1, \cdots, 1) \in I(n, t)\), where \(i_\mu\) is defined in Definition 7.10.
3. \(\lambda^i = (\mu_1^{(i)}, \cdots, \mu_r^{(i)}, 0, \cdots, 0, -\nu_t^{(i)}, \cdots, -\nu_1^{(i)}) \in \mathbb{Z}^q\).
4. \(\lambda_{\mu, \nu} = (\lambda^1, \lambda^2, \cdots, \lambda^k) = (\lambda_1, \lambda_2, \cdots, \lambda_n) \in \mathbb{Z}^n\).
5. \(v_{\lambda_{\mu, \nu}} = m \otimes v_1 \otimes v_1^\ast\), where \(\hat{\lambda}_{\mu, \nu} = \sum_{i=1}^n \lambda_i \epsilon_i + \delta_c\) and \(\delta_c\) is in Assumption 3.6.
6. \(v_{t, d, \kappa_d} = v_{\lambda_{\mu, \nu}} e^f w_{\mu, \nu} y_{(\nu')} d(t) dx^{b_1}, (t, d, \kappa_d) \in \delta(f, \mu', \nu'), \) and \(w_{\mu, \nu} = w_{\mu, \nu} \).

Theorem 7.16. Suppose \(r + t \leq \min\{q_1, q_2, \ldots, q_k\}\).

1. There is a bijection between \(\Lambda_{k,r,t}\) and the set of \(p\)-dominant weights \(\lambda\) such that \(M_{c,t}^r\) contains at least a highest weight vector with highest weight \(\lambda\). Moreover, the map sends \((f, \mu, \nu)\) to \(\hat{\lambda}_{\mu, \nu}\) which is defined in Definition 7.17 (5).
2. The \(C\)-space \(V_{\lambda_{\mu, \nu}}\) of \(g\)-highest weight vectors of \(M_{c,t}^r\) with highest weight \(\hat{\lambda}_{\mu, \nu}\) has a basis \(S = \{v_{t, d, \kappa_d} | (t, d, \kappa_d) \in \delta(f, \mu', \nu')\}\).
**Proof.** (a) follows from Lemma 7.5 and (b). By Theorem 7.8 and Corollary 7.11, we have (b) when \( f = 0 \). Since \((v_i \otimes v_j^*)e_1 = \delta_{i,j} \sum_{i=1}^{n} v_i \otimes v_i^*\), \(e_{i,i+1} \sum_{j=1}^{n} v_j \otimes v_j^* = 0\), for all possible \( j \). So, it suffices to show that \(e_{i,i+1}\) acts on \( m \otimes v_{\mu} \otimes v_{\nu}^* w_{\mu',\nu'} y_{(\nu')'}\) trivially if \( f > 0 \). This follows from our previous result on \( f = 0 \). It remains to prove that \( S \) is linearly independent.

Define

\[
v_{\mu} = v_{\mu} \circ w_{\mu} y_{\mu'} \text{ and } v_{\nu}^* = v_{\nu}^* \circ w_{\nu} y_{(\nu')'},\tag{7.18}
\]

where \( \mu' \) (resp., \( (\nu')' \)) is the composition of \( r - f \) (resp., \( t - f \)) obtained from \( \mu' \) (resp., \( (\nu')' \)) by concatenation. So,

\[
v_{t.d,k.d} = m \otimes v_{1}^{(f)} \otimes v_{\mu} \otimes v_{\nu}^* \otimes (v_{1}^*)^{(f)} = \sum_{i} \mu_{[\mu]'} \pi_{[\mu]'} (\nu_{(\nu')'})' e^{f} d(t) dx^{k.d}.
\]

Let \( D_{t.d} \) be the walled Brauer diagram with respect to \( e^{f} d(t) d \). Note that \( \pi_{[\mu]'} \) (resp., \( \pi_{(\nu_{(\nu')'})'} \)) contains a unique term say, \( F_{\mu} = x_{1}^{\alpha_{1}} x_{2}^{\alpha_{2}} \cdots x_{t}^{\alpha_{t}} \) (resp., \( F_{\nu} = \pi_{(\nu_{(\nu')'})'} = x_{1}^{\beta_{1}} x_{2}^{\beta_{2}} \cdots x_{t}^{\beta_{t}} \)) with highest degree \( \sum_{i} \alpha_{i} \) (resp., \( \sum_{i} \beta_{i} \)), where \( \alpha \in N_{k}^{r-f} \) (resp., \( \beta \in N_{k}^{(t-f)} \)). So, \( \pi_{[\mu]'} \pi_{(\nu_{(\nu')'})'} = F_{\mu} F_{\nu} \) up to some terms of lower degrees. Let

- \( a) \ p_{\mu} = \{ i \mid i \in I(n, r-f), v_{i} \text{ appears in } v_{\mu} \text{ with a non-zero coefficient} \}, \)
- \( b) \ p_{\nu} = \{ i \mid i \in I(n, t-f), \nu_{i} \text{ appears in } v_{\nu}^* \text{ with a non-zero coefficient} \}. \)

Suppose \((l, m) \in p_{\mu} \times p_{\nu} \) and \( d = s_{r-f+1, i_{r-f+1}} s_{r-f+1, i_{r-f+1}} \cdots s_{r-k, i_{r-k}} \in D_{t.d} \). We define a labeled walled Brauer diagram \( D_{t.d} \) with respect to \( D_{t.d} \) so as to describe the action of \( F_{\mu} F_{\nu} D_{t.d} \) on \( m \otimes v_{1}^{(f)} \otimes v_{l} \otimes v_{m}^{*} \otimes (v_{1}^*)^{(f)} \). Recall that \( \kappa_{i} \) is called the \( i \)th component of \( \kappa_{d} \) if \( \kappa_{d} = (\kappa_{1}, \kappa_{2}, \ldots, \kappa_{r}) \in N_{k}^{r} \). First, we insert some beads at some vertices of \( D_{t.d} \) as follows.

- \( a) \) There are \( \alpha_{i} \) (resp., \( \beta_{j} \)) beads at the \( i \)th (resp., \( j \)th) vertex at the top row of \( D_{t.d} \);
- \( b) \) There are \( a \) beads at the \( i_{r-h+1} \)th vertex at the bottom row of \( D_{t.d} \), \( 1 \leq h \leq f \), if the \( i_{r-h+1} \)th component of \( \kappa_{d} \) is \( a \).

Secondly, we label some positive integers at all vertices of \( D_{t.d} \) as follows.

- \( a) \) The vertices \((r, r-1, \ldots, 1; t, \overline{t} \cdots, \overline{t}) \) at the top row of \( D_{t.d} \) are labeled with positive integers according to the sequence \( b:=(b_{1}; b_{2}) \), where

\[
(b_{1}; b_{2}) = (1, \overline{1}, \cdots, 1, l_{r-f}, \ldots, l_{1}; m_{1}, \cdots, m_{t-f}, \overline{1}, \cdots, 1).
\]

- \( b) \) The vertices \((\overline{t}_{t-f+1}, \cdots, \overline{t}_{t-1}, \overline{t}_{t}) \) at the bottom row of \( D_{t.d} \) are labeled according to the sequence \( (p_{k-1} + |\mu(k)| + 1, p_{k-1} + |\mu(k)| + 2, \cdots, p_{k-1} + |\mu(k)| + f) \) of positive integers. Moreover, if the labeling of \( \overline{t}_{t-l} \) is \( p - \sum_{j=1}^{h} q_{k-j} \), provided that there are \( h \) beads at the vertex \( i_{t-l} \).

- \( c) \) Suppose \([i, j] \) is an edge of \( D_{t.d} \) and \( i \) is a vertex at the top row. Suppose there are \( h \) beads at the vertex \( i \), and the labeling of \( i \) is \( p \in p_{i} \), then the labeling of \( j \) is \( p - \sum_{m=1}^{h} q_{m} \).
d) Suppose \([\overline{t}, \overline{j}]\) is an edge of \(D_{t,d}\) and \(\overline{t}\) is a vertex at the top row. Suppose there are \(h\) beads at the vertex \(\overline{t}\), and the labeling of \(\overline{t}\) is \(p \in p_l\), then the labeling of \(\overline{j}\) is \(p + \sum_{m=1}^{h} q_{m-1}\).

Since we are assuming \(r + t \leq \min\{q_1, q_2, \ldots, q_k\}\), the above is well-defined. For pairs \((l_i, \alpha_i)\) and \((m_j, \beta_j)\) (determined by the labeled walled Brauer diagram defined above), define

\[
\begin{align*}
\mathcal{Y}_{l_i, \alpha_i, i} &= e_{l_i - q_i - l_i - q_{i-1} - l_i - q_{i-2} \ldots e_{l_i - \sum_{j=1}^{\alpha_i-1} q_j - l_i - \sum_{j=1}^{\alpha_i-1} q_j} = 1, \\
\hat{\mathcal{Y}}_{m_j, \beta_j, j} &= e_{m_j + q_{o+1} + m_j + q_{o+1} + q_{o+2} + m_j + q_{o+1} + \ldots e_{m_j + \sum_{i=1}^{\beta_j} q_{o+i}}}
\end{align*}
\]

(7.19)

if \(l_i \in p_l\) and \(m_j \in p_o\). For the vertex \(i_{r-l}\) at the bottom row of \(D_{t,d}\), we have the pair say, \((p - \sum_{j=1}^{h} q_{k-j}, h)\) if there are \(h\) beads at the vertex \(i_{r-l}\). Define

\[
\hat{\mathcal{Y}}_{p - \sum_{j=1}^{h} q_{k-j}, h} = e_{p - q_{k-1} - p - q_{k-1} - p - q_{k-2} - \ldots e_{p - \sum_{j=1}^{h} q_{k-j} - p - \sum_{j=1}^{h} q_{k-j}}}
\]

(7.20)

Consider the ordered product \(\mathcal{Y} = \prod_{i=1}^{\ell} \mathcal{Y}_{l_i, \alpha_i, i} \prod_{j=1}^{f} \hat{\mathcal{Y}}_{m_j, \beta_j, j} \prod_{l=1}^{d} \mathcal{Y}_{m_k, \mu, l} \prod_{q=1}^{h} \mathcal{Y}_{p - \sum_{j=1}^{h} q_{k-j}, h}\). By Lemma 3.10 and Remark 3.16, the coefficient of \(\mathcal{Y}_{m} \otimes v_{n_1} \otimes v_{n_2}^*\) in \(v_{t,d,k_d}\) is \(\delta(t,d), (t,d')\) up to a non-zero multiple, where \(n = (n_1; n_2)\) is the sequence of positive integers obtained by reading the labeling of vertices at the bottom row of the labeled walled Brauer diagram \(\tilde{D}_{t,d}\) from left to right. So, \(S\) in (2) is \(\mathbb{C}\)-linear independent. \(\square\)

**Example 7.17.** Assume \((q_1, q_2, q, r, t, f) = (11, 12, 2, 5, 6, 1)\). Suppose \(\mu = ((2), (1, 1))\) and \(\nu = ((2), (2, 1))\), and \(d = \delta_5\), and \(d(t) = s_2 s_3 s_4 s_2 s_3\) and \(\kappa_d = (0^1, 1)\), and \((1; m) = (1, 1, 3, 12; 11, 11, 23, 23, 22)\). Then \(F_\mu = x_1 x_2\) and \(F_\nu = x_1^2 x_2\) and \(x^{\kappa_d} = x_5\). The following diagram is \(\tilde{D}_{t,d}\). In this case,

a) \(b = (1, 1, 1, 13, 12; 11, 11, 23, 23, 22, 1)\),

b) \(n = (3, 1, 1, 1, 2; 22, 23, 22, 22, 14, 23)\),

c) \(\mathcal{Y}_{12, 1, 1} = e_{12, 1}, \mathcal{Y}_{13, 1, 2} = e_{13, 2}, \mathcal{Y}_{3, 1} = e_{14, 3}, \hat{\mathcal{Y}}_{11, 1, 1} = e_{22, 11}\) and \(\hat{\mathcal{Y}}_{11, 1, 2} = e_{22, 11}\).

![Diagram](attachment:diagram.png)

**Theorem 7.18.** For any \((f, \mu, \nu) \in \Lambda_{k,r,t}\), \(\text{Hom}_\mathbb{C}(M^f(\hat{\lambda}_{\mu, \nu}), M^f_{\nu} \otimes \mathcal{O}) \cong C(f, \mu', (\nu')')\) as right \(\mathcal{B}_{k,r,t}\)-modules, where \(\hat{\lambda}_{\mu, \nu}\) is defined in Definition 7.15(5).
Proof. By Lemma 7.14(a) and Theorem 7.16(b), the linear map \( \phi : V_{\lambda,\mu} \to \hat{C}(f,\mu,\nu^0) \) satisfying
\[
\phi(v_{\lambda,\mu,\nu}w_{\mu,\nu}C_{(s_1,c_1),(s_2,c_2)}) = 0
\]
for any \( C_{(s_1,c_1),(s_2,c_2)} \in \mathcal{B}_{k,r,t}^{(f,\mu'},(\nu^0')} \), where \( \mathcal{B}_{k,r,t}^{(f,\mu'},(\nu^0')} \) is the \( \mathbb{C} \)-subspace of \( \mathcal{B}_{k,r,t} \) spanned by all \( C_{(s_1,c_1),(s_2,c_2)} \) with \( (s_1,c_1), (s_2,c_2) \in \delta(f,\mu',(\nu^0')) \) (cf. the proof of Theorem 7.9). Suppose \( s_1 \in \mathcal{T}^{std}(\alpha) \times \mathcal{T}^{std}(\beta) \) such that \( \alpha \in \Lambda_k^+(r-l) \) and \( \beta \in \Lambda_k^+(t-l) \). So either \( l > f \) or \( f = l + \) and either \( \alpha \triangleright \mu' \) or \( \beta \triangleright (\nu^0)' \). In the first case, it’s easy to see that (7.21) holds. The second case follows from the arguments in the proof of Lemma 7.9. Finally, the result follows from Lemma 7.11(b). \( \Box \)

Recall that \( \Lambda_{k,r,t} \) is the poset in (7.16). Let \( \overline{\Lambda}_{k,r,t} \subset \Lambda_{k,r,t} \) such that each cell module \( C(f,\mu,\nu) \) has simple head \( D^{(f,\mu,\nu)} \) for any \( (f,\mu,\nu) \in \overline{\Lambda}_{k,r,t} \). See Proposition 3.7 and Remark 3.8 in [22]. The following is the second main result of this paper.

Theorem 7.19. Suppose \( r + t \leq \min\{q_1,q_2,\ldots,q_k\} \). For any \( (f,\mu,\nu) \in \Lambda_{k,r,t}, (\ell,\alpha,\beta) \in \overline{\Lambda}_{k,r,t} \), \( (T(\hat{\lambda}_{\alpha,\beta}) : M^p(\hat{\lambda}_{\mu,\nu})) = [C(f,\mu',(\nu^0')) : D^{(f,\mu',(\nu^0'))}] \), where \( T(\hat{\lambda}_{\alpha,\beta}) \) is the indecomposable tilting module with respect to \( \hat{\lambda}_{\alpha,\beta} \) in Definition 7.15.

Proof. Since \( M^c_{r,t} \) is a tilting module in \( \mathcal{O} \), it follows from the arguments in Section 5 in [1] that \( \text{End}_\mathcal{O}(M^c_{r,t}) \) is a cellular algebra and for any \( \lambda \in \Lambda_\mathcal{O}^{r,t} \), \( \text{Hom}_\mathcal{O}(M^p(\lambda),M^c_{r,t}) \) (resp., \( \text{Hom}_\mathcal{O}(M_{r,t},N^p(\lambda)) \)) is the corresponding right (resp., left) cell module of \( \text{End}_\mathcal{O}(M^c_{r,t}) \) where \( N^p(\lambda) \) is the dual parabolic Verma module with respect to \( \lambda \). By Theorem 5.17, we have
\[
\text{End}_\mathcal{O}(M^c_{r,t}) \cong \mathcal{B}_{k,r,t}
\]
where \( \mathcal{B}_{k,r,t} \) is defined in 5.12. By Theorem 7.18, \( \text{Hom}_\mathcal{O}(M^p(\hat{\lambda}_{\mu,\nu}),M^c_{r,t}) \cong C(f,\mu',(\nu^0')) \). So we have
\[
\text{Hom}_\mathcal{O}(M^c_{r,t},N^p(\hat{\lambda}_{\mu,\nu})) \cong C(f,\mu',(\nu^0'))
\]
as left cell modules of \( \mathcal{B}_{k,r,t} \). By [1] Proposition 5.4] the indecomposable tilting module \( T(\hat{\lambda}) \) appears as an indecomposable direct summand of \( M^c_{r,t} \) if and only if \( \lambda = \hat{\lambda}_{\alpha,\beta} \) and \( D^{(f,\mu',(\nu^0'))} \) is a simple head of \( C(\ell,\alpha',(\beta^0)'') \). Moreover, we can deduce from the proof of [1] Proposition 5.4] that
\[
\text{Hom}_\mathcal{O}(M^c_{r,t},T(\hat{\lambda}_{\alpha,\beta})) \cong P(\ell,\alpha',(\beta^0)'),
\]
where \( P(\ell,\alpha',(\beta^0)') \) is the projective cover of \( D^{(\ell,\alpha',(\beta^0)')} \). Let \( f := \text{Hom}_\mathcal{O}(M^c_{r,t},?) \) and \( g = M^c_{r,t} \otimes \mathcal{B}_{k,r,t} \). Then by (7.22) and standard arguments (see, e.g., [19] Lemma 5.10)].
\[ \text{gf}(T(\hat{\lambda}_{\alpha,\beta})) \cong T(\hat{\lambda}_{\alpha,\beta}) \text{ for any } (\ell, \alpha, \beta) \in \Lambda_{k,r,t}. \] 

By \((7.23)\)-(7.21),

\[ \text{Hom}_\mathbb{C}(T(\hat{\lambda}_{\alpha,\beta}), N^\mathbb{C}(\hat{\lambda}_{\mu,\nu})) \cong \text{Hom}_{\mathbb{K}_{k,r,t}}(f(T(\hat{\lambda}_{\alpha,\beta})), f(N^\mathbb{C}(\hat{\lambda}_{\mu,\nu}))) \]

\[ \cong \text{Hom}_{\mathbb{K}_{k,r,t}}(P(\ell, \alpha', (\beta'''), C(f, \mu', (\nu'''))). \] 

Comparing the dimensions for both sides of \((7.25)\) yields the result as required. \(\square\)

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