The Mathematical Abstraction Theory,
The Fundamentals for
Knowledge Representation and
Self-Evolving Autonomous Problem Solving Systems

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The intention of the present study is to establish the mathematical fundamentals for automated problem solving essentially targeted for robotics by approaching the task universal algebraically introducing knowledge as realizations of generalized free algebra based nets, graphs with gluing forms connecting in- and out-edges to nodes. Nets are caused to undergo transformations in conceptual level by type wise differentiated intervening net rewriting systems dispersing problems to abstract parts, matching being determined by substitution relations. Achieved sets of conceptual nets constitute congruent classes. New results are obtained within construction of problem solving systems where solution algorithms are derived parallel with other candidates applied to the same net classes. By applying parallel transducer paths consisting of net rewriting systems to net classes congruent quotient algebras are established and the manifested class rewriting comprises all solution candidates whenever produced nets are in anticipated languages liable to acceptance of net automata. Furthermore new solutions will be added to the set of already known ones thus expanding the solving power in the forthcoming - hereby setting a base for a self-evolving autonomous learning system. Moreover special attention is set on universal abstraction, thereof generation by net block homomorphism, consequently multiple order solving systems and the overall decidability of the set of the solutions. By overlapping presentation of nets new abstraction relation among nets is formulated alongside with consequent alphabetical net block renetting system proportional to normal forms of renetting systems regarding the operational power. A new structure in self-evolving problem solving is established via saturation by groups of equivalence relations and iterative closures of generated quotient transducer algebras over the whole evolution.
Introduction

The basis of this thesis can be found in my publications Tirri SI (2013 May) and Tirri SI (2013 Aug).

In all fields of data processing, especially in robotics, physics and overall changing constructions is ever increasing need for knowledge of common structures in creating fast, exact, controllable and sufficiently comprehensive solving algorithms for problems. From René Descartes freely quoted: “there is not very much in results or even in the proofs of them, but the method how they are invented, that is what is the process inventors use to realize proofs”. Restricting data flow to finite cases is often in descriptive models improper in order to achieve sufficient handling with the tasks, e.g. if variables are allowed to be systems themselves as in function representatives of quantum particles. Models in meteorology and models for handling with populations, biological organizations or even combinations in genetic codes call for common approach in problem solving especially in cases where in- or out- data flow volumes are beforehand impossible to predict to be limited in the already known sphere. For connections between neurons in brains, and in more theoretical aspects for allowance of simultaneous “loops”, nets are ideal as formal representations for iterations as e.g. within solutions for powers of higher order differential equations by Picard successive iterants. In robotics strong AI, the abstract mathematical reasoning model, will play the key element in handling data processes in artefacts. In the 1980s in Japan were the first concrete steps taken in robotics trying to imitate human actions; however imitating process is the endless effort to achieve inventiveness which lays its solid ground in the understanding of reasoning itself, thus strong AI is a more effective approach. Infinite ranks (the cardinality of in- or out places in operation relations) are needed as tools for infinite simultaneous data flow into systems (operations) such as in quantum physics where infinite number of different Schrödinger-equation state function solutions compounds a field to be operated. Naturally one can imagine numerous other fields where a mathematical framework for problem solving would be desirable. Within solving any problem an essential thing is to see over details, and one inevitably confronts the necessity of outlining or abstracting the object to be solved to already more familiar forms or to forms easier to be checked – keeping the number of links regarding the environment of the object in hands unchanged (e.g. to be the most comprehensive, the handled data flow would not be allowed to be restricted solely to beforehand computably predictable form).
Lots of studies have been driven to clarify routes between nodes e.g. in process algebra, important topics setting ground to game theories as well as overall in halting problems. On the other hand in more complex dimensional cases ordering definitions in sets of subgraphs have been under vigorous investigations mainly concentrated in tree structures. An amazingly minute portion of studies on graphs concentrates to relations between graphs and abstraction of them and one explanation for this might be that transformations on conceptual levels lead joints to a succinct model proper to syntax as well as to semantic domain requiring combining algebraic structures to loop structured graphs and realizations of them, this requiring symbiosis of abstract syntax and real case sides.

The most remarkable study of human abstraction mechanism yielding a concrete result especially within mathematics in the form of analytical tools has been manifested by French philosopher, mathematician and physicist René Descartes in the 17th century in his work “Regulae ad directionem ingenii, Règles utiles et claires pour la direction de l'Esprit en la recherche de la Vérité (1628)”, freely outlining: “… at first we must organize the things which are the most essential ones in concentrating to do that by simplifying from phase to phase the vague, indefinite original problem. Then we try to understand the relations between those simplified parts and then compare the propositions to be proved i.e. wise versa try to see the connections between the reached relations and the original problem....”. Descartes underlines the importance of the origin of deduction itself namely abstraction by stating ”there is not very much in results or even in the proofs of them, but the method how they are invented, that is what is the process inventors use to realize proofs”. One of the first beneficial syntaxes for the use of infinite sequences in calculus was realized by Gottfried Leibniz in his infinitesimal applications in analytical geometry, but formalizing reasoning accelerated not until the breakthrough ideas of Alan Turing on string language representation methods in 1930s, which adopted in practice during 40s and 50s when the main target was to speed up data handling mainly in the near branch. However this was to push aside the previously nascent contemplation about the essence of reasoning itself and mathematically modeling notion abstraction was to be postponed. After technological revolution having achieved sufficiently strength in 80s Japan has been performing as the most driving force to implement applications which would be prominent vehicles for executing more extensive mathematical reasoning in a variety of situations by constructing
robots. On uppermost have been expert systems and imitation of human actions example wise without innovations towards inventiveness processes behind novel reasoning. Prospective preliminary approaches by implicit abstraction formulation are raised by Plaisted (1981) “clauses mapping” between ground theories and abstract counterparts i.e. relations between syntactical predicates, their evaluations and generalizations to operator evaluations Nayak PP, Levy AY (1995) and “ground language” corresponding to a Boolean graph over variables and a set of ground formulas over possibly more extensive set of variables Giordana A, Saitta L (1990) and abstraction between the whole formal structures: languages, axioms and inference rules Giunchiglia F, Walsh T (1992), “grounded abstraction” combining the whole variety of conceptualized ground entities Saitta L, Zucker J-M (2001). However the implicit nature of those existing models are not sufficiently expressive considering applications on knowledge representation and exact algebraic formalization combining abstraction operators and their semantic counterparts has been waiting for emerging.

The question in automated problem solving basically is how to generate nets from enclosures of a probed net those enclosures being in such a relation with the enclosures in the conceptual nets that the particular relation is invariant under that generating transformation i.e. preserves invariability under class-rewriting. Each perception as itself is able to orchestrate only a rough depict of problem subject under investigation but as posing a conceptual representation and paralleled with other already known concepts liable to the same subject is able to offer an explicit gateway for self-evolvelutional solving systems via algebraic quotient rewriting closure methods. Therefore we handle an idea of automated problem solving as formal inventiveness. In problem solving an essential thing is to see over details, and that is the task we next grip ourselves into by describing ideas such as partitioning nets by normal forms of renetting systems and a connection between partitions by introducing the abstraction relation. We concentrate to construct TD-models for formulas of jungle pairs by conceptualizing ground subjects and then reversing counterparts of existing TD-solutions back to ground level. Then we widen the solution hunting by taking universal partitioning into action in order to allow new links to the environment of applicants. By widening net homomorphism to cover idea of block-altering we obtain TD-transformation generation sets and as a consequence coherent expansion to next order solution levels. Saturation by groups of equivalence relations consistently supplements TD-solution arsenal and by iterating alternately mother nets and solutions in solving system leads to
multidimensional infinite solving process. Furthermore new solutions will be added to the set of already known ones thus expanding the solving power in the forthcoming - hereby establishing a self-evolving autonomous learning system.

Research Methodology and proceedings

Study of the present work will be accomplished by notional mathematics in state of the art, using consecutive theorems and the proofs of them – the main tools fetched from universal algebra, graph theories and formal languages, the grammas of which situated in quotient systems derived via iterative transformations. Formulations are handled in infinitive basis which seems to be the trend in syntax today. In order to avoid future entanglements in embedding problems and limitations in generative power homomorphism is defined in a generic manner fitting to utilized rewrite rules and enabling extensive semantics in high order AI languages.

METHOD AND TARGET

The present study introduces notion net and its operational counterpart for a basic explicit mathematical formalism for syntax of generic notion of knowledge and its semantics for real world cases by offering an algebraic approach to implement consecutive simultaneously looped deterministic and undeterministic operations in generalized universal free algebra and its realizations on different algebras, cf. traditional handling for trees and the evaluations Burris S, Sankappanavar HP (1981), Ohlebusch E (2002) and Denecke K, Wismat SL (2002). Nets can be identified unequivocally by any member of the corresponding net class by root basis as the most appropriate to the occasions. In syntax point of view “hyperedges” Engelfriet J (1997) can be regarded as nodes with in- and outarities of nets and in semantic aspect edge graphs equate with realization process graphs with nodes related to in-/outputs and edges to transformation relations.

The graph transformations from operator to operator are extended to cover more complex cases than appear in trees and to be more proper as premises of substitutions considering matching
properties in general rewriting systems. *Net substitutions* as a special case of more generic net homomorphisms are used to nicely manifest a unified way to express replacements in complicated iterative network in *universal partitioning* compared to the more limited version used in trees Jantzen M (1997).

Graph rewriting are mostly seen to be defined with a sample of rules between elementary entities as edges and nodes thus serving only as implicit tools, not as explicit ones needed for handling more extensive graph structures as a whole. Therefore *Renetting systems* are established as rewriting systems constructed to be adaptable to succinct algebraically represented graphs as well as trees thus avoiding weaker implicit expressions by rules based on exclusively sets of edges and vertices. Renetting systems are equipped with specifics incorporating accounts of the positions in targeted subnets and also the differences between left and right substitutions as well as other more traditional limit demands relating to applied rule orders or simultaneity inter alia cf. Ohlebusch E (2002), Engelfriet J (1997), “priority” by Baeten JCM, Basten T (2001) and Cleaveland R et al. (2001) “probability” Jonsson B et al. (2001). By intervening renetting systems being orchestrated to launch new binding organizations via overlapping partition abstraction to environments of applied entities we expand notional perception to cases where the common origins with already known conceptual counterparts are allowed to possess more limited interface consequently increasing the possibility to find suitable transformation rule sets as solution candidates for target perception entities.

*Transducers* are net realizations with renetting systems as operation vertexes and serve as groups pair wise commutative parallel operations on abstract class quotient algebra composing the closures of solving algorithm structures.

The present study serves as an explicit algebraic system for generic knowledge constitution and problem solving closure structures derived from abstraction classes and class renetting systems over them. Nets are ideal constructions maintaining prime features of operators in intervening rewriting corresponding under special issues of abstract classes manifested by said rewriting to “grounded abstraction” Saitta L, Zucker J-D (2001), where interpretations are organized by shifts inside partition net classes possessing a common origin and in single cases from algebra to its free algebra syntax; consequently widening grounded abstraction notion to deal net class
abstraction instead of only single perception cases. Parallel algebraic transducers serve as more general and explicit counterparts to solution memorizing operators as well as intervening rewriting systems stand for abstract perception operators.

In problem solving area parallel class rewriting covers block graph transformations and the idea of partially matching while net rewriting itself is a generalization to "genetic algorithms" Negnevitsky M (2002) which can be explained as term rewriting within horizontal changes on leaves (terminal letters), in contents as well as in arrangements; nets themselves offering in the realization aspect up-coming streams to influence to the results of operator realizations. By changing indexing inside nets net block is presented to expand net handling alternatives regarding reorganization into environments. Net block homomorphism can be determined to serve as initial abstraction operation via replacing the sets of blocks by letters. By alphabetical net block homomorphism new abstraction relation over the set of the nets is established. As the ground of autonomous solving evolution multiple level abstract algebra equivalent class transducer as operations are constructed of which consequently iteration. From the basis of alphabetical net block homomorphism new renetting systems are obtained to generate normal forms of any renetting system. Saturation by groups of equivalence classes offers next abstraction level to deal targets in the light of wider comprehension. Introducing multilevel iterative abstract quotient transducer classes and assuming mother nets and known solutions be fixed in each level and extending nested processes further exponentially we’ll get autonomous evolution levels and obtain transducer equivalent classes alteration to upgraded abstraction levels and consequently applying the whole string of the achieved process finally manifests self-evolving unrestricted autonomous problem solving formalism.

Inventiveness

The question in automated problem solving basically is how to generate nets from enclosements of a probed net those enclosements being in such a relation with the enclosements in the conceptual nets that the particular relation is invariant under that generating transformation i.e. preserves invariability under class-rewriting. Therefore we handle an idea of automated problem solving as formal inventiveness. In problem solving an essential thing is to see over details, and
that is the task we next grip ourselves into by describing ideas such as partitioning nets by RNS’s and a connection between partitions by introducing the abstraction relation. We concentrate to construct TD-models for formulas of jungle pairs by conceptualizing ground subjects and then reversing counterparts of existing TD-solutions back to ground level. Then we widen the solution hunting by classifying intervening TDG-derivations. Finally implementing generating sets comprising net block homomorphisms and application in multilevel iterative abstract quotient transducer classes finalizes the ultimate stage of self-evolvement serving thus the fundamentals as algorithmical tools for independent problem solving in robotics.
1.§ Preliminaries

This work follows the general custom of the discipline in concern and only necessary symbol definitions are manifested, readers are encouraged to turn to the literature represented in the reference list for the more comprehensive guidance.

1.1. Sets and Relations

We agree that all defined terms are of the cursive style when represented first time.

Definition 1.1.01. We regularly use small letters for elements and capital letters for sets and when necessary bolded capital letters for families of sets. The new defined terms are underlined when represented the first time.

Definition 1.1.02. We use the following convenient symbols for arbitrary element a and set A in the meaning:

\( a \in A \quad \text{“a is an element of A or belongs to A or is in A”} \)

\( a \notin A \quad \text{“a does not belong to A”} \)

\( \exists a \in A \quad \text{“there is such an element a in A that”} \)

\( \exists! a \in A \quad \text{“there is exactly one element a in A”} \)
\[ \exists a \in A \quad \text{“there exists none element a in A”} \]

\[ \forall a \in A \quad \text{“for each a belonging to A”} \]

\[ \Rightarrow \quad \text{“then it follows that”} \]

\[ \Leftrightarrow \quad \text{“if and only if”, shortly “iff”} \]

**Definition 1.1.03.** \{a : *\} or (a : *) means a *conditional* set, the set of all such a-elements which fulfil each condition in sample * of conditions, and *nonconditional*, if sample * does not contain any condition concerning a-elements.

**Definition 1.1.04.** \(\emptyset\) means *empty set*, the set with no elements. A set of sets is called a *family.* For set \(\emptyset\) the notation \{a_i : i \in \emptyset\} is an *indexed set* (over \(\emptyset\)). Set \{a_i : i \in \emptyset\} is \{a\}, if \(a_i = a\) whenever \(i \in \emptyset\). If there is no danger of confusion we identify a set of one element, *singleton*, with its element. It is noticeable that \{\emptyset\} is a singleton set.

**Definition 1.1.05.** For arbitrary sets A and B we use the notations:

\(A \subseteq B\) or \(B \supseteq A\) “A is a *subset* of B (is a part of B or each element of A is in B) or B *includes A*”

\(A \nsubseteq B\) “A is not a part of B (or there is an element in A which is not in B)”

\(A \subset B\) or \(B \supset A\) “A is a *genuine subset* of B” meaning “A \(\subseteq\) B and (\(\exists\) b \(\in\) B) b \(\notin\) A”

\(A \not\subset B\) “A is not a genuine subset of B”

\(A \neq B\) “A is not the same as B “

\(A^c\) or \(\neg A\) “is the *complement* of A” meaning set \{a : a \(\notin\) A\}

\(A \cup B\) “the *union* of A and B” meaning set \{a : a \(\in\) A or a \(\in\) B\}

\(A \cap B\) “the *intersection* of A and B” meaning set \{a : a \(\in\) A, a \(\in\) B\}. If \(A \cap B = \emptyset\), we say that A and B are *distinct* with each other, or *outside each other.*

\(A \setminus B\) “A excluding B” meaning \{a : a \(\in\) A, a \(\notin\) B\}.

**Definition 1.1.06.** The *cardinality* of A, “the number” of the elements in set A, is denoted by |A|.

**Definition 1.1.07.** \(P(A)\) symbolizes the family of all subsets of set A.
Definition 1.1.08. The set of natural numbers \{1,2,...\} is denoted by symbol \(\mathbb{N}\), and \(\mathbb{N}_0 = \mathbb{N} \cup \{0\}\). Maximum of the numbers in subset A of \(\mathbb{N}_0\) is denoted \(\text{max}A\).

Definition 1.1.9. Notice that for sets \(A_1\) and \(A_2\) and samples of conditions \(*_1\) and \(*_2\)

\[
\{a : a \in A_1, *_1\} \subseteq \{a : a \in A_2, *_2\},
\]

if \((A_1 \subseteq A_2\) and \(*_1 = *_2\) or \((A_1 = A_2\) and \(*_2 \subseteq *_1\)\).

Definition 1.1.10. The notation \(\bigcup (A_i : i \in I)\) is the union \(\{a : (\exists i \in I) a \in A_i\}\) and

\[
\bigcap (A_i : i \in I)\]

is the intersection \(\{a : (\forall i \in I) a \in A_i\}\).

for indexed family \(\{A_i : i \in I\}\). For any family \(\mathcal{B}\) we define:

\[
\bigcup \mathcal{B} = \bigcup (B : B \in \mathcal{B})
\]

\[
\bigcap \mathcal{B} = \bigcap (B : B \in \mathcal{B}).
\]

Definition 1.1.11. Set \(\rho\) of ordered pairs \((a,b)\) is a binary relation (shortly relation), where \(a\) is a \(\rho\)-preimage of \(b\) and \(b\) is a \(\rho\)-image of \(a\). The first element of pairs in relations is entitled preimages. \(\text{Dom}(\rho) = \{a : (a,b) \in \rho\}\) is the domain (set) of \(\rho\) (\(\rho\) is over \(\text{Dom}(\rho)\)), and \(\text{Im}(\rho) = \{b : (a,b) \in \rho\}\) is its image (set). Instead of \((a,b) \in \rho\) we often use the notation \(a\rho b\). We also say that \(\rho\) is giving \(b\) from \(a\). If the image set for each element of a domain set is a singleton, the concerning binary relation is called a mapping. For the relations the postfix notation is the basic presumption (\(b = a\rho\)); exceptions are relations with some long expressions in domain set or if we want to point out domain elements, and especially for mappings we use prefix notations (\(b = \rho a\)) or for the sake of clarity \(b = \rho(a)\), if needed. We define \(\rho : A \rightarrow B\), when we want to indicate that \(A = \text{Dom}(\rho)\) and \(B \supseteq \text{Im}(\rho)\), and \(\text{ApB}\), if \((a,b) \in \rho\) whenever \(a \in A\) and \(b \in B\). We also denote \(\text{Ap} = \{ap : a \in A\}\). When defining mapping \(\rho\), we can also use the notation \(\rho : a \rightarrow b, a \in A\) and \(b \in B\). If \(A \supseteq B\), we say that \(\rho\) is a relation in \(A\). When for \(\rho : A \rightarrow B\) we want to restrict \(\text{Dom}(\rho)\) to its subset \(C\) we denote \(\rho|_C\), the restricted mapping of \(\rho\) to \(C\) for which \(\rho|_C = \rho \cap \{(c,b) : c \in C, b \in B\}\).
Set \( \{ b : \alpha \rho b \} \) is called the \( \rho \)-class of \( \alpha \). Let \( \rho : A \rightarrow B \) be a binary relation. We say that \( A' (\subseteq A) \) is closed under \( \rho \), if \( A' \rho \subseteq A' \).

For each binary relations \( \alpha, \chi \) and \( \eta \) we define \( \alpha(\chi, \eta) = \{(a\chi, b\eta) : (a, b) \in \alpha \} \).

For set \( \mathcal{R} \) of relations we denote \( a \mathcal{R} = \{ ar : r \in \mathcal{R} \} \), \( A \mathcal{R} = \{ a : a \in A, r \in \mathcal{R} \} \). If \( \rho(A) \) (=\{\rho(a) : a \in A\}) is \( B \), we call \( \rho \) a surjection. If \( [\rho(x) = \rho(y) \iff x = y ] \), we call \( \rho \) injection. If \( \rho \) is surjection and injection, we say that it is bijection. If \( \rho(x) = x \) whenever \( x \in \text{Dom}(\rho) \), we say that \( \rho \) is an identity mapping (denoted \( \text{Id} \)). The element which is an object for the application of a relation is called an applicant.

For relations \( \rho \) and \( \sigma \) and set \( \mathcal{R} \) of relations we define:

- the catenation \( \rho \sigma = \{(a, c) : \exists b \in (\text{Dom}(\sigma) \cap \mathcal{R}(\rho)) (a, b) \in \rho, (b, c) \in \sigma \} \),
- the inverse \( \rho^{-1} = \{(b, a) : (a, b) \in \rho \} \),
- \( \mathcal{R}^{-1} = \{ \rho^{-1} : \rho \in \mathcal{R} \} \).

Let \( \theta \) be a binary relation in set \( A \). We say that

- \( \theta \) is reflexive, if \( (\forall a \in A) (a, a) \in \theta \),
- \( \theta \) is inversive, if \( \theta^{-1} \subseteq \theta \),
- \( \theta \) is transitive, if \( \theta \theta \subseteq \theta \),
- \( \theta \) is associative, if \( (a \theta b) \theta c = a \theta (b \theta c) \),
- \( \theta \) is an equivalence relation, if it is reflexive, inversive and transitive. If we want to emphasize the domain, say \( A \), where \( \theta \) is relation, we denote \( \theta \in \text{Eq}(A) \).

For sets \( A \) and \( B \) we define

- \( |A| = |B| \), if there is such injection \( \alpha \) that \( \alpha(A) = B \),
- \( |A| < |B| \), if there is such injection \( \alpha \) that \( \alpha(A) \subseteq B \),
- \( |A| \leq |B| \), if \( |A| = |B| \) or \( |A| < |B| \).

Set \( A \) is denumerable, if it is finite or there exists a bijection: \( \mathbb{N} \rightarrow A \); otherwise it is undenumerable.

**Definition 1.1.12.** CARTESIAN POWER. Let \( \mathcal{I} \) be a set of index elements and let \( \{ A_i : i \in \mathcal{I} \} \) be an \( \mathcal{I} \)-indexed family (an index element (shortly index) is incorporated in each element), and let \( \mathcal{B} \) be the set of all the bijections joining each set in the indexed family to exactly one element in that
set and indexing that element (an indexed element) with the index element of that set it includes to.

For any element \( a \) in \( A \), we denote \( \text{index}(a) = i \), and on the other hand for each \( i \in \mathcal{I} \),
\[ \text{elem}(i, \{ A_i : i \in \mathcal{I} \}) = A_i. \]
Family \( \{ \{ r(A_i) : i \in \mathcal{I} \} : r \in \mathcal{I} \} \) (a set of sets consisting indexed elements) is called \([\mathcal{I}]\)-Cartesian power of indexed family \( \{ A_i : i \in \mathcal{I} \} \) and we reserve the notation \( \Pi(A_i : i \in \mathcal{I}) \) for it, and the elements of it are called \([\mathcal{I}]\)-Cartesian elements on \( \{ A_i : i \in \mathcal{I} \} \). The cardinality of \( \mathcal{I} \), \(|\mathcal{I}|\), is called the Cartesian number of the elements of \([\mathcal{I}]\)-Cartesian power. If \( A = A_i \) for each \( i \in \mathcal{I} \), we denote \( A^{[\mathcal{I}]} \) for \([\mathcal{I}]\)-Cartesian power of set \( A \), the elements called \([\mathcal{I}]\)-Cartesian elements of \( A \). In the case index set \( \mathcal{I} \) is \( \mathbb{N} \), we denote \( (a_1, a_2, \ldots) \) as the element of \( |\mathcal{I}|\)-Cartesian power of indexed family \( A = \{ A_i : i \in \mathbb{N} \} \), whenever \( a_1 \in A_1, \ a_2 \in A_2, \ldots \). Any relation from \( \mathcal{I}\)-Cartesian power to a set is called a \([\mathcal{I}]\)-ary relation. For the number of Cartesian element \( \bar{a} \) we reserve the notation \( |\mathcal{I}|(\bar{a}) \).

For finite cases: \( n \in \mathbb{N} \) and sets \( A_1, A_2, \ldots, A_n \) we define \( n\)-Cartesian power
\[ A_1 \times A_2 \times \ldots \times A_n = \{ (a_1, a_2, \ldots, a_n) : a_1 \in A_1, \ a_2 \in A_2, \ldots, a_n \in A_n \}, \]
and call \((a_1, a_2, \ldots, a_n)\) an \( n\)-tuple. If \( \mathcal{I} \) is finite, we can write \( n\)-tuple \((a_1, a_2, \ldots, a_n)\), where \(|\mathcal{I}| = n\), instead of \( \{ a_i : i \in \mathcal{I} \} \) and call that tuple the \( n\)-tuple form of the \([\mathcal{I}]\)-Cartesian element. If \( n = 0 \), \( n\)-Cartesian power is \( \emptyset \). Let \( \mathcal{K} \) be a set of index elements and let \( \{ \mathcal{I}_k : k \in \mathcal{K} \} \) be a family of index sets. We denote \( \otimes(\Pi(A_i : i \in \mathcal{I}_k) : k \in \mathcal{K}) = \{ \{ r(A_i) : i \in \mathcal{I}_k, \ k \in \mathcal{K} \} : r \in \mathcal{I} \} \). For finite \( \mathcal{K} \) we can write
\[ \Pi(A_i : i \in \mathcal{I}_1) \otimes \Pi(A_i : i \in \mathcal{I}_2) \otimes \ldots \otimes \Pi(A_i : i \in \mathcal{I}_{|\mathcal{K}|}) \] instead of \( \otimes(\Pi(A_i : i \in \mathcal{I}_k) : k \in \mathcal{K}) \).

For arbitrary Cartesian element \( s = \{ a_i : i \in \mathcal{I} \} \) we agree on the undressed notation of \( s : (s) = (a_i | i \in \mathcal{I}) \) instead of notation \((\{ a_i : i \in \mathcal{I} \})\), and in finite case \( s = (a_1, a_2, \ldots, a_n) \) instead of \((a_1, a_2, \ldots, a_n)\).

For elements \( (a_i | i \in \mathcal{I}) \) and \( (b_j | j \in \mathcal{J}) \) \( (a_i | i \in \mathcal{I}) = (b_j | i \in \mathcal{I}) \), iff \( \mathcal{I} = \mathcal{J} \) and for each \( (i \in \mathcal{I}) \) \( a_i = b_j \).

We say that relations between two Cartesian powers of indexed families with the same Cartesian number preserve the indexes, if in those relations each projection of each preimage and the same projection of its image have the same index.

**Definition 1.1.13.** PROJECTION. Let \( \mathcal{I} \) and \( \mathcal{J} \) be two arbitrary sets. We call mapping
\[ e[\mathcal{I}] : (\mathcal{I}, \Pi(A_i : i \in \mathcal{I})) \rightarrow \bigcup(\Pi(A_i : i \in \mathcal{I})) \] a projection mapping (reserving that notation for it), where \((\forall j \in \mathcal{J})\) projection element \( e[\mathcal{I}](j, \bar{a}) \) (shortly denoted \( \bar{a}_j \)) is the element indexed with \( j \) in \( \bar{a} \) (belonging to \( A_j \)). We denote simply \( e \), if there is no danger of confusion. We say that a Cartesian element \( a_i \)
another Cartesian element, if and only if each projection element of the former is in the set of the projection elements of the latter and the Cartesian number of the former is less than of the latter.

**Definition 1.1.14. Catenation.** Let \((A_i; i \in \mathcal{S})\) be an indexed set. If each projection element in a \(|\mathcal{S}|\)-Cartesian element of \(\prod (A_i; i \in \mathcal{S})\) is written before or after another we will get an \(|\mathcal{S}|\)-catenation of family \((A_i; i \in \mathcal{S})\) or a catenation over \(\mathcal{S}\), and the projections of the concerning Cartesian element are called members of the catenation. We denote the set of all \(\mathcal{S}\)-catenations of family \((A_i; i \in \mathcal{S})\) by \(\operatorname{Cat}(A_i; i \in \mathcal{S})\). An associative mapping: \(\prod (A_i; i \in \mathcal{S}) \mapsto \operatorname{Cat}(A_i; i \in \mathcal{S})\) joining a \(|\mathcal{S}|\)-catenation to each \(|\mathcal{S}|\)-Cartesian element of \(\prod (A_i; i \in \mathcal{S})\) is called a catenation mapping. Notice that also \(pq\) is a catenation, if \(p\) and \(q\) are catenations, and we say that each member of \(p\) precedes the members of \(q\) and each member of \(q\) succeeds the members of \(p\); thus preceding and succeeding defining catenation order among the members of catenations. If we have a set \(A\) such that for each \(i \in \mathcal{S}\) \(A_i = A\), we speak of an \(|\mathcal{S}|\)-catenation of \(A\) and denote the set of all the \(|\mathcal{S}|\)-catenations of \(A\) by \(A^{[|\mathcal{S}|]}\). E.g. sequence \(a_1a_2\ldots a_n\), \(n \in \mathbb{N}\), \(n > 1\), is a finite catenation. For set \(H\) we define \(H^*\) (the catenation closure of \(H\)) such that \(H^* = \bigcup (H^{[|\mathcal{S}|]} : \mathcal{S} \subseteq \mathcal{S})\). Any catenation of members of catenation \(c\) is called a partial catenation of \(c\).

Such catenation \(d\) which are a catenation of partial catenations of catenation \(c\) and \(d = c\) is called a decomposition of \(c\). For our example, above, \(d_1d_2\), where \(d_1 = a_1a_2\ldots a_i\), \(d_2 = a_{i+1}a_{i+2}\ldots a_n\), is a decomposition of \(a_1a_2\ldots a_n\). Catenation operation \(\oplus\) between sets is defined:

\[ A \oplus B = \{ab: a \in A, b \in B\}. \]

If the members of a catenation closure are relations we speak of a transitive closure of the set of those relations. For set \(A\), index set \(\mathcal{S}\) and set \(\mathcal{R}\) of relations we define:

\[ A \mathcal{R}^\mathcal{S} = (A \mathcal{R}_i) \mathcal{R}^\mathcal{S}, \] whenever \(i \in \mathcal{S}, \mathcal{S} = \mathcal{S} \setminus i\) and \(\mathcal{R}_i = \mathcal{R}\).

**Definition 1.1.15.** For any symbols \(x\) and \(y\) we define replacement \(x \leftarrow y\), which means that \(x\) is replaced with substitute \(y\). Notation \(A(x \leftarrow y \mid C)\) represents an object where each \(x\) occurring in \(A\) is replaced with \(y\) with condition \(C\); and \(A(x \leftarrow \emptyset)\) is an object where \(x\) is deleted.
1.2. § Abstract Data types

1.2.1

Definition 1.2.1.1. ALPHABETS. Let us introduce three types of alphabets (sets of letters) distinct from each other: **frontier alphabet** (the letters called *variables*, often referred “terminal”), **ranked alphabet** (the letters called *operators*, “nonterminals”) Rozenberg G, Salomaa A, ed. (1997) and **arity alphabet** (of *arity letters*) divided in *in-arity* and *out-arity* alphabets, distinct from each other. The distinction for in- and out-arithies is needed in dividing the data flow direction in operations. If there is no danger of confusion symbols X and Y are reserved for frontier alphabets, symbols Σ and Ω are reserved for ranked alphabets and Ξ for the union of in-arity alphabet Ξ_{in} and out-arity alphabet Ξ_{out}. The ranked and frontier letters are called *node letters*, shortly nodes, if there is no danger of confusion, and sometimes for frontier letters synonyms leaves are used. Infinite ranks are needed as tools for infinite simultaneous data flow into systems (operations) such as in quantum physics where infinite number of different state function solutions of a Schrödinger-equation compounds a field to be operated.

Definition 1.2.1.2. OPERATORS. Let \( r \) be a mapping from the union of ranked, frontier and arity alphabets to the set of the ordinals assigning to each letter γ two ordinals, so called *ranks*, an *in-rank* \((\text{in-rank}(\gamma))\) and an *out-rank* \((\text{out-rank}(\gamma))\).

We denote \( \Sigma_{\alpha\beta} = \{ \sigma \in \Sigma : \text{in-rank}(\sigma) = \alpha , \text{out-rank}(\sigma) = \beta \} \) to symbolize the set of all \((\alpha, \beta)\)-*ary operators* in \( \Sigma \). In the special case where the in-rank of an operator of \( \Sigma \) is 0 and out-rank = 1, is called *ground letter*. The in-ranks and out-ranks of frontier letters are 1, and the in-ranks and out-ranks of arity letters are 0. In the following the letters and the operators of them are equated with each other, if there is no danger of confusion. Cf. “signatures” (Rozenberg G, Salomaa A, ed. (1997); Nivat M, Reynolds JC, ed. (1985); Ohlebusch E (2002)).
1.2.2 Nets

1.2.2.1 BASIC DEFINITIONS

Nets describe directed graphs, cf. *model theoretical aspects in consideration of formal descriptive approach for graphs* Thomas W (1997), needed e.g. in computer algorithms and describing connections between neurons in brains, and in more theoretical aspects allowing simultaneous “loops” nets are ideal as formal representations for iterations as e.g. within solutions for powers of higher order differential equations by Picard successive iterants Tirri S, Aurela AM (1989). Without net-formation (differing considerably from trees (Ohlebusch E (2002); Denecke K, Wismat SL (2002)) with only one out-arity) there is no way in a tree to get a return data from any realization of the ranked letter looped to the tree. It is also impossible to cut connection between two parts of one net leaving only subnet and deleting the other part. Also it is impossible to handle simultaneous changes in out-arity connections and furthermore infinite number of out-arities on the whole. Nets allow simultaneous algebraic structures in languages to be recognized by net rewriting automata as would happen in adding the number of saturating term algebra congruence relations by replacing terms with nets and homomorphism relations in tree automata by rewriting systems in Tirri S (1990). Furthermore the results in operation-level in realizations of nets are allowing dependences on coming up streams in carrying nets. By the semantic point of view in process algebra here described nets are concentrating to get in- and output places (filled with arity letters) to ranked letters cf. *tokens* (Best E, Devillers R, Koutny M (2001); Baeten JCM, Basten T (2001)). Some of the preliminary ideas of nets though deviating from knowledge representation, consequently in results, proofs and generalizations are in Tirri SI (2009).

**Definition 1.2.2.1.1. NETS.**

We define $\Sigma X \Xi$-net inductively as follows: Each letter in $\Sigma_0 \cup X \cup \Xi$ is a $\Sigma X \Xi$-net. The letters with in-rank 0 are called ground nets.

$$t = \sigma( r(\xi_i) ; r(\xi_j) \mid i \in \delta, j \in \gamma)$$

is a $\Sigma X \Xi$-basic net, the set of its letters $L(t) = \{\sigma\} \cup (\bigcup(L(r(\xi_k)) : k \in \delta \cup \gamma)$, whenever

(i) $\sigma \in \Sigma$, and $\delta$ and $\gamma$ are such index sets distinct from each other that $|\delta| =$ in-rank($\sigma$)
and $|g| = \text{out-rank}(\sigma)$, and $\{\xi_i : i \in \S\} \subseteq \Xi_{\text{in}}$, $\{\xi_j : j \in \S\} \subseteq \Xi_{\text{out}}$

for each $(m, n \in \S \triangle \S')$, $\xi_m = \xi_n$, iff $m = n$, and

(ii) $\S o \subseteq \S$, $\S o o \subseteq \S$, and $\S' \subseteq \S \triangle \S o$, $\S o o \subseteq \S \triangle \S o$.

(iii) for each $(k \in \S \triangle \S') \alpha_k \in \S$ and $\S k$ and $\S o k$ are such index sets distinct from each other and
from $\S$ and $\S'$ that $|\S k| = \text{in-rank}(\alpha_k)$ and $|\S o k| = \text{out-rank}(\alpha_k)$, and $\{\xi_m : m \in \S k\} \subseteq \Xi_{\text{in}}$.

$\{\xi n : n \in \S o k\} \subseteq \Xi_{\text{out}}$ are such sets of arities that for each $(m, n \in \S k \cup \S o k)$ $\xi_m = \xi_n$, iff $m = n$, and

(iv) $r$ is such a mapping that

(1.) for each $(k \in \S \triangle \S')$

$$r(\xi_k) = \xi_k,$$

$L(r(\xi_k)) = \xi_k$,

where $\xi_k \in \Xi$, and

(2.) for each $(k \in \S o \cup \S o )$

$$r(\xi_k) = \xi_k v_k,$$

$L(r(\xi_k)) = \{\xi_k, v_k\}$,

where $v_k \in X \cup \Sigma_0$, and

(3.) for each $(i \in \S \triangle (\S o \cup \S')$) $(\exists m_{i j} \in \S) \xi_{m_{i j}} \in \Xi_{\text{out}}$ and

$$r(\xi_i) = \xi_i \xi_{m_{i j}} \alpha_i(v_m : \eta_n \mid m \in \S, n \in \S),$$

$L(r(\xi_i)) = \{\xi_i\} \cup L(\alpha(v_m : \eta_n \mid m \in \S, n \in \S))$,

where $L(\alpha(v_m : \eta_n \mid m \in \S, n \in \S)) = \{\alpha_i\} \cup \{v_m, \eta_n : m \in \S, n \in \S\}$,

$\{v_m : m \in \S\} \subseteq \Xi_{\text{in}} \cup X \cup \Sigma_0$, $\{\eta_n : n \in \S\} \subseteq \Xi_{\text{out}} \cup X \cup \Sigma_0$, and

(4.) for each $(j \in \S \triangle \S o \cup \S')$ $(\exists m_{o j} \in \S) \xi_{m_{o j}} \in \Xi_{\text{in}}$ and

$$r(\xi_j) = \xi_j \xi_{m_{o j}} \alpha_j(v_m, \eta_n \mid m \in \S, n \in \S),$$

$L(r(\xi_j)) = \{\xi_j\} \cup L(\alpha_j(v_m : \eta_n \mid m \in \S, n \in \S) )$,

where $L(\alpha_j(v_m : \eta_n \mid m \in \S, n \in \S)) = \{\alpha_j\} \cup \{v_m, \eta_n : m \in \S, n \in \S\}$,

$\{v_m : m \in \S\} \subseteq \Xi_{\text{in}} \cup X \cup \Sigma_0$, $\{\eta_n : n \in \S\} \subseteq \Xi_{\text{out}} \cup X \cup \Sigma_0$. 
We say that for each \((k \in \mathcal{A})\) \(r(\xi_k)\) occupies arity \(\xi_k\) of \(\sigma(v_i; \eta_j \mid i \in \mathcal{A}, j \in \mathcal{B})\), where \(\{v_i, \eta_j ; i \in \mathcal{A}, j \in \mathcal{B}\} \subseteq \Xi \cup X \cup \Sigma_0\), if \(\xi_k \in L(\sigma(v_i; \eta_j \mid i \in \mathcal{A}, j \in \mathcal{B}))\) and \(r(\xi_k) \in X \cup \Sigma_0\). Furthermore we say that for each \((i \in \mathcal{A} \setminus (\mathcal{B} \cup \mathcal{C}))\) out-arity \(\xi_{n_i}\) of \(\alpha(v_m; \eta_n \mid m \in \mathcal{B}, n \in \mathcal{A})\) and in-arity \(\xi_i\) of \(\sigma(\xi_i, r(\xi_h)) \ni h \in \mathcal{B}, h \neq i, k \in \mathcal{C}\) occupy each other in \(t\), and for each \((j \in \mathcal{C} \setminus (\mathcal{B} \cup \mathcal{A}))\) out-arity \(\xi_j\) of \(\sigma(r(\xi_h); \xi_j) \ni h \in \mathcal{B}, k \in \mathcal{C}, k \neq j\) and in-arity \(\xi_{m_j}\) of \(\alpha_j(\xi_{m_j}, \xi_{n_j} \mid m_j \in \mathcal{B}, n_j \in \mathcal{A})\) occupy each other in \(t\). Furthermore using definitions for symbols defined above for \(t\) we define:

\[
s = \sigma(\{r(\xi_p), \xi_{n_s} s_t ; r(\xi_{q_i}), \xi_{n_s} s_j t_j \mid p \in \mathcal{B} \cup \mathcal{C}, q \in \mathcal{C} \setminus \mathcal{A}, i \in \mathcal{A} \setminus (\mathcal{B} \cup \mathcal{C}), j \in \mathcal{C} \setminus (\mathcal{B} \cup \mathcal{A})\})
\]

is a net, and

\[
L(s) = \{\sigma(\xi_p, \xi_{n_s} s_t ; r(\xi_{q_i}), \xi_{n_s} s_j t_j) \mid \sigma \in X \cup \Sigma_0\} \cup (U(L(r(\xi_h)); k \in \mathcal{B} \cup \mathcal{C} \setminus \mathcal{A})) \cup (U(L(s)); i \in \mathcal{A} \setminus (\mathcal{B} \cup \mathcal{C})) \cup (U(L(t)); j \in \mathcal{C} \setminus (\mathcal{B} \cup \mathcal{A}))
\]

is the set of the letters in \(s\),

whenever for each \((i \in \mathcal{A} \setminus (\mathcal{B} \cup \mathcal{C}), j \in \mathcal{C} \setminus (\mathcal{B} \cup \mathcal{A}))\)

(1.) \(s_i\) and \(t_j\) are nets outside \(\Xi \cup X \cup \Sigma_0\), and

(2.) \(\{\xi_{n_s} \mid n \in \mathcal{A} \setminus (\mathcal{B} \cup \mathcal{C})\} \subseteq \Xi_{out}\) and \(\{\xi_{m_j} \mid m \in \mathcal{C} \setminus \mathcal{A}\} \subseteq \Xi_{in}\) are such sets of arities that

for each \((m, n \in \mathcal{A} \setminus (\mathcal{B} \cup \mathcal{C}))\) \(\xi_m = \xi_n\), iff \(m = n\), and

(3.) there is exactly one such index \(n \in \mathcal{A} \setminus (\mathcal{B} \cup \mathcal{C})\) that \(\xi_{n_s}\) is an unoccupied out-arity letter in \(s_i\), and

there is exactly one such index \(m_j \in \mathcal{C} \setminus \mathcal{A}\) that \(\xi_{m_j}\) is an unoccupied in-arity letter in \(t_j\).

We call \(\sigma\) the **root** of \(s\), \(\text{root}(s)\). Net \(\sigma(\xi_i; \xi_j \mid i \in \mathcal{A}, j \in \mathcal{B})\) is called the **ranked net**.

For each net \(u\) we define and reserve for that purpose such **rank index sets** \(\mathcal{A}_{\pi}, \mathcal{B}_{\pi}, \mathcal{C}_{\pi} \ni (\sigma \in L(u) \cap (\Sigma \setminus \Sigma_0))\), distinct from each other \(|\mathcal{A}_{\pi}| = \text{in-rank}(\sigma)\) and \(|\mathcal{B}_{\pi}| = \text{out-rank}(\sigma)\) and the **in-rank index set of net** \(u_{\pi} = U(\mathcal{A}_{\pi} : (\sigma \in L(q) \cap (\Sigma \setminus \Sigma_0))]\) and the **out-rank index set of net** \(u_{\pi} = U(\mathcal{B}_{\pi} : (\sigma \in L(q) \cap (\Sigma \setminus \Sigma_0))]\).

UNO(u) is a notation for the set of the unoccupied arity letters of \(u\) and \(\text{Occ}(u)\) is reserved for the set of all occupied arity letters of \(u\). \(\text{Occ}(\mathcal{A}, t)\) means the set of all those elements in set \(\mathcal{A}\), which are occupied in net \(t\), and \(\text{UNO}(\mathcal{A}, t)\) are reserved for the set of all those which are unoccupied in net \(t\). The index elements in the in-rank index set of net \(u\) for unoccupied arities in \(u\) are called **unoccupied in-arity index elements**, and the index elements in the out-rank index set of net \(u\) for unoccupied arities in \(u\) are called **unoccupied out-arity index elements**. The set of all unoccupied in-arity
The set of the in

are denoted

\( F \). Symbols \( \$ \) and \( \$ \) are reserved for the sets of occupied elements, respectively.

The set of all \( \Sigma \Xi \)-nets is denoted \( F_2(X, \Xi) \). We also denote \( F_\Xi(x)(X) = F_2(X, \Xi) \setminus \Xi \),

\( F_\Xi(x)(\Xi) = F_2(X, \Xi) \setminus X \) and \( F_{\Xi\Xi}(X) = F_2(X, \Xi) \setminus (X \cup \Xi) \) and \( F_{\Xi\Xi}(X) = F_2(X, \Xi) \setminus (X \cup \Xi) \).

**Definition 1.2.2.1.2. TIES and SUBNET.**

Now when we have reached the definition and the sense of unoccupied arities we are ready to give a formulation for nets in accordance with previous given, more convenient later when we are handling substitutions and rewriting. First we introduce tied sets of tied terms (in tied and out tied) of nets

\[
\tilde{F}_{in}(X, \Xi) = \Xi_{in} \cup \{ \xi v : \xi \in \Xi_{in}, v \in X \cup \Sigma_0 \} \cup \{ \xi_1 \xi_2 u : \xi_1 \in \Xi_{in}, \xi_2 \in \Xi_{out}, \xi_2 \in \text{Uno}(u), u \in F_{\Xi\Xi}(\Xi) \} \quad \text{and}
\]

\[
\tilde{F}_{out}(X, \Xi) = \Xi_{out} \cup \{ \xi v : \xi \in \Xi_{out}, v \in X \cup \Sigma_0 \} \cup \{ \xi_1 \xi_2 u : \xi_1 \in \Xi_{out}, \xi_2 \in \Xi_{in}, \xi_2 \in \text{Uno}(u), u \in F_{\Xi\Xi}(\Xi) \}.
\]

We denote \( \tilde{F}_2(X, \Xi) = \tilde{F}_{in}(X, \Xi) \cup \tilde{F}_{out}(X, \Xi) \).

The first set and its elements in the union \( \tilde{F}_{in}(X, \Xi) \) and \( \tilde{F}_{out}(X, \Xi) \) respectively are called 0-tied and the second set and the elements in it are 1-tied, and finally the third set and its elements are 2-tied. For each tied element \( s \) we denote its \( k \)-th member \( s^{(k)} \), \( k = 1, 2, 3 \). The last member of any tied term is called tied net denoted for tied term \( s \) by \( s_l \), and the preceding members of the tied net are tie-arities of \( s_l \), the last arity is a genuine tie-arity of \( s_l \). For 2-tied term \( s \) pair \( (s^{(1)}, s^{(2)}) \) is a 2-tie of \( s_l \); an in-2-tie, if \( s \in F_{in}(X, \Xi) \), and an out-2-tie, respectively, if \( s \in F_{out}(X, \Xi) \). For 1-tied term its first member is the 1-tie of its last member. 0-tied term is its 0-tie itself. For 2-tied term \( s \), \( s^{(1)} \) is the 1-tie (in-1-tie, if \( s^{(1)} \) is an in-arity and out-1-tie, if \( s^{(1)} \) is an out-arity) of \( s_l \). We use names in-tied and in-ties and out-tied and out-ties respectively depending on which one of sets \( F_{in}(X, \Xi) \) and

\( \tilde{F}_{out}(X, \Xi) \) those tied elements belong. The set of the in-k-tied and out-k-tied elements (\( k = 0, 1, 2 \)) are denoted \( F_{in}(X, \Xi)^{(k)} \) and \( F_{out}(X, \Xi)^{(k)} \) respectively, and the union of those sets by \( F_{2}(X, \Xi)^{(k)} \).

The set of the in-ties in net \( s \) is denoted \( IT(s) \), and \( OT(s) \) for the out-ties, respectively.
For any net $s \in \Xi \cup \Sigma_0$, $\xi_s$ is named as an in-gluing form of $s$, where $\xi \in \Xi \cap L(s)$, and if $\xi \in \Xi \cup \Sigma_0$, $\xi_s$ is entitled an out-gluing form of $s$. The set of all in-gluing forms of $s$ is the in-gluing form of $s$, denoted $s_{\mathrm{ing}}$ and the set of all out-gluing forms of $s$ is the out-gluing form of $s$, denoted $s_{\mathrm{outg}}$. The union $s_{\mathrm{glue}} = s_{\mathrm{ing}} \cup s_{\mathrm{outg}}$ is called the gluing form of $s$, and $s$ is denoted $s_{\mathrm{glue}}$. The gluing form of each letter in $X \cup \Sigma_0$ is the letter itself. The arities have no ranks and therefore either no gluing forms.

We define $F_{\mathrm{outg}}(X, \Xi) = \{s_{\mathrm{ing}} : s \in F_{\mathrm{outg}}(X, \Xi)\}$, $F_{\mathrm{in}}(X, \Xi) = \{s_{\mathrm{ing}} : s \in F_{\mathrm{in}}(X, \Xi)\}$ and $F_{\Sigma}(X, \Xi) = F_{\mathrm{in}}(X, \Xi) \cup F_{\mathrm{outg}}(X, \Xi)$. For each $s \in F_{\Sigma}(X, \Xi)$ we denote $NG(s) = \{s_{\mathrm{ing}}, s_{\mathrm{outg}}\}$. The set of all in-ties in net $t$ is denoted $IT(t)$, and $OT(t)$ for the out-ties, respectively. In-ties and out-ties correspond in-coming and respectively out-coming labeled edges and node letters correspond labeled nodes Engelfriet J (1997).

We define for each $s \in \{\sigma\} \cup F_{\Sigma_0}$

$$t = s(\mu_i ; \lambda_j \mid i \in \mathcal{S}_s^{in}, j \in \mathcal{S}_s^{out}, C)$$

is a net, where for each $(i \in \mathcal{S}_s^{in}, j \in \mathcal{S}_s^{out})$ $\mu_i \in F_{\mathrm{in}}(X, \Xi)$, $\lambda_j \in F_{\mathrm{outg}}(X, \Xi)$, $\mu_i$ is replacing in-arity letter $\xi_i$ in $s$ and $\lambda_j$ is replacing out-arity letter $\xi_j$ in $s$, and $\mu_i^{(1)} = \xi_i$, $\lambda_j^{(1)} = \xi_j$, and $C$ is a sample of conditions to be fulfilled (normally assumed to be known) or equivalently

$$s(\mu_i ; \lambda_j \mid C)$$

is a net, whenever $\mu \in F_{\mathrm{in}}(X, \Xi)^{[\mathcal{S}_s^{in}]}$, and $\lambda \in F_{\mathrm{outg}}(X, \Xi)^{[\mathcal{S}_s^{out}]}$, where the first members in each projection elements of $\mu$ and $\lambda$ are in $\mathrm{Uno}(s)$. Nets $\mu_{\mathrm{in}}$, $i \in \mathcal{S}_s^{in}$, are called down-subnets of $t$, respectively $\lambda_{\mathrm{out}}$, $j \in \mathcal{S}_s^{out}$, are called up-subnets of $t$, and for each $(q \in \mathrm{sub}(t)) \mathrm{sub}(q) \subseteq \mathrm{sub}(t)$, where the set of all subnets of net $t$ is denoted $\mathrm{sub}(t)$. Cartesian elements, the projections being nets are Cartesian nets. Nets where the out-ranks of the nodes are 1, are trees, and trees where the in-ranks of the nodes are 1 are chains. We call sets of trees forests. A set of nets is called jungle, and for
jungle \( T \) we agree about \( \text{sub}(T) = \bigcup \{ \text{sub}(t) : t \in T \} \). What is said for nets is in the following generalized for jungles as relations from elements to sets of elements and is denoted respectively. E.g. for jungle \( T \) we denote \( \text{sub}(T) = \bigcup \{ \text{sub}(t) : t \in T \} \) and \( L(T) = \{ L(t) : t \in T \} \). We say that a net is finite, if the cardinalities of the frontier and ranked letters in the net are finite.

Trees can have only nodes with one out-tie at most. The main difference between nets and trees can be demonstrated with the following net containing downstream subnet \( s \) incorporating a node with more than one out-tie:

\[
q(\sigma,\lambda) | i \in \text{sub}_q, j \in \text{sub}_q, (\forall i \in \kappa) s_{\lambda i} = s, |\{s_i : i \in \kappa\}| > 1,
\]

where \( \kappa \subseteq \text{sub}_q, \rho(t,s) \) is defined later in 1.2.2.1.4,

**Definition 1.2.2.1.3.** Links and Net Classes.

Subnets of nets being frontier letters are called leaves of the net, and the set of all leaves in \( v \) is denoted by \( \text{Leav}(v) \). For net \( v \) we denote \( \text{fron}(v) \) as the set of the frontier letters of \( v \), and \( \text{rank}(v) \) is the set of all ranked letters in \( v \).

For \( t = \sigma(\mu_i ; \lambda_j | i \in \text{sub}_\sigma, j \in \text{sub}_\sigma, C) \) net \( \mu_{\lambda i} \) is said to be (next)out-linked to \( s \) by out-tie of \( \mu_{\lambda i} \), called out-arity linkage of \( \mu_{\lambda i} \), respectively \( s \) is said to be (next)in-linked to \( \mu_{\lambda i} \) by in-tie of \( s \), called in-arity linkage of \( s \). An in- and out-linkage of the same node are said to be successive to each other. The linkages between the same two nodes are parallel with each other. If net \( u \) is out-/in-linked to net \( q \) and \( q \) is linked to net \( v \), we say that \( u \) is (successively) out-/in-linked to \( v \). The nets which are not linked to each other are disjoined with each other.

A linkage (comprising of consecutive in- and out-arity linkages) which connects two nodes in a net is an inward linkage connection of the net; the linkages which are not inward connections are outward linkage connections. If a net has no outward linkage connections, it is said to be closed.

Net \( t = \sigma(\mu_i ; \lambda_j | i \in \text{sub}_\sigma, j \in \text{sub}_\sigma, C) \) is called \( \sigma \)-root revealing net, where \( \sigma \in \Sigma \). Linkages in nets can also be defined by using wider parts of nets: for each \( i \in \text{sub}_\sigma \) triple \( (\sigma, \text{root}(\mu_{\lambda i}), \mu_{\lambda i}^{(1)}, \mu_{\lambda i}^{(2)}) \) constitutes node linkage of \( t \), and \( (\text{root}(\mu_{\lambda i}), \sigma, \mu_{\lambda i}^{(1)}, \mu_{\lambda i}^{(2)}) \) is its inverse; respectively for each \( j \in \text{sub}_\sigma \) \( (\sigma, \text{root}(\lambda_{\mu j}), \lambda_{\mu j}^{(1)}, \lambda_{\mu j}^{(2)}) \) is node linkage of \( t \), and \( (\text{root}(\lambda_{\mu j}), \sigma, \lambda_{\mu j}^{(1)}, \lambda_{\mu j}^{(2)}) \) is its inverse. The set of the node linkages of \( t \) we denote \( NL(t) \) and we use notation \( NL^{-1}(t) \) for the set of the inverses of elements in \( NL(t) \). Because inverses exist in the up-subnets of root revealing nets, it is natural that
“writing directions” of the letters in linkages in nets should not determine those nets. Therefore we will give a sensible definition for the identity of nets:

For nets p and q we define \( p = q \), if \( (\forall s \in \text{NL}(p)) s \in \text{NL}(q) \cap \text{NL}^{-1}(q) \) and \( (\forall s \in \text{NL}(q)) s \in \text{NL}(p) \cap \text{NL}^{-1}(p) \). Actually each net defines a class of nets equal with it and for net t we denote that set with \([t]\), its elements entitled \( t\text{-class representatives} \). If there is no danger of confusion, we suppose the appropriate representative to be given.

**Definition 1.2.2.1.4.** POSITION. Next we define locations, positions, of nets in nets using arity letters. Let \( q = \sigma(s_i| t_j \mid i \in S_{\alpha}^{\alpha^*}, j \in S_{\beta}^{\beta^*}) \) be a net. We say that the out-tie of net \( s_iL \) in q is a position of \( s_iL \) in q and \( s_i^{(1)} \) is the position of \( s_iL\text{outg} \) in q, the sets of the described positions are denoted \( p(q,s_iL) \) (\( \subseteq \Pi(S_{\alpha}) \)), \( p(q,s_i\text{outg}) \) (\( \subseteq \Xi_{\text{out}} \cap \text{L}(s_i\text{outg}) \)), respectively, and \( s_iL \) and \( s_i\text{outg} \) are next below q or next down positioned in q, \( i \in S_{\alpha}^{\alpha^*} \), and an in-tie of \( t_jL \) in q is a position of \( t_jL \) in q and \( t_j^{(1)} \) is the position of \( t_j\text{outg} \) in q, the corresponding sets denoted \( p(q,t_jL) \) and \( p(q,t_j\text{outg}) \), and next above q or next up positioned in q, \( j \in S_{\beta}^{\beta^*} \). Furthermore generally for arbitrary nets u and v we define inductively positions as catenations \( p(u,v) = p(u,s)p(s,v) \) and \( p(u,r) = p(u,s)p(s,r) \), whenever \( s \in \text{sub}(u), r \in v_{\text{glue}} \) and \( v \in \text{sub}(s) \), next positioned in s. We also say that v and r are positioned in u. If c is next above h and h is next above u, we define that c is above u. Below is defined analogously. The same terminology is a practice also for the positions of the corresponding nets. Next below/next above is denoted shortly by \( \ll \) , and below/above is denoted by \( < / > \). Let \( P_1 \) and \( P_2 \) be two arbitrary sets of positions. We define and denote that \( P_1 \ll P_2 \), if \( P_1 \) and \( P_2 \) are distinct with each other and \( \forall p_1 \in P_1 \ \exists p_2 \in P_2 \) such that \( p_1 < p_2 \), and \( P_1 \ll P_2 \), if \( \forall p_1 \in P_1 \ p_1 < p_2 \) whenever \( p_2 \in P_2 \).

The set of all positioned elements in t is denoted \( p(t) \). For sets \( T \) and \( S \) of nets or gluing forms we denote \( p(T,S) = \cup(p(t,s) : t \in T, s \in S) \), and \( p(T) = \cup(p(t) : t \in T) \). Furthermore due to the importance of the unoccupied character in nets we take for use notation Unop(t) for the set of the positions of the unoccupied arities in t, and generalize the notation as usual for jungle, say \( T, \) \( \text{Unop}(T) = \cup(\text{Unop}(t) : t \in T) \). Furthermore for jungle \( T \) we denote the cardinality of \( \text{Unop}(T) \) by \( \delta_D(T) \). Cf. “marked letters” Ohlebusch E (2002).
For net v, vlp (an occurrence), is denoted to be the subnet of v having or “topped at” position p in v. A down-/up-frontier net of net v, down-/up-frontnet(v), is such a subnet of v, whose occurrence is next below/next above v (at so called down-/up-frontier position of v). We denote Frd(v) meaning the set of all down-frontier nets of v, and Fru(v) is the set of all up-frontier nets of v, and Fr(v) means the set of all frontier nets of v.

We define the height of net t, hg(t), by the following induction:

1° hg(t) = 0, if t ∈ Ξ ∪ X ∪ Σ

2° hg(t) = 1+max{hg(s): s ∈ Fr(t)}, if t ∈ Fr(X,Ξ) \ (Ξ ∪ X ∪ Σ0).

For arbitrary net t, there is in force equation |[t]| = |{p(t,σ) : σ ∈ L(t) ∩ Σ}|.

Notice that for any net t and its subnet s outside X ∪ Σ0, the positions of s and its gluing form in t are different and that the position of s is unequivocal, but its gluing form can be rearrange in many ways to the context of t next to it (thus forming new nets), depending on which arities of the gluing form is chosen to occupy arities of the context. Trees (owing only one out-arity) have naturally no such difference between the positions of nets and the gluing form of them. We will come to this matter of rearrangement more profoundly later in the chapter of rewriting.

Definition 1.2.2.1.5. ENCLOSUREMENTS. Let t = s(μ;λ) | i ∈ δ^s_UN, j ∈ δ^δ^s_UN) be a net. We call out-gluing forms μ_{pLong}, i ∈ δ^s_UN, s-downstream elements of t for s (s being the upcontext of t for those elements, or for the set of them) and in-gluing forms λ_{pLong}, j ∈ δ^δ^s_UN, s-upstream elements of t for s (s being the down context of t for those elements). If we want to emphasize that net v is the context of net u only for the frontier letters in u, we say that v is the apex of u, apex(u), for those letters; accordingly down and up apex, respectively. We agree of notation apex(T) = {apex(t): t ∈ T}, whenever T is a jungle.

Net s can also be expressed with notation conP(t), where

P = {ρ(t,μ_{pLong}), ρ(t,λ_{pLong}) : i ∈ δ^s_UN, j ∈ δ^δ^s_UN}.

Notice that context conP(t) is the apex of t, if P = {ρ(t,x) : x ∈ X ∩ L(s)}.

The sets of the ties of μ_{pLong} and λ_{pLong} to s, i ∈ δ^s_UN, j ∈ δ^δ^s_UN, are matching arity linkage sets of s to t and the family of all of them is denoted MAL(t,s). We also call s the abover of μ_{pLong} in t, denoted t \ b{μ_{pLong}, i ∈ δ^s_UN}, and each μ_{pLong}, i ∈ δ^s_UN is a belower of s in t, the set of the belowers of context s in t is denoted t \ a{s}.
We say that net $s$ is linked outside to net $t$, if $s$ is linked to $t$ and the set of the arity linkages of $s$ to $t$ differs from set $\text{MAL}(t,s)$.

If $u$ is a subnet of net $v$, we say that $v$ can be divided in two nets: $u$ and the abover of $u$ in $v$. The contexts of the subnets of $t$ are the enclosements of $t$ (we say they are in $t$ or $t$ is embedding them), and the set of all enclosements of $t$ is denoted $\text{enc}(t)$. For jungle $T$ we denote $\text{enc}(T) = \bigcup(\text{enc}(t) : t \in T)$. We say that a net is finite, if the cardinalities of the frontier and ranked letters in the net are finite. The elements in $\text{enc}(t) \backslash t$ are entitled genuine enclosements of $t$ and the set of them is denoted $\text{enc}_g(t)$; for jungle $T$ we have $\text{enc}_g(T) = \bigcup(\text{enc}_g(t) : t \in T)$.

For further need it is worth to notice that because for any net $t$ $\text{NL}(t) = \bigcup(\text{NL}(s) : s \in \text{enc}(t))$, we can write $[\text{enc}(t)] = \text{enc}([t])$.

**Definition 1.2.2.1.6. OVERLAPPING and OMISSION.**

Let $p$ and $q$ be arbitrary nets. If there is such net $t$, each enclosement of which is in both $[p]$ and $[q]$ and has a linkage connection to each other, we say that $p$ and $q$ overlap each other, and $t$ is said to be shared among $p$ and $q$. If $E_{pq}$ is the denotation for the set of all shared nets among $p$ and $q$, the overlapping net of $p$ and $q$ denoted $p \uplus q$ is such a net in $E_{pq}$ (the most “extensive”) that $(\forall k \in E_{pq}) k \in \text{enc}([p \uplus q])$. For jungles $P$ and $Q$ we define $P \uplus Q$ is such a net in $E_{pq} (= \bigcap(E_{pq} : p \in P, q \in Q))$ that $k \in \text{enc}([p \uplus q])$ whenever $k \in E_{pq}$, and $\bigcap Q = Q \uplus Q$. Nets are said to be distinctive from each other, if they do not overlap each other. A jungle is distinctive if all of its nets are distinctive from each other, and furthermore a relation over a distinctive jungle domain is entitled a distinctive relation.

For an arbitrary nets $s$ and $t$ the set of the positions of the outside arities of $t$ in $s$, $(\text{Unop}(t,s))$, means the set of the positions of all those arities of the elements in $L(t \uplus s)$ which are not occupied by anything in net $s$.

Let $s$ and $t$ be two arbitrary nets. Let $s^o$ be the context of such a representative of $[s]$ that the context is for the gluing form of a representative of $[s \uplus t]$, and respectively let $t^o$ be the context of such a representative of $[t]$ that the context is for the gluing form of a representative of $[s \uplus t]$. Jungle
$\{s^0, t^0\}$ is called the omission of $s$ by $t$ or $s$ omitted by $t$, denoted $s \triangleleft t$. Notice that an omission may be broken (cf. “broken jungle” defined later). For arbitrary net $s$ and jungle $S$ we denote $s \triangleleft T = \cap (s \triangleleft t : t \in T)$ and for jungles $S$ and $T$ we use notation $S \sim T = \{s \triangleleft T : s \in S\}$.

Net, say $k$, possessing nets $s$ and $t$ as subnets and for which $k \triangleleft t = s \triangleleft t$ and $k \triangleleft s = t \triangleleft s$ is the assimilation of $s$ and $t$ and we denote $s \sim t$.

### 1.2.2.2 Characteristics of Nets

Here we represent some features typical to nets and the relations between them.

**Definition 1.2.2.2.1. NEIGHBOURING, ISOLATION and BORDER.**

If nets do not overlap each other, but are linked to each other, we say they are neighbouring each other. A set of the neighbouring nets of a net is called a touching surrounding of the net. Nets are said to be isolated from each other, if there is a net neighboured by them. We say that nets being neighboured by each other are linked directly, and nets being isolated from each other are linked via isolation.

If nets are neighbouring each other such that they are not isolated from each other, we say they are closely neighbouring each other.

If nets are isolated from each other, but are not neighbouring each other, we say they are totally isolated from each other.

Net $s$ is $t$-isolated, if the nodes of $t$ are totally isolated from each other by the nodes of $s$, and inversely.

The set of the linkages connecting two nets to each other is called the border between those nets. The border may be empty, too. The union of the set of the borders between a net and all other neighbouring nets is called simply the border of the net.

**Definition 1.2.2.2.2. THE RIM and BROKEN JUNGLE.**

The nets of a jungle which are in-linked inside the jungle, but not out-linked, are out-end nets and at out-end positions in the jungle, and the nets out-linked inside a jungle, but not in-linked, are in-end
nets and at in-end positions in the jungle. The union of the in-end nets and the out-end nets in a jungle is the rim of the jungle.

We call a jungle broken, if each of its nets is disjoined from each other; otherwise it is unbroken. Notice that unbroken jungles are actually nets. Broken jungles, each net having only one letter outside the arity alphabet, are totally broken. E.g. any set, the elements of which are nodes, can be seen as a totally broken jungle and is called degenerated. Because of the close relationship between nets and jungles we often denote jungle by small letter instead of the normal procedure for sets. Comparative study in the form of dependence can be found in Diekert V, Métivier Y (1997).

Notice that even if a net itself is unbroken, an enclosement of it may be a broken jungle.

**Definition 1.2.2.2.3. ROUTES and LOOPS.**

A denumerable route (DR) between nets is defined as follows:

1° any linkage between two nets is a route between those nets, and
2° if P is a DR between net s and t, and Q is a DR between t and net u, then PQ is a DR between s and u.

DR can also be seen as an inversive and transitive relation in the set of the nets, if “linkage” is interpreted as a binary relation in the set of the nets. Any route can also be denoted by the catenation of the nets linked with each other in the route. Cf. paths and cycles Müller J (1997).

We define an in-/out-one-way DR (in-/out-OWR) between nets as transitive relation (“linkage” is a binary relation) among the set of the nets as follows:

1° any linkage which is an in-/out-linkage of net s and on the other hand an out-/in-linkage of net t is an in-/out-OWR from s to t, and
2° if P is an in-/out-OWR from net s to net t, and Q is an in-/out-OWR from t to net u, then PQ is an in-/out-OWR from s to u, and we say that s in-/out-dominates u and u out-/in-dominates s.

Nets s and t are A- or |A|-routed with each other, if A is the set of routes between them. Cf. Trace semantic (van Glabbeek RJ (2001); Aceto L, Fokkink WJ, Verhoef C (2001)).

Triple (N, R, f), where N is a jungle, R is a set of OWR’s and f is a mapping connecting the elements of R to pairs of nets, describes graph (Rozenberg G, Salomaa, A ed. (1997); Müller J (1997)).
If there is no need to distinguish in- and out-arities from each other we write nets simply compounding indexes for in- and out-arities to be one index and interpret out-arities as in-arities.

An DR from a net to itself is a loop of the net, and outside loop, if furthermore in the route there is a linkage to outside the net; otherwise it is an inside loop of the net. The loop where each linkage is among the linkages of the same jungle, is an inside loop of the jungle. OWR’s which are loops (OWR-loops) are called directed loops. A bush is a jungle which has no inside loops and elementary, if it has no parallel linkages between its nets.

The following is an example of equal nets, containing an inside directed loop; \( \sigma, \rho, \lambda \) and \( \mu \) are nets, not including to arities (\( \xi \) stands for in-arity, \( \bar{\xi} \) for out-arity) nor frontier letters:

\[
\begin{align*}
    s &= t(\xi_1 \xi_2 \nu(\xi_1 \xi_2 \xi_1, \xi_2, \xi_1 q ; \xi_1 \xi_2, \lambda); \xi_1 \xi_2 \xi_1 u(\xi_1 \xi_2 \xi_1, \xi_2, \xi_1 s ; \xi_1 \xi_2, \xi_2, \xi_1 t, \lambda)) \\
    q &= u(\xi_1 \xi_2 \xi_1 \rho, \xi_2 \xi_1 s ; \xi_1 \xi_2 \xi_1 \sigma, \xi_2 \xi_2 \xi_2, \xi_2) \\
    r &= v(\xi_1 \xi_2 \xi_1, \xi_2, \xi_2 \xi_1, \xi_2,\xi_1 s, \xi_2 \xi_2 \xi_2, \xi_2) \\

    \text{enc}_e(s) &= \text{enc}\{u, \rho, \sigma, r, q, \mu, \lambda, t, v\} \\
    \text{enc}_e(q) &= \text{enc}\{u, \rho, \sigma, r, s\} \\
    \text{enc}_e(r) &= \text{enc}\{\mu, \lambda, v, s, q\}
\end{align*}
\]

This yields \( s, q \) and \( r \) are enclosures of each other and we have \( s = q = r \).

1.2.3 Realizations, Algebras and Homomorphisms

In this paragraph we introduce the notions of nets in semantic point of view referring to algebras overall. We represent generalization for more common \( \Sigma \)-algebra definition (Aceto L, Fokkink WJ, Verhoef C (2001); Burris S, Sankappanavar HP (1981)) – concerning nets. Net rewriting, represented closely later, can be guided by realizations of nets which realizations can be understood also to correspond on temporal logic and models Gabbay DM, Hogger CJ, Robinson
JA (1995). Due to the generic essentiality relations between algebras in the respect of free generation and morphisms are briefly taken into the consideration.

**Definition 1.2.3.1.** OPERATIONS and $\Sigma X \Xi$-ALGEBRA. The represented definition for nets allows upstream subnets of nets to influence in producing the images of realizations of the roots of the nets. An (upstream related or look-ahead) $\Sigma_{\alpha} X_{\alpha} \Xi_{\alpha}$-algebra $\mathfrak{A}$, is a pair consisting of a set $A$ (including $\Xi_{\alpha_{in}}$), and a mapping, an operation assigning mapping, that assigns to each operator of $\Sigma_{\alpha} \cup X_{\alpha} \cup \Xi_{\alpha}$ ($\Sigma_{\alpha} \subseteq \Sigma$, $X_{\alpha} \subseteq X$, $\Xi_{\alpha} \subseteq \Xi$) $\mathfrak{A}$-operation, to $(\alpha, \beta)$-ary operator $\sigma \in \Sigma_{\alpha}$ a relation, $(\alpha, \beta)$-ary operation $\sigma^{\beta} : A^{\alpha} \otimes \tilde{F}_{outz}(X, \Xi)^{\beta} \rightarrow A$, where $\alpha = \text{in-rank}(\sigma)$ and $\beta = \text{out-rank}(\sigma)$, and to each letter $x \in X_{\alpha}$ operation $x^{\sigma}$ for which $x^{\sigma}(a, t) = a$, whenever $a \in A$.

$t \in \tilde{F}_{outz}(X, \Xi)$, and to each arity letter $\xi \in \Xi_{\alpha_{in}}$ constant operation $\xi^{\sigma}$ in $\Xi_{\alpha_{in}}$. The operations of the ground letters are defined by the constant images in $A$. For simplicity we write $\mathfrak{A} = (A, \Sigma_{\alpha} X_{\alpha} \Xi_{\alpha})$ and assume the operation assigning mapping to be known. We say that $\mathfrak{B} = (B, \Sigma_{\alpha} X_{\alpha} \Xi_{\alpha})$, where $B$ (including $\Xi_{\alpha_{in}}$) is a subset of $A$, is a subalgebra of $\mathfrak{A}$, if $\sigma^{\beta} = \sigma^{\beta}|B^{\alpha} \otimes \tilde{F}_{outz}(X, \Xi)^{\beta}$, and the image set of $\sigma^{\beta}$ is in $B$, whenever $\sigma \in \Sigma_{\alpha}$, and $x^{\beta} = x^{\beta}|B$ and $\xi^{\beta} = \xi^{\beta}|B$. Set $B$ is called a closed subset of $A$. Sub($\mathfrak{A}$) symbolizes the set of all subalgebras of $\mathfrak{A}$. It is worth to mention that up-subnets in realizations of root revealing nets are important as e.g. in TD (defined later), where images of operations may in that way be selected to direct to desired in-arities. For each $\Sigma_{\alpha} X_{\alpha} \Xi_{\alpha}$-algebra $\mathfrak{A} = (A, \Sigma_{\alpha} \cup X_{\alpha} \cup \Xi_{\alpha})$ we define the power algebra of $\mathfrak{A}$, $P(\mathfrak{A}) = (P(A), P(\Sigma_{\alpha}) \cup X_{\alpha \cup \Xi_{\alpha}})$, where for each of its element $A^{\alpha}$ and operation $\sigma^{P(\alpha)}$, $A^{\alpha} \sigma^{P(\alpha)} = \{a \sigma^{\alpha} : a \in A^{\alpha}\}$.

**Definition 1.2.3.2.** $\Sigma X \Xi$-NET ALGEBRA. Algebra $\mathcal{B}_{\Sigma}(X) = (F_{\Sigma}(X, \Xi), \Sigma X \Xi)$ defined so that for each operator $\sigma \in \Sigma$ and $\Sigma X \Xi$-nets $s_{\alpha}$, $i \in S_{\alpha}$, and $\lambda_{j} \in \tilde{F}_{outz}(X, \Xi)$, $j \in \gamma_{\alpha}$

$$\sigma_{\xi}(s_{\alpha}, \lambda_{j} | i \in S_{\alpha}, j \in \gamma_{\alpha}) = \sigma_{\mu}(s_{\alpha}, \lambda_{j} | i \in S_{\alpha}, j \in \gamma_{\alpha})$$

if $\sigma \notin \Sigma_{0}$, and

$$\sigma_{\xi}(s_{\alpha}, \lambda_{j} | i \in S_{\alpha}, j \in \gamma_{\alpha}) = \sigma$$, if $\sigma \in \Sigma_{0}$. 


whenever for each $i \in \mathcal{I}$, $\mu_i \in \tilde{\mathcal{F}}_{\Sigma \Xi}(X, \Xi)$ and $\mu_\mathcal{L} = s_1$, and

$$\gamma^\triangledown_{\Sigma \Xi}(s) = s \quad \text{for each } s \in \Sigma_0 \cup X \cup \Xi \text{ and } \gamma \in X \cup \Xi_{\text{in}},$$

is called the $\Sigma X \Xi$-net algebra or free algebra. If in the $\Sigma X \Xi$-net algebra we interchange in each ranked letter the in-arities and out-arities we will get the co-algebra of the $\Sigma X \Xi$-net algebra.

**Definition 1.2.3.3.** REALIZATION of NETS. Operation $\gamma^\triangledown$ is $\triangledown$-realization of $\gamma$, whenever $\gamma \in \Sigma_0 \cup X \cup \Xi$. Let $t = s(\mu_i : \lambda_j | i \in \mathcal{I}_s^{\text{in}}, j \in \mathcal{J}_s^{\text{in}})$ be a net. Then

$$t^\triangledown = s^\triangledown(\mu_\mathcal{L}^\triangledown, \lambda_j | i \in \mathcal{I}_s^{\text{in}}, j \in \mathcal{J}_s^{\text{in}}, C_{t\triangledown})$$

is $\triangledown$-realization of $t$, where $C_{t\triangledown}$ is a sample of conditions to be fulfilled (normally assumed to be known). If net $t$ is given in the form $t = s(\mu ; \lambda)$, then we can write $\triangledown$-realization of $t$

$$t^\triangledown = s^\triangledown(\mu_\mathcal{L}^\triangledown, \lambda),$$

when $\mu_\mathcal{L}^\triangledown$ is a Cartesian element where each projection of $\mu_\mathcal{L}$ is replaced with its $\triangledown$-realizations.

For jungle $T$ we denote $T^\triangledown = \{ t^\triangledown : t \in T \}$, and the set of $\triangledown$-realizations of all $\Sigma X \Xi$-nets is denoted $\mathcal{F}_{\Sigma \Xi}(X, \Xi)^\triangledown$.

Net $s$ is called the carrier net of $s^\triangledown$. Let $\triangledown = (A, \Sigma A X, \Xi A)$ be a $\Sigma A X, \Xi A$-algebra. We may also use notation $t = (t^\triangledown)^{\triangledown}$, and for jungle $S$, $S = (S^\triangledown)^{\triangledown}$. The set of the $\triangledown$-operations of the nodes in $t$ is entitled $\triangledown$-nest of $t$ or the nest of $t^\triangledown$, $t$ and $t^\triangledown$ being said to be beyond any subset of that nest.

Let $t = s(\mu_i : \lambda_j | i \in \mathcal{I}_s^{\text{ocin}}, j \in \mathcal{J}_s^{\text{ocin}})$ be a net, where $\mathcal{I}_s^{\text{ocin}}$ is such an in-rank index set of $s$ that for each $(i \in \mathcal{I}_s^{\text{ocin}}) \mu_i \in \tilde{\mathcal{F}}_{\Sigma \Xi}(X, \Xi)^{\text{ocin}}(t)$. Let $A_0 = \{ a_i : a_i \in A, i \in \mathcal{I}_s^{\text{ocin}} \}$ be an indexed subset of $A$, a set of inputs.

$$t^\triangledown(A_0) = s^\triangledown(\mu_\mathcal{L}^\triangledown(a_i), \lambda_j | i \in \mathcal{I}_s^{\text{ocin}}, j \in \mathcal{J}_s^{\text{ocin}}, a_i \in A_0, C_{t\triangledown})$$

is called $t^\triangledown$-transformation of $A_0$, the set of outputs of $t^\triangledown$ for $A_0$. Important examples of realizations are equations, where e.g. symbol “=” is the realization of a ranked letter with the in-rank two, and transformations are needed to considering the validity.

Transformations are essential in the context of rewriting systems.

Let $\tau$ be a binary relation in $P(A)$. $\triangledown$-realization $t^\triangledown$ is $\tau$-confluent, if $t^\triangledown(A) \tau t^\triangledown(B)$, whenever $A \tau B$. 
Net realization descriptions

**Lemma 1.2.3.** Each demand or claim can always be presented with realizations of nets.

**PROOF.** Each presentable elementary claim is actually a relation in some algebra. □

**Definition 1.2.3.4.** Let \( \mathcal{A} = (A, \Sigma, X, \Xi) \) be a \( \Sigma X \Xi \)-algebra. Let \( R, S \) and \( T \) be \( \mathcal{A} \)-realizations of some nets. Now we are introducing for only descriptive use some special nets by example wise: Transformer graph (TG) \( \mathcal{T} \) over \( \{R, S, T\} \), denoted \( \text{TG}([R, S, T]) \), is a realization of the net having carrier nets of \( R, S \) and \( T \) among its enclosures. If the set, for which a transformer graph is over, is singleton, we speak simply of transformer. If \( H \) is a set of realizations, set \( K \) being one of the subsets of \( H \), we say that \( \mathcal{T} \) is **beyond** \( K \) whenever \( \mathcal{T} \) is \( \text{TG}(H) \) and we denote \( \text{TG}(|K|) \).

**Realization process graph** (RPG) contains in an addition to TG input and output elements as its nodes for concerning realizations. Cf. model theory Chang CC, Keisler HJ (1973), process graphs van Glabbeek RJ (2001), automata Rozenberg G, Salomaa A, ed. (1997).

Generally speaking: any RPG is a TG-associated net, where the projections of Cartesian elements of \( A \) in the RPG are in- and up-connected, respectively, to at most one \( \mathcal{A} \)-realization in the TG.

**Transformation graph** (TFG) comprises only input and output elements of RPG.

**Definition 1.2.3.5.** GENERATORS. Let \( \mathcal{A} = (A, \Sigma, X, \Xi) \) be an algebra. We say that subset \( H \) of \( A \) is a **generator set of** \( \mathcal{A} \) and is **generating** \( \mathcal{A} \), and we denote \( [H] = A \), if

\[
A = \bigcap \{ B : H \subseteq B, (B, \Sigma, X, \Xi) \in \text{Sub}(\mathcal{A}) \}.
\]

\( H \) is called a **base generator set** of \( \mathcal{A} \), if there is no genuine subset of \( H \) generating \( \mathcal{A} \). Notice that algebra may have several base generator sets.
Definition 1.2.3.6. HOMOMORPHISM. Let $\mathcal{A} = (A, \Sigma, X, \Xi)$ and $\mathcal{B} = (B, \Sigma, Y, \Xi)$ be two algebras. Let $\varphi : A \rightarrow B$ be an indexes preserving relation. The homomorphic extension of $\varphi$ from $\mathcal{A}$ to $\mathcal{B}$, shortly homomorphism, is a relation, denoted $\hat{\varphi} : \mathcal{A} \rightarrow \mathcal{B}$, defined such that

$$\hat{\varphi}(a) = \varphi(a), \text{ whenever } a \in A,$$

and

$$\hat{\varphi}(\sigma^\alpha(a_\lambda : i \in \mathcal{A}, j \in \mathcal{B})) = \sigma^\beta(\hat{\varphi}(a_\lambda : i \in \mathcal{A}, j \in \mathcal{B})), \text{ whenever } \sigma \in \Sigma \text{ and } (a_\lambda : i \in \mathcal{A}, j \in \mathcal{B}) \in A^{\mathcal{A}} \otimes \mathcal{F}_{\text{out}}(X, \Xi)^{\mathcal{B}}. \text{ Homomorphism : } \mathcal{A} \rightarrow \mathcal{B} \text{ is automorphism.}$$

Definition 1.2.3.7. FREE GENERATING. Let $K$ the set of all $\Sigma X \Xi$-algebras. We say that free generating set $A_\alpha$ generates $\mathcal{A} = (A, \Sigma)$ freely over $K$, if there is such subset $A_\alpha \subseteq A$ that

(i) $[A_\alpha] = A$ and

(ii) for each algebra $\mathcal{B} = (B, \Sigma)$ in $K$ and each relation $\varphi : A_\alpha \rightarrow B$, there is the homomorphic extension of $\varphi$ from $\mathcal{A}$ to $\mathcal{B}$.

The next clause represents the well known result for trees, at this time for more general aspects: nets.

Proposition 1.2.3. Set $\Xi \cup X \cup \Sigma_0$ is the base generator set of $\Sigma X \Xi$-net algebra $\mathcal{F}_\Sigma^\Xi(X)$ and generates it freely over all $\Sigma X \Xi$-algebras.

PROOF. First we prove that $\Xi \cup X \cup \Sigma_0$ is the base generator set of $\mathcal{F}_\Sigma^\Xi(X)$ and for that purpose we define for each jungle $Q \subseteq \mathcal{F}_\Sigma^\Xi(X)$ the following sets:

$$G(Q) = Q \cup \{ \sigma^\beta(\xi^\alpha(s_{i, \lambda} : i \in \mathcal{A}, j \in \mathcal{B}) : s_i \in Q, \lambda_j \in \{ \lambda : \lambda \in \mathcal{F}_{\text{out}}(X, \Xi), \lambda_j \in Q \}, \sigma \in \Sigma \setminus \Sigma_0 \},$$

$$G^0(Q) = Q,$$

$$G^{n+1}(Q) = G(G^n(Q)), n \in \mathbb{N}_0,$$

$$G = \bigcup (G^n(Q) : n \in \mathbb{N}_0).$$
Let $\sigma \in \Sigma$ and $\mathcal{F}_0 \subseteq \mathcal{G}_0$ and $\mathcal{F}_\ast \subseteq \mathcal{G}_\ast$, and $s_it_j \in G$, $i \in \mathcal{G}_0$, $j \in \mathcal{G}_\ast$. Then ($\forall \ i \in \mathcal{G}_0$, $j \in \mathcal{G}_\ast$) ($\exists \ m_i,n_j \in \mathbb{N}_0$) $s_i \in G^{m_i}(Q)$ and $t_j \in G^{n_j}(Q)$. Therefore

$$s_i \in G^m(Q), \ i \in \mathcal{G}_0 \text{ and } t_j \in G^m(Q), \ j \in \mathcal{G}_\ast,$$

where $m$ is the maximum of the numbers $n_k$, $k \in \mathcal{G}_0 \cup \mathcal{G}_\ast$. We can write

$$\sigma^{\mathcal{G}_0}(s_i; \lambda_j | i \in \mathcal{G}_0, j \in \mathcal{G}_\ast) \in G(G^m(Q)) = G^{m+1}(Q) \subseteq G,$$

where $\lambda_j = t_j$.

This yields $G$ generates itself. Because $Q \subseteq G$, it is in force

$$[Q] \subseteq G.$$  

On the other hand for each ($n \in \mathbb{N}_0$) $G^n(Q) \subseteq [Q]$. Therefore $G \subseteq [Q]$. If $Q = \Xi \cup X \cup \Sigma_0$, we thus have $G = F_{\Xi}(X, \Xi)$, and finally $[\Xi \cup X \cup \Sigma_0] = F_{\Xi}(X, \Xi)$.

Let then $\\mathcal{I} = (A, \Sigma, \Xi, \Sigma_0)$ be an $\Sigma \Xi \Sigma$-algebra and $\varphi : \Xi \cup X \cup \Sigma_0 \rightarrow A$ an indexes preserving relation. We define relation $\beta : \mathcal{G}_{\Xi}(X) \rightarrow \mathcal{I}$ such that

$$\beta(\gamma) = \varphi(\gamma), \ \text{whenever } \gamma \in \Xi \cup X \cup \Sigma_0, \text{ and}$$

$$\beta(\sigma^{\mathcal{G}_{\Xi}}(s_i; \lambda_j | i \in \mathcal{G}_0, j \in \mathcal{G}_\ast)) = \sigma^{\sigma}(\beta(s_i; \lambda_j | : i \in \mathcal{G}_0, j \in \mathcal{G}_\ast),$$

whenever $\sigma \in \Sigma$ and $(s_i; \lambda_j | i \in \mathcal{G}_0, j \in \mathcal{G}_\ast) \in A^{\mathcal{G}_0} \otimes \mathcal{I}_{\mathcal{G}_{\Xi}}(X, \Xi)|_{\mathcal{G}_0}$. Clearly $\beta$ is the homomorphic extension of $\varphi$ from $\mathcal{G}_{\Xi}(X)$ to $\mathcal{I}$.  

1.3. $\Xi$ Net homomorphism, Substitution and Matching

**Definition 1.3.1.** Net HOMOMORPHISM.

Let $X$ and $Y$ be frontier alphabets, $\Sigma$ and $\Omega$ ranked alphabets and $\Xi_X$ and $\Xi_\Omega$ arity alphabets. We introduce new distinct rank-indexed arity alphabets $E_{in} = \{e_i : i \in \mathcal{E}_{in}\}$ for in-arities and $E_{out} = \{e_i : i \in \mathcal{E}_{out}\}$ for out-arities respectively, disjoint from all other used alphabets.

*Net homomorphism* $h : F_{\Xi}(X, \Xi_X) \cup \mathcal{H}_{\Xi}(X, \Xi_X) \rightarrow F_{\Omega}(Y, \Xi_\Omega) \cup \mathcal{H}_{\Omega}(Y, \Xi_\Omega)$ is a relation defined such that

$$h(t) = \mathcal{H}(\sigma)(h(\mu); h(\lambda) | i \in \mathcal{E}_{in(\sigma)}, j \in \mathcal{E}_{out(\sigma)}) \text{ for each } t = \sigma(\mu; \lambda | i \in \mathcal{G}_0, j \in \mathcal{G}_\ast) \in F_{\Xi}(X, \Xi_X),$$

where $\mathcal{H}$ is the homomorphic extension of $\mathcal{I}_{\mathcal{G}_{\Xi}}(X, \Xi)$ to $\mathcal{I}_{\mathcal{G}_{\Xi}}(X, \Xi)$.
and
\[
h: \Sigma_0 \cup X \cup \Xi_\Sigma \rightarrow \widetilde{F}_\Omega(Y, \Xi_\Omega)^{(1)} \cup \Omega_0 \cup \Xi_\Omega \text{ is an initial rewriting relation, where } h(\xi) \in \Xi_\Omega \text{ for each } \xi \in \Xi_\Sigma \text{ and } h(\sigma) \in \Omega_0 \text{ whenever } \sigma \in \Sigma_0;\]
h|X named the initial manoeuvre rewriting relation, and \( h|\Xi_\Sigma \) the initial arity rewriting relation;

\[
h: \Sigma \rightarrow F_\Omega(Y, \Xi_\Omega \cup E_{in} \cup E_{out}) \cup \Omega \text{ is a } \Sigma\text{-ranked letter rewriting relation ,}
\]
h(u) \in \widetilde{F}_{in}\Omega(Y, \Xi_\Omega)^{(1)}, and \( h(u)_L = h(u)_L \) whenever \( u \in \widetilde{F}_{in}\Omega(X, \Xi_\Sigma)^{(1)} \) and \( u_L \in \Sigma_0 \),
h(u) \in \widetilde{F}_{out}\Omega(Y, \Xi_\Omega)^{(1)}, and \( h(u)_L = h(u)_L \) whenever \( u \in \widetilde{F}_{out}\Omega(X, \Xi_\Sigma)^{(1)} \) and \( u_L \in \Sigma_0 \),
h(u) \in \widetilde{F}_{in}\Omega(Y, \Xi_\Omega)^{(1)} \cup \widetilde{F}_{in}\Omega(Y, \Xi_\Omega)^{(2)}, \text{ whenever } u \in \widetilde{F}_{in}\Omega(X, \Xi_\Sigma)^{(1)} \text{ and } u_L \in X,
h(u) \in \widetilde{F}_{out}\Omega(Y, \Xi_\Omega)^{(1)} \cup \widetilde{F}_{out}\Omega(Y, \Xi_\Omega)^{(2)}, \text{ whenever } u \in \widetilde{F}_{out}\Omega(X, \Xi_\Sigma)^{(1)} \text{ and } u_L \in X,
h(u) \in \widetilde{F}_{in}\Omega(Y, \Xi_\Omega)^{(2)}, \text{ and } h(u)_L = h(u)_L, \text{ whenever } u \in \widetilde{F}_{in}\Omega(X, \Xi_\Sigma)^{(2)}, \text{ and }
h(u) \in \widetilde{F}_{out}\Omega(Y, \Xi_\Omega)^{(2)}, \text{ and } h(u)_L = h(u)_L, \text{ whenever } u \in \widetilde{F}_{out}\Omega(X, \Xi_\Sigma)^{(2)}.

Relation \( h \) is said to be down linear, if the number of the positions of each letter of \( E_{in} \) in \( h_\Sigma(\sigma) \)
is one at most whenever \( \sigma \in \Sigma \); up linear is defined respectively for the letters in \( E_{out} \). Relation \( h \) is
down preserving (otherwise down deleting), if \( |\delta_{in}\Sigma(\sigma)| = |\delta_\sigma| \) for each \( \sigma \in \Sigma \), respectively is defined up
preserving and up deleting. We call \( h \) down alphabetic, if \( h(X \cup \Xi_\Sigma) \subseteq Y \cup \Xi_\Omega \), and for each \( \sigma \in \Sigma \),
\( h_\sigma(\sigma) = 0(\varepsilon_i; \varepsilon_j \mid i \in E_{in}, \ j \in E_{out}) \), where \( \omega \in \Omega \), cf. tree homomorphism Denecke K, Wismat SL (2002).
Notice that because net homomorphism is in its nature “replacing”, it can be seen as a special type of rewriting systems.

**Definition 1.3.2.** **SUBSTITUTION.**

Let \( T \) and \( S \) be arbitrary jungles and \( P \) a family of sets of positions. We define
\[
T(P \uparrow S : *) = \bigcup (v(\mu_i^{(1)}; \nu_i; \lambda_j^{(1)}; \delta_i) \mid t = v(\mu_i; \lambda_j) \mid i \in \delta_i, \, j \in \delta_i, \, s \in \delta_i \), \( \delta_i \in \delta_i \), \( \mu_i, \lambda_j \in P \), \( s \in S \), *, \( \nu_i^{(1)} \in S_{outg} \), \( \delta_i \in S_{ing} \).
\]
That is $T(P \leftarrow S : \ast)$ is the jungle which is obtained by “replacing” (considering conditions $\ast$) all the subnets of each net $t$ in $T$, having the position set in family $P$, by each net in $S$. Notice that the result may be $T$ (that is no execution in replacing), if the arities of the replacing nets and on the other hand the arities of the nets in $T$ are different or $P$ does not represent any positions of subnets in nets of $T$.

If each position set of family $V$ of subnets of each net $t$ in $T$ is wished to be replaced by each of elements in $S$, we write simply $T(V \leftarrow S)$.

Next introduced substitution relation is a special example of net homomorphisms, an essential component in rewriting.

*Net substitution relation* (here $f$) is such a net homomorphism in $\widetilde{F}_2(X, \Xi) \cup \widetilde{F}_2(X, \Xi)$ that each ranked letter rewriting relation is identity relation, as well as the initial arity rewriting relations, and for each $\nu \in \widetilde{F}_2(X, \Xi)^{(2)}$ $f(\nu) = \nu^{(1)} \nu^{(2)} f(\nu_L)$ and for each $\mu \in \widetilde{F}_2(X, \Xi)^{(1)}$ $f(\mu) = \mu^{(1)} f(\mu_L)$.

Let $X_o$ be a set of frontier letters. Net substitution relation $f$ is said to be $X_o$-joining, if

(i) $\{f(x)_L : x \in X_o\}$ is singleton and
(ii) the arity of the gluing form of each letter in $X_o$ is occupying an unoccupied arity of $f(x)_L$, and $|X_o|$ is the cardinality of the set of those unoccupied arities.

It is worth to remember that if an image of net substitution relation for a leaf is empty (set), the arity having been occupied by that leaf is after substitution an unoccupied arity.

Let $P$ and $T$ be arbitrary jungles. If $S$ is a catenation of substitutions such that $T = S(P)$, we say that there is an $S$-substitution route between $P$ and $T$.

**Definition 1.3.3.** *INSTANCE*. Net $t$ is an *instance* of net $s$, if $t = f(s)$ for some net substitution relation $f$. Notice that $s$ is a context of $t$ for the in-glue form of net $f(\nu)_L$, whenever $\nu \in X \cup \Sigma_0$ is in $s$. Notice that $s = \text{con}_P(f(s))$, if $P \in \{ p(f(s), s), \cup (p(f(s), f(x)_L) : x \in X \cap L(s))\}$.

**Definition 1.3.4.** *MATCHING*. Net $s$ is said to *match* $t$ by net substitution relation $f$ in $p(t,s)$, in a so called *matching point*, if $f(s) \in \text{sub}(t)$; thus $\text{apex}(s) \in \text{enc}(t)$. If net $s$ matches net $t$, we say that the genuine tie-arities of $s$ in the linkages between $s$ and $t$ are the *matching arities* of $s$ in $t$, denoted $\text{MA}(t,s)$. 
Definition 1.4.1. For jungle $T$ a type $\rho$ of net (e.g. a tree) being in $\text{enc}(T)$ is of maximal $\rho$-type in $\text{enc}(T)$, if it is not an enclosement of any other $\rho$-type net in $\text{enc}(T)$ than of itself. The other $\rho$-type nets in $\text{enc}(T)$ are genuine.

Definition 1.4.2. COVER of NET. A set of nets is said to be a cover of net $t$, if each node of $t$ is in a net of the set. We denote the set of all covers of net $t$ with $\text{Cov}(t)$, and for jungle, say $T$, we agree $\text{Cov}(T) = \bigcup(\text{Cov}(t): t \in T)$.

Definition 1.4.3. SATURATION of NET. Cover $A$ saturates net $t$, if $A \subseteq \text{enc}(t)$. We denote the set of all saturating covers of net $t$ with $\text{Sat}(t)$, and for jungle, say $T$, we agree $\text{Sat}(T) = \bigcup(\text{Sat}(t): t \in T)$.

E.g. A saturating cover of net $t$ is natural, if each net in the cover is of maximal $\rho$-type, where $\rho$-type net is the net the nodes having only one out-tie (resembling in that respect tree).

Definition 1.4.4. PARTITION of NET. A saturating cover of net $t$ is a partition of $t$, if each node of $t$ is exactly in one net in the cover. We reserve notation $\text{Par}(t)$ as for the set of all partitions of net $t$, and for jungle, say $T$, we agree $\text{Par}(T) = \bigcup(\text{Par}(t): t \in T)$.

For an arbitrary jungle $A$ we define the partition induced by jungle $A$

(denoted $\text{PI}(A) = \{ \bigcap A' \in \{ \bigcap A''': A''' \subseteq A''': A'''' \subseteq A''''' \}: A''' \subseteq P(A) \}$).

We can write the following proposition:

Proposition 1.4. “A correlation between partitions and covers of nets”.
For any net $s$ and jungle $E$

$E \in \text{Cov}(s)$, if and only if $\text{PI}(E) \cap s \in \text{Par}(s)$. 
Notice that if $A$ is a saturating cover of net $t$, then $\Pi(A)$ is a partition of $t$.

### 1.5. \textsection \textbf{REWRITE}

In this chapter we introduce rewriting by using algebraic presentation described earlier regarding edges as ties or linkages which unite node- and on the other hand edge-rewriting (Thomas W (1997); Engelfriet J (1997)). \textit{Term rewriting} as a special case of here presented rewriting can be probed e.g. in (Ohlebusch E (2002); Meseguer J, Goguen JA (1985)).

#### 1.5.1

**Definition 1.5.1.1. \textit{RULES.}** A \textit{rewrite rule} is a set (possibly conditional) of ordered ‘jungle-jungle’-pairs $(S,T)$, the elements of which are entitled \textit{rule preforms}, simply \textit{rules}, if there is no danger of confusion, denoted often by $S \rightarrow T$, $S$ is called the \textit{left side} of pair $(S,T)$ and $T$ is the \textit{right side} of it. We agree that $\text{right}(R)$ is the set of all right sides of rule preforms in each element of set $R$ of rewrite rules; $\text{left}(R)$ is defined accordingly to $\text{right}(R)$. The frontier letters of nets in those rule preforms are called \textit{manoeuvre letters}.

**Types of rewrite rules**

Next we shortly represent some general types of rewrite rules.

**Definition 1.5.1.2.** A rewrite rule is said to be \textit{simultaneous}, if it is not a singleton van Glabbeek RJ (2001). The \textit{inverse rule} of rule $\varphi$, $\varphi^{-1}$, is the set $\{(T,S) : (S,T) \in \varphi\}$. A rule is \textit{single}, if it is singleton.

A rule is an \textit{identity rule}, if the left side is the same as the right side in each rule preform of the rule. A rule is called \textit{monadic}, if there is a net homomorphism connecting the left side to the right side in each rule preform of the rule. If for each rule preform $r$ in rule $\varphi$, $\text{hg(left}(r)) > \text{hg(right}(r))$,
we call $\phi$ _height diminishing_, and if $\text{hg}(\text{left}(r)) < \text{hg}(\text{right}(r))$, $\phi$ is _height increasing_; if $\text{hg}(\text{left}(r)) = \text{hg}(\text{right}(r))$, we call $\phi$ _height saving_.

A rule is _alphabetically diminishing_, if for each rule preform $r$ in the rule there is in force:

(i) right($r$) is a ranked net or $\text{hg}(\text{right}(r)) = 0$ or (ii) $\text{hg}(\text{left}(r)) = 2$, root($\text{right}(r)$) $\in L(\text{left}(r))$ and $\text{hg}(\text{right}(r)) = 1$. For an abbreviation reason for a set of rules $\mathcal{R}$ we may use notation

$\text{left}(\mathcal{R}) = \{ \text{left}(r) : r \in \phi, \phi \in \mathcal{R} \}$ and respectively for the case of right side.

More specifically:

**Definition 1.5.1.3** Any rule and the concerning pairs (i.e. rule preform) in it are said to be

1° _manoeuvre increasing_, if for each of its pairs, $r$, fron(\text{left}(r)) \subset fron(\text{right}(r)),$

2° _manoeuvre deleting_, if for each of its pairs, $r$, fron(\text{left}(r)) \supset fron(\text{right}(r)),$

3° _manoeuvre saving_, if for each of its pairs, $r$, fron(\text{left}(r)) = fron(\text{right}(r)),$

4° _manoeuvre changing_, if at least for one of its pairs, $r$,

$\text{fron}(\text{left}(r)) \not\subset \text{fron}(\text{right}(r))$ and $\text{fron}(\text{right}(r)) \not\subset \text{fron}(\text{left}(r)),$

5° _manoeuvre mightiness saving_, if for each of its pairs, $r$, $|p(\text{left}(r),x)| = |p(\text{right}(r),x)|,$ whenever $x$ is a manoeuvre letter,

6° _arity increasing_, if for each of its pairs, $r$, Uno(\text{left}(r)) \subset Uno(\text{right}(r)),$

7° _arity deleting_, if for each of its pairs, $r$, Uno(\text{left}(r)) \supset Uno(\text{right}(r)),$

8° _arity saving_, if for each of its pairs, $r$, Uno(\text{left}(r)) = Uno(\text{right}(r)),$

9° _arity mightiness saving_, if for each of its pairs, $r$, $|p(\text{left}(r),\xi)| = |p(\text{right}(r),\xi)|,$ whenever $\xi$ is an unoccupied arity letter,

10° (ranked) _letter increasing_, if for each of its pairs, $r$, $L(\text{apex}(\text{left}(r))) \subset L(\text{apex}(\text{right}(r)))$, 

11° (ranked) _letter deleting_, if for each of its pairs, $r$, $L(\text{apex}(\text{left}(r))) \supset L(\text{apex}(\text{right}(r)))$, 

12° (ranked) _letter saving_, if for each of its pairs, $r$, $L(\text{apex}(\text{left}(r))) = L(\text{apex}(\text{right}(r)))$, 

13° (non-arity) _letter mightiness increasing_, if for at least one of its pairs, $r$,

$|\bigcup(p(\text{apex}(\text{left}(r)),z) : z \text{ is a frontier or ranked letter})| < \bigcup(p(\text{apex}(\text{right}(r)),z) : z \text{ is a frontier or ranked letter})|,$ 

14° _X-manoeuvre letter increasing, decreasing, saving_, if
\[ L(\text{left}(r)) \cap X \subset \sqcup, \sqsubseteq = L(\text{right}(r)) \cap X, \text{respectively,} \]

15° \text{-manoeuvre mightiness increasing, decreasing, saving, if for each } x \in X \]
\[ |p(\text{left}(r), x)| <, >, = |p(\text{right}(r), x)|, \text{respectively.} \]

Rule \( \phi \) is \textit{left linear}, if for each \( r \in \phi \) and manoeuvre letter \( x \) there is in force \( |p(\text{left}(r), x)| = 1 \), and \textit{right linear}, if \( |p(\text{right}(r), x)| = 1 \). A rule is \textit{totally linear}, if it is both left and right linear.

### 1.5.2 Renetting systems and Application

We introduce systems using rewrite rules to transform nets, and type wise to define sets of rules with special instructions regarding to apply them.

A set consisting of rewrite rules and of \textit{conditional demands} (possibly none) Ohlebusch E (2002), (for the set of which we reserve symbol \( \mathcal{C} \)) to apply those rules is called \textit{renetting system}, RNS in short, Engelfriet J (1997), and a \( \Sigma X \)-RNS, if its rewrite rules consist exclusively of pairs of \( \Sigma X \)-nets. Conditional demands may concern application orders cf. \textit{process algebra with timing} Baeten JCM, Middelburg CA (2001), \textit{probabilistic processes} Jonsson B, Yi W, Larsen KG (2001), \textit{priority in process algebra} Cleaveland R, Lütggen G, Natarajan V (2001). The objects to be applied may be required to possess certain nodes, linkages or neighbours or to be carrier nets for operations in selected algebras. Desired substitutions may be “context sensitive” i.e. chosen to be of left or right side and matching positions where applications are expected to be seen to happen may also be prerequisites. Notice that rules in RNS’s can be presented also exclusively by net types: pairs of rules in RNS’s defined in accordance with the amount of the arities or nodes possessed by them Engelfriet J (1997), \textit{edge-replacing} Burkart O, Cauca D, Moller F, Steffen B (2001).

**Definition 1.5.2.1.** A \textit{renetting system}, shortly entitled RNS, is \textit{finite}, if the number of rules and \( \mathcal{C} \) in it is finite. A RNS is said to be \textit{limited}, if each rule of it is finite and in each pair of each rule the right side is finite and the heights of the nets in the both sides are finite. For the clarification we may use notation \( \mathcal{C}(\mathcal{R}) \) instead of \( \mathcal{C} \) for RNS \( \mathcal{R} \). A RNS is \textit{conditional} (or \textit{sensitive}), contradicted
nonconditional or free, if its $C$ is not empty. A RNS is simultaneous, contradicted nonsimultaneous, if it has a simultaneous rule.

A RNS is elementary, if for each pair $r$ in each rule of the RNS is monadic or alphabetically diminishing. If each of the rules in a RNS is of the same type, the RNS is said to be of that type, too. For each RNS $R$, we denote $R^{-1} = R(\varphi \leftarrow \varphi^{-1})$.

**Definition 1.5.2.2.** APPLICATION TYPES. For given RNS $R$, jungle $S$ is $R$-rewritten to jungle $T$ (rewrite result), denoted $S \rightarrow^R T$ (called $R$-application), and is reduced under $R$ or by rule $\varphi$ in $R$, and is said to be a rewrite object for $R$ or $\varphi$ respectively, denoting $T = S\varphi$ (the postfix notation is prerequisite), if the following “rewrite” is fulfilled:

$$T = \bigcup (S(p) \leftarrow \text{right}(r)) g : \text{left}(r) \text{ matches } s \text{ in } p \text{ by some net substitution mapping } f_{s}, r \in \varphi, g \in G_{s}, \varphi \in p(s), s \in S, C(R),$$

where $G_{s}$'s are sets of net substitution relations. Mapping $f_{s}$ is called left side substitution relation and each $g$ in $G_{s}$ is right side substitution relation, c.f. under conditional demands “extra variables on right-hand sides” conditional Rewrite Systems Ohlebusch E (2002). We say that RNS is S-instance sensitive (S-INRNS), if for a rule $\varphi \in$RNS and for each $s \in S$, $p \in p(s)$, $G_{s} \neq f_{s}$, and S-mapping instance sensitive (S-MINRNS), if right side substitution relations are mappings. If furthermore all right side substitution mappings are singletons, we entitle SingMINRNS to indicate RNS’s of that nature. If all rules in RNS are obligated to satisfy the demands, instance sensitiveness of RNS is said to be thorough. Notice that for substitution relations, $C(R)$ may contain some orders liable to substituting manoeuvre letters in the rewrite process (substitution order), especially if rewrite objects have outside loops with the apexes of left sides of pairs in rules or $R$ is manoeuvre increasing and instance sensitive. Instructions concerning binding right side substitution relations to specific rules in RNS might also have been included in $C(R)$.

We say that $R$ matches a rewrite object, if the left side of a rule preform matches it. We say that $S$ is a root of $T$ in $R$ and $T$ is a result of $S$ in $R$. Observe that $T = S$, if $R$ does not match $S$; of course $C$ may contain demands for necessary matching. The enclosements at which rewrites can take places (the sets of the apexes of the left sides in the pairs of the rules in RNS’s) satisfying all requirements set on the RNS are called the redexes of the concerning rules or RNS’s in the rewrite objects. For RNS $R$ and jungle $S$ we denote
\[ S \mathcal{R} = \bigcup (S \varphi : \varphi \in \mathcal{R}). \]

Rule \( \varphi \) of \( \mathcal{R} \) is said to be applied to jungle \( S \), if for each \( s \in S \), \( s \) has \( \varphi \)-redexes \( (\text{redexes of } \varphi \text{ in } s) \) fulfilling \( \mathcal{C}(\mathcal{R}) \) and thus \( \varphi \) is applicable to \( S \) and \( S \) is \( \varphi \)-applicable. RNS \( \mathcal{R} \) is applicable to \( S \) and \( S \) is \( \mathcal{R} \)-applicable, if \( \mathcal{R} \) contains a rule applicable to jungle \( S \).

For RNS \( \mathcal{R} \) we define \( \mathcal{R} \)-transformation relation on \( F_{\mathcal{R}}(X, \Xi) \)

\[ \rightarrow_{\mathcal{R}} = \{(s, s') : s \in F_{\mathcal{R}}(X, \Xi)\}. \]

**Lemma 1.5.2.** Any relation can be presented with a RNS and its rewrite objects. On the other hand with any given RNS we have RNS-transformation relation.

**PROOF.** Let \( r \) be a relation. Constructing RNS \( \mathcal{R} = \{a \rightarrow b : (a, b) \in r\} \) we obtain

\[ r = \{(a, a(a \rightarrow b)) : a \rightarrow b \in \mathcal{R}\}. \]

It is quite clear that a net cannot be \( \mathcal{R} \)-rewritten, if \( \mathcal{R} \) is not instance sensitive and the matching points of the left sides of the pairs in the rules of \( \mathcal{R} \) have outside loops to the net, because the apexes of the right sides of those pairs must be enclosed in some images of the right side substitution relations.

**Definition 1.5.2.3.** We call RNS feedbacking in respect to a net, if while applying a rule in it for that net, elements in the image sets of each right side substitution relation regarding to the preforms in the involving rule overlap that net; feedbacking for a rule is total, if the demands concern all elements in the image sets (always total, if the substitution relations are mappings since the image sets are then singletons) and partial, if RNS is feedbacking but not totally. If instead of only overlapping, we claim the enclosure condition for elements in the sets of the right side substitution images, feedbacking RNS is innerly feedbacking - which is e.g. the case in not instance sensitive RNS’s – and if no overlapping is enclosure, RNS is outerly feedbacking. If the net in concern of feedbacking is the applicant for RNS, we speak of self feedbacking. The form of innerly self feedbacking RNS in respect to a net, say \( t \), where for each rule preform \( r \) there is in force equation \( tr \uparrow \text{apex}(\text{right}(r)) = t \uparrow \text{apex}(\text{left}(r)) \), we name environmentally saving in respect to the rewrite object in concern. If all rules in RNS satisfy the feedbacking demands we speak of
It is worth to remind that INRNS’s are capable to join distinct applicable nets.

By the cardinality of the image sets of right side substitution relations and on the other hand for each left side substitution mapping by the cardinality of the set of right side substitution relations regarding to mutual rules, the necessary rules in RNS’s can be compensated so not to exceed finite number – the right side substitutions relations can be defined type wise, i.e. setting their image sets to consist nets of certain type (e.g. limited number of nodes or unoccupied arities). In feedbacking RNS’s right side substitution relations may be regulated to depend on the type of rewrite objects (thus covering large portions of object nets by a limited number of regulations and not needing to raise the amount of rules possibly to infinite), e.g. replacing manoeuvre letters, existing only in the right-handed sides of pairs in rules, by overlapping nets in specific positions, if any.

It is somewhat of worth to mention that RNS’s, not instance sensitive, can own the same rewriting power than INRNS´s, but then we may be compelled to accept infinite number of manoeuvre altering rules – e.g. in the case we have a manoeuvre letter increasing INRNS, where for left side substitution mapping f and right side substitution relation g, g(y), overlaps f(x), for some manoeuvre letters x≠y (i.e. rewrite results are expected to contain loops) and there is expected to be an infinite number of rewrite objects for which RNS is to be constructed, or if the cardinality of set \{f(x) : x∈X\} is infinite.

The following example offers a manifestation of particularity in substitution orders:

Let \(\xi_{a1}, \xi_{b1}, \xi_{c1}, \xi_{d1}\) be out-arities and \(\xi_{a2}, \xi_{b2}, \xi_{c2}, \xi_{d2}\) are in-arities, f standing for a left side and g a right side substitution relation,

- \(f(x), g(x), \in [S], f(x) = g(x), S = d(\xi_{d2}\xi_{c1}c(\xi_{c2}; \xi_{c1}); \xi_{d1})\)
- \(f(x) = \xi_{c2}s_1, s_1 = c(\xi_{c2}; \xi_{c1}\xi_{d2} d(\xi_{d1}; \xi_{d2})) \in [S]\), \(s_1\) is a representative of S,
- \(g(y) = \xi_{d1}d(\xi_{d2}\xi_{c1}t_1; \xi_{d1}), t_1 = c(\xi_{c2}\xi_{b1}t_2; \xi_{c1}\xi_{d2}S), t_2 = b(\xi_{d2}g(y); \xi_{b1})\),

and \(g(y)_i\) and \(f(x)_i\) are overlapping each other, if possible,

\(r = a(\xi_{a2}; \xi_{a1}x) \rightarrow b(\xi_{b2}y; \xi_{b1}x)\).

If \(x\) is substituted first, the result offers fixing point for \(y\)-substitution, yielding a loop structure as a result. If on the other hand \(y\) is firstly substituted, the result is totally of a different nature, where there is a continuously growing chain of iterated nets via \(y\)-substitutions.
For left side substitution mapping \( f \) in loop situations between images must be \( f(x)_L \) overlapping \( f(y)_L \) for some manoeuvre letters \( x \neq y \), and one of them must contain itself as a subnet; illustrated in the next example of an application of manoeuvre cardinality increasing, not instance sensitive rule, with rewrite object containing a loop.

Let \( \xi_{a1}, \xi_{b1}, \xi_{a1} \) be out-arities and \( \xi_{a2}, \xi_{b2}, \xi_{a2} \) are in-arities,

\[
\begin{align*}
f(x) &= \xi_{a2}t_1, \quad t_1 = a(\xi_{a2}; \xi_{a1}\xi_{b2}t_2), \quad t_2 = b(\xi_{b2}; \xi_{b1}\xi_{a2}t_3), \quad t_3 = \alpha(\xi_{a2}; \xi_{a1}f(x)) \\
f(y) &= \xi_{b1}b(\xi_{b2}\xi_{a1}a(\xi_{a2}; \xi_{a1}); \xi_{b1})
\end{align*}
\]

and \( f(x)_L \) and \( f(y)_L \) are overlapping each other. The result via rule \( \alpha(x;y) \rightarrow \beta(x;y,y) \) is unaffected by the substitution orders between \( x \) and \( y \).

In the following our presumption for RNS`s are not to be instance sensitive, if not indicated otherwise.

**Definition 1.5.2.4. **DERIVATION. *Derivation in set \( \mathcal{R} \) of RNS’s is any catenation of applications of RNS’s in \( \mathcal{R} \), say \( \mathcal{D} \), such that the rewrite result of the former part is the rewrite object of the latter part of the consecutive elements in the catenation. The rewrite results of the elements in the catenation are called \( \mathcal{D} \)-derivatives of the rewrite object for the first element, and the catenation of the corresponding rules is entitled deriving sequence in \( \mathcal{R} \), for which in an operational use the postfix notation is the default. We agree of the associativity that for any deriving sequence \( \mathcal{q} \) and any jungle \( S \)

\[
S\mathcal{q} = (S\mathcal{q}_1)\mathcal{q}_2, \text{ if } \mathcal{q} = \mathcal{q}_1\mathcal{q}_2.
\]
1.5.3  Transducers and the Types

**Definition 1.5.3.1.** **TRANSUDER.** For each \( \omega \in \Omega \), \( i \in \mathcal{S}_\omega \), and \( j \in \mathcal{S}_\omega \), let \( r \) be a bijection, \( \text{RNS-attaching mapping} \), joining a set of RNS’s to each triple \( (\omega, i, j) \). Let \( \mathcal{A} = (F_\Xi(X, \Xi), \Omega, \mathcal{Y}_\omega, \Xi) \) be a \( \Omega, \mathcal{Y}_\omega, \Xi \)-algebra, where for each \( \omega \in \Omega \):

\[
\omega^\ell : F_\Xi(X, \Xi)^\ell \otimes \tilde{F}_{\text{out}}(X, \Xi)^\ell \rightarrow F_\Xi(X, \Xi),
\]

where \( \alpha = \text{in-rank}(\omega) \) and \( \beta = \text{out-rank}(\omega) \), is such an operation relation that

\[
\omega^\ell(s_{iL}; \lambda_j | i \in \mathcal{S}_\omega, j \in \mathcal{S}_\omega) \subseteq \bigcup(s_{\text{sd}} r(\omega, i, j) : i \in \mathcal{S}_\omega, j \in \mathcal{S}_\omega).
\]

\( \mathcal{A} \) is called a *renetting algebra*. For any net \( t \in F_\Omega(X, \Xi) \) realization \( t^\ell \) is called \( R \)-transducer (R-TD) over \( \text{RNS-attached family} \ R = \{ r(\omega, i, j) : \omega \in \Omega \cap L(t), i \in \mathcal{S}_\omega, j \in \mathcal{S}_\omega \} \) of sets of RNS’s and it is also entitled an *interaction* between those RNS’s. Notice that \( \omega^\ell \) with \( \text{in-rank}(\omega) = \text{out-rank}(\omega) = 1 \), represents RNS-transformation relation. Referring to a set of TD’s, say \( G \), in concern for the realizations, we use notation \( (F_\Xi(X, \Xi), G) \) for the renetting algebra. We want to notify that samples of possible conditions liable to realizations of upstream subnets in carrier nets of transducers may be used to set extra demands for selecting desired operating RNS’s to influence data flows from targeted in-arities. That notion is expressed in the next lemma.

We say that a TD *matches a rewrite object*, if any of its RNS does it. Let \( \mathcal{S} \) be an arbitrary index set, and for each \( i \in \mathcal{S} \) let \( \mathcal{R}_i \) be a TD, thus we denote Cartesian element \( \mathcal{R}(\mathcal{S}) = (\mathcal{R}_i : i \in \mathcal{S}) \), and \( \tilde{\mathcal{a}}(\mathcal{S}) = (e[\mathcal{S}](i, \tilde{\mathcal{a}}) \mathcal{R}_i : i \in \mathcal{S}) \), whenever \( \tilde{\mathcal{a}} \) is a Cartesian element. For any applicant \( S \) \( \tilde{\mathcal{R}}_S \) is called the *result of S in \( \mathcal{R} \).*

**Lemma 1.5.3.1.** The conditional demands for TD’s can be presented as a TD’s having no demands, and thus any TD, let us say \( \mathcal{R} \), can be given as a TD with no demands and the carrier net of that TD having the enclosures of the carrier net of \( \mathcal{R} \) in its enclosures.

**PROOF.** The claim is following from lemmas 1.2.3 and 1.5.2. \( \square \)

**Definition 1.5.3.2.** **TRANSUDER TYPES.** If each RNS in a TD is of the same type (e.g. manoeuvre saving), we say that the TD is of that type. A TD is said to be *altering*, if while
applying it is changing, e.g. the number of the rules in its RNS’s is changing (thus being *rule number altering*). A TD is entitled *contents expanding*, if some of its RNS’s contain a letter mightiness increasing rule preform. A TD is called *trivial*, if each rewrite objects for it is the same as the result in the TD. A TD is called *upside down tree TD*, if each ranked letter in the carrying net of the TD has only one in-arity.

A TD is a *transducer graph* (TDG) over a set of transducers, if the set of the carrying nets of all transducers in the set is a partition of the carrying net of the TD. I.e. TDG is a special case among transformer graphs. Any transducer graph over set T is denoted TDG(T), and any TDG(T) is *beyond each subset of* T, analogous with TG relevant to that issue.

A TDG(T) is entitled *direct* (in contradiction to *indirect* in other cases), if the only claims for the TDG(T) are those of the TD’s in T.

Any TDG over a set can be visualized as a TG over the same set.

**Lemma 1.5.3.2.** The carrying net of any altering TD can be seen as an enclosure of the larger carrying net of some nonaltering TD.

**PROOF.** Straightforwardly from lemma 1.5.3.1. □

**Definition 1.5.3.4.** TD-TRANSFORMATION RELATION. Let $\mathcal{R}$ be a transducer. We define $\mathcal{R}$-transformation relation $\rightarrow_{\mathcal{R}}$ in the set of the jungles such that

$$\rightarrow_{\mathcal{R}} = \{(t,t') : t \text{ is a jungle}\}.$$  

We say that two transducer $\mathcal{P}$ and $\mathcal{R}$ are the same, $\mathcal{P} = \mathcal{R}$, if $\rightarrow_{\mathcal{P}} = \rightarrow_{\mathcal{R}}$.

**Definition 1.5.3.3.** NORMAL FORM and CATENATION CLOSURE. $\mathcal{O}(\mathcal{R})$ is the notation for the set of all derivations in TD $\mathcal{R}$. If for jungle $S$ and TD $\mathcal{R}$, $S \mathcal{R} = S$, S is entitled $\mathcal{R}$-irreducible or of normal form under $\mathcal{R}$. For the set of all $\mathcal{R}$-irreducible nets we reserve the notation IRR($\mathcal{R}$). For each jungle $S$ and TD $\mathcal{R}$ we denote the following:

$$S^{\mathcal{R}} = S \mathcal{R}^* \cap \text{IRR}(\mathcal{R}),$$

where $\mathcal{R}^*$, the catenation closure of $\mathcal{R}$, is the transitive closure of the rules in $\mathcal{R}$.

Let $\mathcal{R}$ be a TD over family $\mathcal{R}$. We define normal form TD of $\mathcal{R}$, $\mathcal{R}^\wedge$,

$$\mathcal{R}^\wedge = \mathcal{R}(R \rightarrow R^\wedge : R \in \mathcal{R}).$$
1.6. § Equations and decompositions as examples of TD’s

Definition 1.6.1. EXPLICIT and IMPLICIT RNS-CLAUSE. Let $\mathcal{R}$ and $\mathcal{Q}$ be two TD’s. Let $H$ be a list of symbols in $\otimes$, $\mathcal{R}$ and $\mathcal{Q}$, where $\otimes = \{=,\in,\subset,\subseteq\}$. If $\mathcal{R} \otimes e \mathcal{Q}$, where $\otimes e \in \otimes$, we call TDG over $\mathcal{R}, \mathcal{Q}, \otimes$ a RNS-clause (RCl), denoted $\mathcal{E}(\mathcal{R}, \mathcal{Q}, \otimes)$. $\mathcal{E}(\mathcal{R}, \mathcal{Q}, \otimes)$ is of first order in respect to an element of $H$, if that element exists only once in the equation.

RNS-clauses cover also the ‘ordinary’ equations (with no RNS’s), being due to lemma 1.5.2.

Any TD in RNS-clause $\mathcal{E}(\mathcal{R}, \mathcal{Q}, \otimes)$ is called a factor, a left handed factor or a factor of $\mathcal{R}$, if it exists exclusively in $\mathcal{R}$, and a right handed factor or a factor of $\mathcal{Q}$, if it exists exclusively in $\mathcal{Q}$.

Let $K$ be a factor in RNS-clause $\mathcal{E}(\mathcal{R}, \mathcal{Q}, \otimes)$. We say that the RCl is a representation of $K$: specifically an explicit one (in contradiction to implicit in other cases), if $K = \mathcal{R}$ and $K$ is not a factor of $\mathcal{Q}$. The right handed factors are composers of $\mathcal{Q}$ is a compositions of $K$, if $\mathcal{E}(\mathcal{R}, \mathcal{Q}, \otimes)$ is an explicit representation of $K$, and $\otimes$ is $\in$. A composition of $K$ is said to be linear/nonlinear, if it is a direct/an indirect TDG. Because each operation in nets can stands for a simple case of TD’s then that simplyfied RNS-clause equates ordinary equations with operations of variables.

The question in automated problem solving basically is how to generate nets from enclosures of a probed net those enclosures being in such a relation with the enclosures in the conceptual nets that the particular relation is invariant under that generating transformation i.e. preserves invariability under class-rewriting. Therefore in the next three chapters we handle an idea of automated problem solving, as formal inventiveness. In problem solving, an essential thing is to see over details, and that is the task we next grip ourselves into by describing ideas such as partitioning nets by RNS´s and a connection between partitions by introducing the abstraction relation. We concentrate to construct TD-models for formulas of jungle pairs by conceptualizing ground subjects and then reversing counterparts of existing TD-solutions back to ground level. Then we widen the solution hunting by classifying intervening TDG-derivations. Finally we formulate abstract quotient algebra based on congruence class rewriting operations.
2. § Inventiveness

2.1. Recognizers and Languages

Definition 2.1.1 RECOGNIZER and RECOGNITION. Let A and B be sets and let \( \alpha: A \rightarrow B \) be a binary relation and \( A' \) a subset of B. We define recognizer \( \mathcal{A} \) such that \( \mathcal{A} = (\alpha, A') \) entitling \( \alpha \) a recognizer relation and \( A' \) a final set. Element of A, s (probed object), is said to be recognized by recognizer \( \mathcal{A} \), if \( s \in A' \). Language \( L_\mathcal{A} \) is the set of the elements recognized by \( \mathcal{A} \), i.e. \( \alpha \) separates from probed objects those ones, which have property \( A' \). As a special case for nets the recognizer relation can chosen to be \( \chi \eta \), where \( \chi \) is a net homomorphism and \( \eta \) a relation transforming nets to wanted realizations of them, cf. tree automata rules or tree recognizer homomorphisms Gécseg F, Steinby M (1997), hyper tree recognizer hypersubstitutions Denecke K, Wismat SL (2002).

In general: Set \( H \) satisfies transformer \( \mathcal{T} \) via recognizer \( \mathcal{A} \) or is a \( \mathcal{A} \)-model of formula \( \mathcal{T} \), denoted \( H \models^\mathcal{A} \mathcal{T} \), if \( \mathcal{A} \) recognizes \( \mathcal{T} \) –transformation of \( H \), \( \mathcal{T}(H) \). E.g. a recognizer relation (automorphism) in \( \mathcal{A} \) giving desired truth value from \( \mathcal{T}(H) \) we can say that \( H \) is \( \mathcal{A} \)-solution for \( \mathcal{T} \), if the value given by the recognizer mapping is true. I.e. “validity of Boolean inference”: the nest of transformer consisting of elementary logical relations (Boolean) and the concerning recognizer relation being “truth values giving automorphism from truth values of variables in the carrier net of the transformer”, the final set consequently represents the value “true”or “untrue”. A transformer can also be RCl and \( H \) a factor in it Chang CC, Keisler HJ (1973).

For nets \( S, T \) and TD \( \mathcal{R} \) in model theoretical notation \( \mathcal{R} \models^\mathcal{A} (S, T) \) \( \mathcal{R} \) is named a \( \mathcal{A} \)-model of formula \( (S, T) \), or pair \( (S, T) \) is a \( \mathcal{A} \)-model of formula \( \mathcal{R} \) (denoted in that interpretation \( (S, T) \models^\mathcal{A} \mathcal{R} \)), if recognizer \( \mathcal{A} \) (e.g. probing the truth values) is recognizing RCl \( T \subseteq S \mathcal{R} \). Cf. inferring winning game graphs Thomas W (1997).

Definition 2.1.2. Here we introduce a convenient tool, needed later in the context of abstraction relation. Let \( \mathcal{S} \) be an arbitrary index set and for each \( i,j \in \mathcal{S} \) let \( \theta_{ij} \): \( A_i \rightarrow A_j \) be a binary relation from set \( A_i \) to set \( A_j \). Let \( \overline{A} = \Pi(A_i: i \in \mathcal{S}) \) and \( \overline{\theta} = \Pi(\theta_{ij}: (i,j) \in \mathcal{S}_0) \) for some \( \mathcal{S}_0 \subseteq \mathcal{S} \). Let
α: \( \bar{A} \vdash \Pi(\theta_{ij}: (i,j) \in S^2) \) be a binary relation, where \( \bar{a} \alpha = \Pi(\theta_{ij}: (i,j) \in S^2, \text{elem}(i,\bar{a}) \theta_{ij} \text{elem}(j,\bar{a})) \), whenever \( \bar{a} \in \bar{A} \). The language recognized by \( \mathcal{A} = (\alpha, \bar{0}) \) is \( \bar{0} \)-associated over \( S_0 \) (denoted \( L_{S_0} \)); if in \( \bar{0} \) all \( \theta_{ij} \) are the same, say \( \theta \), we speak of \( \theta \)-associated language.

In other words this recognizer picks from among \( \bar{A} \) such elements, the projection elements of which are pair wise in a relation of set \( \{ \theta_{ij}: (i,j) \in S_0 \} \). Notice that \( \theta \)-associated language over a singleton is \( \theta \)-relation itself, if \( |S| = 2 \).

### 2.2 Problem and Solution

**Definition 2.2.1.** Problem \( \mathcal{S} \) is a triple \( (S, \bar{A}, \mathcal{C}) \), where the subject of the problem \( S \) is a jungle, a set of mother nets, \( \bar{A} \) is a recognizer and limit demands \( \mathcal{C} \) (\( \mathcal{C}(\mathcal{S}) \) precise notation, if necessary) is a sample of prerequisites to be satisfied in recognition processes. TD \( \mathcal{V}(\mathcal{S}) \) is a presolution of problem \( \mathcal{S} \), if \( S \mathcal{V}(\mathcal{S}) \in \mathcal{L}_A \), thus \( S \mathcal{V}(\mathcal{S}) \) being called a solution product, and if furthermore \( \mathcal{V}(\mathcal{S}) \) fulfils the demands in set \( \mathcal{C} \), \( \mathcal{V}(\mathcal{S}) \) is a solution of \( \mathcal{S} \). E.g. solution \( \mathcal{V} \) may be a system, by which from certain circumstances \( S \), with some limit demands (e.g. the number of the steps in the process) can be built surrounding \( S \mathcal{V} \), which in certain state \( \alpha(S \mathcal{V}) \) (for morphism \( \alpha \) of a recognizer) has a capacity characterized by the type of the elements in the final set of the recognizer.

We can describe a solution for a problem as wandering in a net:

1. Starting from a given net node (mother net)
2. to the acceptable net (solution product) \( \in \mathcal{L}_A \) of the TFG
3. via the right route in the RPG (TDG-solution) (limit demands accomplishments).

Cf. Aceto L, Fokkink WJ, Verhoef C (2001) mother net \( \vDash \langle \mathcal{T}D \rangle \bar{A} \) .
3. § Parallel Process and Abstract Algebras

(for Automated Problem Solving)

3.1. Partition RNS and Abstraction Relation

Definition 3.1.1. PARTITION RNS. For each jungle (here c) we define a partition RNS (PRNS) \( \mathfrak{w} \) of that jungle as a RNS fulfilling conditions (i)-(iii):

(i) \( \mathfrak{w} \) is manoeuvre mightiness and arity mightiness saving, not instance sensitive,

(ii) \( L(\text{apex}(\text{right}(r))) \setminus \Xi \) is a singleton and its element is outside \( L(c) \), whenever \( r \in \varphi, \varphi \in \mathfrak{w} \), and \( \{(\text{left}(r),\text{right}(r)): r \in \varphi, \varphi \in \mathfrak{w}\} \) is an injection,

(iii) \( \mathfrak{c}(\mathfrak{w}) \supseteq \{L(c) \cap L(c^{\hat{\mathfrak{w}}}) = \emptyset\} \).

In a special case where the left sides of rule preforms do not overlap each other, \( \{\text{apex}(\text{left}(r)): r \in \varphi, \varphi \in \mathfrak{w}\} \) is a partition of jungle c. Furthermore be notified that a characteristically feature of PRNS's is that \( L(c^{\hat{\mathfrak{w}}})(\mathfrak{w}^{-1}) \) is a partition of net c. We say that \( c^{\mathfrak{w}^{-1}} \) is \( \mathfrak{w} \)-partition result from c. Observe that for each PRNS there may be several jungles, the PRNS's of which it is an example of, the nets of those jungles having apexes of left sides of rule preforms of that PRNS in different positions. One of the important factors regarding to the partition result is the independence of reduction ordering Jantzen M (1997), partial matching Körner E, Gewalting M-O, Körner U, Richter A, Rodemann T (1999).

The next characterization clause 3.1 says that the necessary and sufficient condition in order to be the partition result of a PRNS for a rewrite object is that there is a one-to-one correlation between the elements of the partition of that rewrite object and the letters of the result in respect to the cardinality of the positions of the unoccupied arities.

Proposition 3.1. “Characterization Clause”. Let a and b be jungles. Then

\[
(i) \quad (\forall P \in \text{Par}(a)) (\exists n \in \{\delta_D(\alpha): \alpha \in \text{enc}(b), L(\alpha) \setminus \Xi \text{ is a singleton}\} \cup \{\delta_D(t): t \in P\})
\]

\[
|\bigcup(p(P,t): \delta_D(t) = n, t \in P)| \neq |\{c: |L(c) \cap \Xi| = n, c \in \text{enc}(b), L(c) \setminus \Xi \text{ is a singleton}\}|,
\]
or (ii) \((L(a) \setminus \Xi) \cap (L(b) \setminus \Xi) \neq \emptyset\),

if and only if

\[ a \not\equiv b \], whenever \( \mathcal{R} \) is a PRNS.

**Proof.** Our characterization (i) liable to net placing numbers is originated from PRNS definition item (iii), because of manoeuvre mightiness and arity mightiness saving feature of PRNS and characterization (ii) is a subject to definition item (ii). \( \Box \)

**Definition 3.1.2.** **Substance** and **Concept.** If for jungles \( s \) and \( t \) and PRNS \( \mathcal{W} \) of \( s \) there is equation \( s \mathcal{W} \hat{=} t \), we say that rewrite object \( s \) is a *substance* of \( t \) via \( \mathcal{W} \), and rewrite result \( t \) is the *concept* of \( s \) via \( \mathcal{W} \).

**Lemma 3.1.** For each jungle \( c \) and each PRNS \( \mathcal{W} \) of \( c \)

\[ c \mathcal{W} \hat{=} (\mathcal{W}^{-1}) \hat{=} c \]

**Proof.** Straightforward due to non-deleting rules and (iii)-condition in PRNS’s yielding the partition result is independent of reduction orders. \( \Box \)

“The abstraction relation” to be presented next, is needed in the process to refer to a common origin via PRNS between the subjects in problems to be solved and jungles presenting known solutions.

**Definition 3.1.3.** **Abstraction Relation.** The *abstraction relation* (AR) is such a binary relation of the pairs of jungles, where for each pair (here \((s,t)\)) there is such substance \( c \) and intervening PRNS \( \mathcal{W}_1 \) and \( \mathcal{W}_2 \), that

\[ c \mathcal{W}_1 \hat{=} s \quad \text{and} \quad c \mathcal{W}_2 \hat{=} t . \]

Concepts \( s \) and \( t \) are said to be *abstract sisters* with each other and \( c \) is entitled a *common origin* for \( s \) and \( t \).

**Theorem 3.1.** “A characterization of the abstraction relation”. Let \( \theta \) be the abstraction relation, and \( a \) and \( b \) be two jungles. Thus

\[ a \theta b \iff \delta_D(a) = \delta_D(b) . \]
The proof is executed in a finite case and for nets instead of jungles, but that does not diminish the power of the proof. Let $A_1$, $A_2$, $B_1$, $B_2$, and $B_3$ be such jungles that $A_1 \cup A_2$ is a partition of net $a$, and $B_1 \cup B_2 \cup B_3$ is a partition of net $b$. We can construct substance $c$ for $a$ and $b$ as in the following figures, distinguished in two different cases depending on the positions of unoccupied arities.

For border $\mathcal{A}_{12}$ in the partition of net $a$ and borders $\mathcal{B}_{12}$ and $\mathcal{B}_{23}$ in the partition of net $b$ it is to be constructed net $c$ and partitions for it, where

(i) $A'$-partition: $A_1' \cup A_2'$, where $|A_1'| \geq |A_1|$, $|A_2'| \geq |A_2|$, and there is bijection $f_a: A_1' \cup A_2' \rightarrow A_1 \cup A_2$ such that $|L(a')| \geq |L(f_a(a'))|$ whenever $a' \in A_1' \cup A_2'$, and

(ii) $B'$-partition: $B_1' \cup B_2' \cup B_3'$, where $|B_1'| \geq |B_1|$, $|B_2'| \geq |B_2|$, $|B_3'| \geq |B_3|$, and there is bijection $f_b: B_1' \cup B_2' \cup B_3' \rightarrow B_1 \cup B_2 \cup B_3$ such that $|L(b')| \geq |L(f_b(b'))|$ whenever $b' \in B_1' \cup B_2' \cup B_3'$.

For border $\mathcal{A}_{12}$ and borders $\mathcal{B}_{12}$ and $\mathcal{B}_{23}$, a subset of the set of the linkages of the nets in $A'$-partitions " fulfill the equations:

(iii) $|\mathcal{A}_{12}| = |\mathcal{A}_{12}|$, $|\mathcal{B}_{12}| = |\mathcal{B}_{12}|$, $|\mathcal{B}_{23}| = |\mathcal{B}_{23}|$, and

(iv) $A_1$, $A_1'$ and $A_2$, $A_2'$ are sets of unoccupied arities positioned as shown in cases $1^\circ$ and $2^\circ$.

Straightforwardly we thus can construct PRNS $\mathcal{W}_a$ and $\mathcal{W}_b$ of net $c$ such that
\[
A_1' \mathcal{W}_a = A_1, A_2' \mathcal{W}_a = A_2, B_1' \mathcal{W}_b = B_1, B_2' \mathcal{W}_b = B_2, B_3' \mathcal{W}_b = B_3.
\]

Case $1^\circ$ The unoccupied arities are in neighbouring elements in a partition of net $b$.

Case $2^\circ$ The unoccupied arities are in such elements of a partition of net $b$ which are totally isolated from each other.

**PROOF.** $\Rightarrow$:

Let us in contradiction suppose $\delta_D(a) \neq \delta_D(b)$. If $c$ is a substance for net $a$, we have $\delta_D(c) = \delta_D(a)$, because the PRNS between $a$ and $c$ is arity mightiness saving, and from the same reason we are
not able to get any concept to \( c \) with the cardinality of the unoccupied arities differing from that cardinality of \( c \). Therefore \((a, b) \notin \theta. \)

**Corollary 3.1.1.** Any substance and any of its concepts are in the abstraction relation with each other.

**PROOF.** Any substance and its concepts have the same amount of unoccupied arities, because intervening PRNS’s are arity mightiness saving. \( \square \)

**Corollary 3.1.2.** The abstraction relation is an equivalence relation.

**PROOF.** Theorem 3.1. \( \square \)

### 3.2. Altering RNS

“Macros” treated in this chapter are needed in process to get solutions for elements in the subject of the problem in study via known solutions in memories for problems the subject consisting nets with other elements than in the original subject.

**Theorem 3.2. “Altering macro RNS-theorem”.** Let \( \mathcal{R} \) be a RNS, nonconditional for the sake of simplicity, let \( t \) be an arbitrary jungle and \( \mathcal{W} \) a nonconditional PRNS of \( t \). Then there is such RNS \( \mathcal{R}_w \) and such a PRNS \( \mathcal{W}_o \) of \( t \mathcal{R}^\wedge \) that there is in force an implicit equation of first order for unknown \( \mathcal{R}_w^\wedge \), where \( \mathcal{R}_w^\wedge \) is a composer for a linear composition of \( \mathcal{R}^\wedge \):

\[
\begin{align*}
\mathcal{W}^\wedge \mathcal{R}_w^\wedge (\mathcal{W}_o^{-1})^\wedge &= t \mathcal{R}^\wedge.
\end{align*}
\]

We can also solve unknown RNS \( \mathcal{R}^\wedge \) from the explicit equation above for \( \mathcal{R}^\wedge \) with suitable PRNS \( \mathcal{W}_o \) of \( t \mathcal{R}^\wedge \), if PRNS \( \mathcal{W} \) and RNS \( \mathcal{R}_w \) are given.

**PROOF.** Without loosing of generality we present the proof keeping nets as rewrite objects instead of jungles.
1° First we prove the implicit equation.

Let t be an arbitrary net and \( \hat{w} \) be a nonconditional PRNS and \( \mathcal{R} \) an arbitrary nonconditional RNS. Let \( \mathcal{S} \) be such an injection that joins an index element to each rule preforms of any rule, such that \( \varphi = \{a_i \rightarrow B_i : i \in \mathcal{S}(\varphi)\} \), whenever \( \varphi \in \mathcal{R} \). Let us next construct required \( \mathcal{R}_w \), a rule number altering \( \text{macro RNS} \) for \( \mathcal{R} \) in regard to \( \hat{w} \), (thus \( \mathcal{R} \) being entitled as one of the \( \text{micro RNS} \)'s of \( \mathcal{R}_w \)).

We denote jungle \( G_o(t, \hat{w}) = (L(t \hat{w}^\uparrow))(\hat{w}^{-1})^\uparrow \). Hereby on the basis of lemma 3.1.3 we achieve \( G_o(t, \hat{w}) \cap s \in \text{Par}(s) \), whenever \( s \in \text{enc}(t) \). Let us consider rule preform \( a_i \rightarrow B_i \) \( i \in \mathcal{S}(\varphi) \), \( \varphi \in \mathcal{R} \). Let \( c_i \) be the context of a representative in \( [t] \) for apex(\( a_i \)). Next for each \( i \in \mathcal{S}(\varphi) \) we define

\[ Q_i = \{g \in G_o(t, \hat{w}) : g \cap \text{apex}(a_i) \neq \emptyset, g \cap \text{apex}(c_i) \neq \emptyset\} \]

For each \( i \in \mathcal{U}(\mathcal{R}) \) and each \( q_i \in Q_i \) we construct a PRNS of \( q_i \cap \text{apex}(a_i) \), say \( \rho_{q_i a_i} \), and a PRNS of \( q_i \cap \text{apex}(c_i) \), say \( \rho_{q_i c_i} \). Next we define the set of conditional demands \( \mathcal{C}_{oa}(\hat{w}) = \{\text{for each } i \in \mathcal{U}(\mathcal{R}) \text{ and each } q_i \in Q_i \ \rho_{q_i a_i} \text{ is applied only for } q_i \cap \text{apex}(a_i) \} \) and the application order is: first \( \rho_{q_i a_i} \) then \( \hat{w} \). We define PRNS

\[ \hat{w}_{oa} = \bigcup (\rho_{q_i a_i} \cup \hat{w} : i \in \mathcal{U}(\mathcal{R}), q_i \in Q_i, \mathcal{C}_{oa}(\hat{w})) \]

Now let \( d_{q_i} \) be such a representative of such a net class that \( (q_i \cap \text{apex}(a_i)) \rho_{q_i a_i} \) is the context of the \( d_{q_i} \) for \( (q_i \cap \text{apex}(c_i)) \rho_{q_i c_i} \). Let \( \rho_b \) be a PRNS of \( b_i \), \( i \in \mathcal{U}(\mathcal{R}) \), and

\[ \mathcal{C}_{ob}(\hat{w}) = \{\text{for each } i \in \mathcal{U}(\mathcal{R}) \text{ and each } b_i \in B_i \ \rho_b \text{ is applied exclusively for } b_i \text{ in the position where rule preform } a_i \rightarrow b_i \text{ has transformed it and the application order is: first } \rho_b \text{ then } \hat{w} \text{ be a set of conditional demands. Let us define PRNS}
\]

\[ \hat{w}_{oc} = \bigcup (\rho_{q_i c_i} \cup \hat{w} : i \in \mathcal{U}(\mathcal{R}), q_i \in Q_i, \mathcal{C}_{oc}(\hat{w})) \], where

\[ \mathcal{C}_{oc}(\hat{w}) = \{\text{for each } i \in \mathcal{U}(\mathcal{R}) \text{ and each } q_i \in Q_i \ \rho_{q_i c_i} \text{ is applied only for } q_i \cap \text{apex}(c_i) \} \text{ and the application order is: first } \rho_{q_i c_i} \text{ then } \hat{w} \text{ is a set of conditional demands. Further we define PRNS } \hat{w}_o = \bigcup (\rho_b \cup \rho_{q_i c_i} \cup \hat{w} : i \in \mathcal{U}(\mathcal{R}), \mathcal{C}_{ob}(\hat{w}), \mathcal{C}_{oc}(\hat{w})) \}. \]
Now we obtain
\[ t \overset{\circ}{w} \overset{\circ}{R}_{i_{0}} \overset{\circ}{w}_{0}^{-1} = t (a_i \rightarrow B_i) \], because \( \beta_i \), \( i \in \mathcal{U}(\mathcal{R}) \), are manoeuvre mightiness saving. In the next phase we continue the process for net \( t \overset{\circ}{w} \overset{\circ}{R}_{i_{0}} \overset{\circ}{w}_{0}^{-1} \) and obtain \( t \overset{\circ}{w} \overset{\circ}{R}_{j_{1}} \overset{\circ}{w}_{0}^{-1} \overset{\circ}{R}_{j_{1}} \overset{\circ}{w}_{1}^{-1} = t (a_i \rightarrow B_i) (a_j \rightarrow B_j) \), where \( i, j \in \mathcal{U}(\mathcal{R}) \), and \( \overset{\circ}{R}_{i_{0}} \) and \( \overset{\circ}{w}_{0} \) are constructed for net \( t \overset{\circ}{w} \overset{\circ}{R}_{i_{0}} \overset{\circ}{w}_{0}^{-1} \) analogously with \( \overset{\circ}{R}_{i_{0}} \) and \( \overset{\circ}{w}_{0} \) for \( t \).

The continuation of that process concludes our proof for the implicit part of the theory.

2° Now we are ready to move to prove the explicit interpretation of our equation. Let us denote \( \varphi = \{ a_i \rightarrow \beta_i : i \in \mathcal{U}(\varphi) \} \), whenever \( \varphi \in \mathcal{R}_{i_{0}} \). Now we have \( \overset{\circ}{w} \) and \( \overset{\circ}{R}_{i_{0}} \) given. For each \( i \in \mathcal{U}(\mathcal{R}_{i_{0}}) \) let \( \beta_i \) be such a PRNS via which each \( \beta_i \) is a concept. We construct a set of conditional demands \( \mathcal{C}_{\varphi}(\overset{\circ}{w}) = \{ \text{for each } i \in \mathcal{U}(\mathcal{R}_{i_{0}}) \text{ and each } \beta_i \in \mathcal{B}_i, \beta_i \text{ is applied exclusively for } \beta_i \text{ in the position where rule preform } a_i \rightarrow \beta_i \text{ has transformed it and the application order is: first } \beta_i \text{ then } \overset{\circ}{w}. \} \). Further we define PRNS \( \overset{\circ}{w}_{0} = \cup (\beta_i \cup \overset{\circ}{w} : i \in \mathcal{U}(\mathcal{R}_{i_{0}}), \mathcal{C}_{\varphi}(\overset{\circ}{w})) \), and give \( \overset{\circ}{R}_i = \{ a_i(\overset{\circ}{w})^{-1} \rightarrow \{ \beta_i(\overset{\circ}{w})^{-1} : \beta_i \in \mathcal{B}_i \} \} \). Now we can proceed as in 1°. □

It is worthy to observe that any macro/micro depend only on its micros/macros respectively and on the intervening PRNS’s, but not on the rewrite objects which might contain large number or even unlimited number of places for redexes of rules in micros.

3.3. Parallel Process and the Closure of Abstract Languages

**Definition 3.3.1.** Let \( \mathcal{S} \) be an arbitrary set and for each \( i, j \in \mathcal{S} \) let \( \theta_{ij} \) be the abstraction relation, and let
\[ \tilde{\theta} = \Pi(\theta_{ij} : (i, j) \in \mathcal{S}), \] thus \( \tilde{\theta} \)-associated languages is called \( \mathcal{S} \)-abstract language.
**Definition 3.3.2.** MACRO and MICRO TD. Let $\mathcal{R}$ be a set of RNS’s and $\mathcal{K}$ a TD over $\mathcal{R}$ (here singleton set of RNS’s and its element are equalized). We define a **macro TD** of $\mathcal{K}$ in regard to set $\mathcal{W}$ of interacting PRNS’s, denoted $\mathcal{K}_{\mathcal{W}}$, for which $\mathcal{K}_{\mathcal{W}} = \mathcal{K}(\mathcal{R} \rightarrow \mathcal{R}_{\mathcal{W}} : \mathcal{R} \in \mathcal{R}, \mathcal{W} \in \mathcal{W})$, where $\mathcal{R}_{\mathcal{W}}$ is a macro RNS for $\mathcal{R}$ in regard to $\mathcal{W}$. We say that $\mathcal{K}$ is a **micro TD** of $\mathcal{K}_{\mathcal{W}}$, and denote it $(\mathcal{K}_{\mathcal{W}})_{\mathcal{W}^{-1}}$.

The following "parallel"-theorem, one of the direct consequences from "altering macro RNS"-theorem, describes the invariability of the abstraction relation or the closures of abstract languages in class transformation relations, and taking advantage of the equation of "altering macro RNS"-theorem it gives TD-solutions for any problem each mother net of the subject of the problem is an abstract sister to a net which is a mother net of the subject of a problem TD-solutions of which are known. Cf. *class rewriting or confluence modulo* Jantzen M (1997), or TD with possibly freely chosen rules in RNS’s as action cf. *simulation* (Baeten JCM, Basten T (2001); van Glabbeek RJ (2001)), *bisimilarity* Aceto L, Fokkink WJ, Verhoef C (2001). It is worth to mention that there is close connections to game theories, inferring *winning game graphs* Thomas W (1997), *bisimulation equivalence* Burkart O, Cauca D, Moller F, Steffen B (2001), representation changes, abstraction and reformulation in artificial intelligence (Zucker J-D (2003); Holte RC, Choueiry BY (2003)).

**Theorem 3.3.** "Parallel theorem". Let $\mathcal{R}$ be a RNS, $\theta$ the abstraction relation, a and b two such jungles that $a\theta b$, $\mathcal{W}_a$ and $\mathcal{W}_b$ two PRNS’s of such net c that a is a concept of c via $\mathcal{W}_a$ and b a concept of c via $\mathcal{W}_b$. Then we have a valid confluence condition regarding $\theta$ as follows:

1° \(a \overset{\mathcal{R}}{\sim} 0 b(\mathcal{R}_{\mathcal{W}_a^{-1}})_{\mathcal{W}_b}\),

and

2° \(a \overset{\mathcal{R}_{\mathcal{W}_a}}{\sim} 0 b(\mathcal{R}_{\mathcal{W}_b})\).

We call $\mathcal{R}$ and $(\mathcal{R}_{\mathcal{W}_a^{-1}})_{\mathcal{W}_b} parallel with each other, and on the other hand consequently $\mathcal{R}_{\mathcal{W}_a}$ and $\mathcal{R}_{\mathcal{W}_b}$ are also parallel with each other, pairwise preserving \(\theta\)-classes in derivations.
3.4. Abstract Algebras

**Lemma 3.4.1.** All nets in any denumerable class of the abstraction relation have the shared substance (the centre of that class).

**PROOF.** Let $\theta$ be the abstraction relation and let $H$ be a denumerable $\theta$-class. Each substance and its concepts are in the same $\theta$-class in accord to corollary 3.1.1. Because $H$ is an equivalence class being due to corollary 3.1.2, all substances in $H$ are in $\theta$-relation with each other. Repeating the reasoning above for substances of substances and presuming that $H$ is denumerable we will finally obtain the claim of the lemma. □

**Lemma 3.4.2.** Let $\theta$ be the abstraction relation. Furthermore let $R$ be a RNS, and let $Q$ be a distinctive denumerable $\theta$-class with $c$ being its centre. In addition we define a set of macro TD's:

$$R = \{ R^*_W : W \text{ is a PRNS of } c \}.$$ 

Therefore

$$\bigcup (Q \ R \ \theta) = c(R^* \cup \mathcal{U}) \theta,$$

where $\mathcal{U}$ is a trivial TD.

**PROOF.** Because $Q$ is distinctive, for each PRNS $W$ of $c$ $R_W^*$ has redexes exactly in one net of $Q$ and the other nets in $Q$ are in $\text{IRR}(R_W^*)$, our Parallel theorem yields $Q R W^* \leq c(R^* \cup \mathcal{U}) \theta$. Because $\theta$ is an equivalence relation, we get $Q R W^* \theta = c(R^* \cup \mathcal{U}) \theta$ and further

$$Q R = c(R^* \cup \mathcal{U}) \theta.$$ □

**Theorem 3.4.** “Abstraction closure-theorem”.

Let $A$ be the set of the denumerable $\theta$-classes, $\mathcal{A}$ is a set of RNS’s and

$$\mathcal{R} = \bigcup \{ R^*_W : W \text{ is a PRNS of } c \} : R \in \mathcal{A}, c \text{ is the centre of } Q, Q \in A \}$$

be a union of macro TD’s liable to $A$-classes. Then if $\theta$ is the distinctive abstraction relation, pair $(A, \mathcal{R})$ is an algebra, named abstract algebra or net class rewriting algebra.
Lemma 3.4.2 yields our claim, because our presumption for the abstraction relation yields each macro TD set \( \{ R^* \} \) in \( \mathcal{R} \) is matching exactly one \( \theta \)-class, and because the construction of macro yields for each center \( c \) of \( A \) equation

\[
|c \mathcal{R}^*| = |c \mathcal{R}| \text{ and therefore consequently for each } (Q \in A) \ Q\mathcal{R} \text{ is denumerable.} \]

Corollary 3.4. Parallel and Abstraction closure theorems are valid also in cases where the micro RNS is actually a general TD over a set of them and the intervening PRNS is a TD over a set of them as set in definition 3.3.2 (micro - macro TD).

In the next chapter we generalize the idea of PRNS to CRNS, “the cover RNS” where left sides of rule preforms are allowed to match apexes of right sides, and we study how nets are changed under TD’s over CRNS’s, and afterwards turned to be expressed by TD’s over PRNS’s of those rewrite objects. CRNS’s are important in expanding processes to search existing solutions in memory, the subjects of which being in the abstraction relation with the subject of the problem given to be solved.

4.§ Type wise Problem Solving Regarding to Intervening RNS´s

4.1. Cover RNS

In the following we are searching solutions for problems the mother nets having been built up by certain type of parts (elements in covers), this requirement is embedded in cover RNS’s, devoting CRNS as an abbreviation for that particular type of RNS. The apexes of the left sides of the rules in RNS’s in known TD (e.g. the catenation closure of RNS’s) may not be elements in any partition of the mother net of the problem studied, but in some more general cover. Furthermore we expand studies of RNS’s possessing multidimensional rules (G-RNS). The relations between PRNS and GCRNS are especially in focus. We construct generalized
macro/micro (GMA/GMI) TD for GCRNS. Abstraction relation θ is then defined as before except PRNS is replaced with different variations of GCRNS.

**Definition 4.1.1.** For each relation λ we define *relation RNS of* λ, RNS(λ), such that

\[ RNS(\lambda) = \{ s \rightarrow T : s \in \text{Dom}(\lambda), T = s\lambda \}. \]

Notice that in general there is in force equation \((RNS(\lambda))^{-1} = RNS(\lambda^{-1})\).

**Definition 4.1.2.** \textit{COVER RNS.} RNS \(\mathcal{R}\) is a *cover RNS* (CRNS) of jungle s, if it fulfills conditions (i)-(v):

(i) \(\mathcal{R}\) is manoeuvre mightiness and arity mightiness saving, not instance sensitive,

(ii) \(L(\text{apex}(\bigcup(\text{right}(\mathcal{R})))))) \setminus \Xi\) and set \(L(s)\) are distinct with each other,

(iii) There is such jungle \(s'\) for which \(s \subseteq \text{enc}(s')\) and

\(\mathcal{C}(\mathcal{R}) \supseteq \{ L(s') \cap L(s'\mathcal{R}^{-}) = \emptyset \} \) (totally changing the ranked letters of \(s\) and each rule preform of \(\mathcal{R}\) has a redex in \(s'\),

(iv) \(\{(\text{left}(r),\text{right}(r)) : r \in \omega, \omega \in \mathcal{R}\}\) is an injection,

(v) the right side of each rule preform of \(\mathcal{R}\) is a singleton.

Be notified that \(s\) itself may not possess any redex for CRNS. The set of all CRNS`s of jungle \(s\) is denoted CRNS(s). Observe that PRNS`s are examples of CRNS`s. We say that \(s\mathcal{R}^{-}\) is \(\mathcal{R}\)-*cover result* for \(s\).

**Proposition 4.1.1.** “Characterization Clause”. Let \(a\) and \(b\) be two distinct jungles. Then

\[ \delta_D(a) = \delta_D(b) \iff \text{there is such CRNS } \mathcal{R} \text{ that } a\mathcal{R}^{-} = b. \]

**PROOF.** \(\Leftarrow\): CRNS is arity mightiness and manoeuvre mightiness saving, and therefore in the rewrite objects for CRNS the cardinality of the set of the outward linkage connections of the redexes is not changing in derivations.

**PROOF.** \(\Rightarrow\): Choose \(\mathcal{R} = \{a \rightarrow b\} \). \(\square\)
Next we concentrate to make notions adequate to differentiate PRNS and CRNS.

Clearly CRNS is a genuine generalization of PRNS, because PRNS’s do not allow ranked letter mightiness increase and redexes are limited to inside of rewrite objects and genuine overlapping between left and right sides of the rules are excluded.

Because a CRNS rule may have more than one ranked letter in the right side with e.g. different number of inside links within the right side than in the left side, then the family of the unoccupied arity sets of the ranked letters in a rewrite result may deviate from the family of the unoccupied arity set of any partition of the corresponding rewrite object and therefore a CRNS result may not be derived from the same rewrite object by any PRNS.

Proposition 3.1 and the greater expansive nature of CRNS compared to PRNS raise the question: For which jungle $a$ and CRNS $R$ of it there is such PRNS $\hat{w}$ of $a$ that $aR^\hat{w} = a\hat{w}^\hat{w}$? The next proposition gives an answer.

**Proposition. 4.1.2.** Let $t$ be an arbitrary jungle. Let $R$ be a left-right distinct CRNS of $t$ (that is: for each rule preform $r$ apex(left($r$)) and apex(right($r$)) are distinct from each other), and for each rule preform $r$ in $R$ let

\[
(\exists P \in \text{Par}(\text{apex}(\text{left}(r)))) \,(\forall n \in \{\delta_D(\alpha) : \alpha \in \text{enc}(\text{apex}(\text{right}(r))), L(\alpha) \setminus \Xi \text{ is a singleton}\}) \cup \{\delta_D(t) : t \in P\})
\]

\[|\bigcup\{p(P, t) : \delta_D(t) = n, t \in P\}| = |\{c : |L(c) \cap \Xi| = n, c \in \text{enc}(\text{apex}(\text{right}(r))), L(\text{apex}(\text{right}(r))) \setminus \Xi \text{ is a singleton}\}|.
\]

Hence there is such PRNS $\hat{w}$ that $tR^\hat{w} = t\hat{w}^\hat{w}$.

**PROOF.** We apply characterization proposition 3.1 upon the pairs of the left-right sides of the rule preforms in $R$. Being due to our presumptions for the rules of $R$ proposition 3.1 yields that for each rule preform $r$ in $R$ there is such PRNS $\hat{w}$, that apex(right($r$)) is $\hat{w}$-$\text{partition result for apex(left($r$)), furthermore we require that all sets }L(\text{right}(\hat{w}), r \in \omega, \omega \in R), \text{ are distinct from each other. By choosing }\hat{w} = \bigcup\{\hat{w}_r : r \in \omega, \omega \in R, \mathcal{C} = \{\mathcal{C}(\hat{w}) : r \in \omega, \omega \in R\}\text{ we ’ll get a desired PRNS, because }R \text{ is left-right distinct (apex(left(}R)) \text{ being a subset of a partition of }t). \square
In the following definition by generalization we give new types of RNS’s relating to the types of PRNS and CRNS.

**Definition 4.1.3.** GPRNS and GCRNS. GPRNS is RNS which is defined as PRNS but the condition “manoeuvre mightiness saving” is replaced with demand “not manoeuvre deleting and the right sides of the rule preforms are allowed to be also jungles instead of only nets”, and GCRNS is RNS which is defined as CRNS with the above replacement.

Clearly we can generalize theorem 3.2 to be valid also for GPRNS in addition to PRNS.

**Proposition 4.1.3.** Let be a GCRNS of jungle a. If the right sides of the rule preforms among the rules in are distinct from each other (we say is distinct from right sides) (reserving the symbols \( \text{C}_{\text{dRNS}} \) for CRNS and \( \text{GC}_{\text{dRNS}} \) for GCRNS in this respect), then

\[ a\mathcal{R}^*\mathcal{R}^{-1} = a. \]

If is not distinct from right sides, then we have \( a \subseteq a\mathcal{R}^*\mathcal{R}^{-1}. \)

**PROOF.** \( \text{GC}_{\text{dRNS}} \) is not manoeuvre deleting and is totally changing the ranked letters in rewrite objects (condition (iii) in the definition of CRNS). □

Next in the following chapter we prove “Altering Macro RNS”-theorem 3.2 generalized to deal also with the wider intervening RNS-type, cross colouring RNS, and in order to extend problem solving to fit also to that intervening type, a characterization of abstraction relation regarding that type is introduced.

4.2. Generalizing Altering Macro RNS Theorem

Before going to the next theorem we widen notion CRNS embedding it into general RNS’s at overlapping sections between left and right sides of rule preforms.
Definition 4.2.1. CROSS COLOURING RNS, CLRNS. Let \( \mathfrak{W} \) be a RNS. For each net \( r \) we define relations \( O_L \) from pairs \((s,R)\) to nets, where \( s \) is a rule preform in \( \mathfrak{W} \) and \( R \) is a GC\(_d\)RNS recursively:

\[
O_L(s,R) = \bigcup ((\text{apex(left}(s)) \mathcal{R} \text{apex}(r)) \mathcal{R}^*)
\]

\[
O_L(t,R) = \bigcup ((\text{apex(left}(t)) \mathcal{R} O_L(s,R)) \mathcal{R}^*).
\]

Notice that \( O_L \) may not be any mapping due to its potentiality to possess multi-images. We say that \( \mathfrak{W} \) is a cross colouring RNS in respect to net \( r \), CLRNS(\( r \)), if \( O_L(t,R) \) is an enclosement of \( \text{apex(right}(t)) \), whenever \( \text{apex(left}(t)) \mathcal{R} O_L(s,R) \neq \emptyset \) for some \( s \) (\( O_L(t,R) \) thus entitled a coloured jungle whereas \( R \) is a colouring GC\(_d\)RNS of \( \mathfrak{W} \) in respect to \( r \)). If there is such a \( r \)-embedding net \( t \) that \( \text{PI}(\bigcup (\text{apex(left}(\mathfrak{W})))) \mathcal{R} t \in \text{Par}(t) \), we say that CLRNS(\( r \)) \( \mathfrak{W} \) is total.

Definition 4.2.2. MACRO AND MICRO IN REGARD TO GPRNS AND CLRNS.

Let \( R \) be a RNS and \( \mathfrak{W} \) a nonconditional RNS of type \( T \), \( T \in \{ \text{GPRNS,CLRNS} \} \). If there is such a RNS, \( \mathfrak{W} \), and such a nonconditional T-type RNS \( \mathfrak{W}_o \) that there is in force an implicit equation of first order for unknown \( \mathfrak{W}_o \), thus \( R \) being a composer for a linear composition of \( \mathfrak{W} \):

\[
\mathfrak{W}^* R_\mathfrak{W}^* (\mathfrak{W}_o)^{-1} = R^*.
\]

we call \( R_\mathfrak{W} \) a macro of \( R \) in regard to \( \mathfrak{W} \), indicated by MA(\( R,\mathfrak{W} \)). Consequently we entitle \( R \) a micro of \( R_\mathfrak{W} \) in regard to \( \mathfrak{W} \), indicated by MI(\( R,\mathfrak{W} \)).

Theorem 4.2.1. Let \( r \) be a net and \( \mathfrak{R} \) be a RNS. Furthermore let \( \mathfrak{W} \) be a nonconditional total CLRNS(\( r \)) and

\[
a_{\mathfrak{W} r} = \bigcap (t : \mathfrak{r} \in \text{enc}(t), \text{PI}(\bigcup (\text{apex(left}(\mathfrak{W})))) \mathcal{R} t \in \text{Par}(t)),
\]

"the smallest \( r \)-embedding net possessing a partition of \( r \) by \( \mathfrak{W} \)", then such a partition of \( a_{\mathfrak{W} r} \mathfrak{W}^* \), say \( P \), is achieved via \( O_L \)-relations in \( \mathfrak{W} \) that \( P^{(\mathfrak{W}^{-1})^*} \) is a partition of \( r \) and we can obtain MI(\( \mathfrak{R},\mathfrak{W} \)) as well as MA. Notice that in addition in a special case where each colouring GC\(_d\)RNS in respect to net \( r \) in \( \mathfrak{W} \) can be chosen among the set of GPRNS’s, and if \( \mathcal{P} \) is the set of those GPRNS’s, then \( \bigcup \mathcal{P} \) is a GPRNS of \( r \).
PROOF. Analogous with altering macro theorem, the set of the colouring GCₐRNS´s equating GPRNS´s for elements of aₜₐ_partitions as rewrite objects. □

Definition 4.2.3. MACRO AND MICRO IN REGARD TO TD OVER GPRNS´s AND CLRNS´s.

We define macro MA and micro MI in regard to TD over GPRNS´s, respectively over CLRNS´s as previously in the cases over PRNS´s. Consequently we use notations MA(TD,CLRNS) and respectively for MI. Furthermore for each TD ∈ we denote
\[ G_{\Delta}^{+}(T) = \{ R^\sigma : \text{the elements of } R^\sigma \text{ are of type } T \} \]
\[ G_{\Delta}^{-}(T) = \{ (R^\sigma)_{\eta<1} : \text{the elements of } R^\sigma \text{ are of type } T \} , \]
whenever \( T \in \{ \text{GPRNS, CLRNS} \} \). TD´s \( \Delta \) and \( (\Delta, \eta_{<1}) \) are called parallel with each other, denoted also parallel(\( \Delta \)) = (\( \Delta, \eta_{<1} \)) or parallel((\( \Delta, \eta_{<1} \)) = \( \Delta \).

Notice that because CLRNS´s are genuine generalizations of GPRNS´s we have equations
\[ G_{\Delta}^{+}(\text{GPRNS}) \subseteq G_{\Delta}^{+}(\text{CLRNS}) \quad \text{and} \quad G_{\Delta}^{-}(\text{GPRNS}) \subseteq G_{\Delta}^{-}(\text{CLRNS}) . \]

Theorem 4.2.1 yields the following theorem for more general cases:

Theorem 4.2.2. For each nonconditional \( \bar{w} \) of type \( T \), \( T \in \{ \text{GPRNS, CLRNS} \} \), and each string \( \bar{R}_o \) over set of RNS´s, there is \( \bar{R}_{\Delta \bar{w}} \), and such of type \( T \) RNS \( \bar{w}_o \) that
\[ \bar{w} \hat{\bar{R}} \hat{\bar{w}}_{\bar{R}} \hat{\bar{w}}_{\bar{w}_o} = \bar{R} . \]

Corollary 4.2.1. The above theorem can be dressed also somewhat more generally:
Let \( \bar{R}_o \) be a micro TD over set of RNS´s and \( \bar{W}_{\sigma} \) a set of intervening nonconditional RNS´s of type \( T \). Then there is such macro TD of \( \bar{R}_o \), \( \bar{R}_{\Delta M} \), and such set of reversed T-type RNS´s, \( \bar{W}_{\sigma} \), that we have commutativity condition
\[ \bar{W} \hat{\bar{R}} \hat{\bar{W}} \hat{\bar{W}} = \bar{R} , \]
and it manifestates a natural transformation between Functors determining parallel rewriting cf. Theorem 3.3.
Definition 4.2.4. GENERALIZED ABSTRACTION RELATION. The generalized abstraction relation in regard to type T of intervening RNS, denoted GAR(T), T \in \{PRNS, GPRNS, CLRNS, GCdRNS, GCRNS\}, (in short abstraction relation of type T) is such a binary relation in the set of the nets, where for each pair (here (s, t)) there is such net c and intervening RNS W_1 and W_2 of type T, that
\[ cW_1 \simeq s \quad \text{and} \quad cW_2 \simeq t. \]
Nets s and t are said to be abstract sisters of type T with each other, c being a common substance of s and t. Notice that GAR is a genuine generalization for abstraction relation AR, and that AR = GAR(PRNS).

Proposition 4.2.1. “A characterization of abstraction relation GAR(CLRNS).”
Let a and b be two nets and let \( \theta \) be GAR(CLRNS). Then
\[ a \theta b \iff \delta_D(a) = \delta_D(b). \]
PROOF. Theorem 4.2.1 and characterization proposition 4.1.1. 

Remark 4.2. Straightforwardly widening the definition for “parallel” to deal with intervening RNS’s of type GPRNS and CLRNS instead of solely dealing with type PRNS, we clearly have the results for GAR(CLRNS) as is obtained for AR in corollaries 3.1.1 and 3.1.2, result 3.1, parallel-theorem, lemmas 3.4.1 and 3.4.2 and theorem 3.4 and finally consequently results concerning generalizations for TD’s as stated in Corollary 3.4.

Proposition 4.2.2. “Characterization of GAR.”
Let \( T \in \{PRNS, GPRNS, CLRNS, GCdRNS, GCRNS\} \) and let s and t be nets. Then s and t are abstract sisters of type T, if and only if there exist such intervening RNS \( \mathcal{C}_s \) and \( \mathcal{C}_t \) of type T that
\[ (\exists A_s \in \text{Par}(s(\mathcal{C}_s^{-1}))) \quad \text{and} \quad (\exists A_t \in \text{Par}(t(\mathcal{C}_t^{-1}))) \]
there is a bijection between \( A_s \) and \( A_t \).
PROOF. The right sides of rule preforms must pair wise in both of the intervening RNS’s possess the same number of different manoeuvre letters liable to cardinality of the bijection between the related partitions. 

Definition 4.2.5. CONGRUENCE. Let \( \theta \) be a relation in the set of the jungles. We say that \( \theta \) is a congruent relation of TD-type T, if there is in force:
\[ a \theta b \iff a \varphi_a \theta b \varphi_b \text{ whenever } \varphi_a \text{ and } \varphi_b \text{ are TD's of type } T. \]

Each congruent relation of type T, which is an equivalence relation, is entitled *congruence relation of type T*. The set of all congruence relations of type T is denoted Cg(T).

**Theorem 4.2.3.** For each TD-type \( T_m \) \( \text{GAR}(T_m) \in \text{Cg}(T_n) \), \( m \geq n \), \( m, n = 1,2,3 \), where 

\( (T_1, T_2, T_3) = (\text{PRNS}, \text{GPRNS}, \text{CLRNS}) \).

**Proof.** \( \text{GAR}(T) \) is congruent, because any catenation of TD’s is of the same type as the TD of the most general type in that catenation and \( T_m \) is a generalization of \( T_n \), if \( m \geq n \). Proposition 4.2.1 yields the equivalence requirement. \( \square \)

**Syntax of Automated Problem Solving System.**

The mother net of a given problem is first transformed by an intervening CLRNS to concept net for which we construct an abstract sister, one of the substances of which has a partition in bijection with a partition of a substance of the said concept net. Now the known transducer renders possibility to construct a macro for it, the parallel counterpart and finally a micro parallel macro, because the reached concepts guarantee the survival of information of the rules in known TD’s in the process. By iteration we can reach for our original problem a presolution, which finally is a desired solution, if the product is in the anticipated language fulfilling the set of limit demands.

Directly searching a common substance of certain type for a net pair would be substantially more difficult if even impossible than going through pair (macro,parallel macro) in a case where either of the nets in said pair is undenumerable regarding to the cardinalities of the sets of their letters (and actually even if the cardinality of one of them is immense although denumerable).
Part II

Evolution Theory of
Self-Evolving Autonomous Problem Solving Systems

5§ Universal Partitioning

In each phase of parallel rule execution resemblance with memory is important due to construction orders to solve original problem. In problem solving commutative requirement is set to guarantee that the initial resemblance between perceived problem and achieved memory counterparts will be preserved while processing memory in order to give a solution from memory back to the original problem, not something totally different object. The commutative property here is generally comparable to the invariability of equivalence relation in parallel transformations or to generalized congruence as well as to derivations in quotient algebras or obedience to confluence, and actually describes closure properties in systems. Interesting is that these properties are related to symmetry and conservation laws, so very essential in nature. Abstraction relations themselves can be regarded as Functors in the set of Categories, intermediating parallel systems.

In this chapter we introduce notion of new intervening RNS, universal partitioning, encompassing the idea of the widest possible memory hunting: each problem is always depending exclusively on its environment and cannot be understand in any other way; even an arbitrary element can only be noted (as symbol level) but not understood feasibly without the idea of its negation. Without the transaction via said universal RNS´s one cannot even think to search solving RNS´s liable to creation of new outside links to the probed objects.

Definition 5.1. We denote ILC(s) the set of the inward linkage connections of net s and OLC(s) stands for the outward linkage connections.
Definition 5.2. We call cardinality $\text{ORN}(s) = \delta_D(s) \cup |\text{ILC}(s)|$ outward rank number of net $s$, $\delta_D$ standing for the cardinality mapping of the positions of the unoccupied arities.

Definition 5.3. RNS is outward rank number saving, if $\text{ORN}(\text{left}(r)) = \text{ORN}(\text{right}(r))$, whenever $r$ is a rule preform of the RNS in question. This kind of RNS-type is preserving the character of the realization relations in transformations and quarantining the resemblance in memory hunting between original problem-net and the counterpart in the memory.

Definition 5.4. UNIVERSALLY PARTITIONING RNS. For each jungle (here $c$) we define a universally partitioning RNS (UPRNS) $\mathfrak{w}$ of that jungle as a RNS fulfilling conditions (i)-(iii):

(i) $\mathfrak{w}$ is thoroughly totally environmentally saving and outward rank number saving,

(ii) $\mathfrak{c}(\mathfrak{w}) \supseteq \{L(c) \cap L(c^\mathfrak{w}) = \emptyset\}$,

(iii) $L(\text{apex(right}(r)))) \setminus \Xi$ is a singleton and its element is outside $L(c)$, whenever $r \in \varphi, \varphi \in \mathfrak{w}$, and $\{(\text{left}(r),\text{right}(r)): r \in \varphi, \varphi \in \mathfrak{w}\}$ is an injection.

Evidently each GPRNS is UPRNS. An interesting observation can be made: GPRNS are giving resemblances in the memory to nets to be solved such that the mutual order among redexes are preserved, but UPRNS´s give also opportunities to probe the memory among cases where the concerning orders are changed.

Definition 5.5. UNIVERSAL ABSTRACTION RELATION. Universal abstraction relation of type $T$ $(\in \text{ITG} (=\{\text{PRNS,GPRNS,CLRNS,UPRNS}\} \text{ (the symbol reserved for this use)})$, UAR(T), is defined as GAR, but type $T$ is allowed to be also of type UPRNS.

Definition 5.6. UNIVERSAL MACRO TD. Universal macro of TD $\mathfrak{R}$ over a subset $\mathfrak{F}$ of ITG, $\text{UMA}(\mathfrak{F}, \mathfrak{R})$, is TD

$\mathfrak{R}(\mathfrak{R} \rightarrow \mathfrak{R}_{\mathfrak{F}}: \mathfrak{R}_{\mathfrak{F}} \text{ is of type in } \mathfrak{F}, \mathfrak{R} \in \mathfrak{R})$,

where $\mathfrak{R}$ is the set of RNS´s which $\mathfrak{R}$ is over.
Definition 5.7. PARALLEL RNS’s. We generalize in a natural way parallel RNS-definition (cf. Parallel theorem): for abstraction relation 0 of type T (∈ITG) we denoted $\mathcal{R}_1 \bowtie_0 \mathcal{R}_2$ for 0-parallel RNS’s $\mathcal{R}_1$ and $\mathcal{R}_2$. If 0 has no requirements in addition to its type, say T, we denote for the sake of clarity $\mathcal{R}_1 \bowtie_T \mathcal{R}_2$. Clearly $\bowtie_T$ is an equivalence relation in the set of the jungles.

Definition 5.8. PARALLEL TD’s. Let $\mathcal{F}$ be a subset of ITG. We say that TD $\mathcal{R}$ over set of RNS’s, say $\mathcal{R}$, and

$$\mathcal{P} = \text{UMA}(\mathcal{F}, \mathcal{R}, (\mathcal{R} \leftarrow \mathcal{P} : \mathcal{P} \bowtie_T \mathcal{R} \text{, } T \in \mathcal{F}, \mathcal{R} \in \mathcal{R}))$$

are $\mathcal{F}$-parallel TD’s with each other, denoted $\mathcal{P} \bowtie_\mathcal{F} \mathcal{R}$. Clearly $\bowtie_\mathcal{F}$ is an equivalence relation in the set of the TD’s and is named parallel TD-relation over $\mathcal{F}$, denoted $\bowtie_\text{TD}$, if $\mathcal{F}$ is not determined.

Definition 5.9. Let $\mathcal{F} \subseteq \text{ITG}$. We say that transformation relations $\rightarrow_{\mathcal{R}_1}$ and $\rightarrow_{\mathcal{R}_2}$ are $\mathcal{F}$-parallel with each other, if $\mathcal{R}_1 \bowtie_\mathcal{F} \mathcal{R}_2$.

Theorem 5.1. If intervening RNS’s are of the universally partitioning type, parallel RNS´s can be compiled solely in the correspondence with the counterparts in the macro RNS´s in charge.

Proof. Let s be a net, $\emptyset$ a UPRNS, t be $\emptyset$-irreducible image in catenation closure of $\emptyset$-transformation relation from s and $\mathcal{R}_t$ an arbitrary RNS. Let $\mathcal{R}_s$ be such RNS that

$$\exists D \in \text{Par}(s) \exists \text{ bijection between sets } \{\text{ORN}(\sigma) : \sigma \in L(t) \setminus \Xi\} \text{ and } \{|\text{OLC}(d)| : d \in D\}.$$

Therefore $\mathcal{R}_s \bowtie_{\text{UPRNS}} \mathcal{R}_t$. The rest follows from Altering macro RNS-theorem. □.

Next we introduce an extensive example to demonstrate essential features in theorem 5.1

Example. We introduce specifically only RNS-rules for initial and final intervening UPRNS’s $\emptyset$ and $\tilde{\emptyset}$ in the process, each constructed with a little bit different way to depict a variety of alternatives. Let
\[ \varnothing = \{ \varpi_{11}, \varpi_{12} : \text{application order is } \varpi_{11}, \varpi_{12} \}, \]

where arity alphabets in tied terms are designated exclusively for the corresponding net representations and

\[ \varpi_{11} = b(\xi_1x_1, \xi_2; \xi_3y_1 | i = 2,3, j = 1,2) \rightarrow b(\xi_2; \xi_3, \xi_4y_1 | i = 1,2, j = 1,3), \]

where for the left side substitution \( f \) and for the right side substitution \( g \)

\[ f(x_1) = \xi_2 b(\xi_1; \xi_2, \xi_3f(y_1) | i = 1,2,3), \] notice if in the left side of a rule preform there is a loop structure it can alternatively as here is the case be described as an environmental binding to itself by a substitution,

\[ f(y_1) = \xi_1 a(\xi_1; \xi_1 | i = 1,2,3) \]

\[ g(y_1) = \xi_2 a(\xi_1, \xi_2, \xi_3s; \xi_1) , \]

\[ s = \xi_3 \beta(\xi_1; \xi_1g(y_1), \xi_2, \xi_3 | i = 1,2) \]

\[ g(y_3) = \xi_3 a(\xi_1, \xi_3t, \xi_3; \xi_1) , \]

\[ t = \xi_3 \beta(\xi_1; \xi_1g(y_1), \xi_2, \xi_3g(y_3) | i = 1,2) \]

\[ \varpi_{12} = a(\xi_1, \xi_3x_i; \xi_1 | i = 2,3) \rightarrow a(\xi_1x_i; \xi_1 | i =1,2,3), \]

where for the left side substitution \( f \) and for the right side substitution \( g \)

\[ f(x_2) = \xi_2 \beta(\xi_1; \xi_2, \xi_3u | i = 1,2, j = 1,2) , \]

\[ u = \xi_3 a(\xi_1, \xi_2f(x_2); \xi_1 | i = 1,3) \]

\[ f(x_3) = \xi_3 \beta(\xi_1; \xi_2v, \xi_3, \xi_3 | i = 1,2) , \]

\[ v = \xi_2 a(\xi_1, \xi_2f(x_3); \xi_1 | i = 1,2) \]

\[ g(x_1) = \xi_2 \beta(\xi_1; \xi_2, \xi_1p | i = 1,2, j = 2,3) , \]

\[ p = \xi_2 a(\xi_1, \xi_2g(x_i); \xi_1 | i = 1,2,3) \]

\[ g(x_2) = \xi_2 \beta(\xi_1; \xi_2, \xi_2q | i = 1,2, j = 1,3) , \]

\[ q = \xi_1 a(\xi_1, \xi_2g(x_i); \xi_1 | i = 2,3) \]

\[ g(x_3) = \xi_3 a(\xi_3, \xi_3g(x_i); \xi_3 | i = 1,2) \]

\[ \tilde{\varnothing} = \{ \varpi_{31}, \varpi_{32} : \text{application order is } \varpi_{31}, \varpi_{32} \}, \]

where

\[ \varpi_{31} = d(\xi_3x_1, \xi_3; \xi_1, \xi_3y_1 | i = 1,2, j = 2,3) \rightarrow d(\xi_3x_1; \xi_3y_1, \xi_2, \xi_4 | i =1,2, j = 1,3), \]

Notice that \( \varpi_{31} \) cannot occupy in-arity \( \xi_1 \) of \( p \) for which a gluing point to \( c \) is already determined.

\[ f(x_1) = \xi_2 w \]
\[
\begin{align*}
    w &= c(\xi_1 \xi_2 s, \xi_2 \xi_3 s, \xi_1, \xi_2 \xi_1 s, \xi_3 \xi_1 p(\xi_i ; \xi_1 | i = 1, 2)) \\
    s &= d(\xi_1 f(x_i), \xi_2 \xi_1 s, \xi_3 s, \xi_1 \xi_2 s, \xi_3 f(y_j) | j = 2, 3) \\
    f(x_2) &= \xi_1 s \\
    f(y_1) &= \xi_2 s \\
    f(y_2) &= \xi_1 w \\
    f(y_3) &= \xi_2 w \\
    g(x_1) &= \xi_1 \xi_1 u \\
    u &= p(\xi_1 \xi_3 v, \xi_2 \xi_1 t ; \xi_1 \xi_1 t) \\
    t &= \delta(g(x_1), \xi_2 g(x_2) ; \xi_1 g(y_1), \xi_2, \xi_3 g(y_3), \xi_4) \\
    v &= c(\xi_1 \xi_3 t, \xi_2 ; \xi_1 \xi_2 t, \xi_2, \xi_3 \xi_1 u) \\
    g(x_2) &= \xi_1 v \\
    g(y_1) &= \xi_2 u \\
    g(y_3) &= \xi_1 v \\

    \alpha_{i2} &= c(\xi_1 x_i, \xi_2, \xi_j y_j, \xi_2 | j = 1, 3) \rightarrow \chi(\xi_i x_i ; \xi_i y_j, \xi_3 | i = 1, 2, j = 1, 2), \\
    f(x_1) &= \xi_3 t \\
    f(y_1) &= \xi_2 t \\
    f(y_3) &= \xi_1 u \\
    g(x_1) &= \xi_2 m \\
    g(x_2) &= \xi_3 m \\
    g(y_1) &= \xi_1 k \\
    g(y_2) &= \xi_2 m \\
    k &= p(\xi_1 \xi_1 h, \xi_2 \xi_1 m ; \xi_1 \xi_1 m) \\
    h &= \chi(\xi_1 g(x_i), \xi_2 g(x_2) ; \xi_1 g(y_1), \xi_2 g(y_2), \xi_3) \\
    m &= \delta(\xi_1 \xi_1 k, \xi_2 \xi_2 h ; \xi_1 \xi_2 k, \xi_2 \xi_1 h, \xi_3 \xi_2 h)
\end{align*}
\]
Starting net q for the process is
\[ b(\xi_1, \xi_2, b(\xi_i, \xi_1, a(\xi_i, \xi_1) \mid i = 1,2,3), \xi_1) \mid i = 1,2,3), \xi_i, \xi_1 a(\xi_i, \xi_1) \mid i = 1,2,3) \mid i = 2,3) \]
for which we achieve \( q^\sim R^\sim W \approx q^\sim R^\sim W \).

**Remark 5.1.** We get the same results for type UPRNS as for earlier represented intervening types in Abstraction closure-theorem 3.4 and generalized Altering macro RNS-theorem 3.2.

**Definition 5.10.** PARTIALLY QUOTIENT ALGEBRA. Let \( \mathcal{A} = (A,F) \) be an algebra and \( \theta \in \text{Eq}(A) \). We say that pair \((B,G)\) is a \( \theta \)-partially quotient algebra of \( \mathcal{A} \), if \( B = A\theta \) and there is such a bijection \( \alpha : F \mapsto G \) that for each \( f \in F \) there is valid commutation \( \theta \alpha(f) \subseteq f\theta \).

**Theorem 5.2.** Let \( A \) be a distinctive jungle, \( G \) be a set of TD’s, and \( \theta \) a distinctive universal abstraction relation of type in ITG over \( A \). Hence net class rewriting algebra is a partially quotient algebra.

**Proof.** Let \( (A,F) \) be a renetting algebra, and \( \alpha : R \mapsto D_R, D_R \subseteq \{ \mathcal{P} : \mathcal{P}^\circ_0 R, \mathcal{P} \text{ is a TD} \} \), \( \mathcal{R} \in F \), be a mapping. Altering macro Clause for type GPRNS and CLRNS 4.2.2 and Remark 5.1 yield \( R\theta = \theta R \), whenever \( R \in F \) and \( \mathcal{P}^\circ_0 R \). From the distinctive nature of equivalence classes follows \( \alpha \) is a bijection thereby Abstraction closure theorem 3.4 yields \((A\theta, \alpha(F))\) is a \( \theta \)-partially quotient algebra of \((A,F)\).

**Definition 5.11.** ISOMORPHISM. If net homomorphism is a bijection changing at the maximum symbols of letters without altering ranks of ranked letters or cardinalities of letters or positions in the nets in its domain, we speak of net isomorphism.

**Proposition 5.1.** UNIQUENESS OF UAR–CENTER. Centers in the same UAR-class are unique up to net isomorphism.
PROOF. Let us assume in the contradiction that in the same class there are two centers \( d \) and \( d' \) not net isomorphic with each other. Then we have two cases: A.) there is a difference in the ranks of \( d \) and \( d' \); this however via partition yields inequality in the cardinalities of the classes liable to \( d \) and \( d' \), and B.) a difference in cardinalities of positions in \( d \) and \( d' \) leads likewise contradiction with our presupposition of the common class. □

**Definition 5.12.** PRECATEGORY. *Category* is a pair of a set of *objects* and a set of relations on that object set.

**Definition 5.13.** FUNCTOR. *Functor* \( \Gamma \) is such a relation between precategories: \( (A,F) \rightarrow (B,G) \), that the following commutation is valid:

\[ af \Gamma = a \Gamma f \Gamma, \text{ whenever } a \in A \text{ and } f \in F. \]

We call relation \( \Gamma \): object \( \mapsto \) object, \( \Gamma_{\text{ob}} \), the *object projection relation of \( \Gamma \)*, and relation \( \Gamma \): relation \( \mapsto \) relation, \( \Gamma_{\text{re}} \), the *relation projection relation of \( \Gamma \).* It holds a worth to note that Functors are homomorphisms, if the domains and the image sets are algebras. *Normal form \( \mathcal{T}\)-TD-Functor* \( \Gamma \) is such a Functor that the objects are jungles and relations in the pairs of \( \Gamma_{\text{re}} \) are within each other \( \mathcal{T}\)-parallel normal form TD transformation relations.

**Proposition 5.2.** Universal abstraction relation of type \( T \) (\( \in \text{ITG} \)) is the object projection relation of a normal form \( T\)-TD transformation Functor for upside down tree TD’s.
We use net block homomorphism as intervening RNS to implement abstract sisters and commutation. First we extend our net presentation by using overlapping cover blocks.

**Definition 6.1.1. NET NUO-PRESENTATION, a linkage presentation.**

For net \( t = s(\mu_i; \lambda_j \mid i \in I^t, j \in J^t, C) \) set \( \{ s, \mu_i, \lambda_j : i \in I^t, j \in J^t \} \) is entitled block of \( t \).

Let then \( T = \{ s, \mu_i, \lambda_j : i \in I^t, j \in J^t \} \), where \( I^t \subseteq I^s, J^t \subseteq J^s \) and the indexed nets in \( T \) are supposed to occupy indicated arity letters of \( t \) in \( s \), we say that \( s(\mu_i; \lambda_j \mid i \in I^t, j \in J^t, C) \), is a net NUO-representation of \( t \) and we denote \( t = s(\mu_i; \lambda_j \mid i \in I^t, j \in J^t, C) \), and set \( \{ s, \mu_i, \lambda_j : i \in I^t, j \in J^t \} \) is entitled its block. The block of a letter is the letter itself. The NUO-representation of net \( t \) with block \( D \) is indicated by \( D-\text{NUO}(t) \) and NUO\((t)\) is asserted on the establishment for the set of all NUO-representations of \( t \). Notation block\((t)\) is asserted to stand for the family of all block-collections in NUO\((t)\).

Notice that each \( \mu_i \) and \( \lambda_j \) may be nets in \( \text{enc}(t) \) not necessarily totally isolated from \( s \), although \( I^t \cap I^s \) and \( J^t \cap J^s \) may be nonempty, in other words \( \mu_i \) or \( \lambda_j \) may overlap net \( s \) thus comprising the key feature of NUO-representation. Observe also that \( \{ s, \mu_i, \lambda_j : i \in I^t, j \in J^t \} \) is a cover of \( t \), and conversely for any cover of \( t \) there is such a NUO-representation that each element in the said cover stands for a net in the block of a net in NUO\((t)\). If set \( T (\in \text{block}(t)) \) is not distinctive, \( \bigcup\text{block}(p) : p \in T \) is a genuine subset of block\((t)\). For NUO\((t)\)-representations classes of equal nets \( [s], s \in \text{NUO}(t) \), are defined analogous with t-class definition. Furthermore we obtain \( \text{block}([t]) = \bigcup\text{block}(p) : p \in [t] \). If net \( p \) is in \( \text{NUO}(\text{enc}([q])) \), then \( q = [q(p \mid ]) = [p(q \mid ] \).

Because NUO-representation is covering the net definition, in the following our presumption (if not stated other) is simply to use NUO-representation for nets and assume indexes in nets be
subsets of indexes in block elements as described above. For each net t we denote reverse NUO, NUO\(^{-1}\) mapping, by asserting: \(t = \text{NUO}^{-1}(\text{NUO}(t))\).

**Definition 6.1.2. NET BLOCK HOMOMORPHISM.**

Let \(t = s(\mu; \lambda) | i \in \mathbb{S}, j \in \mathbb{F}, C\), where \(\mathbb{S} \subseteq \mathbb{S}_s, \mathbb{F} \subseteq \mathbb{F}_s\), be a net in \(F(\mathbb{X}, \mathbb{E}_\mathbb{X})\) and \(D \subseteq F(\mathbb{X}, \mathbb{E}_\mathbb{X})\). We define *net D-block homomorphism relation* (D-NBH) \(h\) as net homomorphism earlier, but ranked letter rewriting relation \(h\) is replaced by net block rewriting relation \(h : D \mapsto h(\Sigma)\) and

\[
h(t) = h_D(s)(h(\mu_i);h(\lambda_j) | i \in \mathbb{E}_{\text{inh}}(s) \cup \mathbb{E}_{\text{outh}}(s), j \in \mathbb{E}_{\text{inh}}(s) \cup \mathbb{E}_{\text{outh}}(s)).
\]

Jungle D is indicated via corresponding NBH \(h\) by notation block(h). Relation \(h\) is *alphabetically unexpanding* (AlpUnexNBH), if \(h(\mathbb{X} \cup \mathbb{E}_\mathbb{X}) \subseteq Y \cup \mathbb{E}_\mathbb{O}\), and for each \(d \in D \mid \text{rank}(h_D(d))\) is not greater than \(\text{rank}(d)\). Furthermore we say that \(h\) is entitled *right hand side alphabetical* (AlpNBH), if \(h(\mathbb{X} \cup \mathbb{E}_\mathbb{X}) \subseteq Y \cup \mathbb{E}_\mathbb{O}\), and for each \(\sigma \in \Sigma\) \(h(\mathbb{E}_\Sigma)(\sigma) = \omega(\varepsilon_i ; \varepsilon_j | i \in \mathbb{E}_{\text{inh}}, j \in \mathbb{E}_{\text{outh}}\), where \(\omega \in \Omega\) and \(\{\varepsilon_i ; \varepsilon_j | i \in \mathbb{E}_{\text{inh}}, j \in \mathbb{E}_{\text{outh}}\}\) is an arity alphabet. Net block homomorphism is *(overlapping) environment saving*, denoted (D)-(O)ESNBH, and we say that it is *abstracting* (ANBH), if it does not delete the contexts (e.g. \(s\) in our example) of (overlapping) subnets (e.g. \(\{\mu, \lambda : i \in \mathbb{S}, j \in \mathbb{F}\}\)) and preserves at least one linkage between the preimage contexts and each of their (overlapping) subnets. Right hand side alphabetical and environment saving net block homomorphism is called *alphabetically abstracting*, abbreviated (D)-AlpANBH. Furthermore we say that NBH is *(overlapping) linkage cardinality saving* ((D)-(O)LSNBH), if it is (overlapping) environmental saving and additionally preserves the cardinality of linkages between the preimage contexts and each of their (overlapping) subnets. NBH which is both ANBH and LSNBH is entitled *straightforwardly abstracting* denoted SANBH. Notice that for each net in the preimage domain of D-NBH there is a t-saturating subset of D. In the following we denote (T)NBH for meaning the set of all NBH-relations of type T (defined above).

**Proposition 6.1. INVERSE NBH, RESTORING GROUND LEVEL (PERCEPTION)**

For each ANBH there is an inverse ANBH.

**PROOF.** Let \(h \in D\)-ANBH. Because \(h\) is environment saving its net block rewriting relation \(h_D\) is reversible and we can choose such an ANBH, say \(f\), with block(f) = block(Dh) and \(h_D^{-1}\) as its net block rewriting relation that \(hf\) is an identity mapping in the preimage domain of \(h\). \(\Box\)
**Result 6.1.1.** Let $t$ be a net and $\mathfrak{K} \in \text{Cov}(t)$. For each such $Q \subseteq \mathfrak{K}$ that $\cap Q \neq \emptyset$, ESNBH $h$ creates $(|Q|-1) \mid \text{rank}(\cap Q)$ new outward links between nets $h(a), a \in Q$, compared to the counterparts in $Q$.

**Definition 6.1.3. ABSTRACTION RELATION VIA NBH.**

We extend our notion of abstraction relation to comprise also NBH as an intervening operation type and say that two jungles are in NBH-abstraction relation with each other (forming an abstract sister pair), if they are NBH-images of NUO-presentations of the same jungle.

Next theorem will set the general base result that perception regarded as ground basis abstraction of a mother net of given problem can be solved by memory basis abstraction of the same net.

However the question is can we find an implicit solution of an equation, in a resemblance to altering macro RNS theorem, for another of the two abstract sisters (B) when we already have done partition in its origin $K$ in preparation to convert (memory) solution (for the other abstract sister A) to as a solution for $K$, i.e. to find a macro from found micro of memory solution; micro solutions are easier to find.

**Theorem 6.1.1.** Let $\theta$ be AlpANBH-abstraction relation. For each RNS $\mathcal{R}$ there is RNS $\mathcal{P}$ satisfying commutative equation $\theta \mathcal{R} = \mathcal{P} \theta$.

**Proof.** Let (A,B) be an AlpANBH-abstract pair of nets, $\mathcal{W}_1$ and $\mathcal{W}_2$ being of AlpANBH-type intervening NBH’s in concern, A $\mathcal{W}_1$-image of net $s$ and B $\mathcal{W}_2$-image of net $t$ while $s$ and $t$ being NUO-presentations of the same net, say $K$, with block of $s$ $D_s \subseteq \text{block}(\mathcal{W}_1)$ and block of $t$ $D_t \subseteq \text{block}(\mathcal{W}_2)$. Without a loss of generality we make an assertion:

$$s = p(\mu; \lambda, | i \in \mathcal{S}_p, j \in \mathcal{S}_p, C)$$

$$t = q(\mu; \lambda, | i \in \mathcal{S}_q, j \in \mathcal{S}_q, (\exists k \in \mathcal{S}_q^c \cup \mathcal{S}_q^{oc}) p \in \{\mu_k, \lambda_k\}, C).$$
Let \( r \) be in \( A \) a redex possessing rule preform as a part of a known memory solution RNS for \( A \). Next we construct the following rule preforms:

- **micro\((r)\):**

  left side: \( \text{apex(left(micro(r))))W_1 = \text{apex(left(r))} \) and \( \text{micro(r)} \) matches \( K \)

  right side: First we define notion net induced AlpANBH: Let \( t \) be a net. \( N\)–AlpANBH is entitled \( t\)-induced, denoted \( N\)-AlpANBH\(^1\)(\( t \)), if \( N = \alpha(\text{rank}(t)) \) is a cover of a net homomorphism image of \( t \), where \( \alpha \) is a bijection from the rank alphabet to a set of jungles. Because for each \([\text{NUO}(t)]\)-class representative AlpANBH-images are equal for the same AlpANBH, we can now choose \( \text{right(micro(r))} \) to be a net in the preimage domain of an apex(right(r))-induced AlpANBH. From the same reasons for the case “\( r \) is an instance sensitive INRNS-rule”, we can choose for each manoeuvre letter \( x \) in the domain of right side substitution \( g \) of \( r \) the \( x \)-image of right side substitution of \( \text{micro(r)} \) to be a net in the preimage domain of an AlpANBH\(^1\)(\( g(x) \)).

  Therefore there is such an AlpANBH, say \( \tilde{\omega}_{o1} \), that \( s\tilde{\omega}_{o1}r = K(\text{micro(r)})\tilde{\omega}_{o1} \).

- **macro(micro\((r)\)) \((=\{r_1, r_2\})\):**

  First we construct such a new AlpANBH, say \( \tilde{\omega}_3 : K\rightarrow B \) that block(\( \tilde{\omega}_3 \)) is a partition induced by union block(\( \tilde{\omega}_1 \))\( \cup \)block(\( \tilde{\omega}_2 \)):

  \[
  \{ \bigcap D' \setminus \{ \bigcap D'\subset D'', D'' \in \text{P(D)} \} : D' \in \text{P(D)} \}, \text{where} D = \text{block}(\tilde{\omega}_1) \cup \text{block}(\tilde{\omega}_2).
  \]

  1. first executed rule preform \( r_1 : t\tilde{\omega}_{o2} \rightarrow (\text{NUO}^{-1}(t))\tilde{\omega}_3 \), the choice guaranteeing in rewriting processes perseverance of analogous environments in \( K\tilde{\omega}_{o2} \) and \( K \).

  2. secondly executed rule preform \( r_2 : \)

     left side: \( \text{apex(left}(r_2)) = (\text{NUO}^{-1}(\text{apex(left(micro(r))))})\tilde{\omega}_3 \) and \( r_2 \) matches \( (\text{NUO}^{-1}(t))\tilde{\omega}_3 \)

     right side: Following analogously constructing right side of \( \text{micro(r)} \) we are free to choose an intervening AlpANBH, say \( \tilde{\omega}_{o2} \), with the block equal with the block of \( \tilde{\omega}_{o1} \), and accordingly we choose: \( \text{apex(right(micro(r))))\tilde{\omega}_{o2} = \text{apex(right(r))} \).

     Therefore \( t\tilde{\omega}_{o2}r_1r_2 = K(\text{micro(r)})\tilde{\omega}_{o2} \). \( \Box \)
Result 6.1.2. ABSTRACTION RELATIONS INDUCED EQUIVALENCE RELATION OVER THE WHOLE SET OF THE NETS.

Let \( h \) be a NBH and \( T \) be its preimage domain. \( h \)-Abstraction relation \( \theta_h \) is in \( \text{Eq}(T) \), because for each nets \( s \) and \( t \) in \( T \) \( (s,t) \in \theta_h \), iff \( \exists C_s \in \text{Sat}(s) \) and \( C_t \in \text{Sat}(t) \) that \( C_s \cup C_t \subseteq P(T) \). Now we can construct an equivalence relation over the whole set of the nets via \( \text{AlpUnexNBH} \)-abstraction relations:

\[
\bigcup \{ \theta_h : h \text{ is a } \text{D-AlpUnexNBH}, D \in \text{Par}(F_\Sigma(X,_{\Sigma})) \} \in \text{Eq}(F_\Sigma(X,_{\Sigma})).
\]

2° Generating Net Rewriting

Net block homomorphism execution in the set of nets is more general and powerful than single renetting rule. But as this chapter shows normal form in renetting are able to produce the equal results.

NBH execution can be presented by normal form of rewriting, if at first representation change is effected within domains of net classes so that the inward links of contexts will become outward links to context overlapping subnets as shown inductively in the following definition:

Definition 6.2.1. NON-OVERLAPPING NUO-REPRESENTATION RISING MAPPING.

Let \( t = s(\mu_i; \lambda_j \mid i \in \delta, j \in \gamma, C) \) be an arbitrary NUO-representation. We define the non-overlapping NUO-representation rising mapping in the set of nets, denoted \( \text{NORNNO} \), \( f : t \mapsto f(t) = f(s)(\mu_i^{(1)} \mu_i^{(2)} f(\mu_i) ; \lambda_j^{(1)} \lambda_j^{(2)} f(\lambda_j)) \mid i \in \delta, j \in \gamma \) \( (\in [t], f(s) = s \cup \mu_i^{(1)} \mu_i^{(2)} : i \in \delta, j \in \gamma) \),

where \( \lambda \) is a symbol for net omission and \( \text{union} \cup \) stands for the net saturated by the set \( \{ . \} \) of its arguments.

Definition 6.2.2. NET BLOCK HOMOMORPHISM RNS. We define for each index of each index set in each net, say \( i \), a frontier letter \( x_i \). Let \( t = s(\mu_i; \lambda_j \mid i \in \delta, j \in \gamma, C) \). For each consecutive
steps in contexts in block $D$ by $D$-NBH $h$ we establish rules via NORNUO $f$ subject to the same contexts: in $t \mapsto h(s)$, $s \in D$, corresponds rule

$$f(s)(\mu_i \mapsto \mu_i^{(1)} x_i; \lambda_j \mapsto \lambda_j^{(1)} x_j \mid i \in S^{UN}_f(s); j \in S^{UN}_f(s)) \rightarrow h(s)(\mu_i \mapsto \mu_i^{(1)} x_i; \lambda_j \mapsto \lambda_j^{(1)} x_j \mid i \in S^{UN}_f(s); j \in S^{UN}_f(s)),$$

where the left side manoeuvre alphabet is $\{ x_k : k \in S^{UN}_f(s) \cup S^{UN}_f(s) \}$ and $\{ x_k : k \in S^{UN}_f(s) \cup S^{UN}_f(s) \}$ is for the right side respectively. The corresponding renetting system is entitled (with a prospective condition set) \textit{net block homomorphism RNS}, shortly NBHRNS. Consequently we obtain the following theorem.

\textbf{Theorem 6.2.1.} NBH GENERATION BY RNS-\textit{normal form.}

For each NBH $h$ there is such a RNS $\mathcal{R}$ that $h = \hat{\mathcal{R}}$.

\textbf{PROOF.} We choose NBHRNS as an executing RNS-type. \hfill \Box

The set of the rule preforms of rule $r$ in RNS $\mathcal{R}$ is denoted $\text{pre}(r)$ and $\text{pre}(\mathcal{R}) = \{ \text{pre}(r) : r \in \mathcal{R} \}$.

\textbf{Definition 6.2.3.} RENETTING NBH.

Let $\mathcal{R}$ be a RNS and let $D = \text{apex}(\text{left}(\text{pre}(\mathcal{R}))) \cup \text{Dom}(g)$, where $g$ is the right side substitution of $\mathcal{R}$. We define \textit{renetting NBH} (RNNBH) $h$ as NBH with net block rewriting relation $h_D$:

$$\text{apex}(\text{left}(r)) \mapsto \text{right}(r), \quad r \in \text{pre}(\mathcal{R}),$$

$$x \mapsto g(x), \quad x \in \text{Dom}(g).$$

Notice that if the left side and right side substitutions are equal, we simply can define

$$D = \text{apex}(\text{left}(\text{pre}(\mathcal{R}))) \quad \text{and} \quad h_D : \text{apex}(\text{left}(r)) \mapsto \text{apex}(\text{right}(r)), \quad r \in \text{pre}(\mathcal{R}).$$

\textbf{Theorem 6.2.2.} RNS-\textit{normal form GENERATION BY NBH.}

For each RNS $\mathcal{R}$ there is such a NBH $h$ that $\hat{\mathcal{R}} = h$.

\textbf{PROOF.} We choose RNNBH as an executing NBH-type. \hfill \Box
Theorem 6.2.3. The operational efficiency of the set of all NBH’s and on the other hand of all RNS’s equates.

PROOF. Combination of generation theorems above. □

As the conclusion we can form quotient algebra with elements being AlpUnexNBH-abstraction relation classes and with operations being bunches of parallel TD’s over RNS normal forms or NBH’s. And if you know a solution for one element in an abstraction relation class you know solutions for all representatives in that particular class. Knowing solutions in the measure of equivalence relation class cardinality is the only necessity in order to get all conceivable solutions. Furthermore obtaining TD-solutions over a sample of abstraction classes for one representative per class results solutions for all routes derived through these classes.

7§ N:th Order Net Class Rewriting Systems

1° TD-SOLUTION ABSTRACTION

Definition 7.1.1. TD-ABSTRACTION RELATION.
We define *TD-*[[\mathcal{A}\mathcal{L}]]-abstraction relation \( \theta_{\text{TD}}^\mathcal{A} \), \( \mathcal{A} \in \text{ITG} \), in the set of the elements in abstract algebra \( \mathcal{A} \) as follows: Let \( H \) and \( K \) be two \( \mathcal{A} \)-derived TD-operations and we mark renetting algebra by \( \mathcal{N} \).
We define \( H \theta_{\text{TD}}^\mathcal{A} K \iff H^\mathcal{N} \theta_{\text{N}}^\mathcal{N} K^\mathcal{N} \), where \( \theta_{\text{N}}^\mathcal{N} \) is the \( \mathcal{A} \)-abstraction relation in the set of the nets and \( \text{TD}^\mathcal{N} \) stands for the carrier net of TD in concern. If \( \mathcal{A} \) is not specified we write simply \( \theta_{\text{TD}} \) and \( \theta_{\text{N}} \) respectively.

Theorem 7.1.1. Parallel TD-relation classes are in TD-abstraction relation with each other, iff any of the representatives of them are subject to that.
PROOF. The claim follows from the fact that between each pair of carrier nets of parallel TD-relation class representatives there is an intervening linear alphabetic net homomorphism. □

**Corollary 7.1.1.** Parallel TD-relation classes saturate relevant TD-abstraction relation classes.

**PROOF.** The claim follows from theorem 7.1.1, because both of the concerning relations are equivalence relations. □

**CARDINALITIES**

Next we generalize our abstraction relation to cover simultaneously saturated net sets via direct products.

**Definition 7.2.1.** **MULTIDIMENSIONAL ABSTRACTION RELATION.**

Let $\tilde{\Theta}_k \subseteq \text{Eq}(F_2(X, \bar{\Xi}))$. For each $k \in \mathbb{N}$ and $t \in F_2(X, \bar{\Xi})$ we denote $k$-level saturation set of $t$

$$S_k(t) = \{ \{ s \in \theta F_2(X, \bar{\Xi}) : \theta \in \tilde{\Theta}_{i-1} \} : s \in F_2(X, \bar{\Xi}) \} \cap \text{Sat}(t),$$

where *multidimensional abstraction relation set*, $\tilde{\Theta}_k = \{ \tilde{\Theta}_{i_k} : i = 1, \ldots, i_k, C_k \}$, $i_k \in \mathbb{N}$, is defined as the direct product of elements in $\tilde{\Theta}_{i-1}$ such that for each $s$ and $t$ in $F_2(X, \bar{\Xi})$ for each $i = 1, \ldots, i_k$

$$(s, t) \in \tilde{\Theta}_{i_k}, \text{ iff }$$

$$\exists (P \in S_k(s), Q \in S_k(t)) \ (\forall p \in P) (\exists q \in Q) (\exists \theta \in \mathbb{N}(k-1, i)) (p, q) \in \theta, \text{ where } \mathbb{N}(k-1, i) \subseteq \tilde{\Theta}_{i-1}$$

and revised for $q$ and $p$ respectively; $C_k$ being *boundary condition set*, establishing qualifications for applied algebras.

Notice that $(k-1)$-level saturation sets (saturating the classes liable to equivalence relations in the concerned level) saturate each net in $k$-level saturation set and clearly $\tilde{\Theta}_k \subseteq \text{Eq}(F_2(X, \bar{\Xi}))$. We achieve for each $\tilde{\Theta}_{i_k} \subseteq \tilde{\Theta}_k$:

$$(\forall t \in \text{Eq}(F_2(X, \bar{\Xi}))) \ | t \tilde{\Theta}_{i_k} | = \prod (\prod (|q| \theta q : q \in Q, Q \in S_i(t)) : \theta \in \tilde{\Theta}_{i-1}) : i \in \mathbb{N}, i < k,$$
where $\prod$ stands for the multiplication symbol over its argument set.

We denote $\text{nest}(\mathcal{R})$ the nest of TD $\mathcal{R}$, and let $s_{\mathcal{R}}$ stands for a subnet of net $s$ matched by RNS $\mathcal{R}$. Furthermore $\mathcal{H}_{st}$ is reserved to act as the cardinality of the set of the common origins for $\theta_n$-related nets $s$ and $t$. Because for abstract partially quotient algebra

\[(s\theta_n)\mathcal{R}_n \subseteq \{B \subseteq a\theta_n : a \in F_\Xi(X,\Xi)\},\] whenever $\mathcal{R}_n$ is a TD,

we obtain the following claim:

**Claim 7.2.1.** If $\tilde{\Theta}_o$ is chosen to be a singleton comprising $\theta_n$, we obtain an upper limit for the cardinality of $\theta_n$-class $H$ related $\omega$-class of parallel TD’s in $\theta_{\text{td}}$-class $A$ at $k$-level

\[
\mu(H,A,k) = \oplus((\mathcal{H}_{s_{\mathcal{R}}}) : t \theta_n s_{\mathcal{R}}) | t \tilde{\Theta}_k | : s_{\mathcal{R}}, t \in H; F^\omega \leq R_{\mathcal{R}}^\omega, F^\omega \in \text{sub}(\mathcal{R}^\omega), R_{\mathcal{R}} \in \text{nest}(\mathcal{R}), \mathcal{R} \in A),
\]

where $\leq$ stands for “next below” and $\oplus$ for the summation over its argument set.

Even though in the cases intervening RNS’s are of PRNS or CLRNS –types the number of the resulted applicant nets as well as the enclosures in them and consequently the number of the left sides of rule preforms in parallel TD’s may be denumerable, so however as because for each net pair $(s,t) \mathcal{H}_{st}$ and even the number of the alternatives for the right sides of each rule preform may be unlimited and undenumerable, there may – subject to the cardinality of final states in applied recognizers – exist problems we are not able to determine if they can be comprehensively solved i.e. they are manifesting problems of inconsistent or undecidable nature, cf. “The Undecidable” Davis M (1965), Rosser JB (1936).
Multiple Level Abstraction Algebra, Self-Evolving Problem Solving System

In this chapter we iteratively determine PROBLEM SOLVING EVOLUTION structure.

Definition 7.2.1. QUOTIENT RELATION.

Let \( K \) be a set and \( E, G \in \text{Eq}(K), G \subseteq E \). Let \( F \) be a set of operations on \( K \). We define for each \( f \in F \)
\[
f(E/G) = \{ pG : pG \subseteq fE, p \in K \}. 
\]

Definition 7.2.2. THE FIRST ORDER ABSTRACTION ALGEBRA.

Let \( A \) be a jungle, \( F \) a set of TD’s, \( \tilde{\Theta}_k \) multidimensional abstraction relation in \( A \) at \( k \)-level
\((k \in \mathbb{N}_0)\) subject to \( \tilde{\Theta}_0 \) being a singleton comprising \( \theta_N(\in \text{ITG}) \) and \( \sim_{\tilde{\Theta}_k} \) parallel TD-relation in \( F \) subject to \( \tilde{\Theta}_k \). Pair \((A \tilde{\Theta}_k, F \sim_{\tilde{\Theta}_k})\) is called first order abstraction algebra, where for each \( a \in A \) and \( f \) in \( F \) is defined
\[
a \tilde{\Theta}_k f \sim_{\tilde{\Theta}_k} = \{ bp : b \in a \tilde{\Theta}_k, p \in f \sim_{\tilde{\Theta}_k} \}. 
\]
If \( \theta_N \) is distinct then the apexes of the rule performs in each RNS in TD’s are distinct from each other and consequently for each \( k \) relation \( \sim_{\tilde{\Theta}_k} \) is distinct. Therefore
\[
a \tilde{\Theta}_k f \sim_{\tilde{\Theta}_k} \in \{ B \subseteq a \theta_N : a \in A \}. 
\]

Next we agree notation for expanding power set definition for multiple powers: For each set \( K \)
\[
(\forall n \in \mathbb{N}) P^n(K) = P(P^{n-1}(K)) \text{ and } P^0(K) = K. 
\]

Next we define “multiple order abstraction algebra”.
Because \( \sim_{\theta_N} \) saturates \( \theta_m \) by Corollary 7.1, hence we can set the following definition:
Definition 7.2.3. SECOND ORDER RELATIONS IN ABSTRACTION ALGEBRA.

Let $F$ be a set of TD’s, $f, g \in F$ and $k \in \mathbb{N}_0$. We define quotient relation in $F$, second order abstraction relation, $\tilde{\Theta}_{k_{TD}} / ^{\sim_0} \tilde{\Theta}_{k}$ and find out that $f ^{\sim_0} \tilde{\Theta}_{k} \sim_0 ^{\sim_0} \tilde{\Theta}_{k} \sim_0 g ^{\sim_0} \tilde{\Theta}_{k}$, if $f \tilde{\Theta}_{k_{TD}} g$.

Now we define transducer revising binary relation $\odot$ in $F$:

$$((\forall s,t \in F) \quad s \odot t = (s \sim^N t)^N.$$  

Next we define for each $k \in \mathbb{N}_0$ second order parallel relation in $P(F)$, $\sim_0 (\tilde{\Theta}_{k_{TD}} / ^{\sim_0} \tilde{\Theta}_{k})$:

$$((\forall S,T \subseteq F) \quad S \sim_0 (\tilde{\Theta}_{k_{TD}} / ^{\sim_0} \tilde{\Theta}_{k}) T \text{, if } \quad (f ^{\sim_0} \tilde{\Theta}_{k}) \odot S \tilde{\Theta}_{k_{TD}} / ^{\sim_0} \tilde{\Theta}_{k} (g ^{\sim_0} \tilde{\Theta}_{k}) \odot T, \text{ whenever } (f ^{\sim_0} \tilde{\Theta}_{k}, g ^{\sim_0} \tilde{\Theta}_{k}) \in \tilde{\Theta}_{k_{TD}} / ^{\sim_0} \tilde{\Theta}_{k}.$$  

Setting requisite $\theta_n$ is distinct yields $\sim_0 \tilde{\Theta}_{k}$ is distinct and hence we obtain a demonstration of a closure system:

$$(\forall s \in S) \quad s(\tilde{\Theta}_{k_{TD}} / ^{\sim_0} \tilde{\Theta}_{k}) \odot T ^{\sim_0} (\tilde{\Theta}_{k_{TD}} / ^{\sim_0} \tilde{\Theta}_{k}) = \{ h ^{\sim_0} \tilde{\Theta}_{k} \odot t : h ^{\sim_0} \tilde{\Theta}_{k} \in s^{\theta_{TD}}, \ t \in T ^{\sim_0} (\tilde{\Theta}_{k_{TD}} / ^{\sim_0} \tilde{\Theta}_{k}) \} \quad \in \{ B \subseteq k \in f(\tilde{\Theta}_{k_{TD}} / ^{\sim_0} \tilde{\Theta}_{k}) : f \in F \},$$

and we are able to manifest second order abstraction algebra $(F(\tilde{\Theta}_{k_{TD}} / ^{\sim_0} \tilde{\Theta}_{k}), \odot P(F) ^{\sim_0} (\tilde{\Theta}_{k_{TD}} / ^{\sim_0} \tilde{\Theta}_{k}))$.

Next we expand the notion of abstraction algebra to multiple orders and first inductively enumerate abstraction relations in nested order.

Let $A$ be a set. For each $n \in \mathbb{N}_0$ and $B \in \mathcal{P}^n(A)$ we denote inductively

$$\cup^0(B) = B, \quad \cup^1(B) = \cup(\cup^0(B)) \text{ and } \cup^n(B) = \cup^{n-1}(B).$$

For each $n \in \mathbb{N}_0$ we define $k_n \in \mathbb{N}_0$ and $\tilde{\Theta}_{n,k_n_{TD}}$ is such a relation in $P^{n-1}(F)$ that

$$\tilde{\Theta}_{0,k_0_{TD}} = \tilde{\Theta}_{k_{TD}} \text{ and } \quad (\forall H,K \in P^{n-1}(F)) \quad H \tilde{\Theta}_{n,k_n_{TD}} K, \text{ if } (\cup^n(H)) ^{\sim_N} \tilde{\Theta}_{n-1,k_{n-1}TD} (\cup^n(K)) ^{\sim_N}.$$  

Furthermore we agree with the notations:
\[
\begin{align*}
\langle 0/\alpha \rangle^{0,k_0} &= \bar{\Theta}_{0,k_0T_D}, \\
\langle 0/\alpha \rangle^{1,k_1} &= \bar{\Theta}_{1,k_1T_D}/\alpha \bar{\Theta}_{0,k_0T_D}, \\
\langle 0/\alpha \rangle^{n,k_n} &= \bar{\Theta}_{n,k_nT_D}/\alpha \langle \theta/\alpha \rangle^{n-1,k_{n-1}}.
\end{align*}
\]

Next for each \(n \in \mathbb{N}_0\) and \(k_n \in \mathbb{N}_0\) we keep assumed \(\bar{\Theta}_{n,k_nT_D}\) are distinctive.

**Definition 7.2.4.** \(N\)-LEVEL SOLVING.

Let first \(T_n \in \mathcal{P}_n(F)\), \(n \in \mathbb{N}_0\) and \(S_n = S_{n-1} \cup T_n\), \(n \in \mathbb{N}\), \(S_0 = T_0\).

\[
\begin{align*}
\mathcal{R}^{(0)}(S_0, k_0) &= T_0 \bar{\Theta}_{0,k_0T_D} \\
\mathcal{R}^{(1)}(S_1, k_1) &= T_0 \langle \theta/\alpha \rangle^{1,k_1} \odot T_1 \langle \theta/\alpha \rangle^{1,k_1} \quad \in \{ B \subseteq k \in q(\theta/\alpha)^{1,k_1} : q \in F \}, \\
\mathcal{R}^{(2)}(S_2, k_2) &= \mathcal{R}^{(1)}(S_1, k_1) \langle \theta/\alpha \rangle^{2,k_2} \odot T_2 \langle \theta/\alpha \rangle^{2,k_2} \quad \in \{ B \subseteq k \in q(\theta/\alpha)^{2,k_2} : q \in \mathcal{P}(F) \}, \\
\hspace{2cm} \ldots \\
\mathcal{R}^{(n)}(S_n, k_n) &= \mathcal{R}^{(n-1)}(S_{n-1}, k_{n-1}) \langle \theta/\alpha \rangle^{n,k_n} \odot T_n \langle \theta/\alpha \rangle^{n,k_n} \quad \in \{ B \subseteq k \in q(\theta/\alpha)^{n,k_n} : q \in \mathcal{P}^{n-1}(F) \}.
\end{align*}
\]

Finally we are ready to define \(n\)-th order abstraction algebra

\[
\mathcal{R}^{(n-1)}(S, k_{n-1}) \langle \theta/\alpha \rangle^{n,k_n} \odot \mathcal{P}^n(F) \langle \theta/\alpha \rangle^{n,k_n}, \quad \text{where } S = \cup (\mathcal{P}(F) : i = 0, 1, \ldots, n-1),
\]

because of the closure property:

\[
\begin{align*}
\mathcal{R}^{(n-1)}(S, k_{n-1}) \langle \theta/\alpha \rangle^{n,k_n} \odot \mathcal{P}^n(F) \langle \theta/\alpha \rangle^{n,k_n} \\
\subseteq \{ B \subseteq k \in q(\theta/\alpha)^{n,k_n} : q \in \mathcal{P}^{n-1}(F) \} \\
\subseteq \mathcal{R}^{(n-1)}(S, k_{n-1}) \langle \theta/\alpha \rangle^{n,k_n}.
\end{align*}
\]
Families $S_n, n \in \mathbb{N}_0$, correspond to the mother nets of $n$-level problems. $\mathcal{R}^{(n-1)}(S, k_{n-1})(\emptyset / \emptyset)^{\langle n, k_n \rangle}$ corresponds to the set of the mother nets of the $n$-level problems and $\otimes \mathcal{P}^n(F)^{\emptyset / \emptyset}^{\langle 0, k_n \rangle}$ to the set of the solutions at the respective level.

Next if we assume mother nets and known solutions be fixed in each level and extend above $n$:th order processes further exponentially we’ll get autonomous evolution levels and which are of utter importance considering self-developing unrestricted solving processes:

**AUTONOMOUS EVOLUTION OF $m$:TH LEVEL.**

For each $m \in \mathbb{N}_0$ we define $n_m \in \mathbb{N}$ and for each ($n_m \in \mathbb{N}$) $k_{n_m} \in \mathbb{N}$ and furthermore inductively by the mapping of solution sets $m$:th level autonomous evolution level

$$
\mathcal{E}_0 = F(\emptyset / \emptyset)^{\langle 0, k_0 \rangle},
$$

$$
\mathcal{E}_m \equiv \mathcal{E}_m(P_{m,0}, P_{m,1}, \ldots, P_{m,n_m-1}, \mathcal{E}_{m-1})
= \mathcal{R}^{(n_m-1)}(\bigcup(P_{m,i} \subseteq P(\mathcal{E}_{m-1}) : i = 0, 1, \ldots, n_m - 1), k_{n_m-1})(\emptyset / \emptyset)^{\langle n_m, k_m \rangle}.
$$

**Conclusions**

The reached theory serves as such the fundamental algorithmic tools for independency of solving systems in robotics. The study represents a new way to describe knowledge with generalized universal algebra allowing loop structures so very important in AI languages and which gives an extensive variety of notional relations between net entities without restricting the semantic use. Consequently a new syntax model for solving problems defined by said nets is established flexibly utilizing notional similarities with original problems to further match solutions in memory data banks additionally creating transducer graphs of solving rewrite systems and thereof closure system of solving classes. The study introduces universal partitioning to widen environmental attachments subject to abstract relations yielding universal macros form parallel TD-solutions. Net NUO-presentation are delivered providing more
general coverage enabling net block homomorphism to be used for TD-solution generation. A special attention is given to cardinalities of basic solutions. Second order parallel relation is introduced for distinct solution set bases. Finally multiple level abstraction algebra is taken in account for determining self-evolving solving systems. This is reached by tree different stages offering combinational approach in multiple power solution families and iterative solving thus creating solution basis for evolitional levels.

**For the future considerations**

Conceptual graphs constitute equivalence classes as the form of elements in a closed quotient systems, meaning that parallel transformation applied to those classes inevitably drops images back into the set of those particular classes, which guarantees automated problem solving and consequently is in the interest of this research. Incalculably important contributing corollary is consideration of autonomous problem solving systems evolution in ever deepening directions encountering lower level results and deriving new approaches from them. As executions of the present system outside the evident robotics emerges quantum computing in the form of teleportation for new generalized information transportation.

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