Optimal Switching of One-Dimensional Reflected BSDEs, and Associated Multi-Dimensional BSDEs with Oblique Reflection

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Abstract

In this paper, the optimal switching problem is proposed for one-dimensional reflected backward stochastic differential equations (BSDEs, for short) where the generators, the terminal values, and the barriers are all switched with positive costs. The value process is characterized by a system of multi-dimensional reflected BSDEs with oblique reflection, whose existence and uniqueness is by no means trivial and is therefore carefully examined. Existence is shown using both methods of the Picard iteration and penalization, but under some different conditions. Uniqueness is proved by representation either as the equilibrium value process to a stochastic mixed game of switching and stopping, or as the value process to our optimal switching problem for one-dimensional reflected BSDEs.

Keywords. Reflected backward stochastic differential equation, oblique reflection, optimal switching, stochastic game.

MR(2000) Subject Classification. 93E20, 60H10, 90A15

1 Introduction

Let \{W(t), 0 \leq t \leq T\} be a d-dimensional standard Brownian motion defined on some complete probability space \((\Omega, \mathcal{F}, P)\). \{\mathcal{F}_t, 0 \leq t \leq T\} is the natural filtration of the Brownian motion \{W(t), 0 \leq t \leq T\}, augmented by all the \(P\)-null sets of \(\mathcal{F}\). The following notation will be used in the sequel.

\[ S^2 \triangleq \left\{ \phi : \phi \text{ is an } \{\mathcal{F}_t, 0 \leq t \leq T\}-\text{adapted r.c.l.l. process s.t. } E\left[ \sup_{0 \leq t \leq T} |\phi(t)|^2 \right] < \infty \right\}, \]

\[ S^2 \triangleq \{ \phi \in S^2 : \phi \text{ is continuous} \}, \]

\[ N^2 \triangleq \{ \phi \in S^2 : \phi \text{ is increasing and } \phi(0) = 0 \}, \]

\[ N^2 \triangleq \{ \phi \in N^2 : \phi \text{ is continuous} \}, \]

\[ \mathcal{M}^2 \triangleq \{ \phi : \phi \text{ is } \{\mathcal{F}_t, 0 \leq t \leq T\}-\text{predictable and square-integrable on } [0, T] \times \Omega \}. \]

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Let \( \{\theta_j\}_{j=0}^\infty \) be an increasing sequence of stopping times with values in \([0,T]\). \( \forall j, \alpha_j \) is an \( \mathcal{F}_{\theta_j} \)-measurable random variable with values in \( \Lambda \). Assume that \( a.s. \omega \), there exists an integer \( N(\omega) < \infty \) such that \( \theta_N = T \). Then we define a switching strategy as:

\[
a(s) = \alpha_0 \chi_{[\theta_0, \theta_1]}(s) + \sum_{j=1}^{N-1} \alpha_j \chi_{[\theta_j, \theta_{j+1}]}(s).
\]

Denote by \( \mathcal{A}_t \) all the switching strategies with initial data \( \alpha_0 = i \in \Lambda, \theta_0 = t \). For given \( a \in \mathcal{A}_t, \xi \in L^2(\Omega, \mathcal{F}_T, P; R^m) \), and \( S \in (S^2)^m \), consider the following switched reflected backward stochastic differential equation (abbreviated as RBSDE):

\[
\begin{aligned}
U^a(s) &= \xi_{a(T)} + \int_s^T \psi(r, U^a(r), V^a(r), a(r))dr - (L^a(T) - L^a(s)) \\
&- \sum_{j=1}^{N-1} [U^a(\theta_j) - h_{\alpha_{j-1}, \alpha_j}(\theta_j, U^a(\theta_j))] \chi(s,T)(\theta_j) \\
&- \int_s^T V^a(r)dW(r), \quad s \in [t, T]; \\
U^a(s) &\leq S_{a(s)}(s), \quad s \in [t, T]; \\
\int_t^T (U^a(s) - S_{a(s)}(s))dL^a(s) &= 0.
\end{aligned}
\]  

(1.1)

Here and in the following \( \chi \) is an indicator function. The generator \( \psi \), the terminal condition \( \xi \), and the upper barrier \( S \) of RBSDE (1.1) are all switched by \( a \). At each switching time \( \theta_j \) before termination, the value of \( U^a \) will jump by an amount of \( U^a(\theta_j) - h_{\alpha_{j-1}, \alpha_j}(\theta_j, U^a(\theta_j)) \) which can be regarded as a penalty or cost for the switching. RBSDE (1.1) can be solved in a backwardly recursive way in the subintervals \([\theta_{N-1}, T]\) and \([\theta_{j-1}, \theta_j]\) for \( j = N - 1, \cdots, 2, 1 \). To be precise, RBSDE (1.1) in the last subinterval \([\theta_{N-1}, T]\) reads:

\[
\begin{aligned}
U^a(s) &= \xi_{a_{N-1}} + \int_s^T \psi(r, U^a(r), V^a(r), a_{N-1})dr \\
&- (L^a(T) - L^a(s)) - \int_s^T V^a(r)dW(r), \quad s \in [\theta_{N-1}, T]; \\
U^a(s) &\leq S_{a_{N-1}}(s), \quad s \in [\theta_{N-1}, T]; \\
\int_{\theta_{N-1}}^T (U^a(s) - S_{a_{N-1}}(s))dL^a(s) &= 0.
\end{aligned}
\]  

(1.2)

From [3, Theorem 5.2], RBSDE (1.2) has a unique adapted solution on \([\theta_{N-1}, T]\) under Hypothesis 1 (see Section 2 below). Then we have

\[
U^a(\theta_{N-1}-) = h_{a_{N-2}, a_{N-1}}(\theta_{N-1}, U^a(\theta_{N-1})),
\]

which serves as the terminal value of RBSDE (1.1) in \([\theta_{N-2}, \theta_{N-1}]\). In general, in
the subinterval $[\theta_{j-1}, \theta_j]$, RBSDE (1.1) reads

$$
\begin{align*}
U^a(s) &= h_{\alpha_{j-1}, \alpha_j}(\theta_j, U^a(\theta_j)) + \int_s^{\theta_j} \psi(r, U^a(r), V^a(r), \alpha_{j-1}) \, dr \\
&\quad - (L^a(\theta_j) - L^a(s)) - \int_s^{\theta_j} V^a(r) \, dW(r), \quad s \in [\theta_{j-1}, \theta_j);
\end{align*}
$$

(1.3)

for $j = N - 1, \ldots, 2, 1$. Here $U^a(\theta_j)$ is specified in the interval $[\theta_j, \theta_{j+1})$ and we have the following relations:

$$
h_{\alpha_{j-1}, \alpha_j}(\theta_j, U^a(\theta_j)) \leq U^a(\theta_j) \leq S_{\alpha_{j-1}}(\theta_j).$$

The existence and uniqueness of an adapted solution to RBSDE (1.1) in the interval $[0, T]$ is obtained in an obvious way from the existence and uniqueness of an adapted solution to RBSDE (1.1) in all the subintervals $[\theta_{N-1}, T]$ and $[\theta_{j-1}, \theta_j]$ for $j = N - 1, \ldots, 2, 1$.

In this paper, we study the optimal switching problem for one-dimensional RBSDE (1.1), where the generator, the terminal value, and the upper barrier are all switched with positive costs. One-dimensional RBSDEs with fixed single reflecting barrier are a generalization of traditional optimal stopping problems, and they were introduced by El Karoui et al. [3], who gave the first existence and uniqueness result for RBSDEs. Cvitanić and Karatzas [1] generalized the work of El Karoui et al. [3] to the case of fixed double reflecting barriers and linked the solution of one-dimensional RBSDEs with double barriers to the well-known Dynkin games.

The optimal switching problem for RBSDE (1.1) is to maximize $U^a(t)$ over $a \in A_i, i \in \Lambda$. The value process turns out to be described by the following system of multi-dimensional RBSDEs with double reflecting barriers: for $i \in \Lambda \triangleq \{1, \ldots, m\},$

$$
\begin{align*}
Y_i(t) &= \xi_i + \int_t^T \psi(s, Y_i(s), Z_i(s), i) \, ds - \int_t^T dK_i^-(s) \\
&\quad + \int_t^T dK_i^+(s) - \int_t^T Z_i(s) \, dW(s), \quad t \in [0, T];
\end{align*}
$$

(1.4)

$$
S_i(t) \quad \geq \quad Y_i(t) \quad \geq \quad \max_{j \neq i, j \in \Lambda} h_{i,j}(t, Y_j(t)), \quad t \in [0, T]; \\
\int_0^T \left( Y_i(s) - \max_{j \neq i, j \in \Lambda} h_{i,j}(s, Y_j(s)) \right) dK_i^+(s) = 0, \\
\int_0^T (Y_i(s) - S_i(s)) \, dK_i^-(s) = 0.
$$

The last two equalities are respectively called the lower and the upper minimal conditions. Solution of the above RBSDE (1.4) is by no means trivial, and will be examined carefully in this paper. The unusual feature here is that for RBSDE (1.4), the upper barrier is a fixed process, while the lower barrier depends on the unknown process and is therefore implicit, which is different from one-dimensional RBSDEs with fixed double barriers. In contrast to RBSDEs with oblique reflection introduced in Hu and Tang [10], there is an additional fixed upper barrier. This difference will complicate the analysis of the existence and uniqueness for solutions to
RBSDE (1.4). For \( t \in [0, T] \), define
\[
Q(t) \triangleq \{(y_1, \cdots, y_m)^T \in \mathbb{R}^m : h_{i,j}(t, y_j) \leq y_i \leq S_i(t), \forall i, j \in \Lambda, j \neq i\}.
\]
Then the state process \( Y(\cdot) \) of (1.4) is forced to evolve in the time-dependent set \( Q(\cdot) \), thanks to the accumulative action of two increasing processes \( K_i^+ \) and \( K_i^- \).

Existence of the solution of RBSDE (1.4) is proved by two different methods. Assuming that the fixed barrier is super-regular, we firstly prove existence of the solution by the method of Picard iteration, invoking a generalized monotonic limit theorem. As a key condition of the generalized monotonic limit theorem, the comparison of the differential of the increasing process is necessary, which is formulated as our Lemma 2.2. In addition, the proof of the minimal boundary condition is complicated by the appearance of the additional fixed barrier, and our method is very technical and novel. Secondly we consider the case where the inter-connected barrier takes a particular form, and obtain an existence result by a penalty method. A priori estimate plays a crucial role therein.

Uniqueness of the solution to RBSDE (1.4) is proved in Section 5 by linking it either to the value process for our optimal switching of one-dimensional RBSDEs, or to a stochastic game involving both switching and stopping strategies for one-dimensional BSDEs.

Recently, Hu and Tang [10] initially formulated and discussed the optimal switching problem for general one-dimensional BSDEs. The value process is characterized by the solution of multi-dimensional BSDEs with oblique reflection. Later, Hamadene and Zhang [8] generalized the preceding work to a general form of positive cost for switching, with a different method of Picard iteration. The paper is a natural continuation of the two works.

The rest of the paper is organized as follows: In Section 2, we formulate our problem, introduce the generalized monotonic theorem and give some preliminary results on RBSDEs, which will be used in subsequent arguments. In Section 3, we prove existence of the solution by the method of Picard iteration. In Section 4, existence of the solution is shown by the penalty method. Uniqueness of the solution is shown in Section 5.

2 Preliminaries

We make the following assumption on the generator \( \{\psi(\cdot, \cdots, i), i \in \Lambda\} \).

**Hypothesis 1.** The generator \( \psi \) satisfies the following:

(i) The process \( \psi(\cdot, y, z, i) \in \mathcal{M}^2 \) for any \((y, z, i) \in \mathbb{R} \times \mathbb{R}^d \times \Lambda\).

(ii) There is a constant \( C > 0 \) such that for \((y, y', z, z') \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d\) and \((t, i) \in [0, T] \times \Lambda\), we have
\[
|\psi(t, y, z, i) - \psi(t, y', z', i)| \leq C(|y - y'| + |z - z'|).
\]

We make the following assumptions on the function \( \{h_{i,j}, i, j \in \Lambda\} \).

**Hypothesis 2.** For any \((i, j) \in \Lambda \times \Lambda\), the function \( h_{i,j}(t, y) \) is continuous in \((t, y)\), increasing in \( y \), and \( h_{i,j}(t, y) \leq y \).

**Hypothesis 3.** For any \( y_n \in \mathbb{R} \) and any loop \( \{i_k \in \Lambda, k = 1, \cdots, n\} \) such that \( i_1 = i_n \) and \( i_k \neq i_{k+1} \) for \( k = 1, 2, \cdots, n - 1 \), define \( y_k \triangleq h_{i_k, i_{k+1}}(t, y_{k+1}) \) for \( k = 1, \cdots, n - 1 \). Then \( y_1 < y_n \).
In Section 4, we shall specialize the function \( h_{i,j} \) to the form: \( h_{i,j}(t,y) = y - k(i, j) \) for some positively valued function \( k \) defined on \( \Lambda \times \Lambda \). We shall make the following assumption on \( k \), which is standard in the literature.

**Hypothesis 3’**. The function \( k : \Lambda \times \Lambda \to R \) satisfies the following:

(i) \( \forall i, j \in \Lambda, \ k(i, j) > 0 \) for \( i \neq j \), and \( k(i, i) = 0 \).

(ii) \( \forall i, j, l \in \Lambda \) such that \( i \neq j, j \neq l \), \( k(i, j) + k(j, l) \geq k(i, l) \).

**Remark 2.1.** Hypothesis 3 means that there is no free loop of instantaneous switchings. Hypotheses 2 and 3 are satisfied when \( h_{i,j}(t,y) = y - k(i, j) \) for \( (t,y) \in [0,T] \times R \) and \( i, j \in \Lambda \) with the function \( k \) satisfying Hypothesis 3’ (i).

**Definition 2.1.** An adapted solution of system (1.4) is a quadruple

\[
(Y, Z, K^+, K^-) \triangleq \{Y(t), Z(t), K^+(t), K^-(t); 0 \leq t \leq T\} \in (S^m)^2 \times (M^2)^2 \times (N^{2m})^2 \times (N^{2m})^2,
\]

taking values in \( R^m \times R^{m \times d} \times R^m \times R^m \) and satisfying (1.4).

We recall here the generalized monotonic limit theorem [14, Theorem 3.1], which will be used in our method of Picard iteration.

**Lemma 2.1. (generalized monotonic theorem)** We assume the following sequence of Itô processes:

\[
y^i(t) = y^i(0) + \int_0^t g^i(s) \, ds - K^{+i}(t) + K^{-i}(t) + \int_0^t z^i(s) \, dW(s), \quad i = 1, 2, \cdots
\]

satisfy

(i) for each \( i \), \( g^i \in M^2, K^{+i} \in N^2, K^{-i} \in N^{2i} \);

(ii) \( K^{-j}(t) - K^{-j}(s) \geq K^{-i}(t) - K^{-i}(s), \forall 0 \leq s \leq t \leq T, \forall i \leq j \);

(iii) For each \( t \in [0, T] \), \( \{K^{-j}(t)\}_{i=1}^{\infty} \) increasingly converges to \( K^-(t) \) with

\[
E|K^-(T)|^2 < \infty;
\]

(iv) \( (g^i, z^i)_{i=1}^{\infty} \) converges to \((g, z)\) weakly in \( M^2 \);

(v) For each \( t \in [0, T] \), \( \{y^i(t)\}_{i=1}^{\infty} \) increasingly converges to \( y(t) \) with

\[
E \sup_{0 \leq t \leq T} |y(t)|^2 < \infty.
\]

Then the limit \( y \) of \( \{y^i\}_{i=1}^{\infty} \) has the following form

\[
y(t) = y(0) + \int_0^t g(s) \, ds - K^+(t) + K^-(t) + \int_0^t z(s) \, dW(s)
\]

where \( K^+(\text{resp.} \ K^-) \) is the weak (resp. strong) limit of \( \{K^{+i}\}_{i=1}^{\infty} \) (resp. \( \{K^{-i}\}_{i=1}^{\infty} \)) in \( M^2 \) and \( K^+, K^- \in N^{2i} \). Moreover, for any \( p \in [0,2) \),

\[
\lim_{i \to \infty} E \int_0^T |z^i(t) - z(t)|^p \, dt = 0.
\]

If furthermore, \( K^+ \) is continuous, then we have

\[
\lim_{i \to \infty} E \int_0^T |z^i(t) - z(t)|^2 \, dt = 0.
\]
When we apply the above generalized monotonic theorem to RBSDE (1.4), we need to compare the differential of the increasing process $K^i_t$ for $i \in \Lambda$. However, such a kind of consideration does not seem to be available in the literature due to the appearance of the lower barrier, which is implicit and thus varying with the first unknown variable. The following lemma fills in such a gap, which will be used in Section 3.

Assume that $\xi \in L^2(\Omega, \mathcal{F}_T, P)$ and $L$ and $U$ are $\{\mathcal{F}_t, 0 \leq t \leq T\}$-adapted continuous processes satisfying

$$E[\sup_{0 \leq t \leq T} \{|L(t)^+|^2 + |U(t)^-|^2\}] < \infty, \quad L(t) \leq U(t), \quad t \in [0, T].$$

Consider the following one-dimensional RBSDE with fixed double reflecting barriers:

$$\begin{cases}
\hat{Y}(t) = \xi + \int_t^T \psi(s, \hat{Y}(s), \hat{Z}(s))ds - \int_t^T d\hat{K}^-(s) \\
\quad + \int_t^T d\hat{K}^+(s) - \int_t^T \hat{Z}(s)dW(s), \quad t \in [0, T]; \\
L(t) \leq \hat{Y}(t), \quad t \in [0, T]; \\
\int_0^T (\hat{Y}(s) - L(s))d\hat{K}^+(s) = 0, \int_0^T (\hat{Y}(s) - U(s))d\hat{K}^-(s) = 0.
\end{cases} \quad (2.1)$$

**Definition 2.2.** A barrier $U$ is called super-regular if there exists a sequence of processes $\{U^n\}_{n=1}^\infty$ such that

(i) $U^n(t) \geq U^{n+1}(t)$ and $\lim_{n \to \infty} U^n(t) = U(t)$ for $t \in [0, T]$;

(ii) For $n \geq 1$ and $t \in [0, T]$, we have

$$dU^n(t) = u_n(t)dt + v_n(t)dW(t)$$

where $u_n$ is an $\{\mathcal{F}_t, 0 \leq t \leq T\}$-adapted process such that

$$\sup_{n \geq 1} \sup_{0 \leq t \leq T} |u_n(t)| < \infty \quad \text{and} \quad v_n \in \mathcal{M}^2.$$

A barrier $V$ is called sub-regular if the barrier $-V$ is super-regular.

Note that the concept of our super-regular barrier is identical to the definition of the regular upper barrier by Hamadène et al. [5].

**Lemma 2.2.** Assume that $\psi^1$ and $\psi^2$ satisfy Hypothesis 1 and that the barrier $U$ is super-regular. Assume that $\xi^1, \xi^2 \in L^2(\Omega, \mathcal{F}_T, P)$, and $L^1, L^2$, and $U$ are $\{\mathcal{F}_t, 0 \leq t \leq T\}$-adapted continuous processes satisfying

$$E[\sup_{0 \leq t \leq T} \{|L^j(t)^+|^2 + |U(t)^-|^2\}] < \infty, \quad L^j(t) \leq U(t), \quad t \in [0, T], \quad j = 1, 2.$$  

For $j = 1, 2$, let $(\hat{Y}^j, \hat{Z}^j, \hat{K}^{+, j}, \hat{K}^{-, j})$ be the unique adapted solution of RBSDEs (2.1) associated with data $(\xi^j, \psi^j, L^j, U)$. Moreover, assume that

(i) $\xi^1 \leq \xi^2$;

(ii) $\psi^1(t, y, z) \leq \psi^2(t, y, z), \quad \forall (y, z) \in R \times R^d$;

(iii) $L^1 \leq L^2$.

Then we have

(1) $\hat{Y}^1(t) \leq \hat{Y}^2(t), \hat{K}^{-1}(t) \leq \hat{K}^{-2}(t), \quad 0 \leq t \leq T$;

(2) $\hat{K}^{-1}(r) - \hat{K}^{-1}(s) \leq \hat{K}^{-2}(r) - \hat{K}^{-2}(s), \quad 0 \leq s \leq r \leq T$. 

Proof. For $j = 1, 2$, and $n \geq 1$, the following penalized RBSDEs with a single reflecting barrier:

$$
\begin{align*}
\hat{Y}^{j,n}(t) &= \xi^j + \int_t^T \psi^j(s, \hat{Y}^{j,n}(s), \hat{Z}^{j,n}(s)) ds - n \int_t^T (\hat{Y}^{j,n}(s) - U(s))^+ ds \\
&\quad + \int_t^T d\hat{K}^{j,n}(s) - \int_t^T \hat{Z}^{j,n}(s) dW(s), \quad t \in [0,T]; \\
\hat{Y}^{j,n}(t) &\geq L^j(t), \quad t \in [0,T]; \\
\int_0^T (\hat{Y}^{j,n}(s) - L^j(s)) d\hat{K}^{j,n}(s) = 0
\end{align*}
$$

has a unique adapted solution, denoted by $(\hat{Y}^{j,n}, \hat{Z}^{j,n}, \hat{K}^{j,n})$. In view of the comparison theorem [3, Theorem 4.1], we have

$$\hat{Y}^{1,n}(t) \leq \hat{Y}^{2,n}(t), \quad 0 \leq t \leq T.$$ 

Noting that the barrier $U$ is super-regular, by the proof of [5, Theorem 42.2 and Remark 42.3], we have

$$\hat{Y}^j(t) = \lim_{n \to \infty} \hat{Y}^{j,n}(t), \quad \hat{K}^{-j}(t) = \lim_{n \to \infty} n \int_0^t (\hat{Y}^{j,n}(s) - U(s))^+ ds, \quad t \in [0,T].$$

The desired results then follow. \hfill \blacksquare

Remark 2.2. In a symmetric way, assuming that the lower barrier is sub-regular and fixed, we can compare the increment of $\hat{K}^+$ when the upper barrier varies.

The following lemma gives the continuous dependency of RBSDEs with one r.c.l.l. (right continuous and left limited) reflecting barrier, which will be used to prove the lower minimal boundary condition.

Lemma 2.3. Assume that $\xi^j \in L^2(\Omega, \mathcal{F}_T, P)$, $\psi^j$ satisfies Hypothesis 1, and $L^j \in S^2$ for $j = 1, 2$. For $j = 1, 2$, denote by $(Y^j, Z^j, K^j) \in S^2 \times M^2 \times N^2$ the unique adapted solution of the following RBSDE:

$$
\begin{align*}
Y^j(t) &= \xi^j + \int_t^T \psi^j(s, Y^j(s), Z^j(s)) ds + \int_t^T dK^j(s) \\
&\quad - \int_t^T Z^j(s)dW(s), \quad t \in [0,T]; \\
Y^j(t) &\geq L^j(t), \quad t \in [0,T]; \\
\int_0^T (Y^j(s)^-) - L^j(s^-))dK^j(s) = 0
\end{align*}
$$

Set

$$\Delta Y^j(t) \triangleq Y^j(t) - Y^j(t^-), \quad t \in [0,T]$$

for $j = 1, 2$. Then there is a constant $c > 0$ such that

$$
E\left(\sup_{0 \leq t \leq T} |Y^1(t) - Y^2(t)|^2 \right) + E\left(\left|K^1(T) - K^2(T)\right|^2 + \sum_{0 \leq t \leq T} |\Delta Y^1(t) - \Delta Y^2(t)|^2\right) \\
\leq cE\left(\left|\xi^1 - \xi^2\right|^2 + \int_0^T \left|\psi^1(s, Y^1(s), Z^1(s)) - \psi^2(s, Y^1(s), Z^1(s))\right|^2 ds\right) \\
+ cE\left(\sup_{0 \leq t \leq T} \left|L^1(t) - L^2(t)\right|^2\right)^{\frac{1}{2}} \Phi(T)^{\frac{1}{2}}
$$

where $\Phi(t)$ is a suitable function of time.
where
\[ \Phi(T) = \sum_{j=1}^{2} E \left[ |\xi_j|^2 + \int_0^T |\psi^j(t,0,0)|^2 dt + \sup_{0 \leq t \leq T} |L^j(t)|^2 \right]. \]

Remark 2.3. Lemma 2.3 can be extended to the multi-dimensional case where for \( j = 1, 2, Y^j, \xi^j, \psi^j, S^j, \) and \( L^j \) are all \( R^m \)-valued, \( Z^j \) is \( R^{m \times d} \)-valued, and \( |z| \triangleq \sqrt{\text{trace}(zz^T)} \) for \( z \in R^{m \times d} \).

The proof is similar to [3, Proposition 3.6] and is omitted.

3 Existence: the method of Picard iteration

We have the following existence result for RBSDE (1.4).

**Theorem 3.1.** Let Hypotheses 1, 2 and 3 be satisfied. Assume that the upper barrier \( S \) is super-regular with \( S(t) \in Q(t) \) for \( t \in [0,T] \), and that the terminal value \( \xi \in L^2(\Omega, \mathcal{F}_T, P; R^m) \) takes values in \( Q(T) \). Then RBSDE (1.4) has an adapted solution \( (Y, Z, K^+, K^-) \).

**Proof.** We use the method of Picard iteration. The whole proof is divided into the following six steps.

**Step 1. Construction of Picard sequence of solutions** \( \{Y^n_i, Z^n_i, K^n_i; i \in \Lambda \}_{n \geq 0} \).

For \( i \in \Lambda \), the following RBSDE with single reflecting barrier:

\[
\begin{align*}
Y^0_i(t) &= \xi_i + \int_t^T \psi(s, Y^0_i(s), Z^0_i(s), i) \, ds - \int_t^T dK^0_i(s) \\
& \quad - \int_t^T Z^0_i(s) \, dW(s), \quad t \in [0,T]; \\
Y^0_i(t) &\leq S_i(t), \quad t \in [0,T]; \\
\int_0^T (Y^0_i(s) - S_i(s)) \, dK^0_i(s) &= 0
\end{align*}
\]

has a unique adapted solution, denoted by \( (Y^0_i, Z^0_i, K^0_i) \).

For \( n \geq 1 \) and \( i \in \Lambda \), consider the following RBSDE:

\[
\begin{align*}
Y^n_i(t) &= \xi_i + \int_t^T \psi(s, Y^n_i(s), Z^n_i(s), i) \, ds - \int_t^T dK^{-n}_i(s) \\
& \quad + \int_t^T dK^{+n}_i(s) - \int_t^T Z^n_i(s) \, dW(s), \quad t \in [0,T]; \\
S_i(t) &\geq Y^n_i(t) \geq \max_{j \neq i, j \in \Lambda} h_{i,j}(t, Y^{n-1}_j(t)), \quad t \in [0,T]; \\
\int_0^T (Y^n_i(s) - \max_{j \neq i, j \in \Lambda} h_{i,j}(s, Y^{n-1}_j(s))) \, dK^{+n}_i(s) &= 0, \\
\int_0^T (Y^n_i(s) - S_i(s)) \, dK^{-n}_i(s) &= 0.
\end{align*}
\]

Note that
\[
\max_{j \neq i, j \in \Lambda} h_{i,j}(t, Y^{n-1}_j(t)) \leq \max_{j \neq i, j \in \Lambda} h_{i,j}(t, S_j(t)) \leq S_i(t), \quad t \in [0,T]
\]
due to Hypothesis 2 and the assumption that $S(t) \in Q(t)$ for $t \in [0, T]$. In view of [5, Theorem 42.2 and Remark 42.3], RBSDE (3.2) has a unique solution
\[(Y^n_i, Z^n_i, K^{+,n}_i, K^{-,n}_i) \in S^2 \times \mathcal{M}^2 \times \mathcal{N}^2 \times \mathcal{L}^2.\]

**Step 2. Convergence of** \{$Y^n_i, K^{-,n}_i; i \in \Lambda\}_{n \geq 1}. Since h_{i,j}(t, \cdot)$ is increasing, the lower barrier of RBSDE (3.1) is increasing with $n$ by induction.

RBSDE (3.1) can be viewed as having the lower barrier $-\infty$. Then from Lemma 2.2, we have
\[Y^0_i(t) \leq Y^1_i(t), \quad 0 \leq t \leq T.\]

Since $h_{i,j}(t, y)$ is increasing in $y$, using Lemma 2.2 again, we have for $n \geq 1$ and $i \in \Lambda$,
\[Y^n_i(t) \leq Y^{n+1}_i(t), \quad K^{-,n}_i(t) \leq K^{-,n+1}_i(t), \quad 0 \leq t \leq T;\]
\[K^{-,n}_i(r) - K^{-,n}_i(s) \leq K^{-,n+1}_i(r) - K^{-,n+1}_i(s), \quad 0 \leq s \leq r \leq T.\]

Hence, the sequence \{$(Y^n_i(t), K^{-,n}_i(t))\}_{n=1}^{\infty}$ has a limit, denoted by $(Y_i(t), K_i^-(t))$.

Since
\[Y^0_i(t) \leq Y^n_i(t) \leq S_i(t), \quad \forall t \leq T, \quad i \in \Lambda,
\]
we have
\[\sup_{n \geq 1} \mathbb{E} \left[ \sup_{0 \leq t \leq T} |Y^n_i(t)|^2 \right] \leq \mathbb{E} \sup_{0 \leq t \leq T} (|Y^0_i(t)|^2 + |S(t)|^2) < \infty, \quad i \in \Lambda. \quad (3.3)\]

Applying Fatou’s lemma and the dominated convergence theorem, we have
\[\mathbb{E} \left[ \sup_{0 \leq t \leq T} |Y_i(t)|^2 \right] < \infty, \quad \lim_{n \to \infty} \mathbb{E} \int_0^T |Y^n_i(t) - Y_i(t)|^2 dt = 0, \quad i \in \Lambda. \quad (3.4)\]

For $i \in \Lambda$, the following RBSDE
\[
\begin{cases}
\tilde{Y}_i(t) = \xi_i + \int_t^T \psi(s, \tilde{Y}_i(s), \tilde{Z}_i(s), i) \, ds - \int_t^T d\tilde{K}_i^-(s) \\
+ \int_t^T d\tilde{K}_i^+(s) - \int_t^T \tilde{Z}_i(s) \, dW(s), \quad t \in [0, T]; \\
S_i(t) \geq \tilde{Y}_i(t) \geq \max_{j \neq i,j \in \Lambda} h_{i,j}(t, S_j(t)), \quad t \in [0, T]; \\
\int_0^T (\tilde{Y}_i(s) - \max_{j \neq i,j \in \Lambda} h_{i,j}(s, S(s))) \, d\tilde{K}_i^+(s) = 0, \\
\int_0^T (\tilde{Y}_i(s) - S_i(s)) \, d\tilde{K}_i^-(s) = 0
\end{cases}
\quad (3.5)
\]
has a unique adapted solution, denoted by $(\tilde{Y}_i, \tilde{Z}_i, \tilde{K}_i^+, \tilde{K}_i^-) \in S^2 \times \mathcal{M}^2 \times \mathcal{N}^2 \times \mathcal{L}^2$. By Lemma 2.2, we know that for $n \geq 1$ and $i \in \Lambda$,
\[K^{-,n}_i(t) \leq K^{-,n+1}_i(t) \leq \tilde{K}_i^-(t), \quad 0 \leq t \leq T;\]
\[K^{-,n}_i(r) - K^{-,n}_i(s) \leq K^{-,n+1}_i(r) - K^{-,n+1}_i(s) \leq \tilde{K}_i^-(r) - \tilde{K}_i^-(s), \quad 0 \leq s \leq r \leq T.\]

Then, $K_i^-$ is continuous for $i \in \Lambda$. Hence,
\[\sup_{n \geq 1} \mathbb{E} \left[ \sup_{0 \leq t \leq T} |K^{-,n}_i(t)|^2 \right] \leq \mathbb{E} \left[ \sup_{0 \leq t \leq T} |K_i^-(t)|^2 \right] \leq \mathbb{E} \left[ \sup_{0 \leq t \leq T} |\tilde{K}_i^-(t)|^2 \right] < \infty, \quad i \in \Lambda. \quad (3.6)\]
From Dini’s theorem, we have
\[ \lim_{n \to \infty} \sup_{0 \leq t \leq T} |K_i^{-,n}(t) - K_i^{-}(t)|^2 = 0, \quad i \in \Lambda. \]

From the dominated convergence theorem, we have
\[ \lim_{n \to \infty} E\left[ \sup_{0 \leq t \leq T} |K_i^{-,n}(t) - K_i^{-}(t)|^2 \right] = 0, \quad i \in \Lambda. \quad (3.7) \]

**Step 3. Uniform boundedness of \( \{\psi(\cdot, Y_i^n, Z_i^n, i), Z_i^n, K_i^{+,n}; i \in \Lambda\}_{n \geq 1} \) in \( \mathcal{M}^2 \times \mathcal{M}^2 \times \mathcal{N}^2 \).**

Applying Itô’s lemma to \( |Y_i^n(t)|^2 \), we have for \( i \in \Lambda \),
\[
|Y_i^n(t)|^2 + \int_t^T |Z_i^n(s)|^2 ds = \xi_i^2 + 2 \int_t^T Y_i^n(s)\psi(s, Y_i^n(s), Z_i^n(s), i) ds + 2 \int_t^T Y_i^n(s) dK_i^{+,n}(s) \\
- 2 \int_t^T Y_i^n(s)dK_i^{-,n}(s) - 2 \int_t^T Y_i^n(s)Z_i^n(s)dW(s).
\]

Using the Lipschitz property of \( \psi \), the upper minimal boundary condition in (3.2) and the elementary inequality: \( ab \leq \frac{1}{\alpha} a^2 + \alpha b^2 \), we have for any arbitrary positive real number \( \alpha \),
\[
E|Y_i^n(t)|^2 + E\left[ \int_t^T |Z_i^n(s)|^2 ds \right] \leq E\left( \xi_i^2 + 2 \int_t^T Y_i^n(s)\psi(s, 0, 0, i) + C|Y_i^n(s)| + C|Z_i^n(s)|\right) ds \\
+ E\left( 2 \int_t^T Y_i^n(s)dK_i^{+,n}(s) - 2 \int_t^T S_i(s)dK_i^{-,n}(s) \right) \\
\leq E\left( \xi_i^2 + \int_t^T |\psi(s, 0, 0, i)|^2 ds + c \int_t^T |Y_i^n(s)|^2 ds + \frac{1}{3} \int_t^T |Z_i^n(s)|^2 ds \right) \\
+ E\left( \frac{1}{\alpha} \sup_{0 \leq t \leq T} |Y_i^n(t)|^2 + \alpha |K_i^{+,n}(T)|^2 + \sup_{0 \leq t \leq T} |S_i(t)|^2 + |K_i^{-,n}(T)|^2 \right).
\]

Here and in the sequel, \( c \) is a positive constant whose value only depends on the Lipschitz coefficient \( C \) and may change from line to line.

From RBSDE (3.2), we know that for \( i \in \Lambda \),
\[ K_i^{+,n}(T) = Y_i^n(0) - \xi_i - \int_0^T \psi(s, Y_i^n(s), Z_i^n(s), i) ds + K_i^{-,n}(T) + \int_0^T Z_i^n(s)dW(s). \]

Hence,
\[ E|K_i^{+,n}(T)|^2 \leq c \left( 1 + E\int_0^T (|Y_i^n(s)|^2 + |Z_i^n(s)|^2) ds + E|K_i^{-,n}(T)|^2 \right), \quad i \in \Lambda. \quad (3.9) \]

Substituting (3.9) into (3.8) and letting \( \alpha = \frac{1}{2c} \) and \( t = 0 \), we have
\[ E\int_0^T |Z_i^n(s)|^2 ds \leq cE\left( 1 + \sup_{0 \leq t \leq T} |Y_i^n(t)|^2 + \int_0^T |Y_i^n(s)|^2 ds + |K_i^{-,n}(T)|^2 \right), \quad i \in \Lambda. \]
From (3.3) and (3.6), we know

\[
\sup_{n \geq 1} E \int_0^T |Z^n_i(s)|^2 ds < \infty, \quad i \in \Lambda. \tag{3.10}
\]

Then from (3.9), we know

\[
\sup_{n \geq 1} E|K_i^{n}(T)|^2 < \infty, \quad i \in \Lambda. \tag{3.11}
\]

From (3.3), (3.10) and the Lipschitz property of \( \psi \), we know

\[
\sup_{n \geq 1} E \int_0^T |\psi(s, Y^n_i(s), Z^n_i(s), i)|^2 ds < \infty, \quad i \in \Lambda.
\]

Therefore, without loss of generality, we can assume that for \( i \in \Lambda, \{ \psi(\cdot, Y^n_i, Z^n_i, i) \}_{n \geq 0}, \{ Z^n_i \}_{n \geq 0}, \)

and \( \{ K_i^{n+} \}_{n \geq 0} \) converge weakly in \( M^2 \) to \( \psi_i, Z_i, \) and \( K_i^{+} \), respectively.

**Step 4. Verification of the first equation of RBSDE (1.4).**

From the first equation of (3.2), we have

\[
Y^n_i(t) = Y^n_i(0) - \int_0^t \psi(s, Y^n_i(s), Z^n_i(s), i) ds - K_i^{+}(t) + K_i^{-}(t) + \int_0^t Z^n_i(s) dW(s).
\]

All the assumptions of the generalized monotonic limit theorem (see Lemma 2.1) are shown to be satisfied in previous steps. Therefore, for \( i \in \Lambda \), the limit \( Y_i \) is r.c.l.l. and has the form:

\[
Y_i(t) = \xi_i + \int_t^T \psi_i(s) ds - \int_t^T dK_i^{-}(s) + \int_t^T dK_i^{+}(s) - \int_t^T Z_i(s) dW(s),
\]

and \( K_i^{+} \in \mathcal{N}^2 \). Moreover, for any \( p \in [0, 2) \),

\[
\lim_{n \to \infty} E \int_0^T |Z^n_i(s) - Z_i(s)|^p ds = 0, \quad i \in \Lambda.
\]

Hence, we have for \( i \in \Lambda \),

\[
\lim_{n \to \infty} E \int_0^T |\psi(s, Y^n_i(s), Z^n_i(s), i) - \psi(s, Y_i(s), Z_i(s), i)|^p ds = 0;
\]

\[
\psi_i(s) = \psi(s, Y_i(s), Z_i(s), i), \quad a.e., a.s.;
\]

\[
Y_i(t) = \xi_i + \int_t^T \psi(s, Y_i(s), Z_i(s), i) ds - \int_t^T dK_i^{-}(s) + \int_t^T dK_i^{+}(s) - \int_t^T Z_i(s) dW(s). \tag{3.12}
\]

**Step 5. The minimal boundary conditions .**

In view of RBSDE (3.2), we have

\[
\max_{j \neq i, j \in \Lambda} \{ h_{i,j}(t, Y^{n-1}_j(t)) \} \leq Y^{n}_i(t) \leq S_i(t), \quad t \in [0, T], i \in \Lambda.
\]

Passing to the limit, we have

\[
\max_{j \neq i, j \in \Lambda} \{ h_{i,j}(t, Y_j(t)) \} \leq Y_i(t) \leq S_i(t), \quad t \in [0, T], i \in \Lambda. \tag{3.13}
\]
Since
\[
\int_0^T (Y^n_i(s) - S_i(s)) dK_i^{-,n}(s) = 0 \quad \text{and} \quad Y^n_i(s) = Y^n_i(s-) \leq Y_i(s-) \leq S_i(s),
\]
we have
\[
0 = \int_0^T (Y^n_i(s) - S_i(s)) dK_i^{-,n}(s) \leq \int_0^T (Y_i(s-) - S_i(s)) dK_i^{-,n}(s) \leq 0, \quad i \in \Lambda.
\]
Hence,
\[
\int_0^T (Y_i(s-) - S_i(s)) dK_i^{-,n}(s) = 0, \quad i \in \Lambda.
\]
On the other hand, for \(i \in \Lambda\),
\[
0 \leq \int_0^T (S_i(s) - Y_i(s-)) d(K_i^{-}(s) - K_i^{-,n}(s)) \leq \sup_{0 \leq s \leq T} (S_i(s) - Y_i(s-))[K_i^{-}(T) - K_i^{-,n}(T)].
\]
Since
\[
\lim_{n \to \infty} K_i^{-,n}(T) = K_i^{-}(T), \quad i \in \Lambda,
\]
we have
\[
\int_0^T (Y_i(s-) - S_i(s)) dK_i^{-}(s) = \lim_{n \to \infty} \int_0^T (Y_i(s-) - S_i(s)) dK_i^{-,n}(s) = 0, \quad i \in \Lambda.
\]
We have just proved the upper minimal boundary condition. It remains to prove the lower minimal boundary condition. The technique used in [8] is found difficult to be directly applied to our case since the corresponding argument on the smallest \(\psi\)-supermartingale is not true in the case of double barriers. We shall view the RBSDEs with double barriers as RBSDEs with single lower barrier by taking the increasing processes \(K^{-,n}\) as given. For \(i \in \Lambda\) and \(n \geq 1\), the following RBSDE
\[
\begin{align*}
\tilde{Y}^n_i(t) &= \xi_i + \int_t^T \psi(s, \tilde{Y}^n_i(s), \tilde{Z}^n_i(s), i) ds - K_i^{-,n}(T) \\
&+ K_i^{-,n}(t) + \int_t^T d\tilde{K}_i^{+,n}(s) - \int_t^T \tilde{Z}_i^n(s) dW(s), \\
\tilde{Y}^n_i(t) &\geq \tilde{h}_i(t) \triangleq \max_{j \neq i, j \in \Lambda} h_{i,j}(t, Y_j(t)), \quad t \in [0, T], \\
\int_0^T (\tilde{Y}^n_i(s-) - \tilde{h}_i(s-)) d\tilde{K}_i^{+,n}(s) &= 0
\end{align*}
\]
has a unique adapted solution, denoted by \((\tilde{Y}^n_i, \tilde{Z}^n_i, \tilde{K}_i^{+,n})\).

Define
\[
\tilde{X}^n_i \triangleq \tilde{Y}^n_i - K_i^{-,n}, \quad \psi_n(s, y, z, i) \triangleq \psi(s, y + K_i^{-,n}(s), z, i), \quad i \in \Lambda.
\]
Then \((\tilde{X}^n_i, \tilde{Z}^n_i, \tilde{K}_i^{+,n})\) satisfies the following RBSDE:
\[
\begin{align*}
\tilde{X}^n_i(t) &= (\xi_i - K_i^{-,n}(T)) + \int_t^T \psi_n(s, \tilde{X}^n_i(s), \tilde{Z}^n_i(s), i) ds \\
&\quad + \int_t^T d\tilde{K}_i^{+,n}(s) - \int_t^T \tilde{Z}_i^n(s) dW(s), \\
\tilde{X}^n_i(t) &\geq \tilde{h}_i(t) - K_i^{-,n}(t), \quad t \in [0, T], \\
\int_0^T (\tilde{X}^n_i(s-) - \tilde{h}_i(s-) + K_i^{-,n}(s)) d\tilde{K}_i^{+,n}(s) &= 0.
\end{align*}
\]
For \( i \in \Lambda \), let \((\bar{Y}_i, \bar{Z}_i, \bar{K}^+_i)\) be the solution of the following RBSDE:

\[
\begin{align*}
\bar{Y}_i(t) &= \xi_i + \int_t^T \psi(s, \bar{Y}_i(s), \bar{Z}_i(s), i) ds - K^-_i(T) + K^-_i(t) \\
&\quad + \int_t^T d\bar{K}^+_i(s) - \int_t^T \bar{Z}_i(s) dW(s), \quad t \in [0, T]; \\
\bar{Y}_i(t) &\geq \bar{h}_i(t), \quad t \in [0, T]; \\
\int_0^T (\bar{Y}_i(s) - \bar{h}_i(s)) d\bar{K}^+_i(s) &= 0.
\end{align*}
\]

(3.16)

Define

\[
\bar{X}_i \triangleq \bar{Y}_i - K^-_i, \quad \psi_-(s, y, z, i) \triangleq \psi(s, y + K^-_i(s), z, i), \quad i \in \Lambda.
\]

Then \((\bar{X}_i, \bar{Z}_i, \bar{K}^+_i)\) satisfies the following RBSDE:

\[
\begin{align*}
\bar{X}_i(t) &= (\xi_i - K^-_i(T)) + \int_t^T \psi_-(s, \bar{X}_i(s), \bar{Z}_i(s), i) ds \\
&\quad + \int_t^T d\bar{K}^+_i(s) - \int_t^T \bar{Z}_i(s) dW(s), \\
\bar{X}_i(t) &\geq \bar{h}_i(t) - K^-_i(t), \quad t \in [0, T], \\
\int_0^T \left( \bar{X}_i(s) - \bar{h}_i(s) + K^-_i(s) \right) d\bar{K}^+_i(s) &= 0.
\end{align*}
\]

(3.17)

Since

\[
\psi_n(s, y, z, i) = \psi(s, y + K^-_i(s), z, i) \\
&\leq C|K^-_i(s) - K^-_i(s)|,
\]

in view of Lemma 2.3, we have

\[
E\left[ \sup_{0 \leq t \leq T} |\bar{X}^n_i(t) - \bar{X}_i(t)|^2 \right] \\
\leq c\left( E((K^-_i(T) - K^-_i(T))^2) + C \int_0^T |K^-_i(t) - K^-_i(t)|^2 dt \right) + c(E[ \sup_{0 \leq t \leq T} |K^-_i(t) - K^-_i(t)|^2])^{1/2} (\Phi^n(T))^{1/2},
\]

(3.18)

where

\[
\Phi^n(T) \triangleq E\left( (\xi_i - K^-_i(T))^2 + \int_0^T |\psi_n(s, 0, 0, i)|^2 ds \right) \\
+ E[ \sup_{0 \leq t \leq T} ((\bar{h}_i(t) - K^-_i(t))^2)] \\
+ E\left( (\xi_i - K^-_i(T))^2 + \int_0^T |\psi_0(s, 0, 0, i)|^2 ds \right) \\
+ E[ \sup_{0 \leq t \leq T} ((\bar{h}_i(t) - K^-_i(t))^2)].
\]

Since

\[
\psi_n(s, 0, 0, i) \leq \psi(s, 0, 0, i) + CK^-_i(s), \\
\psi_-(s, 0, 0, i) \leq \psi(s, 0, 0, i) + CK^-_i(s),
\]

\[
\begin{align*}
\sup_{0 \leq t \leq T} (\tilde{h}_i(t) - K^{-n}_i(t))^2 &\leq \sup_{0 \leq t \leq T} (\tilde{h}_i(t))^2 \\
&\leq \sup_{0 \leq t \leq T} (\max_{j \neq i, j \in \Lambda} Y_j(t))^2 \\
&\leq \sum_{j \in \Lambda} \sup_{0 \leq t \leq T} |Y_j(t)|^2,
\end{align*}
\]

and
\[
\begin{align*}
\sup_{0 \leq t \leq T} (\tilde{h}_i(t) - K^{-}_i(t))^2 &\leq \sum_{j \in \Lambda} \sup_{0 \leq t \leq T} |Y_j(t)|^2,
\end{align*}
\]

we have
\[
\Phi^n(T) \leq 4E(\xi_i^2 + \int_0^T |\psi(s, 0, 0, i)|^2 ds) + 2 \sum_{j \in \Lambda} E[\sup_{0 \leq t \leq T} |Y_j(t)|^2] + 2(C^2T + 1)E(|K^{-n}_i(T)|^2 + |K^{-}_i(T)|^2).
\]

From (3.3) and (3.6), we see
\[
\sup_{n \geq 1} \Phi^n(T) < \infty. \tag{3.19}
\]

From (3.7), (3.18) and (3.19), we see
\[
\lim_{n \to \infty} E[\sup_{0 \leq t \leq T} |\tilde{X}_i^n(t) - \tilde{X}_i(t)|^2] = 0, \quad i \in \Lambda.
\]

So there is a subsequence of \( \{\tilde{X}_i^n\}_{n \geq 1} \) converging to \( \tilde{X}_i \), a.e., a.s. Without loss of
generality, assume that
\[
\lim_{n \to \infty} \tilde{X}_i^n = \tilde{X}_i, \quad \text{a.e., a.s.,} \quad i \in \Lambda. \tag{3.20}
\]

Set
\[
X_i^n \triangleq Y_i^n - K^{-n}_i, \quad i \in \Lambda.
\]

Then from reflected BSDEs (3.2) we know that \((X_i^n, Z_i^n, K_i^{+,n})\) is the solution of the following reflected BSDEs with single reflecting barrier:
\[
\begin{align*}
X_i^n(t) &= (\xi_i - K^{-n}_i(T)) + \int_t^T \psi_{K^n}(s, X_i^n(s), Z_i^n(s), i) ds \\
&\quad + \int_t^T dK^{+,n}_i(s) - \int_t^T Z_i^n(s) dW(s), \quad t \in [0, T]; \\
X_i^n(t) &\geq \max_{j \neq i, j \in \Lambda} h_{i,j}(t, Y_j^{n-1}(t)) - K^{-n}_i(t), \quad t \in [0, T]; \\
\int_0^T \left( X_i^n(s) - \max_{j \neq i, j \in \Lambda} h_{i,j}(s, Y_j^{n-1}(s)) + K^{-n}_i(s) \right) dK^{+,n}_i(s) &= 0.
\end{align*}
\]

Comparing it with reflected BSDEs (3.15) and using the comparison theorem for r.c.l.l. reflecting barrier [6, Theorem 1.5], we know that
\[
\tilde{X}_i^n(t) \geq X_i^n(t) = Y_i^n(t) - K^{-n}_i(t), \quad (t, i) \in [0, T] \times \Lambda.
\]

In view of (3.20), we have
\[
\tilde{X}_i(t) \geq Y_i(t) - K^{-}_i(t), \quad (t, i) \in [0, T] \times \Lambda. \tag{3.22}
\]

Note that due to the appearance of the additional fixed upper barrier, it is not clear whether the lower barrier of (3.17) is not less than that of (3.21). Such a difficulty is got around by comparing (3.21) and (3.15).
On the other hand, from (3.17) and [11, Theorem 2.1], we know that $\bar{X}_i(\cdot)$ is the smallest $\psi_-$-supermartingale with the lower barrier $\{h_i(t) - K_i^-(t), 0 \leq t \leq T\}$. From (3.12) and (3.13), it can be easily obtained that $\{Y_i(t) - K_i^-(t), 0 \leq t \leq T\}$ is a $\psi_-$-supermartingale with the same lower barrier and terminal value. Hence,

$$\bar{X}_i(t) \leq Y_i(t) - K_i^-(t), \quad \forall t \leq T, i \in \Lambda.$$ 

Together with (3.22), we have

$$\bar{X}_i(t) = Y_i(t) - K_i^-(t), \quad \forall t \leq T, i \in \Lambda.$$

Then

$$\bar{Y}_i(t) = Y_i(t), \quad \forall t \leq T, i \in \Lambda.$$ 

From the uniqueness of the Doob-Meyer Decomposition, it follows that

$$Z_i(t) = Z_i(t), \quad K_i^+(t) = K_i^+(t), \quad \forall 0 \leq t \leq T, i \in \Lambda.$$ 

Hence, for $i \in \Lambda$, $(Y_i, Z_i, K_i^+)$ is a solution to RBSDE (1.4) except that both minimal boundary conditions are replaced by

$$\int_0^T \left( Y_i(s) - \max_{j \neq i, j \in \Lambda} h_{i,j}(s, Y_j(s)) \right) dK_i^+(s) = 0, \quad \int_0^T (Y_i(s) - S(s)) dK_i^-(s) = 0.$$ 

If we further prove the continuity of $Y_i$, then we know $\{(Y_i, Z_i, K_i^+, K_i^-), i \in \Lambda\}$ is an adapted solution of RBSDE (1.4).

**Step 6. The continuity of $Y_i$ and $K_i^+$ for $i \in \Lambda$.**

Since $K_i^-$ is continuous, we know $\Delta Y_i(t) = -\Delta K_i^+(t) \leq 0$. Suppose $\Delta Y_{i_1}(t^*) < 0$ for some $i_1 \in \Lambda$ and some $t^* \in [0, T]$, then $\Delta K_i^+(t^*) > 0$. From the minimal boundary conditions of (4.20), we know

$$Y_{i_1}(t^*) = \max_{j \neq i_1, j \in \Lambda} h_{i_1,j}(t^*, Y_j(t^*)).$$ 

The index attaining the maximum is denoted by $i_2$ and $i_2 \neq i_1$. Hence,

$$h_{i_1,i_2}(t^*, Y_{i_2}(t^*)) = Y_{i_1}(t^*) > Y_{i_1}(t^*) \geq h_{i_1,i_2}(t^*, Y_{i_2}(t^*)),$$

It follows that $\Delta Y_{i_2}(t^*) < 0$. Repeating these arguments, we can obtain that

$$Y_{i_k}(t^*) = h_{i_k,i_{k+1}}(t^*, Y_{i_{k+1}}(t^*)), \quad k = 2, 3, \ldots.$$ 

Since $\Lambda$ is a finite set, there must be a loop in $\Lambda$, without loss of generality, we assume that $i_1 = i_n$. Then for the loop $\{i_n, i_{n-1}, \ldots, i_1\}$, we have

$$Y_{i_{n-1}}(t^*) = h_{i_{n-1},i_n}(t^*, Y_{i_n}(t^*)), \ldots, Y_{i_1}(t^*) = h_{i_1,i_2}(t^*, Y_{i_2}(t^*)), \ldots, Y_{i_n}(t^*) = Y_{i_1}(t^*).$$

This contradicts Hypothesis 3. Therefore, $\Delta Y_i(t) = 0$, $0 \leq t \leq T, \forall i \in \Lambda$. So is $K_i^+$.

### 4 Existence: the penalty method

In Theorem 3.1, every component of the upper barrier $S$ is assumed to be super-regular. Let $k(i, j)$ be the switching cost from state $i$ to state $j$ in the optimal switching problem (see Hu and Tang [10]) satisfying Hypothesis 3’, and let the function $h_{i,j}$ introduced in the preceding section take the particular form $h_{i,j}(t, y) = y - k(i, j)$. Then we can prove by a penalty method the existence of an adapted solution to RBSDE (1.4) without the super-regularity assumption on the upper barrier $S$. 
4.1 Multi-dimensional RBSDEs with fixed single reflecting barrier.

In what follows, we consider the multi-dimensional RBSDEs with fixed single reflecting barrier, show the existence and uniqueness by a penalty method, and give a comparison theorem.

For two $m$-dimensional vectors $x \triangleq (x_1, \ldots, x_m)^T$ and $y \triangleq (y_1, \ldots, y_m)^T$, we mean by $x \leq y$ that $x_i \leq y_i$ for $i \in \Lambda$. For a vector $x \triangleq (x_1, \ldots, x_m)^T$, $x^+$ is defined as the $m$-dimensional vector $(x_1^+, \ldots, x_m^+)^T$.

Consider the following multi-dimensional RBSDE with fixed single reflecting barrier:

\[
\begin{cases}
Y(t) = \xi + \int_t^T \phi(s, Y(s), Z(s)) \, ds - \int_t^T dK(s) \\
- \int_t^T Z(s) \, dW(s), \quad t \in [0, T]; \\
Y(t) \leq S(t), \quad t \in [0, T]; \\
\int_0^T (Y_i(s) - S_i(s)) \, dK_i(s) = 0, \quad i \in \Lambda.
\end{cases}
\]

(4.1)

We make the following assumption on the generator $\phi$, the terminal value $\xi \triangleq (\xi_1, \ldots, \xi_m)^T$, and the barrier $S \triangleq (S_1, \ldots, S_m)^T$.

**Hypothesis 4.**

(i) The process $\phi(\cdot, 0, 0) \in (\mathcal{M}^2)^m$. For $i \in \Lambda$, $\xi_i \in L^2(\Omega, \mathcal{F}_T, P)$ and $S_i \in S^2$ with $\xi_i \leq S_i(T)$.

(ii) There is a constant $C > 0$ such that for any $(t, y, y', z, z') \in [0, T] \times (R^m)^2 \times (R^{m \times d})^2$, we have

\[
|\phi(t, y, z) - \phi(t, y', z')| \leq C(|y - y'| + |z - z'|), \\
-4(y^-, \phi(t, y^+ + y', z) - \phi(t, y', z')) \leq 2 \sum_{i=1}^m \chi_{\{y_i < 0\}}|z_i - z'_i|^2 + C|y^-|^2. \tag{4.2}
\]

We have

**Theorem 4.1.** Let Hypothesis 4 be satisfied. Then RBSDE (4.1) has a unique adapted solution $(Y, Z, K) \in (S^2)^m \times (\mathcal{M}^2)^{m \times d} \times (\Lambda^2)^m$.

**Proof.** For any positive integer $n$, consider the following penalized BSDE:

\[
\begin{align*}
Y^n(t) &= \xi + \int_t^T \phi(s, Y^n(s), Z^n(s)) \, ds - n \int_t^T (Y^n(s) - S(s))^+ \, ds \\
&\quad - \int_t^T Z^n(s) \, dW(s), \quad t \in [0, T].
\end{align*}
\]

(4.3)

From Pardoux and Peng [13], we know that for each $n$, BSDE (4.3) has a unique adapted solution $(Y^n, Z^n) \in (S^2)^m \times (\mathcal{M}^2)^{m \times d}$.

Define for $(t, y, z) \in [0, T] \times R^m \times R^{m \times d}$,

\[
\phi^n(t, y, z) \triangleq \phi(t, y, z) - n(y - S(t))^+ \quad \text{and} \quad K^n(t) \triangleq n \int_0^t (Y^n(s) - S(s))^+ \, ds.
\]
In view of Hypothesis 4 (ii), we have for all \( y \in \mathbb{R}^m, \)
\[
-4(y^-, \phi^n(s, y^+ + y', z) - \phi^{n+1}(s, y', z'))
\leq -4(y^-, \phi(s, y^+ + y', z) - \phi(s, y', z') + (y' - S(s))^+)
\leq -4(y^-, \phi(s, y^+ + y', z) - \phi(s, y', z'))
\leq 2 \sum_{i=1}^m \chi_{\{y_i < 0\}} |z_i - z'_i|^2 + C|y^-|^2.
\]

Applying the comparison theorem of multi-dimensional BSDEs (see [9, Theorem 2.1]), we deduce that
\[
Y^{n+1}(t) \leq Y^n(t), \quad 0 \leq t \leq T.
\]

For \( t \in [0, 1], \) the sequence \( \{Y^n(t)\}_{n \geq 1} \) almost surely admits a limit, which is denoted by \( Y(t) \) below.

Applying Itô's lemma to compute \(|Y^n(t)|^2\), we have
\[
|Y^n(t)|^2 + \int_t^T |Z^n(s)|^2 ds
= |\xi|^2 + 2 \int_t^T \langle Y^n(s), \phi(s, Y^n(s), Z^n(s)) \rangle ds
- \int_t^T \langle Y^n(s), dK^n(s) \rangle - \int_t^T \langle Y^n(s), Z^n(s) dW(s) \rangle.\tag{4.4}
\]

Taking expectation on both sides, in view of the following inequality
\[
\int_t^T \langle Y^n(s), dK^n(s) \rangle \geq \int_t^T \langle S(s), dK^n(s) \rangle,
\]
we have
\[
E|Y^n(t)|^2 + E\int_t^T |Z^n(s)|^2 ds
\leq E|\xi|^2 + 2E\int_t^T |\phi(s, 0, 0)| + C(|Y^n(s)| + |Z^n(s)|)|Y^n(s)| ds
- E\int_t^T \langle S(s), dK^n(s) \rangle \tag{4.5}
\leq E|\xi|^2 + E\int_t^T |\phi(s, 0, 0)|^2 ds + (3C^2 + 2C + 1)E\int_t^T |Y^n(s)|^2 ds
+ \frac{1}{3}E\int_t^T |Z^n(s)|^2 ds + \frac{m}{\alpha}E(\sup_{0 \leq t \leq T} |S^{-}(t)|^2) + \alpha E|K^n(T) - K^n(t)|^2
\]
for an arbitrary positive real number \( \alpha. \)

From (4.3), we have
\[
K^n(T) - K^n(t) = \xi - Y^n(t) + \int_t^T \phi(s, Y^n(s), Z^n(s)) ds - \int_t^T Z^n(s) dW(s).\tag{4.6}
\]

Further, we have
\[
E|K^n(T) - K^n(t)|^2 \leq 4E(|Y^n(t)|^2 + |\xi|^2) + 4(3TC^2 + 1)E\int_t^T |Z^n(s)|^2 ds
+ 12TE\int_t^T (|\phi(s, 0, 0)|^2 + C^2|Y^n(s)|^2) ds. \tag{4.7}
\]
Setting 
\[ \alpha = \frac{1}{12(3TC^2 + 1)} \]
in (4.5), in view of (4.7), we have
\[ \frac{2}{3} E|Y^n(t)|^2 + \frac{1}{3} E \int_t^T |Z^n(s)|^2 ds \leq c(1 + E \int_t^T |Y^n(s)|^2)ds \]
for a constant c which does not dependent on n. It then follows from Gronwall’s inequality that
\[ \sup_{0 \leq t \leq T} E|Y^n(t)|^2 + E \int_0^T |Z^n(s)|^2 ds \leq c, \quad n = 1, 2, \ldots \]
which implies that
\[ E|K^n(T)|^2 \leq c, \quad n = 1, 2, \ldots \] (4.8)
In view of (4.4), applying the Burkholder-Davis-Gundy inequality, we have
\[ E \sup_{0 \leq t \leq T} |Y^n(t)|^2 + E \int_0^T |Z^n(s)|^2 ds + E|K^n(T)|^2 \leq c, \quad n = 1, 2, \ldots \]
Recalling that \( Y(t) = \lim_{n \to \infty} Y^n(t) \), using Fatou’s lemma, we have
\[ E \sup_{0 \leq t \leq T} |Y(t)|^2 \leq c. \]
It then follows from Lebesgue’s dominated convergence theorem that
\[ \lim_{n \to \infty} E \int_0^T |Y^n(s) - Y(s)|^2 ds = 0. \]

For positive integers \( n_1 \) and \( n_2 \), applying Itô’s lemma to compute \( |Y^{n_1}(t) - Y^{n_2}(t)|^2 \), we have for \( t \in [0, T] \),
\[ |Y^{n_1}(t) - Y^{n_2}(t)|^2 + \int_t^T |Z^{n_1}(s) - Z^{n_2}(s)|^2 ds \]
\[ = 2 \int_t^T \langle Y^{n_1}(s) - Y^{n_2}(s), \phi(s, Y^{n_1}(s), Z^{n_1}(s)) - \phi(s, Y^{n_2}(s), Z^{n_2}(s)) \rangle ds \]
\[ -2 \int_t^T \langle Y^{n_1}(s) - Y^{n_2}(s), d(K^{n_1}(s) - K^{n_2}(s)) \rangle \]
\[ -2 \int_t^T \langle Y^{n_1}(s) - Y^{n_2}(s), (Z^{n_1}(s) - Z^{n_2}(s)) dW(s) \rangle. \] (4.9)

Taking expectation and letting \( t=0 \), we have
\[ E|Y^{n_1}(0) - Y^{n_2}(0)|^2 + E \int_0^T |Z^{n_1}(s) - Z^{n_2}(s)|^2 ds \]
\[ \leq 2C(C + 1)E \int_0^T |Y^{n_1}(s) - Y^{n_2}(s)|^2 ds + \frac{1}{2} E \int_0^T |Z^{n_1}(s) - Z^{n_2}(s)|^2 ds \]
\[ + 2E \int_0^T \langle (Y^{n_2}(s) - S(s))^+, dK^{n_1}(s) \rangle + 2E \int_0^T \langle (Y^{n_1}(s) - S(s))^+, dK^{n_2}(s) \rangle. \]
As a consequence,

\[
E \int_0^T |Z^{n_1}(s) - Z^{n_2}(s)|^2 ds \leq 4C(C + 1)E \int_0^T |Y^{n_1}(s) - Y^{n_2}(s)|^2 ds \\
+ 4E \int_0^T \langle (Y^{n_2}(s) - S(s))^+, dK^{n_1}(s) \rangle \\
+ 4E \int_0^T \langle (Y^{n_1}(s) - S(s))^+, dK^{n_2}(s) \rangle.
\]

(4.10)

Following almost the same arguments as in the proof of [3, Lemma 6.1], we have

\[
E \sup_{0 \leq t \leq T} |(Y^n(t) - S(t))^+|^2 \to 0 \quad \text{as} \quad n \to \infty.
\]

(4.11)

Then from (4.8) and (4.11), we know that as \( n_1, n_2 \to \infty \),

\[
E \int_0^T \langle (Y^{n_2}(s) - S(s))^+, dK^{n_1}(s) \rangle + E \int_0^T \langle (Y^{n_1}(s) - S(s))^+, dK^{n_2}(s) \rangle \to 0.
\]

Together with (4.10), we obtain

\[
\lim_{n_1, n_2 \to \infty} E \int_0^T |Z^{n_1}(s) - Z^{n_2}(s)|^2 ds = 0.
\]

In view of (4.9), using the Burkholder-Davis-Gundy inequality, we conclude

\[
\lim_{n_1, n_2 \to \infty} E \sup_{0 \leq t \leq T} |Y^{n_1}(t) - Y^{n_2}(t)|^2 = 0.
\]

In view of (4.3) and the definition of \( K^n \), we know

\[
\lim_{n_1, n_2 \to \infty} E \sup_{0 \leq t \leq T} |K^{n_1}(t) - K^{n_2}(t)|^2 = 0.
\]

From the above convergence, we conclude that there exists \( (Z, K) \in (\mathcal{M}^2)^{m \times d} \times (\mathcal{N}^2)^m \), satisfying

\[
\lim_{n \to \infty} E \int_0^T |Z^n(s) - Z(s)|^2 ds = 0 \quad \text{and} \quad \lim_{n \to \infty} E \sup_{0 \leq t \leq T} |K^n(t) - K(t)|^2 = 0.
\]

Passing to limit in equation (4.3), we know that

\[
(Y, Z, K) \in (S^2)^m \times (\mathcal{M}^2)^{m \times d} \times (\mathcal{N}^2)^m
\]

satisfies the following equation:

\[
Y(t) = \xi + \int_t^T \phi(s, Y(s), Z(s)) ds - \int_t^T dK(s) - \int_t^T Z(s) dW(s).
\]

From (4.11), using Fatou’s lemma, we know that

\[
E \sup_{0 \leq t \leq T} |(Y(t) - S(t))^+|^2 \leq \lim_{n \to \infty} E \sup_{0 \leq t \leq T} |(Y^n(t) - S(t))^+|^2 = 0,
\]

which implies that

\[
Y(t) \leq S(t), \quad 0 \leq t \leq T.
\]

(4.12)
We conclude that \((Y,Z,K)\) is a solution to RBSDE (4.1).

Uniqueness of the solution follows from Lemma 2.3 and Remark 2.3.

\[
\text{Remark 4.1. For the existence and uniqueness for multi-dimensional RBSDEs, we refer the reader to Gegout-Petit and Pardoux [4].}
\]

\[
\text{Remark 4.2. The comparison theorem of multi-dimensional BSDEs is first established by Hu and Peng [9] under the stronger conditions on the generator \(\phi(\cdot,y,z)\) is continuous for any fixed \((y,z)\) and \(\phi(\cdot,0,0) \in (S^2)^m\). By the method of approximation, it can be shown that the comparison theorem still holds if Hypothesis 4 (i) is satisfied.}
\]

Thanks to the above existence and uniqueness result, we can prove the following comparison theorem for multi-dimensional RBSDEs with a fixed single reflecting barrier.

\[
\text{Theorem 4.2. Assume that } (\phi^1,\xi^1), (\phi^2,\xi^2), \text{ and } S \text{ satisfy Hypothesis 4. Further, assume that}
\]

\(\iota \xi^1 \geq \xi^2;\)

\(\iota \) There is a positive constant \(C'\) such that for \((y,y') \in (R^m)^2, (z,z') \in (R^{m \times d})^2,\) and \(t \in [0,T],\)

\[-4(y^-,\phi^1(t,y^+ + y', z) - \phi^2(t,y', z')) \leq 2 \sum_{i=1}^{m} \chi_{\{y_i<0\}} |z_i - z'_i|^2 + C'|y^-|^2. \quad (4.16)\]

\[
\text{For } j = 1,2, \text{ denote by } (Y^j,Z^j,K^j) \text{ the adapted solution of RBSDE (4.1) associate with the data } (\xi^j,\phi^j,S). \text{ Then, we have}
\]

\(1) \ Y^1(t) \geq Y^2(t), \quad K^1(t) \geq K^2(t), \quad 0 \leq t \leq T;\)

\(2) \ K^1(r) - K^1(s) \geq K^2(r) - K^2(s), \quad 0 \leq s \leq r \leq T.\)

\[
\text{Proof. For } j = 1,2 \text{ and positive integer } n, \text{ the following BSDE:}
\]

\[
Y^{j,n}(t) = \xi^j + \int_t^T \phi^j(s,Y^{j,n}(s),Z^{j,n}(s))ds - n \int_t^T (Y^{j,n}(s) - S^j(s))^+ ds
\]

\[
- \int_t^T Z^{j,n}(s)dW(s), \quad t \in [0,T]
\]

\[
(4.17)
\]
has a unique adapted solution, denoted by \((Y^{j,n}, Z^{j,n}, K^{j,n})\). In view of (4.16), we have
\[
-4(y^-, (\phi^1(t, y^+ + y', z) - n(y^+ + y' - S(t))^) - (\phi^2(t, y', z') - n(y' - S(t))^)) \\
\leq -4(y^-, \phi^1(t, y^+ + y', z) - \phi^2(t, y', z') - ny^+)
\leq 2\sum_{i=1}^m \chi_{\{y_i < 0\}}|z_i - z'_i|^2 + C'|y^-|^2.
\]
By the comparison theorem of multi-dimensional BSDEs (see [9, Theorem 2.1]), it follows that
\[
Y^{1,n}(t) \geq Y^{2,n}(t), \quad t \in [0, T], n = 1, 2, \ldots.
\] (4.18)
Then from the proof of Theorem 4.1, we know that for \(t \in [0, T]\) and \(j = 1, 2,\)
\[
Y^j(t) = \lim_{n \to \infty} Y^{j,n}(t), \\
K^j(t) = \lim_{n \to \infty} n \int_0^t (Y^{j,n}(s) - S(s))^+ ds.
\] (4.19)
The desired results then follow from (4.18) and (4.19).

4.2 Multi-dimensional RBSDEs with oblique reflection.
Consider the following RBSDE: for \(i \in \Lambda,\)
\[
\begin{cases}
Y_i(t) = \xi_i + \int_t^T \psi(s, Y_i(s), Z_i(s), i) ds - \int_t^T dK_i^- (s) \\
+ \int_t^T dK_i^+ (s) - \int_t^T Z_i(s) dW(s), \quad t \in [0, T]; \\
S_i(t) \geq Y_i(t) \geq \max_{j \neq i, j \in \Lambda} \{Y_j(t) - k(i, j)\}, \quad t \in [0, T]; \\
\int_0^T \left( Y_i(s) - \max_{j \neq i, j \in \Lambda} \{Y_j(s) - k(i, j)\} \right) dK_i^+ (s) = 0, \\
\int_0^T (Y_i(s) - S_i(s)) dK_i^- (s) = 0.
\end{cases}
\] (4.20)
The following theorem presents the existence of the solution without super-regularity assumption on \(S.\)

**Theorem 4.3.** Let Hypotheses 1 and 3’ be satisfied. Assume that \(S \in (S^2)^m\) with \(S(t) \in Q(t)\) for \(t \in [0, T]\) and \(\xi \in L^2(\Omega, \mathcal{F}_T, P; R^m)\) with \(\xi(\omega) \in Q(T)\). Here we have defined for \(t \in [0, T],\)
\[
\tilde{Q}(t) \triangleq \{(y_1, \ldots, y_m)^T \in R^m : y_j - k(i, j) \leq y_i \leq S_i(t), \forall i, j \in \Lambda, j \neq i\}.
\]
Then RBSDE (4.20) has an adapted solution \((Y, Z, K^+, K^-) \in (S^2)^m \times (\mathcal{M}^2)^{m \times d} \times (\mathcal{N}^2)^{2m}.\)

**Proof.** The proof is divided into four steps.

**Step 1.** The approximating sequence of penalized RBSDEs.
For any positive integer $n$, consider the following RBSDE: \( \forall i \in \Lambda, \)

\[
\begin{cases}
Y^n_i(t) = \xi_i + \int_t^T \psi(s,Y^n_i(s),Z^n_i(s),i) \, ds - \int_t^T dK^{-n}_i(s) \\
\quad + n \sum_{l=1}^m \int_t^T (Y^n_i(s) - Y^n_l(s) + k(i,l))^- \, ds \\
\quad - \int_t^T Z^n_i(s) \, dW(s), \quad t \in [0,T];
Y^n_i(t) \leq S_i(t), \quad t \in [0,T];
\int_0^T (Y^n_i(s) - S_i(s)) dK^{-n}_i(s) = 0.
\end{cases}
\]

They turn out to be RBSDEs well studied in the preceding subsection.

Define for \( (t,y,z,i) \in [0,T] \times R^m \times R^{m \times d} \times \Lambda, \)

\[
\begin{align*}
\bar{\psi}^n(t,y,z,i) &\triangleq \psi(t,y_i,z_i,i) + n \sum_{l=1}^m (y_i - y_l + k(i,l))^- \\
\bar{\psi}^n(t,y,z) &\triangleq (\bar{\psi}^n(t,y,z,1), \ldots, \bar{\psi}^n(t,y,z,m))^T.
\end{align*}
\]

Since

\[
\langle \bar{\psi}^n(t,y,z), y_i^+ + y_i^+ - y_i^+ - y_i^+ + k(i,l) \rangle - \langle \bar{\psi}^n(t,y,z) \rangle \geq 0
\]

for \( y,y' \in R^m \) and \( i,l \in \Lambda \), and \( \bar{\psi}^n(t,y,z,i) \) does not depend on \( z_j \) for \( j \neq i \), we have for \( (y,y') \in (R^m)^2 \), \( (z,z') \in (R^{m \times d})^2 \), and \( (i,l) \in (\Lambda)^2 \),

\[
-4 \langle \bar{\psi}^n(t,y^+ + y', z) - \bar{\psi}^n(t,y', z') \rangle
\]

\[
= -4 \sum_{i=1}^m \langle \bar{\psi}^n(t,y^+_i + y'_i, z_i, i) - \bar{\psi}^n(t,y'_i, z'_i, i) \rangle
\]

\[
-4n \sum_{i,l=1}^m \langle \bar{\psi}^n(t,y^+_i + y'_i - y_l^+ + k(i,l))^- - (y'_l - y'_l + k(i,l))^- \rangle
\]

\[
\leq -4 \sum_{i=1}^m \langle \bar{\psi}^n(t,y^+_i + y'_i, z_i, i) - \bar{\psi}^n(t,y'_i, z'_i, i) \rangle
\]

\[
\leq 2 \sum_{i=1}^m \chi(y_i < 0) |z_i - z'_i|^2 + 2C^2 |y|^2.
\]

It is easy to check that \( \bar{\psi}^n(t,y,z) \) is also Lipschitz continuous in \( (y,z) \). From Theorem 4.1, we know that RBSDE (4.21) has a unique adapted solution \( (Y^n, Z^n, K^{-n}) \) with

\[
Y^n \triangleq (Y^n_1, \ldots, Y^n_m)^T \in (S^2)^m, \quad Z^n \triangleq (Z^n_1, \ldots, Z^n_m)^T \in (M^2)^{m \times d},
\]

and

\[
K^{-n} \triangleq (K^{-n}_1, \ldots, K^{-n}_m)^T \in (N^2)^m.
\]
Moreover, we have from (4.23) that for \((y, y') \in (R^m)^2, (z, z') \in (R^{m \times d})^2\), and \((i, l) \in (\Lambda)^2\),

\[
-4(y^-, \bar{\psi}^n(t, y^+ + y', z) - \bar{\psi}^n(t, y', z'))
\]

\[
= -4(y^-, \bar{\psi}^n(t, y^+ + y', z) - \bar{\psi}^n(t, y', z')) - 4 \sum_{i, l=1}^{m} \langle y_i^-, (y_i^+ + y'_i - y_i^+ - y'_i + k(i, l))^\perp \rangle
\]

\[
\leq -4(y^-, \bar{\psi}^n(t, y^+ + y', z) - \bar{\psi}^n(t, y', z'))
\]

\[
\leq 2 \sum_{i=1}^{m} \chi_{\{y_i < 0\}} |z_i - z'_i|^2 + 2C^2|y^-|^2.
\]

From Theorem 4.2, we know

\[
Y^n(t) \leq Y^{n+1}(t), \quad K_-^{n}(t) \leq K_-^{n+1}(t), \quad 0 \leq t \leq T.
\]

The sequence \(\{(Y^n(t), K_-^n(t))\}_{n \geq 1}\) admits a limit, which is denoted by \((Y(t), K_-(t))\) below with \(Y(t) \triangleq (Y_1(t), \ldots, Y_m(t))^T\) and \(K_-(t) \triangleq (K_1^-(t), \ldots, K_m^-(t))^T\), for \(t \in [0, T]\).

**Step 2. A priori estimate.**

The following lemma is the key to our subsequent arguments.

**Lemma 4.1.** There is a positive constant \(c\) which is independent of \(n\), such that

\[
E \sup_{0 \leq t \leq T} |Y^n(t)|^2 + E|K_-^n(T)|^2 + E \int_0^T |Z^n(s)|^2 ds \leq c,
\]

\[
n^2E \int_0^T |(Y^n_i(s) - Y^n_j(s) + k(i, j))^\perp|^2 ds \leq c, \quad i \in \Lambda.
\]

Its proof follows. Applying Itô-Meyer’s formula [12] to compute \(|(Y^n_i(t) - Y^n_j(t) + k(i, j))^\perp|^2\), we have

\[
|Y^n_i(t) - Y^n_j(t) + k(i, j))^\perp|^2 + \int_t^T \chi_{L_i, j, n}(s) |Z^n_i(s) - Z^n_j(s)|^2 ds
\]

\[
+2n \int_t^T |(Y^n_i(s) - Y^n_j(s) + k(i, j))^\perp|^2 ds
\]

\[
= 2 \int_t^T (Y^n_i(s) - Y^n_j(s) + k(i, j))^\perp [\psi(s, Y^n_j(s), Z^n_j(s), j) - \psi(s, Y^n_i(s), Z^n_i(s), i)] ds
\]

\[
+2 \int_t^T (Y^n_i(s) - Y^n_j(s) + k(i, j))^\perp (Z^n_i(s) - Z^n_j(s)) dW(s)
\]

\[
+2 \int_t^T (Y^n_i(s) - Y^n_j(s) + k(i, j))^\perp d(K_i^{-n}(s) - K_j^{-n}(s))
\]

\[
+2n \int_t^T (Y^n_i(s) - Y^n_j(s) + k(i, j))^\perp (Y^n_j(s) - Y^n_i(s) + k(j, i))^\perp ds
\]

\[
+2n \sum_{l \neq i, l \neq j} \int_t^T (Y^n_i(s) - Y^n_j(s) + k(i, j))^\perp [(Y^n_j(s) - Y^n_i(s) + k(j, l))^\perp
\]

\[-(Y^n_i(s) - Y^n_j(s) + k(i, l))^\perp] ds
\]

where

\[
L_i^{-}, j, n \triangleq \{(t, \omega) : Y^n_i(t) - Y^n_j(t) + k(i, j) < 0\}.
\]
We claim that the last three term of (4.25) are all equal to or less than 0. In fact, due to (4.21), we have
\[ Y_j^n(s) - k(i, j) \leq S_j(s) - k(i, j) \leq S_i(s) \quad \text{(noting } S(s) \in \tilde{Q}(s)) \tag{4.26} \]
and
\[
\int_t^T (Y_i^n(s) - Y_j^n(s) + k(i, j))^- d(K_i^n(s) - K_j^n(s)) \\
\leq \int_t^T (Y_i^n(s) - Y_j^n(s) + k(i, j))^- dK_i^n(s) \\
\leq \int_t^T (Y_i^n(s) - S_i(s))^- dK_i^n(s) \\
= \int_0^T (S_i(s) - Y_i^n(s)) dK_i^n(s) = 0.
\]
In view of Hypothesis 3'(i), we have
\[ \{(y_1, \cdots, y_m)^T \in R^m : y_i - y_j + k(i, j) < 0, y_j - y_i + k(j, i) < 0\} = \emptyset \]
which immediately gives
\[ (Y_i^n(t) - Y_j^n(t) + k(i, j))^- (Y_j^n(t) - Y_i^n(t) + k(j, i))^- = 0. \]
From Hypothesis 3'(ii), using the property that \( x_1^- - x_2^- \leq (x_1 - x_2)^- \), we have
\[
(Y_i^n(s) - Y_j^n(s) + k(i, j))^- [(Y_j^n(s) - Y_i^n(s) + k(j, l))^- \\
- (Y_i^n(s) - Y_l^n(s) + k(i, l))^-] \\
\leq (Y_i^n(s) - Y_j^n(s) + k(i, j))^- (Y_j^n(s) - Y_i^n(s) + k(j, l) - k(i, l))^- \\
\leq (Y_i^n(s) - Y_j^n(s) + k(i, j))^- (Y_j^n(s) - Y_i^n(s) - k(i, j))^- \\
= (Y_i^n(s) - Y_j^n(s) + k(i, j))^- (Y_i^n(s) - Y_j^n(s) + k(i, j))^+ = 0.
\]
Taking expectation on both sides of (4.25), we have
\[
E[(Y_i^n(t) - Y_j^n(t) + k(i, j))^-|^2 + E\int_t^T \chi_{L_{i,j,n}}(s)|Z_i^n(s) - Z_j^n(s)|^2 ds \\
+ 2nE\int_t^T |(Y_i^n(s) - Y_j^n(s) + k(i, j))^-|^2 ds \\
\leq 2E\int_t^T (|Y_i^n(s) - Y_j^n(s) + k(i, j))^-|\psi(s, Y_i^n(s), Z_i^n(s), i) \\
- \psi(s, Y_j^n(s), Z_j^n(s), j)| ds. \tag{4.27}
\]
Noting that
\[
|\psi(s, Y_i^n(s), Z_i^n(s), i) - \psi(s, Y_j^n(s), Z_j^n(s), j)| \\
\leq |\psi(s, Y_i^n(s), Z_i^n(s), i) - \psi(s, Y_j^n(s), Z_j^n(s), j)| \\
+ |\psi(s, Y_i^n(s), Z_i^n(s), j) - \psi(s, Y_j^n(s), Z_j^n(s), j)| \\
\leq c(1 + |\psi(s, 0, 0)| + |Y_i^n(s)| + |Z_i^n(s)| + |Y_j^n(s) - Y_i^n(s)| + |Z_j^n(s) - Z_i^n(s)|) \\
\leq c\left(1 + |\psi(s, 0, 0)| + |Y_i^n(s)| + |Z_i^n(s)| + |Y_j^n(s) - Y_i^n(s) + k(i, j)| \\
+ |Z_i^n(s) - Z_j^n(s)|\right)
\]
for a positive constant $c$ (independent of $n$ and possibly varying from line to line), in view of (4.27), we have

$$E[(Y^n_i(t) - Y^n_j(t) + k(i, j))^2] + E \int_t^T \chi_{L_{i,j,n}}(s) |Z^n_i(s) - Z^n_j(s)|^2 ds$$

$$+ 2nE \int_t^T |(Y^n_i(s) - Y^n_j(s) + k(i, j))^2| ds$$

$$\leq \left( \frac{n}{2} + 2c \right) E \int_t^T |(Y^n_i(s) - Y^n_j(s) + k(i, j))^2| ds + \frac{2c^2}{n} E \int_t^T \chi_{L_{i,j,n}}(s)$$

$$\left( 1 + |\psi(s, 0, 0)|^2 + |Y^n_i(s)|^2 + |Z^n_i(s)|^2 + |Z^n_j(s) - Z^n_j(s)|^2 \right) ds.$$ 

So for sufficiently large $n$,

$$n^2 E \int_0^T |(Y^n_i(s) - Y^n_j(s) + k(i, j))^2| ds \leq c \left( 1 + E \int_t^T (|Y^n_i(s)|^2 + |Z^n_i(s)|^2) ds \right).$$

(4.28)

Applying Itô’s lemma to compute $|Y^n_i(t)|^2$, we have

$$|Y^n_i(t)|^2 + \int_t^T |Z^n_i(s)|^2 ds$$

$$= 2 \int_t^T Y^n_i(s) \left( \psi(s, Y^n_i(s), Z^n_i(s), i) + n \sum_{l=1}^m (Y^n_i(s) - Y^n_j(s) + k(i, l)) \right) ds$$

$$+ \xi_i^2 - 2 \int_t^T Y^n_i(s) dK_i^{-n}(s) - 2 \int_t^T Y^n_i(s) Z^n_i(s) dW(s).$$

(4.29)

Using the elementary inequality:

$$2ab \leq \frac{1}{\alpha} a^2 + ab^2 \quad \text{for} \; a, b, \alpha > 0,$$

we obtain that

$$-2 \int_t^T Y^n_i(s) dK_i^{-n}(s) = -2 \int_t^T S_i(s) dK_i^{-n}(s)$$

$$\leq \frac{1}{\alpha} \sup_{0 \leq t \leq T} |S_i^{-}(t)|^2 + \alpha |K_i^{-n}(T) - K_i^{-n}(t)|^2$$

(4.30)

for an arbitrary positive real number $\alpha$. Then taking expectation on both sides of (4.29), we have

$$E|Y^n_i(t)|^2 + E \int_t^T |Z^n_i(s)|^2 ds$$

$$\leq 2E \int_t^T |Y^n_i(s)|^2 |(\psi(s, Y^n_i(s), Z^n_i(s), i) + n \sum_{l=1}^m (Y^n_i(s) - Y^n_j(s) + k(i, l)) ds$$

$$+ E(\xi_i^2) - 2E \int_t^T Y^n_i(s) dK_i^{-n}(s)$$

$$\leq c_\varepsilon E \int_t^T |Y^n_i(s)|^2 ds + \varepsilon \sum_{l=1}^m n^2 E \int_t^T |(Y^n_i(s) - Y^n_j(s) + k(i, l))^{-2} ds + E(\xi_i)^2$$

$$+ \varepsilon E \int_t^T |Z^n_i(s)|^2 ds + \frac{1}{\alpha} E \sup_{0 \leq t \leq T} |S_i^{-}(t)|^2 + \alpha E|K_i^{-n}(T) - K_i^{-n}(t)|^2$$

(4.31)
for arbitrary positive real numbers $\varepsilon, \alpha$, and a constant $c_{\varepsilon}$ depending on $\varepsilon$. On the other hand, from equation (4.21), we have

$$K^{-n}_i(T) - K^{-n}_i(t) = \xi_i - Y^n_i(t) + \int_t^T \psi(s, Y^n_i(s), Z^n_i(s), i) ds + \sum_{l=1}^m \int_t^T (Y^n_i(s) - Y^n_i(s) + k(i, l))^{-} ds - \int_t^T Z^n_i(s) dW(s).$$

In view of (4.28), we have

$$E|K^{-n}_i(T) - K^{-n}_i(t)|^2 \leq c_{\varepsilon, \alpha}(1 + E \int_t^T |Y^n_i(s)|^2 ds) \leq c.$$

Further, in view of (4.31), we have

$$E|Y^n_i(t)|^2 + \int_t^T |Z^n_i(s)|^2 ds \leq c.$$

The proof of Lemma 4.1 is then complete.

**Step 3. The convergence of penalized BSDEs.**

In view of Lemma 4.1, using Fatou’s lemma, we have

$$E\sup_{0 \leq t \leq T} (|Y(t)|^2 + |K^{-}(t)|^2) \leq \infty.$$
Then applying Lebesgue’s dominated convergence theorem, we have

$$E \int_0^T |Y^n(s) - Y(s)|^2 \, ds + E \int_0^T |K^{-n}(s) - K^{-}(s)|^2 \, ds \to 0 \quad \text{as} \quad n \to \infty. \quad (4.33)$$

For positive integers $n_1$ and $n_2$, applying Itô’s lemma to $|Y_i^{n_1}(t) - Y_i^{n_2}(t)|^2$, we have

$$|Y_i^{n_1}(t) - Y_i^{n_2}(t)|^2 + \int_t^T |Z_i^{n_1}(s) - Z_i^{n_2}(s)|^2 \, ds$$

$$= 2 \int_t^T (\psi(s, Y_i^{n_1}(s), Z_i^{n_1}(s), i) - \psi(s, Y_i^{n_2}(s), Z_i^{n_2}(s), i))(Y_i^{n_1}(s) - Y_i^{n_2}(s)) \, ds$$

$$+ 2n_1 \sum_{l=1}^m \int_t^T (Y_i^{n_1}(s) - Y_i^{n_1}(s) + k(i, l))^{-1}(Y_i^{n_1}(s) - Y_i^{n_2}(s)) \, ds$$

$$- 2n_2 \sum_{l=1}^m \int_t^T (Y_i^{n_2}(s) - Y_i^{n_2}(s) + k(i, l))^{-1}(Y_i^{n_1}(s) - Y_i^{n_2}(s)) \, ds$$

$$= 2 \int_t^T (Y_i^{n_1}(s) - Y_i^{n_2}(s)) d(K_i^{-n_1}(s) - K_i^{-n_2}(s))$$

$$- 2 \int_t^T (Y_i^{n_1}(s) - Y_i^{n_2}(s))(Z_i^{n_1}(s) - Z_i^{n_2}(s)) \, dW(s), \quad \forall i \in \Lambda. \quad (4.34)$$

Since

$$\int_t^T (Y_i^{n_1}(s) - Y_i^{n_2}(s)) d(K_i^{-n_1}(s) - K_i^{-n_2}(s))$$

$$= \int_t^T (S_i(s) - Y_i^{n_2}(s)) dK_i^{-n_1}(s) + \int_t^T (S_i(s) - Y_i^{n_1}(s)) dK_i^{-n_2}(s) \geq 0, \quad i \in \Lambda, \quad (4.35)$$

in view of (4.34), we have for $i \in \Lambda$,

$$E|Y_i^{n_1}(t) - Y_i^{n_2}(t)|^2 + E \int_t^T |Z_i^{n_1}(s) - Z_i^{n_2}(s)|^2 \, ds$$

$$= 2E \int_t^T ((\psi(s, Y_i^{n_1}(s), Z_i^{n_1}(s), i) - \psi(s, Y_i^{n_2}(s), Z_i^{n_2}(s), i))(Y_i^{n_1}(s) - Y_i^{n_2}(s)) \, ds$$

$$+ 2n_1 \sum_{l=1}^m \int_t^T (Y_i^{n_1}(s) - Y_i^{n_1}(s) + k(i, l))^{-1}(Y_i^{n_1}(s) - Y_i^{n_2}(s)) \, ds$$

$$- 2n_2 \sum_{l=1}^m \int_t^T (Y_i^{n_2}(s) - Y_i^{n_2}(s) + k(i, l))^{-1}(Y_i^{n_1}(s) - Y_i^{n_2}(s)) \, ds$$

$$\leq 2C(C + 1)E \int_t^T |Y_i^{n_1}(s) - Y_i^{n_2}(s)|^2 \, ds + \frac{1}{2}E \int_t^T |Z_i^{n_1}(s) - Z_i^{n_2}(s)|^2 \, ds$$

$$+ 2(E \int_t^T |Y_i^{n_1}(s) - Y_i^{n_2}(s)|^2 \, ds)^{\frac{1}{2}} \sum_{l=1}^m (n_1^2E \int_t^T ((Y_i^{n_1}(s) - Y_i^{n_1}(s) + k(i, l))^{-1})^2 \, ds)^{\frac{1}{2}}$$

$$+ 2(E \int_t^T |Y_i^{n_1}(s) - Y_i^{n_2}(s)|^2 \, ds)^{\frac{1}{2}} \sum_{l=1}^m (n_2^2E \int_t^T ((Y_i^{n_2}(s) - Y_i^{n_2}(s) + k(i, l))^{-1})^2 \, ds)^{\frac{1}{2}}.$$

Setting $t = 0$, in view of (4.33) and Lemma 4.1, we have

$$E \int_0^T |Z_i^{n_1}(s) - Z_i^{n_2}(s)|^2 \, ds \to 0, \quad \forall i \in \Lambda, \quad \text{as} \quad n_1, n_2 \to \infty. \quad (4.36)$$
So there exists \( Z \triangleq (Z_1, \cdots, Z_m)^T \in (\mathcal{M}^2)^{m \times d} \) such that

\[
\lim_{n \to \infty} E \int_0^T |Z_i^n(s) - Z_i(s)|^2 ds = 0, \quad \forall i \in \Lambda.
\]

In view of (4.34), applying the Burkholder-Davis-Gundy inequality, we have

\[
E(\sup_{0 \leq t \leq T} |Y_i^{n_1}(t) - Y_i^{n_2}(t)|^2) \to 0, \quad \forall i \in \Lambda \quad \text{as} \quad n_1, n_2 \to \infty. \quad (4.37)
\]

From (4.21), we have

\[
d(K_i^{-,n_1}(t) - K_i^{-,n_2}(t)) = \int_0^t \left( \psi(t, Y_i^{n_1}(t), Z_i^{n_1}(t), i) - \psi(t, Y_i^{n_2}(t), Z_i^{n_2}(t), i) \right) dt
\]

\[
+ \int_0^t (Y_i^{n_1}(s) - Y_i^{n_2}(s)) \left( n_1 \sum_{l=1}^m (Y_i^{n_1}(s) - Y_i^{n_1}(s) + k(i, l))^- ds
\]

\[
- n_2 \sum_{l=1}^m (Y_i^{n_2}(s) - Y_i^{n_2}(s) + k(i, l))^- ds - (Z_i^{n_1}(s) - Z_i^{n_2}(s)) dW(t).
\]

Since the process \( \{(K_i^{-,n_1}(t) - K_i^{-,n_2}(t)), t \in [0, T]\} \) is of finite variation, its quadratic variation is 0. By Itô’s lemma it follows that

\[
(K_i^{-,n_1}(t) - K_i^{-,n_2}(t))^2
\]

\[
= 2 \int_0^t (K_i^{-,n_1}(s) - K_i^{-,n_2}(s))(\psi(s, Y_i^{n_1}(s), Z_i^{n_1}(s), i) - \psi(s, Y_i^{n_2}(s), Z_i^{n_2}(s), i)) ds
\]

\[
+ 2 \int_0^t (K_i^{-,n_1}(s) - K_i^{-,n_2}(s))(n_1 \sum_{l=1}^m (Y_i^{n_1}(s) - Y_i^{n_1}(s) + k(i, l))^- ds
\]

\[
- 2 \int_0^t (K_i^{-,n_1}(s) - K_i^{-,n_2}(s))(n_2 \sum_{l=1}^m (Y_i^{n_2}(s) - Y_i^{n_2}(s) + k(i, l))^- ds
\]

\[
+ 2 \int_0^t (K_i^{-,n_1}(s) - K_i^{-,n_2}(s))dY_i^{n_1}(s) - Y_i^{n_2}(s))
\]

\[
- 2 \int_0^t (K_i^{-,n_1}(s) - K_i^{-,n_2}(s))(Z_i^{-,n_1}(s) - Z_i^{-,n_2}(s))dW(s).
\]

Applying the Burkholder-Davis-Gundy inequality, we have

\[
E(\sup_{0 \leq t \leq T} |K_i^{-,n_1}(t) - K_i^{-,n_2}(t)|^2)
\]

\[
\leq c(E \int_0^T (|K_i^{-,n_1}(s) - K_i^{-,n_2}(s)|^2 + |Y_i^{n_1}(s) - Y_i^{n_2}(s)|^2 + |Z_i^{n_1}(s) - Z_i^{n_2}(s)|^2) ds
\]

\[
+ 2(E \int_0^T |K_i^{-,n_1}(s) - K_i^{-,n_2}(s)|^2 ds)^{1/2} \sum_{l=1}^m (n_1^2 E \int_0^T ((Y_i^{n_1}(s) - Y_i^{n_1}(s) + k(i, l))^- ds)^{1/2}
\]

\[
+ 2(E \int_0^T |K_i^{-,n_1}(s) - K_i^{-,n_2}(s)|^2 ds)^{1/2} \sum_{l=1}^m (n_2^2 E \int_0^T ((Y_i^{n_2}(s) - Y_i^{n_2}(s) + k(i, l))^- ds)^{1/2}
\]

\[
+ E(\sup_{0 \leq t \leq T} \{ 2 \int_0^t (K_i^{-,n_1}(s) - K_i^{-,n_2}(s))dY_i^{n_1}(s) - Y_i^{n_2}(s))
\]

\[
+ \frac{1}{5} E(\sup_{0 \leq t \leq T} |K_i^{-,n_1}(s) - K_i^{-,n_2}(s)|^2
\]

\[
(4.38)
\]
Hence by Itô’s lemma,
we have
\[ \int_0^t (Y^{n_1}_i(s) - Y^{n_2}_i(s)) \, d(K^{-n_1}_i(s) - K^{-n_2}_i(s)) \geq 0. \]

Then as a consequence,
\[ E \sup_{0 \leq t \leq T} \left\{ 2 \int_0^t (K^{-n_1}_i(s) - K^{-n_2}_i(s)) \, d(Y^{n_1}_i(s) - Y^{n_2}_i(s)) \right\} \leq 2 \mathbb{E} \left( \sup_{0 \leq t \leq T} |K^{-n_1}_i(t) - K^{-n_2}_i(t)||Y^{n_1}_i(t) - Y^{n_2}_i(t)| \right) \]
\[ \leq \frac{1}{3} E \sup_{0 \leq t \leq T} |K^{-n_1}_i(t) - K^{-n_2}_i(t)|^2 + 3E \sup_{0 \leq t \leq T} |Y^{n_1}_i(t) - Y^{n_2}_i(t)|^2. \]

Together with (4.33), (4.38), and Lemma 4.1, we have
\[ E( \sup_{0 \leq t \leq T} |K^{-n_1}_i(t) - K^{-n_2}_i(t)|^2) \leq cE \int_0^T (|Y^{n_1}_i(s) - Y^{n_2}_i(s)|^2 + |Z^{n_1}_i(s) - Z^{n_2}_i(s)|^2) \, ds \]
\[ + cE \sup_{0 \leq t \leq T} |Y^{n_1}_i(t) - Y^{n_2}_i(t)|^2. \]

From (4.36) and (4.37), we have
\[ E \sup_{0 \leq t \leq T} |K^{-n_1}_i(t) - K^{-n_2}_i(t)|^2 \rightarrow 0, \quad \forall i \in \Lambda, \quad \text{as} \quad n_1, n_2 \rightarrow \infty. \]

Set
\[ K_i^{+,n}(t) \triangleq n \sum_{l=1}^m \int_0^t (Y^{n}_i(s) - Y^{n}_l(s) + k(i, l) - s) \, ds \]
and
\[ K_i^{+}(t) \triangleq Y_i(0) - Y_i(t) - \int_0^t \psi(s, Y_i(s), Z_i(s), i) \, ds + \int_0^t dK_i^{-}(s) + \int_0^t Z_i(s) \, dW(s). \]

We have
\[ K_i^{+,n}(t) = Y_i^{n}(0) - Y_i^{n}(t) - \int_0^t \psi(s, Y_i^{n}(s), Z_i^{n}(s), i) \, ds + \int_0^t dK_i^{-}(s) + \int_0^t Z_i^{n}(s) \, dW(s) \]
and
\[ \lim_{n \rightarrow \infty} E( \sup_{0 \leq t \leq T} |K_i^{+,n}(t) - K_i^{+}(t)|^2) \leq \lim_{n \rightarrow \infty} cE \left( \sup_{0 \leq t \leq T} \left\{ |Y_i^{n}(t) - Y_i(t)|^2 + |K_i^{-n}(t) - K_i^{-}(t)|^2 \right\} \right) \]
\[ + \int_0^T |Z_i^{n}(s) - Z_i(s)|^2 \, ds \]
\[ = 0. \]
Hence, \( K \triangleq (K^+_1, \cdots, K^+_n) \in (\mathcal{N}^2)^m \), and \((Y, Z, K^+, K^-)\) satisfies the first equation of (4.20). By Lemma 4.1 and Fatou's lemma, we obtain that

\[
E \int_0^T |(Y_i(s) - Y_i(s) + k(i, l))^-|^2 ds \\
\leq \lim_{n \to \infty} E \int_0^T |(Y_i^n(s) - Y_i^n(s) + k(i, l))^-|^2 ds \\
\leq \lim_{n \to \infty} \frac{c}{n^2} = 0,
\]

which implies immediately that

\[
Y_i(t) - Y_i(t) + k(i, l) \geq 0, \quad 0 \leq t \leq T. \tag{4.40}
\]

Since \( Y^n(t) \leq S(t) \) for any \( n \) and \( t \in [0, T] \), we have

\[
Y(t) \leq S(t), \quad 0 \leq t \leq T.
\]

Hence,

\[
Y(t) \in \tilde{Q}(t), \quad 0 \leq t \leq T.
\]

**Step 4. The lower minimal boundary condition.**

For \( i, j, l \in \Lambda, i \neq j, l = i \), it is obvious that

\[
(Y_i^n(s) - Y_j^n(s) + k(i, j))^+ (Y_i^n(s) - Y_i^n(s) + k(i, l))^- = 0.
\]

For \( i, j, l \in \Lambda, i \neq j, l \neq i \), we have

\[
\min_{j \neq i} (Y_i^n(s) - Y_j^n(s) + k(i, j))^+ (Y_i^n(s) - Y_i^n(s) + k(i, l))^- \\
\leq (Y_i^n(s) - Y_i^n(s) + k(i, l))^+ (Y_i^n(s) - Y_i^n(s) + k(i, l))^- \\
= 0.
\]

Hence,

\[
\int_0^T (Y_i^n(s) - \max_{j \neq i} \{Y_j^n(s) - k(i, j)\})^+ dK_i^{+, n}(s) \\
= n \sum_{l=1}^m \int_0^T \min_{j \neq i} \{(Y_i^n(s) - Y_j^n(s) + k(i, j))^+ (Y_i^n(s) - Y_i^n(s) + k(i, j))^- \} ds \\
\leq 0.
\]

On the other hand, since \( K_i^{+, n}(\cdot) \) is increasing, we have

\[
\int_0^T (Y_i^n(s) - \max_{j \neq i} \{Y_j^n(s) - k(i, j)\})^+ dK_i^{+, n}(s) \geq 0.
\]

Therefore,

\[
\int_0^T (Y_i^n(s) - \max_{j \neq i} \{Y_j^n(s) - k(i, j)\})^+ dK_i^{+, n}(s) = 0.
\]

Following the same arguments as in the proof of (4.14) and applying [4, Lemma 5.8], we have

\[
\int_0^T (Y_i(s) - \max_{j \neq i} \{Y_j(s) - k(i, j)\})^+ dK_i^+(s) = 0.
\]

In view of (4.40), we have

\[
\int_0^T (Y_i(s) - \max_{j \neq i} \{Y_j(s) - k(i, j)\}) dK_i^+(s) = 0.
\]
5 Uniqueness

The uniqueness of solution is defined in the following sense: if \((Y', Z', K^+_t, K^-_t)\) is another solution, then \(Y'(t) \equiv Y(t), Z'(t) \equiv Z(t), K^+_t(t) - K^-_t(t) \equiv K^+(t) - K^-(t), \forall 0 \leq t \leq T, a.s.\)

The following Stronger assumption on \(h_{i,j}\) is needed in our proof of the uniqueness result.

**Hypothesis 5.** \(\forall i, j, l \in \Lambda\) such that \(i \neq j, j \neq l, \forall y \in \mathbb{R},\)

\[
h_{i,j}(t, h_{j,l}(t, y)) < h_{i,l}(t, y).
\]

**Remark 5.1.** It is easy to check that Hypothesis 5 implies Hypothesis 3. If \(h_{i,j}(t, y) \triangleq y - k(i, j),\) then Hypothesis 5 reduces to the inequality: \(k(i, j) + k(j, l) > k(i, l)\) for \(i \neq j\) and \(j \neq l.\)

Let \(\{\theta_j\}_{j=0}^{\infty}\) be an increasing sequence of stopping times with values in \([0, T].\) \(\forall j, \alpha_j\) is an \(\mathcal{F}_{\theta_j}\)-measurable random variable with values in \(\Lambda.\) Assume that \(a.s. \omega,\) there exists an integer \(N(\omega) < \infty\) such that \(\theta_N = T.\) Then we define a switching strategy as:

\[
a(s) = \alpha_0 \chi_{[\theta_0, \theta_1]}(s) + \sum_{j=1}^{N-1} \alpha_j \chi_{[\theta_j, \theta_{j+1}]}(s).
\]

We denote by \(\mathcal{A}_t^i\) all the switching strategies with initial data \((\alpha_0, \theta_0) = (i, t) \in \Lambda \times [0, T].\) For given \(a \in \mathcal{A}_t^i,\) consider the following RBSDE:

\[
\begin{cases}
U^a(s) &= \xi_{a(T)} + \int_s^T \psi(r, U^a(r), V^a(r), a(r))dr - (L^a(T) - L^a(s)) \\
&- \sum_{j=1}^{N-1} (U^a(\theta_j) - h_{\alpha_{j-1}, \alpha_j}(\theta_j, U^a(\theta_j)) \chi_{[\theta_j, \theta_{j+1}]}(s)) \\
&- \int_s^T V^a(r)dW(r), \quad s \in [t, T]; \\
&\forall s \in [t, T]; \\
U^a(s) &\leq S_{a(s)}(s), \quad s \in [t, T]; \\
&\int_t^T (U^a(s) - S_{a(s)}(s))dL^a(s) = 0.
\end{cases}
\]

(5.1)

The generator \(\psi\) of RBSDE (5.1) depends on the control \(a\) and at each switching time \(\theta_j\) before termination, the value of \(U^a\) will jump by an amount of \(U^a(\theta_j) - h_{\alpha_{j-1}, \alpha_j}(\theta_j, U^a(\theta_j))\) which can be regarded as a penalty or cost for the switching. In each subinterval divided by the switching times, RBSDE (5.1) evolves as a standard RBSDE with single barrier, which can be solved in a backwardly inductive way.

The optimal control problem for RBSDE (5.1) is to maximize \(U^a(t)\) over \(a \in \mathcal{A}_t^i.\) The solution of RBSDE (1.4) is closely connected with this control problem. Besides, it is also connected to the stochastic game constructed below.

Let \(\tau\) be a stopping time with values in \([t, T].\) For given \(a \in \mathcal{A}_t^i\) and stopping
time \( \tau \), consider the following BSDEs:

\[
U^{a,\tau}(s) = S_{a(\tau)}(\tau)\chi_{\{\tau < T\}} + \xi_{a(\tau)} \chi_{\{\tau = T\}} + \int_s^\tau \psi(r, U^{a,\tau}(r), V^{a,\tau}(r), a(r))dr
- \sum_{j=1}^{N-1} \int_s^\tau [U^{a,\tau}(\theta_j) - h_{\alpha_{j-1},\alpha_j}(\theta_j, U^{a,\tau}(\theta_j))] \chi(s,\tau)\theta_j(\theta_j)
- \int_s^\tau V^{a,\tau}(r)dW(r), \quad s \in [t, \tau].
\]

The terminal value \( S_{a(\tau)}(\tau)\chi_{\{\tau < T\}} + \xi_{a(\tau)} \chi_{\{\tau = T\}} \) of BSDE (5.2) depends on the terminal time \( \tau \). And the value \( U^{a,\tau}(t) \) depends on both the switching strategy \( a \) and terminal time \( \tau \). Using the same arguments as in RBSDE (5.1), we know that BSDE (5.2) is well defined and has a unique solution. Based on that, we construct a zero-sum stochastic game as follows. Suppose there are two players A and B whose benefits are antagonistic. The payoff \( U \) for player A and a cost for player B. Player A chooses a switching strategy \( a \) so as to maximize the reward \( U^{a,\tau}(t) \). Player B chooses the time \( \tau \) to terminate the game and tries to minimize the cost \( U^{a,\tau}(t) \).

The following theorem reveals the connections among the solution of the system (1.4), the above control problem and stochastic game.

**Theorem 5.1.** Let Hypotheses 1 and 5 be satisfied. Assume that \((U^a, V^a, L^a)\) is the unique adapted solution of RBSDE (5.1) for \( a \in A_i \) and \((U^{a,\tau}, V^{a,\tau})\) is the unique solution of BSDE (5.2) for \( a \in A_i \) and stopping time \( \tau \).

Then if \((Y, Z, K^+, K^-)\) is an adapted solution of RBSDE (1.4), we have

\[
Y_i(t) = \text{esssup}_{a \in A_i} U^a(t) = \text{esssup}_{a \in A_i} \text{essinf}_{\tau} U^{a,\tau}(t) = \text{essinf}_{a \in A_i} \text{esssup}_{r} U^{a,\tau}(t), \quad t \in [0, T].
\]

Let \( \hat{\theta}_0 = t \) and \( \hat{\alpha}_0 = i \). Define the sequence \( \{\hat{\theta}_j, \hat{\alpha}_j\}_{j=1}^\infty \) in an inductive way as follows:

\[
\hat{\theta}_j = \inf\{s \geq \hat{\theta}_{j-1} : Y_{\hat{\alpha}_{j-1}}(s) = \max_{k \neq \hat{\alpha}_{j-1}, k \in \Lambda} h_{\hat{\alpha}_{j-1},k}(s, Y_{\hat{\alpha}_{j}}(s))\} \land T.
\]

And if \( \hat{\theta}_j < T \), set \( \hat{\alpha}_j \) be the smallest index in \( \Lambda \) such that

\[
Y_{\hat{\alpha}_{j-1}}(\hat{\theta}_j) = h_{\hat{\alpha}_{j-1},\hat{\alpha}_j}(\hat{\theta}_j, Y_{\hat{\alpha}_j}(\hat{\theta}_j)).
\]

Otherwise, set \( \hat{\alpha}_j \) be an arbitrary index. Define

\[
\hat{a}(s) \triangleq \hat{\alpha}_0 \chi_{[\hat{\theta}_0, \hat{\alpha}_1]}(s) + \sum_{j=1}^{N-1} \hat{\alpha}_j \chi_{[\hat{\theta}_j, \hat{\alpha}_{j+1}]}(s)
\]

and

\[
\tau^* \triangleq \inf\{s \in [t, T) : U^\hat{a}(s) = S_{\hat{a}(s)}(s)\} \land T,
\]

with the convention that \( \inf \emptyset \triangleq +\infty \). Then, we have \( \hat{a} \in A_i \) and

\[
Y_i(t) = U^{\hat{a}}(t) \quad \text{and} \quad Y_i(t) = U^{\hat{a},\tau^*}(t), \quad t \in [0, T].
\]
Proof. For \( t \in [0,T] \), \( a \in \mathcal{A}_t^i \), and \( s \in [t, \theta_N] \), define

\[
Y^a(s) \triangleq \sum_{j=1}^{N} Y_{\alpha_{j-1}}(s) \chi_{[\theta_{j-1}, \theta_j)}(s) + \xi_{\alpha_{N-1}} \chi_{\{s=T\}},
\]

\[
Z^a(s) \triangleq \sum_{j=1}^{N} Z_{\alpha_{j-1}}(s) \chi_{[\theta_{j-1}, \theta_j)}(s),
\]

\[
K^{+,a}(s) \triangleq \sum_{j=1}^{N} (K^{+,a}_{\alpha_{j-1}}(\theta_j \land s) - K^{+,a}_{\alpha_{j-1}}(\theta_{j-1} \land s)),
\]

\[
K^{-,a}(s) \triangleq \sum_{j=1}^{N} (K^{-,a}_{\alpha_{j-1}}(\theta_j \land s) - K^{-,a}_{\alpha_{j-1}}(\theta_{j-1} \land s)).
\]

(5.5)

In view of the jump \( Y_{\alpha_j}(\theta_j) - Y_{\alpha_{j-1}}(\theta_j) = Y^a(\theta_j) - Y^a(\theta_j-) \) at each stopping time \( \theta_j, j = 1, \ldots, N - 1 \), we know that \((Y^a, Z^a, K^{+,a}, K^{-,a})\) satisfies the following RBSDE:

\[
\begin{aligned}
Y^a(s) &= \xi_{a(T)} + \int_s^T \psi(r, Y^a(r), Z^a(r), a(r))dr \\
&\quad - \sum_{j=1}^{N-1} (Y^a(\theta_j) - Y^a(\theta_j-)) \chi_{(s, T]}(\theta_j) + K^{+,a}(T) - K^{+,a}(s) \\
&\quad - (K^{-,a}(T) - K^{-,a}(s)) - \int_s^T Z^a(r)dW(r),
\end{aligned}
\]

(5.6)

And for any stopping time \( \tau \), \((Y^a, Z^a, K^{+,a}, K^{-,a})\) also satisfies the following BSDE:

\[
\begin{aligned}
Y^a(s) &= Y^a(\tau) + \int_s^\tau \psi(r, Y^a(r), Z^a(r), a(r))dr \\
&\quad - \sum_{j=1}^{N-1} (Y^a(\theta_j) - Y^a(\theta_j-)) \chi_{(s, \tau]}(\theta_j) + K^{+,a}(\tau) - K^{+,a}(s) \\
&\quad - (K^{-,a}(\tau) - K^{-,a}(s)) - \int_s^\tau Z^a(r)dW(r),
\end{aligned}
\]

(5.7)

Comparing RBSDE (5.1) with (5.6), in view of the facts that

\[
Y^a(\theta_j-) = Y_{\alpha_{j-1}}(\theta_j) \geq h_{\alpha_{j-1}, \alpha_j}(\theta_j, Y_{\alpha_j}(\theta_j)) = h_{\alpha_{j-1}, \alpha_j}(\theta_j, Y^a(\theta_j))
\]

and

\[
K^{+,a}(T) - K^{+,a}(t) \geq 0,
\]

we deduce from the comparison theorem [3, Theorem 4.1] that

\[
Y^a(s) \geq U^a(s), \quad t \leq s \leq T.
\]

From the definition in (5.5),

\[
Y^a(t) = Y_i(t).
\]

Hence,

\[
Y_i(t) \geq U^a(t), \quad \forall a \in \mathcal{A}_t^i.
\]

(5.8)
For the sequence \( \{ \hat{\theta}_j \}_{j=1}^\infty \), we claim that for a.s. \( \omega \), there exists an integer \( \hat{N}(\omega) < \infty \) such that \( \hat{N} = T \). Otherwise, define \( B \triangleq \bigcap_{j=1}^\infty \{ \omega : \hat{\theta}_j(\omega) < T \} \in \mathcal{F}_T \), then \( P(B) > 0 \). For \( j = 1, 2, \cdots \), we have
\[
Y_{\hat{\alpha}_j-1}(\hat{\theta}_j) = h_{\hat{\alpha}_{j-1},\hat{\alpha}_j}(\hat{\theta}_j, Y_{\hat{\alpha}_j}(\hat{\theta}_j)),
Y_{\hat{\alpha}_j}(\hat{\theta}_{j+1}) = h_{\hat{\alpha}_j,\hat{\alpha}_{j+1}}(\hat{\theta}_{j+1}, Y_{\hat{\alpha}_{j+1}}(\hat{\theta}_{j+1})) , \quad \text{on } B. 
\] (5.9)
Since the sequence \( \{(\hat{\alpha}_{j-1}, \hat{\alpha}_j, \hat{\alpha}_{j+1})\}_{j=1}^\infty \) takes values in \( \Lambda^3 \), which is a finite set, there are a triple \((i_1, i_2, i_3)\) and a subsequence \( j_k \) such that for \( k = 1, 2, \cdots \),
\[
(\hat{\alpha}_{j_{k-1}}, \hat{\alpha}_{j_k}, \hat{\alpha}_{j_{k+1}}) = (i_1, i_2, i_3).
\]
Since the sequence \( \{\hat{\theta}_j\}_{j=1}^\infty \) is increasing and bounded by \( T \), there is a limit \( \hat{\theta}_\infty \). Passing to the limit in (5.9) for the subsequence \( \{j_k\} \), we have
\[
Y_{i_1}(\hat{\theta}_\infty) = h_{i_1,i_2}(\hat{\theta}_\infty, Y_{i_2}(\hat{\theta}_\infty)),
Y_{i_2}(\hat{\theta}_\infty) = h_{i_2,i_3}(\hat{\theta}_\infty, Y_{i_3}(\hat{\theta}_\infty)), \quad \text{on } B. 
\] (5.10)
From Hypothesis 5, we have
\[
Y_{i_1}(\hat{\theta}_\infty) = h_{i_1,i_2}(\hat{\theta}_\infty, h_{i_2,i_3}(\hat{\theta}_\infty, Y_{i_3}(\hat{\theta}_\infty))) < h_{i_1,i_3}(\hat{\theta}_\infty, Y_{i_3}(\hat{\theta}_\infty)) \quad \text{on } B.
\]
This contradicts to the fact that
\[
Y_{i_1}(\hat{\theta}_\infty) \geq \max_{j \neq i_1, j \in \Lambda} h_{i_1, j}(\hat{\theta}_\infty, Y_j(\hat{\theta}_\infty)).
\] (5.11)
This shows \( \hat{\alpha} \in A^\hat{\theta}_\infty \).

It is easy to see that
\[
Y^{\hat{\alpha}}(\hat{\theta}_j-) = h_{\hat{\alpha}_{j-1},\hat{\alpha}_j}(\hat{\theta}_j, Y^{\hat{\alpha}}(\hat{\theta}_j)), \quad K^{+,\hat{\alpha}}(T) - K^{+,\hat{\alpha}}(T) = 0. 
\] (5.12)
Then \( (Y^{\hat{\alpha}}, Z^{\hat{\alpha}}, K^{-,\hat{\alpha}}) \) satisfies RBSDE (5.1). By the uniqueness of the solution of RBSDE (5.1), we have
\[
Y^{\hat{\alpha}}(s) = U^{\hat{\alpha}}(s), \quad t \leq s \leq T.
\]
Hence,
\[
Y_i(t) = U^{\hat{\alpha}}(t).
\]
Noting (5.8), we know
\[
Y_i(t) = \text{esssup}_{a \in A^\hat{\theta}_i} U^a(t), \quad \forall t \in [0, T]. 
\] (5.13)
In view of (5.7), it follows from (5.12) that
\[
Y^{\hat{\alpha}}(s) = Y^{\hat{\alpha}}(\tau) + \int_s^\tau \psi(r, Y^{\hat{\alpha}}(r), Z^{\hat{\alpha}}(r), \hat{\alpha}(r))dr \\
- \sum_{j=1}^{\hat{N}-1} (Y^{\hat{\alpha}}(\hat{\theta}_j) - h_{\hat{\alpha}_{j-1},\hat{\alpha}_j}(\hat{\theta}_j, Y^{\hat{\alpha}}(\hat{\theta}_j))] \chi_{(s,\tau]}(\hat{\theta}_j) \\
- (K^{-,\hat{\alpha}}(\tau) - K^{-,\hat{\alpha}}(s)) - \int_s^\tau Z^{\hat{\alpha}}(r)dW(r). 
\] (5.14)
For any switching strategy \( a \), define

\[
\hat{\tau} \triangleq \inf \{ s \in [t, T) : Y^a(s) = S_{a(s)}(s) \} \land T,
\]

with the convention that \( \inf \emptyset \triangleq +\infty \). From the upper minimal boundary condition in (1.4), we know that

\[
K^{-,a}(\hat{\tau}) - K^{-,a}(s) = 0,
\]

\[
Y^a(\hat{\tau}) = S_{a(\hat{\tau})}(\hat{\tau}) \chi_{\{\hat{\tau} < T\}} + \xi_{a(\hat{\tau})} \chi_{\{\hat{\tau} = T\}}.
\]

In view of (5.7) and (5.14), we have from (5.15) that

\[
Y^a(s) = S_{a(\hat{\tau})}(\hat{\tau}) \chi_{\{\hat{\tau} < T\}} + \xi_{a(\hat{\tau})} \chi_{\{\hat{\tau} = T\}} + \int_s^{\hat{\tau}} \psi(r, Y^a(r), Z^a(r), a(r)) dr
\]

\[
- \sum_{j=1}^{N-1} (Y^a(\theta_j) - Y^a(\theta_j^-)) \chi_{(s, \theta_j)]}(\theta_j) + K^{+,a}(\hat{\tau}) - K^{+,a}(s)
\]

\[
- \int_s^{\hat{\tau}} Z^a(r)dW(r), \quad s \in [t, \hat{\tau}); a \in \mathcal{A}_t^i.
\]

and

\[
Y^{\hat{a}}(s) = S_{\hat{a}(\hat{\tau})}(\hat{\tau}) \chi_{\{\hat{\tau} < T\}} + \xi_{\hat{a}(\hat{\tau})} \chi_{\{\hat{\tau} = T\}} + \int_s^{\hat{\tau}} \psi(r, Y^{\hat{a}}(r), Z^{\hat{a}}(r), \hat{a}(r)) dr
\]

\[
- \sum_{j=1}^{\hat{N}} (Y^{\hat{a}}(\hat{\theta}_j) - h_{\hat{a}_{j-1}, \hat{a}_j}(\hat{\theta}_j, Y^{\hat{a}}(\hat{\theta}_j))) \chi_{(s, \hat{\theta}_j]}(\hat{\theta}_j)
\]

\[
- \int_s^{\hat{\tau}} Z^{\hat{a}}(r)dW(r), \quad s \in [t, \hat{\tau}].
\]

Comparing BSDE (5.2) with (5.16), (5.14) and (5.17), respectively, in view of the facts that

\[
Y^a(\theta_j^-) = Y_{\alpha_{j-1}}(\theta_j) \geq h_{\alpha_{j-1}, \alpha_j}(\theta_j, Y^a(\theta_j)) = h_{\alpha_{j-1}, \alpha_j}(\theta_j, Y^a(\theta_j)),
\]

\[
Y^{\hat{a}}(\tau) \leq S_{\hat{a}(\tau)}(\tau) \chi_{\{\tau < T\}} + \xi_{\hat{a}(\tau)} \chi_{\{\tau = T\}},
\]

and \( K^{+,a} \) and \( K^{-,\hat{a}} \) are increasing processes, we have

\[
Y^a(t) \geq U^{a,\hat{\tau}}(t), \quad Y^{\hat{a}}(t) \leq U^{\hat{a},\tau}(t), \quad Y^a(t) = U^{\hat{a},\hat{\tau}}(t), \quad t \in [0, T].
\]

Noting by definition that

\[
Y^a(t) = Y^{\hat{a}}(t) = Y_t(t), \quad t \in [0, T],
\]

we have

\[
Y_t(t) = U^{\hat{a},\hat{\tau}}(t), \quad U^{a,\hat{\tau}}(t) \leq U^{\hat{a},\hat{\tau}}(t) \leq U^{\hat{a},\tau}(t) \quad \forall t \in [0, T].
\]

This shows that \((\hat{a}, \hat{\tau})\) is a saddle point for the functional \( U^{a,\tau}(t) \) as a functional of \((a, \tau) \in \mathcal{A}_t^i \times \mathcal{T}\), with \( \mathcal{T} \) being the totality of stopping times which take values in \([t, T]\); or equivalently,

\[
Y_t(t) = U^{\hat{a},\hat{\tau}}(t) = \operatorname{esssup}_{a \in \mathcal{A}_t^i} \operatorname{essinf}_{\tau} U^{a,\tau}(t) = \operatorname{essinf}_{a \in \mathcal{A}_t^i} \operatorname{esssup}_{\tau} U^{a,\tau}(t), \quad \forall t \in [0, T].
\]
Theorem 5.1 gives the uniqueness of \( Y \). The uniqueness of other components \( (Z, K^+, K^-) \) of the solution \( (Y, Z, K^+, K^-) \) is a consequence of Doob-Meyer decomposition of \( Y \). We conclude the following results.

**Theorem 5.2.** Let Hypotheses 1, 2 and 5 be satisfied. Assume that the upper barrier \( S \) is super-regular with \( S(t) \in \tilde{Q}(t) \) for \( t \in [0, T] \), and that the terminal value \( \xi \in L^2(\Omega, \mathcal{F}_T; P; R^m) \) takes values in \( Q(T) \). Then RBSDE (1.4) has a unique adapted solution \( (Y, Z, K^+, K^-) \).

**Theorem 5.3.** Let Hypotheses 1 and 3'(i) be satisfied. Assume that the upper barrier \( S \in (S^2)^m \) with \( S(t) \in \tilde{Q}(t) \) for \( t \in [0, T] \), that the terminal value \( \xi \in L^2(\Omega, \mathcal{F}_T; P; R^m) \) takes values in \( Q(T) \), and that \( k(i, j) + k(j, l) > k(i, l) \) for \( i \neq j, j \neq l, i, j, l \in \Lambda \). Then RBSDE (4.20) has a unique adapted solution.

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