First Order Description of $D = 4$ static Black Holes and the Hamilton–Jacobi equation

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Abstract

In this note we discuss the application of the Hamilton–Jacobi formalism to the first order description of four dimensional spherically symmetric and static black holes. In particular we show that the prepotential characterizing the flow coincides with the Hamilton principal function associated with the one-dimensional effective Lagrangian. This implies that the prepotential can always be defined, at least locally in the radial variable and in the moduli space, both in the extremal and non-extremal case and allows us to conclude that it is duality invariant. We also give, in this framework, a general definition of the “Weinhold metric” in terms of which a necessary condition for the existence of multiple attractors is given. The Hamilton–Jacobi formalism can be applied both to the restricted phase space where the electromagnetic potentials have been integrated out as well as in the case where the electromagnetic potentials are dualized to scalar fields using the so-called three-dimensional Euclidean approach. We give some examples of application of the formalism, both for the BPS and the non-BPS black holes.

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1 Introduction

Recently considerable attention has been devoted to the study of black-hole solutions in supergravity theories, especially in connection with the attractor mechanism [1, 2] (see also [3] and references therein). In this respect, of particular relevance is the issue of describing the spatial evolution of the scalar fields and the metric in terms of a first order dynamical system of equations written in terms of a prepotential, also called fake superpotential [4, 5, 6, 7]. In the case of four dimensional spherically symmetric black holes, if we collectively denote the metric and scalar degrees of freedom characterizing the solution by $q^i$ and the radial variable by $\tau$, the issue has been of whether it is possible to define a prepotential $W(q)$, function of $q^i$ and of the quantized electric-magnetic charges, such that the radial evolution of $q^i$ is solution to a system of equations of the following form:

$$\dot{q}^i = G^{ij} \frac{\partial W}{\partial q^j},$$  \hfill (1)

$G_{ij}(q)$ being a suitable non-degenerate metric. Equations (1) are particularly suitable for studying the attractor behavior [1, 2] of four dimensional black-hole solutions [8, 9] as well as higher dimensional black-brane solutions [10]. The reason behind the interest in such a prepotential $W$ goes beyond the simple convenience in writing the black-hole equations in a first order form. This function seems to have a deeper physical meaning also related to the description of the radial evolution of the black-hole fields in terms of a dual RG flow.

So far such a description has been found for specific extremal and non-extremal static four dimensional black-hole solutions and the following questions have been left unanswered:

- For which solutions does the prepotential $W$ exist?
- Is $W$ invariant under the global symmetries (dualities) of the four dimensional theory?
- In ref. [5] it was found for the extremal case that $W$ is a $c$-function, that is always monotonic in $\tau$. Can this property be extended, whenever $W$ exists, also to the non-extremal case?

It is a well established fact that the problem of finding a first order description of the form (1) can be recast, for static solutions, into a Hamilton–Jacobi problem in which the $W$ prepotential is the Hamilton’s characteristic function such that:

$$H(q, \partial_q W) = c^2.$$  \hfill (2)

The Hamilton–Jacobi approach to the first order description of supergravity solutions has been applied to the study of RG flows “dual” to domain walls and cosmological solutions [11, 12, 13, 14] and to extremal black holes [15]. It has also been applied to the general description of extended supergravity solutions in various dimensions [16].

In this note we restrict ourselves to static, spherically symmetric black holes and wish to point out some important consequences of the well known theory of the Hamilton–Jacobi equation which have never been clearly stated in the black-hole literature so far.

We first show that the prepotential, being identified with Hamilton’s characteristic function, can always be given (in both the extremal ($c = 0$) and non-extremal ($c \neq 0$) case) in a local form, in terms of the integral of the Lagrangian along a trajectory which minimizes the
action (characteristic trajectory). We also give, in this framework, a general definition of the “Weinhold metric” \[2\] in terms of which to define a necessary condition \[17\] for the existence of multiple attractors in the extremal case \((c = 0)\). Actually, the expression we find for the prepotential, eq. (14) below, generalizes the one conjectured in \[5\].

Of particular interest to us is the issue of whether \(W(q)\) is duality invariant. If \(W\) is to be associated with some fundamental physical property of the black hole, just like the entropy, it ought to be duality invariant, namely independent on the particular description of its degrees of freedom. The duality invariance of \(W\) is guaranteed by the fact that it can be expressed as the integral of the duality invariant Lagrangian on a characteristic trajectory.

The paper is organized as follows: in Section 2 we briefly review the Hamiltonian description of four dimensional spherically symmetric and static black holes. In Section 3 we introduce the Hamilton–Jacobi problem in \(D = 4\) and the definition of the prepotential \(W\). In Section 4 we write a local form for the solution to the Hamilton–Jacobi equation and discuss some of its properties. In Section 5 we prove duality invariance of \(W\). Finally in Section 6 we generalize the Hamiltonian problem by extending the phase space to include the quantized charges and their conjugate variables, namely the electric-magnetic potentials. This extended formulation has a natural setting in the \(D = 3\) Euclidean theory arising from time reduction of the four dimensional one. We write the Hamilton–Jacobi equation in this enlarged phase space and express its solution \(W_3\) in terms of the four dimensional prepotential \(W\). We discuss cases of interest in which a globally defined solution \(W_3\), and correspondingly \(W\), of the Hamilton–Jacobi equation exists.

## 2 Review of Static \(D = 4\) Black Holes

Let us recall the main facts about the description of a static black hole in \(D = 4\) as solution of a Hamiltonian system. We start from the four dimensional bosonic action of a generic supergravity theory, describing \(n\) scalar fields \(\phi = (\phi^r)\) coupled to \(n_v\) vector fields \(A^A_\mu\):

\[
S_4 = \int d^4x \sqrt{-g} \left[ -\frac{1}{2} R + \frac{1}{2} G_{rs}(\phi) \partial_\mu \phi^r \partial^\mu \phi^s + I_{\Lambda\Sigma}(\phi) F^\Lambda_{\mu\nu} F^{\Sigma\mu\nu} + \frac{1}{2 \sqrt{-g}} e^{\mu\rho\sigma} R_{\Lambda\Sigma}(\phi) F^\Lambda_{\mu\nu} F^{\Sigma}_{\rho\sigma} \right],
\]

where the gauge field-strength 2-form is defined as \(F^\Lambda = dA^\Lambda\) and \(I_{\Lambda\Sigma}, R_{\Lambda\Sigma}\) are the imaginary and real part of the complex kinetic matrix \(N_{\Lambda\Sigma}(\phi)\), with the convention that \(I_{\Lambda\Sigma} < 0\). The general Ansatz for a spherically symmetric static black hole reads \[17, 2, 3\]:

\[
ds_4^2 = g_{\mu\nu} dx^\mu dx^\nu = e^{2U} dt^2 - e^{-2U} \left( \frac{e^4}{\sinh^4(c \tau)} d\tau^2 + \frac{e^2}{\sinh^2(c \tau)} d\Omega^2 \right),
\]

\[
F^\Lambda = m^\Lambda \sin(\theta) d\theta \wedge d\varphi + e^{2U} \ell^A(\phi) dt \wedge d\tau,
\]

where \(\ell^A(\phi) = I^{-1}_{\Lambda\Sigma}(e_\Sigma - R_{\Sigma\Gamma} m^\Gamma), e_\Lambda, m_\Sigma\) being the quantized electric and magnetic charges. The constant \(c\) is the extremality parameter which is related to the temperature and entropy of the black hole through \(c = 2ST\) and identifies the inner and outer horizons corresponding to \(r_\pm = r_0 \pm c\). The dependence of the evolution parameter \(\tau\) on the radial
variable $r$ is implicitly given by the equation 
\[ \frac{c^2}{\sinh^2(c \tau)} = (r - r_0)^2 - c^2, \]
which in the extremal case $c \to 0$ can be written as $\tau = -1/(r - r_0)$.

It is well known that the equations of motion obtained from the above Ansatz read:
\[ \frac{d^2 U}{d\tau^2} \equiv \ddot{U} = V(\phi, e, m) e^{2U}, \]
\[ \frac{D^2 \phi^s}{D\tau^2} \equiv \dddot{\phi}^r + \Gamma^r_{st} \dot{\phi}^s \dot{\phi}^t = g^{rs}(\phi) \frac{\partial V(\phi, e, m)}{\partial \phi^s} e^{2U}, \]
(5)

together with the constraint
\[ \dot{U}^2 + \frac{1}{2} g_{rs}(\phi) \frac{d\phi^r}{d\tau} \frac{d\phi^s}{d\tau} - V(\phi, e, m) e^{2U} = c^2, \]
(6)

where the positive definite geodesic potential $V(\phi, e, m)$ has the following form:
\[ V = -\frac{1}{2} \Gamma^T M(\phi) \Gamma = -\frac{1}{2} \Gamma^T \begin{pmatrix} I + R I^{-1} & -R I^{-1} \\ -R I^{-1} & I^{-1} \end{pmatrix} \Gamma > 0, \]
(7)
in terms of the vector of magnetic-electric charges $\Gamma \equiv (m^\Lambda, e^\Lambda)$. Here the dotted quantities are differentiated with respect to the evolution parameter $\tau$.

The equations of motion (5) can be associated to a one dimensional theory whose Lagrangian is:
\[ \mathcal{L} = \dot{U}^2 + \frac{1}{2} G_{rs}(\phi) \dot{\phi}^r \dot{\phi}^s + e^{2U} V(\phi) = \frac{1}{2} G_{ij}(q) \dot{q}^i \dot{q}^j + \mathcal{V}(q), \]
(8)

where $\mathcal{V}(q) \equiv e^{2U} V(\phi)$ and we introduced the functions $q^i(\tau) = (U(\tau), \phi^r(\tau))$ together with the metric
\[ G_{ij}(q) = \begin{pmatrix} 2 & 0 \\ 0 & G_{rs}(\phi) \end{pmatrix}. \]
(9)

Once the Lagrangian is known we can proceed with an Hamiltonian approach using the phase space that stems from the $q^i(\tau) = (U(\tau), \phi^r(\tau))$ variables, introducing the conjugate momenta to $q^i$:
\[ p_i = \frac{\delta \mathcal{L}}{\delta \dot{q}^i} = G_{ij} \dot{q}^j. \]
(10)

In terms of the phase-space variables $q^i$ and $p_i$ the Hamiltonian $H(p, q)$ then reads:
\[ H(p, q) = \frac{1}{2} p_i G^{ij} p_j - \mathcal{V}(q) = \frac{1}{2} \dot{q}^i G_{ij}(q) \dot{q}^j - \mathcal{V}(q). \]
(11)

This is consistent with the constraint (6) that acquires the meaning of “energy conservation”:
\[ H(p, q) = c^2 \Leftrightarrow \frac{1}{2} \dot{q}^i G_{ij}(q) \dot{q}^j - \mathcal{V}(q) = c^2. \]
(12)

Note that here and in the following by abuse of language we adopt the terms Lagrangian and Hamiltonian even if the evolution variable $\tau$ does not describe the temporal evolution but the radial one.
3 Hamilton–Jacobi Formalism and Static Black Holes

Let us recall how the solutions of the equations of motion can be obtained by applying the machinery of the Hamilton–Jacobi theory. We consider the principal Hamiltonian function $S(q, P, \tau)$ depending on $q^i$ and on new constant momenta $P_i$. It is defined by the set of first order equations:

$$ \frac{\partial S}{\partial q^i} = p_i, \quad \frac{\partial S}{\partial P_i} = Q^i, \quad \frac{\partial S}{\partial \tau} = -H, \quad (13) $$

where $P_i, Q^i$ are new constant canonical variables which can be expressed in terms of the initial values of $q^i$ and $p_i$. From the general theory of canonical transformations, see for instance [19], it is known that the above transformation generated by $S$ always exists locally in the $p, q$ space, in a neighborhood of any point which is not critical, namely in which $(\frac{\partial H}{\partial q}, \frac{\partial H}{\partial p}) \neq (0, 0)$.

We shall leave the dependence of $S$ on the constant $P_i$ implicit and focus on its dependence on the $q^i$.

From the last and the first equations of (13) and from the Hamiltonian constraint (12) we have:

$$ S(q, \tau) = W(q) - c^2 \tau \quad (14) $$

$$ p_i = \frac{\partial W}{\partial q^i}, \quad (15) $$

$$ H(q, \partial_q W) = \frac{1}{2} \partial_i W G^{ij} \partial_j W - V(q) = c^2 \quad (16) $$

where (16) defines the Hamilton–Jacobi equation for the function $W$, usually called Hamilton characteristic function.

From eq. (15) and eq. (10) we finally get

$$ \dot{q}^i = G^{ij} p_j = G^{ij} \frac{\partial W}{\partial q^j}. \quad (17) $$

This shows that, provided a solution to the equations (14)-(16) is found, the evolution of the metric and scalar fields can be described in terms of a dynamical system of the form (1). In the next section we give its unique solution.

Note that the function $S$ in eq. (14) generalizes the expression for the prepotential conjectured in [5] for the general (non-extremal) case. To make contact with the proposal in [5], let us consider the following expression for the principal function $S$:

$$ S(U, \phi^r, \tau) = 2 e^U W(\phi^r, \tau) + c^2 \tau = W(U, \phi^r) - c^2 \tau \quad (18) $$

This expression reproduces the first order equations for the prepotential as given in [4, 5]:

$$ \dot{U} = e^U W, \quad \dot{\phi}^r = 2 e^U G^{rs} \partial_s W, \quad (19) $$

together with the condition found in [5] for the non-extremal case: $\partial_r W = -c^2 e^{-U}$.

We observe, however, that (18) is a rather restrictive Ansatz for the Hamilton's principal function, since the dependence of $W$ on $U$ may not have a factorized form, as it is apparent, for example, in the Reissner–Nordstrom case discussed below. Actually, the present discussion generalizes such an Ansatz.
3.1 Critical points as attractors.

The first order system (17) may have critical points in the space of the $q$'s, namely points $(q_0^i)$ in which its right hand side vanishes: $\partial_i W|_{q_0} = 0$. From the Hamilton–Jacobi equation it is clear that such points may exist only in the extremal case, since the right hand side of the equation

$$ \partial_i W G^{ij} \partial_j W = 2 (e^2 + e^2 U V(\phi)), \quad (20) $$

being $V \geq 0$, may vanish only if $c = 0$. A critical point is an attractor [1] if at that point the Hessian of the geodesic potential is positive. Attractors are reached at $\tau \to -\infty$, where $U(-\infty) \to -\infty$, which corresponds to the horizon, since, for $c = 0$, $\tau = -1/(r - r_0)$.

In the extremal case we can make the Ansatz $W(U, \phi^r) = 2 e^U W(\phi^r)$, which corresponds to taking $W$ in eq. (18) not to depend explicitly on $\tau$, and the Hamilton–Jacobi equation translates in the following equation for $W(\phi^r)$:

$$ W^2 + 2 G^{rs} \frac{\partial W}{\partial \phi^r} \frac{\partial W}{\partial \phi^s} = V(\phi^r). \quad (21) $$

The ADM mass of the solution is given by the value of $\dot{U}$ at radial infinity ($r \to \infty$, $\tau \to 0^-$):

$$ M_{\text{ADM}}(\phi^r_\infty, e, m) = \lim_{r \to 0^-} \dot{U}, \quad \text{where} \quad \phi^r_\infty \text{ are the boundary values of the scalar fields at infinity.} $$

From the first order equations (19) written in terms of $W(\phi^r, e, m)$, we see that, since $\lim_{r \to 0^-} e^U = 1$, the ADM mass of the solution coincides with the value of $W$ as a function of the values of $\phi^r_\infty$: $M_{\text{ADM}}(\phi^r_\infty, e, m) = W(\phi^r_\infty, e, m)$.

**Multiple attractors** For a given set of quantized electric and magnetic charges $\Gamma = (m^A, e_A)$, extremal solutions may have more than one attractors [17, 20]. In [17] a necessary condition was given for the existence of multiple attractors. We want here to give a general proof of this statement for generic (not necessarily BPS) attractors. As in [2] we can define the “Weinhold metric” as the Hessian of $M_{\text{ADM}}(\phi^r_\infty, e, m) = W(\phi^r_\infty, e, m)$ with respect to the boundary values $\phi^r_\infty$:

$$ \hat{W} = (W_{rs}) \equiv \left( \frac{\partial^2 W}{\partial \phi^r \partial \phi^s} \right)_{\phi = \phi^r_\infty}. \quad (22) $$

We can take $\phi^r_\infty$ to coincide with the critical point $\phi^r_0$. In this case $\partial_r V(\phi^r_0) = \partial_r W(\phi^r_0) = 0$ and the scalar fields of the solution will not evolve: $\phi^r(\tau) \equiv \phi^r_0$. This solution is called double extremal. By computing the second derivatives of eq. (21) we can relate the Hessian matrix $(W_{rs})$ of $W$ to the Hessian matrix $\hat{V} \equiv (\partial_r \partial_s V)$ of $V$ at a critical point as follows:

$$ 2 \sqrt{V} \hat{W} + 4 \hat{W} \hat{G}^{-1} \hat{W} = \hat{V}, \quad (23) $$

where $\hat{G}^{-1} \equiv (G^{rs})$. The above matrix equation is solved by

$$ \hat{W} \hat{G}^{-1} = \frac{\sqrt{V}}{4} \left( -\mathbf{1} + \sqrt{1 + \frac{4}{V} \hat{V} \hat{G}^{-1}} \right), \quad (24) $$

where $\mathbf{1} \equiv (\delta^r_0)$. The matrix $\hat{V} \hat{G}^{-1} = (\partial_r \partial_s V G^{rs})|_{\phi_0}$ is the “mass squared” matrix of the fluctuations of $\phi^r$ around $\phi^r_0$. We can consider for instance BPS black holes in $\mathcal{N} = 2$.
supergravity, described by a function $W = |Z|$, $Z$ being the complex central charge of the theory. At the corresponding extremum one can show that $\partial_r \partial_s V(\phi^r_0) = 2 V G_{rs}(\phi^r_0) > 0$ [2], and from eq. (24) one finds

$$\partial_r \partial_s W(\phi^r_0) = \frac{1}{4 \sqrt{V}} \partial_r \partial_s V(\phi^r_0) = \frac{\sqrt{V}}{2} G_{rs}(\phi^r_0) > 0 \quad \text{(BPS attractor)}.$$  \hspace{1cm} (25)\\

In general a critical point $\phi^r_0$ is an attractor iff $\hat{V}(\phi_0)$, or equivalently $\hat{V} \hat{G}^{-1}(\phi_0)$, are positive definite. Eq. (24) then implies that $\phi^r_0$ is an attractor iff $\hat{W}(\phi_0)$, or equivalently $\hat{W}(\phi_0)$, are positive definite. If, for a given choice of the quantized charges $e, m, W$ (or equivalently $V$) has more than one attractor point, $W$ must have more than one minimum and this implies that there should exist at least one critical point in which the Hessian of $W$ has negative eigenvalues. Therefore if the Weinhold metric is positive definite everywhere in the moduli space, there can be no more than one attractor which is the statement made in [17].

4 Solution to the Hamilton–Jacobi equation

We recall that the solution of the set of differential equations (13), (16) in terms of Hamilton’s principal function $S$ is formally given by (see e.g. [21])

$$S(q, \tau) = S_0 + \int_{q_0, \tau_0}^{q, \tau} L(q, \dot{q}) \, d\tau ,$$  \hspace{1cm} (26)\\

where the integral is performed along the characteristic trajectory $\gamma = q^i(\tau)$, i.e. the solution of Hamilton’s equations, such that:

$$q^i(\tau_0) = q^i_0 , \quad q^i(\tau) = q^i .$$  \hspace{1cm} (27)

Few comments here are in order. The above formula provides, in the most general case, only a local definition of $S$: local in $\tau$, to avoid multivaluedness of $S$ [21], and local in the configuration space with coordinates $q^i$s, being $S$ defined only on the points $(q^i)$, for fixed $q^i_0, \tau_0$, for which the interpolating characteristic trajectory, satisfying (27), exists. In fact, as pointed out in Section 3 locally, in the neighborhood of a non-critical point in the phase space, there always exist a complete solution $S(q^i, P_i, \tau)$ to the Hamilton–Jacobi equation and it has the form (26) ($P_i$ can be seen as a complete set of integration constants). In what follows, we shall use eq. (26) bearing its local validity in mind.

In our problem, we can use the Hamiltonian constraint to find the expression of the principal function in terms of the potential, as follows:

$$S(q, \tau) = S_0 + \int_{q_0, \tau_0}^{q, \tau} \left[ c^2 + 2 V(q) \right] \, d\tau ,$$  \hspace{1cm} (28)\\

so that, using (8) and (12), the Hamilton’s characteristic function is given by:

$$W(q) = W_0 + \int_{q_0, \tau_0}^{q, \tau} \left[ c^2 + L(q, \dot{q}) \right] \, d\tau = W_0 + 2 \int_{q_0, \tau_0}^{q, \tau} \left[ c^2 + V(q) \right] \, d\tau .$$  \hspace{1cm} (29)
Actually, the above formula can also be derived from direct integration of eq. (16). Indeed (16) has the form of the eikonal equation for a wave front $\mathcal{W} = \text{const.}$ propagating in a medium of refractive index $n = \sqrt{2(c^2 + \mathcal{V})}$:

$$n^2 = \partial_i \mathcal{W} G^{ij} \partial_j \mathcal{W}. \quad (30)$$

From equation (17) we get that $\partial_i \mathcal{W}$ is tangent to the “light rays” namely the characteristics $\gamma = (q_i(\tau))$. Introducing the proper distance $s$ along a characteristic:

$$ds = \sqrt{q^i G_{ij}(q) q^j} d\tau = \sqrt{2(c^2 + \mathcal{V}(q))} d\tau \quad (31)$$

using (17) we have:

$$\frac{d\mathcal{W}}{ds} = \frac{\partial \mathcal{W}}{dq_i} \frac{d\tau}{ds} = \partial_i \mathcal{W} G^{ij} \partial_j \mathcal{W} \frac{d\tau}{ds} = \sqrt{2(c^2 + \mathcal{V}(q))} \quad (32)$$

that is:

$$d\mathcal{W} = \sqrt{2(c^2 + \mathcal{V}(q))} ds = 2(c^2 + \mathcal{V}(q)) d\tau. \quad (33)$$

From the above equation it follows that $\frac{d\mathcal{W}}{d\tau}$ along $\gamma$ is positive, namely that $\mathcal{W}$ is a monotonic increasing function of $\tau$ along a solution (the same is true for the principal function $S$, since the Lagrangian is non-negative).

**Example** Let us review the construction of $\mathcal{W}$ for the Reissner-Nordström black hole [16]. The $q^i$ variables now consist of the function $U$ alone. This is for instance a solution to $\mathcal{N} = 2$ pure supergravity. With respect to the only vector field of the theory (the graviphoton) the solution can have in general an electric and a magnetic charge $e, m$. The geodesic potential reads:

$$\mathcal{V}(U, e, m) = e^2 U Q^2, \quad Q^2 \equiv \frac{1}{2}(e^2 + m^2). \quad (34)$$

The Hamiltonian constraint and the Hamilton–Jacobi equation read:

$$\dot{U}^2 = (\partial_U \mathcal{W})^2 = c^2 + e^2 U Q^2. \quad (35)$$

We can then readily apply eq. (29) to find, upon changing variables from $\tau$ to $U$:

$$\mathcal{W}(U) = W_0 + 2 \int_{U_0, \tau_0}^{U, \tau} \left[ c^2 + e^2 U Q^2 \right] d\tau = W_0 + 2 \int_{U_0}^{U} \left[ c^2 + e^2 U Q^2 \right] \frac{dU}{\dot{U}} =$$

$$= W_0 + 2 \int_{U_0}^{U} \sqrt{c^2 + e^2 U Q^2} dU =$$

$$= W_0 + 2 \left[ \sqrt{c^2 + e^2 U Q^2} - \frac{c}{2} \log \left( \frac{\sqrt{c^2 + e^2 U Q^2} + c}{\sqrt{c^2 + e^2 U Q^2} - c} \right) \right]. \quad (36)$$
5 Duality invariance of the prepotential \( \mathcal{W} \).

Let us now consider an extended supergravity theory in \( D = 4 \). It is known that the global symmetries of the equations of motion and Bianchi identities are encoded in the isometry group \( G \) of the scalar manifold (if non-empty), whose action on the scalar fields is associated with a simultaneous linear symplectic action on the field strengths \( F^\Lambda \) and their duals \( G^\Lambda \).

The duality action of \( G \) is defined by an embedding \( D \) of \( G \) inside the group \( \text{Sp}(2n_v, \mathbb{R}) \):

\[
g \in G : \begin{cases} \phi^r \rightarrow \phi^r' = g \ast \phi^r \\ \left( F^\Lambda \right) \rightarrow D(g) \cdot \left( F^\Lambda \right) \end{cases},
\]

(37)

where \( g \ast \) denotes the non-linear action of \( g \) on the scalar fields and \( D(g) \) is the \( 2n_v \times 2n_v \) symplectic matrix associated with \( g \).

We are going to prove explicitly that the prepotential \( \mathcal{W}(q) \) is invariant under the duality action of \( G \). Keeping in mind that the metric (and therefore the function \( U \)) is a duality invariant field, we define \( (g \ast q^i) \equiv (U, g \ast \phi^r) \). The on-shell global invariance of the four dimensional theory under \( G \) implies that, if \( \gamma = (q^i(\tau)) \) is a characteristic trajectory of the Lagrangian system \( \mathcal{L} \) with charge parameters \( \Gamma = (m^\Lambda, e_\Lambda) \), then \( g \ast \gamma = (g \ast q^i(\tau)) \) is a trajectory of the Lagrangian system \( \mathcal{L} \) with charge parameters \( D(g) \cdot \Gamma \).

Let us make the dependence of the geodesic potential \( V \) on the electric-magnetic charges explicit by writing \( V(q, \Gamma) \). From general properties of the symplectic matrix \( \mathcal{M}(\phi) \), defined in (7), we have:

\[
\mathcal{M}(g \ast \phi) = D(g)^{-T} \mathcal{M}(\phi) D(g)^{-1}.
\]

(38)

From this it follows that the potential \( V \) is duality invariant, in the sense that:

\[
V(g \ast q, D(g) \cdot \Gamma) = V(q, \Gamma).
\]

(39)

The group \( G \) is then a global symmetry group of the one-dimensional Lagrangian \( \mathcal{L} \) in \( \mathcal{L} \) and thus of both Hamilton’s principal and characteristic functions \( S(q, \tau; \Gamma) \) and \( W(q; \Gamma) \). Indeed from (29) and (39) we have:

\[
\mathcal{W}(q; \Gamma) = W_0 + 2 \int_{q_0, \tau_0}^{q, \tau} \left[ c^2 + V(q, \Gamma) \right] d\tau = W_0 + 2 \int_{g \ast q_0, \tau_0}^{g \ast q, \tau} \left[ c^2 + V(g \ast q, D(g) \cdot \Gamma) \right] d\tau = \mathcal{W}(g \ast q; D(g) \cdot \Gamma).
\]

(40)

6 The extended phase space and time-reductions to \( D = 3 \)

Four-dimensional static and spherically symmetric black holes depend on a number of degrees of freedom which includes, besides the metric and scalars, also the degrees of freedom corresponding to the gauge potentials. The one-dimensional effective Lagrangian \( \mathcal{L} \) encoding the information on the theory can then been formulated as a Lagrangian for a geodesic model describing all the degrees of freedom, as discussed in \[18\], \[2\]. It is then natural to extend the phase space to include as further degrees of freedom the magnetic and electric potentials together with their conjugate momenta. This approach was pioneered in \[22\], in the case of double extremal black holes.
As it is well known, this approach is equivalent, for static black holes, to a time reduction of the four dimensional Lagrangian \[ \text{[18]} \].

To implement the reduction on the time direction, let us consider the following Ansatz for the metric \[ \text{[18]} \]:

\[
ds_{4}^{2} = e^{2U} (dt + A_{i}^{0} \, dx^{i})^{2} - e^{-2U} \, g_{ij}^{(3)} \, dx^{i} \, dx^{j}, \tag{41}\]

where \( x^{i}, i = 1, 2, 3 \), are the coordinates of the final Euclidean space. In this theory all the vectors are dualized to scalar fields so as to obtain a sigma model coupled to gravity. Let us introduce the \( n_{v} \) three dimensional scalars \( \zeta^{\Lambda} = A_{i}^{0} \) which, together with the scalars \( \tilde{\zeta}_{\Lambda} \), form the symplectic vector of electric and magnetic potentials \( \mathcal{Z}^{M} = (\zeta^{\Lambda}, \tilde{\zeta}_{\Lambda}) \). Finally we shall denote by \( a \) the axion dual in \( D = 3 \) to the Kaluza-Klein vector \( A_{i}^{0} \). The final \( D = 3 \) action reads:

\[
S_{3} = \int d^{3}x \sqrt{|g^{(3)}|} \left( \frac{1}{2} R_{3} - \frac{1}{2} G_{IJ}(\Phi) \partial_{i} \Phi^{I} \partial^{i} \Phi^{J} \right),
\]

where \( \Phi^{I} \equiv (U, \phi^{r}, a, \mathcal{Z}^{M}) \), and the sigma model metric reads:

\[
G_{IJ}(\Phi) d\Phi^{I} d\Phi^{J} = dU^{2} + \frac{1}{2} G_{rs} \, d\phi^{r} \, d\phi^{s} + \frac{e^{-4U}}{4} \, \omega^{2} + \frac{e^{-2U}}{2} \, d\mathcal{Z}^{T} \mathcal{M} \, d\mathcal{Z},
\]

where \( \mathcal{M}_{MN} \) is the negative definite matrix defined in \[ \text{[7]} \] and the one-form \( \omega \) is defined as \( \omega = da - Z^{T} C \, d\mathcal{Z}, \) \( C \) being the antisymmetric \( 2n_{v} \times 2n_{v} \) \( \text{Sp}(2n_{v}, \mathbb{R}) \)-invariant matrix, for which we shall use the following form

\[
C = \begin{pmatrix}
0 & -I \\
I & 0
\end{pmatrix}.
\tag{42}\]

The scalar fields \( \Phi^{I} \) span a pseudo-Riemannian manifold \( \mathcal{M}_{3} \) which contains as a submanifold the scalar manifold \( \mathcal{M}_{4} \) of the \( D = 4 \) parent theory spanned by \( \phi^{r} \).

Spherically symmetric black-hole solutions are described by geodesics \( \Phi^{I}(\tau) \) on \( \mathcal{M}_{3} \) parametrized by the radial variable \( \tau \) \[ \text{[18]} \]. We shall restrict ourselves to spherically symmetric solutions with vanishing NUT charge, namely we shall take \( \omega_{r} = 0 \). These solutions will be described by the following effective Lagrangian:

\[
\mathcal{L}_{3} = U^{2} + \frac{1}{2} G_{rs} \, \dot{\phi}^{r} \, \dot{\phi}^{s} + \frac{e^{-2U}}{2} \, \dot{Z}^{T} \mathcal{M} \, \dot{Z} = \frac{1}{2} G_{\alpha\beta} \, \dot{q}^{\alpha} \, \dot{q}^{\beta}.
\]

Let us introduce the following generalized coordinates \( q^{\alpha} \equiv (U, \phi^{r}, \mathcal{Z}^{M}) = (q^{i}, \mathcal{Z}^{M}) \). The conjugate momenta \( p_{\alpha} \) will read:

\[
p_{\alpha} = G_{\alpha\beta}(q^{i}) \, \dot{q}^{\beta} = (p_{i}, p_{M}).
\tag{43}\]

In terms of \( q^{\alpha}, p_{\alpha} \) we write the Hamiltonian:

\[
H_{3}(p, q) = \frac{1}{2} G^{\alpha\beta} p_{\alpha} p_{\beta} = \frac{1}{2} G^{ij}(q^{i}) \, p_{i} \, p_{j} + \frac{e^{2U}}{2} \, p_{M} \, \mathcal{M}_{MN}^{M} \, (\phi^{r}) \, p_{N} = e^{2},
\]

\footnote{This metric describes stationary solutions, in the static case \( A_{i}^{0} = 0 \).}
where $M^{MN}$ denotes the inverse matrix of $M_{MN}$, given by: $M^{MN} = C^{MP} C^{NQ} M_{PQ}$. Since $Z^M$ are cyclic, the corresponding momenta $p_M$ are constants of motion. They are identified with the quantized charges as follows: $p_M = -C_{MN} \Gamma^N$. With this identification, the last term in $H_3$ reads $\frac{1}{2} e^{2U} \Gamma T \cdot M \cdot \Gamma = -V(q^i, \Gamma)$ and $H_3$ coincides with the Hamiltonian $H$ defined in the previous sections. Therefore the resulting equations of motion for $p_i, q^i$ are the same as those discussed earlier. As far as the equation for $Z^M$ is concerned, it reads [13]:

$$\dot{Z}^M = \frac{\partial H_3}{\partial p_M} = -e^{2U} C^{MN} M_{NP} \Gamma^P. \tag{44}$$

This analysis can be viewed as an extension of that given in [22] since it includes in the definition of the phase space also the four dimensional scalar fields.

In this enlarged Hamiltonian system we want now to define Hamilton’s characteristic function, to be denoted by $W_3(q^\alpha, P_\alpha)$. By definition this function generates the canonical transformation to the coordinates $Q^\alpha, P_\alpha$, where $P_\alpha$ are constants of motion. Since also $c^2$ is conserved, it will be a function of the $P_\alpha$, $c^2 = c^2(P)$. It will indeed provide the Hamiltonian in the new coordinates. From the general theory it is known that the coordinates $Q^\alpha$ are linear in $\tau$, i.e. harmonic functions:

$$Q^\alpha = \left( \frac{\partial c^2}{\partial P_\alpha} \right) \tau + Q^\alpha_0. \tag{45}$$

If we choose one of the $P_i$ to coincide with $c^2$, then only the corresponding $Q^i$ will be linear in $\tau$, the other $Q^\alpha$ being constants. The function $W_3(q^\alpha, P_\alpha)$ satisfies the following relations:

$$p_\alpha = \frac{\partial W_3}{\partial q^\alpha}, \tag{46}$$

$$Q^\alpha = \frac{\partial W_3}{\partial P_\alpha}, \tag{47}$$

$$c^2 = H_3(q^\alpha, \frac{\partial W_3}{\partial q^\alpha}), \tag{48}$$

the latter being the Hamilton–Jacobi equation. Since $p_M$ are already constant, $W_3$ should be such that $P_M = p_M$. This function can be expressed in terms of the four-dimensional Hamilton’s characteristic function $W$ as follows:

$$W_3(q^\alpha, P_\alpha) = W(q^i, P_i, P_M) + Z^M P_M, \tag{49}$$

where $W$ was defined in (16). Equation (46), for $\alpha = i$, follows from (15) and, for $\alpha = M$ implies $p_M = P_M$. Therefore the dependence of $W$ on $P_M$ is nothing but the dependence of the four-dimensional $W$ on the quantized charges

$$\Gamma^M = C^{MN} p_N = C^{MN} P_N. \tag{50}$$

Equation (49) can then be rewritten in the form:

$$W_3(q^\alpha, P_\alpha) = W(q^i, P_i, \Gamma^M) - Z^M C_{MN} \Gamma^N. \tag{51}$$

Let us now consider the component $\alpha = M$ of eq. (47):

$$\frac{\partial W_3}{\partial P_M} = \frac{\partial W}{\partial P_M} + Z^M = Q^M. \tag{52}$$
The above equation can also be written, using (50):

$$Z^M - C^{MN} \frac{\partial W}{\partial \Gamma^N} = Q^M. \quad (53)$$

This is a non-trivial equation which implies that the combination on the left hand side is a symplectic vector of harmonic functions. Since the $Q^M$ can be chosen to be constant, we can write:

$$Z^M = C^{MN} \frac{\partial W}{\partial \Gamma^N} + \text{const.} \quad (54)$$

The above equation allows to compute the electric-magnetic potentials once the $W$-prepotential is known on the solution as a function of all the quantized charges.

We shall check below eq. (54) on some specific solutions.

**The BPS Solution for the $\mathcal{N} = 2$ Case.**

We shall refer to the usual $\mathcal{N} = 2$ special geometry notations. Let $z^i$ denote the complex scalar fields on the special Kähler manifold and let $V^M(z, \bar{z})$, $U^M_i(z, \bar{z})$ be the covariantly holomorphic symplectic section and its covariant derivative:

$$V^M = \left( L^\Lambda \over M_\Lambda \right), \quad U^M_i = D_i V^M = \left( f^\Lambda_i \over h_{\Lambda i} \right). \quad (55)$$

The matrix $\mathcal{M} = (\mathcal{M}_{MN})$ is related to the above quantities as follows:

$$\mathbb{C} \mathcal{M} \mathbb{C} = -\mathcal{M}^{-1} = V V^T + \nabla V^T + g^\mathcal{F}_U U_i U_i^T + g^\mathcal{E}_\Upsilon U_i U_i^T. \quad (56)$$

The symplectic section $V^M$ also satisfies the property: $V^T \mathbb{C} V = -i$.

The central charge $Z$ is defined as follows:

$$Z = \Gamma^T \mathbb{C} V = e^\Lambda L^\Lambda - m^\Lambda M_\Lambda, \quad (57)$$

The first order equations describing the spatial evolution of the BPS solution originate from the Killing-spinor equations and read:

$$\dot{U} = e^U |Z|, \quad \dot{z}^i = 2e^U g^{\mathcal{F}} \partial_{\bar{\mathcal{F}}} |Z|. \quad (58)$$

The corresponding prepotential $W$ has the following form $W = 2 e^U |Z|$.

We wish now to verify equation (54) for this class of solutions. To this end we show that the derivative of the right hand side of this equation equals $\dot{Z}$, as given from eq. (44):

$$\frac{d}{d\tau} \frac{\partial W}{\partial \Gamma^M} = -\mathcal{C}_{MN} \dot{Z}^N = -e^{2U} \mathcal{M}_{MN} \Gamma^N. \quad (59)$$

Let us define the quantity:

$$T = H^T \mathbb{C} V = H^\Lambda L^\Lambda - H^\Lambda M_\Lambda, \quad (60)$$

where we have introduced the symplectic vector $H^M$ of harmonic functions $H^M(\tau) \equiv h^M - \sqrt{2} \Gamma^M \tau$. In terms of the above quantities, it was shown in [23] that the BPS solution is defined by the following algebraic equations:

$$\mathbb{T} V^M - T V^M = -\frac{i}{\sqrt{2}} H^M, \quad e^{-U} = |T|, \quad (61)$$
with the condition that $H^T C \dot{H} = 0$. From the above relations and positions one can prove the following properties:

$$\text{Im}(T \mathcal{Z}) = 0, \quad \dot{T} = -Z.$$  
\hline

Differentiating $W = 2 e^U |Z|$ with respect to $\Gamma$ and using (54) we find
\hline

$$Z^M = -2 e^U |Z| \text{Re}(\mathcal{Z} V^M) = -2 e^{2U} \text{Re}(\mathcal{T} V^M).$$  
\hline

Using (58) and (62) one finds:
\hline

$$\frac{d}{d\tau} (\mathcal{T} V^M) = \left( V^M \nabla^N - g^i \mathcal{U}^M_{i} \mathcal{U}^N_{i} \right) C_{NP} \Gamma^P,$$  
\hline

and then

$$\dot{Z}^M = -4 |Z| e^{3U} \text{Re}(\mathcal{T} V^M) - e^{2U} \left( V^M \nabla^N + \nabla^M V^N - g^i \mathcal{U}^M_{i} \mathcal{U}^N_{i} - \right.$$

\hline

Using (61), the first term on the right hand side of the above formula can be rewritten as follows
\hline

$$- 4 |Z| e^{3U} \text{Re}(\mathcal{T} V^M) = 2 e^{2U} \left( V^M \nabla^N + \nabla^M V^N \right) C_{NP} \Gamma^P.$$  
\hline

Finally, using (56) and the above property, equation (65) then yields equation (59).
\hline

The non-BPS extremal solution in $\mathcal{N} = 2$ supergravity with $|Z| \neq 0$.

We shall comment on an interesting class of non-BPS extremal solutions in $\mathcal{N} = 2$ supergravity. These solutions are characterized by a non vanishing value of the central charge $Z$ at the origin. An explicit form of this solution was worked out in [24, 25, 9] for the one-modulus cubic model and the STU model. In those two models the solutions are characterized by three and five independent parameters respectively, which are the duality invariant quantities that can be built out of the central and matter charges. In [5] a simple, duality invariant, form for the $\mathcal{W}$ prepotential was given in terms of all the charges, which, however, did not describe the full duality orbit of solutions. In [4, 6] a relatively simple form for $\mathcal{W}$ is given for cubic models as a function of specific sets of the quantized charges. The corresponding solutions however seem to exhibit all the duality invariant parameters. Let us consider, for the sake of simplicity, the one-modulus $z^3$-model, describing the supergravity multiplet coupled to a vector multiplet. The bosonic sector consists in one complex scalar field $z = x - i y$, $(y > 0)$ and two vectors $A^A_{i} = (A^0_{i}, A^1_{i})$, $n_v = 2$. The components of the covariantly holomorphic section $V^M$ in (55) read

$$L^A = e^{\frac{\dot{A}}{y}}(1, z), \quad M^A = e^{\frac{\dot{A}}{y}}(-z^3, z^2), \quad e^{\frac{\dot{A}}{y}} = \frac{1}{2 \sqrt{2} y^2}. $$  
\hline

The matrix $\mathcal{M}_{MN}$ has the following form:

$$\mathcal{M}_{MN} = \begin{pmatrix}
\frac{(x^2+y^2)^3}{y^3} & \frac{3x(x^2+y^2)^2}{y^4} & \frac{-x^3}{y^3} & \frac{-x^2(x^2+y^2)}{y^2} \\
\frac{3x(x^2+y^2)^2}{y^3} & \frac{-3(x^4+4x^2y^2+y^4)}{y^5} & \frac{3x^2}{y^4} & \frac{3x^3+2xy^2}{y^4} \\
\frac{-x^3}{y^3} & \frac{3x^2}{y^4} & \frac{-y^3}{y^3} & \frac{-x}{y^3} \\
\frac{x^2(x^2+y^2)}{y^3} & \frac{3x^3+2xy^2}{y^4} & \frac{-x}{y^3} & \frac{-(3x^2+y^2)}{3y^4}
\end{pmatrix}. $$  
\hline

13
In [4] the prepotential \( W \), related to \( W \) by \( W(\phi^r) = e^{-U} W/2 \), was given by:

\[
W_0 = W_{e_1,m^0=0} = e^{\frac{K}{2}} \left| e_0 - 3 m^1 \right| z, \tag{69}
\]
\[
W_0 = W_{e_0,m^1=0} = e^{\frac{K}{2}} \left| z (e_1 + m^0 \right| z^2), \tag{70}
\]

for the cases \((e_1, m^0) = 0\) or \((e_0, m^1) = 0\), respectively.

Using \( W_0 \) we cannot fully check eq. (54) since we do not know the derivatives of \( W \) along the charges which are set to zero. We can however check those components of eq. (54) involving the derivatives of \( W \) on the remaining charges. Let us consider the \((e_1, m^0) = 0\) case. The solution exists for \( e_0 m^1 < 0 \). Let us take \( e_0 < 0 \), \( m^1 > 0 \). The first order equations (19), defined from \( W_0 \), read

\[
\begin{align*}
\dot{U} &= -\frac{e^U}{2 \sqrt{2 y^2}} (e_0 - 3 m^1 (x^2 + y^2)) , \\
\dot{y} &= \frac{e^U}{\sqrt{2 y}} (e_0 - m^1 (3 x^2 - y^2)) , \\
\dot{x} &= 2 \sqrt{2 y} e^U m^1 x.
\end{align*}
\tag{71}
\]

The solution in terms of the harmonic functions \( H_0 = h_0 + \sqrt{2 e_0} \tau \), \( H^1 = h^1 - \sqrt{2 m^1} \tau \) reads [24]:

\[
e^{-4U} = H_0 (H^1)^3 - B^2, \quad x = \frac{B}{(H^1)^2}, \quad y = \frac{e^{-2U}}{(H^1)^2},
\tag{72}
\]

where \( B \) is a constant defining the value of the axion at radial infinity (\( \tau = 0 \)). Using the form of \( W_0 \) in eq. (69) and eqs. (71) and (68), it is straightforward to compute \( \frac{\partial W_0}{\partial e_0} \) and \( \frac{\partial W_0}{\partial m^1} \) and check, for these components of \( \frac{\partial W_0}{\partial \Gamma_M} \), that eq. (59) is satisfied. Analogous conclusions are obtained for the case \((e_0, m^1) = 0\).

As far as the existence of \( \mathcal{W}_3 \) is concerned, we can repeat here the comments made in Sections 3 and 4, namely we can express this prepotential \textit{locally} in terms of the integral along characteristic lines (which are now geodesics on \( \mathcal{M}_3 \)) of the Lagrangian. Of special relevance in this respect would be manifolds \( \mathcal{M}_3 \) for which the corresponding \( D = 3 \) Hamiltonian system is \textit{integrable}. In this case a complete solution \( \mathcal{W}_3 \), and thus \( \mathcal{W} \), of the Hamilton–Jacobi equation exists as a function of \textit{globally defined} integration constants \( P_\alpha \) in involution. Candidates for such integrable models are \( D = 3 \) sigma models based on \textit{symmetric} target manifolds \( \mathcal{M}_3 \). This analysis is still work in progress [26].

7 Conclusions

In this note we have analyzed the first order description of static black holes in an Hamiltonian framework, where the prepotential characterizing the flow has a natural interpretation as Hamilton principal function. A local form for the solution is available from the general theory, and was discussed in Section 4 both for the extremal and non extremal cases. We showed that from the general form of the solution also follows that the prepotential is duality invariant.

This kind of analysis, based on the Hamilton–Jacobi formalism, was fruitfully applied in the context of gauge/gravity correspondence from the quantum field theory side in a series of papers pioneered by [11]-[13] and specifically to spherically symmetric, static four
dimensional black holes in [15]. From this analysis emerges the role of the prepotential $\mathcal{W}$, on the dual QFT side, as a c-function characterizing the renormalization group flow towards the conformal fixed point. The results in [15] apply to cases where the principal function does not depend explicitly in the evolution parameter (cut-off) corresponding, on the gravity side, to extremal solutions. It would be interesting to extend such analysis also to non-extremal cases, where no IR critical point exists, but nevertheless, the function $\mathcal{W}$ is defined and should play a role.

It would also be interesting to extend our approach to more general settings which include for instance higher derivative corrections to the supergravity Lagrangian (see [13] and references therein), stationary (non static) solutions and multicenter black holes.

Another relevant issue is related to the duality invariance of the most general $\mathcal{W}$ function in four dimensional supergravity theories, which hints to the fact that $\mathcal{W}$ should be expressed only in terms of duality invariant quantities, constructed in terms of the central and matter charges. This was done in [5] for a broad class of extremal black holes in various supergravities. Such analysis however did not encompass the five-parameter solutions found in [24, 25] and described, for specific choices of electric and magnetic charges, by the prepotentials given in [4, 6]. The prepotentials found in those papers, however, do not exhibit manifest duality invariance, being expressed in terms of a reduced set of charges. We expect that the completion of these prepotentials to a function of a full set of charges will exhibit manifest duality and would be expressible in terms of central and matter charges only. Such completion could be achieved, for instance, by integrating eq. (59). An equivalent approach to the same problem is to use duality transformations on the five parameter solution to generate the remaining charges [9].

Finally $D = 3$ global symmetry transformations which map static black holes into one another, and which are employed in the solution generating techniques [27, 28], should be viewed as canonical transformations in the extended phase space, according to our discussion, and also along the lines of [22]. We leave to a future investigation the study of the relations between the various kinds of static black holes within this new framework. It would be interesting to address, at least for symmetric scalar manifolds $\mathcal{M}_3$, the issue of integrability of the Hamiltonian system in the extended phase space. To this respect, of particular relevance, in the case of symmetric manifolds, is the Lax pair description of the geodesic equations [29, 30, 31], and the existence of a number of conserved Hamiltonians in involution, found in [32].

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