The complexity of recognizing minimally tough graphs

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Abstract: Let \( t \) be a positive real number. A graph is called \( t \)-tough, if the removal of any cutset \( S \) leaves at most \( |S|/t \) components. The toughness of a graph is the largest \( t \) for which the graph is \( t \)-tough. A graph is minimally \( t \)-tough, if the toughness of the graph is \( t \) and the deletion of any edge from the graph decreases the toughness. The complexity class DP is the set of all languages that can be expressed as the intersection of a language in NP and a language in coNP. We prove that recognizing minimally \( t \)-tough graphs is DP-complete for any positive integer \( t \) and for any positive rational number \( t \leq 1/2 \).

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1 Introduction

All graphs considered in this paper are finite, simple and undirected. Let \( \omega(G) \) denote the number of components and \( \alpha(G) \) denote the independence number. For a graph \( G \) and a vertex set \( V \subseteq V(G) \), let \( G[V] \) denote the subgraph of \( G \) induced by \( V \).

The complexity class DP was introduced by C. H. Papadimitriou and M. Yannakakis [4].

Definition 1 A language \( L \) is in the class DP if there exist two languages \( L_1 \in \text{NP} \) and \( L_2 \in \text{coNP} \) such that \( L = L_1 \cap L_2 \).

We mention that DP \( \neq \text{NP} \cap \text{coNP} \), if NP \( \neq \text{coNP} \). Moreover, NP \( \cup \text{coNP} \subseteq \text{DP} \). A language is called DP-hard if all problems in DP can be reduced to it in polynomial time. A language is DP-complete if it is in DP and it is DP-hard.

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A critical-type DP-complete problem is \textsc{CriticalClique} \cite{5}, in our proofs we use an equivalent form of it, \textsc{α-Critical}.

\textbf{CriticalClique}  
\textit{Instance:} a graph $G$ and a positive integer $k$.  
\textit{Question:} is it true that $G$ has no clique of size $k$, but adding any missing edge $e$ to $G$, the resulting graph $G + e$ has a clique of size $k$?

By taking the complement of the graph, we can obtain \textsc{α-Critical} from \textsc{CriticalClique}.

\begin{definition}
A graph $G$ is called \textsc{α-critical}, if $\alpha(G - e) > \alpha(G)$ for all $e \in E(G)$.
\end{definition}

\textbf{α-Critical}  
\textit{Instance:} a graph $G$ and a positive integer $k$.  
\textit{Question:} is it true that $\alpha(G) < k$, but $\alpha(G - e) \geq k$ for any edge $e \in E(G)$?

Since a graph is clique-critical if and only if its complement is \textsc{α-critical}, \textsc{α-Critical} is also DP-complete.

\begin{corollary}
\textsc{α-Critical} is DP-complete.
\end{corollary}

The notion of toughness was introduced by Chvátal \cite{2}.

\begin{definition}
Let $t$ be a positive real number. A graph $G$ is called $t$-tough, if
\[ \omega(G - S) \leq \frac{|S|}{t} \]
for any cutset $S$ of $G$ (i.e. for any $S$ with $\omega(G - S) > 1$). The toughness of $G$, denoted by $\tau(G)$, is the largest $t$ for which $G$ is $t$-tough, taking $\tau(K_n) = \infty$ for all $n \geq 1$.

We say that a cutset $S \subseteq V(G)$ is a tough set if $\omega(G - S) = |S|/\tau(G)$.

For all positive rational number $t$ we can define a separate problem:

\textbf{t-Tough}  
\textit{Instance:} a graph $G$,  
\textit{Question:} is it true that $\tau(G) \geq t$?

Bauer et al. proved the following.

\begin{theorem}[\cite{1}]
For any positive rational number $t$, \textsc{t-Tough} is coNP-complete.
\end{theorem}

The critical form of this problem is minimally toughness.

\begin{definition}
A graph $G$ is minimally $t$-tough, if $\tau(G) = t$ and $\tau(G - e) < t$ for all $e \in E(G)$.
\end{definition}

Given $t$ we define:

\textbf{Min-t-Tough}  
\textit{Instance:} a graph $G$,  
\textit{Question:} is it true that $G$ is minimally $t$-tough?

Our main result is the following.

\begin{theorem}
\textsc{Min-t-Tough} is DP-complete for any positive integer $t$ and for any positive rational number $t \leq 1/2$.
\end{theorem}

First we prove this theorem for $t = 1$, then we generalize that proof for positive integers, and finally we prove it for any positive rational number $t \leq 1/2$. 2
2 Preliminaries

In this section we prove some useful lemmas.

**Proposition 8** Let $G$ be a connected noncomplete graph on $n$ vertices. Then $\tau(G) \in Q^+$, and if $\tau(G) = a/b$, where $a, b$ are positive integers and $(a, b) = 1$, then $1 \leq a, b \leq n - 1$.

**Proof:** By definition,

$$\tau(G) = \min_{S \subseteq V(G)} \frac{|S|}{\omega(G - S)}$$

for a noncomplete graph $G$. Since $G$ is connected and noncomplete, $1 \leq |S| \leq n - 2$ and since $S$ is a cutset, $2 \leq \omega(G - S) \leq n - 1$. □

**Corollary 9** Let $G$ and $H$ be two connected noncomplete graphs on $n$ vertices. If $\tau(G) \neq \tau(H)$, then

$$|\tau(G) - \tau(H)| > \frac{1}{n^2}.$$

**Claim 10** For every positive rational number $t$, Min-$t$-Tough $\in$ DP.

**Proof:** For any positive rational number $t$,

Min-$t$-Tough $= \{G$ graph $| \tau(G) = t$ and $\tau(G - e) < t$ for all $e \in E(G)\} =$

$= \{G$ graph $| \tau(G) \geq t\} \cap \{G$ graph $| \tau(G) \leq t\} \cap$

$\cap\{G$ graph $| \tau(G - e) < t$ for all $e \in E(G)\}.$

Let

$L_{1,1} = \{G$ graph $| \tau(G - e) < t$ for all $e \in E(G)\},$

$L_{1,2} = \{G$ graph $| \tau(G) \leq t\}$

and

$L_2 = \{G$ graph $| \tau(G) \geq t\}.$

$L_2 \in \text{coNP}$, a witness is a cutset $S \subseteq V(G)$ whose removal leaves more than $|S|/t$ components. $L_{1,1} \in \text{NP}$, the witness is a set of cutsets: $S_e \subseteq V(G)$ for each edge $e$ whose removal leaves more than $|S_e|/t$ components.

Now we show that $L_{1,2} \in \text{NP}$, i.e. we can express $L_{1,2}$ in a form of

$L_{1,2} = \{G$ graph $| \tau(G) < t + \varepsilon\},$

which belongs to NP. Let $a, b$ be positive integers such that $t = a/b$ and $(a, b) = 1$, and let $G$ be an arbitrary graph on $n$ vertices. If $G$ is disconnected, then $\tau(G) = 0$, and if $G$ is complete, then $\tau(G) = \infty$, so in both cases $G$ is not minimally $t$-tough. By Proposition 8, if $1 \leq a, b \leq n - 1$ does not hold, then $G$ is also not minimally $t$-tough. So we can assume that $t = a/b$, where $a, b$ are positive integers, $(a, b) = 1$ and $1 \leq a, b \leq n - 1$. With this assumption

$L_{1,2} = \{G$ graph $| \tau(G) \leq t\} = \left\{G$ graph $| \tau(G) < t + \frac{1}{|V(G)|^2}\right\},$

so $L_{1,2} \in \text{NP}$.

Since $L_{1,1} \cap L_{1,2} \in \text{NP}, L_2 \in \text{coNP}$ and Min-$t$-Tough $= (L_{1,1} \cap L_{1,2}) \cap L_2$, we can conclude that Min-$t$-Tough $\in$ DP. □
Claim 11 Let \( t \) be a positive rational number and \( G \) a minimally \( t \)-tough graph. For every edge \( e \) of \( G \),

1. the edge \( e \) is a bridge in \( G \), or
2. there exists a vertex set \( S = S(e) \subseteq V(G) \) with
   \[
   \omega(G - S) \leq \frac{|S|}{t} \quad \text{and} \quad \omega((G - e) - S) > \frac{|S|}{t},
   \]
   and the edge \( e \) is a bridge in \( G - S \).

In the first case, we define \( S = S(e) = \emptyset \).

Proof: Let \( e \) be an arbitrary edge of \( G \), which is not a bridge. Since \( G \) is minimally \( t \)-tough, \( \tau(G - e) < t \).
So there exists a cutset \( S = S(e) \subseteq V(G - e) = V(G) \) in \( G - e \) satisfying \( \omega((G - e) - S) > |S|/t \). On the other hand, \( \tau(G) = t \), so \( \omega(G - S) \leq |S|/t \). This is only possible if \( e \) connects two components of \( (G - e) - S \). \( \Box \)

Finally we cite a Lemma that our proof relies on.

Lemma 12 (Problem 14 of 8 in [3]) If we replace a vertex of an \( \alpha \)-critical graph with a clique, and connect every neighbor of the original vertex with every vertex in the clique, then the resulting graph is still \( \alpha \)-critical.

3 Recognizing minimally 1-tough graphs

To show that \textsc{Min-1-Tough} is DP-hard, we reduce \textsc{\( \alpha \)-Critical} to it.

Theorem 13 \textsc{Min-1-Tough} is DP-complete.

Proof: In Claim 10 we have already proved that \textsc{Min-1-Tough} \( \in \) DP.

Let \( G \) be an arbitrary connected graph on the vertices \( v_1, \ldots, v_n \). Let \( G_\alpha \) be defined as follows. It will be easy to see that it can be constructed from \( G \) in polynomial time. For all \( i \in [n] \), let

\[
V_i = \{v_{i,1}, v_{i,2}, \ldots, v_{i,\alpha}\}
\]

and place a clique on the vertices of \( V_i \). For all \( i, j \in [n] \), if \( v_iv_j \in E(G) \), then place a complete bipartite graph on \( (V_i; V_j) \). For all \( i \in [n] \) and for all \( j \in [\alpha] \) add the vertex \( u_{i,j} \) to the graph and connect it to \( v_{i,j} \). Let

\[
V = \bigcup_{i=1}^{n} V_i
\]

and

\[
U = \{u_{i,j} \mid i \in [n], j \in [\alpha]\}.
\]

Add the vertex set

\[
W = \{w_1, \ldots, w_\alpha\}
\]

to the graph and for all \( j \in [\alpha] \) connect \( w_j \) to \( v_{1,j}, \ldots, v_{n,j} \).
Figure 1: The graph $G_{\alpha}$.

We need to prove that $G$ is $\alpha$-critical with $\alpha(G) = \alpha$ if and only if $G_{\alpha}$ is minimally 1-tough. First we prove the following lemma.

**Lemma 14** Let $G$ be a graph with $\alpha(G) \leq \alpha$. Then $G_{\alpha}$ is 1-tough.

**Proof:** Let $S \subseteq V(G_{\alpha})$ be a cutset. We show that $\omega(G_{\alpha} - S) \leq |S|$.

**Case 1:** $W \subseteq S$. If a vertex of $U$ has only one neighbor in $V(G_{\alpha}) \setminus S$, then we can assume that this vertex is not in $S$. Then there are two types of components in $G_{\alpha} - S$: isolated vertices from $U$ and components containing at least one vertex from $V$. There are at most $\alpha(G)$ components of the second type and (exactly) $|V \cap S| = |S| - \alpha$ components of the first type. Thus $\omega(G_{\alpha} - S) \leq |S| - \alpha + \alpha(G) \leq |S|$.

**Case 2:** $W \not\subseteq S$. First, we make two convenient assumptions for $S$.

1. $U \cap S = \emptyset$.

It is easy to see that if $u_{i,j} \in S$, then we can assume that $v_{i,j} \not\in S$. Now there are two cases.

   **Case 2.1:** $v_{i,j}$ is not isolated in $G_{\alpha} - S$. Then we can consider $S' = (S \setminus \{u_{i,j}\}) \cup \{v_{i,j}\}$ instead of $S$.

   **Case 2.2:** $v_{i,j}$ is isolated in $G_{\alpha} - S$. Since there are no isolated vertices in $G$, there exists $k \in [n]$ such that $v_i v_k \in E(G)$. Then $v_{k,j} \in S$, so $u_{k,j} \not\in S$, which means that $w_j$ is not isolated in $G_{\alpha} - S$, so we can consider $S' = (S \setminus \{u_{i,j}\}) \cup \{w_j\}$ instead of $S$.

2. For all $i \in [n]$, either $V_i \subseteq S$ or $V_i \cap S = \emptyset$.

After the assumption (1), assume that only a proper subset of $V_i$ is contained in $S$. Let $v$ be an element of this subset. We can consider the cutset $S \setminus \{v\}$ instead of $S$, since this decreases the number of components by at most one. So we can repeat this procedure until $V_i \cap S = \emptyset$.

So in $G_{\alpha} - S$ there are isolated vertices from $U$ and one more component containing the remaining vertices of $W$ and $V$. So there are less than $|V \cap S|$ isolated vertices, thus

$$\omega(G_{\alpha} - S) \leq |V \cap S| \leq |S|.$$ 

So $G_{\alpha}$ is 1-tough. $\square$

We show that $G$ is $\alpha$-critical with $\alpha(G) = \alpha$ if and only if $G_{\alpha}$ is minimally 1-tough.
Let us assume that $G$ is $\alpha$-critical with $\alpha(G) = \alpha$. So by Lemma 14 $G_\alpha$ is 1-tough. Let $e \in E(G_\alpha)$ be an arbitrary edge. If $e$ has an endpoint in $U$, then this endpoint has degree 2, so $\tau(G_\alpha - e) < 1$. If $e$ does not have an endpoint in $U$, then it connects two vertices of $V$. By Lemma 12 $G_\alpha[U]$ is $\alpha$-critical, so in $G_\alpha[V] - e$ there exists an independent vertex set $I$ of size $\alpha(G) + 1$. Let $S = (V \setminus I) \cup W$. Then $|S| = (|V| - \alpha(G) - 1) + \alpha = |V| - 1$ and $\omega((G_\alpha - e) - S) = |V|$, so $\tau(G_\alpha - e) < 1$.

Let us assume that $G$ is not $\alpha$-critical with $\alpha(G) = \alpha$. 

Case 1: $\alpha(G) > \alpha$. Let $I$ be an independent vertex set of size $\alpha(G)$ in $G_\alpha[V]$ and let $S = (V \setminus I) \cup W$. Then $|S| = (|V| - \alpha(G)) + \alpha < |V|$ and $\omega(G_\alpha - S) = |V|$, so $\tau(G_\alpha) < 1$, which means that $G_\alpha$ is not minimally 1-tough.

Case 2: $\alpha(G) \leq \alpha$. Since $G$ is not $\alpha$-critical there exists an edge $e \in E(G)$ such that $\alpha(G - e) \leq \alpha$. By Lemma 14 $(G - e)_\alpha$ is 1-tough, but we can obtain $(G - e)_\alpha$ from $G_\alpha$ by edge-deletion, which means that $G_\alpha$ is not minimally 1-tough. □

4 Further results

Theorem 15 For every positive integer $t$, Min-$t$-Tough is DP-complete.

To prove this more general theorem, first we generalize the construction on Figure 1. We follow a similar argument to show that this construction has the required properties. However, due to the more complicated construction, the proof is harder.

The case when $t \leq 1/2$ is also covered in the paper.

Theorem 16 For every positive rational number $t = a/b \leq 1/2$, Min-$t$-Tough is DP-complete.

It is shown that Min-1-Tough can be reduced to this problem. The construction and the proof uses different ideas than the previous proofs.

We were not able to prove the DP-completeness for the remaining $t$ values, but we make the following conjecture.

Conjecture 17 Min-$t$-Tough is DP-complete for any positive rational number $t$.

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