OPEN BOOKS AND EXACT SYMPLECTIC COBORDISMS

MIRKO KLUKAS

Abstract. Given two open books with equal pages we show the existence of an exact symplectic cobordism whose negative end equals the disjoint union of the contact manifolds associated to the given open books, and whose positive end induces the contact manifold associated to the open book with the same page and monodromy given by the concatenation of the given ones. Following an outline by Wendl we show that the complement of the binding of an open book cannot contain any local filling obstruction. Given a contact 3-manifold, according to Eliashberg, there is a symplectic cobordism to a fibration over the circle with symplectic fibres. We extend this result to higher dimensions recovering a result by Dörner–Geiges–Zehmisch. Our cobordisms can also be thought of as the result of the attachment of a generalised symplectic 1-handle.

Introduction

Let $\Sigma$ denote a compact, $2n$-dimensional manifold admitting an exact symplectic form $\omega = d\beta$ and let $Y$ denote the Liouville vector field defined by $i_Y \omega = \beta$. Suppose that $Y$ is transverse to the boundary $\partial \Sigma$, pointing outwards. These properties are precisely the ones requested for $\Sigma$ to be a page of an abstract open book in the contact setting. Given a symplectomorphism $\phi$ of $(\Sigma, \omega)$, equal to the identity near $\partial \Sigma$, one can, following a construction of Thurston and Winkelnkemper [13] or rather its adaption to higher dimensions by Giroux [10] respectively, associate a $(2n+1)$-dimensional contact manifold $M(\Sigma, \omega, \phi)$ to the set of data $(\Sigma, \omega, \phi)$.

The main result of the present paper is part of the author’s thesis [12].

Theorem 1. Given two symplectomorphisms $\phi_0$ and $\phi_1$ of $(\Sigma, \omega)$, equal to the identity near the boundary $\partial \Sigma$, there is an exact symplectic cobordism whose negative end equals the disjoint union of the contact manifolds $M(\Sigma, \omega, \phi_0)$ and $M(\Sigma, \omega, \phi_1)$, and whose positive end equals $M(\Sigma, \omega, \phi_0 \circ \phi_1)$. If, in addition, the page $(\Sigma, \omega)$ is a Weinstein manifold, then so is the cobordism.

For $n = 1$ the above statement is due to Baker–Etnyre–van Horn-Morris [2] and, independently, Baldwin [3]. The general case of Theorem 1 was independently obtained by Avdek in [1], where the cobordism is associated with a so called Liouville connected sum. In [12] I observed that the cobordism in Theorem 1 can also be understood as result of the attachment of a generalised symplectic 1-handle of the form $D^1 \times N(\Sigma)$, where $N(\Sigma)$ denotes a vertically invariant neighbourhood of the symplectic hypersurface $\Sigma$. I will shed some more light on this in §4.

From the methods introduced in the proof of Theorem 1 we can deduce some further applications such as the following. We show the existence of strong fillings for contact manifolds associated with doubled open books, a certain class of fibre
bundles over the circle obtained by performing the binding sum of two open books with equal pages and inverse monodromies (cf. §3).

**Theorem 2.** Any contact manifold associated to a doubled open book admits an exact symplectic filling.

In dimension 3 the above statement appeared in [15], though details of the argument have not been carried out. As outlined by Wendl in [15, Remark 4.1] this statement has the following consequence for local filling obstructions in arbitrary dimensions. Similar results in dimension 3, concerning Giroux- and planar torsion respectively, are presented in [15] and [8].

**Corollary 3.** Let $(B, \pi)$ be an open book decomposition of a $(2n+1)$-dimensional contact manifold $(M, \xi)$ and let $O \subset (M, \xi)$ be any local filling obstruction (e.g. a bLob), then $B$ must intersect $O$ non-trivially.

Our final result will be the following. Let $(M, \xi)$ be a closed, oriented, $(2n+1)$-dimensional contact manifold supported by an open book with page $(\Sigma, \omega)$ and monodromy $\phi$. Suppose further that $(\Sigma, \omega)$ symplectically embeds into a second $2n$-dimensional (not necessarily closed) symplectic manifold $(\Sigma', \omega')$, i.e.

$$(\Sigma, \omega) \subset (\Sigma', \omega').$$

Let $M'$ be the symplectic fibration over the circle with fibre $(\Sigma', \omega')$ and monodromy equal to $\phi$ over $\Sigma \subset \Sigma'$ and equal to the identity elsewhere. The following theorem has previously been proved by Dörner–Geiges–Zehmisch [4]. The proof in the present paper uses slightly different methods (cf. §2).

**Theorem 4.** There is a cobordism $W$ with $\partial W = (-M) \sqcup M'$ and a symplectic form $\Omega$ on $W$ for which $(M, \xi)$ is a concave boundary component, and $\Omega$ induces $\omega'$ on the fibres of the fibration $M' \to S^1$.

For $n=1$ we could, for example, choose $\Sigma'$ to be the closed surface obtained by capping off the boundary components of $\Sigma$. Then Theorem 4 would recover one of the main results (Theorem 1.1) in [5]. The low-dimensional case ($n=1$) of Theorem 4 was, using different methods, already carried out in [16]. One may think of Theorem 4 as an extension of the result in [5], or [16] respectively, to higher dimensions.

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1A bordered Legendrian open book is a generalisation of the notion of an overtwisted disk to higher dimensions [3].
1. Preliminaries

1.1. Symplectic cobordisms. Suppose we are given a symplectic $2n$-manifold $(X, \omega)$, oriented by the volume form $\omega^n$, such that the oriented boundary $\partial X$ decomposes as $\partial X = (-M_-) \cup M_+$, where $-M_-$ stands for $M_-$ with reversed orientation. Suppose further that in a neighbourhood of $\partial X$ there is a Liouville vector field $Y$ for $\omega$, transverse to the boundary and pointing outwards along $M_+$, inwards along $M_-$. The 1-form $\alpha = i_Y \omega$ restricts to $TM_\pm$ as a contact form defining co-oriented contact structures $\xi_\pm$.

We call $(X, \omega)$ a (strong) symplectic cobordism from $(M_-, \xi_-)$ to $(M_+ , \xi_+)$, with convex boundary $M_+$ and concave boundary $M_-$. In case $(M_- , \xi_-)$ is empty $(X, \omega)$ is called a (strong) symplectic filling of $(M_+, \xi_+)$. If the Liouville vector field is defined not only in a neighbourhood of $\partial X$ but everywhere on $X$ we call the cobordism or the filling respectively exact.

A Stein manifold is an affine complex manifold, i.e. a complex manifold that admits a proper holomorphic embedding into $\mathbb{C}^N$ for some large integer $N$. By work of Grauert [11] a complex manifold $(X, J)$ is Stein if and only if it admits an exhausting plurisubharmonic function $\rho: X \to \mathbb{R}$. Eliashberg and Gromov’s symplectic counterparts of Stein manifolds are Weinstein manifolds.

A Weinstein manifold is a quadruple $(X, \omega, Z, \varphi)$, see [7], where $(X, \omega)$ is an exact symplectic manifold, $Z$ is a complete globally defined Liouville vector field, and $\varphi : X \to \mathbb{R}$ is an exhausting (i.e. proper and bounded below) Morse function for which $Z$ is gradient-like. Suppose $(X, \omega)$ is an exact symplectic cobordism with boundary $\partial X = (-M_-) \cup M_+$ and with Liouville vector field $Z$. We call $(X, \omega)$ Weinstein cobordism if there exists a Morse function $\varphi : X \to \mathbb{R}$ which is constant on $M_-$ and on $M_+$, has no boundary critical points, and for which $Z$ is gradient-like.

1.2. Open books. An open book decomposition of an $n$-dimensional manifold $M$ is a pair $(B, \pi)$, where $B$ is a co-dimension 2 submanifold in $M$, called the binding of the open book and $\pi : M \setminus B \to S^1$ is a (smooth, locally trivial) fibration such that each fibre $\pi^{-1}(\varphi)$, $\varphi \in S^1$, corresponds to the interior of a compact hypersurface $\Sigma_\varphi \subset M$ with $\partial \Sigma_\varphi = B$. The hypersurfaces $\Sigma_\varphi$, $\varphi \in S^1$, are called the pages of the open book.

In some cases we are not interested in the exact position of the binding or the pages of an open book decomposition inside the ambient space. Therefore, given an open book decomposition $(B, \pi)$ of an $n$-manifold $M$, we could ask for the relevant data to remodel the ambient space $M$ and its underlying open books structure $(B, \pi)$, say up to diffeomorphism. This leads us to the following notion.

An abstract open books is a pair $(\Sigma, \phi)$, where $\Sigma$ is a compact hypersurface with non-empty boundary $\partial \Sigma$, called the page and $\phi : \Sigma \to \Sigma$ is a diffeomorphism equal to the identity near $\partial \Sigma$, called the monodromy of the open book. Let $\Sigma(\phi)$ denote the mapping torus of $\phi$, that is, the quotient space obtained from $\Sigma \times [0,1]$ by identifying $(x,1)$ with $(\phi(x),0)$ for each $x \in \Sigma$. Then the pair $(\Sigma, \phi)$ determines a closed manifold $M_{(\Sigma, \phi)}$ defined by

$$\begin{equation}
M_{(\Sigma, \phi)} := \Sigma(\phi) \cup_{\text{id}} (\partial \Sigma \times D^2),
\end{equation}$$

where we identify $\partial \Sigma(\phi) = \partial \Sigma \times S^1$ with $\partial(\partial \Sigma \times D^2)$ using the identity map. Let $B \subset M_{(\Sigma, \phi)}$ denote the embedded submanifold $\partial \Sigma \times \{0\}$. Then we can define a
fibration \( \pi: M(\Sigma, \phi) \setminus B \to S^1 \) by

\[
\begin{bmatrix} x, \varphi \\ \theta, r e^{i \pi \varphi} \end{bmatrix} \mapsto [\varphi],
\]

where we understand \( M(\Sigma, \phi) \setminus B \) as decomposed in (1) and \([x, \varphi] \in \Sigma(\phi)\) or \([\theta, r e^{i \pi \varphi}] \in \partial \Sigma \times D^2 \subset \partial \Sigma \times \mathbb{C}\) respectively. Clearly \((B, \pi)\) defines an open book decomposition of \( M(\Sigma, \phi) \).

On the other hand, an open book decomposition \((B, \pi)\) of some \( n \)-manifold \( M \) defines an abstract open book as follows: identify a neighbourhood of \( B \) with \( B \times D^2 \) such that \( B = B \times \{0\} \) and such that the fibration on this neighbourhood is given by the angular coordinate, \( \varphi \) say, on the \( D^2 \)-factor. We can define a 1-form \( \alpha \) on the complement \( M \setminus (B \times D^2) \) by pulling back \( d \varphi \) under the fibration \( \pi \), where this time we understand \( \varphi \) as the coordinate on the target space of \( \pi \). The vector field \( \partial \varphi \) on \( \partial (M \setminus (B \times D^2)) \) extends to a nowhere vanishing vector field \( X \) which we normalise by demanding it to satisfy \( \alpha(X) = 1 \). Let \( \phi \) denote the time-1 map of the flow of \( X \). Then the pair \((\Sigma, \phi)\), with \( \Sigma = \pi^{-1}(0) \), defines an abstract open book such that \( M(\Sigma, \phi) \) is diffeomorphic to \( M \).

### 1.3. Compatibility

Let \( \Sigma \) denote a compact, \( 2n \)-dimensional manifold admitting an exact symplectic form \( \omega = d\beta \) and let \( Y \) denote the Liouville vector field defined by \( i_Y \omega = \beta \). Suppose that \( Y \) is transverse to the boundary \( \partial \Sigma \), pointing outwards. Given such a triple \((\Sigma, \omega, \phi)\) a construction of Giroux [10], cf. also [9, §7.3], produces a contact manifold \( M(\Sigma, \omega, \phi) \) whose contact structure is adapted to the open book in the following sense.

A positive contact structure \( \xi = \ker \alpha \) and an open book decomposition \((B, \pi)\) of an \((2n+1)\)-dimensional manifold \( M \) are said to be compatible, if the 2-form \( d\alpha \) induces a symplectic form on the interior \( \pi^{-1}(\varphi) \) of each page, defining its positive orientation, and the 1-form \( \alpha \) induces a positive contact form on \( B \).

Suppose we are given a contact structure \( \xi = \ker \alpha \) compatible to an open book decomposition \((B, \pi)\). The relevant data to remodel the open book and the compatible contact structure is given as follows: according to [4, Proposition 2] we may assume that in a sufficiently small neighbourhood \( B \times D^2 \) of the binding \( B \) our contact form \( \alpha \), after an isotopic modification, has the form

\[
h_1(r)\alpha|_{TB} + h_2(r)d\varphi,
\]

where \( h_1, h_2 \) are functions satisfying

**Step 1.** According to [4, Proposition 2] we may assume that near \( B \times \{0\} \) our contact form \( \alpha \), after an isotopic modification near \( B \times \{0\} \), has the form

\[
h_1(r)\alpha|_{TB} + h_2(r)d\varphi,
\]

where \( h_1, h_2 \) are functions satisfying **Step 2**.

**Remark 5.**

2. **Concatenation of open books and symplectic fibrations**

The following definitions will turn up in the proofs of all our main results: let \( \Sigma \) denote a compact, \( 2n \)-dimensional manifold admitting an exact symplectic form \( \omega = d\beta \) and let \( Y \) denote the Liouville vector field defined by \( i_Y \omega = \beta \). Suppose that \( Y \) is transverse to the boundary \( \partial \Sigma \), pointing outwards. Denote by \((r, x)\)
coordinates on a collar neighbourhood \((-\varepsilon, 0] \times \partial \Sigma\) induced by the negative flow corresponding to the Liouville vector field \(Y\). Let \(\varrho : \Sigma \to [0, \infty]\) be a function on \(\Sigma\), differentiable over the interior \(\text{Int}(\Sigma)\) of \(\Sigma\) and satisfying the following properties:

- \(\varrho \equiv 0\) over \(\Sigma \setminus ((-\varepsilon, 0] \times \partial \Sigma)\) with coordinates \((r, x)\),
- \(\frac{\partial \varrho}{\partial r} > 0\) and \(\frac{\partial \varrho}{\partial x} \equiv 0\) over \((-\varepsilon, 0] \times \partial \Sigma)\),
- \(\varrho \equiv \infty\) over \(\partial \Sigma\).

Note that over the collar neighbourhood \((-\varepsilon, 0] \times \partial \Sigma\) the vector field \(Y\) is gradient-like for \(\varrho\). In order to define the desired Liouville vector fields in our proofs we will need the following ingredient. Let \(g : [0, \varepsilon] \to \mathbb{R}\) be a functions satisfying the following properties:

- \(g(z) = 1\), for \(z \in [0, \varepsilon]\) near 0,
- \(g(z) = 0\), for \(z \in [0, \varepsilon]\) near \(\varepsilon\),
- \(g'(z) \leq 0\), for each \(z \in [0, \varepsilon]\).

We are now ready to construct the desired exact symplectic cobordism.

**Proof of Theorem 1**. The starting point for the desired cobordisms will be the space \(\Sigma \times \mathbb{R}^2\) with coordinates \((p; t, z)\). This space is symplectic with symplectic form

\[\Omega = \omega + dz \wedge dt.\]

Let \(P = \mathcal{P}_{e,E,C}\) denote the subset of \(\Sigma \times \mathbb{R}^2\) defined by

\[P := \{(p; t, z) : \varrho \leq 0, z^2 + t^2 \leq C^2 \text{ and } (t \mp E)^2 + z^2 \geq \varepsilon^2\},\]

where \(e, E, C \in \mathbb{R}\) are some potentially very large constants satisfying

\[E < (E + \varepsilon) < C.\]

The final choice of these constants will later ensure that our desired Liouville vector field \(Z\) will be transverse to the boundary of the cobordism. Consider the vector field \(Z' = Z'_E\) on \(\Sigma \times \mathbb{R}^2\) defined by

\[Z' = Y + X,\]

where \(X = (1 - f'(t))z \partial_z + f(t) \partial_t\) and \(f : \mathbb{R} \to \mathbb{R}\) is a function satisfying the following properties:

- \(f(\pm E) = f(0) = 0\),
- \(f'\) has exactly two zeros \(\pm t_0\) satisfying \(0 < t_0 < E\),
- \(|f'(t)| < 1\) for each \(t \in \mathbb{R}\), and
- \(\lim_{t \to \pm \infty} f(t) = \pm \infty.\)

An easy computation shows that \(X\) is a Liouville vector field on \((\mathbb{R}^2, dz \wedge dt)\) for any function \(f\). Hence \(Z'\) defines a Liouville vector field on \((\Sigma \times \mathbb{R}^2, \Omega)\).

We are now ready to define the desired symplectic cobordism \(W\). We cut \(P\) along \(\{z = 0\}\) and then reglue with respect to \(\phi_0\) and \(\phi_1\) as follows. Set \(P_\pm := P \cap \{z \geq 0\}\) and \(P_0 = P \cap \{z = 0\}\). Obviously \(P_0\) can be understood as part of the boundary of \(P_+\) as well as of \(P_-\). Now consider

\[P(\phi_0, \phi_1) := (P_+ \sqcup P_-)/_{\sim_\Phi},\]

where we identify with respect to the map \(\Phi : P_0 \to P_0\) (understanding the domain of definition of \(\Phi\) as part of the boundary of \(P_+\) and the target space as part of \(P_-\))
Figure 1. Left: Flow lines of the Liouville vector field $X$. Right: Construction of $P(\phi_0, \phi_1)$

given by

$$
\Phi(p; t, 0) := \begin{cases} 
(\phi_0(p); t, 0), & \text{for } t < -E, \\
(\phi_1^{-1}(p); t, 0), & \text{for } t > E, \\
(p; t, 0), & \text{for } |t| < E.
\end{cases}
$$

Note that, since $\phi_0$ and $\phi_1$ are symplectomorphisms of $(\Sigma, \omega)$ and $\Phi$ keeps the $t$-coordinates fixed $\Omega$ descends to a symplectic form on $P(\phi_0, \phi_1)$ which we will continue to denote by $\Omega$. We are now going to define a Liouville vector field $Z$ on $P(\phi_0, \phi_1)$. Without any loss of generality the symplectomorphisms $\phi_0$ and $\phi_1^{-1}$ can be chosen to be exact (cf. [9]), i.e. we have $\phi_0^* d\beta = d\phi_0$ and $(\phi_1^{-1})^* d\beta = d\phi_1$ defining functions $\phi_0$ and $\phi_1$ on $\Sigma$, unique up to adding a constant. Hence we may assume that $\phi_0$ and $\phi_1$ vanish over a neighbourhood of $\partial \Sigma$. To avoid confusing indices we will write $\Phi^* d\beta = d\phi$ to summarise these facts. Let $g: [0, \varepsilon] \to \mathbb{R}$ be the function as defined at the beginning of the present section. Over $P_+$ we define $Z = Z_E$ to be given as

$$
Z = \left( g(z) (T\Phi^{-1})(Y) + (1 - g(z)) Y \right) + X - g'(z) \varphi(p) \partial_t.
$$

To show that $Z$ is indeed a Liouville vector field we have to take a look at the Lie derivative of $\Omega$ along $Z$. With the help of the Cartan formula we compute

$$
\mathcal{L}_Z \Omega = d(g \Phi^* \beta + (1 - g(z)) \beta) + dz \wedge dt + d(g' \varphi dz)
$$

$$
= (dg \wedge (\Phi^* \beta) - dg \wedge \beta + g(\Phi^* \omega) + (1 - g(z)) \omega) + dz \wedge dt + g' \varphi \wedge dz
$$

$$
= (g' dz \wedge (\Phi^* \beta) - g' dz \wedge \beta + g \omega + (1 - g(z)) \omega) + dz \wedge dt + g' \varphi \wedge dz
$$

$$
= (g' dz \wedge d\varphi + \omega) + dz \wedge dt - g' dz \wedge d\varphi
$$

$$
= \omega + dz \wedge dt
$$

$$
= \Omega.
$$

Observe that we can extend $Z$ over $P_-$ by $Z'$. In particular $Z$ descends to a vector field on $P(\phi_0, \phi_1)$. Let $W' = W_{E,E,C}$ denote the subset of $\Sigma \times \mathbb{R}^2$ defined by

$$
W' := \{(p, z, t) : g^2 + z^2 + t^2 \leq C^2 \text{ and } g^2 + z^2 + (t \pm E)^2 \geq C^2\}
$$

and note that we have $P \subset W'$. Finally we define the symplectic cobordism $W = W_{C,E,E}$ by

$$
W := (W' \setminus P) \cup P(\phi_0, \phi_1).
$$
The boundary of $W$ decomposes as $\partial W = \partial_- W \sqcup \partial_+ W$, where we have

$$\partial_- W = \{ \varrho^2 + z^2 + (t \pm E)^2 = e^2 \} \quad \text{and} \quad \partial_+ W = \{ \varrho^2 + z^2 + t^2 = C^2 \}.$$ 

We do not have to worry about the well-definedness of the function $\varrho$ on $P(\phi_0, \phi_1) \subset W$ since $\phi_0$ and $\phi_1$ can be assumed to equal the identity over $(-\varepsilon, 0] \times \partial \Sigma$, which is the only region where $\varrho$ is non-trivial.

Observe that yet we cannot fully ensure that the Liouville vector field $Z$ is transverse to $\partial W$ pointing inwards along $\partial_- W$ and outwards along $\partial_+ W$. However the only problem is the last term in $Z$, namely the term $g'(z)\varphi(p) \partial_t$. Note that without this term the vector field would be transverse to the boundary as desired. We tame this deviation as follows: up to this point we have not fixed the constants $C, E, e$ yet. By choosing $C, E, e$ sufficiently large the deviation induced by $g'(z)\varphi(p) \partial_t$ becomes non-essential and the Liouville vector field $Z$ becomes transverse to $\partial W$ pointing inwards along $\partial_- W$ and outwards along $\partial_+ W$.

Finally observe that we indeed have $\partial_- W = M(\Sigma, \omega, \phi_0 \circ \phi_1)$ and $\partial_+ W = M(\Sigma, \omega, \phi_0 \circ \phi_1)$, which completes the proof.

\[\square\]

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**Figure 2.** Schematic picture of the symplectic cobordism constructed in Theorem 1.

As mentioned above we will now briefly sketch an alternative approach to Theorem 1 utilising a *generalised symplectic 1-handle* as defined in §4. A similar approach was independently followed by Avdek in [1].

**Sketch of the alternative approach.** Suppose we are given two $(2n+1)$-dimensional contact manifolds $(M_0, \xi_0)$ and $(M_1, \xi_1)$. Suppose further that they are associated with compatible open books $(\Sigma, \omega, \phi_0)$ and $(\Sigma, \omega, \phi_1)$ with equal pages. For $i = 0, 1$ let $\pi_i: M \setminus B \to S^1$ denote the induced fibrations. Note that the subsets $\pi_i^{-1}((-\varepsilon, \varepsilon)) \subset (M_i, \xi_i), i = 0, 1$, (after edge rounding) define an embedding

$$S^0 \times \text{Int} N(\Sigma) \hookrightarrow M_0 \sqcup M_1,$$

where $N(\Sigma)$ denotes a neighbourhood of $\Sigma$ as described in §4. We can understand this as the attaching region $N$ of a generalised 1-handle $H_{\Sigma}$ as described in §4. Attaching $H_{\Sigma}$ to the positive end of a symplectisation of $M_0 \sqcup M_1$ we end up with a cobordism whose positive end equals the contact manifold associated to $(\Sigma, \omega, \phi_0 \circ \phi_1)$, cf. Figure 3. \[\square\]
Note that, as in the previous case, it is possible to obtain the cobordism in Theorem 4 by the attachment of a generalised symplectic 1-handle (§4) to \( \Sigma' \times D^2 \) with symplectic form \( \omega' + dt \wedge dz \) and to the symplectisation of \((M, \xi)\). However in the present paper we provide a direct construction.

Proof of Theorem 4. Let \( \varrho: \Sigma \to [0, \infty) \) be a \( C^\infty \)-function on \( \Sigma \) as defined at the beginning of §2. We can extend this function by \( \infty \) over the rest of \( \Sigma' \). Analogous to the proof of Theorem 1 we consider the symplectic space \( \Sigma' \times R^2 \) with symplectic form \( \Omega = \omega' + dt \wedge dz \). Over \( \Sigma \times R^2 \subset \Sigma' \times R^2 \) we define a Liouville vector field \( Z' = Y + z \partial_z + 2t \partial_t \). Let \( A = A_{E,C} \) denote the subset of \( \Sigma' \times R^2 \) defined by

\[
A := \{(p, t, z): \varrho \leq 0 \text{ and } E^2 \leq z^2 + t^2 \leq C^2\},
\]

where \( E, C \in R \) are some potentially very large constants satisfying \( E < C \).

In analogy of the definition of \( P(\varphi_0, \varphi_1) \) in the proof of Theorem 1 we define \( A(\varphi): \)

\[
A(\varphi) := (A_+ \cup A_-)/\sim,
\]

where we identify with respect to the map \( \Phi: A_0 \to A_0 \) given by

\[
\Phi(p; t, 0) := \begin{cases} 
(\varphi(p); t, 0), & \text{for } t < 0, \\
(p; t, 0), & \text{for } t > 0.
\end{cases}
\]

Let \( W' = W'_{E,C} \) denote the subset of \( \Sigma \times R^2 \) defined by

\[
W' := \{(p, t, z): \varrho^2 + z^2 + t^2 \geq E^2 \text{ and } z^2 + t^2 \leq C^2\}
\]

and note that we have \( A \subset W' \). Finally we define the symplectic cobordism \( W \) by

\[
W := (W' \setminus A) \cup A(\varphi).
\]

Observe that \( \Omega \) descends to a symplectic form on \( W \). Furthermore we indeed have \( \partial W = (-M) \cup M' \).

It remains to define the desired Liouville vector field. The symplectomorphisms \( \varphi \) can be chosen to be exact (cf. [9]), i.e. we have \( \varphi^* \beta - \beta = d\varphi \) defining a function \( \varphi \) on \( \Sigma \), unique up to adding a constant. Hence we may assume that \( \varphi \) vanish over a neighbourhood of \( \partial \Sigma \). Let \( g, h: [-\varepsilon, 0] \to R \) be two functions as defined at the beginning of §2. We define a Liouville vector field \( Z \) on \( W_{\varrho \leq 2} \) by

\[
Z = (g(z) (T\Phi^{-1})(Y) + h(z) Y) + (z \partial_z + 2t \partial_t) - g'(z) \varphi(p) \partial_t.
\]
As in the proof of Theorem 1 for sufficiently large constants $C, E$ the Liouville vector field $Z$ is transverse to the lower boundary $\partial_- W = M(\Sigma, \omega, \phi)$ pointing inwards. Finally observe that $\Omega$ induces $\omega'$ on the fibres of the fibration $M' \to S^1$. □

3. Exact fillings of doubled open books

In the present section we show the strong fillability of contact manifolds obtained by a certain doubling construction. In short, we perform the binding sum of two open books with equal pages and inverse monodromies. To be more precise we do the following: let $(M_0, \xi_0)$ be a contact $(2n + 1)$-manifold supported by an open book $(\Sigma, \omega, \phi)$ and let $(M_1, \xi_1)$ be the contact manifold associated to the open book $(\Sigma, \omega, \phi^{-1})$. Denoting by $B$ the boundary of $\Sigma$ we can form a new contact manifold $(M', \xi')$ as follows: the binding $B$ defines a codimension-2 contact submanifolds $B_i \subset (M_i, \xi_i)$ for $i = 0, 1$. Their normal bundles $\nu B_0$ and $\nu B_1$ admit trivialisations induced by the pages of the respective open book decompositions of $M_0$ and $M_1$. Hence, we can perform the fibre connected sum, cf. [9, §7.4], along each copy of $B$ with respect to these trivialisations of the normal bundles and denote the result by $(M', \xi')$, i.e. denoting by $\Psi$ the fibre orientation reversing diffeomorphism of $B \times D^2 \subset B \times \mathbb{C}$ sending $(b, z)$ to $(b, \bar{z})$, using the notation in [9, §7.4], we define

$$(M', \xi') := (M_0, \xi_0) \#_\Psi (M_1, \xi_1).$$

The result $(M', \xi')$ defines a fibration over the circle with fibre given by

$$\Sigma' = (-\Sigma) \cup_B \Sigma.$$

Note that each fibre $\Sigma'$ defines a convex hypersurface, i.e. there is a contact vector field $X$ on $(M', \xi')$ which is transverse to the fibres. Furthermore for each fibre $\Sigma'$ the contact vector field $X$ is tangent to the contact structure exactly over $B$. We will refer to $(M', \xi')$ as a doubled open book and will sometimes denote it by

$$(\Sigma, \phi) \boxplus (\Sigma, \phi^{-1}).$$

Before we show the existence of the desired symplectic filling in the above statement, we show how it can be utilised to prove Proposition 3. We follow the outline presented in [15, Remark 4.1].
Proof of Corollary 4. Let \((M, \xi)\) be a contact manifold with a compatible open book decomposition \((B, \pi)\). Choose an arbitrarily small neighbourhood of the binding \(N_B \subset (M, \xi)\) and let \((M', \xi')\) denote the doubled open book associated to \((B, \pi)\). Obviously we can understand \((M, \xi) \setminus N_B\) as embedded in \((M', \xi')\). Since by Theorem 2 the doubled open book \((M', \xi')\) admits an exact filling we conclude that \((M, \xi) \setminus N_B\) cannot contain any local filling obstruction. 

It remains to show the existence of a symplectic filling for any doubled open book.

Proof of Theorem 2. Consider the symplectic space \(\Sigma \times \mathbb{R} \times [0, 2\pi]\) with coordinates \((p, t, z)\) and symplectic form \(\Omega = \omega + dz \wedge dt\). Set

\[
W := (\Sigma \times \mathbb{R} \times [0, 2\pi]) / \sim,
\]

where we identify with respect to the map \(\Phi: \Sigma \times \mathbb{R} \times \{0\} \to (\phi(p), t, z)\). Since \(\phi\) is a symplectomorphism of \((\Sigma, \omega)\) the symplectic form \(\Omega\) on \(\Sigma \times \mathbb{R} \times [0, 2\pi]\) descends to a symplectic form on \(W\) which we continue to denote by \(\Omega\). Note that the fibres of the projection \(W \to \mathbb{R}\) on the \(\mathbb{R}\)-coordinate are diffeomorphic to the mapping torus \(\Sigma(\phi)\).

Let \(g: \Sigma \to [0, \infty]\) be a \(C^\infty\)-function on \(\Sigma\) as defined at the beginning of \(\S 2\) and let \(W_C \subset W\), for some constant \(C > 0\), denote the subset defined by

\[
W_C := \{(p, t, z): g^2 + t^2 \leq C^2\}.
\]

The symplectomorphisms \(\phi\) can be chosen to be exact (cf. \([9]\)), i.e. we have \(\phi^*\beta - \beta = d\varphi\) defining a function \(\varphi\) on \(\Sigma\), unique up to adding a constant. Hence we may assume that \(\varphi\) vanishes near a neighbourhood of \(\partial\Sigma\). With \(g, h: [-\varepsilon, 0] \to \mathbb{R}\) as defined at the beginning of \(\S 2\) we define a Liouville vector field \(Z\) on \(W\) by

\[
Z = \left(g(z) (T\Phi^{-1})(Y) + h(z) Y\right) + t \partial_t - g'(z) \varphi(p) \partial_t.
\]

For sufficiently large \(C > 0\) this vector field is transverse to the boundary \(\partial W_C\) of the subset \(W_C\) pointing outwards. Finally observe that \(\partial W_C = M'\) and that \(Z\) indeed induces the contact structure \(\xi'\).

Sketch of the alternative approach. Appearing as a convex boundary of \((\Sigma \times D^2, \omega + dx \wedge dy)'\) the contact manifold \(M_{(\Sigma, \omega, \text{id})}\) associated to a trivial open book \((\Sigma, \omega, \text{id})\) obviously admits a symplectic filling. Recall that \(M_{(\Sigma, \alpha)}\) is given by \((\Sigma \times S^1) \cup_{\text{id}} (\partial\Sigma \times D^2)\). For any symplectomorphism \(\phi\) of \((\Sigma, \omega)\), equal to the identity near \(\partial\Sigma\), the part \((\Sigma \times S^1)\) can be described as

\[
(\Sigma \times S^1) \cong \left((\Sigma \times [0, 1] \cup \Sigma \times [2, 3])\right) / \sim,
\]

where we identify \((x, 3)\) with \((\phi(x), 0)\) and \((\phi(x), 1)\) with \((x, 2)\) for all \(x \in \Sigma\). Consider the subsets (after rounding the edges) \(\Sigma \times (\frac{1}{2} - \varepsilon, \frac{1}{2} + \varepsilon)\) and \(\Sigma \times (\frac{5}{2} - \varepsilon, \frac{5}{2} + \varepsilon)\) of \((\Sigma \times S^1) \subset M_{(\Sigma, \alpha)}\). They define an embedding

\[
S^0 \times \text{Int} N(\Sigma) \hookrightarrow M_{(\Sigma, \text{id})},
\]

where \(N(\Sigma)\) denotes a neighbourhood of \(\Sigma\) as described in \(\S 4\). We can understand this as the attaching region \(N\) of a generalised 1-handle \(H_2\) as described in \(\S 4\).
Attaching $H_\Sigma$ to the positive boundary of a symplectic filling of $M(\Sigma, \omega, \text{id})$ we end up with a cobordism whose positive end is easily identified as the contact manifold associated to the doubled open book $(\Sigma, \phi) \boxplus (\Sigma, \phi^{-1})$, cf. Figure 5.

Figure 5. The attachment of a generalised 1-handle yields a doubled open book.

4. A generalised symplectic 1-handle

We assume that the reader is familiar with the idea behind the symplectic handle constructions due to Eliashberg [6] and Weinstein [14]. For an introduction we point the reader to [9, §6.2]. Consider $\mathbb{R} \times \Sigma \times \mathbb{R}$ with coordinates $(t, p, z)$ and symplectic form $\Omega = \omega + dz \wedge dt$. The vector field $Z = Y + 2z \partial_z - t \partial_t$ defines a Liouville vector field for $\Omega$. Notice that $Z$ is gradient like for the function $g(t, p, z) := \varrho^2 + z^2 - \frac{1}{2}t^2$, where $\varrho: \Sigma \to [0, \infty)$ is a $C^\infty$-function on $\Sigma$ as defined at the beginning of §2. In particular the Liouville vector field $Z$ is transverse to the level sets of $g$ and hence induces contact structures on them. Denote by $N(\Sigma), N_0(\Sigma) \subset \Sigma \times \mathbb{R}$ the subsets defined by

$$N(\Sigma) := \{ \varrho^2 + z^2 \leq 1 \} \quad \text{and} \quad N_0(\Sigma) := \{ \varrho^2 + z^2 = 0 \}.$$ 

Let $\mathcal{N} \cong S^0 \times \text{Int}(N(\Sigma))$ and $\mathcal{N}_0 \cong S^0 \times N_0(\Sigma)$ denote the set of points $(t, p, z) \subset g^{-1}(-1)$ which lie on a flow line of $Z$ through $S^0 \times \text{Int}(N(\Sigma))$ and $S^0 \times N_0(\Sigma)$ respectively. The set $\mathcal{N}$ is going to play the role of the lower boundary. We now define our generalised symplectic 1-handle $H_\Sigma$ as the locus of points $(t, p, z) \in \mathbb{R} \times \Sigma \times \mathbb{R}$ satisfying the inequality

$$-1 \leq g(t, p, z) \leq 1$$

and lying on a flow line of $Z$ through a point of $\mathcal{N}$. Since the Liouville vector field $Y$ is transverse to the level sets of $g$, the 1-form

$$\alpha = i_Y \omega + 2z \, dt + t \, dz$$

induces a contact structure on the lower and upper boundary of $H_\Sigma$.

It is possible to perturb $H_\Sigma$ without changing the contact structure on the lower and upper boundary as follows. Let $\nu: \mathbb{R} \times \Sigma \times \mathbb{R} \to \mathbb{R}$ be an arbitrary function and
let $H_\Sigma^2$ denote the image of $H_\Sigma$ under the time-1 map of the flow corresponding to the vector field $\nu Z$. Sometimes it is more convenient to work with such a perturbed handle.

If we choose $(\Sigma, \omega)$ to be $D^{2n}$ with its standard symplectic form $dx \wedge dy$ and radial Liouville vector field the above 1-handle construction yields an ordinary symplectic 1-handle as described by Eliashberg \cite{eliashberg1992} and Weinstein \cite{weinstein1999}.

4.1. Attachment and of the handle and its result. Let $(M, \xi = \ker \alpha)$ be a $(2n+1)$-dimensional contact manifold. Suppose we are given a strict contact embedding of $\mathcal{N}$, endowed with the contact structure induced by $i_\xi \Omega$, into $(M, \xi = \ker \alpha)$. In the following we will describe the symplectic cobordism $W_{(M, \Sigma)}$ associated to the attachment of the handle $H_\Sigma$.

Note that for each point $x \in \mathcal{N} \setminus N_0$ there is a point $\mu(x) > 0$ in time such that the time-$\mu(x)$ map of the flow corresponding to the Liouville vector field $Z$ maps $x$ to the upper boundary of $H_\Sigma$. This defines a function $\mu: \mathcal{N} \setminus N_0 \to \mathbb{R}^+$ which we may, with respect to the above embedding of $\mathcal{N}$ in $(M, \xi = \ker \alpha)$, extend to a non-vanishing function over $M \setminus N_0$. We continue to denote this map $M \setminus N_0 \to \mathbb{R}^+$ by $\mu$. Consider the symplectisation $(\mathbb{R} \times M, d(e^t \alpha))$ and let $[0, \mu] \times M$ denote the subset defined by

$$[0, \mu] \times M = \{(r, x) \in \mathbb{R} \times M: 0 \leq r \leq \mu(x)\}.$$  

For any point $(0, x) \in \{0\} \times M \setminus N_0$ the time-$\mu(x)$ map of the flow corresponding to the Liouville vector field $\partial_r$ on $(\mathbb{R} \times M, d(e^t \alpha))$ maps $(0, x)$ to $(\mu(x), x)$. We define $W_{(M, \Sigma)}$ as follows: start with the disjoint union

$$[0, \mu] \times (M \setminus N_0) \sqcup H_\Sigma,$$

and define $W_{(M, \Sigma)}$ as the quotient space obtained by identifying $(r, x) \in \mathcal{N} \setminus N_0$ with the image of $x \in \mathcal{N} \subset H_\Sigma$ under the time-$r$ map of the flow corresponding to the Liouville vector field $Z$. This identification does actually respect the symplectic forms (cf. \cite{baldwin2010} Lemma 5.2.4) and we indeed end up with an exact symplectic cobordism $W_{(M, \Sigma)}$. The concave boundary component $\partial_- W_{(M, \Sigma)}$ is equal to $(M, \xi)$ whereas the convex component $\partial_+ W_{(M, \Sigma)}$ equals

$$\# H_\Sigma^2 (M, \xi) := (M, \xi) \setminus \left( S^0 \times \text{Int } N(\Sigma) \right) \cup_{\partial} (D^1 \times \partial N(\Sigma), \eta)$$

where $\eta$ denotes the kernel of the contact form $i_\gamma \omega + dt$. We will refer to $\# H_\Sigma^2 (M, \xi)$ as generalised connected sum. Let us recap the above discussion on the level of contact manifolds and finish with the following statement.

**Proposition 6.** There is an exact symplectic cobordism from $(M, \xi)$ to $\# H_\Sigma^2 (M, \xi)$. Furthermore if $(\Sigma, \omega)$ is Weinstein, then so is the cobordism. In particular we have the following. If $(M, \xi)$ admits a symplectic filling, then so does $\# H_\Sigma^2 (M, \xi).$  

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Mathematisches Institut, Universität zu Köln, Weyertal 86–90, 50931 Köln, Germany
E-mail address: mklukas@math.uni-koeln.de