Killing spinors of some supergravity solutions

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ABSTRACT

We compute explicitly the Killing spinors of some ten dimensional supergravity solutions. We begin with a 10d metric of the form $\mathbb{R}^{1,3} \times \mathcal{Y}_6$, where $\mathcal{Y}_6$ is either the singular conifold or any of its resolutions. Then, we move on to the Klebanov-Witten and Klebanov-Tseytlin backgrounds, both constructed over the singular conifold; and we also study the Klebanov-Strassler solution, built over the deformed conifold. Finally, we determine the form of the Killing spinors for the non-commutative deformation of the Maldacena-Núñez geometry.

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Chapter 1
Introduction

In this work we compute explicitly the Killing spinors of some ten dimensional supergravity solutions. The main interest of these backgrounds comes up in the context of the AdS/CFT correspondence established in [1] (see [2] for a review), for they are dual to four dimensional supersymmetric field theories [3]. Let us recall that the SUSY transformations for a background of ten dimensional type IIB supergravity can be parameterized in terms of a Majorana spinor $\epsilon$ made up of 32 real components, which is the number of charges forming the largest SUSY algebra. For a general background to be supersymmetric, we must require the vanishing of the SUSY transformations of the whole set of bosonic and fermionic fields of the theory. In principle, this will result in a reduction of the number of independent components of $\epsilon$, and therefore, of the supercharges entering the SUSY algebra. It is precisely this resulting spinor, subjected, in general, to some projections relating its components, what we call the Killing spinor of the background. Then, by computing the Killing spinors one can determine the amount of supersymmetry conserved by a certain geometry.

Let us point out that the knowledge of the explicit form of the Killing spinors allows one to apply the kappa symmetry [4] technique when looking for supersymmetric embeddings of different D-brane probes. The addition of D-branes, pioneered by Witten in [5], has become very fruitful in the AdS/CFT field, for it provides a way to uncover different stringy effects in the Yang-Mills (YM) theories. Indeed, by adding different D-brane probes to the supergravity backgrounds, one can study several interesting objects living in the dual field theories. For instance, in ref. [6] it has been shown that D3-brane probes wrapped over three cycles of the internal manifold $T^{1,1}$ in the so-called Klebanov-Witten model [7] (whose geometry, which we will describe in detail in chapter 3, is $AdS_5 \times T^{1,1}$) describe dibaryon operators in the $\mathcal{N} = 1$ superconformal YM theory living on the boundary of $AdS_5$ (see also refs. [8]-[11] for more results on dibaryons in this model and in some orbifold theories). Besides describing other exotic objects as domain walls (by means of D-brane probes of codimension one along the field theory dimensions, see ref. [6]), the addition of D-brane probes permits the introduction of open string degrees of freedom into the gauge/gravity correspondence. One can try to generalize the AdS/CFT correspondence by adding brane probes and identifying the fluctuations of the probe, which correspond to degrees of freedom of open strings connecting the probe and the branes that generated the background, with fundamental hypermultiplets of dynamical matter fields of the dual field theory [12]. Let us
mention that, following this program, in ref. [13] the explicit determination of the Killing spinors of the Klebanov-Witten model allowed us to systematically apply the kappa symmetry technique in order to study the possible supersymmetric embeddings in that background for D3, D5 and D7-brane probes.

The structure of this work is the following: in the next section of this chapter we present the SUSY variations of the IIB SUGRA fermionic fields. In chapter 2 we compute the Killing spinors for the different resolutions of the conifold by using a 10d background (constructed in [14]) arising from the uplift of a certain configuration in 8d gauged supergravity consisting of a D6-brane wrapping an $S^2$. Some results of this chapter, such as the form of the metrics of the singular and deformed conifold and the projections satisfied by their Killing spinors, are used in the following chapters where we deal with 10d SUGRA solutions constructed over the singular conifold or over its deformation. The aim of chapter 3 is to determine the Killing spinors of the Klebanov-Witten solution [7]; we solve the SUSY equations in a frame such that the Killing spinors do not depend on the angular coordinates of the conifold. Chapter 4 is devoted to the Klebanov-Tseytlin model [15]: we briefly introduce it and again we are able to write the Killing spinors in a frame where they do not depend on the angular coordinates of the conifold. In chapter 5 we deal with the Klebanov-Strassler solution [16], we describe it and by computing its Killing spinors we show that the requirement of preserving the same supersymmetries as in the solution corresponding to a D3-brane at the tip of the deformed conifold fixes the values of the three-forms to those found in ref [16]. In chapter 6 the Killing spinors of the non-commutative deformation of the Maldacena-Núñez solution [17] are explicitly computed. This calculation follows closely the one performed in [18] for the commutative case [19, 20] and, in fact, the Killing spinors of the non-commutative background can be written in terms of the ones of the commutative geometry by means of a rotation along the non-commutative plane. Finally, in chapter 7 we summarize our results and give some remarks.

The results of the computations performed in chapters 3 and 5 were published in ref. [13]; as it was said, the knowledge of the Killing spinors of the Klebanov-Witten model was essential for the kappa symmetry analysis carried out there. The Killing spinors of the Klebanov-Strassler model were included in the appendix of [13] as an starting point to extend the study of supersymmetric embeddings to that more interesting solution. The form of the Killing spinors of the non-commutative Maldacena-Núñez solution was published in the appendix of [21] where we studied the addition of open string degrees of freedom to that background.

Last, let us briefly comment on the ten dimensional IIB SUGRA solutions arising from the whole new class of 5d Sasaki-Einstein manifolds $Y^{p,q}$, recently constructed in [22, 23]. It was shown [24, 25] that the 10d backgrounds $AdS_5 \times Y^{p,q}$ are dual to four dimensional superconformal quiver gauge theories. The authors of [26], by performing a similar computation to the ones presented here, determined explicitly the Killing spinors of the 10d background in order to study the addition of brane probes. Recently, in [27] that study was extended to more general backgrounds of the form $AdS_5 \times L^{p,q,r}$, where $L^{p,q,r}$ is the more general family of 5d Sasaki-Einstein manifolds constructed in [28, 29].
1.1 SUSY transformations

In the backgrounds we will consider the fermionic fields (the dilatino and the gravitino) have vanishing expectation values, so the SUSY variations of the bosonic fields are trivially zero. Then, the Killing spinors are obtained by requiring the vanishing of the supersymmetry variations of the fermionic fields of the theory.

In the type IIB theory the spinor $\epsilon$ is actually composed of two Majorana-Weyl spinors $\epsilon_L$ and $\epsilon_R$ of well defined ten-dimensional chirality, which can be arranged as a two-component vector:

$$\epsilon = \begin{pmatrix} \epsilon_L \\ \epsilon_R \end{pmatrix}.$$  \hfill (1.1.1)

However, one can use complex spinors instead of working with the real two-component spinor written in eq. (1.1.1). In terms of $\epsilon_R$ and $\epsilon_L$ the complex spinor is simply:

$$\epsilon = \epsilon_L + i \epsilon_R.$$  \hfill (1.1.2)

For type IIB SUGRA with constant Ramond-Ramond scalar the supersymmetry variations are [30]:

$$\delta \lambda = \frac{i}{2} \partial_N \phi \Gamma^N \epsilon^* - \frac{i}{24} \mathcal{F}^{(3)}_{N_1 N_2 N_3} \Gamma^{N_1 N_2 N_3} \epsilon,$$

$$\delta \psi_M = D_M \epsilon + \frac{i}{1920} \mathcal{F}^{(5)}_{N_1 \cdots N_5} \Gamma^{N_1 \cdots N_5} \Gamma_M \epsilon +$$

$$+ \frac{1}{96} \mathcal{F}^{(3)}_{N_1 N_2 N_3} \left( \Gamma^M \right)_{N_1 N_2 N_3} - 9 \delta^N_M \Gamma^{N_2 N_3} \epsilon^*,$$  \hfill (1.1.3)

where $\Gamma^{N_1 \cdots N_n}$ stands for the antisymmetric product $\Gamma^{[N_1 \cdots N_n]}$. $\lambda(\psi)$ is the dilatino (gravitino), $\phi$ is the dilaton, $F^{(5)}$ is the selfdual Ramond-Ramond (RR) five-form, and $\mathcal{F}^{(3)}$ is the following complex combination of the Neveu-Schwarz-Neveu-Schwarz (NSNS) ($H$) and RR ($F^{(3)}$) three-forms:

$$\mathcal{F}^{(3)}_{N_1 N_2 N_3} = g_s \frac{1}{2} H_{N_1 N_2 N_3} + i g_s^{\frac{1}{2}} F^{(3)}_{N_1 N_2 N_3}.$$  \hfill (1.1.4)
Chapter 2

Killing spinors of the resolutions of the conifold

2.1 The resolutions of the conifold in 10d Supergravity

The conifold is a non-compact Calabi-Yau threefold with a conical singularity. Its metric can be written as $ds^2 = dr^2 + r^2 ds^2_{T^{1,1}}$, where $ds^2_{T^{1,1}}$ is the metric of the $T^{1,1}$ coset $(SU(2) \times SU(2))/U(1)$, which is the base of the cone. The $T^{1,1}$ space is an Einstein manifold whose metric can be written explicitly by using the fact that it is an $U(1)$ bundle over $S^2 \times S^2$. Actually, if $(\theta_1, \phi_1)$ and $(\theta_2, \phi_2)$ are the standard coordinates of the $S^2$'s and if $\psi \in [0, 4\pi)$ parameterizes the $U(1)$ fiber, the metric may be written as

$$ds^2_{T^{1,1}} = \frac{1}{6} \sum_{i=1}^{2} (d\theta_i^2 + \sin^2 \theta_i d\phi_i^2) + \frac{1}{9} \left( d\psi + \sum_{i=1}^{2} \cos \theta_i d\phi_i \right)^2 . \quad (2.1.1)$$

The conical singularity can be resolved in two different ways according to whether an $S^2$ or an $S^3$ is blown up at the singular point [31]. The former is known as the resolved conifold, while the latter is the deformed conifold. Both geometries appear naturally as supergravity duals of D6-branes wrapping an $S^2$. The natural framework for this problem is the eight dimensional Salam-Sezgin gauged supergravity [32] where the D6 become domain walls. The eleven dimensional geometry resulting from uplifting the 8d supersymmetric solution (remember that this 8d SUGRA comes from compactification of the 11d SUGRA on an $SU(2)$ manifold) consists of a fibration of the $S^2$ over the $S^3$ of the compactification due to the twisting that must be performed to get a supersymmetric solution. The resulting 11d metric [34] is of the form $R^{1,4} \times Y_6$, where $Y_6$ is a cone whose base is topologically $S^2 \times S^3$ and the radial coordinate of the cone is the distance to the domain wall in the 8d geometry.

It was shown in [14] that the singular, deformed and resolved conifold (and their generalizations with one additional parameter) are obtained as different solutions of the same system of differential equations, which follows from the vanishing of the 8d SUGRA supersymmetry variations $(\delta \chi_i = \delta \psi_\alpha = 0; \ i = 1, 2, 3; \ \alpha = 0, \ldots, 7)$ for an ansatz of the form:

$$ds^2_8 = e^2f dx^2_{1,4} + e^{2h} d\Omega^2_2 + dr^2 , \quad (2.1.2)$$

$$A^1 = g(r) \sigma^1 , \quad A^2 = g(r) \sigma^2 , \quad A^3 = \sigma^3 , \quad (2.1.3)$$
where \( d\Omega^2 = d\theta^2 + \sin^2 \theta_1 \, d\phi^2 \) is the metric of an \( S^2 \), \( f = f(r) \), \( h = h(r) \), and \( A^i \) \((i = 1, 2, 3)\) is the gauge field along the \( S^2 \), and we have defined \( \sigma^i \) \((i = 1, 2, 3)\) as the Maurer-Cartan one-forms, namely:

\[
\sigma^1 = d\theta_1 , \quad \sigma^2 = \sin \theta_1 \, d\phi_1 , \quad \sigma^3 = \cos \theta_1 \, d\phi_1 ,
\]

which satisfy \( d\sigma^i = -\frac{1}{2} \epsilon_{ijk} \, \sigma^j \wedge \sigma^k \).

When uplifting to eleven dimensions we impose that the unwrapped part of the metric corresponds to flat five dimensional Minkowski spacetime; thus we get the relation \( f = \frac{\phi}{3} \). Let \( \omega^i \) for \( i = 1, 2, 3 \) be a set of \( SU(2) \) left invariant one forms of the external \( S^3 \) satisfying \( d\omega^i = \frac{1}{2} \epsilon_{ijk} \, \omega^j \wedge \omega^k \). Then, the eleven dimensional metric is [32]:

\[
d s^2_{11} = d x^2_{1,4} + e^{2h-2\phi} \, d\Omega^2_2 + e^{-\frac{2\phi}{3}} \, dr^2 + 4e^{\frac{4\phi}{3} - 2\lambda} \left( \omega^1 + g \, \sigma^1 \right)^2 + \\
+ 4e^{\frac{4\phi}{3} + 2\lambda} \left( \omega^2 + g \, \sigma^2 \right)^2 + 4e^{\frac{4\phi}{3} - 4\lambda} \left( \omega^3 + \sigma^3 \right)^2,
\]

where \( \phi = \phi(r) \) is the dilaton of the 8d solution and \( \lambda = \lambda(r) \) is a scalar in the coset \( SL(3, \mathbb{R})/SO(3) \) of the 8d solution [34].

Therefore, once one imposes the vanishing of the 8d gauged SUGRA supersymmetry transformations, this uplifted metric is brought into the form \( \mathbb{R}^{1,4} \times \mathcal{Y}^6 \). \( \mathcal{Y}^6 \) being either the resolved, the deformed, or the singular conifold, according to the different solutions of the aforementioned first order system [14] resulting from the eight dimensional SUSY equations.

By performing a Kaluza-Klein reduction along one of the flat spatial directions of the metric (2.1.5), we get the following ten dimensional ansatz:

\[
d s^2_{10} = d x^2_{1,3} + e^{2h-\frac{2\phi}{3}} \, d\Omega^2_2 + e^{-\frac{2\phi}{3}} \, dr^2 + 4e^{\frac{4\phi}{3} + 2\lambda} \left( \omega^1 + g \, \sigma^1 \right)^2 + \\
+ 4e^{\frac{4\phi}{3} + 2\lambda} \left( \omega^2 + g \, \sigma^2 \right)^2 + 4e^{\frac{4\phi}{3} - 4\lambda} \left( \omega^3 + \sigma^3 \right)^2,
\]

with no fluxes and constant dilaton. The reduction leading to (2.1.6) was performed along one flat spatial direction. Therefore, we expect that by imposing the vanishing of the 10d SUSY transformations for this 10d metric, we will arrive at the same first order system as for the 8d background (2.1.2) [14]. Thus, the metric will be of the form \( \mathbb{R}^{1,3} \times \mathcal{Y}^6 \), where \( \mathcal{Y}^6 \) is the resolved, the deformed or the singular conifold, according to the different solutions of the system of equations. Moreover, since we are working directly in the uplifted 10d background, we will get the explicit form of the Killing spinors for the different ten dimensional metrics \( \mathbb{R}^{1,3} \times \mathcal{Y}^6 \).

### 2.2 Killing spinors

In this section we will compute the Killing spinors of the 10d background (2.1.6). By requiring the vanishing of the SUSY variations written in eq. (1.1.3) we will obtain some projections to be satisfied by the 10d spinor \( \epsilon \), together with some differential equations for the unknown functions entering the ansatz, namely \( g, \phi, \lambda \), and \( h \). The projections imposed on \( \epsilon \) reduce the number of supersymmetries while the different solutions of the differential equations give rise to the different resolutions of the conifold.
The vanishing of the SUSY variations (1.1.3) for the background (2.1.6) (which has no fluxes) results in the following equations:

\[ D_{\tilde{m}} \epsilon = 0 , \]  

where \( \tilde{m} \) runs along the basis formed by the differentials of the coordinates of the geometry and \( \epsilon \) is a 10d spinor. Henceforth we will use indices with tilde when referring to the basis formed by the differentials of the coordinates, i.e. \( e^{\tilde{m}} = dX^{\tilde{m}} \).

Since the geometry (2.1.6) comes up in the framework of 8d gauged supergravity, the Killing spinors should not depend on the coordinates of the \( SU(2) \) group manifold, and, due to the aforementioned SUSY twisting, neither should they depend on the remaining \( S^2 \).

Moreover, the ten dimensional metric can be expressed as the trivial product \( R^{1,3} \times Y_6 \), so the Killing spinors should not depend either on the flat space coordinates. Indeed, let us consider the natural one-form basis \( e^a \) for the ten dimensional metric (2.1.6):

\[ e^{x^\alpha} = dx^\alpha , \ (\alpha = 0, 1, 2, 3) , \ e^r = e^{-\frac{\phi}{3}} dr , \]

\[ e^1 = e^{h - \frac{\phi}{3}} d\theta_1 , \ e^2 = e^{h - \frac{\phi}{3}} \sin \theta_1 d\phi_1 , \]

\[ e^\hat{1} = 2e^{\frac{2\phi}{3} + \lambda} \left( \omega^1 + \sigma^1 \right) , \ e^\hat{2} = 2e^{\frac{2\phi}{3} + \lambda} \left( \omega^2 + \sigma^2 \right) , \]

\[ e^{\hat{3}} = 2e^{\frac{2\phi}{3} - 2\lambda} \left( \omega^3 + \sigma^3 \right) . \]  

Let us point out that the covariant derivative appearing in eq. (2.2.1) can be written as:

\[ D_{\tilde{m}} = \partial_{\tilde{m}} + \frac{1}{4} \omega_{\tilde{m}ab} \Gamma^{ab} , \]

where \( \partial_{\tilde{m}} \) denotes the usual partial derivative with respect to the coordinate \( X^{\tilde{m}} \) and \( \omega_{\tilde{m}ab} \) stands for the components of the spin connection one-form \( \omega^{ab} \), namely:

\[ \omega^{ab} = \omega_{\tilde{m}ab} dX^{\tilde{m}} . \]

The indices \( a, b \) run along the frame (2.2.2). So \( \Gamma^{ab} \) denotes the antisymmetrized product of two constant Dirac matrices \( \Gamma_a \) and \( \Gamma_b \), \( (a, b = x^a, r, 1, 2, \hat{1}, \hat{2}, \hat{3}) \) associated to that frame.

The spin connection one-form \( \omega^{ab} \) is defined by the Cartan equations:

\[ de^a + \omega^a_b \wedge e^b = 0 . \]  

Hence, in order to determine the different components of the spin connection, we insert the derivatives of the one-forms (2.2.2) and a generic ansatz for \( \omega^{ab} \) into eq. (2.2.4). As we will see, it will become useful to write \( \omega^{ab} \) in the frame (2.2.2), it takes the form:

\[ \omega^{x^\alpha} = 0 , \ (\alpha = 0, 1, 2, 3) , \]

\[ \omega^{1r} = e^{\frac{\phi}{3}} \left( h' - \frac{\phi'}{3} \right) e^1 + e^{\frac{2\phi}{3} + \lambda - h} g' e^1 , \ \omega^{2r} = e^{\frac{\phi}{3}} \left( h' - \frac{\phi'}{3} \right) e^2 + e^{\frac{2\phi}{3} + \lambda - h} g' e^2 , \]

\[ \omega^{1r} = e^{\frac{\phi}{3}} \left( \lambda' + \frac{2\phi'}{3} \right) e^1 + e^{\frac{2\phi}{3} + \lambda - h} g' e^1 , \ \omega^{2r} = e^{\frac{\phi}{3}} \left( \lambda' + \frac{2\phi'}{3} \right) e^2 + e^{\frac{2\phi}{3} + \lambda - h} g' e^2 , \]
\[ \omega^{3r} = e^{\frac{\phi}{3}} \left( \frac{2\phi'}{3} - 2\lambda \right) e^3, \quad \omega^{11} = -e^{\frac{4\phi}{3} + \lambda - h} g' e^r, \quad \omega^{22} = -e^{\frac{4\phi}{3} + \lambda - h} g' e^r, \]

\[ \omega^{21} = e^{\frac{\phi}{3} - h} \cot \theta e^2 + e^{\frac{4\phi}{3} - 2\lambda - 2h} \left( g^2 - 1 \right) e^3, \]

\[ \omega^{13} = e^{\frac{\phi}{3} - h} \cosh (3\lambda) g e^2 - \frac{1}{4} e^{\frac{4\phi}{3} - 4\lambda} e^2, \]

\[ \omega^{23} = \frac{1}{4} e^{\frac{4\phi}{3} - 4\lambda} e^3 - e^{\frac{\phi}{3} - h} \cosh (3\lambda) g e^1, \]

\[ \omega^{12} = \omega^{12} = -e^{\frac{\phi}{3} - h} \sinh (3\lambda) g e^3, \]

\[ \omega^{32} = -e^{\frac{\phi}{3} - h} \sinh (3\lambda) g e^1 - e^{\frac{4\phi}{3} - 2\lambda - 2h} \left( g^2 - 1 \right) e^1, \]

\[ \omega^{31} = e^{\frac{\phi}{3} - h} \sinh (3\lambda) g e^2 + e^{\frac{4\phi}{3} - 2\lambda - 2h} \left( g^2 - 1 \right) e^2. \]

(2.2.5)

The prime appearing in these expressions denotes the radial derivative (for instance \( \phi' = \frac{d\phi}{dr} \)). It is not difficult to switch to the basis formed by the differentials of the coordinates; one can write \( \omega^{ab} = E^c_{\hat{m}} \omega^{\hat{m}b} \), where \( E^c_{\hat{m}} \) are the coefficients appearing in the expression of the frame one-forms (2.2.2) in terms of the differentials of the coordinates, i.e. \( e^a = E^a_{\hat{n}} e^{\hat{n}} = E^a_{\hat{n}} dX^{\hat{n}} \).

Once we have determined the spin connection of the geometry, by writing explicitly eqs. (2.2.1) we will get a system of differential equations for \( g, \phi, \lambda, \) and \( h \), together with some algebraic constraints and some projections imposed on \( \epsilon \). Indeed, we start by subjecting the spinor to the following angular projection:

\[ \Gamma_{12} \epsilon = -\Gamma_{12} \epsilon, \]

(2.2.6)

which arises naturally [34] in the framework of the 8d gauged SUGRA when requiring that the D6-brane wraps a two-cycle inside a K3 manifold.

Then, since we are assuming that \( \epsilon \) only depends on \( r \), it will become easier to write the equations (2.2.1) directly in the indices running along the frame (2.2.2), i.e. \( D_a \epsilon = 0 \), resulting:

\[ \omega^{a\hat{b}}_{x\hat{a}} \Gamma_{a\hat{b}} \epsilon = 0, \]

(2.2.7)

\[ \omega^{a\hat{b}}_{\hat{1}} \Gamma_{a\hat{b}} \epsilon = \omega^{a\hat{b}}_{\hat{2}} \Gamma_{a\hat{b}} \epsilon = \omega^{a\hat{b}}_{\hat{3}} \Gamma_{a\hat{b}} \epsilon = \omega^{a\hat{b}}_{\hat{1}} \Gamma_{a\hat{b}} \epsilon = \omega^{a\hat{b}}_{\hat{2}} \Gamma_{a\hat{b}} \epsilon = 0, \]

(2.2.8)

and

\[ e^{\hat{a}} \left( \partial_r + \frac{1}{4} \omega^{a\hat{b}}_{\hat{r}} \Gamma_{a\hat{b}} \right) \epsilon = 0, \]

(2.2.9)
where in the last equation we have used that $D_r \epsilon = (E_r^{-1})^D_r \epsilon$. Since $\omega_a^{\alpha b} = 0$, eqs. (2.2.7) are trivially satisfied. Inserting the spin connection and using the projection (2.2.6) in the first equation of (2.2.8), namely $\omega_1^{ab} \Gamma_{ab} \epsilon = 0$, one gets:

\[
\left( \lambda' + \frac{2}{3} \phi' \right) \epsilon = \left[ e^{\phi + \lambda - h} g' \Gamma_{11} - \frac{1}{4} e^{-\phi - 4 \lambda} \Gamma_r \Gamma_{123} - e^{-h} \sinh (3 \lambda) g \Gamma_r \Gamma_{123} \right] \epsilon. \tag{2.2.10}
\]

The equation $\omega_2^{ab} \Gamma_{ab} \epsilon = 0$ yields again eq. (2.2.10). While from the third equation in (2.2.8) we get:

\[
\left( \frac{2}{3} \phi' - 2 \lambda' \right) \epsilon = \left[ \frac{1}{4} e^{-\phi} \left( e^{-4\lambda} - 2 e^{2\lambda} \right) - e^{\phi - 2\lambda - 2h} \left( g^2 - 1 \right) \right] \Gamma_r \Gamma_{123} \epsilon + 2 e^{-h} \sinh (3 \lambda) g \Gamma_r \Gamma_{123} \epsilon. \tag{2.2.11}
\]

The last two equalities in (2.2.8) render the same equation:

\[
\left( h' - \frac{\dot{\phi}'}{3} \right) \epsilon = \left[ - e^{\phi + \lambda - h} g' \Gamma_{11} + e^{-h} \cosh (3 \lambda) g \Gamma_r \Gamma_{123} + e^{\phi - 2\lambda - 2h} \left( g^2 - 1 \right) \right] \Gamma_r \Gamma_{123} \epsilon. \tag{2.2.12}
\]

One can combine equations (2.2.10) and (2.2.11) to get rid of $\lambda'$, resulting:

\[
\phi' \epsilon + e^{\phi + \lambda - h} g' \Gamma_{11} \epsilon + \left[ \frac{1}{2} e^{\phi - 2\lambda - 2h} \left( g^2 - 1 \right) + \frac{1}{8} e^{-\phi} \left( e^{-4\lambda} + 2 e^{2\lambda} \right) \right] \Gamma_r \Gamma_{123} \epsilon = 0. \tag{2.2.13}
\]

Then, from this last equation it is clear that the 10d spinor $\epsilon$ must satisfy the following projection [14]:

\[
\Gamma_r \Gamma_{123} \epsilon = - \left( \beta + \bar{\beta} \Gamma_{11} \right) \epsilon, \tag{2.2.14}
\]

where $\beta$ and $\bar{\beta}$ are functions of the radial coordinate given by

\[
\phi' = \left[ \frac{1}{2} e^{\phi - 2\lambda - 2h} \left( g^2 - 1 \right) + \frac{1}{8} e^{-\phi} \left( e^{-4\lambda} + 2 e^{2\lambda} \right) \right] \beta, \tag{2.2.15}
\]

\[
e^{\phi + \lambda - h} g' = \left[ \frac{1}{2} e^{\phi - 2\lambda - 2h} \left( g^2 - 1 \right) + \frac{1}{8} e^{-\phi} \left( e^{-4\lambda} + 2 e^{2\lambda} \right) \right] \bar{\beta}. \tag{2.2.16}
\]

Since $(\Gamma_r \Gamma_{123})^2 \epsilon = \epsilon$ and $\{\Gamma_r \Gamma_{123}, \Gamma_{11}\} = 0$, by squaring (2.2.14) one can check that $\beta^2 + \bar{\beta}^2 = 1$ and thus we can represent $\beta$ and $\bar{\beta}$ as

\[
\beta = \cos \alpha, \quad \bar{\beta} = \sin \alpha. \tag{2.2.17}
\]

Hence, the projection (2.2.14) can be written as

\[
\Gamma_r \Gamma_{123} \epsilon = -e^{\alpha \Gamma_{11}} \epsilon, \tag{2.2.18}
\]

and then, solved as

\[
\epsilon = e^{-\frac{\alpha}{2} \Gamma_{11}} \tilde{\epsilon}, \quad \Gamma_r \Gamma_{123} \tilde{\epsilon} = -\tilde{\epsilon}. \tag{2.2.19}
\]
Since we are working in type IIB SUGRA, the 10d spinors have well defined chirality. Then, they verify the following equality: $\Gamma_{x^0 \ldots x^3} \Gamma_r \Gamma_{12\bar{1}\bar{2}\bar{3}} \epsilon = -\epsilon$. Using this identity together with (2.2.17) and the two-cycle projection (2.2.6), the projection (2.2.14) can be rewritten as

$$\Gamma_{x^0 \ldots x^3} (\cos \alpha \Gamma_{12} - \sin \alpha \Gamma_{1\bar{2}}) \epsilon = \epsilon,$$

(2.2.20)

showing that the D6-brane is wrapping a non trivial two-cycle inside the six dimensional manifold $\mathcal{Y}_6$. This cycle mixes the $S^2$ of the eight dimensional geometry (2.1.2) with the external $S^3$ (along which, the reduction to 8d SUGRA was done). Thus, the phase $\alpha$ implements the twisting we mentioned in section 2.1 (below (2.1.1)).

Next, by inserting projection (2.2.14) and equation (2.2.15) into (2.2.11), one gets:

$$\{-2\lambda' + \left[-\frac{2}{3} e^{\phi-2\lambda-2h} \left(g^2 - 1\right) + \frac{1}{3} e^{-\phi} \left(e^{-4\lambda} - e^{2\lambda}\right)\right] \beta + 2e^{-h} \sinh (3\lambda) g \tilde{\beta}\} \epsilon =$$

$$= \left\{2e^{-h} \sinh (3\lambda) g \beta - \left[-e^{-\phi-2\lambda-2h} \left(g^2 - 1\right) + \frac{1}{4} e^{-\phi} \left(e^{-4\lambda} - e^{2\lambda}\right)\right] \tilde{\beta}\right\} \Gamma_{11} \epsilon , $$

(2.2.21)

which consists of an equation for $\lambda'$ and an algebraic constraint:

$$\lambda' = \left[-\frac{1}{3} e^{\phi-2\lambda-2h} \left(g^2 - 1\right) + \frac{1}{6} e^{-\phi} \left(e^{-4\lambda} - e^{2\lambda}\right)\right] \beta + e^{-h} \sinh (3\lambda) g \tilde{\beta} ,$$

(2.2.22)

$$e^{-h} \sinh (3\lambda) g \beta + \left[\frac{1}{2} e^{\phi-2\lambda-2h} \left(g^2 - 1\right) - \frac{1}{8} e^{-\phi} \left(e^{-4\lambda} - e^{2\lambda}\right)\right] \tilde{\beta} = 0 .$$

(2.2.23)

In order to get an equation for $h'$ we can use equation (2.2.13) to eliminate $\phi'$ from equation (2.2.12), hence we get:

$$h' \epsilon = -\frac{2}{3} e^{\phi+\lambda-h} g' \Gamma_{11} \epsilon + e^{-h} \cosh (3\lambda) g \Gamma_r \Gamma_{12\bar{1}\bar{2}\bar{3}} \epsilon +$$

$$+ \frac{1}{6} \left[5 e^{\phi-2\lambda-2h} \left(g^2 - 1\right) - \frac{1}{4} e^{-\phi} \left(e^{-4\lambda} + 2e^{2\lambda}\right)\right] \Gamma_r \Gamma_{1\bar{2}\bar{3}} \epsilon ,$$

(2.2.24)

which after inserting eq. (2.2.14) renders a differential equation for $h'$ and a new algebraic constraint:

$$h' = -e^{-h} \cosh (3\lambda) g \tilde{\beta} + \frac{1}{6} \left[-5 e^{\phi-2\lambda-2h} \left(g^2 - 1\right) + \frac{1}{4} e^{-\phi} \left(e^{-4\lambda} + 2e^{2\lambda}\right)\right] \beta ,$$

(2.2.25)

$$-e^{-h} \cosh (3\lambda) g \beta + \left[\frac{1}{2} e^{\phi-2\lambda-2h} \left(g^2 - 1\right) - \frac{1}{8} e^{-\phi} \left(e^{-4\lambda} + 2e^{2\lambda}\right)\right] \tilde{\beta} = 0 ,$$

(2.2.26)

where we have used eq. (2.2.16) to get rid of $g'$.

To sum up, from equations (2.2.8) we have got a system of differential equations, namely (2.2.15), (2.2.16), (2.2.22), and (2.2.25); two algebraic constraints: (2.2.23) and (2.2.26); and the projection (2.2.14). This projection is compatible with (2.2.6) and both leave unbroken eight supercharges. As it was shown in [14], the algebraic constraints have two different
solutions resulting in two truncations of the system of differential equations and therefore, in two different internal manifolds $Y_6$. One solution leads to the generalized resolved conifold and the other to the generalized deformed conifold.

It remains to determine the radial dependence of the 10d spinor; it will be fixed by equation (2.2.9), which for the spin connection (2.2.5) reduces to:

$$e^{\hat{\phi}} \epsilon' + \frac{1}{2} \left( \omega_{r11}^1 \Gamma_{11} + \omega_{r22}^2 \Gamma_{22} \right) \epsilon = 0 ,$$  \hfill (2.2.27)

where $\epsilon' = \frac{d\epsilon}{dr}$, and we have taken into account that $\omega_{ab}^b = e^{-\hat{\phi}} \omega_{a}^b$. By inserting projection (2.2.6) and the corresponding components of the spin connection into this last equation one arrives at

$$\epsilon' + e^{\phi + \lambda - h} g' \Gamma_{11} \epsilon = 0 ,$$  \hfill (2.2.28)

and after inserting (2.2.19), it results in the two following equations:

$$\epsilon' = 0 ,$$  \hfill (2.2.29)

$$\alpha' = -2e^{\phi + \lambda - h} g' .$$  \hfill (2.2.30)

This last equation determines the radial dependence of the phase $\alpha$, while (2.2.29) implies that the spinor $\hat{\epsilon}$ is independent of $r$. Therefore, the 10d Killing spinor $\epsilon$ can be written as:

$$\epsilon = e^{-\frac{2}{3} \Gamma_{11}} \hat{\epsilon} ,$$  \hfill (2.2.31)

where $\hat{\epsilon}$ is a constant 10d spinor satisfying the projections:

$$\Gamma_r \Gamma_{23} \hat{\epsilon} = -\hat{\epsilon} , \quad \Gamma_{12} \hat{\epsilon} = -\Gamma_{12} \hat{\epsilon} .$$  \hfill (2.2.32)

As mentioned above, both projections are compatible since $[\Gamma_r, \Gamma_{23}, \Gamma_{12}] = 0$. Thus, the 10d SUGRA solution (2.1.6) leaves unbroken eight supersymmetries.

### 2.3 Solving the equations

In this section we will sum up the solutions (obtained in [14]) for the system of differential equations and algebraic constraints we got in the last section (eqs. (2.2.15), (2.2.16), (2.2.22), (2.2.23), (2.2.25) and (2.2.26)). This system determines the geometry of the 6d internal part $Y_6$ of the ten dimensional geometry $R^{1,3} \times Y_6$ (2.1.6). In this approach we get the metrics of the generalized resolved and deformed conifold written in a form which will be very useful in the next chapters when computing the Killing spinors of several 10d backgrounds.

The algebraic constraints (2.2.23) and (2.2.26) can be combined to get:

$$\tan \alpha = \frac{\beta}{\bar{\beta}} = -2e^{\phi + \lambda - h} g = \frac{e^{-3\lambda - h} g}{e^{\phi - 2\lambda - 2h} (g^2 - 1) - \frac{1}{4} e^{\phi - 4\lambda}} .$$  \hfill (2.3.1)

The first part of this equation allows us to write $\alpha$ in terms of the remaining functions, while the last equality yields the following constraint:

$$g \left[ g^2 - 1 + \frac{1}{4} e^{-2\phi - 2\lambda + 2h} \right] = 0 ,$$  \hfill (2.3.2)
which clearly has two solutions. One of them is \( g = 0 \), corresponding to \( \tilde{\beta} = 0 \), \( \beta = 1 \) (then \( \alpha = 0 \)). In this case the system of differential equations reduces to the one studied in [34], whose integral leads to the generalized resolved conifold [37]:

\[
\begin{align*}
\forall \chi \neq 0, \quad \beta = 0, & \quad \tilde{\beta} = \alpha &= 0, \\
\frac{1}{\rho^2} \left[ (\rho^2 + 6a^2) \left( \rho^2 + \sin^2 \theta_1 \rho^2 \right) + \rho^2 \left( \rho^2 + \sin^2 \theta_2 \rho^2 \right) \right],
\end{align*}
\]

(2.3.3)

with \( \kappa(\rho) \) being:

\[
\kappa(\rho) = \frac{\rho^6 + 9a^2 \rho^4 - b^6}{\rho^6 + 6a^2 \rho^4}.
\]

(2.3.4)

where \( a \) and \( b \) are constants of integration. In equation (2.3.3) \( \rho \) is a new radial variable and \( (\theta_2, \phi_2, \psi) \) are the angular coordinates of the external \( S^3 \). We have taken into account that in terms of these coordinates the left-invariant one-forms of the three-sphere (referred to above eq. (2.1.5)) can be written as

\[
\begin{align*}
w^1 &= \sin \psi \sin \theta_2 d\phi_2 + \cos \psi d\theta_2, \\
w^2 &= -\cos \psi \sin \theta_2 d\phi_2 + \sin \psi d\theta_2, \\
w^3 &= d\psi + \cos \theta_2 d\phi_2,
\end{align*}
\]

(2.3.5)

with \( \theta_2 \in [0, \pi] \), \( \phi_2 \in [0, 2\pi) \) and \( \psi \in [0, 4\pi) \). So (2.3.3) can be equivalently written as

\[
\begin{align*}
\forall \chi \neq 0, \quad \beta = 0, & \quad \tilde{\beta} = \alpha &= 0, \\
\frac{1}{\rho^2} \left[ (\rho^2 + 6a^2) \left( \sigma^3 + \omega^3 \right) + \rho^2 \left( \sigma^3 + \omega^3 \right) \right],
\end{align*}
\]

(2.3.6)

The constants of integration \( a \) and \( b \) (appearing in (2.3.4)) provide the generalized resolution of the conifold singularity [35]-[37]. The case \( b = 0 \) corresponds to the resolved conifold: it is easy to see that for \( \rho = 0 \) we get an \( S^2 \) of finite size \( a^2 \) instead of a singularity. For \( a = 0 \), \( b = 0 \) we get back the metric of the singular conifold written in the following form:

\[
\begin{align*}
\forall \chi \neq 0, \quad \beta = 0, & \quad \tilde{\beta} = \alpha &= 0, \\
\frac{1}{\rho^2} \left[ (\rho^2 + 6a^2) \left( \sigma^3 + \omega^3 \right) + \rho^2 \left( \sigma^3 + \omega^3 \right) \right],
\end{align*}
\]

(2.3.7)

The second solution of the constraint (2.3.2) leads to a non trivial relation between \( g \) and the remaining functions of the ansatz, namely:

\[
g^2 = 1 - \frac{1}{4} e^{-2\phi - 2\lambda + 2h}.
\]

(2.3.8)

The corresponding values of \( \beta \) and \( \tilde{\beta} \) are:

\[
\beta = \frac{1}{2} e^{-\phi - \lambda + h}, \quad \tilde{\beta} = -g.
\]

(2.3.9)
Plugging these results into the differential equations (2.2.15), (2.2.22), and (2.2.25) one arrives at the first order system:

\[
\phi' = \frac{1}{8} e^{-2\phi+\lambda+h}, \quad \lambda' = \frac{1}{24} e^{-2\phi+\lambda+h} - \frac{1}{2} e^{3\lambda-h} + \frac{1}{2} e^{-3\lambda-h}, \\
h' = -\frac{1}{12} e^{-2\phi+\lambda+h} + \frac{1}{2} e^{3\lambda-h} + \frac{1}{2} e^{-3\lambda-h},
\]

(2.3.10)

while from (2.2.16) one gets:

\[
g' = -\frac{1}{4} e^{-2\phi+\lambda+h} g.
\]

(2.3.11)

These equations can be straightforwardly solved, resulting:

\[
e^{\phi} = \hat{\mu} (\cosh \tau)^{\frac{3}{2}}, \\
e^{\lambda} = \left(\frac{3}{2}\right)^{\frac{1}{6}} (\cosh \tau)^{\frac{1}{2}} K(\tau)^{\frac{1}{2}}, \\
e^{h} = 2^{\frac{5}{6}} 3^{\frac{1}{9}} \hat{\mu} \frac{\sinh \tau}{(\cosh \tau)^{\frac{1}{3}}} K(\tau)^{\frac{1}{2}}, \\
g = \frac{1}{\cosh \tau},
\]

(2.3.12)

with

\[
K(\tau) = \frac{(\sinh(2\tau) - 2\tau + C)^{\frac{1}{2}}}{2^{\frac{1}{6}} \sinh \tau},
\]

(2.3.13)

\(\hat{\mu}\) and \(C\) are constants of integration and \(\tau\) is a new radial coordinate defined by means of the differential equation:

\[
d\tau = \frac{1}{2} e^{2\lambda-\phi} dr.
\]

(2.3.14)

After inserting the solution (2.3.12) the 10d metric (2.1.6) becomes:

\[
ds_{10}^2 = dx_{1,3}^2 + ds_6^2,
\]

(2.3.15)

with

\[
ds_6^2 = \frac{1}{2} \hat{\mu} \frac{1}{3} K(\tau) \left[ \frac{1}{3 K(\tau)^3} \left( d\tau^2 + (w^1 + \sigma^1)^2 \right) + \frac{\sinh^2 \tau}{2 \cosh \tau} \left( (\sigma^1)^2 + (\sigma^2)^2 \right) + \frac{\cosh \tau}{2} \left[ \left( w^1 + \frac{\sigma^1}{\cosh \tau} \right)^2 + \left( w^2 + \frac{\sigma^2}{\cosh \tau} \right)^2 \right] \right],
\]

(2.3.16)
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which is the metric of the generalized deformed conifold [36]. For $C = 0$ it describes the deformed conifold, with $\mu$ (which is just: $\mu = 2^{\frac{3}{2}} \frac{3}{2} \tilde{\mu}$) being the deformation parameter. It is not difficult to write this 6d metric in the standard form of [16]:

$$ds_6^2 = \frac{1}{2} \mu \frac{4}{3} K(\tau) \left[ \frac{1}{3K(\tau)^3} \left( d\tau^2 + (g^5)^2 \right) + \sinh^2 \left( \frac{\tau}{2} \right) \left( (g^3)^2 + (g^4)^2 \right) + \cosh^2 \left( \frac{\tau}{2} \right) \left( (g^3)^2 + (g^4)^2 \right) \right],$$  

(2.3.17)

where we have defined the following set of one-forms:

$$g^1 = \frac{1}{\sqrt{2}} (\omega^2 - \sigma^2), \quad g^2 = \frac{1}{\sqrt{2}} (\sigma^1 - \omega^1),$$

$$g^3 = \frac{-1}{\sqrt{2}} (\sigma^2 + \omega^2), \quad g^4 = \frac{1}{\sqrt{2}} (\sigma^1 + \omega^1),$$

$$g^5 = \sigma^3 + \omega^3.$$  

(2.3.18)

Furthermore, one can easily see that for $\tau \to 0$ the metric of the deformed conifold degenerates into $d\Omega_3^2 = \frac{1}{2} \mu \frac{4}{3} \left( \frac{\tau}{3} \right)^{\frac{2}{3}} \left[ \frac{1}{2} (g^5)^2 + (g^3)^2 + (g^4)^2 \right]$, which, as expected, is the metric of a round $S^3$.

2.3.1 Killing spinors of the deformed conifold

It will become useful to write down explicitly the Killing spinors of the 10d metric $R^{1,3} \times Y_6$ when $Y_6$ corresponds to the deformed conifold. One just have to insert the particular solution (2.3.12) corresponding to the deformed conifold into the general expression for the Killing spinors written in equation (2.2.31). Thus, one gets:

$$\epsilon = e^{-\frac{\mu}{2} \Gamma_{11}} \eta,$$  

(2.3.19)

where $\eta$ is a constant 10d spinor satisfying the projections

$$\Gamma_7 \Gamma_{123} \eta = -\eta, \quad \Gamma_{12} \eta = -\Gamma_{12} \eta,$$  

(2.3.20)

and the angle $\alpha$ is given by:

$$\sin \alpha = -\frac{1}{\cosh \tau}, \quad \cos \alpha = \frac{\sinh \tau}{\cosh \tau}.$$  

(2.3.21)

As before, $\Gamma_a = \left( x^a, \tau, 1, 2, 1, 2, 3 \right)$ are constant Dirac matrices associated to the frame (2.2.2), which for the particular solution (2.3.12) becomes:

$$e^{\alpha} = dx^\alpha, (\alpha = 0, 1, 2, 3), \quad e^\tau = \frac{\sqrt{6} \mu}{K(\tau)} \, d\tau,$$
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\[ e^i = \frac{\mu^2 \sqrt{K(\tau)}}{2} \frac{\sinh \tau}{\sqrt{\cosh \tau}} \sigma^i, \quad (i = 1, 2), \]

\[ e^\hat{i} = \frac{\mu^2 \sqrt{K(\tau)}}{2} \sqrt{\cosh \tau} \left( w^i + \frac{\sigma^i}{\cosh \tau} \right), \quad (i = 1, 2), \]

\[ e^\hat{3} = \frac{\mu^2}{\sqrt{6} K(\tau)} (w^3 + \sigma^3). \]  

(2.3.22)
CHAPTER 2. KILLING SPINORS OF THE RESOLUTIONS OF THE CONIFOLD
Chapter 3

Killing spinors of the Klebanov-Witten model

3.1 Introduction

In this chapter we will thoroughly present the computation of the Killing spinors of the Klebanov-Witten (KW) model [7]. We will briefly introduce the background and, using some results of the previous chapter, we will construct a one-form frame in which we expect that the Killing spinors do not depend on the angular coordinates of the conifold. Before solving the equations resulting from the vanishing of the SUSY variations, we will have to express the fields of the model in that new frame and also determine the form of the spin connection. Finally, in order to get the explicit form of the Killing spinors of the KW model when global coordinates are used for the $AdS_5$ part of the metric, we will repeat the calculations for the corresponding one-form frame. In both cases we will be able to write the Killing spinors of the theory in terms of a constant 10d spinor satisfying two independent (and compatible) projections, which reduce the number of independent components of the spinor and thus, the number of unbroken supercharges, from 32 to 8 real components as it was expected.

This calculation was schematically published in ref. [13], since the explicit expression of the Killing spinors was essential for the kappa symmetry analysis carried out there.

3.1.1 The Klebanov-Witten model

The so-called Klebanov-Witten background is constructed in ref. [7] by placing a stack of $N$ D3-branes at the apex of the singular conifold. By adding four Minkowski coordinates to the conifold we construct a Ricci flat ten dimensional metric. Let us now place a stack of $N$ coincident D3-branes extended along the Minkowski coordinates and located at the singular point of the conifold. The resulting IIB supergravity solution is the KW model. The corresponding near-horizon metric and Ramond-Ramond selfdual five-form are given by

$$ds_{10}^2 = [h(r)]^{-\frac{1}{2}} dx_{1,3}^2 + [h(r)]^{\frac{1}{2}} (dr^2 + r^2 ds_{T4,1}^2),$$
\[ h(r) = \frac{L^4}{r^4}, \]
\[ g_s F^{(5)} = d^4x \wedge dh^{-1} + \text{Hodge dual}, \]
\[ L^4 = \frac{27}{4} \pi g_s N' \alpha'^2. \] (3.1.1)

By plugging the explicit form of the warp factor into the metric, it can be written as
\[ ds^2_{10} = \frac{r^2}{L^2} dx_{1,3}^2 + \frac{L^2}{r^2} dr^2 + L^2 ds^2_{T^{1,1}}, \] (3.1.2)
which corresponds to the \( AdS_5 \times T^{1,1} \) space.

It was shown in ref. [7] that the gauge theory dual to this supergravity background is an \( \mathcal{N} = 1 \) superconformal field theory with some matter multiplets.

### 3.2 Killing spinors

To obtain the explicit form of the Killing spinors, one has to look at the supersymmetry variations of the dilatino and gravitino (see eq. (1.1.3)). Since the dilaton is constant and there is no three-form flux, the variation of the dilatino vanishes trivially (\( \delta \lambda = 0 \)). We are left with the equations:
\[ \delta \psi_M = D_M \epsilon + \frac{i}{1920} F^{(5)}_{N_1 \cdots N_5} \Gamma^{N_1 \cdots N_5} \Gamma_M \epsilon = 0. \] (3.2.1)

The final result of the calculation is greatly simplified if we choose the basis of the frame one-forms that arises naturally when the \( T^{1,1} \) metric is written as in eq. (2.3.7) of the previous chapter, namely:
\[ ds^2_{T^{1,1}} = \frac{1}{6} \left( (\sigma^1)^2 + (\sigma^2)^2 + (w^1)^2 + (w^2)^2 \right) + \frac{1}{9} (w^3 + \sigma^3)^2, \] (3.2.2)
with the one-forms \( \sigma^i \) and \( \omega^i \) being given by equations (2.1.4) and (2.3.5). Let us recall that this form of writing the \( T^{1,1} \) metric comes up in the framework of the eight dimensional gauged supergravity obtained from a Scherk-Schwarz reduction of eleven dimensional supergravity on an \( SU(2) \) group manifold [32]. Indeed, it was obtained as the gravity dual of D6-branes wrapping an \( S^2 \) inside a K3 manifold [34]. Then, from the consistency of the reduction leading to the gauged supergravity, the Killing spinors should not depend on the coordinates of the \( SU(2) \) external manifold and, actually, in the one-form basis we will use they do not depend on any angular coordinate of the \( T^{1,1} \) space. Accordingly, let us consider the following frame for the ten dimensional metric (3.1.1):
\[ e^{x_\alpha} = \frac{r}{L} dx_\alpha, \quad (\alpha = 0, 1, 2, 3), \quad e^r = \frac{L}{r} dr, \]
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\[ e^i = \frac{L}{\sqrt{6}} \sigma^i, \quad (i = 1, 2), \]

\[ e^3 = \frac{L}{\sqrt{6}} w^i, \quad (i = 1, 2), \]

\[ e^\hat{3} = \frac{L}{3} (w^3 + \sigma^3). \quad (3.2.3) \]

In this frame, the selfdual RR five-form reads:

\[ g_s F^{(5)} = \frac{4}{L} \left( e^{x_0} \wedge e^{x_1} \wedge e^{x_2} \wedge e^{x_3} \wedge e^r + e^1 \wedge e^2 \wedge e^1 \wedge e^2 \wedge e^3 \right) . \quad (3.2.4) \]

3.2.1 Spin connection

Let us recall that, as we have mentioned in the previous chapter, the covariant derivative appearing in the SUSY equations (3.2.1) can be written as

\[ D_\tilde{m} = \partial_\tilde{m} + \frac{1}{4} \omega^a_b \Gamma_{ab}, \quad (3.2.5) \]

in the frame formed by the differentials of the coordinates. So \( \partial_\tilde{m} \) denotes the derivative with respect to \( X^{\tilde{m}} \) and, as before, \( \omega^a_b \) stands for the components of the spin connection one-form in that basis, namely:

\[ \omega^a_b = \omega^a_b dX^{\tilde{m}}. \quad (3.2.6) \]

Then, in order to solve equations (3.2.1) we need the spin connection one-form \( \omega^{a b} \) of the background (where \( a \) and \( b \) are indices running along the one-form basis (3.2.3)). We will compute the spin connection for a metric of the form (3.1.1) but with a generic warp factor \( \tilde{h}(r) \) instead of \( h(r) = \frac{L^4}{r^4} \). Thus, the corresponding frame is:

\[ \tilde{e}^{x^\alpha} = \tilde{h}^{-\frac{1}{4}} dx^\alpha, \quad (\alpha = 0, 1, 2, 3), \quad \tilde{e}^r = \tilde{h}^{\frac{1}{4}} dr, \]

\[ \tilde{e}^i = \tilde{h}^{\frac{1}{4}} \frac{r}{\sqrt{6}} \sigma^i, \quad (i = 1, 2), \]

\[ \tilde{e}^3 = \tilde{h}^{\frac{1}{4}} \frac{r}{\sqrt{6}} w^i, \quad (i = 1, 2), \]

\[ \tilde{e}^\hat{3} = \tilde{h}^{\frac{1}{4}} \frac{r}{3} (w^3 + \sigma^3), \quad (3.2.7) \]

which does not only correspond to the current background (for \( \tilde{h}(r) = \frac{L^4}{r^4} \)), but it also describes the Klebanov-Tseytlin metric, where \( \tilde{h}(r) \) is a more involved function of the radial coordinate as one will see in the next chapter. Let us call \( \tilde{\omega}^{a b} \) to the spin connection corresponding to the generic frame (3.2.7); substituting the derivatives of the one-forms of the frame (3.2.7) together with a generic ansatz for \( \tilde{\omega}^{a b} \) into the Cartan equations (2.2.4) we
get:

\[ \tilde{\omega}^{x^\alpha} = (\tilde{h}^{-\frac{1}{4}})' \tilde{e}^{x^\alpha}, \quad (\alpha = 0, 1, 2, 3), \]

\[ \tilde{\omega}^{s^r} = \tilde{h}^{-\frac{1}{4}} \left( \frac{1}{r} + \frac{1}{4} \tilde{h}' \tilde{h}^{-1} \right) \tilde{e}^s, \quad (s = 1, 2, \hat{1}, \hat{2}, \hat{3}), \]

\[ \tilde{\omega}^{12} = \frac{1}{r} \tilde{h}^{-\frac{1}{4}} \tilde{e}^3 - \frac{\sqrt{6}}{r} \cot \theta_1 \tilde{h}^{-\frac{1}{4}} \tilde{e}^2, \]

\[ \tilde{\omega}^{\hat{1}\hat{2}} = -\frac{1}{r} \tilde{h}^{-\frac{1}{4}} \tilde{e}^2, \quad \tilde{\omega}^{2\hat{3}} = \frac{1}{r} \tilde{h}^{-\frac{1}{4}} \tilde{e}^1, \]

\[ \tilde{\omega}^{\hat{3}\hat{2}} = \frac{1}{r} \tilde{h}^{-\frac{1}{4}} \tilde{e}^1, \quad \tilde{\omega}^{31} = -\frac{1}{r} \tilde{h}^{-\frac{1}{4}} \tilde{e}^2. \quad (3.2.8) \]

We have expressed the resulting one-form in the frame (3.2.7), for, as one will see below, it will be more useful to work directly in that frame. Applying this result to the present background, i.e. plugging \( \tilde{h}(r) = \frac{L}{r^4} \) into (3.2.8), the spin connection of the Klebanov-Witten background, written directly in the frame (3.2.3), reads:

\[ \omega^{x^\alpha} = \frac{1}{L} \tilde{e}^{x^\alpha}, \quad (\alpha = 0, 1, 2, 3), \]

\[ \omega^{12} = \frac{1}{L} \tilde{e}^3 - \frac{\sqrt{6}}{L} \cot \theta_1 \tilde{e}^2, \]

\[ \omega^{\hat{1}\hat{2}} = \frac{2}{L} \tilde{e}^3 - \frac{\sqrt{6}}{L} \cot \theta_1 \tilde{e}^2, \]

\[ \omega^{2\hat{3}} = \frac{1}{L} \tilde{e}^1, \quad \omega^{\hat{2}\hat{3}} = -\frac{1}{L} \tilde{e}^2, \]

\[ \omega^{31} = -\frac{1}{L} \tilde{e}^2, \quad \omega^{3\hat{2}} = \frac{1}{L} \tilde{e}^1. \quad (3.2.9) \]

One should keep in mind that the components in the coordinate basis, i.e. (3.2.6), can be easily computed in terms of the ones in (3.2.9): \( \omega^{a\hat{b}} = E_{\hat{m}}^c \omega^{a\hat{b}} \). \( E_{\hat{m}}^c \) are the coefficients appearing in the expression of the frame one-forms (3.2.3) in terms of the differentials of the coordinates: \( e^a = E_{\hat{m}}^a d\tilde{X}^{\hat{m}} \).

### 3.2.2 Determining the Killing spinors

Once we have computed the form of the spin connection, we can go back to equations (3.2.1). After substituting the selfdual five-form (3.2.4) they become:

\[ D_M \epsilon + \frac{i}{4L} \left( \Gamma^{x^0 x^1 x^2 x^3 r} + \Gamma^{121\hat{2}\hat{3}} \right) \Gamma_M \epsilon = 0. \quad (3.2.10) \]
3.2. KILLING SPINORS

\( \Gamma^a \), \( (a = x^0, r, 1, 2, \hat{1}, \hat{2}, \hat{3}) \) are constant Dirac matrices associated to the frame (3.2.3). Using the identity satisfied by the chiral 10d spinors: \( \Gamma_{x^0 \ldots x^3 \Gamma r} \Gamma_{12\hat{1}\hat{2}\hat{3}} \epsilon = -\epsilon \), these last equations can be written as

\[
D_\mu \epsilon + \frac{i}{2L} \Gamma^{x^0 x^1 x^2 x^3 r} \Gamma_\mu \epsilon = 0 \,, \quad (\mu = x^0, x^1, x^2, x^3, r) \,, \tag{3.2.11}
\]

\[
D_s \epsilon + \frac{i}{2L} \Gamma^{12\hat{1}\hat{2}\hat{3}} \Gamma_s \epsilon = 0 \,, \quad (s = 1, 2, \hat{1}, \hat{2}, \hat{3}) \,, \tag{3.2.12}
\]

working directly in the frame (3.2.3).

Since the spin connection does not mix \( AdS_5 \) with \( T^{1,1} \) components, we can solve these two sets of equations separately, though the projections we get from both sets must be compatible. Let us begin with the \( AdS_5 \) equations (3.2.11); we expect the spinor to depend on the \( AdS_5 \) coordinates so we will need to use the equality:

\[
D_\mu \epsilon = (E^\mu_\nu)^{-1} D_\nu \epsilon \,. \tag{3.2.13}
\]

Then, by inserting the spin connection (3.2.9) and applying this last expression, one can bring equations (3.2.11) into the form:

\[
\partial_{x^\alpha} \epsilon = - \frac{r}{2L^2} \Gamma_{x^\alpha} \Gamma_r (1 - \Gamma_\ast) \epsilon \,, \quad (\alpha = 0, 1, 2, 3) \,, \tag{3.2.14}
\]

\[
\partial_r \epsilon = \frac{1}{2r} \Gamma_\ast \epsilon \,, \tag{3.2.15}
\]

with \( \Gamma_\ast \) being defined as

\[
\Gamma_\ast \equiv i \Gamma_{x^0 x^1 x^2 x^3} \,. \tag{3.2.16}
\]

These equations have two solutions:

\[
\epsilon_1 = \sqrt{r} \epsilon_+ \,, \quad \Gamma_\ast \epsilon_+ = \epsilon_+ \,, \tag{3.2.17}
\]

\[
\epsilon_2 = i \left( \frac{1}{\sqrt{r}} \Gamma_r \Gamma_\ast + \frac{\sqrt{r}}{L^2} x^\alpha \Gamma_{x^\alpha} \right) \epsilon_- \,, \quad \Gamma_\ast \epsilon_- = -\epsilon_- \,, \tag{3.2.18}
\]

where \( \epsilon_\pm \) are 10d spinors independent of the \( AdS_5 \) coordinates. The parameterization of the dependence of \( \epsilon_2 \) on the \( AdS_5 \) coordinates is the same as in ref. [38]. Each solution is obviously \( \frac{1}{2} \) SUSY. After defining \( \eta_- \equiv -i \Gamma_r \epsilon_- \) and \( \eta_+ \equiv \epsilon_+ \), \( \epsilon_1 \) and \( \epsilon_2 \) become:

\[
\epsilon_1 = \sqrt{r} \eta_+ \,, \tag{3.2.19}
\]

\[
\epsilon_2 = \left( \frac{1}{\sqrt{r}} + \frac{\sqrt{r}}{L^2} x^\alpha \Gamma_r \Gamma_{x^\alpha} \right) \eta_- \,, \tag{3.2.20}
\]

with \( \eta_\pm \) being 10d spinors independent of the \( AdS_5 \) coordinates and satisfying:

\[
\Gamma_\ast \eta_\pm = \pm \eta_\pm \,. \tag{3.2.21}
\]
Thus, we have got two independent solutions of the supersymmetry equations for the $AdS_5$ part (3.2.11), each one being $\frac{1}{2}$ SUSY. Notice that whereas for the first solution, the spinor $\epsilon_1$ is independent of the coordinates $x^\alpha$ and satisfies $\Gamma_\ast \epsilon_1 = \epsilon_1$; for the second solution, $\epsilon_2$ does depend on $x^\alpha$ and it is not an eigenvector of $\Gamma_\ast$. Both solutions can be unified in the following expression:

$$\epsilon = r \frac{L}{2} \left[ 1 + \frac{1}{2L^2} x^\alpha \Gamma_\gamma \Gamma_\gamma x^\alpha (1 - \Gamma_\ast) \right] \eta ,$$

(3.2.22)

where $\eta$ is a 10d spinor constant along $AdS_5$ and the dependence on the $AdS_5$ coordinates is parameterized as in ref. [38]. If we decompose $\eta$ according to the different eigenvalues of the matrix $\Gamma_\ast$: $\Gamma_\ast \eta_{\pm} = \pm \eta_{\pm}$, we recover the independent solutions (3.2.19) and (3.2.20).

It remains to solve the second subset of supersymmetry equations (3.2.12), the ones depending on the $T^{1,1}$ part of the metric. Let us insert the solution we have found (3.2.22) into that equations. The $\Gamma$-matrices appearing in (3.2.22) commute with the even number of $T^{1,1}$ $\Gamma$-matrices in (3.2.12), resulting:

$$D_s \eta + \frac{i}{2L} \Gamma^{12\hat{1}\hat{2}\hat{3}} \Gamma_s \eta = 0 , \ (s = 1, 2, \hat{1}, \hat{2}, \hat{3}) .$$

(3.2.23)

One can check, using the spin connection given in (3.2.9), that these five equations are solved by a constant 10d spinor $\eta$ satisfying the usual projections of the $T^{1,1}$ (see [33]):

$$\Gamma_{12} \eta = i \eta , \quad \Gamma_{\hat{1}\hat{2}} \eta = -i \eta .$$

(3.2.24)

Therefore, the Killing spinors of the model are given by the expression (3.2.22) in terms of a constant 10d spinor $\eta$ satisfying the projections (3.2.24). Furthermore, notice that the matrix multiplying $\eta$ in eq. (3.2.22) commutes with $\Gamma_{12}$ and $\Gamma_{\hat{1}\hat{2}}$ so the spinor $\epsilon$ also satisfies the projections (3.2.24), namely:

$$\Gamma_{12} \epsilon = i \epsilon , \quad \Gamma_{\hat{1}\hat{2}} \epsilon = -i \epsilon .$$

(3.2.25)

It is clear from eqs. (3.2.22) and (3.2.24) that our system is $1/4$ supersymmetric, i.e. it preserves 8 supersymmetries, as it corresponds to the supergravity dual of an $\mathcal{N} = 1$ superconformal field theory in four dimensions.

### 3.2.3 Killing spinors using global coordinates

It is also interesting to write down the form of the Killing spinors when global coordinates are used for the $AdS_5$ part of the metric. In these coordinates the ten dimensional metric takes the form:

$$ds_{10}^2 = L^2 \left[ - \cosh^2 \rho \, dt^2 + dp^2 + \sinh^2 \rho \, d\Omega_3^2 \right] + L^2 \, ds_{T^{1,1}}^2 ,$$

(3.2.26)

where $d\Omega_3^2$ is the metric of a unit three-sphere parameterized by three angles $(\alpha^1, \alpha^2, \alpha^3)$:

$$d\Omega_3^2 = (d\alpha^1)^2 + \sin^2 \alpha^1 (d\alpha^2)^2 + \sin^2 \alpha^2 (d\alpha^3)^2 ,$$

(3.2.27)
3.2. KILLING SPINORS

with $0 \leq \alpha^1, \alpha^2 \leq \pi$ and $0 \leq \alpha^3 \leq 2\pi$. In order to write down the Killing spinors in these coordinates, let us choose the following frame for the $AdS_5$ part of the metric:

$$e^t = L \cosh \rho \, dt, \quad e^\rho = L \, d\rho,$$
$$e^{\alpha^1} = L \sinh \rho \, d\alpha^1,$$
$$e^{\alpha^2} = L \sinh \rho \, \sin \alpha^1 \, d\alpha^2,$$
$$e^{\alpha^3} = L \sinh \rho \, \sin \alpha^1 \, \sin \alpha^2 \, d\alpha^3.$$  \hspace{1cm} (3.2.28)

We will continue to use the same frame forms as in eq. (3.2.3) for the $T^{1,1}$ part of the metric. The components of the spin connection corresponding to the $AdS_5$ part become:

$$\omega^{t \rho} = \sinh \rho \, dt, \quad \omega^{\alpha^1 \rho} = \cosh \rho \, d\alpha^1,$$
$$\omega^{\alpha^2 \rho} = \cosh \rho \, \sin \alpha^1 \, d\alpha^2,$$
$$\omega^{\alpha^3 \rho} = \cosh \rho \, \sin \alpha^1 \, \sin \alpha^2 \, d\alpha^3,$$
$$\omega^{\alpha^3 \alpha^1} = \cos \alpha_1 \, d\alpha_2,$$
$$\omega^{\alpha^3 \alpha^2} = \cos \alpha_2 \, d\alpha_3.$$  \hspace{1cm} (3.2.29)

Notice that we have written the spin connection in terms of the differentials of the $AdS_5$ coordinates. The selfdual five-form reads:

$$g_s F^{(5)} = \frac{4}{L} \left( e^1 \wedge e^2 \wedge e^1 \wedge e^2 \wedge e^3 - e^t \wedge e^\rho \wedge e^{\alpha^1} \wedge e^{\alpha^2} \wedge e^{\alpha^3} \right).$$  \hspace{1cm} (3.2.30)

Now we can solve the supersymmetry equations corresponding to the $AdS_5$ part (namely eqs. (3.2.11)) using global coordinates. Written in the frame (3.2.28), they read:

$$D_\mu \epsilon - \frac{i}{2L} \gamma \Gamma_\mu \epsilon = 0, \quad (\mu = t, \rho, \alpha^1, \alpha^2, \alpha^3),$$  \hspace{1cm} (3.2.31)

where we have defined:

$$\gamma \equiv \Gamma^{t \rho \alpha^1 \alpha^2 \alpha^3},$$  \hspace{1cm} (3.2.32)

and $\Gamma_\mu, (\mu = t, \rho, \alpha^1, \alpha^2, \alpha^3)$ are constant Dirac matrices associated to the frame (3.2.28). We expect the Killing spinors to depend on the coordinates so we must proceed as in (3.2.13) to write the covariant derivative in terms of the derivatives of the spinor with respect to the global coordinates. Let us begin with the equation for $\mu = \rho$, which yields:

$$\partial_\rho \epsilon - \frac{i}{2} \Gamma^\rho \gamma \epsilon = 0.$$  \hspace{1cm} (3.2.33)

This can be easily solved as

$$\epsilon = e^{\frac{i}{2} \Gamma^\rho \gamma} \tilde{\epsilon},$$  \hspace{1cm} (3.2.34)

where $\tilde{\epsilon}$ is a ten dimensional spinor independent of $\rho$. The equation for $\mu = t$ renders:

$$\partial_t \epsilon = -\frac{i}{2} \Gamma^t \gamma e^{-i \rho \gamma \Gamma^\rho} \epsilon.$$  \hspace{1cm} (3.2.35)
Chapter 3. Killing Spinors of the Klebanov-Witten Model

Inserting the form of $\epsilon$ written in (3.2.34) into this last equation, we can solve for $\tilde{\epsilon}$ in terms of a spinor $\bar{\epsilon}$ independent of $\rho$ and $t$, namely:

$$\tilde{\epsilon} = e^{-i \frac{t}{2} \Gamma^t \gamma} \bar{\epsilon},$$

(3.2.36)

so we can write $\epsilon$ as

$$\epsilon = e^{i \frac{\rho}{2} \Gamma^\rho \gamma} e^{-i \frac{t}{2} \Gamma^t \gamma} \bar{\epsilon}.$$

(3.2.37)

The equations for the angular components are:

$$\partial_{\alpha_1} \epsilon = -\frac{1}{2} \Gamma^{\alpha_1 \rho} e^{-i \rho \Gamma^\rho \gamma} \epsilon,$$

(3.2.38)

$$\partial_{\alpha_2} \epsilon = -\frac{1}{2} \left( \sin \alpha_1 \Gamma^{\alpha_2 \rho} e^{-i \rho \Gamma^\rho \gamma} - \cos \alpha_1 \Gamma^{\alpha_2 \alpha_1} \right) \epsilon,$$

(3.2.39)

$$\partial_{\alpha_3} \epsilon = -\frac{1}{2} \left( \sin \alpha_1 \sin \alpha_2 \Gamma^{\alpha_3 \rho} e^{-i \rho \Gamma^\rho \gamma} - \cos \alpha_1 \sin \alpha_2 \Gamma^{\alpha_3 \alpha_1} - \cos \alpha_2 \Gamma^{\alpha_3 \alpha_2} \right) \epsilon.$$

(3.2.40)

It is easy to solve these three equations in the order we have written them. After plugging (3.2.37) into the first equation we determine the dependence of $\epsilon$ on $\alpha_1$. Then, the second equation fixes the $\alpha_2$-dependence and finally, from the third equation we get $\epsilon$ in terms of a constant (along $AdS_5$) 10d spinor $\epsilon_0$ [39]:

$$\epsilon = e^{i \frac{\rho}{2} \Gamma^\rho \gamma} e^{-i \frac{t}{2} \Gamma^t \gamma} e^{-\frac{\alpha_1}{2} \Gamma^{\alpha_1 \rho} \Gamma^\rho \gamma} e^{-\frac{\alpha_2}{2} \Gamma^{\alpha_2 \alpha_1} \Gamma^\rho \gamma} e^{-\frac{\alpha_3}{2} \Gamma^{\alpha_3 \alpha_2} \Gamma^\rho \gamma} \epsilon_0.$$

(3.2.41)

As it happened when using cartesian coordinates, all the matrices in this last expression commute with the $\Gamma$-matrices appearing in equations (3.2.12) for the $T^{1,1}$. Hence, $\epsilon_0$ must satisfy the same projections as the ones in (3.2.24), namely:

$$\Gamma_{12} \epsilon_0 = i \epsilon_0, \quad \Gamma_{i\bar{i}} \epsilon_0 = -i \epsilon_0.$$

(3.2.42)

Then, the Killing spinors of the KW model (when using global coordinates for the $AdS_5$ part) are given by the expression (3.2.41) in terms of a 10d constant spinor satisfying the projections (3.2.42). It becomes clear that this solution leaves unbroken eight supersymmetries, as it was expected.
Chapter 4

Killing spinors of the Klebanov-Tseytlin model

4.1 Introduction

The goal of this chapter is to obtain the explicit expression of the Killing spinors of the 10d IIB supergravity solution known as the Klebanov-Tseytlin (KT) model. Proceeding as in the last chapter, we will solve the SUSY equations in a frame such that the Killing spinors are not expected to depend on the compact coordinates of the conifold. We will manage to express them in terms of a constant spinor subjected to three independent (and compatible) projections reducing the number of independent real components from 32 to 4, as it should be for a background that leaves unbroken 4 supercharges.

4.1.1 The Klebanov-Tseytlin model

This solution is constructed in [15] by placing $N$ D3-branes and $M$ fractional D3-branes (wrapped D5-branes) at the singular point of the conifold. The D5-branes wrap a 2-cycle inside $T^{1,1}$ and serve as sources of the magnetic RR three-form flux through the $S^3$ of $T^{1,1}$. The dual field theory is $\mathcal{N} = 1$ SYM with gauge group $SU(N + M) \times SU(N)$ and two chiral multiplets. The non-vanishing three-form flux in the SUGRA solution is the source of the conformal symmetry breaking in the dual field theory. Thus, we expect the corresponding IIB SUGRA solution to have four supersymmetries. The near-horizon metric and the selfdual RR five-form of the solution are:

$$ds^2_{10} = [\hat{h}(r)]^{-\frac{1}{2}} dx_{1,3}^2 + [\hat{h}(r)]^{\frac{1}{2}} (dr^2 + r^2 ds^2_{T^{1,1}}),$$

$$\hat{h}(r) = \frac{27\pi (\alpha')^2}{4r^4} \left[ g_s N + a (g_s M)^2 \ln \left( \frac{r}{r_0} \right) + \frac{a}{4} (g_s M)^2 \right], \quad (4.1.1)$$

with $a = \frac{3}{2\pi}$. The RR selfdual five-form reads:

$$F^{(5)} = 27\pi (\alpha')^2 N_{eff} d\text{Vol} \left( T^{1,1} \right) + \text{Hodge dual}, \quad (4.1.2)$$
where $d\text{Vol}(T^{1,1})$ is the volume five-form of the $T^{1,1}$ space and $N_{\text{eff}}$ is the following function of $r$:

$$N_{\text{eff}} = N + \frac{3}{2\pi} g_s M^2 \ln \left( \frac{r}{r_0} \right), \quad (4.1.3)$$

and one can readily check that

$$\frac{1}{(4\pi^2\alpha')^2} \int_{T^{1,1}} F^{(5)} = N_{\text{eff}}. \quad (4.1.4)$$

Hence, the five-form flux acquires a radial dependence and it is not quantized. It can still be identified with the quantity $N_{\text{eff}}$ defining the gauge group $SU(N_{\text{eff}} + M) \times SU(M)$ only at special radii $r_k = r_0 \exp \left( \frac{2\pi k}{3g_s M} \right)$ where $k$ is an integer, so $N_{\text{eff}} = N - k M$. In fact, the logarithmic decreasing of $N_{\text{eff}}(r)$, related to a continuous reduction in the numbers of degrees of freedom, is known as the RG cascade. This is mapped in the gauge theory side to a Seiberg duality cascade.

The RR and NSNS three-forms can be written as

$$F^{(3)} = \frac{Ma'}{2} \hat{\omega}_3, \quad H = \frac{3g_s a' M}{2r} d r \wedge \hat{\omega}_2, \quad (4.1.5)$$

where $\hat{\omega}_2$ and $\hat{\omega}_3$ are the closed two- and three-forms of the conifold, which in terms of the left invariant $SU(2)$ one-forms (2.3.5) and the Maurer-Cartan one-forms (2.1.4) become:

$$\hat{\omega}_2 = \frac{1}{2} \left( \sigma^1 \wedge \sigma^2 + \omega^1 \wedge \omega^2 \right), \quad \hat{\omega}_3 = \left( \omega^3 + \sigma^3 \right) \wedge \hat{\omega}_2. \quad (4.1.6)$$

As we have said in the subsection 3.2.1 of the previous chapter, the metric of this geometry is described by the one-form basis (3.2.7) simply by changing the generic warp factor $\hat{h}(r)$ to $\tilde{h}(r)$ written in (4.1.1). Then, let us define the following one-form basis:

$$\tilde{e}^a = \tilde{e}^a \left( \tilde{h}(r) \right), \quad \left( a = x^0, ..., x^3, r, 1, 2, \hat{1}, \hat{2}, \hat{3} \right). \quad (4.1.7)$$

where $\tilde{e}^a \left( \tilde{h}(r) \right)$ stands for the one-form frame resulting from the generic one written in eq. (3.2.7), after making $\tilde{h}(r) = \hat{h}(r)$.

### 4.2 Killing spinors

In order to determine the Killing spinors of this solution we have to solve the equations resulting from the vanishing of the SUSY variations (1.1.3). For this model with constant dilaton and three- and five-form fluxes they are:

$$-\frac{i}{24} \mathcal{F}^{(3)}_{N_1N_2N_3} \Gamma^{N_1N_2N_3} \epsilon = 0, \quad (4.2.1)$$

$$D_M \epsilon + \frac{i}{1920} F^{(5)}_{N_1...N_5} \Gamma^{N_1...N_5} \Gamma_M \epsilon + \frac{1}{96} \mathcal{F}^{(3)}_{N_1N_2N_3} \left( \Gamma_M^{N_1N_2N_3} - 9 \delta_M^{N_1} \Gamma^{N_2N_3} \right) \epsilon^* = 0, \quad (4.2.2)$$
where $\mathcal{F}^{(3)}$ is the complex combination of the RR and NSNS three-forms defined in (1.1.4). Let us write the RR five-form and the complex combination of the RR and NSNS three-forms in the one-form basis (4.1.7):

$$
F^{(5)} = -\hat{h}^* \hat{h}^{-\frac{3}{4}} \left( \hat{e}^{x^0} \wedge \hat{e}^{x^1} \wedge \hat{e}^{x^2} \wedge \hat{e}^{x^3} \wedge \hat{e}^r + \hat{e}^1 \wedge \hat{e}^2 \wedge \hat{e}^3 \wedge \hat{e}^r \right),
$$

(4.2.3)

$$
\mathcal{F}^{(3)} = \frac{9M\alpha'}{2r^3} \hat{h}^{-\frac{3}{4}} \left( \hat{e}^1 \wedge \hat{e}^2 + \hat{e}^1 \wedge \hat{e}^3 \right) \wedge \left( \hat{e}^r + i \hat{e}^3 \right),
$$

(4.2.4)

where for simplicity we have taken $g_s = 1$ and we have used that

$$
\hat{h}^* r^5 = -27\pi (\alpha')^2 g_s N_{\text{eff}},
$$

(4.2.5)

which results from differentiating the expression of $\hat{h}(r)$ given in eq. (4.1.1).

Next, in order to write down the spin connection of the background we plug the warp factor $\hat{h}(r)$ into the generic spin connection (3.2.8) we computed in section 3.2.1. Then, the spin connection one-form for the KT model, expressed in the frame (4.1.7), reads:

$$
\omega^{x^a r} = (\hat{h}^{-\frac{1}{4}})^* \hat{e}^{x^a}, \ (\alpha = 0, 1, 2, 3),
$$

$$
\omega^{s r} = \hat{h}^{-\frac{1}{4}} \left( \frac{1}{r} + \frac{1}{4} \hat{h}^* \hat{h}^{-1} \right) \hat{e}^s, \ (s = 1, 2, \hat{1}, \hat{2}, \hat{3}),
$$

$$
\omega^{12} = \frac{1}{r} \hat{h}^{-\frac{1}{4}} \hat{e}^3 - \frac{\sqrt{6}}{r} \cot \theta_1 \hat{h}^{-\frac{1}{4}} \hat{e}^2,
$$

$$
\omega^{12} = \frac{2}{r} \hat{h}^{-\frac{1}{4}} \hat{e}^3 - \frac{\sqrt{6}}{r} \cot \theta_1 \hat{h}^{-\frac{1}{4}} \hat{e}^2,
$$

$$
\omega^{13} = -\frac{1}{r} \hat{h}^{-\frac{1}{4}} \hat{e}^2, \ \omega^{23} = \frac{1}{r} \hat{h}^{-\frac{1}{4}} \hat{e}^1,
$$

$$
\omega^{32} = \frac{1}{r} \hat{h}^{-\frac{1}{4}} \hat{e}^1, \ \omega^{31} = -\frac{1}{r} \hat{h}^{-\frac{1}{4}} \hat{e}^2.
$$

(4.2.6)

We begin by solving the equation $\delta \psi_{x^1} = 0$ for a 10d spinor $\epsilon$ independent of the $x^a$ coordinates. After inserting the three- and five-forms written in (4.2.4) and (4.2.3) and the spin connection we have just computed, one gets:

$$
-\frac{1}{8} \hat{h}^* \hat{h}^{-\frac{3}{4}} \Gamma_{x^1 r} \epsilon + i \frac{1}{8} \hat{h}^* \hat{h}^{-\frac{3}{4}} \Gamma_{x^0 x^1 x^2 x^3} \Gamma_{r x^1} \epsilon +
$$

$$
+ \frac{9M\alpha'}{16} \frac{1}{2r^3} \hat{h}^{-\frac{3}{4}} \Gamma_{x^1} (\Gamma_{12r} + \Gamma_{1\hat{2}r} + i\Gamma_{12\hat{3}} + i\Gamma_{1\hat{1}\hat{2}\hat{3}}) \epsilon^s = 0,
$$

(4.2.7)

where $\Gamma_a, \ (a = x^a, r, 1, 2, \hat{1}, \hat{2}, \hat{3})$ are constant Dirac matrices associated to the frame (4.1.7) and we have inserted the equality $\Gamma_{x^a x^b} \Gamma_r \Gamma_{12\hat{3}} \epsilon = -\epsilon$, following from the well defined chirality of the 10d spinor $\epsilon$. Let us impose the SUSY cycle projection

$$
\Gamma_{12} \epsilon = -\Gamma_{\hat{1}\hat{2}} \epsilon,
$$

(4.2.8)
which is again projection (2.2.6), arising naturally when the conifold is obtained from the 8d gauged SUGRA. Thus, the third term of the supersymmetry equation (4.2.7) vanishes, and from the remaining ones we get the projection

\[ \Gamma_{x^0 x^1 x^2 x^3} \epsilon = -i \epsilon. \]  

(4.2.9)

This is the projection corresponding to a D3-brane extended along the Minkowski space. It can be straightforwardly checked that the remaining equations \( \delta \psi = 0 \) are solved by the same projections (4.2.8) and (4.2.9), which can be inserted in the equality \( \Gamma_{x^0 x^1 x^2 x^3} \Gamma_{12 \hat{1} \hat{2} \hat{3}} \epsilon = -\epsilon \) to get the following useful relation:

\[ \Gamma_{r \hat{1} \hat{2} \hat{3}} \epsilon = -i \epsilon. \]  

(4.2.10)

Now, we try to solve the equations for the angular components of the gravitino assuming that \( \epsilon \) is also independent of the coordinates of the conifold. The equation \( \delta \psi_1 = 0 \) becomes:

\[
\frac{1}{2} \hat{h}^{-\frac{1}{2}} \left[ \left( \frac{1}{r} + \frac{1}{4} \hat{h}' \hat{h}^{-1} \right) \Gamma_{1r} + \frac{1}{r} \Gamma_{32} \right] \epsilon + \frac{i}{8} \hat{h}' \hat{h}^{-\frac{1}{2}} \Gamma_{x^0 x^1 x^2 x^3} \Gamma_{r1} \epsilon + \\
+ \frac{9}{16} \frac{M\alpha'}{r^3} \hat{h}^{-\frac{3}{2}} (\Gamma_{1\hat{1} \hat{2}} - 3\Gamma_2) (\Gamma_r + i\Gamma_3) \epsilon^* = 0,
\]

(4.2.11)

where in the second term we have inserted the total chirality projection \( \Gamma_{x^0 x^1 x^2 x^3} \Gamma_{12 \hat{1} \hat{2} \hat{3}} \epsilon = -\epsilon \). The last term of this equation vanishes after imposing the complex conjugate of eq. (4.2.10), so we are left with

\[
\frac{1}{2r} (1 - \Gamma_{r1\hat{2}\hat{3}}) \epsilon = \frac{1}{8} \hat{h}' \hat{h}^{-1} (i \Gamma_{x^0 x^1 x^2 x^3} - 1) \epsilon,
\]

(4.2.12)

where the right-hand side vanishes when imposing the projection (4.2.9). Hence, the equation renders the projection \( \Gamma_{r1\hat{2}\hat{3}} \epsilon = \epsilon \), which, after making use of (4.2.8), can be written as

\[ \Gamma_{r1\hat{2}\hat{3}} \epsilon = -\epsilon. \]  

(4.2.13)

The equations for the remaining angular components of the gravitino and the equation for the dilatino \( \delta \lambda = 0 \) are easily solved by imposing the three independent projections we have got, namely (4.2.8), (4.2.9) and (4.2.13). Finally, we write down the equation for the radial component of the gravitino assuming that \( \epsilon = \epsilon(r) \). Thus, as we have done in previous chapters, we should use that \( D_r \epsilon = (E_r^\alpha)^{-1} D_r \epsilon \) in order to write the covariant derivative in terms of \( \epsilon' \equiv \partial_r \epsilon \). This time \( E_m^\alpha \) are the coefficients appearing when writing the one-forms (4.1.7) in terms of the differentials of the coordinates. The equation \( \delta \psi_r = 0 \) reads:

\[
\epsilon' + \frac{i}{8} \hat{h}' \hat{h}^{-1} \Gamma_{x^0 x^1 x^2 x^3} \epsilon + \frac{9}{16} \frac{M\alpha'}{2r^3} \hat{h}^{-\frac{3}{2}} (\Gamma_{12} + \Gamma_{\hat{1} \hat{2}}) (-3 + i \Gamma_{r \hat{3}}) \epsilon^* = 0.
\]

(4.2.14)

The third contribution cancels out by virtue of (4.2.8) and if we also impose the projection (4.2.9) we arrive at

\[ \epsilon' + \frac{1}{8} \hat{h}' \hat{h}^{-1} \epsilon = 0. \]  

(4.2.15)
Therefore, the Killing spinor of the Klebanov-Tseytlin model can be expressed in terms of a 10d constant spinor \( \epsilon_0 \) as

\[
\epsilon = \hat{h}^{-\frac{1}{8}} \epsilon_0,
\]

where \( \epsilon_0 \) satisfies three independent projections, namely:

\[
\Gamma_{x^0 x^1 x^2 x^3} \epsilon_0 = -i \epsilon_0, \quad \Gamma_{12 \hat{1} \hat{2}} \epsilon_0 = \epsilon_0, \quad \Gamma_{r \hat{1} \hat{2} \hat{3}} \epsilon_0 = -\epsilon_0.
\]

Thus, the model has 4 independent spinors as it should be for the supergravity dual of a 4d \( \mathcal{N} = 1 \) field theory.

Recalling that \( \Gamma_{x^0 \ldots x^3} \Gamma_r \Gamma_{12 \hat{1} \hat{2}} \epsilon_0 = -\epsilon_0 \), the projections (4.2.17) can be reformulated as

\[
\Gamma_{x^0 x^1 x^2 x^3} \epsilon_0 = -i \epsilon_0, \quad \Gamma_{12} \epsilon_0 = i \epsilon_0, \quad \Gamma_{\hat{1} \hat{2}} \epsilon_0 = -i \epsilon_0.
\]

These projections, which in view of (4.2.16) are also satisfied by \( \epsilon \), can be identified as the projection corresponding to a D3-brane along the Minkowski directions and the two projections of the \( T^{1,1} \) (see [33]). Hence, recalling the results of the previous chapter, one readily notices that these projections are the same as the ones fulfilled by the Killing spinors \( \epsilon_1 \) of eq. (3.2.19) in the last chapter. Those are the four spinors corresponding to the ordinary supersymmetries of the Klebanov-Witten background. In fact, the only difference between \( \epsilon \) written in eq. (4.2.16) and the four Killing spinors \( \epsilon_1 \) of the KW solution relies on the different radial dependence. Therefore, the breaking of conformal invariance due to the addition of the fractional branes in the Klebanov-Tseytlin model, translates into the loss of the four Killing spinors \( \epsilon_2 \) (3.2.20) realizing the superconformal symmetries.
Chapter 5

Killing spinors of the
Klebanov-Strassler model

5.1 Introduction

In this chapter we compute explicitly the Killing spinors of the Klebanov-Strassler (KS) solution [16]. This background has attracted a lot of interest during the last years since it is a gravity dual of $\mathcal{N} = 1$ SYM with very nice features. It is constructed by placing fractional D3-branes and D3-branes on the deformed conifold, so in the UV it approaches the Klebanov-Tseytlin solution described in the last chapter and therefore, it incorporates the logarithmic flow of couplings. On the other hand, in the IR, where the KT model was singular, the deformation of the conifold gives a geometrical realization of chiral symmetry breaking and confinement.

The structure of this chapter is as follows: in the first section we characterize the SUGRA background giving some hints into its construction. In section 5.2 we solve the SUSY equations in a frame where the Killing spinors do not depend on the angular coordinates of the conifold. Finally, in section 5.3 we show that the differential equations for the functions entering the KS ansatz (see below) that we get from the SUSY equations are equivalent to the first order system appearing in ref. [16] plus an extra differential equation.

The results of this calculation were published in [13] as the initial step of an extension of the kappa symmetry analysis carried out there to the more interesting KS background.

5.1.1 The Klebanov-Strassler model

The Klebanov-Tseytlin geometry described in the last chapter becomes singular at sufficiently small $r$, precisely at the end of the RG cascade. Then, in order to construct a SUGRA dual of the IR region of $\mathcal{N} = 1$ SYM, one can substitute the singular conifold by its deformation. Hence, while for large $r$ the geometry approaches the KT solution, at $r = 0$ the geometry does not collapse but degenerates into a finite $S^3$. This fact gives a geometric realization of confinement and chiral symmetry breaking, which are fundamental features of the $\mathcal{N} = 1$ SYM expected at the end of the cascade. The resulting background is the
so-called Klebanov-Strassler \cite{16} solution. The warped 10d metric is:

\[ ds_{10}^2 = [h(\tau)]^{-\frac{1}{2}} ds_{1,4}^2 + [h(\tau)]^{\frac{1}{2}} ds_6^2 , \]  

(5.1.1)

where the six dimensional metric \( ds_6^2 \) is the one corresponding to the deformed conifold and \( \tau \) is the radial coordinate defined in (2.3.14). The metric of the deformed conifold is obtained from the one of the generalized deformed conifold written in eq. (2.3.17) simply by setting \( C = 0 \). Thus, it reads:

\[ ds_6^2 = \frac{1}{2} \mu^4 K(\tau) \left[ \frac{1}{3K(\tau)^3} \left( d\tau^2 + (g^5)^2 \right) + \sinh^2 \left( \frac{\tau}{2} \right) \left( (g^1)^2 + (g^2)^2 \right) + \right. \]
\[ \left. + \cosh^2 \left( \frac{\tau}{2} \right) \left( (g^3)^2 + (g^4)^2 \right) \right] , \]  

(5.1.2)

with

\[ K(\tau) = \frac{\left( \sinh(2\tau) - 2\tau \right)^{\frac{1}{2}}}{2^{\frac{3}{2}} \sinh \tau} , \]  

(5.1.3)

and the one-forms \( g^i \) \((i = 1, \ldots, 5)\) are defined in equation (2.3.18) in terms of the usual angular coordinates.

As we have said in chapter 2, the metric of the deformed conifold reduces to the one of an \( S^3 \) when \( \tau \rightarrow 0 \) while it coincides with the metric of the singular conifold for \( \tau \rightarrow \infty \). Therefore, the RR three-form flux for this model reads:

\[ F^{(3)} = \frac{M \alpha'}{2} \left[ (1 - F) g^5 \wedge g^3 \wedge g^4 + F g^5 \wedge g^1 \wedge g^2 + F' d\tau \wedge (g^1 \wedge g^3 + g^2 \wedge g^4) \right] , \]  

(5.1.4)

with \( F = F(\tau) \) satisfying \( F(0) = 0 \) and \( F(\tau \rightarrow \infty) = \frac{1}{2} \) in order to get an \( F^{(3)} \) lying along the \( S^3 \) when \( \tau \rightarrow 0 \) while being equal to the one written in (4.1.5) when \( \tau \rightarrow \infty \), i.e. equal to the RR three-form flux of the Klebanov-Tseytlin model in the UV. Notice that, as usual, the prime after any radial function (for instance \( F' \)) stands for the radial derivative \( (\frac{d}{d\tau}) \). The NSNS two-form potential \( B \) and its corresponding three-form field strength \( H \) are written as

\[ B = \frac{M \alpha'}{2} \left[ f g^1 \wedge g^2 + k g^3 \wedge g^4 \right] , \]  

(5.1.5)

\[ H = \frac{M \alpha'}{2} \left[ d\tau \wedge (f' g^1 \wedge g^2 + k' g^3 \wedge g^4) + \frac{1}{2} (k - f) g^5 \wedge (g^1 \wedge g^3 + g^2 \wedge g^4) \right] , \]  

(5.1.6)

in terms of two undetermined radial functions \( f = f(\tau) \) and \( k = k(\tau) \). We are taking \( g_s = 1 \). Finally, the five-form flux is constructed by taking:

\[ F^{(5)} = \mathcal{F}^{(5)} + \text{Hodge dual} , \]  

\[ \mathcal{F}^{(5)} = B \wedge F^{(3)} = \frac{M^2 (\alpha')^2}{4} l(\tau) g^1 \wedge g^2 \wedge g^3 \wedge g^4 \wedge g^5 = \]
\[ = 27 M^2 (\alpha')^2 l(\tau) d\text{Vol} \left( T^{1,1} \right) , \]  

(5.1.7)
where we have defined:

\[ l(\tau) \equiv f(\tau) \left( 1 - F(\tau) \right) + k(\tau) F(\tau). \]  

(5.1.8)

The Hodge dual \( \ast \mathcal{F}^{(5)} \) becomes:

\[ \ast \mathcal{F}^{(5)} = \frac{\alpha l(\tau)}{K^2(\tau) h^2 \sinh^2 \tau} \, dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3 \wedge d\tau, \]  

(5.1.9)

with \( \alpha \equiv 4 M^2 (\alpha')^2 \mu^{- 3/2} \).

Therefore, \( F^{(5)} \) satisfies by construction the IIB SUGRA equation of motion \( \dd F^{(5)} = H \wedge F^{(3)} \). However, the ansatz has to verify the remaining equations of motion for the three-forms: \( d \ast F^{(3)} = F^{(5)} \wedge H \), and \( d \ast H = - F^{(5)} \wedge F^{(3)} \), and the constant dilaton condition which implies \( (F^{(3)})^2 = (H)^2 \), together with the Einstein equation. This renders a system of second order differential equations determining the unknown functions of the ansatz \( (F(\tau), f(\tau), k(\tau) \) and \( h(\tau)) \). It is not difficult to find a system of first order differential equations [16] that solves those equations. It reads:

\[ f' = (1 - F) \tanh^2 \left( \frac{\tau}{2} \right), \]

\[ k' = F \coth^2 \left( \frac{\tau}{2} \right), \]

\[ F' = \frac{k - f}{2}, \]  

(5.1.10)

and,

\[ h' = - \frac{\alpha \, l(\tau)}{K^2(\tau) \sinh^2 \tau}. \]  

(5.1.11)

In order to arrive at this system let us recall that if \( \ast F^{(3)} \) satisfies the equation \( \dd \ast F^{(3)} = F^{(5)} \wedge H \), it can be written as \( \ast F^{(3)} = \dd C^{(6)} + C^{(4)} \wedge H \) in terms of the six-form and four-form RR potentials. We will show that the system (5.1.10) results from requiring the vanishing of the six-form RR potential, i.e. \( C^{(6)} = 0 \); which, in view of the last expression of \( \ast F^{(3)} \), is equivalent to:

\[ \ast F^{(3)} = C^{(4)} \wedge H. \]  

(5.1.12)

From eq. (5.1.4) it is straightforward to write down \( \ast F^{(3)} \):

\[ \ast F^{(3)} = \frac{M \alpha'}{2} h^{-1} d^4 x \wedge \left[ (1 - F) \tanh^2 \left( \frac{\tau}{2} \right) d\tau \wedge g^1 \wedge g^2 + F \coth^2 \left( \frac{\tau}{2} \right) d\tau \wedge g^3 \wedge g^4 + F' g^5 \wedge (g^1 \wedge g^3 + g^2 \wedge g^4) \right], \]  

(5.1.13)

with \( d^4 x = dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3 \).

From the equation of motion \( \dd F^{(5)} = H \wedge F^{(3)} \) it is clear that one can write \( F^{(5)} = \dd C^{(4)} + B^{(2)} \wedge F^{(3)} \). In addition, let us write \( C^{(4)} \) as \( C^{(4)} = \hat{C}^{(4)} + \tilde{C}^{(4)} \), with the four-forms \( \hat{C}^{(4)} \) and \( \tilde{C}^{(4)} \) being given by:

\[ \dd \hat{C}^{(4)} = \ast \mathcal{F}^{(5)}, \quad \dd \tilde{C}^{(4)} = \mathcal{F}^{(5)} - F^{(3)} \wedge B, \]  

(5.1.14)
CHAPTER 5. KILLING SPINORS OF THE KLEBANOV-STRASSLER MODEL

so, in view of eq. (5.1.9) it is easy to write down $\hat{C}^{(4)}$:

$$\hat{C}^{(4)} = \hat{f}(\tau) \, dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3,$$

where we have defined $\hat{f}(\tau)$ as a function of the radial coordinate satisfying:

$$\hat{f}'(\tau) = \frac{\alpha \, l(\tau)}{K^2(\tau) \, [h(\tau)]^2 \sinh^2 \tau}.$$  

(5.1.15)

Recalling that $\mathcal{F}^{(5)} \propto d\mathrm{Vol}(T^{1,1})$ and the expressions for $F^{(3)}$ and $B$ (eqs. (5.1.4) and (5.1.5)) one realizes that all the components of $\hat{C}^{(4)}$ are perpendicular to the Minkowski space ($x^0, x^1, x^2, x^3$). Since $H$ neither has components along any Minkowski direction (see eq. (5.1.6)), it becomes clear that $H \wedge \hat{C}^{(4)} = 0$. Hence, one gets:

$$H \wedge C^{(4)} = H \wedge \hat{C}^{(4)} = \frac{M \alpha'}{2} \hat{f}(\tau) \, d^4x \wedge \left[ d\tau \wedge \left( f' \, g^1 \wedge g^2 + k' \, g^3 \wedge g^4 \right) + \frac{1}{2} (k - f) \, g^5 \wedge \left( g^1 \wedge g^3 + g^2 \wedge g^4 \right) \right].$$

(5.1.16)

Inserting this result and the expression of $\star F^{(3)}$ (eq. (5.1.13)) into the equation (5.1.12) one readily obtains the first order system (5.1.10), and the equality $h^{-1} = \hat{f}(\tau)$, which after differentiating yields the differential equation (5.1.11).

### 5.2 Killing spinors

As we have shown in the last section, the Klebanov-Strassler solution is formulated in terms of some functions $F(\tau), \, f(\tau), \, k(\tau)$, and $h(\tau)$ defined by means of a system of first order equations (5.1.10), (5.1.11); which guarantees the fulfilment of the SUGRA equations of motion. But this is not the whole story since we should determine if for any solution of the system we are dealing with a supersymmetric solution of 10d type IIB supergravity. Indeed, we will show by imposing the vanishing of the SUSY variations (1.1.3), that the model is $\frac{1}{8}$ SUSY if the functions $F(\tau), \, f(\tau), \, k(\tau)$ verify the system of first order differential equations (5.1.10) together with an extra algebraic constraint.

As we have seen in the subsection 2.3.1 of the second chapter, if we choose the appropriate one-form basis, the Killing spinors of the 10d solution consisting of adding $\mathbb{R}^{1,3}$ to the deformed conifold do not depend on the angular coordinates of the conifold (see eq. 2.3.19). That basis arises naturally when we write the metric of the deformed conifold as in eq. (2.3.16), namely:

$$ds_6^2 = \frac{1}{2} \mu^4 K(\tau) \left\{ \frac{1}{3K(\tau)^3} \left( d\tau^2 + (w^3 + \sigma^3)^2 \right) + \frac{\sinh^2 \tau}{2 \cosh \tau} \left( (\sigma^1)^2 + (\sigma^2)^2 \right) + \frac{\cosh \tau}{2} \left[ (w^1 + \frac{\sigma^1}{\cosh \tau})^2 + (w^2 + \frac{\sigma^2}{\cosh \tau})^2 \right] \right\}.$$  

(5.2.1)
where $K(\tau)$ is defined in eq. (5.1.3). Thus, it is natural to consider a frame such as (2.3.22), but now including the corresponding powers of the warp factor $h(\tau)$ appearing in the 10d metric (5.1.1). It reads:

$$e^{x^\alpha} = h^{-\frac{1}{4}} dx^\alpha, \quad (\alpha = 0, 1, 2, 3), \quad e^\tau = \frac{\mu^2 h^{\frac{1}{2}}}{\sqrt{6} K(\tau)} d\tau,$$

$$e^i = \frac{\mu^2 h^{\frac{1}{2}}}{2} \left( \frac{\sqrt{K(\tau)}}{\sqrt{\cosh \tau}} \right) \sinh \tau \cosh \frac{\tau}{2} (w^i + \frac{\sigma^i}{\cosh \tau}), \quad (i = 1, 2),$$

$$e^{\hat{i}} = \frac{\mu^2 h^{\frac{1}{2}}}{2} \sqrt{K(\tau)} \left( \frac{\cosh \tau}{\sqrt{\cosh \tau}} \right) (w^i + \frac{\sigma^i}{\cosh \tau}), \quad (i = 1, 2),$$

$$e^{\hat{3}} = \frac{\mu^2 h^{\frac{1}{2}}}{\sqrt{6} K(\tau)} (w^3 + \sigma^3). \quad (5.2.2)$$

Then, the corresponding spin connection one-form will be very similar to the one written in (2.2.5), when this last one is restricted to the solution (2.3.12). Let us write it schematically as

$$\omega^{x^\alpha \tau} = -\frac{1}{4} h' h^{-1} C^{-1} e^{x^\alpha}, \quad (\alpha = 0, 1, 2, 3),$$

$$\omega^{1 \tau} = A' A^{-1} C^{-1} e^1 + \frac{1}{2} g' B A^{-1} C^{-1} e^1,$$

$$\omega^{2 \tau} = A' A^{-1} C^{-1} e^2 + \frac{1}{2} g' B A^{-1} C^{-1} e^1,$$

$$\omega^{1 \tau} = B' B^{-1} C^{-1} e^1 + \frac{1}{2} g' B A^{-1} C^{-1} e^1,$$

$$\omega^{2 \tau} = B' B^{-1} C^{-1} e^2 + \frac{1}{2} g' B A^{-1} C^{-1} e^1,$$

$$\omega^{3 \tau} = C' C^{-2} e^3, \quad \omega^{11} = -\frac{1}{2} g' B A^{-1} C^{-1} e^\tau, \quad \omega^{22} = -\frac{1}{2} g' B A^{-1} C^{-1} e^\tau,$$

$$\omega^{21} = A^{-1} \cot \theta e^2 + \frac{1}{2} C A^{-2} \left( g^2 - 1 \right) e^3,$$

$$\omega^{13} = \frac{1}{2} g A^{-1} \left( B C^{-1} + C B^{-1} \right) e^2 - \frac{1}{2} C B^{-2} e^3,$$

$$\omega^{31} = A^{-1} \cot \theta e^2 + \left( \frac{1}{2} C B^{-2} - C^{-1} \right) e^3,$$

$$\omega^{23} = \frac{1}{2} C B^{-2} e^1 - \frac{1}{2} g A^{-1} \left( B C^{-1} + C B^{-1} \right) e^1,$$
The Killing spinors of the Klebanov-Strassler Model

\omega^{12} = -\omega^{21} = \frac{1}{2} g A^{-1} \left( C B^{-1} - B C^{-1} \right) e^3,

\omega^{32} = \frac{1}{2} g A^{-1} \left( C B^{-1} - B C^{-1} \right) e^1 - \frac{1}{2} C A^{-2} \left( g^2 - 1 \right) e^1,

\omega^{31} = \frac{1}{2} g A^{-1} \left( B C^{-1} - C B^{-1} \right) e^2 + \frac{1}{2} C A^{-2} \left( g^2 - 1 \right) e^2, \quad (5.2.3)

where \( A, B, C \) and \( g \) are the following functions of the radial coordinate:

\[ A = \frac{\mu^\frac{3}{2} h^\frac{1}{2} \sqrt{K(\tau)} \sinh \tau}{2}, \quad B = \frac{\mu^\frac{3}{2} h^\frac{1}{2} \sqrt{K(\tau)}}{2} \sqrt{\cosh \tau}, \quad C = \frac{\mu^\frac{3}{2} h^\frac{1}{2}}{\sqrt{6 K(\tau)}}, \quad g = \frac{1}{\cosh \tau}, \quad (5.2.4) \]

which allow us to write the one-form basis (5.2.2) in the following neat form:

\[ e^\alpha = h^{-\frac{1}{4}} dx^\alpha, \quad (\alpha = 0, 1, 2, 3), \quad e^\tau = C d\tau, \]

\[ e^i = A \sigma^i, \quad (i = 1, 2), \]

\[ e^i = B \left( w^i + g \sigma^i \right), \quad (i = 1, 2), \]

\[ e^3 = C \left( w^3 + \sigma^3 \right). \quad (5.2.5) \]

One can check by substituting the expressions of \( A, B \) and \( C \) above into the components of the spin connection written in (5.2.3) that, except for the terms proportional to \( h'(\tau) \), the resulting one-form is equal term by term (up to \( h(\tau) \) factors) to the spin connection one-form arising from substituting the particular solution (2.3.12) (describing the deformed conifold) into the generalized spin connection written in (2.2.5). We have expressed the spin connection one-form in the frame (5.2.2). One should bear in mind that

\[ \omega^{a b} = \omega^{a b}_{m} dX^m = \omega^{a b}_{c} e^c, \quad (5.2.6) \]

where \( e^c \) refers to the frame (5.2.2). So, when needed, the components \( \omega^{a b}_{m} \) can be easily computed:

\[ \omega^{a b}_{m} = E^{c}_{m} \omega^{a b}_{c}, \quad (5.2.7) \]

with \( E^{c}_{m} \) being the coefficients appearing when one expresses the one-forms (5.2.2) in terms of the differentials of the coordinates, namely: \( e^c = E^{c}_{m} dX^m \).

Let us now write the three- and five-form fluxes in the frame (5.2.2). The selfdual RR five-form takes the form:

\[ F^{(5)} = -\frac{\sqrt{6}}{\mu^\frac{3}{2}} K(\tau) h^{-\frac{1}{4}} h' \left( e^{x0} \wedge e^{x1} \wedge e^{x2} \wedge e^{x3} \wedge e^\tau + e^1 \wedge e^2 \wedge e^3 \wedge e^\tau \wedge e^\hat{3} \right), \quad (5.2.8) \]
the RR three-form (5.1.4) becomes:

\[
F^{(3)} = - \frac{4\sqrt{3} h^{-\frac{3}{4}} M_\alpha'}{\mu^2} \frac{1}{2} \left\{ \frac{1}{2 \cosh \tau} e^3 \wedge e^2 \wedge e^1 + \frac{1 - g - 2F}{2 \sinh \tau} \left( e^3 \wedge e^2 \wedge e^1 + e^3 \wedge e^2 \wedge e^1 \right) + \right. \\
+ \frac{\cosh \tau}{2 \sinh^2 \tau} \left[ 1 + g (g - 2 + 4F) \right] e^3 \wedge e^2 \wedge e^1 + \frac{F'}{\sinh \tau} \left( e^3 \wedge e^2 \wedge e^2 + e^3 \wedge e^2 \wedge e^1 \right) \right\} .
\]

(5.2.9)

The NSNS three-form flux (5.1.6) can be written as

\[
H = - \frac{4\sqrt{3} h^{-\frac{3}{4}} M_\alpha'}{\mu^2} \frac{1}{4} \left\{ \frac{f' + k'}{\cosh \tau} e^\tau \wedge e^2 \wedge e^1 + \frac{1}{\sinh \tau} \left[ (1 - g) k' - (1 + g) f' \right] \left( e^\tau \wedge e^2 \wedge e^1 + \right. \\
+ \left. e^\tau \wedge e^2 \wedge e^1 \right) + \frac{\cosh \tau}{\sinh^2 \tau} \left[ (1 - g)^2 k' + (1 + g)^2 f' \right] e^\tau \wedge e^2 \wedge e^1 + \right. \\
+ \left. \frac{k - f}{\sinh \tau} \left( e^3 \wedge e^2 \wedge e^2 + e^3 \wedge e^2 \wedge e^1 \right) \right\} .
\]

(5.2.10)

Now we are ready to write the equations resulting from the vanishing of the SUSY variations (1.1.3). For this background with constant dilaton and three and five-form fluxes they reduce to the expressions written in the equations (4.2.1) and (4.2.2) of the previous chapter. Since we expect that the Killing spinors will only depend on the radial variable \(\tau\), we will write the SUSY equations directly in the frame (5.2.2).

### 5.2.1 Dilatino SUSY equation

Using the expressions for the three-forms written above, and recalling the definition of \(F^3\) given in (1.1.4), the equation (4.2.1) resulting from the vanishing of the variation of the dilatino takes the form:

\[
\left\{ \frac{f' + k'}{2 \cosh \tau} \Gamma_{\tau 21} + \frac{1}{2 \sinh \tau} \left[ (1 - g) k' - (1 + g) f' \right] \left( \Gamma_{\tau 21} + \Gamma_{\tau 21} \right) + \\
+ \frac{\cosh \tau}{2 \sinh^2 \tau} \left[ (1 + g)^2 f' + (1 - g)^2 k' \right] \Gamma_{\tau 21} + \frac{k - f}{2 \sinh \tau} \left( \Gamma_{321} + \Gamma_{321} \right) + \\
+ \frac{i}{2 \cosh \tau} \Gamma_{32i} + \frac{i}{2 \sinh \tau} \left( 1 - g \right) \left( \Gamma_{321} + \Gamma_{321} \right) + \\
+ \frac{i \cosh \tau}{2 \sinh^2 \tau} \left[ 1 + g \left( 4F - 2 + g \right) \right] \Gamma_{321} + \frac{i F'}{\sinh \tau} \left( \Gamma_{\tau 22} + \Gamma_{\tau 11} \right) \right\} \epsilon = 0 ,
\]

(5.2.11)

where \(\Gamma_a, (a = x^\alpha, \tau, 1, 2, \hat{1}, \hat{2}, \hat{3})\) are constant Dirac matrices associated to the frame (5.2.2) and we have neglected the common factor \(-\frac{4\sqrt{3} h^{-\frac{3}{4}} M_\alpha'}{\mu^2} \frac{1}{2}\).
As we did in chapter 2 (see eq. (2.2.6)) we will impose the angular projection
\[
\Gamma_{12} \epsilon = - \Gamma_{12} \epsilon .
\] (5.2.12)

Furthermore, the Killing spinors of the resolutions of the conifold are subjected to the projection (2.2.14), which for the particular case of the deformed conifold (then, taking into account eq. (2.3.21)) becomes:
\[
\Gamma_{\tau} \Gamma_{123} \epsilon = \left( - \frac{\sinh \tau}{\cosh \tau} + \frac{1}{\cosh \tau} \Gamma_{11} \right) \epsilon .
\] (5.2.13)

We also impose the projection corresponding to a D3-brane extended along the Minkowski directions:
\[
\Gamma_{x_{0}x_{1}x_{2}x_{3}} \epsilon = - i \epsilon .
\] (5.2.14)

We will show below that this projection follows from the vanishing of the gravitino SUSY variation as it happened for the Klebanov-Tseytlin model (see eq. (4.2.9)). Since we are working in type IIB SUGRA, a 10d spinor \( \epsilon \) satisfies the equality:
\[
\Gamma_{x_{0}x_{1}x_{2}x_{3}} \Gamma_{\tau} \Gamma_{123} \epsilon = - \epsilon ,
\] (5.2.15)

which combined with projections (5.2.12) and (5.2.14) gives rise to
\[
\Gamma_{\tau 3} \epsilon = - i \epsilon .
\] (5.2.16)

Using this last projection, the one written in (5.2.13), and some suitable combinations of both ones, equation (5.2.11) becomes:
\[
( - i P_{1} \Gamma_{3} + P_{2} \Gamma_{311} ) \epsilon = 0 ,
\] (5.2.17)

where
\[
P_{1} = - i \frac{\sinh \tau}{2 \cosh^{2} \tau} ( f' + k' + 1 ) + i \frac{1}{2 \sinh \tau} \left[ (1 + g)^{2} f' + (1 - g)^{2} k' + 1 + g (4F - 2 + g) \right] +
\]
\[+ \frac{i}{\sinh \tau \cosh \tau} \left[ (1 - g) k' - (1 + g) f' + 1 - g - 2F \right] ,
\] (5.2.18)

and
\[
P_{2} = - \frac{1}{2 \cosh^{2} \tau} ( f' + k' + 1 ) + \frac{1}{2 \sinh^{2} \tau} \left[ (1 + g)^{2} f' + (1 - g)^{2} k' + 1 + g (4F - 2 + g) \right] +
\]
\[ - \frac{1}{\cosh \tau} \left[ (1 - g) k' - (1 + g) f' + 1 - g - 2F \right] + \frac{1}{\sinh \tau} \left( 2F' - k + f \right) .
\] (5.2.19)

Using that \( g = \frac{1}{\cosh \tau} \) it is not difficult to see that \( P_{1} \) automatically vanishes. Hence, we are left with the equation \( P_{2} = 0 \), which by substituting the value of \( g \) is brought into the form:
\[
2F' + \coth \left( \frac{\tau}{2} \right) f' - \tanh \left( \frac{\tau}{2} \right) k' + 2 \coth \tau F + f - k = \tanh \left( \frac{\tau}{2} \right) .
\] (5.2.20)

So the vanishing of the dilatino SUSY variation results in this differential equation for the functions of the ansatz.
5.2. **KILLING SPINORS**

### 5.2.2 Gravitino Minkowski components

Let us now study the SUSY variation of the gravitino, *i.e.* equations (4.2.2). All the components along the Minkowski directions yield the same equation, so for illustrative purposes we will write the equation corresponding to the \( x^1 \) component, namely \( \delta \psi_{x^1} = 0 \). Looking back at eq. (4.2.2) one can write:

\[
D_{x^1} \epsilon + \frac{i}{1920} F_{N_1 \cdots N_3}^{(5)} \Gamma^{N_1 \cdots N_5} \Gamma_{x^1} \epsilon + \frac{1}{96} F_{N_1 N_2 N_3}^{(3)} (\Gamma_{x^1} N_1 N_2 N_3 - 9 \delta_{x^1}^{N_1} \Gamma^{N_2 N_3}) \epsilon^* = 0, 
\]

(5.2.21)

We will analyze the different pieces of this equation separately. Recalling the expression of the covariant derivative and assuming that \( \epsilon \) does not depend on the Minkowski coordinates, the first piece can be written as \( \frac{1}{4} \omega^{a b} \Gamma_{a b} \epsilon \). Using the spin connection written in (5.2.3), one gets that

\[
D_{x^1} \epsilon = \frac{1}{4} \omega^{a b} \Gamma_{a b} \epsilon = -\frac{1}{8} \sqrt{\mu} K(\tau) h^{-\frac{2}{3}} h^{\prime} \Gamma_{x^1} \epsilon .
\]

(5.2.22)

Inserting the five-form written in (5.2.8) into the second piece of equation (5.2.21) it becomes:

\[
\frac{i}{1920} F_{N_1 \cdots N_3}^{(5)} \Gamma^{N_1 \cdots N_5} \Gamma_{x^1} \epsilon = \frac{i}{8} \sqrt{\mu} K(\tau) h^{-\frac{2}{3}} h^{\prime} \Gamma_{x^1} \epsilon ,
\]

(5.2.23)

where we have used eq. (5.2.15). So in view of these last two equalities (*i.e.* (5.2.22) and (5.2.23)), the first two pieces of eq. (5.2.21) can be written as

\[
D_{x^1} \epsilon + \frac{i}{1920} F_{N_1 \cdots N_3}^{(5)} \Gamma^{N_1 \cdots N_5} \Gamma_{x^1} \epsilon = \frac{1}{8} \sqrt{\mu} K(\tau) h^{-\frac{2}{3}} h^{\prime} \Gamma_{x^1} (1 - i \Gamma_{x^0 x^1 x^2 x^3} \Gamma_{x^1}) \epsilon .
\]

(5.2.24)

In order to make this expression vanish we should impose the projection (5.2.14). Since, as we will see below, the remaining terms in equation (5.2.21) do not mix up with these ones; it becomes clear that we must impose that projection to satisfy the equation (5.2.21).

Let us now look at the third term of equation (5.2.21), the one depending on the three-form \( F^{(3)} \). Taking into account the vanishing of the first two pieces and that the three-form has no components along \( x^1 \), equation (5.2.21) reduces to

\[
F_{N_1 N_2 N_3}^{(3)} \Gamma^{N_1 N_2 N_3} \epsilon^* = 0 ,
\]

(5.2.25)

which is very similar to the equation resulting from the vanishing of the dilatino SUSY variation. In fact, this equation is equal to equation (5.2.11) but with \( \epsilon^* \) instead of \( \epsilon \). Therefore, proceeding as we did there, but using the conjugated projections (for instance \( \Gamma_{x^1} \epsilon^* = i \epsilon^* \) instead of (5.2.16)), we arrive at the following equation:

\[
(i \hat{P}_1 \Gamma_{\hat{3}} + \hat{P}_2 \Gamma_{\hat{3}11}) \epsilon^* = 0 ,
\]

(5.2.26)

where

\[
\hat{P}_1 = \frac{i \sinh \tau}{2 \cosh \tau} (f' + k' - 1) - \frac{i}{2 \sinh \tau} \left[ (1 + g)^2 f' + (1 - g)^2 k' - 1 - g (4 F - 2 + g) \right] - \frac{i}{\sinh \tau \cosh \tau} [(1 - g) k' - (1 + g) f' - 1 + g + 2 F] ,
\]

(5.2.27)
and
\[
\begin{align*}
\hat{P}_2 &= \frac{-1}{2\cosh^2\tau} (f' + k' - 1) + \frac{1}{2\sinh^2\tau} \left[ (1 + g)^2 f' + (1 - g)^2 k' - 1 - g(4F - 2 + g) \right] - \\
&\quad - \frac{1}{\cosh\tau} \left[ (1 - g) k' - (1 + g) f' - 1 + g + 2F \right] - \frac{1}{\sinh\tau} (2f' + k - f).
\end{align*}
\]

Using again that \( g = \frac{1}{\cosh\tau} \) it is not difficult to see that \( \hat{P}_1 \) is identically zero. In fact, if in the expression of \( \hat{P}_1 \) one changes \( f' \rightarrow -f' \) and \( k' \rightarrow -k' \) one arrives at (5.2.18), i.e. at \( P_1 \). Since \( P_1 \) vanishes independently of the form of \( f' \) and \( k' \), then \( \hat{P}_1 \) must also vanish. Then, in order to satisfy the equation \( \delta\psi_{x^1} = 0 \), \( \hat{P}_2 \) must vanish. Inserting the value of \( g \), the equation \( \hat{P}_2 = 0 \) can be written as
\[
2F' - \coth\left(\frac{\tau}{2}\right) f' + \tanh\left(\frac{\tau}{2}\right) k' + 2\coth\tau F - f + k = \tanh\left(\frac{\tau}{2}\right),
\]
which is another differential equation for the unknown functions of the model.

### 5.2.3 Gravitino angular components

We still have to solve the equations resulting from the angular components of the gravitino. We will begin with the equation \( \delta\psi_1 = 0 \), namely:
\[
D_1 \epsilon + \frac{i}{1920} F^{(5)}_{N_1\cdots N_5} \Gamma^{N_1\cdots N_5} \Gamma_1 \epsilon + \frac{1}{96} F^{(3)}_{N_1 N_2 N_3} (\Gamma_1 \Gamma_{N_1 N_2 N_3} - 9 \delta_i^{N_1} \Gamma_{N_2 N_3}) \epsilon^* = 0.
\]

As for the \( x^1 \) equation we will study each contribution to this equation separately. Reading the spin connection from eq. (5.2.3), and assuming that \( \epsilon \) does not depend on the compact coordinates of the \( T^{1,1} \), the first term becomes:
\[
D_1 \epsilon = \frac{1}{4} \omega^{ab}_1 \Gamma_{ab} \epsilon = \frac{1}{8} \frac{\sqrt{6}}{\mu^2} K(\tau) h^{-\frac{4}{3}} h' \Gamma_{1\tau} \epsilon.
\]

We have taken into account that all the terms in \( \omega^{ab}_1 \Gamma_{ab} \epsilon \), except for the one depending on \( h' \), will cancel each other by virtue of eq. (2.2.8) in chapter 2. This is so because, as we have already said, the one-form frame (5.2.2) only differs from the one in eq. (2.3.22) in some \( h \) factors. Indeed, one can write \( \omega^{ab}_1 \Gamma_{ab} \epsilon = h^{-\frac{4}{3}} \omega^\breve{a}\breve{b}_\gamma (1) \Gamma_{\breve{a}\breve{b}} \epsilon + \text{terms (} h' \text{)} \); where \( \omega^\breve{a}\breve{b}_\gamma \) stands for the spin connection of the deformed conifold (eq. (2.2.5) restricted to the particular solution (2.3.12)) and the indices \( \breve{a}, \breve{b} \) refer to the corresponding frame, i.e. (2.3.22). Finally, from the analysis done in chapter 2, it is clear that \( \omega^\breve{a}\breve{b}_\gamma (1) \Gamma_{\breve{a}\breve{b}} \epsilon = 0 \) follows from the more general equation (2.2.8).

Inserting the RR five-form as it is written in eq. (5.2.8) and making use of eq. (5.2.15), the second term of (5.2.30) takes the form:
\[
\frac{i}{1920} F^{(5)}_{N_1\cdots N_5} \Gamma^{N_1\cdots N_5} \Gamma_1 \epsilon = \frac{i}{8} \frac{\sqrt{6}}{\mu^2} K(\tau) h^{-\frac{4}{3}} h' \Gamma_{x^0 x^1 x^2 x^3} \Gamma_{\tau 1} \epsilon,
\]
and by adding it to the first term written in eq. (5.2.31), one gets that

\[
D_1 \epsilon + \frac{i}{1920} F^{(5)}_{N_1 \cdots N_5} \Gamma^{N_1 \cdots N_5} \Gamma_1 \epsilon = \frac{1}{8} \sqrt{\frac{6}{\mu^2}} K(\tau) h^{-\frac{2}{6}} h' \Gamma_{\tau 1} (-1 + i \Gamma_{x^2 x^1 x^2 x^3}) \epsilon, \tag{5.2.33}
\]

which vanishes for a spinor \( \epsilon \) satisfying the projection (5.2.14).

It only remains the third term in eq. (5.2.30). Let us study separately the two terms multiplying the complex three-form \( F^{(3)} \). Reading the three-form fluxes from (5.2.9) and (5.2.10), the first term, i.e. \( \frac{1}{96} F^{(3)}_{N_1 N_2 N_3} \Gamma_1 \Gamma_{N_1 N_2 N_3} \epsilon^* \), becomes:

\[
- \sqrt{\frac{6}{\mu^2}} \frac{M \alpha'}{8} \Gamma_1 \left\{ \frac{f' + k'}{2 \cosh \tau} \Gamma_{\tau 2 1} + \frac{k - f}{2 \sinh \tau} \Gamma_{\tau 3 2 2} + \frac{1}{2 \sinh \tau} [(1 - g) k' - (1 + g) f'] \Gamma_{\tau 2 1} + \right.
\]
\[\left. + \frac{i}{2 \cosh \tau} \Gamma_{\tau 3 2 1} + \frac{i}{2 \sinh \tau} (1 - g - 2 F) \Gamma_{\tau 3 1 2} + \frac{i F'}{\sinh \tau} \Gamma_{\tau 2 2} \right\} \epsilon^*. \tag{5.2.34}
\]

Imposing the projections (5.2.12) and (5.2.16), which after complex conjugation become:

\( \Gamma_{12} \epsilon^* = -\Gamma_{12} \epsilon^* \) and \( \Gamma_{3 \tau} \epsilon^* = i \epsilon^* \), and neglecting the common factor \( -\sqrt{\frac{6}{\mu^2}} \frac{M \alpha'}{8} \Gamma_1 \), equation (5.2.34) takes the form:

\[
\left\{ i \left[ \frac{f' + k'}{2 \cosh \tau} \right] - \frac{1}{2 \cosh \tau} \right\} \Gamma_{\tau 3 \bar{1} 2} - \frac{1}{\sinh \tau} \left( \frac{F' + k - f}{2} \right) \Gamma_{\tau 3 \bar{1} 1} +
\]
\[+ i \left\{ \frac{1}{2 \sinh \tau} [(1 - g) k' - (1 + g) f'] - \frac{1}{2 \sinh \tau} (1 - g - 2 F) \right\} \Gamma_{\tau 3 \bar{1} 2} \right\} \epsilon^*, \tag{5.2.35}
\]

which by using suitable combinations of the complex conjugate of projections (5.2.13) and (5.2.16) can be written as

\[
( Q_1 \Gamma_3 + Q_2 \Gamma_{3 \bar{1} 1} ) \epsilon^*, \tag{5.2.36}
\]

with

\[
Q_1 = -\frac{1}{2} \left\{ \frac{\sinh \tau}{\cosh^2 \tau} (f' + k' - 1) - \frac{1}{\sinh \tau \cosh \tau} [(1 - g) k' - (1 + g) f' - 1 + g + 2 F] \right\}, \tag{5.2.37}
\]

and

\[
Q_2 = -\frac{1}{2} \left\{ \frac{1}{\cosh^2 \tau} (f' + k' - 1) + \frac{1}{\cosh \tau} [(1 - g) k' - (1 + g) f' - 1 + g + 2 F] \right\} -
\]
\[\frac{1}{\sinh \tau} \left( F' + \frac{k - f}{2} \right). \tag{5.2.38}
\]

As we will see, if we do not introduce any extra projection, the remaining terms in equation (5.2.30) will not mix up with these ones. Therefore one must require that \( Q_1 = 0 \), and
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\( Q_2 = 0 \). Furthermore, recalling that \( g = \frac{1}{\cosh \tau} \) one can straightforwardly check that the equation \( Q_1 = 0 \) takes the form:

\[
\coth \left( \frac{\tau}{2} \right) f' + \tanh \left( \frac{\tau}{2} \right) k' - \frac{2F}{\sinh \tau} = \tanh \left( \frac{\tau}{2} \right),
\]

and the equation \( Q_2 = 0 \) can be written as

\[
f' - k' - 2F + 1 - 2 \coth \tau \left( \frac{F' + \frac{k-f}{2}}{2} \right) = 0.
\]

(5.2.39)

We will now write down the second term depending on the complex three-form, namely

\[
-\frac{9}{96} F^{[3]}_{N_1 N_2 N_3} \delta^{N_1}_{\bar{N}1} \Gamma^{N_2 N_3} \epsilon^*,
\]

which after inserting the three-forms (5.2.9) and (5.2.10) becomes:

\[
\frac{\sqrt{6} \, h^{-\frac{1}{2}} \, 3M \alpha'}{\mu^2} \left\{ \frac{i}{2 \sinh \tau} (1 - g - 2F) \Gamma_{\bar{3}2} + \frac{i \cosh \tau}{2 \sinh^2 \tau} \left[ 1 + g \left( g - 2 + 4F \right) \right] \Gamma_{\bar{3}2} + \frac{iF'}{\sinh \tau} \Gamma_{\tau 1} + \frac{1}{2 \sinh \tau} \left[ (1 - g) k' - (1 + g) f' \right] \Gamma_{\tau 2} + \frac{k - f}{2 \sinh \tau} \Gamma_{3i} + \frac{\cosh \tau}{2 \sinh^2 \tau} \left[ (1 - g)^2 k' + (1 + g)^2 f' \right] \Gamma_{13i2} \right\} \epsilon^*,
\]

(5.2.40)

and, after making use of the complex conjugate of projections (5.2.12) and (5.2.16), and neglecting the common factor \( \frac{\sqrt{6} \, h^{-\frac{1}{2}} \, 3M \alpha'}{\mu^2} \), it reads:

\[
\left\{ \frac{i}{2 \sinh \tau} [1 - g - 2F - (1 - g) k' + (1 + g) f'] \Gamma_{\bar{3}2} + \frac{1}{2 \sinh \tau} (2F' + k - f) \Gamma_{3i} + \frac{i \cosh \tau}{2 \sinh^2 \tau} \left[ 1 + g \left( g - 2 + 4F \right) \right] \Gamma_{13i2} \right\} \epsilon^*.
\]

(5.2.41)

By imposing suitable combinations of the complex conjugate of projections (5.2.13) and (5.2.16), this last expression can be written as

\[
\left( \hat{Q}_1 \Gamma_{3i} + \hat{Q}_2 \Gamma_{\bar{3}i} \right) \epsilon^*,
\]

(5.2.42)

where

\[
\hat{Q}_1 = \frac{1}{2 \sinh \tau \cosh \tau} [1 - g - 2F - (1 - g) k' + (1 + g) f'] + \frac{1}{2 \sinh \tau} \left[ 1 + g \left( g - 2 + 4F \right) \right] \Gamma_{13i2},
\]

(5.2.43)

and

\[
\hat{Q}_2 = \frac{1}{2 \cosh \tau} [-1 + g + 2F + (1 - g) k' - (1 + g) f'] + \frac{1}{\sinh \tau} \left( F' + \frac{k-f}{2} \right) + \frac{1}{2 \sinh^2 \tau} \left[ 1 + g \left( g - 2 + 4F \right) \right] \Gamma_{13i2}.
\]

(5.2.44)
In order to satisfy the equation $\delta \psi_1 = 0$ (i.e. eq. (5.2.30)) without imposing new projections on $\epsilon$ we must require that $\hat{Q}_1 = 0$, and $\hat{Q}_2 = 0$. Using that $g = \frac{1}{\cosh \tau}$ one can easily check that $\hat{Q}_1 = 0$ renders the same differential equation as the expression $Q_1 = 0$, namely equation (5.2.39). After inserting the value of $g$, the expression $\hat{Q}_2 = 0$ yields the following differential equation:

$$
\tan^2 \left( \frac{\tau}{2} \right) k' - \coth^2 \left( \frac{\tau}{2} \right) f' + 2 \left( \coth^2 \tau + \csch^2 \tau \right) F + 2 \coth \tau \left( F' + \frac{k - f}{2} \right) = \tan^2 \left( \frac{\tau}{2} \right).
$$

(5.2.46)

Then, from the equation $\delta \psi_1 = 0$, we have got three differential equations for the unknown functions of the Klebanov-Strassler ansatz; these are the equations (5.2.39), (5.2.40), and (5.2.46).

The vanishing of the SUSY variation of $\psi_2$, (i.e. $\delta \psi_2 = 0$) results in the same differential equations as the ones we got above from requiring that $\delta \psi_1 = 0$. As it happened before, the term coming from the covariant derivative and the one containing the RR five-form cancel each other after using the projection (5.2.14). Furthermore, if one imposes suitable combinations of the complex conjugate of projections (5.2.13) and (5.2.16), the terms containing the complex three-form $F^{(3)}$ result to be equal to the ones appearing in $\delta \psi_1 = 0$ and therefore, the arising differential equations are the same ones as before.

Let us now impose the cancellation of the SUSY variation of $\psi_3$. The equation we have to solve is:

$$
\hat{D}_3 \epsilon + \frac{i}{1920} F^{(5)}_{N_1 \ldots N_5} \Gamma^{N_1 \ldots N_5} \hat{\Gamma}_3 \epsilon + \frac{1}{96} F^{(3)}_{N_1N_2N_3} \left( \Gamma^{N_1N_2N_3}_3 - 9 \hat{\delta}_3^{N_1} \Gamma^{N_2N_3}_3 \right) \epsilon^* = 0.
$$

(5.2.47)

We will follow the same steps as for the preceding components of the gravitino. Then, let us write down the form of the first term of this last equation for a spinor $\epsilon$ independent of the compact coordinates of the $T^{1,1}$. It reads:

$$
\hat{D}_3 \epsilon = \frac{1}{4} \omega_3^{ab} \Gamma_{ab} \epsilon = \frac{1}{8} \sqrt{6} \mu_3 K(\tau) h^{-\frac{3}{2}} h' \Gamma_{3\tau} \epsilon,
$$

(5.2.48)

where again we have used the fact that all terms in $\omega_3^{ab} \Gamma_{ab} \epsilon$, apart from the ones depending on $h'$, cancel each other as we have explained below eq. (5.2.31). Reading the RR five-form from eq. (5.2.8) and using eq. (5.2.15), the second term of eq. (5.2.47) becomes:

$$
\frac{i}{1920} F^{(5)}_{N_1 \ldots N_5} \Gamma^{N_1 \ldots N_5} \hat{\Gamma}_3 \epsilon = \frac{i}{8} \sqrt{6} \mu_3 K(\tau) h^{-\frac{3}{2}} h' \Gamma_{x^a x^1 x^2 x^3} \Gamma_{\tau \beta} \epsilon,
$$

(5.2.49)

and one can easily check that by imposing the projection (5.2.14) on $\epsilon$, this last expression cancels the term written in eq. (5.2.48). Thus, as before, the terms coming from the covariant derivative and from the five-form term cancel each other. Then, we are left with the terms containing the complex three-form. Making use of the expressions for the three-forms written in eqs. (5.2.9) and (5.2.10) the first term containing $F^{(3)}$ can be written as

$$
\frac{1}{96} F^{(3)}_{N_1N_2N_3} \Gamma^{N_1N_2N_3}_3 \epsilon^* = -\frac{\sqrt{6} h^{-\frac{3}{2}} M \alpha'}{8} \sqrt{\frac{\tau}{\sinh \tau}} \left( \Gamma_{\tau 22} + \Gamma_{\tau 11} \right) + \frac{f' + k'}{2 \cosh \tau} \Gamma_{\tau 21} +
$$
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\[ + \frac{1}{2 \sinh \tau} [(1 - g) k' - (1 + g) f'] (\Gamma_{r21} + \Gamma_{r21}) + \frac{\cosh \tau}{2 \sinh^2 \tau} [(1 - g)^2 k' + (1 + g)^2 f'] \Gamma_{r21} \] \quad \epsilon^*,

(5.2.50)

and imposing the projections \( \Gamma_{12} \epsilon^* = -\Gamma_{12} \epsilon^* \), and \( \Gamma_{33} \epsilon^* = i \epsilon^* \) (complex conjugate of (5.2.12) and (5.2.16) respectively) it becomes:

\[
\left\{ i \left[ \frac{f' + k'}{2 \cosh \tau} - \frac{\cosh \tau}{2 \sinh^2 \tau} [(1 - g)^2 k' + (1 + g)^2 f'] \right] \Gamma_{312} - \frac{2F'}{\sinh \tau} \Gamma_{311} + \right.

\left. + \frac{i}{\sinh \tau} [(1 - g) k' - (1 + g) f'] \Gamma_{312} \right\} \epsilon^*,
\]

(5.2.51)

where we have neglected the common factor \(-\sqrt{\frac{6}{\mu^2}} \Gamma_{33} \).

Again, if we insert some combinations of the complex conjugate of projections (5.2.13) and (5.2.16), this last expression can be written as the sum of two independent terms:

\[
(M_1 \Gamma_{3} + M_2 \Gamma_{311}) \epsilon^*,
\]

(5.2.52)

with

\[
M_1 = - \frac{\sinh \tau}{2 \cosh^2 \tau} (f' + k') + \frac{1}{2 \sinh \tau} [(1 - g)^2 k' + (1 + g)^2 f'] + \\
+ \frac{1}{\sinh \tau \cosh \tau} [(1 - g) k' - (1 + g) f'],
\]

(5.2.53)

which results to be identically zero after substituting \( g \) by its value, i.e. \( \frac{1}{\cosh \tau} \). On the other hand,

\[
M_2 = - \frac{f' + k'}{2 \cosh^2 \tau} + \frac{1}{2 \sinh^2 \tau} [(1 - g)^2 k' + (1 + g)^2 f'] - \\
- \frac{1}{\cosh \tau} [(1 - g) k' - (1 + g) f'] - \frac{2F'}{\sinh \tau},
\]

(5.2.54)

and, since the remaining terms in \( \delta \psi_3 = 0 \) (eq. (5.2.47)) will not mix up with this last one, one must have \( M_2 = 0 \). Hence, by inserting \( g = \frac{1}{\cosh \tau} \) we get the following differential equation:

\[
\text{coth} \left( \frac{\tau}{2} \right) f' - \text{tanh} \left( \frac{\tau}{2} \right) k' - 2F' = 0.
\]

(5.2.55)

The last term in eq. (5.2.47) is:

\[
- \frac{9}{96} f_{N_1 N_2 N_3}^{(3)} \delta_{\hat{3}}^{N_1} \Gamma_{N_2 N_3} \epsilon^* = \frac{\sqrt{6} h^{-\frac{3}{2}}}{\mu^2} \frac{3M \alpha'}{8} \left\{ \frac{k-f}{2 \sinh \tau} (\Gamma_{22} + \Gamma_{11}) + \frac{i}{2 \cosh \tau} \Gamma_{21} + \\
+ \frac{i}{2 \sinh \tau} (1 - g - 2F) (\Gamma_{21} + \Gamma_{21}) + \frac{i \cosh \tau}{2 \sinh^2 \tau} [1 + g (g - 2 + 4F)] \right\} \epsilon^*,
\]

(5.2.56)
where we have used the expressions of the three-forms written in eqs. (5.2.9) and (5.2.10). Making use of the projection $\Gamma_{12} \epsilon^* = -\Gamma_{12} \epsilon^*$ (see eq. (5.2.12)) and neglecting the common factor $\sqrt{6h^{-3}} \sqrt{3M_{\alpha'}}$, this last expression becomes:

$$\left\{ \frac{k-f}{\sinh \tau} \Gamma_{11} + \frac{i}{\sinh \tau} (1 - g - 2F) \Gamma_{21} + i \left[ \frac{\cosh \tau}{2 \sinh^2 \tau} [1 + g (g - 2 + 4F)] - \frac{1}{2 \cosh \tau} \right] \Gamma_{12} \right\} \epsilon^*, \quad (5.2.57)$$

which by imposing suitable combinations of projections (5.2.13) and (5.2.16) can be written as

$$\left( \hat{M}_1 + \hat{M}_2 \Gamma_{11} \right) \epsilon^*, \quad (5.2.58)$$

where

$$\hat{M}_1 = -\frac{1}{2 \sinh \tau} \left[ 1 + g (g - 2 + 4F) \right] + \frac{\sinh \tau}{2 \cosh^2 \tau} - \frac{1}{\sinh \tau \cosh \tau} (1 - g - 2F), \quad (5.2.59)$$

and,

$$\hat{M}_2 = \frac{1}{2 \sinh^2 \tau} \left[ 1 + g (g - 2 + 4F) \right] - \frac{1}{2 \cosh^2 \tau} - \frac{1}{\cosh \tau} (1 - g - 2F) + \frac{k-f}{\sinh \tau}. \quad (5.2.60)$$

One can readily check that by inserting $g = \frac{1}{\cosh \tau}$, $\hat{M}_1 = 0$ automatically. In order to satisfy the equation $\delta \psi_3 = 0$ we must also have $\hat{M}_2 = 0$, which after substituting $g = \frac{1}{\cosh \tau}$ yields an algebraic relation between the functions entering the Klebanov-Strassler ansatz, namely:

$$2 \coth \tau F + k - f = \tanh \left( \frac{\tau}{2} \right). \quad (5.2.61)$$

Requiring the vanishing of the SUSY variation of the remaining angular components of the gravitino (i.e. $\psi_1$ and $\psi_2$) will not give rise to any new equation relating the functions of the ansatz. Indeed, from the equations $\delta \psi_1 = \delta \psi_2 = 0$ one gets the same equations as from imposing $\delta \psi_1 = 0$; these are eqs. (5.2.39), (5.2.40) and (5.2.46).

### 5.2.4 Gravitino radial component

Finally, we shall look at the SUSY variation of the radial component of the gravitino. Then, we must solve the equation

$$D_\tau \epsilon + \frac{i}{1920} \tilde{F}^{(5)}_{N_1 \ldots N_5} \Gamma_{N_1 \ldots N_5} \tau \epsilon + \frac{1}{96} \tilde{F}^{(3)}_{N_1 N_2 N_3} (\Gamma_{\tau N_1 N_2 N_3} - 9 \delta_{\tau N_1} N_2 N_3) \epsilon^* = 0. \quad (5.2.62)$$

As it obviously implies the radial projection (5.2.13), the spinor $\epsilon$ depends on the radial coordinate. Then, the covariant derivative can be written in terms of $\epsilon' \equiv \frac{d\epsilon}{d\tau}$ as $D_\tau \epsilon = (E_\tau^r)^{-1} \left( \epsilon' + \frac{1}{4} \omega^{ab}_\tau \Gamma_{ab} \epsilon \right)$. Thus, reading the spin connection one-form from eq. (5.2.3), the covariant derivative becomes:

$$D_\tau \epsilon = \frac{\sqrt{6} K(\tau)}{\mu^2 \frac{1}{h^2}} \left( \epsilon' + \frac{1}{2 \cosh \tau} \Gamma_{11} \epsilon \right), \quad (5.2.63)$$

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where we have already imposed the projection (5.2.12) and we have taken into account that
\[ \omega^{\alpha \beta}_\tau = E^\tau_\tau \omega^{\alpha \beta} = \frac{\mu^2}{\sqrt{2}} \frac{h^\tau}{K(\tau)} \omega^{\alpha \beta}. \]

We will show below that the terms in eq. (5.2.62) containing the complex three-form will vanish, so it must happen again that the first two terms in that equation cancel each other. Thus, after inserting the RR five-form (5.2.8), eq. (5.2.15) and projection (5.2.14), we get the following equation:
\[ \epsilon' + \frac{1}{2 \cosh \tau} \Gamma_{11} \epsilon + \frac{1}{8} h^{-1} h' \epsilon = 0. \] (5.2.64)

At this point let us go back to the radial projection written in eq. (5.2.13) and notice that it can be solved as
\[ \epsilon = e^{-\frac{1}{2} \alpha \Gamma_{11}} \epsilon_0, \quad \Gamma_{\tau \hat{1} \hat{2} \hat{3}} \epsilon_0 = -\epsilon_0, \] (5.2.65)
with
\[ \sin \alpha = -\frac{1}{\cosh \tau}, \quad \cos \alpha = \frac{\sinh \tau}{\cosh \tau}. \] (5.2.66)

Plugging (5.2.65) into eq. (5.2.64), one arrives at
\[ e^{-\frac{1}{2} \alpha \Gamma_{11}} \left( \epsilon_0 - \frac{1}{2} \alpha' \Gamma_{11} \epsilon_0 + \frac{1}{2 \cosh \tau} \Gamma_{11} \epsilon_0 + \frac{1}{8} h^{-1} h' \epsilon_0 \right) = 0, \] (5.2.67)
which yields the following two equations:
\[ \alpha' = \frac{1}{\cosh \tau}, \] (5.2.68)
\[ \epsilon'_0 = -\frac{1}{8} h^{-1} h' \epsilon_0. \] (5.2.69)

The first equation is satisfied for \( \alpha \) written in eq. (5.2.66). While eq. (5.2.69) determines the radial dependence of the 10d spinor \( \epsilon_0 \). So finally, the Killing spinors of the Klebanov-Strassler model become:
\[ \epsilon = e^{-\frac{1}{2} \alpha \Gamma_{11}} h^{-\frac{1}{2}} \eta, \] (5.2.70)
with \( \alpha \) being given by eq. (5.2.66) and \( \eta \) being a constant 10d spinor satisfying the following projections:
\[ \Gamma_{\tau \hat{1} \hat{2} \hat{3}} \eta = -\eta, \quad \Gamma_{12} \eta = -\Gamma_{\hat{1} \hat{2}} \eta, \quad \Gamma_{x^0 x^1 x^2 x^3} \eta = -i \eta. \] (5.2.71)

Therefore, the model has 4 independent spinors as it should be for the SUGRA dual of a 4d \( \mathcal{N}=1 \) field theory.

The projections (5.2.71) can be rewritten as
\[ \Gamma_{x^0 x^1 x^2 x^3} \eta = -i \eta, \quad \Gamma_{12} \eta = i \eta, \quad \Gamma_{\hat{1} \hat{2}} \eta = -i \eta, \] (5.2.72)
after making use of the equality \( \Gamma_{x^0 x^1 x^2 x^3} \Gamma_{\tau 12 \hat{1} \hat{2} \hat{3}} \eta = -\eta \). However, the last two projections, corresponding to the ones of the \( T^{1,1} \), are not satisfied by the Killing spinor \( \epsilon \) written in eq. (5.2.65) due to the factor \( e^{-\frac{1}{2} \alpha \Gamma_{11}} \) which anticommutes with them. Only when \( \tau \to \infty \) the angle \( \alpha \) vanishes (see eq. (5.2.66)) and the Killing spinor satisfies the projections corresponding to the \( T^{1,1} \), as it happened for the Klebanov-Tseytlin background in the last
chapter. This must be so, since the KS and the KT solutions are identical in the UV, far away from the tip of the conifold ($\tau = 0$).

Let us show that, as we have said above, the terms in (5.2.62) containing the three-forms effectively vanish. The first term, \( i.e. \frac{1}{96} F^{(3)}_{N_1 N_2 N_3} \Gamma^N_{N_1 N_2 N_3} \Gamma^* \), takes the form:

\[
-\frac{\sqrt{6} h^{-\frac{3}{2}} M \alpha'}{\mu^2} \frac{8}{\sinh \tau} \Gamma \left\{ \frac{k - f}{2 \sinh \tau} (\Gamma_{\bar{3}21} + \Gamma_{311}) + \frac{i}{2 \sinh \tau} (1 - g - 2F) (\Gamma_{\bar{3}21} + \Gamma_{321}) + \right. \\
\left. + \frac{i}{2 \cosh \tau} \Gamma_{\bar{3}21} + \frac{i \cosh \tau}{2 \sinh^2 \tau} [1 + g (g - 2 + 4F)] \Gamma_{321} \right\} \epsilon^*, \quad (5.2.73)
\]

and by using the projection $\Gamma_{12} \epsilon^* = -\Gamma_{12} \epsilon^* \quad (\text{and neglecting the common factor } -\frac{\sqrt{6} h^{-\frac{3}{2}} M \alpha'}{\mu^2} \Gamma_{\tau})$, one can write it as

\[
\left\{ \frac{k - f}{\sinh \tau} \Gamma_{311} \right. + \frac{i}{\sinh \tau} (1 - g - 2F) \Gamma_{\bar{3}21} + \left[ \frac{i \cosh \tau}{2 \sinh^2 \tau} [1 + g (g - 2 + 4F)] - \frac{i}{2 \cosh \tau} \right] \Gamma_{312} \epsilon^*, \quad (5.2.74)
\]

which is nothing else but $\Gamma_{3} \times \text{eq. (5.2.57)}$ and then, it vanishes once we impose the differential equation (5.2.61).

The last term of eq. (5.2.62), \( i.e. -\frac{9}{96} F^{(3)}_{N_1 N_2 N_3} \delta^N_{\tau} \Gamma^N_{N_2 N_3} \epsilon^* \), can be written as

\[
\frac{\sqrt{6} h^{-\frac{3}{2}} M \alpha'}{\mu^2} \frac{3 M \alpha'}{8} \left\{ \frac{f' + k'}{2 \cosh \tau} \Gamma_{2i} + \frac{1}{2 \sinh \tau} [(1 - g) k' - (1 + g) f'] (\Gamma_{21} + \Gamma_{2i}) + \right. \\
\left. + \frac{\cosh \tau}{2 \sinh^2 \tau} [(1 - g)^2 k' + (1 + g)^2 f'] \Gamma_{21} + \frac{i F'}{\sinh \tau} (\Gamma_{22} + \Gamma_{11}) \right\} \epsilon^*, \quad (5.2.75)
\]

which, after using the projection $\Gamma_{12} \epsilon^* = -\Gamma_{12} \epsilon^*$, and neglecting the common factor $\frac{\sqrt{6} h^{-\frac{3}{2}} M \alpha'}{\mu^2}$, becomes:

\[
\left\{ \left[ \frac{\cosh \tau}{2 \sinh^2 \tau} [(1 - g)^2 k' + (1 + g)^2 f'] - \frac{f' + k'}{2 \cosh \tau} \right] \Gamma_{12} + \right. \\
\left. \frac{1}{\sinh \tau} [(1 - g) k' - (1 + g) f'] \Gamma_{21} + \frac{2 i F'}{\sinh \tau} \Gamma_{11} \right\} \epsilon^*. \quad (5.2.76)
\]

Multiplying this last expression by $-i \Gamma_{3}$ one recovers eq. (5.2.51), thus, the differential eq. (5.2.55) implies the vanishing of the last term of eq. (5.2.62).

### 5.3 Differential equations for the KS ansatz

In the last section we have obtained six differential equations (5.2.20), (5.2.29), (5.2.39), (5.2.40), (5.2.46), (5.2.55) and an algebraic constraint (5.2.61) relating the functions $f(\tau)$, $k(\tau)$ and $F(\tau)$ entering the ansatz of the model. We will see that these equations reduce to the system (5.1.10) appearing in [16] together with an extra equation.
Plugging eqs. (5.2.55) and (5.2.61) into eq. (5.2.20), one gets:

$$F' = \frac{k - f}{2},$$  \hspace{1cm} (5.3.1)

which is one of the differential equations entering the system (5.1.10). Inserting this last equation into the algebraic constraint (5.2.61) one gets the following differential equation involving only $F$ and its first derivative:

$$F' + \coth \tau F = \frac{1}{2} \tanh \left( \frac{\tau}{2} \right).$$  \hspace{1cm} (5.3.2)

This equation can be easily integrated, yielding the explicit form of $F$ (see below). In addition, by substituting the value of $F'$ given by this last equation into eq. (5.2.55) one arrives at

$$\coth \left( \frac{\tau}{2} \right) f' - \tanh \left( \frac{\tau}{2} \right) k' = -2 \coth \tau F + \tanh \left( \frac{\tau}{2} \right).$$  \hspace{1cm} (5.3.3)

Looking at equations (5.3.1) and (5.3.2) one can easily write:

$$F' + \frac{k - f}{2} = 2F' = \tanh \left( \frac{\tau}{2} \right) - 2 \coth \tau F.$$  \hspace{1cm} (5.3.4)

Let us substitute this last result into equation (5.2.40). After some calculation we obtain:

$$f' - k' = -\left[ \coth^2 \left( \frac{\tau}{2} \right) + \tanh^2 \left( \frac{\tau}{2} \right) \right] F + \tanh^2 \left( \frac{\tau}{2} \right).$$  \hspace{1cm} (5.3.5)

By combining this equation with eq. (5.3.3), one can solve for $f'$ and $k'$ as functions of $F$, resulting:

$$f' = (1 - F) \tanh^2 \left( \frac{\tau}{2} \right), \hspace{0.5cm} k' = F \coth^2 \left( \frac{\tau}{2} \right).$$  \hspace{1cm} (5.3.6)

These equations, together with eq. (5.3.1) form the first-order system (5.1.10). For the remaining equations one can easily check that eq (5.3.39) is trivially satisfied after substituting (5.1.10), while (5.2.29) and (5.2.46) are verified after inserting (5.1.10) and the new equation for $F'$, i.e. (5.3.2).

Summing up our results; from imposing the cancellation of the SUSY variations of the dilatino and the gravitino we have obtained the first-order system (5.1.10) appearing in [16] and, in addition, we got a new differential equation, namely (5.3.2), or, alternatively, the algebraic relation (5.2.61). The differential equation (5.3.2) can be easily integrated by the method of variation of constants, rendering:

$$F = \frac{1}{2} \frac{\sinh \tau - \tau}{\sinh \tau} + \frac{A}{\sinh \tau},$$  \hspace{1cm} (5.3.7)

where $A$ is a constant, which by requiring regularity of $F$ at $\tau = 0$ gets fixed to the value $A = 0$. Then, it is immediate to integrate the first order equations for $f$ and $k$ (eqs. (5.3.6)). The result is the same as in ref. [16], namely:

$$F = \frac{1}{2} \frac{\sinh \tau - \tau}{\sinh \tau},$$
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\[ f = \frac{1}{2} \frac{\tau \coth \tau - 1}{\sinh \tau} (\cosh \tau - 1), \]

\[ k = \frac{1}{2} \frac{\tau \coth \tau - 1}{\sinh \tau} (\cosh \tau + 1). \]  

(5.3.8)

Therefore, the requirement of preserving the same supersymmetries as in the solution corresponding to a D3-brane at the tip of the deformed conifold (we are imposing the projection corresponding to a D3-brane (5.2.14) together with the projections satisfied by the Killing spinors of the deformed conifold i.e. (5.2.12) and (5.2.13)) fixes the values of the three-forms to those found in ref. [16] (see refs. [40, 41]).
CHAPTER 5. KILLING SPINORS OF THE KLEBANOV-STRASSLER MODEL
Chapter 6

Killing spinors of the non-commutative MN solution

6.1 Introduction

In this chapter we present the construction of the Killing spinors of the non-commutative deformation of the so-called Maldacena-Núñez (MN) background [19, 20]. The commutative background is dual to the large $N$ limit of $\mathcal{N} = 1$ super Yang-Mills theory. This geometry, generated by a fivebrane wrapping a two-cycle, is smooth and leads to confinement and chiral symmetry breaking.

The spatial non-commutative theories are field theories living on a spacetime where two spatial coordinates do not commute, i.e. $[x^i, x^j] = \Theta^{ij} \neq 0$. These theories have been thoroughly studied in recent years after the discovery that they can be obtained as a low energy limit of string theory in the presence of a Neveu-Schwarz $B$-field [42, 43]. In particular, the non-commutative deformation of the MN background was obtained in [17] by means of a chain of string dualities and it corresponds to the decoupling limit of a (D3,D5) bound state with the D3-brane smeared in the worldvolume of the D5 and wrapped on the two-cycle. The corresponding ten dimensional metric breaks four dimensional Lorentz invariance since it distinguishes between the coordinates of the non-commutative plane and the other two Minkowski coordinates. As expected, this solution has a non-vanishing Neveu-Schwarz $B$-field directed along the non-commutative directions.

After reviewing the details of the non-commutative solution in the next subsection, we will compute the Killing spinors of the model in section 6.2. This computation is similar to the one carried out in [18] for the commutative model. As in that case, working in the frame arising naturally when one obtains the MN model as an uplift from 7d gauged supergravity, the Killing spinors do not depend on the internal coordinates of the geometry.

This computation was performed in the context of the work published in [21], where we studied the addition of flavor degrees of freedom to the supergravity dual of the non-commutative deformation of the maximally supersymmetric gauge theories, see refs. [44, 45]. There we have also studied the possibility of adding flavor to non-commutative duals of less supersymmetric theories as it is the case of the MN background. So in order to do that, using the kappa symmetry approach when looking for supersymmetric embeddings of probe
branes, we needed the explicit form of the Killing spinors for that background.

### 6.1.1 The non-commutative Maldacena-Núñez solution

The procedure used in [17] to obtain the non-commutative deformation of the MN solution leads, as we have said, to a metric where the four dimensional Lorentz symmetry is broken. This metric singles out the so-called non-commutative plane along which the NSNS B-field is directed. In the string frame it is given by

$$ds^2 = e^{\phi} \left[ dx_{0,1}^2 + h^{-1} dx_{2,3}^2 + e^{2g} (d\theta_1^2 + \sin^2 \theta_1 d\phi_1^2) + dr^2 + \frac{1}{4} (w^i - A^i)^2 \right],$$

where $\phi$, $h$ and $g$ are functions of the radial coordinate $r$ (see below) which have nothing to do with the functions denoted by the same letters that appeared in previous chapters. $A^i$ is a one-form which can be written in terms of the angles $(\theta_1, \phi_1)$ and a function $a(r)$ as follows:

$$A^1 = -a(r) d\theta_1, \quad A^2 = a(r) \sin \theta_1 d\phi_1, \quad A^3 = -\cos \theta_1 d\phi_1.$$ (6.1.2)

The $\omega^i$'s appearing in eq. (6.1.1) are again the $SU(2)$ left-invariant one-forms defined in (2.3.5). Moreover, the functions $a(r)$, $g(r)$ and $\phi(r)$ are:

$$a(r) = \frac{2r}{\sinh 2r},$$

$$e^{2g} = r \coth 2r - \frac{r^2}{\sinh^2 2r} - \frac{1}{4},$$

$$e^{-2\phi} = e^{-2\phi_0} \frac{2e^g}{\sinh 2r},$$ (6.1.3)

where $\phi_0$ is a constant ($\phi_0 = \phi(r = 0)$). The function $h(r)$, which distinguishes in the metric the coordinates $x^2 x^3$ from $x^0 x^1$, can be written in terms of the function $\phi(r)$ as follows:

$$h(r) = 1 + \Theta^2 e^{2\phi},$$ (6.1.4)

where $\Theta$ is a constant which parameterizes the non-commutative deformation, so when $\Theta \neq 0$ this background is dual to a gauge theory in which the coordinates $x^2$ and $x^3$ do not commute, being $[x^2, x^3] \sim \Theta^2$.

Let us denote by $\hat{\phi}$ the dilaton field of type IIB supergravity. For the solution of ref. [17] this field takes the value:

$$e^{2\hat{\phi}} = e^{2\phi} h^{-1}.$$ (6.1.5)

Notice that, when the non-commutative parameter $\Theta$ is non-vanishing, the dilaton $\hat{\phi}$ does not diverge at the UV boundary $r \to \infty$. Indeed, $e^{\hat{\phi}}$ reaches its maximum value at infinity, where $e^{\hat{\phi}} \to \Theta^{-1}$. This behaviour is in sharp contrast with the one corresponding to the commutative MN background, for which the dilaton blows up at infinity.

This solution of the type IIB supergravity also includes a RR three-form $F^{(3)}$ given by:

$$F^{(3)} = -\frac{1}{4} (w^1 - A^1) \wedge (w^2 - A^2) \wedge (w^3 - A^3) + \frac{1}{4} \sum_a F^a \wedge (w^a - A^a),$$ (6.1.6)
6.2 Killing spinors

where $F^a$ is the field strength of the $SU(2)$ gauge field $A^a$ of eq. (6.1.2), defined as

$$F^a = dA^a + \frac{1}{2} \epsilon_{abc} A^b \wedge A^c .$$

(6.1.7)

The different components of $F^a$ can be obtained by plugging the value of the $A^a$'s on the right-hand side of eq. (6.1.7). One gets:

$$F^1 = -a' dr \wedge d\theta_1 , \quad F^2 = a' \sin \theta_1 dr \wedge d\phi_1 , \quad F^3 = (1 - a^2) \sin \theta_1 d\theta_1 \wedge d\phi_1,$$

(6.1.8)

where the prime denotes derivative with respect to $r$. The NSNS $B$ field is:

$$B = \Theta e^{2\phi} h^{-1} dx^2 \wedge dx^3 .$$

(6.1.9)

It is proportional to the non-commutative parameter $\Theta$ and it is directed along the $x^2x^3$ coordinates spanning the non-commutative plane. Indeed, the introduction of the NSNS magnetic field is the key ingredient in the construction of the non-commutative deformation. The corresponding three-form field strength $H = dB$ reads:

$$H = 2\Theta \phi' e^{2\phi} h^{-2} dr \wedge dx^2 \wedge dx^3 .$$

(6.1.10)

The solution has also a non-vanishing RR five-form $F^{(5)}$, whose expression is:

$$F^{(5)} = B \wedge F^{(3)} + \text{Hodge dual},$$

(6.1.11)

where $B$ and $F^{(3)}$ are given in eqs. (6.1.9) and (6.1.6) respectively. The RR field strengths satisfy the equations:

$$dF^{(3)} = 0 ,$$

$$dF^{(5)} = d^* F^{(5)} = H \wedge F^{(3)} ,$$

$$d^* F^{(3)} = -H \wedge F^{(5)}.$$

(6.1.12)

6.2 Killing spinors

Once again, we will impose the vanishing of the SUSY variations of the IIB SUGRA fermionic fields (1.1.3) in order to arrive at an explicit expression for the Killing spinors of the model.

This computation follows closely a similar analysis done in ref. [18] for the commutative MN background. First of all, it is more convenient to work in Einstein frame, where the metric (6.1.1) becomes:

$$ds_E^2 = e^\frac{2\phi}{h} h^{\frac{1}{4}} \left[ dx_{0,1}^2 + h^{-1} dx_{2,3}^2 + e^{2\phi} (d\theta_1^2 + \sin^2 \theta_1 d\phi_1^2) + dr^2 + \frac{1}{4} (w^i - A^i)^2 \right].$$

(6.2.1)
We shall consider the following basis of frame one-forms:

$$e^{x^{0,1}} = e^{\phi} h^{\phi} dx^{0,1}, \quad e^{x^{2,3}} = e^{\phi} h^{\phi} dx^{2,3},$$

$$e^r = e^{\phi} h^{\phi} dr,$$

$$e^1 = e^{\phi} h^{\phi} e^0 d\theta_1, \quad e^2 = e^{\phi} h^{\phi} e^0 \sin \theta_1 d\phi_1,$$

$$e^i = \frac{1}{2} e^{\phi} h^{\phi} (w^i - A^i), \quad (i = 1, 2, 3). \quad (6.2.2)$$

The corresponding spin connection one-form, which results from solving the Maurer-Cartan equations (2.2.4), reads:

$$\omega^{x^d r} = h^{-\frac{1}{8}} e^{\phi} \left( \frac{\phi'}{4} + \frac{1}{8} h' h^{-1} \right) e^{x^d}, \quad (d = 0, 1),$$

$$\omega^{x^i r} = h^{-\frac{1}{8}} e^{\phi} \left( \frac{\phi'}{4} - \frac{3}{8} h' h^{-1} \right) e^{x^i}, \quad (i = 2, 3),$$

$$\omega^{1 r} = h^{-\frac{1}{8}} e^{\phi} \left[ \left( g' + \frac{\phi'}{4} + \frac{1}{8} h' h^{-1} \right) e^1 + \frac{1}{4} e^{-g} a' e^1 \right],$$

$$\omega^{2 r} = h^{-\frac{1}{8}} e^{\phi} \left[ \left( g' + \frac{\phi'}{4} + \frac{1}{8} h' h^{-1} \right) e^2 - \frac{1}{4} e^{-g} a' e^2 \right],$$

$$\omega^{3 r} = h^{-\frac{1}{8}} e^{\phi} \left[ \left( \frac{\phi'}{4} + \frac{1}{8} h' h^{-1} \right) e^3 + \frac{1}{4} e^{-g} a' e^1 \right],$$

$$\omega^{5 r} = h^{-\frac{1}{8}} e^{\phi} \left[ \left( \frac{\phi'}{4} + \frac{1}{8} h' h^{-1} \right) e^1 - \frac{1}{4} e^{-g} a' e^2 \right],$$

$$\omega^{3 r} = h^{-\frac{1}{8}} e^{\phi} \left( \frac{\phi'}{4} + \frac{1}{8} h' h^{-1} \right) e^3;$$

$$\omega^{11} = \frac{1}{4} h^{-\frac{1}{8}} e^{\phi} e^{-g} a' e^r, \quad \omega^{22} = -\frac{1}{4} h^{-\frac{1}{8}} e^{\phi} e^{-g} a' e^r,$$

$$\omega^{12} = h^{-\frac{1}{8}} e^{\phi} \left[ \frac{1}{4} (1 - a^2) e^{-2g} e^3 - e^{-g} \cot \theta_1 e^2 \right],$$

$$\omega^{31} = h^{-\frac{1}{8}} e^{\phi} \left( e^3 - e^{-g} \cot \theta_1 e^2 \right),$$

$$\omega^{13} = h^{-\frac{1}{8}} e^{\phi} \left( e^3 + e^{-g} a e^2 \right), \quad \omega^{23} = h^{-\frac{1}{8}} e^{\phi} \left( e^{-g} a e^1 - e^1 \right),$$

$$\omega^{32} = h^{-\frac{1}{8}} e^{\phi} \left[ \frac{1}{4} (1 - a^2) e^{-2g} \right] e^1,$$
\[ \omega^{\hat{3}1} = -h^{-\frac{3}{2}} e^{-\frac{3}{2}} \left[ \frac{1}{4} (1 - a^2) e^{-2g} \right] e^2, \]  

(6.2.3)

written directly in the frame \( e^a \) defined in eq. (6.2.2). One should keep in mind that \( \omega^{ab} = \omega_m^{ab} dX^\hat{m} = \omega_c^{ab} e^c \), so \( \omega_m^{ab} = E_c^\hat{a} \omega_c^{ab} \) (and \( e^c = E_c^\hat{a} dX^\hat{m} \) from (6.2.2)). Let us also write the RR and NSNS forms in the frame (6.2.2). The selfdual five-form becomes:

\[
F^{(5)} = \Theta h^{-\frac{3}{2}} e^{\frac{3}{2}} \left[ -2 \left( e^1 \wedge e^2 \wedge e^3 \wedge e^x \wedge e^x + e^0 \wedge e^1 \wedge e^2 \wedge e^3 \right) + \right. \\
+ \left. \frac{1}{2} (1 - a^2) e^{-2g} \left( e^1 \wedge e^2 \wedge e^3 \wedge e^x + e^0 \wedge e^1 \wedge e^x \wedge e^3 \right) \right] \\
- \frac{1}{2} e^{-g} a' \left( e^1 \wedge e^2 \wedge e^3 \wedge e^x + e^0 \wedge e^1 \wedge e^x \wedge e^3 \right) + \\
+ \frac{1}{2} e^{-g} a' \left( e^2 \wedge e^3 \wedge e^x + e^0 \wedge e^1 \wedge e^x \wedge e^3 \right) \right],
\]  

(6.2.4)

and the complex combination of the RR and NSNS three-forms defined in (1.1.4) can be written as

\[
\mathcal{F}^{(3)} = 2\Theta e^{\frac{3g}{4}} h^{-\frac{3}{2}} \phi' e^r \wedge e^x \wedge e^x + i e^{-\frac{g}{2}} h^{-\frac{3}{2}} \left[ -2 e^1 \wedge e^2 \wedge e^3 + \right. \\
+ \left. \frac{1}{2} (1 - a^2) e^{-2g} e^1 \wedge e^2 \wedge e^3 - \frac{1}{2} e^{-g} a' e^r \wedge e^1 \wedge e^1 + \frac{1}{2} e^{-g} a' e^r \wedge e^2 \wedge e^2 \right].
\]  

(6.2.5)

Now we are ready to solve the SUSY equations arising from (1.1.3). Up to now in this work we have worked with complex spinors, however, from now on it will become easier to switch to real two-component spinors. It is straightforward to find the following rules to pass from complex to real spinors:

\[ e^* \leftrightarrow \tau_3 \epsilon , \quad ie^* \leftrightarrow \tau_1 \epsilon , \quad i\epsilon \leftrightarrow -i\tau_2 \epsilon , \]  

(6.2.6)

where \( \tau_i \) (\( i = 1, 2, 3 \)) are Pauli matrices that act on the two dimensional vector \( \left( \begin{array}{c} \epsilon_1 \\ \epsilon_2 \end{array} \right) \).

To begin with, we study the vanishing of the dilatino SUSY variation, which leads to the following equation:

\[
\frac{1}{2} e^{-\frac{g}{2}} h^{-\frac{3}{2}} \phi' \Gamma_r \tau_1 \epsilon - \frac{i}{4} \left\{ 2\Theta e^{\frac{3g}{4}} h^{-\frac{3}{2}} \phi' \Gamma_{r,x_2 x_3} + i e^{-\frac{g}{2}} h^{-\frac{3}{2}} \left[ \frac{1}{2} e^{-g} a' \Gamma_{r23} - \frac{1}{2} e^{-g} a' \Gamma_{r11} \right] + \right. \\
\left. + \left( \frac{1}{2} (1 - a^2) e^{-2g} - 2 \right) \Gamma_{123} \right\} \epsilon = 0 .
\]  

(6.2.7)

\( \Gamma_a \), \( a = x^a, r, 1, 2, \hat{1}, \hat{2}, \hat{3} \) are constant Dirac matrices associated to the frame (6.2.2). We have used that \( \partial_N \hat{\phi} \Gamma^N e^* = (E^r_p)^{-1} \hat{\phi}' \Gamma_r e^* = e^{-\frac{g}{2}} h^{-\frac{3}{2}} \hat{\phi}' \Gamma_r e^* \), and \( \hat{\phi}' = h^{-1} \phi' \), which can be easily checked using eq. (6.1.5).
As it was done in [18] for the commutative MN background, we shall impose the projection:

\[ \Gamma_{12} \epsilon = \Gamma_{12} \epsilon. \]  

(6.2.8)

Thus, after some calculation, one arrives at the equation:

\[ h^{-\frac{1}{2}} \left[ \frac{1}{2} (1 - a^2) e^{-2g} - 2 \right] \Gamma_{123} \tau_1 \epsilon = \]

\[ = \left( -2h^{-1} \phi' + 2\Theta e^\phi h^{-1} \phi' \Gamma_{x_2x_3} \tau_3 + e^{-g} h^{-\frac{1}{2}} a' \Gamma_{11} \tau_1 \right) \epsilon. \]  

(6.2.9)

Let us now introduce the angle \( \alpha \), which also appears in the commutative case, namely:

\[ \cos \alpha = \frac{\phi'}{1 + \frac{1}{e^{-2g}} (a^2 - 1)}, \quad \sin \alpha = \frac{e^{-g} a'}{2 \left( 1 + \frac{1}{e^{-2g}} (a^2 - 1) \right)}. \]  

(6.2.10)

whose value can be obtained from the explicit form (6.1.3) of the solution, resulting:

\[ \cos \alpha = \coth 2r \left( -\frac{2r}{\sinh^2 2r} \right). \]  

(6.2.11)

In addition, we define a new angle \( \beta \) given by:

\[ \cos \beta = h^{-\frac{1}{2}}, \quad \sin \beta = -\Theta e^\phi h^{-\frac{1}{2}}. \]  

(6.2.12)

Notice that \( \beta = 0 \) when \( \Theta = 0 \). Moreover, from the definition of \( h \) one can easily check that \( \sin^2 \beta + \cos^2 \beta = 1 \).

In terms of the angles \( \alpha \) and \( \beta \), the equation (6.2.9) results in a new projection to be imposed on \( \epsilon \), which reads:

\[ \Gamma_{123} \tau_1 \epsilon = \left[ \cos \alpha \left( \cos \beta + \sin \beta \Gamma_{x_2x_3} \tau_3 \right) - \sin \alpha \Gamma_{11} \tau_1 \right] \epsilon. \]  

(6.2.13)

We will now study the SUSY variations of the gravitino. We begin with the components along the Minkowski space. The equation \( \delta \psi_{x^1} = 0 \) is:

\[ D_{x^1} \epsilon + \frac{i}{1920} \mathcal{F}^{(5)}_{N_1 \cdots N_5} \Gamma_{N_1 \cdots N_5} \Gamma_{x^1} \epsilon + \frac{1}{96} \mathcal{F}^{(3)}_{N_1N_2N_3} \left( \Gamma_{x^1} N_1N_2N_3 - 9 \delta_{x^1} N_1N_2N_3 \right) \epsilon^* = 0. \]  

(6.2.14)

Considering a spinor independent of the \( x^\alpha \) coordinates and inserting the corresponding terms of the spin connection (6.2.3), the first term of (6.2.14) takes the form:

\[ D_{x^1} \epsilon = \frac{1}{4} \omega_{x^1}^{ab} \Gamma_{ab} \epsilon = \frac{1}{16} e^{-\frac{\phi}{4}} h^{-\frac{1}{8}} \left( 2\phi' + h^{-1} h' \right) \Gamma_{x^1} \epsilon. \]  

(6.2.15)

We shall plug the five-form (6.2.4) into the second term of (6.2.14). If we also impose the projection (6.2.8), that term becomes:

\[ -\frac{i}{8} \Theta e^{\frac{3\phi}{4}} h^{-\frac{1}{8}} \left\{ \frac{1}{2} (1 - a^2) e^{-2g} - 2 \right\} \Gamma_{123x^2x^3} \Gamma_{x^1} \epsilon - e^{-g} a' \Gamma_{11} \epsilon \right\} \tau_2 \epsilon, \]  

(6.2.16)
where we have also inserted the total chirality projection
\[ \Gamma_x \delta \psi = \epsilon. \] (6.2.17)

Let us now write down the last term of \( \delta \psi_{x} = 0; \) reading the complex three-form from eq. (6.2.5) one arrives at
\[ \frac{1}{8} \Theta e^{\frac{3}{8} h} e^{\frac{1}{8} \phi'} \Gamma_{x_1 x_2 x_3} \tau_3 \epsilon + \frac{1}{16} e^{\frac{1}{8} h} e^{\frac{1}{8} \phi'} \left\{ \frac{1}{2} (1 - a^2) e^{-2g} - 2 \right\} \Gamma_{x_1 i 2 3} - e^{-g} a' \Gamma_{x_1 r 11} \right\} \tau_1 \epsilon. \] (6.2.18)

Then, gathering the three terms written in eqs. (6.2.15), (6.2.16) and (6.2.18), and multiplying the whole equation by \( 8 e^{\frac{3}{8} h} e^{\frac{1}{8} \phi} \Gamma_{x_1 x_1} \), one gets:
\[ \left\{ \frac{1}{2} \left( 2 \phi' + h^{-1} h' \right) + i \Theta e^{\phi} h^{-\frac{1}{2}} \left[ \frac{1}{2} (1 - a^2) e^{-2g} - 2 \right] \Gamma_{x_1 2 3} \Gamma_{x_2 x_3} \tau_2 - \right. \\
- i \Theta e^{\phi} e^{-g} h^{-\frac{1}{2}} a' \Gamma_{1 i 1} \Gamma_{x_2 x_3} \tau_2 + \Theta e^{\phi} h^{-1} \phi' \Gamma_{x_2 x_3} \tau_3 + \\
\left. \frac{1}{2} h^{-\frac{1}{2}} \left[ \frac{1}{2} (1 - a^2) e^{-2g} - 2 \right] \Gamma_{1 2 3} \tau_1 - \frac{1}{2} e^{-g} h^{-\frac{1}{2}} a' \Gamma_{11} \tau_1 \right\} \epsilon = 0, \] (6.2.19)

which, after multiplying by \( \left[ 1 + \frac{1}{4} e^{-2g} (a^2 - 1) \right] \) can be written in terms of the angles \( \alpha \) and \( \beta \), defined in eqs. (6.2.10) and (6.2.12) respectively, as
\[ \left[ (1 + \sin^2 \beta) \cos \alpha + 2i \sin \beta \Gamma_{x_1 2 3} \Gamma_{x_2 x_3} \tau_2 + 2i \sin \alpha \cos \beta \Gamma_{1 i 1} \Gamma_{x_2 x_3} \tau_2 - \\
- \cos \alpha \sin \beta \cos \beta \Gamma_{x_2 x_3} \tau_3 - \cos \beta \Gamma_{x_1 2 3} \tau_1 - \sin \alpha \cos \beta \tau_1 \right] \epsilon = 0. \] (6.2.20)

We have not yet used the projection (6.2.13). Notice that by multiplying that projection by \( -i \Gamma_{x_2 x_3} \tau_3 \), one obtains the following equivalent expression:
\[ \Gamma_{x_1 2 3} \Gamma_{x_2 x_3} \tau_2 \epsilon = i \left[ \cos \alpha \sin \beta - \cos \alpha \cos \beta \Gamma_{x_2 x_3} \tau_3 + \sin \alpha \Gamma_{x_2 x_3} \Gamma_{1 i 1} \tau_3 \tau_1 \right] \epsilon. \] (6.2.21)

So, finally, one can readily check that eq. (6.2.19) is satisfied after imposing (6.2.13) and (6.2.21).

The equation arising from \( \delta \psi_{x_0} = 0 \) is equal to the one resulting from \( \delta \psi_{x_1} = 0 \), while from the other two Minkowski components, namely \( \delta \psi_{x_2} = 0 \) and \( \delta \psi_{x_3} = 0 \), we get an equation slightly different, which also vanishes after imposing the projections (6.2.8) and (6.2.13).

We shall now look at the SUSY variations of the angular components of the gravitino. Let us begin with the equation \( \delta \psi_1 = 0 \):
\[ D_1 \epsilon + \frac{i}{1920} F_{N_1 \cdots N_5}^{(5)} \Gamma_{N_1 \cdots N_5} \Gamma_1 \epsilon + \frac{1}{96} F_{N_1 N_2 N_3}^{(3)} (\Gamma_{N_1 N_2 N_3} - 9 \delta_{N_1}^{N_2} \Gamma_{N_2 N_3}) \epsilon^* = 0. \] (6.2.22)
We assume that $\epsilon$ does not depend on the angular coordinates so, after plugging the corresponding components of the spin connection (6.2.3), the first term of this last equation reads:

$$D_1 \epsilon = \frac{1}{4} \omega^a_b \Gamma_{ab} \epsilon = \frac{1}{2} e^{-\frac{\phi}{2}} h^{-\frac{3}{4}} \left[ \frac{1}{4} e^{\phi} a' \Gamma_{1r} + \left( \frac{\phi'}{4} + \frac{h^{-1}}{8} \right) \Gamma_{1r} - \Gamma_{23} \right] \epsilon. \quad (6.2.23)$$

Inserting the RR five-form (6.2.4), using eq. (6.2.17), and imposing the projection (6.2.8), the second term of (6.2.22) takes the form:

$$\frac{i}{8} \Theta e^{\phi} h^{-\frac{3}{4}} \left[ \frac{1}{2} (1 - a^2) e^{-2g} + 2 \right] \Gamma_{i23} \epsilon. \quad (6.2.24)$$

The first term containing the RR three-form (6.2.5), namely $\frac{1}{16} \mathcal{F}_{N_1 N_2 N_3}^{(3)} \Gamma_{1}^{N_1 N_2 N_3} \epsilon^*$, can be written as

$$\frac{1}{8} \Theta e^{\phi} h^{-\frac{3}{4}} \phi' \Gamma_{1r} \epsilon + \frac{1}{16} e^{-\frac{\phi}{2}} h^{-\frac{3}{4}} \left[ \frac{1}{2} (1 - a^2) e^{-2g} \Gamma_{i312} - \frac{1}{2} e^{\phi} a' \Gamma_{1r11} \right] \tau_1 \epsilon. \quad (6.2.25)$$

where we have imposed the projection (6.2.8). Using that projection, the last term of eq. (6.2.22), i.e. $\frac{9}{96} \mathcal{F}_{N_1 N_2 N_3}^{(3)} \Gamma_{1}^{N_1 N_2 N_3} \epsilon^*$, becomes:

$$\frac{3}{16} e^{-\frac{\phi}{2}} h^{-\frac{3}{4}} \left( 2 \Gamma_{23} + \frac{1}{2} e^{\phi} a' \Gamma_{1r1} \right) \tau_1 \epsilon. \quad (6.2.26)$$

Eventually, we gather the four pieces of $\delta \psi_1 = 0$: (6.2.23), (6.2.24), (6.2.25) and (6.2.26); and we multiply the whole equation by $8 e^{\phi} h^{-\frac{3}{4}} \Gamma_{1r1}$. We arrive at

$$\left\{ \frac{1}{2} \left( 2 \phi' + h^{-1} h' \right) + h^{-\frac{3}{4}} \left[ \frac{1}{4} (1 - a^2) e^{-2g} + 3 \right] \Gamma_{i23} \tau_1 + + i \Theta e^{\phi} h^{-\frac{3}{4}} \left[ \frac{1}{2} (1 - a^2) e^{-2g} + 2 \right] \Gamma_{i23} \Gamma_{x^2 x^3} \tau_2 - 4 \Gamma_{i23} + + \Theta e^{\phi} h^{-\frac{3}{4}} \phi' \Gamma_{x^2 x^3} \tau_3 - e^{-g} a' \Gamma_{1i1} + \frac{1}{2} e^{\phi} h^{-\frac{3}{4}} a' \Gamma_{1i1} \tau_1 \right\} \epsilon = 0. \quad (6.2.27)$$

Let us proceed as before and multiply this last equation by $\left[ 1 + \frac{1}{4} e^{-2g} (a^2 - 1) \right]^{-1}$, which allows us to write it in terms of the angles $\alpha$ and $\beta$. Next we substitute the projection (6.2.13) and its equivalent expression (6.2.21). After some calculation one gets:

$$\left\{ 2 \sin \alpha \cos \beta \Gamma_{1i1} \tau_1 + 2 \sin \alpha \sin \beta \Gamma_{x^2 x^3} \Gamma_{i1i1} \tau_1 \tau_3 - 2 \sin \alpha \Gamma_{i1i1} + + \frac{4}{1 + \frac{1}{4} e^{-2g} (a^2 - 1)} \left( \cos \alpha - \sin \alpha \cos \beta \Gamma_{i1i1} \tau_1 - \sin \alpha \sin \beta \Gamma_{x^2 x^3} \Gamma_{i1i1} \tau_1 \tau_3 - \Gamma_{i1i1} \right) \right\} \epsilon = 0. \quad (6.2.28)$$
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It is clear that in order to satisfy this equation, one must impose an extra projection on $\epsilon$. Indeed, we will see that if we require the term in parentheses to vanish we get a projection that makes the first three terms cancel each other. The term in parentheses in eq. (6.2.28) can be written as

$$\left( \cos \alpha + \sin \alpha \Gamma_{11} \right) \left( 1 - \cos \beta \tau_1 - \sin \beta \Gamma_{x^2x^3} \tau_1 \tau_3 \right) \epsilon ,$$

(6.2.29)

where we have used that $\Gamma_{r123} \epsilon = \left[ \cos \alpha \left( \cos \beta - \sin \beta \Gamma_{x^2x^3} \tau_3 \right) \tau_1 - \sin \alpha \Gamma_{11} \right] \epsilon$, which follows from (6.2.13). If we want eq. (6.2.29) to vanish, $\epsilon$ must satisfy the following equation:

$$\tau_1 \epsilon = \left( \cos \beta + \sin \beta \Gamma_{x^2x^3} \tau_3 \right) \epsilon .$$

(6.2.30)

Let us write the first three terms of (6.2.28) as

$$-2 \sin \alpha \Gamma_{11} \epsilon + 2 \sin \alpha \Gamma_{11} \tau_1 \left( \cos \beta + \sin \beta \Gamma_{x^2x^3} \tau_3 \right) \epsilon ,$$

(6.2.31)

which clearly vanishes after imposing (6.2.30). So the equation $\delta \psi_1 = 0$ is fulfilled by an $\epsilon$ satisfying the known projections (6.2.8) and (6.2.13) together with the new one (6.2.30).

The vanishing of the SUSY variations of the remaining angular components of the gravitino is guaranteed by the three projections we have just mentioned and the first order differential equations satisfied by $g$ and $a$, namely:

$$g' = \frac{1}{2} \left( 1 - a^2 \right) e^{-2g} \cos \alpha - e^{-g} a \sin \alpha ,$$

$$a' = -2a \cos \alpha - \left( 1 - a^2 \right) e^{-g} \sin \alpha .$$

(6.2.32)

$\delta \psi_2 = 0$ takes the same form, up to a global factor, as $\delta \psi_1 = 0$, and $\delta \psi_3 = 0$ also holds if we impose the same projections. The equations $\delta \psi_1 = 0$ and $\delta \psi_2 = 0$ are equal up to a global factor and they vanish if one uses the differential equations (6.2.32) and again the projections (6.2.8), (6.2.13) and (6.2.30).

Notice that the new projection (6.2.30) can be written as

$$\tau_1 \epsilon = e^{\beta \Gamma_{x^2x^3} \tau_3} \epsilon ,$$

(6.2.33)

and, in addition, by using (6.2.30) on the right-hand side of (6.2.13), one arrives at

$$\Gamma_{r123} \epsilon = \left( \cos \alpha - \sin \alpha \Gamma_{11} \right) \epsilon = e^{-a \Gamma_{11}} \epsilon .$$

(6.2.34)

Since $\left[ \Gamma_{x^2x^3} \tau_3 , \Gamma_{11} \right] = \{ \tau_1 , \Gamma_{x^2x^3} \tau_3 \} = \{ \Gamma_{r123} , \Gamma_{11} \} = 0$, we can solve (6.2.8), (6.2.33) and (6.2.34) as follows:

$$\epsilon = e^{\frac{\alpha}{2} \Gamma_{11}} e^{-\frac{\beta}{2} \Gamma_{x^2x^3} \tau_3} \eta ,$$

(6.2.35)

where $\eta$ is a spinor that can only depend on the radial coordinate and satisfies:

$$\Gamma_{12} \eta = \Gamma_{12} \eta , \quad \tau_1 \eta = \eta , \quad \Gamma_{r123} \eta = \eta .$$

(6.2.36)
We still have to write the SUSY variation of the radial component of the dilatino. In principle we suppose that \(\epsilon\) depends on the radial coordinate. The equation \(\delta \psi_r = 0\) will determine such dependence; it takes the form:

\[
D_r \epsilon + \frac{i}{1920} F_{N_1 \cdots N_5}^{(5)} \Gamma^{N_1 \cdots N_5} \Gamma_r \epsilon + \frac{1}{96} F_{N_1 N_2 N_3}^{(3)} (\Gamma_{r r N_1 N_2 N_3} - 9 \delta_{r r}^{N_1} \Gamma^{N_2 N_3}) \epsilon^* = 0. \tag{6.2.37}
\]

Let us write down each term of this equation separately. The covariant derivative can be easily written in terms of \(d\epsilon\) and substitute into the equation the form of \(\eta\) written in eq. (6.2.4), using eq. (6.2.17) and imposing again (6.2.8), takes the form:

\[
D_r \epsilon = e^{-\frac{2}{3}} h^{-\frac{3}{2}} \left( \epsilon' + \frac{1}{4} e^{-g} a' \Gamma_{11} \epsilon \right), \tag{6.2.38}
\]

where we have already used the projection (6.2.8). The second piece of (6.2.37), after inserting the five-form written in eq. (6.2.35), using eq. (6.2.17) and imposing again (6.2.8), takes the form:

\[
\frac{i}{8} \Theta e^{\frac{3g}{2}} h^{-\frac{3}{2}} \left\{ \left[ \frac{1}{2} (1 - a^2) e^{-2g} - 2 \right] \Gamma_{123} \Gamma_r - e^{-g} a' \Gamma_{11} \Gamma_r \right\} \tau_2 \epsilon. \tag{6.2.39}
\]

Next we substitute the RR three-form (6.2.5) into the last terms of eq. (6.2.37) and, after making use of the projection (6.2.8), we arrive at

\[
\frac{1}{96} (\Gamma_{r N_1 N_2 N_3} - 9 \delta_{r r}^{N_1} \Gamma^{N_2 N_3}) \epsilon^* = \frac{1}{8} e^{-\frac{2}{3}} h^{-\frac{3}{2}} \left\{ - \left[ 1 + \frac{1}{4} (a^2 - 1) e^{-2g} \right] \Gamma_{r 123} \tau_1 + 3 \left[ \frac{1}{2} a' e^{-g} \Gamma_{11} \tau_1 - \Theta e^\phi h^{-\frac{3}{2}} \phi' \Gamma_{x^2 x^3} \tau_3 \right] \right\} \epsilon. \tag{6.2.40}
\]

We shall now gather the three contributions to \(\delta \psi_r = 0\) (namely (6.2.38), (6.2.39), (6.2.40)) and substitute into the equation the form of \(\epsilon\) written in eq. (6.2.35). After some calculation, taking into account the projections (6.2.36) satisfied by \(\eta\) and the definitions of \(\alpha\) and \(\beta\) given in (6.2.10) and (6.2.12), we get the following equation:

\[
\eta' + \frac{1}{2} \left[ \alpha' \Gamma_{11} - \beta' \Gamma_{x^2 x^3} \tau_3 + e^{-g} a' \Gamma_{11} - \Theta e^\phi \phi' h^{-1} \Gamma_{x^2 x^3} \tau_3 - \right.
\]

\[
\left. - \frac{1}{2} \Theta^2 e^{2\phi} \phi' h^{-1} - \frac{1}{4} \phi' h^{-1} \right] \eta = 0. \tag{6.2.41}
\]

The fulfillment of this equation requires that the terms proportional to \(\Gamma_{11}\), to \(\Gamma_{x^2 x^3} \tau_3\) and to the identity vanish separately, resulting:

\[
\alpha' + a' e^{-g} = 0, \quad \beta' + \Theta e^\phi \phi' h^{-1} = 0, \tag{6.2.42}
\]

\[
\eta' = \left[ \frac{1}{8} \phi' h^{-1} + \frac{1}{4} \Theta^2 e^{2\phi} \phi' h^{-1} \right]. \tag{6.2.43}
\]
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One can check that (6.2.42) follows automatically from the definitions of $\alpha$ and $\beta$, while (6.2.43) fixes the radial dependence of $\eta$, which can be written as

$$\eta = e^{\frac{\phi}{8}} \frac{1}{h^{16}} \epsilon_0,$$

in terms of a constant spinor $\epsilon_0$ satisfying the same projections as $\eta$, i.e. (6.2.36).

To sum up, the Killing spinors of the non-commutative MN background can be written in terms of a ten dimensional constant spinor satisfying three independent and compatible projections reducing the number of independent spinors from the maximal 32 to 4 as it should be for the gravity dual of a 4d $\mathcal{N} = 1$ supersymmetric theory. In fact, by looking at [18], one can check that the only difference between the Killing spinors of the commutative and non-commutative models is a rotation along the non-commutative plane. This should be expected since those are the directions along which the deformation is performed and the Killing spinors of the MN solution do not depend on those coordinates.
Chapter 7

Conclusions

In this work we have computed explicitly the Killing spinors of five different solutions of ten dimensional type IIB supergravity. We have begun by computing the Killing spinors for backgrounds of the form $\mathbb{R}^{1,3} \times Y_6$, where $Y_6$ is either the singular conifold or any of its resolutions. We have carried out this calculation using a generalized ansatz proposed in [14] resulting from the uplift of a domain wall setup in eight dimensional gauged supergravity corresponding to D6-branes wrapping an $S^2$. As expected, the vanishing of the 10d SUSY variations (1.1.3), besides determining the form of the 10d Killing spinors, resulted in the same system of differential equations for the functions entering the ansatz as in the eight dimensional case. The solutions of that system realize the different resolutions of the conifold. Furthermore, it turns out that when written in the natural frame arising from the uplifted ten dimensional metric, the 10d Killing spinors do not depend on the angular coordinates of the conifold. In fact, we have written them in terms of a constant 10d spinor, by means of a rotation (whose phase depends only on the radial coordinate) along two internal directions. The constant spinor must satisfy two independent and compatible projections reducing its number of independent components from 32 to 8. Thus, as it should be, these backgrounds leave unbroken eight supersymmetries. These results, presented in chapter 2, were very useful for the development of the subsequent chapters, as we will recall in the following paragraphs.

We have studied, in chapter 3, the Killing spinors of the Klebanov-Witten (KW) model [7]. This background arises from placing a stack of $N$ D3-branes at the tip of the singular conifold. After taking the usual decoupling limit, the resulting geometry is $AdS_5 \times T^{1,1}$ (recall that $T^{1,1}$ is the base of the conifold), and it turns out that the computation of the Killing spinors is simplified if we write the $T^{1,1}$ metric in the form (2.3.7) obtained in chapter 2. Indeed, we were able to write the Killing spinors in such a basis that they are independent of the coordinates of the $T^{1,1}$; one suspected this would be so, since that form of the metric comes out as an uplift from 8d gauged supergravity. Therefore, from the consistency of the reduction, the Killing spinors should not depend on any angular coordinate of the group manifold (the $SU(2)$ along which the reduction takes place) but, in addition, the topological twist needed to realize supersymmetry with wrapped branes in the eight dimensional theory results in a fibration of the $SU(2)$ manifold along the remaining $S^2$, reinforcing the conjecture that the Killing spinors will not depend on the $T^{1,1}$ coordinates, when written in the appropriate frame.
CHAPTER 7. CONCLUSIONS

Let us also recall that, contrary to what one could naively expect, the Killing spinors of the KW background do not satisfy the projection corresponding to a D3-brane extended along the Minkowski directions. In fact, we have got two independent solutions, namely (3.2.17) and (3.2.18), each one being $1/8$ supersymmetric, and only the first one satisfies that projection. These are the Killing spinors corresponding to the four ordinary supersymmetries, while the second ones (eq. (3.2.18)) realize the four superconformal symmetries. The complete solution only satisfies the two independent projections corresponding to the $T^{1,1}$ space.

We have already pointed out in the introduction that the explicit knowledge of the Killing spinors of the KW model was essential for the program carried out in [13], where we systematically explored the possibilities of adding different D-brane probes in the Klebanov-Witten background.

The aforementioned form of writing the metric of the $T^{1,1}$, resulting from (2.3.7) in chapter 2, has become useful again for the computation of the Killing spinors of the Klebanov-Tseytlin (KT) background [15] performed in chapter 4. This solution results from adding $M$ fractional D3-branes (wrapped D5-branes) to the setup of the KW model. The three-form flux created by the D5-branes is the source of the conformal symmetry breaking, so the background preserves only the four usual supersymmetries corresponding to a four dimensional $\mathcal{N} = 1$ YM theory. We have written the Killing spinors in a basis where they do not depend on the coordinates of the $T^{1,1}$ and this time, in addition to the two projections of the $T^{1,1}$, they satisfy the projection corresponding to a D3-brane extended along the Minkowski space. Therefore, one has 4 independent spinors standing for the four conserved supersymmetries.

Our next step (chapter 5) was the computation of the Killing spinors of the Klebanov-Strassler (KS) background [16]. This solution is constructed by placing D3-branes and fractional D3-branes at the tip of the deformed conifold. Then, while in the UV this solution approaches the KT model, the deformation of the conifold gives, in the IR, a geometrical realization of confinement and chiral symmetry breaking. Thus, this background is conjectured to provide (in the $\tau \to 0$ limit, where $\tau$ is the holographic coordinate) a dual to the IR region of $\mathcal{N} = 1$ SYM.

The KS background is formulated in terms of several functions of the radial coordinate which are defined by means of a system of first order differential equations solving the equations of motion of type IIB supergravity. By writing the metric of the deformed conifold in the form found in chapter 2 and imposing the projections obtained there for the Killing spinors of the deformed conifold (see section 2.3.1) plus the projection corresponding to a D3-brane along $\mathbb{R}^{1,3}$ we have shown that, in order to have a background preserving some supersymmetry (in particular a $1/8$ supersymmetric solution), those functions defining the KS model must satisfy the mentioned first order system plus an extra algebraic constraint. Then, the Killing spinors of the KS background can be written, in a frame where they do not depend on the angular coordinates of the conifold, in terms of a ten dimensional constant spinor satisfying three independent projections leaving unbroken four supersymmetries.

Finally, in chapter 6 we have studied the Killing spinors of the non-commutative Maldacena-Núñez (MN) background [17]. This solution can be obtained from the commutative geometry [19, 20] by means of a chain of string dualities resulting in a deformed background which singles out the two spatial directions along which the deformation took place (they form the
so-called non-commutative plane). Thus, it breaks the four dimensional Lorentz invariance and the corresponding dual theory has spatial non-commutativity along those directions.

The computation of the Killing spinors goes along the same lines as for the commutative solution (see [18]). Working in the appropriate frame (the one arising when one obtains the commutative MN background as an uplift from 7d gauged SUGRA) it turns out that the Killing spinors do not depend on the internal angular coordinates of the geometry and, in fact, we have written them in terms of a constant spinor satisfying the same three projections as in the commutative case. The only effect of the deformation is the presence of a rotation along the non-commutative plane. This was expected since, when working in the suitable background, the Killing spinors of the commutative solution do not depend on the directions along which the deformation is performed. Let us remark that the solution leaves unbroken four supersymmetries as it corresponds to a four dimensional $\mathcal{N} = 1$ supersymmetric field theory.
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