Lower and upper bounds on nonunital qubit channel capacities

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Classical capacity of unital qubit channels is well known, whereas that of nonunital qubit channels is not. We find lower and upper bounds on classical capacity of nonunital qubit channels by using a recently developed decomposition technique relating nonunital and unital positive qubit maps.

\section{I. INTRODUCTION}

Transmission of classical information through quantum channels is considered in a series of papers \cite{1, 2, 3, 4, 5, 6, 7, 8} and reviews \cite{9, 10, 11, 12}. In brief, if a real number $R$ is an achievable rate of information transmission, then $n$ qubits effectively allow one to transmit $2^{nR}$ classical messages. The encoder assigns an $n$-qubit density operator $\rho_i^{(n)}$ to each message $i$. The $n$-qubit density operator is a positive semidefinite operator with unit trace, which acts on a $2^n$ dimensional Hilbert space $H_2^n$. In the process of information transmission, each qubit is transmitted through a quantum channel $\Phi$, which is a completely positive and trace preserving map. Therefore, the output state of $n$ qubits reads $\Phi^{\otimes n}[\rho_i^{(n)}]$. The decoder represents the measurement device described by a positive operator-valued measure, which assigns a positive-semidefinite operator $M_{ij}^{(n)}$ (acting on $2^n$-dimensional Hilbert space) to each observed outcome $j \in \{1, \ldots, N\}$. Let $p(j|i)$ be the probability of observing outcome $j \in \{0, 1, \ldots, N\}$ if the original message is $i$, then by the quantum-mechanical rule

$$ p(n)(j|i) = \text{tr}[\rho_i^{(n)}M_{ij}^{(n)}]. \quad (1) $$

Condition $\sum_{j=1}^N M_{ij}^{(n)} = I$ guarantees $\sum_{j=1}^N p(n)(j|i) = 1$. The maximum error probability reads

$$ p_{err}(n, N) = \max_{j=1, \ldots, N} \left(1 - p(n)(j|i)\right). \quad (2) $$

$R$ is called an achievable rate of information transmission if

$$ \lim_{n \to \infty} p_{err}(n, 2^nR) = 0. \quad (3) $$

By classical capacity $C(\Phi)$ of quantum channel $\Phi$ we understand the supremum of achievable rates:

$$ C(\Phi) = \sup \left\{ R : \lim_{n \to \infty} p_{err}(n, 2^nR) = 0 \right\}. \quad (4) $$

The Holevo–Schumacher–Westmoreland theorem \cite{2, 3} states that

$$ C(\Phi) = \lim_{n \to \infty} \frac{1}{n} C_\chi(\Phi^{\otimes n}), \quad (5) $$

where the quantity $C_\chi(\Phi)$ is expressed through all possible ensembles of density operators $\{\rho_k, p_k\}$ and von Neumann entropy $S(\rho) = -\text{tr}(\rho \log \rho)$ via formula

$$ C_\chi(\Phi) = \sup_{\{\rho_k, p_k\}} \left[ S \left( \sum_k p_k \rho_k [\rho_k] \right) - \sum_k p_k S(\Phi_\rho_k) \right]. \quad (6) $$

The quantity $C_\chi(\Phi)$ is also known as $\chi$-capacity and the Holevo capacity of quantum channel $\Phi$.

If in formula (1) one restricts to product input states $\rho_i^{(n)} = \bigotimes_{k=1}^n \rho_i^{(1)}(\rho_i)$ and separable measurements $M_{ij}^{(n)} = \bigotimes_{k=1}^n M_{ij}^{(1)}$ with a classical data processing $g: x = (x_1, \ldots, x_n) \mapsto j$, then one obtains a modified capacity $C^{(1)}(\Phi)$. By construction $C^{(1)}(\Phi) \leq C_\chi(\Phi)$. There exists a class of so-called pseudoclassical channels \cite{13} for which $C^{(1)}(\Phi) = C_\chi(\Phi)$. Such a class includes all unital qubit channels and some nonunital qubit channels too. A complete characterization of pseudoclassical qubit channels is given in the paper \cite{13}. In this paper, we consider general nonunital qubit channels without restriction to the class pseudoclassical ones.

Calculation of the $\chi$-capacity $C_\chi(\Phi)$ is complicated even for the case of a general qubit channel \cite{14, 15, 16} and no closed formula is known. Needless to say, the regularized capacity $C(\Phi)$ is even more difficult to estimate due to the fact that additivity hypothesis is not proved for a general qubit channel. In this paper, we partially fill the gap and find lower and upper bounds on the capacity $C(\Phi)$ of qubit channels $\Phi$. We also compare these bounds with the known ones that can be computed with the help of semidefinite programming \cite{10, 20}. We demonstrate that for some channels our upper bound outperforms the previously known upper bounds.

\section{II. RELATION BETWEEN UNITAL AND NONUNITAL QUBIT CHANNELS}

Let $A$ and $B$ be two operators acting on $H_2$. By $\Phi_A$ we denote a completely positive map $\Phi_A[X] = AXA^\dagger$, i.e. a map with a single Kraus operator $A$. Analogously, $\Phi_B[X] = BXB^\dagger$. Hereafter, $\dagger$ denotes Hermitian conjugation.

Suppose that $\Phi$ is a qubit linear map, which belongs to the interior of the cone of positivity preserving maps. Such maps are also referred to as positivity improving ones \cite{21} because $\Phi[g] > 0$ for all $g \geq 0, g \neq 0$. In terms of the paper \cite{22}, $\inf \{\det \Phi[X] \mid X > 0, \det X = 1\}$ is strictly positive and attained. Equivalently, $\Phi$ can be represented as a nontrivial convex combination of some positive map and the tracing map $X \mapsto \text{tr}[X]I$, see Ref. \cite{23}.

If $\Phi$ belongs to the interior of the cone of positivity preserving maps, then by Proposition 2.32 in \cite{21} there exist positive definite operators $A$ and $B$ acting on $H_2$ such that the map

$$ \Upsilon = \Phi_A \circ \Phi \circ \Phi_B \quad (7) $$

trace preserving and unital. Unitality means that $\Upsilon[I] = I$, where $I$ is the identity operator.

The relation (1) is a quantum analogue of Sinkhorn’s theorem for square matrices with strictly positive elements \cite{23}. The Sinkhorn theorem states that if $X$ is an
n \times n matrix with strictly positive elements, then there exist diagonal matrices D_1 and D_2 with strictly positive diagonal elements such that Y = D_1 XD_2 is doubly stochastic. In quantum case, X is replaced by Φ, Y is replaced by Ψ, D_1 and D_2 are replaced by Φ_A and Φ_B, respectively. This is the reason why (7) is sometimes referred to as Sinkhorn’s normal form for positive maps [24]. Historically, the relation (7) was originally observed in [24] rediscovered for positivity improving completely positive maps Φ in [21], and finally clarified in [23, 24].

Suppose that in addition to being positivity improving Φ is also completely positive and trace preserving, then Ψ is completely positive and trace preserving too. For a given nonunital qubit channel Φ the particular form of operators A and B is derived in Refs. [26, 27]. Since A and B are nondegenerate, formula (7) implies that

\[ Φ = Φ_A^{-1} \circ Ψ \circ Φ_B^{-1}, \]

i.e. all non-boundary nonunital qubit channels Φ can be decomposed into a concatenation of three completely positive maps Φ_B^{-1}, Ψ, Φ_A^{-1}, with Ψ being unital.

On the other hand, for any unital qubit channel Υ there exist unitary operators V and W such that [28]

\[ Υ = Φ_W \circ Λ \circ Φ_V, \]

where the quantum channel Λ has so called diagonal form in the basis of conventional Pauli operators I, σ_1, σ_2, σ_3:

\[ Λ[X] = \frac{1}{2} \text{tr}[X] I + \frac{1}{2} \sum_{i=1}^{3} λ_i \text{tr}[σ_i X] σ_i. \]

Parameters λ_1, λ_2, λ_3 in (10) are real and satisfy the constraint 1 ≥ λ_3 ≥ |λ_1 ± λ_2| as Λ is completely positive [28].

Clearly, classical capacities of channels Υ and Λ coincide. Moreover, since the additivity hypothesis holds true for unital qubit channels [29], the classical capacity equals Holevo capacity and reads

\[ C(Υ) = C(Λ) = C_χ(Λ) = 1 - h \left( \frac{1}{2} \left( 1 - \max_{i=1,2,3} |λ_i| \right) \right), \]

where \( h(x) = -x \log_2 x - (1-x) \log_2 (1-x). \)

In what follows, we relate the classical capacity of nonunital qubit channel Φ with the classical capacity of unital qubit channel Υ, which is given by formula (11).

### III. BOUNDS ON CLASSICAL CAPACITY OF NONUNITAL QUBIT CHANNELS

**Theorem 1.** Theorem Suppose Φ is a qubit channel such that the map Ψ = Φ_A \circ Φ \circ Φ_B is a channel (completely positive and trace preserving). Then \( C(Φ) ≥ C(Ψ) - 2 \log_2(∥A||B||). \)

**Proof.** Let \( \{ g_i^{(n)} \}, M_i^{(n)} \}_{i=1}^{N} \) be the optimal code of size \( N = 2^n R_Φ \) for the composite channel Ψ^{⊗n} such that \( \lim_{n \to \infty} P_{\text{err}}(n, 2^n R_Φ) = 0. \)

Consider a set of modified input states

\[ \tilde{g}_i^{(n)} = \frac{B^{⊗n} g_i^{(n)} (B_i)^{⊗n}}{\text{tr}[B^{⊗n} g_i^{(n)} (B_i)^{⊗n}]} \]

and a modified positive operator-valued measure \( \{ j \to \tilde{M}_j^{(n)} \}_{j=0} \) with elements

\[ \tilde{M}_0^{(n)} = I - \sum_{j=1}^{N} \tilde{M}_j^{(n)}, \]

\[ \tilde{M}_j^{(n)} = \frac{(A_1)^{⊗n} M_j^{(n)} A^{⊗n}}{∥A||B||^{2n}}, \]

where \( ∥X∥ = ∥X∥_∞ = \max_{ψ:⟨ψ|ψ⟩=1} ⟨ψ|\sqrt{X}X|ψ⟩ \) is the operator norm. It is not hard to see that \( \tilde{M}_0^{(n)} \) is positive semidefinite.

Using the modified code, let each qubit be transmitted through the channel Φ. Then the probability to observe outcome \( j \neq 0 \) provided input message \( i \) equals

\[ \tilde{p}(n)(j|i) = \frac{\text{tr}[\tilde{g}_i^{(n)} \tilde{M}_j^{(n)}]}{\text{tr}[B^{⊗n} g_i^{(n)} (B_i)^{⊗n}] ||A||^{2n}}, \]

\[ = \frac{p(n)(j|i)}{\text{tr}[B^{⊗n} g_i^{(n)} (B_i)^{⊗n}] ||A||^{2n}}, \]

where \( p(n)(j|i) \) is the probability to get outcome \( j \in \{1, \ldots, N\} \) for the input message \( i \in \{1, \ldots, N\} \) in the original optimal protocol for channel Ψ^{⊗n}.

Observation of the outcome \( j = 0 \) in the modified protocol would be treated as unsuccessful event, whereas observation of the outcome \( j \in \{1, \ldots, N\} \) leads to a successful identification of the message because \( p(n)(j|i) \to δ_{ij} \) if \( n \to ∞. \)

The probability to observe nonzero outcome \( j \) equals

\[ p(n) = \sum_{j=1}^{N} \tilde{p}(n)(j|i) = \frac{1}{\text{tr}[B^{⊗n} g_i^{(n)} (B_i)^{⊗n}] ||A||^{2n}} \geq \frac{1}{(||A|| ||B||)^{2n}}, \]

Utilizing the modified protocol, one can transmit information only in the case of successful events \( j \neq 0 \), so the average number of successfully transmitted messages \( N \) equals

\[ \tilde{N} = p(n) N = p(n) 2^n R_Φ ≥ 2^n (R_Φ - 2 \log_2(∥A||B||)). \]

Therefore, the considered protocol enables one to achieve the rate

\[ \tilde{R} ≥ R_Ψ - 2 \log_2(∥A||B||) \]

by utilizing the channel Φ.

If \( R_Ψ ≤ C(Ψ) \) and one observes the successful event \( j \neq 0 \), then the maximum error probability in the modified protocol

\[ \tilde{p}_{\text{err}}(n, \tilde{N}) = \max_{j=1, \ldots, N} \left( 1 - \frac{\tilde{p}(n)(j|i)}{p(n)} \right) = \max_{j=1, \ldots, N} \left( 1 - p(n)(j|i) \right) \to 0 \text{ if } n \to ∞. \]
Taking supremum on both sides of Eq. (19) with requirement \( \lim_{n \to \infty} \tilde{\rho}_{en}(n, N) = 0 \), we get
\[
C(\Phi) \geq C(\Psi) - 2 \log_2(\|A\||B||B||). \tag{21}
\]

Theorem 1 can be applied to two equalities relating nonunital and unital qubit channels: \( \Upsilon = \Phi_A \circ \Phi \circ \Phi_B \) and \( \Phi = \Phi_{A-1} \circ \Upsilon \circ \Phi_{B-1} \). As a result, we immediately get upper and lower bounds on capacity \( C(\Phi) \).

**Proposition 1.** Proposition Let \( \Phi \) be a unital qubit channel belonging to the interior of positive qubit maps, then there exist positive definite operators \( A \) and \( B \) acting on \( \mathcal{H}_2 \) such that the map \( \Upsilon = \Phi_A \circ \Phi \circ \Phi_B \) is unital and
\[
-2 \log_2(\|A\||B||B||) \leq C(\Phi) - C(\Upsilon) \leq 2 \log_2(\|A^{-1}\||B^{-1}||B^{-1}||). \tag{22}
\]

**Proof.** The statement straightforwardly follows from the decomposition existence [24] and Theorem 1.

To illustrate the obtained results, we consider the following example dealing with 4-parameter nonunital qubit channels.

**Example 1.** Example Consider a nonunital qubit channel of the form
\[
\Phi[X] = \frac{1}{2} \left( \text{tr}[X](I + t_3\sigma_3) + \sum_{j=1}^{3} \lambda_j \text{tr}[\sigma_j \varrho_j \sigma_j] \right), \tag{23}
\]
where \( t_3 \) and \( \lambda_1, \lambda_2, \lambda_3 \) are real parameters, in addition to the condition of complete positivity also satisfy the inequality \( |t_3| + |\lambda_3| < 1 \). It guarantees that \( \Phi \) is an interior point of the cone of positive qubit maps. Ref. [27] provides the explicit form of decomposition \( \Phi = \Phi_{A-1} \circ \Upsilon \circ \Phi_{B^{-1}} \). We further simplify it and obtain
\[
A = \text{diag} \left( \sqrt{(1-t_3)^2 - \lambda_3^2}, \sqrt{(1+t_3)^2 - \lambda_3^2} \right), \tag{24}
\]
\[
B = \frac{\sqrt{2}}{2} \left( 4 - \left( \sqrt{(1-t_3)^2 - \lambda_3^2} - \sqrt{(1+t_3)^2 - \lambda_3^2} \right)^2 \right)^{-1/2}
\times \text{diag}(b_1, b_2), \tag{25}
\]
\[
b_1 = (1+t_3-\lambda_3) \sqrt{(1-t_3)^2 - \lambda_3^2} + (1-t_3+\lambda_3) \sqrt{(1+t_3)^2 - \lambda_3^2},
\]
\[
b_2 = (1+t_3+\lambda_3) \sqrt{(1-t_3)^2 - \lambda_3^2} + (1-t_3-\lambda_3) \sqrt{(1+t_3)^2 - \lambda_3^2}.
\]
The unital qubit map \( \Upsilon = \tilde{\Lambda} \) has the form [11] with parameters
\[
\tilde{\lambda}_1 = \frac{2\lambda_1}{\sqrt{(1+\lambda_3)^2-t_3^2} + \sqrt{(1-\lambda_3)^2-t_3^2}}, \tag{26}
\]
\[
\tilde{\lambda}_2 = \frac{2\lambda_2}{\sqrt{(1+\lambda_3)^2-t_3^2} + \sqrt{(1-\lambda_3)^2-t_3^2}} \tag{27}
\]
\[
\tilde{\lambda}_3 = \frac{4\lambda_3}{\sqrt{(1+\lambda_3)^2-t_3^2} + \sqrt{(1-\lambda_3)^2-t_3^2}}. \tag{28}
\]
Since both \( A \) and \( B \) are diagonal, the norms \( \| A \| = \text{max}(A_{11}, A_{22}), \| B \| = \text{max}(B_{11}, B_{22}), \| A^{-1} \| = 1/\text{min}(A_{11}, A_{22}) \), and \( \| B^{-1} \| = 1/\text{min}(B_{11}, B_{22}) \). Substituting these norms in Proposition 1 we find lower and upper bounds on capacity \( C(\Phi) \). Namely,
\[
C(\Phi) \geq 1 - h \left( \frac{1}{2} \left( 1 - \max_{i=1,2,3} |\tilde{\lambda}_i| \right) \right) - 2 \log_2(\|A\||B||B||), \tag{29}
\]
\[
C(\Phi) \leq 1 - h \left( \frac{1}{2} \left( 1 - \max_{i=1,2,3} |\tilde{\lambda}_i| \right) \right) + 2 \log_2(\|A^{-1}\||B^{-1}||B^{-1}||), \tag{30}
\]
where \( \tilde{\lambda}_i, i = 1, 2, 3 \) are given by formulas (26)–(28).

**Example 2.** Example Let us consider a class of generalized amplitude damping (GAD) qubit channels as a partial case of Example 1. The GAD channel describes the process of qubit dynamics when it exchanges excitations with the thermal environment at finite temperature \( T \).

In this case, parameters of the channel (23) depend on time \( t \geq 0 \) as follows:
\[
\lambda_1 = \lambda_2 = e^{-\gamma t}, \quad \lambda_3 = e^{-2\gamma t}, \quad t_3 = (2p - 1)(1 - e^{-2\gamma t}), \tag{31}
\]
where \( \gamma \) is the energy dissipation rate and \( \text{diag}(p, 1 - p) \) is the equilibrium density operator \((0 \leq p \leq \frac{1}{2} \text{ is the population of the excited state in thermal equilibrium with the environment}) \).

The direct calculation yields
\[
\tilde{\lambda}_1 = \tilde{\lambda}_2 = e^{-\gamma t}, \tag{32}
\]
\[
\| A \||B||B|| = \sqrt{\frac{1}{p} - 1} \frac{1}{\sqrt{f(p, \gamma t)}}, \tag{33}
\]
\[
\| A^{-1} \||B^{-1}||B^{-1}|| = \sqrt{\frac{1}{p} - 1} \frac{1}{\sqrt{f(p, \gamma t)}}, \tag{34}
\]
\[
f(p, \gamma t) = \sqrt{p(1-p)}(1-e^{-2\gamma t}) + \sqrt{1-p + pe^{-2\gamma t}} \sqrt{p + (1-p)e^{-2\gamma t}}. \tag{35}
\]
Substituting (32)–(34) in (29)–(30), we get the following lower and upper bounds:
\[
C(\Phi_{\text{GAD}}) \geq C_{\text{GAD}}^1 = 1 - h \left( \frac{1}{2} \left( 1 - \frac{e^{-\gamma t}}{f(p, \gamma t)} \right) \right) + 2 \log_2 f(p, \gamma t) - \frac{1}{2} \log_2 \frac{1}{p}, \tag{36}
\]
\[
C(\Phi_{\text{GAD}}) \leq C_{\text{GAD}}^1 = 1 - h \left( \frac{1}{2} \left( 1 - \frac{e^{-\gamma t}}{f(p, \gamma t)} \right) \right) + 2 \log_2 f(p, \gamma t) + \frac{1}{2} \log_2 \frac{1}{p}. \tag{37}
\]

Figure 1 illustrates these bounds as well as the \( \chi \)-capacity of the GAD channel. The latter one can be found numerically since the structure of optimal ensemble in formula (6) is known [31, 32]. Note that the GAD channel is not pseudoclassical as it does not satisfy the necessary and sufficient condition of pseudoclassicality (Theorem 23 in [13]) if \( p \in (0, \frac{1}{2}) \), therefore, \( C_{\chi}(\Phi_{\text{GAD}}) < C_{\Phi}(\Phi_{\text{GAD}}) \leq C(\Phi_{\text{GAD}}) \leq C_{\text{GAD}}^1 \).

If \( p \to 0 \), then the bounds (36) and (37) become trivial. This is due to the fact that the GAD channel with \( p = 0 \) is a conventional amplitude damping channel which does not belong to the interior of the cone of positive maps. Therefore, the products \( \| A \||B||B|| \) and \( \| A^{-1} \||B^{-1}||B^{-1}|| \) diverge if \( p \to 0 \).
Example 3. Example Following Ref. [20], we consider a one-parameter qubit channel of the form
\[ \Phi_{\text{mix}} = p A_p + (1 - p) D_p, \]  
where \( 0 \leq p \leq 1 \). \( A_p[X] = K_1 X K_1^\dagger + K_2 X K_2^\dagger \) is the qubit amplitude damping channel with Kraus operators \( K_1 = |0\rangle \langle 0| + \sqrt{1 - p} |1\rangle \langle 1| \) and \( K_2 = \sqrt{p} |0\rangle \langle 1| \). \( D_p \) is the qubit depolarizing channel given by formula \( D_p[X] = (1 - p) X + \frac{p}{3} (X_{zz} + X_{xx} + X_{yy}) \).

\( \Phi_{\text{mix}} \) is a partial case of the 4-parameter channel discussed in Example 1:
\begin{align*}
\lambda_1 &= \lambda_2 = p \sqrt{1 - p} + (1 - p) \left( 1 - \frac{4p}{3} \right), \\
\lambda_3 &= (1 - p) \left( 1 - \frac{p}{3} \right), \\
t_3 &= p^2.
\end{align*}

Substituting these values in equations (38), we readily obtain the lower and upper bounds, \( C_{\text{mix}}^1 \) and \( C_{\text{mix}}^4 \), defined by formulas (29) and (30), respectively. We depict these bounds in Figure 2 and compare them with the previously known lower and upper bounds for \( C(\Phi_{\text{mix}}) \). One can see that our lower bound is not as precise as the \( \chi \)-capacity of \( \Phi_{\text{mix}} \). Nevertheless, our upper bound \( C_{\text{mix}}^4 \) is tighter than the bound in Ref. [19] if \( p < 0.33 \) and tighter than the bound in Ref. [20] if \( p > 0.29 \). Therefore, for \( 0.29 < p < 0.33 \) our upper bound outperforms the previously known upper bounds.

IV. CONCLUSIONS

We have obtained new lower and upper bounds on classical capacities of nonunital qubit channels. The obtained result holds true for the regularized version of Holevo capacity, formula (33). Since the optimal coding procedure is known for unital qubit channels [29] and relies on the use of factorized states, our approach is able to provide the factorized coding (formula (12)), with which one can achieve the transmission rate corresponding to the lower bound on capacity.

FIG. 1: Bounds on capacity \( C(\Phi_{\text{GAD}}) \) of the GAD channel with \( p = 0.475 \) versus dimensionless time \( \gamma t \). Lower bound \( C_{\text{GAD}}(\Phi_{\text{GAD}}) \) is plotted in dotted line. The derived bounds \( C_{\text{GAD}} \) and \( C_{\text{GAD}}^2 \) are given by lower and upper solid lines, respectively.

FIG. 2: Bounds on capacity \( C(\Phi_{\text{mix}}) \) of the mixture channel \( \Phi_{\text{mix}} \) versus dimensionless parameter \( p \). Lower bound \( C_{\text{mix}}(\Phi_{\text{mix}}) \) is plotted in dotted line. The derived bounds (29) and (30) are depicted as lower and upper solid lines, respectively. The upper bound from Ref. [19] is plotted in dash-dotted line, the upper bound from Ref. [20] is plotted in dashed line.

We illustrate our findings by considering a 4-parameter family of qubit channels comprising the set of generalized amplitude damping channels. For some mixtures of amplitude damping and depolarizing qubit channels the derived upper bound (30) outperforms the previously known upper bounds of Refs. [19, 20]. We believe that the derived bounds are particularly useful for such channels \( \Phi \) that slightly deviate from unital qubit channels.

Our proofs are based on the seminal relation between unital and nonunital qubit channels, which was developed in Ref. [24]. Such a relation has already been used in the study of entanglement annihilation [27] and may turn out to be productive in other research areas as well, for instance, in the study of absolutely separating quantum channels [33], divisibility of qubit dynamical maps [34–36] and their tensor products [37]. The most promising application of the decomposition \( \Upsilon = \Phi_A \circ \Phi \circ \Phi_B \) is in the study of quantum capacities \( Q \) of quantum channels [10, 11]. Despite the fact that both \( \Phi_A \) and \( \Phi_B \) are completely positive, one cannot immediately conclude that \( Q(\Upsilon) \) is less or equal to \( Q(\Phi) \), neither the inverse statement is justified. The reason is that the maps \( \Phi_A \) and \( \Phi_B \) are not trace preserving. Moreover, at least one of the maps \( \Phi_A \) and \( \Phi_B \) is not trace decreasing too. The study of these peculiarities in relation with quantum capacity and other types of capacities is an interesting direction for future research.

Acknowledgments

The study is supported by the Russian Foundation for Basic Research under Project No. 16-37-60070 mol-a-dk. The author is grateful to Mark Wilde and Felix Leditzky for useful comments. The author thanks anonymous referees for helpful suggestions to improve the quality of the paper.
