HECKE OPERATORS IN EQUIVARIANT ELLIPTIC COHOMOLOGY
AND GENERALIZED MOONSHINE

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1. Introduction

1.1. Background on generalized Moonshine. We recall Norton’s generalized Moonshine conjecture. Let \( g, h \) be a pair of commuting elements in the Fischer-Griess Monster group \( M \). We denote the centralizer of \( g \) in \( M \) by \( C_M(g) \).

**Conjecture 1.1** (Norton [Mas87]). It is possible to associate with each such pair a modular function \( f(g, h; \tau) \) with the following properties:

(a) Up to a constant factor (which will be a root of unity), there is an equality

\[
  f(g^a h^c, g^b h^d; \tau) = f(g, h, \frac{a\tau + b}{c\tau + d})
\]

whenever \( ad - bc = 1 \),

(b) For any group element \( g \) and nonzero rational number \( l \) the coefficient of \( q^l = e^{2\pi i l \tau} \) in \( f(g, h; \tau) \) is, as a function of \( h \), a character of a central extension\(^1\) of \( C_M(g) \). Note that nonzero characters can occur for nonintegral \( l \), but that generalized characters are not needed.

(c) Conjugation of \( g \) and \( h \) leaves the function unchanged.

(d) Unless \( f(g, h; \tau) \) is a constant function, its invariance group will be a modular group of genus zero, commensurable with the standard modular group \( SL_2(\mathbb{Z}) \).

Norton further explains: “It is the simultaneous action of \( SL_2(\mathbb{Z}) \) on \( \langle g, h \rangle \) and \( \tau \) that seems to suggest an explanation of the Moonshine phenomenon; but this only gives a clue to the actions of the elements of \( SL_2(\mathbb{Z}) \) on \( \tau \), so the reason why the full invariance group should have genus zero remains obscure.”

Such simultaneous actions are well known to topologists: they arise in the context of equivariant elliptic cohomology. There are more indications of a connection between Moonshine and elliptic cohomology:

Generalized Moonshine is supposed to specialize to “classical” Moonshine for pairs of the form \( (1, g) \). This part is often referred to as the “untwisted sector”. In this situation, part (b) of Norton’s conjecture involves only integral powers of \( q \). Moreover, no central extension is needed, i.e., the coefficients are honest characters of \( M = C_M(1) \). In this context, the Moonshine functions are usually denoted

\[
  j_{(g)}(\tau) = f(1, g; \tau) = q^{-1} + a_1 q + a_2 q^2 + \ldots,
\]

since they are generalizations of the \( j \)-function

\[
  j_{(1)} = j - 744 = q^{-1} + 196884 q + 21493760 q^2 + \ldots
\]

and depend only on the cyclic subgroup generated by \( g \).

We recall from [Mas87] the notions of replicable function and twisted Hecke operator. Let \( f \) be a Laurent series in \( q \) of the form

\[
  f(q) = q^{-1} + a_1 q + a_2 q^2 + \ldots.
\]

Then the \( n \)th Faber polynomial \( \Phi_n \) of \( f \) is the unique polynomial of degree \( n \) for which the Laurent series \( \Phi_n(f) \) has the form

\[
  \Phi_n(f) = q^{-n} + b_1 q + b_2 q^2 + \ldots
\]

\(^1\)For a discussion of how characters of central extensions of a group correspond to twisted characters of this group, see for example [Wil].
The series \( f \) is called *replicable*, if there exist *replicates* \( f^{(a)} \) of \( f \), which satisfy the equalities

\[
\Phi_n(f) = \sum_{0 \leq b < d} f^{(a)} \left( \frac{\alpha \tau + b}{d} \right)
\]

for all \( n \in \mathbb{N} \). One shows by induction that the \( f^{(a)} \) are uniquely determined if they exist: assume we know that \( f^{(a)} \) is defined for \( a < n \). By solving the equality (2) for \( f^{(n)}(n \tau) \), we arrive at the question whether a series in \( q \), which we already defined, is indeed a series in \( q^n \).

A replicable series \( f \) is called *completely replicable* if all the replicates of \( f \) are again replicable and their replicates are given by

\[
(f^{(a)})^{(b)} = f^{(ab)}.
\]

Complete replicability was proved to be equivalent to the genus zero property (part (d) of the conjecture) in [CG97] and [Mar96].

If \( f \) is replicable, then the \( n \)th *twisted Hecke operator* acting on \( f \) is defined as

\[
\hat{T}_n(f) := \frac{1}{n} \Phi_n(f).
\]

The classical Moonshine functions are known to be completely replicable, with the replicates given by the Adams operations

\[
j^{(a)}_{(h)}(q) = j_{(he)}(q).
\]

The collection of all classical Moonshine functions is a *McKay-Thompson series*, i.e., it is a series of the form (1) with coefficients in the representation ring \( R(M) \). A function \( f \) satisfying condition (b) of Norton’s conjecture is often referred to as a generalized McKay-Thompson series. The following definition of replicability for McKay-Thompson series can be found in [Mas87]. It is stricter than asking for the individual functions \( j_{(g)} \) to be replicable but seems to be the correct notion in the context of Moonshine.

**Definition 1.2.** A McKay-Thompson series \( f \) is called replicable, if the formula (2) holds when \( f^{(a)} \) is the \( a \)th Adams operation applied to \( f \).

We will rephrase Definition 1.2 in terms of power operations in elliptic cohomology in Section 6.4 below.

1.2. **Background on equivariant elliptic cohomology and statement of results.**

Adams operations were originally defined in the context of (equivariant) \( K \)-theory. Hecke operators in (equivariant) elliptic cohomology are a generalization\(^2\) of the Adams operations [And92], and it seems a natural question to ask for a “topological” explanation for the relationship between Hecke and Adams operations occurring in Moonshine. We will start by explaining the geometry behind the generalized Moonshine conjecture and its relationship to equivariant elliptic cohomology. This discussion is not specific to the Monster and can be formulated for any finite group \( G \). We will translate parts (a) and (c) of Norton’s conjecture to

“*f is an element of the twisted equivariant elliptic cohomology of the one point space, in the sense of Ginzburg, Kapranov and Vasserot: \( f \in \text{Ell}_{G}^{\alpha}(pt) \).*”

\(^2\)More precisely, they are the chromatic level 2 analogue of the Adams operations (cf. also [Gan06]).
Here $\alpha$ is a cocycle representing an element of 

$$H^3(BG; \mathbb{R}/\mathbb{Z}) = H^4(BG, \mathbb{Z}).$$

In other words,

“$f$ is a section of the line bundle $L^\alpha$ over the moduli stack of principal $G$-bundles over (complex) elliptic curves.”

Here $L^\alpha$ is a line bundle associated to $\alpha$ (cf. Section 2.2).

In this context, Ginzburg, Kapranov and Vasserot have defined the action of correspondences. In particular, one can consider the action of the Hecke correspondence. In Section 6.1, I will spell out this geometric definition of the Hecke operators: they act on such sections via pull-backs along isogenies.

We return to Norton’s conjecture. One easily translates part (b) to

“$f$ is an element of the (twisted) Devoto equivariant Tate $K$-theory of the one point space.”

Devoto’s equivariant Tate $K$-theory

$$K_{Dev,g}(X) = \bigoplus_{[g]} K_C^g(X^g)[q^{\frac{1}{h}}]$$

was introduced to study characteristic classes of bundles over orbifold loop spaces. For our statement to be true as stated, we will need to invert $q$. Twisted here means that we will have to replace the centralizers with the appropriate central extensions and $q^{\frac{1}{h}}$ with $q^{\frac{1}{h|g|}}$, where $h$ is the order of $\alpha_{B(g)}$. The equivariant Witten genus takes values here. In [Gan], I defined power operations $P_n$ in an appropriate subring of Devoto equivariant Tate $K$-theory. If a function $f(g,h;\tau)$ satisfies condition (a) of Norton’s conjecture, this implies that $f$ is in this subring. My definitions in [Gan] were guided by the work of Dijkgraaf, Moore, Verlinde and Verlinde on orbifold genera of symmetric powers. The results of [Gan], together with [Dev96], make $K_{Dev,G}$ an equivariant theory with power operations and a Hopkins-Kuhn-Ravenel character theory. In this situation, we have the following combinatorial definition of Hecke operators (cf. [And92], and [Gan06]):

$$T_n(f) = \frac{1}{n} \sum_{[\sigma,\rho] \text{ transitive}} P_n(f)(\sigma, \rho),$$

where the sum runs over all conjugacy classes of pairs of commuting elements $(\sigma, \rho)$ of the symmetric group $\Sigma_n$ with the property that $\langle \sigma, \rho \rangle$ acts transitively on $\{1, \ldots, n\}$.

The main result of this paper can be summarized as follows:

**Theorem 1.3.** The following three definitions of Hecke operators result in exactly the same formula for functions $f(g,h;\tau)$:

1. The geometric definition using isogenies, when $f$ is an element of (twisted) Ginzburg-Kapranov-Vasserot equivariant elliptic cohomology;
2. The combinatorial definition, using the equivariant power operations of [Gan], when $f$ is an element of Devoto’s equivariant Tate $K$-theory on which $P_n$ is defined;
3. for $g = 1$, the definition of twisted Hecke operators acting on McKay-Thompson series.
The formula is
\[ T_n(f)(g, h; \tau) = \sum_{\substack{a \leq b < d \leq n \atop 0 \leq b < d}} f \left( g^d, g^{-b}h^a, \frac{a\tau + b}{d} \right). \]

Similarities between equivariant elliptic cohomology and generalized Moonshine have been studied in the past by Baker, Devoto, and Thomas (cf. e.g. [BT] and [Dev96]). Baker’s work includes a definition of twisted Hecke operators in elliptic cohomology, using Adams operations. The theory of Hecke operators in elliptic cohomology was developed further in Ando’s work, which brought the theory of isogenies of formal groups into the picture [And00].

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2. Principal bundles over complex elliptic curves

Let \( G \) be a finite group, let \( \mathcal{C}(G) \) denote the set of all pairs of commuting elements in \( G \), and consider the right action of \( G \) on \( \mathcal{C}(G) \) given by simultaneous conjugation
\[(g, h) \mapsto (s^{-1}gs, s^{-1}hs).\]

We define the generalized centralizer group \( C_G(g, h) \) of the pair \((g, h)\) by
\[ C_G(g, h) = \text{Stab}_G(g, h). \]

The generalized conjugacy class \([g, h]\) is defined as the orbit of \((g, h)\) under \( G \). The set of all conjugacy classes of pairs of commuting elements in \( G \) is denoted \( \mathcal{C}(G) \).

Let \( E \) be an elliptic curve over \( \mathbb{C} \). Assume that we have picked an isomorphism
\[(4) \quad E \cong \mathbb{C}/\langle \tau, 1 \rangle.\]

Throughout the paper, we will identify \( \langle \tau, 1 \rangle \) with \( \mathbb{Z}^2 \) via the isomorphism
\[
\begin{align*}
\mathbb{Z}^2 & \rightarrow \langle \tau, 1 \rangle \\
(1,0) & \mapsto -1 \\
(0,1) & \mapsto \tau
\end{align*}
\]

Further, we will view \( \mathbb{C} \) with the action of the additive subgroup \( \langle \tau, 1 \rangle \cong \mathbb{Z}^2 \) as a contractible free \( \mathbb{Z}^2 \)-space \( E\mathbb{Z}^2 \), and \( E \) as a classifying space
\[ E = B\mathbb{Z}^2 = E\mathbb{Z}^2/\mathbb{Z}^2. \]

**Lemma 2.1.** The set of isomorphism classes of principal \( G \)-bundles over \( E \) is
\[
\text{Prin}_G(E) \cong [B\mathbb{Z}^2, BG] \cong \text{Hom}(\mathbb{Z}^2, G)/\text{Inn}(G) = \mathcal{C}(G),
\]
the set of all conjugacy classes of pairs of commuting elements of \( G \).
Such a pair of commuting elements may also be viewed as encoding the monodromy of the corresponding principal bundle along the two standard circles of the torus.

Under the isomorphism (5), the map

\[ \varphi: \langle \tau', 1 \rangle \rightarrow \langle \tau, 1 \rangle \]

\[ \tau' \mapsto a\tau + b \]

\[ 1 \mapsto c\tau + d \]

corresponds to the endomorphism of \( \mathbb{Z}^2 \) given by the matrix

\[ \varphi = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}, \]

and

\[ \varphi^\vee: \langle \tau, 1 \rangle \rightarrow \langle \tau', 1 \rangle \]

\[ \tau \mapsto d\tau' - b \]

\[ 1 \mapsto -c\tau' + a \]

becomes the pseudo-inverse of this matrix,

\[ \varphi^\vee = \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \]

The map \( \phi := B\varphi \) is an isogeny

\[ \phi: \mathbb{C}/\langle \tau', 1 \rangle \rightarrow \mathbb{C}/\langle \tau, 1 \rangle \]

with dual isogeny \( \phi^\vee = B(\varphi^\vee) \). Any isogeny of complex elliptic curves can be obtained in this way. If \( ad - bc = 1 \), then \( \phi \) becomes an isomorphism with \( \phi^{-1} = \phi^\vee \). It is the isomorphism obtained when comparing two different choices for (4).

**Proposition 2.2.** The isogeny \( \phi^\vee \) pulls back the principal bundle over \( \mathbb{C}/\langle \tau', 1 \rangle \) which is classified by the pair \( (g, h) \) to the bundle over \( \mathbb{C}/\langle \tau, 1 \rangle \) which is classified by \( (g^a h^c, g^b h^d) \).

**Proof:** After identifying both \( \langle \tau, 1 \rangle \) and \( \langle \tau', 1 \rangle \) with \( \mathbb{Z}^2 \) as in (5), the map \( \varphi^\vee \) sends \((1,0)\) to \((a,c)\) and \((0,1)\) to \((b,d)\). Recall that \((g,h)\) stands for the map from \( \mathbb{Z}^2 \) to \( G \) sending \((1,0)\) to \( g \) and \((0,1)\) to \( h \). If we precompose this map with \( \varphi^\vee \), we obtain

\[ (g,h) \circ \varphi^\vee = (g^a h^c, g^b h^d). \]

\( \square \)

### 2.1. The moduli space of principal \( G \)-bundles.

Let

\[ (s, \gamma) \in G \times \text{SL}_2(\mathbb{Z}) \]

act on

\[ C(G) \times \mathfrak{h} \]

from the right by

\[ (g, h; \tau) \mapsto (s^{-1}(g^a h^c, g^b h^d)s; \gamma^{-1}(\tau)) , \]

where

\[ \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{and} \quad \gamma^{-1}(\tau) = \frac{d\tau - b}{-c\tau + a}. \]
Then the quotient

$$\overline{C}(G) \times \mathcal{H} / \text{SL}_2(\mathbb{Z})$$

is the coarse moduli space $X_G$ of isomorphism classes of principal $G$-bundles over complex elliptic curves. Apart from the roots of unity, conditions (a) and (c) of Norton’s Conjecture are precisely the coherence data satisfied by holomorphic functions on $X_G$ (compare also [Dev96]).

In order to understand the roots of unity, we need to consider the moduli stack

$$\mathcal{M}_G := \mathcal{C}(G) \times \mathcal{H} / G \times \text{SL}_2(\mathbb{Z}).$$

Write

$$S := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{and} \quad T := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

The groupoid $\mathcal{C}(G)//G$ is equivalent to the category of principal $G$-bundles over the torus and isomorphisms between them. To show this, consider the pointed circle $(S^1, \ast)$ and the torus $(S^1, \ast) \times (S^1, \ast)$.

Let $(P, p)$ be a principal $G$-bundle over this torus together with a choice of basepoint over $(\ast, \ast)$. Let $(g, h)$ be the commuting pair given by the monodromy of $P$ around the two circles, starting at $p$. Note that for any two pointed principal bundles $(P, p)$ and $(P', p')$ with the same monodromy, there is a unique basepoint-preserving isomorphism from $P$ to $P'$.

For any commuting pair $(g, h)$ in $G$, fix once and for all a pointed principal $G$-bundle $(P_{g,h}, p)$ over the torus with this monodromy. A (not necessarily basepoint-preserving) morphism from $(P, p)$ to $(P', p')$ is a pair $(f, s)$, where $s \in G$, and $f: (P, ps) \to P'$ is a basepoint-preserving isomorphism. For any $s \in G$, the pointed bundle $(P_{g,h}, ps)$ has monodromy $(s^{-1}gs, s^{-1}hs)$, and there is a unique basepoint-preserving isomorphism

$$f: (P_{g,h}, ps) \to (P_{s^{-1}gs, s^{-1}hs}, p').$$

Note that the automorphism group of $(g, h)$ in $\mathcal{C}(G)//G$ is given by the generalized centralizer group $C_G(g, h)$. The automorphisms of $(g, h; \tau)$ in $\mathcal{M}_G$ might be more complicated: The automorphisms of $P_{g,h}$ over $E$ are parametrized by $C_G(g, h)$, but now there might also be automorphisms of $E$ which pull back $P_{g,h}$ to a bundle isomorphic to $P_{g,h}$. For instance, $(1, ST)$ is an automorphism of $\left( g, g^{-1}, \frac{\sqrt{3}i-1}{2} \right)$ whenever $g$ has order three, and $(s, S)$ is an automorphism of $(g, h; i)$ whenever $s(g, h)s^{-1} = (h^{-1}, g)$.

2.2. Line bundles over $\mathcal{M}_G$.

2.2.1. The cocycle $\alpha$. Fix, once and for all, a cellular decomposition of $BG$. Let $i_{(2)}: BG^{(2)} \to BG$ denote the inclusion of the 2-skeleton in $BG$.

**Definition 2.3** (compare [Wil]). A cocycle $\alpha \in Z^3(BG; \mathbb{R}/\mathbb{Z})$ is called normalized, if $i_{(2)}^*(\alpha) = 0$.

**Lemma 2.4.** Any cocycle $\alpha \in Z^3(BG; \mathbb{R}/\mathbb{Z})$ is cohomologous to a normalized one.
Proof: The statement follows from the commuting diagram

\[
\begin{array}{ccc}
C^2(BG; \mathbb{R}/\mathbb{Z}) & \overset{i^*_2}{\longrightarrow} & C^2(BG^{(2)}; \mathbb{R}/\mathbb{Z}) \\
\downarrow \delta & & \downarrow \delta \\
Z^3(BG; \mathbb{R}/\mathbb{Z}) & \overset{i^*_2}{\longrightarrow} & Z^3(BG^{(2)}; \mathbb{R}/\mathbb{Z}).
\end{array}
\]

Here \( C^* \) are the groups of cochains and \( Z^* \) the groups of cocycles. The upper horizontal arrow is surjective, since \( \mathbb{R}/\mathbb{Z} \) is an injective abelian group. □

We fix a normalized cocycle \( \alpha \in Z^3(BG; \mathbb{R}/\mathbb{Z}) \). Recall that we have

\[ H^4(BG; \mathbb{Z}) \cong H^3(BG; \mathbb{R}/\mathbb{Z}). \]

In the case that \( G \) is the Monster, a candidate for \( [\alpha] \in H^4(BG; \mathbb{Z}) \) was described in [Mas].

2.3. Construction and properties of the Freed-Quinn line bundle \( \mathcal{L}^\alpha \). Inspired by the work of Dijkgraaf, Vafa, Verlinde and Verlinde [DVVV89], Freed and Quinn use \( \alpha \) to define a (possibly degenerate) line bundle \( \mathcal{L}^\alpha = \{L^\alpha_{g,h}\} \) on \( \mathcal{C}(G) \). They proceed to describe the \( SL_2(\mathbb{Z}) \)-action on this line bundle. Their discussion can be interpreted as constructing an \( SL_2(\mathbb{Z}) \)-equivariant line bundle \( \mathcal{H} \times \mathcal{L}^\alpha \rightarrow \mathcal{H} \times \mathcal{C}(G) \), and thus a line bundle \( \mathcal{L}^\alpha \) on the moduli stack \( \mathcal{M}_G \). We briefly recall their construction, which simplifies when \( \alpha \) is normalized.

Construction 2.5. Let \( P \) be a principal \( G \) bundle over \( \mathbb{T}^2 \), and consider the groupoid \( \mathcal{C}_P \) whose objects are cellular classifying maps \( f: P \rightarrow EG \) for \( P \), and whose morphisms from \( f \) to \( f' \) are homotopy classes rel boundary of cellular \( G \)-homotopies

\[ f: [0, 1] \times P \rightarrow EG \]

from \( f \) to \( f' \). We define a functor \( \mathcal{F} \) from \( \mathcal{C}_P \) to the category of metrized complex lines as follows: For every \( f \), one sets\(^3\) \( \mathcal{F}(f) = \mathbb{C} \). Further \( \mathcal{F}(f \xrightarrow{h} f') \) is multiplication with

\[ e^{2\pi i \langle \mathcal{R}^\alpha(x), x \rangle}, \]

where \( x \) is a 3-cycle representing the fundamental class in \([0, 1] \times \mathbb{T}^2\), and \( \mathcal{R} = h/G \). It can be shown that this is independent of the choice of representatives \( h \) and \( x \) and that \( \mathcal{F} \) has no holonomy. The line \( L^\alpha_{\mathbb{R}} \) is then defined as the space of invariant sections of \( \mathcal{F} \).

Let \( \psi \) be an automorphism of \( P \) (not necessarily covering the identity on \( \mathbb{T}^2 \)). Then the action of \( \psi \) on \( L^\alpha_{\mathbb{R}} \) is given by multiplication with the *Chern-Simons* invariant of the glued mapping cylinder

\[ cyl_\psi(P) := [0, 1] \times P / \sim, \quad \text{where} \quad (0, \psi(p)) \sim (1, p). \]

\(^3\)The metrized integration line \( L^\alpha_{\mathbb{T}^2, T} \) of [FQ93] is canonically trivialized if \( \alpha \) is normalized and \( f \) is cellular.
This invariant is computed by choosing a cellular classifying map \( h : \text{cyl}_\psi(P) \to EG \) and a 3-cycle \( x \) representing the fundamental class of \( \text{cyl}_\psi(T^2) \) and then evaluating \( \tilde{H}^1(\alpha) \) at \( x \). More generally, let \( \gamma \in \text{SL}_2(\mathbb{Z}) \) be an automorphism of \( T^2 \). Then the induced isomorphism \( \gamma_\ast \) of metrized integration lines is defined to be the identity
\[
(7) \quad \text{id}_C : F(f) = \mathbb{C} \to \mathbb{C} = F(f \circ \gamma).
\]

Note that \( T^2 \) can be replaced with any Riemann surface. We will often drop \( \alpha \) from the notation and write \( L_P \) for \( L_\alpha P \). If \( P \) is a principal bundle over \( T^2 \) classified by \( (g, h) \), we will sometimes write \( L_{g, h} \) for \( L_P \).

**Proposition 2.6.** Let \( \phi : E' \to E \) be an isogeny of degree \( n \), and let \( P \) be a principal \( G \)-bundle over \( E \). Then there is an isometry of metrized lines,
\[
\phi_\ast : L_{\phi_\ast(P)} \to L_{\phi_\ast}^{\otimes n}.
\]
The construction of \( \phi_\ast \) is natural in \( E \).

**Proof:** Write \( I \) for \([0, 1]\). Let \( A \) be the kernel of \( \phi \), and let \( C_A^3(I \times E') \) denote the group of \( A \)-invariant singular 3-chains in \( I \times E' \). Since \( A \) acts fixed point free on \( I \times E' \), every element \( x \in C_A^3(I \times E') \) is of the form
\[
x = \sum \lambda_i \left( \sum_{a \in A} a \right) s_i,
\]
where the \( \lambda_i \) are integers and the \( s_i \) are singular simplices in \( I \times E' \). Hence the map \( \text{id}_I \times \phi_\ast \) takes values in \( nC_3(I \times E) \). If \( x \in C_A^3(I \times E') \) represents the relative fundamental class of \( I \times E' \), then \( \tilde{\delta} \phi_\ast(x) \) represents the relative fundamental class of \( I \times E \). Let \( C_P \) and \( C_{\phi_\ast P} \) be the groupoids in Construction 2.5. Precomposition with \( \phi \) gives a map of groupoids
\[
\iota : C_P \to C_{\phi_\ast P}.
\]
Let \( F_{P, \alpha} \) and \( F_{\phi_\ast P, \alpha} \) be the functors in Construction 2.5. The discussion above gives rise to a natural isomorphism
\[
F_{P', \alpha} \circ \iota \cong (F_{P, \alpha})^{\otimes n}.
\]
It remains to observe that the invariant sections of \( F_{P', \alpha} \) and \( F_{P', \alpha} \circ \iota \) are canonically isomorphic. \( \square \)

For later reference, we note that the definition of the isomorphism in Proposition 2.6 used only the structure of \( \phi \) as a principal \( A \)-bundle. Note also that the action of \( A \) on \( L_{\phi_\ast P} \) is trivial.

**Lemma 2.7.** Let \( \alpha = 0 \). Then the line bundle \( L^\alpha \) over \( \overline{C}(G) \) is canonically trivial, and the \( \text{SL}_2(\mathbb{Z}) \)-action preserves this trivialization.

**Proof:** Let \( \alpha = 0 \), then the line of the previous construction is canonically trivialized. Explicitly, such a trivialization is given by picking an arbitrary object \( f \in \text{ob}(C_P) \) and evaluating invariant sections of \( F \) at \( f \). Since our choice of trivialization was independent of the choice of \( f \), it follows that
\[
\gamma_\ast = \text{id}_C : L_P^\alpha \to L_{\gamma_\ast P}^\alpha.
\]
\( \square \)
Lemma 2.8. Let \( a: H \to G \) be a map of finite groups. Then \( a \) induces a map of groupoids
\[
\mathcal{C}(a): \mathcal{C}(H)\!\!/H \to \mathcal{C}(G)\!\!/G,
\]
which pulls back \( L^\alpha \) to \( L^{a^*(\alpha)} \) as an \( \text{SL}_2(\mathbb{Z}) \)-equivariant bundle.

PROOF: Fix a cellular classifying map
\[
F: EH[G] \to EG,
\]
and write \( Ba \) for the induced map \( F \) of classifying spaces. Let \( P \) be a principal \( H \)-bundle over \( T^2 \), let \( C_P \) be the groupoid in [FQ93, p.6], and similarly, \( C_{P[G]} \). We have an equivalence of groupoids
\[
\iota: C_P \to C_{P[G]},
\]
\[
f \mapsto F \circ f[G],
\]
\[
h \mapsto F \circ h[G].
\]
Here \( f: P \to EH \) ranges over the classifying maps of \( P \), and \( h \) ranges over the homotopies between such classifying maps. Let \( \mathcal{F}_{P,Ba^*(\alpha)} \) be the functor defined in [FQ93, p.6], and note that
\[
\mathcal{F}_{P,Ba^*(\alpha)} = \mathcal{F}_{P[G],\alpha} \circ \iota.
\]
It follows that the corresponding metrized lines of invariant sections of the two functors are the same:
\[
L^{Ba^*(\alpha)}_P = L^\alpha_{P[G]},
\]
Let now \( F' \) denote a different choice for \( F \) and let \( \iota' \) be the corresponding equivalence of groupoids. Then there exists a cellular homotopy between \( F \) and \( F' \) and hence a natural transformation between \( \iota \) and \( \iota' \), yielding a canonical isomorphism between the spaces of invariant sections of \( \iota^*\mathcal{F} \) and \( (\iota')^*\mathcal{F} \). The construction of the proof is compatible with the functoriality condition in the following sense: Let \( \psi: P \to P' \) be a map of principal \( H \)-bundles covering an orientation-preserving diffeomorphism of the base (for the proof of this lemma it is enough if this isomorphism is the identity of the torus). Then the induced isometries \( \psi_* \) and \( \psi|G|_* \) of the above lines agree.

\[
\square
\]

Corollary 2.9. Let \( G \) be a finite group, and let \( \alpha \) be a normalized 3-cocycle on \( BG \) with values in \( \mathbb{R}/\mathbb{Z} \). Then the line \( L^\alpha_{1,1} \) is canonically trivialized and carries a trivial \( \text{SL}_2(\mathbb{Z}) \)-action.

PROOF: By the previous lemma, it is enough to consider the case where \( G = 1 \) is the trivial group. Let \( id_{T^2}: P \to T^2 \) denote the trivial principal bundle over \( T^2 \). Since trivial bundles can be classified by constant maps, we have an equivalence of groupoids \( \iota: C^{(2)}_P \to C_P \), where \( C^{(2)}_P \) is the subgroupoid whose objects are classifying maps \( f: T^2 \to B1^{(2)} \) and whose morphisms are homotopy classes relative boundary of homotopies \( h: [0,1] \times T^2 \to B1^{(2)} \). Since \( \alpha \) is normalized, it follows that \( \iota^*(L^\alpha_P) \), and hence \( L^\alpha_P \) is canonically trivialized.

\[
\square
\]

Corollary 2.10. Let \( P_{g,h} \) be a principal bundle over the torus which is classified by \( [g,h]_G \).
Let \( n = |g| \cdot |h| \). Then \( \left(L^\alpha_{[g,h]} \right)^{\otimes n} \) is canonically trivial.
Proof: The trivialization is given by Proposition 2.6 applied to the \( n \)-fold covering which wraps the first circle around itself \(|g|\) times and the second one around itself \(|h|\) times. This covering pulls back \( P_{[g,h]} \) to a trivial bundle. \( \square \)

Lemma 2.11. Let \( p \) be a map from the torus to the circle, and let \( P \) be the pull-back along \( p \) of a principal \( G \)-bundle \( Q \) over the pointed circle. Then \( p \) induces a trivialization of \( L^\alpha_p \).

Proof: Let \( f \) be a classifying map for \( Q \). Then \( f \circ p \) is a classifying map for \( P \) and hence yields a trivialization of \( L^\alpha_p \). If \( f' \) is a different classifying map for \( Q \), then there is a cellular homotopy \( h: \mathbb{I} \times S^1 \to EG \), and \( \overline{h} \circ (id \times p)^*(\alpha) = 0 \). Hence our trivialization is independent of our choice of \( f \).

For every \( g \in G \), we fix, once and for all, a principal \( G \)-bundle \( P_g \) over the pointed circle with monodromy \( g \). We will follow the convention to trivialize lines of the form \( L^\alpha_{1,g} \) or \( L^\alpha_{1,g} \) using the projection to the first (respectively second) generating circle of the torus. Let \( d \) be a natural number, and let \( f_g \) be a classifying map for \( P_g \). Precomposing \( f_g \) with the degree \( d \) self-map of the circle gives a classifying map for \( P_{g^d} \). This yields isomorphisms of trivialized lines

\[ i_d: L_{1,g^d} \cong \mathbb{C} = \mathbb{C} \cong L_{1,g}, \]

which are compatible in the sense that \( i_{ad} = i_d \circ i_a \) and \( i_{k|g|+1} = id_{1,g} \). We will occasionally use this isomorphism to identify these lines and write \( L_{(g)} := L_{1,g} \). Here \( \langle g \rangle \) stands for the cyclic subgroup generated by \( g \). Let

\[ \gamma = \begin{pmatrix} a & b \\ nc & d \end{pmatrix} \in \Gamma_0(n). \]

Then \( ad \equiv 1 \mod n \), and \( \langle g^d \rangle = \langle g \rangle \). Hence

\[ \gamma^{-1}: L_{1,g} \to L_{1,g^d} \]

is an automorphism of \( L_{(g)} \). We caution the reader that this action of \( \Gamma_0(n) \) on \( L_{(g)} \) is not independent of the choice of generator \( g \). Instead, replacing \( g \) by \( g^t \) amounts to pulling back \( \alpha|_{B\langle g \rangle} \) along the map \((-)^t \). We will see in the proof of Lemma 2.19 below that restriction along this map raises the action to the \( t \)th power.

Lemma 2.12. Let \( h = |\alpha|_{B\langle g \rangle} \). Then \( T^n \) acts as multiplication by an \( h \)th root of unity on the line \( L^\alpha_{1,g} \). Similarly, any element \( \gamma \in \Gamma_0(n) \) acts as multiplication by a \( h \)th root of unity on the line \( L^\alpha_{(g)} \).

We will see in Corollary 2.15 below, that \( T^n \) actually acts by multiplication with a primitive \( h \)th root of unity. Proof: Let \( \phi \) be the \( n \)-fold covering map from the torus to itself which has degree \( n \) on the first circle and is the identity on the second. Then \( \phi \) pulls back \( P_{g,1} \) to the trivial bundle, and we have

\[ T^n \circ \phi = \phi \circ T. \]

Hence \( \phi \) induces an \( n \)-fold covering of the glued cylinders of Construction 2.5:

\[ \phi: cyl_T(P_{1,1}) \to cyl_{T^n}(P_{g,1}). \]

Just as in the proof of Proposition 2.6, we may conclude that the Chern-Simons invariant of \( cyl_T(P_{1,1}) \) is equal to the \( n \)th power of the Chern-Simons invariant of \( cyl_{T^n}(P_{g,1}) \). The first gives the action of \( T \) on \( L^\alpha_{1,1} \). By Corollary 2.9, this action is 1. The Chern-Simons
invariant of $\text{cycl}_n(P_{g,1})$ gives the action of $T^n$ on $L_{g,1}$. It follows that this action is an $n^{th}$ root of 1. From this point of view, it is also clear why its order divides the order of $\alpha$. Fix $g$, and let $\gamma \in \Gamma_0(n)$ be as above. The same argument, using the degree $n$ map of the second circle, shows that for any two classifying maps $f$ and $f'$ of $P_{1,g^4}$, the corresponding trivializations of $L_{1,g^4}$ differ by an $n^{th}$ root of unity. Writing $f_g$ and $f'_g$ for the classifying maps used to trivialize $L_{1,g}$ and $L_{1,g^4}$, the claim follows when we take $f = f_g$ and $f' = f_g \circ \gamma^{-1}$. □

2.4. Sections of $L^\alpha$. The following definitions and facts are taken from [FQ93, 5]. We let $E$ be the space of sections of $L^\alpha$ over $\mathbb{C}(G)$. In the notation of [FQ93], we have

$$E = \mathcal{L}^2(\mathcal{C}_{\mathbb{S}^1 \times \mathbb{S}^1}, \mathcal{C}_{\mathbb{S}^1 \times \mathbb{S}^1}).$$

Further, Freed and Quinn define a line bundle $\mathcal{L}_{[0,1] \times \mathbb{S}^1}$ on the set $\mathcal{C}_{[0,1] \times \mathbb{S}^1}$ of isomorphism classes of principal $G$-bundles over $[0,1] \times \mathbb{S}^1$ with basepoints chosen over the basepoints of the boundary. They prove that the sections of this line bundle

$$A := \mathcal{L}^2(\mathcal{C}_{[0,1] \times \mathbb{S}^1}, \mathcal{L}_{[0,1] \times \mathbb{S}^1})$$

form a coalgebra, whose underlying vector space is the direct sum of complex lines

$$A \cong \bigoplus_{g,h \in G} L_{g,h}.$$

If we view $A$ as the space of complex valued linear functions on its dual algebra

$$A^* \cong \bigoplus_{g,h \in G} L^*_{g,h},$$

then $E \subseteq A$ is identified with the subspace of central sections.

The algebra $A^*$ is semisimple, and $E$ possesses an orthonormal basis of character functions $\{\chi_\lambda\}$, where $\lambda$ runs through the irreducible representations of $A^*$. The support of $\chi_\lambda$ is

$$\text{supp}(\chi_\lambda) = \{(g, h) \mid g \in [g_0], h \in C_g\}$$

for a conjugacy class $[g_0]$ depending on $\lambda$. For fixed $g$, the restriction of $\chi_\lambda$ to (the units of)

$$\bigoplus_{h \in C_g} L^*_g$$

is a character of the central extension $\hat{C}_g$ of $C_g$ defined by these units. Thus

$$E \cong \mathbb{C}(\chi_\lambda) \cong \bigoplus_{[g] \subseteq G} \mathbb{R}(\hat{C}_g) \otimes \mathbb{C}.$$

Following Freed and Quinn, we write $e_{g,1} \in L_{g,1}$ and $e^*_{g,1}$ for the elements of unit length corresponding to the trivialization we picked above and note that $e^*_{g,1}$ is the unit of the group $\hat{C}_g$. 

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2.4.1. Sections of $\mathcal{L}^\alpha$. By construction, a section of $\mathcal{L}^\alpha$ is equivalent to an $\text{SL}_2(\mathbb{Z})$-equivariant map

$$f: \mathcal{H} \to E.$$  

Fix an element $g \in G$ and consider the $[g]$-component of $f$,

$$f_g: \mathcal{H} \to \mathbb{R}(\hat{\mathcal{C}}_g) \otimes \mathbb{C}.$$  

**Lemma 2.13.** The element $T^n \in \text{SL}_2(\mathbb{Z})$ acts as multiplication by a root of unity on the summand $\mathbb{R}(\hat{\mathcal{C}}_g) \otimes \mathbb{C}$. Its order divides $n$ and the order of $\alpha$. More precisely $T^n$ acts by multiplication with $e^{2\pi i \alpha(x)}$, where $x$ is the 3-cycle

$$x = \sum_{k=0}^{n} (g, g^k, g) \in \mathbb{Z}(B(g)).$$  

**Proof:** The action of $T$ on $\mathbb{A}^*|_{(g)}$ is given by multiplication with the element $e^{\alpha} \in \text{L}^*_{g,1}$ defined in [FQ93, (5.5)]. Hence it is enough to understand the action of $T^n$ on $\text{L}^\alpha_{g,1}$. We saw in Lemma 2.12 that this action is given by multiplication with a root of unity whose order divides $n$ and the order of $\alpha$. To determine it explicitly, we triangulate the fundamental domain $\mathbb{I} \times \mathbb{I}$ of $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ by choosing as 1-simplices $s_1$ from $(0,0)$ to $(0,1)$, $s_2$ from $(0,0)$ to $(1,0)$, $s_3$ from $(0,1)$ to $(1,1)$, $s_4$ from $(1,0)$ to $(1,1)$ and $s_5$ from $(0,1)$ to $(1,0)$. Then $T$ pulls back the corresponding triangulation of $\mathbb{T}^2$ to the one corresponding to the following choice of 1-simplices on $\mathbb{I}^2$: a simplex $t_1$ from $(0,1)$ to $(0,0)$, $t_2$ from $(0,0)$ to $(1,0)$, $t_3$ from $(0,1)$ to $(1,1)$, $t_4$ from $(1,1)$ to $(1,0)$, and $t_5$ from $(0,0)$ to $(1,1)$. Cutting open the pointed bundle $P_{g,1}$ along the two standard circles of $\mathbb{T}^2$ gives a bundle over $\mathbb{I}^2$ with basepoints over the corners and parallel transports $g$ along $s_1$, $s_2$, $s_3$ and $s_4$, and 1 along $s_5$. The same procedure for $T^*P_{g,1} = P_{g,1}$ gives parallel transports $g$ along $t_1$, $t_2$, $t_3$, and $t_5$, and 1 along $t_1$ and $t_4$. We triangulate the faces of the cube $\mathbb{I}^3$ by choosing the first of the above triangulations on $\mathbb{I}^2 \times \{0\}$ and the second one on $\mathbb{I}^2 \times \{1\}$ and adding the 1-simplices $r_1$ from $(1,1,1)$ to $(0,1,0)$, $r_2$ from $(1,1,1)$ to $(1,0,0)$, $r_3$ from $(0,0,1)$ to $(0,1,0)$, $r_4$ from $(0,0,1)$ to $(1,0,0)$,
We add two more degenerate 3-simplices, one along the face $I$ from $(0,0,1)$ to $(0,0,0)$, and note that there exists a bundle $Q_k$ over $I^3$, pointed over the corners, such that the parallel transports along $r_5$ and $r_7$ are $g^{k-1}$ and those along $r_6$ and $r_8$ are $g^k$. The cube $I^3$ is divided into five 3-simplices, one along a classifying map of $Q_k$ are

$$\pm(g^{k-1}, g, 1), \pm(g^{k-1}, 1, g), \pm(1, g, g^{k-1}), \pm(g, 1, g^{k-1}) \text{ and } \pm(g, g^{k-1}, 1).$$

We add two more degenerate 3-simplices, one to the face $[0] \times I^2$ giving the cocycle $\pm(1, g^{k-1}, g)$ and one along the face $I \times [0] \times I$ giving the cocycle $\pm(g, g^{k-2}, g)$. This allows us to glue the first circle of the torus back together via

$$(0, x, y) \sim (1, x, y).$$

This gives a triangulation of the mapping cylinder of $T|_{S^1 \times I}$. Further, we are able to glue $Q_k$ to $Q_{k+1}$ along the ends of the cylinder in such a way that their triangulations fit together. Since $Q_n = Q_0$, we may join $Q_0$ through $Q_{n-1}$ together to get a bundle over the mapping cylinder of $T^n$. Since $\alpha$ is normalized, it vanishes on cycles containing the identity. Hence $T^n$ acts as claimed. \qed

**Lemma 2.14.** The cycle $\sum_{k=0}^{n-1}(g, g^k, g)$ is a generator of the cyclic group $H_3(B(g)) \cong \mathbb{Z}/n\mathbb{Z}$.

**Proof:** We will use the terminology and notation of [Bro94]. Let $Z \leftarrow F_\bullet$ be the standard resolution of $Z$ over $ZG$, and consider the commuting diagram of $Z(g)$-modules

$$Z \leftarrow Z(g) \leftarrow Z(g) \leftarrow \underbrace{\cdots}_{\text{norm}} Z(g) \leftarrow Z(g),$$

$$\downarrow f_0 \quad \downarrow f_1 \quad \downarrow f_2 \quad \downarrow f_3,$$

where $\text{norm}(1) = 1 + g + g^2 + \cdots + g^{n-1}$, and in bar notation, the maps $f_1$ are given by

$$f_0(1) = 1, \quad f_1(1) = [g], \quad f_2(1) = \sum_{k=0}^{n-1} [g^k|g], \quad \text{and} \quad f_3(1) = \sum_{k=0}^{n-1} [g|g^k|g].$$

These are the first maps of an augmentation preserving map of chain complexes $f_\bullet$ from the periodic resolution to the standard resolution. On coinvariants, $f_\bullet$ induces an isomorphism of complexes of the form

$$Z \leftarrow Z \leftarrow Z \leftarrow Z \leftarrow Z \leftarrow Z \leftarrow \cdots,$$

where $1 \in Z(g)$ becomes $1 \in Z$. \qed
Corollary 2.15. The order of the action \( T^n \) on the \( g \)-twisted sector
\[
\bigoplus_{h \in C_g} L_{g,h}
\]
equals the order of \( \alpha|_{(g)} \). In particular, it divides the order of \( \alpha \) and that of \( g \).

Following [CN79], we will write \( h \) for the order of \( \alpha|_{\langle g \rangle} \) and \( n \) for the order of \( g \), and set \( N = nh \).

Corollary 2.16. The map \( f_g \) factors through
\[
\hat{\mathcal{H}} \to \mathbb{C}^\times \\
\tau \mapsto e^{2\pi i \frac{\tau}{N}}.
\]

Corollary 2.17. There is a Fourier expansion
\[
f_g \in \mathbb{R}(\hat{C}_g) \otimes \mathbb{C}[q^{\frac{1}{N}}]
\]
of \( f_g \), where \( q = e^{2\pi i \tau} \). If \( T^n \) acts by multiplication with \( e^{2\pi i \frac{\tau}{N}} \) on \( L_{g,1} \), then \( f_g \) has the form
\[
f_g(\tau) = \sum_k q^{\frac{k}{N}} e^{2\pi i \frac{\tau}{N}}.
\]

Proof: The first part follows immediately from the previous corollary. The second part is proved by a comparison of coefficients of the two sides of the equation
\[
f(q^{\frac{1}{N}} e^{2\pi i \frac{\tau}{N}}) = f(\tau + n) = f(\tau) e^{2\pi i \frac{\tau}{N}} = f(q^{\frac{1}{N}}) e^{2\pi i \frac{\tau}{N}}.
\]

In [DLM00] such a fraction \( \frac{s}{N} \) turns up as the conformal weight of the \( g \)-twisted sector. If \( g = 1 \), only integral powers of \( q \) occur, and the central extension of the centralizer is trivial:
\[
\hat{\mathbb{C}}_1 = \mathbb{U}(1) \times G.
\]

Proposition 2.18. The group \( \Gamma_0(n) \) acts on \( L_{(g)} \) by multiplication with \( h \)th roots of 1. The group \( \Gamma_0(N) \) acts trivially on \( L_{(g)} \).

Proof: The first claim follows immediately from Lemma 2.8 and Lemma 2.12. Let \( \gamma \in \Gamma_0(N) \), and let \( p_h : \mathbb{T}^2 \to \mathbb{T}^2 \) be the degree \( h \) map of the first circle. Then we have
\[
p_h \circ \gamma = \hat{\gamma} \circ p_h
\]
with \( \hat{\gamma} \in \Gamma_0(n) \). As in the proof of Lemma 2.12, it follows that \( \gamma_s = (\hat{\gamma}_s)^h \). But we just saw that \( \hat{\gamma}_s \) acts as an \( h \)th root of unity.

Lemma 2.19. If \([\alpha]\) is an element of order \( h \) in \( H^3(B(g), \mathbb{R}/\mathbb{Z}) \), then the restriction \([\alpha]|_{B(g^h)}\) has order 1 in \( H^3(B(g^h), \mathbb{R}/\mathbb{Z}) \).

Proof: There is a \( \mathbb{Z}\langle g^h \rangle \)-equivariant, augmentation-preserving map between the periodic resolutions, which is the inclusion of \( \mathbb{Z}\langle g^h \rangle \) in \( \mathbb{Z}\langle g \rangle \) in even degrees and sends 1 to \( 1 + g + \cdots + g^{h-1} \) in odd degrees. On coinvariants, this map becomes the identity in even degrees and multiplication with \( h \) in odd degrees.
An alternative proof for this lemma is given as follows: Let $T^t$ denote the transpose of $T$. Note that we can use the degree $h$ map of the second circle to conjugate $(T^t)^n$ into $(T^t)^{hn}$. It follows that the action of $(T^t)^{hn}$ on $L_{(g^h)}$ is the $h$th power of the action of $(T^t)^n$ on $L_{(g)}$.

**Corollary 2.20.** The group $\Gamma_0(\frac{n}{h})$ acts trivially on $L_{(g^h)} = L_{(g)}^{\otimes h}$.

**Definition 2.21** (cf. [CN79]). Let $\Gamma_0(n|h)$ be the subgroup of $\text{SL}_2(\mathbb{R})$ consisting of matrices of the form

\[
\begin{pmatrix}
a & b \\
nc & d
\end{pmatrix},
\]

where $a$, $b$, $c$ and $d$ are integers.

Note that $\Gamma_0(n|h)$ is conjugate to $\Gamma_0(\frac{n}{h})$ via conjugation with

\[
\begin{pmatrix}
h & 0 \\
0 & 1
\end{pmatrix}^{-1} \text{ or with } \begin{pmatrix} 1 & 0 \\ 0 & h \end{pmatrix}.
\]

**Remark 2.22.** The conjecture in [CN79] suggests that $\Gamma_0(n|h)$ acts by $h$th roots of 1 on $L_{(g)}$. While the previous corollary does not tell us how to define this action when $h$ does not divide $b$, it shows that in order for it to be consistent with the rest of the picture, it needs to be an action by $h$th roots of unity.

### 3. Abelian groups and level structures

Let $E = \mathbb{C}/\langle \tau, 1 \rangle$. Since any classifying map $f: \mathbb{Z}^2 \to G$ factors through an abelian subgroup of $G$, it is often enough to study principal bundles over $E$ with abelian structure group. These also are important for the definition of Hecke operators.

#### 3.1. Isogenies

Let $A$ be a finite abelian group. Our first observation is that principal $A$-bundles with connected total space are almost the same as isogenies with kernel $A$. More precisely, let $\phi: E' \to E$ be an isogeny with kernel $A$. Then $A$ acts on $E'$ by addition, making $\phi$ the structure map of a principal $A$-bundle over $E$.

**Proposition 3.1.** Let $f: \mathbb{Z}^2 \to A$ be a group homomorphism. Then $f$ is surjective if and only if the principal $A$-bundle $\phi: P \to E$ over $E$ classified by $Bf$ is isomorphic to an isogeny with kernel isomorphic to $A$, viewed as a principal $A$-bundle in the way just described. This in turn is the case if and only if $P$ is connected.

**Proof:** Let $f: \mathbb{Z}^2 \to A$ be surjective and let $n$ be the order of $A$. Write $\Lambda$ for $\text{ker}(f)$. Then $\Lambda$ is an index $n$ subgroup of $\langle \tau, 1 \rangle$, and therefore a rank two sublattice. The short exact sequence

\[0 \to \Lambda \to \mathbb{Z}^2 \to A \to 0\]

gives rise to a fibre sequence of classifying spaces

\[BA \longrightarrow B\mathbb{Z}^2 \xrightarrow{Bf} BA,\]

whose first map is the principal $A$-bundle classified by $Bf$. It is constructed as the quotient map

\[B\Lambda := E\mathbb{Z}^2 / \Lambda = \mathbb{C}/\Lambda \to \mathbb{C}/\langle \tau, 1 \rangle = B\mathbb{Z}^2,\]
with the $A$ action induced by the action of $\langle \tau, 1 \rangle \cong \mathbb{Z}^2$ on $\mathbb{C} \cong \mathbb{E} \mathbb{Z}^2$. Hence it is an isogeny whose kernel is isomorphic to $A$. Moreover, any such isogeny into $E$ is of the form (8) for some sublattice $\Lambda \subseteq \langle \tau, 1 \rangle$, and therefore classified by the surjective map

$$\mathbb{Z}^2 \cong \langle \tau, 1 \rangle \to \langle \tau, 1 \rangle/\Lambda.$$  

Let now $P$ be connected. We will see in Lemma 4.1 that this implies that $P$ can be viewed as a complex elliptic curve and $\phi$ as an isogeny. More precisely, $P$ is the quotient of $C$ by an index $n$ subgroup $\Lambda \subseteq \langle \tau, 1 \rangle$. Thus $\phi$ is an isogeny with kernel $\langle \tau, 1 \rangle/\Lambda$, and Proposition A.7 implies that $A$ is isomorphic to $\langle \tau, 1 \rangle/\Lambda$. \hfill \Box

**Remark 3.2.** The (base-preserving) automorphisms of a principal $A$-bundle over $E$ are given by multiplication with elements of $A$:

$$\text{Aut}(\mathcal{P}_{a_1, a_2}) \cong C_A(a_1, a_2) = A.$$  

Isogenies, at the other hand, come equipped with a choice of basepoint over zero and therefore have no automorphisms over $E$.

If $f$ is not surjective, we can write $f = i \circ f'$, where $f': \mathbb{Z}^2 \to B$ is surjective, and $i: B \to A$ is injective. Then the $A$-bundle classified by $Bf$ is a collection of isogenies into $E$ with kernel $B$.

3.2. **Duals.** Instead of considering isogenies into $E$, it is often more natural to consider the dual picture, which involves isogenies with source $E$ and the corresponding level structures on $E$. Let $\hat{A} := \text{Hom}(A, \mathbb{S}^1)$ be the Pontrjagin dual of $A$. Since $A$ is finite and abelian, we have $\hat{\hat{A}} = A$. We further know $\hat{\mathbb{Z}} = \mathbb{S}^1$ and $\hat{\mathbb{S}^1} = \mathbb{Z}$.

Let $i_{(\omega_1, \omega_2)}$ denote the isomorphism from $E$ to $\mathbb{S}^1 \times \mathbb{S}^1$ corresponding to the choice of basis $(\omega_1, \omega_2)$. While we had identified $B \mathbb{Z}^2 = \mathbb{S}^1 \times \mathbb{S}^1$ with $E$ in the non-standard way, using $i_{(-1, \tau)}$, we will follow the convention to identify $\hat{\mathbb{Z}}^2 \cong \mathbb{S}^1 \times \mathbb{S}^1$ with $E$ in the standard way, using $i_{(\tau, 1)}$.

**Proposition 3.3** (compare [GKV, (1.4.4)]). These identifications yield an isomorphism

$$\text{Prin}_A(E) \cong \text{Hom}(\hat{A}, E),$$  

where $\text{Prin}_A(E)$ denotes the set of isomorphism classes of principal $A$-bundles over $E$. Let $\phi: E' \to E$ be an isogeny. Then pullback along $\phi^\vee$ on the left-hand side becomes composition with $\phi$ on the right-hand side.

**Proof:** The isomorphism is given by

$$\text{Prin}_A(E) \cong \text{Hom}(\mathbb{Z}^2, A) \cong \text{Hom}(\hat{A}, \mathbb{Z}^2) \cong \text{Hom}(\hat{A}, E).$$  

Here the first isomorphism uses $i_{(-1, \tau)}$ to identify $\mathbb{S}^1 \times \mathbb{S}^1$ with $E$, while the last one uses $i_{(\tau, 1)}$. Let $\varphi: \mathbb{Z}^2 \to \mathbb{Z}^2$ be as in (6) (on page 6), let $\varphi^\vee$ denote its pseudo-inverse, and write $\phi = B \varphi$. Then $\phi$ is viewed as a map from $E'$ to $E$ via the identifications $i_{(-1, \tau')}$ and $i_{(-1, \tau)}$. 

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The first isomorphism sends \((\phi^\vee)^*\) to \((\varphi^\vee)^*\). The second isomorphism maps \((\varphi^\vee)^*\) to \(\hat{\phi}^\vee\). We have a commutative diagram

\[
\begin{array}{ccccccccc}
E' & \xrightarrow{i_{(\tau',1)}} & S^1 \times S^1 & \xrightarrow{\cong} & Z^2 \\
\downarrow{\phi} & & \downarrow{B(\varphi^\vee)^*} & & \downarrow{\varphi^\vee} \\
E & \xrightarrow{i_{(\tau,1)}} & S^1 \times S^1 & \xrightarrow{\cong} & Z^2,
\end{array}
\]

where \((\varphi^\vee)^t\) is the transpose of \(\varphi^\vee\). This completes the proof. \(\square\)

**Lemma 3.4.** Let \(f \in \text{Hom}(\hat{A}, E)\), and let \(\xi_f\) be the principal \(A\)-bundle over \(E\) corresponding to \(f\). Let \(g\) be an automorphism of \(A\). Then the pullback \(g^*(\xi_f)\) of \(\xi_f\) along \(g\) is the principal \(A\)-bundle corresponding to \(f \circ \hat{g}^{-1} \in \text{Hom}(\hat{A}, E)\),

\[
g^*(\xi_f) \cong \xi_{f \circ \hat{g}^{-1}}.
\]

**Proof:** Let \(\hat{f} : Z^2 \to A\) be the dual of \(f\). Then \(\xi_f\) is classified by \(B\hat{f}\), and \(\xi_{f \circ \hat{g}^{-1}}\) is classified by

\[
B(f \circ \hat{g}^{-1}) = B(g^{-1} \circ \hat{f}) = Bg^{-1} \circ B\hat{f}.
\]

The proposition now follows from Corollary A.5. \(\square\)

A homomorphism \(f : Z^2 \to A\) is surjective, if and only if its dual homomorphism \(\hat{f} : \hat{A} \to E\) is injective.

**Proposition 3.5.** Let \(f \in \text{Inj}(\hat{A}, E)\), and let \(\xi_f\) be the principal \(A\)-bundle over \(E\) classified by \(B\hat{f}\). Further, let \(\Psi_f : E \to E'\) be the isogeny with kernel \(f\). Then \(\xi_f\) is isomorphic to the principal bundle defined by the dual isogeny \(\Psi_f^\vee : E' \to E\).

**Proof:** Consider the short exact sequence

\[
0 \to \hat{A} \xrightarrow{\hat{f}} E \xrightarrow{\Psi_f} E' \to 0.
\]

Its Pontrjagin dual is a short exact sequence of the form

\[
0 \to \ker(\hat{f}) \to Z^2 \xrightarrow{\hat{f}} A \to 0.
\]

Write \(\psi\) for the inclusion of \(\ker(\hat{f})\) in \(Z^2\). Then \(\psi\) is such that under the isomorphisms \(i_{(\tau,1)}\) and \(i_{(\tau',1)}\), we have \(\Psi_f = \hat{\psi}\). It follows from the proof of Proposition 3.1 that under the isomorphisms \(i_{(-1,\tau)}\) and \(i_{(-1,\tau')}\), we have \(\xi_f = B\psi\). Finally, we consider the commutative diagram (9) with \(\hat{\xi}_f^\vee\) in the role of \(\phi\) and \(\psi\) in the role of \(\phi^\vee\) to obtain \(\hat{\xi}_f^\vee = \Psi_f\). \(\square\)

**Corollary 3.6.** There is a one-to-one correspondence

\[
\text{Inj}(\hat{A}, E) \leftrightarrow \{\text{principal } A\text{-bundles over } E \text{ with connected total space}\} / \cong
\]

sending \(f\) to \(\Psi_f^\vee\).
The decomposition
\[ \text{Hom}(\hat{A}, E) \cong \coprod_{B \subseteq A} \text{Inj}(\hat{B}, E) \]
is invariant under the action of SL\(_2(\mathbb{Z})\) on \(A \times A\). Hence it induces a decomposition of the coarse moduli space
\[ X_A = \coprod_{B \subseteq A} X_{\text{inj} B}, \]
where \(X_{\text{inj} B}\) is the moduli space of elliptic curves with level \(\hat{B}\)-structures, i.e., of isogenies \(\phi: E \to E'\) together with a choice of isomorphism from their kernel to \(\hat{B}\). Note that
\[ \text{Inj}(\hat{A}, E) \]
is invariant under the action of \(\text{Aut}(A)\) on \(\text{Hom}(\hat{A}, E)\), and that
\[ \text{Inj}(\hat{A}, E)/\text{Aut}(A) \]
corresponds to the set of all isogenies out of \(E\) which allow a choice of isomorphism from \(\hat{A}\) to their kernels. Since no choice of this isomorphism is specified, and \(\hat{A}\) is non-canonically isomorphic to \(A\), this condition is equivalent to allowing an isomorphism from \(A\) to their kernels.

**Definition 3.7.** We will write \(M_A^{\text{conn}}\) for the component of \(M_A\) parametrizing bundles with connected total space and
\[ \widetilde{M}_0(A) := M_A^{\text{conn}} // \text{Aut}(A) \]
for the quotient stack of \(M_A^{\text{conn}}\) by the action of \(\text{Aut}(A)\) on the fibers. We will write \(X_0(A)\) for the corresponding coarse moduli space.

Note that by Remark 3.2 the stack \(\widetilde{M}_0(A)\) is not the same as the moduli-stack \(M_0(A)\) of isogenies allowing an isomorphism from \(A\) to their kernel. However, the forgetful map
\[ M_0(A) \longrightarrow \widetilde{M}_0(A) \]
induces an isomorphism of coarse moduli spaces.

**Definition 3.8.** The Fricke involution \(W_A\) is an involution of \(M_0(A)\). It sends the isogeny \(\phi\) to its dual isogeny
\[ \phi^\vee : E' \to E. \]
If the kernel of \(\phi\) is isomorphic to \(\hat{A}\), then the kernel of \(\phi^\vee\) is (also) isomorphic to \(A\).

3.3. **Cyclic groups.** Let now \(A\) be the cyclic group \(\mathbb{Z}/n\mathbb{Z}\). An injective map from \(\mathbb{Z}/n\mathbb{Z}\) to \(E\) is the same as a choice of point of exact order \(n\) on \(E\). Such a choice is called a \(\Gamma_1(n)\)-level structure on \(E\). Therefore, the space of principal \(\mathbb{Z}/n\mathbb{Z}\) bundles on \(E\) is non-canonically isomorphic to
\[ E[n] \cong \text{Hom}(\mathbb{Z}/n\mathbb{Z}, E) \cong \coprod_{d|n} \text{Level}_{\Gamma_1(d)}(E), \]
and the moduli space \(X_{\mathbb{Z}/n\mathbb{Z}}\) gets (non-canonically) identified with
\[ X_{\mathbb{Z}/n\mathbb{Z}} \cong \coprod_{d|n} X_1(d). \]
Note that
\[ \text{Inj}(\mathbb{Z}/n\mathbb{Z}, E) / \text{Aut}(\mathbb{Z}/n\mathbb{Z}) \cong \text{Level}_{\frac{n}{\mathbb{Z}}} (E). \]
Therefore, we get a (canonical) isomorphism
\[ X_{\mathbb{Z}/n\mathbb{Z}}^{\text{inj}} / \text{Aut}(\mathbb{Z}/n\mathbb{Z}) \cong X_0(n). \]
The Fricke involution \( W_n := W_A \) takes a cyclic subgroup \( C \subseteq E \) of order \( n \) to the cyclic subgroup
\[ E[n]/C \subseteq E/C. \]
Let \( g \in \mathbb{Z}/n\mathbb{Z} \) be a generator. Then the triple \( (1, g; \tau) \) corresponds to the cyclic subgroup \( \langle \frac{1}{n} \rangle \) of \( C/\langle \tau, 1 \rangle \). Therefore \( W_n \) sends it to the cyclic subgroup of \( C/\langle \tau, \frac{1}{n} \rangle \) generated by \( \frac{1}{n} \). This again corresponds to the subgroup generated by \( \frac{1}{n} \) in
\[ C/\langle -\frac{1}{n\tau}, 1 \rangle. \]
Thus
\[ W_n (1, g; \tau) = \left( 1, g; -\frac{1}{n\tau} \right). \]

4. Symmetric groups and coverings

The symmetric groups play a central role in the definition of power operations and will turn out to be important in the theory of replicability. Recall the one-to-one correspondence
\[ \{ \text{Isom. classes of principal } \Sigma_n \text{-bundles over } X \} \leftrightarrow \{ \text{Isom. classes of } n \text{-fold covers of } X \} \]
\[ P \leftrightarrow P \times_{\Sigma_n} n, \]
where \( n \) is a fixed set with \( n \) elements.

**Lemma 4.1.** Let \( \pi: P \to E \) be an \( n \)-fold covering of a complex elliptic curve \( E \) with connected total space. Then \( P \) can be viewed as a complex elliptic curve and \( \pi \) as an isogeny.

**Proof:** By the theory of covering spaces, the universal cover of \( E \),
\[ \mathbb{C} \to \mathbb{C}/\langle \tau, 1 \rangle \cong E, \]
factors through \( \pi \). More precisely, if we choose a basepoint of \( P \) over zero, this identifies \( P \) with the quotient of \( \mathbb{C} \) by an index \( n \) subgroup \( \Lambda \subseteq \langle \tau, 1 \rangle \). Thus \( \pi \) becomes an isogeny. \( \square \)

It follows that every \( \Sigma_n \)-bundle with connected total space is induced from a principal bundle with abelian structure group: Let \( A \) be an abelian group of order \( n \), and pick a bijection between \( A \) and \( n \). Then the action of \( A \) on itself by multiplication defines a map
\[ i: A \to \Sigma_n. \]
Note that a different choice of bijection leads to a different map which is conjugate to \( i \) by an element of \( \Sigma_n \). In particular, this applies to precomposition of \( i \) with any automorphism of \( A \).
Lemma 4.2. Let
\[ \phi: Y \to E \]
be an isogeny together with an isomorphism from \( \ker(\phi) \) to \( A \). Consider the \( n \)-fold covering map underlying \( \phi \), and let \( \xi_Y \) be the corresponding principal \( \Sigma_n \)-bundle. Now consider \( \phi \) as principal \( A \)-bundle. Then the isomorphism class of the \( \Sigma_n \)-bundle \( \phi[\Sigma_n] \) associated to \( \phi \) via \( \iota \) is equal to that of \( \xi_Y \).

Note that (in particular) the isomorphism class of \( \phi[\Sigma_n] \) is independent of the choice of \( \iota \) and of the choice of isomorphism \( \ker(\phi) \cong A \).

Proof: By Lemma A.6 we have
\[ \phi[\Sigma_n] \times \Sigma_n \cong \phi \times_A \Sigma_n, \]
which is the \( n \)-fold covering underlying \( \phi \). Thus \( \xi_Y \) and \( \phi[\Sigma_n] \) correspond to the same covering, which proves the claim. \( \square \)

Take an \( n \)-fold covering \( P \) of \( E \) with connected total space, and view it as a principal \( A \)-bundle. Note that any deck transformation can be realized as \((-) + a\) with \( a \in A \). Hence the automorphisms of \( P \) as an \( n \)-fold covering are the same as the automorphisms of \( P \) as principal \( A \)-bundle. For future reference, we note that their number is \( n = |A| \).

We now direct our attention to coverings whose total spaces are not connected. Recall that if \( P \) and \( P' \) are \( \Sigma_n \)- and \( \Sigma_m \)-principal bundles with associated covering spaces \( Y \) and \( Y' \) respectively, then the associated covering space of \( (P \times P') \times \Sigma_n \times \Sigma_m \Sigma_{n+m} \) is the disjoint union \( Y \cup Y' \). Let now \( Y \) be the disjoint union of connected components
\[ Y = Y_1 \cup \cdots \cup Y_k. \]
Then the \( Y_i \) can be viewed as having abelian structure groups \( A_1, \ldots, A_k \) of orders \( n_1, \ldots, n_k \) with \( n = \sum n_i \). Each of the \( Y_i \) has an associated \( \Sigma_{n_i} \)-principal bundle
\[ P_i = Y_i \times_{A_i} \Sigma_{n_i}, \]
and \( P \) is the bundle
\[ P = \left( \prod_i P_i \right) \times_{\Sigma_1 \times \cdots \times \Sigma_k} \Sigma_n. \]
Note that each \( A_i \) is a quotient of \( \mathbb{Z}^2 \), making the disjoint union
\[ A_1 \cup \cdots \cup A_k \cong \mathbb{n}, \]
into an \( \mathbb{Z}^2 \)-set, whose orbits are the \( A_i \). The \( \mathbb{Z}^2 \)-action is transitive if and only if \( Y \) is connected. We obtain a map
\[ \mathbb{Z}^2 \to \Sigma_n, \]
well defined up to conjugation in \( \Sigma_n \), and this is the pair of commuting elements which classifies \( P \). More precisely, we have the following:

Proposition 4.3. The decomposition of covers into connected components defines an equivalence
\[ e: M_{\Sigma_n} \cong \coprod_{n=|A|\mathbb{N}_A} \prod_{|A|} \tilde{M}_0(A) \wr \Sigma_{\mathbb{N}_A} \]
where the product as well as the sum under the coproduct is over all isomorphism classes of finite abelian groups, and all products are taken over the target maps $\tilde{M}_A \to M_1$.

**Example 4.4.** Let $n = 9$, and consider the decomposition $9 = |\{1\}| + |\mathbb{Z}/4\mathbb{Z}| \cdot 2$. The corresponding summand

$$
\mathcal{M}_1 \times \mathcal{M}_1 \, \tilde{M}_{\mathbb{Z}/4\mathbb{Z}} \uparrow \Sigma_2
$$

parametrizes ninefold covers of elliptic curves that decompose into three connected components, two of which are fourfold covers corresponding to isogenies with kernel $\mathbb{Z}/4\mathbb{Z}$ and one of which is a onefold cover.

**Proof of Proposition 4.3:** We start by considering bundles over a torus. Fix a partition

$$
n = \sum_T |T| \mathbb{N}_T,
$$

where the $T$ are elements of a fixed system of representatives of the set of isomorphism classes of finite transitive $\mathbb{Z}^2$-sets. Such decompositions classify conjugacy classes $[\sigma, \rho]_{\Sigma_n}$ of pairs of commuting elements in $\Sigma_n$, and hence they classify isomorphism classes of principal $\Sigma_n$-bundles over $T^2$. For each $T$, let

$$
A_T := \mathbb{Z}^2 / \text{Stab}_{\mathbb{Z}^2}(T)
$$

denote the corresponding abelian group. Note that the isomorphism class of $T$ is uniquely determined by the surjective map $f_T : \mathbb{Z}^2 \to A_T$, that for any automorphism $g$ of $A_T$, the map $g \circ f_T$ determines the same isomorphism class $[T]$, and that we have

$$
\mathbb{N}_A = \sum_{A_T = A} \mathbb{N}_T.
$$

Now the $n$-fold cover of $T^2$ corresponding to $[\sigma, \rho]_{\Sigma_n}$ decomposes into connected components, $\mathbb{N}_T$ of which are isomorphic to the principal $A_T$-bundle with connected total space that is classified by $f_T$. The automorphism group of this $n$-fold cover over $T^2$ is isomorphic to

$$
C_{\Sigma_n}(\sigma, \rho) = \prod_T A_T^{\mathbb{N}_T} \uparrow \Sigma_{\mathbb{N}_T}.
$$

For each $[A]$, pick, once and for all, an ordering of the classes $[T]$ with $A_T \cong A$. Then, for each pair $[\sigma, \rho]_{\Sigma_n}$ and for each orbit-type $[T]$ occurring in its orbit decomposition, pick an ordering of the orbits isomorphic to $T$. In this way, we assign to $[\sigma, \rho]_{\Sigma_n}$ a point

$$
e(\sigma, \rho) \in \prod_A \left( \text{Surj}(\mathbb{Z}^2, A) / \text{Aut}(A) \right)^{\mathbb{N}_A}.
$$

Our choices also determine an isomorphism

$$
\text{Aut}(e(\sigma, \rho)) \cong C_{\Sigma_n}(\sigma, \rho),
$$

which we use to continue $e$ to automorphisms of $[\sigma, \rho]$. Let now $\gamma \in \text{SL}_2(\mathbb{Z})$. Then $\gamma$ acts from the right on $(\sigma, \rho)$ and on $\text{Surj}(\mathbb{Z}^2, A)$, for each $A$. The transitive $\mathbb{Z}^2$-set $T \cdot \gamma$ classified by $\gamma \circ f_T$ might be different from $T$, but we have

$$
A_{T \cdot \gamma} = A_T.
$$

The action of $\gamma$ on $(\sigma, \rho)$ is compatible with the action of $(\sigma, \rho)$ on the connected components of the corresponding bundle. However, we might have numbered the components belonging
to the same group $A$ differently for $(\sigma, \rho) \cdot \gamma$ than we did for $(\sigma, \rho)$, with the result that they differ by an element $\sigma \gamma \in \Sigma_{N, A}$. To extend the map $e$ to the translation groupoid

$$(\overline{C}(\Sigma_n) \times \mathcal{Y}) \rtimes \text{SL}_2(\mathbb{Z}),$$

we set

$$e((\sigma, \rho; \tau), \gamma) := \prod_{A} (((\sigma, \rho)_{\lambda}; \tau), (\gamma \cdot \sigma_{\gamma})).$$

Here $(\sigma, \rho)_{\lambda}$ stands for the collection of the $(\sigma, \rho)$-orbits that belong to $A$.

By construction, $e$ is fully faithful and essentially surjective. Different choices of orderings would result in a map $e'$, which would differ from $e$ by a natural transformation on its target. Hence, as a map of orbifolds, $e$ is independent of these choices. □

**Definition 4.5.** We will write $\tilde{H}_n$ for the moduli stack of all $n$-fold covers with connected fibre,

$$\tilde{H}_n = \bigsqcup_{|A|=n} \tilde{M}_0(A).$$

Here the disjoint union runs over all isomorphism classes of abelian groups $A$ of order $n$.

Equivalently, $\tilde{H}_n$ is the substack of $\mathcal{M}_{\Sigma_n}$ defined as

$$\tilde{H}_n := \{f : \mathbb{Z}^2 \to \Sigma_n | f \text{ makes } n \text{ into a transitive } \mathbb{Z}^2\text{-set} \} \times \mathcal{Y} \setminus (\Sigma_n \times \text{SL}_2(\mathbb{Z})).$$

The following lemma provides an alternative proof for the fact that the automorphism group of an $n$-fold covering with connected total space is isomorphic to the kernel $A$ of the corresponding isogeny.

**Lemma 4.6.** Let $(\sigma, \rho)$ be a pair of commuting elements of $\Sigma_n$ which acts transitively on $\{1, \ldots, n\}$. Write $A := \langle \sigma, \rho \rangle$. Then $A$ has order $n$, the principal $\Sigma_n$-bundle classified by $(\sigma, \rho)$ corresponds to an isogeny with kernel $A$, and we have

$$C_{\Sigma_n}(\sigma, \rho) = \langle \sigma, \rho \rangle = A.$$

**Proof:** If $(\sigma, \rho)$ acts transitively on $\{1, \ldots, n\}$, then the corresponding $n$-fold covering has connected total space and hence is an isogeny. Write $A'$ for its kernel, then the bundle $P_{(\sigma, \rho)}$ is induced by a principal $A'$-bundle. Hence the classifying map of $P_{(\sigma, \rho)}$ factors through $A'$, yielding an isomorphism to $\Lambda$:

$$\mathbb{Z}^2 \to A' \xrightarrow{\tilde{\pi}} \langle \sigma, \rho \rangle \to \Sigma_n.$$ 

To determine the centralizer of $\langle \sigma, \rho \rangle$, let $\pi \in C_{\Sigma_n}(\sigma, \rho)$. Because of the transitivity of the action of $A$, we can write $\pi(1)$ in the form $\pi(1) = \sigma^c \rho^d(1)$. Further, for $1 \leq x \leq n$, we have $c$ and $d$ such that $x = \sigma^c \rho^d(1)$. It follows that

$$\pi(x) = \pi(\sigma^c \rho^d(1)) = \sigma^c \rho^d(\pi(1)) = \sigma^c \rho^d(\sigma^a \rho^b(1)) = \sigma^a \rho^b(\sigma^c \rho^d(1)) = \sigma^a \rho^b(x).$$

Hence $\pi = \sigma^a \rho^b$. □
5. Hopkins-Kuhn-Ravenel character theory

Let \( \mathcal{A}(G) \) be the category having as objects the abelian subgroups of \( G \) and with morphisms from \( B \) to \( A \) being the \( G \)-equivariant maps from \( G/B \) to \( G/A \). Then \( \mathcal{A}(G) \) is a full subcategory of the standard orbit category. The results of the previous sections can be summarized as follows:

\[
\mathcal{M}_G \simeq \varinjlim_{\mathcal{A}(G)} \mathcal{M}_A.
\]

Here \( \varinjlim \) stands for a weak 2-colimit. Roughly speaking, that means that we take the disjoint union of all the \( \mathcal{M}_A \) and instead of quotienting by the equivalence relation defined by maps in \( \mathcal{A}(G) \), as we would for a classical colimit, we add in extra isomorphisms.

In fact, one can replace \( \mathcal{A} \) by the full subcategory whose objects are abelian groups which are generated by two elements. The equivalence (10) is an analogue of the generalized Artin theorem [HKR00, Thm. A].

5.1. Products. Let \( G \) and \( H \) be finite groups. Then we have an isomorphism of moduli stacks

\[
\mathcal{M}_{G \times H} \cong \mathcal{M}_G \times_{\mathcal{M}_1} \mathcal{M}_H,
\]

which pulls back \( \mathcal{L}^\alpha \otimes \mathcal{L}^\beta \) to \( \mathcal{L}^{\alpha \times \beta} \). This yields external products

\[
\boxtimes: \text{Ell}^\alpha_G(\text{pt}) \otimes \text{Ell}^\beta_H(\text{pt}) \to \text{Ell}^{\alpha \times \beta}_{G \times H}(\text{pt}),
\]

\[
(f_G \boxtimes f_H)(g_1, h_1, g_2, h_2; \tau) = f_G(g_1, g_2) \cdot f_H(h_1, h_2).
\]

In the case that \( G = H \), one composes with pull-back along the diagonal of \( G \times G \) to obtain internal products

\[
\otimes: \text{Ell}^\alpha_G(\text{pt}) \otimes \text{Ell}^\beta_G(\text{pt}) \to \text{Ell}^{\alpha+\beta}_G(\text{pt}).
\]

5.2. Change of groups. Let \( \alpha: H \to G \) be a map of finite groups. We write

\[
\overline{\mathcal{C}}(\alpha): \overline{\mathcal{C}}(H) \to \overline{\mathcal{C}}(G)
\]

for the map of moduli sets induced by \( \alpha \), and note that there is a canonical isomorphism of line bundles

\[
\overline{\mathcal{C}}(\alpha)^*(\mathcal{L}^\alpha) \cong \mathcal{L}^{\alpha(\alpha)}.
\]

For a section \( f \) of \( \mathcal{L}^\alpha \) over \( \overline{\mathcal{C}}_G \), we define the section \( \text{res}|_{\alpha}(f) \) of \( \mathcal{L}^{\alpha(\alpha)} \) over \( \overline{\mathcal{C}}(H) \) by

\[
\text{res}|_{\alpha}(f) := f \circ \overline{\mathcal{C}}(\alpha).
\]

We recall the measure \( \mu \) on \( \overline{\mathcal{C}}(G) \) defined in [FQ93, (2.1)]:

\[
\mu([P_{g,h}]) = \frac{1}{|\text{Aut}(P_{g,h})|} = \frac{1}{|\mathcal{C}_G(g, h)|}.
\]

If \( f \) is a section of \( \mathcal{L}^{\alpha(\alpha)} \) over \( \overline{\mathcal{C}}(H) \), we define a section \( \text{ind}|_{\alpha}(f) \) of \( \mathcal{L}^\alpha \) over \( \overline{\mathcal{C}}(G) \) by

\[
\text{ind}|_{\alpha}(f) \cdot \mu = \int_{\text{fiber}} f \cdot d\mu = \sum_{\text{fiber}} \frac{f([h_1, h_2]_H)}{|\mathcal{C}_H(h_1, h_2)|}.
\]
where at the point \([g_1, g_2]_G\), the sum is over all \(H\)-conjugacy classes \([h_1, h_2]_H\) which satisfy \([h_1, h_2]_G = [g_1, g_2]_G\). We have

\[
\text{ind}|_a(f)([g_1, g_2]_G) = |C_G(g_1, g_2)| \cdot \sum_{[h_1, h_2]_H \in \text{fiber}} \frac{1}{|C_H(h_1, h_2)|} \cdot f([h_1, h_1]_H)
\]

\[
= |C_G(g_1, g_2)| \cdot \sum_{a(h_1, h_2) \sim G(g_1, g_2)} \frac{1}{|H|} \cdot f([h_1, h_1]_H).
\]

Two special cases are important: in the case that \(\alpha\) is an inclusion of groups, we obtain the formula of [HKR00, Thm. D],

\[
\text{ind}|_H^G(f) = \frac{1}{|H|} \sum_{s \in G} f(s^{-1} g_1 s, s^{-1} g_2 s).
\]

In the case that \(\alpha\) is the unique map from \(G\) to the trivial group, we get

\[
\text{ind}|_1^G(f)(1, 1) = \frac{1}{|G|} \sum_{gh=hg} f(g, h),
\]

where the sum is over all pairs of commuting elements of \(G\).

**Definition 5.1.** We define the inner product on \(\text{Ell}_0^G(\text{pt})\) as the composite of the (internal) product with \(\text{ind}|_1^G\),

\[
\langle - , - \rangle_G : \text{Ell}_0^G(\text{pt}) \otimes \text{Ell}_0^G(\text{pt}) \rightarrow \text{Ell}_0^G(\text{pt}) \rightarrow \text{Ell}_1^0(\text{pt}).
\]

We have

\[
\langle f_1, f_2 \rangle_G = \frac{1}{|G|} \sum_{gh=hg} f_1(g, h) \cdot f_2(g, h).
\]

All of these constructions extend to \(\text{SL}_2(\mathbb{Z})\)-equivariant maps from \(\mathcal{H}\) to \(\bigoplus L_{g,h}\).

6. Hecke operators and power operations

6.1. The Hecke correspondence. Let \(\tilde{\mathcal{H}}_{n,G} \subseteq \mathcal{M}_G \times \Sigma_n\) be defined by

\[
\tilde{\mathcal{H}}_{n,G} := \mathcal{M}_G \times_{\mathcal{M}_1} \tilde{\mathcal{H}}_n,
\]

where \(\tilde{\mathcal{H}}_n\) is as in Definition 4.5. Explicitly, we have

\[
\tilde{\mathcal{H}}_{n,G} = \left\{ \begin{array}{c}
\text{P' \rightarrow P} \\
\downarrow \phi \downarrow \text{deg}(\phi) = n
\end{array} \right\} \sim, \quad \text{P' = } \phi^*(P)
\]

where \(P\) and \(P'\) are principal \(G\)-bundles and \(\phi\) is an \(n\)-fold covering with connected total space. Consider the diagram

\[
\begin{array}{c}
\tilde{\mathcal{H}}_{n,G} \\
\downarrow s \quad \downarrow t
\end{array}
\]

\[
\begin{array}{c}
\mathcal{M}_G \\
\mathcal{M}_G
\end{array}
\]

(11)
where the map \(s\) returns \(P' \to E'\), and \(t\) returns \(P \to E\). Note \(s\) is only defined canonically up to canonical natural isomorphism in \(\mathcal{M}_1\). The map \(t\) is induced by the map of groups \(\text{id} \times p: G \times \Sigma_n \to G\), where \(p\) is the unique map from \(\Sigma_n\) to the trivial group.

Note that this is not the usual Hecke correspondence; like \(\widetilde{\mathcal{M}}_0(A)\), the stack \(\widetilde{\mathcal{H}}_n\) has more automorphisms than \(\mathcal{H}_n\) but induces the correct coarse moduli space. Therefore, in the case that \(G\) is the trivial group, the diagram of maps between coarse moduli spaces induced by (11) is the classical Hecke correspondence.

**Definition 6.1.** We define the trace map \(t!\) as the restriction of \(\text{ind}_{\text{id} \times p}\) to \(\widetilde{\mathcal{H}}_{n,G}\).

**Proposition 2.6** yields an isomorphism \(s^*\mathcal{L}^\alpha \cong t^*(\mathcal{L}^\alpha)^\otimes n\) over \(\widetilde{\mathcal{H}}_{n,G}\).

**Definition 6.2.** The \(n\)th Hecke operator \(T_n: \Gamma(\mathcal{L}^\alpha) \to \Gamma(\mathcal{L}^{n\alpha})\) is defined as the composition \(T_n = t! \circ s^*\).

**Proposition 6.3.** Let \(f\) be a section of \(\mathcal{L}^\alpha\). Then \(T_n(f)\) is the following section of \((\mathcal{L}^\alpha)^\otimes n\):

\[
T_n(f)(g, h; \tau) = \frac{1}{n} \sum_{0 \leq b < d} (\phi_{a,b,d})_* f \left( \frac{g^d, g^{-b}h^a; \frac{a\tau + b}{d}}{d} \right),
\]

where \(\phi_{a,b,d}: \mathbb{C}/\langle \tau', 1 \rangle \to \mathbb{C}/\langle \tau, 1 \rangle\) sends \(\tau' = (a\tau + b)/d\) to \(a\tau + b\) and \(1\) to \(d\), and \((\phi_{a,b,d})_*: \mathcal{L}_{g^d, g^{-b}h^a}^\alpha \to (\mathcal{L}_{g,h}^\alpha)^\otimes n\) is the induced isomorphism of line bundles of Proposition 2.6.

**Proof:** Let \(P\) be a principal \(G\)-bundle over \(E \cong \mathbb{C}/\langle \tau, 1 \rangle\), let \(E' \cong \mathbb{C}/\langle \tau', 1 \rangle\), where \(\tau' = \frac{a\tau+b}{d}\) and let \(\phi = \phi_{a,b,d}\). Then \(\phi = B\varphi\), where \(\varphi: \mathbb{Z}^2 \to \mathbb{Z}^2\)

\[
(1, 0) \mapsto (d, 0) \\
(0, 1) \mapsto (-b, a).
\]

Let \(P\) be the principal bundle over \(E\) classified by the pair \((g, h)\). Then the principal bundle \(\Phi^*P\) over \(E'\) is classified by the pair \((g^d, g^{-b}h^a)\). \(\square\)

**Example 6.4.** On the untwisted sector \(g = 1\), the formula of Proposition 6.3 specializes to the formula for the twisted Hecke operators of classical Moonshine in (3).
Example 6.5. Let $G = \mathbb{Z}/l\mathbb{Z}$. Then
\[(e, f) \in (\mathbb{Z}/l\mathbb{Z})^2\]
corresponds to the point $\frac{e\tau + f}{l}$ of $E[l]$. Let $\omega_1$, $\omega_2$ and $\phi$ be as above. We have
\[
\frac{e\tau + f}{l} = de\frac{\omega_1}{l} - be\frac{\omega_2}{l} + af\frac{\omega_2}{l}.
\]
Hence the image of $\frac{e\tau + f}{l}$ under $\phi$ is the point of $E'[l]$ corresponding to $(de, af - be)$.

The action of the twisted Hecke operators on the $q$-expansions of the untwisted sectors is computed as follows:
\[
\hat{T}_n f(\tau) = \frac{1}{n} \sum_{ad-n \atop 0 \leq b < d} f^{(a)} \left( \frac{a\tau + b}{d} \right)
\]
\[
= \frac{1}{n} \sum_{ad=n \atop 0 \leq b < d} c^{(a)}(m) q^{\frac{am}{d}} \zeta_d^{bm}
\]
\[
= \frac{1}{n} \sum_{ad=n \atop m \in \mathbb{Z}} d \cdot c^{(a)}(m) q^{\frac{am}{d}}
\]
\[
= \sum_{m \in \mathbb{Z}} \sum_{a \atop (m,n)} \frac{1}{a} \cdot c^{(a)} \left( \frac{nm}{a^2} \right) q^m.
\]

6.2. Power operations in K-theory. In [Ati66], Atiyah developed the theory of cohomology operations in equivariant $K$-theory (also known as equivariant power operations). In this section, we briefly recall the special cases of Atiyah’s definitions that will serve as a model for our definitions in $E^{\alpha}_G(pt)$. Recall that the coefficient ring of equivariant $K$-theory is the representation ring $K_G(pt) = R(G)$. Atiyah’s definitions, applied to representations, are given as follows:
\[
P_n: R(G) \rightarrow R(G \times \Sigma_n)
\]
\[
[V] \mapsto [V^{\otimes n}],
\]
is called the $n^{th}$ power operation$^4$. For finite groups $G$ and $H$, and $h \in H$, Atiyah and Segal defined the map
\[
\text{Tr}(h|\cdot\cdot): R(G \times H) \rightarrow R(G) \otimes \mathbb{Z}[\zeta]
\]
\[
W \mapsto \sum W_{\eta} \otimes \eta.
\]
Here the sum runs over all eigenvalues $\eta$ of the action of $h$ on $W$, the space $W_{\eta}$ is the eigenspace corresponding to $\eta$, and $\zeta$ is a primitive $|h|^{th}$ root of unity.

$^4$Atiyah’s power operations actually take values in $R(G \wr \Sigma_n)$ rather than $R(G \times \Sigma_n)$, but for our purposes, it is enough to pull them back via the diagonal map of $G^n$.  


The $n$th symmetric power of $V$ is given by
\[
\text{sym}_n(V) = (P_n(V))^{\Sigma_n}
= \frac{1}{n!} \sum_{\sigma \in \Sigma_n} \text{Tr}(\sigma | P_n(V))
= \sum_{[\sigma] \in \Sigma_n} \frac{1}{|C_\sigma|} \cdot \text{Tr}(\sigma | P_n(V)).
\]

Here $(-)^{\Sigma_n}$ picks out the $\Sigma_n$-invariant part of a $G \times \Sigma_n$-representation. All equalities are in $R(G)$, and the second equality can be proved using the splitting principle. Similarly, the $n$th exterior power of $V$ is given by picking out the summand on which $\Sigma_n$ acts by the sign representation:
\[
\lambda_n(V) = \text{Hom}_{G \times \Sigma_n}([CG] \otimes \text{sgn}, P_n(V)).
= \frac{1}{n!} \sum_{\sigma \in \Sigma_n} \text{sgn}(\sigma) \cdot \text{Tr}(\sigma | P_n(V)).
\]

To define the $n$th Adams operation, let $c_n \in \Sigma_n$ be a long cycle, and set
\[
\psi_n(V) = \text{Tr}(c_n | P_n(V)).
\]

Again, one can use the splitting principle to compare this definition with others in the literature. The $\lambda_i$ make $R(G)$ into a $\Lambda$-ring. Writing $\Lambda_t = \sum \lambda_n t^n$ for the total exterior power, and similarly $S_t$ for the total symmetric power, one has the well known equations
\[
(12) \quad \Lambda_{-t}(x) = \exp \left( - \sum \frac{\psi_n(x)}{n} t^n \right) \quad \text{and} \quad S_t(x) = \exp \left( \sum \frac{\psi_n(x)}{n} t^n \right).
\]

Here $t$ is a dummy variable. In [Gan06, 9], I explained that, in the context of Atiyah’s definitions, the equations (12) are generating functions, which encode the cycle decomposition of the elements of the symmetric groups. I also explained that for pairs of commuting elements of the symmetric group, there is an analogous argument using the decomposition of $\mathfrak{n}$ into $(\sigma, \rho)$-orbits, i.e., into transitive $Z$-sets.

To make the analogy with our setup even more explicit, we will view class functions on $G$ as functions on $G/G$, where $G$ acts on itself by conjugation, and note that $G/G$ is the coarse moduli space of principal $G$-bundles on a circle. In this setup, the $n$th Adams operation is given by pull-back along the degree $n$ self-map of the circle. The group of deck transformations of such a map has order $n$.

The $n$th symmetric power is the weighted sum of pull-backs along all isomorphism classes of $n$-fold coverings of $S^1$. The weights are one over the order of the automorphism groups.

From this point of view, the generating function equation for the total symmetric power $S_t$ in (12) encodes the fact that every covering of the circle decomposes into the disjoint union of covers with connected total space. More precisely, the analogue of Proposition 4.3 for covers of the circle is
\[
\prod_{k \geq 0} \text{Cov}_k(S^1) t^k = \prod_{m \geq 0} \left( \prod_{n \geq 1} \text{Cov}_{\text{conn}}^n(S^1) t^n \right) t^{\Sigma_m}.
\]
Here Cov_k(S^1) stands for the groupoid of k-fold covers of S^1 and Cov^conn_k(S^1) for the groupoid of those with connected total space. The second equation in (12) is the weighted sum of the points of both sides of this decomposition.

6.3. **Symmetric and exterior powers on** Ell^G_{\alpha}. In [Gan06], I explained how Atiyah’s theory can be carried over to Ell^G_{\alpha}. The idea is simple: wherever Atiyah uses elements of the group, one replaces them by pairs of commuting elements. Then the Hecke operators take the role of the Adams operations. In this section, I will define the analogues of the symmetric and exterior powers. We will denote them by sym^{(2)} and \lambda^{(2)}_n. Depending on the taste of the reader, the 2 can stand for “pairs of commuting elements”, for the fact that the torus is made from two circles, or for “chromatic level 2”.

In order to define symmetric and exterior powers on Ell^G_{\alpha}, we need to extend the “Hecke correspondence” (11) to all of \mathcal{M}_{G \times \Sigma_n}. We set
\[ t := \mathcal{M}_{\text{id} \times p}: \mathcal{M}_{G \times \Sigma_n} \to \mathcal{M}_G. \]

In order to define s, we recall the equivalence e of Proposition 4.3. For an abelian group A, let
\[ s_A: \mathcal{M}_G \times \mathcal{M}_{A} \to \mathcal{M}_G \]
be the source map. We define s to be e composed with the map which on the component indexed by the decomposition \( \sum N_A|A| = n \) is given by
\[ \prod_A s_A | \Sigma_{N_A}. \]

Over this component, we have
\[ s^* \left( \prod_A L^\alpha | \Sigma_{N_A} \right) \cong t^*(L^\alpha)_{\otimes n}. \]

**Definition 6.6.** We define the n^{th} symmetric power on sections f of L^\alpha by
\[ \text{sym}^{(2)}_n(f) := t_! \circ s^* \left( \prod_{n=\sum|A|} N_A \right) \prod_A f | \Sigma_{N_A}. \]

Here sym^{(2)}_n(f) is a section of (L^\alpha)_{\otimes n}. We define the total symmetric power Sym^{(2)}_n(f) by
\[ \text{Sym}^{(2)}_q(f) := \sum_{n \geq 0} \text{sym}^{(2)}_n(f) \cdot q^n. \]

Here q is a dummy variable.

From now on, t will again denote our dummy variable (as opposed to the target map). The total symmetric power Sym^{(2)}_t takes values in the graded ring
\[ \bigoplus_{n \geq 0} \text{Ell}^n_{G,\alpha}(pt)t^n. \]

In the case that \( \alpha = 0 \), this is the ring of formal power series in t. The series Sym^{(2)}_t(f) has a multiplicative inverse in \( \bigoplus_{n \geq 0} \text{Ell}^n_{G,\alpha}(pt)t^n \), which we will refer to as the total (alternating)
exterior power, denoted
\[ \Lambda^{(2)}(f) = \sum_{n \geq 0} \lambda^{(2)}_n(f)(-t)^n. \]

The \( \lambda^{(2)}_n \) make \( \bigoplus \text{Ell}_G^{-n}(\text{pt}) \) a \( \Lambda \)-ring.

**Definition 6.7.** Let \((\sigma, \rho)\) be a commuting pair in \( \Sigma_n \), and assume that \((\sigma, \rho)\) induces the decomposition \( n = \sum |A|N_A \). Let \( s_{(\sigma,\rho)} \) denote the restriction of \( s \) to the component of \( \mathcal{M}_{\Sigma_n} \) corresponding to this decomposition. We define the power operation \( \psi_{(\sigma,\rho)} \) by
\[ \psi_{(\sigma,\rho)}(f) := s^*_n(\sigma,\rho)(f). \]

**Definition 6.8.** Let \((\sigma, \rho)\) be a pair of commuting elements in \( \Sigma_n \). We define the signature \( \text{sgn}(\sigma, \rho) \) of the pair by
\[ \text{sgn}(\sigma, \rho) := (-1)^\# \text{even orbits of } (\sigma, \rho). \]

We have
\[ \text{sym}^{(2)}_n(f) = \frac{1}{n!} \sum_{\sigma \rho = \rho \sigma} \psi_{(\sigma,\rho)}(f) \]
and
\[ \lambda^{(2)}_n(f) = \frac{1}{n!} \sum_{\sigma \rho = \rho \sigma} \text{sgn}(\sigma, \rho) \cdot \psi_{(\sigma,\rho)}(f). \]

The proof of [Gan06, Prop.9.1] goes through to give the generating function equations
\[ \text{Sym}^{(2)}_t(f) = \exp \left[ \sum_{k \geq 0} T_k(f) \cdot t^k \right] \]
and
\[ \Lambda^{(2)}_t(f) = \exp \left[ -\sum_{k \geq 0} T_k(f) \cdot t^k \right]. \]

**Remark 6.9.** Another proof for the first equation and a detailed discussion of the generating function argument in terms of covers over elliptic curves can be found in [Rot]. In the case that \( G \) is the trivial group and \( f = 1 \), the statement is a special case of Roth’s Lemma 2.9. His discussion is more general: it also applies to (simply branched) coverings of elliptic curves by curves of higher genus.

6.4. **Replicability.** Using these definitions, we can rephrase Definition 1.2 as follows (compare [Bor92, (8.2)]):

**Definition 6.10.** A McKay-Thompson series \( f(q) \) is called **reproducible**, if (over the untwisted sector) it satisfies
\[ (f(t) - f(q)) = t^{-1} \Lambda^{(2)}_t(f(q)). \]

A few words of explanation are in order. The left-hand side of (13) equals
\[ t^{-1} - f(q) + \sum_{n \geq 1} a_n t^n, \]
while the right-hand side is
\[ t^{-1} - f(q) + \sum_{n \geq 1} \lambda_{n+1}^{(2)}(-t)^n. \]

Hence replicability means that for every \( n \geq 1 \), the function \( \lambda_{n+1}^{(2)}(f) \) is a constant, namely \((-1)^n + 1\) times the \( n \)th coefficient of the \( q \)-expansion of \( f \).

7. **The Witten genus**

Let \( X \) be a compact, differentiable, spin manifold of dimension \( 2d \), let \( p_1 \) denote the first Pontrjagin class, and assume that \( \frac{p_1}{2}(X) = 0 \). Let \( T_C \) denote the complexification of the tangent bundle of \( X \), and set \( T_C := T_C - \mathbb{C}^{2k} \). Then the Witten genus of \( X \) is defined as

\[
\Phi_W(X) = \hat{A}\left(X, \bigotimes_{k=1}^{\infty} \text{Sym}_q(T_C)\right).
\]

Here \( \text{Sym}_q = \text{Sym}_q^{(1)} \) stands for the total symmetric power in \( K \)-theory, and \( \hat{A} \) is the \( \hat{A} \)-genus. If the dimension of \( X \) is a multiple of \( 24 \), the condition \( \frac{p_1}{2} = 0 \) (often called “the vanishing of the anomaly”) implies that

\[
\Phi_W(X) = \frac{q^{-2d}}{\Delta^{2d}} \cdot \hat{A}\left(X, \bigotimes_{k=1}^{\infty} \text{Sym}_q(T_C)\right)
\]

becomes a modular function. The Witten genus is linked to Moonshine by Hirzebruch’s prize question [HBJ92, p.86], which was answered affirmatively by Hopkins and Mahowald [MH02]:

**Is there a 24-dimensional manifold \( X \) as above such that \( \hat{A}(X) = 1 \) and \( \hat{A}(X, T_C) = 0 \)?**

For such an \( X \), the expression (14) becomes \( j - 744 \). Hirzebruch explains that if one could find an action of a finite group \( G \) on \( X \) by diffeomorphisms, this action would lift to the tangent bundle and its symmetric powers and make (14) a McKay-Thompson series. In the case that \( G \) is the monster, one could hope for this series to be that of classical Moonshine. In fact, assume that the action of \( G \) lifts to the spin structure and that \( \left(\frac{p_1}{2}(X)\right)_G = 0 \). Then (14) is the \( g = 1 \) part of a generalized McKay Thompson series, \( \Phi_{WG}(X) \), called the equivariant Witten genus of \( X \) [Gan]. Note that the equivariant genus in [Gan] differs from the one usually found in the literature by a normalization factor, so that my \( \Phi_W \) there is actually given by the right-hand side of (14).

However, the setup of [Gan] describes only the \( \alpha = 0 \) case. Also, the condition that the equivariant class

\[
\left(\frac{p_1}{2}\right)_G \in H^4(X \times_G EG; \mathbb{Z})
\]

should be zero turns out to be very restrictive. The following conjecture was explained to me by Matthew Ando, as was much of the next section:

**Conjecture 7.1.** Let \( X \) be a compact, differentiable, \( G \)-equivariant spin manifold of dimension \( 24d \), and assume that \( \frac{p_1}{2}(X) = 0 \) and

\[
\left(\frac{p_1}{2}\right)_G (X) = \pi^*(\alpha),
\]

then...
where \( \pi \) is the map from \( X \) to a point, and \( \alpha \) is an element of \( H^4(BG;\mathbb{Z}) \). Then the equivariant Witten genus of \( X \) is a generalized Thompson series whose transformation behavior under \( SL_2(\mathbb{Z}) \) makes it a section of \( \mathcal{L}^{-\alpha} \).

The idea to interpret non-trivial anomaly as giving a section of a line bundle goes back to Witten and can be found in the work of Freed and Witten [FW99], Liu [Liu95], and Ando [And03]. The next section outlines how this conjecture could follow from the conjectural behavior of the Thom-isomorphism in [GKV].

7.1. Twisted Thom isomorphisms. The Witten genus is closely related to the theory of Thom isomorphisms and transfer maps in elliptic cohomology. Assume that \( V - TX \) is a spin bundle and that we are given a lift of the action of \( G \) to the spin structure. Let \( X^{V-TX} \) denote the equivariant Thom spectrum of the virtual vector bundle \( V - TX \) [LMSM86]. Then \( \text{Ell}^0_G(X^{V-TX}) \) is an invertible \( \text{Ell}^0_G \) module sheaf. Pulling back first along the relative zero section \( X - TX \to X^{V-TX} \) and then along the Pontrjagin-Thom collapse \( S^0 \to X^{-TX} \), we obtain a map

\[
\text{Ell}^0_G(X^{V-TX}) \to \text{Ell}^0_G(X^{-TX}) \to \text{Ell}^0_G(S^0) = \mathcal{O}_{M_G}.
\]

If the equivariant characteristic class

\[
\left( \frac{p_1}{2} \right)_G (V - TX) \in H^4(EG \times_G X;\mathbb{Z})
\]

equals zero, the Thom isomorphism is a trivialization

\[
\text{Ell}^0_G(X) \xrightarrow{\cong} \text{Ell}^0_G(X^{V-TX}).
\]

Composing it with the map above, gives the transfer- (or Gysin-) map along \( \pi: X_+ \to S^0 \),

\[
\pi^Y: \text{Ell}^0_G(X) \to \text{Ell}^0_G(S^0),
\]

and the Witten genus of \( X \) twisted with \( V \) is

\[
\phi_{W,G}(V) = \pi^Y(1).
\]

If \( V \) is trivial, we drop it from the notation and have

\[
\phi_{W,G}(X) = \pi(1).
\]

Assume now that the weaker condition

\[
\left( \frac{p_1}{2} \right)_G (V - TX) = \pi^*(\alpha)
\]

is satisfied, where

\[
\alpha \in H^4(BG;\mathbb{Z}),
\]

and assume that, non-equivariantly, we have

\[
\left( \frac{p_1}{2} \right) (V - TX) = 0 \in H^4(X;\mathbb{Z}).
\]

Then we get a twisted Thom isomorphism

\[
\text{Ell}^0_G(X) \otimes_{\mathcal{O}_{M_G}} \mathcal{L}^\alpha \xrightarrow{\cong} \text{Ell}^0_G(X^{V-TX}),
\]

yielding a transfer map

\[
\pi^Y: \text{Ell}^0_G(X) \to (\mathcal{L}^\alpha)^{-1}.
\]

Hence the Witten genus \( \phi_{W,G}(V) \) takes values in the global sections of \( \mathcal{L}^{-\alpha} \).
7.2. **A few words about physics.** Principal bundles over Riemann surfaces turn up in string theory, when the target space of the theory is an orbifold quotient $X/G$. In string theory with target space $X$, one considers spaces of maps from the circle (a closed string) or from a Riemann surface $\Sigma$ (the worldsheet) to $X$. An orbifold map from $\Sigma$ to $X//G$ turns out to be the same as an equivalence class of $G$-equivariant maps from a principal $G$-bundle over $\Sigma$ to $X$:

$$\text{map}_{\text{orb}}(\Sigma, X//G) = \bigsqcup_{\text{pbdl}} \text{map}_G(P, X)/\sim$$

(cf., e.g., [Sha], [Moe02], [Tam], [LUX]). The space

$$\mathcal{L}(X//G) = \text{map}_{\text{orb}}(S^1, X//G)$$

is called the orbifold loop space of $X//G$, and the space

$$\mathcal{L}^2(X//G) = \text{map}_{\text{orb}}(\mathbb{T}^2, X//G)$$

is the double (orbifold) loop space of $X//G$. They decompose into components

$$\mathcal{L}_g(X//G) := \text{map}_G(P_g, X) \quad \text{and} \quad \mathcal{L}^2_{g,h}(X//G) = \text{map}_G(P_{g,h}, X),$$

often referred to as the twisted sectors and the membrane twisted sectors of the theory.

In the non-equivariant case, the Witten genus $\Phi_W(X)$ is the genus 1 partition function of the supersymmetric non-linear sigma model for the target space $X$ [ST04].

In [dFLNU06], de Fernex, Lupercio, Nevins and Uribe consider a supersymmetric string sigma model whose target space is an orbifold and show that its partition function on a two-dimensional torus is the orbifold Witten genus. The orbifold Witten genus is related to the equivariant Witten genus by

$$\Phi_{\text{W,orb}}(X//G) = \text{ind}_{1/G} \left( \frac{1}{|G|} \sum_{g h = h g} \Phi_W(X)(g, h) \right) = \text{ind}_{1/G} \left( \Phi_{W,G}(X) \right).$$

In [dFLNU06], the $(g, h)$-summand of $\Phi_{\text{W,orb}}$ is the contribution of the corresponding twisted sector to the partition sum, which is a phase integral over the torus with boundary conditions twisted by $g$ and $h$. Hence the discussion of [dFLNU06] fits the equivariant Witten genus (the same thing as in [Gan]) into the framework of non-linear sigma models.

The idea to study maps into an orbifold target space via principal bundles on the worldsheet already turned up in Segal’s celebrated work [Seg04]. In the same paper, Segal prominently announced the existence of a “fairly simple and natural conformal field theory”, whose automorphism group is the Monster. In [FLM85], Frenkel, Lepowski and Meurman constructed a vertex algebra whose automorphism group is the Monster, the so-called Moonshine module. Vertex algebras are often thought of as the mathematical language of two-dimensional conformal field theory. In his famous paper [Bor92], Borcherds proved that the Moonshine module is indeed the McKay-Thompson series of the classical Moonshine conjecture. It is expected and in many cases proved that generalized Moonshine is given by the twisted sectors of the Moonshine module (cf. [DLM00]).

In [DGH88], Dixon, Ginsparg, and Harvey translate the result of [FLM85] into the language of string theory. According to them, the target space of the Monster CFT is the
orbifold $\mathbb{T}_{\text{Leech}}^{24} / (\mathbb{Z}/2\mathbb{Z})$.

Here $\mathbb{T}_{\text{Leech}}^{24} = \mathbb{R}^{24}/\Lambda_{\text{Leech}}$, where $\Lambda_{\text{Leech}}$ is the Leech lattice, and the $\mathbb{Z}/2\mathbb{Z}$ action is induced by multiplication with $-1$ on $\mathbb{R}^{24}$. Its orbifold partition function is $j - 744$. The automorphism group of the theory is the Monster. For a pair of commuting elements $(g, h)$ in the Monster, [DGH88] describes the generalized Moonshine function $f(g, h; \tau)$ as the phase integral over the torus with boundary conditions twisted by $g$ and $h$.

Given these facts, one could hope to reinterpret Hirzebruch’s question and ask whether the orbifold Witten genus of $\mathbb{T}_{\text{Leech}}^{24} / (\mathbb{Z}/2\mathbb{Z})$ was given by $j - 744$. After all, the Witten genus is the partition function of the theory, and the partition function is $j - 744$. Unfortunately, the two theories are different, and in particular, their notions of (orbifold) partition function do not agree. For instance, as a consequence of the rigidity theorem, the contribution of the non-twisted sector of the Witten genus, $\Phi_W(\mathbb{T}_{\text{Leech}}^{24})$, is zero, whereas the analogous contribution to the partition function of the Moonshine module is $j - 720$.

When Witten defines the Witten genus in [Wit87], he explains that different non-linear sigma models lead to different genera.

**Question 7.2.** Is it possible to describe the partition function of the “holomorphic conformal field theory” in [DGH88] as a push-forward construction in equivariant elliptic cohomology, similar to the description of the Witten genus in [Gan], and can we identify the corresponding genus? How does the discussion in [dFLNU06] need to be modified to supply the geometric side of this construction (in terms of orbifold loop spaces)? Ideally, one would want this genus to preserve power operations. In this case, the theory would have a Dijkgraaf-Moore-Verlinde-Verlinde formula.

**7.3. The Witten genus and replicability?** In [Gan], I calculated the total symmetric power in Devoto’s equivariant Tate K-theory applied to a $G$-equivariant vector bundle $V$ and observed that it equals Witten’s exponential characteristic class:

$$\text{Sym}_q^{(2)}(V) = \bigotimes_{k=1}^{\infty} \text{Sym}_q^{(1)}(V).$$

Recall that we have

$$\text{Sym}_q^{(2)}(V) = \Lambda_q^{(2)}(-V).$$

**Question 7.3.** Can one describe replicability (13) in the context of Hirzebruch’s prize question as asking for an equality about iterated total exterior powers at chromatic level 2?

**Question 7.4.** Is it possible to make sense of the notion of replicability for generalized McKay Thompson series? In this case, the coefficient $\lambda_{n+1}^{(2)}(f(q))$ of the right-hand side of (13), is a section of the line bundle $(L^\alpha)^{\otimes n+1}$. What does it mean for such a thing to be a constant? And how should we take the fractional powers of $t$ occurring over the twisted sectors on the left-hand side into account?

Note that these two issues are related: The fractional powers in the $q$-expansions of the twisted sectors come from the fact that $T$ does not act trivially on the lines $L_{g,h}$. Let $L_N$ be the line of invariant sections of the functor $F$ on $\mu_N \rtimes \mu_N$ which is given by $F(\zeta) = \mathbb{C}$, on objects and such that

$$F(\zeta, \xi): F(\zeta) \to F(\zeta \xi)$$
is multiplication with $\xi$. Then there is a $\mathbb{Z}/\mathbb{N}$-equivariant ring isomorphism
\[
\bigoplus_{m \geq -1} L^\otimes_m q^m \rightarrow \mathbb{C}[q^{\frac{1}{N}}]
\]
\[a_m q^m \mapsto a_m (e^{\frac{2\pi i}{N}}) q^m.
\]
Here $1 \in \mathbb{Z}/\mathbb{N}$ acts on $q^{\frac{1}{N}}$ by multiplication with $e^{\frac{2\pi i}{N}}$.

**APPENDIX A. GENERALITIES ON PRINCIPAL BUNDLES**

This appendix collects some well known facts about principal bundles used in the paper. Let $G$ be a finite group.

**Definition A.1.** A principal $G$-bundle $\xi$ consists of a map $\pi: P \rightarrow X$, where $P$ is a right $G$-space such that $G$ acts strictly transitively on the fibers of $\pi$. The space $P$ is called the total space and $X$ the base space of $\xi$.

For simplicity, we will always assume that the base space $X$ is connected.

**Definition A.2.** Let $i: G \hookrightarrow H$ be an inclusion of groups, and let $\xi$ be a principal $G$-bundle. The principal $H$-bundle $\xi[H]$ associated to $\xi$ via $i$ has total space
\[P \times_G H := (P \times H)/\sim,
\]
where $(p, h) \sim (pg, i(g^{-1})h)$. The action of $H$ on $P \times_G H$ is defined by
\[(p, h) \cdot h' := (p, hh').
\]
The projection $\pi[H]$ is defined by $\pi[H](p, h) := \pi(p)$.

If $f: X \rightarrow BG$ classifies the $G$-bundle $\xi$, the associated $H$-bundle $\xi[H]$ is classified by the map $Bi \circ f$.

**Definition A.3.** Let $\xi = (\pi: P \rightarrow X)$ be a principal $G$-bundle with connected total space, and let $\alpha: H \rightarrow G$ be an isomorphism of groups. We define the pull-back of $\xi$ along $\alpha$ to be the principal $H$-bundle $\alpha^*(\xi)$ consisting of the same map $\pi$, where the $H$ action on the total space $P$ is given by
\[p \ast h := p \cdot \alpha(h).
\]
In the case $H = G$, we will refer to this pulled back action as “the $G$-action on $P$ twisted by $\alpha$”.

**Lemma A.4.** The bundle $\alpha^*(\xi)$ is isomorphic to the $G$-bundle associated to $\xi$ via the inclusion $\alpha^{-1}: G \rightarrow G$.

**Proof:** Consider the map
\[f: P \rightarrow P \times_\alpha\negthinspace^{-1} G
\]
\[p \mapsto (p, 1).
\]
It is obviously fiber preserving and bijective. We have
\[f(p) \cdot g = (p, 1) \cdot g = (p, g) \sim (p\alpha(g), \alpha^{-1}(\alpha(g)) \cdot 1) = (p\alpha(g), 1) = f(p\alpha(g)) = f(p \ast g).
\]
Therefore $f$ is $G$-equivariant with respect to the $G$-action on $P$ twisted by $\alpha$. \qed
Corollary A.5. Let $\xi$ be classified by the map $f: X \to BG$. Then $\alpha^*(\xi)$ is classified by $B\alpha^{-1}f$.

Lemma A.6. Let $G \subseteq H$ be an inclusion of groups, let $\xi$ be a principal $H$-bundle, and let $F$ be a left $G$-space. Then
$$\xi \times_H F = \xi[G] \times_G F.$$ 

Proposition A.7. Let $\xi = (\pi: P \to X)$ be a principal $G$-bundle over $X$, and $\zeta = (\rho: Q \to X)$ be a principal $H$-bundle over $X$, and assume that both total spaces $P$ and $Q$ are connected. Assume further that there exists an isomorphism of covering spaces of $X$
$$f: P \to Q.$$ 

Then there exists an isomorphism of groups
$$i: G \to H$$

making $f$ into an isomorphism of principal $G$-bundles
$$f: \xi \xrightarrow{\cong} i^*\zeta.$$ 

Moreover, $i$ is uniquely determined by $f$.

Proof: Fix an element $p \in P$. We define the map $i: G \to H$ as follows: for $g \in G$ let $i(g)$ be the unique element of $H$ mapping $f(p)$ to $f(pg)$. It is clear that $i$ is a bijection. We need to show that $i$ is a map of groups. Fix $g \in G$, and consider the deck transformations $D_g$ of $\pi$ sending $p$ to $pg$ and $D_{i(g)}$ of $\rho$ sending $q$ to $q \cdot i(g)$. Then $D_{i(g)}$ and $f \circ D_g \circ f^{-1}$ both are deck transformations of $Q$ sending $f(p)$ to $f(pg)$. We may conclude that they are the same deck transformation. Now we have
$$f(p) \cdot i(hg) = f(p(hg)) = f((f^{-1}(f(p(h))) \cdot g) = f(ph) \cdot i(g) = (f(p) \cdot i(h)) \cdot i(g).$$ 

Further, for every $g \in G$, we have
$$f \circ D_g = D_{i(g)} \circ f,$$
proving that $f$ is an isomorphism of principal $G$-bundles. Let now $\beta: G \to H$ be an isomorphism making $f$ into an isomorphism between the principal $G$-bundles $\xi$ and $\beta^*(\zeta)$. Then we have for every $g \in G$,
$$f(pg) = f(p)\beta(g),$$ 

implying that $\beta(g) = i(g)$.  

□
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