A West Nile virus nonlocal model with free boundaries and seasonal succession

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Abstract

The paper deals with a West Nile virus (WNv) model, in which the nonlocal diffusion characterizes the long-range movement of birds and mosquitoes, the free boundaries describe their spreading fronts, and the seasonal succession accounts for the effect of the warm and cold seasons. The well-posedness of the mathematical model is established, and its long-term dynamical behaviours, which depend upon the generalized eigenvalues of the corresponding linearized differential operator, are investigated. For both spatially independent and nonlocal WNv models with seasonal successions, the generalized eigenvalues are studied and applied to determine whether the spreading or vanishing occurs. Our results extend those for the case with nonlocal diffusion but no free boundary and those for the case with free boundary but local diffusion, respectively. The generalized eigenvalues reveal that there exists positive correlation between the duration of the warm season and the risk of infection. Moreover, the initial infection length, the initial infection scale and the spreading ability to new areas all play important roles for the long time behaviors of the time dependent solutions.

Keywords West Nile virus · Nonlocal diffusion · Free boundary · Seasonal succession

Mathematics Subject Classification 35K55 · 35K57 · 35R35 · 92D30

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1 Introduction

West Nile virus (WNv) is an emerging mosquito-borne virus that can cause a severe, life-threatening neurological disease in humans and horses, and it is widely distributed throughout the world with considerable impact on both public health and animal health (Beck et al. 2013).

For nearly two decades, many mathematical models for WNv have been proposed and studied. However, most models are focused on the non-spatial transmission dynamics (Bowman et al. 2005; Wonham et al. 2004; Wan and Zhu 2010). In fact, what we should actually do is to consider the spatial spreading, which is an important aspect of the persistence and eradication of WNv. To utilize the cooperative nature of the cross-infection dynamics and estimate the spatial spread speed of the infection, Lewis et al. (2006) proposed the following spatial-dependent WNv model:

\[
\begin{align*}
    u_{1t} &= d_1 \Delta u_1 + \alpha_b \beta_b \frac{(N_b - u_1)}{N_b} u_2 - \gamma_b u_1, \quad t > 0, \quad x \in \Omega, \\
    u_{2t} &= d_2 \Delta u_2 + \alpha_m \beta_b \frac{(A_m - u_2)}{N_b} u_1 - d_m u_2, \quad t > 0, \quad x \in \Omega,
\end{align*}
\]

(1.1)

where \( u_1(t, x) \) and \( u_2(t, x) \) represent the population densities of infected birds and mosquitoes at the location \( x \) and time \( t \geq 0 \), respectively, and the initial conditions satisfy \( 0 < u_1(0, x) \leq N_b, 0 < u_2(0, x) \leq A_m \). The total population of birds (denoted by \( N_b \)) and mosquitoes (denoted by \( A_m \)) are assumed to be positive constants, \( d_1 \) and \( d_2 \) represent the diffusion coefficients for birds and mosquitoes, respectively. \( \alpha_b \) and \( \alpha_m \) account for the WNv transmission probability per bite to birds and mosquitoes. \( \beta_b \) is the biting rate of mosquitoes on birds, \( \gamma_b \) is the recovery rate of birds from WNv, and \( d_m \) is the death rate of adult mosquitoes.

For the sake of clarity, we denote the parameters in (1.1) by

\[
    a_1 = \frac{\alpha_b \beta_b}{N_b}, \quad e_1 = N_b, \quad b_1 = \gamma_b, \quad a_2 = \frac{\alpha_m \beta_b}{N_b}, \quad e_2 = A_m, \quad b_2 = d_m.
\]

Under these new notations, system (1.1) can be rewritten as

\[
\begin{align*}
    u_{1t} &= d_1 \Delta u_1 + a_1 (e_1 - u_1) u_2 - b_1 u_1, \quad t > 0, \quad x \in \Omega, \\
    u_{2t} &= d_2 \Delta u_2 + a_2 (e_2 - u_2) u_1 - b_2 u_2, \quad t > 0, \quad x \in \Omega.
\end{align*}
\]

(1.2)

For the ODE version of (1.2), the authors in Lewis et al. (2006) derived the basic reproduction number

\[
R_0 = \sqrt{\frac{a_1 a_2 e_1 e_2}{b_1 b_2}}
\]

(1.3)

by the next generation matrix method (van den Driessche and Watmough 2002) and they showed that the virus vanishes for \( R_0 < 1 \), while for \( R_0 > 1 \), the disease-endemic equilibrium stabilizes. They further considered the existence of the traveling waves of (1.2) and proved that the spreading speed is equivalent to the minimal wave speed.
While (1.2) can be applied to estimate the speed of the disease transmission, it cannot be used to understand the spreading front of the infected areas. Recently, Lin and Zhu (2017) investigated the following improved version of (1.2) under our notations, in which the spreading fronts are explicitly described as free boundaries:

\[
\begin{align*}
\begin{cases}
  u_1(t, x) &= d_1 u_{1xx} + a_1(e_1 - u_1)u_2 - b_1 u_1, & t > 0, \quad g(t) < x < h(t), \\
  u_2(t, x) &= d_2 u_{2xx} + a_2(e_2 - u_2)u_1 - b_2 u_2, & t > 0, \quad g(t) < x < h(t), \\
  u_1(t, x) &= u_2(t, x) = 0, & t > 0, \quad x \in [g(t), h(t)], \\
  h(0) &= h_0, \quad h'(t) = -\mu u_{1x}(t, h(t)), & t > 0, \\
  g(0) &= -h_0, \quad g'(t) = -\mu u_{1x}(t, g(t)), & t > 0, \\
  u_1(0, x) &= u_{1,0}(x), \quad u_2(0, x) = u_{2,0}(x), \quad -h_0 \leq x \leq h_0,
\end{cases}
\]

where \( x = h(t) \) and \( x = g(t) \) are the moving boundaries to be determined together with \((u_1, u_2), (g(t), h(t))\) is the infected interval, and the initial conditions satisfy

\[
\begin{align*}
\begin{cases}
  u_{i,0}(x) &\in C^2([-h_0, h_0]), & u_{i,0}(-h_0) = u_{i,0}(h_0) = 0, \\
  0 < u_{i,0}(x) \leq e_i, & x \in (-h_0, h_0), \quad i = 1, 2.
\end{cases}
\end{align*}
\]

It is shown in Lin and Zhu (2017) that problem (1.4) has a unique solution which is defined for all \( t > 0 \), and when \( R_0 \leq 1 \), the virus vanishes eventually, and if \( R_0 > 1 \), the spreading-vanishing dichotomy holds. Subsequently, the asymptotic spreading speed of (1.4) was determined in Wang et al. (2019) when the spreading occurs. Some free boundary problems similar to (1.4) have been extensively studied over the past decade; see Wang (2019, 2021), Liu et al. (2019) and the references therein.

We notice that in both problems (1.2) and (1.4), the spatial movement of the birds and mosquitoes is modelled by the random walk, which is referred as the local diffusion, given by \( d_1 u_{1xx} \) and \( d_2 u_{2xx} \), respectively. In the past decades, the problems with local diffusion have been extensively studied in the literatures. However, the classical Laplace operator can not describe all diffusion processes in nature. Murray pointed out in Murray (1998) that the reaction-diffusion equation in the form of (1.2) or (1.4) can only be used to describe the model with sparse density. But in the embryonic development model, the cell density involved is large, and the Laplace operator can not describe their accurate diffusion process, while the convolution operator

\[
(J * u - u)(t, x) := \int_{\mathbb{R}} J(x - y)u(t, y)dy - u(t, x) \quad (1.5)
\]

can overcome this issue. We call (1.5) a nonlocal diffusion operator.

In recent years, nonlocal diffusion models have been widely used in many fields, such as material science (Bates 2006), image processing (Gilboa and Osher 2007), particle systems (Bodnar and Velazquez 2006) and solidification models (Fournier and Laurencot 2006), etc. It has been increasingly recognized that nonlocal diffusion as a long-range process can better model some natural phenomena in spatial ecology. Cao et al. (2019) introduced the free boundary model with nonlocal diffusion, which is a natural extension of a free boundary model with local diffusion in Du and Lin (2010), and they showed that for the spreading-vanishing criteria, the nonlocal diffusion model...
has quite different characteristics in comparison to the corresponding local diffusion model. Subsequently, the spreading speed problem in Cao et al. (2019), when the expansion occurs, was settled by Du et al. (2021). Inspired by the single species free boundary model with nonlocal diffusion in Cao et al. (2019), two species nonlocal diffusion systems with free boundaries have recently been investigated in Du et al. (2022), Wang and Wang (2020a, 2020b) for competition and predator–prey models, and also in Zhao et al. (2020a, 2020b) for epidemic models.

Recently, to better understand the dispersal (especially long-range dispersal) of birds and mosquitoes, Du and Ni (2020) discussed the nonlocal version of (1.4) as follows:

\[
\begin{align*}
  u_{1t} &= d_1 L_1[u_1] + a_1(e - u_1)u_2 - b_1 u_1, & t > 0, & g(t) < x < h(t), \\
  u_{2t} &= d_2 L_2[u_2] + a_2(e_2 - u_2)u_1 - b_2 u_2, & t > 0, & g(t) < x < h(t), \\
  u_1(t, x) &= u_2(t, x) = 0, & t > 0, & x \in \{g(t), h(t)\}, \\
  h'(t) &= \mu \int_{g(t)}^{h(t)} J_1(x - y)u_1(t, x)dydx, & t > 0, \\
  g'(t) &= -\mu \int_{g(t)}^{h(t)} J_1(x - y)u_1(t, x)dydx, & t > 0, \\
  u_i(0, x) &= u_{i,0}(x), i = 1, 2, & -h_0 \leq x \leq h_0,
\end{align*}
\]

where

\[
L_i[u] = L_i[u; g, h](t, x) = \int_{g(t)}^{h(t)} J_i(x - y)u(t, y)dy - u(t, x). 
\]

They showed that problem (1.6) is well-posed and the spreading-vanishing dichotomy holds. At the same time, they gave the criteria when the spreading and vanishing can occur. The most marked difference is that the spreading may have infinite speed (or accelerated spreading) in the nonlocal model, while the spreading speed for the local model is finite whenever the spreading occurs (Wang et al. 2019).

On the other hand, as discussed in Hsu and Zhao (2012), the temporal changes of the environment affect the growth and development of species, and the environmental variations due to alternating seasons in nature not only affect the growth of species but also have crucial impact on the composition of communities. For example, in temperate lakes, phytoplankton and zooplankton have a growing season in warm months, after which species die or enter the dormant period in winter. To investigate the effects of seasonal succession on population dynamics, Steiner et al. (2009) and Hu and Tessier (1995) carried out a lot of experiments to collect data on the effect of seasonal succession on phytoplankton competition, and Klausmeier (2010) considered a well-known Rosenzweig–McArthur model to study the effect of seasonal alternation on the dynamic behavior of the model. Later, the complete dynamic behavior of a class of Lotka–Volterra competitive models with seasonal succession were investigated by Hsu and Zhao (2012). Introducing seasonal succession into the diffusive logistic model with a free boundary, Peng and Zhao (2013) studied the following model:
where \(0 < \tau < 1\), and \(\omega, k\) are positive constants. The parameters \(\omega\) and \(\tau\) account for the period of seasonal succession and the duration ratio of the warm season, respectively. They found the criteria of spreading and vanishing, which determine whether the species \(u\) spatially spreads to infinity or vanishes in a finite interval, and they also illustrated the influence of the duration of the warm season and cold season on the dynamical behavior. Furthermore, the spreading speed of the species was derived when the spreading occurs.

Inspired by the works in Cao et al. (2019), Du and Ni (2020) and Peng and Zhao (2013), we consider a West Nile virus nonlocal model with free boundaries and seasonal succession, which reads as follows:

\[
\begin{aligned}
u_t &= -ku, & \quad m\omega < t \leq m\omega + (1-\tau)\omega, & \quad 0 < x < h(t), \\
u_t - d\frac{\partial u}{\partial x} &= u(a - bu), & \quad m\omega + (1-\tau)\omega < t \leq (m+1)\omega, & \quad 0 < x < h(t), \\
u_x(t, 0) = 0, & \quad u(t, h(t)) = 0, & \quad t > 0, \\
h(t) &= h(m\omega), & \quad m\omega < t \leq m\omega + (1-\tau)\omega, \\
h'(t) &= -\mu u_x(t, h(t)), & \quad m\omega + (1-\tau)\omega < t \leq (m+1)\omega, \\
h(0) = h_0, & \quad u(0, x) = u_0(x), & \quad 0 \leq x \leq h_0,
\end{aligned}
\]

(1.8)

where positive constant \(k \geq b_2\) is the mortality rate in the cold season. The initial time \((t = 0)\) is chosen as the starting time of the warm season of the first year \((m = 0)\), the parameter \(\omega\) is the length of one year, \((1-\delta)\omega\) accounts for the length of the warm season and the initial conditions satisfy

\[
\begin{aligned}
u_{i, 0}(x) &\in C([-h_0, h_0]), & \quad & u_{i, 0}(-h_0) = u_{i, 0}(h_0) = 0, \\
0 < u_{i, 0}(x) &\leq e_i, & \quad & x \in (-h_0, h_0), i = 1, 2.
\end{aligned}
\]

(1.10)

The kernel function \(J_i : \mathbb{R} \to \mathbb{R}\) \((i = 1, 2)\) is nonnegative and continuous, and satisfies

\[
(J) : J_i \in C(\mathbb{R}) \cap L^\infty(\mathbb{R}) \text{ is symmetric, } J_i(0) > 0, \int_{\mathbb{R}} J_i(x)dx = 1, \quad i = 1, 2.
\]

Here and in what follows, unless stated otherwise, we always take \(m = 0, 1, 2, \ldots\).
In (1.9), we take a year as the cycle \([m\omega, (m + 1)\omega]\) and divide a year into the warm season and the cold season. From spring to autumn, because of the warm climate and abundant food, birds and mosquitoes have more opportunities to capture food and reproduce, we define this period as the warm season \((m\omega, m\omega + (1 - \delta)\omega)\), and assume that the spatiotemporal distribution and spreading of species are respectively controlled by the first two equations and free boundary conditions in (1.9). Correspondingly, the cold season \((m\omega + (1 - \delta)\omega, (m + 1)\omega)\) is from autumn to spring of the next year. Due to formidable survival conditions (the cold climate, shortage of resources, or species cannot capture enough food to feed up their offsprings), we assume that the number of mosquitoes follows Malthusian growth rule which says that it decays at an exponential rate (Malthus 1998). During this season, the hypothesis \(h(t) = h(\omega, m\omega + (1 - \delta)\omega, = m(\omega) + (1 - \delta)\omega)\) implies biologically that mosquitoes in winter are in the form of eggs or larvae, and they do not migrate in the space interval \([g(m\omega + (1 - \delta)\omega), h(m\omega + (1 - \delta)\omega)]\). Meanwhile, the infected birds can only spread in the above interval. The free boundaries conditions in (1.9)

\[
\begin{align*}
    h'(t) &= \sum_{i=1}^{2} \mu_i \int_{g(t)}^{h(t)} J_i(x - y)u_i(t, x)dydx, \\
g'(t) &= -\sum_{i=1}^{2} \mu_i \int_{g(t)}^{h(t)} J_i(x - y)u_i(t, x)dydx
\end{align*}
\]

imply that the expansion rate of the common population range for both species (mosquitoes and birds) are directly proportional to the outward flux of two species, where the factors \(\mu_1\) and \(\mu_2\) measure the spreading abilities of birds and mosquitoes to new habitats.

For the convenience of discussions, we introduce some notations. Given \(h_0, \omega > 0\), we denote

\[
C^1_\omega((0, \omega]) := C^1((0, (1 - \delta)\omega]) \cap C^1(((1 - \delta)\omega, \omega])],
\]

\[
C^1_\omega(0, +\infty) := \bigcap_{m=0}^{\infty} \left[ C^1((\omega, m\omega + (1 - \delta)\omega)) \cap C^1((m\omega, m\omega + (1 - \delta)\omega)) \right] 
\]

\[
\mathbb{H}_{h_0, \omega} := \{ h \in C([0, +\infty)) \cap C^1((m\omega, m\omega + (1 - \delta)\omega)) : h(0) = h_0, h(t) \text{ is nondecreasing} \},
\]

\[
\mathbb{G}_{h_0, \omega} := \{ g \in C([0, +\infty)) \cap C^1((m\omega, m\omega + (1 - \delta)\omega)) : -g \in \mathbb{H}_{h_0, \omega} \}.
\]

For any given \(g \in \mathbb{G}_{h_0, \omega}, h \in \mathbb{H}_{h_0, \omega}\) and \(U_0 = (u_{1,0}, u_{2,0})\) satisfying (1.10), set

\[
\Omega^{g,h} := \{(t, x) \in \mathbb{R}^2 : t \in (0, +\infty), g(t) < x < h(t)\},
\]

\[
X_\omega = X^{g,h}_{\omega, U_0} := \{ (\xi_1, \xi_2) : \xi_i \in C(\Omega^{g,h}), \xi_i \geq 0, \xi_i(0, x) = u_{i,0}(x) \}
\]

for \(x \in [-h_0, h_0]\), and \(\xi_i(t, g(t)) = \xi_i(t, h(t)) = 0\) for \(t \in [0, +\infty), i = 1, 2\). Our main results for (1.9) are given in the following.

**Theorem 1.1** (Existence and uniqueness) Suppose that (J) and (1.10) hold. Then problem (1.9) admits a unique positive solution \((u_1(t, x), u_2(t, x); g(t), h(t))\) satisfying
(g, h) ∈ G_{h_0, ω} × H_{h_0, ω}, (u_1, u_2) ∈ X_{ω} for any given ω > 0. Furthermore, g(t) is nonincreasing and h(t) is nondecreasing for t > 0, and 0 < u_i ≤ e_i (i = 1, 2) for t > 0, x ∈ (g(t), h(t)).

**Theorem 1.2** (Spreading-vanishing dichotomy) Suppose that assumptions (J) and (1.10) hold. Let (u_1(t, x), u_2(t, x); g(t), h(t)) be the solution to problem (1.9) and denote

\[ g_∞ := \lim_{t \to ∞} g(t) \quad \text{and} \quad h_∞ := \lim_{t \to ∞} h(t). \]

Then one of the following alternatives must happen for (1.9):

(i) **Spreading:** \(-g_∞ = h_∞ = ∞\) and

\[ \lim_{n \to ∞} (u_1(t + nω, x), u_2(t + nω, x)) = (U_1^Δ, U_2^Δ) \]

uniformly for \( t ∈ [0, ω] \) and locally uniformly for \( x ∈ R \), where \((U_1^Δ, U_2^Δ)(t)\) is the unique positive periodic solutions of the following problem

\[
\begin{cases}
U_{1t} = a_1(e_1 - U_1)U_2 - b_1U_1, & t ∈ (0, (1 - δ)ω], \\
U_{2t} = a_2(e_2 - U_2)U_1 - b_2U_2, & t ∈ (0, (1 - δ)ω], \\
U_{1t} = -b_1U_1, & t ∈ ((1 - δ)ω, ω], \\
U_{2t} = -b_2U_2, & t ∈ ((1 - δ)ω, ω], \\
U_i(0) = U_i(ω)(i = 1, 2).
\end{cases}
\]

(ii) **Vanishing:**

\[ \lim_{t \to ∞} (u_1(t, x), u_2(t, x)) = (0, 0) \] uniformly for \( x ∈ [g(t), h(t)] \).

We mention here that \( h_∞ - g_∞ < ∞ \) when vanishing happens in some special cases, see Lemma 4.7.

**Theorem 1.3** (Spreading-vanishing criteria 1) Suppose that (J) and (1.10) hold, \( J_1(x) = J_2(x) \) and \( 0 ≤ δ < 1 \). Fix \( a_i, e_i, b_2 \) and \( k \). Then the following conclusions hold.

(i) If \( b_1 ≥ b^*_1 := a_1a_2e_1e_2/b_2 \), then vanishing happens; while for \( b_1 ∈ (0, b^*_1) \), there exists \( δ^* ∈ (0, 1) \) such that vanishing happens for \( δ ∈ [δ^*, 1] \).

(ii) If \( b_1 ∈ (0, b^*_1) \) and \( δ ∈ (0, δ^*) \), there exists \( h^*_0 > 0 \) such that spreading happens for \( h_0 ≥ h^*_0 \).

(iii) If \( b_1 ∈ (0, b^*_1), δ ∈ (0, δ^*) \) and \( h_0 ∈ (0, h^*_0) \), then for any given initial datum \((u_{1,0}, u_{2,0})\) satisfying (1.10), there exist \( μ^* ≥ μ^*_0 > 0 \) such that vanishing happens for \( 0 < μ_1 + μ_2 ≤ μ^*_0 \) and spreading happens for \( μ_1 + μ_2 > μ^*_0 \).

Theorem 1.3 implies that, spreading-vanishing of virus depends firstly on the recovery rate \( (b_1 \text{ or } γ_b) \) of birds from WNv, with large recovery rate beneficial for the vanishing of the virus. For small recovery rate, large length \((δω)\) of the cold season
leads to the vanishing of the virus. However, even the recovery rate and the length of the cold season are both small, large initial infected interval \([-h_0, h_0]\) or the expanding factors \((\mu_1, \mu_2)\) can still induce the spreading of WNv.

The following result presents some sufficient conditions for the spreading or vanishing of WNv in the case of \(J_1(x) \neq J_2(x)\).

**Theorem 1.4** (Spreading-vanishing criteria 2) Suppose \((J)\) and \((1.10)\) hold. There exist \(0 \leq \mu_\Delta \leq \mu^\Delta \leq +\infty\) such that vanishing happens for \((\mu_1, \mu_2) < (\mu_\Delta, \mu_\Delta)\), and spreading happens for \((\mu_1, \mu_2) > (\mu_\Delta, \mu_\Delta)\). In particular, if \(0 \leq \delta < 1\) and \(\min\{b_1, k\} \delta > c_1(1 - \delta)\), then \(\mu_\Delta = \mu^\Delta = +\infty\), which means that vanishing happens for any \(\mu_1, \mu_2\), where

\[
c_1 = \frac{-(b_1 + b_2) + \sqrt{(b_1 - b_2)^2 + 4a_1a_2e_1e_2}}{2}.
\]

If \(0 \leq \delta < 1\) and \(\max\{b_1, k\} \delta < c_1(1 - \delta)\), then \(\mu_\Delta = \mu^\Delta = 0\), which means that spreading happens for any \(\mu_1, \mu_2\), provided that \(h_0\) is sufficiently large.

The rest of this paper is organized as follows: In Sect. 2, we introduce the associated eigenvalue problem and present the analysis of principal eigenvalue in a bounded domain. A fixed boundary problem with seasonal succession and the corresponding preliminaries are addressed in Sect. 3. These results pave the way for proving the main results in Sect. 4. While some of our ideas are adopted from Du and Ni (2020), Peng and Zhao (2013), considerable variations are needed as our principal eigenvalue problem here does not satisfy some conditions required in Bao and Shen (2017), and it is also different from the eigenvalue problem in Du and Ni (2020). Section 4 is devoted to the proofs of Theorems 1.1, 1.2, 1.3 and 1.4, respectively. Some ecological discussions of our theoretical results are given in the last section.

## 2 An associated eigenvalue problem

To understand the asymptotical behavior of the solution to problem \((1.9)\), we consider the corresponding problem in the fixed interval \([-L_1, L_2]\), and investigate the following time-periodic eigenvalue problem:

\[
\begin{align*}
\phi_t - d_1\mathcal{L}_1[\phi] &= a_1e_1\psi - b_1\phi + \lambda\phi, &0 < t \leq (1 - \delta)\omega, &-L_1 \leq x \leq L_2, \\
\psi_t - d_2\mathcal{L}_2[\psi] &= a_2e_2\phi - b_2\psi + \lambda\psi, &0 < t \leq (1 - \delta)\omega, &-L_1 \leq x \leq L_2, \\
\phi_t - d_1\mathcal{L}_1[\phi] &= -b_1\phi + \lambda\phi, & (1 - \delta)\omega < t \leq \omega, &-L_1 \leq x \leq L_2, \\
\psi_t + k\psi + \lambda\psi, & (1 - \delta)\omega < t \leq \omega, &-L_1 \leq x \leq L_2, \\
\phi(0, x) = \phi(\omega, x), & \psi(0, x) = \psi(\omega, x), &-L_1 \leq x \leq L_2,
\end{align*}
\]

where

\[
\mathcal{L}_i[u] = \mathcal{L}_i[u; -L_1, L_2](t, x) = \int_{-L_1}^{L_2} J_i(x - y)u(t, y)dy - u(t, x), \quad i = 1, 2.
\]
In order to get some properties of time-periodic eigenvalue problem (2.1), we note that when $J_1(x) = J_2(x)$, (2.1) can be solved by the method of separation variables, and it suffices to consider the spatially dependent eigenvalue problem

$$\mathcal{L}[u(x)] := \int_{-L_1}^{L_2} J(x-y)u(y)dy - u(x) = \lambda u(x), \quad -L_1 \leq x \leq L_2,$$  \hspace{1cm} (2.2)

and the spatially independent time-periodic eigenvalue problem

$$\begin{cases}
    f_1' = a_1 e_1 f_2 - b_1 f_1 + \lambda f_1, & 0 < t \leq (1 - \delta)\omega, \\
    f_2' = a_2 e_2 f_1 - b_2 f_2 + \lambda f_2, & 0 < t \leq (1 - \delta)\omega, \\
    f_1' = -b_1 f_1 + \lambda f_1, & (1 - \delta)\omega < t \leq \omega, \\
    f_2' = -k f_2 + \lambda f_2, & (1 - \delta)\omega < t \leq \omega, \\
    f_1(0) = f_1(\omega), & f_2(0) = f_2(\omega).
\end{cases}$$  \hspace{1cm} (2.3)

The principal eigenvalue of problem (2.2) is known, see Proposition 3.4 in Cao et al. (2019). For (2.3), we investigate a more general spatially independent eigenvalue problem

$$\begin{cases}
    \phi_t = a_1 e_1 \psi - b_1 \phi + \lambda \phi, & 0 < t \leq (1 - \delta)\omega, \\
    \psi_t = a_2 e_2 \phi - b_2 \psi + \lambda \psi, & 0 < t \leq (1 - \delta)\omega, \\
    \phi_t = -k_1 \phi + \lambda \phi, & (1 - \delta)\omega < t \leq \omega, \\
    \psi_t = -k_2 \psi + \lambda \psi, & (1 - \delta)\omega < t \leq \omega, \\
    \phi(0) = \phi(\omega), & \psi(0) = \psi(\omega),
\end{cases}$$  \hspace{1cm} (2.4)

where $b_1$ is replaced by $k_1$ and $k$ is replaced by $k_2$, respectively.

According to the general Caratheodory conditions on the existence and uniqueness of solutions in Hale (1980), for a coupled system, it is easy to obtain the principal eigenvalue and corresponding eigenfunction. However, problem (2.4) is coupled for $\delta \in [0, 1)$, though it is decoupled for the very special situation $\delta = 1$, which means that there are no mosquitoes in the whole year because of coldness.

Grouping the coupled case together with the decoupled case, as in Nadin (2009), we define the generalized principal eigenvalues $\bar{\lambda}_1^O$ and $\underline{\lambda}_1^O$ of problem (2.4) as

$$\bar{\lambda}_1^O := \inf \{ \lambda \in \mathbb{R} \mid \exists \phi, \psi \in C([0, \omega]) \cap C^1_{\omega}((0, \omega]), \quad \phi, \psi > 0, \text{ and} \quad \phi, \psi \text{ are } \omega \text{-periodic so as the inequalities in(2.5) hold} \},$$

where

$$\begin{cases}
    \phi_t \leq a_1 e_1 \psi - b_1 \phi + \lambda \phi, & 0 < t \leq (1 - \delta)\omega, \\
    \psi_t \leq a_2 e_2 \phi - b_2 \psi + \lambda \psi, & 0 < t \leq (1 - \delta)\omega, \\
    \phi_t \leq -k_1 \phi + \lambda \phi, & (1 - \delta)\omega < t \leq \omega, \\
    \psi_t \leq -k_2 \psi + \lambda \psi, & (1 - \delta)\omega < t \leq \omega,
\end{cases}$$  \hspace{1cm} (2.5)
and
\[ \lambda^O_1 := \sup \{ \lambda \in \mathbb{R} \mid \exists \phi, \psi \in C([0, \omega]) \cap C^1_{\omega}((0, \omega]), \phi, \psi > 0 \text{ and } \phi, \psi \text{ are } \omega \text{-periodic so as (2.5) with the reversed inequalities hold} \}. \]

If there exists a positive function pair \((\phi, \psi)\) such that problem (2.4) with \(\lambda = \lambda^O_1\) holds, then \(\lambda^O_1\) is called a principal eigenvalue of problem (2.4).

We first consider the special case \(\delta = 1\). In this case, the spatial-independent problem (2.4) becomes
\[
\begin{align*}
\phi_t &= -k_1 \phi + \lambda \phi, \quad 0 < t \leq \omega, \\
\psi_t &= -k_2 \psi + \lambda \psi, \quad 0 < t \leq \omega, \\
\phi(0) &= \phi(\omega), \quad \psi(0) = \psi(\omega).
\end{align*}
\]

If \(k_1 = k_2\), then \(\bar{\lambda}^O_1 = \lambda^O_1 = \lambda^O_2 = k_1\), and if \(k_1 \neq k_2\), then \(\bar{\lambda}^O_1 = \max\{k_1, k_2\}, \lambda^O_1 = \min\{k_1, k_2\}\) and the two generalized principal eigenvalues are not equal obviously.

Suppose that \(0 \leq \delta < 1\), we claim that \(\lambda^O_1 = \lambda^O_1 = \lambda^O_1\), where \(\lambda^O_1\) is the principal eigenvalue of problem (2.4) with a positive eigenfunction pair \((\phi, \psi)\). In fact, we can further provide a detailed calculation process of principal eigen-pair \((\lambda^O_1, \phi, \psi)\) in the following.

With a view to the periodicity of \(\phi\) and \(\psi\), problem (2.4) can be rewritten as
\[
\begin{align*}
\phi_t &= a_1 e_1 \psi - b_1 \phi + \lambda^O_1 \phi, \quad 0 < t \leq (1 - \delta)\omega, \\
\psi_t &= a_2 e_2 \phi - b_2 \psi + \lambda^O_1 \psi, \quad 0 < t \leq (1 - \delta)\omega, \\
\phi((1 - \delta)\omega) &= \phi(0)e^{(k_1 - \lambda^O_1)\delta\omega}, \\
\psi((1 - \delta)\omega) &= \psi(0)e^{(k_2 - \lambda^O_1)\delta\omega},
\end{align*}
\]

the first two equations in (2.6) are abbreviated as
\[
\begin{pmatrix}
\phi_t \\
\psi_t
\end{pmatrix} = \begin{pmatrix}
-b_1 + \lambda^O_1 & a_1 e_1 a_2 e_2 \\
-a_2 e_2 & -b_2 + \lambda^O_1
\end{pmatrix} \begin{pmatrix}
\phi \\
\psi
\end{pmatrix} := M \begin{pmatrix}
\phi \\
\psi
\end{pmatrix}. \tag{2.7}
\]

Then, we see from the characteristic equation \(|M - \mu E| = 0\) that
\[
\mu_{1,2} = \lambda^O_1 + \frac{-(b_1 + b_2) \pm \sqrt{(b_1 - b_2)^2 + 4a_1 a_2 e_1 e_2}}{2} := \lambda^O_1 + c_{1,2}. \tag{2.8}
\]

Without loss of generality, we assume that \(c_1 \geq c_2\). Direct calculation yields that \(b_1 + c_1 = -(b_2 + c_2) > 0\).

The linearly independent eigenvectors \((k_{11}, k_{12})\) and \((k_{21}, k_{22})\) associated with eigenvalues \(\mu_1\) and \(\mu_2\) satisfy
\[
(k_{i1}, k_{i2}) \begin{pmatrix}
-b_1 + \lambda^O_1 - \mu_i & a_1 e_1 a_2 e_2 \\
-a_2 e_2 & -b_2 + \lambda^O_1 - \mu_i
\end{pmatrix} = (0 0) \tag{2.9}
\]
for \( i = 1, 2 \), and we can take the eigenvectors as

\[
(k_{11}, k_{12}) = (a_2 e_2, b_1 - \lambda_1^0 + \mu_1) = (a_2 e_2, b_1 + c_1)
\]

and

\[
(k_{21}, k_{22}) = (b_2 - \lambda_1^0 + \mu_2, a_1 e_1) = (b_2 + c_2, a_1 e_1).
\]

Subsequently, we consider the following algebraic equations

\[
\begin{pmatrix}
a_2 e_2 & b_1 + c_1 \\
b_2 + c_2 & a_1 e_1
\end{pmatrix}
\begin{pmatrix}
\phi \\
\psi
\end{pmatrix}
= \begin{pmatrix}
e^{\mu_1 t} \\
e^{\mu_2 t}
\end{pmatrix},
\]

(2.10)

whose solution can be explicitly expressed as

\[
(\phi, \psi) = \left( \frac{a_1 e_1 e^{\mu_1 t} - (b_1 + c_1) me^{\mu_2 t}}{C_0}, \frac{-(b_2 + c_2) e^{\mu_1 t} + a_2 e_2 me^{\mu_2 t}}{C_0} \right),
\]

where

\[
C_0 = a_1 a_2 e_1 e_2 - (b_1 + c_1)(b_2 + c_2) = a_1 a_2 e_1 e_2 + (b_1 + c_1)^2 > 0.
\]

(2.11)

Using the third and fourth equations of (2.6), we have

\[
\begin{cases}
a_1 e_1 e^{c_1 (1-\delta) \omega} e^{\lambda_1^0 \omega} - (b_1 + c_1) e^{c_2 (1-\delta) \omega} me^{\lambda_1^0 \omega} + (b_1 + c_1) e^{k_1 \delta \omega} m = a_1 e_1 e^{k_1 \delta \omega}, \\
-(b_2 + c_2) e^{c_1 (1-\delta) \omega} e^{\lambda_1^0 \omega} + a_2 e_2 e^{c_2 (1-\delta) \omega} me^{\lambda_1^0 \omega} - a_2 e_2 e^{k_2 \delta \omega} m = -(b_2 + c_2) e^{k_2 \delta \omega}.
\end{cases}
\]

(2.12)

For convenience, we denote \( \Lambda = e^{\lambda_1^0 \omega} \). Next, we show that problem (2.12) admits a unique solution \((m, \Lambda)\) such that \( \phi > 0 \) and \( \psi > 0 \). For this purpose, we need consider the following three cases.

**Case 1.** If \( k_1 = k_2 \), multiplying both sides of the first equation of (2.12) by \( a_2 e_2 \), and the second equation of that by \((b_1 + c_1)\), then adding the two equations, yields

\[
[a_1 a_2 e_1 e_2 - (b_1 + c_1)(b_2 + c_2)][e^{c_1 (1-\delta) \omega} \Lambda - e^{k_1 \delta \omega}] = 0,
\]

hence problem (2.12) has a unique solution \((0, e^{k_1 \delta \omega - c_1 (1-\delta) \omega})\) since that \([a_1 a_2 e_1 e_2 - (b_1 + c_1)(b_2 + c_2)] = [a_1 a_2 e_1 e_2 + (b_1 + c_1)^2] > 0\), see also Fig. 1. The principal eigenvalue \( \lambda_1^0 \) can be explicitly expressed, that is, \( \lambda_1^0 = (k_1 + c_1) \delta - c_1 \). Moreover, the corresponding eigenfunction pair is

\[
(\phi, \psi) = \left( \frac{a_1 e_1 e^{(k_1 + c_1) \delta t}}{C_0}, \frac{-(b_2 + c_2) e^{(k_1 + c_1) \delta t}}{C_0} \right).
\]
If $k_1 \neq k_2$, for abbreviation, we denote
\[
A_{11} = a_1 e_1 e^{c_1(1-\delta)\omega}, \quad A_{12} = (b_1 + c_1) e^{c_2(1-\delta)\omega}, \\
A_{13} = (b_1 + c_1) e^{k_1\delta\omega}, \quad A_{14} = a_1 e_1 e^{k_1\delta\omega}, \\
A_{21} = -(b_2 + c_2) e^{c_1(1-\delta)\omega}, \quad A_{22} = a_2 e_2 e^{c_2(1-\delta)\omega}, \\
A_{23} = a_2 e_2 e^{k_2\delta\omega}, \quad A_{24} = -(b_2 + c_2) e^{k_2\delta\omega}.
\]

Thus (2.12) becomes
\[
\begin{aligned}
A_{11} \Lambda - A_{12} \Lambda m + A_{13} m &= A_{14}, \\
A_{21} \Lambda + A_{22} \Lambda m - A_{23} m &= A_{24},
\end{aligned}
\]
where $A_{ij}$ are all positive since that $b_1 + c_1 = -(b_2 + c_2) > 0$. By using the elimination method, we can get the quadratic equation with variable $\Lambda$
\[
(A_{11} A_{22} + A_{12} A_{21}) \Lambda^2 - (A_{13} A_{21} + A_{11} A_{23} - A_{22} A_{14} \\
- A_{12} A_{24}) \Lambda + A_{23} A_{14} + A_{13} A_{24} = 0,
\]
and that with variable $m$
\[
(A_{13} A_{22} - A_{12} A_{23}) m^2 - (A_{22} A_{14} + A_{12} A_{24} - A_{13} A_{21} \\
- A_{11} A_{23}) m + A_{11} A_{24} - A_{21} A_{14} = 0.
\]
It follows from Vieta’s theorem that two roots $\Lambda_1, \Lambda_2$ of (2.14) satisfy $\Lambda_1 \Lambda_2 > 0$, and two roots $m_1, m_2$ of (2.15) satisfy

$$m_1m_2 = \frac{A_{11}A_{24} - A_{21}A_{14}}{A_{13}A_{22} - A_{12}A_{23}} = \frac{a_1e_1(b_2 + c_2)e^{(1-\delta)\omega}}{a_2e_2(b_1 + c_1)e^{(1-\delta)\omega}} = \frac{-a_1e_1e^{(1-\delta)\omega}}{a_2e_2e^{(1-\delta)\omega}} < \frac{-a_1e_1}{a_2e_2} < 0.$$

It is not easy to derive the explicit solution of problem (2.13), and we therefore consider the solutions by image method. In fact, problem (2.13) can be rewritten as

$$\begin{cases} 
\Lambda = (A_{14} - A_{13}m)/(A_{11} - A_{12}m), \\
\Lambda = (A_{24} + A_{23}m)/(A_{21} + A_{22}m). 
\end{cases}$$

The first equation in (2.16) shows that its curve must go through points $P_1(\frac{a_1e_1}{b_1+c_1}, 0)$ and $P_2(0, e^{k_1\delta\omega-c_1(1-\delta)\omega})$. What’s more, $\Lambda$ is strictly decreasing with respect to $m$ in the interval $(-\infty, \frac{a_1e_1}{b_1+c_1}e^{(c_1-c_2)(1-\delta)\omega}) \cup (\frac{a_1e_1}{b_1+c_1}e^{(c_1-c_2)(1-\delta)\omega}, +\infty)$, see Fig. 2. Meanwhile, the second equation indicates that $\Lambda$ is strictly increasing on $m$ in $(-\infty, \frac{b_2+c_2}{a_2e_2}e^{(c_1-c_2)(1-\delta)\omega}) \cup (\frac{b_2+c_2}{a_2e_2}e^{(c_1-c_2)(1-\delta)\omega}, +\infty)$, and its curve pass the points $P_3(\frac{b_2+c_2}{a_2e_2}, 0)$ and $P_4(0, e^{k_2\delta\omega-c_1(1-\delta)\omega})$.

**Case 2.** If $k_1 < k_2$, two curves have one intersection in the regions $D_1 := (\frac{b_2+c_2}{a_2e_2}, 0) \times (e^{k_1\delta\omega-c_1(1-\delta)\omega}, e^{k_2\delta\omega-c_1(1-\delta)\omega})$ and $D_2 := (\frac{a_1e_1}{b_1+c_1}e^{(c_1-c_2)(1-\delta)\omega}, +\infty) \times (e^{k_2\delta\omega-c_1(1-\delta)\omega}, e^{k_2\delta\omega-c_2(1-\delta)\omega})$ respectively. That is to say, problem (2.12) have solutions $(m_1, \Lambda_1) \in D_1$ and $(m_2, \Lambda_2) \in D_2$, see Fig. 2a. However, because of $\psi < 0$ in $D_2$, which fails to meet the requirement of the eigenfunction pair, so problem (2.12) has a unique solution, which is in $D_1$, such that $\phi > 0$ and $\psi > 0$.

**Case 3.** If $k_1 > k_2$, the discussion is similar to Case 2. We obtain that problem (2.12) have solutions $(m_3, \Lambda_3) \in D_3 := (-\infty, \frac{b_2+c_2}{a_2e_2}e^{(c_1-c_2)(1-\delta)\omega}) \times (e^{k_1\delta\omega-c_1(1-\delta)\omega}, e^{k_1\delta\omega-c_1(1-\delta)\omega})$ and $(m_4, \Lambda_4) \in D_4 := (0, \frac{a_1e_1}{b_1+c_1}) \times (e^{k_2\delta\omega-c_1(1-\delta)\omega}, e^{k_1\delta\omega-c_1(1-\delta)\omega})$, see Fig. 2b. Noticing that $\phi < 0$ in $D_3$, so problem (2.12) has a unique solution in $D_4$ such that $\phi > 0$ and $\psi > 0$.

In summary, if $0 \leq \delta < 1$, problem (2.12) (i.e. problem (2.4)) has a unique solution with $\phi(t) > 0$ and $\psi(t) > 0$ for $0 \leq t \leq \omega$.

The above discussion enables us to obtain the following conclusions.

**Theorem 2.1** (i) Assume that

$$(H_1) : \begin{cases} 
(a) \ 0 \leq \delta < 1 \ or \ (b) \ \delta = 1, \ k_1 = k_2 
\end{cases}$$

holds. Then $\lambda_1^O = \lambda_1^O = \lambda_1^O$, the eigenvalue problem (2.4) admits a principal eigenvalue $\lambda_1^O$ with a positive eigenfunction pair $(\phi, \psi) \in [C([0, \omega]) \cap C^1(\omega)]^2$. Especially, if $k_1 = k_2$, for any given $\delta \in [0, 1]$, $\lambda_1^O$ can be explicitly expressed as $\lambda_1^O = (k_1 + c_1)\delta - c_1$. 

\[ \text{ Springer} \]
(ii) If $\delta = 1$ and $k_1 \neq k_2$, then $\lambda^O = \max\{k_1, k_2\} = \lambda^O_1$. So the generalized principal eigenvalues $\lambda^O_1$ and $\lambda^O_2$ of the eigenvalue problem (2.4) are not equal.

As in Diekmann et al. (1990), $\lambda^O_1$ is an important threshold of epidemiology models. $\lambda^O_1 < 0 (> 0)$ implies high (low) risk. It follows from Theorem 2.1 that if $\delta = 1$, then $\lambda^O_1 \geq \lambda^O_2 > 0$. We next focus on discussing the sign of $\lambda^O_1$ for $0 \leq \delta < 1$.

**Theorem 2.2** The following statements are valid:

(i) If $\delta = 0$, the necessary and sufficient condition for $\lambda^O_1 = 0$ is that $a_1a_2e_1e_2 = b_1b_2$.

(ii) If $0 < \delta < 1$, then $\lambda^O_1 = 0$ if and only if

$$
\frac{a_1e_1(e^{c_1(1-\delta)\omega} - e^{k_1\delta \omega})}{(b_1 + c_1)(e^{c_2(1-\delta)\omega} - e^{k_2\delta \omega})} = \frac{(b_2 + c_2)(e^{c_1(1-\delta)\omega} - e^{k_2\delta \omega})}{a_2e_2(e^{c_2(1-\delta)\omega} - e^{k_2\delta \omega})},
$$

(2.17)

where $c_i (i = 1, 2)$ is defined in (2.8).

**Proof** (i) We first prove the necessity. Suppose that $\delta = 0$ and $\lambda^O_1 = 0$, then (2.4) becomes

$$
\begin{align*}
\phi_t &= a_1e_1\psi - b_1\phi, & 0 < t \leq \omega, \\
\psi_t &= a_2e_2\phi - b_2\psi, & 0 < t \leq \omega, \\
\phi(0) &= \phi(\omega), & \psi(0) = \psi(\omega).
\end{align*}
$$

(2.18)

Repeating the calculation process from (2.7) to (2.10) with $\lambda^O_1 = 0$ and using the first two equations in (2.18) give

$$
(\phi, \psi) = \left( \frac{a_1e_1e^{c_1t} - (b_1 + c_1)m e^{c_2t}}{C_0}, \frac{(b_2 + c_2)e^{c_1t} + a_2e_2m e^{c_2t}}{C_0} \right).
$$
where \( C_0 \) is defined in (2.11). According to the periodic conditions \( \phi(0) = \phi(\omega) \) and \( \psi(0) = \psi(\omega) \), we have

\[
[a_1a_2e_1e_2 - (b_1 + c_1)(b_2 + c_2)][e^{c_1\omega} - 1] = 0,
\]

therefore, \( e^{c_1\omega} = 1 \), that is, \( c_1 = 0 \), which means that \( a_1a_2e_1e_2 = b_1b_2 \) by using the expression of \( c_1 \) in (2.8).

We next prove the sufficiency. In the case that \( \delta = 0 \), (2.4) becomes

\[
\begin{align*}
\phi_t &= a_1e_1\psi - b_1\phi + \lambda_1^0\phi, & 0 < t \leq \omega, \\
\psi_t &= a_2e_2\phi - b_2\psi + \lambda_1^0\psi, & 0 < t \leq \omega,
\end{align*}
\]

(2.19)

Owing to \( a_1a_2e_1e_2 = b_1b_2 \), we have

\[
c_1 = \frac{-(b_1 + b_2) + \sqrt{(b_1 - b_2)^2 + 4a_1a_2e_1e_2}}{2} = 0,
\]

\[
c_2 = \frac{-(b_1 + b_2) - \sqrt{(b_1 - b_2)^2 + 4a_1a_2e_1e_2}}{2} = -(b_1 + b_2),
\]

and the general solution of problem (2.19) is

\[
(\phi, \psi) = \left( \frac{a_1e_1e^{\lambda_1^0 t} - b_1m[e^{\lambda_1^0 t} - (b_1 + b_2)]t}{C_0}, \frac{-b_1e^{\lambda_1^0 t} + a_2e_2m[e^{\lambda_1^0 t} - (b_1 + b_2)]t}{C_0} \right).
\]

Using the periodicity of \( \phi \) and \( \psi \) yields \( \lambda_1^0 = 0 \) and \( m = 0 \).

(ii) If \( 0 < \delta < 1 \), it can be calculated directly from (2.12) that

\[
\frac{a_1e_1(e^{c_1(1-\delta)\omega}e^{\lambda_1^0\omega} - e^{k_1\delta\omega})}{(b_1 + c_1)(e^{c_2(1-\delta)\omega}e^{\lambda_1^0\omega} - e^{k_2\delta\omega})} = \frac{(b_2 + c_2)(e^{c_1(1-\delta)\omega}e^{\lambda_1^0\omega} - e^{k_2\delta\omega})}{a_2e_2(e^{c_2(1-\delta)\omega}e^{\lambda_1^0\omega} - e^{k_2\delta\omega})}.
\]

(2.20)

therefore, \( \lambda_1^0 = 0 \) if and only if (2.17) holds.

\[\square\]

Now, we give conditions for \( \lambda_1^0 > (\langle)0 \), which means that the risk is low(high).

**Corollary 2.3** The following statements are valid:

(i) If \( \delta = 0 \), then \( \lambda_1^0 > (\langle)0 \) if and only if \( a_1a_2e_1e_2 < (\rangle)b_1b_2 \).

(ii) If \( 0 < \delta < 1 \), then the sufficient condition for \( \lambda_1^0 > 0 \) is \( \min\{k_1, k_2\}\delta > c_1(1-\delta) \), and the necessary condition for \( \lambda_1^0 > 0 \) is \( \max\{k_1, k_2\}\delta > c_1(1-\delta) \). Accordingly, the sufficient condition for \( \lambda_1^0 < 0 \) is \( \max\{k_1, k_2\}\delta < c_1(1-\delta) \), and the necessary condition of \( \lambda_1^0 < 0 \) is \( \min\{k_1, k_2\}\delta < c_1(1-\delta) \), where

\[
c_1 = \frac{-(b_1 + b_2) + \sqrt{(b_1 - b_2)^2 + 4a_1a_2e_1e_2}}{2}.
\]
defined in (2.8). Specially, if \( k_1 = k_2 \), then \( \lambda_1^O > (\cdot <) 0 \) if and only if
\[
k_1\delta > (\cdot <) c_1(1 - \delta).
\]

(iii) If \( \delta = 1 \), then \( \lambda_1^O \geq \lambda_1^O > 0 \).

**Proof**  
(i) The result for \( \lambda_1^O > 0 \) (or \( \lambda_1^O < 0 \)) can be verified by the similar manner as in Theorem 2.2 (i), we omit it here.  
(ii) It can be easily seen from the discussion of Cases 1–3 in Theorem 2.1 (i) that the following results for three cases are valid:

\[
\begin{align*}
(\text{a}) & \quad \text{if } k_1 = k_2, \lambda_1^O = k_1 \delta - c_1(1 - \delta); \\
(\text{b}) & \quad \text{if } k_1 < k_2, k_1 \delta - c_1(1 - \delta) < \lambda_1^O < k_2 \delta - c_1(1 - \delta); \\
(\text{c}) & \quad \text{if } k_1 > k_2, k_2 \delta - c_1(1 - \delta) < \lambda_1^O < k_1 \delta - c_1(1 - \delta).
\end{align*}
\]

As a result, \( \lambda_1^O > 0 \) provided that \( \min\{k_1, k_2\}\delta > c_1(1 - \delta) \), while if \( \lambda_1^O > 0 \), we have \( \max\{k_1, k_2\}\delta > c_1(1 - \delta) \).

(iii) The result for the case \( \delta = 1 \) is directly from Theorem 2.1 (ii).

\[\square\]

Now we study the monotonicity of \( \lambda_1^O \) with respect to parameters of problem (2.4). Especially, we focus on its monotonicity with respect to \( \delta \), which provides a way to study the impact of the length \( \delta \omega \) of cold season on prevention and control of virus.

To emphasize the dependence of the principal eigenvalue \( \lambda_1^O \) on the parameters of problem (2.4), we now denote it by \( \lambda_1^O (\Gamma, b_1, b_2, \delta) \), where \( \Gamma \) is collection of \( a_i, e_i, \omega, k_1 \) and \( k_2 \).

**Lemma 2.4** The following statements are valid:

(i) \( \lambda_1^O (b_1, b_2) \) is strictly increasing in \( b_1 \) and \( b_2 \) for \( b_1, b_2 \geq 0 \);  
(ii) \( \lambda_1^O (\delta) \) is strictly increasing in \( \delta \) for \( \delta \in [0, 1) \) provided that \( k_i \geq b_i (i = 1, 2) \).

**Proof**  
(i) It can easily seen from problem (2.6) that \( \lambda_1^O \) is strictly increasing in \( b_1 \) and \( b_2 \).

(ii) Let \( 0 \leq \delta_1 < \delta_2 < 1 \), and \( (\lambda_1^O (\delta_i), \phi_i(t, x), \psi_i(t, x)) \) is the principal eigen-pair of eigenvalue problem (2.4) with \( \delta = \delta_i (i = 1, 2) \).

Let \( \phi_2^* (t) = \phi_2 (1 - \delta_2) \omega - t \) and \( \psi_2^* (t) = \psi_2 ((1 - \delta_2) \omega - t) \), it follows from (2.4) with \( \delta = \delta_2 \) that
\[
\begin{align*}
-\phi_2^* & = a_1 e_1 \psi_2^* - b_1 \phi_2^* + \lambda_1^O (\delta_2) \phi_2^*, \quad 0 \leq t < (1 - \delta_2) \omega, \\
-\psi_2^* & = a_2 e_2 \phi_2^* - b_2 \psi_2^* + \lambda_1^O (\delta_2) \psi_2^*, \quad 0 \leq t < (1 - \delta_2) \omega, \\
-\phi_2^* & = -k_1 \phi_2^* + \lambda_1^O (\delta_2) \phi_2^*, \quad (1 - \delta_2) \omega < t < \omega, \\
-\psi_2^* & = -k_2 \psi_2^* + \lambda_1^O (\delta_2) \psi_2^*, \quad (1 - \delta_2) \omega < t < \omega, \\
\phi_2^*(0) & = \phi_2^*(\omega), \quad \psi_2^*(0) = \psi_2^*(\omega).
\end{align*}
\]

Multiplying both sides of the first and third equation of (2.21) by \( \phi_1 \), then integrating over \((0, (1 - \delta_2) \omega)\) and \((1 - \delta_2) \omega, \omega)\), respectively, and recalling that
\((\lambda_1^{O}(\delta_1), \phi_1(t, x), \psi_1(t, x))\) is the principal eigen-pair of eigenvalue problem (2.4) with \(\delta = \delta_1\), we obtain

\[
\int_{0}^{(1-\delta_2)\omega} \left[ a_1 e_1 \psi_2^* \phi_1 - b_1 \phi_2^* \phi_1 + \lambda_1^{O}(\delta_2) \phi_2^* \phi_1 \right] dt \\
= -\phi_2^* \phi_1 \big|_{0}^{(1-\delta_2)\omega} + \int_{0}^{(1-\delta_2)\omega} \phi_2^* \left[ a_1 e_1 \psi_1 - b_1 \phi_1 + \lambda_1^{O}(\delta_1) \phi_1 \right] dt \tag{2.22}
\]

and

\[
\int_{(1-\delta_2)\omega}^{\omega} \left[ -k_1 \phi_2^* \phi_1 + \lambda_1^{O}(\delta_2) \phi_2^* \phi_1 \right] dt \\
= -\phi_2^* \phi_1 \big|_{(1-\delta_2)\omega}^{\omega} + \int_{(1-\delta_2)\omega}^{\omega} \phi_2^* \phi_1 dt \\
= -\phi_2^* \phi_1 \big|_{(1-\delta_2)\omega}^{\omega} + \int_{(1-\delta_2)\omega}^{(1-\delta_1)\omega} a_1 e_1 \phi_2^* \psi_1 dt - \int_{(1-\delta_1)\omega}^{(1-\delta_2)\omega} b_1 \phi_1 \phi_2^* dt \\
- \int_{(1-\delta_1)\omega}^{\omega} k_1 \phi_1 \phi_2^* dt + \int_{(1-\delta_2)\omega}^{\omega} \lambda_1^{O}(\delta_1) \phi_1 \phi_2^* dt. \tag{2.23}
\]

Adding (2.22) and (2.23) yields

\[
(\lambda_1^{O}(\delta_2) - \lambda_1^{O}(\delta_1)) \int_{0}^{\omega} \phi_1 \phi_2^* dt \\
= a_1 e_1 \left[ \int_{0}^{(1-\delta_1)\omega} \psi_1 \phi_2^* dt - \int_{0}^{(1-\delta_2)\omega} \phi_1 \psi_2^* dt \right] - \int_{(1-\delta_1)\omega}^{(1-\delta_2)\omega} (k_1 - b_1) \phi_1 \phi_2^* dt. \tag{2.24}
\]

Similarly, multiplying both sides of the second and forth equation of (2.21) by \(\psi_1\), integrating over \((0, (1-\delta_2)\omega)\) and \(((1-\delta_2)\omega, \omega)\), respectively, and then adding them give

\[
(\lambda_1^{O}(\delta_2) - \lambda_1^{O}(\delta_1)) \int_{0}^{\omega} \psi_1 \phi_2^* dt \\
= a_2 e_2 \left[ \int_{0}^{(1-\delta_1)\omega} \phi_1 \psi_2^* dt - \int_{0}^{(1-\delta_2)\omega} \psi_1 \phi_2^* dt \right] \\
- \int_{(1-\delta_1)\omega}^{(1-\delta_2)\omega} (k_2 - b_2) \psi_1 \phi_2^* dt. \tag{2.25}
\]

It follows from (2.24) and (2.25) that
Fig. 3 a, b The contour lines (red colour) of $\lambda_1^O = 0$ when $k_1 \neq k_2$ and when $k_1 = k_2$, respectively

\[
\begin{align*}
&\lambda_1^O(\delta_2) - \lambda_1^O(\delta_1) \geq \max \left\{ \int_0^{(1-\delta_1)\omega} \psi_1 \phi_2^* dt - \int_0^{(1-\delta_2)\omega} \phi_1 \psi_2^* dt, \int_0^{(1-\delta_1)\omega} \phi_1 \psi_2^* dt - \int_0^{(1-\delta_2)\omega} \psi_1 \phi_2^* dt \right\} \\
&> \int_0^{(1-\delta_2)\omega} \left[ \psi_1 \phi_2^* - \phi_1 \psi_2^* + \phi_1 \psi_2^* - \psi_1 \phi_2^* \right] dt = 0
\end{align*}
\]

provided that $k_i \geq b_i (i = 1, 2)$, which implies $\lambda_1^O(\delta_2) > \lambda_1^O(\delta_1)$.

Remark 2.5 The monotonicity of $\lambda_1^O$ in Lemma 2.4 (ii) holds for $\delta \in [0, 1)$. If $\delta = 1$ and $b_1 \neq k$, we have $\lambda_1^O(\delta) > \lambda_1^O(\delta)$, the principal eigenvalue does not exist. However, we can similarly obtain that $\lambda_1^O(\delta)$ and $\lambda_1^O(\delta)$ are strictly increasing in $\delta \in [0, 1]$.

It follows from the above discussions on the monotonicity of $\lambda_1^O$ with respect to $b_1$ and $\delta$, we can get the contour lines of $\lambda_1^O = 0$, see Fig. 3, where $b_i^* = a_1a_2e_1e_2/b_2$.

Remark 2.6 As in our model, let $k_1 = b_1$ and $k_2 = k$, the above conclusions still hold.

Come back to problem (2.1) involving nonlocal diffusion, we define the corresponding generalized principal eigenvalues $\lambda_1^p$ and $\lambda_1^p$ of (2.1) as

\[
\lambda_1^p := \inf \{ \lambda \in \mathbb{R} | \exists \phi, \psi \in C([0, +\infty) \times [-L_1, L_2]), \phi(\cdot, x), \psi(\cdot, x) \in C^1_0(0, \infty), x \in [-L_1, L_2], \\
\text{and } \phi, \psi \text{ are positive } \omega - \text{periodic so as the inequalities in (2.26) hold} \}
\]
where
\[
\begin{align*}
\phi_t - d_1 \mathcal{L}_1[\phi] &\leq a_1 e_1 \psi - b_1 \phi + \lambda \phi, & 0 < t \leq (1 - \delta) \omega, -L_1 \leq x \leq L_2, \\
\psi_t - d_2 \mathcal{L}_2[\psi] &\leq a_2 e_2 \phi - b_2 \psi + \lambda \psi, & 0 < t \leq (1 - \delta) \omega, -L_1 \leq x \leq L_2, \\
\phi_t - d_1 \mathcal{L}_1[\phi] &\leq -b_1 \phi + \lambda \phi, & (1 - \delta) \omega < t \leq \omega, -L_1 \leq x \leq L_2, \\
\psi_t &\leq -k \psi + \lambda \psi, & (1 - \delta) \omega < t \leq \omega, -L_1 \leq x \leq L_2.
\end{align*}
\]

and
\[
\lambda_1^P := \sup \{ \lambda \in \mathbb{R} \mid \exists \phi, \psi \in C([0, +\infty],[-L_1, L_2]), \\
\phi(\cdot, x), \psi(\cdot, x) \in C^1_0(0, \infty), x \in [-L_1, L_2], \\
\text{and } \phi, \psi \text{ are positive } \omega \text{-periodic so as } (2.26) \}
\text{with the reversed inequalities hold}.
\]

If there exists a positive function pair \((\phi, \psi)\) such that problem (2.1) with \(\lambda = \lambda_1^P\) holds, then \(\lambda_1^P\) is called a principal eigenvalue of problem (2.1).

Based the above definitions, we have the following properties.

**Lemma 2.7** (i) \(\lambda_1^P ([\mathcal{L}]) = \lambda_1^P ([0, L_1 + L_2]) \) and \(\lambda_1^P ([\mathcal{L}]) = \lambda_1^P ([0, L_1 + L_2]) \).

(ii) \(\lambda_1^P ([\mathcal{L}]) \) and \(\lambda_1^P ([\mathcal{L}]) \) are decreasing with respect to \(L_1 + L_2\), in the sense, \(\lambda_1^P ([\mathcal{L}]) \leq \lambda_1^P ([\mathcal{L}]) \) and \(\lambda_1^P ([\mathcal{L}]) \leq \lambda_1^P ([\mathcal{L}]) \) if \(L_1 + L_2 > L_3 + L_4\).

**Proof** (i) It follows from the transformation \((t, x) \rightarrow (t, z)\) where \(z = x + L_1\) and \(x \in [-L_1, L_2]\).

(ii) Without loss of generality, we only need to prove that \(\lambda_1^P ([\mathcal{L}]) \) are decreasing with respect to \(L_1 + L_2\), that is, \(\lambda_1^P ([\mathcal{L}]) \leq \lambda_1^P ([\mathcal{L}]) \) if \(L_1 + L_2 > L_3 + L_4\). It suffices to prove \(\lambda_1^P ([\mathcal{L}]) \leq \lambda_1^P ([\mathcal{L}]) \) if \(L_1 + L_2 > L_3 + L_4\) (by (i)).

Let \((2.27, \phi(t, x), \psi(t, x))\) is the generalized principal eigen-pair of (2.1) on \([0, \omega] \times [0, L_1 + L_2]\), and it satisfies

\[
\begin{align*}
\phi_t - d_1 \mathcal{L}_1[\phi] &\geq a_1 e_1 \psi - b_1 \phi + \lambda_1^P \phi, & 0 < t \leq (1 - \delta) \omega, 0 \leq x \leq L_1 + L_2, \\
\psi_t - d_2 \mathcal{L}_2[\psi] &\geq a_2 e_2 \phi - b_2 \psi + \lambda_1^P \psi, & 0 < t \leq (1 - \delta) \omega, 0 \leq x \leq L_1 + L_2, \\
\phi_t - d_1 \mathcal{L}_1[\phi] &\geq -b_1 \phi + \lambda_1^P \phi, & (1 - \delta) \omega \leq t \leq \omega, 0 \leq x \leq L_1 + L_2, \\
\psi_t &\geq -k \psi + \lambda_1^P \psi, & (1 - \delta) \omega \leq t \leq \omega, 0 \leq x \leq L_1 + L_2, \\
\phi(0, x) = \phi(\omega, x), \psi(0, x) = \psi(\omega, x), & 0 \leq x \leq L_1 + L_2.
\end{align*}
\]

(2.27)
Because of

\[-d_1[\int_0^{L_1+L_2} J_1(x-y)\phi(t,y)dy - \phi(t,x)] = -d_1[\int_0^{L_3+L_4} J_1(x-y)\phi(t,y)dy - \phi(t,x)] - d_1[\int_{L_3+L_4}^{L_1+L_2} J_1(x-y)\phi(t,y)dy \leq -d_1[\int_0^{L_3+L_4} J_1(x-y)\phi(t,y)dy - \phi(t,x)]\]

for \( t \in (0, (1 - \delta)\omega) \) and \( x \in [0, L_3 + L_4] \), we have

\[\phi_t - d_1[\int_0^{L_3+L_4} J_1(x-y)\phi(t,y)dy - \phi(t,x)] \geq, \neq a_1 e_1 \psi - b_1 \phi + \lambda_1^P ([0, L_1 + L_2]) \phi\]

for \( t \in (0, (1 - \delta)\omega) \) and \( x \in [0, L_3 + L_4] \), and

\[\phi_t - d_1[\int_0^{L_3+L_4} J_1(x-y)\phi(t,y)dy - \phi(t,x)] \geq, \neq - b_1 \phi + \lambda_1^P ([0, L_1 + L_2]) \phi\]

for \( t \in ((1 - \delta)\omega, \omega] \) and \( x \in [0, L_3 + L_4] \). Furthermore, we have

\[\psi_t - d_2[\int_0^{L_3+L_4} J_2(x-y)\psi(t,y)dy - \psi(t,x)] \geq, \neq a_2 e_2 \phi - b_2 \psi + \lambda_1^P ([0, L_1 + L_2]) \psi\]

for \( t \in (0, (1 - \delta)\omega) \) and \( x \in [0, L_3 + L_4] \), and

\[\psi_t \geq -k \psi + \lambda_1^P ([0, L_1 + L_2]) \psi\]

for \( t \in ((1 - \delta)\omega, \omega] \) and \( x \in [0, L_3 + L_4] \) in the similar manner. Since \( \phi(0, x) = \phi(\omega, x), \psi(0, x) = \phi(\omega, x) \) for \( x \in [0, L_1 + L_2] \), we then have

\[\phi(0, x) = \phi(\omega, x), \psi(0, x) = \phi(\omega, x), x \in [0, L_3 + L_4].\]

It follows from the definition of \( \lambda_1^P ([0, L_3 + L_4]) \) that

\[\lambda_1^P ([0, L_3 + L_4]) \geq \lambda_1^P ([0, L_1 + L_2]).\]

\( \square \)

**Theorem 2.8** Assume (J) and the kernel function \( J_1(x) = J_2(x) \) for \( x \in \mathbb{R} \) hold.
(i) If the following condition holds

\[(H_2): (a) 0 \leq \delta < 1 \text{ or } (b) \delta = 1, \quad b_1 - d_1 \lambda_1^* = k.\]

Then \(\overline{\lambda}_1^P = \lambda_1^P = \lambda_1^P\), where \(\lambda_1^P([L_1, L_2])\) is the principal eigenvalue of eigenvalue problem (2.1) with a positive eigenfunction pair \((\phi, \psi)\). Especially, if \(b_1 - d_1 \lambda_1^* = k\), for any given \(\delta \in [0, 1]\), \(\lambda_1^P\) can be explicitly written as

\[\lambda_1^P = (b_1 - d_1 \lambda_1^* + s_1)\delta - s_1,\]

where \(\lambda_1^*\) is defined in (2.29) and

\[s_1 = \frac{-(b_1 - d_1 \lambda_1^* + b_2 - d_2 \lambda_1^*) + \sqrt{[(b_1 - d_1 \lambda_1^*) - (b_2 - d_2 \lambda_1^*)]^2 + 4a_1a_2e_1e_2}}{2}.\]

(ii) If \(\delta = 1\) and \(b_1 - d_1 \lambda_1^* \neq k\), then

\[\overline{\lambda}_1^P = \max\{b_1 - d_1 \lambda_1^*, k\} > \min\{b_1 - d_1 \lambda_1^*, k\} = \lambda_1^P\]

and the eigenvalue problem (2.1) has unequal generalized principal eigenvalues \(\overline{\lambda}_1^P\) and \(\lambda_1^P\).

**Proof** We notice that the kernel functions \(J_1\) and \(J_2\) satisfy \(J_1(x) = J_2(x)(:= J(x))\) in \(\mathbb{R}\), then \(\mathcal{L}_1 = \mathcal{L}_2(:= \mathcal{L})\). Consider the following eigenvalue problem

\[\mathcal{L}[u(x)] := \int_{-L_1}^{L_2} J(x - y)u(y)dy - u(x) = \lambda u(x), \quad -L_1 \leq x \leq L_2.\]  

(2.29)

It follows from Cao et al. (2019, Proposition 3.4) that problem (2.29) admits a principal eigen-pair \((\lambda_1^*, g(x))\) with \(\lambda_1^*([-L_1, L_2]) < 0, g \in C([-L_1, L_2])\) and \(g(x) > 0\) in \([-L_1, L_2]\).

Suppose that \(0 \leq \delta < 1\), we consider the spatial-independent eigenvalue problem

\[
\begin{align*}
&f_{1t} = a_1e_1f_2 - (b_1 - d_1 \lambda_1^*)f_1 + \lambda_1^O f_1, \quad 0 < t \leq (1 - \delta)\omega, \\
&f_{2t} = a_2e_2f_1 - (b_2 - d_2 \lambda_1^*)f_2 + \lambda_1^O f_2, \quad 0 < t \leq (1 - \delta)\omega, \\
&f_{1t} = -(b_1 - d_1 \lambda_1^*)f_1 + \lambda_1^O f_1, \quad (1 - \delta)\omega < t \leq \omega, \\
&f_{2t} = -k f_2 + \lambda_1^O f_2, \quad (1 - \delta)\omega < t \leq \omega, \\
&f_1(0) = f_1(\omega), \quad f_2(0) = f_2(\omega).
\end{align*}
\]

Theorem 2.1, where \(b_1\) and \(b_2\) are replaced by \(b_1 - d_1 \lambda_1^*\) and \(b_2 - d_2 \lambda_1^*\), implies that the above problem has a principal eigenvalue \(\lambda_1^O(\Gamma, b_1 - d_1 \lambda_1^*, b_2 - d_2 \lambda_1^*)\) with the eigenfunction pair \((f_1(t), f_2(t))\).
Set 

$$\phi(t, x) = f_1(t)g(x), \quad \psi(t, x) = f_2(t)g(x)$$

for $t \in [0, \omega]$, $x \in [-L_1, L_2]$. It is easy to check that 

$$
\begin{align*}
\phi_t - d_1 \Sigma[\phi] &= a_1 e_1 \psi - b_1 \phi + \lambda_1^O \phi, & 0 < t \leq (1 - \delta) \omega, -L_1 \leq x \leq L_2, \\
\psi_t - d_1 \Sigma[\psi] &= a_2 e_2 \phi - b_2 \psi + \lambda_1^O \psi, & 0 < t \leq (1 - \delta) \omega, -L_1 \leq x \leq L_2, \\
\phi_t - d_1 \Sigma[\phi] &= -b_1 \phi + \lambda_1^O \phi, & (1 - \delta) \omega < t \leq \omega, -L_1 \leq x \leq L_2, \\
\psi_t &= -k \psi + \lambda_1^O \psi, & (1 - \delta) \omega < t \leq \omega, -L_1 \leq x \leq L_2, \\
\phi(0, x) = \phi(\omega, x), \quad \psi(0, x) = \psi(\omega, x), & -L_1 \leq x \leq L_2.
\end{align*}
$$

(2.30)

Therefore, \(\lambda_1^p = \lambda_1^p = \lambda_1^p = \lambda_1^O(\Gamma, b_1 - d_1 \lambda_1^*, b_2 - d_2 \lambda_1^*)\). The remaining proof of Theorem 2.8 is similar to that of Theorem 2.1, we omit it here. 

Specially, if \(\delta = 1\), we have to consider two cases. When \(b_1 - d_1 \lambda_1^* = k\), then \(\lambda_1^p = \lambda_1^p = b_1 - d_1 \lambda_1^*\). However, when \(b_1 - d_1 \lambda_1^* \neq k\), we have \(\lambda_1^p = \max\{b_1 - d_1 \lambda_1^*, k\}\) and \(\lambda_1^p = \min\{b_1 - d_1 \lambda_1^*, k\}\), which means that \(\lambda_1^p \neq \lambda_1^p\). \(\square\)

**Corollary 2.9** Similar to the discussions of Theorem 2.1, it follows from Theorem 2.8 that when \(\delta \in [0, 1]\), the following hold true:

(i) if \(k = b_1 - d_1 \lambda_1^*\), \(\lambda_1^p = (b_1 - d_1 \lambda_1^*) + s_1 \delta - s_1\);  

(ii) if \(k > b_1 - d_1 \lambda_1^*\), \((b_1 - d_1 \lambda_1^*)\delta - s_1(1 - \delta) < \lambda_1^p < k\delta - s_1(1 - \delta)\);  

(iii) if \(k < b_1 - d_1 \lambda_1^*\), \(k\delta - s_1(1 - \delta) < \lambda_1^p < (b_1 - d_1 \lambda_1^*)\delta - s_1(1 - \delta)\), where \(\lambda_1^*\) and \(s_1\) are defined in (2.29) and (2.28), respectively.

From Theorem 2.8 and Corollary 2.3, we have the following result.

**Corollary 2.10** Assume (J) and the kernel function \(J_1(x) = J_2(x)\) for \(x \in \mathbb{R}\) hold. The following statements are valid:

(i) If \(\delta = 0\), then 

(a) \(\lambda_1^p = 0\) if and only if \(a_1 a_2 e_1 e_2 = (b_1 - d_1 \lambda_1^*)(b_2 - d_2 \lambda_1^*)\);  

(b) \(\lambda_1^p > (\leq 0)\) if and only if \(a_1 a_2 e_1 e_2 = (>) (b_1 - d_1 \lambda_1^*)(b_2 - d_2 \lambda_1^*)\);  

(ii) If \(0 < \delta < 1\), then  

(a) \(\lambda_1^p = 0\) if and only if (2.17) with \(c_i\) replaced by \(s_i\) and \(b_i\) replaced by \(b_i - d_1 \lambda_1^*\) \((i = 1, 2)\) holds, where \(s_1\) is defined by (2.28) and 

\[
\begin{align*}
\frac{-(b_1 - d_1 \lambda_1^* + b_2 - d_2 \lambda_1^*) - \sqrt{[(b_1 - d_1 \lambda_1^*) - (b_2 - d_2 \lambda_1^*)]^2 + 4a_1 a_2 e_1 e_2}}{2} = s_2
\end{align*}
\]

(2.31)

(b) the sufficient condition for \(\lambda_1^p > 0\) is \(\min\{b_1 - d_1 \lambda_1^*, k\} \delta > s_1(1 - \delta)\), and the necessary condition for \(\lambda_1^p > 0\) is \(\max\{b_1 - d_1 \lambda_1^*, k\} \delta > s_1(1 - \delta)\). Specially, if \(b_1 - d_1 \lambda_1^* = k\), then \(\lambda_1^p > 0\) if and only if \((b_1 - d_1 \lambda_1^*) \delta > s_1(1 - \delta)\).
(c) the sufficient condition for \( \lambda_1^P < 0 \) is \( \max \{ b_1 - d_1 \lambda_1^*, k \} \delta < s_1(1 - \delta) \), and the necessary condition for \( \lambda_1^P < 0 \) is \( \min \{ b_1 - d_1 \lambda_1^*, k \} \delta < s_1(1 - \delta) \). Specially, if \( b_1 - d_1 \lambda_1^* = k \), then \( \lambda_1^P < 0 \) if and only if \( (b_1 - d_1 \lambda_1^*) \delta < s_1(1 - \delta) \).

(iii) If \( \delta = 1 \), then \( \lambda_1^P \geq \lambda_1^P > 0 \) always holds.

To stress the dependence of \( \lambda_1^P \) on the parameters of problem (2.1), we denote it by

\[
\lambda_1^P(\Gamma, [-L_1, L_2], b_1, d_1, \delta),
\]

where \( \Gamma = \{ a_1, e_1, k, \omega \} \). With the above definition, we have the monotonicity of \( \lambda_1^P \) by using Lemma 2.4 and Remark 2.5.

**Corollary 2.11** Assume (J) and the kernel function \( J_1(x) = J_2(x) \) for \( x \in \mathbb{R} \) hold. The following statements are valid:

(i) \( \overline{\lambda}_1^P \) and \( \underline{\lambda}_1^P \) are strictly increasing with respect to \( b_1 \in (0, + \infty) \);

(ii) \( \lambda_1^P(\delta) \) is strictly increasing in \( \delta \in [0, 1] \). Specially, if \( \delta = 1 \), \( \lambda_1^P(\delta) \) is increasing in \( \delta \) when \( b_1 - d_1 \lambda_1^* = k \), and when \( b_1 - d_1 \lambda_1^* \neq k \), then \( \overline{\lambda}_1^P(\delta) \) and \( \underline{\lambda}_1^P(\delta) \) are increasing in \( \delta \).

In particular, if \( \delta = 0 \), the periodic eigenvalue problem (2.1) is without seasonal succession, the following result is known in Du and Ni (2020).

**Lemma 2.12** (Du and Ni 2020, Proposition 2.1) Assume (J) holds. Then problem (2.1) with \( \delta = 0 \) has a principal eigenvalue \( \lambda_1^P \) with a positive eigenfunction pair \( (\phi, \psi)(x) \in [C([-L_1, L_2])]^2 \). Moreover, \( \lambda_1^P \) is an algebraically simple eigenvalue.

Next we present a comparison principle about the generalized principal eigenvalue, which will be used in the sequel.

**Lemma 2.13** (i) Suppose (J) holds and let \( \overline{\lambda}_1^P \) be the generalized principal eigenvalue of (2.1). If there exist two functions \( \tilde{\phi}, \tilde{\psi} \in C([0, \omega] \times [-L_1, L_2]) \) with \( \tilde{\phi}(t, x), \tilde{\psi}(t, x) \geq, \neq 0 \) in \( [0, \omega] \times [-L_1, L_2] \) such that

\[
\begin{align*}
\tilde{\phi}_t - d_1 \mathcal{L}_1[\tilde{\phi}] & \geq a_1 e_1 \tilde{\psi} - b_1 \tilde{\phi} + \lambda_1 \tilde{\phi}, & 0 < t \leq (1 - \delta) \omega, & -L_1 \leq x \leq L_2, \\
\tilde{\psi}_t - d_2 \mathcal{L}_2[\tilde{\psi}] & \geq a_2 e_2 \tilde{\phi} - b_2 \tilde{\psi} + \lambda_1 \tilde{\psi}, & 0 < t \leq (1 - \delta) \omega, & -L_1 \leq x \leq L_2, \\
\tilde{\phi}_t - d_1 \mathcal{L}_1[\tilde{\phi}] & \geq -b_1 \tilde{\phi} + \lambda_1 \tilde{\phi}, & (1 - \delta) \omega < t \leq \omega, & -L_1 \leq x \leq L_2, \\
\tilde{\psi}_t - d_2 \mathcal{L}_2[\tilde{\psi}] & \geq -b_2 \tilde{\psi} + \lambda_1 \tilde{\psi}, & (1 - \delta) \omega < t \leq \omega, & -L_1 \leq x \leq L_2, \\
\tilde{\phi}(0, x) & \geq \tilde{\phi}(\omega, x), & \tilde{\psi}(0, x) & \geq \tilde{\psi}(\omega, x), & -L_1 \leq x \leq L_2 \\
\end{align*}
\]  

(2.32)

for a constant \( \lambda_1 \), then \( \overline{\lambda}_1^P \geq \lambda_1 \) and the equality holds only when \( (\overline{\lambda}_1^P, \tilde{\phi}, \tilde{\psi}) \) is the principal eigen-pair of eigenvalue problem (2.1).

(ii) Suppose (J) holds and let \( \underline{\lambda}_1^P \) be the generalized principal eigenvalue of (2.1). If there exist two functions \( \hat{\phi}, \hat{\psi} \in C([0, \omega] \times [-L_1, L_2]) \) with \( \hat{\phi}(t, x), \hat{\psi}(t, x) \geq, \neq 0 \) in \( [0, \omega] \times [-L_1, L_2] \) such that
\[
\begin{aligned}
\phi_t - d_1 \mathcal{L}_1[\phi] &\leq a_1 e_1 \psi - b_1 \phi + \lambda_2 \phi, & 0 < t \leq (1 - \delta)\omega, -L_1 \leq x \leq L_2, \\
\psi_t - d_2 \mathcal{L}_2[\psi] &\leq a_2 e_2 \phi - b_2 \psi + \lambda_2 \psi, & 0 < t \leq (1 - \delta)\omega, -L_1 \leq x \leq L_2, \\
\phi_t - d_1 \mathcal{L}_1[\phi] &\leq -b_1 \phi + \lambda_2 \phi, & (1 - \delta)\omega < t \leq \omega, -L_1 \leq x \leq L_2, \\
\psi_t &\leq -k \psi + \lambda_2 \psi, & (1 - \delta)\omega < t \leq \omega, -L_1 \leq x \leq L_2, \\
\phi(0, x) &\leq \phi(\omega, x), \quad \psi(0, x) \leq \psi(\omega, x), & -L_1 \leq x \leq L_2 \\
\end{aligned}
\]

(2.33)

for a constant \(\lambda_2\), then \(\tilde{\lambda}_1^P \leq \lambda_2\) and the equality holds only when \((\tilde{\lambda}_1^P, \hat{\phi}, \hat{\psi})\) is the principal eigen-pair of eigenvalue problem (2.1).

**Proof** We only verify the assertion (i), because (ii) can be proved in the same manner. Define

\[
\langle u, v \rangle := \int_{-L_1}^{L_2} u(t, x)v(t, x)dx.
\]

(2.34)

Let \((\phi(t, x), \psi(t, x))\) be positive eigenfunctions corresponding to the generalized principal eigenvalue \(\tilde{\lambda}_1^P\) in (2.1). Set \((\phi_1(t, x) = \phi((1 - \delta)\omega - t, x), \psi_1(t, x) = \psi((1 - \delta)\omega - t, x))\), it is easy to see that

\[
\begin{aligned}
-\phi_{1t} - d_1 \mathcal{L}_1[\phi_1] &\leq a_1 e_1 \psi_1 - b_1 \phi_1 + \tilde{\lambda}_1^P \phi_1, & 0 \leq t < (1 - \delta)\omega, -L_1 \leq x \leq L_2, \\
-\psi_{1t} - d_2 \mathcal{L}_2[\psi_1] &\leq a_2 e_2 \phi_1 - b_2 \psi_1 + \tilde{\lambda}_1^P \psi_1, & 0 \leq t < (1 - \delta)\omega, -L_1 \leq x \leq L_2, \\
-\phi_{1t} - d_1 \mathcal{L}_1[\phi_1] &\leq -b_1 \phi_1 + \tilde{\lambda}_1^P \phi_1, & (1 - \delta)\omega \leq t < \omega, -L_1 \leq x \leq L_2, \\
-\psi_{1t} &\leq -k \psi_1 + \tilde{\lambda}_1^P \psi_1, & (1 - \delta)\omega \leq t < \omega, -L_1 \leq x \leq L_2, \\
\phi_1(0, x) &\equiv \phi_1(\omega, x), \quad \psi_1(0, x) = \psi_1(\omega, x), & -L_1 \leq x \leq L_2.
\end{aligned}
\]

(2.35)

We first restrict \(t \in (0, (1 - \delta)\omega)\), multiplying both sides of the first equation of (2.35) by \(\tilde{\phi}\), and the second equation of that by \(\tilde{\psi}\), then integrating over \([-L_1, L_2]\), yields

\[
\begin{aligned}
-\langle \phi_{1t}, \tilde{\phi} \rangle - d_1 \langle \mathcal{L}_1[\phi_1], \tilde{\phi} \rangle &\leq a_1 e_1 \langle \psi_1, \tilde{\phi} \rangle - b_1 \langle \phi_1, \tilde{\phi} \rangle + \tilde{\lambda}_1^P \langle \phi_1, \tilde{\phi} \rangle, \\
-\langle \psi_{1t}, \tilde{\psi} \rangle - d_2 \langle \mathcal{L}_2[\psi_1], \tilde{\psi} \rangle &\leq a_2 e_2 \langle \phi_1, \tilde{\psi} \rangle - b_2 \langle \psi_1, \tilde{\psi} \rangle + \tilde{\lambda}_1^P \langle \psi_1, \tilde{\psi} \rangle.
\end{aligned}
\]

Since \(\langle \mathcal{L}_1[\phi_1], \tilde{\phi} \rangle = \langle \phi_1, \mathcal{L}_1[\tilde{\phi}] \rangle\) and \(\langle \mathcal{L}_2[\psi_1], \tilde{\psi} \rangle = \langle \psi_1, \mathcal{L}_2[\tilde{\psi}] \rangle\), it follows from (2.32) that

\[
\begin{aligned}
\langle \phi_1, \tilde{\phi}_t - a_1 e_1 \tilde{\psi} - \lambda_1 \tilde{\phi} \rangle &\geq \langle \phi_1, d_1 \mathcal{L}_1[\tilde{\phi}] - b_1 \tilde{\phi} \rangle \\
&\geq -\langle \phi_{1t}, \tilde{\phi} \rangle - a_1 e_1 \langle \psi_1, \tilde{\phi} \rangle - \tilde{\lambda}_1 \langle \phi_1, \tilde{\phi} \rangle, \\
\langle \psi_1, \tilde{\psi}_t - a_2 e_2 \tilde{\phi} - \lambda_1 \tilde{\psi} \rangle &\geq \langle \psi_1, d_2 \mathcal{L}_2[\tilde{\psi}] - b_2 \tilde{\psi} \rangle \\
&\geq -\langle \psi_{1t}, \tilde{\psi} \rangle - a_2 e_2 \langle \phi_1, \tilde{\psi} \rangle - \tilde{\lambda}_1 \langle \psi_1, \tilde{\psi} \rangle,
\end{aligned}
\]

equivalently,
\[(\lambda_1 - \overline{\lambda}_1^p)\langle \phi_1, \tilde{\phi} \rangle \leq a_1 e_1((\psi_1, \tilde{\phi}) - \langle \phi_1, \tilde{\psi} \rangle) + \langle \phi_{1t}, \tilde{\phi} \rangle + \langle \phi_1, \tilde{\phi}_t \rangle, \quad (2.36)\]
\[(\lambda_1 - \overline{\lambda}_1^p)\langle \psi_1, \tilde{\psi} \rangle \leq a_2 e_2((\phi_1, \tilde{\psi}) - \langle \psi_1, \tilde{\phi} \rangle) + \langle \psi_{1t}, \tilde{\psi} \rangle + \langle \psi_1, \tilde{\psi}_t \rangle \quad (2.37)\]
for \(t \in (0, (1 - \delta)\omega)\). For the third equation of (2.35), repeating the above process, we can easily obtain
\[(\lambda_1 - \overline{\lambda}_1^p)\langle \phi_1, \tilde{\phi} \rangle \leq \langle \phi_{1t}, \tilde{\phi} \rangle + \langle \phi_1, \tilde{\phi}_t \rangle \quad (2.38)\]
for \(t \in ((1 - \delta)\omega, \omega)\). For (2.36) and (2.38), integrating over \((0, (1 - \delta)\omega)\) and \(((1 - \delta)\omega, \omega)\) respectively, and adding them together, we obtain
\[
(\lambda_1 - \overline{\lambda}_1^p) \int_0^{\omega} \langle \phi_1, \tilde{\phi} \rangle dt
\leq \int_0^{(1 - \delta)\omega} a_1 e_1((\psi_1, \tilde{\phi}) - \langle \phi_1, \tilde{\psi} \rangle) dt + \int_0^{\omega} \langle \phi_{1t}, \tilde{\phi} \rangle + \langle \phi_1, \tilde{\phi}_t \rangle dt
= \int_0^{(1 - \delta)\omega} a_1 e_1((\psi_1, \tilde{\phi}) - \langle \phi_1, \tilde{\psi} \rangle) dt
+ \int_{-L_1}^{L_2} [\phi_1(\omega, x)\tilde{\phi}(\omega, x) - \phi_1(0, x)\tilde{\phi}(0, x)] dx
\leq \int_0^{(1 - \delta)\omega} a_1 e_1((\psi_1, \tilde{\phi}) - \langle \phi_1, \tilde{\psi} \rangle) dt \quad (2.39)\]
since \(\phi_1(\omega, x) = \phi_1(0, x)\) in (2.35) and \(\tilde{\phi}(\omega, x) \leq \tilde{\phi}(0, x)\) by (2.32). Similarly, multiplying both sides of the forth equation of (2.35) by \(\tilde{\psi}\) and integrating over \([-L_1, L_2]\), yield
\[
-\langle \psi_{1t}, \tilde{\psi} \rangle \leq -k \langle \psi_1, \tilde{\psi} \rangle + \overline{\lambda}_1^p \langle \psi_1, \tilde{\psi} \rangle. \quad (2.40)\]
For (2.37) and (2.40), integrating over \((0, (1 - \delta)\omega)\) and \(((1 - \delta)\omega, \omega)\) respectively, we obtain
\[
(\lambda_1 - \overline{\lambda}_1^p) \int_0^{(1 - \delta)\omega} \langle \psi_1, \tilde{\psi} \rangle dt
\leq \int_0^{(1 - \delta)\omega} a_2 e_2((\phi_1, \tilde{\psi}) - \langle \psi_1, \tilde{\phi} \rangle) dt + \int_0^{(1 - \delta)\omega} \langle \psi_{1t}, \tilde{\psi} \rangle + \langle \psi_1, \tilde{\psi}_t \rangle dt
\leq \int_0^{(1 - \delta)\omega} a_2 e_2((\phi_1, \tilde{\psi}) - \langle \psi_1, \tilde{\phi} \rangle) dt
+ \int_{-L_1}^{L_2} [\tilde{\psi}((1 - \delta)\omega, x)\psi_1((1 - \delta)\omega, x) - \tilde{\psi}(0, x)\psi_1(0, x)] dx,\]
\( \lambda_1 - \lambda_1^P \) \int_{(1-\delta)\omega}^{\omega} \langle \psi_1, \tilde{\psi} \rangle dt \\
\leq \int_{-L_1}^{L_2} [\tilde{\psi}(\omega, x)\psi_1(\omega, x) - \tilde{\psi}((1-\delta)\omega, x)\psi_1((1-\delta)\omega, x)] dx,

and then adding them together yields

\( (\lambda_1 - \lambda_1^P) \int_0^\omega \langle \psi_1, \tilde{\psi} \rangle dt \leq \int_0^{(1-\delta)\omega} a_{2e2}(\langle \phi_1, \tilde{\psi} \rangle - \langle \psi_1, \tilde{\phi} \rangle) dt. \) \hspace{1cm} (2.41)

It follows from (2.39) and (2.41) that

\[ (\lambda_1 - \lambda_1^P) \left[ \frac{1}{a_1 e_1} \int_0^\omega \langle \phi_1, \tilde{\phi} \rangle dt + \frac{1}{a_{2e2}} \int_0^\omega \langle \psi_1, \tilde{\psi} \rangle dt \right] \]
\[ \leq \int_0^{(1-\delta)\omega} (\langle \psi_1, \tilde{\phi} \rangle - \langle \phi_1, \tilde{\psi} \rangle + \langle \phi_1, \tilde{\psi} \rangle - \langle \psi_1, \tilde{\phi} \rangle) dt = 0. \]

Because of \( \langle \phi_1, \tilde{\phi} \rangle > 0 \) and \( \langle \psi_1, \tilde{\psi} \rangle > 0 \), we admit \( \lambda_1^p \geq \lambda_1. \) \hfill \( \square \)

**Lemma 2.14** Assume \((J), (H_2)\) and \(J_1(x) = J_2(x)\) in \(x \in \mathbb{R}\) hold, and let \(\lambda_1^P([-L_1, L_2])\) be the principal eigenvalue of (2.1). Then \(\lambda_1^P([-L_1, L_2])\) is strictly decreasing and continuous in \(L := L_1 + L_2.\)

**Proof** Notice that \(\lambda_1^P([-L_1, L_2])\) is well-defined under the assumptions. It follows from Lemma 2.7 (ii) that \(\lambda_1^P([-L_1, L_2])\) is decreasing in \(L_1 + L_2.\) Moreover, by the definition of the principal eigenvalue, we can see that \(\lambda_1^P([-L_1, L_2])\) is strict decreasing from the proof of Lemma 2.7 (ii).

Next, we consider the continuity of \(\lambda_1^P([-L_1, L_2]).\) It suffices to prove that \(\lambda_1^P([0, L])\) is continuous in \(L\) by Lemma 2.7 (i), that is, for any \(\epsilon > 0,\) there is \(\gamma^* > 0\) such that as long as \(|L_1^* - L| < \gamma^*\),

\[ |\lambda_1^P([0, L_1^*]) - \lambda_1^P([0, L])| < \epsilon. \]

The proof is divided into two cases \(L_1^* \leq L\) and \(L_1^* \geq L\) as done in Du and Ni (2020, Proposition 2.3), we omit the details with some modifications, which characterizes the behaviors in warm and cold seasons. \hfill \( \square \)

Now, we consider the nonlocal free boundary problem (1.9), and denote

\[ \lambda_1^F(t) := \lambda_1^P(\Gamma, [g(t), h(t)], b_i, d_i, \delta) \]
\[ = \lambda_1^D(\Gamma, b_1 - d_1 \lambda_1^*([g(t), h(t)]), b_2 - d_2 \lambda_1^*([g(t), h(t)]), \delta), \]

where \(\lambda_1^*\) is defined in (2.29) with \([-L_1, L_2]\) replaced by \([g(t), h(t)].\)
Lemma 2.15 Assume (J), (H_{2}) and J_{1}(x) = J_{2}(x) in x \in \mathbb{R} hold. Then \lambda_{1}^{F}(t) is nonincreasing for t > 0. Moreover, if 0 \leq \delta < 1, then \lambda_{1}^{F}(t) > \lambda_{1}^{F}(t + \omega) for any t \geq 0.

Proof Recalling that h(t) - g(t) is nondecreasing with respect to t (see Theorem 1.1), we conclude that \lambda_{1}^{F}(t_{1}) \geq \lambda_{1}^{F}(t_{2}) if t_{1} < t_{2} by Lemma 2.14.

Notice that the free boundaries are strictly monotone in the warm season. If 0 \leq \delta < 1, g(t) > g(t + \omega) and h(t) < h(t + \omega) since [t, t + \omega] contains some warm days (\delta = 1, all days are cold), so \lambda_{1}^{F}([g(t), h(t)]) > \lambda_{1}^{F}([g(t + \omega), h(t + \omega)]), that is, \lambda_{1}^{F}(t) > \lambda_{1}^{F}(t + \omega).

Lemma 2.16 Assume (J) and J_{1}(x) = J_{2}(x) in x \in \mathbb{R} hold. The generalized principal eigenvalue of (2.1) satisfied \lambda_{1}^{P}([-L_{1}, L_{2}]) \geq 0 for any L_{1} and L_{2} provided that \lambda_{1}^{O} > 0.

Proof Let (\lambda_{1}^{O}, (\phi(t), \psi(t))) be the generalized principal eigen-pair of eigenvalue problem (2.4) with k_{1} = b_{1} and k_{2} = k, that is

\begin{align}
\phi_{t} \geq a_{1}e_{1}\psi - b_{1}\phi + \lambda_{1}^{O}\phi, & \quad 0 < t \leq 1 - \delta/\omega, \\
\psi_{t} \geq a_{2}e_{2}\phi - b_{2}\psi + \lambda_{1}^{O}\psi, & \quad 0 < t \leq 1 - \delta/\omega, \\
\phi_{t} \geq -b_{1}\phi + \lambda_{1}^{O}\phi, & \quad (1 - \delta/\omega) < t \leq \omega, \\
\psi_{t} \geq -k\phi + \lambda_{1}^{O}\psi, & \quad (1 - \delta/\omega) < t \leq \omega, \\
\phi(0) = \phi(\omega), \psi(0) = \psi(\omega).
\end{align}

We notice that the kernel functions J_{1} and J_{2} satisfy J_{1}(x) = J_{2}(x)(:= J(x)) in \mathbb{R}, problem (2.29) admits a principal eigen-pair (\lambda_{1}^{+}, g(x)) with \lambda_{1}^{+}([-L_{1}, L_{2}]) < 0, g \in C([-L_{1}, L_{2}]) and g(x) > 0 in [-L_{1}, L_{2}]. Let w(t, x) = \phi(t)g(x) and v(t, x) = \psi(t)g(x), then

\begin{align}
w_{t} - d_{1}\xi_{1}[w] & \geq a_{1}e_{1}v - b_{1}w + (d_{1}\lambda_{1}^{+} + \lambda_{1}^{O})w, & \quad 0 < t \leq 1 - \delta/\omega, -L_{1} \leq x \leq L_{2}, \\
v_{t} - d_{2}\xi_{2}[v] & \geq a_{2}e_{2}w - b_{2}v + (d_{2}\lambda_{1}^{+} + \lambda_{1}^{O})w, & \quad 0 < t \leq 1 - \delta/\omega, -L_{1} \leq x \leq L_{2}, \\
w_{t} - d_{1}\xi_{1}[w] & \geq -b_{1}w + (d_{1}\lambda_{1}^{+} + \lambda_{1}^{O})w, & \quad (1 - \delta/\omega) < t \leq \omega, -L_{1} \leq x \leq L_{2}, \\
v_{t} \geq -k\phi + \lambda_{1}^{O}\psi, & \quad (1 - \delta/\omega) < t \leq \omega, -L_{1} \leq x \leq L_{2}, \\
w(0, x) = w(\omega, x), v(0, x) = v(\omega, x).
\end{align}

It follows from Lemma 2.13(i) that \lambda_{1}^{+} \geq \lambda_{1}^{O} since \lambda_{1}^{+} < 0.

Lemma 2.17 Assume (J) and J_{1}(x) = J_{2}(x) in x \in \mathbb{R} hold. If 0 \leq \delta < 1, then the principal eigenvalue \lambda_{1}^{P}([-L_{1}, L_{2}]) = -L_{1} < L_{2} of (2.1) has the following properties:

(i) If \lambda_{1}^{O} \geq 0, then we have \lambda_{1}^{P}([-L_{1}, L_{2}]) > 0 for any L_{1} and L_{2}.
(ii) If \lambda_{1}^{O} < 0 and \lambda_{1}^{P}([-h_{0}, h_{0}]) \leq 0, then \lambda_{1}^{P}([-L_{1}, L_{2}]) < 0 for any L_{1} and L_{2}.
(iii) If \lambda_{1}^{O} < 0 and \lambda_{1}^{P}([-h_{0}, h_{0}]) > 0, then there exists t^{*} > 0 such that \lambda_{1}^{P}([g(t^{*}), h(t^{*})]) = 0 and (t - t^{*})\lambda_{1}^{P}([g(t), h(t)]) < 0 for t \in (0, t^{*} - \omega) \cup (t^{*} + \omega, \infty).
In this section, we first present some known results, such as maximum principle for the corresponding fixed boundary problem with seasonal succession (Lemmas 3.3 and 3.5). If \( \eta \) is a constant function, then

\[
\begin{align*}
\sum^\eta_{\min} &= \{ t \in (0, +\infty) : \exists \epsilon > 0 \text{ such that } \eta(t) < \eta(s) \text{ for } s \in [t - \epsilon, t) \} \quad \text{and} \\
\sum^\eta_{\max} &= \{ t \in (0, +\infty) : \exists \epsilon > 0 \text{ such that } \eta(t) > \eta(s) \text{ for } s \in [t - \epsilon, t) \}.
\end{align*}
\]

If \( \eta \) is strictly decreasing, then \( \sum^\eta_{\min} = (0, +\infty) \), and if \( \eta \) is nondecreasing, then \( \sum^\eta_{\min} = \emptyset \). Particularly, if \( \eta \) is a constant function, then \( \sum^\eta_{\min} = \sum^\eta_{\max} = \emptyset \).
Lemma 3.1 (Maximum principle (Du and Ni 2020, Lemma 3.1)) Let $h_0 > 0$, $g, h \in C([0, +\infty))$ satisfy $g(t) < h(t)$ and $-g(0) = h(0) = h_0$. Suppose that $u_i, u_{ii} \in C(\Omega^{g,h})$, $d_i, c_{ij} \in L^\infty(\Omega^{g,h})$, $d_i \geq 0$ ($i, j = 1, 2, \ldots, n$) and

$$
\begin{align*}
\left\{ \begin{array}{l}
  u_{ii} \geq d_i L_i [u_i] + \sum_{j=1}^{n} c_{ij} u_j & t > 0, \quad g(t) < x < h(t), \\
  u_i(t, g(t)) \geq 0, & t \in \Sigma_{\min}^g, \\
  u_i(t, h(t)) \geq 0, & t \in \Sigma_{\max}^h, \\
  u_i(0, x) \geq 0, & x \in [-h_0, h_0], 
\end{array} \right.
\end{align*}
$$

where $L_i$ is given by (1.7) with every $J_i (i = 1, \ldots, n)$ satisfying (J). Then the following conclusions hold:

(i) If $c_{ij} \geq 0$ ($i \neq j$) for $i, j = 1, \ldots, n$, then $u_i \geq 0$ in $\Omega^{g,h}$.
(ii) If in addition $d_{i0} > 0$ in $\Omega^{g,h}$, and $u_{i0}(0, x) \neq 0$ in $[-h_0, h_0]$, then $u_{i0} > 0$ in $\Omega^{g,h}$.

Applying the maximum principle, we obtain the comparison principle of the problem with seasonal succession, which will be used in the sequel.

Lemma 3.2 Assume (J) holds, and $g \in G_{h_0, \omega}, h \in H_{h_0, \omega}$ for some $\omega$, $h_0 > 0$. Suppose that $u_i, \tilde{u}_i \in C(\Omega^{g,h})$ satisfy

(i) $u_i(\cdot, x), \tilde{u}_i(\cdot, x) \in C^1_{\omega}(0, +\infty), \forall x \in (g(t), h(t))$.
(ii) $0 < u_i \leq e_i, 0 < \tilde{u}_i \leq e_i$.
(iii) $u_i, \tilde{u}_i$ satisfies

$$
\begin{align*}
\left\{ \begin{array}{l}
  \tilde{u}_{1i} \geq d_1 L_1[\tilde{u}_i] + a_1(e_1 - \tilde{u}_i)\tilde{u}_2 - b_1 \tilde{u}_1, & (t, x) \in (m\omega, m\omega + (1 - \delta)\omega] \times (g(t), h(t)), \\
  \tilde{u}_{2i} \geq d_2 L_2[\tilde{u}_2] + a_2(e_2 - \tilde{u}_2)\tilde{u}_1 - b_2 \tilde{u}_2, & (t, x) \in (m\omega, m\omega + (1 - \delta)\omega] \times (g(t), h(t)), \\
  \tilde{u}_1 \geq d_1 L_1[\tilde{u}_i] - b_1 \tilde{u}_1, & (t, x) \in (m\omega + (1 - \delta)\omega, (m + 1)\omega] \times (g(t), h(t)), \\
  \tilde{u}_2 \geq -k\tilde{u}_2, & (t, x) \in (m\omega + (1 - \delta)\omega, (m + 1)\omega] \times (g(t), h(t)).
\end{array} \right.
\end{align*}
$$

(3.1)

(iv) $(u_1, u_2)$ satisfies (3.1) but with the inequalities reversed.
(v) At the boundary,

$$
\begin{align*}
\left\{ \begin{array}{l}
  u_i(t, g(t)) \leq \tilde{u}_i(t, g(t)), & t \in \Sigma_{\min}^g, \\
  u_i(t, h(t)) \leq \tilde{u}_i(t, h(t)), & t \in \Sigma_{\max}^h.
\end{array} \right.
\end{align*}
$$

(vi) At the initial time, $u_i(0, x) \leq \tilde{u}_i(0, x)$ for $x \in [-h_0, h_0]$.

Then we have

$$
u_i(t, x) \leq \tilde{u}_i(t, x) \text{ for } (t, x) \in \Omega^{g,h}, i = 1, 2.$$

\[ \text{Springer} \]
Proof Define

\[ H := \tilde{u}_1 - u_1, \quad V := \tilde{u}_2 - u_2, \]

and

\[
\begin{align*}
  c_{11} & := -(b_1 + a_1 u_2), \quad c_{12} := a_1 (e_1 - \tilde{u}_1) > 0, \\
  c_{21} & := a_2 (e_2 - \tilde{u}_2) > 0, \quad c_{22} := -(b_2 + a_2 u_1)
\end{align*}
\]

for \( t \in (m\omega, m\omega + (1 - \delta)\omega] \),

\[
C_{11} := -b_1, \quad C_{12} := 0, \quad C_{21} := 0, \quad C_{22} := -k, \quad D_1 := d_1, \quad D_2 := 0
\]

for \( t \in (m\omega + (1 - \delta)\omega, (m + 1)\omega] \). It follows from Lemma 3.1 that \( H \geq 0 \) and \( V \geq 0 \) in \( \Omega^{g,h} \).

For subsequent application, we consider the corresponding fixed boundary problem with seasonal succession of (1.9):

\[
\begin{align*}
  u_{1t} &= d_1 \int_{-L_1}^{L_1} J_1(x-y)u_1(t, y)dy - d_1 u_1 + a_1 (e_1 - u_1)u_2 - b_1 u_1, \quad (t, x) \in Q^m_w, \\
  u_{2t} &= d_2 \int_{-L_1}^{L_1} J_1(x-y)u_2(t, y)dy - d_2 u_2 + a_2 (e_2 - u_2)u_1 - b_2 u_2, \quad (t, x) \in Q^m_w, \\
  u_{1t} &= d_1 \int_{-L_1}^{L_1} J_1(x-y)u_1(t, y)dy - d_1 u_1 - b_1 u_1, \quad (t, x) \in Q^m_c, \\
  u_{2t} &= -ku_2, \quad (t, x) \in Q^m_c, \\
  u_i(0, x) &= u_{i,0}(x), \quad i = 1, 2,
\end{align*}
\]

(3.2)

where \( Q^m_w := (m\omega, m\omega + (1 - \delta)\omega] \times (-L_1, L_2) \), \( Q^m_c := (m\omega + (1 - \delta)\omega, (m + 1)\omega] \times (-L_1, L_2) \), \( m = 0, 1, \cdots, u_{i,0}(x) \in C([-L_1, L_2]) \setminus \{0\} \) and \( 0 \leq u_{i,0}(x) \leq e_i (i = 1, 2) \). It is well-known that (3.2) has a unique positive solution which is defined for all \( t > 0 \).

Definition 3.1 A pair of functions \((\tilde{u}_1, \tilde{u}_2)(t, x) \in [C^{1,0}((0, \infty) \times [-L_1, L_2])]^2 \) is called an upper solution to problem (3.2) if

\[
\begin{align*}
  \tilde{u}_{1t} & \geq d_1 \int_{-L_1}^{L_1} J_1(x-y)\tilde{u}_1(t, y)dy - d_1 \tilde{u}_1 + a_1 (e_1 - \tilde{u}_1)\tilde{u}_2 - b_1 \tilde{u}_1, \quad (t, x) \in Q^m_w, \\
  \tilde{u}_{2t} & \geq d_2 \int_{-L_1}^{L_1} J_1(x-y)\tilde{u}_2(t, y)dy - d_2 \tilde{u}_2 + a_2 (e_2 - \tilde{u}_2)\tilde{u}_1 - b_2 \tilde{u}_2, \quad (t, x) \in Q^m_w, \\
  \tilde{u}_{1t} & \geq d_1 \int_{-L_1}^{L_1} J_1(x-y)\tilde{u}_1(t, y)dy - d_1 \tilde{u}_1 - b_1 \tilde{u}_1, \quad (t, x) \in Q^m_c, \\
  \tilde{u}_{2t} & \geq -k\tilde{u}_2, \quad (t, x) \in Q^m_c, \\
  \tilde{u}_i(0, x) & \geq u_{i,0}(x), \quad i = 1, 2, \\
  -L_1 & \leq x \leq L_2.
\end{align*}
\]

Similarly, we call \((\hat{u}_1, \hat{u}_2)(t, x) \) a lower solution of (3.2) if all reversed inequalities in the above places are satisfied.
To understand the asymptotic behavior of solution to the initial value problem (3.2), we now consider the corresponding periodic problem of (3.2)

\[
\begin{aligned}
U_{tt} &= d_1 \int_{-L_1}^{L_1} J_1(x-y)U_t(t, y) dy - d_1 U_t + a_1(e_1 - U_1)U_2 - b_1 U_1, \quad (t, x) \in Q_0^w, \\
U_{2t} &= d_2 \int_{-L_1}^{L_1} J_2(x-y)U_2(t, y) dy - d_2 U_2 + a_2(e_2 - U_2)U_1 - b_2 U_2, \quad (t, x) \in Q_0^w, \\
U_{tt} &= d_1 \int_{-L_1}^{L_1} J_1(x-y)U_t(t, y) dy - d_1 U_t - b_1 U_1, \quad (t, x) \in Q_0^c, \\
U_2t &= -kU_2, \\
U_t(0, x) &= U_t(\omega, x), \quad i = 1, 2,
\end{aligned}
\]

\[(3.3)\]

**Lemma 3.3** Suppose (J) holds, let \((u_1(t, x), u_2(t, x))\) be the unique positive solution of (3.2), \(\lambda^+_1([−L_1, L_2])\) and \(\lambda^-_1([−L_1, L_2])\) are the generalized principal eigenvalue of (2.1). The following conclusions hold.

(i) If \(\lambda^+_1([−L_1, L_2]) \geq 0\), then problem (3.3) admits the unique nonnegative solution \((0, 0)\), and \(\lim_{t \to \infty}(u_1(t, x), u_2(t, x)) = (0, 0)\) uniformly for \(x \in [−L_1, L_2]\).

(ii) If \(\lambda^-_1([−L_1, L_2]) < 0\), then problem (3.3) admits a unique positive periodic solution \((U_1^*(t, x), U_2^*(t, x))\), and \(0 < U_i^* \leq e_i (i = 1, 2)\). Moreover, \(\lim_{n \to \infty}(u_1(t + n\omega, x), u_2(t + n\omega, x)) = (U_1^*(t, x), U_2^*(t, x))\) uniformly for \(x \in [−L_1, L_2]\).

**Proof** Suppose \(\lambda^-_1([−L_1, L_2]) < 0\), we first consider the existence of positive periodic solution \((U_1^*(t, x), U_2^*(t, x))\) to problem (3.3). Let \((\phi, \psi)\) be a positive eigenfunction pair with respect to \(\lambda^-_1([−L_1, L_2])\), we are in a position to check that for a small \(\varepsilon > 0\), \((\varepsilon \phi, \varepsilon \psi)\) and \((e_1, e_2)\) are the lower and upper solutions of (3.3).

Let us denote by \((\overline{U}_1, \overline{U}_2)(t, x)\) the unique positive solution of (3.2) with the initial function pair \((e_1, e_2)\), our next goal is to prove that \((\overline{U}_1, \overline{U}_2)\) is nonincreasing in \(t\). In fact, considering \((e_1, e_2)\) as an upper solution of (3.2), we apply Lemma 3.2, where \((g(t), h(t)) \equiv (−L_1, L_2)\), to assert that

\[\overline{U}_i(t, x) \leq e_i, \quad \text{for} \ (t, x) \in Q_0^c, \ i = 1, 2.\]

It follows from the definition of \((\overline{U}_1, \overline{U}_2)\) and Lemma 3.2 that, for any \(t_1 > 0\),

\[\overline{U}_i(t + t_1, x) \leq \overline{U}_i(t, x) \quad \text{in} \ Q_0^c,\]

which gives the monotonicity in \(t\). Meanwhile, by regarding \((\varepsilon \phi, \varepsilon \psi)\) as a lower solution of (3.2), we can deduce that

\[\overline{U}_1(t, x) \geq \varepsilon \phi \quad \text{and} \quad \overline{U}_2(t, x) \geq \varepsilon \psi \ \text{in} \ Q_0^c.\]

As a consequence, the limitation of \((\overline{U}_1, \overline{U}_2)(t, x)\) exists, denoted by \((U_1^*, U_2^*)(t, x)\), which is a positive periodic solution of (3.3) by the dominated convergence theorem. Furthermore, for any positive periodic solution \((U_1, U_2)(t, x)\) to (3.3) satisfying \(U_i(t, x) \leq e_i\), Lemma 3.2 gives that \(\overline{U}_i(t, x) \geq U_i(t, x)\) for \((t, x) \in Q_0^c\), that is to say, \((U_1^*, U_2^*)(t, x)\) is the maximal positive solution of (3.3).
Next, we proceed to show the uniqueness of the positive periodic solution of (3.3). Assume \((U_{11}, U_{21})\) and \((U_{12}, U_{22})\) are two positive periodic solutions, there is no loss of generality in assuming \((U_{12}, U_{22}) = (U^*_1, U^*_2)\), we have

\[
(0, 0) < (U_{11}, U_{12}) \leq, \neq (U_{12}, U_{22}),
\]

and it follows from (3.3) that

\[
\begin{align*}
\langle U_{11r} - d_1 \xi_1[U_{11}], U_{12} \rangle &= \langle a_1(e_1 - U_{11})U_{21} - b_1 U_{11}, U_{12} \rangle, \\
\langle U_{21r} - d_2 \xi_2[U_{21}], U_{22} \rangle &= \langle a_2(e_2 - U_{21})U_{11} - b_2 U_{21}, U_{22} \rangle, \\
\langle U_{12r} - d_1 \xi_1[U_{12}], U_{11} \rangle &= \langle a_1(e_1 - U_{12})U_{22} - b_1 U_{12}, U_{11} \rangle, \\
\langle U_{22r} - d_2 \xi_2[U_{22}], U_{21} \rangle &= \langle a_2(e_2 - U_{22})U_{12} - b_2 U_{22}, U_{21} \rangle
\end{align*}
\]

hold in warm season \((t \in (0, (1 - \delta)\omega])\), where \(\langle \cdot, \cdot \rangle\) is defined in (2.34) and

\[
\begin{align*}
\langle U_{11r} - d_1 \xi_1[U_{11}], U_{12} \rangle &= \langle -b_1 U_{11}, U_{12} \rangle, \\
\langle U_{21r}, U_{22} \rangle &= \langle -k U_{21}, U_{22} \rangle, \\
\langle U_{12r} - d_1 \xi_1[U_{12}], U_{11} \rangle &= \langle -b_1 U_{12}, U_{11} \rangle, \\
\langle U_{22r}, U_{21} \rangle &= \langle -k U_{22}, U_{21} \rangle
\end{align*}
\]

for \(t \in ((1 - \delta)\omega, \omega]\). Using \(\langle \xi_1[U_{11}], U_{12} \rangle = \langle \xi_1[U_{12}], U_{11} \rangle (i = 1, 2)\) and transferring items yield that

\[
\begin{align*}
\langle U_{11r}, U_{12} \rangle + a_1 \langle (e_1 - U_{12})U_{22}, U_{11} \rangle &= \langle U_{12r}, U_{11} \rangle + a_1 \langle (e_1 - U_{11})U_{21}, U_{12} \rangle, \\
\langle U_{21r}, U_{22} \rangle + a_2 \langle (e_2 - U_{22})U_{12}, U_{21} \rangle &= \langle U_{22r}, U_{21} \rangle + a_2 \langle (e_2 - U_{21})U_{11}, U_{22} \rangle
\end{align*}
\]

(3.4)

for \(t \in (0, (1 - \delta)\omega]\), and

\[
\begin{align*}
\langle U_{11r}, U_{12} \rangle &= \langle U_{12r}, U_{11} \rangle, \\
\langle U_{21r}, U_{22} \rangle &= \langle U_{22r}, U_{21} \rangle
\end{align*}
\]

(3.5)

for \(t \in ((1 - \delta)\omega, \omega]\). Integrating the first equation of (3.4) over \((0, (1 - \delta)\omega]\), the fist equation of (3.5) over \(((1 - \delta)\omega, \omega]\), respectively, then adding them together give

\[
a_1 \int_0^{(1-\delta)\omega} e_1 \langle U_{21}, U_{12} \rangle - \langle U_{22}, U_{11} \rangle dt - \langle U_{11} U_{21}, U_{12} \rangle + \langle U_{12} U_{22}, U_{11} \rangle dt = 0
\]

since \(U_{11}(\omega, x) = U_{11}(0, x)\) and \(U_{12}(\omega, x) = U_{12}(0, x)\) in (3.3). Therefore,

\[
\int_0^{(1-\delta)\omega} [\langle U_{21}, U_{12} \rangle - \langle U_{22}, U_{11} \rangle] dt = \int_0^{(1-\delta)\omega} \frac{1}{e_1} \langle U_{11} U_{21}, U_{21} - U_{22} \rangle dt.
\]

(3.6)
The same procedure for the second equation of (3.4) and that of (3.5) may be easily adapted to obtain

\[ \int_0^{(1-\delta)\omega} [(U_{11}, U_{22}) - (U_{12}, U_{21})]dt = \int_0^{(1-\delta)\omega} \frac{1}{e_2} (U_{21}U_{22}, U_{11} - U_{12})dt. \]  

(3.7)

Adding (3.6) and (3.7) yields

\[ 0 = \int_0^{(1-\delta)\omega} \left[ \frac{1}{e_1} (U_{11}U_{12}, U_{21} - U_{22}) + \frac{1}{e_2} (U_{21}U_{22}, U_{11} - U_{12}) \right] dt < 0 \]

since that \((0, 0) \not\in (U_{11}, U_{12}), \not\equiv (U_{12}, U_{22})\), we then lead to a contradiction. This finishes the proof of the uniqueness of the positive periodic solution.

We notice that \(J^L_{-L} f(x - y) U^*_i(t, y)dy := G_i(i = 1, 2)\) are continuous in \([0, \omega] \times [-L_1, L_2]\), it follows from (3.3) that \(U^*_1\) and \(U^*_2\) can be expressed by a quadratic formula involving \(G_i\) and the constant parameters. Therefore both \(U^*_1\) and \(U^*_2\) are continuous functions in \([0, \omega] \times [-L_1, L_2]\).

Finally, we claim that (3.3) has no positive periodic solution if \(\lambda^P_1([-L_1, L_2]) \geq 0\). Suppose, contrary to our claim, that \((\tilde{U}_1, \tilde{U}_2)\) is a positive periodic solution to problem (3.3) and satisfies

\[
\begin{align*}
\tilde{U}_1(t) - d_1 \xi_1[\tilde{U}_1] &= a_1(e_1 - \tilde{U}_1)\tilde{U}_2 - b_1\tilde{U}_1 < a_1 e_1 \tilde{U}_2 - b_1 \tilde{U}_1, & (t, x) \in Q^0_w, \\
\tilde{U}_2(t) - d_2 \xi_2[\tilde{U}_2] &= a_2(e_2 - \tilde{U}_2)\tilde{U}_1 - b_2\tilde{U}_2 < a_2 e_2 \tilde{U}_1 - b_2 \tilde{U}_2, & (t, x) \in Q^0_v, \\
\tilde{U}_1(t) - d_1 \xi_1[\tilde{U}_1] &= -b_1\tilde{U}_1, & (t, x) \in Q^0_w, \\
\tilde{U}_2(t) &= -k\tilde{U}_2, & (t, x) \in Q^0_v, \\
\tilde{U}_i(0, x) &= \tilde{U}_i(\omega, x), i = 1, 2, & -L_1 \leq x \leq L_2.
\end{align*}
\]  

(3.8)

It follows from the definition of the generalized principal eigenvalue that \(\lambda^P_1([-L_1, L_2]) \leq 0\). If \(\lambda^P_1([-L_1, L_2]) \not\equiv \lambda^P_1([-L_1, L_2])\), then we have \(\lambda^P_1([-L_1, L_2]) < 0\), which is in contradiction with the assumption \(\lambda^P_1([-L_1, L_2]) \geq 0\). If \(\lambda^P_1([-L_1, L_2]) = \lambda^P_1([-L_1, L_2])\), then \(\lambda^P_1([-L_1, L_2])\) is well-defined, and \(\lambda^P_1([-L_1, L_2]) < 0\) by applying Lemma 2.13 to the system (3.8), we also lead to a contradiction.

Through the above discussion, we conclude that problem (3.3) has a unique positive periodic solution if \(\lambda^P_1([-L_1, L_2]) < 0\), while when \(\lambda^P_1([-L_1, L_2]) \geq 0\), \((0, 0)\) is the unique nonnegative periodic solution. Next we prove the stabilities in two cases.

For (i), since \(\lambda^P_1([-L_1, L_2]) \geq 0\), we have \((U^*_1, U^*_2) = (0, 0)\). And then \(\lim_{t \to \infty} (\bar{U}_1, \bar{U}_2) = (0, 0)\) uniformly in \([-L_1, L_2]\), which is due to Dini’s theorem.

We conclude from Lemma 3.2 that \(0 < u_i(t, x) \leq \bar{U}_i(t, x)\) for \((t, x) \in Q^0_w \cup Q^0_v\), hence that \((u_1, u_2) \to (0, 0)\) as \(t \to \infty\).

For (ii), if \(\lambda^P_1([-L_1, L_2]) < 0\), problem (3.3) has a unique positive periodic solution \((U^*_1, U^*_2)\). According to the fact that \((\bar{U}_1, \bar{U}_2)\) is nonincreasing in \(t\) and \(\bar{U}_i\)
is bounded for \((t, x) \in Q_w^0 \cup Q_c^0\). Hence \(\lim_{n \to \infty} (\overline{U}_1, \overline{U}_2)(t + n\omega, x) = (U_1^*, U_2^*)(t, x)\) uniformly for \(x \in [-L_1, L_2]\) by Dini’s theorem.

To find a positive lower solution, now we reset a initial time so that the initial value is positive. Note that \(U_i(1, x) > 0\) for \(x \in [-L, L]\), for the small enough \(\varepsilon > 0\), \(U_i(1, x) \geq \varepsilon \phi, \ U_2(1, x) \geq \varepsilon \psi \in [-L_1, L_2]\). Let \((\underline{U}_1, \underline{U}_2)\) be the solution of (3.2) with initial function pair \((\varepsilon \phi, \varepsilon \psi)\). Then \(\overline{U}_i\) are nondecreasing in \(t\) and by Lemma 3.2, we admit \(\overline{U}_i(t, x) \leq U_i(1 + t, x) \leq \overline{U}_i(1 + t, x)\) in \(Q_w^0 \cup Q_c^0\). Apply similar above arguments to this case, we obtain that \(\lim_{n \to \infty} (\overline{U}_1, \overline{U}_2)(t + n\omega, x) = (U_1^*, U_2^*)(t, x)\) uniformly for \(x \in [-L_1, L_2]\). \(\square\)

**Remark 3.4** It follows from Corollary 2.11 that \(\overline{\lambda}_1^P(\delta)\) and \(\underline{\lambda}_1^P(\delta)\) are strictly increasing in \(\delta\). That is, if \(\delta_1 < \delta_2\), then \(\overline{\lambda}_1^P(\delta_1) < \overline{\lambda}_1^P(\delta_2)\) and \(\underline{\lambda}_1^P(\delta_1) < \underline{\lambda}_1^P(\delta_2)\). Recalling that \(\overline{\lambda}_1^P(1) \geq \underline{\lambda}_1^P(1) > 0\) by Theorem 2.8 (ii), so there exist a \(\delta^*_2\) such that \(\overline{\lambda}_1^P(\delta_2) \geq 0\) for \(\delta \geq \delta^*_2\), which gives

\[ \lim_{t \to \infty} (u_1(t, x), u_2(t, x)) = (0, 0) \]

by Lemma 3.3. That is to say, the bigger \(\delta\) is, the more likely \((u_1, u_2)\) is to become extinct. On the other hand, for small enough \(\delta_1\), we have \(\overline{\lambda}_1^P(\delta_1) < 0\), it follows from Lemma 3.3 that

\[ \lim_{n \to \infty} (u_1(t + n\omega, x), u_2(t + n\omega, x)) = (U_1^*(t, x), U_2^*(t, x)), \]

which implies that the smaller \(\delta\) is, the more likely \((u_1, u_2)\) converges to a positive periodic solution \((U_1^*, U_2^*)\).

For problem (3.3), we consider the corresponding spatial-independent problem

\[
\begin{align*}
U_{1t} &= a_1(e_1 - U_1)U_2 - b_1 U_1, \quad t \in (0, (1 - \delta)\omega], \\
U_{2t} &= a_2(e_2 - U_2)U_1 - b_2 U_2, \quad t \in (0, (1 - \delta)\omega], \\
U_{1t} &= -b_1 U_1, \quad t \in ((1 - \delta)\omega, \omega], \\
U_{2t} &= -k U_2, \quad t \in ((1 - \delta)\omega, \omega], \\
U_i(0) &= U_i(\omega)(i = 1, 2) \tag{3.9}
\end{align*}
\]

Similarly to the proof of Lemma 3.3, we can easily obtain the existence and uniqueness of the positive periodic solution to problem (3.9).

**Lemma 3.5** Suppose (J) and \(J_1(x) = J_2(x)\) hold. If \(0 \leq \delta < 1\) and \(\overline{\lambda}_1^Q < 0\), then

\[ \lim_{L_1, L_2 \to +\infty} (U_1^*[\cdot, [-L_1, L_2]], U_2^*[\cdot, [-L_1, L_2]])(t, x) = (U_1^\Delta, U_2^\Delta)(t) \]

uniformly for \(t \in [0, \omega]\) and locally uniformly for \(x \in \mathbb{R}\), where \((U_1^*[\cdot, [-L_1, L_2]], U_2^*[\cdot, [-L_1, L_2]])(t, x)\) and \((U_1^\Delta, U_2^\Delta)(t)\) are the unique positive periodic solution of problem (3.3) and (3.9), respectively.
Proof If $0 \leq \delta < 1$ and $\lambda_1^O < 0$, it follows from Lemma 2.17 that there exists $T_1$ large enough such that $\lambda_1^O \left( \langle g(T_1), h(T_1) \rangle \right) < 0$. Let $L_0 = \max \{-g(T_1), h(T_1)\}$, using the monotonicity in Lemma 2.14 and the fact $\lambda_1^P \left( \langle g(T_1), h(T_1) \rangle \right) < 0$ gives that $\lambda_1^P \left( [-L_0, L_0] \right) < 0$, which means that problem (3.3) has a unique positive periodic solution $(U_{1,i}^*, [-L, L], U_{2,i}^*, [-L, L])$ for all $L \geq L_0$. To derive the desired result, the proof falls naturally into three parts.

Part 1. Let us first prove that for $0 \leq L$, $U_{1,i}^*$ and $U_{2,i}^*$ are nondecreasing in $L$.

Suppose $L_0 \leq L_1^* \leq L_2^*$, let $(U_{1,1}^*, L_1^*)$, $(U_{2,1}^*, L_2^*)$ be the positive solution of (3.2) with $L_1 = L_2 = L_j^*$ and initial function pair $(U_{1,j}^*(0, x), U_{2,j}^*(0, x))$ $(j = 1, 2)$, and $0 < U_{i,1}^*(0, x) \leq U_{i,2}^*(0, x) \leq e_i (i = 1, 2)$. Thanks to

$$\int_{-L_1^*}^{L_2^*} J_1(x - y)U_{i,1}^*(t, y)dy \geq \int_{-L_1^*}^{L_2^*} J_1(x - y)U_{i,2}^*(t, y)dy,$$

then $(U_{1,1}^*, L_1^*)$, $(U_{2,2}^*, L_2^*)$ is an upper solution over the restriction of $[0, \omega] \times [-L_1^*, L_1^*]$, as a result, $U_{i,1}^*, L_1^* \geq U_{i,2}^*, L_2^*$ in $[0, \omega] \times [-L_1^*, L_1^*]$ by applying comparison principle. Because of $\lambda_1^P < 0$, it follows from Lemma 3.3 (ii) that

$$U_{1,1}^*, L_1^* \leq U_{1,2}^*, L_2^*, U_{2,1}^*, L_1^* \leq U_{2,2}^*, L_2^*$$

for $(t, x) \in (0, \infty) \times [-L_1^*, L_1^*]$. The monotone property implies that the pointwise limit exists and

$$\lim_{L \rightarrow +\infty} (U_{1,1}^*, U_{1,2}^*, L_1^*) = (U_{1}^*, U_{2}^*, L_1^*).$$

Notice that $[-L^M, L^m] \subseteq [-L_1^*, L_2^*] \subseteq [-L^M, L^M]$ with $L^m = \min\{L_1, L_2\}$ and $L^M = \max\{L_1, L_2\}$, we have

$$\lim_{L_1, L_2 \rightarrow +\infty} (U_{1,1}^*, U_{1,2}^*, L_1, L_2) = (U_{1}^*, U_{2}^*, L_1, L_2)$$

by using the monotonicity of positive periodic solution with respect to the interval. It is easily seen that $0 < U_{i}^*(t, x) \leq e_i$ and $(U_{1}^*, U_{2}^*)$ is a positive solution of (3.3) with $(-L_1, L_2)$ replaced by $(-\infty, +\infty)$ by the dominated convergence theorem.

Part 2. We next prove that $U_{i}^*(t, x)$ is independent of $x$. It suffices to show that

$$(U_{1}^*, U_{2}^*)(t, x_1) = (U_{1}^*, U_{2}^*)(t, 0)$$

for any given $x_1 \in \mathbb{R}$. (3.10)

Denote $L^* := |x_1|$. For $L \geq L_0 + 2L^*$, since

$$[-L + 2L^*, L - 2L^*] \subset [-L + L^* - x_1, L - L^* - x_1] \subset [-L, L],$$

it follows from Part 1 that

$$U_{i,1}^*([-L - 2L^*, L - 2L^*], t, x) \leq U_{i,2}^*([-L - L^*, L - L^*], t, x + x_1) \leq U_{i,1}^*([-L, L], t, x) \quad (i = 1, 2).$$
Letting $L \to \infty$, by the definition of $(U_1^*, U_2^*)$, yields

$$(U_1^*, U_2^*)(t, x) = (U_1^*, U_2^*)(t, x + x_1).$$

Take $x = 0$, (3.10) is proved.

**Part 3.** Finally, since that $(U_1^{\Delta}, U_2^{\Delta})$ is the unique positive periodic solution to problem (3.9), the conclusion of Part 2 implies $(U_1^*, U_2^*) = (U_1^{\Delta}, U_2^{\Delta})$, i.e.

$$\lim_{L_1, L_2 \to +\infty} (U_{1,[-L_1, L_2]}^*, U_{2,[-L_1, L_2]}^*)(t, x) = (U_1^{\Delta}, U_2^{\Delta})(t).$$

The convergence is locally uniform in $\mathbb{R}$ by Dini’s theorem.

\[\square\]

### 4 Main results

In this section, the main theorems will be proved. We always assume (J) holds, the initial function pair $(u_{1,0}(x), u_{2,0}(x))$ satisfies (1.10), and $(u_1(t, x), u_2(t, x); g(t), h(t))$ is the unique positive solution of (1.9). Denote

$$g_\infty = \lim_{t \to \infty} g(t) \quad \text{and} \quad h_\infty = \lim_{t \to \infty} h(t). \quad (4.1)$$

**Lemma 4.1** (Comparison principle). Suppose (J) holds and $(u_1(t, x), u_2(t, x); g(t), h(t))$ is the solution of (1.9) with the initial functions satisfy (1.10). Let $\overline{g}, \overline{h} \in C([0, \infty)) \cap C^1((m\omega, m\omega + (1 - \delta)\omega])$. If $(\overline{u}_1, \overline{u}_2) \in [C(\Omega \overline{g}, \overline{h})]^2$ satisfies

$$\begin{align*}
\overline{u}_1 &\geq d_1 \int_{\overline{t}(t)}^{\infty} J_1(x - y)\overline{u}_1(t, y)dy - d_1\overline{u}_1(t, x) + a_{11}(\overline{t})\overline{u}_1 - b_1\overline{u}_1, & t \in (m\omega, m\omega + (1 - \delta)\omega], x \in (\overline{g}(t), \overline{h}(t)), \\
\overline{u}_2 &\geq d_2 \int_{\overline{t}(t)}^{\infty} J_2(x - y)\overline{u}_2(t, y)dy - d_2\overline{u}_2(t, x) + a_{22}(\overline{t})\overline{u}_2 - b_2\overline{u}_2, & t \in (m\omega, m\omega + (1 - \delta)\omega], x \in (\overline{g}(t), \overline{h}(t)), \\
\overline{u}_1 &\geq d_1 \int_{\overline{t}(t)}^{\infty} J_1(x - y)\overline{u}_1(t, y)dy - d_1\overline{u}_1(t, x) - a_1\overline{u}_1 - b_1\overline{u}_1, & t \in (m\omega + (1 - \delta)\omega, (m + 1)\omega], x \in (\overline{g}(t), \overline{h}(t)), \\
\overline{u}_2 &\geq -k\overline{u}_2, & t \in (m\omega + (1 - \delta)\omega, (m + 1)\omega], x \in (\overline{g}(t), \overline{h}(t)), \\
\overline{u}_1(t, x) \geq 0, & t > 0, x \in (\overline{g}(t), \overline{h}(t)), \\
\overline{g}(t) &\leq \sum_{i=1}^{2} \mu_i \int_{\overline{t}(t)}^{\infty} J_i(x - y)\overline{u}_i(t, x)dydx, & t \in (m\omega, m\omega + (1 - \delta)\omega], \\
\overline{h}(t) &\leq \sum_{i=1}^{2} \mu_i \int_{\overline{t}(t)}^{\infty} J_i(x - y)\overline{u}_i(t, x)dydx, & t \in (m\omega, m\omega + (1 - \delta)\omega], \\
\overline{g}(0) &\leq -h_0, \overline{h}(0) \geq h_0, & x \in [\overline{g}(0), \overline{h}(0)].
\end{align*}$$

(4.2)

then $[g(t), h(t)] \subset [\overline{g}(t), \overline{h}(t)]$ and

$$u_i(t, x) \leq \overline{u}_i(t, x), \quad (t, x) \in [0, +\infty) \times [g(t), h(t)].$$
If the fifth inequality in (4.2) is changed to equality and other inequalities are revered, and $(\bar{u}_1, \bar{u}_2; \bar{g}, \bar{h})$ is replaced by $(u_1, u_2; g, h)$, then $[g(t), h(t)] \supset [g(t), h(t)]$ and

$$u_i(t, x) \geq u_i(t, x), \quad (t, x) \in [0, +\infty) \times [g(t), h(t)].$$

**Proof** The proof mainly follows the approach of Cao et al. (2019, Theorem 3.1), we omit the details with some modifications in the cold season.

By using the comparison principle, we present a corollary to show the monotonicity of the unique solution $(u_1, u_2; g, h)$ to (1.9) with respect to the parameters, and denote the solution by $(u_i^\mu, u_i^\mu; g^\mu, h^\mu)$, where $\mu := (\mu_1, \mu_2)$.

**Corollary 4.2** Suppose (J) holds and $(u_i^\mu, u_i^\mu; g^\mu, h^\mu)$ is the solution of (1.9) with the initial functions satisfying (1.10). If $\mu_1^* \leq \mu_1^{**}$ and $\mu_2^* \leq \mu_2^{**}$, we have $u_i^\mu \leq u_i^{**} (i = 1, 2)$ for $t \geq 0$ and $g^{**} \leq x \leq h^{**}$, and $g^{**} \leq g^{**} < h^{**} \leq h^{**}$ for $t \geq 0$.

**Lemma 4.3** If $h_\infty - g_\infty < \infty$, then

$$\lim_{t \to \infty} \|u_1\|_{C([g(t), h(t)])} = \lim_{t \to \infty} \|u_2\|_{C([g(t), h(t)])} = 0$$

and $\lambda_1^P ([g_\infty, h_\infty]) \geq 0$.

**Proof** If $\delta = 1$, problem (1.9) becomes

$$
\begin{align*}
\frac{du_1}{dt} &= d_1 \int_{g(t)}^{h(t)} J_1(x-y)u_1(t, y)dy - u_1 - b_1u_1, \quad t > 0, \quad -h_0 < x < h_0, \\
\frac{du_2}{dt} &= -ku_2, \quad t > 0, \quad -h_0 < x < h_0, \\
u_i(t, x) &= 0, \quad t > 0, \quad x \in \{h_0, h_0\}, \\
u_i(0, x) &= u_i(0, x), \quad -h_0 \leq x \leq h_0, \quad i = 1, 2,
\end{align*}
$$

(4.4)

which is a nonlocal problem in a fixed interval $[-h_0, h_0]$. It is obviously seen that $h_\infty - g_\infty = 2h_0 < \infty$. Let $(\bar{u}_1, \bar{u}_2) = (M_1e^{-h_1t}, M_2e^{-k_1t})$ with $M_i = ||u_i(0)||_{L\infty[-h_0, h_0]}$ as an upper solution of (4.4), we further have (4.3) holds.

If $0 \leq \delta < 1$, the principal eigenvalue $\lambda_1^P ([g_\infty, h_\infty])$ of (2.1) with $[-L_1, L_2]$ replaced with $[g_\infty, h_\infty]$ is well-defined. We firstly prove that $\lambda_1^P ([g_\infty, h_\infty]) \geq 0$. If not, $\lambda_1^P ([g_\infty, h_\infty]) < 0$. Recalling that $J_1(0) > 0$, there exists $\varepsilon_0 \in (0, h_0)$ such that $J_1(x) > 0$ for $x \in [-4\varepsilon_0, 4\varepsilon_0]$. It follows from the assumption and (4.1) that for $\varepsilon_0$ given above, there exists a large positive integer $n$, such that $|g(n\varepsilon_0) - g_\infty| < \varepsilon_0$ and $|h(n\varepsilon_0) - h_\infty| < 4\varepsilon_0$. Using the continuity (Lemma 2.14) of $\lambda_1^P$ gives that $\lambda_1^P ([g(n\varepsilon_0), h(n\varepsilon_0)]) < 0$.

Let $(u_1(0, x), u_2(0, x))$ be the solution of (3.2) with $Q_m^w$ and $Q_c^m$ replaced by $(m\omega, m\omega + (1 - \delta)\omega \times (g(n\omega), h(n\omega)))$ and $(m\omega + (1 - \delta)\omega, (m + 1)\omega \times (g(n\omega), h(n\omega)))$ respectively, and initial functions pair $(u_1(0, x), u_2(0, x)) = (u_1(n\omega, x), u_2(n\omega, x))$. By comparison principle, we admits

$$(u_1, u_2)(t, x) \leq (u_1, u_2)(t + n\omega, x) \quad \text{for} \quad t \geq 0, \quad g(n\omega) \leq x \leq h(n\omega).$$
Then from Lemma 3.3 (ii), we deduce that

$$ (0, 0) < (U_1, U_2)(t, x) = \lim_{t \to \infty} (u_1, u_2)(t, x) \leq \liminf_{t \to \infty} (u_1, u_2)(t, x) $$

with the convergence uniform for $x \in [g(\omega), h(\omega)]$, where $(U_1(t, x), U_2(t, x))$ is the solution of (3.3) with $[-L_1, L_2]$ replaced by $[g(\omega), h(\omega)]$. Therefore, there exists $T_1 \geq n_0$ such that

$$ 0 < \frac{1}{2} U_i(t, x) < u_i(t, x) \quad \text{for} \ t \geq T_1, \ x \in [g(\omega), h(\omega)], \ i = 1, 2. $$

Denote

$$ c_i := \min_{t \in [0, g(\omega)]} U_i(t, x) > 0 \ (i = 1, 2) \quad \text{and} \quad c_3 := \min_{-4e_0 \leq x \leq 4e_0} J_1(x) > 0. $$

Noting that the fact $[h(t) - 2e_0, h(t) - \epsilon_0] \subset [g(\omega), h(\omega)]$ for $t \geq n_0$, together with (1.9) and the above estimations of $u_1$ and $u_2$, yields

$$ h'(t) = \sum_{i=1}^{2} \mu_i \int_{g(t)^i}^{h(t)^i} J_i(x - y) u_i(t, x) dy dx $$

$$ \geq \sum_{i=1}^{2} \mu_i \int_{h(t)^i - 2e_0}^{h(t)^i + e_0} J_i(x - y) u_i(t, x) dy dx $$

$$ \geq 2c_3e_0 \sum_{i=1}^{2} \mu_i \int_{h(t)^i - 2e_0}^{h(t)^i} u_i(t, x) dx \geq 2c_3e_0 \sum_{i=1}^{2} \mu_i \int_{h(t)^i - 2e_0}^{h(t)^i} u_i(t, x) dx $$

$$ \geq 2c_3e_0 \sum_{i=1}^{2} \mu_i \int_{h(t)^i - 2e_0}^{h(t)^i} \frac{1}{2} U_i(t, x) dx \geq c_3e_0 \sum_{i=1}^{2} \mu_i c_i > 0 $$

for $t \geq T_1$, which contradicts with the fact $\lambda_1^p (g(\omega), h(\omega)) \geq 0$.

Let $(\overline{u}_1(t, x), \overline{u}_2(t, x))$ be the solution of (3.2) with $Q_w^m$ and $Q_c^m$ replaced by $(m\omega, m\omega + (1 - \delta)\omega) \times (g(\omega), h(\omega))$ and $(m\omega + (1 - \delta)\omega, m\omega + (m + \delta)\omega) \times (g(\omega), h(\omega))$ respectively, and initial functions $(\overline{u}_{1,0}, \overline{u}_{2,0}) = (e_1, e_2)$. By comparison principle, we have

$$ (0, 0) \leq (u_1, u_2)(t, x) \leq (\overline{u}_1, \overline{u}_2)(t, x) \quad \text{for} \ t \geq 0, \ g(t) \leq x \leq h(t). $$

Since $\lambda_1^p (g(\omega), h(\omega)) \geq 0$, by Lemma 3.3, we have $\lim_{t \to \infty} (\overline{u}_1, \overline{u}_2) = (0, 0)$ uniformly for $x \in [g(\omega), h(\omega)]$, and (4.3) holds.

\[ \square \]

**Remark 4.4** From the above proof, we know that, if $\delta = 1$, then

$$ \lim_{t \to \infty} \| u_1 \|_{\mathbb{C}[g(t), h(t)]} = \lim_{t \to \infty} \| u_2 \|_{\mathbb{C}[g(t), h(t)]} = 0 $$

always holds, and vanishing happens.
Next, we only need to consider the case \( 0 \leq \delta < 1 \). Let \( \lambda_1^p([-h_0, h_0]) \) be the principal eigenvalue of (2.1) with \( L_1 = L_2 = h_0 \).

**Lemma 4.5** Assume \( 0 \leq \delta < 1 \) and \( \lambda_1^p([-h_0, h_0]) > 0 \). If \( \|u_{1,0}\|_{C([-h_0, h_0])} + \|u_{2,0}\|_{C([-h_0, h_0])} \) is sufficiently small, then

\[
\lim_{t \to \infty} \|u_1\|_{C([g(t), h(t))]} = \lim_{t \to \infty} \|u_2\|_{C([g(t), h(t))]} = 0.
\]

**Proof** From Lemma 4.3, we only need to prove \( h_\infty - g_\infty < \infty \). In fact, \( \lambda_1^p([-h_0, h_0]) > 0 \) together with Lemma 2.14 implies that \( \lambda_1^p([-h_1, h_1]) > 0 \) for some \( h_1 = h_0 + \epsilon_0 \) with small enough \( \epsilon_0 > 0 \). Let \( (\phi, \psi) \) be a positive eigenfunction pair corresponding to \( \lambda_1^p([-h_1, h_1]) > 0 \), and \( \phi(t, x) + \psi(t, x) \leq 1 \) for \( 0 \leq t \leq \omega \) and \(-h_1 \leq x \leq h_1 \).

Denote

\[
\gamma := \lambda_1^p([-h_1, h_1])/2, \quad M := \gamma \epsilon_0 \left( \max_{0 \leq t \leq \omega} \int_{-h_1}^{h_1} (\mu_1 \phi + \mu_2 \psi) dx \right)^{-1}.
\]

Take

\[
\begin{align*}
\overline{h}(t) & := h_0 + \epsilon_0 [1 - e^{-\gamma t}], \quad \overline{g}(t) := -\overline{h}(t), \\
\overline{u}_1(t, x) & := Me^{-\gamma t} \phi(t, x), \quad \overline{u}_2(t, x) := Me^{-\gamma t} \psi(t, x)
\end{align*}
\]

for \( t \geq 0, x \in [-h_1, h_1] \). It is easy to see that the above constructions of the functions ensure that the conditions in (4.2) hold except the initial conditions. Further, if we denote

\[
\sigma := \min \left\{ \min_{x \in [-h_0, h_0]} \phi(0, x), \min_{x \in [-h_0, h_0]} \psi(0, x) \right\},
\]

then

\[
\begin{align*}
u_{1,0}(x) & \leq M \phi(0, x) \leq \overline{u}_1(0, x), \quad u_{2,0}(x) \leq M \psi(0, x) \\
& \leq \overline{u}_2(0, x) \quad \text{for } x \in [-h_0, h_0]
\end{align*}
\]

provided that

\[
\|u_{1,0}\|_{C([-h_0, h_0])} + \|u_{2,0}\|_{C([-h_0, h_0])} \leq \sigma M.
\]

Therefore, using Lemma 4.1, we conclude that \((\overline{u}_1, \overline{u}_2; \overline{g}, \overline{h})\) is an upper solution of (1.9) and \([g(t), h(t)] \subset [\overline{g}, \overline{h}]\), hence \(h_\infty - g_\infty \leq \overline{h}_\infty - \overline{g}_\infty = 2h_1\), and the proof is completed. \( \square \)

**Lemma 4.6** If \( 0 \leq \delta < 1 \) and \( \lambda_1^O \geq 0 \), then

\[
\lim_{t \to \infty} (u_1, u_2)(t, x) = (0, 0)
\]

(4.6)
uniformly for \( x \in [g(t), h(t)] \) and vanishing occurs.

**Proof** The result can be proved by using the comparison principle. Let \((\bar{u}_1(t), \bar{u}_2(t))\) be the solution to

\[
\begin{aligned}
\bar{u}_{1t} &= a_1(e_1 - \bar{u}_1)\bar{u}_2 - b_1\bar{u}_1, & m\omega < t \leq m\omega + (1 - \delta)\omega, \\
\bar{u}_{2t} &= a_2(e_2 - \bar{u}_2)\bar{u}_1 - b_2\bar{u}_2, & m\omega < t \leq m\omega + (1 - \delta)\omega, \\
\bar{u}_{1t} &= -b_1\bar{u}_1, & m\omega + (1 - \delta)\omega < t \leq (m + 1)\omega, \\
\bar{u}_{2t} &= -k\bar{u}_2, & m\omega + (1 - \delta)\omega < t \leq (m + 1)\omega, \\
\bar{u}_i(0) &= e_i (i = 1, 2).
\end{aligned}
\]

It is easily seen that

\[
d_i \mathcal{L}_i[\bar{u}_i](t) = d_i \int_{g(t)}^{h(t)} J_i(x - y)\bar{u}_i dy - d_i\bar{u}_i \leq 0
\]

and \( u_i(0, x) \leq e_i = \bar{u}_i(0) \), it follows from Lemma 3.2 that

\[
(u_1, u_2)(t, x) \leq (\bar{u}_1, \bar{u}_2)(t)
\]

for \( x \in [g(t), h(t)] \) and \( t \geq 0 \).

Set

\[
K_1 = a_1e_2 + b_1, \quad K_2 = a_2e_1 + b_2 + k.
\]

Using \((\bar{u}_1^{(0)}, \bar{u}_2^{(0)}) = (e_1, e_2)\), as the initial values, iteration sequence \(\{(\bar{u}_1^{(n)}, \bar{u}_2^{(n)})\}\) is obtained through the following iterative process

\[
\begin{aligned}
\bar{u}_{1t}^{(n)} + K_1\bar{u}_1^{(n)} &= K_1\bar{u}_1^{(n-1)} + a_1(e_1 - \bar{u}_1^{(n-1)})\bar{u}_2^{(n-1)} - b_1\bar{u}_1^{(n-1)}, & t \in (0, (1 - \delta)\omega], \\
\bar{u}_{2t}^{(n)} + K_2\bar{u}_2^{(n)} &= K_2\bar{u}_2^{(n-1)} + a_2(e_2 - \bar{u}_2^{(n-1)})\bar{u}_1^{(n-1)} - b_2\bar{u}_2^{(n-1)}, & t \in (0, (1 - \delta)\omega], \\
\bar{u}_{1t}^{(n)} + K_1\bar{u}_1^{(n)} &= K_1\bar{u}_1^{(n-1)} - b_1\bar{u}_1^{(n-1)}, & t \in ((1 - \delta)\omega, \omega], \\
\bar{u}_{2t}^{(n)} + K_2\bar{u}_2^{(n)} &= K_2\bar{u}_2^{(n-1)} - b_2\bar{u}_2^{(n-1)}, & t \in ((1 - \delta)\omega, \omega], \\
\bar{u}_i^{(n)}(0) &= \bar{u}_i^{(n-1)}(\omega),
\end{aligned}
\]

where \( n = 1, 2, \ldots \), \(\{(\bar{u}_1^{(n)}, \bar{u}_2^{(n)})\}\) is called the maximal sequence. From the comparison principle, it easily follows that the above sequence \(\{(\bar{u}_1^{(n)}, \bar{u}_2^{(n)})\}\) admits the monotone property

\[
(\bar{u}_1^{(n)}, \bar{u}_2^{(n)}) \leq (\bar{u}_1^{(n-1)}, \bar{u}_2^{(n-1)}) \leq \cdots \leq (\bar{u}_1^{(1)}, \bar{u}_2^{(1)}) \leq (\bar{u}_1^{(0)}, \bar{u}_2^{(0)}) = (e_1, e_2)
\]

for \( t \in [0, \omega] \), and therefore the limits exist, denoted by

\[
\lim_{n \to \infty} (\bar{u}_1^{(n)}, \bar{u}_2^{(n)})(t + n\omega) = (U_1, U_2)(t),
\]

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which yields

\[(U_1, U_2) \leq (\bar{u}_1^{(n)}, \bar{u}_2^{(n)}) \leq (\bar{u}_1^{(n-1)}, \bar{u}_2^{(n-1)}) \leq (e_1, e_2).\]

Furthermore, Sobolev imbedding theorem assert that \((U_1, U_2)\) meets with problem \((3.9)\). It is clearly seen that \((U_1, U_2)\) is the solutions of problem \((3.9)\). Since that problem \((3.9)\) has a unique solution \((U_1^\Delta, U_2^\Delta)\), then \((U_1, U_2) = (U_1^\Delta, U_2^\Delta)\), that is,

\[
\lim_{n \to \infty} (\bar{u}_1^{(n)}, \bar{u}_2^{(n)}) = (U_1^\Delta, U_2^\Delta).
\]

Recalling that \((\bar{u}_1, \bar{u}_2)(t) \leq (e_1, e_2) = (\bar{u}_1^{(0)}, \bar{u}_2^{(0)})(t)\) for \(t \in [0, \omega]\), then \((\bar{u}_1, \bar{u}_2)(\omega) \leq (e_1, e_2) = (\bar{u}_1^{(0)}, \bar{u}_2^{(0)})(\omega) = (\bar{u}_1^{(1)}, \bar{u}_2^{(1)})(0)\). Using comparison principle that \((\bar{u}_1, \bar{u}_2)(t + \omega) \leq (\bar{u}_1^{(1)}, \bar{u}_2^{(1)})(t)\). We eventually conclude that for \(t \in [0, \omega]\) and \(n = 0, 1, 2 \ldots\),

\[
(\bar{u}_1, \bar{u}_2)(t + n\omega) \leq (\bar{u}_1^{(n)}, \bar{u}_2^{(n)})(t)
\]

holds by iterating \(\bar{u}_i^{(n)}(0) = \bar{u}_i^{(n-1)}(\omega)\) with respect to \(n\). Therefore

\[
\lim_{n \to \infty} \sup(\bar{u}_1, \bar{u}_2)(t + n\omega) = (U_1^\Delta, U_2^\Delta)(t).
\] (4.9)

Similarly to the proof of Lemmas 2.13 and 3.3, we show that problem \((3.9)\) has the unique nonnegative solution \((0, 0)\) if \(\lambda_1^0 \geq 0\), that is, \((U_1^\Delta, U_2^\Delta) = (0, 0)\). As a result,

\[
\lim_{t \to \infty} (u_1, u_2)(t, x) = (0, 0)
\]

uniformly for \(x \in [g(t), h(t)]\) because of (4.8) and (4.9), which means that vanishing occurs.

\[\square\]

**Lemma 4.7** If \(a_1a_2e_1e_2 \leq b_1b_2\), then \(h_\infty - g_\infty < \infty\) and vanishing happens.

**Proof** In fact, using Lemma 4.3, it suffices to prove that \(h_\infty - g_\infty < \infty\). The result can also be obtained by using the energy-type estimate, see also Du and Ni (2020).

Recalling that \(\int_{\mathbb{R}} J_1(x)dx = 1\) and \(J_1\) is symmetric, we have

\[
\int_{g(t)}^{h(t)} \mathcal{L}_1[u_1](t, x)dx
\]

\[
= \int_{g(t)}^{h(t)} \int_{g(t)}^{h(t)} J_1(x - y)[u_1(t, y) - u_1(t, x)]dydx
\]

\[
- \int_{g(t)}^{h(t)} \int_{h(t)}^{\infty} J_1(x - y)u_1(t, x)dydx
\]

\[
- \int_{g(t)}^{h(t)} \int_{-\infty}^{g(t)} J_1(x - y)u_1(t, x)dydx
\]
Using the aforementioned equations and (1.9) for \( t \in (m \omega, m \omega + (1 - \delta) \omega) \), we obtain

\[
\frac{d}{dt} \int_{g(t)}^{h(t)} [u_1(t, x) + \frac{a_1 e_1}{b_2} u_2(t, x)] dx
\]

\[
= \int_{g(t)}^{h(t)} [u_1(t, x) + \frac{a_1 e_1}{b_2} u_2(t, x)] dx
\]

\[
= \int_{g(t)}^{h(t)} (d_1 \mathcal{L}_1[u_1] + \frac{a_1 e_1 d_2}{b_2} \mathcal{L}_2[u_2]) dx
\]

\[
+ \int_{g(t)}^{h(t)} \left( \frac{a_1 a_2 e_1 e_2}{b_1 b_2} - 1 \right) b_1 u_1 - (a_1 + \frac{a_1 a_2 e_1 e_2}{b_2}) u_1 u_2 \right] dx \leq \int_{g(t)}^{h(t)} (d_1 \mathcal{L}_1[u_1] + \frac{a_1 e_1 d_2}{b_2} \mathcal{L}_2[u_2]) dx
\]

\[
\leq - \min\{d_1, \frac{a_1 e_1 d_2}{b_2}\} \int_{g(t)}^{h(t)} \int_{h(t)}^{\infty} J_1(x - y) u_1(t, x) dy dx
\]

\[
+ \int_{g(t)}^{h(t)} \int_{-\infty}^{g(t)} J_1(x - y) u_1(t, x) dy dx
\]

\[
+ \int_{g(t)}^{h(t)} \int_{h(t)}^{\infty} J_2(x - y) u_2(t, x) dy dx + \int_{g(t)}^{h(t)} \int_{-\infty}^{g(t)} J_2(x - y) u_2(t, x) dy dx
\]

\[
\leq - \frac{\min\{d_1, \frac{a_1 e_1 d_2}{b_2}\}}{\max\{\mu_1, \mu_2\}} [h'(t) - g'(t)] \coloneqq -D[h'(t) - g'(t)]
\]
for the warm season \( t \in (m \omega, m \omega + (1 - \delta) \omega) \).

Denote

\[
S(t) = h(t) - g(t) \quad \text{and} \quad F(t, x) = \frac{1}{D} \left[ u_1 + \frac{a_1 e_1}{b_2} u_2 \right](t, x),
\]

then the inequality (4.13) is rewritten as

\[
S(t) - S(m \omega) \leq - \int_{g(t)}^{h(t)} F(t, x) dx + \int_{g(m \omega)}^{h(m \omega)} F(m \omega, x) dx. \tag{4.14}
\]

For cold season \( t \in ((m - 1) \omega, (m - 1) \omega + (1 - \delta) \omega, m \omega) \),

\[
S(m \omega) - S((m - 1) \omega + (1 - \delta) \omega) = 0. \tag{4.15}
\]

Notice that

\[
\int_{g(m \omega)}^{h(m \omega)} F(m \omega, x) dx - \int_{g((m-1)\omega+(1-\delta)\omega)}^{h((m-1)\omega+(1-\delta)\omega)} F((m-1)\omega+(1-\delta)\omega, x) dx
= \frac{1}{D} \int_{g(m \omega)}^{h(m \omega)} \left[ u_1 + \frac{a_1 e_1}{b_2} u_2 \right](t, x) dx dt
\]
\[
= \frac{1}{D} \int_{g(m \omega)}^{h(m \omega)} \int_{g(t)}^{h(t)} J_1(x, y) [u_1(t, y) - u_1(t, x)] dy dx
\]
\[
= \frac{1}{D} \int_{g(m \omega)}^{h(m \omega)} \int_{g(t)}^{h(t)} J_1(x, y) [u_1(t, y) - u_1(t, x)] dy dx dt
\]
\[
\leq \frac{1}{D} \int_{g(m \omega)}^{h(m \omega)} \int_{g(t)}^{h(t)} J_1(x, y) [u_1(t, y) - u_1(t, x)] dy dx dt
\]
\[
= 0
\]

by using \( h(t) = h(m \omega + (1 - \delta) \omega) \) and \( g(t) = g(m \omega + (1 - \delta) \omega) \) for \( m \omega + (1 - \delta) \omega < t \leq (m + 1) \omega \). It follows from (4.15) and (4.16) that

\[
S(m \omega) - S((m - 1) \omega + (1 - \delta) \omega)
\]
\[
\leq - \int_{g(m \omega)}^{h(m \omega)} F(m \omega, x) dx + \int_{g((m-1)\omega+(1-\delta)\omega)}^{h((m-1)\omega+(1-\delta)\omega)} F((m-1)\omega+(1-\delta)\omega, x) dx. \tag{4.17}
\]

Adding (4.14) and (4.17) yields

\[
S(t) - S((m - 1) \omega + (1 - \delta) \omega)
\]
\[
\begin{aligned}
&\leq -\int_{g(t)}^{h(t)} F(t, x) dx \\
&+ \int_{g((m-1)\omega+(1-\delta)\omega)}^{h((m-1)\omega+(1-\delta)\omega)} F((m-1)\omega + (1 - \delta)\omega, x) dx \\
&
(4.18)
\end{aligned}
\]

for \( t \in (m\omega, m\omega + (1 - \delta)\omega) \).

Repeating the above process, we then have

\[
S(t) - S(0) \leq -\int_{g(t)}^{h(t)} F(t, x) dx + \int_{g(0)}^{h(0)} F(0, x) dx = \int_{g(0)}^{h(0)} \frac{1}{D} [u_1(0, x) + \frac{a_1 e_1}{b_2} u_2(0, x)] dx,
\]

that is,

\[
h(t) - g(t) \leq 2h_0 + \int_{-h_0}^{h_0} \frac{1}{D} [u_1(0, x) + \frac{a_1 e_1}{b_2} u_2(0, x)] dx.
\]

As a result, \( h_\infty - g_\infty < \infty \) and vanishing happens. \( \square \)

**Lemma 4.8** If \( 0 \leq \delta < 1 \) and \( \lambda_1^P([g(t_0), h(t_0)]) \leq 0 \) for some \( t_0 \geq 0 \), then \( g_\infty = -\infty, h_\infty = +\infty \) and

\[
\lim_{n \to \infty} (u_1, u_2)(t + n\omega, x) = (U_1^\Delta, U_2^\Delta)(t)
\]

(4.19)

uniformly for \( t \in [0, \omega] \) and locally uniformly for \( x \in \mathbb{R} \), where \( \lambda_1^P([g(t_0), h(t_0)]) \) is the principal eigenvalue of (3.2) with \([-L_1, L_2]\) replaced by \([g(t_0), h(t_0)]\), and \((U_1^\Delta, U_2^\Delta)(t)\) is the unique positive periodic solution to problem (3.9).

**Proof** Notice that \([g(t_0), h(t_0)] \subsetneq [g(t_1), h(t_1)]\) for \( t_1 > t_0 + \omega \), the monotonicity of \( \lambda_1^P \) gives that

\[
\lambda_1^P([g(t_1), h(t_1)]) < 0 \quad \text{for} \quad t_1 > t_0 + \omega.
\]

(4.20)

Using the above inequality, we get by the similar proof of Lemma 4.3 that \( h_\infty = +\infty \) and \( g_\infty = -\infty \).

Now, we prove (4.19). For big integer \( n \) \( (n \geq \left\lceil \frac{h_0}{\omega} \right\rceil + 1) \), and let \((u_{1,n\omega}(t, x), u_{2,n\omega}(t, x))\) be the solution of (3.2) with \( Q_\omega^m \) and \( Q_\omega^c \) replaced by \((m\omega, m\omega + (1 - \delta)\omega) \times (g(n\omega), h(n\omega))\) and \((m\omega + (1 - \delta)\omega, (m + 1)\omega) \times (g(n\omega), h(n\omega))\) respectively, and initial functions \((u_{1,n\omega}(0, x), u_{2,n\omega}(0, x)) = (u_1(n\omega, x), u_2(n\omega, x))\). By comparison principle, we obtain

\[
(u_{1,n\omega}, u_{2,n\omega})(t + m\omega, x) \leq (u_1, u_2)(t + m\omega + n\omega, x)
\]
for \((t, x) \in [0, \omega] \times [g(n\omega), h(n\omega)], m = 0, 1, \ldots\). Letting \(m \to \infty\) and using Lemma 3.3 (ii) yield

\[
(0, 0) < (U_{1,n\omega}, U_{2,n\omega})(t, x) = \lim_{m \to \infty} (u_{1,n\omega}, u_{2,n\omega})(t + m\omega, x) \\
\leq \liminf_{m \to \infty} (u_1, u_2)(t + m\omega + n\omega, x)
\]

with the convergence uniform for \(t \in [0, \omega]\) and \(x \in [g(n\omega), h(n\omega)]\), where \((U_{1,n\omega}, U_{2,n\omega})(t, x)\) is the positive periodic solution of (3.3) with \([-L_1, L_2]\) replaced by \([g(n\omega), h(n\omega)]\). Noticing the fact that \((g(n\omega), h(n\omega)) \to (-\infty, \infty)\) as \(n \to \infty\), we obtain by Lemma 3.5 that

\[
(U_{1}, U_{2})(t) \leq \liminf_{n \to \infty} (u_1, u_2)(t + n\omega, x) \tag{4.21}
\]

uniformly for \(t \in [0, \omega]\) and locally uniformly for \(x \in \mathbb{R}\). Moreover, it follows from Lemma 4.6 that

\[
\limsup_{n \to \infty}(u_1, u_2)(t + n\omega, x) \leq (U_{1}, U_{2})(t), \tag{4.22}
\]

where \((U_{1}, U_{2})\) is the unique solution of problem (3.9). This completes the proof. \(\Box\)

**Proof of Theorem 1.1** We first consider the solution in the warm season of the first year \((m = 0)\). Denote

\[
\mathbb{H}_{h_0,s} := \{ h \in C([0, s]) : h(0) = h_0, \ h(t) \text{ is strictly increasing}\}, \\
\mathbb{G}_{h_0,s} := \{ g \in C([0, s]) : -g \in \mathbb{H}_{h_0,s}\}.
\]

Similarly as in Du and Ni (2020, Lemma 4.2), for any \((g, h) \in \mathbb{G}_{h_0,1-\delta}\omega} \times \mathbb{H}_{h_0,1-\delta}\omega\), the following problem

\[
\begin{align*}
    u_{1t} = & \quad d_1 L_1[u_1] + a_1(e_1 - u_1)u_2 - b_1 u_1, & 0 < t \leq (1 - \delta)\omega, \ g(t) < x < h(t), \\
    u_{2t} = & \quad d_2 L_2[u_2] + a_2(e_2 - u_2)u_1 - b_2 u_2, & 0 < t \leq (1 - \delta)\omega, \ g(t) < x < h(t), \\
    u_1(t, x) = & \quad u_2(t, x) = 0, & 0 < t \leq (1 - \delta)\omega, \ x \in \{g(t), h(t)\}, \\
    h(0) = & \quad -g(0) = h_0, \\
    u_1(0, x) = & \quad u_{1,0}(x), \ u_2(0, x) = u_{2,0}(x), & -h_0 \leq x \leq h_0.
\end{align*}
\]  

\tag{4.23}

has a unique positive solution for \(t \in (0, s]\) by the contraction mapping theorem, we can further extend the unique solution defined over \([0, s]\) to \([0, (1 - \delta)\omega]\). As a result, problem (4.23) with the initial function pair \((u_{1,0}, u_{2,0})\) admits a unique solution \((u_{1}^{g,h}, u_{2}^{g,h})\) which satisfies

\[
0 < u_{i}^{g,h} \leq e_i \quad \text{for} \quad (t, x) \in (0, (1 - \delta)\omega) \times (g(t), h(t)).
\]
We now define a mapping \( \mathcal{F}(h, g) = (\hat{h}, \hat{g}) \) by

\[
\begin{cases}
  \hat{h}(t) = h_0 + \sum_{i=1}^{2} \mu_i \int_0^t \int_{g(s)}^{h(s)} J_i(x - y)u_i(s, x)dydxds, & 0 < t \leq (1 - \delta)\omega, \\
  \hat{g}(t) = -h_0 - \sum_{i=1}^{2} \mu_i \int_0^t \int_{g(s)}^{h(s)} J_i(x - y)u_i(s, x)dydxds, & 0 < t \leq (1 - \delta)\omega.
\end{cases}
\]

To prove the existence and uniqueness of solution to (1.9) for \( t \in (0, (1 - \delta)\omega) \), it suffices to show that \( \mathcal{F} \) has a unique fixed point in \( \Sigma_{(1-\delta)\omega} := \mathbb{H}_{h_0,(1-\delta)\omega} \times \mathbb{G}_{h_0,(1-\delta)\omega} \). To get the above conclusion, we firstly show that for small \( 0 < \tau < (1 - \delta)\omega \), \( \mathcal{F} \) maps a suitable closed subset of \( \Sigma_\tau \) into itself and is a contraction mapping, which guarantees the existence of a unique solution of (1.9) for \( t \in (0, \tau) \). Then the unique solution can be extended to the range \( t \in (0, (1 - \delta)\omega) \). Since the proof is almost the same as that in Cao et al. (2019, Theorem 2.1), we omit the details here.

On the other hand, over the time interval \(( (1 - \delta)\omega, \omega ] \) (the cold season), the boundaries are fixed, that is, \( g(t) \equiv g((1 - \delta)\omega) \) and \( h(t) \equiv h((1 - \delta)\omega) \) for \( t \in ((1 - \delta)\omega, \omega] \). Then the fixed boundary problem

\[
\begin{aligned}
  u_{1t} &= d_1 \left[ \int_{g((1-\delta)\omega)}^{h((1-\delta)\omega)} J_1(x - y)u_1(t, y)dy - u_1(t, x) \right] - b_1u_1, \quad (1 - \delta)\omega < t \leq \omega, \\
  u_1(t, g((1-\delta)\omega)) &= u_1(t, g((1-\delta)\omega)) = 0, \\
  u_1(0, x) &= u_{1,0}(x), \quad u_2(0, x) = u_{2,0}(x), \\
  g((1 - \delta)\omega) &< x < h((1 - \delta)\omega), \\
  (1 - \delta)\omega &< t \leq \omega, \\
  -h_0 &\leq x \leq h_0.
\end{aligned}
\]

(4.24)

admits a unique solution \( u_1 \) by the result of Cao et al. (2019, Lemma 2.3), moreover \( u_1 \) satisfies

\[ 0 < u_1 \leq e_1 \text{ for } (t, x) \in ((1 - \delta)\omega, \omega] \times (g((1 - \delta)\omega), h((1 - \delta)\omega)). \]

Again, solving the forth equation of (1.9) gives that

\[ u_2(t, x) = u_2((1 - \delta)\omega, x) e^{k(1 - \delta)\omega - t} \leq e_2, \]

for \((t, x) \in [(1 - \delta)\omega, \omega] \times [g((1 - \delta)\omega), h((1 - \delta)\omega))].

By taking \( m = 1, 2, \ldots \), recursively, we therefore obtain the existence and uniqueness of the solution \((u_1, u_2, g, h)\) to (1.9) for \( t \in [0, \infty) \). Moreover, \( -g', h' > 0 \) in the warm season \((m\omega, m\omega + (1 - \delta)\omega]\), and \( g' = h' = 0 \) in the cold season \((m\omega + (1 - \delta)\omega, (m + 1)\omega]\). This completes the proof of Theorem 1.1.

**Proof of Theorem 1.2**

(1) If \( \delta = 1 \), vanishing always happens by Remark 4.4.

(2) If \( 0 \leq \delta < 1 \) and \( \lambda_1^O \geq 0 \), then (4.6) holds by Lemma 4.6 and vanishing happens.

(3) If \( 0 \leq \delta < 1 \) and \( \lambda_1^O < 0 \), by Lemma 2.17 (ii)(iii), we find \( \lambda_1^P (\{g(t_0), h(t_0)\}) < 0 \) for some large \( t_0 > 0 \). According to Lemma 4.8, (4.19) holds and spreading happens.

Therefore, spreading (case (3))-vanishing (cases (1) and (2)) dichotomy holds. \( \square \)
For a better understanding of the spreading-vanishing criteria, we present two lemmas before the proofs of Theorems 1.3 and 1.4.

Lemma 4.9 Suppose \((J, J_1(x) = J_2(x))\) and \(0 \leq \delta < 1\) hold, and the initial functions satisfy (1.10). Let \(\lambda^P_1(\Gamma, b_1 - d_1\lambda^*_1, b_2 - d_2\lambda^*_1, \delta)\) is given by (2.1), and denote

\[
\lambda^O_1(\Gamma, b_1, b_2, \delta) := \lim_{L_1, L_2 \to +\infty} \lambda^P_1(\Gamma, b_1 - d_1\lambda_1([-L_1, L_2]), b_2 - d_2\lambda_1([-L_1, L_2]), \delta),
\]

where \(\Gamma = \{a_i, e_i, \omega, k\}\). The following conclusion is valid.

(i) If \(\lambda^O_1 \geq 0\), then vanishing always happens.

(ii) If \(\lambda^O_1 < 0\) and \(\lambda^P_1([-h_0, h_0]) \leq 0\), then spreading always happens.

(iii) If \(\lambda^O_1 < 0\) and \(\lambda^P_1([-h_0, h_0]) > 0\), then

(a) For any given initial datum \((u_{1,0}, u_{2,0})\) satisfying (1.10), there exists \(\mu^* \geq \mu_* > 0\) such that vanishing happens for \(0 < \mu_1 + \mu_2 \leq \mu_*\) and spreading happens for \(\mu_1 + \mu_2 > \mu^*\).

(b) For fixed \(\mu_1, \mu_2 > 0\), vanishing happens if initial datum \((u_{1,0}, u_{2,0})\) is small enough.

Proof (i) If \(0 \leq \delta < 1\) and \(\lambda^O_1 \geq 0\), then vanishing happens by Lemma 4.6.

(ii) If \(\lambda^O_1 < 0\) and \(\lambda^P_1([-h_0, h_0]) \leq 0\), we derive that \(\lambda^P_1([g(t_0), h(t_0)]) < 0\) for some large \(t_0 > 0\) by continuity in Lemma 2.17, hence spreading occurs by Lemma 4.8.

(iii) According to \(\lambda^P_1([-h_0, h_0]) > 0\), then conclusion (b) is directly proved by Lemma 4.5. It remains to prove (a).

We construct the same upper solution as in the proof of Lemma 4.5, since that

\[
\lim_{(\mu_1, \mu_2) \to +(0, 0)} M := \lim_{(\mu_1, \mu_2) \to (0, 0)} \gamma \varepsilon_0 \left( \max_{0 \leq t \leq \omega} \int_{-h_1}^{h_1} (\mu_1 \phi + \mu_2 \psi) dx \right)^{-1} = \infty,
\]

therefore, for any given initial function pair \((u_{1,0}, u_{2,0})\) satisfying (1.10), there exists \(\mu_* > 0\) such that (4.5) holds for \(0 < \mu_1 + \mu_2 \leq \mu_*\), which indicates that vanishing occurs for problem (1.9).

Next, if \(0 \leq \delta < 1\), we claim that there exists \(\mu^* > 0\) such that spreading occurs when \(\mu_1 + \mu_2 > \mu^*\). To emphasize the dependence of solution on \(\mu := (\mu_1, \mu_2)\), let \((u^\mu_1, u^\mu_2; g^\mu, h^\mu)\) denote the unique positive solution of (1.9). First of all, we show that there exists \(\mu^*_1 > 0\) such that for some large \(t_1 > 0\),

\[
\lambda^P_1([g^\mu_1(t_1), h^\mu_1(t_1)]) < 0. \tag{4.25}
\]

If not, then we assume that for all \(t > 0\) and \(\mu > 0\),

\[
\lambda^P_1([g^\mu(t), h^\mu(t)]) \geq 0.
\]
Recalling that \(-g^\mu(t)\) and \(h^\mu(t)\) are nondecreasing with respect to \(t\), and \(-g^\mu(t)\) and \(h^\mu(t)\) are also nondecreasing in \(\mu > 0\) by Corollary 4.2, therefore the limits

\[
H_\infty := \lim_{t, \mu \to +\infty} h^\mu(t), \\
G_\infty := \lim_{t, \mu \to +\infty} g^\mu(t)
\]

exist. Moreover, \(H_\infty - G_\infty < +\infty\) by Lemma 2.17(iii). Since that \(J_1(0) > 0\), there exist \(\varepsilon_0, \delta_0\) such that \(J_1(x) > \delta_0\) when \(|x| < \varepsilon_0\). Then for the above \(\varepsilon_0\), there exist large enough positive constant \(\mu_0\) and integer \(m_0\) such that

\[
h^\mu(t) > H_\infty - \frac{\varepsilon_0}{4}, \quad \text{when} \quad \mu \geq \mu_0, t \geq m_0\omega.
\]

Thus, for the sixth equation of (1.9), integrating both sides of the equation from \(m_0\omega\) to \(m_0\omega + (1 - \delta)\omega\) with respect to \(t\), gives

\[
h^\mu(m_0\omega + (1 - \delta)\omega) - h^\mu(m_0\omega) = \sum_{i=1}^{2} \mu_i \int_{m_0\omega}^{m_0\omega+(1-\delta)\omega} \int_{h^\mu(t)}^{h^\mu(t)} J_i(x - y) u^\mu_i(\tau, x) dy dx d\tau,
\]

\[
\geq \sum_{i=1}^{2} \mu_i \int_{m_0\omega}^{m_0\omega+(1-\delta)\omega} \int_{h^\mu(t)+\varepsilon_0}^{h^\mu(t)+\varepsilon_0} J_i(x - y) u^\mu_i(\tau, x) dy dx d\tau,
\]

\[
\geq \sum_{i=1}^{2} \mu_i \int_{m_0\omega}^{m_0\omega+(1-\delta)\omega} \int_{h^\mu(t)-\varepsilon_0}^{h^\mu(t)-\varepsilon_0} \int_{h^\mu(t)+\varepsilon_0}^{h^\mu(t)+\varepsilon_0} J_i(x - y) u^\mu_i(\tau, x) dy dx d\tau,
\]

where \(u^\mu_{\min}(t, x) = \min_{m_0\omega < t \leq m_0\omega + (1 - \delta)\omega} \{u^\mu_1(t, x), u^\mu_2(t, x)\}\). Therefore, we have

\[
\mu_1 + \mu_2 \leq \left(\frac{\varepsilon_0}{4}\right) \int_{m_0\omega}^{m_0\omega+(1-\delta)\omega} \int_{h^\mu(t)-\varepsilon_0}^{h^\mu(t)-\varepsilon_0} \left[u^\mu_{\min}(\tau, x) dx d\tau\right]^{-1} [h^\mu(m_0\omega + (1 - \delta)\omega) - h^\mu(m_0\omega)] < +\infty,
\]

which is a contradiction. Therefore, (4.25) holds. We can conclude that spreading occurs when \(\mu = \mu_1^*\) by Lemma 4.8, and so does it when \(\mu \geq \mu_1^*\) by Corollary 4.2. \(\square\)

**Proof of Theorem 1.3** (i) Fix \(a_i, e_1, b_2\) and \(k\), then \(b_1^*: (:= a_1a_2e_1e_2/b_2)\) is determined. It follows from Fig. 3 that \(\lambda_1^0 \geq 0\) if \(b_1 \in [b_1^*, +\infty)\). While if \(b_1 \in (0, b_1^*)\), there

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exists $\delta^* \in (0, 1)$ such that $\lambda_1^O \geq 0$ for $\delta \in [\delta^*, 1]$. Then vanishing always happens from Lemma (4.9) (i).

(ii) If $b_1 \in (0, b_1^*)$ and $\delta \in (0, \delta^*)$, then $\lambda_1^O < 0$. Recalling that $\lambda_1^P([-L_1, L_2])$ is decreasing with respect to $L_1 + L_2$, there exists $h_0^* > 0$ such that $\lambda_1^P([-h_0^*, h_0^*]) \leq 0$ for $h_0 \geq h_0^*$. Therefore, for some $t_0 \geq 0$ such that $g(t_0) \leq -h_0^*$ and $h(t_0) \geq h_0^*$, we have $\lambda_1^P([g(t_0), h(t_0)]) \leq 0$. Hence spreading happens by Lemma (4.9) (ii).

(iii) If $b_1 \in (0, b_1^*)$, $\delta \in (0, \delta^*)$ and $h_0 \in (0, h_0^*)$, then $\lambda_1^O < 0$ and $\lambda_1^P([-h_0^*, h_0^*]) > 0$. The proof is finished by Lemma (4.9) (iii) (a).

**Lemma 4.10** Suppose (J) holds, and the initial functions satisfy (1.10). There exists $0 \leq \mu_\Delta \leq \mu_\Delta^\Delta \leq +\infty$ such that vanishing happens for $(0, 0) < (\mu_1, \mu_2) < (\mu_\Delta, \mu_\Delta)$, and spreading happens for $(\mu_1, \mu_2) > (\mu_\Delta, \mu_\Delta)$. Specially, if $\lambda_1^O \geq 0$, then $\mu_\Delta = \mu_\Delta^\Delta = +\infty$, which means that vanishing happens for any $\mu_1$ and $\mu_2$. While, if $\lambda_1^O < 0$ and $\lambda_1^P([-h_0^*, h_0^*]) \leq 0$, then $\mu_\Delta = \mu_\Delta^\Delta = 0$, which means that spreading happens for any $\mu_1$ and $\mu_2$.

**Proof** Define

$$\mu_\Delta = \sup \{ \mu^* \in [0, +\infty) : \text{vanishing occurs for } (\mu_1, \mu_2) \text{ with } (\mu_1, \mu_2) \leq (\mu^*, \mu^*) \}$$

and

$$\mu_\Delta^\Delta = \inf \{ \mu^* \in (0, +\infty) : \text{spreading occurs for } (\mu_1, \mu_2) \text{ with } (\mu_1, \mu_2) > (\mu^*, \mu^*) \}.$$

It is easy to see that $\mu_\Delta, \mu_\Delta^\Delta$ are well-defined, and $0 \leq \mu_\Delta, \mu_\Delta^\Delta \leq +\infty$. Owing to $h(t) - g(t)$ is nondecreasing in $t$, recalling that $h(\infty) - g(\infty) < \infty$ for the vanishing case and $h(\infty) - g(\infty) = \infty$ for the spreading case yields that $\mu_\Delta \leq \mu_\Delta^\Delta$. Moreover, vanishing happens for $(0, 0) < (\mu_1, \mu_2) < (\mu_\Delta, \mu_\Delta)$, and spreading happens for $(\mu_1, \mu_2) > (\mu_\Delta, \mu_\Delta)$.

If $\lambda_1^O \geq 0$, similarly we can prove that $h_\infty - g_\infty < \infty$ by using the energy-type integral as in Lemma 4.6 and

$$\lim_{t \to \infty} \| u_1 \|_{C([g(t), h(t)])} = \lim_{t \to \infty} \| u_2 \|_{C([g(t), h(t)])} = 0$$

by constructing an upper solution as in Lemma 4.3, so vanishing happens for any $(\mu_1, \mu_2)$, which implies that $\mu_\Delta = \mu_\Delta^\Delta = +\infty$.

If $\lambda_1^O < 0$, we have that $\delta \neq 1$, and the assumptions $\lambda_1^O \leq 0$ and $\lambda_1^P([-h_0^*, h_0^*]) \leq 0$ are equivalent to that $\lambda_1^O < 0$ and $\lambda_1^P([-h_0^*, h_0^*]) \leq 0$, which immediately gives that spreading happens for any $(\mu_1, \mu_2)$ by Lemma 4.8, in this case, $\mu_\Delta = \mu_\Delta^\Delta = 0$. □

**Proof of Theorem 1.4** If $\delta = 1$, then vanishing always happens and $\mu_\Delta = \mu_\Delta^\Delta = +\infty$.

If $0 \leq \delta < 1$ and $\min\{b_1, k\} \delta > c_1(1 - \delta)$, then $\lambda_1^O \geq 0$ from Corollary 2.3. According to Lemma 4.10, $\mu_\Delta = \mu_\Delta^\Delta = +\infty$, which means that vanishing happens for any $\mu_1$ and $\mu_2$.

While if $0 \leq \delta < 1$ and $\max\{b_1, k\} \delta < c_1(1 - \delta)$, then $\lambda_1^O < 0$ by Corollary 2.3. Recalling that $\lambda_1^P([-L_1, L_2])$ is decreasing with respect to $L_1 + L_2$, there exists a
sufficiently large constant $h_0^{**} > 0$ such that $\lambda_1^P([-h_0, h_0]) \leq 0$ for $h_0 \geq h_0^{**}$, which implies that spreading happens for any $\mu_1$ and $\mu_2$ provided $h_0$ is large enough, and therefore $\mu_1 = \mu_2 = 0$. \hfill $\Box$

5 Discussion

As it is well known, WNv usually spreads among people by means of mosquitoes infected with the virus after biting the infected birds. On the one hand, mosquitoes are quite climate sensitive and they breed from June to November (warm season) each year with unknown changing habitat, whose fronts are described by the free boundaries. However, due to the harsh living conditions in cold season, the mosquitoes population is assumed to follow the Malthusian growth law, i.e. decaying at an exponential rate, not to migrate and to stay in a hibernating status. Mathematically, the density ($u_2$) of infected mosquitoes satisfies a nonlocal diffusion in a moving interval in the warm season, while in the cold season, it satisfies some degenerate differential equation without diffusion in a fixed interval.

On the other hand, the ecological pressure, being brought by the increased food-demand for active proliferation during the warm season, causes certain birds to fly to the northern glacier retreating place in summer and then return to the south again to spend the winter as soon as the glaciers recover. In other words, infected birds usually migrate both in the warm season and in the cold season and its density ($u_1$) satisfies a nonlocal diffusion equation.

In this paper, we study the WNv nonlocal model with free boundaries and seasonal succession, which can effectively describe the long-distance diffusion process. Nonlocal diffusion is more suitable than local diffusion for characterizing relatively dense population densities. We define the generalized principal eigenvalues of nonlocal diffusion operators with seasonal succession, and present a detailed analysis of the principal eigenvalue of the corresponding ODE system (Theorem 2.1). Morover, the monotonicity of (generalized) principal eigenvalues with respect to length ($L$) of the interval (Lemmas 2.7 and 2.14), the duration ratio ($\delta$) of the cold season (Lemma 2.4 and Corollary 2.11) are given.

We show that the spreading-vanishing dichotomy (Theorem 1.2) is valid, and the criteria (Theorems 1.3 and 1.4) that completely determine when spreading and vanishing happen are deduced. Our theoretical results show that the longer the warm season, the greater the risk of infection and thus less beneficial for the prevention and control of WNv (Remark 3.4).

In our paper, several results assumed that $J_1(x) = J_2(x)$ holds. While this assumption is rather restrictive from a biological point of view, it enables us to give some fine properties of the generalized principal eigenvalue of problem (2.1) so that the corresponding results for spreading and vanishing are nearly complete mathematically. For the case $J_1(x) \neq J_2(x)$, the principal eigenvalue of problem (2.1) is not fully characterised and it invites further studies.

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