LINEAR THEORY OF WEAKLY AMPLIFIED, PARALLEL PROPAGATING, TRANSVERSE TEMPERATURE-ANISOTROPY INSTABILITIES IN MAGNETIZED THERMAL PLASMAS

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Received 2010 February 1; accepted 2010 May 5; published 2010 June 3
doi:10.1088/0004-637X/716/2/1596

ABSTRACT
A rigorous analytical study of the dispersion relations of weakly amplified transverse fluctuations with wavevectors \( \vec{k} \parallel \vec{B} \) parallel to the uniform background magnetic field \( \vec{B} \) in an anisotropic bi-Maxwellian magnetized electron–proton plasma is presented. A general analytical instability condition is derived that holds for different values of the electron (\( A_e \)) and proton (\( A_p \)) temperature anisotropies. We determine the conditions for which the weakly amplified left-handed (LH) polarized Alfvén proton cyclotron and right-handed (RH) polarized Alfvén Whistler electron cyclotron branches can be excited. For different regimes of the electron plasma frequency phase speed \( w = \omega_{p,e}/(kc) \) these branches reduce to the RH- and LH-polarized Alfvén waves, RH-polarized high and low phase speed Whistler, RH-polarized proton, and LH-polarized electron cyclotron modes. Analytic instability threshold conditions are derived in terms of the combined temperature anisotropy \( A = T_\perp/T_\parallel \), the parallel plasma beta \( \beta_\parallel = 8\pi n k_B T_\parallel /B^2 \) and the electron plasma frequency phase speed \( w = \omega_{p,e}/(kc) \) for each mode. The results of our instability study are applied to the observed solar wind magnetic turbulence at values of \( 90 \leq w \leq 330 \). According to the existence conditions of the different instabilities, only the LH- and RH-polarized Alfvén wave instabilities can operate here. Besides the electron–proton mass ratio \( \mu = 1836 \), the Alfvénic instability threshold conditions are controlled by the single plasma parameter \( w \). The Alfvénic instability diagram explains well the observed confinement limits at small parallel plasma beta values in the solar wind.

Key words: instabilities – magnetic fields – plasmas – solar wind – turbulence

Online-only material: color figures

1. INTRODUCTION

Kinetic plasma relaxation and turbulence generation processes are responsible for the observed properties of the solar wind plasma which is the only cosmic collision poor plasma accessible to detailed in situ satellite observations (Bale et al. 2009). Although the detailed plasma relaxation processes are not understood, the observed electron and proton distribution functions are close to bi-Maxwellian distributions with different temperatures along and perpendicular to the ordered magnetic field direction. Ten years of Wind/SWE data (Kasper et al. 2002) have demonstrated that the proton and electron temperature anisotropies \( A = T_\perp/T_\parallel \) are bounded by mirror and fire-hose instabilities (Hellinger et al. 2006) at large values of the parallel plasma beta \( \beta_\parallel = 8\pi n k_B T_\parallel /B^2 \geq 1 \). In the parameter plane defined by the temperature anisotropy \( A = T_\perp/T_\parallel \) and the parallel plasma beta \( \beta_\parallel \), stable plasma configuration is possible only within a rhomb-like configuration around \( \beta_\parallel \simeq 1 \), whose limits are defined by the threshold conditions for the mirror and fire-hose instabilities. If a plasma would start with parameter values outside this rhomb-like configuration, it immediately would generate fluctuations via the mirror and fire-hose instabilities, which quickly relax the plasma distribution into the stable regime within the rhomb configuration. The plasma parameters of other dilute cosmic plasmas including the interstellar and intracluster medium (Shekochihin et al. 2005) and accretion disks around compact, massive objects (Sharma et al. 2007) are similar to the solar wind plasma, so that the temperature-anisotropy instabilities should also operate in these systems.

The study of linear electromagnetic instabilities in a collisionless, homogeneous, magnetized, electron–proton plasma with temperature anisotropies has a long history; for reviews we refer the interested reader to chapter 7 of the monograph by Gary (1993), as well as Schwartz (1980), Cuperman (1981), and Marsch (2006). Hellinger et al. (2006) considered linear instability calculations for the oblique mirror, fire-hose, and proton cyclotron instabilities, and represented the approximate threshold conditions from the work of Gary et al. (1994, 2001), Gary & Lee (1994), Samsonov et al. (2001), and Pokhotelov et al. (2004) by analytic relations of the form \( A = 1 + a(\beta_\parallel - \beta_0)^{-b} \) where \( a, b, \) and \( \beta_0 \) are fitted parameter values. In order to understand the confinement limits also at small values of the parallel plasma beta \( \beta_\parallel < 1 \), here we rigorously analyze the full linear dispersion relation in a collisionless homogeneous plasma with anisotrop (\( A \neq 1 \)) bi-Maxwellian particle velocity distributions of electrons and protons for electromagnetic fluctuations with wavevectors \( \vec{k} \parallel \vec{B} = 0 \) parallel to the uniform background magnetic field \( \vec{B} \). With a typical solar wind temperature \( T = 10^5 T_5 \) K, the thermal particle energy \( k_B T = 8.6 T_5 \) eV is much less than the electron rest mass \( m_e c^2 = 5.11 \times 10^5 \) eV, so that the use of the nonrelativistic linearized Vlasov/Maxwell equations is justified. For weakly (\( \gamma \ll \omega_B \)) amplified wave solutions, we analytically derive existence and instability conditions. Here, \( \omega_B \) and \( \gamma \) refer to real and imaginary part of the complex frequency \( \omega = \omega_B + i\gamma \). Our analysis closely follows the recent study (Schlickeiser 2010) for equal-mass pair plasmas—hereafter referred to as paper S. Our investigation is restricted to weakly amplified (\( \gamma \ll \omega_B \)) solutions covering the left-handed (LH) polarized Alfvén proton cyclotron branch and the right-handed (RH) polarized Alfvén Whistler electron cyclotron branch. The corresponding analysis of weakly propagating (\( \omega_B \ll \gamma \)) solutions, including mirror, fire-hose, electron
cytropic, and cold magnetized Weibel fluctuations, is the subject of a subsequent paper. As we will demonstrate below, for the case of parallel ($k \times \vec{B} = 0$) wavevectors simple analytical threshold conditions for the two branches can be derived in terms of the combined temperature anisotropy $A = T_\parallel / T_\perp$, the parallel plasma beta $\beta_1 = 8\pi n_e k_B T_\parallel / B^2$, the electron–proton mass ratio, and the electron plasma frequency phase speed $w = \omega_{p,e}/(kc)$.

2. DISPERSION RELATIONS

For a nonzero background magnetic field strength the nonrelativistic dispersion relations for RH- and LH-polarized fluctuations with wavevectors $k \times \vec{B} = 0$ in a thermal electron–proton plasma are (Gary 1993)

$$D_{\text{RH,LH}}(k, \omega) = \omega^2 - k^2 c^2 + \sum_{a=p,e} \omega^2_{p,a} Z \left[ \frac{\omega \pm \Omega_a}{\sqrt{2 k u_{a,\parallel}}} \right] + \frac{1}{2} (1 - A_a) Z \left[ \frac{\omega \pm \Omega_a}{\sqrt{2 k u_{a,\perp}}} \right] = 0,$$  

where we sum over the proton ($p$) and electron ($e$) contributions, and where $k = |k|$. $\omega_{p,e} = (4\pi e^2 n_ee/m_e)^{1/2}$ and $\omega_{p,p} = (m_e/m_p)^{1/2} \omega_{p,e}$ denote the electron and proton plasma frequencies, respectively. $u_{a,\parallel} = (k_B T_\parallel / m_a)^{1/2}$ is the parallel thermal velocity of component $a$, $\Omega_a = e_a B/(m_a e)$ is the nonrelativistic gyrofrequency, and $A_a = T_\parallel / T_a, \parallel$ is the temperature anisotropy of component $a$, where the directional subscripts refer to directions relative to the background magnetic field. The dispersion relations (1) allow for different values of the proton and electron parallel temperatures and temperature anisotropies.

$Z(x)$ and $Z'(x)$ denote the plasma dispersion function (Fried & Conte 1961) and its derivative

$$Z(x) = \pi^{-1/2} \int_{-\infty}^{\infty} dt \frac{e^{-t^2}}{t - x}$$  

with the well-known properties

$$Z'(x) = -2 [1 + x Z(x)],$$

and

$$Z(-x) = 2\pi^{1/2} i e^{-x^2} - Z(x),$$

$$Z(-x) = 4\pi^{1/2} i x e^{-x^2} + Z(x).$$

We will frequently use the asymptotic expansions

$$Z(x) \simeq i \pi^{1/2} e^{-x^2} - 2x \left[ 1 - \frac{2x^2}{3} \right], \ |x| \ll 1$$

and

$$Z(x) \simeq i \sigma \pi^{1/2} e^{-x^2} - \frac{1}{x} \left[ 1 + \frac{1}{2x^2} + \frac{3}{4x^4} \right], \ |x| \gg 1,$$

where $\sigma = 0$ if $\Im(x) > 0$, $\sigma = 1$ if $\Im(x) = 0$, and $\sigma = 2$ if $\Im(x) < 0$.

Paper S has demonstrated that the analytical analysis is enormously facilitated if we work with phase speeds rather than frequencies. We therefore introduce the complex phase speeds

$$f = \frac{\omega}{kc} = \frac{\omega_R + i \gamma}{kc} = R + i S, \ R = \frac{\omega_R}{kc}, \ S = \frac{\gamma}{kc},$$

the plasma frequency phase speed

$$w = \omega_{p,e}/kc,$$

and the absolute value of the electron gyrofrequency phase speed

$$b = |\Omega_e|/kc,$$

where $|\Omega_e| = eB/m_e$ is the absolute value of the electron gyrofrequency. We also introduce the mass ratio

$$\mu = m_p/m_e = 1836,$$

and the dimensionless proton and electron temperatures

$$\Theta_p \equiv \left( \frac{2k_BT_{p,\parallel}}{m_e c^2} \right)^{1/2}, \ \Theta_e \equiv \left( \frac{2k_BT_{e,\parallel}}{m_e c^2} \right)^{1/2}.$$  

Throughout this paper, in the classification of Swanson (1989) we discuss high-density plasmas with $\omega_{p,e} \gg |\Omega_e|$, corresponding to $w \gg b$ which applies to nearly all dilute astrophysical plasmas. These plasmas are dense enough that the electron plasma frequency is much larger than the electron gyrofrequency, but small enough that elastic Coulomb collisions can be neglected.

The two dispersion relations (1) then read

$$0 = \frac{D_{\text{RH,LH}}(k, f)}{k^2 c^2} = A_{\text{RH,LH}}(k, f) = f^2 - 1$$

$$+ \frac{u^2}{\mu} \left[ \frac{f}{\Theta_p} Z \left( \frac{f \pm b}{\Theta_p} \right) + \frac{1}{2} (1 - A_p) Z' \left( \frac{f \pm b}{\Theta_p} \right) \right]$$

$$+ \frac{u^2}{\Theta_e} \left[ \frac{f}{\Theta_e} Z \left( \frac{f \mp b}{\Theta_e} \right) + \frac{1}{2} (1 - A_e) Z' \left( \frac{f \mp b}{\Theta_e} \right) \right].$$

We note the symmetry $\Lambda(-k_\parallel, f) = \Lambda(k_\parallel, f)$ of both dispersion relations allowing us to consider only positive values of the wavenumber $k > 0$. In the following, we will simplify the analysis by considering only equal parallel temperature plasmas ($T_{e,\parallel} = T_{p,\parallel}$) so that $\Theta_e = \Theta$ and $\Theta_p = \Theta/\mu^{1/2}$. Bale et al. (2009), reporting on the observations of the solar wind plasma properties, do not provide information on any measured differences of $T_{e,\parallel}$ and $T_{p,\parallel}$, so we adopt parallel temperature equality which enormously simplifies the calculations.

The dispersion relations (12) can be separated into real and imaginary parts $\Lambda(R, S) = \Re \Lambda(R, S) + i \Im \Lambda(R, S) = 0$, implying the two conditions

$$\Re \Lambda(R, S) = 0, \ \Im \Lambda(R, S) = 0.$$  

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In terms of the complex phase speed \( f = R + i S \) the real and imaginary parts of the two dispersion relations (12) read

\[
0 = \Re \Lambda_{RH,LLH}(R, S) = R^2 - S^2 - 1
\]

\[
+ w^2 \left[ \frac{R}{\Theta} \Re \left[ \frac{1}{\mu} \right] \left( R + i S \pm b \right) \right]
\]

\[
+ \left[ 1 - A_p \right] \Re \left[ \frac{1}{\mu} \right] \left( R + i S \pm b \right)
\]

\[
+ \frac{R}{\Theta} \Re \left[ \frac{R + i S \pm b}{\Theta} \right]
\]

\[
= \Re \Lambda_{RH,LLH}(R, S) = R^2 - S^2 - 1
\]

\[
+ w^2 \left[ \frac{1}{\mu} \Re \left[ \frac{1}{\mu} \right] \left( R + i S \pm b \right) \right]
\]

\[
+ (1 - A_e) \Re \left[ \frac{1}{\mu} \right] \left( R + i S \pm b \right)
\]

\[
+ \frac{w^2 R}{\Theta} \Re \left[ \frac{1}{\mu} \right] \left( R + i S \pm b \right)
\]

\[
+ \Re \left[ \frac{1}{\mu} \right] \left( R + i S \pm b \right)
\]

\[
(14)
\]

and

\[
0 = \Im \Lambda_{RH,LLH}(R, S) = 2RS
\]

\[
+ w^2 \left[ \frac{1}{\mu} \Im \left[ \frac{1}{\mu} \right] \left( R + i S \pm b \right) \right]
\]

\[
+ (1 - A_e) \Im \left[ \frac{1}{\mu} \right] \left( R + i S \pm b \right)
\]

\[
+ w^2 R \left[ \frac{1}{\mu} \Im \left[ \frac{1}{\mu} \right] \left( R + i S \pm b \right) \right]
\]

\[
+ \Im \left[ \frac{1}{\mu} \right] \left( R + i S \pm b \right)
\]

\[
(15)
\]

Here, we consider solutions of the dispersion relations in the weak damping/amplification limit \(|S| \ll R\). As shown in paper S in this limit the real part of the dispersion relation satisfies

\[
\Re \Lambda(R, S = 0) = 0,
\]

whereas the corresponding imaginary part is given by

\[
S = -\frac{\Im \Lambda(R, S = 0)}{\Re \Lambda_{RH,LLH}(R, S = 0)}. \tag{16}
\]

As before we introduce the parallel plasma beta

\[
\beta_p = \frac{\Theta^2 w^2}{b^2} = \frac{8 \pi n_e k_B T_e}{{B^2}}, \tag{18}
\]

which expresses the magnetic field strength as

\[
b = \frac{\Theta w}{\beta_p^{1/2}} \tag{19}
\]

in terms of the electron plasma frequency phase speed \( w \), the parallel temperature \( \Theta \), and the parallel plasma beta \( \beta_p \). Our restriction to high-density plasmas with \( \omega_p e \gg |\Omega| \) or \( w \gg b \) then requires to have parallel plasma betas \( \beta_p \gg \Theta^2 \simeq 3.4 \times 10^{-5} (T_{c,1}/10^3 \text{ K}) \) in the solar wind plasma with typical temperatures \( T_{c,1} \simeq 10^5 \text{ K} \).

3. WEAKLY DAMPED AND AMPLIFIED SOLUTIONS

For weakly damped or amplified fluctuations Equations (14) and (15) read

\[
0 = \Re \Lambda_{RH,LLH}(R, S = 0) = R^2 - 1
\]

\[
+ w^2 \left[ \frac{R}{\Theta} \Re \left[ \frac{1}{\mu} \right] \left( R + i S \pm b \right) \right]
\]

\[
+ \frac{R}{\Theta} \Re \left[ \frac{1}{\mu} \right] \left( R + i S \pm b \right)
\]

\[
+ \frac{1}{\Theta} \Re \left[ \frac{1}{\mu} \right] \left( R + i S \pm b \right)
\]

\[
(20)
\]

and

\[
0 = \Im \Lambda_{RH,LLH}(R, S = 0) = \frac{w^2}{2} \left[ \frac{1}{\mu} \Re \left[ \frac{1}{\mu} \right] \left( R \pm \frac{b}{\mu} \right) \right]
\]

\[
+ (1 - A_e) \Re \left[ \frac{1}{\mu} \right] \left( R \pm \frac{b}{\mu} \right)
\]

\[
+ \frac{w^2 R}{\Theta} \Re \left[ \frac{1}{\mu} \right] \left( R \pm \frac{b}{\mu} \right)
\]

\[
+ \Im \left[ \frac{1}{\mu} \right] \left( R \pm \frac{b}{\mu} \right)
\]

\[
(21)
\]

which have to be investigated for positive values of \( R \geq 0 \).

As noted before, these dispersion relations can be further reduced with the asymptotic expansions (5) and (6), depending on the absolute values of the arguments,

\[
P_{\pm}(R) = \frac{\mu^{1/2}}{\Theta} \left[ |R \pm \frac{b}{\mu}| \right], \quad E_{\pm}(R) = \frac{|R \pm b|}{\Theta}, \tag{22}
\]

of the plasma dispersion function and its derivative being small or large compared to unity. Scaling \( R = bx \), corresponding to \( x = \omega_R/|\Omega| \), the absolute values of the arguments (22) in terms of the parallel plasma beta \( \beta_p \) read

\[
P_{\pm}(x) = \frac{w \mu^{1/2}}{\beta_p^{1/2}} \left[ |x \pm \frac{1}{\mu}| \right], \quad E_{\pm}(x) = \frac{w}{\beta_p^{1/2}} |x \pm 1|, \tag{23}
\]

which are shown in Figure 1 as a function of the normalized real frequency \( x \).
For parallel plasma beta values $\beta_1 \ll (w^2/\mu)$, we note that $P_\parallel(x) \gg 1$, $E_\perp(x) \gg 1$ for all values of $x$, whereas $P_\perp(x) \gg 1$ for $x$ outside the small interval

$$x \notin \left[ \frac{1}{\mu} - \frac{\beta_1^{1/2}}{w \mu^{1/2}} - \frac{1}{\mu} + \frac{\beta_1^{1/2}}{w \mu^{1/2}} \right]$$  \hspace{1cm} (24)$$

around the proton cyclotron frequency, and $E_\perp(x) \gg 1$ for $x$ outside the small interval

$$x \notin \left[ 1 - \frac{\beta_1^{1/2}}{w} , 1 + \frac{\beta_1^{1/2}}{w} \right]$$  \hspace{1cm} (25)$$

around the electron cyclotron frequency.

In the following, we limit our analysis to such values of the parallel plasma beta $\beta_1 \ll (w^2/\mu)$, so that the asymptotic expansion (6) of the plasma dispersion function can be used except near the indicated proton and electron cyclotron frequencies.

Irrespective of the values of $P_\parallel$ and $E_\perp$, we note that both asymptotic expansions (5) and (6) yield the same imaginary part of the dispersion relation

$$\Im \Lambda_{\text{RH,LLH}}(R, S = 0) = \pi^{1/2} w^2 b \left( \frac{\Theta}{\mu^{1/2}} \right) \left[ A_p \left( \frac{R}{b} \pm \frac{1}{\mu} \right) \mp \frac{1}{\mu} \right] \times e^{-\frac{\mu}{2w}(R + \frac{b}{\mu})^2} + \left[ A_e \left( \frac{R}{b} \mp 1 \right) \pm \frac{1}{\mu} \right] \times e^{-\frac{\mu}{2w}(R - \frac{b}{\mu})^2}. \hspace{1cm} (26)$$

4. ALFVÉN, WHISTLER, CYCLOTRON WAVES, AND ELECTROMAGNETIC WAVES

For parallel plasma beta values $\beta_1 \ll (w^2/\mu)$ the asymptotic expansion (6) yields

$$\Im Z \left( \frac{R + b}{\Theta} \right) \approx - \frac{\Theta}{R \pm b} \left[ 1 + \frac{\Theta^2}{2(R \pm b)^2} \right], \hspace{1cm} (27)$$

$$\Im Z \left( \frac{R \pm b}{\Theta} \right) \approx \mp \frac{\Theta^2}{(R \pm b)^2} \left[ 1 + \frac{3\Theta^2}{2(R \pm b)^2} \right], \hspace{1cm} (28)$$

$$\Im Z \left( \frac{\mu^{1/2}}{\Theta} \left[ R \pm \frac{b}{\mu} \right] \right) \approx - \frac{\Theta^2}{\mu^{1/2}(R \pm b)^2} \left[ 1 + \frac{\Theta^2}{2\mu(R \pm \frac{b}{\mu})^2} \right], \hspace{1cm} (29)$$

and

$$\Im Z \left( \frac{\mu^{1/2}}{\Theta} \left[ R \pm \frac{b}{\mu} \right] \right) \approx \pm \frac{\Theta^2}{\mu(R \pm b)^2} \left[ 1 + \frac{3\Theta^2}{2\mu(R \pm \frac{b}{\mu})^2} \right]. \hspace{1cm} (30)$$

We then obtain for the real part of the dispersion relations (20) to lowest order in $\Theta^2 \ll 1$

$$0 = \Re \Lambda_{\text{RH,LLH}}(R, S = 0) = R^2 - 1 - w^2 R \left[ \frac{1}{R \pm b} + \frac{1}{\mu(R \pm \frac{b}{\mu})} \right]$$

$$+ \Theta^2 w^2 \left[ \frac{1 - A_e}{(R \pm b)^2} \mp \frac{R}{2\mu} \right] + \Theta^2 w^2 \left[ \frac{1 - A_p}{(R \pm \frac{b}{\mu})^2} - \frac{R}{2\mu} \right]. \hspace{1cm} (31)$$

With the scaling $R = bx$ and the parallel plasma beta (18) the dispersion relation (31) reads

$$0 = \Re \Lambda_{\text{RH,LLH}}(x, S = 0) = \left[ b^2 + \frac{(1 + \mu)w^2}{(1 + \mu x)(1 \mp x)} \right] x^2 - 1 + \frac{\beta_1}{2} K_{\text{RH,LLH}}. \hspace{1cm} (32)$$

with

$$K_{\text{RH,LLH}} = \frac{1}{(1 \mp x)^3} + \frac{1}{(1 + \mu x)^3} - \frac{A_e}{(1 + \mu x)^2} - \frac{A_p}{(1 + \mu x)^2}. \hspace{1cm} (33)$$

In different limits the solutions of Equations (31) and (32) describe Alfven waves, Whistler waves, cyclotron waves, and electromagnetic light. We consider each case in the next subsections.

4.1. ALFVÉN WAVES AT PHASE SPEEDS $R \ll b/\mu$ FOR $\Theta^2 \ll \beta_1 \ll w^2/\mu$

For phase speeds $R \ll b/\mu$ the dispersion relation (31) simplifies to

$$0 = \Re \Lambda_{\text{RH,LLH}}(R, S = 0)$$

$$= R^2 \left( 1 + \frac{2w^2}{b^2} \right) - 1 + \frac{\beta_1}{2} \left[ 2 - A_e - A_p \right]$$

$$\leq \frac{b}{R} (3 - A_e) \mp \frac{2\mu R}{b} (3 - A_p)$$

$$\simeq R^2 \left( 1 + \frac{c^2}{V_A^2} \right) - 1 + (A - 1) \beta_1, \hspace{1cm} (34)$$

where we introduce the Alfven speed

$$\frac{(1 + \mu)w^2}{b^2} = \frac{c^2}{V_A^2} \hspace{1cm} (35)$$

the parallel plasma beta (18), and the combined plasma temperature anisotropy

$$A = \frac{A_p + A_e}{2}. \hspace{1cm} (36)$$

For high-density plasmas $V_A \ll c$, the dispersion relation (34) yields the LH- and RH-polarized Alfven modes with the same phase speed

$$R \simeq \frac{b}{\sqrt{1 + \mu w^2 + (A + 1)\beta_1}} \hspace{1cm} \left[ 1 + \frac{(1 - A \beta_1)}{(1 + \mu \beta_1)} \right]$$

$$\simeq \frac{V_A}{c} \sqrt{1 + (A - 1)\beta_1}, \hspace{1cm} (37)$$

which either can propagate forward and backward. Note that the condition $R \ll b/\mu$ requires $(w^2/\mu) \gg 1$.

Moreover, the four Alfven modes only exist for temperature anisotropies such that $1 + (A - 1)\beta_1 \geq 0$ corresponding to

$$A \geq \left( 1 - \frac{1}{\beta_1} \right), \hspace{1cm} (38)$$

which includes the isotropic $(A_p = A_e = A = 1)$ plasma temperature case. For small plasma betas $(\beta_1 \ll 1)$ the condition
(38) is always fulfilled whereas for large plasma betas ($\beta_{\parallel} > 1$) the combined anisotropy A has to be larger than $1 - \beta_{\parallel}^{-1}$.

Equation (34) also provides

$$\frac{\partial \eta_{R_{\text{RH}}, \text{LH}}(R, S = 0)}{\partial R} \simeq \frac{2 c^2}{V_{A}^2} R,$$

(39)

so that for all four modes according to Equations (17) and (26) the growth/damping rate is

$$S_{R_{\text{RH}}, \text{LH}} = -\frac{V_{A}^2}{2c^2 R} \lambda_{R_{\text{RH}}, \text{LH}}(R, S = 0)$$

$$- \frac{\pi}{2} \frac{b \delta}{(1 + \mu \delta)} e^{\frac{\pi}{2} \frac{b \omega_{p,e} \delta}{\mu w}} - \frac{A_{e}}{(1 + \mu \delta) e^{\frac{\pi}{2} \frac{b \omega_{p,e} \delta}{\mu w}}}$$

$$\times e^{-\frac{\pi}{2} \frac{b \omega_{p,e} \delta}{\mu w}} (R \pm \frac{1}{2})^2 + \left( A_{e} e^{\frac{\pi}{2} \frac{b \omega_{p,e} \delta}{\mu w}} \right)^2.\] (40)

For isotropic ($A_{p} = A_{e} = 1$) plasma temperatures all four Alfvén modes are damped in agreement with Brinca’s (1990) general theorem on the electromagnetic stability of isotropic plasma populations. The Alfvén damping rate in the isotropic case is given by

$$S_{R_{\text{RH}}, \text{LH}}(A_{p} = A_{e} = 1) = -\frac{\pi^{1/2} b^3}{2(1 + \mu \delta)}$$

$$\times \left( \frac{1}{\mu^{1/2}} e^{-\frac{\pi}{4} b (R \pm \frac{1}{2})^2} + e^{-\frac{\pi b \omega_{p,e}}{\mu w}} \right)^2.\] (41)

In order to drive the Alfvén modes unstable, the growth rate (40) has to be positive requiring that

$$I_{R_{\text{RH}}, \text{LH}}(A_{p}, A_{e}, \beta_{\parallel}) = \pm \left( \frac{A_{p}-1}{1 + \mu \delta} \frac{2}{\mu^{1/2}} e^{\frac{\pi b \omega_{p,e}}{\mu w}} - \frac{A_{e}-1}{1 + \mu \delta} e^{\frac{\pi b \omega_{p,e}}{\mu w}} \right)$$

$$\times \left( \frac{A_{p}}{\mu^{1/2}} e^{-\frac{\pi b \omega_{p,e}}{\mu w}} + A_{e} e^{\frac{\pi b \omega_{p,e}}{\mu w}} \right) < 0.\] (42)

This instability condition is analyzed further in the next section.

4.2. LH-polarized Alfvén Proton Cyclotron and RH-polarized Alfvén Whistler Electron Cyclotron Branches for $\beta_{\parallel} \ll 1$

Because we are primarily interested in low plasma beta plasmas for completeness we consider the solutions of the full dispersion relation (32) for subluminal solutions $R \ll 1$ and small parallel plasma beta $\beta_{\parallel} \ll 1$.

$$0 = \eta_{R_{\text{RH}}, \text{LH}}(x, S = 0) \simeq \frac{(1 + \mu) w^2 x^2}{1 + (\mu - 1) x} - 1\] (43)

yielding the quadratic equation

$$[(1 + \mu) w^2 + \mu] x^2 + (\mu - 1) x = 1.\] (44)

Because $\mu = 1836 \gg 1$ this equation is well approximated by

$$(1 + w^2) x^2 + x = \frac{1}{\mu}\) (45)

with the solutions

$$x_{R_{\text{RH}}, \text{LH}} = \frac{1}{2(1 + w^2)} \left[ \sqrt{1 + \frac{4(1 + w^2)}{\mu}} \mp 1 \right].\] (46)

covering the well-known (e.g., Swanson 1989) LH-polarized Alfvén proton cyclotron and RH-polarized Alfvén Whistler electron cyclotron branches for parallel propagation, as the appropriate limits for small and large values of $w$ demonstrate. For $w \ll \sqrt{(1 + 4) - 1} = 21.4$, corresponding to large wavenumbers $k \gg \omega_{p,e}/21.4c$, we obtain

$$x_{R_{\text{RH}}} \simeq 1 - \frac{1}{w^2},\] (47)

and

$$x_{L_{\text{H}}} \simeq \frac{1}{\mu}\left[ 1 - \frac{4\mu w^2}{\mu + 1}\right].\] (48)

The solution (48), subject to the constraint (24), represents the LH-polarized proton cyclotron waves.

For values of $w \ll 1$ the solution (47) represents the RH-polarized electron cyclotron waves subject to the constraint (25) with the dispersion relation

$$x_{R_{\text{RH}}} \simeq 1 - \frac{1}{w^2},\] (49)

whereas for intermediate values $1 \ll w \ll 21.4$ the solution (47) reduces to

$$x_{R_{\text{RH}}} \simeq \frac{1}{w^2},\] (50)

representing the RH-polarized Whistler waves ($R \simeq b/w^2$).

For large values of $w \gg \sqrt{(1 + 4) - 1} = 21.4$, corresponding to small wavenumbers $k \ll \omega_{p,e}/21.4c$, we obtain for the solution (46)

$$x_{R_{\text{RH}}, \text{LH}} \simeq \frac{1}{\mu(1 + w^2)} \simeq \frac{1}{\mu^{1/2} w},\] (51)

which agrees with the Alfvén wave solutions (37) in the limit $\beta_{\parallel} \ll 1$.

Equation (43) for small plasma beta $\beta_{\parallel} \ll 1$ also provides

$$\frac{\partial \eta_{R_{\text{RH}}, \text{LH}}(R, S = 0)}{\partial R} \simeq \frac{(1 + \mu) w^2 x^2}{b(1 + \mu(1 - 1) x - \mu x^2)}$$

$$= 2 \pm \frac{2(\mu - 1) x}{b(1 + \mu) w^2 x^3}.\] (52)

We note that for all solutions (47)–(51) Equation (52) is positive. According to Equations (17) and (26) the growth/ damping rate then is

$$S_{R_{\text{RH}}, \text{LH}} = -\frac{b(1 + \mu) w^2 x^2}{2(\mu - 1) x} \delta \lambda_{R_{\text{RH}}, \text{LH}}(R, S = 0)$$

$$- \frac{\pi^{1/2}(1 + \mu) b^2 w^3 x^3}{\Theta(2 + (\mu - 1) x)} \left( \frac{A_{p}}{1 + \mu} \mp 1 \right)$$

$$\times e^{-\frac{\pi b \omega_{p,e}}{\mu w}} (R \pm \frac{1}{2})^2 + \left( A_{e} (x + 1) \right) e^{-\frac{\pi b \omega_{p,e}}{\mu w}},\] (53)
which generalizes the Alfvénic growth/damping rate (40) to non-Alfvénic modes. Apart from the different notation in phase speeds, the rate (53) agrees with Equation (7.1.8) of Gary (1993). For isotropic \(\rho = A = 1\) plasma temperatures all modes of the RH and LH branches, including the cyclotron and Whistler modes, are damped in agreement with Brinca’s general theorem. We notice that, in order to drive the cyclotron and Whistler modes unstable, the growth rate (53) has to be positive, requiring again the earlier derived instability condition (42).

4.3. Electromagnetic Waves at Large Frequencies \(R \gg b + \Theta\)

For large frequencies \(R \gg b + \Theta\) the dispersion relation (31) reduces to

\[
0 = \eta \omega_{RLH}(R, S = 0) \simeq R^2 - 1 - \left(1 + \frac{1}{\mu}ight) w^2 \mp \frac{b}{R}
\]

\[- \frac{w^2 b^2}{R^2} \left[1 + \frac{1}{\mu} + \frac{\Theta^2}{2 b^2} \left(A + \frac{A_p}{4 \mu^2}\right)\right]
\]

so that for both polarizations to lowest order in \((b/R)^2\) we obtain the dispersion relation of electromagnetic waves

\[
R^2 \simeq 1 + \left(1 + \frac{1}{\mu}\right) w^2
\]

with the same damping rate

\[
S = -\frac{\pi^{1/2} w^2}{2 \Theta} \left(\frac{A_p}{\mu^{1/2}} e^{-\mu R^2/\Theta^2} + A e^{-R^2/\Theta^2}\right).
\]

5. INSTABILITY CONDITIONS

In order to drive the LH-polarized Alfvén proton cyclotron and the RH-polarized Alfvén Whistler electron cyclotron branches unstable, the condition (42) has to be fulfilled which, after multiplication with \(\mu^{3/4}\), reads

\[
\pm \left(\frac{A_p - 1}{\mu^{3/4}} e^{-X_p^2} - (A_e - 1) \mu^{3/4} e^{-X_e^2}\right)
\]

\[\mp x \mu^{1/2} \left[\frac{A_p}{\mu^{1/4}} e^{-X_p^2} + A e^{-R^2/\Theta^2}\right]
\]

\[= \pm \left(e^{-\frac{1}{2} \ln \mu + (A - 1) - X_e^2} - e^{-\frac{1}{2} \ln \mu + (A - 1) - X_e^2}\right)
\]

\[\mp x \mu^{1/2} \left(e^{-(1 + x) \ln \mu + A_p - X_p^2} + e^{-(1 + x) \ln \mu + A_e - X_e^2}\right) < 0,\] (58)

with

\[X_p = \frac{w}{(\mu \beta c)^{1/2}} (1 \mp \mu x), \quad X_e = \frac{w}{\beta c^{1/2}} (1 \mp x).\] (59)

Both brackets in Equation (58) can be further reduced using the identities

\[e^{a_1 + a_2} - e^{a_1 - a_2} = 2e^{a_1} \sinh(a_2), \quad e^{a_1 + a_2} - e^{a_1 - a_2} = 2e^{a_1} \sinh(a_2),\] (60)

From the first bracket we identify

\[a_1 = \ln \left[(A_p - 1)/(A_e - 1)\right] \frac{X_p^2 + X_e^2}{2},\]

\[a_2 = \frac{3}{4} \ln \mu - \ln \frac{A_e - 1}{A_p - 1} - W^\pm,\] (61)

where

\[W^\pm(x) = \frac{X_p^2 - X_e^2}{2} = \frac{w^2}{2 \beta^3} \left(\pm 4 x + (\mu - 1) \left(x^2 - \frac{1}{\mu}\right)\right).\] (62)

From the second bracket we determine

\[c_1 = \ln \left[A_p A_e\right] \frac{X_p^2 + X_e^2}{2},\]

\[c_2 = -\frac{1}{4} \ln \mu - \ln \frac{A_e - 1}{A_p - 1} - W^\pm.\] (63)

With \(\sinh(-a_2) = -\sinh(a_2)\) and \(\cosh(-c_2) = \cosh(c_2)\) condition (58) becomes

\[\pm (A_p - 1)^{1/2} (A_e - 1)^{1/2} \sinh \left[\frac{3}{4} \ln \mu + \ln \left(A_e - 1\right) \frac{1}{A_p - 1} + W^\pm\right]
\]

\[\mp A_p^{1/2} A_e^{1/2} x \mu^{1/2} \cosh \left[\frac{1}{4} \ln \mu + \ln \left(A_e - 1\right) \frac{1}{A_p - 1} + W^\pm\right]
\]

\[\pm \left(\frac{\mu^{3/4}}{A_p} - \mu^{3/4} (A_e - 1)\right) \cosh W^\pm
\]

\[\pm \left(\frac{\mu^{3/4}}{A_p} + \mu^{3/4} (A_e - 1)\right) \sinh W^\pm
\]

\[- x \mu^{1/2} (\mu^{1/4} A_e + x \mu^{1/4} A_p) \cosh W^\pm
\]

\[- x \mu^{1/2} (\mu^{1/4} A_e - x \mu^{1/4} A_p) \sinh W^\pm > 0,\] (64)

which yields the general instability condition for the two branches in the form

\[\pm (A_e - A_p) [(\mu^{3/2} - 1) + (\mu^{3/2} - 1) \tanh(W^\pm)]
\]

\[+ (x \mu (A_e + A_p) [(\mu^{3/2} - 1) + (\mu^{3/2} - 1) \tanh(W^\pm)])
\]

\[+ \left[(A_e - A_p) [(\mu^{1/2} + 1 + (\mu^{1/2} - 1) \tanh(W^\pm))]
\]

\[> 1.\] (65)

Inserting the general dispersion relation for small plasma betas (46) for \(x\) and \(W^\pm(x)\) then provides the general instability conditions for the LH-polarized Alfvén proton cyclotron and the RH-polarized Alfvén Whistler electron cyclotron branches, which to the best of our knowledge has not been derived before.

For equal electron and proton temperature anisotropies \(A_p = A_e = A_0\) the condition (65) reduces to

\[\pm \left[1 - \frac{1}{A_0}\right] [(\mu^{3/2} - 1) + (\mu^{3/2} - 1) \tanh(W^\pm)]
\]

\[> x \mu [(\mu^{1/2} + 1 + (\mu^{1/2} - 1) \tanh(W^\pm)]\]. (66)

Note that for pair plasmas \((\mu = 1)\) the condition (66) reduces to Equation (56-66) in paper S.

Instead of working with the general dispersion relations (46) used in the general instability condition (65), we will discuss different ranges of the plasma frequency phase speed \(w\), where the dispersion relations (46) reduce to simpler limits, as demonstrated in Section 4.2. For \(w \gg 21.4\) we can use the Alfvénic relation (51), whereas for small values of \(w \ll 21.4\)
we can use the cyclotron and Whistler relations (48)–(50), respectively. We are particularly interested in applying our results to the solar wind observations of Bale et al. (2009), which, as we demonstrate in the next section, cover the phase speed range 90 \leq w \leq 330.

Before investigating the individual modes, we further inspect the instability condition (66). The instability conditions holds for plasma betas \( \beta_0 \ll w^2/\mu \), implying \( w^2/2\beta_0 \gg \mu/2 = 918 \). Consequently, the argument of the tanh-function

\[
W_\pm = \frac{w^2}{2\beta_0} f_\pm(x), \quad f_\pm(x) = \pm 4x + (\mu - 1) \left( x^2 - \frac{1}{\mu} \right)
\]

(67)
is much larger than unity for \( f_\pm(x) \gg 2/\mu \).

The function \( f_\pm(x) \) increases monotonically from its smallest negative value \( f_\pm(0) = -(1 - \mu^{-1}) = -0.9995 \) and becomes zero at

\[
x_{0+} = \frac{\mu + 1}{(\mu - 1)\mu^{1/2}} \left[ 1 - \frac{2\mu^{1/2}}{\mu + 1} \right] = 0.95 \mu^{-1/2} \simeq \mu^{-1/2}.
\]

Likewise, the function \( f_\pm(x) \) attains its negative minimum value \(-1.0016\) at \( x_E = 2/(\mu - 1) = 0.0011 \) and becomes zero at

\[
x_{0-} = \frac{\mu + 1}{(\mu - 1)\mu^{1/2}} \left[ 1 + \frac{2\mu^{1/2}}{\mu + 1} \right] = 1.05 \mu^{-1/2} \simeq \mu^{-1/2}.
\]

Hence, only for values of \( x = R/b \) inside a small interval around \( \mu^{-1/2} \), which for RH-polarized fluctuations lies in the Whistler phase speed range, the functions \( f_\pm(x) \) are smaller than \( 2/\mu \). Both functions are well approximated by

\[
f_\pm(x) \simeq f_0(x) = \mu x^2 - 1,
\]

(70)

which is negative in the range of Alfvén and proton cyclotron waves and positive in the range of electron cyclotron waves, respectively.

6. OBSERVED SOLAR WIND FLUCTUATIONS

The solar wind magnetic fluctuations measured by Bale et al. (2009) near 1 AU have wavenumbers

\[
k \simeq \alpha/\rho_p
\]

(71)

with \( \alpha = 0.56 \pm 0.32 \) and the thermal proton gyroradius \( \rho_p = 4.23 \times 10^6 T_5^{1/2} B_4^{-1} \) cm, where we adopt an interplanetary magnetic field value \( B = 10^{-4} B_4 \) G and a temperature \( T = 10^5 T_5 \) K. With the solar wind particle density \( n_a = 10^9 n_2 \) cm\(^{-3}\), we find that the plasma frequency phase speed (8)

\[
w = \frac{79.5}{\alpha} \frac{(T_5 n_2)^{1/2}}{B_4} = \frac{142}{1 \pm 0.57} \frac{(T_5 n_2)^{1/2}}{B_4}
\]

(72)
covers the interval

\[
90 \frac{(T_5 n_2)^{1/2}}{B_4} \lesssim w \lesssim 330 \frac{(T_5 n_2)^{1/2}}{B_4}.
\]

The electron gyrofrequency phase speed (9)

\[
b = 3.13 \times 10^3 w \frac{B_4}{n_2^{1/2}}
\]

(74)
is indeed much smaller than the electron plasma frequency phase speed \( w \) justifying the high-density plasma approximation \( w \gg b \).

Because of the observed large value (72) of the plasma frequency phase speed only the LH- and RH-polarized Alfvén wave instabilities can operate in the \( w \)-interval (73). We therefore concentrate our analysis of the instability condition (66) on this case.

In terms of the proton plasma frequency phase speed

\[
h = \frac{w}{\mu^{1/2}} = \frac{w}{43},
\]

(75)

the Bale et al. (2009) measurements cover the interval

\[
2.1 \frac{(T_5 n_2)^{1/2}}{B_4} \lesssim h \lesssim 7.7 \frac{(T_5 n_2)^{1/2}}{B_4}.
\]

(76)

7. ALFVÉN WAVES

In the range of Alfvén waves \( x \ll \mu^{-1} \) the functions \( f_\pm(x) \simeq -1 \) are negative and practically constant, so that the instability condition (66) becomes

\[
(1 - \frac{1}{A_0}) \left[ (\mu^{1/2} + 1) \tanh \left( \frac{w^2}{2\beta_0} \right) - (\mu^{1/2} - 1) \right]
\]

\[
> x \mu \left[ (\mu^{1/2} + 1) - (\mu^{1/2} - 1) \tanh \left( \frac{w^2}{2\beta_0} \right) \right].
\]

(77)

Because \( \beta_0 \ll w^2/\mu \) the argument of the tanh-function

\[
\tanh \left( \frac{w^2}{2\beta_0} \right) \simeq 1.
\]

(79)

We then find for condition (77)

\[
\left( 1 - \frac{1}{A_0} \right) > \mu x,
\]

(80)

where the parallel plasma beta dependence and additional \( A_0 \)-dependences enter via the solution \( x(A_0, \beta_0) \) of the dispersion relation (32).

For \( b \ll w \) and \( A_p = A_e = A_0 \) the dispersion relation (32) reads

\[
0 = Re_\pm(LH)(x, S = 0) \simeq \frac{(1 + \mu)w^2x^2}{(1 \pm \mu x)(1 \mp x)} - 1 + \frac{\beta_0}{2} K_{RLH,LH}.
\]

(81)

with

\[
K_{RLH,LH} = \frac{1}{(1 \mp x)^2} + \frac{1}{(1 \pm x)^2} - A_0 \left[ \frac{1}{(1 \mp x)^2} + \frac{1}{(1 \pm x)^2} \right].
\]

(82)

For \( x \ll \mu^{-1} \) we approximate \( K_{RLH,LH} \simeq 2(1 - A_0) \) and

\[
Re_\pm(LH)(x, S = 0) \simeq (1 + \mu)w^2x^2 - 1 - \beta_0(A_0 - 1) = 0,
\]

(83)
yielding the solution (37) and

\[
\mu x \simeq \frac{\mu^{1/2}}{w} \sqrt{1 + (A_0 - 1)\beta_0} = \frac{1}{h} \sqrt{1 + (A_0 - 1)\beta_0},
\]

(84)
in terms of the proton plasma frequency phase speed (75).
7.1. LH-polarized Alfvén Waves

Inserting Equation (84), we obtain for the instability condition of LH-polarized Alfvén waves (80)

\[ 1 - \frac{1}{A_0} > \frac{1}{h} \sqrt{1 + (A_0 - 1)\beta_{\parallel}}, \]

which can only be fulfilled for \( A_0 > 1 \). Condition (85) is equivalent to

\[ \beta_{\parallel} < \frac{1}{A_0 - 1} \left[ \frac{h^2 (A_0 - 1)^2}{A_0} - 1 \right]. \]

which is shown as the upper dashed curve in Figure 2 calculated for the value \( w = 142 \), corresponding to \( h = 3.3 \), representing the Bale et al. (2009) measurements.

For values of \( h > 1 \) the limiting plasma beta value of condition (86) is zero for \( A_0 = h/(h - 1) = 1.43 \) for \( h = 3.3 \). Substituting \( y = A_0/(A_0 - 1) > 1 \) yields for the right-hand side of Equation (86)

\[ F_L(y) = (y - 1) \left( \frac{h^2}{y^2} - 1 \right), \]

and for its first derivative

\[ \frac{dF_L}{dy} = \frac{2h^2}{y^3} - \frac{h^2}{y^2} - 1. \]

The function \( F_L(y) \) exhibits one extreme value at \( y_E = h t \), where

\[ t^3 + t = \frac{2}{h}, \]

yielding

\[ y_E = h^{2/3} \left( \left[ \sqrt{\frac{1}{27} + \frac{h^2}{27}} + 1 \right]^{1/3} - \left[ \frac{1}{27} + \frac{h^2}{27} - 1 \right]^{1/3} \right). \]

For the case \( h = 3.3 \) shown in Figure 2 we obtain \( y_E = 1.61 \), corresponding to \( A_0 = 2.64 \), and the maximum plasma beta \( F_{\text{max}} = 1.95 \), which agree with the plot for LH-polarized Alfvén waves shown in Figure 2.

7.2. RH-polarized Alfvén Waves

Likewise, inserting Equation (84) we obtain for the instability condition of RH-polarized Alfvén waves (80)

\[ \frac{1}{A_0} - 1 > \frac{1}{h} \sqrt{1 - (1-A_0)\beta_{\parallel}}, \]

which can only be fulfilled for \( A_0 < 1 \) and requires the existence condition (38) \( A_0 > 1 - \beta_{\parallel}^{-1} \). Condition (91) is equivalent to

\[ \beta_{\parallel} > \frac{1}{1-A_0} \left[ 1 - h^2 \left( \frac{1-A_0}{A_0} \right)^2 \right]. \]

which is shown as the lower full curve in Figure 2 calculated for the value \( h = 3.3 \), representing the Bale et al. (2009) measurements.

The limiting plasma beta value of condition (92) is zero for \( A_0 = h/(h + 1) = 0.77 \) for \( h = 3.3 \). Substituting \( y = A_0/(1 - A_0) > 0 \) yields for the right-hand side of Equation (92)

\[ F_R(y) = (y + 1) \left( 1 - \frac{h^2}{y^2} \right). \]

The first derivative

\[ \frac{dF_R}{dy} = \frac{2h^2}{y^3} + \frac{h^2}{y^2} + 1 \]

is positive for all values of \( y \), so that the function \( F_R(y) \) is monotonically increasing, in agreement with the plot for RH-polarized Alfvén waves shown in Figure 2.

The properties of the weakly amplified LH- and RH-polarized Alfvén mode are summarized in Tables 1 and 2, respectively.

We emphasize that the Alfvénic instability threshold diagram shown in Figure 2 is controlled by a single observed plasma parameter: the proton plasma frequency phase speed \( h = w/\mu \). The limiting anisotropy values for small plasma betas \( \beta_{\parallel} \rightarrow 0 \) as well as the maximum plasma beta value for LH-polarized Alfvén waves (see Equation (90)) are fixed by the value of \( h \).

8. PROTON CYCLOTRON WAVES

In the range of LH-polarized proton cyclotron waves \( x \leq \mu^{-1} \) the functions \( f_{\pm}(x) \simeq -1 \) are negative and practically constant, so that here the instability condition (80) also holds:

\[ 1 - \frac{1}{A_0} > \mu x, \]
where again the parallel plasma beta dependence and additional $A_0$-dependences enter via the solution $x(A_0, \beta_\parallel)$ of the dispersion relation (81). Because $x > 0$ the proton cyclotron instability condition (95) can only be fulfilled for $A_0 > 1$.

Substituting $s = 1 - \mu x$, the condition (95) becomes

$$A_0 > 1 - \frac{1}{s}. \quad (96)$$

We now need solutions of the LH dispersion relation (83) for values of $s \ll 1$ but, due to the constraint (24), much larger than $s \gg \beta_\parallel^{1/2}/h$, i.e.,

$$\frac{\beta_\parallel^{1/2}}{h} \ll s \ll 1. \quad (97)$$

Because of the smallness of $s$ the LH dispersion relation (83) can be approximated as

$$\frac{h^2}{s} \left[1 + \frac{\beta_\parallel}{2h^2 s^2}\right] = 1 + \frac{\beta_\parallel(A_0 - 1)}{2} + \frac{\beta_\parallel A_0}{2 s^2}. \quad (98)$$

Because of the left inequality in Equation (97), we neglect the second term in the first bracket in Equation (98) to obtain

$$\frac{h^2}{s} \simeq 1 + \frac{\beta_\parallel (A_0 - 1)}{2} + \frac{\beta_\parallel A_0}{2 s^2} \quad (99)$$

with the solution

$$s \simeq \frac{h^2}{2 + \beta_\parallel(A_0 - 1)} \left[1 + \sqrt{1 - \frac{\beta_\parallel A_0[2 + \beta_\parallel(A_0 - 1)]}{h^4}}\right]. \quad (100)$$

which in the limit $\beta_\parallel \to 0$ correctly reproduces solution (48). This solution exists provided that

$$\beta_\parallel A_0[2 + \beta_\parallel(A_0 - 1)] \ll h^4. \quad (101)$$

For $\beta_\parallel A_0[2 + \beta_\parallel(A_0 - 1)] \ll h^4$, which corresponds to the first constraint

$$A_0 \ll \frac{1}{\beta_\parallel} \left[\sqrt{\left(\frac{\beta_\parallel - 2}{2}\right)^2 + h^4 + \frac{\beta_\parallel - 2}{2}}\right], \quad (102)$$

the solution (101) simplifies to

$$s \simeq \frac{2h^2}{2 + \beta_\parallel(A_0 - 1)}. \quad (103)$$

The restrictions (97) on $s$ require two additional constraints

$$1 - \frac{2(1 - h^2)}{\beta_\parallel} \ll A_0 \ll 1 + \frac{2}{\beta_\parallel^{1/2}} \left[h^3 - \beta_\parallel^{1/2}\right]. \quad (104)$$

which requires $\beta_\parallel \ll h^6$. Then using the solution (103) we obtain for the instability condition (96)

$$A_0 > \frac{2 - \beta_\parallel}{2h^2 - \beta_\parallel}. \quad (105)$$

The requirement $A_0 > 1$ then implies small proton plasma frequency phase speeds $h < 1$, and consequently also small values of $\beta_\parallel < h^6 \ll 1$. In this case, the left-hand side of the constraint (104) is negative and therefore automatically fulfilled. Its right-hand side yields

$$A_0 \ll \frac{2h^3}{\beta_\parallel^{3/2}}. \quad (106)$$

Moreover, Equations (102) and (105) simplify to

$$A_0 \ll \frac{1}{\beta_\parallel} \left[\sqrt{1 + h^4} - 1\right] \simeq \frac{h^4}{2\beta_\parallel} \quad (107)$$

and

$$A_0 > \frac{1}{2h^2} \left[1 + \frac{\beta_\parallel}{2h^2}(1 - h^6)\right]. \quad (108)$$

At plasma betas $\beta_\parallel < h^6$ the constraint (106) is stronger than the constraint (107), so that we are left with

$$\frac{1}{2h^2} \left[1 + \frac{\beta_\parallel}{2h^2}(1 - h^6)\right] < A_0 \ll \frac{2h^3}{\beta_\parallel^{3/2}} \quad (109)$$

as instability conditions for proton cyclotron waves subject to $h < 1$ and $\beta_\parallel < h^6 < 1$. The final conditions (109) are shown in Figure 3 for a value of $h = 2^{-1/2}$. Because of the required small value $h < 1$ this instability cannot operate in the regime (76) observed by Bale et al. (2009).

The properties of the weakly amplified proton cyclotron mode are summarized in Table 3.
9. ELECTRON CYCLOTRON WAVES

In the range of electron cyclotron waves $\mu^{-1/2} \ll x < 1$ the function $f_*(x) \simeq \mu x^2 \gg 1$ is positive and much larger than unity. The instability condition (66) together with $W_* = w^2 f_*(x)/(2\beta\parallel) = \mu w^2 x^2/(2\beta\parallel)$ then becomes

$$\left(1 - \frac{1}{A_0}\right) \left[ (\mu^{3/2} - 1) + (\mu^{3/2} + 1) \tanh \left[ \frac{\mu w^2 x^2}{2\beta\parallel} \right] \right] > x\mu \left[ (\mu^{1/2} + 1) + (\mu^{1/2} - 1) \tanh \left[ \frac{\mu w^2 x^2}{2\beta\parallel} \right] \right],$$  \hspace{1cm} (110)

Again the argument of the tanh-function is very large compared to unity so that

$$\tanh \left[ \frac{\mu w^2 x^2}{2\beta\parallel} \right] \simeq \tanh \left[ \frac{w^2}{2\beta\parallel} \right] \simeq 1,$$  \hspace{1cm} (111)

yielding for the instability condition (110)

$$A_0 > \frac{1}{1 - x},$$  \hspace{1cm} (112)

where again the parallel plasma beta dependence and additional $A_0$-dependencies enter via the solution $x(A_0, \beta\parallel)$ of the dispersion relation (81). Because $x < 1$ the electron cyclotron instability condition (112) can only be fulfilled for $A_0 > 1$.

Substituting $y = 1 - x$ we need solutions of the RH dispersion relation (83) for values of $y \ll 1$ but, due to the constraint (25), much larger than $y \gg \beta\parallel^2/w$, i.e.,

$$\frac{\beta\parallel^{1/2}}{w} \ll y \ll 1.$$  \hspace{1cm} (113)

Because of the smallness of $y$ the RH dispersion relation (83) can be approximated as

$$\frac{w^2}{y} \left[ 1 + \frac{\beta\parallel}{2w^2 y^2} \right] = 1 + \frac{\beta\parallel A_0}{2y^2}.$$  \hspace{1cm} (114)

Because of the left inequality in Equation (113), we neglect the second term in the first bracket in Equation (114) to obtain

$$\frac{w^2}{y} \simeq 1 + \frac{\beta\parallel A_0}{2y^2}.$$  \hspace{1cm} (115)

with the solution

$$y = \frac{w^2}{2} \left[ 1 + \sqrt{1 - \frac{2\beta\parallel A_0}{w^2}} \right].$$  \hspace{1cm} (116)

which in the limit $\beta\parallel \to 0$ correctly reproduces solution (49). This solution exists provided that

$$2\beta\parallel A_0 \ll w^4.$$  \hspace{1cm} (117)

The restrictions (113) on $s$ require $w^2 \ll 1$ and $\beta\parallel \ll w^8$. Then using the solution (117) we obtain for the instability condition (112)

$$A_0 w^2 \left[ 1 - \frac{\beta\parallel A_0}{2w^2} \right] > 1,$$  \hspace{1cm} (120)

or with $\beta\parallel \ll w^6$

$$A_0 > \frac{w^4}{\beta\parallel} \left[ 1 - \sqrt{1 - \frac{2\beta\parallel}{w^2}} \right] \simeq \frac{1}{w^2} \left[ 1 + \frac{\beta\parallel}{2w^6} \right].$$  \hspace{1cm} (121)

The dual instability requirements subject to $w \ll 1$ and $\beta\parallel \ll w^6$ are shown in Figure 4 for a value of $w = 0.1$. Because of the required small value $w \ll 1$ this instability cannot operate in the regime (72) observed by Bale et al. (2009).

The properties of the weakly amplified RH-polarized electron cyclotron modes are summarized in Table 4.

10. SUMMARY AND CONCLUSIONS

We rigorously studied the dispersion relations of weakly amplified fluctuations with wavevectors $\vec{k} \times \vec{B}_0 = 0$ in an anisotropic bi-Maxwellian magnetized proton–electron plasma by the appropriate Taylor expansion of the plasma dispersion
function. Apparently for the first time, for equal parallel electron and proton temperatures, a general analytical instability condition (65) is derived that holds for different values of the electron and proton temperatures, a general analytical instability condition. Apparently for the first time, for equal parallel electron and proton plasma beta range \( \Theta \) is derived that holds for different values of the electron and proton temperatures, a general analytical instability condition.

Table 4

| Real phase speed range | \( b \mu^{-1/2} \ll R \ll b \) |
|------------------------|-----------------------------|
| Dispersion relation    | \( x = R/b = 1 - w^2 \left[ 1 - \beta_i/A_0 \right] \) |
| Existence condition    | \( w \ll 1 \) |
| Parallel plasma beta range | \( \Theta \ll \beta \) |
| Instability condition  | \( \frac{1}{\omega_p} \left[ 1 + \frac{\beta_i}{\omega_p} \right] \ll A_0 \ll \frac{\omega_p}{\pi \beta} \) |

For large values of \( w \gg 21.4 \), corresponding to small wavenumbers \( k \ll \omega_p/(21.4 c) \), RH- and LH-polarized Alfvén waves are excited in different regions of the temperature anisotropy \( A = T_L/T_H \), the parallel plasma beta \( \beta_i = 8 \pi n_e k_B T_i/B^2 \) and the electron plasma frequency phase speed \( w = \omega_p/(kc) \) for each mode.

For appropriate temperature anisotropies, RH-polarized low and high phase speed Whistler waves are excited for intermediate values of \( 6.5 \ll w \ll 21.4 \) and \( 1 \ll w \ll 6.5 \), respectively. At values \( w \ll 21.4 \) LH-polarized proton cyclotron waves can be excited, whereas RH-polarized electron cyclotron waves can be excited at small values of \( w \ll 1 \). In agreement with Brinca’s (1990) general theorem on the electromagnetic stability of isotropic plasma populations none of these modes can be excited for isotropic plasma distributions \( A_e = A_p = 1 \).

We apply the results of our instability study to the observed solar wind magnetic turbulence (Bale et al. 2009), corresponding to values of \( 90 \leq w \leq 330 \). According to the existence conditions of the different instabilities, only the LH and RH-polarized Alfvén wave instabilities can operate here. The Alfvénic instability threshold conditions are controlled by a single observed plasma parameter: the proton plasma frequency phase speed \( h = w/\mu^{1/2} \). All asymptotic and characteristic values of the threshold anisotropies and the characteristic plasma beta values are solely determined by the value of \( h \). The Alfvénic instability diagram explains well the observed confinement limits at small parallel plasma beta values in the solar wind. It remains to be investigated in future work how weakly propagating and/or obliquely propagating weakly amplified modes add to this diagram.

This work was supported by the Deutsche Forschungsgemeinschaft through grants Schl 201/19-1 and Schl 201/21-1.

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