Existence of infinitely many solutions for a nonlocal elliptic PDE involving singularity

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Abstract
In this article, we will prove the existence of infinitely many positive weak solutions to the following nonlocal elliptic PDE.

\[-\Delta^s u = \lambda \frac{u^\gamma}{u^\gamma} + f(x, u) \quad \text{in} \quad \Omega,
\]
\[u = 0 \quad \text{in} \quad \mathbb{R}^N \setminus \Omega,
\]

where \(\Omega\) is an open bounded domain in \(\mathbb{R}^N\) with Lipschitz boundary, \(N > 2s, s \in (0, 1), \gamma \in (0, 1)\). We will employ variational techniques to show the existence of infinitely many weak solutions of the above problem.

Keywords Elliptic PDE · Genus · PS condition · Mountain Pass Theorem

Mathematics Subject Classification 35J20 · 35J35 · 35J60 · 35J75

1 Introduction

We consider the following nonlocal problem involving singularity.

\[-\Delta^s u = \frac{\lambda}{u^\gamma} + f(x, u) \quad \text{in} \quad \Omega,
\]
\[u = 0 \quad \text{in} \quad \mathbb{R}^N \setminus \Omega,
\]
\[u > 0 \quad \text{in} \quad \Omega,
\]
where,

\[ (-\Delta)^s u(x) = -\frac{1}{2} \int_{\mathbb{R}^N} \frac{(u(x+y) + u(x-y) - 2u(x))}{|x-y|^{N+2s}} dy, \quad \text{for} \quad x \in \mathbb{R}^N, \]

\( \lambda > 0, \ s \in (0, 1), \ \gamma \in (0, 1) \) and \( \Omega \) be an open, bounded subset of \( \mathbb{R}^N, \ N \geq 2. \)

The study of nonlocal elliptic PDEs is important to both, from the mathematical research as well as from the real world application, point of views. Some of the applications are in the probability theory, the obstacle problem, optimization, finance, phase transitions, soft thin films, conservation laws, minimal surfaces, material science and water waves. The application in probability theory, in particular to the Lévy process, can be found in [2] and that in the field of finance, one can refer to [10]. For a fruitful note on the application of fractional Laplacian one may refer [29] and the references therein. Recently, the study of singular elliptic PDE has drawn a great attention to many researchers. One of the earliest study on the existence and regularity of a weak solution was made by Lazer and Mckenna [21] to the following problem.

\[ -\Delta u = \frac{p(x)}{u^{\gamma}} \quad \text{in} \quad \Omega, \]

\[ u = 0 \quad \text{in} \quad \partial \Omega, \]

\[ u > 0 \quad \text{in} \quad \Omega, \quad (1.2) \]

where \( p : \Omega \rightarrow \mathbb{R} \) is a nonnegative bounded function. In [21], the authors proved that for a \( C^{2+\gamma} \) boundary, the problem (1.2) has a \( H^1_0(\Omega) \) solution iff \( \gamma < 3 \) and if \( \gamma > 1 \), the problem cannot have \( C^1(\overline{\Omega}) \) solution. The following problem have been studied for existence, uniqueness and regularity of solutions for \( p = 2 \) in [14] and for \( 1 < p < \infty \) in [8], where \( 0 < s < 1. \)

\[ (-\Delta_p)^s u = \frac{\lambda a(x)}{u^{\gamma}} \quad \text{in} \quad \Omega, \]

\[ u = 0 \quad \text{in} \quad \mathbb{R}^N \setminus \Omega, \]

\[ u > 0 \quad \text{in} \quad \Omega, \quad (1.3) \]

where \( a : \Omega \rightarrow \mathbb{R} \) is a nonnegative bounded function. The author in [14], guaranteed the existence of a unique \( C^{2,\alpha}(\Omega), \ (0 < \alpha < 1) \) solution for \( \lambda a(x) \equiv 1. \) Canino et al. [8], had proved the existence and uniqueness of solution to the problem (1.3) by dividing \( \lambda \) into three cases \( 0 < \lambda < 1, \ \lambda = 1 \) and \( \lambda > 1. \) A few more noteworthy study involving singularity for both Laplacian and fractional Laplacian operators can be found in [5,11,17,24] and the references therein.

Multiplicity of solutions to the following type of problem has been widely studied by many authors, a few of them are in [15,16,23,25] and the references therein.
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\[ (-\Delta_p)^s u = \lambda \frac{a(x)}{u^\gamma} + f(x, u) \quad \text{in} \quad \Omega, \]
\[ u = 0 \quad \text{in} \quad \mathbb{R}^N \setminus \Omega, \]
\[ u > 0 \quad \text{in} \quad \Omega. \]  

(1.4)

Here \( N > ps, M \geq 0, a : \Omega \to \mathbb{R} \) is a nonnegative bounded function. The authors in [16,25], have used a variational technique to guarantee the existence of multiple solutions. Nehari manifold method has been used to prove the multiplicity result in [15,23]. In most of these studies, the authors obtained two distinct weak solutions.

The existence results of infinitely many solutions to both Laplacian and fractional Laplacian with a nonsingular, nonlinear data have been studied widely with Dirichlet boundary condition. In most of these studies the authors proved the existence result with the help of the symmetric Mountain Pass Theorem [19,20]. One of the earliest attempt to show the existence of infinitely many solutions was made by Ambrosetti and Rabinowitz [1] to the following problem.

\[ -\Delta u = f(x, u) \quad \text{in} \quad \Omega, \]
\[ u = 0 \quad \text{in} \quad \partial \Omega, \]
\[ u > 0 \quad \text{in} \quad \Omega, \]  

(1.5)

where \( f \) is superlinear but subcritical near infinity. The authors in [19] have used the symmetric Mountain Pass Theorem to guarantee the existence. A few more similar type of studies made are in [20,22,30] and the references therein.

Recently, Gu et al. [18] has guaranteed the existence of infinitely many solutions to a nonlocal problem of the following type with a sublinear growth of \( f \).

\[ (-\Delta)^s u = f(x, u) \quad \text{in} \quad \Omega, \]
\[ u = 0 \quad \text{in} \quad \mathbb{R}^N \setminus \Omega, \]
\[ u > 0 \quad \text{in} \quad \Omega. \]  

(1.6)

For further details to the problem (1.6), we refer the readers to [3,4,13,27,28] and the references therein.

In the literature the study to obtain infinitely many solutions for the problems of the type (1.5), (1.6), the authors have considered either a sublinear or a superlinear growth on \( f \). To our knowledge, the study of the problem (1.1) is very new to the literature due to the presence of a singular term \( u^{-\gamma} \). Motivated from [18], we will prove the existence of infinitely many weak solutions to the problem (1.1). We assume the following growth conditions on \( f \).

(A1) \( f \in C(\Omega \times \mathbb{R}, \mathbb{R}) \) and there exists a \( \delta > 0 \) such that \( \forall x \in \Omega \) and \( |t| \leq \delta \), \( f(x, -t) = -f(x, t) \).

(A2) \( \lim_{t \to 0} \frac{f(x, t)}{t} = +\infty \) uniformly on \( \Omega \).

(A3) There exists \( r > 0 \) and \( \alpha \in (1 - \gamma, 2) \) such that \( \forall, x \in \Omega \) and \( |t| \leq r \), \( t f(x, t) \leq \alpha F(x, t) \), where \( F(x, t) = \int_0^t f(x, \tau) d\tau \).

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Prior to stating the main theorem, we will define the necessary function spaces and the associated notations. Consider the space 
\[ X = \left\{ u : \mathbb{R}^N \rightarrow \mathbb{R} \text{ is measurable}, u|_\Omega \in L^2(\Omega) \text{ and } \frac{|u(x) - u(y)|}{|x - y|^{\frac{N+2s}{2}}} \in L^2(Q) \right\} \]
equipped with the Gagliardo norm
\[ \|u\|_X = \|u\|_2 + \left( \int_Q \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} \, dy \right)^{\frac{1}{2}}. \]
where \( \Omega \subset \mathbb{R}^N \), \( Q = \mathbb{R}^{2N} \setminus ((\mathbb{R}^N \setminus \Omega) \times (\mathbb{R}^N \setminus \Omega)) \). Here, \( \|u\|_2 \) refers to the \( L^2 \)-norm of \( u \). A frequently used space in this article will be the subspace \( X_0 \) of \( X \) defined as
\[ X_0 = \left\{ u \in X : u = 0 \text{ a.e. in } \mathbb{R}^N \setminus \Omega \right\}, \]
equipped with the norm
\[ \|u\| = \left( \int_Q \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} \, dy \right)^{\frac{1}{2}}. \]
The space \((X_0, \|\|)\) is a Hilbert space [26]. The best Sobolev constant is defined as
\[ S = \inf_{u \in X_0 \setminus \{0\}} \frac{\int_Q \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} \, dy}{\left( \int_\Omega |2x^s| \, dx \right)^{\frac{2s}{N}}} \quad (1.7) \]
We now define a weak solution to the problem (1.1).

**Definition 1.1** A function \( u \in X_0 \) is a weak solution to the problem (1.1), if \( u > 0 \), \( \phi u^{-\nu} \in L^1(\Omega) \) and
\[ \int_Q \frac{(u(x) - u(y))(\phi(x) - \phi(y))}{|x - y|^{N+2s}} \, dy - \int_\Omega \left( \frac{\lambda}{u^{\nu}} + f(x, u) \right) \phi \, dx = 0, \quad \forall \phi \in X_0. \quad (1.8) \]
The energy functional \( I : X_0 \rightarrow (-\infty, \infty] \) associated with the problem (1.1) is defined as
\[ I(u) = \frac{1}{2} \int_Q \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} \, dy - \frac{\lambda}{1 - \gamma} \int_\Omega u^{1-\nu} \, dx - \int_\Omega F(x, u) \, dx \quad (1.9) \]
We will now state our main result.
Theorem 1.2 Let the assumptions (A1) — (A3) hold, then there exists $\Lambda < \infty$ and for every $\lambda \in (0, \Lambda)$, the problem (1.1) has a sequence of nonnegative weak solutions $\{u_n\} \subset X_0 \cap L^\infty(\Omega)$ such that $I(u_n) < 0$, $I(u_n) \to 0^-$ and $u_n \to 0$ in $X_0$.

We will use the symmetric Mountain Pass Theorem version due to Clark [9], to guarantee the existence of distinct infinitely many weak solutions. There exists two versions of the symmetric Mountain Pass Theorem. One of them gives a sequence of critical values diverging to infinity for the superlinear data. The other one provides a sequence of critical values converging to zero for the sublinear data. In the present article, we will use the theorem for the sublinear case. To state the symmetric Mountain Pass Theorem, we will define the notion of genus, which will be used to prove our main Theorem 1.2.

Definition 1.3 (Genus) Let $X$ be a Banach space and $A \subset X$. A set $A$ is said to be symmetric if $u \in A$ implies $-u \in A$. Let $A$ be a close, symmetric subset of $X$ such that $0 \not\in A$. We define a genus $\gamma(A)$ of $A$ by the smallest integer $k$ such that there exists an odd continuous mapping from $A$ to $\mathbb{R}^k\{0\}$. We define $\gamma(A) = \infty$, if no such $k$ exists.

Let us consider the following set,

$$\Gamma_n = \{A_n \subset X : A_n \text{ is closed, symmetric and } 0 \not\in A_n \text{ such that the genus } \gamma(A_n) \geq n\}.$$

The following version of the symmetric Mountain Pass Theorem has been taken from [20].

Theorem 1.4 Let $X$ be an infinite dimensional Banach space and $\tilde{I} \in C^1(X, \mathbb{R})$ satisfies the following

(i) $\tilde{I}$ is even, bounded below, $\tilde{I}(0) = 0$ and $\tilde{I}$ satisfies the $(PS)_c$ condition.
(ii) For each $n \in \mathbb{N}$, there exists an $A_n \in \Gamma_n$ such that $\sup_{u \in A_n} \tilde{I}(u) < 0$.

Then for each $n \in \mathbb{N}$, $c_n = \inf_{A \in \Gamma_n} \sup_{u \in A} \tilde{I}(u) < 0$ is a critical value of $\tilde{I}$.

In order to apply the symmetric Mountain Pass Theorem, we modify the problem (1.1) to

$$(-\Delta)^s u = \frac{\text{sign}(u)}{|u|^{\gamma}} + f(x, u) \quad \text{in} \quad \Omega,$$

$$u = 0 \quad \text{in} \quad \mathbb{R}^N\setminus\Omega,$$

We define the energy functional $J : X_0 \to (-\infty, \infty]$ associated with the problem (1.10) as

$$J(u) = \frac{1}{2} \int_\Omega \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} dx dy - \frac{\lambda}{1-\gamma} \int_\Omega |u|^{1-\gamma} dx - \int_\Omega F(x, u) dx$$

We now give the definition of a weak solution to the problem (1.10).
Definition 1.5 A function $u \in X_0$ is a weak solution of (1.10), if $\phi|u|^{-\gamma} \in L^1(\Omega)$ and

$$
\int_Q \frac{(u(x) - u(y))(\phi(x) - \phi(y))}{|x - y|^{N+2s}} dxdy - \int_{\Omega} \left( \frac{\lambda \text{sign}(u)}{|u|^\gamma} + f(x, u) \right) \phi dx = 0,
$$

(1.12)

for all $\phi \in X_0$.

It is easy to see that, if $u$ is a weak solution to the problem (1.10) and $u > 0$ a.e., then $u$ is also a weak solution to the problem (1.1). We use a cutoff technique given in [9] to guarantee the existence of infinitely many positive weak solutions to the problem (1.10). Let us choose $l$ to be small, such that $0 < l \leq \frac{1}{2} \min\{\delta, r\}$, where $\delta$ and $r$ are same as in the assumptions on $f$. Let us define a $C^1$ function $\xi : \mathbb{R} \to \mathbb{R}^+$ such that $0 \leq \xi(t) \leq 1$ and

$$
\xi(t) = \begin{cases} 1, & \text{if } |t| \leq l \\ \xi & \text{is decreasing, if } l \leq t \leq 2l \\ 0, & \text{if } |t| \geq 2l. 
\end{cases}
$$

Since our main objective is to prove the existence of positive solutions, we will define $f(x, t) = 0$ for $t \leq 0$. Let us consider the following cutoff problem.

$$
(-\Delta)^s u = \frac{\lambda \text{sign}(u)}{|u|^\gamma} + \tilde{f}(x, u) \quad \text{in } \Omega,
$$

$$
u = 0 \quad \text{in } \mathbb{R}^N \setminus \Omega,
$$

where,

$$
\tilde{f}(x, u) = \begin{cases} f(x, u)\xi(u), & \text{for } u \geq 0 \\ 0, & \text{for } u \leq 0.
\end{cases}
$$

One can easily see that if $u$ is a weak solution to (1.13) with $\|u\|_{\infty} \leq l$, then $u$ is also a weak solution to (1.10). We will investigate the existence of infinitely many weak solutions to the problem (1.13). Moreover, to achieve our goal we will prove that $\|u\|_{\infty} \leq l$ and the solutions to (1.13) are positive.

The energy functional $\tilde{I} : X_0 \to (-\infty, \infty]$ associated with the problem (1.13) is defined as

$$
\tilde{I}(u) = \frac{1}{2} \int_Q \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} dxdy - \lambda \int_{\Omega} |u|^{1-\gamma} dx - \int_{\Omega} \tilde{F}(x, u) dx
$$

(1.14)

A weak solution to the problem (1.13) is given by the following definition.
Definition 1.6 A function \( u \in X_0 \) is a weak solution of (1.13), if \( \phi|u|^{-\gamma} \in L^1(\Omega) \) and
\[
\int_\Omega \frac{(u(x) - u(y))(\phi(x) - \phi(y))}{|x - y|^{N+2s}} dxdy - \int_\Omega \left( \frac{\lambda \text{sign}(u)}{|u|^{\gamma}} + \bar{f}(x, u) \right) \phi dx = 0, \tag{1.15}
\]
for all \( \phi \in X_0 \).

Henceforth, a weak solution will be referred to as a solution.

2 Existence and multiplicity of solutions

We begin this section by proving that
\[
\Lambda = \inf \{ \lambda > 0 : \text{The problem (1.1) has no weak solution} \}.
\]
is a finite, nonnegative real number.

Lemma 2.1 Assume \( 0 < \gamma < 1 \) and (A1) – (A3) holds. Then \( 0 \leq \Lambda < \infty \).

Proof It is clear that \( \Lambda \geq 0 \) from its definition. Let \( \lambda_1 \) be the principal eigenvalue of the fractional Laplacian operator \( (-\Delta)^s \) in \( \Omega \) and let \( \phi_1 > 0 \) be the associated eigenfunction [6]. Therefore, we have
\[
(-\Delta)^s \phi_1 = \lambda_1 \phi_1 \quad \text{in} \quad \Omega, \\
\phi_1 > 0 \quad \text{in} \quad \Omega, \\
\phi = 0 \quad \text{in} \quad \mathbb{R}^N \setminus \Omega. \tag{2.16}
\]

By choosing \( \phi_1 \) as a test function in the Definition 1.1, we get
\[
\lambda_1 \int_\Omega u \phi_1 dx = \int_\Omega (-\Delta)^s \phi_1 u dx = \int_\Omega \left( \frac{\lambda}{|u|^{\gamma}} + f(x, u) \right) \phi_1 dx. \tag{2.17}
\]

Let us now choose, any arbitrary constant \( \tilde{\Lambda} \) such that \( \tilde{\Lambda} t^{-\gamma} + f(x, t) > 2\lambda_1 t \forall t > 0 \). This contradicts to the Eq. (2.17). Hence, we get \( \Lambda < \infty \). \( \square \)

Remark 2.2 In fact, we will finally prove that \( \Lambda > 0 \).

We will now prove the following Lemma, to obtain one of the hypothesis of the symmetric Mountain Pass Theorem.

Lemma 2.3 The functional \( \tilde{I} \) is bounded below and satisfies \((PS)_c\) condition.
Proof By using the definition of $\xi$ and Hölder’s inequality, we get

$$\tilde{I}(u) \geq \frac{1}{2} \int_{Q} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} dxdy - \lambda C\|u\|^{1-\gamma} - C_1$$

$$\geq \frac{1}{2} \|u\|^2 - \lambda C\|u\|^{1-\gamma} - C_1$$

where, $C, C_1$ are non negative constants. This implies that $\tilde{I}$ is coercive and bounded below in $X_0$. Let $\{u_n\} \subset X_0$ be a Palais Smale sequence for $\tilde{I}$. Then $\{u_n\}$ is bounded in $X_0$ due to the coerciveness of $\tilde{I}$. Therefore, we may assume, $u_n \rightharpoonup u$ in $X_0$ up to a subsequence. Thus, we have

$$\int_{Q} \frac{(u_n(x) - u_n(y))(\phi(x) - \phi(y))}{|x - y|^{N+2s}} dxdy \rightarrow \int_{Q} \frac{(u(x) - u(y))(\phi(x) - \phi(y))}{|x - y|^{N+2s}} dxdy$$

(2.18)

for all $\phi \in X_0$. By the embedding result [26], we can assume

$$u_n \rightarrow u \quad \text{in} \quad L^p(\Omega), \quad (2.19)$$

$$u_n(x) \rightarrow u(x) \quad \text{a.e.} \quad L^p(\Omega). \quad (2.20)$$

Therefore, from Lemma A.1 [31], we get that there exists $g \in L^p(\Omega)$ such that

$$|u_n(x)| \leq g(x) \quad \text{a.e. in } \Omega, \quad \forall \ n \in \mathbb{N}. \quad (2.21)$$

Hence, by using (2.19), (2.20), (2.21) and the Lebesgue dominated convergence theorem, we get

$$\int_{\Omega} \tilde{f}(x, u_n)udx \rightarrow \int_{\Omega} \tilde{f}(x, u)udx \quad \text{and} \quad \int_{\Omega} \tilde{f}(x, u_n)u_ndx \rightarrow \int_{\Omega} \tilde{f}(x, u)udx$$

(2.22)

Again, on using the Hölder’s inequality and passing to the limit $n \rightarrow \infty$, we get

$$\int_{\Omega} u_n^{1-\gamma} dx \leq \int_{\Omega} u^{1-\gamma} dx + \int_{\Omega} |u_n - u|^{1-\gamma} dx$$

$$\leq \int_{\Omega} u^{1-\gamma} dx + C\|u_n - u\|_{L^{2}(\Omega)}^{1-\gamma}$$

$$= \int_{\Omega} u^{1-\gamma} dx + o(1) \quad (2.23)$$
Similarly, we have
\[
\int_{\Omega} u^{1-\gamma} \, dx \leq \int_{\Omega} u_n^{1-\gamma} \, dx + \int_{\Omega} |u_n - u|^{1-\gamma} \, dx \\
\leq \int_{\Omega} u_n^{1-\gamma} \, dx + C \|u_n - u\|_{L^2(\Omega)}^{1-\gamma} \\
= \int_{\Omega} u_n^{1-\gamma} \, dx + o(1) \tag{2.24}
\]
Therefore,
\[
\int_{\Omega} u_n^{1-\gamma} \, dx = \int_{\Omega} u^{1-\gamma} \, dx + o(1) \tag{2.25}
\]
Now, since \( \langle \tilde{I}(u_n), u_n \rangle \to 0 \), we have
\[
\int_{Q} \frac{|u_n(x) - u_n(y)|^2}{|x - y|^{N+2s}} \, dxdy - \lambda \int_{\Omega} |u_n|^{1-\gamma} \, dx - \int_{\Omega} \tilde{f}(x, u_n)u_n \, dx \to 0 \tag{2.26}
\]
Therefore, by (2.22), (2.25) and (2.26), we get
\[
\int_{Q} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} \, dxdy \to \lambda \int_{\Omega} |u|^{1-\gamma} \, dx - \int_{\Omega} \tilde{f}(x, u)u \, dx \tag{2.27}
\]
Moreover,
\[
\langle \tilde{I}(u_n), u \rangle = \int_{Q} \frac{(u_n(x) - u_n(y))(u(x) - u(y))}{|x - y|^{N+2s}} \, dxdy \\
- \lambda \int_{\Omega} \text{sign}(u_n)|u_n|^{-\gamma} \, udx - \int_{\Omega} \tilde{f}(x, u_n)u \, dx \tag{2.28}
\]
Note that
\[
\langle \tilde{I}(u_n), u \rangle \longrightarrow 0, \text{ as } n \to \infty.
\]
Hence, taking \( \phi = u \) in (2.18) and by using (2.25)–(2.27), we get
\[
\int_{Q} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} \, dxdy = \lambda \int_{\Omega} |u|^{1-\gamma} \, dx + \int_{\Omega} \tilde{f}(x, u)u \, dx \tag{2.29}
\]
Therefore, we conclude \( \|u_n\| \to \|u\| \) and this completes the proof.

We will now prove a Lemma which will guarantee that, for each \( n \in \mathbb{N} \), the set \( \Gamma_n \neq \phi \), where \( \phi \) is the empty set.
Lemma 2.4 For any $n \in \mathbb{N}$, there exists a closed, symmetric subset $A_n \subset X_0$ with $0 \notin A_n$ such that the genus $\gamma(A_n) \geq n$ and $\sup_{u \in A_n} \tilde{I}(u) < 0$.

Proof We will first guarantee an existence of a closed, symmetric subset $A_n$ over every finite dimensional subspace of $X_0$ such that $\gamma(A_n) \geq n$. Let $X_k$ be a subspace of $X_0$ such that $\dim(X_k) = k$. We know that every norm over a finite dimensional norm linear space are equivalent. Therefore, there exists a positive constant $L = L(k)$ such that $\|u\| \leq L\|u\|_{L^2(\Omega)}$ for all $u \in X_k$.

Claim There exists a positive constant $R$ such that

$$\frac{1}{2} \int_\Omega |u|^2 \, dx \geq \int_{\{|u| > l\}} |u|^2 \, dx, \quad \forall \, u \in X_k \text{ such that } \|u\| \leq R. \quad (2.30)$$

The proof is by contradiction. Let $\{u_n\}$ be a sequence in $X_k \setminus \{0\}$ such that $u_n \to 0$ in $X_0$ and

$$\frac{1}{2} \int_\Omega |u_n|^2 \, dx < \int_{\{|u_n| > l\}} |u_n|^2 \, dx. \quad (2.31)$$

Choose, $v_n = \frac{u_n}{\|u_n\|_{L^2(\Omega)}}$. Then

$$\frac{1}{2} < \int_{\{|u_n| > l\}} |v_n|^2 \, dx. \quad (2.32)$$

Now, since $\dim(X_k) = k$, we can assume $v_n \to v$ in $X_0$ up to a subsequence. Therefore, $v_n \to v$ also in $L^2(\Omega)$. Further, observe that,

$$m\{x \in \Omega : |u_n| > l\} \to 0 \quad \text{as} \quad n \to \infty$$

since, $u_n \to 0$ in $X_0$. This is a contradiction to (2.32). Therefore, the claim is proved. Now, from the assumption (A2), we can choose $0 < l \leq 1$ such that,

$$\tilde{F}(x, t) = F(x, t) \geq 2L^2t^2, \quad \forall \, (x, t) \in \Omega \times [0, l].$$

Hence, for all $u \in X_k \setminus \{0\}$ such that $\|u\| \leq R$ and by using (2.30), we get

$$\tilde{I}(u) \leq \frac{1}{2} \int_\Omega \frac{|u(x) - u(y)|^2}{|x - y|^{N+2\gamma}} \, dx dy - \frac{\lambda}{1 - \gamma} \int_\Omega |u|^{1 - \gamma} \, dx - \int_{\{|u| \leq l\}} \tilde{F}(x, u) \, dx$$

$$\leq \frac{1}{2} \|u\|^2 - \frac{\lambda}{1 - \gamma} \int_\Omega |u|^{1 - \gamma} \, dx - 2L^2 \int_{\{|u| \leq l\}} |u|^2 \, dx$$

$$= \frac{1}{2} \|u\|^2 - \frac{\lambda}{1 - \gamma} \int_\Omega |u|^{1 - \gamma} \, dx - 2L^2 \left( \int_{\Omega} |u|^2 \, dx - \int_{\{|u| > l\}} |u|^2 \, dx \right)$$

$$\leq \frac{1}{2} \|u\|^2 - \frac{\lambda}{1 - \gamma} \int_\Omega |u|^{1 - \gamma} \, dx - L^2 \int_{\Omega} |u|^2 \, dx.$$
\[
\frac{1}{2} \|u\|^2 - \frac{\lambda}{1 - \gamma} \int_{\Omega} |u|^{1-\gamma} dx < 0.
\]

Let us now choose, \(0 < \rho \leq R\) and \(A_n = \{u \in X_n : \|u\| = \rho\}\). This serves the purpose of showing that \(\Gamma_n \neq \phi\). Since \(A_n\) is symmetric, closed with \(\gamma(A_n) \geq n\) such that \(\sup_{u \in A_n} \tilde{I}(u) < 0\). This completes the proof. \(\square\)

The following Lemmas will be proved to guarantee the boundedness of the solutions to the problem (1.13).

**Lemma 2.5** Let \(g : \mathbb{R} \to \mathbb{R}\) be a convex \(C^1\) function. Then for every \(a, b, A, B \in \mathbb{R}\) with \(A, B > 0\) the following inequality holds.

\[
(g(a) - g(b))(A - B) \leq (a - b)(Ag'(a) -Bg'(b))
\]

**Proof** Since, \(g\) is convex, we have

\[
g(b) - g(a) \geq g'(a)(b - a) \text{ and } g(a) - g(b) \geq g'(b)(a - b)
\]

Therefore, using (2.34), we get

\[
(a - b) \left[ Ag'(a) - Bg'(b) \right] = A(a - b)g'(a) - B(a - b)g'(b) \\
\quad \geq A \left[ g(a) - g(b) \right] - B \left[ g(a) - g(b) \right] \\
\quad = (A - B)(g(a) - g(b))
\]

\(\square\)

**Lemma 2.6** Let \(h : \mathbb{R} \to \mathbb{R}\) be an increasing function, then for \(a, b, \tau \in \mathbb{R}\) with \(\tau \geq 0\) we have

\[
[H(a) - H(b)]^2 \leq (a - b)(h(a) - h(b))
\]

where, \(H(t) = \int_0^t \sqrt{h'(\tau)} d\tau\), for \(t \in \mathbb{R}\).

**Proof**

\[
(a - b)(h(a) - h(b)) = (a - b) \int_b^a h'(\tau) d\tau \\
\quad = (a - b) \int_b^a (H'(\tau))^2 d\tau \\
\quad \geq \left( \int_b^a H'(\tau) d\tau \right)^2 \text{ by Jensen's inequality} \\
\quad = [H(a) - H(b)]^2
\]

This completes the proof. \(\square\)
Lemma 2.7 Let \( u \in X_0 \) be a positive weak solution to the problem in (1.13), then \( u \in L^\infty(\Omega) \).

**Proof** We follow the steps from Brasco and Parini [6]. For every small \( \epsilon > 0 \), let us define the smooth function

\[
g_\epsilon(t) = (\epsilon^2 + t^2)^{\frac{1}{2}}
\]

Observe that the function \( g_\epsilon \) is convex and Lipschitz. For all positive \( \psi \in C^\infty_c(\Omega) \), we take \( \phi = \psi g'_\epsilon(u) \) to be the test function in (1.13). By choosing \( a = u(x), b = u(y), A = \psi(x) \) and \( B = \psi(y) \) in Lemma 2.5, we have

\[
\int_Q \frac{(g_\epsilon(u(x)) - g_\epsilon(u(y)))((\psi(x) - \psi(y)))}{|x - y|^{N + 2s}} \, dx \, dy \leq \int_{\Omega} \left( |\lambda u^{-\gamma} + \tilde{f}(x, u)| \right) |g'_\epsilon(u)| \psi \, dx
\]  

(2.36)

The function \( g_\epsilon(t) \to |t| \) as \( t \to 0 \) and hence \( |g'_\epsilon(t)| \leq 1 \). Therefore, on using Fatou’s Lemma and passing the limit \( \epsilon \to 0 \) in (2.36), we obtain

\[
\int_Q \frac{(|u(x)| - |u(y)|)((\psi(x) - \psi(y)))}{|x - y|^{N + 2s}} \, dx \, dy \leq \int_{\Omega} \left( |\lambda u^{-\gamma} + \tilde{f}(x, u)| \right) \psi \, dx
\]

(2.37)

for all \( \psi \in C^\infty_c(\Omega) \) with \( \psi > 0 \). The inequality (2.37) remains true for all \( \psi \in X_0 \) with \( \psi \geq 0 \). We define the cutoff function \( u_k = \min((u - 1)^+, k) \in X_0 \) for \( k > 0 \).

Now for any given \( \beta > 0 \) and \( \delta > 0 \), we choose \( \psi = (u_k + \delta)^\beta - \delta^\beta \) as the test function in (2.37) and get

\[
\int_Q \frac{(|u(x)| - |u(y)|)((u_k(x) + \delta)\beta - (u_k(y) + \delta)\beta)}{|x - y|^{N + 2s}} \, dx \, dy \leq \int_{\Omega} \left( |\lambda u^{-\gamma} + \tilde{f}(x, u)| \right) ((u_k + \delta)\beta - \delta^\beta) \, dx
\]  

(2.38)

Now applying the Lemma 2.6 to the function \( h(u) = (u_k + \delta)^\beta \), we get

\[
\int_Q \frac{|(u_k(x) + \delta)^\frac{\beta + 1}{2} - (u_k(y) + \delta)^\frac{\beta + 1}{2}|^2}{|x - y|^{N + 2s}} \, dx \, dy \\
\leq \frac{(\beta + 1)^2}{4\beta} \int_Q \frac{(|u(x)| - |u(y)|)((u_k(x) + \delta)^\beta - (u_k(y) + \delta)\beta)}{|x - y|^{N + 2s}} \, dx \, dy \\
\leq \frac{(\beta + 1)^2}{4\beta} \int_{\Omega} \left( |\lambda u^{-\gamma} + \tilde{f}(x, u)| \right) ((u_k + \delta)\beta - \delta^\beta) \, dx \\
\leq \frac{(\beta + 1)^2}{4\beta} \int_{\Omega} \left( |\lambda u^{-\gamma}| + |\tilde{f}(x, u)| \right) ((u_k + \delta)\beta - \delta^\beta) \, dx \\
= \frac{(\beta + 1)^2}{4\beta} \int_{|u| \geq 1} \left( |\lambda u^{-\gamma}| + |\tilde{f}(x, u)| \right) ((u_k + \delta)\beta - \delta^\beta) \, dx
\]
\[
\leq \frac{(\beta + 1)^2}{4\beta} \int_{\{u \geq 1\}} (|\lambda| + (|c_1| + |c_2| |u|^\alpha)) ((u_k + \delta)^\beta - \delta^\beta) \, dx
\]
\[
\leq C_1 \frac{(\beta + 1)^2}{4\beta} \int_{\{u \geq 1\}} (1 + |u|^\alpha) ((u_k + \delta)^\beta - \delta^\beta) \, dx
\]
\[
\leq 2C_1 \frac{(\beta + 1)^2}{4\beta} \int_{\{u \geq 1\}} |u|^\alpha ((u_k + \delta)^\beta - \delta^\beta) \, dx
\]
\[
\leq C \frac{(\beta + 1)^2}{4\beta} |u|_{L^2}^\alpha \|(u_k + \delta)^\beta\|_q
\]
where, \( q = \frac{2^*}{2^* - \alpha} \) and \( C = \max\{1, |\lambda|\} \). Now by using the Sobolev inequality for \( X_0 \), given by [12], we get
\[
\int_{\Omega} \left[ \int_{\Omega} \left( (u_k(x) + \delta)^{\frac{\beta + 1}{2}} - (u_k(y) + \delta)^{\frac{\beta + 1}{2}} \right)^2 |x - y|^{N + 2s} \, dx \, dy \right] \geq C_{N,s} |(u_k + \delta)^{\frac{\beta + 1}{2}} - \delta^{\frac{\beta + 1}{2}}|^{2}_{2^*}
\]
where, \( C_{N,s} \) is the embedding constant. The triangle inequality and \((u_k + \delta)^{\beta + 1} \geq \delta(u_k + \delta)^{\beta}\) implies
\[
\left[ \int_{\Omega} \left( (u_k + \delta)^{\frac{\beta + 1}{2}} - \delta^{\frac{\beta + 1}{2}} \right)^2 \, dx \right]^{\frac{2}{2^*}} \geq \frac{\delta}{2} \left( \int_{\Omega} (u_k + \delta)^{2^*} \, dx \right)^{\frac{2}{2^*}} - \delta^{\beta + 1} |\Omega|^{\frac{2}{2^*}}
\]
Therefore, on using (2.41) in (2.40) and then applying (2.40) to the last inequality of (2.39), we get
\[
|(u_k + \delta)^{\frac{\beta}{2}}|_{2^*}^2 \leq C \left( \frac{2(\beta + 1)^2}{4\beta \delta C_{N,s}} |u|_{L^2}^\alpha |(u_k + \delta)^\beta|_q + \delta^\beta |\Omega|^{\frac{2}{2^*}} \right)
\]
\[
\leq C \left( \frac{2(\beta + 1)^2}{4\beta \delta C_{N,s}} |u|_{L^2}^\alpha |(u_k + \delta)^\beta|_q + \frac{(\beta + 1)^2}{4\beta} |\Omega|^{1 - \frac{1}{q} - \frac{2^*}{2^*}} |(u_k + \delta)^\beta|_q \right)
\]
\[
\leq C \frac{(\beta + 1)^2}{4\beta} |(u_k + \delta)^\beta|_q \left( \frac{|u|_{L^2}^\alpha}{\delta C_{N,s}} + |\Omega|^{1 - \frac{1}{q} - \frac{2^*}{2^*}} \right)
\]
where, \( C = C(N, s) > 0 \).

We will now choose, \( \delta = \frac{|u|_{L^2}^\alpha}{\delta C_{N,s}|\Omega|^{1 - \frac{1}{q} - \frac{2^*}{2^*}}} > 0 \) and \( \beta \geq 1 \) such that \( \left( \frac{\beta + 1}{2\beta} \right)^2 \leq 1 \).

Let us now choose \( \eta = \frac{2^*}{2^* - \alpha} > 1 \) and \( \tau = q\beta \) and then rewrite the inequality (2.42) as
\[
|(u_k + \delta)|_{\eta \tau} \leq C |\Omega|^{1 - \frac{1}{q} - \frac{2^*}{2^*}} \left( \frac{\tau}{q} \right)^{\frac{2^*}{2^*}} |(u_k + \delta)|_{\tau}
\]
We now iterate the inequality (2.43) by setting the following sequence.

\[ \tau_0 = q \quad \text{and} \quad \tau_{n+1} = \eta \tau_n = \eta^{n+1} q \]

Hence, after \( n \) iteration, the inequality (2.43) reduces to

\[
| (u_k + \delta) |_{\tau_{n+1}} \leq \left( C |\Omega|^{1-\frac{1}{q}} \frac{q}{2^{q}} \right) \left( \sum_{i=0}^{n} \frac{\tau_i}{q} \right) \prod_{i=0}^{n} \left( \frac{\tau_i}{q} \right)^{\frac{q}{2}} | (u_k + \delta) |_{q} 
\]

(2.44)

Since \( \eta > 1 \), we have

\[
\sum_{i=0}^{\infty} \frac{q}{\tau_i} = \sum_{i=0}^{\infty} \frac{1}{\eta^i} = \frac{\eta}{\eta - 1}
\]

and

\[
\prod_{i=0}^{\infty} \left( \frac{\tau_i}{q} \right)^{\frac{q}{2}} = \eta^{\frac{n}{(q-1)^2}}.
\]

Now, by taking limit \( n \rightarrow \infty \) in (2.44), we get

\[
|u_k|_{\infty} \leq \left( C |\Omega|^{1-\frac{1}{q}} \frac{q}{2^{q}} \right) \frac{\eta}{(q-1)^2} | (u_k + \delta) |_{q} 
\]

(2.45)

Therefore, by using \( u_k \leq (u - 1^+) \) and the triangle inequality in (2.45), we get

\[
|u_k|_{\infty} \leq C \eta^{\frac{n}{(q-1)^2}} \left( |\Omega|^{1-\frac{1}{q}} \frac{q}{2^{q}} \right) \frac{n}{(q-1)^2} \left( | (u - 1)^+ |_{q} + \delta |\Omega|^{\frac{1}{q}} \right) \quad \text{(2.46)}
\]

Now letting \( k \rightarrow \infty \) in (2.46), we have

\[
|(u - 1)^+|_{\infty} \leq C \eta^{\frac{n}{(q-1)^2}} \left( |\Omega|^{1-\frac{1}{q}} \frac{q}{2^{q}} \right) \frac{n}{(q-1)^2} \left( | (u - 1)^+ |_{q} + \delta |\Omega|^{\frac{1}{q}} \right) \quad \text{(2.47)}
\]

Hence, we conclude that \( u \in L^\infty(\Omega) \). \( \square \)

**Proof of Theorem 1.2** From the definition of \( \xi \) and the assumption (A1), we have \( \tilde{I} \) is even and \( \tilde{I}(0) = 0 \). Therefore, Lemmas 1.4, 2.3 and 2.4 guarantees that \( \tilde{I} \) has sequence of critical points \( \{u_n\} \) such that \( \tilde{I}(u_n) < 0 \) and \( \tilde{I}(u_n) \rightarrow 0^- \).

**Claim** Suppose \( u_n \) is a critical point of \( \tilde{I} \), then for each \( n \in \mathbb{N} \), \( u_n \geq 0 \) a.e. in \( X_0 \).

**Proof** Let us consider, \( \Omega = \Omega^+ \cup \Omega^- \), where \( \Omega^+ = \{ x \in X_0 : u_n(x) \geq 0 \} \) and \( \Omega^- = \{ x \in X_0 : u_n(x) < 0 \} \). We define \( u_n(x) = u_n^+ - u_n^- \), where \( u_n^+(x) = \max\{u_n(x), 0\} \) and \( u_n^-(x) = \max\{-u_n(x), 0\} \). Suppose, \( u_n < 0 \) a.e. in \( \Omega \), then on
taking, \( \phi = u_n^- \) as the test function in the Eq. (1.6) in conjunction with the inequality 
\( (a - b)(a^- - b^-) \leq -(a^- - b^-)^2 \), we get
\[
\int_{\Omega} \left( \lambda \frac{\text{sign}(u_n^-)u_n^-}{|u_n|^\gamma} + \tilde{f}(x, u_n)u_n^- \right) \, dx = \int_{Q} \frac{(u_n(x) - u_n(y))(u_n^-(x) - u_n^-(y))}{|x - y|^{N+2\alpha}} \, dxdy \\
\Rightarrow \lambda \int_{\Omega^-} |u_n^-|^{1-\gamma} \, dx \leq -\|u_n^-\|^2 < 0.
\]
This implies \( |\Omega^-| = 0 \), which is a contradiction to the assumption \( u_n^- < 0 \) a.e. in \( \Omega \). This proves our claim.

Moreover, from the definition of \( \tilde{I} \), we have
\[
\frac{1}{\alpha} \langle \tilde{I}(u_n), u_n \rangle - \tilde{I}(u_n) = \frac{1}{\alpha} \left[ \|u_n\|^2 - \int_{\Omega} \left( \frac{\lambda \text{sign}(u_n)u_n}{|u_n|^\gamma} + \tilde{f}(x, u_n)u_n \right) \, dx \right] \\
- \left[ \frac{1}{2} \|u_n\|^2 - \int_{\Omega} \left( \frac{\lambda}{1-\gamma} |u_n|^{1-\gamma} + \tilde{F}(x, u_n) \right) \, dx \right] \\
= \left( \frac{1}{\alpha} - \frac{1}{2} \right) \|u_n\|^2 - \lambda \left( \frac{1}{\alpha} - \frac{1}{1-\gamma} \right) \int_{\Omega} |u_n|^{1-\gamma} \, dx \\
+ \frac{1}{\alpha} \int_{\Omega} (\alpha \tilde{F}(x, u_n) - \tilde{f}(x, u_n)) \, dx \\
\geq \left( \frac{1}{\alpha} - \frac{1}{2} \right) \|u_n\|^2
\]
It can be easily seen that, since
\[
\frac{1}{\alpha} \langle \tilde{I}(u_n), u_n \rangle - \tilde{I}(u_n) = o_n(1) \\
\Rightarrow \left( \frac{1}{\alpha} - \frac{1}{2} \right) \|u_n\|^2 \leq o_n(1),
\]
hence, we get \( u_n \to 0 \) in \( X_0 \). Now by using Moser iteration and Lemma 2.7, we can assume that as \( n \to \infty \), \( \|u_n\|_{L^\infty(\Omega)} \leq l \). Therefore, the problem (1.10) has infinitely many solutions. Further, due to the nonnegativity of \( u_n \) and \( \tilde{I}(u_n) < 0 \), we conclude that the problem (1.1) has infinitely many weak solutions and hence the Theorem 1.2 is proved.

**Remark 2.8** \( \Lambda > 0 \), because the solution to the problem (1.1) exists for some \( \lambda > 0 \).

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