WONG-ZAKAI APPROXIMATIONS AND ASYMPOTOTIC
BEHAVIOR OF STOCHASTIC GINZBURG-LANDAU
EQUATIONS

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Abstract. In this paper we discuss the Wong-Zakai approximations given
by a stationary process via the Wiener shift and their associated long term
pathwise behavior for stochastic Ginzburg-Landau equations driven by a white
noise. We first apply the Galerkin method and compactness argument to prove
the existence and uniqueness of weak solutions. Consequently, we show that
the approximate equation has a pullback random attractor under much weaker
conditions than the original stochastic equation. At last, when the stochastic
Ginzburg-Landau equation is driven by a linear multiplicative noise, we estab-
lish the convergence of solutions of Wong-Zakai approximations and the upper
semicontinuity of random attractors of the approximate random system as the
size of approximation approaches zero.

1. Introduction. The generalized complex Ginzburg-Landau equation is one of
the most important equations in mathematical physics, which can describe turbulent
dynamics and has a long history in physics as a generic amplitude equation near
the onset of instabilities in fluid mechanical systems, as well as in the theory of
phase transitions and superconductivity[3, 9]. The global existence and long time
behavior of Ginzburg-Landau equation were studied in [10, 17, 30]. The random
attractors of stochastic Ginzburg-Landau equations were got in [29, 28, 31, 44].

In this paper we study the Wong-Zakai approximations given by a stationary
process via the Wiener shift and their associated long term pathwise behavior for
the stochastic Ginzburg-Landau equations driven by a white noise. Specifically, we
discuss the dynamics of the following stochastic Ginzburg-Landau equations defined
in a bounded domain $\mathcal{O}$:

$$
\frac{\partial u}{\partial t} - (1 + i\lambda)\Delta u + \rho u = f(u) + g(t, x) + h(t, x, u) \circ \frac{dW}{dt},
$$

(1)

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where \( u = u(x,t) \) is a unknown complex-valued function, \( i \) is the imaginary unit, \( \lambda \in \mathbb{R}, \rho > 0 \), the nonlinear term \( f(u) \) is a complex-valued function, \( g(t,x) \in L^2_{\text{loc}}(\mathbb{R}, L^2(\mathcal{O})) \), \( h \) is a nonlinear function with certain properties, \( W \) is a two-sided real-valued wiener process on a probability space. The symbol \( \circ \) indicates that the equation is understood in the sense of Stratonovich’s integration.

We are interested in studying the dynamics of Eq.(1) for almost all sample paths, including the existence and uniqueness of random attractors. For this purpose, one needs to define a random dynamical system (or cocycle) based on the solution operator of the equation. However, the existence of such a random dynamical system is unknown in general for a nonlinear function \( h \) in (1) (see, e.g., [11, 15]), which results in a serious challenge for studying the pathwise dynamics of the equation. At present, the existence of random attractors for (1) has been established only when \( h \) is \( u \) or independent of \( u \) (see, e.g., [29, 28, 31, 44]). In the nonlinear case, some progress has been made for a class of stochastic PDEs driven by a fractional Brownian motion by using rough path analysis, see Gao et al. [14]. Due to the difficulty directly dealing with the stochastic equation, in this paper, we investigate the Wong-Zakai approximations of (1) by using a stationary process via the Wiener shift. Consequently, the approximate equation generates a random dynamical system. Thus one can discuss its sample-wise (or pathwise) dynamics.

The standard probability space \( (\Omega, \mathcal{F}, \mathbb{P}) \) will be used in this paper, where

\[
\Omega = C_0(\mathbb{R}, \mathbb{R}) := \{ \omega \in C(\mathbb{R}, \mathbb{R}) : \omega(0) = 0 \}
\]

with the open compact topology, \( \mathcal{F} \) is its Borel \( \sigma \)-algebra, and \( \mathbb{P} \) is the Wiener measure. The Brownian motion has the form \( W(t, \omega) = \omega(t) \). Consider the Wiener shift \( \theta_t \) defined on the probability space \( (\Omega, \mathcal{F}, \mathbb{P}) \) by

\[
\theta_t \omega(\cdot) = \omega(t + \cdot) - \omega(t).
\]

From Arnold [1], the probability measure \( \mathbb{P} \) is an ergodic invariant measure for \( \theta_t \) and \( (\Omega, \mathcal{F}, \mathbb{P}, \{\theta_t\}_{t \in \mathbb{R}}) \) forms a metric dynamical system.

For each \( \delta \in \mathbb{R} \), let \( G_\delta : \Omega \rightarrow \mathbb{R} \) denote the random variable

\[
G_\delta(\omega) = \frac{1}{\delta} \omega(\delta).
\]

Then we have

\[
G_\delta(\theta_t \omega) = \frac{1}{\delta} (\omega(t + \delta) - \omega(t)). \tag{2}
\]

By the properties of Brownian motions, it implies that \( G_\delta(\theta_t \omega) \) is a stationary stochastic process with a normal distribution and is unbounded in \( t \) for almost all \( \omega \). \( G_\delta(\theta_t \omega) \) may be viewed as an approximation of white noise. Indeed, we will give in Lemma 2.1 that

\[
W_\delta(t, \omega) = \int_0^t G_\delta(\theta_s \omega) ds
\]

converges to \( W(t, \omega) \) almost surely uniformly on any finite time interval as \( \delta \rightarrow 0 \), which suggests that the dynamics of stochastic equation (1.1) could be approached by the following Wong-Zakai approximation of the equation as \( \delta \rightarrow 0 \):

\[
\frac{\partial u}{\partial t} - (1 + i\lambda) \Delta u + \rho u = f(u) + g(t,x) + h(t,x,u)G_\delta(\theta_t \omega). \tag{3}
\]

We will show, for a wide class of nonlinearity \( h \), the random Eq.(3) generates a continuous random dynamical system in \( L^2(\mathcal{O}) \) and it has a unique tempered random attractor (see Theorem 2.5). This is in contrast with the stochastic equation
where the existence of random attractors is only known when $h$ is linear in its third argument or independent of $u$. Furthermore, we will prove that when Eq. (1) is driven by a linear multiplicative noise (i.e., $h(t, x, u) = u$), the solutions of (3) converge to that of (1) in $L^2(\Omega)$ as $\delta \to 0$ (see Corollary 3). As for the long term dynamics, we will prove the random attractors of Eq. (3) approach that of (1) in terms of the Hausdorff semi-distance in $L^2(\Omega)$ as $\delta \to 0$ (see Theorems 3.10).

Using deterministic differential equations to approximate stochastic differential equations was introduced by Wong and Zakai in their pioneer work [60, 59] in which they discussed both piecewise linear approximations and piecewise smooth approximations for one-dimensional Brownian motions. Their work was later generalized to stochastic differential equations of higher dimensions, for example, by McShane [37], Stroock and Varadhan [45], Sussmann [46, 47], Ikeda et al. [22], Ikeda and Watanabe [23], and recently by Kelly and Melbourne [24], and Shen and Lu [43] in which the same approximations as this paper were studied. The results of the Wong-Zakai approximations have also been extended to stochastic differential equations driven by martingales and semimartingales, see for example, Nakao and Yamato [39], Konecny [25], Protter [41], Nakao [38], and Kurtz and Protter [26, 27].

There are also a lot of publications on Wong-Zakai approximations of solutions for stochastic partial differential equations, see for example, Brzezniak et al. [4], Gyongy [20, 18], Twardowska [52, 51, 50, 53], Bally et al. [2], Brzezniak and Flandoli [5], Grecksch and Schmalfuß [16], Gyongy and Shmatkov [20], Nowak [40], Tessitore and Zabczyk [49], Deya et al. [8], Ganguly [13], and Hairer and Pardoux [21].

In the current paper, we use $W_\delta(t, \omega) = \int_0^t G_\delta(\theta_s \omega) ds$ to approximate the Brownian motion $W(t, \omega)$. For such approximations, the corresponding approximate Eq. (3) generates a random dynamical system, which allows us to investigate the pathwise dynamics such as random attractors. Such approximation was also used in Lu and Wang [36, 35], Wang et al. [58], and Shen et al. [43] where they studied the chaotic behavior of random differential equations driven by a multiplicative noise of $G_\delta(\theta_t \omega)$ and long term behavior of stochastic reaction-diffusion equations driven by multiplicative noise.

The concept of random attractor for autonomous stochastic equations was introduced in Crauel and Flandoli [7], Flandoli and Schmalfuss [12], and Schmalfuss [42]. After then, there is an extensive literature on this subject for autonomous SPDEs. In the non-autonomous case, random attractors have been studied, for example, in Caraballo et al. [6], Li [32, 33] and Wang [57, 56, 55]. In this paper we will follow the framework of Wang [57], Lu and Wang [35] to deal with the non-autonomous random attractors of (1) and (3).

When we prove the existence and uniqueness of random attractors in $L^2(\Omega)$, the nonlinearity $f(u)$ is special form, i.e., $f(u) = -(1 + i\mu)|u|^2 u, \mu \in \mathbb{R}$, which is consistent with the general physical background for the Ginzburg-Landau equation [3, 9].

This paper is organized as follows. In Section 2, we show the existence of random attractors for the Wong-Zakai approximate Eq. (3) in $L^2(\Omega)$ with general nonlinear function $h$. We first apply Galerkin method to prove the existence and uniqueness of solutions for non-autonomous stochastic Ginzburg-Landau equation. And then we derive uniform estimates for solutions and the existence of a pullback random attractor is proved. In Section 3, we consider the case $h(t, x, u) = u$ for $t \in \mathbb{R}$, $x \in \mathcal{O}$ and $u \in \mathbb{R}$, and show that the solutions of (3) converge to that of the stochastic
equation (1) in $L^2(\mathcal{O})$ as $\delta \to 0$. We also obtain the upper semicontinuity of random attractors of (3) in this case.

2. Wong-Zakai approximations of equation (1).

In this section, to define a continuous cocycle for the Wong-Zakai approximate Eq.(3) which is a non-autonomous stochastic Ginzburg-Landau equation driven by a stationary process, we first apply Galerkin method to prove the existence and uniqueness of solutions for the equation, and then prove the existence of pullback random attractors in $L^2(\mathcal{O})$ for a wide class of nonlinear functions $h$.

2.1. Definition of continuous cocycles. Let $\mathcal{O}$ be a bounded domain in $\mathbb{R}^n$ and $\tau, \delta \in \mathbb{R}$ with $\delta \to 0$. Consider the non-autonomous stochastic Ginzburg-Landau equation defined in $\mathcal{O}$ for $t > \tau$,

$$\frac{\partial u}{\partial t} - (1 + i\lambda) \Delta u + \rho u = f(u) + g(t, x) + h(t, x, u) \circ \frac{dW}{dt},$$

with boundary condition

$$u(t, x) = 0, \ x \in \partial \mathcal{O}, \ t > \tau,$$

and initial condition

$$u(\tau, x) = u_{\tau}(x), \ x \in \mathcal{O},$$

where $f(u) = -(1 + i\mu)|u|^2 u$, $g(t, x) \in L^2_{loc}(\mathbb{R}, L^2(\mathcal{O}))$, $W$ is the two-sided real-valued Wiener process mentioned before. Let $h : \mathbb{R} \times \mathcal{O} \times \mathbb{C} \to \mathbb{C}$ be continuous such that for all $t \in \mathbb{R}$, $s \in \mathbb{C}$ and $x \in \mathcal{O}$,

$$|h(t, x, s)| \leq c_1 |s|^{q-1} + \psi_1(t, x),$$

$$|\frac{\partial h}{\partial s}(t, x, s)| \leq \psi_2(t, x),$$

where $2 \leq q < 4$, $c_1$ is a nonnegative constant, $\psi_1 \in L^4_{loc}(\mathbb{R}, L^4(\mathcal{O}))$ and $\psi_2 \in L^\infty_{loc}(\mathbb{R}, L^\infty(\mathcal{O}))$.

Recall that for $\delta \neq 0$, the random variable $G_\delta$ is defined by

$$G_\delta(\omega) = \frac{\omega(\delta)}{\delta}, \ \text{for all} \ \omega \in \Omega.$$  

From [1], It follows that there exists a $\theta_t$-invariant set $\tilde{\Omega} \subseteq \Omega$ of full $P$ measure such that for each $\omega \in \tilde{\Omega}$,

$$\frac{\omega(t)}{t} \to 0, \ t \to \pm \infty.$$  

Hereafter, for simplicity, we will write $\tilde{\Omega}$ as $\Omega$. From (9) we find

$$G_\delta(\theta_t \omega) = \frac{\omega(t + \delta) - \omega(t)}{\delta} \quad \text{and} \quad \int_0^t G_\delta(\theta_s \omega) ds = \int_t^{t+\delta} \frac{\omega(s)}{\delta} ds + \int_0^\delta \frac{\omega(s)}{\delta} ds.$$  

By (11) and the continuity of $\omega$, we obtain for all $t \in \mathbb{R}$,

$$\lim_{\delta \to 0} \int_0^t G_\delta(\theta_s \omega) ds = \omega(t) - \omega(0) = \omega(t).$$  

In fact, the convergence (12) holds true uniformly with respect to $t$ on any finite interval $[\tau, \tau + T]$ with $\tau \in \mathbb{R}$ and $T > 0$ which is stated below.
Lemma 2.1 ([35]). Let \( \tau \in \mathbb{R}, \omega \in \Omega, T > 0 \). Then for every \( \varepsilon > 0 \), there exists \( \delta_0 = \delta_0(\varepsilon, \tau, \omega, T) > 0 \), such that for all \( 0 < |\delta| < \delta_0 \) and \( t \in [\tau, \tau + t] \), we have

\[
| \int_0^t \mathcal{G}_\delta(\theta_s \omega) ds - \omega(t) | < \varepsilon.
\]

Note that the continuity of \( \omega(t) \) on \( [\tau, \tau + T] \) indicates that there exists \( c = c(\tau, \omega, T) > 0 \), such that

\[
|\omega(t)| \leq c \quad \text{for all} \quad t \in [\tau, \tau + T].
\]

From (14) and Lemma 2.1, it follows that there exists \( \delta = \delta(\tau, \omega, T) > 0 \) such that for all \( 0 < |\delta| < \delta \) and \( t \in [\tau, \tau + T] \),

\[
| \int_0^t \mathcal{G}_\delta(\theta_s \omega) ds | \leq | \int_0^t \mathcal{G}_\delta(\theta_s \omega) ds - \omega(t) | + |\omega(t)| \leq c.
\]

This inequality proves useful in later sections.

By (11) we may rewrite Eq.(4) as

\[
\frac{\partial u}{\partial t} - (1 + i\lambda)\Delta u + \rho u = f(u) + g(t, x) + h(t, x, u)\mathcal{G}_\delta(\theta_t \omega).
\]

To define a continuous cocycle for the Wong-Zakai approximate Eq.(3), we need to prove the existence and uniqueness of solutions for the equation. In fact, we will prove that under conditions (7)-(8), for every \( \tau \in \mathbb{R}, \omega \in \Omega \) and \( u_\tau \in L^2(\Omega) \), the solution \( u \) of (16) supplemented with (5)-(6) is well posed in \( L^2(\Omega) \).

Theorem 2.2. Suppose (7)-(8) hold. Then for every \( \tau \in \mathbb{R}, \omega \in \Omega \) and \( u_\tau \in L^2(\Omega) \), Eq.(16) supplemented with (5)-(6) has a unique solution

\[
u(t, \tau, \omega, T) \in C([\tau, \infty), L^2(\Omega)) \bigcap \tilde{L}^2_{loc}(\Omega, H^1(\Omega)).\]

In addition, this solution is continuous with respect to initial data in \( L^2(\Omega) \) and is \( (\mathcal{F}, \mathcal{B}(L^2(\Omega))) \)-measurable in \( \omega \in \Omega \).

In this following, we will prove this theorem by the Galerkin method and compactness argument, see [9, 28].

Proof: The proof proceeds in four distinct steps. We first construct a sequence of approximate solutions from finite-dimensional systems, then derive uniform estimates, and further take the limit of those approximate solutions, finally prove the uniqueness.

Step (1): Approximate solutions. Given \( n \in \mathbb{N} \), let \( \{e_1, e_2, \ldots, e_n, \ldots\} \) be an orthogonal basis of \( L^2(\Omega) \) and \( X_n \) be the subspace spanned by \( \{e_j : j = 1, 2, \ldots, n\} \) and \( P_n : L^2(\Omega) \rightarrow X_n \) be the orthogonal projection.

Consider the following system for \( u_n \in X_n \) defined for \( t > \tau \):

\[
\frac{\partial u_n}{\partial t} - (1 + i\lambda)\Delta u_n + \rho u_n = P_n f(u_n) + P_n g(t, x) + P_n h(t, x, u_n)\mathcal{G}(\theta_t \omega), t > \tau,
\]

with initial condition

\[
u_n(\tau) = P_n u_\tau.
\]

It follows that for every \( \tau \in \mathbb{R}, \omega \in \Omega \) and \( u_\tau \in L^2(\Omega) \), system (18)-(19) has a maximal solution \( u_n \) for some \( T > 0 \), which is measurable in \( \omega \in \Omega \). Next, we derive uniform estimates on \( u_n \), which implies the solutions are globally defined, that is, \( T = \infty \).
Step (2): Uniform estimates. By (18) we get
\[
\frac{1}{2} \frac{d}{dt} \|u_n\|^2 + \|\nabla u_n\|^2 + \rho \|u_n\|^2 = -\|u_n\|_{L^4}^4 + Re \int_{\Omega} g(t, x) \bar{u}_n dx
\]
\[+ Re G_{\delta} \int_{\Omega} h(t, x, u_n) \bar{u}_n dx. \tag{20}\]

By (7) we obtain
\[
Re G_{\delta} \int_{\Omega} h(t, x, u_n) \bar{u}_n dx \leq c_1 |G_{\delta}| \int_{\Omega} |u_n|^q dx + \int_{\Omega} |G_{\delta} \psi_1 u_n| dx
\]
\[\leq \frac{1}{2} \|u_n\|_{L^4}^4 + c |\Omega| \|G_{\delta}\|^\frac{2}{q-2} + |G_{\delta}|^\frac{4}{2} \|\psi_1\|^\frac{4}{q \frac{2}{3}}. \tag{21}\]

By Young’s inequality we get
\[
Re \int_{\Omega} g(t, x) \bar{u}_n dx \leq \frac{\rho}{2} \|u_n\|^2 + \frac{1}{2\rho} \|g(t, x)\|^2. \tag{22}\]

By (20)-(22) we get
\[
\frac{d}{dt} \|u_n\|^2 + \rho \|u_n\|^2 + 2 \|\nabla u_n\|^2 + 2 \|u_n\|_{L^4}^4 \leq \frac{1}{\rho} \|g(t, x)\|^2 + c \|G_{\delta}\|^\frac{2}{q-2} + c |G_{\delta}|^\frac{4}{2} \|\psi_1\|^\frac{4}{q \frac{2}{3}}. \tag{23}\]

Multiplying (23) by $e^{\rho t}$ and then integrating over $(\tau, t)$ with $t > \tau$, we get, for every $\omega \in \Omega$,  
\[
|u_n(t, \tau, \omega, u_\tau)|^2 + 2 \int_{\tau}^{t} e^{\rho(s-t)} \|\nabla u_n(s, \tau, \omega, u_\tau)\|^2 ds
\]
\[+ \int_{\tau}^{t} e^{\rho(s-t)} \|u_n(s, \tau, \omega, u_\tau)\|_{L^4}^4 ds
\]
\[\leq e^{\rho(\tau-t)} \|u_\tau\|^2 + \frac{1}{\rho} \int_{\tau}^{t} e^{\rho(s-t)} \|g(s)\|^2 ds + c \int_{\tau}^{t} e^{\rho(s-t)} \|G_{\delta}(\theta_\omega)\|^\frac{2}{q-2}
\]
\[+ |G_{\delta}(\theta_\omega)|^\frac{4}{2} \|\psi_1\|^\frac{4}{q \frac{2}{3}} ds. \tag{24}\]

Note that $G_{\delta}(\theta_\omega)$ is continuous in $t$ for fixed $\omega$. Therefore, for every $\tau \in \mathbb{R}, \omega \in \Omega$ and $T > \tau$, we obtain from (24) that
\[
\{ u_n \}_{n=1}^\infty \text{ is bounded in } L^2((\tau, T), H^1_0(\Omega)) \bigcap L^4((\tau, T), L^4(\Omega)). \tag{25}\]

By definition we have
\[
Re \int_{\tau}^{T} \int_{\Omega} |f(u_n)|^\frac{q}{2} dx dt = Re \int_{\tau}^{T} \int_{\Omega} \left| -(1 + i\mu) |u_n|^2 u_n \right|^\frac{q}{2} dx dt
\]
\[\leq \int_{\tau}^{T} \int_{\Omega} |u_n|^4 dx dt. \tag{26}\]

By (7) we have
\[
Re \int_{\tau}^{T} \int_{\Omega} |h(t, x, u_n)|^\frac{q}{2} dx dt \leq \int_{\tau}^{T} \int_{\Omega} \left( |c_1|^\frac{q}{2} |u_n|^\frac{q}{2(q-1)} + |\psi_1(t, x)|^\frac{q}{2} dx dt
\]
\[\leq c \int_{\tau}^{T} \int_{\Omega} |u_n|^4 dx dt + c \int_{\tau}^{T} \int_{\Omega} |\psi_1(t, x)|^\frac{q}{2} dx dt + c. \tag{27}\]
Thus (26) and (27) along with (25) imply that
\[
\{f(u_n)\}_{n=1}^\infty \text{ and } \{h(t, x, u_n)G(\theta t \omega)\}_{n=1}^\infty \text{ is bounded in } L^2((\tau, T), L^2(\mathcal{O})).
\] (28)

By (25) and (28) we get
\[
\{ \frac{du_n}{dt} \} \text{ is bounded in } L^2((\tau, T), H^{-1}(\mathcal{O}))+L^4((\tau, T), L^4(\mathcal{O})).
\] (29)

**Step (3): Existence of solutions.** It follows from (25), (28) and (29) that there exist \( \tilde{u} \in L^2(\mathcal{O}), u \in L^2((\tau, T), H^1_0(\mathcal{O})) \cap L^4((\tau, T), L^4(\mathcal{O})), \chi_1 \in L^2((\tau, T), L^2(\mathcal{O})), \chi_2 \in L^2((\tau, T), L^2(\mathcal{O})) \) such that, up to a subsequence,
\[
u_n \to u \text{ in } L^2((\tau, T), H^1(\mathcal{O})),
\] (30)
\[
u_n \to u \text{ in } L^4((\tau, T), L^4(\mathcal{O})),
\] (31)
\[
f(u_n) + G_\delta(\theta t \omega)h(t, x, u_n) \to \chi_1 \text{ in } L^2((\tau, T), L^2(\mathcal{O})),
\] (32)
\[
\frac{du_n}{dt} \to \chi_2 \text{ in } L^2((\tau, T), H^{-1}(\mathcal{O}))+L^4((\tau, T), L^4(\mathcal{O})),
\] (33)
\[
u_n(t_0, \tau, \omega, x) \to \tilde{u} \text{ in } L^2(\mathcal{O}) \text{ for a fixed } t_0 \in [\tau, T].
\] (34)

By a standard procedure (see [28, 58]), we can check that \( \chi_1 = f(u) + G_\delta(\theta t \omega)h(t, x, u), \chi_2 = \frac{du}{dt} \text{ and } \tilde{u} = u(t_0). \) Therefore, \( u \) is a weak solution of (16) supplement with initial and boundary condition (5)-(6).

**Step (4): Uniqueness of solutions.** Let \( u_1 \) and \( u_2 \) be solutions of (4)-(6) and \( \tilde{u} = u_1 - u_2. \) Then we have
\[
\frac{\partial \tilde{u}}{\partial t} - (1+i\lambda)\Delta \tilde{u} + \rho \tilde{u} = -(1+i\mu)(|u_1|^2 u_1 - |u_2|^2 u_2) + G_\delta(\theta t \omega)(h(t, x, u_1) - h(t, x, u_2)).
\] (35)

From (35), we have
\[
\frac{d}{dt} ||\tilde{u}||^2 + 2||\nabla \tilde{u}||^2 + 2\rho ||\tilde{u}||^2 = -2 \int_\Omega |u_1|^2 |\tilde{u}|^2 dx - Re(1+i\mu) \int_\Omega (|u_1|^2 - |u_2|^2)(u_2 \tilde{u} + \bar{u}_2 \tilde{u}) dx + 2ReG_\delta(\theta t \omega) \int_\Omega (h(t, x, u_1) - h(t, x, u_2)) \tilde{u} dx.
\] (36)

Applying the interpolation inequality
\[
||\tilde{u}||_{L^3} \leq c||\tilde{u}||^{1-\theta}||\tilde{u}||^{\theta}_{H^1}, \theta = \frac{1}{4},
\]
the first term on the right hand side of (36) can be bounded by
\[
-2 \int_\Omega |u_1|^2 |\tilde{u}|^2 dx \leq c||u_1||_{L^3} ||\tilde{u}||_{L^4}^2 \leq c||u_1||_{L^3}^2 ||\tilde{u}||^{2(1-\theta)}_{L^4} ||\tilde{u}||^{2\theta}_{H^1}
\leq c||u_1||_{L^3}^2 ||\tilde{u}||^2 + c||u_1||_{L^3}^{2\theta} ||\tilde{u}||^2 + ||\nabla \tilde{u}||^2.
\] (37)

Noting that for some \( \delta \in (0, 1) \)
\[
||u_1||^2 - |u_2|^2 | \leq 2|\delta|u_1| + (1-\delta)|u_2|) ||(|u_1| - |u_2|) | \leq c(||u_1|| + ||u_2||)||\tilde{u}|.\]
so the second term on the right hand side of (36) can be bounded by

\[-Re(1 + i\mu) \int_\Omega (|u_1|^2 - |u_2|^2)(u_2 \overline{u} + \overline{u_2}u)dx \leq c \int_\Omega (|u_1| + |u_2|)|\overline{u}| |u_2| dx\]

\[\leq c \left( \int_\Omega (|u_1| + |u_2|)^4 dx \right)^{\frac{1}{2}} \left( \int_\Omega |\overline{u}|^4 dx \right)^{\frac{1}{2}} \quad (38)\]

\[\leq c \|V\|^2_{L^4} \|\overline{u}\|^2 + c \|V\|^2_{L^4} \|\overline{u}\|^2 + \|\nabla \overline{u}\|^2,
\]

where \(|V| = |u_1| + |u_2|\). By (8) we obtain

\[2ReG_\delta(\theta, \omega) \int_\Omega \left( h(t, x, u_1) - h(t, x, u_2) \right) \overline{u} dx \leq c|G_\delta(\theta, \omega)||\overline{u}|^2 \|\psi_2(t, x)\|_{L^\infty}. \quad (39)\]

By (36)-(39) we have

\[\frac{d}{dt} \|\overline{u}\|^2 \leq c(\|V\|^2_{L^4} + \|V\|^2_{L^4} + |G_\delta(\theta, \omega)||\psi_2(t, x)||L^\infty) \|\overline{u}\|^2. \quad (40)\]

which along with the Gronwall Lemma, the continuity of \(\omega(t)\) in \(t\) and (17) imply the uniqueness and continuity of solutions in the initial data in \(L^2(\mathcal{O})\).

Finally, we prove the measurability of solutions in \(\omega \in \Omega\). By (34) and the uniqueness of solutions, we find that for every \(\omega \in \Omega\), the whole sequence \(u_n(t_0, \tau, \omega, u_\tau) \rightarrow u(t_0, \tau, \omega, u_\tau)\) weakly in \(L^2(\mathcal{O})\) for any fixed \(t_0 \in [\tau, T]\) and \(\omega \in \Omega\). Then the measurability of \(u(t, \tau, \omega, u_\tau)\) follows from that of \(u_n(t, \tau, \omega, u_\tau)\). This completes the proof.

\[\square\]

The following result is useful when proving the asymptotic compactness of solutions.

**Lemma 2.3.** Given \(\tau \in \mathbb{R}, t > \tau\) and \(\omega \in \Omega\), the solution operator \(u(t, \tau, \omega, \cdot) : L^2(\mathcal{O}) \rightarrow L^2(\mathcal{O})\) of problem (16) is compact, that is, for every bounded set \(B\) in \(L^2(\mathcal{O})\), the image \(u(t, \tau, \omega, B)\) is precompact in \(L^2(\mathcal{O})\).

**Proof.** Since the embedding \(H^1_0(\mathcal{O}) \hookrightarrow L^2(\mathcal{O})\) is compact, by (25), (29) and the compactness result in [34] we infer that there exists \(v \in L^2((\tau, T), L^2(\mathcal{O}))\) such that, up to a subsequence,

\[u(\cdot, \tau, \omega, u_{0,n}) \rightarrow v \quad \text{in} \quad L^2((\tau, T), L^2(\mathcal{O})). \quad (41)\]

By choosing a further subsequence (not relabeled again), we get from (41)

\[u(s, \tau, \omega, u_{0,n}) \rightarrow v(s) \quad \text{in} \quad L^2(\mathcal{O}) \quad \text{for almost all} \quad s \in (\tau, T). \quad (42)\]

Since \(\tau < t < T\), by (42) we must have \(s \in (\tau, t)\) such that the convergence (41) holds true for this \(s\). Then by the continuity of solutions in initial data in \(L^2(\mathcal{O})\), we obtain from (42)

\[u(t, \tau, \omega, u_{0,n}) = u(t, s, \omega, u(s, \tau, \omega, u_{0,n})) \rightarrow u(t, s, \omega, v(s)),\]

as desired. This completes the proof.

\[\square\]

Next, we define a cocycle \(\Phi : \mathbb{R}^+ \times \mathbb{R} \times \Omega \times L^2(\mathcal{O}) \rightarrow L^2(\mathcal{O})\) such that for all \(t \in \mathbb{R}^+, \tau \in \mathbb{R}, \omega \in \Omega\) and \(u_\tau \in \mathbb{R}\),

\[\Phi(t, \tau, \omega, u_\tau) = u(t + \tau, \tau, \theta_{-\tau} \omega, u_\tau). \quad (43)\]

Then \(\Phi\) given by (43) is a continuous cocycle on \(L^2(\mathcal{O})\) over \((\Omega, \mathcal{F}, P, \{\theta_t\}_{t \in \mathbb{R}})\).
We will study tempered random attractors of $\Phi$ in $L^2(\mathcal{O})$. Let $D = \{D(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\}$ be a family of bounded nonempty subsets of $L^2(\mathcal{O})$. Such a family $D$ is called tempered if for every $c > 0, \tau \in \mathbb{R}$ and $\omega \in \Omega$,

$$\lim_{t \to -\infty} e^{ct} \|D(\tau + t, \theta_t\omega)\| = 0, \quad (44)$$

where we use the notation $\|D\| = \sup_{\omega \in D} \|u\|_{L^2(\mathcal{O})}$ for a subset $D$ of $L^2(\mathcal{O})$. Hereafter, we will use $D$ to denote the collection of all tempered families of bounded nonempty subsets of $L^2(\mathcal{O})$, i.e.,

$$D = \{D = \{D(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} : D \text{satisfies (44)}\}. \quad (45)$$

When deriving uniform estimates, we assume that, for every $L$ nonempty subsets of $\mathbb{R}$, after, we will use $\|u\|_{L^\infty}L^2(\mathcal{O})$, for every $c^r = D$, $\|D\| < \infty$ is called tempered if for every $\tau \in \mathbb{R}$ and $\omega \in \Omega$, $D$ is a family of bounded nonempty subsets of $L^2(\mathcal{O})$.

For deriving the existence of tempered random attractors, we further assume for every $\tau \in \mathbb{R}$,

$$\int_{-\infty}^{0} e^{ps} \|g(s, \cdot)\|^2 ds < \infty. \quad (46)$$

For deriving the existence of tempered random attractors, we further assume for every $c > 0$,

$$\lim_{t \to -\infty} e^{ct} \int_{-\infty}^{0} e^{ps} \|g(s + t, \cdot)\|^2 ds = 0. \quad (47)$$

We find that both conditions (46) and (47) do not require that $g$ be bounded with respect to their first argument in $L^2(\mathcal{O})$ at $\pm \infty$.

2.2. Existence of pullback random attractors. In this subsection, we establish the existence of tempered absorbing sets for random (16) in $L^2(\mathcal{O})$ as well as the pullback asymptotic compactness of solutions. We finally prove the existence and uniqueness of tempered random attractors for the equation. Throughout this section, we assume $\psi_1 \in L^\infty(\mathbb{R}, L^4(\mathcal{O}))$

**Lemma 2.4.** Suppose (7)-(8) and (46) hold. Then for every $\sigma \in \mathbb{R}, \tau \in \mathbb{R}, \omega \in \Omega$ and $D = \{D(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in D$, there exists $T = T(\tau, \omega, D, \sigma) > 0$ such that for all $t \geq T$, the solution $u$ of Eq. (16) satisfies

$$\|u(\sigma, \tau - t, \theta_{t-\omega}, u_{t-\tau})\|^2 + \int_{-\infty}^{\sigma} e^{p(s-\sigma)} \|\nabla u(s, \tau - t, \theta_{t-\omega}, u_{t-\tau})\|^2 ds$$

$$+ \int_{-\infty}^{\sigma} e^{p(s-\sigma)} \|u(s, \tau - t, \theta_{t-\omega}, u_{t-\tau})\|^4_{L^4} ds$$

$$\leq M \int_{-\infty}^{\sigma} \rho(s - \tau - \omega) \left(\|g(s + \tau\| + |G(\theta_t\omega)|^{1/\gamma} + |G(\theta_t\omega)|^{1/\gamma})ds. \quad (48)\right.$$

where $u_{t-\tau} \in D(\tau - t, \theta_{t-\omega})$, $M$ is a positive constant dependent of $\rho$ and independent of $\tau, \omega, D, \sigma$.

**Proof.** Multiplying (16) with $\overline{u}$ and taking the real part, we have

$$\frac{1}{2} \frac{d}{dt} \|u\|^2 + \|\nabla u\|^2 + |\rho \cdot u|^2 = -\|u\|^2_{L^4} + Re \int_{\mathcal{O}} g(t, x) \overline{u} dx + ReG_\delta \int_{\mathcal{O}} h(t, x, u) \overline{u} dx. \quad (49)$$

By (7) we have

$$ReG_\delta \int_{\mathcal{O}} h(t, x, u) \overline{u} dx \leq c|G_\delta| \int_{\mathcal{O}} |u|^4 dx + \int_{\mathcal{O}} |G_\delta \psi_1 u| dx$$

$$\leq \frac{1}{2} \|u\|^4_{L^4} + c|G_\delta| |G_\delta(\theta_t\omega)|^{1/\gamma} + c|G_\delta(\theta_t\omega)|^{1/\gamma} \|\psi_1(t)\|_{L^\infty(\mathcal{O})}^{1/\gamma}. \quad (50)$$
By Young’s inequality we obtain
\[ \Re \int_{\Omega} g(t, x) \bar{u} \, dx \leq \frac{\rho}{2} \|u\|^2 + \frac{1}{2\rho} \|g(t, x)\|^2. \tag{51} \]

By (49)-(51) we get
\[ \frac{d}{dt} \|u\|^2 + \rho \|u\|^2 + 2 \|\nabla u\|^2 + \|u\|^4_{L^4} \leq \frac{1}{\rho} \|g(t, x)\|^2 + c_1 |G_\delta(\theta_1 \omega)|^{1/2} \]
\[ + c_1 |G_\delta(\theta_1 \omega)|^{1/4} \|\psi_1(t)\|^{1/2}_{L^2}. \tag{52} \]

Multiplying (52) by \( e^{\rho t} \) and then integrating over \((r, \sigma)\) with \( \sigma \geq r \), we get, for every \( \omega \in \Omega \),
\[ \|u(\sigma, r, \omega, u_r)\|^2 + 2 \int_{r}^{\sigma} e^{\rho(s-\sigma)} \|\nabla u(s, r, \omega, u_r)\|^2 ds + \int_{r}^{\sigma} e^{\rho(s-\sigma)} \|u(s, r, \omega, u_r)\|^4_{L^4} ds \]
\[ \leq e^{\rho(r-\sigma)} \|u_r\|^2 + \frac{1}{\rho} \int_{r}^{\sigma} e^{\rho(s-\sigma)} \|g(s)\|^2 ds \]
\[ + c_2 \int_{r}^{\sigma} e^{\rho(s-\sigma)} (|G_\delta(\theta_1 \omega)|^{3/4} + |G_\delta(\theta_2 \omega)|^{1/2}) ds. \tag{53} \]

Replacing \( r \) by \( \tau - t \) and \( \omega \) by \( \theta_{-t} \omega \) in (53) we obtain
\[ \|u(\sigma, \tau - t, \theta_{-t} \omega, u_{\tau-t})\|^2 + 2 \int_{\tau-t}^{\sigma} e^{\rho(s-\sigma)} \|\nabla u(s, \tau - t, \theta_{-t} \omega, u_{\tau-t})\|^2 ds \]
\[ + \int_{\tau-t}^{\sigma} e^{\rho(s-\sigma)} \|u(s, \tau - t, \theta_{-t} \omega, u_{\tau-t})\|^4_{L^4} ds \]
\[ \leq e^{\rho(\tau-t-\sigma)} \|u_{\tau-t}\|^2 + \frac{1}{\rho} \int_{\tau-t}^{\sigma} e^{\rho(s-\sigma)} \|g(s)\|^2 ds \]
\[ + c \int_{\tau-t}^{\sigma} e^{\rho(s-\sigma)} (|G_\delta(\theta_{-t} \omega)|^{3/4} + |G_\delta(\theta_{-t} \omega)|^{1/2}) ds \]
\[ \leq e^{\rho(\tau-t-\sigma)} \|u_{\tau-t}\|^2 + \frac{1}{\rho} \int_{-\infty}^{\sigma-\tau} e^{\rho(s+\tau-\sigma)} \|g(s + \tau)\|^2 ds \]
\[ + c \int_{-\infty}^{\sigma-\tau} e^{\rho(s+\tau-\sigma)} (|G_\delta(\theta_2 \omega)|^{3/4} + |G_\delta(\theta_2 \omega)|^{1/2}) ds. \tag{54} \]

Note that the last two integrals in (54) are well defined due to (10), (11) and (46). On the other hand, since \( u_{\tau-t} \in D(\tau - t, \theta_{-t} \omega) \) and \( D \in D \), we find
\[ e^{\rho(\tau-t-\sigma)} \|u_{\tau-t}\|^2 \leq e^{\rho(\tau-t-\sigma)} \|D(\tau - t, \theta_{-t} \omega)\|^2 \to 0, \]
as \( t \to \infty \). Thus, there exists \( T = T(\tau, \omega, D, \sigma) > 0 \) such that for all \( t \geq T \),
\[ e^{\rho(\tau-t-\sigma)} \|u_{\tau-t}\|^2 \leq \int_{-\infty}^{\sigma-\tau} e^{\rho(s+\tau-\sigma)} (|G_\delta(\theta_2 \omega)|^{3/4} + |G_\delta(\theta_2 \omega)|^{1/2}) ds. \tag{55} \]

From (54)-(55) the lemma follows. \[ \square \]

As an immediate consequence of Lemma 2.4, we obtain the existence of \( D \)-pullback absorbing sets for Eq. (16).
Corollary 1. Suppose (7)-(8) and (46)-(47) hold. Then the cocycle $\Phi$ associated with Eq. (16) possesses a closed measurable $D$-pullback absorbing set $K = \{K(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in D$ in $L^2(O)$ where for every $\tau \in \mathbb{R}$ and $\omega \in \Omega$, 
\[ K(\tau, \omega) = \{ u \in L^2(O) : \| u \|^2 \leq R(\tau, \omega) \} \] 
and 
\[ R(\tau, \omega) = M \int_{-\infty}^{0} e^{\rho(s+\tau-\sigma)}(\| g(s+\tau) \|^2 + |G_\delta(\theta_s \omega)|^{\frac{4}{\tau_0}} + |G_\delta(\theta_s \omega)|^{\frac{2}{\tau_0}}) ds \]
with $M$ being as in (48).

Proof. Given $\tau \in \mathbb{R}, \omega \in \Omega$ and $D \in D$, it follows from Lemma 2.4 with $\sigma = \tau$ that there exists $T = T(\tau, \omega, D) > 0$ such that for all $t \geq T$,
\[ \Phi(t, \tau - t, \theta_{-t} \omega, D(\tau - t, \theta_{-t} \omega)) = u(\tau, \tau - t, \theta_{-t} \omega, D(\tau - t, \theta_{-t} \omega)) \subseteq K(\tau, \omega). \] 
Next, we show $K$ belongs to $D$. Let $\beta$ be an arbitrary positive number and conside
\[ \lim_{t \to -\infty} e^{\beta t} \| K(\tau + t, \theta_t \omega) \|^2 = \lim_{t \to -\infty} e^{\beta t} R(\tau + t, \theta_t \omega) \] 
\[ = \lim_{t \to -\infty} Me^{\beta t} \int_{-\infty}^{0} e^{\rho s}(\| g(s + \tau + t) \|^2) ds + \] 
\[ \lim_{t \to -\infty} Me^{\beta t} \int_{-\infty}^{0} e^{\rho s}(|G_\delta(\theta_{s+t} \omega)|^{\frac{4}{\tau_0}} + |G_\delta(\theta_{s+t} \omega)|^{\frac{2}{\tau_0}}) ds. \] 
By (47) we get
\[ \lim_{t \to -\infty} Me^{\beta t} \int_{-\infty}^{0} e^{\rho s}(\| g(s + \tau + t) \|^2) ds = Me^{-\beta} \lim_{t \to -\infty} e^{\beta t} \int_{-\infty}^{0} e^{\rho s}(\| g(s + t) \|^2) ds. \] 
Let $\gamma = \min\{\rho, \beta\}$. Then for the last term in (58) we have for every $t \leq 0$,
\[ Me^{\beta t} \int_{-\infty}^{0} e^{\rho s}(|G_\delta(\theta_{s+t} \omega)|^{\frac{4}{\tau_0}} + |G_\delta(\theta_{s+t} \omega)|^{\frac{2}{\tau_0}}) ds \] 
\[ \leq M \int_{-\infty}^{0} e^{\gamma(s+t)}(|G_\delta(\theta_{s+t} \omega)|^{\frac{4}{\tau_0}} + |G_\delta(\theta_{s+t} \omega)|^{\frac{2}{\tau_0}}) ds \] 
\[ \leq M \int_{-\infty}^{t} e^{\gamma s}(|G_\delta(\theta_s \omega)|^{\frac{4}{\tau_0}} + |G_\delta(\theta_s \omega)|^{\frac{2}{\tau_0}}) ds. \] 
By (10) and (11) we know $\int_{-\infty}^{0} e^{\gamma s}(|G_\delta(\theta_s \omega)|^{\frac{4}{\tau_0}} + |G_\delta(\theta_s \omega)|^{\frac{2}{\tau_0}}) ds < \infty, \omega,$ and hence by (60) we get
\[ \lim_{t \to -\infty} Me^{\beta t} \int_{-\infty}^{0} e^{\rho s}(|G_\delta(\theta_s \omega)|^{\frac{4}{\tau_0}} + |G_\delta(\theta_s \omega)|^{\frac{2}{\tau_0}}) ds = 0. \] 
It follows from (58), (59) and (61) that $\lim_{t \to -\infty} e^{\beta t} \| K(\tau + t, \theta_t \omega) \|^2 = 0$ for any $\beta > 0$. This along with (57) completes the proof. \(\square\)

Note that Lemma 2.3 implies the $D$-pullback asymptotic compactness of $\Phi$ in $L^2(O)$ more precisely, we have the following result.

Corollary 2. Suppose (7)-(8) and (46) hold. Then for every $\tau \in \mathbb{R}$, $\omega \in \Omega$ and $D = \{D(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in D$, the sequence $\Phi(t_n, \tau - t_n, \theta_{-t_n} \omega, u_{0,n})$ has a convergent subsequence in $L^2(O)$ provided $t_n \to \infty$ and $u_{0,n} \in D(\tau - t, \theta_{-t} \omega)$. 
Proof. It follows from Lemma 2.4 with $\sigma = \tau - 1$ that there exist $T = T(\tau, \omega, D) > 0$ and $c = c(\tau, \omega) > 0$ such that for all $t \geq T$ and $u_0 \in D(t-t, \theta_{-\tau}\omega)$,

$$|u(t-1, \tau-t, \theta_{-\tau}\omega, u_0)| \leq c(\tau, \omega). \quad (62)$$

Since $\tau_n \to \infty$ and $u_{0, n} \in D(t-t, \theta_{-\tau}\omega)$, by (62) we find that there is $N = N(\tau, \omega, D) > 0$ such that for all $n \geq N$,

$$\|u(t-1, \tau - t_n, \theta_{-\tau}\omega, u_{0, n})\| \leq c(\tau, \omega).$$

This shows that

$$\{u(t-1, \tau - t_n, \theta_{-\tau}\omega, u_{0, n})\}_{n=1}^\infty \text{ is bounded in } L^2(\mathcal{O}). \quad (63)$$

By (63) and Lemma 2.3 we know the sequence

$$u(t, \tau - t_n, \theta_{-\tau}\omega, u_{0, n}) = u(t, t-1, \theta_{-\tau}\omega, u(t-1, t-t_n, \theta_{-\tau}\omega, u_{0, n}))$$

is precompact in $L^2(\mathcal{O})$, which along with $\Phi(t_n, \tau - t_n, \theta_{-\tau}\omega, u_{0, n}) = u(t, \tau - t_n, \theta_{-\tau}\omega, u_{0, n})$ completes the proof.

We are now ready to prove the existence of $\mathcal{D}$-pullback attractors of $\Phi$.

**Theorem 2.5.** Suppose (7)-(8) and (46)-(47) hold. Then the cocycle $\Phi$ associated with Eq. (16) has a unique $\mathcal{D}$-pullback attractor $A = \{A(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$ in $L^2(\mathcal{O})$ which is characterized by, for each $\tau \in \mathbb{R}$ and $\omega \in \Omega$,

$$A(\tau, \omega) = \Omega(K, \tau, \omega) = \bigcup_{B \in \mathcal{D}} \Omega(B, \tau, \omega)$$

$$= \{\psi(0, \tau, \omega) : \psi \text{ is a } \mathcal{D} \text{- complete orbit of } \Phi\}$$

$$= \{\xi(\tau, \omega) : \xi \text{ is a } \mathcal{D} \text{- complete quasi - solution of } \Phi\},$$

where $K$ is the $\mathcal{D}$-pullback absorbing set of $\Phi$ as given by (56).

If, in addition, there exist $T > 0$ such that $h, g$ are all $T$-periodic in their first argument, then the attractor $A$ is also $T$-periodic, that is, for every $\tau \in \mathbb{R}$ and $\omega \in \Omega$.

**Proof.** Since $\Phi$ has a closed measurable $\mathcal{D}$-pullback absorbing set $K$ by Corollary 1 and is $\mathcal{D}$-pullback asymptotically compact in $L^2(\mathcal{O})$ by Corollary 2, then the existence of $\mathcal{D}$-pullback attractor $A$ of $\Phi$ follows from Proposition 2.10 in [56] immediately. Moreover, this attractor is unique and its structure is given as above.

If $h$ and $g$ are $T$-periodic in their first argument, then the cocycle $\Phi$ is also $T$-periodic, i.e., $\Phi(t, \tau + T, \omega, \cdot) = \Phi(t, \tau, \omega, \cdot)$ for all $t \in \mathbb{R}^+, \tau \in \mathbb{R}$ and $\omega \in \Omega$. If $g$ is $T$-periodic in the first argument, then by (56) we have $K(\tau + T, \omega) = K(\tau, \omega)$ for all $\tau \in \mathbb{R}$ and $\omega \in \Omega$. Therefore the $T$-periodicity of $A$ follows from the periodicity of $\Phi$ and $K$ according to Proposition 2.11 in [56].

3. **Stochastic ginzburg-landau equations driven by linear multiplicative noise.**

In this section, we consider Eq. (1) with multiplicative noise:

$$\frac{\partial u}{\partial t} - (1 + i\lambda)\Delta u + \rho u = f(u) + g(t, x) + u \circ \frac{dW}{dt}, \quad (64)$$

which is supplemented with boundary condition (5) and initial condition (6). To discuss random attractors in this case, we need to convert (64) into a pathwise
deterministic equation that can be done via the standard transformation \( v(t, \tau, \omega) = e^{-\omega(t)}u(t, \tau, \omega) \). From (64) we find \( v \) satisfies
\[
\frac{\partial v}{\partial t} - (1 + i\lambda) \Delta v + \rho v = e^{-\omega(t)}f(e^{\omega(t)}v) + e^{-\omega(t)}g(t, x),
\]
with boundary condition
\[
v(t, x) = 0, \quad x \in \partial \mathcal{O} \quad \text{and} \quad t > \tau,
\]
and initial condition
\[
v(t, x) = v_\tau(x), \quad x \in \mathcal{O},
\]
where \( v_\tau(x) = e^{-\omega(\tau)}u_\tau(x) \). Given \( \omega \in \Omega, \tau \in \mathbb{R} \) and \( v_\tau \in L^2(\mathcal{O}) \), problem (65)-(67) is a deterministic system, and hence, as in the previous section, one can prove, this system has a unique solution
\[
v(\cdot, \tau, \omega, v_\tau) \in C([\tau, \infty), L^2(\mathcal{O})) \cap L^2_{loc}((\tau, \infty), H^1_0(\mathcal{O})).
\]
In addition, \( v(\cdot, \tau, \omega, v_\tau) \) is continuous in \( v_\tau \) with respect to the norm of \( L^2(\mathcal{O}) \) and \( (\mathcal{F}, \mathcal{B}(L^2(\mathcal{O}))) \)-measurable in \( \omega \in \Omega \). This enables us to define a cocycle \( \Phi_0 : \mathbb{R}^+ \times \mathbb{R} \times \Omega \times L^2(\mathcal{O}) \rightarrow L^2(\mathcal{O}) \) for the stochastic equation (64) via the solutions of (65). Given \( t \in \mathbb{R}^+, \tau \in \mathbb{R}, \omega \in \Omega \) and \( u_\tau \in L^2(\mathcal{O}) \), let
\[
\Phi_0(t, \tau, \omega, u_\tau) = u(t + \tau, \tau, \theta_{-\tau} \omega, u_\tau) = e^{\omega(\tau)}u(t + \tau, \tau, \theta_{-\tau} \omega, v_\tau),
\]
where \( v_\tau = e^{-\omega(\tau)}u_\tau \). Then we find that \( \Phi_0 \) given by (68) is a continuous cocycle on \( L^2(\mathcal{O}) \) over \( (\Omega, \mathcal{F}, P, \{\theta_t\}_{t \in \mathbb{R}}) \). We want to show the stochastic equation (64) has a \( \mathcal{D} \)-pullback attractor in \( L^2(\mathcal{O}) \). To that end, we must derive uniform estimates of the solutions which are given below.

**Lemma 3.1.** Suppose (46) hold. Then for every \( \tau \in \mathbb{R}, \omega \in \Omega \) and \( D = \{D(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D} \), there exists \( T = T(\tau, \omega, D) > 0 \) such that for all \( t \geq T \), the solution \( u \) of Eq. (64) satisfies
\[
\|u(\tau, t, \theta_{-\tau} \omega, u_{\tau-t})\|^2 \leq \frac{4}{\rho} \int_{-\infty}^{0} e^{\frac{3}{4} \rho s - 2\omega(s)} \|g(s + \tau)\|^2 ds,
\]
where \( u_{\tau-t} \in D(\tau - t, \theta_{-\tau} \omega) \).

**Proof.** From (65) we have
\[
\frac{1}{2} \frac{d}{dt} \|v\|^2 + \|\nabla v\|^2 + \rho \|v\|^2 = -e^{2\omega(t)}\|v\|^4_{L^4} + \text{Re} \int_{\mathcal{O}} e^{-\omega(t)}g(t, x)\nabla v dx. \tag{69}
\]
By Young’s inequality we get
\[
\text{Re} \int_{\mathcal{O}} e^{-\omega(t)}g(t, x)\nabla v dx \leq \frac{\rho}{4} \|v\|^2 + \frac{1}{\rho} e^{-2\omega(t)}\|g(t)\|^2. \tag{70}
\]
By (69)-(70) we find
\[
\frac{d}{dt} \|v\|^2 + 2\|\nabla v\|^2 + \frac{3}{2} \rho \|v\|^2 \leq -2e^{2\omega(t)}\|v\|^4_{L^4} + \frac{2}{\rho} e^{-2\omega(t)}\|g(t)\|^2. \tag{71}
\]
Solving (71) we get for every \( \tau \in \mathbb{R}, t \in \mathbb{R}^+ \) and \( \omega \in \Omega \),
\[
\|v(\tau, t, \omega, v_{\tau-t})\|^2 \leq e^{-\frac{3}{4} \rho t} \|v_{\tau-t}\|^2 + \frac{2}{\rho} \int_{\tau-t}^{\tau} e^{\frac{3}{4} \rho s - 2\omega(s)} \|g(s)\|^2 ds. \tag{72}
\]
By definition we have
\[ \| u(\tau, \tau - t, \theta_{t-\tau}\omega, u_{\tau-t}) \|^2 = e^{-\omega(-\tau)} v(\tau, \tau - t, \theta_{t-\tau}\omega, v_{\tau-t}) \]
with \( v_{\tau-t} = e^{\omega(-\tau)-\omega(-t)} u_{\tau-t} \),
which along with (72) yields
\[ \| u(\tau, \tau - t, \theta_{t-\tau}\omega, u_{\tau-t}) \|^2 \]
\[ \leq e^{-\frac{3}{2}pt-2\omega(-t)} \| u_{\tau-t} \|^2 + \frac{2}{\rho} \int_{\tau-t}^{\tau} e^{\frac{3}{2}pt-2\omega(s-\tau)} \| g(s) \|^2 ds \]
\[ \leq e^{-\frac{3}{2}pt-2\omega(-t)} \| u_{\tau-t} \|^2 + \frac{2}{\rho} \int_{-\infty}^{0} e^{\frac{3}{2}pt-2\omega(s)} \| g(s + \tau) \|^2 ds. \] (73)

By (10) and (46) we see the following integral is convergent:
\[ \frac{2}{\rho} \int_{-\infty}^{0} e^{\frac{3}{2}pt-2\omega(s)} \| g(s + \tau) \|^2 ds < \infty. \] (74)

Note that \( u_{\tau-t} \in D(\tau - t, \theta_{t-\tau}\omega) \) and \( D \in D \). Therefore, by (10) we have
\[ \lim_{t \to \infty} \sup_{t \geq T} e^{-\frac{3}{2}pt-2\omega(-t)} \| u_{\tau-t} \|^2 \leq \lim_{t \to \infty} \sup_{t \geq T} e^{-\frac{3}{2}pt-2\omega(-t)} \| D(\tau - t, \theta_{t-\tau}\omega) \|^2 = 0, \]
which shows that there exists \( T = T(\tau, \omega, D) > 0 \) such that for all \( t \geq T \),
\[ e^{-\frac{3}{2}pt-2\omega(-t)} \| u_{\tau-t} \|^2 \leq \frac{2}{\rho} \int_{-\infty}^{0} e^{\frac{3}{2}pt-2\omega(s)} \| g(s + \tau) \|^2 ds. \] (75)

Then (73)-(75) imply the desired estimates.

As a consequence of inequality (71) we have the following useful estimates.

**Lemma 3.2.** Suppose (46) hold. Then for every \( \tau \in \mathbb{R}, \omega \in \Omega \) and \( T > 0 \), there exists \( \rho = c(\tau, \omega, T) > 0 \) such that for all \( t \in [\tau, \tau + T] \), the solution \( v \) of system (65)-(67) satisfies
\[ \| v(t, \tau, \omega, v_\tau) \|^2 + \int_{\tau}^{t} \| v(s, \tau, \omega, v_\tau) \|_{L^4}^4 ds \leq c(\| v_\tau \|^2 + \int_{\tau}^{t} \| g(s) \|^2 ds). \]

**Proof.** Multiplying (71) by \( e^{\frac{3}{2}pt} \) and then integrating over \((\tau, t)\) we get
\[ \| v(t, \tau, \omega, v_\tau) \|^2 + 2 \int_{\tau}^{t} e^{\frac{3}{2}p(s-\tau)} \| v(s, \tau, \omega, v_\tau) \|_{L^4}^4 ds \]
\[ \leq e^{\frac{3}{2}p(t-\tau)} \| v_\tau \|^2 + \frac{2}{\rho} \int_{\tau}^{t} e^{\frac{3}{2}pt-2\omega(s)} \| g(s) \|^2 ds, \]
which together with the continuity of \( \omega \) on \([\tau, \tau+T]\) yields the desired estimates. \( \square \)

Now we will prove the \( D \)-pullback asymptotic compactness of solutions of (64).

**Lemma 3.3.** Suppose (46) hold. Then the cocycle \( \Phi_0 \) associated with the stochastic equation (64) is \( D \)-pullback asymptotically compact in \( L^2(\Omega) \), that is, for every \( \tau \in \mathbb{R}, \omega \in \Omega \) and \( D = \{ D(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega \} \in D \), the sequence \( \Phi_0(t_n, \tau - t_n, \theta_{t_n-\tau}\omega, u_{0,n}) \) has a convergent subsequence in \( L^2(\Omega) \) provided \( t_n \to \infty \) and \( u_{0,n} \in D(\tau - t, \theta_{t_n-\tau}\omega) \).
Proof. The proof is similar to that of Corollary 2.6. First, by the argument of
Lemma 3.1, we can show that there exist $T=T(\tau, \omega, D)>0$ and $c = c(\tau, \omega)>0$
such that for all $t \geq T$ and $u_0 \in D(\tau - t, \theta \omega)$,
\[ |u(\tau - 1, \tau - t, \theta \omega, u_0)| \leq c(\tau, \omega). \]
Therefore, there is $N = N(\tau, \omega, D)>0$ such that for all $n \geq N$,
\[ |u(\tau - 1, \tau - t_n, \theta \omega, u_{0,n})| \leq c(\tau, \omega). \quad (76) \]
Following the proof of Lemma 2.3 we find the solution operators of Eq. (64) are
compact in $L^2(\mathcal{O})$, i.e., the map $v(\sigma, s, \omega, \cdot) : L^2(\mathcal{O}) \to L^2(\mathcal{O})$ is compact for every
$\sigma > s$ and $\omega \in \Omega$. This implies the compactness of the solution operators of (64) in
$L^2(\mathcal{O})$, i.e., the map $u(\sigma, s, \omega, \cdot) : L^2(\mathcal{O}) \to L^2(\mathcal{O})$ is compact for every $\sigma > s$ and
$\omega \in \Omega$. By definition we have
\[ \Phi_0(t_n, \tau - t_n, \theta \omega, u_{0,n}) = u(\tau - 1, \theta \omega, u_{0,n}) = u(\tau, \tau - 1, \theta \omega, u_{0,n}), \]
which together with (76) concludes the proof. \qed

In the following, we present the existence of $D$-pullback attractors of Eq. (64).

**Theorem 3.4.** Suppose (46)-(47) hold. Then the cocycle $\Phi_0$ associated with Eq.
(64) has a unique $D$-pullback attractor $A_0 = \{A_0(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in D$
in $L^2(\mathcal{O})$. If, in addition, there exists $T > 0$ such that $g$ is $T$-periodic in their first
argument, then the attractor $A_0$ is also $T$-periodic.

**Proof.** Given $\tau \in \mathbb{R}$ and $\omega \in \Omega$, define a subset $K_0(\tau, \omega)$ by
\[ K_0(\tau, \omega) = \{ u \in L^2(\mathcal{O}) : \|u\|^2 \leq \frac{4}{\rho} \int_{-\infty}^{0} e^{\frac{3}{2} \rho(s - 2\omega(s))} \|g(s + \tau)\|^2 ds \}. \]
Then by Lemma 3.1 we find that for every $\tau \in \mathbb{R}, \omega \in \Omega$ and $D = \{D(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in D$, there exists $T = T(\tau, \omega, D)>0$ such that for all $t \geq T$,
\[ \Phi_0(t, \tau - t, \theta \omega, D(\tau - t, \theta \omega)) = u(t, \tau - t, \theta \omega, D(\tau - t, \theta \omega)) \subseteq K_0(\tau, \omega). \quad (77) \]
Moreover, by (47), one can check $K_0$ is tempered in $L^2(\mathcal{O})$, i.e., $K_0 \in D$. Therefore,
$K_0$ is a closed measurable $D$-pullback absorbing set, which along with Lemma 3.3
implies the existence and uniqueness of $D$-pullback attractor $A_0$ of $\Phi_0$. In the case
$g$ is $T$-periodic in the first argument, we find $\Phi_0$ and $K_0$ are also $T$-periodic. Thus
by Proposition 2.11 in [56], it follows that the attractor $A_0$ is $T$-periodic. \qed

We now approximate the solutions of the stochastic Ginzburg-Landau equation
(64) by the following pathwise Wong-Zakai approximated equation:
\[ \frac{\partial u_\delta}{\partial t} - (1 + i\lambda) \Delta u_\delta + \rho u_\delta = f(u_\delta) + g(t, x) + \mathcal{G}_\delta u_\delta \quad (78) \]
This approximation together with the boundary condition (5) and initial condition (6). Note that the
solution of Eq. (78) is written as $u_\delta$ from now on to imply its dependence on $\delta$. By
the previous section, we see for every $\delta \neq 0$, Eq. (78) defines a continuous cocycle
$\Phi_\delta$ in $L^2(\mathcal{O})$ which has a unique $D$-pullback attractor $A_\delta$. In what follows, we
will investigate the convergence of solutions of (78) as $\delta \to 0$; more precisely, we
will show that the solutions of Eq. (78) approach that of the stochastic Ginzburg-
Landau equation (64) in $L^2(\mathcal{O})$ as $\delta \to 0$. Furthermore, we will obtain the upper
semicontinuity of random attractors $A_\delta$ as $\delta \to 0$. These results partially justify the
idea to approximate a stochastic Ginzburg-Landau equation by replacing the white
noise by a process $G_\delta(\theta, \omega)$ with small $\delta$, and thus, under certain circumstances, we could use the methods of deterministic dynamical systems to discuss the dynamics of stochastic Ginzburg-Landau equations.

To better understand the relations between the solutions of (64) and (78), we need a similar transformation for (78) as we did for (64). Let

$$v_\delta(t, \tau, \omega) = e^{-\int_0^\tau G_\delta(\theta, \omega) d\theta} u_\delta(t, \tau, \omega).$$

Then we have from (78):

$$\frac{\partial v_\delta}{\partial t} - (1 + i\lambda) \Delta v_\delta + \rho v_\delta = e^{\int_0^\tau G_\delta(\theta, \omega) d\theta} f(e^{\int_0^\tau G_\delta(\theta, \omega) d\theta} v_\delta) + e^{\int_0^\tau G_\delta(\theta, \omega) d\theta} g(t, x),$$

with boundary and initial conditions

$$v_\delta(t, x) = 0 \quad \text{for} \quad x \in \partial \Omega \quad \text{and} \quad v_\delta(\tau, x) = v_{\delta, \tau}(x) \quad \text{for} \quad x \in \Omega.$$

Next we derive the following estimates on the solutions of system (80)-(81) on a finite time interval.

**Lemma 3.5.** Suppose (48) hold. Then for every $\tau \in \mathbb{R}$, $\omega \in \Omega$ and $T > 0$, there exists $\delta_0 = \delta_0(\tau, \omega, T) > 0$, $c = c(\tau, \omega, T) > 0$ such that the solution $v_\delta$ of system (80) - (81) satisfies, for all $0 < \delta < \delta_0$ and $t \in [\tau, \tau + T]$,

$$\|v_\delta(t, \tau, \omega, v_{\delta, \tau})\|^2 + \int_\tau^t \|\nabla v_\delta(s, \tau, \omega, v_{\delta, \tau})\|^2 ds + \int_\tau^t \|v_\delta(s, \tau, \omega, v_{\delta, \tau})\|_{L^4}^4 ds$$

$$\leq c(\|v_{\delta, \tau}\|^2 + \int_\tau^t \|g(s)\|^2 ds).$$

**Proof.** From (80) we get

$$\frac{1}{2} \frac{d}{dt} \|v_\delta\|^2 + \|\nabla v_\delta\|^2 + \rho \|v_\delta\|^2 = -e^{2\int_0^\tau G_\delta(\theta, \omega) d\theta} \|v_\delta\|_{L^4}^4$$

$$+ \text{Re} \int_\Omega e^{\int_0^\tau G_\delta(\theta, \omega) d\theta} g(t, x) \overline{v_\delta} dx.$$  \hfill (83)

For the last term on the right-hand side of (83) we obtain

$$\text{Re} \int_\Omega e^{\int_0^\tau G_\delta(\theta, \omega) d\theta} g(t, x) \overline{v_\delta} dx \leq \frac{\rho}{4} \|v_\delta\|^2 + \frac{1}{\rho} e^{-2\int_0^\tau G_\delta(\theta, \omega) d\theta} \|g(t)\|^2.$$  \hfill (84)

By (83)-(84) we have

$$\frac{d}{dt} \|v_\delta\|^2 + 2\|\nabla v_\delta\|^2 + \frac{3}{2} \rho \|v_\delta\|^2 \leq -2e^{2\int_0^\tau G_\delta(\theta, \omega) d\theta} \|v_\delta\|_{L^4}^4 + \frac{2}{\rho} e^{-2\int_0^\tau G_\delta(\theta, \omega) d\theta} \|g(t)\|^2.$$  \hfill (85)

From (85), we find that for all $\tau \in \mathbb{R}$, $t > \tau$ and $\omega \in \Omega$,

$$\|v_\delta(t, \tau, \omega, v_{\delta, \tau})\|^2 + 2 \int_\tau^t e^{\frac{2}{\rho}(s-t)} \|\nabla v_\delta(s, \tau, \omega, v_{\delta, \tau})\|_{L^4}^4 ds$$

$$+ 2 \int_\tau^t e^{\frac{2}{\rho}(s-t)+2\int_0^\tau G_\delta(\theta, \omega) d\theta} \|v_\delta(s, \tau, \omega, v_{\delta, \tau})\|_{L^4}^4 ds$$

$$\leq e^{\frac{2}{\rho}(\tau-t)} \|v_{\delta, \tau}\|^2 + \frac{2}{\rho} \int_\tau^t e^{\frac{2}{\rho}(s-t)-2\int_0^\tau G_\delta(\theta, \omega) d\theta} \|g(s)\|^2 ds,$$

which along with (15) yields (82).
Lemma 3.6. Suppose (46) hold. Then for every \( \delta \neq 0, \tau \in \mathbb{R}, \omega \in \Omega \) and \( D = \{ D(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega \} \in \mathbb{D} \), there exist \( T = T(\tau, \omega, D, \delta) > 0 \) such that for all \( t \geq T \), the solution \( u_\delta \) of Eq. (78) satisfies

\[
\|u_\delta(\tau, t, \theta_{-\tau}, u_{\delta, t-})\|^2 \leq \frac{4}{\rho} \int_{-\infty}^{\tau} e^{2\rho s + 2 \int_0^s G_\delta(\theta_s, \omega) ds} \|g(s + \tau)\|^2 ds,
\]

where \( u_{\delta, t-} \in D(\tau - t, \theta_{-\tau}) \).

Proof. By (85) we have for every \( \tau \in \mathbb{R}, t \in \mathbb{R}^+ \) and \( \omega \in \Omega \)

\[
\|v_\delta(\tau, t, \theta_{-\tau}, u_{\delta, t-})\|^2 \leq e^{-2\rho t} \|v_{\delta, t-}\|^2 + \frac{2}{\rho} \int_{\tau-t}^{\tau} e^{2\rho(s-\tau)} e^{2 \int_0^{s-\tau} G_\delta(\theta_s, \omega) ds} \|g(s)\|^2 ds.
\]

By (79) and (87) we have

\[
\|u_\delta(\tau, t, \theta_{-\tau}, u_{\delta, t-})\|^2 = e^{2 \int_0^t G_\delta(\theta_s, \omega) ds} \|v_\delta(\tau, t, \theta_{-\tau}, u_{\delta, t-})\|^2 \leq e^{-2\rho t} \|v_{\delta, t-}\|^2 + \frac{2}{\rho} \int_{\tau-t}^{\tau} e^{2\rho(s-\tau)} e^{2 \int_0^{s-\tau} G_\delta(\theta_s, \omega) ds} \|g(s)\|^2 ds \leq e^{-2\rho t} \|v_{\delta, t-}\|^2 + \frac{2}{\rho} \int_{\tau-t}^{\tau} e^{2\rho(s-\tau)} e^{2 \int_0^{s-\tau} G_\delta(\theta_s, \omega) ds} \|g(s)\|^2 ds.
\]

By (10) and the ergodic theory, we obtain

\[
\lim_{s \to \pm \infty} \frac{1}{s} \int_0^s G_\delta(\theta_s, \omega) ds = E(G_\delta(\omega)) = 0.
\]

which together with (46) implies

\[
\int_{-\infty}^{\tau} e^{2\rho s + 2 \int_0^s G_\delta(\theta_s, \omega) ds} \|g(s + \tau)\|^2 ds < \infty.
\]

On the other hand, since \( u_{\delta, t-} \in D(\tau - t, \theta_{-\tau}) \) and \( D \in \mathbb{D} \), by (89) we see that there exists \( T = T(\tau, \omega, D, \delta) > 0 \) such that for all \( t \geq T \),

\[
e^{-2\rho t} e^{2 \int_0^t G_\delta(\theta_s, \omega) ds} \|u_{\delta, t-}\|^2 \leq e^{-2\rho t} \|D(\tau - t, \theta_{-\tau})\|^2 \leq \frac{2}{\rho} \int_{-\infty}^{\tau} e^{2\rho s + 2 \int_0^s G_\delta(\theta_s, \omega) ds} \|g(s + \tau)\|^2 ds.
\]

Finally, by (88)-(91) we have (86). \( \square \)

As a result of Lemma 3.6, we get a \( \mathbb{D} \)-pullback absorbing set of Eq. (78) immediately.

Lemma 3.7. Suppose (46)-(47) hold. Then the continuous cocycle \( \Phi_\delta \) associated with Eq. (78) possesses a \( \mathbb{D} \)-pullback absorbing set \( K_\delta \in \mathbb{D} \) which is given by, for every \( \tau \in \mathbb{R} \) and \( \omega \in \Omega \),

\[
K_\delta(\tau, \omega) = \{ u \in L^2(\mathcal{O}) : \|u\|^2 \leq R_\delta(\tau, \omega) \}
\]

with

\[
R_\delta(\tau, \omega) = \frac{4}{\rho} \int_{-\infty}^{\tau} e^{2\rho s + 2 \int_0^s G_\delta(\theta_s, \omega) ds} \|g(s + \tau)\|^2 ds.
\]
In addition, for every \( \tau \in \mathbb{R} \) and \( \omega \in \Omega \),
\[
\lim_{\delta \to 0} R_\delta(\tau, \omega) = \frac{4}{\rho} \int_{-\infty}^{0} e^{\frac{2}{\rho}s-2\omega(s)}\|g(s + \tau)\|^2 ds. \tag{94}
\]

Proof. Note that \( K_\delta = \{ K_\delta(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega \} \) given by \( \Phi_0(\omega, D) \) is a closed measurable random set in \( L^2(O) \). Moreover, by Lemma 3.6 we find that for every \( \tau \in \mathbb{R} \), \( \omega \in \Omega \) and \( D \in D \), there exists \( T_0 = T_0(\tau, \omega, D) > 0 \) such that
\[
\Phi_0(t, \tau - t, \theta_{-\omega}(\omega, D)) \subseteq K_\delta(\tau, \omega) \quad \text{for all} \quad t > T_0. \tag{95}
\]

We next show \( K_\delta \) is tempered. Assume that \( \beta \) be an arbitrary positive number and consider
\[
e^{\beta t}\|K_\delta(\tau + t, \theta_{t}\omega)\|^2 = e^{\beta t} R_\delta(\tau + t, \theta_{t}\omega) = \frac{4}{\rho} e^{\beta t} \int_{-\infty}^{0} e^{\frac{2}{\rho}s+2\int_{0}^{s} \rho \omega(\theta_{s}\omega) dr} \|g(s + \tau + t)\|^2 ds
\]
\[
= \frac{4}{\rho} e^{\beta t} \int_{-\infty}^{0} e^{\frac{2}{\rho}s-\frac{1}{2}\int_{s}^{s+\delta} \omega(r + t) dr - \frac{1}{2}\int_{s}^{s+\delta} \omega(r + t) dr} \|g(s + \tau + t)\|^2 ds
\]
where we have used \( (11) \). Put \( c = \min\{ \frac{\beta}{2}, \frac{\beta}{8} \} \). By \( (10) \) we see that there exists \( T_1 = T_1(\omega) < 0 \) such that
\[
|\omega(t)| \leq -ct \quad \text{for all} \quad t > T_1. \tag{97}
\]

By the mean value theorem we find that for every \( s \leq 0 \), there exists \( r_0 \) between \( s \) and \( s + \delta \) such that \( e^{-\frac{2}{\rho}\int_{s}^{s+\delta} \omega(r + t) dr} = e^{-2\omega(r_0 + t)} \). In this case, we obtain \( r_0 \leq |\delta| \), and therefore from \( (97) \) we get that for all \( t \leq T_1 - |\delta| \),
\[
|\omega(r_0 + t)| \leq -c(r_0 + t). \tag{98}
\]

Note that \( s - r_0 \leq |\delta| \), and thus from \( (98) \) we get
\[
e^{-\frac{2}{\rho}\int_{s}^{s+\delta} \omega(r + t) dr} = e^{-2\omega(r_0 + t)} \leq e^{-2\omega(s) - 2ct + 2c|\delta|} \quad \text{for all} \quad t \leq T_1 - |\delta|. \tag{99}
\]

Similarly, one can prove that
\[
e^{-\frac{2}{\rho}\int_{r_0}^{r_0+t} \omega(r + t) dr} \leq e^{-2ct + 2c|\delta|} \quad \text{for all} \quad t \leq T_1 - |\delta|. \tag{100}
\]

By \( (96)-(97) \) and \( (99)-(100) \) we have for all \( t \leq T_1 - |\delta| \),
\[
e^{\beta t}\|K_\delta(\tau + t, \theta_{t}\omega)\|^2 \leq \frac{4}{\rho} e^{\frac{2}{\rho}|\delta|} e^{\frac{2}{\rho} \int_{-\infty}^{0} e^{\rho s} \|g(s + \tau + t)\|^2 ds}
\]
\[
\leq \frac{4}{\rho} e^{\frac{2}{\rho}|\delta|} e^{\frac{2}{\rho} \int_{-\infty}^{0} e^{\rho s} \|g(s + \tau + t)\|^2 ds}. \tag{101}
\]

By \( (47) \) and \( (101) \) we obtain
\[
\lim_{t \to -\infty} e^{\beta t}\|K_\delta(\tau + t, \theta_{t}\omega)\|^2 = 0 \quad \text{for all} \quad \beta > 0 \quad \text{and} \quad \delta \neq 0,
\]
and so \( K_\delta \) belongs to \( D \). This together with \( (95) \) shows that \( K_\delta \) is a closed measurable \( D \)-pullback absorbing set of \( \Phi_\delta \).

We now prove \( (94) \). By \( (11) \) we get
\[
2 \int_{s}^{0} G_\delta(\theta_{t}\omega) dr = -2 \int_{s}^{s+\delta} \frac{\omega(r)}{\delta} dr + 2 \int_{0}^{\delta} \frac{\omega(r)}{\delta} dr. \tag{102}
\]
Note that \( \lim_{\delta \to 0} \int_0^\delta \frac{\omega(r)}{\delta} dr = 0 \), there exists \( \delta_1 = \delta_1(\omega) > 0 \) such that

\[
|2 \int_0^\delta \frac{\omega(r)}{\delta} dr| \leq 1 \quad \text{for all} \quad 0 < |\delta| < \delta_1.
\]

By the mean value theorem we find \(-2 \int_0^{\delta + \delta_1} \frac{\omega(r)}{\delta} dr \leq -2\omega(r_2)\) for some \( r_2 \) between \( s \) and \( s + \delta \), which implies \( |s - r_2| < |\delta| \). So if \( |\delta| \leq 1 \) and \( s \leq T_1 - 1 \), then we get \( r_2 < s + |\delta| < T_1 \). Thus, by (97) with \( c \leq \frac{\rho}{4} \) we have

\[
|\omega(r_2)| \leq -\frac{\rho}{4} r_2 \leq -\frac{\rho}{4} |\delta| \leq \frac{\rho}{4} s \leq \frac{\rho}{4} s.
\]

Let \( \delta_2 = \min \{1, \delta_1\} \). By (102)-(104) we obtain for all \( 0 < |\delta| < \delta_2 \) and \( s \leq T_1 - 1 \)

\[
2 \int_0^\delta G_\delta(\theta, \omega) dr \leq \frac{\rho}{2} - \frac{\rho}{2} s + 1.
\]

Given \( s \in \mathbb{R}, \delta \neq 0, \tau \in \mathbb{R} \) and \( \omega \in \Omega \), put

\[
\tilde{R}_\delta(\tau, \omega, s) = \frac{4}{\rho} e^{2\rho s - 2\omega(s)} \|g(s + \tau)\|^2 ds.
\]

By (12) we find

\[
\lim_{\delta \to 0} \tilde{R}_\delta(\tau, \omega, s) = \frac{4}{\rho} e^{2\rho s - 2\omega(s)} \|g(s + \tau)\|^2 ds.
\]

By (93) we have

\[
R_\delta(\tau, \omega) = \int_{-\infty}^0 \tilde{R}_\delta(\tau, \omega, s) ds = \int_{T_1 - 1}^{T_1} \tilde{R}_\delta(\tau, \omega, s) ds + \int_0^{T_1 - 1} \tilde{R}_\delta(\tau, \omega, s) ds.
\]

Next, we consider the limits of the last two terms in (108) as \( \delta \to 0 \). By (105)-(106) we have, for all \( 0 < |\delta| < \delta_2 \) and \( s \leq T_1 - 1 \),

\[
\tilde{R}_\delta(\tau, \omega, s) = \frac{4}{\rho} e^{1 + \frac{3}{2} \rho s} \|g(s + \tau)\|^2 ds.
\]

By (46) we see that the right-hand side of (109) is integrable over \( (-\infty, T_1 - 1) \), which together with (107), (109) and the Lebesgue dominated convergence theorem yields

\[
\lim_{\delta \to 0} \int_{T_1 - 1}^{T_1} \tilde{R}_\delta(\tau, \omega, s) = \frac{4}{\rho} \int_{-\infty}^{T_1 - 1} e^{2\rho s - 2\omega(s)} \|g(s + \tau)\|^2 ds.
\]

On the other hand, by Lemma 2.1 we find that \( 2 \int_0^\delta G_\delta(\theta, \omega) dr \) converges to \(-2\omega(s)\) uniformly on \([T_1 - 1, 0]\) as \( \delta \to 0 \), and hence one can check

\[
\lim_{\delta \to 0} \int_{T_1 - 1}^0 \tilde{R}_\delta(\tau, \omega, s) = \frac{4}{\rho} \int_{T_1 - 1}^0 e^{2\rho s - 2\omega(s)} \|g(s + \tau)\|^2 ds.
\]

Finally, by (108), (110) and (111), we have (94). This completes the proof. \( \square \)

Next, we consider the convergence of solutions of (78) as \( \delta \to 0 \).

**Lemma 3.8.** Suppose (46) hold. If \( u_{\delta} \) and \( u \) are the solutions of (78) and (64) with initial data \( u_{\delta, \tau} \) and \( u_{\tau} \), respectively, then for every \( \tau \in \mathbb{R}, \omega \in \Omega, T > 0 \) and \( \varepsilon \in (0, 1) \), there exists \( \delta_0 = \delta_0(\tau, \omega, T, \varepsilon) > 0 \) and \( c = c(\tau, \omega, T) > 0 \) such that for all \( 0 < |\delta| < \delta_0 \) and \( t \in [\tau, \tau + T] \),

\[
\|u_{\delta}(t, \tau, \omega, u_{\delta, \tau} - u(t, \tau, \omega, u_{\tau})\|^2
\]

\[
\leq c\|u_{\delta, \tau} - u_{\tau}\|^2 + c\varepsilon(1 + \|u_{\tau}\|^2 + \|u_{\delta, \tau}\|^2 + \int_{\tau}^{t} \|g(s)\|^2 ds).
\]

(112)
Proof. Let $\xi = v_\delta - v$. By (65) and (80) we have
\[
\frac{d\xi}{dt} - (1 + i\lambda)\Delta \xi + \rho \xi = e^{-\int_0^t g_{\delta}(\theta, \omega)\,dr} f(e_{0}^{t} g_{\delta}(\theta, \omega)\,dr \, v_{\delta}) - e^{-\omega(t)} f(e^{\omega(t)} v) + (e_{0}^{t} g_{\delta}(\theta, \omega)\,dr - e^{-\omega(t)}) g(t, x). \tag{113}
\]
From (113) we obtain
\[
\frac{1}{2} \frac{d}{dt} \|\xi\|^2 + \|\nabla \xi\|^2 + \rho \|\xi\|^2 = (e^{-\int_0^t g_{\delta}(\theta, \omega)\,dr} - e^{-\omega(t)}) Re \int_{\Omega} g(t, x) \xi \, dx + Re \int_{\Omega} (e^{-\int_0^t g_{\delta}(\theta, \omega)\,dr} f(e_{0}^{t} g_{\delta}(\theta, \omega)\,dr \, v_{\delta}) - e^{-\omega(t)} f(e^{\omega(t)} v)) \xi \, dx. \tag{114}
\]
For the last term on the right hand side of (114), we find
\[
Re \int_{\Omega} (e^{-\int_0^t g_{\delta}(\theta, \omega)\,dr} f(e_{0}^{t} g_{\delta}(\theta, \omega)\,dr \, v_{\delta}) - e^{-\omega(t)} f(e^{\omega(t)} v)) \xi \, dx
= Re \int_{\Omega} e^{-\int_0^t g_{\delta}(\theta, \omega)\,dr} f(e_{0}^{t} g_{\delta}(\theta, \omega)\,dr \, v_{\delta}) - f(e_{0}^{t} g_{\delta}(\theta, \omega)\,dr \, v)) \xi \, dx
+ Re \int_{\Omega} (e^{-\int_0^t g_{\delta}(\theta, \omega)\,dr} - e^{-\omega(t)}) f(e_{0}^{t} g_{\delta}(\theta, \omega)\,dr \, v)) \xi \, dx
+ Re \int_{\Omega} e^{-\omega(t)} (f(e_{0}^{t} g_{\delta}(\theta, \omega)\,dr \, v) - f(e^{\omega(t)} \xi)) \, dx. \tag{115}
\]
By (14)-(15) and Lemma 2.1, we see that for every $\varepsilon > 0$, there exists $\delta_1 = \delta_1(\varepsilon, \tau, \omega, T) > 0$ such that for all $0 < |\delta| < \delta_1$ and $t \in [\tau, \tau + T]$,
\[
|e^{-\int_0^t g_{\delta}(\theta, \omega)\,dr} - e^{-\omega(t)}| < \varepsilon. \tag{116}
\]
It yields from (115)-(116) that there exists $c_1 = c_1(\tau, \omega, T) > 0$ such that for all $0 < |\delta| < \delta_1$ and $t \in [\tau, \tau + T]$,
\[
Re \int_{\Omega} (e^{-\int_0^t g_{\delta}(\theta, \omega)\,dr} f(e_{0}^{t} g_{\delta}(\theta, \omega)\,dr \, v_{\delta}) - e^{-\omega(t)} f(e^{\omega(t)} v)) \xi \, dx
\leq c_1 |\xi|^2 + c_1 \varepsilon + c_1 \varepsilon \int_{\Omega} (|v|^4 + |v|^4) \, dx. \tag{117}
\]
By (116) we also get, for all $0 < |\delta| < \delta_1$ and $t \in [\tau, \tau + T]$,
\[
|e^{-\int_0^t g_{\delta}(\theta, \omega)\,dr} - e^{-\omega(t)} Re \int_{\Omega} g(t, x) \xi \, dx| \leq \frac{\varepsilon}{2} |\xi|^2 + \frac{\varepsilon}{2} |\xi|^2. \tag{118}
\]
By (114), (117) and (118) we have that for every $\varepsilon > 0$, there exists $c_2 = c_2(\tau, \omega, T) > 0$ such that for all $\varepsilon \in (0, 1), 0 < |\delta| < \delta_1$ and $t \in [\tau, \tau + T]$,
\[
\frac{d}{dt} |\xi|^2 \leq c_2 |\xi|^2 + c_2 \varepsilon (1 + \|v_{\delta}\|^2_{L^4} + \|v\|^2_{L^4} + \|g(t)^2\|^2). \tag{119}
\]
Solving (119) we see that for all $0 < |\delta| < \delta_1$ and $t \in [\tau, \tau + T]$,
\[
|\xi(t)|^2 \leq c_2^{2(t-\tau)} |\xi(\tau)|^2 + c_2 \varepsilon c_2^{2(t-\tau)} \int_{\tau}^{t} (1 + \|v_{\delta}(s, \tau, \omega, v_{\delta}, \tau)\|^2_{L^4} + \|v\|^2_{L^4} + \|g(s)^2\|^2) \, ds. \tag{120}
\]
By (120) and Lemmas 3.2 and 3.5 we see that there exist \( \delta_2 \in (0, \delta_1) \) and \( c_3 = c_3(\tau, \omega, T) > 0 \) such that for all \( 0 < \delta \leq \delta_1 \) and \( t \in [\tau, \tau + T] \),

\[
\|v_\delta(t, \tau, \omega, v_{\delta, \tau}) - v(t, \tau, \omega, v_\tau)\|^2 \\
\leq e^{c_2(\tau-t)}\|v_{\delta, \tau} - v_\tau\|^2 + c_3 e^{c_2(\tau-t)}(1 + \|v_\tau\|^2 + \|v_{\delta, \tau}\|^2 + \int_\tau^t \|g(s)\|^2 ds).
\]

(121)

Note that

\[
u_\delta(t, \tau, \omega, v_{\delta, \tau}) - u(t, \tau, \omega, u_\tau) = e^{\int_0^t \mathcal{G}_\delta(\theta, \omega)dr} v_\delta(t, \tau, \omega, v_{\delta, \tau}) - e^{\omega(t)}v(t, \tau, \omega, v_\tau)
\]

\[
= e^{\int_0^t \mathcal{G}_\delta(\theta, \omega)dr} (v_\delta(t, \tau, \omega, v_{\delta, \tau}) - v(t, \tau, \omega, v_\tau)) + (e^{\int_0^t \mathcal{G}_\delta(\theta, \omega)dr} - e^{\omega(t)})v(t, \tau, \omega, v_\tau),
\]

(122)

where \( v_{\delta, \tau} = e^{-\int_0^t \mathcal{G}_\delta(\theta, \omega)dr} v_\delta, \tau \) and \( v_\tau = e^{-\omega(t)}u_\tau \). From (14)-(15) and (122), we see that there exist \( \delta_3 \in (0, \delta_2) \) and \( c_4 = c_4(\tau, \omega, T) > 0 \) such that for all \( 0 < \delta \leq \delta_3 \) and \( t \in [\tau, \tau + T] \),

\[
\|u_\delta(t, \tau, \omega, v_{\delta, \tau}) - u(t, \tau, \omega, u_\tau)\|
\leq c_4 \|v_\delta(t, \tau, \omega, v_{\delta, \tau}) - v(t, \tau, \omega, v_\tau)\| + c_4 e^{\int_0^t \mathcal{G}_\delta(\theta, \omega)dr - \omega(t)} - 1\|v(t, \tau, \omega, v_\tau)\|
\]

which together with (116), (121) and Lemma 3.2 implies (122).

As a direct consequence of (78) as \( \delta \to 0 \).

**Corollary 3.** Suppose (46)-(47) hold and \( \delta_n \to 0 \). Let \( u_{\delta_n} \) and \( u \) be the solutions of (78) and (64) with initial data \( u_{\delta_n, \tau} \) and \( u_\tau \), respectively. If \( u_{\delta_n, \tau} \to u_\tau \) in \( L^2(\Omega) \) as \( n \to \infty \), then for every \( \tau \in \mathbb{R}, \omega \in \Omega \) and \( t > \tau \),

\[
u_{\delta_n}(t, \tau, \omega, u_{\delta_n, \tau}) \to u(t, \tau, \omega, u_\tau) \text{ in } L^2(\Omega) \text{ as } n \to \infty.
\]

Recall that for each \( \delta \neq 0 \), \( A_\delta \) is the unique \( \mathcal{D} \)-pullback attractor of \( \Phi_\delta \) in \( L^2(\Omega) \). To prove the upper semicontinuity of these attractors as \( \delta \to 0 \), we need the following compactness result.

**Lemma 3.9.** Suppose (46)-(47) hold. Let \( \tau \in \mathbb{R} \) and \( \omega \in \Omega \) be fixed. If \( \delta_n \to 0 \) and \( u_n \in A_{\delta_n}(\tau, \omega) \), then the sequence \( \{u_n\}_{n=1}^\infty \) has a convergent subsequence in \( L^2(\Omega) \).

**Proof.** Since \( u_n \in A_{\delta_n}(\tau, \omega) \) by assumption, it yields from the invariance of \( A_{\delta_n} \), that there exists \( \bar{u}_n \in A_{\delta_n}(\tau - 1, \theta - 1, \omega) \) such that

\[
u_n = \Phi_{\delta_n}(1, \tau - 1, \theta - 1, \omega, \bar{u}_n) = \delta_n(\tau, \tau - 1, \theta - 1, \omega, \bar{u}_n).
\]

(123)

From (94), there exists \( N_1 = N_1(\tau, \omega) \geq 1 \) such that for all \( n \geq N_1 \),

\[
R_{\delta_n}(\tau - 1, \theta - 1, \omega) \leq 1 + \frac{4}{\rho} \int_{-\infty}^0 e^{\frac{2}{\rho s - 2(\theta - 1, \omega)}(s)} \|g(s + \tau - 1\|^2 ds.
\]

(124)

Since \( \bar{u}_n \in A_{\delta_n}(\tau - 1, \theta - 1, \omega) \subseteq K_{\delta_n}(\tau - 1, \theta - 1, \omega) \), by (92) and (124) we have, for all \( n \geq N_1, \)

\[
\|\bar{u}_n\|^2 \leq 1 + \frac{4}{\rho} \int_{-\infty}^0 e^{\frac{2}{\rho s - 2(\theta - 1, \omega)}(s)} \|g(s + \tau - 1\|^2 ds.
\]

(125)

By (79) we get

\[
u_{\delta_n}(s, \tau - 1, \theta - 1, \omega, \bar{u}_n) = e^{-\int_0^s \mathcal{G}_{\delta_n}(\theta - 1, \omega)dr} u_{\delta_n}(s, \tau - 1, \theta - 1, \omega, \bar{u}_n),
\]

(126)
By (12) we find
\[ \lim_{n \to \infty} e^{-\int_0^r g_{\xi_n}(\theta_{r-s})} = e^{\omega(-\tau) - \omega(-1)}. \]
which together with (125)-(126) implies that the sequence \( \{\tilde{v}_n\}_{n=1}^\infty \) is bounded in \( L^2(O) \). Therefore, by Lemma 3.5 we see that
\[ v_{\delta_n}(\cdot, \tau-1, \theta_{-\tau} \omega, \tilde{v}_n) \text{ is bounded in } L^2((\tau-1, \tau), H^1_0(O)) \cap L^4((\tau-1, \tau), L^4(O)). \]
\[ (127) \]
By (80) and (127) we can prove
\[ \frac{d}{ds} v_{\delta_n}(\cdot, \tau-1, \theta_{-\tau} \omega, \tilde{v}_n) \text{ is bounded in } L^2((\tau-1, \tau), H^{-1} + L^4((\tau-1, \tau), L^4(O)). \]
\[ (128) \]
By (127)-(128), it yields from [34] that there exists \( \tilde{v} \in L^2((\tau-1, \tau), L^2(O)) \) such that, up to a subsequence,
\[ v_{\delta_n}(\cdot, \tau-1, \theta_{-\tau} \omega, \tilde{v}_n) \to \tilde{v} \text{ in } L^2((\tau-1, \tau), L^2(O)), \]
which indicates, up to a further subsequence,
\[ v_{\delta_n}(s, \tau-1, \theta_{-\tau} \omega, \tilde{v}_n) \to \tilde{v}(s) \text{ in } L^2(O) \text{ for almost all } s \in (\tau-1, \tau). \]
\[ (129) \]
By (12), (126) and (129) we have
\[ u_{\delta_n}(s, \tau-1, \theta_{-\tau} \omega, \tilde{u}_n) \to e(\theta_{-\tau} \omega(s)) \tilde{v}(s) \text{ in } L^2(O) \text{ for almost all } s \in (\tau-1, \tau). \]
\[ (130) \]
Since \( \delta_n \to 0 \), it infers from Corollary 3.9 and (130) that
\[ u_{\delta_n}(\tau, s, \theta_{-\tau} \omega, u_{\delta_n}(s, \tau-1, \theta_{-\tau} \omega, \tilde{u}_n)) \to u(\tau, s, \theta_{-\tau} \omega, e(\theta_{-\tau} \omega(s)) \tilde{v}(s)) \text{ in } L^2(O), \]
\[ (131) \]
where \( u \) is the solution of (64). Note that
\[ u_{\delta_n}(\tau, s, \theta_{-\tau} \omega, u_{\delta_n}(s, \tau-1, \theta_{-\tau} \omega, \tilde{u}_n)) = u_{\delta_n}(\tau, \tau-1, \theta_{-\tau} \omega, \tilde{u}_n). \]
Therefore, by (131) we obtain
\[ u_{\delta_n}(\tau, \tau-1, \theta_{-\tau} \omega, \tilde{u}_n) \to u(\tau, s, \theta_{-\tau} \omega, e(\theta_{-\tau} \omega(s)) \tilde{v}(s)) \text{ in } L^2(O), \]
which together with (123) completes the proof. \( \square \)

We finally give the upper semicontinuity of random attractors as \( \delta \to 0 \).

Theorem 3.10. Suppose (46)-(47) hold. Then for every \( \tau \in \mathbb{R} \) and \( \omega \in \Omega \),
\[ \lim_{\delta \to 0} \text{dist}_{L^2(O)}(A_\delta(\tau, \omega), A_0(\tau, \omega)) = 0. \]
\[ (132) \]
Proof. Given \( \tau \in \mathbb{R} \) and \( \omega \in \Omega \), put
\[ K_0(\tau, \omega) = \{ u \in L^2(O) : \| u \|^2 \leq R_0(\tau, \omega) \} \]
where \( R_0(\tau, \omega) \) is given by
\[ R_0(\tau, \omega) = \frac{4}{\rho} \int_{-\infty}^0 e^{\frac{1}{2}g(s+\tau)} \| g(s+\tau) \|^2 ds. \]
By (47) we find that the family \( K_0 = \{ K_0(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega \} \) belongs to \( \mathcal{D} \). Moreover, by (94) we get
\[ \lim_{\delta \to 0} \| K_\delta(\tau, \omega) \| = \| K_0(\tau, \omega) \| \text{ for all } \tau \in \mathbb{R} \text{ and } \omega \in \Omega. \]
\[ (133) \]
Thus by (133), Corollary 3 and Lemma 3.9, we have (132) from Theorem 3.1 in [54] immediately. \( \square \)
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