Ward Identities in Two-Dimensional String Theory.

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ABSTRACT

I study the Ward identities of the $w_{\infty}$ symmetry of the two-dimensional string theory. It is found that, not just an isolated vertex operator, but also a number of vertex operators colliding at a point can produce local charge non-conservation. The structure of all such contact terms is determined. As an application, I calculate all the non-vanishing bulk tachyon amplitudes directly through the Ward identities for a Virasoro subalgebra of the $w_{\infty}$. 

Recently, a considerable effort has been devoted to understanding the role of discrete states in two-dimensional string theory. In the context of the $c = 1$ matrix model they were first noticed in refs. [1, 2], while in the continuum approach they have been known for a long time [3]. These physical states, that are present only for quantized values of momentum, are the remnants of transverse string excitations [4]. They appear, somewhat surprisingly, in the two-dimensional string model and make it quite non-trivial. In a number of recent papers it was shown that the discrete states generate large symmetry groups, such as the area preserving diffeomorphisms. In the matrix model this was done in ref. [5-8], while in the continuum – in ref. [9, 10]. A better understanding of these symmetries may hold the key to a complete solution of the theory. In this paper I derive the symmetry Ward identities and show how they determine the non-vanishing “tachyon” correlation functions. I will restrict myself to the non-compact closed string case and to the “bulk” correlation functions that satisfy the energy sum rule [4, 11].

The symmetry currents were constructed by Witten [9] who first understood the important physical role of the BRST invariant discrete states of ghost number zero [12]. The currents assume the form [9]

$$W_{J,m} = \Psi_{J+1,m}(z)\bar{O}_{J,m}(\bar{z})$$
$$\bar{W}_{J,m} = \bar{\Psi}_{J+1,m}(\bar{z})O_{J,m}(z)$$

(1)

where $J = 0, \frac{1}{2}, 1, \ldots$, and $m = -J, -J + 1, \ldots, J - 1, J$. The fields $\Psi_{j,m}$ are the gravitationally dressed primary fields of the $c = 1$ matter system,

$$\Psi_{j,m}(z) = \psi_{j,m}(z)e^{(1-j)\phi(z)}$$

(2)

The crucial point in the construction of the currents is the existence of the BRST
invariant operators $O_{J,m}$ that form a ground ring [9] generated by

$$
O_{J,\frac{1}{2}} = (cb + \frac{i}{2} \partial X - \frac{1}{2} \partial \phi) e^{\frac{i}{2}(iX + \phi)} \\
O_{J,\frac{1}{2}} = (cb - \frac{i}{2} \partial X - \frac{1}{2} \partial \phi) e^{\frac{i}{2}(-iX + \phi)}
$$

These operators are necessary in eq. (1) to balance the right and left momenta, so that the currents are good quantum fields. Up to BRST commutators, the ring fusion rule is [9]

$$O_{J_1,m_1}O_{J_2,m_2} = O_{J_1+J_2,m_1+m_2}$$

(4)

From the currents $W$ we can form the charges

$$Q_{J,m} = \frac{1}{2\pi i} \oint dz W_{J,m}(z) .$$

(5)

Putting together eqns. (3), (4) and charges, we find the charge algebra

$$[Q_{J_1,m_1},Q_{J_2,m_2}] = 2\{(J_1 + 1)m_2 - (J_2 + 1)m_1\}Q_{J_1+J_2,m_1+m_2}$$

(6)

This is known as the wedge subalgebra of $w_{\infty}$ [13]. There are also charges $\bar{Q}_{J,m}$ constructed from the currents $\bar{W}_{J,m}$, which satisfy the same algebra as eq. (6). For the charges to be conserved we should have

$$\bar{\partial}W_{J,m} = \partial \bar{W}_{J,m} = 0$$

Obviously, $\partial \bar{W}_{J+1,m} = 0$ but $O_{J,m}$ seems to depend on $z$. However, we have

$$\partial O = [L_{-1},O] = \{Q_{BRST},b_{-1}\}, O] = \{Q_{BRST},[b_{-1},O]\} .$$

(7)

Thus, if $O$ acts on the vacuum state, then there is no dependence on $z$. Inside correlation functions the dependence of $O$ on $z$ comes only from boundary terms on the moduli space.
The symmetry Ward identities on correlation functions are encoded in
\[
\left\langle \int d^2 z \bar{\partial} W_{J,m} \prod_i O_i \right\rangle = \left\langle \int d^2 z \partial \bar{W}_{J,m} \prod_i O_i \right\rangle = 0 \tag{8}
\]
where \( O_i \) are the vertex operators, three of which are fixed and the rest integrated.

Due to the left-right symmetry on the world sheet, we only need to consider the Ward identities due to the currents \( W \). The currents \( \bar{W} \) give identical constraints.

We will concentrate on the correlation functions of the “tachyon” operators

\[
T_{k}^{\pm} = e^{ikX + (-1 \pm k)\phi} \tag{9}
\]
where the superscript labels the chirality.

Let us begin with the role of the charges \( Q_{m,m} \) and \( Q_{-m,m} \). These sets of charges are special because they form two separate Virasoro sub-algebras\(^*\)

\[
[Q_{n,n}, Q_{m,m}] = 2(m - n)Q_{m+n,n+m} , \quad m = 0, \frac{1}{2}, 1, \ldots \tag{10}
\]

\[
[Q_{-n,n}, Q_{-m,m}] = 2(m - n)Q_{m-n,n+m} , \quad -m = 0, \frac{1}{2}, 1, \ldots \tag{11}
\]

In fact, we have only the non-negative Virasoro generators in each case. Note that \( W_{m,m} \) carries momentum \( k = m \) and Liouville energy \( \epsilon = m \). Thus, the action of \( Q_{m,m} \) should shift the momentum and energy of a tachyon by the same amount \( m \), \( i.e. \) it should convert \( T_{k}^{+} \) into \( T_{k+m}^{+} \). After an explicit calculation we indeed find

\[
W_{m,m}(z) \, \bar{c}c T_{k}^{+}(0) = \frac{1}{z} F_{m}(k) \, \bar{c}c T_{k+m}^{+}(0) + \ldots \tag{12}
\]
where the first term on the right-hand side is the leading singularity. The deter-

\(^*\) This was pointed out to me by H. Verlinde.
mination of the function $F_m(k)$ proceeds in two steps. On the $\bar{z}$ side we obtain

$$\hat{O}_{1/2,1/2}^m \bar{c} e^{ikX+(-1+k)\phi} = (2k)(2k+1)\ldots(2k+2m-1)\bar{c} e^{i(k+m)X+(-1+k+m)\phi},$$

(13)

which is easily derived by a repeated application of

$$\hat{O}_{1/2,1/2} \bar{c} e^{ikX+(-1+k)\phi} = 2k\bar{c} e^{i(k+1/2)X+(-1+k+1/2)\phi}.$$  

(14)

On the $z$ side we find

$$\Psi_{m+1,m}(z) \bar{c} c e^{ikX+(-1+k)\phi}(0) = \frac{1}{z} (2k+2m)! (-1)^{2m \bar{c} e^{i(k+m)X+(-1+k+m)\phi}(0)}$$

(15)

which can be derived using

$$\psi_{m+1,m}(z) = -(2m+1)! H_-(z) e^{i(m+1)X(z)}$$

$$H_-(z) = \oint du \frac{2\pi i}{\bar{c} e^{-iX(u+z)}}$$

(16)

Putting together eqs. (13) and (15), we get

$$F_m(k) = (-1)^{2m}(2k)^2(2k+1)^2 \ldots (2k+2m-1)^2(2k+2m)$$

(17)

Eq. (12) can be simplified if we introduce specially normalized vertex operators

$$\hat{T}_k^+ = \frac{\Gamma(2k)}{\Gamma(1-2k)} T_k^+.$$ 

(18)

Then, from eqs. (12) and (17) we find simply

$$W_{m,m}(z) \bar{c} c \hat{T}_k^+(0) = \frac{1}{z} (2k+2m) \bar{c} c \hat{T}_{k+m}^+(0) + \ldots$$

(19)

which implies

$$[Q_{m,m}, \bar{c} c \hat{T}_k^+] = 2(k+m) \bar{c} c \hat{T}_{k+m}^+.$$ 

(20)

This result is consistent with the Jacobi identity

$$[Q_{m,m}, [Q_{n,n}, \bar{c} c \hat{T}_k^+]] - [Q_{n,n}, [Q_{m,m}, \bar{c} c \hat{T}_k^+]] = [[Q_{m,m}, Q_{n,n}], \bar{c} c \hat{T}_k^+] \cdot$$

(21)

Now let us insert $Q_{m,m}$ into the correlation functions of type $(N,1)$, i.e. with
$N$ tachyons of chirality $+$ and 1 tachyon of chirality $-$. Eq. (19) determines how
the charge conservation is violated by each of the $+$ vertex operators. To complete
the Ward identity, we need to consider the action of $Q_{m,m}$ on $T^-$ which, by the
sum rules, carries momentum $p = -\frac{1}{2}(N-1)$. This is one of the discrete momenta
where $T^-$ can be considered a special state, and we find

$$W_{m,m}(z) \ c\bar{c} T^+_p(0) \sim \frac{1}{z} c\bar{c} \Psi_{-p+m,p+m} \bar{X}_{-p+m,p+m},$$

(22)

where $\bar{X}$ appears to be a mixture of $\bar{c}\bar{\Psi}$ and another state that is not in the relative
cohomology [14]. The Ward identity for the current $W_{m,m}$, eq. (8), assumes the
form

$$A_{N,D}(k_1, \ldots, k_N) + (2k_1 + 2m)A_{N,1}(k_1 + m, k_2, \ldots, k_N) +
(2k_2 + 2m)A_{N,1}(k_1, k_2 + m, \ldots, k_N) + \ldots + (2k_N + 2m)A_{N,1}(k_1, k_2, \ldots, k_N + m) = 0$$

(23)

The first term is the correlation function of the special state $D = c\Psi \bar{X}$, found
in eq. (22), and $N$ tachyons of chirality $+$; all the remaining terms are tachyon
correlators of type $(N,1)$. Eq. (23) is reminiscent of the Virasoro constraints in
$c < 1$ matrix models [15].

There is an interesting subtlety in the derivation of eq. (23). In eq. (12) we
derived the violation of charge conservation near a fixed vertex operator supplied
with the factor $cc$. When we study a moving operator, then naively the factor
$2k + 2m$ is replaced by another function, which would lead to nonsensical results.
The resolution of this problem $^\star$ is that

$$\partial W_{J,m} = (\partial \Psi_{J+1,m}) \bar{O}_{J,m} + \Psi_{J+1,m} \partial \bar{O}_{J,m},$$

and that the second term in the equation cannot be neglected. In fact, by eq. (7),
its insertion reduces to boundary terms, and each moving vertex operator gives an

$^\star$ This puzzle was resolved by A. M. Polyakov.
extra contribution to charge non-conservation. In this fashion, as expected, the symmetry between the fixed and moving vertex operators is restored, and each one introduces the factor $2k + 2m$ into the Ward identity.

The Ward identities can be rephrased in a Fock space notation for tachyons. The state created by inserting a number of tachyon operators onto the Riemann surface will be denoted by $|k_1, k_2, \ldots, k_N; p_1, p_2, \ldots, p_M>$, where $k_i$ are the momenta of the $+$ tachyons, and $p_i$ are the momenta of the $-$ tachyons. The results above can be stated as

$$Q_{m,m}|k_1, k_2, \ldots, k_N> = (2k_1 + 2m)|k_1 + m, k_2, \ldots, k_N> +$$
$$+ (2k_2 + 2m)|k_1, k_2 + m, \ldots, k_N> + \ldots + (2k_N + 2m)|k_1, k_2, \ldots, k_N + m>.$$  

The charges $Q_{-m,m}$ act analogously on the $-$ tachyons. The correlation functions can be written as

$$<0|S|k_1, k_2, \ldots, k_N; p_1, p_2, \ldots, p_M>$$

where $S$ is the S-matrix. A compact statement of the Ward identities is to insert $Q_{J,m}$ into eq. (25), and use $[Q_{J,m}, S] = 0$ to show that the result is zero.

So far we have found that $Q_{m,m}$ acting on $+$ tachyons and $Q_{-m,m}$ acting on $-$ tachyons do not change the particle number, but simply shift their momenta sequentially by a quantized amount. However, as suggested by Witten [16], in this theory one generally expects the symmetry charges to alter the particle number. This expectation comes true in a very interesting way. It turns out that the charge $Q_{m+n,m}$ acts to reduce the number of $+$ tachyons by $n$,

$$Q_{m+n,m}|k_1, k_2, \ldots, k_{n+1} > = f_{m+n,m}(k_i)|k > ,$$

$$k = m + \sum_{i=1}^{n+1} k_i .$$

To show that this equation is allowed, let us count the Liouville energy of the state
on the right-hand side,
\[
\epsilon = m + n + \sum_{i=1}^{n+1}(k_i - 1) = -1 + m + \sum_{i=1}^{n+1}k_i = -1 + k.
\] (27)

Thus, \(k\) and \(\epsilon\) of the resulting state are appropriate for a single + tachyon. Of course, it remains to show that \(f_{m+n,m}(k_i) \neq 0\), but there is no general reason for it to vanish. We will now calculate it directly for the special case \(m = -\frac{1}{2}, n = 1\), using the explicit form
\[
W_{\frac{1}{2}, -\frac{1}{2}} = \left[(\partial X)^2 - i\partial^2 X][\bar{c}\bar{b} - \frac{i}{2}\bar{\partial}X - \frac{1}{2}\bar{\partial}\phi]\right]e^{\frac{1}{2}(-iX + \phi)}. \] (28)

We need to calculate the perturbed operator product
\[
W_{\frac{1}{2}, -\frac{1}{2}}(z) \ c\bar{c}T^{+}_{k_1}(0) \int d^2wT^{+}_{k_2}(w, \bar{w}) = \frac{1}{z}I(k_1, k_2) \ c\bar{c}T^{+}_{k_1+k_2-\frac{1}{2}}(0) + \ldots \] (29)

After all the contractions, we find an integral which can be evaluated using the results of ref. [17],
\[
I(k_1, k_2) = \int d^2u|u|^4(k_1+k_2-1)|1-u|^{-4k_2} \left(8k_1k_2 + 2k_1(2k_1 - 1)(1-u) + \frac{2k_2(2k_2 - 1)}{1-u}\right)
= 2\pi(2k_1 + 2k_2 - 1) \frac{\Gamma(1-2k_1) \Gamma(1-2k_2) \Gamma(-1+2k_1+2k_2)}{\Gamma(2k_1) \Gamma(2k_2) \Gamma(2-2k_1-2k_2)}. \] (30)

Changing normalization from \(T^+\) to \(\tilde{T}^+\) removes all the cumbersome factors, and we find
\[
f_{\frac{1}{2}, -\frac{1}{2}}(k_1, k_2) = I(k_1, k_2) \frac{\Gamma(2k_1)}{\Gamma(1-2k_1)} \frac{\Gamma(2k_2)}{\Gamma(1-2k_2)} \frac{\Gamma(2-2k_1-2k_2)}{\Gamma(-1+2k_1+2k_2)} = 2\pi(2k_1+2k_2-1). \]

To summarize, we have found that
\[
Q_{\frac{1}{2}, -\frac{1}{2}}|k_1, k_2 > = 2\pi(2k_1 + 2k_2 - 1)|k_1 + k_2 - \frac{1}{2} > . \] (31)

We could attempt a general calculation of \(f_{m+1,m}\) along the same lines, but it is
easier to use the algebra (6) to deduce

\[ Q_{m+1,m}|k_1, k_2 > = \frac{1}{4m+3} [Q_{\frac{1}{2}, \frac{1}{2}}, Q_{m+\frac{1}{2}, m+\frac{1}{2}}]|k_1, k_2 >= 4\pi(k_1+k_2+m)|k_1+k_2+m > . \]  

(32)

From this formula we can determine the action of all the charges \( Q_{m+n,m} \). For example, for \( n = 2 \) we may use

\[ [Q_{s+1,s}, Q_{t+1,t}]|k_1, k_2, k_3 >= 4(t-s)Q_{s+t+2,s+t}|k_1, k_2, k_3 > . \]  

(33)

Thus, we recursively derive

\[ Q_{m+n,m}|k_1, k_2, \ldots, k_{n+1} >= 2\pi^n(n+1)!(m + \sum_{i=1}^{n+1} k_i)|m + \sum_{i=1}^{n+1} k_i > \]  

(34)

This formula is a mnemonic for how the charge conservation is violated when all the \( n+1 \) vertex operators collide at a point. Thus, it is implicit that at most one of them is of the fixed type. If a state contains \( N > n+1 \) tachyons, then \( Q_{m+n,m} \) acts to convert it into a sum of \( (N-n) \)-particle states by turning every possible set of \( n+1 \) particles into one particle according to eq. (34). If there are \( N < n+1 \) tachyons, then \( Q_{m+n,m} \) appears to annihilate the state. We should keep in mind, however, that we are considering the renormalized tachyons \( \tilde{T}^+_k \) that, at discrete \( k \), are related to \( T^+_k \) by an infinite factor. If we do not renormalize the field, then the discrete momenta need to be treated specially. At these momenta there may be additional contributions to the Ward identity, proportional to the special states. Eq. (23) is an example of this.

Finally, we note that all the above formulae can be modified for the \(-\) tachyons by a simple parity flip. If we now introduce oscillators \( a(k) \) for the renormalized \(+\) tachyons, and \( b(p) \) for the renormalized \(-\) tachyons, then the charges can be
represented as

\[
Q_{J,m} = 2\pi^{J-m} \int \prod_{i=1}^{J-m+1} dk_i \sum_{k=1}^{J-m+1} k a_i^\dagger (k) \prod_{s=1}^{J-m+1} a(k_s) \delta \left( \sum_{l=1}^{J-m+1} k_l + m - k \right) \\
+ (-1)^{2m} 2\pi^{J+m} \int dp \prod_{i=1}^{J+m+1} dp_i p b_i^\dagger (p) \prod_{s=1}^{J+m+1} b(p_s) \delta \left( \sum_{l=1}^{J+m+1} p_l + m - p \right).
\]  

(35)

Now that we have determined how the charges act on the vertex operators, we can derive powerful constraints on the correlation functions. As an example, I will show that the correlators of type \((N,1)\) are completely determined by the Ward identities once we have set \(A_{2,1} = 0\) to fix the normalization of the string coupling constant. Note that the \(\text{--}\) tachyon is kept unrenormalized because its momentum is discrete. As a warm-up, let us calculate the 4-point function from the identity

\[
<0|S|Q_{\frac{3}{2},-\frac{1}{2}}|k_1, k_2, k_3; p> = 0
\]  

(36)

We chose \(Q_{\frac{3}{2},-\frac{1}{2}}\) because it can shift the momentum of the \(\text{--}\) tachyon, and it can also convert two + tachyons into a single one. This is precisely what we need to express the \((3,1)\) amplitude in terms of the \((2,1)\) amplitude.

First, using the sum rules, we find that \(k_1 + k_2 + k_3 = 1, p = -\frac{1}{2}\) in eq. (36). Then, writing out the action of the charge, we get

\[
-2 <0|S|k_1, k_2, k_3; -1> + 2\pi(2k_1 + 2k_2 - 1) <0|S|k_1 + k_2 - \frac{1}{2}, k_3; -\frac{1}{2}> \\
+ 2\pi(2k_1 + 2k_3 - 1) <0|S|k_1 + k_3 - \frac{1}{2}, k_2; -\frac{1}{2}> \\
+ 2\pi(2k_2 + 2k_3 - 1) <0|S|k_2 + k_3 - \frac{1}{2}, k_1; -\frac{1}{2}> = 0.
\]  

(37)

It follows that

\[
A_{3,1}(k_1, k_2, k_3) = \pi(4(k_1 + k_2 + k_3) - 3) = \pi.
\]

This agrees with the results of refs. [4, 11, 18] for the amplitude of the renormalized + tachyons.
This procedure can be iterated. Inserting $Q_{1, -\frac{1}{2}}$ into the $(4, 1)$ amplitude, we get an expression for $A_{4,1}$ in terms of $A_{3,1}$. Solving it, we find $A_{4,1} = \pi^2/2$. Repeating the steps, it is not hard to show recursively that

$$A_{N,1}(k_1, k_2, \ldots, k_N) = \frac{\pi^{N-2}}{(N-2)!},$$

in agreement with refs. [4, 11, 18]. Alternatively, $A_{N,1}$ can be found in a single step by using

$$<0|SQ_{-m,m}|k_1, k_2, \ldots, k_N; p> = 0$$

where $m = 1 - \frac{N}{2}$, $p = -\frac{1}{2}$. This charge can shift the momentum $p$ of the $-$ tachyon, and it can also reduce the number of $+$ tachyons from $N$ to 2. Writing out eq. (39), we find

$$[(N-2)!]^2(N-1)A_{N,1} = \pi^{N-2}(N-1)! \left( N(2-N) + 2(N-1) \sum_{i=1}^{N} k_i \right) = \pi^{N-2}(N-1)!$$

so that eq. (38) again follows. Thus, the $(N, 1)$ amplitudes are determined by the Ward identities for the Virasoro subalgebra (11). Similarly, the $(1, N)$ amplitudes are determined by the other Virasoro subalgebra (10).* Instead of analyzing the formidable multiple integrals, we have calculated the tachyon amplitudes in a few lines, relying essentially only on the algebraic structure of the theory. This encourages us to believe that all the correlation functions are determined by the symmetries of the theory. Further work is needed to investigate this question.

As I was finishing this work, I received a number of papers [19-21] addressing various closely related issues.

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* Of course, the two sets of amplitudes are related by parity inversion.
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