NECKLACE RINGS AND LOGARITHMIC FUNCTIONS

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Abstract. In this paper, we develop the theory of the necklace ring and the logarithmic function. Regarding the necklace ring, we introduce the necklace ring functor $N^r$ from the category of special $\lambda$-rings into the category of special $\lambda$-rings and then study the associated Adams operators. As far as the logarithmic function is concerned, we generalize the results in Bryant’s paper (J. Algebra. 253 (2002); no.1, 167-188) to the case of graded Lie (super)algebras with a group action by applying the Euler-Poincaré principle.

1. Introduction

Let $V$ be a finite-dimensional vector space over an arbitrary field. Many mathematicians have long intensively studied the free Lie algebra $\mathfrak{L}(V)$ generated by $V$ for its remarkable connections with combinatorics. In this paper, we study two algebraic objects which originate from the combinatorics of free Lie algebras. The necklace ring is related to the well-known fact that the dimension of the $n$-th homogeneous component in $\mathfrak{L}(V)$, can be computed by counting the number of primitive necklaces of length $n$ out of $\dim V$ letters. The second, the logarithmic function, is related to the “Lazard elimination theorem” ([5]).

The notion of the necklace ring was first introduced by Metropolis and Rota. The necklace ring has many interesting and significant algebraic features. For example, it is isomorphic to the universal ring of Witt vectors over a certain class of commutative rings ([21]). In particular, the necklace ring over $\mathbb{Z}$ was explicitly realized as the Burnside-Grothendieck ring, $\hat{\Omega}(C)$, of isomorphism classes of almost finite cyclic sets. Here, the notation $C$ represents the multiplicative infinite cyclic group. The isomorphism between the universal ring of Witt vectors, $\mathcal{W}(\mathbb{Z})$, and $\hat{\Omega}(C)$ has been called the Teichmüller map ([18]).

The classical construction of Witt vectors can be understood as a special case of a more general construction. More precisely, Dress and Siebenicher constructed a covariant functor, $\mathcal{W}_G$, such that $\mathcal{W}_C$ coincides with the classical Witt-ring functor, $\mathcal{W}$, where $C$ denotes a profinite group and $\hat{C}$ the profinite completion of $C$. In this case, the isomorphism between $\mathcal{W}_G(\mathbb{Z})$ and $\hat{\Omega}(G)$ has been called the extended Teichmüller map ([9]). Generalizing this, Graham constructed a functor, $F_G$, which shares many properties with $\mathcal{W}_G$, for every group $G$. Recently, Brun showed that for any finite group, $G$, the functor $\mathcal{W}_G$ coincides with the left adjoint of the algebraic functor from the category of $G$-Tambara functors to the category of commutative rings with an action of $G$ ([17],[12]).

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Motivated by the work of Dress and Siebeneicher, we introduced a new ring, \( \hat{\Omega}_R(G) \), which coincides with \( \hat{\Omega}(G) \), if \( G = \hat{\mathbb{C}} \) for a special \( \lambda \)-ring \( R \). Furthermore, we constructed a map,

\[ \tau_R : \mathbb{W}_G(R) \to \hat{\Omega}_R(G), \]

which is analogous to the extended Teichmüller map. This map, which is a ring isomorphism in cases where \( R \) is torsion-free, has been called the \( R \)-Teichmüller map \( (25) \).

The necklace ring of \( G \) over \( R \), denoted by \( \mathcal{N}_R(G) \), is the ring obtained from \( \hat{\Omega}_R(G) \) via the interpretation map (int for short) which is nothing but the map reading of the coefficients of the elements in \( \hat{\Omega}_R(G) \). Pictorially, this result can be illustrated in the following:

\[ \mathbb{W}(\mathbb{Z}) \xrightarrow{\tau(R)} \hat{\Omega}(\mathbb{C}) \quad \text{if } R \text{ is torsion-free} \]

\[ \mathbb{W}(R) \xrightarrow{\tau_R} \mathcal{N}_R(G) \xrightarrow{\text{int}} \hat{\Omega}_R(G) \]

Very recently, developing the above construction, we have constructed a functor, \( \Delta_G \), such that (i) \( \Delta \) is equivalent to \( \mathbb{W}_G \), and (ii) \( \Delta_G(\mathbb{Z}) = \hat{\Omega}_R(\mathbb{Z}) \) \( (25) \).

On this topic, we first study a general theory of the Witt-Burnside ring and the necklace ring of a profinite group. For example, we will show that the \( R \)-Teichmüller map, \( \tau_R : \mathbb{W}_G(R) \to \mathcal{N}_R(G) \), is a ring isomorphism for arbitrary special \( \lambda \)-rings \( R \). Next, we apply the above general theory to the special case where \( G = \hat{\mathbb{C}} \). From the fact that \( \tau_R \) is a ring homomorphism we obtain that

\[ \mathbb{W}(R) \cong \mathcal{N}_R(R) \cong \Lambda_1(R) \quad \text{(as rings)} \quad (1.1) \]

for any special \( \lambda \)-ring \( R \). Here, \( \Lambda_1(R) \) is the modified Grothendieck ring of formal power series with constant term 1.

Grothendieck showed in 1956 that \( \Lambda_1(A) \) (originally \( \Lambda(A) \)) has a special \( \lambda \)-ring structure for any unital commutative ring \( A \). In view of Eq. 1.1, this implies that if \( R \) is a special \( \lambda \)-ring, then it is possible to make \( \mathbb{W}(R) \) and \( \mathcal{N}_R(R) \) into special \( \lambda \)-rings by transporting the special \( \lambda \)-ring structure of \( \Lambda_1(R) \). Based on this observation, we will show that \( \mathbb{W} \) and \( \mathcal{N}_R \) can be viewed as functors from the category of special \( \lambda \)-rings into the category of special \( \lambda \)-rings. In this paper, we will focus particularly on their Adams operations rather than their \( \lambda \)-operations since the former often are much easier to handle than the latter. Indeed, the Adams operations coincide with the well-known Frobenius operators (Section 2 and Section 3).

Next, we study necklace polynomials. In particular, we investigate the class of rings over which the necklace polynomials

\[ M(x,n) := \frac{1}{n} \sum_{d|n} \mu(d) x^{\frac{n}{d}}, \quad n \in \mathbb{N}, \quad (1.2) \]

remain valid.

The second topic relates to the logarithmic function. The concept of the logarithmic function was first introduced by R. M. Bryant \( (7) \) to explain the properties of Lie module functions of free Lie algebras. Let \( G \) be a group, \( K \) a field, and \( V \) a \( KG \)-module. Letting \( \mathfrak{L}(V) \) be the free Lie algebra generated by \( V, G \) acts on \( \mathfrak{L}(V) \)
by Lie algebra automorphisms. In particular, \( \mathfrak{L}(V)_n \), the \( n \)-th homogeneous components in \( \mathfrak{L}(V) \), are themselves \( KG \)-modules. The \textit{Lie module function} of \( \mathfrak{L}(V) \), denoted by \( [\mathfrak{L}(V)] \), is a formal \( q \)-series,

\[
\sum_{n \geq 1} [\mathfrak{L}(V)_n] q^n,
\]

where the coefficients \([\mathfrak{L}(V)_n]\) are the elements of the Green algebra, \( \Gamma_K(G) \) of \( G \) over \( K \) (Section 5.2). Bryant proved that for all \( G \) and \( K \), there exists a unique logarithmic function, \( D \), on \( q\Gamma_K(G)[[q]] \), such that

\[
[\mathfrak{L}(V)] = D([V]) \quad (1.3)
\]

for every \( \mathbb{N} \)-graded \( KG \)-module \( V \). However, despite of the existence of such a logarithmic function, we have no explicit idea of how it looks in general. Consequently, he introduced the concept of the Grothendieck Lie module function of \( \mathfrak{L}(V) \), denoted by \( \mathcal{L}(V) \), instead of the Lie module function \([\mathfrak{L}(V)]\), when \( G \) is a finite group.

The \textit{Grothendieck Lie module function}, \( \mathcal{L}(V) \), is a formal \( q \)-series defined by

\[
\sum_{n \geq 1} \mathcal{L}(V)_n q^n,
\]

where the coefficients \( \mathcal{L}(V)_n \) are the elements of the Grothendieck algebra \( \Gamma_K(G) \) of \( G \) over \( K \) (Section 5.2). In this situation, he not only showed that for every finite group, \( G \), and every field, \( K \), there exists a unique logarithmic function, \( \mathcal{D} \), on \( q\Gamma_K(G)[[q]] \), such that

\[
\mathcal{L}(V) = \mathcal{D}(V) \quad (1.4)
\]

for all \( KG \)-modules \( V \), but he also provided its explicit form.

In this paper, we generalize Bryant’s results to the case of graded Lie superalgebras with a group action. Let \( \hat{\Gamma} \) be a free abelian group with finite rank, and let \( \Gamma \) be a countable (usually infinite) sub-semigroup in \( \hat{\Gamma} \), such that every element, \( \alpha \in \Gamma \), can be written as a sum of elements of \( \Gamma \) in only a finite number of ways. Consider \((\Gamma \times \mathbb{Z}_2)\)-graded Lie superalgebras, \( \mathfrak{L} = \bigoplus_{(\alpha,a) \in \Gamma \times \mathbb{Z}_2} \mathfrak{L}_{(\alpha,a)} \), with \( \dim \mathfrak{L}_{(\alpha,a)} < \infty \) for all \( (\alpha,a) \in (\Gamma \times \mathbb{Z}_2) \). Suppose \( G \) acts on \( \mathfrak{L} \), preserving the \((\Gamma \times \mathbb{Z}_2)\)-gradation. Our goal in this work is to find the condition on \( G \) and \( K \) for which there exists a unique logarithmic function on \( \Gamma_K(G)[[\Gamma \times \mathbb{Z}_2]] \) (resp., \( \Gamma_K(G)[[\Gamma \times \mathbb{Z}_2]] \)) satisfying an identity such as (1.3) (resp., (1.4)). We also seek to compute the explicit form of this logarithmic function.

This paper is organized as follows. In Section 2 we introduce the basic definitions and notation used in this paper. We also introduce a \( q \)-deformation of the Grothendieck ring of formal power series with constant term 1, which is isomorphic to the \( q \)-deformation of the universal ring of Witt vectors. In Section 3 we present the main results on the necklace polynomial and the necklace ring. In Section 4 we prove the main theorems of a logarithmic function (Theorems 4.7, 4.11, and 4.15) for graded Lie superalgebras. As corollaries of these theorems, we obtain closed formulas for the homogeneous components, \([\mathfrak{L}_{(\alpha,a)}]\) and \( [\mathfrak{L}_{(\alpha,a)}] \) (Corollaries 4.8 and 4.16). The final section is devoted to applications. In Section 5.1 we interpret the symmetric power map, \( \tilde{s}_k \), using plethysm and then make a few remarks on symmetric functions. In Section 5.2 we present several generating sets of supersymmetric functions. Section 5.3 contains recursive formulas for computing
 Frequently, recursive formulas are more efficient than closed formulas for this purpose, so, we will give recursive formulas for $|L_{(\alpha,a)}|$ and $|L_{(\alpha,a)}|$ (Proposition 5.7). Section 5.4 discusses a new interpretation of completely replicable functions from a viewpoint of logarithmic functions and $\lambda$-ring structures (Propositions 5.17 and 5.18).

2. PRELIMINARIES

2.1. $\lambda$-rings and Grothendieck ring of formal power series. A $\lambda$-ring $R$ is a unital commutative ring with operations $\lambda^n : R \to R$, $(n = 0, 1, 2, \cdots)$ such that

\begin{enumerate}
  \item $\lambda^0(x) = 1$,
  \item $\lambda^1(x) = x$,
  \item $\lambda^n(x + y) = \sum_{r=0}^{n} \lambda^r(x)\lambda^{n-r}(y)$.
\end{enumerate}

If $t$ is an indeterminate, we let

$$\lambda_t(x) := \sum_{n=0}^{\infty} \lambda^n(x)t^n, \quad x \in R.$$ 

By the third condition, it is straightforward that $\lambda_t(x+y) = \lambda_t(x)\lambda_t(y)$. An element $x \in R$ is said to be $n$-dimensional if $\lambda_t(x)$ is a polynomial of degree $n$ in $t$. For further information refer to [2, 13, 17].

Grothendieck, in 1956, introduced a functor, $\Lambda$, from the category of unital commutative rings to the category of special $\lambda$-rings. $\Lambda(A)$ is the ring whose underlying set is

$$1 + A[[t]]^+ := \{1 + \sum_{n=1}^{\infty} x_n t^n : x_n \in A\}.$$ 

Its ring structure is determined in the following way: Let $x_1, x_2, \cdots; y_1, y_2, \cdots$ be indeterminates. We define $s_i$ (resp. $\sigma_j$) to be the elementary symmetric functions in variables $x_1, x_2, \cdots$ (resp. $y_1, y_2, \cdots$), that is,

\begin{align*}
(1 + s_1 t + s_2 t^2 + \cdots) &= \prod_{i=1}^{\infty} (1 + x_i t), \\
(1 + \sigma_1 t + \sigma_2 t^2 + \cdots) &= \prod_{i=1}^{\infty} (1 + y_i t).
\end{align*}

Set $P_n(s_1, \cdots, s_n; \sigma_1, \cdots, \sigma_n)$ to be the coefficient of $t^n$ in

$$\prod_{i,j \geq 1} (1 + x_i y_j t),$$

and $P_{n,m}(s_1, \cdots, s_{mn})$ the coefficient of $t^n$ in

$$\prod_{i_1 < i_2 < \cdots < i_m} (1 + x_{i_1} \cdots x_{i_m} t).$$

With this notation, let us define the $\lambda$-ring structure on $\Lambda(A)$ by

\begin{enumerate}
  \item $\oplus :$ Addition is just the multiplication of power series.
  \item $\star :$ Multiplication is given by
    \begin{equation*}
    (1 + \sum a_n t^n) \star (1 + \sum b_n t^n) = 1 + \sum P_n(a_1, \cdots, a_n; b_1, \cdots, b_n)t^n.
    \end{equation*}
\end{enumerate}
(c) $\Lambda^n(1 + \sum a_t t^n) = 1 + \sum P_{n,m}(a_1, \cdots, a_{nm}) t^n$.

The operations of $\Lambda(A)$ can be understood better in terms of symmetric functions. Let $X = \{x_i : i \geq 1\}$ and $Y = \{y_j : j \geq 1\}$ be the infinite sets of commuting indeterminates $x_i$’s and $y_j$’s respectively. We call these sets alphabets. Now, let us introduce the following operations on alphabets:

$$X + Y = \{x_i, y_i : i \geq 1\},$$
$$X \cdot Y = \{x_i y_j : i, j \geq 1\},$$
$$\Lambda^m(X) = \{x_{i_1} \cdots x_{i_m} : i_1 < i_2 < \cdots < i_m\}.$$

Exploiting the notation

$$E(X, t) = \prod_{i=1}^\infty (1 + x_i t),$$

then the above conditions (a) through (c) can be regarded as expressing the identities

$$E(X, t) \oplus E(Y, t) = E(X + Y, t),$$
$$E(X, t) \star E(Y, t) = E(X \cdot Y, t),$$
$$\Lambda^m(E(X, t)) = E(\Lambda^m(X), t).$$

Similarly, we can introduce another ring structures on $1 + A[[t]]^+$ via the bijective maps

$$\theta_0 : \Lambda(A) \rightarrow 1 + A[[t]]^+, \quad f(t) \mapsto f(-t),$$
$$\theta_1 : \Lambda(A) \rightarrow 1 + A[[t]]^+, \quad f(t) \mapsto \frac{1}{f(-t)}.$$

Denote by $\Lambda_i(A)$ ($i = 0, 1$) the ring whose ring structure is induced from $\theta_i$, ($i = 0, 1$). In this paper, we mainly deal with the ring $\Lambda_1(A)$ rather than $\Lambda(A)$ and $\Lambda_0(A)$. Let $\oplus$ (resp., $\star$) denote the addition (resp., the multiplication) of $\Lambda_1(A)$. Define $\bar{s}_i$ (resp., $\bar{\sigma}_j$) to be the symmetric functions in variables $x_1, x_2, \cdots$ (resp., $y_1, y_2, \cdots$) determined by the equations

$$(1 + \bar{s}_1 t + \bar{s}_2 t^2 + \cdots) = \prod_{i=1}^\infty \frac{1}{1 - x_i t},$$
$$(1 + \bar{\sigma}_1 t + \bar{\sigma}_2 t^2 + \cdots) = \prod_{i=1}^\infty \frac{1}{1 - y_i t}.$$

Set $\bar{P}_n(\bar{s}_1, \cdots, \bar{s}_n; \bar{\sigma}_1, \cdots, \bar{\sigma}_n)$ to be the coefficient of $t^n$ in

$$\prod_{i,j} \frac{1}{1 - x_i y_j t},$$

and $\bar{P}_{n,m}(\bar{s}_1, \cdots, \bar{s}_{mn})$ the coefficient of $t^n$ in

$$\prod_{i_1 < i_2 < \cdots < i_m} \frac{1}{1 - x_{i_1} \cdots x_{i_m} t}.$$
Indeed, the $\lambda$-ring structure on $\Lambda_1(A)$ is given by

(a') $\oplus$ : Addition is just multiplication of power series.

(b') $\star_1$ : Multiplication is given by

\[(1 + \sum a_n t^n) \star_1 (1 + \sum b_n t^n) = 1 + \sum P_n(a_1, \cdots, a_n; b_1, \cdots, b_n) t^n.\]

(c') $\tilde{\Lambda}^m(1 + \sum a_n t^n) = 1 + \sum \tilde{P}_{n,m}(a_1, \cdots, a_{nm}) t^n.$

As in the previous paragraph, if we let

\[H(X, t) = \prod_{i=1}^{\infty} \frac{1}{1 - x_i t},\]

then the above conditions (a') through (c') can be regarded as expressing the identities

\[H(X, t) \oplus H(Y, t) = H(X + Y, t),\]
\[H(X, t) \star_1 H(Y, t) = H(X \cdot Y, t),\]
\[\tilde{\Lambda}^m(H(X, t)) = H(\Lambda^m(X), t).\]

**Definition 2.1.** A $\lambda$-ring $R$ is said to be special if $\lambda_1 : R \to \Lambda(R)$ is a $\lambda$-homomorphism, that is, a ring homomorphism commuting with the $\lambda$-operations.

2.2. Adams operations and binomial rings. Let $R$ be a $\lambda$-ring. We define the $n$-th Adams operation, $\Psi^n : R \to R$, by

\[\frac{d}{dt} \log \lambda_t(x) =: \sum_{n=0}^{\infty} (-1)^n \Psi^{n+1}(x) t^n \quad (2.1)\]

for all $x \in R$. If $R$ is a special $\lambda$-ring, it is well-known that $\Psi^n$ are $\lambda$-ring homomorphisms and $\Psi^n \circ \Psi^m = \Psi^{mn}$ for all $m, n \geq 1$. On the other hand, if we define the $n$-th symmetric power operations by

\[\mathcal{S}_t(x) = \sum_{n=0}^{\infty} S^n(x) t^n := \frac{1}{\lambda_{-1}(x)}, \quad (2.2)\]

then Eq. (2.1) can be rewritten as

\[\frac{d}{dt} \log \mathcal{S}_t(x) = \sum_{n=0}^{\infty} \Psi^{n+1}(x) t^n. \quad (2.3)\]

With this notation, it is easy to show that $R$ is special if and only if $\mathcal{S}_t : R \to \Lambda_1(R)$ is a $\lambda$-homomorphism.

Frequently, Adams operations completely determine the structure of the associated $\lambda$-ring. A ring $R$ is said to be a $\Psi$-ring if it is a unital commutative ring with a set of operations $\Psi^n : R \to R$ for all $n \geq 1$, satisfying

\[\Psi^1(a) = a,\]
\[\Psi^n(a + b) = \Psi^n(a) + \Psi^n(b)\]

for all $a, b \in R$. A special $\Psi$-ring is defined to be a $\Psi$-ring satisfying

\[\Psi^n(1) = 1,\]
\[\Psi^n(ab) = \Psi^n(a) \Psi^n(b),\]
\[\Psi^n(\Psi^m(a)) = \Psi^{nm}(a)\]
for all $m, n \geq 1$ and all $a, b \in R$. Note that if $R$ is a unital commutative ring, then it becomes a special $\Psi$-ring by setting $\Psi^n = id$ for all $n \geq 1$.

**Theorem 2.2.** (33) Let $R$ be a special $\Psi$-ring which has no $\mathbb{Z}$-torsion and such that $\Psi^n(a) = a^n \mod pR$ if $p$ is a prime. Then, there is a unique special $\lambda$-ring structure on $R$ such that the $\Psi^n$ are the associated Adams operations.

As an easy application of Theorem 2.2, we have

**Corollary 2.3.** Let $R$ be a commutative ring with unity which has no $\mathbb{Z}$-torsion and such that $a^n = a \mod pR$ if $p$ is a prime. Then, $R$ has a unique special $\lambda$-ring structure with $\Psi^n = id$ for all $n \geq 1$.

**Proof.** For all $n \in \mathbb{N}$ let $\Psi^n$ be the identity map on $R$. Then, it is clear that $R$ is a special $\Psi$-ring with regard to the operators $\{\Psi^n : n \in \mathbb{N}\}$. Now, Theorem 2.2 implies our assertion. \(\square\)

**Example 2.4.** Let $R$ be a $\mathbb{Q}$-algebra, $\mathbb{Z}$, or $\mathbb{Z}(r)$ the ring of integers localized at $r$. Then one can easily verify that it satisfies the condition of Corollary 2.3.

**Definition 2.5.** A special $\lambda$-ring in which $\Psi^n = id$ for all $n \geq 1$ will be called a binomial ring.

### 2.3. $q$-deformation of Grothendieck ring of formal power series.

It is quite interesting to note that the universal ring of Witt vectors has a $q$-analogue for every integer $q$. Recall that the universal ring $\mathbb{W}(A)$ of Witt vectors over $A$ is isomorphic to $\Lambda_1(A)$. Thus, it would be natural to think over the $q$-analogue of $\Lambda_1(A)$ which corresponds to the $q$-deformation of $\mathbb{W}(A)$ (refer to [25]). In this section, we deal with the question very briefly.

We start with remarking that notations associated with formal group laws used in this paper can be found in [18]. In the theory of formal group laws, it is well-known that $\Lambda_1(A)$ is isomorphic (as abelian groups) to the group of curves,

$$C(F_1, A) = \{ \sum_{n \geq 1} a_n t^n : a_n \in A \},$$

in the multiplicative formal group law $F_1(X, Y) = X + Y - XY$ via the isomorphism

$$\beta : \gamma(t) \mapsto \frac{1}{1 - \gamma(t)}.$$

More generally, we can verify that the group of curves $C(F_q, A)$ in the formal group law $F_q(X, Y) = X + Y - qXY$, ($q \in \mathbb{Z} \setminus \{0\}$), is isomorphic (as abelian groups) to $\Lambda_1(A)$ via the isomorphism

$$\beta^q : \gamma(t) \mapsto \frac{1}{1 - q\gamma(t)}$$

if $A$ is a ring in which $q$ is invertible. On the other hand, $C(F_q, A)$ is isomorphic to $\mathbb{W}^{F_q}(A)$, the universal ring of Witt vectors over $A$ associated with $F_q$, via the Artin-Hasse type exponential map

$$H^{F_q} : \mathbb{W}^{F_q} \rightarrow C(F_q, A), \quad \alpha \mapsto \sum_{n \geq 1} F_q^{n} \alpha_n t^n.$$
Here, it should be remarked that if we endow $C(F_q, A)$ with the ring structure via $H^{F_q}$ then $\beta^q$ is no longer a ring homomorphism. In order to make it into a ring homomorphism, we need to modify the multiplication of $\Lambda_1(A)$ so as to satisfy
\[
\prod_{i=1}^{\infty} \left( \frac{1}{1 - x_i t} \right)^q \ast_q \prod_{j=1}^{\infty} \left( \frac{1}{1 - y_j t} \right)^q = \prod_{i,j} \left( \frac{1}{1 - x_i y_j t} \right)^q.
\] (2.4)

Here, the notation $\ast_q$ represents the modified multiplication. This can be verified by observing that the map, $E^{F_q} : gh(A) \to C(F_q, A)$, is given by
\[
(\beta^q)^{-1} \circ \exp \circ \iota_q
\]
since
\[
\log_q(X) = \sum_{n \geq 1} \frac{q^{n-1}}{n} X^n = \frac{1}{q} \log \left( \frac{1}{1 - qX} \right).
\]
Here, the map $\iota_q : gh(A) \to A[[t]]$ is defined by
\[(a_n)_n \mapsto \sum_{n \geq 1} \frac{1}{n} a_nt^n.
\]
Combining the relation
\[
(\beta^q \circ E^{F_q})^{-1} \left( \prod_{i=1}^{\infty} \left( \frac{1}{1 - x_i t} \right)^q \right) = (p_n(X))_n
\]
with the well-known identity
\[
(p_n(X) \cdot p_n(Y) = p_n(X \cdot Y)
\]
yields the identity (2.4). The notation $p_n(X)$ represents the $n$-th power sum of the alphabet $X$. From this it follows that
\[
\prod_{i=1}^{\infty} \left( \frac{1}{1 - x_i t} \right)^q \ast_q \prod_{i=1}^{\infty} \left( \frac{1}{1 - y_i t} \right)^q = (\beta^q \circ E^{G_m})(p_n(X \cdot Y)_{n \geq 1}).
\]

Denote by $\Lambda_q(A)$ the ring induced from Eq. (2.4). In conclusion, if $A$ is a commutative ring in which $q$ is invertible, then
\[
\mathcal{W}^{F_q}(A) \cong C(F_q, A) \cong \Lambda_q(A) \text{ (as rings)}.
\]

2.4. Green algebras and Grothendieck algebras. We end this section by recalling some terminologies which will be exploited in the part of logarithmic functions.

Let $G$ be a group and $K$ be a field. We consider a set $\{I_\lambda : \lambda \in \Lambda\}$ consisting of representatives from each isomorphism class of finite dimensional indecomposable $KG$-modules. If $V$ is any finite dimensional $KG$-module, then we write $[V]$ for the element of $\sum_{\alpha_\lambda} I_\lambda$, where $\alpha_\lambda$ is the number of summands isomorphic to $I_\lambda$ in an unrefinable direct sum decomposition of $V$. The free abelian group generated by $\{I_\lambda : \lambda \in \Lambda\}$, denoted by $R_K(G)$, is called the Green ring (or representation ring) of $G$ over $K$, where multiplication on $R_K(G)$ is defined by tensor products. Note that for all finite dimensional $KG$-modules $U$ and $V$, $[U] + [V] = [U \oplus V]$ and $[U][V] = [U \otimes V]$. The scalar extension $C \otimes_K R_K(G)$, which becomes a commutative $C$-algebra, is called the Green algebra of $G$ over $K$ and denoted by $\Gamma_K(G)$. In particular, when $K = \mathbb{C}$, we use the notations $R(G)$, $\Gamma(G)$ instead of $R_K(G)$, $\Gamma_K(G)$. 

From now on, let $G$ be a finite group and $K$ be an arbitrary field. Let $I$ be the subspace of $\Gamma_K(G)$ spanned by all the elements of the form $[V] - [U] - [W]$, where $U,V, W$ are finite dimensional $KG$-modules occurring in a short exact sequence $0 \rightarrow U \rightarrow V \rightarrow W \rightarrow 0$. Note that $I$ is an ideal. We consider $\Gamma_K(G)/I$, the quotient of $\Gamma_K(G)$ by $I$, which is called the Grothendieck algebra. Similarly we define the Grothendieck group $\Gamma_K(G)$. Then, we have $\Gamma_K(G) = \mathbb{C} \otimes \Gamma_K(G)$. If $V$ is any finite dimensional $KG$-module, then we write $V$ for the element of $\Gamma_K(G)$ and $\Gamma_K(G)$.

3. NECKLACE RINGS

3.1. The Witt-Burnside ring and the necklace ring of a profinite group.
In this section, we start with recalling a functor denoted by $N_{rG}$. This functor was first introduced as a generalization of the necklace ring functor of Metropolis and Rota (see [21, 23]). In the next section, we focus on the case where $G$ is $\hat{C}$, the profinite completion of the multiplicative infinite cyclic group $C$.

Let $G$ be a profinite group. For two subgroups $U, V$ of $G$, we say that $U$ is subconjugate to $V$ if $U$ is a subgroup of some conjugates of $V$. This is a partial order on the set of the conjugacy classes of open subgroups of $G$, and will be denoted by $[V] \preceq [U]$. Consider (and fix) an enumeration of this poset satisfying the condition

$$[V] \preceq [U] \Rightarrow [V] \text{ precedes } [U].$$

By abuse of notation denote this poset by $O(G)$.

Let $R$ be a special $\lambda$-ring. With the above notation, let us define the necklace ring of $G$ over $R$, denoted by $N_{rG}(R)$, by the ring whose underlying set is

$$\prod_{[U] \in O(G)} R.$$ 

Its addition is defined componentwise. On the contrary, its multiplication is somewhat complicated. For $x = (x_{[V]})_{[V]}$ and $y = (y_{[W]})_{[W]}$ define

$$(x \cdot y)_{[U]} := \sum_{[V], [W]} \sum_{g \in G} \Psi_{Z(g, V, W)}(x_{[V]})\Psi_{Z(g, V, W)}(y_{[W]}).$$

Here the notation $Z(g, V, W)$ represents $V \cap gWg^{-1}$ and the notation $(V : U)$ means the index of $U$ in $gVg^{-1}$.

For an open subgroup $U$ of $G$ let us introduce the induction map,

$$\text{Ind}^{G}_{U}(R) : N_{rU}(R) \rightarrow N_{rG}(R), \quad (x_{[V]})_{[V] \in O(U)} \mapsto (y_{[W]})_{[W] \in O(G)}, \quad (3.1)$$

where

$$y_{[W]} = \sum_{[V] = [W] \text{ in } O(G)} x_{[V]}.$$ 

Also, we introduce the exponential map

$$\tau^{G}_{R} : R \rightarrow N_{rG}(R), \quad r \mapsto \left( \sum_{[V] \in O(G)} M_{G}(r, [V]) \right)_{[V]}.$$ 

Here the coefficients $M_{G}(r, V)$ are determined in the following manner. First, we write $r$ as a sum of one-dimensional elements, say, $r_{1} + r_{2} + \cdots + r_{m}$. Now, consider
\(\mathcal{C}(G, r)\) the set of continuous maps from \(G\) to the topological space
\[
\tau := \{ r_1, r_2, \cdots, r_m \}
\]
with regard to the discrete topology with trivial \(G\)-action. It is well-known that
\(\mathcal{C}(G, r)\) becomes a \(G\)-space with regard to the compact-open topology via the
following standard \(G\)-action
\[
(g \cdot f)(x) = f(g^{-1} \cdot x)
\]
(see [8]). Write \(\mathcal{C}(G, r)\) as the disjoint union of \(G\)-orbits, say,
\[
\bigcup_h G \cdot h,
\]
where \(h\) runs through a system of representatives of this decomposition. After
writing \(G/G_h = \bigcup_{1 \leq i \leq (G:G_h)} G_h \cdot h\), where \(G_h\) represents the isotropy subgroup of \(h\),
we let
\[
[h] := \prod_{i=1}^{(G:G_h)} h(w_i).
\]
Clearly, this is well-defined since \(h\) is \(G_h\)-invariant. With this notation, we define
\(M_G(r, V)\) by
\[
\sum_h [h],
\]
where \(h\) is taken over the representatives such that \(Gh\) is isomorphic to \(G/V\).

Recently we have constructed a map, called \(R\)-Teichmüller map,
\[
\tau_R : \mathbb{W}_G(R) \rightarrow Nr_G(R), \quad \alpha \mapsto \sum_{[U] \in \mathcal{O}(G)} \text{Ind}_U^G(\tau_R^U(\alpha([U]))),
\]
and then have shown that this map is bijective, and a ring homomorphism if \(R\) is
torsion-free (see [23]).

**Remark 3.1.** Note that we have identified \(Nr_G(R)\) with \(\hat{\Omega}_G(R)\) via the interpre-
tation map (for short, int) which is given by
\[
\text{interpretation} : \hat{\Omega}_R(G) \rightarrow Nr_G(R), \quad \sum_{[U]} a_{[U]}[G/U] \mapsto (a_{[U]})_{[U]}.
\]

The following lemma is immediate by the definition of the isomorphism \(\tau_R\).

**Lemma 3.2.** Let \([W] \in \mathcal{O}(G)\). Then, the \([W]\)-th component of \(\tau_R(\alpha), \alpha \in \mathbb{W}_G(R)\),
is given by
\[
\sum_{[U] \in \mathcal{O}(G)} \sum_{[V] \in \mathcal{O}(U)} M_U(\alpha([U]), V),
\]
where \([V]\) ranges over the elements \([Z] \in \mathcal{O}(U)\) such that \([Z] = [W]\) in \(\mathcal{O}(G)\).

**Theorem 3.3.** \(\tau_R\) is a ring isomorphism.

**Proof.** It was shown in [23] that \(\tau_R\) is bijective for every special \(\lambda\)-ring. So, for
our purpose we have only to show that it is a ring homomorphism. Given a unital
commutative ring \(R\), Dress and Siebeneicher [21] showed that for \(\alpha, \beta \in \mathbb{W}_G(R)\)
\[
\alpha + \beta = (s_U(\alpha_{[V]}, \beta_{[V]} \mid [G] \preceq [V] \preceq [U]))_{[U] \in \mathcal{O}(G)}
\]
and
\[
\alpha \cdot \beta = (p_U(\alpha_{[V]}, \beta_{[V]} \mid [G] \preceq [V] \preceq [U]))_{[U] \in \mathcal{O}(G)}
\]
for some integral polynomials $s_U$ and $p_U$ for every $[U] \in \mathcal{O}(G)$. Observe that for every $[Z] \in \mathcal{O}(G)$, the $[Z]$-th component of $\tau_R(x)$, $x = (x_{[U]})_{[U] \in \mathcal{O}(G)}$, is an integral polynomial in $\lambda^k(x_{[U]})$'s for $1 \leq k \leq (U : Z)$, $[G] \leq [U] \leq [Z]$.

This follows from Lemma 3.2. More precisely, this is because a necklace polynomial $M_U(\alpha([U]), Z)$ becomes a symmetric polynomial in one-dimensional elements after writing $\alpha([U])$ as a sum of one-dimensional elements. Applying the elementary theory of symmetric functions, one can show that

$$M_U(\alpha([U]), Z)$$

can be expressed as an integral polynomial in $\lambda^k(\alpha([U]))$'s, $1 \leq k \leq (U : Z)$.

Combining these two observations, we can conclude that the identities

$$\tau_R(\alpha + \beta) = \tau_R(\alpha) + \tau_R(\beta),$$

$$\tau_R(\alpha \cdot \beta) = \tau_R(\alpha) \cdot \tau_R(\beta)$$

(3.2)

hold universally. In other words, two equations in (3.2) are always true for arbitrary special $\lambda$-rings. This completes the proof. \qed

As a byproduct of Theorem 3.3, we obtain an trivial, but very significant corollary. Suppose that $R$ has two kinds of special $\lambda$-ring structures. Then, each structure produces different necklace rings, say, $N_{rG,1}$ and $N_{rG,2}$.

**Corollary 3.4.** $N_{rG,1}(R)$ is canonically isomorphic to $N_{rG,2}(R)$.

**Proof.** Consider the map

$$N_{rG}(id_{12}) := \tau_{R,2} \circ \tau_{R,1}^{-1} : N_{rG,1}(R) \to N_{rG,2}(R).$$

Clearly $N_{rG}(id_{12})$ is a ring-isomorphism by Theorem 3.3. \qed

From now on, we investigate inductions and restrictions on $\hat{\Omega}_R(G)$, $\hat{\mathcal{W}}_G(R)$, and $R^{\mathcal{O}(G)}$ which have played a crucial role in in the theory of necklace rings and Witt-Burnside rings. In the next section, we show that if $G = \hat{C}$ then restriction maps on these rings coincide with Adams operations associated with the special $\lambda$-structure induced from that of $\Lambda_1(R)$.

First, let us recall inductions and restrictions on $\hat{\Omega}_R(G)$, equivalently on $N_{rG}(R)$ (see 23, 24). For an open subgroup $U$ of $G$, the induction $\text{Ind}_{U}^{G}$, as defined in Eq. 3.1, is an additive homomorphism from $\hat{\Omega}_R(U)$ to $\hat{\Omega}_R(G)$ given by

$$\text{Ind}_{U}^{G} \left( \sum_{[W] \in \mathcal{O}(U)} b_{[W]}[U/W] \right) = \sum_{[V] \in \mathcal{O}(G)} \left( \sum_{[W] \in \mathcal{O}(U)} b_{[W]} \right) [G/V].$$

While, the restriction

$$\text{Res}_{U}^{G} : \hat{\Omega}_R(G) \to \hat{\Omega}_R(U)$$

is defined by the rule

$$\sum_{[V]} b_{[V]}[G/V] \mapsto \sum_{[V]} \sum_{g} \Psi(V; Z_{\hat{g}(U, V)}) = \sum_{[W] \in \mathcal{O}(U)} \left( \sum_{[V] \in \mathcal{O}(G) \mid Z_{\hat{g}(U, V)} = [W] \in \mathcal{O}(U)} b_{[V]} \right) [U/W].$$
Here, \( g \) ranges over a set of representatives of \( U \)-orbits of \( G/V \). One of many significant properties of restrictions is that they are indeed ring homomorphisms. Also, it is worthwhile noting that for open subgroups \( U \leq V \leq G \),

\[
\text{Ind}_V^G \circ \text{Ind}_U^V = \text{Ind}_U^G,
\]

\[
\text{Res}_U^V \circ \text{Res}_V^G = \text{Res}_U^G.
\]  

(3.3)

In [23, Lemma 3.13] it has been shown that the diagram

\[
\begin{array}{ccc}
W_G(R) & \xrightarrow{\tau_R} & \hat{\Omega}_G(R) \\
\Phi \downarrow & & \downarrow \hat{\psi} \\
\hat{\rho} \downarrow & & \downarrow \rho^\lambda(G)
\end{array}
\]

is commutative. Here,

\[
\Phi(\alpha) = \left( \sum_{[G] \leq [V] \leq [U]} \varphi_U(G/V) \cdot \alpha([V])^{(V:U)} \right)_{[U]}
\]

and

\[
\hat{\varphi}_U \left( \sum_{[V] \in \Omega(G)} b_{[V]}[G/V] \right) = \left( \sum_{[G] \leq [V] \leq [U]} \varphi_U(G/V)\Psi^{(V:U)}(b_{[V]}) \right)_{[U]}
\]

The notation \( \varphi_U(X) \) means the cardinality of the set \( X^U \) of \( U \)-invariant elements of \( X \) and let \( G/U \) denote the \( G \)-space of left cosets of \( U \) in \( G \). Now, by the transport of inductions and restrictions on \( \hat{\Omega}_G(R) \) via the map \( \tau_R \), we will define the operators \( \bar{v}_U, \bar{f}_U \) on \( \mathcal{W}_G(R) \). Indeed, it was already shown in [3] that for an open subgroup \( U \) of \( G \), one has well-defined natural transformations \( v_U : \mathcal{W}_U(-) \to \mathcal{W}_G(-) \) and \( f_U : \mathcal{W}_G(-) \to \mathcal{W}_U(-) \) satisfying all relations which are known to hold generally for the restriction \( \text{res}_U^G : \hat{\Omega}(G) \to \hat{\Omega}(U) \) and the induction \( \text{ind}_U^G : \hat{\Omega}(U) \to \hat{\Omega}(G) \).

**Theorem 3.5.** Let \( U \) be an open subgroup of \( G \). Regard \( \mathcal{W}_G(-) \) and \( \mathcal{W}_U(-) \) as the functors from the category of special \( \lambda \)-rings to the category of commutative rings with identity. Then, as a natural transformation from \( \mathcal{W}_G(-) \) to \( \mathcal{W}_U(-) \), \( \bar{f}_U \) coincides with \( f_U \). Similarly, \( \bar{v}_U \) coincides with \( v_U \).

Actually, the proof of Theorem 3.5 can be done essentially in the same way as in [3]. It is based on the following lemmas.

**Lemma 3.6.** (cf. [3, Lemma (2.12.12)]) For any two open subgroups \( U, V \leq G \) and \( \alpha \in R \) one has

\[
\hat{\varphi}_U(\text{Ind}_V^G(\tau_R^V(\alpha))) = \hat{\varphi}_U(G/V) \alpha^{(V:U)}.
\]
Proof. Note that
\[ \tilde{\phi}_U(\text{Ind}^G_U(\tau^V_R(\alpha))) = \sum_{gV \in (G/V)^U} \tilde{\phi}_U \circ \text{Res}^V_U(g)(\tau^V_R(\alpha)) \quad \text{(by Proposition 3.10 (c))} \]
\[ = \sum_{gV \in (G/V)^U} \tilde{\phi}_U(\tau^U_R(\alpha(V:U))) \quad \text{(by Lemma 3.11)} \]
\[ = \tilde{\phi}_U(G/V)\alpha(V:U). \]
□

Lemma 3.7. (cf. [9, Lemma (3.2.2)]) With the notation in [9, Lemma (3.2.2)], we obtain that for any \( \alpha, \beta \in R \)
\[ \tau^G_R(\alpha + \beta) = \sum_{A \in G \setminus U(G)} \text{Ind}^G_U(\tau^U_R(\alpha^A \cdot \beta^{G-A})). \quad (3.5) \]

Proof. First, we assume that \( R \) is torsion-free. For all open subgroups \( U \leq G \), if we take \( \tilde{\phi}_U \) on the right side of Eq. (3.5), one has
\[ \tilde{\phi}_U(\sum_{A \in G \setminus U(G)} \text{Ind}^G_U(\tau^U_R(\alpha^A \cdot \beta^{G-A}))) \]
\[ = \sum_{A \in G \setminus U(G)} \tilde{\phi}_U(G/U_A)(\alpha^A \cdot \beta^{G-A})(U_A:U) \quad \text{(by Lemma 3.6)} \]
\[ = \sum_{A \in U(G), U \leq U_A} \alpha^A(U/A), \beta^{G-A}/U \]
\[ = (\alpha + \beta)^G(U) \]
\[ = \tilde{\phi}_U(\tau^G_R(\alpha + \beta)). \]
Since \( \tilde{\phi}_U \) is injective, we have the desired result. However, in case where \( R \) is not torsion-free, \( \tilde{\phi}_U \) is no longer injective. In this case, we note that the \( [U] \)-th components appearing in both sides of Eq. (3.5) are integral polynomials in \( \lambda^k(\alpha) \)'s and \( \lambda^l(\beta) \)'s for \( 1 \leq k, l \leq [G : U] \). This implies that Identity (3.5) holds regardless of torsion. □

Lemma 3.8. (cf. [9, Lemma (3.2.5)]) For some \( k \in \mathbb{N} \) let \( V_1, \ldots, V_k \leq G \) be a sequence of open subgroups of \( G \). Then, for every open subgroups \( U \leq G \) and every sequence \( \varepsilon_1, \ldots, \varepsilon_k \in \{ \pm 1 \} \) there exists a unique polynomial \( \xi_U = \xi_U(\varepsilon_1, \ldots, \varepsilon_k) = \xi_U(x_1, \ldots, x_k) \in \mathbb{Z}[x_1, \ldots, x_k] \) such that for all \( \alpha_1, \ldots, \alpha_k \in R \) one has
\[ \tau^{-1}_R(\sum_{i=1}^k \varepsilon_i \cdot \text{Ind}^G_{V_i}(\tau^V_R(\alpha_i)))(U) = \xi_U(\alpha_1, \ldots, \alpha_k). \]

Proof. The proof can be done in the exactly same way of that of [9, Lemma (3.2.5)]. □
Note that the polynomial \( \xi_U(x_1, \ldots, x_k) \) must coincide with that of [9, Lemma (3.2.5)] since \( R \) contains \( \mathbb{Z} \) and
\[ \tau_R = \tau, \quad \text{Ind}^G_U = \text{ind}^G_U \quad \text{if} \ R = \mathbb{Z}. \]
Proof of Theorem 3.5 For \( \alpha \in \mathcal{W}_G(R) \) one has
\[
\text{Res}_U^G(\tau_R(\alpha)) = \sum_{[V]} \text{Res}_U^G \cdot \text{Ind}_V^G(\tau_R^V(\alpha([V])))
\]
\[
= \sum_{[V]} \sum_{UgV \leq G} \text{Ind}_U^{U \cap gVg^{-1}} \cdot \text{Res}_U^V \tau_R^V(g)(\alpha([V]))
\]
\[
= \sum_{[V]} \sum_{UgV \leq G} \text{Ind}_U^{U \cap gVg^{-1}}(\alpha([V]));\tau_R^{U \cap gVg^{-1}}(\alpha([V]));V:U \cap gVg^{-1}^{-1})
\]

Hence, with the same notation as in [9, Eq. (3.3.9)], it follows from Lemma 3.8
that for any open subgroup \( W \) of \( U \)
\[
(\tau_R^{-1} \circ \text{Res}_U^G \circ \tau_R)(\alpha)([W]) = \xi_{[W;W_1,\ldots,W_k;1]}(\alpha([V_1]),\cdots,\alpha([V_k]));(V_1,\cdots,V_k).
\]
(3.6)
Similarly, one can show \((\tau_R^{-1} \circ \text{Ind}_U^G \circ \tau_R)(\alpha)([W])\) is a polynomial with integral coefficients in those \( \alpha([V])'s \) \( (V \ an \ open \ subgroup \ of \ U \ to \ which \ W \ is \ sub-conjugate \ in \ G) \), which clearly coincides with the polynomial in [9]. Thus, we complete the proof.

By definition of \( \tilde{f}_U \), we have
\[
\text{Res}_U^G \circ \tau_R = \tau_R \circ \tilde{f}_U.
\]
(3.7)
Let us define the operator \( \mathcal{F}_U : \mathcal{O}(G) \to \mathcal{O}(U) \) by
\[
(b_{[V]});[V] \in \mathcal{O}(G) \mapsto (c_{[W]});[W] \in \mathcal{O}(U)
\]
where
\[
c_{[W]} := \begin{cases} b_{[V]} & \text{if } [W] = [V] \in \mathcal{O}(G), \\ 0 & \text{otherwise.} \end{cases}
\]
From [23] it follows that
\[
\mathcal{F}_U \circ \tilde{\varphi} = \tilde{\varphi} \circ \text{Res}_U^G.
\]
(3.8)
In view of Eq. (3.7) and Eq. (3.8), we have
\[
\mathcal{F}_U \circ \Phi = \Phi \circ \tilde{f}_U.
\]
Consequently, we can complete the following commutative diagrams:
\[
\begin{array}{ccccccc}
\mathcal{W}_G(R) & \xrightarrow{\tau_R} & \hat{\Omega}_R(G) & \xrightarrow{\hat{\phi}} & \mathcal{O}(G) & \xrightarrow{\Phi} & \mathcal{W}_G(R) \\
\downarrow f_U & & \downarrow \text{Res}_U^G & & \downarrow \mathcal{F}_U & & \downarrow f_U \\
\mathcal{W}_U(R) & \xrightarrow{\tau_R} & \hat{\Omega}_R(U) & \xrightarrow{\hat{\phi}} & \mathcal{O}(U) & \xrightarrow{\Phi} & \mathcal{W}_U(R)
\end{array}
\]

The above diagrams are also valid with regard to induction operators. To begin with, by definition of \( \tilde{v}_U \), we have
\[
\text{Ind}_U^G \circ \tau_R = \tau_R \circ \tilde{v}_U.
\]
(3.9)
It was shown in [23] that if we define
\[
\nu_U : \mathcal{O}(U) \to \mathcal{O}(G), \quad (b_{[V]});[V] \in \mathcal{O}(U) \mapsto (c_{[W]});[W] \in \mathcal{O}(G),
\]
where

\[ c_{[W]} = \sum_{\{V \mid \exists c(V) \in \mathcal{O}(G) \} \mid V \neq [W] \in \mathcal{O}(G)} [N_G(W) : N_U(V)] b_{[V]}, \]

then it holds that

\[ \tilde{\varphi} \circ \text{Ind}_U^G = \nu_U \circ \tilde{\varphi}^{-1}. \]  

Here, the notation \( N_G(W) \) represents the normalizer of \( W \) in \( G \). In view of Eq. (3.12) and Eq. (3.14), we have

\[ \nu_U \circ \Phi = \Phi \circ \tilde{\nu}_U. \]

Consequently, we have the following commutative diagrams:

\[
\begin{array}{cccc}
\mathbb{W}_U(R) & \overset{\tau_R}{\longrightarrow} & \tilde{\Omega}_R(U) & \overset{\tilde{\varphi}}{\longrightarrow} & R^0(U) \\
\downarrow \tilde{\nu}_U & & \downarrow \text{Ind}_U^G & & \downarrow \nu_U \\
\mathbb{W}_G(R) & \overset{\tau_R}{\longrightarrow} & \tilde{\Omega}_R(G) & \overset{\tilde{\varphi}}{\longrightarrow} & R^0(G)
\end{array}
\]

3.2. Necklace polynomials. In [21], Metropolis and Rota introduced the necklace polynomials,

\[ M(x, n) = \frac{1}{n} \sum_{d | n} \mu(d) x^\frac{n}{d}, \quad n \in \mathbb{N}, \]  

and then asked on what class of rings the necklace polynomials in Eq. (3.11) remain valid. The reason why they gave this question is because over this class the ring of Witt vectors becomes isomorphic to the necklace ring. In this section, we give the answer to this question. To do this, we will modify the definition of the necklace polynomials in Eq. (1.2) and study the properties of the modified ones.

An alphabet is a set of commuting variables so that, for example, \( \{x_1, x_2, \ldots, x_m\} \) is the alphabet of variables \( x_1, x_2, \ldots, x_m \). For alphabets \( X = \{x_1, x_2, \ldots, x_m\} \) and \( Y = \{y_1, y_2, \ldots, y_n\} \), \( X \cdot Y \) denotes an alphabet \( \{x_a y_b : 1 \leq a \leq m, 1 \leq b \leq n\} \). The notation \( \Psi^*(X) \), which is introduced to be consistent with that of a \( \lambda \)-ring, will denote an alphabet \( \{x_1^a, x_2^b, \ldots, x_m^c\} \). The elements of an alphabet \( X \) is called letters, and a word of \( X \) is a finite juxtaposition of letters of \( X \). The length of a word is the number of letters, where the product of two or more words is juxtaposition. Two words \( u \) and \( w \) are said to be conjugate when \( u = uv \) and \( w = vu \) where \( u \) and \( v \) are words. The identity in the monoid of words is the empty words. An equivalence class of words under the equivalence relation of conjugacy will be called a necklace. If \( w = u' \), then we say that the word of \( w \) has period \( n/i \), where \( n \) is the length of \( w \). The smallest \( j \) such that \( w = v^n/j \) for some \( v \) is called the primitive period of the word \( w \). A word of primitive period \( n \) is said to be aperiodic, and an equivalence class of aperiodic words will be called a primitive necklace.

Given a word \( w \), let \( m(w) \) be the monomial \( x_1^{a_1} x_2^{a_2} \cdots x_m^{a_m} \) where \( a_i \) is the number of \( x_i \)'s in \( w \). Define the necklace polynomial of degree \( k \) over \( X \) to be

\[ M(X, k) := \sum_{w} m(w), \]  

where the sum is over the primitive necklaces \( w \) out of \( X \) such that the degree of \( m(w) \) equals \( k \). Then, one easily computes

\[ M(X, k) = \frac{1}{k} \sum_{d | k} \mu(d) p_d(X)^{\frac{k}{d}}, \]
where \( p_d \) is the \( d \)-th power sum, that is, \( p_d(X) = x_1^d + \cdots + x_m^d \). For positive integers \( i \) and \( j \), we let \([i, j]\) be the least common multiple and \((i, j)\) the greatest common divisor of \( i \) and \( j \). With this notation, one can get the identities analogous to those in [21 Section 3].

**Theorem 3.9.**

(a) For alphabets \( X = \{x_1, x_2, \ldots, x_m\} \) and \( Y = \{y_1, y_2, \ldots, y_n\} \),

\[
M(X \cdot Y, k) = \sum_{[i,j]=k} (i, j)M(\Psi_{\frac{i}{k}}(X), i)M(\Psi_{\frac{j}{k}}(Y), j).
\]

(b) For an alphabet \( X = \{x_1, x_2, \ldots, x_m\} \),

\[
M(X^r, k) = \sum_{[r,j]=kr} \frac{j}{k}M(\Psi_{\frac{j}{k}}(X), j).
\]

(c) For an alphabet \( X = \{x_1, x_2, \ldots, x_m\} \) and \( Y = \{y_1, y_2, \ldots, y_n\} \), we have

\[
(r, s)M(X^{\frac{r}{kr}} Y^{\frac{s}{ks}}, k) = \sum_{i,j} (ri, sj)M(\Psi_{\frac{ri}{ks}}(X), i)M(\Psi_{\frac{sj}{kr}}(Y), j),
\]

where \( i, j \) range over positive integers such that \( ij/(ri, sj) = k/(r, s) \).

**Proof.** The proof can be done by a slight modification of that in [21 Section 3]. So, we will prove only (a). For a primitive word \( w \) of degree \( n \) out of the alphabet \( XY \), write it as a monomial in \( x_i, y_j \)'s, say

\[
x_1^{a_1}x_2^{a_2}\cdots x_m^{a_m}y_1^{b_1}y_2^{b_2}\cdots y_n^{b_n}.
\]

Let \( w' \) be the word \( x_1^{a_1}x_2^{a_2}\cdots x_m^{a_m} \) and \( w'' \) be the word \( y_1^{b_1}y_2^{b_2}\cdots y_n^{b_n} \). From [21 Theorem 1, Section 3] it follows that the word \( w \) is primitive if and only if \( w' \) has primitive period \( i \), the word \( w'' \) has primitive period \( j \), and \([i,j]=n\). Write \( w' = u^\frac{i}{k} \) and \( w'' = v^\frac{j}{k} \) for some \( i, j \) satisfying \([i,j]=n\). Thus, we have \( w = u^\frac{i}{k} v^\frac{j}{k} \).

Now, the bijectivity of this correspondence implies our assertion. \( \square \)

Another important interpretation of our necklace polynomial \( M(X, k) \) can be described in the following manner. For a function \( f \) from \( \mathbb{Z}/k\mathbb{Z} \) to \( X \), we let

\[
[f] := \prod_{i=0}^{k-1} f(\bar{i}).
\]

Let us define \( \mathbb{Z}/k\mathbb{Z} \)-action on \( f \) by \( \bar{n} \cdot f(\bar{i}) := f(\bar{i-n}) \). Then, the following proposition is almost straightforward.

**Proposition 3.10.** With the above notation, we have

\[
M(X, k) = \sum_{f} [f],
\]

where the sum is over \( f \)'s on which \( \mathbb{Z}/k\mathbb{Z} \) acts freely.

**Remark 3.11.** The concept of our necklace polynomial has a nice generalization associated with a profinite group \( G \) and its open subgroups \( V \). In detail, given a datum \((G, V, X)\), the polynomial \( M_G(V, X) \) such that \( M_G(\hat{C}^k, X) = M(X, k) \) was introduced. Here, \( X \) represents an alphabet. For the complete information refer to [20].
Let $R$ be a special $\lambda$-ring. For $r \in R$, $n \in \mathbb{N}$, we let

$$M(r, n) := \frac{1}{n} \sum_{d | n} \mu(d) \Psi^d(r^n).$$

In order to show this definition to be consistent with Eq. (3.12), write $r$ as a sum of one-dimensional elements, say, $r = r_1 + r_2 + \cdots + r_m$. In fact, this expression is possible by virtue of the splitting principle for special $\lambda$-rings. Now, consider the alphabet $X_r$ consisting of $r_i$'s, $1 \leq i \leq m$. Then one can easily verify that

$$M(r, n) = M(X_r, n),$$

and which says that if $R$ is a binomial ring, then $M(r, n)$ coincides with $M(r, n)$ (see [23]).

**Theorem 3.12.** Let $R$ be a unital commutative torsion-free ring. Then, $R$ is a binomial ring if and only if $M(r, n) \in R$ for all $r \in R$ and $n \in \mathbb{N}$.

**Proof.** The “if” part follows from Corollary [23]. More precisely, letting $n$ be a prime $p$, the assumption implies that $M(r, p) = \frac{1}{p}(r^p - r) \in R$.

Hence, $R$ satisfies the condition in Corollary [23]. For the “only if” part, assume that $r = r_1 + r_2 + \cdots + r_m$, where $r_i$'s are 1-dimensional. From the fundamental theorem of symmetric functions it follows that the symmetric function

$$M(X, k) = \frac{1}{k} \sum_{d | k} \mu(d) p_d(X)$$

is an integral polynomial in the elementary symmetric functions defined using the product of $(1 + x_i t)$’s. Here, $X$ denotes the alphabet $\{x_1, x_2, \cdots, x_m\}$. Now, the specialization that $x_i = r_i, 1 \leq i \leq m$, yields the desired result since the $n$-th elementary symmetric function is same to $\lambda^n(r)$’s after this specialization. □

3.3. **Covariant functor $Nr$ and its Verschiebung and Frobenius operators.**

In this section, we will show that the functor, $\hat{N}r_\hat{C}$, is isomorphic to the functors $\mathbb{W}$ and $\Lambda_1$ if these are viewed as the functors from the category of special $\lambda$-rings into itself. As before, $\hat{C}$ denotes the profinite completion of the multiplicative infinite cyclic group $C$. From now on, we use the notation $Nr$ instead of $\hat{N}r_\hat{C}$.

**Remark 3.13.** (a) It should be noted that our notation $Nr$ is different from the one in [21]. Actually, the latter coincides with the functor $\hat{N}r_\hat{C}$ in [25].

(b) Very often, the terminologies “Verschiebung and Frobenius operators” are used instead of the inductions and the restrictions in case $G = \hat{C}$.

We begin with investigating the structure of the necklace ring $Nr(R)$ over $R$ in more detail, where $R$ is a special $\lambda$-ring. $Nr(R)$ is the ring whose underlying set is $\prod_n R$, whose addition is defined componentwise, and whose multiplication is given by

$$\langle b \cdot c \rangle_n := \sum_{[i, j] = n} (i, j) \Psi^\hat{\lambda}(h_i) \Psi^\hat{\lambda}(c_j)$$

(3.13)
for two sequences \( b = (b_1, b_2, \ldots, b_n, \ldots) \) and \( c = (c_1, c_2, \ldots, c_n, \ldots) \). Especially, if \( R \) is a binomial ring, then the multiplication in (3.13) reduces to
\[
(b \cdot c)_n = \sum_{|i,j|=n} (i,j) \cdot b_i c_j.
\]

**Lemma 3.14.** ([13, Proposition (17.2.9)]) For every commutative ring \( A \) with identity, the map
\[
E_A : \mathbb{W}(A) \to \Lambda_1(A), \quad (a_n)_{n \geq 1} \mapsto \prod_{n=1}^{\infty} \left( \frac{1}{1 - a_n t^n} \right)
\]
(3.14) is a ring isomorphism.

For a special \( \lambda \)-ring \( R \) let us consider the map
\[
\tilde{s}_t : N_r(R) \to \Lambda_1(R), \quad (b_1, b_2, \ldots) \mapsto \prod_{n=1}^{\infty} \left( \sum_{r=0}^{\infty} S^r(b_n)t^{nr} \right).
\]
(3.15)

As mentioned in Eq. (2.2), the notation \( S_n \) represents the \( n \)-th symmetric power operation associated with the given special \( \lambda \)-ring structure. It has been shown in [23] that \( \tilde{s}_t \) is a bijective map, and a ring homomorphism if \( R \) is torsion-free.

### Theorem 3.15.
Let \( R \) be a special \( \lambda \)-ring. Then, \( \tilde{s}_t \) is a ring isomorphism.

**Proof.** It follows from [23, Lemma 3.13] that
\[
\tilde{s}_t \circ \tau_R = E_R
\]
(3.16) (refer to Eq. (3.14)). Combining Theorem [23] and Lemma 3.14 implies the desired assertion. \( \Box \)

**Remark 3.16.** Let \( R \) be a binomial ring. Then, the mapping (3.15) reduces to
\[
\tilde{s}_t(b_1, b_2, \ldots) = \prod_{n=1}^{\infty} \left( \frac{1}{1 - t^n} \right)^{b_n},
\]
which was dealt with intensively in (21).

We now can make \( N_r(R) \) and \( \mathbb{W}(R) \) into special \( \lambda \)-rings by virtue of the isomorphisms \( \tilde{s}_t \) and \( E_A \). The \( \lambda^m \)-operation on \( N_r(R) \) is defined by
\[
\lambda_m(\tilde{s}_t^{-1}(f(t))) = \tilde{s}_t^{-1}(\tilde{\Lambda}_m f(t)).
\]

Since
\[
\tilde{\Lambda}_m \left( \prod_{i=1}^{\infty} \frac{1}{1 - x_i t} \right) = \prod_{i_1 < i_2 < \cdots < i_m} \frac{1}{1 - x_{i_1} \cdots x_{i_m} t}
\]
and
\[
\tilde{s}_t(M(x_i)) = \frac{1}{1 - x_i t},
\]
(3.17)
the \( \lambda_m \)-operation on \( N_r(R) \) must satisfy
\[
\lambda_m \left( \sum_{n=1}^{\infty} M(x_i) \right) = \sum_{i_1 < i_2 < \cdots < i_m} M(x_{i_1} \cdots x_{i_m}).
\]
(3.18)
Actually, we can verify easily that Eq. (3.18) determines the $\lambda$-operations completely. For example, let
\[(c_1, c_2, c_3, \cdots) := \lambda^2(b_1, b_2, b_3, \cdots).\]
In order to compute $c_1$ we may assume that $b_1 = x_1 + x_2$ and $x_i = 0$ for $i \geq 3$. Then, from Eq. (3.17) it follows that
\[b_2 = \frac{1}{2}(x_1^2 - \Psi^2(x_1)) + \frac{1}{2}(x_2^2 - \Psi^2(x_2)) = \frac{1}{2}(b_1^2 - 2c_1 - \Psi^2(b_1)).\]
So, we conclude that $c_1 = -b_2 + \frac{1}{2}(b_1^2 - \Psi^2(b_1))$.

Similarly, the $\lambda_m$-operation on $\mathbb{W}(R)$ is defined by
\[\lambda_m E_R^{-1}(f(t)) = E_R^{-1}(\Lambda_m f(t)).\]
Then, from Eq. (3.18) it follows that
\[\lambda_m \left( \sum_{n=1}^{\infty} (x_{i_1}, 0, 0, \cdots) \right) = \sum_{i_1 < i_2 < \cdots < i_m} (x_{i_1} \cdots x_{i_m}, 0, 0, \cdots), \quad (3.19)\]
where the summation is being done in $\mathbb{W}(R)$. Note that Eq. (3.19) also determines the $\lambda$-operations of $\mathbb{W}(R)$ completely. For example, letting
\[(d_1, d_2, d_3, \cdots) := \lambda^2(e_1, e_2, e_3, \cdots),\]
then it is easy to show that $d_1 = -e_2$.

Now, let us investigate morphisms. For a special $\lambda$-ring homomorphism, $f : A \to B$, consider the homomorphisms
\[
\Lambda(f) : \Lambda_1(A) \to \Lambda_1(B), \quad 1 + \sum a_n t^n \mapsto 1 + \sum f(a_n) t^n,
\]
\[\text{Nr}(f) : \text{Nr}(A) \to \text{Nr}(B), \quad (b_n)_n \mapsto (f(b_n))_n,
\]
\[\mathbb{W}(f) : \mathbb{W}(A) \to \mathbb{W}(B), \quad (a_n)_n \mapsto (f(a_n))_n.
\]
It is well-known that $\Lambda(f)$ is a special $\lambda$-ring homomorphism. Therefore, $\text{Nr}(f)$ and $\mathbb{W}(f)$ also are special $\lambda$-ring homomorphisms by the definition of their $\lambda$-operations. In addition, following the same way as in [25], we can show that
\[
\tau_B \circ \mathbb{W}(f) \circ \tau_A^{-1} = \tilde{s}_t^{-1} \circ \Lambda(f) \circ \tilde{s}_t,
\]
\[\text{Nr}(f) = \tau_B \circ \mathbb{W}(f) \circ \tau_A^{-1}.
\]

The discussion until now can be illustrated in the following commutative diagram:
\[
\begin{array}{ccc}
\mathbb{W}(A) & \xrightarrow{\tau_A} & \text{Nr}(A) & \xrightarrow{\tilde{s}_t} & \Lambda_1(A) \\
\downarrow \mathbb{W}(f) & & \downarrow \text{Nr}(f) & & \downarrow \Lambda(f) \\
\mathbb{W}(B) & \xrightarrow{\tau_B} & \text{Nr}(B) & \xrightarrow{\tilde{s}_t} & \Lambda_1(B)
\end{array}
\]
Next, let us discuss Verschiebung and Frobenius operators on $\text{Nr}(R)$ and $\Lambda_1(R)$ more closely. Indeed, they are nothing but inductions and restrictions respectively with $G = C$. The $r$-th Verschiebung operator, $V_r$, on $\text{Nr}(R)$ is defined to be
\[V_r \alpha = \beta, \quad \text{where } \beta_n = \frac{1}{n} \Psi^\frac{1}{n} (\alpha_j),\]

The $r$-th Frobenius operator, $F_r$, on $\text{Nr}(R)$ is defined to be
\[F_r \alpha = \beta, \quad \text{where } \beta_n = \sum_{[r,j]=rn} \frac{1}{n} \Psi^\frac{1}{n} (\alpha_j).\]
Note that $F_r$ is a ring isomorphism, whereas $V_r$ is just additive. For $r \in R$, we let

$$M(r) := (M(r, 1), M(r, 2), \cdots, M(r, n), \cdots)$$

(refer to [23]).

**Proposition 3.17.** (cf. [21]) For $a, b \in R$, $\alpha \in Nr(R)$, and $r, s \in \mathbb{N}$, we have

(a) $V_r V_r = V_{rs}$.
(b) $F_r F_s = F_{rs}$.
(c) $F_r V_r(\alpha) = r\alpha$.
(d) $V_r M(a) \cdot V_s M(b) = (r, s) V_{[r,s]} M(a^{\frac{1}{r \cdot s}} b^{\frac{1}{r \cdot s}})$.
(e) $F_r M(a) = M(a^r)$.
(f) $F_r V_s = (r, s) V_{[r,s]} F_{[r,s]}$.

**Proof.** (a) and (b) follow from Eq. (3.3).
(c) Letting $b = F_r V_r(\alpha)$, then

$$b_n = \sum_{[r, j] = nr} \frac{j}{n} \Psi_n^\frac{1}{n} (V_r(\alpha)_j).$$

For $j = ri$ the condition $[r, j] = nr$ implies that $i = n$. Thus, we have the desired result.
(d) Letting $c = (c_n)_n = V_r M(\alpha) \cdot V_s M(\beta)$, then

$$c_n = \sum_{[i, j] = n} (i, j) \Psi_n^\frac{1}{n} (M(\alpha; \frac{j}{r})) \Psi_n^\frac{1}{n} (M(\beta; \frac{j}{s})).$$

If we substitute $\frac{j}{r} = i'$ and $\frac{j}{s} = j'$, then the desired result is immediate.
(e) By definition of $F_r$ we have

$$F_r M(a) = \sum_{[r, j] = rn} \frac{j}{n} \Psi_n^\frac{1}{n} (M(a; j)) = M(a^r, j) \quad \text{by Theorem 3.9(b).}$$

(f) This identity follows from [23] Proposition 3.10].

We now suppose that $R$ has two different kinds of special $\lambda$-ring structures. Write their $\lambda$-operations and Adams operations as

$$(\lambda_1^n, \Psi_1^n; n \geq 1) \quad \text{and} \quad (\lambda_2^n, \Psi_2^n; n \geq 1).$$

In Corollary 4.4 we have shown that there exists a canonical isomorphism

$${\text{Nr(id}_{12})} = \tau_{R,2} \circ \tau_{R,1}^{-1} : Nr_1(R) \to Nr_2(R),$$

$$\sum_{n=1}^{\infty} V_n M_1(q_n) \mapsto \sum_{n=1}^{\infty} V_n M_2(q_n),$$

where

$$M_i(r) := (M_i(r, 1), M_i(r, 2), \cdots)$$

with

$$M_i(r, k) = \frac{1}{k} \sum_{d \mid k} \mu(d) \Psi_i^d (r^\frac{1}{d}).$$

The map $Nr(id_{12})$ behaves nicely for Verschiebung and Frobenius operators.
Proposition 3.18. The isomorphism $\text{Nr}(\text{id}_{12})$ preserves Verschiebung and Frobenius operators, that is,
(a) $\text{Nr}(\text{id}_{12})(V_r \alpha) = V_r \text{Nr}(\text{id}_{12})(\alpha), \quad \alpha \in \text{Nr}_1(R)$.
(b) $\text{Nr}(\text{id}_{12})(F_r \alpha) = F_r \text{Nr}(\text{id}_{12})(\alpha), \quad \alpha \in \text{Nr}_1(R)$.

Proof. Since $\tau_{R,i}, i = 1, 2$ preserve inductions and restrictions by Eq. (3.7) and Eq. (3.9), so does $\text{Nr}(\text{id}_{12})$. □

The $n$-th Verschiebung operator, $V^A_n$, is defined by
$$V^A_n(1 + a_1 t + a_2 t^2 + \cdots) := 1 + a_1 t^n + a_2 t^{2n} + \cdots.$$ In order to define the $n$-th Frobenius operator, $F^A_n$, we define $a_i$'s by the equation
$$1 + a_1 t + a_2 t^2 + \cdots = \prod_{i=1}^{\infty} \frac{1}{1 - x_i t}.$$

Let $Q_{n,k}(a_1, \cdots, a_{nk})$ be the coefficient of $t^k$ in
$$\prod_{i=1}^{\infty} \frac{1}{1 - x_i^n t}.$$

Now, we define
$$F^A_n(1 + \sum c_i t^i) = 1 + \sum_{k=1}^{\infty} Q_{n,k}(c_1, \cdots, c_{nk}) t^k.$$

Proposition 3.19. (cf. Hazewinkel [13]) The $n$-th Frobenius operator $F^A_n$ coincides with the $n$-th Adams operator $\Psi^n$.

Proof. This assertion was proved for the ring $\Lambda_0(A)$ in [13]. Hence, applying the isomorphism
$$\iota : \Lambda_0(A) \cong \Lambda_1(A), \quad f(t) \mapsto \frac{1}{f(t)},$$
we obtain the desired result. □

Proposition 3.20. Let $R$ be a special $\lambda$-ring. Then, the map $\tilde{s}_i$ in Eq. (3.15) preserves Verschiebung and Frobenius operators.

Proof. In order to prove the assertion, it is enough to show that
$$\tilde{s}_i(V_r V_s M(\alpha)) = V^A_r \tilde{s}_i(V_s M(\alpha)),$$
$$\tilde{s}_i(F_r V_s M(\alpha)) = F^A_r \tilde{s}_i(V_s M(\alpha)).$$

For the first identity, let us combine Eq. (3.16) with Proposition 3.17 (a) to get
$$\tilde{s}_i(V_r V_s M(\alpha)) = \tilde{s}_i(V_{rs} M(\alpha)) = \frac{1}{1 - \alpha t^{rs}} = V^A_r \frac{1}{1 - \alpha t^s}.$$

Since $\tilde{s}_i(V_s M(\alpha)) = \frac{1}{1 - \alpha t^s}$, we are done. For the second one, note that
$$\tilde{s}_i(F_r \circ V_s M(\alpha)) = \tilde{s}_i((r, s) V_{rs, 1} F_{rs, 1} M(\alpha))$$
$$= \tilde{s}_i((r, s) V_{rs, 1} M(\alpha)_{[r,s]}) \quad \text{(by Proposition 3.17 (c))}$$
$$= \left( \frac{1}{1 - \alpha t_{[r,s]}} \right)^{[r,s]}.$$
On the other hand,
\[ F^α_r \tilde{s}_t(V_s M(\alpha)) = F^α_r V^α_r \tilde{s}_t(M(\alpha)) \]
\[ = (r, s) V^α_r \left( \frac{1}{1 - \alpha \frac{r}{s} \tilde{t}} \right) \]
\[ = \left( \frac{1}{1 - \alpha \frac{r}{s} \tilde{t}} \right)^{(r, s)}. \]

This completes the proof. \[ \square \]

The results in this section may be summarized as follows. The diagram
\[
\begin{array}{cccc}
\mathbb{W}(R) & \xrightarrow{\tau_R(\xi)} & Nr(R) & \xrightarrow{\tilde{s}_t(\xi)} & \Lambda_1(R) \\
\downarrow \Phi & & \downarrow \tilde{\varphi} & & \downarrow \Phi + \log \\
\text{gh}(R) & \xrightarrow{id} & \text{gh}(R) & \xrightarrow{\text{identification}} & R[[t]]
\end{array}
\]

is commutative and all the maps appearing in this diagram preserve Verschiebung and Frobenius operators. Here, \( \text{gh}(R) \) represents the ghost ring of \( R \), which is the set \( \prod_n R \) with the addition and the multiplication defined componentwise.

We end this section by introducing two significant properties of \( \tilde{\varphi} \).

The first is that for all \( n \geq 1 \) the maps \( \tilde{\varphi}_n \) provide natural transformations from the functor \( Nr \) to the identity functor, which follows from the commutative diagram
\[
\begin{array}{cccc}
Nr(A) & \xrightarrow{Nr(f)} & Nr(B) \\
\uparrow \tau_A & & \uparrow \tau_B \\
\mathbb{W}(A) & \xrightarrow{W(f)} & \mathbb{W}(B) \\
\downarrow \Phi_n & & \downarrow \Phi_n \\
A & \xrightarrow{f} & B
\end{array}
\]

Here, \( \Phi_n \) is the projection of \( \Phi \) to the \( n \)-th component. Observe that
\[
f \circ \tilde{\varphi}_n = f \circ \Phi_n \circ \tau_A^{-1} \\
= \Phi_n \circ \mathbb{W}(f) \circ \tau_A^{-1} \\
= \Phi_n \circ \tau_B^{-1} \circ Nr(f) \\
= \tilde{\varphi}_n \circ Nr(f).
\]

The second is related to a generalization of Theorem 3.9 (a). Let \( R \) be a \( \mathbb{Q} \)-algebra and \( a = (a_n)_n, b = (b_n)_n \in \prod_n R \). Set
\[
E(a, n) = \frac{1}{n} \sum_{d|n} \mu(d) \Psi^d(a_\frac{n}{d}).
\]

Since \( \tilde{\varphi} \) is a ring homomorphism we can derive the identity
\[
E(ab, n) = \sum_{[i, j]=n} (i, j) \Psi^\tilde{\varphi}(E(a, i)) \Psi^\tilde{\varphi}(E(b, j)).
\]

In particular, considering the case \( a = (r, r^2, r^3, \ldots) \) and \( b = (s, s^2, s^3, \ldots) \), we can recover Theorem 5.4 (a).
4. LOGARITHMIC FUNCTIONS ASSOCIATED WITH GRADED LIE SUPERALGEBRAS

4.1. Definition. Let $\hat{\Gamma}$ be a free abelian group with finite rank and let $\Gamma$ be a countable (usually infinite) sub-semigroup in $\hat{\Gamma}$ such that every element $\alpha \in \Gamma$ can be written as a sum of elements of $\Gamma$ in only finitely many ways. Given a commutative algebra $R$ with unity over $\mathbb{C}$, consider

$$R[[\Gamma \times \mathbb{Z}_2]] = \{ \sum_{(\lambda, a) \in \Gamma \times \mathbb{Z}_2} \zeta(\lambda, a)E^{(\lambda, a)} : \zeta(\lambda, a) \in R \},$$

the completion of the semi-group algebra $R[\Gamma \times \mathbb{Z}_2]$. Here $E^{(\lambda, a)} = (-1)^ae^{(\lambda, a)}$ and $e^{(\lambda, a)}$ are the usual basis elements of $R[\Gamma \times \mathbb{Z}_2]$ with the multiplication given by $e^{(\lambda, a)}e^{(\mu, b)} = e^{(\lambda+\mu, a+b)}$ for $(\lambda, a), (\mu, b) \in \Gamma \times \mathbb{Z}_2$. Then it is easy to show that $\{ E^{(\lambda, a)} : (\lambda, a) \in \Gamma \times \mathbb{Z}_2 \}$ is also a basis of $R[\Gamma \times \mathbb{Z}_2]$ with the multiplication $E^{(\lambda, a)}E^{(\mu, b)} = E^{(\lambda+\mu, a+b)}$.

For each $\alpha = \sum_{i=1}^r k_i\alpha_i \in \Gamma$, we define the height of $\alpha$, denoted by $ht(\alpha)$, to be the number $\sum_{i=1}^r k_i$. We write $R[[\Gamma \times \mathbb{Z}_2]]^{(\alpha)}$ for the set consisting of all elements

$$\sum \zeta(\beta, b)E^{(\beta, b)},$$

where $\zeta(\beta, b) = 0$ whenever $ht(\beta) < n$.

It is clear that $R[[\Gamma \times \mathbb{Z}_2]]^{(\alpha)}$ is an ideal of $R[[\Gamma \times \mathbb{Z}_2]]$, and therefore we have a filtration of ideals

$$R[[\Gamma \times \mathbb{Z}_2]] = R[[\Gamma \times \mathbb{Z}_2]]^{(1)} \supset R[[\Gamma \times \mathbb{Z}_2]]^{(2)} \supset R[[\Gamma \times \mathbb{Z}_2]]^{(3)} \supset \cdots.$$  

For $f \in R[[\Gamma \times \mathbb{Z}_2]]$ and $c \in \mathbb{C}$, define $(1 + f)^c$ by

$$1 + cf + \frac{c(c-1)}{2!}f^2 + \frac{c(c-1)(c-2)}{3!}f^3 + \cdots.$$  

**Definition 4.1.** (cf. [1]) We call a function $L : R[[\Gamma \times \mathbb{Z}_2]] \rightarrow R[[\Gamma \times \mathbb{Z}_2]]$ logarithmic if it satisfies the following properties.

1. $L(R[[\Gamma \times \mathbb{Z}_2]]^{(n)}) \subseteq R[[\Gamma \times \mathbb{Z}_2]]^{(n)}$ for all $n \in \mathbb{N}$,
2. $L(f) + L(g) = L(f + g - fg)$ for all $f, g \in R[[\Gamma \times \mathbb{Z}_2]]$,
3. $cL(f) = L(1 - (1 - f)^c$ for all $c \in \mathbb{C}, f \in R[[\Gamma \times \mathbb{Z}_2]]$.

Typical examples of logarithmic functions are Log and Exp, which are defined to be

$$\text{Log}(f) = -\log(1 - f) = \sum_{k \geq 1} \frac{1}{k}f^k, \quad (4.1)$$

$$\text{Exp}(f) = 1 - \exp(-f) = -\sum_{k \geq 1} \frac{(-1)^k}{k}f^k \quad (4.2)$$

for all $f \in R[[\Gamma \times \mathbb{Z}_2]]$. One can easily show that they are mutually inverses to each other.

Given an element $f = \sum \zeta(\alpha, a)E^{(\alpha, a)} \in R[[\Gamma \times \mathbb{Z}_2]]$, the coefficient of $E^{(\alpha, a)}$ in Log($f$) can be computed as follows. Let

$$P(f) = \{ (\alpha, a) \in \Gamma \times \mathbb{Z}_2 : \zeta(\alpha, a) \neq 0 \},$$

and let $\{ (\beta_i, b_j) | i, j = 1, 2, 3, \cdots \}$ be a fixed enumeration of $P(f)$. For $(\alpha, a) \in P(f)$, set

$$T(\alpha, a) = \{ s = (s_{ij}) : s_{ij} \geq 0, \sum s_{ij}(\beta_i, b_j) = (\alpha, a) \}.$$
And then we set

\[ W(\alpha, a) = \sum_{s \in T(\alpha, a)} \frac{(|s| - 1)!}{s!} \prod \zeta(\beta_i, b_j)^{s_{ij}}, \]

(4.3)

where \(|s| = \sum i_j s_{ij}\) and \(s! = \prod_{i,j} s_{ij}!\). By direct computation of the right hand side of Eq. (4.1) we can derive the following expansion formula:

\[ \Log(f) = \sum W(\alpha, a)E^{(\alpha, a)}. \]

Two basic properties of logarithmic functions were given by Bryant in the case where the rank of \(\Gamma\) is one (see [7, Theorem 2.2 and 2.3]). Actually one can easily generalize those properties to the general case following Bryant’s approach. Let \(\{v_\lambda| \lambda \in \Lambda\}\) be a \(\mathbb{C}\)-basis of \(R\). Then, as a \(\mathbb{C}\)-vector space, \(R[[\Gamma \times \mathbb{Z}_2]]\) has a basis

\[ \{v_\lambda E^{(\alpha, a)}: \lambda \in \Lambda, (\alpha, a) \in \Gamma \times \mathbb{Z}_2\}. \]

To each \(v_\lambda E^{(\alpha, a)}\) we assign an arbitrary element \(f_{\lambda,(\alpha, a)} \in R[[\Gamma \times \mathbb{Z}_2]]^{(\text{int}(n))}\).

**Proposition 4.2.**

(a) There exists a unique logarithmic function \(L : R[[\Gamma \times \mathbb{Z}_2]] \rightarrow R[[\Gamma \times \mathbb{Z}_2]]\) such that

\[ L(v_\lambda E^{(\alpha, a)}) = f_{\lambda,(\alpha, a)}, (\alpha, a) \in \Gamma \times \mathbb{Z}_2. \]

(b) A function \(L\) is logarithmic if and only if \(L = \Phi \circ \Log\) for some \(\mathbb{C}\)-linear function \(\Phi\) satisfying \(\Phi(R[[\Gamma \times \mathbb{Z}_2]]^{(n)}) \subseteq R[[\Gamma \times \mathbb{Z}_2]]^{(n)}\) for all \(n \geq 1\).

For later use we introduce a shift operator on \(R[[\Gamma \times \mathbb{Z}_2]]\). Given each positive integer \(k\), let

\[ \Theta_k : R[[\Gamma \times \mathbb{Z}_2]] \rightarrow R[[\Gamma \times \mathbb{Z}_2]], \quad \sum \zeta(\beta, b)E^{(\beta, b)} \rightarrow \sum \zeta(\beta, b)E^{k(\beta, b)}. \]

Very often we use \(\Gamma\)-grading instead of \((\Gamma \times \mathbb{Z}_2)\)-grading. In this case we let

\[ \zeta(\alpha) = \zeta(\alpha, \mathbf{0}) + \zeta(\alpha, \mathbf{1}), \quad E^{(\alpha, \mathbf{0})} = E^{(\alpha, \mathbf{1})} = E^{\alpha}, (\alpha \in \Gamma). \]

Also, \(R[[\Gamma \times \mathbb{Z}_2]]\) will be replaced by

\[ R[[\Gamma]] = \{ \sum_{\lambda \in \Gamma} \zeta(\lambda)E^{\lambda}: \zeta(\lambda) \in R\}. \]

**4.2. Graded Lie superalgebra and its Lie module denominator identity.**

Let \(G\) be a group and \(K\) be a field with \(\text{char} K \neq 2\). Throughout this section we assume that \(\Gamma_K(G)\) is a special \(\lambda\)-ring. Consider a \((\Gamma \times \mathbb{Z}_2)\)-graded \(K\)-vector space

\[ V = \bigoplus_{(\alpha, a) \in \Gamma \times \mathbb{Z}_2} V_{(\alpha, a)} \quad \text{with dim} V_{(\alpha, a)} < \infty \quad \text{for all} \quad (\alpha, a) \in (\Gamma \times \mathbb{Z}_2). \]

Furthermore, if \(G\) acts on \(V\) preserving the \(\Gamma \times \mathbb{Z}_2\)-gradation, we define

\[ [V] := \sum_{(\alpha, a)} [V_{(\alpha, a)}]E^{(\alpha, a)} \in \Gamma_K(G)[[\Gamma \times \mathbb{Z}_2]]. \]

If we set \(|V_{(\alpha, a)}| := (-1)^a|V_{(\alpha, a)}|\), then it is easy to show that

\[ [V] = \sum_{(\alpha, a)} |V_{(\alpha, a)}| E^{(\alpha, a)}. \]

We call \([V]\) the \(G\)-module function of \(V\) over \(K\).
On the other hand, a \( \mathbb{Z}_2 \)-graded \( K \)-vector space \( \mathfrak{L} = \mathfrak{L}_{\mathbb{P}} \oplus \mathfrak{L}_{\mathbb{T}} \) is called a Lie superalgebra if there exists a bilinear map \([ , ] : \mathfrak{L} \times \mathfrak{L} \rightarrow \mathfrak{L}\), called the bracket, such that

\[
[\mathfrak{L}_a, \mathfrak{L}_b] \subseteq \mathfrak{L}_{a+b},
\]

\[
[x, y] = (-1)^{ab}[y, x],
\]

\[
[x, [y, z]] = [[x, y], z] + (-1)^{ab}[y, [x, z]]
\]

for all \( x \in \mathfrak{L}_a, y \in \mathfrak{L}_b, a, b \in \mathbb{Z}_2 \). If char \( K = 3 \), then additionally \([x, [x, x]] = 0\) for \( x \in \mathfrak{L}_{\mathbb{T}}\).

The homogeneous elements of \( \mathfrak{L}_{\mathbb{P}} \) (resp. \( \mathfrak{L}_{\mathbb{T}} \)) are called even (resp. odd). Consider a \((\Gamma \times \mathbb{Z}_2)\)-graded Lie superalgebra \( \mathfrak{L} = \bigoplus_{(\alpha, a) \in \Gamma \times \mathbb{Z}_2} \mathfrak{L}_{(\alpha, a)} \) with \( \dim \mathfrak{L}_{(\alpha, a)} < \infty \) for all \((\alpha, a) \in (\Gamma \times \mathbb{Z}_2)\). In addition we suppose that \( G \) acts on \( \mathfrak{L} \) preserving the \((\Gamma \times \mathbb{Z}_2)\)-gradation.

**Remark 4.3.** When we deal with Lie algebras instead of Lie superalgebras, \( K \) may have characteristic 2. In this case, we assume that \([x, x] = 0\) for all \( x \in \mathfrak{L} \). All the results for Lie superalgebras appearing in this section may be carried over to Lie algebras over an arbitrary field, in particular, of characteristic 2.

For each \((\Gamma \times \mathbb{Z}_2)\)-graded Lie superalgebra \( \mathfrak{L} \), the homology groups of \( \mathfrak{L} \) are defined as the torsion groups of its universal enveloping algebra viewed as a supplemented algebra (see [13]).

Let \( \mathfrak{L} = \mathfrak{L}_{\mathbb{P}} \oplus \mathfrak{L}_{\mathbb{T}} \) be a Lie superalgebra and \( U = U(\mathfrak{L}) \) be its universal enveloping algebra. For each \( k \geq 0 \), define

\[
C_k = C_k(\mathfrak{L}) = \bigoplus_{p+q=k} A^p(\mathfrak{L}_{\mathbb{P}}) \otimes S^q(\mathfrak{L}_{\mathbb{T}}).
\]

Consider the following chain complex \((M_k, \partial_k) (k \geq -1)\), where

\[
M_k = \begin{cases} 
U \otimes_K C_k(\mathfrak{L}) & \text{if } k \geq 0, \\
K & \text{if } k = -1,
\end{cases}
\]

and the differentials \( \partial_k : M_k \rightarrow M_{k-1} \) are given by

\[
\partial_k (u \otimes (x_1 \wedge \cdots \wedge x_p) \otimes (y_1 \cdots y_q)) = \\
\sum_{1 \leq s < t \leq p} (-1)^{s+t} u \otimes ([x_s, x_t] \wedge x_1 \wedge \cdots \wedge \hat{x}_s \wedge \cdots \wedge \hat{x}_t \wedge \cdots \wedge x_p) \otimes (y_1 \cdots y_q) \\
+ \sum_{s=1}^p \sum_{t=1}^q (-1)^{s+t} u \otimes (x_1 \wedge \cdots \wedge \hat{x}_s \wedge \cdots \wedge x_p) \otimes ([x_s, y_t] y_1 \cdots \hat{y}_t \cdots y_q) \\
- \sum_{1 \leq s < t \leq q} u \otimes ([y_s, y_t] x_1 \wedge \cdots \wedge x_p) \otimes (y_1 \cdots \hat{y}_s \cdots \hat{y}_t \cdots y_q) \\
+ \sum_{s=1}^p (-1)^{s+1} (u \cdot x_s) \otimes (x_1 \wedge \cdots \wedge \hat{x}_s \wedge \cdots \wedge x_p) \otimes (y_1 \cdots y_q) \\
+ (-1)^p \sum_{t=1}^q (u \cdot y_t) \otimes (x_1 \wedge \cdots \wedge x_p) \otimes (y_1 \cdots \hat{y}_t \cdots y_q)
\]

for \( k \geq 1 \), \( \partial_0 \) is the augmentation map extracting the constant term, and \( \partial_{-1} = 0 \). Then one can easily verify that \( \partial_{k-1} \circ \partial_k = 0 \).
In particular, when $\mathfrak{L}$ is a $\Gamma$-graded Lie algebra and $U = U(\mathfrak{L})$ is its universal enveloping algebra over an arbitrary field $K$, then

$$C_k = C_k(\mathfrak{L}) = \Lambda^k(\mathfrak{L}),$$

$$M_k = \begin{cases} U \otimes_K C_k(\mathfrak{L}) & \text{if } k \geq 0, \\ K & \text{if } k = -1, \end{cases}$$

and the differentials $\partial_k : M_k \to M_{k-1}$ reduce to

$$\partial_k(u \otimes (x_1 \wedge \cdots \wedge x_k))$$

$$= \sum_{1 \leq s < t \leq k} (-1)^{s+t} u \otimes ([x_s, x_t] \wedge x_1 \wedge \cdots \wedge \hat{x}_s \wedge \cdots \wedge \hat{x}_t \wedge \cdots \wedge x_k)$$

$$+ \sum_{s=1}^k (-1)^{s+1} (u \cdot x_s) \otimes (x_1 \wedge \cdots \wedge \hat{x}_s \wedge \cdots \wedge x_k).$$

**Proposition 4.4.** (\S\ [L3])

(a) Let $\mathfrak{L}$ be a Lie algebra over an arbitrary field $K$. Then, the chain complex $M = (M_k, \partial_k)$ is a free resolution of the trivial 1-dimensional left $U$-module $K$.

(b) Let $\mathfrak{L}$ be a Lie superalgebra over $\mathbb{C}$. Then, the chain complex $M = (M_k, \partial_k)$ is a free resolution of the trivial 1-dimensional left $U$-module $\mathbb{C}$.

**Remark 4.5.** In Proposition 4.4(b), $\mathbb{C}$ may be replaced by a field of characteristic zero. However, it does not seem to be known yet whether this statement remains true over an arbitrary field.

Let $K$ be the trivial one dimensional $\mathfrak{L}$-module and consider the chain complex $(K \otimes_U M_k, 1_k \otimes_U \partial_k)$. Then, it is straightforward that the homology modules of this chain complex, denoted by $H_k(\mathfrak{L}) = H_k(\mathfrak{L}, K)$, are determined from the complex

$$\cdots \to C_k(\mathfrak{L}) \xrightarrow{d_k} C_{k-1}(\mathfrak{L}) \xrightarrow{d_{k-1}} \cdots \to C_1(\mathfrak{L}) \xrightarrow{d_1} C_0(\mathfrak{L}) \to 0,$$

where $C_k(\mathfrak{L})$ are defined by

$$C_k(\mathfrak{L}) = \bigoplus_{p+q=k} \Lambda^p(\mathfrak{L}_\Gamma) \otimes S^q(\mathfrak{L}_\Gamma)$$

and the differentials $d_k : C_k(\mathfrak{L}) \to C_{k-1}(\mathfrak{L})$ are given by

$$d_k((x_1 \wedge \cdots \wedge x_p) \otimes (y_1 \cdots y_q))$$

$$= \sum_{1 \leq s < t \leq p} (-1)^{s+t} ([x_s, x_t] \wedge x_1 \wedge \cdots \wedge \hat{x}_s \wedge \cdots \wedge \hat{x}_t \wedge \cdots \wedge x_p) \otimes (y_1 \cdots y_q)$$

$$+ \sum_{s=1}^p \sum_{t=1}^q (-1)^s (x_1 \wedge \cdots \wedge \hat{x}_s \wedge \cdots \wedge x_p) \otimes ([x_s, y_t]y_1 \cdots \hat{y}_s \cdots y_q)$$

$$- \sum_{1 \leq s < t \leq q} ([y_s, y_t] \wedge x_1 \wedge \cdots \wedge x_p) \otimes (y_1 \cdots \hat{y}_s \cdots \hat{y}_t \cdots y_q)$$

for $k \geq 2$, $x_i \in \mathfrak{L}_0$, $y_j \in \mathfrak{L}_\Gamma$, and $d_1 = 0$. 


Let
\[
C(\mathfrak{L}) = \sum_{k=0}^{\infty} (-1)^k C_k(\mathfrak{L}) = C \oplus \mathfrak{L} \oplus C_2(\mathfrak{L}) \oplus \cdots ,
\]
\[
\Lambda(\mathfrak{L}_0) = \sum_{k=0}^{\infty} (-1)^k \Lambda^k(\mathfrak{L}_0) = C \oplus \mathfrak{L}_0 \oplus \Lambda^2(\mathfrak{L}_0) \oplus \cdots ,
\]
\[
S(\mathfrak{L}_1) = \sum_{k=0}^{\infty} (-1)^k S^k(\mathfrak{L}_1) = C \oplus \mathfrak{L}_1 \oplus S^2(\mathfrak{L}_1) \oplus \cdots ,
\]
\[
H(\mathfrak{L}) = \sum_{k=1}^{\infty} (-1)^k H_k(\mathfrak{L}) = H_1(\mathfrak{L}) \oplus H_2(\mathfrak{L}) \oplus H_3(\mathfrak{L}) \oplus \cdots
\]
be the alternating direct sums of \((\Gamma \times \mathbb{Z}_2)\)-graded vector spaces. Clearly
\[
C(\mathfrak{L}) = \Lambda(\mathfrak{L}_0) \otimes S(\mathfrak{L}_1).
\]

Let \(t = (t(\alpha, a), \alpha, a) \in \Gamma \times \mathbb{Z}_2\) be a sequence of nonnegative integers indexed by \((\Gamma \times \mathbb{Z}_2)\) with only finitely many nonzero terms, and set \(|t| = \sum t(\alpha, a)\). Since the \(k\)-th exterior power \(\Lambda^k(\mathfrak{L}_0)\) is decomposed as
\[
\Lambda^k(\mathfrak{L}_0) = \bigoplus_{|t| = k} \left( \bigotimes_{\alpha \in \Gamma} \Lambda^{t(\alpha, a)}(\mathfrak{L}_{(\alpha, a)}) \right)
\]
as a \((\Gamma \times \mathbb{Z}_2)\)-graded \(KG\)-module, we have
\[
\Lambda(\mathfrak{L}_0) = \prod_{\alpha \in \Gamma} \left( \sum_{m=0}^{\infty} (-1)^m \left[ \Lambda^m(\mathfrak{L}_{(\alpha, 0)}) \right] e^m(\alpha, 0) \right).
\]

By the definition of Adams operations, we have
\[
\Lambda(\mathfrak{L}_0) = \prod_{\alpha \in \Gamma} \exp \left( -\sum_{r=1}^{\infty} \frac{1}{r} \Psi^r(\mathfrak{L}_{(\alpha, 0)}) e^r(\alpha, 0) \right).
\]

Similarly, we have
\[
S(\mathfrak{L}_1) = \prod_{\alpha \in \Gamma} \left( \sum_{m=0}^{\infty} (-1)^m \left[ S^m(\mathfrak{L}_{(\alpha, 1)}) \right] e^m(\alpha, 1) \right).
\]

By the definition of symmetric power operations, we have
\[
S(\mathfrak{L}_1) = \prod_{\alpha \in \Gamma} \exp \left( \sum_{r=1}^{\infty} \frac{(-1)^r}{r} \Psi^r(\mathfrak{L}_{(\alpha, 1)}) e^r(\alpha, 1) \right).
\]
It follows that
\[
[C(L)] = \sum_{k=0}^{\infty} (-1)^k [C_k(L)] = [A(L_\gamma)] \cdot [S(L_\gamma)]
\]
\[
= \prod_{\alpha \in \Gamma} \exp \left( - \sum_{r=1}^{\infty} \frac{1}{r} \Psi^r(|L_{(\alpha,a)}|) e^{r(\alpha,T)} \right) \times \prod_{\alpha \in \Gamma} \exp \left( \sum_{r=1}^{\infty} \frac{(-1)^r}{r} \Psi^r(|L_{(\alpha,1)}|) e^{r(\alpha,T)} \right).
\]
Replacing \(|L_{(\alpha,a)}| = (-1)^a[L_{(\alpha,a)}]| and \(E^{(\alpha,a)} = (-1)^a e^{(\alpha,a)}\), the above identity reduces to
\[
[C(L)] = \prod_{(\alpha,a) \in \Gamma \times Z_2} \exp \left( - \sum_{r=1}^{\infty} \frac{1}{r} \Psi^r(|L_{(\alpha,a)}|) E^{r(\alpha,a)} \right).
\]
Hence, by the Euler-Poincaré principle, we obtain the Lie module denominator identity for the graded Lie superalgebra \(L = \bigoplus_{(\alpha,a) \in \Gamma \times Z_2} L_{(\alpha,a)}\).

**Proposition 4.6.** For every \((\Gamma \times Z_2)\)-graded Lie superalgebras \(L\), we have
\[
\prod_{(\alpha,a) \in \Gamma \times Z_2} \exp \left( - \sum_{r=1}^{\infty} \frac{1}{r} \Psi^r(|L_{(\alpha,a)}|) E^{r(\alpha,a)} \right) = 1 - [H(L)].
\]

**4.3. Main results on logarithmic functions.** To begin with, we introduce the \(C\)-linear operators on \(\Gamma_K(G)[[\Gamma \times Z_2]]\) such as
\[
\omega = \sum_{k \geq 1} \frac{\mu(k)}{k} \Theta_k \circ \Psi^k, \quad \eta = \sum_{k \geq 1} \frac{1}{k} \Theta_k \circ \Psi^k,
\]
where \(\mu\) is the Möbius inverse function. Recall that the maps \(\Theta_k\) are defined in Section 4.1 and the maps
\[
\Psi^k : \Gamma_K(G)[[\Gamma \times Z_2]] \to \Gamma_K(G)[[\Gamma \times Z_2]]
\]
are the induced algebra homomorphisms defined by the action of Adams operations on coefficients. Applying the Möbius inversion formula one can easily show that \(\omega\) and \(\eta\) are mutually inverses to each other. Observe that
\[
\eta([L]) = \sum_{(\gamma,a) \in \Gamma \times Z_2} \eta(\gamma,a) E^{(\gamma,a)}
\]
is the formal power series whose coefficients are given by
\[
\eta(\gamma, \overline{\gamma}) = \sum_{d | \gamma} \frac{1}{d} \Psi^d \left( |L_{(\gamma,d)}| \right) + \sum_{d | \gamma: even} \frac{1}{d} \Psi^d \left( |L_{(\gamma,d)}| \right),
\]
\[
\eta(\gamma, T) = \sum_{d | \gamma: odd} \frac{1}{d} \Psi^d \left( |L_{(\gamma,d)}| \right).
\]
Let
\[
D = \omega \circ \Log.
\]
By definition \(\omega\) is \(C\)-linear and
\[
\omega(R[[\Gamma \times Z_2]]^{(n)}) \subseteq R[[\Gamma \times Z_2]]^{(n)}.
\]
Therefore, we conclude that $\mathcal{D}$ is logarithmic by Proposition 4.2.

**Theorem 4.7.** $\mathcal{D}$ is the unique logarithmic function on $\Gamma_K(G)[[\Gamma \times \mathbb{Z}_2]]$ such that

$$[\mathcal{L}] = \mathcal{D}([H(\mathcal{L})]).$$

(4.6)

for every $(\Gamma \times \mathbb{Z}_2)$-graded Lie superalgebra $\mathcal{L}$. Moreover $\omega \circ \text{Log} = \text{Log} \circ \omega$.

**Proof.** If we take the logarithm on both sides of the Lie module denominator identity (4.4), then we obtain the equality

$$\sum_{(\alpha, a)} \eta(\alpha, a) E^{(\alpha, a)} = \text{Log}([H(\mathcal{L})])$$

(4.7)

where

$$\eta(\alpha, a) = \sum_{r \mid (\alpha, a)} \frac{1}{r} \Psi^{r} \left(\mathcal{L}_{(\alpha, a)}\right).$$

Equivalently,

$$\eta([\mathcal{L}]) = \text{Log}([H(\mathcal{L})]).$$

Taking $\omega$ on both sides, we have the desired result. On the other hand, since the operators $\Theta_k$ and $\Psi^k$ commute with Log, so does $\omega$.

In order to prove the uniqueness of $\mathcal{D}$ assume that $\mathcal{D}'$ is another logarithmic function satisfying Eq. (4.6). Let us choose a $\mathbb{C}$-basis of $\Gamma_K(G)[[\Gamma \times \mathbb{Z}_2]]$, say $\{I_\lambda E^{(\alpha, a)} : \lambda \in \Lambda, (\alpha, a) \in (\Gamma \times \mathbb{Z}_2)\}$, such that $I_\lambda$ is an actual finite dimensional $KG$-module for every $\lambda \in \Lambda$. Given $\lambda$ and $(\beta, b)$, let $V = \bigoplus_{(\alpha, a) \in (\Gamma \times \mathbb{Z}_2)} V(\alpha, a)$, where

$$V(\alpha, a) = \begin{cases} I_\lambda & \text{if } (\alpha, a) = (\beta, b), \\
0 & \text{otherwise.} \end{cases}$$

Then $\mathcal{D}'$ must map $[V]$ to $[\mathcal{L}(V)]$ which coincides with $\mathcal{D}([V])$ by the property (1.6). In other words,

$$\mathcal{D}(I_\lambda E^{(\beta, b)}) = \mathcal{D}'(I_\lambda E^{(\beta, b)})$$

for every $(\beta, b) \in (\Gamma \times \mathbb{Z}_2)$. It is obvious $[\mathcal{L}(V)] \in \Gamma_K(G)[[\Gamma \times \mathbb{Z}_2]]^{(ht(\beta))}$. Such a logarithmic function is uniquely determined by Proposition 4.2(a). So, $\mathcal{D} = \mathcal{D}'$. □

Comparing the coefficients of both sides of Eq. (4.7), we can derive that $\eta(\alpha, a)$ equals to $W(\alpha, a)$, which is defined in (4.3), for all $(\alpha, a) \in (\Gamma \times \mathbb{Z}_2)$. It follows that

$$\mathcal{D}([\mathcal{L}]) = \omega \circ \text{Log}([H(\mathcal{L})]) = \omega(\sum W(\alpha, a) E^{(\alpha, a)}).$$

Immediately the above identity provide a close formula for the homogeneous component $[\mathcal{L}(\alpha, a)]$.

**Corollary 4.8.** For every $(\alpha, a) \in (\Gamma \times \mathbb{Z}_2)$

$$[\mathcal{L}(\alpha, a)] = \sum_{d=0}^{d(\alpha, a)} \frac{\mu(d)}{d!} \Psi^d(W(\alpha, a)),$$

(4.8)

where

$$W(\alpha, a) = \sum_{s \in I_{(\alpha, a)}} (|s| - 1)! \prod_{i} |H(\mathcal{L})_{(\beta_i, b_j)}|^{s_i}.$$
Corollary 4.9. Let $V = \bigoplus_{(\alpha, \tau) \in \Gamma \times \mathbb{Z}/2} V_{(\alpha, \tau)}$ be a $(\Gamma \times \mathbb{Z}/2)$-graded $K$-vector space with finite dimensional homogeneous subspaces, and let $\mathfrak{L}(V) = \bigoplus_{(\alpha, \tau) \in \Gamma \times \mathbb{Z}/2} V_{(\alpha, \tau)}$ be the free Lie superalgebra generated by $V$. Suppose $G$ acts on $V$ preserving $(\Gamma \times \mathbb{Z}/2)$-gradation. Then, $\mathfrak{D}$ is the unique logarithmic function on $\Gamma(G)[[\Gamma \times \mathbb{Z}/2]]$ such that

$$[\mathfrak{L}(V)] = \mathfrak{D}([V]) \quad (4.9)$$

for every $V$.

Proof. By close inspection of the proof of [15, Corollary 3.2], we know that

$$H_k(\mathfrak{L}(V)) = \begin{cases} V & \text{if } k = 1, \\ 0 & \text{otherwise} \end{cases}$$

for every field $K$, char$K \neq 2$. Hence, (4.6) is reduced to

$$[\mathfrak{L}(V)] = \mathfrak{D}([V]).$$

The uniqueness of $\mathfrak{D}$ also follows from Theorem 4.10.

For example, let us consider rank=1 case. If we use the notation, $[V] := [V_0] - [V_1]$ and $[\mathfrak{L}(V)] := [\mathfrak{L}(V)_{(\alpha, \tau)}] - [\mathfrak{L}(V)_{(\alpha, \tau)}]$, then Eq. (4.9) implies that

$$[\mathfrak{L}(V)] = \frac{1}{n} \sum_{d|n} \mu(d) \Psi^d ([V]^{\pi}).$$

Similarly, if we use the notation $\{\mathfrak{L}(V)_{(\alpha, \tau)}\} := [\mathfrak{L}(V)_{(\alpha, \tau)}] + [\mathfrak{L}(V)_{(\alpha, \tau)}]$, then Eq. (4.1) implies the formula

$$\{\mathfrak{L}(V)_{(\alpha, \tau)}\} = \frac{1}{n} \sum_{d|n} \mu(d) \Psi^d ([V]^{\pi} - (-1)^d [V]^{\pi}).$$

From now on, we assume that $K$ is an arbitrary field with char$K \neq 2$. Recall that we showed in Theorem 4.11 that if $\Gamma_K(G)$ is a special $\lambda$-ring, then there exists a unique logarithmic function $\mathfrak{D}$ on $\Gamma_K(G)[[\Gamma \times \mathbb{Z}/2]]$ satisfying

$$[\mathfrak{L}] = \mathfrak{D}([H(\mathfrak{L})])$$

for every $(\Gamma \times \mathbb{Z}/2)$-graded Lie superalgebra $\mathfrak{L}$. But when $\Gamma_K(G)$ is not a special $\lambda$-ring, we do not know whether this is true. However, in the case of free Lie superalgebras, we can derive an analogue of Theorem 4.11. In showing this the subsequent lemma, which can be obtained by applying the Lazard elimination theorem, plays an essential role.

Lemma 4.10. (cf. [4, 17]) Let $G$ be a group and $K$ be any field with char$K \neq 2$. For $(\Gamma \times \mathbb{Z}/2)$-graded $KG$-modules $U$ and $V$, we have

(a) $\mathfrak{L}(U \oplus V) = \mathfrak{L}(U) \oplus \mathfrak{L}(V \cap U)$, 

where $V \cap U$ is the space spanned by all products $[v, u_1, \cdots, u_m]$ with $m \geq 1$, $v \in V$ and $u_i \in U$ for all $i$, with grading induced by that of $\mathfrak{L}(U \oplus V)$.

(b) $[V \cap U] = [V] (1 - [U])^{-1}$.

Theorem 4.11. Let $G$ be a group and $K$ be any field with char$K \neq 2$. Then, there exists a unique logarithmic function $\mathfrak{D}$ on $\Gamma_K(G)[[\Gamma \times \mathbb{Z}/2]]$ such that

$$[\mathfrak{L}(V)] = \mathfrak{D}([V]) \quad (4.10)$$

for every $(\Gamma \times \mathbb{Z}/2)$-graded $KG$-module $V$. 
Proof. Choose \( \{ I_\lambda E^{(\alpha, a)} : \lambda \in \Lambda, (\alpha, a) \in (\Gamma \times \mathbb{Z}_2) \} \), a \( \mathbb{C} \)-basis of \( \Gamma_K(G)[[\Gamma \times \mathbb{Z}_2]] \), such that \( I_\lambda \) is an actual finite dimensional \( KG \)-module for every \( \lambda \in \Lambda \). Given \( I_\lambda E^{(\beta, b)} \), consider the graded \( KG \)-module \( V(\lambda, (\beta, b)) = \bigoplus_{(\alpha, a) \in \Gamma \times \mathbb{Z}_2} V_{(\alpha, a)} \), where \( V_{(\beta, b)} = I_\lambda \) and \( V_{(\alpha, a)} = 0 \) for all \( (\alpha, a) \neq (\beta, b) \). Then, by Proposition 4.10, we conclude that there exists a unique logarithmic function \( D \) such that
\[
D(I_\lambda E^{(\beta, b)}) = [\mathcal{L}(V(\lambda, (\beta, b)))].
\] (4.11)
It is obvious that \( [\mathcal{L}(V(\lambda, (\beta, b)))] \in \Gamma_K(G)[[\Gamma \times \mathbb{Z}_2]](ht(\beta)) \). For any graded \( KG \)-module \( V \), write \( [V] = \sum a_{\alpha, (\beta, b)} I_\lambda E^{(\beta, b)} \). We claim that \( D([V]) = [\mathcal{L}(V)] \). To show this observe that
\[
D([U] + [V]) = D([U]) + D([V])(1 - [U])^{-1}.
\] (4.12)
for \( (\Gamma \times \mathbb{Z}_2) \)-graded \( KG \)-modules \( U \) and \( V \). Indeed this property follows from the definition of logarithmic function easily. Combining (4.12) with Eq. (4.11) and Lemma 4.10, we conclude that \( D([V]) = [\mathcal{L}(V)] \). □

It is very worthwhile remarking that Theorem 4.11 guarantees the existence of such a logarithmic function \( D \), but provides no information on its explicit form. However, using Grothendieck algebra \( \Gamma\Gamma_K(G) \) instead of Green algebra \( \Gamma_K(G) \), we can write out such a logarithmic function explicitly.

From now on, let \( G \) be a finite group and \( K \) be an arbitrary field. Let \( G_p' \) be the set of all elements of \( G \) of order not divisible by \( p \). And we let \( C \) be the \( \mathbb{C} \)-algebra consisting of all class functions from \( G_p' \) to \( \mathbb{C} \), that is, functions from \( G_p' \) to \( \mathbb{C} \) such that \( f \in C, f(g) = f(g') \) whenever \( g \) and \( g' \) are conjugate in \( G \). Denote the algebraic closure of \( K \) by \( \bar{K} \). We choose and fix a primitive \( e \)-th root of unity \( \xi \) in \( \bar{K} \) and \( \omega \in \bar{C} \), where \( e \) denotes the least common multiple of the orders of the elements of \( G_p' \). For a \( KG \)-module \( V \), we define \( ch_V \) be the function from \( G_p' \) to \( \bar{C} \) such that, for \( a \in G_p', ch_V(a) = \omega^{k_1} + \cdots + \omega^{kr} \), where \( \xi^{k_i}(1 \leq i \leq r) \) are eigenvalues of \( a \) in its action on \( \bar{K} \otimes V \) (see [7]).

Lemma 4.12. ([7])

(a) There exists an injective \( \mathbb{C} \)-algebra homomorphism \( \tau : \Gamma_K(G) \to \Gamma_{\bar{K}}(G) \) such that \( \tau(V) = \bar{K} \otimes V \).

(b) The \( \mathbb{C} \)-algebra homomorphism \( ch : \Gamma_{\bar{K}}(G) \to C \), defined by \( ch(\tau, \lambda) = ch_{\tau, \lambda} \), is an isomorphism.

Thus we may regard \( \Gamma_K(G) \) as a subalgebra of \( \Gamma_{\bar{K}}(G) \). Then \( C \) becomes a special \( \Psi\)-ring for the operations \( \Psi^n(f)(g) = f(g^n) \) for all \( g \in G_p' \) and \( n \geq 1 \). From Theorem 2.2, it follows that \( C \) is a special \( \lambda \)-ring. Moreover, it is easy to verify that
\[
ch_{\Psi^n(x)} = \Psi^n(ch(x))
\]
for \( x \in \Gamma_K(G) \). Also, we can easily show that \( \Gamma_K(G) \), when regarded as a subalgebra of \( C \), is invariant under \( \Psi^n \) for all \( n \).

Proposition 4.13. Let \( G \) be a finite group and \( K \) be any field. Then, \( \Gamma_K(G) \) is a special \( \lambda \)-ring.

Proof. Since \( \Gamma_K(G) \), when regarded as a subalgebra of \( C \), is invariant under \( \Psi^n \) for all \( n \), it becomes a special \( \lambda \)-ring by Theorem 2.2. □

Remark 4.14. In the case where \( G \) is finite and \( K \) has characteristic \( 0 \) or characteristic not dividing the order of \( |G| \), it is well known that \( \Gamma_K(G) \) may be identified
with \( \Gamma_K(G) \) (see \( [4,7] \)). However, for \((G,K)\) such that \(G\) is finite and \(\text{char}K \mid |G|\), we have no idea whether \(\Gamma_K(G)\) is a special \(\lambda\)-ring or not.

Consider a \((\Gamma \times \mathbb{Z}_2)\)-graded \(K\)-vector space \(V = \bigoplus_{(\alpha,a) \in (\Gamma \times \mathbb{Z}_2)} V_{(\alpha,a)}\) with \(\dim V_{(\alpha,a)} < \infty\) for all \((\alpha,a) \in \Gamma \times \mathbb{Z}_2\). Suppose \(G\) acts on \(V\) preserving the \((\Gamma \times \mathbb{Z}_2)\)-gradation.

We define
\[
\mathcal{V} := \sum_{(\alpha,a)} V_{(\alpha,a)} e^{(\alpha,a)} \in \Gamma_K(G)[[\Gamma \times \mathbb{Z}_2]].
\]

By setting \(|V_{(\alpha,a)}| = (-1)^a V_{(\alpha,a)}\), we have \(\mathcal{V} = \sum_{(\alpha,a)} |V_{(\alpha,a)}| E^{(\alpha,a)}\).

Let us consider a \((\Gamma \times \mathbb{Z}_2)\)-graded Lie superalgebra \(L = \bigoplus_{(\alpha,a) \in (\Gamma \times \mathbb{Z}_2)} L_{(\alpha,a)}\) with \(\dim L_{(\alpha,a)} < \infty\) for all \((\alpha,a) \in (\Gamma \times \mathbb{Z}_2)\) over \(K\). Suppose \(G\) acts on \(L\) preserving the \((\Gamma \times \mathbb{Z}_2)\)-gradation. Then, we can derive an identity analogous to the denominator identity Eq. (4.4).

\[
\prod_{(\alpha,a) \in \Gamma \times \mathbb{Z}_2} \exp \left( -\sum_{r=1}^{\infty} \frac{1}{r} \Psi^r \left( |L_{(\alpha,a)}| \right) E^{(\alpha,a)} \right) = 1 - H(L). \tag{4.13}
\]

We call this the Grothendieck Lie module denominator identity of \(L\). Set \(\overline{\mathcal{D}} = \omega \circ \text{Log}\).

Theorem 4.15. \(\overline{\mathcal{D}}\) is the unique logarithmic function on \(\Gamma_K(G)[[\Gamma \times \mathbb{Z}_2]]\) such that
\[
\overline{\mathcal{D}}(\overline{L}) = \overline{\mathcal{D}(\overline{H(L)})} \tag{4.14}
\]
for all every \((\Gamma \times \mathbb{Z}_2)\)-graded Lie superalgebra \(L\). Moreover \(\omega \circ \text{Log} = \text{Log} \circ \omega\).

Proof. This can be exactly in the same way as in the proof of Theorem 4.7. \(\square\)

We call \(\overline{\mathcal{D}}\) the Grothendieck Lie module function of \(L\). Comparing the coefficients of both sides of Eq. (4.14) provides the following closed formula analogous to Eq. (4.16).

Corollary 4.16. For every \((\alpha,a) \in (\Gamma \times \mathbb{Z}_2)\) we have
\[
|\overline{L_{(\alpha,a)}}| = \sum_{d \geq 0} \frac{\mu(d)}{d} \Psi^d \left( W(\tau,b) \right),
\]
where
\[
W(\alpha,a) = \sum_{s \in \Gamma(\alpha,a)} \frac{(|s| - 1)!}{s!} \prod_{\beta \neq 0} \overline{H(\overline{L}_{(\beta,b)})}. \tag{4.17}
\]

Corollary 4.17. Let \(V = \bigoplus_{(\alpha,a) \in (\Gamma \times \mathbb{Z}_2)} V_{(\alpha,a)}\) be a \((\Gamma \times \mathbb{Z}_2)\)-graded \(K\)-vector space with finite dimensional homogeneous subspaces, and let \(\mathfrak{L}(V)\) be the free Lie superalgebra generated by \(V\). Suppose \(G\) acts on \(V\) preserving \((\Gamma \times \mathbb{Z}_2)\)-gradation. Then, \(\overline{\mathcal{D}}\) is the unique logarithmic function on \(\Gamma_K(G)[[\Gamma \times \mathbb{Z}_2]]\) such that
\[
\overline{\mathcal{D}(L)} = \overline{\mathcal{D}(V)}
\]
for every \(V\).
5. APPLICATIONS

5.1. New interpretation of the symmetric power map, \( s_t \), using plethysm. Since Atiyah and Tall suggested the "splitting principle" and "verification principle", the theory of special \( \lambda \) -rings has been developed with a close connection with the theory of symmetric functions (2). In particular, some identities and properties of symmetric functions seem to be more natural in our framework. In this section, we reformulate plethystic equations in the context of symmetric functions as relations among the ring of Witt vectors, the necklace ring, and the Grothendieck ring of formal power series.

Example 5.1. Let \( R \) be the ring of symmetric functions in infinitely many variables \( x_1, x_2, \cdots \). We also let \( e_n \) be the \( n \)-th elementary symmetric functions defined using the product \( \prod_i (1 + x_i t) \), \( h_n \) be the \( n \)-th complete symmetric functions defined using the product \( \prod_i 1 - x_i t \), and \( p_n \) be the \( n \)-th power sum symmetric function. Then, one can check that \( R \) has a \( \lambda \)-ring structure if we set \( \lambda^n(e_1) = e_n \), or equivalently \( \Psi^n(p_1) = p_n \) for all \( n \geq 1 \). Observe that over the ring \( R \otimes \mathbb{Q} \), the identity

\[
\sum_{n=0}^{\infty} h_n t^n = \prod_{n=1}^{\infty} \frac{1}{1 - q_n t^n} = \exp \left( \sum_{n=1}^{\infty} \frac{p_n}{n} t^n \right)
\]

can be rewritten as

\[
(q_1, q_2, \cdots) \xrightarrow{\Phi} (p_1, p_2, \cdots),
\]

\[
(h_1, h_2, \cdots) \xrightarrow{\log} (p_1, p_2, \cdots).
\]

The symmetric function \( q_n(n \geq 1) \) enjoy the peculiar property such that \(-q_n\), \( n \geq 2 \), are Schur-positive (see [29]). Similarly, if we define \((t_n)\) by

\[
\sum_{n=0}^{\infty} c_n t^n = \prod_{n=1}^{\infty} \frac{1}{1 - t_n t^n} = \exp \left( \sum_{n=1}^{\infty} \frac{(-1)^n p_n}{n} t^n \right)
\]

\((-t_n)\), \( n \geq 2 \) also turn out to be Schur-positive. Indeed, this follows from the observation

\[
t_n = \begin{cases} q_n & \text{if } n \text{ is odd}, \\ \text{sgn} S_n \otimes q_n & \text{if } n \text{ is even}. \end{cases}
\]

Let the base ring be

\[ \mathbb{Q}[[\Psi^m(d_n) : m, n \geq 1]]. \]

Given the following commutative diagram

\[
\begin{array}{ccc}
(c_n)_n & \xrightarrow{\tau_n} & (b_n)_n \\
\downarrow \Phi & & \downarrow \phi \\
(d_n)_n & \xrightarrow{id} & (d_n)_n \\
\end{array}
\]

\[
\begin{array}{ccc}
\xrightarrow{\text{id}} & \xrightarrow{s_t} & \sum_{n \geq 0} a_n t^n \\
\xrightarrow{\text{identification}} & & \sum_{n \geq 0} d_{n+1} t^n,
\end{array}
\]

we let

\[
A = \sum_{n \geq 1} a_n, \quad B = \sum_{n \geq 1} b_n, \quad C = \sum_{n \geq 1} c_n.
\]
Set the degree of \( d_n \) to be \( n \) for all \( n \geq 1 \). Then, \( A \) and \( C \) can be viewed as the elements in
\[
\mathbb{Q}[[d_j : j \geq 1]]
\]
since \( a_n \) and \( c_n \) can be expressed as homogeneous polynomials of degree \( n \) with \( \mathbb{Q} \)-coefficients in variables \( d_j, 1 \leq j \leq n \). And
\[
B = \sum_{n=1}^{\infty} \frac{1}{n} \sum_{d|n} \mu(d) \Psi^d(d_{\frac{n}{d}})
\]
since
\[
b_n = \frac{1}{n} \sum_{d|n} \mu(d) \Psi^d(d_{\frac{n}{d}}).
\]
In order to generalize the notion of plethysm for elements
\[
\xi(d_j : j \geq 1) \in \mathbb{Q}[[d_j : j \geq 1]]
\]
and
\[
\zeta(\Psi^i(d_j) : i, j \geq 1) \in \mathbb{Q}[[\Psi^m(d_n) : m, n \geq 1]]
\]
let us define \( \xi \oplus \zeta \) by
\[
\xi \oplus \zeta := \xi(\zeta_1, \zeta_2, \cdots), \tag{5.1}
\]
where
\[
\zeta_k = \zeta(\Psi^k(d_j) : i, j \geq 1).
\]
With this notation we can obtain another characterization of \( \tilde{s}_t \).

**Theorem 5.2.** (a) With the above notation, we have
\[
A \oplus B = A.
\]
(b) Set the degree of \( \Psi^n(d_j) \) to be \( nj \) for all \( n, j \geq 1 \). Let \( Y = \sum_{n=1}^{\infty} y_n \), where \( y_n \) is a homogeneous polynomial in \( \Psi^d(d_{\frac{n}{d}}) \) with \( d, n \geq 1 \), and \( d \mid n \). Then, \( A \oplus Y = A \) if and only if \( Y = B \). Equivalently, \( A \oplus Y = A \) if and only if \( \tilde{s}_t(y_1, y_2, \cdots) = 1 + \sum_n a_n t^n \).

**Proof.** (a) By the definition \[5.1\] we obtain
\[
A \oplus B = \exp \left( \sum_{n=1}^{\infty} \frac{d_n}{n} \right) \oplus B
= \exp \left( \sum_{n=1}^{\infty} \frac{B_n}{n} \right).
\]
Here,
\[
B_n = \sum_{i=1}^{\infty} \frac{1}{i} \sum_{d|i} \mu(d) \Psi^{nd}(d_{\frac{n}{d}}).
\]
On the other hand,
\[
\lim_{n \to \infty} \frac{B_n}{n} = \sum_{n=1}^{\infty} \sum_{d|n} \frac{1}{n} \frac{\mu(d) \Psi^d(d^{\frac{a}{m}})}{m^{\frac{a}{m}}}
\]
\[
= \sum_{s,t} \Psi^s(d_t) \sum_{d|s} \frac{1}{s^t} \mu(d)
\] (letting \(dn = s, \frac{m}{d} = t\))
\[
= \sum_{t=1}^{\infty} \frac{d_t}{t}
\]
Since
\[
A = \exp \left( \sum_{n=1}^{\infty} \frac{d_n}{n} \right),
\]
the desired result follows.

(b) From the proof of (a) it follows that
\[
\sum_{d|n} \frac{1}{d} \Psi^d(y_n) = \frac{d_n}{n}.
\]
Applying Möbius inversion formula implies that \(y_n = b_n\) for all \(n \geq 1\).

Consider the case \(d_n = p_n\) for all \(n \geq 1\). In this case, \(A \circ B\) coincides with the usual plethysm of \(A\) and \(B\), denoted by \(A \circ B\). Let \(H = \sum_{n \geq 0} h_n\) and \(L = \sum_{n \geq 1} l_n\), where
\[
l_n = \frac{1}{n} \sum_{d|n} \mu(d) p_d^{\frac{a}{m}}.
\]
Viewing \(H\) and \(L\) as functions in variables \(p_n\)'s, \(n \geq 1\), Theorem 5.2 (a) implies that
\[
H \circ L = \frac{1}{1 - p_1}.
\]
Indeed, Joyal proved this identity using PBW theorem. On the contrary, Reutenauer showed it using the plethysm (refer to [27]).

Let
\[
L(k) = \sum_{n=1}^{\infty} \frac{1}{n} \sum_{d|n} c(k, d) \Psi^d(d^{\frac{a}{m}}),
\]
where \(c(k, d)\) is the sum of the \(k\)-th powers of the primitive \(d\)-roots of unity. Then, by the same argument as in [24], we have
\[
A \circ L(k) = \prod_{l|k} \frac{1}{d_l}.
\]
Imitating the proof of Theorem 5.2 we can provide some interesting plethystic equations. Let
\[
\tilde{e}_n = \frac{1}{n} \sum_{d|n} (-1)^{\frac{a}{d}} \mu(d) p_d^{\frac{a}{m}}.
\]
Regarding \(H, L, E = \sum_{n \geq 0} e_n, \) and \(\tilde{L} = \sum_{n \geq 1} \tilde{e}_n\) as functions in \(p_n\)'s, \(n \geq 1\), we can provide the following plethystic equations in the same way as Reutenauer did.
Corollary 5.3.

(a) \( H \circ \tilde{L} = \frac{1}{1 + p_1} \)
(b) \( E \circ L = \frac{1 - p_1}{1 - p_2} \)
(c) \( E \circ \tilde{L} = \frac{1 + p_1}{1 + p_2} \)

Proof. Since the identities (a) throughout (c) can be proven in the same way, we will prove only (c).

\[
E \circ L = \exp \left( \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \frac{1}{m} \mu(d) \right) \circ \tilde{L} = \exp \left( \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \tilde{L}_n \right).
\]

Thus, by definition of \( \tilde{l}_n \) we have

\[
\sum_{n=1}^{\infty} (-1)^n \frac{\tilde{L}_n}{n} = \sum_{n=1}^{\infty} \sum_{m \geq 1} (-1)^n \frac{1}{n} \frac{1}{m} \mu(d) \frac{m}{dn}
\]

\[
= \sum_{s, t} (-1)^t \frac{p_1^t}{s} \sum_{d \mid s} \frac{1}{s} \frac{1}{d} \mu(d) \quad \text{(letting } dn = s, \frac{m}{d} = t)\]

\[
= - \sum_{t=1}^{\infty} \frac{p_1^t}{t} + \sum_{t=1}^{\infty} \frac{p_2^t}{t} = \log \left( \frac{1 + p_1}{1 + p_2} \right).
\]

The third equality follows from

\[
\sum_{d \mid n} \mu(n)(-1)^\frac{m}{d} = \begin{cases} 
-1 & \text{if } n = 1 \\
2 & \text{if } n = 2 \\
0 & \text{otherwise.}
\end{cases}
\]

Therefore, we have the desired result. \( \square \)

5.2. Generators of supersymmetric functions. In this section, we are going to show that the diagram \[3.24\], being applied to supersymmetric polynomials, provides several generating sets of the set of supersymmetric polynomials. We recall the definition of supersymmetric polynomials briefly (see \[32\]). Let \( K \) be a field of characteristic 0, and let \( x_1, \ldots, x_a, y_1, \ldots, y_b, t \) be independent indeterminates. A polynomial \( p \) in

\[
K[X, Y] := K[x_1, \ldots, x_a, y_1, \ldots, y_b]
\]

is said to be supersymmetric if

(1) \( p \) is invariant under permutations of \( x_1, \ldots, x_a \),
(2) \( p \) is invariant under permutations of \( y_1, \ldots, y_b \),
(3) when the substitution \( x_1 = t, y_1 = t \) is made in \( t \), the resulting polynomial is independent of \( t \).

Let \( T(a, b) \) denote the set of supersymmetric polynomials in \( K[X, Y] \). In \[32\] Stembridge provided two generating sets of \( T(a, b) \) such as

\[
\{ \sigma_{a,b}^{(n)}(x; y) : n \geq 1 \}
\]

and

\[
\{ \tau_{a,b}^{(n)}(x; y) : n \geq 1 \}.
\]
Here, \( \sigma_{a,b}^{(n)}(x;y) := (x_1^n \cdots + x_n^n) - (y_1^n \cdots + y_b^n) \) and \( \tau_{a,b}^{(n)}(x;y) \) is defined by the equation
\[
\sum_{n=0}^{\infty} \tau_{a,b}^{(n)}(x;y)t^n := \prod_{i,j} \frac{1-x_i t}{1-y_j t}.
\]

Now, let us consider the element \( r-s \in R \) where \( r \) is \( a \)-dimensional and \( s \) \( b \)-dimensional. After decomposing \( r,s \) into the sum of 1-dimensional elements we can write \( r = x_1 + \cdots + x_a \) and \( s = y_1 + \cdots + y_b \). Considering the following diagram
\[
\begin{array}{c}
(q_{a,b})_n \\ \Phi \downarrow \Phi \downarrow \\
(\Psi^n(r))_n \rightarrow (\Psi^n(r))_n \rightarrow \end{array}
\]
we obtain supersymmetric polynomials \( q_{a,b}^{(n)}(x;y) \) and \( h_{a,b}^{(n)}(x;y) \) satisfying
\[
\sum_{n=0}^{\infty} h_{a,b}^{(n)}(x;y)t^n = \prod_{n=1}^{\infty} \frac{1}{1-q_{a,b}^{(n)}(x;y)t^n} = \exp \left( \sum_{n=1}^{\infty} \frac{\sigma_{a,b}^{(n)}(x;y)}{n} t^n \right).
\]
(5.2)

Note that \( h_{a,b}^{(n)}(x;y) \) is defined by the equation
\[
\sum_{n=0}^{\infty} h_{a,b}^{(n)}(x;y)t^n := \prod_{i,j} \frac{1-y_j t}{1-x_i t}.
\]

Similarly, we can consider supersymmetric polynomials \( t_{a,b}^{(n)}(x;y) \) and \( e_{a,b}^{(n)}(x;y) \) such that
\[
\sum_{n=0}^{\infty} e_{a,b}^{(n)}(x;y)t^n = \prod_{n=1}^{\infty} \frac{1}{1-t_{a,b}^{(n)}(x;y)t^n} = \exp \left( \sum_{n=1}^{\infty} \frac{(-1)^n \sigma_{a,b}^{(n)}(x;y)}{n} t^n \right),
\]
(5.3)

where \( e_{a,b}^{(n)}(x;y) \) is defined by the equation
\[
\sum_{n=0}^{\infty} e_{a,b}^{(n)}(x;y)t^n := \prod_{i,j} \frac{1+x_i t}{1+y_j t}.
\]

Finally, we let
\[
I_{a,b}^{(n)}(x;y) = \frac{1}{n} \sum_{d|n} \mu(d) \sigma_{a,b}^{(d)}(x;y) \frac{x_i t}{1+y_j t},
\]
(5.4)

In view of Eq. 5.22 through Eq. 5.31 we can obtain several generating sets.

**Proposition 5.4.** The sets of supersymmetric polynomials \( \{ h_{a,b}^{(n)}(x;y) : n \geq 1 \} \), \( \{ e_{a,b}^{(n)}(x;y) : n \geq 1 \} \), \( \{ q_{a,b}^{(n)}(x;y) : n \geq 1 \} \), \( \{ t_{a,b}^{(n)}(x;y) : n \geq 1 \} \), and \( \{ I_{a,b}^{(n)}(x;y) : n \geq 1 \} \) generate the algebra \( T(a,b) \) respectively.
5.3. Recursive formulas for $|\Sigma_{(\alpha,a)}|$ and $[\Sigma_{(\alpha,a)}]$. In Corollary 18 and Corollary 19, we provided closed formulas which enable us to compute the homogeneous components $|\Sigma_{(\alpha,a)}|$ and $[\Sigma_{(\alpha,a)}]$ for a graded Lie superalgebra $\Sigma$ if we know information on the homology $H(\Sigma)$. On the contrary, we focus on recursive formulas associated with (Grothendieck)Lie module denominator identities in this section. For this end, we introduce some formal $R$-linear operators on $R[[\Gamma \times \mathbb{Z}_2]]$, where $R$ is a commutative algebra with unity over $\mathbb{C}$.

Let $\hat{\Gamma}_C = \mathbb{C} \otimes_\mathbb{Z} \hat{\Gamma}$ be the complexification of $\hat{\Gamma}$. Choose a non-degenerate symmetric bilinear form $(\cdot \mid \cdot)$ on $\hat{\Gamma}_C$ and fix a pair of dual bases $\{u_i\}$ and $\{u^i\}$ of $\hat{\Gamma}_C$. We define a partial ordering $\geq$ on $\hat{\Gamma}_C$ by $\lambda \geq \mu$ if and only if $\lambda - \mu \in \Gamma$ or $\lambda = \mu$. We will denote by $\lambda > \mu$ if $\lambda \geq \mu$ and $\lambda \neq \mu$. In particular, if $\zeta(\lambda, a) \in \mathbb{C}$, it means that $\zeta(\lambda, a) \cdot 1$.

Definition 5.5. (10)
(a) The partial differential operators $\partial_i$ and $\partial^i$ are defined by
\[
\partial_i(E^{(\lambda,a)}) = (\lambda|u_i)E^{(\lambda,a)}, \quad \partial^i(E^{(\lambda,a)}) = (\lambda|u^i)E^{(\lambda,a)}.
\]
(b) For an element $\rho \in \hat{\Gamma}_C$, we define the $\rho$-directional derivative $D_\rho$ by
\[
D_\rho(E^{(\lambda,a)}) = \sum (\rho|u_i)\partial^i(E^{(\lambda,a)}) = (\rho|\lambda)E^{(\lambda,a)}.
\]
(c) The Laplacian $\Delta$ is defined to be
\[
\Delta(E^{(\lambda,a)}) = \sum \partial^i\partial_i(E^{(\lambda,a)}) = (\lambda|\lambda)E^{(\lambda,a)}.
\]

Recall the definition of $\eta$ given in Section 3.1. Define
\[
\eta^*([\Sigma]) = \sum_{(\gamma,a) \in \Gamma \times \mathbb{Z}_2} \eta^*(\gamma, a)E^{(\gamma,a)}
\]
to be the formal power series whose coefficients are given by
\[
\eta^*(\gamma, a) = (\gamma|\gamma)\eta(\gamma, a) - \sum_{(\gamma,a) = (\gamma',a') + (\gamma'',a'')} (\gamma'|\gamma'')\eta(\gamma', a')\eta(\gamma'', a '').
\]
Similarly, we write
\[
\eta([\Sigma]) = \sum_{(\gamma,a) \in \Gamma \times \mathbb{Z}_2} \eta(\gamma, a)E^{(\gamma,a)}
\]
and
\[
\eta^*([\Sigma]) = \sum_{(\gamma,a) \in \Gamma \times \mathbb{Z}_2} \eta^*(\gamma, a)E^{(\gamma,a)}.
\]

With this notation,

**Proposition 5.6.** For $(G, K)$ such that $\text{char}K \neq 2$ and $\Gamma_K(G)$ is a special $\lambda$-ring, we have
(a) $D_\rho(1 - [H(\Sigma)]) = -D_\rho(\eta([\Sigma]))(1 - [H(\Sigma)])$.
(b) $\Delta(1 - [H(\Sigma)]) = -\eta^*([\Sigma])(1 - [H(\Sigma)])$.

**Proof.** By applying the following formal differential identities to the Lie module denominator identity (5.5)
\[
D_\rho(\log D) = \frac{D_\rho(D)}{D},
\]
\[
\frac{\Delta(D)}{D} = \sum_i \partial_i \left( \frac{\partial^i D}{D} \right) + \sum_i \left( \frac{\partial_i D}{D} \right) \left( \frac{\partial^i D}{D} \right)
\]
(5.5)
and then comparing the coefficients of both sides, we can complete the proof. □

Recall that under the condition that $G$ is a finite group and $K$ a field with $\text{char}K \neq 2$, we obtained the Grothendieck Lie module denominator identity (1.13) for every $(\Gamma \times \mathbb{Z}_2)$-graded Lie superalgebra $\mathfrak{L}$. By applying (1.13) to Eq. (4.13), we can derive the following relations.

**Proposition 5.7.** Let $G$ be a finite group and $K$ be any field with $\text{char}K \neq 2$. Then, we have

(a) $D_\rho(1 - \frac{H(\mathfrak{L})}{\mathfrak{L}}) = -D_\rho(\eta(\mathfrak{L}))(1 - \frac{1}{\mathfrak{L}})$.

(b) $\Delta(1 - \frac{H(\mathfrak{L})}{\mathfrak{L}}) = -\eta(\mathfrak{L})(1 - \frac{1}{\mathfrak{L}})$.

**Remark 5.8.** If $g$ is a Borcherds superalgebra, the operator $(\Delta + 2D_\rho)$ plays an essential role. In fact, in this case, we have $(\Delta + 2D_\rho)(1 - |\mathfrak{g}_-|) = 0$, where $\rho$ is a Weyl vector. Exploiting this fact, we can obtain Peterson’s and Freudenthal’s formulas for the $[\mathfrak{g}_a]$.

Comparing the coefficient of $E^{(\alpha,a)}$ in Proposition 5.6 (a) and (b), we have

**Corollary 5.9.**

(a) $\rho(\alpha)\eta(\alpha,a) - \sum_{\beta<\alpha}(\rho(\beta)\eta(\beta,b)|H(\mathfrak{L})_{(\alpha-\beta,a-b)}|) = (\rho(\alpha)|H(\mathfrak{L})_{(\alpha,a)}|).

(b) $\eta^*(\alpha,a) - \sum_{\beta<\alpha}\eta^*(\beta,b)|H(\mathfrak{L})_{(\alpha-\beta,a-b)}|) = (\alpha|H(\mathfrak{L})_{(\alpha,a)}|).

Similarly, it follows from Proposition 5.7 (a) and (b) that

**Corollary 5.10.**

(a) $\rho(\alpha)\overline{\eta}(\alpha,a) - \sum_{\beta<\alpha}(\rho(\beta)\overline{\eta}(\beta,b)|\overline{H(\mathfrak{L})}_{(\alpha-\beta,a-b)}|) = (\rho(\alpha)|\overline{H(\mathfrak{L})}_{(\alpha,a)}|).

(b) $\overline{\eta}^*(\alpha,a) - \sum_{\beta<\alpha}\overline{\eta}^*(\beta,b)|\overline{H(\mathfrak{L})}_{(\alpha-\beta,a-b)}|) = (\alpha|\overline{H(\mathfrak{L})}_{(\alpha,a)}|).

For example, let $M$ be the Monster simple group and $\mathfrak{L} = \bigoplus_{(m,n)}\mathfrak{L}_{(m,n)}$ be the Monster Lie algebra (see [4]). Recall that $\mathfrak{L}_{(m,n)} \simeq V_{m,n}$ for $(m,n) \neq 0$, where $V_\mathbb{Z} = \bigoplus_{n \geq 1}V_n$ is the Moonshine Module constructed by Frenkel et al. Here, we use $\Gamma$-grading, not the $\Gamma \times \mathbb{Z}_2$-grading. Applying Corollary 5.9 (a), we can recover the identity

$$[\mathfrak{L}_{(m,n)}] = - \sum_{k \mid (m,n), \ k \geq 1} \frac{1}{k} \Psi^k \left( [\mathfrak{L}_{(\frac{m}{k},\frac{n}{k})}] \right) + \sum_{l \leq k < m, l \leq l < n} \frac{k}{m} c(k,l) [H(\mathfrak{L})_{(m-k,n-l)}] + [H(\mathfrak{L})_{(m,n)}],$$

where $c(m,n) = \sum_{k \mid (m,n), \ k \geq 1} \frac{1}{k} \Psi^k \left( [\mathfrak{L}_{(\frac{m}{k},\frac{n}{k})}] \right)$.

### 5.4. Replicable functions from the viewpoint of logarithmic functions.

The concept of replicable functions was first introduced by Norton as a generalization of the replication formulae. In their famous Moonshine conjecture Conway and Norton suggested replication formulae as an important family of character identities that are satisfied by the Thompson series of the Monster simple group (see...
Later, Borcherds has proven this conjecture completely by showing that the Thompson series are indeed replicable functions.

Let \( F(q) = q^{-1} + \sum_{n \geq 1} f(n)q^n \) be a normalized q-series. By setting \( q = e^{2\pi iz} \) with \( \text{Im} z > 0 \), we often write \( F = F(z) \) so that the notation can be consistent with the Fourier expansion of modular functions. Note that for each \( m \geq 1 \), there exists a unique polynomial \( X_m(t) \in \mathbb{C}[t] \) such that

\[
X_m(F) \equiv \frac{1}{m} q^{-m} \mod q\mathbb{C}[[q]].
\]

We write

\[
X_m(F) = \frac{1}{m} q^{-m} + \sum_{n \geq 1} H_{m,n} q^n.
\]

In [22], Norton has shown that the coefficients \( H_{m,n} \) satisfy the identity

\[
\sum_{m,n \geq 1} H_{m,n} p^m q^n = -\log \left( 1 - pq \sum_{i=1}^{\infty} f(i) \frac{p^i - q^i}{p - q} \right).
\]

Indeed, one can show that Eq. (5.6) is equivalent to the product identity

\[
p^{-1} \prod_{m=1}^{\infty} \exp \left( -X_m(F(q))p^m \right) = F(p) - F(q).
\]

We recall the definition of replicable functions (1, 22).

**Definition 5.11.** A normalized q-series \( F(q) = q^{-1} + \sum_{n \geq 1} f(n)q^n \) is said to be replicable if \( H_{a,b} = H_{c,d} \) whenever \( ab = cd \) and \( (a,b) = (c,d) \).

The replicable functions can be characterized as follows.

**Proposition 5.12.** (1, 22) A normalized q-series \( F(q) = q^{-1} + \sum_{n \geq 1} f(n)q^n \) is replicable if and only if for all \( m > 0 \) and \( a|m \), there exist normalized q-series \( F^{(a)}(q) = q^{-1} + \sum_{n=1}^{\infty} f^{(a)}(n)q^n \) such that

\[
F^{(1)}(1), \quad X_m(F) = \frac{1}{m} \sum_{a \mid m \atop 0 < b < d} F^{(a)} \left( \frac{az + b}{d} \right).
\]

where \( q = e^{2\pi iz}, \text{Im} z > 0 \).

The normalized q-series \( F^{(a)} \) is called the \( a \)-th replicate of \( F \). If \( F^{(a)} \) are also replicable for all \( a \geq 1 \), then \( F \) is said to be completely replicable.

Let \( \Gamma = \mathbb{Z}_{>0} \times \mathbb{Z}_{>0} \) and consider the formal power series ring

\[
\mathbb{C}[[\Gamma]] = \{ \sum_{m,n \geq 0 \atop (m,n) \neq (0,0)} a(m,n)p^m q^n \mid a(m,n) \in \mathbb{C} \}.
\]

Set

\[
T(m,n) = \{ s = (s_{ij})_{i,j \geq 1} \mid s_{ij} \in \mathbb{Z}_{\geq 0}, \sum s_{ij}(i,j) = (m,n) \},
\]

and define

\[
W(m,n) := \sum_{s \in T(m,n)} \frac{(|s| - 1)!}{s!} \prod_{i,j} f(i + j - 1)^{s_{ij}}.
\]
Then, we have

$$\text{Log} \left( \sum_{i,j \geq 1} f(i + j - 1)p^i q^j \right) = \sum_{m,n \geq 1} W(m,n)p^m q^n. \quad (5.8)$$

**Proposition 5.13.** A normalized q-series $F$ replicable if and only if $W(a,b) = W(c,d)$ whenever $ab = cd$ and $(a,b) = (c,d)$.

**Proof.** From Eq. (5.7) one can derive that

$$\prod_{m,n=1}^{\infty} \exp \left( -H_{m,n}p^m q^n \right) = 1 - \sum_{m,n=1}^{\infty} f(m+n-1)p^m q^n. \quad (5.9)$$

If we take the logarithm on both sides of Eq. (5.9), then it follows from (5.8) that $H_{m,n} = W_{m,n}$. This completes the proof. □

From now on, we discuss how to obtain completely-replicable functions in a unified way.

First, we consider monstrous functions appearing in Moonshine conjecture. The Monster Lie algebra $L$ is a $I_{1,1}$-graded representation of the Monster simple group $M$ acting by automorphisms of $L$ for $(m,n) \neq (0,0)$ as $M$-modules. In particular,

$$\text{tr}(g|L_{(m,n)}) = \text{tr}(g|V_{mn}) := c_g(mn)$$

where $V$ is the Moonshine module $V^\sharp = \bigoplus_{n \geq -1} V_n$ constructed by Frenkel et al. and $c_g(n)$ is the coefficient of $q^n$ of the elliptic modular function $J(q) = j(q) - 744$.

From the denominator identity of $L$ in [4], we obtain

$$\prod_{m,n \geq 1} \exp \left( -\sum_{d|(m,n)} 1/d \Psi_d([L_{mn}])p^m q^n \right) = 1 - \sum_{i,j} [V_{i+j-1}]p^i q^j.$$ 

Let

$$\Gamma(M)[[\Gamma]] = \{ \sum_{m,n \geq 0, (m,n) \neq (0,0)} a(m,n)p^m q^n | a(m,n) \in \Gamma(M) \}.$$

Define $D : \Gamma(M)[[\Gamma]] \to \Gamma(G)[[\Gamma]]$ by $D = \omega \circ \text{Log}$, where $\omega = \sum_{k \geq 1} \frac{\mu(k)}{k} \Theta_k \circ \Psi_k$.

Then, we have

**Proposition 5.14.** $D$ is a logarithmic function on $\Gamma(M)[[\Gamma]]$ such that

$$\sum_{m,n \geq 1} [V_{mn}]p^m q^n = D \left( \sum_{m,n \geq 1} [V_{m+n-1}]p^m q^n \right). \quad (5.10)$$

Taking ch$_g$ on both sides the above identity (5.10), we have

$$\sum_{m,n \geq 1} \text{tr}(g|V_{mn})p^m q^n = D \left( \sum_{m,n \geq 1} \text{tr}(g|V_{m+n-1})p^m q^n \right), \quad (5.11)$$

which says that Thompson series $T_g$ are replicable functions.

As for replicable functions, in particular non-monstrous functions, the above method is no more effective. In these cases we need a different approach.
For every positive integer $r$, consider the formal power series in $q$
\[ h^{(r)} = q^{-1} + \sum_{m=1}^{\infty} x^{(r)}_m q^m \]
whose coefficients are the indeterminates $x^{(r)}_m$. Let
\[ \Xi = \mathbb{C}[\cdots, x^{(r)}_m, \cdots | m, r \geq 1]. \]
For a fixed $r$ and an arbitrary $m \geq 1$ consider the family of equations
\[ X_m(h^{(r)}(q)) - \frac{1}{m} \sum_{d \leq m, 0 \leq b < d} h^{(ra)}(\exp(2\pi i b d) q^d) = 0. \] (5.12)
Expand \(5.12\) in a $q$-series and then consider the coefficient of $q^n$ for every $n \geq 1$.

Let $I^{(r)}$ be the ideal in $\Xi$ generated by them. Let $I$ be the ideal in $\Xi$ generated by $\bigcup_{r=1}^{\infty} I^{(r)}$.

Consider the semigroup $\Delta$ of $GL^+(2, \mathbb{Q})$ given by
\[ \Delta = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} | a, b, d \in \mathbb{Z}, a > 0 \right\}. \]
For $A = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in \Delta$, set
\[ e||_A = e, \quad x^{(m)}_m||_A = x^{(rm)}_m, \quad q||_A = \exp(2\pi i b d) q^d \]
for all $e \in \mathbb{C}$, $m, m \geq 1$. We extend the mapping $||_A$ to the $\mathbb{C}$-algebra homomorphism from $\Xi[[q]] \to \Xi[[q]]$. Since the ideal $I$ is stable under $||_A$, it induces a homomorphism from $\Xi/I$ to $\Xi/I$.

**Definition 5.15.** Let $n$ be the positive integer. We define $\Psi^n : \Xi/I \to \Xi/I$ to be $\Psi^n(x) = x||_A$ for all $x \in \Xi/I$ where $A = \begin{pmatrix} n & 0 \\ 0 & n \end{pmatrix}$.

**Proposition 5.16.** $\Xi/I$ is a special $\lambda$-ring.

**Proof.** It is easy to verify that $\Xi/I$ is a special $\Psi$-ring. By Theorem 2.2 we conclude that $\Xi/I$ is a special $\lambda$-ring. \(\square\)

Define $D : \Xi/I[[\Gamma]] \to \Xi/I[[\Gamma]]$ by $D = \omega \circ \log$, where $p = e^{(1, 0)}$, $q = e^{(0, 1)}$ and $\omega = \sum_{k \geq 1} \frac{\mu(k)}{k} \Theta_k \circ \Psi^k$. Then, we have

**Proposition 5.17.** $D$ is a logarithmic function on $\Xi/I[[\Gamma]]$ such that
\[ \sum_{m, n \geq 1} x^{(r)}_{mn} p^m q^n = D \left( \sum_{m, n \geq 1} x^{(r)}_{m+n-1} p^m q^n \right) \] (5.13)
for every positive integer $r$.

**Proof.** By the identity $\text{1.7}$, we have
\[ p^{-1} \prod_{m=1}^{\infty} \exp \left( -X_m(h^{(r)}(q)) p^m \right) = h^{(r)}(p) - h^{(r)}(q) \]
for all \( r \geq 1 \). Substituting
\[
\frac{1}{m} \sum_{\substack{a \leq b \leq d \leq c \leq m}} h^{(r)}(a,b) \left( \exp \left( 2\pi i \frac{b}{d} \right) q^a \right)
\]
for \( X_m(h^{(r)}(q)) \) gives rise to the following product identity
\[
\prod_{m,n=1}^{\infty} \exp \left( -\sum_{k=1}^{\infty} \frac{1}{k} \Psi_k \left( x_{mn}^{(r)} \right) p^{km} q^{kn} \right) = 1 - \sum_{m,n=1}^{\infty} x_{m+n-1}^{(r)} p^{mn} q^n.
\]
Taking the logarithmic function \( \mathcal{D} \) on both sides, we get Eq. (5.13).
\( \square \)

Let \( F(q) = q^{-1} + \sum_{n=1}^{\infty} f(n) q^n \) be a completely replicable function and \( F^{(a)}(q) = q^{-1} + \sum_{n=1}^{\infty} f^{(a)}(n) q^n \) be its \( a \)-th replicates for all \( a \geq 1 \). Then, we obtain a \( \mathbb{C} \)-algebra homomorphism, \( \psi_F : \Xi/I \rightarrow \mathbb{C} \), such that \( \psi_F(x^{(a)}_n) = f^{(a)}(n) \). Conversely, if we have a \( \mathbb{C} \)-algebra homomorphism, \( \psi : \Xi/I \rightarrow \mathbb{C} \), we get a completely-replicable function \( F_{\psi}^{(1)} \) and its replicates by setting the q-series \( F_{\psi}^{(r)}(q) := q^{-1} + \sum_{n \geq 1} \psi(x^{(r)}_n) q^n \). Define \( \Phi \) by the function from the set completely replicable functions to the set of \( \mathbb{C} \)-algebra homomorphisms from \( \Xi/I \) to \( \mathbb{C} \) sending \( F \) to \( \psi_F \), and \( \Upsilon \) by the function from the set of \( \mathbb{C} \)-algebra homomorphisms from \( \Xi/I \) to \( \mathbb{C} \) to the set completely replicable functions sending \( \psi \) to \( F_{\psi} \). Then, we have

**Proposition 5.18.** There is a natural one-to-one correspondence between the set of completely replicable functions and the set of \( \mathbb{C} \)-algebra homomorphisms from \( \Xi/I \) to \( \mathbb{C} \).

**Proof.** It suffices to show that \( \Phi \circ \Upsilon = \text{id} \) and \( \Upsilon \circ \Phi = \text{id} \), equivalently
\[
F_{\psi_F} = F \text{ and } \psi_{F_{\psi}} = F;
\]
which follows from the definition of \( F_{\psi} \) and \( \psi_F \) immediately. \( \square \)

In the above proposition, if we replace \( I \) by \( I^{(1)} \), then we have

**Proposition 5.19.** There is a natural one-to-one correspondence between the set of replicable functions and the set of \( \mathbb{C} \)-algebra homomorphisms from \( \Xi/I^{(1)} \) to \( \mathbb{C} \).

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**References**

[1] D. Alexander, C. Cummings, J. McKay, C. Simons, Completely replicable functions, in “Groups, Combinatorics, and Geometry”, Durham Symposium, 1990”, London Math. Soc. Lecture Note Ser., Vol. 165, 87-98, Cambridge Univ. Press, Cambridge, 1992.
[2] M. F. Atiyah, D. O. Tall, Group representations, \( \lambda \)-rings and the \( J \)-homomorphism, Topology 8 (1969), 253-297.
[3] D. J. Benson, Representation and cohomology I, Cambridge Univ. Press, Cambridge, 1995.
[4] R. E. Borcherds, Monstrous moonshine and monstrous Lie superalgebras, Invent. Math. 109 (1992), 405-444.
[5] N. Bourbaki, Lie groups and Lie algebras, Part 1 : Ch. I-III, Hermann, Paris, 1975.
[6] M. Brun, Witt vectors and Tambara functors, Adv. Math. 193 (2005), 233-256.
[7] R. M. Bryant, Free Lie algebras and formal power series, J. Algebra. 253(1) (2002), 167-188.
[8] H. Cartan, S. Eilenberg, Homological algebra, Princeton Mathematics Series, Princeton University, 1956.
A. Dress, C. Siebeneicher, The Burnside ring of profinite groups and the Witt vectors construction, Adv. Math. 70 (1988), 87-132.

A. Dress, C. Siebeneicher, The Burnside ring of the infinite cyclic group and its relation to the necklace algebra, \( \lambda \)-ring and the Universal ring of the Witt vectors, Adv. Math. 78 (1989), 1-41.

D. B. Fuks, Cohomology of infinite dimensional Lie algebras, Consultant Bureau, New York, 1986.

J. J. Graham, Generalized Witt vectors, Adv. Math. 99 (1993), 248-263.

M. Hazewinkel, Formal groups and applications, Academic Press, New York, 1978.

S. -J. Kang, C. H. Kim, J. K. Koo, Y. -T. Oh, Graded Lie superalgebras and super-replicable functions, J. Algebra. 285 (2005), 531-573.

S. -J. Kang, J. -H. Kwon, Graded Lie superalgebras, supertrace formula, and orbit Lie superalgebra, Proc. London Math. Soc. 81(3) (2000), 675-724.

S. -J. Kang, J. -H. Kwon, Y. -T. Oh, Peterson-type Dimension formulas for graded Lie superalgebras, Nagoya Math. J. 163 (2001), 107-144.

D. Knutson, \( \lambda \)-rings and the representation theory of the symmetric group, Lecture notes in Math. 308, Springer-Verlag, 1973.

C. Lenart, Formal group-theoretic generalization of the necklace algebra, including a \( q \)-deformation, J. Algebra. 199 (1998), 703-732.

I. G. Macdonald, Symmetric functions and Hall polynomials, Second Edition, Oxford Univ. Press, 1995.

Y. Martin, On modular invariance of completely replicable functions, in “Moonshine, the Monster, and Related Topics”, Contemp. Math. 193 (1996), 263-286.

N. Metropolis, G.-C. Rota, Witt vectors and the algebra of necklaces, Adv. Math. 50 (1983), 95-125.

S. P. Norton, More on moonshine, in “Computational Group theory”, Academic Press, 1984, 185-193.

Y. -T. Oh, \( R \)-analogue of the Burnside ring of profinite groups and free Lie algebras, Adv. Math. 190 (2005), 1-46.

Y. -T. Oh, Corrigendum to \( R \)-analogue of the Burnside ring of profinite groups and free Lie algebras: [Adv. Math. 190 (2005) 1-46], Adv. Math. 192 (2005), 226-227.

Y. -T. Oh, Generalized Burnside-Grothendieck ring functor and aperiodic ring functor associated with profinite groups, J. Algebra, in press.

Y. -T. Oh, \( q \)-deformation of Witt-Burnside rings, arXiv:math.RA/0411353.

C. Reutenauer, On symmetric functions related to Witt vectors and the free Lie algebra, Adv. Math. 110 (1995), 234-246.

C. Seubert, The theory of Lie superalgebras, Lecture notes in Math. 716, Springer-Verlag, Berlin, 1979.

M. Ronco, Free Lie algebra and Lambda-ring structure, Bull. Australian Math. Soc. 50 (1995), 373-382.

T. Scharf, J. -V. Thibon, On Witt vectors and symmetric functions, Algebra Colloq. 3(3) (1996), 231-238.

T. Scharf, J. -V. Thibon, On Witt vectors and symmetric functions, Adv. Math. 192 (2005), 226-227.

C. Wilkerson, \( \lambda \)-rings, binomial domains, and vector bundles over \( CP(\infty) \), Comm. Alg. 10 (1982), 311-328.

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