ON A CLASS OF UNIVALENT FUNCTIONS DEFINED BY SÁLÁGEAN INTEGRO-DIFFERENTIAL OPERATOR

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Abstract. In this paper we consider the \( L^n : \mathcal{A} \to \mathcal{A} \), \( L^n f(z) = (1 - \lambda) P^n f(z) + \lambda I^n f(z) \) linear operator, where \( P^n \) is the Sálágean differential operator and \( I^n \) is the Sálágean integral operator. We study several differential subordinations generated by \( L^n \). We introduce a class of holomorphic functions \( L^m_n(\beta) \), and obtain some subordination results.

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1. PRELIMINARIES

Let \( U \) be the unit disk in the complex plane:
\[
U = \{ z \in \mathbb{C} : |z| < 1 \}.
\]

Let \( \mathcal{H}(U) \) be the space of holomorphic functions in \( U \) and let
\[
\mathcal{A}_m = \{ f \in \mathcal{H}(U) : f(z) = z + a_{m+1}z^{m+1} + \cdots, z \in U \}
\]
with \( \mathcal{A}_1 = \mathcal{A} \). For \( a \in \mathbb{C} \) and \( m \in \mathbb{N} \), \( \mathbb{N}_0 = \mathbb{N} \cup \{0\} \), \( \mathbb{N} = \{1,2,\ldots\} \) let
\[
\mathcal{H}[a,m] = \{ f \in \mathcal{H}(U) : f(z) = a + a_mz^m + a_{m+1}z^{m+1} + \cdots, z \in U \}.
\]

Denote by
\[
K = \left\{ f \in \mathcal{A} : \Re \frac{zf''(z)}{f'(z)} + 1 > 0, z \in U \right\}
\]
the class of normalized convex functions in \( U \).

Definition 1 ([5], def. 3.5.1). Let \( f \) and \( g \) be analytic functions in \( U \). We say that the function \( f \) is subordinate to the function \( g \), if there exists a function \( w \), which is analytic in \( U \) and \( w(0) = 0; |w(z)| < 1; z \in U \), such that \( f(z) = g(w(z)) \); \( \forall z \in U \). We denote by \( \prec \) the subordination relation. If \( g \) is univalent, then \( f \prec g \) if and only if \( f(0) = g(0) \) and \( f(U) \subseteq g(U) \).
Let \( \psi : \mathbb{C}^3 \times U \rightarrow \mathbb{C} \) be a function and let \( h \) be univalent in \( U \). If \( p \) is analytic in \( U \) and satisfies the (second-order) differential subordination
\[
(i) \quad \psi \left( p(z), z p'(z), z^2 p''(z); z \right) < h(z), \quad (z \in U)
\]
then \( p \) is called a solution of the differential subordination. The univalent function \( q \) is called a dominant of the solution of the differential subordination, or more simply a dominant, if \( p < q \) for all \( p \) satisfying \((i)\). A dominant \( \tilde{q} \), which satisfies \( \tilde{q} < q \) for all dominants \( q \) of \((i)\) is said to be the best dominant of \((i)\). The best dominant is unique up to a rotation of \( U \). In order to prove the original results we use the following lemmas.

**Lemma 1** (Hallenbeck and Ruscheweyh, [2]). Let \( h \) be a convex function with \( h(0) = a \), and let \( \gamma \in \mathbb{C}^* \) be a complex number with \( \Re \gamma \geq 0 \). If \( p \in \mathcal{H}[a, n] \) and
\[
p(z) + \frac{1}{\gamma} z p'(z) < h(z), \quad z \in U
\]
then
\[
p(z) < q(z) < h(z), \quad z \in U
\]
where
\[
q(z) = \frac{\gamma}{n \gamma^{\gamma/n}} \int_0^z h(t) t^{\gamma/n-1} dt, \quad z \in U.
\]

**Lemma 2** (Miller and Mocanu, [3]). Let \( q \) be a convex function in \( U \) and let \( h(z) = q(z) + n \alpha z q'(z), \quad z \in U \)
where \( \alpha > 0 \) and \( n \) is a positive integer. If
\[
p(z) = q(0) + p_n z^n + p_{n+1} z^{n+1} + \ldots, \quad z \in U
\]
is holomorphic in \( U \) and
\[
p(z) + n \alpha z p'(z) < h(z), \quad z \in U
\]
then
\[
p(z) < q(z)
\]
and this result is sharp.

**Definition 2** ([8]). For \( f \in \mathcal{A}, n \in \mathbb{N}_0 \), the Sălăgean differential operator \( \mathcal{D}^n \) is defined by \( \mathcal{D}^n : \mathcal{A} \rightarrow \mathcal{A}, \)
\[
\mathcal{D}^0 f(z) = f(z),
\]
\[
\mathcal{D}^n f(z) = z \left( \mathcal{D}^{n-1} f(z) \right)', z \in U
\]
Remark 1. If \( f \in \mathcal{A} \) and \( f(z) = z + \sum_{k=2}^{\infty} a_k z^k \), then
\[
\mathcal{D}^n f(z) = z + \sum_{k=2}^{\infty} k^n a_k z^k, z \in U.
\]

Definition 3 ([8]). For \( f \in \mathcal{A}, n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, \mathbb{N} = \{1, 2, \ldots\}\), the operator \( I^n \) is defined by
\[
I^n f(z) = f(z),
\]
\[
I^n f(z) = I(I^{n-1} f(z)), z \in U
\]

Remark 2. If \( f \in \mathcal{A} \) and \( f(z) = z + \sum_{k=2}^{\infty} a_k z^k \), then
\[
I^n f(z) = z + \sum_{k=2}^{\infty} \frac{a_k}{k^n} z^k.
\]
\( z \in U, (n \in \mathbb{N}_0) \) and \( z (I^n f(z))' = I^{n-1} f(z) \).

Definition 4. Let \( \lambda \geq 0, n \in \mathbb{N} \). Denote by \( \mathcal{L}^n : \mathcal{A} \rightarrow \mathcal{A} \),
\[
\mathcal{L}^n f(z) = (1-\lambda) \mathcal{D} f(z) + \lambda I^n f(z), z \in U.
\]

Remark 3. If \( f \in \mathcal{A} \) and \( f(z) = z + \sum_{k=2}^{\infty} a_k z^k \), then
\[
\mathcal{L}^n f(z) = z + \sum_{k=2}^{\infty} \left[k^n (1-\lambda) + \lambda \frac{1}{k^n}\right] a_k z^k, z \in U.
\]

2. MAIN RESULTS

Theorem 1. Let \( q \) be a convex function, \( q(0) = 1 \) and let \( h \) be the function
\[
h(z) = q(z) + z q'(z), z \in U.
\]
If \( f \in \mathcal{A}, \lambda \geq 0, n \in \mathbb{N} \) and satisfies the differential subordination
\[
[\mathcal{L}^n f(z)]' < h(z), z \in U
\]
then
\[
\frac{\mathcal{L}^n f(z)}{z} < q(z), z \in U
\]
and this result is sharp.
Proof. Let

\[ p(z) = \frac{\mathcal{L}^n f(z)}{z} = \frac{z + \sum_{k=2}^{\infty} \left( k^n - \frac{1}{k^n} \right) a_k z^k }{z} = 1 + p_n z^n + p_{n+1} z^{n+1} + \cdots \]

(2.2)

\( z \in U \). From (2.2) we have \( p \in \mathcal{H} [1, 1] \). Let

\[ \mathcal{L}^n f(z) = z p(z), z \in U. \]  

(2.3)

Differentiating (2.3), we obtain

\[ [\mathcal{L}^n f(z)]' = p(z) + z p'(z), z \in U. \]

(2.4)

Then (2.1) becomes

\[ p(z) + z p'(z) < h(z), z \in U. \]

(2.5)

By using Lemma 2, we have

\[ p(z) < q(z), z \in U, \]

i.e.

\[ \frac{\mathcal{L}^n f(z)}{z} < q(z), z \in U. \]

\( \square \)

Remark 4. If \( \lambda = 0 \) we get Theorem 4 from Oros [6] and for \( \lambda = 1 \) we get Theorem 4 from Bălaet ı [1].

Example 1. For \( \lambda = 0, n = 1, f \in \mathcal{A} \) we deduce that

\[ f'(z) + z f''(z) < \frac{1}{(1-z)^2}, z \in U \]

implies

\[ f'(z) < \frac{1}{1-z}, z \in U. \]

Example 2. For \( \lambda = 1, n = 1, f \in \mathcal{A} \) we deduce that

\[ \frac{f(z)}{z} < \frac{1}{(1-z)^2}, z \in U \]

implies

\[ \frac{\int_0^z f(t) t^{-1} dt}{z} < \frac{1}{1-z}, z \in U. \]
Theorem 2. Let $q$ be a convex function, $q(0) = 1$ and let $h$ be the function

$$h(z) = q(z) + zq'(z), z \in U.$$  

If $f \in A$, $\lambda \geq 0$, $n \in \mathbb{N}$ and satisfies the differential subordination

$$\left( \frac{z \mathcal{L}^{n+1} f(z)}{\mathcal{L}^n f(z)} \right)' < h(z), \; z \in U$$  

(2.6)

then

$$\frac{\mathcal{L}^{n+1} f(z)}{\mathcal{L}^n f(z)} < q(z), \; z \in U$$

and this result is sharp.

Proof. Let

$$p(z) = \frac{\mathcal{L}^{n+1} f(z)}{\mathcal{L}^n f(z)} = \frac{z + \sum_{k=2}^{\infty} \left[ k^{n+1} (1-\lambda) + \lambda \frac{1}{k^{n+1}} \right] a_k z^k}{a_1 + \sum_{k=2}^{\infty} \left[ k^n (1-\lambda) + \lambda \frac{1}{k^n} \right] a_k z^k}.$$  

We have

$$p'(z) = \frac{(\mathcal{L}^{n+1} f(z))'}{\mathcal{L}^n f(z)} - p(z) \frac{(\mathcal{L}^n f(z))'}{\mathcal{L}^n f(z)}$$  

and

$$p(z) + z p'(z) = \left( \frac{z \mathcal{L}^{n+1} f(z)}{\mathcal{L}^n f(z)} \right)'.$$  

Relation (2.6) becomes

$$p(z) + z p'(z) < h(z) = q(z) + zq'(z), \; z \in U.$$  

By using Lemma 2 we have

$$p(z) < q(z) \; \text{i.e.} \; \frac{\mathcal{L}^{n+1} f(z)}{\mathcal{L}^n f(z)} < q(z), \; z \in U.$$  

Theorem 3. Let $q$ be a convex function, $q(0) = 1$ and let $h$ be the function

$$h(z) = q(z) + zq'(z), z \in U.$$  

If $f \in A$, $\lambda \geq 0$, $n \in \mathbb{N}$ and satisfies the differential subordination

$$\left( \mathcal{L}^{n+1} f(z) \right)' + \lambda \left[ (I^{n-1} f(z))' - (I^{n+1} f(z))' \right] < h(z), \; z \in U$$  

(2.7)

then

$$\mathcal{L}^n f(z)' < q(z), \; z \in U$$

and this result is sharp.
Proof. By using the properties of operator \( \mathcal{L}^n \), we obtain
\[
\mathcal{L}^{n+1} f(z) = (1 - \lambda) \mathcal{D}^{n+1} f(z) + \lambda I^{n+1} f(z), \quad z \in U.
\] (2.8)
Then (2.7) becomes
\[
[(1 - \lambda) \mathcal{D}^{n+1} f(z) + \lambda I^{n+1} f(z)]' + \lambda \left[ (I^{n-1} f(z))' - (I^{n+1} f(z))' \right] < h(z), \quad z \in U.
\] (2.9)
After computation we get
\[
(1 - \lambda) \left[ (\mathcal{D}^n f(z))' + \lambda (I^n f(z))' \right] < h(z)
\]
or equivalently
\[
(1 - \lambda) \left[ z (\mathcal{D}^n f(z))' + \lambda z (I^n f(z))' \right] < h(z).
\]
The above relation is equivalent to
\[
(1 - \lambda) \left[ (\mathcal{D}^n f(z))' + z (\mathcal{D}^n f(z))'' \right] + \lambda \left[ (I^n f(z))' + z (I^n f(z))'' \right] < h(z)
\]
or
\[
[\mathcal{L}^n f(z)]' + z [\mathcal{L}^n f(z)]'' < h(z), \quad z \in U.
\] (2.10)
Let
\[
p(z) = (1 - \lambda) \left[ \mathcal{D}^n f(z) \right]' + \lambda [I^n f(z)]' = [\mathcal{L}^n f(z)]', \quad z \in U
\] (2.11)
\[
= (1 - \lambda) \left[ z + \sum_{k=2}^{\infty} k^n a_k z^k \right]' + \lambda \left[ z + \sum_{k=2}^{\infty} \frac{1}{k^n} a_k z^k \right]'
\]
\[
= (1 - \lambda) \left[ 1 + \sum_{k=2}^{\infty} k^{n+1} a_k z^{k-1} \right] + \lambda \left[ 1 + \sum_{k=2}^{\infty} \frac{1}{k^n} a_k z^{k-1} \right] = 1 + \sum_{k=2}^{\infty} \left[ k^{n+1} (1 - \lambda) + \lambda \frac{1}{k^n} \right] a_k z^{k-1} = 1 + p_1 z + p_2 z^2 + \cdots
\]
In view of (2.11), we deduce that \( p \in \mathcal{H} [1, 1] \). Using the notation in (2.11), the (2.10) differential subordination becomes
\[
p(z) + z p'(z) < h(z) = q(z) + z q'(z), \quad z \in U.
\]
By using Lemma 2 we have
\[
p(z) < q(z) \quad \text{i.e.} \quad [\mathcal{L}^n f(z)]' < q(z), \quad z \in U.
\]

Remark 5. If \( \lambda = 0 \) we get Theorem 2 from Oros [6] and for \( \lambda = 1 \) we get Theorem 2 from Bălăeţi [1].
Example 3. For $\lambda = 0, n = 1, f \in A$ we deduce that
\[ f'(z) + 3zf''(z) + z^2 f'''(z) < 1 + 2z, \ z \in U \]
implies
\[ f'(z) + zf''(z) < 1 + z, \ z \in U. \]

**Theorem 4.** Let $h \in H(U)$ such that $h(0) = 1$ and
\[ \Re \left[ 1 + \frac{zh''(z)}{h'(z)} \right] > -\frac{1}{2}, \ z \in U. \]
If $f \in A$ satisfies the differential subordination
\[ (\mathcal{L}^{n+1} f(z))' + \lambda \left[ (I^{n-1} f(z))' - (I^{n+1} f(z))' \right] < h(z), \ z \in U \tag{2.12} \]
then
\[ \left[ \mathcal{L}^n f(z) \right]' < q(z), \ z \in U \]
where $q$ is given by $q(z) = \frac{1}{z} \int_0^z h(t)dt$. The function $q$ is convex and is the best dominant.

**Proof.** If we use the differential subordination technique we can see that the function $g$ is convex.[3], p. 66 By using (2.11) we obtain
\[ (\mathcal{L}^{n+1} f(z))' + \lambda \left[ (I^{n-1} f(z))' - (I^{n+1} f(z))' \right] = p(z) + z p'(z), \ z \in U \]
Then (2.12) becomes
\[ p(z) + z p'(z) < h(z), \ z \in U. \]
Since $p \in H[1,1]$, we deduce that $p(z) < q(z)$, i.e.
\[ \left[ \mathcal{L}^n f(z) \right]' < q(z) = \frac{1}{z} \int_0^z h(t)dt, \ z \in U \]
and $q$ is the best dominant. \hfill \Box

**Remark 6.** If $\lambda = 0$ we get Theorem 3 from Oros [6].

**Example 4.** For $\lambda = 0, n = 0, h(z) = \frac{1+z}{1-z}$ we deduce that
\[ f'(z) + z f''(z) < \frac{1+z}{1-z}, \ z \in U, \]
implies
\[ f'(z) < 1 - \frac{2}{z} \ln (1-z), \ z \in U. \]
Theorem 5. Let \( h \in \mathcal{H}(U) \) such that \( h(0) = 1 \) and
\[
\Re \left[ 1 + \frac{zh''(z)}{h'(z)} \right] > -\frac{1}{2}, \quad z \in U.
\]
If \( f \in \mathcal{A} \) satisfies the differential subordination
\[
\left[ \mathcal{L}^n f(z) \right]' < h(z), \quad z \in U \tag{2.13}
\]
then
\[
\mathcal{L}^n f(z) < q(z), \quad z \in U
\]
where \( q \) is given by \( q(z) = \frac{1}{z} \int_0^z h(t)dt \). The function \( q \) is convex and is the best dominant.

Proof. If we use the differential subordination technique we can see that the function \( g \) is convex. [3], p. 66. Differentiating both sides in (2.2) we obtain
\[
\left[ \mathcal{L}^n f(z) \right]' = p(z) + zp'(z), \quad z \in U
\]
Then (2.13) becomes
\[ p(z) + zp'(z) < h(z), \quad z \in U. \]
Since \( p \in \mathcal{H}[1,1] \), we deduce that \( p(z) < q(z) \), i.e.
\[
\frac{\mathcal{L}^n f(z)}{z} < q(z) = \frac{1}{z} \int_0^z h(t)dt, \quad z \in U
\]
and \( q \) is the best dominant. \( \square \)

Remark 7. If \( \lambda = 0 \) we get Theorem 5 from Oros [6] and for \( \lambda = 1 \) we get Theorem 5 from Bǎlaeƫi [1].

Example 5. For \( \lambda = 0, n = 1, h(z) = \frac{1}{1+z} \) we deduce that
\[
f'(z) < \frac{1}{(1+z)^2}, \quad z \in U,
\]
implies
\[
\frac{f(z)}{z} < \frac{1}{1+z}, \quad z \in U.
\]
We get the same result as [4].

Definition 5 ([7], [9], [1], [6]). If \( 0 \leq \beta < 1 \) and \( n \in \mathbb{N} \), we let \( L^m_n(\beta) \) stand for the class of functions \( f \in \mathcal{A}_m \), which satisfy the inequality
\[
\Re \left[ \mathcal{L}^n f(z) \right]' > \beta, \quad (z \in U).
\]

Remark 8. For \( n = 0 \) we obtain \( \Re f'(z) > \beta \).
Theorem 6. The set $L_n^m(\beta)$ is convex.

Proof. Let the function

$$f_i(z) = z + \sum_{k=2}^{\infty} a_{k_i} z^k, \quad i = 1, 2, \quad z \in U$$

be in the class $L_n^m(\beta)$. It is sufficient to show that the function

$$h(z) = \mu_1 f_1(z) + \mu_2 f_2(z)$$

with $\mu_1, \mu_2 \geq 0$ and $\mu_1 + \mu_2 = 1$ is in $L_n(\beta)$. Since

$$h(z) = z + \sum_{k=2}^{\infty} (\mu_1 a_{k_1} + \mu_2 a_{k_2}) z^k, \quad z \in U$$

then

$$\mathcal{L}_n h(z) = z + \sum_{k=2}^{\infty} \left[ k^n (1 - \lambda) + \lambda \frac{1}{k^{n-1}} \right] (\mu_1 a_{k_1} + \mu_2 a_{k_2}) z^{k-1}, \quad z \in U. \quad (2.14)$$

Differentiating (2.14), we get

$$\left[ \mathcal{L}_n h(z) \right]' = 1 + \sum_{k=2}^{\infty} \left[ k^{n+1} (1 - \lambda) + \lambda \frac{1}{k^{n-1}} \right] (\mu_1 a_{k_1} + \mu_2 a_{k_2}) z^{k-1}. \quad (2.15)$$

Hence

$$\Re \left[ \mathcal{L}_n h(z) \right]' = 1 \Re \left\{ \mu_1 \sum_{k=2}^{\infty} \left[ k^{n+1} (1 - \lambda) + \lambda \frac{1}{k^{n-1}} \right] a_{k_1} z^{k-1} \right\} +$$

$$+ \Re \left\{ \mu_2 \sum_{k=2}^{\infty} \left[ k^{n+1} (1 - \lambda) + \lambda \frac{1}{k^{n-1}} \right] a_{k_2} z^{k-1} \right\}. \quad (2.15)$$

Since $f_1, f_2 \in L_n^m(\beta)$, we obtain

$$\Re \left\{ \mu_i \sum_{k=2}^{\infty} \left[ k^{n+1} (1 - \lambda) + \lambda \frac{1}{k^{n-1}} \right] a_{k_i} z^{k-1} \right\} > \mu_i (\beta - 1), \quad i = 1, 2. \quad (2.16)$$

Using (2.16) we get from (2.15)

$$\Re \left[ \mathcal{L}_n h(z) \right]' > 1 + \mu_1 (\beta - 1) + \mu_2 (\beta - 1),$$

and since $\mu_1 + \mu_2 = 1$, we deduce

$$\Re \left[ \mathcal{L}_n h(z) \right]' > \beta, \quad (z \in U)$$

i.e. $L_n^m(\beta)$ is convex. □
Theorem 7. If $0 \leq \beta < 1$ and $m, n \in \mathbb{N}$ then we have

$$L^m_n (\beta) \subset L^m_{n+1} (\delta),$$

where $\delta (\beta, m) = 2\beta - 1 + 2(1 - \beta) \frac{1}{m} \sigma \left( \frac{1}{m} \right)$ and $\sigma (x) = \int_0^x \frac{t^{x-1}}{1+t} \, dt$. The result is sharp.

Proof. Assume that $f \in L^m_n (\beta)$. Let $L^m_n f (z) = z p(z), z \in U$. Differentiating, we obtain

$$\left[ L^m_n f (z) \right]' = p(z) + z p'(z), z \in U.$$

Since $f \in L^m_n (\beta)$, from Definition 5 we have

$$\Re \left( p(z) + z p'(z) \right) > \beta, z \in U$$

which is equivalent to

$$p(z) + z p'(z) < \frac{1 + (2\beta - 1)z}{1+z} = h(z), z \in U$$

By using Lemma 1, we have:

$$p(z) < q(z) < h(z), z \in U,$$

where

$$q(z) = \frac{1}{mz^\frac{1}{m}} \int_0^z \frac{1 + (2\beta - 1)t}{1+t} \frac{1}{t^\frac{1}{m} - 1} \, dt =$$

$$= \frac{1}{mz^\frac{1}{m}} \int_0^z \left[ 2\beta - 1 + 2(1 - \beta) \frac{1}{1+t} \right] \frac{1}{t^\frac{1}{m} - 1} \, dt =$$

$$= \frac{1}{mz^\frac{1}{m}} \int_0^z (2\beta - 1) \frac{t^\frac{1}{m} - 1}{t^\frac{1}{m}} \, dt + \frac{2(1 - \beta)}{mz^\frac{1}{m}} \int_0^z \frac{1}{1+t} \, dt =$$

$$= 2\beta - 1 + 2(1 - \beta) \frac{1}{m} \sigma \left( \frac{1}{m} \right) \frac{1}{z^\frac{1}{m}}, z \in U.$$

The function $q$ is convex and is the best dominant. From $p(z) < q(z)$ follows that

$$\Re p(z) > \Re q (1) = \delta (\beta, m) = 2\beta - 1 + 2(1 - \beta) \frac{1}{m} \sigma \left( \frac{1}{m} \right),$$

from which we deduce that $L^m_n (\beta) \subset L^m_{n+1} (\delta)$. \qed

Remark 9. If $\lambda = 0$ we get Theorem 1 from Oros [6] and for $\lambda = 1$ we get Theorem 1 from Bălătei [1].
Theorem 8. Let \( q \) be a convex function in \( U \) with \( q(0) = 1 \) and let

\[
h(z) = q(z) + \frac{1}{c+2} z q'(z), \quad z \in U,
\]

where \( c \) is a complex number, with \( \Re c > -2 \).

If \( f \in L_n^m(\beta) \) and \( F = I_c(f) \), where

\[
F(z) = I_c(f)(z) = \frac{c+2}{c+1} \int_0^z t^c f(t) dt, \quad \Re c > -2,
\]

then

\[
\left[ \mathcal{L}^n f(z) \right]' < h(z), \quad z \in U,
\]

implies

\[
\left[ \mathcal{L}^n F(z) \right]' < q(z), \quad z \in U,
\]

and this result is sharp.

Proof. From (2.17), we have

\[
z^{c+1} F(z) = (c+2) \int_0^z t^c f(t) dt, \quad \Re c > -2, \quad z \in U.
\]

Differentiating, with respect to \( z \), we obtain

\[
(c + 1) F(z) + z F'(z) = (c + 2) f(z), \quad z \in U
\]

and

\[
(c + 1) \mathcal{L}^n F(z) + z \left[ \mathcal{L}^n F(z) \right]' = (c + 2) \mathcal{L}^n f(z), \quad z \in U.
\]

Differentiating (2.20), we obtain

\[
\left[ \mathcal{L}^n F(z) \right]' + \frac{z}{c+2} \left[ \mathcal{L}^n F(z) \right]'' = \left[ \mathcal{L}^n f(z) \right]', \quad z \in U.
\]

Using (2.21), the differential subordination (2.18) becomes

\[
\left[ \mathcal{L}^n F(z) \right]' + \frac{1}{c+2} z \left[ \mathcal{L}^n F(z) \right]' < h(z) = q(z) + \frac{1}{c+2} z q'(z), \quad z \in U.
\]

Let

\[
p(z) = \left[ \mathcal{L}^n F(z) \right]' = \left\{ z + \sum_{k=2}^{\infty} \left[ k^n (1-\lambda) + \frac{1}{k^n} \right] a_k z^k \right\}' = 1 + p_1 z + p_2 z^2 + \cdots, \quad z \in U, \quad p \in \mathcal{H}[1,1].
\]

Replacing (2.23) in (2.22) we obtain

\[
p(z) + \frac{1}{c+2} z p'(z) < h(z) = q(z) + \frac{1}{c+2} z q'(z), \quad z \in U
\]

Using Lemma 1, we obtain \( p(z) < q(z) \) i.e.

\[
\left[ \mathcal{L}^n F(z) \right]' < q(z), \quad z \in U
\]

and \( q \) is the best dominant.
Remark 10. If $\lambda = 0$ we get Theorem 2.2 from Tăut et alii [9].

Example 6. If we take $c = 1 + 2i$ and $q(z) = \frac{1+z}{1-z}$ then

$$h(z) = \frac{(1-z^2)(3+2i)+2z}{(3+2i)(1-z)^2}.$$  

From Theorem 8 we deduce

$$\left[ \mathcal{L}^n f(z) \right] \left( 1 \right) > \frac{(1-z^2)(3+2i)+2z}{(3+2i)(1-z)^2}, \quad z \in U,$$

implies

$$\left[ \mathcal{L}^n F(z) \right] \left( 1 \right) > \frac{1+z}{1-z}, \quad z \in U,$$

where $F$ is given by (2.17).

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