Exponential Orlicz Spaces: New Norms and Applications.

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Abstract

The aim of this paper is investigating of Orlicz spaces with exponential \( N \) function and correspondence Orlicz norm: we introduce some new equivalent norms, obtain the tail characterization, study the product of functions in Orlicz spaces etc.

We consider some applications: estimation of operators in Orlicz spaces and problem of martingales convergence and divergence.

Key words: Orlicz spaces, \( \Delta_2 \) condition, martingale, slowly varying function, absolute continuous norm.

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1. INTRODUCTION.

Let \( (\Omega, F, \mathcal{P}) \) be a probability space. Introduce the following set of \( N \) - Orlicz functions:

\[
LW = \{ N = N(u) = N(W, u) = \exp(W(\log u)) \}, \quad u \geq e^2
\]

where \( W \) is a continuous strictly increasing convex function in domain \( [2, \infty) \) such that \( u \to \infty \Rightarrow \lim_{u \to \infty} W_-(u) = \infty \); here \( W_-(u) \) denotes the left derivative of the function \( W \).

We define the function \( N(W(u)) \) arbitrary for the values \( u \in [0, e^2) \) but so that \( N(W(u)) \) will be continuous convex strictly increasing and such that

\[
u \to 0+ \Rightarrow N(W(u)) \sim C(W)u^2,
\]

for some \( C(W) = \text{const} \in (0, \infty) \). For \( u < 0 \) we define as usually \( N(W(u)) = N(W(|u|)) \).
We denote the set of all those $N$ - functions as $\text{ENF} : \text{ENF} = \{N(W(\cdot))\}$ (Exponential N - Functions) and denote also the correspondence Orlicz space as $\text{EOS}(W) = \text{Exponential Orlicz Space}$ or simple $\text{EW}$ with Orlicz (or, equally, Luxemburg) norm

$$||\eta||_L(N) = ||\eta||_L(N(W)) = \inf_{v>0}\{v^{-1}(1 + \mathbb{E}_N(W(\cdot), v\eta))\},$$

where $\mathbb{E}, \mathbb{D}$ denote the expectation and variance with respect to the probability measure $P$:

$$\mathbb{E}_\eta = \int_\Omega \eta(\omega) \ P(d\omega).$$

Let us introduce the following important function:

$$\psi(p) = \psi(W, p) = \exp(\frac{W^*(p)}{p}), \ p \geq 2, \quad (1)$$

where

$$W^*(p) \overset{def}{=} \sup_{z \geq 2} (pz - W(z)) -$$

is the Young - Fenchel transform of $W$. The function $p \to p \log \psi(\cdot) = W^*(\cdot)$ is continuous convex increasing and such that

$$p \uparrow \infty \Rightarrow p \log \psi(p) \uparrow \infty.$$

We denote the set of all those functions $\{\psi\} = \{\psi(W, p)\} = \{\exp(W^*(p)/p)\}$ by the symbol $\Psi$:

$$\Psi = \cup_W\{\psi(W, p)\} = \{\exp(W^*(p)/p)\}.$$

Inversely, $W(\cdot)$ may be constructed by means of $\Psi$:

$$W(p) = (p \log \psi(p))^*$$

(theorem of Fenchel - Moraux).

**Definition.** We introduce the so - called $G(\psi)$ norm for arbitrary function $\psi(\cdot) \in \Psi$ and the correspondent $G(\psi)$ space: $\eta \in G(\psi)$ iff

$$||\eta||_{G(\psi)} \overset{def}{=} \sup_{p \geq 2} |\eta|_p/\psi(p) < \infty,$$

where $| \cdot |_p$ denotes the classical $L_p$ norm:

$$|\eta|_p = \mathbb{E}^{1/p}|\eta|^p, \ p \geq 1.$$
In particular, $|\eta|^2 = D\eta + E^2\eta$. It is easy to prove (see, for example, [3], p. 373; [12], p. 67) that $G(\psi)$ is some (full) Banach space.

Note that if there exists a family of measurable functions $\{\eta_\alpha\}, \alpha \in A$ so that

$$\sup_\alpha |\eta_\alpha|_p < \infty,$$

then there exists a $G(\psi)$ space, $\psi \in \Psi$ such that $\forall \alpha \Rightarrow \eta_\alpha \in G(\psi)$. For example, put

$$\psi(p) = \sup_\alpha |\eta_\alpha|_p.$$

**Remark 1.** In this paper the letters $C, C_j$ will denote positive finite various constants which may differ from one formula to another and which do not depend on the essential parameters: $x, u, p, z, \lambda$.

### 2. MAIN RESULTS.

**Theorem 1.** The norms $|||\eta|||_{L(N(W))}$ and $|||\eta|||_{G(\psi)}$ are equivalent. Further, $\eta \neq 0, \eta \in G(\psi)$ iff $\exists C, C_1 \in (0, \infty) \Rightarrow \forall u > 2C$

$$P(|\eta| > u) \leq C_1 \exp(-W(\log(u/C))). \quad (2)$$

**Remark 2.** This result is a little generalization of [3]; see also [16], p.305, as long as we do not suppose that the function $\eta = \eta(\omega), \omega \in \Omega$ to be exponential integrable:

$$\exists \lambda > 0 \Rightarrow \mathbf{E} \exp(\lambda|\eta|) < \infty$$

(so - called Kramer condition).

**Proof.** a). Suppose $\eta \in EW, \eta \neq 0$. Then for some $C \in (0, \infty) \Rightarrow \mathbf{E} \exp(W(\log C|\eta|)) < \infty$. Proposition (2) follows from Chebyshev inequality.

b). Inversely, let us assume that $\eta(\omega)$ is a measurable function such that

$$P(|\eta| > u) \leq \exp(-W(\log u)), \quad u \geq e^2.$$ 

Then, by virtue of properties $W(\cdot)$ we have:

$$\mathbf{E}N(W(|\eta|/e^2)) = \int_\Omega N(W, |\eta(\omega)|/e^2) \ P(d\omega) \leq$$

$$C_1 + \sum_{k=2}^{\infty} \int_{e^k < |\eta| \leq e^{k+1}} \exp((W(|\eta|/e^2)) \ P(d\omega) \leq C_1 +$$
\[
\sum_{k=2}^{\infty} \exp((W(k-1)) \ P(|\eta| > k) \leq C_2 + \sum_{k=2}^{\infty} \exp(W(k) - W(k+1)) < \infty.
\]

Hence \( ||\eta||_{L(N(W))} < \infty \).

c) Let \( \eta \in G(\psi) \); without loss of generality we can assume that \( ||\eta||_{G(\psi)} = 1 \). We deduce: \( E|\eta|^p \leq \psi^p(p) \),

\[
P(|\eta| > x) \leq x^{-p} \psi^p(p) = \exp(-p \log x + p \log \psi(p)), \ x > e^2;
\]

and, after the minimization over \( p : x \geq e^2 \) \( \Rightarrow \)

\[
P(|\eta| > x) \leq \exp\left(-\sup_{p \geq 2}(p \log x - p \log \psi(p))\right) = \exp(-W(\log x)).
\]

d) Let us next assume that \( P(|\eta| > x) \leq \exp(-W(\log x)), x > e^2 \). We have:

\[
E|\eta|^p \leq C^p + \int_{e^2}^{\infty} px^{p-1} \exp(-W(\log x)) \ dx =
\]

\[
C^p + p \int_{e^2}^{\infty} \exp(py - W(y)) \ dy, \ p \geq 2.
\]

Using Laplace’s method and theorem of Fenchel - Moraux we get:

\[
E|\eta|^p \leq C^p + C_4^p \exp\left(\sup_{y \geq 2}(py - W(y))\right) = C^p +
\]

\[
C_2^p \exp(W^*(p)) = C^p + C_2^p \exp(p \log \psi(p)) \leq C_3^p \psi^p(p).
\] (3)

Finally, \( ||\eta||_{G(\psi)} \leq C_3 < \infty \).

Remark 3. If conversely

\[
P(|\eta| > x) \geq \exp(-W(\log x)), \ x \geq e^2;
\]

then for sufficiently large values of \( p; \ p \geq p_0 = p_0(W) \geq 2 \)

\[
|\eta|_p \geq C_0(W)\psi(p), \ C_0 \in (0, \infty).
\]

For arbitrary Orlicz spaces \( L(N), EOS(W) = EW, G(\psi) \) we denote correspondently \( L(N)^0, EW^0, G^0(\psi) \) a closure in \( L(N), EW, G(\psi) \) norm the set
of all bounded measurable functions. It is known (see, for example, \[15\], p.75, \[10\], p.138) that in our conditions (\(P(\Omega) = 1, N(W_u) \sim Cu^2\) etc.)

\[LN(W)^0 = \{\eta : \forall k > 0 \Rightarrow E_N(W, |\eta|/k) < \infty,\}\]

**Theorem 2.** Let \(\psi \in \Psi\). We assert that \(\eta \in EW^0\), or, equally, \(\eta \in G^0(\psi)\) if and only if

\[
\lim_{p \to \infty} |\eta|_p/\psi(p) = 0. \tag{4}
\]

**Proof.** It is sufficient by virtue of theorem 1 to consider only the case of the \(G(\psi)\) spaces.

1. Denote \(GB^0(\psi) = \{\eta : \lim_{p \to \infty} |\eta|_p/\psi(p) = 0,\eta \neq 0\}\). Let \(\eta \in G^0(\psi), \eta \neq 0\). Then for arbitrary \(\delta = \text{const} > 0\) there exists a constant \(K \in (0, \infty)\) such that

\[
||\eta - \eta I(|\eta| \leq K)||G(\psi) \leq \delta/2,
\]

where for any event \(A \in F I(A) = 1, \omega \in A, I(A) = 0\) if \(\omega \notin A\). Since \(|\eta I(|\eta| \leq K)| \leq K\), we deduce

\[
|\eta I(|\eta| \leq K)|_p/\psi(p) \leq K/\psi(p).
\]

Using the triangular inequality we obtain for sufficiently large values \(p : p > p_0(\delta) = p_0(\delta, K)\):

\[
|\eta|_p/\psi(p) \leq \delta/2 + K/\psi(p) < \delta,
\]

as long as \(\psi(p) \to \infty\) as \(p \to \infty\). Therefore, \(G^0(\psi) \subset GB^0(\psi)\).

(The set \(GB^0(\psi)\) is a closed subspace of \(G(\psi)\) with respect to the \(G(\psi)\) norm and contains all bounded random variables).

2. Conversely, assume \(\eta \in GB^0(\psi)\). Let us denote \(\eta(K) = \eta I(|\eta| > K), K \in (0, \infty)\). We deduce:

\[
\forall Q \geq 2 \Rightarrow \lim_{K \to \infty} |\eta(K)|_Q = 0.
\]

Further,

\[
||\eta(K)||G(\psi) = \sup_{p \geq 2} |\eta(K)|_p/\psi(p) \leq \max_{p \in [2, Q]} |\eta(K)|_p/\psi(p) + \sup_{p > Q} |\eta(K)|_p/\psi(p) \overset{\text{def}}{=} \sigma_1 + \sigma_2;
\]

5
\[ \sigma_2 = \sup_{p > Q} |\eta(K)|_p / \psi(p) \leq \sup_{p \geq Q} (|\eta|_p / \psi(p)) \leq \delta / 2 \]

for sufficiently large \( Q \) as long as \( \eta \in GB^0(\psi) \). Further,

\[ \sigma_1 \leq \max_{p \in [2, Q]} |\eta(K)|_p / \psi(2) \leq |\eta(K)|_Q / \psi(2) \leq \delta / 2 \]

for sufficiently large \( K = K(Q) \). Therefore,

\[
\lim_{K \to \infty} \|\eta(K)\|_{G(\psi)} = 0, \quad \Rightarrow \eta \in G^0(\psi).
\]

Hence \( GB^0(\psi) \subset G^0(\psi) \).

Let now \( \{\eta_a\}, a \in A \) be some family of functions from the \( G^0(\psi) \) space.

**Theorem 3.** Let \( \psi \in \Psi \). In order to a family \( \{\eta_a\} \) of a function belonging to the \( LG(\psi) \) space has the uniform absolute continuous norm in this space, briefly: \( \{\eta_a\} \in UCN(G(\psi)) \), it is necessary and sufficient:

\[
\lim_{p \to \infty} \sup_{a \in A} |\eta_a|_p / \psi(p) = 0. \tag{5}
\]

**Proof.** Recall that by definition \( \{\eta_a\} \in UCN(G(\psi)) \) if

\[
\lim_{\delta \to 0^+} \sup_{V: P(V) < \delta} \sup_{a \in A} ||\eta_a I(V)||_{G(\psi)} = 0.
\]

1. Let the condition (5) be satisfied, then there exists a function \( \epsilon = \epsilon(p) \to 0 \) as \( p \to \infty \) such that \( \forall a \in A \) and \( \forall p \geq 2 \Rightarrow |\eta_a|_p \leq \epsilon(p) \psi(p) \).

It follows that for all \( Q \geq 2 \) the family of functions \( |\eta_a|^Q \) is uniform integrable. Let \( V \) be an arbitrary measurable set: \( V \in F \) with sufficiently small measure: \( P(V) \leq \delta, \delta \in (0, 1/2) \). We have:

\[
\sup_a ||\eta_a I(V)||_{G(\psi)} \leq \sup_a \max_{p \leq Q} |\eta_a|_p / \psi(p) + \sup_a \sup_{p > Q} |\eta_a|_p / \psi(p) \overset{\text{def}}{=} s_1 + s_2;
\]

\[
s_2 \leq \sup_{p \geq Q} \epsilon(p) \to 0, \quad Q \to \infty;
\]

\[
s_1 \leq \sup_a |\eta I(V)|_Q / \psi(2) \to 0, \quad \delta \to 0^+.
\]

Hence the family \( \{\eta_a\} \) has the uniform absolute continuous norms in our Orlicz space \( G(\psi) \).
2. Assume now the family \( \{ \eta_a \}, \eta_a \in G^0(\psi) \) belongs to \( UCN(G(\psi)) \): \( \{ \eta_a \} \in UCN(G(\psi)) \). Then this family is uniformly finitely approximate in \( G(\psi) \) norm:
\[
\lim_{K \to \infty} \sup_a \| \eta_a \| G(\psi) = 0.
\]

Further proof is the same as in the theorem 3.

Recall here the definition for two \( N \)–functions \( M = M(u), N = N(u) \) (see, for example, [8], chapter 2, section 13, [9], p.144): the function \( N(\cdot) \) is called essentially greater than \( M(\cdot) : N = N(\cdot) >> M = M(\cdot) \) or equally \( M(\cdot) \) decreases essentially more rapidly then \( N(\cdot) = M(\cdot) \), if
\[
\forall \lambda > 0 \Rightarrow \lim_{u \to \infty} M(\lambda u)/N(u) = 0.
\]

**Theorem 4.** Let \( \psi(\cdot) = \psi_N(\cdot), \nu(\cdot) = \nu_M(\cdot) \) be two functions of the classes \( \Psi \) with correspondent \( N \) – Orlicz functions \( N(\cdot), M(\cdot) \):
\[
N(u) = N_\psi(u) = \exp\{[p \log \psi(p)]^*(\log u)\},
M(u) = N_\nu(u) = \exp\{[p \log \nu(p)]^*(\log u)\}.
\]

We assert that \( \lim_{p \to \infty} \psi(p)/\nu(p) = 0 \) if and only if \( N >> M \).

**Proof.** 1. Assume that \( \lim_{p \to \infty} \psi(p)/\nu(p) = 0 \). Denote \( \epsilon(p) = \psi(p)/\nu(p) \), then \( \epsilon(p) \to 0, \ p \to \infty \).

Let \( \{ \eta_a \}, a \in A \) be arbitrary bounded in the \( G(\psi) \) sense set of functions:
\[
\sup_a ||\eta_a||_G = \sup_a \sup_{p \geq 2} |\eta_a|/\psi(p) = C < \infty,
\]

then
\[
\sup_a |\eta_a|_p/\nu(p) \leq C \epsilon(p) \to 0, \ p \to \infty.
\]

It follows from theorem 3 that \( \forall a \in A \ \eta_a \in G^0(\nu) \) and that the family \( \{ \eta_a \} \) in the space \( G^0(\nu) \) has the uniform absolute continuous norm: \( \{ \eta_a \} \in UCN(G(\nu)) \). Our assertion it follows from lemma 13.3 in the book [8].

2). Inversely, let \( N >> M \). Let us consider the measurable function \( \eta : \Omega \to R^1 \) such that \( \forall x \geq C \)
\[
C_1 \exp(-C_2[p \log \psi(p)]^*(\log x)) \leq P(|\eta| > x) \leq C_3 \exp(-C_4[p \log \psi(p)]^*(\log x)).
\]
It follows from theorem 1 that \( \eta \in G(\psi) \) and \( C_5 \psi(p) \leq |\eta|_p \leq C_6 \psi(p) \), \( p \geq 2 \). Since \( ||\eta||G(\psi) < \infty \), \( M \ll N \), we deduce that \( \eta \in G^0(\nu) \) ([8], theorem 13.4). It follows from theorem 2 that

\[
\lim_{p \to \infty} |\eta|_p / \nu(p) = 0.
\]

Thus \( \lim_{p \to \infty} \psi(p) / \nu(p) = 0 \).

Obviously, if for some \( C_1, C_2 \): \( 0 < C_1 \leq C_2 < \infty \) and for all \( p \geq 2 \)

\[
C_1 \leq \psi(p) / \nu(p) \leq C_2,
\]
then the norms \( || \cdot ||_{G(\psi)} \) and \( || \cdot ||_{G(\nu)} \), or, equally, \( || \cdot ||_{L(N_\psi)} \) and \( || \cdot ||_{L(N_\nu)} \) are equivalent.

**Theorem 5.** Let \( N \in LW \) and \( \eta \) be a random variable (measurable function \( \eta : \Omega \to \mathbb{R}^1 \)) such that for sufficiently large values \( u : u \geq C_0 \Rightarrow \)

\[
C_1 \exp(-W(\log u/C)) \leq \mathbb{P}(|\eta| > u) \leq C_2 \exp(-W(\log u/C_3)), \tag{6}
\]
then \( \eta \in EW \setminus EW^0 \).

**Proof.** It follows from the right - side of inequality (6) in accordance to the theorem 1 that \( \eta \in EW = G(\psi) \). Further, as well as in the proof of theorem 4 we obtain

\[
\lim_{p \to \infty} |\eta|_p / \psi(p) > 0.
\]

It follows from theorem 2 that \( \eta \notin G^0(\psi) \).

Let us consider some concrete examples. The so - called \( EL_m, m = const > 0 \) spaces are very important subclasses of the \( EW \) spaces. By definition of the \( EL_m \) spaces,

\[
W(\log u) = W_{L,m}(\log u) \stackrel{def}{=} u^m L(u), \ u \geq e^2;
\]
and correspondently \( N(u) = \exp(u^m L(u)) \), where \( L(\cdot) \) is a slowly varying at \( u \to \infty \) continuous function such that \( L(0) > 0 \), \( u^m L(u) \uparrow \infty \) at \( u \uparrow \infty \) and

\[
\lim_{u \to \infty} L(u/L(u))/L(u) = 1. \tag{7}
\]

We denote the set of all those functions as \( SV_m, SV \stackrel{def}{=} \cup_L \{L(\cdot)\} \).

It is known ([12], p. 25) that in the case \( m > 1 \), \( L \in SV_m \) the centered random variable \( \eta, \mathbb{E}\eta = 0 \) belongs to the space \( L_m : \eta \in EL_m \) if and only if \( \exists C \in (0, \infty), \forall \lambda \geq 2 \)
\[ \mathbf{E} \exp(\pm \lambda \eta) \leq \exp \left( C \lambda^{m/(m-1)} L^{-1/(m-1)}(\lambda^{1/(m-1)}) \right). \] (8)

The following particular cases of \( EL_m \) spaces are very convenient in the practical using. Define for \( p, u \geq 2, m > 0, r \in R^1 \)

\[ \psi_{m,r} = \psi_{m,r}(p) = p^{1/m} \log^* p, \quad G_{m,r} = G(\psi_{m,r}), G_m = G_{m,0}, \]

\[ ||\eta||_{m,r} = ||\eta||_r G(\psi_{m,r}), \quad ||\eta||_m = ||\eta||_{m,0}, \]

\[ N_{m,r}(u) = \exp \left( u^m \left( \log(C(m,r) + u) \right)^{-mr} \right). \]

The Orlicz spaces \( (\Omega, F, \mathbf{P}; L(N_{m,r}(\cdot))) \) and \( G_{m,r} \) are isomorphic and the correspondence norms are equivalent

\[ C_1(m, r)||\eta||_{m,r} \leq ||\eta||_r L(N_{m,r}) \leq C_2(m, r)||\eta||_{m,r}. \]

Moreover, \( \eta \in G_{m,r} \) iff \( \forall x \geq 2 \)

\[ \mathbf{P}(||\eta|| > x) \leq \exp \left( -C_3(m, r)x^m(\log(C(m, r) + x))^{-mr} \right). \]

Another example of \( EOS(W) \) spaces are so-called \( V(Z, \beta) \) spaces:

\[ V(Z, \beta) = G \left( \psi^{(Z, \beta)} \right), \quad \psi^{(Z, \beta)}(p) = \exp \left( Zp^{\beta} \right), \quad Z, \beta = \text{const} > 0. \]

From theorem 1 follows that \( \eta \in V(Z, \beta), \eta \neq 0 \) iff \( \exists C \in (0, \infty) \Rightarrow \forall x \geq 2C \)

\[ \mathbf{P}(||\eta|| > x) \leq 2 \exp \left( -Z^{-1/\beta}(1 + \beta)^{1+1/\beta}(\log x/C)^{1+1/\beta} \right). \]

We continue investigating the properties of \( EL_m \) spaces. Let \( \epsilon(k) \) be a Rademacher sequence, i.e. the sequence of independent random variables with distributions:

\[ \mathbf{P}(\epsilon(k) = 1) = \mathbf{P}(\epsilon(k) = -1) = 1/2. \]

Let also \( B = \text{const} \in (0.5; 1) \) and \( L = L(u) \) be a function belonging to the class \( SV : L(\cdot) \in SV. \) Introduce the random variable \( \xi \) by the formula

\[ \xi = \sum_{k=2}^{\infty} k^{-B} L(k) \epsilon(k). \] (9)
Denote
\[ \tilde{L}(u) = L^{-1/(1-B)} \left( u^{1/(1-B)} \right). \]

**Theorem 6.** There exist \( C_1, C_2 \in (0, \infty) \), \( C_1 \leq C_2 \) so that \( \forall u \geq 2 \)
\[
\exp \left( -C_2 u^{1/(1-B)} \tilde{L}(u) \right) \leq P(|\xi| \geq u) \leq \exp \left( -C_1 u^{1/(1-B)} \tilde{L}(u) \right).
\]

**(10)**

**Proof.** Since
\[
\sum_{k=2}^{\infty} D \left( k^{-B} L(k) \epsilon(k) \right) < \infty,
\]
then there exists the r.v. \( \xi \) and has a symmetric distribution.

Introduce for all values \( \lambda \in \mathbb{R}^1 \) the function
\[
\varphi(\lambda) \overset{def}{=} \log \mathbb{E} \exp(\lambda \xi) = \sum_{k=2}^{\infty} \log \cosh \left( k^{-B} L(k) \lambda \right).
\]
We have at \( \lambda \to +\infty : \varphi(\lambda) \sim \int_{2}^{\infty} \log \cosh \left( x^{-B} L(x) \lambda \right) dx \sim \lambda^{1/B} \left( \int_{0}^{\infty} \log \cosh \left( z^{-B} L \left( \frac{\lambda^{1/B}}{B} \right) \right) dz \right) \sim \lambda^{1/B} \int_{0}^{\infty} \log \cosh \left( z^{-B} \right) dz \in (0, \infty).
\]

since \( B \in (0.5; 1) \).

Assertion (10) follows from the main result of paper [2].

**Theorem 7.** Let \( \xi, \eta \) be a random variables belonging to the \( EL_m \) space, \( m > 0, L(\cdot) \in SL \). Denote \( \tau = \xi \eta \).

**(A).** We assert that for \( x \geq 2 \)
\[
P(|\tau| > x) \leq \exp \left( -C x^{m/2} L(\sqrt{x}) \right).
\]

**(11)**

**(B).** Inversely, assume that the random variables \( \xi, \eta \) are independent, identically symmetrical distributed and \( \exists L(\cdot) \in SV, \forall x \geq 2 \Rightarrow \)
\[
\exp \left( -C_2 x^m L(x) \right) \leq P(|\xi| > x) =
\]


$P(|\eta| > x) \leq \exp (-C_1 x^m L(x)),$

where $0 < C_1 < C_2 < \infty$ (the case $C_1 = C_2$ is trivial). Statement: $x \geq 2 \Rightarrow$

$P(|\tau| > x) \geq \exp \left(-C_3 x^{m/2} L(\sqrt{x})\right).$ \hfill (12)

**Proof.** As long as

$P(\xi^2 > x) \leq \exp \left(-C_1 x^{m/2} L(\sqrt{x})\right),$

the random variables $\xi^2, \eta^2$ belong to the Orlicz space $EL_{m/2}$ with $L$ function $L(\sqrt{x}), x \geq 2$. The first assertion (A) it follows from the linear properties of the $EL_{m/2}$ spaces and from the elementary relation: $2\xi\eta = (\xi + \eta)^2 - \xi^2 - \eta^2$.

Now we are going to prove the assertion (B). Let the value $x$ be sufficiently large. We have for some $Y = const \in \left(1, \sqrt[2m]{C_2/C_1}\right)$ and $k = 2, 3, \ldots,$ using the full probability formula, since the function $L(\cdot)$ is slowly varying:

$P := P(|\tau| > x) \geq$

$\sum_k P(|\xi\eta| > x)/\left(Y^k \leq |\eta| < Y^{k+1}\right) \times P(|\eta| \in \left[Y^k, Y^{k+1}\right]) \geq$

$\sum_k P(|\xi| > xY^{-k-1}) \times P(|\eta| \in \left[Y^k, Y^{k+1}\right]) \geq$

$\sum_k \exp \left(-C_2 Y^{-m}(x/Y^k)^m L(x/Y^{k+1})\right) \times$

$\times \left[\exp \left(-C_2 Y^k\right) - \exp \left(-C_1 Y^{k+1} L(Y^{k+1})\right)\right] \geq$

$\sum_k \exp \left(-C_4 (x/Y^k)^m L(x/Y^k) - C_5 Y^k L(Y^k)\right)$

Choosing in this sum the member with $k = k_1(x) = \text{Ent}[\log_Y(\sqrt{x})]$, where $\text{Ent}[z]$ is an integer part of $z$, we obtain for sufficiently large values $x$

$\bar{P} \geq \exp \left(-C_6 x^{m/2} L(\sqrt{x})\right).$

3. APPLICATIONS TO THE OPERATOR’S THEORY.
Let \((\Omega_1, F_1, \mu_1)\) and \((\Omega_2, F_2, \mu_2)\) be two probability spaces, \(L^j_p = L^j_p(\Omega_j), G^j_m = G_m(\Omega_j), j = 1, 2;\)

\[ |f|^{(j)} = |f|L^j_p, \quad ||f||^{(j)}_m = ||f||G^j_m, \]

and \(Q = Q[f], f : \Omega_1 \to R^1, Q[f] : \Omega_2 \to R^1\) be an operator, not necessary linear, defined on the set

\[ \text{Dom}(Q) = \cap_{p \geq 2} L^1_p \]

with image

\[ \text{Im}(Q) = \{Q[f], f \in \text{Dom}(Q)\} \subset \cup_{p \geq 2} L^2_p = L^2. \]

**Theorem 8.** Suppose that there exist some constants \(a \geq 0, b, d, C, C_1 \in (0, \infty)\) such that \(\forall p \geq p_0 = \text{const} \geq 2 \Rightarrow\)

\[ |Q[f]|^{(2)}_p \leq C_1 p^a [||f||^{(1)}_{C^p^b}]^d. \]  

(13)

Then for all values \(m > 0\) the operator \(Q\) may be defined on the Orlicz space \(G^1_m\) with values into Orlicz space \(G^2_n, n = m/(am + bd)\) and \(\forall f \in G^1_m \Rightarrow\)

\[ ||Qf||^{(2)}_n \leq C [||f||^{(1)}_m]^d. \]  

(14)

**Proof** is very simple. Let \(f \in G^1_m, ||f||^{(1)}_m = 1, \) then \(\forall p \geq 2 \Rightarrow |f|_p \leq C_2p^{1/m}\). It follows from condition (13) that

\[ |Q[f]|^{(2)}_p \leq C_3 p^a \cdot \left(p^{bd/m}\right)^d [||f||^{(1)}_m]^d = C_3p^{1/n}. \]

Our statement (14) follows from theorem 1.

There are many operators satisfying the condition (13), for example: Hilbert operator ([5], p.119), singular integral operators of a type Kalderon - Zygmund, Hardy operator ([18], p. 42), operator of solution of linear or non - linear evolution equation [6] and many others. In all known cases the values \(a\) are only \(a = 0; a = 1/2\) and \(a = 1\).

A very simple example of operator \(Q[\cdot]\) with \(d \neq 1 : Q[f](\omega) = f^d(\omega)\).

Let us consider now in detail partial Fourier sums in the bounded domain \([0; 1]\) with usually Lebesque measure \(\mu :\)

\[ S_N[f](x) = 0.5a_0 + \sum_{k=1}^N (a_k \cos(2\pi x) + b_k \sin(2\pi x)), \]

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where $a_k, b_k$ are Fourier's coefficients of the (integrable) function $f$. It is well-known (theorem of M. Riesz):

$$p \geq 2 \Rightarrow |H[f]|_p \leq C_p \|f\|_p,$$

where the symbol $H[\cdot]$ denotes the Hilbert transform for a function defined on the interval $(0, 1)$ and, equally, (see, for example, [5], p. 119 - 121) for some other $C \in (0, \infty)$

$$\sup_{N \geq 1} |S_N[f]|_p \leq C_p \|f\|_p,$$

where $C$ is an absolute constant, i.e. here $a = 1$ uniformly on $N$. It is proved in the paper [14] that this estimate is exact at $p \to \infty$.

By virtue of theorem 8 we conclude: \( \forall m > 0 \text{ and } \forall f(\cdot) \in G_m \Rightarrow \)

$$||H[f]||_{m/(m+1)} \leq C_0(m) \|f\|_m,$$

$$\sup_{N \geq 1} ||S_N[f]||_{m/(m+1)} \leq C_1(m) ||f||_m.$$

**Lemma 1.** The previous constant $m/(m + 1)$ is optimal in the case $m \geq 1$. In detail, \( \forall m \geq 1 \exists g \in G_m \Rightarrow \forall \Delta > 0 \)

$$||H[g]||_{(m+\Delta)/(m+1)} = \infty.$$

**Proof.** Let us introduce the function

$$g(x) = g_m(x) = \sqrt[\sqrt]{|\log x|}.$$ 

Since 

$$\forall u \geq 0 \Rightarrow \mu \{x : g_m(x) > u\} = \exp(-Cu^m),$$

we conclude: \( g_m(\cdot) \in G_m \setminus G_0 \) (theorem 5). Further, it is very simple to verify using the explicit view of Hilbert transform ([5], p. 112) that

$$C_1 \left( |\log x|^{(m+1)/m} + 1 \right) \leq |H[g_m](x)| \leq C_2 \left( |\log x|^{(m+1)/m} + 1 \right).$$

Hence \( \forall u \geq 2 \)

$$\exp \left( -C_3u^{m/(m+1)} \right) \leq \mu \{x, H[g_m](x) > u\} \leq \exp \left( -C_4u^{m/(m+1)} \right)$$
and again follows from theorem 5:

$$H[g_m] \in G_m/(m+1) \setminus G_m^0/(m+1).$$

Hence $H[g_m](\cdot)$ does not belong to the space $G_{(m+\Delta)/(m+1)}$ for all values $\Delta \in (0, \infty)$.

Consequently, the Fourier sums $S_N[f]$ converge to $f$ for all functions $f \in G_m$ in the other Orlicz space $G(\nu)$:

$$\lim_{N \to \infty} ||S_N[f] - f||G(\nu) = 0$$

if for example

$$\lim_{p \to \infty} \nu(p)^{(1+1/m)} = \infty$$

Now we construct the examples (for all values $m > 0$) of the function $g \in G_m$ such that

$$\lim_{N \to \infty} ||S_N[g] - g||_m \neq 0. \tag{15}$$

Put again $g(x) = g_m(x)$, but here $m \in (0, \infty)$. It is evident that $g(\cdot) \in G_m \setminus G_m^0$ (theorem 5). As long as the trigonometrical system is bounded, the assertion (15) is true.

4. APPLICATIONS TO THE THEORY OF MARTINGALES.

Let $(\Omega, F, \mathbf{P})$ be an arbitrary probability space with some filtration (flow of $\sigma$-algebras) $(\emptyset, \Omega) = F_0 \subset F_1 \subset F_2 \ldots \subset F_n \ldots \subset F$ and $(S_n, F_n)$ be a centered martingale: $\forall n = 1, 2, \ldots \mathbf{E}S_n = 0$, $\mathbf{E}|S_n| < \infty$, and

$$\mathbf{E}S_n/F_{n-1} = S_{n-1} \pmod{\mathbf{P}}, \ n = 1, 2, \ldots.$$

It is well known ([7], p.18; [13]) that if for some $p > 1$ sup$_n |S_n|_p < \infty$ then the martingale $S_n$ converges a.e. and in $L_p$ norm: $\exists S = \lim S_n \pmod{\mathbf{P}}$, at $n \to \infty$; $S \in L_p$ and

$$\lim_{n \to \infty} |S_n - S|_p \to 0.$$

Some generalizations of this statement are obtained in the publications [13], [11], p. 217; in particular, on the Orlicz spaces $(\Omega, F, \mathbf{P}, N(\cdot))$ with $N$ function (convex, even etc.) belonging to the $\Delta_2 \cap \nabla_2$ class:

$$\Delta_2 = \{N : \exists (u_0 > 0, \beta < \infty), \forall u \geq u_0 \Rightarrow N(2u) \leq \beta N(u)\},$$
\( \nabla_2 = \{ N : \exists (u_0 > 0, l > 1), \forall u \geq u_0 \Rightarrow N(u) \leq N(\frac{lu}{2l}) \} \).

For example, if \( N(u) = N_m(L), m > 1 \), then \( N(\cdot) \in \Delta_2 \cap \nabla_2 \).

Recall that if \( N(\cdot) \in \Delta_2 \cap \nabla_2 \) and the measure \( \mathbf{P} \) is diffuse that the Orlicz space \( L(N) \) is separable and reflexive.

(Note that the considered above Exponential Orlicz \( N \) – functions \( N \in LW \) do not satisfy the \( \Delta_2 \) condition.)

It is proved, more exactly, in [13], [11], p. 217 that if \( N \in \Delta_2 \cap \nabla_2 \), then for arbitrary martingale \((S_n, F_n)\)

\[
\sup_n \|S_n\|_{L(N)} < \infty \Rightarrow \lim_{n \to \infty} \|S_n - S\|_{L(N)} = 0.
\]

We investigate in this section the convergence \( S_n \) to \( S \) in the Orlicz spaces with \( N \)– function \( M(\cdot) \) without \( \Delta_2 \) condition. Namely, we describe for any fixed exponential \( N \)– function \( N = N(\cdot) \in ENF \) the set of other \( N \)– functions \( M = M(u) \) such that for arbitrary martingale \((S_n, F_n)\) holds the following implication:

\[
\sup_n \|S_n\|_{L(N)} < \infty \Rightarrow \lim_{n \to \infty} \|S_n - S\|_{L(M)} = 0. \tag{16}
\]

Generally speaking, in the considered case \( N \in ENF \Rightarrow \lim_{u \to 0^+} M(u)/N(u) = 0 \). Further we construct some examples when functions \( M, N \) are not equivalent.

It follows from the theory of Orlicz spaces ([8], theorem 13.4) that the condition \( M(\cdot) \ll N(\cdot) \) is sufficient for the implication (16). Further we find some other conditions (necessary conditions and sufficient conditions).

For some \( \psi \) – function \( \psi \in \Psi \) mentioned above, \( p \geq 2, \delta \in (0, 1) \) we define a new function \( R = R(\delta, p, \psi(\cdot)) = \)

\[
\inf \left[ \delta^{2/(p\beta+2)} \psi^{p\beta/(p\beta+2)}(\alpha p) : \alpha, \beta > 1, 1/\alpha + 1/\beta = 1 \right],
\]

where \( \inf \) is calculated over all values \((\alpha, \beta)\) such that \( \alpha, \beta > 1, 1/\alpha + 1/\beta = 1 \).

**Theorem 9.** Let \( \nu = \nu(p) \) be some function, \( \nu \in \Psi \), so that

\[
\lim_{\delta \to 0^+} \sup_{p \geq 2} R(\delta, p, \psi(\cdot))/\nu(p) = 0. \tag{17}
\]

Then for all martingales \((S_n, F_n)\) such that

\[
\sup_n \|S_n\|_{G(\psi)} < \infty \tag{18}
\]
follows the implication:

\[ n \to \infty \Rightarrow ||S_n - S||G(\nu) \to 0, \quad (19) \]

i.e. the martingale \( S_n \) converges in the sense of \( G(\nu) \) norm.

**Proof.** From our condition (and classical theorem of Doob) follows that there exists \( S = \lim_{n \to \infty} S_n \) a.e. and \( \sigma_n^2 \to \sigma^2 \), where \( \sigma_n^2 = DS_n \leq \sigma^2 = DS \).

Let us denote \( \gamma^2 = \gamma_n^2 = \sigma^2 - \sigma_n^2 = |S - S_n|^2 \), \( (\gamma_n \to 0, \ n \to \infty) \);

\[ K = \sup_n ||S_n||G(\psi) < \infty. \]

We obtain using Chebyshev inequality:

\[ P(|S_n - S| > \varepsilon) \leq (\sigma^2 - \sigma_n^2)/\varepsilon^2 = \gamma_n^2/\varepsilon^2. \]

Put \( \xi_n = S_n - S \). We have by virtue of Doob’s inequality, since \( \sup_n ||S_n||G(\psi) < \infty \):

\[ \| \max_{1 \leq n} S_l \|G(\psi) \leq \sup_{p \geq 2} \{|p/(p - 1)| \ |S_n|\psi(p)\} \leq 2 \sup_{p \geq 2} |S_n|\psi(p) = 2 ||S_n||G(\psi) \leq 2K, \]

hence

\[ ||S||G(\psi) \leq ||S_n||G(\psi) \leq 4K, \]

\[ \sup_n ||\xi_n||G(\psi) \leq \sup_n ||S_n||G(\psi) + ||S||G(\psi) \leq 5K. \]

Further we have for all \( p \geq 2, \varepsilon > 0 \)

\[ E|\xi_n|^p = \int_\Omega |\xi_n|^p d\mathbf{P} \leq \varepsilon^p + \int_\Omega |\xi_n|^p I(|\xi_n| \geq \varepsilon) d\mathbf{P}. \]

We get estimating the right side by the Hölder inequality

\[
\int_\Omega |\xi_n|^p I(|\xi_n| > \varepsilon) d\mathbf{P} \leq \left( \int_\Omega |\xi_n|^\alpha d\mathbf{P} \right)^{1/\alpha} \times
\]

\[ \times (P(|\xi_n| > \varepsilon)^{1/\beta} \leq |\xi_n|_\alpha \times (\gamma_n/\varepsilon)^{2/\beta} \leq 5^p K^p \psi^p(\alpha \varepsilon) (\gamma_n/\varepsilon)^{2/\beta}, \]

where as above \( \alpha, \beta > 1, 1/\alpha + 1/\beta = 1 \). Therefore

\[ E|\xi_n|^p \leq \varepsilon^p + 5^p K^p \psi^p(\alpha \varepsilon) \gamma_n^{2/\beta} \varepsilon^{-2/\beta}. \]
After the minimization of the right side over $\varepsilon > 0$ and $(\alpha, \beta)$ we obtain
\[ E|\xi_n|^p \leq 2 \cdot 5^p R^p(\gamma_n, p, K \times \psi(\cdot)), \quad |\xi_n|^p \leq 5\sqrt{2} R(\gamma_n, p, K \times \psi(\cdot)) \]
and
\[ ||\xi_n||G(\nu) \leq 5\sqrt{2} \sup_{p \geq 2} V(\gamma_n, p, K \times \psi)/\nu(p) \rightarrow 0 \]
at $n \rightarrow \infty$ by virtue of condition (17). This completes the proof.

Note that theorem 9 gives the concrete estimation $||S_n - S||G(\nu)$ in the term of $D(S_n - S) = |S_n - S|_2^2$: $||S_n - S||G(\nu) \leq 5\sqrt{2} \sup_{p \geq 2} V(|S_n - S|_2, p, [\sup_n ||S_n||G(\psi)] \times \psi(\cdot)).$

By virtue of Doob’s inequality we can obtain analogous maximal inequality for the values $\tau_n = || \sup_{t \geq n} |S_t - S||G(\nu)$:
\[ \tau_n \leq 10\sqrt{2} \sup_{p \geq 2} V(|S_n - S|_2, p, [\sup_n ||S_n||G(\psi)] \times \psi(\cdot))/\nu(p). \]

For example suppose that for some $m > 0$ $\sup_n ||S_n||_m = C_1 < \infty$. Let $\Delta = const > 0.$ Let us denote $\gamma_n = |S_n - S|_2.$ We obtain:
\[ ||S_n - S||_{m/(m\Delta + 1)} \leq C(\Delta) \log \gamma_n^-\Delta, \]
and, moreover,
\[ ||\sup_{t \geq n} |S_t - S||_{m/(m\Delta + 1)} \leq C(\Delta) \log \gamma_n^-\Delta. \]

**Corollary 1.** Since
\[ V(\delta, p, \psi) \leq C \delta^{1/(p+1)} \psi^{(p+1)/(2p)}, \]
we obtain the following sufficient condition for $\nu(\cdot)$: if
\[ \lim_{\delta \rightarrow 0+} \sup_{p \geq 2} \delta^{1/(p+1)} \psi^{(p+1)/(2p)}/\nu(p) = 0, \tag{20} \]
then for any martingale $(S_n, F_n)$ such that $\sup_n ||S_n||G(\psi) < \infty$ it follows $||S_n - S||G(\nu) \rightarrow 0$ at $n \rightarrow \infty$.

**Corollary 2.** If $\psi(\cdot) \in \Delta_2$, then the condition
\[ \lim_{p \rightarrow \infty} \psi(p)/\nu(p) = 0, \tag{21} \]
or, equally, $N_{\nu}(\cdot) \ll N_{\psi}(\cdot)$ is sufficient for the implication (16).

For instance, if $\psi(p) = \psi_{m,r}(p) = p^{1/m} \log^r p$, $m > 0, r \in \mathbb{R}$, then $\nu(p)$ may be, for example,

$$\nu(p) = p^{\Delta+1/m} \log^r p, \quad \nu(p) = p^{1/m} \log^{\Delta+r} p,$$

$$\nu(p) = p^{1/m} \log^r p \log^{\Delta}(2 + \log p), \quad \Delta = \text{const} > 0.$$ 

etc. In particular, if for some martingale $(S_n, F_n)$ there exists $m > 0$ such that $\sup_n ||S_n||_m < \infty$, then $\forall \Delta \in (0, m)$

$$\lim_{n \to \infty} ||S_n - S||_{m-\Delta} = 0.$$ 

Further we obtain the necessary and sufficient conditions for implication (16) in the case if $\psi \in EL_m$.

**Theorem 10.** Let $\psi \in EL_m$ for some $m > 0$; let $\nu(\cdot)$ be some function belonging $\Psi$. The following implication is true:

[$\forall G(\psi) - \text{bounded martingale} (S_n, F_n)$:

$$\sup_n ||S_n|| G(\psi) < \infty \Rightarrow \lim_n ||S_n - S|| G(\nu) = 0]$$

if and only if

$$\lim_{p \to \infty} \psi(p)/\nu(p) = 0,$$

or $N_{\nu} \ll N_{\psi}$.

**Proof.** Sufficientness it follows immediately from theorem 7 and corollary 2. In order to prove necessity of condition (24) we must prove that if condition (24) is not satisfied then there exists a martingale $(S_n, F_n)$ (which may be defined on some probability space) such that $\sup_n ||S_n|| L(N) < \infty$ but $S_n$ does not converge in the $L(M)$ norm. Here $N = N_\psi, M = N_\nu$. Put for some $L_0 \in SL$ and $B = \text{const} \in (0, 0.5; 1)$

$$S_n = \sum_{k=2}^{n} k^{-B} L_0(k) \epsilon(k), \quad S = \sum_{k=2}^{\infty} k^{-B} L_0(k) \epsilon(k),$$

where $\{\epsilon(k)\}$ is again Rademacher sequence, $F_n = \sigma\{\epsilon(i), i \leq n\}$. We choose $B$ and $L_0(\cdot)$ in the case $m > 2$ such that

$$B = m/(m+1), \quad L_0^{-1/(1-B)} \left(u^{1/(1-B)}\right) = L(u),$$

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where \( N_\psi(u) = \exp(u^m L(u)), \ u \geq 2. \)

From theorem 5 follows that
\[
S \in G(\psi_N) \setminus G^0(\psi_N).
\]

Assume converse for condition (24), i.e. that
\[
\lim_{p \to \infty} \psi(p)/\nu(p) > 0.
\]

Since \(|S_n - S||G(\nu) \to 0\) as \(n \to \infty\) and since \(S_n\) is bounded:
\[
vraisup_{\omega \in \Omega} |S_n| \leq \sum_{k=2}^{n} k^{-B} L_0(k),
\]

the random variable \(S\) belongs to the space \(G^0(\nu)\). Therefore (see theorem 2)
\[
\lim_{p \to \infty} |S|_p/\nu(p) = 0.
\]

But according to our conditions it follows that
\[
\lim_{p \to \infty} |S|_p/\nu(p) > 0.
\]

This contradictions proves theorem 10, but only in the case \(m > 2\). In order to prove our statement for the values \(m \in (1, 2]\), we consider a new martingale
\[
S^{(2)}_n = \sum_{i,j=1,2,...,n;i \neq j} i^{-B} L_0(i) j^{-B} L_0(j) \epsilon(i, 1) \epsilon(j, 2),
\]

with correspondence \(\sigma - \text{flow} \ F_n = \sigma\{\epsilon(i, 1), \epsilon(j, 2); \ i, j \leq n\}\), where \(\epsilon(i, s)\) are independent sequences (over \(s = 1, 2, \ldots\)) of Rademacher series, \(B\) again belongs to the interval \((0.5; 1)\). It follows from theorem 7 and the representation \(S_n^{(2)} = \)
\[
= \sum_{i=1}^{n} i^{-B} L_0(i) \epsilon(i, 1) \times \sum_{j=1}^{n} j^{-B} L_0(j) \epsilon(j, 2) - \sum_{i=1}^{n} i^{-2B} L^2_0(i) \epsilon(i, 1) \epsilon(i, 2)
\]

that for all \(x \geq 2\)
\[
\exp(-C_1 x^{m/2} L(\sqrt{x})) \leq P(|S^{(2)}_n| > x) \leq \exp(-C_1 x^{m/2} L(\sqrt{x})).
\]
As above we conclude that if
$$\lim_{p \to \infty} \psi_N(p)/\nu_M(p) > 0,$$
then $S^{(2)} \in EOS(M_{m/2,L(v)})$ \ EOS$^0(M_{m/2,L(v)})$. Therefore
$$\lim_{n \to \infty} ||S_n^{(2)} - S^{(2)}||L(N_{m/2,L(v)}) \neq 0$$
and now $m/2 \in (1; \infty)$. Analogously the case $m \in (0.5; \infty)$ etc. may be considered.

**Remark 4.** The same result as in theorem 10 is also true in the case $V(Z, \beta)$ spaces. Namely, for all martingales $(S_n, F_n)$ from condition
$$\sup_n ||S_n||G(\psi^{(Z, \beta)}) < \infty$$
follows that for some $\nu(\cdot) \in \Psi$
$$\lim_{n \to \infty} ||S_n - S||G(\nu) = 0$$
if and only if $\lim_{p \to \infty} \psi^{(Z, \beta)}(p)/\nu(p) = 0$.

The conclusion "if" follows from theorem 9, the counterexample in the spirit of theorem 10 may be constructed by formula
$$Y_n = \sum_{d=1}^{\infty} C(d) Y(d, n), \ Y = a.e. \lim_{n \to \infty} Y_n.$$
Now let us choose $C(d) : C(d) = d^{-\gamma}, \gamma = \text{const} > 0$. We obtain from the triangular inequality:

$$|Y_n|_p \leq \sum_{d=1}^{\infty} C_d \left( p^{1/\gamma} \right)^d d^{-d\gamma} \leq \exp \left( C_5(\gamma, m)p^{1/(m \gamma)} \right)$$  \hfill (25)

(upper bound). Now we obtain the low bound for $|Y_n|_p$. Since the martingales $\{Y(d, n)\}$ are independent $|Y_n|_p \geq \sup_{d} |Y(d, n)|_p \geq \sup_{d} C_d \left( p^{1/\gamma} \right)^d d^{-d\gamma} \geq \exp \left( C_4(\gamma, m)p^{1/(m \gamma)} \right)$.

From estimations (25), (26) follows that

$$Y \in V(Z, \beta) \setminus V^0(Z, \beta), \quad \beta = 1/(m\gamma), \quad \exists Z \in (0, \infty).$$

Therefore, if

$$\lim_{p \to \infty} \psi^{(Z, \beta)}(p)/\nu(p) > 0$$

then

$$\lim_{n \to \infty} ||Y_n - Y||_{G(\nu)} = 0.$$

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REFERENCES

1. H. Aimar, E. Harbour and B. Iaffel. Boundedness of Convolution Operators with smooth Kernels on Orlicz Spaces. *Studia Math.*, 151(3), (2002), 195 - 206.
2. D. R. Bagdasarov, E. I. Ostrovsky. Reversion of Chebyshev’s Inequality. *Probab. Theory Appl.*, v. 40 N° 4, 737 - 742.
3. V. V. Buldygin, D. I. Mushtary, E. I. Ostrovsky, M. I. Puchalsky. New Trends in Probability Theory and Statistic. MOKSLAS, 1992, Amsterdam, New York.
4. V. V. Buldygin, Ju. V. Kozachenko. Metric Characterization of Random Variables and Random Processes. AMS, 678, 2000, Providence, RI.
5. R. E. Edwards. Fourier Series. A Modern Introduction. Springer Verlag, 1982, Berlin, Heidelberg, Hong Kong.
6. Y. Giga, H. Sohr. Abstract $L^p$ Estimations for Cauchy Problem with Applications to the Navier - Stokes Equations in Exterior Domains. *J. Funct. Anal.*, 102, N° 1, (1991), 72 - 94.
7. P. Hall, C. C. Heyde. Martingale Limit Theory and its Applications. Academic Press, 1979, New York, London, Toronto, Sydney, San Francisco.
8. M. A. Krasnoselsky, Ya. B. Routisky. Convex Functions and Orlicz Spaces. P. Noordhoff Ltd, 1961, Groningen.
9. A. Kufner, O. John, S. Fuchik. Function Spaces. Academia, Prague and Noordhoff Ltd, 1979, Groningen.
10. J. Lindenstrauss, L. Tsafriri. Classical Banach Spaces. Springer Verlag, 1977, Berlin - Heidelberg, New York.
11. J. Neveu. Discrete - Parameter Martingales. North - Holland Publishing Company, 1975, Amsterdam - Oxford - New York.
12. E. I. Ostrovsky. Exponential Estimations for the Random Fields. OINPE, 1999, Obninsk (in Russian).
13. G. Peshkir. Maximal Inequalities of Kahane - Khintchin type in Orlicz Spaces. Preprint Series N° 33, Institute of Mathematics, 1992, University of Aarhus (Danemark).
14. S. K. Pichorides. On the best values of the constant in the theorem of M. Riesz, Zygmund and Kolmogorov. *Studia Math.*, 44, (1972), 165 - 179.
15. M. M. Rao, Z. D. Ren. Theory of Orlicz Spaces. Marcel Dekker Inc., 1991, New York, Basel, Hong Kong.
16. M. M. Rao, Z. D. Ren. Applications of Orlicz Spaces. Marcel Dekker Inc., 2002, New York, Basel.
17. E. Seneta. Regularly Varying Functions. Springer Verlag, 1976, Berlin,
Heidelberg, New York.
18. E.M.Stein. Singular Integrals and Differentiability Properties of Functions. Princeton Univ. Press, 1970, Princeton, New York.

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Exponential Orlicz Spaces: New Norms and Applications.

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Abstract.

Let $(\Omega, F, P)$ be probability space, $N = N(u)$ be an exponential $N$ – Orlicz function. We introduce in the Orlicz space $(\Omega, F, P, N)$ other norms which are equivalent to the classical Orlicz norm $\| \cdot \|_{L(N)}$, for example, by means of all moments:

$$\| \eta \|_G(\psi) = \sup_{p \geq 2} |\eta|_p / \psi(p), \; |\eta|_p = E^{1/p} |\eta|^p,$$

and show convenience of their applications in the theory of Orlicz spaces, in the operator theory, in the theory of Fourier series and in the theory of martingales.

For instance, let $(S_n, F_n)$ be a martingale over some probability space and $N = N(u)$ be some exponential Orlicz function such that

$$\sup_n \| S_n \|_{L(N)} < \infty, \; N \notin \Delta_2.$$

where $\| \cdot \|$ is the classical Luxemburg norm. We study all new $N$ functions $M = M(u)$ such that for all martingales $(S_n, F_n)$

$$\lim_{n \to \infty} \| S_n - S \|_{L(M)} = 0, \; S = a.e. \lim_{n \to \infty} S_n.$$

and show that in general case of exponential Orlicz function $N(u)$ the condition $M(\cdot) \ll N(\cdot)$ is necessary and sufficient for this implication.

References: 18 publications.