On a characterization of idempotent distributions on discrete fields and on the field of $p$-adic numbers

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Abstract

We prove the following theorem. Let $X$ be a discrete field, $\xi$ and $\eta$ be independent identically distributed random variables with values in $X$ and distribution $\mu$. The random variables $S = \xi + \eta$ and $D = (\xi - \eta)^2$ are independent if and only if $\mu$ is an idempotent distribution. A similar result is also proved in the case when $\xi$ and $\eta$ are independent identically distributed random variables with values in the field of $p$-adic numbers $Q_p$, where $p > 2$, assuming that the distribution $\mu$ has a continuous density.

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1 Introduction

According to the classical Kac–Bernstein theorem if $\xi$ and $\eta$ are independent random variables and their sum $\xi + \eta$ and difference $\xi - \eta$ are independent, then the random variables $\xi$ and $\eta$ are Gaussian. This characterization of a Gaussian measure remains true if instead of the difference $\xi - \eta$ we consider its square $(\xi - \eta)^2$, but assume that the random variables $\xi$ are $\eta$ identically distributed. This characterization of a Gaussian measure is a particular case of the well-known Geary theorem: if $\xi_1, \xi_2, \ldots, \xi_n, n \geq 2$, are independent identically distributed random variables such that the sample mean $\bar{\xi} = \frac{1}{n} \sum_{j=1}^{n} \xi_j$ and the sample variance $s^2 = \frac{1}{n} \sum_{j=1}^{n} (\xi_j - \bar{\xi})^2$ are independent, then the random variables $\xi_j$ are Gaussian ([8], [12], [11], [17]).

A lot of research are devoted to generalizations of the classical characterization theorems of mathematical statistics to different algebraic structures, first of all to locally compact Abelian groups (see e.g. [1], [11], [6], [7], [9], [13], [15], and also [5], where one can find additional references). So far they dealt only with linear forms of independent random variables with values in a group, where coefficients of the linear forms were topological automorphisms of the group (the Kac–Bernstein theorem, the Skitovich–Darmois theorem, etc).

In this article we study the simplest "nonlinear" characterization problem. Let $X$ be a locally compact field. Let $\xi$ and $\eta$ be independent identically distributed random variables with values in $X$ and distribution $\mu$. Assume that the random variables $S = \xi + \eta$ and $D = (\xi - \eta)^2$ are independent. Assuming that $X$ is a discrete field, we prove that then $\mu$ is idempotent distribution. A similar result we prove in the case, when $X$ is the field of $p$-adic numbers $Q_p$, where $p > 2$, assuming that $\mu$ has a density with respect to a Haar measure on $Q_p$, and this density is continuous. Since on totally disconnected locally compact Abelian groups, in particular, on discrete groups and on $Q_p$ idempotent distributions are a natural analogue of Gaussian measures, the obtained results one can consider as a natural analogue of the corresponding characterization theorem both for discrete fields and the field $Q_p$.

Let $X$ be a locally compact field. Denote by $e$ the unit of the field $X$. The additive group of the field $X$ is a locally compact Abelian group. We also denote this group by $X$. Denote by $(x, y)$, $x, y \in X$, elements of the group $X^2$. Denote by $T$ the mapping $T : X^2 \rightarrow X^2$ defined by the formula $T(x, y) = (x + y, (x - y)^2)$. Let $A$ be a subset of $X$. Put $A[2] = \{x \in X : x = t^2, t \in A\}$. Let $G$ be an arbitrary locally compact Abelian group. Denote by $m_G$ a Haar measure on $G$. If $G$ is a compact group, then we assume that $m_G$ is a distribution, i.e. $m_G(G) = 1$. Denote by $I(X)$ the set of all
idempotent distributions on $X$, i.e. the set of all shifts of Haar distributions $m_K$ of compact subgroup $K$ of the group $X$. If $x \in X$ denote by $E_x$ the degenerate distribution concentrated at the point $x$. If $\xi$ and $\eta$ are random variables with values in $X$, then we denote by $\mu_\xi$ the distribution of the random variable $\xi$, and by $\mu_{(\xi, \eta)}$ the distribution of the random vector $(\xi, \eta)$.

2 Characterization of idempotent distributions on discrete fields

We study in this section the case when independent random variables take values in a discrete field $X$. First we consider the case when independent random variables take values in a discrete field $X$. Taking into account the independence of the random variable $T$ we study in this section the case when independent random variables take values in a discrete field $X$. Note that if the characteristic of a field $X$ differs from 0, and then the case when the characteristic of the field $X$ is equal to 0.

Moreover (6) implies that $\mu_D(0) \neq 0$, because otherwise $\mu(x) = 0$ for all $x \in X$, contrary to the condition of the theorem. Find $\mu_S(u)$ from (6) and substitute it to (7). We get

\[
\mu^2 \left( \frac{u}{2} \right) \mu_D(t^2) = 2\mu_D(0)\mu \left( \frac{u + t}{2} \right) \mu \left( \frac{u - t}{2} \right), \quad u \in X, \ t \neq 0.
\]
Rewrite (8) in the form
\[ \mu^2(u)\mu_D(4t^2) = 2\mu_D(0) (\mu(u + t) \mu(u - t)), \quad u \in X, \ t \neq 0. \] (9)

Putting in (9) \( u = 0 \), we obtain
\[ \mu_D(4t^2) = \frac{2\mu_D(0)\mu(t)\mu(-t)}{\mu^2(0)}, \quad t \neq 0. \] (10)

It follows from (9) and that (11)
\[ \mu^2(u)\mu(t)\mu(-t) = \mu^2(0)\mu(u + t)\mu(u - t), \quad u \in X, \ t \neq 0. \] (11)

Note that (11), obviously, holds true also when \( t = 0 \). Therefore the function \( \mu(x) \) satisfies equation (1). So, we proved the necessity.

To prove sufficiency we first note that random variables \( \xi_1 \) and \( \xi_2 \) with values in a discrete space \( X \) are independent if and only if there exist functions \( a(x) \) and \( b(x) \) on \( X \) such that
\[ \mu_{\xi_1,\xi_2}(x, y) = a(x)b(y), \quad x, y \in X. \] (12)

Let \( \xi \) and \( \eta \) be independent identically distributed random variables with values in \( X \) and distribution \( \mu \) such that \( \mu(0) > 0 \). It is obvious that if \( v \notin X^{[2]} \), then \( \mu_{(S,D)}(u, v) = 0 \). It follows from (11) and (12) that we have a representation
\[ \mu_{(S,D)}(u, v) = \begin{cases} \mu^2\left(\frac{u}{2}\right), & u \in X, v = 0; \\ 2\mu\left(\frac{u}{2}\right)\mu\left(\frac{u}{2}\right), & u \in X, v = t^2, t \neq 0; \\ 0, & u \in X, v \notin X^{[2]}. \end{cases} \]

This representation implies that if a function \( \mu(x) \) satisfies equation (1), then the random variables \( S \) and \( D \) are independent, because in this case we can put in (12) \( a(x) = \mu^2\left(\frac{x}{2}\right) \) and
\[ b(x) = \begin{cases} 1, & x = 0; \\ 2\mu\left(\frac{x}{2}\right)\mu\left(-\frac{x}{2}\right), & x = t^2, t \neq 0; \\ 0, & x \notin X^{[2]}. \end{cases} \]

**Lemma 2.** Let \( X \) be an Abelian group with unique division by 2. Let \( \mu \) be a function on \( X \), satisfying equation (1) and the condition \( \mu(0) > 0 \). Then the set \( K = \{x \in X : \mu(x) \neq 0\} \) is a subgroup of \( X \).

**Proof.** Assume that \( \mu(x) \neq 0 \) at a point \( x \in X \). Put in (11) \( u = v = \frac{x}{2} \). We get
\[ \mu^3\left(\frac{x}{2}\right)\mu\left(-\frac{x}{2}\right) = \mu^3(0)\mu(x). \] (13)

Since \( \mu(0) \neq 0 \) and \( \mu(x) \neq 0 \), it follows from (13) that
\[ \mu\left(\frac{x}{2}\right)\mu\left(-\frac{x}{2}\right) \neq 0. \] (14)

Put in (11) \( u = v = -\frac{x}{2} \). We obtain
\[ \mu^3\left(-\frac{x}{2}\right)\mu\left(\frac{x}{2}\right) = \mu^3(0)\mu(-x). \] (15)

Taking into account (13), we imply from (15) that \( \mu(-x) \neq 0 \). So, we proved that if \( \mu(x) \neq 0 \), then also \( \mu(-x) \neq 0 \). When it is considered, (11) implies that the set \( K = \{x \in X : \mu(x) \neq 0\} \) is a subgroup of \( X \).
First we prove a theorem on characterization of idempotent distributions for a discrete fields of nonzero characteristic.

**Theorem 1.** Let $X$ be a countable discrete field of characteristic $p$, where $p > 2$. Let $\xi$ and $\eta$ be independent identically distributed random variables with values in $X$ and distribution $\mu$. In order that the random variables $S = \xi + \eta$ and $D = (\xi - \eta)^2$ be independent it is necessary and sufficient that $\mu \in I(X)$.

**Proof.** Necessity. Replacing if it is necessary the random variables $\xi$ and $\eta$ by new independent random variables $\xi + x$ and $\eta + x$, we can assume from the beginning that $\mu(0) > 0$. Then Lemma 1 implies that the function $\mu(x)$ satisfies equation (1). Since the characteristic of the field $X$ is greater than 2, $X$ is an Abelian group with unique division by 2. Hence, by Lemma 2 the set $K = \{x \in X : \mu(x) > 0\}$ is a subgroup of $X$. Let $x_0 \in K$, $x_0 \neq 0$. Denote by $L$ the subgroup generating by $x_0$. Then $L \cong \mathbb{Z}(p)$, where $\mathbb{Z}(p)$ is the group of residue modulo $p$, and $L \subset K$. Consider the restriction of equation (1) to $L$. Put $\varphi(x) = \log \mu(x)$, $x \in L$. It follows from (1) that

$$2\varphi(u) + \varphi(v) + \varphi(-v) = 2\varphi(0) + \varphi(u + v) + \varphi(u - v), \quad u, v \in L.$$ 

Integrate both sides of this equality by the measure $dm_L(v)$. We get that $\varphi(u) = \varphi(0)$ for all $u \in L$. It means that $\mu(x) = \mu(0)$ for all $x \in L$, and hence, $\mu(x) = \mu(0)$ for all $x \in K$. This implies that $K$ is a finite subgroup and $\mu = m_K$. The necessity is proved.

Sufficiency. Let $K$ be a finite subgroup of $X$. Denote by $|K|$ the number of elements of the subgroup $K$. Let $\xi$ and $\eta$ be independent identically distributed random variables with values in $X$ and distribution $\mu = m_K$. Since all nonzero elements of the field $X$ have order $p$, and $p > 2$, the following statement holds: if $2x \in K$, then $x \in K$. This easily implies that the function

$$\mu(x) = m_K(x) = \begin{cases} |K|^{-1}, & x \in K; \\ 0, & x \notin K. \end{cases}$$

satisfies equation (1), and hence, by Lemma 1 the random variables $S$ and $D$ are independent.

**Remark 1.** Let $X$ be a discrete field of characteristic 2. Let $\xi$ and $\eta$ be independent random variables with values in $X$ and distributions $\mu$ and $\nu$. If $E \mu \nu$ the random variables $S = \xi + \eta$ and $D = (\xi - \eta)^2$ are independent, then $\mu$ and $\nu$ are degenerate distributions. Indeed, in a field of characteristic 2 the equality $(\xi - \eta)^2 = (\xi + \eta)^2$ holds, i.e. $D = S^2$. But as it easily seen, in a filed of characteristic 2 if $\mu_S$ is a nondegenerate distribution, then $\mu_S^2$ is also a nondegenerate distribution. Hence, in a field of characteristic 2 independence of $S$ and $S^2$ implies that $S$ has a degenerate distribution (compare below with Remark 2). But then $\mu$ and $\nu$ are also degenerate distributions.

We prove now a theorem on characterization of idempotent distributions for a discrete field of characteristic 0.

**Theorem 2.** Let $X$ be a countable discrete field of characteristic 0. Let $\xi$ and $\eta$ be independent identically distributed random variables with values in $X$ and distribution $\mu$. If the random variables $S = \xi + \eta$ and $D = (\xi - \eta)^2$ are independent, then $\mu$ is a degenerate distribution.

**Proof.** Replacing if it is necessary the random variables $\xi$ and $\eta$ by new independent random variables $\xi + x$ and $\eta + x$, we can assume from the beginning that $\mu(0) > 0$. Then Lemma 1 implies that the function $\mu(x)$ satisfies equation (1). Since the characteristic of the field $X$ is 0, $X$ is an Abelian group with unique division by 2. Then by Lemma 2 the set $K = \{x \in X : \mu(x) > 0\}$ is a subgroup of $X$.

Assume that $\mu(x_0) > 0$ at a point $x_0 \in X$, $x_0 \neq 0$. Consider a subgroup of $X$ of the form

$$G = \left\{ \frac{m x_0}{2^n} : m = 0, \pm 1, \pm 2, \ldots, n = 0, 1, \ldots \right\}.$$

Taking into account that $K$ is a subgroup, equation (1) implies that $\mu(x) > 0$ for all $x \in G$. Consider the restriction of equation (1) to the subgroup $L$. Put $\varphi(x) = \log \mu(x)$, $x \in G$. It follows from (1) that

$$2\varphi(0) + \varphi(u + v) + \varphi(u - v) = 2\varphi(u) + \varphi(v) + \varphi(-v), \quad u, v \in G.$$

(16)
Let $h$ be an arbitrary element of the group $G$. Substituting in (16) $u$ for $u + h$ and $v$ for $v + h$, we obtain
\[2\varphi(0) + \varphi(u + v + 2h) + \varphi(u - v) = 2\varphi(u + h) + \varphi(v + h) + \varphi(-v - h), \quad u, v, h \in G.\] (17)
Subtracting (16) from (17) we find
\[\Delta_{2h}\varphi(u + v) = 2\Delta_h\varphi(u) + \Delta_h\varphi(v) + \Delta_{-h}\varphi(-v), \quad u, v, h \in G.\] (18)
Putting in (18) $u = 0$, we get
\[\Delta_{2h}\varphi(v) = 2\Delta_h\varphi(0) + \Delta_h\varphi(v) + \Delta_{-h}\varphi(-v), \quad v, h \in G.\] (19)
Subtracting (19) from (18), we obtain
\[\Delta_{2h}\Delta_u\varphi(v) = 2\Delta_h\Delta_u\varphi(0), \quad u, v, h \in G.\] (20)
Let $k$ be an arbitrary element of the group $G$. Substituting in (20) $v$ for $v + k$ and subtracting from the obtained equation equation (20), we find
\[\Delta_{2h}\Delta_u\Delta_k\varphi(v) = 0, \quad u, v, h, k \in G.\] (21)
Since $u$, $h$ and $k$ are arbitrary elements of $G$, it follows from (21) that the function $\varphi(x)$ satisfies the equation
\[\Delta^3_u\varphi(v) = 0, \quad u, v \in G.\] (22)
Let $x$ be an arbitrary element of the group $G$. Then $x = rx_0$, where $r = \frac{m}{n}$. As it easily follows from (22) $\varphi(x) = ar^2 + br + c$, where $a, b, c$ are some constants. Hence, $\mu(x) = e^{ar^2+br+c}$. But this is impossible because
\[\sum_{x \in G} \mu(x) \leq 1.\]
Thus, there not exists a point $x_0 \in X$, $x_0 \neq 0$, such that $\mu(x_0) > 0$, and hence $\mu = E_0$.

Remark 2. Let $X$ be a countable discrete field of characteristic $p$, where $p \neq 2$. Let $\xi$ and $\eta$ be independent random variables with values in $X$ and distributions $\mu$ and $\nu$. Assume that the random variables $S = \xi + \eta$ and $D = (\xi - \eta)^2$ are independent. Generally speaking this not implies that both distributions $\mu$ and $\nu$ are idempotent. There is a corresponding example.

Since $p \neq 2$, we have $e \neq -e$. Let $\xi$ and $\eta$ be independent random variables with values in $X$ and distributions $\mu$ and $\nu$ such that $\mu = \frac{1}{2}(E_{-e} + E_e)$, $\nu = E_0$. Then $S = \xi$ $D = \xi^2$. Since $\mu_D$ is a degenerate distribution, the random variables $S$ and $D$ are independent.

3 Characterization of idempotent distributions on the field of $p$-adic numbers

Let $p$ be a fixed prime number. Denote by $\mathbb{Q}_p$ the field of $p$-adic numbers. We need some simple properties of the field of $\mathbb{Q}_p$. As a set $\mathbb{Q}_p$ coincides with a set of doubly infinite sequences of positive integers
\[x = (\ldots, x_{-n}, x_{-n+1}, \ldots, x_{-1}, x_0, x_1, \ldots, x_n, x_{n+1}, \ldots), \quad x_n \in \{0, 1, \ldots, p - 1\}\]
, being $x_n = 0$ for $n < n_0$, where the number $n_0$ depends on $x$. We correspond to each element $x \in \mathbb{Q}_p$ a series $\sum_{k=-\infty}^{\infty} x_kp^k$. Addition and multiplication of series are defined in a natural way and define the
operations of addition and multiplication on $\mathbb{Q}_p$. With respect to these operations $\mathbb{Q}_p$ is a field. Denote by $\mathbb{Z}_p$ a subset of $\mathbb{Q}_p$ consisting of all $x \in \mathbb{Q}_p$ such that $x_n = 0$ for $n < 0$. The set $\mathbb{Z}_p$ is a ring and is called the ring of $p$-adic integers. Denote by $\mathbb{Z}_p^\times$ a subset of $\mathbb{Q}_p$ of the form $\mathbb{Z}_p^\times = \{x \in \mathbb{Q}_p : x_n = 0$ for $n < 0, x_0 \neq 0\}$. The subset $\mathbb{Z}_p^\times$ coincides with the group of invertible elements of the ring $\mathbb{Z}_p$.

Moreover each element $x \in \mathbb{Q}_p$ is uniquely represented in the form $x = p^l e$, where $l$ is an integer, and $e \in \mathbb{Z}_p^\times$. Denote by $e$ the unity of the field $\mathbb{Q}_p$. One can define a norm $|x|_p$ on $\mathbb{Q}_p$ by the following way. If $x \in \mathbb{Q}_p$, $x = p^l e$, where $l$ is an integer, and $e \in \mathbb{Z}_p^\times$, we put $|x|_p = p^{-l}$, $|0|_p = 0$. The norm $|x|_p$ satisfies the conditions: $|x + y|_p \leq \max(|x|_p, |y|_p)$, $|xy|_p = |x|_p |y|_p$ and defines a topology on $\mathbb{Q}_p$.

The field $\mathbb{Q}_p$ with respect to this topology is locally compact, noncompact and totally disconnected, and the ring $\mathbb{Z}_p$ is compact. Choose a Haar measure $m_{\mathbb{Q}_p}$ on $\mathbb{Q}_p$ in such a way that $m_{\mathbb{Q}_p}(\mathbb{Z}_p) = 1$. Then $m_{\mathbb{Q}_p}(p^k \mathbb{Z}_p) = p^{-k}$. We will also assume that $m_{\mathbb{Q}_p^2} = m_{\mathbb{Q}_p} \times m_{\mathbb{Q}_p}$.

To prove the main theorem of this section we need some lemmas.

**Lemma 3.** Consider the field $\mathbb{Q}_p$. Then on the set $\mathbb{Q}_p^2$ there exists a continuous function $s(x)$ satisfying the equation

$$s^2(x) = x, \quad x \in \mathbb{Q}_p^2.$$  

(23)

The set $\mathbb{Q}_p^2 \setminus \{0\}$ is a disjoin union of balls in each of which the function $s(x)$ is expressed by a convergent power series.

**Proof.** The representation

$$\mathbb{Q}_p = \{0\} \cup \bigcup_{l=-\infty}^{\infty} p^l \mathbb{Z}_p^\times$$

implies that

$$\mathbb{Q}_p^2 = \{0\} \cup \bigcup_{l=-\infty}^{\infty} p^{2l} (\mathbb{Z}_p^\times)^2.$$  

(24)

Assume first that $p > 2$. It is well known that in the multiplicative group $\mathbb{Z}_p^\times$ there exists an element $\varepsilon$ of order $p - 1$, and elements $0, \varepsilon, \varepsilon^2, \ldots, \varepsilon^{p-1} = e$ form a complete set of coset representatives of the subgroup $p \mathbb{Z}_p$ in the group $\mathbb{Z}_p$. Put $A_k = \varepsilon^k + p \mathbb{Z}_p = \{x : |x - \varepsilon^k|_p \leq \frac{1}{p}\}, k = 1, 2, \ldots, p - 1$.

Then $\mathbb{Z}_p^\times = \bigcup_{k=1}^{p-1} A_k$. First define the function $s(x)$ on the set $(\mathbb{Z}_p^\times)^2$. Since $(e + p \mathbb{Z}_p)^2 = e + p \mathbb{Z}_p$, we have $A_k^2 = A_{2k}$, if $k = 1, 2, \ldots, \frac{p-1}{2}$, and $A_k^2 = A_{2k+p-1}$, if $k = \frac{p+1}{2}, \frac{p+3}{2}, \ldots, p - 1$. Note that

$$(\mathbb{Z}_p^\times)^2 = \bigcup_{k=1}^{\frac{p-1}{2}} A_{2k},$$

and define the function $s(x)$ on each coset $A_{2k}$. Take $x \in A_{2k}$. Then the equation $x = t^2$ has two solutions $t_1 \in A_k$ and $-t_1 \in A_{k + \frac{p-1}{2}}$. These solutions belong to different cosets. The coset $A_k$ is a compact set, and the function $g(x) = x^2$ is continuous on $A_k$ and it is one-to-one mapping of the set $A_k$ on $A_{2k}$. This implies that the inverse to $g(x)$ mapping $s_k : A_{2k} \mapsto A_k$ is also continuous, and hence is a homeomorphism between $A_{2k}$ and $A_k$. Put $s(x) = s_k(x)$, if $x \in A_{2k}$, $k = 1, 2, \ldots, \frac{p-1}{2}$. Since $A_{2k}$ is an open set in $\mathbb{Q}_p$, the function $s(x)$ is continuous and satisfies equation (23) on $(\mathbb{Z}_p^\times)^2$.

Taking into account (24), put

$$s(x) = \begin{cases} p^l s(c), & x = p^{2l} c, c \in (\mathbb{Z}_p^\times)^2; \\ 0, & x = 0. \end{cases}$$

It is not difficult to verify that the constructed function $s(x)$ in each ball $A_{2k}$ for $k = 1, 2, \ldots, \frac{p-1}{2}$ is expressed by a convergent power series

$$s(x) = \varepsilon^k \left(e + \frac{\varepsilon^{-2k}}{2}(x - \varepsilon^{2k}) + \sum_{n=2}^{\infty} (-1)^{n-1} \frac{(2n - 3)!!}{2n!!} \varepsilon^{-2kn}(x - \varepsilon^{2k})^n \right).$$
This implies that in each ball \( p^{2l}A_{2k}, l = \pm 1, \pm 2, \ldots \) the function \( s(x) \) is also expressed by a convergent power series. It is obvious that the set \( Q^2_p \setminus \{0\} \) is a disjoint union of balls \( p^{2l}A_{2k}, k = 1, 2, \ldots, \frac{p-1}{2}, l = 0, \pm 1, \pm 2, \ldots, \) and \( s(x) \) is the required function.

If \( p = 2 \), the reasoning is changed slightly. Put \( B_k = ke + 4Z_2, k = 0, 1, 2, 3 \). Then the elements 0, 2e, 3e form a complete set of coset representatives of the subgroup \( 4Z_2 \) in the group \( Z_2 \). We have \((2Z_2^2)^2 = B_1^2 = B_3^2 = e + 8Z_2\). Take \( x \in e + 8Z_2 \). Then the equation \( x = t^2 \) has two solutions \( t_1 \in B_1 \) and \( -t_1 \in B_3 \). These solutions belong to different cosets. The rest part of the proof is similar to the case when \( p > 2 \).

**Lemma 4.** Consider the field \( Q_p \), where \( p > 2 \). Take \((x_0, y_0) \in Q^2_p \) such that \(|x_0 - y_0|_p = p^{-l}\). Then for \( k \geq l + 1 \) the following equality

\[
T\{(x_0, y_0) + (p^kZ_p)^2\} = (x_0 + y_0, (x_0 - y_0)^2) + (p^kZ_p) \times (p^{k+1}Z_p)
\]

(25)

holds.

**Proof.** Since \(|x_0 - y_0|_p = p^{-l}\), we have \( x_0 - y_0 = p^l c \), where \( c \in Z_p^x \). Note that on the one hand,

\[
T\{(x_0, y_0) + (p^kZ_p)^2\} = T\{(x_0 + p^k x, y_0 + p^k y) : x, y \in Z_p\} = \{(x_0 + y_0 + p^k(x + y), (x_0 - y_0)^2 + 2p^k(x_0 - y_0)(x - y) + p^{2k}(x - y)^2) : x, y \in Z_p\} = \{(x_0 + y_0 + p^k s, (x_0 - y_0)^2 + 2p^k s + p^{2k} t^2 + p^k t^2) : s, t \in Z_p\}
\]

(26)

holds true for any \( k \). On the other hand,

\[
\{2ct + p^{k-l}t^2 : t \in Z_p\} = Z_p
\]

(27)

holds true for \( k \geq l + 1 \).

Indeed, note that \((e + p^mZ_p^2)^2 = e + p^mZ_p^2\) is fulfilled for any \( m \geq 1 \). This implies that \((c + p^mZ_p^2)^2 = c^2 + p^mZ_p^2\) for all \( c \in Z_p^x \), i.e. \( \{c^2 + 2cp^m t + p^{2m} t^2 : t \in Z_p\} = c^2 + p^mZ_p^2\), and hence, \( \{2ct + p^{m} t^2 : t \in Z_p\} = Z_p \). For \( k \geq l + 1 \) this equality implies (27). Taking into account (27), (25) follows from (4).

**Lemma 5.** Consider the field \( Q_p \), where \( p > 2 \). Let a function \( s \) be as constructed in the proof of Lemma 3. Consider the mappings \( S_j \) from \( Q_p \times Q^2_p \) to \( Q^2_p \) of the form

\[
S_1(u, v) = \left( \frac{u + s(v)}{2}, \frac{u - s(v)}{2} \right), \quad S_2(u, v) = \left( \frac{u - s(v)}{2}, \frac{u + s(v)}{2} \right).
\]

Let \((u_0, v_0) \in Q_p \times Q^2_p \) and \(|s(v_0)|_p = p^{-l}\). Put \( E_k = \{(u_0, v_0) + (p^kZ_p) \times (p^{k+1}Z_p)\} \). Then for \( k \geq l + 1 \) the following statements are valid:

(i) \( E_k \subset Q_p \times Q^2_p \),

(ii) \( S_1(E_k) \cap S_2(E_k) = \emptyset \),

(iii) \[
\int_{S_j(E_k)} \Phi_j(x, y) dm_{Q^2_p}(x, y) = \int_{E_k} \Phi_j(S_j(u, v)) |s(v)|^{-1}_p dm_{Q^2_p}(u, v), \quad j = 1, 2,
\]

for any continuous function \( \Phi_j(x, y) \) on \( S_j(E_k) \).

**Proof.** Note that \(|v_0|_p = p^{-2l}\). It follows from the proof of Lemma 3 that if \( w_0 \in Q^2_p \) and \(|w_0|_p = p^{-2l}\), then \( w_0 + w \in Q^2_p \) for \( w \in p^{2l+1}Z_p \). Since \( k \geq l + 1 \), from what has been said it follows that (i) is fulfilled.
To prove (ii) assume that $S_1(E_k) \cap S_2(E_k) \neq \emptyset$. Then as easily seen, there exist elements $v_1, v_2 \in \mathbb{Z}_p$ such that

$$s(v_0 + p^{k+1} v_1) + s(v_0 + p^{k+1} v_2) \in p^{k} \mathbb{Z}_p.$$  

(28)

Since $v_0 \in \mathbb{Q}_p^{[2]}$ and $|v_0|_p = p^{-2l}$, we have $v_0 = p^{2l} c$, where $c \in (\mathbb{Z}_p^{\times})^{[2]}$ and $v_0 + p^{k+1} v_1 = p^{2l}(c + p^{k-l} v_1)$, \(i = 1, 2\). This implies that $s(v_0 + p^{k+1} v_1) + s(v_0 + p^{k+1} v_2) = p^l(s(c + p^{k-l} v_1) + s(c + p^{k-l} v_2))$. It follows from the definition of the function $s$ that the elements $s(c + p^{k-l} v_1), i = 1, 2$, are at the same coset of the subgroup $p\mathbb{Z}_p$ in the group $\mathbb{Z}_p$, and hence, $s(c + p^{k-l} v_1) + s(c + p^{k-l} v_2) \notin p\mathbb{Z}_p$. Therefore $s(v_0 + p^{k+1} v_1) + s(v_0 + p^{k+1} v_2) \notin p^{l+1}\mathbb{Z}_p$. But this contradicts (23) for $k \geq l + 1$. So, we proved (ii).

Let us prove (iii). We will prove that equality (iii) holds true for $S_1$. For $S_2$ the reasoning is similar. Put $(x_0, y_0) = S_1(u_0, v_0) = \left( \frac{u_0 + s(v_0)}{2}, \frac{u_0 - s(v_0)}{2} \right)$ and check that

$$S_1(E_k) = (x_0, y_0) + (p^k \mathbb{Z}_p)^2.$$  

(29)

We have $S_1(u_0 + u, v_0 + v) = \left( \frac{u_0 + u + s(v_0 + v)}{2}, \frac{u_0 + u - s(v_0 + v)}{2} \right)$. It is easily seen that

$$\frac{|u_0 + u + s(v_0 + v)|}{2} - \frac{|u_0 + s(v_0)|}{2} \leq \max \{|u|_p, |s(v_0 + v) - s(v_0)|_p\}. \quad (30)$$

Under the condition of the theorem

$$|u|_p \leq p^{-k}. \quad (31)$$

Since $s^2(x) = x$, we have

$$|s(v_0 + v) - s(v_0)|_p = \frac{|v|_p}{|s(v_0 + v) + s(v_0)|_p}. \quad (32)$$

Note that $v_0 + v = p^{2l} c + p^{k+l} t$ for some $t \in \mathbb{Z}_p$. This implies that $s(v_0 + v) = p^l s(c + p^{k-l} t)$ and moreover, $s(v_0) = p^l s(c)$. Inasmuch as the points $s(c + p^{k-l} t)$ and $s(c)$ are at the same coset of the subgroup $p\mathbb{Z}_p$ in the group $\mathbb{Z}_p$, so is $|s(c + p^{k-l} t) + s(c)|_p = 1$, and hence, $|s(v_0 + v) + s(v_0)|_p = p^{-l}$. If it is remembered that $|v|_p \leq p^{-k-l}$, we get from (32) that

$$|s(v_0 + v) - s(v_0)|_p \leq p^{-k}. \quad (33)$$

Taking into account (31) and (33), it follows from (30) that inequality

$$\left| \frac{u_0 + u + s(v_0 + v)}{2} - \frac{u_0 + s(v_0)}{2} \right|_p \leq p^{-k}$$

holds, and hence

$$S_1(E_k) \subset (x_0, y_0) + (p^k \mathbb{Z}_p)^2. \quad (34)$$

We note that if $T(a, b) = T(a', b')$, then either $(a, b) = (a', b')$, or $(a, b) = (b', a')$. Since $|x_0 - y_0|_p = |s(v_0)|_p = p^{-l}$, and $k \geq l + 1$, the restriction of the mapping $T$ to the set $(x_0, y_0) + (p^k \mathbb{Z}_p)^2$ is injective. Taking this into account, (23) follows from Lemma 4 and (34). Moreover it follows from has been said the mappings $T$ and $S_1$ are inverse homeomorphisms of the sets $S_1(E_k)$ and $E_k$.

Observe that the set $E_k$ is a product of the balls $E_k = \left\{ u : |u - u_0|_p \leq \frac{1}{p^{l+1}} \right\} \times \left\{ v : |v - v_0|_p \leq \frac{1}{p^{l+1}} \right\}$. Put $x(u, v) = \frac{u + s(u)}{2}$, $y(u, v) = \frac{u - s(u)}{2}$, $(u, v) \in E_k$. The mapping $S_1$ is a homeomorphism of the open compacts $E_k$ and $S_1(E_k)$, and the functions $x(u, v)$ and $y(u, v)$ by Lemma 3 on $E_k$ are expressed by convergent power series in $u$ and $v$. We have $\frac{dx}{du} = \frac{du}{du} = \frac{1}{2}$, $\frac{dx}{dv} = -\frac{1}{4s(u)}, \frac{dy}{dv} = \frac{1}{4s(u)}$. It follows from this that

$$\left| \frac{1}{p^{l+1}} \right|_{p} \neq 0, \quad (u, v) \in E_k. \quad (35)$$

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It is obvious that (iii) follows from the change of variables formula in integrals ([16] §4) and (35).

**Lemma 6.** Consider the field \( \mathbf{Q}_p \), where \( p > 2 \). Let \( \xi \) and \( \eta \) be independent identically distributed random variables with values in \( \mathbf{Q}_p \) and distribution \( \mu \). Assume that \( \mu \) has the density \( \rho \) with respect to \( m_{\mathbf{Q}_p} \) such that \( \rho \) is continuous and \( \rho(0) > 0 \). The random variables \( S = \xi + \eta \) and \( D = (\xi - \eta)^2 \) are independent if and only if the density \( \rho \) satisfies the equation

\[
\rho^2(u)\rho(v)\rho(-v) = \rho^2(0)\rho(u + v)\rho(u - v), \quad u, v \in \mathbf{Q}_p.
\]  

**Proof.** Inasmuch as \( \mu_{(S,D)} = T(\mu_{(\xi,\eta)}) \) and the distribution \( \mu_{(\xi,\eta)} \) is absolutely continuous with respect to \( m_{\mathbf{Q}_p^2} \), so is \( \mu_{(\xi,\eta)} \{ (t, t) : t \in \mathbf{Q}_p \} = 0 \). Therefore the distribution \( \mu_{(S,D)} \) concentrated at the set \( \mathbf{Q}_p \times (\mathbf{Q}_p^2 \setminus \{0\}) \). Let the mappings \( S_j \) and the sets \( E_k \) be the same as in Lemma 5. Take \((u_0, v_0) \in \mathbf{Q}_p \times (\mathbf{Q}_p^2 \setminus \{0\}) \) and represent the element \( \varsigma(v_0) \) in the form \( \varsigma(v_0) = p^j c \), where \( c \in \mathbf{Z}_p^\times \). By Lemma (i) and (ii) are fulfilled for \( k \geq l + 1 \). We have

\[
\mu(\varphi, \varphi) \{ E_k \} = T(\mu_{(\xi,\eta)}) \{ E_k \} = \mu_{(\xi,\eta)} \{ T^{-1}(E_k) \} = \int_{T^{-1}(E_k)} \rho(x)\rho(y)dm_{\mathbf{Q}_p^2}(x, y) = \int_{S_1(E_k)} \rho(x)\rho(y)dm_{\mathbf{Q}_p^2}(x, y) + \int_{S_2(E_k)} \rho(x)\rho(y)dm_{\mathbf{Q}_p^2}(x, y).
\]

Taking into account equalities (iii) of Lemma 5, we transform integrals in the right-hand side of equality (3) and obtain

\[
\int_{S_1(E_k)} \rho(x)\rho(y)dm_{\mathbf{Q}_p^2}(x, y) = \int_{E_k} \rho \left( \frac{u + \varsigma(v)}{2} \right) \rho \left( \frac{u - \varsigma(v)}{2} \right) \left| \varsigma(v) \right|_p^{-1}dm_{\mathbf{Q}_p^2}(u, v),
\]

\[
\int_{S_2(E_k)} \rho(x)\rho(y)dm_{\mathbf{Q}_p^2}(x, y) = \int_{E_k} \rho \left( \frac{u - \varsigma(v)}{2} \right) \rho \left( \frac{u + \varsigma(v)}{2} \right) \left| \varsigma(v) \right|_p^{-1}dm_{\mathbf{Q}_p^2}(u, v).
\]

Then (3) implies that

\[
\mu(\varphi, \varphi) \{ E_k \} = 2\int_{E_k} \rho \left( \frac{u + \varsigma(v)}{2} \right) \rho \left( \frac{u - \varsigma(v)}{2} \right) \left| \varsigma(v) \right|_p^{-1}dm_{\mathbf{Q}_p^2}(u, v).
\]

This equality means that the distribution \( \mu_{(S,D)} \) has a density with respect to \( m_{\mathbf{Q}_p^2} \), and this density is equal to

\[
2\rho \left( \frac{u + \varsigma(v)}{2} \right) \rho \left( \frac{u - \varsigma(v)}{2} \right) \left| \varsigma(v) \right|_p^{-1}, \quad u \in \mathbf{Q}_p, \ v \in \mathbf{Q}_p^2 \setminus \{0\}.
\]  

(38)

Note that when we got representation (38) for the density of distribution \( \mu_{(S,D)} \), we did not use the independence of the random variables \( S \) and \( D \).

By the condition of the lemma the random variables \( S \) and \( D \) are independent. Therefore there exist integrable with respect to \( m_{\mathbf{Q}_p^2} \) functions \( r_j \) on \( \mathbf{Q}_p \) such that the equality

\[
r_1(u)r_2(v) = 2\rho \left( \frac{u + \varsigma(v)}{2} \right) \rho \left( \frac{u - \varsigma(v)}{2} \right) \left| \varsigma(v) \right|_p^{-1}
\]

holds true almost everywhere on \( \mathbf{Q}_p \times (\mathbf{Q}_p^2 \setminus \{0\}) \) with respect to \( m_{\mathbf{Q}_p^2} \). Since the function in the right-hand side of equality (39) is continuous, we can assume without loss of generality that the functions \( r_j \)
are also continuous, and equality \(39\) holds true everywhere on \(Q_p \times (Q_p[2] \setminus \{0\})\). Since \(\rho(0) > 0\), it is easily seen that \(r_1(0) > 0\). Put \(v = t^2\), \(t \neq 0\). It follows from \(39\) that

\[
\rho^2(0) = 2r_1(0) \rho \left( \frac{s(t^2)}{2} \right) \rho \left( -\frac{s(t^2)}{2} \right) |s(t^2)|^{-1}, \quad t \in Q_p, \quad t \neq 0.
\]

Note that \(39\) and \(40\) imply the equality

\[
r_1(u) \rho \left( \frac{s(t^2)}{2} \right) \rho \left( -\frac{s(t^2)}{2} \right) = r_1(0) \rho \left( \frac{u + s(t^2)}{2} \right) \rho \left( \frac{u - s(t^2)}{2} \right),
\]

\((u, t) \in Q_p^2, \quad t \neq 0.\)

It follows from the continuity of \(\rho\) and \(r_1\) that equality \(39\) holds true for all \(u, t \in Q_p\). Put \(43\) in \(39\).

We deduce from the last equality that

\[
r_1(u) = \frac{r_1(0)}{\rho^2(0)} \rho^2 \left( \frac{u}{2} \right), \quad u \in Q_p.
\]

Substituting \(42\) into \(39\), we find that

\[
\rho(0) \left( \frac{s(t^2)}{2} \right) \rho \left( -\frac{s(t^2)}{2} \right) = \rho^2(0) \rho \left( \frac{u + s(t^2)}{2} \right) \rho \left( \frac{u - s(t^2)}{2} \right),
\]

\(u, t \in Q_p.\)

Since either \(s(t^2) = t\), or \(s(t^2) = -t\), the equalities

\[
\rho \left( \frac{s(t^2)}{2} \right) \rho \left( -\frac{s(t^2)}{2} \right) = \rho \left( \frac{t}{2} \right) \rho \left( -\frac{t}{2} \right), \quad t \in Q_p
\]

and

\[
\rho \left( \frac{u + s(t^2)}{2} \right) \rho \left( \frac{u - s(t^2)}{2} \right) = \rho \left( \frac{u + t}{2} \right) \rho \left( \frac{u - t}{2} \right), \quad u, t \in Q_p
\]

are fulfilled. Substituting \(44\) and \(45\) into \(39\), we get that the density \(\rho\) satisfies equation \(36\). The necessity is proved.

Let us prove the sufficiency. Taking into account \(38\) and \(45\) we have the following representation for the density \(\rho\) of the distribution \(\mu(S,D)\):

\[
\rho(u, v) = \begin{cases} 
2\rho \left( \frac{u + t}{2} \right) \rho \left( \frac{u - t}{2} \right) |s(t^2)|^{-1}, & u \in Q_p, \quad v = t^2, \quad t \neq 0; \\
0, & u \in Q_p, \quad v \notin (Q_p[2] \setminus \{0\}).
\end{cases}
\]

If a density \(\rho\) satisfies equation \(36\), it is easily seen that the density \(\rho(u, v)\) is represented as a product of a function of \(u\) and a function of \(v\). This implies the independence of \(S\) and \(D\).

Now we can prove the main theorem of this section.

**Theorem 3.** Consider the field \(Q_p\), where \(p > 2\). Let \(\xi\) and \(\eta\) be independent identically distributed random variables with values in \(Q_p\) and distribution \(\mu\). Assume that \(\mu\) has the density \(\rho\) with respect to \(m_{Q_p}\) such that \(\rho\) is continuous. In order that the random variables \(S = \xi + \eta\) and \(D = (\xi - \eta)^2\) be independent it is necessary and sufficient that \(\mu \in L(Q_p)\).

**Proof.** Necessity. We reason as in the proof of Theorem 1. It is obvious that replacing if it is necessary the random variables \(\xi\) and \(\eta\) by new independent random variables \(\xi + x\) and \(\eta + x\), we can assume from the beginning that \(\rho(0) > 0\). Applying Lemma 6 we find that the density \(\rho\) satisfies equation \(36\). Since the field \(Q_p\) is an Abelian group with unique division by \(2\), by Lemma 2 the set
$K = \{ x \in \mathbb{Q}_p : \rho(x) > 0 \}$ is a subgroup of $\mathbb{Q}_p$. Obviously, $K$ is an open subgroup, and hence a closed one.

Assume first that $K \neq \mathbb{Q}_p$. Since $K \neq \{0\}$, the subgroup $K$ is of the form $K = p^n \mathbb{Z}_p$ for some integer $m$, and hence, $K$ is a compact group. Consider the restriction of equation (36) to $K$. Put $\varphi(x) = \log \rho(x)$, $x \in K$. It follows from (36) that

$$2\varphi(u) + \varphi(v) + \varphi(-v) = 2\varphi(0) + \varphi(u + v) + \varphi(u - v), \quad u, v \in K.$$ Integrating both sides of this equality by the measure $dm_K(v)$, we get that $\varphi(u) = \varphi(0)$ for all $u \in K$. Hence, $\rho(x) = \rho(0)$ for all $x \in K$, but this means that $\mu = m_K$.

If $K = \mathbb{Q}_p$, then we consider the restriction of equation (36) to an arbitrary subgroup $G = p^n \mathbb{Z}_p$ of $\mathbb{Q}_p$. The reasoning above shows that $\rho(x) = \rho(0)$ for all $x \in G$, and hence, $\rho(x) = \rho(0)$ for all $x \in \mathbb{Q}_p$, contrary to the integrability of the density $\rho$. So, the case when $K = \mathbb{Q}_p$, is impossible. The necessity is proved.

Sufficiency. Let $K$ be a nonzero compact subgroup of $\mathbb{Q}_p$. As has been stated above, $K$ is of the form $K = p^n \mathbb{Z}_p$ for some integer $m$. Let $\xi$ and $\eta$ be independent identically distributed random variables with values in $\mathbb{Q}_p$ and distribution $\mu = m_K$. Since $p > 2$, we have $\frac{2}{p} \in K$ for any element $x$ of the subgroup $K$. It is well known that this implies that the random variables $\xi + \eta$ and $\xi - \eta$ are independent ([5, §7])). Hence, the random variables $S$ and $D$ are also independent.

**Remark 3.** Let us discuss the case of the field $\mathbb{Q}_2$. A lemma similar to Lemma 4 holds also true for the field $\mathbb{Q}_2$. Unlike equality (25) for $k \geq l + 2$ the equality

$$T\{ (x_0, y_0) + (2^k \mathbb{Z}_2)^2 \} = (x_0 + y_0, (x_0 - y_0)^2) + (2^k \mathbb{Z}_2) \times (2^{k+l+1} \mathbb{Z}_2)$$

(47)

holds true. Respectively, taking (17) into account, one can also reformulate Lemma 5. It allows to prove Lemma 6 for the field $\mathbb{Q}_2$. Next reasoning as in the proof of the necessity in Theorem 3, we get that if $\xi$ and $\eta$ are independent identically distributed random variables with values in $\mathbb{Q}_2$ and distribution $\mu = m_K$, where the density $\rho$ is continuous, $\rho(0) > 0$ and the random variables $S = \xi + \eta$ and $D = (\xi - \eta)^2$ are independent, then $\mu = m_K$, where $K = 2^m \mathbb{Z}_2$ for some integer $m$. But in this case the corresponding density

$$\rho(x) = \begin{cases} 2^m, & x \in 2^m \mathbb{Z}_2; \\ 0, & x \notin 2^m \mathbb{Z}_2. \end{cases}$$

does not satisfies equation (36). To see this, put in (36) $u = v \in 2^{m-1} \mathbb{Z}_2 \setminus 2^m \mathbb{Z}_2$. Then the left-hand side of (36) is equal to zero, but the right-hand side does not. Thus, do not exist independent identically distributed random variables $\xi$ and $\eta$ with values in the field $\mathbb{Q}_2$ and distribution $\mu$, such that $\mu$ has the density $\rho$ with respect to a Haar measure $m_{\mathbb{Q}_2}$, where the density $\rho$ is continuous and the random variables $S = \xi + \eta$ and $D = (\xi - \eta)^2$ are independent.

4 Comments and unsolved problems

Let $\xi$ and $\eta$ be independent random variables with values in the field of real numbers $\mathbb{R}$. According to the Kac-Bernstein theorem the independence of $\xi + \eta$ and $\xi - \eta$ implies that the random variables $\xi$ and $\eta$ are Gaussian. The similar result holds also true for the considering in the article fields. Indeed, let $\xi$ and $\eta$ be independent random variables with values either in a countable discrete field or in the field $\mathbb{Q}_p$ and with distributions $\mu$ and $\nu$. Since in the both cases the connected component of zero of the field does not contain elements of order 2, the independence of $\xi + \eta$ and $\xi - \eta$ implies that $\mu$ and $\nu$ are idempotent distributions ([5, Theorem 7.10]). In so doing we do not assume that $\xi$ and $\eta$ are identically distributed, and in the case when $\xi$ and $\eta$ take values in the field $\mathbb{Q}_p$ we do not assume that $\mu$ and $\nu$ have continuous densities.
The situation changes if instead of $\xi + \eta$ and $\xi - \eta$ we will consider $S = \xi + \eta$ and $D = (\xi - \eta)^2$. As has been proved in [10], if one consider independent random variables with values in the field of real numbers $\mathbb{R}$, then without the condition of identically distributivity of $\xi$ and $\eta$ the independence of $S$ and $D$, generally speaking, does not imply that both distributions of random variables $\xi$ and $\eta$ are Gaussian. In [10] was constructed an example, when one of the random variables has a Gaussian distribution, but the distribution of the second random variable is a mixture of two Gaussian distributions. It follows from Remark 2 that in Theorems 1 and 2 one can not omit the condition of identically distributivity of independent random variables $\xi$ and $\eta$. The problem, if we can not omit the condition of identically distributivity of independent random variables $\xi$ and $\eta$ in Theorem 3 is unsolved.

The problem, if we can prove Theorem 3 without assumption that the distribution $\mu$ has a density with respect to $m_{\mathbb{Q}_p}$ and this density is continuous, is also unsolved.

Finally, the theorems proved in the article are analogues for the considering fields of a particular case of the theorem Geary. It is interesting to find out if the theorem Geary holds true for the considering fields in the general case, i.e. when $n \geq 2$.

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