Free-parafermionic $Z(N)$ and free-fermionic $XY$ quantum chains

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The relationship between the eigenspectrum of Ising and $XY$ quantum chains is well known. Although the Ising model has a $Z(2)$ symmetry and the $XY$ model a $U(1)$ symmetry, both models are described in terms of free-fermionic quasi-particles. The fermionic quasi-energies are obtained by means of a Jordan-Wigner transformation. On the other hand, there exist in the literature a huge family of $Z(N)$ quantum chains whose eigenspectra, for $N > 2$, are given in terms of free parafermions and they are not derived from the standard Jordan-Wigner transformation. The first members of this family are the $Z(N)$ free-parafermionic Baxter quantum chains. In this paper we introduce a family of $XY$ models that beyond two-body also have $N$-multispin interactions. Similarly to the standard $XY$ model they have a $U(1)$ symmetry and are also solved by the Jordan-Wigner transformation. We show that with appropriate choices of the $N$-multispin couplings, the eigenspectra of these $XY$ models are given in terms of combinations of $Z(N)$ free-parafermionic quasi-energies. In particular all the eigenenergies of the $Z(N)$ free-parafermionic models are also present in the related free-fermionic $XY$ models. The correspondence is established via the identification of the characteristic polynomial which fixes the eigenspectrum. In the $Z(N)$ free-parafermionic models the quasi-energies obey an exclusion circle principle that is not present in the related $N$-multispin $XY$ models.

I. INTRODUCTION

The first step toward understanding an interacting system is to consider the simplest models where the eigenenergies are given in terms of free-particle quasi-energies. These are the so-called free systems and they play an important role in condensed matter, statistical physics and quantum information theory.

The most studied free quantum chains are the quantum Ising model in a transverse field and the $XY$ quantum chains, also known as $XX$ quantum chains \cite{1,2}. They describe the dynamics of spin-$1/2$ quantum spins attached to the sites of a lattice. Although the Ising quantum chain has a simpler $Z(2)$ symmetry and the $XY$ model has a larger $U(1)$ symmetry, the eigenspectra of both models are related. The $2^L$ eigenenergies of the $XY$ model with $L$ sites ($L$ even) are obtained from those of two decoupled Ising quantum chains with $L/2$ sites. In \cite{3} this correspondence is shown for the general case of the Ashkin-Teller model and the XXZ quantum chain. These models for the vanishing anisotropy give us the $XY$ model and two decoupled Ising models.

An interesting extension of these free-fermionic models are the $Z(N)$ free-parafermionic Baxter chains \cite{4,11}. They are generalizations of the Ising models described in terms of $N \times N$ matrices satisfying a $Z(N)$ algebra. On a lattice of size $L$ the $N^L$ eigenenergies $E_{s_1,...,s_L}$ are obtained by combining $L$ distinct pseudo-energies $\varepsilon_k$ ($k = 1,\ldots,L$):

$$E_{s_1,...,s_L} = -\sum_{i=1}^{L} e^{i2\pi s_k/N} \varepsilon_k,$$  

where $s_k = 0,1,\ldots,N-1$.

We have in \cite{11} a $Z(N)$-circle exclusion since any quasi-energy $\varepsilon_k$ enters with one and only one possible value $s_k = 0,1,2,\ldots,N-1$.

The $N = 2$ case recovers the free-fermionic quantum Ising chain, and the fermion exclusion principle translates into the $Z(2)$-circle exclusion. On each eigenlevel the appearance of the energy $\varepsilon_k$ excludes the appearance of $e^{i2\pi/2} \varepsilon_k = -\varepsilon_k$. In the related free-fermionic $XY$ model, with $2L$ sites, the same quasi-energies $\varepsilon_k$ ($k = 1,\ldots,L$) appear, but now the energies may or may appear, independently, and we do not have the $Z(2)$-circle exclusion observed in the quantum Ising chain.

The aim of this paper is to introduce a new family of $U(1)$-symmetric $XY$ models that extend the observed Ising-$XY$ correspondence to the $Z(N)$ free-parafermionic Baxter chains, with $N > 2$. Toward this end, we introduce an extension of the $XY$ model that beyond the usual two-body interactions also contains $N$-multispin interactions ($N = 2,3,\ldots$). The $N = 2$ case recovers the standard $XY$ model (or the $XX$ model), but for $N > 2$ the models have a free-fermionic eigenspectra with complex quasi-energies which are associated with the ones appearing in a $Z(N)$ free-parafermionic Baxter chain, but with no $Z(N)$-circle exclusion. Although they share eigenvalues with a quantum chain with $Z(N)$ symmetry, these models have an $U(1)$ symmetry, since the $z$-component...
of the total magnetization is a good quantum number.

Recently, a new family of $Z(N)$ free-parafermionic quantum chains with $(p + 1)$-multispin interactions were introduced ($p = 1, 2, \ldots$) \cite{12, 13}. In the $N = 2$ case these multispin models are also related to a free-fermionic model in a frustration graph \cite{14}. The $p = 1$ case recovers the $Z(N)$ free-parafermionic Baxter chains. These models for $N > 2$ are non-Hermitian and exhibit multicritical points belonging to new universality classes of critical behavior. In the cases where $N \geq (p + 1)$ we were able to derive $N$-multispin XY quantum chains whose eigen-spectra are given in terms of the quasi-energies of these new $Z(N)$ multispin models. That is, the quasi-energies forming the eigenspectra of both models are the same. In particular, all the eigenvalues of these free parafermionic multispin models are also present in the eigenspectra of the free-fermionic XY quantum chains with $N$-multispin interactions.

The correspondences reported in this paper allow us to give the exact ground-state energy and the critical exponents at the multicritical points of the generalized multispin XY quantum chains.

This paper is organized as follows. In the next section we review the $(p + 1)$-multispin $Z(N)$ quantum chains introduced in \cite{12, 13}. In Sec. III, we present the generalized XY models with $N$-multispin interactions. In Sec. IV, we give the correspondence of the generalized XY models with the $Z(N)$ free-parafermionic Baxter chains. Finally, in Sec. V we present our conclusions. In Appendix A, we extend the results of Sec. IV by giving the XY models related to the new families \cite{12, 13} of $Z(N)$ free-parafermionic multispin models of Sec. II. In Appendix B, we provide some simple numerical comparisons of the quasi-energies of the multispin XY and $Z(N)$ quantum chains.

II. GENERAL FREE-FERMIONIC AND FREE-ParaFERMIonic QUANTUM CHAINS

Recently, in \cite{12, 13}, a large family of quantum chains were introduced whose eigenspectra are given in terms of free quasi-particles. Here we collect some key results. On their general formulation these Hamiltonians are written as a sum of $M$ generators \{$h_i$\},

$$H^{(N,p)}_M(\lambda_1, \ldots, \lambda_M) = -\sum_{i=1}^M h_i, \quad (2)$$

where $N = 2, 3, \ldots$, and $p = 1, 2, \ldots$. The generators satisfy the $Z(N)$ exchange algebra:

$$h_i h_{i+m} = \omega h_{i+m} h_i \quad \text{for} \quad 1 \leq m \leq p; \quad \omega = e^{i2\pi/N},$$

$$[h_i, h_j] = 0 \quad \text{for} \quad |j - i| > p, \quad (3)$$

with the closure relation

$$h_i^N = \lambda_i^N, \quad (4)$$
given in terms of the scalars \{$\lambda_i$\}. For any representation \cite{3} and \cite{4} with integer values of $N \geq 2$ and $p \geq 1$ the Hamiltonian has a free-particle spectrum. For $N = 2$ we have a free-fermionic spectrum and for $N > 2$ we have a free-parafermionic one.

The standard quantum Ising model in a lattice with $L = (M + 1)/2$ sites, and arbitrary couplings \{$\lambda_i$\} give us a representation for $N = 2, p = 1$ and odd $M$, namely

$$H^{(2,1)}_{2L-1}(\lambda_1, \ldots, \lambda_{2L-1}) = -\sum_{i=1}^L \lambda_{2i-1} \sigma_i^x - \sum_{i=1}^{L-1} \lambda_{2i} \sigma_i^z \sigma_{i+1}^z, \quad (5)$$

where $\sigma_i^x, \sigma_i^z$ are the standard Pauli matrices acting on the $i$-th site of the chain. Another representation for the fermionic case $N = 2$, but for $p = 2$, is given by the 3-spin interacting Fendley model \cite{15}, where

$$H^{(2,2)}_M(\lambda_i) = -\sum_{i=1}^M \lambda_i \sigma_i^z \sigma_{i+1}^z \sigma_{i+2}^z. \quad (6)$$

The general integrability condition for interacting multispin quantum chains, as presented in \cite{16}, also explains the integrability of \cite{3, 4}.

A possible $Z(2)$ representation of \cite{3} and \cite{4}, with arbitrary values of the integer parameter $p \geq 1$ is given by the $(p + 1)$-interacting quantum chain

$$H^{(2,p)}_M(\lambda_i) = -\sum_{i=1}^M \lambda_i \sigma_i^z \sigma_{i+1}^z \cdots \sigma_{i+p-1}^z \sigma_{i+p}^z, \quad (7)$$
on a lattice with $M + p$ sites and open boundary conditions.

For the $N > 2$ cases the representations, due to \cite{3} and \cite{4}, are given in terms of the $Z(N)$ generalized Pauli matrices. They are $N \times N$ matrices satisfying

$$XZ = \omega ZX, \quad X^N = Z^N = 1, \quad Z^d = Z^{N-1}, \quad (8)$$

and, as before, $\omega = \exp(i2\pi/N)$. In the $Z$-basis they are given by the $Z(N)$-cyclic matrices

$$Z_{i,j} = \omega^{i-1} \delta_{i,j}, \quad X_{i,j} = \delta_{j-i+1} + \delta_{i-j-N-1}; \quad i, j = 1, \ldots, N.$$ 

The $Z(N)$ generalization of the Ising representation \cite{5} for $p = 1$, gives us the free-parafermionic Baxter quantum chain \cite{4}, with the Hamiltonian given by

$$H^{(N,1)}_{2L-1} = -\sum_{i=1}^L \lambda_{2i-1} X_i - \sum_{i=1}^{L-1} \lambda_{2i} Z_i Z_{i+1}, \quad (9)$$

while the generalization of \cite{7}, for arbitrary $p$, leads to

$$H^{(N,p)}_M = -\sum_{i=1}^M \lambda_i Z_i Z_{i+1} \cdots Z_{i+p-1} X_{i+p}. \quad (10)$$
The non-vanishing eigenenergies of all the Hamiltonians, \[5\]-\[7\], \[9\] and \[10\], apart from degeneracies, are given by
\[
E_{s_1,\ldots,s_M} = -\sum_{i=1}^M \omega^s_i \varepsilon_i, \quad (11)
\]
where \(s_i\) takes one and only one of the possible values \(s_i = 0, 1, \ldots, N - 1\). The number \(M\) of quasi-energies \(\varepsilon_i\) is given by
\[
M \equiv \text{int} \left( \frac{M + p}{p + 1} \right) = \left\lfloor \frac{M + p}{p + 1} \right\rfloor, \quad (12)
\]
where hereafter we use \([a]\) to denote the integer part of \(a\).

Each quasi-energy enters a single time in the composition of the eigenenergies \((11)\). We can visualize the eigenenergies in \((11)\) as in Fig. 1, where we draw concentric circles of radius \(\varepsilon_i\) in the complex plane. The intersections of the circles with the radial lines at angle \(\frac{2\pi j}{N}\) \((j = 0, 1, \ldots, N - 1)\) give us the possible contributions of the quasi-energies (open circles in Fig. 1) to the eigenenergies. Each circle gives one and only one contribution to \((11)\) (filled circles in Fig. 1). This means that the quasi-particles \(\varepsilon_i\) obey a \(Z(N)\)-circle exclusion principle that in the \(N = 2\) case is similar to the Pauli exclusion principle.

The quasi-energies in \((11)\)
\[
\varepsilon_i = 1/z_i^{1/N}, \quad (13)
\]
as shown in \[15\] for \(p = N = 2\), and for general values of \(p\) and \(N\) in \[12\]-\[13\], are obtained from the roots \(z_i\) \((i = 1, \ldots, M)\) of the polynomials
\[
P_M^{(p)}(z) = \sum_{\ell=0}^M C_M(\ell) z^\ell, \quad P_M^{(p)}(z_i) = 0, \quad (14)
\]
satisfying the recurrence relations
\[
P_M^{(p)}(z) = P_{M-1}^{(p)}(z) - \lambda_M^{N-1} P_{M-(p+1)}^{(p)}(z), \quad (15)
\]
for \(M \geq 1\), with the initial condition
\[
P_M^{(p)}(z) = 1, \text{ for } M \leq 0. \quad (16)
\]
For real \(\{\lambda_i\}\), we have checked that the roots of \((14)\) are real.

The ground state energy \(E_0\) of the general \((N,p)\) models is real and it is obtained by taking \(s_1 = \cdots = s_M = 0\) in \((11)\).
\[
E_0 = E_{0,0,\ldots,0} = -\sum_{i=1}^M \varepsilon_i. \quad (17)
\]
While the ground state is real, the energy levels are in general complex. We define the lower excited states as those with the lowest real part eigenvalue. In particular, the first excited states are obtained by taking the value \(s_1 \neq 0\) and \(s_k = 0\) for \(k = 2, \ldots, M\) in \((11)\), that is, \(E_{\text{exc},j} = E_0 + (1 - \omega^j) \varepsilon_j\) \((j = 1, \ldots, N - 1)\). Accordingly, the mass gap is defined as,
\[
\text{Re}(\text{gap}_j) = (1 - \cos(\frac{2\pi j}{N})) \varepsilon_1; \quad j = 1, \ldots, N - 1 \quad (18)
\]
In Fig. 1 the configuration for \(N = 2\) is the ground-state while the one for \(N = 3\) is one of the excited states that give the smallest gap.

In \[12\]-\[13\] the \((N,p)\) models are shown to be critical at their isotropic point \(\{\lambda_i\} = 1\). The ground-state energy per site, at this critical point, is obtained analytically. It is given \[12\] in terms of the Lauricella hypergeometric series \(F^{(p-1)}_D\) \[18\]. Also, the gaps at these critical points have the finite-size leading dependence
\[
\text{Re}(\text{gap}) \sim \frac{1}{M^z}, \quad z = \frac{p + 1}{N}, \quad (19)
\]
giving us the dynamical critical exponent \(z\). The specific heat exponent \(\alpha\), for \(p = 1\) is known analytically: \(\alpha = 1 - 2/N\) \[10\]. For \(p > 1\), the numerical solutions for the polynomial roots of \((15)-(16)\) for large lattice sizes, give us \[13\]
\[
\alpha = \max\{0, 1 - \frac{p + 1}{N}\}. \quad (20)
\]

### III. THE XY QUANTUM CHAINS WITH MULTISPIN INTERACTIONS

We introduce in this section the general \(U(1)\)-symmetric XY models, which contain, in addition to nearest-neighbor interactions, \(N\)-multispin interactions \((N = 2, 3, \ldots)\). In their general form the Hamiltonians are given by
\[
H_M^{N,XY}(\{\mu_i\}, \{\gamma_i\}) = \sum_{i=1}^{M+N-2} \mu_i \sigma_i^+ \sigma_{i+1}^- + \sum_{i=1}^{M} \sigma_i^+ \left( \prod_{j=i+1}^{i+N-2} \sigma_j^- \right) \sigma_{i+N-1}^+, \quad (21)
\]
where \( \sigma^\pm = (\sigma^x \pm i\sigma^y)/2 \) and \( \{\mu_i\}, \{\gamma_i\} \) are real coupling constants. We can simplify the above Hamiltonian by performing the canonical transformation

\[
\sigma_i^+ \to \sigma_i^+, \sigma_i^- \to \left( \prod_{j=1}^{i-1} \mu_j \right)^{\pm 1} \sigma_i^+, (i = 2, \ldots, M + N - 1),
\]

\[
\sigma_i^- \to \sigma_i^+ (i = 1, \ldots, M + N - 1),
\]

which gives

\[
\begin{align*}
H_M^{(N,XY)}(\{\lambda_i\}) &= \sum_{i=1}^{M+N-2} \sigma_i^+ \sigma_{i+1}^- \\
&+ \sum_{i=1}^{M} \lambda_i^N \sigma_i^+ \left( \prod_{j=i+1}^{i+N-2} \sigma_j^- \right) \sigma_{i+N-1}^+,
\end{align*}
\]

(23)

where we have the effective parameters

\[
\lambda_i^N = \gamma_i \left( \prod_{j=i}^{i+N-2} \mu_j \right).
\]

(24)

In Fig. 2 we draw the interactions in the Hamiltonian \( H_M^{(N,XY)} \) for some values of \( N \).

![Diagram of Hamiltonian interactions](image)

We can verify that in the \( N = 2 \) case, the Hamiltonian \( H_M^{(2,XY)} \), after the canonical transformation

\[
\sigma_i^+ \to \sigma_i^+, \sigma_i^- \to \left( \prod_{j=1}^{i-1} \lambda_j \right)^{\pm 1} \sigma_i^+, (i = 2, \ldots, M + N - 1),
\]

\[
\sigma_i^- \to \sigma_i^+ (i = 1, \ldots, M + N - 1),
\]

(25)

recovers the standard two-body non-uniform XY quantum chain (or XX quantum chain):

\[
H_M^{(2,XY)}(\{\lambda_i\}) = \sum_{i=1}^{M-1} \frac{\lambda_i^2}{2} (\sigma_i^x \sigma_{i+1}^x + \sigma_i^y \sigma_{i+1}^y),
\]

(26)

having a real eigenspectrum due to its Hermiticity. For \( N > 2 \), similarly to the \( Z(N) \) free-parafermionic quantum chains of the preceding section, the Hamiltonians \( H_M^{(N,XY)} \) are not Hermitian. However for all values of \( N \), instead of being \( Z(N) \) invariant the models have a larger \( U(1) \) symmetry due to their commutations \([H_M^{(N,XY)}, \sigma_i^z] = 0\), with the \( z \)-magnetization \( S^z = \sum_{i=1}^{M+N-1} \sigma_i^z \).

In order to proceed let us introduce spinless fermionic operators through the Jordan-Wigner transformation \( \{\lambda_i\} \)

\[
c_i = \sigma_i^+ \prod_{j=1}^{i-1} \sigma_j^+, c_i^\dagger = \sigma_i^- \prod_{j=1}^{i-1} \sigma_j^-, \quad (i = 1, \ldots, M + N - 1),
\]

(27)

for \( i = 1, \ldots, M + N - 1 \), which satisfy the fermionic algebra

\[
\{c_i, c_j^\dagger\} = \delta_{i,j}, \quad \{c_i, c_j\} = 0.
\]

(28)

The Hamiltonian \( \hat{H} \), in terms of these fermionic operators, has the bilinear form

\[
\hat{H} = -\sum_{i,j=1}^{M+N-1} c_i^\dagger \hat{A}_{i,j} c_j,
\]

(29)

where the connectivity matrix \( \hat{A} \) has the simple banded matrix form (tridiagonal for \( N = 2 \)):

\[
\hat{A}_{i,j} = \delta_{j+i+1} + \lambda_j^N \delta_{j+i-1-N}.
\]

(30)

Consider the transformation \( \{c_i, c_i^\dagger\} \to \{\eta_i, \eta_i^\dagger\} \):

\[
\eta_k = \sum_i L_{i,k} c_i, \quad \eta_k^\dagger = \sum_i L_{i,k}^T c_i^\dagger,
\]

(31)

where \( L_{i,j} \) and \( R_{i,j} \) are the components \( i \) of the left and right eigenvector \( j \) of the matrix \( A \), with eigenvalue \( \Lambda_j \), i.e.,

\[
\sum_{k=1}^{M+N-1} \Lambda_{i,k} L_{k,i} = \sum_{k=1}^{M+N-1} \Lambda_{i,k} R_{k,i} = \Lambda_j L_{i,j}^T = \Lambda_j R_{i,j}^T.
\]

Since \( A \) is diagonalizable \( L_i^T R = 1 \), that also implies \( R_i^T T = 1 \). These relations and the fermionic relations \( \{c_i, c_j\} = 0 \) imply that \( \eta_k, \eta_k^\dagger \) are also fermionic operators:

\[
\{\eta_k, \eta_{k'}^\dagger\} = \delta_{k,k'}, \quad \{\eta_k, \eta_{k'}\} = 0.
\]

(32)

Inverting \( \{31\} \) we obtain

\[
c_i = \sum_{k=1}^{M+N-1} R_{i,k} \eta_k, \quad c_i^\dagger = \sum_{k=1}^{M+N-1} L_{i,k} \eta_k^\dagger.
\]

(33)

Inserting \( \{33\} \) in \( \{29\} \) we obtain

\[
\hat{H} = -\sum_{k=1}^{M+N-1} \Lambda_k \eta_k^\dagger \eta_k.
\]

(34)

This implies that all the \( 2^{M+N-1} \) eigenvalues of \( \hat{H} \) are obtained from the \( M + N - 1 \) quasi-energies \( \Lambda_k \) given by the eigenvalues of the matrix \( A \), i.e.,

\[
E_{s_1,\ldots,s_{M+N-1}} = -\sum_{k=1}^{M+N-1} s_k \Lambda_k; \quad s_k = 0, 1.
\]

(35)
The quasi-energies $\Lambda_k$ are obtained by solving $\det(A - \Lambda_k \mathbb{I}) = 0$. Apart from the zero modes ($\Lambda_0 = 0$), which will change the degeneracy of the eigenspectrum, these quasi-energies are obtained from the zeros $\{z_k\}$ of the characteristic polynomial $P^{(N-1)}(z)$, given by

$$P^{(N-1)}_M(u) \equiv \det(\mathbf{1} - Au), \quad P^{(N-1)}_M(u_k) = 0,$$

where $u_k = \frac{1}{\varepsilon}$. These polynomials, as a consequence of the Laplace cofactor’s rule for determinants, satisfy the recurrence relation

$$P^{(N-1)}_M(z) = P^{(N-1)}_{M-1}(z) - \varepsilon z^N P^{(N-1)}_{M-N}(z),$$

where $z = (1/\Lambda)^N$, and

$$P^{(N-1)}_M(z) = 1, \text{ for } M \leq 0.$$  

From (37) we can see that the order of the polynomial is $[(M + N - 1)/N]$. Comparing (37) and (38) with (15) and (16), we can see that the polynomials $P^{(N-1)}_M(z)$ are the same as those fixing the eigenspectra of the $Z(N)$ multispins chains with $N = p + 1$ multispin interactions. The same roots $P^{(N-1)}_M(z_i) = 0 \ (i = 1, \ldots, [(M + N - 1)/N])$ that give the quasi-energies $\varepsilon_i = 1/z_i^{1/N}$ of the $Z(N)$ free-parafermionic multispin models also give us the quasi-energies of the fermionic XY chains with $N$-multispin interactions. Each root $z_i$ gives us $N$ fermionic quasi-energies

$$\Lambda_{j,i} = e^{i \frac{2\pi}{N} j \varepsilon_i}, \quad i = 1, \ldots, \left[\frac{M + N - 1}{N}\right]; \quad j = 0, 1, \ldots, N - 1.$$  

Since the total number of fermionic quasi-energies in (35) is $M + N - 1$, we should have

$$N_z = M + N - 1 - N \left[\frac{N + M - 1}{N}\right]$$

zero modes, producing a $2^{N_z}$-degeneracy in the whole eigenspectrum. Apart from this degeneracy the eigenlevels of the $N$-multispin interacting XY models are given by

$$E_{\{s_{i,j}, r_{i,j}\}} = -\sum_{i=1}^{\left\lfloor\frac{M+N-1}{N}\right\rfloor} \left(\sum_{j=0}^{N-1} r_{i,j} \omega^{s_{i,j}}\right) \varepsilon_i,$$

where for each $i = 1, \ldots, \left\lfloor\frac{M+N-1}{N}\right\rfloor$, $s_{i,j}$ takes one of the possible values $s_{i,j} = 0, 1, \ldots, N - 1$ and $r_{i,j} = 0, 1$.

In the next section we compare the eigenspectrum of the $Z(N)$ free-parafermionic quantum chains and the $N$-multispin XY models.

IV. THE CORRESPONDENCE OF THE XY MODEL WITH $N$-MULTISPIN INTERACTIONS AND THE $Z(N)$ FREE-PARAFERMIONIC BAXTER CHAIN

We show in this section that the eigenspectrum of the $N$-multispin interacting XY models, defined in the previous section, contains the eigenspectrum of the $Z(N)$ free-parafermionic Baxter quantum chains. This extends the known correspondence of the eigenspectra of the standard Ising and XY quantum chains.

Let us consider initially the case $N = 2$. By comparing (37) and (38) with (15) and (16), we see that the polynomials $P^{(1)}_M(z)$, whose roots $\varepsilon_i = 1/z_i^{1/2}$ fix the eigenspectra of the Ising and XY models are the same. This means from (11) that, while a given quasi-energy contributes to the eigenlevels with $\varepsilon_i$ or $-\varepsilon_i$, in the Ising chain, the contribution in the XY model can be of four possible values: $0, -\varepsilon_i, +\varepsilon_i, \varepsilon_i - \varepsilon_i \equiv -\varepsilon_i, 0, 0, +\varepsilon_i$.

In Fig. 3 we show pictorially the contribution of a given quasi-energy $\varepsilon_i$ to the eigenspectra of both models. We can see from the figure that while in the Ising case we have a $Z(2)$ circle exclusion for the quasi-energy, in the XY model there is no such exclusion. An independent check of this correspondence comes from the algebraic properties of $\sigma^x_i \sigma^x_{i+1}$ and $\sigma^y_i \sigma^y_{i+1}$ operators in (26). The XY model (26) and two decoupled Ising models with coupling $\{\lambda_i/2\}$ (see (3)) are given in terms of density-energy operators satisfying the same algebraic rules. This implies that the contributions of the eigenenergies are $(\varepsilon_i/2, \varepsilon_i/2) \oplus (-\varepsilon_i/2, \varepsilon_i/2)$ i.e., $(-\varepsilon_i, 0, 0, \varepsilon_i)$.

Let us now consider the general cases $N \geq 2$. We split the $M$ coupling constants $\{\lambda_1, \ldots, \lambda_M\}$ in cells of size $N$:

$$M = N \left[\frac{M}{N}\right] + \ell_M,$$

and in each cell only the two first coupling constants are nonzero, i.e.,

$$\lambda_j = \lambda_{k+1}N + j = 0, \text{ for } j = 3, \ldots, N.$$  

We redefine the nonzero coupling constants

$$\tilde{\lambda}_\ell = \tilde{\lambda}_{2(k-1)+j} \equiv \lambda_{(k-1)N+j}$$

so that $\ell = 1, 2, \ldots, \tilde{M}$, and the number of nonzero $N$-multispin couplings in the chain is

$$\tilde{M} = 2 \left[\frac{M}{N}\right] + \min(\ell_M, 2),$$

while $\ell_M$ is given in (42). The set of coupling constants, from (43), are $\{\tilde{\lambda}_1, \tilde{\lambda}_2, 0, \ldots, 0; \lambda_3, \lambda_4, 0, \ldots, 0; \cdots\}$. 

FIG. 3. The two and four possible contributions of the quasi-energy $\varepsilon_i$ for the Ising chains and for the XY model, respectively.
The $N$-multispin XY model \cite{23} is now given by

$$H_{N,2}^{(N,XY)}(\{\tilde{\lambda}_i\}) = \sum_{i=1}^{M+N-2} \sigma_i^+ \sigma_{i+1}^- + \sum_{\ell=1}^{\tilde{M}} \tilde{\lambda}_{\ell}^N \sigma_{\ell}^+ \left( \prod_{j=\ell+1}^{\ell+N-2} \sigma_j^z \right) \sigma_{\ell+N-1}^+,$$

(46)

where

$$\tilde{\ell} = \ell + (N-2) \left[ \frac{\ell - 1}{2} \right].$$

(47)

For example for $N = 3$ and $M = 4$ we have

$$H_{4,2}^{(3,XY)} = \sum_{i=1}^{5} \sigma_i^+ \sigma_{i+1}^- + \tilde{\lambda}_1^3 \sigma_1^- \sigma_2^z \sigma_3^+ + \lambda_2^3 \sigma_2^z \sigma_3^+ \sigma_4^+ + \tilde{\lambda}_3 \sigma_4^+ \sigma_5^z \sigma_6^+.$$  

(48)

To proceed, let us insert (43) in the recurrence relation (37), and we obtain for $j = 3, \ldots, N$

$$P_{i-1}^{(N-1)}(z) = P_{(k-1)N+j}^{(N-1)},$$

and for $j = 1, 2$ we define similarly as in (44)

$$\widetilde{P}_{\ell}^{(1)} = P_{i-1}^{(N-1)}(z) = \tilde{P}_{(k-1)N+j}^{(N-1)}.$$  

(50)

with $\ell = (k-1)2+j$. It follows from the above relations that

$$P_{i-1}^{(N-1)} = \tilde{P}_{\ell-1}^{(1)},$$

(51)

$$P_{i-1}^{(N-1)} = \tilde{P}_{(k-2)N+j}^{(1)}.$$  

(52)

Inserting (51) and (52) in the recurrence relations (37) and (38), we obtain

$$P_{\ell}^{(1)}(z) = \widetilde{P}_{(\ell-1)}^{(1)}(z) - z\tilde{\lambda}_N \tilde{P}_{\ell-2}^{(1)}(z),$$

(53)

for $\ell = 1, \ldots, \tilde{M}$, with the initial condition

$$\tilde{P}_{\ell}^{(1)}(z) = 1, \text{ for } \ell \leq 0.$$  

(54)

The above recurrence relations are precisely the ones for the polynomial $P_{M}^{(p)}(z)$ given in (15) and (16) for the $Z(N)$ multispin chains with $p = 1$ and the identification $\tilde{M} \leftrightarrow M$ and $\tilde{\lambda}_{\ell} \leftrightarrow \lambda_{\ell}$. The roots $z_i$ of these polynomials give us the quasi-energies $\varepsilon_i = 1/z_i^{1/N}$ of the models. This implies that the $N$-multispin XY model with Hamiltonian $H_{N,2}^{(N,XY)}$ given in (44) and (48) has the same quasi-energies as the $Z(N)$ multispin free-parafermionic Hamiltonian $H_{N,2}^{(N,1)}$ given in (10). In the particular case where $\tilde{M} = 2L - 1$ is an odd number, the XY model \cite{47} is related to the $Z(N)$ free-parafermionic Baxter chain, with the Hamiltonian given in (9). The quasi-energies $\varepsilon_i$ ($i = 1, \ldots, L$), from (41) give the eigenspectra of $H_{N,2}^{(N,XY)}$

$$E_{\{s_{i,j}, r_{i,j}\}} = - \sum_{i=1}^{L} \left( \sum_{j=0}^{N-1} r_{i,j} \omega^{s_{i,j}} \right) \varepsilon_i,$$

(55)

where $s_{i,j} = 0, 1, \ldots, N-1$ and $r_{i,j} = 0, 1$. From (11) the same quasi-energies also give the eigenspectra of $H_{2L-1}^{(N,1)}$

$$E_{s_1, \ldots, s_L} = - \sum_{i=1}^{L} \omega^{s_i} \varepsilon_i,$$

(56)

where $s_i = 0, 1, \ldots, N-1$ and $\omega = \exp(i2\pi/N)$.

In (56) a given quasi-energy $\varepsilon_i$ contributes to an eigenvalue with one of the $N$ possible values $\omega^{s_i} \varepsilon_i$ ($s_i = 0, 1, \ldots, N-1$), while in (55) the same root contributes with $2^N$ distinct values. In Fig. 4 we draw pictorially the contributions of a given root $\varepsilon_i$ to both models with $N = 3$. In the free-parafermionic $Z(N)$ chain (see Fig. 4) we have a $Z(N)$ circle repulsion for the quasi-energies, that is absent in the $N$-multispin XY quantum chain. These results generalize the correspondence of the Ising-XY models \cite{23} (see Fig. 3) to the $Z(N)$ free-parafermionic models.

All the eigenergies of the $Z(N)$ free-parafermionic model also appear in the $U(1)$ sector of the $XY$ model with $L$ particles. In particular, the ground-state energy and the mass gap are the same. In Fig. 5 we show for the model with symmetry $Z(4)$ the quasi-energies forming the ground state and one of the lowest eigenenergy states of both models. In Appendix B, we show a numerical comparison of the quasi-energies of the $N = 4$ multispin $XY$ and the $Z(4)$ Baxter chain.

From the exact results of the $Z(N)$ free-parafermionic chains, at the critical isotropic point $\{\lambda_i = 1\}$ \cite{12,13} the...
dynamical and specific heat exponents for the XY model \( (46) \) are given by \( (19) \) and \( (20) \), respectively.

We can also obtain the XY models with multispin interactions that share the quasi-energies with the \( Z(4) \) free-fermionic model and for the \( N = 4 \)-multispin interactions in a lattice size \( L = 3 \).

Since the exact values for some of the exponents are known for the \( Z(N) \) models, they are also exact for these generalized XY models.

V. CONCLUSIONS

The relationship between the standard Ising model and XY quantum chains is well known. The eigenvalues of the XY quantum chain can be obtained from the ones of two decoupled Ising Hamiltonians. Both models have a free-fermionic eigenspectra. In this paper we show that the eigenspectra of several quantum chains with a free-parafermionic quasi-particle eigenspectra can also be obtained from generalized XY quantum chains. These XY models for \( N = 2 \) recover the standard XY quantum chain. For \( N > 2 \) they are non-Hermitian, like the known free-parafermionic \( Z(N) \) models. These generalized XY models contain, in addition to two-body interactions, also \( N \)-multispin interactions.

The \( Z(N) \) symmetry in the free-parafermionic chains is enhanced to an \( U(1) \) symmetry in the generalized XY models. The spectrum of the generalized XY model is given in terms of linear combinations of all the quasi-energies of the free parafermionic \( Z(N) \) models; see Eq.(45). In particular, the special \( U(1) \) sector with \( \overline{M} \) fermions contains all the \( N\overline{M} \) eigenvalues of the corresponding free parafermionic model. The ground-state and the low-lying excited states have the same energy in both models for \( N \leq 4 \), implying that at their critical point they share the same values for the critical exponents. For \( N > 4 \), differently from the \( Z(N) \) free-parafermionic models, where the ground-state is always real, in the related XY models the eigenenergy with the lowest real part is complex.

In Appendix A, we present these models. The construction of the equivalent XY models we present, only works for the models where \( N \geq p + 1 \). Differently from the case \( p = 1 \), where we can derive the \( N\)-XY model for arbitrary \( Z(N) \) symmetry, in the cases where \( p > 1 \) we only found the correspondence for \( N \geq p + 1 \). This means, for example, that for the multispin model with \( p = 2 \) we only found the related \( N\)-XY models with \( N = 3, 4, \ldots \). This excludes, for example, the \( Z(2) \) fermionic 3-spin interacting Fendley model \( [15] \). In Appendix B, we show a numerical comparison of the quasi-energies of the \( N = 4 \)-multispin XY and the \( Z(4) \) free-parafermionic model with 3-multispin interactions (\( p = 2 \)).

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Appendix A: The correspondence of the generalized XY model and the \(Z(N)\) free-parafermionic models with \((p+1)\)-multispin interactions

In this appendix we generalize the results of Sec. 4, to obtain the \(N\)-XY models that contain the eigenspectra of the \(Z(N)\) free-parafermionic quantum chains with \((p+1)\)-multispin interactions \[12\] \[13\]. The generalization we present only works for \(N \geq p + 1\).

As in \[42\], we split the \(M\) coupling constants of \[23\] \(\{\lambda_1, \ldots, \lambda_M\}\) in cells containing \(N\) lattice sites. In each cell we cancel the coupling constants
\[
\lambda_i = \lambda_{(k-1)N+j} = 0, \quad \text{for } j = p+2, \ldots, N, \tag{A1}
\]
and we define
\[
\hat{\lambda}_\ell = \hat{\lambda}_{(p+1)(k-1)+j} \equiv \lambda_{(k-1)N+j}, \quad \text{for } j = 1, \ldots, p+1,
\]
so that \(\ell = 1, 2, \ldots, \hat{M}\), and the number of non-zero couplings is
\[
\tilde{M} = (p + 1) \left( \left\lfloor \frac{M}{N} \right\rfloor + \min(\ell_M, p + 1) \right), \tag{A2}
\]
with \(\ell_M\) given in \[42\]. The \(N\)-multispin XY model \[23\] is now given by
\[
H^{(N,XY)}_{N,p+1}(\{\hat{\lambda}_\ell\}) = \sum_{i=1}^{M+N-2} \sigma_i^+ \sigma_{i+1}^- + \frac{\tilde{M}}{N} \sum_{\ell=1}^{\hat{M}} \hat{\lambda}_\ell^N \sigma_\ell^+ \left( \prod_{j=1}^{\ell-1} \sigma_j \right) \sigma_{\ell+N-1}^+ \tag{A3}
\]
where
\[
\hat{\ell} = \ell + (N - p - 1) \left\lfloor \frac{\ell - 1}{p + 1} \right\rfloor. \tag{A4}
\]
As an example, for \(N = 4, M = 6\) and \(p = 2\), we have
\[
H^{(4,XY)}_{4,3} = \sum_{i=1}^{8} \sigma_i^+ \sigma_{i+1}^- + 4 \hat{\lambda}_1^4 \sigma_1^+ \sigma_2^+ \sigma_3^+ \sigma_4^+ + 4 \hat{\lambda}_2^4 \sigma_2^+ \sigma_3^+ \sigma_4^+ \sigma_5^+ + 4 \hat{\lambda}_3^4 \sigma_3^+ \sigma_4^+ \sigma_5^+ \sigma_6^+ + 4 \hat{\lambda}_4^4 \sigma_4^+ \sigma_5^+ \sigma_6^+ \sigma_7^+ + 4 \hat{\lambda}_5^4 \sigma_5^+ \sigma_6^+ \sigma_7^+ \sigma_8^+.
\]
Inserting \[A1\] in the recurrence relations \[37\] we obtain for \(j = p + 2, \ldots, N\)
\[
P_i^{(N-1)} = P_{(k-2)(N+j)}^{(N-1)} = P_{(k-2)(p+1)+j}^{(p+1)}, \tag{A6}
\]
and for \(j = 1, \ldots, p+1\) we define
\[
\tilde{P}_\ell^{(p)} = P_i^{(N-1)} = P_{(k-1)N+j}^{(N-1)}, \tag{A7}
\]
with \(\ell = (k-1)(p+1) + j\). From the above relations we obtain the generalization of \[41\], \[42\]:
\[
P_i^{(N-1)} = \tilde{P}_\ell^{(p)}, \tag{A8}
\]
\[
P_i^{(N-1)} = P_{(k-2)(N+j)}^{(N-1)} = \tilde{P}_{(k-2)(p+1)+j}^{(p+1)} = \tilde{P}_{(k-1)N+j}^{(N-1)} = \tilde{P}_\ell^{(p)}. \tag{A9}
\]
Inserting the last relations in the recurrence \[37\], we finally obtain
\[
\tilde{P}_\ell^{(p)}(z) = \tilde{P}_{\ell-1}^{(p)}(z) - \frac{z}{N} \tilde{P}_\ell^{(p)}(z), \tag{A10}
\]
for \(\ell = 1, \ldots, \hat{M}\), with the initial condition
\[
\tilde{P}_\ell^{(p)}(z) = 1, \quad \text{for } \ell \leq 0. \tag{A11}
\]
These relations are the same as the ones of the \(Z(N)\) \((p+1)\)-multispin free-parafermionic models given in \[15\]-\[19\]. This means that the XY model \[A3\]-\[A4\] are given by a free-particle eigenspectrum with the same quasi-energies.

Appendix B: Comparison of quasi-energies of the \(Z(N)\) parafermionic chains and the \(N\)-multispin chains for small lattice sizes

In this appendix, we give two simple examples of the quasi-eigenenergies of the \(Z(N)\) free-parafermionic with \((p+1)\)-multispin interaction and the \(N\)-multispin free-fermionic XY quantum chains. In both examples we consider \(M = 5, N = 4\).

In the first example the non-zero coupling constants are \[44\]:
\[
\hat{\lambda}_1 = 1, \hat{\lambda}_2 = 2, \hat{\lambda}_3 = 3. \tag{B1}
\]
In this case the XY model is (see \[46\])
\[
H^{(4,XY)}_{4,2}(\hat{\lambda}_1, \hat{\lambda}_2, \hat{\lambda}_3). \tag{B2}
\]
Its quasi-energies are related to the ones in the \(Z(4)\)-Baxter chain \[9\]. From \[45\], the number of density operators in the Hamiltonian is \(M = 3\), and it corresponds to the \(Z(4)\) Baxter chain \[9\] with \(L = (M+1)/2 = 2\) sites, with the coupling constants \(\hat{\lambda}_1, \hat{\lambda}_2, \hat{\lambda}_3\). The polynomial \[53\]-\[54\] is given by
\[
\tilde{P}_3^{(1)}(z) = 1 - (\hat{\lambda}_1^N + \hat{\lambda}_2^N + \hat{\lambda}_3^N)z + \hat{\lambda}_1^N \hat{\lambda}_2^N \hat{\lambda}_3^N z^2. \tag{B3}
\]
The roots \(z_{\pm}\) of this polynomial give the quasi-energies
\[
\varepsilon_1 = \frac{1}{z_{1/4}^{1/4}} = 3.139634, \quad \varepsilon_2 = \frac{1}{z_{1/4}^{1/4}} = 0.955525. \tag{B3}
\]
We show in the first three columns of Table I the numerical values of the quasi-energies appearing in the XY and \(Z(4)\) related quantum chains.

The second example is for the multispin XY model related to the \(Z(4)\) free-parafermionic quantum chain with 3-multispin interactions \((p = 2)\). The non-zero coupling constants in \[23\] are now:
\[
\hat{\lambda}_1 = 1, \hat{\lambda}_2 = 2, \hat{\lambda}_3 = 3, \hat{\lambda}_4 = 4 = 5.
\]
The XY model is now (see appendix A), \[H^{(4,XY)}_{4,3}(\hat{\lambda}_1, \hat{\lambda}_2, \hat{\lambda}_3, \hat{\lambda}_4)\], with \(\tilde{M} = 4\). The polynomial \[A10\]-\[A11\] is of degree \(((\tilde{M} + p)/(p + 1)) = 2\)
\[
\tilde{P}_4^{(2)}(z) = 1 - (\hat{\lambda}_1^N + \hat{\lambda}_2^N + \hat{\lambda}_3^N + \hat{\lambda}_4^N)z + \hat{\lambda}_1^N \hat{\lambda}_2^N \hat{\lambda}_3^N \hat{\lambda}_4^N z^2,
\]
TABLE I. Quasi-energies for the related free-fermionic N-multispin XY and the free-parafermionic Z(N) quantum chains, for $M = 5$, $N = 4$ and $\omega = \exp(2\pi i/4)$. The quasi-energies for the XY models are the ones of the Hamiltonian $H^{N,XY}$ given in [A3], and for the free-parafermionic $Z(N)$ cases they are the ones of the Baxter [9] and 3-spin interaction chains [10].

with roots $z_\pm$, giving us the quasi-energies

$$\varepsilon_1 = \frac{1}{z_+^{1/4}} = 4.335391, \quad \varepsilon_2 = \frac{1}{z_-^{1/4}} = 0.922639.$$  

(B4)

In the last three columns of Table I we show the numerical values of the quasi-energies in the XY and Z(4) quantum chains with 3-spin interactions.

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