ON 2 ACYCLIC SIMPLE GRAPHOIDAL COVERING OF BICYCLIC GRAPHS

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Abstract. A 2—simple graphoidal cover of G is a set ψ of (not necessarily open) paths in G such that every edge is in exactly one path in ψ and every vertex is an internal vertex of at most two paths in ψ and any two paths in ψ has at most one vertex in common. The minimum cardinality of the 2—simple graphoidal cover of G is called the 2—simple graphoidal covering number of G and is denoted by η2s. A 2—simple graphoidal cover ψ of a graph G is called 2—acyclic simple graphoidal cover if every member of ψ is a path. The minimum cardinality of a 2—acyclic simple graphoidal cover of G is called the 2—acyclic graphoidal covering number of G and is denoted by η2as. This paper discusses 2—acyclic simple graphoidal cover on bicyclic graphs.

Keywords: bicyclic graphs; 2-simple graphoidal cover; 2-acyclic simple graphoidal cover.

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1. Introduction

In graph theory, Graph Decomposition is the one of the fastest-growing research topics and plays a major role in Road Network, Block design and so on. A decomposition of a graph G is a collection of edge disjoint subgraphs H1, H2, …, Hn of G such that every edge of G is in exactly one Hi. Several authors [1][3][7][9][12] impose different conditions and parameters to

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find out different types of decomposition of the graphs. The above motivates the definition of Graphoidal covers.

2. Preliminaries

All the graph $G = (V, E)$ in this paper is a nontrivial, simple-connected, and undirected graphs. The number of elements of $V$ is said to be the order of $G$ is expressed by $p$ and the number of elements in the $E$ are said to be the size of $G$ is expressed by $q$. For graph theoretic terminology, Harary [8] is referred. The vertices $u_0$ and $u_l$ are called external vertices of $P$ and $u_1, u_2, \ldots, u_{l-1}$ are internal vertices of $P$, where $P = (u_0, u_1, u_2, \ldots, u_{l-1}, u_l)$ is a path or cycle in $G$. Two paths $P_1$ and $P_2$ are said to be internally disjoint if no vertex of $G$ is an internal vertex of both $P_1$ and $P_2$. The graphoidal cover introduced and discussed by Acharya and Sampath Kumar [1] [2]. 2−graphoidal path cover introduced by Nagarajan et.al [9]. 2−graphoidal cover extensively studied and discussed by Das and Singh [7]. The authors [12] discuss about 2−acyclic simple graphoidal cover and discusses 2−acyclic simple graphoidal covering on standard graph. In this paper, the authors determine the 2−acyclic simple graphoidal cover on bicyclic graphs.

Definition 2.1 (1). A graphoidal cover of $G$ is a set $\psi$ of (not necessarily open) paths in $G$ satisfying the following conditions.

(i) Every path in $\psi$ has at least two vertices.

(ii) Every vertex of $G$ is an internal vertex of at most one path in $\psi$.

(iii) Every edge of $G$ is in exactly one path in $\psi$.

The minimum cardinality of a graphoidal cover of $G$ is called the graphoidal covering number of $G$ and is denoted by $\eta(G)$.

Definition 2.2 (9). An 2−graphoidal cover of a graph $G$ is a collection $\psi$ of paths (not necessarily open) in $G$ such that

(i) Every path in $\psi$ has at least two vertices.

(ii) Every edge is exactly in one path $\psi$.

(iii) Every vertex is an internal vertex of at most two paths in $\psi$. 
The minimum cardinality of a 2−graphoidal cover of $G$ is called the 2−graphoidal covering number of $G$ and is denoted by $\eta_2(G)$.

**Definition 2.3** (12). A 2−simple graphoidal cover of a graph $G$ is a 2−graphoidal cover $\psi$ of $G$ such that any two paths in $\psi$ have at most one vertex in common. The minimum cardinality of a 2−simple graphoidal cover of $G$ is called the 2−simple graphoidal covering number of $G$ and is denoted by $\eta_{2s}(G)$.

**Definition 2.4** (12). A 2−acyclic simple graphoidal cover of $G$ is said to be 2− simple graphoidal cover $\psi$ of $G$ such that every member $\psi$ of $G$ is a path. The minimum cardinality of a 2−acyclic simple graphoidal cover of $G$ is called the 2−acyclic simple graphoidal covering number of $G$ and is denoted by $\eta_{2as}(G)$.

**Definition 2.5.** Let $\psi$ be a collection of internally disjoint paths in $G$. A vertex $v$ of $G$ is said to be an interior vertex of $\psi$ if it is an internal vertex of some path in $\psi$. Otherwise, it is said to be an exterior vertex.

**Notations 2.6** (9). Let $\psi$ be a 2−acyclic simple graphoidal cover of $G$. The following notations are used in the theorems. Here $i_\psi(P), t_1(\psi), t_2(\psi), t_\psi$ denotes the number of internal vertices of the path $P$, the number of vertices appear as internal vertex exactly in one path $\psi$, the number of vertices appears as internal vertex exactly in two paths of $\psi$ and the number of vertices are not internal in $\psi$ respectively.

If 2−acyclic simple graphoidal cover $\psi$ of $G$ is minimum, then it is clear that $t_1(\psi), t_2(\psi)$ should be maximum and $t(\psi)$ should be minimum. We define $t_i = \max t_i(\psi) \ (i = 1, 2)$ where the maximum is taken over all 2−acyclic simple graphoidal covers of $\psi$ of $G$ and $t = \min t_\psi$ where the minimum is taken over all 2−acyclic simple graphoidal cover $\psi$ of $G$.

**Theorem 2.7** (12). For any $(p,q)$ graph, $\eta_{2as}(G) = q - p - t_2 + t$.

**Corollary 2.8** (12). There exists a 2−acyclic simple graphoidal cover $\psi$ of $G$ in which every vertex is internal vertex in exactly 2 paths in $\psi$ of $G$ if and only if $\eta_{2as}(G) = q - 2p$. 
Theorem 2.9 (12). Let $G$ be a unicycle graph with $n$ pendant vertices. Let $C$ be the unique cycle on $G$. Let $l$ be the number of vertices of degree greater than 2 on $C$. Then

$$\eta_{2as}(G) = \begin{cases} 
3 & \text{if } l=0 \\
(n+1) - m & \text{if } l=2 \\
(n+2) - m & \text{if } l=1 \\
n - m & \text{otherwise}
\end{cases}$$

where $m$ is the total number of vertices of degree $\geq 4$ on $G$.

Definition 2.10 (6). A connected $(p, p+1)$ graph $G$ is called a Bicyclic graph.

Definition 2.11 (6). A one-point union of two cycles is a simple graph obtained from two cycles, say $C_l$ and $C_m$ where $l, m \geq 3$, by identifying one and the same vertex from both cycles. Without loss of generality, we assume $C_l = (u_0, u_1, \ldots, u_{l-1}, u_0)$ and $C_m = (u_0, u_l, u_{l+1}, \ldots, u_{m+l-2}, u_0)$. This graph is denoted by $U(l : m)$.

Definition 2.12 (6). A long dumbbell graph is a simple graph obtained by joining two cycles $C_l$ and $C_m$ where $l, m \geq 3$, with a path of length $i, i \geq 1$. Without loss of generality, we may assume $C_l = (u_0, u_1, \ldots, u_{l-1}, u_0)$, $P_i = (u_{l-1}, u_l, \ldots, u_{l+i-1})$ and $C_m = (u_{l+i-1}, u_{l+i}, \ldots, u_{l+m+i-2}, u_{l+i-1})$. This graph is denoted by $D(l : m : i)$.

3. Main Results

Theorem 3.1. Let $G$ be a bicyclic graph with $n$ pendant vertices. Also let $U(l : m)$ be the unique bicycle in $G$ and let $l$ be the number of vertices of degree greater than 2 on $C$. Then
Let $G = U(l : m)$

$$
\eta_{2as}(G) = \begin{cases} 
4 & \text{if } G = U(l : m) \\
(n+5) - m & \text{if } (l = 1 \text{ and } \deg(u_0) \geq 5) \\
(n+4) - m & \text{if } (l = 2 \text{ and } \deg(u_0) \geq 4, \deg(v) \geq 3) \\
(n+3) - m & \text{if } (l = 3 \text{ and } \deg(u_0) \geq 4, \deg(v) \geq 3, \deg(u) \geq 3) \\
& \text{Or } (l = 4 \text{ and } \deg(u_0) \geq 4, \deg(u) \geq 3, \deg(v) \geq 4, \deg(u) \geq 3, \deg(v) \geq 3, \deg(w) \geq 3, \deg(x) \geq 3, \{u, v, w, x \in C_i\}) \\
(n+2) - m & \text{if } (l = 4 \text{ and } \deg(u_0) \geq 4, \deg(u) \geq 3, \deg(v) \geq 3, \deg(u) \geq 3, \deg(v) \geq 3, \deg(w) \geq 3, \deg(x) \geq 3, \{u, v, w, x \in C_i\}) \\
(n+1) - m & \text{otherwise}
\end{cases}
$$

Proof: Let $C_i = (u_0, u_1, \ldots, u_{l-1}, u_0)$ and $C_m = (u_0, u_l, \ldots, u_{l+m-2}, u_0)$ be two cycles sharing a common vertex say $u_0$ with $q = p + 1$.

**Case 1.** Suppose $G = U(l : m)$

Let $P_1 = (u_2, u_3, \ldots, u_0, u_1, \ldots, u_{l+m-5})$, $P_2 = (u_{l+m-5}, u_{l+m-4})$, $P_3 = (u_{l+m-4}, u_{l+m-3}, u_{l+m-2}, \ldots, u_{l-1}, u_0)$, $P_4 = (u_1, u_2)$ is a minimum 2–acyclic simple graphoidal cover of $G$ so that $\eta_{2as}(G) = 4$.

**Case 2.** When $l = 1$ and $\deg(u_0) \geq 5$

Let $P = (w, x)$, where $P$ be a path on $U(l : m)$ and $w, x \in C_m$. Take $G_1 = G - P$ is a unicyclic graph with $(n + 2)$ pendent vertices and $m$ vertices is of degree $\geq 4$ with $l = 1$. Hence by theorem 2.9, $\eta_{2as}(G_1) = (n + 2) + 2 - m = (n + 4) - m$. Let $\psi_1$ be the minimum 2–acyclic simple graphoidal cover of $G_1$. Then $\psi = \psi_1 \cup \{P\}$ is a minimum 2–acyclic simple graphoidal cover of $G$, hence $\eta_{2as}(G) \leq (n + 5) - m$. For any 2–acyclic simple graphoidal cover of $G$,
n pendent vertices and at least four vertices in $U(l : m)$ are external and at most $m$ vertices are internal twice. Therefore $t_\psi \geq (n + 4), t_2(\psi) \leq m$. Hence $t \geq (n + 4), t_2 \leq m$ so that $\eta_{2as}(G) = q - p - t_2 + t = (p + 1) - p - m + (n + 4) \geq (n + 5) - m$. Thus $\eta_{2as}(G) = (n + 5) - m$.

**Case 3.** When $l = 2$ and let $u_0, v$ be the two vertices is of degree greater than two on $U(l : m)$ and $\deg(u_0) \geq 4, \deg(v) \geq 3, v \in C_l$

Take $G_1 = G - P$ where $P = (w, x)$ be a path on $U(l : m)$ and $w, x \in C_m$. It is clear that $G_1$ is a unicyclic graph with $(n + 2)$ pendent vertices and $m$ vertices is of degree $\geq 4$ with $l = 2$. By theorem 2.9, $\eta_{2as}(G_1) = (n + 2) + 1 - m = (n + 3) - m$. Let $\psi_1$ be the minimum $2$–acyclic simple graphoidal cover of $G_1$. Then $\psi = \psi_1 \cup \{P\}$ is a minimum $2$–acyclic simple graphoidal cover of $G$, hence $\eta_{2as}(G) \leq (n + 4) - m$. For any $2$–acyclic simple graphoidal cover of $G$, $n$ pendent vertices and at least three vertices in $U(l : m)$ are external and at most $m$ vertices are internal twice. Therefore $t_\psi \geq (n + 3), t_2(\psi) \leq m$. Hence $t \geq (n + 3), t_2 \leq m$ so that $\eta_{2as}(G) = q - p - t_2 + t = (p + 1) - p - m + (n + 3) \geq (n + 4) - m$. Thus $\eta_{2as}(G) = (n + 4) - m$.

**Case 4.** When $l = 3$ and let $u_0, u, v$ be the vertices is of degree greater than two on $U(l : m)$ and $\deg(u_0) \geq 4, \deg(u) \geq 3, \deg(v) \geq 3$. Then there are two subcases.

**Subcase 4.1.** Suppose $u, v \in C_l$

Take $G_1 = G - P$ where $P = (x, y)$ be a path on $U(l : m)$ and $x, y \in C_m$. It is clear that $G_1$ is a unicyclic graph with $(n + 2)$ pendent vertices and $m$ vertices is of degree $\geq 4$ with $l = 3$. By theorem 2.9, $\eta_{2as}(G_1) = (n + 2) - m$. Let $\psi_1$ be the minimum $2$–acyclic simple graphoidal cover of $G_1$. Then $\psi = \psi_1 \cup \{P\}$ is a minimum $2$–acyclic simple graphoidal cover of $G$, hence $\eta_{2as}(G) \leq (n + 3) - m$. For any $2$–acyclic simple graphoidal cover of $G$, $n$ pendent vertices and at least two vertices in $U(l : m)$ are external and at most $m$ vertices are internal twice. Therefore $t_\psi \geq (n + 2), t_2(\psi) \leq m$. Hence $t \geq (n + 2), t_2 \leq m$ so that $\eta_{2as}(G) = q - p - t_2 + t = (p + 1) - p - m + (n + 2) \geq (n + 3) - m$. Thus $\eta_{2as}(G) = (n + 3) - m$.

**Subcase 4.2.** Suppose $u \in C_l, v \in C_m$, then there are two subcases.

**Subcase 4.2.1.** Suppose $\deg(v) = 3$ (Or) $\deg(v) \geq 5$

Take $G_1 = G - P$ where $P = (v, x)$ be a path on $U(l : m)$ and $x \in C_m$. It is clear that $G_1$ is a unicyclic graph with $(n + 1)$ pendent vertices and $m$ vertices is of degree $\geq 4$ with $l = 2$. By theorem 2.9, $\eta_{2as}(G_1) = (n + 2) - m$. Let $\psi_1$ be the minimum $2$–acyclic simple graphoidal
Subcase 4.2.2. Suppose $\deg(v) = 4$

Take $G_1 = G - P$ where $P = (v,x)$ be a path on $U(l : m)$ and $x \in C_m$. It is clear that $G_1$ is a unicyclic graph with $(n + 1)$ pendent vertices and $(m - 1)$ vertices is of degree $\geq 4$ with $l = 2$. By theorem 2.9, $\eta_{2as}(G_1) = (n + 1) + 1 - (m - 1) = (n + 3) - m$. Let $\psi_1$ be the minimum 2–acyclic simple graphoidal cover of $G_1$. Let $P_1$ be a path in $\psi_1$ in which $v$ is an external. Then $\psi = (\psi_1 - P_1) \cup \{P_1 P\}$ is a 2–acyclic simple graphoidal cover of $G$, hence $\eta_{2as}(G) \leq ((n + 3) - m) - 1 + 1 = (n + 3) - m$. For any 2–acyclic simple graphoidal cover of $G$, $n$ pendent vertices and atleast two vertices in $U(l : m)$ are external and atmost $m$ vertices are internal twice. Therefore $t_\psi \geq (n + 2), t_2(\psi) \leq m$. Hence $t \geq (n + 2), t_2 \leq m$ so that $\eta_{2as}(G) = q - p - t_2 + t = (p + 1) - p - m + (n + 2) \geq (n + 3) - m$. Thus $\eta_{2as}(G) = (n + 3) - m$.

Case 5. Suppose $l = 4$ and let $u_0, u, v, w$ be the vertices is of degree greater than two on $U(l : m)$ and $\deg(u_0) \geq 4, \deg(u) \geq 3, \deg(v) \geq 3, \deg(w) \geq 3$, then there are two subcases.

Subcase 5.1. Suppose $u, v, w \in C_l$

Take $G_1 = G - P$, where $P = (y, z)$ be a path on $U(l : m)$ and $y, z \in C_m$. It is clear that $G_1$ is a unicyclic graph with $(n + 2)$ pendent vertices and $m$ vertices is of degree $\geq 4$ with $l = 3$. By theorem 2.9, $\eta_{2as}(G_1) = (n + 2) - m$. Let $\psi_1$ be the minimum 2–acyclic simple graphoidal cover of $G_1$. Then $\psi = \psi_1 \cup \{P\}$ is a 2–acyclic simple graphoidal cover of $G$, hence $\eta_{2as}(G) \leq (n + 3) - m$. For any 2–acyclic simple graphoidal cover of $G$, $n$ pendent vertices and atleast two vertices in $U(l : m)$ are external and atmost $m$ vertices are internal twice. Therefore $t_\psi \geq (n + 2), t_2(\psi) \leq m$. Hence $t \geq (n + 2), t_2 \leq m$ so that $\eta_{2as}(G) = q - p - t_2 + t = (p + 1) - p - m + (n + 2) \geq (n + 3) - m$. Thus $\eta_{2as}(G) = (n + 3) - m$.

Subcase 5.2. Suppose $u, v \in C_l$ and $w \in C_m$, then there are two subcases.

Subcase 5.2.1. Suppose $\deg(w) = 3$ (Or) $\deg(w) \geq 5$
Take $G_1 = G - P$, where $P = (w, y)$ be a path on $U(l : m)$ and $y \in C_m$. It is clear that $G_1$ is a unicyclic graph with $(n + 1)$ pendent vertices and $m$ vertices is of degree $\geq 4$ with $l = 3$. By theorem 2.9, $\eta_{2as}(G_1) = (n + 1) - m$. Let $\psi_1$ be the minimum 2-acyclic simple graphoidal cover of $G_1$. Then $\psi = \psi_1 \cup \{P\}$ is a 2-acyclic simple graphoidal cover of $G$, hence $\eta_{2as}(G) \leq (n + 2) - m$. For any 2-acyclic simple graphoidal cover of $G$, $n$ pendent vertices and at least one vertex in $U(l : m)$ are external and at most $m$ vertices are internal twice. Therefore $t_\psi \geq (n + 1), t_2(\psi) \leq m$. Hence $t \geq (n + 2), t_2 \leq m$ so that $\eta_{2as}(G) = q - p - t_2 + t = (p + 1) - p - m + (n + 1) \geq (n + 2) - m$. Thus $\eta_{2as}(G) = (n + 2) - m$.

Subcase 5.2.2. Suppose $deg(w) = 4$

Take $G_1 = G - P$, where $P = (w, y)$ be a path on $U(l : m)$ and $y \in C_m$. It is clear that $G_1$ is a unicyclic graph with $(n + 1)$ pendent vertices and $(m - 1)$ vertices is of degree $\geq 4$ with $l = 3$. By theorem 2.9, $\eta_{2as}(G_1) = (n + 1) - (m - 1) = (n + 2) - m$. Let $\psi_1$ be the minimum 2-acyclic simple graphoidal cover of $G_1$. Let $P_1$ be a path in $\psi_1$ in which $x$ is an external vertex. Then $\psi = (\psi_1 - P_1) \cup \{P_1P\}$ is a 2-acyclic simple graphoidal cover of $G$, hence $\eta_{2as}(G) \leq ((n + 2) - m) - 1 + 1 = (n + 2) - m$. For any 2-acyclic simple graphoidal cover of $G$, $n$ pendent vertices and at least one vertex in $U(l : m)$ are external and at most $m$ vertices are internal twice. Therefore $t_\psi \geq (n + 1), t_2(\psi) \leq m$. Hence $t \geq (n + 2), t_2 \leq m$ so that $\eta_{2as}(G) = q - p - t_2 + t = (p + 1) - p - m + (n + 1) \geq (n + 2) - m$. Thus $\eta_{2as}(G) = (n + 2) - m$.

Case 6. When $l \geq 5$ and let $u_0, u, v, w, x$ be the vertices is of degree greater than two on $U(l : m)$ and Suppose $deg(u_0) \geq 4, deg(u) \geq 3, deg(v) \geq 3, deg(w) \geq 3, deg(x) \geq 3$, then there are three subcases.

Subcase 6.1. Suppose $u, v, w, x \in C_l$

Take $G_1 = G - P$, where $P = (y, z)$ be a path on $U(l : m)$ and $y, z \in C_m$. It is clear that $G_1$ is a unicyclic graph with $(n + 2)$ pendent vertices and $m$ vertices is of degree $\geq 4$ with $l \geq 5$. By theorem 2.9, $\eta_{2as}(G_1) = (n + 2) - m$. Let $\psi_1$ be the minimum 2-acyclic simple graphoidal cover of $G_1$. Then $\psi = \psi_1 \cup \{P\}$ is a 2-acyclic simple graphoidal cover of $G$, hence $\eta_{2as}(G) \leq (n + 3) - m$. For any 2-acyclic simple graphoidal cover of $G$, $n$ pendent vertices and at least two vertices in $U(l : m)$ are external and at most $m$ vertices are internal twice. Therefore $t_\psi \geq (n + 2), t_2(\psi) \leq m$. Hence $t \geq (n + 2), t_2 \leq m$ so that $\eta_{2as}(G) = q - p - t_2 + t =
\[(p + 1) - p - m + (n + 2) \geq (n + 3) - m. \text{ Thus } \eta_{2as}(G) = (n + 3) - m.\]

**Subcase 6.2.** When \(u, v, w \in C_l\) and \(x \in C_m\) and let \(P = (x, y)\) be a path in \(U(l : m)\) and \(y \in C_m\). Then there are two subcases.

**Subcase 6.2.1.** Suppose \(\deg(x) = 3\) (Or) \(\deg(x) \geq 5\)

Take \(G_1 = G - P\) is a unicyclic graph with \((n + 1)\) pendent vertices and \(m\) vertices is of degree \(\geq 4\) with \(l \geq 4\). By theorem 2.9, \(\eta_{2as}(G_1) = (n + 1) - m\). Let \(\psi_1\) be the minimum 2-acyclic simple graphoidal cover of \(G_1\). Then \(\psi = \psi_1 \cup \{P\}\) is a 2-acyclic simple graphoidal cover of \(G\), hence \(\eta_{2as}(G) \leq \eta_{2as}(G_1) = (n + 2) - m\). For any 2-acyclic simple graphoidal cover of \(G\), \(n\) pendent vertices and atleast one vertex in \(U(l : m)\) are external and atmost \(m\) vertices are internal twice.

Therefore \(t \geq (n + 1), t_1(\psi) \leq m\). Hence \(t \geq (n + 1), t_2 \leq m\) so that \(\eta_{2as}(G) = q - p - t_2 + t = (p + 1) - p - m + (n + 1) \geq (n + 2) - m.\) Thus \(\eta_{2as}(G) = (n + 2) - m.\)

**Subcase 6.2.2.** Suppose \(\deg(x) = 4\)

Take \(G_1 = G - P\) is a unicyclic graph with \((n + 1)\) pendent vertices and \((m - 1)\) vertices is of degree \(\geq 4\) with \(l \geq 4\). By theorem 2.9, \(\eta_{2as}(G_1) = (n + 1) - (m - 1) = (n + 2) - m\). Let \(\psi_1\) be the minimum 2-acyclic simple graphoidal cover of \(G_1\). Let \(P_1\) be a path in \(\psi_1\) in which \(x\) is an external vertex. Then \(\psi = (\psi_1 - P_1) \cup \{P_1 \cup P\}\) is a 2-acyclic simple graphoidal cover of \(G\), hence \(\eta_{2as}(G) \leq ((n + 2) - m) - 1 + 1 = (n + 2) - m\). For any 2-acyclic simple graphoidal cover of \(G\), \(n\) pendent vertices and atleast one vertex in \(U(l : m)\) are external and atmost \(m\) vertices are internal twice. Therefore \(t \geq (n + 1), t_2(\psi) \leq m\). Hence \(t \geq (n + 1), t_2 \leq m\) so that \(\eta_{2as}(G) = q - p - t_2 + t = (p + 1) - p - m + (n + 1) \geq (n + 2) - m.\) Thus \(\eta_{2as}(G) = (n + 2) - m.\)

**Subcase 6.3.** Suppose \(u, v \in C_l\) and \(w, x \in C_m\) and let \(P = (w, x)\) be a path in \(U(l : m)\). Then there are three subcases.

**Subcase 6.3.1.** When \((\deg(w) = \deg(x) = 3)\) (Or) \((\deg(w) = 3 \text{ and } \deg(x) \geq 5)\) (Or) \((\deg(w) \geq 5 \text{ and } \deg(x) \geq 5)\)

Take \(G_1 = G - P\) is a unicyclic graph with \(n\) pendent vertices and \(m\) vertices is of degree \(\geq 4\) with \(l = 3\). By theorem 2.9, \(\eta_{2as}(G_1) = n - m\). Let \(\psi_1\) be the minimum 2-acyclic simple graphoidal cover of \(G_1\). Then \(\psi = \psi_1 \cup \{P\}\) is a 2-acyclic simple graphoidal cover of \(G\), hence \(\eta_{2as}(G) \leq (n + 1) - m\). For any 2-acyclic simple graphoidal cover of \(G\), \(n\) pendent vertices are external and atmost \(m\) vertices are internal twice. Therefore \(t_\psi \geq n, t_2(\psi) \leq m.\)
Hence $t \geq n, t_2 \leq m$ so that $\eta_{2as}(G) = q - p - t_2 + t = (p + 1) - p - m + n \geq (n + 1) - m$. Thus $\eta_{2as}(G) = (n + 1) - m$.

**Subcase 6.3.2.** When $(\deg(w) = 3$ and $\deg(x) = 4)$ (Or) $(\deg(w) \geq 5$ and $\deg(x) = 4)$

Take $G_1 = G - P$ is a unicyclic graph with $n$ pendent vertices and $(m - 1)$ vertices is of degree $\geq 4$ with $l = 3$. By theorem 2.9, $\eta_{2as}(G_1) = n - (m - 1) = (n + 1) - m$. Let $\psi_1$ be the minimum 2–acyclic simple graphoidal cover of $G_1$. Let $P_1$ be a path in $\psi_1$ in which $x$ is an external vertex. Then $\psi = (\psi_1 - P_1) \cup \{PP_1\}$ is a 2–acyclic simple graphoidal cover of $G$, hence $\eta_{2as}(G) \leq ((n + 1) - m) - 1 + 1 = (n + 1) - m$. For any 2–acyclic simple graphoidal cover of $G$, $n$ pendent vertices are external and atmost $m$ vertices are internal twice. Therefore $t_\psi \geq n, t_2(\psi) \leq m$. Hence $t \geq n, t_2 \leq m$ so that $\eta_{2as}(G) = q - p - t_2 + t = (p + 1) - p - m + n \geq (n + 1) - m$. Thus $\eta_{2as}(G) = (n + 1) - m$.

**Subcase 6.3.3.** When $\deg(w) = 4$ and $\deg(x) = 4$

Take $G_1 = G - P$ is a unicyclic graph with $n$ pendent vertices and $(m - 2)$ vertices is of degree $\geq 4$ with $l = 3$. By theorem 2.9, $\eta_{2as}(G_1) = n - (m - 2) = (n + 2) - m$. Let $\psi_1$ be the minimum 2–acyclic simple graphoidal cover of $G_1$. Let $P_1$ be a path in $\psi_1$ in which $x$ is an external vertex and $P_2$ be a path in $\psi_1$ in which $w$ is an external vertex. Then $\psi = (\psi_1 - P_1 - P_2) \cup \{P_2PP_1\}$ is a 2–acyclic simple graphoidal cover of $G$, hence $\eta_{2as}(G) \leq ((n + 2) - m) - 2 + 1 = (n + 1) - m$. For any 2–acyclic simple graphoidal cover of $G$, $n$ pendent vertices are external and atmost $m$ vertices are internal twice. Therefore $t_\psi \geq n, t_2(\psi) \leq m$. Hence $t \geq n, t_2 \leq m$ so that $\eta_{2as}(G) = q - p - t_2 + t = (p + 1) - p - m + n \geq (n + 1) - m$. Thus $\eta_{2as}(G) = (n + 1) - m$. \qed
Example 3.2. Consider the Bicyclic graph $U(l : m)$ shown in Figure 1

Here $p = 33$, $q = 34$, $n = 14$, $l = 6$ and $m = 4$, then

$$\eta_{2as}(G) = \{ \langle v_{16}, v_6, v_5, v_3, v_{19} \rangle, \langle v_{20}, v_3, v_{14}, v_{13}, v_{31} \rangle, \langle v_{13}, v_{12}, v_{11}, v_{21}, v_{25}, v_{26} \rangle, \langle v_{25}, v_{27} \rangle, \langle v_{24}, v_{21}, v_{22}, v_{28} \rangle, \langle v_{22}, v_{23}, v_{29} \rangle, \langle v_{23}, v_{30} \rangle, \langle v_{11}, v_{10}, v_9, v_{33} \rangle, \langle v_{32}, v_9, v_{15}, v_3 \rangle, \langle v_3, v_2, v_1, v_8, v_{18} \rangle, \langle v_8, v_7, v_6, v_{17} \rangle \} = 11 = (n + 1) - m$$

Theorem 3.3. Let $G$ be a bicyclic graph with $n$ pendant vertices. Also let $D(l : m : i)$ be the unique bicycle in $G$ and let $l$ be the number of vertices of degree greater than 2 on cycles on $D(l : m : i)$ Then
\[
\eta_{2as}(G) = \begin{cases} 
5 & \text{if } G = D(l : m : i) \\
(n + 5) - m & \text{if } (l = 2 \text{ and } \deg(u_{l-1}) \geq 4, \deg(u_{l+i-1}) \geq 3) \\
(n + 4) - m & \text{if } (l = 3 \text{ and } \deg(u_{l-1}) \geq 3, \deg(u_{l+i-1}) \geq 3, \\
& \deg(u) \geq 3) \\
(n + 3) - m & \text{if } (l = 4 \text{ and } \deg(u_{l-1}) \geq 3, \deg(u_{l+i-1}) \geq 3, \deg(u) \geq 3, \deg(v) \geq 3, u, v, w \in C_l) \\
(n + 2) - m & \text{if } (l = 5 \text{ and } \deg(u_{l-1}) \geq 3, \deg(u_{l+i-1}) \geq 3, \deg(u) \geq 3, \\
& \deg(v) \geq 3, \deg(w) \geq 3, u, v \in C_l, w \in C_m) \text{ Or } (l > 5, \\
& \deg(u_{l-1}) \geq 3, \deg(u_{l+i-1}) \geq 3, \deg(u) \geq 3, \deg(v) \geq \\
& \deg(w), \deg(x) \geq 3, u, v, x \in C_l, w \in C_m) \\
(n + 1) - m & \text{otherwise}
\end{cases}
\]

Proof. Let \( C_l = (u_0, u_1, \ldots, u_{l-1}, u_0), P_i = (u_{l-1}, u_i, \ldots, u_{l+i-1}) \) and \( C_m = (u_{l+i-1}, u_{l+i}, \ldots, u_{l+m+i-2}, u_{l+i-1}) \).

**Case 1.** Suppose \( G = D(l : m : i) \)

Let \( P_1 = (u_2, u_3, \ldots, u_{l-3}), P_2 = (u_{l-3}, u_{l-2}, u_{l-1}, u_i, \ldots, u_{l+i-1}, u_{l+i}), P_3 = (u_{l+i}, u_{l+i+1}), P_4 = (u_{l+i+1}, u_{l+i+2}, \ldots, u_{l+i-1}), P_5 = (u_{l-1}, u_0, u_1, u_2) \) is a minimum 2-acyclic simple graphoidal cover of \( G \) so that \( \eta_{2as}(G) = 5 \).

**Case 2.** Suppose \( l = 2 \) and let \( P \) denote \((u_{l+i-1}, w)\) section of \( C_m \) such that it has at least one internal vertex say \( u_i \) and \( w \in C_m \). Let \( P_1 \) and \( P_2 \) denote the \((u_{l+i-1}, u_i)\) and \((u_i, w)\) section of \( P \) respectively. Then there are two subcases.

**Subcase 2.1.** When \( \deg(u_{l-1}) \geq 4 \) and \((\deg(u_{l+i-1}) = 3 \text{ (Or } \deg(u_{l+i-1}) \geq 5)\)

Take \( G_1 = G - \{P\} \) is a unicyclic graph with \((n + 1)\) pendent vertices and \( m \) vertices is of degree \( \geq 4 \) with \( l = 1 \). Hence by the theorem 2.9, \( \eta_{2as}(G_1) = ((n + 1) + 2) - m = (n + 3) - m \).
Let $\psi_1$ be a minimum 2–acyclic simple graphoidal cover of $G_1$. Then $\psi = \psi_1 \cup \{P_1\} \cup \{P_2\}$ is a 2–acyclic simple graphoidal cover of $G$, hence $\eta_{2\text{ar}}(G) \leq (n+5) - m$. For any 2–acyclic simple graphoidal cover of $G$ at least $n$ pendent vertices and four vertices in $D(l : m : i)$ are external and at most $m$ vertices are internal twice. Therefore $t_\psi \geq (n+4), t_2(\psi) \leq m$. Hence $t \geq (n+4), t_2 \leq m$ so that $\eta_{2\text{ar}}(G) = q - p - t_2 + t = (p + 1) - p - m + (n+4) \geq (n+5) - m$. Thus $\eta_{2\text{ar}}(G) = (n+5) - m$.

**Subcase 2.2.** When $\text{deg}(u_{l-1}) \geq 4$ and $\text{deg}(u_{l+i-1}) = 4$

Take $G_1 = G - \{P\}$ is a unicyclic graph with $(n+1)$ pendent vertices and $(m-1)$ vertices is of degree $\geq 4$ with $l = 1$. Hence by the theorem 2.9, $\eta_{2\text{ar}}(G_1) = ((n+1) + 2) - (m-1) = (n+4) - m$. Let $\psi_1$ be a minimum 2–acyclic simple graphoidal cover of $G_1$. Let $P_3$ be a path in $\psi_1$ in which $u_{l+i-1}$ is an external vertex. Take $\psi = (\psi_1 - P_3) \cup \{P_3 P_1\} \cup \{P_2\}$ is a 2–acyclic simple graphoidal cover of $G$ and $\eta_{2\text{ar}}(G) \leq ((n+4) - m) + 1 + 2 = (n+5) - m$. For any 2–acyclic simple graphoidal cover of $G$ at least $n$ pendent vertices and four vertices in $D(l : m : i)$ are external and at most $m$ vertices are internal twice. Therefore $t_\psi \geq (n+4), t_2(\psi) \leq m$. Hence $t \geq (n+4), t_2 \leq m$ and hence $\eta_{2\text{ar}}(G) = q - p - t_2 + t = (p + 1) - p - m + (n+4) \geq (n+5) - m$. Thus $\eta_{2\text{ar}}(G) = (n+5) - m$.

**Case 3.** When $l = 3$ and let $u$ be the only vertex is of degree greater than 2 other than $u_{l-1}, u_{l+i-1}$. Let $P$ denote $(u_{l+i-1}, w)$ section of $C_m$ such that it has at least one internal vertex say $u_i$ and $w \in C_m$. Let $P_1$ and $P_2$ denote the $(u_{l+i-1}, u_i)$ and $(u_i, w)$ section of $P$ respectively. Suppose $\text{deg}(u_{l-1}) \geq 3, \text{deg}(u) \geq 3, u \in C_m$, then there are two subcases.

**Subcase 3.1.** When $\text{deg}(u_{l+i-1}) = 3$ (Or $\text{deg}(u_{l+i-1}) \geq 5$)

Take $G_1 = G - \{P\}$ is a unicyclic graph with $(n+1)$ pendent vertices and $m$ vertices is of degree $\geq 4$ with $l = 2$. Hence by the theorem 2.9, $\eta_{2\text{ar}}(G_1) = ((n+1) + 1) - m = (n+2) - m$. Let $\psi_1$ be the minimum 2–acyclic simple graphoidal cover of $G_1$. Then $\psi = \psi_1 \cup \{P_1\} \cup \{P_2\}$ is a 2–acyclic simple graphoidal cover of $G$, hence $\eta_{2\text{ar}}(G) \leq (n+4) - m$. For any 2–acyclic simple graphoidal cover of $G$ at least $n$ pendent vertices and three vertices in $D(l : m : i)$ are external and at most $m$ vertices are internal twice. Therefore $t_\psi \geq (n+3), t_2(\psi) \leq m$. Hence $t \geq (n+3), t_2 \leq m$ and hence $\eta_{2\text{ar}}(G) = q - p - t_2 + t = (p + 1) - p - m + (n+3) \geq (n+4) - m$. Hence $\eta_{2\text{ar}}(G) = (n+5) - m$.
Subcase 3.2. When \( \deg(u_{i-1}) = 4 \)

Take \( G_1 = G - \{P\} \) is a unicyclic graph with \((n + 1)\) pendent vertices and \((m - 1)\) vertices is of degree \( \geq 4 \) with \( l = 2 \). Hence by the theorem 2.9, \( \eta_{2as}(G_1) = ((n + 1) - (m - 1)) = (n + 3) - m. \) Let \( \psi_1 \) be the minimum 2–acyclic simple graphoidal cover of \( G_1 \). Let \( P_3 \) be a path in \( \psi_1 \) in which \( u_{i-1} \) is external vertex. Take \( \psi = (\psi_1 - P_3) \cup \{P_3P_1 \} \cup \{P_2\} \) is a 2–acyclic simple graphoidal cover of \( G \) and \( \eta_{2as}(G) \leq ((n + 3) - m) - 1 + 2 = (n + 4) - m. \) For any 2–acyclic simple graphoidal cover of \( G \) at least \( n \) pendent vertices and three vertices in \( D(l : m : i) \) are external and at most \( m \) vertices are internal twice. Therefore \( t_{\psi} \geq (n + 3), t_2(\psi) \leq m. \) Hence \( t \geq (n + 3), t_2 \leq m \) and hence \( \eta_{2as}(G) = (p + 1) - p - m + (n + 3) \geq (n + 4) - m. \) Thus \( \eta_{2as}(G) = (n + 4) - m. \)

Case 4. When \( l = 4 \) and \( u, v \) be the only vertices is of degree greater than 2 other than \( u_{i-1}, u_{i+1-1} \). Then there are four sub cases.

Subcase 4.1. When \( \deg(u_{i-1}) \geq 3, \deg(u_{i+1-1}) = 3, \) then there are two subcases.

Subcase 4.1.1. When \( \deg(u) \geq 3, \deg(v) \geq 3 \) and \( u, v \in C_l \)

Take \( G_1 = G - P \), where \( P \) denotes \((u_{i-1}, w)\) section of \( C_m \) such that it has at least one internal vertex say \( u_i \) and \( w \in C_m \). Let \( P_1 \) and \( P_2 \) denote the \((u_{i-1}, u_i)\) and \((u_i, w)\) section of \( P \) respectively. It is clear that \( G_1 \) is a unicyclic graph with \((n + 1)\) pendent vertices and \( m \) vertices is of degree \( \geq 4 \) with \( l = 2 \). Therefore by theorem 2.9, \( \eta_{2as}(G_1) = (n + 1) - m. \) Let \( \psi_1 \) be the minimum 2–acyclic simple graphoidal cover of \( G_1 \). Take \( \psi = \psi_1 \cup \{P_1\} \cup \{P_2\} \) is a 2–acyclic simple graphoidal cover of \( G \), hence \( \eta_{2as}(G) \leq (n + 1) - m + 2 = (n + 3) - m. \) For any 2–acyclic simple graphoidal cover of \( G \) at least \( n \) pendent vertices and two vertices in \( D(l : m : i) \) are external and at most \( m \) vertices are internal twice. Therefore \( t_{\psi} \geq (n + 2), t_2(\psi) \leq m. \) Hence \( t \geq (n + 2), t_2 \leq m \) and hence \( \eta_{2as}(G) = (p + 1) - p - m + (n + 2) \geq (n + 3) - m. \) Thus \( \eta_{2as}(G) = (n + 3) - m. \)

Subcase 4.1.2. When \( \deg(u) \geq 3 \) and \( \deg(v) \geq 3 \), \( u \in C_l, v \in C_m \) and let \( P \) denotes \((u_{i-1}, v)\) section of \( C_m \) such that it has at least one internal vertex say \( u_i \). Let \( P_1 \) and \( P_2 \) denote the \((u_{i-1}, u_i)\) and \((u_i, v)\) section of \( P \) respectively. Then there are two subcases.

Subcase 4.1.2.1. When \( \deg(v) = 3 \) (Or) \( \deg(v) \geq 5 \)
Take $G_1 = G - P$ is a unicyclic graph with $n$ pendant vertices and $m$ vertices is of degree $\geq 4$ with $l = 2$. Therefore by theorem 2.9, $\eta_{2as}(G_1) = (n + 1) - m$. Let $\psi_1$ be the minimum 2–acyclic simple graphoidal cover of $G_1$. Take $\psi = \psi_1 \cup \{P_1\} \cup \{P_2\}$ is a 2–acyclic simple graphoidal cover of $G$, hence $\eta_{2as}(G) \leq (n + 1) - m + 2 = (n + 3) - m$. For any 2–acyclic simple graphoidal cover of $G$ atleast $n$ pendant vertices and two vertices in $D(l : m : i)$ are external and atmost $m$ vertices are internal twice. Therefore $t_\psi \geq (n + 2), t_2(\psi) \leq m$. Hence $t \geq (n + 2), t_2 \leq m$ and hence $\eta_{2as}(G) = q - p - t_2 + t = (p + 1) - p - m + (n + 2) \geq (n + 3) - m$. Thus $\eta_{2as}(G) = (n + 3) - m$.

**Subcase 4.1.2.2.** When $\deg(v) = 4$

Take $G_1 = G - P$ is a unicyclic graph with $n$ pendant vertices and $(m - 1)$ vertices is of degree $\geq 4$ with $l = 2$. Hence by theorem 2.9, $\eta_{2as}(G_1) = (n + 1) - (m - 1) = (n + 2) - m$. Let $\psi_1$ be the minimum 2–acyclic simple graphoidal cover of $G_1$. Let $P_3$ be a path in $\psi_1$ in which $v$ is an external vertex. Take $\psi = \psi_1 \cup \{P_1\} \cup \{P_2P_3\}$ is a 2–acyclic simple graphoidal cover of $G$ and $\eta_{2as}(G) \leq (n + 2) - m + 1 + 2 = (n + 3) - m$. For any 2–acyclic simple graphoidal cover of $G$ atleast $n$ pendant vertices and two vertices in $D(l : m : i)$ are external and atmost $m$ vertices are internal twice. Therefore $t_\psi \geq (n + 2), t_2(\psi) \leq m$. Hence $t \geq (n + 2), t_2 \leq m$ and hence $\eta_{2as}(G) = q - p - t_2 + t = (p + 1) - p - m + (n + 2) \geq (n + 3) - m$. Thus $\eta_{2as}(G) = (n + 3) - m$.

**Subcase 4.2.** When $\deg(u_{l-1}) \geq 4$ and $\deg(u_{l+i-1}) \geq 4$, then there are three subcases.

**Subcase 4.2.1.** When $\deg(u) \geq 3, \deg(v) \geq 3, u, v \in C_l$

Take $G_1 = G - P$, where $P$ denotes $(x, y)$ section of $C_m$ and $x, y \in C_m$. It is clear that $G_1$ is a unicyclic graph with $(n + 2)$ pendant vertices and $m$ vertices is of degree $\geq 4$ with $l = 3$. Hence by theorem 2.9, $\eta_{2as}(G_1) = (n + 2) - m$. Let $\psi_1$ be the minimum 2–acyclic simple graphoidal cover of $G_1$. Take $\psi = \psi_1 \cup \{P\}$ is a 2–acyclic simple graphoidal cover of $G$, hence $\eta_{2as}(G) \leq (n + 2) - m + 1 = (n + 3) - m$. For any 2–acyclic simple graphoidal cover of $G$ atleast $n$ pendant vertices and two vertices in $D(l : m : i)$ are external and atmost $m$ vertices are internal twice. Therefore $t_\psi \geq (n + 2), t_2(\psi) \leq m$. Hence $t \geq (n + 2), t_2 \leq m$ and hence $\eta_{2as}(G) = q - p - t_2 + t = (p + 1) - p - m + (n + 2) \geq (n + 4) - m$. Thus $\eta_{2as}(G) = (n + 4) - m$.

**Subcase 4.2.2.** When $\deg(u_{l+i-1}) = 4, \deg(u) \geq 3, \deg(v) \geq 3, u \in C_l, v \in C_m$ and let $P$ denotes $(u_{l+i-1}, v)$ section of $C_m$ such that it has atleast one internal vertex say $u_i$. Let $P_1$ and $P_2$ denote
the \((u_{l+i-1}, u_i)\) and \((u_i, v)\) section of \(P\) respectively. Then there are two subcases.

**Subcase 4.2.2.1.** When \(\deg(v) = 3\) (Or) \(\deg(v) \geq 5\)

Take \(G_1 = G - P\) is a unicyclic graph with \(n\) pendent vertices and \((m - 1)\) vertices is of degree \(\geq 4\) with \(l = 2\). Hence by theorem 2.9, \(\eta_{2as}(G_1) = (n + 1) - (m - 1) = (n + 2) - m\). Let \(\psi_1\) be the minimum 2–acyclic simple graphoidal cover of \(G_1\). Let \(P_4\) be a path in \(\psi_1\) in which \(u_{l+i-1}\) is an external vertex. Take \(\psi = (\psi_1 - P_4) \cup \{P_4P_1\} \cup \{P_2\}\) is a 2–acyclic simple graphoidal cover of \(G\), hence \(\eta_{2as}(G) \leq (n + 2) - m - 1 + 2 = (n + 3) - m\). For any 2–acyclic simple graphoidal cover of \(G\) atleast \(n\) pendent vertices and atleast two vertices in \(D(l : m : i)\) are external and atmost \(m\) vertices are internal twice. Therefore \(t\psi \geq (n + 2), t_2(\psi) \leq m\). Hence \(t \geq (n + 2), t_2 \leq m\) so that \(\eta_{2as}(G) = q - p - t_2 + t = (p + 1) - p - m(n + 2) \geq (n + 3) - m\). Thus \(\eta_{2as}(G) = (n + 3) - m\).

**Subcase 4.2.2.2.** When \(\deg(v) = 4\)

Take \(G_1 = G - P\) is a unicyclic graph with \(n\) pendent vertices and \((m - 2)\) vertices is of degree \(\geq 4\) with \(l = 2\). Hence by theorem 2.9, \(\eta_{2as}(G_1) = (n - 1) - (m - 2) = (n + 3) - m\). Let \(\psi_1\) be the minimum 2–acyclic simple graphoidal cover of \(G_1\). Let \(P_3\) be a path in \(\psi_1\) in which \(v\) is an external vertex and let \(P_4\) be a path in \(\psi_1\) in which \(u_{l+i-1}\) as the external vertex. Take \(\psi = (\psi_1 - P_3 - P_4) \cup \{P_4P_1\} \cup \{P_2P_3\}\) is a 2–acyclic simple graphoidal cover of \(G\), hence \(\eta_{2as}(G) \leq (n + 3) - m - 2 + 2 = (n + 3) - m\). For any 2–acyclic simple graphoidal cover of \(G\) at least \(n\) pendent vertices and at least two vertices in \(D(l : m : i)\) are external and at most \(m\) vertices are internal twice. Therefore \(t\psi \geq (n + 2), t_2(\psi) \leq m\). Hence \(t \geq (n + 2), t_2 \leq m\) and hence \(\eta_{2as}(G) = q - p - t_2 + t = (p + 1) - p - m(n + 2) \geq (n + 3) - m\). Thus \(\eta_{2as}(G) = (n + 3) - m\).

**Subcase 4.3.** When \(\deg(u_{l+i-1}) \geq 5\) and \(\deg(u) \geq 3\), \(\deg(v) \geq 3\), \(u \in C_l, v \in C_m\) and let \(P\) denotes \((u_{l+i-1}, v)\) section of \(C_m\) such that it has at least one internal vertex say \(u_i\). Let \(P_1\) and \(P_2\) denote the \((u_{l+i-1}, u_i)\) and \((u_i, v)\) section of \(P\) respectively. Then there are two subcases.

**Subcase 4.3.1.** When \(\deg(v) = 3\) Or \(\deg(v) \geq 5\)

Take \(G_1 = G - P\) is a unicyclic graph with \(n\) pendent vertices and \(m\) vertices is of degree \(\geq 4\) with \(l = 2\). Hence by theorem 2.9, \(\eta_{2as}(G_1) = (n + 1) - m\). Let \(\psi_1\) be the minimum 2–acyclic simple graphoidal cover of \(G_1\). Let \(P_4\) be a path in \(\psi_1\) in which \(u_{l+i-1}\) is an external vertex. Take \(\psi = \psi_1 \cup \{P_1\} \cup \{P_2\}\) is a 2–acyclic simple graphoidal cover of \(G\), hence \(\eta_{2as}(G) \leq \).
(n + 3) − m. For any 2–acyclic simple graphoidal cover of $G$ at least $n$ pendant vertices and at least two vertices in $D(l : m : i)$ are external and atmost $m$ vertices are internal twice. Therefore $t_\psi \geq (n + 2), t_2(\psi) \leq m$. Hence $t \geq (n + 2), t_2 \leq m$ and hence $\eta_{2as}(G) = q - p - t_2 + t = (p + 1) - p - m(n + 2) \geq (n + 3) - m$. Thus $\eta_{2as}(G) = (n + 3) - m$.

**Subcase 4.3.2.** When $\deg(v) = 4$

Take $G_1 = G - P$ is a unicyclic graph with $n$ pendant vertices and $(m - 1)$ vertices is of degree $\geq 4$ with $l = 2$. Hence by theorem 2.9, $\eta_{2as}(G_1) = (n + 1) - (m - 1) = (n + 2) - m$. Let $\psi_1$ be the minimum 2–acyclic simple graphoidal cover of $G_1$. Let $P_3$ be a path in $\psi_1$ in which $v$ is an external vertex. Take $\psi = (\psi_1 - P_3) \cup \{P_1\} \cup \{P_2P_3\}$ is a 2–acyclic simple graphoidal cover of $G$ and $\eta_{2as}(G) \leq (n + 2) - m - 1 + 2 = (n + 3) - m$. For any 2–acyclic simple graphoidal cover of $G$ at least $n$ pendant vertices and at least two vertices in $D(l : m : i)$ are external and at most $m$ vertices are internal twice. Therefore $t_\psi \geq (n + 2), t_2(\psi) \leq m$. Hence $t \geq (n + 2), t_2 \leq m$ and hence $\eta_{2as}(G) = q - p - t_2 + t = (p + 1) - p - m(n + 2) \geq (n + 3) - m$. Thus $\eta_{2as}(G) = (n + 3) - m$.

**Case 5.** When $l = 5$ and $u, v, w$ be the only vertices is of degree greater than 2 other than $u_{l-1}, u_{l+i-1}$. Then there are three subcases.

**Subcase 5.1.** Suppose $\deg(u_{l-1}) \geq 3, \deg(u_{l+i-1}) = 3$, then there are two subcases.

**Subcase 5.1.1.** When $\deg(u) \geq 3, \deg(v) \geq 3, \deg(w) \geq 3, u, v, w \in C_l$

Take $G_1 = G - P$, where Let $P = (x, y)$ be a path in $C_m$ and $x, y \in C_m$. It is clear that $G_1$ is a unicyclic graph with $(n + 2)$ pendant vertices and $m$ vertices is of degree $\geq 4$ with $l > 3$. Hence by theorem 2.9, $\eta_{2as}(G_1) = (n + 2) - m$. Let $\psi_1$ be the minimum 2–acyclic simple graphoidal cover of $G_1$. Take $\psi = \psi_1 \cup \{\cup P\}$ is a 2–acyclic simple graphoidal cover of $G$, hence $\eta_{2as}(G) \leq ((n + 2) - m) + 1 = (n + 3) - m$. For any 2–acyclic simple graphoidal cover of $G$ at least $n$ pendant vertices and two vertices in $D(l : m : i)$ are external and at most $m$ vertices are internal twice. Therefore $t_\psi \geq (n + 2), t_2(\psi) \leq m$. Hence $t \geq (n + 2), t_2 \leq m$ and hence $\eta_{2as}(G) = q - p - t_2 + t = (p + 1) - p - m(n + 2) \geq (n + 3) - m$. Thus $\eta_{2as}(G) = (n + 3) - m$.

**Subcase 5.1.2.** When $\deg(u) \geq 3$, $\deg(v) \geq 3$, $\deg(w) \geq 3$ and $u, v \in C_l, w \in C_m$. Let $P$ denotes $(u_{l+i-1}, w)$ section of $C_m$ such that it has at least one internal vertex say $u_i$. Let $P_1$ and $P_2$ denote
the \((u_{l+i-1}, u_l)\) and \((u_i, w)\) section of \(P\) respectively. Then there are two subcases.

**Subcase 5.1.2.1.** When \(\text{deg}(w) = 3\) (Or) \(\text{deg}(w) \geq 5\)

Take \(G_1 = G - P\) is a unicyclic graph with \(n\) pendent vertices and \(m\) vertices is of degree \(\geq 4\) with \(l = 3\). Hence by theorem 2.9, \(\eta_{2as}(G_1) = n - m\). Let \(\psi_1\) be the minimum 2–acyclic simple graphoidal cover of \(G_1\). Take \(\psi = \psi_1 \cup \{P_1\} \cup \{P_2\}\) is a 2–acyclic simple graphoidal cover of \(G\), hence \(\eta_{2as}(G) \leq (n + 2) - m\). For any 2–acyclic simple graphoidal cover of \(G\) at least \(n\) pendent vertices and at least one vertex in \(D(l : m : i)\) are external and at most \(m\) vertices are internal twice. Therefore \(t_{\psi} \geq (n + 1), t_{\psi}(\psi) \leq m\). Hence \(t \geq (n + 1), t_2 \leq m\) and hence \(\eta_{2as}(G) = q - p - t_2 + t = (p + 1) - p - m + (n + 1) \geq (n + 2) - m\). Thus \(\eta_{2as}(G) = (n + 2) - m\).

**Subcase 5.1.2.2.** When \(\text{deg}(w) = 4\)

Take \(G_1 = G - P\) is a unicyclic graph with \(n\) pendent vertices and \((m - 1)\) vertices is of degree \(\geq 4\) with \(l = 3\). Hence by theorem 2.9, \(\eta_{2as}(G_1) = n - (m - 1) = (n + 1) - m\). Let \(\psi_1\) be the minimum 2–acyclic simple graphoidal cover of \(G_1\). Let \(P_3\) be a path in \(\psi\) in which \(w\) as the external vertex. Take \(\psi = (\psi_1 - P_3) \cup \{P_1\} \cup \{P_2P_3\}\) is a 2–acyclic simple graphoidal cover of \(G\) and \(\eta_{2as}(G) \leq (n + 1) - m - 1 + 2 = (n + 2) - m\). For any 2–acyclic simple graphoidal cover of \(G\) at least \(n\) pendent vertices and at least one vertex in \(D(l : m : i)\) are external and at most \(m\) vertices are internal twice. Therefore \(t_{\psi} \geq (n + 1), t_{\psi}(\psi) \leq m\). Hence \(t \geq (n + 1), t_2 \leq m\) and hence \(\eta_{2as}(G) = q - p - t_2 + t = (p + 1) - p - m + (n + 1) \geq (n + 2) - m\). Thus \(\eta_{2as}(G) = (n + 2) - m\).

**Subcase 5.2.** When \(\text{deg}(u_{l-1}) \geq 4\) and \(\text{deg}(u_{l+i-1}) \geq 4\), then there are three subcases.

**Subcase 5.2.1.** Suppose \(\text{deg}(u_{l+i-1}) \geq 4\), \(\text{deg}(u) \geq 3\), \(\text{deg}(v) \geq 3\), \(\text{deg}(w) \geq 3\), \(u, v, w \in C_l\)

Take \(G_1 = G - P\), where let \(P = (y, z)\) be a path in \(C_m\) and \(y, z \in C_m\). It is clear that \(G_1\) is a unicyclic graph with \((n + 2)\) pendent vertices and \(m\) vertices is of degree \(\geq 4\) with \(l > 3\). Hence by theorem 2.9, \(\eta_{2as}(G_1) = (n + 2) - m\). Let \(\psi_1\) be the minimum 2–acyclic simple graphoidal cover of \(G_1\). Take \(\psi = \psi_1 \cup \{P\}\) is a 2–acyclic simple graphoidal cover of \(G\) so that \(\eta_{2as}(G) \leq ((n + 2) - m) + 1 = (n + 3) - m\). For any 2–acyclic simple graphoidal cover of \(G\) at least \(n\) pendent vertices and two vertices in \(D(l : m : i)\) are external and at most \(m\) vertices are internal twice. Therefore \(t_{\psi} \geq (n + 2), t_{\psi}(\psi) \leq m\). Hence \(t \geq (n + 2), t_2 \leq m\) and hence \(\eta_{2as}(G) = q - p - t_2 + t = (p + 1) - p - m + (n + 2) \geq (n + 3) - m\). Thus \(\eta_{2as}(G) = (n + 3) - m\).
**Subcase 5.2.2.** Suppose $\deg(u_{l+i-1}) = 4$, $\deg(u) \geq 3$, $\deg(v) \geq 3$, $\deg(w) \geq 3$, $u, v, w \in C_l, w \in C_m$ and let $P$ denotes $(u_{l+i-1}, w)$ section of $C_m$ such that it has atleast one internal vertex say $u_i$. Let $P_1$ and $P_2$ denote the $(u_{l+i-1}, u_i)$ and $(u_i, w)$ section of $P$ respectively. Then there are two subcases.

**Subcase 5.2.2.1.** When $\deg(w) = 3$ (Or) $\deg(w) \geq 5$

Take $G_1 = G - P$ is a unicyclic graph with $n$ pendant vertices and $(m - 1)$ vertices is of degree $\geq 4$ with $l = 3$. Hence by theorem 2.9, $\eta_{2as}(G_1) = (n + 1) - m$. Let $\psi_1$ be the minimum $2$–acyclic simple graphoidal cover of $G_1$. Let $P_4$ be a path in $\psi_1$ in which $u_{l+i-1}$ is an external vertex. Take $\psi = (\psi_1 - P_4) \cup \{P_4 P_1\} \cup \{P_2\}$ is a $2$–acyclic simple graphoidal cover of $G$, hence $\eta_{2as}(G) \leq (n + 1) - m - 1 + 2 = (n + 2) - m$. For any $2$–acyclic simple graphoidal cover of $G$ at least $n$ pendent vertices and at least one vertex in $D(l : m : i)$ are external and at most $m$ vertices are internal twice. Therefore $t_{\psi} \geq (n + 1), t_2(\psi) \leq m$. Hence $t \geq (n + 1), t_2 \leq m$ and so $\eta_{2as}(G) = q - p - t + t = (p + 1) - p - m + (n + 1) \geq (n + 2) - m$. Thus $\eta_{2as}(G) = (n + 2) - m$.

**Subcase 5.2.2.2.** When $\deg(w) = 4$

Take $G_1 = G - P$ is a unicyclic graph with $n$ pendant vertices and $(m - 2)$ vertices is of degree $\geq 4$ with $l = 3$. Hence by theorem 2.9, $\eta_{2as}(G_1) = (n + 2) - m$. Let $\psi_1$ be the minimum $2$–acyclic simple graphoidal cover of $G_1$. Let $P_3$ be a path in $\psi_1$ in which $w$ is an external vertex and let $P_4$ be a path in $\psi_1$ in which $u_{l+i-1}$ is an external vertex. Take $\psi = (\psi_1 - P_3 - P_4) \cup \{P_4 P_1\} \cup \{P_2 P_3\}$ is a $2$–acyclic simple graphoidal cover of $G$, hence $\eta_{2as}(G) \leq (n + 2) - m - 2 + 2 = (n + 2) - m$. For any $2$–acyclic simple graphoidal cover of $G$ at least $n$ pendent vertices and at least one vertex in $D(l : m : i)$ are external and at most $m$ vertices are internal twice. Therefore $t_{\psi} \geq (n + 1), t_2(\psi) \leq m$. Hence $t \geq (n + 1), t_2 \leq m$ and hence $\eta_{2as}(G) = q - p - t + t = (p + 1) - p - m + (n + 1) \geq (n + 2) - m$. Thus $\eta_{2as}(G) = (n + 2) - m$.

**Subcase 5.2.3.** When $\deg(u_{l+i-1}) \geq 5$, $\deg(u) \geq 3$, $\deg(v) \geq 3$, $\deg(w) \geq 3$ and $u, v, w \in C_l, w \in C_m$ and let $P$ denotes $(u_{l+i-1}, w)$ section of $C_m$ such that it has atleast one internal vertex say $u_i$. Let $P_1$ and $P_2$ denote the $(u_{l+i-1}, u_i)$ and $(u_i, w)$ section of $P$ respectively. Then there are two subcases.

**Subcase 5.2.3.1.** When $\deg(w) = 3$ Or $\deg(w) \geq 5$
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Take $G_1 = G - P$ is a unicyclic graph with $n$ pendant vertices and $m$ vertices is of degree $\geq 4$ with $l = 3$. Hence by theorem 2.9, $\eta_{2as}(G_1) = n - m$. Let $\psi_1$ be the minimum 2–acyclic simple graphoidal cover of $G_1$. Take $\psi = \psi_1 \cup \{P_1\} \cup \{P_2\}$ is a 2–acyclic simple graphoidal cover of $G$, hence $\eta_{2as}(G) \leq (n + 2) - m$. For any 2–acyclic simple graphoidal cover of $G$ atleast $n$ pendent vertices and atleast one vertex in $D(l : m : i)$ are external and atmost $m$ vertices are internal twice. Therefore $t_{\psi} \geq (n + 1), t_2(\psi) \leq m$. Hence $t \geq (n + 1), t_2 \leq m$ and so $\eta_{2as}(G) = q - p - t_2 + t = (p + 1) - p - m(n + 1) \geq (n + 2) - m$. Thus $\eta_{2as}(G) = (n + 2) - m$.

Subcase 5.2.3.2. When $deg(w) = 4$

Take $G_1 = G - P$ is a unicyclic graph with $n$ pendant vertices and $(m - 1)$ vertices is of degree $\geq 4$ with $l > 3$. Hence by theorem 2.9, $\eta_{2as}(G_1) = (n + 1) - m$. Let $\psi_1$ be the minimum 2–acyclic simple graphoidal cover of $G_1$. Let $P_3$ be a path in $\psi_1$ in which $w$ is an external vertex. Take $\psi = (\psi_1 - P_3) \cup \{P_1\} \cup \{P_2P_3\}$ is a 2–acyclic simple graphoidal cover of $G$, hence $\eta_{2as}(G) \leq (n + 1) - m - 2 = (n + 2) - m$. For any 2–acyclic simple graphoidal cover of $G$ atleast $n$ pendent vertices and atleast one vertex in $D(l : m : i)$ are external and atmost $m$ vertices are internal twice. Therefore $t_{\psi} \geq (n + 1), t_2(\psi) \leq m$. Hence $t \geq (n + 1), t_2 \leq m$ and so $\eta_{2as}(G) = q - p - t_2 + t = (p + 1) - p - m(n + 1) \geq (n + 2) - m$. Thus $\eta_{2as}(G) = (n + 2) - m$.

Case 6. When $l \geq 5$ and $u, v, w, x$ be the vertices is of degree greater than 2 on $D(l : m : i)$ other than $u_{l-1, l_{l+i-1}}$. Then there are three subcases.

Subcase 6.1. Suppose $deg(u_{l-1}) \geq 3$ and $deg(u_{l+i-1}) = 3$, then there are three sub cases.

Subcase 6.1.1. When $deg(u) \geq 3$, $deg(v) \geq 3$, $deg(w) \geq 3$ and $deg(x) \geq 3$, $u, v, w, x \in C_l$

Take $G_1 = G - P$ where $P = (y, z)$ be a path in $C_m$ and $y, z \in C_m$. It is clear that $G_1$ is a unicyclic graph with $(n + 2)$ pendant vertices and $m$ vertices is of degree $\geq 4$ with $l > 3$. Hence by therorem 2.9, $\eta_{2as}(G_1) = (n + 2) - m$. Let $\psi_1$ be the minimum 2–acyclic simple graphoidal cover of $G_1$. Take $\psi = \psi_1 \cup \{P\}$ is a 2–acyclic simple graphoidal cover of $G$, hence $\eta_{2as}(G) \leq ((n + 2) - m) + 1 = (n + 3) - m$. For any 2–acyclic simple graphoidal cover of $G$ atleast $n$ pendent vertices and two vertices in $D(l : m : i)$ are external and atmost $m$ vertices are internal twice. Therefore $t_{\psi} \geq (n + 2), t_2(\psi) \leq m$. Hence $t \geq (n + 2), t_2 \leq m$ so that $\eta_{2as}(G) = q - p - t_2 + t = (p + 1) - p - m(n + 2) \geq (n + 3) - m$. Thus $\eta_{2as}(G) = (n + 3) - m$.

Subcase 6.1.2. When $deg(u) \geq 3$, $deg(v) \geq 3$, $deg(w) \geq 3$ and $deg(x) \geq 3$, $u, v, w \in C_l$, $x \in C_m$
and let \( P \) denotes \((u_{i+l-1}, x)\) section of \( C_m \) such that it has atleast one internal vertex say \( u_i \).

Let \( P_1 \) and \( P_2 \) denote the \((u_{i+l-1}, u_i)\) and \((u_i, x)\) section of \( P \) respectively. Then there are two subcases.

**Subcase 6.1.2.1.** When \( \deg(x) = 3 \) (Or) \( \deg(x) \geq 5 \)

Take \( G_1 = G - P \) is a unicyclic graph with \( n \) pendant vertices and \( m \) vertices is of degree \( \geq 4 \) with \( l > 3 \). Hence by theorem 2.9, \( \eta_{2as}(G_1) = n - m \). Let \( \psi_1 \) be the minimum 2–acyclic simple graphoidal cover of \( G_1 \). Take \( \psi = \psi_1 \cup \{P_1\} \cup \{P_2\} \) is a 2–acyclic simple graphoidal cover of \( G \), hence \( \eta_{2as}(G) \leq (n + 2) - m \). For any 2–acyclic simple graphoidal cover of \( G \) atleast \( n \) pendent vertices and atleast one vertex in \( D \) with \( \psi \) denotes the section of \( G \), therefore \( \eta_{2as}(G) = q - p - t_2 + t = (p + 1) - p - m + (n + 1) \geq (n + 2) - m \). Thus \( \eta_{2as}(G) = (n + 2) - m \).

**Subcase 6.1.2.2.** When \( \deg(x) = 4 \)

Take \( G_1 = G - P \) is a unicyclic graph with \( n \) pendant vertices and \( (m - 1) \) vertices is of degree \( \geq 4 \) with \( l = 3 \). Hence by theorem 2.9, \( \eta_{2as}(G_1) = (n + 1) - m \). Let \( \psi_1 \) be the minimum 2–acyclic simple graphoidal cover of \( G_1 \). Take \( P_3 \) be a path in \( \psi_1 \) in which \( w \) is an external vertex. Take \( \psi = (\psi_1 - P_3) \cup \{P_1\} \cup \{P_2 \cup P_3\} \) is a 2–acyclic simple graphoidal cover of \( G \), hence \( \eta_{2as}(G) \leq (n + 1) - m - 1 + 2 = (n + 2) - m \). For any 2–acyclic simple graphoidal cover of \( G \) atleast \( n \) pendent vertices and atleast one vertex in \( D \) with \( \psi \) denote the section of \( G \). Therefore \( t_\psi \geq (n + 1), t_2(\psi) \leq m \). Hence \( t \geq (n + 1), t_2 \leq m \) so that \( \eta_{2as}(G) = q - p - t_2 + t = (p + 1) - p - m + (n + 1) \geq (n + 2) - m \). Thus \( \eta_{2as}(G) = (n + 2) - m \).

**Subcase 6.1.3.** When \( \deg(u) \geq 3, \deg(v) \geq 3, \deg(w) \geq 3 \) and \( \deg(x) \geq 3 \), \( u, v, \in C_1, w, x \in C_m \) and let \( P \) denotes \((w, x)\) section of \( C_m \). Then there are three subcases.

**Subcase 6.1.3.1.** When \((\deg(w) = \deg(x) = 3) \) (Or) \((\deg(w) \geq 5, \deg(x) \geq 5) \) (or) \((\deg(w) = 3, \deg(x) \geq 5) \)

Take \( G_1 = G - P \) is a unicyclic graph with \( n \) pendant vertices and \( m \) vertices is of degree \( \geq 4 \) with \( l > 3 \). Hence by theorem 2.9, \( \eta_{2as}(G_1) = n - m \). Let \( \psi_1 \) be the minimum 2–acyclic simple graphoidal cover of \( G_1 \). Take \( \psi = \psi_1 \cup \{P\} \) is a 2–acyclic simple graphoidal cover of \( G \), hence \( \eta_{2as}(G) \leq (n + 1) - m \). For any 2–acyclic simple graphoidal cover of \( G \) atleast \( n \) pendent vertices are external and atmost \( m \) vertices are internal twice. Therefore \( t_\psi \geq n, t_2(\psi) \leq m \).
Hence \( t \geq n, t_2 \leq m \) so that \( \eta_{2as}(G) = q - p - t_2 + t = (p + 1) - p - m + n \geq (n + 1) - m. \) Thus \( \eta_{2as}(G) = (n + 1) - m. \)

**Subcase 6.1.3.2.** When \( \deg(w) = \deg(x) = 4 \)

Take \( G_1 = G - P \) is a unicyclic graph with \( n \) pendent vertices and \( (m - 2) \) vertices is of degree \( \geq 4 \) with \( l \geq 3. \) Hence by theorem 2.9, \( \eta_{2as}(G_1) = (n + 2) - m. \) Let \( \psi_1 \) be the minimum 2–acyclic simple graphoidal cover of \( G_1. \) Let \( P_3 \) be a path in \( \psi_1 \) in which \( w \) is an external vertex and let \( P_4 \) be a path in \( \psi \) in which \( x \) is an external vertex. Take \( \psi = (\psi_1 - P_3 - P_4) \cup \{P_3PP_4\} \) is a 2–acyclic simple graphoidal cover of \( G, \) hence \( \eta_{2as}(G) \leq (n + 2) - m - 2 + 1 = (n + 1) - m. \) For any 2–acyclic simple graphoidal cover of \( G \) at least \( n \) pendent vertices are external and atmost \( m \) vertices are internal twice. Therefore \( t_\psi \geq n, t_2(\psi) \leq m. \) Hence \( t \geq n, t_2 \leq m \) so that \( \eta_{2as}(G) = q - p - t_2 + t = (p + 1) - p - m + n \geq (n + 1) - m. \) Thus \( \eta_{2as}(G) = (n + 1) - m. \)

**Subcase 6.1.3.3.** When \( \deg(w) = 3, \deg(x) = 4 \) Or \( \deg(w) \geq 5, \deg(x) = 4 \)

Take \( G_1 = G - P \) is a unicyclic graph with \( n \) pendent vertices and \( (m - 1) \) vertices is of degree \( \geq 4 \) with \( l \geq 3. \) Hence by theorem 2.9, \( \eta_{2as}(G_1) = n - (m - 1) = (n + 1) - m. \) Let \( \psi_1 \) be the minimum 2–acyclic simple graphoidal cover of \( G_1. \) Let \( P_3 \) be a path in \( \psi_1 \) in which \( w \) is an external vertex. Take \( \psi = (\psi_1 - P_3) \cup \{PP_3\} \) is a 2–acyclic simple graphoidal cover of \( G, \) hence \( \eta_{2as}(G) \leq (n + 1) - m - 1 + 1 = (n + 1) - m. \) For any 2–acyclic simple graphoidal cover of \( G \) at least \( n \) pendent vertices are external and atmost \( m \) vertices are internal twice. Therefore \( t_\psi \geq n, t_2(\psi) \leq m. \) Hence \( t \geq n, t_2 \leq m \) so that \( \eta_{2as}(G) = q - p - t_2 + t = (p + 1) - p - m + n \geq (n + 1) - m. \) Thus \( \eta_{2as}(G) = (n + 1) - m. \)

**Subcase 6.2.** When \( \deg(u_{l-1}) \geq 4 \) and \( \deg(u_{l+i-1}) \geq 4, \) then there are three sub cases.

**Subcase 6.2.1.** When \( \deg(u_{l+i-1}) \geq 4, \deg(u) \geq 3, \deg(v) \geq 3, \deg(w) \geq 3 \) and \( \deg(x) \geq 3, u, v, w, x \in C_l \)

Take \( G_1 = G - P \) where \( P = (y, z) \) be a path in \( C_m \) and \( y, z \in C_m. \) It is clear that \( G_1 \) is a unicyclic graph with \( (n + 2) \) pendent vertices and \( m \) vertices is of degree \( \geq 4 \) with \( l > 3. \) Hence by theorem 2.9, \( \eta_{2as}(G_1) = (n + 2) - m. \) Let \( \psi_1 \) be the minimum 2–acyclic simple graphoidal cover of \( G_1. \) Take \( \psi = \psi_1 \cup \{P\} \) is a 2–acyclic simple graphoidal cover of \( G, \) hence \( \eta_{2as}(G) \leq ((n + 2) - m) + 1 = (n + 3) - m. \) For any 2–acyclic simple graphoidal cover of \( G \) at least \( n \) pendent vertices and two vertices in \( D(l : m : i) \) are external and atmost \( m \) vertices are
internal twice. Therefore \( t_\psi \geq (n+2), t_2(\psi) \leq m \). Hence \( t \geq (n+2), t_2 \leq m \) so that \( \eta_{2as}(G) = q - p - t_2 + t = (p + 1) - p - m(n+2) \geq (n+3) - m \). Thus \( \eta_{2as}(G) = (n+3) - m \).

**Subcase 6.2.2.** Suppose \( \text{deg}(u_{l+i-1}) = 4 \) and \( \text{deg}(u) \geq 3, \text{deg}(v) \geq 3, \text{deg}(w) \geq 3 \) and \( \text{deg}(x) \geq 3, u, v, w \in C_i, x \in C_m \) and let \( P \) denotes \((u_{l+i-1}, x)\) section of \( C_m \) such that it has atleast one internal vertex say \( u_i \). Let \( P_1 \) and \( P_2 \) denote the 
\((u_{l+i-1}, u_i)\) and \((u_i, x)\) section of \( P \) respectively. Then there are two subcases.

**Subcase 6.2.2.1.** When \( \text{deg}(x) = 3 \) Or \( \text{deg}(x) \geq 5 \)

Take \( G_1 = G - P \) is a unicyclic graph with \( n \) pendent vertices and \((m-1)\) vertices is of degree \( \geq 4 \) with \( l > 3 \). Hence by theorem 2.9, \( \eta_{2as}(G_1) = (n+1) - m \). Let \( \psi_1 \) be the minimum 2--acyclic simple graphoidal cover of \( G_1 \). Let \( P_4 \) be a path in \( \psi_1 \) in which \( u_{l+i-1} \) is an external vertex. Take \( \psi = (\psi_1 - P_4) \cup \{P_4 \cup P_1 \} \cup \{P_2 \} \) is a 2--acyclic simple graphoidal cover of \( G \), hence \( \eta_{2as}(G) \leq (n+1) - m - 1 + 2 = (n+2) - m \). For any 2--acyclic simple graphoidal cover of \( G \) atleast \( n \) pendent vertices and atleast one vertex in \( D(l : m : i) \) are external and atmost \( m \) vertices are internal twice. Therefore \( t_{\psi} \geq (n+1), t_2(\psi) \leq m \). Hence \( t \geq (n+1), t_2 \leq m \) and hence \( \eta_{2as}(G) = q - p - t_2 + t = (p + 1) - p - m + (n+1) \geq (n+2) - m \). Thus \( \eta_{2as}(G) = (n+2) - m \).

**Subcase 6.2.2.2.** When \( \text{deg}(x) = 4 \)

Take \( G_1 = G - P \) is a unicyclic graph with \( n \) pendent vertices and \((m-2)\) vertices is of degree \( \geq 4 \) with \( l > 3 \). Hence by theorem 2.9, \( \eta_{2as}(G_1) = (n+2) - m \). Let \( \psi_1 \) be the minimum 2--acyclic simple graphoidal cover of \( G_1 \). Let \( P_3 \) be a path in \( \psi_1 \) in which \( x \) is an external vertex and let \( P_4 \) be a path in \( \psi_1 \) in which \( u_{l+i-1} \) is an external vertex. Take \( \psi = (\psi_1 - P_3 - P_4) \cup \{P_4 \cup P_1 \} \cup \{P_2 P_3 \} \) is a 2--acyclic simple graphoidal cover of \( G \) and \( \eta_{2as}(G) \leq (n+2) - m - 2 + 2 = (n+2) - m \). For any 2--acyclic simple graphoidal cover of \( G \) atleast \( n \) pendent vertices and atleast one vertex in \( D(l : m : i) \) are external and atmost \( m \) vertices are internal twice. Therefore \( t_{\psi} \geq (n+1), t_2(\psi) \leq m \). Hence \( t \geq (n+1), t_2 \leq m \) and hence \( \eta_{2as}(G) = q - p - t_2 + t = (p + 1) - p - m + (n+1) \geq (n+2) - m \). Thus \( \eta_{2as}(G) = (n+2) - m \).

**Subcase 6.2.3.** When \( \text{deg}(u_{l+i-1}) \geq 5 \) and \( \text{deg}(u) \geq 3, \text{deg}(v) \geq 3, \text{deg}(w) \geq 3 \) and \( \text{deg}(x) \geq 3, u, v, w \in C_i, x \in C_m \) and let \( P \) denotes \((u_{l+i-1}, x)\) section of \( C_m \) such that it has atleast one internal vertex say \( u_i \). Let \( P_1 \) and \( P_2 \) denote the 
\((u_{l+i-1}, u_i)\) and \((u_i, x)\) section of \( P \) respectively. Then
there are two subcases.

**Subcase 6.2.3.1.** When \( \deg(x) = 3 \) (Or) \( \deg(x) \geq 5 \)

Take \( G_1 = G - P \) is a unicyclic graph with \( n \) pendent vertices and \( m \) vertices is of degree \( \geq 4 \) with \( l > 3 \). Hence by theorem 2.9, \( \eta_{2as}(G_1) = (n + 1) - m \). Let \( \psi_1 \) be the minimum 2–acyclic simple graphoidal cover of \( G_1 \). Let \( P_4 \) be a path in \( \psi_1 \) in which \( u_{l+i-1} \) is an external vertex. Take \( \psi = \psi_1 \cup \{P_1\} \cup \{P_2\} \) is a 2–acyclic simple graphoidal cover of \( G \), hence \( \eta_{2as}(G) \leq (n+2) - m \). For any 2–acyclic simple graphoidal cover of \( G \) atleast \( n \) pendent vertices and atleast one vertex in \( D(l : m : i) \) are external and atmost \( m \) vertices are internal twice. Therefore \( t_{\psi} \geq (n + 1), t_2(\psi) \leq m \). Hence \( t \geq (n + 1), t_2 \leq m \) so that \( \eta_{2as}(G) = q - p - t_2 + t = (p + 1) - p - m(n + 1) \geq (n + 2) - m \). Thus \( \eta_{2as}(G) = (n + 2) - m \).

**Subcase 6.2.3.2.** When \( \deg(x) = 4 \)

Take \( G_1 = G - P \) is a unicyclic graph with \( n \) pendent vertices and \( (m - 1) \) vertices is of degree \( \geq 4 \) with \( l > 3 \). Hence by theorem 2.9, \( \eta_{2as}(G_1) = (n + 1) - m \). Let \( \psi_1 \) be the minimum 2–acyclic simple graphoidal cover of \( G_1 \). Let \( P_3 \) be a path in \( \psi_1 \) in which \( x \) is an external vertex. Take \( \psi = (\psi_1 - P_3) \cup \{P_1\} \cup \{P_2P_3\} \) is a 2–acyclic simple graphoidal cover of \( G \), hence \( \eta_{2as}(G) \leq (n + 1) - m + 2 = (n + 2) - m \). For any 2–acyclic simple graphoidal cover of \( G \) atleast \( n \) pendent vertices and atleast one vertex in \( D(l : m : i) \) are external and atmost \( m \) vertices are internal twice. Therefore \( t_{\psi} \geq (n + 1), t_2(\psi) \leq m \). Hence \( t \geq (n + 1), t_2 \leq m \) so that \( \eta_{2as}(G) = q - p - t_2 + t = (p + 1) - p - m + (n + 1) \geq (n + 2) - m \). Thus \( \eta_{2as}(G) = (n + 2) - m \).

**Subcase 6.2.4.** When \( \deg(u_{l+i-1}) \geq 4, \deg(u) \geq 3, \deg(v) \geq 3, \deg(w) \geq 3 \) and \( \deg(x) \geq 3, u, v \in C_l, w, x \in C_m \) and let \( P \) denotes \((w, x)\) section of \( C_m \). Then there are three subcases.

**Subcase 6.2.4.1.** When \( \deg(w) = \deg(x) = 3 \) Or \( \deg(w) = 3 \) and \( \deg(x) \geq 5 \) (Or) \( \deg(w) \geq 5 \) and \( \deg(x) \geq 5 \)

Take \( G_1 = G - P \) is a unicyclic graph with \( n \) pendent vertices and \( m \) vertices is of degree \( \geq 4 \) with \( l > 3 \). Hence by theorem 2.9, \( \eta_{2as}(G_1) = n - m \). Let \( \psi_1 \) be the minimum 2–acyclic simple graphoidal cover of \( G_1 \). Take \( \psi = \psi_1 \cup \{P\} \) is a 2–acyclic simple graphoidal cover of \( G \), hence \( \eta_{2as}(G) \leq (n + 1) - m \). For any 2–acyclic simple graphoidal cover of \( G \) atleast \( n \) pendent vertices are external and atmost \( m \) vertices are internal twice. Therefore \( t_{\psi} \geq n, t_2(\psi) \leq m \). Hence \( t \geq n, t_2 \leq m \) and hence \( \eta_{2as}(G) = q - p - t_2 + t = (p + 1) - p - m + n \geq (n + 1) - m \).
Thus $\eta_{2as}(G) = (n+1) - m$.

**Subcase 6.2.4.2.** When $(\deg(w) = 4$ and $\deg(x) = 3$) Or $(\deg(w) = 4$ and $\deg(x) \geq 5$)

Take $G_1 = G - P$ is a unicyclic graph with $n$ pendant vertices and $(m - 1)$ vertices is of degree $\geq 4$ with $l \geq 3$. Hence by theorem 2.9, $\eta_{2as}(G_1) = (n+1) - m$. Let $\psi_1$ be the minimum $2$–acyclic simple graphoidal cover of $G_1$.

Let $\psi_1$ be the minimum $2$–acyclic simple graphoidal cover of $G$ and $\eta_{2as}(G) \leq (n+1) - m - 1+1 = (n+1) - m$. For any $2$–acyclic simple graphoidal cover of $G$ at least $n$ pendant vertices are external and at most $m$ vertices are internal twice. Therefore $t_{\psi} \geq n, t_{2}(\psi) \leq m$.

Hence $t \geq n, t_{2} \leq m$ so that $\eta_{2as}(G) = q - p - t_{2} + t = (p+1) - p - m + n \geq (n+1) - m$. Thus $\eta_{2as}(G) = (n+1) - m$.

**Subcase 6.2.4.3.** When $\deg(w) = \deg(x) = 4$

Take $G_1 = G - P$ is a unicyclic graph with $n$ pendant vertices and $(m - 2)$ vertices is of degree $\geq 4$ with $l \geq 3$. Hence by theorem 2.9, $\eta_{2as}(G_1) = (n+2) - m$. Let $\psi_1$ be the minimum $2$–acyclic simple graphoidal cover of $G_1$. Let $P_3$ be a path in $\psi_1$ in which $w$ is an external vertex and let $P_4$ be a path in $\psi$ in which $x$ as the external vertex. Take $\psi = (\psi_1 - P_3 - P_4) \cup \{P_3P_4\}$ is a $2$–acyclic simple graphoidal cover of $G$, hence $\eta_{2as}(G) \leq (n+2) - m - 2 + 1 = (n+1) - m$.

For any $2$–acyclic simple graphoidal cover of $G$ at least $n$ pendant vertices are external and at most $m$ vertices are internal twice. Therefore $t_{\psi} \geq n, t_{2}(\psi) \leq m$. Hence $t \geq n, t_{2} \leq m$ so that $\eta_{2as}(G) = q - p - t_{2} + t = (p+1) - p - m + n \geq (n+1) - m$. Thus $\eta_{2as}(G) = (n+1) - m$. □
**Example 3.4.** Consider the Bicyclic graph $D(l : m : i)$ shown in Figure 2

Here $p = 31$, $q = 32$, $n = 12$, $l = 8$ and $m = 6$, then $\eta_{2ax}(G) = \{(v_{28}, v_{6}, v_{5}, v_{4}, v_{3}, v_{26}), (v_{29}, v_{6}, v_{5}, v_{31}), (v_{29}, v_{6}, v_{7}, v_{8}, v_{30}), (v_{8}, v_{1}, v_{2}, v_{9}, v_{10}, v_{11}, v_{12}, v_{13}, v_{20}), (v_{21}, v_{13}, v_{14}, v_{15}, v_{16}, v_{17}, v_{23}), (v_{22}, v_{17}, v_{18}, v_{24}), (v_{25}, v_{18}, v_{19}, v_{12})\} = 7 = (n + 1) - m.$

**CONFLICT OF INTERESTS**

The author(s) declare that there is no conflict of interests.

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