ON THE EXTENSIONS OF THE LEFT MODULES FOR A MEROMORPHIC OPEN-STRING VERTEX ALGEBRA

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Abstract. We study the extensions of two left modules $W_1, W_2$ for a meromorphic open-string vertex algebra $V$. We show that the extensions satisfying some technical but natural convergence conditions are in bijective correspondence to the first cohomology classes associated to the $V$-bimodule $H_N(W_1, W_2)$ constructed in [HQ]. When $V$ is a grading-restricted vertex algebra containing a nice vertex subalgebra $V_0$, those convergence conditions hold automatically. In addition, we show that $\text{Ext}^1(W_1, W_2)$ is finite dimensional, with its dimension bounded above by the fusion rule $N_{V_0}^{W_1}$ of $V_0$-modules. When $V$ is the Virasoro VOA that does not contain any nice subalgebras, we also give an example of an abelian category consisting of certain modules where the convergence conditions hold for every object.

1. Introduction

In the representation theory of various algebras (including, but not limited to, commutative associative algebras, associative algebras, Lie algebras, etc.), one of the main tools is the cohomological method. The powerful method of homological algebra often provides a unified treatment of many results in representation theory, giving not only solutions to open problems, but also conceptual understandings to the results.

Vertex operator algebras (VOAs hereafter) arose naturally in both mathematics and physics (see [BPZ], [B], and [FLM]) and are analogous to both Lie algebras and commutative associative algebras. In [H1], Yi-Zhi Huang introduced two cohomology theories for grading-restricted vertex algebras and modules that are analogous to Hochschild and Harrison cohomology for commutative associative algebras. Huang’s construction uses linear maps from the tensor powers of the vertex algebra to suitable spaces of “rational functions valued in the algebraic completion of a V-module” satisfying some natural conditions, including a technical convergence condition. Geometrically, this cohomology theory is consistent with the geometric and operadic formulation of VOA. Algebraically, the first and second cohomologies are given by (grading-preserving) derivations and first-order deformations, similarly to those for commutative associative algebras.

In [Q2], the author generalized Huang’s work on the Hochschild-like cohomology in [H1], and introduced a Hochschild-like cohomology theory for meromorphic open-string vertex algebras (a non-commutative generalization to VOAs) and their bimodules, analogously to the Hochschild cohomology for (not-necessarily commutative) associative algebras and the bimodules. As an application of the cohomology theory, in an early version of [HQ], Huang and the author proved that if $V$ is a meromorphic open-string vertex algebra such that the first cohomology $\tilde{H}^1(V, W) = 0$
for every $V$-bimodule $W$, then every left $V$-module satisfying a technical but natural convergence condition is completely reducible. See also the author’s Ph.D. thesis \[Q3\].

We remark that the proof in the early version of [HQ] and in \[Q3\] essentially uses only a particular type of $V$-bimodule $W$, namely the bimodule $\mathcal{H}_N(W_1, W_2)$ whose construction takes over the major part of the paper. Thus it is not necessary to require $\hat{H}^1(V, W) = 0$ for every $V$-bimodule. Indeed, after the author’s graduation, we found that $\hat{H}^1(V, W)$ is nonzero for a large class of vertex operator algebras whose module category is semisimple. In these cases, $\hat{H}^1(V, W)$ is given by the space $Z^1(V, W)$ of zero-mode derivations. The final version of [HQ] proves the same conclusion with the revised assumption: for every $V$-bimodule $W$, the first cohomology $\hat{H}^1(V, W)$ = $Z^1(V, W)$. It is conjectured that if every $V$-module is semisimple, then $\hat{H}^1(V, W)$ = $Z^1(V, W)$. The conjecture is supported by \[Q5\], which computes $\hat{H}^1(V, W)$ for the most common examples of VOA $V$ and $V$-modules $W$.

The current paper pushes forward our understanding of the cohomology theory of a MOSVA $V$. Let $W_1, W_2$ be two left $V$-modules. We study the space $\text{Ext}^1_{\mathcal{H}_N}(W_1, W_2)$ consisting of (equivalence classes of) extensions of $W_1$ by $W_2$ satisfying the technical and natural convergence conditions imposed in [HQ], namely, the composability condition and the $N$-weight-degree condition (where the subscript $N$ comes from). We establish a bijective correspondence between $\text{Ext}^1_{\mathcal{H}_N}(W_1, W_2)$ and the first cohomology $\hat{H}^1(V, \mathcal{H}_N(W_1, W_2))$. With this bijective correspondence, it is clear that if every $V$-module is semisimple, then $\hat{H}^1(V, \mathcal{H}_N(W_1, W_2)) = 0$. This corollary provides a converse to the conclusion in [HQ] (though weaker than the conjecture). It also explains why we totally ignored the zero-mode derivations in the early version of [HQ] and in [Q3]: they simply do not exist in such context. This result means a lot to the author, who views it as a final defense to his thesis [Q3].

In [HQ], it is shown that if $V$ is a grading-restricted vertex algebra containing a nice subalgebra $V_0$, then there exists some number $N$ such that the composability and the $N$-weight-degree conditions automatically holds. The current paper further shows that under the same assumption, the convergence conditions hold automatically for every extension of $W_2$ and $W_1$. In other words, the $\text{Ext}^1(W_1, W_2)$ to the usual sense coincides with $\text{Ext}^1_{\mathcal{H}_N}(W_1, W_2)$ for some $N$. Moreover, the dimension of $\text{Ext}^1(W_1, W_2)$ is bounded above by the fusion rule $N(W_2^{W_1})$ in the category of $V_0$-modules. The cohomology classes corresponding to the extensions are all given by intertwining operators of type $\left(\frac{W_2}{VW_1}\right)$. In particular, if all fusions rules in $V_0$ are finite, then $\text{Ext}^1(W_1, W_2)$ is finite-dimensional.

The current paper also shows that if the convergence conditions hold for a module $W$, then they hold for every submodule and (sub)quotients of the module $W$. This leads us to consider the category of modules satisfying the convergence conditions. However, generally speaking, such a category is not necessarily abelian as it is not closed under direct sums. Nevertheless, we construct a concrete abelian category consisting of quotients of finite direct sums of Verma modules for the
Virasoro VOA of all types except for the type $III_{\pm}$ (as in Feigin-Fuchs classification in [A], or the Case $1^\pm$ as in [IK], or the thick block as in [BNW]). We show that the convergence conditions hold automatically for every object in the category. So in this case, the convergence conditions hold automatically even when the VOA $V$ does not contain any nice subalgebras. This hints that the convergence conditions we need might not be a serious obstruction for us to apply results in this paper, though more studies are necessary to understand them further.

The paper is organized as follows: Section 2 recalls the necessary prerequisites, including the definitions of MOSVAs and modules, cohomology theory, and the construction of the bimodule $H_N(W_1,W_2)$ for two left $V$-modules $W_1,W_2$. Section 3 defines $\text{Ext}^1_N(W_1,W_2)$ and establishes the bijective correspondence to $\hat{H}^1(V, H_N(W_1,W_2))$. Section 4 first investigates category $C_N$ of modules satisfying the convergence conditions, then show that $C_N$ coincides with the module category when $V$ contains a nice subalgebra $V_0$, finally proves the finite-dimensionality of $\text{Ext}^1(W_1,W_2)$. Section 5 discusses the example of Virasoro VOA and gives the example of the abelian category where the convergence conditions hold for every object.

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## 2. Preliminaries

### 2.1. Meromorphic open-string vertex algebra and its modules.

We briefly review the definitions of the MOSVA, its left module, right module, and bimodules. Please find further details in [H3], [Q1] and [Q4].

**Definition 2.1.** A **meromorphic open-string vertex algebra** (hereafter MOSVA) is a $\mathbb{Z}$-graded vector space $V = \bigoplus_{n \in \mathbb{Z}} V(n)$ (graded by weights) equipped with a **vertex operator map**

$$Y : V \otimes V \rightarrow V[[x,x^{-1}]]$$

$$u \otimes v \mapsto Y(u,x)v,$$

and a **vacuum** $1 \in V$, satisfying the following axioms:

1. Axioms for the grading:
   
   (a) **Lower bound condition**: When $n$ is sufficiently negative, $V(n) = 0$.

   (b) **d-commutator formula**: Let $d_V : V \rightarrow V$ be defined by $d_v v = nv$ for $v \in V(n)$. Then for every $v \in V$

   $$[d_V,Y(v,x)] = x \frac{d}{dx} Y(v,x) + Y(d_v v,x).$$


(2) Axioms for the vacuum:
   (a) Identity property: Let $1_V$ be the identity operator on $V$. Then $Y(1, x) = 1_V$.
   (b) Creation property: For $u \in V$, $Y(u, x)1 \in V[[x]]$ and $\lim_{x \to 0} Y(u, x)1 = u$.

(3) $D$-derivative property and $D$-commutator formula: Let $D_V : V \to V$ be the operator given by
   \[ D_V v = \lim_{x \to 0} \frac{d}{dx}Y(v, x)1 \]
   for $v \in V$. Then for $v \in V$,
   \[ \frac{d}{dx}Y(v, x) = Y(D_V v, x) = [D_V, Y(v, x)]. \]

(4) Weak associativity with pole-order condition: For every $u_1, v \in V$, there exists $p \in \mathbb{N}$ such that for every $u_2 \in V$,
   \[ (x_0 + x_2)^p Y(u_1, x_0 + x_2)Y(u_2, x_2)v = (x_0 + x_2)^p Y(Y(u_1, x_0)u_2, x_2)v. \]

Proposition-Definition 2.2. Let $(V, Y, 1)$ be a MOSVA. Define the skew-symmetry vertex operator as follows:
   \[ Y^s : V \otimes V \to V[[x, x^{-1}]] \]
   \[ u \otimes v \mapsto e^{xD_V}Y(-v, x)u. \]
   Then $(V, Y^s, 1)$ is also a MOSVA, called the opposite MOSVA of $(V, Y, 1)$, denoted by $V^{op}$.
   Clearly $(V^{op})^{op} = V$.

Definition 2.3. Let $V$ be a MOSVA. A left $V$-module is a $\mathbb{C}$-graded vector space $W = \coprod_{m \in \mathbb{C}} W[m]$ (graded by weights), equipped with a vertex operator map
   \[ Y^L_W : V \otimes W \to W[[x, x^{-1}]] \]
   \[ u \otimes w \mapsto Y^L_W(u, x)w, \]
   an operator $d_W$ of weight 0 and an operator $D_W$ of weight 1, satisfying the following axioms:

(1) Axioms for the grading:
   (a) Lower bound condition: When Re($m$) is sufficiently negative, $W[m] = 0$.
   (b) $d$-grading condition: for every $w \in W[m]$, $d_W w = mw$.
   (c) $d$-commutator formula: For $u \in V$,
   \[ [d_W, Y^L_W(u, x)] = Y^L_W(d_V u, x) + x \frac{d}{dx} Y^L_W(u, x). \]

(2) The identity property: $Y^L_W(1, x) = 1_W$.

(3) The $D$-derivative property and the $D$-commutator formula: For $u \in V$,
   \[ \frac{d}{dx} Y^L_W(u, x) = Y^L_W(D_V u, x) \]
   \[ = [D_W, Y^L_W(u, x)]. \]
Let Definition 2.5. proved to be equivalent to Definition 2.7.

Definition 2.4. Let $W_1$, $W_2$ be two left $V$-modules. A linear map $f : W_1 \to W_2$ is a homomorphism if $fd_{W_1} = d_{W_2}f$, $fD_{W_1} = D_{W_2}f$, and for every $v \in V$, $fY_{W_1}(v, x) = Y_{W_2}(v, x)f$.

Definition 2.5. Let $V$ be a MOSVA. A right $V$-module is a graded vector space $W = \prod_{m \in \mathbb{C}} W_m$ (graded by weights), equipped with a vertex operator

$$Y_{W}^{s(R)} : V \otimes W \to W[[x, x^{-1}]]$$

$$u \otimes w \mapsto Y_{W}^{s(R)}(u, x)w,$$

an operator $d_W$ of weight 0 and an operator $D_W$ of weight 1, such that $(W, Y_{W}^{s(R)}, d_W, D_W)$ forms a left $V^{op}$-module.

Remark 2.6. Conceptually, a right module should be defined with a vertex operator $Y_{W}^{R}$: $W \otimes V \to W[[x, x^{-1}]]$ that is analogous to an intertwining operator of type $(V^W)$. But this is proved to be equivalent to Definition 2.5.

Definition 2.7. Let $(V, Y, 1)$ be a MOSVA. A $V$-bimodule is a vector space equipped with a left $V$-module structure and right $V$-module structure such that these two structures are compatible. More precisely, a $V$-bimodule is a graded vector space

$$W = \prod_{n \in \mathbb{C}} W_n$$

equipped with two vertex operators

$$Y_{W}^{L} : V \otimes W \to W[[x, x^{-1}]]$$

$$u \otimes w \mapsto Y_{W}^{L}(u, x)v,$$

and linear operators $d_W, D_W$ on $W$ satisfying the following conditions.

1. $(W, Y_{W}^{L}, d_W, D_W)$ is a left $V$-module.
2. $(W, Y_{W}^{s(R)}, d_W, D_W)$ is a left $V^{op}$-module.
3. Compatibility with pole-order condition: For every $v_1, v_2 \in V$, there exists $p \in \mathbb{N}$ such that for every $w \in W$,

$$(x_1 - x_2)^p Y_{W}^{L}(v_1, x_1)Y_{W}^{s(R)}(v_2, x_2)w = (x_1 - x_2)^p Y_{W}^{s(R)}(v_2, x_2)Y_{W}^{L}(v_1, x_1)w.$$
converges absolutely in the region

\[ |z_1| > \cdots > |z_l| > |z_{l+1}| > \cdots > |z_{l+r}| > 0 \]

to a rational function in \( z_1, \ldots, z_{l+1}, \ldots, z_{l+r} \) with the only possible poles at \( z_i = 0 \) \((i = 1, \ldots, l+r)\) and \( z_i = z_j \) \((1 \leq i < j \leq l + r)\). Please see [Q1] for further details.

2.2. Cohomology theory of meromorphic open-string vertex algebras. We briefly recall
the definition of cohomologies for MOSVA. Please see [H1] and [Q2] for further details.

**Definition 2.9.** For \( n \in \mathbb{Z}_+ \), we consider the configuration space

\[ F_n \mathbb{C} = \{(z_1, \ldots, z_n) \in \mathbb{C}^n : z_i \neq z_j, i \neq j\} \]

A \( \overline{W} \)-valued rational function in \( z_1, \ldots, z_n \) with the only possible poles at \( z_i = z_j, i \neq j \) is a map

\[ f : F_n \mathbb{C} \to \overline{W} = \prod_{m \in \mathbb{C}} W_{[m]} \]

\[(z_1, \ldots, z_n) \mapsto f(z_1, \ldots, z_n)\]

such that

1. For any \( w' \in W' \),

\[ \langle w', f(z_1, \ldots, z_n) \rangle \]

is a rational function in \( z_1, \ldots, z_n \) with the only possible poles at \( z_i = z_j, i \neq j \).

2. There exists integers \( p_{ij}, 1 \leq i < j \leq n \) and a formal series \( g(x_1, \ldots, x_n) \in W[[x_1, \ldots, x_n]] \),

such that for every \( w' \in W' \) and \((z_1, \ldots, z_n) \in F_n \mathbb{C} \),

\[ \prod_{1 \leq i < j \leq n} (z_i - z_j)^{p_{ij}} \langle w', f(z_1, \ldots, z_n) \rangle = \langle w', g(z_1, \ldots, z_n) \rangle \]

as a polynomial function.

The space of all such functions will be denoted by \( \overline{W}_{z_1, \ldots, z_n} \).

Similarly, by modifying (1), we define \( \overline{W} \)-valued rational function with only possible poles at \( z_i = 0 \) \((i = 1, \ldots, n)\) and at \( z_i = z_j \) \((1 \leq i < j \leq n)\). We denote the space of all such functions by \( \hat{W}_{z_1 \ldots z_n} \).

**Notation 2.10.** For a series \( f(z_1, \ldots, z_n) \in W[[z_1, z_1^{-1}, \ldots, z_n, z_n^{-1}]] \) that converges absolutely to a \( \overline{W} \)-valued rational function, we will use the notation \( \bar{E}(f(z_1, \ldots, z_n)) \) to denote the limit. For example, let \( V \) be a MOSVA and \( W \) be a bimodule. Then for \( v_1, v_2 \in V, w \in W \) and \( w' \in W' \), both

\[ \langle w', Y^L_W(v_1, z_1)Y^{s(R)}_W(v_2, z_2)w \rangle \]  \hspace{1cm} (1)

and

\[ \langle w', Y^{s(R)}_W(v_2, z_2)Y^L_W(v_1, z_1)w \rangle \]  \hspace{1cm} (2)
converges absolutely to the same rational function in \( z_1, z_2 \) with poles at \( z_1 = 0, z_2 = 0 \) and \( z_1 = z_2 \). Since the series (1) and (2) converge in disjoint regions, we cannot equate them. But we can say that

\[
E(Y^L_W(v_1, z_1)Y^s_W(v_2, z_2)w) = E(Y^s_W(v_2, z_2)Y^L_W(v_1, z_1)w)
\]
as elements in \( \hat{W}_{z_1 z_2} \).

**Notation 2.11.** The cohomology theory of MOSVA is built upon the linear maps \( \Phi : V^\otimes n \to \hat{W}_{z_1...z_n} \). We will use the notation

\[
\Phi(v_1 \otimes \cdots v_n; z_1, ..., z_n)
\]
to denote the image of \( v_1 \otimes \cdots \otimes v_n \) in \( \hat{W}_{z_1...z_n} \).

**Definition 2.12.** A linear map \( \Phi : V^\otimes n \to \hat{W}_{z_1,...,z_n} \) is said to have the D-derivative property if

(1) For \( i = 1, ..., n, v_1, ..., v_n \in V, w' \in W' \),

\[
\langle w', \Phi(v_1 \otimes \cdots \otimes v_i-1 \otimes D_V v_i \otimes v_{i+1} \otimes \cdots \otimes v_n; z_1, ..., z_n) \rangle = \frac{\partial}{\partial z_i} \langle w', \Phi(v_1 \otimes \cdots \otimes v_n; z_1, ..., z_n) \rangle
\]

(2) For \( v_1, ..., v_n \in V, w' \in W' \),

\[
\langle w', D_W(\Phi(v_1 \otimes \cdots \otimes v_n; z_1, ..., z_n)) \rangle = \left( \frac{\partial}{\partial z_1} + \cdots + \frac{\partial}{\partial z_n} \right) \langle w', \Phi(v_1 \otimes \cdots \otimes v_n; z_1, ..., z_n) \rangle
\]

**Definition 2.13.** A linear map \( \Phi : V^\otimes n \to \hat{W}_{z_1,...,z_n} \) is said to have the d-conjugation property if for \( v_1, ..., v_n \in V, w' \in W', (z_1, ..., z_n) \in F_n \mathbb{C} \) and \( z \in \mathbb{C}^\times \) so that \( (zz_1, ..., zz_n) \in F_n \mathbb{C} \),

\[
\langle w', z^{dw} \Phi(v_1 \otimes \cdots \otimes v_n; z_1, ..., z_n) \rangle = \langle w', \Phi(z^{dv} v_1 \otimes \cdots \otimes z^{dv} v_n; zz_1, ..., zz_n) \rangle
\]

**Definition 2.14.** Let \( \Phi : V^\otimes n \to \hat{W}_{z_1,...,z_n} \) be a linear map. Let \( m \in \mathbb{Z}_+ \). \( \Phi \) is said to be composable with \( m \) vertex operators if for every \( \alpha_0, \alpha_1, ..., \alpha_n \in \mathbb{Z}_+ \) such that \( \alpha_0 + \cdots + \alpha_n = m + n \), every \( l_0 = 0, ..., \alpha_0 \), and every \( v_1(0), ..., v_{\alpha_0}(0), v_1^{(1)}, ..., v_{\alpha_1}(1), ..., v_{\alpha_n}(n) \in V \), the series of \( \mathbb{W} \)-valued rational functions

\[
Y^L_W(v_1^{(0)}, z_1^{(0)}) \cdots Y^L_W(v_{\alpha_0}^{(0)}, z_{\alpha_0}^{(0)})Y^s_W(u_0^{(0)}, z_0^{(0)})Y^s_W(u_{l_0+1}^{(0)}, z_{l_0+1}^{(0)}) \cdots Y^s_W(u_{l_0}^{(0)}, z_{l_0}^{(0)})
\]

\[
\cdot \Phi(Y(v_1^{(1)}, z_1^{(1)} - \zeta_1) \cdots Y(v_{\alpha_1}^{(1)}, z_{\alpha_1}^{(1)} - \zeta_1) 1) \]

\[
\otimes \cdots \otimes Y(v_1^{(n)}, z_1^{(n)} - \zeta_n) \cdots Y(v_{\alpha_n}^{(n)}, z_{\alpha_n}^{(n)} - \zeta_n) 1; \zeta_1, ..., \zeta_n
\]
Let's start by analyzing the equation and its implications:

\[
\sum_{k_1^{(0)}, \ldots, k_{n+1}^{(0)}, k_1^{(n)}, \ldots, k_{n+1}^{(n)} \in \mathbb{Z}} \Phi \left( Y_{k_1^{(1)}}^{(1)}(Z_1) \cdots Y_{k_{n+1}^{(n)}}^{(n)}(u_{\alpha_1}^{(1)}) \right) \cdot Y_{k_1^{(2)}}^{(2)}(u_1^{(2)}) \cdots Y_{k_{n+1}^{(n)}}^{(n)}(u_{\alpha_1}^{(2)}) \cdot \cdots \cdot Y_{k_1^{(n)}}^{(n)}(u_1^{(n)}) \cdots Y_{k_{n+1}^{(n)}}^{(n)}(u_{\alpha_1}^{(n)})
\]

This equation appears to be a summation over various indices and functions, involving vertex operators and their compositions. The summation involves terms of the form \(Y_{k_1^{(i)}}^{(i)}(Z_i)\) and \(Y_{k_{n+1}^{(i)}}^{(i)}(u_{\alpha_i}^{(i)})\), where \(k_i^{(i)}\) are integers and \(Z_i\) and \(u_{\alpha_i}^{(i)}\) are variables or constants. The summation is over all possible combinations of these indices.

The goal is to understand how these terms converge absolutely (in certain regions) to \(V\)-valued rational functions depending only on \(z_1, \ldots, z_{n+1}\) with the only possible poles at \(z_i = z_j\) (for \(1 \leq i < j \leq n+1\)).

**Proposition-Definition 2.16.** Let \(m, n \in \mathbb{Z}_+\), \(\Phi \in \hat{C}_\infty(V, W)\). Then for every \(i = 1, \ldots, n\), every \(v_1, \ldots, v_{n+1} \in V\), the following series

\[
\Phi(v_1 \otimes \cdots \otimes Y(v_i, z_i - \zeta)Y(v_{i+1}, z_{i+1} - \zeta)1 \otimes v_{i+2} \otimes \cdots \otimes v_{n+1}; z_1, \ldots, z_{i-1}, \zeta, z_{i+2}, \ldots, z_{n+1})
\]

converge absolutely (in certain regions) to \(\hat{V}\)-valued rational functions depending only on \(z_1, \ldots, z_{n+1}\) with the only possible poles at \(z_i = z_j\) (for \(1 \leq i < j \leq n+1\)). We will then define the maps

\[
\Phi \otimes E_{V}^{(2)} : V \otimes (n+1) \rightarrow \hat{W}_{z_1, \ldots, z_{n+1}}
\]
\[ E_W^{(1,0)} \circ_2 \Phi : V^{\otimes (n+1)} \to \tilde{W}_{z_1, \ldots, z_{n+1}} \]
\[ E_W^{(0,1)} \circ_2 \Phi : V^{\otimes (n+1)} \to \tilde{W}_{z_1, \ldots, z_{n+1}} \]
sending \( v_1 \otimes \cdots \otimes v_{n+1} \) to the respective \( \tilde{W} \)-valued rational functions given by the limits of the series \((3), (4), (5)\).

**Definition 2.17.** For \( n \in \mathbb{Z}_+ \), we define the coboundary operator as follows

\[ \hat{\delta}^n : \hat{C}_n^\infty(V,W) \to \hat{C}_n^{n+1}(V,W) \]

by

\[ \hat{\delta}^n \Phi = E_W^{(1,0)} \circ_2 \Phi + \sum_{i=1}^n (-1)^i \Phi \circ E_V^{(2)} + (-1)^{n+1} E_W^{(0,1)} \circ_2 \Phi \]

For \( n = 0 \), we define \( \hat{\delta}^0 : \hat{C}_0^\infty(V,W) \to \hat{C}_1^\infty(V,W) \) by the following: for \( w \in \hat{C}_0^\infty(V,W) \)

\[ [(\hat{\delta}^0(w))(v)](z) = E(Y_W^L(v, z)w - Y_W^R(v, z)w) \]

**Theorem 2.18.** For every \( n \in \mathbb{Z}_+ \),

\[ \hat{\delta}^0(\hat{C}_n^\infty(V,W)) \subseteq \hat{C}_n^1(V,W), \]
\[ \hat{\delta}^n(\hat{C}_n^\infty(V,W)) \subseteq \hat{C}_n^{n+1}(V,W). \]

Moreover, for every \( n \in \mathbb{Z}_+ \), we have \( \hat{\delta}^1 \circ \hat{\delta}^0 = 0 \), and \( \hat{\delta}^n_0 \circ \hat{\delta}^n_\infty = 0 \). The sequence

\[ \hat{C}_0^\infty(V,W) \xrightarrow{\hat{\delta}^0} \hat{C}_1^\infty(V,W) \xrightarrow{\hat{\delta}^1} \hat{C}_2^\infty(V,W) \xrightarrow{\hat{\delta}^2} \hat{C}_3^\infty(V,W) \to \cdots \]

forms a cochain complex.

**Definition 2.19.** For every \( n \in \mathbb{N} \), the \( n \)-th cohomology group is defined as

\[ \tilde{H}_n^\infty(V,W) = \text{Ker} \hat{\delta}_\infty^n / \text{Im} \hat{\delta}_\infty^{n-1}. \]

Elements in \( \text{Ker} \hat{\delta}_\infty^n \) are called \( n \)-cocycles. Elements in \( \text{Im} \hat{\delta}_\infty^{n-1} \) are called \( n \)-coboundaries.

**Example 2.20.** We describe the first cohomology explicitly.

1. Let \( \Phi : V \to \tilde{W}_z \) be a 1-cocycle. Then the map \( V \to W \) given by \( v \mapsto \Phi(v; 0) \) is a grading-preserving derivation, i.e.,

\[ \Phi(Y(u, x)v; 0) = Y_W^L(u, x)\Phi(v; 0) + e^{x Dw} Y_W^R(v, -x)\Phi(v; 0). \]

Conversely, let \( f : V \to W \) be a grading-preserving derivation, then the map \( v \mapsto e^{x Dw} f(v) \) gives a 1-cocycle.

2. Let \( \Phi : V \to \tilde{W}_z \) be a 1-coboundary. Then the map \( V \to W \) given by \( v \mapsto \Phi(v; 0) \) is a grading-preserving inner-derivation, i.e., there exists a vacuum-like vector \( w \in W \), such that

\[ \Phi(v; 0) = \lim_{x \to 0} Y_W^L(v, x)w - \lim_{x \to 0} Y_W^R(v, x)w. \]

Conversely, let \( f : V \to W \) be a grading-preserving inner derivation, then the map \( v \mapsto e^{x Dw} f(v) \) gives a 1-coboundary.
(3) Thus \( \hat{H}^l_\infty(V, W) = \{\text{grading-preserving derivations}\}/\{\text{grading-preserving inner derivations}\} \).

2.3. The bimodule \( \mathcal{H}_N(W_1, W_2) \). We briefly review the definition of the \( V \)-bimodule \( \mathcal{H}_N(W_1, W_2) \) associated with two left \( V \)-modules \( W_1 \) and \( W_2 \). Please find further details in \( \text{[QM]} \) and \( \text{[HQ]} \).

**Proposition-Definition 2.21.** Let \( V \) be a MOSVA. Let \( W_1 \) and \( W_2 \) be two left \( V \)-modules. Let \( \widehat{W_2}_\zeta \) be the space of \( \overline{W_2} \)-valued rational functions with the only possible pole at \( \zeta = 0 \).

Recall that \( \widehat{W_2}_\zeta \) is the space of \( \overline{W_2} \)-valued holomorphic functions. Thus \( \widehat{W_2}_\zeta \supset \widehat{W_2}_\cdot \).

Let \( \mathcal{H}_N(W_1, W_2) \) be the subspace of \( \text{Hom}_\mathbb{C}(W_1, \widehat{W_2}_\zeta) \) spanned by elements, denoted by

\[
\phi : w_1 \mapsto \phi(\zeta)w_1 \in \widehat{W_2}_\zeta
\]

satisfying the following conditions:

1. The \( D \)-derivative property: For \( w_1 \in W_1 \),

\[
\frac{d}{d\zeta}(\phi(\zeta)w_1) = D_{W_2}(\phi(\zeta)w_1) - \phi(\zeta)D_{W_1}w_1,
\]

where \( D_{W_2} \) is the natural extension of \( D_{W_2} \) on \( W_2 \) to \( \overline{W_2} \).

2. The \( d \)-conjugation property: There exists \( n \in \mathbb{Z} \) (called the weight of \( \phi \) and denoted by \( \text{wt}\phi \)) such that for \( a \in \mathbb{C}^\times \) and \( w_1 \in W_1 \),

\[
\phi^{d_{W_2}}(\phi(\zeta)w_1) = a^n(\phi(a\zeta))a^{d_{W_1}}w_1.
\]

3. The composability condition: For \( k, l \in \mathbb{N} \) and \( v_1, \ldots, v_{k+l} \in V \), \( w_1 \in W_1 \) and \( w'_2 \in W'_2 \), the series

\[
\langle w'_2, Y_{W_2}(v_1, z_1) \cdot \cdot \cdot Y_{W_2}(v_k, z_k)\phi(\zeta)Y_{W_2}(v_{k+1}, z_{k+1}) \cdot \cdot \cdot Y_{W_2}(v_{k+l}, z_{k+l})w_1 \rangle
\]

is absolutely convergent in the region \( |z_1| > \cdots > |z_k| > |\zeta| > |z_{k+1}| > \cdots > |z_{k+l}| > 0 \) to a rational function

\[
f(z_1, \ldots, z_k, \zeta, z_{k+1}, \ldots, z_{k+l})
\]

in \( z_1, \ldots, z_{k+l} \) and \( \zeta \) with the only possible poles \( z_i = 0 \) for \( i = 1, \ldots, k+l \), \( \zeta = 0 \), \( z_i = z_j \) for \( i, j = 1, \ldots, k+l, i \neq j \) and \( z_i = \zeta \) for \( i = 1, \ldots, k+l \). Moreover, there exist \( r_i \in \mathbb{N} \) depending only on the pair \( (v_i, w_1) \) for \( i = 1, \ldots, k+l \), \( m \in \mathbb{N} \) depending only on the pair \( (\phi, w_1) \), \( p_{ij} \in \mathbb{N} \) depending only on the pair \( (v_i, v_j) \) for \( i, j = 1, \ldots, k+l, i \neq j \), \( s_i \in \mathbb{N} \) depending only on the pair \( (v_i, \phi) \) for \( i = 1, \ldots, k+l \) and \( g(z_1, \ldots, z_{k+l}, \zeta) \in W_2[[z_1, \ldots, z_{k+l}, \zeta]] \) such that for \( w'_2 \in W'_2 \),

\[
\zeta^m \prod_{i=1}^{k+l} z_i^{r_i} \prod_{1 \leq i < j \leq k+l} (z_i - z_j)^{p_{ij}} \prod_{i=1}^{k+l} (z_i - \zeta)^{s_i} \cdot f(z_1, \ldots, z_k, \zeta, z_{k+1}, \ldots, z_{k+l})
\]

is a polynomial and is equal to \( \langle w'_2, g(z_1, \ldots, z_{k+l}, \zeta) \rangle \).

4. The \( N \)-weight-degree condition: Expand \( f(z_1, \ldots, z_k, \zeta, z_{k+1}, \ldots, z_{k+l}) \) in the region \( |z_{k+l}| > |z_{k+l-1} - z_{k+l}| > \cdots > |z_k - z_{k+l}| > |z_{k+1} - z_{k+l}| > \cdots > |z_1 - z_{k+l}| > |z_1 - z_{k+l}| > |z_{k+l} - z_{k+l}| > 0 \) as a Laurent series in \( z_i - z_{k+l} \) for \( i = 1, \ldots, k+l - 1 \) and \( \zeta - z_{k+l} \) with Laurent polynomials.
in $z_{k+l}$ as coefficients. Then the total degree of each monomial in $z_i - z_{k+l}$ for $i = 1, \ldots, k+l-1$ and $\zeta - z_{k+l}$ (that is, the sum of the powers of $z_i - z_{k+l}$ for $i = 1, \ldots, k+l-1$ and $\zeta - z_{k+l}$) in the expansion is greater than or equal to $N - \sum_{i=1}^{k+l} wtv_i - w\phi$.

Let $\mathcal{H}_N(W_1, W_2)_{[n]}$ be the subspace of $\mathcal{H}_N(W_1, W_2)$ consisting of elements of weight $n$. Then

$$\mathcal{H}_N(W_1, W_2) = \bigoplus_{n \in \mathbb{Z}} \mathcal{H}_N(W_1, W_2)_{[n]}.$$ We define a $V$-bimodule structure using the left and right vertex operator maps:

$$Y^L_\mathcal{H} : V \otimes \mathcal{H}_N(W_1, W_2) \rightarrow \mathcal{H}_N(W_1, W_2)[[x, x^{-1}]]$$

\[v \otimes \phi \mapsto Y^L_\mathcal{H}(v, x)\phi \]

$$Y^{s(R)}_\mathcal{H} : V \otimes \mathcal{H}_N(W_1, W_2) \rightarrow \mathcal{H}_N(W_1, W_2)[[x, x^{-1}]]$$

\[v \otimes \phi \mapsto Y^{s(R)}_\mathcal{H}(v, x)\phi \]

defined by

\[
\begin{align*}
(Y^L_\mathcal{H}(v, x)\phi)(\zeta)w_1 &= \iota_{\zeta^x}E(Y_{W_2}(v, x + \zeta)\phi(\zeta)w_1), \\
(Y^{s(R)}_\mathcal{H}(v, x)\phi)(\zeta)w_1 &= \iota_{\zeta^x}E(\phi(\zeta)Y_{W_1}(v, x + \zeta)w_1).
\end{align*}
\]

(8) and (9)

Define also the operators $d_\mathcal{H}$ and $D_\mathcal{H}$ on $\mathcal{H}_N(W_1, W_2)$

$$d_\mathcal{H}\phi = n\phi \text{ for } \phi \in \mathcal{H}_N(W_1, W_2)_{[n]}.$$ 

$$(D_\mathcal{H}\phi)(\zeta)w_1 = \frac{\partial}{\partial \zeta}(\phi(\zeta)w_1)$$

for $\phi \in \mathcal{H}_N(W_1, W_2)$, $w_1 \in W_1$. Then $(\mathcal{H}_N(W_1, W_2), Y^L_\mathcal{H}, Y^{s(R)}_\mathcal{H}, d_\mathcal{H}, D_\mathcal{H})$ forms a $V$-bimodule with lowest $d_\mathcal{H}$-weight being $N$.

**Remark 2.22.** The convergence conditions mentioned in the abstract are precisely Condition (3) and (4) in Proposition-Definition 2.21. See also Definition 4.1 and Remark 4.3.

### 3. Extensions and the first cohomology

#### 3.1. Extensions of left modules

**Definition 3.1.** Let $W_1, W_2$ be two left $V$-modules. Let $U$ be a left $V$-module that fits in the exact sequence

$$0 \rightarrow W_2 \rightarrow U \xrightarrow{p} W_1 \rightarrow 0.$$ We call $U$ the extension of $W_1$ by $W_2$. We will say that $W_2$ is a submodule of $U$ without distinguishing $W_2$ and its image in $U$. Two extensions $U^{(1)}, U^{(2)}$ are equivalent, if there exists a homomorphism $T : U^{(1)} \rightarrow U^{(2)}$, such that the following diagram commutes

\[
\begin{array}{ccc}
0 & \rightarrow & W_2 & \rightarrow & U^{(1)} & \xrightarrow{p^{(1)}} & W_1 & \rightarrow & 0 \\
& & & & \downarrow{T} & & \downarrow{T} & & \\
0 & \rightarrow & W_2 & \rightarrow & U^{(2)} & \xrightarrow{p^{(2)}} & W_1 & \rightarrow & 0 \\
\end{array}
\]

(10)
Definition 3.2. Let $\text{Ext}^N_N(W_1, W_2)$ be the set of equivalence classes of extensions $U$ satisfying the following additional property: there exists a section $\psi : W_1 \rightarrow U$, such that $U = W_2 \Pi \psi(W_1)$ as vector spaces, and for every $v \in V$, the map

$$\pi_2 Y_U(v, \zeta) \psi \in \mathcal{H}_N(W_1, W_2).$$

where $\pi_2 : W_2 \Pi \psi(W_1) \rightarrow W_2$ is the projection operator.

Remark 3.3. We will keep the $\oplus$ notation for direct sums of $V$-modules. The notation $W_1 \Pi W_2$ means we have vector space direct sum of $W_1$ and $W_2$ that does not respect the $V$-actions.

Proposition 3.4. Let $U, \psi, \pi_2$ be as in Definition 3.2. Let $\pi_1 = 1 - \pi_2 : U \rightarrow \psi(W_1)$. Then $(\psi(W_1), \pi_1 Y_U, d_U|_{\psi(W_1)}, \pi_1 D_U|_{\psi(W_1)})$ forms a left $V$-module. Moreover, the map $\psi : (W_1, Y_{W_1}, d_{W_1}, D_{W_1}) \rightarrow (\psi(W_1), \pi_1 Y_U, d_U|_{\psi(W_1)}, \pi_1 D_U|_{\psi(W_1)})$ is a left $V$-module isomorphism.

Proof. For the first part, we only verify the $D$-commutator formula and weak associativity. Other axioms follow trivially.

From the $D$-commutator formula, we have

$$\frac{d}{dx} Y_U(v, x) \psi(w_1) = D_U Y_U(v, x) \psi(w_1) - Y_U(v, x) D_U \psi(w_1)$$

Apply $\pi_1$ on both sides of (11) and using the fact that $\pi_1 + \pi_2 = 1$, we have

$$\frac{d}{dx} \pi_1 Y_U(v, x) \psi(w_1) = \pi_1 D_U (\pi_1 + \pi_2) Y_U(v, x) \psi(w_1) - \pi_1 Y_U(v, x) \pi_1 D_U \psi(w_1)$$

$$= \pi_1 D_U \pi_1 Y_U(v, x) \psi(w_1) - \pi_1 Y_U(v, x) \pi_1 D_U \psi(w_1)$$

$$+ \pi_1 D_U \pi_2 Y_U(v, x) \psi(w_1) - \pi_1 Y_U(v, x) \pi_2 D_U \psi(w_1)$$

Since $W_2$ is a submodule, $Y_U(v, x) W_2 \subset W_2[[x, x^{-1}]]$, $D_U W_2 \subset W_2$. Thus $\pi_1 Y_U(v, x) W_2 = 0, \pi_1 D_U W_2 = 0$. So (13) is zero. Thus we verified that

$$\frac{d}{dx} \pi_1 Y_U(v, x) \psi(w_1) = \pi_1 D_U \pi_1 Y_U(v, x) \psi(w_1) - \pi_1 Y_U(v, x) \pi_1 D_U \psi(w_1)$$

So $\pi_1 D_U$ satisfies the $D$-commutator formula.

From the weak associativity on $U$, for every $u \in U, w_1 \in W_1$, we have $p \in \mathbb{N}$, such that for every $v \in V$,

$$(x_0 + x_2)^p Y_U(Y(u, x_0) v, x_2) \psi(w_1) = (x_0 + x_2)^p Y_U(u, x_0 + x_2) Y_U(v, x_2) \psi(w_1)$$

Apply $\pi_1$ on both sides of (14). Then

$$(x_0 + x_2)^p \pi_1 Y_U(Y(u, x_0) v, x_2) \psi(w_1) = (x_0 + x_2)^p \pi_1 Y_U(u, x_0 + x_2) (\pi_1 + \pi_2) Y_U(v, x_2) \psi(w_1)$$

$$= (x_0 + x_2)^p \pi_1 Y_U(u, x_0 + x_2) \pi_1 Y_U(v, x_2) \psi(w_1)$$

$$+ (x_0 + x_2)^p \pi_1 Y_U(u, x_0 + x_2) \pi_2 Y_U(v, x_2) \psi(w_1)$$

However, since $W_2$ is a submodule, $\pi_1 Y_U(u, x) w_2 = 0$ for every $w_2 \in W_2$. Thus (16) is zero. Thus we see that

$$(x_0 + x_2)^p \pi_1 Y_U(Y(u, x_0) v, x_2) \psi(w_1) = (x_0 + x_2)^p \pi_1 Y_U(u, x_0 + x_2) \pi_2 Y_U(v, x_2) \psi(w_1).$$
So we have shown the weak associativity.

Now we show that \( \psi \) is an isomorphism. Let \( p : U \to W_1 \) be the homomorphism in the exact sequence. Then

\[
pY_U(v, x)\psi(w_1) = Y_{W_1}(v, x)p\psi(w_1) = Y_{W_1}(v, x)w_1
\]

Note that the left-hand-side can also be written as

\[
p(\pi_1 + \pi_2)Y_U(v, x)\psi(w_1) = p\pi_1Y_U(v, x)\psi(w_1)
\]

because \( p \) restricted to \( W_2 \) is zero. Thus we see that \( p\pi_1Y_U(v, x)\psi(w_1) = Y_{W_1}(v, x)w_1 \). Similarly we see that \( p\pi_1D_U = D_{W_1} \). Thus \( p|\psi(W_1) \) is an isomorphism from \( \psi(W_1) \) to \( W_1 \). Obviously \( \psi \) is the inverse of \( p|\psi(W_1) \). Thus we see that \( \psi \) is an isomorphism. \( \square \)

3.2. Derivation from an extension in \( \text{Ext}^1_N(W_1, W_2) \).

**Proposition 3.5.** Let \( U \) be an extension of \( W_1 \) by \( W_2 \), \( \psi : W_1 \to U \) be the section of \( p : U \to W_1 \), \( \pi_2 : U \to W_2 \) be the projection operator as in Definition 3.2. Then the map

\[
F : V \to \mathcal{H}_N(W_1, W_2)
\]

\[
v \mapsto \pi_2Y_U(v, \zeta)\psi
\]

is a derivation.

**Proof.** From the weak associativity on \( U \), for every \( u \in U, w_1 \in W_1 \), we have \( q \in \mathbb{N} \), such that for every \( v \in V \),

\[
(x_0 + x_2)^qY_U(Y(u, x_0)v, x_2)\psi(w_1) = (x_0 + x_2)^qY_U(u, x_0 + x_2)Y_U(v, x_2)\psi(w_1)
\]

Apply \( \pi_2 \) on both sides.

\[
(x_0 + x_2)^q\pi_2Y_U(Y(u, x_0)v, x_2)\psi(w_1) = (x_0 + x_2)^q\pi_2Y_U(u, x_0 + x_2)Y_U(v, x_2)\psi(w_1)
\]

\[
= (x_0 + x_2)^q\pi_2Y_U(u, x_0 + x_2)\pi_2Y_U(v, x_2)\psi(w_1) + (x_0 + x_2)^q\pi_2Y_U(u, x_0 + x_2)(1 - \pi_2)Y_U(v, x_2)\psi(w_1)
\]

(17)

(18)

We have seen Proposition 3.4 that for \( \pi_1 = 1 - \pi_2, (\psi(W_1), \pi_1Y_U, d_U|\psi(W_1), \pi_1D_U|\psi(W_1) \) is a \( V \)-module isomorphic to \( W_1 \). Thus (18) can be simplified as

\[
(x_0 + x_2)^q\pi_2Y_U(u, x_0 + x_2)Y_U(v, x_2)\psi(w_1) = (x_0 + x_2)^q\pi_2Y_U(u, x_0 + x_2)\psi(Y_{W_1}(v, x_2)w_1).
\]

Notice also that \( \pi_2Y_U(v, x)\pi_2 = Y_U(v, x)\pi_2 \) since \( W_2 \) is a submodule, thus we can remove the first \( \pi_2 \) in (17). Combining our knowledge for (17) and (18), we see that

\[
(x_0 + x_2)^q\pi_2Y_U(Y(u, x_0)v, x_2)\psi(w_1) = (x_0 + x_2)^q\pi_2Y_U(u, x_0 + x_2)\pi_2Y_U(v, x_2)\psi(w_1)
\]

\[
+ (x_0 + x_2)^q\pi_2Y_U(u, x_0 + x_2)\psi(Y_{W_1}(v, x_2)w_1)
\]

(19)
Let \( f(x_0, x_2) \) to be the lower-truncated series in \( W_2((x_0, x_2)) \) representing both sides of (19), then
\[
\pi_2 Y_U(Y(u, x_0)v, x_2)\psi(w_1) = \iota_{x_2x_0} \frac{f(x_0, x_2)}{(x_2 + x_0)^q} \\
= \iota_{x_2x_0} E(Y_U(u, x_0 + x_2)\pi_2 Y_U(v, x_2)\psi(w_1)) \\
+ \iota_{x_2x_0} E(\pi_2 Y_U(u, x_0 + x_2)\psi(Y_W(v, x_2)w_1))
\]
Now we substitute \( x_2 = \zeta \). With the knowledge of the definition of \( Y_{\mathcal{H}}^L \) and \( Y_{\mathcal{H}}^{s(R)} \), we immediately see that
\[
\pi_2 Y_U(Y(u, x_0)v, \zeta)\psi(w_1) = (Y_{\mathcal{H}}^L(u, x_0)\pi_2 Y_U(v, \zeta)\psi) (w_1) + \left( e^{x_0D_H} Y_{\mathcal{H}}^{s(R)}(v, -x_0)\pi_2 Y_U(u, \zeta)\psi \right) (w_1)
\]
Thus the map \( v \mapsto \pi_2 Y_U(v, \zeta)\psi \) is a derivation from \( V \) to \( \mathcal{H}_N(W_1, W_2) \). \( \square \)

**Proposition 3.6.** Let \( U^{(1)}, U^{(2)} \) be two equivalent extensions with \( T : U^{(1)} \to U^{(2)} \) be the \( V \)-module isomorphism fitting in the diagram (10). Let \( \psi^{(i)} : W_1 \to U^{(i)} \) be the section and \( \pi_2^{(i)} : U^{(i)} \to W_2 \) be projection as in Definition 3.2 (\( i = 1, 2 \)). Then
\[
\pi_2^{(1)} Y_U(v, \zeta)\psi^{(1)} - \pi_2^{(2)} Y_U(v, \zeta)\psi^{(2)}
\]
is an inner derivation corresponding to the element \( \phi(\zeta) \in \mathcal{H}_N(W_1, W_2) \) given by
\[
\phi(\zeta) w_1 = \pi_2^{(2)} T\psi^{(1)} w_1.
\]
Here \( \phi(\zeta) \) is a constant \( W_2 \)-valued rational function.

**Proof.** For \( i = 1, 2 \), let \( p^{(i)} : U^{(i)} \to W_1 \) be the map as in (10); write \( U^{(i)} = W_2 \ll \psi^{(i)}(W_1) \) where \( \psi^{(i)} \) is the section of \( p^{(i)} \) as in Definition 3.2; let \( \pi_2^{(i)} \) be the projection as in Definition 3.2; \( \pi_1^{(i)} = 1 - \pi_2^{(i)} \). Since (10) is commutative, we have
\[
T(w_2, 0) = (w_2, 0), p^{(1)}(w_2, \psi^{(1)}(w_1)) = p^{(2)} T(w_2, \psi^{(1)}(w_1)),
\]
which tells us that
\[
Tw_2 = w_2,
\]
and
\[
w_1 = p^{(2)} T\psi^{(1)} w_1.
\]
If we write \( T\psi^{(1)} w_1 = \pi_2^{(2)} T\psi^{(1)} w_1 + \pi_1^{(2)} T\psi^{(1)} w_1 \), then since \( p^{(2)}(W_2) = 0 \), we have
\[
w_1 = p^{(2)} \pi_1^{(2)} T\psi^{(1)} w_1.
\]
From Proposition 3.4 we see that \( \psi^{(2)} \) is the inverse of \( p^{(2)} \). Thus we have
\[
\psi^{(2)} w_1 = \pi_1^{(2)} T\psi^{(1)} w_1.
\]
In summary, we have
\[
T(w_2, \psi^{(1)} w_1) = (w_2 + \pi_2^{(2)} T\psi^{(1)} w_1, \psi^{(2)} w_1).
\]
Since $T$ is an isomorphism, we know that
\[ TY_U^{(1)}(v, x) = Y_U^{(2)}(v, x)T. \] (21)

Apply the left-hand-side of (21) to $(w_2, \psi^{(1)}w_1)$, we have
\[ TY_U^{(1)}(v, x)(w_2, \psi^{(1)}w_1) = T(Y_U^{(1)}(v, x)w_2 + \pi_2^{(1)}Y_U^{(1)}(v, x)\psi^{(1)}w_1, \pi_1^{(1)}Y_U^{(1)}(v, x)\psi^{(1)}w_1) \]
\[ = T(Y_U^{(1)}(v, x)w_2 + \pi_2^{(1)}Y_U^{(1)}(v, x)\psi^{(1)}w_1, \psi^{(1)}(Y_{W_1}(v, x)w_1) \]
\[ = (Y_U^{(1)}(v, x)w_2 + \pi_2^{(1)}Y_U^{(1)}(v, x)\psi^{(1)}w_1 + \pi_2^{(2)}T\psi^{(1)}Y_{W_1}(v, x)w_1, \psi^{(2)}Y_{W_1}(v, x)w_1) \] (22)

Here (22) follows from $\psi^{(1)}$ being a homomorphism (cf. Proposition 3.3); (23) follows from (20).

Apply the right-hand-side of (21) to $(w_2, \psi^{(1)}w_1)$, we have
\[ Y_U^{(2)}(v, x)T(w_2, \psi^{(1)}w_1) \]
\[ = Y_U^{(2)}(v, x)(w_2 + \pi_2^{(2)}T\psi^{(1)}w_1, \psi^{(2)}w_1) \]
\[ = (Y_U^{(2)}(v, x)w_2 + \pi_2^{(2)}Y_U^{(2)}(v, x)\psi^{(2)}w_1 + \pi_2^{(2)}Y_U^{(2)}(v, x)\psi^{(2)}w_1) \]
\[ = (Y_U^{(2)}(v, x)w_2 + Y_U^{(2)}(v, x)\pi_2^{(2)}T\psi^{(1)}w_1 + \pi_2^{(2)}Y_U^{(2)}(v, x)\psi^{(2)}w_1, \psi^{(2)}Y_{W_1}(v, x)w_1) \] (24)

Here (21) follows from (20); (25) follows from $\psi^{(2)}$ being a homomorphism (cf. Proposition 3.3). Notice also that
\[ Y_U^{(1)}(v, x)w_2 = Y_U^{(2)}(v, x)w_2 = Y_{W_2}(v, x)w_2, \]
thus the equality of (23) and (25), after substituting $x = \zeta$, implies that
\[ \pi_2^{(1)}Y_U^{(1)}(v, \zeta)\psi^{(1)}w_1 - \pi_2^{(2)}Y_U^{(2)}(v, \zeta)\psi^{(2)}w_1 = \pi_2^{(2)}T\psi^{(1)}Y_{W_1}(v, \zeta)w_1 - Y_{W_2}(v, \zeta)\pi_2^{(2)}T\psi^{(1)}w_1. \] (26)

If we now set $\phi(\zeta) = \pi_2^{(2)}T\psi^{(1)}$ as an element in $\mathcal{H}_N(W_1, W_2)$ that is constant in $\zeta$, then
\[ (Y^L_{\mathcal{H}}(v, x)\phi)(\zeta)w_1 = i_\xi E(Y_{W_1}(v, x + \zeta)\phi(\zeta))w_1 = Y_{W_2}(v, \zeta + x)\pi_2^{(2)}T\psi^{(1)}w_1 \]
\[ (Y^{s(R)}_{\mathcal{H}}(v, x)\phi)(\zeta)w_1 = i_\xi E(\phi(\zeta)Y_{W_1}(v, x + \zeta))w_1 = \pi_2^{(2)}T\psi^{(1)}Y_{W_2}(v, \zeta + x)w_1 \]
Clearly, $D_H\phi(\zeta) = 0$. This means that $\phi$ is a vacuum-like element in $\mathcal{H}_N(W_1, W_2)$. Thus we see that right-hand-side of (26) is precisely
\[ \lim_{x \to 0} \left( (Y^L_{\mathcal{H}}(v, x)\phi)(\zeta) - (Y^{s(R)}_{\mathcal{H}}(v, x)\phi)(\zeta) \right) \]
This shows that the left-hand-side of (26) is an inner derivation associated to $\phi(\zeta) = \pi_2^{(2)}T\psi^{(1)}$.

\[ \square \]

**Remark 3.7.** In particular, Proposition 3.6 shows that the cohomology class obtained Proposition 3.5 is independent of the choice of the section $\psi^{(1)}$.

**Corollary 3.8.** The map $F : \text{Ext}^1_N(W_1, W_2) \to \hat{H}^1(V, \mathcal{H}_N(W_1, W_2))$ sending $U$ to the derivation $v \mapsto \pi_2Y_U(v, \zeta)\psi$ is well-defined. Here $U, \pi_2$ and $\psi$ are as in Definition 3.2.
3.3. 1-cocycles in the bimodule $\mathcal{H}_N(W_1, W_2)$. To understand the converse map, we need some convergence results regarding 1-cocycles. We will start from the following lemmas concerning complex series.

**Lemma 3.9.**  
1. Let $n$ be a positive integer. Let $f$ be a rational function in $z_1, \ldots, z_n$. Let $T$ be a connected multicircular domain on which the lowest power of $z_n$ in the Laurent series expansion of $f(z_1, \ldots, z_n)$ is the same as the negative of the order of pole $z_n = 0$. Let $S$ be a nonempty open subset of $T$ and $S'$ be the image of $S$ via the projection $(z_1, \ldots, z_n) \mapsto (z_1, \ldots, z_{n-1})$. Assume that for each fixed $k_n \in \mathbb{Z}$, the series

$$
\sum_{k_1, k_2, \ldots, k_{n-1} \in \mathbb{Z}} a_{k_1 k_2 \ldots k_{n-1}} z_1^{k_1} z_2^{k_2} \cdots z_{n-1}^{k_{n-1}}
$$

converges absolutely for every $(z_1, z_2, \ldots, z_{n-1}) \in S'$, and

$$
\sum_{k_n \in \mathbb{Z}} \left( \sum_{k_1, k_2, \ldots, k_{n-1} \in \mathbb{Z}} a_{k_1 k_2 \ldots k_{n-1}} z_1^{k_1} z_2^{k_2} \cdots z_{n-1}^{k_{n-1}} \right) z_n^{k_n}, \tag{27}
$$

viewed as a series whose terms are

$$
\sum_{k_1, k_2, \ldots, k_{n-1} \in \mathbb{Z}} a_{k_1 k_2 \ldots k_{n-1}} z_1^{k_1} z_2^{k_2} \cdots z_{n-1}^{k_{n-1}} \cdot z_n^{k_n},
$$

is lower-truncated in $z_n$ and converges to $f(z_1, \ldots, z_n)$ for every $(z_1, z_2, \ldots, z_{n-1}, z_n) \in S$. Then the corresponding Laurent series

$$
\sum_{k_1, k_2, \ldots, k_{n-1}, k_n \in \mathbb{Z}} a_{k_1 k_2 \ldots k_n} z_1^{k_1} z_2^{k_2} \cdots z_{n-1}^{k_{n-1}} z_n^{k_n}, \tag{28}
$$

converges absolutely to $f(z_1, \ldots, z_n)$ for every $(z_1, \ldots, z_n) \in T$

2. Let $n$ be a positive integer. Let $f$ be a rational function in $z_1, \ldots, z_n$. Let $T$ be a connected multicircular domain on which the highest power of $z_n$ in the Laurent series expansion of $f(z_1, \ldots, z_n)$ is the same as the negative of the order of pole $z_n = \infty$. Let $S$ be a nonempty open subset of $T$ and $S'$ be the image of $S$ via the projection $(z_1, \ldots, z_n) \mapsto (z_1, \ldots, z_{n-1})$. Assume that for each fixed $k_n \in \mathbb{Z}$, the series

$$
\sum_{k_1, k_2, \ldots, k_{n-1} \in \mathbb{Z}} a_{k_1 k_2 \ldots k_{n-1}} z_1^{k_1} z_2^{k_2} \cdots z_{n-1}^{k_{n-1}}
$$

converges absolutely for every $(z_1, z_2, \ldots, z_{n-1}) \in S'$, and

$$
\sum_{k_n \in \mathbb{Z}} \left( \sum_{k_1, k_2, \ldots, k_{n-1} \in \mathbb{Z}} a_{k_1 k_2 \ldots k_{n-1}} z_1^{k_1} z_2^{k_2} \cdots z_{n-1}^{k_{n-1}} \right) z_n^{k_n},
$$

viewed as a series whose terms are

$$
\sum_{k_1, k_2, \ldots, k_{n-1} \in \mathbb{Z}} a_{k_1 k_2 \ldots k_{n-1}} z_1^{k_1} z_2^{k_2} \cdots z_{n-1}^{k_{n-1}} \cdot z_n^{k_n},
$$

is upper-truncated in $z_n$ and converges to $f(z_1, \ldots, z_n)$ for every $(z_1, z_2, \ldots, z_{n-1}, z_n) \in S$. Then the corresponding Laurent series

$$
\sum_{k_1, k_2, \ldots, k_{n-1}, k_n \in \mathbb{Z}} a_{k_1 k_2 \ldots k_n} z_1^{k_1} z_2^{k_2} \cdots z_{n-1}^{k_{n-1}} z_n^{k_n},
$$

is
converges absolutely to \( f(z_1, ..., z_n) \) for every \((z_1, ..., z_n) \in T\).

**Proof.** An exposition can be found in [Q1], Lemma 4.5 and Lemma 4.7. \(\square\)

**Proposition 3.10.** Let \( \Phi : V \to \overline{\mathcal{H}_N(W_1, W_2)} \) be a 1-cocycle. For \( v \in V \), let \( \phi(v, \zeta) = (\Phi(v; 0))(\zeta) \in \mathcal{H}_N(W_1, W_2) \). Then for every \( v_1 \in V, w_1 \in W_1 \), the series

\[
(Y^L_H(v_1, z_1)\Phi(v_2; z_2))(\zeta)w_1 \quad \text{and} \quad (Y^R_H(v_2, z_2)\Phi(v_1; z_1))(\zeta)w_1
\]

converge absolutely when \( |\zeta| > |z_1| > |z_2| \) respectively to the \( \overline{W}_2 \)-valued rational functions

\[
E(Y_{W_2}(v_1, \zeta + z_1)\phi(v_2, \zeta + z_2)w_1)
\]

\[
E(\phi(v_2, \zeta + z_2)Y_{W_1}(v_1, \zeta + z_1)w_1)
\]

\[
E(Y_{W_2}(v_1, \zeta + z_1)\phi(v_2, \zeta + z_2)w_1) + E(\phi(v_2, \zeta + z_2)Y_{W_1}(v_1, \zeta + z_1)w_1)
\]

**Proof.** We start with the following observation: Since \( \Phi \) satisfies the \( D_H \)-derivative properties, we have

\[
\Phi(v; z) = e^{zD_H} \Phi(v; 0)
\]

for every \( v \in V \) and every complex number \( z \). Acting both sides on \( w_1 \in W \) with the parameter \( \zeta \), we see that

\[
(\Phi(v; z))(\zeta)w_1 = e^{zD_H} (\Phi(v; 0))(\zeta)w_1 = e^{zD_H} \phi(v, \zeta)w_1 = \sum_{n=0}^{\infty} \left( \phi^{(n)}(v, \zeta)w_1 \right) \frac{z^n}{n!},
\]

where \( \phi^{(n)}(v, \zeta) \) is the \( n \)-th derivative of \( \phi(v, \zeta) \). Now we fix \( v_1, v_2 \in V, w_1 \in W_1 \) and analyze each term in the cocycle condition.

(1) For the left action term in the cocycle condition, we have

\[
(Y^L_H(v_1, z_1)\Phi(v_2; z_2))(\zeta)w_1 = \sum_{n=0}^{\infty} \left( (Y^L_H(v_1, z_1)\phi^{(n)})(v_2, \zeta)w_1 \right) \frac{z^n}{n!}
\]

\[
= \sum_{n=0}^{\infty} \left( \iota_{\zeta z_1} E(Y_{W_2}(v_1, \zeta + z_1)\phi^{(n)}(v_2, \zeta)w_1) \right) \frac{z^n}{n!}.
\]

We now show that (29) converges to the same \( \overline{W}_2 \)-valued rational function as

\[
Y_{W_2}(v_1, \zeta + z_1)\phi(v_2, \zeta + z_2)w_1.
\]

From the definition of \( \mathcal{H}_N(W_1, W_2) \), we know that for every \( w'_2 \in W'_2 \),

\[
\langle w'_2, Y_{W_2}(v_1, \zeta + z_1)\phi(v_2, \zeta + z_2)w_1 \rangle
\]

(31)
converges absolutely in the region $|\zeta + z_1| > |\zeta + z_2| > 0$ to a rational function $f(z_1, z_2, \zeta)$ with the only possible poles at $z_1 - z_2 = 0$, $z_1 + \zeta = 0$, $z_2 + \zeta = 0$. From the formal Taylor theorem, (31) equals to

$$\sum_{n=0}^{\infty} \langle w'_2, Y_{W_2}(v_1, \zeta + z_1)\phi(n)(v_2, \zeta)w_1 \rangle \frac{z'^n}{n!}$$

(32)

in the region $|\zeta| > |z_2|, |\zeta + z_1| > |\zeta + z_2| > 0$. Regard (32) as a series in $\zeta + z_1$, $\zeta$ and $z_2$ that is globally truncated in $z_2$, take

$$S = \{(\zeta + z_1, \zeta, z_2) : |\zeta| > |z_2|, |\zeta + z_1| > |\zeta + z_2| > 0\}$$

$$S' = \{(\zeta + z_1, \zeta, z_2) : |\zeta| > 0, |\zeta + z_1| > |\zeta| > 0\}$$

$$T = \{(\zeta + z_1, \zeta, z_2) : |\zeta| > |z_2|, |\zeta + z_1| > |\zeta| + |z_2|\}$$

and apply Lemma 3.9 (1), we see that (32) converges absolutely to $f(z_1, z_2, \zeta)$ when $|\zeta| > |z_2|, |\zeta + z_1| > |\zeta| + |z_2|$.

Now that for every fixed power $n$ of $z_2$, the coefficient

$$\langle w'_2, Y_{W_2}(v_1, \zeta + z_1)\phi(n)(v_2, \zeta)w_1 \rangle$$

(33)

of the series of (32) converges absolutely to the rational function

$$\langle w'_2, E(Y_{W_2}(v_1, \zeta + z_1)\phi(n)(v_2, \zeta)w_1) \rangle$$

with the only possible pole at $\zeta = 0, z_1 = 0, z_1 + \zeta = 0$. If we expand the negative powers of $\zeta + z_1$ in (33) as power series in $z_1$, we see that (33) equals to

$$\iota_{\zeta z_1} \langle w'_2, E(Y_{W_2}(v_1, \zeta + z_1)\phi(n)(v_2, \zeta)w_1) \rangle,$$

when $|\zeta + z_1| > |\zeta| > |z_1|$. Thus we know (32) equals to the series

$$\sum_{n=0}^{\infty} \left(\iota_{\zeta z_1} \langle w'_2, E(Y_{W_2}(v_1, \zeta + z_1)\phi(n)(v_2, \zeta)w_1) \rangle \right) \frac{z'^n}{n!}$$

(34)

when $|\zeta| > |z_2|, |\zeta + z_1| > |\zeta| + |z_2|, |\zeta| > |z_1|$. Regard (34) as a series in $z_1, z_2, \zeta$ that is globally truncated in $z_2$ and $z_1$, take

$$S = \{(z_1, \zeta, z_2) : |\zeta| > |z_2|, |\zeta + z_1| > |\zeta| + |z_2|, |\zeta| > |z_1|\},$$

$$S' = \{(z_1, \zeta) : |\zeta + z_1| > |\zeta| > |z_1|\},$$

$$T = \{(z_1, \zeta, z_2) : |\zeta| > |z_1| > |z_2|\},$$

and apply Lemma 3.9 (1), we see that (34) converges absolutely to $f(z_1, z_2, \zeta)$ when $|\zeta| > |z_1| > |z_2|$. Thus the series (29) converges absolutely to the $W_2$-valued rational function (30).

(2) For the right action term in the cocycle condition, we know that

$$(Y_{H}^{s(R)}(v_2, z_2)\Phi(v_1; z_1))(\zeta)w_1 = \sum_{n=0}^{\infty} \left((Y_{H}^{s(R)}(v_2, z_2)\phi(n))(v_1, \zeta)w_1 \right) \frac{z'^n}{n!}$$
\[
E(\Phi(Y(v_1, z_1 - \eta)Y(v_2, z_2 - \eta),1, \eta))
\]

converges absolutely to something in $W_2$ in the region $|z_1 - \eta| > |z_2 - \eta| > 0$
that is independent of the choice of $\eta$. If we evaluate $\eta = z_2$ then apply to $w_1$, we obtain
the following iterated series

\[
(\Phi(Y(v_1, z_1 - z_2)v_2, z_2))(\zeta)w_1 = \sum_{n=0}^{\infty} \left( (\phi^{(n)}(Y(v_1, z_1 - z_2)v_2, \zeta))w_1 \right) \frac{z_2^n}{n!}
\]

\[
= \sum_{n=0}^{\infty} \left( \sum_{m \in \mathbb{Z}} (\phi^{(n)}((v_1)_m v_2, \zeta))w_1 \right) (z_1 - z_2)^{-1-m} \frac{z_2^n}{n!}
\]

(37)

If we only assumed the composable condition of $\Phi$, then it is not necessary for $E$ to converge to a rational function. However, $\Phi$ is now a 1-cocycle in the bimodule $H_N(W_1, W_2)$, thus we have

\[
E(\Phi(Y(v_1, z_1 - z_2)v_2, z_2))(\zeta)w_1) = E((Y^{(n)}_N(v_1, z_1)\Phi)(v_2, z_2)w_1) + E((Y^{(n)}_N(v_2, z_2)\Phi)(v_1, z_1)w_1).
\]

(38)

Since the $W_2$-valued rational function on the right-hand-side are given by $Y_{W_2}(v_1, \zeta + z_1)\phi(v_2, \zeta + z_2)w_1$ and $\phi(v_1, \zeta + z_1)Y_{W_1}(v_2, \zeta + z_2)w_1$, from the definition of $H_N(W_1, W_2)$, there exists constants $p_1, p_2, p_{12}$ depending only on the pairs $(v_1, w_1), (v_2, w_2)$ and $(v_1, v_2)$ respectively, such that

\[
(z_1 - z_2)^{p_{12}}(z_1 + \zeta)^{p_1}(z_2 + \zeta)^{p_2}(\Phi(Y(v_1, z_1 - z_2)v_2, z_2))(\zeta)w_1 \in W_2[[z_1, z_2, \zeta]]
\]

(39)

Therefore, the series $E$ is precisely the product of the series with the expansions of

\[
t_{z_2}^{z_2}(\zeta + z_2)^{-p_2} = \sum_{n=0}^{\infty} (-p_2) \frac{z_2^n}{n!}
\]

and

\[
t_{z_2}^{z_2+z_1}(\zeta + z_1)^{-p_2} = \sum_{n=0}^{\infty} (-p_2) \frac{z_2^n}{n!} \sum_{i=0}^{n} \frac{n_i}{i} (z_1 - z_2)^{n-i} z_2^i.
\]
Consequently, the series \( \sum_{n=0}^{\infty} \left( Y_{s}(\Phi(v_1, z_1))\right)(v_1, z_1)w_1 \) converges absolutely to a \( \mathbb{W}_2 \)-valued rational function with the only possible poles at \( z_1 + \zeta = 0, z_2 + \zeta = 0, z_1 - z_2 = 0 \) when \( |\zeta| > |z_1 - z_2| + |z_2|, |z_1 - z_2| > 0 \).

Similarly, we see that

\[
(Y^s_{s}(R)(v_2, z_2)\Phi(v_1; z_1))(\zeta)w_1 = \sum_{n=0}^{\infty} \left( Y^s_{s}(R)(v_2, z_2)\phi^{(n)}(v_1, \zeta)\right)(v_1, z_1)w_1 \frac{z^n_1}{n!} = \sum_{n=0}^{\infty} \zeta_2(\phi^{(n)}(v_1, \zeta)Y_{\mathbb{W}_1}(v_2, \zeta + z_2)w_1) \frac{z^n_1}{n!}
\]

\[
\Phi(Y(v_1, z_1 - \eta)Y(v_2, z_2 - \eta)1, \eta) = \sum_{n=0}^{\infty} \phi^{(n)}(Y(v_1, z_1 - \eta)Y(v_2, z_2 - \eta)1; \eta) \frac{\eta^n}{n!}
\]

\[
Y^L_{s}(v_1, z_1)\Phi(v_2; z_2) - \Phi(Y(v_1, z_1 - \eta)Y(v_2, z_2 - \eta)1; \eta) + Y^s_{s}(R)(v_2, z_2)\Phi(v_1, z_1) = 0
\]
as an \( \mathcal{H}_N(W_1, W_2) \)-valued rational function. \( \square \)

**Remark 3.11.** The proof of the convergence unfortunately does not generalize to higher cohomologies. Details will be discussed in the subsequent paper [Q6].

**Proposition 3.12.** Let \( \Phi : V \to \overline{\mathcal{H}_N(W_1, W_2)} \) be a 1-cocycle. For \( v \in V \), let \( \phi(v, \zeta) = (\Phi(v; 0))(\zeta) \in \mathcal{H}_N(W_1, W_2) \). Then for every \( v_1 \in V, w_1 \in W_1 \), there exists \( p \in \mathbb{N} \), such that for every \( v_2 \in V \),

\[
(z_1 + \zeta)^{p}\phi(Y(v_1, z_1)v_2, \zeta)w_1 = (z_1 + \zeta)^{p}Y_{\mathbb{W}_2}(v_1, z_1 + \zeta)\phi(v_2, \zeta)w_1 + (z_1 + \zeta)^{p}\phi(v_1, \zeta + z_1)Y_{\mathbb{W}_1}(v_2, \zeta)w_1
\]
in \( W_2((z_1, \zeta)) \).

**Proof.** From the cocycle condition \( \Phi \) together with the conclusion of Proposition 3.10 we have

\[
E(\Phi(Y(v_1, z_1 - z_2)v_2, z_2))w_1 = E(Y_{\mathbb{W}_2}(v_1, z_1 + z_2)\phi(v_2, z_2 + \zeta)w_1) + E(\phi(v_1, z_1 + \zeta)Y_{\mathbb{W}_1}(v_2, z_2 + \zeta)w_1)
\]

Multiplying both side by \( (z_1 - z_2)^{p_1}(z_1 + \zeta)^{p_2}(z_2 + \zeta)^{p_2} \) where \( p_1, p_2, p_12 \) are numbers depending only on the pairs \((v_1, w_1), (v_2, w_2), (v_1, v_2)\) (see the proof of Proposition 3.10), then evaluate \( z_2 = 0 \) on both sides, we have

\[
z_1^{p_12}(z_1 + \zeta)^{p_1}(z_2 + \zeta)^{p_2}\phi(Y(v_1, z_1)v_2, \zeta)w_1 = z_1^{p_12}(z_1 + \zeta)^{p_1}(z_2 + \zeta)^{p_2}Y_{\mathbb{W}_2}(v_1, z_1 + \zeta)\phi(v_2, \zeta)w_1 + z_1^{p_12}(z_1 + \zeta)^{p_1}(z_2 + \zeta)^{p_2}\phi(v_1, \zeta + z_1)Y_{\mathbb{W}_1}(v_2, \zeta)w_1
\]
as series in \( W_2[[z_1, z_2, \zeta]] \). The conclusion then follows by dividing both sides by \( z_1^{p_12} \) and \( z_2^{p_2} \). \( \square \)

**Proposition 3.13.** Let \( \Phi : V \to \overline{\mathcal{H}_N(W_1, W_2)} \) be a 1-coboundary. For \( v \in V \), let \( \phi(v, \zeta) = (\Phi(v; 0))(\zeta) \in \mathcal{H}_N(W_1, W_2) \). Then \( \phi(v, \zeta) = Y_{\mathbb{W}_2}(v, \zeta)\phi_{-1} - \phi_{-1}Y_{\mathbb{W}_1}(v, \zeta) \) for some linear map \( \phi_{-1} : W_1 \to W_2 \).
Proof. Since $\Phi$ is a 1-coboundary, there exists some $\phi \in \mathcal{H}_N(W_1, W_2)$ satisfying $D_H\phi = 0$, such that,

$$\Phi(v; z) = Y^L_H(v, z)\phi - Y^{s(R)}_H(v, z)\phi$$

(40)

Note that since $D_H\phi = 0$ means for every $w_1 \in W_1$, $(\partial/\partial \zeta)\phi(\zeta)w_1 = 0$, which further implies that $\phi(\zeta)w_1 = \phi_{-1}(w_1) \in W_2$. Moreover, for every $v \in V$,

$$Y^L_H(v, z)\phi(\zeta)w_1 = \iota_{\zeta}Y_{W_2}(v, \zeta + z)\phi_{-1}w_1$$

(41)

$$Y^{s(R)}_H(v, z)\phi(\zeta)w_1 = \iota_{\zeta}\phi_{-1}Y_{W_1}(v, \zeta + z)w_1$$

(42)

Since $D_H\phi = 0$, we are allowed to evaluate $z = 0$ in (40), (41) and (42), so as to conclude that

$$(\Phi(v; 0))(\zeta)w_1 = Y_{W_2}(v, \zeta)\phi_{-1}w_1 - \phi_{-1}Y_{W_1}(v, \zeta),$$

the left-hand-side of which is precisely $\phi(v, \zeta)w_1$.

\[\square\]

3.4. Extension in $\text{Ext}^1_N(W_1, W_2)$ from a derivation.

**Proposition 3.14.** Let $F : V \rightarrow \mathcal{H}_N(W_1, W_2)$ be a derivation. Then for every $v \in V$, $F(v)$ is a map $W_1 \rightarrow (\overline{W_2})$. We use the notation $F(v, \zeta)w_1$ to denote the image of $w_1$ via the map $F(v)$ in $(\overline{W_2})$. Then the vector space

$$U = W_2 \Pi W_1$$

equipped with the vertex operator map

$$Y_U(v, x)(w_2, w_1) = (Y_{W_2}(v, x)w_2 + F(v, x)w_1, Y_{W_1}(v, x)w_1),$$

the operator $d_U = d_{W_2} \Pi d_{W_1}$, and the operator $D_U = D_{W_2} \Pi D_{W_1}$ forms a left $V$-module that fits in the exact sequence

$$0 \longrightarrow W_2 \longrightarrow U \longrightarrow^{p} W_2 \longrightarrow 0,$$

where the map $p : U = W_2 \Pi W_1 \rightarrow W_1$ is precisely the projection. Moreover, $U$ satisfies the conditions in Definition 3.2

**Proof.** We only give a sketch here. The grading on $U$ is given by the gradings on $W_1$ and $W_2$, with the homogeneous subspaces being $U_{[m]} = (W_2)_{[m]} \Pi (W_1)_{[m]}$. Thus the grading is lower bounded. The $d$-commutator formula follows from the $d$-conjugation property in Proposition-Definition 3.2. The identity property follows from $F(1) = 0$, due to $F$ being a derivation. The $D$-derivative-commutator formula also follows from

$$F(D_Fv, \zeta) = \frac{d}{d\zeta}F(v, \zeta) = D_{W_2}F(v, \zeta) - F(v, \zeta)D_{W_1}.$$

To show the weak associativity, we notice that from Proposition 3.12, $v_1 \in V, w_1 \in W_1$, there exists $v_2 \in V$, such that

$$(x_0 + \zeta)^pF(Y(v_1, x_0)v_2, \zeta)w_1 = (x_0 + \zeta)^pY_{W_2}(v_1, x_0 + \zeta)F(v_2, \zeta)w_1$$
+ (x_0 + \zeta)^p F(v_1, x_0 + \zeta) Y_{W_1}(v_2, \zeta) w_1.

This implies that the vertex operator \( Y_U \) satisfies the weak associativity with pole-order condition. The relation \( pY_U(v, x) = Y_{W_1}(v, x)p \) follows trivially from the definition. The conditions of Definition 3.2 are satisfied with the section \( \psi : W_1 \rightarrow W_2 \Pi W_1, w_1 \mapsto (0, w_1) \) and the projection \( \pi_2 : W_2 \Pi W_1 \rightarrow W_2, (w_2, w_1) \mapsto w_2 \). Indeed, \( \pi_2 Y_U(v, x) \psi = F(v, x) \) for every \( v \in V \).

**Proposition 3.15.** Let \( F^{(1)}, F^{(2)} : V \rightarrow \mathcal{H}_N(W_1, W_2) \) be two derivations that differ by an inner derivation. Let \( U^{(1)} \) and \( U^{(2)} \) be the extensions given respectively by \( F^{(1)} \) and \( F^{(2)} \). Then \( U^{(1)} \) and \( U^{(2)} \) are equivalent extensions.

**Proof.** We only give a sketch here. From Proposition 3.13 we see that

\[
F^{(1)}(v, \zeta) - F^{(2)}(v, \zeta) = Y_{W_1}(v, \zeta) \phi_1 - \phi_1 Y_{W_1}(v, \zeta)
\]

for some linear homogeneous map \( \phi_1 : W_1 \rightarrow W_2 \). Define \( T : U^{(1)} = W_2 \Pi W_1 \rightarrow W_2 \Pi W_1 = U^{(2)} \) by

\[
T(w_2, w_1) = (w_2 + \psi_1(w_1), w_1)
\]

It is straightforward to check that for every \( v \in V \),

\[
TY^{(1)}_U(v, x) = Y^{(2)}_U(v, x)T
\]

where \( Y^{(1)}_U \) and \( Y^{(2)}_U \) be the vertex operators so constructed in Proposition 3.14.

**Corollary 3.16.** The map \( \mathcal{G} : \hat{H}^1(V, \mathcal{H}_N(W_1, W_2)) \rightarrow \text{Ext}^1_N(W_1, W_2) \) sending a derivation \( F \) to equivalence class of the module \( U = W_2 \Pi W_1 \) given in Proposition 3.14 is well-defined.

3.5. **Proof of the main theorem.**

**Theorem 3.17.** \( \text{Ext}^1_N(W_1, W_2) \) is in one-to-one correspondence to \( \hat{H}^1(V, \mathcal{H}_N(W_1, W_2)) \).

**Proof.** Recall the maps \( \mathcal{F} \) and \( \mathcal{G} \) from Corollary 3.8 and Corollary 3.16. We first show that \( \mathcal{F} \circ \mathcal{G} = \text{Id} \). Starting from a 1-cohomology class \( F \), we have obtained an extension \( U = \mathcal{G}(F) \) satisfying conditions in Definition 3.2. From Remark 3.7 we see that the image \( \mathcal{F}(U) = \mathcal{G}(\mathcal{F}(F)) \) is independent of the choice of the section. If we use the section \( \psi \) as described at the end of the proof of Proposition 3.14 the derivation obtained in Proposition 3.5 is precisely \( F \). So we conclude that \( \mathcal{F}(\mathcal{G}(F)) = F \), i.e., \( \mathcal{F} \circ \mathcal{G} = \text{Id} \).

Now we show that \( \mathcal{G} \circ \mathcal{F} = \text{Id} \). Starting from an extension \( U \) satisfying the condition in Definition 3.2, we obtained a 1-cohomology class corresponding to \( F(v, x) = \pi_2 Y_U(v, x) \psi^{(1)} \). If we use this \( F \) to construct the module \( \mathcal{G}(\mathcal{F}(U)) \), we see that

\[
Y_{\mathcal{G}(\mathcal{F}(U))}(v, x)(w_2, w_1) = (Y_{W_2}(v, x)w_2 + \pi_2 Y_U(v, x) \psi^{(1)} w_1, Y_{W_1}(v, x) w_1),
\]

It is straightforward to check that \( \text{Id} : \hat{H}^1(V) \rightarrow \text{Ext}^1_N(W_1, W_2) \) sending \( (w_2, w_1) \) to \( (w_2, \psi^{(1)} w_1) \) is an isomorphism. So \( \mathcal{G}(\mathcal{F}(U)) \) represents the same element in \( \text{Ext}^1_N(W_1, W_2) \) as \( U \) does.
Remark 3.18. The bimodule $\mathcal{H}_N(W_1, W_2)$ is the vertex algebraic analogue of the bimodule $\text{Hom}_C(M_1, M_2)$ associated with the two left modules $M_1, M_2$ for an associative algebra. What we have shown in Theorem 3.17 is just the analogue of the bijection \[ \text{Ext}^1(M_1, M_2) \cong HH^1(A, \text{Hom}_C(M_1, M_2)) \] where $HH^1(A, \text{Hom}_C(M_1, M_2))$ is the first Hochschild cohomology with respect to the bimodule $\text{Hom}_C(M_1, M_2)$.

3.6. Some comments on reductivity.

Theorem 3.19. Let $V$ be a MOSVA such that every left $V$-module is completely reducible, then for every left $V$-module $W_1$ and $W_2$,

$$\widehat{H}^1(V, \mathcal{H}_N(W_1, W_2)) = 0$$

Proof. If every $V$-module is completely reducible, then the only possible extension of $W_1$ by $W_2$ is represented by the $V$-module direct sum $W_1 \oplus W_2$. If we pick $\psi$ to be the embedding of $W_1$ to $W_1 \oplus W_2$, then clearly $\pi(X, W_1 \oplus W_2) = 0$. Thus every 1-cocycle is cohomologous to 0. \[ \square \]

Remark 3.20. In [HQ], we studied the left $V$-module $W$ satisfying some technical but natural conditions convergence conditions. The conditions can be reformulated as: for every $V$-submodule $W_2$ of $U$, we require the existence of a complementary subspace $W_1$, such that the projection operators $\pi_{W_2} : U \to W_2$ and $\pi_{W_1} : U \to W_1$ makes $\pi_{W_2} Y_U(v, z) \pi_{W_1} \in \mathcal{H}_N(W_1, W_2)$ (here $W_1$ is a $V$-module with the vertex operator $\pi_{W_1}$. We proved that such a left $V$-module $U$ is completely reducible if for every $V$-bimodule, the first cohomology is given by the zero-mode derivations. It should be remarked here that proof in [HQ] used only the bimodule $\mathcal{H}_N(W_1, W_2)$ and the cohomology $\widehat{H}^1(V, \mathcal{H}_N(W_1, W_2))$. Therefore, Theorem 3.19 can be viewed as a converse of the result in [HQ].

Remark 3.21. In the early version of [HQ] and in the author’s PhD thesis [Q3], the reductivity was proved with the assumption that $\widehat{H}^1(V, W) = 0$ for every $V$-bimodule $W$. Theorem 3.19 explains why we ignored the zero-mode derivations: $\mathcal{H}_N(W_1, W_2)$ does not have non-trivial zero-mode derivations. Indeed, one can directly show that every zero-mode derivation $V \to \mathcal{H}_N(W_1, W_2)$ is an inner derivation. We decide not to include the detail here as it is too technical and much less conceptual than what is shown here.

4. Category $\mathcal{C}_N$
to develop the homological methods of vertex algebras. In case $V$ is a grading-restricted vertex algebra containing a nice subalgebra, we show that every $V$-module is in $C_N$, and the dimension of the space of extensions is finite.

4.1. **Category $C_N$ and some general facts.**

**Definition 4.1.** Let $C_N$ be the category whose objects are left $V$-modules $W$ such that for every $V$-submodule $W_2 \subseteq W$, there exists a vector subspace $W_1 \subseteq W$ satisfying the following conditions:

1. $W = W_1 \oplus W_2$ as vector spaces with projection operator $\pi_{W_1} : W \to W_1, \pi_{W_2} : W \to W_2$.
2. For every $v \in V$, $\pi_{W_2}Y_W(v, x)\pi_{W_1} \in \mathcal{H}(W_1, W_2)$.

Here $W_1$ is regarded as $V$-module with the vertex operator $v \mapsto \pi_{W_1}Y_W(v, x)$. Morphisms of $C_N$ are the left $V$-module homomorphisms.

**Remark 4.2.** If $W_1$ is a complement of $W_2$, i.e., the vector space $W_1$ is a also a submodule of $W$, then for every $v \in V$, $\pi_{W_2}Y_W(v, x)\pi_{W_1} = 0$. The conditions (2) holds trivially. Consequently, if $V$ is a strongly rational vertex operator algebra, then for every $N \in \mathbb{Z}$, $C_N$ coincides with the category of $V$-modules.

**Remark 4.3.** The composability and $N$-weight-degree conditions in Proposition-Definition 2.21 can also be formulated in terms of formal variables.

3. The composability condition: For every $k, l \in \mathbb{N}$, every $u_1, \ldots, u_{k+l} \in V$, there exists $p \in \mathbb{N}, p_i, q_i \in \mathbb{N}(i = 1, \ldots, k + l), q_{ij} \in \mathbb{N}(1 \leq i < j \leq k + l)$, and a formal power series $g(x_1, \ldots, x_k, x, x_{k+1}, \ldots, x_{k+l}) \in W_2[[x_1, \ldots, x_k, x, x_{k+1}, \ldots, x_{k+l}]]$, such that

$$Y_{W_2}(u_1, x_1) \cdots Y_{W_2}(u_k, x_k)\phi(x)Y_W(u_{k+1}, x_{k+1}) \cdots \pi_{W_1}Y_W(x_{k+l}, x_{k+l})w_1 =$$

$$\left(\frac{x^p \prod_{i=1}^{k+l} x_i^{q_i} \prod_{i=1}^{k} (x_i - x)^{p_i} \prod_{i=k+1}^{k+l} (x_i - x)^{p_i} \prod_{1 \leq i < j \leq k+l} (x_i - x_j)^{q_{ij}}}{x^{k+l} \prod_{i=1}^{k+l} (x_i + x_j)^{q_{ij}} \prod_{i=1}^{k} (x_i - x)^{p_i} \prod_{1 \leq i < j \leq k+l} (x_i - x_j)^{q_{ij}}}ight)$$

where the right-hand-side series is in $W_2[[x_1, x_1^{-1}, \ldots, x_k, x_k^{-1}, x, x^{-1}, x_{k+1}, x_{k+1}^{-1}, \ldots, x_{k+l}, x_{k+l}^{-1}]]$ with all negative powers of binomials expanded as power series of the second variable in accordance with the binomial expansion convention.

4. $N$-weight-degree condition: With same notations as in (3), the series

$$g(x_1 + x_{k+1}, \ldots, x_k + x_{k+l}, x + x_{k+l}, x_{k+1}, \ldots, x_{k+l-1} + x_{k+l}, x_{k+1} + x_{k+l})$$

in $W_2[[x_1, x_1^{-1}, \ldots, x_k, x_k^{-1}, x, x^{-1}, x_{k+1}, x_{k+1}^{-1}, \ldots, x_{k+l}, x_{k+l}^{-1}]]$, viewed as a series in

$$(W_2[[x_{k+l}, x_{k+l}^{-1}]])[[x_1, x_1^{-1}, \ldots, x_k, x_k^{-1}, x, x^{-1}, x_{k+1}, x_{k+1}^{-1}, \ldots, x_{k+l-1}, x_{k+l-1}^{-1}]]$$

has total degree at least as large as $N - \text{wt}(u_1) - \cdots - \text{wt}(u_{k+1}) - \text{wt}(v)$. 

Proposition 4.4. The category $\mathcal{C}_N$ is closed under the operation of taking submodules and quotients.

Proof. Let $T \subseteq W$ be a submodule of a left $V$-module $W \in \mathcal{C}_N$. Fix any $V$-submodule $T_2$ of $T$. Then $T_2$ is also a left $V$-submodule of $W$. Since $W \in \mathcal{C}_N$, there exists a vector subspace $W_1$, such that $W = W_1 \sqcup T_2$, and for every $v \in V$, the corresponding projection operator $\pi_{T_2}$ and $\pi_{W_1}$ makes the map $\pi_{T_2} Y_T(v, x) \pi_{W_1}$ infinitely composable with $Y_{T_2}$ and $\pi_{W_1} Y_W$ and satisfy the $N$-weight-degree condition. Let $T_1 = W_1 \cap T$. Then $T = T_1 \sqcup T_2$. Moreover, for every $v \in V$,

$$\pi_{T_2} Y_T(v, x) \pi_{T_1} = \pi_{T_2} Y_W(v, x) \pi_{W_1} |_{T_1},$$

and

$$\pi_{T_1} Y_T(v, x) = \pi_{W_1} Y_W(v, x) |_{T_1}.$$  

Thus for every $k, l \in \mathbb{N}$, every $u_1, \ldots, u_{k+l} \in V$, every $t'_2 \in W'_2, t_1 \in T_1$,

$$(t'_2, Y_{W_2}(u_1, z_1) \cdots Y_{W_2}(u_k, z_k) \pi_{W_2} Y_{W_2}(v, z) \pi_{W_1} Y_{W}(u_{k+1}, z_{k+1}) \cdots \pi_{W_1} Y_{W}(u_{k+l}, z_{k+l}) t_1)$$

$$= (t'_2, Y_{T_2}(u_1, z_1) \cdots Y_{T_2}(u_k, z_k) \pi_{T_2} Y_{T}(v, z) \pi_{T_1} Y_{T}(u_{k+1}, z_{k+1}) \cdots \pi_{T_1} Y_{T}(u_{k+l}, z_{k+l}) t_1)$$

Since the left-hand-side series converges absolutely to a rational function satisfying the $N$-weight-degree condition, so is the right-hand-side series. Thus we verified that for every fixed $V$-submodule $T_2$ of $T$, there exists a vector subspace $T_1$ of $T$ such that $T = T_1 \sqcup T_2$, and for every $v \in V$, the corresponding projection operators $\pi_{T_2}$ and $\pi_{T_1}$ makes the map $\pi_{T_2} Y_T(v, x) \pi_{T_1}$ satisfy the composability and the $N$-weight-degree conditions. Thus $T \in \mathcal{C}_N$.

Let $T \subseteq W$ be a left $V$-module and consider the quotient module $W/T$. For every $v \in V$, the vertex operator on $W/T$ is given naturally by

$$Y_{W/T}(v, x)(w + T) = Y_W(v, x)w + T.$$

Fix any submodule $W_2/T \subseteq W/T$. Then $W_2 + T$ is a submodule of $W$. Since $W \in \mathcal{C}_N$, there exists a vector subspace $W_1$ of $W$ such that $W = (W_2 + T) \sqcup W_1$, and for every $v \in V$, the corresponding projection operators $\pi_{W_2 + T}$ and $\pi_{W_1}$ makes the map $\pi_{W_2 + T} Y_W(v, x) \pi_{W_1}$ satisfy the composability (with $Y_{W_2 + T}$ and $\pi_{W_1} Y_W$) and the $N$-weight-degree conditions. We use the vector space $(W_1 + T)/T$, so that $W/T = (W_1 + T)/T \sqcup W_2/T$. Clearly, for every $w \in W$, from $T \cap W_1 = 0$, we have

$$\pi_{(W_1 + T)/T}(w + T) = \pi_{W_1} w + T$$

Therefore, for every $v \in V$ and $w \in W$, the corresponding projection operators $\pi_{(W_1 + T)/T}$ and $\pi_{W_2/T}$ satisfy

$$\pi_{W_2/T} Y_{W/T}(v, x) \pi_{(W_1 + T)/T}(w + T) = \pi_{W_2} Y_W(v, x) \pi_{W_1} w + T,$$

and

$$\pi_{(W_1 + T)/T} Y_{W/T}(v, x)(w + T) = \pi_{W_1} Y_W(v, x)w + T.$$
These identifications allow us to pass the Conditions (3) and (4) in Remark 4.3 satisfied by the map \( \pi_{W_2/Y} \) from \( W_2 \) to \( W_2/T \). Then, we see that the map \( \pi_{W_2/T} \) also satisfies Conditions (3) and (4). Thus we proved that for every submodule \( W_2/T \) of \( W/T \), there exists a vector subspace \( (W_1 + T)/T \) of \( W/T \), such that \( W/T = (W_1 + T)/T \oplus W_2/T \), and corresponding projections makes the map \( \pi_{W_2/T} \) infinitely composable and satisfy the \( N \)-weight-degree condition. So \( W/T \in C_N \). □

**Remark 4.5.** In general, we cannot prove that \( C_N \) is closed under direct sums. Thus \( C_N \) generally does not form an abelian category. However, we will see in Section 4.2 that if \( V \) is a grading restricted vertex algebra containing a nice vertex subalgebra, then \( C_N \) coincides with the category of grading-restricted (generalized) \( V \)-modules.

**Remark 4.6.** Even if \( V \) does not contain a nice subalgebra, it is still possible that certain \( V \)-modules forms an abelian category whose objects are in \( C_N \). In Section 5 we will illustrate such an example using certain modules for the Virasoro VOA.

### 4.2. When \( V \) contains a nice subalgebra \( V_0 \)

Let \( V \) is a grading-restricted vertex algebra and \( V_0 \) is a vertex subalgebra of \( V \). Note that the grading and the operator \( D_{V_0} \) for the vertex subalgebra \( V_0 \) of \( V \) are induced from those for \( V \). In particular, when \( V \) is a vertex operator algebra such that \( d_V = L_V(0) \) and \( D_V = L_V(-1) \) and \( V_0 \) is a vertex subalgebra with a different conformal vector, \( d_{V_0} = L_{V_0}(0)|_{V_0} \) and \( D_{V_0} = L_{V_0}(-1)|_{V_0} \) are in general different from \( L_{V_0}(0) \) and \( L_{V_0}(-1) \), respectively.

We start by interpreting Proposition 6.3, Theorem 6.4 in [HQ] to the following form.

**Theorem 4.7.** Let \( V \) is a grading-restricted vertex algebra and \( V_0 \) is a vertex subalgebra of \( V \) satisfying the following conditions:

1. Every grading-restricted left \( V_0 \)-module (or generalized \( V_0 \)-module satisfying the grading-restriction conditions in the terminology in [HLZ]) is completely reducible.

2. For any \( n \in \mathbb{Z}_+ \), products of \( n \) intertwining operators among grading-restricted left \( V_0 \) modules evaluated at \( z_1, \ldots, z_n \) are absolutely convergent in the region \( |z_1| > \cdots > |z_n| > 0 \) and can be analytically extended to a (possibly multivalued) analytic function in \( z_1, \ldots, z_n \) with the only possible singularities (branch points or poles) \( z_i = 0 \) for \( i = 1, \ldots, n \) and \( z_i = z_j \) for \( i, j = 1, \ldots, n, i \neq j \).

3. The associativity of intertwining operators among grading-restricted left \( V_0 \) modules holds.

Let \( U \) be an extension of \( W_1 \) by \( W_2 \). Then \( U \) is in \( C_N \).

**Proof.** The proofs are basically identical to Proposition 6.3 and Theorem 6.4 in [HQ]. We shall not repeat here. □
Remark 4.8. The proof of Proposition 6.3 and Theorem 6.4 in [HQ] was discovered by Yi-Zhi Huang. The proof uses the theory of intertwining operators and the tensor category theory for module categories for grading-restricted vertex algebras satisfying suitable conditions. It hints deep connections between the cohomology theory and the theory of intertwining operators or the tensor category theory.

Recall that the fusion rule $N_{W_2/VW_1}$ is the dimension of the space of intertwining operators of type $(W_2/VW_1)$.

Theorem 4.9. If the fusion rule $N_{W_2/VW_1}$ is finite, then both $\text{Ext}^1(W_1, W_2)$ and the first cohomology $\widehat{H}^1(V, \mathcal{H}_N(W_1, W_2))$ are finite dimensional.

Proof. Let $U \in \text{Ext}^1(W_1, W_2)$ and consider the linear map $U \mapsto (I : v \mapsto \pi_2 Y_U(v, z) \psi_1)$. Clearly $I$ is an intertwining operator of type $(W_2/VW_1)$. The map is well defined since the trivial extension $W_1 \oplus W_2$ has the image zero. Conversely, assume $U$ has image zero in the space of intertwining operators. Thus for every $v \in V$, $\pi_2 Y_U(v, z) \psi_1 = 0$. But this implies that the cohomology class given by $\pi_2 Y_U(v, \zeta + z) \psi_1 = 0$ in $\widehat{H}^1(V, \mathcal{H}_N(W_1, W_2))$. From the bijective correspondence proved in Theorem 3.17, we see that $U$ is the trivial extension $W_1 \oplus W_2$. Thus the map $U \mapsto I$ is injective. Thus $\text{Ext}^1(W_1, W_2)$ embeds into the space of intertwining operators of type $(W_2/VW_1)$. Hence

$$\dim \mathbb{C} \text{Ext}^1(W_1, W_2) = \dim \mathbb{C} \widehat{H}^1(V, \mathcal{H}_N(W_1, W_2)) < N_{W_2/VW_1} < \infty.$$ 

Remark 4.10. Though the inclusion of a nice vertex subalgebra $V_0$ is a very strong condition, there is no further requirements. In particular, we do not require $\dim V(0) = 1$ (as in [HKL]), or the conformal element of $V$ coincides with the conformal element of $V(0)$. We expect Theorem 4.9 to be a useful tool to study the category of modules for the extensions of $V_0$.

5. An example from the Virasoro VOA

In this section, we demonstrate an example of an abelian category of modules for the Virasoro VOA satisfying the conditions Definition 4.1. Note that in this case, the VOA does not contain any nice subalgebra as in Section 4.2. This suggest that the conditions in Definition 4.1 might not be a serious obstruction for the application of cohomology theory developed in this paper.

5.1. Review of Virasoro algebra and Feigin-Fuchs theory. Let $Vir$ be the Virasoro algebra, i.e., $Vir = \bigoplus_{n \in \mathbb{Z}} \mathbb{C}L_n \oplus \mathbb{C}k$, with

$$[k, L_n] = 0, [L_m, L_n] = (m - n)L_{m+n} + \delta_{m+n,0} \frac{m^3 - m}{12} k,$$
for every \( m, n \in \mathbb{Z} \). Fix \( c, h \in \mathbb{C} \). Consider the one-dimensional space \( \mathbb{C} \cdot 1_{c,h} \) where \( k \) acts by the scalar \( c \), \( L_0 \) acts by the scalar \( h \), and \( L_n \) acts trivially for every \( n > 0 \). The induced module

\[
M(c, h) = U(Vir) \otimes_{\mathbb{C}[n \geq 0] C L_n \oplus \mathbb{C} k} \mathbb{C} \cdot 1_{c,h}
\]
is called the Verma module (aka, universal restricted module) associated with central charge \( c \) and lowest weight \( h \). For every \( n \in \mathbb{Z} \), we will use the notation \( L(n) \) to denote the action of \( L_n \) on \( M(c, h) \). It is known that \( M(c, h) \) is graded by the \( L(0) \)-eigenvalues. For each \( n \in \mathbb{N} \), the eigenspace \( M(c, h)_{(h+n)} \) has a basis of vectors of the following form

\[
L(-n_1) \cdots L(-n_s) 1_{c,h}, n_1 \geq \cdots \geq n_s \geq -1, n_1 + \cdots + n_s = n.
\]

It is also known that \( M(c, h) \) has a unique irreducible quotient, denoted by \( L(c, h) \). Define a partial order among the pairs \( (c, h) \):

\[
(c_1, h_1) \preceq (c_2, h_2) \text{ iff } L(c_2, h_2) \text{ is a subquotient of } M(c_1, h_1).
\]

We use \( \sim \) to denote the equivalence relation given by transitive closure of \( \preceq \), i.e., \( (c_1, h_1) \sim (c_2, h_2) \) iff there exists \((c^{(1)}, h^{(1)}), \ldots, (c^{(n)}, h^{(n)})\) such that \((c^{(1)}, h^{(1)}) = (c_1, h_1), (c^{(n)}, h^{(n)}) = (c_2, h_2)\), and for every \( i = 1, \ldots, n - 1 \), either \((c^{(i)}, h^{(i)}) \preceq (c^{(i+1)}, h^{(i+1)})\) or \((c^{(i+1)}, h^{(i+1)}) \preceq (c^{(i)}, h^{(i)})\). The equivalence class of \( (c, h) \) is called the block, denoted by \([c, h]\).

Here we list all the previously known results regarding the Verma modules, their submodules, and structure of blocks (see [A], [BNW], [IK], [FF], [MP]). For simplicity, we follow the notation and classification in [BNW].

1. For any \( c, h \in \mathbb{C} \), the submodules of \( M(c, h) \) is generated by at most two singular vectors, i.e., vectors \( w \in M(c, h) \) with \( L(n)w = 0 \) for every \( n > 0 \).
2. For any \( c, h \in \mathbb{C} \) and any \( n \in \mathbb{N} \), there exists at most one singular vector in the homogeneous subspace \( M(c, h)_{(h+n)} \).
3. The submodule generated by a singular vector of weight \( h' \) is isomorphic to \( M(c, h') \).
4. For fixed \( c, h \in \mathbb{C} \)

\[
\nu = \frac{c - 13 + \sqrt{(c - 1)(c - 25)}}{12}, \beta = \frac{\sqrt{-4 c h + (\nu + 1)^2}}{c}
\]

and the line in the \( rs \)-plane

\[
\mathcal{L}_{c,h} : r + \nu s + \beta = 0.
\]

(a) Suppose \( \mathcal{L}_{c,h} \) passes through no integer points or one integer point \((r, s)\) with \( rs = 0 \), then \( M(c, h) \) is irreducible. The block \([c, h] = \{(c, h)\}\).

(b) Suppose \( \mathcal{L}_{c,h} \) passes through no integer points or one integer point \((r, s)\) with \( rs \neq 0 \).

Then the block \([c, h] \) is given by \( \{(c, h), (c, h + rs)\}\).

(c) Suppose \( \mathcal{L}_{c,h} \) passes through infinitely many integer points and crosses an axis at an integer point. Label these points by \( (r_i, s_i) \) so that

\[
\cdots < r_{-2}s_{-2} < r_{-1}s_{-1} < 0 = r_0s_0 < r_1s_1 < r_2s_2 < \cdots
\]
The block $[c, h]$ is given by $[c, h] = \{(c, h + r_i s_i) : i \in \mathbb{Z}\}$.

(d) Suppose $\mathcal{L}_{c,h}$ passes through infinitely many integer points and does not cross either axis at an integer point. Label these points by $(r_i, s_i)$ so that

$$\cdots < r_{-2}s_{-2} < r_{-1}s_{-1} < r_0s_0 < 0 < r_1s_1 < r_2s_2 < \cdots$$

Also consider the auxiliary line $\hat{\mathcal{L}}_{c,h}$ with the same slope as $\mathcal{L}_{c,h}$ passing through the point $(-r_1, s_1)$. Label the integer points on $\hat{\mathcal{L}}_{c,h}$ by $(r'_i, s'_i)$ in the same way as $\mathcal{L}_{c,h}$. The block $[c, h]$ is given by $[(c, h)] = \{(c, h_i), (c, h'_i) : i \in \mathbb{Z}\}$, where

$$h_i = \begin{cases} h + r_is_i & i \text{ odd} \\ h + r_is_1 + r'_is'_i & i \text{ even} \end{cases}, h'_i = \begin{cases} h + r_{i+1}s_{i+1} & i \text{ odd} \\ h + r_is_1 + r'_{i+1}s'_{i+1} & i \text{ even} \end{cases}$$

(5) If $[c_1, h_1] \neq [c_2, h_2]$, then $\text{Ext}^1(M(c_1, h_1), M(c_2, h_2)) = 0$.

Let $V = V(c, 0)$ be the quotient of $M(c, 0)$ by the submodule generated by $L(-1)1_{c,0}$. It is known that $V(c, 0)$ is a vertex operator algebra, and every $M(c, h)$ is a $V$-module. Let $W$ be a $V$-module which is a quotient of a finite direct sum of Verma modules $M(c, h)$ appearing in the blocks described by (a), (b) and (c). The goal of this section is to show that $W$ satisfies Definition 4.1. Clearly, such modules form an abelian category. Thus the objects of the category $\mathcal{C}$ formed by such modules are all in $\mathcal{C}_N$.

5.2. Verma module case. We first study the simplest case where $W$ is the Verma module $M(c, h)$ appearing in the blocks (a), (b) and (c). We would omit the lowest weight vector and write

$$W = \text{span}\{L(-n_1) \cdots L(-n_s) | n_1, \ldots, n_s \in \mathbb{Z}_+, n_1 \geq \cdots \geq n_s \geq 1\}.$$ 

Note that the vectors in the spanning set are linearly independent when $h \neq 0$. In case $h = 0$, the vectors in the spanning set with $n_1 \geq \cdots \geq n_s \geq 2$ are linearly independent. For the vector $L(-n_1) \cdots L(-n_s)$, the number $\sum_{i=1}^s n_i$ is called the level.

Every submodule of $M(c, h)$ is generated by singular vectors. We know from [A] that for each $N \in \mathbb{Z}_+$, there exists at most one singular vector of level $N$ (up to a scalar), and the coefficient of $L(-1)^N$ is nonzero.

Let $W_2 = M(c, h_1)$ be a submodule. By our choice of $W$, $W_2$ is generated by one singular vector

$$s_{h_1} = L(-1)^N + \sum_{L<M_1, M_1 \geq \cdots \geq M_L \geq 1 \atop M_1 + \cdots + M_L = N} a_{M_1 \cdots M_L} L(-M_1) \cdots L(-M_L)$$

of some fixed level $N$. Then $h_1 = h + N$, and

$$W_2 = \text{span}\{L(-n_1) \cdots L(-n_s) L(-1)^{N-N} s_{h_1} | n_1, \ldots, n_s \in \mathbb{Z}_+, n_1 \geq \cdots \geq n_s \geq 2, n \geq N\}.$$ 

It follows from the Kac determinant formula that $h_1 \neq 0$ (see details in [Q5], Proposition 4.9). Note also that the vectors in the spanning set are linearly independent. We now choose

$$W_1 = \text{span}\{L(-n_1) \cdots L(-n_s) L(-1)^n | n_1, \ldots, n_s \in \mathbb{Z}_+, n_1 \geq \cdots \geq n_s \geq 2, n < N\}.$$
Clearly, $W = W_1 \coprod W_2$ as a vector space. Let $\pi_{W_2} : W \to W_2$ be the projection of $W$ onto $W_2$, i.e.,

$$\pi_{W_2}|_{W_2} = Id_{W_2}, \pi_{W_2}|_{W_1} = 0.$$ 

Then, for $n_1 \geq \cdots \geq n_s \geq 2$, we have

$$\pi_{W_2}(L(-n_1) \cdots L(-n_s)L(-1)^N) = L(-n_1) \cdots L(-n_s)s_h.$$ 

Moreover,

$$\pi_{W_2}(L(-n_1) \cdots L(-n_s)L(-1)^n) \neq 0$$

only when $n \geq N$.

**Definition 5.1.** Every element $w \in W$ can be written uniquely as

$$w = \sum_{p_i \geq \cdots \geq p_i \geq 2} \sum_{q \geq 0} a_{p_1 \cdots p_t,q}L(-p_1) \cdots L(-p_t)L(-1)^q.$$ 

We will refer this summation as the standard expression of $w$. Clearly the sum is finite, i.e., there exists finitely many $p_1, \ldots, p_t$ and $q$, such that $a_{p_1 \cdots p_t,q} \neq 0$. We define the index of $w$ as

$$\text{Ind}(w) = \max\{q|a_{p_1 \cdots p_t,q} \neq 0, p_1 \geq \cdots \geq p_t \geq 2\}.$$ 

If $w = 0$, we define $\text{Ind}(w) = -1$. Note that $\text{Ind}(w_1 + w_2) = \max\{\text{Ind}(w_1), \text{Ind}(w_2)\}$. Note also that if $\text{Ind}(w) < N$, then $\pi_{W_2}(w) = 0$.

**Proposition 5.2.**

1. For every $m \geq 2$,

$$\text{Ind}(L(-m)w) = \text{Ind}(w).$$

2. Fix $n \in \mathbb{Z}_+$ and $n_1, \ldots, n_s \geq 2$ (not necessarily decreasing). Then

$$\text{Ind}(L(m)L(-n_1) \cdots L(-n_s)L(-1)^n) \leq n + 1.$$ 

The equality holds only for finitely many $m$. Moreover, if we restrict that $m \geq -1$, then only finitely many $m$ makes

$$\text{Ind}(L(m)L(-n_1) \cdots L(-n_s)L(-1)^n) \geq n.$$ 

**Proof.** These follow easily from the commutator relation

$$L(m)L(n) - L(n)L(m) = (m - n)L(m + n) + \delta_{m+n,0} \frac{m^3 - m}{12}, m, n \in \mathbb{Z}.$$

\(\Box\)

**Definition 5.3.** Let $l \in \mathbb{Z}_+$ and $p \in \{1, \ldots, l\}$. We call $\alpha_1, \ldots, \alpha_p$ an increasing sequence in $\{1, \ldots, l\}$ if $1 \leq \alpha_1 < \cdots < \alpha_p \leq l$. The number $p$ is called the length of the sequence $\alpha_1, \ldots, \alpha_p$. Sometimes we will also use the notation $(\alpha_1, \ldots, \alpha_p)$ to denote an increasing sequence. For a fixed increasing sequence $\alpha_1, \ldots, \alpha_p$, there exists a unique increasing sequence $\alpha_1^c, \ldots, \alpha_{l-p}^c$ such that $\{\alpha_1, \ldots, \alpha_p\} \coprod \{\alpha_1^c, \ldots, \alpha_{l-p}^c\} = \{1, \ldots, l\}$. We call the increasing sequence $\alpha_1^c, \ldots, \alpha_{l-p}^c$ the
complement of $\alpha_1, ..., \alpha_p$. Note that when $p = l$, then an increasing sequence is uniquely $1, ..., l$ whose complement is empty.

**Corollary 5.4.** Let $n \in \mathbb{Z}_+$ and $n_1 \geq \cdots \geq n_s \geq 2$. Let $l \in \mathbb{Z}_+$, $p \in \{1, ..., l\}$, $\alpha_1, ..., \alpha_p$ be an increasing sequence in $\{1, ..., l\}$, $\alpha_1^c, ..., \alpha_{l-p}^c$ be its complement. Let $m_1, ..., m_l \in \mathbb{Z}_+$ such that $m_{\alpha_i} \geq 1, i = 1, ..., p; m_{\alpha_j^c} < -1, j = 1, ..., l-p$. Then

$$\text{Ind}(L(m_1) \cdots L(m_l)L(-n_1) \cdots L(-n_s)L(-1)^n) \leq n + p.$$ 

Moreover, for fixed $m_{\alpha_1^c} < -1, ..., m_{\alpha_{l-p}^c} < -1$, there exists only finitely many choices of $m_{\alpha_1} \geq -1, ..., m_{\alpha_p} \geq 1$, such that

$$\text{Ind}(L(m_1) \cdots L(m_l)L(-n_1) \cdots L(-n_s)L(-1)^n) \geq n.$$

**Proof.** We proceed with induction of $p$. For the base case $p = 1$, we know from Proposition 5.2

that

$$\text{Ind}(L(m_1)L(m_{\alpha_p})L(m_{\alpha_p+1}) \cdots L(m_l)L(-n_1) \cdots L(-n_s)L(-1)^n) = \text{Ind}(L(m_{\alpha_p})L(m_{\alpha_p+1}) \cdots L(m_l)L(-n_1) \cdots L(-n_s)L(-1)^n) \leq n + 1.$$

The index is greater than or equal to $n$ only for finitely many $m_{\alpha_p} \geq -1$. So the base case is proved.

For the general $p$, we start by rewriting

$$L(m_1) \cdots L(m_{\alpha_p-1})L(m_{\alpha_p})L(m_{\alpha_p+1}) \cdots L(m_l)L(-n_1) \cdots L(-n_s)L(-1)^n = \sum_{q \geq 0} \sum_{l \geq 0} \sum_{q_1 \geq q_2, q \geq 2} a_{q_1 \cdots q_s} \cdot L(-n_1) \cdots L(-n_s)L(-1)^q.$$

From the knowledge of the base case,

$$\max\{q | a_{q_1 \cdots q_s} \neq 0\} \leq n + 1$$

with $n + 1$ achieved only for finitely many choices of $m_{\alpha_p}$. Now among $m_1, ..., m_{\alpha_p-1}$, only $p - 1$ of them are greater than $-1$. Induction hypothesis applies, showing that the index is no larger than $n + 1 + p - 1 = n + p$. Moreover, only finitely many $m_{\alpha_1} \geq -1, ..., m_{\alpha_{p-1}} \geq -1$ make the index to be greater than or equal to $\max\{q | a_{q_1 \cdots q_s} \neq 0\}$, while $\max\{q | a_{q_1 \cdots q_s} \neq 0\} \geq n$ for only finitely many $m_{\alpha_p} \geq -1$. Thus the index is greater than or equal to $n$ for finitely many $m_{\alpha_1}, ..., m_{\alpha_p} \geq -1$. □

**Remark 5.5.** Since

$$\text{Ind}(w) < N \Rightarrow \pi_{W_2}(w) = 0,$$

Corollary 5.4 shows that for fixed $m_{\alpha_1^c}, ..., m_{\beta_1^c} < -1$ and $n_1, ..., n_s \geq 2$, if $n < N$, then only finitely many $m_{\alpha_1} \geq -1, ..., m_{\alpha_p} \geq -1$ makes

$$\pi_{W_2}(L(m_1) \cdots L(m_l)L(-n_1) \cdots L(-n_s)L(-1)^n) \neq 0.$$
To conclude the convergence we need for the composability condition, we also need the following fact regarding $\pi_{W_2}$.

**Proposition 5.6.** For every $w \in W$ and every integer $m \geq 2$,

$$\pi_{W_2}(L(-m)w) = L(-m)\pi_{W_2}(w).$$

**Proof.** It suffices to consider

$$w = L(-n_1) \cdots L(-n_s)L(-1)^n$$

with $n_1 \geq \cdots \geq n_s \geq 2$ and $n \geq N$ (when $n < N$ the conclusion holds trivially). Clearly, in the universal enveloping algebra of the Virasoro algebra, the standard expression of $L(-m)L(-n_1) \cdots L(-n_s)$ does not have any summands that contains factors of $L(-1)$. In other words,

$$L(-m)L(-n_1) \cdots L(-n_s) = \sum_{t \geq 0, p_1 \geq \cdots \geq p_t \geq 2} b_{p_1 \cdots p_t} L(-p_1) \cdots L(-p_s).$$

Now we argue by induction. For $n = N$, we have

$$\pi_{W_2}(L(-m)L(-n_1) \cdots L(-n_s)L(-1)^N) = \pi_{W_2} \left( \sum_{t \geq 0, p_1 \geq \cdots \geq p_t \geq 2} b_{p_1 \cdots p_t} L(-p_1) \cdots L(-p_s)L(-1)^N \right)$$

$$= \sum_{t \geq 0, p_1 \geq \cdots \geq p_t \geq 2} b_{p_1 \cdots p_t} L(-p_1) \cdots L(-p_s)s_{h_1}$$

$$= L(-m)L(-n_1) \cdots L(-n_s)s_{h_1}$$

$$= L(-m)\pi_{W_2}(L(-n_1) \cdots L(-n_s)L(-1)^N),$$

where the second and fourth equal signs follow from the definition of $\pi_{W_2}$, the third equal sign follows from the equality in the universal enveloping algebra. For general $n$, we have

$$\pi_{W_2}(L(-m)L(-n_1) \cdots L(-n_s)L(-1)^n) = \pi_{W_2} \left( L(-m)L(-n_1) \cdots L(-n_s)L(-1)^{n-N}s_{h_1} \right)$$

$$- \pi_{W_2} \left( L(-m)L(-n_1) \cdots L(-n_s)L(-1)^{n-N} \sum_{L \leq N, M_1 \geq \cdots \geq M_L \geq 2 \atop M_1 + \cdots + M_L = N} a_{M_1 \cdots M_L} L(M_1) \cdots L(-M_L) \right)$$

From the definition of $\pi_{W_2}$, the first term is precisely

$$L(-m)L(-n_1) \cdots L(-n_s)L(-1)^{n-N}s_{h_1} = L(-m)\pi_{W_2}(L(-n_1) \cdots L(-n_s)L(-1)^{n-N}L(-1)^N).$$

For the second sum, since the standard expression of

$$L(-n_1) \cdots L(-n_2)L(-1)^{n-N}L(-M_1) \cdots L(-M_L)$$

for each fixed $M_1, ..., M_L$ has index strictly smaller than $n$, the induction hypothesis applies, showing that

$$\pi_{W_2}(L(-m)L(-n_1) \cdots L(-n_2)L(-1)^{n-N}L(-M_1) \cdots L(-M_L))$$
for every $M_1, ..., M_L$ appearing in the sum. The conclusion then follows.

\[= L(-m)\pi_{W_2}(L(-n_1) \cdots L(-n_2)L(-1)^{n-N}L(-M_1) \cdots L(-M_L))\]

Remark 5.7. In general we do not have $\pi_{W_2}(L(m)w) = L(m)\pi_{W_2}(w)$ for $m \geq -1$. The simplest counter-example is $m = -1, w = L(-1)^{N-1}$. Left-hand-side is nonzero while the right-hand-side is zero.

Proposition 5.8. Fix $w_1 \in W_1$. Then

$$\pi_{W_2}Y_W(\omega, x)w_1 \in W_2[x, x^{-1}].$$

Moreover, $\pi_{W_2}Y_W(\omega, x)w_1 \neq 0$ only when $\text{Ind}(w_1) = N - 1$.

Proof. We focus on a basis vector

$$w_1 = L(-n_1) \cdots L(-n_s)L(-1)^n, n_1 \geq \cdots \geq n_s \geq 2, n < N.$$ 

Then

$$\pi_{W_2}Y_W(\omega, x)w_1 = \sum_{m \in \mathbb{Z}} \pi_{W_2}L(m) \cdot L(-n_1) \cdots L(-n_s)L(-1)^n x^{-m-2}.$$

We will proceed with induction on $s$.

The base case is $s = 0$. We show that in this case, $\pi_{W_2}Y_W(\omega, x)L(-1)^n$ is nonzero only when $n = N - 1$. In fact,

$$\pi_{W_2}Y_W(\omega, x)L(-1)^n = \sum_{m < -1} \pi_{W_2}L(m)L(-1)^n x^{-m-2} + L(-1)^{n+1} x^{-1} + \sum_{m > -1} \pi_{W_2}L(m)L(-1)^n x^{-m-2}.$$

Since $n < N$, the first sum is zero. The second term is nonzero only when $n + 1 = N$, i.e., $n = N - 1$. For the third sum, we know that $L(m)L(-1)^n$ is a linear combination of $L(-1)^t$ with $t < n$. Thus the third sum is constantly zero. Therefore we conclude that

$$\pi_{W_2}Y_W(\omega, x)L(-1)^n = \begin{cases} sh_1 x^{-1} & \text{if } n = N - 1, \\ 0 & \text{otherwise.} \end{cases}$$

Thus $\pi_{W_2}Y_W(\omega, x)L(-1)^n \in W_2[x, x^{-1}]$. It is nonzero only when $n = N - 1$.

Now we consider the case with a general $s$. In this case,

$$\pi_{W_2}Y(\omega, x)w = \sum_{m \in \mathbb{Z}} \pi_{W_2}L(m)L(-n_1) \cdots L(-n_s)L(-1)^n x^{-m-2}$$

$$= \sum_{m < -1} \pi_{W_2}L(m)L(-n_1) \cdots L(-n_s)L(-1)^n x^{-m-2} \tag{44}$$

$$+ \pi_{W_2}L(-1)L(-n_1) \cdots L(-n_s)L(-1)^n x^{-1} \tag{45}$$

$$+ \sum_{m > -1} \pi_{W_2}L(m)L(-n_1) \cdots L(-n_s)L(-1)^n x^{-m-2}. \tag{46}$$

For (44), we see that if we express each summand as a linear combination of basis vectors of $W$, then the number of $L(-1)$’s is strictly less than $N$. Therefore the (44) is zero.
For \( \textbf{[15]} \), notice that
\[
\pi_{W_2}L(-1)L(-n_1) \cdots L(-n_s)L(-1)^n = \sum_{i=1}^{s} L(-n_1) \cdots [L(-1), L(-n_i)] \cdots L(-n_s)L(-1)^n
\]
\[
+ L(-n_1) \cdots L(-n_s)L(-1)^{n+1}.
\]
Thus in the final expression as a linear combination of basis vectors in \( W \), the number of \( L(-1) \)'s in the first sum is strictly \( n \) which is less than \( N \), while the number of \( L(-1) \)'s in the second sum increases by 1. Thus \( \textbf{[45]} \) is simply
\[
\pi_{W_2}(L(-n_1) \cdots L(-n_s)L(-1)^{n+1})x^{-1}
\]
which is nonzero only when \( n + 1 = N \).

For \( \textbf{[16]} \), notice that for each \( m > -1 \)
\[
\sum_{m > -1} \pi_{W_2}L(m)L(-n_1)L(-n_2) \cdots L(-n_s)L(-1)^n x^{-m-2}
\]
\[
= \sum_{m > -1} \pi_{W_2}L(-n_1)L(m)L(-n_2) \cdots L(-n_s)L(-1)^n x^{-m-2}
\] (47)
\[
+ \sum_{m > -1} \pi_{W_2}(m + n_1)L(m - n_1)L(-n_2) \cdots L(-n_s)L(-1)^n x^{-m-2}
\] (48)
\[
+ \sum_{m > -1} \pi_{W_2}\delta_{m,n_1} \frac{m^3 - m}{12}L(-n_2) \cdots L(-n_s)L(-1)^n x^{-m-2}.
\] (49)
Clearly, \( \textbf{[19]} \) is zero by definition of \( \pi_{W_2} \). To study \( \textbf{[47]} \) and \( \textbf{[48]} \), note that the induction hypothesis yields that
\[
\pi_{W_2}Y(\omega, x)L(-n_2) \cdots L(-n_s)L(-1)^n = \sum_{m > -1} \pi_{W_2}L(m)L(-n_2) \cdots L(-n_s)L(-1)^n x^{-m-2}
\]
is a finite Laurent polynomial in \( x \), nonzero only when \( n = N - 1 \). In particular, there exists only finitely many \( m \), such that the number of \( L(-1) \)'s in the final expression of
\[
L(m)L(-n_2) \cdots L(-n_s)L(-1)^n,
\] (50)
is as large as \( N \). This shows that \( \textbf{[48]} \) is a finite sum. Since adding \( L(-n_1) \) in the front of \( \textbf{[50]} \) does not change the number of \( L(-1) \)'s in the final expression, we see that \( \textbf{[47]} \) is also a finite sum. Therefore, we conclude that
\[
\pi_{W_2}Y(\omega, x)L(-n_1) \cdots L(-n_s)L(-1)^n
\]
is a finite sum of elements in \( W_2 \) together with certain power of \( x \), i.e.,
\[
\pi_{W_2}Y(\omega, x)L(-n_1) \cdots L(-n_s)L(-1)^n \in W_2[x, x^{-1}].
\]
\[\square\]

**Theorem 5.9.** Fix \( w_1 \in W_1 \). Then for every \( l \in \mathbb{Z}_+ \), every \( p = 1, \ldots, l \), every increasing sequence \( \alpha_1, \ldots, \alpha_p \) in \( \{1, \ldots, l\} \), there exists a Laurent polynomial
\[
f_{\alpha_1 \cdots \alpha_p}(x_1, \ldots, x_n) \in W_2[x_{\alpha_1}, x_{\alpha_1}^{-1}, \ldots, x_{\alpha_p}, x_{\alpha_p}^{-1}],
\]
together with
\[ v^{(1)}_{\alpha_1, \ldots, \alpha_p}, \ldots, v^{(l-p)}_{\alpha_1, \ldots, \alpha_p} \in \text{span}\{L(-1)^q \omega | q \in \mathbb{N}\} \]
and integers
\[ P^i_{\alpha_1, \ldots, \alpha_p} \in \mathbb{N}, i = 1, \ldots, p, j = 1, \ldots, l-p, \]
such that
\[ \pi_{W_2} Y^l_1(\omega, x) \cdots Y^l_1(\omega, x)w_1 = \sum_{p=1}^{l} \sum_{1 \leq \alpha_1 < \cdots < \alpha_p \leq l} Y^l_1(v^{(1)}_{\alpha_1, \ldots, \alpha_p}, x_{\alpha_1^1}) \cdots Y^l_1(v^{(l-p)}_{\alpha_1, \ldots, \alpha_p}, x_{\alpha_p^{l-p}}) \ell_1 \cdots n \left( \frac{f_{\alpha_1, \ldots, \alpha_p}(x_1, \ldots, x_n)}{\prod_{i=1}^{p} \prod_{j=1}^{l-p} (x_{\alpha_i^1} - x_{\alpha_j^1})^{P^i_{\alpha_1, \ldots, \alpha_p}}} \right). \]
Here \( Y^l_1(\omega, x) = \sum_{m=-1}^{l} L(m)x^{-m-2} \) is the regular part of \( Y^l_1(\omega, x) \), \( \alpha_1^1, \ldots, \alpha_{l-p}^1 \) is the complement of \( \alpha_1, \ldots, \alpha_p \), \( \ell_1, \ldots, n \) is the formal series expansion that expands negative powers of \( x_i - x_j \) as a power series of \( x_j \) for every \( 1 \leq i < j \leq n \).

**Proof.** We will proceed with induction on \( l \). The base case \( l = 1 \) has been proved in Proposition [5.8]. Here we argue the induction step. Again we focus on a basis vector
\[ w_1 = L(-n_1) \cdots L(-n_s)L(-1)^n, n_1 \geq \cdots \geq n_s \geq 2, n < N. \]
Then
\[ \pi_{W_2} Y^l_1(\omega, x) \cdots Y^l_1(\omega, x)w_1 = \sum_{m_1, \ldots, m_l \in \mathbb{Z}} \pi_{W_2} L(m_1) \cdots L(m_l) \cdot L(-n_1) \cdots L(-n_s)L(-1)^n x_1^{-m_1-2} \cdots x_l^{-m_l-2} \]
If \( m_1 < -1, \ldots, m_l < -1 \), the index of \( L(m_1) \cdots L(m_l)L(-n_1) \cdots L(-n_s)L(-1)^n \) is strictly less than \( N \). Thus the image of \( \pi_{W_2} \) is then zero.

Now let’s study the terms where not all \( m_i \)'s are strictly less than \( 1 \). More precisely, we fix some \( p \in \{1, \ldots, l\} \), some increasing sequence \( \alpha_1, \ldots, \alpha_p \) together with its complement \( \alpha_1^c, \ldots, \alpha_{l-p}^c \).
We will study the partial sum
\[ \sum_{m_1, \ldots, m_{l-p} \geq 1} \sum_{1 \leq \alpha_i < \cdots < \alpha_j \leq l} \pi_{W_2} L(m_1) \cdots L(m_l) \cdot L(-n_1) \cdots L(-n_s)L(-1)^n x_1^{-m_1-2} \cdots x_l^{-m_l-2} \]
Introduce a total-ordering among the increasing sequences in \( \{1, \ldots, l\} \) of length \( p \): we say that
\[ (\alpha_1, \ldots, \alpha_p) < (\beta_1, \ldots, \beta_p) \]
if
\[ \exists i \in [1, l], \alpha_i < \beta_i, \alpha_{i+1} = \beta_{i+1}, \ldots, \alpha_p = \beta_p. \]
We first show the following intermediate result:

**Claim.** ([51]) is of the form
\[ Y^l_1(\omega, x_1) \cdots Y^l_1(\omega, x_{l-p})^l_{l-p+1 \cdots l}(x_{l-p+1}, \ldots, x_l) \]
+ \sum_{q > p} \sum_{1 \leq \beta_1 < \cdots < \beta_q \leq l} Y_W^+(v_{\beta_1 \cdots \beta_q}) \cdot Y_W^+(v_{\beta_1 \cdots \beta_q}) \cdot \cdots \cdot Y_W^+(v_{\beta_1 \cdots \beta_q}) \cdot f_{\beta_1 \cdots \beta_q}(x_{\beta_1}, \ldots, x_{\beta_q}) \cdot \prod_{i=1}^{q} x_{\beta_i} \cdot x_{\beta_q}^{p \cdot j \cdots \beta_q} \\cdot \prod_{j=1}^{l-q} \left( x_{\beta_i} - x_{\beta_j} \right) \cdot p \cdot j \cdots \beta_q \cdot \beta_1 \cdots \beta_q \tag{52}

We start by considering the case where the increasing sequence is the largest among those of length \( p \), i.e., \( (\alpha_1, \ldots, \alpha_p) = (l - p + 1, \ldots, l) \). Using Proposition 5.0, we see that \( \text{(51)} \) is simply

\[
\sum_{m_i < -1} L(m_1) \cdots L(m_i) x_1^{-m_i - 2} \cdots x_1^{-m_i - 2} \left( \sum_{m_j \geq -1} \pi W_2 L(m_{l-p+1}) \cdots L(m_l) x_1^{-m_l - 2} \right) \tag{53}
\]

From Corollary 5.4, the series in the parenthesis is a finite sum and thus can be represented by a Laurent polynomial \( f_{l-p+1,l}(x_{l-p+1}, \ldots, x_l) \in W_2[x_{l-p+1}, x_{l-p+1}, \ldots, x_l, x_{l-1}] \). The summation outside the parenthesis is precisely \( Y_W^+(\omega, x_1) \cdots Y_W^+(\omega, x_{l-p}) \). Thus we see that in this case, \( \text{(51)} \) is of the form

\[
Y_W^+(\omega, x_1) \cdots Y_W^+(\omega, x_{l-p}) f_{l-p+1,l}(x_1, \ldots, x_n).
\]

Now fix a general increasing sequence \( (\alpha_1, \ldots, \alpha_p) < (l - p + 1, \ldots, l) \). We assume the claim is proved for every larger increasing sequence in \( \{1, \ldots, l\} \). We also assume that the claim is proved for every increasing sequence in every proper subset of \( \{1, \ldots, l\} \).

Pick \( k \) in \( [1, l] \), such that \( \alpha_k + 1 \) is not a member of the increasing sequence. Clearly, such an \( k \) exists, with either \( \alpha_k + 1 < \alpha_{k+1} \) or \( k = p, \alpha_p < l \). In the first case, \( (\alpha_1, \ldots, \alpha_p) < (\alpha_1, \ldots, \alpha_{k-1}, \alpha_k + 1, \alpha_{k+1}, \ldots, \alpha_p) \). In the second case, \( (\alpha_1, \ldots, \alpha_p) < (\alpha_1, \ldots, \alpha_{p-1}, \alpha_p + 1) \). Using the commutator formula of \( L(m_{\alpha_k}) \) and \( L(m_{\alpha_{k+1}}) \), we rewrite \( \text{(51)} \) as

\[
\sum_{m_\alpha < -1} \sum_{m_\alpha \geq -1} \pi W_2 L(m_1) \cdots L(m_{\alpha_{k+1}}) L(m_{\alpha_k}) \cdots L(m_l) \cdot w_1 x_1^{-m_1 - 2} \cdots x_l^{-m_l - 2} \tag{54}
\]

\[
+ \sum_{m_\alpha < -1} \sum_{m_\alpha \geq -1} (m_{\alpha_{k+1}} - m_{\alpha_{k+1}}) \pi W_2 L(m_1) \cdots L(m_{\alpha_{k+1}} + m_{\alpha_{k+1}}) \cdots L(m_l) \cdot w_1 x_1^{-m_1 - 2} \cdots x_l^{-m_l - 2} \tag{55}
\]

It suffices to check individually that \( \text{(53)}, \text{(54)} \) and \( \text{(55)} \) are of the form \( \text{(52)} \).

- For \( \text{(53)} \), the claim follows immediately from the induction hypothesis since \( (\alpha_1, \ldots, \alpha_{k-1}, \alpha_k + 1, \alpha_{k+1}, \ldots, \alpha_p) \) is larger.
- For \( \text{(54)} \), we single out the part

\[
\sum_{m_\alpha < -1} \sum_{m_\alpha \geq -1} (m_{\alpha_{k+1}} - m_{\alpha_{k+1}}) \pi W_2 L(m_1) \cdots L(m_{\alpha_{k+1}} + m_{\alpha_{k+1}}) \cdots L(m_l) \cdot w_1 x_\alpha^{-m_\alpha - 2} x_{\alpha_{k+1}}^{-m_{\alpha_{k+1}} - 2} \tag{56}
\]
and reparametrize: set \( m' = m_{\alpha_k} + m_{\alpha_k+1} \). Notice that
\[
\sum_{m_{\alpha_k+1} < -1} \sum_{m_{\alpha_k} \geq -1} = \sum_{m' \geq 1} \sum_{m_{\alpha_k} \geq m' - 2} + \sum_{m' = -2} \sum_{m_{\alpha_k} \geq 0} + \sum_{m' = -1} \sum_{m_{\alpha_k} \geq 1} \tag{56}
\]
while the summands becomes
\[
(2m_{\alpha_k} - m') \pi_{W_2} L(m_1) \cdots L(m_{\alpha_k-1}) L(m') L(m_{\alpha_k+2}) \cdots L(m_t) \cdot w_1 x_{\alpha_k}^{-m_{\alpha_k} - 2} x_{\alpha_k+1}^{-m_{\alpha_k+1}} - 2m' + m_{\alpha_k} - 2
\]
\[
= (2m_{\alpha_k} - m') \pi_{W_2} L(m_1) \cdots L(m_{\alpha_k-1}) L(m') L(m_{\alpha_k+2}) \cdots L(m_t) \cdot w_1 x_{\alpha_k}^{-m_{\alpha_k} - 2} x_{\alpha_k+1}^{-m_{\alpha_k+1}} - 2m' - m_{\alpha_k} - 2
\]

We will handle the three summations to the right-hand-side of (56) individually and show that they are of the form (52).

For first summation on the right-hand-side of (56), using the general fact that
\[
\sum_{n \geq m' + 2} (2 - m') y_1^{-n-2} y_2^n = 2 y_2 \frac{\partial}{\partial y_2} \left( \sum_{n \geq m' + 2} y_1^{-n-2} y_2^n \right) - m' \sum_{n \geq m' + 2} y_1^{-n-2} y_2^n
\]
\[
= 2 y_2 \frac{\partial}{\partial y_2} \left( y_1^{-m' - 2} y_2^{m' + 2} y_1 y_2 \left( \frac{y_1}{y_1 - y_2} \right) \right) - m' \left( y_1^{-m' - 2} y_2^{m' + 2} y_1 y_2 \left( \frac{y_1}{y_1 - y_2} \right) \right)
\]

we obtain the following expression:
\[
\sum_{m' \geq 1} (\pi_{W_2} L(m_1) \cdots L(m_{\alpha_k-1}) L(m') L(m_{\alpha_k+2}) \cdots L(m_t) \cdot w_1) \cdot (m' + 2) x_{\alpha_k}^{-m_{\alpha_k} - 2} \cdot \frac{1}{x_{\alpha_k} - x_{\alpha_k+1}}
\]
\[
\sum_{m' \geq 1} (\pi_{W_2} L(m_1) \cdots L(m_{\alpha_k-1}) L(m') L(m_{\alpha_k+2}) \cdots L(m_t) \cdot w_1) \cdot x_{\alpha_k}^{-m_{\alpha_k} - 2} \cdot \frac{2}{(x_{\alpha_k} - x_{\alpha_k+1})^2}. \tag{57}
\]

If we sum up (56) together with \( x_1^{-m_{\alpha_k} - 2} \cdots x_{\alpha_k}^{-m_{\alpha_k} - 2} x_{\alpha_k+2}^{-m_{\alpha_k+2}} \cdots x_t^{-m_t - 2} \) with respect to all other \( m_1, \ldots, m_{\alpha_k-1}, m_{\alpha_k+2}, \ldots, x_t \), from the induction hypothesis with \( l - 1 \) vertex operators, we see that the sum is of the form
\[
Y_{W}^+(\omega, x_1) \cdots Y_{W}^+(\omega, x_{\alpha_k+1}) \cdots Y_{W}^+(\omega, x_{l-p}) f_{l-p+1, \ldots, l} (x_{\alpha_k+1}, \ldots, x_l) \cdot \frac{2}{(x_{\alpha_k} - x_{\alpha_k+1})^2}
\]
\[
+ \sum_{q > p} \sum_{1 \leq 1 \leq q \leq l} Y_{W}^+(v_{\beta_1}^{(1)}, x_{\beta_1}^{(1)}, \ldots, x_{\beta_q}^{(1)}) \cdot \frac{2}{(x_{\alpha_k} - x_{\alpha_k+1})^2}
\]
Here the terms with hat are excluded from the list. Thus this summation is of the form (52).
If we sum up \([57]\) together with \(x_1^{-m_1-2} \cdots x_{\alpha_k-1}^{-m_{\alpha_k-1}-2} x_{\alpha_k+2}^{-m_{\alpha_k+2}} \cdots x_l^{-m_l-2}\) with respect to all other \(m_1, ..., m_{\alpha_k-1}, m_{\alpha_k+2}, ..., x_l\), what we obtain is precisely

\[
- \frac{\partial}{\partial x_{\alpha_k}} \left( Y_{\omega}^+(\omega, x_1) \cdots Y_{\omega}^+(\omega, x_{\alpha_k+1}) \cdots Y_{\omega}^+(\omega, x_{l-p+1}) f_{l-p+1, \cdots, l}(x_{l-p+1}, \cdots, x_{\alpha_k+1}, \cdots, x_l) \right) \cdot \frac{1}{(x_{\alpha_k} - x_{\alpha_k+1})} \cdot \frac{1}{(x_{\alpha_k} - x_{\alpha_k+1})^2}.
\]

But the partial derivative can be incorporated either into the vertex operator by adding an \(L(-1)\) on \(Y_{\omega}^+(\cdot, x_{\alpha_k})\), or into the derivative of the rational function, which remains a rational function of the same form. Thus we showed that the first summation to the right-hand-side of \((57)\) is of the form \((52)\).

For the second summation on the right-hand-side of \((56)\), using the same argument, we obtain the following expression

\[
(\pi_{W_2} L(m_1) \cdots L(m_{\alpha_k-1}) L(-2) L(m_{\alpha_k+2}) \cdots L(m_l) \cdot w_1) \cdot \frac{2}{(x_{\alpha_k} - x_{\alpha_k+1})^2}.
\]

To show the summation of this term together with \(x_1^{-m_1-2} \cdots x_{\alpha_k-1}^{-m_{\alpha_k-1}-2} x_{\alpha_k+2}^{-m_{\alpha_k+2}} \cdots x_l^{-m_l-2}\) with respect to \(m_1, ..., m_{\alpha_k-1}, m_{\alpha_k+2}, ..., m_l\) is also of the form \((52)\), it suffices to focus on the following part

\[
\sum_{m_{\alpha_k} < -1} \sum_{m_{\alpha_j} \geq -1} (\pi_{W_2} L(m_1) \cdots L(m_{\alpha_k-1}) L(-2) L(m_{\alpha_k+2}) \cdots L(m_l) \cdot w_1) \cdot x_1^{-m_1-2} \cdots x_{\alpha_k-1}^{-m_{\alpha_k-1}-2} x_{\alpha_k+2}^{-m_{\alpha_k+2}} \cdots x_l^{-m_l-2}.
\]

Note that aside from \(L(-2)\), the number of \(L(m_i)\) operators involved in the summation involves is \(l - 2\) number of \(L(m_i)\) operators aside from \(L(-2)\). We will prove the following slightly stronger conclusion: if for any \(p \in [1, l - 2]\), any increasing sequence \(\alpha_1, ..., \alpha_p\) in \(\{1, ..., l - 2\}\), and every \(w_i \in W_1\), the series

\[
\sum_{m_{\alpha_i} < -1} \sum_{m_{\alpha_j} \geq -1} (\pi_{W_2} L(m_1) \cdots L(m_{l-2}) \cdot w_1) \cdot x_1^{-m_1-2} \cdots x_{l-2}^{-m_{l-2}-2}
\]

is of form \((52)\), then so is the series

\[
\sum_{m_{\alpha_i} < -1} \sum_{m_{\alpha_j} \geq -1} (\pi_{W_2} L(m_1) \cdots L(m_i) L(m_{i+1}) L(m_{l-2}) \cdot w_1) \cdot x_1^{-m_1-2} \cdots x_{l-2}^{-m_{l-2}-2}
\]

for every \(m' \leq -2\) and every \(i = 1, ..., l - 2\).
If $i = l - 2$, then it follows from Proposition 5.6 that $L(-m')w_1 \in W_1$. The conclusion follows directly. For $i < l - 2$, we assume the conclusion holds for every larger $i$ and for every $i$ with smaller $l$.

If $i + 1$ is not a member of the increasing sequence, then (59) is
\[
\sum_{m_{i+1} < -1 \atop i=1, \ldots, l-2-p} \sum_{m_{i+1} \geq -1} (\pi W_2 L(m_1) \cdots L(m_i)L(m_i+1) L(m_{l-2}) \cdot w_1) \cdot x_1^{-m_{i+1} -2} \cdots x_{l-2}^{-m_{l-2} -2} \\
+ \sum_{m_{i+1} < -1 \atop i=1, \ldots, l-2-p} \sum_{m_{i+1} \geq -1} (m' - m_{i+1}) (\pi W_2 L(m_1) \cdots L(m_i)L(m_i+1 + m') L(m_{l-2}) \cdot w_1) \cdot x_1^{-m_{i+1} -2} \cdots x_{l-2}^{-m_{l-2} -2}
\]

The first sum is taken care by the induction hypothesis. For the second sum, we focus on the summation with respect to $m_{i+1}$ and reparametrize:
\[
\sum_{m_{i+1} < -1} (m' - m_{i+1}) L(m_{i+1} + m') x_{i+1}^{-m_{i+1} -2} \\
= \sum_{m_{i+1} < -1 \atop m' < -1 + m'} (2m' - m_{i+1}) L(m_{i+1}) x_{i+1}^{-m_{i+1} +m' -2} \\
= \sum_{m_{i+1} < -1} (2m' - m_{i+1}) L(m_{i+1}) x_{i+1}^{-m_{i+1} +m' -2} - \sum_{-1+m' < m_{i+1} < -1} (2m' - m_{i+1}) L(m_{i+1}) x_{i+1}^{-m_{i+1} +m' -2}
\]

The first sum is a linear combination of $\sum_{m_{i+1} < -1} L(m_{i+1}) x_{i+1}^{-m_{i+1} -2}$ and its derivative (with a correction power of $x_{i+1}$ sitting in the coefficient). Thus for the total summation, using the induction hypothesis together with $L(-1)$-derivative property, we see that it is of form (52). The second sum is finite. Using the induction hypothesis, we see that the total sum is a finite sum of series of form (52).

If $i + 1$ is a member of the increasing sequence, then (59) is
\[
\sum_{m_{i+1} < -1 \atop i=1, \ldots, l-2-p} \sum_{m_{i+1} \geq -1} (\pi W_2 L(m_1) \cdots L(m_i)L(m_i+1) L(m_{l-2}) \cdot w_1) \cdot x_1^{-m_{i+1} -2} \cdots x_{l-2}^{-m_{l-2} -2} \\
+ \sum_{m_{i+1} < -1 \atop i=1, \ldots, l-2-p} \sum_{m_{i+1} \geq -1} (m' - m_{i+1}) (\pi W_2 L(m_1) \cdots L(m_i)L(m_i+1 + m') L(m_{l-2}) \cdot w_1) \cdot x_1^{-m_{i+1} -2} \cdots x_{l-2}^{-m_{l-2} -2} \\
+ \sum_{m_{i+1} < -1 \atop i=1, \ldots, l-2-p} \sum_{m_{i+1} \geq -1} \delta_{m'+m_{i+1},0} \frac{(m')^3 - m'}{12} c (\pi W_2 L(m_1) \cdots L(m_i)L(m_{l-2}) \cdot w_1) \cdot x_1^{-m_{i+1} -2} \cdots x_{l-2}^{-m_{l-2} -2}
\]

Again the first sum is taken care by the induction hypothesis. For the second sum, we focus on the summation with respect to $m_{i+1}$ and reparametrize similarly
\[
\sum_{m_{i+1} \geq -1} (m' - m_{i+1}) L(m_{i+1} + m') x_{i+1}^{-m_{i+1} -2} \\
= \sum_{m_{i+1} \geq -1 \atop m' < -1 + m'} (2m' - m_{i+1}) L(m_{i+1}) x_{i+1}^{-m_{i+1} +m' -2}
\]
We analyze the total sums associated with these two sums similarly (for the first sum, ...). Details shall not be repeated here. For the third sum, the summation with respect to \( m_{i+1} \) is simply
\[
\sum_{m_{i+1} \geq -1} \delta_{m'+m_{i+1},0} \frac{(m')^3 - m' c_{x_{i+1}}^{-m_{i+1}} - 2}{12} = \frac{(m')^3 - m' c_{x_{i+1}}^{-m_{i+1}} - 2}{12}.
\]

It follows from the induction hypothesis with smaller \( l \) that the total sum associated with this line is of form \([52]\).

> For the third summation of the \([50]\), using the general fact that
\[
\sum_{n \geq -1} (2n - m') y_1^{-n-2} y_2^n
\]
\[
= (-2 - m') y_1^{-1} y_2^{-1} + 2y_2 \frac{\partial}{\partial y_2} \left( \sum_{n \geq 0} y_1^{-n-2} y_2^n \right) - m' \sum_{n \geq 0} y_1^{-n-2} y_2^n
\]
\[
= (-2 - m') y_1^{-1} y_2^{-1} + 2y_2 \frac{\partial}{\partial y_2} \left( \frac{y_1^{-2} - y_1 y_2}{y_1 - y_2} \right) - m' y_1^{-2} y_1 y_2 \frac{y_1}{y_1 - y_2}
\]
\[
= (-2 - m') y_1^{-1} y_2^{-1} + 2y_1^{-1} y_2 \left( \frac{1}{(y_1 - y_2)^2} \right) - m' y_1^{-1} y_1 y_2 \frac{1}{y_1 - y_2},
\]
we obtain the following expression:
\[
\frac{\partial}{\partial x_{\alpha_{k+1}}} \left( \sum_{m' < -1} \pi_{W_2} L(m_1) \cdots L(m_{\alpha_k-1}) L(m_{\alpha_k+1}) L(m_{\alpha_k+2} \cdots L(m_l) \cdot w_1 x_{\alpha_{k+1}}^{-m_{l+2}} \right) \cdot \frac{1}{x_{\alpha_k} - x_{\alpha_{k+1}}^{m-1}}
\]
\[
+ \frac{2}{(x_{\alpha_k} - x_{\alpha_{k+1}})^2}
\]
(60)

By an argument similar to those for \([57]\) and \([58]\), we see that both \([60]\) and \([61]\) are of form \([52]\). For brevity, we will not repeat the details here.

> Thus we verified \([54]\) is of the form \([52]\).

- For \([55]\), we rewrite it as
\[
\sum_{m_{\alpha_k} > -1} \sum_{m_{\alpha_k} > -1} \sum_{i=1, \ldots, l-p} \sum_{j=1, \ldots, p} \frac{m_{\alpha_k}^3 - m_{\alpha_k} - m_{\alpha_k+1}}{12} c \pi_{W_2} L(m_1) \cdots L(m_{\alpha_k-1}) L(m_{\alpha_k+2}) \cdots L(m_l) \cdot w_1 x_{\alpha_k}^{-m_{l+2}} \cdots x_{\alpha_k}^{-m_{l+2}}
\]
\[
= \sum_{m_{\alpha_k} > -1} \sum_{m_{\alpha_k} > -1} \sum_{i=1, \ldots, l-p} \sum_{j=1, \ldots, p} \pi_{W_2} L(m_1) \cdots L(m_{\alpha_k-1}) L(m_{\alpha_k+2}) \cdots L(m_l) \cdot w_1 x_{\alpha_k}^{-m_{l+2}} \cdots x_{\alpha_k}^{-m_{l+2}} \cdot \frac{m_{\alpha_k}^3 - m_{\alpha_k} - m_{\alpha_k+2} - m_{\alpha_k+1}}{12} x_{\alpha_{k+1}}^{-m_{l+2}}
\]
\[
= \sum_{m_{\alpha_k} > -1} \sum_{m_{\alpha_k} > -1} \sum_{i=1, \ldots, l-p} \sum_{j=1, \ldots, p} \frac{m_{\alpha_k}^3 - m_{\alpha_k} - m_{\alpha_k+2} - m_{\alpha_k+1}}{12} x_{\alpha_{k+1}}^{-m_{l+2}}
\]
(61)
Clearly,
\[
\sum_{m_{\alpha_k} \geq -1} \frac{m_{\alpha_k}^3 - m_{\alpha_k} - m_{\alpha_k} - 2 m_{\alpha_k} - 2}{12} \cdot x_{\alpha_k} \cdot x_{\alpha_k+1} = \frac{c}{12} \frac{\partial^3}{\partial x_{\alpha_{k+1}}^3} \sum_{m_{\alpha_k} \geq 2} x_{\alpha_k}^{m_{\alpha_k}-2} x_{\alpha_k+1}^{m_{\alpha_k}+1} = \frac{c}{12} \frac{\partial^3}{\partial x_{\alpha_{k+1}}^3} l_{1\ldots n} \cdot \left( x_{\alpha_k} - x_{\alpha_{k+1}} \right)^3.
\]

Also from the induction hypothesis, the summation regarding the other \( m_{\alpha_i} \) and \( m_{\alpha_j} \) is of form \((52)\). Multiply it by \((62)\), it will still be of the form \((52)\). Thus \((55)\) is also of the form of the form \((52)\).

Therefore, we proved the claim that for every \( p \in \{1, \ldots, l\} \) and every increasing sequence \( \alpha_1, \ldots, \alpha_p \) in \( \{1, \ldots, l\} \), the partial sum \((51)\) is of the form \((52)\). To conclude the proof, note that
\[
\pi_{W_2} Y_W(\omega, x_1) \cdots Y_W(\omega, x_l)
\]
is precisely the sum of \((51)\) with respect to \( p = 1, \ldots, l \) and the increasing sequences \( \alpha_1, \ldots, \alpha_p \) in \( \{1, \ldots, l\} \). Since every summand is of form \((52)\) and the sum is finite, so is the sum. \(\square\)

**Proposition 5.10.** For every \( l_1, l_2 \in \mathbb{N} \), every \( v_1, \ldots, v_m \in V \), every \( w' \in W' \) and \( w \in W \), the series
\[
\langle w', Y_W(v_1, z_1) \cdots Y_W(v_l, z_l) \pi_{W_2} Y_W(\omega, z_{l+1}) \cdots Y_W(\omega, z_{l+1}) w \rangle
\]
converges absolutely in the region
\[
|z_1| > \cdots > |z_l| > |z_{l+1}| > \cdots > |z_{l+1}| > 0
\]
to a rational function whose only possible poles are at \( z_i = 0 \) \((i = 1, \ldots, l_1 + l_2)\), and at \( z_i = z_j \) \((1 \leq i < j \leq l_1 + l_2)\).

**Proof.** Write \( w = w_1 + w_2 \). Then the series in question is written as the following sum
\[
\langle w', Y_W(v_1, z_1) \cdots Y_W(v_l, z_l) \pi_{W_2} Y_W(\omega, z_{l+1}) \cdots Y_W(\omega, z_{l+1}) w_1 \rangle \tag{63}
\]
\[
+ \langle w', Y_W(v_1, z_1) \cdots Y_W(v_l, z_l) \pi_{W_2} Y_W(\omega, z_{l+1}) \cdots Y_W(\omega, z_{l+1}) w_2 \rangle \tag{64}
\]
The convergence of \((64)\) is clear since \( \pi_{W_2} \) can be removed. We focus on \((63)\). Apply Theorem \(5.9\) and rewrite the polynomial
\[
f_{\alpha_1 \cdots \alpha_p}(x_{\alpha_1}, \ldots, x_{\alpha_p}) = \sum_{i_1, \ldots, i_p} w_{\alpha_1 \cdots \alpha_p} x_{\alpha_1}^{i_1} \cdots x_{\alpha_p}^{i_p},
\]
we know that
\[
Y_W(v_1, x_1) \cdots Y_W(v_l, x_l) \pi_{W_2} Y_W(\omega, x_{l+1}) \cdots Y_W(\omega, x_{l+1}) w
\]
is a finite linear combination of series of the form
\[
Y_W(v_1, x_1) \cdots Y_W(v_l, x_l)
\]
Theorem 5.11. For every $l_1, l_2 \in \mathbb{N}$, every $v_1, ..., v_{l_1+l_2} \in V$, every $w' \in W'$ and $w \in W$, the series

$$
\langle w', Y_W(v_1, z_1) \cdots Y_W(v_{l_1}, z_{l_1}) \pi_{l_2} W_2 Y_W(v_{l_1+1}, z_{l_1+1}) \cdots Y_W(v_{l_1+l_2}, z_{l_1+l_2}) w \rangle
$$

(69)

converges absolutely in the region

$$
|z_1| > \cdots > |z_{l_1}| > |z_{l_1+1}| > \cdots > |z_{l_1+l_2}| > 0
$$

(70)

to a rational function whose only possible poles are at $z_i = 0$ ($i = 1, ..., l_1 + l_2$), and at $z_i = z_j$ $(1 \leq i < j \leq l_1 + l_2)$.
Proof. It suffices to focus on the case where
\[ v_{l+i} = L(-n_1^{(i)}) \cdots L(-n_k^{(i)})1. \]
We will argue by induction of \( i \) (from \( l_2 \) to 1) and \( s_i \). For brevity, we only show the inductive step here: suppose the conclusion holds for the series of the form
\[ \langle w', Y_W(v_1, z_1) \cdots Y_W(v_{l+i}, z_{l+i}) \rangle \]
then it also holds for the series of the form
\[ \langle w', Y_W(v_1, z_1) \cdots Y_W(v_{l+i}, z_{l+i}) \rangle \]
for every \( n \geq 2 \). The idea is to use the iterate formula
\[ Y_W(L(-n)v_{l+i}, z_{l+i}) \]
\[ = \text{Res}_z \left( \left( z - z_{l+i} \right)^{-n-1} Y_W(\omega, z) Y(v_{l+i}, z_{l+i}) - \left( z - z_{l+i} \right)^{-n-1} Y(v_{l+i}, z_{l+i}) Y_W(\omega, z) \right) \]
From the induction hypothesis, we know that the series
\[ t_{z,z_{l+i}} \left( z - z_{l+i} \right)^{-n-1} \langle w', Y_W(v_1, z_1) \cdots Y_W(v_{l+i}, z_{l+i}) \rangle \]
converges absolutely in the region
\[ |z_1| > \cdots > |z_{l+i-1}| > |z| > |z_{l+i}| > \cdots > |z_{l+i}+2| > 0 \]
to a rational function \( f_1(z_1, \ldots, z_{l+i+2}, z) \) with poles at \( z = 0, z_i = 0, z_i = z \ (i = 1, \ldots, l_1 + l_2) \), \( z_i = z_j \ (1 \leq i < j \leq l_1 + l_2) \). Applying \( \text{Res}_z \) to \((73)\) amounts to integrating \( f_1(z_1, \ldots, z_{l+i+2}, z) \) along the curve \( z = re^{i\theta}, \theta \in [0, 2\pi] \) with \( |z_{l+i-1}| > r > |z_{l+i}| \). From Cauchy integral theorem, \( \text{Res}_z \) of \((73)\) converges absolutely in the region
\[ |z_1| > \cdots > |z_{l+i-1}| > |z_{l+i}| > \cdots > |z_{l+i}+2| > 0 \]
(which is precisely the region \((70)\)) to the function
\[ g_1(z_1, \ldots, z_{l+i+2}) = \int_{z = re^{i\theta}} f_1(z_1, \ldots, z_{l+i+2}, z) dz \]
\[ = \sum_{j=i}^{l_2} \int_{C(z_{l+i+j})} f_1(z_1, \ldots, z_{l+i+2}, z) dz \]
\[ = \sum_{j=i}^{l_2} \text{Res}_{z=z_{l+i+j}} f_1(z_1, \ldots, z_{l+i+2}, z) dz \]
where for each index \( j \) in the sum, \( \text{Res}_{z=z_{l+i+j}} f_1(z_1, \ldots, z_{l+i+2}, z) \) is the coefficient of \( (z - z_{l+i+j})^{-1} \) in the expansion of the rational function \( f_1(z_1, \ldots, z_{l+i+2}, z) \) where
- negative powers of \( z = z_{l+i+j} + (z - z_{l+i+j}) \) are expanded as a power series in \( (z - z_{l+i+j}) \);
• negative powers of \( z_i - z = (z_i - z_{1+i}) - (z - z_{1+j}) \) are expanded as a power series in \((z - z_{1+j})\), for every \( i \neq 1 + j \).

So each summand in (75) is a rational function with possible poles at \( z_i = 0 \) \((i = 1, ..., l_1 + l_2)\), \( z_i = z_j \) \((1 \leq i < j \leq l_1 + l_2)\). Thus \( g_1(z_1, ..., z_{1+l_2}) \) is also such a rational function.

Using an almost identical argument, we see that \( \text{Res}_z \) of the series

\[
\ell_{z_{1+i}}(-z_{1+i} + z)^{-n-1}(w', Y_W(v_1, z_1) \cdots Y_W(v_l, z_l)\pi_W v_1 Y_W(\omega, z_{l_1+1}) \cdots Y_W(\omega, z_{l_1+i-1}) \\
\cdot Y_W(v_l, z_{l_1+i}) Y_W(\omega, z) \cdots Y_W(v_{l_1+l_2}, z_{l_1+l_2}) w)
\] (76)

converges in the same region to a rational function \( g_2(z_1, ..., z_{1+l_2}) \) with the same possible poles as in \( g_1(z_1, ..., z_{1+l_2}) \). Using the iterate formula, we see that (72) converges absolutely in the region (70) to the rational function \( g_1 + g_2 \) with possible poles at \( z_i = 0 \) \((i = 1, ..., l_1 + l_2)\) and \( z_i = z_j \) \((1 \leq i < j \leq l_1 + l_2)\). So the inductive step is proved.

\[ \square \]

**Corollary 5.12.** For every \( v \in V \), the map \( \pi_{W_2} Y_W(v, x)\pi_W : W_1 \to W_2((x)) \) is in \( \mathcal{H}_N(W_1, W_2) \).

**Proof.** Clearly from Theorem 5.11 for every \( k, l \in \mathbb{N} \), every \( u_1, ..., u_{k+l} \in V \), every \( w'_2 \in W'_2, w_1 \in W_1 \),

\[
\langle w'_2, Y_W(u_1, z_1) \cdots Y_W(u_k, z_k)\pi_W Y_W(v, z) Y_W(u_{k+1}, z_{k+1}) \cdots Y_W(u_{k+l}, z_{k+l}) w_1 \rangle
\] (77)

converges absolutely in the region

\[
|z_1| > \cdots > |z_k| > |z| > |z_{k+1}| > \cdots > |z_{k+l}| > 0
\]

to a rational function \( f(z_1, ..., z_k, z, z_{k+1}, ..., z_{k+l}) \) with the only possible poles at \( z_i = 0 \) \((i = 1, ..., k + l)\), \( z = 0, z_i = z_j \) \((1 \leq i \neq j \leq k + l)\), \( z_i = z \) \((i = 1, ..., k + l)\). To squeeze \( \pi_{W_1} \) in between, we start by rewriting (74) as

\[
\langle w'_2, Y_W(u_1, z_1) \cdots Y_W(u_k, z_k)\pi_W Y_W(v, z)\pi_W Y_W(u_{k+1}, z_{k+1}) \cdots Y_W(u_{k+l}, z_{k+l}) w_1 \rangle \\
+ \langle w'_2, Y_W(u_1, z_1) \cdots Y_W(u_k, z_k)\pi_W Y_W(v, z)\pi_W Y_W(u_{k+1}, z_{k+1}) \cdots Y_W(u_{k+l}, z_{k+l}) w_1 \rangle
\] (78)

Note that since \( W_2 \) is a submodule, \( \pi_{W_2} Y_W(v, z) \pi_{W_2} = Y_W(v, z) \pi_{W_2} \). Thus the convergence of (79) follows from Theorem 5.11. Therefore (78) converges. We can repeat the same process consecutively to squeezing \( \pi_{W_1} \) between the vertex operators and show convergence. Details should not be repeated here.

For the \( N \)-weight-degree condition, we first see from the associativity of vertex operators that the condition holds without \( \pi_{W_2} \) and \( \pi_{W_1} \). Note that inserting \( \pi_{W_2} \) and \( \pi_{W_1} \) does not change the weights of \( v_i \) and the weight of \( \pi_{W_2} Y_W(v, x)\pi_W \). From Theorem 5.9 and the subsequent discussions, it is clear after inserting the projection operators \( \pi_{W_2} \) and \( \pi_{W_1} \), the order of every pole of the resulted rational function does not get larger. Consequently, the lowest possible sum of powers of \( z_i - z_{k+l} \) and \( z - z_{k+l} \) in the expansions of the resulted rational function cannot get lower. Therefore, the \( N \)-weight-degree condition holds for \( \pi_{W_2} Y_W(v, x)\pi_W \). \[ \square \]
Remark 5.13. For the Verma module $M(c, h)$ with $[c, h]$ in the block (d), we currently do not know how to show the convergence when the submodule is generated by two singular vectors. Some preliminary computation shows that the situation is much more complicated than what we observer here.

5.3. General case. From Proposition 4.13 it suffices that we consider the module $\bigoplus_{i=1}^{n} M(c, h_i)$. We start by classifying the submodules.

Proposition 5.14. Let $M_i = M(c, h_i)$ be Verma modules, $i = 1, \ldots, n$. Let $W$ be a submodule of $M = \bigoplus_{i=1}^{n} M_i$. Then $W$ is generated by singular vectors. Moreover, there exists submodules $N_i \subseteq M_i$ for each $i = 1, \ldots, n$, such that $W$ is isomorphic to $\bigoplus_{i=1}^{n} N_i$.

Proof. The idea of the proof is suggested by Kenji Iohara. The conclusion clearly holds if $n = 1$. We proceed by induction. Assume the conclusion holds for smaller $n$, write $M = P \oplus M_n, P = \bigoplus_{i=1}^{n-1} M_i$. Consider the projection operator $\pi_P : M = P \oplus M_n \to P$. Then $\pi_P$ is a homomorphism of $V$-modules. Clearly, the sequence

$$0 \to W \cap M_n \to W \to \pi_P W \to 0$$

is exact.

If $\pi_P W = 0$, then $W \subseteq M_h$. The conclusion follows from the base case (certainly it is possible that $W = 0$). We simply take $N_1 = \cdots = N_{n-1} = 0, N_n = W$. If $W \cap M_n = 0$, then $\pi_P$ is an isomorphism. $\pi_P^{-1}$ sends the generators of $\pi_P W$ (which are singular vectors by the induction hypothesis) to generators of $W$. We take $N_1, \ldots, N_{n-1}$ from induction hypothesis applied to $\pi_P W$, and take $N_n = 0$.

We focus on the case when $W \cap M_n$ and $\pi_P W$ are both nonzero. From the induction hypothesis, $\pi_P W$, as a submodule of $P$, is also generated by singular vectors. Let $u_1, \ldots, u_{l-1}$ be singular vectors generating $\pi_P W$. Let $U_1, \ldots, U_{l-1}$ be submodules generated by $u_1, \ldots, u_{l-1}$ in $\pi_P W$. Let $w_1, \ldots, w_{l-1} \in W$ such that $\pi_P w_i = u_i, i = 2, \ldots, l$. Let $h$ be the lowest weight of $W$. Restricting the exact sequence to the weight-$h$ homogeneous subspace, we have

$$0 \to (W \cap M_n)_h \to W_h \to (\pi_P W)_h \to 0.$$

Consider the following cases:

(1) $(\pi_P W)_h \neq 0$. Fix any $i$ such that $\text{wt}(u_i) = h$. Necessarily, $w_i \in W_h$. Since $h$ is the lowest weight of $W$, $L(n)w_i = 0$ for every $n \in \mathbb{Z}_+$. Then $w_i - u_i = w_i - \pi_P w_i$ is a singular vector in $M_n$.

In case $(W \cap M_n)_h \neq 0$. Then since $h$ is the lowest weight of $W$, there exists a singular vector $w_n \in W \cap M_n$ of weight $h$. From the uniqueness of singular vectors, we know that $w_i - u_i$ is a scalar multiple $w_n$. In particular, we see that $u_i$ is an element of $W$. Now fix any $j$ such that $\text{wt}(u_j) > h$. We show that $u_j \in W$. We first note that
of i.e., any submodule of $Z$ for every $(\ldots)$. If $u_j \notin W$, then $w_j - u_j \notin W$ while $w_j - u_j \in M_n$. So there exists a submodule of $M_n$ containing $w_j - u_j$. Let $Z$ be the minimal submodule of $M_n$ containing $w_j - u_j$, i.e., any submodule of $Z$ does not contain $w_j - u_j$. Then $L(m)(w_j - u_j) \in Z \cap W \cap M_n$ for every $m \in \mathbb{Z}_+$. 

(a) If $Z$ is generated by one singular vector, then $Z$ is a Verma module which has a unique irreducible quotient. From the choice of $Z$, we know that $w_j - u_j$ is nonzero in the irreducible quotient of $Z$. However for every $m \in \mathbb{Z}_+$, since $L(m)(w_j - u_j)$ is an element of $Z \cap W \cap M_n$, a proper submodule of $Z$, it follows that $L(m)(w_j - u_j)$ is zero in the irreducible quotient. Consequently, $w_j - u_j$ is zero in the irreducible quotient, a contradiction. So in this case, $u_j \in W$.

(b) If $Z$ is generated by two singular vectors, write $Z = Z_1 + Z_2$ where $Z_1, Z_2$ are Verma modules. Then $w_j - u_j \notin Z_1$ and $w_j - u_j \notin Z_2$. From the second fundamental theorem of homomorphisms, we see that $w_j - u_j$ is nonzero in the irreducible quotients of both $Z_1$ and $Z_2$. If $W \cap M_n \subset Z_1 \cap Z_2$, then for every $m \in \mathbb{Z}_+$, $L(m)(w_j - u_j)$ is zero in the irreducible quotient of both $Z_1$ and $Z_2$. This leads to a similar contradiction, showing $u_j \in W$. If $W \cap M_n$ is not a subset of $Z_1 \cap Z_2$, then either $W \cap M_n = Z_1$ or $W \cap M_n = Z_2$. Then for every $m \in \mathbb{Z}_+$, $L(m)(w_j - u_j)$ is zero in the irreducible quotient of either $Z_2$ or $Z_1$, which again leads to a similar contradiction.

In conclusion, we showed that if $(W \cap M_n)_{(h)} \neq 0$, then $u_1, \ldots, u_{l-1} \in W$. So $\pi_p W \subset W$. A dimension counting argument shows that

$$W = (W \cap M_n) \oplus \pi_p W$$

is generated by singular vectors from $W \cap M_n$ and $\pi_p W$.

Now we study the case when $(W \cap M_n)_{(h)} = 0$. Since we assumed that $W \cap M_n \neq 0$, there exists a smallest integer $p$ such that $(W \cap M_n)_{(h+p)} \neq 0$. Then for every $i = 1, \ldots, l-1$ with $\text{wt}(u_i) < h + p$, from the fact that $L(m)(w_i - u_i) = L(m)w_i \in (M_n \cap W)_{(\text{wt}(u_i) - m)} = 0$ for every $m \in \mathbb{Z}_+$, we see that $w_i$ are singular vectors. Thus such $w_i$ generates a submodule $W_i$ of $W$ that is isomorphic to $U_i$ in $\pi_p W$. For $i = 2, \ldots, l$ with $\text{wt}(u_i) > h + n$, repeating our analysis above, we can similarly show $u_i \in W$. A dimension counting argument shows that

$$W = (W \cap M_n) \oplus \sum_{2 \leq i \leq l, \text{wt}(u_i) < h + m} W_i + \sum_{2 \leq i \leq l, \text{wt}(u_i) \geq h + m} U_i$$

is generated by singular vectors of $W \cap M_n$, $w_i$ and $u_i$ (with $i$ in suitable ranges). Note that sum is direct if all $M_i$ are in the blocks (a), (b) and (c). Note also that the module $\sum_{2 \leq i \leq l, \text{wt}(u_i) < h + m} W_i + \sum_{2 \leq i \leq l, \text{wt}(u_i) \geq h + m} U_i$ is isomorphic to $\pi_p W$ (but not necessarily equal).
(2) \((\pi_p W)_{(h)} = 0\). Then \((W \cap M_n)_{(h+p)} \neq 0\) for every \(p \in \mathbb{N}\). By a similar argument as above, we show that \(u_i \in W\). So in this case,

\[
W = W \cap M_n \oplus \pi_p W
\]

is also generated by singular vectors from \(W \cap M_n\) and \(\pi_p W\).

To conclude the inductive step, it remains to choose \(N_1, \ldots, N_n\). Since the induction hypothesis holds for \(\pi_p W\) as a submodule for \(\bigoplus_{i=1}^{n-1} M_i\), we simply choose \(N_1, \ldots, N_{n-1}\) from the knowledge of \(\pi_p W\). \(N_n\) can simply be chosen as \(W \cap M_n\). \(\square\)

**Remark 5.15.** In case \(M_1, \ldots, M_n\) are in the blocks (a), (b) and (c), then all the \(N_i\)'s are Verma modules generated by one singular vector. In this case, the proof actually gives a "row echelon form" of singular vectors, which will be useful in the proof of the convergence.

To find this "row echelon form", we start from investigating the lowest weight of \(\pi_i W\). Without loss of generality, we rearrange the index so that

\[
\pi_1 W = (\pi_1 + \pi_2)W \cdots = \left( \sum_{i=1}^{i_0} \pi_i \right) W = 0, \pi_j W \neq 0, j > i_0.
\]

Then clearly, we can pick \(N_1 = \cdots = N_i_0 = 0\). Then, we compare the lowest weights of \(\pi_j W\) for \(j = i_0 + 1, \ldots, n\). Rearrange the indices, so that

\[
\pi_{i_0+1} W \text{ is of minimal lowest weight.}
\]

Clearly, the lowest weight of \(\pi_{i_0+1} W\) should coincide with the lowest weight of \(W\). We pick \(N_{i_0+1} = \pi_{i_0+1} W\). Let \(w_{i_0+1}\) be a singular vector in \(W\) such that \(\pi_{i_0+1} w_{i_0+1}\) generates \(\pi_{i_0+1} W\). If we write

\[
w_{i_0+1} = (w_{i_0+1}^{(1)}, \ldots, w_{i_0+1}^{(i_0)}, w_{i_0+1}^{(i_0+1)}, \ldots, w_{i_0+1}^{(n)}),
\]

then necessarily, \(w_{i_0+1}^{(1)} = \cdots = w_{i_0+1}^{(i_0)} = 0, w_{i_0+1}^{(i_0+1)} \neq 0\). So we have

\[
w_{i_0+1} = (0, \ldots, 0, w_{i_0+1}^{(i_0+1)}, \ldots, w_{i_0+1}^{(n)}).
\]

Now we further rearrange the indices, so that

\[
\left[ \left( \sum_{i=1}^{i_0+1} \pi_i \right) W \right] \cap M_{i_0+1} = \cdots = \left[ \left( \sum_{i=1}^{i_1} \pi_i \right) W \right] \cap M_{i_1} = 0.
\]

Then clearly, we can pick \(N_{i_0+2} = \cdots = N_{i_1} = 0\). Now compare the lowest weights of \(\left[ \left( \sum_{i=1}^{i_1} \pi_i + \pi_j \right) W \right] \cap M_j \neq 0, j = i_1 + 1, \ldots, n\). Rearrange the indices, so that

\[
\left[ \left( \sum_{i=1}^{i_1} \pi_i + \pi_{i_1+1} \right) W \right] \cap M_{i_1+1} \text{ is of minimal lowest weight.}
\]

From the proof of Proposition 5.14, we take \(N_{i_1+1} = \left[ \left( \sum_{i=1}^{i_1} \pi_i + \pi_{i_1+1} \right) W \right] \cap M_{i_1+1}\). Let \(w_{i_1+1}\) be the singular vector in \(W\) such that \(\sum_{i=1}^{i_1+1} \pi_i w\) generates \(N_{i_1+1}\). If we write

\[
w_{i_1+1} = (w_{i_1+1}^{(1)}, \ldots, w_{i_1+1}^{(i_1)}, w_{i_1+1}^{(i_1+1)}, \ldots, w_{i_1+1}^{(i_1+1)}),
\]

then necessarily, \(w_{i_1+1}^{(1)} = \cdots = w_{i_1+1}^{(i_1)} = 0, w_{i_1+1}^{(i_1+1)} \neq 0\). So we have

\[
w_{i_1+1} = (0, \ldots, 0, w_{i_1+1}^{(i_1+1)}, \ldots, w_{i_1+1}^{(i_1+1)}).
\]
then necessarily, \( w_{i_1+1}^{(1)} = \cdots = w_{i_1+1}^{(i_1)} = 0, w_{i_1+1}^{(i_1+1)} \). So we have
\[
w_{i_1+1} = (0, \ldots, 0, 0, \ldots, 0, w_{i_1+1}^{(i_1+1)}, \ldots, w_{i_1+1}^{(n)}).
\]

Repeating this process until there is no index left, we will obtain singular vector
\[
w_{i_0+1} = \left( 0, \ldots, 0, w_{i_0+1}^{(i_1+1)}, \ldots, w_{i_0+1}^{(i_1)}, \ldots, w_{i_0+1}^{(i_2)}, \ldots, w_{i_0+1}^{(i_2+1)}, \ldots, w_{i_0+1}^{(n)} \right).
\]
\[
w_{i_1+1} = \left( 0, \ldots, 0, 0, \ldots, 0, w_{i_1+1}^{(i_1+1)}, \ldots, w_{i_1+1}^{(i_2)}, \ldots, w_{i_1+1}^{(i_2+1)}, \ldots, w_{i_1+1}^{(n)} \right),
\]
\[\vdots\]
\[
w_{i_{r-1}+1} = \left( 0, \ldots, 0, 0, \ldots, 0, 0, \ldots, 0, w_{i_{r-1}+1}^{(i_1+1)}, \ldots, w_{i_{r-1}+1}^{(i_1)}, \ldots, w_{i_{r-1}+1}^{(i_2)}, \ldots, w_{i_{r-1}+1}^{(i_2+1)}, \ldots, w_{i_{r-1}+1}^{(n)} \right),
\]
with \( w_{i_{r-1}+1}^{(i_{r-1}+1)} \neq 0 \) for \( i = 0, \ldots, r-1 \).

**Theorem 5.16.** Let \( M = \bigoplus_{i=1}^n M(c, h_i) \) where each \( M(c, h_i) \) is in block (a), (b) or (c). Then \( M \) satisfies the condition in Definition 4.11

**Proof.** Let \( M_i = M(c, h_i) \). We arrange the indices \( i = 1, \ldots, n \) as in Remark 5.15 so that the isomorphic image of \( N_i \) in \( W \) is generated by the singular vector \( w_i \) of the form \( (0, \ldots, 0, w_i^{(i_1)}, \ldots, w_i^{(n)}) \), for \( i = \alpha + 1, \ldots, n \). With the same notations as in Remark 5.15 \( w_i \neq 0 \) only when \( i = i_0 + 1, \ldots, i_{r-1} + 1 \). For each such \( i \), since \( N_i \) is a submodule of \( M_i \) while \( M_i \) is a Verma module in block (a), (b) or (c), \( N_i \) itself is a Verma module. As a submodule of \( M \), \( N_i \) is generated by the singular vector \( (0, \ldots, 0, w_i^{(i_1)}, \ldots, 0) \). \( w_i^{(j)} \) is a (possibly zero) singular vector in \( M_j \) for every \( j \geq i \).

From what we proved Section 5.2 each \( N_i \) admits a complementary space \( P_i \) in \( M_i \) (for \( i = 1, \ldots, \alpha \), \( P_i \) can simply be chosen as \( M_i \)), such that for every \( v \in V, \pi_{N_i} Y_{M_i}(v, x)|_{P_i} \in \mathcal{H}(P_i, N_i) \).

Moreover, if \( N_i \neq 0 \), then \( N_i \) is spanned by the vectors
\[
(0, \ldots, 0, L(-n_1) \cdots L(-n_s) w_i^{(i_1)}, 0, \ldots, 0)
\]
where \( n_1 \geq \cdots \geq n_s \geq 1 \); \( P_i \) is chosen as the subspace spanned by the vectors
\[
(0, \ldots, 0, L(-n_1) \cdots L(-n_s) L(-1)^m, 0, \ldots, 0)
\]
where \( n_1 \geq \cdots \geq n_s \geq 2, m \leq \text{Ind}(w_i^{(i_1)}) \). Now we set
\[
P = \bigoplus_{i=1}^n P_i.
\]

A dimension counting argument shows that \( P \) is also a complementary space of \( W \). Let \( \rho_0 : \bigoplus_{i=1}^n N_i \rightarrow W \) be the \( V \)-module isomorphism, such that
\[
\rho_0(0, \ldots, 0, w_i^{(i_1)}, \ldots, 0, 0) = (0, \ldots, 0, w_i^{(i_1)}, \ldots, w_i^{(i_1+1)}, \ldots, w_i^{(n)}).
\]

Then clearly, for every \( n_1, \ldots, n_s \geq 1 \),
\[
\rho_0(0, \ldots, 0, L(-n_1) \cdots L(-n_s) w_i^{(i)}), 0, \ldots, 0)
\]
We extend $\rho_0$ to a linear isomorphism $\rho : W \to W$ by setting $\rho|_P = id_P, \rho|_{N_i} = \rho_0|_{N_i}$ (note that the extended $\rho$ will not be a $V$-module homomorphism). Clearly, with respect to the above basis consisting of vector of form (80) and (81), the linear map $\rho$ is triangular with all diagonal entries being 1. Thus, so is the matrix of $\rho$ the extended $\rho$ commutes with $L(-m)$ for every $m \geq 2$, since $L(-m)P \subseteq P$.

Now we set

$$\pi_N = \sum_{i=1}^{n} \pi_{N_i}$$

where $\pi_{N_i}$ is extended from the map $M_i \to N_i$ constructed in Section 5.2 to $\bigoplus_{i=1}^{n} M_i$ such that $\pi_{N_i}|_{M_j} = 0$ for every $1 \leq i \neq j \leq n$. We note that since every $\pi_{N_i}$ commutes with $L(-m)$ for $m \geq 2$, so is $\pi_N$. Let

$$\pi_W = \rho \circ \pi_N \circ \rho^{-1}$$

Clearly, $\pi_W : M \to W$ is a projection. We will show that for every $v \in V$,

$$\pi_W Y_M(v, x)|_P \in H_N(P, W).$$

From the arguments in Section 5.2 it suffices to generalize Theorem 5.9 to the current context.

For simplicity, consider $p = (0, ..., 0, p^{(i)}, 0, ..., 0) \in P_i$.

1. If $i \neq i_0 + 1, ..., i_{r-1} + 1$, then for every $q \in \mathbb{N}$,

$$\pi_W Y_M(\omega, x_1) \cdots Y_M(\omega, x_q)p = \rho \pi_N \rho^{-1}(0, ..., 0, Y_{M_i}(\omega, x_1) \cdots Y_{M_i}(\omega, x_q)p^{(i)}, 0, ..., 0)
= \rho \pi_N(0, ..., 0, Y_{M_i}(\omega, x_1) \cdots Y_{M_i}(\omega, x_q)p^{(i)}, 0, ..., 0)
= 0$$

where (82) follows from the fact that $\rho^{-1}|_P = id_P$. (83) follows from $\pi_N|_P = 0$. So in this case, the convergence is trivial.

2. If $i = i_j + 1$ for some $j = 1, ..., r - 1$, then for every $q \in \mathbb{N}$,

$$\pi_W Y_M(\omega, x_1) \cdots Y_M(\omega, x_q)p = \rho \pi_N \rho^{-1}(0, ..., 0, Y_{M_i}(\omega, x_1) \cdots Y_{M_i}(\omega, x_q)p^{(i)}, 0, ..., 0)
+ \rho \pi_N \rho^{-1}(0, ..., 0, \pi_P Y_{M_i}(\omega, x_1) \cdots Y_{M_i}(\omega, x_q)p^{(i)}, 0, ..., 0)$$

Similarly as in (1), (86) is zero. To analyze (85), we notice from Theorem 5.9 that

$$\pi_N Y_{M_i}(\omega, x_1) \cdots Y_{M_i}(\omega, x_q)p^{(i)}$$

$$= \sum_{t=1}^{q} \sum_{1 \leq \alpha_1 < \ldots < \alpha_t \leq q} Y_{M_i}^{(i)}(v^{(i)}_{\alpha_1, ..., \alpha_t}, x_{\alpha_t}) \cdots Y_{M_i}^{(i)}(v^{(i)}_{\alpha_1, ..., \alpha_t}, x_{\alpha_1}) u_{1...q} \frac{f_{\alpha_1, ..., \alpha_t}(x_1, ..., x_q)}{\prod_{i=1}^{l} f_{\alpha_i}(x_{\alpha_i} - x_{\alpha_j}) P_{\alpha_1, ..., \alpha_t}^{(i)}}$$

for some Laurent polynomial $f_{\alpha_1, ..., \alpha_t}(x_1, ..., x_q) \in N_i[x_{\alpha_1}, x_{\alpha_1}^{-1}, ..., x_{\alpha_t}, x_{\alpha_t}^{-1}]$, some $v^{(i)}_{\alpha_1, ..., \alpha_t}, ..., v^{(i)}_{\alpha_1, ..., \alpha_t} \in V$, and some integers $P_{\alpha_1, ..., \alpha_t}^{(i)} \in \mathbb{N}, i = 1, ..., p, j = 1, ..., l-p$. Using the fact that $\rho, \pi_N, \rho^{-1}$
commutes with $L(-m)$ for every $m \geq 2$, we rewrite (55) as

$$
\sum_{t=1}^{q} \sum_{1 \leq \alpha_1 < \cdots < \alpha_t \leq q} Y_{\alpha_1}^{(1)}(x_1, \ldots, x_{\alpha_t}) \cdots Y_{\alpha_t}^{(q-t)}(x_{\alpha_t}, x_{\alpha_{t-1}}) \prod_{i=1}^{q-t} (x_{\alpha_i} - x_{\alpha_i})^{-\rho_{(\alpha_1, \ldots, \alpha_t)}^{(i)}} \rho_{\pi_N}^{-1} \left(0, \ldots, 0, f_{(\alpha_1, \ldots, \alpha_t)}^{(i)}(x_1, \ldots, x_q), 0, \ldots, 0\right).
$$

To handle the term $\rho_{\pi_N}^{-1} \left(0, \ldots, 0, f_{(\alpha_1, \ldots, \alpha_t)}^{(i)}(x_1, \ldots, x_q), 0, \ldots, 0\right)$, we first rewrite

$$
f_{(\alpha_1, \ldots, \alpha_t)}(x_1, \ldots, x_q) = \sum_{\beta_1, \ldots, \beta_q} \sum_{\text{finite } n_1 \geq \cdots \geq n_q} \prod_{\text{finite } n_1 \geq \cdots \geq n_q, m \geq 0} \rho_{\pi_N}^{-1}(0, \ldots, 0, f_{(\alpha_1, \ldots, \alpha_t)}^{(i)}(x_1, \ldots, x_q), 0, \ldots, 0) \cdot \rho_{\pi_N}^{-1}(0, \ldots, 0, f_{(\alpha_1, \ldots, \alpha_t)}^{(i)}(x_1, \ldots, x_q), 0, \ldots, 0).
$$

The action of $\rho^{-1}$ changes the zero entries in each $k$-th spot of the vector $(0, \ldots, 0, f_{(\alpha_1, \ldots, \alpha_t)}^{(i)}(x_1, \ldots, x_q), 0, \ldots, 0)$ to some nonzero entries for each $k > i$. The action of $\pi_N$ kills all the nonzero entries except for those at $i_j + 1$-th spot, $i_{j+1} + 1$-th spot, ..., and $i_{r+1} + 1$-th spot. As a result, we have

$$
\pi_N \rho^{-1}(0, \ldots, 0, f_{(\alpha_1, \ldots, \alpha_t)}^{(i)}(x_1, \ldots, x_q), 0, \ldots, 0) = (0, \ldots, 0, f_{(\alpha_1, \ldots, \alpha_t)}^{(i)}(x_1, \ldots, x_q), 0, \ldots, 0, f_{(\alpha_1, \ldots, \alpha_t)}^{(i+1)}(x_1, \ldots, x_q), 0, \ldots, 0).
$$

Finally, the action of $\rho$ on this vector clearly generates a vector consisting of Laurent polynomials in $W[x_{\alpha_1}, x_{\alpha_1}^{-1}, \ldots, x_{\alpha_t}, x_{\alpha_t}^{-1}]$. Thus finishes the generalization of Theorem 5.9 to the current context. The rest of the proof is a simple repetition of the arguments in Section 5.2 and shall not be repeated.

**Remark 5.17.** The situation for Verma modules of block (d) is much more complicated yet interesting. The preliminary attempts hints that we might need a completely different approach to show the convergence. But what we discovered in this section hints that the convergence requirements might not be a serious obstruction for the application of cohomology theory. We conjecture that category of finite length modules for the Virasoro VOA should all be in $C_N$. There should not be any restrictions on the central charges and the lowest weight. The conjecture should be attempted in future works.
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