ON THE DECAY PROPERTIES OF SOLUTIONS TO A CLASS OF
SCHRÖDINGER EQUATIONS

L. DAWSON, H. MCGAHAGAN, AND G. PONCE

Abstract. We construct a local in time, exponentially decaying solution of the onedimensional variable coefficient Schrödinger equation by solving a nonstandard boundary value problem. A main ingredient in the proof is a new commutator estimate involving the projections $P_\pm$ onto the positive and negative frequencies.

1. Introduction

In [5], T. Kato showed that the semigroup $\{e^{-it\partial_x^3} : t \geq 0\}$ in the space $L^2(e^{2\beta x}dx)$ with $\beta > 0$ is formally equivalent to the semigroup $e^{-t(\partial_x-\beta)^3}$ in $L^2(\mathbb{R})$. Among the immediate consequences of this result is that if $u \in C([0,T] : H^1(\mathbb{R}))$ is a strong solution of the $k$-generalized Korteweg de Vries (KdV) equation,

$$\partial_t u + \partial_x^3 u + u^k \partial_x u = 0, \quad k = 1, 2, \ldots$$

(1.1)

with data $u_0 \in L^2(e^{2\beta x}dx)$, then $u \in C([0,T] : L^2(e^{2\beta x}dx)) \cap C^\infty(\mathbb{R} \times (0,T])$. In other words, the solution $u = u(x,t)$ satisfies the persistence property $e^{\beta x}u \in C([0,T] : L^2(\mathbb{R}))$ and a “parabolic” regularization, $u \in C^\infty(\mathbb{R} \times (0,T])$.

Since results for solutions of the $k$-generalized KdV equation and Schrödinger equations of the type

$$(a) \quad \partial_t u - i\Delta u = f(|u|)u, \quad (b) \quad \partial_t u - i(\Delta u + W(x,t)u) = F(x,t),$$

(1.2)

run parallel – for instance, solutions of both satisfy Strichartz estimates, local smoothing effects of the Kato type, and persistence properties in $H^s(\mathbb{R})$, the weighted spaces $H^s(\mathbb{R}) \cap L^2(|x|^k)$, and the Schwartz space – one may ask what the equivalent result to that described above for the KdV equation is in the case of Schrödinger equations. One first notices that even for the free Schrödinger group $\{e^{it\Delta} : t \in \mathbb{R}\}$, both of the above properties fail: assuming we are in $\mathbb{R}^1$ ($\Delta = \partial_x^2$) for simplicity, we can construct initial data $u_0 \in L^2(\mathbb{R}) \cap L^2(e^{2\beta x}dx)$ such that $e^{it\partial_x^3}u_0 \notin L^2(e^{2\beta x}dx) \cup C^\infty(\mathbb{R})$ for any $t > 0$.

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Roughly, the difficulty lies in the fact that if \( u(x, t) = e^{i\beta x}u_0(x) \), then \( v(x, t) := e^{\beta x}u(x, t) \) formally solves the equation

\[
\begin{aligned}
\partial_t v - i(\partial_x - \beta)^2 v &= \partial_t v - i\partial_x^2 v + 2i\beta \partial_x v - i\beta^2 v = 0,
\end{aligned}
\]

whose associated initial value problem (IVP) is ill-posed in \( L^2(\mathbb{R}) \). However, the operator \( 2i\beta \partial_x \), whose symbol is \(-2\beta \xi\), introduces a parabolic structure in the negative frequency for positive time and in the positive frequency for negative time. Thus, to find \( L^2 \)-solutions of equation (1.3) in the time interval \([0, T]\), one needs to consider a “boundary value problem” for (1.3) where

\[
\begin{aligned}
v_-(x, 0) &= P_- v(x, 0) := (\chi_{[-\infty, 0]}(\xi)\hat{\varphi}(\xi, 0))^\vee(x), \\
v_+(x, T) &= P_+ v(x, T) := (\chi_{(0, \infty]}(\xi)\hat{\varphi}(\xi, T))^\vee(x)
\end{aligned}
\]

are prescribed. In this case, one finds the solution

\[
v(x, t) = e^{(i\partial_x^2 - 2\beta \partial_x + i\beta^2)}v_-(x, 0) + e^{-(T-t)(i\partial_x^2 + 2\beta \partial_x + i\beta^2)}v_+(x, T),
\]

with \( D_x h(x) := (-\partial_x^2)^{1/2}h(x) = (c|\xi|\hat{h}(\xi))^\vee(x) \). Then,

\[
\sup_{[0, T]} \|v(t)\|_2 \leq c(\|v_+(x, T)\|_2 + \|v_-(x, 0)\|_2),
\]

c independent of \( \beta > 0 \) and \( T \), and \( v \in C^\infty(\mathbb{R} \times (0, T)) \). We observe that in formula (1.5), the positive and negative frequencies do not interact and, also, that \( u(x, t) := e^{-\beta x}v(x, t) \) is not necessarily an \( L^2 \)-solution of the free Schrödinger equation.

The following estimate established in [7] of the type described in (1.6) for a linear Schrödinger equation with lower order variable coefficients (1.2) (b) was a key step in the proof of the unique continuation results obtained in [7] and [4].

**Lemma.** [7] There exists \( \epsilon > 0 \) such that if \( W : \mathbb{R}^n \times [0, T] \to \mathbb{C} \) satisfies \( \|W\|_{L^1_t L^\infty_x} \leq \epsilon \) and \( u \in C([0, T] : L^2_x(\mathbb{R}^n)) \) is a strong solution of the equation (1.2) (b) with

\[
\begin{aligned}
u_0 = u(\cdot, 0), \ u_T &\equiv u(\cdot, T) \in L^2(e^{2\beta x} \, dx), \quad F \in L^1([0, T] : L^2_x(e^{2\beta x} \, dx))
\end{aligned}
\]

for some \( \beta \in \mathbb{R} \), then there exists \( c \) independent of \( \beta \) such that

\[
\sup_{0 \leq t \leq T} \|e^{\beta x}u(\cdot, t)\|_2 \leq c \left( \|e^{\beta x}u_0\|_2 + \|e^{\beta x}u_T\|_2 + \int_0^T \|e^{\beta x}F(\cdot, t)\|_2 \, dt \right).
\]

Notice that in the above result, one assumes the existence of a reference solution \( u(x, t) \) of equation (1.2) (b) and shows that under hypothesis (1.7), exponential decay in the time interval \([0, T]\) is preserved.

The \( L^2 \)-well-posedness of the IVP associated to the equation

\[
\partial_t w = i\Delta w + b(x) \cdot \nabla_x w + f(x, t),
\]
has been extensively studied. In particular, S. Mizohata [8] gives the following necessary condition for the IVP associated to (1.8) to be well-posed in $L^2(\mathbb{R}^n)$:

\begin{equation}
\sup_{x \in \mathbb{R}^n, \omega \in S^{n-1}, R > 0} \left| \text{Im} \int_0^R b(x + r\omega) \cdot \omega \, dr \right| < \infty.
\end{equation}

The gain of regularity of solutions to the variable coefficient Schrödinger equation

$$\partial_t u - i \partial_x (a_j(x) \partial_{x_k} u) + W(x) u = 0$$

as a consequence of its dispersive character and the decay assumptions on the data has also been studied in several works; see [2], [3], and references therein.

In this note, we shall combine the above ideas with some new commutator estimates to construct an exponentially decaying solution to the one-dimensional variable coefficient Schrödinger equation

\begin{equation}
\partial_t u = i(\partial_x (a(x, t) \partial_x u) + W(x, t) u). \tag{1.10}
\end{equation}

More precisely, we are interested in a solution $u \in C([0, T] : L^2(\mathbb{R}) \cap L^2(e^{2\beta x} dx))$.

To ensure that we construct $u \in L^2(\mathbb{R})$, we will need to refer to the following function $\varphi_\beta(x)$: for $\beta > 0$ we denote by $\varphi(x) = \varphi_\beta(x)$ a $C^4(\mathbb{R})$ function such that $\varphi(x) = 1$ if $x \leq 0$, $\varphi(x) = e^{\beta x}$ if $x \geq 10\beta$, and $\varphi(x)$ is strictly increasing on $(0, 10\beta)$.

**Theorem 1.1.** Let $a : \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R}$ be such that

\begin{equation}
\begin{aligned}
a &\in C^2(\mathbb{R} \times \mathbb{R}^+) \cap L^1_t(\mathbb{R}^+ : L^\infty_x(\mathbb{R})), \\
\langle x \rangle \partial_x^j a &\in L^1_t(\mathbb{R}^+ : L^\infty_x(\mathbb{R})), \quad j = 1, 2, \\
a(x, t) &\geq \lambda \geq 0, \quad \forall (x, t) \in \mathbb{R} \times \mathbb{R}^+. 
\end{aligned} \tag{1.11}
\end{equation}

Let $W : \mathbb{R} \times \mathbb{R}^+ \to \mathbb{C}$ be such that

\begin{equation}
W \in L^1_t(\mathbb{R}^+ : L^\infty_x(\mathbb{R})). \tag{1.12}
\end{equation}

Then given $(f, g) \in P_- L^2(\mathbb{R}) \times P_+ L^2(\mathbb{R})$, there exists $T = T(\beta; \|a\|_1; \|W\|_{L^1_t L^\infty_x}) > 0$ such that (1.10) has a unique solution $u \in C([0, T] : L^2(\mathbb{R}))$ with $e^{\beta x} u \in C([0, T] : L^2(\mathbb{R}))$ and with $P_-(\varphi(x) u(x, 0)) = f(x)$ and $P_+(\varphi(x) u(x, T)) = g(x)$.

If in addition $a, W \in C^\infty(\mathbb{R} \times \mathbb{R}^+) ; \lambda > 0$ with

$\beta \lambda \geq c(\| \langle x \rangle \partial_x a \|_{L^\infty(\mathbb{R} \times \mathbb{R}^+)} + \| \langle x \rangle \partial_x^2 a \|_{L^\infty(\mathbb{R} \times \mathbb{R}^+)}),$ \hspace{1cm} and

$\partial_t^k \partial_x^j a, \partial_t^k \partial_x^j W \in L^\infty(\mathbb{R} \times \mathbb{R}^+)$ for any $k, j \in \mathbb{Z}^+$, then $u \in C^\infty(\mathbb{R} \times (0, T))$.

We use the notation $\langle x \rangle := (1 + |x|^2)^{1/2}$. Also, $\|a\|_1$ denotes the sum of the $L^1_t L^\infty_x$-norms of the expressions involving the function $a$ described in (1.11):

$$\|a\|_1 := \|a\|_{L^1_t L^\infty_x} + \sum_{j=1}^2 \| \langle x \rangle \partial_x^j a \|_{L^1_t L^\infty_x}.$$
Under the assumptions of Theorem 1.1 we do not know if the dependence on the parameter $\beta$ of the time interval $[0,T]$ can removed as was done in [7]. Also, here we shall restrict ourselves to the one-dimensional case.

To prove Theorem 1.1 we consider a system describing the time evolution of the projection of the weighted function $v:=\varphi u$ into the positive and negative frequencies. Since our equation has variable coefficients, this becomes a coupled system. It will be essential in our arguments that the coupled terms are, roughly speaking, of “order zero.” We will show this using commutator estimates such as the following: for all $p \in (1, \infty), l, m \in \mathbb{Z}^+$ there exists $c = c(p; l, m) > 0$ such that

$$\|\partial_x^l [P_+; a] \partial_x^m f\|_p \leq c \|\partial_x^{l+m} a\|_\infty \|f\|_p.$$  

Clearly, the inequality (1.13) holds with $P_-$ or $H$, the Hilbert transform, in place of $P_+$. In the case $l+m = 1$, (1.13) is Calderón’s first commutator estimate [1]. A related version of estimate (1.13) was obtained in [9] for general positive derivatives, but did not involve the $L^\infty$-norm.

2. Proof of Theorem 1.1

Consider the equation

$$\partial_t u = i(\partial_x (a(x,t) \partial_x u) + W(x,t)u).$$  

We wish to construct a solution $u \in L^2((1 + e^{2\beta x})dx)$ for a fixed $\beta > 0$. Recall the definition of the function $\varphi(x) = \varphi_\beta(x)$, and define $\phi(x) := \varphi'(x)/\varphi(x)$. Notice that $\phi(x) = \beta \chi_{\mathbb{R}^+}(x)$ except on the interval $0 < x < 10\beta$ and that $\|\phi\|_\infty = \beta$.

Let $v(x,t) := \varphi(x) u(x,t)$. Then, multiplying (2.1) by $\varphi(x)$ and using the fact that $[\varphi; \partial_x] = -\phi \varphi$, we have that

$$\partial_x v = i((\partial_x - \phi(x))(a(x,t)(\partial_x - \phi(x))v + W(x,t)v$$

$$= i\partial_x^2 (a\partial_x v) - 2ia\phi \partial_x v + i((\phi^2 - \partial_x \phi)a - \phi \partial_x a)v + iWv.$$  

We will construct a solution $v \in L^2(\mathbb{R})$ of (2.2). This suffices since the definition of $\varphi$ then guarantees that $u$ defined by $u(x) = v(x)$ on $x \leq 0$ and $u(x) = \varphi^{-1}(x)v(x)$ on $x > 0$ will be in $L^2((1 + e^{2\beta x})dx)$, and $u$ will solve (2.1).

Applying the projection operators $P_\pm$ to equation (2.2), we obtain

$$\partial_t v_\pm = i\partial_x (a\partial_x v_\pm) - 2i\phi a \partial_x v_\pm + P_\pm(i((\phi^2 - \partial_x \phi)a - \phi \partial_x a)v) + P_\pm(iWv)$$

$$+ i\partial_x ([P_\pm; a]\partial_x v) - 2i[P_\pm; a\phi]\partial_x v,$$

where $v_\pm := P_\pm v$. We can rewrite this as the following coupled system:

$$\partial_t v_+ = i\partial_x (a\partial_x v_+) - 2i\phi a \partial_x v_+ + \Lambda_+(v_+, v_-)$$

$$\partial_t v_- = i\partial_x (a\partial_x v_-) - 2i\phi a \partial_x v_- + \Lambda_-(v_+, v_-),$$  

where $\Lambda_\pm(v_+, v_-)$ are the commutators of $P_\pm$ and $a\phi$. These commutators can be estimated using the Hardy-Littlewood inequality and the fact that $\phi(x) = \beta \chi_{\mathbb{R}^+}(x)$ except on the interval $0 < x < 10\beta$. This completes the proof of Theorem 1.1.
where
\[
\Lambda_{\pm}(v_+, v_-) := P_{\pm}(i((\phi^2 - \partial_x \phi)a - \phi \partial_x a)(v_+ + v_-)) + P_{\pm}(iW(v_+ + v_-)) + i\partial_x([P_{\pm}; a] \partial_x (v_+ + v_-)) - 2i[P_{\pm}; a \phi] \partial_x (v_+ + v_-).
\]

Notice that once we construct functions \(v_+\) and \(v_-\) that solve this system, \(v = v_+ + v_-\) will be the desired solution of (2.2).

Taking the \(L^2\) norm of \(\Lambda_{\pm}\) and applying Lemma 3.1, it follows that \(\Lambda_{\pm}\) can be written as a sum of linear operators in \((v_+, v_-)\) of “order zero”:

\[
\|\Lambda_{\pm}(v_+, v_-)\|_2 \leq c \left( \|(\phi^2 - \partial_x \phi)a - \phi \partial_x a\|_\infty + \|W\|_\infty + \|\partial_x^2 a\|_\infty \right.
\]
\[
+ \|\partial_x(a \phi)\|_\infty \right) \|v_+ + v_-\|_2 \leq K(t) (\|v_+\|_2 + \|v_-\|_2),
\]

with
\[
K(t) := c \left( \sum_{j=0}^2 \beta^j \|\partial_x^{-j} a(t)\|_\infty + \|a(t)\|_\infty + \|W(t)\|_\infty \right).
\]

To prove the existence of a solution \((v_+, v_-) \in L^2\) to (2.3), we will establish a priori estimates and local existence for a related uncoupled system, and then find \((v_+, v_-)\) as a limit of these solutions.

First, we fix the time interval on which we will solve the equation. Define
\[
c_{a,\beta}(t) := c \left( \|a(t)\|_\infty + (1 + \beta)\|\langle x \rangle \partial_x a(t)\|_\infty + \beta\|\langle x \rangle \partial_x^2 a(t)\|_\infty \right),
\]
and let \(T = T(\beta; \|a\|_1; \|W\|_{L^1_t L^\infty_x}) > 0\) be such that
\[
e^4 \int_0^T c_{a,\beta}(t) dt \leq 2/3 \quad \text{and} \quad \int_0^T K(t) dt \leq 1/8.
\]

These inequalities must hold for some \(T > 0\) by hypotheses (1.11) and (1.12). Also, we define the norm \(\|v\|_T := \sup_{[0,T]} \|v_+(t)\|_2 + \sup_{[0,T]} \|v_-(t)\|_2\), and letting \(\delta := \|v_+(T)\|_2 + \|v_-(0)\|_2\), we define the space
\[
X_T := \{ v : \mathbb{R} \times [0, T] \to \mathbb{C} : \|v\|_T \leq 4\delta \}.
\]

Next, using standard energy estimates, we obtain a priori bounds for the solutions of both of the following (uncoupled) equations on \(\mathbb{R} \times [0, T]\):

\[
\partial_t v_+ = i\partial_x(a \partial_x v_+) - 2ia\phi \partial_x v_+ + F_+(x, t)
\]
\[
\partial_t v_- = i\partial_x(a \partial_x v_-) - 2ia\phi \partial_x v_- + F_-(x, t),
\]
with functions $F_\pm \in L^1_t(L^2_x(\mathbb{R}))$. Multiplying (2.8) by \( \overline{v}_- \), integrating in the \( x \)-variable, and taking the real part, we have that

\[
\frac{1}{2} \frac{d}{dt} \|v_-(t)\|_2^2 = \text{Re} \left\{ -2i \int a\phi \partial_x v_- \overline{v}_- \, dx + \int F_-(x,t) \overline{v}_- \, dx \right\}.
\]

Using the definition of $D_x^\alpha$ and the fact that \( \hat{v}_- \) is supported on \( \mathbb{R}^- \), we compute

\[
-2i \int a\phi \partial_x v_- \overline{v}_- \, dx = -2 \int a\phi |D_x^{1/2}v_-|^2 \, dx - 2 \int (D_x^{1/2}[D_x^{1/2}; a\phi]v_-) v_- \, dx;
\]

therefore,

\[
(2.9) \quad \frac{d}{dt} \|v_-(t)\|_2^2 + 4 \int a\phi |D_x^{1/2}v_-|^2 \leq 4c_{\alpha,\beta}(t) \|v_-(t)\|_2^2 + 2\|F_-(t)\|_2 \|v_-(t)\|_2,
\]

where the final inequality follows from combining the estimate from Proposition 3.2 in the appendix and the Gagliardo-Nirenberg inequality to see that

\[
\|D_x^{1/2}[D_x^{1/2}; a\phi]v_-\|_2 \leq c \|J\partial_x(a\phi)\|_q \|v_-\|_2 \leq c \|\partial_x(a\phi)\|^{1-\delta}_q \|J\partial_x(a\phi)\|^{\delta}_q \|v_-\|_2
\]

\[
\leq c \left( \|\partial_x(a\phi)\|_q + \|\partial_x^2(a\phi)\|_q \right) \|v_-\|_2 \leq c_{\alpha,\beta}(t) \|v_-(t)\|_2,
\]

where we take \( q < \infty \) and \( 0 < \delta < 1 \) such that both \( \delta > 1/q \) and \( \delta > 1 - 1/q \), and also \( q \) large enough that \( \|\langle x \rangle\|_q < \infty \). Bounding \( \frac{d}{dt} \|v_-(t)\|_2 \) from (2.9), we find that

\[
\|v_-(t)\|_2 \leq \left( \|v_-(0)\|_2 + \int_0^T \|F_-(t)\|_2 \right) e^{2 \int_0^T c_{\alpha,\beta}(\tau) \, d\tau} \text{ for all } t \in [0,T].
\]

Putting this back into (2.9) in order to bound \( \int_0^T \int a\phi |D_x^{1/2}v_-|^2 \, dx \, dt \), we obtain the estimate

\[
\sup_{t \in [0,T]} \|v_-(t)\|_2 + 2 \left( \int_0^T \int a(x,t)\phi(x)|D_x^{1/2}v_-|^2 \, dx \, dt \right)^{1/2}
\]

\[
\leq 3 \left( \|v_-(0)\|_2 + \int_0^T \|F_-(t)\|_2 \, dt \right) e^{4 \int_0^T c_{\alpha,\beta}(\tau) \, d\tau}.
\]

A similar argument applied to the equation for \( v_+ \) (2.7) shows that

\[
\frac{d}{dt} \|v_+(t)\|_2^2 - 4 \int a\phi |D_x^{1/2}v_+|^2 \, dx \geq -4c_{\alpha,\beta}(t) \|v_+(t)\|_2^2 - 2\|F_+(t)\|_2 \|v_+(t)\|_2.
\]
Integrating from $t$ to $T$, we estimate $\|v_+(t)\|_2 \leq \left(\|v_+(T)\|_2 + \int_0^T \|F_+\|_2\right) e^{2\int_0^T c_{a,\beta}}$, and then, it follows that

$$\sup_{t \in [0,T]} \|v_+(t)\|_2 + 2\left(\int_0^T \int a(x,t)\phi(x)|D_x^{1/2}v_+|^2 \,dx\,dt\right)^{1/2} \leq 3\left(\|v_+(T)\|_2 + \int_0^T \|F_+(t)\|_2 \,dt\right) e^{4\int_0^T c_{a,\beta}(\tau)d\tau}. \tag{2.11}$$

To establish the first part of Theorem 1.1, we apply the contraction principle in the space $X_T$ (2.6) with $(v^m_+, v^m_-)$ for $m \in \mathbb{N}$ the iteratively defined solution of the system

$$\begin{cases}
\partial_t v^m_+ = i\partial_x (a\partial_x v^m_+) - 2ia\phi \partial_x v^m_+ + \Lambda_+(v^{m-1}_+, v^{m-1}_-), \\
\partial_t v^-_m = i\partial_x (a\partial_x v^m_-) - 2ia\phi \partial_x v^m_- + \Lambda_-(v^{m-1}_+, v^{m-1}_-), \\
v^m_+(x,T) = g(x), \quad v^m_-(x,0) = f(x),
\end{cases} \tag{2.12}$$

where $v^0_+ = v^0_- := 0$. The above equations are of the form (2.7) and (2.8), and the existence of solutions in $C([0,T] : L^2(\mathbb{R}))$ will be proven below. Letting $\|v(t)\|_2 := \|v_+(t)\|_2 + \|v_-(t)\|_2$, we have, from the energy estimates (2.10) and (2.11), that

$$\sup_{t \in [0,T]} \|v^{m+1}(t)\|_2 \leq 3(\delta + 2 \sup_{t \in [0,T]} \|v^m(t)\|_2 \int_0^T K(t) \,dt) e^{4\int_0^T c_{a,\beta}(\tau)d\tau} \tag{2.13}$$

for $m \in \mathbb{N}$. From our choice of $T$ in (2.5), $\sup_{t \in [0,T]} \|v^1(t)\|_2 \leq 3\delta e^{4\int_0^T c_{a,\beta}} \leq 2\delta$, and if we assume $\sup_{t \in [0,T]} \|v^m(t)\|_2 \leq 4\delta$, then the energy estimate (2.13) yields

$$\sup_{t \in [0,T]} \|(v^{m+1}(t))_2 \leq 3(\delta + 2(4\delta)(1/8))2/3 = 4\delta.$$

Repeating the derivation of the energy estimates for the equations for the differences $v^{m+1}_+ - v^+_m$ and $v^{m+1}_- - v^-_m$ and using (2.5) yields the estimate

$$\sup_{t \in [0,T]} \|(v^{m+1} - v^m)(t)\|_2 \leq \frac{1}{2} \sup_{t \in [0,T]} \|(v^m - v^{m-1})(t)\|_2.$$
family of equations
\begin{equation}
\partial_t v_\epsilon^- = -\epsilon \partial_x^4 v_\epsilon^- + i \partial_x(a \partial_x v_\epsilon^-) - 2i\alpha \partial_x v_\epsilon^- + F_- = -\epsilon \partial_x^4 v_\epsilon^- + \Phi(v_\epsilon^-), \quad t > 0.
\end{equation}

By Duhamel's principle, the solution $v_\epsilon^-(t)$ satisfies
\[ v_\epsilon^-(t) = e^{-\epsilon t \partial_x^4} v_\epsilon^-(0) + \int_0^t e^{-\epsilon (t-t') \partial_x^4} \Phi(v_\epsilon^-(t')) \, dt'. \]

We have the inequality (by computing max$_{\xi \in \mathbb{R}} \xi^2 e^{-\epsilon \xi^4} = c_j(\epsilon t)^{-j/4}$, with $c_0 = 1$),
\begin{equation}
\| \partial_x e^{-\epsilon t \partial_x^4} f \|_2 \leq c_j(\epsilon t)^{-j/4} \| f \|_2 \quad j = 0, 1, 2, 3.
\end{equation}

Therefore, formally,
\[ \| v_\epsilon^-(t) \|_2 \leq \| v_\epsilon^-(0) \|_2 + \int_0^t \| e^{-\epsilon (t-t') \partial_x^4} \{ \partial_x^2 (av_\epsilon^-) - \partial_x(a \partial_x v_\epsilon^-) + 2a\alpha \partial_x v_\epsilon^- + (2\partial_x(a\phi)v_\epsilon^- + F_-) \} \|_2 \, dt' \]
\[ \leq \| v_\epsilon^-(0) \|_2 + c \int_0^t \left\{ \left( \frac{1}{(\epsilon(t-t'))^{1/2}} + \frac{1}{(\epsilon(t-t'))^{1/4}} + 1 \right) \| v_\epsilon^-(t') \|_2 + \| F_- \|_2 \right\} \, dt' \]
\[ \leq \| v_\epsilon^-(0) \|_2 + c \left( \frac{T^{1/2}}{\epsilon^{1/2}} + \frac{T^{3/4}}{\epsilon^{1/4}} + T \right) \sup_{t \in [0,T]} \| v_\epsilon^-(t) \|_2 + \int_0^T \| F_- \|_2 \, dt. \]

A standard argument then shows the existence of a solution $v_\epsilon^- \in C([0, T_\epsilon] : L^2(\mathbb{R}))$ to (2.14), with $T_\epsilon \downarrow 0$ as $\epsilon \downarrow 0$. Using the a priori estimate (2.9), which holds uniformly in $\epsilon > 0$, we reapply the above local argument to extend the solution $v_\epsilon^-$ to the time interval $[0, T]$, with $T$ as in (2.5), for all $\epsilon \in (0, 1)$. Letting $\epsilon \rightarrow 0$ in an appropriate manner, we find the desired solution.

Since $v(x, t) = \varphi(x)u(x, t)$, both $u$ and $e^{\beta x}u$ are in $C([0, T] : L^2(\mathbb{R}))$, with $u$ solving (2.1) in $C([0, T] : H^{-2}(\mathbb{R}))$. Also, notice that
\[ w(x, t) := e^{\beta x}u(x, t) \in C([0, T] : L^2(\mathbb{R})) \]
is a solution of the equation
\begin{align*}
\partial_t w &= i((\partial_x - \beta)a(\partial_x - \beta)w + W(x, t)w(x, t)) \\
&= i\partial_x(a\partial_x w) - 2i\beta a\partial_x w + i(\beta^2 a - \beta \partial_x a)w + iWw,
\end{align*}
with $w_-(x, 0) = P_-(e^{\beta x}u(x, 0))$, and $w_+(x, T) = P_+(e^{\beta x}u(x, T))$.

To prove the second part of Theorem 1.1 we project the above equation onto the positive and negative frequencies, obtaining a coupled system for $w_\pm := P_\pm w$, from which we find the energy estimate
\begin{equation}
\beta \int_0^T \int a(x, t)(|D_x^{1/2} w_+|^2 + |D_x^{1/2} w_-|^2) \, dx dt \leq c(\| w_-(0) \|_2^2 + \| w_+(T) \|_2^2).
\end{equation}
Therefore, from the hypothesis $a \geq \lambda > 0$, we see that $w \in L^2([0, T] : H^{1/2}(\mathbb{R}))$.

We observe that formally $z(x, t) = D_x^{1/2}w(x, t)$ satisfies the equation

$$\partial_t z = i\partial_x (a\partial_x z) - 2i\beta a\partial_x z + i\partial_x [D_x^{1/2}; a]\partial_x w - 2i\beta [D_x^{1/2}; a]\partial_x w + \Gamma(z, w),$$

where $\Gamma(z, w)$ denotes a linear operator of “order zero” in $(z, w)$. Applying the projection operators, we obtain

$$\partial_t z_{\pm} = i\partial_x (a\partial_x z_{\pm}) - 2i\beta a\partial_x z_{\pm} + i\partial_x [P_{\pm}; a]\partial_x z - 2i\beta [P_{\pm}; a]\partial_x z$$

$$+ P_{\pm} (i\partial_x [D_x^{1/2}; a]\partial_x w - 2i\beta [D_x^{1/2}; a]\partial_x w + \Gamma(z, w)).$$

Noticing that $\partial_x = D_x^{1/2}HD_x^{1/2}$, where $H$ is the Hilbert transform $(\mathcal{H}f(\xi) := i\text{sgn}(\xi)\hat{f}(\xi))$, and using Proposition 3.2, it follows that both

$$|\int (P_{\pm}[D_x^{1/2}; a]\partial_x w)z_{\pm} dx| = |\int ([D_x^{1/2}; a]D_x^{1/2}H z)P_{\pm}z_{\pm} dx|$$

$$\leq c||J^\delta \partial_x a||_q \|z\|_2 \|z_{\pm}\|_2,$$

$$|\int (P_{\pm}(\partial_x [D_x^{1/2}; a]\partial_x w))z_{\pm} dx| = |\int (D_x^{1/2}[D_x^{1/2}; a]D_x^{1/2}H z)D_x^{1/2}H z_{\pm} dx|$$

$$\leq c||J^\delta \partial_x a||_q \|D_x^{1/2}z\|_2 \|D_x^{1/2}z_{\pm}\|_2$$

where we take $0 < \delta < 1$ and $1 < q < \infty$ such that $\delta > 1/q$. Since we know that $\|z\|_{L^q L^2} = \|D_x^{1/2}w\|_{L^q L^2} \leq C_o$ ($C_o$ denoting a constant that depends on the data $\|w_-(0)\|_2$ and $\|w_+(T)\|_2$), we have that $\|z(t)\|_{L^2} < \infty$ for a.e. $t$. Therefore, for every $\epsilon > 0$, we can find $t^0_\epsilon \in (0, \epsilon)$ and $t^1_\epsilon \in (T - \epsilon, T)$ such that $\|z(t^i_\epsilon)\|_{L^2} \leq C_o(\epsilon)$ for $i = 0, 1$. From the equations (2.17), we obtain the following energy estimate for $z$:

$$\beta \lambda \int_{t^0_\epsilon}^{t^1_\epsilon} \int |D_x^{1/2}z|^2 dx dt \leq \beta \int_{t^0_\epsilon}^{t^1_\epsilon} a(x, t)(|D_x^{1/2}z_+|^2 + |D_x^{1/2}z_-|^2) dx dt$$

$$\leq C_o(\epsilon) + c||J^\delta \partial_x a||_{L^\infty L^2} \int_{t^0_\epsilon}^{t^1_\epsilon} \|D_x^{1/2}z\|_2^2.$$

By the hypothesis on the size of $\beta\lambda$, we can absorb the term on the right-hand side that arose from (2.19) into the left-hand side. This allows us to conclude that

$$w \in C((0, T) : H^{1/2}(\mathbb{R})), \quad D_xw \in L^2(\mathbb{R} \times [t^e_\epsilon, t^1_\epsilon]) \quad \text{for every } \epsilon > 0.$$

Reapplying this argument, it follows that $w = e^{\beta_\epsilon x}u \in C^\infty(\mathbb{R} \times (0, T))$. 

3. Appendix

Lemma 3.1. Let $T$ denote one of the following operators: $P_+, P_-$, or $H$, the Hilbert transform. Then for any $p \in (1, \infty)$ and any $l, m \in \mathbb{Z}^+$ there exists $c = c(p; l; m) > 0$ such that

$$\| \partial_x^l [T; a] \partial_x^m f \|_p \leq c \| \partial_x^{l+m} a \|_\infty \| f \|_p. \tag{3.1}$$

Proof. Without loss of generality we take $T = P_+$ and observe that

$$\partial_x^l [P_+; a] h = \sum_{j=0}^l c_j, [P_+; \partial_x^j a] \partial_x^{l-j} h,$$

so it suffices to prove (3.1) in the case $l = 0$. Also since

$$[P_+; a] \partial_x^m f = P_+(a \partial_x^m f) - a P_+ \partial_x^m f = P_+(a P_- \partial_x^m f) + P_+(a P_+ \partial_x^m f) - a P_+ \partial_x^m f$$

$$= P_+(a P_- \partial_x^m f) - (1 - P_+)(a P_+ \partial_x^m f) = P_+(a P_- \partial_x^m f) - P_-(a P_+ \partial_x^m f),$$

it suffices to show the inequality

$$\| P_+(a P_- \partial_x^m f) \|_p \leq c \| \partial_x^m a \|_\infty \| f \|_p \tag{3.2}$$

and the corresponding inequality for $P_-(a P_+ \partial_x^m f)$, the proof of which we omit as it is similar to the proof of (3.2). As we commented earlier, an inequality related to that in (3.2) was proved in [6].

To establish (3.2), we will use the Littlewood-Paley decomposition, following the approach and the notation given in [6]. First, we define functions $\eta$ and $\tilde{\eta}$ centered at the frequencies $\pm 1$. Let $\eta \in C_0^\infty(\mathbb{R}), \eta \geq 0, \text{supp } \eta \subseteq \pm (1/2, 2)$ with the condition

$$\sum_{\xi} \eta(2^{-k}\xi) = 1 \text{ for } \xi \neq 0. \quad \text{Let } \tilde{\eta} \in C_0^\infty(\mathbb{R}), \tilde{\eta} \geq 0, \text{supp } \tilde{\eta} \subseteq \pm (1/8, 8) \text{ with } \tilde{\eta}(\xi) = 1 \text{ for } \xi \in \pm [1/4, 4].$$

Then, define the associated multiplication operators $Q_k$ and $\tilde{Q}_k$ as follows:

$$(Q_k f)^\wedge(\xi) := \eta(2^{-k}\xi) \hat{f}(\xi) \text{ and } (\tilde{Q}_k f)^\wedge(\xi) := \tilde{\eta}(2^{-k}\xi) \hat{f}(\xi).$$

Let $P_k f := \sum_{j \leq k-3} Q_j f$; therefore, $(P_k f)^\wedge(\xi) = p(2^{-k}\xi) \hat{f}(\xi)$ with $p(0) = 1$ and $\text{supp } p \subseteq (1/4, 1/4)$.

Finally, define the cutoff function $\tilde{p} \in C_0^\infty(\mathbb{R})$ with $\tilde{p}(\xi) = 1$ for $\xi \in [-10, 10]$ and let $(\tilde{P}_k f)^\wedge(\xi) = \tilde{p}(2^{-k}\xi) \hat{f}(\xi)$.

Using that $(Q_k f)^\wedge$ is supported on $\pm (2^{-k-1}, 2^{k+1})$ and that $(P_k f)^\wedge$ is supported on $(-2^{-k-2}, 2^{k-2})$, we can compute that $\text{supp } (Q_k f P_k g)^\wedge \subseteq \pm (2^{-k-2}, 2^{k+2})$; therefore,

$$Q_k f P_k g = \tilde{Q}_k (Q_k f P_k g). \tag{3.3}$$

Also, since $\tilde{P}_k f = f$ if $\text{supp } \hat{f} \subset (-10 \cdot 2^k, 10 \cdot 2^k)$, we see that for $|j| \leq 2$,

$$Q_k f Q_k g = \tilde{P}_k (Q_k f Q_k g). \tag{3.4}$$
To prove the needed estimate (3.2), we first take the dyadic decomposition of the functions on the left-hand side and split the double sum into three parts \((l - k \leq -3, l - k \geq 3, \text{and } |l - k| \leq 2)\):

\[
P_+(a P_0 \partial_x^m f) = P_+ \left( \sum_{k,l} Q_k a P_-(Q_l \partial_x^m f) \right) = P_+ \left( \sum_k Q_k a P_-(P_k \partial_x^m f) \right) + \\
P_+ \left( \sum_k P_k a P_-(Q_k \partial_x^m f) \right) + P_+ \left( \sum_{|j| \leq 2} \sum_k Q_k a P_-(Q_{k-j} \partial_x^m f) \right) =: I + II + III.
\]

Since for all \(k \in \mathbb{Z}, \) \(\text{supp } (P_k a Q_k (P_0 \partial_x^m f))^\wedge \subset (-\infty, 0)\) it follows that \(II = 0\). To estimate \(I\), we use (3.3) to write

\[
I = \sum_k P_+ (Q_k a P_k (P_0 \partial_x^m f)) = \sum_k \tilde{Q}_k^+ (Q_k a P_k (P_0 \partial_x^m f))
\]

\[
=c \sum_k \int \int e^{ix(\xi + \mu)} \tilde{\eta}^+(2^{-k}(\xi + \mu)) \eta(2^{-k}\xi) p(2^{-k}\mu) \mu^m a(\xi) \chi_{\mathbb{R}^+}(\mu) \hat{f}(\mu) d\xi d\mu
\]

\[
=c \sum_k \int \int e^{ix(\xi + \mu)} m_k(\xi, \mu) \tilde{\mu}_m a(\xi) \chi_{\mathbb{R}^+}(\mu) \hat{f}(\mu) d\xi d\mu,
\]

where \(m_k(\xi, \mu) := m(2^{-k}\xi, 2^{-k}\mu)\), and \(m(\xi, \mu) := \tilde{\eta}^+(\xi + \mu) \eta(\xi) p(\mu) \left( \frac{\mu}{\xi} \right)^m\).

Let \(q, h \in C_0^\infty(\mathbb{R})\) with \(q \equiv 1\) on \(\text{supp } \eta, h \equiv 1\) on \(\text{supp } p\), \(\text{supp } h \subset (-1/2, 1/2)\), and \(\text{supp } q \subset \pm (1/4, 4)\), so that \(m(\xi, \mu) = \tilde{\eta}^+(\xi + \mu) \eta(\xi) p(\mu) \tau(\xi, \mu)\), with \(\tau(\xi, \mu) := q(\xi) h(\mu) \mu^{m-1}/\xi^m \in C_0^\infty(\mathbb{R}^2)\). Thus, we can write the function \(\tau\) as the Fourier transform of a Schwartz function:

\[
\tau(\xi, \mu) = c \int \int e^{i(\xi \theta + \mu \nu)} r(\theta, \nu) d\theta d\nu, \quad \text{for some } r \in S(\mathbb{R}^2).
\]

Hence,

\[
I = \int \nu \int_\theta \sum_k \tilde{Q}_k (Q_k^0 (\partial_x^m a) P_k^\nu (P_- f)) r(\theta, \nu) d\theta d\nu,
\]

where the symbols of \(Q_k^0\) and \(P_k^\nu\) are \(e^{i\theta_2-k \xi} \eta(2^{-k} \xi)\) and \(e^{i\nu_2-k \mu} 2^{-k} \mu p(2^{-k} \mu)\), respectively, which belong to the class considered in [9] (page 607). So using Lemma A.3 in [6] and
the Hardy-Littlewood maximal function $M$, it follows that
\[
\| \sum_k \tilde{Q}_k(Q_k^a(\partial_x^m a) P_k^\nu(P-f)) \|_p \leq c \| (\sum_k |Q_k^\nu(\partial_x^m a) P_k^\nu(P-f)|^2)^{1/2} \|_p \\
\leq c \| \sup_k |Q_k^\nu(\partial_x^m a)| (\sum_k |P_k^\nu(P-f)|^2)^{1/2} \|_p \\
\leq c \| M(\partial_x^m a) \|_\infty \| (\sum_k |P_k^\nu(P-f)|^2)^{1/2} \|_p \leq c \| \partial_x^m a \|_\infty \| f \|_p.
\]
(3.5)

Finally, note that $III = 0$ if $j = -2, -1, or 0$. Then, using (3.4), we find that
\[
III = P_+ \left( \sum_{j=1}^2 \sum_k Q_k(a) Q_{k-j}(P-\partial_x^m f) \right) = \sum_{j=1}^2 \sum_k \tilde{P}_k^+(Q_k(\partial_x^m a) Q_{k-j}^*(P-f)),
\]
where the operators $Q_k^*$ and $Q_{k-j}^*$ for $j = 1, 2$ are given by
\[
\tilde{Q}_k^* h(\xi) := \frac{\eta(2^{-j} \xi)}{(2^{-j} \xi)^m} h(\xi), \quad \tilde{Q}_{k-j}^* h(\xi) := (2^{-j} \xi)^m \eta(2^{-(j-j)} \xi) \tilde{h}(\xi).
\]
The symbols of these multipliers lie in the class considered in [6] and $\tilde{P}_k$ is uniformly bounded in $L^p$, so an argument similar to (3.5) provides the desired inequality. \qed

**Proposition 3.2.** Let $\alpha \in [0, 1)$, $\beta \in (0, 1)$ with $\alpha + \beta \in [0, 1]$. Then for any $p, q \in (1, \infty)$ and for any $\delta > 1/q$ there exists $c = c(\alpha; \beta; p; q; \delta) > 0$ such that
\[
\| D_\alpha^\delta[D_\beta^\delta a] D_\alpha^{1-(\alpha+\beta)} f \|_p \leq \| J_\alpha^\delta \partial_x a \|_q \| f \|_p,
\]
where $J := (1 - \partial_x^2)^{1/2}$.

Note. The inequality (3.6) still holds with the same proof for $\tilde{D}_\alpha^\delta = H D_\alpha^\delta$ in place of $D_\alpha^\delta$.

Also, in the case $\beta = 1$, we can use $[D_x; a] f = [H; a] \partial_x f + H(\partial_x a f)$ and (3.1) to obtain the inequality (3.6) with $q = \infty$ and $\delta = 0$.

**Proof.** We observe that
\[
D_\alpha^\delta[D_\beta^\delta a] D_\alpha^{1-(\alpha+\beta)} f = [D_x^\alpha a] D_\alpha^{1-(\alpha+\beta)} f - [D_x^\alpha a] D_\alpha^{1-\alpha} f.
\]
Therefore, it suffices to consider the case $\alpha = 0$. But the proof of this case follows by combining the argument in Proposition A.2, Lemma A.3, and Theorem A.8 in the appendix of [6] with $\alpha = 1$ and the Sobolev inequality, so it will be omitted. \qed
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