ENUMERATION OF RATIONAL CONTACT CURVES VIA TORUS ACTIONS

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ABSTRACT. We prove that some Gromov-Witten numbers associated to rational contact (Legendrian) curves in any contact complex projective space with arbitrary incidence conditions are enumerative. Also, we use Bott formula on the Kontsevich space to find the exact value of those numbers. As an example, the numbers of rational contact curves of low degree in $\mathbb{P}^3$ and $\mathbb{P}^5$ are computed. The results are consistent with existing results.

1. INTRODUCTION

A contact structure on a smooth complex manifold $X$ of odd dimension is a corank 1 non-integrable holomorphic distribution. For example, projective spaces $\mathbb{P}^{2n+1}$ carry a canonical contact structure. Roughly speaking, this structure can be defined by the association, for each point $p \in \mathbb{P}^{2n+1}$, of a hyperplane $H_p \subset \mathbb{P}^{2n+1}$ containing $p$.

Contact structures attracted a lot of interest in recent years, both from mathematicians and physicists, and their classification is still ongoing. For example, Ye proved in [Ye94] that a contact threefold is either $\mathbb{P}^3$ or $\mathbb{P}(T_M)$ for some smooth surface $M$. Druel [Dru99] proved that toric contact manifolds are either $\mathbb{P}^{2n+1}$ or $\mathbb{P}(T_{\mathbb{P}^1 \times \ldots \times \mathbb{P}^1})$. Finally, [Dru98] and [KPSW00] partially classify contact manifolds by the positivity of the canonical divisor using Mori theory.

A submanifold of a contact manifold $X$ is called contact (or Legendrian) if it is point-wise tangent to the contact distribution. Contact curves play a central role in the classification of contact manifolds, as well as in the theory of superminimal Riemann surfaces (see [Bry82]). In this article we are interested in rational contact curves $C \subset \mathbb{P}^{2n+1}$. Despite their importance, enumerative invariants of contact curves are largely unknown, even for projective spaces. The first paper to address this problem is [LV11], where mainly planar curves in $\mathbb{P}^3$ are analyzed. But the first work studying this problem using torus actions is Amorim’s Ph.D. thesis [Amo14]. The main result is the computation of the equivariant class of the class of contact curves in the Kontsevich moduli space $\overline{M}_{0,0}(\mathbb{P}^3, d)$. Using Atiyah-Bott formula, he computes an invariant related to the number of degree 3 (resp. 4) rational contact curves meeting 7 (resp. 9) general lines in $\mathbb{P}^3$.

In this article, we will prove that the moduli space of rational contact stable maps is of the expected dimension (Lemma 3.7). In particular, Gromov-Witten invariants of rational contact curves are enumerative (Corollary 3.9). Further, we will use torus...
actions and equivariant cohomology to provide virtually all enumerative invariants of rational contact curves of degree $d$ in any projective space meeting an arbitrary number of linear subspaces. In the last section, we list such invariants for $\mathbb{P}^3$ and $\mathbb{P}^5$, and low values of $d$ (Table 5.1, 5.2 and 5.3). More invariants will be in [MS].

The results of this article can be used to find any enumerative invariants of rational contact curves. These results are based on Atiyah-Bott formula, which usually requires to perform tedious computations. I think that quantum cohomology can be used to find a generating function giving recursively all enumerative invariants of rational contact curves. Such generating function would look like the generating function for usual rational curves in $\mathbb{P}^3$, see [FP97, Section 9]. This may be the topic of a future work.

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2. Contact structures

In this section we recall basic facts about complex contact structures. All varieties are defined over $\mathbb{C}$.

**Definition 2.1** ([Kob59]). Let $X$ be a $(2n + 1)$-dimensional complex manifold. A contact structure on $X$ is a open cover $\{U_i\}_i$ of $X$ together with 1-forms $\alpha_i$ on each $U_i$ such that:

1. at every point of $U_i$, the form $\alpha_i \wedge (d\alpha_i)^n$ is non-zero, and
2. if the intersection $U_i \cap U_j$ is non empty, then there exists a non-vanishing holomorphic function $f_{ij}$ on $U_i \cap U_j$ such that $\alpha_i = f_{ij} \alpha_j$ on $U_i \cap U_j$.

**Definition 2.2.** Let $X$ be a manifold with a contact structure $\{U_i, \alpha_i\}$. A curve $C \subset X$ is a contact curve if any local 1-form $\alpha_i$ vanishes at every smooth point of $C \cap U_i$.

All odd-dimensional projective spaces $\mathbb{P}^{2n+1} = \mathbb{P}(\mathbb{C}^{2n+2})$ are contact. Indeed, let $\{x_i\}_{i=0}^{2n+1}$ be local coordinates in $\mathbb{C}^{2n+2}$, and let

$$\alpha := \sum_{i=0}^{n} x_{2i}dx_{2i+1} - x_{2i+1}dx_{2i}$$

be a 1-form on $\mathbb{C}^{2n+2}\{0\}$. Let $\{U_i\}_i$ be an open cover of $\mathbb{P}^{2n+1}$, and for every index $i$ let $r_i$ be a holomorphic section over $U_i$ of the standard projection $\mathbb{C}^{2n+2}\{0\} \to \mathbb{P}^{2n+1}$. Then $\{U_i, r_i^*\alpha\}$ is a contact structure. The line given by the equations $x_i = 0$, $i = 1, \ldots, 2n$ is an example of a contact curve. The moduli space of contact lines in $\mathbb{P}^{2n+1}$ has a very explicit description.

**Example 2.3.** Let us consider the symplectic form $d\alpha \in \wedge^2(\mathbb{C}^{2n+2})^\vee$. Let $S$ be the tautological rank 2 vector bundle on the Grassmannian $G := G(2, 2n+2)$. The canonical inclusion $S \subset \mathbb{C}^{2n+2} \times O_G$ induces a surjective map:

$$\wedge^2(\mathbb{C}^{2n+2})^\vee \times O_G \to \wedge^2 S^\vee,$$

where $\wedge^2 S^\vee = O_G(1)$ is the Plücker line bundle. Hence $d\alpha$ induces a global section of $O_G(1)$, whose zero locus is exactly the locus of 2-dimensional subspaces of $\mathbb{C}^{2n+2}$.
isotropic with respect to $d\alpha$. Such locus is usually denoted by $SG(2,2n+2)$ and parameterizes contact projective lines in $\mathbb{P}^{2n+1}$.

**Remark 2.4.** It is not hard to see that the definition of contact manifold is equivalent to the following: A contact structure on $X$ is a pair $(X,L)$ where $L$ is a line subbundle of $\Omega_X^1$ such that if $s$ is a non trivial local section of $L$, then $s \wedge (ds)^{\wedge n}$ is everywhere non-zero. So, contact curves are exactly those curves $C$ such that $s|_{T_C} = 0$.

**Example 2.5.** Any symplectic form on $\mathbb{C}^{2n+2}$ induces in a natural way a subbundle $\mathcal{O}_{\mathbb{P}^{2n+1}}(-2) \hookrightarrow \Omega_{\mathbb{P}^{2n+1}}^1$. All contact structures of $\mathbb{P}^{2n+1}$ can be constructed this way, and they are equivalent by a change of coordinates, see [OSSG80, 1.4.2].

**Proposition 2.6.** Let $X$ be a manifold with a contact structure $\{U_i,\alpha_i\}_i$, and let $C \subset X$ be a curve. Let $L$ be the contact line bundle and let $\mathcal{D} := \ker(T_X \to L^\vee)$. The following are equivalent.

- $C$ is a contact curve.
- $T_{C,p} \subset \mathcal{D}_p$ for every smooth point $p \in C$.

Moreover if $p \in C \cap U_i$, then $T_{C,p}$ is an isotropic vector subspace of the symplectic space $(\mathcal{D}_p, d\alpha_i)$.

**Proof.** By Definition 2.1 it follows that for every 1-form $\alpha$ the 2-form $d\alpha$ restricted to $\mathcal{D}_p$ is a symplectic form.

Let $i : C \to X$ be the inclusion. The condition for $C$ to be contact becomes $i^* \alpha_i \equiv 0$, that is $T_{C,p} \subset \mathcal{D}_p$. On the other hand, if $T_{C,p} \subset \mathcal{D}_p$ then $C$ is contact.

Finally, $i^* \alpha \equiv 0$ implies $i^* d\alpha \equiv 0$ so that $T_{C,p}$ is an isotropic subspace of $(\mathcal{D}_p, d\alpha_i)$. \qed

3. **Contact stable maps**

**Definition 3.1.** A stable map of $\mathbb{P}^n$ is the datum $(C,f,p_1,\ldots,p_m)$ where $C$ is a projective, connected, nodal curve of arithmetic genus 0, the markings $p_1,\ldots,p_m$ are distinct nonsingular points of $C$, and $f : C \to \mathbb{P}^n$ is a morphism such that $f_*(-([C]))$ is a 1-cycle of degree $d$. Moreover, for every rational component $E \subset C$ mapped to a point, $E$ contains at least three points among marked points and nodal points. Two stable maps, $(C,f,p_1,\ldots,p_m)$ and $(C',f',p_1',\ldots,p_m')$, are equivalent if there exists an isomorphism $\varphi : C \to C'$ such that $f = f' \circ \varphi$ and $\varphi(p_i) = p_i'$ for all $i = 1,\ldots,m$.

For any non-negative integer $m$, we denote by $\overline{M}_{0,m}(\mathbb{P}^n,d)$ the moduli stack of stable maps of degree $d$, with coarse moduli space $\overline{M}_{0,m}(\mathbb{P}^n,d)$. Moreover, we denote by

$$
\U_m \xrightarrow{ev_{m,0}} \mathbb{P}^n
$$

the universal family. For $i = 1,\ldots,m$ we denote by $ev_i : \overline{M}_{0,m}(\mathbb{P}^n,d) \to \mathbb{P}^n$ the evaluations maps. Further, there are two interesting subschemes of $\overline{M}_{0,m}(\mathbb{P}^n,d)$. They are $M_{0,m}(\mathbb{P}^n,d)$ and $\overline{M}_{0,m}(\mathbb{P}^n,d)$, with the following properties:

- $M_{0,m}(\mathbb{P}^n,d)$ parameterizes stable maps with irreducible domain,
• $\overline{M}_{0,m}(\mathbb{P}^n, d)$ is smooth and it is a fine moduli space of automorphisms-free stable maps.

Both of them are open and dense in $\overline{M}_{0,m}(\mathbb{P}^n, d)$. See [FP97] for more details.

As mentioned in Example 2.5, the space $\mathbb{P}^{2n+1}$ has an essentially unique contact structure with contact line bundle $L = \mathcal{O}_{\mathbb{P}^{2n+1}}(-2)$.

**Definition 3.2.** Let $C$ be a projective, connected, nodal curve of arithmetic genus $0$. A contact morphism is a morphism $f : C \to \mathbb{P}^{2n+1}$ such that for any local section $s$ of $L$, the map $f^* s$ vanishes at every point of $C$.

A morphism $\mathbb{P}^1 \to \mathbb{P}^{2n+1}$ of degree $d$ is given by $2n+2$ sections of $H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(d))$, modulo a multiplicative constant. That is, a point

$$[s_0 : \ldots : s_{2n+1}] \in \mathbb{P}(V),$$

where $V$ is the $(2n+2)$-fold product of $H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(d))$. Such morphism is contact exactly when the pull back of the 1-form (2.1) vanishes. That is, when $\sum_{i=0}^{n} s_{2i} d s_{2i+1} - s_{2i+1} d s_{2i} = 0$. We will use the following result of Kobak and Loo.

**Proposition 3.3.** Let $\mathcal{M}_{n, d} \subset \mathbb{P}(V)$ be the moduli space of contact morphisms $\mathbb{P}^1 \to \mathbb{P}^{2n+1}$ of degree $d$. Then $\mathcal{M}_{n, d}$ is a connected quasi-projective reduced variety of dimension $2n(d+1) + 2$.

**Proof.** See [KL98, PROPOSITION 4.3].

Note that the property of being contact is clearly invariant by automorphisms of $C$.

**Definition 3.4.** A contact stable map is a stable map $(C, f, p_1, \ldots, p_m)$ of $\mathbb{P}^{2n+1}$ such that $f : C \to \mathbb{P}^{2n+1}$ is a contact morphism for some (hence, any) representative of $(C, f, p_1, \ldots, p_m)$. We denote by $S_m(d)$ the space of all contact stable maps of degree $d$ and $m$ marks, with canonical inclusion $j : S_m(d) \to \overline{M}_{0,m}(\mathbb{P}^{2n+1}, d)$. For every $i = 1, \ldots, m$, we have maps $\overline{ev}_i : S_m(d) \to \mathbb{P}^{2n+1}$ given by composition $\overline{ev}_i := ev_i \circ j$.

**Remark 3.5.** If $C$ is a contact curve contained in $\mathbb{P}^{2n+1}$, and $f : C \to \mathbb{P}^{2n+1}$ is the embedding, then $(C, f, p_1, \ldots, p_m)$ is a contact stable map. Anyway, not all contact stable maps are of this form. For example, if $f$ is a finite cover of a contact curve, then $(C, f, p_1, \ldots, p_m)$ is a contact stable map.

The moduli space of contact stable maps in $\mathbb{P}^{2n+1}$ is the zero locus of a section of a vector bundle on $\overline{M}_{0,m}(\mathbb{P}^{2n+1}, d)$, that is:

**Theorem 3.6.** $S_m(d)$ is the zero locus of a section of

$$E_{d,m} := \pi_*(\omega_{\pi} \otimes ev^*_{m,0} L^\vee).$$

In particular, the expected dimension of $S_m(d)$ is $2n(d+1) - 1 + m$.

**Proof.** We follow the approach of [LV11] 3.2. Let $(C, f, p_1, \ldots, p_m)$ be a stable map. By Proposition 2.6 (C, f, p_1, . . ., p_m) is contact if and only if the map $T_C \to f^* T_{\mathbb{P}^{2n+1}}$ factors through $f^* D$, i.e., the composition $T_C \to f^* L^\vee$ is the zero map. The locus of such maps is the locus where the map $T_\pi \to ev^*_{m,0} L^\vee$ vanishes along the whole fiber of $\pi$ over $(C, f, p_1, \ldots, p_m)$. The sheaf $E_{d,m}$ is locally free by [Har77] III.12.9 as $H^1(C, \omega_C \otimes f^* L^\vee) = h^0(C, f^* L) = 0$. It follows that the locus of contact
stable maps is given by a section of $\mathcal{E}_{d,m}$ by [AK77] (2.3). Note that the argument in Levcovitz-Vainsencher is made for $m = 0$. In the case $m > 0$ we may have some component $C' \subset C$ contracted by $f$. But in that case the map $T_{C'} \to f^* T_{\mathbb{P}^{2n+1}}$ is zero, so the result is still valid.

Since the rank of $\mathcal{E}_{d,m}$ is $\dim (\mathbb{P}^1, \omega_{\mathbb{P}^1} \otimes f^* \mathcal{L}) = \dim (\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(2d-2)) = 2d-1$, the expected dimension of $S_m(d)$ is $\dim \overline{M}_{0,m}(\mathbb{P}^{2n+1}, d) - (2d-1) = 2n(d+1) - 1 + m$. □

The following results are useful for enumerative applications.

**Lemma 3.7.** The moduli space $S_m(d)$ is purely of the expected dimension.

**Proof.** Let $X = \mathbb{P}^{2n+1}$. If $\delta : \overline{M}_{0,m+1}(X, d) \to \overline{M}_{0,m}(X, d)$ is the forgetful morphism of the last mark, we see for every $m \geq 0$ we have $S_{m+1}(d) = \delta^{-1}(S_m(d))$. Indeed, let $(D, g, p_1, \ldots, p_m, p_{m+1}) \in \overline{M}_{0,m+1}(X, d)$ and

$$\delta((D, g, p_1, \ldots, p_m, p_{m+1})) = (C, f, p_1, \ldots, p_m) \in \overline{M}_{0,m}(X, d).$$

By construction, there exists a map $\hat{\delta} : D \to C$ such that $g = f \circ \hat{\delta}$ and $\hat{\delta}$ is, in every component of $D$, either a contraction or an isomorphism. If $s$ is any local section of $L$, $s^*$'s vanishes at some point if and only if $\hat{\delta}^* (f^* s)$ vanishes. It follows that $(D, g, p_1, \ldots, p_m, p_{m+1})$ is contact if and only if $(C, f, p_1, \ldots, p_m)$ is contact.

Since all fibers of the forgetful map are connected and 1-dimensional, it follows that if $S_0(d)$ is purely of the expected dimension for some $d$, then the same holds for $S_m(d)$ for all $m \geq 0$.

We will prove the theorem by induction on $d$. We know that $S_0(1)$ is purely of the expected dimension by Example 2.3.

Since $S_m(d)$ is the zero section of a vector bundle, each component is at least of the expected dimension. Consider the moduli space $\mathcal{M}_{n,d}$ of Proposition 3.3. By the universal property of $\overline{M}_{0,0}(X, d)$, there exists a map $\mathcal{M}_{n,d} \to \overline{M}_{0,0}(X, d)$ which sends a contact morphism $f : \mathbb{P}^1 \to X$ to the contact stable map $(\mathbb{P}^1, f)$. We denote the image by $S_0(d)^\circ$. It is clear that $S_0(d)^\circ$ is the scheme of all contact stable maps with irreducible domain, i.e., $S_0(d)^\circ = S_0(d) \cap M_{0,0}(X, d)$. Moreover, the fiber of any point $(\mathbb{P}^1, f) \in S_0(d)^\circ$ is the orbit

$$(3.1) \{ \varphi \circ f : \mathbb{P}^1 \to X / \forall \varphi \in \text{Aut} (\mathbb{P}^1) \}.$$

Note that (3.1) is isomorphic to $\text{Aut}(\mathbb{P}^1)$ when $(\mathbb{P}^1, f)$ is automorphisms-free. Otherwise, it is a quotient of $\text{Aut}(\mathbb{P}^1)$ by a finite group. It follows that the dimension of each component of $S_0(d)^\circ$ is at most $2n(d+1) + 2 - 3 = 2n(d+1) - 1$, as expected.

So, the closure of $S_0(d)^\circ$ is purely of the expected dimension.

The space $S_0(d)$ may have components supported in the complement of $M_{0,0}(X, d)$. That complement is the union of boundary divisors of the form

$$(3.2) \overline{M}_{0,1}(X, d_1) \times_X \overline{M}_{0,1}(X, d_2),$$

for $d_1$ and $d_2$ positive integers such that $d_1 + d_2 = d$. Since the property of being contact is a local condition, a stable map contained in the variety of display (3.2) is contact if and only if each component is contact. It follows that the moduli space of contact stable maps contained in (3.2) is the fiber product:

$$S_1(d_1) \times_X S_1(d_2).$$
By induction, it has dimension:
\[ 2n(d_1 + 1) + 2n(d_2 + 1) - \dim X = 2n(d_1 + d_2 + 1) + 2n - 2n - 1 = 2n(d + 1) - 1. \]
That is, the same as \( S_0(d) \). So \( S_0(d) \) is purely of the expected dimension. \( \square \)

There exist results in other contexts similar to Theorem 3.6 and Lemma 3.7. See [Bry91, Proposition 5.1] and [Ye, 2.3&2.4].

**Theorem 3.8.** Let \( \Gamma_1, \ldots, \Gamma_m \) be subvarieties of \( \mathbb{P}^{2n+1} \), such that \( \sum_{i=1}^m \text{codim}(\Gamma_i) = 2n(d+1) - 1 + m \). Let \( G \) be the group of invertible complex matrices of order \( 2n+2 \). Then for a general \( \sigma = (g_1, \ldots, g_m) \in G^m \), the scheme-theoretic intersection
\[
S_m(d) \cap \text{ev}^{-1}_1(g_1 \Gamma_1) \cap \cdots \cap \text{ev}^{-1}_m(g_m \Gamma_m)
\]
is a finite number of reduced points supported in \( \overline{M}_{0,m}(\mathbb{P}^{2n+1}, d) \). Further, the number of points in this intersection is
\[
\int_{\overline{M}_{0,m}(\mathbb{P}^{2n+1}, d)} \text{ev}_1^*(\Gamma_1) \cdots \text{ev}_m^*(\Gamma_m) \cdot c_{\text{top}}(E_{d,m}).
\]

**Proof.** We may apply the celebrated Kleiman-Bertini Theorem [Kle74]: Let \( G \) be a connected algebraic group, and \( X \) a homogeneous \( G \)-variety, \( Y \) and \( Z \) varieties mapping to \( X \):

\[
\begin{array}{ccc}
Z & \xrightarrow{\beta} & X \\
Y & \xrightarrow{\alpha} & X
\end{array}
\]
For \( \sigma \in G \), denote by \( Y^\sigma \) the image of the composition \( Y \xrightarrow{\sim} X \xrightarrow{\sigma} X \). Then

(1) There is a dense open subscheme \( G^\circ \) of \( G \) such that, for \( \sigma \in G^\circ \), \( Y^\sigma \times_X Z \) is either empty, or of pure dimension \( \dim Y + \dim Z - \dim X \).

(2) Further, if \( Y \) and \( Z \) are nonsingular, then \( G^\circ \) can be found so that \( Y^\sigma \times_X Z \) is nonsingular.

Let us denote \( X = \mathbb{P}^{2n+1} \), which is a homogeneous \( G \)-variety with respect to the group of invertible matrices of order \( 2n + 2 \). If we apply Kleiman-Bertini to the diagram

\[
\begin{array}{ccc}
S_m(d) & \xrightarrow{(\text{ev}_1 \times \cdots \times \text{ev}_m)} & \Gamma_1 \times \cdots \times \Gamma_m =: \Gamma \xrightarrow{=} X^m \times m
\end{array}
\]
we deduce that, for a general \( \sigma \in G^m \), \( \Gamma^\sigma \times_X S_m(d) \) is either empty, or of dimension 0. We used that \( S_m(d) \) has the expected dimension. We assume, without loss of generality, that \( \Gamma^\sigma \) is smooth by applying again Kleiman-Bertini. In the same way we may prove\(^1\) that the intersection in (3.3) is supported in the intersection between \( \overline{M}_{0,m}(X, d) \) and the smooth part of \( S_m(d) \).

\(^1\)Details can be found in [Tho98 Lemma p. 5] or [FP97 Lemma 14].
So, when $\Gamma^\sigma \times_{X \times \ast} S_m(d)$ is not empty, it is reduced of dimension 0. Standard arguments of intersection theory tell us that $\text{(3.3)}$ is the degree of the zero cycle
\[
\int_{S_m(d)} \bar{ev}^*_1(\Gamma_1) \cdots \bar{ev}^*_m(\Gamma_m),
\]
in the Chow ring of $S_m(d)$. The equivalence between $\text{(3.4)}$ and $\text{(3.5)}$ follows from the fact that $j_*[S_m(d)] = c_{\text{top}}(\mathcal{E}_{d,m})$ by Theorem $\text{3.6}$.

**Corollary 3.9.** Let $\Gamma_1, \ldots, \Gamma_m$ be subvarieties of $\mathbb{P}^{2n+1}$ in general position, such that $\sum_{i=1}^m \text{codim}(\Gamma_i) = 2n(d+1) - 1 + m$. Then the number of contact curves in $\mathbb{P}^{2n+1}$ of degree $d$ meeting all $\Gamma_1, \ldots, \Gamma_m$ is
\[
\int_{\overline{M}_{0,m}(\mathbb{P}^{2n+1}, d)} \bar{ev}^*_1(\Gamma_1) \cdots \bar{ev}^*_m(\Gamma_m) \cdot c_{\text{top}}(\mathcal{E}_{d,m}).
\]

**Proof.** The zero cycle $\text{(3.6)}$ equals the number of contact stable maps $(C, f, p_1, \ldots, p_m)$ satisfying $f(p_i) \in g_i \Gamma_i$ for some general $g_i$. The property of being in general position is preserved by the action of a general $\sigma \in G^{x,m}$, so we can substitute all $\Gamma_i$ with $g_\sigma \Gamma_i$. Since these stable maps are automorphisms-free, the set of all curves $f(C) \subset \mathbb{P}^{2n+1}$ is the set of all contact curves meeting all $\Gamma_1, \ldots, \Gamma_m$.

**Remark 3.10.** We have seen that $S_m(d)$ is purely of the expected dimension, but it is far from being irreducible. Indeed, the components of $S_m(d)$ are nasty fiber products
\[
S_{m_1}(d_1) \times_{\mathbb{P}^{2n+1}} \cdots \times_{\mathbb{P}^{2n+1}} S_{m_k}(d_k),
\]
where $(d_1, \ldots, d_k)$ is a partition of $d$. Since all of them have the same dimension, all of them have non trivial contribution to $\text{(3.4)}$. So Equation $\text{(3.4)}$ considers all irreducible contact curves as well as all reducible ones.

We can reduce to the case $m = 0$ thanks to the following

**Lemma 3.11.** Let $\Gamma_1, \ldots, \Gamma_m$ be subvarieties of $X = \mathbb{P}^{2n+1}$, such that $\sum_{i=1}^m \text{codim}(\Gamma_i) = 2n(d+1) - 1 + m$. Let $\delta : \overline{M}_{0,1}(X,d) \to \overline{M}_{0,0}(X,d)$ be the forgetful map. Then
\[
\int_{\overline{M}_{0,m}(X,d)} \bar{ev}^*_1(\Gamma_1) \cdots \bar{ev}^*_m(\Gamma_m) \cdot c_{\text{top}}(\mathcal{E}_{d,m})
= \int_{\overline{M}_{0,0}(X,d)} \delta_* (\bar{ev}^*_1(\Gamma_1)) \cdots \delta_* (\bar{ev}^*_m(\Gamma_m)) \cdot c_{\text{top}}(\mathcal{E}_{d,0})
\]

**Proof.** Suppose $m = 1$ and consider the natural diagram of stacks
\[
\begin{array}{ccc}
\mathcal{U}_1 & \xrightarrow{\delta} & \mathcal{U}_0 \\
\pi' \downarrow & & \downarrow \pi \\
\overline{M}_{0,1}(X,d) & \xrightarrow{\delta} & \overline{M}_{0,0}(X,\beta)
\end{array}
\]
where $\tilde{\delta}$ is the natural map such that $\bar{ev}_{1,0} = ev_{0,0} \circ \tilde{\delta}$. The square is clearly Cartesian, so that $\delta^* \omega_\pi = \omega_{\pi'}^*$ and $\delta^* \mathcal{E}_{d,0} = \mathcal{E}_{d,1}$. By $\text{[Vis80]} \text{(3.9) Lemma}$ and
projection formula we have:

\[
\int_{\overline{M}_{0,1}(X,d)} \text{ev}_1^*(\Gamma_1) \cdot c_{\text{top}}(\mathcal{E}_{d,1}) = \delta_s \int_{\overline{M}_{0,1}(X,d)} \text{ev}_1^*(\Gamma_1) \cdot c_{\text{top}}(\delta^* \mathcal{E}_{d,0}) = \delta_s \int_{\delta^*(\overline{M}_{0,0}(X,d))} \text{ev}_1^*(\Gamma_1) \cdot \delta^* c_{\text{top}}(\mathcal{E}_{d,0}) = \int_{\overline{M}_{0,0}(X,d)} \delta_s(\text{ev}_1^*(\Gamma_1)) \cdot c_{\text{top}}(\mathcal{E}_{d,0})
\]

Since the intersection

\[
\int_{\overline{M}_{0,1}(X,d)} \text{ev}_1^*(\Gamma_1) \cdot c_{\text{top}}(\mathcal{E}_{d,1})
\]

is supported in the automorphisms-free locus, we get Eq. \(3.7\). In order to prove the general case, it is enough to repeat the same argument. \(\square\)

4. Equivariant classes in \(\mathbb{P}^n\)

Let us recall [CK99] Chapter 9. The standard action of \(T = (\mathbb{C}^*)^{n+1}\) on \(\mathbb{P}^n\) has \(n + 1\) fixed points \(\{q_i\}_{i=0}^n\) and \(n(n + 1)/2\) coordinate (invariant) lines. We fix an isomorphism

\[
H^*(BT) \cong \mathbb{C}[\lambda_0, \ldots, \lambda_n],
\]

where \(\lambda_i\) is the corresponding weight of the \(T\)-action on \(\mathcal{O}_{\mathbb{P}^n}(-1)|q_i\). There is an induced \(T\)-action on \(\overline{M}_{0,m}(\mathbb{P}^n, d)\), with fixed points given by stable maps \((C, f, p_1, \ldots, p_m)\) such that each component of \(C\) is either mapped by \(f\) to some \(q_i\) or it is the cover of a coordinate line. Moreover marked points, nodes of \(C\) and ramification points of \(f\) are mapped to fixed points. In particular, if \(C'\) is a component of \(C\) and a degree \(d'\) cover of a coordinate line via \(f\), then locally \(f\) is represented by homogeneous coordinates \((x_0, x_1) \mapsto (x_0^{d'}, x_1^{d'})\).

For each fixed stable map \((C, f, p_1, \ldots, p_m)\) we associate a colored tree \(\Gamma\). The vertices \(v\) are in one-to-one correspondence with the connected components \(C_v\) of \(f^{-1}(\{q_0, \ldots, q_n\})\). Each vertex \(v\) is labeled by the point \(q_v \in \{q_i\}_{i=0}^n\) such that \(f|_{C_v} = q_v\). Moreover, the edges correspond to those irreducible components mapped by \(f\) onto some coordinate line. We associate to each edge \(e\) the degree \(d_e\) of the cover it represents. We define the set \(S_v\) consisting of those indexes \(i \in \{1, \ldots, m\}\) for which \(p_i \in C_v\). An isomorphism of such trees is an isomorphism of graphs preserving the labels \(q_v, S_v, d_v\).

The connected components of the fixed point locus of \(\overline{M}_{0,m}(\mathbb{P}^n, d)\) are parameterized by all possible trees \(\Gamma\), modulo isomorphisms. For each equivariant vector bundle \(V\) on \(\overline{M}_{0,m}(\mathbb{P}^n, d)\), and each polynomial \(P(V)\) in the Chern classes of \(V\), we will denote by \(P^T(V)\) the corresponding equivariant polynomial of \(P(V)\). For example, \(c_{\text{top}}^T(V)\) is the equivariant Euler class of \(V\). We use the following notation:

- \(i_\Gamma : \overline{M}_\Gamma \hookrightarrow \overline{M}_{0,m}(\mathbb{P}^n, d)\) is the substack of fixed maps with corresponding graph \(\Gamma\),
- \(P^T(V)(\Gamma) = i_\Gamma^* \left( P^T(V) \right)\) is the pull back in equivariant cohomology
- \(i_\Gamma^* : H^*_T(\overline{M}_{0,m}(\mathbb{P}^n, d)) \otimes \mathbb{C}(\lambda_0, \ldots, \lambda_n) \to H^*_T(\overline{M}_\Gamma) \otimes \mathbb{C}(\lambda_0, \ldots, \lambda_n)\),
- \(G(d)\) is the set of all possible trees, modulo isomorphisms, corresponding to fixed maps.

\(^2\)A tree is a connected, undirected, simple graph.
$V_1$ is the set of all vertices of $\Gamma$,
$E_1$ is the set of all edges of $\Gamma$,
for each $v \in V_1$, $\lambda_v \in \mathbb{C}[\lambda_0, \ldots, \lambda_n]$ is the weight of the $T$-action on $O_{\mathbb{P}^n}(-1)|_{q_v}$,
for each $e \in E_1$, $d_e$ and $(e(i), e(j))$ are, respectively, the degree of the cover corresponding to $e$ and the pair of labels of its vertices.

In order to compute the number of contact curves, we will use Bott formula (see Bot67, Kon95):

\begin{equation}
\int_{\overline{M}_{0,m}(\mathbb{P}^n,d)} P(V) = \sum_{\Gamma \in G(d)} \frac{1}{a_\Gamma} \frac{P^T(V)(\Gamma)}{c^T_{\text{top}}(N_\Gamma)(\Gamma)},
\end{equation}

where $N_\Gamma$ is the normal bundle of $\overline{M}_\Gamma$, and

$$a_\Gamma := |\text{Aut}(\Gamma)| \cdot \prod_{e \in E_1} d_e,$$

where $|\text{Aut}(\Gamma)|$ is the order of the group of automorphisms of $\Gamma$.

Once we can compute $P^T(V)(\Gamma)$ and $a_\Gamma$, we can apply formula (4.1) using a computer. In the case $m = 0$ (so, no marked points), in order to find $G(d)$ we have to find all trees colored with $\{q_0, \ldots, q_n\}$, and where each edge $e$ has a weight $d_e \geq 1$ such that $\sum_e d_e = d$. For example, $G(1)$ contains the following trees:

$q_i \quad 1 \quad q_j \quad 0 \leq i < j \leq n, \quad a_\Gamma = 1.$

On the other hand, the set $G(2)$ contains the following trees:

$q_i \quad 1 \quad q_j \quad 1 \quad q_k \quad 0 \leq i < k \leq n, \quad 0 \leq j \leq n, \quad j \neq i, k, \quad a_\Gamma = 1.$

$q_i \quad 1 \quad q_j \quad 1 \quad q_i \quad 0 \leq i, j \leq n, \quad i \neq j, \quad a_\Gamma = 2.$

$q_i \quad 2 \quad q_j \quad 0 \leq i < j \leq n, \quad a_\Gamma = 2.$

We will use Bott formula only in the case without marked points thanks to Lemma 3.11. Since we are interested in the number of contact curves meeting a certain number of linear subspaces, we need to compute both the equivariant classes $c^T_{\text{top}}(\mathcal{E},d_0)$ and the equivariant class of the cycle of curves meeting a linear subspace.

4.1. Contact curves. We will denote $\mathcal{E}_{d,0}$ by $\mathcal{E}_d$. The fiber of $\mathcal{E}_d$ at a point $(C, f)$ of $\overline{M}_{0,0}(\mathbb{P}^n, d)$ is the vector space $H^0(C, \omega_C \otimes f^*O_{\mathbb{P}^n}(2))$, where $\omega_C$ is the dualizing sheaf of $C$. There exists a canonical isomorphism $H^0(C, \omega_C \otimes f^*O_{\mathbb{P}^n}(2)) \cong H^1(C, f^*O_{\mathbb{P}^n}(-2))^\vee$. So the weights of $H^0(C, \omega_C \otimes f^*O_{\mathbb{P}^n}(2))$ are the weights of $H^1(C, f^*O_{\mathbb{P}^n}(-2))$ multiplied by $(-1)$.

Lemma 4.1. Let $\Gamma \in G(d)$. Suppose that there exist two trees $\Gamma_1 \in G(d_1)$, $\Gamma_2 \in G(d_2)$ such that $\Gamma$ is the union of $\Gamma_1$ and $\Gamma_2$ at a vertex $v$. Then

\begin{equation}
c^T_{\text{top}}(\mathcal{E}_d)(\Gamma) = c^T_{\text{top}}(\mathcal{E}_{d_1})(\Gamma_1) c^T_{\text{top}}(\mathcal{E}_{d_2})(\Gamma_2)(2\lambda_v).
\end{equation}

Proof. Let $(C, f)$ be a fixed stable map corresponding to $\Gamma$. Let $(C_1, f_1, q_1)$ and $(C_2, f_2, q_2)$ be fixed stable maps corresponding to, respectively, $\Gamma_1$ and $\Gamma_2$ such that the marked point is in $C_v$. In particular, $f_1(q_1) = f_2(q_2) = q_v$. The standard gluing map

$$\overline{M}_{0,1}(\mathbb{P}^n, d_1) \times_{\mathbb{P}^n} \overline{M}_{0,0}(\mathbb{P}^n, d_2) \to \overline{M}_{0,0}(\mathbb{P}^n, d).$$
sends \(((C_1, f_1, q_1^0), (C_2, f_2, q_2^0))\) to \((C, f)\). Let us consider the normalization exact sequence tensorized by \(\mathcal{O}_{\P^n}(-2)\) (see, e.g. [ACG11] pag 88):

\[
0 \to f^* \mathcal{O}_{\P^n}(-2) \to f_1^* \mathcal{O}_{\P^n}(-2) \oplus f_2^* \mathcal{O}_{\P^n}(-2) \to \mathcal{O}_{\P^n}(-2)|_{q'} \to 0.
\]

If we take the dual of the exact sequence given by the spaces of global sections, we get the following short exact sequence:

\[
0 \to H^1(C_1, f_1^* \mathcal{O}_{\P^n}(-2))^\vee \oplus H^1(C_2, f_2^* \mathcal{O}_{\P^n}(-2))^\vee \to H^1(C, f^* \mathcal{O}_{\P^n}(-2))^\vee \to 0.
\]

This immediately implies Eq. (4.2). \(\square\)

**Proposition 4.2.** Let \(\Gamma \in G(d)\) be a tree. Then

\[
c^\top_{\text{top}}(E_d)(\Gamma) = \left( \prod_{e \in E_\Gamma} 2d_e - 1 \prod_{\alpha = 1}^{d_e - 1} \frac{\alpha \lambda_{e(i)} + (2d_e - \alpha) \lambda_{e(j)}}{d_e} \right) \prod_{v \in V_\Gamma} (2\lambda_{v})^{\text{val}(v) - 1}
\]

where \(\text{val}(v)\) is the number of edges connected to \(v\).

Proposition 4.2 was first proved in [Amo14, Proposition 3.1.3] with the following strategy. Let \(f : \P^1 \to \P^3\) be a stable curve of degree \(d\). If we take the Euler exact sequence of \(\P^1\) [Har77, II.8.13] and tensor it by \(\mathcal{O}_{\P^1}(2d) = f^* \mathcal{O}_{\P^3}(2)\), we get the vector space \(H^0(\P^1, \omega_{\P^1} \otimes f^* \mathcal{O}_{\P^3}(2))\) as the kernel of the surjective map

\[
H^0(\P^1, \mathcal{O}_{\P^1}(2d - 1) \otimes f^* \mathcal{O}_{\P^3}(2)) \to H^0(\P^1, \mathcal{O}_{\P^1}(2d)).
\]

He found the weights of the action on \(H^0(\P^1, \omega_{\P^1} \otimes f^* \mathcal{O}_{\P^3}(2))\) from the weights of \(H^0(\P^1, \mathcal{O}_{\P^1}(2d - 1) \otimes f^* \mathcal{O}_{\P^3}(2))\) and \(H^0(\P^1, \mathcal{O}_{\P^1}(2d))\), and from the map in (4.3).

**Proof of Proposition 4.2.** Suppose that \(\Gamma = e\) is the graph with only one edge and labels \((q_0, q_1)\) corresponding to a degree \(d\) map \((\P^1, f)\). One can show that a basis for \(H^1(\P^1, f^* \mathcal{O}_{\P^n}(-2))\) is given by the Čech cocycles:

\[
\zeta_0^{-\alpha} \zeta_1^{-(2d - \alpha)}, 1 \leq \alpha \leq 2d - 1,
\]

see [Har77, III.5.5.1]. We know that \(x_0, x_1 \in H^0(\P^1, \mathcal{O}_{\P^1}(1))\) have weights \(\lambda_i, \lambda_j\). Since \(x_k = \zeta_k^d\) via \(f\), it follows that \(\zeta_0, \zeta_1\) have weights \(\frac{\lambda_i}{d}, \frac{\lambda_j}{d}\). Hence the cocycles in (4.4) have weights

\[
-\frac{\alpha \lambda_i + (2d - \alpha) \lambda_j}{d}, 1 \leq \alpha \leq 2d - 1.
\]

So the equivariant Euler class of \(E_d\) is

\[
\prod_{\alpha = 1}^{2d-1} (-1)^{-\frac{\alpha \lambda_i + (2d - \alpha) \lambda_j}{d}} = \prod_{\alpha = 1}^{2d-1} \frac{\alpha \lambda_i + (2d - \alpha) \lambda_j}{d}.
\]

We can prove the proposition for any \(\Gamma\) by induction on the number of edges and using Lemma 4.1. This concludes the proof. \(\square\)
4.2. Incidence to a linear subspace. The cycle of stable maps whose image meets a general codimension $r + 1$ linear subspace of $\mathbb{P}^n$ is known to be $\pi_*(h^{r+1})$ where $h = c_1(\mathcal{O}_{\mathbb{P}^n}(1))$. In this section, we will represent $\pi_*(h^{r+1})$ as a linear combination of polynomials in the Chern classes of vector bundles. It follows that the equivariant class of $\pi_*(h^{r+1})$, which we denote by $[\pi_*(h^{r+1})]^T$, is the linear combination of the equivariant version of the same polynomials. At the very end the result will be the following.

**Theorem 4.3.** Let $\Gamma \in G(d)$. The equivariant class for the incidence to a codimension $r + 1$ linear subspace of $\mathbb{P}^n$ in the component $\overline{\mathcal{M}}_{\Gamma}$ is

$$\sum_{e \in E_{\Gamma}} d_e \sum_{t=0}^{r} X_{e(i)}^t X_{e(j)}^{r-t}.$$

See [Amo14, Proposição 3.1.2] for the case $r = 1$, with a similar strategy.

**Lemma 4.4.** Let $d$, $r$ be integers. Let $V$ be the matrix of order $r + 1$ whose coefficients are $(V)_{a,k} = a^k$. Let $p(x) = \sum_i p_i x^i$ be a polynomial of degree at most $r + 1$ with $p(0) = 0$. Let $V_{p(x)}$ be the matrix of order $r + 1$ such that $(V_{p(x)})_{a,k} = (V)_{a,k}$ for $k \neq r + 1$, and $(V_{p(x)})_{a,r+1} = p(ad)$. That is,

$$V_{p(x)} := \begin{pmatrix}
1 & 1^2 & 1^3 & 1^4 & \cdots & 1^r & p(d) \\
2 & 2^2 & 2^3 & 2^4 & \cdots & 2^r & p(2d) \\
\vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\
[r+1] & (r+1)^2 & (r+1)^3 & (r+1)^4 & \cdots & (r+1)^r & p((r+1)d) \\
\end{pmatrix}.$$

Then $\det V_{p(x)} = d^{r+1} p_{r+1} \det V$.

**Proof.** If $p(x) = p_k x^k$ is a monomial with $1 \leq k \leq r$, we see that the last column of $V_{p(x)}$ is a multiple of the $k^{th}$ column, then $\det V_{p(x)} = 0$. So can suppose that $p(x) = p_{r+1} x^{r+1}$, and the result follows easily. $\square$

**Lemma 4.5.** The cycle $\pi_*(h^{r+1})$ is a linear combination of Chern characters of equivariant vector bundles.

**Proof.** Let $a$ be an integer, consider the sheaf $\pi_*(\mathcal{L}(a))$ on $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^n, d)$, where $\mathcal{L}(a) := \mathcal{O}_{\mathbb{P}^n}(a))$. Given a stable map $(C, f)$, using Riemann-Roch we can see that $h^1(C, f^* \mathcal{O}_{\mathbb{P}^n}(a)) = 0$ for every $a \geq 0$, hence for those values of $a$ we have that $\pi_*(\mathcal{L}(a))$ is a vector bundle and $\pi_*(\mathcal{L}(a)) = \pi_*(\mathcal{L}(a))$. Let $td(T_\pi)$ be the Todd’s class of the coherent sheaf $T_\pi$, and denote by $td(T_\pi)_k$ its $k$-codimensional component. We apply Grothendieck-Riemann-Roch Theorem (GRR) [Ful98 Chapter 18] (see also [HM06, 3E])

$$\text{ch}(\pi_*(\mathcal{L}(a))) = \pi_* (\text{ch}(\mathcal{L}(a)) \cdot td(T_\pi)) = \pi_* \left( \left(1 + ah + \frac{(ah)^2}{2!} + \cdots \right)(td(T_\pi)_0 + td(T_\pi)_1 + td(T_\pi)_2 + \ldots) \right) = \pi_* \left( [td(T_\pi)_0] + [td(T_\pi)_1 + ah] + \ldots \right).$$

Moreover, since $\text{ch}(\pi_*(\mathcal{L}(0))) = 0$ for $i \geq 1$, by GRR

$$\begin{align*}
\text{ch}(\pi_*(\mathcal{L}(0))) & = \pi_*(\text{ch}(\mathcal{L}(0)) \cdot td(T_\pi)) \\
& = \pi_*(td(\mathcal{L}(0) \otimes T_\pi)) \\
& = \pi_*(td(T_\pi)),
\end{align*}$$

$$\begin{align*}
\text{ch}(\pi_*(\mathcal{L}(a))) & = \pi_* (\text{ch}(\mathcal{L}(a)) \cdot td(T_\pi)) \\
& = \pi_* \left( \left(1 + ah + \frac{(ah)^2}{2!} + \cdots \right)(td(T_\pi)_0 + td(T_\pi)_1 + td(T_\pi)_2 + \ldots) \right) = \pi_* \left( [td(T_\pi)_0] + [td(T_\pi)_1 + ah] + \ldots \right).$$

Then $\det V_{p(x)} = d^{r+1} p_{r+1} \det V$.

**Proof.** If $p(x) = p_k x^k$ is a monomial with $1 \leq k \leq r$, we see that the last column of $V_{p(x)}$ is a multiple of the $k^{th}$ column, then $\det V_{p(x)} = 0$. So can suppose that $p(x) = p_{r+1} x^{r+1}$, and the result follows easily. $\square$
We can apply Cramer’s rule and get an explicit expression

\[ \text{ch}_r(\pi_*c(L)) = \sum_{k=1}^{r+1} a^k \cdot \pi_* \left( \frac{\text{td}(T_{\pi})(r+1-k)h^k}{k!} \right) \]

(4.5)

Now we use a little bit of linear algebra. In (4.5), we consider

\( \pi_* \left( \frac{\text{td}(T_{\pi})(r+1-k)h^k}{k!} \right) \) as

unknowns, so for \( a = 1, 2, 3, \ldots, r + 1 \) we get \( r + 1 \) linearly independent equations. We can apply Cramer’s rule and get an explicit expression

\[
\frac{1}{(r+1)!} \pi_* (h^{r+1}) = \frac{1}{\det V} \begin{vmatrix}
1 & 1^2 & \cdots & 1^r & \text{ch}_r(\pi_*c(1)) \\
2 & 2^2 & \cdots & 2^r & \text{ch}_r(\pi_*c(2)) \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
r + 1 & (r+1)^2 & \cdots & (r + 1)^r & \text{ch}_r(\pi_*c(r + 1))
\end{vmatrix},
\]

where \( V \) is the matrix of the linear system, i.e., \( (V)_{a,k} = a^k \) for \( a, k = 1, \ldots, r + 1 \).

\[ \square \]

Lemma 4.6. Let \( \Gamma \in G(d) \). Suppose that there exist two trees \( \Gamma_1 \in G(d_1), \Gamma_2 \in G(d_2) \) such that \( \Gamma \) is the union of \( \Gamma_1 \) and \( \Gamma_2 \). Then

\[ [\pi_* (h^{r+1})]^T(\Gamma) = [\pi_* (h^{r+1})]^T(\Gamma_1) + [\pi_* (h^{r+1})]^T(\Gamma_2). \]

Proof. We will argue like in Lemma 4.1. Let \( v \) be the common vertex of \( \Gamma_1 \) and \( \Gamma_2 \). From the normalization exact sequence tensorized by \( O_{\mathbb{P}^n}(a) \), we get the short exact sequence

\[
0 \to H^0(C, f^*O_{\mathbb{P}^n}(a)) \to H^0(C_1, f_1^*O_{\mathbb{P}^n}(a)) \oplus H^0(C_2, f_2^*O_{\mathbb{P}^n}(a)) \to H^0(q_v, O_{\mathbb{P}^n}(a)|_{q_v}) \to 0.
\]

This immediately implies that

\[
\text{ch}_r^{T}(\pi_*c(L))(\Gamma) = \text{ch}_r^{T}(\pi_*c(L))(\Gamma_1) + \text{ch}_r^{T}(\pi_*c(L))(\Gamma_2) - \text{ch}_r^{T}(\pi_*c(L))(v).
\]

Since \( \pi_* (h^{r+1}) \) is a linear combination of \( \{\text{ch}_r(\pi_*c(L))\}_{r+1} \), in order to get \( [\pi_* (h^{r+1})]^T(\Gamma) \) we just need to substitute in the expression of \( \pi_* (h^{r+1}) \) all \( \text{ch}_r(\pi_*c(L)) \) by \( \text{ch}_r^{T}(\pi_*c(L))(\Gamma) \).

Since

\[
\text{ch}_r^{T}(\pi_*c(L))(v) = a^r \lambda^r_v,
\]

by Lemma 4.4 we see that \( \text{ch}_r^{T}(\pi_*c(L))(v) \) has no influence on the value of \( [\pi_* (h^{r+1})]^T(\Gamma) \), so by linearity we get Eq. (4.6).

\[ \square \]

Finally, we can prove the theorem.

Proof of Theorem 4.3. Suppose that \( \Gamma = e \) is the graph with only one edge and labels \( (q, q_j) \) corresponding to a degree \( d \) map \( (\mathbb{P}^1, f) \). The equivariant polynomial \( r!\text{ch}_r^{T}(\pi_*c(L))(\Gamma) \) is given by the sum of the \( r \)-powers of all weights of the action
on $H^0(\mathbb{P}^1, f^*\mathcal{O}_{\mathbb{P}^1}(a))$. That is

$$
\text{ch}_T^*(\pi_*\mathcal{L}(a))(\Gamma) = \frac{1}{r!} \sum_{k=0}^{ad} \left( \frac{(ad-k)\lambda_i + k\lambda_j}{d} \right)^r
$$

$$
= \frac{1}{d^r r!} \sum_{k=0}^{ad} \left( \sum_{t=0}^r \binom{r}{t} (ad-k)^t k^{r-t} \lambda_i^t \lambda_j^{r-t} \right)
$$

(4.7)

$$
= \frac{1}{d^r r!} \sum_{t=0}^r \binom{r}{t} \left( \sum_{k=0}^{ad} (ad-k)^t k^{r-t} \right) \lambda_i^t \lambda_j^{r-t}.
$$

Equation (4.7) tells us that each $\text{ch}_T^*(\pi_*\mathcal{L}(a))(\Gamma)$ is a linear combination of monomials $\lambda_i^t \lambda_j^{r-t}$ for $t = 0, \ldots, r$. This implies that also $[\pi_*(hr+1)]^T(\Gamma)$ is a linear combination of the same monomials. By linearity, the coefficient of the monomial $\lambda_i^t \lambda_j^{r-t}$ in $[\pi_*(hr+1)]^T(\Gamma)$ is

$$
1 \frac{1}{d^r r!} \sum_{k=0}^{ad} (ad-k)^t k^{r-t} \lambda_i^t \lambda_j^{r-t}.
$$

(4.8)

In order to prove the theorem, we need to show that the value of (4.8) is $d$.

**Claim 4.7.** There exists a degree $r+1$ polynomial $p(x) = \sum_i p_i x^i \in \mathbb{Q}[x]$ such that $p(0) = 0$, $p_{r+1} = (r+1)^{-1}$ and

$$
p(ad) = \binom{r}{t} \sum_{k=0}^{ad} (ad-k)^t k^{r-t}, \; 1 \leq a \leq r + 1.
$$

(4.9)

**Proof.** For every integer $q \geq 0$, let us denote by $S^q(x)$ the Faulhaber polynomial of degree $q+1$. That is, a polynomial with leading coefficient $(q+1)^{-1}$ such that for every integer $n \geq 0$, $S^q(n) = \sum_{k=1}^{n} k^q$. If we define

$$
p(x) := \binom{r}{t} \sum_{j=0}^{t} (-1)^j \binom{t}{j} x^{t-j} S^{r+j-t}(x),
$$

See [Ber09], or [Knu77, pag 503] for a modern reference.
it is clear that \( p(x) \) has degree \( r + 1 \) and \( p(0) = 0 \). Moreover, it satisfies (4.9), indeed:

\[
p(ad) = \binom{r}{t} \sum_{j=0}^{\frac{t}{2}} (-1)^j \binom{t}{j} (ad)^{t-j} S^{r+j-t}(ad)
\]

\[
= \binom{r}{t} \sum_{j=0}^{\frac{t}{2}} (-1)^j \binom{t}{j} (ad)^{t-j} \left( \sum_{k=0}^{r+j-t} k r^r j^r \right)
\]

\[
= \binom{r}{t} \sum_{k=0}^{ad} \left( \sum_{j=0}^{t} (-1)^j \binom{t}{j} (ad)^{t-j} k r^r j^r \right) k^r - t
\]

\[
= \binom{r}{t} \sum_{k=0}^{ad} (ad - k)^r k^r - t.
\]

Let us compute the leading coefficient of \( p(x) \):

\[
p_{r+1} = \binom{r}{t} \sum_{j=0}^{t} (-1)^j \binom{t}{j} \frac{1}{r + j - t + 1}
\]

\[
= \frac{1}{r + 1} \sum_{j=0}^{t} (-1)^j \binom{r}{t} \binom{r + 1}{t} \frac{r + 1}{r + j - t + 1}
\]

\[
= \frac{1}{r + 1} \sum_{j=0}^{t} (-1)^j \binom{r\! -\! t\! +\! j}{j} \binom{r + 1}{t - j}.
\]

If we apply Eq. (17) and (21) of [Knu97, §1.2.6], we get

\[
\frac{1}{r + 1} \sum_{j=0}^{t} (-1)^j \binom{r\! -\! t\! +\! j}{j} \binom{r + 1}{t - j} = \frac{1}{r + 1} \sum_{j=0}^{t} \binom{t - r - 1}{j} \binom{r + 1}{t - j}
\]

\[
= \frac{1}{r + 1} \binom{t}{j},
\]

that is, \( p_{r+1} = (r + 1)^{-1} \). Hence, the claim is proved.

Following the notation of Lemma 4.4 we can denote the matrix appearing in (4.8) by \( V_{p(x)} \), where \( p(x) \) is the polynomial of Claim 4.7. The value of (4.8) is:

\[
\frac{1}{d^r} \frac{1}{r!} \det V \det V_{p(x)} = \frac{1}{d^r} \frac{1}{r!} \det V \frac{1}{r + 1} \det V
\]

\[
= d.
\]

Finally,

\[
[\pi_\ast(h^{r+1})]^T(\Gamma) = d(\lambda_1^r + \lambda_2^{r-1} \lambda_1 + \lambda_2^{r-2} \lambda_1^2 + \lambda_2^{r-3} \lambda_1^3 + \ldots + \lambda_1 \lambda_2^{r-1} + \lambda_2^r).
\]

We can prove the theorem for any \( \Gamma \) by induction on the number of edges and using Lemma 4.6. This concludes the proof.
Remark 4.8. Note that by linearity of push forward, the equivariant class of curves meeting a subvariety of codimension \( r \) and degree \( k \) is \( k \)-times the class of the linear subspace of the same codimension. For this reason, it is not restrictive to consider only linear subspaces in the enumeration of contact curves.

5. Final remarks

The explicit number of contact curves of degree \( d \) in \( \mathbb{P}^{2n+1} \) can be calculated combining the result of the previous sections. We know, by Example 2.3, that the subscheme of contact lines is a section of the Plücker line bundle of the Grassmannian of lines. So all characteristic numbers of contact lines can be deduced by classical Schubert calculus. For curves of higher degree, no generating formula is known.

Example 5.1. Suppose we want to compute the number of contact lines in \( \mathbb{P}^3 \), meeting \( a_2 \) lines and \( a_3 \) points. By Corollary 3.9, Lemma 3.11 and Equation (4.1), we know that

\[
(a_2 + 2a_3 = 3,
\]

and the number of such lines is

\[
N_1(a_2, a_3) = \sum_{\Gamma \in G(1)} \frac{1}{a_\Gamma} c_{\text{top}}(\mathcal{E}_1)(\pi_\ast (h^2))^a_2 (\pi_\ast (h^3))^{a_3} c_{\text{top}}(N_1)(\Gamma)
\]

\[
= \sum_{0 \leq i < j \leq 3} \frac{(\lambda_i + \lambda_j)(\lambda_i + \lambda_j)^{a_2}(\lambda_i^7 + \lambda_i \lambda_j + \lambda_j^7)^{a_3}}{\prod_{k \neq i, j}(\lambda_i - \lambda_k)(\lambda_j - \lambda_k)}.
\]

The value of \( c_{\text{top}}(N_1)(\Gamma) \) is known thanks to [Kon95]. We expect that the above sum equals an integer number. For curves of higher degree, we argue in the same way.

In Table 5.1, we listed the number of rational contact curves of degree \( d \), denoted by \( N_d(a_2, a_3) \), for all possible values of \( (a_2, a_3) \) and \( d \leq 3 \), computed using Bott formula and implemented in Wolfram Mathematica. Anyway, our results can be generalized \textit{mutatis mutandis} to every odd dimensional projective space. To my knowledge, characteristic numbers of contact curves in spaces different from \( \mathbb{P}^3 \) have never been investigated.

The code used here is available on author's website, or can be provided upon request.

| \( d \) | \( (a_2, a_3) \) | \( N_1(a_2, a_3) \) | \( (a_2, a_3) \) | \( N_2(a_2, a_3) \) | \( (a_2, a_3) \) | \( N_3(a_2, a_3) \) |
|-------|----------------|--------------------|----------------|--------------------|----------------|--------------------|
| 1     | (3, 0)         | 2                  | (5, 0)         | 40                 | (7, 0)         | 4160               |
|       | (1, 1)         | 1                  | (3, 1)         | 8                  | (5, 1)         | 512                |
|       |                 |                    | (1, 2)         | 2                  | (3, 2)         | 72                 |
|       |                 |                    |                |                    | (1, 3)         | 12                 |

Table 5.1. Number of contact curves in \( \mathbb{P}^3 \) of degree \( d \leq 3 \).

The values of \( N_2(5, 0) \) and \( N_3(7, 0) \) were already found in [LV11, Amo14]. One may try to get the number of smooth contact curves of some degree by throwing out
the reducible ones. For example, contact conics are union of lines (see, e.g., [LM07 Proposition 17]). Hence, also reducible contact cubics must be union of lines. Using the fact that $N_1(1,1) = 1$, it can be easily shown that there are exactly 9 reducible contact cubics through 3 points and a line. Since $N_3(1,3) = 12$, it means that we have exactly 3 smooth contact cubics through these points and that line, as proved in [Kal20 Corollary 6.3] with other techniques. The problem of finding the number of smooth contact cubics is interesting in its own and we hope to address these questions elsewhere.

In Table 5.2 and 5.3 we listed the number of contact curves in $\mathbb{P}^5$ of degree 1 and 2 meeting $a_i$ linear subspaces of codimension $i$ for $i = 2, 3, 4, 5$.

| $a = (a_2, a_3, a_4, a_5)$ | $N_1(a)$ | $(a_2, a_3, a_4, a_5)$ | $N_1(a)$ |
|-----------------------------|----------|------------------------|----------|
| $(7, 0, 0, 0)$              | 14       | $(1, 3, 0, 0)$          | 4        |
| $(5, 1, 0, 0)$              | 9        | $(1, 1, 0, 1)$          | 1        |
| $(4, 0, 1, 0)$              | 4        | $(1, 0, 2, 0)$          | 2        |
| $(3, 2, 0, 0)$              | 6        | $(0, 2, 1, 0)$          | 2        |
| $(2, 1, 1, 0)$              | 3        | $(0, 0, 1, 1)$          | 1        |

Table 5.2. Number of contact curves in $\mathbb{P}^5$ of degree 1.

| $a = (a_2, a_3, a_4, a_5)$ | $N_2(a)$ | $(a_2, a_3, a_4, a_5)$ | $N_2(a)$ | $(a_2, a_3, a_4, a_5)$ | $N_2(a)$ |
|-----------------------------|----------|------------------------|----------|------------------------|----------|
| $(11, 0, 0, 0)$             | 103876   | $(4, 2, 1, 0)$          | 624      | $(1, 5, 0, 0)$          | 288      |
| $(9, 1, 0, 0)$              | 30864    | $(4, 0, 1, 1)$          | 541      | $(1, 3, 0, 1)$          | 28       |
| $(8, 0, 1, 0)$              | 5798     | $(3, 4, 0, 0)$          | 912      | $(1, 2, 2, 0)$          | 48       |
| $(7, 2, 0, 0)$              | 9420     | $(3, 2, 0, 1)$          | 76       | $(1, 1, 0, 2)$          | 4        |
| $(7, 0, 0, 1)$              | 544      | $(3, 1, 2, 0)$          | 152      | $(1, 0, 2, 1)$          | 8        |
| $(6, 1, 1, 0)$              | 1898     | $(3, 0, 0, 2)$          | 8        | $(0, 4, 1, 0)$          | 64       |
| $(5, 3, 0, 0)$              | 2924     | $(2, 3, 1, 0)$          | 200      | $(0, 2, 1, 1)$          | 8        |
| $(5, 1, 0, 1)$              | 202      | $(2, 1, 1, 1)$          | 22       | $(0, 1, 3, 0)$          | 12       |
| $(5, 0, 2, 0)$              | 436      | $(2, 0, 3, 0)$          | 44       | $(0, 0, 1, 2)$          | 2        |

Table 5.3. Number of contact curves in $\mathbb{P}^5$ of degree 2.

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