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Singular Nonsymmetric Jack Polynomials for Some Rectangular Tableaux

Charles F. Dunkl

Department of Mathematics, University of Virginia, Charlottesville, VA 22904-4137, USA; cfd5z@virginia.edu

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Abstract: In the intersection of the theories of nonsymmetric Jack polynomials in $N$ variables and representations of the symmetric groups $S_N$ one finds the singular polynomials. For certain values of the parameter $\kappa$ there are Jack polynomials which span an irreducible $S_N$-module and are annihilated by the Dunkl operators. The $S_N$-module is labeled by a partition of $N$, called the isotype of the polynomials. In this paper the Jack polynomials are of the vector-valued type, i.e., elements of the tensor product of the scalar polynomials with the span of reverse standard Young tableaux of the shape of a fixed partition of $N$. In particular, this partition is of shape $(m, m, \ldots, m)$ with $2k$ components and the constructed singular polynomials are of isotype $(mk, mk)$ for the parameter $\kappa = 1/(m + 2)^2$. This paper contains the necessary background on nonsymmetric Jack polynomials and representation theory and explains the role of Jucys–Murphy elements in the construction. The main ingredient is the proof of uniqueness of certain spectral vectors, namely the list of eigenvalues of the Jack polynomials for the Cherednik–Dunkl operators, when specialized to $\kappa = 1/(m + 2)^2$. The paper finishes with a discussion of associated maps of modules of the rational Cherednik algebra and an example illustrating the difficulty of finding singular polynomials for arbitrary partitions.

Keywords: Dunkl operators; standard modules

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1. Introduction

In the study of polynomials in several variables there are two approaches, one is algebraic, which may involve symmetry groups generated by permutations of coordinates and sign changes, for example, and the analytic approach, which includes orthogonality with respect to weight functions and related calculus. The two concepts are combined in the theory of Dunkl operators, which form a commutative algebra of differential-difference operators, determined by a reflection group $G$ and a parameter, and which are an analog of partial derivatives. The relevant weight functions are products of powers of linear functions vanishing on the mirrors and which are invariant under the reflection group $G$. In the particular case of the objects of our study, namely the symmetric groups $S_N$, an orthogonal basis of polynomials (called nonsymmetric Jack polynomials) is constructed as the set of simultaneous eigenfunctions of the Cherednik–Dunkl operators. This is a commutative set of operators, self-adjoint for an inner product related to the weight function. The inner product is positive-definite for an interval of parameter values but for a discrete set of values there exist null polynomials (that is, $\langle p, p \rangle = 0$). It is these parameter values that concern us here. The set of such polynomials of minimal degree has an interesting algebraic structure: in general it is a linear space and an irreducible module of $S_N$. The theory for scalar polynomials is by now well understood [1], and the open problems concern vector-valued polynomials whose values lie in irreducible modules. That is, the symmetric group $S_N$ acts not only on the domain but also the range of the polynomials. The key device for dealing with the representation theory is to analyze when a polynomial is a simultaneous...
eigenfunction of the Cherednik–Dunkl operators and of the Jucys-Murphy elements with the same respective eigenvalues. In Etingof and Stoica [2] there is an analysis of the vanishing properties, that is, the zero sets, of singular polynomials of the groups $S_N$ as well as results on singular polynomials associated with minimal values of the parameter for general modules of $S_N$ and for the exterior powers of the reflection representation of any finite reflection group $G$ (see also [3]). Their methods do not involve Jack polynomials. Feigin and Silantyev [4] found explicit formulas for all singular polynomials which span a module isomorphic to the reflection representation of $G$.

This paper concerns polynomials taking values in the representation of the symmetric group corresponding to a rectangular partition. In particular for $\tau = \left( \begin{smallmatrix} m^2k \\ \end{smallmatrix} \right)$ (the superscript indicates multiplicity) we construct nonsymmetric Jack polynomials in $2mk$ variables which are singular (annihilated by the Dunkl operators) for the parameter $\frac{1}{m+\tau}$ and which span a module isomorphic to the representation $\sigma = (mk, mk)$.

In Section 2 we present the basic definitions of operators, combinatorial objects used in the representation theory of the symmetric groups, and vector-valued nonsymmetric Jack polynomials. Section 2.1 is a concise treatment of the formulas for the transformations of the Jack polynomials under simple reflections $s_i$ (so $s_i^2 = 1$), generate $S_N$, i.e., $w_1 (w_2 p) (x) = (w_2 p) (w_1 x) = p (w_1 w_2) (x)$ for all $w_1, w_2 \in S_N$. Our structures depend on a transcendental (formal) parameter $\kappa$, which may be specialized to a specific rational value $k_0$.

Furthermore, $S_N$ is generated by reflections in the mirrors $\{x : x_i = x_j\}$ for $1 \leq i < j \leq N$. These are transpositions, denoted by $(i, j)$, so that $x (i, j)$ denotes the result of interchanging $x_i$ and $x_j$. Define the $S_N$-action on $\alpha \in \mathbb{Z}^N$ so that $(xw)^\alpha = x^{\alpha w}$

$$
(xw)^\alpha = \prod_{i=1}^N x_{w(i)}^{\alpha_i} = \prod_{j=1}^N x_{w^{-1}(j)}^{\alpha_j},
$$

that is $(w \alpha) = \alpha_{w^{-1}(j)}$ (so $\alpha$ is taken as a column vector and $\alpha w = [w] \alpha$).

The simple reflections $s_i := (i, i+1), 1 \leq i \leq N-1$, generate $S_N$. They are the key devices for applying inductive methods, and satisfy the braid relations:

$$
s_is_j = s_js_i \text{ if } |i - j| \geq 2,
$$

$$
s_is_{i+1}s_i = s_{i+1}s_is_{i+1}.
$$

We consider the situation where the group $S_N$ acts on the range as well as on the domain of the polynomials. We use vector spaces, called $S_N$-modules, on which $S_N$ has an irreducible
orthogonal representation. See James and Kerber [5] for representation theory, including a discussion of Young’s methods.

Denote the set of partitions
\[ \mathbb{N}_0^{N,+} := \left\{ \lambda \in \mathbb{N}_0^N : \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N \right\}. \]

An irreducible representation \( \tau \) of \( S_N \) corresponds to a partition of \( N \) given the same label, that is \( \tau \in \mathbb{N}_0^{N,+} \) and \( \left| \tau \right| = N \). The length of \( \tau \) is \( \ell(\tau) = \max \{ i : \tau_i > 0 \} \). There is a Ferrers diagram of shape \( \tau \) (also given the same label), with boxes at points \((i, j)\) with \(1 \leq i \leq \ell(\tau)\) and \(1 \leq j \leq \tau_j\). A tableau of shape \( \tau \) is a filling of the boxes with numbers, and a reverse standard Young tableau (RSYT) is a filling with the numbers \(\{1, 2, \ldots, N\}\) so that the entries decrease in each row and each column. The set of RSYT of shape \( \tau \) (also given the same label), with boxes at points \((i, j)\) with \(1 \leq i \leq \ell(\tau)\) and \(1 \leq j \leq \tau_j\).

Definition 1. The Dunkl and Cherednik–Dunkl operators are \(1 \leq i \leq N, p \in \mathbb{P}, T \in \mathcal{Y}(\tau)\)

\[ D_i(p(x) \otimes T) := \frac{\partial p(x)}{\partial x_i} \otimes T + \sum_{j \neq i} \frac{p(x) - p(x(i, j))}{x_i - x_j} \otimes \tau((i, j)) T, \]

\[ U_i(p(x) \otimes T) := D_i(x, p(x) \otimes T) - \sum_{j=1}^{i-1} p(x(i, j)) \otimes \tau((i, j)) T, \]

extended by linearity to all of \( \mathcal{P}_\tau \).

The commutation relations analogous to the scalar case hold, i.e.,

\[ D_i D_j = D_j D_i, \quad U_i U_j = U_j U_i, \quad 1 \leq i, j \leq N \]

\[ w D_i = D_{w(i)} w, \quad \forall w \in S_N; \quad s_i U_j = U_{s_i j}, \quad j \neq i - 1, i; \]

\[ s_i U_{s_i j} = U_{i+1} + k s_i, \quad U_{s_i j} = s_i U_{i+1} + k, \quad U_{i+1 s_i} = s_i U_i - k. \]

The simultaneous eigenfunctions of \( \{U_i\} \) are called (vector-valued) nonsymmetric Jack polynomials (NSJP). They are the type \( A \) special case of the polynomials constructed by Griffith [6] for the complex reflection groups \( G(n, p, N) \). For generic \( \kappa \) these eigenfunctions form a basis of \( \mathcal{P}_\tau \) (generic means that \( \kappa \neq \frac{m}{n} \) where \( m, n \in \mathbb{Z} \) and \( 1 \leq n \leq N \)). They have a triangularity property with respect to the partial order \( \triangleright \) on compositions, which is derived from the dominance order:

\[ \alpha \prec \beta \iff \sum_{j=1}^{i} \alpha_j \leq \sum_{j=1}^{i} \beta_j, \quad 1 \leq i \leq N, \quad \alpha \neq \beta, \]

\[ \alpha \prec \beta \iff (|\alpha| = |\beta|) \land \left[ (\alpha^{+} \prec \beta^{+}) \lor (\alpha^{+} = \beta^{+} \land \alpha \prec \beta) \right]. \]

There is a subtlety in the leading terms, which relies on the rank function \( r_\alpha \):
Definition 2. For $\alpha \in \mathbb{N}_0^N, 1 \leq i \leq N$

$$r_\alpha (i) = \# \{ j : a_j > a_i \} + \# \{ j : 1 \leq j \leq i, a_j = a_i \},$$

then $r_\alpha \in S_N$.

A consequence is that $r_\alpha a = a^+$, the nonincreasing rearrangement of $a$, for any $\alpha \in \mathbb{N}_0^N$. For example if $a = (1, 2, 1, 5, 4)$ then $r_\alpha = [4, 3, 5, 1, 2]$ and $r_\alpha a = a^+ = (5, 4, 2, 1, 1)$ (recall $wa_i = a_{w^{-1}(i)}$). Also $r_\alpha = I$ if and only if $\alpha \in \mathbb{N}_0^{N,+}$.

For each $\alpha \in \mathbb{N}_0^N$ and $T \in \mathcal{Y}(\tau)$ there is a NSJP $J_{\alpha,T}$ with leading term $x^a \otimes \tau (r_\alpha^{-1}) T$, i.e.,

$$J_{\alpha,T} (x) = x^a \otimes \tau (r_\alpha^{-1}) T + \sum_{\alpha \neq \beta} x^\beta \otimes v_{\alpha,\beta,T} (\kappa) \quad (1)$$

where $v_{\alpha,\beta,S} (\kappa) \in V_T$; the coefficients are rational functions of $\kappa$. These polynomials satisfy

$$U_i I_{\alpha,S} = \zeta_{\alpha,S} (i) I_{\alpha,S},$$

$$\zeta_{\alpha,S} (i) := a_i + 1 + \kappa c (r_\alpha (i), S), \quad 1 \leq i \leq N.$$

For detailed proofs see [7]. The commutation

$$D_i x_i = x_i D_i + 1 + \kappa \sum_{j \neq i} (i,j)$$

implies

$$U_i = x_i D_i + 1 + \kappa \sum_{j > i} (i,j). \quad (2)$$

This introduces the definition of \textit{Jucys–Murphy elements} in the group algebra $\mathbb{R}S_N$

$$\omega_i = \sum_{i=1+1}^N (i,j), \quad 1 \leq i < N; \quad \omega_N = 0;$$

which satisfy

$$\omega_i \omega_j = \omega_j \omega_i,$$

$$s_i \omega_i = \omega_i s_i, \quad |i-j| \geq 2,$$

$$s_i \omega_j s_i = \omega_{i+1} + s_j.$$

They act on $V_\tau$ by $\tau (\omega_i) T = \sum_{j>i} \tau ((i,j)) T = c (i,T) T$. We will use the modified operators

$$U_i' = \frac{1}{\kappa} (U_i - 1) = \frac{1}{\kappa} x_i D_i + \omega_i.$$ 

The associated spectral vector is

$$\zeta_{\alpha,i} (i) := \frac{a_i}{\kappa} + c (r_\alpha (i), T),$$

so that $U_i' J_{\alpha,T} = \zeta_{\alpha,i} (i) J_{\alpha,T}$ for $1 \leq i \leq N$.

Throughout we use the phrase “at $\kappa = \kappa_0$” where $\kappa_0$ is a rational number to mean that the operators $U_i$ and polynomials $J_{\alpha,S}$ are evaluated at $\kappa = \kappa_0$. The transformation formulas and eigenvalue properties are polynomial in $x$ and rational in $\kappa$. Thus, the various relations hold provided there is no pole. Hence to validly specialize to $\kappa = \kappa_0$ it is necessary to prove the absence of poles.

Suppose $p \in \mathcal{P}_T$ and $1 \leq i \leq N$ then $D_i p = 0$ if and only if $U_i' p = \omega_i p$ at $\kappa = \kappa_0$ (obvious from (2)). The polynomial $p$ is said to be \textit{singular} and $\kappa_0$ is a \textit{singular value}. From the representation theory of $S_N$ it is known that an irreducible $S_N$-module is isomorphic to an abstract space whose
basis consists of RSYT’s of shape $\sigma$, a partition of $N$. The eigenvalues of $\{\omega_i\}$ form content vectors which uniquely define an RSYT. Suppose $\sigma$ is a partition of $N$ then a basis $\{p_S : S \in \mathcal{Y}(\sigma)\}$ (of an $S_N$-invariant subspace) is called a basis of isotype $\sigma$ if each $\omega_ip_S = c(i, S) p_S$ for $1 \leq i \leq N$ and each $p_S$. If some $p \in \mathcal{P}_\tau$ is a simultaneous eigenfunction of $\{\omega_i\}$ with $\omega_ip = \gamma_ip$ for $1 \leq i \leq N$ then the representation theory of $S_N$ implies that $\{\gamma_i\}_{i=1}^N$ is the content vector of a uniquely determined RSYT of shape $\sigma$ for some partition $\sigma$ of $N$; this allows specifying the isotype of a single polynomial without referring to a basis. The key point here is when a subspace does have a basis of isotype $\sigma$ made up of NSJP’s, specialized to a fixed rational $\kappa = \kappa_0$.

In this paper, we construct singular polynomials for the partition $m^{2\kappa}$ of $N = 2mk$ for the singular value $\kappa_0 = \frac{1}{\overline{m}+2}$ and which are of isotype $\sigma = (mk, mk)$, with $m \geq 1, k \geq 2$. To show that the nonsymmetric Jack polynomials in the construction have no poles at $\kappa = \frac{1}{\overline{m}+2}$ we use the devices for proving uniqueness of spectral vectors and performing valid transformations of the polynomials. The proof of singularity will follow once we show the relevant polynomials are eigenfunctions of the Jucys–Murphy operators $\omega_i$. These properties are proven by a sort of induction using the simple reflections $s_i$. For this purpose we describe the effect of $s_i$ on $J_{a,T}$.

One key device is to consider the related tableaux as a union of $k$ rectangles of shape $2 \times m$, which we call bricks.

2.1. Review of Transformation Formulas

We collect formulas for the action of $s_i$ on $J_{a,T}$. They will be expressed in terms of the spectral vector $b'_{a,T} = [\frac{a_i}{\kappa} + c(r_a(i), T)]_{i=1}^{2mk}$ and (for $1 \leq i < 2mk$)

$$b_{a,T}(i) = \frac{1}{b'_{a,T}(i) - b'_{a,T}(i+1)} \kappa_{a_i - a_{i+1} + \kappa(c(r_a(i), T) - c(r_a(i+1), T))}.$$ 

The formulas are consequences of the commutation relationships: $s_jU'i' = U'i's_j$ for $j < i - 1$ and $j > i$; $s_iU'j's_i = U'_{i+1} + s_i$ for $1 \leq i < 2mk$. Observe that the formulas manifest the equation

$$(s_i - b_{a,T}(i)) (s_i - b_{a,T}(i)) = 1 - b_{a,T}(i)^2.$$ 

2.1.1. Case: $a_{i+1} > a_i$

$$(s_i - b_{a,T}(i)) J_{a,T} = J_{a,T} s_i,$$

$$(s_i + b_{a,T}(i)) J_{a,T} = \left(1 - b_{a,T}(i)^2\right) J_{a,T},$$

2.1.2. Case: $a_i > a_{i+1}$

$$(s_i - b_{a,T}(i)) J_{a,T} = \left(1 - b_{a,T}(i)^2\right) J_{a,T},$$

$$(s_i + b_{a,T}(i)) J_{a,T} = J_{a,T}.$$ 

2.1.3. Case: $a_i = a_{i+1}, r_a(i) = j$

In this case $b_{a,T}(i) = 1/(c(j, T) - c(j+1, T))$. Then if

1. $b_{a,T}(i) = 1, (\text{row } (j, T) = \text{row } (j+1, T)) s_i J_{a,T} = J_{a,T}$,
2. $b_{a,T}(i) = -1, (\text{col } (j, T) = \text{col } (j+1, T)) s_i J_{a,T} = -J_{a,T}$,
3. \(0 < b_{a,T}(i) \leq \frac{1}{2} (\text{col}(j,T) > \text{col}(j+1,T), \text{row}(j,T) < \text{row}(j+1,T))\)

\[
(s_i - b_{a,T}(i)) I_{a,T} = I_{a,T}^{(i)},
\]

\[
(s_i + b_{a,T}(i)) I_{a,T} = \left(1 - b_{a,T}(i)^2\right) I_{a,T},
\]

4. \(-\frac{1}{2} \leq b_{a,T}(i) < 0 (\text{col}(j,T) < \text{col}(j+1,T), \text{row}(j,T) > \text{row}(j+1,T))\)

\[
(s_i - b_{a,T}(i)) I_{a,T} = \left(1 - b_{a,T}(i)^2\right) I_{a,T}^{(i)},
\]

\[
(s_i + b_{a,T}(i)) I_{a,T}^{(i)} = I_{a,T}.
\]

Remark 1. The previous four formulas when restricted to \(\{1 \otimes T : T \in \mathcal{Y}(\tau)\}\) (so that \(\alpha = (0,0,\ldots)\), \(\mathfrak{r}_a(i) = i\) and \(I_{a,T} = 1 \otimes T\)) describe the action of \(\tau\) on \(V_T\); in this situation \(U'_T(1 \otimes T) = c(i,T)(1 \otimes T)\) for \(1 \leq i \leq 2mk\).

There is an important implication when \(a_i > a_{i+1}\) and \(b_{a,T}(i) = \pm 1\) (at \(\kappa = \kappa_0\)) the general relation \((s_i - b_{a,T}(i)) I_{a,T} = \left(1 - b_{a,T}(i)^2\right) I_{a,T}\) becomes \(s_I I_{a,T} = b_{a,T}(i) I_{a,T}\) provided that \(I_{a,T}\) does not have a pole. Our device for proving this is to show uniqueness of the spectral vector of \(I_{a,T}\) or of another polynomial \(p_{\beta,S}\) which can be transformed to \(I_{a,T}\) by a sequence of invertible \((\zeta_{\gamma,S}(i) - \zeta'_{\gamma,S}(i + 1)) \geq 2\) steps using the simple reflections \(\{s_i\}\).

3. Properties of Bricks

A brick is a \(2 \times m\) tableau which is one of the \(k\) congruent rectangles making up the Ferrers diagram of \(\sigma\) or \(\tau\). Since it is clear from the context we can use the same name for the appearance in \(\sigma\) or \(\tau\). Let \(0 \leq \ell \leq k-1\), then \(B_\ell\) is the part \(\{[i,j] : i = 1,2,m\ell < j \leq (m+1)\ell\}\) of \(\sigma\) or the part \(\{[i,j] : i = 2\ell + 1,2\ell + 2,1 \leq j \leq m\}\) of \(\tau\). The standard brick \(B_\ell\) has the entries entered column by column:

\[
\bar{B}_\ell = \begin{bmatrix}
2m (k - \ell) & \cdots & 2m (k - \ell - 1) + 2 \\
2m (k - \ell) - 1 & \cdots & 2m (k - \ell - 1) + 1
\end{bmatrix}.
\]

In this section, we use bricks to construct for each \(S \in \mathcal{Y}(\sigma)\) a pair \((\beta,T) \in \mathbb{N}^{2mk} \times \mathcal{Y}(\tau)\) such that \((m+2)\beta_i + c(r_{\beta}(i),T) = c(i,S)\) for \(1 \leq i \leq 2mk\). Later on we will prove uniqueness of \((\beta,T)\).

For the partition \(\sigma\) we use the distinguished RSYT \(S_0\) formed by entering \(2mk, 2mk - 1, \ldots, 2,1\) column by column, i.e., \(S_0\) is the concatenation of \(\bar{B}_{\ell_1} \bar{B}_{\ell_2} \cdots \bar{B}_{\ell_{k-1}}\). Observe \#\(\mathcal{Y}(\sigma) = \frac{1}{(m+1)(2mk)}\), a Catalan number. The contents of \(B_\ell\) in \(\sigma\) are given by:

\[
\begin{bmatrix}
m\ell & \cdots & m(\ell + 1) - 1 \\
m\ell - 1 & \cdots & m(\ell + 1) - 2
\end{bmatrix}
\]

Form the distinguished RSYT \(T_0\) of shape \(\tau\) by stacking the standard bricks, from \(\bar{B}_0\) at the top (rows \#1 and \#2) to \(\bar{B}_{k-1}\) at the bottom (rows \#\((2k - 1)\) and \#\((2k)\)). The location of \(\bar{B}_{k-1}\) in \(T_0\) has corners \([2\ell + 1,1], [2\ell + 1,m], [2\ell + 2,1], [2\ell + 2,m]\) (and the entries are \(2m,2m-1,\ldots,2,1\) entered column by column). Thus, \(T_0\) has the numbers \(2mk, 2mk - 1, \ldots, 2,1\) entered column by column in each brick; here is the example \(m = 3, k = 2\):

\[
T_0 = \begin{bmatrix}
12 & 10 & 8 \\
11 & 9 & 7 \\
6 & 4 & 2 \\
5 & 3 & 1
\end{bmatrix}
\]
The contents for $B_\ell$ in $T_0$ are
\[
\begin{bmatrix}
-2\ell & \cdots & -2\ell + m - 1 \\
-2\ell - 1 & \cdots & -2\ell + m - 2
\end{bmatrix}.
\]

Let $\lambda = \binom{(k-1)^{2m}, (k-2)^{2m}, \ldots, 1^m, 0^m}$. 

**Proposition 1.** The spectral vector at $\kappa = \frac{1}{m+2}$ of $J_{\lambda, T_0}$ equals the content vector of $S_0$.

**Proof.** Since $\lambda \in \mathbb{N}_0^{2mk}$ if $i \in \tilde{B}_\ell$ then $\lambda_i = \ell$ and $\zeta^T_{\lambda, T_0}(i) = (m+2)\ell + c(i, T_0)$. By the structure of $\tilde{B}_\ell$ it suffices to check the value at one corner, say the top left one. Taking $i = 2m(k-\ell)$ with $c(i, T_0) = -2\ell$ we obtain $\zeta^T_{\lambda, T_0}(i) = (m+2)\ell - 2\ell = m\ell$. \hfill $\Box$

Suppose $S \in \mathcal{Y}(\sigma)$ then there is a permutation $\beta$ of $c$ and an RSYT of shape $\tau$ such that $\zeta^T_{\beta, \tau}(i) = c(i, S)$ at $\kappa = \frac{1}{m+2}$ for $1 \leq i \leq 2mk$. The construction is described in the following.

**Definition 3.** Suppose $S \in \mathcal{Y}(\sigma)$ and $0 \leq \ell \leq k-1$; and suppose the part of $S$ in the brick $B_\ell$ is
\[
\begin{bmatrix}
n_1 & n_2 & \cdots & n_m \\
n_{m+1} & n_{m+2} & \cdots & n_{2m}
\end{bmatrix}.
\]

The entries decrease in each column and in each row. Define $\beta \in \mathbb{N}_0^{2mk}$ by $\beta_{n_i} = \ell$ for $1 \leq i \leq 2m$. Set up a local rank function (for $1 \leq i \leq 2m$): $\rho_i = \# \{ j : n_j \geq n_i, 1 \leq j \leq 2m \}$, then $\rho_{2m} = 1$ and $\rho_1 = 2m$. Picturesquely, form a brick-shaped tableau by replacing $n_i$ by $\rho_i$ and then adding $2m\ell$ to each entry. Then stack these tableaux to form an RSYT of shape $\tau$. Specifically set $T[2\ell + 1, i] = \rho_i + 2(k-1-\ell)m$, $T[2\ell + 2, i] = \rho_i + m + 2(k-1-\ell)m$ for $1 \leq i \leq m$. Perform this construction for each $\ell$ with $0 \leq \ell \leq k-1$.

Denote $\beta, T$ constructed in the Definition by $\beta \{ S \}, T \{ S \}$, or by the abbreviation $\pi \{ S \}$.

**Proposition 2.** Suppose $S \in \mathcal{Y}(\sigma)$ and $\beta = \beta \{ S \}, T = T \{ S \}$ then $(m+2)\beta_i + c(\beta_i, T) = c(i, S)$ for all $i$.

**Proof.** Suppose $i$ is in brick $B_\ell$ and $i = n_j$ for some $j$. If $1 \leq j \leq m$ then row $(i, S) = 1$ and $c(i, S) = m\ell + j - 1$, while if $m < j \leq 2m$ then row $(i, S) = 2$ and $c(i, S) = m\ell + j - m - 2$. Then $\# \{ s : \beta_s = \beta_i = \ell, s \leq i \} = \{ s : n_s \leq n_i \} = \rho_i$. Also $\# \{ s : \beta_s > \ell \} = 2m(k-1-\ell)$ and thus $r_\beta(i) = \rho_i + 2m(k-1-\ell)$.

If $1 \leq j \leq m$ then $T[2\ell + 1, j] = r_\beta(i)$, $c(\beta(i), T) = j - 2\ell - 1$ and $(m+2)\beta_i + c(\beta(i), T)$ $= m\ell + j - 1 = c(i, S)$. If $m + 1 \leq j \leq 2m$ then $T[2\ell + 2, j - m] = r_\beta(i)$, $c(\beta(i), T) = j - m - 2\ell - 2$ and $(m+2)\beta_i + c(\beta(i), T) = m\ell + (j - m - 2) = c(i, S)$. \hfill $\Box$

Here is an example for $m = 3, k = 3, \kappa = \frac{1}{5}$

\[
S = \begin{bmatrix}18 & 17 & 13 & 14 & 10 & 8 & 7 & 6 & 3 \\ 16 & 15 & 12 & 11 & 9 & 5 & 4 & 2 & 1\end{bmatrix},
\]

$\beta \{ S \} = (2, 2, 2, 2, 1, 2, 2, 1, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0)$

with the values of the local rank $\rho$

\[
\begin{bmatrix}6 & 5 & 2 \\ 4 & 3 & 1\end{bmatrix}, \begin{bmatrix}6 & 4 & 2 \\ 5 & 3 & 1\end{bmatrix}, \begin{bmatrix}6 & 5 & 3 \\ 4 & 2 & 1\end{bmatrix}
\]
now add 12, 6, 0 respectively and combine to form

$$T \{S\} = \begin{bmatrix} 18 & 17 & 14 \\ 16 & 15 & 13 \\ 12 & 10 & 8 \\ 11 & 9 & 7 \\ 6 & 5 & 3 \\ 4 & 2 & 1 \end{bmatrix}.$$ 

For example $\beta_{10} = 1, r_\beta (10) = 10$ and $c (10, T) = -1$ thus $\zeta_{\beta, T} (10) = 5 - 1 = 4 = c (10, S)$. Essentially what is left to do for the singularity proofs is to show span $\{ \pi \{ S \} : S \in \mathcal{Y} (\pi) \}$ is closed under $\{ s_i : 1 \leq i < 2mk \}$ and that $\omega_i \pi \{ S \} = c (i, S) \pi \{ S \}$ for all $i$. Here is a small example of the impending difficulty: let $m = 2, k = 2 \left( \kappa = \frac{1}{4} \right)$ and

$$S = \begin{bmatrix} 8 & 6 & 5 & 2 \\ 7 & 4 & 3 & 1 \end{bmatrix}, T = T \{ S \} = \begin{bmatrix} 8 & 6 \\ 7 & 5 \\ 4 & 2 \\ 3 & 1 \end{bmatrix},$$

$$\beta = \beta \{ S \} = (1, 1, 0, 0, 0, 0).$$

What is the result of applying $ss$? Interchanging 5 and 6 in $S$ results in a tableau violating the condition of decreasing entries in each row (thus outside the span), and the general transformation formula in Section 2.1.2 $(\beta_5 > \beta_6)$ says

$$(ss - b_{\beta, T} (5)) I_{\beta, T} = \left( 1 - b_{\beta, T} (5)^2 \right) I_{ss\beta, T}$$

with $ss\beta = (1, 1, 0, 0, 1, 0, 0)$ and $b_{\beta, T} (5)^{-1} = \frac{1}{2} (1 - 0) + c (4, T) - c (6, T) = \frac{1}{2} + (-2 - 1)$, thus $b_{\beta, T} (5) = 1$ at $\kappa = \frac{1}{4}$. To show that the formula gives $ss I_{\beta, T} = I_{\beta, T}$ it is necessary to show $I_{ss\beta, T}$ has no pole at $\kappa = \frac{1}{4}$. These proofs comprise a large part of the sequel.

4. Action of Jucys–Murphy Elements

The Jucys–Murphy elements satisfy $s_j \omega_i = \omega_i s_j$ for $j \neq i - 1, i$ and $s_i \omega_i s_i = \omega_i + s_i$ for $i < 2mk$. Suppose there is a subset $\mathcal{Z} \subset \mathbb{N}_0^{2mk} \times \mathcal{Y} (\tau)$ with the properties (spectral vectors at $\kappa = \frac{1}{m+2}$, recall $b_{\lambda, T} (i) = \left( \zeta_{\lambda, T}^\tau (i) - \zeta_{\lambda, T}^\tau (i + 1) \right)^{-1}$):

1. $(\beta, T) \in \mathcal{Z}$ and $|b_{\beta, T} (i)| \leq \frac{1}{2}$ implies $(s_i - b_{\beta, T} (i)) I_{\beta, T} = \gamma I_{\beta', T'}$, for some $\gamma \neq 0$ and $(\beta', T') \in \mathcal{Z}$; also $\zeta_{\beta', T'}^\tau = s_i \zeta_{\beta, T}^\tau$;
2. $(\beta, T) \in \mathcal{Z}$ and $b_{\beta, T} (i) = 1$ implies $s_i I_{\beta, T} = b_{\beta, T} (i) I_{\beta, T}$;
3. $(\beta, T) \in \mathcal{Z}$ implies $\beta_{2mk} = 0$ and thus $\zeta_{\beta, T}^\tau (2mk) = 0$.

The following is a basic theorem on representations of $S_N$ and we sketch the proof.

**Theorem 1.** If $\mathcal{Z} \subset \mathbb{N}_0^{2mk} \times \mathcal{Y} (\tau)$ satisfies these properties then $(\beta, T) \in \mathcal{Z}$ implies $\omega_i I_{\beta, T} = \zeta_{\beta, T}^\tau (i) I_{\beta, T}$ for $1 \leq i \leq 2mk$. 

**Proof.** Arguing by induction suppose \( \omega_1 j_{\beta,T} = \zeta_{\beta,T}^i (j) j_{\beta,T} \) for all \((\beta,T) \in \mathcal{Z}\) and \(i < j \leq 2mk\). The start \(i = 2mk - 1\) is given in the hypotheses. Let \((\beta,T) \in \mathcal{Z}\) and suppose that \(b_{\beta,T} (i) = \pm 1\), then

\[
\omega_1 j_{\beta,T} = (s_i \omega_{i+1}s_i + s_i) j_{\beta,T} = \left\{ b_{\beta,T} (i)^2 \zeta_{\beta,T}^i (i+1) + b_{\beta,T} (i) \right\} j_{\beta,T} = \zeta_{\beta,T}^i (i) j_{\beta,T}.
\]

Next suppose \( |b_{\beta,T} (i)| \leq \frac{1}{2} \) and set \( p = \gamma j_{\beta,T} = (s_i - b_{\beta,T} (i)) j_{\beta,T} \), thus \( \omega_{i+1} p = \zeta_{\beta,T}^i (i) p \) (inductive hypothesis). Then

\[
\omega_1 j_{\beta,T} = \left( p + b_{\beta,T} (i) j_{\beta,T} \right) = \left( \zeta_{\beta,T}^i (i) s_i + 1 \right) p + b_{\beta,T} (i) \left( \zeta_{\beta,T}^i (i+1) s_i + 1 \right) j_{\beta,T} = \left\{ \zeta_{\beta,T}^i (i) \right\} j_{\beta,T}.
\]

This completes the induction. \( \square \)

We want to show that \( \{(\beta \{S\}, T \{S\}) : S \in \mathcal{Y}(\mathcal{S})\} \) (as in Definition 3) satisfies the hypotheses of Theorem 1. From the construction it is clear that \( \beta \{S\}_{2mk} = 0 \) because \( S [1,1] = 2mk \) and this cell is in \( B_0 \). Fix \( S \in \mathcal{Y}(\mathcal{S}) \) and \( i < 2mk \). Abbreviate \( \beta = \beta \{S\}, T = T \{S\}, b = b_{\beta,T} (i) \). There are several cases:

1. \( |c(i,S) - c(i+1,S)| \geq 2 \) then \( |b| \leq \frac{1}{2} \) and \( (s_i - b) j_{\beta,T} = \gamma j_{\beta,T} \), where \( \gamma = 0 \) and \( \zeta_{\beta,T}^i (i) = c(i,S) \) for all \( j \). Specifically if \( \beta_i \neq \beta_{i+1} \) implying that \( i \) and \( i + 1 \) are in different bricks then \( b' = s_i \beta \) and \( T' = T \), while if \( \beta_i = \beta_{i+1} = \ell \) then \( i, i + 1 \in B_\ell \) and \( T' \) is formed from \( T \) by transforming the part of \( T \) in \( B_\ell \) interchanging \( r_\beta (i) \) and \( r_\beta (i + 1) \).

2. \( c(i,S) - c(i+1,S) = -1 \) (col \( i, S \) = col \( i+1, S \)) then by construction \( \beta_i = \beta_{i+1} = \ell \) and \( i, i + 1 \in B_\ell \); suppose that \( i + 1 = n_j \) in the notation of Definition 3. By hypothesis \( i = n_{j+m} \), \( \rho_j = 1 \). Then \( T [2\ell + 1,j] = \rho_j + 2 (k - 1 - \ell) m \) and \( T [2\ell + 2,j] = T [2\ell + 1,j] - 1 \). This implies \( s_i j_{\beta,T} = \rho_j + 2 (k - 1 - \ell) m \) and \( s_i j_{\beta,T} = \rho_j - 1 \).

3. \( c(i,S) - c(i+1,S) = 1 \) (row \( i, S \) = row \( i+1, S \)) and \( \beta_i = \beta_{i+1} = \ell \); then \( i, i + 1 \in B_\ell \); using Definition 3 \( i = n_j \) and \( i + 1 = n_{j+1} \) for some \( j \) with \( 2 \leq j \leq m \) or \( m + 2 \leq j \leq 2m \) and \( \rho_j = 1 \). Thus, \( r_\beta (i) = \rho_j + 2 (k - 1 - \ell) m \). In the first case \( T [2\ell + 1,j] = T [2\ell + 1,j - 1] - 1 \) and \( T [2\ell + 2,j - 1] = T [2\ell + 2,j - 1] - 1 \) and thus \( s_i j_{\beta,T} = \rho_j - 1 \).

4. \( c(i,S) - c(i+1,S) = 1 \) (row \( i, S \) = row \( i+1, S \)) and \( \beta_i = \beta_{i+1} \); then \( i, i + 1 \in B_{\ell - 1} \); in \( B_\ell \) (because the entries of \( S \) are decreasing in each row). Thus, \( i + 1 \) is in position \( n_m \) or \( n_{m-1} \); in \( B_{\ell - 1} \) and \( i = n_1 \) or \( n_{m+1} \) respectively. The relevant transformation formula is in Section 2.1.2: \( (s_i - b_{\beta,T} (i)) j_{\beta,T} = (1 - b_{\beta,T} (i)^2) j_{\beta,T} \). To allow \( \kappa = \frac{1}{\ell + 2} \) in this equation and conclude \( (s_i - b_{\beta,T} (i)) j_{\beta,T} \) is \( 0 \) is necessary to show \( s_i j_{\beta,T} \) has no poles there.

To complete the proof that \( \omega_1 j_{B\{S\},T\{S\}} = c(i,S) j_{B\{S\},T\{S\}} \) for \( 1 \leq i \leq 2mk \) and \( S \in \mathcal{Y}(\mathcal{S}) \) (at \( \kappa = \frac{1}{\ell + 2} \)) we will show each \( j_{B\{S\},T\{S\}} \) and \( s_i j_{\beta,T} \) (as described in (4) above) has no poles at \( \kappa = \frac{1}{\ell + 2} \).

In the next section, we show that it suffices to analyze \( 1 + 2 (k - 1) \) specific tableaux.

### 5. Reduction Theorems

Suppose some \( j_{\beta,T} \) was shown to be defined at \( \kappa = \frac{1}{\ell + 2} \) (no poles) and \( \zeta_{\beta,T}^i (i) - \zeta_{\beta,T}^i (i + 1) \geq 2 \) then \( j_{\beta,T} \) where \( \zeta_{\beta,T}^i (i) = s_i \zeta_{\beta,T}^i (i) \) is also defined (recall \( (s_i - b_{\beta,T} (i)) j_{\beta,T} \) is a nonzero multiple of \( s_i j_{\beta,T} \) if
\[ \beta_i \neq \beta_{i+1} \] or of \( I_{\beta, T}(i) \) if \( \beta_i = \beta_{i+1} \) and \( j = r_{\beta}(i.) \) and the process is invertible. In other words if \( \zeta_{\beta, T}' \) is a valid spectral vector and \( |\zeta_{\beta, T}'(i) - \zeta_{\beta, T}'(i+1)| \geq 2 \) then \( s_i \zeta_{\beta, T}' \) is also a valid spectral vector (valid.

We consider column-strict tableaux \( S \) of shape \( \sigma \) which are either RSYT or \( S \) differs by one row-wise transposition from being an RSYT. Their content vectors are used in the argument. Column-strict means that the entries in each column are decreasing.

**Definition 4.** Suppose \( 1 \leq n < m \) and \( j = 1, 2 \) then \( R_{j,n} \) is the set of tableaux \( S \) of shape \( \sigma \) such that \( S \) is column-strict and \( S' \) defined by \( S'[j, n] = S[j, n+1], S'[j, n+1] = S[j, n] \) and \( S'[a, b] = S[a, b] \) for \( b \neq n, n+1 \) is an RSYT.

Suppose \( \zeta_{\beta, T}'(u) = c(u, S) \) for \( 1 \leq u \leq 2mk \), row \( (i, S) = 2 \), row \( (i+1, S) = 1 \) and \( \text{col} (i, S) < \text{col} (i+1, S) \) then \( \zeta_{\beta, T}'(i) - \zeta_{\beta, T}'(i+1) \leq -2 \) and \( s_i \zeta_{\beta, T}' \) is a spectral vector associated with \( S^{(i)} \). Call this a permissible step. In fact, the inequality \( \zeta_{\beta, T}'(i) - \zeta_{\beta, T}'(i+1) \leq -2 \) is equivalent to the row and column property just stated. If \( S \in \mathcal{Y}(\sigma) \) then \( S^{(i)} \in \mathcal{Y}(\sigma) \) and if \( S \in R_{j,n} \) then \( S^{(i)} \in R_{j,n} \) (because any row or column orderings do not change). For counting permissible steps we define

\[
\text{inv}(S) = \# \{ (a, b) : a < b, \text{row } (a, S) < \text{row } (b, S) \}
\]

(3)

A permissible step \( S \to S^{(i)} \) adds 1 to \( \text{inv}(S) \). The reduction process aims to apply permissible steps until an inv-maximal tableau is reached. In \( \mathcal{Y}(\sigma) \) the inv-maximal element is \( S_0 \) and \( \text{inv}(S_0) = \binom{mk}{2} \).

**Definition 5.** For \( 1 \leq n < m \) and \( j = 1, 2 \) define a distinguished element \( S_{j,n}(\mu) \) of \( R_{j, n} \) by \( S_{j,n}(1, \mu) = 2mk + 2 - 2i, S_{j,n}(2, \mu) = 2mk + 1 - 2i \) for \( i \neq n, n+1 \) and for \( a = 1, 2, b = n, n+1 \)

\[
\begin{align*}
S_{1,n}(a, b) &= \begin{cases} 2mk - 2n + 2 & 2mk - 2n + 1 \\ 2mk - 2n & 2mk - 2n - 1 \end{cases} \\
S_{2,n}(a, b) &= \begin{cases} 2mk - 2n + 2 & 2mk - 2n + 1 \\ 2mk - 2n - 1 & 2mk - 2n \end{cases}
\end{align*}
\]

Then \( \text{inv} S_{j,n}(\mu) = \binom{mk}{2} - 1 \). Here are two examples with \( mk = 6 \):

\[
\begin{align*}
S_{1,3} &= \begin{bmatrix} 12 & 10 & 7 & 8 & 4 & 2 \\ 11 & 9 & 6 & 5 & 3 & 1 \end{bmatrix}, \quad S_{2,2} &= \begin{bmatrix} 12 & 10 & 9 & 6 & 4 & 2 \\ 11 & 7 & 8 & 5 & 3 & 1 \end{bmatrix}
\end{align*}
\]

Any \( S \in \mathcal{Y}(\sigma) \) can be transformed by a sequence of permissible steps to \( S_0 \) (this is a basic fact in representation theory but the explanation is useful to motivate the argument for \( R_{j,n} \)), and any \( S \in R_{j,n} \) can be transformed in this way to \( S_{j,n}(\mu) \). For convenience, replace \( mk \) by \( N \) since only the number of columns is relevant. Suppose \( S \in \mathcal{Y}(\sigma) \) and by permissible steps was transformed to \( S' \) with \( S'[1, i] = 2N - 2r - 2i, S'[2, i] = 2N + 1 - 2i \) for \( i \leq r < N - 1 \) (the inductive argument starts with \( r = 0 \)). From the definition of \( \mathcal{Y}(\sigma) \) it follows that \( S'[1, i] < S'[1, 2r + 1] \) and \( S'[2, i] < S'[2, r + 1] < S'[1, r + 1] \) for all \( i > r + 1 \). This implies \( S'[1, r + 1] = 2N - 2r \) and \( S'[2, r + 1] = 2N - 2r - u \) with \( u \geq 1 \). Then the list of entries \( S'[1, \ell] \) for \( \ell \leq r + 1 \) equals \( [2N - 2r, 2N - 2r - 2, \ldots, 2N - 2r - u + 1] \) Apply \( s_{2N - 2r - u} s_{2N - 2r - u + 1} \cdots s_{2N - 2r - 2} \) in this order (if \( u = 1 \) then already done). Each one is a permissible step, with \( t \) in \( [2, r + 1] \) and \( t + 1 \) in \( [2, n'] \) with \( n' > r + 1 \). This produces \( S' \) satisfying \( S'[1, i] = 2N - 2r - 2i, S'[2, i] = 2N + 1 - 2i \) for \( i \leq r < N \). The inductive argument starts at \( r = 1 \). From the
The definition of $R_{j,n}$ it follows that $S'[1,i] < S'[1,2r+1]$ and $S'[2,i] < S'[2,r+1] < S'[1,r+1]$ for all $i > r + 1$. This implies $S'[1,r + 1] = 2N - 2r$ and $S'[2,r + 1] = 2N - 2r - u$ with $u \geq 1$. The numbers $2N - 2r - u + 1, 2N - 2r - u + 2, \ldots, 2N - 2r$ are in $\{S'[1,\ell] : r + 1 \leq \ell \leq r + u\}$. As in the RSYT case apply $s_{2N - 2r - u}, s_{2N - 2r - u + 1}, \ldots, s_{2N - 2r - 2}$ in this order (if $u = 1$ then already done). It is possible that one pair of adjacent entries is out of order (when $j = 1$) but the argument is still valid. Here is a small example with $N = 4, n = 2, j = 1, r = 0$.

$$S = \begin{bmatrix} 8 & 6 & 7 & 3 \\ 5 & 4 & 2 & 1 \end{bmatrix} \xrightarrow{s_5} \begin{bmatrix} 8 & 5 & 7 & 3 \\ 6 & 4 & 2 & 1 \end{bmatrix} \xrightarrow{s_5} \begin{bmatrix} 8 & 5 & 6 & 3 \\ 7 & 4 & 2 & 1 \end{bmatrix}.$$ 

The inductive process can be continued until $r = n - 1$ and the result is a tableau $S' \in R_{j,n}$ with the entries $2N - 2n + 3, \ldots, 2N$ in the first $n - 1$ columns. Thus, the entries $1, 2, \ldots, 2N - 2n + 2$ are in the remaining $N - n + 1$ columns.

The next part of the process is to start from the last column and work forward. Suppose $S'$ by permissible steps was transformed to $S''$ with $S''[1,N + 1 - i] = 2i, S''[2,N + 1 - i] = 2i - 1$ for $i \leq r < N - n - 2$ (the first step is with $r = 0$). As before $S''[1,i] > S''[1,N - r] > S''[2,N - r]$ and $S''[2,i] > S''[2,N - r]$ for $i \leq N - r - 1$. This implies $S''[2,N - r] = 2r + 1$ and $S''[1,N - r] = 2r + 1 + u$ with $u \geq 1$. The numbers $2r + 1, 2r + 2, \ldots, 2r + u$ are in $\{S''[2,\ell] : N - r - u + 1 \leq \ell \leq N - r\}$. This range of cells has contents $N - r - u - 1, N - r - u, \ldots, N - r - 3$ (excluding $[2,N - r]$). Possibly one pair of adjacent entries is out of order (when $j = 2$). In terms of contents while $c(2r + 1 + u, S'') = N - r - 1$ so the steps $s_{2r+u}, s_{2r+u-1}, \ldots, s_{2r+u}$ are permissible in that order resulting in $S'''$ with $S'''[1,N - r] = 2r + 2$. The process stops at $N - r = n + 2$. The result is $S'''[1,i] = 2N - 2 + 2i, S'''[2,i] = 2N + 1 - 2i$ for $1 \leq i < n$ and $n + 1 < i \leq N$. Thus, the entries in columns $n$ and $n + 1$ are $2N - 1 - 2n, \ldots, 2N + 2 - 2n$. The definition of $R_{j,n}$ forces the position of these entries:

$$R_{1,n} : \begin{bmatrix} 2N - 2n + 1 & 2N - 2n + 2 \\ 2N - 2n & 2N - 2n - 1 \end{bmatrix}, R_{2,n} : \begin{bmatrix} 2N - 2n + 2 & 2N - 2n + 1 \\ 2N - 2n - 1 & 2N - 2n \end{bmatrix}.$$ 

Thus, we showed that any $S' \in R_{j,n}$ can be transformed by permissible steps to $S_{(j,n)}$ an inv-maximal tableau.

In the above example $r = 0$ and $u = 2$ and the action of $s_2$ suffices to obtain the desired tableau:

$$S'' = \begin{bmatrix} 8 & 5 & 6 & 3 \\ 7 & 4 & 2 & 1 \end{bmatrix} \xrightarrow{s_2} \begin{bmatrix} 8 & 5 & 6 & 2 \\ 7 & 4 & 3 & 1 \end{bmatrix};$$

no more permissible steps are possible.

In our applications $N = mk$ and $n = \ell m$ with $1 \leq \ell \leq k - 1$.

6. Uniqueness Theorems

This section starts by showing how uniqueness of spectral vectors is used to prove that specific Jack polynomials exist for some $\kappa = \kappa_0$, that is, there are no poles there.

**Proposition 3.** Suppose $(\beta, T) \in N_0^N \times \mathcal{Y}(\tau)$ has the property that $(\gamma, T') \in N_0^N \times \mathcal{Y}(\tau), \gamma \leq \beta$ and $x_{T',\gamma} = x_{\beta,T} (i)$ for $1 \leq i \leq N$ at $\kappa = \kappa_0$ implies $(\gamma, T') = (\beta, T)$ then $I_{\beta,T}$ is defined at $\kappa = \kappa_0$, in the sense that the generic expression for $I_{\beta,T}$ can be specialized to $\kappa = \kappa_0$ without poles.

**Proof.** From the $\triangleright$-triangular nature of (1) it follows that the inversion formulas are also triangular, in particular

$$x^\otimes \tau \left( r_{\beta}^{-1} \right) T = I_{\beta,T} + \sum_{\gamma < \beta, T' \in \mathcal{Y}(\tau)} u(\beta, \gamma, T', T'; \kappa) I_{\gamma,T'},$$
where \( u(\beta, \gamma, T, T'; \kappa) \) is a rational function of \( \kappa \). By hypothesis for each \( \gamma < \beta \) and \( T' \in \mathcal{V}(\tau) \) there is an index \( i [\gamma, T'] \) such that \( \zeta_{\gamma, T'}^\prime (i [\gamma, T']) \neq \zeta_{\gamma, T'} (i [\gamma, T']) \) at \( \kappa = \kappa_0 \). Recall that the generic spectral vector \( \zeta_{\gamma, T'}^\prime \) uniquely determines \((\gamma, T')\), since \([\gamma_1, \ldots, \gamma_N]\) is found from the coefficients of \( \frac{1}{\kappa} \) and the remaining terms of \( \zeta_{\gamma, T'}^\prime \) determine the content vector of \( T' \). Define an operator on \( \mathcal{P}_\tau \) by

\[
\mathcal{T}_{\beta, T} = \prod_{\gamma < \beta, T' \in \mathcal{V}(\tau)} \frac{u(\beta, \gamma, T', \kappa) - \zeta_{\gamma, T'}^\prime (i [\gamma, T'])}{\zeta_{\gamma, T'} (i [\gamma, T']) - \zeta_{\gamma, T'} (i [\gamma, T'])}.
\]

Then \( \mathcal{T}_{\beta, T} \) annihilates each \( J_{\gamma, T'} \) with \( \gamma < \beta \) and maps \( J_{\beta, T} \) to itself. Thus \( \mathcal{T}_{\beta, T} (x^\beta \otimes \tau (r_\beta) T) = J_{\beta, T} \) and by construction the right hand side has no poles at \( \kappa = \kappa_0 \). \( \square \)

The condition in the Proposition is sufficient, not necessary. There is an example in the concluding remarks to support this statement.

We introduce a simple tool for the analysis of a pair \( \beta, T \), namely the tableau \( \lambda_{\beta, T} \) with the entries being pairs \((i, \beta_{\gamma(i)}^{-1})\) such that the tableau of just the first entries coincides with \( T \), i.e., if \( T[a, b] = i \) then \( \lambda_{\beta, T}[a, b] = (i, \beta_i^+) \). As example let

\[
T = \begin{bmatrix}
12 & 11 & 10 & 6 \\
9 & 8 & 5 & 2 \\
7 & 4 & 3 & 1 \\
\end{bmatrix}, \beta = (120201303121), \beta^+ = (33221110000)
\]

\[
\lambda_{\beta, T} = \begin{bmatrix}
(12, 0) & (11, 0) & (10, 0) & (6, 1) \\
(9, 1) & (8, 1) & (5, 2) & (2, 3) \\
(7, 1) & (4, 2) & (3, 2) & (1, 3) \\
\end{bmatrix}.
\]

The tableau \( \lambda_{\beta, T} \) has order properties: in each row the first entries decrease and the second entries nondecrease (weakly increase), and the same holds for each column.

The first part is to assume \( \lambda \supseteq \beta \) and \((m + 1) \beta_j + c (r_\beta (j), T) = c (j, S_0)\) (called the fundamental equation) for \( 1 \leq i \leq 2mk \) and to deduce that \( \beta = \lambda \) and \( T = T_0 \).

Our approach to the uniqueness proofs is to work one brick at a time, and in each brick alternating between even and odd indices showing the values of \( \beta_i \) and \( T' \) agree with those of \( \lambda, T_0 \). For each cell we use the fundamental equation and the order properties of \( \lambda_{\beta, T} \) to set up inequalities which lead to a contradiction if \( \beta \neq \lambda \).

**Theorem 2.** Suppose \((\beta, T) \in \mathbb{N}_0^{2mk} \times \mathcal{V}(\tau), \beta \subseteq \lambda \) and \((m + 2) \beta_j + c (r_\beta (j), T) = c (j, S_0)\) for \( 1 \leq j \leq 2mk \) then \( \beta = \lambda \) and \( T = T_0 \).

**Proof.** This is an inductive argument alternating between even and odd indices to prove the desired equalities for brick \( B_0 \). Then the argument is applied to the tableaux with one less brick. Suppose we showed \( \beta_{2n-j} = 0 \) for \( 0 \leq j \leq 2n - 1 \leq 2m - 3 \) (thus \( n \leq m - 1 \)) and \( T[1, i + 1] = 2mk - 2i \) for \( 0 \leq i \leq n - 1 \) and \( T[2, i + 1] = 2mk - 2i - 1 \) for \( 0 \leq i \leq n - 1 \). The start of the induction is \( n = 0 \) so the previous conditions are vacuous. Suppose \( \beta_{2mk-2n} = \ell, r_\beta (2mk - 2n) = \rho \) and \( \lambda_{\beta, T}[a, b] = (\rho, \ell) \). Then

\[
\ell (m + 2) + c (\rho, T) = c (2mk - 2n, S_0) = n,
\]

\[
b - a = c (\rho, T) = n - \ell (m + 2),
\]

\[
a = b - n + \ell (m + 2)
\]

\[
\geq (\ell - 1) (m + 2) + m + 3 - n.
\]
Thus, if \( \ell > 0 \) and \( n \leq m - 1 \) then \( a \geq 4 \). Let \( X'_{\beta,T} \) with \( d > \rho \), \( j < \ell \) and \( r_{\beta}(e) = d \) (thus \( \beta_{e} = j \)); furthermore by the inductive hypothesis and \( a - 1 \geq 3 \) it follows that \( e \leq 2mk - 2n \). Then

\[
j (m + 2) + c (d, T) = j (m + 2) + b - a + 1 = c (e, S_{0}) \]
\[
c (e, S_{0}) = j (m + 2) + n + 1 - \ell (m + 2) = n + 1 - (\ell - j) (m + 2) \leq n - m - 1.
\]

However, \( \min (c (i, S_{0}) : i \leq 2mk - 2n) = n - 1 \) and there is a contradiction. Thus, \( \beta_{2mk - 2n - 1} = 0 \), \( \rho = 2mk - 2n \) and \( T [1, n + 1] = 2mk - 2n \) (the other possibility for the entry \( 2mk - 2n \) in \( T \) is \( [3, 1] \), ruled out by the content value). The start \( n = 0 \) forces \( T [1, 1] = 2mk \). The last step is with \( n = m - 1 \) and results in \( T [1, m] = 2mk - 2m + 2 \).

Suppose we showed \( \beta_{2mk - j} = 0 \) for \( 0 \leq j \leq 2n \leq 2m - 2 \) and \( T [1, i + 1] = 2mk - 2i \) for \( 0 \leq i \leq n \) and \( T [2, i + 1] = 2mk - 2i - 1 \) for \( 0 \leq i < n \) (the first step is with \( n = 0 \)). Suppose \( \beta_{2mk - 2n - 1} = \ell \), \( r_{\beta}(2mk - 2n - 1) = \rho \) and \( X'_{\beta,T} \) with \( (\rho, \ell) \) then

\[
\ell (m + 2) + c (\rho, T) = c (2mk - 2n - 1, S_{0}) = n - 1,
\]
\[
b - a = c (\rho, T) = n - 1 - \ell (m + 2),
\]
\[
a = b - n + 1 + \ell (m + 2) \geq (\ell - 1) (m + 2) + m - n + 4.
\]

If \( \ell > 0 \) and \( n \leq m - 1 \) then \( a \geq 5 \) and \( X'_{\beta,T} \) with \( d > \rho \), \( j < \ell \) (because \( \beta_{2mk - 2n - 1} \) is the last appearance of \( \ell \) in \( \beta \)) and \( r_{\beta}(e) = d \) (thus \( \beta_{e} = j \)). By the inductive hypothesis and \( a - 1 \geq 3 \) it follows that \( e \leq 2mk - 2n - 1 \). Then

\[
(m + 2) j + c (d, T) = (m + 2) j + b - a + 1 = c (e, S_{0}) \]
\[
c (e, S_{0}) = (m + 2) j + n - (\ell - j) (m + 2) \leq n - m - 2.
\]

However, \( \min (c (i, S_{0}) : i \leq 2mk - 2n - 1) = n - 1 > n - m - 2 \) and this is a contradiction. Thus, \( \beta_{2mk - 2n - 1} = 0 \) and \( T [2, n + 1] = 2mk - 2n - 1 \) (the other possibilities for the entry \( 2mk - 2n - 1 \) in \( T \) are \([1, n + 2]\) and \([3, 1]\), both ruled out by their content values). The last step of the induction is for \( n = m - 1 \).

Replace the original problem by a smaller one: let \( \lambda' = [\lambda_{i - 1}]_{i=1}^{2m(k - 1)} \), \( \beta' = [\beta_{i - 1}]_{i=1}^{2m(k - 1)} \) the tableaux \( S'_{0} \) of shape \( 2 \times (k - 1) \) with entries \( S'_{0} [i, j] = S_{0} [i, j + m] \) for \( i = 1, 2 \) and \( 1 \leq j \leq m (k - 1) \) and the tableaux \( T'_{0} \) and \( T' \) of shape \( 2 \times (k - 1) \) with entries \( T'_{0} [i, j] = T_{0} [i + 2, j] \), \( T' [i, j] = T [i + 2, j] \) for \( 1 \leq i \leq 2 (k - 1) \) and \( 1 \leq j \leq m \). The consequences of these definitions are with \( 1 \leq i \leq 2m (k - 1) \)

\[
c (i, T') = c (i, T) + 2, c (i, T'_{0}) = c (i, T_{0}) + 2,
\]
\[
c (i, S'_{0}) = c (i, S_{0}) - m, \quad r_{\beta'} (i) = r_{\beta} (i), \lambda' \geq \beta'.
\]

Then

\[
(m + 2) \beta' + c (r_{\beta'} (i), T') = (m + 2) (\beta_{i - 1}) + c (r_{\beta} (i), T) + 2
\]
\[
= (m + 2) \beta_{i} + c (r_{\beta} (i), T) - m
\]
\[
= c (i, S_{0}) - m = c (i, S'_{0}) .
\]
and the same argument as before shows that \((\beta, T)\) agrees with \((\lambda, T_0)\) in the first four rows (the first two bricks). Repeat this process \((k - 1)\) times arriving at \(\beta''_i = 0\) for \(1 \leq i \leq 2m\) and the entries of the remaining \(T''\) are \(2m, 2m - 1, \ldots, 1\) entered column by column

\[
\begin{bmatrix}
2m & \cdots & 4 & 2 \\
2m - 1 & \cdots & 3 & 1
\end{bmatrix}.
\]

Thus, \((\beta, T) = (\lambda, T_0)\) and the spectral vector of \((\lambda, T)\) is unique. \(\square\)

We set up the same argument for removing the last brick \(B_{k-1}\). Intuitively this is already done: rotate the tableaux through 180° and replace the entry \(r\) by \(2mk + 1 - r\). This idea guides the proof. The property \(\beta \subseteq \lambda\) implies \(\beta_i \leq k - 1\) for all \(i\). Here the inductive argument alternates between odd and even indices.

**Remark 2.** The following is an alternate proof of Theorem 2.

**Proof.** Suppose we showed \(\beta_j = k - 1\) for \(1 \leq j \leq 2n\), \(T[2k, m - j] = 2j + 1\) for \(0 \leq j \leq n - 1\) and \(T[2k - 1, m - j] = 2j + 2\) for \(0 \leq j \leq n - 1\) (the first step is at \(n = 0\) with vacuous conditions on \(T\), the last at \(n = m - 1\)). Suppose \(\beta_{2n + 1} = \ell, r_{\beta} (2n + 1) = \rho\) and \(X_{\beta, T}[a, b] = (\rho, \ell)\) then (using \(b \leq m\))

\[
\ell (m + 2) + c (\rho, T) = c (2n + 1, S_0) = mk - n - 2,
\]

\[
b - a = mk - n - 2 - \ell (m + 2),
\]

\[
a = b - mk + n + 2 + \ell (m + 2)
\]

\[
\leq 2k - 3 - (m - 1 - n) - (k - 2 - \ell) (m + 2).
\]

If \(\ell < k - 1\) and \(n \leq m - 1\) then \(a \leq 2k - 3\). Let \(X_{\beta, T}[a + 1, b] = (d, j)\) with \(d < \rho, j > \ell\), and \(r_{\beta} (e) = d\) (so that \(\beta_d = j\)); furthermore by the inductive hypothesis and \(a + 1 \leq 2k - 2\) it follows that \(e \geq 2n + 1\), then

\[
(m + 2) j + c (d, T) = (m + 2) j + b - a - 1 = c (e, S_0)
\]

\[
c (e, S_0) = mk - n - 3 - \ell (m + 2) + j (m + 2)
\]

\[
\geq mk + m - n - 1
\]

However, \(\max \{c (i, S_0) : i \geq 2n + 1\} = mk - n - 1 < mk + m - n - 1\), a contradiction, thus \(\beta_{2n + 1} = k - 1\) and \(\rho = 2n + 1\) which implies \(T[2k, m - n] = 2n + 1\) (the other possible location for the entry \(2n + 1\) in \(T\) is \([2k - 2, m]\) but \(m - 2k + 2 \neq m - n - 2k\)). The start is \(T[2k, m] = 1\) (forced by definition of RSYT) and \(\beta_1 = k - 1\). The last step results in \(T[2k, 1] = 2m - 1\).

Suppose we showed \(\beta_i = k - 1\) for \(1 \leq i \leq 2n - 1\), \(T[2k, m - j] = 2j + 1\) for \(0 \leq j \leq n - 1\) and \(T[2k - 1, m - j] = 2j + 2\) for \(0 \leq j \leq n - 2\) (the first step is at \(n = 1\), the last at \(n = m\)). Suppose \(\beta_{2n} = \ell, r_{\beta} (2n) = \rho\) and \(X_{\beta, T}[a, b] = (\rho, \ell)\) then

\[
\ell (m + 2) + c (\rho, T) = c (2n, S_0) = mk - n,
\]

\[
b - a = mk - n - \ell (m + 2),
\]

\[
a = b - mk + n + \ell (m + 2)
\]

\[
\leq 2k - 4 - (m - n) - (k - 2 - \ell) (m + 2).
\]
Thus, if $\ell < k - 1$ and $n \leq m$ then $a \leq 2k - 4$. Let $X_{\beta,T} [a+1,b] = (d, j)$ with $d < \rho$, $j > \ell$, and $r_\beta (e) = d$ (so that $\beta_e = j$) then

\[
(m + 2) j + c (d, T) = b - a - 1 + (m + 2) j = c (e, S_0) \geq mk - n - 1 - \ell (m + 2) + j (m + 2)
\]

However, $\max \{ c (i, S_0) : i \geq 2n \} = mk - n - 1 < mk + m - n + 1$ and there is a contradiction. Thus, $\beta_{2m} = k - 1, T_0, T_0 (2n) = 2n$ and $T [2k - 1, m - 1, n] = 2n$; the other possible locations for the entry $2n$ in $T$ are $[2k, m, n]$, $[2k - 2, m]$ ruled out by their content values.

The inductive process concludes by showing $\beta_i = k - 1$ for $1 \leq i \leq 2m$ and $T [i, j] = T_0 [i, j]$ for $i = 2k - 1, 2k$ and $1 \leq j \leq m$. As before the original problem can be reduced to a smaller one by removing the last brick. This is implemented by defining $\lambda', \beta', S'_0, T'_0, T'$ as follows:

\[
\lambda'_i = \lambda_{i + 2m}, \beta'_i = \beta_{i + 2m}, r'_\beta (i) = r_\beta (i + 2m) - 2m, 1 \leq i \leq 2m (k - 1),
\]

\[
S'_0 [i, j] = S_0 [i, j] - 2m, i = 1, 2, 1 \leq j \leq m (k - 1),
\]

\[
T'_0 [i, j] = T_0 [i, j] - 2m, T' [i, j] = T [i, j] - 2m, 1 \leq i \leq 2m - 1, 1 \leq j \leq m.
\]

Clearly $\lambda' \geq \beta'$ and the hypothesis $(m + 2) \beta'_i + c (r'_\beta (i), T') \geq c (j, S'_0)$ holds for $1 \leq j \leq 2m (k - 1)$. So the bricks can be removed in the order $k - 1, k - 2, \ldots$. At the end there is only one brick $B_0, \beta_i = 0, r_\beta (i) = i$ and $c (i, T) = c (i, S_0)$ (for $2mk - 2m < i \leq 2mk$ and the corresponding parts (brick $B_0$) of $T$ and $S_0$ are identical, and thus to $T_0$. □

The second part is to prove uniqueness for the spectral vectors derived from the content vectors of the tableaux $S_{(j,n)}$ (see Definition 5) where $n = ms$, at the edge of brick $B_{s-1}$ adjacent to the edge of $B_s$. To prove this we use the previous arguments to remove the bricks above and below bricks $s - 1$ and $s$ leaving us with a straightforward argument where only two values of $\lambda$ play a part. To obtain the hypothetically unique spectral vectors we apply reflections to $J_{\lambda,T_0}$. For brevity let $i_0 = 2m (k - s)$.

First compute $s_0 J_{\lambda,T_0}$, a nonzero multiple of $J_{\lambda,T_0}$, this is a permissible step and hence this polynomial is defined for $k = \frac{1}{m+2}$. Then form $s_{i_0+1} s_0 J_{\lambda,T_0}$ which produces the polynomial labeled by $\left( a^{(1)}, T_0 \right)$ whose spectral vector equals the content vector of $S_{(1,ms)}$. Also form $s_{i_0-1} s_0 J_{\lambda,T_0}$ with label $\left( a^{(2)}, T_0 \right)$ associated with $S_{(2,ms)}$. Here are tables of values of $\lambda, s_0 \lambda, a^{(1)} = s_{i_0+1} s_0 \lambda, a^{(2)} = s_{i_0-1} s_0 \lambda$ in the zone of relevance $(i_0 - 1 \leq i \leq i_0 + 2)$:

\[
\begin{array}{cccccc}
  i = & i_0 - 1 & i_0 & i_0 + 1 & i_0 + 2 \\
  \lambda = & s & s & s - 1 & s - 1 \\
  s_0 \lambda = & s & s - 1 & s - 1 & s \\
  a^{(1)} = & s & s - 1 & s - 1 & s \\
  a^{(2)} = & s - 1 & s & s - 1 & s \\
\end{array}
\]

and the corresponding spectral vectors $\beta_i (m + 2) + c (r_\beta (i), T_0)$, denoted by $v (\beta, i)$ for convenience,

\[
\begin{array}{cccccc}
  i = & i_0 - 1 & i_0 & i_0 + 1 & i_0 + 2 \\
  v (\lambda, \cdot) = & sm - 1 & sm & sm - 2 & sm - 1 \\
  v (s_0 \lambda, \cdot) = & sm - 1 & sm - 2 & sm & sm - 1 \\
  v (a^{(1)}, \cdot) = & sm - 1 & sm - 2 & sm - 1 & sm \\
  v (a^{(2)}, \cdot) = & sm - 2 & sm - 1 & sm & sm - 1 \\
\end{array}
\]
The respective cells in $T_0$ are $T_0[2s+2,1] = i_0-1, T_0[2s+1,1] = i_0, T_0[2s,m] = i_0+1, T_0[2s-1,m] = i_0+2$ with respective contents $-1-2s, -2s, m-2s, m+1-2s$. Except for these four locations $v(\lambda,i) = v(a^{(1)},i) = v(a^{(2)},i) = c(i, S_0)$ so in the bricks $B_j$ for $0 \leq j < s-1$ and $s < j \leq k-1$ the previous proofs can be applied; the various $\max\{v(\lambda,i); i > b\}$ and $\min\{v(\lambda,i); i < b\}$ values apply verbatim.

**Theorem 3.** Suppose $u = 1$ or $2$ and $(\beta, T) \in N_0^{2mk} \times V(\tau)$ such that $\beta \leq a^{(u)}$ and $(m+2)\beta_j + c(r_\beta(i), T) = v(a^{(u)},i)$ for $1 \leq i \leq 2mk$ then $(\beta, T) = (a^{(u)}, T_0)$.

**Proof.** By the previous arguments we show $T[i,j] = T_0[i,j]$ for $1 \leq j \leq m, 1 \leq i \leq 2s-2$ (using the proof for Theorem 2) and $2s+3 \leq i \leq 2k$ (using the alternate proof following Remark 2). This leaves just two bricks and we can assume $s = 1, k = 2$. Reducing $v(a^{(u)}, \cdot)$ to $s = 1, k = 2$ results in (with $1 \leq i \leq 2m$ and $i \neq m, m+1$)

$$v(a^{(u)}_{2}, i) = 2m - i,$$
$$v(a^{(u)}_{2i-1}, i) = 2m - i - 1,$$
$$v(a^{(1)}_{i}, i)_{2m+2}^{i = 2m-1} = [m - 2, m - 1, m - 1],$$
$$v(a^{(2)}_{i}, i)_{2m+2}^{i = 2m-1} = [m - 1, m - 2, m - 1].$$

The property $\beta \leq a^{(u)}$ implies that $\beta$ is a permutation of $(1^{2m}, 0^{2m})$. The entries $2m + 1, 2m + 2, \ldots, 4m$ in $T$ are all in $B_0$ and the entries $1, 2, \ldots, 2m$ are in $B_1$. This follows from $X_{\beta,T}[a,b] = (r,0)$ implies $1 \leq a \leq 2$ for $a = 3$ or $4$ then the ordering property of $X_{\beta,T}$ implies $X_{\beta,T}[3,1] = (r',0)$ or $X_{\beta,T}[4,1] = (r',0)$ but $c(r', T) = -2$ or $-3$ is impossible (as values of $v(a^{(u)}, i)$). Thus, the $2m$ pairs \{$(r,0) : 2m + 1 \leq r \leq 4m$\} fill \{${X_{\beta,T}[a,b] : a = 1, 2, 1 \leq b \leq m}$\}. The next few steps are for $m \geq 2$; if $m = 1$ then there are just 4 cells left and the last part of the proof suffices. By using the previous arguments we show $\beta_i = 1$ for $1 \leq i \leq 2m - 2$ and $\beta_i = 0$ for $2m + 3 \leq i \leq 4m$, and also that for $1 \leq j \leq m - 1$ and $m + 2 \leq j \leq 2m$

$$T[1,j] = T_0[1,j] = 4m + 2 - 2j,$$
$$T[2,j] = T_0[2,j] = 4m + 1 - 2j,$$

and for $2 \leq j \leq m$

$$T[3,j] = T_0[3,j] = 2m + 2 - 2j,$$
$$T[4,j] = T_0[4,j] = 2m + 1 - 2j.$$

As example of the steps of the proof let $\beta_1 = \ell$ then $(m + 2) \ell + c(r_\beta(1), T) = 2m - 2$ and let $X_{\beta,T}[a,b] = (r_\beta(1), \ell)$. Then

$$b - a = 2m - 2 - (m + 2) \ell,$$
$$a = b - 2m + 2 + (m + 2) \ell$$

$$\leq 2 - m + (m + 2) \ell$$

However, if $\ell = 0$ then $a \leq 0$ (since $m \geq 2$) which is impossible; thus $\beta_1 = 1, a = 4$ and $T[4,m] = 1$. Similarly consider $\beta_{4m} = \ell$ and $X_{\beta,T}[a,b] = (r_\beta(4m), \ell)$ then
Then \( r \kappa \) Symmetry \( \sigma \)

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but if \( \ell = 1 \) then \( a \geq 5 \) which is impossible; thus \( \beta_{4m} = 0 \) and \( T[1, 1] = 4m \).

All but four entries were accounted for and thus \( T[2, m] = 2m + 1 \), the rank of the first zero in \( \beta \), and \( T[3, 1] = 2m \), the rank of the last \( 1 \) in \( \beta \). Thus, \( T = T_0 \). The relevant part of the content vector is

\[
[c(i, T)]_{i=2m-1}^{2m+2} = [-3, -2, m - 2, m - 1]
\]

The remaining equations are

\[
\begin{align*}
(m + 2) \beta_{2m-1} + c(r_{\beta}(2m - 1), T) &= m - 1 \quad m - 2 \\
(m + 2) \beta_{2m} + c(r_{\beta}(2m), T) &= m - 2 \quad m - 1 \\
(m + 2) \beta_{2m+1} + c(r_{\beta}(2m + 1), T) &= m - 1 \quad m \\
(m + 2) \beta_{2m+2} + c(r_{\beta}(2m + 2), T) &= m \quad m - 1
\end{align*}
\]

Let \( \beta_{j_1} = \beta_{j_2} = 1 \) with \( 2m - 1 < j_1 < j_2 \leq 2m + 2 \). Then \( r_{\beta}(j_1) = 2m - 1, m + 2 \) and \( c(r_{\beta}(j_2), T) = m - 1 \) and \( \beta_{j_1} = \beta_{j_2} = 0 \) with \( 2m - 1 < j_3 < j_4 \leq 2m + 2 \). Then \( r_{\beta}(j_3) = 2m + 1, c(r_{\beta}(j_4), T) = m - 2 \) and \( r_{\beta}(j_4) = 2m + 2, c(r_{\beta}(j_4), T) = m - 1 \).

Case \( a^{(1)} \): From the table we see that \( j_3 = 2m \) and \( j_2 = 2m + 2 \). This implies \( j_k = 2m + 1 \) and \( j_{i_1} = 2m - 1 \) thus \( \beta = a^{(1)} \), with central entries \((1, 0, 0, 1)\).

Case \( a^{(2)} \): From the table we see that \( j_3 = 2m - 1 \) and \( j_2 = 2m + 2 \). This implies \( j_k = 2m \) and \( j_{i_1} = 2m + 2 \). Thus \( \beta = a^{(2)} \), with \((0, 1, 1, 0)\) being the central entries.

This concludes the proof. \( \Box \)

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The algebra generated by \( D_i \) and multiplication by \( x_i \) for \( 1 \leq i \leq 2nk \) along with \( w \in S_{2nk} \) is the rational Cherednik algebra \( A_{2nk-1} \) and \( \mathcal{P}_T \) is called the standard module associated with \( \tau \), denoted \( \Delta_\kappa (\tau) \). In this section, we construct a homomorphism from the module \( \mathcal{P}_T \) to \( \mathcal{P}_T \) when the parameter \( \kappa = \frac{1}{1 + \tau} \). In the notation of Definition 3 for each \( S \in \mathcal{Y}(\tau) \) there is a pair \((\beta(S), T(S))\) such that the spectral vector \( \zeta_{\beta(S), T(S)}(i) = c(i, S) \) for \( 1 \leq i \leq 2nk \) at \( \kappa = \frac{1}{1 + \tau} \). However the polynomials \( I_{\beta(S), T(S)} \) need to be rescaled so that they transform under \( w \) with the same matrix as \( \mathcal{Y}(\tau) \). Recall the formula for \( ||S||^2 \) which is derived from the requirement that \( \{S : S \in \mathcal{Y}(\tau)\} \) is an orthogonal basis and each \( \sigma(w) \) is an isometry (and we use this requirement for \( ||I_{\beta(S), T(S)}||^2 \) as well)

\[
||S||^2 = \prod_{1 \leq i < j \leq 2km} \left( 1 - \frac{1}{c(i, S) - c(j, S)^2} \right).
\]

By the construction of \( S_0 \) (column by column) either \( i \) is odd and \( j > i \) implies \( c(i, S_0) \leq c(i, S_0) - c(j, S_0) \geq -1 \) or \( i \) is even and \( j > i \) implies \( c(j, S_0) < c(i, S_0) \) and \( c(i, S_0) - c(j, S_0) \geq 0 \), thus \( ||S_0||^2 = 1 \).

Suppose row \( (i, S) < \text{row} \ (i + 1, S), \text{col} \ (i, S) > \text{col} \ (i + 1, S) \) so that

\[
\begin{align*}
c(i, S) &= \text{col} \ (i, S) - \text{row} \ (i, S) \geq (\text{col} \ (i + 1, S) + 1) - (\text{row} \ (i + 1, S) - 1) \\
&= c(i + 1, S) + 2,
\end{align*}
\]
and the transformation rule Section 2.1.3 yields (with \(b_1 (S) = (c (i, S) - c (i + 1, S))^{-1}\)):

\[
\sigma (s_i) S = S^{(i)} + b_S (i) S
\]

\[
\|S\|^2 = \|\sigma (s_i) S\|^2 = \|S^{(i)}\|^2 + b_S (i)^2 \|S\|^2
\]

\[
\|S^{(i)}\|^2 = \left(1 - b_S (i)^2\right) \|S\|^2.
\]

We need two rules for the NSJP \(J_{a,T}\):

1. \(a_i = a_{i+1}, j = r_a (i)\) and \(c (j, T) - c (j + 1, T) \geq 2\) (Section 2.1.3)

\[
s_iJ_{a,T} = b_{a,T} (i) J_{a,T} + J_{a,T^{(i)}}
\]

\[
\|J_{a,T}\|^2 = \|s_iJ_{a,T}\|^2 = b_{a,T} (i)^2 \|J_{a,T}\|^2 + \|J_{a,T^{(i)}}\|^2
\]

\[
\|J_{a,T^{(i)}}\|^2 = \left(1 - b_{a,T} (i)^2\right) \|J_{a,T}\|^2
\]  (4)

2. \(a_i > a_{i+1}\) (Section 2.1.2)

\[
s_iJ_{a,a,T} = -b_{a,T} (i) J_{a,a,T} + J_{a,T}
\]

\[
\|J_{a,a,T}\|^2 = \|s_iJ_{a,a,T}\|^2 = b_{a,T} (i)^2 \|J_{a,a,T}\|^2 + \|J_{a,T}\|^2
\]

\[
\|J_{a,a,T}\|^2 = \left(1 - b_{a,T} (i)^2\right)^{-1} \|J_{a,T}\|^2.
\]  (5)

Recall the abbreviation \(\pi \{S\} = \beta \{S\}, T \{S\}\). The following discussion of \(\|J_{a,T}\|^2\) applies only to \(\text{span} \{I_{\pi (S)}: S \in \mathcal{Y} (\sigma)\}\) at \(\kappa = \frac{1}{\|\pi\|^2}\), which is an irreducible \(S_{2mk}\)-module, isomorphic to \(V_\sigma\). We use the normalization \((\lambda = \beta \{S_0\}, T_0 = T \{S_0\})\)

\[
\|J_{\lambda,T_0}\|^2 = 1
\]

and this determines the other norms.

**Definition 6.** For \(S \in \mathcal{Y} (\sigma)\) let \(\gamma_S = \|S\| / \|I_{\pi (S)}\|\). By convention \(\gamma_{S_0} = 1\).

**Proposition 4.** Suppose \(S \in \mathcal{Y} (\sigma)\) then

\[
\gamma_S = \prod_{1 \leq i < j \leq 2mk, \beta \{S_i\} < \beta \{S_j\}} \left(1 - \frac{1}{(c (i, S) - c (j, S))^2}\right).
\]

**Proof.** We argue by induction on \(\text{inv} (S)\) (see (3)). Suppose the formula holds for each \(S \in \mathcal{Y} (\sigma)\) with \(\text{inv} (S) \geq \text{inv} (S_0) - u\) for some \(u\); the start is \(u = 0\). Suppose \(\text{inv} (S') = \text{inv} (S_0) - u - 1\) and \(c (i, S') - c (i + 1, S') \leq -2\), i.e., \(\text{row} (i, S') = 2, \text{row} (i + 1, S') = 1\) and \(\text{col} (i, S') < \text{col} (i + 1, S')\).

Then \(S' = S^{(i)}\) and \(\text{inv} (S) = \text{inv} (S_0) - u\). Also \(\|S^{(i)}\|^2 = \left(1 - b_i (S)^2\right) \|S\|^2\). For convenience let \(\beta = \beta \{S\}, T = T \{S\}\). (Recall that \(S'_{\beta \{S\}, T \{S\}} (j) = c (j, S)\) for all \(j\)). There are two cases for the relative locations of \(i\) and \(i + 1\) in \(S\).
If $i, i + 1 \in B_{\ell}$ for some $\ell$ then by definition $\beta_i = \beta_{i+1} = \ell$, $r_\beta (i+1) = j + 1 = r_\beta (i) + 1$ and $j, j + 1$ are in the same brick $B_{\ell}$ in $T$. Then $T \{ S^{(i)} \} = T^{(i)}, \beta \{ S^{(i)} \} = \beta$. By Formula (4)

$$\| I_{\pi} \{ S^{(i)} \} \| = \left( 1 - b_i (S)^2 \right) \| I_{\pi} \{ S \} \|^2$$

$\gamma_{S'} = \gamma_S$.

Because $\beta_i = \beta_{i+1}$ the product in $\gamma_S$ is invariant under the replacement $S \rightarrow S^{(i)}$.

If $i$ and $i + 1$ are in different bricks then $\beta_i > \beta_{i+1}$ because $\text{col} (i, S) > \text{col} (i+1, S)$. Then $\beta \{ S' \} = s_i \beta, T \{ S' \} = T$ and by Formula (5)

$$\| J_{\beta_i T} \|^2 = \left( 1 - b_{\beta_i T} (i)^2 \right)^{-1} \| J_{\beta T} \|^2$$

$$\| S^{(i)} \|^2 = \left( 1 - b_{\beta_i T} (i)^2 \right) \| S \|^2$$

$$\gamma_{S'} = \left( 1 - b_{\beta_i T} (i)^2 \right) \gamma_S.$$

The product in $\gamma_S$ is over pairs $(a, b)$ with $a < b$ and $\beta_a < \beta_b$. Changing $S$ to $S^{(i)}$ leaves the pairs with $(a, b) \cap \{ i, i + 1 \} = \emptyset$ alone and interchanges the pairs $(a, i), (a, i + 1)$ and $(i, b), (i + 1, b)$ respectively. The pair $(i, i + 1)$ is added to the product since $\beta \{ S^{(i)} \} = \beta \{ S^{(i)} \} = \beta$, and thus

$$\gamma_{S'} = \prod_{1 \leq a < b \leq 2mk} \left( 1 - \frac{1}{(c (a, S') - c (b, S'))^2} \right).$$

This completes the induction. \( \square \)

For $w \in S_{2nk}$ let $A (w)$ denote the matrix of the action of $\sigma (w)$ on the basis $\{ S : S \in \mathcal{Y} (\sigma) \}$, so that $\sigma (w) S = \sum_{S'} A (w)_{S', S} S'$. These matrices are generated by the $A (s_i)$ which are specified in the transformation formulas in Section 2.1.3. The polynomial $\gamma_{S \pi \{ S \}}$ is a simultaneous eigenfunction of $\{ \omega_i \}$ with the same respective eigenvalues as $S$ and it has the same length, thus it satisfies

$$w \gamma_{S \pi \{ S \}} = \sum_{S'} A (w)_{S', S} \gamma_{S'} \pi \{ S' \}, w \in S_{2nk}, S \in \mathcal{Y} (\sigma).$$

Note $w \gamma_{S \pi \{ S \}} (x) = \tau (w) \gamma_{S \pi \{ S \}} (xw)$.

**Definition 7.** The linear map $\mu : \mathcal{P}_\sigma \rightarrow \mathcal{P}_\pi$ is given by

$$\mu \left( \sum_S f_S (x) \otimes S \right) = \sum_S f_S (x) \gamma_{S \pi \{ S \}} (x),$$

where each $f_S \in \mathcal{P}$.

**Proposition 5.** The map $\mu$ commutes with multiplication by $x_i$ and with the action of $S_{2nk}$ for $1 \leq i \leq 2nk$ and $w \in S_{2nk}$.
Theorem 4. The map $\mu$ commutes with $\mathcal{D}_i$ for $1 \leq i \leq 2mk$. 

**Proof.** Let $g(x) = f(x) \otimes S$ for some $f \in \mathcal{P}$ and $S \in \mathcal{Y}(\sigma)$. Then

$$w(g(x)) = f(xw) \otimes \sigma(w)S = \sum_{s'} A(w)_{s',s} f(xw) \otimes S',$$

$$\mu(w(g(x))) = \sum_{s'} A(w)_{s',s} f(xw) \gamma_{s'} I_{\pi(s')} (x).$$

Also

$$w(\mu(g(x))) = f(xw) w_{\gamma_{s'}} I_{\pi(s')} (x) = f(xw) \sum_{s'} A(w)_{s',s} \gamma_{s'} I_{\pi(s')} = \mu(w(g(x))).$$

\[\square\]

Recall the key fact: $\mathcal{D}_i I_{\pi(s),\pi(s')} = 0$ for $1 \leq i \leq 2mk$, at $\kappa_0 = \frac{1}{n+2}$.

The Proposition and the Theorem together show that $\mu$ is a map of modules of the rational Cherednik algebra (with parameter $\kappa = \frac{1}{m+2}$). In fact the map can be reversed: define...
which are of isotype $\sigma$.

The polynomial $p_{\sigma,T}(x)$ by the $\gamma$'s is $p_{\sigma,T}(x) = \sum_{T \in \mathcal{Y}(\tau)} p_{\sigma,T}(x) \otimes T$ then it can be shown (fairly straightforwardly) that

$$q_T(x) = \sum_S p_{\sigma,T}(x) \frac{||T||^2}{||S||^2} \otimes S$$

is a singular polynomial in $\mathcal{P}_\tau$ for $\kappa = -\frac{1}{m+2}$ and is of isotype $\tau$. So one can define a map analogous to $\mu$ from $\mathcal{P}_\tau \to \mathcal{P}_\nu$. There are general results about duality and maps of modules of the rational Cherednik algebra in ([8] Sect. 4). In [9] there are theorems about the existence of maps between standard modules in the context of complex reflection groups.

8. Further Developments and Concluding Remarks

The construction of singular polynomials in $\mathcal{P}_\tau$ which are of isotype $\sigma$ is easily extendable to $\kappa = \frac{n}{m+2}$ with $n \geq 1$ and $\gcd(n,m+2) = 1$. Define $\lambda' = n\lambda$ then $I_{\lambda',T_0}$ is singular for $\kappa = \frac{n}{m+2}$.

This is valid because the uniqueness theorems can be derived from the $n = 1$ case: suppose $(\beta, T) \in \mathbb{N}_0^{m\kappa} \times \mathcal{Y}(\tau)$ such that $\beta \leq n\lambda$ and $\frac{m+2}{n} \beta_i + c \left( r_{\beta}(i), T \right) = c \left( i, S_0 \right)$, then $\frac{m+2}{n} \beta_i \in \mathbb{N}_0$ which implies $\beta_i = n\beta_i'$, for each $i$. Further $\beta' \leq \lambda$ and the uniqueness of $\beta'$ shows the same for $\beta$.

It should be possible to extend our analysis to the situation where the top brick is truncated, that is, $\sigma = (mk + \ell, mk + \ell)$ and $\tau = \left( m^{2k}, \ell \right)$ with $1 \leq \ell < m$, but we leave this for another time.

In this paper, we constructed singular nonsymmetric Jack polynomials in $\mathcal{P}_\tau$ which are of isotype $\sigma$. In general, suppose $\tau$ and $\sigma$ are partitions of $N$ and there are singular polynomials in $\mathcal{P}_\tau$ for $\kappa = \kappa_0$ which are of isotype $\sigma$, then it can be shown that there are singular polynomials in $\mathcal{P}_\nu$ for $\kappa = -\kappa_0$ which are of isotype $\tau$. This idea was sketched in Section 7. It turns out that interesting new problems may arise. In the present work, we used uniqueness theorems about spectral vectors to show the validity of specializing NSJP's to $\kappa = \kappa_0$ (some fixed rational) to obtain singular polynomials. However it is possible that some singular polynomial is a simultaneous eigenfunction of $\{U_{\ell}\}$ but is not the specialization of an NSJP.

Our example is for $N = 5$, $\tau = (3,1,1)$, $\sigma = (5)$ and $\kappa = \frac{1}{2}$. The singular polynomials for $\mathcal{P}_\nu$ (scalar polynomials) are well-known ([1]). In particular $I_{(3,2,0,0,0)}$ has no pole at $\kappa = -\frac{1}{2}$ and is singular there, furthermore it is of isotype $\tau = (3,1,1)$. From the general result there are singular polynomials in $\mathcal{P}_\tau$ of isotype $\sigma$ (that is, invariant) for $\kappa = \frac{1}{2}$. The uniqueness approach fails here. Let

$$T = \begin{bmatrix} 5 & 4 & 3 \\ 2 & 1 \\ 0 \end{bmatrix}, T' = \begin{bmatrix} 5 & 3 & 2 \\ 1 \end{bmatrix},$$

$$[c \left( \cdot, T \right)] = [-2, -1, 2, 1, 0], [c \left( \cdot, T' \right)] = [-2, 2, 1, -1, 0],$$

$$\kappa = (3,2,0,0,0), \beta = (1,1,2,1,0).$$

The spectral vectors $[2\gamma_i + c \left( r_{\gamma}(i), T \right)]_{i=1}^5$ are (note $r_{\beta} = (2,3,1,4,5)$)

$$\zeta_{\beta,T} = (6 - 2, 4 - 1, 2, 1, 0) = (4,3,2,1,0),$$

$$\zeta_{\beta,T'} = (2 + 2, 2 + 1, 4 - 2, 2 - 1, 0) = (4,3,2,1,0).$$

By direct (symbolic computation assisted) calculation we find that both $I_{\kappa,T}$ and $I_{\kappa,T'}$ are defined (no pole) at $\kappa = \frac{1}{7}$, neither is singular or of isotype (5) (invariant under each $s_1$) but

$$D_i \left( I_{\kappa,T} + 2I_{\kappa,T'} \right) = 0, 1 \leq i \leq 5, \kappa = \frac{1}{2}.$$

Also $I_{\kappa,T} + 2I_{\kappa,T'}$ is invariant and $U_{\ell} \left( I_{\kappa,T} + 2I_{\kappa,T'} \right) = (5-i) \left( I_{\kappa,T} + 2I_{\kappa,T'} \right)$ for $1 \leq i \leq 5$. The polynomial $I_{\kappa,T}$ is a sum of 100 monomials in $x$, with coefficients in $V_\tau$. 

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We suspect that our results benefitted from the fact that $\tau$ and $\sigma$ are rectangular partitions, and that the analysis of singular polynomials for other partitions (hook tableaux for example) becomes significantly more difficult.

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