POSITIVE SOLUTIONS OF QUASILINEAR ELLIPTIC EQUATIONS WITH EXPONENTIAL NONLINEARITY COMBINED WITH CONVECTION TERM

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Abstract. We establish the existence of positive solutions for a nonlinear elliptic Dirichlet problem in dimension $N$ involving the $N$-Laplacian. The nonlinearity considered depends on the gradient of the unknown function and an exponential term. In such case, variational methods cannot be applied. Our approach is based on approximation scheme, where we consider a new class of normed spaces of finite dimension. As a particular case, we extended the result achieved by De Araujo and Montenegro [2016] for any $N > 2$.

1. Introduction

Let $\Omega \subset \mathbb{R}^N$ be a smooth bounded domain and $p > 1$. Consider the following problem

\begin{equation}
\begin{cases}
-\Delta_p u = g(x, u, \nabla u) & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega.
\end{cases}
\end{equation}

Here, the operator $-\Delta_p : W^{1,p}_0(\Omega) \to W^{-1,p'}(\Omega)$, where $\frac{1}{p} + \frac{1}{p'} = 1$, is defined by

$$
\langle -\Delta_p u, v \rangle = \int_\Omega |\nabla u|^{p-2} \nabla u \nabla v dx \text{ for all } u, v \in W^{1,p}_0(\Omega),
$$

and the forcing term $g$ has the form of a convection term, that is, it depends also on the gradient of the unknown function. Due to the presence of the gradient $\nabla u$ in the term $g(x, u, \nabla u)$, problem (1.1) does not have, in general, variational structure. This kind of problems are usually studied by means of topological degree, the method of sub-supersolutions, fixed point theory, approximation techniques and iterative scheme. For instance, we would like to cite [2, 5, 8, 12–14, 25]. In particular, in [13], via an approximation on finite dimensional subspaces, the authors proved the existence of a positive solution for the following problem

\begin{equation}
\begin{cases}
-\Delta_p u - \mu \Delta_q u = g(x, u, \nabla u) & \text{in } \Omega, \\
u > 0 & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega,
\end{cases}
\end{equation}

where $\mu \geq 0$, $g : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is a continuous function satisfying the growth condition:

\begin{enumerate}
\item[(G)] $b_0 |t|^{r_0} \leq f(x, t, \xi) \leq b_1 (1 + |t|^{r_1} + |\xi|^{r_2})$ for all $(x, t, \xi) \in \Omega \times \mathbb{R} \times \mathbb{R}^N$, with constants $b_0, b_1 > 0$, $r_1, r_2 \in [0, p - 1]$, $r_0 \in [0, p - 1]$ if $\mu = 0$, and $r_0 \in [0, q - 1]$ if $\mu > 0$.
\end{enumerate}
On the other hand, elliptic problems of the type

\[
\begin{cases}
-\Delta_N v = g(x, v) & \text{in } \Omega, \\
v = 0 & \text{on } \partial\Omega,
\end{cases}
\]

where \(\Omega \subset \mathbb{R}^N\) and \(g(x, v)\) is continuous and behaves like \(\exp(\alpha |v|^{N/(N-1)})\) as \(|v| \to +\infty\) have been studied by many authors, we would like to cite \([3, 6, 9-11, 19, 26]\). One of the main ingredients is the Trudinger-Moser inequality introduced in \([22, 28]\). Namely, given \(u \in W^{1,N}_{0}(\Omega)\), then

\[
e^{\sigma|u|^\frac{N}{N-1}} \in L^1(\Omega) \text{ for every } \sigma > 0,
\]

and there exists a positive constant \(L(N)\) which depends on \(N\) only, such that

\[
\sup_{\|u\|_{W^{1,N}_{0}(\Omega)} \leq 1} \int_{\Omega} e^{\sigma|u|^\frac{N}{N-1}} \, dx \leq L(N)|\Omega| \text{ for every } \sigma \leq \alpha_N,
\]

where \(|\Omega| = \int_{\Omega} dx\), \(\alpha_N = Nw_{N-1}^\frac{1}{N-1}\) and \(w_{N-1}\) is the \((N-1)\)-dimensional measure of the \((N-1)\)-sphere.

In particular, in \([6]\) the authors proved existence of solutions for the following problem

\[
\begin{cases}
-\Delta u = \lambda u^q + f(u) & \text{in } \Omega, \\
u > 0 & \text{in } \Omega, \\
u = 0 & \text{on } \partial\Omega,
\end{cases}
\]

where \(\Omega \subset \mathbb{R}^2\), \(\lambda > 0\) is a small enough parameter, \(0 < q < 1\) and \(f : [0, \infty) \to \mathbb{R}\) is a continuous function satisfying the growth condition:

\((H)\) \(0 \leq tf(t) \leq C|t|^r \exp(\alpha t^2)\) where \(\alpha > 0\) and \(r > 2\).

In \([7]\), still considering \(N = 2\), the authors proved existence of solutions for an elliptic system with arguments based in \([6]\) and with nonlinearities satisfying the growth condition \((H)\).

In this work we are concerned with the existence of positive solutions for the problem:

\((P)\)

\[
\begin{cases}
-\Delta_N u = \lambda(a_1 u^{r_1} + a_2 |\nabla u|^{r_2}) + f(u) & \text{in } \Omega, \\
u = 0 & \text{on } \partial\Omega,
\end{cases}
\]

where \(\Omega \subset \mathbb{R}^N\) is a bounded domain with a \(C^{1,\alpha}\)-boundary \(\partial\Omega\), for some \(0 < \alpha \leq 1\), \(\lambda > 0\) is a parameter, \(0 < r_i < N - 1\), for \(i = 1, 2\), \(a_1 > 0\), \(a_2 \geq 0\), and \(f : [0, \infty) \to \mathbb{R}\) is a nonegative continuous function. The main assumption on the function \(f\) is the following, which will be referred throughout the paper as \((F)\):

\((F)\) \(0 \leq tf(t) \leq a_3 t^{r_3 + 1} \exp(\alpha t^N)\) where \(a_3, \alpha > 0\), and \(r_3 > N - 1\).

Most of the papers, to prove existence results for the problem, assume Ambrosetti-Rabinowitz conditions (or some additional conditions) to obtain Palais-Smale or Cerami compactness condition. Notice that in this paper we don’t need to impose such extra hypothesizes.

An interesting problem related to \((P)\), by considering a more general operator, was treated by \([29]\). The authors studied a \((N, q)\)-Laplacian problem with a critical Trudinger-Moser
nonlinearity as following

\[
\begin{aligned}
- \Delta_N u - \Delta_q u &= \mu |u|^{q-2} u + \lambda |u|^{N-2} u e^{u|N/(N-1)} \quad \text{in } \Omega \\
u &= 0 \quad \text{on } \partial \Omega,
\end{aligned}
\]

where \( N > q > 1, \mu \in \mathbb{R} \) and \( \lambda > 0 \). By using a critical point theorem, based on a cohomological index, they proved the existence of solution for \( \mu \) interacting with the first eigenvalue of the \(-\Delta_q u, W_0^{1,q}(\Omega)\) operator and for \( \lambda \) sufficiently large.

Here we extend the results of [6] for the general dimension case \( N > 1 \) \((a_2 = 0)\). In order to prove the existence of positive solutions for \((P)\), we borrow some ideas from [6] and [13]. Due to the presence of the supercritical term \(\exp(\alpha|v|^{N/(N-1)})\), along with the convection term, we had to overcome some problems. For example, in \(W_0^{1,N}(\Omega)\) we need to assume a Schauder basis instead of the Hilbert basis (like in [6]), which becomes some additional difficulty. By comparing with [13], due to the presence of the term \(\exp(\alpha|v|^{N/(N-1)})\), a suitable modification on the approximating approach had to be done. Although in [13] the authors used the Schauder basis, we could not obtain the necessary estimates for this approach by considering the approximate spaces used there. To do this, we consider a new class of normed spaces of finite dimension.

Our main result reads as follows:

**Theorem 1.1.** Suppose that \( f : [0, \infty) \to \mathbb{R} \) is a continuous function satisfying the assumption \((F)\). Then there exists \( \lambda^* > 0 \) such that for every \( \lambda \in (0, \lambda^*) \) problem \((P)\) admits a (positive) weak solution \( u \in W_0^{1,N}(\Omega) \).

2. Preliminary results

The Sobolev space \(W_0^{1,N}(\Omega)\) is endowed with the norm

\[
\|u\|_{W_0^{1,N}(\Omega)} = \left( \int_\Omega |\nabla u|^N \, dx \right)^{1/N}.
\]

To prove Theorem 1.1 we approximate \( f \) by Lipschitz functions \( f_k : \mathbb{R} \to \mathbb{R} \) defined by

\[
f_k(s) = \begin{cases} 
-k[G(-k - \frac{s}{k}) - G(-k)], & \text{if } s \leq -k, \\
-k[G(s - \frac{s}{k}) - G(s)], & \text{if } -k \leq s \leq -\frac{1}{k}, \\
k^2s[G(-\frac{s}{k}) - G(-\frac{1}{k})], & \text{if } -\frac{1}{k} \leq s \leq 0, \\
k^2s[G(\frac{s}{k}) - G(\frac{1}{k})], & \text{if } 0 \leq s \leq \frac{1}{k}, \\
k[G(s + \frac{1}{k}) - G(s)], & \text{if } \frac{1}{k} \leq s \leq k, \\
k[G(k + \frac{1}{k}) - G(k)], & \text{if } s \geq k,
\end{cases}
\]

where \( G(s) = \int_0^s f(\xi) \, d\xi \).

The following (approximation) result was proved in [27] and uses the explicit expression of the sequence \((2.1)\).

**Lemma 2.1.** Let \( f : \mathbb{R} \to \mathbb{R} \) be a continuous function such that \( sf(s) \geq 0 \) for every \( s \in \mathbb{R} \). Then there exists a sequence \( f_k : \mathbb{R} \to \mathbb{R} \) of continuous functions satisfying

(i) \( sf_k(s) \geq 0 \) for every \( s \in \mathbb{R} \);
(ii) \( \forall k \in \mathbb{N} \exists c_k > 0 \) such that \( |f_k(\xi) - f_k(\eta)| \leq c_k|\xi - \eta| \) for every \( \xi, \eta \in \mathbb{R} \);
(iii) \( f_k \) converges uniformly to \( f \) in bounded subsets of \( \mathbb{R} \).

The sequence \( f_k \) of the previous lemma has some additional properties.
Lemma 2.2. Let \( f : \mathbb{R} \to \mathbb{R} \) be a continuous function satisfying (F) for every \( s \in \mathbb{R} \). Then the sequence \( f_k \) of Lemma 2.1 satisfies

(i) \( \forall k \in \mathbb{N}, 0 \leq sf_k(s) \leq C_1|s|^r \exp\left(2N^{\frac{N}{N-1}} \alpha |s|^\frac{N}{N-1}\right) \) for every \( |s| \geq \frac{1}{k} \);

(ii) \( \forall k \in \mathbb{N}, 0 \leq sf_k(s) \leq C_2|s|^2 \exp\left(2N^{\frac{N}{N-1}} \alpha |s|^\frac{N}{N-1}\right) \) for every \( |s| \leq \frac{1}{k} \),

where \( C_1 \) and \( C_2 \) are positive constants independent of \( k \).

Proof. Everywhere in this proof the constant \( a_3 \) is the one of (2.1).

**First step.** Suppose that \( -k \leq s \leq -\frac{1}{k} \).

By the mean value theorem, there exists \( \eta \in \left(s - \frac{1}{k}, s\right) \) such that

\[
f_k(s) = -k[G(s - \frac{1}{k}) - G(s)] = -kG'(\eta)(s - \frac{1}{k} - s) = f(\eta)
\]

and

\[
sf_k(s) = sf(\eta).
\]

Since \( s - \frac{1}{k} < \eta < s < 0 \) and \( f(\eta) < 0 \), we have \( sf(\eta) \leq \eta f(\eta) \). Therefore,

\[
sf_k(s) \leq sf(\eta) \leq a_3|\eta|^r \exp(\alpha |\eta|^\frac{N}{N-1})
\]

\[
\leq a_3|s - \frac{1}{k}|^r \exp(\alpha |s|\frac{1}{k})^\frac{N}{N-1}
\]

\[
\leq a_3(|s| + \frac{1}{k})^r \exp(\alpha (|s| + \frac{1}{k})^\frac{N}{N-1})
\]

\[
\leq a_3(2|s|)^r \exp(\alpha (2|s|)^\frac{N}{N-1})
\]

\[
= a_32^r|s|^r \exp(2\frac{N}{N-1} \alpha |s|^\frac{N}{N-1}).
\]

**Second step.** Assume \( \frac{1}{k} \leq s \leq k \).

By the mean value theorem, there exists \( \eta \in (s, s + \frac{1}{k}) \) such that

\[
f_k(s) = k[G(s + \frac{1}{k}) - G(s)] = kG'(\eta)(s + \frac{1}{k} - s) = f(\eta)
\]

and

\[
sf_k(s) = sf(\eta).
\]

Since \( 0 < s < \eta < s + \frac{1}{k} \) and \( f(\eta) > 0 \), we have \( sf(\eta) \leq \eta f(\eta) \). Therefore,

\[
sf_k(s) \leq sf(\eta) \leq a_3|\eta|^r \exp(\alpha |\eta|^\frac{N}{N-1})
\]

\[
\leq a_3|s + \frac{1}{k}|^r \exp(\alpha |s + \frac{1}{k}|^\frac{N}{N-1})
\]

\[
\leq a_3(2|s|)^r \exp(\alpha (2|s|)^\frac{N}{N-1})
\]

\[
= a_32^r|s|^r \exp(2\frac{N}{N-1} \alpha |s|^\frac{N}{N-1}).
\]

**Third step.** Suppose that \( |s| \geq k \), then

\[
f_k(s) = \begin{cases} 
-k[G(-k - \frac{1}{k}) - G(-k)], & \text{if } s \leq -k \\
 k[G(k + \frac{1}{k}) - G(k)], & \text{if } s \geq k.
\end{cases}
\]

If \( s \leq -k \), by the mean value theorem, there exists \( \eta \in (-k - \frac{1}{k}, -k) \) such that

\[
f_k(s) = k[G(-k - \frac{1}{k}) - G(-k)] = -kG'(\eta)(-k - \frac{1}{k} - (-k)) = f(\eta)
\]

and

\[
sf_k(s) = sf(\eta).
\]
Since \(-k - \frac{1}{k} < \eta < -k < 0\) and \(k < |\eta| < k + \frac{1}{k}\), we conclude that
\[
s_f(s) = \frac{s}{|\eta|}\eta f(\eta) \leq \frac{|s|}{|\eta|}a_3|\eta|^{r_3} \exp(\alpha |\eta|^{\frac{N}{N-2}}) = a_3|s||\eta|^{r_3-1} \exp(\alpha |\eta|^{\frac{N}{N-2}}) \\
\leq a_3|s|(k + \frac{1}{k})^{r_3-1} \exp(\alpha (k + \frac{1}{k})^{\frac{N}{N-2}}) \\
\leq a_3|s|(|s| + \frac{1}{k})^{r_3-1} \exp(\alpha (|s| + \frac{1}{k})^{\frac{N}{N-2}}) \\
\leq a_3|s|(2|s|)^{r_3-1} \exp(\alpha (2|s|)^{\frac{N}{N-2}}) \\
\leq a_32^{r_3-1}|s|^{r_3} \exp(2\frac{N}{N-2} \alpha |s|^{\frac{N}{N-2}}).
\] (2.3)

If \(s \geq k\), by the mean value theorem, there exists \(\eta \in (k, k + \frac{1}{k})\) such that
\[
f_k(s) = \frac{s}{k}G(k + \frac{1}{k}) = kG(\eta)(k + \frac{1}{k} - k) = f(\eta).
\]

By computations similar to conclude (2.3) one has
\[
s_f(s) = s \frac{\partial f}{\partial \eta} f(\eta) \leq \frac{|s|}{|\eta|}a_3|\eta|^{r_3} \exp(\alpha |\eta|^{\frac{N}{N-2}}) \leq a_32^{r_3-1}|s|^{r_3} \exp(2\frac{N}{N-2} \alpha |s|^{\frac{N}{N-2}}).
\]

**Fourth step.** Assume \(-\frac{1}{k} \leq s \leq \frac{1}{k}\), then
\[
f_k(s) = \left\{ \begin{array}{ll} k^2 s[G(-\frac{2}{k}) - G(-\frac{1}{k})], & \text{if } -\frac{1}{k} \leq s \leq 0 \\
 k^2 s[G(\frac{2}{k}) - G(\frac{1}{k})], & \text{if } 0 \leq s \geq \frac{1}{k}. \end{array} \right.
\] (2.4)

If \(-\frac{1}{k} \leq s \leq 0\), by the mean value theorem, there exists \(\eta \in (-\frac{2}{k}, -\frac{1}{k})\) such that
\[
f_k(s) = k^2 s[G(-\frac{2}{k}) - G(-\frac{1}{k})] = k^2 sG'(\eta)(-\frac{2}{k} - (-\frac{1}{k})) = -ks f(\eta).
\]

Therefore
\[
s_f(s) = -ks^2 f(\eta) = -k \frac{s^2}{|\eta|} \eta f(\eta) \leq k \frac{s^2}{|\eta|} \eta f(\eta) \leq a_32^{r_3-1}|s|^{r_3-1} \exp(2\frac{N}{N-2} \alpha |s|^{\frac{N}{N-2}}) \leq a_32^{r_3-1}|s|^{r_3} \exp(2\frac{N}{N-2} \alpha |s|^{\frac{N}{N-2}}).
\] (2.5)

If \(0 \leq s \leq \frac{1}{k}\), by the mean value theorem, there exists \(\eta \in (\frac{1}{k}, \frac{2}{k})\) such that
\[
f_k(s) = k^2 s[G(\frac{2}{k}) - G(\frac{1}{k})] = k^2 sG'(\eta)(\frac{2}{k} - \frac{1}{k}) = ks f(\eta).
\]

By similar computations to conclude (2.5) one obtains
\[
s_f(s) = ks^2 f(\eta) = k \frac{s^2}{|\eta|} \eta f(\eta) \leq a_32^{r_3-1} \exp(2\frac{N}{N-2} \alpha |s|^{\frac{N}{N-2}}) \leq a_32^{r_3-1} \exp(2\frac{N}{N-2} \alpha |s|^{\frac{N}{N-2}}).
\]

The proof of the lemma follows by taking \(C_1 = a_32^{r_3}\) ad \(C_2 = a_32^{r_3-1}C2^{r_3-1}\) exp(2\(\frac{N}{N-2}\)\(\alpha\)) where \(a_3\) is given in (F).  

Before concluding this section, we will enunciate a comparison principle due to Faria, Miyagaki and Motreanu [13, Theorem 2.2].

Consider the Dirichlet problem
\[
\left\{ \begin{array}{ll} -\Delta_p u - \mu \Delta_q u = g(u) & \text{in } \Omega \\
u = 0 & \text{on } \partial \Omega, \end{array} \right.
\] (2.6)
where \( 1 < q < p < +\infty, \mu \geq 0 \) and \( g : \mathbb{R} \to \mathbb{R} \) is a continuous function.

We recall that \( u_1 \in W^{1,p}(\Omega) \) is a subsolution of problem (2.6) if \( u_1 \geq 0 \) a.e. on \( \partial \Omega \) and

\[
\int_{\Omega} (|\nabla u_1|^{p-2}\nabla u_1 \nabla \varphi + \mu |\nabla u_1|^{q-2}\nabla u_1 \nabla \varphi) dx \leq \int_{\Omega} g(u_1)\varphi dx
\]

for all \( \varphi \in W^{1,p}_0(\Omega) \) with \( \varphi \geq 0 \) a.e. in \( \Omega \), while \( u_2 \in W^{1,p}(\Omega) \) is a supersolution of (2.6) if the reversed inequalities are satisfied with \( u_2 \) in place of \( u_1 \) for all \( \varphi \in W^{1,p}_0(\Omega) \) with \( \varphi \geq 0 \) a.e. in \( \Omega \).

**Theorem 2.3.** Let \( g : \mathbb{R} \to \mathbb{R} \) be a continuous function such that \( t^{1-q}g(t) \) is nonincreasing for \( t > 0 \) if \( \mu > 0 \), and \( t^{1-p}g(t) \) is nonincreasing for \( t > 0 \) if \( \mu = 0 \). Assume that \( u_1 \in W^{1,p}_0(\Omega) \) and \( u_2 \in W^{1,p}(\Omega) \) are a positive subsolution and a positive supersolution of problem (2.6), respectively. If \( u_i \in L_\infty(\Omega) \cap C^{1,\alpha}(\Omega), \Delta_\mu u_i \in L_\infty(\Omega), u_i/u_j \in L_\infty(\Omega) \) for \( i, j = 1, 2 \), then \( u_2 \geq u_1 \) in \( \Omega \).

### 3. APPROXIMATION PROBLEM

For each \( n \in \mathbb{N} \), we define the auxiliary problem \((P_n)\) by

\[
\begin{aligned}
-\Delta u &= \lambda(a_1(u_+)^r + a_2|\nabla u|^2) + f_n(u) + \frac{1}{n} \quad \text{in} \quad \Omega \\
|u| &= 0 \quad \text{on} \quad \partial \Omega,
\end{aligned}
\]

where \( f_n \) are given by Lemma 2.1 and Lemma 2.2, and \( u_+ = \max\{u, 0\} \).

To prove Theorem 1.1 we first show the existence of a solution of problem \((P_n)\) by using the Galerkin method. We would like to cite [1] as the seminal paper in this type of approach.

### 3.1. Finite-Dimensional Spaces

Let \( \mathcal{B} = \{w_1, w_2, \ldots, w_n, \ldots\} \) be a Schauder basis of \( W^{1,N}_0(\Omega) \) (see [4,15]). For each positive integer \( m \), let

\[
W_m = [w_1, w_2, \ldots, w_m]
\]

be the \( m \)-dimensional subspace of \( W^{1,N}_0(\Omega) \) (generated by \( \{w_1, w_2, \ldots, w_m\} \)) with norm induced from \( W^{1,N}_0(\Omega) \). Let \( \xi = (\xi_1, \ldots, \xi_m) \in \mathbb{R}^m \), notice that

\[
|\xi|_m = \| \sum_{j=1}^{m} \xi_j w_j \|_{W^{1,N}_0(\Omega)},
\]

defines a norm in \( \mathbb{R}^m \). In fact, let \( \xi^i = (\xi^i_1, \ldots, \xi^i_m) \in \mathbb{R}^m, i = 1, 2 \), and let \( \lambda \in \mathbb{R} \).

(i) \( |\xi^1 + \xi^2|_m \leq |\xi^1|_m + |\xi^2|_m \):

\[
|\xi^1 + \xi^2|_m = \| \sum_{j=1}^{m} \xi^1_j w_j + \sum_{j=1}^{m} \xi^2_j w_j \|_{W^{1,N}_0(\Omega)}
\]
\[
\leq \| \sum_{j=1}^{m} \xi^1_j w_j \|_{W^{1,N}_0(\Omega)} + \| \sum_{j=1}^{m} \xi^2_j w_j \|_{W^{1,N}_0(\Omega)}
\]
\[
= |\xi^1|_m + |\xi^2|_m.
\]
(ii) $|\lambda \xi^1|_m = |\lambda||\xi^1|_m$:

$$|\lambda \xi^1|_m = \|\lambda \sum_{j=1}^{m} \xi^1_j w_j\|_{W_0^{1,N}(\Omega)} = |\lambda||\sum_{j=1}^{m} \xi^1_j w_j\|_{W_0^{1,N}(\Omega)} = |\lambda||\xi^1|_m.$$  

(iii) $|\xi^1|_m = 0 \Leftrightarrow \xi^1 = 0$:

$(\Rightarrow) 0 = |\xi^1|_m = \sum_{j=1}^{m} |\xi^1_j w_j|_{W_0^{1,N}(\Omega)}$ implies $\sum_{j=1}^{m} \xi^1_j w_j = 0$. By uniqueness of the representation (using a Schauder basis) of the null vector, we conclude that $\xi^1 = 0$.

$(\Leftarrow)$ It is trivial.

By using the above notation, we can identify the normed spaces $(W_0^{1,N}(\Omega), \| \cdot \|_{W_0^{1,N}(\Omega)})$ and $(\mathbb{R}^m, | \cdot |_m)$ by the isometric linear transformation

$$v = \sum_{j=1}^{m} \xi_j w_j \in V_m \mapsto \xi = (\xi_1, \ldots, \xi_m) \in \mathbb{R}^m. \tag{3.1}$$

The lemma below is a consequence of Brouwers Fixed Point Theorem and its proof can be found in Kesavan [17].

**Lemma 3.1.** Let $F : \mathbb{R}^d \to \mathbb{R}^d$ be a continuous function such that $(F(\xi), \xi) \geq 0$ for every $\xi \in \mathbb{R}^d$ with $|\xi| = r$ for some $r > 0$. Then, there exists $\xi_0$ in the closed ball $B_r(0)$ such that $F(\xi_0) = 0$.

**3.2. Existence.** The following result is concerning the existence result for the auxiliary problem $(P_n)$.

**Lemma 3.2.** There exists $\lambda^* > 0$ and $n^* \in \mathbb{N}$ such that $(P_n)$ admits a (positive) weak solution $\xi \in W_0^{1,N}(\Omega) \cap C^{1,\alpha}(\Omega)$, for some $0 < \alpha < 1$, for every $\lambda \in (0, \lambda^*)$ and $n \geq n^*$.

**Proof.** Let $\mathcal{B} = \{w_1, w_2, \ldots, w_n, \ldots\}$ be a Schauder basis of $W_0^{1,N}(\Omega)$. For each positive integer $m$, let $W_m = [w_1, w_2, \ldots, w_m]$. By using the isometric linear transformation (3.1), define the function $F : \mathbb{R}^m \to \mathbb{R}^m$ such that $F(\xi) = (F_1(\xi), F_2(\xi), \ldots, F_m(\xi))$, where

$$F_j(\xi) = \int_{\Omega} \nabla u |^{N-2} \nabla u \nabla w_j dx - \lambda \left( a_1 \int_{\Omega} (u_+)^{r_1} w_j dx + a_2 \int_{\Omega} |\nabla u|^{r_2} w_j dx \right) dx$$

$$- \int_{\Omega} f_n(u_+) w_j - \frac{1}{n} \int_{\Omega} w_j dx, \quad j = 1, \ldots, m.$$  

Therefore,

$$\langle F(\xi), \xi \rangle = \int_{\Omega} |\nabla u|^N dx - \lambda \left( a_1 \int_{\Omega} (u_+)^{r_1} u dx + a_2 \int_{\Omega} |\nabla u|^{r_2} u dx \right) - \int_{\Omega} f_n(u_+) u dx - \frac{1}{n} \int_{\Omega} u dx. \tag{3.2}$$

Given $u \in W_m$, we define

$$\Omega^+_n = \{ x \in \Omega : |u(x)| \geq \frac{1}{n} \}$$

and

$$\Omega^-_n = \{ x \in \Omega : |u(x)| < \frac{1}{n} \}.$$
Thus, we rewrite (3.2) as
\[ \langle F(\xi), \xi \rangle = \langle F(\xi), \xi \rangle_P + \langle F(\xi), \xi \rangle_N, \]
where
\[ \langle F(\xi), \xi \rangle_P = \int_{\Omega_n^+} |\nabla u|^N dx - \lambda \left( a_1 \int_{\Omega_n^+} (u_+)^{r_1} dx + a_2 \int_{\Omega_n^+} |\nabla u|^{r_2} dx \right) - \int_{\Omega_n^+} f_n(u_+) u_+ dx - \frac{1}{n} \int_{\Omega_n^+} u_+ dx \]
and
\[ \langle F(\xi), \xi \rangle_N = \int_{\Omega_n^+} |\nabla u|^N dx - \lambda \left( a_1 \int_{\Omega_n^+} (u_+)^{r_1} dx + a_2 \int_{\Omega_n^+} |\nabla u|^{r_2} dx \right) - \int_{\Omega_n^+} f_n(u_+) u_+ dx - \frac{1}{n} \int_{\Omega_n^+} u_+ dx. \]

**Step 1.** Since \( 0 < r_i < N - 1 \), for \( i = 1, 2 \), then
\[ \int_{\Omega_n^+} (u_+)^{r_1+1} dx \leq \int_{\Omega} (u_+)^{r_1+1} dx \leq \int_{\Omega} |u|^{r_1+1} dx = \|u\|_{L^{r_1+1}(\Omega)}^{r_1+1} \leq C_1 \|u\|_{W_0^{1,N}(\Omega)}^{r_1+1}. \]
By virtue of Lemma 2.2 (i) we get
\[ \int_{\Omega_n^+} f_n(u_+) u_+ dx \leq C_1 \int_{\Omega_n^+} |u|^{r_3+1} \exp\left(2 \frac{N}{N-1} \alpha |u|^\frac{N}{N-1}\right) dx \]
\[ \leq a_3 \left( \int_{\Omega} |u|^{N(r_3+1)} dx \right)^{\frac{1}{N(r_3+1)}} \left( \int_{\Omega} \exp\left(2 \frac{N}{N-1} \alpha |u|^\frac{N}{N-1}\right) dx \right)^{\frac{1}{N}}, \]
where \( \frac{1}{N} + \frac{1}{N(r_3+1)} = 1. \)

It follows from (3.3) and (3.4) that
\[ \langle F(\xi), \xi \rangle_P \geq \int_{\Omega_n^+} |\nabla u|^N dx - \lambda \left( a_1 \|u\|_{W_0^{1,N}(\Omega)}^{r_1+1} + a_2 \int_{\Omega_n^+} |\nabla u|^{r_2} dx \right) - C_3 \|u\|_{W_0^{1,N}(\Omega)}^{r_3+1} \left( \int_{\Omega} \exp\left(2 \frac{N}{N-1} \alpha |u|^\frac{N}{N-1}\right) dx \right)^{\frac{1}{N}} - \frac{C_4}{n} \|u\|_{W_0^{1,N}(\Omega)}, \]
where \( C_0, C_1 \) and \( C_3 \) are constants not depending \( n \) and \( m. \)

**Step 2.** Since \( 0 < r_i < N - 1 \), for \( i = 1, 2 \), then
\[ \int_{\Omega_n^+} (u_+)^{r_1+1} \leq \int_{\Omega_n^+} |u|^{r_1+1} \leq |\Omega|^\frac{1}{n(r_1+1)}. \]

By virtue of Lemma 2.2 (ii) we get
\[ \int_{\Omega_n^+} f_n(u_+) u_+ dx \leq C_2 \int_{\Omega_n^+} |u|^2 \exp\left(2 \frac{N}{N-1} \alpha |u|^\frac{N}{N-1}\right) dx \leq C_2 \exp\left(2 \frac{N}{N-1} \alpha |\Omega|^\frac{1}{n^2} \right), \]
It follows from (3.6) and (3.7) that
\[ \langle F(\xi), \xi \rangle_N \geq \int_{\Omega_n^+} |\nabla u|^N - \lambda \left( a_1 |\Omega|^\frac{1}{n(r_1+1)} + a_2 \int_{\Omega_n^+} |\nabla u|^{r_2} dx \right) - C_3 \exp\left(2 \frac{N}{N-1} \alpha |\Omega|^\frac{1}{n^2} \right) \frac{1}{n^2} - |\Omega|^\frac{1}{n^2}. \]

Since
\[ \int_{\Omega_n^+} |\nabla u|^{r_2} dx + \int_{\Omega_n^+} |\nabla u|^{r_2} dx = \int_{\Omega} |\nabla u|^{r_2} dx \]
and
\[ (3.9) \quad \int_{\Omega} |\nabla u|^2 |u| \, dx \leq \left( \int_{\Omega} |\nabla u|^N \, dx \right)^{r_2/N} \left( \int_{\Omega} |u|^{N/(N-r_2)} \, dx \right)^{(N-r_2)/N} \leq C \|u\|_{W^{1,N}_0(\Omega)}^{r_2+1}.
\]

Thus (3.5), (3.8) and (3.9) imply
\[ (3.10) \quad \langle F(\xi), \xi \rangle \geq \|u\|_{W^{1,N}_0(\Omega)}^N - \lambda \left( a_1 C_1 \|u\|_{W^{1,N}_0(\Omega)}^{r_1+1} + a_2 C_2 \|u\|_{W^{1,N}_0(\Omega)}^{r_2+1} \right)
- C_3 \|u\|_{W^{1,N}_0(\Omega)}^{r_3+1} \left( \int_{\Omega} \exp(N2^{N-1} \alpha |u|^{N-1}) \, dx \right)^{\frac{1}{N-1}}
- \lambda a_1 |\Omega| \frac{1}{n^{r_1+1}} - C_5 \exp(2^{N-1} \alpha) |\Omega| \frac{1}{n^2} - |\Omega| \frac{1}{n^2}.
\]

Assume now that \( \|u\|_{W^{1,N}_0(\Omega)} = r \) for some \( r > 0 \) to be chosen later. We have
\[ (3.11) \quad \int_{\Omega} \exp(N2^{N-1} \alpha |u|^{N-1}) \, dx = \int_{\Omega} \exp \left( N2^{N-1} \alpha r^{N-1} \left( \frac{|u|}{\|u\|_{W^{1,N}_0(\Omega)}} \right)^{N-1} \right) \, dx
\]
and in order to apply the Trudinger-Moser inequality (1.4) we must have \( N2^{N-1} \alpha r^{N-1} \leq \alpha_N \). Consequently,
\[ r \leq \frac{1}{2} \left( \frac{\alpha_N}{N^2} \right)^{\frac{N-1}{N}}.
\]

Then
\[ \sup_{\|u\|_{W^{1,N}_0(\Omega)} \leq 1} \int_{\Omega} \exp \left( N2^{N-1} \alpha r^{N-1} \left( \frac{|u|}{\|u\|_{W^{1,N}_0(\Omega)}} \right)^{N-1} \right) \, dx \leq L(N)|\Omega|.
\]

Hence,
\[ \langle F(\xi), \xi \rangle \geq r^N - \lambda (a_1 C_1 r^{r_1+1} + a_2 C_2 r^{r_2+1}) - C_3 r^{r_3+1} L^{1/N}(N) - \frac{C_4}{n} r
- \lambda a_1 |\Omega| \frac{1}{n^{r_1+1}} - C_5 \exp(2^{N-1} \alpha) |\Omega| \frac{1}{n^2} - |\Omega| \frac{1}{n^2}.
\]

We need to choose \( r \) such that
\[ r^N - C_3 L(N) \frac{1}{N} r^{r_3+1} \geq \frac{r^N}{2},
\]
in other words,
\[ r \leq \frac{1}{2(2C_3 L(N) \frac{1}{N})^{3+1-\frac{2}{N}}}.
\]

Thus, let \( r = \min \left\{ \frac{1}{2(2C_3 L(N) \frac{1}{N})^{3+1-\frac{2}{N}}}, \frac{1}{2} \left( \frac{\alpha_N}{N^2} \right)^{\frac{N-1}{N}} \right\} \), hence
\[ \langle F(\xi), \xi \rangle \geq \frac{r^N}{2} - \lambda (a_1 C_1 r^{r_1+1} + a_2 C_2 r^{r_2+1}) - \frac{C_4}{n} r - \lambda a_1 |\Omega| \frac{1}{n^{r_1+1}} - C_5 \exp(2^{N-1} \alpha) |\Omega| \frac{1}{n^2} - |\Omega| \frac{1}{n^2}.
\]

Now, defining \( \rho = \frac{r^N}{2} - \lambda (a_1 C_1 r^{r_1+1} + a_2 C_2 r^{r_2+1}) \), we choose \( \lambda^* > 0 \) such that \( \rho > 0 \) for \( \lambda < \lambda^* \). Since \( 0 < r_i < N - 1 \), for \( i = 1, 2 \), we can choose
\[ \lambda^* = \frac{1}{2} \frac{r^N}{2a_1 C_1 r^{r_1+1} + 2a_2 C_2 r^{r_2+1}}.
\]
Now we choose \( n^* \in \mathbb{N} \) such that
\[
\frac{C_4}{n} r + \lambda a_1 |\Omega| \frac{1}{n^{r+1}} + C_5 \exp\left(2^{N-1} a|\Omega| \frac{1}{n^2} + |\Omega| \frac{1}{n^2} \right) < \frac{\rho}{2},
\]
for every \( n \geq n^* \). Let \( \xi \in \mathbb{R}^m \), such that \(|\xi| := \|\sum_{i=1}^{m} \xi_i w_i\|_{W_0^{1,N}(\Omega)} = r\), then for \( \lambda < \lambda^* \) and \( n \geq n^* \) we obtain
\[
\langle F(\xi), \xi \rangle \geq \frac{\rho}{2} > 0.
\]

Then by Lemma 3.1, for every \( m \in \mathbb{N} \) there exists \( y \in \mathbb{R}^m \) (with \(|y| \leq r\)) such that \( F(y) = 0 \). Therefore, there exists \( u_m \in W_m \) verifying
\[
\|u_m\|_{W_0^{1,N}(\Omega)} \leq r, \text{ for every } m \in \mathbb{N},
\]
and such that
\[
\int_{\Omega} |\nabla u_m|^{N-2} \nabla u_m \nabla w dx = \lambda \left( a_1 \int_{\Omega} (u_m^+)^{r_1} w dx + a_2 \int_{\Omega} |\nabla u_m|^{r_2} w dx \right)
+ \int_{\Omega} f_n(u_m+) w dx + \frac{1}{n} \int_{\Omega} w dx, \forall w \in W_m.
\]
Since \( W_m \subset W_0^{1,N}(\Omega) \) \( \forall m \in \mathbb{N} \) and \( r \) does not depend on \( m \), then \( (u_m) \) is a bounded sequence in \( W_0^{1,N}(\Omega) \). Then, for some subsequence, there exists \( u_n \in W_0^{1,N}(\Omega) \) (to simplify the notation, until the end of this section we will omit the subscript \( n \) of the variable \( u \)) such that
\[
u_m \to u \text{ weakly in } W_0^{1,N}(\Omega)
\]
and
\[
u_m \to u \text{ in } L^N(\Omega) \text{ and a.e. in } \Omega.
\]

Notice that
\[
\|u\|_{W_0^{1,N}(\Omega)} \leq \liminf_{m \to \infty} \|u_m\|_{W_0^{1,N}(\Omega)} \leq r, \forall n \in \mathbb{N},
\]
and \( r \) does not depend on \( n \). We claim that
\[
u_m \to u \text{ in } W_0^{1,N}(\Omega).
\]
Using the fact that \( B = \{w_1, w_2, \ldots, w_n, \ldots\} \) is a Schauder basis of \( W_0^{1,N}(\Omega) \), for every \( u \in W_0^{1,N}(\Omega) \) there exists a unique sequence \( (\alpha_n)_{n \geq 1} \) in \( \mathbb{R} \) such that \( u = \sum_{j=1}^{\infty} \alpha_j w_j \), so
\[
\psi_m := \sum_{j=1}^{m} \alpha_j w_j \to u \text{ in } W_0^{1,N}(\Omega) \text{ as } m \to \infty.
\]
Using as test function \( w = (u_m - \psi_m) \in W_m \) in (3.14), we get
\[
\int_{\Omega} |\nabla u_m|^{N-2} \nabla u_m \nabla (u_m - \psi_m) dx = \lambda \left( a_1 \int_{\Omega} (u_m^+)^{r_1} (u_m - \psi_m) dx + a_2 \int_{\Omega} |\nabla u_m|^{r_2} (u_m - \psi_m) dx \right)
+ \int_{\Omega} f_n(u_m+) (u_m - \psi_m) dx + \frac{1}{n} \int_{\Omega} (u_m - \psi_m) dx.
\]

By continuity of \( f_n \), (3.15), (3.16), (3.19) and hypothesis \( (F) \), we get
(3.21) \[ \lim_{m \to \infty} \frac{1}{n} \int_{\Omega} (u_m - \psi_m) \, dx = 0, \]

(3.22) \[ \lim_{m \to \infty} a_1 \int_{\Omega} (u_m^+) \tau_1 (u_m - \psi_m) \, dx = 0, \]

(3.23) \[ \lim_{m \to \infty} a_2 \int_{\Omega} |\nabla u_m|^2 (u_m - \psi_m) \, dx = 0, \]

and

(3.24) \[ \lim_{m \to \infty} \int_{\Omega} f_n ((u_m^+)) (u_m - \psi_m) \, dx = 0. \]

Notice that (3.21), (3.22) and (3.23) are immediately. Let us verify (3.24). By continuity of \( f_n \) and (3.16) we obtain

\[ f_n ((u_m^+))^N' \to f_n (u^+)^N' \text{ a.e. in } \Omega \]

and by Lemma 2.1 and (3.13), we obtain

\[ \int_{\Omega} f_n ((u_m^+))^N' \, dx \leq c_n^N' \int_{\Omega} |u_m|^N' = \|u_m\|^N'_{L^N'(\Omega)} \leq c_n^N' \|u_m\|^N'_{W^{1,N}_{0}} \leq c_n^N' C r^{N'}. \]

Hence, [16, Theorem 13.44] leads to

(3.25) \[ f_n ((u_m^+)) \to f_n (u^+) \text{ weakly in } L^{N'}(\Omega). \]

Applying (3.16), (3.19) and (3.25), we conclude that (3.24) holds.

By (3.13) and (3.15), we obtain

(3.26) \[ \lim_{m \to \infty} \int_{\Omega} |\nabla u_m|^{N-2} \nabla u_m \nabla (u - \psi_m) \, dx = 0. \]

By (3.21) – (3.24) and (3.26), we obtain

(3.27) \[ \lim_{m \to \infty} \int_{\Omega} |\nabla u_m|^{N-2} \nabla u_m \nabla (u_m - u) \, dx = 0. \]

Now it is sufficient to apply the \((S_{+})-\) property of \(-\Delta_p\) (see, e.g., [23, Proposition 3.5.]) for obtaining (3.18).

Let \( k \in \mathbb{N} \), then for every \( m \geq k \) we obtain

\[ \int_{\Omega} |\nabla u_m|^{N-2} \nabla u_m \nabla w_k \, dx = \lambda \left( a_1 \int_{\Omega} u_m^+ \tau_1 w_k \, dx + a_2 \int_{\Omega} |\nabla u_m|^2 w_k \, dx \right) + \int_{\Omega} f_n ((u_m^+)) w_k \, dx \]

\[ + \frac{1}{n} \int_{\Omega} w_k \, dx, \quad \forall w_k \in W_k. \]

Letting \( m \to \infty \), on account of (3.18) we arrive at

\[ \int_{\Omega} |\nabla u|^{N-2} \nabla u \nabla w_k \, dx = \lambda \left( a_1 \int_{\Omega} u^+ \tau_1 w_k \, dx + a_2 \int_{\Omega} |\nabla u|^2 w_k \, dx \right) + \int_{\Omega} f_n (u^+) w_k \, dx \]

\[ + \frac{1}{n} \int_{\Omega} w_k \, dx, \quad \forall w_k \in W_k. \]
Since $[W_k]_{k \in \mathbb{N}}$ is dense in $W_0^{1,N}(\Omega)$ we conclude that
\[
\int_\Omega |\nabla u|^{N-2} \nabla u \nabla w dx = \lambda \left( a_1 \int_\Omega (u^+) r_1 dx + a_2 \int_\Omega |\nabla u|^{r_2} dx \right) + \int_\Omega f_n(u^+) dx + \frac{1}{n} \int_\Omega w dx, \quad \forall w \in W_0^{1,N}(\Omega).
\]
Furthermore, $u \geq 0$ in $\Omega$. In fact, since $u_- \in W_0^{1,N}(\Omega)$ then from (3.2) we obtain
\[
\int_\Omega |\nabla u|^{N-2} \nabla u \nabla u_- dx = \lambda \left( a_1 \int_\Omega (u^+) r_1 u_- dx + a_2 \int_\Omega |\nabla u|^{r_2} u_- dx \right) + \int_\Omega f_n(u^+) u_- dx + \frac{1}{n} \int_\Omega u_- dx.
\]
Hence
\[-\|u_-\|_{W_0^{1,N}(\Omega)}^N \geq \lambda \left( a_1 \int_\Omega (u^+) r_1 u_- dx + a_2 \int_\Omega |\nabla u|^{r_2} u_- dx \right) + \int_\Omega f_n(u^+) u_- dx + \frac{1}{n} \int_\Omega u_- dx \geq 0,
\]
because $\int_\Omega f_n(u^+) u_- dx = 0$. Then $u_- \equiv 0$ a.e. in $\Omega$.

The first inequality in hypothesis (F) and the equation in $(P_n)$ guarantee that $u \neq 0$. Here the presence of $\frac{1}{n} > 0$ is needed. Next, we observe that hypothesis (F) allows us to refer to [18, Theorem 7.1] from which we infer that $u \in L^\infty(\Omega)$. Furthermore, the regularity result up to the boundary in [20, Theorem 1] and [21, p. 320] ensures that $u \in C^{1,\beta}(\Omega)$ with some $\beta \in (0,1)$. We also note that we may apply the strong maximum principle in [24, Theorem 5.4.1]. We are thus in a position to apply [24, Theorem 5.4.1] concluding that $u > 0$ in $\Omega$ because we know that $u \geq 0$, $u \neq 0$, thereby $u$ is a solution of problem $(P_n)$. This completes the proof. \hfill \Box

**Remark 3.3.** To apply [18, Theorem 7.1] and infer that $u \in L^\infty(\Omega)$, notice that it is necessary to consider the approximating functions $f_n$ (given by Lemma 2.1) instead of $f$. In fact, since
\[ F_n(v, p) = \lambda(a_1 v^{r_1} + a_2 |p|^{r_2}) + f_n(v) + \frac{1}{n} \]
satisfies the inequality (7.2) in [18], because
\[ \text{sign}(v).F_n(v, p) \leq \lambda(a_1 |v|^{r_1} + a_2 |p|^{r_2}) + c_n |v| + \frac{1}{n}. \]

While $F(v, p) = \lambda(a_1 v^{r_1} + a_2 |p|^{r_2}) + f(v)$ does not necessarily satisfy such a hypothesis, in fact
\[ \text{sign}(v).F(v, p) \leq \lambda(a_1 |v|^{r_1} + a_2 |p|^{r_2}) + a_3 |u|^{r_3} \exp(\alpha |v|^N). \]

4. **Proof of the main result**

In this section we will prove Theorem 1.1. Consider the following problem
\[
\begin{cases}
-\Delta_N v = \lambda a_1 v^{r_1} & \text{in } \Omega \\
v > 0 & \text{in } \Omega \\
v = 0 & \text{on } \partial\Omega
\end{cases}
\]
where $\lambda, a_1$ and $r_1$ were given in Theorem 1.1. This problem admits a solution $v_0 \in C_0^1(\Omega)$, see for instance [13, Lemma 4.1]. The function $v_0$ allows us to bound from below the solutions of $(P_n)$. 

For each $\lambda \in (0, \lambda^*)$ and $n \in \mathbb{N}$ sufficiently large, by using Lemma 3.2, we get that equation 
\[(P_n)\] has a weak solution $u_n \in W^{1,N}_0(\Omega) \cap C^{1,\alpha}(\Omega)$ for some $\alpha \in (0, 1)$.

In view of (3.17), we can argue as for (3.18), to find a subsequence $n \to \infty$ such that the corresponding sequence $\{u_n\}$ is strongly convergent:
\[
\int_\Omega |\nabla u_n|^{N-2}\nabla u_n \nabla w \geq \lambda a_1 \int_\Omega (u_n)^{r_1} w, \forall w \in W^{1,N}_0(\Omega) \text{ with } w \geq 0.
\]

Since
\[
u_n \to u \text{ a.e. in } \Omega,
\]
we have
\[
f_n(u_n(x)) \to f(u(x)) \text{ a.e. in } \Omega,
\]
by the uniform convergence of Lemma 2.1 (iii).

By Lemma 2.2
\[
\int_\Omega f_n(u_n)^{N'} dx = \int_{\Omega_n^+} f_n(u_n)^{N'} dx + \int_{\Omega_n^-} f_n(u_n)^{N'} dx,
\]
and
\[
\int_{\Omega_n^+} f_n(u_n)^{N'} dx \leq C_1^{N/2} \int_{\Omega_n^+} |u_n|^{(r_3-1) N/2} \exp(\frac{N}{N-1} 2^{N/2} \alpha |u_n|^{N/2}) dx
\]
\[
\leq C_1^{N/2} \left( \int_{\Omega_n^+} |u_n|^{(r_3-1) N/2} dx \right)^{N/2} \left( \int_{\Omega} \exp(2^{N/2} \alpha |u_n|^{N/2}) dx \right)^{1/2}
\]
\[
= C_1^{N/2} \|u_n\|_{W^{1,N}_0(\Omega)}^{(r_3-1) N/2} \left( \int_{\Omega} \exp(2^{N/2} \alpha |u_n|^{N/2}) dx \right)^{1/2}
\]
\[
\leq C\|u_n\|_{W^{1,N}_0(\Omega)}^{(r_3-1) N/2} \left( \int_{\Omega} \exp(2^{N/2} \alpha |u_n|^{N/2}) dx \right)^{1/2}
\]
and
\[
\int_{\Omega_n^-} f_n(u_n)^{N'} dx \leq C_2^{N/2} \int_{\Omega_n^-} |u_n|^N \exp(\frac{N}{N-1} 2^{N/2} \alpha |u_n|^N) dx
\]
\[
\leq C_2^{N/2} \exp(\frac{\frac{N}{N-1} 2^{N/2} \alpha |\Omega|}{n^{N-1}}) \frac{1}{n^{N-1}}.
\]

Since $\|u_n\|_{W^{1,N}_0(\Omega)} \leq r$, by the estimates before, we obtain
\[
\int_\Omega f_n(u_n)^{N'} dx \leq C,
\]
for each $n$. Since $f_n(u_n(x)) \to f(u(x))$ a.e. in $\Omega$, [16, Theorem 13.44] leads to
\[
f_n(u_n) \to f(u) \text{ weakly in } L^{N'}(\Omega).
\]
Recall from (3.2) that, for all \( w \in W_0^{1,N}(\Omega) \),
\[
(4.7) \quad \int_{\Omega} |\nabla u_n|^{N-2} \nabla u_n \nabla w = \lambda \left( a_1 \int_{\Omega} (u_n)^{r_1} w + a_2 \int_{\Omega} |\nabla u_n|^{r_2} w \right) + \int_{\Omega} f_n(u_n) w + \frac{1}{n} \int_{\Omega} w.
\]
Taking \( w = u_n - u \) in (4.7), we obtain
\[
(4.8) \quad \lim_{m \to \infty} \int_{\Omega} |\nabla u_n|^{N-2} \nabla u_n \nabla (u_n - u) dx = 0.
\]
Now it is sufficient to apply the \((S_+)-\) property of \(-\Delta_p\) for obtaining (4.2).

Then from (4.2) and (4.6) and the fact that \( u_n \) solves \((P_n)\), by passing to limit when \( n \to +\infty \) we get that
\[
(4.9) \quad \int_{\Omega} |\nabla u|^{N-2} \nabla u \nabla w = \lambda \left( a_1 \int_{\Omega} (u)^{r_1} w + a_2 \int_{\Omega} |\nabla u|^{r_2} w \right) + \int_{\Omega} f(u) w, \quad \forall w \in W_0^{1,N}(\Omega).
\]

Now, we are going to check that \( u > 0 \) in \( \Omega \). Notice that, by (4.1) and (4.4), for each \( n \) sufficiently large \( u_n \) is a supersolution and \( v_0 \) is a subsolution of Problem (4.1). In order to apply Theorem 2.3, we need to check that \( \frac{u_n}{v_0}, \frac{v_0}{u_n} \in L^\infty(\Omega) \). This follows by using Hopf boundary point lemma (in the strong maximum principle for both Dirichlet problems (4.1) and \((P_n)\) with corresponding solutions \( v_0 \) and \( u_n \), regularity up to the boundary and L’Hôpital theorem (see [13] for details). Therefore, \( u_n(x) \geq v_0(x) > 0 \) for all \( x \in \Omega \). Thus, by passing to the limit, we conclude that \( u \) is a positive solution of problem (P) and the proof of the theorem is thus complete. \( \square \)

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