A NOTE ON THE LOCAL WEYL FORMULA ON COMPACT LIE GROUPS

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ABSTRACT. In this note we investigate a local Weyl type formula for an arbitrary pseudo-differential operator on a compact Lie group in terms of its matrix-valued symbol. We link our formula to the principal symbol of the operator, to the representation theory of the group, and to the problem of the quantum unique ergodicity.

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1. Introduction

As it was pointed out in [17] by Robert Strichartz, what is “spectral asymptotics” if not the study of the asymptotic behavior as $\lambda \to \infty$ of the eigenvalue counting function

$$N(\lambda) = | \{ j : |\lambda_j| \leq \lambda \} |$$

of some operator. So, without the usual assumptions of ellipticity, given a pseudo-differential operator $A$ of order zero on a compact Lie group $G$, in this note we address the problem of computing its local Weyl formula in terms of its matrix-valued symbol (which in this setting is equivalent to computing it in terms of the matrix-valued symbol of $A$).

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representation theory of these groups). Similar formulae are obtained for operators of positive order. Formulae of this type involving the representation theory of the group are only known for the case of the torus \( G = \mathbb{T}^n \) (where the symbols are scalar-valued), see e.g. Zelditch [20] and here we deal with its extension to the non-commutative setting.

1.1. Motivations: the quantum ergodicity problem. On a compact Riemannian manifold \((M, g)\), the local Weyl law of Zelditch establishes that for a pseudo-differential operator \( A \in \Psi^0(M) \), of order 0, in terms of the normalised spectral data \( \{ \phi_j, \lambda_j^2 \} \) of the positive Laplacian \(-\Delta_g\) on \((M, g)\) (that are, its system of eigenfunctions \( \phi_j \) with corresponding eigenvalues \( \lambda_j^2 \)) the following asymptotic expansion is valid (see Zelditch [20])

\[
\sum_{\lambda_j \leq \lambda} (A\phi_j, \phi_j) = \left( \int_{T^*S(M)} a_0(x, \xi') d\mu_L(x, \xi') \right) \lambda^n + O(\lambda^{n-1}),
\]

for \( \lambda \) big enough where \( a_0 \in \Gamma(T^*M) \) is the (section of the cotangent bundle \( T^*M \) that provides the) principal symbol of the operator \( A \), and \( d\mu_L(x, \xi) \) is the Liouville measure on the spherical cotangent bundle \( T^*S(M) = \{(x, \xi) \in T^*M : \|\xi\|_g = 1\} \).

In this note, when \( M = G \) is a compact Lie group, we address the problem of computing the left-hand side of (1.1) in terms of the representation theory of the group. To do this (in our main Theorem 1.1), and by following the modern techniques of the theory of pseudo-differential operators, we will use the formalism of a matrix-valued global symbol as developed in [11]. Indeed, according to such a matrix-valued quantisation, to a pseudo-differential operator, one can associate a unique and globally defined symbol on the non-commutative phase space \( G \times \hat{G} \), with \( \hat{G} \) being the unitary dual of the group. Thus, the main problem in computing the left-hand side of (1.1) is to translate the spectral information \((A\phi_j, \phi_j)\) in terms of the algebraic one that provides the unitary dual \( \hat{G} \), that encodes the representation theory of the group.

Before presenting the contribution of this note in the form of Theorem 1.1, we present our main motivation, which arose from the understanding of the local Weyl formula and its relation with the quantum ergodicity problem on compact manifolds.

Indeed, let us explain the connection of (1.1) with the quantum ergodicity problem. We have denoted the eigenvalues of the Laplacian by \( \lambda_j^2 \). They are usually understood as different levels of energies \( E_j = \lambda_j^2 \). Their square roots are the frequencies \( \lambda_j \geq 0 \), and the most basic quantities testing the asymptotics of eigenfunctions are the matrix elements \((A\phi_j, \phi_j)\). These matrix elements measure the expected value of the observable \( A \) (when the order of this operator is zero) in the energy state \( \phi_j \). Much of the work in quantum ergodicity since the pioneering work of Schnirelman [15] is devoted to the study of the limits of \((A\phi_j, \phi_j)\). Although difficult to determine, these limits are still the most accessible aspects of eigenfunctions. Indeed, a real number \( Q \) for which there exists a subsequence of \((A\phi_j, \phi_j)\) with \( Q \) as its limit point is called a quantum limit of the system \((\phi_j)\) with respect to the observable \( A \). A priority problem in quantum mechanics (and then, in spectral geometry) is to determine whether or not, the sequence \((A\phi_j, \phi_j)\) has a unique quantum limit. Even, it is also important to answer whether there exists a sparse exceptional sequence that has a singular concentration. This is known as the quantum unique ergodicity problem.
The local Weyl formula is connected with the quantum unique ergodicity problem. Indeed, if \( N(\lambda) = \{ j : \lambda_j \leq \lambda \} \) is the eigenvalue counting function of \( \sqrt{-\Delta_g} \), then (1.1) implies that

\[
\lim_{\lambda \to \infty} \frac{1}{N(\lambda)} \sum_{\lambda_j \leq \lambda} (A\phi_j, \phi_j) = \int_{T^*\mathbb{S}(M)} a_0(x, \xi') d\mu_L(x, \xi).
\] (1.2)

So, if the observable \( A \) is a self-adjoint operator of order zero, there is a sub-sequence of matrix elements \( (A\phi_j, \phi_j) \) which converges to \( Q_L := \int_{T^*\mathbb{S}(M)} a_0(x, \xi') d\mu_L(x, \xi) \). This proves that the quantity in the right hand side of (1.2) is a quantum limit of the sequence \( (A\phi_j, \phi_j) \). In particular, a sequence of eigenfunctions \( \phi_j \) is ergodic if the limit of \( (A\phi_j, \phi_j) \) is the quantity \( \int_{T^*\mathbb{S}(M)} a_0(x, \xi') d\mu_L(x, \xi) \). Applications of quantum ergodicity to problems in nodal geometry can be found in Zelditch [19]. In this context, it is natural for analysts to ask how one can use such matrix elements to study classical problems on eigenfunctions such as growth and distributions of zeros and critical points. Numerical evidence allows one to deduce that ergodicity causes maximal oscillation in the real and complex domain of the eigenfunctions and gives rise to special distributions of zeros and (conjecturally) to critical points.

Although we do not discuss it here, a good portion of the literature on quantum chaos is dedicated to numerics and physicists’ heuristics. Some classics are Berry [2] and Heller [8, 9]. The article of A. Barnett [1] is a relatively recent discussion of the numerical results in quantum ergodicity. There is a wealth of phenomenology standing behind the rigorous results and heuristic proofs in this field. Much of it still lies far beyond the scope of the current mathematical techniques. For an excellent expository work on the subject we refer the reader to Zelditch [19].

1.2. Contributions of the note. We refer the reader to Subsection 3.1 for the notations used here. In particular, in the case of a compact Lie group \( G \), we denote by \( L_G \) the positive Laplace-Beltrami operator. Its discrete spectrum \( \{ \mu_\xi : [\xi] \in \widehat{G} \} \) can be enumerated with the unitary dual \( \widehat{G} \). Then, we use the notations \( |\xi| := \sqrt{\mu_\xi} \) and \( N(\lambda) = \{ [\xi] \in \widehat{G} : |\xi| \leq \lambda \} \) for the eigenvalues of \( \sqrt{L_G} \) as well as for its Weyl eigenvalue counting formula, respectively. The following theorem is our main result.

**Theorem 1.1.** Let \( G \) be a compact Lie group of dimension \( n \), and let \( A \in \Psi^0(G) \) be a pseudo-differential operator of order zero. In terms of the following data:

- first, with \( \sigma_{loc} \) being the (Hörmander) principal symbol of \( A \) (which is a section of the cotangent bundle \( T^*G \)),
- denoting by \( \sigma_{glob} \) the matrix-valued symbol of \( A \) defined on \( G \times \widehat{G} \),
- and with \( \mu_L \) denoting the Liouville measure on the spherical vector bundle \( T^*\mathbb{S}(G) \),

for any \( \lambda > 0 \), the partial trace of \( A \) admits the asymptotic expansion

\[
\sum_{[\xi] \in \widehat{G} : |\xi| \leq \lambda} d_\xi \int_G \text{Tr}(\sigma_{glob}(x, \xi)) dx = (2\pi)^{-n} \left( \int_{T^*\mathbb{S}(G)} \sigma_{loc}(x, \xi') d\mu_L(x, \xi') \right) \lambda^n + O(\lambda^{n-1}).
\] (1.3)
Moreover, the convergence at infinity of its average with respect to the eigenvalue counting function \( N(\lambda) := \{ [\xi] \in \hat{G} : |\xi| \leq \lambda \} \) is given by the quantum limit
\[
\lim_{\lambda \to \infty} \frac{1}{N(\lambda)} \sum_{[\xi] \in \hat{G} : |\xi| \leq \lambda} d_{\xi} \int_G \text{Tr}(\sigma_{\text{glob}}(x, \xi)) dx = C_n \int_{T^*S(G)} \sigma_{\text{loc}}(x, \xi') d\mu_L(x, \xi').
\] (1.4)

Now, we present the following application of Theorem 1.1 to quantum limits.

**Corollary 1.2.** Let \( G \) be a compact Lie group of dimension \( n \), and let \( A \in \Psi^0(G, \text{loc}) \) be a pseudo-differential operator of order zero. Assuming that \( A \) is self-adjoint (an observable) on \( L^2(G) \), there exists a sequence of unitary representations \( \xi_j \) of the group such that the sequence
\[
\{ d_{\xi_j} \int_G \text{Tr}(\sigma_{\text{glob}}(x, \xi_j)) dx \}
\]
converges to the quantum limit \( \text{QL} := \int_{T^*S(G)} \sigma_{\text{loc}}(x, \xi') d\mu_L(x, \xi') \).

The following Theorem 1.3 is an extension of the local Weyl formula for pseudo-differential operators of positive order. There we relax the ellipticity condition on the operator that one usually requires in the standard Weyl type formulae (see e.g. the discussion in Strichartz [17]) by imposing the positivity of its matrix-valued symbol.

**Theorem 1.3.** Let \( A \in \Psi^m(G) \) be a pseudo-differential operator of order \( m > 0 \) and assume that on every representation space the trace of its matrix-valued symbol is positive, i.e.
\[
\forall (x, [\xi]) \in G \times \hat{G}, \text{Tr}(\sigma_{\text{glob}}(x, [\xi])) \geq 0.
\] (1.5)

In terms of the following data:
- first, with \( \sigma_{\text{loc}} \) being the (Hörmander) principal symbol of \( A \) (which is a section of the cotangent bundle \( T^*G \)),
- and with \( \mu_L \) denoting the Liouville measure on the spherical vector bundle \( T^*S(G) \),
for any \( \lambda > 0 \), the partial trace of \( A \) satisfies the growth estimate
\[
\sum_{[\xi] \in \hat{G} : 0 < |\xi| \leq \lambda} d_{\xi} \int_G \text{Tr}(\sigma_{\text{glob}}(x, \xi)) dx \leq C_{n,A} \lambda^{n+m} + O(\lambda^{n+m-1}),
\] (1.6)
where
\[
C_{n,A} := (2\pi)^{-n} \int_{T^*S(G)} \sigma_{\text{loc}}(x, \xi') d\mu_L(x, \xi').
\] (1.7)

**Remark 1.4.** In Theorem 1.3 the matrix-valued symbol of the operator \( A \in \Psi^m(G) \) satisfies the trace condition (1.5) if e.g. on every representation space the matrix-valued symbol is a positive definite matrix i.e. if \( \forall (x, [\xi]) \in G \times \hat{G}, \sigma_{\text{glob}}(x, [\xi]) \geq 0 \).

**Remark 1.5.** In [4], [5], [6], and [7], the second and the third author have studied the \( L^p \)-nuclearity and the membership in Schatten-von-Neumann classes of Fourier multipliers and of pseudo-differential operators on compact Lie groups. It follows from the analysis in these references that any pseudo-differential operator \( A \in \Psi^m(G) \) belongs to the Schatten-von-Neumann class \( S_r(L^2(G)) \) on \( L^2(G) \), with \( r > 0 \), provided
that $m < -n/r$. In particular, $S_1(L^2(G))$ is the ideal of the trace class operators on $L^2(G)$, and the order condition $m < -n$ implies that $A \in S_1(L^2(G))$.

In this setting, for any $A \in \Psi^m(G)$, $m < -n$, the trace formula

$$\text{Tr}(A) = \sum_{[\xi] \in \hat{G}} \frac{d_{\xi}}{G} \int \text{Tr}(\sigma_{\text{glob}}(x, \xi))dx = \lim_{\lambda \to \infty} \sum_{[\xi] \in \hat{G} : |\xi| \leq \lambda} \frac{d_{\xi}}{G} \int \text{Tr}(\sigma_{\text{glob}}(x, \xi))dx \quad (1.8)$$

holds and it agrees with the sum of the eigenvalues $\lambda_n(A)$ of the trace class operator $A$. Since the order condition $m < -n$ is sharp, the trace formula in (1.8) is not valid if $A \in \Psi^0(G) \setminus \Psi^{-\varepsilon}(G)$, where $0 < \varepsilon \leq n$. That is the case if e.g. $A \in \Psi^0$ is an elliptic operator. However, the local Weyl formula in Theorem 1.1 gives us information about the growth of the partial sums

$$S_n(A) := \sum_{[\xi] \in \hat{G} : |\xi| \leq \lambda} \frac{d_{\xi}}{G} \int \text{Tr}(\sigma_{\text{glob}}(x, \xi))dx,$$

as well as the growth estimate (1.6) in Theorem 1.3.

On the other hand, from the proof of (1.4) (see Section 3) and from the Weyl eigenvalue counting formula for the operator $\sqrt{\mathcal{L}}_G$ one has that (c.f. [17]) $N(\lambda) \sim \text{Vol}(G)\lambda^n$, so that the constant $C_n$ in (1.4) is proportional to $(2\pi)^{-n}/\text{Vol}(G)$ where $n$ is the dimension of the group $G$.

2. Preliminaries

2.1. The Fourier analysis of a compact Lie group. Let $dx$ be the Haar measure on a compact Lie group $G$. The Hilbert space $L^2(G)$ will be endowed with the inner product

$$(f, g) = \int_G f(x)\overline{g(x)}dx.$$ 

According to the Peter-Weyl theorem the spectral decomposition of $L^2(G)$ can be done in terms of the entries of unitary representations on a compact Lie group $G$. To present such a theorem we will give some preliminaries.

Definition 2.1 (Unitary representation of a compact Lie group). A continuous and unitary representation of $G$ on $\mathbb{C}^\ell$ is any continuous mapping $\xi \in \text{Hom}(G, U(\ell))$, where $U(\ell)$ is the Lie group of unitary matrices of order $\ell \times \ell$. The integer number $\ell = \text{dim}\xi$ is called the dimension of the representation $\xi$ since it is the dimension of the representation space $\mathbb{C}^\ell$.

Remark 2.2 (Irreducible representations). A subspace $W \subset \mathbb{C}^{d_{\xi}}$ is called $\xi$-invariant if for any $x \in G$, $\xi(x)(W) \subset W$, where $\xi(x)(W) := \{\xi(x)v : v \in W\}$. The representation $\xi$ is unitary if its only invariant subspaces are $W = \emptyset$ and $W = \mathbb{C}^{d_{\xi}}$, the trivial ones. On the other hand, any unitary representation $\xi$ is a direct sum of unitary irreducible representations. We denote it by $\xi = \xi_1 \otimes \cdots \otimes \xi_j$, with $\xi_i$ being irreducible representations on factors $\mathbb{C}^{d_{\xi_i}}$ that decompose the representation space

$$\mathbb{C}^{d_{\xi}} = \mathbb{C}^{d_{\xi_1}} \otimes \cdots \otimes \mathbb{C}^{d_{\xi_j}}.$$

Definition 2.3 (Equivalent representations). Two unitary representations

$$\xi \in \text{Hom}(G, U(d_{\xi})) \text{ and } \eta \in \text{Hom}(G, U(d_{\eta}))$$

are equivalent if $\xi = \eta$.
are equivalent if there exists a linear mapping \( S : \mathbb{C}^{d_\xi} \to \mathbb{C}^{d_\eta} \) such that for any \( x \in G \), \( S\xi(x) = \eta(x)S \). The mapping \( S \) is called an intertwining operator between \( \xi \) and \( \eta \). The set of all the intertwining operators between \( \xi \) and \( \eta \) is denoted by \( \text{Hom}(\xi, \eta) \).

**Remark 2.4 (Schur Lemma).** In view of the 1905’s Schur lemma, if \( \xi \in \text{Hom}(G, U(d_\xi)) \) is irreducible, then \( \text{Hom}(\xi, \xi) = \mathbb{C}I_{d_\xi} \) is formed by scalar multiples of the identity matrix \( I_{d_\xi} \) of order \( d_\xi \).

**Definition 2.5 (The unitary dual).** The relation \( \sim \) on the set of unitary representations \( \text{Rep}(G) \) defined by: \( \xi \sim \eta \) if and only if \( \xi \) and \( \eta \) are equivalent representations, is an equivalence relation. The quotient \( \hat{G} := \text{Rep}(G)/\sim \) is called the unitary dual of \( G \).

The unitary dual encodes all the Fourier analysis on the group. The Fourier transform is defined as follows.

**Definition 2.6 (Group Fourier transform).** If \( \xi \in \text{Rep}(G) \), the Fourier transform \( \mathcal{F}_G \) associates to any \( f \in C^\infty(G) \) a matrix-valued function \( \mathcal{F}_G f \) defined on \( \text{Rep}(G) \) as follows

\[
(\mathcal{F}_G f)(\xi) \equiv \hat{f}(\xi) = \int_G f(x)\xi(x)^*dx, \; \xi \in \text{Rep}(G).
\]

**Remark 2.7 (The Fourier inversion formula on a compact Lie group).** The discrete Schwartz space \( \mathcal{S}(\hat{G}) := \mathcal{F}_G(C^\infty(G)) \) is the image of the Fourier transform on the class of smooth functions. This operator admits a unitary extension from \( L^2(G) \) into \( \ell^2(\hat{G}) \), with

\[
\ell^2(\hat{G}) = \left\{ \phi : \forall [\xi] \in \hat{G}, \phi(\xi) \in \mathbb{C}^{d_\xi \times d_\xi} \text{ and } \|\phi\|_{\ell^2(\hat{G})} := \left( \sum_{[\xi] \in \hat{G}} d_\xi \|\phi(\xi)\|_{\text{HS}}^2 < \infty \right)^{\frac{1}{2}} \right\}.
\]

The norm \( \|\phi(\xi)\|_{\text{HS}} \) is the standard Hilbert-Schmidt norm of matrices. The Fourier inversion formula takes the form

\[
f(x) = \sum_{[\xi] \in \hat{G}} d_\xi \text{Tr}[\xi(x)\hat{f}(\xi)], \; f \in L^1(G),
\]

where the summation is understood in the sense that from any equivalence class \([\xi]\) we choose randomly a unitary representation.

### 2.2. The quantisation formula

Let \( A : C^\infty(G) \to C^\infty(G) \) be a continuous linear operator with respect to the standard Fréchet structure on \( C^\infty(G) \). There is a way of associating to the operator \( A \) a matrix-valued function \( \sigma_A \) defined on the noncommutative phase space \( G \times \hat{G} \) to rewrite the operator \( A \) in terms of the Fourier inversion formula and in terms of the Fourier transform. Such an expression is called the quantisation formula. To introduce it we require the following definition.
2.5

Definition 2.8 (Right convolution kernel of an operator). The Schwartz kernel theorem associates to \( A \) a kernel \( K_A \in (\mathcal{C}^\infty(G), \mathcal{D}'(G)) \) such that

\[
Af(x) = \int_G K_A(x, y)f(y)dy, \quad f \in \mathcal{C}^\infty(G).
\]

The distribution defined via \( R_A(x, xy^{-1}) := K_A(x, y) \) that provides the convolution identity

\[
Af(x) = \int_G R_A(x, xy^{-1})f(y)dy, \quad f \in \mathcal{C}^\infty(G),
\]

is called the right-convolution kernel of \( A \).

Remark 2.9 (The quantisation formula). Now, we will associate a global symbol \( \sigma_A : G \times \hat{G} \to \cup_{\ell \in \mathbb{N}} \mathbb{C}^{\ell \times \ell} \) to \( A \). Indeed, in view of the identity \( Af(x) = (f * R_A(x, \cdot))(x) \), and after taking the Fourier transform with respect to \( x \in G \), we get

\[
\hat{A}f(x) = \hat{R}_A(x, \xi)\hat{f}(\xi).
\]

Then, the Fourier inversion formula gives the following representation of the operator \( A \) in terms of the Fourier transform,

\[
Af(x) = \sum_{[\xi] \in \hat{G}} d_\xi \text{Tr}[\xi(x)\hat{R}_A(x, \xi)\hat{f}(\xi)], \quad f \in \mathcal{C}^\infty(G). \tag{2.3}
\]

In view of the identity (2.3), from any equivalence class \([\xi] \in \hat{G}\), we can choose one and only one irreducible unitary representation \( \xi_0 \in [\xi] \), such that the matrix-valued function

\[
\sigma_A(x, [\xi]) \equiv \sigma_A(x, \xi_0) := \hat{R}_A(x, \xi_0), \quad (x, [\xi]) \in G \times \hat{G}, \tag{2.4}
\]

such that

\[
Af(x) = \sum_{[\xi] \in \hat{G}} d_\xi \text{Tr}[\xi_0(x)\sigma_A(x, [\xi])\hat{f}(\xi_0)], \quad f \in \mathcal{C}^\infty(G). \tag{2.5}
\]

The representation in (2.5) is independent of the choice of the representation \( \xi_0 \) from any equivalent class \([\xi] \in \hat{G}\). This is a consequence of the Fourier inversion formula.

In the following quantisation theorem we observe that the distribution \( \sigma_A \) in (2.5) is unique and can be written in terms of the operator \( A \), see Theorems 10.4.4 and 10.4.6 of [11, Pages 552-553].

Theorem 2.10. Let \( A : \mathcal{C}^\infty(G) \to \mathcal{C}^\infty(G) \) be a continuous linear operator. The following statements are equivalent.

- The distribution \( \sigma_A(x, [\xi]) : G \times \hat{G} \to \cup_{\ell \in \mathbb{N}} \mathbb{C}^{\ell \times \ell} \) satisfies the quantisation formula

\[
\forall f \in \mathcal{C}^\infty(G), \forall x \in G, \quad Af(x) = \sum_{[\xi] \in \hat{G}} d_\xi \text{Tr}[\xi(x)\sigma_A(x, [\xi])\hat{f}(\xi)]. \tag{2.6}
\]

- \( \forall (x, [\xi]) \in G \times \hat{G}, \sigma_A(x, \xi) = \hat{R}_A(x, \xi) \).

- \( \forall (x, [\xi]), \sigma_A(x, \xi) = \xi(x)^*A\xi(x), \) where \( A\xi(x) := (A_{ij}(x))_{i,j=1}^{d_\xi} \).
Remark 2.11. In view of the quantisations formulae (2.5) and (2.6), a symbol \( \sigma_A \) can be considered as a mapping defined on \( G \times \hat{G} \) or as a mapping defined on \( G \times \text{Rep}(G) \) by identifying all the values \( \sigma_A(x, \xi) = \sigma_A(x, \xi') = \sigma(x, [\xi]) \) when \( \xi', \xi \in [\xi] \).

Example 2.12 (The symbol of a Borel function of the Laplacian). Let \( X = \{X_1, \ldots, X_n\} \) be an orthonormal basis of the Lie algebra \( \mathfrak{g} \). The positive Laplacian on \( G \) is the second order differential operator

\[
\mathcal{L}_G = -\sum_{j=1}^{n} X_j^2.
\]

The operator \( \mathcal{L}_G \) is independent of the choice of the orthonormal basis \( X \) of \( \mathfrak{g} \). The \( L^2 \)-spectrum of \( \mathcal{L}_G \) is a discrete set that can be enumerated in terms of the unitary dual \( \hat{G} \)

\[
\text{Spect}(\mathcal{L}_G) = \{\mu_{[\xi]} : [\xi] \in \hat{G}\}.
\]

For a Borel function \( f : \mathbb{R}_0^+ \to \mathbb{C} \), the right-convolution kernel \( R_{f(\mathcal{L}_G)} \) of the operator \( f(\mathcal{L}_G) \) (defined by the spectral calculus) is determined by the identity

\[
f(\mathcal{L}_G)\phi(x) = \phi \ast R_{f(\mathcal{L}_G)}(x), \quad x \in G.
\]

This kernel satisfies the identity

\[
\hat{R}_{f(\mathcal{L}_G)}([\xi]) = f(\mu_{[\xi]})I_{d_\xi}.
\]

Then the matrix-valued symbol of \( f(\mathcal{L}_G) \) can be determined e.g. using Theorem 2.10 as follows

\[
\sigma_{f(\mathcal{L}_G)}(x, \xi) = \hat{R}_{f(\mathcal{L}_G)}([\xi]).
\]

Since the operator \( f(\mathcal{L}_G) \) is central the symbol \( \sigma_{f(\mathcal{L}_G)}(\xi) = \sigma_{f(\mathcal{L}_G)}(x, \xi) \) does not depend of the spatial variable \( x \in G \). Of particular interest for us will be the Japanese bracket function

\[
\langle t \rangle := (1 + t)^{\frac{1}{2}}, \quad t \geq -1.
\]

In particular the symbol of the operator \( \langle \mathcal{L}_G \rangle \) is given by

\[
\sigma_{\langle \mathcal{L}_G \rangle}([\xi]) := \langle \xi \rangle I_{d_\xi}, \quad \langle \xi \rangle := \langle \mu_\xi \rangle.
\]

2.3. Global Hörmander classes on compact Lie groups. In this section we denote for any linear mapping \( T \) on \( \mathbb{C}^l \) by \( \|T\|_{\text{op}} \) the standard operator norm

\[
\|T\|_{\text{op}} = \|T\|_{\text{End}(\mathbb{C}^l)} := \sup_{v \neq 0} \|Tv\|_{\mathbb{C}^l}/\|v\|_{\mathbb{C}^l},
\]

where \( \| \cdot \|_e \) is the Euclidean norm.

For introducing the Hörmander classes on compact Lie groups we have to measure the growth of derivatives of symbols in the group variable, for this we use vector fields \( X \in T(G) \). To derivate symbols with respect to the discrete variable \( [\xi] \in \hat{G} \) we use difference operators. Before introducing the Hörmander classes on compact Lie groups we have to define these differential/difference operators.
Definition 2.13 (Left-invariant canonical differential operators). If \( \{X_1, \ldots, X_j\} \) is an arbitrary family of left-invariant vector fields, we will denote by
\[
X^\alpha_x := X^\alpha_{1,x} \cdots X^\alpha_{n,x}
\]
an arbitrary canonical differential operator of order \( m = |\alpha| \).

Also, we have to take derivatives with respect to the “discrete” frequency variable \( \xi \in \text{Rep}(G) \). To do this, we will use the notion of difference operators introduced in [13]. Indeed, the frequency variable in the symbol \( \sigma_A(x, [\xi]) \) of a continuous and linear operator \( A \) on \( C^\infty(G) \) is discrete. This is since \( \hat{G} \) is a discrete space.

Definition 2.14 (Canonical difference operators on the dual \( \hat{G} \)). If \( \xi_1, \xi_2, \ldots, \xi_k \), are fixed irreducible and unitary representation of \( G \), which not necessarily belong to the same equivalence class, then each coefficient of the matrix
\[
\xi_\ell(g) - I_{d\xi_\ell} = [\xi_\ell(g)_{ij} - \delta_{ij}]_{i,j=1}^{d\xi_\ell}, \quad g \in G, \ 1 \leq \ell \leq k,
\]
that is each function \( q_\ell^{ij}(g) := \xi_\ell(g)_{ij} - \delta_{ij}, \ g \in G \), defines a difference operator
\[
\mathbb{D}_{\xi_\ell,i,j} := \mathcal{F}_G(\xi_\ell(g)_{ij} - \delta_{ij})\mathcal{F}^{-1}_G.
\]
We can fix \( k \geq \dim(G) \) of these representations in such a way that the corresponding family of difference operators is admissible, that is,
\[
\text{rank}\{\nabla q_\ell^{ij}(e) : 1 \leq \ell \leq k\} = \dim(G).
\]
To define higher order difference operators of this kind, let us fix a unitary irreducible representation \( \xi_\ell \). Since the representation is fixed we omit the index \( \ell \) of the representations \( \xi_\ell \) in the notation that will follow. Then, for any given multi-index \( \alpha \in \mathbb{N}^{d\xi_\ell} \), with \( |\alpha| = \sum_{i,j=1}^{d\xi_\ell} \alpha_{i,j} \), we write
\[
\mathbb{D}^\alpha := \mathbb{D}^{\alpha_{11}}_{1,1} \cdots \mathbb{D}^{\alpha_{d\xi_\ell,d\xi_\ell}}_{d\xi_\ell,d\xi_\ell}
\]
for a difference operator of order \( m = |\alpha| \).

Now, we are ready for introducing the global Hörmander classes on compact Lie groups.

Definition 2.15 (Global \((\rho, \delta)\)-Hörmander classes in the whole range \( 0 \leq \delta, \rho \leq 1 \)). We say that \( \sigma \in S^{m}_{\rho,\delta}(G \times \hat{G}) \) if the following symbol inequalities
\[
\|X^\beta_\xi \mathbb{D}^\alpha \sigma(x, \xi)\|_{\text{op}} \leq C_{\alpha,\beta} \langle \xi \rangle^{m-\rho|\gamma|+\delta|\beta|},
\]
are satisfied for all \( \beta \) and \( \gamma \) multi-indices and for all \( (x, [\xi]) \in G \times \hat{G} \), where \( \langle \xi \rangle \) denotes the Japanese bracket function at \( \lambda_\xi \) defined in 2.13.

The class \( \Psi^{m}_{\rho,\delta}(G \times \hat{G}) \equiv \text{Op}(S^{m}_{\rho,\delta}(G \times \hat{G})) \) is defined by those continuous and linear operators on \( C^\infty(G) \) such that \( \sigma_A \in S^{m}_{\rho,\delta}(G \times \hat{G}) \).

In the next theorem we describe some fundamental properties of the global Hörmander classes of pseudo-differential operators [11].
Theorem 2.16. Let $\rho, \delta \in [0, 1]$ be such that $0 \leq \delta \leq \rho \leq 1$, $\rho \neq 1$. Then

$$\Psi_{\rho,\delta}^\infty(G) := \bigcup_{m \in \mathbb{R}} \Psi_{\rho,\delta}^m(G)$$

is an algebra of operators stable under compositions and adjoints, that is:

- The mapping $A \mapsto A^* : \Psi_{\rho,\delta}^m(G \times \hat{G}) \to \Psi_{\rho,\delta}^m(G \times \hat{G})$ is a continuous linear mapping between Fréchet spaces.
- The mapping $(A_1, A_2) \mapsto A_1 \circ A_2 : \Psi_{\rho,\delta}^{m_1}(G \times \hat{G}) \times \Psi_{\rho,\delta}^{m_2}(G \times \hat{G}) \to \Psi_{\rho,\delta}^{m_1+m_2}(G \times \hat{G})$ is a continuous bilinear mapping between Fréchet spaces.

Moreover, any operator in the class $\Psi_{\rho,\delta}^0(G \times \hat{G})$ admits a bounded extension from $L^2(G)$ to $L^2(\hat{G})$.

Remark 2.17. With $0 \leq \delta < \rho < 1$ such that $\rho \geq 1 - \delta$, the condition $A \in \Psi_{\rho,\delta}^m(G \times \hat{G})$ where $m \in \mathbb{R}$, is equivalent to the fact that, when microlocalising the operator $A$ into a local coordinate system $U$, the operator $A$ takes the form

$$\forall f \in C_0^\infty(U), \forall x \in \mathbb{R}^n, \quad Af(x) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{2\pi i(x-y) \cdot \xi} a(x, \xi) f(y) dy d\xi,$$

where the function $a = a_U$, is such that for every compact subset $K \subset U$ and for all $\alpha, \beta \in \mathbb{N}_0^n$, the inequalities

$$|\partial_\alpha^\beta \partial_\xi^\alpha a(x, \xi)| \leq C_{\alpha,\beta,\delta}(1 + |\xi|)^{m - |\rho\alpha| + |\delta\beta|}, \quad (2.17)$$

hold uniformly in $(x, \xi) \in K \times \mathbb{R}^n$. This characterisation of the Hörmander classes on $G$ was proved in [12]. So, for any compact Lie group $G$, the classes $\Psi_{\rho,\delta}^m(G \times \hat{G})$ agree with the ones introduced by Hörmander [10] when $0 \leq \delta < \rho \leq 1$ and $\rho \geq 1 - \delta$.

3. Local Weyl formula on compact Lie groups

3.1. Notations. We will use the following notations taken from [11, 20].

- $(M, g)$ is a compact Riemannian manifold without boundary. We are interested when $M = G$ is a compact Lie group and $g$ is an Ad-invariant Riemannian metric. We will say in this case that $g(X, Y) := (X, Y)_g$ is $G$-invariant.
- For all $m \in \mathbb{R}$, $\Psi^m(G) := \Psi_{1,0}^m(G)$ denotes the Kohn-Nirenberg class of pseudo-differential operators of order $m$.
- $T^*G$ is the cotangent bundle of $G$, and $T^*G \setminus \{0\}$ is the cotangent bundle where the zero section is removed.
- $g_{ij} = g\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right)$, $[g^{ij}]$ is the inverse matrix to $[g_{ij}]$. Then,

$$\|\xi\|_g = \sqrt{\sum_{i,j=1}^n g^{ij}(x) \xi_i \xi_j} : T^*G \setminus \{0\} \to \mathbb{R}^+$$

denotes the length of a (co)-vector.
- The quantization of the Hamiltonian $\|\xi\|_g$ is the square root $\sqrt{-\Delta_g}$ of the positive Laplacian $-\Delta_g$ of $(G, g)$. It generates the wave operator

$$U_t = e^{it\sqrt{-\Delta}},$$
which is a group of Fourier integral operators which propagate singularities along geodesics. We denote by $|\xi| := \sqrt{\mu_\xi}$, $[\xi] \in \hat{G}$, the system of eigenvalues of $\sqrt{-\Delta_g}$.

- The unit (co-) ball bundle is denoted by $T^*\mathbb{B}(G) = \{(x, \xi) : \|\xi\|_g \leq 1\}$. Its boundary $T^*\mathbb{S}(G) = \{\|\xi\|_g = 1\}$ is the spherical vector bundle.
- $d\mu_L$ is the Liouville measure on $T^*\mathbb{S}(G)$, i.e. the surface measure $d\mu = \frac{dx d\xi}{dH}$ induced by the Hamiltonian $H = \|\xi\|_g$ and by the symplectic volume measure $dx d\xi$ on $T^*G$.
- $\hat{G}$ denotes the unitary dual of a compact Lie group, that is formed of all equivalence classes of unitary, irreducible and continuous representations $\xi : G \to \text{End}(\mathbb{C}^\ell)$, where $\ell := \dim(\xi)$ is the dimension of the representation space.
- We denote by $\sigma_{\text{glob}} : G \times \hat{G} \to \bigcup_{\ell \times \xi \in \hat{G}} \text{End}(\mathbb{C}^\ell)$ the matrix-valued global symbol of a pseudo-differential operator $A \in \Psi^0(G, \text{loc})$ in the standard Hörmander class of order zero. Then, by Theorem 2.10

$$\forall [\xi] \in \hat{G}, \forall x \in G, \sigma_{\text{glob}}(x, \xi) = \xi(x)^* A \xi(x).$$

### 3.2. Local Weyl formula for zero order operators

Now we prove our main theorem. In this subsection $A \in \Psi^0(G)$ denotes a pseudo-differential operator of order zero.

**Proof of Theorem 1.1.** Let us choose one and only one unitary and irreducible representation $\xi$ from each equivalence class $[\xi] \in \hat{G}$. Then, any $\xi : G \to \text{End}(\mathbb{C}^\ell)$ is a continuous mapping and $\ell = \dim(\xi)$ is the dimension of its representation space. Moreover, for any $x \in G$, $\xi(x)$ is a unitary linear operator. For any $x \in G$, and in terms of a basis $B_\xi := \{e^\xi_i : 1 \leq i \leq \ell = \dim(\xi)\}$ of the representation space, $\xi(x) : \mathbb{C}^\ell \to \mathbb{C}^\ell$, can be identified with a unitary matrix $\xi(x) \equiv (\xi_{ij}(x))_{i,j=1}^{d_\xi}$, where, for any $(i, j)$,

$$\xi_{ij}(x) := (e^\xi_i, \xi(x)e^\xi_j),$$

is the $(i, j)$-entry of the unitary operator $\xi(x)$ with respect to the matrix $B_\xi$, and $d_\xi = \dim(\xi)$. We have denoted by $(\cdot, \cdot)$ the standard inner product on $\mathbb{C}^\ell$.

Let us consider the system of functions

$$B := \{d^\frac{1}{2}_\xi \xi_{ij} : 1 \leq i, j \leq d_\xi, [\xi] \in \hat{G}\}$$

formed by $d^\frac{1}{2}_\xi$ times the matrix entries $\xi_{ij}$ of the unitary representations $\xi$. By the Peter-Weyl theorem, $B$ is a system of eigenfunctions of the positive Laplacian that provides an orthonormal basis of $L^2(G)$. In view of the local Weyl formula, see Zelditch [20], we have the asymptotic formula

$$\sum_{|\xi| \leq \lambda} \sum_{i,j=1}^{d_\xi} \left( A(d^\frac{1}{2}_\xi \xi_{ij}, d^\frac{1}{2}_\xi \xi_{ij}) = (2\pi)^{-n} \int_{T^*\mathbb{S}(G)} \sigma_{\text{loc}}(x, \xi') d\mu_L(x, \xi') \right) \lambda^n + O(\lambda^{n-1})$$

for any $\lambda > 0$, where $\sigma_{\text{loc}} \in \Gamma(T^*G)$ is the (Hörmander) principal symbol of $A$, and $\mu_L$ denotes the Liouville measure on the spherical vector bundle $T^*\mathbb{S}(G)$.
Now, we want to prove that the left-hand side of (3.3) agrees with the left-hand side of (1.3). So, let us use (3.1)

$$\sigma_{\text{glob}}(x, \xi) = \xi(x)^* A\xi(x), \ x \in G, \ [\xi] \in \widehat{G}. \tag{3.4}$$

In consequence,

$$A\xi(x) = \xi(x)\sigma_{\text{glob}}(x, \xi), \ x \in G, \ [\xi] \in \widehat{G}. \tag{3.5}$$

Let us re-write this algebraic information. Note that for any $1 \leq i, j \leq d$,

$$A\xi_{ij}(x) = \xi_{ij}(x) \sigma_{\text{glob}}(x, \xi), \ x \in G, \ [\xi] \in \widehat{G}. \tag{3.6}$$

The inner product can be written as

$$\langle A(d_{\xi}^\frac{1}{2}\xi_{ij}), d_{\xi}^\frac{1}{2}\xi_{ij} \rangle = d_{\xi} \int_G A\xi_{ij}(x)\overline{\xi_{ij}(x)} \, dx. \tag{3.7}$$

Plugging (3.6) into (3.7) we obtain that

$$\sum_{|\xi| \leq \lambda} \sum_{i,j=1}^{d_{\xi}} d_{\xi} \langle A\xi_{ij}, \xi_{ij} \rangle = d_{\xi} \int_G \sum_{i,j,k=1}^{d_{\xi}} \xi_{ik}(x)\sigma_{\text{glob}}(x, \xi)_{kj}(x) \overline{\xi_{ij}(x)} \, dx$$

$$= \sum_{|\xi| \leq \lambda} d_{\xi} \int_G \text{Tr}[\xi(x)\sigma_{\text{glob}}(x, \xi)\xi(x)^*] \, dx. \tag{3.8}$$

Now, taking into account that the spectral trace is invariant under unitary transformations, we have that

$$\sum_{|\xi| \leq \lambda} d_{\xi} \int_G \text{Tr}[\xi(x)\sigma_{\text{glob}}(x, \xi)\xi(x)^*] \, dx = \sum_{|\xi| \leq \lambda} d_{\xi} \int_G \text{Tr}[\sigma_{\text{glob}}(x, \xi)] \, dx. \tag{3.9}$$

The previous analysis shows the validity of (1.3). To prove the equality

$$\lim_{\lambda \to \infty} \frac{1}{N(\lambda)} \sum_{|\xi| \leq \lambda} d_{\xi} \int_G \text{Tr}[\sigma_{\text{glob}}(x, \xi)] \, dx = (2\pi)^{-n} \int_{T^*\mathbb{S}(G)} \sigma_{\text{loc}}(x, \xi^*) d\mu_L(x, \xi^*), \tag{3.10}$$

where $N(\lambda) := \{[\xi] : |\xi| \leq \lambda\}$ is the spectral function of the Bessel potential $(1 + \mathcal{L})^{\frac{1}{2}}$, just note that

$$\frac{1}{N(\lambda)} \int_G d_{\xi} \text{Tr}[\sigma_{\text{glob}}(x, \xi)] \, dx = \frac{1}{(2\pi)^n} \left( \int_{T^*\mathbb{S}(G)} \sigma_{\text{loc}}(x, \xi^*) d\mu_L(x, \xi^*) \right) \frac{\lambda^n}{N(\lambda)} + O(\lambda^{-1}). \tag{3.11}$$

Since

$$\lambda^n \times N(\lambda)^{-1} \to c_n > 0, \ \lambda \to \infty,$$

the term of the right hand side can be neglected since it goes to zero when $\lambda \to \infty$ proving (1.4). \qed
3.3. Local Weyl formula for operators of positive order. In this subsection $A \in \Psi^m(G)$ denotes a pseudo-differential operator of order $m > 0$.

**Proof of Theorem 1.3.** Let $(\cdot, \cdot)_g$ be a $G$-invariant inner product on $\mathfrak{g}$. If $X_1, \ldots, X_n$ is an orthonormal basis of $\mathfrak{g}$ with respect to the inner product $(\cdot, \cdot)_g$, then the positive Laplacian on $G$ is given by $\mathcal{L}_G = -\sum_{i=1}^n X_i^2$. Note that $(\cdot, \cdot)_g$ induces an inner product on the dual $\mathfrak{g}^*$. Below we continue with the notation in Subsection 3.1.

Since $T^*G$ is parallelizable we can make the identification $T^*G \cong G \times \mathfrak{g}^*$. Then, the (Hörmander) principal symbol of $\mathcal{L}_G$ is given by (see e.g. Wallach [18, Page 16])

$$\forall (x, \theta) \in (T^*G \setminus \{0\}) \cong (G \times \mathfrak{g}^* \setminus \{0\}), \quad \sigma_{\text{loc}, \mathcal{L}_G}(x, \theta) = (\theta, \theta)_g =: \|\theta\|_g^2. \quad (3.10)$$

Moreover, using Seeley’s functional calculus [14], we have that for any $\alpha \in \mathbb{R},$

$$\sqrt{\mathcal{L}_G^z} \in \Psi^\alpha(G)$$

is a pseudo-differential operator of order $\alpha \in \mathbb{R}$. Moreover, for any $z \in \mathbb{C}$ with $\text{Re}(z) < 0$, one has that (see Stein [16])

$$\sqrt{\mathcal{L}_G} \in \Psi^{\text{Re}(z)}(G), \quad \sqrt{\mathcal{L}_G} \left(\sqrt{\mathcal{L}_G} + \text{Proj}_{\text{Ker}(\sqrt{\mathcal{L}_G})}\right)^z - \text{Proj}_{\text{Ker}(\sqrt{\mathcal{L}_G})}. \quad (3.11)$$

Define the pseudo-differential operator

$$\tilde{A} = A \sqrt{\mathcal{L}_G}^{-m} \in \Psi^0(G). \quad (3.12)$$

In view of Theorem 1.1 we have the local Weyl formula

$$\sum_{|\xi| \in \mathcal{G}, |\xi| \leq \lambda} d\xi \int_G \text{Tr}(\tilde{\sigma}_\text{glob}(x, \xi)) dx = (2\pi)^{-n} \left( \int_{T^*S(G)} \tilde{\sigma}_\text{loc}(x, \xi') d\mu_L(x, \xi') \right) \lambda^n + O(\lambda^{n-1}), \quad (3.13)$$

where $\tilde{\sigma}_\text{glob}$ and $\tilde{\sigma}_\text{loc}$ denote the matrix-valued symbol and the (Hörmander) local principal symbol of $\tilde{A}$, respectively. Note that on the co-sphere, $\|\theta\|_g = 1$ for all $\theta \in \mathfrak{g}^* \setminus \{0\}$, and

$$\forall (x, \theta) \in (T^*G \setminus \{0\}) \cong G \times \mathfrak{g}^* \setminus \{0\}, \quad \tilde{\sigma}_\text{loc}(x, \theta) = \sigma_{\text{loc}}(x, \theta)\|\theta\|_g^{-m} = \sigma_{\text{glob}}(x, \theta). \quad (3.14)$$

On the other hand, note that the matrix-valued symbol of $\tilde{A}$ on every representation space is given by

$$\forall (x, \xi) \in G \times (\mathcal{G} \setminus \{1_G\}), \quad \tilde{\sigma}_\text{glob}(x, \xi) = \sigma_{\text{glob}}(x, \xi)|\xi|^{-m} \quad (3.15)$$

where $1_G$ denotes the irreducible trivial representation (which is one-dimensional, and therefore unique up to isomorphism). Indeed, since $|\xi| = \sqrt{\mu}_\xi$, only for the trivial representation $\xi_0 = 1_G$ we have $|\xi_0| = 0$. Then, we have

$$\sum_{|\xi| \in \mathcal{G}, 0 < |\xi| \leq \lambda} d\xi |\xi|^{-m} \int_G \text{Tr}(\sigma_{\text{glob}}(x, \xi)) dx = (2\pi)^{-n} \left( \int_{T^*S(G)} \tilde{\sigma}_\text{loc}(x, \xi') d\mu_L(x, \xi') \right) \lambda^n + O(\lambda^{n-1})$$

$$- \int_G \text{Tr} [\tilde{\sigma}_{\text{glob}}(x, 1_G)] dx$$

$$= (2\pi)^{-n} \left( \int_{\{(x, \xi') : \|\xi'\|_g = 1\}} \sigma_{\text{loc}}(x, \xi') \|\xi'\|_g^{-m} d\mu_L(x, \xi') \right) \lambda^n$$
is complete.

The analysis above proves (1.6). Indeed, note that \( f_G \text{Tr}[\tilde{\sigma}_{\text{glob}}(x, 1_G)] = 0 \). To see this note that \( 1 \in \text{Ker}(\mathcal{L}_G) \), and the functional calculus implies that for any function \( f \in \text{Ker}(\mathcal{L}_G) \),

\[ \sqrt{\mathcal{L}_G}^{-m} f \equiv 0. \]

Then, since \( 1_G(x) = 1 \) for any \( x \in G \), we have that

\[ \tilde{\sigma}_{\text{glob}}(x, 1_G) = 1_G(x)A\sqrt{\mathcal{L}_G}^{-m}1_G(x) = A\sqrt{\mathcal{L}_G}^{-m}1 = 0. \]

The proof of Theorem 1.3 is complete.

3.4. Expected value of multiplication operators in the energy states \( \phi_j \).

Now, let us analyse a simple prototype of a pseudo-differential operator. Consider \( \varphi \in C^\infty(G) \) be a smooth function. Let

\[ A_\varphi : L^2(G) \to L^2(G), \quad A_\varphi f := \varphi f, \quad (3.16) \]

be the corresponding multiplication operator by \( \varphi \). In general, to analyse the behaviour of the matrix-coefficients \( (A_\varphi \phi_j, \phi_j) = (\varphi \phi_j, \phi_j) \) for an arbitrary normalised system of eigenfunctions \( \phi_j \) of the positive Laplacian \( -\Delta_g \) is a difficult task. Let us analyse the behaviour of the expected values

\[ (A(d_\xi^2 \xi_{ij}), d_\xi^2 \xi_{ij}) = d_\xi(A\xi_{ij}, \xi_{ij}) = d_\xi(\varphi \xi_{ij}, \xi_{ij}) \quad (3.17) \]

where any \( \phi_{ij, [\xi]} := d_\xi^2 \xi_{ij} \) is an energy state that belongs to the basis \( B \) defined in (3.2). To analyse the matrix-coefficients in (3.17) we consider the convergence of its
average, that according with the proof of Theorem 1.1 satisfies the identity

\[
\frac{1}{N(\lambda)} \sum_{|\xi| \leq \lambda} d_\xi \left( A \xi_{ij}, \xi_{ij} \right) = \frac{1}{N(\lambda)} \sum_{|\xi| \leq \lambda} d_\xi \int_G \text{Tr}(\sigma_{\text{glob}}(x, \xi)) dx,
\]

(3.18)

where \( \sigma_{\text{glob}}(x, \xi) \) is the global symbol of \( A_\xi \). Now, by using the global quantisation formula, note that

\[
\sigma_{\text{glob}}(x, \xi) = \xi(x)^* \sigma(x) \xi(x) = \sigma(x) I_\xi, \quad [\xi] \in \hat{G}, \quad x \in G,
\]

where \( I_\xi \) is the identity operator in \( C^d_\xi \). Consequently,

\[
\lim_{\lambda \to \infty} \frac{1}{N(\lambda)} \sum_{|\xi| \leq \lambda} d_\xi \int_G \text{Tr}(\sigma_{\text{glob}}(x, \xi)) dx = \lim_{\lambda \to \infty} \frac{1}{N(\lambda)} \sum_{|\xi| \leq \lambda} d_\xi \int_G \sigma(x) dx = \int_G \sigma(x) dx.
\]

(3.19)

In view of Theorem 1.1 we deduce that following simplified version of (1.4)

\[
C_n \left( \int_{T^*S(G)} \sigma_{\text{op}}(x, \xi') d\mu_L(x, \xi') \right) = \int_G \sigma(x) dx.
\]

(3.20)

Let us use a numerable correspondence between \( N \) and the unitary dual \( \hat{G} = \{ [\xi] : j \in \mathbb{N} \} \). Let us define the sequence

\[
A_n := d_\xi \int_G \text{Tr}[\sigma_{\text{op}}(x, \xi_n)] dx.
\]

Let \( S_n \) be the average sums of the sequence \( \{ A_n \}_{n \in \mathbb{N}} \). Note that from Theorem 1.1 we have the identity

\[
\lim_{n \to \infty} \frac{1}{N(n)} \sum_{[\xi_n] : |\xi_n| \leq n} A_j = (2\pi)^{-n} \lim_{n \to \infty} \frac{1}{N(n)} \sum_{[\xi_n] : |\xi_n| \leq n} S_{N(n)} = \int_{T^*S(G)} \sigma_{\text{loc}}(x, \xi') d\mu_L(x, \xi').
\]

(3.21)

Then, there exists a sub-sequence \( \{ A_{n_j} \} \) such that when \( j \to \infty \),

\[
A_{n_j} \to \int_{T^*S(G)} \sigma_{\text{loc}}(x, \xi') d\mu_L(x, \xi').
\]

(3.22)

The proof of Corollary 1.2 is complete. \( \square \)
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