Spin 0 and spin 1/2 particles in a constant scalar-curvature background

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Abstract
We study the Klein-Gordon and Dirac equations in the presence of a background metric $ds^2 = -dt^2 + dx^2 + e^{-2g_2} (dy^2 + dz^2)$ in a semi-infinite lab ($x > 0$). This metric has a constant scalar curvature $R = 6g^2$ and is produced by a perfect fluid with equation of state $p = -\rho/3$. The eigenfunctions of spin-0 and spin-1/2 particles are obtained exactly, and the quantized energy eigenvalues are compared. It is shown that both of these particles must have nonzero transverse momentum in this background. We show that there is a minimum energy $E_{\text{min}}^2 = m^2c^4 + g^2c^2\hbar^2$ for bosons ($E_{\text{KG}} > E_{\text{min}}$), while the fermions have no specific ground state ($E_{\text{Dirac}} > mc^2$).

1 Introduction
Understanding the connection between the quantum mechanics and gravity has been one of the main purposes of physics, from the early birth of quantum mechanics, and many efforts have been made in this area [1-6]. These investigations have a vast spectrum, from the simplest case of the Schrodinger equation in the presence of constant gravity [1], to the most complicated case of studying the Berry phase of spin-1/2 particles moving in a space-time with torsion [7]. Although, in almost all cases, the quantum gravitational effects are weak, but they can be measured experimentally even for the weak gravity of our earth, see for example [8, 9] for the earliest recent experiments.

One of the interesting and important questions which arises in this connection is that how much the spin of the quantum particles is important in the quantum-gravity phenomena. For example the problem of equivalence principle have been studied for spin-1/2 particles in [10]. It has been shown that there is some difference between the Dirac Hamiltonian in a Schwarzschild background and in a uniformly accelerating Minkowski frame, which can be a signature of violating the equivalence principle for spin-1/2 particles. This violation does not exist for spin-0 particles.

From another point of view, the spectrum of spin-0 and spin-1/2 particles in a constant gravitational field has been studied in [11], and it is shown that they differ by an amount of $mg\hbar c$, where $m$ is the mass of particles (fermions and

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bosons) and $g$ is the acceleration of gravity. This is an important, although weak, difference which shows the influence of the spin in gravitational interaction. In this paper, we want to study this effect for a more complicated case.

The situation considered in [11] is a semi-infinite laboratory ($x > 0$) with metric $ds^2 = u^2(x)(-dt^2 + dx^2) + dy^2 + dz^2$ and an infinite barrier in $x < 0$ region. For a constant gravity in a small region, $u(x) = 1+gx$, the eigenfunctions have been obtained for the Klein-Gordon and Dirac equation in a form of power series, and the eigenvalues have been obtained in high energy and in vertically fall cases.

In this article, we are going to study the same semi-infinite lab, but with a more complicated metric. We first consider the metric $ds^2 = -dt^2 + dx^2 + u^2(x)(dy^2 + dz^2)$, in which there are coupling terms between $x - y$ and $x - z$ coordinates. To make it simple, after some steps, we take an exponential form for $u(x)$, that is $u(x) = \exp(-gx)$. We show that this metric corresponds to a perfect-fluid, with negative pressure, and has a constant scalar-curvature $R = 6g^2$. We solve the Klein-Gordon and Dirac equations in this constant curvature background (in $x > 0$ region) and find the eigenfunctions exactly. In some energy regions, the energy eigenvalues of spin-0 and spin-1/2 particles are obtained and it is shown that they do not coincide. One of the main features of spectrum is that the spin-0 particles have a ground state energy $E_{\text{min}}^2 = m^2 c^4 + g^2 c^2 \hbar^2$, while the fermions’ kinetic energy can be any positive value, determined by their momentum. This is another interesting signature which shows the importance of spin in quantum-gravity phenomena.

The plan of the paper is as follows: In section 2, the Klein-Gordon (KG) and Dirac equations are obtained in $ds^2 = -dt^2 + dx^2 + u^2(x)(dy^2 + dz^2)$ background. For simplifying the results, we take $u'(x)/u(x)$ as a constant, which restricts us to $u(x) = \exp(-gx)$. In sections 3 and 4 we calculate the eigenfunctions of KG and Dirac equations, respectively, and in section 5 the eigenvalues are compared. Finally, to understand the physics behind this metric and to justify the falling behavior of trajectories, we calculate the energy-momentum tensor and equation of state of the matter corresponds to this metric and also the geodesics of it.

## 2 Relativistic quantum equations in arbitrary $u(x)$ background

In a spacetime with metric $g_{\mu\nu}$, the Klein-Gordon equation is

$$\left(\frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^\mu} \sqrt{-g} g^{\mu\nu} \frac{\partial}{\partial x^\nu} - m^2\right) \psi_{\text{KG}} = 0,$$

where $g := \det |g_{\mu\nu}|$ (in $c = \hbar = 1$ unit). The Dirac equation in a curved spacetime is

$$[\gamma^a (\partial_a + \Gamma_a) - m] \psi_{\text{D}} = 0,$$

in which the spin connections $\Gamma_a$ can be obtained from tetrads $e^a$ through

$$de^a + \Gamma^a_{\ b} \wedge e^b = 0,$$

$$\Gamma^a_{\ b} := \Gamma^a_{\ c b} e^c,$$

$$\Gamma_a := -\frac{1}{8} [\gamma_b, \gamma_c] \Gamma^c_{\ a b}.$$
We consider a gravitational field which is represented by the metric
\[ ds^2 = -dt^2 + dx^2 + u^2(x) \left( dy^2 + dz^2 \right). \] (1)
For this metric, the Klein-Gordon equation reads
\[
\left[ -\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial x^2} + \frac{2u}{u} \frac{\partial}{\partial x} + \frac{1}{u^2} \left( \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) - m^2 \right] \psi_{\text{KG}} = 0. \] (2)
To write the Dirac equation, we need the spin connections. For metric (1), the nonvanishing \( \Gamma^a_{bc} \)'s are \( \Gamma_2^1 = -\Gamma_1^2 = (u'/u)e^2 \) and \( \Gamma_3^1 = -\Gamma_1^3 = (u'/u)e^3 \). Therefore \( \Gamma_{21}^2 = -\Gamma_{12}^2 = \Gamma_{31}^3 = -\Gamma_{13}^3 = u'/u, \) from which
\[ \Gamma_2 = -(u'/2u)\gamma_1\gamma_2 \text{ and } \Gamma_3 = -(u'/2u)\gamma_1\gamma_3. \]
Noting that \( \gamma^a \partial_a = \gamma^a e_{a \mu} \partial_\mu \), where for metric (1) we have \( e_{a \mu} = \text{diag}(1, 1/2, 1, 1/2) \), the Dirac equation becomes:
\[
\left[ \gamma^0 \frac{\partial}{\partial t} + \gamma^1 \frac{\partial}{\partial x} + \frac{u'}{u} \gamma^1 + \frac{1}{u} \left( \gamma^2 \frac{\partial}{\partial y} + \gamma^3 \frac{\partial}{\partial z} \right) - m \right] \psi_D = 0. \] (3)
For a semi-infinite lab, which there is an infinite potential barrier at \( x = 0 \), we must impose the suitable boundary condition at \( x = 0 \). In nonrelativistic case, the Schrodinger equation, the boundary condition is \( \lim_{x \to 0^-} \psi_{\text{Sch}} = 0 \), which emerges from the fact that the Schrodinger equation is second order in \( x \), so \( \psi_{\text{Sch}} \) must be continuous at \( x = 0 \). For Klein-Gordon equation, which is also second order in \( x \), the suitable boundary condition is
\[ \psi_{\text{KG}}(0) = 0. \] (4)
But the Dirac equation is of first order and therefore \( \psi_D \) can be discontinuous at \( x = 0 \), if the potential goes to infinity there. This problem has been investigated carefully in [11], and it has been shown that the suitable boundary condition for infinite potential barrier is
\[ (\gamma^1 - 1) \psi_D(0) = 0. \] (5)
So our problem is to solve eqs. (2) and (3) with boundary conditions (4) and (5), respectively. As is clear from (2) and (3), these differential equations becomes more simple if we choose \( u'/u = \text{cte} \), with solution \( u(x) = \exp(cx) \). To avoid divergency in \( x \to \infty \) region (note that \( x < 0 \) is not in our physical region, i.e. semi-infinite laboratory), we must take \( c \) negative, \( c = -g \), which restricts us to
\[ u(x) = e^{-gx}. \] (6)

3 The Klein-Gordon equation

Since \( u(x) \) in eq. (1) does not depends on \( t, y, \) and \( z \), we seek a solution for the Klein-Gordon equation (2), with \( u(x) \) defined in (6), whose functional form is
\[ \psi_{\text{KG}}(x, y, z, t) = \exp(-iEt + ip_2y + ip_3z)\psi_{\text{KG}}(x). \]
Then \( \psi_{\text{KG}}(x) \) satisfies
\[
\left[ E^2 + \frac{d^2}{dx^2} - 2g \frac{d}{dx} - (p_2^2 + p_3^2) \right] \psi_{\text{KG}}(x) = 0. \] (7)
Defining $\phi(x)$ through
\[ \psi_{\text{KG}}(x) := e^{gx} \phi(x), \] (8)
we get
\[ \frac{d^2 \phi(x)}{dx^2} + \left[ (\lambda^2 - 1) g^2 - p^2 e^{2gx} \right] \phi(x) = 0, \] (9)
in which
\[ p^2 := p_2^2 + p_3^2, \]
\[ \lambda^2 := \frac{E^2 - m^2}{g^2}. \] (10)
Changing variable from $x$ to $X = (p/g) \exp(gx)$ (which is possible only if $p \neq 0$), eq. (9) reduces to
\[ X^2 \frac{d^2 \phi(X)}{dX^2} + X \frac{d\phi(X)}{dX} - (X^2 + 1 - \lambda^2) \phi(X) = 0, \] (11)
which is the modified Bessel equation with solutions $K_{\sqrt{1-\lambda^2}}(X)$ and $I_{\sqrt{1-\lambda^2}}(X)$. But the wavefunctions must satisfy $\lim_{x \to \infty} \psi_{\text{KG}}(x) = 0$, which restricts us to consider the modified Bessel function $K_{\nu}(X)$ ($\nu = \sqrt{1-\lambda^2}$) as solution. Now the wavefunction of a spin-0 particle in a semi-infinite lab must satisfy (4). But $K_{\nu}(X)$ becomes zero only when $\nu$ is pure imaginary, [12], which means that we have solutions only when $\lambda^2 > 1$, or we have a ground state with energy:
\[ E^2 \geq E_{\text{min}}^2 = m^2 c^4 + g^2 \epsilon^2 \hbar^2. \] (12)
The wavefunctions are
\[ \psi_{\text{KG}}(x) = e^{gx} K_{i\sqrt{1-\lambda^2}}(\frac{p}{g} e^{gx}), \] (13)
and the energy eigenvalues can be obtained by the following equation
\[ K_{i\sqrt{1-\lambda^2}}(\frac{p}{g}) = 0. \] (14)
If $p = 0$, the previous change of variable is forbidden. But in this case, the differential equation (11) becomes
\[ \frac{d^2 \phi(x)}{dx^2} + (\lambda^2 - 1) g^2 \phi(x) = 0. \] (15)
For $\lambda^2 < 1$, the solutions of (15) are $\exp(\pm \alpha x)$, ($\alpha^2 = g^2(1 - \lambda^2)$), which no combinations of them can be found so that $\psi_{\text{KG}}(x \to \infty) \to 0$ and $\psi_{\text{KG}}(0) = 0$. For $\lambda^2 > 1$, the solutions are $\exp(\pm ikx)$, ($k^2 = g^2(\lambda^2 - 1)$), which again there exist no combinations to satisfy the desired boundary conditions. So the solution of the Klein-Gordon equation exists only when the transverse momentum of particle is different from zero:
\[ p \neq 0. \] (16)
4 The Dirac equation

Like the previous section, we again take $\psi_D(x, y, z, t) = \exp(-iEt + ip_2y + ip_3z)\psi(x)$ in eq. (3) with $u(x)$ defined in (10). The result is:

$$\left[ -iE\gamma^0 - g\gamma^1 + \gamma^1 \frac{d}{dx} + ie^{gx} \left( p_2\gamma^2 + p_3\gamma^3 \right) - m \right] \psi_D(x) = 0. \quad (17)$$

Defining $\tilde{\psi}_D$ through

$$\psi_D(x) = e^{gx} \tilde{\psi}_D(x), \quad (18)$$

then it satisfies

$$\left[ e^{-gx}\gamma^1 \frac{d}{dx} - iEe^{-gx}\gamma^0 + i \left( p_2\gamma^2 + p_3\gamma^3 \right) - me^{-gx} \right] \tilde{\psi}_D(x) = 0. \quad (19)$$

Considering the following two operators:

$$O_1 = e^{-gx} \left( \gamma^1 \frac{d}{dx} - iE\gamma^0 - m \right) \gamma^1 \gamma^0 = e^{-gx} \left( \gamma^0 \frac{d}{dx} - iE\gamma^1 - m\gamma^1\gamma^0 \right),$$

$$O_2 = i \left( p_2\gamma^2 + p_3\gamma^3 \right) \gamma^1 \gamma^0, \quad (20)$$

it can be easily seen that the eq. (19) is reduced to following equation for $\phi_D(x) = -\gamma^0\gamma^1 \tilde{\psi}_D(x)$:

$$(O_1 + O_2) \phi_D(x) = 0. \quad (21)$$

But it can be seen that:

$$[O_1, O_2] = 0, \quad (22)$$

so $O_1$ and $O_2$ have the same eigenspinors. Noting that $\gamma^0 = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}$ and $\gamma^k = \begin{pmatrix} 0 & \sigma^k \\ \sigma^k & 0 \end{pmatrix}$, the eigenvalues of $O_2$ are found to be $\pm ip$, with $p$ defined in (10).

Let us focus on $ip$ eigenvalue, which is two-fold degenerate with eigenspinors:

$$\psi_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \psi_2 = \begin{pmatrix} 0 \\ 0 \\ ip(p_2 + p)/p_3 \end{pmatrix}. \quad (23)$$

One can choose $\phi_D$ as $\phi_1'\psi_1 + \phi_2'\psi_2$, with arbitrary functions $\phi_1'(x)$ and $\phi_2(x)$, which is therefore the eigenspinor of $O_2$ with eigenvalue $ip$, i.e. $O_2\phi_D = ip\phi_D$.

If we write $\phi_D = \phi_1'\psi_1 + \phi_2'\psi_2$ as

$$\begin{pmatrix} \phi_1 \\ \phi_1' \\ \phi_2 \\ \phi_2' \end{pmatrix}, \quad (24)$$

then $\phi_1$ and $\phi_2'$ are related to $\phi_1'$ and $\phi_2$ as following:

$$\phi_1 = \frac{i(p_2 - p)}{p_3} \phi_1', \quad (25)$$

$$\phi_2' = \frac{i(p_2 + p)}{p_3} \phi_2. \quad (26)$$
Now $\phi_1$ and $\phi_2$ (and from them $\phi_1'$ and $\phi_2'$) are obtained by imposing $\phi_D$ satisfies (21), which results the following coupled-differential equations:

$$\frac{d\phi_1}{dx} = pe^{gx} \phi_1 - (E + m) \phi_2, \quad (27)$$
$$\frac{d\phi_2}{dx} = (E - m) \phi_1 - pe^{gx} \phi_2. \quad (28)$$

If one differentiates these two equations, they become decoupled, and if then changes variable from $x$ to $X = (2p/g)\exp(gx)$ (which is again possible if $p \neq 0$), the resulting equation for $\phi_1$ is:

$$X^2 \frac{d^2\phi_1(X)}{dX^2} + X \frac{d\phi_1(X)}{dX} + \left( -\frac{1}{2} X - \frac{1}{4} X^2 + \lambda^2 \right) \phi_1(X) = 0, \quad (29)$$

where $\lambda$ is defined in (10). Defining $\tilde{\phi}_1$ through

$$\phi_1 = X^{-1/2} \tilde{\phi}_1, \quad (30)$$

eq (29) becomes

$$\frac{d^2\tilde{\phi}_1}{dX^2} + \left[ -\frac{1}{4} - \frac{1/2}{X} + \frac{(1/4 + \lambda^2)}{X^2} \right] \tilde{\phi}_1(X) = 0. \quad (31)$$

The above equation is Whittaker differential equation with solution:

$$\tilde{\phi}_1(X) = e^{-X/2} X^{i\lambda+1/2} [c_1 M(1 + i\lambda, 1 + 2i\lambda, X) + c_2 U(1 + i\lambda, 1 + 2i\lambda, X)], \quad (32)$$

where $M(a, c, x)$ and $U(a, c, x)$ are confluent hypergeometric functions. Since the asymptotic behavior of $M(a, c, x)$ is $e^x/x^{c-a}$, so $c_1 = 0$. Therefore $\phi_1(X)$ is equal to

$$\phi_1(X) = e^{-X/2} X^{i\lambda} U(1 + i\lambda, 1 + 2i\lambda, X). \quad (33)$$

$\phi_2(X)$ can be obtained by eq. (27), with result:

$$\phi_2(X) = \frac{g}{m + E} e^{-X/2} X^{i\lambda} [(1 - \frac{i\lambda}{X}) U(1 + i\lambda, 1 + 2i\lambda, X)$$

$$+ (1 + i\lambda) U(2 + i\lambda, 2 + 2i\lambda, X)], \quad (34)$$

in which we use the equality:

$$\frac{d}{dx} U(a, c, x) = -a U(a + 1, c + 1, x).$$

The boundary condition (5) implies the following boundary condition on $\phi_1$ and $\phi_2$ :

$$\left( \phi_1 + \phi_2 \right) |_{x=0} = 0. \quad (35)$$

This relation comes from the fact that $\psi_D(0) = \tilde{\psi}_D(0) = \gamma^1 \gamma^0 \phi_D(0)$. The above equation can be used to specify the energy eigenvalues. If $p = 0$, the change of variable $x \rightarrow X = (2p/g) e^{gx}$ is not applicable, and the problem must be solved again. We first note that for $p_2 = p_3 = 0$, the operator $O_2$ is equal to zero. So eq. (21) is reduced to $O_1 \phi_D(x) = 0$, with result $\exp(\pm ip_x x)$ for components of...
spinor $\phi_D$. Therefore $\psi_D$ in eq. (18) becomes proportional to $e^{ix}$ which does not fulfill the condition $\psi_D(x \to \infty) \to 0$. So like the Klein-Gordon equation, the Dirac equation in the space-time considered here, has no solution with $p = 0$ and the transverse momentum is always different from zero:

$$p \neq 0.$$ 

5 Comparing the spectrums

The allowed energy eigenvalues of spin-0 particles are obtained by eq. (14):

$$K_i \sqrt{\lambda^2 - 1} \left( \frac{p}{g} \right) = 0,$$

(36)

while for spin-1/2 particles, it can be calculated from eq. (35):

$$\phi_1 \left( \frac{2p}{g} \right) + \phi_2 \left( \frac{2p}{g} \right) = 0,$$

(37)

where $\phi_1$ and $\phi_2$ are given by eqs. (33) and (34), respectively. Unfortunately, none of these equations can be solved analytically and only in the first case, eq. (36), the approximate solutions can be obtained for large $\lambda$ values.

Obtaining the roots of eqs. (36) and (37) numerically, for a fixed value of $p/g$ (or $p/(gh)$ in ordinary units), shows the differences of the spectrums of $\lambda_D$ and $\lambda_{KG}$ of Dirac and Klein-Gordon particles with same mass (see Table 1, as a specific example). It can also be seen that for any values of $p/(gh)$, $\lambda_{KG}$ is always greater than one (as predicted by eq. (12)), but $\lambda_D$ can be less than one (see Table 2).

| Table 1: The first ten values of $\lambda_D$ and $\lambda_{KG}$ for $g \equiv 1$ (meter$^{-1}$), $m = m_{electron}$, and $p/(gh) = 50$. |
|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| $\lambda_D$    | 26.07           | 26.54           | 27.01           | 27.48           | 28.41           | 28.88           | 29.34           | 29.79           | 30.23           | 30.63           |
| $\lambda_{KG}$ | 26.05           | 26.52           | 26.99           | 27.46           | 28.39           | 28.84           | 29.30           | 29.74           | 30.15           | 30.49           |

| Table 2: The first ten values of $\lambda_D$ and $\lambda_{KG}$ for $g \equiv 1$ (meter$^{-1}$), $m = m_{electron}$, and $p/(gh) = 0.001$. |
|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| $\lambda_D$    | 0.46            | 0.87            | 1.25            | 1.61            | 1.97            | 2.31            | 2.66            | 2.99            | 3.33            | 3.66            |
| $\lambda_{KG}$ | 1.08            | 1.28            | 1.54            | 1.83            | 2.14            | 2.45            | 2.76            | 3.08            | 3.40            | 3.71            |

In $p/(gh) \gg 1$ region, we can use the asymptotic form of the zeros of the function $K_{\alpha}(s)$, to obtain an approximate values of spectrum of Klein-Gordon equation. It is known that the relation between $\alpha$ and the zeros of $K_{\alpha}(s)$ is

$$\alpha_n = s + \beta_n s^{1/3} + O(s^{-1}),$$

(38)
in which $\beta_n$ is the $n$-th zero of the Airy function $Ai(-2^{1/3}\beta)$. In our case (eq. 36), $\alpha_n = \sqrt{\lambda_n^2 - 1}$ in which $\lambda_n^2 = (E_n^2 - m^2)/g^2$ (eq. 10) and $s = p/g$. So the energy eigenvalues for large $p/(g\hbar)$ is:

$$E_n^2 = m^2c^4 + g^2c^2\hbar^2\{1 + \left[p/\sqrt{gh} + \beta_n(p/\sqrt{gh})^{1/3}\right]^2 + O(g\hbar/p)\}. \quad (39)$$

6 Classical characteristics of the metric

In this section we are going to investigate the physical and geometrical properties of the metric:

$$ds^2 = -dt^2 + dx^2 + e^{-2gx}(dy^2 + dz^2). \quad (40)$$

Firstly we obtain the properties of matter corresponds to this metric, and secondly the geodesics of the above metric. The latter is important because we have considered semi-infinite laboratory with a potential barrier at $x = 0$. This situation (considering $x = 0$ as the floor of the laboratory) is physical if the classical particles, in this background metric, strike the barrier, i.e., the classical trajectories have a falling behavior in -x (downward) direction.

6.1 Equation of state

The non-vanishing Christoffel symbols of the metric (40) are:

$$\Gamma^1_{22} = \Gamma^1_{43} = ge^{-2gx},$$
$$\Gamma^2_{12} = \Gamma^2_{21} = \Gamma^3_{13} = \Gamma^3_{31} = -g. \quad (41)$$

Therefore the components of Ricci tensor, which are different from zero, are $R_{11} = 2g^2$ and $R_{22} = R_{33} = 2g^2e^{-2gx}$, and the scalar curvature of the metric is:

$$R = g_{\mu \nu}R^{\mu \nu} = 6g^2. \quad (42)$$

Considering the Einstein field equation $R_{\mu \nu} - (1/2)g_{\mu \nu}R = 8\pi GT_{\mu \nu}$, the energy-momentum tensor becomes:

$$T_{\mu \nu} = \kappa \text{ diag } (3, -1, -e^{-2gx}, -e^{-2gx}), \quad (43)$$

in which:

$$\kappa = \frac{g^2}{8\pi G}. \quad (44)$$

Now consider the energy-momentum tensor of a perfect fluid:

$$T_{\mu \nu} = (p + \rho)U_\mu U_\nu + pg_{\mu \nu}, \quad (45)$$

where $U^\mu = (\gamma, \gamma v)$ is the fluid four-velocity, and $p$ and $\rho$ are pressure and energy density, respectively. Comparing eqs. (43) and (45), shows $U^\mu = (1, 0)$, or $U_\mu U_\nu = \delta_{\mu \nu} \delta_{\alpha \beta}$ (i.e., the fluid is at rest). But $g_{\mu \nu} = \text{diag } (-1, 1, e^{-2gx}, e^{-2gx})$, so $T_{00} = \rho = 3\kappa$ and $T_{ij} = pg_{ij} = -\kappa g_{ij}$. Therefore the matter which is the origin of the metric (40), is a perfect fluid which is at rest, has a negative pressure $p = -\kappa$, and its equation of state is:

$$p = -\frac{\rho}{3}. \quad (46)$$
6.2 The geodesics

The equation of geodesics is:

$$\frac{d^2x^\mu}{ds^2} + \Gamma^\mu_{\nu\sigma} \frac{dx^\nu}{ds} \frac{dx^\sigma}{ds} = 0.$$  (47)

Using the Christoffel symbols, we arrive at:

$$\frac{d^2t}{ds^2} = 0,$$  (48)

$$\frac{d^2x}{ds^2} + ge^{-2g_x} \left[ \left( \frac{dy}{dx} \right)^2 + \left( \frac{dz}{dx} \right)^2 \right] = 0,$$  (49)

$$\frac{d^2y}{ds^2} - 2g \frac{dx}{ds} \frac{dy}{ds} = 0,$$  (50)

$$\frac{d^2z}{ds^2} - 2g \frac{dx}{ds} \frac{dz}{ds} = 0.$$  (51)

Equation (48) results (c is the speed of light):

$$s = ct.$$  (52)

Differentiating (40) with respect to s, and using eqs. (49) and (52), we find:

$$\frac{du}{2 - u^2} = -gds,$$  (53)

where $u := \frac{dx}{ds}$. Noting that $v_x = \frac{dx}{dt} < c$ or $u < 1$, integrating of the above equation results:

$$u = \frac{1}{c} v_x = \sqrt{\frac{k e^{-\alpha t} - 1}{k e^{-\alpha t} + 1}},$$  (54)

where $\alpha = 2\sqrt{2gc}$ and $k$ is determined by the initial condition:

$$k = \frac{c + v_{0x}}{c - v_{0x}},$$  (55)

where for $v_{0x} > 0$, $k$ is greater than one. Once more integration of (54), specifies $x(t)$ as following:

$$x(t) = x_0 - \frac{1}{2g} \left[ \ln \left( e^{\alpha t} + k \right) \left( ke^{-\alpha t} + 1 \right) - 2 \ln \left( 1 + k \right) \right].$$  (56)

At $t = t^* = \left( \ln k \right)/\alpha$, $v_x$ becomes zero and the particle begins to fall down, i.e. $v_x(t > t^*) < 0$. It can be also seen that at $t = 2t^*$, $x(2t^*) = x_0$ and $v_x(2t^*) = -v_{0x}$. Therefore everything shows that we have a falling particle in -x direction.

Equations (50) and (51) can be written as:

$$\frac{dv}{ds} = 2guv,$$

$$\frac{dw}{ds} = 2guv,$$  (57)
where \( v = \frac{dy}{ds} \) and \( w = \frac{dz}{ds} \). Dividing these two equations, results \( \frac{dv}{v} = \frac{dw}{w} \), or \( v = k_1 w \), from which:

\[
y = k_1 z + k_2, \tag{58}
\]

where \( k_1 \) and \( k_2 \) are some constants. Inserting \( k_1 \) in \( k_2 \), gives \( v_y \) and \( v_z \) as following:

\[
v_i = \frac{(1 + k)^2 v_0}{(e^{\alpha t} + k)(ke^{-\alpha t} + 1)}, \tag{59}
\]

with \( i = y, z \). So \( v_\perp = 0 \) if \( v_{0\perp} = 0 \).

References

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