On the conformal properties of topological terms in even dimensions

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Abstract. Conformal properties of the topological gravitational terms in $D=2$, $D=4$ and $D=6$ are discussed. It is shown that in the last two cases the integrands of these terms, when being settled into the dimension $D-1$ and multiplied by a scalar, become conformal invariant. Furthermore we present a simple covariant derivation of Paneitz operator in $D=4$ and formulate two general conjectures concerning the conformal properties of topological structures in even dimensions.

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1 Introduction

Conformal operators and conformal properties of topological terms in different space-time dimensions $D$ are important issues, especially due to the applications in quantum theory. The solution in $D=2$ is a very well-known Polyakov action [1], while the conformal operator is just a two-derivative $\Delta_2 = \Box$. The conformal operator $\Delta_4$ acting on scalar in $D=4$ was first obtained by Paneitz [2] and independently by Riegers, Fradkin and Tseytlin [3]. This operator has four derivatives and acts on the conformally inert scalar field. One can easily obtain a generalization to the $D \neq 4$, when the corresponding scalar gains a non-trivial transformation law proportional to the difference $D-4$, see Ref. [4].
In order to integrate conformal anomaly in $D \geq 6$ and explore the allegedly general universality properties, it would be very useful to have similar operators in general even dimensions $D = 6, 8, \ldots$. In the mathematical literature one can find a general theory for constructing conformal operators [5, 6, 7, 8], which can be used to obtain explicit examples. For instance, the analog of Paneitz operator with six derivatives, $\Delta_6$, in $D = 6$ can be found in [9], consequent paper [10] and in [11], where the generalization to $D \neq 6$ was also obtained.

It is important to remember that the generalization of the results of [1, 3] to dimensions $D \geq 6$ requires not only constructing the corresponding conformal operators, but also their relations to the topological terms. By using both things and also the relation between surface terms in the anomaly and local finite terms in the effective action, one can expect to obtain compact and useful expressions for the anomaly-induced effective action of gravity, such as Polyakov action in $D = 2$ and analogous expression in $D = 4$ [3] (see also [13] and [14] for the reviews and description of recent results in this direction).

In the present work we describe some preliminary results related to the conformal properties of topological structures in even dimensions and their relation to the conformal invariants in odd dimensions. Furthermore, we formulate two conjectures about possible relation between the integrands of topological structures and existence of higher derivative conformal operators, which may be valid (or not) in even dimensions. The verification of these conjectures will be hopefully presented soon in a separate paper. The material presented here is very simple and some part of it may be not completely new. However we believe that it may have some interest for those who work on the related subjects. In order to make the manuscript brief we skip many details concerning conformal transformations of curvature tensor(s) and their contractions. One can consult the previous paper [15] for all intermediate formulas. At the same time, all relevant final expressions are presented for the convenience of the reader.

The paper is organized as follows. In Sect. 2 one can find some covariant calculations, which includes a new way of deriving the Paneitz operator in $D = 4$. Sect. 3 is devoted to the conformal transformation of the integrands of topological invariants in $D = 2$, $D = 4$ and $D = 6$ dimensions. As a by-product we arrive at the new conformal invariants in $D - 1$ dimensions for all three cases. In Sect. 4 the two conjectures about conformal operators and conformal properties of topological structures are formulated. Finally, in Sect. 5 we draw our conclusions.

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1This was recently discussed in [12], where one the our conjectures from Sect. 4 was formulated for the particular case of a flat background. Here we approach the problem in a partially different way and consider an arbitrary curved metric.
2 Covariant derivation of Paneitz operator

Let us start by reviewing terms which are topological in $D=2$ (Einstein-Hilbert) and $D=4$ (Gauss-Bonnet term). Previously, the last case has been discussed in some works devoted to quantum gravity [16, 17], where one can find more detailed consideration.

In $D=2$ one has to start from the Einstein-Hilbert action

$$S_2 = \int d^Dx \sqrt{-g} R, \quad D = 2.$$  \hspace{1cm} (1)

The equations of motion boil down to

$$R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R = 0,$$  \hspace{1cm} (2)

which is an identity in $D=2$.

It is interesting to see whether something similar occurs in $D=4$. In this case the topological action has the form

$$S_4 = \int d^Dx \sqrt{-g} E_4,$$  \hspace{1cm} (3)

where $E_4 = R^{\alpha\beta\rho\sigma} R_{\alpha\beta\rho\sigma} - 4 R^{\mu\nu} R_{\mu\nu} + R^2$ is Euler characteristic in $D=4$.

It proves useful to define the integrals of the squares of curvatures,

$$I_1(D) = \int d^Dx \sqrt{-g} R_{\mu\nu\alpha\beta}^2, \quad I_2(D) = \int d^Dx \sqrt{-g} R_{\mu\nu}^2, \quad I_3(D) = \int d^Dx \sqrt{-g} R^2$$  \hspace{1cm} (4)

and the surface term $I_4(D) = \int d^Dx \sqrt{-g} \Box R$. The variation with respect to the metric in $D = 4$ yields

$$\frac{1}{\sqrt{-g}} \frac{\delta I_1(4)}{\delta g_{\mu\nu}} = \frac{1}{2} g^{\mu\nu} R_{\rho\sigma\alpha\beta}^2 - 2 R^{\mu\rho\alpha\beta} R_{\rho\sigma\alpha\beta} - 4 R^{\rho\mu\alpha\beta} R_{\rho\alpha\beta} + 4 R^{\rho\mu} R^{\rho\alpha}$$

$$+ 2 \nabla^\mu \nabla^\nu R - 4 \Box R^{\mu\nu},$$  \hspace{1cm} (5)

$$\frac{1}{\sqrt{-g}} \frac{\delta I_2(4)}{\delta g_{\mu\nu}} = \frac{1}{2} g^{\mu\nu} R_{\rho\sigma}^2 - 2 R^{\mu\rho\alpha\beta} R_{\rho\alpha\beta} + \nabla^\mu \nabla^\nu R - \frac{1}{2} g^{\mu\nu} \Box R - \Box R^{\mu\nu},$$  \hspace{1cm} (6)

$$\frac{1}{\sqrt{-g}} \frac{\delta I_3(4)}{\delta g_{\mu\nu}} = \frac{1}{2} g^{\mu\nu} R^2 + 2 \nabla^\mu \nabla^\nu R - 2 g^{\mu\nu} \Box R - 2 R R^{\mu\nu}.$$  \hspace{1cm} (7)

It is not easy to show that the linear combination of these expressions corresponding to the action (3) is identically zero, as it was discussed in [16]. At the same time the traces of the combinations corresponding to the Weyl action $I_1 - 2 I_2 + I_3/3$ and to the Gauss-Bonnet topological term (3) can be immediately observed to vanish.
For $D \neq 4$ the Gauss-Bonnet term $\mathcal{B}$ is not topological. It is easy to derive the trace of the corresponding equation,

$$\frac{1}{\sqrt{-g}} g_{\mu \nu} \frac{\delta}{\delta g_{\mu \nu}} \int d^D x \sqrt{-g} E_4 = \frac{(D - 4)}{2} E_4. \quad (8)$$

Consider equations of motion for the action

$$I_E (D) = I_1 (D) - 4 I_2 (D) + I_3 (D) \quad (9)$$
on a special de Sitter background, when Riemann and Ricci tensors can be presented as

$$R_{\mu \nu \alpha \beta} = \frac{1}{D (D - 1)} \Lambda (g_{\mu \alpha} g_{\nu \beta} - g_{\mu \beta} g_{\nu \alpha}), \quad R_{\mu \nu} = \frac{1}{D} \Lambda g_{\mu \nu}, \quad \Lambda = \text{const.} \quad (10)$$

After a small algebra we arrive at the following results:

$$\frac{1}{\sqrt{-g}} \frac{\delta I_1}{\delta g_{\mu \nu}} \bigg|_{dS} = \frac{(D - 4)}{D^2 (D - 1)} \Lambda^2 g^{\mu \nu}, \quad (11)$$

$$\frac{1}{\sqrt{-g}} \frac{\delta I_2}{\delta g_{\mu \nu}} \bigg|_{dS} = \frac{(D - 4)}{2 D^2} \Lambda^2 g^{\mu \nu}, \quad (12)$$

$$\frac{1}{\sqrt{-g}} \frac{\delta I_3}{\delta g_{\mu \nu}} \bigg|_{dS} = \frac{(D - 4)}{2 D} \Lambda^2 g^{\mu \nu}. \quad (13)$$

Consequently, for the $D$-dimensional version of the Gauss-Bonnet term we obtain

$$\frac{1}{\sqrt{-g}} \frac{\delta I_E (D)}{\delta g_{\mu \nu}} \bigg|_{dS} = \frac{(D - 2) (D - 3) (D - 4)}{2 D^2 (D - 1)} \Lambda^2 g^{\mu \nu}. \quad (14)$$

Naturally, when $D = 4$, the equation $\mathcal{B}$ becomes zero, but the same also occurs in $D = 2$ and $D = 3$ cases, where the Gauss-Bonnet term is not topological. One can note that the expressions $\mathcal{B}$ do vanish only at $D = 4$, so the cancelation for $D = 2$ and $D = 3$ takes place only for the topological term. Later on we obtain more detailed form of this relation.

The next exercise is to obtain the Paneitz operator $\mathcal{B}$ in $D = 4$ in a covariant way. The usual definition of this operator is through the conformal transformation,

$$g_{\mu \nu} = \bar{g}_{\mu \nu} e^{2 \sigma}, \quad \text{where} \quad \sigma = \sigma (x) \quad (15)$$

and $\bar{g}_{\mu \nu}$ is a fiducial metric. We can assume that $\sigma$ is a scalar field and then $\bar{g}_{\mu \nu}$ is a second rank tensor. It is important that $\bar{g}_{\mu \nu}$ does not depend on $\sigma$ and this can be achieved, e.g., by using the covariant non-local construction of $\mathcal{B}$.
For the conformally inert scalar $\varphi = \bar{\varphi}$ the Paneitz operator $\Delta_4$ has to be Hermitian and provide the invariance of the bilinear expression,

$$\int d^4x \sqrt{-\bar{g}} \varphi \Delta_4 \varphi = \int d^4x \sqrt{-\bar{g}} \bar{\varphi} \Delta_4 \bar{\varphi}.$$  \hspace{1cm} (16)

Here the bar means that the operator is constructed with the $\bar{g}_{\mu\nu}$ metric. The solution for $\Delta_4$ has been found in [2] (see also [3] and generalization to other dimensions in [4]), but we shall solve the same problem in a completely covariant way, without explicit use of the transformation (15).

We start from a simple observation about the variational derivative with respect to $\sigma$ in (15). For a functional of a metric $A = A(g_{\mu\nu})$ we have

$$\frac{\delta A}{\delta \sigma} = \frac{\delta g_{\mu\nu}}{\delta \sigma} \cdot \frac{\delta A}{\delta g_{\mu\nu}} = 2 \bar{g}_{\mu\nu} e^{2\sigma} \frac{\delta A}{\delta g_{\mu\nu}} = 2 g_{\mu\nu} \frac{\delta A}{\delta g_{\mu\nu}}.$$  \hspace{1cm} (17)

This simple calculation shows that everything that is linear in $\sigma$ can be obtained by taking the trace of covariant equations of motion for the metric.

In order to obtain the Paneitz operator $\Delta_4$ in a covariant way, one can define new actions which depend on an additional conformally inert scalar field $\varphi = \bar{\varphi}$,

$$I^\varphi_1 = \int d^4x \sqrt{-g} \varphi R^2, \quad I^\varphi_2 = \int d^4x \sqrt{-g} \varphi R_{\mu\nu}^2, \quad I^\varphi_3 = \int d^4x \sqrt{-g} \varphi R_{\mu\nu\alpha\beta}^2, \quad I^\varphi_4 = \int d^4x \sqrt{-g} \varphi \Box R.$$  \hspace{1cm} (18)

The equations of motion have the form

$$\frac{1}{\sqrt{-g}} \frac{\delta I^\varphi_1}{\delta g_{\mu\nu}} = \frac{1}{2} g^{\mu\nu} R^2 \varphi + 2 \nabla^\alpha \nabla^\mu (R \varphi) - 2 g^{\mu\nu} \Box (R \varphi) - 2 R^{\mu\nu} (R \varphi),$$  \hspace{1cm} (20)

$$\frac{1}{\sqrt{-g}} \frac{\delta I^\varphi_2}{\delta g_{\mu\nu}} = \frac{1}{2} g^{\mu\nu} R_{\rho\sigma}^2 \varphi - 2 R^{\nu\alpha} R^\rho_\alpha \varphi + 2 \nabla^\lambda \nabla_\mu (R^\lambda_\nu \varphi)$$

$$- g^{\mu\nu} \nabla_\beta \nabla_\alpha (R^{\alpha\beta}_\nu \varphi) - \Box (R^{\mu\nu} \varphi),$$  \hspace{1cm} (21)

$$\frac{1}{\sqrt{-g}} \frac{\delta I^\varphi_3}{\delta g_{\mu\nu}} = \frac{1}{2} g^{\mu\nu} R^{2}_{\alpha\beta\rho\sigma} \varphi + 4 \nabla_\beta \nabla_\alpha (R^{\mu\nu\beta\lambda} \varphi) - 2 (R^{\mu}_{\beta\alpha\rho} R^{\nu\beta\alpha \varphi}),$$  \hspace{1cm} (22)

$$\frac{1}{\sqrt{-g}} \frac{\delta I^\varphi_4}{\delta g_{\mu\nu}} = \frac{1}{2} g^{\mu\nu} \Box R \varphi + \nabla^\mu \nabla^\nu \Box \varphi - g^{\mu\nu} \Box^2 \varphi - R^{\mu\nu} \Box \varphi - R \nabla^\mu \nabla^\nu \varphi$$

$$+ \nabla_\lambda (R g^{\lambda\varphi} (\varphi) - \frac{1}{2} \nabla_\lambda (g^{\mu\nu} R \nabla^\lambda \varphi)).$$  \hspace{1cm} (23)

Using this formulas it is easy to check that the Weyl term with an additional scalar is still conformal invariant,

$$\frac{1}{\sqrt{-g}} g_{\mu\nu} \frac{\delta}{\delta g_{\mu\nu}} \int d^4x \sqrt{-g} \varphi C^2 = 0.$$
For the Gauss-Bonnet term with additional scalar we obtain
\[
\frac{1}{\sqrt{-g}} g_{\mu \nu} \frac{\delta}{\delta g_{\mu \nu}} \int d^4 x \sqrt{-g} \varphi E_4 = 4 R^\mu{}_{\nu} (\nabla_\mu \nabla_\nu \varphi) - 2 R \Box \varphi.
\] (24)

Let us now make an important step and introduce the “corrected” term
\[
\tilde{E}_4 = E_4 - \frac{2}{3} \Box R.
\] (25)

Taking into account (24) and (23), after a very small algebra we arrive at
\[
\frac{1}{2\sqrt{-g}} g_{\mu \nu} \frac{\delta}{\delta g_{\mu \nu}} \int d^4 x \sqrt{-g} \varphi \tilde{E}_4
\]
\[
= \left[ \Box^2 + 2 R^\mu{}_{\nu} \nabla_\mu \nabla_\nu - \frac{2}{3} R \Box + \frac{1}{3} (\nabla_\lambda R) \nabla^\lambda \right] \varphi = \Delta_4 \varphi,
\] (26)

where $\Delta_4$ is exactly the Paneitz operator which we are looking for.

Two observations are in order. First, the derivation of the conformal operator $\Delta_2 = \Box$ in $D = 2$ can be performed in the very same way as we just did in $D = 4$, but in the two-dimensional case there is no need to introduce an additional term to $E_2 = R$. Since the calculation is quite trivial, we skip the details. The second point is that there is no regular way to derive the coefficient in Eq. (25), so the origin of its value $-2/3$ looks mysterious. In the full conformal derivation (see, e.g., [15]) this coefficient provides cancellation of all but linear terms in $\sigma$ in the transformation of $\sqrt{-g} \tilde{E}_4$, but (as far as we know) there is no other way to obtain this coefficient except by an explicit calculation.

3 Conformal transformation of topological terms

Let us explore the conformal properties of topological terms in even dimensions. We will be mainly concerned with the $D = 6$ case which attracted considerable interest in the recent literature (see, e.g., [11, 20, 21, 22] and references therein). According to the standard classification [23] (see also earlier work [24]), the anomalous terms that correspond to the non-local part of effective action are either conformal invariants or topological terms. Hence it is very important to better understand the conformal properties of the topological terms, in particular for the case of $D = 6$.

The conformal transformation is defined as a parametrization (15) of the metric tensor. It makes sense to analyze the transformations of Euler densities not only in the dimensions in which these quantities are topological, but also in other dimensions. Euler density in even dimensions $D = 2n$ ($n = 1, 2, ...$) is well-known (see e.g. [25]),
\[
E_{2n} = \frac{1}{2n} \varepsilon^{\alpha_1 \beta_1 ... \alpha_n \beta_n} \varepsilon^{\gamma_1 \delta_1 ... \gamma_n \delta_n} R_{\alpha_1 \beta_1 \gamma_1 \delta_1} ... R_{\alpha_n \beta_n \gamma_n \delta_n}.
\] (27)
It is instructive to consider a few examples. For $D = 2$, the definition \((27)\) gives
\[
E_2 = \frac{1}{2} \varepsilon^{\mu\nu} \varepsilon^{\alpha\beta} R_{\mu\nu\alpha\beta} = R. \tag{28}
\]
In $D = 4$ case Eq. \((27)\) provides the Gauss-Bonnet term \((3)\),
\[
E_4 = \frac{1}{4} \varepsilon^{\mu\nu\lambda\tau} \varepsilon^{\alpha\beta\rho\sigma} R_{\mu\nu\alpha\beta} R_{\lambda\tau\rho\sigma}. \tag{29}
\]
In $D = 6$ the evaluation of \((27)\) is more cumbersome, the result is (see, e.g., \([26]\))
\[
E_6 = \frac{1}{8} \varepsilon^{\mu\nu\alpha\beta\lambda\xi} \varepsilon^{\rho\sigma\kappa\omega} \eta\xi R_{\mu\nu\rho\sigma} R_{\alpha\beta\kappa\omega} R_{\lambda\xi\eta\omega} \\
= 4R_{\mu\nu}^{\alpha\beta} R_{\alpha\beta}^{\lambda\tau} R_{\mu\nu}^{\lambda\tau} - 8R_{\mu\nu\alpha\beta} R_{\alpha\beta}^{\lambda} R_{\lambda\tau\rho\sigma} - 24R_{\mu\nu}^{\alpha\beta\lambda\rho} R_{\alpha\beta\lambda\rho} R_{\mu\nu}^{\lambda\rho} \\
+ 24R_{\mu\nu\alpha\beta} R_{\mu\nu}^{\alpha\beta} + 16R_{\mu\nu}^{\alpha\beta} R_{\alpha\beta}^{\rho\sigma} + 3R_{\mu\nu\alpha\beta}^{2} - 12R_{\mu\nu}^{2} + R^{3}. \tag{30}
\]
Consider how these three quantities behave under $D$-dimensional conformal transformation \((15)\). The transformations of Riemann, Ricci tensor and scalar and of the $\Box$ can be found, e.g., in \([15]\), so let us skip the intermediate formulas and present only the final results,
\[
\sqrt{-g} E_2 = \sqrt{-\bar{g}} e^{(D-2)\sigma} \left\{ \bar{E}_2 - (D - 1) \left[ 2\Box\sigma + (D - 2)(\nabla\sigma)^2 \right] \right\}, \tag{31}
\]
where we multiplied the expression by $\sqrt{-g}$ for convenience.

Similarly, the $E_4$ calculation yields
\[
\sqrt{-g} E_4 = \sqrt{-\bar{g}} e^{(D-4)\sigma} \left\{ \bar{E}_4 + (D - 3) \chi_{(4)} \right\}, \tag{32}
\]
where
\[
\chi_{(4)} = 8R_{\mu\nu}^{\sigma\mu\nu} - 8R_{\mu\nu}^{\sigma\mu\nu} - 4\Box\sigma - 2(D - 4)(\nabla\sigma)^2 + (D - 2) \left[ 8\sigma_{\mu\nu}^{\sigma\mu\nu} \\
- 4\sigma_{\mu\nu}^{2} + (D - 4)(D - 1)(\nabla\sigma)^2 + 4(\Box\sigma)^2 + 4(D - 3)\Box\sigma(\nabla\sigma)^2 \right]. \tag{33}
\]
Here we used the condensed notations $\sigma_\alpha = \bar{\nabla}_\alpha \sigma$, $\sigma_{\alpha\beta} = \bar{\nabla}_\alpha \bar{\nabla}_\beta \sigma$, $\Box = \bar{g}^{\alpha\beta} \sigma_{\alpha\beta}$, $(\nabla\sigma)^2 = \bar{g}^{\alpha\beta} \sigma_\alpha \sigma_\beta$, also all indices are raised and lowered with the fiducial metric $\bar{g}^{\alpha\beta}$ and its inverse.

One can note that the conformally non-covariant part of the r.h.s. of \((31)\) is proportional to $D - 3$, which is a non-linear generalization of the previously considered Eq. \((14)\).

One can see that the non-linear expression \((32)\) is conformally non-covariant at $D = 2$ (some related observations can be found in \([15]\)), but is covariant at $D = 3$.

Finally, consider the case of $E_6$. The corresponding calculations were performed by using Cadabra software \([27, 28]\) and the result reads
\[
\sqrt{-g} E_6 = \sqrt{-\bar{g}} e^{(D-6)\sigma} \left\{ E_6 + (D - 5) \chi_{(6)} \right\}, \tag{34}
\]
where

\[ \chi(6) = - \left[ 6 \Box \sigma + 3(D - 6)(\nabla \sigma)^2 \right] E_4 + 24(2 \tilde{R}_{\alpha \beta} \tilde{R}^{\alpha \beta \mu \nu} - \tilde{R}_{\alpha \beta} \tilde{R}^{\alpha \beta \mu \nu} - \tilde{R}^{\mu \nu} + 2 \tilde{R}^{\mu \alpha} \tilde{R}_{\alpha}^{\nu}) (\sigma_\mu \sigma_\nu - \sigma^{\mu \nu}) + 12(D - 2) \tilde{R} [ (\Box \sigma)^2 - \sigma_{\mu \nu}^2 + 2 \sigma_\mu \sigma^{\mu \nu} ] \]

An interesting feature of Eq. (34) is that the conformally non-covariant part of this expression vanish in $D = 5$ dimension. As we have seen before, this is similar to $E_2$ and $E_4$. As a consequence one can construct new conformal invariants in odd dimensions $2n - 1$. Consider an auxiliary scalar field $\Phi$ which transforms according to

\[ \Phi = e^\sigma \tilde{\Phi} \] (36)

simultaneously with (15). Then we meet

\[ \int d^{2n-1}x \sqrt{-g} \Phi E_{2n} = \int d^{2n-1}x \sqrt{-g} \tilde{\Phi} E_{2n}, \] (37)

where $n = 1, 2, 3$ and the expressions (37) provides the set of conformally invariant actions. Of course this is a trivial statement for $D = 2$, but in the cases of $D = 4$ and $D = 6$ we can claim that the topological invariants in these even dimensions give rise to the new conformal invariants (37) in three- and five-dimensional spaces, correspondingly.

4 Conjectures

Taking our previous experiences into account, let can formulate the following two conjectures concerning the topological terms (27):

Conjecture 1. For any even dimension $D = 2n$, $n = 1, 2, 3, 4, ...$, the expressions (37) are conformal invariant if the scalar $\Phi$ transforms according to (36). This means we arrive at the chain of conformal actions

\[ S_{2n-1}^c = \int d^{2n-1}x \sqrt{-g} \Phi E_{2n} \] (38)
Conjecture 2. For any even dimension $D = 2n$ there is such a metric-dependent vector function $\chi^\mu_{2n}$ that the “corrected” topological invariant

$$E_{2n} + \nabla_\mu \chi^\mu_{2n},$$

possesses linear conformal transformation,

$$\sqrt{-g}(E_{2n} + \nabla_\mu \chi^\mu_{2n}) = \sqrt{-\bar{g}}(\bar{E}_{2n} + \nabla_\mu \bar{\chi}^\mu_{2n} + c \cdot \bar{\Delta}_{2n} \sigma).$$

(40)

Here $c$ is some unknown constant and operator $\Delta_{2n} = \Box^n + \ldots$ is conformal, in the sense

$$\int d^{2n} \sqrt{-g} \varphi \Delta_{2n} \varphi = \int d^{2n} \sqrt{-\bar{g}} \varphi \bar{\Delta}_{2n} \varphi.$$

(41)

Let us remember that all quantities with bars are constructed on the basis of the fiducial metric $\bar{g}_{\mu \nu}$ in (15). In the case of $D = 2$ we know that $\chi^\mu_{2} = 0$ and for $D = 4$ we know that $\chi^\mu_{4} = -(2/3) \nabla^\mu R$. The verification of this conjecture for six dimensions requires a significant calculational work and we expect to report on the result soon.²

An important step towards a general understanding of the second Conjecture would be explanation of the $-2/3$ coefficient in the four-dimensional case. At the moment we are not able to give such an explanation and rely on a direct computations.

5 Conclusions and discussions

Since the conformal anomaly is one of the main sources of our knowledge of the semiclassical corrections to the gravitational action (see, e.g., [13, 31, 32, 33]), it would be useful to have better understanding of the conformal properties of the terms which constitute this anomaly. In this relation it is a challenging problem to establish conformal properties of the topological invariants and their relations to the conformal operators acting on conformally inert scalars.

At the moment we know such relations for the two- and four-dimensional spaces. However, there is no real understanding of the fundamental reasons of why these relations take place in $D = 4$ and whether similar relations exist for higher even dimensions. In this respect it would be most interesting to verify the second Conjecture formulated above (see also previous work [12] on the same subject). A practical realization of this program is

²After submitting the first version of this work we learned that the flat limit of the relation (40) and an incomplete form of anomaly-induced action was recently obtained in a very interesting paper [29]. Our project can be seen as presenting the result in a covariant form of relation (40).
a necessary step in integrating conformal anomaly in $D = 6$ and higher even dimensions, and also may help to approaching the solution of one of the mathematical puzzles related to conformal anomaly. It is important that integrating the trace anomaly requires not only conformal operator [8, 11] (see also [7, 34, 35, 36] for other publications on the subject), but also the relation between conformal operators and topological structures, e.g., expressed in the form (40). This kind of formula is critically important for integrating anomaly in $D = 2$ and $D = 4$ and hence the proof of the Conjecture 2 would be a decisive step forward in completing the same program in higher even dimensions.

After the proof of the second Conjecture, the problem will not be solved yet. The reason is that there a third type of terms in the anomaly, which go beyond the known classification of [23] and come from the renormalization of surface terms. The experience which we have from the $D = 4$ shows that these terms should be taken seriously, in particular they emerge in a direct calculation via adiabatic regularization (see, e.g., [37]). In order to neglect these terms one needs at least to be sure that the finite anomaly-generating terms in the effective action are local. The detailed constructive proof of this statement would open the door for deriving anomaly-induced action in $D = 6, 8, ...$ and to the corresponding physical and mathematical applications.

Appendix. The case of conformally flat metric

In order to show how the Conjecture 2 described in Sect. 4 works, let us consider a relatively simple example of the conformally flat metric. Earlier the same method has been used in [12], but we shall go into more details and obtain a slightly more general result.

Let us remember that our ansatz assumes adding only total second derivatives terms to $E_6$. Using some reduction procedure, one can show that the general form of the corrected topological term is

$$
\tilde{E}_6 = E_6 + \alpha_1 \Box^2 R + \alpha_2 \Box R^2 + \alpha_3 \Box R_{\mu}^2 + \alpha_4 \Box R_{\mu\alpha\beta}^2 + \alpha_5 \nabla_\mu \nabla_\nu (R^\mu_{\lambda\alpha\beta} R^{\nu\lambda\alpha\beta}) \\
+ \alpha_6 \nabla_\mu \nabla_\nu (R_{\alpha\beta} R^{\mu\alpha\beta}) + \alpha_7 \nabla_\mu \nabla_\nu (R^\mu R^{\nu\lambda}) + \alpha_8 \nabla_\mu \nabla_\nu (RR_{\mu\nu}). \tag{42}
$$

Our interest is to find the values of the parameters $\alpha_{1,2,\ldots,8}$ for which the relation (40) takes place. In order to do it, we can choose any background metric, which makes the solution simpler. One of the possibilities is to take the metric $g_{\mu\nu} = e^{2\sigma} \eta_{\mu\nu}$, with $\sigma = \sigma(\eta)$, where $\eta$ is conformal time. After that one has to evaluate the conformal transformation for each of the nine terms of (12).
After performing these steps we found the combination of $\alpha$’s which satisfies the requested condition. The most general form depends on four parameters $a, b, c, d$ as follows:

\[
\tilde{E}_6 = E_6 + \frac{3}{5} \Box^2 R + a \Box R^2 + b \Box R_{\mu\nu}^2 + c \Box R_{\mu\nu\alpha\beta}^2 + d \nabla_\mu \nabla_\nu (R^\mu_{\lambda\alpha\beta} R^{\mu\alpha\beta}) \\
- \left( \frac{21}{5} + 20a + 8b + 6c + \frac{d}{2} \right) \nabla_\mu \nabla_\nu (R_{\alpha\beta} R^{\mu\alpha\beta}) + \left( 3 - 2b - 2c - \frac{d}{2} \right) \nabla_\mu \nabla_\nu (R_{\lambda}^\mu R^{\mu\lambda}) \\
+ \frac{1}{5} \left( -3 + 6b + 6c + \frac{d}{2} \right) \nabla_\mu \nabla_\nu (RR^{\mu\nu}).
\] (43)

This particular form of (42) eliminates all terms of second and higher orders in $\sigma$ in a conformal transformation of the action (43). Let us note that the relation (43) possesses this property for any $a, b, c, d$.

Indeed, the cancelation of the second- and higher-orders in $\sigma$ on flat background does not guarantee that the desired relation (40) holds on an arbitrary background. One can suppose that this relation on an arbitrary background will require to impose some constraints on the parameters $a, b, c, d$. Anyway, in order to have a chance to achieve the result, it is certainly important to have a most general form, such as (43).

Let us consider some particular cases of (43). For instance, one can take the parameters $a, b, c$ and $d$ in such a way that the terms with Riemann tensor vanish, that means one-parametric solution,

\[ c = d = 0, \quad a = \frac{123}{100} - \frac{1}{2} \zeta, \quad b = \frac{5}{4} \zeta - \frac{18}{15}, \] (44)

that corresponds to the result of [12]. Another, much simpler case is obviously $a = b = c = d = 0$, corresponding to

\[
\sqrt{-g} \left\{ E_6 + \frac{3}{5} \Big[ \Box^2 R + \nabla_\mu \nabla_\nu (-7 R_{\alpha\beta} R^{\mu\alpha\beta} + 5 R^\mu_{\alpha} R^{\mu\alpha} - RR^{\mu\nu}) \Big] \right\} = -6 \Delta_6 \sigma
\] (45)
on a flat metric background.

The next phase of the work includes trying to find similar relations for an arbitrary fiducial metric. In this way one can expect to remove the remaining uncertainty of the coefficients $a, b, c$ and $d$ in (43). In this way one can establish a general form of $\Delta_6$ which can be used for comparison with [9, 10, 11], verification of Conjecture 2 and finally for deriving the anomaly-induced effective action in $D = 6$.

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References

[1] A. M. Polyakov, Phys. Lett. B 103 (1981) 207.

[2] S. Paneitz, MIT preprint, 1983; SIGMA 4 (2008) 036.

[3] R. J. Riegert, Phys. Lett. B 134 (1984) 56.
   E. S. Fradkin and A. A. Tseytlin, Phys. Lett. B 134 (1984) 187.

[4] A. de Barros and I. L. Shapiro, Phys. Lett. B 412 (1997) 242.

[5] C. R. Graham, R. Jenne, L. J. Mason and G. A. J. Sparling, J. London Math. Soc. s2-46 (3) (1992) 557.

[6] R. J. Baston and M. G. Eastwood, Invariant operators, Twistors in mathematics and physics, London Math. Soc. Lecture Notes 156 (ed. T. Bailey and R. Baston, University Press, Cambridge, 1990).

[7] T. P. Branson, Comm. Part. Diff. Equations 1 (1982) 393. Supp. Rend. Cir. Mat. Palermo, II 21 (1989) 115.

[8] T. P. Branson, Math. Scand. 57 (1985) 293.

[9] T. Arakelyan, D. R. Karakhanyan, R. P. Manvelyan and R. L. Mkrtchyan, Phys. Lett. B 353 (1995) 52.

[10] K. Hamada, Prog. Theor. Phys. 105 (2001) 673.

[11] H. Osborn and A. Stergiou, JHEP 1504 (2015) 157.

[12] D. Anselmi, Nucl. Phys. B 567 (2000) 331.

[13] I. L. Shapiro, Class. Quant. Grav. 25 (2008) 103001.

[14] P. O. Mazur and E. Mottola, Phys. Rev. D 64 (2001) 104022.

[15] D. F. Carneiro, E. A. Freitas, B. Gonçalves, A. G. de Lima and I. L. Shapiro, Grav. and Cosm. 40 (2004) 305.
[16] D. M. Capper and D. Kimber, J. Phys. A 13 (1980) 3671.
[17] G. de Berredo-Peixoto and I. L. Shapiro, Phys. Rev. D 70 (2004) 044024; Phys. Rev. D 71 (2005) 064005.
[18] F. Englert, C. Truffin and R. Gastmans, Nucl. Phys. B 117 (1976) 407.
[19] E. S. Fradkin and G. A. Vilkovisky, Phys. Lett. B 73 (1978) 209.
[20] B. Grinstein, A. Stergiou and D. Stone, JHEP 1311 (2013) 195.
[21] A. F. Astaneh and S. N. Solodukhin, arXiv:hep-th/1504.01653.
[22] C. Cordova, T. T. Dumitrescu, K. Intriligator, arXiv:hep-th/1506.03807.
[23] S. Deser and A. Schwimmer, Phys. Lett. B 309 (1993) 279.
[24] S. Deser, M. J. Duff and C. J. Isham, Nucl. Phys. B 111 (1976) 45.
[25] Y. Décanini and A. Folacci, Class. Quant. Grav. 24 (2007) 4777.
[26] R. C. Myers and B. Robinson, JHEP 1008 (2010) 067.
[27] K. Peeters, Comput. Phys. Commun. 176 (2007) 550.
[28] K. Peeters, SPIN-06-46, ITP-UU-06-56, arXiv:hep-th/0701238.
[29] H. Elvang, D. Z. Freedman, L. -Y. Hung, M. Kiermaier, R. C. Myers and S. Theisen, JHEP 1210 (2012) 011.
[30] F. M. Ferreira and I. L. Shapiro, work in progress.
[31] N. D. Birell and P. C. W. Davies, Quantum Fields in Curved Space, (Cambridge University Press, Cambridge, 1982).
[32] M. J. Duff, Class. Quant. Grav. 11 (1994) 1387.
[33] S. Deser, Phys. Lett. B 479 (2000) 315; Helv. Phys. Acta 69 (1996) 570.
[34] V. Wünsh, Math. Nahr. 129 (1986) 269.
[35] A. R. Gover and L. J. Peterson, Comm. Math. Phys. 235 (2003) 339.
[36] M. Grigoriev and A. Waldron, Nucl. Phys. B 853 (2011) 291.
[37] L. Parker, D. Toms, Quantum Field Theory in Curved Spacetime, (Cambridge University Press, Cambridge, 2009).