State Complexity Approximation

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In this paper, we introduce the new concept of state complexity approximation, which is a further development of state complexity estimation. We show that this new concept is useful in both of the following two cases: the exact state complexities are not known and the state complexities have been obtained but are in incomprehensible form.

1 Introduction

The state complexity of combined operations has been studied in, e.g., [12, 4, 3]. It has been shown that the state complexity of combined operations is at least as important and practical as the state complexity of individual operations. There is only a limited number of individual operations on regular languages. However, the number of combined operations on regular languages is unlimited and each of them is not simply a mathematical composition of the state complexities of their component individual operations. It appears that the exact state complexity of each combined operation has to be studied specifically.

There are at least the following two problems concerning the state complexities for combined operations. First, the state complexities of many combined operations are extremely difficult to compute. Second, a large proportion of results that have been obtained are pretty complex and impossible to comprehend. For example, the state complexity of the catenation for four regular languages accepted by $m, n, p, q$ states, respectively, is

$$9(2m - 1)2^{n+p+q-5} - 3(m - 1)2^{p+q-2} - (2m - 1)2^{n+q-2} + (m - 1)2^q + (2m - 1)2^{n-2}.$$

It is clear that close estimations of state complexities are good enough in many automata applications. In [13, 3], estimations of state complexity of combined operations have been proposed and studied. In this paper, we go further in the direction of the study in [13, 3] and introduce the concept of state complexity approximation. Briefly speaking, an approximation of a state complexity is an estimate of the state complexity with a ratio bound clearly defined. The ratio bound gives a precise measurement on the quality of the estimate.

The idea of state complexity approximation is from the notion of approximation algorithms which was formalized in early 1970’s by David S. Johnson et al. [5, 9, 10]. Many polynomial-time approximation algorithms have been designed for a quite large number of NP-complete problems, which include the well-known travelling-salesman problem, the set-covering problem, and the subset-sum problem. Obtaining an optimal solution for an NP-complete problem is considered intractable. Near optimal solutions are often good enough in practice. Assuming that the problem is a maximization or a minimization problem, an approximation algorithm is said to have a ratio bound of $\rho(n)$ if for any input of size $n$, the cost $C$ of the solution produced by the algorithm is within a factor of $\rho(n)$ of the cost $C^*$ of an optimal solution [11]:

$$\max \left( \frac{C}{C^*}, \frac{C^*}{C} \right) \leq \rho(n).$$
The concept of state complexity approximation is in many ways similar to that of approximation algorithms. A state complexity approximation is close to the exact state complexity and normally not equal to it. The ratio bound shows the error range of the approximation. In addition to the property of having a small ratio bound in general, we also consider that a state complexity approximation should be in a simple and intuitive form.

In spite of the similarities, there are fundamental differences between a state complexity approximation and an approximation algorithm. The efforts in the area of approximation algorithms are in finding polynomial algorithms for NP-complete problems such that the results of the algorithms approximate the optimal results. In comparison, the efforts in the state complexity approximation are in searching directly for the estimations of state complexities such that they satisfy certain ratio bounds. The aim of designing an approximation algorithm is to transform an intractable problem into one that is easier to compute and the result is acceptable although not optimal. In comparison, a state complexity approximation result may have two different effects: (1) it gives a reasonable estimation of certain state complexity, with some bound, the exact value of which is difficult or impossible to compute; or (2) it gives a simpler and more comprehensible formula that approximates a known state complexity.

In the next section, we give some basic definitions and notation including the formal definition of state complexity approximation. In Section 3, we show the state complexity approximation results on four basic combined operations: the star of union, the star of intersection, the star of catenation, and the star of reversal. In Section 4, we show that state complexity approximation results can be easily obtained for some operations the exact state complexities of which may be very difficult to obtain. In Section 5, we show that certain state complexity can be very complex in formulation. A state complexity approximation is clearly more intuitive and comprehensible than the exact state complexity. In Section 6, we conclude the paper.

2 Preliminaries

A deterministic finite automaton (DFA) is denoted by a 5-tuple \( A = (Q, \Sigma, \delta, s, F) \), where \( Q \) is the finite and nonempty set of states, \( \Sigma \) is the finite and nonempty set of input symbols, \( \delta : Q \times \Sigma \rightarrow Q \) is the state transition function, \( s \in Q \) is the initial state, and \( F \subseteq Q \) is the set of final states. A DFA is said to be complete if \( \delta \) is a total function. Complete DFAs are the basic model for considering state complexity. Without specific mentioning, all DFAs are assumed to be complete in this paper.

A nondeterministic finite automaton (NFA) is also denoted by a 5-tuple \( M = (Q, \Sigma, \delta, s, F) \), where \( Q, \Sigma, s, \) and \( F \) are defined the same way as in a DFA and \( \delta : Q \times \Sigma \rightarrow 2^Q \) maps a pair consisting of a state and an input symbol into a set of states rather than, more restrictively, a single state. An NFA may have multiple initial states, in which case an NFA is denoted \( (Q, \Sigma, \delta, S, F) \) where \( S \) is the set of initial states.

The reader may refer to [8] [11] [14] for a rather complete background knowledge in automata theory.

State complexity ([14]) is a descriptional complexity measure for regular languages based on the deterministic finite automaton model. So, by state complexity we mean the deterministic state complexity.

The state complexity of a regular language \( L \), denoted \( sc(L) \), is the number of states in the minimal complete DFA accepting \( L \). When we speak about the state complexity of a (combined) operation on regular languages, we mean the worst case state complexity of the languages resulting from the operation as a function of the state complexity of the regular operand languages. So, without specific mentioning, by state complexity we mean the worst-case state complexity in the following.

If the above definition is based on minimal NFA rather than minimal complete DFA, we have the
nondeterministic state complexity, which has been studied in \cite{6, 7}.

Let \( \xi \) be a combined operation on \( k \) regular languages. Assume that the state complexity of \( \xi \) is \( \theta \). We say that \( \alpha \) is a state complexity approximation of the operation \( \xi \) with the ratio bound \( \rho \) if, for any large enough positive integers \( n_1, \ldots, n_k \), which are the numbers of states of the DFAs that accept the argument languages of the operation, respectively,

\[
\max \left( \frac{\alpha(n_1, \ldots, n_k)}{\theta(n_1, \ldots, n_k)} \right) \leq \rho(n_1, \ldots, n_k).
\]

Note that in many cases, \( \rho \) is a constant. Since state complexity is a worst-case complexity, an approximation that is not smaller than the actual state complexity is preferred, which is the case for every approximation result in this paper.

### 3 Some basic results on state complexity approximation

In \cite{13}, an estimation method through nondeterministic state complexities was introduced for the (deterministic) state complexities of certain types of combined operations. The method is described in the following.

Assume we are considering the combination of a language operation \( g_i \) with \( k \) arguments together with operations \( g_j^i \), \( i = 1, \ldots, k \). The nondeterministic estimation upper bound, or NEU-bound for the deterministic state complexity of the combined operation \( g_i(g_j^1, \ldots, g_j^k) \) is calculated as follows:

(i) Let the arguments of the operation \( g_j^i \) be DFAs \( A_j^i \) with \( m_j^i \) states, \( i = 1, \ldots, k, j = 1, \ldots, r_i, r_i \geq 1 \).

(ii) The nondeterministic state complexity of the combined operation is at most the composition of the individual state complexities, and hence the language

\[
g_i(g_j^1(L(A_1^1), \ldots, L(A_1^{r_1})), \ldots, g_j^k(L(A_1^k), \ldots, L(A_1^{r_k})))
\]

has an NFA with at most

\[
nsc(g_i)(nsc(g_j^1)(m_1^1, \ldots, m_1^{r_1}), \ldots, nsc(g_j^k)(m_1^k, \ldots, m_1^{r_k}))
\]

states, where \( nsc(g) \) is the nondeterministic state complexity (as a function) of the language operation \( g \).

(iii) Consequently, the deterministic state complexity of the combined operation \( g_i(g_j^1, \ldots, g_j^k) \) is upper bounded by

\[
2^{nsc(g_i)(nsc(g_j^1)(m_1^1, \ldots, m_1^{r_1}), \ldots, nsc(g_j^k)(m_1^k, \ldots, m_1^{r_k}))}
\]

The nondeterministic state complexity of the basic individual operations on regular languages has been investigated in \cite{6, 7, 2}.

In the following we show that this estimation method can produce nice approximation results for the state complexities of certain combined operations. The table below shows the actual state complexities and their corresponding NEU-bounds of the four combined operations \cite{13}: (1) star of union, (2) star of intersection, (3) star of catenation, and (4) star of reversal.

| Operations | State Complexity | NEU-bound |
|------------|-----------------|-----------|
| \((L(A) \cup L(B))^*\) | \(2^{m+n-1} - 2^{m-1} - 2^{n-1} + 1\) | \(2^{m+n+2}\) |
| \((L(A) \cap L(B))^*\) | \(3/4 2^{mn}\) | \(2^{m+n+1}\) |
| \((L(A)L(B))^*\) | \(2^{m+n-1} + 2^{m+n-4} - 2^{m-1} - 2^{n-1} + m + 1\) | \(2^{m+n+1}\) |
| \((L(B)^R)^*\) | \(2^n\) | \(2^{n+2}\) |
The next table shows clearly that each NEU-bound in the previous table gives a very good approximation to its corresponding state complexity.

| Operations                        | Ratio bounds of the approximation |
|-----------------------------------|-----------------------------------|
| \((L(A) \cup L(B))^*\)           | \(\approx 8\)                     |
| \((L(A) \cap L(B))^*\)           | \(8/3\)                           |
| \((L(A)L(B))^*\)                 | \(4\)                             |
| \((L(B)^R)^*\)                   | \(4\)                             |

In the above cases, although the exact state complexities have been obtained, the approximation results with small ratio bounds are good enough for practical purposes, and they clearly have the advantage of being more intuitive and simpler in formulation.

4 Approximation without knowing actual state complexity

In this section, we consider two combined operations: (1) star of left quotient and (2) left quotient of star. For each of the combined operations, we do not have the exact state complexity; however, an approximation with a good ratio bound is obtained.

Let \(R\) and \(L\) be two languages over the alphabet \(\Sigma\). Then the left quotient of \(R\) by \(L\), denoted \(L \setminus R\), is the language

\[
\{y \mid xy \in R \text{ and } x \in L\}.
\]

In the following, we assume that all languages are over an alphabet of at least two letters.

4.1 The state complexity approximation of star of left quotient

**Theorem 1.** Let \(R\) be a language accepted by an \(n\)-state DFA \(M\), \(n > 0\), and \(L\) be an arbitrary language. Then there exists a DFA of at most \(2^n\) states that accepts \((L \setminus R)^*\).

**Proof:** Let \(M = (Q, \Sigma, \delta, s, F)\) be a complete DFA of \(n\) states and \(R = L(M)\). For each \(q \in Q\), denote by \(L(M_q)\) the set \(\{w \in \Sigma^* \mid \delta(s, w) = q\}\). We construct an NFA \(M'\) with multiple initial states to accept \((L \setminus R)^*\) as follows. \(M'\) is the same as \(M\) except that the initial state \(s\) of \(M\) is replaced by the set of initial states \(S = \{q \mid L(M_q) \cap L \neq \emptyset\}\) and \(\varepsilon\)-transitions are added from each final state to the states in \(S\). By using subset construction, we can construct a DFA \(A'\) of no more than \(2^n - 1\) states that is equivalent to \(M'\). Note that \(\emptyset\) is not a state of \(A'\). From the DFA \(A'\), we construct a new DFA \(A\) by just adding a new initial state that is also a final state and the transitions from this new state that are the same as the transitions from the original initial state of \(A'\). It is easy to see that \(L(A) = (L \setminus R)^*\) and \(A\) has \(2^n\) states.

This result gives an upper bound for the state complexity of the combined operation: star of left quotient.

**Theorem 2.** For any integer \(n \geq 2\), there exist a DFA \(M\) of \(n\) states and a language \(L\) such that any DFA accepting \((L \setminus L(M))^*\) needs at least \(2^{n-1} + 2^{n-2}\) states.

**Proof:** For \(n = 2\), it is clear that \(R = \{w \in \{a, b\}^* \mid \#_a(w)\ \text{is odd}\}\) is accepted by a two-state DFA, and

\[
\{\varepsilon\} \setminus R^* = \varepsilon \cup \{w \in \{a, b\}^* \mid \#_a(w) \geq 1\}
\]
cannot be accepted by a DFA with less than three states.

For \( n > 2 \), let \( M = (Q, \Sigma, \delta, s, F) \) where \( Q = \{0, 1, \ldots, n - 1\} \), \( \delta(i, a) = i + 1 \mod n \), \( i = 0, 1, \ldots, n - 1 \), \( \delta(0, a) = 0 \) and \( \delta(j, b) = j + 1 \mod n \), \( j = 1, \ldots, n - 1 \).

It has been proved in [15] that the minimal DFA accepting \( L(M)^* \) has \( 2^{n-1} + 2^{n-2} \) states. Let \( L = \{\varepsilon\} \). Then \((L\backslash L(M))^* = L(M)^*\). So, any DFA accepting \((L\backslash L(M))^*\) needs at least \( 2^{n-1} + 2^{n-2} \) states.

This result gives a lower bound for the state complexity of star of left quotient. Clearly, the lower bound does not coincide with the upper bound. We still do not know the exact state complexity for this combined operation, yet, which could be difficult to obtain. However, we can easily obtain a good state complexity approximation for the operation. Let \( 2^n \) the approximation. Then the ratio bound would be

\[
\frac{2^n}{2^{n-1} + 2^{n-2}} = \frac{4}{3}.
\]

### 4.2 The state complexity approximation of left quotient of star

Here we consider the combined operation: left quotient of star.

**Theorem 3.** Let \( R \) be a language accepted by an \( n \)-state DFA \( M \) and \( L \) an arbitrary language. Then there exists a DFA of at most \( 2^{n+1} - 1 \) states that accepts \( L \setminus R^* \).

**Proof:** Let \( M = (Q, \Sigma, \delta, s, F) \) be a complete DFA of \( n \) states and \( R = L(M) \). Then we can easily construct an \((n+1)\)-state NFA \( M' = (Q \cup \{s'\}, \Sigma, \delta', s', F \cup \{s'\})\) such that \( L(M') = R^* \) by adding a new initial state \( s' \) and transitions \( \delta'(s', \varepsilon) = s \) and \( \delta'(f, \varepsilon) = s' \) for each final state \( f \in F \). For each \( q \in Q \cup \{s'\} \), we denote by \( L(M_q) \) the set \( \{w \in \Sigma^* \mid q \in \delta'(s', w)\} \). We construct an NFA \( N \) with multiple initial states to accept \( L \setminus L(M') = L \setminus R^* \) as follows. \( N \) is the same as \( M' \) except that the initial state \( s' \) of \( M' \) is replaced by the set of initial states \( S = \{q \mid L(M_p) \cap L \neq \emptyset\} \). By using subset construction, we can verify that there exists a DFA \( A \) of no more than \( 2^{n+1} - 1 \) states that is equivalent to \( N \). Note that \( \emptyset \) is not a state of \( A \). It is easy to see that

\[
L(A) = L(N) = L \setminus L(M') = L \setminus R^*.
\]

So, \( 2^{n+1} - 1 \) is an upper bound of the state complexity of left quotient of star. \( \square \)

**Theorem 4.** For any integer \( n \geq 2 \), there exist a DFA \( M \) of \( n \) states and a language \( L \) such that any DFA accepting \( L \setminus L(M)^* \) needs at least \( 2^{n-1} + 2^{n-2} \) states.

**Proof:** For \( n = 2 \), we still use \( R = \{w \in \{a, b\}^* \mid \#_a(w) \text{ is odd}\} \) which is accepted by a two-state DFA. \( \{\varepsilon\}\setminus R^* = R^* \) cannot be accepted by a DFA with less than three states.

Again we use the same DFA \( M \) defined in the proof of Theorem 2 for any integer \( n > 2 \). As stated before, it has been proved that the minimal DFA accepting \( L(M)^* \) has \( 2^{n-1} + 2^{n-2} \) states. So any DFA accepting \( L \setminus L(M)^* \) needs at least \( 2^{n-1} + 2^{n-2} \) states. \( \square \)

For this combined operation, we choose \( 2^{n+1} \) to be an approximation of its state complexity. Then the ratio bound can be calculated easily as follows:

\[
\frac{2^{n+1}}{2^{n-1} + 2^{n-2}} = \frac{8}{3}.
\]
5 State complexity approximation of the catenation of regular languages

As we know, the state complexity of the catenation of an \( n_1 \)-state DFA language and an \( n_2 \)-state DFA language, \( n_1 \geq 1 \) and \( n_2 \geq 2 \), is \( n_1 2^{n_2} - 2^{n_2-1} \) \([15]\). The state complexity of multiple catenations has been studied in \([3]\) and the following estimate was obtained.

Claim 1. Let \( R_1, \ldots, R_k \), \( k \geq 2 \), be regular languages accepted by DFAs of \( n_1, \ldots, n_k \) states, respectively. Then the state complexity of \( R_1 \cdots R_k \) is no more than

\[
n_1 2^{n_2 + \cdots + n_k} - 2^{n_2 + \cdots + n_{k-1}} - 2^{n_3 + \cdots + n_{k-1}} - \cdots - 2^{n_k} - 1.
\]

The exact state complexity of the catenations of three and four regular languages was also obtained in \([3]\). In this section, we prove the exact state complexities of the catenation of \( k \) regular languages for arbitrary \( k \geq 2 \). Note that this is not a state complexity in the normal definition that is for only one specific (combined) operation. This is a state complexity (formula) for a class of (combined) operations.

After we prove this state complexity, we show an approximation of the complexity and state why the approximation is useful in this case.

We first consider a lower bound.

Theorem 5. For any integers \( n_i \geq 2 \), \( 1 \leq i \leq k \), there exist DFA \( A_i \) of \( n_i \) states, respectively, such that any DFA accepting \( L(A_1) \cdots L(A_k) \) needs at least

\[
n_1 2^{n_2 + \cdots + n_k} - D - \sum_{i=1}^{k-1} E_i
\]

states, where

\[
D = n_1(2^{n_3 + \cdots + n_k} - 1) + n_1(2^{n_2} - 1)(2^{n_3 + \cdots + n_k} - 1) + \cdots + n_1(2^{n_2} - 1)(2^{n_k - 1})(2^{n_k} - 1);
\]

\[
E_1 = 1 + (2^{n_2 - 1} - 1)(1 + (2^{n_3} - 1)(1 + (2^{n_4} - 1) \cdots (1 + (2^{n_k - 1} - 1)(2^{n_k}) \cdots));
\]

\[
E_2 = (n_1 - 1)2^{n_2 - 1}(1 + (2^{n_3 - 1} - 1)(1 + (2^{n_4} - 1) \cdots (1 + (2^{n_k - 1} - 1)(2^{n_k}) \cdots)) + 2^{n_2 - 2}(1 + (2^{n_3 - 1} - 1)(1 + (2^{n_4} - 1) \cdots (1 + (2^{n_k - 1} - 1)(2^{n_k}) \cdots));
\]

\[
\vdots
\]

\[
E_i = (n_1 - 1)(2^{n_2 - 1} - 1) \cdots 2^{n_{i-1} - 1}(1 + (2^{n_{i+1} - 1} - 1)(1 + (2^{n_{i+2} - 1} \cdots (1 + (2^{n_k - 1} - 1)(2^{n_k}) \cdots)) \cdots + 2^{n_{i-2} - 2} \cdots 2^{n_{i-1} - 2}(1 + (2^{n_{i+1} - 1} - 1)(1 + (2^{n_{i+2} - 1} \cdots (1 + (2^{n_k - 1} - 1)(2^{n_k}) \cdots)).
\]

Proof: Let \( \Sigma = \{a_j \mid 1 \leq j \leq 2k - 1\} \). Define a DFA \( A_1 = (Q_1, \Sigma, \delta_1, 0, F_1) \), where

\[
Q_1 = \{0, 1, \ldots, n_1 - 1\};
\]

\[
F_1 = \{n_1 - 1\};
\]

\[
\delta_1(t, a_1) = t + 1 \mod n_1, \ 0 \leq t \leq n_1 - 1;
\]

\[
\delta_1(t, a_{2k-2}) = 0, \ 0 \leq t \leq n_1 - 1;
\]

\[
\delta_1(t, b) = t, \ b \in \Sigma - \{a_1, a_{2k-2}\}, \ 0 \leq t \leq n_1 - 1.
\]

Let DFA \( A_i = (Q_i, \Sigma, \delta_i, 0, F_i) \), \( 2 \leq i \leq k \), where

\[
Q_i = \{0, 1, \ldots, n_i - 1\};
\]

\[
F_i = \{n_i - 1\};
\]

\[
\delta_i(t, a_{2i-2}) = t + 1 \mod n_i, \ 0 \leq t \leq n_i - 1;
\]

\[
\delta_i(t, a_{2i-1}) = 1, \ 0 \leq t \leq n_i - 1;
\]

\[
\delta_i(t, b) = t, \ b \in \Sigma - \{a_{2i-2}, a_{2i-1}\}, \ 0 \leq t \leq n_i - 1.
\]
For each \( x \in \{a_1, a_2, a_4, \ldots, a_{2k-2}\}^* \) and \( 2 \leq s \leq k \), we define
\[
P_s(x) = \{ p | x = u_1u_2 \ldots u_s, u_l \in L(A_l), 1 \leq l \leq s - 1, \text{ and } p = \#_{a_{2s-2}}(u_s) \mod n_s \}.
\]

Consider that \( x, y \in \{a_1, a_2, a_4, \ldots, a_{2k-2}\}^* \) such that \( P_s(x) \neq P_s(y) \). Let \( c \in P_s(x) - P_s(y) \) (or \( P_s(y) - P_s(x) \)) and \( w = a_{2s+1}a_{2s+1}a_{2s+1} \ldots a_{2k-1}a_{2k-1} \). Then it is clear that \( xw \in L(A_1) \ldots L(A_k) \) but \( yw \notin L(A_1) \ldots L(A_k) \). So, \( x \) and \( y \) are in different equivalence classes of the right-invariant relation induced by \( L(A_1) \ldots L(A_k) \).

For each \( x \in \{a_1, a_2, a_4, \ldots, a_{2k-2}\}^* \), define
\[
P_1(x) = \#_{a_1}(z) \text{ where } x = ydz, \ y \in \{a_1, a_2, a_4, \ldots, a_{2k-2}\}^*, \ z \in \{a_1, a_2, a_4, \ldots, a_{2k-4}\}^*, \text{ if } a_{2k-2} \text{ occurs in } x;
\]
\[
P_1(x) = \#_{a_1}(x), \text{ otherwise.}
\]

Consider \( u, v \in \{a_1, a_2, a_4, \ldots, a_{2k-2}\}^* \) such that \( P_1(u) \mod n_1 > P_1(v) \mod n_1 \).

Let \( t = P_1(u) \mod n_1 \) and \( w = a_{1}^{n_1-1}a_3a_{2}^{n_2-1} \ldots a_{k-1}a_{2k-2}^{n_k-1} \). Then clearly \( uw \in L(A_1) \ldots L(A_k) \) but \( vw \notin L(A_1) \ldots L(A_k) \).

Notice that there does not exist a word \( w \) such that \( 0 \notin P_2(w) \) and \( P_1(w) = n_1 - 1 \), since \( P_1(w) = n_1 - 1 \) guarantees that \( 0 \in P_2(w) \). Because of the same reason, there does not exist a word \( w \) such that \( n_1 - 1 \in P_1(w) \) and \( 0 \notin P_1(w), \ 2 \leq t \leq k - 1. \) It is also impossible that \( P_1(w) = \emptyset \) but \( P_{k+1}(w) \neq \emptyset \).

For each subset \( s = \{d_1, s, \ldots, d_{e_{s}}, s\} \) of \( \{0, \ldots, n_s \} - \{1\} \) where \( d_1, s > \cdots > d_{e_{s}}, s \) and \( 2 \leq s \leq k \), and an integer \( p_1 \in \{0, \ldots, n_1 - 1\} \), except the cases we mentioned above, there exists a word
\[
x = a_1^{n_1}a_2^{n_2}a_3^{n_3} \cdots a_{k-1}^{n_{k-1}}a_{2k-2}^{n_{2k-2}}a_{1}^{n_1}a_2^{n_2}a_3^{n_3} \cdots a_{2k-2}^{n_{2k-2}} \cdots
\]
\[
a_1^{n_1}a_2^{n_2}a_3^{n_3} \cdots a_{k-1}^{n_{k-1}}a_{2k-2}^{n_{2k-2}}a_1^{n_1}a_2^{n_2}a_3^{n_3} \cdots a_{2k-2}^{n_{2k-2}} \cdots
\]
\[
a_1^{n_1}a_2^{n_2}a_3^{n_3} \cdots a_{k-1}^{n_{k-1}}a_{2k-2}^{n_{2k-2}}a_1^{n_1}a_2^{n_2}a_3^{n_3} \cdots a_{2k-2}^{n_{2k-2}}a_1^{n_1}a_2^{n_2} \cdots a_1^{n_1}a_2^{n_2}a_3^{n_3} \cdots a_{2k-2}^{n_{2k-2}} \cdots
\]
\[
a_1^{n_1}a_2^{n_2}a_3^{n_3} \cdots a_{k-1}^{n_{k-1}}a_{2k-2}^{n_{2k-2}} \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots
\]

such that \( P_1(x) = p_1 \) and \( P_s(x) = p_s \).

In total, there are \( n_12^{n_2}2^{n_3} \cdots 2^{n_k} \) classes. There are
\[
D = n_1(2^{n_2} + \cdots + 2^{n_k}) - 2^{n_2}2^{n_3} \cdots 2^{n_k} \]

classes with both \( p_t = \emptyset \) and \( p_{k+1} \neq \emptyset, 2 \leq t \leq k - 1. \) There are
\[
E_1 = (1 + 2^{n_2} + 1)(1 + 2^{n_3} - 1)(1 + 2^{n_4} - 1) \cdots (1 + 2^{n_k} - 1)(1 + 2^{n_k} \cdots 2^{n_k})
\]

classes with both \( p_1 = n_1 - 1 \) and \( p_2 \notin p_2. \) There are
\[
E_2 = (n_1 - 1)^2(1 + 2^{n_2} - 1)(1 + 2^{n_4} - 1) \cdots (1 + 2^{n_k} - 1)(1 + 2^{n_k} - 1)(2^{n_k})
\]

classes with both \( n_2 - 1 \in p_2 \) and \( 0 \notin p_3, \) which are not in \( E_1. \) We omit the other similar classes until the \( i \)th group of classes. There are
\[
E_i = (n_1 - 1)(2^{n_2} - 1) \cdots 2^{n_i} \cdots 1 + 2^{n_i} - 1)(1 + 2^{n_i} - 1) \cdots (1 + 2^{n_i} - 1)(2^{n_k}) \cdots
\]

classes with both \( n_i - 1 \in p_i \) and \( 0 \notin p_{i+1}, \) which are not in \( E_1, E_2, \ldots, E_{i-1}. \)

Thus, there are at least \( n_12^{n_2}+\cdots+2^{n_k} - D - E_1 - E_2 - \cdots - E_{k-1} \) distinct equivalence classes.
Theorem 6. Let $A_i$, $1 \leq i \leq k$ be $k$ DFAs of $n_i$, respectively, where $A_i$ has $f_i$ final states, $0 < f_i < n_i$. Then there exists a DFA of

$$n_12^{n_2+\cdots+n_k} - D - \sum_{i=1}^{k-1} E_i$$

states that accepts $L(A_1)\cdots L(A_k)$, where

$$D = n_1(2^{n_2+\cdots+n_k} - 1) + n_2(2^{n_2} - 1)(2^{n_2+\cdots+n_k} - 1) + \cdots + n_k(2^{n_k} - 1)(2^{n_2+\cdots+n_k} - 1);$$

$$E_1 = f_1(1 + 2^{n_2-1} - 1)(1 + 2^{n_3-1} - 1)(1 + 2^{n_4-1} - 1)(1 + 2^{n_k-1} - 1);$$

$$E_2 = n_1(1 - f_1)(2^{n_2-1} - 1)(1 + 2^{n_3-1} - 1)(1 + 2^{n_4-1} - 1)(1 + 2^{n_k-1} - 1);$$

\[\vdots\]

$$E_i = n_1(1 - f_1)(2^{n_2-i} - 1)(1 + 2^{n_3-i} - 1)(1 + 2^{n_4-i} - 1)(1 + 2^{n_k-i} - 1);$$

\[\vdots\]

$$E_{k-1} = n_1(1 - f_1)(2^{n_2-k} - 1)(1 + 2^{n_3-k} - 1)(1 + 2^{n_4-k} - 1)(1 + 2^{n_k-k} - 1);$$

Proof: Let DFA $A_i = (Q_i, \Sigma, \delta_i, 0, F_i)$, $1 \leq i \leq k$.

Construct $E = (Q_E, \Sigma, \delta_E, q_0, F_E)$ such that

$$Q_E = Q_1 \times 2^{Q_2} \times 2^{Q_3} \times \cdots \times 2^{Q_k} - D' - \sum_{i=1}^{k-1} E'_i;$$

$$q_0 = \begin{cases} (0, \emptyset, \ldots, \emptyset), & \text{if } 0 \notin F_i, 1 \leq i \leq k; \\ (0, \{0\}, \ldots, \emptyset), & \text{if } 0 \in F_i \text{ and } 0 \notin F_i, 2 \leq i \leq k; \\ \vdots & \\ (0, \{0\}, \ldots, \{0\}), & \text{if } 0 \in F_i, 1 \leq i \leq k - 1; \end{cases}$$

$$F_E = \{ (u_1, u_2, \ldots, u_k) \in Q_E \mid u_k \cap F_k \neq \emptyset \};$$

$$\delta_E : \delta_E((u_1, u_2, \ldots, u_k), a) = (u'_1, u'_2, \ldots, u'_k),$$

for $a \in \Sigma$, where

$$u'_1 = \delta_{A_1}(u_1, a),$$

$$u'_2 = \begin{cases} \delta_{A_2}(u_2, a) \cup \{0\}, & \text{if } u'_1 \in F_1, \\ \delta_{A_2}(u_2, a), & \text{otherwise}, \end{cases}$$

$$u'_i = \begin{cases} \delta_{A_i}(u_i, a) \cup \{0\}, & \text{if } u'_{i-1} \cap F_{i-1} \neq \emptyset, \\ \delta_{A_i}(u_i, a), & \text{otherwise}, \end{cases}$$

for $3 \leq i \leq k$,

where

$$D' = Q_1 \times \{0\} \times (2^{Q_2} - \{0\}) \times \cdots \times (2^{Q_k} - \{0\}) + Q_1 \times (2^{Q_2} - \{0\}) \times (2^{Q_3} - \{0\}) \times \cdots \times (2^{Q_k} - \{0\}) + \cdots + Q_1 \times (2^{Q_2} - \{0\}) \times \cdots \times (2^{Q_k} - \{0\}) \times (2^{Q_k} - \{0\});$$

$$E'_1 = F_1 \times (\emptyset)^{k-1} \cup (2^{Q_2} - \{0\}) \times (\emptyset)^{k-2} \cup (2^{Q_3} - \{0\}) \times \cdots \times (\emptyset)^{k-3} \cup (2^{Q_4} - \{0\}) \times \cdots \times (\emptyset)^{k-4} \cup (2^{Q_5} - \{0\}) \times \cdots \times (\emptyset)^{k-5} \cup (2^{Q_6} - \{0\}) \times \cdots \times (\emptyset)^{k-6} \cup (2^{Q_7} - \{0\}) \times \cdots \times (\emptyset)^{k-7} \cup (2^{Q_8} - \{0\}) \times \cdots \times (\emptyset)^{k-8} \cup (2^{Q_9} - \{0\}) \times \cdots \times (\emptyset)^{k-9} \cup (2^{Q_{10}} - \{0\}) \times \cdots \times (\emptyset)^{k-10} \cup (2^{Q_{11}} - \{0\}) \times \cdots \times (\emptyset)^{k-11} \cup (2^{Q_{12}} - \{0\}) \times \cdots \times (\emptyset)^{k-12} \cup (2^{Q_{13}} - \{0\}) \times \cdots \times (\emptyset)^{k-13} \cup \cdots \cup (2^{Q_{k-1}} - \{0\}) \times \emptyset + F_1 \times (\emptyset)^{k-1} \cup (2^{Q_2} - \{0\}) \times (\emptyset)^{k-2} \cup (2^{Q_3} - \{0\}) \times \cdots \times (\emptyset)^{k-3} \cup (2^{Q_4} - \{0\}) \times \cdots \times (\emptyset)^{k-4} \cup (2^{Q_5} - \{0\}) \times \cdots \times (\emptyset)^{k-5} \cup (2^{Q_6} - \{0\}) \times \cdots \times (\emptyset)^{k-6} \cup (2^{Q_7} - \{0\}) \times \cdots \times (\emptyset)^{k-7} \cup (2^{Q_8} - \{0\}) \times \cdots \times (\emptyset)^{k-8} \cup (2^{Q_9} - \{0\}) \times \cdots \times (\emptyset)^{k-9} \cup (2^{Q_{10}} - \{0\}) \times \cdots \times (\emptyset)^{k-10} \cup (2^{Q_{11}} - \{0\}) \times \cdots \times (\emptyset)^{k-11} \cup (2^{Q_{12}} - \{0\}) \times \cdots \times (\emptyset)^{k-12} \cup (2^{Q_{13}} - \{0\}) \times \cdots \times (\emptyset)^{k-13} \cup \cdots \cup (2^{Q_{k-1}} - \{0\}) \times \emptyset;$$

\[\vdots\]

$$E'_{k-1} = (Q_1 - F_1) \times (2^{Q_2} - F_1) \times \cdots \times (2^{Q_k} - F_1) \times \emptyset + \cdots + F_1 \times (\emptyset)^{k-1} \cup (2^{Q_2} - \{0\}) \times (\emptyset)^{k-2} \cup (2^{Q_3} - \{0\}) \times \cdots \times (\emptyset)^{k-3} \cup (2^{Q_4} - \{0\}) \times \cdots \times (\emptyset)^{k-4} \cup (2^{Q_5} - \{0\}) \times \cdots \times (\emptyset)^{k-5} \cup (2^{Q_6} - \{0\}) \times \cdots \times (\emptyset)^{k-6} \cup (2^{Q_7} - \{0\}) \times \cdots \times (\emptyset)^{k-7} \cup (2^{Q_8} - \{0\}) \times \cdots \times (\emptyset)^{k-8} \cup (2^{Q_9} - \{0\}) \times \cdots \times (\emptyset)^{k-9} \cup (2^{Q_{10}} - \{0\}) \times \cdots \times (\emptyset)^{k-10} \cup (2^{Q_{11}} - \{0\}) \times \cdots \times (\emptyset)^{k-11} \cup (2^{Q_{12}} - \{0\}) \times \cdots \times (\emptyset)^{k-12} \cup (2^{Q_{13}} - \{0\}) \times \cdots \times (\emptyset)^{k-13} \cup \cdots \cup (2^{Q_{k-1}} - \{0\}) \times \emptyset;$$

\[\vdots\]
Intuitively, $Q_E$ is a set of $k$-tuples whose first component is a state in $Q_1$ and the $i$th component is a subset of states in $Q_i$, $2 \leq i \leq k$.

$Q_E$ does not contain those $k$-tuples whose $i$th component is $\emptyset$ and whose $j$th component is not $\emptyset$, when $1 < i < j \leq k$. $D'$ is the set of them.

$Q_E$ does not contain those $k$-tuples whose first component is an element of $F_1$ and whose second component is not $\emptyset$ (if it is $\emptyset$ then all the elements afterward have to be $\emptyset$) and does not contain 0, either. $E'_1$ is the set of them.

$Q_E$ does not contain those $k$-tuples whose $i$th component contains one or more final states of DFA $A_i$ and whose $(i + 1)$th component is not $\emptyset$ (if it is $\emptyset$ then all the elements afterward have to be $\emptyset$) and does not contain 0, when $2 \leq i \leq k - 1$, either. $E'_i$ is the set of them.

Clearly, $L(E) = L(A_1) \cdots L(A_k)$. Let $|Q_{A_i}| = n_i$ and $|F_{A_i}| = f_i$, $1 \leq i \leq k$.

Then $E$ has $n_12^{n_2 + \cdots + n_k} - D - E_1 - E_2 - \cdots - E_{k-1}$ states. $\square$

Note that when each $A_i$, $1 \leq i \leq k$, has one final state, this upper bound is exactly the same as the lower bound stated in Theorem 5. Thus, this bound is tight and is the state complexity of the catenation of $k$ regular languages.

Although we have proved that this state complexity is tight, it is too long and complex to be intuitive and comprehensible. Let $SC_{CAT}(n_1, \ldots, n_k)$ denote the state complexity of catenation of $k$ languages accepted by $n_1$-state, $\ldots$, $n_k$-state DFAs, respectively, $n_1, \ldots, n_k \geq 2$. By observing the structure of the result, we can see that $n_12^{n_2 + \cdots + n_k}$ is a good approximation with the ratio bound

$$\frac{n_12^{n_2 + \cdots + n_k}}{SC_{CAT}(n_1, \ldots, n_k)} < 4.$$  

However, all our experiments show that the ratio bound for this approximation is less than 3, but we have not been able to prove it.

6 Conclusion

The new concept of state complexity approximation is introduced. It further advances the idea of state complexity estimation by including the ratio bound. The ratio bound gives a precise and intuitive measurement on the “quality” of the estimation.

We show that state complexity approximation can play useful roles in two different cases. In the first case, the exact state complexities have not been obtained. They may be very difficult to obtain. However, approximation results with low ratio bounds can be obtained rather easily and they are good enough for practical purposes in general. In the second case, the exact state complexities have been proved. The approximations of those results with low ratio bounds can simplify the formulae of the complexities and make them more intuitive and easier to apply.

Clearly, the state complexity approximation is a useful and important concept. We expect many new results on state complexity approximation will come out in the near future.

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