Bose–Hubbard models with photon pairing in circuit-QED

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Abstract
In this work, we study a family of bosonic lattice models that combine an on-site repulsion term with a nearest-neighbour pairing term, \( \sum_i a_i^\dagger a_i^\dagger + H.c. \). Like the original Bose–Hubbard model, the nearest-neighbour term is responsible for the mobility of bosons and it competes with the local interaction, inducing two-mode squeezing. However, unlike a trivial hopping, the counter-rotating terms form a pairing that cannot be studied with a simple mean-field theory and does not present a quantum phase transition in phase space. Instead, we show that there is a di-over from a pure insulator to long-range correlations that start up as soon as the two-mode squeezing is switched on. We also show how this model can be naturally implemented using coupled microwave resonators and superconducting qubits.

(Some figures may appear in colour only in the online journal)

1. Introduction

The physics of lattice bosons is an interesting topic that has experienced several revivals associated with the proposal and realization of new experimental setups. These include, for instance, the study of phase transitions in Josephson-junction arrays [1, 2], the ground-breaking studies of quantum phase transitions with bosonic atoms in optical lattices [3] and, more recently, the proposals for polariton physics in coupled cavities and photonic systems [4–6]. The previous represents a curious round trip where a very interesting physical model is introduced in the context of condensed matter physics; we learn how to tame it and control it with the tools of quantum simulation, and the final and most promising platform for studying its physics seems to be, once more, a superconducting setup [7, 8].

The use of superconducting quantum circuit setups to implement bosonic lattice models has various advantages over atomic implementations, but the most important one is the access to a wider variety of interactions. Using photons as particles, we can easily engineer nonlinearities [4–6], but we also have access to controlled dissipation [9–11], easily customizable geometries [8] and effective dispersion relations [12] and, most importantly, the possibility of increasing the coupling to a point at which counter-rotating interactions become relevant [13, 14].

In this work, we explore the possibility of engineering exotic interactions in circuit-QED, studying models with photons that do not preserve the number of particles. Our study is centred around a model of coupled cavities with pairing and on-site interaction (\( \hbar = 1 \)):

\[
H = \frac{g}{2} \sum_i (e^{i\phi} a_i^\dagger a_{i+1}^\dagger + H.c.) + U \sum_i a_i^\dagger a_i a_i^\dagger a_i^\dagger + \sum_i \omega_i a_i^\dagger a_i. \tag{1}
\]

In contrast with the original Bose–Hubbard model (BHM)

\[
H = -\frac{t}{2} \sum_i (e^{i\phi} a_i^\dagger a_{i+1}^\dagger + H.c.) + U \sum_i a_i^\dagger a_i a_i^\dagger a_i^\dagger, \tag{2}
\]

our problem facilitates the mobility of bosons through a pairing term, \( a_i^\dagger a_{i+1}^\dagger \), that does not preserve the number of particles. We will show that the competition between pairing, \( g \), and on-site repulsion, \( U \), does not have a phase transition associated with it. Instead, for any small value of \( g \), we find a cross-over mechanism that establishes long-range entanglement and squeezing in the lattice, through a process that cannot be described by a trivial mean-field theory.

Our work is structured in three sections. In section 2, we propose a quantum simulation of equation (1) using an array of superconducting microwave resonators which are coupled through periodically driven SQUIDs [15]. We will consider important issues, such as state preparation and tomography of...
the correlations that we estimate in the next two sections. Here, we study this model in two different regimes. In section 3, we adopt the regime $U = 0$ of non-interacting bosons and derive the squeezing and entanglement properties of the resulting array of linear cavities. We will show that the ground state in momentum space is a collection of paired photons with different momenta. The paired momenta can be controlled using the phase $\phi_i$ in equation (1) and the whole system approaches criticality when $g \approx \omega$. The competition between this long-range entanglement and the local interaction is the topic of section 4. There, we will add a nonzero value of $U$ and study the problem using the methods of infinite matrix product states or iTEBD [16]. We will show that now $U$ prevents a breakdown of the model for large $g$ and that it also suppresses entanglement. However, unlike in the BHM, the transition from the insulator regime to the squeezed multimode system does not happen through a phase transition but through a cross-over. Finally, in section 5, we summarize our results and discuss possible extensions of this work.

2. Quantum simulation

The model that we have studied so far (1) is not likely to appear in nature: the counter-rotating terms normally appear together with rotating terms, in such a way that they dominate the dynamics. However, the fact that certain interactions are not commonly available does not mean that we cannot study them in a physical setup. Instead, we can find a highly tunable quantum mechanical system and implement such interactions by controlling its dynamics, in what is known as a ‘quantum simulation’ (see [17] and accompanying articles). In essence, this is the spirit behind all recent proposals about coupled cavities and polariton-type physics [4–6], and it will be the method we suggest for the implementation of those models.

For the simulation of our coupled-cavity model, we suggest using a one-dimensional setup of coupled microwave resonators. As shown in figure 1, such a setup consists basically of a number of superconducting segments (coplanar waveguides or striplines) joined by a nonlinear element called SQUID. Each piece of cable supports a number of quantized standing waves, of which we will only focus on the fundamental mode, $\omega_0 a_i^\dagger a_i$, with a frequency that will change from cavity to cavity. As shown elsewhere [15], by joining the cables through SQUIDs, we actually implement a tunable coupling element

$$H = \sum_i \omega_0 a_i^\dagger a_i + \sum_i g[\Phi_i(t)](a_i^\dagger + a_i)(a_{i+1}^\dagger + a_{i+1}),$$

which can be controlled using the magnetic flux $\Phi$ that runs through the Josephson-junction loop.

To activate the counter-rotating (rotating) terms in the previous equation, we only need to drive the flux on each cavity periodically [15],

$$\Phi_i(t) \approx a_i + h_i \cos(\nu t + \phi_i),$$

with arbitrary $a_i$, $h_i$ and phase. If the magnetic flux $\Phi$ is driven with $\nu_i$ being on resonance with the counter-rotating terms, the propagation of the rotating terms may be averaged out and vice versa. For this, we only need to fabricate cavities with alternating frequencies

$$\omega_i = \omega_0 + (-1)^i \delta \omega/2,$$

and choose a driving $\nu_i = 2\omega_0/\delta \omega$ (alternatively choose $\nu_i = \delta \omega(-1)^i$ in order to activate rotating terms).

Note also that in addition to the resonance, we can also control the phase. The time origin of the driving propagates through a unitary transformation to the effective model that results, creating the full pairing term from equation (1).

The last two ingredients in our system are the interaction and the possible external driving. The latter is easy to achieve, because the inductive or capacitive coupling between a cavity and a nearby cable will introduce the possibility of driving photons in and out of the system. The nonlinearity is a bit trickier, as it involves a nontrivial photon–photon interaction inside the cavity. We expect that such terms could be ported from existing proposals based on nonlinear resonators [18], but a more interesting approach is that of polariton physics [4–6], where the interaction with a nearby qubit can provide both attractive and repulsive nonlinearities.

The measurement of properties in this setup is also an interesting topic. By nature, the circuit-QED setups tend to be rather closed, as any additional cable or probe may be regarded as a source of decoherence. However, following [19], we envision an alternative that consists of a single transmission line running parallel to our setup of coupled cavities. Ensuring a weak coupling between both systems, we can watch in real time the out-coupling of photons from the cavities into the line,
thereby probing the frequency- and momentum-dependent correlations that we will discuss in the next section (cf. figure 2). In addition to this, it would be interesting to probe the transport properties of these models, to see whether the pairing may assist in the mobility of photons, transmitting correlations through the chain, and probe the lack of insulating phases.

3. Linear cavities

In this section, we will study a uniform model of coupled cavities ($\omega_0 = \omega$) without on-site interactions ($U = 0$),

$$H_0 = \frac{g}{2} \sum_i (e^{i\phi} a_i^\dagger a_{i+1} + \text{H.c.}) + \omega \sum_i a_i^\dagger a_i. \quad (6)$$

Just as in the BHM, our aim is to find the state of the bosons in the limit in which they are free particles. Since this is a quadratic problem in Fock operators, we will use the formalism of Gaussian states, analysing the entanglement that arises in the system which, as we will see, is not condensed but forms a multimode squeezed state.

In section 3.1, we will first Fourier transform $H_0$ in order to rewrite it in momentum space, where we will see that it can be expressed as the direct sum of two-mode Hamiltonians. Then, in section 3.2, we will be able to diagonalize it and study its stability conditions. In section 3.3, we will analyse the effect of adding a phase to each site that grows linearly with the lattice. In section 3.4, we will study entanglement between photonic modes, i.e. between modes in momentum space.

3.1. The reciprocal lattice

For simplicity, we will assume that the model in equation (6) describes a one-dimensional lattice with periodic boundary conditions and assume $\phi_i = 0$. Due to the translational invariance of the problem, it makes sense to rewrite the same model in Fourier space. We perform the transformation

$$a_i = \frac{1}{\sqrt{N}} \sum_{k \in I} e^{-i q_k i} b_{q_k}. \quad (7)$$

Here the lattice sites run over the indices $i = 0, \ldots, N - 1$, and $q_k = 2\pi k/N \in (-\pi, \pi]$ is the quasimomentum, and we choose the quasimomentum index to run over $I = \{ -\frac{N + 1}{2}, \ldots, \frac{N + 1}{2} \}$ if $N$ is even or over $I = \{ -\frac{N + 1}{2}, \ldots, \frac{N - 1}{2} \}$ if it is odd, forming a symmetric Brillouin zone. This unitary transformation transforms $H_0$ to a sum of Hamiltonians $H_0 = \sum_{q} H_0^{(q)}$ with

$$H_0^{(q)} = \omega (n_q + n_{-q}) + g \cos(q) (b_{q}b_{-q} + \text{H.c.}). \quad (8)$$

Note how, except for $q_k = 0$ and $q_k = \pi$ (if $N$ is even), $H_0^{(q)}$ represents the usual two-mode squeezing interaction between bosonic modes. In the following, we will analyse the entanglement that results from this coupling, neglecting the point-like singularities at $q_k = 0, \pi$, which do not add much to the physics.

3.2. Two-mode squeezing

We diagonalize the momentum Hamiltonian (8) using the squeezing transformation $[20]$

$$S_q(\xi_q) = \exp \left[ \frac{i}{2} \left( \xi_q^* b_q b_{-q} - \xi_q b_q^\dagger b_{-q}^\dagger \right) \right], \quad \xi_q = r_q e^{i\phi_q}, \quad (9)$$

obtaining an operator

$$\tilde{H}_0^{(q)} = S_q^\dagger(\xi_q) H_0^{(q)} S_q(\xi_q) = \epsilon_q (b_q^\dagger b_q + b_{-q}^\dagger b_{-q}) + \epsilon_q - \omega. \quad (10)$$

The dispersion relation of this problem is

$$\epsilon_q = \sqrt{\omega^2 - g^2 \cos(q)^2}. \quad (11)$$

In order to warrant the stability of the setup, we must constrain the photon–photon interaction to be $|g| \leq \omega$, because otherwise there would be no vacuum for this problem. Doing so, the squeezing parameter is defined as the oscillating function

$$\tanh r_q = \frac{|g \cos(q)|}{\omega}, \quad (12)$$

which is maximum around $q = 0$ and $\pm \pi$. Since we have chosen $\phi_q = 0$, the parameter $\xi_q$ is real, and the phase $\varphi_q$ just absorbs its sign.
3.3. Momentum boosts

In the full model, we considered the possibility of having a site-to-site phase difference, $\phi_i$. If this phase is constant, it may be eliminated through a global phase transformation of the bosonic operators. Instead, we will consider it to grow linearly through the lattice. Since we wish to preserve translational invariance and periodic boundary conditions, we find that the dependence of the phase must be

$$\phi_i = \pi \frac{d}{N} (2i + 1), \quad d, i \in [0, \ldots, N - 1].$$

We can engineer a unitary gauge transformation that absorbs this phase:

$$a_i = e^{i \pi d/N} \tilde{a}_i. \quad (14)$$

The original Hamiltonian $H_0$ is expressed now in terms of $[\tilde{a}_i, \tilde{a}^\dagger_i]$, but with $\phi = 0$. The difference is that the quasimomentum operators for this transformed Hamiltonian,

$$\tilde{b}_q = \frac{1}{\sqrt{N}} \sum_i e^{i m q} \tilde{a}_i = \frac{1}{\sqrt{N}} \sum_i e^{i (q - \pi d/N) a} a_i, \quad (15)$$

correspond to the quasimomenta $q_k = -\pi d/N$ from the original model. In other words, the phase $\phi = 2\pi d/N$ corresponds to a momentum boost of amount $-\pi d/N$. Note that, to achieve a similar effect in the BHM, we needed a Peierls transformation with uniform phase.

3.4. Entangled photon pairs

Since the Hamiltonian splits into a sum of commuting terms, each of them acting on a separate pair of momenta, we may regard the total density matrix as a tensor product $\rho = \otimes \rho_q$ of Gaussian states, $\rho_q$, for each of the momenta. Each of these density matrices is best characterized by the covariance matrix (CM).

The CM is written in terms of first- and second-order statistical moments using the field quadrature operators $x_q = (b_q + b^\dagger_q)/\sqrt{2}$ and $p_q = i(b^\dagger_q - b_q)/\sqrt{2}$. Defining $Q^{(q)} = (x_q, p_q, x_{-q}, p_{-q})^T$, the CM is the $4 \times 4$ matrix that results from the expectation values $C_{nm} = \frac{1}{2} \langle Q_n Q_m + Q_m Q_n \rangle - \langle Q_n \rangle \langle Q_m \rangle$. In complete agreement with any quantum characterization of two-mode Gaussian states [21], we find a structure of $2 \times 2$ blocks

$$C = \begin{pmatrix} \alpha & \gamma \\ \gamma^T & \beta \end{pmatrix}, \quad (16)$$

where $\alpha = \beta = \frac{1}{2} \cosh q_r \cdot 1_2$ and $\gamma = \frac{1}{2} \sinh q_r \cdot 1_2$. Out of these terms, only $\gamma$ encodes correlations between modes.

According to the Peres–Horodecki–Simon criterion [22–24], the positivity of the partial transpose $C^{Tr} = \Lambda^T C \Lambda$, $\Lambda = \text{diag}(1, 1, 1, -1)$ represents a sufficient condition for the separability of the two-mode state. The entanglement monotone associated with this criterion is the logarithmic negativity [21, 25],

$$E_N = -\frac{1}{2} \sum_i^4 \log_2 \left[ \min[1, 2|v_i|] \right]. \quad (17)$$

where $v_i \in \{-v_\pm, +v_\pm\}$ are the four symplectic eigenvalues of the CM

$$v_\pm = \frac{1}{2} \frac{\Delta_1 \pm \sqrt{\Delta_1^2 - 4 \Delta_2}}{2}, \quad (18)$$

where we substituted the invariants $\Delta_1 = \det \alpha + \det \beta - 2 \det \gamma$ and $\Delta_2 = \det \mathcal{C}$ for their values in this particular problem.

The immediate consequence is that the negativity is nonzero for any amount of two-mode squeezing

$$E_{N,q} = \frac{1}{2} \log_2 \left( \frac{1 + |g \cos(q)/\omega|}{1 - |g \cos(q)/\omega|} \right). \quad (19)$$

In other words, as was intuitively expected, the state is not separable and the entanglement grows with increasing coupling strength, $g$.

In figure 2, we show a typical distribution of the entanglement on the first Brillouin zone, both without and with the phase $\phi$. In both cases, we have pair entanglement between different momenta, which is maximal around two points. The maxima are displaced with the help of the momentum boost $\phi$. Using $\phi$, we can switch from entangling photons that travel in opposite directions but carry the same energy, to entangling co-propagating photons that differ in frequency, as is the case of $\phi = 0$ in figure 2(b). Note also the similarity between the shape of the entanglement and that of the squeezing (cf equation (12)).

4. Pair-hopping tunnelling

So far, we have seen that in the linear case, the counter-rotating terms create entanglement and long-range correlations. We want to analyse how these correlations compete with an on-site repulsion such as the $U$ term from our full model in equation (1).

We already know from the physics of the BHM (cf equation (2)) a similar competition [26]: a kinetic term which preserves the number of particles fights for their delocalization against an on-site repulsion that hinders mobility. The result is a quantum phase transition between a superfluid regime, for $t \gg U$, and an insulator regime, $U \gg t$, which has been repeatedly observed using bosonic atoms in optical lattices [3].

In the model from equation (1), the kinetic term is implemented with a pairing term that, by allowing the creation and destruction of pairs of particles in nearby sites, also allows the mobility of particles. This is obvious from the previous results, the energy bands and the delocalized wavefunctions that result. It is obvious that such a term would compete with the on-site repulsion $U$, but it is not clear under which constraints this competition happens.

A trivial inspection of equation (1) reveals that, should we try a similar mean-field treatment as for the Bose–Hubbard, that is replacing $a$ with a nonzero expectation value, $\langle a \rangle$, we would obtain the same phase diagram as for the original model. More precisely, assuming a product state ansatz $|\psi\rangle = \otimes_{i=1}^N |\phi\rangle$, the energy per site reads

$$\frac{1}{N} \langle H \rangle_{\omega = 0} = \frac{g}{2} \langle |a| \rangle^2 + \text{H.c.} + U \langle |a| \rangle^4, \quad (20)$$
which is minimized for a nonzero expectation value of $a$. Yet, we know from the linear model that $\langle a \rangle = 0$ always because of the creation and destruction of pairs of particles. This opens the door to a different type of phase transition, but it also points out the need for a more rigorous and more sophisticated treatment of the problem.

In the following, we will show that indeed there is no such phase transition, but instead we find a continuous growth of entanglement along the lattice as the pairing $g$ increases and $U$ decreases. This is proven using three different tools: the iTEBD method developed in [27] and further explained in [16], DMRG or MPS simulations for the same model [28] and the perturbation theory in the weak coupling regime, $g \ll U$.

**4.1. Bosonic phase diagram**

The iTEBD method [16, 27] poses a translationally invariant tensor-product state ansatz for a one-dimensional quantum system. Such an ansatz may be qualitatively written as the contraction of four tensors, which are repeated with period 2 along the lattice,

$$|\psi\rangle = \sum \Gamma_{A_0}^{A_0} \Gamma_{B_0}^{B_0} \Gamma_{A_1}^{A_1} \ldots |i_0, i_1, i_2, \ldots\rangle \quad (21)$$

Here, the $|i_k\rangle$ represent the physical states of the $k$th lattice sites, which in our case is the occupation number of the cavities, cut-off to a reasonable number: $i_k = 0, 1, 2, \ldots, n_{\text{max}}$. The tensors $\Gamma$ have dimension $\chi^{2(n_{\text{max}}+1)}$, where $\chi$ is another cut-off, this time for the amount of entanglement that the ansatz may host. Simulations have been done ensuring convergence with respect to both cut-offs.

Taking into account the obvious limitations of the ansatz, it is nevertheless possible to get qualitative and even quantitatively accurate answers to many of the physical properties of the ground state. For that, we make use of two important properties of equation (21). The first one is that the expectation values of local operators, or of products of them (i.e. two- and $n$-body correlators) can be efficiently computed. The second one is that given an energy functional which is expressed as the sum of local operators, the ansatz above can be optimized to minimize that energy. Both procedures are described in detail in [16] as to how these quantities are computed through a suitable manipulation and contraction of the tensors in equation (21).

We have applied the iTEBD method for studying the properties of the pair-hopping model. The study in this section focuses on the ground state of the free energy functional

$$F = H_{\text{sub}} - \mu N. \quad (22)$$

The choice of this functional is motivated by the need for a fair comparison with earlier studies of the BHM, such as those describing cold atom setups. In this context, the chemical potential $\mu$ is a Lagrange multiplier that enforces an average number of bosons in the system, typically fixed during the loading of atoms in the optical lattice. In the photonic system, enforcing an average number of bosons through an adiabatic preparation is harder to achieve, because the total number of photons in the system is not conserved. It is for this reason that in section 4.2 we offer an alternative study based on a simpler mechanism to inject photons in the lattice.

Minimizing the mean free energy per site over the set of iTEBD states with fixed bond dimension, $\chi$, we have produced the pictures shown in figure 3, where we plot the density, the number fluctuations $\langle \Delta n \rangle^2 = \langle (a_i^\dagger a_i)^2 \rangle - \langle a_i^\dagger a_i \rangle^2$ and the pairing between nearest neighbours, all for the ground state. In these pictures we appreciate the fact that the lobes of the BHM are present in the growth of the number fluctuations (cf figure 3(b)). However, the system no longer exhibits plateaus of the density, and there is no sharp phase transition between

![Figure 3. Phase diagram of the free energy $H - \mu N$, for the model in equation (1). We plot (a) the number of photons, (b) fluctuations of the photons and (c) the nearest-neighbour pairing term, versus the chemical potential $\mu$ and the photon–photon pairing $g$. The usual boson delocalization $(a_i^\dagger a_{i+1})$ is zero over the whole diagram. An occupation number cut-off $n_{\text{max}} = 10$ for the states $|i_k\rangle$ (cf equation (21)) and $\chi = 20$ has been chosen.](image-url)
the insulator and the squeezed or entangled regime. The order parameter of a typical superfluid, which is the largest eigenvalue of the single-particle density matrix, is now zero because \( \langle a_i^\dagger a_{i+1}\rangle = 0 \) everywhere. In the paired case, we know that the limit \( U \to 0 \) is characterized by the two-mode squeezing and long-distance correlations. While we do not expect symmetry breaking even in the presence of interactions, we introduce the pairing observable, \( \langle a_i^\dagger a_{i+1}\rangle \), as a witness or ‘order parameter’ of this squeezed phase. Our simulations confirm that indeed this correlator increases smoothly with the interaction \( g/U \), as shown in figure 3(c).

The continuity of the transition can be further confirmed by studying weak perturbations around an insulator state. We start with a bare state that contains exactly \( n \) photons per cavity \( |\psi_0\rangle = \bigotimes_{N=0}^{n=1} |n\rangle \) and apply the perturbation theory up to second order, obtaining (cf the appendix)

\[
\langle a_i^\dagger a_i \rangle = n + \frac{g^2}{8} \left( f_{n+1}^2 - f_n^2 \right) + O \left( g^3 \right),
\]

\[
\langle a_i^\dagger a_i \rangle = n^2 + \frac{g^2}{8} \left[ (2n+1) f_{n+1}^2 - (2n-1) f_n^2 \right] + O \left( g^3 \right),
\]

\[
\langle a_i a_{i+1} \rangle = e^{i\phi_n} \frac{g}{4} \left[ n f_n - (n+1) f_{n+1} \right] + O \left( g^3 \right),
\]

\[
\langle a_i^\dagger a_{i+1} \rangle = 0 + O(g^2),
\]

expressed in terms of the function \( f_n = n \left[ 2U(n-1) - \mu \right]^{-1} \).

Using these expressions, we can approximate all observables, including the density, the number fluctuation, etc and compare with the exact results. This is done in figure 4, where we find excellent agreement all the way up to \( g = 0.1 \), for a variety of filling factors. It is important to remark that the perturbation theory confirms the lack of superfluid order, accompanied by the fast growth of the nearest-neighbour squeezing (the quantity that grows fastest of all for small \( g \)).

In the previous figures and in the perturbative analysis, we have seen that the density and number fluctuations grow smoothly with the coupling strength and the chemical potential, interpolating from the insulating phase to the squeezed phase. In order to provide further evidence of this continuous behaviour, we now add three types of studies: (i) cuts along the phase diagram that show the lack of plateaus, (ii) comparisons with the BHM to show the differences and the lack of non-analyticities and (iii) further studies involving the long-distance correlations in the system, to appreciate whether a phase transition might be hidden in the nonlocal observables.

We start by singling out a fixed photon–photon pairing \( g \) and comparing it with the original BHM in equation (2). This is done in figure 5 where we plot the photon number, the photon number fluctuation and the compressibility \( \partial \langle n \rangle / \partial \mu \) for the Bose–Hubbard and the pair-hopping models for the same chemical potential \( \mu \). In the BHM, one can clearly distinguish the superfluid and insulator regions where the photon number \( \langle n \rangle \) (number fluctuation \( \Delta n \)) remains constant and the formation of plateaus becomes evident. The vanishing compressibility within the plateaus confirms the presence of an insulator state followed by a sharp transition into the superfluid regime. This is in sharp contrast with our model where not even quantities such as the compressibility exhibit sharp jumps other than when approaching the line \( g = 0 \). In our model, the former plateaus of the BHM now exhibit weak compressibility that is monotonically increasing with the photon–photon pairing \( g \). The only valid insulator regime may be observed for the otherwise trivial case \( g = 0 \).

We may further stress the lack of plateaus in figure 6 where we plot the two different order parameters, that is the photon pairing \( \langle a_i^\dagger a_{i+1} \rangle \) for the model in equation (1) and the boson delocalization \( \langle a_i^\dagger a_i \rangle \) for the model in equation (2), together with the photon–photon correlation \( \langle n_i n_{i+1} \rangle - \langle n_i \rangle \langle n_{i+1} \rangle \) between nearest neighbours of both models. As pointed out before, the cross-over in the pair-hopping model is clearly driven by squeezing, whereas in the BHM, the boson delocalization acts as an adequate order parameter for a superfluid regime. The photon–photon correlation is constant

\[ \text{In all rigour, the actual order parameter would be the largest eigenvalue of the reduced density matrix of the single-particle reduced density matrix } \rho_{ij} = |a_i^\dagger a_j|, \text{ but since that eigenvalue is associated with a macroscopic population of the zero-momentum state, this, as a consequence, also results in the growth of the nearest-neighbour correlator. We expect similar behaviour for the squeezing.} \]
Where there are plateaus in (b), there is no compressibility in (c) (the solid line for the compressibility for the cases of long-range order in some observables or correlators. To discard this possibility, we have independently studied the density correlations are continuously differentiable functions everywhere.

All local observables that we have examined so far show differentiable behaviour typical of a cross-over. However, a phase transition might be hidden in the sudden appearance of long-range order in some observables or correlators. To discard this possibility, we have independently studied the correlation length of the squeezing witness \( \langle a_i^\dagger a_j^\dagger \rangle \) using finite-size simulations with DMRG. It can be shown that in the limit \( U \to 0 \), this correlator decays algebraically as \( 1/|i-j| \), with a divergent correlation length. To probe how the photons approach this regime, we have computed an average correlation length \( \xi(i) \), defined as

\[
\xi(i) = \frac{\sum_j (|i-j| \langle a_i^\dagger a_j^\dagger \rangle - \langle a_i^\dagger \rangle \langle a_j^\dagger \rangle)}{\sum_j \langle a_i^\dagger a_j^\dagger \rangle - \langle a_i^\dagger \rangle \langle a_j^\dagger \rangle},
\]

(27)

for any lattice site \( i \). In figure 7, we show the correlation length \( \xi \) for a finite-size system of 100 lattice sites. We trace two lines of the phase diagram: a line of fixed chemical potential \( \mu \) (figure 7(a)) and that of fixed photon pairing \( g \) (figure 7(b)). In both cases, the correlation length grows fast when one leaves the quasi-lobes of the bosonic phase diagram in figure 3, but we appreciate no sharp divergence or a boundary. This is most obvious for the fixed chemical potential \( \mu \). For a fixed photon pairing \( g \), the transition among the lobes occurs continuously, with a peak in the region between apparent lobes. This maximum correlation length, however, is never infinite, and it grows slowly with \( g \).

The previous approach relied on a condensed matter physics phase diagram that combines the interaction with the chemical potential. The chemical potential describes the balance of particles between the system and a reservoir which is put in contact with it. Thus, this approach is very suitable for describing experiments with atoms or electrons, such as lattice models that result from loading atoms in periodic potentials. In a photonic setup, though, we lack superselection rules for the number of photons, and indeed, these particles are quickly lost after a given time. It is for this reason that instead of achieving equilibrium with a reservoir of photons, it seems more practical to fill the lattice using other mechanisms, such as a coherent drive that injects energy into one or more cavities.
Figure 6. Comparison of the photon pairing model (BH pairing) in equation (1) and the original BHM in equation (2). We plot (a) the pairing term $\langle a_i^\dagger a_{i+1}^\dagger \rangle$ and the boson delocalization $\langle a_i^\dagger a_{i+1} \rangle$ and (b) the photon–photon correlation between two neighbouring lattice sites, versus the chemical potential $\mu$ for four different values of the coupling strength $g$. An occupation number cut-off $n_{\text{max}} = 10$ for the states $|\{i\rangle$ (cf equation (21)) and $\chi = 20$ has been chosen.

Figure 7. Correlation length $\xi$ for a finite-size system of 100 lattice sites folded to a ring for the pairing model in equation (1). We plot the correlation length for (a) three fixed values of the chemical potential $\mu$ versus the photon–photon pairing $g$ and (b) three fixed values of the photo–photon pairing $g$ versus the chemical potential $\mu$. An occupation number cut-off $n_{\text{max}} = 10$ for the states $|\{i\rangle$ (cf equation (21)) and $\chi = 70$ has been chosen.
4.2. Photonic phase diagram

Following this reasoning, and in order to recreate a simpler experiment with superconducting circuits, we have studied an alternative Hamiltonian,

\[ H_{\text{driven}} = H + \sum_i \Omega (a_i + a_i^\dagger), \]

that introduces an external displacement of the cavity. This displacement, of strength \( \Omega/\sqrt{\omega} \) in the non-interacting case, is implemented through an external source, which in our case can be a time-independent flux source (i.e. an inductor).

An unfortunate consequence of this formulation is that we lose the lobe structure. For fixed \( \omega_0 = 1, U = 1 \), figure 8 shows the same observables as before (see also figure 9 for comparison of the model without driving). We appreciate the fact that the only insulating region is now around the vacuum, at \( g = \Omega = 0 \). Pairing continues to be a good order parameter, but it has to be properly defined, eliminating the
equilibrium values due to the injection, \( \langle a_i^+ a_i^{++} \rangle - \langle a_i^+ \rangle \langle a_i^{++} \rangle \). If we do so, then the pairing is relatively insensitive to \( \Omega \) and the average number of photons, and grows rapidly with the coupling strength \( g \).

5. Conclusions

Summing up, in this work, we have studied a lattice model of bosons coupled entirely through counter-rotating pairing terms. Unlike its relative model, the Bose–Hubbard model, the resulting model parameter \( \langle a_i^+ a_i^{++} \rangle \) terms. Unlike its relative model, the Bose–Hubbard model, the resulting model parameter \( \langle a_i^+ a_i^{++} \rangle \) does not experience any quantum phase transition. Instead, for any value of the pairing, correlations are established among cavities, which we quantify using the pairing order parameter \( \langle a_i^+ a_i^{++} \rangle \).

We have shown that in the non-interacting model \( U = 0 \), this pairing is maximal in momentum space, where it leads to the establishment of two-mode squeezing between different momenta. The selection of paired momenta can be controlled through the phase of the driving or the phase in equation (1).

All simulations demonstrate that the introduction of the nonlinear term \( U \) transforms our squeezed state, reducing in a continuous way the fluctuations in the number of particles per site, and also the squeezing and the correlation length. The system remains compressible and, with a sufficient amount of long-distance entanglement, of experimental interest.

All these properties, including the numerically computed correlators and two-photon CM, can be explored in a setup that consists of superconducting resonators coupled by driven SQUIDs. An open problem which escapes the tools and details.

The state \( |u_n^0\rangle \) is not degenerate, and hence we may refer to the standard Schrödinger theory [29] for the perturbative expansion of the eigenstate in first order in \( g \). Here, \( |e_i\rangle \) refers to the rather trivial eigenstates of \( H_0 \) in equation (A.2) with eigenenergies \( e_i \), i.e. \( H_0 |e_i\rangle = e_i |e_i\rangle \).

Consequently, the latter provides

\[
|u_n^0\rangle = \sum_i \frac{(e_i|H_1|u_n^0)}{E_n^0 - e_i}|e_i\rangle,
\]

(4.4)

we may obtain a correction for the eigenstate in first order in \( g \).

From equation (A.5), a second-order correction for the energy is immediately found to be

\[
E_n = \frac{n}{2(U(n - 1) - \mu)},
\]

(6.6)

and the notation \( \{|a_1, a_2, \ldots, a_n\rangle \} \) marks the position \( i = 0, \ldots, N - 1 \) of the \( m \)-tuple \( a_1, a_2, \ldots, a_n \) in the otherwise equally loaded state \( |u_n^0\rangle \). For example, taking into account periodic boundary conditions on the ring, it is \( |n - 1, n - 1\rangle_{N-1} = |n - 1, n, \ldots, n - 1\rangle \).

Equation (A.5), a second-order correction for the energy is immediately found to be

\[
E_n^2 = \frac{N}{8}[nf_n - n(n + 1)f_{n+1}].
\]

(7.7)

Since there is no first-order correction in energy, that is \( \langle u_n^0|H_1|u_n^0\rangle = 0 \), one may rewrite the second-order correction for the state as

\[
|u_n^0\rangle = \sum_i \frac{(e_i|H_1|u_n^0)}{E_n^0 - e_i}|e_i\rangle - \frac{N}{32}(f_n^2 + f_{n+1}^2)|u_n^0\rangle.
\]

(8.8)

Since we are interested in expectation values that lead to first- and second-order statistical moments, only the second term in the latter really matters up to second order in \( g \). The first term requires at least higher moments in order to have an effect in the same order of \( g \). We hence abridge providing further details.

Appendix. Perturbation theory up to second order

Motivated by the numerical result, we probe perturbatively the state

\[
|u_n^0\rangle = \bigotimes_{i=0}^{N-1} |n_i\rangle
\]

(A.1)

for an equally loaded lattice with uniform population \( n \) on each lattice site \( i = 0, \ldots, N - 1 \). Considering \( N \) lattice sites, the unperturbed Hamiltonian, \( H_0 \), and the perturbative term, \( H_1 \), are given by

\[
H_0 = U \sum_{i=0}^{N-1} a_i^+ a_i + \mu \sum_{i=0}^{N-1} a_i^+ a_i,
\]

\[
H_1 = \frac{1}{2} \sum_{i=0}^{N-1} \left( e^{i\phi} a_i^+ a_{i+1}^+ + \text{H.c.} \right),
\]

(A.2)

with \( H = H_0 + gH_1 \). Its unperturbed energy is given by

\[
E_n^0 = Nn(\mu n - \mu).
\]

(A.3)

The state \( |u_n^0\rangle \) is not degenerate, and hence we may refer to the standard Schrödinger theory [29] for the perturbative expansion of the eigenstate in first order in \( g \). Here, \( |e_i\rangle \) refers to the rather trivial eigenstates of \( H_0 \) in equation (A.2) with eigenenergies \( e_i \). Consequently, the latter provides

\[
|u_n^0\rangle = \sum_i \frac{(e_i|H_1|u_n^0)}{E_n^0 - e_i}|e_i\rangle,
\]

(A.4)

we may obtain a correction for the eigenstate in first order in \( g \).

From equation (A.5), a second-order correction for the energy is immediately found to be

\[
E_n^2 = \frac{N}{8}[nf_n - n(n + 1)f_{n+1}].
\]

(A.7)

Since there is no first-order correction in energy, that is \( \langle u_n^0|H_1|u_n^0\rangle = 0 \), one may rewrite the second-order correction for the state as

\[
|u_n^0\rangle = \sum_i \frac{(e_i|H_1|u_n^0)}{E_n^0 - e_i}|e_i\rangle - \frac{N}{32}(f_n^2 + f_{n+1}^2)|u_n^0\rangle.
\]

(A.8)

Since we are interested in expectation values that lead to first- and second-order statistical moments, only the second term in the latter really matters up to second order in \( g \). The first term requires at least higher moments in order to have an effect in the same order of \( g \). We hence abridge providing further details.

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