Indexed Nörlund Summability Factor of Improper Integrals

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Abstract. In this paper, we have defined summability of improper integrals and have established a theorem on indexed absolute Nörlund summability factors of improper integrals under sufficient conditions, generalizing the result of Ozgen [9] and Mishra et al. [8].

1. Literature Survey

Considering the \((N, P_n)\) and \((K, 1, \alpha)\) summability, Parashar [10] obtained the minimum set of conditions for an infinite series to be \((K, 1, \alpha)\) summable. In 1986, Bor [1] found a relationship between two summability techniques \((C, 1)_k\), and \(|N, p_n|_k\), and in [2], he used the \(|N, p_n|_k\) for generalization of a theorem based on minimal set of sufficient conditions for infinite series. In 2016, Sonker and Munjal [11] determined a theorem on generalized absolute Cesaro summability with the sufficient conditions for infinite series and in [12], they used the concept of triple matrices for obtaining the minimal set of sufficient conditions of infinite series to be bounded. In 2017, Sonker and Munjal [13] found the approximation of the function \(f \in Lip(\alpha, p)\) using infinite matrices of Cesaro submethod and in [14], they obtained boundness conditions of absolute summability factors. In this way by using the advanced summability method, we can improve the quality of the filters. Borwein [3] extended many results on ordinary and absolute summability methods of integral. Canak [4] and Totur [16] worked on the concept of Cesaro summability with a very interesting result for integrals. In the same direction, we extended the results of Mazhar [7] with the help of some new generalized conditions and absolute Nörlund summability \(|N, p_n|_k\) factor for integrals.
2. Introduction
Let \( \sum a_n \) be a given infinite series with sequence of partial sums \( (s_n) \). Let
\[
\sigma_n = \frac{1}{n} \sum_{k=1}^{n} s_k
\] (1)
The series \( \sum a_n \) is said to be \((C, 1)\) summable, if
\[
\lim_{n \to \infty} \sigma_n = s
\] (2)
where \( s' \) is a finite number. The series \( \sum a_n \) is said to be \(|C, 1|_k, k \geq 1\), summable, if
\[
\sum_{n=1}^{\infty} n^{k-1} |\sigma_n - \sigma_{n-1}|^k < \infty
\] (3)
Let \( f \) be a real valued continuous function defined in the interval \([0, \infty)\) and \( s(x) = \int_0^x f(t) dt \).
We define the Cesaro mean of \( s(x) \), denoted by \( \tau(x) \), as
\[
\tau(x) = \frac{1}{x} \int_0^x s(t) dt
\] (4)
\[
\tau(x) = \frac{1}{x} \int_0^x (x - t)f(t) dt
\] (5)
The integral \( \int_0^\infty f(t) dt \) is said to be summable \(|C, 1|\), if
\[
\int_0^\infty |\tau'(x)| dx < \infty
\] (6)
and is said to be summable \(|C, 1|_k, k \geq 1\), if
\[
\int_0^\infty x^{k-1} |\tau'(x)|^k dx < \infty.
\] (7)
Let \( p(x) \) be a real valued continuous function defined in the interval \([0, \infty)\) and \( P(x) = \int_0^x p(t) dt \).
We define the Nörlund mean or \((N, p)\) mean of \( s(x) \) as a function \( t(x) \) given by
\[
t(x) = \frac{1}{P(x)} \int_0^x p(t) s(t) dt
\] (8)
The integral \( \int_0^\infty f(t) dt \) is said to be summable \(|N, p|\), if
\[
\int_0^\infty |t'(x)| dx < \infty
\] (9)
and is said to be summable \(|N, p|_k, k \geq 1\), if
\[
\int_0^\infty \left( \frac{P(x)}{p(x)} \right)^{k-1} |t'(x)|^k dx < \infty
\] (10)
Clearly, we have \( s(x) - \tau(x) = \frac{1}{\tau(x)} \int_0^x P(t)f(t) dt \).
Let
\[
s(x) - \tau(x) = \nu(x)
\] (11)
Then (10) can be written as
\[
\int_0^\infty p(x) \left( \frac{P'(x)}{p(x)} \right)^k |\nu(x)|^k dx < \infty
\] (12)
3. Known Results
Concerning absolute Cesaro summability $|C, 1|_k$ factors of integrals, Ozgen\[9\] obtained the following results:

3.1. Theorem
Let $\gamma(x)$ be a positive monotonic non-decreasing function such that
$$\gamma(x) \gamma(x) = O(1), \text{ as } x \to \infty \quad (13)$$
$$\int_0^x u|\lambda''(u)| \gamma(u)du = O(1), \quad (14)$$
$$\int_0^x \frac{|\nu(u)|^k}{u} \, du = O(\gamma(x)), \text{ as } x \to \infty. \quad (15)$$
Then the integral $\int_0^\infty f(t)\, dt$ is summable $|C, 1|_k$, $k \geq 1$.

Recently, Mishra et al.\[8\] extended Theorem-3.1 to $|C, 1|_k, \delta$, $\delta > 0$, $\delta k \leq 1$ summability by establishing the following theorem:

3.2. Theorem
Let $\chi(x)$ be a positive non-decreasing function and ther be two functions $\beta(x)$ and $\varepsilon(x)$ such that
$$|\varepsilon'(x)| \leq \beta(x), \quad (16)$$
$$\beta(x) \to 0, \text{ as } x \to \infty \quad (17)$$
$$\int_0^\infty u|\beta'(u)| \chi(u)du < \infty, \quad (18)$$
$$|\varepsilon(x)| \chi(x) = O(1), \quad (19)$$
and
$$\int_0^x u^{\delta k - 1}|\nu(u)|^k \, du = O(\chi(x)), \text{ as } x \to \infty. \quad (20)$$
Then the integral $\int_0^\infty \varepsilon(t)f(t)\, dt$ is summable $|C, 1|_k$, for $k \geq 1$, $\delta k \leq 1$.

Prior to above result, dealing with N"orlund summability of improper integrals, Sonker and Munjal \[14\] has established the following results.

3.3. Theorem
Let $p(0) > 0$, $p(x) \geq 0$ and a non-increasing function. Let $\chi(x)$ be a positive non-decreasing function and there be two functions $\beta(x)$ and $\varepsilon(x)$
$$|\varepsilon'(x)| \leq \beta(x), \quad (21)$$
$$\beta(x) \to 0, \text{ as } x \to \infty \quad (22)$$
$$\int_0^\infty u|\beta'(u)| \chi(u)du < \infty, \quad (23)$$
$$|\varepsilon(x)| \chi(x) = O(1), \quad (24)$$
and
$$\int_0^x \frac{|\nu(u)|^k}{u} \, du = O(\chi(x)), \text{ as } x \to \infty. \quad (25)$$
Then the integral $\int_0^\infty \varepsilon(t)f(t)\, dt$ is summable $|N, p_n|_k$, for $k \geq 1$. 

3 \.
4. Main Result
In 1965, Kishore[6] had proved a result that when a series is summable \(|C,1|\), then it is summable \(|N,p_n|\) under certain condition. While proving their theorem, Sonker and Munjal had used the result of Kishore. In fact it seems that it is not true in general. However, we establish the following result independently. We prove:

4.1. Theorem
Let \(\chi(x)\) be a positive non-decreasing function and there be two functions \(\beta(x)\) and \(\varepsilon(x)\) such that

\[|\varepsilon'(x)| \leq \beta(x),\]
\[\beta(x) \to 0, \text{ as } x \to \infty\]
\[|\varepsilon(x)| \chi(x) = O(1),\]

\[
\frac{1}{(p(x))^{k-1}} \int_0^x P(u) |\beta'(u)| \chi(u) du = O(1), \text{ as } x \to \infty, \tag{29}
\]

\[
\frac{1}{(p(x))^{k-1}} \int_0^x P'(u) |\beta(u)| \chi(u) du = O(1), \text{ as } x \to \infty, \tag{30}
\]

and

\[
\int_0^x p(t)|P'(t)|^k \nu(t)^k = O(\chi(x)) \text{ as } x \to \infty \tag{31}
\]

Then the integral \(\int_0^\infty \varepsilon(t)f(t)dt\) is summable \(|N,p|\), for \(k \geq 1\).

Note:
The above theorem can be proved by using the concept of example that \(\int_0^\infty x|\beta'(x)|\chi(x)dx < \infty\) is weaker \(\int_0^\infty x|\varepsilon''(x)|\chi(x)dx < \infty\) and hence the introduction of the function \{\beta(x)\} is justified. It may be possible to choose the function \(\beta(x)\) such that \(|\varepsilon'(x)| \leq \beta(x)\). When \(\varepsilon'(x)\) oscillates, \(\beta(x)\) may be chosen such that \(|\beta(x)| < |\varepsilon''(x)|\). Hence \(|\beta'(x)| < |\varepsilon''(x)|\). So that \(\int_0^\infty x|\beta'(x)|\chi(x)dx < \infty\) is a weaker requirement that \(\int_0^\infty x|\varepsilon''(x)|\chi(x)dx < \infty\).

5. Proof of the Theorem
Let \(T(x)\) be the \((N,p_n)\) mean of the integral \(\int_0^\infty \varepsilon(t)f(t)dt\). The integral \(\int_0^\infty \varepsilon(t)f(t)dt\) is \(|N,p_n|\) summable, if

\[
\int_0^x \left( \frac{P(x)}{p(x)} \right)^{k-1} |T'(t)|dt = O(1), \text{ as } x \to \infty, \tag{32}
\]

where \(T(x)\) is given by

\[
T(x) = \frac{1}{P(x)} \int_0^x p(t) \left( \int_0^t \varepsilon(u)f(u)du \right) dt
= \frac{1}{P(x)} \int_0^x \varepsilon(u)f(u)du \int_u^x p(t) dt
= \frac{1}{P(x)} \int_0^x (P(x) - P(u)) \varepsilon(u)f(u) du
= \int_0^x \left( 1 - \frac{P(u)}{P(x)} \right) \varepsilon(u)f(u) du \tag{33}
\]
Further, by Hölder’s inequality, we have

\[
\int_0^x \left( \frac{P(t)}{p(t)} \right)^{(k-1)} |T_1(t)|^k \, dt
\]

\[
= \int_0^x \left( \frac{P(t)}{p(t)} \right)^{(k-1)} |P(t)|^k |\nu(t)|^k |\varepsilon(t)|^k \, dt
\]

\[
\leq \int_0^x \frac{p(t)|P'(t)|^k}{P(t)p(t)} |\nu(t)|^k |\varepsilon(t)|^k \, dt
\]

\[
= |\varepsilon(x)| \int_0^x \frac{p(t)|P'(t)|^k}{P(t)p(t)} |\nu(t)|^k \, dt - \int_0^x |\varepsilon'(t)| \left( \int_0^t \frac{p(y)|P'(y)|^k}{P(y)p(y)} |\nu(y)|^k \, dy \right) \, dt
\]

\[
= O(1)|\varepsilon(x)| \chi(x) - \int_0^x \beta(t) \chi(t) \, dt
\]

\[
= O(1) - \beta(x) \int_0^x \chi(u) \, du + \int_0^x \beta'(t) \left( \int_0^t \chi(u) \, du \right) \, dx
\]

\[
\leq O(1) - \beta(x) \int_0^x \chi(u) \, du + \int_0^x t \beta'(t) \chi(t) \, dt
\]

\[
= O(1), \text{ as } x \to \infty. \quad (36)
\]

By virtue of the hypothesis of Theorem-3.1,

\[
\int_0^x \frac{P(t)}{p(t)} \left( \frac{(k-1)}{p(t)} \right) |T_2(t)|^k \, dt
\]

\[
= \int_0^x \frac{P(t)}{p(t)} \left( \frac{(k-1)}{p(t)} \right) \int_0^t P(u)\varepsilon'(u)\nu(u) \, du \, dt
\]

\[
\leq \int_0^x \frac{P'(t)}{(p(t))^{k-1}(P(t))^2} \left( \int_0^t |P(u)|^k |\varepsilon'(u)|^k |\nu(u)|^k \, du \right) \left( \frac{1}{P(t)} \int_0^t P''(u) \, du \right)^{-1} \, dt
\]

\[
= \int_0^x |P(u)|^k |\varepsilon'(u)|^k |\varepsilon(u)|^k |\nu(u)|^k \, du \int_0^x \frac{P'(t)}{(p(t))^{k-1}(P(t))^2} \, dt.
\]
So
\[ \int_{0}^{x} \left( \frac{P(t)}{p(t)} \right)^{(k-1)} |T_2(t)|^k dt = O(1), \quad \text{as } x \to \infty. \] (38)

On collecting (33) – (38), we have
\[ \int_{0}^{x} \left( \frac{P(t)}{p(t)} \right)^{(k-1)} |T'(t)|^k dt = O(1), \quad \text{as } x \to \infty. \]

6. Conclusion
The main result of this research article is an attempt to formulate the problem of absolute summability factor of integrals which make a more modified filter. Through the investigation, we concluded that the improper integral is absolute Nörlund summable under the minimal sufficient conditions. Further, this study has a number of direct applications in rectification of signals in FIR filter (Finite Impulse Response Filter) and IIR filter (Infinite Impulse Response Filter). In a nut shell absolute summability method is a motivation for the researchers, interested in studies of improper integrals.

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