We study a solution of Einstein’s equations that describes a straight cosmic string with a variable angular deficit, starting with a $2\pi$ deficit at the core. We show that the coordinate singularity associated to this defect can be interpreted as a traversible wormhole lodging at the the core of the string. A negative energy density gradually decreases the angular deficit as the distance from the core increases, ending, at radial infinity, in a Minkowski spacetime. The negative energy density can be confined to a small transversal section of the string by gluing to it an exterior Gott’s like solution, that freezes the angular deficit existing at the matching border. The equation of state of the string is such that any massive particle may stay at rest anywhere in this spacetime. In this sense this is 2+1 spacetime solution.

A generalization, that includes the existence of two interacting parallel wormholes is displayed. These wormholes are not traversible.

Finally, we point out that a similar result, flat at infinity and with a $2\pi$ defect (or excess) at the core, has been recently published by Dyer and Marleau. Even though theirs is a local string fully coupled to gravity, our toy model captures important aspects of this solution.

I. INTRODUCTION AND SUMMARY

Cosmic strings are static cylindrically symmetric objects generated by phase transition of a self interacting scalar field minimally coupled to a gauge field, at the beginning of the universe. They may have left a print behind in the form of growing density inhomogeneities (see for instance [1] or [2]). So far there is no observation that precludes the existence of cosmic strings in the early universe. The effects of these primordial inhomogeneities remain inside the allowed band of density fluctuations established by the COBE results [3].

There exists several authoritatives reviews that describe the different scenarios that give rise to the formation of cosmic strings and include the description of its topological and physical properties [4],[5]. For this reason we will not go through a detailed introduction here.

The search for an exact analytic solutions of a gauge field coupled to gravity has proved to be a hard one. An exact analytic solution for a global string, static and with cylindrical symmetry was found by Cohen and Kaplan [6]. It describes the outer region of the string and, as pointed by Vilenkin and Shellard [5], it has a singularity at a finite radius of its core. The linear approximation to the field equations, developed by Harari and Sikivie [7] has been used to understand the geometry behind the Cohen and Kaplan model. However, the study of a cosmic string using a linearized solution of Einstein’s equations rapidly drags in a non-physical singularity [7], [8] around the string.

Concerning the singularity mentioned above, Gregory [9] and later Gibbons et al. [10], concluded that the metric associated with a global string must develop a singularity at a finite distance of the string. This was confirmed by a numerical approach to this example developed by Laguna and Garfinkle [11].

Laguna-Castillo and Matzner [12], using a numerical approach, have proved that for a wide range of the vacuum expectation value of the scalar field $\eta$, with $\eta < 10^{18}$ Gev, the solution at radial infinity resembles a Minkowski spacetime minus a wedge. Laguna and Garfinkle [11], studied a supermassive cosmic string finding Kasner-like metric away from the core of the string. Ortiz [13], extended this result finding a new solution with a singularity at a finite distance from the core of this supermassive cosmic string. The energy density of a supermassive cosmic string takes the value $\mu \sim \eta^2 \approx 10^{-2}$,
where the usual value for the symmetry breaking energy scale is $\mu \sim 10^{-6}$. A comprehensive and critical summary of these results appears in Raychaudhuri \[14\].

Along this line, the work of Dyer and Marleau \[15\] is the more relevant for our results. They found strings solutions with an angle surplus at the core and asymptotically Minkowskian (no wedge at radial infinity) using a numerical approach. Our work reaches a similar result, working analytically with a simplified model of a cosmic string.

The simplified approach we just mentioned refers to models where the radial and angular components of the stress energy tensor of the string has not been considered. Simultaneously, a fixed width to the string is assigned and usually is matched to an empty spacetime \[16\], \[17\], \[18\]. This is a generalization of the local cosmic string first introduced by Vilenkin \[5\].

From the previous account we conclude that the search for exact solutions of local scalar theories coupled to gravity seems to be unsurmountable analytically. For this reason in this work we have adopted a simplified stress energy tensor of the kind mentioned in the last paragraph: keeping only the $T_{zz}$ and $T_{tt}$ components as different from zero. This model has been studied previously (see for instance \[16\], \[18\]) and has succeeded in raising interesting questions about the physics taking place around a cosmic string. However, as Raychaudhuri has showed \[14\], this approach is not a cosmic string in the sense that the boundary conditions set the gauge field and the complex scalar field to zero. However, as becomes clear from the previous references, the existence of a model, although simple, susceptible to an analytic study may be of great help to understand the nature of more complicated and realistic objects. Is in this context that the results exhibited by Dyer and Maleau interest to us, since they have similar boundary conditions at the core of the string and reach the same geometry at radial infinity.

Another recent work in the same spirit that ours belongs to Clément \[19\], who obtained a solution for a multiwormhole associated with a scalar field in 2+1 dimensions and extended it to a cylindrical traversable wormholes in four dimensions. In \[19\], he has been able to glue several cosmic strings in a flat background to form a wormhole in the spirit of the Wheeler-Misner \[21\].

In our case, we will explore the physical consequences of the existence of a wormhole at the core of a simplified model for a cosmic string, already described. Our solution arises when we allow a different boundary condition at the core of the string. The curvature in our solution decreases slowly from the core of the string to the radial infinity, this a difference with Clément’s model.

We display an exact solution of Einstein’s equations with cylindrical symmetry. It is asymptotically minkowskian in the radial direction and the stress energy tensor is such that $T_{zz} = T_{tt}$ and $T_{\theta\theta} = T_{rr} = 0$. It has an angular deficit at the core, that slowly vanishes as we move toward infinity. The radial patch of coordinates does not have a curvature singularity in $1 \leq r < \infty$. At $r = 1$, it contains a logarithmic coordinate singularity, that we interpret as the location of a traversible wormhole connecting two identical universes along the string. This is the only extension of this metric we have studied, though it is not the only one possible.

We made an effort to find a family of solutions that would break the cylindrical symmetry and, simultaneously, would show the existence of (at least) a couple of wormholes, keeping the asymptotic flatness of the metric. However, we only found a solution where the wormholes cannot be crossed in a finite proper time.

Since the source of this cosmic string has a negative energy density, and it is advisable to keep the transversal cross section of this string as small as possible \[22\]. We have added a solution that can be matched to the wormhole metric and has a positive energy density. However, for all our computations we use the original metric since we are interested in the physics occurring in the neighborhood of the wormhole throat.

It is worth to mention that since our background metric is not empty, our solution differs from the one found by Clément \[21\].

The location of the wormhole is arbitrarily set at $r = 1$. This number has no physical meaning; we only have established an arbitrary scale for the coordinates.

We use the signature conventions and the definitions of the Riemann and Ricci tensor appearing in Wald’s book, \[23\]. Also $G = \hbar = c = 1$. Greek indexes run from $1...4$.

II. A TRAVERSIBLE WORMHOLE
A. The metric

Let’s consider the following metric:

\[
  ds^2 = -dt^2 + \left(1 + \frac{B}{\ln\left(r/r_0\right)}\right) dr^2 + r^2 d\varphi^2 + dz^2.
\]

The only components of the Einstein’s equations:

\[
  G_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R = 8\pi T_{\alpha\beta},
\]

associated with this metric, setting \( r_0 \equiv 1 \), are:

\[
  G_{tt} = G_{zz} = \frac{B}{2r^2 \left[ \ln(r) + B \right]^2} r^2, \quad T^t_t = T^z_z = -\rho, \quad \text{with} \quad \rho > 0.
\]

The trace of the Einstein’s tensor is:

\[
  G_{\mu\mu} = -R = \frac{B}{r^2 \left[ \ln(r) + B \right]^2}.
\]

This solution of the Einstein’s equations has a stress energy tensor with pressure in the z-direction and an equation of state \( p_z + \rho = 0 \). If we take \( B > 0 \), for the reasons we state below, the energy density becomes negative, \( \rho < 0 \), as seen from the equation (3). Both, the pressure and the density, decay slowly to zero at radial infinity. As showed by Morris and Thorne [22], the existence of a wormhole needs a negative energy density in a region around it. As pointed out in the same reference [22], it is necessary to keep the negative energy density reduced to a small region around the wormhole. We do not ignore this here, however, for simplicity, all our computations adopt the metric (1). In section IV, we will display a matching of this geometry to the Gott’s exterior solution [16] to keep the negative energy density reduced to a small region around the wormhole.

The coordinates patch valid for equation (1), extends from \( r = 1 \) to \( r = \infty \), and there are no physical singularities in this region. The invariants associated with this metric adopt two expressions: they are proportional to a power of the Ricci scalar or vanish. For instance, for the above metric the square of the Riemann tensor is

\[
  R_{\mu\nu\sigma\tau}R^{\mu\nu\sigma\tau} = \left[ \frac{B^2}{r^4} \left( \ln(r) + B \right)^4 \right] = R^2.
\]

The Ricci scalar has singularities at \( r = 0 \) and \( \ln(r) = -B \), both of them remain outside the patch of the radial coordinate \( r \geq 1 \), if \( B > 0 \). Thus, under these conditions, all the invariants are finite. For \( B < 0 \), we have a positive energy density and a flat spacetime at infinity, however we drag in a naked singularity at \( \ln(r) = |B| \) that hinders any physical interpretation of the source of this metric.

In the following section we will show that the coordinate singularity that lies at \( r = 1 \), allows us to naturally plug in another mirror spacetime there. This throat, that connects two twin spacetimes, is what we have called here a wormhole.

B. Geodesics

The geodesics equations for a lightlike vector associated with this metric (1) are:

\[
  -\dot{t}^2 + \left(1 + \frac{B}{\ln r}\right) \dot{r}^2 + -r^2 \dot{\varphi}^2 = 0, \quad -\dot{t} = E, \quad \dot{\varphi} r^2 = L.
\]

Here, the prime indicates a radial derivative and the dot, a derivative with respect to the affine parameter. We also have assumed that the motion takes place on the \( z = \text{constant} \), plane. Solving these equations for \( \dot{r} \), we have:

\[
  \dot{r} = \pm \frac{\sqrt{\left( E^2 - \frac{L^2}{r^2} \right)}}{\sqrt{\left(1 + B/\ln r\right)}}.
\]
The coordinate patch becomes singular at $r = 1$. To study the trajectory of test particles near the wormhole throat, we introduce a new set of coordinates that avoids this singularity. Defining a radial coordinate as

$$
\chi = \pm \int \sqrt{1 + \frac{B}{\ln(r)}} \, \frac{1}{r} \, dr,
$$

(7)

here each sign maps a different side of the wormhole, welding both sides at $r = 1$. The explicit dependence of $\chi$ on $r$, obtained from (6), is:

$$
\chi = \pm \left[ \sqrt{\ln(r)} \sqrt{\ln(r) + B} + B \ln \left( \sqrt{\ln(r) + \sqrt{\ln(r) + B}} \right) - \frac{1}{2} B \ln(B) \right],
$$

(8)

where $\chi \in (-\infty \ldots \infty)$ and we have added a constant to set $\chi|_{r=1} = 0$. In these coordinates the metric takes the expression:

$$
ds^2 = P(\chi) (d\chi^2 + d\phi^2) + dz^2 - dt^2,
$$

(9)

Assuming a geodesic equation with just a radial dependence $\chi$, we obtain the following general solution (with $z =$constant):

$$
l^a = \left( E, \pm \sqrt{\frac{D - L^2}{P(\chi)}}, \frac{L}{P(\chi)}, 0 \right), \text{ where } D = \begin{cases} \frac{E^2}{E^2} & \text{if } l_al^a = 0, \\ \frac{E^2}{E^2 - 1} & \text{if } l_al^a = -1. \end{cases}
$$

(10)

Since there is not a closed form expression for $r = r(\chi)$, we study the $r \approx 1$ region. There, the equation (6) can be approximated, to the lowest non-trivial term, as $\chi \approx \pm 2\sqrt{B} \sqrt{r - 1}$, obtaining $P(\chi) = r^2 \approx 1 + \chi^2 / 2B$.

Using this approximation and the equation (10), the evolution of any kind of test particles near the wormhole throat, is given by the 4-vector:

$$
l^a = (E, \pm \sqrt{D - L^2}, L, 0).
$$

(11)

This result points to the existence of particles orbiting at $\chi = 0$, if $D = L^2$. Otherwise, if $D > L^2$, the particle crosses the wormhole throat, going from one universe to the other. We picture this situation as the test particle crossing the wormhole and appearing in a twin universe, an analytic continuation of the original spacetime.

On the opposite case, if $D < L^2$, the test particle bounces at some radius larger than unity ($\chi > 0$), avoiding the throat.

One of the geodesics that belongs to this family is pictured in Figure 2. It corresponds to an embedding of the conical structure of the throat in a three dimensional space. The line threading this surface is a lightlike geodesic obtained through a numerical computation of the orbit.

To prove that the wormhole is traversible, we need to compute the time a particle spends crossing the wormhole. In the absence of a closed form for $r = r(\chi)$, we use the approximation established for the neighborhood of $r = 1$ or $\chi = 0$. Let’s compute the time it takes an infalling particle to reach $r = 1$ from another point close to it:

$$
\frac{d\chi}{dt} = \frac{1}{E} \sqrt{\frac{DP(\chi) - L^2}{P^2(\chi)}}, \text{ therefore } t = -\int_{\chi}^{0} E \sqrt{\frac{P^2(\chi)}{DP(\chi) - L^2}} \, d\chi,
$$

(12)

the minus sign corresponds to an infalling particle. This integral can be evaluated exactly (with the approximation $P(\chi) \approx 1 + \chi^2 / (2B)$ already defined). The result is finite unless $D - L^2$ vanishes. This computation shows that this wormhole is traversible.

For $D - L^2 = 0$, the integral is proportional to $\ln \chi$, so for an infalling particle (massless or not) it takes an infinite time to reach the throat under this conditions. This result points to a possible appearance of an instability at the throat in case that part of the background radiation from the surrounding get stored, orbiting at $r = 1$ or near it. We think that most probably the back reaction generated by this matter may close the wormhole. There is a hint in this direction, the spherical symmetric Lorentzian wormhole spacetimes has been unable to support a quantized massive scalar field [2].
Note that a massive particle may remain at rest at any value of the \((r, \phi)\) coordinates, as it can be seen from equation (10). In this case we have \(E = 1, \dot{t} = 1\) and \(L = 0\). In this sense this string is the equivalent of the Vilenkin cosmic string. It also reflects its 2+1 connection.

If we set a new radial coordinate as \(R = \int_1^r \frac{d r}{\sqrt{1 + B/\ln(r)}}\) we get:

\[
d s^2 = -dt^2 + dR^2 + r(R)^2 d\phi^2 + dz^2,
\]

the particle does not feel any gravity and remains at rest. Of course we can reach the same conclusion from \(g_{tt} = -1\), there is not a gravitational potential in this metric. We also learn that as soon as we introduce a coordinate dependence in \(g_{tt}\) there will appear a force and the particle will not be able to stand at rest.

### C. Total Mass

The total energy density associated with this string is:

\[
T^t_t = \frac{1}{8\pi} \frac{B}{2r^2(B + \ln(r))^2} = T^z_z, \quad T_{rr} = T_{\phi\phi} = 0.
\]

To set the singularity inside \(r = 1\), we need \(B > 0\), as a consequence we have a negative energy density \(\rho\). The total mass per unit length obtained after integrating the density from \(r = 1\) to \(r = \infty\) is \(\mu = -\frac{c^2}{4G}\).

A cosmic string with a positive mass per unit of length that adopts this density, \(\mu = \frac{c^2}{4G}\), represents a supermassive string \([3], [4]\). For this spacetime, a foliation with \(t\) and \(z\) constants, starts flat at the center and goes as a cylinder at infinity (see for instance reference \([6]\)). Using this result, we can offer an intuitive picture of our solution as follows: we start with a cylindrical geometry at the throat \((r = 1)\), generated by a so called topological mass equal to \(\frac{c^2}{4G}\), then, as we move out, there are shells of negative energy added. Given the absence of radial and angular component imposed in the stress energy tensor, the excess angle introduced by the negative mass added as we move out, slowly cancels the \(2\pi\) defect that lies at the throat.

Attempting to make contact with the work of Dyer and Marleau \([15]\), we added an extra term to the temporal component of the metric: \(g_{00} = -1 + [C \ln(r)]/r\). With this extra term the spacetime remains asymptotically flat, but the energy density depends on the relative values of the constants \(B\) and \(C\). In this case it is possible to adjust these constants to have a positive mass density in the neighborhood of \(r = 1\). Also the extra term added turns on the radial and angular component of the stress energy. Even though with this addition we eliminated the naked singularity mentioned in the above paragraph for \(B < 0\), we were not able to give a physical interpretation to the stress energy associated with this term.

### III. THE KLEIN GORDON EQUATION

To study the dynamics of a scalar field in this background geometry constitutes the first step to learn about the stability of the wormhole \([3]\). Besides, some of the properties observed in this solution may reappear in other integer spin fields. In the \((t, r, \phi)\) coordinates, the Klein Gordon equation takes the following form:

where \(\mu\) is defined as the mass of the scalar field. We have not included a wave propagating along the \(z\)-axis to keep the contact with the 2+1 spacetime. We concentrate ourselves in the behavior of cylindrical waves in this background metric. We look for solutions of the kind:

\[
\Psi \equiv \exp\{i[\omega t + m \phi]\} \psi_m(r),
\]

with \(m\) an integer. The differential equation for \(\psi_m\) is:

\[
\left\{ \frac{1}{r \sqrt{q(r)}} \frac{\partial}{\partial r} \left( r \sqrt{q(r)} \frac{\partial}{\partial r} \right) + \left( \kappa^2 - \frac{1}{r^2} m^2 \right) \right\} \psi_m(r) = 0
\]
where $\kappa^2 \equiv \omega^2 - \mu^2$.

This equation (16), does not seem to have a global analytical solution, however it is possible to extract some information out of it. For instance, for $r \to \infty$ or $q(r) \to 1$, it turns to a Bessel differential equation, as expected, given the cylindrical symmetry of the problem and its asymptotic Minkowskian behavior. Using the coordinates introduced in (7) and (8), near the throat of the wormhole, the Klein-Gordon equation becomes:

$$
\left( \frac{1}{1 + \frac{\chi^2}{2B}} \left[ \frac{\partial^2}{\partial \chi^2} + \frac{\partial^2}{\partial \phi^2} \right] + \frac{\partial^2}{\partial z^2} - \frac{\partial^2}{\partial t^2} \right) \Psi = \mu^2 \Psi.
$$

(17)

Using the same decomposition of $\Psi$ as the one defined previously in (15), we have

$$
d^2 \psi_m + \left[ \frac{\kappa^2}{2B} \chi^2 - m^2 + \kappa^2 \right] \psi_m = 0.
$$

(18)

The solution of this differential equation can be expressed as a combination of two confluent hypergeometric functions (see for instance [26]):

$$
\psi_m = e^{-u/2} \left[ A_{1} F_{1} \left( p, 1/2, u \right) + D u^{1/2} \frac{1}{\sqrt{2B}} \frac{3}{2} \psi_1 \left( p + 1/2, \frac{3}{2}, u \right) \right]
$$

(19)

with $A$ and $D$ constants, $u \equiv i(\kappa/\sqrt{2B}) \chi^2$ and

$$
4p = i \left[ \frac{\left( \kappa^2 - m^2 \right) \sqrt{2B}}{\sqrt{\kappa^2}} + 1 \right].
$$

(20)

For high frequencies: $\omega^2 > \mu^2$, the equation (18) is similar to the Schrödinger equation for the harmonic oscillator with an inverted potential $V(x) = -kx^2/2$, where the energy is represented by $(\kappa^2 - m^2)$. If this last expression is positive, the scalar wave crosses the throat and reaches the other twin universe. In the opposite case the wave bounces at the potential barrier, the angular momentum generates a centrifugal barrier that reflects the wave preventing it from crossing the throat. Finally, for low frequency waves $\kappa^2 < 0$, the wave tunnels through to reach the other side of the throat.

### IV. SQUEEZING THE WORMHOLE

It is advisable [22] to reduce the size of the spacetime where the energy conditions are not obeyed. Inspired in the work of Gott [16], we enclose this solution with an empty spacetime endowed with a conical singularity. We can visualize the idea as follows: keep $t$ and $z$ constants, and embed the remaining 2-surface in a three dimensional space, there we have a smooth surface with a continuously decreasing angular deficit (as displayed in Figure 2) and at a certain radius, namely $r = r^*$, we freeze the angular deficit and extend it to radial infinity with the same angular deficit.

To set up this program we have found convenient to introduce a new set of coordinates for this metric (1). The new coordinate is:

$$
R = \exp \left( \sqrt{\ln r} \sqrt{\ln(r) + B} + B \ln(\sqrt{\ln(r)} + \sqrt{\ln(r) + B}) \right),
$$

(21)

and the metric (1), becomes:

$$
ds^2 = A(r) R^{-\lambda(r)} \left( dR^2 + R^2 d\phi^2 \right) + dz^2 - dt^2
$$

(22)

with the functions $A(r)$ and $\lambda(r)$ defined as follows:

$$
\lambda(r) = 2 \left[ 1 - \left( \frac{1 + \frac{B}{\ln r}}{\sqrt{\ln r}} \right)^{-1/2} \right], \quad \text{and} \quad A(r) = \frac{r^2}{R^2(r)} \left( 1 + \frac{B}{\ln r} \right)^{-1/2},
$$

(23)

where $\lambda(r)$ is proportional to the difference between the so called topological mass that hides at $r = 1$ and the amount of negative mass we added till $r$. 

6
At a certain radius \( r \equiv r^* \), we freeze the values of both constants \( A = A(r^*) \) and \( \lambda(r) = \lambda(r^*) \), and continue the metric with these values. The metric for \( r > r^* \) becomes:

\[
ds^2 = A(r^*) R^{-\lambda(r^*)} \left( dR^2 + R^2 d\phi^2 \right) + dz^2 - dt^2,
\]

which corresponds to a cosmic string metric as first introduced by Vilenkin \[5\]. In this way, after this matching, an outside observer will not not be able to see the wormhole inside. As remarked by Dyer and Marleau, the angular surplus is not known to the rest of the universe.

V. TWO PARALLEL WORMHOLES

We have found a solution for two parallel and interconnected wormholes in the sense of Misner-Wheeler \[21\]. Unfortunately, the solution found is not traversable as we show below.

We looked for solutions with no essential singularities and that turned to a Minkowski spacetime at infinity. This constraint was imposed to keep the same boundary conditions as obtained for the previous example.

Let’s consider the following metric:

\[
ds^2 = -dt^2 + dz^2 + h(r, \phi) \left[ dr^2 + r^2 d\phi^2 \right],
\]

where the coordinate ranges are \( r \in [0, \infty] \) and \( \phi \in [0, \pi] \). The Einstein’s equations are:

\[
G_{\mu \nu} = G_r^r = -\frac{1}{2} R,
\]

The expression for the Ricci scalar is:

\[
R = \frac{-1}{h(r, \phi)} \nabla^2 \left[ \ln(h(r, \phi)) \right], \quad \text{where} \quad \nabla^2 \equiv \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2}{\partial \phi^2}.
\]

Also, the metric \[24\] obeys the following identity, that characterizes the 2+1 geometries:

\[
R_{\mu \nu \alpha \beta} R^{\mu \nu \alpha \beta} = R^2.
\]

This result indicates that the relevant physics occurs in the \( z \) = constant, plane. To reach a Minkowski spacetime at infinity, we need: \( \lim_{r \to \infty} h(r, \phi) = 1 \).

The usual approach to this problem is to concentrate all the curvature as a singularity at the throat of the wormhole and set the rest of the (two dimensional) spacetime flat. This is equivalent to set \( \nabla^2 h(r, \phi) = 0 \). This is the approach followed, for example, by Clément \[22\].

Looking for a solution with two singularities that represents two parallel wormholes, we have found an expression for \( h(r, \phi) \) that fulfills (in part) the conditions we required:

\[
h(r, \phi) = \frac{1}{\left( 1 - \frac{2\pi a^2 \cos^2 \phi c c}{(r^2 + a^2)^2} \right)^2},
\]

where \( a \) and \( c \) are two independent parameters. For this choice of \( h(r, \phi) \) \[28\], the Einstein’s tensor is:

\[
G_{tt} = G_{zz} = -8 \left[ a^4 r^8 + \left( 2c^4 - 4a^4 \cos 2\phi \right) r^6 + 2a^6 \left( c^4 - 2a^4 \sin^2 2\phi + 3a^4 \right) r^4 \right.
\]

\[
- \left. \left( 2c^4(1 - 4 \cos^2 \phi) + 4a^4 \cos 2\phi \right) a^4 r^2 + a^2 \left( a^4 - c^4 \right)^2 \right] / \left( r^2 + a^2 \right)^6.
\]

The structure of the coordinate singularity of equation \[28\] is given by:

\[
[r^2 + (a^2 - c^2)] \left[ r^2 + (a^2 + c^2) \right] - 4r^2 a^2 \cos^2 \phi = 0.
\]

The curve described by this equation depends on the relative values of the parameters \( a \) and \( c \). Solving equation \[24\], we obtain
\[ r^2 = 2a^2 \cos 2\phi \pm \sqrt{4a^4 \cos^2 2\phi - (a^4 - c^4)} \]  

(31)

For \( a < c \), the square root in equation (31) is larger than the first term, therefore there exists a solution for \( \phi = 0 \) and \( \phi = \pi/2 \); the curve adopts peanut shape as shown in Figure 3. If \( a \) starts growing, we reach the equality \( a = c \), here the peanut collapse to a pair of touching loops. It is easier to see this result from the previous equation (31): for \( a = c \), there are two solutions \( r = 0 \) for any value of \( \phi \) and \( r^2 = 2a^2 \cos 2\phi \), valid for a limited range of \( \phi \), in agreement with the curve displayed in Figure 4. For \( a > c \), there is not solution for \( \phi = \pi/2 \) because the loops detached from each other as indicated in Figure 5, where we have drawn this curve for \( c = 1 \) and \( a = 1.01 \), generating two strings separated by a certain distance, that mimics two parallel wormholes.

Note that the two egglike figures obtained in this case (Figure 5), turn to a pair of circles as we increase the distance that mediates between them. What happens in this case is the following: as we move the strings apart, the interaction between them becomes weaker and, when the relative distance increases enough, both strings behave like two independent cosmic strings.

Other condition needed for the existence of a wormhole, is the absence of curvature singularities in this neighborhood. This requirement is also satisfied by the eqs. (25) and (29), as well as the other invariants (27).

It is straightforward to write the geodesic equations for this geometry. Since they do not have an analytic solution, in Figure 6 we have plotted a family of infalling lightlike geodesics which are purely radial at \( r = 3 \). We have used \( a = 1 \) and \( c = 1.1 \) to characterize the coordinate singularity and we have embedded the geodesics in a three dimensional space. The geodesics displayed in Figure 6 yields information about the geometry of this spacetime near the coordinate singularity.

Given the bifocal symmetry of the problem, there only exist geodesics independent of the coordinate \( \phi \), for those cases where \( \phi = 0 \) and \( \partial h(r, \phi)/\partial \phi = 0 \), as it can be shown from the geodesic equations. These conditions are saved precisely for \( \phi = \frac{\pi}{2} \). To prove that in this case it takes an infinite time to reach the throat, we compute the time spent for a particle radially falling. The radial velocity is given by

\[ \frac{dr}{dt} \propto \frac{1}{\sqrt{h(r, \phi = 0)}}, \quad \text{therefore} \quad t \propto \int \frac{dr}{\sqrt{r^2 - (2a^2 + \sqrt{3a^4 - c^4})}}. \]

extracting the singular term from this integral, we obtain

\[ t \propto \ln \left( r - \sqrt{2a^2 - \sqrt{3a^4 - c^4}} \right). \]  

(32)

The border of the throat is characterized by the vanishing of the expression between the parenthesis, therefore it becomes clear that it takes an infinite time to reach the throat of this wormhole for this particular case. We assume that the rest of the orbits behave in a similar way.

VI. CONCLUSIONS AND FINAL REMARKS

We have studied a solution of Einstein’s equations with cylindrical symmetry with a wormhole at its core. The negative energy density region associated with this wormhole has been minimized and confined inside a cylinder around the core, properly matching it to an exterior Gott’s like metric. In this way we have hidden the wormhole away from an observer at infinity. Nevertheless, we used the negative energy density solution throughout this work to benefit from its simplicity and because our main interest has been the geometry in the neighborhood of the wormhole.

The continuous change that experiments the value of the angular deficit as we move with respect to the core of the string, allows the presence of geodesics that go around the string. Some of them bounce back to infinity, if \( D < L^2 \), as if there existed a centrifugal barrier, or cross the wormhole through, if \( D > L^2 \). In this last case, the time it takes to cross it, is finite, either for a lightlike or a timelike particle. There exist closed orbits at \( r = 1 \) if \( D = L^2 \). For these particular conditions, the time it takes to reach the throat is infinite, therefore the particle can be considered as effectively trapped in the neighborhood of the wormhole. This is a source of a classical type of instability, as opposed to the quantum one mentioned in the text, in the following sense: the concentration of
many particles bounded to reach the throat will change the sign of the energy density evaporating
the wormhole from this region.

The existence of a solution to the Klein Gordon equation in the neighborhood of the throat, opens
up the possibility to study the stability of this wormhole in a more qualitative way.

After building consistently two parallel wormholes in the same spacetime, we discover that they
are not traversible. It takes an infinite time to get close to the throat. It seems that the negative
energy stored in the region between the strings stretches to infinity the path to the throat. The
interaction between these two strings is seen through the shape of the cross sections of the strings we
have drawn in the Figures 4, 5 and 6. When the wormholes are separated the interaction decreases
and each of them get the appearance of a single isolated one.

A solution of the kind we have displayed here has been mentioned in the work of Dyer and Marleau
[15]. They have solved numerically the full set of equations of the global string coupled to gravity. By
relaxing the boundary conditions at the core, they have seen strings solutions that have an angular
surplus (deficit) at the axis and are asymptotically Minkowski at infinite. We claim that our solution
constitutes a simple analytic partner of theirs.

The work of Clément [20] is similar to ours, but the curvature appears as a singularity at the
union between the flat space and the mouth of the wormhole. In our case, the curvature is smooth
and everywhere different from zero.

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FIGURE CAPTIONS

Figure 1: Here appears the matching of the two spacetime connected through $r = 1$. Each of the surfaces is an embedding on a three dimensional space of the coordinates $r$ and $\phi$ of the wormhole.

Figure 2: The trajectory of a null geodesic is depicted here. The angular momentum is smaller than the energy $L < E$, therefore this geodesic crosses the throat of the wormhole as shown by our calculations.

Figure 3: The shape of the singularity for the case $a = 1$ and $c = 1.1$. The abscissa corresponds to $r \cos \phi$ and the ordinate to $r \sin \phi$.

Figure 4: The loops that appear in the Figure corresponds to $a = c$. In particular, for this case we have used $a = 1$. The point where the loops converge $r = 0$, is singular.

Figure 5: The more interesting case corresponds to $a > c$. For this relative values of $a$ and $c$, we obtain two disconnected loops that represent two parallel wormholes. The Figure corresponds to $a = 1.01$ and $c = 1$.

Figure 6: This Figure represents a family of radially infalling lightlike geodesics. We notice that, in their fall they mimic the wormhole throat shape. Also we observe that the ones located at positions where the wormhole throat is faced symmetrically remain radial along the throat.
FIG. 2.

FIG. 3.
