Universal solutions for interacting bosons in one-dimensional harmonic traps

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Abstract

We consider systems of interacting bosons confined to one-dimensional harmonic traps. In the limit of perturbatively weak two-body interactions the system exhibits several universal states that are exact solutions for a large class of two-body interactions. These states are closely related to the exact solutions found previously in rotating Bose-Einstein condensates.

I. INTRODUCTION

Bose-Einstein condensation in dilute atomic gases has received considerable attention in the past few years [1]. A recent experiment reports the realization of a quasi one-dimensional Bose-Einstein condensate [2] in a highly anisotropic, cigar-shaped, three-dimensional trap [3]. Corresponding theoretical studies predict qualitatively different regimes for the condensate depending on the density and the strength of the two-body interaction. For high densities and strong repulsive interaction, the Thomas-Fermi regime applies. Presently, this regime is accessed in the experiment [3]. For low densities and sufficiently strong repulsive interactions, theory predicts a Girardeau-Tonks gas of impenetrable hard cores [4–6]. A necessary condition for this regime is that the interaction energy per particle exceeds the oscillator spacing [5]. If this condition is violated, one approaches the regime of a gas with perturbatively weak interactions.

In this work, we want to consider the regime of low densities and perturbatively weak interactions. Within this regime the interaction simply lifts the enormous degeneracy of the harmonically trapped $N$-boson system. We will show that the Hilbert space and the structure of the Hamiltonian is closely related to the problem of rotational states in weakly interacting harmonically trapped Bose systems in two spatial dimensions [7–9]. This allows us to transfer exact solutions from the latter problem to the present case. We will present universal wave functions that are exact solutions for a wide class of two-body interactions.

This paper is divided as follows. In the following section we state the problem in first quantization. Next, we establish the close relationship to rotating Bose-systems. This allows us to present exact solutions for the one-dimensional system in analogy to the case of rotating systems. In the fourth section we investigate the evolution of the spectrum under change of a parameter and find the coexistence of exact solvability and nonintegrability. We finally give a summary.
Consider $N$ bosons in a one-dimensional harmonic trap that interact via two-body contact interactions. The Hamiltonian is

$$H = \frac{1}{2} \sum_{j=1}^{N} \left( -\frac{\partial^2}{\partial x_j^2} + x_j^2 \right) + g \sum_{1 \leq i < j \leq N} \delta(x_i - x_j),$$

where $g$ is a coupling constant. For vanishing $g = 0$ the spectrum consists of degenerate sets of levels that differ by multiples of the oscillator spacing. For simplicity we set the oscillator spacing to one and set the ground state energy to zero. Then the energies are simply given by integer values $E = 0, 1, 2, \ldots$. Many-body basis functions are completely symmetrized products $\phi_{n_1}(x_1) \ldots \phi_{n_N}(x_N)$ of single-particle oscillator wave functions $\phi_n(x)$, $n = 0, 1, 2, \ldots$ and fulfill $\sum_{j=1}^{N} a_j = E$. At energy $E$ the number of degenerate states equals the number of partitions of $E$ into at most $N$ integers: Hilbert space is partition space. Switching on the two-body interaction lifts the degeneracy between the levels. For sufficiently weak interaction $N|g| \ll 1$ the quasi-degenerate levels are much more closely spaced than the oscillator spacing, and levels with different energies $E$ do not mix. This is the regime we are interested in. In first order perturbation theory, the spectrum can be obtained by diagonalizing the interaction in the space of degenerate levels.

We analytically treat the problem in first quantization. Let $\hat{a}_j$ and $\hat{a}_{\dagger j}$ create and annihilate an excitation of boson $j$, i.e. $\hat{a}_j \phi_n(x_j) = \sqrt{n+1} \phi_{n+1}(x_j)$, and $\hat{a}_{\dagger j} \phi_n(x_j) = \sqrt{n} \phi_{n-1}(x_j)$. These operators fulfill the usual commutation relation $[\hat{a}_i, \hat{a}_{\dagger j}] = \delta_{ij}$. It is useful to express the contact interaction in terms of creation and annihilation operators:

$$\delta(x_k - x_l) = \delta \left( \frac{\hat{a}_{\dagger k} - \hat{a}_{\dagger l}}{\sqrt{2}} + \frac{\hat{a}_k - \hat{a}_l}{\sqrt{2}} \right)$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} dt \exp \left[ i \frac{t}{\sqrt{2}} \left( \hat{a}_{\dagger k} - \hat{a}_{\dagger l} + \hat{a}_k - \hat{a}_l \right) \right]$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} dt \exp \left( -\frac{t^2}{2} \right) \exp \left[ i \frac{t}{\sqrt{2}} \left( \hat{a}_{\dagger k} - \hat{a}_{\dagger l} \right) \right] \exp \left[ i \frac{t}{\sqrt{2}} \left( \hat{a}_k - \hat{a}_l \right) \right]$$

$$= \frac{1}{2\pi} \sum_{m,n=0}^{\infty} \frac{\Gamma \left( \frac{m+n+1}{2} \right)}{m! n!} \left( \hat{a}_{\dagger k} - \hat{a}_{\dagger l} \right)^m (\hat{a}_k - \hat{a}_l)^n \tag{2}.$$
\[ \hat{V} = \sum_{m=0}^{\infty} c_m \hat{A}_m, \]  
(4)

with

\[ \hat{A}_m = \sum_{1 \leq k < l \leq N} \left( \hat{a}_k \dagger - \hat{a}_l \dagger \right)^m \left( \hat{a}_k - \hat{a}_l \right)^m. \]  
(5)

The coefficients \( c_m \) depend on the details of the specific two-body interaction under consideration. One finds, e.g., that a monomial interaction of the form \((x_k - x_l)^n\) yields coefficients

\[ c_m = \frac{(2n)! \cdot 2^{-n} \cdot \left( \begin{array}{c} n+m-1 \end{array} \right)}{(n-m)! \cdot (m)!^2} \]

for \( m \leq n \) and \( c_m = 0 \) otherwise.

This result can be derived by employing the characteristic function that generates the moments of the position variable in the harmonic oscillator. Thus, zero range interactions and interactions that are analytical in the two-particle distance are of the general form (4) and (5).

The operator

\[ \hat{E} = \sum_{k} \hat{a}_k \dagger \hat{a}_k \]  
(6)

counts the number \( E \) of excited quanta and clearly commutes with the interaction (4). In harmonic systems with interactions that depend only on the two-particle distances, one may separate the motion of the center-of-mass mode

\[ \hat{a}_c \dagger = \frac{1}{N} \sum_{k=1}^{N} \hat{a}_k \dagger. \]  
(7)

The energy associated with this mode is given by

\[ \hat{E}_c = N \hat{a}_c \dagger \hat{a}_c. \]  
(8)

This operator commutes with the interaction (4) and the energy (6). It has integer quantum numbers denoted by \( E_c \), and \( E_c = 0, 1, 2, \ldots, E - 2, E \) for fixed energy \( E \).

### III. One-Dimensional Case and Yrast Line Problem

When stated in terms of the operators \( \hat{V}, \hat{E} \) and \( \hat{E}_c \) presented in equations (4), (6), and (8), respectively, the problem of interacting bosons confined by a one-dimensional harmonic potential is very similar to the Yrast line problem of rotating bosons in spherically symmetric two-dimensional harmonic traps [7–9]. The Yrast line problem possesses analytical solutions for several wave functions and energies [9–16]. We may thus transfer these solutions to the present case. Before we do so, we briefly remind the reader of the Yrast line problem and its results that are relevant for us.

\[ ^1 \text{The terminology “Yrast line” is borrowed from nuclear physics.} \]
The Yrast line problem consists of finding the ground states of a rotating many-body system as a function of total angular momentum $L$. Let us assume that the bosons are in their ground state with respect to excitations in the $z$-direction. The problem thus becomes essentially two-dimensional. Again, one considers perturbatively weak interactions that simply lift the degeneracy of the harmonic trap. Single-particle wave functions are $\psi_l(z) = (2\pi l!)^{-1/2} z^l$, where $z = x + iy$ is the coordinate, and we have omitted the Gaussian in the wave function. Many-body basis states are symmetrized products of single-particle wave functions subject to the requirements that the single-particle angular momenta fulfill $\sum_{j=1}^N l_j = L$. The Hilbert space is thus partition space and isomorphic to the Hilbert space of weakly interacting bosons in one-dimensional traps once we identify $L = E$. Let us now turn to operators in the Yrast line problem. Clearly, $z_j$ and $\frac{\partial}{\partial z_j}$ are creation and annihilation operators, respectively, when acting on the single-particle state $\psi_l(x)$. Upon substitution

$$\hat{a}_j^\dagger \leftrightarrow z_j, \quad \hat{a}_j \leftrightarrow \frac{\partial}{\partial z_j}, \quad (9)$$

we indeed obtain all relevant operators for the Yrast line problem [14]: Two-body interactions $\hat{V}$ have the form of eq. (4), the total angular momentum corresponds to the operator $\hat{E}$ in eq. (6), and the angular momentum of the center of mass is given by the operator $\hat{E}_c$ in eq. (8). Thus, there is a correspondence between the Yrast line problem and interacting bosons in one-dimensional harmonic traps. This correspondence is based on the isomorphism of the Hilbert spaces and a formal identity between operators. The Yrast line wave functions

$$\Psi_L(z_1, \ldots, z_N) = \sum_{1 \leq p_1 < p_2 < \cdots < p_L \leq N} (z_{p_1} - z_c)(z_{p_2} - z_c) \cdots (z_{p_L} - z_c) \prod_{j=1}^N e^{-\frac{1}{2} |z_j|^2} \quad (10)$$

are exact solutions for a large class of two-body interactions for total angular momentum $L$, and $2 \leq L \leq N$ [13][14][16]. Here $z_c = N^{-1} \sum_{j=1}^N z_j$ denotes the center of mass. The associated eigenenergy is $\frac{1}{2} N(N-1)c_0 + NL(c_1 + 2c_2)$, and the coefficients $c_0, c_1$ and $c_2$ are determined by the specific interaction (3) under consideration. Let us transfer this important result to the present case. It is a peculiarity of the Yrast line problem that the creation operators, single-particle coordinates and single-particle wave functions are all denoted in terms of $z_j, j = 1, \ldots, N$. Clearly, the Gaussian in the wave function (10) involves only coordinates and no operators. Taking the correspondence between operators (9), we thus find from eq. (10) that

$$\Phi_E(x_1, \ldots, x_N) = \sum_{1 \leq p_1 < p_2 < \cdots < p_E \leq N} (\hat{a}_{p_1}^\dagger - \hat{a}_c^\dagger) \cdots (\hat{a}_{p_E}^\dagger - \hat{a}_c^\dagger) \prod_{j=1}^N e^{-\frac{1}{2} x_j^2} \quad (11)$$

is an eigenfunction of interaction (3). The corresponding energy is [14]

$$\epsilon(N, E) = \frac{1}{2} N(N-1)c_0 + NE(c_1 + 2c_2), \quad (12)$$

for $2 \leq E \leq N$. This is the main result of this work. A direct proof of this statement may be obtained by repeating the derivation in ref. [14] and using the correspondence (9). It is
based on the observation that the state (11) is an eigenfunction of the operators $\hat{A}_m$. The eigenvalues are $\frac{1}{2}N(N-1), NE$ and $2NE$ for $\hat{A}_0, \hat{A}_1$ and $\hat{A}_2$, respectively. The operators $\hat{A}_m$ with $m > 2$ annihilate this state and thus have zero eigenvalue. By construction, the state $\Phi_E$ is totally symmetric in the single-particle coordinates, involves $E$ excited quanta, and does not excite the center-of-mass mode (7), i.e. $\hat{E}_c \Phi_E = 0$. Note that the wave function (11) does not depend on the details of the interaction (4) since there is no reference to the coefficients $c_j$. It is therefore a universal solution for a large class of two-body interactions (4). The corresponding eigenenergy (12), however, does depend on the first three coefficients $c_0, c_1$ and $c_2$.

Note that we may also establish a correspondence between basis wave functions of the Yrast line problem and the present problem. In the former system, wave functions are Gaussians multiplied by homogeneous polynomials of degree $L$ that are totally symmetric in the single-particle coordinates $z_j, j = 1, \ldots, N$. Corresponding wave functions for the present case are obtained by using the correspondence rule (4) and act with the resulting operator on the Gaussian ground state. In this basis, both wave functions (11) and (11) have identical structure, i.e. identical expansion coefficients. These wave functions are also identical in second quantization.

We next ask the question of whether other states in the spectrum can be written as exact algebraic functions. Leaving aside center-of-mass excitations, there are at most two states with $E_c = 0$ in the spectrum when $E < 6$ and they are both algebraic. The representation of the second state is presented in ref. [14]. We believe that there are no additional universal states when $E \geq 6$, as will be seen in the next section.

**IV. OTHER UNIVERSAL SOLUTIONS?**

Let us investigate whether there are further universal wave functions. To this purpose we may take an interaction (4) that depends on a parameter $\tau$ and consider the spectrum as a function of the parameter. Avoided crossings and level repulsion would certainly indicate nonintegrability. We may further monitor the wave function structure as a function of the parameter and thereby search for other universal states. This requires numerical calculations.

Let us consider the interaction

$$\hat{W}(\tau) \equiv (1 - \tau) \hat{V}_{1d} + \tau \hat{V}_{pert}$$

(13)

which interpolates between the contact interaction $\hat{V}_{1d}$ in one-dimensional systems and the interaction $\hat{V}_{pert}$ as $\tau$ evolves from zero to one. We take $\hat{V}_{pert} = 10^{-4} \hat{A}_4$ for definiteness. Numerical computations are most conveniently done in second quantization. Let $\hat{b}^\dagger_n$ and $\hat{b}_n$ create and annihilate a boson in the single particle state $|n\rangle$ with energy $n$, respectively. Clearly, we have $\langle x|n\rangle = \phi_n(x)$. At fixed energy $E$, Hilbert space is spanned by many-body states $|n_0, n_1, \ldots, n_E\rangle$ with $\sum_{j=0}^E n_j = N$ and $\sum_{j=0}^E j n_j = E$. The matrix elements for the contact interaction $\hat{V}_{1d}$ involve an integral over four oscillator functions and can be calculated analytically [17]. The second quantized form of the operator (3) reads

$$\hat{V}_{1d} = \frac{1}{2} \sum_{ijkl} v_{ijkl} \hat{b}^\dagger_i \hat{b}^\dagger_j \hat{b}_k \hat{b}_l,$$

(14)
with

\[ v_{ijkl} = 2^{-\frac{1}{2}} \frac{k! l!}{i! j!} \sum_{t=0}^{\min(k,l)} \frac{\Gamma\left(t + \frac{1}{2}\right) \Gamma\left(i - t + \frac{1}{2}\right) \Gamma\left(j - t + \frac{1}{2}\right)}{t! (k - t)! (l - t)!} \delta_{i+j+t}. \]

The operator \( \hat{A}_m \) has similar structure and matrix elements

\[ a_{ijkl}^{(m)} = \sum_{\nu = \max(0,m-k,l-i)}^{\min(m,l,m-i+l)} \binom{m}{\nu} \binom{m}{i-l+\nu} \frac{(-1)^{i-l}}{(l-\nu)!} \frac{\sqrt{i! j! k! l!}}{(k-m+\nu)!} \delta_{i+j+l}. \]

We compute the spectrum of \( \hat{W}(\tau) \) for \( 0 \leq \tau \leq 1 \). We have fixed \( E = 12, N = 30 \) and have restricted ourselves to the states with \( E_c = 0 \). The lowest energy is set to zero. Fig. 1 shows several avoided crossings as the parameter \( \tau \) evolves, and there are no crossings. This indicates that the operator \( \hat{W}(\tau) \) is nonintegrable. Inspection of the eigenstates shows that only the lowest energy state is independent of \( \tau \). On the one hand, this finding demonstrates the universality of the wave function (11). On the other hand, it shows that there are no further states that are universal solutions. The system of bosons in one-dimensional harmonic traps, as well as the Yrast line problem, thus exhibits exact solvability in coexistence with level repulsion and nonintegrability.

Due to their nonintegrability, the systems considered in this work differ from other exactly solvable many-body systems. We recall that the Calogero-Sutherland models with long-ranged inverse square potentials are completely integrable [18,19]. Other examples are the one-dimensional system of zero-range interacting bosons confined to a box which is exactly solvable for all interaction strengths [20] or the Tonks-Girardeau gas of hard-core bosons in one-dimension [21,22]. However, the present work shows that harmonically confined Bose systems display a few universal wave functions that are exact solutions for a large class of perturbatively weak two-body interactions. In this sense these fall into the class of partially solvable quantum many-body problems [23].

Let us finally compare the spectra of the contact interactions in one-dimensional traps and the Yrast line problem. To this purpose we choose \( \hat{V}_{\text{pert}} = \hat{V}_{\text{Yrast}} \) and repeat the numerical calculation. The contact interaction \( \hat{V}_{\text{Yrast}} \) for the Yrast line problem is of identical structure as the operator \( \hat{V}_{1d} \) in eq. (14) but has matrix elements [8]

\[ w_{ijkl} = \frac{(k + l)!}{2^{k+l} \sqrt{i! j! k! l!}} \delta_{i+j+l}. \]

We compute the spectrum of \( \hat{W}(\tau) \) for \( 0 \leq \tau \leq 1 \). We choose \( E = 14, N = 100 \) and restrict ourselves to the states with \( E_c = 0 \). Fig. 2 shows the result. The lowest energy is set to zero, and the spectra at \( \tau = 0 \) and \( \tau = 1 \) are scaled to cover similar spectral ranges for display purposes. Clearly, the operators \( \hat{V}_{1d} \) and \( \hat{V}_{\text{Yrast}} \) yield very similar spectra and are in this sense close together. This suggests that one may transfer several results from the Yrast problem to the present case. E.g., the structure of low-lying excitations [8,24,28] are expected to be similar in both problems. Though the spectrum does not undergo dramatic changes, we found only one universal state and a few avoided crossings as well.
V. SUMMARY

We have established a close relationship between the problem of interacting bosons in one-dimensional harmonic traps and the Yrast state problem for interacting bosons in two-dimensional isotropic harmonic traps. The Hilbert spaces of both problems are isomorphic to each other and the important operators have identical structure. This allowed us to transfer several analytical results concerning eigenstates from the Yrast problem to the one-dimensional case. In particular, the one-dimensional problem exhibits universal wave functions that are exact solutions for a large class of two-body interactions, too. Among these are the lowest-lying excitations of the Bose-Einstein condensate. It is remarkable that one can thus learn something about rotational states from wave functions in one-dimensional, non-rotating systems. We observed level repulsion in a parametric Hamiltonian. This indicates that the system is nonintegrable.

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REFERENCES

[1] F. Dalfovo, S. Giorgini, L. P. Pitaevskii, and S. Stringari, Rev. Mod. Phys. 71 (1999) 463.
[2] W. Ketterle and N. J. van Druten, Phys. Rev. A 54 (1996) 656.
[3] A. Görlitz, J. M. Vogels, A. E. Leanhardt, C. Raman, T. L. Gustavson, J. R. Abo-Shaeer, A. P. Chikkatur, S. Gupta, S. Inouye, T. P. Rosenband, D. E. Pritchard, and W. Ketterle, cond-mat/0104549.
[4] D. S. Petrov, G. V. Shlyapnikov, and J. T. M. Walraven, Phys. Rev. Lett. 85 (2000) 3745.
[5] V. Dunjko, V. Lorent, and M. Olshanii, Phys. Rev. Lett. 86 (2001) 5413.
[6] M. D. Girardeau, E. M. Wright and J. M. Triscari, Phys. Rev. A 63 (2001) 033601.
[7] N. K. Wilkin, J. M. F. Gunn, R. A. Smith, Phys. Rev. Lett. 80 (1998) 2265.
[8] B. Mottelson, Phys. Rev. Lett. 83 (1999) 2695.
[9] G. F. Bertsch and T. Papenbrock, Phys. Rev. Lett. 83 (1999) 5412.
[10] G. M. Kavoulakis and A. D. Jackson, Phys. Rev. Lett. 85 (2000) 2854.
[11] R. A. Smith and N. K. Wilkin, Phys. Rev. A 62 (2000) 061602.
[12] T. Papenbrock and G. F. Bertsch, Phys. Rev. A 63 (2001) 023616.
[13] W.-J. Huang, Phys. Rev. A 63 (2001) 015602.
[14] T. Papenbrock and G. F. Bertsch, J. Phys. A 34 (2001) 603.
[15] Y. Wu, X. Yang, and Y. Xiao, Phys. Rev. Lett. 86 (2001) 2200.
[16] M. S. Hussein and O. K. Vorov, cond-mat/0102505.
[17] I. W. Busbridge, J. London Math. Soc. 23 (1948) 135.
[18] F. Calogero, J. Math. Phys. 10 (1969) 2191, 2197; 12 (1971) 419.
[19] B. Sutherland, J. Math. Phys. 12 (1971) 246; 12 (1971) 251.
[20] E. H. Lieb and W. Lininger, Phys. Rev. 130 (1963) 1605.
[21] L. Tonks, Phys. Rev. 50 (1936) 955.
[22] M. Girardeau, J. Math. Phys. 1 (1960) 516.
[23] F. Calogero, J. Math. Phys. 40 (1999) 4208.
[24] G. M. Kavoulakis, B. Mottelson, and C. J. Pethick, Phys. Rev. A 62 (2000) 063605.
[25] T. Nakajima and M. Ueda, Phys. Rev. A 63 (2001) 043610.
[26] V. Bardek, L. Jonke, and S. Meljanac, cond-mat/0012438, to appear in Phys. Rev. A.
[27] G. M. Kavoulakis, B. Mottelson, and S. M. Reimann, Phys. Rev. A 63 (2001) 055602.
[28] M. Ueda and T. Nakajima, cond-mat/0012458.
FIG. 1. Spectrum of \((1 - \tau) \hat{V}_{1d} + 10^{-4} \tau \hat{A}_4\) as a function of \(\tau\). There are several avoided crossings.
FIG. 2. Spectrum of $(1 - \tau) \hat{V}_{1d} + \tau \hat{V}_{Yrast}$ as a function of $\tau$. The spectrum exhibits only a few avoided crossings.