HAANTJES ALGEBRAS AND DIAGONALIZATION

PIERGIULIO TEMPESTA AND GIORGIO TONDO

Abstract. We propose the notion of Haantjes algebra, which consists of an assignment of a family of fields of operators over a differentiable manifold, with vanishing Haantjes torsion and satisfying suitable compatibility conditions among each others. Haantjes algebras naturally generalize several known interesting geometric structures, arising in Riemannian geometry and in the theory of integrable systems. At the same time, they play a crucial role in the theory of diagonalization of operators on differentiable manifolds. Whenever the elements of an Haantjes algebra are semisimple and commute, we shall prove that there exists a set of local coordinates where all operators can be diagonalized simultaneously. Moreover, in the non-semisimple case, they acquire simultaneously a block-diagonal form.

CONTENTS

1. Introduction 1
2. Nijenhuis and Haantjes operators 3
3. The geometry of Haantjes operators 8
4. Haantjes algebras
   4.1. Main Definition 14
   4.2. Characteristic Haantjes coordinates 16
   4.3. Cyclic Haantjes algebras 18
5. Cyclic-type Haantjes operators, triangular and Jordan: Examples 21
   5.1. Construction of a Haantjes chart 22
   5.2. A triangular but not Jordan-type operator 24
   5.3. Jordan form 25
6. A comparison with other algebraic structures: Haantjes manifolds and Killing-Stäckel algebras 25

Acknowledgement
References

1. Introduction

The purpose of this paper is to introduce a new geometric-algebraic structure, that we shall call Haantjes algebra, based on the notion of Haantjes torsion. This notion was introduced in 1955 by J. Haantjes in [13], and represents a natural generalization of the torsion defined by Nijenhuis in [20]. Although the theory of tensors with vanishing Nijenhuis torsion has been intensively investigated in the last forty years, mainly due to its applications in the theory of integrable systems.
and separation of variables (where they are usually called recursion operators), quite surprisingly the relevance of Haantjes’s differential-geometric work has not been recognized for a long time, with the exception of some notable applications to Hamiltonian systems of hydrodynamic type [6, 9]. For a nice review of classical and more recent results about the theory of Nijenhuis and Haantjes tensors, see [14].

Our work is inspired, from one side, by the notion of Haantjes manifolds proposed in [16, 17], from the other side by the concept of symplectic-Haantjes manifold or $\omega H$ manifold that we have recently introduced in [22] in connection with the theory of classical integrable systems. Indeed, $\omega$ represents a symplectic two-form; besides, the $\omega H$ structure provides us with a natural theoretical framework for dealing with the integrability and separability properties of mechanical systems, and represents a formulation that parallels and in some aspects completes the one offered by the Nijenhuis geometry.

In this article, we will extend the previous constructions by proposing a very general and abstract setting. The resulting geometric-algebraic structure will be called Haantjes algebra. It consists essentially of a differentiable manifold $M$ endowed with a family of endomorphisms of the tangent bundle with vanishing Haantjes torsions and compatible among each others. The existence of an underlying symplectic structure is no longer required.

From one hand, this simple geometric structure is a flexible tool that in principle can be specialized to treat many different interesting constructions in a natural and unified language. Needless to say, Magri’s Haantjes manifolds are a specially relevant instance of Haantjes algebras. Another important class of Haantjes algebras is represented by the Killing–Stäckel algebras introduced in [3] over a $n$-dimensional Riemannian manifold, in order to characterize separation of variables in classical Hamiltonian systems.

From the other hand, our main motivation, apart the intrinsic interest of a geometric structure combining the Haantjes geometry with that of symplectic (or Riemannian) manifolds, is the abstract problem of diagonalization of operators on a differentiable manifold. Indeed, we shall prove that the algebras of Haantjes fields of operators introduced below can be diagonalized simultaneously in an appropriate local coordinate system, that we shall call a Haantjes chart. Note that no hermiticity assumption is made on our operators: we only require that they are point-wise diagonalizable and that their Haantjes torsion vanishes. Being the latter a fourth-degree requirement in such operators (see formula (4) below), it is easy to ascertain that a very large class of tensor fields do satisfy it.

Our statement concerning diagonalization, proved in Theorem 39, is the following

**Main Result.** Given a semisimple Abelian Haantjes algebra $(M, \mathcal{H})$, there exists a set of local coordinates in which every $K \in \mathcal{H}$ can be simultaneously diagonalized. Conversely, let $(K_1, \ldots, K_m)$ be a set of $m$ linearly independent fields of operators. If they share a set of local coordinates in which they take simultaneously a diagonal form, then they generate a semisimple Abelian Haantjes algebra of rank $m$.

The previous result can also be extended to the very general but largely unexplored case of **non-semisimple** Haantjes operators. Indeed, for this class we shall
prove that there exists a local coordinate system where all the operators of a Haantjes algebra acquire simultaneously a quasi-diagonal (i.e. block-diagonal) form.

We also wish to mention that in this article a new, infinite “tower” of generalized Nijenhuis torsions of level \( n \) for all \( n \in \mathbb{N} \) is also defined. The geometrical meaning of our notion, which naturally generalizes both the classical Nijenhuis and Haantjes torsions, deserves to be further investigated and will be discussed in detail elsewhere.

The paper is organized as follows. After a discussion, proposed in Section 2, of the main algebraic structures needed in this work, including the generalized torsions, we shall present in Section 3 a brief introduction to the Nijenhuis and Haantjes geometries. In Section 4, the formal construction of a Haantjes algebra is proposed; the main result of the theory and other relevant properties of our Haantjes algebras are also proved. In Section 5, we present some examples illustrating the main theorems, the theory of cyclic generators of semisimple Haantjes algebras and the case of non-semisimple Haantjes operators. A comparison with other related geometric structures is proposed in the final Section 6.

2. Nijenhuis and Haantjes Operators

The natural frames of vector fields associated with local coordinates on a differentiable manifold, being obviously integrable, can be characterized in a tensorial manner as eigen-distributions of a suitable class of \((1,1)\) tensor fields, i.e. the ones with vanishing Nijenhuis or Haantjes torsion. In this section, we review some basic algebraic results concerning the theory of such tensors. For a more complete treatment, see the original papers \([13, 20]\) and the related ones \([21, 10]\).

Let \( M \) be a real differentiable manifold and \( L : TM \rightarrow TM \) be a \((1,1)\) tensor field, i.e., a field of linear operators on the tangent space at each point of \( M \).

**Definition 1.** The Nijenhuis torsion of \( L \) is the skew-symmetric \((1,2)\) tensor field defined by

\[
\mathcal{T}_L(X,Y) := L^2[X,Y] + [LX,LY] - L([X,LY] + [LX,Y]),
\]

where \( X, Y \in TM \) and \([\ , \ ]\) denotes the commutator of two vector fields.

In local coordinates \( x = (x^1, \ldots, x^n) \), the Nijenhuis torsion can be written in the form

\[
(\mathcal{T}_L)_{jk} = \sum_{\alpha=1}^{n} \left( \frac{\partial L^i_\alpha}{\partial x^\alpha} L^\alpha_j - \frac{\partial L^i_j}{\partial x^\alpha} L^\alpha_\alpha + \left( \frac{\partial L^i_\alpha}{\partial x^\alpha} - \frac{\partial L^i_j}{\partial x^\alpha} \right) L^\alpha_j \right),
\]

amounting to \( n^2(n-1)/2 \) independent components.

**Definition 2.** The Haantjes torsion associated with \( L \) is the \((1,2)\) tensor field defined by

\[
\mathcal{H}_L(X,Y) := L^2 \mathcal{T}_L(X,Y) + \mathcal{T}_L(LX,LY) - L \left( \mathcal{T}_L(X,LY) + \mathcal{T}_L(LX,Y) \right).
\]

Explicitly one can also write \([14]\)

\[
\mathcal{H}_L(X,Y) = L^4[X,Y] + [L^2X,L^2Y] - 2L^3 \left( [X,LY] + [LX,Y] \right) + L^2 \left( [X,L^2Y] + 4[LX,LY] + [L^2X,Y] \right) - 2L \left( [LX,L^2Y] + [L^2X,LY] \right).
\]
The skew-symmetry of the Nijenhuis torsion implies that the Haantjes torsion is also skew-symmetric. Its local expression is

\[(\mathcal{H}_L)^i_{jk} = \sum_{\alpha,\beta=1}^{n} \left( L_\alpha^i L_\beta^j (\mathcal{T}_L)^\beta_{jk} + (\mathcal{T}_L)^\alpha_{jk} L_\beta^i - L_\alpha^i \left( (\mathcal{T}_L)^\alpha_{jk} L_\beta^i + (\mathcal{T}_L)^\alpha_{jk} L_\beta^i \right) \right), \]

or in explicit form

\[(\mathcal{H}_L)^i_{jk} = \sum_{\alpha=1}^{n} \left( -2(L_\alpha^i \partial_j L_\alpha^k) + (L_\alpha^i)^j + \sum_{\beta=1}^{n} L_\beta^i \partial_{\alpha \beta} (L_\alpha^k) \right). \]

Here for the sake of brevity we have used the notation \( \partial_j := \frac{\partial}{\partial x^j} \) and the indices between square brackets are to be skew-symmetrized, except those in \(| \cdot |\).

The notion of Haantjes torsion can be easily generalized by means of a recursive procedure. Indeed, one can introduce a “tower” of generalized torsions of Nijenhuis type.

**Definition 3.** We define the generalized Nijenhuis torsion of level \( n \) as the \((1,2)\)-tensor field given by

\[\tau^{(n)}_L(X,Y) := L^2 \tau^{(n-1)}_L(X,Y) + \tau^{(n-1)}_L(LX, LY) - L \left( \tau^{(n-1)}_L(X, LY) + \tau^{(n-1)}_L(LX, Y) \right), \]

where \( \tau^{(0)}_L(X,Y) = [X,Y] \), \( X, Y \in T \mathcal{M} \). Here \( \tau^{(1)}_L = \tau_L \) and \( \tau^{(2)}_L = \mathcal{H}_L \).

The expression of the \( n \)-th level torsion in local coordinates is given by

\[\left( \tau^{(n)}_L \right)^i_{jk} = \sum_{\alpha,\beta=1}^{n} \left( L_\alpha^i L_\beta^j (\tau^{(n-1)}_L)^\beta_{jk} + (\tau^{(n-1)}_L)^i_{\alpha j} L_\beta^k L_\alpha^j - L_\alpha^i \left( (\tau^{(n-1)}_L)^\alpha_{jk} L_\beta^i + (\tau^{(n-1)}_L)^\alpha_{jk} L_\beta^i \right) \right). \]

We stress that the notion of \( n \)-th order generalized Nijenhuis torsion was proposed in a completely independent way also in [14]. In that algebraic construction, one considers generalized torsions of order \( n \) associated more generally to an arbitrary vector-valued skew symmetric bilinear map on a real vector space. In our geometric approach, which is a recursive one, we work on a tangent bundle and fix the initial condition of the recurrence with the standard choice of the Lie bracket between vector fields.

We shall first consider some specific cases, in which the construction of the Nijenhuis and Haantjes torsions will be particularly simple.

**Example 4.** Let \( L \) be a field of operators that takes a diagonal form

\[L(x) = \sum_{i=1}^{n} l_i(x) \frac{\partial}{\partial x^i} \otimes dx^i,\]

in some local chart \( x = (x^1, \ldots, x^n) \). Its Nijenhuis torsion is given by

\[\left( \mathcal{T}_L \right)^i_{jk} = (l_j - l_k) \left( \frac{\partial l_j}{\partial x^k} \delta^i_j + \frac{\partial l_k}{\partial x^j} \delta^i_k \right). \]

It is evident that \( \left( \mathcal{T}_L \right)^i_{jk} = 0 \) if \( i, j \) and \( k \) are distinct or if \( j = k \). Thus, we can limit ourselves to analyze the \( n(n-1) \) components

\[\left( \mathcal{T}_L \right)^i_{jk} = (l_j - l_k) \frac{\partial l_j}{\partial x^k}, \quad j \neq k. \]
If \( \frac{\partial l_j}{\partial x_k} \neq 0 \), each component vanishes if and only if \( l_j(x) \equiv l_k(x) \). Therefore, we can state the following

**Lemma 5.** Let \( L \) be the diagonal field of operators (9), and suppose that its Nijenhuis torsion vanishes. Let us denote with \( (i_1, \ldots, i_j, \ldots, i_r) \), \( r \leq n \) an ordered subset of \( (1, 2, \ldots, n) \). If the \( j \)-th eigenvalue of \( L \) depends on the variables \( (i_1, \ldots, i_j, \ldots, i_r) \), then

\[
(12) \quad l_j(i_1, \ldots, i_j, \ldots, i_r) \equiv l_{i_1} \equiv l_{i_2} \equiv \ldots = l_{i_r}.
\]

From the other side, apart when each eigenvalue is constant, we can distinguish several cases, ensuring that the Nijenhuis torsion of a diagonal operator vanishes. For instance,

i) \( l_j(x) = \lambda_j(x) \) \( j = 1, \ldots, n \Rightarrow n \) simple eigenvalues

ii) \( l_j(x) = \lambda(x) \) \( j = 1, \ldots, n \Rightarrow 1 \) eigenvalue of multiplicity \( n \)

represent the extreme cases. An exhaustive analysis of all intermediate possibilities is left to the reader.

**Example 6.** Let \( \dim M = 2 \). Then, it easy to prove by a straightforward computation that the Haantjes torsion of any field of smooth operators vanishes.

**Example 7.** Let \( L \) be the diagonal operator of Example 4. Its Haantjes torsion reads

\[
(13) \quad (H_L)^i_{jk} = (l_i - l_j)(l_i - l_k)(T_L)^i_{jk},
\]

where \( (T_L)^i_{jk} \) is given by eq. (10).

The following proposition is a direct consequence of eqs. (10) and (13).

**Proposition 8.** Let \( L \) be a smooth field of operators. If there exists a local coordinate chart \( (x^1, \ldots, x^n) \) where \( L \) takes the diagonal form (9), then the Haantjes torsion of \( L \) vanishes.

Due to the relevance of the Haantjes (Nijenhuis) vanishing condition, we propose the following definition.

**Definition 9.** A Haantjes (Nijenhuis) operator is a field of operators whose Haantjes (Nijenhuis) torsion identically vanishes.

For subsequent purposes, we recall that the transposed operator \( L^T : T^*M \rightarrow T^*M \) is defined as the transposed linear map of \( L \) with respect to the natural pairing between a vector space and its dual space

\[
< L^T \alpha, X > = < \alpha, LX > \quad \alpha \in T^*M, \quad X \in TM.
\]

The torsionless condition of a Nijenhuis operator \( N \) can be written in the following equivalent manner \[11\], through the Lie derivative along the flow of any vector field \( X \in TM \)

\[
(14) \quad \mathcal{L}_X(N) = N \mathcal{L}_X(N).
\]

Analogously, the vanishing of the Haantjes torsion \[3\] of a field of operators \( L \) is equivalent to the following condition

\[
(15) \quad \mathcal{L}_{L^2X}(L)L = L^3 \mathcal{L}_X(L) - L^2 \big( 2 \mathcal{L}_X(L) + \mathcal{L}_X(L)L \big) + L \big( \mathcal{L}_{L^2X}(L) + 2 \mathcal{L}_X(L)L \big).
\]
Lemma 10. Jacobi Formula. Let \( L : TM \rightarrow TM \) a field of operators. For any vector field \( X \in T M \), holds true that

\[
\mathcal{L}_X (\det L) = \text{Trace}(\text{Cof}(L) \mathcal{L}_X (L)),
\]

where \( \text{Cof}(L) \) the cofactor operator and \( \text{Trace} \) the trace of an operator.

A nice intrinsic proof can be found in the textbook [25].

Proposition 11. [7] Let \( N : TM \rightarrow TM \) a Nijenhuis operator. It holds true that

\[
N^T d(\det N) = \det N d(\text{Trace}(N))
\]

Proof. If \( \det N = 0 \), the thesis is obvious. Assume that \( \det N \neq 0 \). Then, substituting \( \text{Cof}(N) = \det(N)N^{-1} \) and \( X = NY \) in the Jacobi formula (16), one gets

\[
\mathcal{L}_{NY} (\det N) = \text{Trace}(\det(N)N^{-1} \mathcal{L}_{NY}(N)) = \det(N) \text{Trace}(N^{-1}N \mathcal{L}_Y(N)) = \det(N) \text{Trace}(\mathcal{L}_Y(N)) = \det(N) \mathcal{L}_Y(\text{Trace}(N))
\]

where the torsionless condition (14) and the fact that \( \text{Trace} \) commutes with the Lie derivative have been used. \( \square \)

It is well known (see for instance [11]) that given an invertible Nijenhuis operator, its inverse is also a Nijenhuis operator. The same holds true for a Haantjes operator.

Proposition 12. [4]. Let \( L \) be a Haantjes operator in \( M \). If \( L^{-1} \) exists, it is also a Haantjes operator.

Proof. Its a consequence of the following identity that can be easily recovered by Eq. (4)

\[
\mathcal{H}_{L^{-1}}(X,Y) = L^{-4} \mathcal{H}_L(L^{-2}X, L^{-2}Y).
\]

For an alternative proof, see Proposition 2, p. 257 of [4]. \( \square \)

The product of a Nijenhuis operator with a generic function is no longer a Nijenhuis operator, as is proved by the following identity

\[
\mathcal{H}_{fL}(X,Y) = f^2 \mathcal{H}_L(X,Y) + f \left( (LX)(f)LY - (LY)(f)LX + Y(f)L^2X - X(f)L^2Y \right),
\]

where \( X(f) \) denotes the Lie derivative of an arbitrary function \( f \in C^\infty(M) \) with respect to the vector field \( X \). Instead, the differential and algebraic properties of a Haantjes operator are much richer, as follows from these remarkable results.

Proposition 13. [4]. Let \( L \) be a field of operators. The following identity holds

\[
\mathcal{H}_{f+gL}(X,Y) = g^4 \mathcal{H}_L(X,Y),
\]

where \( f, g : M \rightarrow \mathbb{R} \) are \( C^\infty(M) \) functions, and \( I \) denotes the identity operator in \( TM \).

Proof. See Proposition 1, p. 255 of [4]. \( \square \)
Proposition 14. [5]. Let $L$ be a Haantjes operator in $M$. Then for any polynomial in $L$ with coefficients $a_j \in C^\infty(M)$, the associated Haantjes torsion vanishes, i.e.

$$(21) \quad H_L(X, Y) = 0 \implies H(\sum_j a_j(x)L^j)(X, Y) = 0.$$ 

Proof. See Corollary 3.3, p. 1136 of [5].

Propositions 13 and 14 imply that a single Haantjes operator generates an algebra of Haantjes operators over the ring of smooth functions on $M$. This is not the case for a Nijenhuis operator $N$ since a polynomial in $N$ with coefficients $a_j \in C^\infty(M)$, it is not necessarily a Nijenhuis operator.

Let us introduce an interesting example of Nijenhuis and Haantjes operators drawn from the realm of Rational Mechanics.

Example 15. Let $M = \{(P_\gamma, m_\gamma) \in (E, \mathbb{R})\}$ be a finite system of mass points (possibly with $m_\gamma < 0$) in the $n$-dimensional affine Euclidean space $E_n$. Let us consider the $(1,1)$ tensor field defined by

$$(22) \quad E_P(\vec{v}) = \sum_\gamma m_\gamma \left( (P_\gamma - P) \cdot \vec{v} \right) (P_\gamma - P) \quad \vec{v} \in T_P E_n \equiv \mathbb{E}_n ,$$ 

called the planar inertia tensor (or Euler tensor in Continuum Mechanics), and the inertia tensor field, given by

$$(23) \quad I_P(\vec{v}) = \sum_\gamma m_\gamma \left( |P_\gamma - P|^2 \vec{v} - \left( (P_\gamma - P) \cdot \vec{v} \right) (P_\gamma - P) \right) .$$

They are related by the formulas

$$(24) \quad I_P = \text{Trace}(E_P)I_n - E_P , \quad E_P = \frac{\text{Trace}(I_P)}{n-1}I_n - I_P , \quad n > 1 ,$$

where $I_n$ is the identity operator in $\mathbb{E}_n$. Both of them are symmetric w.r.t. the Euclidean scalar product, so that they are diagonalizable at any point of $E_n$. Furthermore, by virtue of (24) they commute; consequently, they can be simultaneously diagonalized.

If $G$ is the center of mass of $M$, defined by

$$G - P = \frac{1}{m} \sum_\gamma (P_\gamma - P) \quad m := \sum_\gamma m_\gamma \quad m \in \mathbb{R} \setminus \{0\} ,$$

the following Huygens-Steiner transposition formulas hold

$$(25) \quad E_P(\vec{v}) = E_G(\vec{v}) + m((P - G) \cdot \vec{v}) (P - G) ,$$

$$(26) \quad I_P(\vec{v}) = I_G(\vec{v}) + m|P - G|^2 - m((P - G) \cdot \vec{v}) (P - G) .$$

From eqs. (25) and (26) it follows that in the Cartesian coordinates $(x^1, \ldots, x^n)$ with origin in $G$, defined by the common eigen-directions of $E_G$ and $I_G$, we have

$$(27) \quad (E_P)^{ij}_j = \lambda_i(G)\delta^i_j + m x^i x_j ,$$

$$(28) \quad (I_P)^{ij}_j = l_i(G)\delta^i_j + m \left( \sum_{\alpha=1}^n x^\alpha x_\alpha - x^i x_j \right) \quad i, j = 1, \ldots, n .$$
Here \( x_\alpha = \delta_{\alpha\beta} x^\beta \), and \( \lambda_i (G) \) and \( l_j (G) \) denote the eigenvalues of the tensor fields \( E \) and \( I \) respectively, both evaluated at the point \( G \). In \[1, 2\] it has been proved that the Nijenhuis torsion of \( E \) vanishes
\[
(T_E)_{ij}^k = \sum_{\alpha=1}^n \left( x^i (\delta_{\alpha k} E_j^\alpha - \delta_{\alpha j} E_i^\alpha) + (\delta_{ij} - \delta_{jk}) x^\alpha E_i^\alpha \right) = 0 ,
\]
thus its Haantjes torsion also vanishes. Furthermore, we observe that the torsion of \( I \) reads
\[
(T_I)_{ij}^k = 2 \sum_{\alpha=1}^n \left( x_\alpha (\nabla^\alpha \delta^k_i - \nabla^\alpha \delta^k_j) + x_k \nabla^i_j - x_j \nabla^i_k \right) ,
\]
i.e. it is not identically zero, although its Haantjes torsion vanishes as a consequence of the identity \[20\], applied to the relation \[24\].

Other relevant examples of Haantjes operators in terms of Killing tensors in a Riemannian manifold can be found in \[23\].

3. The Geometry of Haantjes Operators

As stated in Proposition 8, the Haantjes torsion \( \mathcal{H}_L \) of a field of operators \( L \) has a relevant geometrical meaning: its vanishing is a necessary condition for the eigen-distributions of \( L \) to be integrable. To clarify this point, first we need to recall that a reference frame is a set of \( n \) vector fields \( \{Y_1, \ldots, Y_n\} \) such that, at each point \( x \) belonging to an open set \( U \subseteq M \), they form a basis of the tangent space \( T_x U \). Two frames \( \{X_1, \ldots, X_n\} \) and \( \{Y_1, \ldots, Y_n\} \) are said to be equivalent if \( n \) nowhere vanishing smooth functions \( f_i \) exist such that
\[
X_i = f_i(x) Y_i , \quad i = 1, \ldots, n .
\]
A natural frame is the frame associated to a local chart \( \{U, (x^1, \ldots, x^n)\} \) and denoted as \( \{\partial/\partial x^1, \ldots, \partial/\partial x^n\} \).

Definition 16. An integrable frame is a reference frame equivalent to a natural frame.

Proposition 17. \[3\] A reference frame in a manifold \( M \) is an integrable frame if and only if it satisfies one of the two equivalent conditions:
- each distribution generated by any two vector fields \( \{Y_i, Y_j\} \) is Frobenius integrable;
- each distribution \( E_i \) generated by all the vector fields except \( Y_i \) is Frobenius integrable.

Definition 18. A field of operators \( L \) is said to be semisimple (or diagonalizable) if, in each open neighborhood \( U \subseteq M \) there exists a reference frame formed by (proper) eigenvector fields of \( L \). This frame will be called an eigen-frame of \( L \). Moreover, \( L \) is said to be simple if all of its eigenvalues are point-wise distinct, namely if \( l_i(x) \neq l_j(x) \), \( i, j = 1, \ldots, n, \forall x \in M \).

An important question is to ascertain under which conditions eigen-frames of \( L \) are integrable, according to Definition \[16\].

Proposition \[8\] amounts to say that if an operator admits a local chart in which it takes a diagonal form, then its Haantjes torsion necessarily vanishes, therefore the natural frame associated is an eigen-frame that is (trivially) integrable. In 1955,
Haantjes proved in [13] that the vanishing of the Haantjes torsion of a semisimple operator $L$ is also a sufficient condition to ensure the integrability of each of its eigen-distributions (with constant rank) and the existence of local coordinate charts in which $L$ takes a diagonal form. We call such coordinates Haantjes coordinates for $L$. Furthermore, he stated that the vanishing of the Haantjes torsion of an operator with real eigenvalues $L$ is also a sufficient (but not necessary) condition to ensure the integrability of each of its generalized eigen-distributions (with constant rank). An equivalent statement of the above-mentioned results is that each Haantjes operator with real eigenvalues admits a generalized eigen-frame that is an integrable frame.

Let us denote with $\text{Spec}(L) := \{l_1(x), l_2(x), \ldots, l_s(x)\}$ the set of the distinct eigenvalues of an operator $L$, which we shall always assume to be real in all the forthcoming considerations. Also, we denote with

\begin{equation}
D_i = \text{Ker}\left( L - l_i(x)I \right)^{\rho_i}, \quad i = 1, \ldots, s
\end{equation}

the $i$-th generalized eigen-distribution, that is the distribution of all the generalized eigenvector fields corresponding to the eigenvalue $l_i(x)$. In eq. (31), $\rho_i$ stands for the Riesz index of $l_i$, namely the minimum integer such that

\begin{equation}
\text{Ker}\left( L - l_i(x)I \right)^{\rho_i} \equiv \text{Ker}\left( L - l_i(x)I \right)^{\rho_i+1}.
\end{equation}

When $\rho_i = 1$, $D_i$ is a proper eigen-distribution.

**Definition 19.** A generalized eigen-frame of a field of operators $L$ is a frame of generalized eigenvectors of $L$.

**Theorem 20.** [13]. Let $L$ be a field of operators, and assume that the rank of each generalized eigen–distribution $D_i$ is independent of $x \in M$. The vanishing of the Haantjes torsion

\begin{equation}
H_{L}(X, Y) = 0 \quad \forall X, Y \in TM
\end{equation}

is a sufficient condition to ensure the integrability of each generalized eigen–distribution $D_i$ and of any direct sum $D_i \oplus D_j \oplus \ldots \oplus D_k$ (where all indices $i, j, \ldots, k$ are different). In addition, if $L$ is semisimple, condition (33) is also necessary.

In the original paper by Haantjes, the proof of Theorem 20 is explicitly made only for the case of a semisimple operator. Below, within a different approach we present the proof for the more general case of an operator admitting generalized eigenvectors with arbitrary Riesz index.

Without loss of generality, we focus only on two eigenvalues of $L$, $\mu$ and $\nu$, possibly coincident. Let us denote by $X_\alpha, Y_\beta$ two fields of generalized eigenvectors with indices $\alpha$ and $\beta$, corresponding to the eigenvalues $\mu = \mu(x)$ and $\nu = \nu(x)$, and belonging to a Jordan chain in $D_\mu, D_\nu$, respectively:

\begin{align}
LX_\alpha &= \mu X_\alpha + X_{\alpha-1}, \\
LY_\beta &= \nu Y_\beta + Y_{\beta-1}, \quad 1 \leq \alpha \leq \rho_\mu, \quad 1 \leq \beta \leq \rho_\nu,
\end{align}

where $X_0$ and $Y_0$ are agreed to be null vector fields. Then, it holds true that

\begin{align}
X_\alpha &\in \text{Ker}\left( L - \mu I \right)^{\alpha}, \\
Y_\beta &\in \text{Ker}\left( L - \nu I \right)^{\beta}.
\end{align}

Evaluating the Nijenhuis torsion on such eigenvector fields, we get
\[ T_L(X_\alpha, Y_\beta) = \left( L - \mu I \right) \left( L - \nu I \right) [X_\alpha, Y_\beta] + (\mu - \nu) \left( X_\alpha (\nu) Y_\beta + Y_\beta (\mu) X_\alpha \right) \]

(36) 

\[ - \left( L - \mu I \right) [X_\alpha, Y_{\beta - 1}] - \left( L - \nu I \right) [X_{\alpha - 1}, Y_\beta] + [X_{\alpha - 1}, Y_{\beta - 1}] \]

\[ - \left( X_\alpha (\nu) Y_{\beta - 1} + Y_{\beta - 1} (\mu) X_\alpha \right) + \left( X_{\alpha - 1} (\nu) Y_\beta + Y_\beta (\mu) X_{\alpha - 1} \right). \]

The analogous relation for the Haantjes torsion is

\[ \mathcal{H}_L(X_\alpha, Y_\beta) = \left( L - \mu I \right) \left( L - \nu I \right) T_L(X_\alpha, Y_\beta) + \]

(37) 

\[ - \left( L - \mu I \right) T_L(X_\alpha, Y_{\beta - 1}) - \left( L - \nu I \right) T_L(X_{\alpha - 1}, Y_\beta) + T_L(X_{\alpha - 1}, Y_{\beta - 1}) \]

Substituting (36) into (37) we find

(38) \[ \mathcal{H}_L(X_\alpha, Y_\beta) = \sum_{i,j=0}^{2} (-1)^{i+j} \binom{2}{i} \binom{2}{j} \left( L - \mu I \right)^{2-i} \left( L - \nu I \right)^{2-j} [X_{\alpha - i}, Y_{\beta - j}]. \]

**Lemma 21.** Let \( L \) be a field of operators and \( X_\alpha, Y_\beta \) be two of its fields of generalized eigenvectors in \( D_\mu \), belonging to possibly different Jordan chains. If

(39) \[ \mathcal{H}_L(D_\mu, D_\mu) = 0, \]

then their commutator satisfies the relation

(40) \[ [X_\alpha, Y_\beta] \in Ker \left( L - \mu I \right)^{\alpha + \beta + 2} = Ker \left( L - \mu I \right)^{\min(\alpha + \beta + 2, \rho_\mu)} \subseteq Ker \left( L - \mu I \right)^{\rho_\mu}, \]

where \( \min(\cdot, \cdot) \) stands for the minimum of its arguments.

**Proof.** If \( \alpha = \beta = 1 \) and \( \mu = \nu \), eq. (38) implies that \( [X_1, Y_1] \in Ker \left( L - \mu I \right)^4 \). By induction over \( (\alpha + \beta) \), and applying the operator \( \left( L - \mu I \right)^{\alpha + \beta - 2} \) to both members of eq. (38) it follows that \( [X_\alpha, Y_\beta] \in Ker \left( L - \mu I \right)^{\alpha + \beta + 2} \).

**Proposition 22.** Let \( L \) be a field of operators. Each of its eigen-distributions \( D_\mu \) with Riesz index \( \rho_\mu \) is integrable if

(41) \[ \mathcal{H}_L(D_\mu, D_\mu) = 0. \]

In the case \( \rho_\mu = 1 \), the converse is also true.

**Proof.** Lemma 21 immediately implies that the Frobenius integrability condition for \( D_\mu \)

(42) \[ [D_\mu, D_\mu] \subseteq D_\mu \]

is fulfilled. In particular, if \( \rho_\mu = 1 \), every \( \mu \)-eigenvector of \( L \) is a proper eigenvector, and from eq. (38) one infers that

\[ \mathcal{H}_L(D_\mu, D_\mu) = 0 \iff [X_1, Y_1] \in Ker \left( L - \mu I \right)^4 = Ker \left( L - \mu I \right) = D_\mu. \]

\[ \square \]
Lemma 23. Let $L$ be a Haantjes operator. The commutator of two generalized eigenvector fields of $L$, with different eigenvalues $\mu$, $\nu$, fulfills the relation

$$\begin{align*}
[X_\alpha, Y_\beta] & \in Ker\left( L - \mu I \right)^{\alpha+1} \oplus Ker\left( L - \nu I \right)^{\beta+1} \\
& \equiv Ker\left( L - \mu I \right)^{\min(\alpha+1, \rho_{\mu})} \oplus Ker\left( L - \nu I \right)^{\min(\beta+1, \rho_{\nu})} \\
& \subseteq Ker\left( L - \mu I \right)^{\rho_{\mu}} \oplus Ker\left( L - \nu I \right)^{\rho_{\nu}},
\end{align*}$$

with $1 \leq \alpha \leq \rho_{\mu}$, $1 \leq \beta \leq \rho_{\nu}$.

Proof. If $\alpha = \beta = 1$ and $\mu \neq \nu$, eq. (43) implies that $[X_1, Y_1] \in Ker\left( L - \mu I \right)^{2} \oplus Ker\left( L - \nu I \right)^{2}$. By induction over $(\alpha + \beta)$, applying the operator $\left( L - \mu I \right)^{\alpha-1}\left( L - \nu I \right)^{\beta-1}$ to both members of (43), the assertion follows. \qed

It is immediate to ascertain that the above Lemma implies $[D_{\mu}, D_{\nu}] \subset D_{\mu} \oplus D_{\nu}$, so that the following result holds

Proposition 24. Let $L$ be a Haantjes operator, and $D_{\mu}$, $D_{\nu}$ be two eigen-distributions with Riesz indices $\rho_{\mu}$ and $\rho_{\nu}$, respectively. Then, the distribution

$$D_{\mu} \oplus D_{\nu} \equiv Ker\left( L - \mu I \right)^{\rho_{\mu}} \oplus Ker\left( L - \nu I \right)^{\rho_{\nu}}, \quad \mu \neq \nu$$

is integrable.

The Haantjes Theorem 20 is a direct consequence of Propositions 22 and 24.

In \cite{8} and \cite{12}, the integrability of the eigen-distributions of a Nijenhuis operator with generalized eigenvectors of Riesz index 2 was proved. However, the case of Haantjes operators was not considered. On the other hand, to the best of our knowledge, the proofs of the Haantjes theorem available in the literature (see for instance \cite{10}, \cite{11}) are based on the more restrictive assumption that the Haantjes operator be diagonalizable.

In the particular case of a Nijenhuis operator, we are able to prove new invariance properties of its eigenvalues. They generalize the results proved in \cite{8} for Nijenhuis operators with Riesz index equal to 2 to the case of an arbitrary Riesz index. By exploiting the identity (40), by induction over $(\alpha + \beta)$, we have proved the following identity

$$\begin{align*}
&\left( L - \mu I \right)^{\alpha-1}\left( L - \nu I \right)^{\beta-1} T_L(X_\alpha, Y_\beta) + \left( L - \mu I \right)^{\alpha-1}\left( L - \nu I \right)^{\beta-2} T_L(X_\alpha, Y_{\beta-1}) + \\
&\left( L - \mu I \right)^{\alpha-2}\left( L - \nu I \right)^{\beta-1} T_L(X_{\alpha-1}, Y_\beta) + \left( L - \mu I \right)^{\alpha-2}\left( L - \nu I \right)^{\beta-2} T_L(X_{\alpha-1}, Y_{\beta-1}) \\
= &\left( L - \mu I \right)^{\alpha}\left( L - \nu I \right)^{\beta} \left[ X_\alpha, Y_\beta \right] - \beta(\nu - \mu)^{\alpha} X_\alpha(\nu) Y_1 + \alpha(\mu - \nu)^{\beta} Y_\beta(\mu) X_1 .
\end{align*}$$

Consequently, the following result holds.

Proposition 25. Let $N$ be a Nijenhuis operator. Each of its eigenvalues $\mu$ is constant along the flow of the generalized eigenvectors belonging to $\mathcal{E}_{\mu}$, that is

$$Y_\beta(\mu) = 0 \quad \forall \ Y_\beta \in Ker\left( N - \nu I \right)^{\rho_{\nu}}, \quad \mu \neq \nu .$$
Moreover,

\[ [X_\alpha, Y_\beta] \in \text{Ker}(N - \mu I)^\alpha \oplus \text{Ker}(N - \nu I)^\beta, \quad \mu \neq \nu \]  

(46)

\[ [X_\alpha, Y_\beta] \in \text{Ker}(N - \mu I)^{\alpha + \beta}, \quad \mu = \nu. \]  

(47)

**Proof.** If the torsion of \( N \) vanishes, the left side of (44) vanishes as well. Taking \( \alpha = \rho \mu, \) from the Haantjes theorem it follows that the first addend of the r.h.s. belongs to \( D_\nu, \) therefore the third addend vanishes and \( Y_\beta(\mu) = 0, \) if \( \mu \neq \nu. \) Consequently, each addend in (44) vanishes as well and relation (47) holds. \( \square \)

Let us show in detail how to determine a coordinate system that, under the assumptions of Theorem 20, provides a quasi-diagonal form for a Haantjes operator \( L. \) Denote by

\[ \mathcal{E}_i := \text{Im}(L - l_i I)^{\rho_i} = \bigoplus_{j=1, j \neq i}^s D_j, \quad i = 1, \ldots, s \]  

the distribution of rank \( (n - r_i) \) spanned by all of the generalized eigenvectors of \( L, \) except those associated with the eigenvalue \( l_i. \) Such a distribution will be called a **characteristic distribution** of \( L. \) Let \( \mathcal{E}_i^\perp \) denote the annihilator of the distribution \( \mathcal{E}_i. \) Since \( L \) has real eigenvalues by hypothesis, the tangent and cotangent spaces of \( M \) can be locally decomposed as

\[ T_x M = \bigoplus_{i=1}^s D_i(x), \quad T_x^* M = \bigoplus_{i=1}^s \mathcal{E}_i^\perp(x). \]  

(49)

Moreover, each characteristic distribution \( \mathcal{E}_i \) is integrable by virtue of Theorem 20. We shall denote by \( E_i \) the foliation of \( \mathcal{E}_i \) and by \( E_i(x) \) the connected leave through \( x \) belonging to \( E_i. \) Thus, the set \( (E_1, E_2, \ldots, E_s) \) generates as many foliations \( (E_1, E_2, \ldots, E_s) \) as the number of distinct eigenvalues of \( L. \) This set of foliations will be referred to as the **characteristic web** of \( L \) and the leaves \( E_i(x) \) of each foliation \( E_i \) as the **characteristic fibers** of the web.

**Definition 26.** A collection of \( r_i \) smooth functions will be said to be adapted to a foliation \( E_i \) of the characteristic web of a Haantjes operator \( L \) if the level sets of such functions coincide with the characteristic fibers of the foliation.

**Definition 27.** A parametrization of the characteristic web of a Haantjes operator \( L \) is an ordered set of \( n \) independent smooth functions \((f_1, \ldots, f_n)\) such that each ordered subset \((f_{i_1}, \ldots, f_{i_r})\) is adapted to the \( i-th \) characteristic foliation of the web:

\[ f_{ik}|_{E_i(x)} = \text{const} \quad \forall E_i(x) \in E_i, \quad k = 1, \ldots, r, \quad i_r = i_1 + r_i. \]  

(50)

In this case, we shall say that the collection of functions is adapted to the web and that each of them is a characteristic function.

**Lemma 28.** The vanishing of the Haantjes torsion of \( L \) is sufficient to ensure that it admits an equivalence class of integrable generalized eigen-frames, where \( L \) takes a quasi-diagonal form. Furthermore, if \( L \) is semisimple it takes a diagonal form and the vanishing of its Haantjes torsion is also a necessary condition. In addition, if \( L \) is simple each eigen-frame is integrable.
Proof. Since each characteristic distribution $\mathcal{E}_i$ is integrable by virtue of the Haantjes Theorem \ref{theo:haantjes}, in the corresponding annihilator $\mathcal{E}_i^\circ$ one can find $r_i$ exact one-forms $(d^i_1, \ldots, d^i_{r_i})$ that provide functions $(x^i_1, \ldots, x^i_{r_i})$ adapted to the characteristic foliation $\mathcal{E}_i$. Collecting together all these functions, one can construct a set of $n$ independent coordinates, that we rename $(x^1, \ldots, x^n)$, adapted to the characteristic web.

Any natural frame $\{\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n}\}$ turns out to be a generalized eigen-frame. In fact, as

\begin{equation}
\mathcal{D}_i^\circ = \bigoplus_{j=1, j\neq i} E_j^\circ,
\end{equation}

any generalized eigenvector $W \in \mathcal{D}_i$ leaves invariant all the coordinate functions except at most the characteristic functions $(x^i_1, \ldots, x^i_{r_i})$ of $\mathcal{E}_i$. Thus, we have that $W = \sum_{k=1}^{r_i} W(x^{i,k}) \frac{\partial}{\partial x^{i,k}}$, therefore

\begin{equation}
\mathcal{D}_{i|U} = \left\langle \frac{\partial}{\partial x^i_1}, \ldots, \frac{\partial}{\partial x^i_{r_i}} \right\rangle,
\end{equation}

(hereafter the symbol $<>$ denotes the $C^\infty(M)$-linear span of the considered vector fields) and each frame equivalent to $\{\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n}\}$ is an integrable eigen-frame of generalized eigenvector fields. Consequently, there exists an equivalence class of integrable frames and associated local charts where the operator $L$ takes a block-diagonal form due to the invariance of its eigen-distributions. Moreover, if $L$ is semisimple, then its generalized eigen-frames are proper eigen-frames and the block-diagonal form collapses into a diagonal one.

Conversely, if there exists a local chart where $L$ takes a diagonal form, the corresponding natural frame is obviously an integrable one. Thus, due to Proposition \ref{prop:diagonal} the Haantjes torsion of $L$ vanishes. Finally, if $L$ is simple each eigen-distribution has rank 1, thus each natural eigen-frame fulfills the conditions of Proposition \ref{prop:simple}.

\begin{definition}
Let $M$ be a differentiable manifold and $L$ be a Haantjes operator in $M$. A local chart $\{U, (x^1, \ldots, x^n)\}$ whose natural frame is a generalized eigen-frame for $L$ will be called a Haantjes chart for $L$.
\end{definition}

A Haantjes chart for $L$ can also be computed by using the transposed operator $L^T$. Let us denote with

\begin{equation}
\text{Ker} \left( L^T - l_i(x) I \right)^{\rho_i}
\end{equation}

the $i$-th distribution of the generalized eigen 1-forms with eigenvalue $l_i(x)$, which fulfills the property

\begin{equation}
\text{Ker} (L^T - l_i I)^{\rho_i} = \left( \text{Im} \left( L - l_i I \right)^{\rho_i} \right)^{\circ} = \mathcal{E}_i^\circ
\end{equation}

Such a property implies that each generalized eigenform of $L^T$ annihilates all generalized eigenvectors of $L$ with different eigenvalues. Moreover, it allows to prove that

\begin{proposition}
Let $L$ be a Haantjes operator. The differentials of the characteristic coordinate functions are exact generalized eigenforms for the transposed operator $L^T$.
\end{proposition}
operator $L^T$. Conversely, each (locally) exact generalized eigenform of $L^T$ provides a characteristic function for the Haantjes web of $L$.

The characteristic functions of a Haantjes operator are characterized by the following simple property.

**Proposition 31.** A function $h$ on $M$ is a characteristic function of a Haantjes operator associated with the eigenvalue $l_i$ if and only if, given a set of local coordinates adapted to the characteristic web $(x^1, \ldots, x^n)$, $h$ depends, at most, on the subset of coordinates $(x^{i_1}, \ldots, x^{i_{r_i}})$ that are constant over the leaves of the foliation $E_i$.

**Proof.** If $h = h(x^{i_1}, \ldots, x^{i_{r_i}})$, it is constant on the leaves of $E_i$, then $dh \in \mathcal{E}_i^\circ$. Viceversa, if we assume that $dh \in \mathcal{E}_i^\circ$, then it can be expressed in terms of a linear combination (with function coefficients) of $\{dx^{i_1}, \ldots, dx^{i_{r_i}}\}$ only. The thesis follows from the exactness of $dh$. \qed

In the special case of a Haantjes operator that is also a Nijenhuis operator, the search for its characteristic functions is simplified by the following

**Proposition 32.** Let $N$ be a Nijenhuis operator. Its (not constant) eigenvalues $\lambda_i$ are characteristic functions for the Haantjes web of $N$, as

$$d\lambda_i \in \text{Ker}(N_T - \lambda_i I)^{\rho_i}.$$  

In particular, the eigenvalues $\lambda_i$ with Riesz index $\rho_i = 1$, satisfy

$$N_T d\lambda_i = \lambda_i d\lambda_i.$$  

**Proof.** The invariance property (45) and the identity (54) implies (55). From Proposition 30 the thesis follows. \qed

**Remark 33.** Let us suppose that a generic field of operators $L$ admits a symmetry, i.e. a vector field $X$ such that

$$\mathcal{L}_X (L) = 0.$$  

In this case, the operator $L$ will be called a recursion operator for $X$. Then, the eigenvalues of $L$ as well are invariant along the flow of $X$ and the corresponding generalized eigen-distributions are stable, i.e.

$$\mathcal{L}_X (l_i) = 0, \quad \mathcal{L}_X (\mathcal{D}_i) \subseteq \mathcal{D}_i, \quad \mathcal{L}_X (\mathcal{E}_i^\circ) \subseteq \mathcal{E}_i^\circ \quad \forall i = 1, 2, \ldots, s.$$  

4. **Haantjes Algebras**

In this section we define the new notion of Haantjes algebra.

4.1. **Main Definition.**

**Definition 34.** A Haantjes algebra of rank $m$ is a pair $(M, \mathcal{H})$ which satisfies the following conditions:

- $M$ is a differentiable manifold of dimension $n$;
- $\mathcal{H}$ is a set of Haantjes operators $K_i : TM \rightarrow TM$, that generates
  - a free module of rank $m$ over the ring of smooth functions on $M$;

$$\mathcal{H}(\{fK_i + gK_i\})(X, Y) = 0, \quad \forall X, Y \in TM, \quad \forall f, g \in C^\infty(M);$$
HAANTJES ALGEBRAS AND DIAGONALIZATION

– a ring w.r.t. the composition operation

\[ H(K, K')(X, Y) = H(K', K)(X, Y) = 0, \quad \forall K, K' \in \mathcal{H}, \quad \forall X, Y \in TM. \]

In addition, if

\[ K, K' \in \mathcal{H}, \quad \forall K, K' \in \mathcal{H}, \quad \forall X, Y \in TM, \]

the algebra \( \mathcal{H} \) will be said to be an Abelian Haantjes algebra.

Moreover, if the identity operator \( I \in \mathcal{H} \), then \( \mathcal{H} \) will be called a Haantjes algebra with identity.

We observe that the assumptions (59), (60) ensure that the set \( \mathcal{H} \) generates an associative algebra of Haantjes operators. Besides, whenever \( K, K' \in \mathcal{H} \),

\[ K_i K_j = K_j K_i \quad \forall K_i, K_j \in \mathcal{H}, \]

Let us consider the minimal polynomial of an operator \( K \in \mathcal{H} \)

\[ m_K(x, \lambda) = \prod_{i=1}^{s} (\lambda - l_i(x))^\rho_i = \lambda^r + \sum_{j=1}^{r} c_j(x) \lambda^{r-j}, \quad r = \sum_{i=1}^{s} \rho_i, \]

where \( l_i(x) \) \( i = 1, \ldots, s \) are the point-wise distinct eigenvalues of \( K \). Then, the observations above imply the following

**Lemma 35.** Let \( \mathcal{H} \) be a Haantjes algebra of rank \( m \). Then, if \( I \in \mathcal{H} \) the degree \( r \) of the minimal polynomial of each \( K \in \mathcal{H} \) is not greater than \( m \); if \( I \notin \mathcal{H} \) \( r \leq (m + 1) \).

**Lemma 36.** Let \( \mathcal{H} \) be a Haantjes algebra without identity. Then, every \( K \in \mathcal{H} \) is not invertible.

**Proof.** Let \( m_K(\lambda) \) be the minimal polynomial of \( K \) in eq. (62). Let us consider the operator \( c_r(x)I = -(K^r + \sum_{j=1}^{r-1} c_j(x)K^{r-j}) \) that, at the points \( x \in M \) where \( c_r(x) \neq 0 \), belongs to \( \mathcal{H} \). Then, at the same points, also \( I \) should belong to \( \mathcal{H} \); but this is absurd. Therefore \( c_r(x) = (-1)^r \prod_{i=1}^{r} l_i^\rho_i(x) \) vanishes at any point of \( M \). Consequently, \( K \) is not invertible in any point of \( M \). □

**Remark 37.** The class of \( \omega \mathcal{H} \) manifolds introduced and discussed in \[22\] is nothing but a family of symplectic manifolds of dimension \( 2n \), endowed with an Abelian Haantjes algebra of rank \( m = n \) that fulfills an additional compatibility condition with \( \omega \).

The conditions of Definition \[24\] are apparently demanding and difficult to solve. However, a class of natural solutions is given, in a local chart \( \{ U, x = (x^1, \ldots, x^n) \} \), by each operator of the form

\[ K = \sum_{k=1}^{n} l_k(x) \frac{\partial}{\partial x^k} \otimes dx^k. \]

The diagonal operators \( K \) have their Haantjes torsion vanishing and satisfy the differential compatibility condition (59) by virtue of Proposition \[8\]. Moreover, they form a commutative ring, according to eqs. (60). In fact, such operators generate an algebraic structure that we name diagonal Haantjes algebra. In Section 4.3 we shall present another relevant class of Haantjes algebras, namely the cyclic algebras,
generated by the powers of a Haantjes operator. Also, examples of non-diagonal Haantjes algebras will be shown in Section 5.

4.2. Characteristic Haantjes coordinates. We shall present below some of our main results, concerning the existence of charts of coordinates in which an algebra of Haantjes operators takes a diagonal form in the semisimple case, and a block-diagonal form in the non-semisimple one.

**Definition 38.** A Haantjes algebra \((M, \mathcal{H})\) is said to be semisimple if each operator \(K \in \mathcal{H}\) is semisimple.

Let us recall that if the Haantjes algebra \((M, \mathcal{H})\) is Abelian, then there exist reference frames in which all \(K \in \mathcal{H}\) are simultaneously diagonalized. A relevant problem is to ascertain whether one can find, among such common eigen-frames, integrable eigen-frames.

**Theorem 39.** Given a semisimple Abelian Haantjes algebra \((M, \mathcal{H})\), there exists a set of local coordinates in which every \(K \in \mathcal{H}\) can be simultaneously diagonalized. Conversely, let \(\{K_1, \ldots, K_m\}\) be a commuting set of \(m C^\infty(M)\)–linearly independent operators. If they share a set of local coordinates in which they take a diagonal form, then they generate a semisimple Abelian Haantjes algebra of rank \(m\).

**Proof.** Let \(m\) be the rank of the Haantjes algebra \(\mathcal{H}\) and \(\{K_1, \ldots, K_m\}\) one of its basis. Let us consider the operator \(K_1\). It is semisimple by assumption, therefore the Haantjes Theorem 20 assures that a local Haantjes chart \(\{U, (x^1, \ldots, x^n)\}\) exists in which it takes the diagonal form \((63)\). Such a chart is adapted to the decomposition of each tangent space by the eigen-spaces

\[
T_x M = \bigoplus_{i_1=1}^{s_1} D^{(1)}_{i_1}(x)
\]

associated to the spectrum \(\text{Spec}(K_1) := \{l^{(1)}_1(x), l^{(1)}_2(x), \ldots, l^{(1)}_{s_1}(x)\}\) of \(K_1\). Precisely, we have

\[
D^{(1)}_{i_1}(x) = \langle \frac{\partial}{\partial x^{r_{i_1}^1}}, \ldots, \frac{\partial}{\partial x^{r_{i_1}^1}} \rangle, \quad r_{i_1} = \text{rank}(D^{(1)}_{i_1})
\]

where we have renamed the coordinates \((x^1, \ldots, x^n)\) according to the decomposition \((62)\). Now, let us consider the Haantjes operator \(K_2\) which, by assumption, commutes with \(K_1\). Therefore, the eigen-distributions \(D^{(1)}_{i_1}\) of \(K_1\) are invariant w.r.t. \(K_2\). Thus, \(K_2\) can be restricted to the integral leaves \(D^{(1)}_{i_1}\) of \(D^{(1)}_{i_1}\). In these submanifolds, the following properties hold true:

- \(K_1|_{D^{(1)}_{i_1}} = l^{(1)}_{i_1} I|_{D^{(1)}_{i_1}}\);
- \(K_2|_{D^{(1)}_{i_1}}\) can be point-wise diagonalized;
- \(\mathcal{H}_{K_2}(X, Y) = 0 \quad \forall X, Y \in TD^{(1)}_{i_1}\).

Then, the Haantjes Theorem 20 still holds for the restriction of \(K_2\) to \(D^{(1)}_{i_1}\). Therefore, there exists a transformation of coordinates

\[
\Phi : M \rightarrow M, \quad (x^{1,1}, \ldots, x^{1,r_{1}}; \ldots; x^{i_1,1}, \ldots, x^{i_1,r_{i_1}}; \ldots; x^{s_1,1}, \ldots, x^{s_1,r_{s_1}}) \mapsto (\bar{x}^{1,1}, \ldots, \bar{x}^{1,r_{1}}, \ldots, \bar{y}^{n,1}, \ldots, \bar{y}^{i_1,r_{i_1}}, \ldots; \bar{x}^{s_1,1}, \ldots, \bar{x}^{s_1,r_{s_1}})
\]

\[
\bar{x}^{i_1} = x^{i_1} + \sum_{r_{i_1}} \frac{\partial l^{(1)}_{i_1}}{\partial x^{r_{i_1}}} \quad \text{and} \quad \bar{y}^{i_1} = y^{i_1} + \sum_{r_{i_1}} \frac{\partial l^{(1)}_{i_1}}{\partial x^{r_{i_1}}}.
\]
such that

\[(65) \quad K_2|_{D_1^{(i)}} = \sum_{j=1}^{r_1} l_{i_1,j}^{(2)}(y) \frac{\partial}{\partial y^{i_1,j}} \otimes dy^{i_1,j},\]

where the eigenvalues \(l_{i_1,j}^{(2)}\) may not be distinct. Performing such transformations for \(i_1 = 1, \ldots, s_1\), one can construct a set of local coordinates in \(M\)

\[(66) \quad (y^{1,1}, \ldots, y^{1,r_1}; \ldots; y^{s_1,1}, \ldots, y^{s_1,r_{s_1}})\]

adapted to the decomposition

\[(67) \quad T_xM = \bigoplus_{s_1,s_2} D_{i_1}^{(1)}(x) \cap D_{i_2}^{(2)}(x),\]

in which \(K_1\) and \(K_2\) take simultaneously a diagonal form. Obviously, each addend in the decomposition \(67\) is an integrable distribution being an intersection of integrable distributions. Moreover, each direct sum of two or more addends of \(67\) is also an integrable distribution, since it is generated by the constant vector fields of the natural frame associated to the chart \(66\).

Restricting \(K_3\) to each addend of the decomposition \(67\) and applying again the previous reasoning one can construct a local chart in which \(K_1, K_2\) and \(K_3\) take a diagonal form. Iterating this argument for \(K_4, \ldots, K_m\), one gets the decomposition

\[(68) \quad T_xM = \bigoplus_{s_1,\ldots,s_m} D_{i_1}^{(1)}(x) \cap \cdots \cap D_{i_m}^{(m)}(x)\]

Let us denote by \(\mathcal{V}_a\) the nontrivial addends in the direct sum \(68\), by \(v\) their number \(v \leq n\) and by \(r_a\) their rank \(\sum_{a=1}^{v} r_a = n\). Then, by virtue of the previous reasoning there exists a local chart

\[(69) \quad \{U, (y^{a,j_a})\}, \quad a = 1, \ldots, v, \quad j_a = 1, \ldots, r_a,\]

adapted to the decomposition \(68\) such that

\[(70) \quad \mathcal{V}_a = \left\{ \frac{\partial}{\partial y^{a,1}}, \ldots, \frac{\partial}{\partial y^{a,r_a}} \right\},\]

where the natural frame \(\left\{ \frac{\partial}{\partial y^{a,1}}, \ldots, \frac{\partial}{\partial y^{a,r_a}} \right\}\) over the leaves of \(\mathcal{V}_a\) is formed by common eigenvector fields of the basis \(\{K_1, \ldots, K_m\}\). The distributions \(\mathcal{V}_a, \ a = 1, \ldots, v\) are obviously integrable together with each direct sum of the form \(\mathcal{V}_a \oplus \mathcal{V}_b \oplus \ldots\) as they are spanned by constant vector fields belonging to the natural frame of the local chart \(69\). Thus, the first part of the thesis follows.

The converse statement follows from the fact that if \(\{K_1, \ldots, K_m\}\) is a set of operators that share local coordinates in which they all take a diagonal form, due to Proposition 8 they are all Haantjes operators and generate an Abelian Haantjes algebra. Indeed, the set of diagonal matrices is closed under \(C^\infty(M)\)-linear combinations and matrix products.

The first statement of Theorem 39 can be generalized to the case of non-semisimple Haantjes algebras. Precisely, in the following we shall prove the existence of a local chart where a non-semisimple operator takes a quasi-diagonal form. The converse statement does not seem to admit any straightforward generalization.
Proposition 40. Let \((M, \mathcal{H})\) be an Abelian Haantjes algebra of non-semisimple Haantjes operators. Then, there exists a set of local coordinates in which every \(K \in \mathcal{H}\) can be simultaneously set in a quasi-diagonal form.

Proof. Since the proof is essentially based on the same arguments of Lemma 28 and Theorem 39, we will just sketch it. Consider a basis \(\{K_1, \ldots, K_m\}\) of \(\mathcal{H}\).

Due to the invariance of the addends of the decomposition (68) with respect to any \(K \in \mathcal{H}\), one can restrict all of these operators to a set of common integral leaves of their eigen-distributions, which decompose the tangent bundle into a direct sum. By virtue of the Haantjes theorem applied to the restricted operators, one infers the existence of a set of local coordinates, adapted to the previous decomposition, in which all the operators \(\{K_1, \ldots, K_m\}\) acquire simultaneously a block-diagonal form. □

Definition 41. Let \((M, \mathcal{H})\) be a Haantjes algebra. A local chart \(\{U, (x^1, \ldots, x^n)\}\) whose natural frame is a generalized eigen-frame for each \(K \in \mathcal{H}\) will be called a Haantjes chart for \(\mathcal{H}\).

4.3. Cyclic Haantjes algebras. An especially relevant class of Haantjes algebras is represented by those generated by a single Haantjes operator \(L : TM \rightarrow TM\).

In fact, one can consider the Haantjes algebra \(L\) of any powers of \(L\)

\[
L(L) := \langle I, L, L^2, \ldots, L^{n-1}, \ldots \rangle,
\]

that we shall call a cyclic Haantjes algebra. Its rank is equal to the degree of the minimal polynomial of \(L\) that is not greater than \(n\), due to the Cayley–Hamilton theorem.

A natural question is to establish when a given Haantjes algebra can be generated by a single Haantjes operator, giving rise to a cyclic Haantjes algebra. To investigate this problem, the following definition will be useful.

Definition 42. Let \((M, \mathcal{H})\) be a Haantjes algebra with identity, of rank \(m\). An operator \(L\) having its minimal polynomial of degree \(m\) will be called a cyclic generator of \(\mathcal{H}\) if

\[
\mathcal{H} \equiv L(L).
\]

The set

\[
B_{cyc} = \{I, L, L^2, \ldots, L^{m-1}\}
\]

will be called a cyclic basis of \(\mathcal{H}\).

A cyclic basis allows us to represent each Haantjes operator \(K \in \mathcal{H}\) as a polynomial field in \(L\) of degree at most \((m-1)\), i.e.

\[
K = p_K(x, L) = \sum_{k=0}^{m-1} a_k(x) L^k,
\]

where \(a_k(x)\) are smooth functions in \(M\).

Let us consider the following proposition that holds for each semisimple field of operators with real eigenvalues.

Proposition 43. Let \(L\) be a semisimple operator with \(m\) eigenvalues \(\{\lambda_1(x), \ldots, \lambda_m(x)\}\) point-wise distinct, and \(K\) be another field of operators possessing \(s\) point-wise distinct eigenvalues, with \(s \leq m\). The following conditions are equivalent:
• $K$ belongs to the cyclic algebra of rank $m$ generated by $L$, i.e.

\[(73)\]  
\[K \in \mathcal{L}(L) ;\]

• there exists a polynomial field $p_K(x, \lambda)$ in $\lambda$ of degree at most $m - 1$ such that

\[(74)\]  
\[K = p_K(x, L) ;\]

• each eigen-distribution of $L$ is included in a single eigen-distribution of $K$,

\[(75)\]  
\[\mathcal{C}_{\lambda_i} := \ker(L - \lambda_i I) \subseteq \mathcal{D}_{\lambda_i} := \ker(K - l_i I),\]

where it is understood that the eigenvalues \{\(l_1(x), \ldots, l_m(x)\)\} of $K$ might not be all distinct.

**Proof.** The equivalence between \((73)\) and \((74)\) is due to the fact that the minimal polynomial of $L$ is

\[m_L(x, \lambda) = \Pi_{i=1}^m (\lambda - \lambda_i(x))\]

and the basis \((71)\) is a basis of $\mathcal{L}(L)$.

Condition \((74)\) implies \((73)\) as every eigen-vector field $X$ of $L$ belonging to $\mathcal{C}_{\lambda_i}$, is also an eigenvector field of $K$ with eigenvalue $l_i(x) = p_K(x, \lambda_i)$

\[K X = p_K(x, L)X = p_K(x, \lambda_i)X .\]

Vice versa, if condition \((75)\) is fulfilled, it suffices to show that there exist a field of polynomials $p_K(x, \lambda)$, $\lambda \in \mathbb{R}$, such that $K$ and $p_K(x, L)$ agree on a basis adapted to the decomposition

\[T_x M = \bigoplus_{i=1}^m \mathcal{C}_{\lambda_i}(x) .\]

To this aim, we must solve the following system

\[(76)\]  
\[l_i(x) = p_K(x, \lambda_i) = \sum_{k=0}^{m-1} a_k(x)\lambda_i^k(x) , \quad i = 1, \ldots, m ,\]

The equations \((76)\) can be solved by means of the $m$ Lagrange interpolation polynomials of degree $(m - 1)$

\[\pi_i(\lambda) = \frac{\Pi_{j \neq i}^m (\lambda - \lambda_j)}{\Pi_{j \neq i}^m (\lambda_i - \lambda_j)} , \quad i = 1, \ldots, m \]

which yield the expressions

\[p_K(x, \lambda) = \sum_{i=1}^m l_i(x) \pi_i(\lambda) .\]

Therefore,

\[(77)\]  
\[K = \sum_{i=1}^m p_K(x, \lambda_i) \pi_i(L) ,\]

where $\mathcal{B}_{int} = \{\pi_1(L), \ldots, \pi_m(L)\}$ will be said to be a Lagrange interpolation basis of $\mathcal{L}(L)$. \(\square\)
Corollary 44. If one of the equivalent conditions of Proposition 43 is satisfied and \( L \) is a Haantjes operator, then the operator \( K \) is also a Haantjes operator which commutes with \( L \). In addition, every eigen-basis of \( L \) is also an eigen-basis for \( K \). Therefore,

\[
L |_{C_j} = \lambda_j I |_{C_j}, \quad K |_{C_j} = p_K(x, \lambda_j) I |_{C_j},
\]

where \( C_j \) denotes an integral leaf of the eigen-distribution \( C_{\lambda_j} \) of \( L \).

Thus, given a semisimple Haantjes operator \( K \) with \( s \) point-wise distinct eigenvalues \( \{l_1(x), \ldots, l_s(x)\} \), one can always construct another Haantjes operator \( L \) fulfilling the condition (74), by considering a finer (or at least no coarser) decomposition than the spectral decomposition of \( K \)

\[
T_x M = \bigoplus_{i=1}^s D_{l_i}(x),
\]

according to which

\[
K = \sum_{i=1}^s l_i P_i,
\]

being \( P_i \) the projection operator onto \( D_{l_i} \). By way of an example, one can consider the further decomposition

\[
D_{l_i}(x) = \bigoplus_{j_i=1}^{r_i} C_{i,j_i}(x), \quad r_i = \text{rank } D_{l_i},
\]

in terms of one-dimensional Lie subalgebras \( C_{i,j_i} \subseteq D_{l_i} \). Then, one can construct the operator

\[
L = \sum_{i=1,j_i=1}^{s,r_i} \lambda_{i,j_i} \pi_{i,j_i},
\]

where \( \pi_{i,j_i} \) are projection operators onto the subalgebras \( C_{i,j_i} \), and \( \lambda_{i,j_i} \) are arbitrarily chosen (but point-wise distinct) functions, numbered with respect to the finer decomposition (79) of \( T_x M \), which will play the role of eigenvalues of \( L \). Consequently, we have

\[
P_i = \sum_{j_i=1}^{r_i} \pi_{i,j_i}.
\]

Thus, \( p(\lambda_{i,j_i}) = l_i, \ i = 1, \ldots, s, \ j_i = 1, \ldots, r_i \).

As a consequence of the Proposition 43 we have the following result.

Proposition 45. Let \((M, \mathcal{H})\) be a semisimple Abelian Haantjes algebra of rank \( m \). Consider the spectral decomposition (80)

\[
T_x M = \bigoplus_{a=1}^v V_a(x).
\]

Then the Haantjes algebra \((M, \mathcal{H})\) is cyclic if \( v \leq m \). Also, assume that the set

\[
\{U_i(y^{a,j_a})\}, \quad a = 1, \ldots, m, \ j_a = 1, \ldots, r_a
\]
is a Haantjes chart adapted to the decomposition \( (80) \) whenever \( v = m \), or to a decomposition finer than \( (80) \) if \( v < m \). Then each operator of the form

\[
L = \sum_{a=1}^{m} \lambda_a(x) \sum_{j_a=1}^{r_a} \frac{\partial}{\partial y^{a,j_a}} \otimes dy^{a,j_a},
\]

is a cyclic generator of \( \mathcal{H} \) provided that its \( m \) eigenvalues \( \{\lambda_1(x), \ldots, \lambda_m(x)\} \) are smooth functions, arbitrary but distinct at any point of \( U \). In particular, if the eigenvalues of \( L \) are chosen to be

\[
\lambda_a(x) = \lambda_a(y^{a,1}, \ldots, y^{a,r_a}) \quad a = 1, \ldots, m,
\]

then \( L \) is a cyclic Nijenhuis generator, that is its Nijenhuis torsion identically vanishes.

**Proof.** Due to Theorem 39, there exist Haantjes charts adapted to the decomposition \( (80) \) and they are of the form \( (69) \). If \( v = m \), the eigen-distributions of \( (82) \) are given by \( V_a \); consequently, by construction they fulfill condition \( (75) \). Moreover, as the eigenvalues of \( L \) are distinct, this operator satisfies the assumptions of Proposition 43.

When \( v < m \), a cyclic generator can still be constructed, because we can further decompose some of the distributions \( V_a \), into a direct sum of sub-distributions

\[
V_a = \left( \frac{\partial}{\partial y^{a,1}}, \ldots, \frac{\partial}{\partial y^{a,r_a}} \right) = \bigoplus_{i_a=1}^{\tilde{r}_a} \left( \frac{\partial}{\partial y^{a,1}}, \ldots, \frac{\partial}{\partial y^{a,i_a}} \right) = \bigoplus_{i_a=1}^{\tilde{r}_a} C_{a,i_a},
\]

with \( \sum \tilde{r}_a = r_a \), in such a way that the number of addends into the following decomposition

\[
T_x M = \bigoplus_{a=1, i_a=1}^{u, \tilde{r}_a} C_{a,i_a}
\]

equals again \( m \).

Finally, if the eigenvalues of \( (82) \) are chosen according to the assumption \( (83) \), then the Nijenhuis torsion of \( L \) vanishes due to Lemma 5. \( \square \)

To conclude this discussion, we observe that in the case \( v > m \), no cyclic operator exists for a Haantjes algebra \( \mathcal{H} \) of rank \( m \). However, it is possible to immerse \( \mathcal{H} \) into a larger Haantjes algebra, of rank equal to \( v \), by adding \( (v - m) \) suitable diagonal operators not belonging to \( \mathcal{H} \), independent among each others. Thus, such an enlarged algebra admits a cyclic generator according to Proposition 45 (for an example of this procedure, see e.g. Section 5.1).

5. **Cyclic-type Haantjes operators, triangular and Jordan: Examples**

We shall present here some examples illustrating several interesting aspects of the Haantjes geometry previously discussed.

In the first example, as an application of the main theorems, we develop the procedure of constructing a Haantjes chart for a semisimple Haantjes algebra of rank 2. Also, a cyclic generator for a suitably enlarged algebra is constructed.

Instead, the second and third examples deal with non-semisimple Haantjes algebras. In this context, an important open problem is to establish, if possible,
the normal form of non-semisimple Haantjes operators that admit a local reference frame in which they could take a triangular form. In order to illustrate the different possibilities that may arise, we propose two paradigmatic case studies. In fact, the second example consists of a Haantjes operator assuming in a local chart a triangular but not Jordan-type form. The third one admits local charts in which it takes more specifically a Jordan-type form.

5.1. Construction of a Haantjes chart. Let us consider the affine 3D space $\mathcal{A}_3$, with a cartesian coordinates system $\{O; (x, y, z)\}$ and the two operators $K_1 : TM \to TM$

$$K_1 := y^2 \frac{\partial}{\partial x} \otimes dx + x^2 \frac{\partial}{\partial y} \otimes dy - xy \left( \frac{\partial}{\partial x} \otimes dy + \frac{\partial}{\partial y} \otimes dx \right)$$

$$K_2 := (y^2 + z^2) \frac{\partial}{\partial x} \otimes dx + (x^2 + z^2) \frac{\partial}{\partial y} \otimes dy + (y^2 + z^2) \frac{\partial}{\partial z} \otimes dz - xy \left( \frac{\partial}{\partial x} \otimes dy + \frac{\partial}{\partial y} \otimes dx \right) - xz \left( \frac{\partial}{\partial x} \otimes dz + \frac{\partial}{\partial z} \otimes dx \right) - yz \left( \frac{\partial}{\partial y} \otimes dz + \frac{\partial}{\partial z} \otimes dy \right)$$

that generate an Abelian Haantjes algebra of rank 2. Let us note that $K_2$ is a special case of the inertia tensor $[23]$, for of a single point $P_\gamma \equiv O$ of unitary mass, and $n = 3$. The spectra of $K_1$ and $K_2$ are

$$\text{Spec}(K_1) = \{ l_1^{(1)} = x^2 + y^2, l_1^{(2)} = 0 \}$$

$$\text{Spec}(K_2) = \{ l_2^{(1)} = 0, l_2^{(2)} = x^2 + y^2 + z^2 \} ,$$

and their eigen–distributions

$$\mathcal{D}_1^{(1)} = \langle Y_1 \rangle , \quad \mathcal{D}_2^{(1)} = \langle Y_2, Y_3 \rangle , \quad Y_1 := -y \frac{\partial}{\partial x} + x \frac{\partial}{\partial y} , \quad Y_2 := x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} , \quad Y_3 := \frac{\partial}{\partial z} ,$$

$$\mathcal{D}_1^{(2)} = \langle Z_1 \rangle , \quad \mathcal{D}_2^{(2)} = \langle Z_2, Z_3 \rangle , \quad Z_1 := Y_2 + zY_3 , \quad Z_2 := Y_1 , \quad Z_3 := -z \frac{\partial}{\partial x} + x \frac{\partial}{\partial z} .$$

Haantjes charts can be constructed finding coordinates adapted to the decomposition (85). To this aim, we observe that

$$\mathcal{D}_1^{(1)} \cap \mathcal{D}_1^{(2)} = \langle 0 \rangle , \quad \mathcal{D}_1^{(1)} \cap \mathcal{D}_2^{(2)} = \langle Z_2 \rangle , \quad \mathcal{D}_1^{(2)} \cap \mathcal{D}_1^{(2)} = \langle Z_1 \rangle ,$$

$$\mathcal{D}_2^{(1)} \cap \mathcal{D}_2^{(2)} = \langle Z_4 \rangle , \quad Z_4 := -xz \frac{\partial}{\partial x} - yz \frac{\partial}{\partial y} + (x^2 + y^2) \frac{\partial}{\partial z} ,$$

so that the tangent spaces are decomposed by three addends as

$$(85) \quad T_x M = \langle Z_1 \rangle (x) \oplus \langle Z_2 \rangle (x) \oplus \langle Z_4 \rangle (x) .$$

A set of local coordinates adapted to such decompositions is given (for $x \neq 0$) by the functions

$$x_1 = \frac{y}{x} \Rightarrow dx_1 \in \left( \langle Z_1 \rangle \oplus \langle Z_4 \rangle \right)^{\circ}$$

$$x_2 = \sqrt{x^2 + y^2 + z^2} \Rightarrow dx_2 \in \left( \langle Z_2 \rangle \oplus \langle Z_4 \rangle \right)^{\circ}$$

$$x_3 = \frac{z}{\sqrt{x^2 + y^2 + z^2}} \Rightarrow dx_3 \in \left( \langle Z_1 \rangle \oplus \langle Z_2 \rangle \right)^{\circ} .$$

Such coordinates are related to spherical coordinates $(\rho, \varphi, \theta)$ in $\mathcal{A}_3$ as

$$x_1 = \tan \varphi , \quad x_2 = \rho , \quad x_3 = \cos \theta .$$
They are characteristic functions of the spherical web whose fibers are the three foliations generated by:

- \( \langle Z_1 \rangle \oplus \langle Z_4 \rangle \), half-planes issued from the \( z \) axis;
- \( \langle Z_2 \rangle \oplus \langle Z_4 \rangle \), spheres centered in \( O \);
- \( \langle Z_1 \rangle \oplus \langle Z_2 \rangle \) one-folded circular cones with axis \( z \) and vertex \( O \).

In these coordinates, the Haantjes operators \( K_1 \) and \( K_2 \) take the diagonal form

\[
K_1 = x_2^2 \left( 1 - x_3^2 \right) \frac{\partial}{\partial x_1} \otimes dx_1,
\]

\[
K_2 = x_2^2 \left( \frac{\partial}{\partial x_1} \otimes dx_1 + \frac{\partial}{\partial x_3} \otimes dx_3 \right).
\]

The Haantjes algebra \( \mathcal{H} \), whose basis is \( \{ K_1, K_2 \} \), is not a cyclic algebra according to Definition 42, as the identity operator does not belong to \( \mathcal{H} \). Indeed, its spectral decomposition (85) does not fulfill the assumption of Proposition 45 as \( v = 3 > m = 2 \). Moreover, coherently with Lemma 36, each element of \( \mathcal{H} \) is not invertible.

However, if we extend the algebra simply by adding the identity operator \( I \), we shall get a cyclic Haantjes algebra of rank three. For instance, a cyclic generator with three distinct eigenvalues, for the extended algebra of rank 3, is given by

(87)  \[ L = K_1 + K_2, \]

with

\[
\text{Spec}(L) = \{ \lambda_1 = -x_2^2(x_3^2 - 2), \lambda_2 = 0, \lambda_3 = x_2^2 \}.
\]

In fact

\[
K_1 = \frac{x_3^2 - 3}{x_3^2 - 2} L - \frac{1}{\lambda_1} L^2,
\]

and

\[
K_2 = -\frac{\lambda_3}{\lambda_1} L + \frac{1}{\lambda_1} L^2.
\]

Thus, \( \mathcal{H} \) turns out to be a Haantjes subalgebra (but not a cyclic one) of the enlarged cyclic Haantjes algebra whose basis is

(88)  \[ \{ I, K_1, K_2 \}. \]

The eigenvalues of \( L \) have been chosen in such a way that \( L \) in the original cartesian coordinates \( (x, y, z) \) takes a rational simple form, precisely

\[
L = (2y^2 + z^2) \frac{\partial}{\partial x} \otimes dx + (2x^2 + z^3) \frac{\partial}{\partial y} \otimes dy + (x^2 + y^2) \frac{\partial}{\partial z} \otimes dz
\]

\[
-2xy \left( \frac{\partial}{\partial x} \otimes dy + \frac{\partial}{\partial y} \otimes dx \right) - xz \left( \frac{\partial}{\partial x} \otimes dz + \frac{\partial}{\partial z} \otimes dx \right)
\]

\[
-yz \left( \frac{\partial}{\partial y} \otimes dz + \frac{\partial}{\partial z} \otimes dy \right).
\]

Let us note that cyclic Nijenhuis generators of the cyclic Haantjes algebra with basis (88) do exist as well, for instance

\[
N = x_1 \frac{\partial}{\partial x_1} \otimes dx_1 + x_2^2 \frac{\partial}{\partial x_2} \otimes dx_2 + x_3^2 \frac{\partial}{\partial x_3} \otimes dx_3.
\]

However, although its form is again a rational one in the original cartesian coordinates, it turns out to be much more complicated than the form of \( L \). In fact, the numerators of its components are polynomials up to degree 9.
5.2. A triangular but not Jordan-type operator. We will show an example of a non-semisimple Haantjes operator, which does not admit local charts where it assumes a Jordan form, nor a generalized Jordan form. However it admits, by its own, a local chart where it takes a triangular form.

Let us consider the affine 4D space $A_4$, the cartesian coordinates $(x_1, x_2, x_3, x_4)$ and the Haantjes operator

$$K := (x_3 + x_1 x_2 + \frac{1}{3} x_4^3) \left( \frac{\partial}{\partial x_1} \otimes dx_1 + \frac{\partial}{\partial x_2} \otimes dx_2 + \frac{\partial}{\partial x_3} \otimes dx_3 + \frac{\partial}{\partial x_4} \otimes dx_4 \right)$$

$$+ \left( x_2 + \frac{1}{2} x_4^2 \right) \left( \frac{\partial}{\partial x_1} \otimes dx_2 + \frac{\partial}{\partial x_2} \otimes dx_4 \right) + x_1 \left( \frac{\partial}{\partial x_1} \otimes dx_3 + \frac{\partial}{\partial x_2} \otimes dx_4 \right)$$

$$+ \frac{\partial}{\partial x_1} \otimes dx_4 .$$

Such operator appears in [15], in connection with the theory of hydrodynamic-type systems. The spectrum of $K$ is given by one eigenvalue

$$Spec(K) = \{ l = x_3 + x_1 x_2 + \frac{1}{3} x_4^3 \}$$

with a proper one-dimensional eigen-distribution

$$Ker(K - lI) = \left( \frac{\partial}{\partial x_1} \right) ,$$

and a four-dimensional generalized eigen-distribution of Riesz index 4

$$\mathcal{D}_l = Ker(K - lI)^4 = \left( \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}, \frac{\partial}{\partial x_4} \right) \equiv \mathcal{T}A_4 .$$

Now, we search for a coordinates transformation $\Phi : A_4 \to A_4, (x_1, x_2, x_3, x_4) \mapsto q_i = \phi_i(x_1, x_2, x_3, x_4), i = 1, 2, 3, 4$, such that $(q_1, q_2, q_3, q_4)$ be a local chart in which the Haantjes operator $K$ takes the classical Jordan form.

$$K = l \left( \frac{\partial}{\partial q_1} \otimes dq_1 + \frac{\partial}{\partial q_2} \otimes dq_2 + \frac{\partial}{\partial q_3} \otimes dq_3 + \frac{\partial}{\partial q_4} \otimes dq_4 \right)$$

$$+ \left( \frac{\partial}{\partial q_1} \otimes dq_2 + \frac{\partial}{\partial q_2} \otimes dq_3 + \frac{\partial}{\partial q_3} \otimes dq_4 \right) .$$

This request leads to a system of PDE for the functions $(\phi_1, \phi_2, \phi_3, \phi_4)$ that admits only the solution

$$\phi_1 = \phi_1(x_4), \quad \phi_2 = c_2, \quad \phi_3 = c_3, \quad \phi_4 = c_4 ,$$

where $c_2, c_3, c_4$ are arbitrary constant. Thus, we can conclude that a local chart in which $K$ takes a classical Jordan form does not exist. Furthermore, an analogous result can be obtained if one searches for a local chart in which $K$ assumes a generalized Jordan form, that is a form of the type

$$K = l \left( \frac{\partial}{\partial q_1} \otimes dq_1 + \frac{\partial}{\partial q_2} \otimes dq_2 + \frac{\partial}{\partial q_3} \otimes dq_3 + \frac{\partial}{\partial q_4} \otimes dq_4 \right)$$

$$+ f \frac{\partial}{\partial q_1} \otimes dq_2 + g \frac{\partial}{\partial q_2} \otimes dq_3 + h \frac{\partial}{\partial q_3} \otimes dq_4 ,$$

where $f, g, h$ are arbitrary smooth functions on $A_4$. 
5.3. Jordan form. We present an example of a non-semisimple operator that admits
different charts in which it takes a Jordan-type form.
Let us consider the affine 4D space \( A_4 \), the cartesian coordinates \((x_1, x_2, x_3, x_4)\)
and the Haantjes operator
\[
K : = 6x_4 \left( \frac{\partial}{\partial x_1} \otimes dx_1 + \frac{\partial}{\partial x_2} \otimes dx_2 + \frac{\partial}{\partial x_3} \otimes dx_3 + \frac{\partial}{\partial x_4} \otimes dx_4 \right) \\
- 9x_1 \left( \frac{\partial}{\partial x_1} \otimes dx_4 + \frac{\partial}{\partial x_2} \otimes dx_3 \right) + 3x_3 \left( \frac{\partial}{\partial x_2} \otimes dx_1 + \frac{\partial}{\partial x_3} \otimes dx_4 \right).
\]
The spectrum of \( K \) is given by one eigenvalue
\[
\text{Spec}(K) = \{ l = 6x_4 \}
\]
with a proper two-dimensional eigen-distribution
\[
\text{Ker}(K - lI) = \langle Y_1, Y_2 \rangle, \quad Y_1 := \frac{\partial}{\partial x_2}, \quad Y_2 := -3x_1 \frac{\partial}{\partial x_1} + x_3 \frac{\partial}{\partial x_3},
\]
and a four-dimensional generalized eigen-distribution of Riesz index 2
\[
\mathcal{D}_1 = \text{Ker}(K - lI)^2 = \langle Y_1, W_1, Y_2, W_2 \rangle, \quad W_1 := \frac{\partial}{\partial x_1}, \quad W_2 := \frac{1}{3} \frac{\partial}{\partial x_1}.
\]
Note that the reference frame \((Y_1, W_1, Y_2, W_2)\) is a generalized Jordan frame as
\[
KY_i = l Y_i, \quad KW_i = l W_i + f_i Y_i \quad i = 1, 2,
\]
with \( f_1 = 3x_3, \ f_2 = 1 \). Such a particular frame satisfies also the property to be integrable according to Proposition 17 as every pair of vector fields commute except for \([Y_2, W_1] = 3W_1\). Therefore, each annihilator of the distributions of rank 3 generated by three of the vector fields \((Y_1, W_1, Y_2, W_2)\) contains an exact one form. For instance,
\[
\langle Y_1, W_2, Y_2 \rangle^o = \langle d(x_1^{1/3} x_3) \rangle, \quad \langle Y_2, W_1, W_2 \rangle^o = \langle dx_2 \rangle, \\
\langle Y_1, W_1, Y_2 \rangle^o = \langle dx_3 \rangle, \quad \langle Y_1, W_1, Y_2 \rangle^o = \langle dx_4 \rangle.
\]
This fact implies that in the local chart \( \{ (q_1 = x_2, q_2 = x_1^{1/3}, q_3 = x_3, q_4 = x_4) \} \)
the Haantjes operator \( K \) takes the following generalized Jordan form
\[
K = 6q_4 \left( \frac{\partial}{\partial q_1} \otimes dq_1 + \frac{\partial}{\partial q_2} \otimes dq_2 + \frac{\partial}{\partial q_3} \otimes dq_3 + \frac{\partial}{\partial q_4} \otimes dq_4 \right) \\
+ 9q_2^2 \left( \frac{\partial}{\partial q_1} \otimes dq_2 \right) + 3q_3 \left( \frac{\partial}{\partial q_3} \otimes dq_4 \right).
\]

6. A comparison with other algebraic structures: Haantjes manifolds and Killing-Stäckel algebras

It is interesting to compare our definition of Haantjes algebras over a differentiable manifold
classified in Section 5 with the notion of Haantjes manifolds recently introduced by Magri [17].
The main difference between the two constructions resides on the distinct degree of generality of them, which obviously reflects
the variety of our motivations.

In our construction, we are mainly concerned with the abstract, more general theory of commuting Haantjes operators defining an Haantjes algebra, without
any reference to additional geometric structures like \( 1 \)-forms or symmetry
vector fields, that in Magri’s theory are essential to construct Lenard complexes of commuting vector fields or exact 1-forms [18–19].

In other words, although Magri’s Haantjes manifolds possess a richer axiomatic structure than our Haantjes algebras, we prefer to choose a minimal number of requirements in order to have a flexible structure, which in a subsequent step can be made more suitable for the study of specific problems, as separation of variables in the context of integrability or Riemannian geometry. To this aim, we postpone the introduction of additional geometric structures (as Magri–Haantjes chains of exact 1-forms, symplectic forms [22], Poisson bivectors [21] or Riemannian metrics [23]) to a further stage of the theory.

The notion of **Killing-Stackel algebra** in an $n$-dimensional Riemannian manifold, due to Benenti et al. [3], can be naturally interpreted in terms of Haantjes algebras of rank $n$. In order to compare the two notions, it is useful to observe that the cyclic generator (87) enjoys a special property: In fact, its contravariant form is a Killing two-tensor with respect to the Euclidean metric of the affine space $A_3$. So, we shall call it a **Killing-Haantjes cyclic generator**; it can be identified with a **characteristic tensor** (CKT) of the **Killing-Stäckel algebra** (88). Thus, the theory of Haantjes cyclic algebras over a Riemannian manifold makes contact with the theory of Killing-Stäckel algebras, which offers a geometrical setting for the classical theory of separation of variables for Hamiltonian systems going back to Eisenhart, Stäckel, Jacobi, etc. The main difference between the two algebraic structures is that Killing-Stäckel algebras are vector spaces of Killing two-tensors, closed w.r.t. linear combinations with real constant coefficients. By contrast, Haantjes algebras are modules of Haantjes operators, closed w.r.t. linear combinations with (smooth) functions.

At the same time, it is interesting to notice that, starting from a Killing-Haantjes cyclic generator, one can choose suitable functions to generate other Killing-Haantjes two-tensors, that is, elements of Killing-Stäckel algebras. The conditions which such functions must obey are under investigation. To conclude the comparison among these geometric structures, we can distinguish three different scenarios.

i) When dealing with Killing-Stäckel algebras, we are realizing a specific Haantjes algebra with constant coefficients, which typically does not possess a cyclic Killing-Haantjes generator. In fact, although a CKT does exist, it is not a cyclic generator, since to reconstruct the full algebra one would need to combine the powers of the CKT with suitable functions. Instead, in Killing-Stäckel algebras only linear combinations with constant coefficient are allowed (by definition).

ii) Let us then consider a generalization of the Killing-Stäckel algebra, obtained combining the powers of the CKT by means of functions. In this case one obtains a larger algebra, which is a full cyclic Haantjes algebra, defined over the ring of smooth functions.

iii) The most general case is that of Haantjes algebras that do not come from Killing tensors.

**Acknowledgement**

The authors wish to thank F. Magri for having drawn our attention to the theory of Haantjes manifolds.
We also thank heartily Y. Kosmann-Schwarzbach for letting us know the unpublished preprint [14] when the present work was in its final stages. G. T. also wishes to thank B. Konopelchenko for many useful discussions about the paper [15]. P. T. acknowledges the support of the research project FIS2015-63966, MINECO, Spain, and by the ICMAT Severo Ochoa project SEV-2015-0554 (MINECO). P. T. and G. T. are members of Gruppo Nazionale di Fisica Matematica (GNFM) of INDAM.

References

[1] S. Benenti, *Inertia tensors and Stackel systems in the Euclidean spaces*, Rend. Sem. Mat. Univ. Politec. Torino 50, 315–341 (1992).
[2] S. Benenti, *Orthogonal separable dynamical Systems*, Math. Publ., Silesian Univ. Opava 1, 163–184 (1993).
[3] S. Benenti, C. Chanu, and G. Rastelli, *Remarks on the connection between the additive separation of the Hamilton-Jacobi equation and the multiplicative separation of the Schrödinger equation. I. The completeness and Robertson conditions*, J. Math. Phys. 43, 5183–5222 (2002).
[4] O. I. Bogoyavlenskij, *Necessary Conditions for Existence of Non-Degenerate Hamiltonian Structures*, Commun. Math. Phys. 182, 253-290 (1996).
[5] O. I. Bogoyavlenskij, *General algebraic identities for the Nijenhuis and Haantjes torsions*, Izvestya Mathematics 68, 1129-1141 (2004).
[6] O. I. Bogoyavlenskij, *Schouten tensor and bi-Hamiltonian systems of hydrodynamic type*, J. Math. Phys. 47, paper n. 023504, pp. 14 (2006).
[7] A.V. Bolsinov and V.S. Matveev, *Geometrical interpretation of Benenti systems*, J. Geom. Phys. 44, 489-506 (2003).
[8] M. De Filippo, G. Vilasi, and M. Salerno *A geometrical approach to the integrability of soliton equations*, Lett. Math. Phys. 9, 85-91 (1985).
[9] E. V. Ferapontov and D. G. Marshall, *Differential-geometric approach to the integrability of hydrodynamics chains: the Haantjes tensor*, Mat. Ann. 339, 61–99 (2007).
[10] A. Frolicher and A. Nijenhuis, *Theory of Vector-Valued Differential Forms. Part I*, Indag. Mathematicae 18, 338–359 (1956).
[11] Gerdjikov V.S., Vilasi G. and Yanovski A.B.: *Integrable Hamiltonian Hierarchies*. Lect. Not. Phys., Vol. 748 Berlin Heidelberg: Springer, 2008
[12] D. Gutzkin, *G-Cohomologie et opérateurs de récursion*, Ann. Inst. Henri Poincaré 47, 355-366 (1987).
[13] J. Haantjes, *On $X_{n-1}$-forming sets of eigenvectors*, Indag. Mathematicae 17, 158–162 (1955).
[14] Y. Kosmann-Schwarzbach, *Beyond recursion operators*, Preprint, arXiv: 1712.08908 (2017), Proceedings of the XXXVI Workshop on Geometric Methods in Physics, Białowieża, Poland.
[15] Y. Kodama and B.G. Konopelchenko, *Confluence of hypergeometric functions and integrable hydrodynamic-type systems*, Theor. Math. Phys. 188, 1334–1357 (2016).
[16] F. Magri, *Recursion operators and Frobenius manifolds*, SIGMA 8, paper 076, 7 pp. (2012).
[17] F. Magri, *Haantjes manifolds*, Journal of Physics: Conference Series 482, paper 012028, 10 pp. (2014).
[18] F. Magri, *WDVV Equations*, II Nuovo Cimento C 38, paper 166, 10 pp. (2015).
[19] F. Magri, *Haantjes manifolds and Veselov Systems*, Theor. Math. Phys. 189, 1486–1499 (2016).
[20] A. Nijenhuis, $X_{n-1}$-forming sets of eigenvectors, Indag. Mathematicae 54, 200-212 (1951).
[21] A. Nijenhuis, *Jacobi-type identities for bilinear differential concomitants of certain tensor fields I,II*, Indag. Math 17, 390-397, 398-403 (1955).
[22] P. Tempesta, G. Tondo, *Haantjes Manifolds and Classical Integrable Systems*, Preprint arxiv: 1405.5118v2, 2016.
[23] G. Tondo, P. Tempesta, *Haantjes structures for the Jacobi-Calogero model and the Benenti Systems*, SIGMA 12, 023, 18 pp. (2016)
[24] G. Tondo, *Haantjes Algebras of the Lagrange Top*, Preprint (2017).
[25] S. Winitzki, *Linear Algebra via Exterior Products*, Lulu Press, https://sites.google.com/site/winitzki/linalg (2010).

Departamento de Física Teórica II, Facultad de Físicas, Universidad Complutense, 28040 – Madrid, Spain and Instituto de Ciencias Matemáticas, C/ Nicolás Cabrera, No 13–15, 28049 Madrid, Spain.

E-mail address: p.tempesta@fis.ucm.es, piergiulio.tempesta@icmat.es

Dipartimento di Matematica e Geoscienze, Università degli Studi di Trieste, piazzale Europa 1, I–34127 Trieste, Italy.

E-mail address: tondo@units.it