Outage Analysis of Uplink Two-tier Networks

Zolfa Zeinalpour-Yazdi and Shirin Jalali

Abstract—Employing multi-tier networks is among the most promising approaches to address the rapid growth of the data demand in cellular networks. In this paper, we study a two-tier uplink cellular network consisting of femtocells and a macrocell. Femto base stations, and femto and macro users are assumed to be spatially deployed based on independent Poisson point processes. We consider an open access assignment policy, where each macro user based on the ratio between its distances from its nearest femto access point (FAP) and from the macro base station (MBS) is assigned to either of them. By tuning the threshold, this policy allows controlling the coverage areas of FAPs. For a fixed threshold, femtocell coverage areas depend on their distances from the MBS; Those closest to the fringes will have the largest coverage areas. Under this open-access policy, ignoring the additive noise, we derive analytical upper and lower bounds on the outage probabilities of femto users and macro users that are subject to fading and path loss. We also study the effect of the distance from the MBS on the outage probability experienced by the users of a femtocell. In all cases, our simulation results comply with our analytical bounds.

Index Terms—Heterogeneous networks, Uplink communication, Outage, Open access policy, Poisson point process

I. INTRODUCTION

Wireless cellular networks, originally designed for voice communications, are nowadays commonly used for surfing the Internet or communicating image, audio or video files. This massive unpredicted overhead load has urged communication engineers to develop new approaches to design and employment of cellular communication systems. One of such relatively new techniques, which has been proved to be successful, is employing multi-tier networks. For instance, in the case of two-tier networks, the existing cellular network is overlaid by femtocells, which are employed by users in an ad-hoc manner at their homes or offices.

Analytical performance evaluation of cellular networks has always been a complicated task. Modeling various aspects of cellular networks, such as the physical channel itself, has been a cornerstone of such analysis and therefore the subject of extensive research for many years. Modeling the users’ and cells’ locations is another aspect of a cellular network that also plays a major role in analytical evaluations. Traditionally, the idealized grid model has been employed to model the locations of the cells and their coverage areas. This model, although simple to describe, is intractable for most analytical evaluations and is also arguably not very accurate. This is especially true in heterogenous networks with ad hoc employment of small cells. More comprehensive and recent models for spatial distributions of the cells and users are models based on stochastic geometric tools such as Poisson point process (PPP). Such models are advantageous from two main perspectives: first, they provide a more realistic model of cellular networks compared to the traditional grid-based models, second, they make the analysis more tractable.

In this paper, we analyze the outage performance of an uplink two-tier network with a MBS located at the center of a circle representing its coverage area; macro users (MU), femto users (FU) and FAPs are assumed to be spatially distributed within the circle randomly and independently according to PPPs with different densities. We consider the open access policy studied in [1], [2] for downlink communication. This model covers closed access policy as a special case and allows optimizing the coverage areas of the FAPs when the system parameters vary. Using this model, we derive tight upper and lower bounds on the outage probabilities of users covered by the MBS and also FUs and MUs that are covered by FAPs. To achieve this goal we first derive upper and lower bounds on the Laplace transform of the number of MUs covered by a FAP located at a specific distance from the MBS. Employing the Laplace transforms of the number of users in each group, plus some geometric analysis, we bound our desired outage probabilities. Our simulation results confirm and comply with our bounds.

A. Related work

While employing PPP as a stochastic model for users or access points distributions was originally proposed in 1997 in [3]–[5], it was not until recently that this model was used for analyzing the performance of wireless cellular networks. (Refer to [6] for a review of this model.) The stochastic geometric-based models such as PPP was employed by Baccelli et al. in [7] to analyze large mobile ad hoc network (MANET) and by Andrews et al. in [8] to study the downlink performance of cellular networks. Later, this model was used for analyzing the downlink performance of multi-tier networks [9]–[11]. However, as mentioned in [12], similar analysis for uplink communication has been missing until very recently.

Multi-tier networks have been studied from different perspectives such as power control [13], [14], spectrum allocation [15], [16], and exploiting cognitive radio techniques [17], [18]. These are just few examples of some related work and by no means are meant to be a comprehensive review of the literature. (See [12], [19], [20] and the references therein for a relatively comprehensive review of the literature.)

Analytic study of the outage performance of a single-tier network with nodes distributed according to a PPP is done in [21]. Uplink performance of two-tier networks has been studied in the literature under different models and approximations.
While most of the work on this topic has been on traditional grid model, recently there has been several results on analyzing uplink performance of two-tier networks under PPP model for users and access points. Chandrasekhar et al. study outage probabilities of femto and MUs distributed according to PPPs in a reference macrocell in [22]. The authors consider a CDMA-based model under closed access and approximate the outage probability. Xia et al. in [23] compare closed access versus open access policy in an uplink communication. In their analysis, they consider a reference macrocell with the base station located at the center, and a single FAP located at a specific distance from the base station. The MUs are assumed to be distributed independently at random. They suggest that while for orthogonal multiple access schemes such as TDMA or OFDMA the choice of open versus closed depends on the users density, in non-orthogonal schemes such as CDMA open access is strictly better than closed access.

The uplink performance of macrocells overlaid with femtocells is also studied in [24]. There, while the authors consider PPP spatial distribution for MUs, FUs and FAPs, the users assignment policy is closed access, and by assuming a TDMA scheme they limit the number of active users in each femtocell per time slot to one. In [25], the authors study the distribution of the signal to interference plus noise ratio (SINR) in both uplink and downlink, when time division duplex (TDD) is employed. In their setup, the users of each tier are distributed according to a PPP and each user connects to the closest base station.

In an independent work, which the authors became aware of right before submitting this paper, Bao et al. analyze the interference and outage performance of a two-tier uplink network under closed access policy [26]. The authors of [26] also study the open access policy in a subsequent paper [27]. While the ultimate goals in [27] and this paper are the same, there are some major differences between the two. First, unlike this paper, in [27], each femtocell is assumed to have a fixed coverage area, and a MU is handed off to the FAP if it falls within that fixed coverage area. Here, we consider a different open access policy, where each MU decides to connect to either its closest FAP or the MBS, based on its distances from them. This policy leads to FAPs having different coverage areas, depending on their distances to the MBS. This assignment policy introduces new geometrical aspects to our outage analysis. Second, unlike [27], we derive closed-form expressions for our upper and lowers bounds on the outage probabilities of MUs and FUs. For a MU serviced by the MBS, we study and bound its outage performance as a function of its distance of the FAP from the MBS. Finally, the system model considers multi-carrier frequency hopping modulation, which provides a decentralized alternative to OFDM. In [27], the authors consider a single shared channel for all users. Finally, one of the reviewers pointed us to the work of ElSawy et al. [28], which has appeared on Arxiv after our initial submission. In [28], the authors study the uplink performance of a multi-tier network under a different access policy where each user connects to its closest access point (femto or macro).

B. Notation

Calligraphic letters such as $\mathcal{X}$ and $\mathcal{Y}$ represent sets. The size of set $\mathcal{X}$ is denoted by $|\mathcal{X}|$. Given sample space $\Omega$ and event $E \subseteq \Omega$, $\mathbb{1}_E$ is an indicator random variable that is one when event $E$ happens. For $1 \leq i \leq j \leq n$, $x_{i:j} \triangleq (x_i, x_{i+1}, \ldots, x_j)$. Also, for simplicity $x^j = x_1^j$. Uppercase letter characters such as $X$ and $Y$ are used for matrices and random variables.

C. Paper organization

The organization of the paper is as follows. Section II reviews the network model studied in this paper from various perspectives: modulation technique, spatial distributions of users, channel model and access policy. In Section III we study the users density distributions. The results of this section is used extensively in the following sections in analyzing the performance of the system. In Section IV, which contains the main results of the paper, we analyze the femto and MUs outage probabilities. Section V presents the simulation results and compares them with our analytical bounds. Finally, Section VI concludes the paper.

II. System model

A. MCFH technique

Orthogonal frequency division multiplexing (OFDM) is a widely popular multiple access method in wireless networks, and has received a lot of attention in recent years. In an OFDM-based multiple access system, the carrier frequencies are assigned by the central node. (Three different methods for assigning frequencies are described in [29].) However, this centralized frequency assignment is not quite desirable for emerging decentralized wireless cellular networks such as femtocells, where, due to practical challenges’, it is preferred to minimize the coordination between the central and the femto base stations.

Multicarrier frequency-hopping (MCFH) modulation introduced in [30], and later analyzed by various researchers [31–34], provides a decentralized alternative to OFDM modulation. In MCFH, similar to OFDM, all sub-carriers are orthogonal to each other. However, unlike OFDM, in a multi-user setup, the carriers are not assigned to the users by a central node, and the users are allowed to randomly and independently select their carriers. In addition to being decentralized, another advantage of MCFH to OFDM, as will become clear throughout the paper, is that it makes the model more amenable to direct analysis. The results of such analysis will provide insight on how to select the systems’ parameters in an OFDM-based system as well. In this paper, we assume that all users adopt MCFH modulation. While MCFH is clearly different from OFDM, most of our results will continue to hold for OFDM-based systems with some mild adjustments.

In MCFH, the available bandwidth is divided into $n_s$ non-overlapping adjacent subbands. Each subband respectively is divided into $n_h$ equispaced frequencies. Hence, overall, there will be $n_s n_h$ available subchannels. (It is usually said that the system’s processing gain $G$ is equal to $n_s n_h$. At each time, each user uniformly at random selects one of the $n_h$ carriers in
each subband. Fig. 1 shows the carrier selections in a simple MCFH system with $n_s = 3$ and $n_k = 4$ and two users. As shown in the figure, since unlike OFDM, users select their carriers independently with no coordination, it is possible that two users send data over the same frequency simultaneously.

### B. Spatial distribution

For spatial distribution of MUs, FUs and FAPs, we follow the model introduced in [1]. We consider a MBS $b_m$ located at the center of a circle of radius $R$ denoted by $S_m$. FAPs $A_f$ are distributed according to a PPP with density $\lambda_f$. Therefore, the number of FAPs ($|A_f|$) is distributed as $\text{Pois} (\tilde{n}_{\text{fap}})$, where $\tilde{n}_{\text{fap}} = \pi R^2 \lambda_f$. Conditioned on $|A_f| = m$, the locations of the $m$ FAPs are uniformly distributed over $S_m$. Independently, MUs $U_m$ are distributed based on a PPP with density $\mu_m$. Note that “MUs” are users that are not inside a home, office, etc. that is equipped with a FAP. However, a MU might be served by a FAP based on its distance from the MBS and the locations of surrounding FAPs. Finally, FUs of FAP $a_f \in A_f$ are distributed according to a PPP with density $\mu_f$ restricted to a ring of internal radius $r_f$ and width $\Delta$ centered at $a_f$. For FAP $a_f \in A_f$, let $U_f(a_f)$ and $U_m(a_f)$ denote the set of FUs and MUs serviced by the FAP $a_f$, respectively. Clearly, $U_{a_f} \subseteq U_f(a_f)$. Thus, it is possible that $U_m(a_f) \cap U_{a_f}$ is non-empty.

Various studies indicate that open access policies have superior performance both from the perspective of the FUs in a random setting, in which there is only one MBS. In a random setting, the MBSs employ one of the known frequency reuse methods, the $\Delta$-partition determined by the locations of MBSs. Assuming that it is reasonable to assume that each MU connects to its base station, which is equivalent to having a closed access assignment policy.

Remark 1: In our analysis we consider a single MBS located at the center of a circle of radius $R$. In reality of course there are more MBSs. The placement of the macrocells can be modeled either as a deterministic process or random based on an independent PPP with density $\lambda_m$. In both cases, it is reasonable to assume that each MU connects to its closest MBS, and hence divide the plane based on the Voronoi partition determined by the locations of MBSs. Assuming that the MBSs employ one of the known frequency reuse methods, then, without loss of generality, in the analysis one can focus on the case where there is only one MBS. In a random setting, where MBSs are employed according to a PPP of density $\lambda_m$, it is proved in [35] that the expected number of FAPs in a “typical” macrocell becomes equal to $\lambda_f/\lambda_m$. Based on this result, choosing $\lambda_m = 1/(\pi R^2)$, the expected number of FAPs in a “typical” macrocell is equal to $\lambda_f/\lambda_m = \pi R^2 \lambda_f$, which is consistent by our model in this paper. By controlling radius $R$, we can study the effect MBSs’ density $\lambda_m$ on the performance.

### C. Channel Model

We consider both small scale fading and path loss. Let $h_{u,a_f}$ and $b_{u,b_m}$ denote the fading coefficients corresponding to the channel in subband $i \in [1:n_s]$ from user $u$ to FAP $a_f$ and to MBS $b_m$, respectively. We consider a slow-fading channel model, and assume that the fading coefficients remain constant during the whole coding block. Furthermore, we assume that the coefficients corresponding to different subbands and also different channels are all independent. The channel coefficients are assumed to have a Rayleigh distribution. That is, the power attenuation coefficient $|h_{u,a_f}|^2$, where $a \in \{a_f, b_m\}$, is exponentially distributed as $P(|h_{u,a_f}|^2 > x) = e^{-x/\sigma^2}$, for $x > 0$. The path loss affecting the signal transmitted by user $u$ to base station $a$, $a \in \{a_f, b_m\}$, is modeled as $PL_{u,a} = L_0 d_u^{\alpha}$, where $L_0$ is path loss at unit distance, and $\alpha > 2$ denotes the attenuation factor [3].

### D. Access policy

Consider macro user $u \in U_m$. For user $u$ and FAP or MBS $a$, let $d(u,a)$ denote their Euclidean distance. Further, let $d_u^{(f)}$ denote the distance between user $u$ and its nearest FAP, i.e., $d_u^{(f)} = \min_{a \in A_f} d(u,a_f)$. As mentioned earlier, we focus on open access policy, where MUs can also be serviced by FAPs. We consider the following open access policy, which was considered in [1]. Let $\kappa < 1$ be a parameter of the system. Then, according to this assignment policy,

1. if $d_u^{(f)} < \kappa d(u_m,b_m)$, then MU $u_m$ is assigned to its closest FAP.
2. if $d_u^{(f)} \geq \kappa d(u_m,b_m)$, then MU $u_m$ is assigned to the MBS $b_m$.

Letting $\kappa = 0$, requires all MUs to be serviced by the base station, which is equivalent to having a closed access assignment policy. $\kappa$ controls the coverage areas of FAPs and increasing it enlarges the coverage areas.

As defined earlier, $U_m(a_f) \subset U_m$ denotes the set of MUs that are serviced by FAP $a_f \in A_f$. Let $U_m^{(f)} \equiv U_m \backslash \bigcup_{a_f \in A_f} U_m(a_f)$. We consider both small scale fading and path loss. Let $h_{u,a_f}$ and $b_{u,b_m}$ denote the fading coefficients corresponding to the channel in subband $i \in [1:n_s]$ from user $u$ to FAP $a_f$ and to MBS $b_m$, respectively. We consider a slow-fading channel model, and assume that the fading coefficients remain constant during the whole coding block. Furthermore, we assume that the coefficients corresponding to different subbands and also different channels are all independent. The channel coefficients are assumed to have a Rayleigh distribution. That is, the power attenuation coefficient $|h_{u,a_f}|^2$, where $a \in \{a_f, b_m\}$, is exponentially distributed as $P(|h_{u,a_f}|^2 > x) = e^{-x/\sigma^2}$, for $x > 0$. The path loss affecting the signal transmitted by user $u$ to base station $a_f$, $a \in \{a_f, b_m\}$, is modeled as $PL_{u,a} = L_0 d_u^{\alpha}$, where $L_0$ is path loss at unit distance, and $\alpha > 2$ denotes the attenuation factor [3].

### III. USERS DENSITY DISTRIBUTION

Before analyzing the signal to interference ratios (SIR) experienced by different users in different groups, in this section we study the distributions of $N_{f}^{\alpha}$, $|U_f(a_f)|$, $N_{m}^{\alpha}$ and $|U_m(a_f)|$. By our assumption, the FUs are distributed in a ring of width $\Delta$ and internal radius $r_f$. Hence, $N_f^{\alpha} \sim \text{Pois} (\tilde{n}_{fu})$, where $\tilde{n}_{fu} = \pi (r_f + \Delta)^2 - r_f^2 \mu_f$ denotes the expected number of FUs in each FAP, and its Laplace transform $\Phi_{N_f^{\alpha}}(s)$ is equal to

$$\Phi_{N_f^{\alpha}}(s) = \mathbb{E}[e^{-s N_f^{\alpha}}] = e^{\tilde{n}_{fu}(e^{-s} - 1)}. \tag{1}$$

Also, let $\tilde{n}_{mu} = \pi R^2 \mu$ denote the expected number of all MUs.

Consider MBS $b_m$ and FAP $a_f$ shown in Fig. 2 that are located at distance $r$ from each other. MU $u_m$ is served by FAP $a_f$ instead of $b_m$, if $d(u_m,a_f) < \kappa d(u_m,b_m)$, where $\kappa < 1$. Translating this condition into cartesian coordinate dimensions...
with the origin located at $b_m$ and the $x$-axis along the line connection $b_m$ to $af$, we obtain $(x-r)^2 + y^2 \leq \kappa^2 (x^2 + y^2)$, or $(1 - \kappa^2) x^2 + (1 - \kappa^2) y^2 - 2 rx + r^2 \leq 0$. In other words, 
\[
(x - \frac{r}{1 - \kappa^2})^2 + y^2 \leq r^2 \left(\frac{1}{(1 - \kappa^2)^2} - \frac{1}{1 - \kappa^2}\right),
\]
which is equivalent to a circle of radius $r_c = \frac{r}{(1 - \kappa^2)}$ centered at $(r/(1 - \kappa^2), 0)$. Therefore, the coverage area of each FAP depends on its distance from the base station. As the distance increases, the coverage area, and the expected number of covered MUs increase as well. In summary, given FAP $af$ located at distance $df = d(b_m, af)$, the coverage area of $af$ for MUs, i.e., the area in which MUs are serviced by FAP $af$ is a circle of radius $\sqrt{\gamma}df$, where 
\[
\gamma = \frac{\kappa^2}{(1 - \kappa^2)^2}.
\]
Therefore, $N_{af}^m \sim \text{Poisson}(\pi \gamma df^2 \mu_m)$, where $\bar{n}_{mu}^f = \pi \gamma df^2 \mu_m$, and
\[
\Phi_{N_{af}^m}(s|df) = \mathbb{E}[e^{-s N_{af}^m}] = e^{\bar{n}_{mu}^f (e^{-s} - 1)}. 
\]
In some cases, in our analysis we are interested in the distribution of $N_{af}^m - 1$, conditioned on $N_{af}^m \geq 1$. For that we define,
\[
\Phi_{N_{af}^m}^+(s|df) = \mathbb{E}[e^{-s(N_{af}^m - 1)}] = e^{\bar{n}_{mu}^f(e^{-s} - 1) - \bar{n}_{mu}^f} (1 - e^{-s\bar{n}_{mu}^f}).
\]
Finally, we study $N_{m}^b$. Conditioned on $A_f$ (locations of FAPs), $N_{m}^b$ is a Poisson random variable of mean $\mu_m S_f$, where $S_f$ denotes the area excluding covered only by $b_m$ and not FAPs. Therefore, $\Phi_{N_{m}^b}(s) = \mathbb{E}[e^{s N_{m}^b}]$.

**Theorem 1:** For $s \geq 0$, the Laplace transform of $N_{m}^b$, $\Phi_{N_{m}^b}(s)$, satisfies,
\[
\Phi_{N_{m}^b}(s) \geq e^\bar{n}_{mu}^f(e^{-s} - 1) \leq e^\bar{n}_{mu}^f(e^{-s} - 1) + e^{\gamma - 1 - \bar{n}_{m} + \gamma - 1 \log(\bar{n}_{m})} \tau(s),
\]
where 
\[
\tau(s) \triangleq \frac{e^{(1 - e^{-s})\bar{n}_{mu}^f - 1}}{(1 - e^{-s})\bar{n}_{mu}^f},
\]
and $\gamma$ is defined in (2).

**Proof:** Since $N_{m}^b \geq \mathcal{U}_m$ always holds, for $s \geq 0$, $e^{-s N_{m}^b} \geq e^{-s \mathcal{U}_m}$. Therefore, $\Phi_{N_{m}^b}(s) \geq \mathbb{E}[e^{s \mathcal{U}_m}] = e^{\bar{n}_{mu}^f(e^{-s} - 1)}$. If the coverage areas of the FAPs do not overlap, then $S_f = \pi (R^2 - \gamma \sum_{a_f \in A_f} d^2(a_f, b_m))$. In general, the regions might overlap, and therefore $S_f \geq \pi (R^2 - \gamma \sum_{a_f \in A_f} d^2(a_f, b_m))$. This lower bound clearly is a function of the locations of the FAPs, and can be negative. Let $E$ denote the event that $|A_f| \leq \gamma^{-1}$. However, if $E$ holds, then $\gamma \sum_{a_f \in A_f} d^2(a_f, b_m) \leq \gamma R^2 |A_f| \leq R^2$, and $\pi (R^2 - \gamma \sum_{a_f \in A_f} d^2(a_f, b_m)) \geq 0$. We employ this observation to derive an upper bound on $\Phi_{N_{m}^b}(s)$. By the law of total expectation 
\[
\Phi_{N_{m}^b}(s) = \mathbb{E}[e^{\mu_m S_f(e^{-s} - 1)} | E] \mathbb{P}(E) + \mathbb{E}[e^{\mu_m S_f(e^{-s} - 1)} | E^c] \mathbb{P}(E^c),
\]

(6)

On the other hand,
\[
\mathbb{E}[e^{\mu_m S_f(e^{-s} - 1)} | E] \leq \mathbb{E}[e^{\mu_m \pi (R^2 - \gamma \sum_{a_f \in A_f} d^2(a_f, b_m)) (e^{-s} - 1)} | E]
\]
\[
= \mathbb{E}[e^{\mu_m \pi (R^2 - \gamma \sum_{a_f \in A_f} d^2(a_f, b_m)) (e^{-s} - 1)}] \mathbb{P}(E) + \mathbb{P}(E^c).
\]

(7)

By the Chernoff bound, for $x > 0$, $\mathbb{P}(E^c) = \mathbb{P}(|A_f| > \gamma^{-1}) \leq \frac{\mathbb{E}[e^{\gamma |A_f|}]^{\gamma^{-1}}}{e^{\gamma^2/2}} \leq \frac{e^{\gamma \bar{n}_{mu}^f (e^{-s} - 1)}}{e^{\gamma^2/2}}$. Optimizing the bound by choosing $x$ as the solution of $\bar{n}_{m} = \gamma^{-1} = 0$, we obtain
\[
\mathbb{P}(E^c) \leq e^{\gamma - 1 - \bar{n}_{m} \tau(s)}
\]

(8)

Combining (6), (7) and (8) yields the desired result.

In our outage analysis presented in the proceeding sections, in some cases we study the case where we know that there exists one FAP $af$ at distance $df$ from $b_m$. In those cases, it will be useful to define the conditional Laplace transform of $N_{m}^b$ as $\Phi_{N_{m}^b}(s|df)$. 

**Theorem 2:** For $s \geq 0$ and $df \in (0, R)$, we have $\Phi_{N_{m}^b}(s|df) \geq e^{\mu_m (e^{-s} - 1)}$, and
\[
\Phi_{N_{m}^b}(s|df) \leq \frac{e^{\mu_m (e^{-s} - 1)}}{e^{\gamma - 1 - \bar{n}_{m} + \gamma - 1 \log(\bar{n}_{m})}} \tau(s),
\]

(9)

where $\tau(s)$ and $\gamma$ are defined in (5) and (2), respectively, and as before $\bar{n}_{mu}^f = \pi \gamma df^2 \mu_m$, $\bar{n}_{mu} = \pi R^2 \mu_m$, and $\bar{n}_{m} = \pi R^2 \lambda_f$.

**Proof:** The proof closely follows the steps of the proof of Theorem 1. The only difference is that here we know that $|A_f| \geq 1$ and that one FAP is located at distance $df$ from $b_m$. Therefore, $S_f = \pi (R^2 - \gamma \sum_{a_f \in A_f} d^2(a_f, b_m))$. 

![Diagram](image-url)
Define event $E$ as before. Then, again by the law of total expectation, 
$$\Phi_n^{[s]}(d_f) \leq E[e^{\mu S_r f(e^{-s}-1)}|E, |A_f| > 0]P(E| |A_f| > 0) + P(E^c| |A_f| > 0).$$

Note that
$$E[e^{\mu S_r f(e^{-s}-1)}|E, d_f] = P(E| |A_f| > 0) e^{\left(n_{mf} f-a_{mf}(e^{-s}-1)\right)} \sum_{n=1}^{\infty} P(|A_f| = n) \frac{m_{mf} f}{n_{mf} f} \left(1-e^{-n_{mf} f}\right)^{n-1} \left(1-e^{-n_{mf} f}\right)^{n-1} \left(1-e^{-n_{mf} f}\right)^{n-1} \frac{P(E^c| |A_f| > 1)}{1-e^{-n_{mf} f}}.$$

Furthermore, $P(E^c| |A_f| > 1) = \frac{P(E^c| |A_f| \geq 1)}{1-e^{-n_{mf} f}}.$

IV. OUTAGE ANALYSIS

In this section we analyze the outage performance of MUs and FUs in the uplink network described in Section II. We assume that every user equipment employs power control to compensate for the effect of path loss. By power control, MUs serviced by the MBS intend to achieve received power levels of $P_m$. Similarly, FUs and MUs serviced by FAPs adjust their transmitted power to achieve received power of $P_f$. We further assume that the performance of the users is primarily limited by the interference caused by other users of both tiers. Therefore we ignore the effect of additive Gaussian noise in our analysis.

To bound the outage probability, in each case, we first compute the signal to interference ratio (SIR) experienced by user equipments. The derived SIRs are probabilistic and depend on channel coefficients, and user locations. Then, we bound the outage probabilities by employing results we proved in the previous section.

A. MU served by a FAP

Consider FAPs $a_f$ and $a_m f \in A_f \setminus \{a_f\}$. The distance between user $u$ that is covered by $a_f$ is usually much smaller than the distance between $u$ and $a_m$. That is, $d(u, a_f) \ll d(u, a_m)$, or $\frac{d(u, a_m)}{d(u, a_f)} \ll 1$. Therefore, in evaluating the performance of users covered by $a_f$, unless the density of FAPs $|A_f|$ is very large, the term corresponding to the interference caused by users (macro or femto) covered by other FAPs is negligible compared to the other terms. Making this approximation, the upload SIR experienced by user $u_m \in U_m(a_f)$ in subband $i \in \{1, 2, \ldots, n_s\}$ is equal to
$$\text{SIR}_{m,f} = \frac{P_f |b_{m,a_f}^i|^2}{I_{m,f}},$$
where
$$I_{m,f} = \sum u_f \in U_f(a_f) \frac{P_f |h_{m,a_f}^i|^2}{G} + \sum u_m \in U_m(a_f) \frac{P_f |h_{m,a_f}^i|^2}{G} + \sum \frac{d(\hat{u}_m, b_m)}{d(\hat{u}_m, a_f)} \frac{P_m |b_{m,a_f}^i|^2}{G}.$$

In (11), the interference terms are caused by the FUs of FAP $a_f$, the other MUs of FAP $a_f$, and the MUs serviced by the MBS, respectively. In our model, from the perspective of the outage performance of MUs served by a FAP, there is no difference between MUs and FUs covered by that FAP. Therefore, statistically, (10) and (11) also describe the performance experienced by FUs of $a_f$.

Remark 2: While it might seem that we have assumed the same attenuation factor $\alpha$ for all different links in the network, in fact, the results do not change in general where the path-loss exponent of outdoor and cross-wall (outdoor-indoor) transmissions are assumed to be equal (1) and larger than the path-loss exponent of indoor transmissions (2). To observe this, note that in (10), exponent $\delta$ only affects users in $U_f(a_f)$. However, due to our power control assumption, terms like $d^2(u_f, a_f)$ do not appear in the interference $I_{m,f}$. The same is true for our analysis presented in the next section corresponding to MUs served by the MBS.

Consider FAP $a_f \in A_f$ positioned at distance $d_f = d(a_f, b_m)$ from MBS $b_m$. In the rest of this section, we derive upper and lower bounds on the outage probability of MU $u_m \in U_m(a_f)$ as a function of $d_f$. As just mentioned, the same bounds hold for FUs covered by $a_f$, as well.

For MU $\hat{u}_m \in U_m(a_f)$, since $\hat{u}_m$ is directly served by MBS $b_m$, instead of one of the FAPs such as $a_f$, we must have $\kappa d(\hat{u}_m, b_m) \leq d_f(\hat{u}_m, a_f) \leq d(\hat{u}_m, a_f)$. Therefore, $\frac{d(\hat{u}_m, b_m)}{d(\hat{u}_m, a_f)} \leq \frac{\kappa}{\kappa'}. $

Let $\delta_{\hat{u}_m} \triangleq \frac{d(\hat{u}_m, b_m)}{d(\hat{u}_m, a_f)}$, where, as a reminder, $b_m$ and $a_f$ denote the base station and the FAP at distance $d_f$ from $b_m$, respectively. As we just argued, $\delta_{\hat{u}_m} \leq \kappa^{-1}$, for all $\hat{u}_m \in U_m(a_f)$. Given the complicated distribution of $\delta_{\hat{u}_m}$, in order to characterize the outage probability, we quantize $\delta_{\hat{u}_m}$.

Consider the setup shown in Fig. 3 where FAP $a_f$ is located at distance $d_f$ from $b_m$. The coverage area of $b_m$ is shown by the black circle of radius $R$ centered at $b_m$. The green circle on the right represents points with $\delta_{\hat{u}_m} = \kappa^{-1}$. Similarly, the points on the green circle on the left have $\delta_{\hat{u}_m} = \kappa$. The other pairs of circles correspond to some other values of $\kappa' > \kappa$. Points on the black line have $\delta_{\hat{u}_m} = 1$. Note that, by our assumption, all MUs are located inside the black circle. Therefore, the parts of colored circles that are outside of the black circle have zero probability. Consider
respectively. Similarly, define and, unlike \( \kappa \) and the coverage area of \( MBS \) area that is not covered by FAPs.) Also, Lemma 1: Proof: Let \( S \) denote the whole circuit of radius \( R \) minus the coverage area of \( af \). Hence the area of \( S \) is equal to \( s = s_0 + \sum_{i=1}^t (s_i + s_{i-1}) \). For \( i = -t, \ldots, -1 \), we have

\[
P(\hat{u}_m = \kappa - i) = \mathbb{E}[\mathbf{1}_{\hat{u}_m \in S_i} | A_f] = \mathbb{E}[\mathbf{1}_{\hat{u}_m \in S_i} | A_f] = \mathbb{E}\left[\frac{s_i - S_{i,m}}{s - S_m} \right] ,
\]

where \( S_m \) and \( S_m \) denote the region in \( S \) that is covered by the MBS \( b_m \), and its area, respectively. (This is of course the area that is not covered by FAPs.) Also, \( S_{i,m} \) denotes the area of \( S \). Note that \( S_{i,m} \) and \( S_m \) are both random variables that depend on the locations of the FAPs. To derive the desired result, we employ the tower property one more time:

\[
P(\hat{u}_m = \kappa_i) = \mathbb{E}\left[\frac{s_i - S_{i,m}}{s - S_m} | S_m\right] = \mathbb{E}\left[\frac{s_i - (s_i/s) S_m}{s - S_m}\right] = \frac{s_i}{s} - 1
\]

The proof of the rest of the theorem follows from the same argument.

Let \( P_{out}^{uf}(d_f) \) denote the outage probability experienced by a MU covered by a FAP located at distance \( d_f \) of \( b_m \). We employ Lemma 1 and our upper-bounding and lower-bounding quantizations of \( \delta_{\hat{u}_m} \) to derive the following theorem that presents both an upper bound and a lower bound on \( P_{out}^{uf}(d_f) \).

Theorem 3: Let \( T_h \equiv T_h/m \). For \( t \in \mathbb{N}^+ \), define

\[
q_u(s) \triangleq \frac{p_0}{1 + s \kappa_0/\eta} + \sum_{i=1}^t \left( \frac{p_i}{1 + s/(\eta \kappa_i/\eta)} + \frac{p_{-i}}{1 + s \kappa_i/\eta} \right) ,
\]

and

\[
q_u(s) \triangleq p_0 + \sum_{i=1}^t \left( \frac{p_i}{1 + s/(\eta \kappa_i/\eta)} + \frac{p_{-i}}{1 + s \kappa_i/\eta} \right) ,
\]

where \( (p_{-t}, \ldots, p_i) \) are defined and characterized in Lemma \( \eta \triangleq P_f/P_m \), and \( \tau_0 = \tau - \log q_u(\hat{u}) \), with \( \tau \) defined in \( \hat{u} \). Then,

\[
P_{out}^{uf}(d_f) \leq 1 - \left( 1 + T_h \right)(e^{\eta \mu_0/1+T_h} - 1) - e^{-\eta_0 T_h/(1+T_h)}
\]

and

\[
P_{out}^{uf}(d_f) \geq 1 - \left( 1 + T_h \right)(e^{\eta_0 T_h/(1+T_h) - 1}) - e^{-\eta_0 T_h/(1+T_h)}
\]

Proof: The details of the proof is presented in Appendix \( \hat{u} \) but the outline of the proof is as follows. First, we derive upper and lower bounds on the upload SIR experienced by macro user \( u_m \in U_m(af) \), namely SIR_{m,f}. To achieve this goal, we employ the quantizations of \( \delta_{\hat{u}_m} \) defined in \( \hat{u} \) and \( \hat{u} \). Then, we connect the outage probability with the Laplace transform of the bounds on SIR. Finally, we use the fact that the support sets of the locations of MUs served by MBSs and FAPs do not overlap to prove that independence of the number of interfering users in different groups.

Remark 3: In Theorem \( \hat{u} \), \( t \) corresponds to the number of partition levels of the MBS’s coverage area, and is a parameter that can be selected arbitrarily. In other words, the bounds hold for any \( t \in \mathbb{N}^+ \), but choosing higher value of \( t \) leads to tighter upper and lower bounds.

Remark 4: In Theorem \( \hat{u} \), the terms in the bounds that depend on distance \( d_f \) are \( \eta_0 T_h/(\sigma^2) \) and \( q_u(T_h/\sigma^2) \). As we will see in the numerical results presented in Section \( \hat{u} \) both the upper and lower bounds are not monotonic in \( d_f \). This follows from the non-monotonic behavior of \( q_u(T_h/\sigma^2) \) and \( q_u(T_h/\sigma^2) \). The term \( (1+T_h)(e^{\eta_0 T_h/(1+T_h) - 1})/e^{\eta_0 T_h/(1+T_h)} \), which appears in both upper and lower bound, is usually very close to one, and is monotonically decreasing in \( d_f \). Therefore, the other terms in each bound are the dominant terms.

Remark 5: To gain more insight on the effect of different parameters on the bounds in Theorem \( \hat{u} \) we can consider their approximate values for typical set of parameters, when
the number of carriers is large. As mentioned in Remark 4, $(1 + T_h)(e^{\gamma_{\text{th}}}/(1 + T_h) - 1)/(e^{\gamma_{\text{th}}} - 1)$ is close to one, especially if the number of carriers is large. We can also approximate $\tau_0$ as $1 - e^{-\log((q_i/\sigma^2)\gamma_{\text{th}})}$ by approximating of $e^{-s}$ in $\tau(s)$ as $1 - s$. Employing these approximations, and ignoring the other non-dominant terms, the upper and lower bound can be simplified to $1 - e^{-\epsilon_{\text{th}} T_h - \epsilon_{\text{ms}}(1 - q_i(T_h/\sigma^2))}$ and $1 - e^{-\epsilon_{\text{th}} T_h - \epsilon_{\text{ms}}(1 - q_i(T_h/\sigma^2))}/\epsilon_{\text{ms}} - 1$. When the FAP gets close to the MBS, i.e., $d_f \ll R$, $\delta_{\text{ms}} \approx 1$, and hence, $q_i(T_h/\sigma^2), q_i(T_h/\sigma^2) = 1 + T_h/(\sigma^2\eta)$, or $1 - q_i(T_h/\sigma^2), 1 - q_i(T_h/\sigma^2) \approx T_h/(\sigma^2\eta)$, which further simplifies the upper and lower bounds to $1 - e^{-T_h(\epsilon_{\text{ms}} + \epsilon_{\text{th}}\gamma_{\text{th}})}$, and $1 - e^{-T_h(\epsilon_{\text{ms}} + \epsilon_{\text{th}}\gamma_{\text{th}})}\approx T_h/(\sigma^2\eta)$.

B. MU served by the MBS

The upload SIR experienced by user $u_m \in U_m(-f)$ at the MBS $b_m$ in subband $i$ is equal to

$$\text{SIR}_{m,m} = \frac{P_m|\hat{h}_{m,b_m}|^2}{I_{m,m}}$$

where, for $i \in \{1, \ldots, n_s\}$,

$$I_{m,m} = \sum_{a_f \in A_f} \sum_{u_m \in U_m(a_f)} \left(\frac{d(u_m, a_f)}{d(u_m, b_m)}\right)^\alpha |h_{m,b_m}|^2 E[1_{c_{u_m}}(i) = c_{u_m}]$$

$$= \sum_{u_m \in U_m(a_f)} \sum_{a_f \in A_f} \left(\frac{d(u_m, a_f)}{d(u_m, b_m)}\right)^\alpha |h_{u_m,b_m}|^2 P_f \quad \text{(17)}$$

In deriving (17), we have ignored the interference caused by the FUs. The reason is that in most cases the distance between FU $u_f$ and its FAP is much smaller than the distance between $u_f$ and $b_m$.

**Theorem 4:** Let $P^n_{\text{out}} \triangleq \text{P}(\text{SIR}_{m,m} < T)$. Then, $P^n_{\text{out}} \geq 1 - (1 + T_h)(e^{\gamma_{\text{th}} T_h/(1 + T_h) + \epsilon_{\text{ms}}(\tau_0 - 1)} + e^{-\epsilon_{\text{ms}} T_h/(1 + T_h) + \epsilon_{\text{ms}}(\tau_0 - 1)})$, and

$$P^n_{\text{out}} \leq 1 - (1 + T_h)(e^{\gamma_{\text{th}} T_h/(1 + T_h) - 1})$$

where $\tau_0' = \frac{e^{\gamma_{\text{th}} T_h/(1 + T_h) - 1}}{e^{\gamma_{\text{th}} T_h/(1 + T_h) - 1}}$.

**Proof:** The proof is relegated to Appendix C. Similar to the proof of Theorem 3, here too, we derive upper and lower bound on the experienced SIR, SIR$_{m,m}$. In this case, when a macro user is served by the MBS, $d(u_m, a_f) \gg d(u_m, b_m)$. Employing this bound, yields a lower bound on SIR$_{m,m}$. To derive the upper bound, we only consider the interference caused by the other MUs served by the MBS.

**Remark 6:** Typically, the second term in the upper bound is negligible compared to the first term and can be ignored. Approximating by the first-order Taylor expansion, $\tau_0'$ can be approximated as $\tau_0' \approx 1 + 0.5\gamma_{\text{th}} T_h$, which holds for large number of carriers per subband. Using these approximations, the lower bound can be simplified to $1 - e^{-\epsilon_{\text{ms}} T_h(1 - 1/2)}$.

**V. NUMERICAL RESULTS**

In this section, to investigate the uplink network performance and to verify our upper and lower bounds, we present some simulation results. Monte-Carlo computer simulations with $10^5$ realizations are carried out to validate our analytical bounds and illustrate the accuracy of our approximations. The considered scenario is a two-tier network in a circle of radius $R = 1$ Km with the MBS located at the center. In the ensuing plots, we use the default values in Table I unless otherwise stated. Fig. 4 shows a realization of the network with the specified parameters. Note that the size of the coverage area of a femtocell depends on its distance from the MBS. In our model, if a MU falls in the coverage areas of more than one user, it is served by the closest one.

**Table I**

| Parameter | Description | Default Values |
|-----------|-------------|----------------|
| $\lambda_f$ | density of FAPs | $5 \times 10^{-5}$ m$^{-2}$ |
| $\mu_m$ | density of macrocell users | $15 \times 10^{-5}$ m$^{-2}$ |
| $\mu_f$ | density of FUs | $0.01$ m$^{-2}$ |
| $\Delta$ | ring width of FUs placement | $5$ m |
| $\beta_f$ | ring internal radius of FUs placement | $10$ m |
| $\alpha$ | path loss exponent | $4$ |
| $d_f$ | distance between considered FAP and MBS | $700$ m |
| $\beta$ | SIR threshold level | $2$ |
| $n_s$ | number of subbands | $32$ |
| $n_{th}$ | number of subchannels in each subband | $256$ |
| $\eta$ | power ratio between FAPs and MBS | $25$ |
| $\kappa$ | handover parameter | $0.1$ |
different MUs densities. Figs. 5 and 6 reflect that our analytical upper and lower bounds (solid and dot curves) are reasonable approximations for all considered SIR thresholds.

As expected, increasing the threshold level increases the probability of outage. Clearly this does not imply that the performance can be improved by lowering $T$, as its reduction decreases the achieved rate as well. In general, there is a trade-off between expected capacity [35], [36] and threshold $T$. The problem of maximizing the expected rate by optimizing $T$ is studied in [37] for a single-tier MCFH system. We leave extending those results to multi-tier networks for future research.

Fig. 7 demonstrates how the average outage probabilities of MUs vary with handover parameter $\kappa$. Here too the bounds are consistent with the simulation results, but the gap increases slightly as $\kappa$ increases. In contrast to the downlink scenario [11], where the outage probability is not monotonic in $\kappa$, here, increasing $\kappa$ improves the performance for both MUs and FUs. This result is consistent with [23], where the authors argue that in non-orthogonal setups, open access is strictly better than closed access policy. The difference between uplink and downlink arises from the fact that in the downlink scenario as the MUs get farther away from the MBS, their received powers and hence SIRs decrease. On the other hand, in the uplink scenario, as they become farther away from the MBS, due to power control, their transmit powers increase as well to compensate for the path loss. Naturally, increasing the handover parameter leads to more MUs being covered by FAPs and hence to lower co-tier interference.

Note that for plotting the average probability experienced by MUs served by FAPs, we have taken the expected value of the upper and lower bounds mentioned in Theorem 3 by considering the randomness in $d_f$.

Fig. 8 shows the average outage performance of MUs as a function of $\eta = P_f/P_m$, the power ratio between FAPs and MBS. In these plots we have fixed the transmit power of MBS and because of this, the outage curves of the MUs served by the MBS are almost constant. Obviously the outage of MUs served by FAPs improves by increasing FAPs transmit powers. Note that although increasing FAPs powers increases the interferences level, but its effect is not significant for MUs,
whose performance is mainly limited by other MUs and not FUs.

Fig. 9 illustrates the conditional outage probability of MUs served by a FAP, as a function of the FAP’s normalized distance from the MBS. As it can be observed from the figure, at first, the outage probability increases as the MU gets farther form the MBS. In fact because of the assumption of constant received power by the MBS in the uplink scenario, as the MU gets farther from the MBS, it will transmit at a higher power, which leads to the degradation in the performance of FUs and also MUs served by FAPs. However, as the femtocells get close to the fringes of the cell, their users outage probabilities start to improve as well. The reason is that femtocells that are far away from the MBS have larger coverage areas and therefore, in those regions most MUs are serviced by nearby FAPs, which results in lower interference caused by them.

Figs. 10 and 11 show the conditional outage probability of MUs served by a FAP located at distance $d_f = 700$ m, and the average outage probability of MUs served by the MBS, respectively, as a function of MUs density $\mu_m$. Obviously, increasing the macrocell users density will increase their outage probabilities as well, because of more co-tier interferences. However their performance can be greatly improved by boosting the number of available sub-channels as can be seen in the figure, too.

VI. CONCLUSIONS

In this paper we investigated the uplink performance of two-tier networks consisting of a macrocell overlaid by femtocells. We considered a stochastic spatial distribution for MUs, FUs and FAPs, and assumed that they are generated by independent PPPs. For cell association, we considered an open access policy, where each MU is assigned to its nearest FAP if their distance is less than its distance from the MBS times some factor $\kappa < 1$. Under this model we studied the outage performance of the system and derived analytical upper and lower bounds on the outage probabilities of both FUs and MUs. The bounds were shown to be tight by our simulations.

Throughout the paper we considered a fixed threshold $\kappa$ for all MUs. A more general model is when $\kappa$ is not fixed and depends on the FAP. In other words, since $\kappa$ determines the coverage area of FAPs, it is conceivable to consider a scenario where FAPs are heterogeneous and can choose their coverage areas. For instance, a FAP can move toward a closed access policy by lowering its corresponding $\kappa$, i.e., by only accepting MUs that are very close.

FAPs are connected to a central gateway via wired connections. The capacity constraints imposed by this backhaul wired network can potentially affect the cell selection procedure and may impede some femtocells to service all MUs that fell in their coverage area. Characterizing the effect of this constraint on the system’s performance is another interesting question that is left for future research.

ACKNOWLEDGMENTS

The authors would like to thank the anonymous reviewers for their helpful comments and suggestions, especially one of the reviewers who pointed us to reference [38].

Fig. 9. Conditional outage probability of a MU served by a FAPs as a function of the normalized distance of FAP from the MBS.

Fig. 10. Conditional outage probability of MUs served by a FAP located at distance $d_f = 700$ m, and the average outage probability of MUs served by the MBS versus their density.

Fig. 11. Average outage probability of MUs served by MBS versus their density.
Appendix A
Derivation of $s_i$

As defined in Section IV, $s_i$, $i = 1, \ldots, t$, denotes the area of the region corresponding to $\delta^i_{u_1} = \kappa_i^{-1}$ and $\delta^i_{u_m} = \kappa_i^{-1}$ or equivalently $\kappa_i^{-1} \leq \delta_{u_m} \leq \kappa_i^{-1}$. We showed earlier that the points that satisfy $\delta_{u_m} = \kappa_i^{-1}$, are located on a circle of radius $\frac{\kappa_i^{-1}}{1 - \kappa_i^2}$ centered at $(\frac{d_i}{1 - \kappa_i^2}, 0)$. Hence, $s_i$ is the macrorcell coverage area surrounded by two such circles with $\kappa = \kappa_i^{-1}$ and $\kappa = \kappa_i^{-1}$. Therefore, for $i = 1, 2, \ldots, t - 1$,

$$s_i = \pi \kappa_i^2 \left(1 - \kappa_i^2\right) - \frac{1}{\kappa_i^2} \sum_{j=0}^{i-1} s_j,$$

and $s_t = R^2 (\theta - \frac{1}{2} \sin(2\theta)) - \sum_{j=0}^{t-1} s_j$, where $\theta = \arccos(\frac{d_i}{2R})$.

And

$$f(a, b, c) \triangleq a^2 \sec^{-1} \left( \frac{2ac}{b^2 - a^2 - c^2} \right) - b^2 \sec^{-1} \left( \frac{2bc}{b^2 + c^2 - a^2} \right) + \frac{1}{2} \sqrt{(a + b + c)(b + c - a)(c + a - b)(a + b - c)}$$

[39]. Similarly, $s_i$, for $i = -t, \ldots, -1$ denotes the area of the region where $\kappa_{i-1}^{-1} \leq \delta_{u_m} \leq \kappa_{i-1}^{-1}$, and $\delta_{u_m} = \kappa_{i-1}^{-1}$. Here, the geometrical location of the points satisfying $\delta_{u_m} = \kappa$ is a circle with the origin at $(\frac{\kappa d_i}{1 - \kappa^2}, 0)$ and radius $\frac{\kappa d_i}{1 - \kappa^2}$. Therefore the area of the macrorcell zone enclosed by two such circles with $\kappa = \kappa_{i-1}$ and $\kappa = \kappa_{i-1}^{-1}$ is given by

$$s_i = \pi \kappa_{i-1}^2 \left(1 - \kappa_{i-1}^2\right) - \frac{1}{\kappa_{i-1}^2} \sum_{j=0}^{i-1} s_j,$$

for $i = -t + 1, \ldots, -1$, and $s_{-t} = R^2 (\pi - \theta + \frac{1}{2} \sin(2\theta)) - \sum_{j=0}^{-t+1} s_j$, where again $\theta = \arccos(\frac{d_i}{2R})$.

Appendix B
Proof of Theorem

Define event $\mathcal{E} = \{d(a_f, b_m) = df, N_{m,f}^{a_f} \geq 1\}$. Then, by definition, $P_{\text{out}}(df) = P(\text{SIR}_{m,f} < T|\mathcal{E})$. Combining the quantizations defined in [12] and [13] with [10] and [11], we have

$$\frac{n_h |h_{u_m,a_f}^i|^2}{K_u} \leq \text{SIR}_{m,f} \leq \frac{n_h |h_{u_m,a_f}^i|^2}{K_l} \quad \text{(B.1)}$$

where

$$K_u \triangleq \sum_{a_f \in \mathcal{D}(a_f)} |h_{u_f,a_f}^i|^2 + \frac{1}{\eta} \sum_{\hat{u}_m \in \mathcal{U}(a_f)} |h_{u_m,a_f}^i|^2 (\tilde{\delta}_{u_m}^i)\eta,$$

and

$$K_l \triangleq \sum_{a_f \in \mathcal{D}(a_f)} |h_{u_f,a_f}^i|^2 + \frac{1}{\eta} \sum_{\hat{u}_m \in \mathcal{U}(a_f)} |h_{u_m,a_f}^i|^2 (\tilde{\delta}_{u_m}^i)\eta,$$

and

$$P_{\text{out}}(df) \leq P \left( \frac{n_h |h_{u_m,a_f}^i|^2}{K_u} < T \right| \mathcal{E} \right) = E \left[ \mathbb{1}[|h_{u_m,a_f}^i|^2 < \frac{K_u}{n_h} |\mathcal{E}, K_u] \right] \geq 1 - E \left[ e^{-\frac{2\kappa_{m,f}^2}{T n_h \sigma^2}} \right] = 1 - \Phi_{K_u} \left( \frac{T}{n_h \sigma^2} |df \right), \quad \text{(B.4)}$$

where (a) follows from our assumption that channel coefficient $|h_{u_m,a_f}^i|$ has a Rayleigh distribution. Here $\Phi_{K_u}(s|df) \triangleq E[e^{-sK_u}|\mathcal{E}]$ denotes the conditional Laplace transform of $K_u$ defined in [2]. Since conditioned on the number of users in each category, the channel coefficients and $\{\delta_{u_m} \hat{u}_m \in \mathcal{U}(\cdot |df)\}$ are all independent of each other, it follows that

$$\Phi_{K_u}(s|df) = E \left[ \left( \frac{1}{1 + s\sigma^2} \right)^{N_{m,f}^a + N_{m,f}^b - 1} \cdot q_u(s)^{N_{m,f}^{a\mathcal{U}}} \bigg| \mathcal{E} \right]. \quad \text{(B.5)}$$

where $q_u(s)$ is defined in [15]. To derive the lower bound, again from (B.1), and following a similar steps as in (B.4), we derive

$$P_{\text{out}}(df) \geq P \left( \frac{n_h |h_{u_m,a_f}^i|^2}{K_l} < T \right| \mathcal{E} \right) = 1 - E \left[ e^{-\frac{2\kappa_{m,f}^2}{T n_h \sigma^2}} \right]$$

$$= 1 - \Phi_{K_l} \left( \frac{T}{n_h \sigma^2} |df \right), \quad \text{(B.6)}$$

where $\Phi_{K_l}(s|df) \triangleq E[e^{-sK_l}|\mathcal{E}]$. Also, as argued before in deriving (B.5), we have

$$\Phi_{K_l}(s|df) = E \left[ \left( \frac{1}{1 + s\sigma^2} \right)^{N_{m,f}^a + N_{m,f}^b - 1} \cdot q_l(s)^{N_{m,f}^{a\mathcal{U}}} \bigg| \mathcal{E} \right], \quad \text{(B.7)}$$

where $q_l(s)$ is defined in [15].

Conditioned on the location of $a_f$, $N_{m,f}^{a_f}$, $N_{m,f}^{b_f}$ and $N_{m,f}^{b_m}$ are independent random variables. The independence of $N_{m,f}^{a_f}$ and $(N_{m,f}^{a_m}, N_{m,f}^{b_m})$ follows from our initial assumption that the process of drawing MUs and FUs are independent. To see the independence of $N_{m,f}^{a_m}$ and $N_{m,f}^{a_m}$, note that $U_m(a_f)$ denotes the users that are located in a circle of radius $df \sqrt{1 - \kappa^2} - 1$. (Refer to Fig. 2). On the other hand, conditioned on the location of $a_f$, $U_m(\cdot |df)$ denotes users that are not located in any of the circles corresponding to different FAPs, one of which is the mentioned circle corresponding to $a_f$. Therefore, the support sets of the locations of MUs in $U_m(\cdot |df)$ and the MUs in $U_m(a_f)$ do not have any overlap. Therefore, since the macro users are generated by a PPP process, $N_{m,f}^{a_f}$ and $N_{m,f}^{b_m}$ are independent random variables. As a result,

$$\Phi_{K_u}(s|df) = \Phi_{N_{m,f}^{a_f}}(\log(1 + s\sigma^2)) \Phi_{N_{m,f}^{b_f}}(\log(1 + s\sigma^2)) df \cdot \Phi_{N_{m,f}^{b_m}}(-\log q_u(s)|df). \quad \text{(B.8)}$$

Similarly,

$$\Phi_{K_l}(s|df) = \Phi_{N_{m,f}^{a_f}}(\log(1 + s\sigma^2)) \Phi_{N_{m,f}^{b_f}}(\log(1 + s\sigma^2)|df) \cdot \Phi_{N_{m,f}^{b_m}}(-\log q_l(s)|df). \quad \text{(B.9)}$$

Combining the bounds derived for $\Phi_{N_{m,f}^{a_m}}(s|df)$ in Theorem 2.
with (1), (3), (B.4), (B.6), (B.8) and (B.9) completes the proof of Theorem 5.

APPENDIX C
PROOF OF THEOREM 4
For user \( u_m \in U_k(\alpha_f) \), by our assignment policy, we should have \( d(\alpha_f, u_m, \alpha_f) \leq \kappa d(\alpha_f, u_m, \beta_m) \). Therefore, \( I_{m,m} \) can be upper-bounded as

\[
I_{m,m} \leq \sum_{\alpha_f \in A} \sum_{\alpha_m \in U_m(\alpha_f)} K_{\alpha_m} E[e^{\kappa h_{\alpha_m}^\beta}] \leq \sum_{\alpha_m \in U_m^\beta} \sum_{\alpha_f \in A} \frac{P_m}{G} h_{\alpha_m}^\beta \left| 1 + \frac{1}{\kappa} \right|, \tag{C.10}
\]

where the last line follows since \( \kappa < 1 \). Also, clearly, \( I_{m,m} \geq \sum_{\alpha_m \in U_m^\beta} \sum_{\alpha_f \in A} \frac{P_m}{G} h_{\alpha_m}^\beta \left| 1 + \frac{1}{\kappa} \right| \). Hence,

\[
\frac{n_h}{K_u} h_{\alpha_m}^\beta \left| 1 + \frac{1}{\kappa} \right| \leq \text{SIR}_{m,m} \leq \frac{n_h}{K_u} h_{\alpha_m}^\beta \left| 1 + \frac{1}{\kappa} \right|
\]

where \( K_u = \sum_{\alpha_m \in U_m^\beta} h_{\alpha_m}^\beta \left| 1 + \frac{1}{\kappa} \right| \), and

\[
1 - \Phi_{K_u}(\frac{T}{n_h \sigma^2}) \leq P_{\text{out}} \leq 1 - \Phi_{K_u}(\frac{T}{n_h \sigma^2}).
\]

To derive the upper bound, note that \( \Phi_{K_u}(s) = E[e^{-s K_u}] = E[(E[e^{-s h_{\alpha_m}^\beta} | u_m = 1])^{\bar{m}}] = \frac{\alpha^{n_h}}{\bar{m}^{n_h}}, \) where \( \alpha = \frac{1}{1 + \sigma^2} \). For the lower bound, we employ the upper bound on \( \Phi_{K_u}(s) \) presented in Theorem 4. The only difference here compared to Theorem 4 is that here we need to condition on \( N_m^\beta \geq 1 \). However, \( E[e^{-s N_m^\beta}] = \sum_{i=1}^{n_h} \frac{p_i}{1 + \sigma^2} \geq 0 \). Combining this with Theorem 4 yields the lower bound.

REFERENCES

[1] W.C. Cheung, T.Q.S. Quek, and M. Kountouris. Throughput optimization, spectrum allocation, and access control in two-tier femtocell networks. Sel. Areas in Comm., IEEE Journal on, 30(5):561–574, 2012.
[2] H.S. Jo, Y.J. Sang, P. Xia, and J.G. Andrews. Heterogeneous cellular networks with flexible cell association: A comprehensive downlink SINR analysis. Wireless Communications, IEEE Trans. on, 11(10):3484–3495, 2012.
[3] F. Baccelli and S. Zuyev. Stochastic geometry models of mobile communication networks. Frontiers in queueing, pages 227–243, 1997.
[4] F. Baccelli, M. Klein, M. Lebourges, and S. Zuyev. Stochastic geometry and architecture of communication networks. Telecommunication Systems, 7(1-3):209–227, 1997.
[5] T.X. Brown. Cellular performance bounds via shotgun cellular systems. Sel. Areas in Comm., IEEE Journal on, 18(11):2443–2455, 2000.
[6] M. Haenggi, J. G. Andrews, F. Baccelli, O. Dousse, and J. Franceschetti. Stochastic geometry and random graphs for the analysis and design of wireless networks. Sel. Areas in Comm., IEEE Journal on, 27(7):1029–1046, 2009.
[7] F. Baccelli, P. Mühlethaler, and B. Blaszczyszyn. Stochastic analysis of spatial and opportunistic Aloha. Sel. Areas in Comm., IEEE Journal on, 27(7):1105–1119, 2009.
[8] J.G. Andrews, F. Baccelli, and R.K. Ganti. A tractable approach to coverage and rate in cellular networks. Comm., IEEE Trans. on, 59(11):3122–3134, Nov. 2011.
hopping spread spectrum systems. *Wireless Communications, IEEE Trans. on*, 7(3):888–897, 2008.

[35] M. Effros and A. Goldsmith. Capacity of general channels with receiver side information. In *Proc. IEEE Int. Symp. Inform. Theory*, page 39, 1998.

[36] M. Effros, A. Goldsmith, and Y. Liang. Generalizing capacity: New definitions and capacity theorems for composite channels. *IEEE Trans. Inform. Theory*, 56(7):3069–3087, 2010.

[37] Z. Zeinalpour-Yazdi and S. Jalali. On expected capacity of multicarrier frequency hopping systems. In *Comm. and Info. Theory (IWCIT), 2013 Iran Workshop on*, pages 1–6. IEEE, 2013.

[38] S.G. Foss and S.A. Zuyev. On a voronoi aggregative process related to a bivariate poisson process. *Advances in Applied Probability*, pages 965–981, 1996.

[39] Eric W. Weisstein. Lune. *From MathWorld–A Wolfram Web Resource*. http://mathworld.wolfram.com/Lune.html.