Lie-Hopf algebras and their Hopf cyclic cohomology

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Abstract

The correspondence between Lie algebras, Lie groups, and algebraic groups, on one side and commutative Hopf algebras on the other side are known for a long time by works of Hochschild-Mostow and others. We extend this correspondence by associating a noncommutative noncocommutative Hopf algebra to any matched pair of Lie algebras, Lie groups, and affine algebraic groups. We canonically associate a modular pair in involution to any of these Hopf algebras. More precisely, to any locally finite representation of a matched pair object as above we associate a SAYD module to the corresponding Hopf algebra. At the end, we compute the Hopf cyclic cohomology of the associated Hopf algebra with coefficients in the aforementioned SAYD module in terms of Lie algebra cohomology of the Lie algebra associated to the matched pair object relative to an appropriate Levi subalgebra with coefficients induced by the original representation.

1 Introduction

Lie groups and Lie algebras are well-known for a long time and their correspondence has been adopted to many other algebraic structures such as algebraic groups and their Lie algebras. This correspondence was advanced by the work of Hochschild and Mostow when they defined the notion of

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representative functions for Lie groups, Lie algebras, and algebraic groups. They showed that the algebra of representative functions on these objects forms a commutative Hopf algebra whose coalgebra structure encodes the group structure of the group or the Lie bracket of the Lie algebra respectively [21, 20]. They also defined a cohomology theory, for Lie algebras, Lie groups, and affine algebraic groups, called the representative cohomology due to the fact that it is based on the representatively injective resolutions [19]. They proved their van Est type theorem, see [8], by computing the representative cohomology in terms of relative Lie algebra cohomology for suitable pair of Lie algebras for each case [19]. The important point for us is to see this cohomology as the cohomology of the underlaying coalgebra of the commutative Hopf algebra with coefficients in the comodule induced by the representation in question.

We extend the work of Hochschild-Mostow by associating a not necessarily commutative or cocommutative Hopf algebra to any matched pair of Lie algebras, Lie groups, and algebraic groups. The resulting Hopf algebra is a bicrossed product Hopf algebra. Such a Hopf algebra is made of two Hopf algebras in such a way that both algebra and coalgebra structures interact with each other. We refer the reader to [26] for a comprehensive account on these Hopf algebras. One of the interesting examples of bicrossed product Hopf algebras is $\mathcal{H}_n$, the Hopf algebra of general transverse symmetry on $\mathbb{R}^n$, defined by Connes and Moscovici [5]. It is shown in [22, 44] that $\mathcal{H}_1$ is a bicrossed product Hopf algebra. In [29], Moscovici and the first author associated to each infinite primitive Lie pseudogroup a bicrossed product Hopf algebra by means of transverse symmetries.

One of our minor aims in this paper is to develop a short method for proving that a Hopf algebra is of the form of bicrossed product. We refer the reader to [29, 10] for other solutions to this problem. We introduce the notion of Lie-Hopf algebras as the responsible objects for this method.

We extend the whole theory developed in [29, 30] including coefficients, the Chevalley-Eilenberg bicomplex, and van Est isomorphism to the case of abstract Lie-Hopf algebras. In [29] the total Hopf algebra was first constructed by means of an algebra of operators on a crossed product algebra and then it was shown to be of the form of a bicrossed product Hopf algebra. However in this paper we do not assume to have the total Hopf algebra from the beginning. This forces us to glue two different Hopf algebras together by dealing with them conceptually.
Hopf cyclic cohomology was defined by Connes and Moscovici in [5] and then was generalized by Hajac, Khalkhali, Sommerhauser, and the first author in [11, 12] to include the appropriate coefficients.

In Section 2 we first define the notion of Lie-Hopf algebras. In three separated subsections, we explain carefully how to construct Lie-Hopf algebras by having a matched pair of Lie algebras, a matched pair of Lie groups, and a matched pair of affine algebraic groups.

In Section 3 we deal with coefficients. In Subsection 3.1 we canonically associate a modular pair in involution to any Lie-Hopf algebra. In Subsection 3.2 we classify a subcategory of Yetter-Drinfeld modules over the bicrossed product Hopf algebras under the name of induced module.

In Section 4 we study the Hopf cyclic cohomology of commutative Hopf algebras. We define the notion of Hopf-Levi decomposition and prove a general van Est isomorphism by computing the Hopf cyclic cohomology of such a Hopf algebra with coefficients in an induced module.

The Section 5 contains the main results and is in fact the main motivation of the paper. In Subsection 5.1 we advance the machinery developed by Moscovici and the first author in [28, 29, 30] to cover all Lie-Hopf algebras. We prove that for any \( g \)-Hopf algebra \( \mathcal{F} \) and any induced module \( M \) there is a bicomplex computing the Hopf cyclic cohomology of \( \mathcal{F} \triangleright \triangleleft U(g) \). We then prove Theorem 5.9 in its full generality as is stated here.

**Theorem 5.9.** Let \((g_1, g_2)\) be a matched pair of Lie algebras and \( \mathcal{F} \) be a \((g_1, g_2)\)-related Hopf algebra. Let assume that \( g_2 = h \ltimes l \) is a \( \mathcal{F} \)-Levi decomposition such that \( h \) is \( g_1 \)-invariant and the natural action of \( h \) on \( g_1 \) is given by derivations. Then for any \( \mathcal{F} \)-comodule and \( g_1 \)-module \( M \), the map \( \mathcal{V} \), defined in (5.41), is a map of bicomplexes and induces an isomorphism between Hopf cyclic cohomology of \( \mathcal{F} \triangleright \triangleleft U(g_1) \) with coefficients in \( \sigma M_\delta \) and the Lie algebra cohomology of \( a := g_1 \bowtie g_2 \) relative to \( h \) with coefficients in the \( a \)-module induced by \( M \). In other words,

\[
HP^\bullet(\mathcal{F} \triangleright \triangleleft U(g_1), \sigma M_\delta) \cong \bigoplus_{i = \ast \mod 2} H^i(a, h, M). \tag{5.1}
\]

We get three corollaries by specializing this theorem to our geometric cases.
2 Geometric noncommutative Hopf algebras

In this paper all Lie algebras, Lie groups, and affine algebraic groups are over complex numbers. All groups are assumed to be connected and all Lie
algebras are finite dimensional.

In Subsection 2.1 we recall the needed definitions and basics of bicrossed and double crossed Hopf algebras. In Subsection 2.2 we introduce Lie-Hopf algebras and proves that they are equivalent to bicrossed product Hopf algebras. In Subsection 2.3 we consider a matched pair of Lie algebra \((g_1, g_2)\) and construct the bicrossed product Hopf algebra \(R(g_2) \triangleright U(g_1)\), where \(R(g_2)\) is the Hopf algebra of representative functions on \(U(g_2)\). In Subsection 2.4 we associate the Hopf algebra \(R(G_2) \triangleright U(g_1)\) to a matched pair of Lie groups \((G_1, G_2)\), where \(R(G_2)\) is the Hopf algebra of representative smooth functions on \(G_2\) and \(g_1\) is the Lie algebra of \(G_1\). We finish this section by Subsection 2.5 where we construct the Hopf algebra \(P(G_2) \triangleright U(g_1)\) for a matched pair of connected affine algebraic group \((G_1, G_2)\), where \(P(G_2)\) is the Hopf algebra of all polynomial representative functions on \(G_2\) and \(g_1\) is the Lie algebra of \(G_1\).

2.1 Bicrossed and double crossed product Hopf algebras

It is always helpful to decompose complicated algebraic structures into product of less complicated objects. In the category of Hopf algebras there are many of such decompositions. The first one helping us in this paper is bicrossed product and the second one is double crossed product. We refer the interested reader to [26] for a comprehensive account, however we briefly recall below the most basic notions concerning the bicrossed and double crossed product construction.

Let \(U\) and \(F\) be two Hopf algebras. A linear map

\[
\triangleright: U \rightarrow U \otimes F, \quad \triangleright u = u_{(0)} \otimes u_{(1)},
\]

defines a right coaction, and thus equips \(U\) with a right \(F\)–comodule coalgebra structure, if the following conditions are satisfied for any \(u \in U\):

\[
u_{(1)} \otimes \nu_{(2)} \otimes v_{(1)} \otimes \nu_{(2)} \otimes u_{(1)} \otimes u_{(2)} = u(1) \otimes u(2) \otimes u_{(1)} \otimes u_{(2)} \otimes \nu_{(1)} \otimes \nu_{(2)},
\]

\[
\epsilon(u_{(1)}) v_{(1)} = \epsilon(u) 1.
\]

One then forms a cocrossed product coalgebra \(F \triangleright U\), that has \(F \otimes U\) as
underlying vector space and the following coalgebra structure:

\[
\Delta(f \rhd u) = f_{(1)} \rhd u_{(1) <0>} \otimes f_{(2)} u_{(1) <1>} \rhd u_{(2)},
\]

\[\epsilon(f \rhd u) = \epsilon(f) \epsilon(u).\]  

(2.3)  

(2.4)

In a dual fashion, \( F \) is called a left \( U \)-module algebra, if \( U \) acts from the left on \( F \) via a left action \( \triangleright : F \otimes U \rightarrow F \)

which satisfies the following conditions for any \( u \in U \), and \( f, g \in F \):

\[ u \triangleright 1 = \epsilon(u) 1 \]  

\[ u \triangleright (fg) = (u_{(1)} \triangleright f)(u_{(2)} \triangleright g). \]  

(2.5)  

(2.6)

This time we can endow the underlying vector space \( F \otimes U \) with an algebra structure, to be denoted by \( F \rhd U \), with \( 1 \rhd 1 \) as its unit and the product given by

\[(f \rhd U)(g \rhd U) = f u_{(1)} \triangleright g u_{(2)} v \]

\[ (2.7) \]

\( U \) and \( F \) are said to form a matched pair of Hopf algebras if they are equipped, as above, with an action and a coaction which satisfy the following compatibility conditions for any \( u \in U \), and any \( f \in F \).

\[ \epsilon(u \triangleright f) = \epsilon(u) \epsilon(f), \]  

\[ \Delta(u \triangleright f) = u_{(1) <0>} \triangleright f_{(1)} \otimes u_{(1) <1>} (u_{(2)} \triangleright f_{(2)}), \]

\[ \nabla(1) = 1 \otimes 1, \]  

\[ \nabla(uv) = u_{(1) <0>} v_{<0>} \otimes u_{(1) <1>} (u_{(2)} \triangleright v_{<1>}), \]

\[ u_{(2) <0>} \otimes (u_{(1)} \triangleright f) u_{(2) <1>} = u_{(1) <0>} \otimes u_{(1) <1>} (u_{(2)} \triangleright f). \]  

(2.8)  

(2.9)  

(2.10)  

(2.11)  

(2.12)

One then forms a new Hopf algebra \( F \rhd U \), called the bicrossed product of the matched pair \( (F, U) \); it has \( F \rhd U \) as underlying coalgebra, \( F \rhd U \) as underlying algebra and the antipode is defined by

\[ S(f \rhd u) = (1 \rhd S(u_{<0>}))(S(f_{u_{<1>}}) \rhd 1), \quad f \in F; \ u \in U. \]  

(2.13)

On the other hand, we need to recall the notion of double crossed product Hopf algebra \([26]\). Let \( U \) and \( V \) be two Hopf algebras such that \( V \) is a
right $\mathcal{U}$–module coalgebra and $\mathcal{U}$ is left $\mathcal{V}$–module coalgebra. We call them mutual pair if their actions satisfy the following conditions.

$$v \triangleright (u^1 u^2) = (v^{(1)} \triangleright u^{1\langle 1 \rangle})(v^{(2)} \triangleright u^{1\langle 2 \rangle}) \triangleright u^2), \quad 1 \triangleright u = \varepsilon(u),$$  \hspace{1cm} (2.14)

$$v^1 \triangleright u = (v \triangleright (v^{2\langle 1 \rangle} \triangleright u^{\langle 1 \rangle}))(v^{2\langle 2 \rangle} \triangleright u^{\langle 2 \rangle}), \quad v \triangleright 1 = \varepsilon(v),$$  \hspace{1cm} (2.15)

$$\sum v^{(1)} \triangleright u^{(1)} \otimes v^{(2)} \triangleright u^{(2)} = \sum v^{(2)} \triangleright u^{(2)} \otimes v^{(1)} \triangleright u^{(1)}. \hspace{1cm} (2.16)$$

Having a mutual pair of Hopf algebras, one constructs the double crossed product Hopf algebra $\mathcal{U} \triangleright \triangleleft \mathcal{V}$. As a coalgebra $\mathcal{U} \triangleright \triangleleft \mathcal{V}$ is $\mathcal{U} \otimes \mathcal{V}$, however its algebra structure is defined by the following rule together with $1 \triangleright \triangleleft 1$ as its unit.

$$(u^1 \triangleright v^1)(u^2 \triangleright v^2) := u^1(v^{1\langle 1 \rangle} \triangleright u^{2\langle 1 \rangle}) \triangleright \triangleleft (v^{2\langle 2 \rangle} \triangleright u^{\langle 2 \rangle})v$$ \hspace{1cm} (2.17)

The antipode of $\mathcal{U} \triangleright \triangleleft \mathcal{V}$ is defined by

$$S(u \triangleright v) = (1 \triangleright \triangleleft S(v))(S(u) \triangleright \triangleleft 1) = S(v^{\langle 1 \rangle}) \triangleright \triangleleft S(u^{\langle 1 \rangle}) \triangleright \triangleleft S(v^{\langle 2 \rangle}) \triangleright \triangleleft S(u^{\langle 2 \rangle}). \hspace{1cm} (2.18)$$

### 2.2 Lie-Hopf algebras and associated bicrossed product Hopf algebras

It is now clear that bicrossed product Hopf algebras play a crucial rôle in many places, especially when one tries to compute Hopf cyclic cohomology of nontrivial Hopf algebras \[28, 29, 30\]. It is always a lengthy task to show that a particular Hopf algebra is of the form of bicrossed product. In this subsection, we introduce the minimum criteria by which one verifies a Hopf algebra is of this form. Based on our interest, we restrict ourself to the case that one of the building block Hopf algebra is commutative and the other one is enveloping algebra of a Lie algebra. It is clear that everything can be extended to more general situations but we do not try it here.

Let $\mathcal{F}$ be a commutative Hopf algebra on which a Lie algebra $\mathfrak{g}$ acts by derivations. We endow the vector space $\mathfrak{g} \otimes \mathcal{F}$ with the following bracket.

$$[X \otimes f, Y \otimes g] = [X, Y] \otimes fg + Y \otimes \varepsilon(f)X \triangleright g - X \otimes \varepsilon(g)Y \triangleright f. \hspace{1cm} (2.19)$$

**Lemma 2.1.** Let $\mathfrak{g}$ act on a commutative Hopf algebra $\mathcal{F}$ and $\varepsilon(X \triangleright f) = 0$ for any $X \in \mathfrak{g}$ and $f \in \mathcal{F}$. Then the bracket defined in \(2.19\) endows $\mathfrak{g} \otimes \mathcal{F}$ with a Lie algebra structure.
Proof. It is obvious that the bracket is antisymmetric. We need to check the Jacobi identity. Indeed, after routine computation we observe that,

\[
[[X \otimes f, Y \otimes g], Z \otimes h] = [[X, Y], Z] \otimes fgh + Z \otimes \varepsilon(fg)[X, Y] \triangleright h-
\]
\[
[X, Y] \otimes \varepsilon(h)Z \triangleright (fg) + [Y, Z] \otimes \varepsilon(f)hX \triangleright g - Y \otimes \varepsilon(h)\varepsilon(f)Z \triangleright (X \triangleright g)-
\]
\[
[X, Z] \otimes \varepsilon(g)hY \triangleright f + X \otimes \varepsilon(h)\varepsilon(g)Z \triangleright (Y \triangleright f).
\]

(2.20)

\[
[[Y \otimes g, Z \otimes h], X \otimes f] = [[Y, Z], X] \otimes fgh + X \otimes \varepsilon(gh)[Y, Z] \triangleright f-
\]
\[
[Y, Z] \otimes \varepsilon(f)X \triangleright (gh) + [Z, X] \otimes \varepsilon(g)fY \triangleright h - Z \otimes \varepsilon(f)\varepsilon(g)X \triangleright (Y \triangleright h)-
\]
\[
[Y, X] \otimes \varepsilon(h)fZ \triangleright g + Y \otimes \varepsilon(f)\varepsilon(h)X \triangleright (Z \triangleright g).
\]

(2.21)

\[
[[Z \otimes h, X \otimes f], Y \otimes g] = [[Z, X], Y] \otimes fgh + Y \otimes \varepsilon(hf)[Z, X] \triangleright g-
\]
\[
[Z, X] \otimes \varepsilon(g)Y \triangleright (hf) + [X, Y] \otimes \varepsilon(h)gZ \triangleright f - X \otimes \varepsilon(h)\varepsilon(g)Y \triangleright (Z \triangleright f)-
\]
\[
[Z, Y] \otimes \varepsilon(f)gX \triangleright h + Z \otimes \varepsilon(g)\varepsilon(f)Y \triangleright (X \triangleright h).
\]

(2.22)

Summing up (2.20), (2.21), and (2.22) and using the fact that \( g \) is a Lie algebra acting on \( F \) by derivations, we get

\[
[[X \otimes f, Y \otimes g], Z \otimes h] + [[Y \otimes g, Z \otimes h], X \otimes f] + [[Z \otimes h, X \otimes f], Y \otimes g] = 0. \tag{2.23}
\]

\[\square\]

Now let assume that \( F \) coacts on \( g \) from right via \( \nabla_\theta : \ g \rightarrow g \otimes F \). We define the first-order matrix coefficients \( f_{ij}^i \in F \) of \( \nabla_\theta \) by,

\[
\nabla_\theta(X_i) = \sum_j X_i \otimes f_{ij}^i. \tag{2.24}
\]

One use the fact that \( \nabla_\theta \) is a coaction to observe

\[
\Delta(f_{ij}^i) = \sum_k f_{ij}^k \otimes f_{ik}^k. \tag{2.25}
\]

We define the second-order matrix coefficients by

\[
f_{j,k}^i := X_k \triangleright f_{ij}^i. \tag{2.26}
\]
Let $C^k_{i,j}$ stand for the structure constants of the Lie algebra $\mathfrak{g}$, i.e,

$$[X_i, X_j] = \sum_k C^k_{i,j} X_k$$  \hspace{1cm} (2.27)

**Definition 2.2.** We say that a coaction $\nabla_\mathfrak{g} : \mathfrak{g} \to \mathfrak{g} \otimes F$ satisfies the structure identity of $\mathfrak{g}$ if

$$f^k_{j,i} - f^k_{i,j} = \sum_{s,r} C^k_{s,r} f^s_i f^r_j + \sum_l C^l_{i,j} f^k_l.$$  \hspace{1cm} (2.28)

**Lemma 2.3.** The coaction $\nabla_\mathfrak{g} : \mathfrak{g} \to \mathfrak{g} \otimes F$ satisfies the structure identity of $\mathfrak{g}$ if and only if $\nabla_\mathfrak{g} : \mathfrak{g} \to \mathfrak{g} \otimes F$ is a Lie algebra map.

**Proof.** Indeed we just need to check for two basis elements.

$$\nabla_\mathfrak{g} ([X_i, X_j]) = \nabla_\mathfrak{g} (C^k_{i,j} X_k) = C^k_{i,j} X_l \otimes f^l_k.$$  \hspace{1cm} (2.29)

On the other hand, by using (2.28) we observe

$$[\nabla_\mathfrak{g} (X_i), \nabla_\mathfrak{g} (X_j)] = [X_p \otimes f^p_i, X_q \otimes f^q_j] =$$

$$[X_p, X_q] \otimes f^p_i f^q_j + X_q \otimes \varepsilon(f^q_i) X_p \triangleright f^q_j - X_p \otimes \varepsilon(f^p_i) X_q \triangleright f^p_i =$$

$$C^k_{r,s} X_k \otimes f^r_i f^s_j + X_k \otimes X_i \triangleright f^k_j - X_k \otimes X_j \triangleright f^k_i =$$

$$C^k_{r,s} X_k \otimes f^r_i f^s_j + X_k \otimes (f^k_{j,i} - f^k_{i,j}) =$$

$$C^k_{r,s} X_k \otimes f^r_i f^s_j + C^k_{s,r} X_k \otimes f^s_i f^r_j + C^k_{i,j} X_l \otimes f^l_k = C^k_{i,j} X_l \otimes f^l_k.$$  \hspace{1cm} (2.30)

The converse argument is similar to the above. \hfill \Box

One uses the action of $\mathfrak{g}$ on $F$ and the coaction of $F$ on $\mathfrak{g}$ to define an action of $\mathfrak{g}$ on $F \otimes F$ by

$$X \bullet (f^1 \otimes f^2) = X_{\langle 0 \rangle} \triangleright f^1 \otimes X_{\langle 1 \rangle} f^2 + f^1 \otimes X \triangleright f^2.$$  \hspace{1cm} (2.31)

**Definition 2.4.** Let a Lie algebra $\mathfrak{g}$ act on a commutative Hopf algebra $F$ by derivations. We say that $F$ is a $\mathfrak{g}$-Hopf algebra if

1. $F$ coacts on $\mathfrak{g}$ and its coaction satisfies the structure identity of $\mathfrak{g}$.

2. $\Delta$ and $\varepsilon$ are $\mathfrak{g}$-linear, i.e, $\Delta(X \triangleright f) = X \bullet \Delta(f), \quad \varepsilon(X \triangleright f) = 0, \quad f \in F$ and $X \in \mathfrak{g}$. 

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Let $\mathcal{F}$ be a $\mathfrak{g}$-Hopf algebra. Then $U(\mathfrak{g})$ acts on $\mathcal{F}$ in the obvious way and makes it a $U(\mathfrak{g})$-module algebra. We extend the coaction $\nabla_\mathfrak{g}$ of $\mathcal{F}$ on $\mathfrak{g}$ to a coaction $\nabla$ of $\mathcal{F}$ on $U(\mathfrak{g})$ inductively via the rule (2.11) and $\nabla(1) = 1 \otimes 1$.

**Lemma 2.5.** The extension of $\nabla_\mathfrak{g} : \mathfrak{g} \to \mathfrak{g} \otimes \mathcal{F}$ to $\nabla : U(\mathfrak{g}) \to U(\mathfrak{g}) \otimes \mathcal{F}$ via (2.11) is well-defined.

**Proof.** It is necessary and sufficient to prove that $\nabla_\mathfrak{g}([X,Y]) = \nabla(XY - YX)$. Using Lemma 2.3, the rule (2.11) and the fact that $\mathcal{F}$ is commutative, we see that

$$\nabla(XY - YX) = [X_{<0>}, Y_{<0>}] \otimes X_{<1>} Y_{<1>} + Y_{<0>} \otimes X \triangleright Y_{<1>} - X_{<0>} \otimes Y \triangleright X_{<1>} = \nabla([X,Y]). \quad (2.32)$$

**Theorem 2.6.** Let $\mathcal{F}$ be a commutative $\mathfrak{g}$-Hopf algebra. Then via the coaction of $\mathcal{F}$ on $U(\mathfrak{g})$ defined above and the natural action of $U(\mathfrak{g})$ on $\mathcal{F}$, the pair $(U(\mathfrak{g}), \mathcal{F})$ becomes a matched pair of Hopf algebras. Conversely, for a commutative Hopf algebra $\mathcal{F}$, if $(U(\mathfrak{g}), \mathcal{F})$ is a matched pair of Hopf algebras then $\mathcal{F}$ is a $\mathfrak{g}$-Hopf algebra.

**Proof.** We need to verify that the matched pair conditions are satisfied. The axioms (2.8) and (2.10) are held by the definition. We prove the other two. By definition of the coaction $\nabla : U(\mathfrak{g}) \to U(\mathfrak{g}) \otimes \mathcal{F}$, the axiom (2.11) is held for any $u, v \in U(\mathfrak{g})$.

Next we check (2.9). By Definition 2.4 (2) we observe that (2.9) is satisfied for any $X \in \mathfrak{g}$ and $f \in \mathcal{F}$. Let assume that it is held for $u, v \in U(\mathfrak{g})$, and any $f \in \mathcal{F}$. By using (2.11) we see that

$$\begin{align*}
(uv)_{(1)}_{<0>} \otimes (uv)_{(1)}_{<1>} \otimes (uv)_{(2)} &= u_{(1)}_{<0>} v_{(1)}_{<0>} \otimes u_{(1)}_{<1>} (u_{(2)} \triangleright v_{(1)}_{<1>}) \otimes u_{(3)} v_{(2)}.
\end{align*} \quad (2.33)$$

Using (2.33) and the fact that $\mathcal{F}$ is $U(\mathfrak{g})$-module algebra we prove our claim.

$$\begin{align*}
\Delta(uv \triangleright f) &= u_{(1)}_{<0>} \triangleright (v \triangleright f)_{(1)} \otimes u_{(1)}_{<1>} (u_{(2)} \triangleright (v \triangleright f)_{(2)}) = u_{(1)}_{<0>} \triangleright (v_{(1)}_{<0>} \triangleright f_{(1)}) \otimes u_{(1)}_{<1>} (u_{(2)} \triangleright (v_{(1)}_{<1>} (v_{(2)} \triangleright f_{(2)} = u_{(1)}_{<0>} \triangleright (v_{(1)}_{<0>} \triangleright f_{(1)}) \otimes u_{(1)}_{<1>} (u_{(2)} \triangleright (v_{(1)}_{<1>} (u_{(3)} v_{(2)} \triangleright f_{(2)} = (uv)_{(1)}_{<0>} \triangleright f_{(1)} \otimes (uv)_{(1)}_{<1>} ((uv)_{(2)} \triangleright f_{(2})).
\end{align*} \quad (2.34)$$
Finally we check that \( U(\mathfrak{g}) \) is a \( \mathcal{F} \)-comodule coalgebra i.e., we verify (2.11) which is obvious for any \( X \in \mathfrak{g} \). Let assume that (2.11) is satisfied for \( u, v \in U(\mathfrak{g}) \), and prove it is so for \( uv \). Indeed, by using (2.11) and the fact that \( U(\mathfrak{g}) \) is comocommutative, \( \mathcal{F} \) is commutative, and \( \mathcal{F} \) is \( U(\mathfrak{g}) \)-module algebra, we observe that

\[
(uv)_{<0>} \otimes (uv)_{<0>} \otimes (uv)_{<1>} (uv)_{<1>} =
\]

\[
(u_1v_1)_{<0>} \otimes (u_2v_2)_{<0>} \otimes (u_1v_1)_{<1>} (u_2v_2)_{<1>} =
\]

\[
u_{<0>} (u_{<0>}) \otimes u_{<0>} v_{<0>} \otimes u_{<1>} (u_{<1>} v_{<1>}) u_{<1>} (u_{<1>} v_{<1>}) =
\]

\[
u_{<0>} (u_{<0>}) \otimes (u_{<0>}) \otimes (u_{<1>}) (u_{<1>}) v_{<1>} =
\]

\[
u_{<0>} (u_{<0>}) \otimes (u_{<0>}) (u_{<1>}) (u_{<1>}) v_{<1>} =
\]

(2.35)

Conversely, let \( \mathcal{F} \) be a commutative Hopf algebra and \((U(\mathfrak{g}), \mathcal{F})\) be a matched pair of Hopf algebras. Let us denote the coaction of \( \mathcal{F} \) on \( U(\mathfrak{g}) \) by \( \bigtriangledown_{\mathcal{F}} \). First we prove that the restriction of \( \bigtriangledown_{\mathcal{F}} : U(\mathfrak{g}) \to U(\mathfrak{g}) \otimes \mathcal{F} \) on \( \mathfrak{g} \) lands in \( \mathfrak{g} \otimes \mathcal{F} \). Indeed, since \( U(\mathfrak{g}) \) is \( \mathcal{F} \)-comodule coalgebra, we see that

\[
X_{<0>} \otimes 1 \otimes X_{<1>} + 1 \otimes X_{<0>} \otimes X_{<1>} =
\]

\[
x_{<0>} \otimes x_{<0>} \otimes x_{<1>} =
\]

\[
x_{<0>} \otimes x_{<0>} \otimes x_{<1>} =
\]

(2.36)

This shows that for any \( X \in \mathfrak{g} \), \( \bigtriangledown_{\mathcal{F}}(X) \) belongs to \( P \otimes \mathcal{F} \), where \( P \) is the Lie algebra of primitive elements of \( U(\mathfrak{g}) \). But we know that \( P = \mathfrak{g} \) by [15]. So we get a coaction \( \bigtriangledown_{\mathcal{F}} : \mathfrak{g} \to \mathfrak{g} \otimes \mathcal{F} \) which is the restriction of \( \bigtriangledown_{\mathcal{F}} \). Since \( \mathcal{F} \) is a \( U(\mathfrak{g}) \)-module algebra, \( \mathfrak{g} \) acts on \( \mathcal{F} \) by derivations. An argument like the one in the proof of Lemma [2.5] shows that the coaction \( \bigtriangledown_{\mathcal{F}} : \mathfrak{g} \to \mathfrak{g} \otimes \mathcal{F} \) is a coalgebra map. Finally [2.9] implies that \( \Delta \) is \( \mathfrak{g} \)-equivariant. So we have proved that \( \mathcal{F} \) is a \( \mathfrak{g} \)-Hopf algebra.

2.3 Matched pair of Hopf algebras associated to matched pair of Lie algebras

In this subsection we associate a bi-crossed product Hopf algebra to any matched pair of Lie algebras \((\mathfrak{g}_1, \mathfrak{g}_2)\). Let us recall the notion of matched
A pair of Lie algebras \((g_1, g_2)\) is called a matched pair if there are linear maps

\[ \alpha : g_2 \otimes g_1 \to g_2, \quad \alpha_X(\zeta) = \zeta \triangleright X, \quad \beta : g_2 \otimes g_1 \to g_1, \quad \beta_\zeta(X) = \zeta \triangleleft X, \quad (2.37) \]

satisfying the following conditions,

\[ \begin{align*}
[\zeta, \xi] \triangleright X & = \zeta \triangleright (\xi \triangleright X) - \xi \triangleright (\zeta \triangleright X), \\
\zeta \triangleleft [X, Y] & = (\zeta \triangleright X) \triangleleft Y - (\zeta \triangleleft Y) \triangleright X, \\
\zeta \triangleright [X, Y] & = [\zeta \triangleright X, Y] + [X, \zeta \triangleright Y] + (\zeta \triangleright X) \triangleright Y - (\zeta \triangleright Y) \triangleright X, \\
[\zeta, \xi] \triangleleft X & = [\zeta \triangleleft X, \xi] + [\zeta, \xi \triangleleft X] + \zeta \triangleleft (\xi \triangleright X) - \xi \triangleleft (\zeta \triangleright X).
\end{align*} \quad (2.38)-(2.41) \]

Given a matched pair of Lie algebras \((g_1, g_2)\), one defines a double crossed sum Lie algebra \(g_1 \bowtie \bowtie g_2\). Its underlying vector space is \(g_1 \oplus g_2\) and its Lie bracket is defined by:

\[ [X \oplus \zeta, Z \oplus \xi] = ([X, Z] + \zeta \triangleright Z - \xi \triangleright X) \oplus ([\zeta, \xi] + \zeta \triangleleft Z - \xi \triangleleft X). \quad (2.42) \]

One checks that both \(g_1\) and \(g_2\) are Lie subalgebras of \(g_1 \bowtie \bowtie g_2\) via obvious inclusions. Conversely, if for a Lie algebra \(g\) there are two Lie subalgebras \(g_1\) and \(g_2\) so that \(g = g_1 \oplus g_2\) as vector spaces, then \((g_1, g_2)\) forms a matched pair of Lie algebras and \(g \cong g_1 \bowtie \bowtie g_2\) as Lie algebras \([26]\). In this case the actions of \(g_1\) on \(g_2\) and \(g_2\) on \(g_1\) for \(\zeta \in g_2\) and \(X \in g_1\) are uniquely determined by

\[ [\zeta, X] = \zeta \triangleright X + \zeta \triangleleft X. \quad (2.43) \]

**Proposition 2.7** \([26]\). Let \(g = g_1 \bowtie \bowtie g_2\) be a double crossed sum of Lie algebras. Then the enveloping algebras \((U(g_1), U(g_2))\) becomes a mutual pair of Hopf algebras. Moreover, \(U(g)\) and \(U(g_1) \bowtie \bowtie U(g_2)\) are isomorphic as Hopf algebras.

In terms of the inclusions

\[ i_1 : U(g_1) \to U(g_1 \bowtie \bowtie g_2) \quad \text{and} \quad i_2 : U(g_2) \to U(g_1 \bowtie \bowtie g_2) \quad (2.44) \]

the isomorphism mentioned in the above proposition is

\[ \mu \circ (i_1 \otimes i_2) : U(g_1) \bowtie \bowtie U(g_2) \to U(g) \quad (2.45) \]
Here $\mu$ is the multiplication on $U(\mathfrak{g})$. One easily observes that there is a linear map

$$\Psi : U(\mathfrak{g}_2) \bowtie U(\mathfrak{g}_1) \to U(\mathfrak{g}_1) \bowtie U(\mathfrak{g}_2),$$

(2.46)
satisfying

$$\mu \circ (i_2 \otimes i_1) = \mu \circ (i_1 \otimes i_2) \circ \Psi .$$

(2.47)

The mutual actions of $U(\mathfrak{g}_1)$ and $U(\mathfrak{g}_2)$ are defined as follows

$$\triangleright := (\text{id}_{U(\mathfrak{g}_2)} \otimes \varepsilon ) \circ \Psi \quad \text{and} \quad \triangleleft := (\varepsilon \otimes \text{id}_{U(\mathfrak{g}_1)}) \circ \Psi .$$

(2.48)

We now recall the definition of $R(\mathfrak{g})$, the Hopf algebra of representative functions on $U(\mathfrak{g})$, for a Lie algebra $\mathfrak{g}$.

$$R(\mathfrak{g}) = \{ f \in \text{Hom}(U(\mathfrak{g}), \mathbb{C}) \mid \exists \text{ a finite codimensional ideal } I \subseteq \ker f \}$$

The finite codimensionality condition in the definition of $R(\mathfrak{g})$ guarantees that for any $f \in R(\mathfrak{g})$ there exist a finite number of functions $f'_i, f''_i \in R(\mathfrak{g})$ such that for any $u^1, u^2 \in U(\mathfrak{g})$.

$$f(u^1 u^2) = \sum_i f'_i(u^1)f''_i(u^2).$$

(2.49)

The Hopf algebraic structure of $R(\mathfrak{g})$ is summarized by:

$$\mu : R(\mathfrak{g}) \otimes R(\mathfrak{g}) \to R(\mathfrak{g}), \quad \eta : \mathbb{C} \to R(\mathfrak{g}), \quad \Delta : R(\mathfrak{g}) \to R(\mathfrak{g}) \otimes R(\mathfrak{g}) ,$$

$$\Delta(f) = \sum_i f'_i \otimes f''_i, \quad \text{if} \quad f(u^1 u^2) = \sum_i f'_i(u^1)f''_i(u^2),
$$

$$S : R(\mathfrak{g}) \to R(\mathfrak{g}), \quad S(f)(u) = f(S(u)).$$

(2.50)

(2.51)

(2.52)

(2.53)

Let $< X_1, \ldots, X_n >$ be a basis for $\mathfrak{g}_1$ and $< \theta^1, \ldots, \theta^n >$ be its dual basis for $\mathfrak{g}_1^*$, that is $\langle \theta^i, X_j \rangle = \delta^i_j$, where $\delta^i_j$ are Kronecker’s delta function. We define the following linear functionals on $U(\mathfrak{g}_2)$,

$$f^j_i(v) = < v \triangleright X_i, \theta^j >,$$

(2.54)

equivalently

$$v \triangleright X_i = f^j_i(v)X_j.$$

(2.55)
Lemma 2.8. For any $1 \leq i, j \leq n$, the functions $f^j_i$ are representative.

Proof. For $1 \leq i \leq n$, we set $I_i = \{v \in U(g_2) \mid v \triangleright X_i = 0\}$. Since $g_1$ is finite dimensional, $I_i$ is a finite codimensional left ideal of $U(g_2)$ sitting in the null space of $f^j_i$ for any $j$.

As a result, we have the following coaction admitting $f^j_i$’s as the first order matrix coefficients, c.f. (2.24).

\[ \nabla_{\text{Alg}} : g_1 \to g_1 \otimes R(g_2), \quad \nabla_{\text{Alg}}(X_i) = \sum_{k=1}^{n} X_j \otimes f^j_i, \]  

(2.56)

By the work of Harish-Chandra \[13\] we know that $R(g_2)$ separates elements of $U(g_2)$. This results in a nondegenerate Hopf pairing between $R(g_2)$ and $U(g_2)$.

\[ \langle f, v \rangle := f(v). \]  

(2.57)

We use the pairing (2.57) to define the natural action of $U(g_1)$ on $R(g_2)$,

\[ U(g_1) \otimes R(g_2) \to R(g_2), \quad \langle u \triangleright f, v \rangle = \langle f, v \triangleright u \rangle. \]  

(2.58)

We define the new elements of $R(g_2)$ by

\[ f_{i_1, \ldots, i_k}^j := X_{i_k} \cdots X_{i_2} \triangleright f_{i_1}^j. \]  

(2.59)

Lemma 2.9. The coaction $\nabla_{\text{Alg}}$ satisfy the structure identity of $g_1$.

Proof. We should prove that

\[ f_{j,i}^k - f_{i,j}^k = \sum_{s,r} C_{s,t}^{k,i} f^s_r f^t_j + \sum_i C_{i,j}^{t} f^t_i, \]  

(2.60)

where $C_{i,j}^{t}$’s are the structure constants of $g_1$.

We first use (2.14) to observe that

\[ v \triangleright [X, Y] = [v \triangleright X, v \triangleright Y] + (v \triangleleft X) \triangleright Y - (v \triangleleft Y) \triangleright X. \]  

(2.61)

We apply two hand sides of (2.28) on an arbitrary element of $v \in U(g_2)$ and use the above observation to finish the proof.

\[ C_{i,j}^{t} f^t_i(v)X_k = C_{i,j}^{t} v \triangleright X_i = v \triangleright [X_i, X_j] = \]
\[ [v \triangleright X_i, v \triangleright X_j] + (v \triangleleft X_i) \triangleright X_j - (v \triangleleft X_j) \triangleright X_i = \]
\[ f^t_j(v \triangleright X_i)X_k + f^t_i(v \triangleleft X_j)X_k - f^k_i(v \triangleleft X_j)X_k = \]
\[ C_{r,s}^{t} f^s_r f^t_j(v)X_k + f^t_j(v \triangleleft X_i)X_k - f^k_j(v \triangleleft X_i)X_k. \]  

(2.62)
Proposition 2.10. For any matched pair of Lie algebras $(\mathfrak{g}_1, \mathfrak{g}_2)$, the Hopf algebra $R(\mathfrak{g}_2)$ is a $\mathfrak{g}_1$-Hopf algebra.

Proof. Lemma 2.9 proves that the coaction $\nabla_{\text{Alg}}$ satisfies the structure identity of $\mathfrak{g}_1$. So, by using Theorem 2.6 it suffices to prove that $\varepsilon(X \triangleright f) = 0$ and $\Delta(X \triangleright f) = X \bullet \Delta(f)$. We observe that $\varepsilon(X \triangleright f) = (X \triangleright f)(1) = f(1 \bowtie X) = 0$. Then it remains to show that $\Delta(X \triangleright f) = X \bullet \Delta(f)$. Indeed,

$$\begin{align*}
\Delta(X \triangleright f)(v^1 \otimes v^2) &= X \triangleright f(v^1 v^2) = f(v^1 v^2 \triangleleft X) = \\
f(v^1 \triangleleft (v^2(1) \triangleright X)v^2(2)) + f(v^1 (v^2 \triangleright X)) &= \\
f(v^1 \triangleleft (X_{<1>})(v^2(1)X_{<0>})f(2)(v^2(2)) + f(v^1) f(2)(v^2 \triangleleft X) = \\
(X_{<0>} \triangleright f(1)) (v^1)_X (v^2(1)f(2)(v^2(2)) + f(v^1)X \triangleright f(2)(v^2) = \\
(X \bullet \Delta(f))(v^1 \otimes v^2). 
\end{align*}$$

We summarize the main result of this section as follows.

Theorem 2.11. Let $(\mathfrak{g}_1, \mathfrak{g}_2)$ be a matched pair Lie algebras. Then, via the canonical action and coaction defined in (2.58) and (2.56) respectively, $(U(\mathfrak{g}_1), R(\mathfrak{g}_2))$ is a matched pair of Hopf algebras.

Proof. By Proposition 2.10, $R(\mathfrak{g}_2)$ is a $\mathfrak{g}_1$-Hopf algebra. One then applies Theorem 2.6.

As a result of Theorem 2.11 to any matched pair of Lie algebras $(\mathfrak{g}_1, \mathfrak{g}_2)$, we canonically associate the Hopf algebra

$$\mathcal{H}(\mathfrak{g}_1, \mathfrak{g}_2) := R(\mathfrak{g}_2) \triangleright \bowtie U(\mathfrak{g}_1).$$

2.4 Matched pair of Hopf algebras associated to matched pair of Lie groups

In this subsection, our aim is to associate a bicrossed product Hopf algebra to any matched pair of Lie groups. We first recall the notion of matched pair of Lie groups. Let $(G_1, G_2)$ be a pair of Lie groups with mutual smooth actions $\triangleright : G_2 \times G_1 \to G_1$ and $\bowtie : G_2 \times G_1 \to G_2$. Then $(G_1, G_2)$ is called a matched
pair of Lie groups provided for any \( \varphi, \varphi_1, \varphi_2 \in G_1 \) and any \( \psi, \psi_1, \psi_2 \in G_2 \) the following compatibilities are satisfied

\[
\begin{align*}
\psi \triangleright \varphi_1 \varphi_2 &= (\psi \triangleright \varphi_1)((\psi \triangleleft \varphi_1) \triangleright \varphi_2), & \psi \triangleright e &= e \\
\psi_1 \psi_2 \triangleleft \varphi &= (\psi_1 \triangleleft (\psi_2 \triangleright \varphi))(\psi_2 \triangleleft \varphi), & e \triangleleft \varphi &= e
\end{align*}
\] (2.65)

Let \( g_1 \) and \( g_2 \) be the Lie algebras of \( G_1 \) and \( G_2 \) respectively. One defines the action of \( g_1 \) on \( g_2 \) by taking derivative of the action of \( G_1 \) on \( G_2 \). Similarly, one defines the corresponding action of \( g_2 \) on \( g_1 \) and shows with a routine argument that if \((G_1, G_2)\) is a matched pair of Lie groups, then \((g_1, g_2)\) is a matched pair of Lie algebras.

Let \( G \) be a Lie group and \( \rho: G \to GL(V) \) a finite dimensional smooth representation. Then, the composition \( \pi \circ \rho \) is called a representative function of \( G \), where \( \pi \in \text{End}(V)^* \) is a linear functional.

We have the following characterization for the representative function due to [20]. For a smooth function \( f: G \to \mathbb{C} \) the following are equivalent: 1) \( f \) is a representative function. 2) The right translates \( \{ f \cdot \psi | \psi \in G \} \) span a finite dimensional vector space over \( \mathbb{C} \). 3) The left translates \( \{ \psi \cdot f | \psi \in G \} \) span a finite dimensional vector space over \( \mathbb{C} \). Here the left and the right translation actions of \( G_2 \) on \( R(G_2) \) are defined by

\[
\psi \cdot f := f_{(1)} f_{(2)}(\psi), \quad f \cdot \psi := f_{(1)}(\psi) f_{(2)}.
\] (2.66)

It is well-known that the representative functions form a commutative Hopf algebra via

\[
\begin{align*}
(f^1 f^2)(\psi) &= f^1(\psi) f^2(\psi), & 1_{R(G)}(\psi) &= 1, \\
\Delta(f)(\psi_1, \psi_2) &= f(\psi_1 \psi_2), & \varepsilon(f) &= f(e), \\
S(f)(\psi) &= f(\psi^{-1}).
\end{align*}
\] (2.67)-(2.68)

Our main objective now is to prove that \( R(G_2) \) is a \( g_1 \)-Hopf algebra.

We first define a right coaction of \( R(G_2) \) on \( g_1 \). To do so, we differentiate the right action of \( G_2 \) on \( G_1 \) to get a right action of \( G_2 \) on \( g_1 \). We then introduce the functions \( f^j_i : G_2 \to \mathbb{C} \)

\[
\psi \triangleright X_i = X_j f^j_i(\psi), \quad X \in g_1, \psi \in G_2.
\] (2.70)

**Lemma 2.12.** The functions \( f^j_i : G_2 \to \mathbb{C} \) defined above are representative functions.
Proof. For $\psi_1, \psi_2 \in G_2$, we observe
\[
X_j f^j_i (\psi_1 \psi_2) = \psi_1 \psi_2 \triangleright X_i = \psi_1 \triangleright (\psi_2 \triangleright X_i) = \\
\psi_1 \triangleright X_i f^j_i (\psi_2) = X_j f^j_i (\psi_1) f^j_i (\psi_2),
\]
(2.71)
Therefore,
\[
\psi_2 \cdot f^j_i = f^j_i (\psi_2) f^j_i \quad (2.72)
\]
In other words, $\psi_2 \cdot f^j_i \in \text{span}\{f^j_i\}$ for any $\psi_2 \in G_2$. \hfill \Box
As a direct consequence of Lemma 2.12, the equation (2.70) defines a coaction:

**Proposition 2.13.** The map $\nabla_{Gr} : g_1 \to g_1 \otimes R(G_2)$ defined by
\[
\nabla_{Gr}(X_i) := X_j \otimes f^j_i \quad (2.73)
\]
is a right coaction of $R(G_2)$ on $g_1$.

Let us recall the natural left action of $G_1$ on $C^\infty(G_2)$ defined by $\varphi \triangleright f(\psi) := f(\psi \triangleleft \varphi)$, and define the derivative of this action by
\[
X \triangleright f := \left. \frac{d}{dt} \right|_{t=0} \exp(tX) \triangleright f, \quad X \in g_1, f \in R(G_2). \quad (2.74)
\]
In fact, considering $R(G_2) \subseteq C^\infty(G_2)$, this is nothing but $d_{\psi} \rho(X)|_{R(G_2)}$, derivative of the representation $\rho : G_1 \to GL(C^\infty(G_2))$ at identity.

**Lemma 2.14.** For any $X \in g_1$ and any $f \in R(G_2)$, we have $X \triangleright f \in R(G_2)$. Moreover we have
\[
\Delta(X \triangleright f) = X_{<0>} \triangleright f_{(1)} \otimes X_{<1>} \triangleright f_{(2)} + f_{(1)} \otimes X \triangleright f_{(2)}. \quad (2.75)
\]
Proof. For any $\psi_1, \psi_2 \in G_2$, by using the fact that $(\triangleright) \circ \exp = \exp \circ d_{\psi} \triangleright$, we
observe

\[(X \triangleright f)(\psi_1 \psi_2) = \frac{d}{dt} \bigg|_{t=0} f(\psi_1 \psi_2 < \exp(tX)) =\]

\[\frac{d}{dt} \bigg|_{t=0} f((\psi_1 < (\psi_2 \triangleright \exp(tX))) \psi_2 < \exp(tX))) =\]

\[\frac{d}{dt} \bigg|_{t=0} f((\psi_1 < (\psi_2 \triangleright \exp(tX)) \psi_2) + \frac{d}{dt} \bigg|_{t=0} f(\psi_1 \psi_2 < \exp(tX))) =\]

\[\frac{d}{dt} \bigg|_{t=0} (\psi_2 \cdot f)(\psi_1 < (\psi_2 \triangleright \exp(tX))) + \frac{d}{dt} \bigg|_{t=0} (f \cdot \psi_1)(\psi_2 < \exp(tX)) =\]

\[f_{(2)}(\psi_2)((\psi_2 \triangleright X) \triangleright f_{(1)})(\psi_1) + f_{(1)}(\psi_1)(X \triangleright f_{(2)})(\psi_2).\]

(2.76)

Which shows

\[\psi_2 \cdot (X \triangleright f) = f_{(2)}(\psi_2)(\psi_2 \triangleright X) \triangleright f_{(1)} + (X \triangleright f_{(2)})(\psi_2) f_{(1)} .\]  

(2.77)

We conclude that \(\psi_2 \cdot (X \triangleright f) \in \text{span}\{X_{<0>} \triangleright f_{(1)}, f_{(1)}\}\), that is, left translates of \(X \triangleright f\) span a finite dimensional vector space. \(\mathfrak{g}_1\)-linearity of \(\Delta\) is shown by

\[f_{(2)}(\psi_2)((\psi_2 \triangleright X) \triangleright f_{(1)})(\psi_1) + f_{(1)}(\psi_1)(X \triangleright f_{(2)})(\psi_2) =\]

\[(X_{<0>} \triangleright f_{(1)})(\psi_1) X_{<12}(\psi_2) f_{(2)}(\psi_2) + f_{(1)}(\psi_1)(X \triangleright f_{(2)})(\psi_2).\]

Proposition 2.15. The map \(\mathfrak{g}_1 \otimes R(G_2) \to R(G_2)\) is a left action.

Proof. Using that fact that \(\text{Ad} \circ \exp = \exp \circ \text{ad}\), we prove the compatibility
of the action and the bracket,
\[
[X,Y] \triangleright f = \left. \frac{d}{dt} \right|_{t=0} \exp([tX,Y]) \triangleright f =
\]
\[
\left. \frac{d}{dt} \right|_{t=0} \exp(tX) \triangleright f =
\]
\[
\left. \frac{d}{dt} \right|_{t=0} \exp(tX) \triangleright f = \left. \frac{d}{ds} \right|_{s=0} \exp(sY) \triangleright f =
\]
\[
\left. \frac{d}{dt} \right|_{t=0} \exp(sY) \triangleright f =
\]
\[
X \triangleright (Y \triangleright f) - Y \triangleright (X \triangleright f).
\]

(2.78)

Lemma 2.16. The coaction \( \Box_{G_2} : \mathfrak{g}_1 \to \mathfrak{g}_1 \otimes R(G_2) \) satisfies the structure identity of \( \mathfrak{g}_1 \).

Proof. We will prove that
\[
f^k_{j,i} - f^k_{i,j} = \sum_{r,s} C^k_{s,r} f^r_i f^s_j + \sum_l C^l_{i,j} f^k_l.
\]

(2.79)

Realizing the elements of \( \mathfrak{g}_1 \) as local derivations on \( C^\infty(G_1) \),
\[
(\psi \triangleright [X_i, X_j])(\hat{f}) = \left. \frac{d}{dt} \right|_{t=0} \hat{f}(\exp(t(\psi \triangleright [X_i, X_j]))) =
\]
\[
\left. \frac{d}{dt} \right|_{t=0} \hat{f}(\psi \triangleright \exp(ad(tX_i))(X_j)) =
\]
\[
[\psi \triangleright X_i, \psi \triangleright X_j](\hat{f}) + \left. \frac{d}{dt} \right|_{t=0} ((\psi \triangleleft \exp(tX_i)) \triangleright X_j)(\hat{f}) -
\]
\[
\left. \frac{d}{ds} \right|_{s=0} ((\psi \triangleleft \exp(sX_j)) \triangleright X_i)(\hat{f}),
\]

(2.80)

for any \( \psi \in G_2, \hat{f} \in C^\infty(G_1) \) and \( X_i, X_j \in \mathfrak{g}_1 \). Hence we conclude that
\[
\psi \triangleright [X_i, X_j] = [\psi \triangleright X_i, \psi \triangleright X_j] + \left. \frac{d}{dt} \right|_{t=0} ((\psi \triangleleft \exp(tX_i)) \triangleright X_j) -
\]
\[
\left. \frac{d}{ds} \right|_{s=0} ((\psi \triangleleft \exp(sX_j)) \triangleright X_i).
\]

(2.81)
Now, writing $[X_i, X_j] = C^l_{i,j} X_l$ and recalling that by the definition of the coaction we have $\psi \triangleright [X_i, X_j] = X_k C^l_{i,j} f^k_l (\psi)$. Similarly,

$$[\psi \triangleright X_i, \psi \triangleright X_j] = [X_i f^i_l (\psi), X_j f^j_l (\psi)] = C^k_{r,s} X_k f^i_l (\psi) f^j_l (\psi) = X_k C^k_{r,s} (f^i_l f^j_l) (\psi).$$

Finally

$$\left. \frac{d}{dt} \right|_{t=0} ((\psi \wedge \exp(tX_i)) \triangleright X_j) = \left. \frac{d}{dt} \right|_{t=0} X_k f^k_l (\psi \wedge \exp(tX_i)) = X_k (X_i \triangleright f^k_l) (\psi) = X_k f^k_{j,i} (\psi),$$

and in the same way

$$\left. \frac{d}{ds} \right|_{s=0} ((\psi \wedge \exp(sX_j)) \triangleright X_i) = X_k f^k_{j,i} (\psi).$$

Theorem 2.17. Let $(G_1, G_2)$ be a matched pair of Lie groups. Then by the action (2.74) and the coaction (2.73), the pair $(U(g_1), R(G_2))$ is a matched pair of Hopf algebras.

Proof. According to Theorem 2.6 we need to prove that the Hopf algebra $R(G_2)$ is a $g_1$–Hopf algebra. Considering the Hopf algebra structure of $R(G_2)$, we see that,

$$\varepsilon(X \triangleright f) = (X \triangleright f)(e) = \left. \frac{d}{dt} \right|_{t=0} f(e \wedge \exp(tX)) = \left. \frac{d}{dt} \right|_{t=0} f(e) = 0. \quad (2.85)$$

By the Lemma 2.16 we know that $\triangledown_{G_2}$ satisfies the structure identity of $g_1$. The equation (2.75) proves that $\Delta(X \triangleright f) = X \bullet \Delta(f)$. \qed

As a result, to any matched pair of Lie groups $(G_1, G_2)$, we associate the Hopf algebra

$$\mathcal{H}(G_1, G_2) := R(G_2) \triangleright U(g_1). \quad (2.86)$$

We proceed by providing the relation between the Hopf algebras $\mathcal{H}(G_1, G_2)$ and $\mathcal{H}(g_1, g_2)$. To this end, we first introduce a map

$$\theta : R(G_2) \to R(g_2), \quad \pi \circ \rho \to \pi \circ d_e \rho \quad (2.87)$$
for any finite dimensional representation $\rho : G_2 \to GL(V)$, and linear functional $\pi : End(V) \to \mathbb{C}$. Here we identify the representation $d_{e, \rho} : g_2 \to gl(V)$ of $g_2$ and the unique algebra map $d_{e, \rho} : U(g_2) \to gl(V)$ making the following diagram commutative

\[
\begin{array}{ccc}
g_2 & \xrightarrow{d_{e, \rho}} & \mathfrak{g}l(V) \\
\downarrow & & \downarrow \\
U(g_2) & \xleftarrow{d_{e, \rho}} & \mathfrak{g}l(V)
\end{array}
\]

Let $G$ and $H$ be two Lie groups, where $G$ is simply connected. Let also $g$ and $h$ be the corresponding Lie algebras respectively. Then, a linear map $\sigma : g \to h$ is the differential of a map $\rho : G \to H$ of Lie groups if and only if it is a map of Lie algebras [9]. Therefore, in case of $G_2$ to be simply connected, the map $\theta : R(G_2) \to R(g_2)$ is bijective.

We can express $\theta : R(G_2) \to R(g_2)$ explicitly. The map $d_{e, \rho} : U(g_2) \to gl(V)$ sends $1 \in U(g_2)$ to $Id_V \in gl(V)$, hence for $f \in R(G_2)$

\[
\theta(f)(1) = f(e)
\]  

(2.88)

and since it is multiplicative, for any $\xi_1, ..., \xi_n \in g_2$

\[
\theta(f)(\xi_1 ... \xi_n) = \frac{d}{dt} \bigg|_{t=0} ... \frac{d}{ds} \bigg|_{s=0} f(\exp(t_1 \xi_1) ... \exp(t_n \xi_n))
\]  

(2.89)

**Proposition 2.18.** The following is a map of Hopf algebras

\[
\Theta : H(G_1, G_2) \to H(g_1, g_2), \quad \Theta(f \triangleright u) = \theta(f) \triangleright u.
\]  

(2.90)

Moreover, $H(G_1, G_2) \cong H(g_1, g_2)$ provided $G_2$ is simply connected.

**Proof.** First we show that $\Theta$ is an algebra map. To this end, we need to prove that $\theta$ is a map of $U(g_1)$-module algebras. It is easy to observe that $\theta : R(G_2) \to R(g_2)$ is a map of Hopf algebras. However, we prove that $\theta$ is a $U(g_1)$-module map. Indeed for any $X \in g_1$ and any $\xi \in g_2$,

\[
\theta(X \triangleright f)(\xi) = \frac{d}{dt} \bigg|_{t=0} \frac{d}{ds} \bigg|_{s=0} f(\exp(s \xi) \triangleleft \exp(tX)) = \frac{d}{dt} \bigg|_{t=0} f(\exp(t(\xi \triangleleft X))) = X \triangleright \theta(f)(\xi).
\]  

(2.91)
Next we prove that the following diagram is commutative

\[
\begin{array}{ccc}
g_1 \otimes R(G_2) & \xrightarrow{\Theta} & g_1 \otimes R(g_2) \\
\downarrow^{\text{Gr}} & & \downarrow^{\text{alg}} \\
g_1 \otimes R(G_2) & \xrightarrow{\Theta} & g_1 \otimes R(g_2)
\end{array}
\]

Indeed, by valuating on \( \xi \in g_2 \) we have

\[
X^j \theta(g^j_i)(\xi) = \frac{d}{dt} \bigg|_{t=0} X^j g^j_i(\exp(t\xi)) = \frac{d}{dt} \bigg|_{t=0} (\exp(t\xi) \triangleright X^i) = \xi \triangleright X^i = X^j f^j_i(\xi).
\]

This shows that \( \Theta \) is a map of coalgebras and hence the proof is complete. \( \square \)

### 2.5 Matched pair of Hopf algebras associated to matched pair of affine algebraic groups

In this subsection we aim to associate a bicrossed product Hopf algebra to any matched pair of affine algebraic groups. Let \( G_1 \) and \( G_2 \) be two affine algebraic groups. We assume that there are maps of affine algebraic sets

\[
\triangleright : G_2 \times G_1 \rightarrow G_1 \quad \text{and} \quad \triangleleft : G_2 \times G_1 \rightarrow G_2,
\]

which means the existence of the following maps [31, Chap 22]

\[
\mathcal{P}(\triangleright) : \mathcal{P}(G_1) \rightarrow \mathcal{P}(G_2 \times G_1) = \mathcal{P}(G_2) \otimes \mathcal{P}(G_1)
\]

\( f \mapsto f^{<1>} \otimes f^{<0>} \) such that \( f^{<1>}(\psi)f^{<0>}(\varphi) = f(\psi \triangleright \varphi) \) (2.95)

and

\[
\mathcal{P}(\triangleleft) : \mathcal{P}(G_2) \rightarrow \mathcal{P}(G_2 \times G_1) = \mathcal{P}(G_2) \otimes \mathcal{P}(G_1)
\]

\( f \mapsto f^{<0>} \otimes f^{<1>} \) such that \( f^{<0>}(\psi)f^{<1>}(\varphi) = f(\psi \triangleleft \varphi) \) (2.96)

We say \((G_1,G_2)\) is a matched pair if they satisfy (2.65). Then we define mutual actions

\[
f \triangleleft \psi := f^{<1>}(\psi)f^{<0>}, \quad \text{and} \quad \varphi \triangleright f := f^{<0>}(\varphi)f^{<1>}(\varphi)
\]

(2.97)

to get representations of \( G_2 \) on \( \mathcal{P}(G_1) \) and \( G_1 \) on \( \mathcal{P}(G_2) \). We denote the action of \( G_1 \) on \( \mathcal{P}(G_2) \) by \( \rho \).

In a similar fashion to the Lie group case, we define the action \( g_1 \) as the derivative of the action of \( G_1 \). Here \( \rho^\circ \) being the derivative of \( \rho \). By [16] \( X \triangleright f := \rho^\circ(X)(f) = f^{<0>} X(f^{<1>}). \) (2.98)
Remark 2.19. In the case of Lie groups, assuming that the action
\[ \triangleleft : G_2 \times G_1 \to G_2, \]  
induces a map
\[ R(\triangleleft) : R(G_2) \to R(G_2 \times G_1) = R(G_2) \otimes R(G_1), \]
we arrive
\[ (X \triangleright f)(\psi) = f^{(0)}(\psi)(f^{(1)}(\exp_t X)) = \frac{d}{dt} \bigg|_{t=0} f^{(0)}(\psi)(f^{(1)}(\exp_t X)) = \frac{d}{dt} f(\psi \triangleleft \exp_t X). \]
That is, we derive exactly the same action as we got in the \[ (2.74) \].

The action of \( G_2 \) on \( g_1 \) is also defined as before
\[ \psi \triangleright X := (L_\psi)^\circ(X) \]
where
\[ L_\psi : G_1 \to G_1, \quad \varphi \mapsto \psi \triangleright \varphi. \]

Now we prove that \( \mathcal{P}(G_2) \) coacts on \( g_1 \). In view of Lemma 1.1 of [104], there exists a basis \( \{X_1, \ldots, X_n\} \) of \( g_1 \) and a corresponding subset \( S = \{f^1, \ldots, f^n\} \subseteq \mathcal{P}(G_1) \) such that \( X_j(f^i) = \delta_{ij} \). Using the left action of \( G_2 \) on \( g_1 \) and dual basis \( \{\theta^1, \ldots, \theta^n\} \) for \( g_1^* \), we introduce the functions \( f_i^j : G_2 \to \mathbb{C} \) exactly as before
\[ f_i^j(\psi) := \langle \psi \triangleright X_i, \theta^j \rangle, \]
that is
\[ \psi \triangleright X_i = f_i^j(\psi)X_j. \]

Lemma 2.20. The functions \( f_i^j : G_2 \to \mathbb{C} \) defined above are polynomial functions.
Proof. On one hand,
\[
(\psi \triangleright X_i)(f) = X_i(f \triangleright \psi) = X_i(f^{<\,1>}(\psi)f^{<\,0>}) = f^{<\,1>}(\psi)X_i(f^{<\,0>}),
\]
(2.106)
while on the other
\[
(\psi \triangleright X_i)(f) = f^{j\,i}_i(\psi)X_j(f).
\]
(2.107)
Hence, for \(f = f^k \in S\) we have
\[
f^k_i(\psi) = (f^k)^{<\,1>}(\psi)X_i((f^k)^{<\,0>}),
\]
that is,
\[
f^k_i = X_i((f^k)^{<\,0>})(f^k)^{<\,1>} \in \mathcal{P}(G_2).
\]
As before, by the very definition of \(f^{i\,j}_i\)'s, we have the following coaction.
\[
\nabla_{pol} : g_1 \to g_1 \otimes \mathcal{P}(G_2), \quad X_i \mapsto X_j \otimes f^{j\,i}_i,
\]
(2.108)
and the second order matrix coefficients,
\[
X_k \triangleright f^{j\,i}_i = f^{j\,k}_{i,k}.
\]
(2.109)

**Proposition 2.21.** The coaction \(\nabla_{pol} : g_1 \to g_1 \otimes \mathcal{P}(G_2)\) satisfies the structure identity of \(g_1\).

Proof. We have to show that
\[
f^{k\,j}_{j,i} - f^{i\,j}_{i,j} = \sum_{r,s} C^{k\,r}_{i,s} f^{r\,s}_i + \sum_{l} C^{l\,i}_{i,j} f^{l\,j}_i.
\]
(2.110)

We first observe
\[
(f \triangleleft \psi)^{(1)} (\varphi)(f \triangleleft \psi)^{(2)} (\varphi') = (f \triangleleft \psi)(\varphi \varphi') = (f \triangleright \varphi \varphi') = f^{(1)}(\psi \triangleright \varphi)f^{(2)}((\psi \triangleleft \varphi) \triangleright \varphi') =
\]
\[
(f^{(2)})^{<\,1> \cdot <\,0>}(\psi)((f^{(1)} \triangleright \psi) \cdot (f^{(2)})^{<\,1> \cdot <\,1>})(\varphi)(f^{(2)})^{<\,0>}(\varphi'),
\]
(2.111)
which implies that
\[
(f \triangleleft \psi)^{(1)} \otimes (f \triangleleft \psi)^{(2)} = (f^{(2)})^{<\,1> \cdot <\,0>}(\psi)(f^{(1)} \triangleleft \psi) \cdot (f^{(2)})^{<\,1> \cdot <\,1> \otimes (f^{(2)})^{<\,0>}.\]
(2.112)
Next, by using (2.112) we have
\[
(\psi \triangleright [X_i, X_j])(f) = [X_i, X_j](f \triangleleft \psi) = (X_i \cdot X_j - X_j \cdot X_i)(f \triangleleft \psi) = X_i((f \triangleleft \psi)(1))X_j((f \triangleleft \psi)(2)) - X_j((f \triangleleft \psi)(1))X_i((f \triangleleft \psi)(2)) = (f_{(2)})^{<1><0>}(\psi)[X_i(f_{(1)} \triangleleft \psi)(f_{(2)})^{<1><1>} (e_1) + (f_{(1)} \triangleleft \psi)(e_1)X_i((f_{(2)})^{<1><1>})X_j((f_{(2)})^{<0>}) - (f_{(2)})^{<1><0>}(\psi)[X_j(f_{(1)} \triangleleft \psi)(f_{(2)})^{<1><0>} (e_1) + (f_{(1)} \triangleleft \psi)(e_1)X_j((f_{(2)})^{<1><1>})X_i((f_{(2)})^{<0>}) = [\psi \triangleright X_i, \psi \triangleright X_j](f) + f^{<1><0>}(\psi)X_i(f^{<1><1>})X_j(f^{<0>}) - f^{<1><0>}(\psi)X_j(f^{<1><1>})X_i(f^{<0>}).
\]
\[(2.113)\]

We finally notice that
\[
(f^{<1><0>})(\psi) = f^{<1>}X_j(f^{<0>})(f) = (X_k(f^{<1>})(\psi))(f) = (X_k f_j^k)(\psi).
\]
\[(2.114)\]

Hence,
\[
f^{<1><0>}(\psi)X_i(f^{<1><1>})X_j(f^{<0>}) = (X_i \triangleright f^{<1><0>})(\psi)X_j(f^{<0>}) = (X_i \triangleright f^{<1><0>})(\psi)(f) = (X_k(X_i \triangleright f_j^k)(\psi))(f)
\]
\[(2.115)\]

Similarly
\[
f^{<1><0>}(\psi)X_j(f^{<1><1>})X_i(f^{<0>}) = (X_k(X_j \triangleright f_i^k)(\psi))(f).
\]
\[(2.116)\]

So we have observed the following
\[
\psi \triangleright [X_i, X_j] = [\psi \triangleright X_i, \psi \triangleright X_j] + X_k(X_i \triangleright f_j^k)(\psi) - X_k(X_j \triangleright f_i^k)(\psi),
\]
\[(2.117)\]

which implies the structure equality immediately. \[\square\]

We now express the main result of this subsection.

**Theorem 2.22.** Let \((G_1, G_2)\) be a matched pair of affine algebraic groups. Then by the action (2.98) and the coaction (2.108) defined above, \((U(g_1), P(G_2))\) is a matched pair of Hopf algebras.
Proof. In view of Theorem 2.6, it is enough to show that $\mathcal{P}(G_2)$ is a $\mathfrak{g}_1$–Hopf algebra. In Proposition 2.21 we prove that the structure identity is satisfied. Therefore, here we need to prove

$$\Delta(X \triangleright f) = X \bullet \Delta(f) \quad \text{and} \quad \varepsilon(X \triangleright f) = 0.$$  \hspace{1cm} (2.118)

First we observe that

$$f^{<0>}_{(1)}(\psi_1)f^{<0>}_{(2)}(\psi_2)f^{<1>}(\varphi) = f^{<0>}_{(1)}(\psi_1\psi_2)f^{<1>}(\varphi) =$$

$$f(\psi_1\psi_2 < \varphi) = f_{(1)}(\psi_1 < (\psi_2 \triangleright \varphi))f_{(2)}(\psi_2 < \varphi) =$$

$$(f_{(1)})^{<0>}(\psi_1)((f_{(1)})^{<1>}\cdot (f_{(2)})^{<0>})(\psi_2)((f_{(1)})^{<1>}\cdot (f_{(2)})^{<1>})(\varphi),$$  \hspace{1cm} (2.119)

which shows that

$$f^{<0>}_{(1)} \otimes f^{<0>}_{(2)} \otimes f^{<1>} =$$

$$(f_{(1)})^{<0>} \otimes (f_{(1)})^{<1>}\cdot (f_{(2)})^{<0>} \otimes (f_{(1)})^{<1>}\cdot (f_{(2)})^{<1>}.$$  \hspace{1cm} (2.120)

Therefore,

$$\Delta(X \triangleright f)(\psi_1, \psi_2) = (X \triangleright f)(\psi_1\psi_2) = f^{<0>}_{(1)}(\psi_1)f^{<0>}_{(2)}(\psi_2)X(f^{<1>}) =$$

$$(f_{(1)})^{<0>}(\psi_1)(\psi_2 \triangleright X)((f_{(1)})^{<1>})f_{(2)}(\psi_2) + f_{(1)}(\psi_1 < e)(X \triangleright f_{(2)})(\psi_2) =$$

$$((\psi_2 \triangleright X) \triangleright f_{(1)})(\psi_1)f_{(2)}(\psi_2) + f_{(1)}(\psi_1)(X \triangleright f_{(2)})(\psi_2) =$$

$$(X_{<0>} \triangleright f_{(1)})(\psi_1)(X_{<1>} \cdot f_{(2)})(\psi_2) + f_{(1)}(\psi_1)(X \triangleright f_{(2)})(\psi_2) =$$

$$(X \bullet \Delta(f))(\psi_1, \psi_2).$$  \hspace{1cm} (2.121)

Next, we want to prove that $\varepsilon(X \triangleright f) = 0$. To this end, we notice that

$$\varepsilon(X \triangleright f) = (X \triangleright f)(e_2) = f^{<0>}(e_2)X(f^{<1>}) = X(f^{<0>}(e_2)f^{<1>}) = 0,$$  \hspace{1cm} (2.122)

because $f^{<0>}(e_2)f^{<1>} \in \mathcal{P}(G_1)$ is constant. The proof is done. \hfill \Box

Now we construct the following pairing

$$\langle , \rangle : \mathcal{P}(G_2) \times U(\mathfrak{g}_2) \rightarrow \mathbb{C}$$

$$(f, v) \mapsto \langle f, v \rangle := f^*(v) \quad \text{where} \quad f^*(v) := (v \cdot f)(e_2)$$  \hspace{1cm} (2.123)

Here, the left action $U(\mathfrak{g}_2)$ comes from the derivative of the left translation by $G_2$.  

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Proposition 2.23. The pairing \( \langle \cdot, \cdot \rangle \) defines a Hopf duality between \( \mathcal{P}(G_2) \) and \( U(g_2) \). In other words

\[
\langle f, v^1 v^2 \rangle = \langle f(v_1) v^2, v^2 \rangle, \quad \langle f, 1 \rangle = \varepsilon(f), \tag{2.124}
\]

\[
\langle fg, v \rangle = \langle f, v(1) \rangle \langle g, v(2) \rangle, \quad \langle 1, v \rangle = \varepsilon(v). \tag{2.125}
\]

Proof. Let \( v^1 = \xi_1 \cdots \xi_n \) and \( v^2 = \xi'_1 \cdots \xi'_m \) for \( \xi_1, \cdots, \xi_n, \xi'_1, \cdots, \xi'_m \in g_2 \). Since left translation is given by \( \psi \triangleright f = f(\psi) \), we observe \( \xi \cdot f = f(\xi) (f(\psi)) \), which evidently implies that \( \langle f, \xi_1 \cdots \xi_n \rangle = \xi_1(f(\psi)) \cdots \xi_n(f(\psi)) \).

Therefore

\[
\langle f, v^1 v^2 \rangle = f^*(v^1 v^2) = \xi_1(f(\psi)) \cdots \xi_n(f(\psi)) \xi'_1(f(\psi)) \cdots \xi'_m(f(\psi)) =
\]

\[
\langle f, v^1 \rangle \langle f, v^2 \rangle = f^*(v^1) \langle f, v^2 \rangle = \varepsilon(f).
\]

We easily see that

\[
\langle f, 1 \rangle = (1 \cdot f)(e_2) = f(e_2) = \varepsilon(f).
\]

To show the other compatibility, we first observe that

\[
\langle fg, \xi \rangle = (fg)^*(\xi) = \xi(f \cdot g) =
\]

\[
\langle f, g(\xi) \rangle + f(e_1) \xi(g) = f^*(\xi) g^*(1) + f^*(1) g^*(\xi) = \langle f, \xi(1) \rangle \langle g, \xi(2) \rangle
\]

and by induction

\[
\langle fg, v \xi \rangle = (fg)^*(v \xi) =
\]

\[
(f(\psi) g(\xi)) (v(\psi)) + f^*(v(\xi)) g^*(v(\xi)) = \varepsilon((v \xi)(1)) (v \xi)(2) = \langle f, (v \xi) \rangle \langle g, (v \xi) \rangle.
\]

Finally we prove the following proposition.

Proposition 2.24. The pairing \( \langle \cdot, \cdot \rangle : \mathcal{P}(G_2) \times U(g_2) \to \mathbb{C} \) is \( U(g_1)-balanced \). In other words

\[
\langle f, v \triangleright X \rangle = \langle X \triangleright f, v \rangle.
\]
Proof. The right action of $G_1$ on $g_2$ leads to the coaction

$$\nabla : g_2 \rightarrow \mathcal{P}(G_1) \otimes g_2,$$

$$\xi \mapsto \xi_{<1>} \otimes \xi_{<0>},$$

such that $\xi_{<1>} (\varphi) \xi_{<0>} = \xi \varphi$. \hfill (2.131)

We first observe that

$$\xi_{<1>} (\varphi) \xi_{<0>} (f) = (\xi \varphi) (f) = \xi (\varphi f) = \xi (f_{<0>} f_{<1>} (\varphi)) = \xi (f_{<0>}) f_{<1>} (\varphi),$$

that is,

$$\xi_{<1>}, \xi_{<0>} (f) = \xi (f_{<0>}) f_{<1>}.$$ \hfill (2.132)

Therefore,

$$\langle f, \xi \ll X \rangle = \xi (f_{<0>}) X (f_{<1>}) = \xi (f_{<0>}) X (f_{<1>}) = \xi (X \triangleright f) = \langle X \triangleright f, \xi \rangle.$$ \hfill (2.133)

We finish the proof by induction as follows

$$\langle X \triangleright f, v \xi \rangle = (X \triangleright f) \ast (v \xi) = ((X \triangleright f)_{(1)}) \ast (v) ((X \triangleright f)_{(2)}) \ast (\xi) =$$

$$(X_{<0>} \triangleright f_{(1)}) \ast (v)(X_{<1>} \cdot f_{(2)}) \ast (\xi) + (f_{(1)}) \ast (v)(X \triangleright f_{(2)}) \ast (\xi) =$$

$$(X_{<0>} \triangleright f_{(1)}) \ast (v) \xi (X_{<1>} \cdot f_{(2)}) + (f_{(1)}) \ast (v) \xi (X \triangleright f_{(2)}) =$$

$$(X_{<0>} \triangleright f_{(1)}) \ast (v)[\xi (X_{<1>})] f_{(2)} (e_2) + X_{<1>} (e_2) \xi (f_{(2)})] +$$

$$(f_{(1)}) \ast (v) (\xi \ll X) (f_{(2)})$$

$$f \ast (v \ll (\xi \gg X)) + f \ast ((v \ll X) \xi) = f \ast (v (\xi \ll X)) = f \ast (v \xi \ll X) =$$

$$\langle f, v \xi \ll X \rangle.$$ \hfill (2.134)

\hfill \blacksquare

3 Hopf cyclic coefficients

In this section we first recall the definition of modular pair in involution (MPI) and stable-anti-Yetter-Drinfeld (SAYD) module from \cite{7} and \cite{11} respectively. In Subsection 3.1 we canonically associate a modular pair in involution over $\mathcal{F} \bowtie U(g)$ to any $g$-Hopf algebra $\mathcal{F}$. This modular pair in involution plays an important rôle in the last section. In Subsection 3.2 we characterize a subcategory of the category of SAYD modules over a Lie-Hopf
algebra and call them induced SAYD modules. In Subsection 3.3, we com-
pletely determine the induced modules over the geometric Hopf algebras that
we constructed in Subsections 2.3, 2.4, and 2.5, based on the modules over
the geometric object on which the Hopf algebra is constructed.

Let $H$ be a Hopf algebra. By definition, a character $\delta : H \to \mathbb{C}$ is an
algebra map. A group-like $\sigma \in H$ is the dual object of the character, i.e,
$\Delta(\sigma) = \sigma \otimes \sigma$. The pair $(\delta, \sigma)$ are called modular pair in involution \[7\] if

$$\delta(\sigma) = 1, \quad \text{and} \quad S^2_{\delta} = Ad_{\sigma},$$

where $Ad_{\sigma}(h) = \sigma h \sigma^{-1}$ and $S_{\delta}$ is defined by

$$S_{\delta}(h) = \delta(h_{(1)}) S(h_{(2)}).$$

We recall from \[11\] the definition of a right-left stable-anti-Yetter-Drinfeld
module over a Hopf algebra $H$. Let $M$ be a right module and left comodule
over a Hopf algebra $H$. We say it is stable-anti-Yetter-Drinfeld (SAYD) if

$$\nabla(m \cdot h) = S(h_{(3)}) m_{<12>} h_{(1)} \otimes m_{<3>} h_{(2)}, \quad m_{<0>} m_{<12>} = m,$$

for any $m \in M$ and $h \in H$. It is shown in \[11\] that any MPI defines a one
dimensional SAYD module and all one dimensional SAYD modules come this
way.

### 3.1 Canonical MPI associated to Lie-Hopf algebras

In this subsection, we associate a canonical modular pair in involution to any
Hopf algebra we constructed in the previous section. More generally let $F$
be a $g$-Hopf algebra. We associate a canonical modular pair in involution to
the Hopf algebra $U(g)$. We define

$$\delta_{g} : g \to \mathbb{C}, \quad \delta_{g}(X) = Tr(Ad_{X}).$$

It is known that $\delta_{g}$ is a derivation of Lie algebras. So we extend $\delta_{g}$ to an
algebra map on $U(g)$ which is again denoted by $\delta_{g}$. We extend $\delta_{g}$ to an
algebra map $\delta := \varepsilon \triangleleft \delta_{g} : F \triangleright U(g) \to \mathbb{C}$ by

$$\delta(f \triangleright u) = \varepsilon(f) \delta_{g}(u).$$
Let us introduce a canonical element $\sigma_\mathcal{F} \in \mathcal{F}$ which plays an important rôle in our study. For a $\mathfrak{g}$-Hopf algebra $\mathcal{F}$, we recall $f_j^i \in \mathcal{F}$ defined in (2.24) for a fixed basis $X_1, \ldots, X_m$ of $\mathfrak{g}$, i.e,

$$\nabla(X_j) = \sum_{i=1}^{m=\dim \mathfrak{g}} X_i \otimes f_j^i. \quad (3.6)$$

As it is shown, $f_j^i$ are independent of $\{X_1, \ldots, X_m\}$ as a basis for $\mathfrak{g}$. Now we define,

$$\sigma_\mathcal{F} := \det M_f = \sum_{\pi \in S_m} (-1)^{\pi} f_{\pi(1)}^{\pi(1)} \cdots f_{\pi(m)}^{\pi(m)}.$$

(3.7)

where $M_f \in M_m(\mathcal{F})$ is the matrix $[f_j^i] \in M_m(\mathcal{F})$.

**Lemma 3.1.** The element $\sigma_\mathcal{F} \in \mathcal{F}$ is group-like.

**Proof.** We know that $\Delta(f_j^i) = \sum_k f_k^j \otimes f_k^i$. So,

$$\Delta(\sigma_\mathcal{F}) = \sum_{\pi \in S_m} \sum_{i_1, \ldots, i_m} (-1)^{\pi} f_{i_1}^{\pi(1)} \cdots f_{i_m}^{\pi(m)} \otimes f_1^{i_1^1} \cdots f_m^{i_m^m} = \sum_{\pi \in S_m} \sum_{i_1, \ldots, i_m} (-1)^{\pi} f_{i_1}^{\pi(1)} \cdots f_{i_m}^{\pi(m)} \otimes f_1^{i_1^1} \cdots f_m^{i_m^m} + \sum_{\pi \in S_m} \sum_{i_1, \ldots, i_m} (-1)^{\pi} f_{i_1}^{\pi(1)} \cdots f_{i_m}^{\pi(m)} \otimes f_1^{i_1^1} \cdots f_m^{i_m^m}.$$

One uses the fact that $\mathcal{F}$ is commutative to prove that the second sum is zero. We now deal with the the first sum. We associate a unique permutation $\mu \in S_m$ to each distinct $m$-tuple $(i_1, \ldots, i_m)$ by the rule $\mu(j) = i_j$. So we have

$$\Delta(\sigma_\mathcal{F}) = \sum_{\pi \in S_m} \sum_{\mu \in S_m} (-1)^{\pi} f_1^{\pi(1)} \cdots f_m^{\pi(m)} \otimes f_1^{\mu(1)} \cdots f_m^{\mu(m)} = \sum_{\pi \in S_m} \sum_{\mu \in S_m} (-1)^{\pi} (-1)^{\mu} f_1^{\pi+1(1)} \cdots f_m^{\pi+1(m)} \otimes f_1^{\mu+1(1)} \cdots f_m^{\mu+1(m)} = \sum_{\eta, \mu \in S_m} (-1)^{\eta} f_1^{\eta(1)} \cdots f_m^{\eta(m)} \otimes f_1^{\mu(1)} \cdots f_m^{\mu(m)} = \sigma_\mathcal{F} \otimes \sigma_\mathcal{F}.$$

$\square$
One easily sees that $\sigma := \sigma_F \triangleright \bowtie 1$ is a group-like element in the Hopf algebra $\mathcal{F} \triangleright \bowtie U(g)$.

**Theorem 3.2.** For any $g$-Hopf algebra $\mathcal{F}$, the pair $(\delta, \sigma)$ is a modular pair in involution for the Hopf algebra $\mathcal{F} \triangleright \bowtie U(g)$.

**Proof.** Let us first do some preliminary computations. For an element $1 \triangleright \bowtie X_i \in \mathcal{H} := \mathcal{F} \triangleright \bowtie U(g)$, the action of iterated comultiplication $\Delta^{(2)}$ is calculated by

$$\Delta^{(2)}(1 \triangleright \bowtie X_i) = (1 \triangleright \bowtie X_i)_{(1)} \otimes (1 \triangleright \bowtie X_i)_{(2)} \otimes (1 \triangleright \bowtie X_i)_{(3)}$$

$$= 1 \triangleright \bowtie X_i_{<0>} \otimes X_{i<1>} \otimes 1 \otimes X_{i<2>} \otimes \triangleright \bowtie 1 + 1 \triangleright \bowtie 1 \otimes 1 \triangleright \bowtie 1 \otimes 1 \triangleright \bowtie X_i.$$

By definition of the antipode [2,13], we observe

$$S(1 \triangleright \bowtie X_i) = (1 \triangleright \bowtie S(X_{i<0>}))(S(X_{i<1>}) \triangleright \bowtie 1) = -(1 \triangleright \bowtie X_j)(S(f_i^j) \triangleright \bowtie 1)$$

$$= -X_{j(1)} \triangleright \bowtie S(f_i^j) \triangleright \bowtie X_{j(2)} = -X_j \triangleright \bowtie S(f_i^j) \triangleright \bowtie 1 - S(f_i^j) \triangleright \bowtie X_j,$$

and hence

$$S^2(1 \triangleright \bowtie X_i) = -S(X_j \triangleright \bowtie S(f_i^j)) \triangleright \bowtie 1 - (1 \triangleright \bowtie S(X_{j<0>}) \cdot (S(f_i^j)X_{j<1>}) \triangleright \bowtie 1)$$

$$= -S(X_{j<1>})(X_{j<0>} \triangleright \bowtie f_i^j) \triangleright \bowtie 1 + (1 \triangleright \bowtie X_k) \cdot (f_i^jS(f_j^k) \triangleright \bowtie 1)$$

$$= -S(f_j^k)(X_k \triangleright \bowtie f_i^j) \triangleright \bowtie 1 + 1 \triangleright \bowtie X_i = -S(f_j^k)f_i^j \triangleright \bowtie 1 + 1 \triangleright \bowtie X_i.$$

Finally, for the twisted antipode $S_\delta : \mathcal{H} \rightarrow \mathcal{H}$, we simplify its square action by

$$S_\delta^2(h) = \delta(h_{(1)})\delta(S(h_{(2)}))S^2(h_{(2)}), \quad h \in \mathcal{H}.$$  \hspace{1cm} (3.8)

We aim to prove that

$$S_\delta^2 = \text{Ad}_\sigma.$$  \hspace{1cm} (3.9)

Since the twisted antipode is anti-algebra map, it is enough to prove (3.9) is held for the elements of the form $1 \triangleright \bowtie X^i$ and $f \triangleright \bowtie 1$. For the latter elements, it is seen that $S_\delta(f \triangleright \bowtie 1) = S(f) \triangleright \bowtie 1$. Hence there is nothing to prove, since $S^2(f) = f = \sigma f \sigma^{-1}$. According to the above preliminary calculations, we observe that

$$S_\delta^2(1 \triangleright \bowtie X_i) = \delta_\delta(X_j)(f_i^j \triangleright \bowtie 1) + (1 \triangleright \bowtie X_i) - S(f_j^k)f_i^j \triangleright \bowtie 1 - \delta_\delta(X_i)1 \triangleright \bowtie 1.$$  \hspace{1cm} (3.10)
Multiplying \( S(f^k_j) \) to both hand sides of the structure identity (2.28) which is recalled here,

\[
f^j_{i,k} - f^j_{k,i} = \sum_{r,s} C^j_{s,r} f^s_k f^r_i - \sum_l C^j_{i,k} f^l_j,
\]

we obtain the following expression

\[
-S(f^k_j) f^j_{i,k} = -S(f^k_j) f^j_{k,i} + \sum_{r,s} C^j_{s,r} S(f^k_j) f^r_i f^s_s - \sum_l C^j_{i,k} S(f^k_j) f^l_j
\]

\[
= -S(f^k_j) f^j_{i,k} + \sum_{s,j} C^j_{s,j} f^s_i - \sum_l C^j_{i,l} 1_F = -S(f^k_j) f^j_{i,k} + \delta_g(X_s) f^s_s - \delta_g(X_l) 1_F.
\]

(3.12)

Combining (3.12) and (3.10) we get

\[
S_2 f^j_{i,k} = -S(f^k_j) f^j_{k,i} + 1_F + 1_F
\]

(3.13)

On the other hand, since \( g \) acts on \( F \) by derivation, we see that

\[
0 = X_i \triangleright (S(f^k_j) f^j_{i,k}) = f^j_k (X_i \triangleright S(f^k_j)) + S(f^k_j)(X_i \triangleright f^j_k)
\]

\[
= f^j_k (X_i \triangleright S(f^k_j)) + S(f^k_j) f^j_{i,k}.
\]

(3.14)

From (3.14) and (3.13) we deduce that

\[
S_2 f^j_{i,k} (X_i \triangleright S(f^k_j)) = f^j_k (X_i \triangleright S(f^k_j)) + 1_F + 1_F X_i
\]

(3.15)

Now we consider the element

\[
\sigma^{-1} = det[S(f^k_j)] = \sum_{\pi \in S_m} (-1)^\pi S(f^{\pi(1)}_1) S(f^{\pi(2)}_2) \cdots S(f^{\pi(m)}_m),
\]

(3.16)

and by using the fact that \( g \) acts on \( F \) by derivation we observe that

\[
X_i \triangleright \sigma^{-1} = X \triangleright det[S(f^k_j)]
\]

\[
= \sum_{1 \leq j \leq m, \pi \in S_m} (-1)^\pi S(f^{\pi(1)}_1) S(f^{\pi(2)}_2) \cdots X_i \triangleright S(f^{\pi(j)}_j) \cdots S(f^{\pi(m)}_m).
\]

(3.17)

Since \( \sigma = det[f^j_i] \) and \( F \) is commutative, we observe \( \sigma = det[f^j_i]^T \). Here \( [f^j_i]^T \) is meant the transpose of the matrix \( [f^j_i] \). We can then conclude that

\[
\sigma (\sum_{\pi \in S_m} (-1)^\pi [X_i \triangleright S(f^{\pi(1)}_1)] S(f^{\pi(2)}_2) \cdots S(f^{\pi(m)}_m)) = f^1_k (X_i \triangleright S(f^k_1)),
\]

(3.18)
which implies
\[ \sigma(X_i \triangleright \sigma^{-1}) = f_k^j(X_i \triangleright S(f_j^k)). \] (3.19)

Finally, by
\[ \text{Ad}_\sigma(1 \bowtie X_i) = (\sigma \bowtie 1)(1 \bowtie X_i)(\sigma^{-1} \bowtie 1) = \sigma X_i \triangleright \sigma^{-1} \bowtie 1 + 1 \bowtie X_i, \]
followed we substitute (3.19) in (3.15) finishes the proof verifying
\[ S_\delta^2(1 \bowtie X_i) = \text{Ad}_\sigma(1 \bowtie X_i). \] (3.20)

3.2 Induced Hopf cyclic coefficients

We investigate the category of SAYD modules over the Hopf algebra associated to a \( g \)-Hopf algebra \( F \), i.e., \( \mathcal{H} := F \bowtie U(g) \). We determine a subcategory of it whose objects are called by us induced SAYD modules over \( \mathcal{H} \). Our strategy is to find a YD module over \( \mathcal{H} \), simple enough to work with and rich enough to give us a representation of the geometric ambient object i.e., the Lie algebra, Lie group, and algebraic group that we started with. Next, we tensor the induced YD module with the canonical modular pair in involution and use [11, Lemma 2.3] to get our desired SAYD module.

Let \( M \) be a left \( g \)-module and a right \( \mathcal{F} \)-comodule via \( \nabla_M : M \to M \otimes \mathcal{F} \). We say that \( M \) is an induced \((g, \mathcal{F})\)-module if
\[ \nabla_M(X \cdot m) = X \cdot \nabla_M(m). \] (3.21)

Here, as before, \( X \cdot (m \otimes f) = X_{<0>} m \otimes X_{<1>} f + m \otimes X \triangleright f \). One extend the action of \( g \) on \( M \) to an action of \( U(g) \) on \( M \) in the natural way. We observe that

\textbf{Lemma 3.3.} The coaction \( \nabla : M \to F \otimes M \) is \( U(g) \)-linear. In other words,
\[ \nabla_M(u \cdot m) = u \cdot \nabla_M(m) := u^{(1)}_{<0>} \cdot m_{<0>} \otimes u^{(1)}_{<1>} \cdot (u^{(2)} \triangleright m_{<1>}). \] (3.22)

\textit{Proof.} For any \( X \in g \), the condition (3.22) is obviously satisfied. Let assume that it is satisfied for \( u^1, u^2 \in U(g) \) and any \( m \in M \) we show it is also held.
for $u^1 u^2$ and any $m \in M$. Using (2.11) we observe

\[
\nabla_M(u^1 u^2 \cdot m) = u^{1(1)}(u^2 \cdot m)_{<0>} \otimes u^{1(2)}_{<1>} (u^{2(1)} \triangleright (u^{2(1)} \triangleright (u^{2(1)} \triangleright m_{<1>}))) = \\
\]

\[
u^{1(1)}_{<0>} u^{2(1)}_{<0>} \cdot m_{<0>} \otimes u^{1(2)}_{<1>} (u^{2(1)} \triangleright u^{2(1)}_{<1>} (u^{2(1)} \triangleright m_{<1>}))) = \\
\]

\[
u^{1(1)}_{<0>} u^{2(1)}_{<0>} \cdot m_{<0>} \otimes u^{1(2)}_{<1>} (u^{2(1)} \triangleright u^{2(1)}_{<1>} (u^{1(3)} u^{2(1)} \triangleright m_{<1>})) = \\
(u^{1u^2})_{<0>} \cdot m_{<0>} \otimes (u^{1u^2})_{<1>} (u^{1u^2})_{<2>} \triangleright m_{<1>}.
\]

Now we let $\mathcal{H}$ act on $M$ from left via

\[
\mathcal{H} \otimes M \rightarrow M, \quad (f \triangleright u)m = \varepsilon(f)um.
\]

Since $\mathcal{F}$ is a Hopf subalgebra of $\mathcal{H}$ one easily extend the coaction of $\mathcal{F}$ on $M$ to a right coaction of $\mathcal{H}$ on $M$ via,

\[
\nabla_M : M \rightarrow M \otimes \mathcal{H}, \quad \nabla_M(m) = m_{<0>} \otimes m_{<1>} \triangleright 1.
\]

For a Hopf algebra $\mathcal{H}$, a left module and right comodule $M$, via $\nabla_M(m) = m_{<1>} \otimes m_{<0>}$ is called Yetter-Drinfeld module (YD for short) if the following condition satisfied

\[
(h_{(2)} \cdot m)_{<0>} \otimes (h_{(2)} \cdot m)_{<1>} h_{(1)} = h_{(1)} \cdot m_{<0>} \otimes h_{(2)} m_{<1>}.
\]

The above condition equivalent to the following one in case $\mathcal{H}$ has invertible antipode.

\[
\nabla_M(h \cdot m) = h_{(2)} \cdot m_{<0>} \otimes h_{(3)} m_{<1>} S^{-1}(h_{(1)}).
\]

**Proposition 3.4.** Let $\mathcal{F}$ be a $\mathfrak{g}$-Hopf algebra and $M$ an induced $(\mathfrak{g}, \mathcal{F})$-module. Then, via the action and coaction defined in (3.24) and (3.25), $M$ is a YD-module over $\mathcal{F} \triangleright U(\mathfrak{g})$.

**Proof.** Since condition (3.26) is multiplicative, it suffices to check it for $f \triangleright 1$ and $1 \triangleright u$. We check it for the latter element as for the former it is obviously satisfied. We see that $\Delta(1 \triangleright u) = 1 \triangleleft u_{(1)}_{<0>} \otimes u_{(1)}_{<1>} \triangleright u_{(2)}$. We use (3.27) and the fact that $U(\mathfrak{g})$ is cocommutative to observe that for $h = 1 \triangleright u$ we have

\[
\]

\[
(h_{(2)} \cdot m)_{<0>} \otimes (h_{(2)} \cdot m)_{<1>} h_{(1)} = \\
((u_{(1)}_{<1>} \triangleright u_{(2)}) \cdot m)_{<0>} \otimes ((u_{(1)}_{<1>} \triangleright u_{(2)}) \cdot m)_{<1>} = \\
u_{(2)}_{<0>} \cdot m_{<0>} \otimes u_{(2)}_{<1>} (u_{(3)} \triangleright m_{<1>}) \triangleright u_{(1)} = \\
h_{(1)} \cdot m_{<0>} \otimes h_{(2)} m_{<1>}.
\]
Now, one knows that by composing $S$ with the action and $S^{-1}$ with the coaction, we change a left-right YD module over $\mathcal{H}$ into a right-left YD module over $\mathcal{H}$. However, since the coaction always lands in $\mathcal{F}$ and $S^{-1}(f ▶ u) = S^{-1}(f) ▶ u$, we conclude that the following defines a right-left YD module over $\mathcal{H}$.

\[
M \otimes \mathcal{H} \rightarrow M, \quad m(f ▶ u) = \varepsilon(f) S(u) \cdot m. \tag{3.29}
\]

\[
\nabla_M : M \rightarrow \mathcal{H} \otimes M, \quad \nabla_M(m) = S(m_{<_1}) ▶ 1 \otimes m_{<_0}. \tag{3.30}
\]

It means that the above right $\mathcal{H}$-module and left $\mathcal{H}$-comodule $M$ satisfies

\[
h(2)(m \cdot h(1))_{<_1} \otimes (m \cdot h(1))_{<_0} = m_{<_1} h(1) \otimes m_{<_0} \cdot h(2), \tag{3.31}
\]

or equivalently,

\[
\nabla_M(m \cdot h) = S^{-1}(h(3)) m_{<_1} h(1) \otimes m_{<_0} \cdot h(2). \tag{3.32}
\]

The main aim of this section is to provide a class of SAYD module over the Hopf algebra $\mathcal{F} ▶ U(\mathfrak{g})$ for a $\mathfrak{g}$-Hopf algebra $\mathcal{F}$.

It is known that tensor product of an AYD module and a YD module is an AYD \cite{11}. Let us give a proof for this fact here. Let $N$ be a right-left AYD module over a Hopf algebra $\mathcal{H}$ and $M$ be a right-left YD module over $\mathcal{H}$. We endow $N \otimes M$ with the the action $(n \otimes m) \cdot h = n h(2) \otimes m h(1)$ we also endow $N \otimes M$ with the coaction $\nabla_{N \otimes M}(n \otimes m) = n_{<_1} m_{<_1} \otimes n_{<_0} \otimes m_{<_0}$. We observe that

\[
\nabla_{N \otimes M}((n \otimes m) \cdot h) = \nabla_{N \otimes M}(n \cdot h(2) \otimes m \cdot h(1)) = \\
S(h(6)) m_{<_1} h(4) S^{-1}(h(3)) m_{<_1} h(1) \otimes n_{<_0} \otimes m_{<_0} \cdot h(2) = \\
S(h(6)) m_{<_1} m_{<_1} h(1) \otimes (n_{<_0} \otimes m_{<_0}) \cdot h(2). \tag{3.33}
\]

In general, even if $N$ and $M$ are stable, there is no grantee that $N \otimes M$ becomes stable. However, our case is special and one easily sees that $N \otimes M$ is stable. Since any MPI is a one dimensional AYD module, $\delta C_{\delta} \otimes M$ is an AYD module over $\mathcal{F} ▶ U(\mathfrak{g})$. We denote $\delta C_{\delta} \otimes M$ by $\delta M_{\delta}$. We simplify the action of $\mathcal{F} ▶ U(\mathfrak{g})$ as follows.

\[
\delta M_{\delta} \otimes \mathcal{F} ▶ U(\mathfrak{g}) \rightarrow \delta M_{\delta}, \quad m \cdot \delta h(u) := \varepsilon(f) \delta(u(0)) S(u(0)) m \tag{3.34}
\]
Its coaction is given by,
\[ \sigma \triangledown_\delta : \sigma M_\delta \rightarrow \mathcal{F} \triangleright \triangleleft U(\mathfrak{g}) \otimes \sigma M_\delta, \]
\[ \triangledown_{(\delta, \sigma)}(m) := \sigma S(m_{<1>}) \triangleright \triangleleft 1 \otimes m_{<0>}. \]
(3.35)

We summarize this subsection in the following.

**Theorem 3.5.** For any \( \mathfrak{g} \)-Hopf algebra \( \mathcal{F} \) and any induced \((\mathfrak{g}, \mathcal{F})\)-module \( M \), there is a SAYD module structure on \( \sigma M_\delta := \sigma \mathbb{C}_\delta \otimes M \) over \( \mathcal{F} \triangleright \triangleleft U(\mathfrak{g}) \) defined in (3.34) and (3.35). Here, \((\delta, \sigma)\) is the canonical modular pair in involution associated to \((\mathfrak{g}, \mathcal{F})\).

### 3.3 Induced Hopf cyclic coefficients in geometric cases

In Section 2, we associated a Hopf algebra to any matched pair of Lie algebras, Lie groups, and affine algebraic groups. These geometric objects admit their own representations. In this section, to such a representation we associate an induced module over the corresponding matched pair and hence a SAYD module over the Hopf algebra in question.

We start with the \( \mathfrak{g}_1 \)-Hopf algebra \( R(\mathfrak{g}_2) \), where \((\mathfrak{g}_1, \mathfrak{g}_2)\) is a matched pair of Lie algebras. A left module \( M \) over the double crossed sum Lie algebra \( \mathfrak{g}_1 \triangleright \triangleleft \mathfrak{g}_2 \) is naturally a left \( \mathfrak{g}_1 \)-module as well as a left \( \mathfrak{g}_2 \)-module. In addition, it satisfy the following compatibility condition.
\[ \zeta \cdot (X \cdot m) - X \cdot (\zeta \cdot m) = (\xi \triangleright X) \cdot m + (\zeta \triangleleft X) \cdot m. \]
(3.36)

Conversely, if \( M \) is left module over \( \mathfrak{g}_1 \) and \( \mathfrak{g}_2 \) satisfying (3.36), then \( M \) is a \( \mathfrak{g}_1 \triangleright \triangleleft \mathfrak{g}_2 \)-module in its natural way, i.e. \( (X \oplus \zeta) \cdot m := X \cdot m + \zeta \cdot m \). This is generalized in the following lemma whose proof is elementary and is omitted.

**Lemma 3.6.** Let \( \mathcal{U} \) and \( \mathcal{V} \) be a mutual pair of Hopf algebras and \( M \) a left module over both Hopf algebras. Then \( M \) is a left \( \mathcal{U} \triangleright \triangleleft \mathcal{V} \)-module via the action
\[ (u \triangleright \triangleleft v) \cdot m := u \cdot (v \cdot m), \]
(3.37)
if and only if
\[ (v_{(1)} \triangleright u_{(1)}) \cdot ((v_{(2)} \triangleleft u_{(2)}) \cdot m) = v \cdot (u \cdot m), \]
(3.38)
for any \( u \in \mathcal{U}, v \in \mathcal{V} \) and \( m \in M \). Conversely, every left module over \( \mathcal{U} \triangleright \triangleleft \mathcal{V} \) comes this way.
Now let $M$ be a left $g := g_1 ∗ g_2$-module such that the restriction of action $M$ is locally finite over $g_2$. We let $U(g_1)$ and $U(g_2)$ act from left on $M$ in the natural way. Hence the resulting actions satisfy the condition (3.38). We know that the category of locally finite $g_2$ modules is equivalent with the category of $R(g)$-comodules [21]. The functor between these two categories is as follows. Let $M$ be a locally finite left $U(g_2)$-module and define the coaction $\nabla_M : M → M ⊗ R(g_2)$ by

$$\nabla(m) = m_{<0>} ⊗ m_{<1>}, \quad \text{if and only if, } \quad v · m = m_{<1>} (v) m_{<0>}. \quad (3.39)$$

Conversely, let $M$ be a right coaction via $\nabla_M : M → M ⊗ R(g_2)$. Then one defines the left action of $U(g_2)$ on $M$ by

$$v · m := m_{<1>} (v) m_{<0>}. \quad (3.40)$$

**Proposition 3.7.** Let $M$ be a left $g := g_1 ∗ g_2$-module such that the restriction of the action results a locally finite $g_2$-module. Then, via the $g_1$ action on $M$, by restriction, and the coaction defined in (3.39), $M$ becomes an induced $(g_1, R(g_2))$-module. Conversely, every induced $(g_1, R(g_2))$-module comes this way.

**Proof.** Let $M$ satisfies the criteria of the proposition. We prove that it is an induced $(g_1, R(g_2))$-module, i.e. $\nabla_M (X ▷ m) = X • \nabla(m)$. Using the compatibility condition (3.36) for $v ∈ U(g_2)$ and $X ∈ g_1$, we observe that

$$(v_{(1)} ▷ X) · (v_{(2)} · m) + (v ↪ X) · m = v · (X · m). \quad (3.41)$$

Translating (3.41) via (3.39), we observe

$$\begin{align*}
(X • \nabla_M(m))(v) &= (X_{<0>} · m_{<0>}) ⊗ (X_{<1>} m_{<1>})(v) + m_{<0>} ⊗ (X ▷ m_{<1>})(v) \\
&= X_{<1>} (v_{(1)}) m_{<1>} (v_{(2)}) X_{<0>} · m_{<0>} + (X ▷ m_{<1>})(v) m_{<0>} \\
&= m_{<1>} (v_{(2)}) (v_{(1)} ▷ X) · m_{<0>} + (X ▷ m_{<1>})(v) m_{<0>} \\
&= (v_{(1)} ▷ X) · (v_{(2)} · m) + m_{<0>} m_{<1>} (v ▪ X) \\
&= (v_{(1)} ▷ X) · (v_{(2)} · m) + (v ▪ X) · m = v ▷ (X · m) \\
&= (X · m)_{<0>} (X · m)_{<1>} (v) = \nabla_M (X · m)(v).
\end{align*} \quad (3.42)$$

Conversely, from a comodule over $R(g_2)$ one obtains a locally finite module over $g_2$ by (3.40). One shows that the compatibility (3.36) follows from $\nabla_M(X · m) = X • \nabla_M(m)$ via (3.42).
Now we investigate the same correspondence for matched pair of Lie groups. Let \((G_1, G_2)\) be a matched pair of Lie groups. Then \((C G_1, C G_2)\) is a mutual pair of Hopf algebras and for \(G = G_1 \bowtie G_2\) we have \(C G = C G_1 \bowtie C G_2\). Therefore, as it is indicated by Lemma 3.6 a module \(M\) over \(G_1\) and \(G_2\) is a module over \(G\) with

\[
(\varphi \cdot \psi) \cdot m = \varphi \cdot (\psi \cdot m), \quad \varphi \in G_1, \psi \in G_2
\] (3.43)

if and only if

\[
(\psi \triangleright \varphi) \cdot ((\psi \triangleleft \varphi) \cdot m) = \psi \cdot (\varphi \cdot m).
\] (3.44)

Now let \(g_1\) and \(g_2\) be the corresponding Lie algebras of the Lie groups \(G_1\) and \(G_2\) respectively. We define the \(g_1\)-module structure of \(M\) as follows

\[
X \cdot m = \frac{d}{dt} \bigg|_{t=0} \exp(tX) \cdot m, \quad \text{for any} \ X \in g_1, m \in M.
\] (3.45)

Assuming \(M\) to be locally finite as a left \(G_2\)-module, we define the right coaction \(\nabla : M \to M \otimes R(G_2)\) of \(R(G_2)\) on \(M\) as usual.

\[
\nabla(m) = m_{<0>} \otimes m_{<1>}, \quad \text{if and only if} \quad \psi \cdot m = m_{<1>}(\psi)m_{<0>}.
\] (3.46)

In this case, we can express an infinitesimal version of the compatibility condition (3.44) as follows

\[
(\psi \triangleright X) \cdot (\psi \cdot m) + \psi \cdot m_{<0>}(X \triangleright m_{<1>})(\psi) = \psi \cdot (X \cdot m),
\] (3.47)

for any \(\psi \in G_2\), \(X \in g_1\) and \(m \in M\).

Now we state the following proposition whose proof is similar to that of Proposition 3.7.

**Proposition 3.8.** For a matched pair of Lie groups \((G_1, G_2)\), let \(M\) be a left \(G = G_1 \bowtie G_2\)-module by (3.43) such that \(G_2\) acts locally finitely. Then by the \(g_1\) action (3.45) and the \(R(G_2)\) coaction (3.46) defined, \(M\) becomes an induced \((g_1, R(G_2))\)-module. Conversely, every induced \((g_1, R(G_2))\)-module comes this way.
We conclude by discussing the case of affine algebraic groups. Let \((G_1, G_2)\) be a matched pair of affine algebraic groups and \(M\) a locally finite polynomial representation of \(G_1\) and \(G_2\). Let us write the dual comodule structures as

\[
\nabla_1 : M \to M \otimes \mathcal{P}(G_1), \quad m \mapsto m_{(0)} \otimes m_{(1)}, \tag{3.48}
\]

and

\[
\nabla_2 : M \to M \otimes \mathcal{P}(G_2), \quad m \mapsto m_{<0>} \otimes m_{<1>}, \tag{3.49}
\]

On the other hand, if we call the \(G_1\)-module structure as

\[
\rho : G_1 \to \text{GL}(M), \tag{3.50}
\]

then we have the \(g_1\)-module structure

\[
\rho^\circ : g_1 \to \mathfrak{gl}(M). \tag{3.51}
\]

The compatibility condition is the same as the Lie group case and the main result is as follows.

**Theorem 3.9.** Let \(M\) be a \(G = G_1 \cdot G_2\)-module as well as a locally finite polynomial representation of \(G_1\) and \(G_2\). Then via the action \(\rho^\circ\) and the coaction \(\nabla_2\), \(M\) becomes an induced \((g_1, \mathcal{P}(G_2))\)-module. Conversely, any induced \((g_1, \mathcal{P}(G_2))\)-module comes this way.

**Proof.** For arbitrary \(\psi \in G_2\), \(X \in g_1\) and any \(m \in M\)

\[
\nabla_2(X \cdot m)(\psi) = \nabla_2(m_{(0)})(\psi)X(m_{(1)}) = m_{(0)}_{<0>} m_{<1>} m_{(1)} \cdot \psi X(m_{(1)}). \tag{3.52}
\]

On the other hand, for any \(\varphi \in G_1\),

\[
m_{(0)}_{<0>} m_{(0)}_{<1>} (\psi)m_{(1)}(\varphi) = (\varphi \cdot m)_{<0>} (\varphi \cdot m)_{<1>} (\psi) = \psi \cdot (\varphi \cdot m) =
\]

\[
(\psi \triangleright \varphi) \cdot ((\psi \triangleleft \varphi) \cdot m) = ((\psi \triangleright \varphi) \cdot m)_{<0>} m_{<1>} (\psi) \triangleleft \varphi =
\]

\[
m_{<0>} (m_{<0>} \triangleleft \psi)(\varphi)m_{<1>} (\psi)m_{<1>} (\varphi) =
\]

\[
m_{<0>} (m_{<1>}) \triangleleft \psi((m_{<0>} \triangleleft \psi) \cdot m_{<1>})(\varphi)
\]

Therefore,

\[
m_{(0)}_{<0>} \otimes m_{(0)}_{<1>} \otimes m_{(1)} = m_{<0>} (m_{(0)}_{<1>} \otimes m_{(1)}) \triangleleft \psi \cdot m_{<1>}. \tag{3.54}
\]
Plugging this result in

\[\nabla_2(X \cdot m)(\psi) = m_{<0>}^{<0>}(\psi)X\left((m_{<0>}^{<1>}(\psi) \cdot m_{<1>}^{<1>})\right) =
\]

\[m_{<0>}^{<0>}(\psi)X((m_{<0>}^{<1>}(\psi))m_{<1>}^{<1>}(e_1) +
\]

\[m_{<0>}^{<0>}(\psi)(m_{<0>}^{<1>}(\psi)(e_1)X(m_{<1>}^{<1>}) =
\]

\[m_{<0>}^{<0>}(\psi)(\psi \triangleright X)(m_{<0>}^{<1>}) + m_{<0>}(X \triangleright m_{<1>})(\psi) =
\]

\[(\psi \triangleright X) \cdot (\psi \cdot m) + m_{<0>}(X \triangleright m_{<1>})(\psi).\]

Also as before, we evidently have

\[(X \bullet \nabla_2(m))(\psi) = (\psi \triangleright X) \cdot (\psi \cdot m) + m_{<0>}(X \triangleright m_{<1>})(\psi). \quad (3.56)\]

So, the equality

\[\nabla_2(X \cdot m) = X \bullet \nabla_2(m), \quad m \in M, \, X \in g_1, \quad (3.57)\]

is proved. \qed

## 4 Hopf cyclic cohomology of commutative geometric Hopf algebras

In this subsection, we compute the Hopf cyclic cohomology of the commutative Hopf algebras \(R(G), \ P(G), \) and \(R(g)\) with coefficients in a suitable comodule.

### 4.1 Preliminaries about Hopf cyclic cohomology

Hopf cyclic cohomology was first defined by Connes and Moscovici as a computational tool to compute the index cocycle defined by the local index formula \([5]\). The original definition of Hopf cyclic cohomology involved only a Hopf algebra and a character with involutive twisted antipode. However, they completed the picture in \([7]\) by bringing the notion of modular pair in involution which is recalled in \([11]\). The theory has had rather rapidly developments in different directions in the last decade. One of the development was to enlarge the space of coefficients to arbitrary dimension and also replace the Hopf algebra with a (co)algebra upon which the Hopf algebra
(co)acts [12]. The suitable coefficients for this development are defined in [11] and called SAYD modules which are also recalled in (3.3). Here, we do not recall the Hopf cyclic cohomology in its full scope, it suffices to mention the case of Hopf algebras and SAYD modules will be. Let $M$ be a right-left SAYD module over a Hopf algebra $H$. Let

$$C^q(H, M) := M \otimes H^{\otimes q}, \quad q \geq 0.$$  \hspace{1cm} (4.1)

We recall the following operators on $C^*(H, M)$

- **face operators** $\partial_i : C^q(H, M) \to C^{q+1}(H, M), \quad 0 \leq i \leq q + 1$
- **degeneracy operators** $\sigma_j : C^q(H, M) \to C^{q-1}(H, M), \quad 0 \leq j \leq q - 1$
- **cyclic operators** $\tau : C^q(H, M) \to C^q(H, M)$

by

\begin{align*}
\partial_0(m \otimes h^1 \otimes \cdots \otimes h^q) &= m \otimes 1 \otimes h^1 \otimes \cdots \otimes h^q, \\
\partial_i(m \otimes h^1 \otimes \cdots \otimes h^q) &= m \otimes h^1 \otimes \cdots \otimes h^i(1) \otimes h^i(2) \otimes \cdots \otimes h^q, \\
\partial_{q+1}(m \otimes h^1 \otimes \cdots \otimes h^q) &= m_{\sigma_\geq} \otimes h^1 \otimes \cdots \otimes h^q \otimes m_{\sigma_{\leq<}} , \\
\sigma_j(m \otimes h^1 \otimes \cdots \otimes h^q) &= m \otimes h^1 \otimes \cdots \otimes \varepsilon(h^{j+1}) \otimes \cdots \otimes h^q, \\
\tau(m \otimes h^1 \otimes \cdots \otimes h^q) &= m_{\sigma_\geq} h^1(1) \otimes S(h^1(2)) \cdot (h^2 \otimes \cdots \otimes h^q \otimes m_{\sigma_{\leq<}}),
\end{align*}

(4.2)

where $H$ acts on $H^{\otimes q}$ diagonally.

The graded module $C^*(H, M)$ endowed with the above operators is then a cocyclic module [12], which means that $\partial_i$, $\sigma_j$ and $\tau$ satisfy the following identities

\begin{align*}
\partial_j \partial_i &= \partial_i \partial_{j-1}, & \text{if} & \quad i < j, \\
\sigma_j \sigma_i &= \sigma_i \sigma_{j+1}, & \text{if} & \quad i \leq j, \\
\sigma_j \partial_i &= \begin{cases} \\
\partial_i \sigma_{j-1}, & \text{if} & \quad i < j \\
\text{Id} & \text{if} & \quad i = j \text{ or } i = j + 1 \\
\partial_{i-1} \sigma_j & \text{if} & \quad i > j + 1,
\end{cases} \\
\tau \partial_i &= \partial_{i-1} \tau, & 1 \leq i \leq q \\
\tau \partial_0 &= \partial_{q+1}, & \tau \sigma_i &= \sigma_{i-1} \tau, & 1 \leq i \leq q \\
\tau \sigma_0 &= \sigma_n \tau^2, & \tau^{q+1} = \text{Id}.
\end{align*}

(4.3)
One uses the face operators to define the Hochschild coboundary

\[ b : C^q(\mathcal{H}, M) \to C^{q+1}(\mathcal{H}, M), \quad \text{by} \quad b := \sum_{i=0}^{q+1} (-1)^i \partial_i \] (4.4)

It is known that \( b^2 = 0 \). As a result, one obtains the Hochschild complex of the coalgebra \( \mathcal{H} \) with coefficients in bicomodule \( M \). Here, the right co-module defined trivially. The cohomology of \((C^\bullet(\mathcal{H}, M), b)\) is denoted by \( H^{\text{coalg}}(H, M) \).

One uses the rest of the operators to define the Connes boundary operator,

\[ B : C^q(\mathcal{H}, M) \to C^{q-1}(\mathcal{H}, M), \quad \text{by} \quad B := \left( \sum_{i=0}^{q} (-1)^i \tau_i \right) \sigma_{q-1} \tau \] (4.5)

It is shown in [3] (can be found also in [4]) that for any cocyclic module \( b^2 = B^2 = (b + B)^2 = 0 \). As a result, one defines the cyclic cohomology of \( \mathcal{H} \) with coefficients in SAYD module \( M \), which is denoted by \( HC^\bullet(\mathcal{H}, M) \), as the total cohomology of the bicomplex

\[ C^{p,q}(\mathcal{H}, M) = \begin{cases} M \otimes \mathcal{H}^ \otimes q, & \text{if } 0 \leq p \leq q, \\ 0, & \text{otherwise.} \end{cases} \] (4.6)

One also defines the periodic cyclic cohomology of \( \mathcal{H} \) with coefficients in \( M \), which is denoted by \( HP^\bullet(\mathcal{H}, M) \), as the total cohomology of direct sum total of the following bicomplex

\[ C^{p,q}(\mathcal{H}, M) = \begin{cases} M \otimes \mathcal{H}^ \otimes q, & \text{if } p \leq q, \\ 0, & \text{otherwise.} \end{cases} \] (4.7)

It can be seen that the periodic cyclic complex and hence cohomology is \( \mathbb{Z}_2 \) graded.

### 4.2 Lie algebra cohomology and Hopf cyclic cohomology

In this subsection, we recall the Lie algebra cohomology and relate it to the Hopf cyclic cohomology of commutative Hopf algebras.
Let \( g \) be a finite dimensional Lie algebra. Let also \( \{ \theta^i \} \) and \( \{ X_i \} \) be a pair of dual basis for \( g^* \) and \( g \). Assume that \( V \) is a right \( g \)-module. The Chevalley-Eilenberg complex of the \((g,V)\) is defined by

\[
\begin{array}{ccc}
V & \xrightarrow{\partial_0} & C^1(g,V) \xrightarrow{\partial_1} C^2(g,V) \xrightarrow{\partial_2} \cdots ,
\end{array}
\]

where \( C^n(g,V) = \text{Hom}(\wedge^n g^*, V) \) is the vector space of all alternating linear maps on \( g^\otimes n \) with values in \( V \).

\[\partial_q(\omega)(X_0, \ldots, X_q) = \sum_{i<j} (-1)^{i+j} \omega([X_i, X_j], X_0 \ldots \widehat{X_i}, \ldots, \widehat{X_j}, \ldots, X_q) + \sum_i (-1)^i \omega(X_0, \ldots, \widehat{X_i}, \ldots X_q)X_i.\]

(4.9)

Alternatively, one identifies \( C^q(g,V) \) with \( V \otimes \wedge^n g^* \) and the coboundary \( \partial_\bullet \) with the following one

\[
\begin{align*}
\partial_0(v) &= vX_i \otimes \theta^i, \\
\partial_q(v \otimes \omega) &= vX_i \otimes \theta^i \wedge \omega + v \otimes \partial_{dR}(\omega).
\end{align*}
\]

(4.10)

Here \( \partial_{dR} : \wedge^n g^* \to \wedge^{n+1} g^* \) is the de Rham coboundary which is a derivation of degree 1 and recalled here by

\[\partial_{dR}(\theta^k) = \frac{1}{2} C_{i,j}^{k} \theta^i \wedge \theta^j.\]

(4.11)

We denote the cohomology of \((C^\bullet(g,V), \partial)\) by \( H^\bullet(g,V) \) and refer to it as the Lie algebra cohomology of \( g \) with coefficients in \( V \). For a Lie subalgebra \( h \subseteq g \) one defines the relative cochains by

\[
C^q(g,h,V) = \{ \omega \in C^q(g,V) | \ i_X \omega = \mathcal{L}_X(\omega) = 0, \ X \in h \},
\]

(4.12)

where,

\[
\begin{align*}
i_X(\omega)(X_1, \ldots, X_q) &= \omega(X, X_1, \ldots, X_q), \\
\mathcal{L}_X(\omega)(X_1, \ldots, X_q) &= \sum (-1)^i \omega([X, X_i], X_1, \ldots, \widehat{X_i}, \ldots, X_q) + \omega(X_1, \ldots, X_q)X.
\end{align*}
\]

(4.13)

(4.14)
One also identifies $C^q(g, h, V)$ with $\text{Hom}_h(\wedge^q(g/h), V)$ which is $(V \otimes \wedge^q(g/h))^h$, where the action of $h$ on $g/h$ is induced by the adjoint action of $h$ on $g$.

It is checked in [2] that the Chevalley-Eilenberg coboundary $\pi_\bullet$ is well defined on $C^n(g, h, V)$. We denote the cohomology of $(C^\bullet(g, h, V), \partial)$ by $H^\bullet(g, h, V)$ and refer to it as the relative Lie algebra cohomology of $h \subseteq g$ with coefficients in $V$.

Let a coalgebra $C$ and an algebra $A$ be in duality, i.e., there is a pairing between $\langle \cdot, \cdot \rangle : C \otimes A \to \mathbb{C}$ compatible with product and coproduct i.e,

$$\langle c, ab \rangle = \langle c_1, a \rangle \langle c_2, b \rangle, \quad \langle c, 1 \rangle = \varepsilon(c). \quad (4.15)$$

By using the duality, one turns any bicomodule $V$ over $C$ into a bimodule over $A$ via

$$av := \langle v_{<1>}, a \rangle v_{<0>}, \quad va := \langle v_{<-1>}, a \rangle v_{<0>}. \quad (4.16)$$

Now we define the following map

$$\theta_{(C,A)} : V \otimes C^\otimes q \to \text{Hom}(A^\otimes q, V),$$

$$\theta_{(C,A)}(v \otimes c^1 \otimes \cdots \otimes c^q)(a_1 \otimes \cdots \otimes a_q) = \langle c^1, a_1 \rangle \cdots \langle c^q, a_q \rangle v. \quad (4.17)$$

**Lemma 4.1.** For any algebra $A$, coalgebra $C$ with a pairing and any $C$-bicomodule $V$, the map $\theta_{(C,A)}$ defined in (4.17) is a map of complexes between Hochschild complex of the coalgebra $C$ with coefficients in the bicomodule $V$ and the Hochschild complex of the algebra $A$ with coefficients in the $A$-bimodule induced by $V$.

**Proof.** The proof is elementary and uses only the pairing property (4.15). \qed

Now let $\mathcal{F}$ be a commutative Hopf algebra with a Hopf pairing with $U(g_2)$, the enveloping Hopf algebra of some Lie algebra $g_2$

$$\langle \cdot, \cdot \rangle : \mathcal{F} \otimes U(g) \to \mathbb{C}, \quad (4.18)$$

satisfying (4.15) and

$$\langle f^1 f^2, u \rangle = \langle f^1, u_{(1)} \rangle \langle f^2, u_{(2)} \rangle, \quad \langle 1, u \rangle = \varepsilon(u), \quad (4.19)$$

$$\langle S(f), u \rangle = \langle f, S(u) \rangle. \quad (4.20)$$

In addition, we assume that $g_2 = h \ltimes l$, where the Lie subalgebra $h$ is reductive and every finite dimensional representation of $h$ is semisimple, and $l$ is an ideal of $g$.
For a \( g_2 \)-module \( V \), we observe that the Lie algebra inclusion \( I \hookrightarrow g_2 \) induces a map of Hochschild complexes, where \( I \) acts on \( V \) by restriction of the action of \( g_2 \)

\[
\pi_1 : \text{Hom}(U(g_2)^{\otimes q}, V) \to \text{Hom}(U(I)^{\otimes I}, V) \tag{4.21}
\]

One uses the antisymmetrization map

\[
\alpha : \text{Hom}(U(I)^{\otimes q}, V) \to C^q(I, V) := V \otimes \Lambda^q I^*, \tag{4.22}
\]

\[
\alpha(\omega)(X_1, \ldots, X_q) = \sum_{\sigma \in S_q} (-1)^{\sigma} \omega(X_{\sigma(1)}, \ldots, X_{\sigma(q)}).
\]

It is known that \( \alpha \) is a map of complexes between Hochschild cohomology of \( U(I) \) with coefficients in \( V \) and the Lie algebra cohomology of \( I \) with coefficients in \( V \).

One then uses the fact that \( h \) acts semisimply to decompose the complex \( C^\bullet(I, V) \) into the weight spaces

\[
C^\bullet(I, V) = \bigoplus_{\mu \in h^*} C^\bullet_\mu(I, V). \tag{4.23}
\]

Since \( h \) acts on \( I \) by derivations, one observes that each \( C^\bullet_\mu(I, V) \) is a complex for its own and the projection \( \pi^\mu : C^\bullet(I, V) \to C^\bullet_\mu(I, V) \) is a map of complexes.

Composing \( \theta_{\mathcal{F}, U(g)} \), \( \pi_1 \) and \( \pi^\mu \) we get a map of complexes

\[
\theta_{\mathcal{F}, U(g), I, \mu} := \pi^\mu \circ \pi_1 \circ \theta_{\mathcal{F}, U(g)} : C^\bullet_{\text{coalg}}(\mathcal{F}, V) \to C^\bullet_\mu(I, V). \tag{4.24}
\]

**Definition 4.2.** Let a commutative Hopf algebra \( \mathcal{F} \) be in a Hopf pairing with \( U(g) \), the enveloping Hopf algebra of \( g \). A decomposition of Lie algebras \( g = h \ltimes I \) is called a \( \mathcal{F} \)-Levi decomposition if the map \( \theta_{\mathcal{F}, I, \mu} \) is a quasi isomorphism for \( \mu = 0 \) and any \( \mathcal{F} \)-comodule \( V \).

**Theorem 4.3.** Let a commutative Hopf algebra \( \mathcal{F} \) be in duality with the enveloping Hopf algebra of a Lie algebra \( g \), and assume that \( g = h \ltimes I \) is an \( \mathcal{F} \)-Levi decomposition. Then the map \( \theta_{\mathcal{F}, I, 0} \) induces an isomorphism between Hopf cyclic cohomology of \( \mathcal{F} \) with coefficients in \( V \) and the relative Lie algebra cohomology of \( h \subseteq g \) with coefficients in \( V \). In other words,

\[
HP^\bullet(\mathcal{F}, V) \cong \bigoplus_{\bullet = i \mod 2} H^i(g, h, V). \tag{4.25}
\]
Proof. First, one uses [24] to observe that for any commutative Hopf algebra $F$ and trivial comodule, the Connes boundary $B$ vanishes in the level of Hochschild cohomology. The same proof works for any comodule and hence we have
\[ HP^* (F, V) \cong \bigoplus_{* = k \mod 2} H^i_{\text{coalg}} (F, V). \] (4.26)
Since $g = \mathfrak{h} \ltimes \mathfrak{l}$ is assumed to be a $F$-Levi decomposition, the map of complexes $\theta_{F, l, 0}$ induces an isomorphism in the level of cohomologies. \hfill \square

4.3 Hopf cyclic cohomology of $R(G)$

In this subsection, we compute the Hopf cyclic cohomology of the commutative Hopf algebra $F := R(G)$, the Hopf algebra of all representative functions on a Lie group $G$, with coefficients in a right comodule $V$ over $F$. Indeed, let $V$ be a right comodule over $F$, or equivalently a representative left $G$-module. Let us recall from [19] that a representative $G$-module is a locally finite $G$-module such that for any finite-dimensional $G$-submodule $W$ of $V$, the induced representation of $G$ on $W$ is continuous. The representative $G$-module $V$ is called representatively injective if for every exact sequence
\[ 0 \rightarrow A \overset{\rho}{\rightarrow} B \overset{\alpha}{\rightarrow} C \overset{\beta}{\rightarrow} V \rightarrow 0 \] (4.27)
of $G$-module homomorphisms between representative $G$-modules $A$, $B$, and $C$, and for every $G$-module homomorphism $\alpha : A \rightarrow V$, there is a $G$-module homomorphism $\beta : B \rightarrow V$ such that $\beta \circ \rho = \alpha$. A representatively injective resolution of the representative $G$-module $V$ is an exact sequence of $G$-module homomorphisms
\[ 0 \rightarrow X_0 \rightarrow X_1 \rightarrow \cdots, \] (4.28)
where each $X_i$ is a representatively injective $G$-module. The representative cohomology group of $G$ with value in the representative $G$-module $V$ is then defined to be the cohomology of
\[ X^G_0 \rightarrow X^G_1 \rightarrow \cdots, \] (4.29)
where $X^G_i$ are the elements of $X_i$ which are fixed by $G$. We denote this cohomology by $H^*_{\text{rep}} (G, V)$. 

Proposition 4.4. For any Lie group $G$ and any representative module $V$, the representative cohomology groups of $G$ with value in $V$ coincide with the coalgebra cohomology groups of the coalgebra $R(G)$ with coefficients in the induced comodule by $V$.

Proof. In [20] it is shown that

$$V \xrightarrow{d_{-1}} V \otimes \mathcal{F} \xrightarrow{d_0} V \otimes \mathcal{F}^\otimes 2 \xrightarrow{d_1} \cdots,$$

(4.30)

is a representatively injective resolution for the representative $G$-module $V$. Here $G$ acts on $V \otimes \mathcal{F}^\otimes n$ by

$$\gamma(v \otimes f^1 \otimes \cdots \otimes f^q) = \gamma v \otimes f^1 \cdot \gamma^{-1} \otimes \cdots \otimes f^q \cdot \gamma^{-1},$$

(4.31)

where $G$ acts on $\mathcal{F}$ by right translation. One look at $V \otimes \mathcal{F}^\otimes q$ as a group cochain with value in the $G$-module $V$ by embedding $V \otimes \mathcal{F}^\otimes q$ into $\mathcal{F}(G^\times q, V)$, the space of all continuous maps from $G \times \cdots \times G$ $q$ times to $V$, by

$$(v \otimes f^1 \otimes \cdots \otimes f^q)(\gamma_1, \ldots, \gamma_q) = f^1(\gamma_1) \cdots f^q(\gamma_q)v.$$

(4.32)

The coboundaries $d_i$ are defined by

$$d_{-1}(v)(\gamma) = v,$$

$$d_i(\phi)(\gamma_1, \ldots, \gamma_{q+1}) = \sum_{i=0}^{q+1} \phi(\gamma_0, \ldots, \hat{\gamma}_i, \ldots, \gamma_q).$$

(4.33)

One then identifies $(V \otimes \mathcal{F}^\otimes q)^G$ with $V \otimes \mathcal{F}^\otimes q-1$

$$v \otimes f^1 \otimes \cdots \otimes f^q \mapsto \varepsilon(f^1)v \otimes f^2 f^3(1) \cdots \otimes f^q(1) \otimes f^q(2) \cdots f^q(q-1) \otimes f^q(q) \otimes f^q(q+1).$$

(4.34)

The complex of the $G$-fixed part of the resolution is

$$V \xrightarrow{\delta_0} V \otimes \mathcal{F} \xrightarrow{\delta_1} \cdots,$$

(4.35)

where the coboundaries $\delta_i$ are defined by

$$\delta_0 : V \rightarrow V \otimes \mathcal{F}, \quad \delta(v) = v_{<0>} \otimes v_{<1>} - v \otimes 1,$$

$$\delta_i : V \otimes \mathcal{F}^\otimes q \rightarrow V \otimes \mathcal{F}^\otimes q+1,$$

$$\delta_i(v \otimes f^1 \otimes \cdots \otimes f^q) = v_{<0>} \otimes v_{<1>} \otimes f^1 \otimes \cdots \otimes f^q + \sum (-1)^i v \otimes f^1 \otimes \cdots \otimes f^i(1) \otimes f^i(2) \otimes \cdots \otimes f^q + (-1)^{q+1} v \otimes f^1 \otimes \cdots \otimes f^q \otimes 1.$$
which is the complex who computes the coalgebra cohomology of $\mathcal{F}$ with coefficients in $\mathcal{F}$-comodule in $V$.

One of course has the version of the above proposition for right $G$-module and corresponding left $\mathcal{F}$-comodule.

Let us recall from [20] that a nucleus of a Lie group $G$ is a simply connected solvable closed normal Lie subgroup $L$ of $G$ such that $G/L$ is reductive. It means that $G/L$ has a faithful representation and every finite dimensional analytic representation of $G/L$ is semisimple. In this case one proves that $G = S \ltimes L$, where $S \coloneqq G/L$ is reductive. Let, in addition, $s \subseteq g$ be Lie algebras of $S$ and $G$ respectively.

For a representative $G$-module $V$ one defines the following map.

$$D_{Gr} : V \otimes \mathcal{F}^\otimes q \to C^q(g, h, V),$$

$$D_{Gr}(v \otimes f_1 \otimes \cdots \otimes f_q)(X_1, \ldots, X_q) = \sum_{\mu \in S_q} (-1)^\mu \left. \frac{d}{dt_1} \right|_{t_1=0} \cdots \left. \frac{d}{dt_q} \right|_{t_q=0} f_1(\exp(t_1 X_{\mu(1)})) \cdots f_q(\exp(t_q X_{\mu(q)}))v.$$  

(4.36)

**Theorem 4.5.** Let $G$ be a Lie group, $V$ a representative $G$-module, $L$ a nucleus of $G$ and $s \subseteq g$ the Lie algebras of $S \coloneqq G/L$ and $G$ respectively. Then the map $D$ defined in (4.36) induces an isomorphism of vector spaces between the periodic Hopf cyclic cohomology of $R(G)$, the Hopf algebra of all representative functions on $G$, with coefficients in the comodule induced by $V$, and the relative Lie algebra cohomology of $s \subseteq g$ with coefficients in $g$-module induced by $V$. In other words,

$$HP^*(R(G), V) \cong \bigoplus_{+ = i \mod 2} H^i(g, s, V).$$  

(4.37)

**Proof.** One knows that $R(G)$ and $U(g)$ are in Hopf duality via

$$\langle f, X \rangle = \left. \frac{d}{dt} \right|_{t=0} f(\exp(tX)), \quad X \in g, f \in R(G).$$  

(4.38)

On the other hand it is easy to see that $D_{Gr}$ is nothing but $\theta_{R(G), U(g), t_0}$ and hence, by applying Proposition [4.4] a map of complexes between complex of representative group cochains of $G$ with value in $V$ and Chevalley-Eilenberg
complex of the Lie algebras $\mathfrak{s} \subseteq \mathfrak{g}$ with coefficients in the $\mathfrak{g}$-module induced by $V$. With a slight modification of the same proof as in [19, Theorem 10.2], one shows that $\mathcal{D}_{Gr}$ induces a quasi-isomorphism. So $\mathfrak{g} = \mathfrak{s} \ltimes \mathfrak{l}$ is a $R(G)$-Levi decomposition and hence the rest follows from Theorem 4.3.

4.4 Hopf cyclic cohomology of $R(\mathfrak{g})$

Let $\mathfrak{g}$ be a Lie algebra and $R(\mathfrak{g})$ the commutative Hopf algebra of representative functions on $U(\mathfrak{g})$ recalled in Subsection 2.3. Let $V$ be a locally finite $\mathfrak{g}$-module or equally an $R(\mathfrak{g})$-comodule. In this subsection we compute the Hopf cyclic cohomology of $R(\mathfrak{g})$ with coefficients in $V$. To this end, we need some new cohomology theory similar to the representative cohomology of Lie groups. We assume all modules are locally finite. The $\mathfrak{g}$-module $V$ is called injective if for every exact sequence
\[
0 \to A \xrightarrow{\rho} B \xrightarrow{\alpha} C \xrightarrow{\beta} V \xrightarrow{\gamma} 0
\]
of $\mathfrak{g}$-module homomorphisms between $\mathfrak{g}$-modules $A$, $B$, and $C$, and for every $\mathfrak{g}$-module homomorphism $\alpha : A \to V$, there is a $\mathfrak{g}$-module homomorphism $\beta : B \to V$ such that $\beta \circ \rho = \alpha$. An injective resolution of the $\mathfrak{g}$-module $V$ is an exact sequence of $\mathfrak{g}$-module homomorphisms
\[
0 \to X_0 \xrightarrow{\rho} X_1 \xrightarrow{\alpha} \cdots
\]
where each $X_i$ is an injective $\mathfrak{g}$-module. The representative cohomology groups of $G$ with value in the $\mathfrak{g}$-module $V$ is then defined to be the cohomology of
\[
X_0^\mathfrak{g} \xrightarrow{\rho} X_1^\mathfrak{g} \xrightarrow{\alpha} \cdots
\]
where $X_i^\mathfrak{g}$ are the the coinvariant elements of $X_i$ i.e,
\[
X_i^\mathfrak{g} = \{ \xi \in X_i \mid X\xi = 0 \text{ for all } X \in \mathfrak{g} \}.
\]
We denote this cohomology by $H^*_\text{rep}(\mathfrak{g}, V)$.
Since the category of locally finite $\mathfrak{g}$-modules and the category of $R(\mathfrak{g})$-comodules are equivalent, we conclude that $H^*_\text{rep}(\mathfrak{g}, V)$ is the same as $\text{Cotor}^*_R(\mathfrak{g})(V, \mathbb{C})$ which is by definition $H^*_\text{calg}(R(\mathfrak{g}), V)$. 49
Let \( l \) be the solvable radical of \( g \), i.e., \( l \) is the unique maximal solvable ideal of \( g \). Levi decomposition implies that \( g = s \ltimes l \), where \( s \) is a semisimple subalgebra of \( g \) called a Levi subalgebra.

We now consider the following map

\[
D_{\text{Alg}} : V \otimes R(g)^{q} \to (V \otimes \wedge^{q} l^{*})^{s},
\]

\[
D_{\text{Alg}}(v \otimes f^{1} \otimes \cdots \otimes f^{q})(X_{1}, \ldots, X_{q}) = \sum_{\sigma \in S_{q}} (-1)^{\sigma} f^{1}(X_{\sigma(1)}) \cdots f^{q}(X_{\sigma(q)})v.
\]

(4.43)

**Proposition 4.6.** Let \( g \) be a Lie algebra with \( g = s \ltimes l \) as a Levi decomposition. Then for any finite dimensional \( g \)-module \( V \), the map \( D_{\text{Alg}} \) induces an isomorphism between the representative cohomology of \( H_{\text{rep}}(g, V) \) and the relative Lie algebra cohomology \( H(g, s, V) \).

**Proof.** First one notes that \( D_{\text{Alg}} \) induces a map of complexes. Now one lets \( G \) be the simply connected Lie group of the Lie algebra \( g \). The Levi decomposition \( g = s \ltimes l \) induces a nucleus of \( G \) and \( G = S \ltimes L \). Since \( G \) is simply connected the representation of \( g \) and representation of \( G \) coincides and any injective resolution of \( g \) is induced by an representatively injective resolution of \( G \). It means that the obvious map \( H_{\text{rep}}(G, V) \to H_{\text{rep}}(g, V) \) is surjective. Since \( V \) is finite dimensional, \( D_{\text{Gr}} : H_{\text{rep}}^{*}(G, V) \to H^{*}(g, s, V) \) is an isomorphism and it factors through \( D_{\text{Alg}} : H_{\text{rep}}(g, V) \to H^{*}(g, s, V) \), the latter map is an isomorphism. \( \square \)

Finally we summarize the result of this subsection as the following theorem.

**Theorem 4.7.** Let \( g \) be a finite dimensional Lie algebra with a Levi decomposition \( g = s \ltimes l \). Then for any finite dimensional \( g \)-module \( V \), the map \( D_{\text{Alg}} \) defined in (4.43) induces an isomorphism of vector spaces between the periodic Hopf cyclic cohomology of \( R(g) \), the Hopf algebra of all representative functions on \( g \), with coefficients in the comodule induced by \( V \), and the relative Lie algebra cohomology of \( s \subseteq g \) with coefficients in \( V \). In other words,

\[
HP^{*}(R(g), V) \cong \bigoplus_{* = i \text{ mod } 2} H^{i}(g, s, V).
\]

(4.44)

**Proof.** We know that \( R(g) \) and \( U(g) \) are in (nondegenerate) Hopf duality via

\[
\langle f, u \rangle = f(u), \quad u \in U(g), f \in R(g).
\]

(4.45)
On the other hand, it is easy to see that $D_{\text{Alg}}$ is $\theta R(g, U(g), t_0)$ and hence, by applying Proposition 4.4, is a map of complexes between the complex of representative group cochains of $G$ with value in $V$ and the Chevalley-Eilenberg complex of the Lie algebras $s \subseteq g$ with coefficients in the $g$-module $V$. By Proposition 4.6, $g = s \rtimes l$ is a $R(g)$-Levi decomposition. Hence the proof is completed by applying Theorem 4.3.

4.5 Hopf cyclic cohomology of $P(G)$

In this section, we compute the Hopf cyclic cohomology of $P(G)$, the Hopf algebra of all complex polynomial functions of an affine algebraic group $G$ over $\mathbb{C}$.

Let $V$ be a finite dimensional polynomial right $G$-module. One defines the polynomial group cohomology of $G$ with coefficients in $V$ as the cohomology of the following complex

$$C^0_{\text{pol}}(G, V) \xrightarrow{\delta} C^1_{\text{pol}}(G, V) \xrightarrow{\delta} \cdots \quad (4.46)$$

Here $C^0_{\text{pol}}(G, V) = V$, and

$$C^q_{\text{pol}}(G, V) = \{ \phi : G \times \cdots \times G \to V \mid \text{$\phi$ is polynomial} \} \quad (4.47)$$

The coboundary $\delta$ is the usual group cohomology coboundary which is recalled here by

$$\delta : V \to C^1_{\text{pol}}(G, V), \quad \delta(v)(\gamma) = v - v \cdot \gamma,$$

$$\delta : C^q_{\text{pol}}(G, V) \to C^{q+1}_{\text{pol}}(G, V), \quad \delta(\phi)(\gamma_1, \ldots, \gamma_{q+1}) = \delta(\phi)(\gamma_2, \ldots, \gamma_{q+1}) +$$

$$\sum_{i=1}^{q} (-1)^i \phi(\gamma_1, \ldots, \gamma_i \gamma_{i+1}, \ldots, \gamma_{q+1}) + (-1)^{q+1} \phi(\gamma_1, \ldots, \gamma_q) \cdot \gamma_{q+1} \quad (4.48)$$

One identifies $C^q_{\text{pol}}(G, V)$ with $V \otimes P(G)^{\otimes q}$ via

$$v \otimes f^1 \otimes \cdots \otimes f^q(\gamma_1, \ldots, \gamma_q) = f^1(\gamma_1) \cdots f^q(\gamma_q)v. \quad (4.49)$$

and observe that the coboundary $\delta$ is identified with the Hochschild coboundary of the coalgebra $P(G)$ with value in the bicomodule $V$, where the right comodule is trivial and the left comodule is induced by the right $G$-module.
The cohomology \( C^\ast_{\text{pol}}(G, V), \delta \) is denoted by \( H_{\text{pol}}(G, V) \). One notes that \( H(G, V) \) can be also calculated by the means of polynomially injective resolutions [17]. Let us recall here polynomially injective resolutions. A polynomial module \( V \) over an affine algebraic group \( G \) is called polynomially injective if for any exact sequence of polynomial modules over \( G \\
0 \longrightarrow A \xrightarrow{\rho} B \xrightarrow{\alpha} C \xrightarrow{\beta} 0 \\
\downarrow \quad \quad \downarrow \quad \quad \downarrow \quad \quad \downarrow \\
V \quad \quad \quad \quad B \quad \quad \quad \quad C 
\) there is a polynomial \( G \)-module homomorphism \( \beta : B \rightarrow V \) such that \( \beta \circ \rho = \alpha \). A polynomially injective resolution for a polynomial module \( V \) over \( G \) is an exact sequence of polynomially injective modules over \( G \\
0 \longrightarrow V \longrightarrow X_0 \longrightarrow X_1 \longrightarrow \cdots , 
\) (4.51)

It is shown in [17] that the \( H_{\text{pol}}(G, V) \) is computed by the cohomology of the \( G \)-fixed part of any polynomially injective resolution i.e., the following complex

\[ X^G_0 \longrightarrow X^G_1 \longrightarrow \cdots . \]

The most natural polynomially injective resolution of a polynomial \( G \)-module \( V \) is the resolution of differential polynomial forms with value in \( V \) which is \( V \otimes \wedge^\ast(\mathfrak{g}^\ast) \otimes \mathcal{P}(G) \), where \( G \) acts by \( \gamma \cdot (u \otimes f) = u \otimes (f \cdot \gamma^{-1}) \), and \( G \) acts on \( \mathcal{P}(G) \) by right translations. This yields the following map of complexes

\[
\mathcal{D}_{\text{Pol}} : C^q_{\text{pol}}(G, V) \rightarrow C^q_{\text{pol}}(\mathfrak{g}, \mathfrak{g}^{\text{red}}, V), \\
\mathcal{D}_{\text{Pol}}(v \otimes f^1 \otimes \cdots \otimes f^q)(X_1, \ldots, X_q) = \\
\sum_{\sigma \in S_q} (-1)^\sigma (X_{\sigma(1)} \cdot f^1)(e) \cdots (X_{\sigma(q)} \cdot f^q)(e) \cdot v .
\]

(4.53)

Here, we identify \( \mathfrak{g} \) the Lie algebra of \( G \) by the left \( G \)-invariant derivations of \( \mathcal{P}(G) \).

Here \( G = G^{\text{red}} \ltimes G^{u} \), is a Levi decomposition of an affine algebraic group \( G \), where \( G^{u} \) is the unipotent radical of \( G \) and \( G^{\text{red}} \) is the maximal reductive subgroup of \( G \). We also use \( \mathfrak{g}^{\text{red}} \) and \( \mathfrak{g}^{u} \) to denote the Lie algebra of \( G^{\text{red}} \) and \( G^{u} \) respectively.
Theorem 4.8. Let $G$ be an affine algebraic group over $\mathbb{C}$ and $V$ be a finite dimensional polynomial $G$-module. Let $G = G^\text{red} \ltimes G^u$ be a Levi decomposition of $G$ and $g^\text{red} \subseteq g$ be the Lie algebras of $G^\text{red}$ and $G$ respectively. Then the map $D_{\text{Pol}}$ defined in 4.53 induces and isomorphism between Hopf cyclic cohomology of $P(G)$, the Hopf algebra of polynomial functions on $G$, with coefficients in the comodule induced by $V$, and the Lie algebra cohomology of $g$ relative to $g^\text{red}$ with coefficients in the $g$-module $V$. In other words

$$HP^\bullet(P(G), V) \cong \bigoplus_{i=0, \text{mod} 2} H^i(g, g^\text{red}, V) \quad (4.54)$$

Proof. It is shown in [17] that $V \otimes \wedge^\bullet g^* \otimes P(G)$ is a polynomially injective resolution for $V$. The comparison between this resolution and the standard resolution, i.e. $V \otimes P(G)^{*,1}$, results the map $D_{\text{Pol}}$. The complete proof which shows that the map $D_{\text{Pol}}$ is an isomorphism between $H^\bullet_{\text{pol}}(G, V)$ and $H^\bullet(g, g^\text{red}, V)$ is Theorem 2.2 in [25]. On the other hand the map $D_{\text{pol}}$ equals to $\theta_{P(G), U(g), g^u, 0}$, where we naturally pair $P(G)$ and $U(g)$ by

$$\langle v, f \rangle = v \cdot f(e). \quad (4.55)$$

This shows that $g = g^\text{red} \ltimes g^u$ is a $P(G)$-Levi decomposition. One then applies the Theorem 4.3 to finish the proof. \qed

5 Hopf cyclic cohomology of noncommutative geometric Hopf algebras

We use the machinery developed for computing the Hopf cyclic cohomology of bicrossed product Hopf algebra by Moscovici and the first author in [29, 30] to compute the Hopf cyclic cohomology of the geometric bicrossed product Hopf algebras we constructed in Subsections 2.3, 2.4, and 2.5. Since most of the improvements done in [30] are for special cases, we need first to advance the machinery to cover the case of Lie-Hopf algebras in general.

5.1 Bicocyclic module associated to Lie Hopf algebras

Let $g$ be a Lie algebra and $F$ a commutative $g$-Hopf algebra. We denote the bicrossed product Hopf algebra $F \rhd \bowtie U(g)$ by $\mathcal{H}$. Let the character $\delta$ and the group-like $\sigma$ be the canonical modular pair in involution defined in (3.3)
and \((5.7)\). In addition, let \(M\) be an induced \((g,F)\)-module and \(\sigma M_\delta\) be the associated SAYD module over \(H\) defined in \((3.34)\) and \((3.35)\).

The Hopf algebra \(U := U(g)\) admits the following right action on \(\sigma M_\delta \otimes F^{\otimes q}\), which plays a key role in the definition of the next bicocyclic module:

\[
(m \otimes \tilde{f})u = \delta_g(u_{(1)}) S(u_{(2)}) \cdot m \otimes S(u_{(3)}) \bullet \tilde{f},
\]

(5.1)

Where \(\tilde{f} := f^1 \otimes \cdots \otimes f^q\), and the left action of \(U\) on \(F^\otimes q\) is defined by

\[
u \bullet (f^1 \otimes \cdots \otimes f^n) := u_{(1)} <_{0>} \triangleright f^1 \otimes u_{(2)} <_{1>} \triangleright u_{(3)} <_{2>} \triangleright \cdots \triangleright u_{(n-1)} <_{n-1>} \triangleright u_{(n)} \triangleright f^n.
\]

(5.2)

One then defines a bicocyclic module \(C^{\bullet \bullet}(U,F,M)\), where

\[
C^{\sigma q}(U,F,\sigma M_\delta) := \sigma M_\delta \otimes U^{\otimes p} \otimes F^\otimes q, \quad p, q \geq 0,
\]

(5.3)

whose horizontal morphisms are given by

\[
\begin{align*}
\overrightarrow{\partial}_0(m \otimes \tilde{u} \otimes \tilde{f}) &= m \otimes 1 \otimes u^1 \otimes \cdots \otimes u^p \otimes \tilde{f} \\
\overrightarrow{\partial}_j(m \otimes \tilde{u} \otimes \tilde{f}) &= m \otimes u^1 \otimes \cdots \otimes \Delta(u^i) \otimes \cdots \otimes u^p \otimes \tilde{f} \\
\overrightarrow{\partial}_{p+1}(m \otimes \tilde{u} \otimes \tilde{f}) &= m \otimes u^1 \otimes \cdots \otimes u^p \otimes 1 \otimes \tilde{f} \\
\overrightarrow{\sigma}_j(m \otimes \tilde{u} \otimes \tilde{f}) &= m \otimes u^1 \otimes \cdots \otimes \epsilon(u^{j+1}) \otimes \cdots \otimes u^p \otimes \tilde{f} \\
\overrightarrow{\tau}(m \otimes \tilde{u} \otimes \tilde{f}) &= \\
&= \delta_g(u_{(1)}) S(u_{(2)}) \cdot m \otimes S(u^1) \cdot (u^2 \otimes \cdots \otimes u^p \otimes 1) \otimes S(u^3) \bullet \tilde{f},
\end{align*}
\]

(5.4)

while the vertical morphisms are defined by

\[
\begin{align*}
\uparrow \partial_0(m \otimes \tilde{u} \otimes \tilde{f}) &= m \otimes \tilde{u} \otimes 1 \otimes \tilde{f}, \\
\uparrow \partial_j(m \otimes \tilde{u} \otimes \tilde{f}) &= m \otimes \tilde{u} \otimes f^1 \otimes \cdots \otimes \Delta(f^j) \otimes \cdots \otimes f^q, \\
\uparrow \partial_{q+1}(m \otimes \tilde{u} \otimes \tilde{f}) &= m_{<q>} \otimes \tilde{u}_{<q>} \otimes \tilde{f} \otimes S(\tilde{u}_{<1>}, m_{<1>}) \sigma, \\
\uparrow \sigma_j(m \otimes \tilde{u} \otimes \tilde{f}) &= m \otimes \tilde{u} \otimes f^1 \otimes \cdots \otimes \epsilon(f^{j+1}) \otimes \cdots \otimes f^n, \\
\uparrow \sigma(m \otimes \tilde{u} \otimes \tilde{f}) &= \\
&= m_{<q>} \otimes \tilde{u}_{<q>} \otimes S(f^1) \cdot (f^2 \otimes \cdots \otimes f^n \otimes S(\tilde{u}_{<1>}, m_{<1>}) \sigma).
\end{align*}
\]

(5.5)

One notes that, by definition, a bicocyclic module is a bigraded module whose rows and columns form cocyclic modules which have their own Hochschild
coboundary and Connes boundary maps. These boundaries and coboundaries are denoted by \( \overrightarrow{B}, \uparrow B, \overrightarrow{b}, \) and \( \uparrow b \), which are demonstrated in the following diagram. We refer the reader to \([29, 30]\) for details on the construction of \( C^{••}(U, F, \sigma M_δ) \).

\[
\begin{array}{c}
\sigma M_δ \otimes U^2 \xrightarrow{\overrightarrow{b}} \sigma M_δ \otimes U^2 \otimes F \xrightarrow{\overrightarrow{b}} \sigma M_δ \otimes U^2 \otimes F^2 \xrightarrow{\overrightarrow{b}} \cdots \\
\uparrow b \downarrow B \uparrow b \downarrow B \uparrow b \downarrow B \uparrow b \downarrow B \cdots \\
\sigma M_δ \otimes \mathcal{U} \xrightarrow{\overrightarrow{b}} \sigma M_δ \otimes \mathcal{U} \otimes \mathcal{F} \xrightarrow{\overrightarrow{b}} \sigma M_δ \otimes \mathcal{U} \otimes \mathcal{F}^2 \xrightarrow{\overrightarrow{b}} \cdots \\
\uparrow b \downarrow B \uparrow b \downarrow B \uparrow b \downarrow B \uparrow b \downarrow B \cdots \\
\sigma M_δ \otimes \mathcal{F} \xrightarrow{\overrightarrow{b}} \sigma M_δ \otimes \mathcal{F}^2 \xrightarrow{\overrightarrow{b}} \cdots
\end{array}
\]

\[(5.6)\]

In the next move, we identify the standard Hopf cocyclic module \( C^{••}(\mathcal{H}, \sigma M_δ) \) with the diagonal subcomplex \( D^{••} \) of \( C^{••}(\mathcal{H}) \). This is achieved by means of the map \( \Psi_{\eta} : D^{••} \to C^{••}(\mathcal{H}, \sigma M_δ) \) together with its inverse \( \Psi_{\eta}^{-1} : C^{••}(\mathcal{H}, \sigma M_δ) \to D^{••} \). They are explicitly defined as follows:

\[
\begin{align*}
\Psi_{\eta}(m \otimes u^1 \otimes \cdots \otimes u^n \otimes f^1 \otimes \cdots \otimes f^n) &= m \otimes f^1 \bowtie u^1_{<0>} \otimes \cdots \otimes \cdots \otimes f^n_{<n-1>} \bowtie u^n_{<n-1>} \\
&= m \otimes u^1_{<0>} \otimes \cdots \otimes u^{n-1}_{<0>} \otimes \otimes f^1 \otimes \\
&\otimes f^2 S(u^1_{<n-1>}) \otimes f^3 S(u^1_{<n-2>} u^2_{<n-2>}) \otimes \cdots \otimes f^n S(u^1_{<1>} \cdots u^{n-1}_{<1>}).
\end{align*}
\]

\[(5.7)\]

respectively

\[
\begin{align*}
\Psi_{\eta}^{-1}(m \otimes f^1 \bowtie u^1 \otimes \cdots \otimes f^n \bowtie u^n) &= m \otimes u^1_{<0>} \otimes \cdots \otimes u^{n-1}_{<0>} \otimes \otimes f^1 \otimes \\
&\otimes f^2 S(u^1_{<n-1>}) \otimes f^3 S(u^1_{<n-2>} u^2_{<n-2>}) \otimes \cdots \otimes f^n S(u^1_{<1>} \cdots u^{n-1}_{<1>}).
\end{align*}
\]

\[(5.8)\]

The bicocyclic module \( (5.6) \) can be further reduced to the bicomplex

\[
C^{••}(g, F, \sigma M_δ) := \sigma M_δ \otimes \wedge^• g \otimes F^{••}, \quad (5.9)
\]

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obtained by replacing the tensor algebra of $\mathcal{U}(g)$ with the exterior algebra of $g$. To this end, recall the action (5.10), which is restricted to $g$ reads

$$X \cdot (f_1 \otimes \cdots \otimes f^q) = (5.10)$$

and define $\partial\delta : \sigma M_\delta \otimes \wedge^p g \otimes \mathcal{F}^\otimes q \to \sigma M_\delta \otimes \wedge^{p-1} g \otimes \mathcal{F}^\otimes q$ as the Lie algebra homology boundary with respect to the action of $g$ on $\sigma M_\delta \otimes \mathcal{F}^\otimes q$ defined by

$$(m \otimes \tilde{f}) \triangleleft X = m \partial g(X) \otimes \tilde{f} - X \cdot m \otimes \tilde{f} - m \otimes X \cdot \tilde{f}.$$  

Via the antisymmetrization map

$$\tilde{\alpha}_g : \sigma M_\delta \otimes \wedge^p g \otimes \mathcal{F}^\otimes q \to \sigma M_\delta \otimes \mathcal{U}^\otimes q \otimes \mathcal{F}^\otimes p,$$  

the pullback of the vertical $b$-coboundary in (5.6) vanishes, while the vertical $B$-coboundary becomes $\partial\sigma g$ (cf. [5, Proposition 7]).

On the other hand, since $\mathcal{F}$ is commutative, the coaction $\nabla : g \to g \otimes \mathcal{F}$, extends from $g$ to a unique coaction $\nabla : \wedge^p g \to \wedge^p g \otimes \mathcal{F}$. After tensoring with the right coaction of $\sigma M_\delta$,

$$\nabla_{\sigma M_\delta \otimes g}(m \otimes X^1 \wedge \cdots \wedge X^q) =$$

$$m_{<o>} \otimes X^1_{<o>} \wedge \cdots \wedge X^q_{<o>} \otimes \sigma^{-1}m_{<1>}X^1_{<1>} \cdots X^q_{<1>}.$$  

The coboundary $b_\mathcal{F}$ is given by

$$b_\mathcal{F}(m \otimes \alpha \otimes f^1 \otimes \cdots \otimes f^p) = m \otimes \alpha \otimes 1 \otimes f^1 \otimes \cdots \otimes f^p +$$

$$\sum_{m \geq 1 \geq p} (-1)^i m \otimes \alpha \otimes f^1 \otimes \cdots \otimes \Delta(f^i) \otimes \cdots \otimes f^p +$$

$$(-1)^{p+1}m_{<o>} \otimes \alpha_{<o>} \otimes f^1 \otimes \cdots \otimes f^p \otimes S(\alpha_{<1>})S(m_{<1>})\sigma,$$  

while the $B$-boundary is

$$B_\mathcal{F} = \left(\sum_{i=0}^{q-1} (-1)^{(q-1)i} \tau_\mathcal{F}\right) \sigma\tau_\mathcal{F},$$  

where

$\tau_\mathcal{F}(m \otimes \alpha \otimes f^1 \otimes \cdots \otimes f^q) =$

$$m_{<o>} \otimes \alpha_{<o>} \otimes S(f^1) \cdot (f^2 \otimes \cdots \otimes f^q \otimes S(\alpha_{<1>})S(m_{<1>})\sigma),$$

$$\sigma(m \otimes \alpha \otimes f^1 \otimes \cdots \otimes f^q) = \varepsilon(f^q)m \otimes \alpha \otimes f^1 \otimes \cdots \otimes f^{q-1}.$$  

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Actually, since $F$ is commutative and $F$ acts on $\sigma M_\delta \otimes \wedge^q g$ trivially, by [24, Theorem 3.22] $B_F$ vanishes in Hochschild cohomology and therefore can be omitted.

We arrive at the bicomplex $C_{**}(g, F, \sigma M_\delta)$, described by the diagram

\[
\begin{array}{ccc}
\vdots & \vdots & \vdots \\
\sigma M_\delta \otimes \wedge^2 g & \xrightarrow{b_F} & \sigma M_\delta \otimes \wedge^2 g \otimes F \\
\downarrow \partial_0 & & \downarrow \partial_0 \\
\sigma M_\delta \otimes g & \xrightarrow{b_F} & \sigma M_\delta \otimes g \otimes F \\
\downarrow \partial_0 & & \downarrow \partial_0 \\
\sigma M_\delta & \xrightarrow{b_F} & \sigma M_\delta \otimes F \\
\end{array}
\]

Referring to [29, Prop. 3.21 and §3.3] for more details, we state the conclusion as follows.

**Proposition 5.1.** The map (5.12) induces a quasi-isomorphism between the total complexes $\text{Tot} C_{**}(g, F, \sigma M_\delta)$ and $\text{Tot} C_{**}((U, F, \sigma M_\delta)$.

In order to convert the Lie algebra homology into Lie algebra cohomology, we shall resort to the Poincaré isomorphism

\[
\mathcal{D}_g = \text{Id} \otimes \partial_g \otimes \text{Id} : M \otimes \wedge^q g^* \otimes F^{\otimes p} \to M \otimes \wedge^m g^* \otimes \wedge^{m-q} g \otimes F^{\otimes p}
\]

\[
\partial_g(\eta) = \varpi^* \otimes \iota(\eta) \varpi,
\]

(5.17)

where $\varpi$ is a covolume element and $\varpi^*$ is the dual volume element. The contraction operator is defined as follows: for $\lambda \in g^*$, $\iota(\lambda) : \wedge^* g \to \wedge^{*-1} g$ is the unique derivation of degree $-1$ which on $g$ is the evaluation map

\[
\iota(\lambda)(X) = \langle \lambda, v \rangle, \quad \forall X \in g,
\]

while for $\eta = \lambda_1 \wedge \cdots \wedge \lambda_q \in \wedge^q g^*$, $\iota(\eta) : \wedge^* g \to \wedge^{*-q} g$ is given by

\[
\iota(\lambda_1 \wedge \cdots \wedge \lambda_q) := \iota(\lambda_q) \circ \cdots \circ \iota(\lambda_1), \quad \forall \lambda_1, \ldots, \lambda_q \in g^*.
\]
Noting that the coadjoint action of \( \mathfrak{g} \) induces on \( \bigwedge^m \mathfrak{g}^* \) the action

\[
\text{ad}^*(X)\varpi^* = \delta_g(X)\varpi^*, \quad \forall X \in \mathfrak{g},
\]

we shall identify \( \bigwedge^m \mathfrak{g}^* \) with \( \mathbb{C}_\delta \) as \( \mathfrak{g} \)-modules.

Let \( \{X_i\} \) be a basis for \( \mathfrak{g} \) and \( \{\theta^i\} \) be its dual basis for \( \mathfrak{g}^* \). We use \( \nabla_\mathfrak{g}(X_i) = X_j \otimes f^i_j \) to define the following left coaction \( \nabla^*_\mathfrak{g} : \mathfrak{g}^* \to \mathcal{F} \otimes \mathfrak{g}^* \) which can be seen as the transpose of the original right coaction \( \nabla_\mathfrak{g} \).

\[
\nabla^*_\mathfrak{g}(\theta^i) = \sum_j f^i_j \otimes \theta^j.
\]

Let us check that it is a coaction. We know from (2.25) that \( \Delta(f^i_j) = f^i_k \otimes f^k_j \).

\[
((\text{Id} \otimes \nabla^*_\mathfrak{g}) \circ \nabla^*_\mathfrak{g})(\theta^i) = \sum_{j,k} f^i_k \otimes f^k_j \otimes \theta^j = ((\Delta \otimes \text{Id}) \circ \nabla^*_\mathfrak{g})(\theta^i).
\]

We extend this coaction on \( \bigwedge^\bullet \mathfrak{g}^* \) diagonally and observe that the result is a left coaction just because \( \mathcal{F} \) is commutative. For \( \alpha := \theta^{i_1} \wedge \cdots \wedge \theta^{i_k} \), it is recorded below by

\[
\alpha_{<1>} \otimes \alpha_{<0>} = \sum_{1 \leq l_1 \leq m} f^{i_1}_{l_1} \cdots f^{i_k}_{l_k} \otimes \theta^{i_1} \wedge \cdots \wedge \theta^{i_k}.
\]

One easily sees that we have \( \nabla^*_\mathfrak{g}(\varpi^*) = \sigma \otimes \varpi^* \). In other words, as a right module and left comodule,

\[
\bigwedge^\dim \mathfrak{g}^* = \sigma \mathbb{C}_\delta.
\]

One uses the antipode of \( \mathcal{F} \) to turn \( \nabla^*_\mathfrak{g} \) into a right coaction, we denote resulting right coaction by \( \nabla_\mathfrak{g}^* \). We then apply this right coaction to endow \( M \otimes \bigwedge^p \mathfrak{g}^* \) with a right coaction as follows,

\[
\nabla_{M \otimes \mathfrak{g}^*} : M \otimes \bigwedge^p \mathfrak{g}^* \to M \otimes \bigwedge^p \mathfrak{g}^* \otimes \mathcal{F},
\]

\[
\nabla_{M \otimes \mathfrak{g}^*}(m \otimes \alpha) = m_{<0>} \otimes \alpha_{<1>} \otimes m_{<1>} S(\alpha_{<1>}).
\]

**Lemma 5.2.** The Poincaré isomorphism connects \( \nabla_{\sigma M \otimes \mathfrak{g}} \) and \( \nabla_{M \otimes \mathfrak{g}^*} \) in the following sense,

\[
\nabla_{M \otimes \mathfrak{g}^*}(m \otimes \omega) = \mathcal{D}_\mathfrak{g}^{-1} \circ \nabla_{\sigma M \otimes \mathfrak{g}} \circ \mathcal{D}_\mathfrak{g}(m \otimes \omega).
\]
Proof. Without loss of generality let $\omega := \theta^{p+1} \land \cdots \land \theta^m$. We observe that

$$\mathcal{D}_g(m \otimes \omega) = m \otimes \omega^* \otimes X_1 \land \cdots \land X_l.$$  \hfill (5.25)

Applying $\nabla^*_{M_2 \land \Theta}$ we get,

$$\nabla^*_{M_2 \land \Theta}(\mathcal{D}_g(m \otimes \omega)) =$$

$$\sum_{1 \leq t_1, t_2 \leq m} m_{<t_1} \otimes \Theta^1 \land \cdots \land \Theta^m \otimes X_{t_1} \land \cdots \land X_{t_2} \otimes m_{<t_2} S(f_{t_1}^1) \cdots S(f_{t_2}^m) f_{t_1}^1 \cdots f_{t_2}^p =$$

$$\sum_{1 \leq t_1, t_2 \leq m} m_{<t_1} \otimes \Theta^1 \land \cdots \land \Theta^p \land \Theta^{p+1} \land \cdots \land \Theta^m \otimes X_{t_1} \land \cdots \land X_{t_2} \otimes m_{<t_2} S(f_{t_1}^1) \cdots S(f_{t_2}^m)$$

$$\sum_{\mu \in S_m} (-1)^\mu m_{<\mu} \otimes \omega^* \otimes X_{\mu(1)} \land \cdots \land X_{\mu(p)} \otimes m_{<\mu} S(f_{\mu(1)}^{p+1}) \cdots S(f_{\mu(m)}^m) =$$

$$\sum_{1 \leq t_1, \ldots, t_p \leq m, \mu \in S_{m-p}} (-1)^\mu m_{<\mu} \otimes \omega^* \otimes X_{t_1} \land \cdots \land X_{t_p} \otimes m_{<t_1} S(f_{t_1}^{p+1}) \cdots S(f_{t_p}^m).$$  \hfill (5.26)

Here in the last part we denote by $\{j_1 < j_2 < \cdots < j_{m-p}\}$ the complement of $\{l_1 < l_2 < \cdots < l_p\}$ in $\{1, \ldots, m\}$. On the other hand,

$$\mathcal{D}_g(\nabla_{M \land \Theta^*}(m \otimes \omega)) =$$

$$\sum_{t_1, \ldots, t_{m-p}} \mathcal{D}_g(m_{<t_1} \otimes \Theta^1 \land \cdots \land \Theta^{m-p} \otimes m_{<t_{m-p}} S(f_{t_1}^{p+1}) \cdots S(f_{t_{m-p}}^m)) =$$

$$\sum_{t_1, \ldots, t_{m-p}} \mathcal{D}_g(m_{<t_1} \otimes \Theta^1 \land \cdots \land \Theta^{m-p} \otimes m_{<t_{m-p}} S(f_{t_1}^{p+1}) \cdots S(f_{t_{m-p}}^m) =$$

$$\sum_{1 \leq j_1, \ldots, j_{m-p} \leq m, \mu \in S_{m-p}} (-1)^\mu \mathcal{D}_g(m_{<\mu} \otimes \Theta^{j_1} \land \cdots \land \Theta^{j_{m-p}} \otimes m_{<\mu} S(f_{j_1}^{p+1}) \cdots S(f_{j_{m-p}}^m) =$$

$$\sum_{1 \leq l_1, \ldots, l_p \leq 1, \mu \in S_{m-p}} (-1)^\mu m_{<\mu} \otimes \omega^* \otimes X_{l_1} \land \cdots \land X_{l_p} \otimes m_{<l_1} S(f_{l_1}^{p+1}) \cdots S(f_{l_p}^m).$$  \hfill (5.27)
By transfer of structure, the bicomplex (5.16) becomes $\left( C^{\ast \ast}(g^*, F, M), \partial_{g^*}, b_F^* \right)$,

\[
\begin{array}{c}
\vdots \\
\partial_{g^*} \\
M \otimes \wedge^2 g^* \xrightarrow{b_F^*} M \otimes \wedge^2 g^* \otimes F \\
\partial_{g^*} \\
M \otimes \wedge^q g^* \xrightarrow{b_F^*} M \otimes \wedge^q g^* \otimes F \otimes F^2 \\
\partial_{g^*} \\
M \xrightarrow{b_F^*} M \otimes F \\
\partial_{g^*} \\
\vdots
\end{array}
\]

(5.28)

The vertical coboundary $\partial_{g^*} : C^{p,q} \rightarrow C^{p,q+1}$ is the Lie algebra cohomology coboundary of the Lie algebra $g$ with coefficients in $M \otimes F^\otimes p$, where the action of $g$ is

\[
(m \otimes f^1 \otimes \cdots \otimes f^p) \triangleright X = \\
- X \cdot m \otimes f^1 \otimes \cdots \otimes f^p - m \otimes X \bullet (f^1 \otimes \cdots \otimes f^p). \quad (5.29)
\]

The horizontal $b$-coboundary $b_F^*$ is precisely defined by

\[
b_F^*(m \otimes \alpha \otimes f^1 \otimes \cdots \otimes f^q) = \\
m \otimes \alpha \otimes 1 \otimes f^1 \otimes \cdots \otimes f^q + \sum_{i=1}^{q} (-1)^i m \otimes \alpha \otimes f^1 \otimes \cdots \otimes \Delta(f^i) \otimes \cdots \otimes f^q + \\
(-1)^{q+1} m_{<0>} \otimes \alpha_{<1>} \otimes f^1 \otimes \cdots \otimes f^i \otimes S(m_{<1>})\alpha_{<-1>}.
\]

(5.30)

For future reference, we record the conclusion.

**Proposition 5.3.** The map $\left(\delta_{\ast\ast}\right)$ induces a quasi-isomorphism between the total complexes $\text{Tot} C^{\ast \ast}(U, F, \sigma M_\delta)$ and $\text{Tot} C^{\ast \ast}(g^*, F)$.

### 5.2 Hopf cyclic cohomology of $F \triangleright U(g_1)$, for $F = R(G_2), R(g_2), \text{ and } \mathcal{P}(G_2)$

As we have seen for a matched pair of Lie algebras $(g_1, g_2)$ there is a right action of $g_1$ on $g_2$ and a left action of $g_2$ on $g_1$ satisfying the compatibility conditions (2.38), . . . , (2.41).
Given such a matched pair, one defines the double crossed sum Lie algebra whose underlying vector space is $g_1 \oplus g_2$ by setting

$$[X \oplus \zeta, Z \oplus \xi] = ([X, Z] + \zeta \triangleright Z - \xi \triangleright X) \oplus ([\zeta, \xi] + \zeta \triangleleft Z - \xi \triangleleft X).$$

Conversely, given a Lie algebra $a$ and two Lie subalgebras $g_1$ and $g_2$ so that $a = g_1 \oplus g_2$ as vector spaces, then $(g_1, g_2)$ forms a matched pair of Lie algebras and $a \cong g_1 \bowtie g_2$ as Lie algebras. In this case, the actions of $g_1$ on $g_2$ and $g_2$ on $g_1$ for $\zeta \in g_2$ and $X \in g_1$ are uniquely determined by

$$[\zeta, X] = \zeta \triangleright X + \zeta \triangleleft X.$$

Let $h \subseteq g_2$ be a $g_1$-invariant subalgebra. Then one easily sees that $a/h \cong g_1 \oplus g_2/h$. In addition, we let $h$ act on $a/h$ by the induced adjoint action i.e.,

$$Ad_\zeta(Z \oplus \bar{\xi}) = [0 \oplus \zeta, Z \oplus \bar{\xi}] = \zeta \triangleright Z \oplus [\zeta, \bar{\xi}] . \quad (5.31)$$

For simplicity we denote $g_2/h$ by $l$. We like to make sure that the Chevalley-Eilenberg coboundary of $g_1$ with coefficients in $M \otimes \wedge^q l^*$ is $h$-linear. To do so, we observe the restriction of the action of $g_2$ on $g_1$ induces an action of $h$ on $g_1$. We assume that this action of $H$ on $g_1$ is given by derivations.

We now introduce the following bicomplex

$$(M \otimes \wedge^2 g_1^*)^b \xrightarrow{\partial} (M \otimes \wedge^2 g_1^* \otimes l^*)^b \xrightarrow{\partial} (M \otimes \wedge^2 g_1^* \otimes \wedge^2 l^*)^b \xrightarrow{\partial} \cdots$$

$$(M \otimes g_1^*)^b \xrightarrow{\partial} (M \otimes g_1^* \otimes l^*)^b \xrightarrow{\partial} (M \otimes g_1^* \otimes \wedge^2 l^*)^b \xrightarrow{\partial} \cdots$$

$$M^h \xrightarrow{\partial} (M \otimes l^*)^b \xrightarrow{\partial} (M \otimes \wedge^2 l^*)^b \xrightarrow{\partial} \cdots \quad (5.32)$$

Here $\partial$ is the relative Lie algebra cohomology of the pair $(g_2, h)$ with coefficients in $M \otimes \wedge^p g_1^*$, where $g_2$ acts on $M$ by restriction and on $g_1$ by its natural action. The vertical coboundary $\uparrow \partial$ is the Lie algebra cohomology of $g_1$ with coefficients on $M \otimes \wedge^q l^*$, where $g_1$ acts on $M$ in the obvious way,
i.e., restriction of the action of $a$ on $M$, and on $I$ by the induced action of $g_1$ on $g_2$. One notes that since action of $h$ on $g_1$ is given by derivations then the vertical coboundary is well-defined.

One identifies

$$(M \otimes \wedge^s(a/h)^*)^h \xrightarrow{\sharp} \bigoplus_{p+q=s} (M \otimes \wedge^p g_1^* \otimes \wedge^q I)^h$$

(5.33)

This isomorphism is implemented by the map

$$\sharp : C^s(g_1 \bowtie g_2, h, M) \to \bigoplus_{p+q=s} (M \otimes \wedge^p g_1^* \otimes \wedge^q I)^h,$$

$$\sharp(\omega)(Z_1, \ldots, Z_p | \zeta_1, \ldots, \zeta_q) = \omega(Z_1 \oplus 0, \ldots, Z_p \oplus 0 \oplus \zeta_1, \ldots, 0 \oplus \zeta_q),$$

whose inverse is given by

$$\sharp^{-1}(\mu \otimes \nu)(Z_1 \oplus \zeta_1, \ldots, Z_{p+q} \oplus \zeta_{p+q}) = \sum_{\sigma \in Sh(p,q)} (-1)^\sigma \mu(Z_{\sigma(1)}, \ldots, Z_{\sigma(p)}) \nu(\zeta_{\sigma(p+1)}, \ldots, \zeta_{\sigma(p+q)}).$$

**Lemma 5.4.** The map $\sharp$ is an isomorphism of complexes.

**Proof.** We see that $\sharp$ is induced by $a^* = g_1^* \oplus g_2^*$. Then one uses (5.31) to show that the vertical and horizontal coboundaries of (5.32) are just restriction of the Chevalley-Eilenberg coboundary of $a$ with coefficients in $V$. It is routine to show that the map $\sharp$ is an isomorphism of complexes between the relative Lie algebra cohomology of the pair $(a, h)$ with coefficients in the $a$-module $M$ and the total complex of the bicomplex (5.32). However, we refer the reader to Lemma 2.7 in [30] for a proof in a very similar situation. \qed

**Definition 5.5.** Let a $g_1$-Hopf algebra $F$ be in Hopf duality with $U(g_2)$ for a matched pair of Lie algebras $(g_1, g_2)$. Then we say $F$ is $(g_1, g_2)$-related if

1. The pairing is $U(g_1)$-balanced, i.e

$$\langle v, u \triangleright f \rangle = \langle v \langle u, f \rangle, \quad f \in F, \quad v \in U(g_2), \quad u \in U(g_1).$$

(5.34)

2. The action of $U(g_2)$ on $U(g_1)$ is compatible with the coaction of $F$ on $U(g_1)$ via the pairing, i.e

$$u_{<0>} \langle v, u_{<1>} \rangle = v \triangleright u, \quad u \in U(g_1), \quad v \in U(g_2).$$

(5.35)
One uses the right action of $\mathfrak{g}_1$ on $\mathfrak{g}_2$ to induce a right action of $U(\mathfrak{g}_1)$ on $U(\mathfrak{g}_2)^{\otimes q}$ as follows,

\[(v^1 \otimes \cdots \otimes v^q) \ast u = v^1 \triangleleft (v^2(1) \cdots v^q(1) \triangleright u(1)) \otimes \cdots \otimes v^{q-1}(q-1) \triangleleft (v^q(q-1) \triangleright u(q-1)) \otimes v^q(q) \triangleright u(q)\]

(5.36)

**Lemma 5.6.** The equation (5.36) defines an action of $U(\mathfrak{g}_1)$ on $U(\mathfrak{g}_2)^{\otimes q}$.

**Proof.** We need to prove that for $\tilde{v} := v^1 \otimes \cdots \otimes v^q \in U(\mathfrak{g}_2)^{\otimes q}$, and $u^1, u^2$ in $U(\mathfrak{g}_1)$, we have $(\tilde{v} \ast u^1) \ast u^2 = (\tilde{v}) \ast (u^1 u^2)$. Indeed, first we use the fact that $U(\mathfrak{g}_2)$ is $U(\mathfrak{g}_1)$-module coalgebra and (2.14) to observe that

\[
\begin{align*}
(v^1 \triangleleft (v^2(1) \triangleright u^1(1)) \otimes v^2(2) \triangleleft u^1(2)) \ast u^2 &= v^1 \triangleleft ((v^2(1) \triangleright u^1(1))(v^2(2) \triangleleft u^1(2))) \otimes (v^2(3) \triangleleft u^1(3) u^2(3)) = \\
v^1 \triangleleft ((v^2(1) \triangleright (u^1(1) u^2(2))) \otimes v^2(2) \triangleleft (u^1(2) u^2(2))) &= (v^1 \otimes v^2) \ast u^1 u^2.
\end{align*}
\]

(5.37)

Then for $\tilde{v} \in U(\mathfrak{g}_2)^{\otimes m}$ and $\tilde{w} = w^1 \otimes \cdots \otimes w^l \in U(\mathfrak{g})^{\otimes l}$, we observe that

\[(\tilde{v} \otimes \tilde{w}) \ast u = \tilde{v} \ast (w^1(1) \cdots w^l(1) \triangleright u(1)) \otimes (w^1(2) \otimes \cdots \otimes w^l(2)) \ast u(2).
\]

(5.38)

This observation and (5.37) completes the proof. \qed

**Proposition 5.7.** Let $(\mathfrak{g}_1, \mathfrak{g}_2)$ be a matched pair of Lie algebras and $\mathcal{F}$ be a $(\mathfrak{g}_1, \mathfrak{g}_2)$-related Hopf algebra. Then the map $\theta_{\mathcal{F}, U(\mathfrak{g}_2)}$ defined in (4.17) is $U(\mathfrak{g}_1)$-linear provided $\mathfrak{g}_1$ acts on $\mathcal{F}^{\otimes q}$ by $\bullet$ defined in (5.10), and on $U(\mathfrak{g}_2)^{\otimes q}$ by $\ast$ defined in (5.36).

**Proof.** Without loss of generality we assume that $V = \mathbb{C}$. We use the Hopf
pairing properties, \((4.34)\), and \((5.35)\) to observe that

\[
\theta_{F, U(g_2)}(u \bullet (f^1 \otimes \cdots \otimes f^q)) (v^1 \otimes \cdots \otimes v^q) = \\
\langle v^1, u(1)_{<0>} \triangleright f^1 \rangle \langle v^2, u(1)_{<12>} (u(2)_{<00>} \triangleright f^1) \rangle \cdots \langle v^q, u(1)_{<q-1>} \cdots u(q-1)_{<1,1>}, (u(q+1) \triangleright f^1) \rangle = \\
\langle v^1, u(1)_{<0>} \triangleright f^1 \rangle \langle v^2(1), u(1)_{<11>} \rangle \langle v^2(2), u(2)_{<00>} \triangleright f^2 \rangle \cdots \\
\langle v^q(1), u(1)_{<q-1>} \rangle \langle v^q(2), u(2)_{<q-2>} \rangle \cdots \langle v^q_{<q-1>}, u(q-1)_{<1,1>} \rangle \langle v^q_{<q,11>}, u(q)_{<q+1>} \triangleright f^q \rangle = \\
\langle v^1, u(1)_{<0>} \triangleright f^1 \rangle \langle v^2(1) \cdots v^q(1)_{<q,11>} \rangle \langle v^2(2)_{<00>} \triangleright f^2 \rangle \langle v^3(2) \cdots v^q(2), u(2)_{<q-2>} \rangle \cdots \\
\cdots \langle v^q-1_{<q-1}, v(q-1)_{<00} \triangleright f^{q-1} \rangle \langle v^q_{q-1} \rangle \langle v^q_{<q,11>}, u(q)_{<q+1>} \triangleright f^q \rangle = \\
\langle v^1 \triangleleft (v^2(1) \cdots v^q(1)_{<q,11>}, v(q-1)_{<q-1}, f^1) \rangle \langle v^2(2) \triangleleft (v^3(2) \cdots v^q(2), u(2)_{<q-2>}, f^2) \rangle \cdots \\
\cdots \langle v^q-1_{<q-1}, v(q-1)_{<q-1} \triangleright u(q-1)_{<00} \rangle \langle v^q_{q-1} \rangle \langle v^q, u(q), f^q \rangle = \\
\theta_{F, U(g_2)}(f^1 \otimes \cdots \otimes f^q)((v^1 \otimes \cdots \otimes v^q) \ast u).
\]

\((5.39)\)

\[\square\]

**Proposition 5.8.** For a matched pair of Lie algebras \((g_1, g_2)\), let \(g_1\) act on \(U(g_2)^{\otimes q}\) by \(\ast\) defined in \((5.36)\), and on \(\wedge^q g_2\) by the intrinsic right action of \(g_1\) on \(g_2\). Then the antisymmetrization map is a \(g_1\)-linear map of complexes between normalized Hochschild cochains of \(U(g_2)\) and Lie algebra cochains of \(g_2\).

**Proof.** It is known that the antisymmetrization map \(\alpha : \text{Hom}(U(g_2)^{\otimes q}, V) \to V \otimes \wedge^q g_2^*\) defined in \((1.22)\), is a map of complexes. In the next proposition we prove that it is actually \(g_1\)-linear. One uses the fact that \(v \triangleleft 1 = \varepsilon(v)\), and that the elements of the Lie algebra is primitive in its enveloping Hopf algebra to see that for any \(\xi^1, \ldots, \xi^q \in g_2\), any \(X \in g_1\), and any normalized

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cochain $\phi$ we have

$$\phi((\xi_1 \otimes \cdots \otimes \xi^q) \ast X) = \phi(\xi^1 \triangleleft (\xi^2_{(1)} \cdots \xi^q_{(1)} \triangleright X) \otimes \xi^2_{(2)} \triangleleft (\xi^3_{(2)} \cdots \xi^q_{(2)} \triangleright X) \otimes \cdots \\otimes \xi^{q-1}_{(q-1)} \triangleleft (\xi^q_{(q-1)} \triangleright 1) \otimes \xi^q_{(q)}) + \cdots$$

$$\xi^1 \triangleleft (\xi^2_{(1)} \cdots \xi^q_{(1)} \triangleright 1) \otimes \xi^2_{(2)} \triangleleft (\xi^3_{(2)} \cdots \xi^q_{(2)} \triangleright X) \otimes \cdots \\otimes \xi^{q-1}_{(q-1)} \triangleleft (\xi^q_{(q-1)} \triangleright 1) \otimes \xi^q_{(q)}) + \cdots + \xi^1 \triangleleft (\xi^2_{(1)} \cdots \xi^q_{(1)} \triangleright 1) \otimes \xi^2_{(2)} \triangleleft (\xi^3_{(2)} \cdots \xi^q_{(2)} \triangleright 1) \otimes \cdots \\otimes \xi^{q-1}_{(q-1)} \triangleleft (\xi^q_{(q-1)} \triangleright 1) \otimes \xi^q_{(q)})$$

$$\sum_{i=1}^q \phi(\xi^1 \otimes \xi^2 \otimes \cdots \otimes \xi^i \triangleleft (\xi^{i+1}_{(1)} \cdots \xi^q_{(1)} \triangleright X) \otimes \xi^{i+1}_{(2)} \otimes \cdots \otimes \xi^q_{(2)}) =$$

$$\sum_{i=1}^q \phi(\xi^1 \otimes \xi^2 \otimes \cdots \otimes \xi^i \triangleleft (\xi^{i+1}_{(1)} \cdots \xi^q_{(1)} \triangleright X) \otimes \xi^{i+1}_{(2)} \otimes \cdots \otimes \xi^q_{(2)}) +$$

$$\sum_{i=1}^q \phi(\xi^1 \otimes \xi^2 \otimes \cdots \otimes \xi^i \triangleleft (\xi^{i+1}_{(1)} \cdots \xi^q_{(1)} \triangleright X) \otimes \xi^{i+1}_{(2)} \otimes \cdots \otimes \xi^q_{(2)})$$

$$\sum_{i=1}^q \phi(\xi^1 \otimes \xi^2 \otimes \cdots \otimes \xi^i \triangleleft X \otimes \xi^{i+1} \otimes \cdots \otimes \xi^q) + 0.$$  

(5.40)

Now let $(g_1, g_2)$ be a matched pair of Lie algebras and $\mathcal{F}$ a $(g_1, g_2)$-related Hopf algebra. In addition, let $g_2 = h \ltimes l$ be a semi-crossed-sum of Lie algebras, where $h$ is reductive and every $h$-module is semisimple. We define the following map between the bicomplexes (5.28) and (5.32)

$$\mathcal{V} := \theta_{\mathcal{F}, l_0} : M \otimes \wedge^q g_1^* \otimes \mathcal{F}^\otimes q \rightarrow (M \otimes \wedge^p g_1 \otimes \wedge^q l^*)^h$$  

(5.41)

In other words,

$$\mathcal{V}(m \otimes \omega \otimes f^1 \cdots \otimes f^q) \langle X^1, \ldots, X^p | \xi^1, \ldots, \xi^q \rangle =$$

$$\omega(X^1, \ldots, X^p) \sum_{\sigma \in S_q} (-1)^\sigma \langle \xi^{\sigma(1)}, f^1 \rangle \ldots \langle \xi^{\sigma(q)}, f^q \rangle m.$$  

(5.42)
Theorem 5.9. Let \((g_1, g_2)\) be a matched pair of Lie algebras and \(F\) a \((g_1, g_2)\)-related Hopf algebra. Assume that \(g_2 = h \ltimes l\) is a \(F\)-Levi decomposition such that \(h\) is \(g_1\)-invariant and the natural action of \(h\) on \(g_1\) is given by derivations. Then for any \(F\)-comodule and \(g_1\)-module \(M\), the map \(V\) defined in (5.41), is a map of bicomplexes and induces an isomorphism between Hopf cyclic cohomology of \(F \rhd U(g_1)\) with coefficients in \(\sigma M_\delta\) and the Lie algebra cohomology of \(a := g_1 \bowtie g_2\) relative to \(h\) with coefficients in the \(a\)-module induced by \(M\). In other words,

\[
HP^\bullet(F \rhd U(g_1), \sigma M_\delta) \cong \bigoplus_{i=\cdot \mod 2} H^i(g_1 \bowtie g_2, h, M). \tag{5.43}
\]

Proof. First we have to prove that \(V\) commutes with both of the boundaries of our bicomplexes. The commutation of \(V\) with horizontal coboundaries is guaranteed by the fact that \(\theta_{F, U(g_2)}\) is a complex map for \(V = M \otimes \wedge^p g_1^*\). Since \(V\) does not have any affect on \(M \otimes \wedge^p g_1^*\), by the equivalent definition of Chevalley-Eilenberg coboundary in (4.10), to prove that \(V\) commutes with horizontal coboundaries it is necessary and sufficient to show that \(V\) is \(g_1\)-linear. Since \(V\) is made of antisymmetrization map \(\alpha\) and \(\theta_{F, U(g_2)}\), Proposition 5.7 and Proposition 5.8 prove that \(V\) commutes with horizontal coboundaries which are both Lie algebra coboundaries of \(g_1\). Finally, one uses the assumption that \(g_2 = h \ltimes l\) is a \(F\)-Levi decomposition which implies, by definition, that \(V\) induces an isomorphism in the first term of the spectral sequence of the bicomplexes. We then conclude that \(V\) induces an isomorphism in the level of total cohomology of total complexes. The proof is complete since total complex of the (5.28) computes the Hopf cyclic cohomology by Proposition 5.3 and the total complex of (5.32) computes the relative Lie algebra cohomology by Lemma 5.4.

Corollary 5.10. Let \((g_1, g_2)\) be a matched pair of finite dimensional Lie algebras. Assume that \(g_2 = h \ltimes l\) is a \(F\)-Levi decomposition of \(g_2\) such that \(h\) is \(g_1\)-invariant and the action of \(h\) on \(g_1\) is given by derivations. Then for any finite dimensional \(g_1 \bowtie g_2\)-module \(M\) we have

\[
HP^\bullet(R(g_2) \rhd U(g_1), \sigma M_\delta) \cong \bigoplus_{i=\cdot \mod 2} H^i(g_1 \bowtie g_2, h, M). \tag{5.44}
\]

Proof. The main task here to prove that the criteria of Theorem 5.9 are satisfied for \(F := R(g_2)\). It is shown in Proposition 2.10 that \(R(g_2)\) is a \(g_1\)-Hopf algebra.
We know that $\mathcal{F}$ and $U(\mathfrak{g}_2)$ are in a Hopf pairing via (2.57), i.e,
$$\langle v, f \rangle_{\text{Alg}} = f(v), \quad f \in R(\mathfrak{g}_2), \quad v \in U(\mathfrak{g}_2).$$
(5.45)

The pairing (5.45) is by definition, as it is defined in (2.58), $U(\mathfrak{g}_1)$-balanced. The equation (5.45) shows that the coaction of $R(\mathfrak{g}_2)$ on $U(\mathfrak{g}_1)$ is compatible with the pairing. So $R(\mathfrak{g}_2)$ is $(\mathfrak{g}_1, \mathfrak{g}_2)$-related. Finally, Theorem 4.7 shows that any Levi decomposition of Lie algebra $\mathfrak{g}_2$ implies a $R(\mathfrak{g}_2)$-Levi decomposition.

Corollary 5.11. Let $(G_1, G_2)$ be a matched pair of Lie groups. Assume that $L$ is a nucleus of $G$. Let $\mathfrak{h}$, $\mathfrak{g}_1$ and $\mathfrak{g}_2$ denote the Lie algebras of $H := G/L$, $G_1$ and $G_2$ respectively. Let also assume that $\mathfrak{h}$ is $\mathfrak{g}_1$-invariant and the natural action of $\mathfrak{h}$ on $\mathfrak{g}_1$ is given by derivations. Then for any representative $G_1 \rtimes G_2$ module $M$, we have
$$HP^\bullet(R(\mathfrak{g}_2) \triangleright U(\mathfrak{g}_1), \, \sigma M) \cong \bigoplus_{i=\bullet \mod 2} H^i(\mathfrak{g}_1 \ltimes \mathfrak{g}_2, \mathfrak{h}, M).$$
(5.46)

Proof. The Hopf algebra map $\theta : R(\mathfrak{g}_2) \rightarrow R(\mathfrak{g}_2)$ defined in (2.87) and the Hopf duality between $R(\mathfrak{g}_2)$ and $U(\mathfrak{g}_2)$ defined in (2.57) guarantee the desired Hopf duality between $R(\mathfrak{g}_2)$ and $U(\mathfrak{g}_2)$ as it is recalled here by
$$\langle f, \xi \rangle_{\mathfrak{g}_2} = \langle \theta(f), \xi \rangle_{\text{Alg}} = \frac{d}{dt} \bigg|_{t=0} f(\exp(t\xi)), \quad \xi \in \mathfrak{g}_2, \quad f \in R(\mathfrak{g}_2).$$
(5.47)

By (2.91), the map $\theta$ is $U(\mathfrak{g}_1)$-linear and hence the pairing (5.47) is $U(\mathfrak{g}_1)$-balanced since the pairing (5.45) is $U(\mathfrak{g}_1)$ balanced. Let us use the Sweedler notation $u_{<0>} \otimes u_{<1>}$ for the coaction of $R(\mathfrak{g}_2)$ on $U(\mathfrak{g}_1)$, and $u_{<\mathfrak{g}_2>} \otimes u_{<\mathfrak{g}_1>}$ for the coaction of $R(\mathfrak{g}_2)$ on $U(\mathfrak{g}_1)$. Now one uses the commutativity of the diagram (2.92), and the compatibility of the coaction of $R(\mathfrak{g}_2)$ on $U(\mathfrak{g}_2)$ with the pairing $\langle \cdot, \cdot \rangle_{\text{Alg}}$ to observe that
$$u_{<0>}(v, u_{<1>})_{\mathfrak{g}_2} = u_{<0>} \langle v, \theta(u_{<0>}) \rangle_{\text{Alg}} = u_{<\mathfrak{g}_2>} \langle v, u_{<\mathfrak{g}_1>} \rangle_{\text{Alg}} = v \triangleright u.$$  (5.48)

So far we have proved that $R(\mathfrak{g}_2)$ is $(\mathfrak{g}_1, \mathfrak{g}_2)$-related. Finally, Theorem 4.7 shows that $\mathfrak{g}_2 = \mathfrak{h} \ltimes \mathfrak{i}$ is a $R(\mathfrak{g}_2)$-Levi decomposition. Here $\mathfrak{i} \subseteq \mathfrak{g}_2$ is the Lie algebra of $L$. We are now ready to apply Theorem 5.9.
Corollary 5.12. Let \((G_1, G_2)\) be a matched pair of connected affine algebraic groups and \(G_2 = G_2^{\text{red}} \times G_2^{\text{u}}\) be a Levi decomposition of \(G_2\). Let \(g_1, g_2^{\text{red}} \subseteq g_2\) be the Lie algebras of \(G_1, G_2^{\text{red}}\) and \(G_2\) respectively. We assume that \(g_2^{\text{red}}\) is \(g_1\) invariant and the natural action of \(g_2^{\text{red}}\) on \(g_1\) is given by derivations. Then for any finite dimensional polynomial module \(M\) over \(G_1 \triangleright G_2\), we have

\[
HP^\bullet(\mathcal{P}(G_2) \triangleright U(g_1), M) \cong \bigoplus_{i=\bullet \mod 2} H^i(g_1 \triangleright g_2, g_2^{\text{red}}, M).
\] (5.49)

Proof. We need to prove that the criteria of Theorem 5.9 are satisfied. To this end, we first observe that \(\mathcal{P}(G_2)\) is in a Hopf duality with \(U(g_2)\) via the pairing defined in (2.123) recalled here by

\[
\langle v, f \rangle_{\text{pol}} = f^*(v) = (v \cdot f)(e), \quad f \in \mathcal{P}(G_2), \ v \in U(g_2). \] (5.50)

By Proposition 2.24, the pairing \(\langle \cdot, \cdot \rangle_{\text{pol}}\) is \(U(g_1)\)-balanced. Since the coaction of \(\mathcal{P}(g_2)\) on \(U(g_1)\) is obtained by the action of \(G_2\) on \(U(g_1)\), we conclude that the Hopf algebra \(\mathcal{P}(G_2)\) is \((g_1, g_2)\)-related. Finally, \(g_2 = g_2^{\text{red}} \ltimes g_2^{\text{u}}\) is a \(\mathcal{P}(G_2)\)-Levi decomposition by Theorem 4.8. Here \(g_2^{\text{u}}\) is the Lie algebra of \(G_2^{\text{u}}\). \(\square\)

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