Quantum-gravitational corrections to the hydrogen atom and harmonic oscillator

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It is shown that the rate of corrections to the hydrogen atom and harmonic oscillator due to profound quantum-gravitational effect of space-time dimension running/reduction coincides well with those obtained by means of the minimum-length deformed quantum mechanics. The rate of corrections are pretty much the same within the accuracy by which we can judge the quantum-gravitational corrections at all. Such a convergence of results makes the concept of space-time dimension running more appreciable. As a remarkable distinction, the energy shift due to dimension reduction has the opposite sign as compared with the correction obtained by means of the minimum-length modified quantum mechanics. Thereby, the sign of total quantum-gravitational correction remains obscure.

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I. INTRODUCTION

Along the development of various scenarios for quantum gravity (QG), the study of QG phenomenology becomes important. One of the first intensive streams for such phenomenological study was triggered by treating the general relativity as an effective field theory\textsuperscript{1}. This effective field theory approach, that is, to consider general relativity as an effective low energy approximation to some as yet unknown fundamental theory of quantum gravity, offers a way to get round the familiar renormalization difficulties of general relativity in the low energy regime\textsuperscript{1}. Using this approach one can make reliable predictions about the radiative QG corrections to various physical quantities in the low energy limit \((E \ll E_P)\)\textsuperscript{1,2}.

Another inspiration for a systematic study of QG phenomenology was the minimum-length modified quantum mechanics\textsuperscript{3} that stems from the generalized uncertainty relations. Generalized uncertainty relations naturally arise in string theory\textsuperscript{4} as well as in some heuristic QG considerations\textsuperscript{5}.

A new door for a versatile study of the QG phenomenology is open by a profound QG effect of space-time dimension running/reduction discovered recently in two different approaches to quantum gravity\textsuperscript{6}. In this paper we study the QG corrections to the hydrogen atom and harmonic oscillator due to dimension running and compare the results with those obtained by means of the minimum-length modified quantum mechanics. Throughout the paper we assume the system of units \(c = 1\). The paper is organized as follows. First, in sections II and III we consider two distinct qualitative approaches for estimating of QG energy shift to the hydrogen atom that gives good understanding of the order of magnitude for this correction. Then, in section IV we review the results obtained for hydrogen atom in the framework of minimum-length modified quantum mechanics. In this approach different results were reported for the hydrogen atom that reflects the technical difficulty of using the KMM quantum mechanics\textsuperscript{3} in practice. So the further detailed elucidating study of these results is in order. Next, in section V we provide a simple consideration of the effect of QG dimension running/reduction without resorting to any particular model of QG, but rather on the basis of a finite space-time resolution that is implied by all QG scenarios. Throughout this consideration a simple analytic expression of running dimension emerges. Using this expression of running dimension we estimate the corrections to the hydrogen atom and also consider how the dimension running affects the relativistic and QED radiative corrections. From the very outset let us notice that comparing all approaches considered throughout this paper, an unique result emerges for the rate of QG energy shift to the hydrogen atom, but the signs of corrections that come in the framework of minimum-length modified quantum mechanics and due to dimension running respectively, are opposite. Further, in section VI we consider in a similar manner the corrections to the harmonic oscillator due to minimum-length modified quantum mechanics and the dimension running. Then follows the concluding remarks.

II. SEMI-CLASSICAL TREATMENT

The hydrogen like atom has the well known non-relativistic spectrum

\[ E_n = -\frac{E_R}{n^2}, \quad \text{where} \quad E_R = \frac{Z^2 e^4 m}{2\hbar^2}, \]

with the orbital radii

\[ a_n = n^2 a, \quad \text{where} \quad a = \frac{\hbar^2}{Ze^2}. \]
Throughout the paper we will tacitly assume that the condition of non-relativistic motion of the electron $e^2Z \ll n$ is satisfied. As is well known \[7\], we can successfully estimate the ground state parameters of the hydrogen atom simply by using the position-momentum uncertainty relation

$$\delta x \delta p \geq \frac{\hbar}{2}.$$ 

Certainly, all that should be expected in the framework of this semiclassical treatment is an order-of-magnitude estimate.\[1\] Namely, the minimum of hydrogen energy

$$E = \frac{p^2}{2m} - \frac{Ze^2}{r},$$

(1)

where on the grounds of uncertainty relation $p$ is understood as $p = \hbar \delta p$, occurs at $r = a$ that results in the ground state energy $E = -E_R$. Following this line of discussion let us see how the ground state parameters will change for QG modified uncertainty relation \[2\]

$$\delta x \geq \frac{\hbar}{2 \delta p} + \frac{(\delta x_{\min})^2}{2\hbar} \delta p,$$

which results from the commutation relation

$$[\hat{x}, \hat{p}] = i\hbar(1 + \beta \hat{p}^2), \quad \text{where} \quad \delta x_{\min} = \hbar \sqrt{\beta}.$$ 

Solving for the momentum uncertainty in terms of the distance uncertainty, the equation

$$\delta x = \frac{\hbar}{2 \delta p} + \frac{(\delta x_{\min})^2}{2\hbar} \delta p,$$

gives

$$\delta p = \hbar \frac{\delta x - \sqrt{(\delta x)^2 - (\delta x_{\min})^2}}{(\delta x_{\min})^2} \approx \frac{\hbar}{2 \delta x} + \frac{\hbar (\delta x_{\min})^2}{8(\delta x)^3}.$$ 

Now minimizing the hydrogen energy \[1\] with this expression of momentum, $p = 2\delta p$, where $\delta x$ is replaced by $r$, from $dE/dr = 0$ one finds

$$\frac{mZe^2}{r^2} - \frac{\hbar}{r} + \hbar \frac{(\delta x_{\min})^2}{4 r^3} \left( \frac{\hbar}{r^2} + 3\hbar (\delta x_{\min})^2 \right) = 0.$$ 

To the lowest order in $\delta x_{\min}$ this equation reduces to

$$mZe^2 r^3 - h^2 r^2 - h^2 (\delta x_{\min})^2 = 0.$$ 

The solution of this equation to the lowest order in $\delta x_{\min}$ looks like

$$r = \frac{\hbar^2}{mZe^2} + \frac{mZe^2 (\delta x_{\min})^2}{h^2} = a + \frac{(\delta x_{\min})^2}{a},$$

that leads to the ground state energy

$$E = -E_R \left[ 1 - \frac{1}{2} \left( \frac{\delta x_{\min}}{a} \right)^2 \right].$$ 

So from minimum-length modified quantum mechanics we get $\delta E \sim E_R \langle \delta x_{\min}/a \rangle^2$.

## III. ONE MORE QUALITATIVE TREATMENT

Instructive example from QED. – In early days of the systematic treatment of QED infinities, Welton gave a simple qualitative description of the Lamb-Retherford effect \[9\] by considering the interaction of electron with the vacuum fluctuations of the electromagnetic field \[10, 11\]. Upon treating the electron classically and non-relativistically, the equation governing the fluctuations of the electron takes the form

$$\frac{d^2(\delta r)}{dt^2} = e\mathcal{E},$$

(2)

where $\mathcal{E}$ denotes the fluctuating electric field. In its vacuum state, the electromagnetic field is characterized by the mean-square fluctuations attributed to the zero-point oscillations,

$$\left\langle \frac{\mathcal{E}_\omega^2}{4\pi} \right\rangle = \frac{\hbar \omega}{2}.$$ 

Under the influence of a given Fourier component of the electric field $\mathcal{E}$ with the frequency $\omega$, electron is subject to the oscillations which have the amplitude

$$\delta r_\omega = -\frac{e\mathcal{E}_\omega}{m\omega^2}.$$ 

For the mean-square fluctuations one finds $\langle (\delta r_\omega)^2 \rangle = e^2 (\mathcal{E}_\omega^2)/m^2\omega^4$ and, respectively,

$$\langle (\delta r)^2 \rangle = \frac{2e^2}{m^2} \int \frac{d^2k}{(2\pi)^3} \frac{2\mathcal{E}_\omega^2}{\omega^4} = \frac{2e^2\hbar}{\pi m^2} \int \frac{d\omega}{\omega},$$

(3)

where we have summed over polarizations as well. To give a concrete meaning to this formal expression, we need to specify the integration limits for the frequency integral \[3\]. As the system under consideration has the binding energy $\sim (Z^2 e^4 m)/\hbar^2$, the natural lower limit on the fluctuation frequencies is set up by this binding energy, because for lower frequencies the electron can not be considered as a free and the equation \[2\] becomes invalid. On the other hand, as the electron can not be probed beneath its Compton wavelength, there is a natural upper limit on the integration set by the electron mass. Hence we get

$$\langle (\delta r)^2 \rangle \approx \frac{2e^2\hbar}{\pi m^2} \int_{\epsilon^4 Z^2 m}/\hbar^3} \frac{m}{h} d\omega = \frac{2e^2\hbar}{\pi m^2} \ln \frac{\hbar^2}{e^4 Z^2}.$$ 

\[1\] Let us notice that by taking into account that the radial wave function with the principal quantum number $n$ has $(n-1)$ nodes and therefore the electron in this state can be considered as localized in the spatial region $\delta x \sim r/n$, that is, $p \sim (\hbar n)/r$ we will get exact result for hydrogen spectrum \[3\].
As the electron is forced to fluctuate around the equilibrium position, it sees the Coulomb potential to be somewhat smeared out. The second term of the Taylor expansion

\[ V(\mathbf{r} + \delta \mathbf{r}) = V(\mathbf{r}) + \nabla V \cdot \delta \mathbf{r} + \frac{1}{2} \sum_{i,j} \delta r^i \delta r^j \frac{\partial^2 V}{\partial r^i \partial r^j} + \cdots, \]

gives the zero average effect. Therefore, the first non-trivial correction has the form

\[ \delta V = \frac{1}{6} \left\langle (\delta r)^2 \right\rangle \frac{-e^2 Z}{r} = \frac{2\pi Z e^2}{3} \left\langle (\delta r)^2 \right\rangle \delta(r). \]

The average effect of this smearing of the potential in a given eigenstate of the atom will result in the energy shift

\[ \delta E_n = \int \delta V |\psi_n(\mathbf{r})|^2 d^3r = \frac{2\pi Z e^2}{3} \left\langle (\delta r)^2 \right\rangle |\psi_n(0)|^2. \]

The wave function at the origin \(|\psi_n(0)|\) vanishes for all states with non-zero angular momentum. For \(l = 0\), after inserting in this equation the values of \(\left\langle (\delta r)^2 \right\rangle\) and \(|\psi_n(0)|^2 \approx Z^3 m^3 e^6 / \pi n^3 h^6\) we get

\[ \delta E_n \approx \frac{8mZ^4 e^{10}}{3\pi n^3 h^6} \ln \frac{\hbar}{e^2 Z}. \]

This result accounts well for the experimental observations. In fact, for \(l \neq 0\) states the experimentally measured energy shifts are not precisely zero, but they are much smaller than the energy shift of the \(2s_1/2\) level.

**Applying QG setup.** – In the case of QG treatment, the situation is somewhat simplified as we know \(\delta r\) from the very outset. Namely, in QG the position of electron can not be specified better than \(\delta x_{min}\), thus for position fluctuation of the electron one simply finds \(\delta r = \delta x_{min}\).

For \(l = 0\), the QG induced energy shift takes the form

\[ \delta E_n \approx \frac{2}{3} Z e^2(\delta x_{min})^2 \left( \frac{Z m^2 e^2}{\hbar^2} \right)^3 = \frac{4 E_R}{3 n^3} \left( \frac{\delta x_{min}}{a} \right)^2. \]

We see the correction is of the same order as that one considered in previous section.

**IV. MINIMUM-LENGTH MODIFIED QUANTUM MECHANICAL TREATMENT**

When the system has several degrees of freedom the minimum-length modified commutation relation considered in section II generalizes to

\[ [\hat{x}_i, \hat{p}_j] = i \hbar \left( \delta_{ij} + \beta \hat{p}^2 \delta_{ij} + \beta \hat{p}_i \hat{p}_j \right). \]

The minimum length which follows from these commutation relations is (for more details see [12])

\[ \delta x_{min} = h \sqrt{3\beta + \beta'}. \]

In a particular case \(\beta' = 2\beta\), i.e.,

\[ [\hat{x}_i, \hat{p}_j] = i \hbar (\delta_{ij} + \beta \hat{p}^2 \delta_{ij} + 2\beta \hat{p}_i \hat{p}_j), \]

the realization of this algebra to the linear order in \(\beta\) can be done in a simple way in terms of the standard position and momentum operators \([\hat{x}_i, \hat{p}_j] = i \hbar \delta_{ij} [13]\). Indeed, defining

\[ \hat{x}_i = \hat{x}_i^0, \quad \hat{p}_i = \hat{p}_i^0 \left[ 1 + \beta (\hat{p}^0)^2 \right], \]

one easily finds that to the linear order in \(\beta\) these operators satisfy the algebra [3, 12]. Working to this accuracy one gets the following universal QG correction to the Hamiltonian [13]

\[ \hat{H} = \frac{\hat{p}^2}{2m} + V(\hat{r}) = \frac{\left(\hat{p}^0\right)^2}{2m} + V(\hat{r}^0) + \frac{\beta}{m} (\hat{p}^0)^4 + O(\beta^2). \]

Written in this way, the only appeal to the dynamical system under consideration is to supply us with the Hamiltonian \(\hat{H}^0\). Denoting by \(\psi^0, E^0_n\) the eigenfunctions and eigenvalues of the unperturbed operator \(\hat{H}^0\)

\[ \hat{H}^0 \psi^0_n = \left(\frac{\left(\hat{p}^0\right)^2}{2m} + V(\hat{r}^0)\right) \psi^0_n = E^0_n \psi^0_n, \]

for the first order correction to the eigenvalue \(E^0_n\) one finds [14]

\[ \delta E_n = \frac{\beta}{m} \langle \psi^0_n | (\hat{p}^0)^4 | \psi^0_n \rangle. \]

From now on we will work in the coordinate representation \(\hat{r}^0 = \hat{r}, \hat{p}^0 = -i \hbar \partial_r\). Using the Eq. [3] and the fact that \(\hat{p}^0\) is a Hermitian operator, one finds

\[ \frac{\beta}{m} \langle \psi^0_n | (\hat{p}^0)^4 | \psi^0_n \rangle = 2\beta \langle \psi^0_n | (\hat{p}^0)^2 | E^0_n - V(\hat{r}) \rangle | \psi^0_n \rangle = 2\beta \langle (\hat{p}^0)^2 \psi^0_n \left| (E^0_n - V(\hat{r})) \right| \psi_n^0 \rangle = 4\beta m \langle \psi^0_n \left| (E^0_n - V(\hat{r}))^2 \right| \psi^0_n \rangle. \]
From Eq. (9), for the Hydrogen atom we get

$$\delta E_n = \frac{\beta}{m} \langle \psi_n^0 | (\hat{p}^0)^4 | \psi_n^0 \rangle \equiv 4\beta \int dr r^2 R_{nl}^4 \left[ \frac{mZ^2e^4}{2\hbar^2n^2} - \frac{Ze^2}{r} \right]^2,$$

where $$\psi_n^0 = R_{nl}(r)Y_{nm}(\theta, \phi)$$ are eigenfunctions for the Coulomb problem [14].

Using the Bohr radius $$a = h^2/mZe^2$$ as a length unit, one can write the Eq. (10) in the form

$$\delta E_n = 4\beta m \left( \frac{mZ^2e^4}{\hbar^2} \right)^2 \int_0^\infty d\xi \xi^2 R_{nl}^4 \left[ \frac{1}{2n^2} - \frac{1}{\xi} \right]^2,$$

where

$$\xi = r/a, \quad R_{nl} = a^{3/2} R_{nl}.$$

Now the integrand is written in dimensionless quantities and the $$R_{nl}$$ function has an explicit form [14]

$$R_{nl} = -\frac{2}{n^2} \sqrt{\frac{(n - \ell - 1)!}{[(n + \ell)!]^3}} e^{-\xi/n} \left( \frac{2\xi}{n} \right)^\ell L_n^{2\ell+1} \left( \frac{2\xi}{n} \right).$$

Using the normalization condition for $$R_{nl}$$

$$\int_0^\infty \tilde{R}_{nl}^2(\xi) \xi^2 d\xi = 1,$$

and the well known mean values [14, 13]

$$\bar{\xi} = \frac{1}{n^2}, \quad \bar{\xi}^{-2} = \frac{1}{n^3 (\ell + \xi)},$$

where

$$\bar{\xi}^k = \int_0^\infty \tilde{R}_{nl}^2(\xi) \xi^{2+k} d\xi,$$

we reproduce the result of [13]:

$$E_{nl} = -E_R \left[ \frac{1}{n^2} - 2 \frac{4n}{5(\ell + 1/2)} \left( \frac{\delta \bar{x}_{min}}{a_n} \right)^2 \right].$$

(12)

Therefore the QG correction works with a positive sign.

The use of KMM quantum mechanics [3] that exactly satisfies the modified commutation relations [16] is technically more complicated. Using KMM quantum mechanics, the posterior study of the hydrogen atom led authors of the paper [16] to the following result for $$\ell = 0$$ energy levels

$$E_n = \frac{E_R}{n^2} \left[ 1 + 2(\beta + \beta') \frac{m^2Ze^4}{\ell^2n^2} \right],$$

implying in the case $$\beta' = 2\beta$$

$$\delta E_n \approx -\frac{6}{5} \frac{E_R}{a_n} \left( \frac{\delta \bar{x}_{min}}{a_n} \right)^2.$$

(13)

Albeit the rate of the correction is about the same the sign is opposite as compared with the Eq. (12). The subsequent consideration of the hydrogen atom in the KMM formalism [17], exhibits positive QG correction in agreement with the Eq. (12) but unfortunately this paper could neither account for the discrepancy nor reproduce the results of Eqs. (12, 13). Further study for a final clarification of the sign of QG correction in the KMM approach is desired.

It is noteworthy that in the KMM construction the inverse distance operator $$\hat{r}^{-1}$$ is non-local [3] and, therefore, in this approach the Coulomb potential smearing considered in section III is automatically taken into account. While the above described scheme for an approximate realization of the minimum-length modified commutation relations, Eq. (10), misses this effect.

V. HYDROGEN ATOM IN VIEW OF THE QG RUNNING/REDUCTION OF SPACE-TIME DIMENSION

QG running/reduction of space-time dimension

Because of quantum gravity the dimension of space-time appears to depend on the size of region, it is somewhat smaller than four at small scales and monotonically decreases with increasing the size of the region [6]. We can account for this effect in a simple and physically clear way that allows us to write simple analytic expression for space-time dimension running. Let us consider a subset $$\mathcal{F}$$ of four dimensional Euclidean space $$\mathbb{R}^4$$, and let $$l^4$$ be a smallest box containing this set, $$\mathcal{F} \subseteq l^4$$. For estimating the dimension of $$\mathcal{F}$$ we have to cover it by $$\epsilon^4$$ cells and counting the minimal number of such cells, $$N(\epsilon)$$, we determine the dimension, $$d \equiv \dim(\mathcal{F})$$ as a limit

$$d = d(\epsilon \to 0),$$

where $$n(\epsilon) = N$$ and $$n = l/\epsilon$$. For more details see [18]. This definition is referred to as a box-counting dimension and can be written in a more familiar form as

$$d = \lim_{\epsilon \to 0} \frac{\ln N(\epsilon)}{\ln \frac{l}{\epsilon}}.$$

Certainly, in the case when $$\mathcal{F} = l^4$$, by taking the limit $$d(\epsilon \to 0)$$ we get the dimension to be 4. From the fact that we are talking about the dimension of a set embedded into the four dimensional space, $$\mathcal{F} \subseteq \mathbb{R}^4$$, it automatically follows that its dimension can not be greater than 4, $$d \leq 4$$. We see that the volume of a fractal $$\mathcal{F}$$ uniformly filling the box $$l^4$$ is reduced

$$V(\mathcal{F}) = \lim_{\epsilon \to 0} N(\epsilon)\epsilon^4 = \lim_{\epsilon \to 0} \frac{n(\epsilon)^4}{\epsilon^4},$$

in comparison with the four dimensional value $$l^4$$. Introducing $$\delta N = n(\epsilon)^4 - N(\epsilon)$$, the reduction of dimension
Apart from the less important numerical (logarithmic) factors, the rate of correction \( \sim E_R/|\delta x_{\text{min}}/a|^2 \) is in perfect agreement with the results of previous sections. Indeed, the factor \( \ln|1/\delta x_{\text{min}}| \) is of less significance as for the length scale \( l \) at which the QG corrections become important the ratio \( 1/\delta x_{\text{min}} \) is not very large. So, in view of the precision we can pretend to in study of the QG corrections this factor is really less important. We see that the correction due to QG dimension reduction works with a negative sign, that is, it increases the binding energy.

The angular momentum eigenvalues also change because of dimension reduction \[ L = \sqrt{\ell (\ell + D - 2)} \hbar \approx \sqrt{\ell (\ell + 1)} \hbar - \frac{\varepsilon \hbar \ell}{2 \sqrt{\ell (\ell + 1)}} . \]

It should be noticed that in this discussion we have not taken into account the modification of Coulomb potential \( \sim 1/r \) due to dimension reduction \( \sim 1/r^{D-2} \).

### Shift of radiative and relativistic corrections due to dimension reduction

The dimensional regularization approach allows one to simply estimate how the dimension reduction affects the well known QED radiative corrections \cite{22, 23, 26}. Namely, in calculating QED radiative corrections to the photon propagator or to the photon-electron vertex, we have due to integration in momentum space the well known \( \Gamma(\varepsilon/2) \) factor (see, for example \cite{27}). Now, keeping simply the terms linear in \( \varepsilon \) in the radiative corrections, that come from the decomposition of the \( \Gamma(\varepsilon/2) \) function alluded to above, \[ \Gamma \left( \frac{\varepsilon}{2} \right) = \frac{2}{\varepsilon} - \gamma + \frac{\varepsilon}{4} \left( \frac{\pi^2}{6} + \gamma^2 \right) + O(\varepsilon^2) , \quad \gamma \approx 0.5772 , \]

one easily infers that due to dimension reduction we will have the shift of the standard QED radiative corrections which are of the order \( \sim \varepsilon \times \text{QED radiative corrections} \).

In this way one finds that the corrections due to QG dimension reduction to the electron anomalous magnetic moment and the Lamb shift are of the order of \( \sim \varepsilon e^3/m \) and \( \sim \varepsilon Z^2 e^6 E_R/\hbar^3 n^3 \) respectively, see \cite{26}. For this energy shift we see that because of factor \( e^6/\hbar^3 \approx (1/137)^3 \) it is by about six orders of magnitude smaller than the leading QG correction \( \sim \varepsilon E_R/n^3 \) found above.

To see how the dimension reduction affects the relativistic corrections to the hydrogen energy levels, one can use formally the solution of the Dirac equation in \( D + 1 \) space-time dimensions for the Coulomb potential \cite{28}:

\[ \tilde{E}_n = m \left( 1 + \frac{(Ze^2/\hbar)^2}{\sqrt{K^2 - (Ze^2/\hbar)^2} + n - \ell - 1} \right)^{-1/2} \quad \text{for} \quad \ell \in \{0, 1, 2, \ldots, n - 3\} . \]
where $K = (2l + D - 1)/2$ and assume $D = 3 - \varepsilon$. Expanding Eq. (15) in powers of $(Ze^2/\hbar)^2$, one finds
\[
\bar{E}_n = m - \frac{m(Ze^2/\hbar)^2}{2(n + D - 3/2)^2} - \frac{m(Ze^2/\hbar)^4}{2(n + D - 3/2)^4} \left( \frac{2n + D - 3}{2\ell + D - 1} \right),
\]
where the first term on the right-hand side represents the rest energy $m$, the second term, which was discussed above, comes from the Schrödinger equation, and the third one describes relativistic corrections. Substituting $D = 3 - \varepsilon$, where $\varepsilon$ is estimated at the atomic scale $l = a_n$, one easily finds the shift of relativistic corrections due to dimension reduction. This shift of relativistic correction due to dimension reduction is suppressed in comparison with the leading QG correction $\sim \varepsilon E_R/n^3$ by the factor $(Ze^2/\hbar)^2$, that is by about four orders of magnitude for hydrogen atom.

Again, the modification of the Coulomb potential because of dimension reduction $1/r \rightarrow 1/r^{D-2}$ is ignored throughout this discussion.

### VI. HARMONIC OSCILLATOR

Assuming the modified commutation relations (4), for $D$-dimensional harmonic oscillator
\[
\hat{\mathcal{H}} = \frac{\hat{p}^2}{2m} + \frac{m\omega^2 n^2}{2},
\]
one finds
\[
E_n = \hbar \omega \left[ \left( n + \frac{D}{2} \right) \sqrt{1 + \left\{ \beta^2 \frac{L^2}{\hbar^2} + \frac{(D\beta + \beta')^2}{4} \right\} m^2 \hbar^2 \omega^2} \right.
\]
\[
+ \left\{ (\beta + \beta') \left( n + \frac{D}{2} \right)^2 + (\beta - \beta') \left( \frac{L^2}{\hbar^2} + \frac{D^2}{4} \right) + \beta' \frac{D}{2} \right\} m \hbar \omega \right],
\]
where $L^2 = \ell(\ell + D - 2)\hbar^2$. With this expression one easily considers the minimum-length modified quantum mechanical correction to the harmonic oscillator against the correction coming from QG reduction of dimension. For $D = 3$, the energy spectrum to the lowest order in $\beta$, $\beta'$ takes the form
\[
E_n \approx \hbar \omega \left[ \left( n + \frac{3}{2} \right) \right. + \frac{1}{2} \left( k^2 + k'^2 \right) \left( n + \frac{3}{2} \right)^2
\]
\[
+ (k^2 - k'^2) \left( \ell(\ell + 1) + \frac{9}{4} \right) + k'^2 \frac{3}{2} \right], \tag{16}
\]
where $k^2 = \beta \hbar \omega$, $k'^2 = \beta' \hbar \omega$.

To estimate the QG corrections due to dimension running/reduction
\[
E_n = \hbar \omega \left( n + \frac{D}{2} \right) = \hbar \omega \left( n + \frac{3}{2} \right) - \hbar \omega \frac{\varepsilon}{2},
\]
first we have to evaluate the size of a localization region for the system. One can do this in a simple semiclassical way by equating
\[
m \omega^2 r^2 = \hbar \omega \left( n + \frac{3}{2} \right) \Rightarrow r_n = \left[ \frac{2\hbar}{m \omega} \left( n + \frac{3}{2} \right) \right]^{1/2},
\]
where $\delta x_{min} = \ln \left( \frac{\delta x_{min}}{r_n} \right)^2 = \frac{\hbar \omega (\beta + \beta')}{2 \left( n + \frac{3}{2} \right)}$.

We see that as in the case of hydrogen atom, the QG corrections to the harmonic oscillator due to dimension reduction and minimum-length modified quantum mechanics have the same rate up to less important logarithmic factors but work with opposite signs. QG correction to the harmonic oscillator due to dimension reduction works with negative sign as in the case of hydrogen atom.

### VII. CONCLUDING REMARKS

The qualitative treatments considered in sections II and III give us a reliable idea about the rate of QG correction to the energy spectrum of the hydrogen atom but probably tell little about the proper sign of the correction. From the physical point of view, one can identify three distinct sources of the QG corrections to the hydrogen atom. The first one is related to the modification of electron dynamics due to existence of the minimum
length. Corresponding correction can be estimated qualitatively with the use of modified uncertainty relations, (Section II). The second type of QG correction results from the smearing of Coulomb potential at the electron radius because the minimum length effectively implies a non-zero size for the electron. This correction can also be estimated qualitatively along the Welton’s discussion of Lamb shift, (Section III). Both corrections are of the same order of magnitude.

The third correction is related to the quantum gravity running/reduction of space-time dimension. The same minimum length gives a clear physical understanding of space-time dimension reduction. Physically the existence of minimum length implies the presence of uncontrollable fluctuations of the background metric as the point in space-time can not be determined to a better accuracy than $\delta x_{\text{min}}$. Because of these fluctuations four volume also undergoes fluctuation that under assumption of the four-dimensionality of the background space-time immediately indicates an effective reduction of dimension, (Section V). Presently the concept of QG running/reduction of space-time dimension is no longer the subject of intuition, it is well established in two different approaches to QG [6]. Nevertheless, the present physical discussion allows us to get a simple analytic expression of space-time dimension running [19,21]. Corresponding QG correction to the hydrogen atom can be estimated by using $D$-dimensional Schrödinger (or Dirac) equation for the Coulomb potential. This type of correction is pretty much of the same order as the previous ones. For more definiteness we notice that it is deceptive to count on the suppression of QG correction due to logarithmic term that appears in the dimension reduction approach, $1/\ln(l/\delta x_{\text{min}})$, because when the QG corrections become important, that is, when the ratio $l/\delta x_{\text{min}}$ is not extremely large, this term is within the precision which we can require for QG corrections no matter what the particular approach is. Let us notice once more that throughout this paper we neglected the modification of Coulomb potential $\sim 1/r$ due to dimension reduction $\sim 1/r^{D-2}$.

After determining that the rate of QG corrections due to dimension reduction are pretty much of the same order as those coming from the minimum-length modified quantum mechanics, there remains a subtle question of their signs. The considerations based on the minimum-length modified quantum mechanical approach including the qualitative discussions of sections II and III exhibit the energy shifts with a positive sign. While, we see that for hydrogen atom as well as harmonic oscillator energy spectrum the QG correction due to dimension reduction works with a negative sign. Therefore the sign of total QG contribution remains obscure.

In a recent paper [30] the QG correction to the Lamb shift due to modified commutation relations was estimated as

$$\delta E \sim E_R \left(\frac{\delta x_{\text{min}}}{a}\right)^2 \frac{E_R e^2}{\hbar m}.$$ 

It agrees well with the above-found correction of Lamb shift due to dimension reduction in QED radiative corrections. We recall that it is by about six orders of magnitude smaller than the leading QG correction $\sim E_R (\delta x_{\text{min}}/a)^2$. Let us note that the correction to the hydrogen energy spectrum due to dimension reduction (taking account of the replacement $1/r \rightarrow 1/r^{D-2}$ as well) was estimated long ago in [31] as $\delta E \sim -\varepsilon E_R/6$, which perfectly agrees with our estimate.

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