COMPACT SUPPORT PROPERTY OF SUPERBROWNIAN MOTION IN RANDOM ENVIRONMENTS

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ABSTRACT. In this paper, we prove the compact support property for a class of nonlinear SPDE including the equation that the density of one-dimensional Super-Brownian Motion in random environment satisfies.

1. Introduction

It’s well known that the one-dimensional density of classic Super-Brownian Motion satisfies the following SPDE,

$$\partial_t u(t, x) = \Delta u(t, x) + \sqrt{u(t, x)} \dot{B}(t, x)$$

(1)

Here we replace $\frac{1}{2} \Delta$ with $\Delta$ for simplicity and $\dot{B}(t, x)$ is time-space white noise. The above equation can be written as following form:

$$\partial_t u(t, x) = \Delta u(t, x) + \sqrt{u(t, x)} \varphi_k(x) \beta^k_t$$

where $\{\varphi_k\}$ is an orthonormal basis of $L^2(\mathbb{R})$ and $\{\beta^k_t\}$ is a sequence of independent Brownian Motions. The solutions to this equation has compact support property, roughly speaking, if the initial date $u(0, x)$ have compact support, then for all $t > 0$, $u(t, \cdot)$ has compact support almost surely. In [3], the authors proved the compact support property for the solutions of a large class of SPDEs including (1). Later Krylov given a simpler proof in [2] by using his $L^p$ theory.

One the other hand, from 1990’s many experts started to study superprocesses in random environments. In [4], Mytnik introduced models of superprocesses in random environments. We give a brief description below:

Let $\{\xi_k(x), k \in \mathbb{N}\}$ be a sequence of independent identically distributed random fields on $\mathbb{R}^d$ satisfying:

$$E\xi_k(x) = 0, \quad E\xi_k(x)\xi_k(y) = g(x, y), \quad \sup_x E|\xi_k(x)|^3 < \infty, \quad x \in \mathbb{R}^d, k \in \mathbb{N}.$$ 

(2)

Where $g$ is the covariance function which satisfies $\sup_{x, y \in \mathbb{R}} |g(x, y)| \leq C < \infty$, $g(x, \cdot) \in C_0(\mathbb{R}^d)$ $(\forall x \in \mathbb{R}^d)$, we further assume $g \in C^2_b(\mathbb{R}^d)$ in this paper. $\{\xi_k(x), k \in \mathbb{N}\}$ serves as the random environments. For each fixed $n \in \mathbb{N}$, consider a particle system in which there are $K_n \geq 1$ particles located in $\mathbb{R}^d$, each of them moves independent as a copy of Brownian motion(with generator $\Delta$) until time $t = 1/n$. Given $\{\xi_k(x), k \in \mathbb{N}\}$, at time $\frac{1}{n}$, each particle split into two particles with probability $\frac{1}{2} + \frac{1}{2\sqrt{n}} [(-\sqrt{n}) \vee \xi_1(x) \wedge (\sqrt{n})]$ or dies with probability $\frac{1}{2} - \frac{1}{2\sqrt{n}} [(-\sqrt{n}) \vee \xi_1(x) \wedge (\sqrt{n})]$. The new particles then moves in space independently as Brownian motions(with generator $\Delta$) starting at their place of birth, during the time interval $[1/n, 2/n)$. In general, at time $\frac{i}{n}$, each surviving particle split into two particles with probability $\frac{1}{2} + \frac{1}{2\sqrt{n}} [(-\sqrt{n}) \vee \xi_i(x) \wedge (\sqrt{n})]$ or dies with
probability $\frac{1}{2} - \frac{1}{\sqrt{n}}((-\sqrt{n}) \lor \xi_i(x) \land (\sqrt{n})$), and in the time interval $[i/n, (i+1)/n]$ particles independently according to Brownian motions (with generator $\Delta$). Let $X^n_t$ be the measure-valued Markov process, defined as

$$X^n_t(B) = \frac{\text{number of particles in } B \text{ at time } t}{n}.$$ 

where $B \in \mathcal{B}(\mathbb{R}^d)$ are Borel sets in $\mathbb{R}^d$. Let $C^k_b(\mathbb{R}^d)$ (respectively $C^\infty_b(\mathbb{R}^d)$) denote the collection of all bounded continuous functions on $\mathbb{R}^d$ with bounded continuous derivatives up to order $k$ (respectively with bounded derivatives of all orders). For all bounded measurable $f$, let $\mu(f) = \langle f, \mu \rangle$ denote the integral of $f$ with respect to the measure $\mu$ on $\mathbb{R}^d$. For any measurable functions $f, h$ on $\mathbb{R}^d$, let $f \otimes h$ denote an $\mathbb{R}^{2d}$ valued function defined by $(f \otimes h)(x, y) := f(x)h(y)$, $x, y \in \mathbb{R}^d$. It was proved in [5] that if $X^n_t$ converge weakly to a finite measure $\mu$ on $\mathbb{R}^d$, then the processes $X^n = \{X^n_t, t \geq 0\}$ converges weakly to a measure-valued process $X = \{X_t, t \geq 0\}$, where $X$ is the unique solution to the following martingale problem:

$$(MP) : \begin{cases} 
\text{for all } f \in C^2_b, \quad M^f_t = \langle f, X_t \rangle - \langle f, \mu \rangle - \int_0^t \langle \Delta f, X_s \rangle \\
\text{is a continuous square-integrable martingale with quadratic variation} \\
\langle M^f \rangle_t = \int_0^t \langle f^2, X_s \rangle ds + \int_0^t \langle g \cdot f, X_s \otimes X_s \rangle ds.
\end{cases}$$

(3)

However, there only very few works about the properties of these processes. As far as to my knowledge, the most interesting work is [5], in which the authors studies the local extinction property. Under some assumption, it shows the super-Brownian motion in random environments will extinct locally in any dimension which is quite different from the classic case (see Theorem 1.1, 1.2 of [5] for more details).

In this paper, we show the density of one-dimensional super-Brownian motion in random environments satisfies:

$$\partial_t u = \Delta u + \sqrt{u} \varphi_k \beta^k_t + uh_k w^k_t.$$ 

Here $\{w^k_t\}$ and $\{\beta^k_t\}$ are two sequences of independent standard Brownian motions. In order to do that we give the second order moment formula of superprocesses in Section 2. In Section 3, using Krylov’s $L^p$ theory for SPDEs (see [1]), we prove the compact support property for more general stochastic partial differential equations under some reasonable assumptions.

2. Moment Formula

In this section, we give the first and second order moment formula for $X_t$.

The following Lemma is a simple application of Stone-Weierstrass theorem.

**Lemma 2.1.** For any $f(x, y) \in C_b(\mathbb{R}^{d_1+d_2})$ $(x \in \mathbb{R}^{d_1}, y \in \mathbb{R}^{d_2})$, there exist a sequence of smooth functions $\{f_n(x, y)\}$ with form $f_n(x, y) = \sum_{i=1}^n \phi_i^n(x) \psi_i^n(y)$ such that $f_n \to f$ uniformly on any compact subset in $\mathbb{R}^{d_1+d_2}$ and $\|f_n\|_{\infty} \leq \|f\|_{\infty}$.

In order to get the second moment formula, we need the following generalization of Lemma 2.1.
Lemma 2.2. For any $f(x, y) \in C_{b,1}^{k_2}([-d_2, d_2])$ ($x \in \mathbb{R}^{d_1}, y \in \mathbb{R}^{d_2}$), there exist a sequence of smooth functions $\{f_n(x, y)\}$ with form $f_n(x, y) = \sum_{i=1}^{n} \phi^n_i(x) \psi^n_i(y)$ such that for any $a = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}^d$, $\beta = (\beta_1, \ldots, \beta_d) \in \mathbb{N}^d$ and $|\alpha| = \alpha_1 + \cdots + \alpha_d \leq k_1$, $|\beta| = \beta_1 + \cdots + \beta_d \leq k_2$. $\partial^a_x \partial^b_y f_n \rightarrow \partial^a_x \partial^b_y f$ uniformly on any compact subset and $\|f_n\|_{C^{k_1,k_2}} \leq C\|f\|_{C^{k_1,k_2}}$.

Proof. To keep the proof simple, we assume $k_1, d_i = 1$. Let $\varphi(x) \in C_c^\infty(\mathbb{R})$ and $\varphi(x) = 1$ if $|x| \leq 1/2$, $\varphi(x) = 0$ if $|x| \geq 1$. Fixed $R > 0$, let $f_R(x, y) = f(x, y) \varphi(\frac{x}{R}) \varphi(\frac{y}{R})$. Using standard diagonal argument, in order to prove the Lemma, we only need to show there exists a sequence of smooth functions $f_{R,n}(x, y) = \sum_{i=1}^{n} \phi^n_i(x) \psi^n_i(y)$ such that $\|f_{R,n} - f_R\|_{C^1} \rightarrow 0$ $(n \rightarrow 0)$. Now define

$$f_R^{m}(x, y) = \int_{R^2} m^2 \varphi(m \xi) \varphi(m \eta) f_R(x - \xi, y - \eta) d\xi d\eta \in C_c^\infty(\mathbb{R}^2).$$

Since $f_R \in C^1_c$, we have

$$\lim_{m \rightarrow 0} \|f_R^m - f_R\|_{C^1} = 0. \quad (4)$$

On the other hand, we have

$$f_R^{m}(x, y) = \int_{-R}^{y} \int_{-R}^{x} \partial_x \partial_y f_R^{m}(\xi, \eta) d\xi d\eta,$$

by Lemma 2.1, there exist a sequence of smooth functions $g^{m,n}_R(x, y) = \sum_{i=1}^{n} \phi^{m,n}_i(x) \psi^{m,n}_i(y)$ such that $g^{m,n}_R(x, y) \rightarrow \partial_x \partial_y f_R^m$ uniformly as $n \rightarrow \infty$. Let $f^{m,n}_R = \int_{-R}^{y} \int_{-R}^{x} g^{m,n}_R(\xi, \eta) d\xi d\eta$, then

$$\lim_{n \rightarrow \infty} \|f^{m,n}_R - f_R^m\|_{C^1} = 0. \quad (5)$$

Combining (4) and (5), using standard diagonal argument, we can find a sequence of function $f_{R,n} = \sum_{i=1}^{n} \phi^n_i(x) \psi^n_i(y) \rightarrow f_R$ in $C^1$.

Before giving the second moment formula, we first prove an estimate for $\mathbb{E}_\mu \langle f, X_t \rangle^2$.

Lemma 2.3.

$$\mathbb{E}_\mu \langle f, X_t \rangle = \langle P_t f, \mu \rangle;$$

$$\mathbb{E}_\mu \langle f, X_t \rangle^2 \leq \left\{ \langle P_t f, \mu \rangle^2 + \int_0^t \langle \mu, P_s [(P_{t-s} f)^2] \rangle ds \right\} \exp(\|g\|_{\infty} t).$$

Where $\{P_t\}$ is the semigroup whose generator is $\Delta$.

Proof. Just as the proof of Proposition II 5.7 of [1], if $\phi_t(x) \in C^{1,2}(\mathbb{R}_+ \times \mathbb{R}^d)$, we can prove

$$\langle \phi_t, X_t \rangle - \langle \phi_0, \mu \rangle - \int_0^t \langle \phi_s + \Delta \phi_s, X_s \rangle ds$$

is a martingale with quadratic variation $\langle M \rangle_t = \int_0^t \langle \phi_s^2, X_s \rangle ds + \int_0^t \langle g \cdot \phi_s \otimes \phi_s, X_s \otimes X_s \rangle ds$. In addition, there exists a martingale measure $\mathbb{M}(dt, dx)$ such that for any $f \in B(\mathbb{R}^d)$,

$$\langle f, X_t \rangle = \langle \mu, P_t f \rangle + \int_0^t \int_{\mathbb{R}^d} P_{t-s} f(x) \mathbb{M}(ds, dx)$$
\{ \int_0^t p_{t-r} f(x) M(dr, dx) \}_{s \leq t} \) is a martingale from time 0 to \( t \) with
\[
\mathbb{E}_\mu(\int_0^t p_{t-r} f(x) M(dr, dx))_s = \int_0^s \int_{\mathbb{R}^d} (p_{t-r} f(x))^2 X_r(dx)dr + \int_0^s \int_{\mathbb{R}^d} g(x, y) p_{t-r} f(x) p_{t-r} f(y) X_r(dx)X_r(dy)dr
\]
Hence
\[
\mathbb{E}_\mu(\langle f, X_t \rangle) = \langle p_t f, \mu \rangle,
\]
\[
\mathbb{E}_\mu[\langle p_t s f, X_s \rangle^2] = \langle p_t f, \mu \rangle^2 + \mathbb{E}_\mu \int_0^s \int_{\mathbb{R}^d} (p_{t-r} f(x))^2 X_r(dx)dr
\]
\[
+ \mathbb{E}_\mu \int_0^s \int_{\mathbb{R}^d} g(x, y) p_{t-r} f(x) p_{t-r} f(y) X_r(dx)X_r(dy)dr
\]
\[
\leq \langle p_t f, \mu \rangle^2 + \int_0^s \langle \mu, p_r [(p_t f)^2] \rangle dr + \|g\|_\infty \int_0^s \mathbb{E}_\mu[\langle p_t r f, X_r \rangle^2] ds
\]
By Gronwall’s Inequality, we obtain
\[
E_\mu[\langle f, X_t \rangle^2] \leq \left\{ \langle p_t f, \mu \rangle^2 + \int_0^t \langle \mu, p_r [(p_t s f)^2] \rangle ds \right\} \exp(\|g\|_\infty t)
\]
Now we are in a position to prove the second moment formula:

**Theorem 2.4.** Let \( Q_t = p_t \otimes p_t \) and \( Q_t^0 \) be the semigroup generated by \( \Delta + g \). Then for all \( \phi \in \mathcal{C}_b(\mathbb{R}^d) \),
\[
\mathbb{E}_\mu(X_t(\phi))^2 = \langle Q_t^0(\phi \otimes \phi), \mu \otimes \mu \rangle + \int_0^t \langle \mu, p_s [(\pi Q^0_{t-s}(\phi \otimes \phi))] ds, \mu \rangle
\]
Here
\[
(\pi f)(x) = f(x, x); \ f \in \mathcal{B}(\mathbb{R}^{2d}), \ x \in \mathbb{R}^d.
\]

**Proof.** We assume \( \mu = \delta_0 \) for simple. The proof for general \( \mu \in \mathcal{M}_F(\mathbb{R}^d) \) is similar. \( \forall \phi_s, \psi_s \in \mathcal{C}_b^{1,2}(\mathbb{R}^+ \times \mathbb{R}^d) \), using Ito’s Formula,
\[
d(X_s(\phi_s)X_s(\psi_s)) = X_s(\phi_s)d(M_s^\phi + V_s^\phi) + X_s(\psi_s)d(M_s^\phi + V_s^\phi) + d(M^\phi, M^\psi)_s
\]
Where
\[
V_s^\phi = \langle \phi, \mu \rangle + \int_0^s \langle \phi_r + \Delta \phi_r, X_r \rangle dr, \ V_s^\psi = \langle \psi, \mu \rangle + \int_0^s \langle \psi_r + \Delta \psi_r, X_r \rangle dr.
\]
\[
M_s^\phi = \langle \phi_s, X_s \rangle - V_s^\phi, \ M_s^\psi = \langle \psi_s, X_s \rangle - V_s^\psi.
\]
By linearity, identity holds for all
\[
\mathbb{E}_{\delta_0}(X_t(\phi_t)X_t(\psi_t)) = \phi_0(0)\psi_0(0) + \mathbb{E}_{\delta_0}\int_0^t X_s(\phi_s)dV_s^\phi + \mathbb{E}_{\delta_0}\int_0^t X_s(\psi_s)dV_s^\psi + \mathbb{E}_{\delta_0}\langle M^\phi, M^\psi \rangle_t
\]
\[
= \phi_0(0)\psi_0(0) + \mathbb{E}_{\delta_0}\int_0^t [X_s(\phi_s)X_s(\psi_s + \Delta \psi_s) + X_s(\psi_s)X_s(\phi_s + \Delta \phi_s)]ds
\]
\[
+ X_s \otimes X_s(g \cdot \phi_s \otimes \psi_s)ds + \mathbb{E}_{\delta_0}\int_0^t X_s(\phi_s\psi_s)ds
\]
\[
= \phi_0(0)\psi_0(0) + \mathbb{E}_{\delta_0}\int_0^t X_s \otimes X_s(\partial_s(\phi_s \otimes \psi_s) + (\Delta + g)(\phi_s \otimes \psi_s))ds
\]
\[
+ \mathbb{E}_{\delta_0}\int_0^t X_s(\pi \phi_s \otimes \psi_s)ds
\]
holds for all \( f(t, x, y) = \sum_{i=1}^n \phi_i(t, x)\psi_i(t, y) \).

For any \( \| \phi \| \) and \( \| \psi \| \), let \( \delta \) be such that \( \| \phi \| \) and \( \| \psi \| \) are \( \delta \)-close. Define \( I_R(x) = I_{[-R, R]^d}(x) \), \( I_R'(x) = 1 - I_R(x) \). Then we can find functions \( f^n(s, x, y) = \sum_{i=1}^n \phi_i^n(s, x)\psi_i^n(s, y) \) such that for any fixed \( R \geq 0 \),
\[
\lim_{n \to 0} \| f^n - f \|_{C^{1,2}(Q_R)} = 0
\]
and \( \| f^n \|_{C^{1,2}_b} \leq C \| f \|_{C^{1,2}_b} \). Here \( Q_R = [0, R] \times [-R, R]^{2d} \) and \( C \) is independent with \( n \).

Using Lemma 2.3,
\[
\mathbb{E}_{\delta_0} X_s \otimes X_s(|\delta^n(s)|) \leq C \mathbb{E}_{\delta_0} (X_s(I_R)X_s(I'_R) + X_s(I'_R)^2) + c_n \mathbb{E}_{\delta_0} X_s(I_R)^2
\]
\[
\leq C \left\{ \mathbb{E}_{\delta_0} X_s(I'_R)^2 + \mathbb{E}_{\delta_0}(X_s(I_R)^2) \right\} + c_n \mathbb{E}_{\delta_0} X_s(I_R)^2.
\]

Here \( c_n = \| \delta^n \|_{L^\infty(Q_R)} \to 0 \) as \( n \to \infty \).

For any \( \epsilon > 0 \), choose \( R \) so large, such that
\[
\frac{1}{(\sqrt{2\pi}t)^d} \int_{|y|<\frac{R}{2}} e^{-|y|^2/2t} dy < \epsilon^2.
\]

Then
\[
\sup_{s \leq t} \mathbb{E}_{\delta_0} X_s(I'_R)^2 \leq C \left\{ [P_s I_R']^2(0) + \int_0^t P_r([P_{s-r} I'_R]^2)(0) dr \right\}
\]

For any \( \epsilon > 0 \), choose \( R \) so large, such that
\[
\frac{1}{(\sqrt{2\pi}t)^d} \int_{|x|>R} e^{-x^2/2t} dx < \epsilon^2.
\]
\[[P_{s-R}I']^2(x) \leq \left( I^2_R(x) \int_{|y| > R} e^{(x-y)^2/2(s-r)} dy + I^2_R(x) \right)^2 \leq C e^2 I^2_R(x) + C I'_R(x)\]

Hence
\[\int_0^s P_r[(P_{s-r}I')^2](0)\,dr \leq C e^2 s + C \int_0^s \frac{1}{\sqrt{2\pi r}} \int_{|y| > R} e^{y^2/2r} dy\,dr \leq C e^2\]

(10)

So by (7), (8), (9), (10), we have
\[\lim_{n \to \infty} \sup_{s \leq t} E_{\delta_0} X_s \otimes X_s(|\delta^n(s)|) = 0.\]  

(11)

By the same argument we can prove
\[\lim_{n \to \infty} \left| E_{\delta_0} \int_0^t X_s \otimes X_s (\partial_s f^n(s) + (\Delta + g) f^n(s))ds - E_{\delta_0} \int_0^t X_s \otimes X_s (\partial_s f(s) + (\Delta + g) f(s))ds \right| \]
\[\leq \lim_{n \to \infty} \int_0^t E_{\delta_0} X_s \otimes X_s (|\partial_s \delta^n(s) + (\Delta + g) \delta^n(s)|) ds \]
\[\leq C ||f||_{C^{1,2}([0,t] \times \mathbb{R}^2)} \lim_{R \to \infty} \int_0^t E_{\delta_0} (X_s(I_R)X_s(I'_R) + X_s(I'_R)^2) ds \]
\[+ \lim_{R \to n \to \infty} \|\delta^n\|_{C^{1,2}(Q_R)} \int_0^t E_{\delta_0} X_s(I_R)^2 ds \]
\[= 0\]  

(12)

Similarly,
\[\lim_{n \to \infty} \left| E_{\delta_0} \int_0^t X_s(\pi f^n(s))ds - E_{\delta_0} \int_0^t X_s(\pi f(s))ds \right| = 0.\]  

(13)

So we obtain (6) holds for all \(f \in C^{1,2}_b(\mathbb{R}_+ \times \mathbb{R}^d)\).

Suppose \(\phi \in C_c^\infty\), let \(f_s = Q_{t-s}^\phi \otimes \phi\) (define \(f_s = f_t\) if \(s > t\)), then \(f \in C^{1,2}_b(\mathbb{R}_+ \times \mathbb{R}^d)\). Hence for all \(\phi \in C_c^\infty(\mathbb{R})\), we have the formula
\[E_{\delta_0}(X_t(\phi)^2) = (Q_t^\phi(\phi \otimes \phi))(0,0) + \int_0^t [P_s(\pi Q_{t-s}^\phi(\phi \otimes \phi))](0)\,ds.\]

A simple approximation argument shows for all \(\phi \in C_b(\mathbb{R}^d)\)
\[E_{\delta_0}(X_t(\phi)^2) = \langle Q_t^\phi(\phi \otimes \phi), \delta_0 \otimes \delta_0 \rangle + \int_0^t \langle P_s(\pi Q_{t-s}^\phi(\phi \otimes \phi))ds, \delta_0 \rangle\]
\[\square\]

Remark 2.5. Indeed, we can also using conditional Laplace transform introduced by [5] to get the same formula. However, the proof presented here is more elementary.
In this section, we first use the moment formula to get the equation that the density of Super-Brownian motion satisfies and then prove the compact support property for a class of parabolic SPDEs.

**Lemma 3.1.** Suppose $\mu(\mathbb{R}) < \infty$, then the density of the 1-d Super-Brownian Motion in random environments satisfies the following SPDE:

$$\partial_t u = \Delta u + \sqrt{u} \phi_k \beta^k + uh \omega^k$$

in weak sense, which means for any $\phi \in C^\infty_c(\mathbb{R})$ we have

$$\int_\mathbb{R} \phi(x)u(t,x)dx = \int_\mathbb{R} \phi(x)u(0,x) + \int_0^t \int_\mathbb{R} u(s,x)\phi''(x)dxds$$

$$+ \sum_{k=1}^\infty \int_0^t \int_\mathbb{R} \phi(x)\phi_k(x)\sqrt{u(s,x)}dx\beta^k_s + \sum_{k=1}^\infty \int_0^t \int_\mathbb{R} \phi(x)h_k(x)u(s,x)dxdw^k_s.$$  

Where $\beta^k, w^k$ are independent Brownian Motion.

**Proof.** By the second moment formula, we have

$$E_\mu(\langle X_t, \phi \rangle \langle X_t, \psi \rangle) = (\mu \otimes \mu)(Q_t^d(\phi \otimes \psi)) + \int_0^t \mu[\pi Q^d_{t-s}(\phi \otimes \psi)]$$

Denote $q^\phi_t$ be the density of $Q^\phi_t$. Using the above equation,

$$E_\mu(\langle X_t, p(\epsilon, x - \cdot) \rangle \langle X_t, p(\epsilon', x - \cdot) \rangle)$$

$$= \int_{\mathbb{R}^2} \mu(y_1)\mu(y_2) \int_{\mathbb{R}^2} q^\phi_t((y_1, y_2), (z_1, z_2))p(\epsilon, x - z_1)p(\epsilon', x - z_2)dz_1dz_2$$

$$+ \int_0^t ds \int_{\mathbb{R}^2} \mu(dx) \int_{\mathbb{R}^3} p(s, w - y)q^\phi_{t-s}((y, w), (z_1, z_2))p(\epsilon, x - z_1)p(\epsilon', x - z_2)dz_1dz_2dy$$

$$= I(t, x) + II(t, x)$$

(15)

It’s not hard to prove that

$$\int_0^T dt \int_{\mathbb{R}} I(t, x)dx$$

$$= \int_0^T dt \int_{\mathbb{R}} dx \int_{\mathbb{R}^2} \mu(dy_1)\mu(dy_2) \int_{\mathbb{R}^2} q^\phi_t((y_1, y_2), (z_1, z_2))p(\epsilon, x - z_1)p(\epsilon', x - z_2)dz_1dz_2$$

$$\xrightarrow{\epsilon, \epsilon' \rightarrow 0} \int_0^T dt \int_{\mathbb{R}} dx \int_{\mathbb{R}^2} \mu(dy_1)\mu(dy_2) \int_{\mathbb{R}^2} q^\phi_t((y_1, y_2), (x, x))$$

$$\leq C \int_0^T dt \int_{\mathbb{R}^2} \mu(dy_1)\mu(dy_2) \int_{\mathbb{R}} p(t, x - y_1)p(t, x - y_2)dx$$

$$\leq C \mu(\mathbb{R})^2 \int_0^T \frac{1}{\sqrt{t}}dt < \infty$$

(16)
By the same argument, 
\[ \int_0^T dt \int_\mathbb{R} I(t, x) dx \xrightarrow{\epsilon' \to 0} \int_0^T dt \int_\mathbb{R} \mu(dw) \int_\mathbb{R} p(s, w - y) q^2_{t-s}((y, y), (x, x)) dy \]
\[ \leq C \mu(\mathbb{R}) \int_0^T dt \int_0^t \frac{1}{\sqrt{t-s}} ds < \infty \]
(17)

By (15), (16), (17), we get if \( \mu(\mathbb{R}) < \infty \), \( \{\langle p(\epsilon, x - \cdot), X_t \rangle\} \) is a Cauchy sequence in \( L^2(\Omega \times [0, T] \times \mathbb{R}) \), define \( u_t(x) = \lim_{\epsilon \to 0} \langle p(\epsilon, x - \cdot), X_t \rangle \)
we have
\[ M_t^\phi = (u_t, \phi) - (u_0, \phi) - \int_0^t (u_s, \Delta \phi) ds \]
is a martingale with quadratic variation
\[ \langle M^\phi \rangle_t = \int_0^t (\phi^2, u_s) ds + \int_0^t ds \int_\mathbb{R}^2 \sum_k h_k(x) h_k(y) \phi(x) \phi(y) dx dy \]

By martingale representation theorem, there exist independent Brownian sheet \( B(t, x) \) and time-white, space-colored Guassian noise \( W(t, x) \) with \( \mathbf{E}(W(t, x)W(s, y)) = (s \wedge t)g(x, y) \) such that
\[ M_t^\phi = \int_0^t \int_\mathbb{R} \phi(x) \sqrt{u_s(x)} B(ds, dx) + \int_0^t \int_\mathbb{R} \phi(x) u_s(x) W(ds, dx) \]
The rightside of above equation can be written as
\[ \int_0^t (\phi, \sqrt{u_s} \varphi_k) d\beta^k_s + \int_0^t (\phi, u_s \varphi_k) dw^k_s \]
where \( \{\varphi_k\} \) is an orthonormal basis of \( L^2(\mathbb{R}) \), \( \sum_k h_k(x) h_k(y) = g(x, y) \) and \( \beta^k_t, w^k_t \) are independent Brownian motions. Hence
\[ (u_t, \phi) = (u_0, \phi) + \int_0^t (u_s, \phi'') ds + \int_0^t (\sqrt{u_s} \varphi_k, \phi) d\beta^k_s + \int_0^t (u_s \varphi_k, \phi) dw^k_s \]
\[ \square \]

Now we begin to consider the compact support property of following parabolic SPDEs:
\[ \partial_t u = \Delta u + u^\gamma \varphi_k \beta^k_t + u h_k \dot{w}^k_t \]
\[ (u_t, \phi) = (u_0, \phi) + \int_0^t (u_s, \phi'') ds + \int_0^t (u_s^\gamma \varphi_k, \phi) d\beta^k_s + \int_0^t (u_s h_k, \phi) dw^k_s \]
(18)
Here \( \gamma \in [1/2, 1) \), \( \{\varphi_k\} \) is the standard orthonormal basis of \( L^2(\mathbb{R}) \), \( \{h_k\} \ satisfies \)
\[ \sup_{x \in \mathbb{R}} \sum_k h^2_k(x) < \infty. \]
Define \( C_{tem} = \left\{ f \in C(\mathbb{R}) : \int_\mathbb{R} e^{-|x|} |f(x)| dx < \infty; \forall \lambda > 0 \right\} \)
The next lemma is standard.

**Lemma 3.2.** If \( u_t(x) \in C_{tem} \) is the weak solution to Equation (18) with initial data \( u_0 \in C_{tem} \), then \( u_t(x) \) satisfies the following equation:

\[
\begin{align*}
    u_t(x) &= p_t * u_0(x) + \int_0^t \left[ \int_{\mathbb{R}} p_{t-s}(x-y) u_s^2(y) \varphi_k(y) dy \right] d\beta^k_s \\
    &\quad + \int_0^t \left[ \int_{\mathbb{R}} p_{t-s}(x-y) u_s(y) h_k(y) dy \right] dw^k_s
\end{align*}
\]

(19)

**Lemma 3.3.** Suppose \( u_0 \in C_{tem}^+ \), there exists an \( \{ \mathcal{B}_t \} \)-space-time white noise \( \hat{B}(t, x) \), an independent \( \{ \mathcal{W}_t \} \)-time white space colored noise \( W(t, x) \) and a \( C(\mathbb{R}_{+};C_{tem}^+) \cap C(\mathbb{R}_{+} \times \mathbb{R}) \) solution \( u(t, \cdot) \in \mathcal{F}_t = \mathcal{B}_t \lor \mathcal{W}_t \lor \mathcal{B}_t \lor \mathcal{W}_t \) to (18) on a suitable probability space with filtration \( (\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P}) \). What’s more, for any \( \lambda > 0 \),

\[
\sup_{t \leq T} \mathbb{E} \int_{\mathbb{R}} |u_t(x)|^p e^{-\lambda|x|} dx \leq C(T, \lambda) \left\{ 1 + \int_{\mathbb{R}} u_0^p(x) e^{-\lambda|x|} dx \right\}
\]

(20)

**Proof.** The proof for existence of \( C_{tem}^+ \) solution to (18) is similar with Theorem 2.6 in [7], so we only prove (20) here.

Taking \( p \)'s power in both side of (19) then taking expectation, using BDG inequality and Minkowski inequality, we obtain

\[
\begin{align*}
    \mathbb{E}|u_t(x)|^p &\leq C \left\{ |p_t * u_0|^p + \mathbb{E} \left[ \int_0^t ds \int_{\mathbb{R}} p_{t-s}^2(x-y) u_s^{2\gamma}(y) dy \right]^{p/2} \\
    &\quad + \mathbb{E} \left[ \int_0^t ds \sum_k \left( \int_{\mathbb{R}} p_{t-s}(x-y) u_s(y) h_k(y) dy \right)^{p/2} \right] \right\} \\
    &\leq C \left\{ |p_t * u_0|^p + \mathbb{E} \left[ \int_0^t (t-s)^{-1/2} ds \int_{\mathbb{R}} (t-s)^{1/2} p_{t-s}^2(x-y)(1 + u_s^2(y)) dy \right]^{p/2} \\
    &\quad + \int_0^t ds \int_{\mathbb{R}} p_{t-s}(x-y) \mathbb{E} u_s^p(y) dy \right\} \\
    &\leq C \left\{ |p_t * u_0|^p + \int_0^t (t-s)^{-1/2} ds \int_{\mathbb{R}} (t-s)^{1/2} p_{t-s}^2(x-y)(1 + \mathbb{E} u_s^p(y)) dy \right\} \\
    &\quad + \int_0^t ds \int_{\mathbb{R}} p_{t-s}(x-y) \mathbb{E} u_s^p(y) dy \right\}
\end{align*}
\]

Hence, for any \( t \leq T \) we have

\[
\begin{align*}
    \int_{\mathbb{R}} e^{-\lambda|x|} \mathbb{E}|u_t(x)|^p dx &\leq C \left\{ 1 + \int_{\mathbb{R}} u_0^p(y) dy \int_{\mathbb{R}} p_t(x-y) e^{-\lambda|x|} dx \\
    &\quad + \int_0^t (t-s)^{-1/2} ds \int_{\mathbb{R}} \mathbb{E} u_s^p(y) dy \int_{\mathbb{R}} (t-s)^{1/2} p_{t-s}^2(x-y) e^{-\lambda|x|} dx \\
    &\quad + \int_0^t ds \int_{\mathbb{R}} \mathbb{E} u_s^p(y) dy \int_{\mathbb{R}} p_{t-s}(x-y) e^{-\lambda|x|} dx \right\} \\
    &\leq C \left\{ 1 + \int_{\mathbb{R}} u_0^p(x) e^{-\lambda|x|} dx + \int_0^t \frac{ds}{\sqrt{t-s}} \int_{\mathbb{R}} e^{-\lambda|y|} u_s^p(y) dy \right\}
\end{align*}
\]
In the last inequality, we use the element inequality: \( \sup_{t \leq T} e^{\lambda |y|} \int_{\mathbb{R}} e^{-\lambda |x|} |u_s(x)|^p dx \leq C. \)

Denote \( f(t) = \sup_{s \leq t} \int_{\mathbb{R}} e^{-\lambda |x|} E|u_s(x)|^p dx, A = 1 + \int_{\mathbb{R}} u_0^n(x)e^{-\lambda |x|} dx \) then,

\[
f(t) \leq CA + C \int_0^t f(s) \frac{ds}{\sqrt{t-s}} \leq CA + C \int_0^t \int_0^s f(r) \frac{dr}{\sqrt{s-r}} \frac{ds}{\sqrt{t-s}} \leq CA + C \int_0^t f(r) dr \int_r^t \frac{ds}{\sqrt{(t-s)(s-r)}} \leq CA + C \int_0^t f(s) ds
\]

Using Gronwall’s inequality, we obtain \( [20] \)

**Corollary 3.4.** Suppose \( u \in C(\mathbb{R}_+; C^+_{\text{tem}}) \) is a solution to \( [18] \) with \( u_0(x) = 0 \ (x \geq 0) \) then for any \( T > 0, \)

\[
a_p(T) \triangleq \sup_{n \in \mathbb{N}} E \int_0^T dt \int_n^{n+1} u_t^n(x) dx < \infty
\]

**Proof.** For any \( n \in \mathbb{N} \), let \( v_t(x) = u_t(n + x) \in C([0, T], C^+_{\text{tem}}) \), \( v \) satisfies the equation

\[
\partial_t v_t(x) = \Delta v_t(x) + v_t^\gamma(x) \varphi_k(n + x) \beta_k^x + v_t(x) h_k(n + x) \dot{w}_k
\]

Since \( \{\varphi_k(n + \cdot)\} \) is again the orthonormal basis of \( L^2(\mathbb{R}) \), \( \{h_k(n + \cdot)\} \) satisfies the same condition with \( \{h_k\} \). By Lemma \((3,3)\) we have

\[
E \int_0^T \int_0^1 v_t^n(x) dx dt \leq C(T) \left\{ 1 + \int_{-\infty}^0 e^{-|x|} u_0(n + x) dx \right\} \leq C(T) \left\{ 1 + \int_{-\infty}^{-n} e^{-|x|} u_0(x) dx \right\} \leq C(T)
\]

The last constant is independent with \( n. \)

The following theorem is our main result.

**Theorem 3.5.** If \( u_0 \in C^+_{\text{tem}}, u_0(x) = 0 \ (x \geq 0), u_t(x) \in C(\mathbb{R}_+, C^+_{\text{tem}}) \) is the solution to equation \( [18], \) then \( \exists N(\omega), \) such that \( u(t, x, \omega) = 0 \ \forall x \geq N(\omega). \)

Before proving the main theorem, we need some simple estimates. Define \( P_t f(x) = (4\pi t)^{-1/2} \int_{\mathbb{R}} e^{-|x-y|^2/4t} f(y) dy, \)

\[
\mathcal{F} \left( \int_0^\infty t^{\delta/2-1} e^{-t} P_t f dt \right) (\xi) = \mathcal{F} f(\xi) \int_0^\infty t^{\delta/2-1} e^{-t} t^{\xi^2}$ \)

Hence

\[
(1 - \Delta)^{-\delta/2} f = c(\delta) \int_0^\infty t^{\delta/2-1} e^{-t} P_t f dt
\]

Define

\[
R_\delta(x) = c(\delta) \int_0^\infty t^{\delta/2-1} (2\pi t)^{-1/2} e^{-t} e^{-|x|^2/4t} dt = c(\delta) \int_0^\infty t^{\delta/2-3/2} e^{-t} e^{-|x|^2/4t} dt
\]
Suppose $\delta < 1$, if $|x| \ll 1$ then

$$
\int_0^\infty t^{\delta/2-3/2}e^{-t}e^{-|x|^2/4t}dt \leq C|x|^\delta \int_0^\infty s^{-\delta/2-1/2}e^{-s}ds \leq C|x|^\delta
$$

If $|x| \gg 1$,

$$
\int_0^\infty t^{\delta/2-3/2}e^{-t}e^{-|x|^2/4t}dt \leq Ce^{-|x|} \int_1^\infty t^{\delta/2-3/2}dt + C\int_0^1 e^{-1/2t}dt \leq Ce^{-|x|}
$$

Hence, $R_\delta \in L^p(\mathbb{R})$ with $p < 1/(1-\delta)$;

Suppose $\delta = 1$, if $x \ll 1$, then

$$
R_1(x) = c(\delta) \int_0^\infty t^{-1}e^{-t}e^{-|x|^2/4t}dt \leq C \left( \int_0^{|x|} t^{-1}e^{-|x|^2/t}dt + \int_{|x|}^\infty t^{-1}e^{-t}dt \right) \leq -C \log |x|
$$

and $R_1 \leq Ce^{-c|x|}$ when $|x| \to \infty$. Hence $R_1 \in L^p(\mathbb{R})$ ($p < \infty$);

Suppose $\delta > 1$, then $R_\delta$ is bounded and not greater than $Ce^{-|x|}$ when $|x| \to \infty$. Hence $R_\delta \in L^p(\mathbb{R})$ ($p \leq \infty$).

By the same argument we have $R_{\delta+1}' \in L^p(\mathbb{R})$ with $p(1-\delta) < 1$.

**Lemma 3.6.** Suppose $u \in C(\mathbb{R}_+; C^+_\text{tem})$ is a solution to (18) with $u_0(x) = 0$ for any $x > 0$, then

$$
E \sup_{t \leq T} \left( \int_0^\infty u_t(x)dx \right)^2 \leq Ce^{CT}; \quad E \int_0^T dt \int_0^\infty u_t^2(x)dx \leq Ce^{CT}.
$$

**Proof.** Choose

$$
\phi_n(x) = \begin{cases} 
0 & x \leq 0 \text{ or } x > n \\
\frac{1}{2}[1 + \sin \pi(x - \frac{1}{2})] & 0 < x \leq 1 \\
1 & 1 < x < n - 1 \\
\frac{1}{2}[1 + \cos \pi(x - n + 1)] & n - 1 < x \leq n
\end{cases}
$$

For convenience we omit the subindex $n$. And all the estimates below are independent with $n$.

By definition

$$(u_t, \phi) = \int_0^t (u_s, \phi''')ds + \int_0^t (u_s^2, \phi) d\beta_s + \int_0^t (u_s h_s, \phi) dw_s.$$
By Doob’s inequality

\[
E \sup_{t \leq T} (u_t, \phi)^2 \leq C \left\{ E \int_0^T \sum_k (u_t^k, \varphi_k)^2 dt + E \int_0^T \sum_k (u_t\phi, h_k)^2 dt + E \left( \int_0^T \left| (u_t, \phi'') \right| dt \right)^2 \right\}
\]

\[
\leq C \left\{ E \int_0^T dt \int_{\mathbb{R}} u_t^2(x)\phi^2(x) dx + E \int_0^T dt \left( \int_{\mathbb{R}} u_t(x)\phi(x) dx \right)^2 + E \left( \int_0^T (u_t, |\phi''|) dt \right)^2 \right\}
\]

\[
\leq C \left\{ a(T) + E \int_0^T dt \int_{\mathbb{R}} [u_t(x)\phi(x) + u_t^2(x)\phi^2(x)] dx + E \int_0^T dt \left( \int_{\mathbb{R}} u_t(x)\phi(x) dx \right)^2 \right\}
\]

\[
\leq C \left\{ 1 + E \int_0^T dt \int_{\mathbb{R}} u_t^2(x)\phi^2(x) dx + E \int_0^T \textbf{E}(u_t, \phi)^2 dt \right\}
\]

(21)

Let \( v_t = u_t\phi, \) \( v_t \) satisfies the following equation

\[
\partial_t v = \Delta v + \phi^{1-\gamma}v^\gamma \varphi_k \hat{\beta}^k_t + vh_k \hat{v}^k_t
\]

where \( \kappa = -2(u\phi')' + u\phi''. \)

\[
\|(1 - \Delta)^{-1/2}q_1\|^2_{L^2(I)} = E \int_0^T \int_{\mathbb{R}} |R_1 \ast \varphi_k|^2 dx dt
\]

\[
\leq C \left\{ E \int_0^T dt \int_{\mathbb{R}} |R_2 \ast (u_t\phi')(x)|^2 dx + E \int_0^T dt \int_{\mathbb{R}} |R_2 \ast (u_t\phi'')(x)|^2 dx \right\}
\]

\[
\leq C \left\{ E \int_0^T dt \left( \int_{\mathbb{R}} |u_t\phi'(x)| dx \right)^2 + E \int_0^T dt \left( \int_{\mathbb{R}} |u_t\phi''(x)| dx \right)^2 \right\}
\]

\[
\leq C \left\{ E \int_0^T dt \int_0^1 u_t(x)^2 dx + E \int_0^T dt \int_{n-1}^n u_t(x)^2 dx \right\}
\]

\[
\leq a_2(T)
\]

\[
\|(1 - \Delta)^{-1/2}q_1\|^2_{L^2(I)} = E \int_0^T dt \int_{\mathbb{R}} |R_1 \ast \phi^{1-\gamma}v^\gamma \varphi_k|^2 dx
\]

\[
\leq C E \int_0^T dt \int_{\mathbb{R}} dx \int_{\mathbb{R}} R_1^2(x - y)v^{2\gamma}(y) dy
\]

\[
\leq C E \int_0^T dt \int_{\mathbb{R}} v_t^{2\gamma}(x) dx
\]

\[
\leq C K E \int_0^T dt \int_{\mathbb{R}} v_t(x) dx + C \epsilon E \int_0^T dt \int_{\mathbb{R}} v_t^2(x) dx
\]

\[
\leq C K E \int_0^T dt \left( \int_{\mathbb{R}} v_t(x) dx \right)^2 + C \epsilon E \int_0^T dt \int_{\mathbb{R}} v_t^2(x) dx
\]
\[ \|(1 - \Delta)^{-1/2} v h_k\|_{L^2(\mathbb{R})}^2 = \mathbb{E} \int_0^T dt \int_\mathbb{R} \sum_k |R_1 * v_t h_k(x)|^2 dx \]
\[ = \mathbb{E} \int_0^T dt \int_\mathbb{R} dx \left[ \sum_k \left( \int_\mathbb{R} R_1(x - y) v_t(y) h_k(y) dy \right)^2 \right] \]
\[ \leq \mathbb{E} \int_0^T dt \int_\mathbb{R} dx \left\{ \int_\mathbb{R} \left[ |R_1(x - y) v_t(y)|^2 \sum_k h_k^2(y) \right]^{1/2} dy \right\}^2 \]
\[ \leq C \mathbb{E} \int_0^T dt \|R_1 * v_t\|_2^2 \leq C \mathbb{E} \int_0^T dt \|R_1\|_2^2 \|v_t\|_1^2 \]
\[ \leq C \mathbb{E} \int_0^T dt \left( \int_\mathbb{R} v_t(x) dx \right)^2 \]

By [1] Theorem 5.1,
\[ \|v\|_{L^2(T)}^2 \leq C \left\{ \|(1 - \Delta)^{-1/2} \phi \|_{L^2(\mathbb{R})}^2 + \|(1 - \Delta)^{-1/2} \phi \|_{L^2(\mathbb{R})}^2 + \|v \|_{L^2(\mathbb{R})}^2 + \|v \|_{L^2(\mathbb{R})}^2 \right\} \]
\[ \leq C \left[ 1 + K' \mathbb{E} \int_0^T dt \left( \int_\mathbb{R} v_t(x) dx \right)^2 \right] + C \epsilon \mathbb{E} \int_0^T dt \int_\mathbb{R} v_t^2(x) dx \]

Choose \( \epsilon \) small, such that \( C \epsilon \leq 1/2 \). Since \( \|v\|_{L^2(T)}^2 \geq \mathbb{E} \int_0^T dt \int_\mathbb{R} v_t^2(x) dx \), we have
\[ \mathbb{E} \int_0^T dt \int_\mathbb{R} v_t^2(x) dx \leq C \left[ 1 + K' \mathbb{E} \int_0^T dt \left( \int_\mathbb{R} v_t(x) dx \right)^2 \right] \quad (22) \]

Combining (21), (22) we get
\[ \mathbb{E} \sup_{t \leq T} \left( \int_\mathbb{R} v_t(x) dx \right)^2 \leq C \left[ 1 + \mathbb{E} \int_0^T dt \left( \int_\mathbb{R} v_t(x) dx \right)^2 \right] \]

Using Gronwall’s inequality,
\[ \mathbb{E} \sup_{t \leq T} \left( \int_\mathbb{R} v_t(x) dx \right)^2 \leq C e^{CT} \]

Since our estimates independent with \( n \), we can let \( n \to \infty \), we get
\[ \mathbb{E} \sup_{t \leq T} \left( \int_0^\infty u_t(x) dx \right)^2 \leq C e^{CT} \]
\[ \mathbb{E} \int_0^T dt \int_0^\infty u_t^2(x) dx \leq C \left[ 1 + \mathbb{E} \sup_{t \leq T} \left( \int_0^\infty u_t(x) dx \right)^2 \right] \leq C e^{CT} \]

\[ \square \]

**Lemma 3.7.** Suppose \( u \in C(\mathbb{R}^+; C_{tem}^+) \) is a solution to (18) satisfying \( u_0(x) = 0 \) on \( \mathbb{R}_+ \). Then
\[ \| u \|_{C^\infty([0,T] \times \mathbb{R}_+)} < \infty \ \text{a.s.}, \quad (23) \]
for some $\alpha \in (0, 1)$ and
\[
E \sup_{t \leq T} \int_0^\infty x u_t(x) dx < \infty
\] (24)

Proof. Let
\[
\zeta(x) = \begin{cases} 
0 & x \leq 0 \\
\frac{1}{2}[1 + \sin \pi(x - \frac{1}{2})] & 0 < x \leq 1 \\
1 & x > 1
\end{cases}
\]
Like the proof of Lemma 3.6 we define $v_t = u_t \zeta$, then $v_t$ satisfies the equation
\[
\partial_t v = \Delta v + \kappa + \zeta^{1-\gamma} \varphi_k \beta_t^k + vh_k \omega_t^k; \ v_0 = 0
\]
where $\kappa = -2(u\zeta')' + u\zeta''$. Let
\[
T_n = n \land \inf \left\{ t \geq 0 : \int_0^t \left( \int_\mathbb{R} v_s(x) dx \right)^p ds \geq n \right\}.
\]
then $T_n \to \infty$ a.s.. Since $R_2$ behaviors like $-\log |x|$ near the original and decreases exponentially, as $|x| \to \infty$ as before we can prove
\[
\|(1 - \Delta)^{-1} \kappa\|_{L^p(T)} < \infty \quad (\forall p \in \mathbb{N}).
\]
We claim for any $p \geq 2$,
\[
E \int_0^T \|v_t\|_p^p dt < \infty.
\]
(25)
Let $p_k = 2/\gamma_k$, if we have
\[
E \int_0^T \|v_t\|_{p_k}^{p_k} dt < \infty.
\]
then
\[
\|(1 - \Delta)^{-1/2} \zeta^{1-\gamma} \varphi_k\|_{L^{p_k+1}(T^2)} \leq C E \int_0^T dt \int_\mathbb{R} dx \left( \int_\mathbb{R} R_1^2(x - y)v_t^{2\gamma}(y) dy \right)^{p_k+1/2}
\]
\[
\leq C E \int_0^T dt \int_\mathbb{R} v_t^{p_k}(x) dx < \infty
\]
\[
\|(1 - \Delta)^{-1/2} v h_k\|_{L^{p_k+1}(T^2)} \leq C E \int_0^T dt \int_\mathbb{R} dx \left( \sum_k \left( \int_\mathbb{R} R_1(x - y)v_t h_k(y) \right)^2 \right)^{p_k+1/2}
\]
\[
\leq C E \int_0^T dt \int_\mathbb{R} dx \left( \int_\mathbb{R} R_1(x - y)v_t h_k(y) \right)^{p_k+1}
\]
\[
\leq C E \int_0^T dt \left( \int_\mathbb{R} v_t(x) dx \right)^{p_k+1} < \infty
\]
Hence, if (25) holds, by [1, Theorem 5.1], we obtain
\[
E \int_0^T \|v_t\|_{p_k+1}^{p_k+1} dt \leq \|v\|_{L^{p_k+1}(T)}^{p_k+1} < \infty
\]
Hence, for any $p \geq 2$, inequality (25) holds. Like the argument in the Lemma 1.5 of [2], choose $1/2 < \delta < 1$, for any $p \geq 2$,

$$\left( \int_{\mathbb{R}} |R_{\delta+1} \ast \kappa_t(x)|^p dx \right)^{1/p} \leq C \left( \int_{\mathbb{R}} |R'_{\delta+1} \ast (\phi'u_t)(x)|^p dx \right)^{1/p} + C \left( \int_{\mathbb{R}} R_{\delta+1} \ast (\phi''u_t)(x) dx \right)^{1/p}$$

$$\leq C(\|R'_{\delta+1}\|_p + \|R_{\delta+1}\|_p) \int_{[0,1]\cup[n-1,n]} u_t(x) dx$$

Hence

$$\mathbb{E} \int_0^T dt \int_{\mathbb{R}} |R_{\delta+1} \ast \kappa_t(x)|^p dx \leq C \mathbb{E} \int_0^T dt \left( \int_{[0,1]\cup[n-1,n]} u_t(x) dx \right)^p < \infty$$

$$\| (1 - \Delta)^{-\delta/2} \phi^{1-\gamma} v^{\gamma} \varphi_k \|^p_{L^p(\mathbb{R})} \leq C \mathbb{E} \int_0^T dt \int_{\mathbb{R}} dx \left[ \int_{\mathbb{R}} R^2_{\sigma}(x-y) v^{2\gamma}(y) dy \right]^{p/2}$$

$$\leq C \| R_{\delta} \|^2 \mathbb{E} \int_0^T dt \int_{\mathbb{R}} v^{\gamma p}(x) dx < \infty$$

$$\| (1 - \Delta)^{-\delta/2} vh_k \|^p_{L^p(\mathbb{R})} \leq C \mathbb{E} \int_0^T dt \int_{\mathbb{R}} dx \left\{ \sum_k \left[ \int_{\mathbb{R}} R_{\delta}(x-y) v h_k(y) \right]^2 \right\}^{p/2}$$

$$\leq C \mathbb{E} \int_0^T dt \int_{\mathbb{R}} dx \left[ \int_{\mathbb{R}} R_{\delta}(x-y) v h_k(y) \right]^{p}$$

$$\leq C \| R_{\delta} \|^p \mathbb{E} \int_0^T dt \| v_t \|^p < \infty$$

By [1, Theorem 5.1], we have $v \in \mathcal{H}_{p}^{1-\delta}$. By choosing for instance $\delta = 0.6$, $p = 33$, we have $v \in \mathcal{H}_{p}^{1-\delta}(T)$, and by Sobolev’s embedding theorem, $C^{1/10}([0, T] \times \mathbb{R}) \subset \mathcal{H}_{p}^{1-\delta}(T)$ we get

$$\| u \|_{C^{1/10}([0, T] \times \mathbb{R})} < \infty \ \ a.s.$$  

Now we prove (24). Choose $\eta_n(x) \in C_c(\mathbb{R})$ and $supp \eta_n \in \mathbb{R}_+$, $\eta_n(x) = x$ when $x \in [1, n]$ and $sup_{x,n} |\eta''_n(x)| < \infty$.

$$0 \leq \int_1^n x u_t(x) dx \leq \int \eta_n(x) u_t(x) dx$$

$$= \int_0^t \int u_s(x) \eta''_n(x) dx ds + M^n_t$$

$$[M^n]_t = \int_0^t u^{2\gamma}_s(x) \eta''_n(x) dx ds + \sum_k \int_0^t ds \left( \int u_s(x) h_k(x) \eta_n(x) dx \right)^2$$

$$\leq C \int_0^t \left( \int u_s(x) dx \right)^2 ds + \int_0^t \int u^{2\gamma}_s(x) dx ds \in L^1(\mathbb{P})$$

Hence $M^n_t$ is a martingale, taking expectation and let $n \to \infty$, we get

$$\mathbb{E} \sup_{t \leq T} \int_0^\infty x u(t,x) dx \leq C \mathbb{E} \sup_{t \leq T} \int_0^\infty u_t(x) dx < \infty$$
The proof of Theorem 3.5 follows the idea of [2], we present here for reader’s convenience.

**Proof Of Main Theorem.** We follow the proof in [2].

**Step 1.** For $\psi \in C_c(\mathbb{R})$, if $\psi'' = \nu$ is a finite measure on $\mathbb{R}$, then equation (18) also holds. (see [2, Lemma 3.1])

**Step 2.** On the set $\{\omega : \int_0^T u_s(0, \omega) ds = 0\}$, $u(t, x) = 0 \forall x > 0, t \in [0, T]$. Let $\psi_n = x_n \phi(x/n)$, 

$$0 \leq \int_0^\infty \psi_n(x) u_t(x) dx = \int_0^t u_s(0) ds + \int_0^t \int_0^\infty u_s(x) \psi''_n(x) dx ds + M^n_t$$

where $M^n_t$ is a local martingale with 

$$[M^n]_t = \int_0^t \int_0^\infty \psi^2_n(x) u^2_n(x) dx ds + \sum_k \int_0^t ds \left( \int_{\mathbb{R}} \psi_n(x) h_k(x) u_s(x) dx \right)^2 \in L^1(P).$$

Hence $M^n_t$ is a martingale, so $E|M^n_t| = 2E(M^n_\tau)^-$ for any bounded stopping time $\tau$. Let 

$$V^n_t = \int_0^t u_s(0) ds + \int_0^t \int_0^\infty u_s(x) \psi''_n(x) dx ds.$$ 

Using (26), 

$$(M^n_\tau)^- \leq V^n_t \leq C \left\{ \int_0^t u_s(0) ds + \frac{1}{n} \int_0^t \int_0^\infty u_s(x) dx ds + \frac{1}{n^2} \int_0^t \int_0^\infty x u_s(x) dx ds \right\}$$

Hence for any bounded stopping time $\tau$, 

$$E|M^n_\tau| = 2E(M^n_\tau)^- \leq 2EV^n_\tau$$

By the generalized Ito’s inequality, we get for any $0 < \alpha < 1$ and any bounded stopping time $\tau$,

$$E \left( \int_0^\tau \int_0^\infty \psi^2_n(x) u^2_n(x) dx ds \right)^{\alpha/2} \leq E[M^n]_\tau^{\alpha/2} \leq C E(\sup_{s \leq \tau} |M^n_s|)^{\alpha}$$

$$\leq C E(V^n_\tau)^{\alpha} \leq C E V^n_\tau$$

$$\leq C \left\{ E \int_0^\tau u_s(0) ds + \frac{1}{n} E \int_0^\tau \int_0^\infty u_s(x) dx ds \right.\right.$$ 

$$\left. + \frac{1}{n^2} E \int_0^\tau \int_0^\infty x u_s(x) dx ds \right\}$$

Let $n \to \infty$, we get 

$$E \left( \int_0^\tau \int_0^\infty x^2(x) u^2_n(x) dx ds \right)^{\alpha/2} \leq C \int_0^\tau u_s(0) ds$$

Now let $\tau = \inf\{t : \int_0^t u_s(0) ds > 0\}$, we obtain the conclusion.

**Step 3.** Following the proof of Lemma 2.1 of [2] one can show: If $\gamma \in [\frac{1}{2}, 1)$, then for any $p, q > 0$, $0 < r \leq 1$, $0 < \alpha < 1$ there exists a point $x \in [r, 2r]$ such that 

$$P \left( \int_0^T u_s(x) ds \geq p \right) \leq P \left( \int_0^T u_s(0) ds \geq q \right) + C r^{-3\alpha/2} \left( \frac{q}{p^\gamma} \right)^\alpha$$

\[\square\]
and here $C$ is independent with $p, q, r.$

Step 4. Now the Theorem can be prove just as Theorem 1.7 of [2].

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