On the small–amplitude approximation to the
differential equation $\ddot{x} + (1 + \dot{x}^2)x = 0$

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Abstract

We obtain the radius of convergence of the small–amplitude approximation to the period of the nonlinear oscillator $\ddot{x} + (1 + \dot{x}^2)x = 0$ with the initial conditions $x(0) = A$ and $\dot{x}(0) = 0$ and show that the inverted perturbation series appears to converge smoothly from below.

There has recently been great interest in the study of the period of the nonlinear oscillator

$$\ddot{x}(t) + [1 + \dot{x}(t)^2]x(t) = 0$$

$$x(0) = A, \quad \dot{x}(0) = 0$$

(1)

as a function of the amplitude $A$. Apparently, it aroused from the fact that the first–order harmonic balance method yielded the approximate frequency [1]

$$\omega_{HB}(A) = \frac{2}{\sqrt{4 - A^2}}$$

(2)

that is not defined for $A > 2$.

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By straightforward analysis of the dynamical trajectories in the $x - y$ plane, where $y = \dot{x}$, Beatty and Mickens [2] concluded that such a restriction is merely an artifact of the harmonic balance method.

Later, Mickens [3] derived an explicit expression for the period

$$T(A) = 4A \int_{0}^{1} \frac{du}{\sqrt{e^{4\rho(1-u^2)} - 1}}$$

where $u = x/A$. By means of this expression he proved that $dT/dA < 0$ and obtained upper and lower bounds to the period.

Kalmár–Nagy and Erneux [4] derived the behaviour of the period for small and large values of $A$

$$T(A) \approx 2\pi \left(1 - \frac{A^2}{8}\right), \quad A \ll 1$$
$$T(A) \approx \frac{2\pi}{A}, \quad A \gg 1$$

as well as most interesting approximations to the periodic orbits in both limits. In particular, they showed that the trajectory $u(t)$ satisfies the equation

$$\ddot{u} + \frac{dV}{du} = 0,$$
$$V(u) = \frac{1 - e^{\rho(1-u^2)}}{2\rho}, \quad \rho = A^2$$

that leads to the same expression for the period [3] derived earlier by Mickens [3].

The results of those authors clearly show that the period $T(A)$ does not exhibit singular points for real values of $A$ but they do not explain why the harmonic balance fails as shown in equation [2] [1]. A possible explanation is that the harmonic balance is reflecting a singular point in the complex $A$–plane. If it exists, then the small–amplitude expansion will have a finite radius.
of convergence.

In order to derive the small–amplitude expansion we change the integration variable in equation (3) to \( u = \cos \theta \) so that the period becomes

\[
T(\rho) = 4 \int_0^{\pi/2} \frac{d\theta}{\sqrt{F(\rho \sin^2 \theta)}}
\]  
(6)

where

\[
F(z) = \frac{e^z - 1}{z} = \sum_{j=0}^{\infty} \frac{z^j}{(j+1)!}
\]  
(7)

If we substitute the expansion

\[
\frac{1}{\sqrt{F(z)}} = \sum_{j=0}^{\infty} c_j z^j = 1 - \frac{z}{4} + \frac{z^2}{96} + \frac{z^3}{384} + \ldots
\]  
(8)

into the equation (6) we obtain as many coefficients as desired of the small–amplitude series

\[
T(A) = T_0 + T_1 \rho + T_2 \rho^2 + \ldots \\
= 2\pi \left( 1 - \frac{\rho}{8} + \frac{\rho^2}{256} + \ldots \right)
\]  
(9)

The function \( 1/\sqrt{e^z - 1} \) has two complex–conjugate singular points closest to the origin at \( z = \pm 2\pi i \); therefore

\[
\lim_{j \to \infty} \left| \frac{c_j}{c_{j+1}} \right| = 2\pi
\]  
(10)

If we take into account that

\[
I_j = \int_0^{\pi/2} \sin^{2j} \theta \, d\theta = \frac{\sqrt{\pi} \Gamma(j + 1/2)}{2 \Gamma(j + 1)}
\]  
(11)
then we conclude that

$$\lim_{j \to \infty} \left| \frac{c_j I_j}{c_{j+1} I_{j+1}} \right| = \lim_{j \to \infty} \left| \frac{c_j}{c_{j+1}} \right| = 2\pi$$

(12)

In other words, the $\rho$–power series has a finite radius of convergence $R_\rho = 2\pi$ because of a pair of complex conjugate singular points at $\rho_c = \pm 2\pi i$.

Fig. 1 shows the first partial sums $S_T^{[N]}(\rho) = T_0 + T_1 \rho + \ldots + T_N \rho^N$ and the accurate numerical values of $T(A)$. It dramatically illustrates the effect of the nonzero convergence radius $R_A = \sqrt{2\pi}$ of the small–amplitude expansion determined by the singular points of $T(A)$ closest to the origin in the complex $A$–plane.

We can provide another argument about the location of the singular points of $T(A)$. First, note that the effective potential–energy function $V(u)$ given in equation (5) exhibits a minimum $V(0) = (1 - e^\rho)/(2\rho) < 0$ and that the energy of the oscillatory motion is $E = \dot{u}^2/2 + V(u) = 0$ for the given initial conditions. Therefore, we expect a critical value of $\rho$ given by $V(0) = 0$ that yields $\rho_c = \pm 2\pi i$ in agreement with the analysis above based on the small–amplitude series.

In order to verify those exact analytical results in a numerical way we constructed Padé approximants $[N, N](\rho)$ from the partial sums $S_T^{[2N]}(\rho)$ and looked for the complex zeroes of the denominator. A sequence of such zeroes appeared to converge to a limit quite close to $\pm 6.3i$ with a small real part that was negligible compared to the errors of the estimates. Besides, assuming that there is an algebraic singular point closest to origin of the form $(z - z_0)^\alpha$ we carried out the same Padé analysis, but now on $T^{-1}dT/dA$ (as a function of $\rho$), and obtained roughly the same complex numbers that are quite close to $\pm 2\pi i$. Therefore, there appears to be no doubt that the radius of convergence of the $\rho$–power series is in fact $R_\rho = 2\pi$ and is due to complex conjugate singular points located on the imaginary axis of the complex $\rho$–plane at $\pm 2\pi i$. 

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Recently, Amore and Fernández [5] investigated the possible advantages of the inverted perturbation series that in the present case takes the form

\[
\rho = \rho_1 \Delta T + \rho_2 \Delta T^2 + \ldots
\]

\[
= -\frac{4\Delta T}{\pi} + \frac{\Delta T^2}{2\pi^2} - \frac{13\Delta T^3}{24\pi^3} + \ldots
\]

\[
\Delta T = T - 2\pi
\]

Fig. 2 shows that the partial sums for the inverted series \( S_\rho^{[N]}(\Delta T) = \rho_1 \Delta T + \rho_2 \Delta T^2 + \ldots + \rho_N \Delta T^N \) converge smoothly from below towards the accurate numerical values of \( T(A) \). We are presently unable to prove such most interesting feature of the inverted series rigorously.

**Summarizing:** Earlier studies on the nonlinear oscillator \([1, 2, 3]\) have clearly shown that the period is finite for all values of the amplitude. However, they did not cast any light on the failure of the harmonic balance (with the ansatz \( x^{HB}(t) = A \cos(\omega t) \)) that predicts a singularity for \( A = 2 \). In this paper we suggest that the harmonic balance may be reflecting the singular points that determine the radius of convergence of the small–amplitude series for the period. We have exactly calculated the location of those singular points and concluded that the series converge for \( 0 < A < \sqrt{2\pi} \). This result may help to understand similar difficulties in future applications of the harmonic balance. We expect that a harmonic–balance approach with more terms will give a frequency with only complex singular points.

In addition to what was mentioned above, we have shown that for this problem the inverted perturbation series appears to converge smoothly from below and it is therefore preferable to the original small–amplitude expansion. It is an old and well–known approach that may, in some cases, lead to surprisingly accurate results [5].
References

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Fig. 1. Accurate numerical period (solid line) and partial sums for the small–amplitude approximation (dashed lines).

Fig. 2. Accurate numerical period (solid line) and partial sums for the inverse perturbation series (dashed lines).