Higher-Order Topological Insulator in a Dodecagonal Quasicrystal

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(Dated: January 22, 2020)

Higher-order topological insulators (HOTIs) are a newly discovered class of topological insulators, exhibiting unconventional bulk-boundary correspondence. Very recently, the concept of HOTIs has been extended to quasicrystalline systems, where a novel HOTI phase protected by an eightfold rotational symmetry was identified. Here we propose a distinct quasicrystalline HOTI in a dodecagonal quasicrystal. We observe twelfold-symmetric zero-energy corner modes located at the boundary of a quasicrystal dodecagon, which are the hallmark feature of the quasicrystalline HOTI. These zero-energy corner modes are protected by the combination of a twelfold rotational symmetry and a mirror symmetry as well as particle-hole symmetry, which have no crystalline counterpart.

Introduction.— Since the discovery of topological insulator phase, the exploration of various topological phases of matter has become a major goal of research in condensed matter physics [1–6]. According to the three fundamental non-spatial symmetries containing particle-hole symmetry (PHS), time-reversal symmetry (TRS), and chiral symmetry, the fully gapped free fermionic systems have been classified into the ten Altland-Zirnbauer symmetry classes [7–11]. Later, with the notion of “topological crystalline insulators” [12, 13] being put forward, the influence of space-group symmetries on topological phases has attracted extensive attention and enriches the topological classification of crystalline solids [12–19]. Very recently, a new class of topological crystalline insulators, known as higher-order topological insulators (HOTIs) [20–84], was discovered. HOTIs are protected by space-group symmetries, such as mirror, inversion, and rotational symmetries, but exhibit unconventional bulk-boundary correspondence. For example, a second-order topological insulator in two dimensions has topological gapless boundary states at its zero-dimensional boundary corners, in contrast to the conventional two-dimensional (2D) first-order topological insulators which have one-dimensional (1D) gapless edge states.

Up to now, most of the published research on topological phases is done in crystalline systems. Interestingly, the quasicrystalline systems, which lack the translational symmetry but possess forbidden symmetries in crystals, are found to host topological phases [48, 49, 85–102]. Even more striking is that a kind of HOTI protected by an eightfold rotational symmetry can be realized in the Ammann-Beenker (AB) tiling octagonal quasicrystal [48, 49], which can not be found in crystals because an eightfold rotational symmetry is incompatible with translational symmetry. An intriguing question is whether there are other quasicrystalline HOTIs which do not have counterpart in crystals.

Another representative type of quasicrystal is the dodecagonal quasicrystal with twelfold rotational symmetry. The first dodecagonal quasicrystalline lattice (QL) was proposed by P. Stampfli [103]. Recently, the dodecagonal QL was realized in the twisted bilayer graphene rotated exactly 30° [104–108]. Moreover, Park et al. [109] proposed that a crystalline HOTI protected by a mirror symmetry and a sixfold rotational symmetry can be realized in the twisted bilayer graphene with the twist angle 21.78°.

In this work, we investigate the 2D HOTI phase in the Stampfli-tiling dodecagonal quasicrystal. The dodecagonal quasicrystal is tiled using squares, triangles, and rhombuses (see Fig. 1). The construction process of the dodecagonal QL is shown in the Supplemental Material [110] (see Fig. S1). We uncover a HOTI protected by the combination of the twelfold rotational symmetry $C_{12}$ and mirror symmetry $m_z$ as well as PHS in the Stampfli-tiling dodecagonal QL. To obtain the HOTI, we start with a first-order topological insulator model with TRS on the dodecagonal QL, which supports counter-propagating 1D edge modes. Then, we introduce an additional mass term that breaks both TRS and $C_{12}$ to gap out the 1D edge modes and form an edge mass domain. We found twelve zero-energy modes symmetrically localized at the twelve corners of the regular Stampfli-tiling quasicrystal dodecagon. These zero-energy corner modes are robust against any symmetry-preserving perturbations.

FIG. 1. Schematic illustrations of the Stampfli-tiling quasicrystal dodecagon. The quasicrystal consists of three types of primitive tiles: square tiles (green), regular triangle tiles (blue), and rhombus tiles (red) with the small angle 30°.
Counterpropagating edge modes on the dodecagonal QL.

We first consider a first-order topological insulator tight-binding model on the dodecagonal QL. The lattice sites are located on the vertices of the Stampfli-tiling as shown in Fig. 1. The nearest-neighbor sites are connected by short diagonals of rhombuses, and the next-nearest-neighbor sites are connected by the sides of the three primitive tiles, etc. The model Hamiltonian is given by [48]

\[
H_0 = -\sum_{j<k} \frac{f(r_{jk})}{2} c_{j\uparrow}^\dagger [i t_1 (\sigma_z \tau_x \cos \phi_{jk} + \sigma_0 \tau_y \sin \phi_{jk}) + t_2 \sigma_0 \tau_z] c_{k\uparrow} + \sum_j (M + 2t_2) c_{j\uparrow}^\dagger \sigma_0 \tau_z c_j,
\]

where the basis is \( c_{j\alpha}^\dagger = (c_{j\alpha \uparrow}^\dagger, c_{j\alpha \downarrow}^\dagger, c_{j\beta \uparrow}^\dagger, c_{j\beta \downarrow}^\dagger) \), \( \alpha \) and \( \beta \) are different orbital degree of freedom, \( \uparrow \) and \( \downarrow \) represent electron spin, and \( j \) and \( k \) denote lattice sites running from 1 to \( N \). \( \sigma_0 \) and \( \tau_0 \) are the \( 2 \times 2 \) identity matrices. \( \sigma_x, y, z \) and \( \tau_x, y, z \) are the Pauli matrices acting on the spin and orbital degrees of freedom, respectively. \( M \) is the mass that determines the topological insulator phase, \( t_1 \) and \( t_2 \) are hopping parameters. \( \phi_{jk} \) is the polar angle of bond connecting sites \( j \) and \( k \) with respect to the horizontal direction. \( f(r_{jk}) = e^{1-r_{jk}/\xi} \) is the spatial decay factor of hoppings with the decay length \( \xi \), and \( r_{jk} = |r_j - r_k| \). In subsequent calculations, the energy unit is set as \( t_2 \), the next-nearest-neighbor lattice distance is used as the length unit, and the spatial decay length \( \xi \) is fixed as 1.

Before we move to the energy spectrum of edge states in the topological insulator, we would like to discuss the symmetries of the Hamiltonian \( H_0 \). The Hamiltonian (1) satisfies

\[
P H_0 P^{-1} = -H_0, \quad T H_0 T^{-1} = H_0, \quad S H_0 S^{-1} = -H_0, \quad \tag{2}
\]

Here \( P \), \( T \), \( S \) are PHS, TRS and chiral symmetry operators, respectively, and they are expressed by

\[
P = \sigma_x \tau_y \mathcal{I} K, \quad T = i \sigma_y \tau_0 \mathcal{I} K, \quad S = PT, \quad \tag{3}
\]

where \( K \) is the complex conjugate operator, and \( \mathcal{I} \) is the \( N \times N \) identity matrix. Therefore, \( H_0 \) possesses PHS, TRS, and chiral symmetry, and belongs to symmetry class DIII [7–11]. In addition, the Hamiltonian (1) has a mirror symmetry about the \( x-y \) plane \( m_x \), and satisfies \([H_0, m_x] = 0\) with \( m_z = \sigma_z \tau_0 \mathcal{I} \). Simultaneously, for the Stampfli-tiling quasicrystal dodecagon, the QL obeys a global twofold rotational symmetry, therefore the Hamiltonian (1) also satisfies \([H_0, C_{12}] = 0\). Here the twofold rotational symmetry operator is \( C_{12} = e^{-i \pi \sigma_z \tau_y \mathcal{R}_{12}} \), where \( \mathcal{R}_{12} \) is an orthogonal matrix permuting the sites of the QL to rotate the whole system by an angle of \( \pi/6 \). We also give more details of symmetry analysis for the Hamiltonian (1) in the Supplemental Material [110] (see Tab. S1).

To study the topological edge states, we directly diagonalize the Hamiltonian (1) on a quasicrystal dodecagon. The energy spectrum versus the eigenvalue index \( n \) is shown in Fig. 2(a). The red circles correspond to eigenvalues of edge states. Note that all the eigenvalues of edge states are doubly degenerate counterpropagating modes due to TRS. Figure 2(b) shows the spatial probability density of doubly degenerate eigenstates near zero energy marked by the black arrow in (a). (c) Energy spectrum of the HOTI Hamiltonian \( H \) versus the eigenvalue index \( n \) for the TRS breaking mass \( g/t_2 = 2 \). The inset at top shows the color circle of the effective edge mass. The green and violet regions denote the two regions of the edge orientation with opposite sign of the effective edge mass. The inset at lower right shows the enlarged section of 12 zero-energy modes marked by the red dots. (d) The probability density of zero-energy modes in (c). The color map shows the values of the probability density. We take the model parameters \( t_1/t_2 = 2, M/t_2 = 1 \), and lattice site number \( N = 2569 \).

Corner states on the dodecagonal QL.—To obtain a 2D HOTI, we can gap out 1D edge state by introducing a symmetry-breaking mass that gives rise the effective edge mass domain structure. Recently, this approach has been also proposed in the fivefold Penrose tiling QL [86, 87] and the eightfold AB tiling QL [48, 90].

Counterpropagating edge modes on the dodecagonal QL.

FIG. 2. (a) Energy spectrum of the first-order topological insulator Hamiltonian \( H_0 \) on a quasicrystal dodecagon versus the eigenvalue index \( n \). Red circles mark all the edge states. (b) The probability density of doubly degenerate eigenstates near zero energy marked by the black arrow in (a). (c) Energy spectrum of the HOTI Hamiltonian \( H \) versus the eigenvalue index \( n \) for the TRS breaking mass \( g/t_2 = 2 \). The inset at top shows the color circle of the effective edge mass. The green and violet regions denote the two regions of the edge orientation with opposite sign of the effective edge mass. The inset at lower right shows the enlarged section of 12 zero-energy modes marked by the red dots. (d) The probability density of zero-energy modes in (c). The color map shows the values of the probability density. We take the model parameters \( t_1/t_2 = 2, M/t_2 = 1 \), and lattice site number \( N = 2569 \).
types of HOTIs in the AB tiling quasicrystal, which has been discussed in Ref. [48]. When \( \eta = 6 \), the mass term \( H_m \) in the dodecagonal QL breaks the rotational symmetry \( C_{12} \) and mirror symmetry \( m_z \) in addition to TRS.

After introducing the mass term \( H_m \), we numerically diagonalize the HOTI Hamiltonian \( H \) on a Stampfli-tiling quasicrystal dodecagon. The energy spectrum versus the eigenvalue index \( n \) is shown in Fig. 2(c). We found that \( H_m \) opens an energy gap in the edge spectrum, and twelve zero-energy modes appear in the edge energy gap. The spatial probability density of zero-energy modes is shown in Fig. 2(d). We can see that the zero-energy modes are symmetrically localized at the twelve corners of the regular quasicrystal dodecagon. The twofold symmetric zero-energy corner modes are a hallmark feature of the quasicrystalline HOTI on the dodecagonal QL.

The corner states are protected by the combination of rotational symmetry \( C_{12} \) and mirror symmetry \( m_z \) as well as PHS. This kind of symmetry-protected corner states in the dodecagonal quasicrystal is quite distinct from that in crystalline systems, because the rotational symmetry \( C_{12} \) is a forbidden rotational symmetry in a crystal. In a word, we identify another type of quasicrystalline HOTI, comparing with the one in the AB tiling quasicrystal [48, 49], which requires a distinct spatial symmetry for protection. More details of symmetry analysis for the system are given in the Supplementary Material [110].

Stability of corner states.— Here, we use some symmetry-breaking perturbations to test the robustness of the zero-energy corner modes. As mentioned above, the mass term \( H_m \), needed by the quasicrystalline HOTI breaks TRS, the mirror symmetry \( m_z \), and the rotational symmetry \( C_{12} \). However, the combined symmetries \( C_{12}m_z \) and \( C_{12}T \) are preserved. In the following, we use on-site potential perturbations in the calculations, which can be written as

\[
\Delta H^{p'q'} = U \sum_j c_j^\dagger \sigma_p \tau_{q'} c_j, \tag{5}
\]

where \( U \) is the potential strength, \( p' \) and \( q' \) denote the identity matrix and the three components of the Pauli matrices, respectively. Therefore, there exist sixteen kinds of on-site potentials in total.

In the main text, we adopt two of the sixteen kinds of perturbation terms, which are \( \Delta H^{zz} = U \sum_j c_j^\dagger \sigma_z \tau_z c_j \) and \( \Delta H^{yx} = U \sum_j c_j^\dagger \sigma_y \tau_x c_j \), to test the stability of the corner states in the quasicrystal dodecagon. \( \Delta H^{zz} \) breaks the combined symmetry \( C_{12}T \) but preserves the combined symmetry \( C_{12}m_z \) and PHS, while \( \Delta H^{yx} \) breaks all the three symmetries. The energy spectra of the quasicrystalline HOTI Hamiltonian under \( \Delta H^{zz} \) and \( \Delta H^{yx} \) are shown in Figs. 3(a) and 3(b), respectively. Figure 3(a) shows that the twelve zero-energy corner states remain stable in the presence of \( \Delta H^{zz} \) because \( C_{12}m_z \) and PHS are preserved. In contrast, in Fig. 3(b), we can see that the zero-energy corner states are gapped out by the perturbation term. This is because the symmetries that protect the zero-energy corner states are broken by the perturbation \( \Delta H^{yx} \). In addition, we summarize the results of other kinds of perturbations in the Supplemental Material [110].

Physical mechanism of the HOTI on the dodecagonal QL.— The physical explanation of the zero-energy corner states in HOTIs can be given by the Jackiw-Rebbi mechanism [111]. In this mechanism, a topological zero-energy mode appears when a mass domain wall forms. In the present case, the mass term \( H_m \), relying on the polar angle of the bond \( \phi_{jk} \), can result in an effective edge mass domain structure. However, it is not easy to derive an explicitly analytic expression of the effective mass for the edge states on the dodecagonal QL, owing to the lack of translational symmetry. Here, as a rough approximation, we treat the sides of a quasicrystal polygon as a long “bond” and the sign of the effective mass for the edge state depends on the polar angle of the sides \( \theta_{edge} \) [48]. For a given side of quasicrystal polygons, the effective edge mass on the side is determined by the factor \( \cos(6\theta_{edge}) \), which controls the sign of effective mass by varying \( \theta_{edge} \).

The green and violet in the top inset of Fig. 2(c) define different regions with opposite signs of the effective edge mass. The green region is determined by \( \theta_{edge} \in (-\frac{\pi}{12} + \frac{n\pi}{3}, \frac{\pi}{12} + \frac{n\pi}{3}) \) and the violet region is \( \theta_{edge} \in (\frac{\pi}{12} + \frac{n\pi}{3}, \frac{\pi}{12} + \frac{n\pi}{3}) \), where \( n = 0, 1, 2, 3, 4, 5 \). According to this rule, we can tell whether there is a zero-energy mode at a boundary corner. For the quasicrystal dodecagon, all the adjacent sides lie in two different regions as shown in the top inset of Fig. 2(c), so that an effective mass domain wall occurs at all the corners of the regular dodecagon, resulting in twelve zero-energy modes.

To further illustrate the validity of the physical mechanism, now we consider the dodecagonal QL with a different boundary shape in Fig. 4. This geometric structure can be obtained by cutting off four vertices/corners of a quasicrystal dodecagon. The energy spectrum of the Hamiltonian \( H \) for this geometric structure is shown in Fig. 4(a). We found that eight zero-energy modes appear in the energy gap of edge spectrum. Figure 4(b) shows the spatial probability density of the eight zero-energy modes, we can see that they are localized at the eight vertices of the original quasicrystal dodecagon. The corner states are absent at the two rightmost vertices of the quasicrystal polygon. We can also use the Jackiw-Rebbi mecha-
nism to explain why there are eight zero-energy modes in a quasicrystal polygon with ten vertices. For the two rightmost corners, all adjacent sides fall into the regions with the same effective mass sign, therefore, no mass domain wall forms at these two corners and no zero-energy mode appears. Note that, for this quasicrystal polygon, we can not define a global rotational symmetry, thus the corner states lack the protection available for the previous twelvefold symmetric corner states. Moreover, we also show the results on the dodecagonal QL under different boundary shapes in the Supplemental Material [110]. We found that the numerical results are in good agreement with the rough approximation of the Jackiw-Rebbi mechanism.

Conclusion.—In this work, we study the first-order topological insulator and HOTI on the dodecagonal QL. The first-order time-reversal invariant topological insulator supports gapless edge states. The HOTI is obtained by introducing an additional mass term, which is protected by PHS and the combination of the twelvefold rotational symmetry and mirror symmetry. For a finite-sized quasicrystal dodecagon, the quasicrystalline HOTI hosts twelvefold symmetric corner states, which are robust against any symmetry-preserving perturbations.

Acknowledgments.—R.C. and D.-H.X. were supported by the NSFC (Grant No. 11704106). D.-H.X. also acknowledges the financial support of the Chutian Scholars Program in Hubei Province. R.C. was supported by the Project funded by China Postdoctoral Science Foundation (Grant No. 2019M661678).

Note added.—A few days prior to the completion of this manuscript, we became aware of a complementary study [112], which addresses similar problems from a different perspective.
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