Dynamical generation of phase-squeezed states in a two-component Bose-Einstein condensates

G. R. Jin, Y. An, T. Yan, and Z. S. Lu
Department of Physics, Beijing Jiaotong University, Beijing 100044, China

As an “input” state of a linear (Mach-Zehnder or Ramsey) interferometer, phase-squeezed state proposed by Berry and Wiseman exhibits the best sensitivity approaching to the Heisenberg limit [Phys. Rev. Lett. 85, 5098 (2000)]. Similar with the Berry and Wiseman’s state, we find that two kinds of phase-squeezed states can be generated dynamically with atomic Bose-Einstein condensates confined in a symmetric double-well potential, which shows the squeezing along spin operator $\hat{S}_y$ and the anti-squeezing along $\hat{S}_z$, leading to sub-shot-noise phase estimation.

PACS numbers: 03.75.Mn, 05.30.Jp, 42.50.Lc

I. INTRODUCTION

Recently, atom interferometer with a trapped Bose-Einstein condensates (BEC) has attracted much attention due to its potential applications in quantum information and quantum metrology [1–7]. In particular, a two-component BEC with the ‘one-axis twisting’ (OAT) interaction $\chi \hat{S}_x^2$ is widely believed to be a useful resource for preparing the OAT-type spin squeezed state $|\Psi_{\text{OAT}}\rangle$ [8–11], as well as many-partite entangled state [12, 13]. Using maximally entangled (i.e., the N00N) state rather than product coherent spin states (CSS), phase sensitivity of a linear (Mach-Zehnder, or equivalently Ramsey) interferometer can be improved from the so-called shot-noise limit (SNL) $\Delta \Phi \sim 1/\sqrt{N}$ to the Heisenberg limit (HL) $\sim 1/N$ [14–17], where $N$ is total particle number. However, it is very difficult to create the N00N state.

As a special case of spin squeezed state, Gaussian number-squeezed state (NSS) has been demonstrated in several experiments [18–23], which exhibits sub-Poissonian atom number distribution [18–21] and long coherence time [22, 23] due to the spin squeezing of relative-number operator $\hat{S}_z$. These features manifest it as a promising candidate for the Heisenberg-limited metrology [24, 25]. Dynamical generation of the NSS has been proposed in a two-component BEC by exploiting the interplay of nonlinear interaction and atomic tunneling [24, 25]. It was found that the variance of $\hat{S}_z$ has an exact relationship with the visibility of the Ramsey signal (i.e., the phase coherence) [26, 27].

In this paper, we further investigate dynamical generation of phase-squeezed states (PSS) with atomic BEC confined in a symmetric double-well potential. We focus on a symmetric input state of a linear interferometer, with which we derive easily the phase sensitivity and its relation with spin squeezing parameter. Two experimentally realizable schemes are proposed to generate the PSS. Similarly with the Berry and Wiseman’s (BW) state [30], we find that it exhibits the squeezing of the spin operator $\hat{S}_y$ and the anti-squeezing of $\hat{S}_z$. Numerically, we calculate power rules of the optimal squeezing with particle number $N$. Although the achievable squeezing is worse than that of the BW state, dynamically generated PSS are still useful to gain sub-shot-noise limited phase sensitivity [31].

This paper is arranged as follows. In Sec. II we review quantum metrology protocol based upon the linear interferometer. For a symmetric input state, we derive the relation between phase sensitivity and the spin squeezing parameter [14]. In Sec. III, we investigate general features of the optimal phase-squeezed state proposed by Berry and Wiseman [30]. Two routes for creating the PSS are proposed with the two-component BECs, which utilize a cooperation between the nonlinear interaction and the Josephson tunneling of ultra-cold atoms confined in a symmetric double-well potential. Scaling rules of the optimal coupling and the resultant spin squeezing are obtained to compare with the BW state. Finally, we conclude in Sec. IV with main results of our work.

II. A SYMMETRIC INPUT STATE OF LINEAR INTERFEROMETER

Let us firstly review the achievable sensitivity of a linear interferometer, which can be described by multiple rotations to the input state [32], namely

$$|\Psi\rangle_{\text{out}} = e^{-i\tilde{\Phi}\hat{S}_z}e^{-i\phi\hat{S}_y}e^{-i\frac{\Phi}{2}\hat{S}_x}|\Psi\rangle_{\text{in}} = e^{-i\phi\hat{S}_y}|\Psi\rangle_{\text{in}},$$

(1)

where the three unitary transformations (from left to right) represent the output 50:50 beam splitter (or equivalently a $\pi/2$-pulse), the linear phase shifter, and the input beam splitter, respectively. In the Schwinger representation, the angular momentum operator $\hat{S} = (\hat{S}_x, \hat{S}_y, \hat{S}_z)$ with $\hat{S}_x = \frac{1}{2} (\hat{a}_L^\dagger \hat{a}_R + \hat{a}_R^\dagger \hat{a}_L)$, $\hat{S}_y = \frac{1}{2} (\hat{a}_L^\dagger \hat{a}_R - \hat{a}_R^\dagger \hat{a}_L)$, and $\hat{S}_z = \frac{1}{2} (\hat{a}_L^\dagger \hat{a}_L - \hat{a}_R^\dagger \hat{a}_R)$, with $\hat{a}_\mu$ denoting the annihilation operator for two Bosonic modes $\mu = L, R$. Via a detection of $\hat{S}_z$ (i.e., the light-intensity difference or population imbalance) at the output ports, the dimensionless phase shift $\tilde{\Phi}$ could be estimated with its precision quantified by

$$\Delta \tilde{\Phi} = \frac{(\Delta \hat{S}_z)_{\text{out}}}{|d(S_z)_{\text{out}}/d\Phi|},$$

(2)
where \( \langle \hat{A} \rangle_{\text{out}} \) and \((\Delta \hat{A})_{\text{out}} = [\langle \hat{A}^2 \rangle_{\text{out}} - \langle \hat{A} \rangle_{\text{out}}^2]^{1/2} \) denote the expectation value and the variance of an operator \( \hat{A} \) with respect to the output state \( |\Psi\rangle_{\text{out}} \).

In terms of common eigenstates of \( \hat{S}^2 \) and \( \hat{S}_z \), \( |s,n \rangle = \left(a_1^\dagger a_2^\dagger a_R^\dagger \right)^{s-n} |0 \rangle / \sqrt{(s+n)!(n-s)!} \equiv |s+n,n-s \rangle_{\text{LR}} \), symmetric input state of a linear interferometer can be expanded as

\[
|\Psi\rangle_{\text{in}} = \sum_{n=-s}^{s} c_n |s,n \rangle ,
\]

where \( s = N/2 \) and the probability amplitudes \( c_{-n} = c_n \). It is easy to prove that the mean spin \( \langle \hat{S}_z \rangle_{\text{in}} = \langle \langle \hat{S}_z \rangle_{\text{in}} \rangle \) and \( \langle \hat{S}_x \rangle_{\text{in}} \) is oriented in the \( x \) direction. Special case: the mean spin \( \langle \hat{S}_z \rangle_{\text{in}} = 0 \) due to \( c_{-n} = c_n \). In addition, expectation value of the ladder operator \( \hat{S}_+^n = \hat{S}^n_1 \hat{a} \) is given by

\[\langle \hat{S}_+^n \rangle_{\text{in}} = \sum_{n=-s}^{s-1} X_{n} c_n^* c_{n+1} = 2 \sum_{n=0}^{s} X_{-n} \text{Re} \left[ c_n c_{n+1}^* \right],\]

where \( X_{n} = (s+n)^{1/2}(s-n+1)^{1/2} \), satisfying \( X_{-n} = X_{n} \). Due to real \( \langle \hat{S}_x \rangle_{\text{in}} \), we have \( \langle \hat{S}_y \rangle_{\text{in}} = \text{Im} \langle \hat{S}_z \rangle_{\text{in}} = 0 \) and \( \langle \hat{S}_x \rangle_{\text{in}} = \text{Re} \langle \hat{S}_z \rangle_{\text{in}} \neq 0 \). Similarly, we can prove that \( \langle \hat{S}_x \rangle_{\text{in}} = \langle \hat{S}_y \rangle_{\text{in}} = \text{Im} \langle \hat{S}_z \rangle_{\text{in}} = 0 \) and \( \langle \hat{S}_x \rangle_{\text{in}} = \text{Re} \langle \hat{S}_z \rangle_{\text{in}} \).

The above results, valid also for odd \( N \) case, enable us to derive the phase sensitivity \( \Delta \Phi \) in a simple way. For instance, we get immediately the output signal \( \langle \hat{S}_x \rangle_{\text{out}} = -\langle \hat{S}_x \rangle_{\text{in}} \sin \Phi \) and the variance \( (\Delta \hat{S}_x)_{\text{out}}^2 = (\Delta \hat{S}_x)_{\text{in}}^2 \cos^2 \Phi + (\Delta \hat{S}_z)_{\text{in}}^2 \sin^2 \Phi / 2 \), where we have used the relation: \( e^{i\Phi \hat{S}_x} e^{-i\Phi \hat{S}_x} = \langle \hat{S}_x \rangle_{\text{in}} \cos \Phi - \hat{S}_x \sin \Phi \) for \( k = 1, 2 \). Inserting into Eq. (2), we arrive at

\[
\Delta \Phi = \frac{\sqrt{(\Delta \hat{S}_z)^2_{\text{in}} + (\Delta \hat{S}_x)^2_{\text{in}}} \tan^2 \Phi}{\langle \hat{S}_x \rangle_{\text{in}}} ,
\]

which is minimized for the phase shift \( \Phi = 0 \), with the best sensitivity \( (\Delta \Phi)_{0} = (\Delta \hat{S}_z)^2_{\text{in}} / \langle \hat{S}_x \rangle_{\text{in}} \). Here, the subscript “0” denotes the achievable sensitivity for \( \Phi = 0 \). Following Windeland et al. \[14\], we introduce the squeezing parameter

\[
\zeta_S = \frac{(\Delta \Phi)_{0}}{(\Delta \Phi)_{\text{cos}}} = \frac{\sqrt{N} (\Delta \hat{S}_z)_{\text{in}}}{\langle \hat{S}_x \rangle_{\text{in}}},
\]

where \( (\Delta \Phi)_{\text{cos}} = 1/\sqrt{N} \), known as the shot-noise limit, represents the best sensitivity with product coherent spin states \[33\]:

\[
|s,\pm s \rangle_x = e^{-i\hat{S}_y} |s,\pm s \rangle_x.
\]

Note that the mean spin \( |s,\pm s \rangle_x \) is also parallel with the \( x \)-axis due to \( \hat{S}_x |s,\pm s \rangle_x = \pm s |s,\pm s \rangle_x \). In addition, the two CSS satisfy minimal-uncertainty relationship \[33\]: \( (\Delta \hat{S}_z)^2 = (\Delta \hat{S}_y)^2 = (\langle \hat{S}_x \rangle_{\text{in}}) / 2 \), with the length of the mean spin \( \langle \hat{S}_x \rangle_{\text{in}} = s \). Therefore, as an input of linear interferometer, the coherent spin states lead to phase estimation limited by the shot noise \( 1/\sqrt{N} \) (i.e., \( \zeta_S = 1 \)).

How to improve the sensitivity beyond the shot-noise limit \( (\zeta_S < 1) \) is one of the main subjects of quantum metrology \[31,32\]. It has been found that several nonclassical states can reach the Heisenberg-limit \( (\Delta \Phi)_{\text{H}} \propto 1/N \), such as the N00N state \( N/\sqrt{2} |s,s \rangle + |s,-s \rangle \) \[14,17\], and the twin-Fock state \( |s,0 \rangle = \sqrt{N/2} |N/2,0 \rangle_{\text{LR}} \) for even \( N \) \[33\]. However, experimental realization of these states is not an easy task. In this paper, we present experimentally available schemes for preparing the phase-squeezed state and its application in the quantum metrology.

### III. DYNAMICAL GENERATION OF PHASE-SQUEEZED STATE

Previously, Berry and Wiseman have proposed an optimal phase-squeezed state \( |\Psi\rangle_{\text{BW}} = \sum_n c_n |s,n \rangle \) with the probability amplitude \[30\]

\[
c_n = \frac{1}{\sqrt{s+1}} \cos \left( \frac{n\pi}{2s+2} \right).
\]

Obviously, the BW state has a symmetric probability distribution about \( n = 0 \) because of \( c_{-n} = c_n \), which in turn leads to the \( x \)-polarized mean spin \( \langle \hat{S}_y \rangle = \langle \langle \hat{S}_x \rangle \rangle = 0 \). As shown in Fig. 1 one can also find that spin squeezing of the BW state is along \( y \)-axis and the anti-squeezing along \( z \)-axis, i.e.,

\[
(\Delta \hat{S}_y)^2 < s/2 \quad \text{and} \quad (\Delta \hat{S}_z)^2 > s/2.
\]

Here, the value \( s/2 \) \( (= N/4) \) denotes the standard quantum limit (SQL).

![FIG. 1: (Color online) (a) Probability distribution \( |c_n|^2 \) for the Berry and Wiseman’s state, with \( c_n \) given by Eq. (6); (b) Quasi-probability distribution \( Q(\theta, \phi) = |\langle \theta, \phi |\Psi \rangle_{\text{BW}}|^2 \) on a Bloch sphere, for \( s = N/2 = 20 \) and the coherent spin state \( |\theta, \phi \rangle \) given by Eq. (9).](image-url)
$|\Psi\rangle_{\text{out}} = e^{-i\hat{S}_z} e^{i\Omega \hat{S}_y} |\Psi\rangle_{\text{in}}$. Inserting it into Eq. (2), we obtain the best sensitivity $(\Delta \Phi)_0 = (\Delta \hat{S}_y)_{\text{PSS}}/(|\bar{\Delta} \hat{S}_y|)$ and hence the squeezing parameter $\zeta_S = \sqrt{N(\Delta \Phi)_0}$, where $(\Delta \hat{S}_y)_{\text{PSS}}$ and $(\Delta \hat{S}_y)$ denote the expectation value and the mean-square root of the variance with respect to $|\Psi\rangle_{\text{PSS}}$. As an optimal PSS, the BW state gives rise to the Heisenberg-limited sensitivity $(\Delta \Phi)_0 \sim 2\sqrt{2}/N$ and thus $\zeta_S \sim 2\sqrt{2}N^{-1/2}$ for large enough particle number $(N > 10^2)$.

In order to prepare the PSS, we now consider atomic Bose–Einstein condensate, which is tightly confined in a double well with $Ω > Nχ$. By introducing two bosonic operators $a_{L(R)}$ for left (right) mode-function $ϕ_{L(R)}$ and the Schwinger’s angular momentum operators as Eq. (1), quantum dynamics of the BEC system can be described by an effective spin Hamiltonian with $h = 1$:

$$\hat{H}_γ = \delta \hat{S}_z + \Omega \hat{S}_γ + \chi \hat{S}_z^2,$$

where $δ$ denotes the potential bias, $Ω$ the amplitude of Josephson coupling, and $χ$ the $s$-wave interaction strength. In the second term of the Hamiltonian, an equatorial spin operator $\hat{S}_γ = \hat{S}_x \cos γ + \hat{S}_y \sin γ$ is introduced to simulate Josephson tunneling of the atoms with Rabi frequency $Ω e^{iγ}$, where both the amplitude $Ω$ and the phase $γ$ are tunable. The third term, arising from inter- and intraspecies nonlinear interaction, gives rise to the OAT-induced spin squeezing and many-particle entanglement.

Following original spin-squeezing scheme proposed by Kitagawa and Ueda [8], and more recently by Riedel et al. [9], we assume that a $π/2$-pulse is applied to the BEC to prepare the initial states $|s, ±s\rangle_x = e^{-i\hat{S}_z} |s, ±s\rangle$, which are coherent spin states:

$$|\theta, φ\rangle = \exp[-i\theta(\hat{S}_y \cos φ - \hat{S}_z \sin φ)] |s, ±s\rangle,$$

with polar angle $θ = π/2$ and azimuth angle $φ = 0$ or $π$. During the pulse, the two-mode functions $ϕ_{L(R)}$ merge with each other such that $δ \approx χ \approx 0$ and hence the Hamiltonian $\hat{H}_{π/2} \approx Ω\hat{S}_γ$, where the phase and the duration of the coupling are adjusted as $γ = Ωt_{π/2} = π/2$ with $Ω \gg Nχ$ [4, 7]. Next, the system evolves under the Hamiltonian $\hat{H}_0 = Ω\hat{S}_y + χ\hat{S}_z^2$ for a duration $t_{\text{min}}$. Nonzero $χ$ can be realized by splitting the two-mode BEC that allows for a suitable tunneling with the amplitude $Ω$ and the phase $γ = 0$ (or $π$), which is fulfilled by applying a second pulse $[4]$. The interplay of the linear tunneling and the nonlinear interaction results in dynamical generation of Gaussian number-squeezed state $|s, ±s\rangle_x$, and Schrödinger’s cat state (i.e., the NNN0 state) [26, 29]. Here, we show that it is also possible to create phase-squeezed state in the two-mode BEC. To understand how it works, we consider the Hamiltonian $\hat{H}_0$ with time-independent $χ$ and $Ω$. For a fixed particle number $N (= 2s)$, the system’s energy is conserved, i.e., $d(\hat{H}_0)/dt = 0$, which yields $(\hat{S}_z(t)) = C - Ω(\hat{S}_z(t))/\chi$ with a constant of integral $C = s/2 ± Ωs/\chi$ for the initial CSS $|s, ±s\rangle_x$, that is

$$\langle \hat{S}_z(t) \rangle = \frac{s}{2} \left[ 1 \pm \frac{2Ω}{\chi} \left( 1 ± \frac{(\hat{S}_z(t))}{s} \right) \right],$$

where the upper sign corresponds to $|s, +s\rangle_x$ and the lower sign for $|s, -s\rangle_x$. It is interesting to note that the variance $(\Delta \hat{S}_z)^2 = (\hat{S}_z^2) > s/2$ only if the generated state is symmetric and evolved from $|s, +s\rangle_x$ (or $|s, -s\rangle_x$) under $\hat{H}_0$ with $Ωχ > 0$ ($< 0$). Except the so-called ‘over-squeezing’ [51], for which both $(\Delta \hat{S}_z)^2$ and $(\Delta \hat{S}_y)^2$ are larger than the SQL, the enhanced spin fluctuation of relative atom number implies the appearance of a PSS with $(\Delta \hat{S}_y)^2 < s/2$. For brevity, we will assume that both $Ω$ and $χ$ are real and positive.

A. Direct method to prepare the PSS

Firstly, let us consider dynamical evolution of the initial CSS $|s, +s\rangle_x$ under the system Hamiltonian $\hat{H}_0$. To distinguish with previous scheme [17], we adopt relatively larger coupling ratio $Ω/χ ≃ 1.1N$ (see blow Fig. 3), which leads to a completely different dynamics due to the fact that the initial state, i.e., Eq. (9) with $θ = π/2$ and $φ = 0$, now corresponds to a stable fixed point in phase space. To confirm it, we plot time evolution of probability distribution $|c_n(t)|^2$. As shown in Fig. 2(a), the generated state $|Ψ(t)\rangle$ shows symmetric probability distribution at any time $t$. For large $N$, the initial CSS $|s, +s\rangle_x$ is a Gaussian [52], namely

$$|c_n(0)|^2 = \frac{1}{2\pi s} \frac{2s}{s + n} ≃ \frac{1}{\sqrt{π}s} e^{-n^2/2},$$

with its width determined by the variance $(\Delta \hat{S}_z) = \sqrt{s/2}$ (see blue line of Fig. 2(b), also Ref. 28). After a certain duration $t_{\text{min}} ≃ \log_2(N)/(2Nχ)$, the initial state will evolve into a phase-squeezed state, which can be inferred from an increased width of probability distribution $(\Delta \hat{S}_z) > s/2$ (red line of Fig. 2). Such a result can be explained from Eq. (10). Indeed, we can prove that the mean spin of $|Ψ(t)\rangle$ aligns with the $x$-axis and its length $(\hat{S}_z(t)) < s$. Therefore, from Eq. (10) we have $(\Delta \hat{S}_z)^2 > s/2$ and hence $(\Delta \hat{S}_y)^2 < s/2$, except for the over-squeezing.

The appearance of the PSS can be illuminated by calculating quasi-probability distribution $Q(\theta, φ) = |⟨(θ, φ) |Ψ(t)⟩|^2$ on the Bloch sphere $\mathbb{S}$, where the CSS $|θ, φ\rangle$ is defined by Eq. (9), with the polar angle $0 ≤ θ ≤ π$ and the azimuth angle $0 ≤ φ ≤ 2π$. As depicted in the left plot of Fig. 3(a), an elliptic shape of the $Q$ function is found for the spin state $|Ψ(t)\rangle$ at $t_{\text{min}}$, which shows the squeezing along $\hat{S}_y$ and the anti-squeezing along $\hat{S}_z$, similar with that of the BW state. In Fig. 3(b), we plot time evolution of the squeezing parameter $\zeta_S$ (solid red line)
and the fidelity between the prepared state with the BW state, \( F(t) = |\langle \Psi(t) | \Psi(t) \rangle|^2 \) (dotted line). For instance at \( t = 0 \), the fidelity is given by
\[
F(0) \approx \frac{2(\pi s)^{1/2}}{s + 1} \exp \left[ -\frac{s}{2} \left( \frac{\pi}{1 + s} \right)^2 \right], \tag{12}
\]
where we used Eq. (9) and the approximate result of Eq. (11). For the case \( N = 2s = 40 \) and \( \Omega/\chi = 1.134N \), our analytical result predicts \( F(t) \approx 0.675 \), coincident with exact numerical simulation 0.672. From Fig. 3(b), we find that the generated PSS at time \( t_{\text{min}} = 6.95 \times 10^{-2} \chi^{-1} \) shows a high fidelity \( F = 0.982 \) and a local minimum of the squeezing parameter \( \zeta_S = 0.507 \) (-2.949 dB), which is the optimal squeezing that this scheme can reach for \( N = 40 \).

It is necessary to investigate the optimal coupling ratio \( \Omega/\chi \) and the minimal value of \( \zeta_S \) (i.e., the maximal squeezing) for any finite \( N \). As an example, we consider the exactly solvable case with \( N = 2 \). Analytical results can be obtained as the following: \( \langle \hat{S}_x \rangle = 1 - \frac{x^2}{2x} \sin^2(\mathcal{E}t) \) and \( (\Delta \hat{S}_y)^2 = \frac{1}{4} [1 - \frac{\Omega^2}{\chi^2} \sin^2(\mathcal{E}t)] \), where \( \mathcal{E} = \sqrt{\Omega^2 + (\chi/2)^2} \), denoting half of level-spacing between the ground state and the second-excited state. Therefore, we get the maximal squeezing \( \zeta_S^2 = (4x^2 + 1)/(2x + 1)^2 \) with \( x = \Omega/\chi \), which occurs at times \( t_k = \left( k + \frac{1}{2} \right) \pi/\mathcal{E} \) for an integer \( k = 0, 1, \ldots \). Minimizing \( \zeta_S^2 \) with respect to \( x \), we further obtain the optimal coupling ratio \( \Omega/\chi = 1/2 \) and thus the best squeezing \( \zeta_S = 1/\sqrt{2} \). For three-particle case, \( \langle \hat{S}_z \rangle = \frac{3}{2}x \) and \( (\Delta \hat{S}_y)^2 = \frac{3}{4} [1 - 2x(\Omega/\chi)^2 \sin^2(\mathcal{E}t)] \), where \( \mathcal{E} = \sqrt{\Omega^2 - \chi^2} \). The maximal squeezing also occurs at times \( t_k \), with
\[
\zeta_S^2 = \frac{(x^2 - x + 1)(x^2 - 3x + 3)}{x^2(x - 1)^2}, \tag{13}
\]
from which we find that the optimal squeezing \( \zeta_S = 0.762 \) is attainable for the coupling ratio \( x = \Omega/\chi = 2.794 \) (\( \sim 0.931N \) with \( N = 3 \)). Counter-intuitively, a slightly weaker spin squeezing is obtained in a comparison with previous \( N = 2 \) case. This is because the generated PSS in the two-particle system is also maximally-entangled N00N state, which results in the Heisenberg-limited sensitivity \( \Delta \Phi \) and the optimal squeezing \( \zeta_S = 1/(\sqrt{N} \chi) \) (with \( N = 2 \)).

The optimal squeezing for \( N > 3 \) has to be determined numerically. Due to computation limit, only finite particle number \( 2 \leq N \leq 200 \) are shown in Fig. 4. Our numerical simulations show that the optimal squeezing with \( \zeta_S \sim N^{-0.21} \) can be obtained for the coupling ratio \( \Omega/\chi \sim 1.1N \) (see balls of Fig. 4). Although the achievable squeezing is worse than that of the BW state (empty circles) \( 35 \), it is still useful to achieve the sub-shot noise in the phase estimation due to \( \zeta_S < 1 \).

Next, we consider dynamical evolution of another initial CSS \( |s, -s \rangle_x \) under governed by the Hamiltonian \( \hat{H}_0 \). According to Law et al. \( 26 \), it is possible to generate dynamically a Gaussian number-squeezed state, which shows the squeezing along \( \hat{S}_z \) and the anti-squeezing along \( \hat{S}_y \) (see middle plot of Fig. 3(a)). During time evolution, the generated state \( |\Psi(t)\rangle \) is always symmetric around \( n = 0 \). More specially, the probability amplitudes obey \( c_{-n}(t) = c_n(t) \) for even \( N \) case, or \( c_{-n}(t) = -c_n(t) \) for odd \( N \) case, leading to the z-polarized mean spin \( 27 \).

Previously, the optimal coupling ratio \( \Omega/\chi \) as a function of \( N \) has been obtained by minimizing the reduced variance over the SQL: \( \xi_N^2 = 2(\Delta \hat{S}_z)^2/N \), which was referred as the number-squeezing parameter \( 20, 22, 40 \).

\[\text{FIG. 2: (Color online) (a) Probability distribution } |c_n|^2 \text{ of the spin state evolved from } |s, +s \rangle_x \text{ as a function of time } t \text{ (in units of } \chi^{-1}) \text{; (b) Snapshot of } |c_n|^2 \text{ for the initial state (blue line) and the PSS at time } t_{\text{min}} = 6.95 \times 10^{-2} \chi^{-1} \text{ (red line with points). Other parameters: } \Omega/\chi = 1.134N \text{ and } N = 2s = 40.\]

\[\text{FIG. 3: (Color online) (a) Quasi-probability distribution } Q(\theta, \phi) \text{ of the dynamical generated PSS (left), the NSS (middle), and the PSS: } e^{(\pi/2)\hat{S}_z} |\Psi(0)\rangle \text{ (right); (b) The squeezing parameter } \zeta_S \text{ (solid red) and the Fidelity } F(t) \text{ (dotted) as a function of time } t \text{ (in units of } \chi^{-1}) \text{ for the initial states } |s, +s \rangle_x \text{ with } s = N/2 = 20 \text{ and } \Omega/\chi = 1.134N \text{. The dashed blue line corresponds to evolution of } \zeta_S \text{ from the initial CSS } |s, -s \rangle_x \text{, with } s = 20 \text{ and } \Omega/\chi = 2.664.\]

\[\text{B. Indirect method to prepare the PSS via a rotation of the NSS}\]
and is measurable by a detection of the phase coherence $g^{(1)} = |\langle S_z \rangle|/s$. For the symmetric state, Eq. (11) gives us an exact relationship between $\xi_n$ and $g^{(1)}$.

To confirm whether the prepared NSS is useful for metrology or not, we have to investigate the squeezing parameter $\xi_n = \xi_{N}/g^{(1)}$ [20]. As shown by dashed blue line of Fig. 3(b), one can find that the prepared state $|\Psi(t)\rangle$ at time $t_{\text{min}} = 0.150\chi^{-1}$ shows the optimal squeezing $\xi_n = 0.322$ (about $-5$dB) for the case $N = 40$ and $\Omega/\chi = 2.664$. In Fig. 4 we investigate power rules of the optimal coupling ratio $\Omega/\chi$ and the best squeezing $\xi_n$. Our numerical results (the red crosses) show that $\Omega/\chi \sim 0.91N^{0.27}$ and $\xi_n \sim 1.24N^{-0.37}$. Such a scaling is worse than that of the BW state $\xi_n \sim N^{-0.5}$. However, as an input state of linear interferometer, it is sufficient to achieve the sub-shot-noise limited metrology.

Before closing, it should be mentioned that the Gaussian number-squeezed state has been prepared in several experiments [18–23], which exhibits sub-Poissonian atom number distribution [18, 21] and relatively longer coherence time compared with that of the product CSS [22]. These features manifest it as a promising candidate for quantum metrology [24, 22]. Recently, Uys and Meystre have found that the NSS gives rise to the Heisenberg-limited phase sensitivity [24] (see also Ref. [23]). If a Gaussian NSS is fed into the interferometer as Eq. (1), the action of the input beam splitter $e^{i(\pi/2)S_z}$, equivalent with a $\pi/2$-pulse, rotates the prepared NSS about negative $x$-axis by an angle $\pi/2$, which results in a PSS: $|\Psi_{\text{pss}}\rangle = e^{i(\pi/2)S_z}|\Psi_{\text{nss}}\rangle$, as shown by right plot of Fig. 3(a). This is another method to generate phase-squeezed state [3, 4].

IV. CONCLUSION

In summary, we have demonstrated that it is possible to generate phase-squeezed state in a two-mode Bose-Einstein condensate. Considering a symmetric input state of linear interferometer, we show that it is easy to derive the best sensitivity and its relation with the squeezing parameter $\xi_n$. As an example, we have discussed general features of the Berry and Wiseman’s state: the $x$-polarized mean spin and the Heisenberg-limited phase sensitivity. Two experimentally realizable schemes are proposed to generate the phase-squeezed states. We find that it can be prepared from an initial coherent spin state $|s, +s\rangle_x$, without any rotation. Alternatively, starting with $|s, -s\rangle_x$, a number-squeezed state is prepared and then transformed into phase-squeezed state via a $\pi/2$-pulse. The optimal coupling and the maximal squeezing are determined based upon a wide range of numerical simulations. Our results show that the generated phase-squeezed states can achieve sub-shot-noise-limited quantum metrology.

Acknowledgments

We thank Prof. L. You and Dr. Y. C. Liu for helpful discussion. This work is supported by the NSFC (Contract No. 10804007). One of the authors (Y. An) is partially supported by National Innovation Experiment Program for University Students (BJTU No. 1070006).

[1] T. Schumm, S. Hoferberth, L. M. Andersson, S. Wildermuth, S. Groth, I. Bar-Joseph, J. Schmiedmayer, and P. Krüger, Nature Phys. 1, 57 (2005).
[2] Y. Shin, C. Sanner, G.-B. Jo, T. A. Pasquini, M. Saba, W. Ketterle, and D. E. Pritchard, Phys. Rev. A 72, 021604(R) (2005).
[3] L. Pezzé, A. Smerzi, G. P. Berman, A. R. Bishop, and L. A. Collins, Phys. Rev. A 74, 033610 (2006).
[4] J. Grond, G. von Winckel, J. Schmiedmayer, and U. Hoehnester, Phys. Rev. A 80, 053625 (2009).
[5] A. D. Cronin, J. Schmiedmayer, and D. E. Pritchard, Rev. Mod. Phys. 81, 1051 (2009).
[6] C. Gross, T. Zibold, E. Nicklas, J. Estève, and M. K. Oberthaler, Nature (London) 464, 1165 (2010).
[7] M. F. Riedel, P. Böhi, Y. Li, T. W. Hänsch, A. Sinatra, and P. Treutlein, Nature (London) 464, 1170 (2010).
[8] M. Kitagawa and M. Ueda, Phys. Rev. A 47, 5138 (1993).
[9] A. Sorensen, L.-M. Duan, J. I. Cirac, and P. Zoller, Nature (London) 409, 63 (2001).
[10] X. Wang and B. C. Sanders, Phys. Rev. A 68, 012101 (2003).
[11] G. R. Jin, Y. C. Liu, and W. M. Liu, New J. Phys. 11, 073049 (2009).
[12] K. Mølmer and A. Sørensen, Phys. Rev. Lett. 82, 1835.
