Noncommutative Chern-Simons theory on the quantum sphere $S^3_\theta$

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Abstract

We consider the $\theta$-deformed quantum three sphere $S^3_\theta$ and study its Chern-Simons theory from a spectral point of view. We first construct a spectral triple on $S^3_\theta$ as a generalization of the Dirac geometry on $S^3$. Since the choice of Dirac operator is not unique, we give two more natural spectral triples on $S^3_\theta$ related to the standard round metric. We then compute the Chern-Simons action with respect to the three spectral triples, it turns out that it is not a topological invariant, that is, it depends on the choice of Dirac operator. Finally, we compute the partition function of the quantum Chern-Simons field theory.

1 Introduction

In order to understand the abstract theory of noncommutative geometry, it is better to test the theory on different examples. Besides the standard noncommutative tori, it is natural to consider some noncommutative sphere. In fact, there are a variety of quantum spheres proposed by authors from different point of view in the literature [9].

Our main interest in this paper is the quantum 3-sphere $S^3_\theta$, which was first introduced by Connes and Landi in [7] from a K-theoretic consideration. $S^3_\theta$ is a special case of a more general class of noncommutative 3-spheres considered in [6], and it also coincides with the quantum spheres discussed in [14, 16]. By definition $S^3_\theta$ is a $\theta$-deformed C*-algebra and its K-groups are computable, indeed $K_0(S^3_\theta) \cong \mathbb{Z}$, $K_1(S^3_\theta) \cong \mathbb{Z}$. The 4-dimensional quantum

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sphere $S^4_\theta$ considered in [7] is the suspension of $S^3_\theta$, so it is possible to get a Dirac operator on $S^4_\theta$ by reduction from that on $S^4_\theta$. The quantum 3-sphere $S^3_\theta$ is similar to $SU_q(2)$ ($0 < q < 1$), but with a complex parameter $\lambda = e^{2\pi i \theta} \in U(1)$, and we will see later that the noncommutative 2-torus $T^2_\theta$ is naturally embedded inside $S^3_\theta$.

The Chern-Simons form was first introduced as a boundary term when the authors were computing the first Pontryagin number of a 4-manifold [2]. It can be defined as a secondary characteristic class by the transgression of the Chern character on principal bundles. Let $M$ be a closed oriented 3-manifold and $G$ a compact simply connected Lie group, for example $SU(2)$, with Lie algebra $\mathfrak{g}$. If $P \to M$ is a principal $G$-bundle and $A \in \mathcal{A}_P \subset \Omega^1_P(\mathfrak{g})$ is a $\mathfrak{g}$-valued connection 1-form on $P$, in practice we take $P$ as the trivial $G$-bundle. The Chern-Simons action is defined by the integral,

$$CS(A) = \frac{1}{8\pi^2} \int_M tr \left( A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right)$$

where $tr$ is an invariant bilinear form on $\mathfrak{g}$. Denote by $G$ all the gauge transformations,

$$A \mapsto A^g = g^{-1} A g + g^{-1} dg, \quad g : M \to G$$

the Chern-Simons action is well-known for its gauge invariance modulo addition of winding numbers,

$$CS(A^g) = CS(A) + \frac{1}{24\pi^2} \int_M tr(g^{-1} dg)^3$$

Based on the path integral quantization, one considers the partition function as a formal integral over the moduli space of connections $\mathcal{A}_P/G$ for some $k \in \mathbb{Z}$,

$$Z(k) = \int_{\mathcal{A}_P/G} D[A] e^{2\pi i k CS(A)}$$

This quantum Chern-Simons theory is a topological quantum field theory. On the 3-sphere $Z(k)$ is exactly solvable,

$$Z_{S^3}(k) = \sqrt{\frac{2}{k+2}} \sin \left( \frac{\pi}{k+2} \right)$$
and the Jones polynomial can be recovered by surgery and conformal field theory [21]. In the same paper, Witten also proposed an asymptotic formula of $Z(k)$ near flat connections in the weak coupling limit, i.e. as $k \to \infty$,

$$Z(k) = e^{2\pi i k CS(A_0)} \frac{\det \Delta_0}{\sqrt{|\det L_-|}} e^{-i\pi \eta(L_-)/4}$$  \hspace{1cm} (5)

In the above one loop approximation, $A_0$ is some flat connection, and

$$\frac{\det \Delta_0}{\sqrt{|\det L_-|}}$$

is the square root of the Ray-Singer analytic torsion, i.e. a topological invariant. In addition, the $\eta$-invariant $\eta(L_-) = \eta(L_-, 0)$ is used to regularize the signature,

$$\eta(L_-, s) = \sum \text{sign} \lambda_n \cdot |\lambda_n|^{-s}$$

the reader is referred to the original paper for more details about the operators $\Delta_0$ and $L_-$ [21].

There are different proposals for the definition of Chern-Simons action in noncommutative geometry and the difficulty is in gauge invariance. For instance, Chamseddine and Fröhlich defined the noncommutative Chern-Simons action based on the idea of transgression in [1], Krajewski used the Dixmier trace instead of the classical integral over 3-manifolds as a generalization in [13]. Connes and Chamseddine introduced the Chern-Simons action as the integral relative to a cyclic 3-cocycle, they obtained the variation of the spectral action under inner fluctuations as a Yang-Mills action plus a Chern-Simons action assuming that the tadpole graph does not contribute [5]. The above mentioned noncommutative Chern-Simons actions are not gauge invariant in general.

In [17] Pfante gave a definition of noncommutative Chern-Simons action for 3-summable spectral triples, which is gauge invariant modulo addition of a Fredholm index based on the local index formula [8]. In this new action besides a 3-cocycle $\phi_3$ there exists a 1-cocycle $\phi_1$ so that $(\phi_1, \phi_3)$ forms a $(b, B)$-cocycle, when $\phi_1$ is zero it coincides with Connes and Chamseddine’s definition. As examples Pfante computed the Chern-Simons action over the quantum compact group $SU_q(2)$ [17] and the noncommutative 3-torus $T^3_\Theta$ [18]. In the case of $SU_q(2)$ the $\phi_1$ term contributes to the action, while on $T^3_\Theta$ the $\phi_1$ term vanishes.
In this paper we first recall the noncommutative local index formula and the definition of Chern-Simons action in section 2. After introducing the quantum 3-sphere $S^3_\theta$, we explicitly construct the first spectral triple generalizing the Dirac geometry on $S^3$ in section 3. The dimension spectrum of the first spectral triple is discussed in section 4 and further the Chern-Simons action is computed, in particular the linear term $\phi_1$ vanishes on $S^3_\theta$. However, there are two more natural Dirac operators on $S^3_\theta$ related to the round metric on $S^3$, one is a reduction from the Dirac operator on $S^4_\theta$ and the other is defined based on a natural orthogonal framing in Hopf fibration. In section 5 we compute the Chern-Simons action with respect to these two spectral triples and different Chern-Simons actions are compared. It turns out that the choice of Dirac operator determines the Chern-Simons action, in other words it is not a topological invariant on $S^3_\theta$, which was also observed on $SU_q(2)$ in [17]. Given the gauge invariance, in section 6 we consider the Feynman path integral as a quantum Chern-Simons field theory based on the generalized Dirac geometry on $S^3_\theta$. In order to set up the integral without residual gauge, the ghost and anti-ghost fields are introduced and the standard BRST gauge-fixing method is applied. The partition function is computed with the help of Gaussian and Grassmann integrals and zeta regularization. The result could be compared with the one loop approximation of the partition function (5) in the classical case.

2 NC Chern-Simons action

In order to fix the notations we briefly recall the definition of noncommutative Chern-Simons action following [17] in dimension three and its relation to the noncommutative local index formula [8, 10] in the odd case. The local index formula on $SU_q(2)$ has been studied in [4, 20] and the Chern-Simons action on $SU_q(2)$ was discussed in [17].

A noncommutative odd-dimensional Riemannian manifold is described by an odd spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$. $\mathcal{A}$ is a unital associative algebra with involution, in practice we take it as a pre-$C^*$-algebra closed under holomorphic functional calculus. $\mathcal{A}$ acts on the separable Hilbert space $\mathcal{H}$ as bounded operators through a faithful representation $\pi : \mathcal{A} \rightarrow B(\mathcal{H})$. $\mathcal{D}$ is an unbounded self-adjoint operator with compact resolvent such that $[\mathcal{D}, a]$ is bounded for any $a \in \mathcal{A}$. Furthermore, the Dirac-type operator $\mathcal{D}$ determines the metric
on the state space of $\mathcal{A}$,

$$d(\phi, \psi) = \sup_{a \in \mathcal{A}} \{ |\phi(a) - \psi(a)|; \|D, a\| \leq 1 \}$$

The prototype of a spectral triple is given by $(C^\infty(M), L^2(M, \mathcal{D}), \mathcal{D})$, i.e. the Dirac geometry on a closed Riemannian spin manifold $(M, g)$.

Given an odd spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$, let $\mathcal{F} = \mathcal{D}|\mathcal{D}|^{-1}$ be the sign of $\mathcal{D}$, i.e. $\mathcal{F}^2 = 1$, and $P = (\mathcal{F} + 1)/2$ be the projection onto the +1 eigenspace of $\mathcal{F}$ in $\mathcal{H}$. For a unitary operator $u \in U(\mathcal{A})$, $PuP : \mathcal{PH} \to \mathcal{PH}$ is a Fredholm operator with its index given by,

$$\text{Index}(PuP) = \dim \ker PuP - \dim \ker Pu^*P$$  \hspace{1cm} (6)

That is, the spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ determines an additive map by the Fredholm index,

$$\text{ind}_\mathcal{D} : K_1(\mathcal{A}) \to \mathbb{Z}; \quad [u] \mapsto \text{Index}(PuP)$$ \hspace{1cm} (7)

We call $(\mathcal{A}, \mathcal{H}, \mathcal{F})$ the associated Fredholm module over $\mathcal{A}$, which can be viewed as an abstract elliptic operator in K-homology. $(\mathcal{A}, \mathcal{H}, \mathcal{F})$ is called $p$-summable if for every integer $n \geq p$ the following product is in the trace class $L^1 \subset K$,

$$[\mathcal{F}, a_0][\mathcal{F}, a_1] \cdots [\mathcal{F}, a_n] \in L^1, \quad \forall a_i \in \mathcal{A}$$

A closely related concept is the dimension of the spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$, which is defined as the smallest integer $p$ such that the characteristic values $\mu_n$ of $\mathcal{D}^{-1}$ behave like

$$\mu_n(\mathcal{D}^{-1}) = O(n^{-1/p}), \quad \text{as} \quad n \to \infty$$

If the dimension $p$ of $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is finite, we call it a $p$-summable spectral triple. In particular, the dimension of $(C^\infty(M), L^2(M, \mathcal{D}), \mathcal{D})$ is exactly the dimension of $M$.

The index map $\text{ind}_\mathcal{D}$ can be computed by pairing $K_1(\mathcal{A})$ with the odd Connes-Chern character in cyclic cohomology. Denote by $C^n(\mathcal{A})$ the space of $(n+1)$-linear functionals $\phi : \mathcal{A}^{n+1} \to \mathbb{C}$ such that $\phi(a_0, a_1, \cdots, a_n) = 0$ if $a_j = 1$ for some $j \geq 1$. The coboundary map $b : C^n(\mathcal{A}) \to C^{n+1}(\mathcal{A})$ is defined by

$$b\phi(a_0, \cdots, a_{n+1}) = \sum_{j=0}^n (-1)^j \phi(a_0, \cdots, a_j a_{j+1}, \cdots, a_{n+1}) + (-1)^{n+1} \phi(a_{n+1} a_0, \cdots, a_n)$$
Since $b^2 = 0$, one defines the Hochschild cohomology groups of $\mathcal{A}$ by the cohomology of the Hochschild complex $(C^*(\mathcal{A}), b)$, denoted by $HH^n(\mathcal{A})$. In addition, $\phi : \mathcal{A}^{\otimes n+1} \to \mathbb{C}$ is said to be cyclic if $\phi = \lambda \phi$, where

$$\lambda \phi(a_0, \ldots, a_n) = (-1)^n \phi(a_n, a_0, \ldots, a_{n-1})$$

Then one has the cyclic complex, denoted by $(C^\lambda(\mathcal{A}), b)$, as a subcomplex of $(C^*(\mathcal{A}), b)$, similarly one defines the cyclic cohomology groups $HC^n(\mathcal{A})$.

**Theorem 1 ([3]).** Let $(\mathcal{A}, \mathcal{H}, \mathcal{F})$ be a $p$-summable odd Fredholm module and let $n = 2k + 1 \geq p$, then the following cochain

$$\phi(a_0, \ldots, a_n) = \frac{1}{2} Tr(C[a_0][a_1] \cdots [a_n])$$

defines a cyclic cocycle such that its pairing with a unitary $u \in U(\mathcal{A})$ gives the Fredholm index up to a normalization constant,

$$\phi(u, u^*, \ldots, u, u^*) = (-1)^{(n+1)/2} 2^n \text{Index}(PuP)$$

One step further, we need the periodic cyclic cohomology groups to pair with $K$-groups. Define Connes' boundary map by the composition $B = N \circ B_0 : C^1(\mathcal{A}) \to C^0(\mathcal{A}) \to C^1(\mathcal{A})$, more precisely,

$$B_0 \phi(a_0, \ldots, a_{n-1}) = \phi(1, a_0, \ldots, a_{n-1}) - (-1)^n \phi(a_0, \ldots, a_{n-1}, 1);$$

$$N \phi(a_0, \ldots, a_{n-1}) = \sum_{j=0}^{n-1} \lambda \phi = \sum (-1)^{(n-1)j} \phi(a_j, a_{j+1}, \ldots, a_{j-1});$$

$$B \phi(a_0, \ldots, a_{n-1}) = \sum_{j=0}^{n} (-1)^n \phi(1, a_j, a_{j+1}, \ldots, a_{j-1})$$

Since $b^2 = Bb + bB = B^2 = 0$, we obtain the Connes' $(b, B)$-bicomplex, denoted by $B(\mathcal{A})$, and define the periodic cyclic cohomology $HP^*(\mathcal{A})$ as the cohomology of the total complex $(TotB(\mathcal{A}), b + B)$.

For example, an odd $(b, B)$-cocycle $\phi \in HP^1(\mathcal{A})$ is given by

$$\phi = (\phi_1, \phi_3, \phi_5, \ldots), \quad \text{s.t.} \quad b\phi_{2k-1} + B\phi_{2k+1} = 0$$

In fact, the map

$$\begin{align*}
(C^\lambda(\mathcal{A}), b) & \to (TotB(\mathcal{A}), b + B) \\
\phi_n & \mapsto (0, \ldots, 0, \phi_n, 0, \ldots)
\end{align*}$$

induces a quasi-isomorphism of complexes.
Definition 1. Let \((A, H, F)\) be a \(p\)-summable odd Fredholm module and let 
\(n = 2k + 1 \geq p\), the odd Connes-Chern character is defined by 
\[
C_h(a_0, \cdots, a_n) = \frac{\Gamma(1 + n/2)}{2 \cdot n!} Tr(F[F, a_0][F, a_1] \cdots [F, a_n]),
\]
which is a cyclic cocycle and its periodic cyclic cohomology class is independent of the choice of \(n\).

A spectral triple \((A, H, D)\) is regular if \(A\) and \([D, A]\) both belong to 
\(OP^0 = \bigcap_{n \geq 1} Dom\delta^n\), the domain of all derivations \(\delta^n\) with 
\(\delta(T) := \|[D], T\|\). Let \(\mathcal{L}^{1, \infty}\) be the set of compact operators having finite 
\(\|\cdot\|_{1, \infty}\)-norm, where

\[
\|T\|_{1, \infty} = \sup_N \frac{\sum_{i=1}^N \mu_i(T)}{\log N}
\]

There exists a well-defined trace functional on \(\mathcal{L}^{1, \infty}\), the so-called Dixmier trace 
\(Tr_\omega : \mathcal{L}^{1, \infty} \to \mathbb{C}\).

The Hochschild character theorem tells us that we can compute the Connes-Chern character by a Hochschild cohomology class.

Theorem 2 ([3]). Let \((A, H, D)\) be a regular odd spectral triple, assume 
\(a \cdot |D|^{-n} \in \mathcal{L}^{1, \infty}\) for every \(a \in A\) and some odd positive integer \(n\), then the 
Connes-Chern character is cohomologous to the Hochschild cocycle 
\[
\Phi(a_0, \cdots, a_n) = \frac{\Gamma(1 + n/2)}{n \cdot n!} Tr_\omega(a_0[D, a_1] \cdots [D, a_n]|D|^{-n})
\]

In the commutative triple \((C^\infty(M), L^2(M, \mathcal{H}), \mathcal{D})\), the Hochschild character \(\Phi\) is computable by translating the Dixmier trace into a classical integral over \(M\). In higher dimensions, we have the Connes-Chern character cohomologous to a \((b, B)\)-cocycle defined by Wodzicki residue by Connes and Moscovici [8], which is exactly the noncommutative local index formula. Now we state the 3-dimensional local index theorem as follows.

Theorem 3 ([8]). If \((A, H, D)\) is a regular 3-summable spectral triple and 
\(u \in U(A)\) is a unitary operator, let 
\(F = D|D|^{-1}\) be the sign of \(D\) and \(P\) be the projection \((F + 1)/2\), then the Fredholm index can be computed by pairing 
\(K_1(A)\) with a \((b, B)\)-cocycle \((\phi_1, \phi_3)\),

\[
\text{Index}(PuP) = \phi_1(u^*, u) - \phi_3(u^*, u, u^*, u)
\]
where
\[
\phi_1(a^0, a^1) = \tau_0(a^0 da^1|D|^{-1}) - \frac{1}{4} \tau_0(a^0 \nabla(da^1)|D|^{-3}) - \frac{1}{2} \tau_1(a^0 \nabla(da^1)|D|^{-3}) + \frac{1}{8} \tau_0(a^0 \nabla^2(da^1)|D|^{-5}) + \frac{1}{3} \tau_1(a^0 \nabla^2(da^1)|D|^{-5}) + \frac{1}{12} \tau_2(a^0 \nabla^2(da^1)|D|^{-5})
\] (13)

and
\[
\phi_3(a^0, a^1, a^2, a^3) = \frac{1}{12} \tau_0(a^0 da^1 da^2 da^3|D|^{-3}) + \frac{1}{6} \tau_1(a^0 da^1 da^2 da^3|D|^{-3})
\] (14)

with the notations
\[
\tau_k(a) = \text{Res}_{z=0} z^k Tr(a|D|^{-z})
\]
\[
da = [D, a] \quad \text{and} \quad \nabla(a) = [D^2, a].
\]

In order to define the Chern-Simons action, we first need the definition of connections in noncommutative geometry and we follow that in inner fluctuations of the spectral action [5]. Formally one defines the space of 1-forms over \(\mathcal{A}\) as the bimodule,
\[
\Omega^1_D(\mathcal{A}) = \left\{ \sum_i a_i[D, b_i] ; \ a_i, b_i \in \mathcal{A} \right\}
\] (15)

then a connection 1-form \(A \in \Omega^1_D(\mathcal{A})\) is a self-adjoint element, i.e. \(A = A^*\).

**Definition 2** ([17]). Let \((\mathcal{A}, \mathcal{H}, D)\) be a regular 3-summable spectral triple and \(A \in \Omega^1_D(\mathcal{A})\) be a connection 1-form, the noncommutative Chern-Simons action is defined by
\[
S_{CS}(A) = 3\phi_3(AdA + \frac{2}{3} A^3) - \phi_1(A)
\] (16)

for the \((b, B)\)-cocycle \((\phi_1, \phi_3)\) given in the 3-dimensional local index formula.

When the linear term \(\phi_1\) vanishes, this definition coincides with that of Connes and Chamseddine introduced in [5]. For example, \(\phi_1\) does not vanish in the case of \(SU_q(2)\), but it does vanish for \(T^3_\Theta\) or \(S^3_\theta\).

**Theorem 4** ([17]). The Chern-Simons action is well defined up to the addition of a Fredholm index under gauge transformations by unitaries \(u \in U(\mathcal{A})\),
\[
S_{CS}(u^*Au + u^*du) = S_{CS}(A) + \text{Index}(PuP)
\] (17)

This can be verified directly by the property of the \((b, B)\)-cocycle \((\phi_1, \phi_3)\) and the local index formula, so the gauge invariance of the Chern-Simons action was established, more details about the proof can be found in [17].
3 Quantum 3-sphere \( S_\theta^3 \)

In this section we first recall the Dirac geometry on the 3-sphere \( S^3 \cong SU(2) \), the Dirac spectrum can be computed from different approaches [11, 12, 15]. The quantum 3-sphere \( S_\theta^3 \) can be defined as a \( \theta \)-deformed \( C^* \)-algebra [7], and a spectral triple on \( S_\theta^3 \) will be constructed as a noncommutative analogue of the Dirac geometry of \( S^3 \).

On the unit 3-sphere \( S^3 = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^2 = 1 \} \), the Hopf action is the isometric circle action, \( S^1 \times S^3 \to S^3 \), \( (e^{i\omega}, (z_1, z_2)) \mapsto (e^{i\omega} z_1, e^{i\omega} z_2) \) or equivalently, it is the matrix multiplication on \( SU(2) \),

\[
\begin{pmatrix}
  e^{i\omega} & 0 \\
  0 & e^{-i\omega}
\end{pmatrix}
\begin{pmatrix}
  z_1 & z_2 \\
  -\bar{z}_2 & \bar{z}_1
\end{pmatrix}
= 
\begin{pmatrix}
  e^{i\omega} z_1 & e^{i\omega} z_2 \\
  -e^{-i\omega} \bar{z}_2 & e^{-i\omega} \bar{z}_1
\end{pmatrix}
\]

In addition, the Hopf map is defined by

\[ h : S^3 \to S^2; \ (z_1, z_2) \mapsto (|z_1|^2 - |z_2|^2, 2z_1 \bar{z}_2), \]

which induces the Hopf fibration \( S^1 \hookrightarrow S^3 \xrightarrow{h} S^2 \).

In real coordinates \( S^3 = \{(x_0, x_1, x_2, x_3) \in \mathbb{R}^4 : x_0^2 + x_1^2 + x_2^2 + x_3^2 = 1 \} \), there is a canonical choice of orthonormal right invariant vector fields in the tangent space \( T_e S^3 \cong \mathfrak{su}(2) \) for example at \( e = (1, 0, 0, 0) \),

\[
X = -x_3 \partial_0 - x_2 \partial_1 + x_1 \partial_2 + x_0 \partial_3 \\
Y = -x_2 \partial_0 + x_3 \partial_1 + x_0 \partial_2 - x_1 \partial_3 \\
Z = -x_1 \partial_0 - x_0 \partial_1 + x_3 \partial_2 + x_2 \partial_3
\]

where \( Z = \partial_\omega \) is the velocity field of the rotation in the Hopf action. The Dirac operator on the left trivialization of the spinor bundle \( L^2(S^3, \mathcal{F}) \) was defined in [12] by

\[ D = \frac{3}{2} I_2 + iX \sigma_1 + iY \sigma_2 + iZ \sigma_3 \]

with Pauli matrices

\[
\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
\]
For convenience, the Dirac operator without constant is denoted by $D'$,

$$D' = iX\sigma_1 + iY\sigma_2 + iZ\sigma_3$$

If one identifies $z_1 = x_0 + ix_1$, $z_2 = x_2 + ix_3$, then in complex coordinates

$$X = -i(\bar{z}_2\partial_{\bar{z}_1} - z_2\partial_{z_1} - \bar{z}_1\partial_{\bar{z}_2} + z_1\partial_{z_2})$$
$$Y = -(\bar{z}_2\partial_{z_1} + z_2\partial_{\bar{z}_1} - \bar{z}_1\partial_{z_2} - z_1\partial_{\bar{z}_2})$$
$$Z = (z_1\partial_{\bar{z}_1} - \bar{z}_1\partial_{z_1} + z_2\partial_{z_2} - \bar{z}_2\partial_{\bar{z}_2})$$

It is convenient to define the ladder operators,

$$L_- = X + iY = -2i(\bar{z}_2\partial_{z_1} - z_2\partial_{\bar{z}_1})$$
$$L_+ = X - iY = 2i(z_2\partial_{\bar{z}_1} - \bar{z}_2\partial_{z_1})$$

They satisfy the commutation relations

$$[Z, L_+] = 2iL_+, \quad [Z, L_-] = -2iL_-, \quad [L_+, L_-] = 4iZ$$

In other words, $H = -iZ/2$, $E = iL_+/2\sqrt{2}$, $F = iL_-/2\sqrt{2}$ give a representation of $\mathfrak{su}(2)$, i.e.

$$[H, E] = E, \quad [H, F] = -F, \quad [E, F] = H$$

In the Hopf coordinates, whose geometric picture is the join operation $S^1 \star S^1 = S^3$,

$$z_1 = e^{i\xi_1} \cos \eta, \quad z_2 = e^{i\xi_2} \sin \eta, \quad \xi_i \in [0, 2\pi], \quad \eta \in [0, \pi/2]$$

the vector fields can be written as

$$L_+ = -e^{i(\xi_1 + \xi_2)}[(\tan \eta \partial_{\xi_1} - \cot \eta \partial_{\xi_2}) + 2i\partial_{\eta}]$$
$$L_- = -e^{-i(\xi_1 + \xi_2)}[(\tan \eta \partial_{\xi_1} - \cot \eta \partial_{\xi_2}) - 2i\partial_{\eta}]$$
$$Z = \partial_{\xi_1} + \partial_{\xi_2}$$

The Casimir operator for $\mathfrak{su}(2)$ is

$$C = H^2 + FE + EF = -\frac{1}{4}[Z^2 + (L_+L_- + L_-L_+)/2]$$
$$= -\partial_{\eta}^2 - \frac{1}{4}(\sec^2 \eta \partial_{\xi_1}^2 + \csc^2 \eta \partial_{\xi_2}^2)$$

and the invariant Dirac Laplacian is given by

$$D'^2 = -4\partial_{\eta}^2 - \sec^2 \eta \partial_{\xi_1}^2 - \csc^2 \eta \partial_{\xi_2}^2 = 4C$$
The coefficient 4 in front of $\partial^2_\eta$ can be absorbed by using $\cos 2\eta$ and $\sin 2\eta$ in the Hopf coordinates so that

$$D'^2 = -\partial^2_\eta - \sec^2 2\eta \partial^2_{\xi_1} - \csc^2 2\eta \partial^2_{\xi_2}$$

Because of this discrepancy, we have to make a small modification to the coefficients in the ladder operators.

**Definition 3.** The Dirac operator on $S^3$ in the Hopf coordinates is defined by

$$D = \frac{3}{2} I + i \left( \begin{array}{cc} Z & \tilde{L}_+ \\ \tilde{L}_- & -Z \end{array} \right)$$

where

$$\tilde{L}_+ = -ie^{i(\xi_1+\xi_2)}[\partial_\eta - i(\tan \eta \partial_{\xi_1} - \cot \eta \partial_{\xi_2})]$$

$$\tilde{L}_- = ie^{-i(\xi_1+\xi_2)}[\partial_\eta + i(\tan \eta \partial_{\xi_1} - \cot \eta \partial_{\xi_2})]$$

$$Z = \partial_{\xi_1} + \partial_{\xi_2}$$

The convenient notation for the Dirac operator without the constant term is also used as before,

$$D' = i \left( \begin{array}{cc} Z & \tilde{L}_+ \\ \tilde{L}_- & -Z \end{array} \right)$$

then

$$D'^2 = -\partial^2_\eta - \sec^2 \eta \partial^2_{\xi_1} - \csc^2 \eta \partial^2_{\xi_2}$$

corresponds to the round metric on $S^3$,

$$ds^2 = d\eta^2 + \cos^2 \eta d\xi_1^2 + \sin^2 \eta d\xi_2^2$$

By the Peter-Weyl theorem, one has an orthogonal Hilbert basis for $L^2(SU(2),d\mu)$ with $d\mu$ the standard Haar measure on $SU(2)$,

$$\phi_{i,j}^m(g) = \left( \begin{array}{c} m \\ i \end{array} \right)^{-1/2} \left( \begin{array}{c} m \\ j \end{array} \right)^{-1/2} \sum_{s+t=i} \left( \begin{array}{c} m-j \\ s \\ t \end{array} \right) z_1^s (-z_2)^{j-t} z_2^m z_1^{-j-s}$$

where $m \geq 0$, $0 \leq i, j \leq m$ such that

$$\int_{SU(2)} \phi_{i,j}^m(g) \overline{\phi}_{k,l}^n(g) d\mu(g) = \frac{1}{m+1} \delta_{mn} \delta_{ik} \delta_{jl}$$
Denote the coefficients by
\[ c_{i,j}^m = \binom{m}{i}^{-1/2} \binom{m}{j}^{-1/2}, \quad b_{s,t}^{m,j} = \binom{m-j}{s} \binom{j}{t} \]
in the Hopf coordinates,
\[ \phi_{i,j}^m = c_{i,j}^m \sum_{s+t=l} (-1)^{j-t} \phi_{k,l}^{m,j} e^{i(l+j-m)} e^{i(l-j)} (\cos \eta)^{m-j-s+t} (\sin \eta)^{j-t+s} \]
It is easy to check that
\[ Z \phi_{i,j}^m = i(2l - m) \phi_{i,j}^m \]
\[ \tilde{L}_+ \phi_{i,j}^m = 2i \sqrt{l + 1} \sqrt{m - l} \phi_{l+1,j}^m \]
\[ \tilde{L}_- \phi_{i,j}^m = 2i \sqrt{l} \sqrt{m + 1} - l \phi_{l-1,j}^m \]
and the Dirac Laplacian has eigenvalues \( m(m+2) \) with multiplicity \((m+1)^2\),
\[ \mathcal{D}^2 \phi_{i,j}^m = - [Z^2 + (\tilde{L}_+ \tilde{L}_- + \tilde{L}_- \tilde{L}_+) / 2] \phi_{i,j}^m = (m^2 + 2m) \phi_{i,j}^m \]
One constructs the orthonormal eigenspinors in \( L^2(S^3, \mathfrak{g}) \) for the left trivialization as in [12],
\[ \Phi_{k,\ell}^m = \left( \frac{-\sqrt{k} \phi_{m+1-k,\ell}^m}{\sqrt{m+1-k} \phi_{m-k,\ell}^m} \right) \quad (0 \leq k \leq m+1, \ 0 \leq \ell \leq m) \]
\[ \Phi_{-k,\ell}^{-m} = \left( \frac{\sqrt{m+1-k} \phi_{m+1-k,\ell}^{-m}}{\sqrt{k+1} \phi_{m-k,\ell}^{-m}} \right) \quad (0 \leq k \leq m, \ 0 \leq \ell \leq m+1) \]
Similarly one can define eigenspinors based on left invariant vector fields and the right trivialization of \( L^2(S^3, \mathfrak{g}) \). It is easy to check that
\[ \mathcal{D}' \Phi_{k,\ell}^m = i \left( \frac{Z}{\tilde{L}_+ - Z} \right) \left( \frac{-\sqrt{k} \phi_{m+1-k,\ell}^m}{\sqrt{m+1-k} \phi_{m-k,\ell}^m} \right) = m \Phi_{k,\ell}^m \]
\[ \mathcal{D}' \Phi_{-k,\ell}^{-m} = i \left( \frac{Z}{\tilde{L}_- - Z} \right) \left( \frac{\sqrt{m+1-k} \phi_{m+1-k,\ell}^{-m}}{\sqrt{k+1} \phi_{m-k,\ell}^{-m}} \right) = -(m+3) \Phi_{-k,\ell}^{-m} \]
Together with the Frobenius reciprocity, the spinor bundle \( L^2(S^3, \mathfrak{g}) \) has a decomposition
\[ L^2(S^3, \mathfrak{g}) = H^+ \oplus H^- = (\oplus E_{-m}) \oplus (\oplus E_m) \]
where \( E_m \) (resp. \( E_{-m} \)) is the eigenspace of \( \mathcal{D} \) with eigenvalue \( m+3/2 \) (resp. \(-m+3/2\)). In addition, the multiplicity of the eigenvalues \( \pm (m+3/2) \) is equal to the dimension of \( E_{\pm m} \), i.e. \( \dim E_{\pm m} = (m+1)(m+2) \). Our concrete construction is parallel to the representation theoretic approach in [12].
Definition 4. The quantum 3-sphere $S_3^{\theta}$ is defined as the $C^*$-algebra generated by operators $\alpha$ and $\beta$ satisfying the relations,

$$\alpha \beta = \lambda \beta \alpha, \quad \alpha^* \beta = \lambda^* \alpha, \quad \beta \lambda \alpha = \alpha \beta^* = \beta^* \beta, \quad \alpha \alpha^* + \beta \beta^* = 1$$

for the complex parameter $\lambda = e^{2\pi i \theta}$ and irrational $\theta \in \mathbb{R} \setminus \mathbb{Q}$.

In other words, $S_3^{\theta}$ is the $C^*$-algebraic version of the $\lambda$-deformed $SU(2)$, i.e.

$$\left( \begin{array}{cc} \alpha & \beta \\ -\lambda \beta^* & \alpha^* \end{array} \right) \in S_3^{\theta}$$

$S_3^{\theta}$ was first introduced in [7], it is a special case of a more general class of noncommutative 3-spheres considered in [6]. The K-groups of this quantum 3-sphere are simply given by,

$$K_0(S_3^{\theta}) \cong \mathbb{Z}, \quad K_1(S_3^{\theta}) \cong \mathbb{Z}$$

It is also possible to generate $S_3^{\theta}$ by self-adjoint operators and more details can be found in [6].

There exists a natural parametrization of the generators in $S_3^{\theta}$ by Hopf coordinates,

$$\alpha = u \cos \psi, \quad \beta = v \sin \psi, \quad \psi \in [0, \pi/2]$$

where $u, v$ are the generators of the noncommutative 2-torus $T_\theta^2$ satisfying $uv = \lambda vu$. One can define the Hopf circle action as usual and the Hopf map

$$h : (u \cos \psi, v \sin \psi) \mapsto (\cos 2\psi, uv^* \sin 2\psi)$$

gives rise to a quantum principal $U(1)$-Hopf fibration.

Over the quantum 3-sphere $S_3^{\theta}$, we define the Dirac operator as

$$D_1 = \frac{3}{2} I_2 + i \left( \begin{array}{cc} X_3 & X^+ \\ X^- & -X_3 \end{array} \right)$$

where

$$X_3 = i \delta_1 + i \delta_2$$
$$X^+ = -iuv[\partial_\psi + (\tan \psi \delta_1 - \cot \psi \delta_2)]$$
$$X^- = i(\lambda uv)^*[\partial_\psi - (\tan \psi \delta_1 - \cot \psi \delta_2)]$$

and $\delta_i$ are the canonical derivations on $T_\theta^2$:

$$\delta_1(u) = u, \quad \delta_1(v) = 0, \quad \delta_2(u) = 0, \quad \delta_2(v) = v$$
In order to get the same Dirac spectrum we actually have to distinguish between left and right multiplications. More precisely, let us use $L$ (resp. $R$) to indicate the left (resp. right) multiplication, the ladder operators in the Dirac operator should be defined as

$$
X^+ = -iL(u)R(v)[\partial_\psi + (\tan \psi \delta_1 - \cot \psi \delta_2)] \\
X^- = iL(u^*)R(v^*)[\partial_\psi - (\tan \psi \delta_1 - \cot \psi \delta_2)]
$$

(23)

As expected, we have the same eigenvalues as before if these operators are applied to $	ilde{\phi}_{l,j}^m = c_{l,j}^m \sum_{s+t=l} b_{s,t}^{m,j} (-1)^{j-t} u^{l+j-m} v^{l-j} (\cos \psi)^{m-j-s+t} (\sin \psi)^{j-t+s}$

The eigenspinors $\tilde{\Phi}_{l,j}^m$ can be defined similarly so that $D_1$ has the same Dirac spectrum as in the Dirac geometry of $S^3$. In other words, we have easily obtained the Hilbert space $L^2(S^3, \Theta)$ of spinors $\tilde{\Phi}_{l,j}^m$ with complex coordinates replaced by the generators of $S^3_\Theta$.

Denote by $C^\infty(S^3_\Theta)$ the pre-$C^*$-algebra of smooth elements $a \in C^\infty(S^3_\Theta)$ of rapid decay, i.e.

$$
a = \sum_{(n,m) \in \mathbb{Z}^2} a_{mn} \alpha^n \beta^m, \quad s.t. \quad \{|n|^p|m|^q|a_{mn}|\}_{(n,m) \in \mathbb{Z}^2} \subset B_d
$$

the sequence is bounded for any $p, q > 0$.

Assemble these parts together, $(C^\infty(S^3_\Theta), L^2(S^3_\Theta, \Theta), D_1)$ is defined as the spectral triple generalizing the Dirac geometry $(C^\infty(S^3), L^2(S^3, \Theta), \partial)$. An alternative way to construct the same spectral triple is to introduce a Moyal product into the commutative triple $(C^\infty(S^3), L^2(S^3, \Theta), \partial)$. More precisely, define a star product so that $(C^\infty(S^3), \ast_\Theta) = C^\infty(S^3_\Theta)$, then the spectral triple consists of the same Dirac operator and Hilbert space but a new non-commutative smooth algebra $(C^\infty(S^3), \ast_\Theta)$. More details about such Moyal star-product deformation can be found in the work by Rieffel [19].

4 Chern-Simons action on $S^3_\Theta$

In this section we first check that the spectral triple $(C^\infty(S^3_\Theta), L^2(S^3_\Theta, \Theta), D_1)$ satisfies the conditions of the local index theorem and has simple dimension spectrum, then we compute the Chern-Simons action.
For later convenience, we write the Dirac operator as

\[ D_1 = \frac{3}{2} I_2 + \left( \begin{array}{cc} \delta_3^- & \delta_3^+ \\ \delta_3^- & -\delta_3^+ \end{array} \right) = \frac{3}{2} I_2 + \phi_1 \sigma_1 + \phi_2 \sigma_2 + \phi_3 \sigma_3 \]

where

\[ \delta^- = -L(u^*) R(v^*) [\partial_{\psi} - (\tan \psi \delta_1 - \cot \psi \delta_2)] \]
\[ \delta^+ = L(u) R(v) [\partial_{\psi} + (\tan \psi \delta_1 - \cot \psi \delta_2)] \]
\[ \delta_3 = - (\delta_1 + \delta_2) \]

and

\[ \delta_1 = \frac{1}{2}(\delta^+ + \delta^-), \quad \delta_2 = \frac{i}{2}(\delta^+ - \delta^-) \]

It is easy to see that

\[ [D_1, \alpha] = \beta^*(\sigma_1 - i \sigma_2) - \alpha \sigma_3 = \begin{pmatrix} -\alpha & 0 \\ 2\beta^* & \alpha \end{pmatrix} \]
\[ [D_1, \beta] = -\alpha^*(\sigma_1 - i \sigma_2) - \beta \sigma_3 = \begin{pmatrix} -\beta & 0 \\ -2\alpha^* & \beta \end{pmatrix} \]
\[ [D_1, \alpha^*] = -\beta(\sigma_1 + i \sigma_2) + \alpha^* \sigma_3 = \begin{pmatrix} \alpha^* & -2\beta \\ 0 & -\alpha^* \end{pmatrix} \]
\[ [D_1, \beta^*] = \alpha(\sigma_1 + i \sigma_2) + \beta^* \sigma_3 = \begin{pmatrix} \beta^* & 2\alpha \\ 0 & -\beta^* \end{pmatrix} \]

so the commutator \([D_1, a] \) for any \( a \in A = C^\infty(S^3) \) is a bounded operator. Furthermore, \((C^\infty(S^3), L^2(S^3, S), D_1)\) is a 3-summable spectral triple since the Dirac operator \( D_1 \) has the same spectrum as in the Dirac geometry.

For the pseudo-differential calculus, we use the conventional notations,

\[ OP^0 = \cap_{n=1}^\infty Dom \delta^n, \quad OP^k = |D|^k OP^0, \quad OP^{-\infty} = \cap_{k>0} OP^{-k} \]

As for the regularity condition, i.e. \( A \subset OP^0 \) and \([D_1, A] \subset OP^0\), it is enough to check it on the generators of \( C^\infty(S^3) \). Let \( F = D_1 |D_1|^{-1} \) be the sign of \( D_1 \), for example, \( \delta(\alpha) = [\|D_1\|, \alpha] = F[D_1, \alpha] + [F, \alpha]D_1 \) and \([F, \alpha] \) belongs to the two sided ideal \( OP^{-\infty} \subset OP^0 \). Similarly \( \delta([D_1, \alpha]) = [FD_1, [D_1, \alpha]] \) and the order of \([D_1, \alpha] \) is zero. Together with the results of commutators with the generators, it is obvious that the spectral triple \((C^\infty(S^3), L^2(S^3, S), D_1)\) is regular.
Let $\mathcal{B}$ be the algebra generated by the spaces $\delta^n(\mathcal{A})$ and $\delta^n([\mathcal{D}_1, \mathcal{A}])$ for all $n \geq 0$, define the zeta function for each $b \in \mathcal{B}$,
\[
\zeta_b(z) = \text{Tr}(b|\mathcal{D}_1|^{-z})
\] (24)
which is analytic for $\text{Re}(z) > 3$. Recall that the dimension spectrum of a spectral triple is the discrete singular points $\Sigma \subset \mathbb{C}$ of the meromorphic function $\zeta_b(z)$ after analytic continuation for all $b \in \mathcal{B}$.

\[
\text{Tr}(b|\mathcal{D}_1|^{-z}) = \sum_{m \geq 0} <\tilde{\Phi}_{k,l}^m, b|\mathcal{D}_1|^{-z}\tilde{\Phi}_{k,l}^m> + <\tilde{\Phi}_{k,l}^{-m}, b|\mathcal{D}_1|^{-z}\tilde{\Phi}_{k,l}^{-m}>
\]

$b \in \mathcal{B}$ is a general $2 \times 2$ matrix with entries being functions in the generators of $S^3_\theta$. So this reduces the problem to consider $<\phi_{k,l}^m, O\phi_{k,l}^m>$ for an arbitrary operator valued function $O(\alpha, \beta)$, but only the constant term contributes, i.e. for some $b_0 \in \mathbb{C},$

\[
\text{Tr}(b|\mathcal{D}_1|^{-z}) = \sum_{m \geq 0} (m+1)(m+2)(m+3/2)^{-z}b_0
\]

\[
= b_0[\zeta_H(z-2,3/2) - \frac{1}{4}\zeta_H(z,3/2)]
\]

where $\zeta_H(z,a)$ is the Hurwitz zeta function. By the property of Hurwitz zeta function, the spectral triple $(C^\infty(S^3_\theta), L^2(S^3_\theta, \mathcal{S}), \mathcal{D}_1)$ has simple dimension spectrum $\{1,3\}$.

Since the dimension spectrum is simple, the $(b, B)$-cocycle $(\phi_1, \phi_3)$ can be simplified further. With the notation of noncommutative integral,

\[
\int \text{T} = \text{Res}_{z=0} \text{Tr} T|\mathcal{D}|^{-z}
\] (25)

we have

\[
\phi_1 = \int a^0 da^1|\mathcal{D}_1|^{-1} - \frac{1}{4}\int a^0 \nabla(da^1)|\mathcal{D}_1|^{-3} + \frac{1}{8}\int a^0 \nabla^2(da^1)|\mathcal{D}_1|^{-5}
\] (26)

\[
\phi_3 = \frac{1}{12}\int a^0 da^1 da^2 da^3|\mathcal{D}_1|^{-3}
\] (27)

$da = [\mathcal{D}_1, a]$ and $\nabla(a) = [\mathcal{D}_1^2, a]$ as understood.

**Proposition 1.** The first cochain $\phi_1$ in the Chern-Simons action vanishes for the spectral triple $(C^\infty(S^3_\theta), L^2(S^3_\theta, \mathcal{S}), \mathcal{D}_1)$.
We have already seen $\phi_1(a^0 da^1)$, if we further compute $\nabla(da), \nabla^2(da)$, there are still Pauli matrices in front of every term just like in $da$. Thus taking a trace gives zero, in other words, we have proved that the linear term $\phi_1(a^0 da^1)$ vanishes.

Since the linear term disappears on $S^3$, the noncommutative Chern-Simons action is a direct generalization of the classical Chern-Simons form integrated over 3-sphere.

Proof. Without loss of generality, we assume $A$ is a self-adjoint element in the bimodule $\Omega^1_\mathcal{D}_1(S^3)$.

$$S_{CS}(A) = -\sum_{(p', q', p, q) \in \mathbb{Z}^2} \lambda^q(p' + q')(p + q) \bar{a}_{p'q'}b_{p'q'}a_{pq}b_{pq}$$

**Theorem 5.** The Chern-Simons action on the quantum 3-sphere $S^3$ with respect to the spectral triple $(C^\infty(S^3), L^2(S^3), \mathcal{D}_1)$ is given by

$$S_{CS}(A) = \phi_3(3A \wedge [D_1, A] + 2A \wedge A \wedge A)$$

where $a_{mn}, b_{pq}$ are coefficients of rapid decay, and we expect the powers will be canceled out finally. Then the components of the connection $A$ are

$$A_1 = a[\bar{\theta}_1, b] = (p \tan \psi - q \cot \psi)au^*bv^*$$

$$A_2 = a[\bar{\theta}_2, b] = i(q \cot \psi - p \tan \psi)au^*bv^*$$

$$A_3 = a[\bar{\theta}_3, b] = -(p + q)ab$$

here we omit the summation and keep $a, b$ intact for brevity. By direct computation we have

$$A_1[\bar{\theta}_2, A_3] = i(p + q)(p' \tan \psi - q' \cot \psi)$$

$$[(m + p) \tan \psi - (n + q) \cot \psi]au^*bv^*u^*abv^*$$
Here we use the fact on the connection assumption $A$ and the residue trace is

\[
\begin{align*}
A_1[\partial_3, A_2] &= i(m + p + n + q - 2)(p' \tan \psi - q' \cot \psi) \\
(p \tan \psi - q \cot \psi)au^*bv^*au^*bv^* \\
A_2[\partial_1, A_3] &= i(p + q)(p' \tan \psi - q' \cot \psi) \\
((m + p) \tan \psi - (n + q) \cot \psi)au^*bv^*u^*abv^* \\
A_2[\partial_3, A_1] &= i(m + p + n + q - 2)(p' \tan \psi - q' \cot \psi) \\
(p \tan \psi - q \cot \psi)au^*bv^*au^*bv^* \\
A_3[\partial_1, A_2] &= i(p' + q')(p + q)abuau^*b + i(p' + q')(q(n + q - 1) \cot^2 \psi \\
+ p(m + p - 1) \tan^2 \psi - p(n + q) - q(m + p))abu^*au^*b(v^*)^2 \\
A_3[\partial_2, A_1] &= -i(p' + q')(p + q)abuau^*b + i(p' + q'q(n + q - 1) \cot^2 \psi \\
+ p(m + p - 1) \tan^2 \psi - p(n + q) - q(m + p))abu^*au^*b(v^*)^2
\end{align*}
\]

Putting together, we have the Chern-Simons action as

\[
CS[p'q'] = \frac{1}{2}\int_M \sum_{m,n} \sum_{p,q} \lambda^m(p' + q') (p + q) a_m b_{pq} \lambda^p \lambda^q a_{pq}
\]

Putting together, we have the Chern-Simons action as

\[
S_{CS}(A) = -2 \sum_{p',q',p,q} \lambda^p(p' + q') (p + q) \tilde{a}_{p'q'} b_{p'q'} \tilde{a}_{pq} b_{pq}
\]

We will see in next section that the Chern-Simons action not only depends on the connection $A$, but the choice of Dirac operator as well.
5 Choice of Dirac operator

In this section we give another two spectral triples on $S^3_\theta$ with the same Dirac Laplacian spectrum as in the classical 3-sphere. The Chern-Simons action will be computed and we conclude that it depends on the choice of Dirac operator.

If we consider the round metric on $S^3_\theta$ in Hopf coordinates,

$$G = d\psi^2 + \cos^2 \psi du^2 + \sin^2 \psi dv^2$$

we get another Dirac operator by direct computation,

$$\mathcal{D}_2 = \sec \psi \delta_1 \sigma_1 + \csc \psi \delta_2 \sigma_2 + i(\partial_\psi + \frac{1}{2}(\cot \psi - \tan \psi))\sigma_3$$

$\mathcal{D}_2$ can also be obtained by restricting the Dirac operator over $S^4_{\theta}$ [7] onto the equator $S^3_\theta$ when we fix the second angle to be a constant. Notice that the classical Laplace-Beltrami operator corresponds to

$$\mathcal{D}_2' = \sec^2 \psi \delta_1^2 + \csc^2 \psi \delta_2^2 - \partial_\psi^2 - 2 \cot(2\psi) \partial_\psi$$

and its eigenvalues are easy to get,

$$\mathcal{D}_2' \tilde{\phi}^m_{l,j} = (m^2 + 2m)\tilde{\phi}^m_{l,j}$$

with multiplicity $(m + 1)^2$.

Now we have a second spectral triple $(C^\infty(S^3_\theta), L^2(S^3_\theta), \mathcal{D}_2)$ on the quantum 3-sphere, and one could double it and consider the augmented spectral triple $(C^\infty(S^3_\theta) \otimes M_2(\mathbb{C}), L^2(S^3_\theta) \otimes \mathbb{C}^2, \mathcal{D}_2 \otimes I_2)$. However, in order to compare with $(C^\infty(S^3_\theta), L^2(S^3_\theta) \otimes \mathbb{C}^2, \mathcal{D}_1)$ on the same footing, we consider the spectral triple only with the Hilbert space augmented, i.e. $(C^\infty(S^3_\theta), L^2(S^3_\theta) \otimes \mathbb{C}^2, \mathcal{D}_2)$. Further assume that $L^2(S^3_\theta) \otimes \mathbb{C}^2$ is equipped with a Hilbert basis $\tilde{\phi}^m_{l,j} \otimes e_i$ where $\{e_i\} (i = 1, 2)$ is the standard basis in $\mathbb{C}^2$.

The commutators of the Dirac operator $\mathcal{D}_2$ with the generators are

$$[\mathcal{D}_2, \alpha] = u\sigma_1 - iu \sin \psi \sigma_3 = \begin{pmatrix} -iu \sin \psi & u \\ u & iu \sin \psi \end{pmatrix}$$

$$[\mathcal{D}_2, \beta] = u\sigma_2 + iv \cos \psi \sigma_3 = \begin{pmatrix} iv \cos \psi & -iv \\ iv & -iv \cos \psi \end{pmatrix}$$
\[ [\mathcal{D}_2, \alpha^*] = -u^* \sigma_1 - iu^* \sin \psi \sigma_3 = \begin{pmatrix} -iu^* \sin \psi & -u^* \\ -u^* & iu^* \sin \psi \end{pmatrix} \]

\[ [\mathcal{D}_2, \beta^*] = -v^* \sigma_2 + iv^* \cos \psi \sigma_3 = \begin{pmatrix} iv^* \cos \psi & iv^* \\ -iv^* & -iv^* \cos \psi \end{pmatrix} \]

\([\mathcal{D}_2, a]\) is a bounded operator for any \(a \in C^\infty(S^3_0)\) and the spectral triple \((C^\infty(S^3_0), L^2(S^3_0) \otimes \mathbb{C}^2, \mathcal{D}_2)\) is also 3-summable regular since it gives an isospectral deformation.

**Lemma 1.** The spectral triple \((C^\infty(S^3_0), L^2(S^3_0) \otimes \mathbb{C}^2, \mathcal{D}_2)\) has simple dimension spectrum \(\{3\}\).

**Proof.** Let us look at the spectral zeta function,

\[
Tr(b|\mathcal{D}_2|^{-z}) = \sum_{m,k,\ell < \tilde{\phi}_m^{m,k,\ell}} b(\mathcal{D}^2_2)^{-z/2} \tilde{\phi}_m^{m,k,\ell} > \\
= \sum_{m \geq 0} (m+1)^2 (m^2 + 2m - \cot^2 2\psi)^{-z/2} < \tilde{\phi}_m^{m,k,\ell} b(\mathcal{D}^2_2)^{-z/2} \\
= b_0 \sum_{m \geq 0} (m+1)^2 (m^2 + 2m - \cot^2 2\psi)^{-z/2} \\
= b_0 \sum_{m \geq 0} (m+1)^2 (m^2 + 2m - \csc^2 2\psi)^{-z/2} \\
= b_0 \sum_{n \geq 1} n^2 (n^2 - \csc^2 2\psi)^{-z/2}
\]

For fixed \(\psi\), there exists a smallest \(n_0(\psi)\) such that \(n_0^2 > \csc^2 2\psi\). On the other hand, we know the binomial expansion for \(|w| < 1\),

\[
(1 - w)^{-s} = \sum_{k=0}^{\infty} \frac{\Gamma(s+k)}{\Gamma(s)k!} w^k
\]

We could modify the first \(n_0\) terms since they don’t change the singular points and residues of the spectral zeta function, and we write it in terms of Riemann zeta function,

\[
Tr(b|\mathcal{D}_2|^{-z}) = b_0 \left( \sum_{1 \leq n \leq n_0} + \sum_{n > n_0} \right) n^2 (n^2 - \csc^2 2\psi)^{-z/2} \\
\sim b_0 \sum_{n > n_0} n^2 (n^2 - \csc^2 2\psi)^{-z/2} \\
= b_0 \sum_{n \geq n_0} n^{2-z} (1 - \csc^2 2\psi/n^2)^{-z/2} \\
= b_0 \sum_{k \geq 0} \csc^2 2\psi \frac{\Gamma(k+z/2)}{\Gamma(z/2)k!} \sum_{n \geq n_0} n^{2-z-2k} \\
\sim b_0 \sum_{k \geq 0} \csc^2 2\psi \frac{\Gamma(k+z/2)}{\Gamma(z/2)k!} \zeta(z + 2k - 2)
\]

So the only pole is at \(z = 3\) by the property of Riemann zeta function. \(\square\)
Theorem 6. The Chern-Simons action on $S^3$ with respect to the spectral triple $(C^\infty(S^3), L^2(S^3) \otimes \mathbb{C}^2, D_2)$ is given by

$$S_{CS}(A) = -2 \sum_{p,q,p',q'} (p'q \sec^2 \psi + pq' \csc^2 \psi) \bar{a}_{p'q'} b_{p'q'} a_{pq} b_{pq}$$

(34)

Proof. Let $\partial_1 = \sec \psi_1$, $\partial_2 = \csc \psi_2$, and $\partial_3 = i(\partial_\psi + \cot 2\psi)$, and a connection $A = a_1 D_2, b]$ with $a = a_{mn} \beta^n \alpha^m$ and $b = b_{pq} \alpha^p \beta^q$ as before. The components of the connection are

$$A_1 = p \sec \psi \alpha \beta$$
$$A_2 = q \csc \psi \alpha \beta$$
$$A_3 = i(q \cot \psi - p \tan \psi) \alpha \beta$$

and direct computation gives

$$A_1[\partial_2, A_3] = ip'[(n + q) \cos \psi - p \tan \psi] \sec \psi \csc \psi \alpha \beta \alpha \beta$$
$$A_1[\partial_3, A_2] = ip'q[(n + q - 1) \cos \psi - (m + p) \tan \psi] \sec \psi \csc \psi \alpha \beta \alpha \beta$$
$$A_2[\partial_1, A_3] = iq'(m + p) \cos \psi - p \tan \psi \sec \psi \csc \psi \alpha \beta \alpha \beta$$
$$A_2[\partial_3, A_1] = ipq'[(n + q) \cos \psi - (m + p - 1) \tan \psi] \sec \psi \csc \psi \alpha \beta \alpha \beta$$
$$A_3[\partial_1, A_2] = iq(m + p) \cos \psi - p' \tan \psi \sec \psi \csc \psi \alpha \beta \alpha \beta$$
$$A_3[\partial_2, A_1] = ip(n + q) \cos \psi - p' \tan \psi \sec \psi \csc \psi \alpha \beta \alpha \beta$$

It is easy to see that

$$\varepsilon^{ijk} A_i[\partial_j, A_k] = i(p'q \csc^2 \psi + pq' \sec^2 \psi) \alpha \beta \alpha \beta \alpha \beta, \quad \varepsilon_{ijk} A_i A_j A_k = 0$$

Since the residue of Riemann zeta function at its simple pole $s = 1$ is given by $Res_{s=1} \zeta(s) = 1$, the residue trace is computable,

$$Res_{s=0} Tr(\varepsilon^{ijk} A_i[\partial_j, A_k]|D_2|^{-3-s}) = iRes_{s=0} Tr((p'q \csc^2 \psi + pq' \sec^2 \psi) \alpha \beta \alpha \beta \alpha \beta |D_2|^{-3-s})$$
$$= 2i \sum_{p,q,p',q'} (p'q \csc^2 \psi + pq' \sec^2 \psi) a_{pq} b_{pq}$$

$$Res_{s=0} \sum_m (m + 1)^2 < \delta^m, (m^2 + 2m - \cot^2 2\psi)^{-\frac{4m}{2}} abab\delta^m >$$
$$= 2i \sum_{p,q,p',q'} (p'q \csc^2 \psi + pq' \sec^2 \psi) a_{pq} b_{pq}$$

$$Res_{s=0} \sum_k \csc^{2k} 2\psi \Gamma(k + (3 + z)/2) \zeta_R(z + 2k + 1)$$
$$= 4i \sum_{p,q,p',q'} (p'q \csc^2 \psi + pq' \sec^2 \psi) a_{pq} b_{pq}$$

$$= 4i \sum_{p,q,p',q'} (p'q \csc^2 \psi + pq' \sec^2 \psi) a_{pq} b_{pq}$$

$\square$
From the first spectral triple, we have seen the orthogonal framing of $T_e S^3$ in Hopf coordinates,

\[ \{ \partial_{\xi_1} + \partial_{\xi_2}, \partial_\eta, \tan \eta \partial_{\xi_1} - \cot \eta \partial_{\xi_2} \} \] (35)

where the first vector field is tangent to the Hopf fiber as mentioned before. It is possible to define a third Dirac operator on $S^3_{\theta}$ by

\[ D_3 = i \partial_\psi \sigma_1 - (\tan \psi \delta_1 - \cot \psi \delta_2) \sigma_2 - (\delta_1 + \delta_2) \sigma_3 \] (36)

and its Dirac Laplacian corresponds to the round metric as well,

\[ D_3^2 = -\partial_\psi^2 + \sec^2 \eta \delta_1^2 + \csc^2 \eta \delta_2^2 \] (37)

Thus a third spectral triple can be defined as $(C^\infty(S^3_{\theta}), L^2(S^3_{\theta}) \otimes \mathbb{C}^2, D_3)$, which is a 3-summable regular spectral triple with simple dimension spectrum \{3\} as in the second spectral triple.

**Theorem 7.** The Chern-Simons action on $S^3_{\theta}$ with respect to the spectral triple $(C^\infty(S^3_{\theta}), L^2(S^3_{\theta}) \otimes \mathbb{C}^2, D_3)$ is given by

\[ S_{CS}(A) = -2 \sum_{p,q,p',q'} (p' + q')(p \sec^2 + q \csc^2) \bar{a}_{p'q'}b_{p'q}' \bar{a}_{pq}b_{pq} \] (38)

**Proof.** Let $\hat{\partial}_1 = i \partial_\psi$, $\hat{\partial}_2 = -(\tan \psi \delta_1 - \cot \psi \delta_2)$ and $\hat{\partial}_3 = -(\delta_1 + \delta_2)$, and a connection $A = a[D_3, b]$ with $a = a_{mn,\beta}^m \alpha^m$ and $b = b_{pq} \alpha^p \beta^q$ as before. The components of the connection are given by

\[ A_1 = i(q \cot \psi - p \tan \psi)ab \]
\[ A_2 = (q \cot \psi - p \tan \psi)ab \]
\[ A_3 = -(p + q)ab \]

It is easy to see that

\[ A_1[\hat{\partial}_2, A_3] = i(p + q)(q' \cot \psi - p' \tan \psi)[(m + p) \tan \psi - (n + q) \cot \psi]abab \]
\[ A_1[\hat{\partial}_3, A_2] = i(m + p + n + q)(q' \cot \psi - p' \tan \psi)(p \tan \psi - q \cot \psi)abab \]
\[ A_2[\hat{\partial}_1, A_3] = i(p + q)(p' \tan \psi - q' \cot \psi)[(n + q) \cot \psi - (m + p) \tan \psi]abab \]
\[ A_2[\hat{\partial}_3, A_1] = i(m + p + n + q)(p \tan \psi - q \cot \psi)(q' \cot \psi - p' \tan \psi)abab \]
\[ A_3[\hat{\partial}_1, A_2] = -i(p' + q') [q(n + q - 1) \cot^2 \psi + p(m + p - 1) \tan^2 \psi \\
- p(n + q + 1) - q(m + p + 1)] \text{abab} \]
\[ A_3[\hat{\partial}_2, A_1] = i(p' + q') (p \tan \psi - q \cot \psi) [(n + q) \cot \psi - (m + p) \tan \psi] \text{abab} \]

Combine these terms together,
\[ \varepsilon^{ijk} A_i [\hat{\partial}_j, A_k] = i(p' + q') (p \sec^2 + q \csc^2) \text{abab}, \quad \varepsilon^{ijk} A_i A_j A_k = 0 \]

The residue trace is then,
\[ \text{Res}_{z=0} \text{Tr}(\varepsilon^{ijk} A_i [\hat{\partial}_j, A_k] |D_3|^{3-z}) \]
\[ = i \text{Res}_{z=0} \text{Tr}((p' + q')(p \sec^2 + q \csc^2) \text{abab} |D_3|^{3-z}) \]
\[ = 2i \sum_{p,q,p',q'} (p' + q') (p \sec^2 + q \csc^2) a_{-p'-q'} b_{p'q'} a_{-p-q} b_{pq} \]
\[ \text{Res}_{z=0} \sum_{m} (m + 1)^2 (m^2 + 2m)^{-\frac{3+z}{2}} \]
\[ = 4i \sum_{p,q,p',q'} (p' + q') (p \sec^2 + q \csc^2) a_{-p'-q'} b_{p'q'} a_{-p-q} b_{pq} \]
\[ \text{Res}_{z=0} \sum_{k \geq 0} \frac{1}{(1+\frac{3+z}{2})!} \zeta_R(z + 2k + 1) \]
\[ = 4i \sum_{p,q,p',q'} (p' + q') (p \sec^2 + q \csc^2) a_{-p'-q'} b_{p'q'} a_{-p-q} b_{pq} \]

We have seen that these three spectral triples are all related to the round metric, comparison between their Chern-Simons actions confirms that the Chern-Simons action is not a topological invariant.

**Proposition 2.** The noncommutative Chern-Simons action on the quantum 3-sphere \( S^3_\theta \) depends on the choice of Dirac operator, more generally the choice of spectral triple.

### 6 Quantum field theory

In this section we consider the quantum Chern-Simons field theory in the noncommutative setting, the partition function is shown to be comparable with the one loop approximation in the abelian Chern-Simons field theory in the classical case.

Thanks to the gauge invariance, we quantize the Chern-Simons action by path integral and define the partition function as the formal integral over connection 1-forms \( A \in \Omega^1_D(A) \) modulo gauge transformations,

\[ Z(k) = \int DA e^{2\pi i k \text{CS}(A)} \]
where $k \in \mathbb{Z}$ is the Chern-Simons level. If $A = a[D, b]$, then the measure $DA$ is assumed to be the infinite product of Lebesgue measures,

$$DA = \prod_{(m,n) \in \mathbb{Z}^2} da_{mn} \prod_{(p,q) \in \mathbb{Z}^2} db_{pq}$$

We apply the BRST method to fix the gauge and define the effective action functional as

$$S = 2\pi ikS_{CS}(A) - i \int \delta_B V |D|^{-3}$$

where

$$V = \bar{c} \left( \frac{1}{2} \xi B - d^\mu A_\mu \right)$$

with the $R_\xi$ gauge. More precisely, if we introduce the ghost and anti-ghost fields $c$ and $\bar{c}$ and an auxiliary field $B$, then the partition function under gauge-fixing is given by

$$Z'(k) = \int D\bar{c} Dc DA exp\{2\pi ikS_{CS}(A) - i \int \left( \frac{1}{2} \xi^{-1} d^\mu A_\mu d^\nu A_\nu + d^\mu \bar{c} D_\mu c \right) |D|^{-3} \}$$

(39)

The ghost and anti-ghost fields $c, \bar{c}$ are treated as independent Grassmann variables, suppose they have formal decompositions,

$$c = \sum c_{pq} \alpha^p \beta^q, \quad \bar{c} = \sum c_{mn} \beta^m \alpha^n$$

and the associated measures are

$$dc = \prod_{p,q} dc_{pq}, \quad d\bar{c} = \prod_{m,n} dc_{mn}$$

**Theorem 8.** With respect to the spectral triple $(C^\infty(S^3_\theta), L^2(S^3_\theta, S), \mathcal{D}_1)$, the partition function is formally given by

$$Z'(k) = -\frac{e^{i\pi/4}}{2\pi \sqrt{k}} \prod_{q \in \mathbb{Z}} \sqrt{\frac{\lambda^{2q}}{1 + \lambda^q}}$$

(40)

under the specific gauge chosen when $\xi = \frac{2}{\pi k}$. 24
Proof. By the above computation in \((C^\infty(S^3_\theta), L^2(S^3_\theta, S), D_1)\), we have
\[
d^\mu A_\mu = 2(p + q)uau^*b + (p + q)(m + p + n + q)ab
\]
which gives rise to another quadratic term,
\[
\frac{1}{2} \xi^{-1} \int d^\mu A_\mu d^\nu A_\nu |D_1|^{-3} = 4 \xi^{-1} \sum_{p,q} (p + q)^2 \lambda^2 a^2_{pq} b^2_{pq}
\]
Combine the quadratic terms and take a convenient gauge when \(\xi = 2/\pi k\),
\[
2\pi ikS_{CS}(A) - i(2\xi)^{-1} \int d^\mu A_\mu d^\nu A_\nu |D_1|^{-3}
= -4\pi ik \sum_{p,q} (p + q)^2 a^2_{pq} b^2_{pq} - 8i\xi^{-1} \sum_{p,q} (p + q)^2 \lambda^2 a^2_{pq} b^2_{pq}
= -4\pi ik \sum_{p,q} (\lambda^q + \lambda^{2q})(p + q)^2 a^2_{pq} b^2_{pq}
\]
By direct computation,
\[
d^\mu \bar{c} D_\mu c = (m + n)(p + q)(\bar{c}c + \bar{c}abc)
\]
and the last term involving ghost and anti-ghost is
\[
\int d^\mu \bar{c} D_\mu c |D_1|^{-3} = 2 \sum_{m,n,p,q} (m + n)(p + q) c'_{mn}c_{-m-n}(1 + \bar{a} b_{pq})
\]
Putting together,
\[
Z'(k) = \int D\bar{c} Dc D\bar{A} A \exp\{2\pi ikS_{CS}(A) - i \int \frac{1}{2\pi} d^\mu A_\mu d^\nu A_\nu + d^\mu \bar{c} D_\mu c |D_1|^{-3}\}
= \int D\bar{c} Dc D\bar{A} A \exp\{-4\pi ik \sum_{p,q} (\lambda^q + \lambda^{2q})(p + q)^2 a^2_{pq} b^2_{pq}
- 2i \sum_{m,n,p,q} (m + n)(p + q) c'_{mn}c_{-m-n}(1 + \bar{a} b_{pq})\}
\]
Recall the useful Gaussian integral and Grassmann integral,
\[
\int \frac{dx}{\sqrt{2\pi}} \exp\{-wx^2/2\} = \frac{1}{\sqrt{w}}, \int d\bar{c} dc \exp\{w\bar{c}c\} = w
\]
then the partition function is given by
\[
Z'(k) = -e^{i\pi/4} \frac{1}{\sqrt{k}} \prod \frac{(m + n)}{\sqrt{\lambda^q + \lambda^{-2q}}}
\]
The formal product \(\prod (m + n) = 1/2\pi\) can be obtained as a Laplacian determinant and zeta regularization
\[
\prod \lambda_i = \exp(-\zeta'(0)), \quad \zeta(s) = \sum \lambda_i^{-s}
\]
combined with the property of Riemann zeta function \( \zeta'_R(0) = -\frac{1}{2}\ln(2\pi) \).

Finally, we have

\[
Z'(k) = \frac{-e^{i\pi/4}}{2\pi\sqrt{k}} \prod \sqrt{\frac{\lambda^{2q}}{1 + \lambda^q}}
\]

since \( \lambda \) is a complex parameter the square root is a multivalued function. \( \square \)

If we add some normalization constant in front of the gauge-fixing term in the effective action, then the convenient gauge could be taken to be another multiple of \( k^{-1} \).

7 Discussion

Recall that \( K_1(S^3_q) \cong \mathbb{Z} \) is generated by the unitary operator \( U \in M_2(S^3_q) \),

\[
U = \begin{pmatrix} \alpha & \beta \\ -\lambda \beta^* & \alpha^* \end{pmatrix}, \quad U^* = \begin{pmatrix} \alpha^* & -\bar{\lambda}\beta \\ \beta^* & \alpha \end{pmatrix}
\]

The local index theorem (3) can be verified directly by pairing the generator \( U \) with the \((b,B)\)-cocycle \((\phi_1, \phi_3)\) in the spectral triple \((C^\infty(S^3_q), L^2(S^3_q, S), D_1)\), or work on the \( \eta \)-cochain as the authors did for \( SU_q(2) \) in [20] using a possible variation of proposition 2 of [4].

It is well known that one has the canonical trace on the noncommutative 2-torus \( T^2_q \),

\[
\tau_{T^2_q}(a_{mn}u^m v^n) = a_{00}
\]

and the Hilbert space \( \mathcal{H}_\tau \) is constructed by GNS representation, i.e. the completion of \( T^2_q \) under the inner product

\[
\langle a, b \rangle = \tau(b^* a)
\]

However, over \( C^\infty(S^3_q) \) we cannot make

\[
\tau_{S^3_q}(\sum_{(i,j) \in \mathbb{Z}^2} c_{ij} \alpha^i \beta^j) := c_{00}
\]

into a well-defined trace since \( \alpha \) and \( \beta \) are not unitaries, i.e. we have to take care of the extra parameter \( \psi \in [0, \pi/2] \) in the Hopf coordinates.

There are natural knots and links in the quantum 3-sphere, we would try to introduce Wilson loop operators and consider knot invariants related
to two-point functions in the perturbative picture. On the other hand, our assumptions on the spectral triples and connection 1-form are too simple, in general the cubic term $A \wedge A \wedge A$ cannot be zero, we would consider two-loop contributions involving gauge bosons and ghost fields in the future investigation.

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