McKean-Vlasov multivalued stochastic differential equations with oblique subgradients and related stochastic control problems

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July 26, 2022

Abstract

In this article, we prove the existence of weak solutions as well as the existence and uniqueness of strong solutions for McKean-Vlasov multivalued stochastic differential equations with oblique subgradients (MVMSDEswOS, for short) by means of the equations of Euler type and Skorohod’s representation theorem. For this type of equation, compared with the method in [19, 13], since we can’t use the maximal monotony property of its constituent subdifferential operator, some different specific techniques are applied to solve our problems. Afterwards, we give an example for MVMSDEswOS with time-dependent convex constraints, which can be reduced to MVMSDEswOS. Finally, we consider an optimal control problem and establish the dynamic programming principle for the value function.

AMS Subject Classification: 60H10; 60F10.
Keywords: McKean-Vlasov; Oblique subgradients; Dynamic programming principle; Subdifferential operators; Weak solution and strong solution;

1 Introduction

As is well known, multivalued stochastic differential equations (MSDEs) are widely applied to model stochastic systems in different branches of science and industry. Stability, boundedness
and applications of the solution are the most popular research topic in the field of stochastic dynamic systems and control. The form of these equations is as follows:

\[
\begin{align*}
X(t) + A(X(t))dt &\ni f(X(t))dt + g(X(t))dW(t), t > t_0 \\
X(t_0) &= x_0 \in \overline{D(A)},
\end{align*}
\]  

where \(A\) is a multivalued maximal monotone operator on \(\mathbb{R}^m\), \(\overline{D(A)}\) is the closure of \(D(A)\) and \(W\) is a standard Brownian motion. Compared with the usual stochastic differential equations (SDEs), i.e., \(A = 0\), most of difficulties for MSDEs come from the presence of the finite-variation process \(\{k(t), t \in [0, T]\}\). One only knows that \(\{k(t), t \in [0, T]\}\) is a continuous process with finite total variation and can not prove any further regularity. This type of equations was first studied by Cépa \([3, 4]\). Later, Zhang \([24]\) extended Cépa’s results to the infinite-dimensional case. Ren et al. \([11]\) studied the large deviations for MSDEs which solved the moderate deviation problem for the above equations. In particular, if \(A\) is taken as some subdifferential operator, the corresponding MSDEs can be used to solve a class of stochastic differential equations with reflecting boundary conditions. For MSDEs, more applications can be found in \([10, 19, 17, 7]\) and references therein.

Recently, Gassous et al. \([13]\) built a fundamental framework for the following MSDEs with generalized subgradients:

\[
\begin{align*}
der_x(t) + H(x(t))\partial\Pi(x(t))dt &\ni f(x(t))dt + g(x(t))dB(t), t \in [t_0, \infty), \\
x(t) &= x_0, t = t_0,
\end{align*}
\]

where the new quantity \(H(\cdot, \cdot) : \Omega \times \mathbb{R}^m \to \mathbb{R}^{m \times m}\) acts on the set of subgradients and \(\partial\Pi(\cdot)\) is a subdifferential operator. The product \(H(x(t))\partial\Pi(x(t))\) will be called, from now on, the set of oblique subgradients. The problem becomes challenging due to the presence of this new term, since this new term preserves neither the monotonicity of the subdifferential operator nor the Lipschitz property of the matrix involved. Some different specific techniques have been applied to solve the existence and uniqueness results for this type of equations. Later, Gassous et al. \([11]\) and Maticiuc and Rotenstein \([12]\) investigated backward stochastic differential equations (BSDEs) with oblique subgradients. But there are few applications of MSDEs with oblique subgradients.

On the other hand, many researchers are interested in studying the following equations, which are called McKean-Vlasov stochastic differential equations (MVSDEs):

\[
\begin{align*}
x(t) &= x_0 + \int_0^t f(x(s), \mu_s)ds + g(x(s), \mu_s)dB(s), t \in [t_0, \infty), \\
\mu_t := \text{the probability distribution of } x(t).
\end{align*}
\]

Obviously, the coefficients involved depend not only on the state process but also on its distribution. MVSDEs, being clearly more involved than Itô’s SDEs, arise in McKean \([16]\), who was inspired by Kac’s Programme in Kinetic Theory \([18]\), as well as in some other areas of high interest such as propagation of chaos phenomenon, PDEs, stability, invariant probability measures, social science, economics, engineering, etc. (see e.g. \([2, 8, 20, 24, 14, 21, 22, 13, 3, 6, 9]\)).
Motivated by the above articles, we shall study the following McKean-Vlasov multivalued stochastic differential equations with oblique subgradients (MVMSDEswOS):

\[
\begin{aligned}
&\frac{dx(t)}{dt} + H(x(t), \mu_t)\partial \Pi(x(t)) dt \ni f(x(t), \mu_t) dt + g(x(t), \mu_t) dB(t), \ t \in [t_0, T], \\
&x(t_0) = x_0,
\end{aligned}
\tag{1.2}
\]

where \( H(\cdot, \cdot) : \Omega \times \mathbb{R}^m \times \mathcal{P}_2(\mathbb{R}^m) \to \mathbb{R}^{m \times m} \), \( \mu_t \) is the distribution of \( x(t) \) and \( x_0 \in \mathbb{R}^m \). The appearance of \( H(\cdot, \cdot) \) leads to \( H \partial \Pi \) does not inherit the maximal monotonicity of the subdifferential operator. As a consequence, some specific techniques when approaching the above problem are mandatory.

The main contributions of the paper are as follows:

- Since the coefficients involved depend not only on the state process but also on its distribution, the method in [13] is ineffective. We use the equations of Euler type and Skorohod’s representation theorem to prove the existence and uniqueness result.

- We present an example to illustrate our theory. The example indicates that a class of SDEs with time-dependent constraints are equivalent to some MVMSDEswOS.

- In [13], Gassous et al. studied MSDEswOS, but they did not give the applications for this type of equations. As far as we known, there are no works on optimal control problem for MSDEswOS. We will investigate an optimal control problem MVMSDEswOS and establish the dynamic programming principle for the value function.

We close this part by giving our organization in this article. In Section 2, we introduce some necessary notations, subdifferential operators. In Section 3, We give our main results and an example to illustrate our theory. In Section 4, we consider an optimal control problem and establish the dynamic programming principle for the value function. Furthermore, we make the following convention: the letter \( C(\eta) \) with or without indices will denote different positive constants which only depends on \( \eta \), whose value may vary from one place to another.

2 Notations, Subdifferential operators

2.1 Notations

Throughout this paper, let \((\Omega, \mathcal{F}, \mathbb{F}, P)\) be a complete probability space with filtration \( \mathbb{F} := \{ \mathcal{F}_t \}_{t \geq 0} \) satisfying the usual conditions (i.e., it is increasing and right continuous, \( \mathcal{F}_0 \) contains all \( P \)-null sets) taking along a standard \( d \)-Brownian motion \( B(t) \). For \( x, y \in \mathbb{R}^m \), we use \(|x|\) to denote the Euclidean norm of \( x \), and use \( \langle x, y \rangle \) to denote the Euclidean inner product. For \( M \in \mathbb{R}^{m \times d} \), \(|M|\) represents \( \sqrt{\text{Tr}(MM^*)} \). Denote \( \mathcal{B}(\mathbb{R}^d) \) by the Borel \( \sigma \)-algebra on \( \mathbb{R}^d \). Let \( \mathcal{P}(\mathbb{R}^m) \) be the space of all probability measures, and \( \mathcal{P}_p(\mathbb{R}^m) \) denotes the space of all probability measures defined on \( \mathcal{B}(\mathbb{R}^m) \) with finite \( p \)th moment:

\[
W_p(\mu) := \left( \int_{\mathbb{R}^m} |x|^p \mu(dx) \right)^{\frac{1}{p}} < \infty.
\]
For $\mu, \nu \in \mathcal{P}_p(\mathbb{R}^m)$, we define the Wasserstein distance for $p \geq 1$ as follows:

$$W_p(\mu, \nu) := \inf_{\pi \in \Pi(\mu, \nu)} \left\{ \int_{\mathbb{R}^m \times \mathbb{R}^m} |x - y|^p \pi(dx, dy) \right\}^{\frac{1}{p}},$$

where $\Pi(\mu, \nu)$ denotes the family of all couplings for $\mu, \nu$. Next, we define several spaces for future use.

$C([t_0, T]; \mathbb{R}^m)$ stands for the space of all continuous functions from $[t_0, T]$ to $\mathbb{R}^m$, which is endowed with the uniform norm $|\varphi|_{C([t_0, T])} = \sup_{t_0 \leq t \leq T} |\varphi(t)|$.

$V_0$ denotes the set of all continuous functions $k : [0, T] \to \mathbb{R}^m$ with finite variation and $k(0) = 0$. $\int \downarrow k \downarrow t$ stands for the variation of $k$ on $[s, t]$, and denote $\int \downarrow k \downarrow t = \int \downarrow k \downarrow s$.

$L^2(0, T; \mathbb{R}^m) := \left\{ \varphi \text{ is square integrable stochastic process i.e. } |\varphi|_{M^2} := \left( \mathbb{E} \int_0^T |\varphi(s)|^2 ds \right)^{\frac{1}{2}} < \infty \right\}$.

$B(x, R)$ represents the ball centered at $x$ with the radius $R$ and $\bar{B}(x, R)$ represents the closed ball centered at $x$ and with the radius $R$.

### 2.2 Subdifferential operators

We recall the definition of the subdifferential operators of a proper lower semicontinuous convex functions $\Pi$ (l.s.c. for short) and the Moreau-Yosida approximation of the function $\Pi$.

**Definition 2.1.** Assume $\Pi : \mathbb{R}^m \to (-\infty, +\infty)$ is a proper lower semicontinuous convex functions such that $\Pi(x) \geq \Pi(0) = 0, \forall x \in \mathbb{R}^m$. Denote $D(\Pi) = \{ x \in \mathbb{R}^m : \Pi(x) < \infty \}$. The set

$$\partial \Pi(x) = \{ u \in \mathbb{R}^m : \langle u, v - x \rangle + \Pi(v) \leq \Pi(x), \forall v \in \mathbb{R}^m \}$$

is called the subdifferential operator of $\Pi$. Denote its domain by $D(\partial \Pi) = \{ x \in \mathbb{R}^m : \partial \Pi(x) \neq \emptyset \}$, and denote $Gr(\partial \Pi) = \{ (a, b) \in \mathbb{R}^{2m} : a \in \mathbb{R}^m, b \in \partial \Pi(a) \}$, $\mathcal{A} := \{ (x, k) : x \in C([0, T]; \mathbb{R}^m), k \in V_0, dk(t) \in \partial \Pi(x(t))dt, and \langle x(t) - a, dk(t) - bd(t) \rangle \geq 0 \text{ for any } (a, b) \in Gr(\partial \Pi) \}$.

The corresponding Moreau-Yosida approximation of $\Pi$ is defined as follows:

$$\Pi_\epsilon(x) = \inf \left\{ \frac{1}{2\epsilon} |z - x|^2 + \Pi(z) : z \in \mathbb{R}^d \right\}.$$
Lemma 2.1. Suppose that \( \Pi_\epsilon \) is a convex \( C^1 \)-class function. For any \( x \in \mathbb{R}^m \), denote \( J_\epsilon x = x - \epsilon \nabla \Pi_\epsilon(x) \), where \( \nabla \) is gradient operator. Then we have \( \Pi_\epsilon(x) = \frac{1}{2\epsilon} |x - J_\epsilon x|^2 + \Pi(J_\epsilon x) \). We present some useful properties on the above approximation tools (for more details, see, e.g., [12]).

Definition 2.2. A pair of continuous processes \((x, k)\) is called a strong solution of Eq. (1.2) if

(i) \( \Pi_\epsilon(x) = \frac{1}{2} |\nabla \Pi_\epsilon(x)|^2 + \Pi(J_\epsilon x) \).

(ii) \( \nabla \Pi_\epsilon(x) \in \partial \Pi(J_\epsilon x) \).

(iii) \( |\nabla \Pi_\epsilon(x) - \nabla \Pi_\epsilon(y)| \leq \frac{1}{\epsilon} |x - y| \).

(iv) \( \langle \nabla \Pi_\epsilon(x) - \nabla \Pi_\epsilon(y), x - y \rangle \geq 0 \).

(v) \( \Pi_\epsilon(0) \leq \Pi_\epsilon(x) \), \( J_\epsilon(0) = \nabla \Pi_\epsilon(0) = 0 \).

(vi) \( \frac{1}{2} |\nabla \Pi_\epsilon(x)|^2 \leq \Pi_\epsilon(x) \leq \langle \nabla \Pi_\epsilon(x), x \rangle, \forall x \in \mathbb{R}^d \).

Lemma 2.1. Suppose that \( \text{Int}(D(\partial \Pi)) \neq \emptyset \), where \( \text{Int}(D(\partial \Pi)) \) denotes the interior of \( D(\partial \Pi) \). Then for any \( a \in \text{Int}(D(\partial \Pi)) \), there exists constants \( \lambda_1 > 0, \lambda_2 \geq 0, \lambda_3 \geq 0 \) such that for any \((x, k) \in \mathcal{A} \) and \( 0 \leq s \leq t \leq T \),

\[
\int_s^t \langle x(r) - a, dk(r) \rangle \geq \lambda_1 \int_s^t 1 - \lambda_2 \int_s^t |x(r) - a| dr - \lambda_3 (t - s).
\]

2.3 Strong and weak solutions of Eq. (1.2)

Definition 2.2. A pair of continuous processes \((x, k)\) is called a strong solution of (1.2) if

(i) \( P(x(t_0) = x_0) = 1 \);

(ii) \( x(t) \) is \( \mathcal{F}_t \)-adapted.

(iii) \( (x, k) \in \mathcal{A}, a.s., P \).

(iv) \( \int_{t_0}^T (|f(x(t), \mu_t)| + |g(x(t), \mu_t)|^2) dt < +\infty, a.s., P \).

(v) For \( y \in C([t_0, +\infty); \mathbb{R}^m) \) and \( t_0 \leq s \leq t < \infty \), it holds that

\[
\int_s^t \langle y(r) - x(r), dk(r) \rangle + \int_s^t \Pi(x(r)) dr \leq \int_s^t \Pi(y(r)) dr.
\]

(vi) \((x, k)\) satisfies the following equation

\[
x(t) + \int_{t_0}^t H(x(s), \mu_s) dk(s) = x_0 + \int_{t_0}^t f(x(s), \mu_s) ds + \int_{t_0}^t g(x(s), \mu_s) dB(s), t \in [t_0, T].
\]
Definition 2.3. We say that Eq. (1.2) admits a weak solution if there exists a filtered probability space \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \in [0,T]}, \tilde{P})\) taking along a standard Brownian motion \(\tilde{B}(t)\) as well as a pair of continuous processes \((\tilde{x}, \tilde{k})\) defined on \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \in [0,T]}, \tilde{P})\) such that

(i) \(\tilde{P}(\tilde{x}(t_0) = x_0) = 1\);

(ii) \(\tilde{x}(t)\) is \(\mathcal{F}_t\)-adapted.

(iii) \((\tilde{x}, \tilde{k}) \in \mathcal{A}, \text{a.s., } \tilde{P}\).

(iv) \(\int_0^T (|f(\tilde{x}, \mu_t)| + |g(\tilde{x}, \mu_t)|^2)dt < +\infty, \text{a.s.}, \tilde{P}\).

(v) For \(y \in C(t_0, +\infty; \mathbb{R}^m)\) and \(t_0 \leq s \leq t < \infty\), it holds that

\[
\int_s^t (y(r) - \tilde{x}(r), d\tilde{k}(r)) + \int_s^t \Pi(\tilde{x}(r))dr \leq \int_s^t \Pi(\tilde{y}(r))dr,
\]

(vi) \((\tilde{x}, \tilde{k})\) satisfies the following equation

\[
\dot{\tilde{x}}(t) + \int_{t_0}^t H(\tilde{x}, \mu_s)\tilde{d}k(s) = \tilde{x}_0 + \int_{t_0}^t f(\tilde{x}, \mu_s)ds + \int_{t_0}^t g(\tilde{x}, \mu_s)dB(s), t \in [t_0, T].
\]

Remark 2.1. We make the following convention: For a pair of process \((x, k)\) satisfying \(dk(t) \in \partial \Pi(x(t))dt\), we denote that \(U(t)\) is a process such that \(dk(t) = U(t)dt\).

3 Main Results

Before giving our main results for Eq. (1.2). For the sake of simplicity, we assume \(f(0, \delta_0) = g(0, \delta_0) = 0\), where \(\delta_x\) stands for the Dirac measure at \(x\). Now, we make the following assumptions:

(A1) The coefficients \(f, g\) satisfy that for some positive constant \(L > 0\) and all \(x, y \in \mathbb{R}^d, \mu, \nu \in \mathcal{P}_2(\mathbb{R}^m)\),

\[
|f(x, \mu) - f(y, \nu)| + |g(x, \mu) - g(y, \nu)| \leq L(|x - y| + W_2(\mu, \nu)),
\]

Furthermore, from the above assumptions, one has

\[
|f(x, \mu)| \leq L(|x| + W_2(\mu, \delta_0)), |g(x, \mu)| \leq L(|x| + W_2(\mu, \delta_0)).
\]

(A2) \(H(\cdot, \cdot) = (a_{ij}(\cdot, \cdot))_{m \times m} : \mathbb{R}^m \times \mathcal{P}_2(\mathbb{R}^m) \rightarrow \mathbb{R}^{m \times m}\) is a continuous mapping and for any \((x, \mu) \in \mathbb{R}^m \times \mathcal{P}_2(\mathbb{R}^m), H(x, \mu)\) is a invertible symmetric matrix. Moreover, there exist two positive \(a_H, b_H\) such that

(i) \(a_H|u|^2 \leq \langle H(x, \mu)u, u \rangle \leq b_H|u|^2, \forall u \in \mathbb{R}^m\).
(ii) For all \( x, y \in \mathbb{R}^m, \mu, \nu \in \mathcal{P}_2(\mathbb{R}^m) \),
\[
|H(x, \mu) - H(y, \nu)| + |H^{-1}(x, \mu) - H^{-1}(y, \nu)| \leq L(|x - y| + W_2(\mu, \nu)),
\]
where \( |H(x, \mu)| := \left( \sum_{i,j=1}^{m} |a_{ij}(x, \mu)|^2 \right)^{\frac{1}{2}} \) and \( H^{-1}(x, \mu) \) the inverse of the matrix \( H(x, \mu) \).

**Lemma 3.1.** Let \( I \) be an arbitrary set of indexes. For each \( i \in I \), suppose that \( (\Omega^i, \mathcal{F}^i, P^i, \{\mathbb{F}^i_t\}_{t \geq 0}, B^i, x^i, k^i) \) is a weak solution of the equation
\[
\left\{
\begin{aligned}
&dx^i(t) + H_i \partial \Pi(x^i(t))dt \\
&\exists f^i(x^i(t), \mu^i_t)dt + g^i(x^i(t), \mu^i_t)dB^i(t), t \in [t_0, T], \\
x(t_0) = x_0^i \in \mathbb{R}^m.
\end{aligned}
\right.
\]
where \( f^i, g^i, H_i \) satisfy (A1)–(A2) and \( H_i \) is independent on \( x, \mu \). If \( \sup \mathbb{E}[\sup_{0 \leq t \leq T} |x^i(t)|^2] < \infty \), then \( (x^i, k^i)_{i \in I} \) is tight in \( C([t_0, T]; \mathbb{R}^m) \times C([t_0, T]; \mathbb{R}^m) \).

**Proof.** Set \( \dot{x}^i(t) = H_i^{\frac{1}{2}}x^i(t) \). From (3.1), we know that \( \dot{x}^i(t) \) satisfies the following equation:
\[
\left\{
\begin{aligned}
&d\dot{x}^i(t) + H_i^{\frac{1}{2}}\partial \Pi(x^i(t))dt \\
&\exists H_i^{\frac{1}{2}}f^i(x^i(t), \mu^i_t)dt + H_i^{\frac{1}{2}}g^i(x^i(t), \mu^i_t)dB^i(t), t \in [t_0, T], \\
x(t_0) = x_0^i.
\end{aligned}
\right.
\]
When we apply Itô’s formula to \( |\dot{x}^i(t)|^2 \) for the above equation, \( H_i^{\frac{1}{2}} \) will disappear. Then, we can use the maximal monotonity property of the subdifferential operator \( \partial \Pi \). Using the similar method in [23, Theorem 3.2], we can get the desired result.

**Theorem 3.2.** Let \( f, g \) satisfy the following linear growth conditions:
\[
|f(x, \mu)| \leq L(|x| + W_2(\mu, \delta_0)), \quad |g(x, \mu)| \leq L(|x| + W_2(\mu, \delta_0)),
\]
and assume (A2) holds. Then Eq. (3.2) has a weak solution.

**Proof.** For fixed \( n \), take \( t_n = 2^{-n} \lfloor 2^nt \rfloor \), where \( \lfloor z \rfloor \) denotes the integer part of a real number \( z \). In addition, denote \( \mu_{t_0} = \delta_{x_0} \). Consider the following approximation equation for \( n \geq 2 \):
\[
\left\{
\begin{aligned}
&dx^n(t) + H(x^{n-1}(t_n), \mu_{t_n}^{n-1})\partial \Pi(x^n(t))dt \\
&\exists f(x^{n-1}(t_n), \mu_{t_n}^{n-1})dt + g(x^{n-1}(t_n), \mu_{t_n}^{n-1})dB(t), t \in [t_0, T], \\
x(t_0) = x_0.
\end{aligned}
\right.
\]
By solving a deterministic Skorohod problem (see [3] for more details), we can obtain the solution of this equation step by step. Thus, there exists \( (x^n, k^n) \) such that
\[
\left\{
\begin{aligned}
x^n(t) &= x^n(t) + \int_{t_0}^{t} H(x^{n-1}(s_n), \mu_{s_n}^{n-1})dk^n(s) \\
&= \int_{t_0}^{t} f(x^{n-1}(s_n), \mu_{s_n}^{n-1})ds + \int_{t_0}^{t} g(x^{n-1}(s_n), \mu_{s_n}^{n-1})dB(s), t \in [t_0, T], \\
x(t_0) &= x_0.
\end{aligned}
\right.
\]
For given $a \in Int(D(\partial \Pi))$, set $\tilde{x}^n(t) = x^n(t) - a, \tilde{\pi}^n(t) = H^{-\frac{1}{2}}(x^{n-1}(t_n), \mu_{s_n}^{n-1})\tilde{x}^n(t)$. Then, $\tilde{x}^n$ satisfies the following equation:

$$\tilde{x}^n(t) = \int_{t_0}^t H^{-\frac{1}{2}}(x^{n-1}(s_n), \mu_{s_n}^{n-1})d\tilde{x}^n(s) + |\tilde{x}^n(t_0)|^2$$

$$= \int_{t_0}^t H^{-\frac{1}{2}}(x^{n-1}(s_n), \mu_{s_n}^{n-1})f(x^{n-1}(s_n), \mu_{s_n}^{n-1})ds - \int_{t_0}^t H^{\frac{1}{2}}(x^{n-1}(s_n), \mu_{s_n}^{n-1})dk^n(s)$$

$$+ \int_{t_0}^t H^{-\frac{1}{2}}(x^{n-1}(s_n), \mu_{s_n}^{n-1})g(x^{n-1}(s_n), \mu_{s_n}^{n-1})dB(s) + |\tilde{x}^n(t_0)|^2. \quad (3.5)$$

Using Itô’s formula, we have

$$|\tilde{x}^n(t)|^2 = |\tilde{x}^n(t_0)|^2 + 2 \int_{t_0}^t \langle \tilde{x}^n(s), H^{-\frac{1}{2}}(x^{n-1}(s_n), \mu_{s_n}^{n-1})f(x^{n-1}(s_n), \mu_{s_n}^{n-1}) \rangle ds$$

$$+ \int_{t_0}^t |H^{-\frac{1}{2}}(x^{n-1}(s_n), \mu_{s_n}^{n-1})g(x^{n-1}(s_n), \mu_{s_n}^{n-1})|^2 ds$$

$$+ 2 \int_{t_0}^t \langle \tilde{x}^n(s), H^{-\frac{1}{2}}(x^{n-1}(s_n), \mu_{s_n}^{n-1})g(x^{n-1}(s_n), \mu_{s_n}^{n-1}) \rangle dB(s)$$

$$- 2 \int_{t_0}^t \langle x^n(s) - a, dk^n(s) \rangle$$

$$=: |\tilde{x}^n(t_0)|^2 + \sum_{i=1}^4 I_i. \quad (3.6)$$

Firstly, we estimate $I_1$.

$$I_1 = \int_{t_0}^t \langle \tilde{x}^n(s), H^{-\frac{1}{2}}(x^{n-1}(s_n), \mu_{s_n}^{n-1})f(x^{n-1}(s_n), \mu_{s_n}^{n-1}) \rangle ds$$

$$\leq 2 \int_{t_0}^t |\tilde{x}^n(s)|^2 ds + a_H^{-1} \int_{t_0}^t |f(x^{n-1}(s_n), \mu_{s_n}^{n-1})|^2 ds$$

$$\leq (1 + 8a_H^{-1}b_H L^2) \int_{t_0}^t \sup_{1 \leq i \leq n, t_0 \leq r \leq s} |\tilde{x}^i(s)|^2 ds + 8L^2 a^2 a_H^{-1} T.$$
Finally, we calculate $I_4$. By Gronwall’s inequality, we have

$$
\leq 32E\left(\int_{t_0}^{t} |\bar{x}^n(s)|^2 |H^{-\frac{1}{2}}(x^{n-1}(s), \mu_{s_n}^{n-1})g(x^{n-1}(s), \mu_{s_n}^{n-1})|^2 ds\right)^{\frac{1}{2}} \\
\leq 32E\left[ \sup_{t_0 \leq s \leq t} |\bar{x}^n(s)| \left( \int_{t_0}^{t} |H^{-\frac{1}{2}}(x^{n-1}(s), \mu_{s_n}^{n-1})g(x^{n-1}(s), \mu_{s_n}^{n-1})|^2 ds\right)^{\frac{1}{2}} \right] \\
\leq \frac{1}{l}E[ \sup_{t_0 \leq s \leq t} |\bar{x}^n(s)|^2] + 32lE \int_{t_0}^{t} |H^{-\frac{1}{2}}(x^{n-1}(s), \mu_{s_n}^{n-1})g(x^{n-1}(s), \mu_{s_n}^{n-1})|^2 ds \\
\leq \frac{32}{l}E[ \sup_{1 \leq i \leq n \ t_0 \leq s \leq t} |\bar{x}^{i}(s)|^2 ] + 256lL^2a_H^1b_H E \int_{t_0}^{t} \sup_{1 \leq i \leq n \ t_0 \leq r \leq s} |\bar{x}^{i}(r)|^2 ds + 8L^2a^2a_H^{-1}T.
$$

Finally, we calculate $I_4$. By Lemma 2.1, we have

$$
-2 \int_{0}^{t} (x^n(s) - a, dk^n(s)) \leq \lambda_2 \int_{s}^{t} |x^n(s) - a| ds + \lambda_3 T \\
\leq \lambda_2 b_H T \int_{s}^{t} \sup_{1 \leq i \leq n \ t_0 \leq r \leq s} |\bar{x}^{i}(r)|^2 ds + \lambda_3 T.
$$

Take $l = 64$. Combining the above calculations, we have

$$
E[ \sup_{1 \leq i \leq n \ t_0 \leq s \leq t} |\bar{x}^{i}(s)|^2 ] \leq (2^{18}a_H^1b_H L^2 + 2^5a_H^1b_H L^2 + 2\lambda_2 b_H T + 2)E \int_{t_0}^{t} \sup_{1 \leq i \leq n \ t_0 \leq r \leq s} |\bar{x}^{i}(r)|^2 ds \\
+ 8L^2a^2a_H^{-1}T + 2\lambda_3 T + |\bar{x}^{n}(t_0)|^2.
$$

By Gronwall’s inequality, we have

$$
\sup_{n \geq 0} E[ \sup_{t_0 \leq s \leq t} |\bar{x}^{n}(s)|^2 ] \leq C(a_H, b_H, T, \lambda_2, \lambda_3).
$$

Then,

$$
\sup_{n \geq 0} E[ \sup_{t_0 \leq s \leq t} |x^{n}(s)|^2 ] \leq C(a_H, b_H, T, \lambda_2, \lambda_3).
$$

Furthermore, from Lemma 2.1 and (3.6), we derive

$$
\sup_{n \geq 0} E[ \sup_{t_0 \leq s \leq t} |k^{n}(s)|^2 ] \leq C(a_H, b_H, T, \lambda_2, \lambda_3).
$$

Using the above estimates, by Lemma 3.1, $(x^n, k^n)_{n \in \mathbb{N}}$ is tight in $C([t_0, T]; \mathbb{R}^{2m})$. By the Prohorov theorem there exists a subsequence which we still denote by $(x^n, k^n, \downarrow k^n, \downarrow B)$ such that, as $n \to \infty$,

$$(x^n, k^n, \downarrow k^n, \downarrow B) \Rightarrow (x, k, \downarrow k, \downarrow B).$$

By Skorohod representation theorem, we can choose a probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$ and some quadruples $(\tilde{x}^n, \tilde{k}^n, \tilde{V}^n, \tilde{B}^n)$ and $(\tilde{x}, \tilde{k}, \tilde{V}, \tilde{B})$ defined on $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$, having the same laws as $(x^n, k^n, \downarrow k^n, \downarrow B)$ and $(x, k, \downarrow k, \downarrow B)$, respectively, such that , in $C([t_0, T]; \mathbb{R}^{2m+1+d})$, as $n \to \infty$,

$$(\tilde{x}^n, k^n, \tilde{V}^n, \tilde{B}^n) \Rightarrow (\tilde{x}, \tilde{k}, \tilde{V}, \tilde{B}), a.e., n \to \infty.$$
Since \((x^n, k^n, \downarrow k^n \downarrow, B) \Rightarrow (\tilde{x}, \tilde{k}, \tilde{V}, \tilde{B})\), then by Proposition 16 in [13], we have that, for all \(t_0 \leq s < t\),
\[
\tilde{x}(t_0) = x_0, \quad \tilde{k}(t_0) = 0, \quad |\tilde{k}|_\nu(t) - |\tilde{k}|_\nu(s) \leq \tilde{V}(t) - \tilde{V}(s), 0 = \tilde{V}(t_0) \leq \tilde{V}(s) \leq \tilde{V}(t), \quad \tilde{P} - a.e. \tag{3.7}
\]
Furthermore, since \(\forall t_0 \leq s < t\),
\[
\int_s^t \Pi(x^n(r))dr \leq \int_s^t \Pi(y(r))dr - \int_s^t (y(r) - x^n(r), dk^n(r)),
\]
and [13 Proposition 16], one can see that
\[
\int_s^t \Pi(\tilde{x}(r))dr \leq \int_s^t \Pi(y(r))dr - \int_s^t (y(r) - \tilde{x}(r), d\tilde{k}(r)). \tag{3.8}
\]
Combining (3.7) and (3.8), it infer
\[
d\tilde{k}(r) \in \partial \Pi(\tilde{x}(r))(dr).
\]
Using the Lebesgue theorem, we derive
\[
\tilde{M}^n(\cdot) = x_0 + \int_{t_0}^t f(\tilde{x}^{n-1}(s_n), \mu_{s_n}^{n-1})ds + \int_{t_0}^t g(\tilde{x}^{n-1}(s_n), \mu_{s_n}^{n-1})dB(s)
\rightarrow \tilde{M}(\cdot) = x_0 + \int_{t_0}^t f(\tilde{x}(s), \mu_s)ds + \int_{t_0}^t g(\tilde{x}(s), \mu_s)d\tilde{B}(s) \text{ in } L^2(t_0, T; \mathbb{R}^m).
\]
By Proposition 17 in [13], it follows that
\[
\mathcal{L}(\tilde{x}^n, \tilde{k}^n, \tilde{B}^n, \tilde{M}^n) = \mathcal{L}(x^n, k^n, B^n, M^n) \text{ in } C([t_0, T]; \mathbb{R}^{3n+d}),
\]
where \(\mathcal{L}(\cdot)\) is denoted by the probability law of the random variables. Since
\[
x^n(t) + \int_{t_0}^t H(x^{n-1}(s_n), \mu_{s_n}^{n-1})dk^n(s) - M^n(t) = 0,
\]
we have
\[
x^n(t) + \int_{t_0}^t H(\tilde{x}^{n-1}(s_n), \mu_{s_n}^{n-1})d\tilde{k}(s) - \tilde{M}^n(t) = 0.
\]
Letting \(n \to \infty\), one has
\[
\tilde{x}(t) + \int_{t_0}^t H(\tilde{x}(s), \mu_s)d\tilde{k}(s) - \tilde{M}(t) = 0.
\]
This means
\[
\tilde{x}(t) + \int_{t_0}^t H(\tilde{x}(s), \mu_s)d\tilde{k}(s) = x_0 + \int_{t_0}^t f(\tilde{x}(s), \mu_s)ds + \int_{t_0}^t g(\tilde{x}(s), \mu_s)d\tilde{B}(s).
\]
Thus, \(((\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P}), \mathcal{F}_t^\tilde{B, \tilde{x}}, \tilde{x}(t), \tilde{k}(t), \tilde{B}(t))_{t \geq t_0}\) is a weak solution. The proof is complete.
**Theorem 3.3.** Let (A1) – (A2) hold and \( H \) is independent of \( \mu \). The Eq. (1.2) has a unique strong solution.

Based on Theorem (3.2), it suffices to prove the pathwise uniqueness. Assume that \( (x, k) \) and \( (x', k') \) are two solutions of (1.2). Let \( Q(t) := H^{-1}(x(t)) + H^{-1}(x'(t)) \). Then, we have

\[
dQ^\frac{1}{2}(t) = dN(t) + \sum_{i=1}^{d} \beta_i(t)dB_i(t),
\]

where \( N \) is an \( \mathbb{R}^{m \times m} \)-valued bounded variation continuous stochastic process with \( N(t_0) = 0 \), and \( \beta_i, i = 1, 2, \cdots, d \) is an \( \mathbb{R}^{m \times m} \)-valued stochastic process such that \( \mathbb{E} \int_0^T |\beta_i(t)|^2 dt < \infty \).

Set \( \hat{x}(t) = Q^\frac{1}{2}(t)(x(t) - x'(t)) \). Then, \( \hat{x}(t) \) satisfies the following equation:

\[
d\hat{x}(t) = [dQ^\frac{1}{2}(t)](x(t) - x'(t)) + Q^\frac{1}{2}(t)d(x(t) - x'(t)) + \sum_{i=1}^{d} \beta_i(t)(g(x(t), \mu_i) - g(x'(t), \mu'_i))e_i dt
\]

\[
=: dF(t) + G(t)dB(t),
\]

where

\[
dF(t) = [dN(t)]Q^{-\frac{1}{2}}(t)\hat{x}(t) + Q^\frac{1}{2}(t)(H(x(t))dk(t) - H(x'(t))dk'(t))
\]

\[
+ Q^\frac{1}{2}[f(x(t), \mu_i)dk(t) - f(x'(t), \mu'_i)]dt + \sum_{i=1}^{d} \beta_i(t)(g(x(t), \mu_i) - g(x'(t), \mu'_i))e_i dt,
\]

and

\[
G(t) = \Gamma(t) + Q^\frac{1}{2}(g(x(t), \mu_i) - g(x'(t), \mu'_i))e_i,
\]

and \( \Gamma(t) \) is an \( \mathbb{R}^{m \times d} \) matrix with the columns \( \beta_1(t)(x(t) - x'(t)), \cdots, \beta_d(t)(x(t) - x'(t)) \). Using the properties of matrices \( H, H^{-1} \) and the following relation

\[
Q(t)(H(x'(t))dk'(t) - H(x(t))dk(t)) = (H^{-1}(x(t)) - H^{-1}(x'(t)))[H(x'(t))dk'(t) + H(x(t))dk(t)] + 2(dk'(t) - dk(t)),
\]

we have

\[
\langle \hat{x}(t), Q^\frac{1}{2}(t)(H(x(t))dk(t) - H(x'(t))dk'(t)) \rangle
\]

\[
= \langle x(t) - x'(t), (H^{-1}(x(t)) - H^{-1}(x'(t)))(H(x(t))dk(t) + H(x'(t))dk'(t)) \rangle
\]

\[
+ 2\langle x(t) - x'(t), dk(t) - dk'(t) \rangle
\]

\[
\leq C(L, b_H)|x(t) - x'(t)|^2(\mu \uparrow k \downarrow \mu + d \uparrow k' \downarrow \mu).
\]

Thus,

\[
\langle \hat{x}(t), dF(t) \rangle + \frac{1}{2}|G(t)|^2 dt \leq |\hat{x}(t)|^2 dV(t),
\]

11
where
\[ dV(t) = C(L, b_H) \frac{d}{dt} N \frac{d}{dt} + d \frac{d}{dt} k \frac{d}{dt} + d \frac{d}{dt} k' \frac{d}{dt} + C(L, b_H) \sum_{i=1}^{d} |\beta_i(\cdot)|^2 dt. \]

By [13, Proposition 14], it yields
\[ \mathbb{E} \frac{e^{-2V(t)}|\hat{x}(t)|^2}{1 + 2e^{-2V(t)}|\hat{x}(t)|} \leq \mathbb{E} \frac{e^{-2V(0)}|\hat{x}(0)|^2}{1 + 2e^{-2V(0)}|\hat{x}(0)|} = 0. \]

Then,
\[ Q^\frac{1}{2}(t)(x(t) - x'(t)) = 0, \text{ for all, } t \geq t_0. \]

Consequently,
\[ x(t) = x'(t), \text{ for all, } t \geq t_0. \]

Then, the uniqueness holds. The proof is complete. \( \square \)

The following examples illustrate the theories about existence and uniqueness.

Example 3.4. Assume that \( \mathcal{O} \) is a closed convex subset of \( \mathbb{R} \), and that \( I_\mathcal{O} \) is the indicator function of \( \mathcal{O} \), i.e.,
\[ I_\mathcal{O}(x) = \begin{cases} +\infty, & x \notin \mathcal{O}, \\ 1, & x \in \mathcal{O}. \end{cases} \]

Then, the subdifferential operator of \( I_\mathcal{O} \) given by
\[ \partial I_\mathcal{O}(x) = \begin{cases} \emptyset, & x \notin \mathcal{O}, \\ \{0\}, & x \in \text{Int}(\mathcal{O}), \\ \Lambda_x, & x \in \partial \mathcal{O}, \end{cases} \]

where \( \Lambda_x \) is the exterior normal cone at \( x \) and \( \text{Int}(\mathcal{O}) \) is the interior of \( \mathcal{O} \). For any \( (x, \mu) \in \mathbb{R}^2 \times \mathcal{P}_2(\mathbb{R}^2) \), set
\[ H(x, \mu) = \begin{pmatrix} \sin x + 5 + \cos(W_2(\mu, \delta_0)) \\ 0 \\ e^{\cos x} + 4 + (W_2(\mu, \delta_0) \wedge 1) \end{pmatrix}, \]
\[ f(x, \mu) = \sqrt{|x|^2 + 5} + W_2(\mu, \delta_0), g(x, \mu) = e^{1|\cdot|} + \sin(W_2(\mu, \delta_0)). \]

Obviously, \( H(\cdot, \cdot), f(\cdot, \cdot), g(\cdot, \cdot) \) satisfy (A1) and (A2). Then, the following equation has a unique weak solution:
\[ \begin{cases} \frac{dx(t) + H(x(t), \mu_t)\partial \Pi(x(t))dt = f(x(t), \mu_t)dt + g(x(t), \mu_t)dB(t), t \in [t_0, T],} \\ x(t_0) = x_0. \end{cases} \]
Moreover, if we take
\[ H(x) = \begin{pmatrix} \sin x + 5 & 0 \\ 0 & e^{x^2} + 4 + \cos x \end{pmatrix}, \]
then following equation has a unique strong solution:
\[
\begin{cases}
  \mathrm{d}x(t) + H(x(t), \mu_t)\partial \Pi(x(t)) \mathrm{d}t \ni f(x(t), \mu_t) \mathrm{d}t + g(x(t), \mu_t) \mathrm{d}B(t), t \in [t_0, T], \\
x(t_0) = x_0.
\end{cases}
\]

**Example 3.5.** In this example, we shall show that some MVMSDEs with time-dependent constraints can be transferred to some suitable MVMSDEs wOS with oblique subgradients. More precisely, consider the following MVMSDEs wOS with time-dependent constraints:
\[
\begin{cases}
  \mathrm{d}x(t) + \partial I_{H(t)\Xi}(x(t)) \mathrm{d}t \ni f(x(t), \mu_t \circ H(t)) \mathrm{d}t + g(x(t), \mu_t \circ H(t)) \mathrm{d}B(t), t \in [t_0, T], \\
x(t) = x_0,
\end{cases}
\]
where \( \mu_t \) is the distribution of \( x(t), \Xi \in \mathbb{R}^d \) is a convex set and a deterministic time-dependent matrix \( H : [0, T] \to \mathbb{R}^{m \times m} \) satisfying (A2). Assume the coefficients \( f, g \) satisfy the condition (A1). Furthermore, for any \( \Theta \in \mathcal{B}(\mathbb{R}^m), H(t)(\Theta) := \{H(t)a | a \in \Theta\}, \mu \circ H(t)(\Theta) := \mu(H(t)(\Theta)) \). Since \( \partial I_{H(t)\Xi}(y) = \partial I_{\Xi}(H^{-1}(t)y), \forall y \in D(\partial I_{H(t)\Xi}) \), we can easily verify that \( x \) is a solution for Eq.(3.9) if only if \( \bar{x} = H^{-1}x \) is a solution for the following GSDEs with oblique subgradients:
\[
\begin{cases}
  \mathrm{d}\bar{x}(t) + (H^{-1}(t))^2 \partial I_{\Xi}(\bar{x}(t)) \mathrm{d}t \ni \bar{f}(\bar{x}(t), \bar{\mu}_t) \mathrm{d}t + \bar{g}(\bar{x}(t), \bar{\mu}_t) \mathrm{d}B(t), t \in [t_0, T], \\
  \bar{x}(t_0) = H^{-1}(t_0)x_0.
\end{cases}
\]

All assumptions in ((A1)) and (A2) are satisfied for coefficients of equation (3.10). Consequently, using Theorem 4.13 Eq.(3.10) admits a unique solution. Thus, Eq.(3.9) has a solution.

### 4 Stochastic principle of optimality

In this section, we will investigate optimal control for Eq.(4.1) below. The aim is to show that the value function satisfies the dynamic programming principle (DPP, for short). Let \( \mathbf{U} \) be a separable metric space. For a control process \( u(\cdot, \cdot) : [0, T] \times \Omega \to \mathbf{U} \), we consider the following stochastic controlled system:
\[
\begin{cases}
  \mathrm{d}x(t) + H(t)\partial \Pi(x(t)) \mathrm{d}t \ni f(x(t), \mu_t, u(t)) \mathrm{d}t + g(x(t), \mu_t, u(t)) \mathrm{d}B(t), t \in [s, T], \\
x(s) = x_0,
\end{cases}
\]
where
\[ f(\cdot, \cdot, \cdot) : \mathbb{R}^m \times \mathcal{P}(\mathbb{R}^m) \times \mathbb{U} \rightarrow \mathbb{R}^m, g(\cdot, \cdot, \cdot) : \mathbb{R}^m \times \mathcal{P}(\mathbb{R}^m) \times \mathbb{U} \rightarrow \mathbb{R}^{m \times d}. \]

For the sake of simplicity, we assume \( f(0, \delta_0, u) = g(0, \delta_0, u) = 0 \) and make the following assumptions:

(A3) The coefficients \( f \) and \( g \) satisfy that for some positive constant \( L > 0 \) and all \( x, y \in \mathbb{R}^d, \mu, \nu \in \mathcal{P}(\mathbb{R}^m), u \in \mathbb{U} \),
\[
|f(x, \mu, u) - f(y, \nu, u)| + |g(x, \mu, u) - g(y, \nu, u)| \leq L(|x - y| + W^2(\mu, \nu)).
\]

From the above assumptions, one can see that
\[
|f(x, \mu, u)| \leq L(|x| + W^2(\mu, \delta_0)), \quad |g(x, \mu, u)| \leq L(|x| + W^2(\mu, \delta_0)).
\]

Define the cost functional as follows:
\[
J(s, x_0; u) = \mathbb{E}\left[ \int_s^T b(x^{s,x_0,u}(t), u(t)) \, dt + \alpha(x^{s,x_0,u}(T)) \right], \tag{4.2}
\]
where \( x^{s,x_0,u} \) is the solution of Eq.(4.1) associated with \( s, x_0, u \) and \( b : \mathbb{R}^m \times \mathbb{U} \rightarrow \mathbb{R}, \alpha : \mathbb{R}^m \rightarrow \mathbb{R} \) satisfy the following conditions:

(A4) For some positive constant \( L > 0 \) and all \( x, y \in \mathbb{R}^m, \mu, \nu \in \mathcal{P}(\mathbb{R}^m), u \in \mathbb{U} \),
\[
|b(x, u) - b(y, u)| + |\alpha(x) - \alpha(y)| \leq L|x - y|,
\]

For the sake of simplicity, we assume \( b(0, u) = \alpha(0) = 0 \). This implies
\[
|b(x, u)| \leq L|x|, \quad |\alpha(x)| \leq L|x|.
\]

We give the associated valued function as the infimum among all \( u \in \mathcal{U}[s,T] \):
\[
V(s, x_0) := \inf_{u \in \mathcal{U}[s,T]} J(s, x_0; u), (s, x_0) \in [0, T) \times \mathbb{R}^m, \tag{4.3}
\]
where \( \mathcal{U}[s,T] \) denotes the set of all the processes \( u(\cdot, \cdot) : \Omega \times [s, T] \rightarrow \mathbb{U} \) satisfying
\[
\mathbb{E}\left[ \int_s^T |b(x^{s,x_0,u}(r), u(r))| \, dr + |\alpha(x^{s,x_0,u}(T))| \right] < \infty.
\]

**Definition 4.1.** If for any \( (s, x_0) \in [0, T) \times \mathbb{R}^m \), it holds that
\[
V(s, x_0) = \inf_{u \in \mathcal{U}[s,T]} \mathbb{E}\left[ \int_s^\tau b(x^{s,x_0,u}(t), u(t)) \, dt + V(\tau, x^{s,x_0,u}(\tau)) \right], \tag{4.4}
\]
for any \( \tau \in [s,T] \). Then, we say that the value function \( V \) satisfies the DDP.
In order to prove that the value function for Eq. (4.1) fulfill DDP, we consider the following penalized equation:

\[
\begin{cases}
    x^{\epsilon, s, x_0, u}(t) + \int_s^t H(r) \nabla \Pi(x^{\epsilon, s, x_0, u}(r)) dr = x_0 + \int_s^t f(x^{\epsilon, s, x_0, u}(r), \mu_t, u(r)) dr \\
    + \int_s^t g(x^{\epsilon, s, x_0, u}(r), \mu_t, u(r)) dB(r), t \in [s, T],
\end{cases}
\]  

(4.5)

and for any \((t, x_0) \in [0, T) \times \mathbb{R}^m\) the penalized valued function is defined as follows:

\[
V_\epsilon(s, x_0) = \inf_{u \in \mathcal{U}[s, T]} \mathbb{E} \left[ \int_s^T b(x^{\epsilon, s, x_0, u}(t), u(t)) dt + \alpha(x^{\epsilon, s, x_0, u}(T)) \right],
\]  

(4.6)

and set

\[
J_\epsilon(s, x_0; u) = \mathbb{E} \left[ \int_s^T b(x^{\epsilon, s, x_0, u}(t), u(t)) dt + \alpha(x^{\epsilon, s, x_0, u}(T)) \right].
\]  

(4.7)

**Lemma 4.1.** Assume (A2) – (A4). Let \((x^{s, x_0, u}, k^{s, x_0, u})\) and \((x^{s', x_0', u}, k^{s', x_0', u})\) be the solutions of Eq. (4.1) corresponding to the initial date \((s, x_0)\) and \((s', x'_0)\) respectively. Then, we have the following estimates:

\[
\mathbb{E}[ \sup_{s \leq r \leq T} |x^{s, x_0, u}(r)|^2 ] \leq C(a_H, b_H, x_0, L, T),
\]  

(4.8)

and

\[
\mathbb{E}[ \sup_{s \leq s' \leq r \leq T} |x^{s, x_0, u}(r) - x^{s', x_0', u}(r)|^2 ] \leq C(a_H, b_H, x_0', L, T) \mathbb{E}[|x_0 - x_0'|^2] + C(a_H, b_H, x_0', L, T) |s - s'|.
\]  

(4.9)

**Proof.** Assume that \(U^{s, x_0, u}, U^{s', x_0', u}\) are two processes such that

\[
dk^{s, x_0, u}(t) = U^{s, x_0, u}(t) dt, dk^{s', x_0', u}(t) = U^{s', x_0', u}(t) dt.
\]

Using Itô’s formula, (4.8) can be easily derived. We only prove (4.9). Assume that \(s \geq s'\). For any \(t \in [s, T]\), we have

\[
x^{s, x_0, u}(t) + \int_s^t H(r) U^{s, x_0, u}(r) dr = x^{s, x_0, u}(s) + \int_s^t f(x^{s, x_0, u}(r), \mu_t, u(r)) dr \\
+ \int_s^t g(x^{s, x_0, u}(r), \mu_t, u(r)) dB(r), t \in [s, T],
\]

(4.10)

and

\[
x^{s', x_0', u}(t) + \int_s^t H(r) U^{s', x_0', u}(r) dr = x^{s', x_0', u}(s) + \int_s^t f(x^{s', x_0', u}(r), \mu'_t, u(r)) dr \\
+ \int_s^t g(x^{s', x_0', u}(r), \mu'_t, u(r)) dB(r), t \in [s, T],
\]

(4.11)
\[ + \int_s^t g(x^{s',x_0,u}(r), \mu'_r, u(r))dB(r), \ t \in [s, T]. \] (4.11)

Set \( N(s) = -\frac{d}{ds}(H^{-\frac{1}{2}}(s)) = \frac{1}{2}H^{-\frac{3}{2}}(s) \frac{dH}{ds}(s). \) By (A2), we have
\[
|N(s)| \leq \frac{1}{2} \left| H^{-\frac{3}{2}}(s) \right| \left| \frac{dH}{ds}(s) \right| \leq \frac{1}{2}a_H^{-\frac{3}{2}}M, \tag{4.12}
\]
where \( M := \sup_{0 \leq s \leq T} \left| \frac{dH}{ds}(s) \right| \). Set \( \hat{x}(t) = x^{s,x_0,u}(t) - x^{s',x_0,u}(t), \) \( \tilde{x}(t) = H^{-\frac{1}{2}}(t)\hat{x}(t). \) Then, \( \tilde{x}(\cdot) \) satisfies the following equation:
\[
\tilde{x}(t) = \tilde{x}(s) + \int_s^t \tilde{x}(r) dH^{-\frac{1}{2}}(r) + \int_s^t H^{-\frac{1}{2}}(r) d\tilde{x}(r)
+ \int_{t_0}^t R(r) dr + \int_s^t H^{-\frac{1}{2}}(r) \left( g(x^{s,x_0,u}(r), \mu_r, u(r)) - g(x^{s',x_0,u}(r), \mu'_r, u(r)) \right) dB(r). \tag{4.13}
\]

where
\[
R(r) := \tilde{x}(r)N(r) - H^{-\frac{1}{2}}(r)[U^{s,x_0,u}(r) - U^{s',x_0,u}(r)]
+ H^{-\frac{1}{2}}(s)[f(x^{s,x_0,u}(r), \mu_r, u(r)) - f(x^{s',x_0,u}(r), \mu'_r, u(r))].
\]

Applying Itô’s formula, we have
\[
|\tilde{x}(t)|^2 = |\tilde{x}(s)|^2 + 2 \int_s^t \langle \tilde{x}(r), R(r) \rangle dr
+ \int_s^t \left| H^{-\frac{1}{2}}(r) \left( g(x^{s,x_0,u}(r), \mu_r, u(r)) - g(x^{s',x_0,u}(r), \mu'_r, u(r)) \right) \right|^2 dr
+ 2 \int_s^t \langle \tilde{x}(r), H^{-\frac{1}{2}}(r) \left( g(x^{s,x_0,u}(r), \mu_r, u(r)) - g(x^{s',x_0,u}(r), \mu'_r, u(r)) \right) \rangle dB(r)
\leq |\tilde{x}(s)|^2 + (1 + b_H^{-\frac{3}{2}}M + 4L^2a_H^{-1}b_HL^2) \int_{t_0}^t \sup_{t_0 \leq s \leq s} |\tilde{x}(r)|^2 ds
+ 4L^2a_H^{-1}b_HL^2 \int_s^t W_s^2(\mu_r, \mu'_r) dr
+ 2 \int_s^t \langle \tilde{x}(r), H^{-\frac{1}{2}}(s) \left( g(x^{s,x_0,u}(r), \mu_r, u(r)) - g(x^{s',x_0,u}(r), \mu'_r, u(r)) \right) \rangle dB(r). \tag{4.14}
\]

From BDG’s inequality, for any \( l > 0, \) we have
\[
E \sup_{s \leq r \leq l} \left| \int_s^r \langle \tilde{x}(u), H^{-\frac{1}{2}}(u) \left( g(x^{s,x_0,u}(u), \mu_u, u(u)) \right) \right| du \]

16
This together with (4.14) implies for taking \( l = 2 \),

\[
\mathbb{E}[\sup_{s \leq r \leq t} |\tilde{x}(r)|^2] 
\leq \mathbb{E}[|\tilde{x}(s)|^2] + (2 + 2b_H^{-\frac{5}{2}}M + (2^{11} + 2^5)L^2a_H^{-1}b_H) \int_s^t \mathbb{E}[\sup_{s \leq u \leq r} |\tilde{x}(u)|^2]du. \tag{4.15}
\]

Furthermore,

\[
\mathbb{E}[|\tilde{x}(s)|^2] 
\leq 3a_H^{-1}|x_0 - x_0'|^2 + 3a_H^{-1}\mathbb{E}\left|\int_{s'}^{s} f'(r)dr\right|^2 + 3a_H^{-1}\mathbb{E}\int_{s'}^{s} |g'(r)|^2dr 
\leq 3a_H^{-1}|x_0 - x_0'|^2 + 3a_H^{-1}(s - s')\mathbb{E}\int_{s'}^{s} |f'(r)|^2dr + 3a_H^{-1}\mathbb{E}\int_{s'}^{s} |g'(r)|^2dr 
\leq 3a_H^{-1}|x_0 - x_0'|^2 + C(a_H, b_H, x_0', L, T)(s - s'), \tag{4.16}
\]

where \( f'(r) := f(x'(r), \mu'_r), g'(r) := g(x'(r), \mu'_r) \). Combining (4.15) and (4.16), we have

\[
\mathbb{E}[\sup_{s \leq r \leq t} |\tilde{x}(r)|^2] \leq \mathbb{E}[|\tilde{x}(s)|^2] 
+ (2 + 2b_H^{-\frac{5}{2}}M + (2^{11} + 2^5)L^2a_H^{-1}b_H) \int_s^t \mathbb{E}[\sup_{s \leq u \leq r} |\tilde{x}(u)|^2]du 
\leq (2b_H^{-\frac{5}{2}}M + 8L + \sigma^2(6L^2 + 12L^2a_H^{-1}b_H)) \int_s^t \mathbb{E}[\sup_{s \leq u \leq r} |\tilde{x}(u)|^2]du 
+ 3a_H^{-1}|x_0 - x_0'|^2 + C(a_H, b_H, x_0', L, T, \overline{\sigma})(s - s'). \tag{4.17}
\]
Assume solution \( G \) of Eq. \( (4.3) \) leads to
\[ M \]
where
\[ Q \]

Theorem \( 4.3 \). Assume \( f \) is a continuous function on \( \mathbb{R}^m \). Then, \( \tilde{\eta} \) satisfies the following equation:
\[ \tilde{\eta}_t = 2H_t^{-1}(s)\tilde{g}(s) - 2H_t^{-1}(s)\tilde{\eta}_t\tilde{\eta}_s - 2H_t^{-1}(s)\tilde{\eta}_t\tilde{\eta}_s \quad \text{for all } t \leq T. \]

The following lemma shows that \( x^{\epsilon,s,x_0,u} \) is a Cauchy sequence in \( L^2(s,T;\mathbb{R}^m) \).

Lemma \( 4.2 \). Assume \( (A2) - (A4) \). Let \( x^{\epsilon,s,x_0,u} \) be the solutions of Eq. \( (4.5) \) corresponding to the initial date \((s,x_0)\). Then, we have the following estimates:
\[ \mathbb{E}[\sup_{t_0 \leq t \leq T} |x^{\epsilon,s,x_0,u}(t) - x^{\epsilon',s,x_0,u}(t)|^2] \leq (a_H,b_H,x_0,L,T)(\epsilon + \epsilon'). \]

Proof. Set \( N(s) = -\frac{4}{ds}H^{-\frac{1}{2}}(s) = \frac{1}{2}H^{-\frac{1}{2}}(s)\frac{dH}{ds}(s) \). By (A2), we have
\[ |N(s)| \leq \frac{1}{2}|H^{-\frac{1}{2}}(s)| \left| \frac{dH}{ds}(s) \right| \leq \frac{1}{2}a_H^{-\frac{1}{2}}M, \]
where \( M := \sup_{0 \leq s \leq T} \left| \frac{dH}{ds}(s) \right| \). Set \( \tilde{\epsilon}^{\epsilon',s}(s) = x^{\epsilon,s,x_0,u}(s) - x^{\epsilon',s,x_0,u}(s), \tilde{\eta}_t(s) = H^{-\frac{1}{2}}(s)\tilde{\eta}(s) \).

Then, \( \tilde{\epsilon}^{\epsilon',s}(s) \) satisfies the following equation:
\[ \tilde{\epsilon}^{\epsilon',s}(t) = -\int_t^0 \tilde{\epsilon}^{\epsilon',s}(s) H^{-\frac{1}{2}}(s) \, ds - \int_t^0 H^{-\frac{1}{2}}(s) d\tilde{\eta}_t^{\epsilon',s}(s) \]
\[ = \int_t^0 Q(s) \, ds + \int_t^0 H^{-\frac{1}{2}}(s)\left[ g(x^{\epsilon,s,x_0,u}(s),\mu_t^\epsilon) - g(x^{\epsilon',s,x_0,u}(s),\mu_t^\epsilon') \right] dB(s) \]

where \( Q(s) := \tilde{\epsilon}^{\epsilon',s}(s)N(s) - H^{-\frac{1}{2}}(s)[\nabla \Pi_t(x^{\epsilon}(s)) - \nabla \Pi_t(x^{\epsilon'}(s))], H^{-\frac{1}{2}}(s)[f(x^{\epsilon,s,x_0,u}(s),\mu_t^\epsilon) - f(x^{\epsilon',s,x_0,u}(s),\mu_t^\epsilon')]. \) Applying Itô’s formula and the property \((\epsilon)\) of \( \nabla \Pi_t(\cdot) \), we have
\[ |\tilde{\epsilon}^{\epsilon',s}(t)|^2 = 2\int_t^0 \langle \tilde{\epsilon}^{\epsilon',s}(s), Q(s) \rangle \, ds + \int_t^0 |H^{-\frac{1}{2}}(s)\left[ g(x^{\epsilon,s,x_0,u}(s),\mu_t^\epsilon) - g(x^{\epsilon',s,x_0,u}(s),\mu_t^\epsilon') \right] dB(s) + 2\int_t^0 \langle \tilde{\epsilon}^{\epsilon',s}(s), H^{-\frac{1}{2}}(s)\left[ g(x^{\epsilon,s,x_0,u}(s),\mu_t^\epsilon) - g(x^{\epsilon',s,x_0,u}(s),\mu_t^\epsilon') \right] dB(s) \]
\[\leq C(a_H, b_H, \xi, t_0, L, T) \int_{t_0}^t \sup_{t_0 \leq r \leq s} \left| \tilde{x}^{\epsilon, \epsilon}(r) \right|^2 ds + (\epsilon + \epsilon') \int_{t_0}^t \left| \nabla I(x'(s)) \right|^2 ds \]

This together with (4.22) implies for taking \(l > 0\), we have

\[
\mathbb{E} \sup_{0 \leq t \leq l} \left| \int_{t_0}^t \langle \tilde{x}^{\epsilon, \epsilon}(s), H^{-\frac{1}{2}}(s)(g(x^{\epsilon, s, x_0, u}(s), \mu_s^\epsilon) - g(x^{\epsilon, s, x_0, u}(s), \mu_s^\epsilon')) dB(s) \right|
\leq \mathbb{E} \left( \int_{t_0}^t \left| \tilde{x}^{\epsilon, \epsilon}(s) \right|^2 \mathbb{E} \left[ \sup_{t_0 \leq s \leq t} \left| \tilde{x}^{\epsilon, \epsilon}(s) \right|^2 \right] + l \mathbb{E} \left( \int_{t_0}^t \left| H^{-\frac{1}{2}}(s)(g(x^{\epsilon, s, x_0, u}(s), \mu_s^\epsilon) - g(x^{\epsilon, s, x_0, u}(s), \mu_s^\epsilon') dB(s) \right|^2 ds \right) \right)
\leq \frac{1}{l} \mathbb{E} \left[ \sup_{t_0 \leq s \leq t} \left| \tilde{x}^{\epsilon, \epsilon}(s) \right|^2 \right] + l C(a_H, b_H, \xi, t_0, L, T) \mathbb{E} \int_{t_0}^t \left| \tilde{x}^{\epsilon, \epsilon}(s) \right|^2 ds,
\]

This together with (4.22) implies for taking \(l = 2\),

\[
\mathbb{E} \left[ \sup_{0 \leq t \leq T} \left| \tilde{x}^{\epsilon, \epsilon}(s) \right|^2 \right]
\leq C(a_H, b_H, \xi, t_0, L, T) \int_{t_0}^t \mathbb{E} \left[ \sup_{t_0 \leq r \leq s} \left| \tilde{x}^{\epsilon, \epsilon}(r) \right|^2 \right] ds + 2(\epsilon + \epsilon') \mathbb{E} \int_{t_0}^t \left| \nabla I(x'(s)) \right|^2 ds + 2(\epsilon + \epsilon') \mathbb{E} \int_{t_0}^t \left| \nabla I(x'(s)) \right|^2 ds
\leq C(a_H, b_H, \xi, t_0, L, T) \int_{t_0}^t \mathbb{E} \left[ \sup_{t_0 \leq r \leq s} \left| \tilde{x}^{\epsilon, \epsilon}(r) \right|^2 \right] ds + (\epsilon + \epsilon') C(a_H, b_H, x_0, L, T).
\]

Gronwall’s inequality leads to

\[
\mathbb{E} \left[ \sup_{0 \leq t \leq T} \left| \tilde{x}^{\epsilon, \epsilon}(s) \right|^2 \right] \leq C(a_H, b_H, x_0, L, T)(\epsilon + \epsilon').
\]

The proof is therefore complete.

Lemma 4.4. Let (A2) – (A4) hold. The following equality holds for a subsequence of \(\epsilon\), which we still denote by \(\epsilon\),

\[
\lim_{\epsilon \to 0} \mathbb{E} \int_{r_0}^T \left| x^{\epsilon, s, x_0, u}(r) - x^{s, x_0, u}(r) \right|^2 dr = 0.
\]
Lemma 4.5. Assume (A2) – (A4). The following estimates hold:
\[ |V(s, x_0)| \leq C(a_H, b_H, x_0, L, T), \quad (4.24) \]
\[ |V(s, x_0) - V(s', x_0')| \leq C(a_H, b_H, x_0', L, T)|x_0 - x_0'| + C(a_H, b_H, x_0', L, T)|s - s'|^{\frac{1}{2}}. \quad (4.25) \]
Lemma 4.6. Assume (A2) – (A4). Let \( x^{s, x_0, u} \) be the solution of (4.5). Then, for any \((t, x_0) \in [0, T) \times \mathbb{R}^m\), it holds that

\[
V_t(s, x_0) = \inf_{u \in \mathcal{U}[s, T]} \mathbb{E} \left[ \int_s^T b(x^{s, x_0, u}(t), u(t)) dt + V(\tau, x^{s, x_0, u} \tau) \right],
\]

for every stopping time \( \tau \in [s, T] \).

Now, we are going to prove that Eq. (4.11) satisfies the DDP

Proposition 4.1. Assume (A2) – (A4). Then it holds that:

\[
|V_t(s, x_0) - V(s, x_0)| \leq C(a_H, b_H, x_0, L, T)\epsilon^{\frac{1}{2}}.
\]
Proof. By Lemma 4.4 taking the limit in (4.19), we deduce that

$$
\mathbb{E}\left[ \sup_{s \leq r \leq T} \left| x^{s, x_0, u}(r) - x^{s, x_0, u}(r) \right|^2 \right] \leq C(a_H, b_H, x_0, L, T) \epsilon.
$$

With the same argument as in the proof of Lemma 4.5, we obtain

$$
|J_\epsilon(s, x_0, u) - J(s, x_0, u)| \leq C(a_H, b_H, x_0, L, T) \epsilon^\frac{1}{2}.
$$

Consequently, we obtain

$$
|V_\epsilon(s, x_0) - V(s, x_0)| \leq \sup_{u \in \mathcal{U}[s, T]} |J_\epsilon(s, x_0, u) - J(s, x_0, u)| \leq C(a_H, b_H, x_0, L, T) \epsilon^\frac{1}{2}.
$$

The second conclusion follows.

From the above lemma, the following result immediately holds.

**Lemma 4.7.** $V_\epsilon(\cdot, \cdot)$ is uniformly convergent on compacts to the value function $V(\cdot, \cdot)$ on $[0, T] \times \overline{D(\Pi)}$.

Now, we state our main result.

**Theorem 4.8.** Under the assumptions (A2) – (A4), the value function $V$ satisfies the DPP (4.4).

*Proof.* Taking any $(t, x_0) \in [0, T] \times \overline{D(\Pi)}$, we have, for any $\epsilon > 0$, $u \in \mathcal{U}[s, T], \tau \in [s, T]$, and $R > 0$,

$$
\begin{align*}
\mathbb{E}[|V_\epsilon(\tau, x^{\epsilon, s, x_0, u}(\tau)) - V(\tau, x^{s, x_0, u}(\tau))|] \\
&\leq \mathbb{E}[|V_\epsilon(\tau, x^{\epsilon, s, x_0, u}(\tau)) - V_\epsilon(\tau, x^{s, x_0, u}(\tau))|] \\
&\quad + \mathbb{E}[|V_\epsilon(\tau, x^{s, x_0, u}(\tau)) - V(\tau, x^{s, x_0, u}(\tau))|]\{1_{A_1} + 1_{A_2}\} \\
&\leq \mathbb{E}[|V_\epsilon(\tau, x^{\epsilon, s, x_0, u}(\tau)) - V_\epsilon(\tau, x^{s, x_0, u}(\tau))|] + \sup_{(t, y) \in B} \mathbb{E}[|V_\epsilon(t, y) - V(t, y)|] \\
&\quad + \mathbb{E}[|V_\epsilon(\tau, x^{s, x_0, u}(\tau)) - V(\tau, x^{s, x_0, u}(\tau))|]1_{A_2},
\end{align*}
$$

(4.29)

where

$$
A_1 := \{\omega : |x^{s, x_0, u}(\tau)| \leq R\}, \quad A_2 := \{\omega : |x^{s, x_0, u}(\tau)| > R\},
$$

and

$$
B := [0, T] \times (\overline{D(\Pi)} \cap \overline{B(0, R)}).
$$

With the same argument as in the proof of Lemma 4.5, we have

$$
|J_\epsilon(\tau, x^{\epsilon, s, x_0, u}(\tau)) - J_\epsilon(\tau, x^{s, x_0, u}(\tau))| \leq C(a_H, b_H, \xi, L, T) \epsilon^\frac{1}{2}.
$$

(4.30)

Now, we look at the term $\mathbb{E}[|V_\epsilon(\tau, x^{s, x_0, u}(\tau)) - V(\tau, x^{s, x_0, u}(\tau))|1_{A_2}]$. By definition of $A_2$, we have

$$
\begin{align*}
\mathbb{E}[|V_\epsilon(\tau, x^{s, x_0, u}(\tau)) - V(\tau, x^{s, x_0, u}(\tau))|1_{A_2}] \\
&\leq (\mathbb{E}[|V_\epsilon(\tau, x^{s, x_0, u}(\tau)) - V(\tau, x^{s, x_0, u}(\tau))|^2])^{\frac{1}{2}}(\mathbb{E}1_{A_2})^{\frac{1}{2}}
\end{align*}
$$

22
\[ \leq \sqrt{2}(\mathbb{E}[|V(\tau, x_{10}^{s}u(\tau))|^2] + \mathbb{E}[|V(\tau, x_{10}^{s}u(\tau))|^2])^{\frac{1}{2}} \leq \frac{C(a_H, b_H, x_0, L, T)}{R}. \]  

Combining (4.29), (4.30) and (4.31), we have

\[
\begin{align*}
\mathbb{E}[|V(\tau, x_{10}^{s}u(\tau)) - V(\tau, x_{10}^{s}u(\tau))|] \\
\leq \mathbb{E}[|V(\tau, x_{10}^{s}u(\tau)) - V_{\epsilon}(\tau, x_{10}^{s}u(\tau))|] + \sup_{(t,y) \in B} \mathbb{E}[|V_{\epsilon}(t, y) - V(t, y)|] \\
+ \frac{C(a_H, b_H, x_0, L, T)}{R} \\
\leq C(a_H, b_H, x_0, L, T)\epsilon^{\frac{1}{2}} + \sup_{(t,y) \in B} \mathbb{E}[|V_{\epsilon}(t, y) - V(t, y)|] \\
+ \frac{C(a_H, b_H, x_0, L, T)}{R}. \quad (4.32)
\end{align*}
\]

In addition,

\[
\begin{align*}
V(t, \xi) \leq V_{\epsilon}(t, x_0) + |V_{\epsilon}(t, x_0) - V(t, x_0)| \\
\leq \mathbb{E} \left[ \int_s^\tau b(x_{r}^{s, x_{10}^{s}u}(r), x_{r}^{s, x_{10}^{s}u}, u(r))dr + V(\tau, x_{10}^{s}u(\tau)) \right] + |V_{\epsilon}(t, x_0) - V(t, x_0)| \\
\leq \mathbb{E} \left[ \int_s^\tau b(x_{r}^{s, x_{10}^{s}u}, u(r))dr + V(\tau, x_{10}^{s}u(\tau)) \right] + |V_{\epsilon}(t, x_0) - V(t, x_0)| \\
+ \mathbb{E} \left[ \int_s^\tau b(x_{10}^{s}u(\tau), u(r))dr - \int_s^\tau b(x_{10}^{s, x_{10}^{s}u}(r), u(r))dr \right] \\
+ |V_{\epsilon}(\tau, x_{10}^{s}u(\tau)) - V_{\epsilon}(\tau, x_{10}^{s}u(\tau))| \\
\leq \mathbb{E} \left[ \int_s^\tau b(x_{10}^{s, x_{10}^{s}u}(r), u(r))dr + V(\tau, x_{10}^{s}u(\tau)) \right] \\
+ C(a_H, b_H, x_0, L, T)\epsilon^{\frac{1}{2}} + \sup_{(t,y) \in B} \mathbb{E}[|V_{\epsilon}(t, y) - V(t, y)|] \\
+ \frac{C(a_H, b_H, \xi, L, T)}{R}. \quad (4.33)
\end{align*}
\]

Passing to the limit for \( \epsilon \to 0 \) and \( R \to \infty \), we obtain

\[ V(t, x_0) \leq \mathbb{E} \left[ \int_s^\tau b(x_{10}^{s}u(\tau), u(r))dr + V(\tau, x_{10}^{s}u(\tau)) \right]. \]

Conversely, for any \( \delta > 0 \), since \( V_{\epsilon} \) satisfies the DPP, there exists \( u_\delta \in \mathcal{U}[t, T] \) such that

\[ V_{\epsilon}(t, x) + \frac{\delta}{2} \geq \mathbb{E} \left[ \int_s^\tau b(x_{10}^{s, u_\delta}(r), u_\delta(r))dr + V_{\epsilon}(\tau, x_{10}^{s, u_\delta}(\tau)) \right]. \]
Using the above inequality, we have

\[ V(t, x) + \delta \geq V(\tau, x) + \delta - |V\epsilon(t, x) - V(t, x)| \]

\[ \geq \mathbb{E} \left[ \int_{s}^{T} b(x^{s,x_0,u_\delta}(r), u_\delta(r)) dr + V(\tau, x^{s,x_0,u_\delta}(\tau)) \right] + \frac{\delta}{2} - |V\epsilon(t, x) - V(t, x)| \]

\[ \geq \mathbb{E} \left[ \int_{s}^{T} b(x^{s,x_0,u_\delta}(r), u_\delta(r)) dr + V(\tau, x^{s,x_0,u_\delta}(\tau)) \right] + \frac{\delta}{2} - |V\epsilon(t, x_0) - V(t, x_0)| \]

\[ - V\epsilon(\tau, x^{s,x_0,u_\delta}(\tau)) - V(\tau, x^{s,x_0,u_\delta}(\tau)) \]

\[ \geq \mathbb{E} \left[ \int_{s}^{T} b(x^{s,x_0,u_\delta}(r), x^{s,x_0,u_\delta}(r), u_\delta(r)) dr + V(\tau, x^{s,x_0,u_\delta}(\tau)) \right] + \frac{\delta}{2} - |V\epsilon(t, x_0) - V(t, x_0)| \]

\[ - \mathbb{E} \left[ \int_{s}^{T} b(x^{s,x_0,u_\delta}(r), x^{s,x_0,u_\delta}(r), u_\delta(r)) dr \right] + \frac{\delta}{2} + \sup_{(t,y) \in B} \mathbb{E}[|V\epsilon(t, y) - V(t, y)|] \]

\[ \geq \mathbb{E} \left[ \int_{s}^{T} b(x^{s,x_0,u_\delta}(r), x^{s,x_0,u_\delta}(r), u_\delta(r)) dr + V(\tau, x^{s,x_0,u_\delta}(\tau)) \right], \]

where the last inequality follows by letting \( \epsilon \to 0 \) and \( R \to \infty \). The proof is complete.

\[ \square \]

**Funding**

This research was supported by the National Natural Science Foundation of China (Grant nos. 61876192, 11626236), the Fundamental Research Funds for the Central Universities of South-Central University for Nationalities (Grant nos. CZY15017, KTZ20051, CZT20020).

**Availability of data and materials**

Not applicable.

**Competing interests**

The author declare they have no competing interests.

**Authors’ contributions**

All authors conceived of the study and participated in its design and coordination. All authors read and approved the final manuscript.

24
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