On the Separation of Variables for the Modular XXZ Magnet and the Lattice Sinh-Gordon Models

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Abstract. We construct the generalised eigenfunctions of the entries of the monodromy matrix of the $N$-site modular XXZ magnet and show, in each case, that these form a complete orthogonal system in $L^2(\mathbb{R}^N)$. In particular, we develop a new and simple technique, allowing one to prove the completeness of such systems. As a corollary of our analysis, we prove the Bytsko–Teschner conjecture relative to the structure of the spectrum of the $\mathcal{B}(\lambda)$-operator for the odd length lattice Sinh-Gordon model.

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1. Introduction

The quantum separation of variables has been developed by Sklyanin in [27–29] and was applied, since then, to obtain the spectra and eigenstates of numerous quantum integrable models having finite- and infinite-dimensional local spaces [5,9,10,19,20,23,24]. The very essence of the method consists in mapping the original Hilbert space $\mathcal{H}_{\text{org}}$ where a quantum integrable model is formulated onto an auxiliary Hilbert space $\mathcal{H}_{\text{aux}}$. This mapping is done in a very specific way that strongly simplifies the original spectral problem for the model’s transfer matrix. When formulated on the level of the original Hilbert space $\mathcal{H}_{\text{org}}$, the spectral problem for the transfer matrix is a genuine multidimensional and multi-parameter spectral problem. However, upon implementing the separation of variables transform, the latter is re-expressed in terms of a multi-parameter but one-dimensional spectral problem which takes the form of a scalar $T−Q$ equation. Thus, the method provides an outstanding simplification.

An important ingredient of the whole construction consists in establishing that the spaces $\mathcal{H}_{\text{org}}$ and $\mathcal{H}_{\text{aux}}$ are unitarily equivalent, a necessary step
for guaranteeing the equivalence between the two Hilbert space descriptions of
the spectral problem. In the finite-dimensional setting, one may establish this
unitarity by a direct counting of dimension arguments. However, the situation
is disproportionately harder in the case of integrable models associated with
infinite-dimensional local Hilbert spaces. Indeed, even in the simplest possible
case, the quantum $N$-particle Toda chain [15], establishing the unitarity of
the transform is equivalent to proving the completeness and orthogonality of
the system of Whittaker functions for $\mathfrak{gl}_N$. This topic has a very long history.
The satisfactory and full resolution of the problem was first achieved through
rather evolved tools of harmonic analysis on non-compact groups [31]. These
techniques, on top of their complexity, were completely unrelated to the alge-
braic structures usually dealt with in quantum integrable models. Hence, their
generalisation to more complex models was hardly imaginable. A first impor-
tant progress towards a quantum integrability-based proof of unitarity was
achieved in [9]. That paper proposed a method for constructing a pyramidal—
the so-called Gauss–Givental [14]—representation for the integral kernel of
the separation of variables transform in the case of the non-compact XXX
$\mathfrak{sl}(2,\mathbb{C})$ chain. In fact, the integral kernel appeared as a generalised system of
eigenfunctions of one of the entries of the model’s monodromy matrix. The
work [9] developed a technique allowing one to establish orthogonality of this
system, which is “half” of the proof of unitarity. The mentioned technique
was later applied to other models subordinate to the rational $R$-matrix: the
non-compact XXX $\mathfrak{sl}(2,\mathbb{R})$ [10] or to the Toda chain [26]. A quantum inverse
scattering-based method for proving completeness was developed in [22], where
all handlings leading to the proof of unitarity were set in a rigorous framework
of distribution theory. Finally, one should mention the work [11] where, on
the example of the $\mathfrak{sl}(2,\mathbb{C})$ chain, the technique of iterative construction of
eigenfunctions was perfected and set in a very natural way to the quantum
inverse scattering method picture.

The purpose of the present paper is to push these developments fur-
ther. On the one hand, we extend the method of constructing Gauss–Givental
representations to the case of a non-compact, rank 1, model associated with
a trigonometric $R$-matrix. On the other hand, we propose a simple method
for proving completeness. Indeed, as observed in [22], Mellin–Barnes integral
representations are well suited for proving completeness of the unitarity trans-
form’s kernel. On a technical level, doing so demands to have at one’s disposal a
certain integral representation for the symmetric delta function in $N$-variables.
A technique for proving this integral representation was developed in [22] and
was based on a fine analysis of the consequences of the orthogonality relations.
While providing the desired results, this method is not based on a direct argu-
ment. In the present paper, we develop a direct and systematic technique for
establishing the integral identities of interest. We believe this to be an impor-
tant contribution of our work. Below, we focus our study on the case of the
$N$-site modular $XXZ$ magnet. The $N = 1$ specialisation of our results already
appeared earlier in the literature. Namely, for $N = 1$, the unitary equivalence
was established in [16] in the context of solving the spectral problem for the
Dehn twist operators in the quantum Teichmüller theory. Later, the same problem was addressed in [30] and solved by other means.

The \( N \)-site modular XXZ magnet is closely connected with the lattice Sinh-Gordon model. Thus, as a by-product of our analysis, we prove the Bytsko–Teschner conjecture [5] relative to the structure of the spectrum of the \( B \)-operator in this model. Finally, at the time where this manuscript was being finalised, the work [25] appeared on the ArXiV and dealt, by slightly different means with a related problem in the context of the \( q \)-Toda chain. It would be interesting to compare the methods.

The paper is organised as follows. In Sect. 2, we introduce the modular XXZ chain’s Lax matrix and discuss some of the fundamental objects related to this model. The Gauss–Givental integral representation for the system of eigenfunctions associated with entries of the monodromy matrix is constructed in Sect. 3. It immediately allows us to establish the orthogonality of such a system. Section 4 is devoted to the proof of the completeness of the eigensystems. We derive the Mellin–Barnes integral representation for the eigenfunctions and evaluate some auxiliary integrals. Using these results, we prove the completeness of the orthogonal systems associated with entries of the monodromy matrix. Then, in Sect. 5, we discuss properties of the complete orthogonal system associated with entries of the monodromy matrix. Finally, in Sect. 6, we apply our results to the case of the lattice Sinh-Gordon model. The paper contains two appendices. Appendix A reviews the various notations introduced in the paper. Appendix B reviews the special functions used throughout the work.

2. The Modular XXZ Magnet

2.1. The Model

The modular XXZ magnet’s Hilbert space has the tensor product decomposition \( \mathcal{H} = \bigotimes_{n=1}^{N} \mathcal{H}_n \) where the Hilbert space \( \mathcal{H}_n \) associated with the \( n \)-th site of the model is isomorphic to \( L^2(\mathbb{R}, dx) \). The \( n \)-th-site Lax matrix takes the form

\[
L_n^{(\kappa)}(\lambda) = 2 \begin{pmatrix}
-i \sinh \left( \frac{\pi}{\omega_1} (\lambda - p_n) \right) & \left( \frac{\pi}{\omega_1} (p_n + \kappa - \omega_2 p_n) \right) e^{-\pi \omega_2 x_n} \\
\left( \frac{\pi}{\omega_1} (p_n - \kappa + \omega_2 p_n) \right) e^{\pi \omega_2 x_n} & -i \sinh \left( \frac{\pi}{\omega_1} (\lambda + p_n) \right)
\end{pmatrix}.
\] (1)

Here and in the following, \( \omega_1, \omega_2 \) and \( \kappa \) are three real parameters such that \( \omega_1 > 0 \) and \( \kappa \neq 0 \). Further, \( x_n \) and \( p_n \) are operators on \( \mathcal{H}_n \) satisfying the canonical commutation relations \( [x_n, p_m] = \delta_{n,m} \frac{i}{\pi} \), where \( \delta_{n,m} \) is the Kronecker symbol. In the following, we shall work in a representation where \( x_n \) is the multiplication operator by the \( n \)-th coordinate

\[
(x_n f)(x_N) = x_n f(x_N) \quad \text{with} \quad x_N = (x_1, \ldots, x_N)
\]

so that

\[
(p_n f)(x_N) = -\frac{i}{2\pi} \partial_{x_n} f(x_N).
\] (2)
The model’s monodromy matrix takes the form

$$T_N(\lambda) = L_1^{(\kappa)}(\lambda) \cdots L_N^{(\kappa)}(\lambda) = \begin{pmatrix} A_N(\lambda) & B_N(\lambda) \\ C_N(\lambda) & D_N(\lambda) \end{pmatrix}.$$  \(3\)

The transfer matrix

$$t(\lambda) = \text{tr}[T(\lambda)] = (-i)^N e^{\frac{\pi i}{12} N \Lambda} \sum_{k=0}^{N} (-e^{-\frac{2\pi i}{N} \lambda})^k T_k$$  \(4\)

provides us with a commutative algebra of positive self-adjoint operators \(\{ T_k \}_{1}^{N} \) on \(\mathfrak{h}\), c.f. [5] for more details.

### 2.2. The Elementary Operator Relations of the Model

The \(\mathcal{R}\)-operator for the modular XXZ magnet was first constructed in [4]. Its non-trivial part was given as a special function of a positive self-adjoint operator cooked up from a representation of the coproduct of the Casimir of \(U_q(\mathfrak{sl}_2)\). That characterisation of \(\mathcal{R}\) was rather implicit. The paper [6] proposed an alternative way to construct this \(\mathcal{R}\)-operator. This allowed for a much simpler and more explicit form of \(\mathcal{R}\). This construction was based on the existence of an alternative factorisation of the model’s Lax matrix

$$L_n^{(\kappa)}(\lambda) = -i M_{u_2}(x_n) \cdot H(p_n) \cdot N_{u_1}(x_n)$$  \(5\)

where

$$M_{u_2}(x) = \begin{pmatrix} U_2 & -U_2^{-1} \\ -U_2^{-1} e^{2\pi i \omega_2 x} & U_2 e^{2\pi i \omega_2 x} \end{pmatrix}, \quad N_{u_1}(x) = \begin{pmatrix} -U_1 & U_1^{-1} e^{-2\pi i \omega_2 x} \\ -U_1^{-1} & U_1 e^{-2\pi i \omega_2 x} \end{pmatrix}$$  \(6\)

and \(H(p) = e^{-\frac{\pi i}{2}(p - i\frac{\Omega}{2})}\sigma_3\). Here, \(\sigma_3 = \text{diag}(1, -1)\) and we used the notation

$$U_a = e^{\frac{\pi i}{12} u_a} \quad \text{with} \quad \begin{cases} u_1 = \frac{1}{2} \cdot (\lambda + \kappa - i\frac{\tau}{2}) \\ u_2 = \frac{1}{2} \cdot (\lambda - \kappa - i\frac{\tau}{2}) \end{cases}$$  \(7\)

and, agree, from now on, to denote

$$\Omega = \omega_1 + \omega_2, \quad \tau = \omega_2 - \omega_1.$$  \(8\)

The factorisation (5) allows one to interpret the Lax matrix as a function of the parameters \(u_1\) and \(u_2\), \textit{viz}.

$$L_n^{(\kappa)}(\lambda) \equiv L_n(u_1, u_2).$$  \(9\)

We shall adopt this notation in the following. Note that changing \(\kappa \rightarrow -\kappa\) produces an exchange of the parameters \(u_1\) and \(u_2\). It is well known that the operator \(D_{-\kappa}(p)\), the function \(D_\alpha\) being defined in Appendix B c.f. (248), is an intertwining operator between the \(\kappa\) and \(-\kappa\) representations, meaning that

$$D_{u_2-u_1}(p_1) \cdot L_1(u_1, u_2) = L_1(u_2, u_1) \cdot D_{u_2-u_1}(p_1) \cdot D_{u_2-u_1}(p_1).$$  \(10\)

In fact, the function \(D_\alpha\) coincides with the Boltzmann weight of the Faddeev–Volkov model, c.f. [2,3]. The operator \(D_\alpha(p)\) acts as a multiplication operator in the momentum representation where \(p\) acts as a multiplication operator and can be represented as an integral operator in the position representation where
x acts as a multiplication operator. The latter property can be obtained by means of the Fourier transform (256):

\[ [D_\alpha(p) \cdot f](x) = \sqrt{\omega_1 \omega_2} \mathcal{A}(\alpha) \int_{\mathbb{R}} D_{\alpha^*}(\omega_1 \omega_2(x - x')) f(x') \cdot dx' \] (11)

where we have introduced

\[
\alpha^* = -\alpha - i \frac{\Omega}{2} \quad \text{and} \quad \mathcal{A}(\alpha) = \varpi(\alpha - \alpha^*) = \varpi(2\alpha + i \frac{\Omega}{2}).
\] (12)

Also, \(\varpi\) refers to the quantum dilogarithm whose definition is recalled in Appendix B c.f. (241).

On the basis of the factorisation (5), it was established in [6] that the following relation holds:

\[ D_{u_1-v_2}(\omega_1 \omega_2 x_{12}) L_1(u_1, u_2) L_2(v_1, v_2) = L_1(v_2, u_2) L_2(v_1, u_1) D_{u_1-v_2}(\omega_1 \omega_2 x_{12}). \] (13)

Above and in the following, for any quantities \(\alpha_a, \alpha_b\) we adopt the notation \(\alpha_{ab} = \alpha_a - \alpha_b, \ e.g. \ x_{12} = x_1 - x_2\).

The two above intertwining relations thus ensure that the operator

\[ R_{12}(u_1, u_2 | v_2) = D_{u_2-v_1}(\omega_1 \omega_2 x_{12}) \cdot D_{u_2-v_2}(p_1) \cdot D_{u_1-v_2}(\omega_1 \omega_2 x_{12}) \] (14)

\[ = D_{u_1-v_2}(p_1) \cdot D_{u_2-v_2}(\omega_1 \omega_2 x_{12}) \cdot D_{u_2-u_1}(p_1) \] (15)

realises the intertwining

\[ R_{12}(u_1, u_2 | v_2) L_1(u_1, u_2) L_2(v_1, v_2) = L_1(v_2, u_2) L_2(v_1, u_1) R_{12}(u_1, u_2 | v_2). \] (16)

The two factorisations of \(R_{12}\) stem from the two possible ways of decomposing the permutation

\[ (u_1, u_2, v_1, v_2) \mapsto (u_1, v_2, v_1, u_2) \] (17)

into a product of 2-cycles. Note that the equality between the two factorisations is, in fact, a consequence of the three-term integral relation (260) satisfied by the \(D\) functions which, in an operator form, is given in (264). This relation was first obtained in [16], see also [17]. Also, we stress that \(R_{12}\) does not correspond to the \(\mathfrak{R}\) operator of the XXZ modular magnet. Nonetheless, it corresponds to half of its building block. We refer to [6] for more details.

Finally, observe that one has the identities

\[ v_\epsilon^t \cdot L_n(u_1, u_2) \cdot e^{-2i\pi \epsilon (v_2-u_2)x_n} = e^{-2i\pi \epsilon (v_2-u_2)x_n} \cdot v_\epsilon^t \cdot L_n(u_1, u_2) \] (18)

and

\[ L_n(u_1, u_2) v_\epsilon \cdot e^{2i\pi \epsilon (u_1-v_1)x_n} = e^{2i\pi \epsilon (u_1-v_1)x_n} \cdot L_n(u_1, u_2) v_\epsilon \] (19)

where we have introduced

\[ v_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad v_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \] (20)
3. The Gauss–Givental Integral Representation and Orthogonality

3.1. The $\Lambda$-Operator

We have now introduced enough notations so as to introduce the $\Lambda^{(N)}_{y,\epsilon}$ operators and their adjoints. These operators play a crucial role in constructing the generalised eigenfunctions of the operators $A_N$ and $B_N$ defined in (3). Below, we shall establish some of their exchange relations which are important for our further purposes. These should be understood in the weak sense, and when one considers sufficiently regular functions. More precisely, any operator identity $O = 0$ should be understood as $(f, O[g]) = 0$ for any compactly supported $f, g$.

See [22] for more details.

Definition 3.1. Let $y_\pm$ stand for

$$y_{\pm} = \frac{1}{2} \left( y \pm \kappa - i\frac{\Omega}{2} \right).$$  \hspace{1cm} (21)

The continuous operators $\Lambda^{(N)}_{y,\epsilon} : \mathcal{L}^\infty(\mathbb{R}^{N-1}) \mapsto \mathcal{L}^\infty(\mathbb{R}^N)$ are defined by

$$\Lambda^{(N)}_{y,\epsilon} = \frac{e^{2i\pi y \cdot x_1}}{(\mathcal{A}(y_+) \sqrt{\omega_1 \omega_2})^{N-1}} \cdot \tilde{J}_N(y_+ - y_-) \cdot U_N(y_+, y_-) \cdot e^{2i\pi \epsilon y_+ \cdot x_N} 1_N$$

$$= \frac{e^{2i\pi y \cdot x_1}}{(\mathcal{A}(y_+) \sqrt{\omega_1 \omega_2})^{N-1}} \cdot U_N(y_+, y_-) \cdot J_N(y_- - y_+) \cdot e^{2i\pi \epsilon y_+ \cdot x_N} 1_N$$

(22)

with

$$\tilde{J}_N(y) = \prod_{a=1}^{N-1} D_y(\omega_1 \omega_2 x_{a+1}) \quad , \quad J_N(y) = \prod_{a=1}^{N-1} D_y(p_a)$$

as well as

$$U_N(y_+, y_-) = \prod_{a} \left\{ D_{y_+}(p_a) D_{y_-}(\omega_1 \omega_2 x_{a+1}) \right\}.$$ \hspace{1cm} (24)

Finally, the operator $1_N$ stands for the constant function on the $N$th space and the formula is to be understood as the partial action of the chain of operators on this function.

Above and in the following, we agree to denote the ordered products as

$$\prod_{a}^{1 \wedge N} 0_a = 0_1 \cdots 0_N \quad \text{and} \quad \prod_{a}^{N \wedge 1} 0_a = 0_N \cdots 0_1.$$ \hspace{1cm} (26)

Furthermore, the equality between (22) and (23) follows from a multiple application of the relation (264).
Owing to the conjugation property of the $D_\alpha$ function c.f. (248), and those of the quantum dilogarithm, the adjoint operators

$$\Lambda_{y,\epsilon}^{(N)} \dagger = \left( \frac{\mathcal{A}(\lambda y_2^\dagger)}{\sqrt{\omega_1 \omega_2}} \right)^{N-1} \cdot \frac{1}{N} e^{-2i\pi \epsilon (y_1^\dagger)^*} \cdot x_N \cdot \mathcal{U}_N^\dagger (y_-, y_+) \cdot \mathcal{J}_N^{-1} (y_- - y_+) e^{-2i\pi \epsilon y_+^* x_1}$$

$$= \left( \frac{\mathcal{A}(\lambda y_2^\dagger)}{\sqrt{\omega_1 \omega_2}} \right)^{N-1} \cdot \frac{1}{N} e^{-2i\pi \epsilon (y_1^\dagger)^*} \cdot x_N \cdot \mathcal{J}_N^{-1} (y_- - y_+) \cdot \mathcal{U}_N^\dagger (y_+, y_-) e^{-2i\pi \epsilon y_+^* x_1}$$

(27)

give rise to continuous operators $L^\infty (\mathbb{R}^N) \to L^\infty (\mathbb{R}^{N-1})$. Here, $^*$ stands for the complex conjugation while $\dagger$ for the transform (12). Moreover, $1_N^\dagger$ represents the operation of integration versus the $N$th coordinate. Finally, we have set

$$\mathcal{U}_N^\dagger (y_+, y_-) = \prod_a D_{-y_-^a} (\omega_1 \omega_2 x_{a a+1}) D_{-y_+^a} \left( p_a \right).$$

(29)

We stress that the $\Lambda$-operators are in fact very closely related to the partial $\mathbb{R}$-operators in that they can be expressed as

$$\Lambda_{y,\pm}^{(N)} = \frac{e^{2i\pi \epsilon y_{\pm} x_1}}{\left( \mathcal{A}(y_-) \sqrt{\omega_1 \omega_2} \right)^{N-1}} \cdot R_{12}(u_1, u_2 | v_2) \cdot \cdots \cdot R_{N-1N}(u_1, u_2 | v_2) \cdot e^{2i\pi \epsilon y_+^* x_N},$$

(30)

where $y_\pm$ are as defined in (21).

The first interesting property of the operators $\Lambda_{y,\pm}^{(N)}$ concerns their exchange relations with the operators $A_N, B_N$.

**Lemma 3.1.** The operators $\Lambda_{y,\epsilon}^{(N)}$ satisfy

$$A_N (\lambda) \cdot \Lambda_{y,-}^{(N)} = -2i \sinh \left[ \frac{\pi}{\omega_1} (\lambda - y) \right] \cdot \Lambda_{y,-}^{(N)} \cdot A_{N-1} (\lambda) \quad \quad (31)$$

$$B_N (\lambda) \cdot \Lambda_{y,+}^{(N)} = -2i \sinh \left[ \frac{\pi}{\omega_1} (\lambda - y) \right] \cdot \Lambda_{y,+}^{(N)} \cdot B_{N-1} (\lambda). \quad \quad (32)$$

**Proof.** Introduce the notation

$$T_N (u_1, u_2) \equiv T_N (\lambda) = L_1 (u_1, u_2) \cdot L_2 (u_1, u_2) \cdot \cdots \cdot L_N (u_1, u_2),$$

(33)

where $u_a$’s are given as in (7). Then, by virtue of (16), (18), (19), for arbitrary $v_1, v_2$, the operators

$$0_{\epsilon',\epsilon}^{(N)} (u_1, u_2 | v_1, v_2) = e^{-2i\pi \epsilon' (v_2 - u_2) x_1} \cdot R_{12}(u_1, u_2 | v_2) \cdot \cdots \cdot R_{N-1N}(u_1, u_2 | v_2) \cdot e^{2i\pi \epsilon (u_1 - v_1) x_N}$$

(34)

satisfy to the exchange relation

$$v_{\epsilon'}^t \cdot T_N (u_1, u_2) \cdot v_\epsilon \cdot 0_{\epsilon',\epsilon}^{(N)} (u_1, u_2 | v_2) = 0_{\epsilon',\epsilon}^{(N)} (u_1, u_2 | v_1, v_2) \cdot v_{\epsilon'}^t \cdot T_{N-1} (u_1, u_2) \cdot L_N (v_1, v_2) \cdot v_\epsilon. \quad \quad (35)$$
Upon adopting the parameterisation of \( v_1, v_2 \) in the form
\[
v_1 = \frac{1}{2}(\mu + s - i\frac{\tau}{2}) \quad \text{and} \quad v_2 = \frac{1}{2}(\mu - s - i\frac{\tau}{2}),
\]
and the notation \( \chi_{N';\epsilon}(x_N) = e^{2i\pi\epsilon x_N(s+i\frac{\Omega}{2})} \), it is readily checked that
\[
\left( v^+_1 T_{N-1}(u_1, u_2) \cdot L_N(v_1, v_2) v_\epsilon \cdot \chi_{N';\epsilon}(x_N) \right)(x_N)
= -2i \sinh \left[ \frac{\pi}{\omega_1}(\mu - s - i\frac{\tau}{2}) \right] \left( \Lambda_{N-1}^\delta(\lambda) \delta_{\epsilon,1} + B_{N-1}(\lambda) \delta_{\epsilon,-1} \right) \cdot \chi_{N';\epsilon}(x_N).
\]
Finally, consider the re-parameterisation
\[
v_2 = \frac{1}{2}(\lambda - y + i\omega_1) \quad \text{and} \quad v_1 = v_2 + s \quad \text{so that} \quad \begin{cases} v_2 - u_2 = -y_- \\ u_1 - v_1 + s + i\frac{\Omega}{2} = -y_+ \end{cases}
\]
where \( y_\pm \) are as defined in (21). These then lead to the rewriting
\[
R_{12}(u_1, u_2 \mid v_2) = D_{y_-,y_+}(\omega_1\omega_2 x_{12}) \cdot D_{y_-(p_1)} \cdot D_{y_+}(\omega_1\omega_2 x_{12})
= D_{y_+}(p_1) \cdot D_{y_-}(\omega_1\omega_2 x_{12}) \cdot D_{y_-(p_1)},
\]
which entails
\[
0^{(N)}_{+,\epsilon}(u_1, u_2 \mid v_1, v_2) \cdot \chi_{N';\epsilon} = \left( \mathcal{A}(y_+) \sqrt{\omega_1 \omega_2} \right)^{N-1} \cdot \Lambda_{y_-,\epsilon}^{(N)}.
\]
All together with (35)–(37) yields the two representations given in (31)–(32).

We now establish exchange relations between the \( \Lambda_{y,\pm}^{(N)} \) operators on the one hand and, on the other hand, between these operators and their duals. Below, we shall adopt hypergeometric-like notations for product of functions, as defined in (240).

**Proposition 3.2.** Given \( y, t \in \mathbb{R} \), let \( y_\pm, t_\pm \) be defined according to (21).

For \( y \neq t \), the operators \( \left( \Lambda_{y,\epsilon}^{(N)} \right)^\dagger \) and \( \Lambda_{t,\epsilon}^{(N)} \) satisfy to the exchange relations
\[
\left( \Lambda_{y,\epsilon}^{(N)} \right)^\dagger \cdot \Lambda_{t,\epsilon}^{(N)} = \frac{1}{\omega_1 \omega_2} \cdot \mathcal{A}(t^*_+ - (y^*_+)^*, t_- - y_-^*) \cdot \Lambda_{t,\epsilon}^{(N-1)} \cdot \left( \Lambda_{y,\epsilon}^{(N-1)} \right)^\dagger
\]
and enjoy the pseudo-commutation relations
\[
\Lambda_{y,\epsilon}^{(N)} \Lambda_{t,\epsilon}^{(N-1)} = \Lambda_{t,\epsilon}^{(N)} \Lambda_{y,\epsilon}^{(N-1)}.
\]

**Proof.** In order to establish the pseudo-commutativity, observe that
\[
\Lambda_{y,\epsilon}^{(N)} \Lambda_{t,\epsilon}^{(N-1)} = \left( \mathcal{A}(t^+, y^+) \right)^{2-N} \left( \frac{2}{\omega_1 \omega_2} \right)^{N-2} \mathcal{A}(t^+, y^+) \cdot \mathcal{P}(y, t) \cdot \mathcal{J}_{N-1}(t_- - t_+) \cdot 1_N \otimes 1_{N-1}
\]
where
\[
\mathcal{P}(y, t) = \mathcal{A}(t^+) e^{2i\pi y_{-x_1}} \cdot U_N(y_-, y_+) \cdot e^{2i\pi y^* x_N} \cdot e^{2i\pi t_{-x_1}} \cdot U_{N-1}(t_+, t_-) \cdot e^{2i\pi \epsilon x_{N-1}}.
\]
Since $y_+ - y_- = t_+ - t_-$, it is enough to establish that $P(y, t)$ is symmetric under the exchange $t \leftrightarrow y$.

One can recast $P(y, t)$ in the product form

$$P(y \pm, t \pm) = e^{2i\pi y \cdot x_1} D_{y\pm}(p_1) \cdot D_{y\pm}(\omega_1 \omega_2 x_{12}) e^{2i\pi t \cdot x_1} D_{t\pm}(p_1)$$

$$\times \prod_{\alpha}^{2\times N-2} \left\{ D_{y\pm}(p_\alpha) \cdot D_{y\pm}(\omega_1 \omega_2 x_{\alpha a+1}) D_{t\pm}(\omega_1 \omega_2 x_{\alpha a-1}) D_{t\pm}(p_\alpha) \right\} \times A(t\pm) \left[ D_{y\pm}(p_{N-1}) \cdot D_{y\pm}(\omega_1 \omega_2 x_{N-1N}) D_{t\pm}(\omega_1 \omega_2 x_{N-2N-1}) e^{2i\pi (t\pm^{x_N} + y\pm^{x_N})} \right] .$$

(45)

At this stage, it remains to invoke the relations (266), (267) and (268) so as to conclude that $P(y \pm, t \pm)$ is symmetric in $t \leftrightarrow y$.

The exchange relation (41) for the adjoint can be established by means of the integral relations (262)–(263) for the $D_\alpha$ functions. Also, one should use the integral representation

$$\left[ e^{2i\pi t \cdot x_1} \cdot U_N(t_-, t_+) \cdot e^{2i\pi \epsilon x_1} 1_N \cdot f \right] (x'_{N})$$

$$= \int_{\mathbb{R}^N} e^{2i\pi t \cdot x_1} \cdot \prod_{a=1}^{N-1} \left\{ \tilde{A}(t_-) \cdot D_{t\pm} (\omega_1 \omega_2 (x'_a - z_a) \right\} \cdot D_{t\pm} (\omega_1 \omega_2 (x'_{a+1} - z_a)) \} \cdot e^{2i\pi \epsilon x_1} f(z'_{N-1}) \cdot d^{N-1} z$$

(46)

where $\tilde{A}(\alpha) = \sqrt{\omega_1 \omega_2} A(\alpha)$, which holds provided $f$ is the regular function. Analogously, for $g$ regular enough, one has

$$\left[ 1_N e^{-2i\pi \epsilon (y'_+)^* x_N} \cdot U_N^\dagger(y_-, y_+) \cdot e^{-2i\pi \epsilon y'_+ x_1} \cdot g \right] (x'_{N-1})$$

$$= \int_{\mathbb{R}^N} e^{-2i\pi \epsilon (y'_+)^* x_N} \cdot \prod_{a=1}^{N-1} \left\{ \tilde{A}(-y'_-) \cdot D_{(-y'_-)\dagger} (\omega_1 \omega_2 (x'_a - z_a) \right\} \cdot D_{-y'_-} (\omega_1 \omega_2 (z'_{a+1} - x'_a)) \} \cdot e^{-2i\pi y'_+ x_1} g(z'_{N-1}) \cdot d^{N-1} z .$$

(47)

Above, we denote by $x_k$ the $k$-dimensional vector $x_k = (x_1, \ldots, x_k)$. The claim then follows after a longish but straightforward calculation based on the integral identities (262)–(263).

3.2. Further Properties of the $\Lambda$ Operators

**Proposition 3.3.** It holds

$$B_N(y) \cdot \Lambda^{(N)}_{y, -} = \left\{ b(y) \right\}^N \cdot \Lambda^{(N)}_{y + i\omega_2, -} \quad \text{with} \quad b(\lambda) = -2i \sinh \left[ \frac{\pi}{\omega_1} \left( \lambda + \kappa + i \frac{\Omega}{2} \right) \right]$$

and
\[ C_N(y) \cdot \Lambda_{y, -}^{(N)} = \left\{ c(y) \right\}^N \cdot \Lambda_{y - i\omega_2, -} \quad \text{with} \quad c(\lambda) = -2i \sinh \left[ \frac{\pi}{\omega_1} (\lambda - \kappa - i\frac{\Omega}{2}) \right]. \] (49)

Furthermore, dual relations hold for the dual objects.

One possible way to establish the above relations is based on the use of the gauge transformation initially suggested in the paper [12] for the derivation of the Baxter $T - Q$ equation in the case of the Toda chain and adapted to the case of the XXX-spin chain in [8] and later used in [1, 9–11]. Here, however, we shall present a proof which adapts, to the situation of interest, the reasoning introduced in [7]. Although we do not provide these here, analogous relations can be obtained, within the same technique, for the action of the $A_N, D_N$ operators on the $\Lambda_{y, +}^{(N)}$ operator.

Proof. The intertwining relation (16) may be recast in the form
\[ R_{12}(u_1, u_2 \mid v_2) L_1(u_1, u_2) = L_1(u_1, v_2) L_2(u_1, u_2) R_{12}(u_1, u_2 \mid v_2) \cdot L_2^{-1}(u_1, v_2). \] (50)

Then, by using the elementary decomposition of the L matrices (5), one gets the identity
\[ R_{12}(u_1, u_2 \mid v_2) L_1(u_1, u_2) = -i M_{v_2}(x_1) \cdot G \cdot M_{v_2}^{-1}(x_2) \] (51)

with
\[ G = H(p_1) \cdot N_{u_1}(x_1) \cdot M_{u_2}(x_2) \cdot H(p_2) \cdot R_{12}(u_1, u_2 \mid v_2) \cdot H^{-1}(p_2). \] (52)

Upon multiplication by the gauge matrices
\[ Z_k = \begin{pmatrix} e^{2\pi \omega_2 x_k} & 0 \\ 0 & 1 \end{pmatrix}, \] (53)

the above relation takes the form
\[ Z_1 \cdot R_{12}(u_1, u_2 \mid v_2) L_1(u_1, u_2) \cdot Z_2^{-1} = \begin{pmatrix} \tilde{G}_{11} & \tilde{G}_{12} \cdot [V_2 - V_2^{-1}]^{-1} \\ [V_2 - V_2^{-1}] \cdot \tilde{G}_{21} & \tilde{G}_{22} \end{pmatrix}, \] (54)

where $V_2$ is defined analogously to (7), viz. $V_2 = e^{2\pi \omega_2 v_2}$, and we have set
\[ \tilde{G} = \frac{-i}{V_2 + V_2^{-1}} \begin{pmatrix} V_2 & -V_2^{-1} \\ -V_2^{-1} e^{2\pi \omega_2 x_1} & V_2 e^{2\pi \omega_2 x_2} \end{pmatrix} \cdot G \cdot \begin{pmatrix} 1 & -V_2^{-1} e^{-2\pi \omega_2 x_2} \\ -1 & V_2 e^{-2\pi \omega_2 x_2} \end{pmatrix}. \] (55)

The matrix elements of $\tilde{G}$ are smooth functions in $V_2$ belonging to a vicinity of 1. Furthermore, the lhs of (54) has a well-defined limit when $v_2 \to 0$. Thus, the 12-entry of the rhs of (54) admits a $V_2 \to 1$ limit. Furthermore, the 21-entry vanishes in this limit. Hence, one obtains
\[ Z_1 \cdot R_{12}(u_1, u_2 \mid 0) L_1(u_1, u_2) \cdot Z_2^{-1} = \left[ \begin{pmatrix} \tilde{G}_{11} & \tilde{G}_{12}^* \\ 0 & \tilde{G}_{22} \end{pmatrix} \right]_{v_2=0}. \] (56)

In order to compute the entries $\tilde{G}_{11}$ and $\tilde{G}_{22}$, it appears more convenient to slightly reorganise their expressions.
Substituting the second expression for $R_{12}$ given in (15) and using its independence on $p_2$, one moves the operators $D_{\alpha}(p_1)$ to the almost extreme left and right sides of (56), what can be achieved by using the intertwining relation (10). Then, simplifying factors issuing from the adjoint action of $L_2$ operators from its decomposition (5) one gets

$$Z_1 \cdot R_{12}(u_1, u_2 | 0) L_1(u_1, u_2) \cdot Z_2^{-1} = Z_1 \cdot D_{u_1 - v_2}(p_1) \cdot L_1(v_2, u_1) \cdot M_{u_2}(x_2) \cdot D_{u_2 - v_2} \left( \omega_1 \omega_2 x_{12} - i \frac{\omega_2}{2} \sigma_3 \right) \times M_{v_2}^{-1}(x_2) \cdot Z_2^{-1} \cdot D_{u_2 - v_1}(p_1).$$

(57)

In order to read out from that the expression for $\tilde{G}_{11}$ one should contract the rhs with the co-vectors $v_+^t$ and vectors $v_+$ which were introduced in (20). For doing so, it is useful to remark that

$$v_+^t Z_1 \cdot D_{u_1 - v_2}(p_1) \cdot L_1(v_2, u_1) = D_{u_1 - v_2}(p_1) \left( U_1, -U_1^{-1} \right)$$

and

$$M_{v_2}^{-1}(x_2) \cdot Z_2^{-1} v_+ = \frac{1}{V_2 + V_2^{-1}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$  

(58)

There, $U_1$ is as defined in (7). Also, one has

$$N_0(x_1) \cdot M_{u_2}(x_2) \cdot D_{u_2 - v_2} \left( \omega_1 \omega_2 x_{12} - i \frac{\omega_2}{2} \sigma_3 \right) \begin{pmatrix} 1 \\ -1 \end{pmatrix} = - \begin{pmatrix} m \\ m \end{pmatrix}$$

with

$$m = \left( U_2 + U_2^{-1} e^{2\pi \omega_2 x_{21}} \right) \cdot D_{u_2} \left( \omega_1 \omega_2 x_{12} - i \frac{\omega_2}{2} \right) + \left( U_2^{-1} + U_2 e^{2\pi \omega_2 x_{21}} \right)$$

$$\times D_{u_2} \left( \omega_1 \omega_2 x_{12} + i \frac{\omega_2}{2} \right).$$

(60)

(61)

By using the transmutation properties of $D_\alpha$ functions (253), one recasts $m$ in the form

$$m = 2 e^{\pi \omega_2 x_{21}} \cdot D_{u_2 + i \frac{\omega_2}{2}} \left( \omega_1 \omega_2 x_{12} \right).$$

(62)

All of this leads to

$$\tilde{G}_{11} |_{v_2 = 0} = iD_{u_1}(p_1) \left( U_1, -U_1^{-1} \right) e^{-\frac{\pi}{\omega_1} (p_1 - i \frac{\omega_1}{2}) \sigma_3} \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{\pi \omega_2 x_{21}}$$

$$\times D_{u_2 + i \frac{\omega_2}{2}} \left( \omega_1 \omega_2 x_{12} \right) \cdot D_{u_2 - v_1}(p_1).$$

(63)

By writing the scalar products explicitly and then using the transmutation properties (253), one arrives to

$$\tilde{G}_{11} |_{v_2 = 0} = -e^{-\pi \omega_2 x_1} R_{12} \left( u_1 + i \frac{\omega_2}{2}, u_2 + i \frac{\omega_2}{2} | 0 \right) \cdot e^{\pi \omega_2 x_2}.$$  

(64)

Quite similarly, one obtains

$$\tilde{G}_{22} |_{v_2 = 0} = 4 \sinh \left[ \frac{2\pi}{\omega_1} u_1 \right] \sinh \left[ \frac{2\pi}{\omega_1} u_2 \right] \cdot e^{\pi \omega_2 x_1} R_{12} \left( u_1 - i \frac{\omega_2}{2}, u_2 - i \frac{\omega_2}{2} | 0 \right) \cdot e^{-\pi \omega_2 x_2}.$$  

(65)
The triangular structure in (56) allows one to explicitly evaluate the products and leads to the relation

\[ Z_1 \left( R_{12} \cdots R_{N0} \right) (u_1, u_2 | v_2) T_N(u_1, u_2) Z_0^{-1} = e^{-\pi \omega_2 x_1 \sigma_3} \left( \begin{array}{c} C_1^* \\ \text{C}_1 \end{array} \right) e^{\pi \omega_2 x_0 \sigma_3}. \]  

(66)

There, \( T_N(u_1, u_2) = L_1(u_1, u_2) \cdots L_N(u_1, u_2) \) is the monodromy matrix and we denote

\[ C_\uparrow = (-1)^N \left( R_{12} \cdots R_{N0} \right) \left( u_1 + i \frac{\omega_2}{2}, u_2 + i \frac{\omega_2}{2} | 0 \right) \]

(67)

and

\[ C_\downarrow = \left\{ 4 \sinh \left[ \frac{2 \pi}{\omega_1} u \right] \sinh \left[ \frac{2 \pi}{\omega_1} u_2 \right] \right\}^N \left( R_{12} \cdots R_{N0} \right) \left( u_1 - i \frac{\omega_2}{2}, u_2 - i \frac{\omega_2}{2} | 0 \right). \]

(68)

In order to deduce from (66) the action of the \( B_N \) operator, one needs to exchange the position of the monodromy matrix and the string of partial \( R \)-operators. The intertwining relation (16) leads to

\[ \left( R_{12} \cdots R_{N0} \right) (u_1, u_2 | v_2) L_1(u_1, u_2) \cdots L_N(u_1, u_2) L_0(u_1, v_2) \]

\[ = L_1(u_1, v_2) L_2(u_1, u_2) \cdots L_0(u_1, u_2) \left( R_{12} \cdots R_{N0} \right) (u_1, u_2 | v_2). \]

(69)

Then, upon using the relation (18) one obtains

\[ e^{-2i \pi \epsilon (v_2 - u_2) x_1} \cdot \left( R_{12} \cdots R_{N0} \right) (u_1, u_2 | v_2) \cdot v^t \cdot T_N(u_1, u_2) L_0(u_1, v_2) \]

\[ = v^t \cdot T_N(u_1, u_2) L_0(u_1, u_2) e^{-2i \pi \epsilon (v_2 - u_2) x_1} \left( R_{12} \cdots R_{N0} \right) (u_1, u_2 | v_2). \]

(70)

Finally, upon using the independence on \( p_0 \) of \( R_{N0} \) and inserting the explicit expression for the \( L_0 \) operators, one arrives to

\[ e^{-2i \pi \epsilon (v_2 - u_2) x_1} \cdot \left( R_{12} \cdots R_{N0} \right) (u_1, u_2 | v_2) \cdot v^t \cdot T_N(u_1, u_2) \]

\[ = v^t \cdot T_N(u_1, u_2) e^{-2i \pi \epsilon (v_2 - u_2) x_1} \left( R_{12} \cdots R_{N-1} \right) (u_1, u_2 | v_2) \]

\[ \times D_{u_1 - v_2} (p_N) \cdot M_{u_2}(x_0) \cdot D_{u_2 - v_2} \left( \omega_1 \omega_2 x_{N0} - i \frac{\omega_2}{2} \sigma_3 \right) \]

\[ \times M^{-1}_{u_2}(x_0) \cdot D_{u_2 - u_1} (p_N). \]

(71)

This last identity implies that one may recast (66) in the form

\[ v^t \cdot T_N(u_1, u_2) e^{2i \pi u_2 x_1} \left( R_{12} \cdots R_{N-1} \right) (u_1, u_2 | 0) \]

\[ = e^{i \pi (2u_2 + i \omega_2) x_1} \left( C_\uparrow, \right) e^{\pi \omega_2 x_0 \sigma_3} \]

\[ \times D_{u_1 - u_2} (p_N) \cdot Z_0 \cdot M_{0}(x_0) \cdot D_{-u_2} \left( \omega_1 \omega_2 x_{N0} - i \frac{\omega_2}{2} \sigma_3 \right) \cdot M^{-1}_{u_2}(x_0) \]

\[ \times D_{-u_1} (p_N). \]

(72)

Evaluating this expression on \( v_- \), acting with it on the function 1 in the space 0 and using that
\[
\lim_{x_0 \to +\infty} \left\{ e^{-\pi \omega_2 x_0} \cdot D_{u_2+i \frac{\omega_2}{2}} \left( \omega_1 \omega_2 x_{N0} \right) \cdot D_{-u_2} \left( \omega_1 \omega_2 x_{N0} - i \frac{\omega_2}{2} \sigma_3 \right) \right\} = e^{-\pi \omega_2 x_N} U_2^{-\sigma_3},
\]
leads to
\[
B_N(y - i \omega_1) \cdot \gamma_N(u_1, u_2) = (-1)^{N+1} \gamma_N \left( u_1 + i \frac{\omega_2}{2}, u_2 + i \frac{\omega_2}{2} \right) \times D_{u_1+i \frac{\omega_2}{2}} (p_N) e^{-\pi \omega_2 x_N} D_{-u_1} (p_N),
\]
where we used that the condition \( v_2 = 0 \) is equivalent to \( \lambda = y - i \omega_1 \) and we have set
\[
\gamma_N(u_1, u_2) = e^{2i \pi u_2 x_1} \left( R_{12} \cdots R_{N-1} \right) (u_1, u_2 | 0).
\]
Finally, acting on the function \( e^{-2i \pi y_+ x_N} \) on the \( N \)th space produces the \( \Lambda \)-operator on the \( \text{lhs} \) and one gets, by recombining the factors,
\[
B_N(y - i \omega_1) \cdot \Lambda_{y_-, -}^{(N)} = (-1)^{N+1} \left( \frac{\mathcal{A}( (y + i \omega_2)_+) }{\mathcal{A}(y_+) } \right)^{N-1}
\]
\[
D_{u_1+i \frac{\omega_2}{2}} \left( y^*_+ - i \frac{\omega_2}{2} \right) D_{-u_1} (y^*_+ \cdot \Lambda_{y_+ + i \omega_2, -}^{(N)}
\]
Upon observing that one has \( u_1 = y_+ \), with \( y_+ \) as in (21), a straightforward calculation leads to
\[
B_N(y - i \omega_1) \cdot \Lambda_{y_-, -}^{(N)} = (-1)^{N-1} b(y) \Lambda_{y_- + i \omega_2, -}^{(N)}
\]

hence yielding the claim upon observing that \( B_N(y - i \omega_1) = (-1)^{N-1} B_N(y) \).

For the statement of the next result, it appears convenient to keep track of the \( \kappa \) dependence of the operator \( \Lambda_{y_-, -}^{(N)} \) and denote it as \( \Lambda_{y_-, \kappa, -}^{(N)} \).

**Lemma 3.2.** One has the exchange relations
\[
J_{N+1} (y_--y_+) \cdot \Lambda_{y_-, \kappa, -}^{(N)} = D_N^\kappa(y) \cdot \Lambda_{y_-, \kappa, -}^{(N)} \cdot \mathcal{W}_N(y_--y_+) \cdot J_N(y_--y_+) \quad (78)
\]
where we have introduced
\[
J_{N+1}(\alpha) = \prod_{a=1}^N D_\alpha(p_a) \quad \text{and} \quad \mathcal{W}_{N+1}(\alpha) = J_{N+1}(\alpha) e^{2i \pi \alpha x_N}
\]
\[
\prod_{a=1}^N D_\alpha \left( \omega_1 \omega_2 x_{a} + 1 \right) \cdot e^{2i \pi \alpha x_1}.
\]

Also, it holds
\[
\mathcal{W}_{N+1}(y_--y_+) \cdot \Lambda_{y_-, \kappa, -}^{(N)} = D_N^\kappa(y) \cdot \Lambda_{y_-, \kappa, -}^{(N)} \cdot \mathcal{W}_N(y_--y_+) \quad (80)
\]

Note that, in the above definition, it is understood that
\[
\mathcal{W}_1(\alpha) = J_1(\alpha) = D_\alpha(p_1) \quad (81)
\]

**Proof.** This is a direct consequence of a multiple application of the star-triangle relation and of the interchange relation of \( D \)-s and exponentials (262). \( \square \)
3.3. Elementary Properties of a Basic Complete Orthogonal System

In this subsection, we discuss the complete orthogonal system of generalised eigenfunctions of the one-site operator $B_1(\lambda)$ and establish its properties under specific action of $\Lambda_{y,-}^{(2)}$ operators.

**Lemma 3.3.** The functions

$$\phi_y(x) = \int_{\mathbb{R}} \varpi(\kappa + t)e^{2i\pi t(x+y)} \cdot dt$$

(82)

are self-dual under the exchange $\omega_1 \leftrightarrow \omega_2$ and satisfy to

$$B_1(\lambda) \cdot \phi_y(x) = e^{2\pi\omega_2 y} \phi_y(x).$$

(83)

The family $\{\phi_y(x)\}$ forms a complete orthogonal system on $L^2(\mathbb{R})$, viz.

$$\int_{\mathbb{R}} \phi_y^*(x) \cdot \phi_y(x) \cdot dx = \delta(y'-y) \quad \text{and} \quad \int_{\mathbb{R}} \phi_y^*(x') \cdot \phi_y(x) \cdot dy = \delta(x'-x).$$

(84)

**Proof.** The generalised spectral problem for the operator $B_1(\lambda)$ associated with the generalised eigenvalue $e^{2\pi\omega_2 y}$ can be recast in the form of the below finite-difference equation

$$2i \sinh \left[ \frac{\pi}{\omega_1} \left( \kappa + \frac{p + i \tau}{2} \right) \right] \cdot \phi_y(x) = e^{2\pi\omega_2 (x+y)} \phi_y(x).$$

(85)

One possible solution is given by $\phi_y(x) = \phi_0(x + y)$. Thus, we focus on $\phi_0$. Passing to the Fourier space, one gets the below finite-difference equation satisfied by the Fourier transform

$$\mathcal{F}[\phi_0](t + i\omega_2) = 2i \sinh \left[ \frac{\pi}{\omega_1} \left( \kappa + t + \frac{i \tau}{2} \right) \right] \mathcal{F}[\phi_0](t).$$

(86)

This equation, along with its dual, is easily solved in terms of the dilogarithm leading, all-in-all, to the claim. *c.f.* (242). The completeness and orthogonality follows from straightforward handlings. In particular, these entail that each generalised eigenvalue of $B_1(\lambda)$ appears with multiplicity one. $\square$

The relation (84) can be rephrased in the form of the unitary of a map $L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})$. More precisely, consider the transform

$$\mathcal{U}_1^{(+)}[f](y) = \int_{\mathbb{R}} dx \phi_y(x) f(x)$$

(87)

first defined for $f \in C_0^{\infty}(\mathbb{R})$ and then extended by continuity to the whole of $L^2(\mathbb{R})$. The relation (84) ensures that $\mathcal{U}_1^{(+)} : L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})$ is unitary.

**Lemma 3.4.** For $y, t, t_0 \in \mathbb{R}$, it holds

$$\left( \Lambda_{y,+}^{(2)} \right)^{\dagger} \cdot \Lambda_{t,+}^{(2)} \cdot \phi_{t_0} = \delta(y - t) \frac{e^{-2\pi\Omega t_0}}{(\omega_1\omega_2)^{\frac{1}{2}}} \phi_{t_0}.$$ 

(88)
Proof. One has the representation
\[
\left( \Lambda_{y,+}^{(2)} \right)^{\dagger} \cdot \Lambda_{t,+}^{(2)} = \frac{1}{\omega_1 \omega_2} \mathcal{A} \left( \frac{-y_+^*}{t_+} \right)
\times \text{tr}_2 \left[ e^{-2i \pi \left( y_+^* \right)^* x_2} D_{-y_+^*} (\omega_1 \omega_2 x_{12}) D_{-y_-} \left( p_1 \right) \cdot e^{2i \pi \left( t_- - y_-^* \right) x_1} \right.
\times \left. D_{t_-} \left( p_1 \right) \cdot D_{t_+} (\omega_1 \omega_2 x_{12}) e^{2i \pi t_+^* x_2} \right] \tag{89}
\]
where the trace runs through the second quantum space. Here, we remind that we use the hypergeometric-like notation (240). In virtue of (265), the expression can be recast as
\[
\left( \Lambda_{y,+}^{(2)} \right)^{\dagger} \cdot \Lambda_{t,+}^{(2)} \cdot \mathcal{A} \left( \frac{t_+}{-y_+^*} \right) \cdot \omega_1 \omega_2
= \text{tr}_2 \left[ e^{2i \pi \left[ t_- x_1 - \left( y_+^* \right)^* x_2 \right]} D_{t_+} \left( p_1 \right) D_{t_- - y_-^*} (\omega_1 \omega_2 x_{12}) \right.
\times \left. D_{t_+} (\omega_1 \omega_2 x_{12}) e^{2i \pi \left( t_+^* x_2 - y_-^* x_1 \right)} \right] \tag{90}
\]
Then, invoking the star–triangle relation (264) one obtains
\[
\left( \Lambda_{y,+}^{(2)} \right)^{\dagger} \cdot \Lambda_{t,+}^{(2)} \cdot \mathcal{A} \left( \frac{t_+}{-y_+^*} \right) \cdot \omega_1 \omega_2
= \text{tr}_2 \left[ e^{2i \pi \left[ t_- x_1 - \left( y_+^* \right)^* x_2 \right]} D_{t_+} \left( p_1 \right) D_{t_- - y_-^*} (\omega_1 \omega_2 x_{12}) \right.
\times \left. D_{t_+} (\omega_1 \omega_2 x_{12}) e^{2i \pi \left( t_+^* x_2 - y_-^* x_1 \right)} \right]
= e^{2i \pi t_- x_1} \cdot D_{t_+} \left( p_1 \right) \cdot e^{-2i \pi \left( y_+^* \right)^* x_1}
\times \text{tr}_2 \left[ e^{-2i \pi \left[ t_+^* - \left( y_+^* \right)^* \right] x_2} D_{t_- - y_-^*} (\omega_1 \omega_2 x_{12}) \right.
\times \left. e^{2i \pi t_+^* x_2} \cdot D_{t_- - y_-^*} \left( p_1 \right) \cdot e^{-2i \pi y_-^* x_1} \right]. \tag{91}
\]
At this stage, one can already take the trace over the second space. However, for this purpose, one needs to regularise the integral appropriately. Indeed, one ends up with the below integral
\[
\text{tr}_2 \left[ e^{-2i \pi \left[ t_+^* - \left( y_+^* \right)^* \right] x_2} D_{t_- - y_-^*} (\omega_1 \omega_2 x_{12}) \right] = \lim_{\eta \to 0^+} \int_{\mathbb{R}} e^{2i \pi \left[ t_+^* - \left( y_+^* \right)^* + \frac{\eta}{2} \right] x} D_{t_- - y_-^* + \frac{i \eta}{2}} (\omega_1 \omega_2 x) \cdot dx. \tag{92}
\]
The above regularisation is such that \( f_{\eta} \in L^1 (\mathbb{R}) \). Indeed, one has
\[
\left\{ \begin{array}{l}
t_+^* - \left( y_+^* \right)^* = \frac{1}{2} (y - t) - i \frac{\eta}{2} \quad \text{viz.} \quad \left[ t_- - y_-^* + \frac{i \eta}{2} \right]^* = \frac{y - t}{2} - i \frac{\eta}{2} \\
t_- - y_-^* = \frac{1}{2} (t - y) - i \frac{\eta}{2}
\end{array} \right. \quad (93)
\]
We are now in position to present a set of generalised eigenfunctions asso-
ciated with the operators $A_N(\lambda)$ and $B_N(\lambda)$. Then, we establish that these
eigenfunctions form, in each case, an orthogonal system.

Finally representing $\phi_{t_0}$ as

$$\phi_{t_0}(x) = \int_{\mathbb{R}} \varpi(\kappa + (y + i\Omega)) e^{2i\pi(y + i\Omega)(x + t_0)} \, dy$$

and moving the action of $(\Lambda_{y, +}^{(2)})^{\dagger} \cdot \Lambda_{t, +}^{(2)}$ under the integral sign one gets the
claim.

3.4. The Orthogonal Sets of Eigenfunctions of $A_N(\lambda)$, $B_N(\lambda)$

We are now in position to present a set of generalised eigenfunctions asso-
ciated with the operators $A_N(\lambda)$ and $B_N(\lambda)$. Then, we establish that these
eigenfunctions form, in each case, an orthogonal system.

Proposition 3.4. The functions

$$\varphi_{y_N}^{(-)}(x_N) = \left( \Lambda_{y_N, -}^{(N)} \cdots \Lambda_{y_1, -}^{(1)} \right)(x_N)$$

(99)
and
\[
\varphi^{(+)}_{y_0,y_{N-1}}(x_N) = e^{(N-1)\pi \Omega y_0} \cdot (\omega_1 \omega_2)^{\frac{3}{2}(N-1)} \cdot (\Lambda^{(N)}_{y_{N-1},+} \cdots \Lambda^{(2)}_{y_{1},+} \cdot \varphi_{y_0})(x_N)
\]  
(100)

are, respectively, symmetric functions of \(y_N\) and \(y_{N-1}\) and satisfy to

\[
A_N(\lambda) \cdot \varphi_{y_N}^{(-)}(x_N) = \prod_{a=1}^{N} \left\{ -2i \sinh \left[ \frac{\pi}{\omega_1} (\lambda - y_a) \right] \right\} \cdot \varphi_{y_N}^{(-)}(x_N)
\]

and

\[
B_N(\lambda) \cdot \varphi^{(+)}_{y_0,y_{N-1}}(x_N) = e^{2\pi \omega_2 y_0} \prod_{a=1}^{N-1} \left\{ -2i \sinh \left[ \frac{\pi}{\omega_1} (\lambda - y_a) \right] \right\} \cdot \varphi^{(+)}_{y_0,y_{N-1}}(x_N).
\]

(102)

**Proof.** The symmetry property in the \(y\)-variables follows from commutativity of the \(\Lambda\) operators established in Proposition 3.2. The form of the action of \(B_N\) and \(A_N\) operators is a consequence of Lemmata 3.3 and 3.1 and the fact that

\[
(\Lambda_1(\lambda) \cdot \Lambda_{y_1}^{(1)}(x_1) = -2i \sinh \left[ \frac{\pi}{\omega_1} (\lambda - y) \right] \Lambda_{y,1}^{(1)}(x_1) \quad \text{where} \quad \Lambda_{y,1}^{(1)}(x_1) = e^{2i\pi y x}.
\]

(103)

**Proposition 3.5.** The families

\[
\{ \varphi_{y_N}^{(-)}(x_N) \}_{y_N \in \mathbb{R}^N} \quad \text{and} \quad \{ \varphi_{y_0,y_{N-1}}^{(+)}(x_N) \}_{(y_0,y_{N-1}) \in \mathbb{R}^N}
\]

(104)

form an orthogonal system in \(L^2(\mathbb{R}^N, d^N x)\), viz.

\[
\int_{\mathbb{R}^N} \left( \varphi_{y_N}^{(-)}(x_N) \right)^* \cdot \varphi_{y_N}^{(-)}(x_N) \cdot d^N x = \frac{1}{\mu_N(y_N)} \cdot \delta_\text{sym}(y_N - y_N')
\]

(105)

and

\[
\int_{\mathbb{R}^N} \left( \varphi_{y_0,y_{N-1}}^{(+)}(x_N) \right)^* \cdot \varphi_{y_0,y_{N-1}}^{(+)}(x_N) \cdot d^N x
\]

\[
= \frac{1}{\mu_{N-1}(y_{N-1})} \cdot \delta(y_0 - y_0) \cdot \delta_\text{sym}(y_{N-1} - y_{N-1}')
\]

(106)

where

\[
\mu_N(y_N) = \prod_{a < b} \left\{ 4 \omega_1 \omega_2 \sinh \left[ \frac{\pi}{\omega_1} (y_a - y_b) \right] \cdot \sinh \left[ \frac{\pi}{\omega_2} (y_a - y_b) \right] \right\}
\]

(107)

and

\[
\delta_\text{sym}(y_N - y_N') = \sum_{\sigma \in \mathcal{S}_N} \prod_{a=1}^{N} \delta(y_a - y_{\sigma(a)}).
\]

(108)
The \( N = 1 \) specialisation of the above proposition was obtained earlier in [16,30].

Note that the orthogonality may be recast in operators terms. Let \( d\mu_N(y_N) = \mu_N(y_N)\,d^N y \) and denote by \( L^2_{\text{sym}}(\mathbb{R}^N, d\mu_N) \) the space of symmetric functions on \( \mathbb{R}^N \) square integrable with respect to \( d\mu_N(y_N) \). Finally, let \( L^2_{\times \text{sym}}(\mathbb{R} \times \mathbb{R}^{N-1}, dy_0 \otimes d\mu_{N-1}) \) denote the space of functions on \( \mathbb{R}^N \) that are symmetric with respect to the last \( N - 1 \) variables and square integrable with respect to \( dy_0 \otimes d\mu_{N-1} \).

Define the integral transform \( \mathcal{U}^{(-)}_N : L^2_{\text{sym}}(\mathbb{R}^N, d\mu_N) \rightarrow L^2(\mathbb{R}^N, d^N x) \)

\[
\mathcal{U}^{(-)}_N[f](x_N) = \frac{1}{\sqrt{N!}} \int_{\mathbb{R}^N} \varphi^{(-)}_{y_N}(x_N)f(y_N)d\mu_N(y_N)
\]  

(109) and \( \mathcal{U}^{(\pm)}_N : L^2_{\times \text{sym}}(\mathbb{R} \times \mathbb{R}^{N-1}, dy_0 \otimes d\mu_{N-1}) \rightarrow L^2(\mathbb{R}^N, d^N x) \)

\[
\mathcal{U}^{(\pm)}_N[f](x_N) = \frac{1}{\sqrt{(N-1)!}} \int_{\mathbb{R}^N} \varphi^{(\pm)}_{y_0, y_{N-1}}(x_N)f(y_0, y_{N-1})dy_0\,d\mu_N(y_{N-1}).
\]  

(110)

The transforms are first defined on smooth compactly supported functions and then extended to the whole of the base space by continuity which is ensured by the relations (105)–(106). In fact, the relations (105)–(106) are recast, in terms of \( \mathcal{U}^{(\pm)}_N \) as

\[
(\mathcal{U}^{(\pm)}_N)^\dagger \cdot \mathcal{U}^{(\pm)}_N = \text{id}.
\]  

(111)

They thus provide one with “half” of the relations necessary for \( \mathcal{U}^{(\pm)}_N \) to be unitary maps.

**Proof.** We start by discussing the family \( \varphi^{(-)}_{y_N} \) which is slightly easier to deal with. The orthogonality integral, which is to be understood in the sense of distributions, can be recast as

\[
\int_{\mathbb{R}^N} \left( \varphi^{(-)}_{x_N}(x_N) \right)^* \varphi^{(-)}_{y_N}(x_N) \,d^N x = \left( \Lambda_{t;1}^{(1)} \right)^\dagger \cdots \left( \Lambda_{t;N}^{(1)} \right)^\dagger \cdot \Lambda_{y^1;1}^{(N)} \cdots \Lambda_{y^N;1}^{(1)}.
\]  

(112)

One now moves the Hermitian conjugate operators through the string of \( \Lambda \) operators by means of the exchange relations (41). However, in order to exchange the operator \( \left( \Lambda_{t;1}^{(1)} \right)^\dagger \) with \( \Lambda_{y^i;1}^{(N)} \) one needs to ensure that \( y \neq t \). This can be achieved by first, introducing the compactly supported smooth function \( \varrho_\epsilon \) such that

\[
\varrho_\epsilon = 1 \text{ on } [-\epsilon; \epsilon] \text{ and } \varrho_\epsilon = 0 \text{ on } \mathbb{R} \setminus [-2\epsilon; 2\epsilon].
\]  

(133)

Then, one considers

\[
g(y_N, t_N) = \sum_{\sigma \in \mathcal{E}_N} \prod_{a \neq b} N \left( 1 - \varrho_\epsilon(y_a - t_{\sigma(b)}) \right)
\]  

(114)
which is uniformly away from 0 on $D_\epsilon \times D_\epsilon$ with

$$D_\epsilon = \left\{ y_N \in \mathbb{R}^N : \min_{a \neq b} |y_a - y_b| \geq 7\epsilon \right\}.$$ 

Thus, one gets a partition of the unity on $D_\epsilon \times D_\epsilon$:

$$s_\epsilon(y_N, t_N) = \frac{1}{g(y_N, t_N)} \prod_{a \neq b} (1 - \varrho_\epsilon(y_a - t_{\sigma(b)})) \text{so that} \sum_{\sigma \in \varnothing_N} s_\epsilon(y_N, t_N) = 1. \quad (115)$$

Thus, one gets on $D_\epsilon \times D_\epsilon$:

$$\left( \Lambda_{t_{1}; -}^{(1)} \right)^\dagger \cdots \left( \Lambda_{t_{N}; -}^{(N)} \right)^\dagger \cdot \Lambda_{y_{1}; -}^{(1)} \cdots \Lambda_{y_{N}; -}^{(N)}$$

$$= \sum_{\sigma \in \varnothing_N} s_{\text{id}}(y_N, t_N^{(\sigma)}) \left( \Lambda_{t_{\sigma(1)}; -}^{(1)} \right)^\dagger \cdots \left( \Lambda_{t_{\sigma(N)}; -}^{(N)} \right)^\dagger \cdot \Lambda_{y_{\sigma(1)}; -}^{(1)} \cdots \Lambda_{y_{\sigma(N)}; -}^{(N)} \quad (116)$$

where $t_N^{(\sigma)} = (t_{\sigma(1)}, \cdots, t_{\sigma(N)})$. One may already apply the exchange relation on the level of each summand (41) since the presence of $s_{\text{id}}(y_N, t_N^{(\sigma)})$ does ensure an appropriate distinctiveness of the variables, c.f. [22] for a rigorous treatment of this feature. This leads, in the $\epsilon \to 0^+$ limit, to

$$\int_{\mathbb{R}^N} \left( \varphi_{t_N}^{-}(x_N) \right)^* \cdot \varphi_{y_N}^{-}(x_N) \cdot d^N x$$

$$= \sum_{\sigma \in \varnothing_N} \prod_{a < b} \left\{ \frac{1}{\omega_1 \omega_2} A \left( (t_{\sigma(a)})_+^* - ((y_b)_-^*)^* - (t_{\sigma(a)})_- - ((y_b)_-^*) \right) \right\}$$

$$\times \prod_{a=1}^N \left( \Lambda_{t_{\sigma(a)}; -}^{(1)} \right)^\dagger \cdot \Lambda_{y_{\sigma(a)}; -}^{(1)}. \quad (117)$$

At this stage, it solely remains to observe that for real $t, y$ one has

$$A \left( t_+^* - (y_+)^*, t_- - y_+^* \right) = A \left( \frac{y - t}{2} - \frac{i \Omega}{2}, \frac{t - y}{2} - \frac{i \Omega}{2} \right)$$

$$= \left\{ 4 \sinh \left[ \frac{\pi}{\omega_1} (y - t) \right] \cdot \sinh \left[ \frac{\pi}{\omega_2} (y - t) \right] \right\}^{-1} \quad (118)$$

and also that

$$\left( \Lambda_{t_-; -}^{(1)} \right)^\dagger \cdot \Lambda_{y_-; -}^{(1)} = \delta(y - t). \quad (119)$$

In the above handlings, we have omitted all technical details that need to be discussed so as to deal rigorously with the various steps of the calculations in the sense of distributions. The appropriate theory for that has been developed in [22] and we refer to this work for all the details leading to a rigorous treatment.

In the case of $\varphi_{y_0, y_{N-1}}^{(+)}$, the same kind of operations leads to

$$\int_{\mathbb{R}^N} \left( \varphi_{t_0, t_{N-1}}^{(+)}(x_N) \right)^* \cdot \varphi_{y_0, y_{N-1}}^{(+)}(x_N) \cdot d^N x$$

$$= e^{(N-1)\pi \Omega (t_0 + y_0)} \cdot \left( \frac{\omega_1 \omega_2}{2} \right)^\frac{3}{2} (N-1)$$
In this section, we construct the Mellin–Barnes representation for a family \( \psi_{y_{N-1}} \) of generalised eigenfunctions of the operator \( A_N(\lambda) \), resp. \( B_N(\lambda) \). We then use this integral representation to show that \( \{ \psi_{y_{N-1}} \} \) forms a complete system on \( L^2(\mathbb{R}^N, d^N x) \) in the sense that

\[
\frac{1}{N!} \int_{\mathbb{R}^N} \left( \psi_{y_{N-1}}^{(-)}(x_N) \right)^* \psi_{y_{N-1}}^{(-)}(x_N) \cdot d\mu_N(y_N) = \prod_{a=1}^N \delta(x_a - x'_a),
\]

and

\[
\frac{1}{(N-1)!} \int_{\mathbb{R}^N} \left( \psi_{y_{N-1}}^{(+)}(x_N) \right)^* \psi_{y_{N-1}}^{(+)}(x_N) \cdot d\mu_{N-1}(y_{N-1}) = \prod_{a=1}^N \delta(x_a - x'_a).
\]
Proposition 4.1. Let \( y_N \in \mathbb{R}^N \). Define the functions \( \psi_{y_N}^{(-)}(x_N) \) inductively by

\[
\psi^{(-)}_{y_N}(x_N) = \int_{C_{y_N}^{N-1}} \psi^{(-)}_{w_{N-1}}(x_{N-1}) \psi^{(-)}_{y_N-w_{N-1}}(x_N) \Phi(w_{N-1} \mid y_N) \cdot \frac{d^{N-1}w}{(N-1)!}.
\]

and

\[
\psi^{(-)}_{y}(x) = e^{2i\pi xy}.
\]

Here

\[
\Phi(w_{N-1} \mid y_N) = \left( \frac{1}{\omega_1 \omega_2} \right)^{N-1} \prod_{a=1}^{N} \frac{\mathcal{C}(y_a - \kappa)}{\mathcal{C}(y_a - \kappa)} \cdot \prod_{a=1}^{N} \frac{\mathcal{C}^{N-1}(w_a + \kappa)}{\mathcal{C}(y_N - w_{N-1} - \kappa)}
\]

\[
\times \prod_{a=1}^{N} \prod_{b=1}^{N} \mathcal{C}(w_b - w_a - i \frac{\Omega}{2})
\]

and \( C_{y_N} \) is a curve corresponding to small deformation of the real axis such that all the simple poles for each \( w \)-variable at

\[
w = y_b + i\ell \omega_1 + ik \omega_2, \quad \text{with } (\ell, k) \in \mathbb{N}^2 \text{ and } b \in [1; N]
\]

are located above it.

Then, the functions \( \psi^{(-)}_{y_N} \) are well defined, self-dual under \( \omega_1 \leftrightarrow \omega_2 \) and satisfy

\[
A_N(\lambda) \cdot \psi^{(-)}_{y_N}(x_N) = a(\lambda \mid y_N) \cdot \psi^{(-)}_{y_N}(x_N) \quad \text{with}
\]

\[
a(\lambda \mid y_N) = \prod_{b=1}^{N} \left\{ -2i \sinh \left[ \frac{\pi}{\omega_1}(\lambda - y_b) \right] \right\}
\]

as well as

\[
B_N(\lambda) \cdot \psi^{(-)}_{y_N}(x_N) = \sum_{k=1}^{N} \left\{ b(y_k) \right\} \prod_{p=1}^{N} \left. \sinh \left[ \frac{\pi}{\omega_1}(\lambda - y_p) \right] \right/ \prod_{p \neq k} \left. \sinh \left[ \frac{\pi}{\omega_1}(y_k - y_p) \right] \right.
\]

\[
\times \psi^{(-)}_{y_N+i\omega_2e_k}(x_N)
\]

\[
C_N(\lambda) \cdot \psi^{(-)}_{y_N}(x_N) = \sum_{k=1}^{N} \left\{ c(y_k) \right\} \prod_{p=1}^{N} \left. \sinh \left[ \frac{\pi}{\omega_1}(\lambda - y_p) \right] \right/ \prod_{p \neq k} \left. \sinh \left[ \frac{\pi}{\omega_1}(y_k - y_p) \right] \right.
\]

\[
\times \psi^{(-)}_{y_N-i\omega_2e_k}(x_N)
\]

where

\[
b(\lambda) = -2i \sinh \left[ \frac{\pi}{\omega_1}(\lambda + \kappa + i \frac{\Omega}{2}) \right] \quad \text{and} \quad c(\lambda) = 2i \sinh \left[ \frac{\pi}{\omega_1}(\lambda - \kappa - i \frac{\Omega}{2}) \right].
\]

Further, given \( (y_0, y_{N-1}) \in \mathbb{R}^N \), the functions \( \psi^{(+)}_{y_0, y_{N-1}}(x_N) \) defined as
\[ \Psi(w_N | y_0, y_{N-1}) = \left( \frac{1}{\sqrt{\omega_1 \omega_2}} \right)^{N-1} \cdot e^{2i\pi y_0 w_N} \omega(w_N + \kappa) \cdot \prod_{a=1}^{N-1} \frac{\omega(y_a - \kappa)}{\omega^{N-1}(y_a + \kappa)} \times \prod_{a=1}^{N-1} \frac{\omega(w_a + w_N + i \frac{\Omega}{2}) \cdot \prod_{b=1}^{N-1} \omega(y_b - w_a - i \frac{\Omega}{2})}{\prod_{a \neq b} \omega(w_b - w_a - i \frac{\Omega}{2})}, \]  

in which

\[ \Psi(w_N | y_0, y_{N-1}) = \left( \frac{1}{\sqrt{\omega_1 \omega_2}} \right)^{N-1} \cdot e^{2i\pi y_0 w_N} \omega(w_N + \kappa) \cdot \prod_{a=1}^{N-1} \frac{\omega(y_a - \kappa)}{\omega^{N-1}(y_a + \kappa)} \times \prod_{a=1}^{N-1} \frac{\omega(w_a + w_N + i \frac{\Omega}{2}) \cdot \prod_{b=1}^{N-1} \omega(y_b - w_a - i \frac{\Omega}{2})}{\prod_{a \neq b} \omega(w_b - w_a - i \frac{\Omega}{2})}, \]  

satisfy to

\[ B_N(\lambda) \cdot \psi_{y_0; y_{N-1}}^{(+)}(x_N) = e^{2\pi \omega_2 y_0} a(\lambda | y_{N-1}) \cdot \psi_{y_0; y_{N-1}}^{(+)}(x_N) \]  

and

\[ A_N(\lambda) \cdot \psi_{y_0; y_{N-1}}^{(+)}(x_N) = -ia(\lambda | y_{N-1}) \cdot \left\{ e^{\frac{\pi}{\omega_1}(\lambda + \gamma_{N-1})} \psi_{y_0 + \frac{\Omega}{2\omega_1} \cdot y_{N-1}}^{(+)}(x_N) \right. \]

\[ -6^{-\frac{\pi}{\omega_1}(\lambda + \gamma_{N-1})} \psi_{y_0 - \frac{\Omega}{2\omega_1} \cdot y_{N-1}}^{(+)}(x_N) \]

\[ + \sum_{k=1}^{N-1} \left\{ b(y_k) \right\} \prod_{p=1}^{N-1} \sinh \left[ \frac{\pi}{\omega_1} (\lambda - y_p) \right] \cdot \psi_{y_0, y_{N-1} + i \omega_2 e_k}(x_N) \]  

(135)

**Proof.** The proof goes by induction on \( N \). One builds \( \psi_{y_N}^{(-)}(x_N) \) as in (124) and assumes that the properties (128), (129) and (130) hold up to \( N-1 \). Then, decomposing the monodromy matrix \( T_N(\lambda) = T_{N-1}(\lambda) L_N^{(\kappa)}(\lambda) \) allows one to compute the action of the operator \( A_N, B_N \) and \( C_N \) on \( \psi_{y_N}^{(-)}(x_N) \). The system of first-order difference equations satisfied by \( \Phi(w_{N-1} | y_N) \) then allows one to conclude that, indeed, the claimed form of the actions holds. See [29] or, in a closer setting to the present one, [18,20]. Self-duality of \( \psi_{y_N}^{(-)}(x_N) \) is manifest. The convergence of the integral can be established along the lines discussed in [13] for the analogous representation arising in the case of the quantum Toda chain.

\[ \Box \]

### 4.2. An Auxiliary Integral

Prior to proving completeness, as already discussed in Introduction, we need to establish an auxiliary integral representation for the symmetric \( \delta \)-function in many variables. This identity appears as a central tool allowing to prove, in a simple and direct manner, the completeness of a system of functions defined through a Mellin–Barnes-like representation. The technique for establishing this identity that we develop below thus emerges as an important tool.
for studying quantum integrable models with non-compact local spaces and solvable by the quantum separation of variables method.

Given \( \mathbf{v}_{N-1}, \mathbf{w}_{N-1} \in \mathbb{R}^{N-1} \) and \( \varepsilon_N \in \mathbb{R}^N \) such that \( \varepsilon_a > 0 \) for all \( a \) and is small enough. Define \( \mathcal{C} \) to be a curve obtained from \( \mathbb{R} \) by a small deformation of \( \mathbb{R} \) such that, for any \( a = 1, \ldots, N-1 \) and \( b = 1, \ldots, N \):

- the lattice \( v_a - i \varepsilon_b + i \omega_1 N + i \omega_2 N \) lies entirely above \( \mathcal{C} \);
- the lattice \( w_a + i \varepsilon_b - i \omega_1 N - i \omega_2 N \) lies entirely below of \( \mathcal{C} \).

Such a curve always exists provided that \( ||\varepsilon_N|| \) is small enough and the \( \varepsilon_a \)'s are generic.

A very useful object occurring in the proof of completeness is given by the below family of integrals parameterised by \( \varepsilon_N \) as above:

\[
\mathcal{J}_{\varepsilon_N}^{(m)}(x \mid \mathbf{w}_{N-1}, \mathbf{v}_{N-1}) = \int_\mathcal{C} d^m y \int_{\mathbb{R}^{N-m}} \prod_{a=m+1}^N dy_a \cdot \mathcal{I}_{\varepsilon_N}^{(m)}(\varepsilon_N \mid x, y_N; \mathbf{w}_{N-1}, \mathbf{v}_{N-1}),
\]

with

\[
\mathcal{I}_{\varepsilon_N}^{(m)}(\varepsilon_N \mid x, y_N; \mathbf{w}_{N-1}, \mathbf{v}_{N-1}) = \frac{e^{2i \pi N x} \prod_{a=1}^{N-1} \prod_{b=1}^N \varpi(y_b - w_a - i \Omega \frac{2}{N} + i \zeta^{(m)}_b, v_a - y_b - i \Omega \frac{2}{N} + i \zeta^{(m)}_b)}{N! (\omega_1 \omega_2)^{N-1} \prod_{a < b} \varpi(y_{ba} - i \Omega \frac{1}{N} + i \zeta^{(m)}_a + i \zeta^{(m)}_b, y_{ab} - i \Omega \frac{1}{N} - i \zeta^{(m)}_b + i \zeta^{(m)}_a)}.
\]

(137)

The sequence \( \zeta^{(m)}_a \) is defined as

\[
\zeta^{(m)}_a = -\varepsilon_a, \quad a = 1, \ldots, m \quad \text{and} \quad \zeta^{(m)}_a = (\tilde{\varepsilon}^{(m)}_N)_a = \varepsilon_{N+m+1-a}, \quad a = m+1, \ldots, N.
\]

(138)

Also, we remind that \( y_{ab} = y_a - y_b \) and we used hypergeometric-like notation for products of \( \varpi \) factors, c.f. (240).

**Proposition 4.2.** For any \( N \geq 1 \) and \( m \in \{0; N\} \), it holds, in the sense of distributions,

\[
\lim_{\varepsilon_N \to 0^+} \left\{ \mathcal{J}_{\varepsilon_N}^{(m)}(x \mid \mathbf{w}_{N-1}, \mathbf{v}_{N-1}) \right\} = \prod_{a \neq b} \varpi(w_a - w_b - i \Omega \frac{2}{N}) \cdot \delta(x) \times \delta_{\text{sym}}(\mathbf{w}_{N-1} - \mathbf{v}_{N-1}) \cdot \begin{cases} 1 & m = 0 \\ \frac{1}{N} & m = 1 \\ 0 & m \in \{2; N\} \end{cases}.
\]

(139)

**Proof.** The result is established by a triangular induction.

- **Initialisation**
  It is obvious that

\[
\lim_{\varepsilon_1 \to 0^+} \left\{ \mathcal{J}_{\varepsilon_1}^{(1)}(x \mid \emptyset, \emptyset) \right\} = \delta(x), \quad \text{for } a \in \{0, 1\}.
\]

(140)
Next, assume that the claim holds up to $N - 1$ and for any $m \in \llbracket 0 ; N - 1 \rrbracket$.

- **Initialisation step at $N$: the integral $\mathcal{I}^{(N);N}_{\varepsilon}$**

First of all, observe that the appropriate choice of signs of the regulators $\zeta_a^{(N)}$ in (137)–(138) ensures that the integrand

$$I^{(N)} \left( \varepsilon_N \mid x, y_N ; w_{N-1}, v_{N-1} \right)$$

(141)

decays as a power law in each of the variables $y_a$ along the lines $\mathbb{R} \pm i r$, with $r > 0$. This property allows one to compute the integral by taking residues in the upper or lower complex plane—relatively to any variables—and conclude that, in the sense of distributions, the contours at $\infty$ do not contribute to the integral. Thus, in the sense of distributions, the integral can be taken by computing the residues of all the poles lying above $\mathcal{C}$ -if $x \geq 0$- or below $\mathcal{C}$ -if $x \leq 0$. Here, we only discuss the case when $x \geq 0$. The integrand has simple poles in each variables $y_a$ at

$$y_a = v_b - i \varepsilon_a + i \ell_a \omega_1 + i k_a \omega_2 \quad \text{with} \quad \ell_a, k_a \in \mathbb{N} \quad \text{and} \quad b \in \llbracket 1 ; N \rrbracket .$$

(142)

Thus, one gets that, in the sense of distributions,

$$\mathcal{I}^{(N);N}_{\varepsilon} \left( x \mid w_{N-1}, v_{N-1} \right)$$

$$= \sum_{\alpha \in \mathcal{M}_N} \sum_{\ell_N, k_N \in \mathbb{N}^N} \text{Res} \left( I^{(N)} \left( \varepsilon_N \mid x, y_N ; w_{N-1}, v_{N-1} \right) \cdot d^N y, \right.$$

$$\left. y_N = v^{(\alpha)} + i \ell_N \omega_1 + i k_N \omega_2 - i \varepsilon_N \right) .$$

(143)

The sums in (143) run through all maps $\alpha \in \mathcal{M}_N$, where $\mathcal{M}_N = \left\{ \alpha : \llbracket 1 ; N \rrbracket \rightarrow \llbracket 1 ; N-1 \rrbracket \right\}$. Furthermore, given a map $\alpha : \llbracket 1 ; N \rrbracket \rightarrow \llbracket 1 ; N-1 \rrbracket$, we denote

$$v^{(\alpha)} \in \mathbb{R}^N \quad \text{the vector given by} \quad \left( v^{(\alpha)} \right)_a = v_{\alpha(a)} \quad \text{for} \quad a = 1, \ldots, N .$$

(144)

Once the residues are computed, one may send $\varepsilon_a \to 0^+$ for any $a = 1, \ldots, N$. The expression for the residues of the dilogarithm (247) leads to

$$\text{Res} \left( I^{(N)} \left( 0 \mid x, y_N ; w_{N-1}, v_{N-1} \right) \cdot d^N y, \right.$$

$$\left. y_N = v^{(\alpha)} + i \ell_N \omega_1 + i k_N \omega_2 \right) = P^{(\alpha)}_{\ell_N, k_N} \cdot G^{(\alpha)}_{\ell_N, k_N} \cdot \gamma^{(\alpha)}_{\ell_N, k_N} ,$$

(145)

in which
\[ \gamma_{\ell_n,k_N}^{(\alpha)} = \frac{2^{N(N-1)} N! \cdot (\omega_1 \omega_2)^{N-1}}{\prod_{a=1}^{N} \prod_{b=1}^{N-1} \prod_{c \neq a,b} \prod_{d \neq a,b,c} (-1)^{k_a \ell_a} (2i)^{\ell_a + k_a} \prod_{p=1}^{k_a} \sinh \left[ i p \pi \frac{\omega_2}{\omega_1} \right] \times \prod_{p=1}^{\ell_a} \sinh \left[ i p \pi \frac{\omega_1}{\omega_2} \right] \}^{-1}, \] 

(146)

\[ P_{\ell_n,k_N}^{(\alpha)} = e^{2i\pi x \left( \varpi(N-1) + iK_1 \omega_1 + iK_2 \omega_2 \right)} \times \prod_{a=1}^{N-1} \prod_{b=1}^{N} \omega \left( v_{\alpha(b)} - w_a - i \frac{\Omega}{2} - i \ell_b \omega_1 - i k_b \omega_2 \right). \] 

(147)

Note that above, we made use of the notation (239). Finally, one has

\[ G_{\ell_n,k_N}^{(\alpha)} = \prod_{b=1}^{N} \prod_{a=1}^{N-1} \prod_{c \neq a,b} \prod_{d \neq a,b,c} \sinh \left( \frac{\pi}{\omega_1} (v_{\alpha(a)}(b) - v_{\alpha(b)}(a) - i \frac{\Omega}{2} - i \ell_b \omega_1 - i k_b \omega_2) \right) \times \prod_{a<b} \sinh \left( \frac{\pi}{\omega_1} (v_{\alpha(a)}(b) + i \ell_b \omega_1 + i k_b \omega_2) \right) \times \sinh \left( \frac{\pi}{\omega_2} (v_{\alpha(a)}(b) + i \ell_b \omega_1 + i k_b \omega_2) \right). \] 

(148)

Let \( \alpha \in \mathcal{M}_N \) be given. Then, there exists \( a \neq b \) such that \( (w^{(\alpha)})_a = (w^{(\alpha)})_b \). A longish but straightforward calculation then shows that one has the factorisation

\[ G_{\ell_n,k_N}^{(\alpha)} = (-1)^{(\ell_n k_a + \ell_n k_b)(N-2)} \sinh \left( \frac{i \pi}{\omega_1} k_{ab} \right) \cdot \sinh \left( \frac{i \pi}{\omega_2} \ell_{ab} \right) \cdot \mathcal{G}_{\ell_n,k_N}^{(\alpha;\alpha;b)}, \] 

in which \( \mathcal{G}_{\ell_n,k_N}^{(\alpha;\alpha;b)} \) is symmetric under the permutation \( (\ell_a, \ell_b) \leftrightarrow (\ell_b, \ell_a) \) or, independently, the permutation \( (k_a, k_b) \leftrightarrow (k_b, k_a) \). It takes the explicit form

\[ \mathcal{G}_{\ell_n,k_N}^{(\alpha;\alpha;b)} = (-1)^{(\ell_N + K_1)(N-3)} \prod_{c<d}^{N} \sinh \left( \frac{\pi}{\omega_1} (v_{\alpha(c)}(d) + ik_{cd} \omega_2) \right) \times \sinh \left( \frac{\pi}{\omega_2} (v_{\alpha(c)}(d) + i \ell_{cd} \omega_1) \right) \times \prod_{d=1}^{N} \prod_{k \in \{k_a,k_b\}} \left\{ i \sinh \left( \frac{\pi}{\omega_1} (v_{\alpha(a)}(d) + i \omega_1 (k - k_d)) \right) \right\} \times \prod_{d=1}^{N} \prod_{k \in \{k_a,k_b\}} \left\{ i \sinh \left( \frac{\pi}{\omega_2} (v_{\alpha(a)}(d) + i \omega_1 (\ell - \ell_d)) \right) \right\} \times \prod_{d=1}^{N-1} \prod_{k \in \{k_a,k_b\}} \left\{ i \omega_1 (v_{\alpha(a)}(d) - i \omega_2) \right\} \]
\[
\times \prod_{\ell \in \{\ell_a, \ell_b\}} \prod_{p=1}^{\ell} \sinh \left[ \frac{\pi}{\omega_2} \left( v_{d\alpha(a)} - ip\omega_1 \right) \right]^{-1} \\
\times \prod_{c=1}^{N} \prod_{d=1}^{N-1} \mathbb{w} \left( v_{d\alpha(c)} - i\frac{\Omega}{2} - i\ell_c\omega_1 - ik_c\omega_2 \right) .
\]

Likewise, one gets that
\[
P_{\ell_n; k_N}^{(\alpha)} = (-1)^{(\ell_a k_a + \ell_b k_b)(N-1)} . \mathcal{P}_{\ell_n; k_N}^{(\alpha; a, b)},
\]
where
\[
\mathcal{P}_{\ell_n; k_N}^{(\alpha; a, b)} = e^{2i\pi \left( v_{\ell_n\omega_1} + iN\omega_1 \right)} \left( -2i \right)^{(k_a + k_b + \ell_a + \ell_b)(N-1)}
\]
\[
\times \prod_{c=1}^{N} \prod_{d=1}^{N-1} \mathbb{w} \left( v_{\alpha(a)} - w_d - i\frac{\Omega}{2} - i\ell_c\omega_1 - ik_c\omega_2 \right)
\]
\[
\times \prod_{c=1}^{N-1} \left\{ \mathbb{w}^2 \left( v_{\alpha(a)} - w_c - i\frac{\Omega}{2} \right) \right\} \prod_{k \in \{\ell_a, k_a\}}^{k=1} \prod_{p=0}^{\ell-1} \sinh \left[ \frac{\pi}{\omega_1} \left( v_{\alpha(a)} - w_c + ip\omega_1 \right) \right].
\]

Finally, one also gets \(\Gamma_{\ell_n; k_N}^{(\alpha; a, b)} = (-1)^{(\ell_a k_a + \ell_b k_b)} \cdot \Gamma_{\ell_n; k_N}^{(\alpha; a, b)}\), where
\[
\Gamma_{\ell_n; k_N}^{(\alpha; a, b)} = \frac{2^{N(N-1)}}{N! \left( \omega_1\omega_2 \right)^{\frac{N}{2} - 1}} \prod_{c=1}^{N} \left\{ (2i)^{\ell_c + k_c} \cdot \prod_{p=1}^{k_c} \sinh \left[ \frac{ip\omega_2}{\omega_1} \right] \cdot \prod_{p=1}^{\ell_c} \sinh \left[ \frac{i\pi\omega_1}{\omega_2} \right] \right\}^{-1}
\]
\[
\times \prod_{c=1}^{N} \left\{ (-1)^{k_c\ell_c} \right\} .
\]

Note that \(\Gamma_{\ell_n; k_N}^{(\alpha; a, b)}\) and \(\mathcal{P}_{\ell_n; k_N}^{(\alpha; a, b)}\) enjoy the same symmetry properties as \(\mathcal{G}_{\ell_n; k_N}^{(\alpha; a, b)}\). Thus, in the end, the residue expansion of the integral can be cast into the form
\[
\operatorname{Lim}_{\varepsilon \rightarrow N-0+} \left\{ \mathcal{G}_{\varepsilon N}^{(N; N)} \left( x \mid w_{N-1}, v_{N-1} \right) \right\}
\]
\[
= \sum_{\alpha \in \mathcal{M}_N} \mathcal{G} \cdot \mathcal{P} \cdot \Gamma_{\ell_n; k_N}^{(\alpha; a, b)}
\]
\[
\times \sinh \left[ \frac{i\pi\omega_2}{\omega_1} k_{a\alpha} b_{a\alpha} \right] \sinh \left[ \frac{i\pi\omega_1}{\omega_2} \ell_{a\alpha} \right] .
\]

Above, for a given \(\alpha\), we denote by \(a_{\alpha}, b_{\alpha}\) any two distinct integers such that \(\alpha(a_{\alpha}) = \alpha(b_{\alpha})\). Due to the symmetry properties of \(\mathcal{G}, \mathcal{P}\) and \(\Gamma\), the summand in (154) is antisymmetric under \(\ell_{a\alpha} \leftrightarrow \ell_{b\alpha}\) as well as under \(k_{a\alpha} \leftrightarrow k_{b\alpha}\). This entails that, in the sense of distributions, the last line in (154) vanishes, \textit{viz.}
\[
\lim_{\varepsilon \to 0^+} \left\{ \mathcal{H}_{\varepsilon_N}^{(N;N)}(x \mid w_{N-1}, v_{N-1}) \right\} = 0. \quad (155)
\]

- **The fundamental induction equation**

Let \( m \in \{0, N - 1\} \) and assume the \( \varepsilon_a \)'s to be small enough. Then, in \( \mathcal{H}_{\varepsilon_N}^{(N;m)} \) one deforms the \( y_N \) integration contour to \( \mathbb{R} - i\alpha \) with \( \alpha > 0 \) small enough and such that \( \alpha > \max\{\|\varepsilon_a\|\} \). One crosses the poles at

\[
y_N = w_a - i\zeta_N^{(m)} = w_a - i\varepsilon_{m+1}, \quad \text{with} \quad a = 1, \ldots, N - 1.
\quad (156)
\]

This yields

\[
\mathcal{H}_{\varepsilon_N}^{(N;m)}(x \mid w_{N-1}, v_{N-1})
= \int_{\mathcal{Q}_m} d^m y \int_{\mathbb{R} - i\alpha} \prod_{a=m+1}^{N-1} dy_a \int_{\mathbb{R} - i\alpha} dy_N \cdot \mathcal{I}_{N-1}^{(m)}(\varepsilon_N \mid x, y_N; w_{N-1}, v_{N-1})
+ \sum_{k=1}^{N-1} \int_{\mathcal{Q}_m} d^m y \int_{\mathbb{R} - i\alpha} \prod_{a=m+1}^{N-1} dy_a \cdot e^{2i\pi(w_k - i\varepsilon_{m+1})x} \frac{\prod_{a=1}^{N-1} \varpi(w_{ka} - i\frac{\Omega}{2}) \prod_{a=1}^{N-1} \varpi(v_a - w_k - i\frac{\Omega}{2} + 2i\varepsilon_{m+1})}{\prod_{a=1}^{N-1} \varpi(w_k - y_a - i\frac{\Omega}{2} + i\zeta_a^{(m)}), y_a - w_k - i\frac{\Omega}{2} + i\zeta_a^{(m)}}}
\times \frac{1}{N\sqrt{\omega_1\omega_2}} \cdot \mathcal{I}_{N-1}^{(m)}(\varepsilon_N; x, y_N; w_{N-1}, v_{N-1}).
\quad (157)
\]

Above, the vector \( \varepsilon_{N;m} \in \mathbb{R}^{N-1} \) is defined by

\[
\begin{align*}
\left( \varepsilon_{N;m} \right)_a & = \varepsilon_a, \quad a = 1, \ldots, m \quad \text{and} \\
\left( \varepsilon_{N;m} \right)_a & = \varepsilon_{a+1}, \quad a = m + 1, \ldots, N - 1.
\end{align*}
\quad (158)
\]

The fact that the last line of (157) does indeed take the form as written follows from the below identification:

\[
\begin{align*}
\zeta_a^{(m)} & \equiv -\left( \varepsilon_{N;m} \right)_a = -\varepsilon_a, \quad a = 1, \ldots, m \\
\zeta_a^{(m)} & \equiv \left( \varepsilon_{N;m} \right)^{(m)}_a = \varepsilon_{N+m+1-a}, \quad a = m + 1, \ldots, N - 1.
\end{align*}
\quad (159)
\]

in which \( \zeta_a^{(m)} \) is the sequence built up from \( \varepsilon_{N;m} \). Hence, (159) ensures that \( \zeta_a^{(m)} = \zeta_a^{(m)} \) for \( a = 1, \ldots, N - 1 \).

Define \( P_a \) as the operator \( \left(P_a \cdot \varepsilon_N\right)_b = (-1)^{\delta_{ab}}\varepsilon_b \). Let \( \overline{\zeta}_a^{(m)} \) be the sequence built up from the vector \( P_{m+1} \cdot \varepsilon_N \), namely

\[
\begin{align*}
\overline{\zeta}_a^{(m)} & = -\varepsilon_a, \quad a = 1, \ldots, m; \quad \overline{\zeta}_N^{(m)} = -\varepsilon_{m+1} \quad \text{and} \\
\overline{\zeta}_a^{(m)} & = \varepsilon_{N+m+1-a}, \quad a = m + 1, \ldots, N - 1.
\end{align*}
\quad (160)
\]

Thus, one has

\[
\begin{align*}
\overline{\zeta}_a^{(m)} & = \zeta_a^{(m+1)}, \quad a = 1, \ldots, m; \quad \overline{\zeta}_N^{(m)} = \zeta_N^{(m+1)} \quad \text{and}
\end{align*}
\]
\[ \zeta_a^{(m)} = \zeta_a^{(m+1)}, \quad a = m + 1, \ldots, N - 1. \] (161)

Upon defining
\[ \overline{y}_a = y_a, \quad a = 1, \ldots, m; \quad \overline{y}_{m+1} = y_N \quad \text{and} \quad \overline{y}_{a+1} = y_a, \quad a = m + 1, \ldots, N - 1, \] (162)

one gets the below product identities
\[
\prod_{b=1}^{N} \varpi \left( y_b - w - \frac{i \Omega}{2} + i \zeta_b^{(m)}, \ v - y_b - \frac{i \Omega}{2} + i \zeta_b^{(m)} \right)
\]
\[
= \prod_{b=1}^{N} \varpi \left( \overline{y}_b - w - \frac{i \Omega}{2} + i \zeta_b^{(m+1)}, \ v - \overline{y}_b - \frac{i \Omega}{2} + i \zeta_b^{(m+1)} \right)
\] (163)

and
\[
\prod_{a<b}^{N} \varpi \left( y_{ab} - i \frac{\Omega}{2} + i \zeta_a^{(m)} + i \zeta_b^{(m)}, \ y_{ab} - i \frac{\Omega}{2} + i \zeta_a^{(m)} + i \zeta_b^{(m)} \right)
\]
\[
= \prod_{a<b}^{N} \varpi \left( \overline{y}_{ab} - i \frac{\Omega}{2} + i \zeta_a^{(m+1)} + i \zeta_b^{(m+1)}, \overline{y}_{ab} - i \frac{\Omega}{2} + i \zeta_a^{(m+1)} + i \zeta_b^{(m+1)} \right)
\]
\[
\times \prod_{a=m+2}^{N} \varpi \left( \overline{y}_{m+1a} - i \frac{\Omega}{2} + i \zeta_{m+1}^{(m+1)} + i \zeta_a^{(m+1)}, \overline{y}_{m+1a} - i \frac{\Omega}{2} + i \zeta_{m+1}^{(m+1)} + i \zeta_a^{(m+1)} \right)
\]. (164)

Thus, define
\[
\delta \mathcal{J}^{(N;m)}_{\varepsilon_N}(x \mid w_{N-1}, v_{N-1}) = \int_{\mathcal{C}} d^m y \int_{\mathbb{R}^{N-m-1}} \prod_{a=m+1}^{N-1} dy_a \int_{\mathbb{R} - i \alpha} dy_N
\]
\[
\times \left\{ \mathcal{I}_N^{(m)}(\varepsilon_N \mid x, y_N; w_{N-1}, v_{N-1}) - \mathcal{I}_N^{(m)}(P_m+1 \varepsilon_N \mid x, y_N; w_{N-1}, v_{N-1}) \right\},
\] (165)

Then, in the counter-term integral, \textit{viz.} the one associated with \( P_{m+1} \varepsilon_N \) in (165), one deforms the \( y_N \)-integration curve \( \mathbb{R} - i \alpha \) to \( \mathcal{C} \). One picks up poles at \( w_k + i \varepsilon_{m+1} = w_k - i \zeta_a^{(m)} \) with \( k = 1, \ldots, N - 1 \), and this yields
\[
\int_{\mathcal{C}} d^m y \int_{\mathbb{R}^{N-m-1}} \prod_{a=m+1}^{N-1} dy_a \int_{\mathbb{R} - i \alpha} dy_N \cdot \mathcal{I}_N^{(m)}(\varepsilon_N \mid x, y_N; w_{N-1}, v_{N-1})
\]
\[
= \delta \mathcal{J}^{(N;m)}_{\varepsilon_N}(x \mid w_{N-1}, v_{N-1}) + \tilde{\mathcal{J}}^{(N;m+1)}_{\varepsilon_N}(x \mid w_{N-1}, v_{N-1})
\]
\[
- \sum_{k=1}^{N-1} \int_{\mathcal{C}} d^m y \int_{\mathbb{R}^{N-m-1}} \prod_{a=m+1}^{N-1} dy_a \ e^{2i\pi(w_k+i\varepsilon_{m+1})x}
\]
\[
\times \left( \prod_{a=1}^{N-1} \varpi \left( w_{ka} - i \frac{\Omega}{2} \right) \prod_{a=1}^{N-1} \varpi \left( v_a - w_k - i \frac{\Omega}{2} - 2i \varepsilon_{m+1} \right) - \prod_{a=1}^{N-1} \varpi \left( w_k - y_a - i \frac{\Omega}{2} + i \zeta_a^{(m)}, y_a - w_k - i \frac{\Omega}{2} + i \zeta_a^{(m)} \right) \right)
\]
\begin{equation}
\times \frac{1}{N \sqrt{\omega_1 \omega_2}} \cdot I_{N-1}^{(m)} (\varepsilon_{N;m} \mid x, y_{N-1}; w_{N-1}, v_{N-1}).
\end{equation}

Above, we have introduced
\begin{equation}
\tilde{f}_{\varepsilon}(N;m+1) (x \mid w_{N-1}, v_{N-1})
= \int_{\mathbb{R}^{N-m-1}} \prod_{a=m+2}^{N} dy_a \cdot I_{N-1}^{(m+1)} (\varepsilon_N \mid x, y_N; w_{N-1}, v_{N-1})
\times \prod_{a=m+2}^{N} \omega \left( y_{m+1} - i \frac{\Omega}{2} + \zeta^{(m+1)}_{m+1} \right) \left( y_{m+1} - i \frac{\Omega}{2} - \zeta^{(m+1)}_{m+1} \right). \tag{167}
\end{equation}

Thus, all-in-all, one gets
\begin{align*}
\left( \tilde{f}_{\varepsilon}(N;m) - \delta \tilde{f}_{\varepsilon}(N;m) - \tilde{f}_{\varepsilon}(N;m+1) \right) (x \mid w_{N-1}, v_{N-1})
&= \sum_{k=1}^{N-1} \frac{e^{2\pi \omega_{1} x}}{N \sqrt{\omega_1 \omega_2}} \int d^m y \\
&\quad \int_{\mathbb{R}^{N-m-1}} \prod_{a=m+1}^{N-1} dy_{a} \cdot I_{N-1}^{(m)} (\varepsilon_{N;m} \mid x, y_{N-1}; w_{N-1}; [k], v_{N-1})
\times \prod_{a=1}^{N-1} \omega \left( w_{ka} - i \frac{\Omega}{2} \right) \left( w_{ka} - y_{a} - i \frac{\Omega}{2} + \zeta^{(m)}_{a} \right) \cdot \mathcal{D}_{\varepsilon}(v_{N-1}, w_{k}). \tag{168}
\end{align*}

Here, \( w_{N-1;[k]} \) is as defined through (238),
\begin{equation}
\mathcal{D}_{\varepsilon}(v_{N-1}, w_{k}) = e^{2\pi \varepsilon \tau} \prod_{a=1}^{N-1} \omega \left( v_{a} - w_{k} - i \frac{\Omega}{2} + 2i \varepsilon \right) - (\varepsilon \leftrightarrow -\varepsilon). \tag{169}
\end{equation}

Upon using that
\begin{equation}
\omega \left( \lambda - i \frac{\Omega}{2} \right) = \frac{i}{2} \frac{\omega \left( \lambda + i \frac{\Omega}{2} \right)}{\sinh \left( \frac{\pi}{\omega_1} \lambda \right)} \tag{170}
\end{equation}

and applying the pole expansion
\begin{equation}
e^{\frac{2\pi \omega_{1} x}{\omega}} \prod_{a=1}^{N-1} \left\{ \frac{1}{\sinh \frac{\omega}{\omega_1} (v_{a} - x)} \right\}
= \sum_{\ell=1}^{N-1} e^{\frac{2\pi \omega_{1} x}{\omega}} \prod_{a=1}^{N-1} \frac{1}{\sinh \frac{\omega}{\omega_1} (v_{a} - v_{\ell})} \tag{171}
\end{equation}

one obtains the decomposition
\begin{equation}
\mathcal{D}_{\varepsilon}(v_{N-1}, w_{k}) = \left( \frac{i}{2} \right) \sum_{\ell=1}^{N-1} \prod_{a=1}^{N-1} \omega \left( v_{a} - w_{k} - i \frac{\Omega}{2} + 2i \varepsilon \right) \prod_{a=1}^{N-1} \frac{1}{\sinh \frac{\omega}{\omega_1} (v_{a} - v_{\ell})} \tag{171}
\end{equation}
This decomposition entails that, in the sense of distributions,

$$\mathcal{D}_\varepsilon(v_{N-1}, w_k) \underset{\varepsilon \to 0^+}{\longrightarrow} \sqrt{\omega_1 \omega_2} \sum_{\ell=1}^{N-1} \delta(v_\ell - w_k) \prod_{a=1 \atop a \neq \ell}^{N-1} \varpi(v_a - v_\ell - i \Omega/2).$$  \hspace{1cm} (173)

Thus, all-in-all

$$\lim_{\varepsilon_{m+1} \to 0^+} \left( \mathcal{J}_\varepsilon^{(N;m)} - \delta \mathcal{J}_\varepsilon^{(N;m)} - \mathcal{J}_\varepsilon^{(N;m+1)} \right)(x \mid w_{N-1}, v_{N-1})$$

$$= \frac{1}{N} \sum_{k,\ell=1}^{N-1} \delta(v_\ell - w_k) \cdot e^{2i\pi w_k x} \cdot \prod_{a=1 \atop a \neq \ell}^{N-1} \varpi(v_a - i \Omega/2)$$

$$\times \prod_{a=1 \atop a \neq k}^{N-1} \varpi(\omega_{ka} - i \Omega/2) \mathcal{J}_\varepsilon^{(N;m)}(x \mid w_{N-1};[k], v_{N-1};[\ell]).$$  \hspace{1cm} (174)

Finally, one observes that the difference structure of the integrand of $\delta \mathcal{J}_\varepsilon^{(N;m)}$ entails that

$$\lim_{\varepsilon \to 0^+} \{ \delta \mathcal{J}_\varepsilon^{(N;m)}(x \mid w_{N-1}, v_{N-1}) \} \quad = \quad 0. \hspace{1cm} (175)$$

Likewise, since the additional $\varepsilon_N$ dependence present in $\mathcal{J}_\varepsilon^{(N;m+1)}(x \mid w_{N-1}, v_{N-1})$ arises in the regular part of the integrand and since the singularities arising in the $\varepsilon_N \to 0^+$ limit of the integrand generate at most Sokhotski–Plemelj distributions, one has

$$\lim_{\varepsilon_N \to 0^+} \{ \mathcal{J}_\varepsilon^{(N;m+1)}(x \mid w_{N-1}, v_{N-1}) \}$$

$$= \lim_{\varepsilon_N \to 0^+} \{ \mathcal{J}_\varepsilon^{(N;m+1)}(x \mid w_{N-1}, v_{N-1}) \}. \hspace{1cm} (176)$$

All of the above leads to the relation among the various integrals:

$$\lim_{\varepsilon_N \to 0^+} \{ \mathcal{J}_\varepsilon^{(N;m)}(x \mid w_{N-1}, v_{N-1}) \} - \lim_{\varepsilon_N \to 0^+} \{ \mathcal{J}_\varepsilon^{(N;m+1)}(x \mid w_{N-1}, v_{N-1}) \}$$

$$= \frac{1}{N} \sum_{k,\ell=1}^{N-1} \delta(v_\ell - w_k) \cdot e^{2i\pi w_k x} \cdot \prod_{a=1 \atop a \neq \ell}^{N-1} \varpi(v_a - i \Omega/2)$$

$$\times \prod_{a=1 \atop a \neq k}^{N-1} \varpi(\omega_{ka} - i \Omega/2) \lim_{\varepsilon_{N-1} \to 0^+} \{ \mathcal{J}_{\varepsilon_{N-1}}^{(N;m)}(x \mid w_{N-1};[k], v_{N-1};[\ell]) \},$$  \hspace{1cm} (177)

which holds for any $m \in [0; N - 1]$.

- The induction step
Upon setting $m = N - 1$ and then decreasing the value of $m$ down to $m = 2$, one infers from (155) and (177) that

$$\lim_{\varepsilon N \to 0^+} \left\{ \mathcal{J}^{(N;m)}_{\varepsilon N} (x \mid w_{N-1}, v_{N-1}) \right\} = 0 \quad \text{for any} \quad m \in \{2, N\}.$$ 

(178)

When $m = 1$, the previous results and the induction hypothesis lead to

$$\lim_{\varepsilon N \to 0^+} \left\{ \mathcal{J}^{(N;m)}_{\varepsilon N} (x \mid w_{N-1}, v_{N-1}) \right\} = \frac{1}{N(N-1)} \sum_{k, \ell=1}^{N-1} \delta(v_\ell - w_k) \cdot e^{2\pi w_k x} \times \prod_{a=1 \atop a \neq \ell}^{N-1} \mathcal{O}(v_a - i\frac{\Omega}{2}) \times \prod_{a=1 \atop a \neq k}^{N-1} \mathcal{O}(w_{ka} - i\frac{\Omega}{2}) \times \prod_{a=1 \atop a \neq b \atop a, b \neq k}^{N-1} \delta(x \cdot \delta_{\text{sym}}(w_{N-1;k} - v_{N-1;\ell}))$$

$$= \frac{1}{N} \delta(x) \prod_{a \neq b}^{N-1} \mathcal{O}(w_{ab} - i\frac{\Omega}{2}) \cdot \sum_{k=1}^{N-1} \frac{1}{N-1} \sum_{\ell=1}^{N-1} \sum_{\sigma(k) = \ell}^{N-1} \delta(v_{\sigma(a)} - w_a)$$

$$= \frac{1}{N} \delta(x) \cdot \delta_{\text{sym}}(v_{N-1} - w_{N-1}) \cdot \prod_{a \neq b}^{N-1} \mathcal{O}(w_{ab} - i\frac{\Omega}{2}).$$

(179)

An analogous reasoning establishes the induction hypothesis for $m = 0$, hence entailing the claim. \hfill \Box

### 4.3. The Completeness

We are finally in position to establish the completeness of the system of eigenfunctions introduced in Proposition 4.1.

**Proposition 4.3.** The family $\{\psi_{y_N}^{(-)}\}_{y_N \in \mathbb{R}^N}$ forms a complete system on $L^2(\mathbb{R}^N, d^N x)$ in the sense that

$$\frac{1}{N!} \int_{\mathbb{R}^N} (\psi_{y_N}^{(-)}(x')^* \cdot \psi_{y_N}^{(-)}(x)) \cdot d\mu_N(y_N) = \prod_{a=1}^{N} \delta(x_a - x'_a),$$

(180)

where

$$d\mu_N(y_N) = \mu_N(y_N) d^N y$$

(181)

and $\mu_N(y_N)$ is as defined in (107).

The family $\{\psi_{y_0,y_{N-1}}^{(+)}\}_{y_0 \in \mathbb{R}, y_{N-1} \in \mathbb{R}^{N-1}}$ forms a complete system on $L^2(\mathbb{R}^N, d^N x)$

$$\frac{1}{(N-1)!} \int_{\mathbb{R}^N} (\psi_{y_0,y_{N-1}}^{(+)}(x')^* \cdot \psi_{y_0,y_{N-1}}^{(+)}(x)) \cdot dy_0 \cdot d\mu_{N-1}(y_{N-1}) = \prod_{a=1}^{N} \delta(x_a - x'_a).$$

(182)
The $N = 1$ specialisation of the above proposition was obtained earlier in [16,30]. The relations (180)–(182) are recast, in terms of $\mathcal{U}_N^{(\pm)}$, c.f. (110) and (109), as
\[
\mathcal{U}_N^{(\pm)} \cdot \left( \mathcal{U}_N^{(\pm)} \right)^\dagger = \text{id}.
\] (183)

They thus provide one with the other “half” of the relations necessary for the operators $\mathcal{U}_N^{(\pm)}$ to be unitary maps.

**Proof.** Assume that completeness, in the above sense, holds for $\{\psi_{y_{N-1}}\} y_{N-1} \in \mathbb{R}^{N-1}$. Then, by using the Mellin–Barnes representation and the form of the regularisation one has to impose for the integral representation to make sense, one gets that (c.f. [22] for a rigorous treatment of the various steps in the sense of distributions)

\[
\frac{1}{N!} \int_{\mathbb{R}^N} \left( \psi_{y_{N}}^{(-)} (x_N') \right)^* \psi_{y_{N}}^{(-)} (x_N) \cdot d\mu_N(y_N) = \int_{\mathbb{R}^{N-1}} \frac{d\mu_{N-1}(w_{N-1})}{(N-1)!} \int_{\mathbb{R}^{N-1}} \frac{d^{N-1}v}{(N-1)!} \cdot \left( \psi_{v_{N-1}}^{(-)} (x_{N-1}') \right)^* \psi_{v_{N-1}}^{(-)} (x_{N-1}) \times \lim_{\varepsilon_N \to 0^+} \chi_N^{(\varepsilon_N)} (w_{N-1}, v_{N-1}; x_N, x_{N}')
\] (184)

where

\[
\chi_N^{(\varepsilon_N)} (w_{N-1}, v_{N-1}; x_N, x_{N}') = \prod_{a=1}^{N-1} \left\{ \frac{\varpi (w_a + \kappa)}{\varpi (v_a + \kappa)} \right\}^{N-1-\varepsilon} \cdot \frac{\psi_{v_{N-1}}^{(-)} (x_{N-1})}{\psi_{w_{N-1}}^{(-)} (x_{N-1})} \cdot \frac{\mathcal{K}_N^{(\varepsilon_N)} (w_{N-1}, v_{N-1}; x_N - x_{N}')}{\prod_{a \neq b} \varpi (v_{ab} - i \frac{\Omega}{2})}
\] (185)

while

\[
\mathcal{K}_N^{(\varepsilon_N)} (w_{N-1}, v_{N-1}; x) = \int_{\mathbb{R}^N} e^{2i\pi x \varpi_N} \prod_{a=1}^{N} \varpi (y_{a} - w_{a} - i \frac{\Omega}{2} + i \varepsilon_{N+1-a}) \varpi (v_{a} - v_{a} - i \frac{\Omega}{2} + i \varepsilon_{N+1-a}) \cdot d^N y.
\] (186)

The use of the integral representation

\[
\frac{\varpi (\varpi_N - \varpi_{N-1} - \kappa)}{\varpi (\varpi_N - \varpi_{N-1} - \kappa)} = \int_{\mathbb{R}} dt \int_{\mathbb{R}} ds \frac{\varpi (s - \varpi_{N-1} - \kappa)}{\varpi (s - \varpi_{N-1} - \kappa)} e^{2\pi t(s - \varpi_N)}
\] (187)

allows one to recast the previous expression as
Theorem 5.1. Each of the four operator entries of the monodromy matrix \( T_{N} \) (3) admits a complete and orthogonal system of eigenfunctions, \( \{ \Phi_{y_{N}}^{(E)} \}_{y_{N} \in \mathbb{R}^{N}} \) for \( E \in \{ A, D \} \) and \( \{ \Phi_{y_{0},y_{N-1}}^{(E)} \}_{y_{0} \in \mathbb{R}, \ y_{N-1} \in \mathbb{R}^{N-1}} \) for \( E \in \{ B, C \} \).

Let \( d\mu_{N}(y_{N}) = \mu_{N}(y_{N}) d^{N}y \) with \( \mu_{N}(y_{N}) \) as defined in (107).

- The system \( \{ \Phi_{y_{N}}^{(E)} \}_{y_{N} \in \mathbb{R}^{N}}, E \in \{ A, D \} \), satisfies:

\[
\frac{1}{N!} \int_{\mathbb{R}^{N}} \left( \Phi_{y_{N}}^{(E)}(x') \right)^{*} \Phi_{y_{N}}^{(E)}(x_{N}) \cdot d\mu_{N}(y_{N}) = \prod_{a=1}^{N} \delta(x_{a} - x'_{a})
\]  

(191)

along with

\[
\int_{\mathbb{R}^{N}} \left( \Phi_{y_{N}}^{(E)}(x_{N}) \right)^{*} \Phi_{y_{N}}^{(E)}(x_{N}) \cdot d^{N}x = \frac{1}{\mu_{N}(y_{N})} \cdot \delta_{\text{sym}}(y_{N} - y'_{N}).
\]

(192)

Thus, upon inserting this result in (185) and then (184), one obtains:

\[
\frac{1}{N!} \int_{\mathbb{R}^{N}} \left( \Psi_{y_{N}}^{(-)}(x'_{N}) \right)^{*} \Psi_{y_{N}}^{(-)}(x_{N}) \cdot d\mu_{N}(y_{N})
\]

\[
= \frac{\delta(x_{N} - x'_{N})}{(N-1)!} \int_{\mathbb{R}^{N-1}} \left( \Psi_{y_{N-1}}^{(-)}(x'_{N-1}) \right)^{*} \Psi_{y_{N-1}}^{(-)}(x_{N-1}) \cdot d\mu_{N-1}(y_{N-1})
\]

\[
= \prod_{a=1}^{N} \delta(x_{a} - x'_{a}).
\]

(190)

5. Complete and Orthogonal System of Eigenfunctions of

\[ [T_{N}(\lambda)]_{ab} \]

In this section, we summarise the results established in the previous two sections.

Theorem 5.1. Each of the four operator entries of the monodromy matrix \( T_{N}(\lambda) \) (3) admits a complete and orthogonal system of eigenfunctions, \( \{ \Phi_{y_{N}}^{(E)} \}_{y_{N} \in \mathbb{R}^{N}} \) for \( E \in \{ A, D \} \) and \( \{ \Phi_{y_{0},y_{N-1}}^{(E)} \}_{y_{0} \in \mathbb{R}, \ y_{N-1} \in \mathbb{R}^{N-1}} \) for \( E \in \{ B, C \} \).

Let \( d\mu_{N}(y_{N}) = \mu_{N}(y_{N}) d^{N}y \) with \( \mu_{N}(y_{N}) \) as defined in (107).

- The system \( \{ \Phi_{y_{N}}^{(E)} \}_{y_{N} \in \mathbb{R}^{N}}, E \in \{ A, D \} \), satisfies:

\[
\frac{1}{N!} \int_{\mathbb{R}^{N}} \left( \Phi_{y_{N}}^{(E)}(x') \right)^{*} \Phi_{y_{N}}^{(E)}(x_{N}) \cdot d\mu_{N}(y_{N}) = \prod_{a=1}^{N} \delta(x_{a} - x'_{a})
\]  

(191)

along with

\[
\int_{\mathbb{R}^{N}} \left( \Phi_{y_{N}}^{(E)}(x_{N}) \right)^{*} \Phi_{y_{N}}^{(E)}(x_{N}) \cdot d^{N}x = \frac{1}{\mu_{N}(y_{N})} \cdot \delta_{\text{sym}}(y_{N} - y'_{N}).
\]

(192)
Furthermore, the generalised eigenvalue equation takes the form

\[ E_N(\lambda) \cdot \Phi^{(E)}_{yN}(x_N) = \prod_{a=1}^{N} \left\{ -2i \sinh \left[ \frac{\pi}{\omega_1} (\lambda - y_a) \right] \right\} \cdot \Phi^{(E)}_{yN}(x_N). \]  

(193)

- The system \( \left\{ \Phi^{(E)}_{y_0,y_{N-1}} \right\}_{y_0 \in \mathbb{R}, y_{N-1} \in \mathbb{R}^{N-1}}, E \in \{ B, C \} \) satisfies:

\[ \frac{1}{(N-1)!} \int_{\mathbb{R}^N} \left( \Phi^{(E)}_{y_0,y_{N-1}}(x_N') \right)^* \cdot \Phi^{(E)}_{y_0,y_{N-1}}(x_N) \cdot \delta(y_0 \cdot d\mu_{N-1}(y_{N-1}) = \prod_{a=1}^{N} \delta(x_a - x_a') \]  

(194)

along with

\[ \int_{\mathbb{R}^N} \left( \Phi^{(E)}_{y_0,y_{N-1}}(x_N) \right)^* \cdot \Phi^{(E)}_{y_0,y_{N-1}}(x_N) \cdot d^N x = \frac{1}{\mu_{N-1}(y_{N-1})} \cdot \delta(y_0 - y_0') \cdot \delta_{\text{sym}}(y_{N-1} - y_{N-1}'). \]  

(195)

Finally, the generalised eigenvalue equation takes the form

\[ E_N(\lambda) \cdot \Phi^{(E)}_{y_0,y_{N-1}}(x_N) = e^{2\pi \omega_2 y_0} \prod_{a=1}^{N-1} \left\{ -2i \sinh \left[ \frac{\pi}{\omega_1} (\lambda - y_a) \right] \right\} \cdot \Phi^{(E)}_{y_0,y_{N-1}}(x_N). \]  

(196)

The generalised eigenfunctions admit Gauss–Givental and Mellin–Barnes integral representations:

\[ \Phi^{(A)}_{yN}(x_N) = \psi_y^{(-)}(x_N) = c_A \varphi_y^{(-)}(x_N) \quad \text{and} \quad \Phi^{(B)}_{y_0,y_{N-1}}(x_N) = \psi^{(+)}_{y_0,y_{N-1}}(x_N) = c_B \varphi^{(+)}_{y_0,y_{N-1}}(x_N) \]  

(197)

Above, \( c_A, \text{resp. } c_B, \) are \( y_N, \) resp. \( y_0, y_{N-1}, \) independent constants equal to \( \pm 1. \)

We conjecture that, in fact, the proportionality constants in (197) equal 1.

**Proof.** We shall only establish the properties of the generalised eigenfunctions of the operator \( A_N(\lambda), \) as the case of the \( B_N(\lambda) \) operator can be dealt with similarly. Furthermore, the case of the operators \( D_N(\lambda) \) and \( C_N(\lambda) \) is a direct consequence of the results relative to the operators \( A_N(\lambda) \) and \( B_N(\lambda). \)

Indeed, observe that upon introducing the operator \( \Omega_a \) such that

\[ \Omega_a x_a \Omega_a = -x_a \quad \text{and} \quad \Omega_a p_a \Omega_a = -p_a, \]  

(198)

one has the relation

\[ \Omega_n \cdot L_n^{(\kappa)}(\lambda) \cdot \Omega_n = \sigma^x \cdot L_n^{(\kappa)}(\lambda) \cdot \sigma^x \text{i.e.} \prod_{a=1}^{N} \Omega_a \cdot T_N(\lambda) \cdot \prod_{a=1}^{N} \Omega_a = \sigma^x \cdot T_N(\lambda) \cdot \sigma^x. \]  

(199)
This entails that
\[
\prod_{a=1}^{N} \Omega_a \cdot D_N(\lambda) \cdot \prod_{a=1}^{N} \Omega_a = A_N(\lambda) \quad \text{and} \quad \prod_{a=1}^{N} \Omega_a \cdot C_N(\lambda) \cdot \prod_{a=1}^{N} \Omega_a = B_N(\lambda) .
\] (200)

Thus, if \( \{ \Phi_{y_{N}}^{(A)} \}_{y_{N} \in \mathbb{R}^{N}} \), resp. \( \{ \Phi_{y_{0},y_{N}-1}^{(B)} \}_{y_{0} \in \mathbb{R},y_{N-1} \in \mathbb{R}^{N-1}} \), is the complete and orthogonal system of generalised eigenfunctions of the \( A_N(\lambda) \), resp. \( B_N(\lambda) \), operator then
\[
\left\{ \prod_{a=1}^{N} \Omega_a \cdot \Phi_{y_{N}}^{(A)} \right\}_{y_{N} \in \mathbb{R}^{N}}, \quad \text{resp.} \quad \left\{ \prod_{a=1}^{N} \Omega_a \cdot \Phi_{y_{0},y_{N}-1}^{(B)} \right\}_{y_{0} \in \mathbb{R},y_{N-1} \in \mathbb{R}^{N-1}} ,
\] (201)
is the complete and orthogonal system of generalised eigenfunctions of the \( D_N(\lambda) \), resp. \( C_N(\lambda) \), operator. Furthermore, the form of the generalised eigenvalues is a direct consequence of the conjugation relation between the operators (200) and the form of the generalised eigenvalues of the operators \( A_N(\lambda) \) and \( B_N(\lambda) \).

**Proportionality of \( \varphi_{y_{N}}^{(-)} \) and \( \psi_{y_{N}}^{(-)} \)**

We first show that, for any fixed \( y_{N} \in \mathbb{R}^{N} \), the functions \( \varphi_{y_{N}}^{(-)} \) and \( \psi_{y_{N}}^{(-)} \) are non-identically zero. Given \( f \in L_{\text{sym}}^{2}(\mathbb{R}^{N},d\mu_{N}(y_{N})) \), smooth and compactly supported, define
\[
\mathcal{U}_{N}[f](x_{N}) = \frac{1}{N!} \int_{\mathbb{R}^{N}} \varphi_{y_{N}}^{(-)}(x_{N})f(y_{N})d\mu_{N}(y_{N}) .
\] (202)

Then, the orthogonality relation satisfied by the functions \( \varphi_{y_{N}}^{(-)}(x_{N}) \) ensures that, for any such \( f \), it holds
\[
\int_{\mathbb{R}^{N}} d^{N}x \left( \varphi_{y_{N}}^{(-)}(x_{N}) \right)^{*} \cdot \mathcal{U}_{N}[f](x_{N}) = f(y_{N}) .
\] (203)

Thus, \( \varphi_{y_{N}}^{(-)} \) cannot be identically zero.

Regarding \( \psi_{y_{N}}^{(-)} \), as in [21], one may compute the \( x_{N} \to \infty, x_{a+1} - x_{a} \to +\infty \) of \( \psi_{y_{N}}^{(-)}(x_{N}) \) staring from its Mellin–Barnes integral representation by pushing the various integration contours slightly into the upper-half plane. This shows that the function is non-vanishing in this asymptotic regime and hence is non-identically zero.

Since
\[
\begin{align*}
A_N(\lambda)\psi_{y_{N}}^{(-)}(\lambda | y_{N}) &= a(\lambda | y_{N})\psi_{y_{N}}^{(-)}(\lambda) \quad \text{with} \quad a(\lambda | y_{N}) = \prod_{\alpha=1}^{N} \left\{ -2i \sinh \left[ \frac{\pi}{\omega_{1}}(\lambda - y_{\alpha}) \right] \right\} , \\
\psi_{y_{N}}^{(-)} &\text{ is non-identically zero,} \\
\text{the system } &\{ \psi_{y_{N}}^{(-)} \}_{y_{N} \in \mathbb{R}^{N}} \text{ forms a complete system,}
\end{align*}
\]
each generalised eigenvalue \( a(\lambda \mid y_N) \) of \( A_N(\lambda) \) appears with exactly multiplicity 1. Since it also holds \( A_N(\lambda)\varphi_y^{(-)} = a(\lambda \mid y_N)\varphi_y^{(-)} \), with \( \varphi_y^{(-)} \) non-identically zero, it follows that there exists a constant \( c_N(y_N, \kappa) \in \mathbb{C}^* \) such that

\[
\psi_y^{(-)} = c_N(y_N, \kappa)\varphi_y^{(-)} .
\] (204)

The proportionality constant may, in principle, depend on \( y_N \) and the representation parameter \( \kappa \). Below, we establish various properties that ought to be satisfied by these constants. This will strongly constrain its value. Since for each \( y_N \) the function \( x_N \mapsto \varphi_y^{(-)}(x_N) \) is not-identically zero and also continuous in \( y_N \), it follows from (204) that \( c_N(y_N, \kappa) \) is continuous in \( y_N \).

- \( c_N(y_N, \kappa) \) is unimodular. We first establish that the proportionality constant is unimodular. The orthogonality relation for \( \varphi_y^{(-)} \) leads to the relation

\[
\int_{\mathbb{R}^N} \left( \psi_y^{(-)}(x_N) \right)^* \psi_y^{(-)}(x_N) \cdot d^N x = \frac{|c_N(y_N, \kappa)|^2}{\mu_N(y_N)} \cdot \delta_{\text{sym}}(y_N - y_N') .
\] (205)

Integrating both sides of this equation versus

\[
\frac{d\mu_N(y_N')}{N!} \otimes \frac{d\mu_N(y_N)}{N!} \psi_y^{(-)}(x_N') \left( \psi_y^{(-)}(x_N'') \right)^*
\] (206)

and using completeness of the \( \psi_y^{(-)} \) one is led to

\[
\prod_{a=1}^{N} \delta(x'_a - x''_a) = |c_N(y_N, \kappa)|^2 \prod_{a=1}^{N} \delta(x'_a - x''_a) .
\] (207)

Thus, indeed, one has that \(|c_N(y_N, \kappa)| = 1\).

- \( c_N(y_N, \kappa) \) is invariant under \( \kappa \)-reflections

It follows from (10) that

\[
\prod_{a=1}^{N} D_{-\kappa}(p_a) \cdot T_N(\lambda; \kappa) = T_N(\lambda; -\kappa) \cdot \prod_{a=1}^{N} D_{-\kappa}(p_a)
\] (208)

where we have explicitly stressed the dependence of the monodromy matrix on the parameter \( \kappa \). In the following, it will be convenient to make also explicit the dependence of the functions \( \psi_y^{(-)} \), \( \varphi_y^{(-)} \) on \( \kappa \), \( \nu \), \( \psi_y^{(-)}(x_N; \kappa) \), \( \varphi_y^{(-)}(x_N; \kappa) \). The intertwining of the monodromy matrix suggests that analogous relations should exist between the generalised eigenfunctions. These will be established below.

The Mellin–Barnes integral representation induction immediately leads to
\[
\psi_{y_N}^{(-)}(x_N; \kappa) = \prod_{s=1}^{N-1} \left\{ \int \frac{d^{N-s}w(s)}{(N-s)!} \right\} 
\times \prod_{s=1}^{N} \psi_{w_{N-s+1}^{-1}w_{N-s}^{-1}}^{(-)} \left( x_{N-s+1} \right) \prod_{s=1}^{N} \Phi (w_{N-s}^{(s)} \mid w_{N-s+1}^{(s-1)})
\]

which can be reorganised as

\[
\psi_{y_N}^{(-)}(x_N; \kappa) = \sigma(y_N; \kappa) \prod_{s=1}^{N-1} \left\{ \int \frac{d^{N-s}w(s)}{(N-s)!} \right\} 
\times \prod_{s=1}^{N} \psi_{w_{N-s+1}^{-1}w_{N-s}^{-1}}^{(-)} (x_{N-s+1}) \cdot \xi_{\kappa} \left( \{ w_{N-s}^{(s)} \}_{s=1}^{N} \right)
\]

where

\[
\xi_{\kappa} \left( \{ w_{N-s}^{(s)} \}_{s=1}^{N} \right) = \frac{\prod_{s=1}^{N-1} \prod_{a=1}^{N-s} \varpi \left( w_{a}^{(s)} - \kappa, w_{a}^{(s)} + \kappa \right)}{\prod_{s=1}^{N} \varpi \left( w_{N-s+1}^{(s-1)} - w_{N-s}^{(s)} - \kappa \right)} 
\times \prod_{s=1}^{N-1} \left\{ \prod_{a=1}^{N-s} \prod_{b=1}^{N-s+1} \varpi \left( w_{b}^{(s-1)} - w_{a}^{(s)} - i \frac{\Omega}{2} \right) \right\}
\]

and

\[
\sigma(y_N; \kappa) = \prod_{a=1}^{N} \frac{\varpi(y_a - \kappa)}{\varpi(y_a + \kappa)} \right\}^{N-1}. \tag{212}
\]

The relation

\[
\left( D_{-\kappa}(p) \cdot \psi_{y}^{(-)} \right)(x) = D_{-\kappa}(y) \psi_{y}^{(-)}(x)
\]

and the fact that

\[
\xi_{\kappa} \left( \{ w_{N-s}^{(s)} \}_{s=1}^{N} \right) = \prod_{s=1}^{N} \varpi \left( w_{N-s+1}^{(s-1)} - w_{N-s}^{(s)} + \kappa \right) \varpi \left( w_{N-s+1}^{(s-1)} - w_{N-s}^{(s)} - \kappa \right) \cdot \xi_{-\kappa} \left( \{ w_{N-s}^{(s)} \}_{s=1}^{N} \right) \tag{214}
\]

then immediately lead to

\[
\prod_{a=1}^{N} D_{-\kappa}(p_a) \cdot \psi_{y_N}^{(-)}(x_N; \kappa) = \prod_{a=1}^{N} D_{-\kappa}(y_a) \cdot \psi_{y_N}^{(-)}(x_N; -\kappa). \tag{215}
\]

The same property holds for \( \varphi_{y_N}^{(-)} \), namely

\[
\prod_{a=1}^{N} D_{-\kappa}(p_a) \cdot \varphi_{y_N}^{(-)}(x_N; \kappa) = \prod_{a=1}^{N} D_{-\kappa}(y_a) \cdot \varphi_{y_N}^{(-)}(x_N; -\kappa). \tag{216}
\]

Indeed, this identity is a direct consequence of an inductive application of Lemma 3.2 on the level of the recursive construction of \( \varphi_{y_N}^{(-)}(x_N; \kappa) \).
This entails that $c_N(y_N, \kappa) = c_N(y_N, -\kappa)$.

**Complex conjugation of $c_N(y_N, \kappa)$**

Finally, the $\Lambda^{(N)}_{y,\epsilon}$ operator can be represented as

$$\left(\Lambda^{(N)}_{y,\epsilon} \cdot f\right)(x_N) = \int_{\mathbb{R}^{N-1}} \Lambda^{(N)}_{y,\epsilon}(x_N, x'_{N-1}; \kappa) f(x'_{N-1}) \cdot d^{N-1}x'$$

(217)

where the integral kernel takes the form

$$\Lambda^{(N)}_{y,\epsilon}(x_N, x'_{N-1}; \kappa) = \frac{e^{2i\pi y_1 x_1} e^{2i\pi y_+^* x_N}}{(\sqrt{\omega_1 \omega_2 \mathcal{A}(y_+)})^{N-1}} \times \prod_{a=1}^{N-1} \left\{ D_{y_--y_+} (\omega_1 \omega_2 x_{a+1} \epsilon) D_{y'_-} (\omega_1 \omega_2 (x_a - x'_a)) D_{y_+} (\omega_1 \omega_2 (x_{a+1} - x'_a)) \right\}.$$  

(218)

It is then immediate to check that

$$\left(\Lambda^{(N)}_{y,\epsilon}(x_N, x'_{N-1}; \kappa)\right)^* = \Lambda^{(N)}_{-y,\epsilon}(x_N, x'_{N-1}; -\kappa).$$

(219)

This relation then leads to

$$\left(\varphi_{y_N}^{(-)}(x_N; \kappa)\right)^* = \varphi_{-y_N}^{(-)}(x_N; -\kappa).$$

(220)

Now, starting from the Mellin–Barnes representation (210) and upon using that for

$$w_{N-s}^{(s)} \in \mathbb{R}^{N-s} \text{ and } \alpha_s = \alpha_s(1, \ldots, 1) \in \mathbb{R}^{N-s} \text{ with } \alpha_s \text{ real},$$

(221)

it holds

$$\left(\xi_{-\kappa} \left( \{ - w_{N-s}^{(s)} - i\alpha_s \}_{s=1}^N \right) \right)^* = \xi_{-\kappa} \left( \{ w_{N-s}^{(s)} - i\alpha_s \}_{s=1}^N \right),$$

(222)

one entails that, as well

$$\left(\psi_{y_N}^{(-)}(x_N; \kappa)\right)^* = \psi_{-y_N}^{(-)}(x_N; -\kappa).$$

(223)

This entails that $\left(c_N(y_N, \kappa)\right)^* = c_N(-y_N, -\kappa)$. Hence, by invoking the $\kappa$-reflection property, one infers that the constant $c_N$ behaves under complex conjugation as

$$\left(c_N(y_N, \kappa)\right)^* = c_N(-y_N, \kappa).$$

(224)
• $c_N(y_N, \kappa)$ is $y_N$-independent

It remains to prove that the constant does not depend on $y_N$. As follows from Propositions 3.3 and 3.2, $\varphi^{(-)}_{y_N}$ satisfies identically the same Eqs. (129)–(130) as $\psi^{(-)}_{y_N}$. By projecting these on a given variable $y_N$, this yields that

$$c_N(y_N, \kappa) = c_N(y_N + \omega_2 e_k, \kappa)$$

(225)

where $e_k$ is the $k$th unit vector in $\mathbb{R}^N$. The same equation holds for $\omega_1 \leftrightarrow \omega_2$ since both $\varphi^{(-)}_{y_N}$ and $\psi^{(-)}_{y_N}$ satisfy the dual Eqs. to (129)–(130) as well. Since $c_N(y_N, \kappa)$ is continuous in $y_N$, it must be $y_N$ independent.

Thus, all-in-all, we get that $(c_N(y_N, \kappa))^* = c_N(y_N, \kappa)$, which along with $|c_N(y_N, \kappa)| = 1$ implies that $c_N(y_N, \kappa) = \pm 1$ uniformly in $y_N$. □

6. The Sinh-Gordon Model

The Lax matrix of the lattice Sinh-Gordon model takes the form [4]:

$$L_n^{(SG)}(\lambda) = \sigma_x L_n^{(\kappa)}(\lambda).$$

(226)

The monodromy matrix of the $N$-site lattice Sinh-Gordon model thus takes the form

$$T_N^{(SG)}(\lambda) = L_1^{(SG)}(\lambda) \cdots L_N^{(SG)}(\lambda) = \left( \begin{array}{cc} A_N^{(SG)}(\lambda) & B_N^{(SG)}(\lambda) \\ C_N^{(SG)}(\lambda) & D_N^{(SG)}(\lambda) \end{array} \right).$$

(227)

The analysis of the system of eigenfunctions of the operators $A_N$ and $B_N$ that was carried out in the previous sections allows us to access the system of eigenfunctions of the operator $B_N^{(SG)}(\lambda)$ and characterise its spectrum. In this manner, we prove the conjecture raised in [4] relatively to the spectrum of this operator. In order to state the result, it is convenient to recall the operators $\Omega_a$ introduced in (198) which enjoy the exchange relations:

$$\Omega_a x_a \Omega_a = -x_a \quad \text{and} \quad \Omega_a p_a \Omega_a = -p_a.$$  

(228)

**Theorem 6.1.** The operator $B_N^{(SG)}(\lambda)$ admits a complete and orthogonal system of eigenfunctions:

- for $N$ odd, this system is \{ $\Phi_{y_N}$ $\}_{y_N \in \mathbb{R}^N}$ and it satisfies :

$$\frac{1}{N!} \int_{\mathbb{R}^N} \left( \Phi_{y_N}(x'_N) \right)^* \Phi_{y_N}(x_N) \cdot d\mu_N(y_N) = \prod_{a=1}^{N} \delta(x_a - x'_a)$$

(229)

along with

$$\int_{\mathbb{R}^N} \left( \Phi_{y_N}(x_N) \right)^* \Phi_{y_N}(x_N) \cdot d^N x = \frac{1}{\mu_N(y_N)} \cdot \delta_{\text{sym}}(y_N - y'_N).$$

(230)

Furthermore, the generalised eigenvalue equation takes the form

$$B_N^{(SG)}(\lambda) \cdot \Phi_{y_N}(x_N) = \prod_{a=1}^{N} \left\{ -2i \sinh \left[ \frac{\pi}{\omega_1}(\lambda - y_a) \right] \right\} \cdot \Phi_{y_N}(x_N).$$

(231)
• for $N$ even, this system is $\{\Phi_{y_0, y_{N-1}}\}_{y_0 \in \mathbb{R}, y_{N-1} \in \mathbb{R}^{N-1}}$ and it satisfies:

$$\frac{1}{(N-1)!} \int_{\mathbb{R}^N} (\Phi_{y_0', y_{N-1}}(x'_N))^* \cdot \Phi_{y_N}(x_N) \cdot dy_0 \cdot d\mu_{N-1}(y_{N-1}) = \prod_{a=1}^{N} \delta(x_a - x'_a)$$

(232)

along with

$$\int_{\mathbb{R}^N} (\Phi_{y_0', y_{N-1}}(x_N))^* \cdot \Phi_{y_0, y_{N-1}}(x_N) \cdot d^N x = \frac{1}{\mu_{N-1}(y_{N-1})} \cdot \delta(y_0 - y'_0) \cdot \delta_{\text{sym}}(y_{N-1} - y'_N).$$

(233)

Finally, the generalised eigenvalue equation takes the form

$$E_{N}^{(SG)}(\lambda) \cdot \Phi_{y_0, y_{N-1}}(x_N) = e^{2\pi \omega_2 y_0} \prod_{a=1}^{N-1} \{-2i \sinh \left( \frac{x}{\omega_1}(\lambda - y_a) \right) \} \cdot \Phi_{y_0, y_{N-1}}(x_N).$$

(234)

**Proof.** The relation between the Lax matrices (226) implies that the monodromy matrix of the Sinh-Gordon model is related to the one of the modular XXZ chains as

$$T_{N}^{(SG)}(\lambda) = \begin{cases} \prod_{a=1}^{N/2} \Omega_{2a-1} \cdot T_N(\lambda) \cdot \prod_{a=1}^{N/2} \Omega_{2a-1} & \text{if } N \text{ is even} \\ \prod_{a=1}^{(N+1)/2} \Omega_{2a-1} \cdot T_N(\lambda) \sigma^x \cdot \prod_{a=1}^{(N+1)/2} \Omega_{2a-1} & \text{if } N \text{ is odd} \end{cases}$$

(235)

Indeed, this relation is a simple consequence of the local identity

$$\sigma^x L_n^{(\kappa)}(\lambda) \sigma^x = \Omega_n L_n^{(\kappa)}(\lambda) \Omega_n.$$  

(236)

The representation (235) then ensures that the system of generalised eigenfunctions of the operator $E_{N}^{(SG)}(\lambda)$ is given by

$$\begin{cases} \prod_{a=1}^{N/2} \Omega_{2a-1} \cdot \Phi_{y_N}^{(A)} & \text{if } N \text{ is even} \\ \prod_{a=1}^{(N+1)/2} \Omega_{2a-1} \cdot \Phi_{y_0, y_{N-1}}^{(B)} & \text{if } N \text{ is odd} \end{cases}$$

(237)

where $\Phi_{y_N}^{(A)}$, $\Phi_{y_0, y_{N-1}}^{(B)}$ are as introduced in Theorem 5.1. The rest follows from the stated properties of the functions $\Phi_{y_N}^{(A)}$, $\Phi_{y_0, y_{N-1}}^{(B)}$ in that theorem. $\square$

**Conclusion**

In this work, we have constructed the eigenfunctions of the entries of the $N$-site monodromy matrix of the modular XXZ magnet. We have established that
each system associated with a given entry forms a complete and orthogonal system. These properties ensure that the integral transform subordinate to such a system provides a unitary equivalence between the appropriate Hilbert spaces. The proof of the orthogonality was achieved by means of handlings on the level of the Gauss–Givental representation for these eigenfunctions. The proof of the completeness was carried out on the level of the Mellin–Barnes representation. We stress that we have proposed a new and simple method for proving the completeness. The technique we developed is general and applicable to a wide variety of quantum integrable models solvable by the separation of variables method. As a by-product of our analysis, we have proved the conjectures raised by Bytsko–Teschner on the spectrum of the $B$-operator for the lattice Sinh-Gordon model.

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Appendix A. Main Notations

- $N$-dimensional vectors are denoted as $x_N = (x_1, \ldots, x_N)$;
- $N - 1$ dimensional vectors built from an $N$-dimensional vector $x_N$ with the removed $m$th coordinate are denoted as $x_{N;[m]}$ and read

$$x_{N;[m]} = (x_1, \ldots, x_{m-1}, x_{m+1}, \ldots, x_N).$$

(238)

- Given an $N$-dimensional vector $x_N$, we denote

$$\overline{x}_N = \sum_{a=1}^{N} x_a.$$

(239)

- Ratios of products of one variable functions appearing with multi-component entries are denoted using the hypergeometric notations, e.g.

$$f\left(\begin{array}{c} a_1, \ldots, a_n \\ b_1, \ldots, b_m\end{array}\right) = \frac{\prod_{k=1}^{n} f(a_k)}{\prod_{k=1}^{m} f(b_k)}.$$  

(240)

- Given indexed symbols $x_a, x_b$, we denote $x_{ab} = x_a - x_b$.
- Given $y \in \mathbb{C}$, $y^*$ stands for its complex conjugate and $y^* = -y - i \frac{\Omega}{2}$. 
Appendix B. Properties of the Auxiliary Special Functions

B.1. The Quantum Dilogarithm and the $D_\alpha$ Functions

The quantum dilogarithm $\varpi$ is a meromorphic function on $\mathbb{C}$ which admits the integral representation

$$\varpi(\lambda) = \exp \left\{ \pm \frac{i\pi}{2\omega_1\omega_2} \left( \lambda^2 + \frac{\omega_1^2 + \omega_2^2}{12} \right) - i \int_{\mathbb{R}_{\pm i0^+}} \frac{dt}{4t} \sinh (t\omega_1) \cdot \sinh (t\omega_2) \right\},$$

valid for $|\Im(\lambda)| < \Omega/2$.

This function is self-dual and satisfies the first-order finite-difference equations

$$\varpi(\lambda + i\omega_2) = 2i \sinh \left[ \frac{\pi}{\omega_1}(\lambda + i\frac{\tau}{2}) \right] \cdot \varpi(\lambda)$$

and

$$\varpi(\lambda + i\omega_1) = 2i \sinh \left[ \frac{\pi}{\omega_2}(\lambda - i\frac{\tau}{2}) \right] \cdot \varpi(\lambda).$$

From there, one entails that

$$\varpi \left( \lambda - i\frac{\Omega}{2} + i\ell\omega_1 + ik\omega_2 \right) = (-1)^{k\ell} \left( -2i \right)^{\ell+k} \prod_{p=0}^{k-1} \sinh \left[ \frac{\pi}{\omega_1}(\lambda + ip\omega_2) \right]$$

$$\times \prod_{p=0}^{\ell-1} \sinh \left[ \frac{\pi}{\omega_2}(\lambda + ip\omega_1) \right] \cdot \varpi \left( \lambda - i\frac{\Omega}{2} \right)$$

and symmetrically,

$$\varpi \left( \lambda - i\frac{\Omega}{2} - i\ell\omega_1 - ik\omega_2 \right) = (-1)^{k\ell} \left( \frac{1}{2} \right)^{\ell+k} \prod_{p=1}^{k} \left\{ \sinh \left[ \frac{\pi}{\omega_1}(\lambda - ip\omega_2) \right] \right\}^{-1}$$

$$\times \prod_{p=1}^{\ell} \left\{ \sinh \left[ \frac{\pi}{\omega_2}(\lambda - ip\omega_1) \right] \right\}^{-1} \cdot \varpi \left( \lambda - i\frac{\Omega}{2} \right).$$

The quantum dilogarithm has only simple poles and zeroes. These are located at

$$\varpi(x) = 0 \quad \text{iff} \quad x \in i\frac{\Omega}{2} + iN\omega_1 + iN\omega_2$$

and

$$\varpi^{-1}(x) = 0 \quad \text{iff} \quad x \in -i\frac{\Omega}{2} - iN\omega_1 - iN\omega_2.$$

(245)

$\varpi$ satisfies the inversion identity $\varpi(\lambda)\varpi(-\lambda) = 1$ and $(\varpi(\lambda^*))^* = \varpi^{-1}(\lambda)$.

One can also establish that

$$\text{Res} \left( \varpi \left( \lambda - i\frac{\Omega}{2} \right) \cdot d\lambda, \lambda = 0 \right) = \frac{i}{2\pi} \sqrt{\omega_1\omega_2}$$

and

$$\varpi \left( \frac{1}{2}\tau \right) = \sqrt{\frac{\omega_2}{\omega_1}}.$$

(246)
The above entails that, for \((k, \ell) \in \mathbb{N}^2\),
\[
\text{Res}\left(\varpi(\lambda - i \frac{\Omega}{2}) \cdot d\lambda, \lambda = -i \ell \omega_1 - i k \omega_2\right) = \frac{i}{2\pi} \sqrt{\omega_1 \omega_2} (-1)^k \left(\frac{1}{2i}\right)^{\ell+k} \times \left\{ \prod_{p=1}^{k} \sinh \left[ i p \pi \frac{\omega_2}{\omega_1} \right] \prod_{p=1}^{\ell} \sinh \left[ i p \pi \frac{\omega_1}{\omega_2} \right] \right\}^{-1}.
\]
(247)

The function \(D_\alpha\) is defined by the below ratio of dilogarithms
\[
D_\alpha(\lambda) = \frac{\varpi(\lambda + \alpha)}{\varpi(\lambda - \alpha)} \quad \text{so that} \quad D_\alpha(\lambda) = \left(D_{-\alpha}(\lambda^*)\right)^*.
\]
(248)

The function \(D_\alpha(\lambda)\) is a meromorphic function on \(\mathbb{C}\) that admits a holomorphic determination of the logarithm on
\[
S_{\frac{\alpha}{2} - |\Im(\alpha)|}(\mathbb{R}) = \left\{ \lambda \in \mathbb{C} : |\Im(\lambda)| < \frac{1}{2} (\omega_1 + \omega_2) - |\Im(\alpha)| \right\}
\]
given by
\[
\ln D_\alpha(\lambda) = \mp \frac{2i\pi}{\omega_1 \omega_2} \alpha \lambda + i \int_{\mathbb{R}^{\pm 10^1}} dt \frac{e^{2i\lambda t} \sin(2\alpha t)}{2t \sinh(t \omega_1) \cdot \sinh(t \omega_2)}.
\]
(250)

The function \(D_\alpha(\lambda)\) satisfies the properties
\begin{itemize}
  \item \(D_\alpha(\lambda)\) is self-dual, namely invariant under the exchange \(\omega_1 \leftrightarrow \omega_2\);
  \item for \(\omega_2 > \omega_1\) it admits the asymptotic behaviours
    \[
    D_\alpha(\lambda) = e^{\mp \frac{2i\pi}{\omega_1 \omega_2} \lambda \alpha} \cdot \left( 1 + O\left( e^{\mp \frac{2\pi}{\omega_2} \alpha \sinh\left[ \frac{2\pi}{\omega_2} \right] } \right) \right)
    \]
    when \(\lambda \to \infty\), \(|\arg(\pm \lambda)| < \frac{\pi}{2}\);
    (251)
  \item \(D_\alpha(\lambda)\) satisfies the difference equation
    \[
    \frac{D_\alpha(\lambda + i \omega_2)}{D_\alpha(\lambda)} = \frac{\cosh \left[ \frac{\pi}{\omega_1} \left( \lambda + i \frac{\omega_2}{2} + \alpha \right) \right]}{\cosh \left[ \frac{\pi}{\omega_1} \left( \lambda + i \frac{\omega_2}{2} - \alpha \right) \right]}
    \]
    as well as its dual \(\omega_1 \leftrightarrow \omega_2\);
  \item \(D_\alpha(\lambda)\) enjoys the transmutation properties
    \[
    D_\alpha\left(\lambda \mp i \frac{\omega_2}{2}\right) = \frac{D_{\alpha - i \frac{\omega_2}{2}}(\lambda)}{2 \cosh \left[ \frac{\pi}{\omega_1} (\lambda \pm \alpha) \right]};
    \]
    (253)
  \item \(D_\alpha(\lambda)\) has simple zeroes at
    \[
    \pm \left\{ - \alpha + i \frac{\Omega}{2} + in\omega_1 + im\omega_2 : (n, m) \in \mathbb{N}^2 \right\};
    \]
    (254)
  \item \(D_\alpha(\lambda)\) has simple poles at
    \[
    \pm \left\{ \alpha + i \frac{\Omega}{2} + in\omega_1 + im\omega_2 : (n, m) \in \mathbb{N}^2 \right\}.
    \]
    (255)
\end{itemize}
B.2. Integral Identities

The function $D_\alpha$ admits the Fourier transform

$$\mathcal{F}[D_\alpha](t) = \sqrt{\omega_1 \omega_2} \cdot \mathcal{A}(\alpha) \cdot D_\alpha \left( \frac{\omega_1 \omega_2 t}{2\pi} \right) \quad (256)$$

with

$$\mathcal{F}[f](t) = \int_{\mathbb{R}} e^{itx} f(x) \cdot dx \quad \text{for} \quad f \in L^1(\mathbb{R}, dx). \quad (257)$$

Here, we remind that

$$\mathcal{A}(\alpha) = \varpi \left( 2\alpha + i \frac{\Omega}{2} \right) \quad \text{and} \quad \alpha^* = -\alpha - i \frac{\Omega}{2}. \quad (258)$$

(256) entails that

$$\lim_{\alpha \to 0} \left\{ \mathcal{A}(\alpha)D_\alpha^*(t) \right\} = \delta(t). \quad (259)$$

The $D_\alpha$ functions satisfy the three-term integral identity [16]

$$\int_{\mathbb{R}} D_\alpha(\omega_1 \omega_2(x - u)) \cdot D_\beta(\omega_1 \omega_2(x - v)) \cdot D_\gamma(\omega_1 \omega_2(x - w)) \cdot dx$$

$$= \mathcal{A}(\alpha, \beta, \gamma) \cdot \frac{1}{\sqrt{\omega_1 \omega_2}} D_\alpha^*(\omega_1 \omega_2(w - v)) \cdot D_\beta^*(\omega_1 \omega_2(u - w))$$

$$\times D_\gamma^*(\omega_1 \omega_2(u - v)) \quad (260)$$

provided that $\alpha + \beta + \gamma = -i\Omega$ and [5]

$$\int_{\mathbb{R}} D_\alpha(\omega_1 \omega_2(x - u)) \cdot D_\beta(\omega_1 \omega_2(x - v)) \cdot D_\gamma(\omega_1 \omega_2(x - w)) \cdot D_\delta(\omega_1 \omega_2(x - z)) \cdot dx$$

$$= \mathcal{A}(\alpha, \beta, \gamma, \delta) \cdot \frac{1}{\sqrt{\omega_1 \omega_2}} \left( \frac{\omega_1 \omega_2(u - v)}{\omega_1 \omega_2(w - z)} \right)$$

$$\times \int_{\mathbb{R}} D_\alpha^*(\omega_1 \omega_2(x - v)) \cdot D_\beta^*(\omega_1 \omega_2(x - u)) \cdot D_\gamma^*(\omega_1 \omega_2(x - z))$$

$$\times D_\delta^*(\omega_1 \omega_2(x - w)) \cdot dx \quad (261)$$

provided that $\alpha + \beta + \gamma + \delta = -i\Omega$.

Sending one of the integration variables to infinity provides one with the auxiliary identities

$$\int_{\mathbb{R}} D_\alpha(\omega_1 \omega_2(x - u)) \cdot e^{\pm 2\pi i \beta x} \cdot D_\gamma(\omega_1 \omega_2(x - w)) \cdot dx$$

$$= \frac{\mathcal{A}(\alpha, \beta, \gamma)}{\sqrt{\omega_1 \omega_2}} \cdot e^{\pm 2\pi i (\alpha^* w + \gamma^* u)} D_\beta^*(\omega_1 \omega_2(u - w)) \quad (262)$$

provided that $\alpha + \beta + \gamma = -i\Omega$ and
\[
\int_{\mathbb{R}} D_\alpha(\omega_1 \omega_2(x - u)) \cdot D_\beta(\omega_1 \omega_2(x - v)) \cdot D_\gamma(\omega_1 \omega_2(x - w)) \cdot e^{\pm 2i\pi \delta x} \cdot dx
\]

\[
= \mathcal{A}(\alpha, \beta, \gamma, \delta)e^{\pm 2i\pi(\alpha + \beta + i\frac{\Omega}{2})} D_{\alpha + \beta + i\frac{\Omega}{2}}(u - v)
\]

\[
\times \int_{\mathbb{R}} D_{\alpha^*}(\omega_1 \omega_2(x - v)) \cdot D_{\beta^*}(\omega_1 \omega_2(x - u))
\]

\[
\times D_{\delta^*}(\omega_1 \omega_2(x - w)) \cdot e^{\pm 2i\pi \gamma^* x} \cdot dx \quad (263)
\]

provided that \(\alpha + \beta + \gamma + \delta = -i\Omega\).

The three-term integral relation can be recast in an operator form as

\[
D_u(p) \cdot D_{u+v}(\omega_1 \omega_2 x) \cdot D_v(p) = D_v(\omega_1 \omega_2 x) \cdot D_{u+v}(\omega_1 \omega_2 x) \quad \text{(264)}
\]

see e.g. [17], whereas its degenerate form can be recast as

\[
D_\alpha(p) \cdot e^{\pm 2i\pi \beta x} \cdot D_\gamma(p) = e^{\pm 2i\pi \gamma x} \cdot D_\beta(p) \cdot e^{\pm 2i\pi \alpha x} \quad \text{(265)}
\]

where \(\alpha, \beta, \gamma\) fulfil the constraint \(\beta = \alpha + \gamma\).

Let \(y_\pm, t_\pm\) be parameters as in (21). Then, the integral identity involving four \(D\) functions can be recast in the operator form as

\[
D_{y_-(p_2)} \cdot D_{y_+(\omega_1 \omega_2 x_{23})} \cdot D_{t_-(\omega_1 \omega_2 x_{12})} \cdot D_{t_+(p_2)}
\]

\[
= D_{y_+ + t_+ + i\frac{\Omega}{2}}(\omega_1 \omega_2 x_{12}) \cdot D_{t_+}(p_2) \cdot D_{t_-}(\omega_1 \omega_2 x_{23}) \cdot D_{y_-}(\omega_1 \omega_2 x_{12}) \cdot D_{y_+}(p_2)
\]

\[
\times D_{y_- + t_- + i\frac{\Omega}{2}}(\omega_1 \omega_2 x_{23}) \quad (266)
\]

Likewise, its exponential degenerate form can be recast as

\[
e^{2i\pi y_- x_1} \cdot D_{y_-(p_2)} \cdot D_{y_+(\omega_1 \omega_2 x_{12})} \cdot e^{2i\pi t_- x_1} \cdot D_{t_+(p_2)}
\]

\[
e^{2i\pi t_- x_1} \cdot D_{t_-(p_2)} \cdot D_{t_+}(\omega_1 \omega_2 x_{12}) \cdot e^{2i\pi y_- x_1} \cdot D_{y_+}(p_2)
\]

\[
\times D_{y_- + t_- + i\frac{\Omega}{2}}(\omega_1 \omega_2 x_{21}) \quad (267)
\]

Finally, the exponential degenerate form of the four \(D\) function integrals (263) can be also recast as

\[
\mathcal{A}(t_+) D_{y_-}(p_2) \cdot D_{y_+(\omega_1 \omega_2 x_{23})} \cdot e^{\pm 2i\pi(t_+ x_2 + y_+ x_3)} \cdot D_{t_-}(\omega_1 \omega_2 x_{12})
\]

\[
= \mathcal{A}(y_+) D_{y_+ + t_+ + i\frac{\Omega}{2}}(\omega_1 \omega_2 x_{12}) D_{t_+}(p_2) \cdot D_{t_-}(\omega_1 \omega_2 x_{23})
\]

\[
\times e^{\pm 2i\pi(y_+ x_2 + t_+ x_3)} \cdot D_{y_-}(\omega_1 \omega_2 x_{12}) \quad (268)
\]

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