Mathematical Structure of Rabi Oscillations
in the Strong Coupling Regime

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Abstract

In this paper we generalize the Jaynes–Cummings Hamiltonian by making use of
some operators based on Lie algebras su(1,1) and su(2), and study a mathematical
structure of Rabi floppings of these models in the strong coupling regime. We show
that Rabi frequencies are given by matrix elements of generalized coherent operators
(quant–ph/0202081) under the rotating–wave approximation.

In the first half we make a general review of coherent operators and generalized
coherent ones based on Lie algebras su(1,1) and su(2). In the latter half we carry out
a detailed examination of Frasca (quant–ph/0111134) and generalize his method,
and moreover present some related problems.

We also apply our results to the construction of controlled unitary gates in Quantum
Computation. Lastly we make a brief comment on application to Holonomic
Quantum Computation.

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1 Introduction

Coherent states or generalized coherent states play an important role in quantum physics, in particular, quantum optics, see [1] and [2]. They also play an important one in mathematical physics. See the textbook [3]. For example, they are very useful in performing stationary phase approximations to path integral, [4], [5], [6].

Coherent operators which produce coherent states are very useful because they are unitary and easy to handle. The basic reason is probably that they are subject to the elementary Baker-Campbell-Hausdorff (BCH) formula. Many basic properties of them are well-known, see [3] or [8].

Generalized coherent operators which produce generalized coherent states are also useful. But they are not so easy to handle in spite of having the disentangling one corresponding to the elementary BCH formula. In [7] and [14] the author determined all matrix elements of generalized coherent operators based on Lie algebras su(1,1) and su(2). They are interesting by themselves, but moreover have a very interesting application.

In [12] Frasca dealt with the Jaynes–Cummings model which describes a two-level atom interacting with a single radiation mode (see [10] for a general review) in the strong coupling regime (not weak coupling one!) and showed that Rabi frequencies are obtained by matrix elements of coherent operator under the rotating–wave approximation. His aim was to explain the recent experimental finding on Josephson junctions [11].

This is an interesting result and moreover his method can be widely generalized. See also [13] for an another example dealt with in the strong coupling regime.

In this paper we generalize the Jaynes–Cummings Hamiltonian by making use of some operators based on Lie algebras su(1,1) and su(2), and study a mathematical structure of Rabi flossings of these extended models in the strong coupling regime.

We show that (generalized) Rabi frequencies are also given by matrix elements of generalized coherent operators under the rotating–wave approximation. We believe that the results will give a new aspect to Quantum Optics or Mathematical Physics.

We also apply our results to the construction of controlled unitary gates in Quantum
Computation in the last section.

Lastly we discuss an application to Holonomic Quantum Computation, but our discussion is not complete.

2 Coherent and Generalized Coherent Operators

2.1 Coherent Operator

Let \( a(a^\dagger) \) be the annihilation (creation) operator of the harmonic oscillator. If we set \( N \equiv a^\dagger a \) (: number operator), then

\[
\begin{align*}
[N, a^\dagger] &= a^\dagger, \\
[N, a] &= -a, \\
[a^\dagger, a] &= -1.
\end{align*}
\]

(1)

Let \( \mathcal{H} \) be a Fock space generated by \( a \) and \( a^\dagger \), and \( \{ |n\rangle \mid n \in \mathbb{N} \cup \{0\} \} \) be its basis. The actions of \( a \) and \( a^\dagger \) on \( \mathcal{H} \) are given by

\[
\begin{align*}
a|n\rangle &= \sqrt{n}|n-1\rangle, \\
a^\dagger|n\rangle &= \sqrt{n+1}|n+1\rangle, \\
N|n\rangle &= n|n\rangle
\end{align*}
\]

(2)

where \( |0\rangle \) is a normalized vacuum (\( a|0\rangle = 0 \) and \( \langle 0|0 \rangle = 1 \)). From (2) state \( |n\rangle \) for \( n \geq 1 \) are given by

\[
|n\rangle = \frac{(a^\dagger)^n}{\sqrt{n!}}|0\rangle.
\]

(3)

These states satisfy the orthogonality and completeness conditions

\[
\langle m|n\rangle = \delta_{mn}, \quad \sum_{n=0}^{\infty} |n\rangle\langle n| = 1.
\]

(4)

Definition We call a state

\[
|z\rangle = e^{za}\bar{z}|0\rangle \equiv U(z)|0\rangle \quad \text{for} \quad z \in \mathbb{C}
\]

(5)

the coherent state.

2.2 Generalized Coherent Operator Based on \( su(1, 1) \)

Let us state generalized coherent operators and states based on \( su(1, 1) \).
We consider a spin $K \ (> 0)$ representation of $su(1, 1) \subset sl(2, \mathbb{C})$ and set its generators
$\{K_+, K_-, K_3\}$ ($\langle K_+ \rangle^\dagger = K_-$),

\[
[K_3, K_+] = K_+, \quad [K_3, K_-] = -K_-, \quad [K_+, K_-] = -2K_3.
\] (6)

We note that this (unitary) representation is necessarily infinite dimensional. The Fock space on which $\{K_+, K_-, K_3\}$ act is $\mathcal{H}_K \equiv \{ |K, n\rangle | n \in \mathbb{N} \cup \{0\} \}$ and whose actions are

\[
K_+ |K, n\rangle = \sqrt{(n + 1)(2K + n)} |K, n + 1\rangle, \quad K_- |K, n\rangle = \sqrt{n(2K + n - 1)} |K, n - 1\rangle,
\]

\[
K_3 |K, n\rangle = (K + n) |K, n\rangle,
\] (7)

where $|K, 0\rangle$ is a normalized vacuum ($K_- |K, 0\rangle = 0$ and $\langle K, 0 |K, 0\rangle = 1$). We have written $|K, 0\rangle$ instead of $|0\rangle$ to emphasize the spin $K$ representation, see [4]. From (7), states $|K, n\rangle$ are given by

\[
|K, n\rangle = \frac{(K_+)^n}{\sqrt{n!(2K)_n}} |K, 0\rangle,
\] (8)

where $(a)_n$ is the Pochammer’s notation $(a)_n \equiv a(a + 1) \cdots (a + n - 1)$. These states satisfy the orthogonality and completeness conditions

\[
\langle K, m |K, n\rangle = \delta_{mn}, \quad \sum_{n=0}^{\infty} |K, n\rangle \langle K, n| = 1_K.
\] (9)

Now let us consider a generalized version of coherent states :

**Definition** We call a state

\[
|z\rangle = V(z)|K, 0\rangle \equiv e^{zK_+ - \bar{z}K_-} |K, 0\rangle \quad \text{for} \quad z \in \mathbb{C}.
\] (10)

the generalized coherent state (or the coherent state of Perelomov’s type based on $su(1, 1)$ in our terminology).

Here let us construct an example of this representation. First we set

\[
K_+ \equiv \frac{1}{2} \left( a^\dagger \right)^2, \quad K_- \equiv \frac{1}{2} a^2, \quad K_3 \equiv \frac{1}{2} \left( a^\dagger a + \frac{1}{2} \right),
\] (11)

then it is easy to check that these satisfy the commutation relations (6). That is, the set $\{K_+, K_-, K_3\}$ gives a unitary representation of $su(1, 1)$ with spin $K = 1/4$ and $3/4$, [3].

Now we also call an operator

\[
S(z) = e^{\frac{1}{2}(za^\dagger)^2 - \bar{z}a^2} \quad \text{for} \quad z \in \mathbb{C}
\] (12)
the squeezed operator, see the book [3].

2.3 Generalized Coherent Operator Based on $su(2)$

Let us state generalized coherent operators and states based on $su(2)$.

We consider a spin $J (> 0)$ representation of $su(2) \subset sl(2, \mathbb{C})$ and set its generators \( \{ J_+, J_-, J_3 \} \) \( (J_+)^\dagger = J_- \),

\[
\begin{align*}
[J_3, J_+] &= J_+, \quad [J_3, J_-] = -J_- , \quad [J_+, J_-] = 2J_3 .
\end{align*}
\] (13)

We note that this (unitary) representation is necessarily finite dimensional. The Fock space on which \( \{ J_+, J_-, J_3 \} \) act is \( \mathcal{H}_J \equiv \{ |J, n\rangle | 0 \leq n \leq 2J \} \) and whose actions are

\[
\begin{align*}
J_+|J, n\rangle &= \sqrt{(n+1)(2J-n)}|J, n+1\rangle , \quad J_-|J, n\rangle = \sqrt{n(2J-n+1)}|J, n-1\rangle , \\
J_3|J, n\rangle &= (-J + n)|J, n\rangle ,
\end{align*}
\] (14)

where \( |J, 0\rangle \) is a normalized vacuum \( \langle J, 0| J, 0\rangle = 0 \) and \( \langle J, 0| J, 0\rangle = 1 \). We have written \( |J, 0\rangle \) instead of \( |0\rangle \) to emphasize the spin \( J \) representation, see [4]. From (14), states \( |J, n\rangle \) are given by

\[
|J, n\rangle = \frac{(J_+)^n}{\sqrt{n!2J^n}}|J, 0\rangle .
\] (15)

These states satisfy the orthogonality and completeness conditions

\[
\langle J, m|J, n\rangle = \delta_{mn} , \quad \sum_{n=0}^{2J} |J, n\rangle \langle J, n| = 1_J .
\] (16)

Now let us consider a generalized version of coherent states:

Definition We call a state

\[
|z\rangle = W(z)|J, 0\rangle \equiv e^{zJ_+ - \bar{z}J_-}|J, 0\rangle \quad \text{for} \quad z \in \mathbb{C} .
\] (17)

the generalized coherent state (or the coherent state of Perelomov’s type based on $su(2)$ in our terminology).

A comment is in order. We can construct the spin \( K \) and \( J \) representations by making use of Schwinger’s boson method. But we don’t repeat here, see for example [7].
3 Matrix Elements of Coherent and Generalized Coherent Operators · · · [14]

3.1 Matrix Elements of Coherent Operator

We list matrix elements of coherent operators $U(z)$.

The Matrix Elements

The matrix elements of $U(z)$ are:

(i) $n \leq m$ \[ \langle n|U(z)|m \rangle = e^{-\frac{1}{2}|z|^2} \sqrt{\frac{n!}{m!}} (-\bar{z})^{m-n} L_n^{(m-n)}(|z|^2), \] (18)

(ii) $n \geq m$ \[ \langle n|U(z)|m \rangle = e^{-\frac{1}{2}|z|^2} \sqrt{\frac{m!}{n!}} \bar{z}^{n-m} L_m^{(n-m)}(|z|^2), \] (19)

where $L_n^{(\alpha)}$ is the associated Laguerre’s polynomial defined by

\[ L_k^{(\alpha)}(x) = \sum_{j=0}^{k} (-1)^j \binom{k+\alpha}{k-j} \frac{x^j}{j!}. \] (20)

In particular $L_k \equiv L_k^{(0)}$ is the usual Laguerre’s polynomial and these are related to diagonal elements of $U(z)$.

3.2 Matrix Elements of Coherent Operator Based on $su(1, 1)$

We list matrix elements of $V(z)$ coherent operators based on $su(1, 1)$. In this case it is always $2K > 1$ ($2K = 1$ under some regularization).

The Matrix Elements

The matrix elements of $V(z)$ are:

(i) $n \leq m$ \[ \langle K, n|V(z)|K, m \rangle = \sqrt{\frac{n!m!}{(2K)_n(2K)_m}} (-\bar{\kappa})^{m-n}(1 + |\kappa|^2)^{-\frac{n+m}{2}} \times \]
\[ \sum_{j=0}^{n} (-1)^{n-j} \frac{\Gamma(2K+m+n-j)}{\Gamma(2K)(m-j)!(n-j)!j!} (1 + |\kappa|^2)^{j}(|\kappa|^2)^{n-j}, \] (21)

(ii) $n \geq m$ \[ \langle K, n|V(z)|K, m \rangle = \sqrt{\frac{n!m!}{(2K)_n(2K)_m}} \kappa^{n-m}(1 + |\kappa|^2)^{-\frac{n+m}{2}} \times \]
\[ \sum_{j=0}^{m} (-1)^{m-j} \frac{\Gamma(2K+m+n-j)}{\Gamma(2K)(m-j)!(n-j)!j!} (1 + |\kappa|^2)^{j}(|\kappa|^2)^{m-j}, \] (22)

where

\[ \kappa \equiv \frac{\sinh(|z|)}{|z|} \bar{z} = \cosh(|z|) \zeta. \] (23)
The author doesn’t know whether or not the right hand sides of (21) and (22) could be written by making use of some special functions such as generalized Laguerre’s functions in (20). Therefore we set temporarily

\[ F_m(n-m)(x:2K) = \sum_{j=0}^{m} (-1)^{m-j} \frac{\Gamma(2K + m + n - j)}{\Gamma(2K)(m-j)!(n-j)!j!} (1+x)^j x^{m-j} \]  

and \( F_m^{(0)}(x;2K) = F_m(x;2K) \).

### 3.3 Matrix Elements of Coherent Operator Based on \textit{su}(2)

We list matrix elements of \( W(z) \) coherent operators based on \textit{su}(2). In this case it is always \( 2J \in \mathbb{N} \).

**Matrix Elements**

The matrix elements of \( W(z) \) are:

(i) \( n \leq m \) \( \langle J,n|W(z)|J,m \rangle = \sqrt{\frac{n!m!}{2J P_{n2J}P_m}} (-\bar{\kappa})^{m-n}(1-|\kappa|^2)^{\frac{n+m}{2}} \times \sum_{j=0}^{n} (-1)^{n-j} (2J)! \frac{(2J-m-n+j)!(m-j)!}{(m-j)!j!} (1-|\kappa|^2)^j (|\kappa|^2)^{n-j} \) \hspace{1cm} \text{(25)}

(ii) \( n \geq m \) \( \langle J,n|W(z)|J,m \rangle = \sqrt{\frac{n!m!}{2J P_{n2J}P_m}} |\kappa|^{n-m}(1-|\kappa|^2)^{\frac{n+m}{2}} \times \sum_{j=0}^{m} (-1)^{m-j} (2J)! \frac{(2J-m-n+j)!(m-j)!}{(m-j)!j!} (1-|\kappa|^2)^j (|\kappa|^2)^{m-j} \) \hspace{1cm} \text{(26)}

where

\[ \kappa \equiv \frac{\sin(|z|)}{|z|} z = \cos(|z|) \eta. \]  

Here \( \sum_{*} \) means a summation over \( j \) satisfying \( 2J - m - n + j \geq 0 \).

The author doesn’t know whether or not the right hand sides of (25) and (26) could be written by making use of some special functions. We set temporarily

\[ F_m(n-m)(x:2J) = \sum_{j=0}^{m} (-1)^{m-j} \frac{(2J)!}{(2J-m-n+j)!(m-j)!j!} (1-x)^j x^{m-j} \]  

and \( F_m^{(0)}(x;2J) = F_m(x;2J) \).
4 Jaynes–Cummings Models in the Strong Coupling Regime

In [12] Frasca treated the Jaynes–Cummings model and developed some method to calculate Rabi frequencies in the strong coupling regime. We in this section generalize the model and method, and show that Rabi frequencies in our extended model are given by matrix elements of generalized coherent operators under the rotating-wave approximation. This gives a unified approach to them.

Let \( \{\sigma_1, \sigma_2, \sigma_3\} \) be Pauli matrices and \( 1_2 \) a unit matrix:

\[
\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad 1_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{29}
\]

The Hamiltonian adopted in [12] is

\[
(H_N) \quad H_N = \omega 1_2 \otimes a^\dagger a + \frac{\Delta}{2} \sigma_3 \otimes 1 + g \sigma_1 \otimes (a^\dagger + a) \tag{30}
\]

where \( \omega \) is the frequency of the radiation mode, \( \Delta \) the separation between the two levels of the atom, \( g \) the coupling between the radiation field and the atom.

Moreover we want to treat the following Hamiltonians (our extension)

\[
(H_K) \quad H_K = \omega 1_2 \otimes K_3 + \frac{\Delta}{2} \sigma_3 \otimes 1_K + g \sigma_1 \otimes (K_+ + K_-), \tag{31}
\]

\[
(H_J) \quad H_J = \omega 1_2 \otimes J_3 + \frac{\Delta}{2} \sigma_3 \otimes 1_J + g \sigma_1 \otimes (J_+ + J_-). \tag{32}
\]

To treat these three cases at the same time we set

\[
\{L_+, L_-, L_3\} = \begin{cases} (H_N) & \{a^\dagger, a, N\} \\ (H_K) & \{K_+, K_-, K_3\} \\ (H_J) & \{J_+, J_-, J_3\} \end{cases} \tag{33}
\]

and

\[
H = H_0 + V = \omega 1_2 \otimes L_3 + \frac{\Delta}{2} \sigma_3 \otimes 1_L + g \sigma_1 \otimes (L_+ + L_-) \tag{34}
\]

where we have written \( H \) instead of \( H_L \) for simplicity.
Mysteriously enough we cannot solve these simple models completely (maybe non-integrable), nevertheless we have found these models have a very rich structure.

For these (non-integrable) models we usually have two perturbation approaches:

**Weak Coupling Regime** \( (0 < g \ll \Delta) \)

\[
H_0 = \omega \mathbf{1}_2 \otimes L_3 + \frac{\Delta}{2} \sigma_3 \otimes \mathbf{1}_L, \quad V = g \sigma_1 \otimes (L_+ + L_-).
\] (35)

**Strong Coupling Regime** \( (0 < \Delta \ll g) \)

\[
H_0 = \omega \mathbf{1}_2 \otimes L_3 + g \sigma_1 \otimes (L_+ + L_-), \quad V = \frac{\Delta}{2} \sigma_3 \otimes \mathbf{1}_L.
\] (36)

In the following we consider only the strong coupling regime (see [10] for the weak one).

First let us solve \( H_0 \) which is a relatively easy task.

Let \( W \) be a Walsh–Hadamard matrix

\[
W = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = W^{-1}
\]

then we can diagonalize \( \sigma_1 \) by using this \( H \) as \( \sigma_1 = W \sigma_3 W^{-1} \). The eigenvalues of \( \sigma_1 \) is \( \{1, -1\} \) with eigenvectors

\[
|1\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad |-1\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad \Rightarrow \quad |\lambda\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ \lambda \end{pmatrix}.
\]

We note that

\[
|1\rangle\langle 1| = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = W \begin{pmatrix} 1 \\ 0 \end{pmatrix} W^{-1},
\]

\[
|-1\rangle\langle -1| = \frac{1}{2} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} = W \begin{pmatrix} 0 \\ 1 \end{pmatrix} W^{-1},
\]

\[
\Rightarrow \quad |\lambda\rangle\langle \lambda| = \frac{1}{2} \begin{pmatrix} 1 & \lambda \\ \lambda & 1 \end{pmatrix} = W \begin{pmatrix} \frac{1+\lambda}{2} \\ \frac{1-\lambda}{2} \end{pmatrix} W^{-1}.
\]

Then we have

\[
H_0 = (W \otimes \mathbf{1}_L) (\omega \mathbf{1}_2 \otimes L_3 + g \sigma_3 \otimes (L_+ + L_-)) (W^{-1} \otimes \mathbf{1}_L)
\]

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\[
(W \otimes 1_L) \begin{pmatrix}
\omega L_3 + g(L_+ + L_-) \\
\omega L_3 - g(L_+ + L_-)
\end{pmatrix} (W^{-1} \otimes 1_L).
\]
\[
= |1\rangle \langle 1| \otimes \{\omega L_3 + g(L_+ + L_-)\} + |-1\rangle \langle -1| \otimes \{\omega L_3 - g(L_+ + L_-)\}
\]
\[
= \sum_\lambda |\lambda\rangle \langle \lambda| \otimes \{\omega L_3 + \lambda g(L_+ + L_-)\}
\]
\[
= \sum_\lambda |\lambda\rangle \langle \lambda| \otimes \left\{ e^{-\frac{I}{2}(L_+ - L_-)} \left( \Omega L_3 \right) e^{\frac{I}{2}(L_+ - L_-)} \right\}
\]
\[
= \sum_\lambda \left( |\lambda\rangle \otimes e^{-\frac{I}{2}(L_+ - L_-)} \right) \left( \Omega L_3 \right) \left( |\lambda\rangle \otimes e^{\frac{I}{2}(L_+ - L_-)} \right) \tag{37}
\]

where we have used the following

**Key Formulas**

For \(\lambda = \pm 1\) we have

\[
(N) \quad \omega a^+ a + \lambda g(a^+ + a) = \Omega e^{-\frac{I}{2}(a^+ - a)} \left( N - \frac{g^2}{\omega^2} \right) e^{\frac{I}{2}(a^+ - a)}
\]
where \(\Omega = \omega, \quad x = 2g/\omega\), \(\tag{38}\)

\[
(K) \quad \omega K_3 + \lambda g(K_+ + K_-) = \Omega e^{-\frac{I}{2}(K_3 - K_-)} K_3 e^{\frac{I}{2}(K_3 - K_-)}
\]
where \(\Omega = \omega \sqrt{1 - (2g/\omega)^2}, \quad x = \tanh^{-1}(2g/\omega)\), \(\tag{39}\)

\[
(J) \quad \omega J_3 + \lambda g(J_+ + J_-) = \Omega e^{-\frac{I}{2}(J_3 - J_-)} J_3 e^{\frac{I}{2}(J_3 - J_-)}
\]
where \(\Omega = \omega \sqrt{1 + (2g/\omega)^2}, \quad x = \tan^{-1}(2g/\omega)\). \(\tag{40}\)

The proof is not difficult, so we leave it to the readers. That is, we could diagonalize the Hamiltonian \(H_0\). This is two-fold degenerate and its eigenvalues and eigenvectors are given respectively

\[
\text{(Eigenvalues, Eigenvectors)} = \begin{cases} 
(N) & \omega n - \frac{g^2}{\omega}, \quad \lambda \otimes e^{-\frac{I}{2}(a^+ - a)} |n\rangle \\
(K) & \Omega(K + n), \quad \lambda \otimes e^{-\frac{I}{2}(K_3 - K_-)} |K, n\rangle \\
(J) & \Omega(-J + n), \quad \lambda \otimes e^{-\frac{I}{2}(J_3 - J_-)} |J, n\rangle
\end{cases} \tag{41}
\]

for \(\lambda = \pm 1\) and \(n \in \mathbb{N} \cup \{0\}\). For the latter convenience we set

\[
\text{Eigenvalues} = \{E_n\}, \quad \text{Eigenvectors} = \{|\{\lambda, n\}\rangle\}. \tag{42}
\]

Then \((37)\) can be written as

\[
H_0 = \sum_\lambda \sum_n E_n |\lambda, n\rangle \langle \lambda, n| \tag{43}
\]
Next we would like to solve the following Schrödinger equation:

\[ \frac{d}{dt} \Psi = H \Psi = \left( H_0 + \frac{\Delta}{2} \sigma_3 \otimes 1_L \right) \Psi, \]  

where we have set \( \hbar = 1 \) for simplicity. To solve this equation we appeal to the method of constant variation. First let us solve

\[ \frac{d}{dt} \Psi = H_0 \Psi, \]  

which general solution is given by

\[ \Psi(t) = U_0(t) \Psi_0 = e^{-itH_0} \Psi_0 \]  

where \( \Psi_0 \) is a constant state. It is easy to see from (43) \( U_0(t) = e^{-itH_0} = \sum_\lambda \sum_n e^{-itE_n} \langle \{\lambda,n\} | \{\lambda,n\} \rangle \).

The method of constant variation goes as follows. Changing like \( \Psi_0 \rightarrow \Psi_0(t) \), we insert (46) into (42). After some algebra we obtain

\[ \frac{d}{dt} \Psi_0 = \frac{\Delta}{2} U_0^\dagger (\sigma_3 \otimes 1_L) U_0 \Psi_0. \]  

We have only to solve this equation. If we set

\[ H_F = \frac{\Delta}{2} U_0^\dagger (\sigma_3 \otimes 1_L) U_0, \]  

then we have easily from (47)

\[ H_F = \frac{\Delta}{2} \sum_{\lambda,\mu} \sum_{m,n} e^{it(E_{m} - E_{n})} \langle \{\lambda,m\} | (\sigma_3 \otimes 1_L) | \{\mu,n\} \rangle \langle \{\lambda,m\} | \{\mu,n\} \rangle \]

\[ = \frac{\Delta}{2} \sum_{\lambda} \sum_{n} e^{it\Omega m-n} \langle \{m | e^{it(L_+ - L_-)} | n\} \rangle | \{\lambda,m\} \rangle \langle \{-\lambda,n\} \rangle \]

where we have used the relation \( \langle \lambda | \sigma_3 = (-\lambda) \). Remind that \( |n\rangle \) is respectively

\[
|n\rangle = \begin{cases} 
(N) & |n\rangle \\
(K) & |K,n\rangle \\
(J) & |J,n\rangle 
\end{cases}
\]
In this stage we meet **matrix elements of the coherent and generalized coherent operators** \( e^{\lambda x (L_+ - L_-)} \) in section 3 \((z = \bar{z} = \lambda x)\).

Here we divide \( H_F \) into two parts

\[
H_F = H_F^\prime + H_F^\prime\prime
\]

where

\[
H_F^\prime = \frac{\Delta}{2} \sum_\lambda \sum_n \langle \langle n | e^{\lambda x (L_+ - L_-)} | n \rangle \rangle \{ \{ \lambda, n \} \} \langle \{ -\lambda, n \} \rangle,
\]

\[
H_F^\prime\prime = \frac{\Delta}{2} \sum_\lambda \sum_{m,n} e^{i\Omega (m-n)} \langle \langle m | e^{\lambda x (L_+ - L_-)} | n \rangle \rangle \{ \{ \lambda, m \} \} \langle \{ -\lambda, n \} \rangle.
\]

Noting

\[
\langle \langle n | e^{x (L_+ - L_-)} | n \rangle \rangle = \langle \langle n | e^{-x (L_+ - L_-)} | n \rangle \rangle
\]

by the results in section 3, \( H_F^\prime \) can be written as

\[
H_F^\prime = \frac{\Delta}{2} \sum_n \langle \langle n | e^{x (L_+ - L_-)} | n \rangle \rangle \{ \{ 1, n \} \} \{ \{ -1, n \} \} + \{ \{ -1, n \} \} \{ \{ 1, n \} \},
\]

from which we can diagonalize \( H_F^\prime \) as

\[
H_F^\prime = \frac{\Delta}{2} \sum_n \sum_\sigma \langle \langle n | e^{x (L_+ - L_-)} | n \rangle \rangle \sigma \{ \{ \sigma, \psi_n \} \} \langle \{ \sigma, \psi_n \} \rangle
\]

if we define a new basis

\[
\{ \{ \sigma, \psi_n \} \} = \frac{1}{\sqrt{2}} (\{ \{ 1, \psi_n \} \} - \{ \{ -1, \psi_n \} \}), \quad \sigma = \pm 1.
\]

These states can be seen as so-called Schrödinger cat states, \([15]\). From these we have

\[
\{ \{ 1, n \} \} = \frac{1}{\sqrt{2}} \{ \{ 1, \psi_n \} \} - \{ \{ -1, \psi_n \} \},
\]

\[
\{ \{ -1, n \} \} = \frac{1}{\sqrt{2}} \{ \{ 1, \psi_n \} \} + \{ \{ -1, \psi_n \} \}.
\]

Inserting these equations into \((52)\) and taking some algebras we obtain

\[
H_F^\prime\prime = \frac{\Delta}{2} \sum_{m,n} \sum_{\sigma,\sigma'} e^{i\Omega (m-n)} \left\{ \langle \langle m | e^{x (L_+ - L_-)} | n \rangle \rangle \frac{\sigma}{2} \{ \{ \sigma, \psi_m \} \} \{ \{ \sigma', \psi_n \} \} + \langle \langle m | e^{x (L_+ - L_-)} | n \rangle \rangle \frac{\sigma'}{2} \{ \{ \sigma, \psi_m \} \} \{ \{ \sigma', \psi_n \} \} \right\}.
\]
For simplicity in (53) we set in the following

\[ E_{n,\sigma} = \frac{\Delta}{2} \sigma \langle \langle n | e^{x(L_+ - L_-)} | n \rangle \rangle, \]  

(55)

then

\[ E_{n,\sigma} = \begin{cases} 
(N) & \frac{\Delta}{2} \sigma e^{-\frac{\omega^2 t \Delta^2}{2}} L_n \left( \frac{4\omega^2}{\omega^2} \right) \\
(K) & \frac{\Delta}{2} \sigma e^{-\frac{\omega^2 t \Delta^2}{2}} L_n \left( \frac{4\omega^2}{\omega^2} \right) \left( 1 - |\kappa|^2 \right) \left( \kappa + : 2K \right) \\
(J) & \frac{\Delta}{2} \sigma e^{-\frac{\omega^2 t \Delta^2}{2}} L_n \left( \frac{4\omega^2}{\omega^2} \right) \left( 1 - |\kappa|^2 \right) \left( \kappa + : 2J \right) 
\end{cases} \]  

(56)

from (41) and the results in section 3.1. Now let us solve (48)

\[ \frac{d}{dt} \Psi_0 = \frac{\Delta}{2} H_F \Psi_0 = \frac{\Delta}{2} (H_F' + H_F'') \Psi_0. \]

For that if we set \( \Psi_0(t) \) as

\[ \Psi_0(t) = \sum_{\sigma} \sum_{n} e^{-itE_{n,\sigma}} a_{n,\sigma}(t) \{|\sigma, \psi_n\} \rangle, \]

(57)

then we have a set of complicated equations with respect to \( \{a_{n,\sigma}\} \), see [12]. But it is almost impossible to solve them. Therefore we make a daring assumption : for \( m < n \)

\[ \Psi_0(t) = \sum_{\sigma} e^{-itE_{m,\sigma}} a_{m,\sigma}(t) \{|\sigma, \psi_m\} \rangle + \sum_{\sigma} e^{-itE_{n,\sigma}} a_{n,\sigma}(t) \{|\sigma, \psi_n\} \rangle. \]

(58)

That is, we consider only two terms with respect to \( \{n|n \geq 0\} \). After some algebras we obtain

\[ i \frac{d}{dt} a_{m,\sigma} = \frac{\Delta}{2} \sum_{\sigma'} e^{-it(E_{m,\sigma} - E_{m,\sigma})} e^{it\Omega(m-n)} \left\{ \langle \langle m | e^{x(L_+ - L_-)} | n \rangle \rangle \right\} \frac{\sigma}{2} a_{n,\sigma} \]

(59)

But we cannot still solve the above equations exactly (see Appendix), so let us make so-called rotating-wave approximation. The resonance condition is

\[ -(E_{n,\sigma'} - E_{m,\sigma}) + (m - n)\Omega = 0 \quad \Rightarrow \quad E_{n,\sigma'} - E_{m,\sigma} = (m - n)\Omega \]

(60)

for some \( \sigma \) and \( \sigma' \), and we reject the remaining term in (59). Then we obtain simple equations :
Interband Transition Case \((\sigma \neq \sigma')\)

\[
E_{n,-\sigma} - E_{m,\sigma} = (m - n)\Omega
\]

\[
\begin{align*}
\frac{d}{dt} a_{m,\sigma} &= \frac{\Delta}{2} \left\{ \frac{\langle \langle m | e^{x(L_+ - L_-)} | n \rangle \rangle^{\sigma}}{2} - \frac{\langle \langle m | e^{-x(L_+ - L_-)} | n \rangle \rangle^{\sigma}}{2} \right\} a_{n,-\sigma} \\
&= \frac{\Delta}{2} \sigma \langle \langle m | \sinh (x(L_+ - L_-)) | n \rangle \rangle a_{n,-\sigma},
\end{align*}
\]

\[
(61)
\]

\[
\begin{align*}
\frac{d}{dt} a_{n,-\sigma} &= \frac{\Delta}{2} \sum_{\sigma'} \left\{ -\frac{\langle \langle n | e^{x(L_+ - L_-)} | m \rangle \rangle^{\sigma}}{2} + \frac{\langle \langle n | e^{-x(a^{1\dagger} - a)} | m \rangle \rangle^{\sigma}}{2} \right\} a_{m,\sigma} \\
&= -\frac{\Delta}{2} \sigma \langle \langle n | \sinh (x(L_+ - L_-)) | m \rangle \rangle a_{m,\sigma}.
\end{align*}
\]

Intraband Transition Case \((\sigma = \sigma')\)

\[
\begin{align*}
\frac{d}{dt} a_{m,\sigma} &= \frac{\Delta}{2} \left\{ \frac{\langle \langle m | e^{x(L_+ - L_-)} | n \rangle \rangle^{\sigma}}{2} + \frac{\langle \langle m | e^{-x(L_+ - L_-)} | n \rangle \rangle^{\sigma}}{2} \right\} a_{n,\sigma} \\
&= \frac{\Delta}{2} \sigma \langle \langle m | \cosh (x(L_+ - L_-)) | n \rangle \rangle a_{n,\sigma},
\end{align*}
\]

\[
(62)
\]

\[
\begin{align*}
\frac{d}{dt} a_{n,\sigma} &= \frac{\Delta}{2} \sum_{\sigma'} \left\{ \frac{\langle \langle n | e^{x(L_+ - L_-)} | m \rangle \rangle^{\sigma}}{2} + \frac{\langle \langle n | e^{-x(L_+ - L_-)} | m \rangle \rangle^{\sigma}}{2} \right\} a_{m,\sigma} \\
&= \frac{\Delta}{2} \sigma \langle \langle n | \cosh (x(L_+ - L_-)) | m \rangle \rangle a_{m,\sigma}.
\end{align*}
\]

For simplicity we set

\[
\mathcal{R} = \Delta \langle \langle n | \sinh (x(L_+ - L_-)) | m \rangle \rangle, \quad \mathcal{R}' = \Delta \langle \langle n | \cosh (x(L_+ - L_-)) | m \rangle \rangle,
\]

\[
(63)
\]

then

\[
\Delta \langle \langle m | \sinh (x(L_+ - L_-)) | n \rangle \rangle = -\mathcal{R}, \quad \Delta \langle \langle m | \cosh (x(L_+ - L_-)) | n \rangle \rangle = \mathcal{R}'.
\]

These are two Rabi frequencies as shown in the following. It is important that Rabi frequencies in our models are given by matrix elements of coherent and generalized coherent operators!

By making use of the results in section 3 and \((38), (39), (40)\) we have

\[
(N) \begin{cases}
\mathcal{R} = \frac{\Delta}{2} \sqrt{\frac{m!}{n!}} \left( \frac{2a}{\omega} \right)^{n-m} e^{-2a^2/\omega^2} L_m^{(n-m)} \left( \frac{4a^2}{\omega^2} \right) \{ 1 - (-1)^{n-m} \} \\
\mathcal{R}' = \frac{\Delta}{2} \sqrt{\frac{m!}{n!}} \left( \frac{2a}{\omega} \right)^{n-m} e^{-2a^2/\omega^2} L_m^{(n-m)} \left( \frac{4a^2}{\omega^2} \right) \{ 1 + (-1)^{n-m} \}
\end{cases}
\]

(64)
(K) 
\[ \mathcal{R} = \frac{\Delta}{2} \sqrt{\frac{n!m!}{(2K)_n(2K)_m}} \kappa^{n-m} (1 + |\kappa|^2)^{-K - \frac{a+m}{2}} F_{m}^{(n-m)}(|\kappa|^2 : 2K) \{1 - (-1)^{n-m}\} \]
\[ \mathcal{R}' = \frac{\Delta}{2} \sqrt{\frac{n!m!}{(2K)_n(2K)_m}} \kappa^{n-m} (1 + |\kappa|^2)^{-K - \frac{a+m}{2}} F_{m}^{(n-m)}(|\kappa|^2 : 2K) \{1 + (-1)^{n-m}\} \]

where \( \kappa = \sinh (x) \) with \( x = \tanh^{-1} \left( \frac{2g}{\omega} \right) \)

(J) 
\[ \mathcal{R} = \frac{\Delta}{2} \sqrt{\frac{n!m!}{2J P_{n+m} P_{m}}} \kappa^{n-m} (1 - |\kappa|^2)^{J - \frac{a+m}{2}} F_{m}^{(n-m)}(|\kappa|^2 : 2J) \{1 - (-1)^{n-m}\} \]
\[ \mathcal{R}' = \frac{\Delta}{2} \sqrt{\frac{n!m!}{2J P_{n+m} P_{m}}} \kappa^{n-m} (1 - |\kappa|^2)^{J - \frac{a+m}{2}} F_{m}^{(n-m)}(|\kappa|^2 : 2J) \{1 + (-1)^{n-m}\} \]

where \( \kappa = \sin (x) \) with \( x = \tan^{-1} \left( \frac{2g}{\omega} \right) \)

From these we find a constraint between \( m \) and \( n \):

**Interband Case** \( n - m = 2N - 1 \implies n = m + 2N - 1 \) for \( N \in \mathbb{N} \),

**Intraband Case** \( n - m = 2N \implies n = m + 2N \) for \( N \in \mathbb{N} \).

Now let us solve (61) and (62).

\[ i \frac{d}{dt} \begin{pmatrix} a_{m,\sigma} \\ a_{n,-\sigma} \end{pmatrix} = \begin{pmatrix} 0 & -\sigma \frac{\kappa}{2} \\ -\sigma \frac{\kappa}{2} & 0 \end{pmatrix} \begin{pmatrix} a_{m,\sigma} \\ a_{n,-\sigma} \end{pmatrix}, \]

\[ \begin{align*}
&= \begin{pmatrix} 0 & \sigma \frac{\kappa'}{2} \\ \sigma \frac{\kappa'}{2} & 0 \end{pmatrix} \begin{pmatrix} a_{m,\sigma} \\ a_{n,\sigma} \end{pmatrix}, \\
&= \begin{pmatrix} \cos(\frac{\kappa}{2}t) & i\sigma \sin(\frac{\kappa}{2}t) \\ i\sigma \sin(\frac{\kappa}{2}t) & \cos(\frac{\kappa}{2}t) \end{pmatrix} \begin{pmatrix} a_{m,\sigma}(0) \\ a_{n,-\sigma}(0) \end{pmatrix}, \\
&= \begin{pmatrix} \cos(\frac{\kappa'}{2}t) & -i\sigma \sin(\frac{\kappa'}{2}t) \\ -i\sigma \sin(\frac{\kappa'}{2}t) & \cos(\frac{\kappa'}{2}t) \end{pmatrix} \begin{pmatrix} a_{m,\sigma}(0) \\ a_{n,\sigma}(0) \end{pmatrix}.
\]

We have obtained some solutions under the rotating–wave approximation. Now it may be suited to compare our results with a recent experimental finding in [11], but this is beyond our scope. See [12].

Let us conclude this section by a comment. Our ansatz (58) to solve the equation is too restrictive. We want to use (57) to solve the equation, but it is very hard at this stage.
Problem  Find more dynamic methods!

5 Quantum Computation

Let us reconsider the results in the preceding section in the light of Quantum Computation. Remind once more that the following arguments are based on the rotating-wave approximation.

Interband Case ($\sigma = 1$)

\[
\begin{pmatrix}
  a_{m,1} \\
  a_{m,-1} \\
  a_{n,1} \\
  a_{n,-1}
\end{pmatrix}
= 
\begin{pmatrix}
  0 & 0 & 0 & \frac{-R}{2} \\
  0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 \\
  -\frac{R}{2} & 0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
  a_{m,1} \\
  a_{m,-1} \\
  a_{n,1} \\
  a_{n,-1}
\end{pmatrix}. 
\tag{69}
\]

The solution is

\[
\begin{pmatrix}
  a_{m,1}(t) \\
  a_{m,-1}(t) \\
  a_{n,1}(t) \\
  a_{n,-1}(t)
\end{pmatrix}
= 
\begin{pmatrix}
  \cos\left(\frac{R}{2}t\right) & 0 & 0 & i\sin\left(\frac{R}{2}t\right) \\
  0 & 1 & 0 & 0 \\
  0 & 0 & 1 & 0 \\
  i\sin\left(\frac{R}{2}t\right) & 0 & 0 & \cos\left(\frac{R}{2}t\right)
\end{pmatrix}
\begin{pmatrix}
  a_{m,1}(0) \\
  a_{m,-1}(0) \\
  a_{n,1}(0) \\
  a_{n,-1}(0)
\end{pmatrix}. 
\tag{70}
\]

Interband Case ($\sigma = -1$)

\[
\begin{pmatrix}
  a_{m,1} \\
  a_{m,-1} \\
  a_{n,1} \\
  a_{n,-1}
\end{pmatrix}
= 
\begin{pmatrix}
  0 & 0 & 0 & 0 \\
  0 & 0 & \frac{R}{2} & 0 \\
  0 & \frac{R}{2} & 0 & 0 \\
  0 & 0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
  a_{m,1} \\
  a_{m,-1} \\
  a_{n,1} \\
  a_{n,-1}
\end{pmatrix}. 
\tag{71}
\]
The solution is
\[
\begin{pmatrix}
a_{m,1}(t) \\
a_{m,-1}(t) \\
a_{n,1}(t) \\
a_{n,-1}(t)
\end{pmatrix} =
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & \cos(\frac{\kappa t}{2}) & -\sin(\frac{\kappa t}{2}) & 0 \\
0 & -\sin(\frac{\kappa t}{2}) & \cos(\frac{\kappa t}{2}) & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
a_{m,1}(0) \\
a_{m,-1}(0) \\
a_{n,1}(0) \\
a_{n,-1}(0)
\end{pmatrix},
\]  
(72)

Intraband Case ($\sigma = 1$)

\[
\frac{id}{dt} \begin{pmatrix}
a_{m,1} \\
a_{m,-1} \\
a_{n,1} \\
a_{n,-1}
\end{pmatrix} =
\begin{pmatrix}
0 & 0 & \frac{\kappa'}{2} & 0 \\
0 & 0 & 0 & 0 \\
\frac{\kappa'}{2} & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
a_{m,1} \\
a_{m,-1} \\
a_{n,1} \\
a_{n,-1}
\end{pmatrix}.
\]  
(73)

The solution is
\[
\begin{pmatrix}
a_{m,1}(t) \\
a_{m,-1}(t) \\
a_{n,1}(t) \\
a_{n,-1}(t)
\end{pmatrix} =
\begin{pmatrix}
\cos(\frac{\kappa' t}{2}) & 0 & -\sin(\frac{\kappa' t}{2}) & 0 \\
0 & 1 & 0 & 0 \\
-\sin(\frac{\kappa' t}{2}) & 0 & \cos(\frac{\kappa' t}{2}) & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
a_{m,1}(0) \\
a_{m,-1}(0) \\
a_{n,1}(0) \\
a_{n,-1}(0)
\end{pmatrix}.
\]  
(74)

Intraband Case ($\sigma = -1$)

\[
\frac{id}{dt} \begin{pmatrix}
a_{m,1} \\
a_{m,-1} \\
a_{n,1} \\
a_{n,-1}
\end{pmatrix} =
\begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & -\frac{\kappa'}{2} \\
0 & 0 & 0 & 0 \\
0 & -\frac{\kappa'}{2} & 0 & 0
\end{pmatrix}
\begin{pmatrix}
a_{m,1} \\
a_{m,-1} \\
a_{n,1} \\
a_{n,-1}
\end{pmatrix}.
\]  
(75)

The solution is
\[
\begin{pmatrix}
a_{m,1}(t) \\
a_{m,-1}(t) \\
a_{n,1}(t) \\
a_{n,-1}(t)
\end{pmatrix} =
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & \cos(\frac{\kappa' t}{2}) & 0 & \sin(\frac{\kappa' t}{2}) \\
0 & 0 & 1 & 0 \\
0 & \sin(\frac{\kappa' t}{2}) & 0 & \cos(\frac{\kappa' t}{2})
\end{pmatrix}
\begin{pmatrix}
a_{m,1}(0) \\
a_{m,-1}(0) \\
a_{n,1}(0) \\
a_{n,-1}(0)
\end{pmatrix}.
\]  
(76)
If we can identify (58) with an element in two-qubit space

$$a_{m,1}(t)|00⟩ + a_{m,-1}(t)|01⟩ + a_{n,1}(t)|10⟩ + a_{n,-1}(t)|11⟩ \in C^2 \otimes C^2 \quad (77)$$

where \(C^2 = \text{Vect}_C\{|0⟩, |1⟩\}\), then the solutions (70), (72), (74), (76) are kinds of controlled unitary operations (gates) which play a crucial role in Quantum Computation, see for example [10]. For example, (76) is just one of controlled unitary gates expressed graphically as

![Controlled Unitary Gate](image)

We note here that controlled unitary gates above are written down as

$$C\text{-Unitary} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & u_{11} & 0 & u_{12} \\ 0 & 0 & 1 & 0 \\ 0 & u_{21} & 0 & u_{22} \end{pmatrix} \quad (78)$$

for

$$U = \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix} \in U(2).$$

A comment is in order. [17] (and [18]) is considering the same subject. However the authors in them treated it in the weak coupling regime, while we treated it in the strong coupling regime. It is very interesting to investigate a deep relation (connection) between them.
6 Discussion

One of motivations of this study is to apply our results to Holonomic Quantum Computation developed by Italian group (Pachos, Rasetti and Zanardi) and the author, see [13], [20], [21], [22] and [23]—[27] and recent [28], [29].

In this theory we usually use the effective Hamiltonian of a single–mode field of Kerr medium

\[ H_0 = X N (N - 1), \quad N = a^\dagger a \quad \text{where} \quad X \quad \text{is a constant} \]  

(79)

as a background and the real Hamiltonian is in one–qubit case given by

\[ H(z, w) = W(z, w) H_0 W^{-1}(z, w) \]  

(80)

where \( W \) is a product of coherent operator \( U(z) \) and squeezed one \( S(w) \) in section 2. In the above Hamiltonian \( H_0 \) the zero–eigenvalue is two–fold degenerate whose eigenvectors are \( |0\rangle \) and \( |1\rangle \). We set \( |\text{vac}\rangle = (|0\rangle, |1\rangle) \). Then we can construct a connection form \( \mathcal{A} \) on the parameter space \( \{(z, w) \in \mathbb{C}^2\} \) as

\[ \mathcal{A} = \langle \text{vac}|W^{-1}dW|\text{vac}\rangle \]  

(81)

from (80) where \( d = dz \frac{\partial}{\partial z} + dw \frac{\partial}{\partial w} \). By making use of this connection we can construct a holonomy group \( \text{Hol}(\mathcal{A}) (\subseteq U(2)) \) which is in this case equal to \( U(2) \). In Holonomic Quantum Computation we use this holonomy group as unitary operations in Quantum Computation. The point at issue is that we use not full property of the Hamiltonian but only property of the zero–eigenvalue.

By the way, the Hamiltonian \( H_F \) in (49)

\[ H_F = \frac{\Delta}{2} U_0^{-1} (\sigma_3 \otimes 1_L) U_0 \quad \text{where} \quad L = (N) \quad \text{or} \quad (K) \quad \text{or} \quad (J) \]

is very similar to (80). This system is always two–fold degenerate. Then a natural question arises:

**Problem** Is it possible to perform a holonomic quantum computation by combining the systems \{ (N), (K), (J) \}?
This is a very interesting and challenging problem.

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Appendix

On Equations (59)

Here let us write down full equations of (59) with matrix equation form:

\[
\frac{d}{dt} \begin{pmatrix} a_m \\ a_n \end{pmatrix} = \begin{pmatrix} 0 & A \\ A^\dagger & 0 \end{pmatrix} \begin{pmatrix} a_m \\ a_n \end{pmatrix}
\]

(82)

where

\[
a_k = \begin{pmatrix} a_{k,1} \\ a_{k,-1} \end{pmatrix}
\]

for \( k = m, n \)

and

\[
A \equiv A(t) = \begin{pmatrix} \frac{R'}{2} e^{it(-E_{n,1}+E_{m,1}+\Omega(m-n))} & -\frac{R}{2} e^{it(-E_{n,-1}+E_{m,1}+\Omega(m-n))} \\ \frac{R}{2} e^{it(-E_{n,1}+E_{m,-1}+\Omega(m-n))} & -\frac{R'}{2} e^{it(-E_{n,-1}+E_{m,-1}+\Omega(m-n))} \end{pmatrix}
\]

(83)

because \( E_{k,-\sigma} = -E_{k,\sigma} \).

We can give (82) a formal solution by infinite series (called Dyson series in Theoretical Physics). Then we meet secular terms.

For example let us consider the following simple equation:

\[
\frac{d}{dt} a = e^{i\omega t} a \quad \text{with} \quad a(0) = c.
\]

The solution is given by

\[
a(t) = \begin{cases} 
  c \exp\left(\frac{\text{e}^{i\omega t} - 1}{i\omega}\right) & \omega \neq 0 \\
  c e^t & \omega = 0
\end{cases}
\]
That is, we meet the secular term.

By the way, we have known how to handle (simple) secular terms called Renormalization Group Method (Approach), see [30] for a general introduction.

Frasca in [31] has applied this method to the above equation. The conclusion is interesting, but seems to be rather involved. We are now reconsidering his approach. Therefore let us present

**Problem**  Solve this matrix equation completely!

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