ON A CERTAIN METAPLECTIC EISENSTEIN SERIES AND THE TWISTED SYMMETRIC SQUARE $L$-FUNCTION

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ABSTRACT. In our earlier paper, based on a paper by Bump and Ginzburg, we used an Eisenstein series on the double cover of $\text{GL}(r)$ to obtain the integral representation of the twisted symmetric square $L$-function of $\text{GL}(r)$. Using that, we showed that the (incomplete) twisted symmetric square $L$-function of $\text{GL}(r)$ is holomorphic for $\text{Re}(s) > 1$. In this paper, we will determine the possible poles of this Eisenstein series more precisely and show that the (incomplete) twisted symmetric square $L$-function is entire except possible simple poles at $s = 0$ and $s = 1$.

1. Introduction

Let $\pi \cong \otimes_v \pi_v$ be an irreducible cuspidal automorphic representation of $\text{GL}_r(\mathbb{A})$ and $\chi$ a unitary Hecke character on $\mathbb{A} \times \mathbb{A}$, where $\mathbb{A}$ is the ring of adeles over a number field $F$. By the local Langlands correspondence by Harris-Taylor [HT] and Henniart [He], each $\pi_v$ corresponds to an $r$-dimensional representation $\text{rec}(\pi_v)$ of the Weil-Deligne group $WD_{F_v}$ of $F_v$. We can also consider the twist of $\text{rec}(\pi_v)$ by $\chi_v$, namely,

$$\text{rec}(\pi_v) \otimes \chi_v : WD_{F_v} \to \text{GL}_r(\mathbb{C}),$$

where $\chi_v$ is viewed as a character of $WD_{F_v}$ via local class field theory. Now for each homomorphism $\rho : \text{GL}_r(\mathbb{C}) \to \text{GL}_N(\mathbb{C})$, one can associate the local $L$-factor $L_v(s, \pi_v, \rho \circ \text{rec}(\pi_v) \otimes \chi_v)$ of Artin type. Then one can define the automorphic $L$-function by

$$L(s, \pi, \rho \otimes \chi) := \prod_v L_v(s, \pi_v, \rho \circ \text{rec}(\pi_v) \otimes \chi_v).$$

In particular in this paper, we consider the case where $\rho$ is the symmetric square map $Sym^2 : \text{GL}_r(\mathbb{C}) \to \text{GL}_{\frac{r(r+1)}{2}}(\mathbb{C})$, namely we consider the twisted symmetric square $L$-function $L(s, \pi, Sym^2 \otimes \chi)$. By the Langlands-Shahidi method, it can be shown that the $L$-function $L(s, \pi, Sym^2 \otimes \chi)$ admits meromorphic continuation and a functional equation. (See [Sh1, Theorem 7.7].)

The Langlands-Shahidi method, however, is unable to determine the locations of the possible poles of $L(s, \pi, Sym^2 \otimes \chi)$. The main theme of this paper is to determine them though we consider only the incomplete $L$-function $L^S(s, \pi, Sym^2 \otimes \chi)$. To be more specific, let $S$ be a finite set of places that contains all the archimedean places and non-archimedean places where $\pi$ or $\chi$ ramifies. For $v \notin S$, each $\pi_v$ is parameterized by a set of $r$ complex numbers $\{\alpha_{v,1}, \ldots, \alpha_{v,r}\}$ known as the Satake parameters. Then we have

$$L_v(s, \pi_v, Sym^2 \otimes \chi_v) = \prod_{1 \leq j \leq r} \frac{1}{1 - \chi_v(\omega_v)\alpha_{v,j}\alpha_{v,j}q_v^{-s}},$$
where \( v \) is the uniformizer of \( F \), and \( q_v \) is the order of the residue field, and we set
\[
L^S(s, \pi, \text{Sym}^2 \otimes \chi) = \prod_{v \notin S} L_v(s, \pi_v, \text{Sym}^2 \otimes \chi_v).
\]

As our main theorem (Theorem 7.1) we will prove

**Theorem.** Let \( \pi \) be a cuspidal automorphic representation of \( \text{GL}_r(\mathbb{A}) \) with unitary central character \( \omega_\pi \) and \( \chi \) a unitary Hecke character. Then the incomplete twisted symmetric square \( L \)-function
\[
L^S(s, \pi, \text{Sym}^2 \otimes \chi)
\]
is holomorphic everywhere except that it has a possible pole at \( s = 0 \) and \( s = 1 \). Moreover there is no pole if \( \chi^r \omega_\pi^2 = 1 \). (Here the set \( S \) can be taken to be exactly the finite set of places containing all the archimedean places, places dividing \( 2 \), and the non-archimedean places where \( \pi \) or \( \chi \) is ramified.)

Indeed, in our previous work ([T1]), based on works of various people such as Patterson and Piatetski-Shapiro ([PP]), Gelbart and Jacquet ([GJ]), and most originally Shimura ([Shi]), we showed the \( L \)-function
\[
L^S(s, \pi, \text{Sym}^2 \otimes \chi)
\]
is holomorphic for \( \text{Re}(s) > 1 \). (Actually what we showed in [T1] is slightly more than this. See [T1] for more details.) In [T1], however, we were unable to show the holomorphy for \( \text{Re}(s) < 1 \). This was because we were unable to determine the locations of possible poles of certain Eisenstein series on the metaplectic double cover \( \tilde{\text{GL}}_r \) of \( \text{GL}_r \) for all \( s \in \mathbb{C} \). For the sake of explaining it, let us assume \( r \) is odd here. Then in [T1], the twisted symmetric square \( L \)-function
\[
L^S(s, \pi, \text{Sym}^2 \otimes \chi)
\]
is represented by Rankin-Selberg integrals of the form
\[
Z(\phi, \Theta, f^*) = \int_{Z(\mathbb{A}) \text{GL}_r(F) \backslash \text{GL}_r(\mathbb{A})} \phi(g) \Theta(g) E(\kappa(g), s; f^*) dg,
\]
where \( \phi \) is a cusp form in \( \pi \), \( \Theta \) is an automorphic form on the twisted exceptional representation of \( \text{GL}_r(\mathbb{A}) \), and \( E(-, s; f^*) \) is the Eisenstein series on \( \tilde{\text{GL}}_r(\mathbb{A}) \) associated with the section \( f^* \) in the global induced space \( \text{Ind}_{Q(\mathbb{A})}^{\tilde{\text{GL}}_r(\mathbb{A})} \theta \otimes \delta_Q^s \), where \( Q \) is the \( (r - 1, 1) \)-parabolic of \( \text{GL}_r \) and \( \theta \) is the exceptional representation of the Levi part \( \tilde{\text{GL}}_{r-1}(\mathbb{A}) \times \tilde{\text{GL}}_1(\mathbb{A}) \). (Those exceptional representations will be recalled in later sections.) Then the holomorphy of the twisted symmetric square \( L \)-function can essentially be reduced to the holomorphy of the normalized Eisenstein series
\[
E^*(-, s; f^*) = L^S(r(2s + \frac{1}{2}) \chi^r \omega_\pi^2) E(-, s; f^*)
\]
Indeed, the bulk of this paper is devoted to showing the following result on the normalized Eisenstein series, which is Theorem 5.2 with the notation adjusted.

**Theorem.** The normalized Eisenstein series above is holomorphic for all \( s \in \mathbb{C} \) except that if \( \chi^r \omega_\pi^2 = 1 \) it has a possible simple pole at \( s = \frac{1}{2} \) and \(-\frac{1}{4}\).

Let us note that the possible pole at \( s = \frac{1}{2} \) (resp. \( s = -\frac{1}{4} \)) for the normalized Eisenstein series gives the one at \( s = 1 \) (resp. \( s = 0 \)) for the \( L \)-function.

Determination of the location of possible poles of (normalized) Eisenstein series (especially degenerate Eisenstein series for classical groups) has been done in various places such as [PSR] [GPSR] [KR] [KJ], and we essentially follow their approach, in which we determine possible poles of the Eisenstein series by computing a constant term of the Eisenstein series and poles of intertwining operators. Our Eisenstein series, however, is on the metaplectic group \( \tilde{\text{GL}}_r(\mathbb{A}) \), which requires extra care, and for this reason we have developed the theory of metaplectic tensor products for automorphic representations in our earlier paper [T2].
Even though the theory of metaplectic groups is an important subject in representation theory and automorphic forms, it has an unfortunate history of numerous technical errors and as a result published literatures in this area are often marred by those errors which compromise their reliability. For this reason, we try to make this paper as self-contained as possible and supply as detailed proofs as possible. In particular, we will not use any of the results in [BG] (though many of the ideas in this paper are borrowed from [BG]) except one proposition ([BG, Proposition 7.3]) on \( \widetilde{GL}_2 \) for which the proof there is detailed enough to be reliable.

The following is the structure of the paper. In the next section, we will recall the theory of the metaplectic double cover \( \text{GL}_r \) of \( GL_r \) both locally and globally and quote the results from \( \text{[T2]} \) on the metaplectic tensor product, which will be needed in later sections. In Section 3, we recall the notion of the exceptional representation on \( \text{GL}_r \), which was originally developed in [KP] for the non-twisted case, [BH] for the local twisted case, and finally in \( \text{[T1]} \) for the general case. The exceptional representation is used to define our Eisenstein series. In Section 4, we define the induced representation that gives rise to our Eisenstein series, and examine analytic properties of the intertwining operators on it, and in Section 5 we will determine the possible poles of the (unnormalized) Eisenstein series for \( \text{Re}(s) \geq 0 \). Those two sections comprise the main part of the paper. Then in Section 6, we will determine the possible poles of the normalized Eisenstein series. Finally in Section 7, we will give the main theorem on the twisted symmetric square \( L \)-function.

### Notations

Throughout the paper, \( F \) is a local field of characteristic zero or a number field. If \( F \) is a number field, we denote the ring of adeles by \( \mathbb{A} \). Both locally and globally, we denote by \( \mathcal{O}_F \) the ring of integers of \( F \). For each algebraic group \( G \) over a global \( F \), and \( g \in G(\mathbb{A}) \), by \( g_v \) we mean the \( v \)-th component of \( g \), and so \( g_v \in G(F_v) \).

If \( F \) is local, the symbol \( (-,-)_F \) denotes the Hilbert symbol of \( F \). If \( F \) is global, we let \( (-,-)_{\mathbb{A}} := \prod_v (-,-)_{F_v} \), where the product is finite. We sometimes write simply \( (-,-) \) for the Hilbert symbol when there is no danger of confusion.

Throughout the paper we write

\[
\mathcal{r} = \begin{cases} 2q \\ 2q + 1 \end{cases}
\]

depending on the parity of \( r \). For a partition \( r_1 + \cdots + r_k = r \) of \( r \), we let

\[
M = \text{GL}_{r_1} \times \cdots \times \text{GL}_{r_k} \subseteq \text{GL}_r
\]

and assume it is embedded diagonally as usual. Let \( P = P_1 \times \cdots \times P_k \) be a parabolic subgroup of \( M \) where each \( P_i \) is a parabolic subgroup of \( \text{GL}_{r_i} \). Further assume that the Levi factor of \( P_i \) is \( \text{GL}_{l_{i_1}} \times \cdots \times \text{GL}_{l_{i_{l_i}}} \), where \( l_{i_1} + \cdots + l_{i_{l_i}} = r_i \). Then we write

\[
P = P_{l_{i_1}, \ldots, l_{i_{l_i}}}^{r_1, \ldots, r_k}
\]

namely the superscript indicates the ambient group \( M \), and the subscript indicates the Levi part. For example, \( B_{1,1,1}^{2,2,2} \) means the parabolic subgroup of \( \text{GL}_2 \times \text{GL}_2 \times \text{GL}_1 \) whose Levi part is \( \text{GL}_1 \times \text{GL}_2 \times \text{GL}_1 \times \text{GL}_1 \times \text{GL}_1 \). For the minimal parabolic of \( M \), we write \( B^{r_1, \ldots, r_k} \), namely \( B^{r_1, \ldots, r_k} = P_{1,1,1}^{r_1, \ldots, r_k} \). Also if \( M = \text{GL}_r \), we usually omit the superscript and simply write \( P_{l_1, \ldots, l_m} \) for the \( (l_1, \ldots, l_m) \)-parabolic of \( \text{GL}_r \). In particular \( B \) denotes the Borel subgroup of \( \text{GL}_r \). For a parabolic subgroup \( P \), we denote its Levi part by \( M_P \) and unipotent radical by \( N_P \). We use the same convention...
for the subscripts and superscripts for the unipotent radical. For example, \( N^{2, r-2}_{1,1, r-2} \) denotes the unipotent radical of the parabolic \( P^{2, r-2}_{1,1, r-2} \).

For any group \( G \) and subgroup \( H \subseteq G \), and for each \( g \in G \), we define \( \vartheta H = g g^{-1} \). Then for a representation \( \pi \) of \( H \), we define \( \vartheta \pi \) to be the representation of \( \vartheta H \) defined by \( \vartheta \pi(h') = \pi(g^{-1} h' g) \) for \( h' \in g g^{-1} \).

For each \( r \), we denote the \( r \times r \) identity matrix by \( I_r \). We let \( W \) be the set of all \( r \times r \) permutation matrices, so for each element \( w \in W \) each row and each column has exactly one 1 and all the other entries are 0. The Weyl group of \( GL_r \) is identified with \( W \). Also for a Levi \( M = GL_{r_1} \times \cdots \times GL_{r_k} \), we let \( W_M \) be the subset of \( W \) that only permutes the \( GL_r \)-blocks of \( M \). Namely \( W_M \) is the collection of block matrices

\[
W_M := \{ (\delta_{\sigma(i), j} I_{r_i}) \in W : \sigma \in S_k \},
\]

where \( S_k \) is the permutation group of \( k \) letters. Though \( W_M \) is not a group in general, it is in bijection with \( S_k \). Accordingly we sometimes use the permutation notation for the Weyl group element. For example, \((12 \ldots k) \in S_k \) corresponds to the longest element in \( W_M \).

We use the usual notation for the roots of \( GL_r \). Namely \( e_i \) is the character on the maximum torus defined by \( (t_1, \ldots, t_r) \mapsto t_i \). Then each root is of the form \( e_i - e_j \) and each positive root is of the form \( e_i - e_j \) with \( i < j \). Let \( P = MN \) be a parabolic subgroup whose Levi is \( M \). We let \( \Phi_P(C) \) be the \( C \)-vector space spanned by the roots of \( M \). So in particular if \( M = GL_r \), then \( \Phi_P(C) \cong \mathbb{C}^{r-1} \) and each \( \nu \in \Phi_P(C) \) is for the form \( \nu = s_1 e_1 + \cdots + s_r e_r \) with \( s_1 + \cdots + s_r = 0 \). We let \( \rho_P \) be half the sum of the positive roots of \( M \).

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2. The metaplectic double cover \( \widetilde{GL}_r \) of \( GL_r \)

In this section, we review the theory of the metaplectic double cover \( \widetilde{GL}_r \) of \( GL_r \) for both local and global cases, which was originally constructed by Kazhdan and Patterson in [KP] and the metaplectic tensor product for the Levi part developed by Mezo ([Mc]) and the author ([T2]).

2.1. The local metaplectic double cover \( \widetilde{GL}_r(F) \). Let \( F \) be a (not necessarily non-archimedean) local field of characteristic 0. In this paper, by the metaplectic double cover \( GL_r(F) \) of \( GL_r(F) \), we mean the central extension of \( GL_r(F) \) by \{ ±1 \} as constructed in [KP] by Kazhdan and Patterson. (Kazhdan and Patterson considered more general \( n \)th covers \( \widetilde{GL}_r^{(c)}(F) \) with a twist by \( c \in \{ 0, \ldots, n-1 \} \). But we only consider the non-twisted double cover, i.e. \( n = 2 \) and \( c = 0 \).) Later, Banks, Levy, and Sepanski ([BLS]) gave an explicit description of a 2-cocycle

\[
\sigma_r : GL_r(F) \times GL_r(F) \to \{ \pm 1 \}
\]

which defines \( \widetilde{GL}_r(F) \) and shows that their 2-cocycle is “block-compatible”, by which we mean the following property of \( \sigma_r \): For the standard \((r_1, \ldots, r_k)\)-parabolic \( P \) of \( GL_r \), so that its Levi \( M_P \) is of the form \( GL_{r_1} \times \cdots \times GL_{r_k} \) which is embedded diagonally into \( GL_r \), we have

\[
\sigma_r \left( \begin{array}{cccc} g_1 & \cdots & \cdots & g_k \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \end{array} \right), \left( \begin{array}{cccc} g'_1 & \cdots & \cdots & g'_k \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \end{array} \right) = \prod_{i=1}^{k} \sigma_{r_i}(g_i, g'_i) \prod_{1 \leq i < j \leq k} (\det(g_i), \det(g'_i))_F,
\]

where
for all $g, g' \in \text{GL}_r(F)$ (see [BLS, Theorem 11, §3]), where $(-,-)_F$ is the Hilbert symbol for $F$. The 2-cocycle of [BLS] generalizes the well-known cocycle given by Kubota [Kub] for the case $r = 2$. Note that $\tilde{\text{GL}}_r(F)$ is not the $F$-rational points of an algebraic group, but this notation seems to be standard.

We define $\sigma \tilde{\text{GL}}_r(F)$ to be the group whose underlying set is

$$\sigma \tilde{\text{GL}}_r(F) = \text{GL}_r(F) \times \{\pm 1\} = \{(g, \xi) : g \in \text{GL}_r(F), \xi \in \{\pm 1\}\},$$

and the group law is defined by

$$(g_1, \xi_1) \cdot (g_2, \xi_2) = (g_1 g_2, \sigma_r(g_1, g_2) \xi_1 \xi_2).$$

Since we would like to emphasize the cocycle being used, we write $\sigma \tilde{\text{GL}}_r(F)$ instead of $\tilde{\text{GL}}_r(F)$.

To use the block-compatible 2-cocycle of [BLS] has obvious advantages. In particular, it can be explicitly computed and, of course, it is block-compatible. However it does not allow us to construct the global metaplectic cover $\tilde{\text{GL}}_r(\mathbb{A})$. Namely one cannot define an adelic block-combatible 2-cocycle simply by taking the product of the local block-combatible 2-cocycles over all the places. This can be already observed for the case $r = 2$. (See [F, p.125].)

For this reason, we will use a different 2-cocycle $\tau_r$ which works nicely with the global metaplectic cover $\tilde{\text{GL}}_r(\mathbb{A})$. To construct such $\tau_r$, first assume $F$ is non-archimedean. It is known that an open compact subgroup $K$ splits in $\text{GL}_r(F)$, and moreover if the residue characteristic of $F$ is odd, $K = \text{GL}_r(\mathcal{O}_F)$. (See [KP] Proposition 0.1.2.) Also for $k, k' \in K$, we have $(\det(k), \det(k'))_F = 1$. Hence one has a continuous map $s_r : \text{GL}_r(F) \to \{\pm 1\}$ such that $\sigma_r(g, g') s_r(g) s_r(g') = s_r(gg')$ for all $g, g' \in K$. Then define our 2-cocycle $\tau_r$ by

$$(\tau_r(g, g'))(g, \xi) := \sigma_r(g, g') s_r(g) s_r(g')/s_r(gg')$$

for $g, g' \in \text{GL}_r(F)$. If $F$ is archimedean, we set $\tau_r = \sigma_r$.

The choice of $s_r$ and hence $\tau_r$ is not unique. But there is a canonical choice with respect to the splitting of $K$ in the sense explained in [T2]. With this choice of $s_r$, the section $K \to \sigma \tilde{\text{GL}}_r(F)$ defined by $k \mapsto (k, s_r(k))$ is what is called the canonical lift in [KP] which is denoted by $\kappa^*$ there. Also if $r = 2$, our choice of $\tau_2$ is equal to the cocycle denoted by $\beta$ in [F], which can be shown to be block compatible. Indeed, the restriction of $\tau_2$ to $B^2 \times B^2$ where $B^2$ is the Borel subgroup of $\text{GL}_2$ coincides with $\sigma_2$.

Using $\tau_r$, we realize $\tilde{\text{GL}}_r(F)$ to be

$$\tilde{\text{GL}}_r(F) = \text{GL}_r(F) \times \{\pm 1\},$$

as a set and the group law is given by

$$(g, \xi) \cdot (g', \xi') = (gg', \tau_r(g, g') \xi \xi').$$

Note that we have the exact sequence

$$0 \longrightarrow \{\pm 1\} \longrightarrow \tilde{\text{GL}}_r(F) \overset{p}{\longrightarrow} \text{GL}_r(F) \overset{\kappa}{\longrightarrow} 0$$

given by the obvious maps, where we call $p$ the canonical projection.

We define a set theoretic section

$$\kappa : \text{GL}_r(F) \to \tilde{\text{GL}}_r(F), \ g \mapsto (g, 1).$$

Note that $\kappa$ is not a homomorphism. But by our construction of the cocycle $\tau_r$, $\kappa|_K$ is a homomorphism if $F$ is non-archimedean and $K$ is a sufficiently small open compact subgroup, and moreover if the residue characteristic is odd, one has $K = \text{GL}_r(\mathcal{O}_F)$.

Also we define another set theoretic section

$$s : \text{GL}_r(F) \to \tilde{\text{GL}}_r(F), \ g \mapsto (g, s_r(g)^{-1})$$
where \( s_r(g) \) is as above. We have the isomorphism
\[
\widetilde{\text{GL}}_r(F) \rightarrow \sigma \widetilde{\text{GL}}_r(F), \quad (g, \xi) \mapsto (g, s_r(g)\xi),
\]
which gives rise to the commutative diagram
\[
\begin{array}{ccc}
\widetilde{\text{GL}}_r(F) & \xrightarrow{s} & \sigma \widetilde{\text{GL}}_r(F) \\
\downarrow{g \mapsto (g, 1)} & & \\
\text{GL}_r(F) & &
\end{array}
\]
of set theoretic maps, \( i.e. \) maps which are not necessarily homomorphisms. Also note that the elements in the image \( s(\text{GL}_r(F)) \) “multiply via \( s_r \)” in the sense that for \( g, g' \in \text{GL}_r(F) \), we have
\[
(2.3) \quad (g, s_r(g)^{-1})(g', s_r(g')^{-1}) = (gg', s_r(g)g's_r(g')^{-1}).
\]

For a subgroup \( H \subseteq \text{GL}_r(F) \), whenever the cocycle \( s_r \) is trivial on \( H \times H \), the section \( s \) splits \( H \) by \((2.3)\). We often denote the image \( s(H) \) by \( H^* \) or sometimes simply by \( H \) when it is clear from the context. Particularly important is that by \[BLS\] Theorem 7 (f), \( s \) splits \( N_B \), the unipotent radical of the Borel subgroup \( B \) of \( \text{GL}_r(F) \), and accordingly we denote \( s(N_B) = N_B^* \).

Assume \( F \) is non-archimedean of odd residue characteristic. By \[KP\] Proposition 0.1.3 we have
\[
(2.4) \quad \kappa|_{T \cap K} = s|_{T \cap K}, \quad \kappa|_W = s|_W, \quad \kappa|_{N_B \cap K} = s|_{N_B \cap K},
\]
where \( W \) is the Weyl group and \( K = \text{GL}_r(O_F) \). In particular, this implies \( s_r|_{T \cap K} = s_r|_W = s_r|_{N_B \cap K} = 1 \). Also note that \( s_r(1) = 1 \).

Now assume \( F \) is any local field \( F \). For each element \( w \in W \), we denote \( s(w) \) by \( w \), which is equal to \((w, 1)\) if the residue characteristic of \( F \) is odd or \( F = \mathbb{C} \), when it is clear from the context. However it is important to note that \( s \) does not split \( W \) if the residue characteristic of \( F \) is even or \( F = \mathbb{R} \). Indeed, \( s \) splits \( W \) if and only if \((-1, -1)_F = 1 \).

Note that \( \widetilde{\text{GL}}_1 = \text{GL}_1(F) \times \{ \pm 1 \} \), where the product is the direct product, \( i.e. \) \( \sigma_1 \) is trivial. (See \[BLS\] Corollary 8, \( \S 3 \).) Also we define \( F^\times \) to be \( F^\times = F^\times \times \{ \pm 1 \} \) as a set but the product is given by \((a, \xi) \cdot (a', \xi') = (aa', a\xi' a^{-1} \xi') \). (It is known that \( F^\times \) is isomorphic to \( \widetilde{\text{GL}}_1 \) if and only if \((-1, -1)_F = 1 \). It is our understanding that this is due to J. Klose (see \[KP\] p.42], though we do not know where his proof is written. See \[Ad\] for a proof for a more general statement.)

For each subgroup \( H(F) \subseteq \text{GL}_r(F) \), we denote the preimage \( p^{-1}(H(F)) \) of \( H(F) \) via the canonical projection \( p \) by \( \widetilde{H}(F) \) or sometimes simply by \( \widetilde{H} \) when the base field is clear from the context. We call it the “metaplectic preimage” of \( H(F) \).

If \( P \) is a parabolic subgroup of \( \text{GL}_r \) whose Levi is \( M_P = \text{GL}_{r_1} \times \cdots \times \text{GL}_{r_k} \), we often write
\[
\widetilde{M}_P = \text{GL}_{r_1} \times \cdots \times \text{GL}_{r_k}
\]
for the metaplectic preimage of \( M_P \). One can check
\[
\widetilde{P} = \widetilde{M}_P N_P
\]
and \( N_P \) is normalized by \( \widetilde{M}_P \). Hence if \( \pi \) is a representation of \( \widetilde{M}_P \), one can consider the parabolically induced representation \( \text{Ind}^{\text{GL}_r}_{\widetilde{M}_P N_P} \pi \) as usual by letting \( N_P \) act trivially.

Next let
\[
\text{GL}_r^{(2)} = \{ g \in \text{GL}_r : \det g \in F^\times 2 \},
\]
and $\widetilde{\text{GL}}_r^{(2)}$ its metaplectic preimage. Also we define

$$M'_P^{(2)} = \{ \begin{pmatrix} g_1 & \cdots & \cdots & g_k \\ \vdots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ g_k & \cdots & \cdots & g_1 \end{pmatrix} \in M_P : \det g_i \in F^\times \}$$

and often denote its preimage by

$$\widetilde{M'}_P^{(2)} = \widetilde{\text{GL}}_r^{(2)} \times \cdots \times \widetilde{\text{GL}}_r^{(2)}.$$

We write $P^{(2)} = M'_P^{(2)} N_P$ and denote its preimage by $\widetilde{P}^{(2)}$. Then we have

$$\widetilde{P}^{(2)} = \widetilde{M'}_P^{(2)} N^*_P.$$

Let us mention the following important fact. Let $Z_{\text{GL}_r} \subset \text{GL}_r$ be the center of $\text{GL}_r$. Then the preimage $\widetilde{Z}_{\text{GL}_r}$, though abelian, is not the center of $\widetilde{\text{GL}}_r$ in general. It is the center only when $r = 2q + 1$ or $F = \mathbb{C}$. If $r = 2q$, the center $Z_{\text{GL}_r}$ is

$$Z_{\text{GL}_r} = \{(aI_r, \xi) : a \in F^\times, \xi \in \{\pm 1\}\}.$$  

From (2.1), one can compute

$$\sigma_r(aI_r, a'I_r) = \prod_{1 \leq i < j \leq r} (a, a')_F = (a, a')_{F^r}^{(r\delta - 1)}.$$  

Hence for either $r = 2q$ or $r = 2q + 1$, $\widetilde{Z}_{\text{GL}_r}$ is isomorphic to $\widetilde{F}$ if $q$ is odd, and isomorphic to $\widetilde{\text{GL}}_1$ if $q$ is even. Also note that for $r = 2q$ we have $\widetilde{Z}_{\text{GL}_r} \subset \widetilde{\text{GL}}_r^{(2)}$ and it is the center of $\widetilde{\text{GL}}_r^{(2)}$.

Let $\pi$ be an admissible representation of a subgroup $\widetilde{H} \subset \widetilde{\text{GL}}_r$. We say $\pi$ is “genuine” if each element $(1, \xi) \in \widetilde{H}$ acts as multiplication by $\xi$. On the other hand, if $\pi$ is a representation of $H$, one can always view it as a (non-genuine) representation of $\widetilde{H}$ by pulling back $\pi$ via the canonical projection $\widetilde{H} \to H$, which we denote by the same symbol $\pi$. In particular, for a parabolic subgroup $P$, we view the modular character $\delta_P$ as a character on $\widetilde{P}$ in this way.

2.2. The global metaplectic double cover $\widetilde{\text{GL}}_r(\mathbb{A})$. In this subsection we consider the global metaplectic group. So we let $F$ be a number field and $\mathbb{A}$ the ring of adeles. We shall define the 2-fold metaplectic cover $\widetilde{\text{GL}}_r(\mathbb{A})$ of $\text{GL}_r(\mathbb{A})$. (Just like the local case, we write $\widetilde{\text{GL}}_r(\mathbb{A})$ even though it is not the adelic points of an algebraic group.) The construction of $\text{GL}_r(\mathbb{A})$ has been done in various places such as [KP, FK].

First define the adelic 2-cocycle $\tau_r$ by

$$\tau_r(g, g') := \prod_{v} \tau_{r,v}(g_v, g'_v),$$

for $g, g' \in \text{GL}_r(\mathbb{A})$, where $\tau_{r,v}$ is the local cocycle defined in the previous subsection. By definition of $\tau_{r,v}$, we have $\tau_{r,v}(g_v, g'_v) = 1$ for almost all $v$, and hence the product is well-defined.

We define $\text{GL}_r(\mathbb{A})$ to be the group whose underlying set is $\text{GL}_r(\mathbb{A}) \times \{\pm 1\}$ and the group structure is defined as in the local case, i.e.

$$(g, \xi) \cdot (g', \xi') = (gg', \tau_r(g, g')\xi\xi'),$$

for $g, g' \in \text{GL}_r(\mathbb{A})$, and $\xi, \xi' \in \{\pm 1\}$. Just as the local case, we have

$$0 \longrightarrow \{\pm 1\} \longrightarrow \text{GL}_r(\mathbb{A}) \longrightarrow \text{GL}_r(\mathbb{A}) \longrightarrow 0,$$
where we call \( p \) the canonical projection. Define a set theoretic section \( \kappa : \GL_r(\mathbb{A}) \to \tilde{\GL}_r(\mathbb{A}) \) by \( g \mapsto (g, 1) \).

It is well-known that \( \GL_r(F) \) splits in \( \tilde{\GL}_r(\mathbb{A}) \). However the splitting is not via \( \kappa \) but via the product of all the local sections \( s_{r,v} \). Namely one can define the map

\[
s : \GL_r(F) \to \tilde{\GL}_r(\mathbb{A}), \quad g \mapsto (g, s_r(g)^{-1}),
\]

where

\[
s_r(g) := \prod_v s_{r,v}(g)
\]

makes sense for all \( g \in \GL_r(F) \) and the splitting is implied by the “product formula” for the block-compatible 2-cocycle.

Unfortunately, however, the expression \( \prod_v s_{r,v}(g_v) \) does not make sense for every \( g = \prod_v g_v \in \GL_r(\mathbb{A}) \) because one does not know whether \( s_{r,v}(g_v) = 1 \) for almost all \( v \). But whenever the product \( \prod_v s_{r,v}(g_v) \) makes sense we denote the element \( (g, \prod_v s_{r,v}(g_v)^{-1}) \) by \( s(g) \). This defines a partial global section \( s : \GL_r(\mathbb{A}) \to \tilde{\GL}_r(\mathbb{A}) \).

It is shown in \( \text{(1.9)} \) that the section \( s \) is defined and splits the groups \( \GL_r(F) \) and \( N_B(\mathbb{A}) \). Also \( s \) is defined, though not a homomorphism, on \( B(\mathbb{A}) \) thanks to \( \text{(2.4)} \). And the following will be used later

**Lemma 2.5.** Let \( g \in \GL_r(F) \) and \( n, n' \in N_B(\mathbb{A}) \), the section \( s \) is defined on \( ngn' \), and moreover we have

\[
s(ngn') = s(n)s(g)s(n').
\]

*Proof.* Let us first note that in \( \text{(1.9)} \), Lemma 1.9], it is shown that both \( s(ng) \) and \( s(gn') \) are defined and moreover \( s(ng) = s(n)s(g) \) and \( s(gn') = s(g)s(n') \). Namely the lemma holds for \( n = 1 \) or \( n' = 1 \). Hence it suffices to show \( s(ngn') \) is defined and \( s(ngn') = s(n)s(gn') \). But if \( s_r(ngn') \) is defined,

\[
s_r(ngn') = s_r(n, gn') s_r(gn') / \tau_r(n, gn'),
\]

Note that here all of \( s_r(n) \), \( s_r(gn') \) and \( \tau_r(n, gn') \) are defined. Moreover, locally \( s_r(n, gn') = 1 \) for all \( v \) by \( \text{[BLS] Theorem 7, p.153} \). Hence \( s_r(ngn') \) is defined. Thus \( s(ngn') \) is defined. Moreover, since \( s_r(ngn') = 1 \), we have \( s(ngn') = s(n)s(gn') \). \( \square \)

Analogously to the local case, if the partial global section \( s \) is defined on a subgroup \( H \subseteq \GL_r(\mathbb{A}) \) and \( s|_H \) is a homomorphism, we denote the image \( s(H) \) by \( H^* \) or simply by \( H \) when there is no danger of confusion.

We define the groups like \( \tilde{\GL}_r(\mathbb{A}) \), \( \tilde{\GL}_r(\mathbb{A}) \), \( \tilde{\GL}_r(\mathbb{A}) \), etc completely analogously to the local case. Also \( \tilde{\mathbb{A}}^\times \) is a group whose underlying set is \( \mathbb{A}^\times \times \{1, \pm 1\} \) and the group structure is given by the global Hilbert symbol analogously to the local case. Also just like the local case, the preimage \( \tilde{\tilde{Z}}_{\tilde{\GL}_r}(\mathbb{A}) \) of the center \( Z(\tilde{\mathbb{A}}) \) is the center of \( \tilde{\GL}_r(\mathbb{A}) \) only if \( r = 2q + 1 \). If \( r = 2q \), then the center of \( \tilde{\GL}_r(\mathbb{A}) \) is the set of elements of the form \( (aI_r, \xi) \) where \( a \in \tilde{\mathbb{A}}^{x^2} \) and \( \xi \in \{1, \pm 1\} \), and \( Z_{\tilde{\GL}_r}(\mathbb{A}) \) is the center of only \( \tilde{\GL}_r(\mathbb{A}) \).

Let \( \pi \) be a representation of \( \tilde{H}(\mathbb{A}) \subseteq \tilde{\GL}_r(\mathbb{A}) \). Just like the local case, we call \( \pi \) genuine if \( (1, \xi) \in \tilde{H}(\mathbb{A}) \) acts as multiplication by \( \xi \). If \( \pi \) is a genuine automorphic representation of \( \tilde{\GL}_r(\mathbb{A}) \), then for each automorphic form \( f \in \pi \) we have \( f(g, \xi) = \xi f(g, 1) \) for all \( (g, \xi) \in \tilde{\GL}_r(\mathbb{A}) \). Also any representation of \( H(\mathbb{A}) \) is viewed as a representation of \( \tilde{H}(\mathbb{A}) \) by pulling it back by the canonical projection \( p \), which we also denote by \( \pi \). In particular, this applies to the modular character \( \delta_P \) for each parabolic \( P(\mathbb{A}) \).
We would like to construct a representation of a set and the group law is given by this cocycle $\tau$.

Namely the cocycle $\tau$ is not known to be block-compatible. To get around it, one needs to introduce an intermediate $\ker\rho$ of the global metaplectic double cover $\prod_v\tilde{\Pi}_v$ by $\Pi_v$ so that $g_v \in K_v$ and $\xi_v = 1$ for almost all $v$, we have the surjection

$$\rho : \prod_v\tilde{\Pi}_v(F_v) \to \tilde{\Pi}_v(A), \quad \Pi_v(g_v, \xi_v) \mapsto (\Pi_v g_v, \Pi_v \xi_v).$$

This is a group homomorphism by our definition of $\tilde{\Pi}_v(F_v)$ and $\tilde{\Pi}_v(A)$. We have

$$\prod_v\tilde{\Pi}_v(F_v)/\ker\rho \cong \tilde{\Pi}_v(A),$$

where $\ker\rho$ consists of the elements of the form $\Pi_v(1, \xi_v)$ with $\xi_v = -1$ at an even number of $v$.

Suppose we are given a collection of irreducible admissible representations $\pi_v$ of $\tilde{\Pi}_v(F_v)$ such that $\pi_v$ is $\kappa(K_v)$-spherical for almost all $v$. Then we can form an irreducible admissible representation of $\prod_v\tilde{\Pi}_v(F_v)$ by taking a restricted tensor product $\otimes_v\pi_v$ as usual. Suppose further that $\ker\rho$ acts trivially on $\otimes_v\pi_v$, which is always the case if each $\pi_v$ is genuine. Then it descends to an irreducible admissible representation of $\tilde{\Pi}_v(A)$, which we denote by $\otimes_v\pi_v$, and call it the “metaplectic restricted tensor product”. Let us emphasize that the space for $\otimes_v\pi_v$ is the same as that for $\otimes_v\pi_v$. Conversely, if $\pi$ is an irreducible admissible representation of $\tilde{\Pi}_v(A)$, it is written as $\otimes_v\pi_v$ where $\pi_v$ is an irreducible admissible representation of $\tilde{\Pi}_v(F_v)$, and for almost all $v$, $\pi_v$ is $\kappa(K_v)$-spherical. (To see it, view $\pi$ as a representation of the restricted product $\prod_v\Pi_v(F_v)$ by pulling it back by $\rho$ and apply the usual tensor product theorem for the restricted product, which gives $\otimes_v\pi_v$, and it descends to $\otimes_v\pi_v$.)

2.3. The block-compatibility for $\tilde{\Pi}_v(A)$. We need to address an issue on the block-compatibility of the global metaplectic cover $\tilde{\Pi}_v(A)$. As we already mentioned, one cannot define $\tilde{\Pi}_v(A)$ by using the block-compatible local cocycles $\sigma_v$, but instead one needs to introduce the cocycle $\tau_v$ which is not known to be block-compatible. To get around it, one needs to introduce an intermediate cocycle $\tau_P$ for each parabolic subgroup $P$.

Let $P(A) \subseteq \tilde{\Pi}_v(A)$ be a parabolic subgroup whose Levi part is $M_P(A) = GL_{r_1}(A) \times \cdots \times GL_{r_k}(A)$. We define a 2-cocycle $\tau_P$ on $M_P(A)$ by

$$\tau_P \left( \begin{pmatrix} g_1 \\ \vdots \\ g_k \end{pmatrix}, \begin{pmatrix} g_1' \\ \vdots \\ g_k' \end{pmatrix} \right) = \prod_{i=1}^k \tau_v(g_i, g_i') \prod_{1 \leq i < j \leq k} (\det(g_i), \det(g_j'))_A,$$

where $(-,-)_A$ is the global Hilbert symbol. We define the group $c\tilde{M}_P(A)$ to be $M_P(A) \times \{-1\}$ as a set and the group law is given by this cocycle $\tau_P$. Then it is shown in [T2] that $c\tilde{M}_P \cong \tilde{M}_P$.

Namely the cocycle $\tau_P$ is cohomologous to $\tau_v|_{M_P(A) \times M_P(A)}$.

2.4. The metaplectic tensor product. In this subsection, we assume $F$ is either a number field or a local field. Let $P \subseteq \tilde{\Pi}_v$ be a parabolic subgroup whose Levi part is $M_P = GL_{r_1} \times \cdots \times GL_{r_k}$. Given irreducible admissible representations (or automorphic representations) $\pi_1, \ldots, \pi_k$ of $\tilde{\Pi}_{r_1}, \ldots, \tilde{\Pi}_{r_k}$, we would like to construct a representation of $\tilde{M}_P$ that can be called the “metaplectic tensor product” of $\pi_1, \ldots, \pi_k$. However unlike the non-metaplectic case, the construction is far from trivial, because $\tilde{M}_P$ is not the direct product $\tilde{\Pi}_{r_1} \times \cdots \times \tilde{\Pi}_{r_k}$, and even worse there is no natural map between
them. The construction of the metaplectic tensor product for the local case was carried out by Mezo in [Mc] and the global case was carried out by the author in [T2]. In what follows, we will briefly recall this construction.

Assume \( F \) is local. Let \( \pi_i^{(2)} \) be an irreducible constituent of the restriction \( \pi_i|_{\widetilde{GL}_{r_i}^{(2)}(F)} \). Then the (usual) tensor product \( \pi_1^{(2)} \otimes \cdots \otimes \pi_k^{(2)} \), which is a representation of the direct product \( \widetilde{GL}_{r_1}^{(2)}(F) \times \cdots \times \widetilde{GL}_{r_k}^{(2)}(F) \), descends to an irreducible admissible representation \( \pi_1^{(2)} \otimes \cdots \otimes \pi_k^{(2)} \) of \( \widetilde{M}^{(2)}_P(F) = \widetilde{GL}_{r_1}^{(2)}(F) \times \cdots \times \widetilde{GL}_{r_k}^{(2)}(F) \). Let \( \omega \) be a character on the center \( Z_{\widetilde{GL}_{r}}(F) \) such that \( \omega \) agrees with \( \pi_i^{(2)} \otimes \cdots \otimes \pi_k^{(2)} \) on the overlap \( Z_{\widetilde{GL}_{r}}(F) \cap \widetilde{M}^{(2)}_P(F) \), so that we can extend \( \pi_1^{(2)} \otimes \cdots \otimes \pi_k^{(2)} \) to a representation of \( Z_{\widetilde{GL}_{r}}(F) \widetilde{M}^{(2)}_P(F) \) by letting \( Z_{\widetilde{GL}_{r}}(F) \) act by \( \omega \), which we denote by \( \omega(\pi_1^{(2)} \otimes \cdots \otimes \pi_k^{(2)}) \).

Now extend it to a representation of some subgroup \( \widetilde{H}(F) \subseteq \widetilde{M}_P(F) \), so that the induced representation \( \text{Ind}_{\widetilde{H}(F)}^{\widetilde{M}_P(F)} \omega(\pi_1^{(2)} \otimes \cdots \otimes \pi_k^{(2)}) \) is irreducible. Then Mezo has shown that this induced representation is independent of all the choices made except the character \( \omega \). We denote this induced representation by

\[
\pi_\omega := (\pi_1 \tilde{\otimes} \cdots \tilde{\otimes} \pi_k)_\omega,
\]

and call it the metaplectic tensor product of \( \pi_1, \ldots, \pi_k \) with respect to the character \( \omega \). Moreover one can show that the induced representation \( \text{Ind}_{\widetilde{H}(F)}^{\widetilde{M}_P(F)} Z_{\widetilde{GL}_{r}}(F) \widetilde{M}_P^{(2)}(F) \omega(\pi_1^{(2)} \tilde{\otimes} \cdots \tilde{\otimes} \pi_k^{(2)}) \) not only contains \( \pi_\omega \) but any of its constituent is (isomorphic to) \( \pi_\omega \). Let us mention that if \( r \) is even we have the situation \( Z_{\widetilde{GL}_{r}}(F) \subseteq \widetilde{M}_P^{(2)}(F) \), in which case there is no choice for \( \omega \) and the metaplectic tensor product is canonical and we sometimes write simply \( \pi_1 \tilde{\otimes} \cdots \tilde{\otimes} \pi_k \).

Next assume \( F \) is global, and assume all the \( \pi_i \) are irreducible unitary automorphic representations of \( \widetilde{GL}_{r_i}(A) \). Let \( \pi_i^{(2)} \) be an irreducible constituent of the representation of \( \widetilde{GL}_{r_i}^{(2)}(A) \) obtained by restricting the automorphic forms in \( \pi_i \) to \( \widetilde{GL}_{r_i}^{(2)}(A) \). One can construct an “automorphic representation” \( \pi_1^{(2)} \tilde{\otimes} \cdots \tilde{\otimes} \pi_k^{(2)} \) analogously to the local case. Let \( \omega \) be a Hecke character of \( Z_{\widetilde{GL}_{r}}(A) \) such that \( \omega \) agrees with \( \pi_1^{(2)} \tilde{\otimes} \cdots \tilde{\otimes} \pi_k^{(2)} \) on the overlap \( Z_{\widetilde{GL}_{r}}(A) \cap \widetilde{M}_P^{(2)}(A) \). Then essentially in the analogous way to the local case, one can construct an automorphic representation \( \pi_\omega \) of \( \widetilde{M}_P(A) \), which is independent of all the choices made except \( \omega \), such that

\[
\pi_\omega \equiv \tilde{\otimes}_r \pi_\omega,
\]

i.e. it is the restricted metaplectic tensor product of the local metaplectic tensor products \( \pi_\omega \). Just like the local case we write \( \pi_\omega = (\pi_1 \tilde{\otimes} \cdots \tilde{\otimes} \pi_k)_\omega \).

In [T2] it is shown that the metaplectic tensor product behaves just like the usual tensor product for the non-metaplectic case. First of all, the cuspidality and square-integrability are preserved.

**Proposition 2.8.** Assume \( F \) is global. If each \( \pi_i \) is square-integrable modulo center (resp. cuspidal), then the tensor product \( (\pi_1 \tilde{\otimes} \cdots \tilde{\otimes} \pi_k)_\omega \) is square-integrable modulo center. (resp. cuspidal).

The metaplectic tensor product behaves as expected under the action of the Weyl group element. Namely,

**Proposition 2.9.** Let \( w \in W_M \) be a Weyl group element of \( GL_r \) that only permutes the \( GL_{r_i} \)-factors of \( M \). Namely for each \( (g_1, \ldots, g_k) \in GL_{r_1} \times \cdots \times GL_{r_k} \), we have \( w(g_1, \ldots, g_k)w^{-1} = (g_{\sigma(1)}, \ldots, g_{\sigma(k)}) \) for a permutation \( \sigma \in S_k \) of \( k \) letters. Then both locally and globally, we have

\[
w(\pi_1 \tilde{\otimes} \cdots \tilde{\otimes} \pi_k)_\omega \equiv (\pi_{\sigma(1)} \tilde{\otimes} \cdots \tilde{\otimes} \pi_{\sigma(k)})_\omega,
\]

where the left hand side is the twist of \( (\pi_1 \tilde{\otimes} \cdots \tilde{\otimes} \pi_k)_\omega \) by \( w \).
The metaplectic tensor product is compatible with parabolic induction.

**Proposition 2.10.** Both locally and globally, let $P = MN \subseteq \text{GL}_r$ be the standard parabolic subgroup whose Levi part is $M = \text{GL}_{r_1} \times \cdots \times \text{GL}_{r_k}$. Further for each $i = 1, \ldots, k$ let $P_i = M_i N_i \subseteq \text{GL}_{r_i}$ be the standard parabolic of $\text{GL}_{r_i}$ whose Levi part is $M_i = \text{GL}_{r_{i_1}} \times \cdots \times \text{GL}_{r_{i_l}}$. For each $i$, we are given a representation

$$\sigma_i := (\tau_{i,1} \otimes \cdots \otimes \tau_{i,l_i})_\omega,$$

of $\tilde{M}_i$, which is given as the metaplectic tensor product of the representations $\tau_{i,1}, \ldots, \tau_{i,l_i}$ of $\text{GL}_{r_{i_1}}, \ldots, \text{GL}_{r_{i_l}}$, respectively. Assume that $\pi_i$ is an irreducible constituent of the induced representation $\text{Ind}_{P_i}^{\text{GL}_{r_i}} \sigma_i$. Then the metaplectic tensor product

$$\pi_\omega := (\pi_1 \otimes \cdots \otimes \pi_k)_\omega$$

is an irreducible constituent of the induced representation

$$\text{Ind}_{Q}^{\tilde{M}} (\tau_{1,1} \otimes \cdots \otimes \tau_{1,l_1} \otimes \cdots \otimes \tau_{k,1} \otimes \cdots \otimes \tau_{k,l_k})_\omega,$$

where $Q$ is the standard parabolic subgroup of $M$ whose Levi part is $M_1 \times \cdots \times M_k$. (Here “irreducible constituent” can be replaced by “irreducible quotient” or “irreducible subrepresentation”, and the analogous proposition still holds.)

The global metaplectic tensor product behaves nicely with restriction to a smaller Levi in the following sense.

**Proposition 2.11.** Assume $F$ is global.

(a) Let

$$M_2 = \text{GL}_{r_2} \times \cdots \times \text{GL}_{r_k} \subseteq M = \text{GL}_{r_1} \times \text{GL}_{r_2} \times \cdots \times \text{GL}_{r_k}$$

be the natural embedding in the lower right corner. Then there exists a realization of the metaplectic tensor product $\pi_\omega = (\pi_1 \otimes \cdots \otimes \pi_k)_\omega$ such that for each $f \in \pi$ and the restriction $f|_{\widetilde{M}_2(\mathbb{A})}$ we have

$$f|_{\widetilde{M}_2(\mathbb{A})} \in \bigoplus_{\delta \in \text{GL}_{r_1}(F)} m_\delta (\pi_2 \otimes \cdots \otimes \pi_k)_\omega,$$

where $(\pi_2 \otimes \cdots \otimes \pi_k)_\omega$ is the metaplectic tensor product of $\pi_2, \ldots, \pi_k$, $\omega_\delta$ is a certain character twisted by $\delta \in \text{GL}_{r_1}(F)$ and $m_\delta \in \mathbb{Z}_{\geq 0}$ is a multiplicity.

(b) Let

$$M'_2 = \text{GL}_{r_1} \times \cdots \times \text{GL}_{r_{k-1}} \subseteq M = \text{GL}_{r_1} \times \cdots \times \text{GL}_{r_{k-1}} \times \text{GL}_{r_k}$$

be the natural embedding in the upper left corner. Then there exists a realization (possibly different from the above) of the metaplectic tensor product $\pi_\omega = (\pi_1 \otimes \cdots \otimes \pi_k)_\omega$ such that for each $f \in \pi$ and the restriction $f|_{\widetilde{M}'_2(\mathbb{A})}$ we have

$$f|_{\widetilde{M}'_2(\mathbb{A})} \in \bigoplus_{\delta' \in \text{GL}_{r_k}(F)} m_{\delta'} (\pi_1 \otimes \cdots \otimes \pi_{k-1})_\omega,$$

where $(\pi_1 \otimes \cdots \otimes \pi_{k-1})_\omega$ is the metaplectic tensor product of $\pi_1, \ldots, \pi_{k-1}$, $\omega_\delta'$ is a certain character twisted by $\delta' \in \text{GL}_{r_k}(F)$ and $m_{\delta'} \in \mathbb{Z}_{\geq 0}$ is a multiplicity.

Finally let us mention that the uniqueness of the metaplectic tensor product.
Proposition 2.12. Let $F$ be global (resp. local). Let $\pi_1, \ldots, \pi_k$ and $\pi'_1, \ldots, \pi'_k$ be unitary automorphic representations (resp. irreducible admissible representations) of $\GL_{r_1}, \ldots, \GL_{r_k}$. They give rise to isomorphic metaplectic tensor products with a Hecke character (resp. character) $\omega$, i.e.
\[(\pi_1 \otimes \cdots \otimes \pi_k)_\omega \cong (\pi'_1 \otimes \cdots \otimes \pi'_k)_\omega,
\]
if and only if for each $i$ there exists a Hecke character (resp. character) $\omega_i$ of $\widetilde{\GL}_{r_i}$ trivial on $\widetilde{\GL}_{r_i}^{(2)}$ such that $\pi_i \cong \omega_i \otimes \pi'_i$.

3. Exceptional representations of $\widetilde{\GL}_r$

In this section, we review the theory of the exceptional representation of $\widetilde{\GL}_r$, a special case of which is the Weil representation on $\GL_2$. Throughout the section $\chi$ will denote a unitary character on $F^\times$ when $F$ is local or a unitary Hecke character on $\A^\times$ when it is global.

3.1. The Weil representation of $\GL_2$. First let us review the theory of the Weil representation of $\GL_2$.

Local case:
Let us consider the local case, and hence $F$ will be a (not necessarily non-archimedean) local field of characteristic 0. Everything stated below without any specific reference is found in [GPS §2] for the non-archimedean case and in [G §4] for the archimedean case. Let $S(F)$ be the space of Schwartz-Bruhat functions on $F$, i.e. smooth functions with compact support if $F$ is non-archimedean, and functions with all the derivatives rapidly decreasing if $F$ is archimedean. Let $r_\psi$ denote the representation of $\SL_2(F)$ on $S(F)$ such that
\[
\begin{align*}
(3.1) \quad r_\psi(s \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix})f(x) &= \gamma(\psi)x(x) \\
(3.2) \quad r_\psi(s \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix})f(x) &= \psi(bx^2)f(x), \quad b \in F \\
(3.3) \quad r_\psi(s \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix})f(x) &= |a|^{1/2}\mu_\psi(a)f(ax), \quad a \in F^\times \\
(3.4) \quad r_\psi^\xi(1, \xi)f(x) &= \xi f(x),
\end{align*}
\]
where $\hat{f}(x) = \int f(y)\psi(2xy)dy$ with the Haar measure $dy$ normalized in such a way that $\hat{f}(x) = f(-x)$. Also $\gamma(\psi)$ is the Weil index of $\psi$, and $\mu_\psi(a) = \gamma(\psi_a)/\gamma(\psi)$. (See [R] Appendix for the notion of Weil index.) It is well-known that $r_\psi$ is reducible and written as $r_\psi = r_+^\psi \oplus r_-^\psi$, where $r_+^\psi$ (resp. $r_-^\psi$) is an irreducible representation realized in the subspace of even functions (resp. odd functions) in $S(F)$.

If $\chi(-1) = 1$ (resp. $\chi(-1) = -1$), one can extend $r_+^\psi$ (resp. $r_-^\psi$) to a representation $r_+^\chi$ of $\widetilde{\GL}_2^{(2)}(F)$ by letting
\[(3.5) \quad r_+^\chi(s \begin{pmatrix} 1 & 0 \\ 0 & a^2 \end{pmatrix})f(x) = \chi(a)|a|^{-1/2}f(ax^{-1}x).
\]
This is indeed a well-defined irreducible representation of $\widetilde{\GL}_2^{(2)}(F)$ and call it the Weil representation of $\widetilde{\GL}_2^{(2)}(F)$ associated with $\chi$. We denote by $S_\chi(F)$ the subspace of $S(F)$ in which $r_+^\chi$ is realized,
which is the space of even functions if $\chi(−1) = 1$ and odd functions if $\chi(−1) = −1$. Note that

\begin{equation}
(3.6) \quad r^\psi(s\left(\begin{array}{cc}
a & 0 \\
0 & a
\end{array}\right))f(x) = \chi(a)\mu_\psi(a)f(x).
\end{equation}

The Weil representation $r_\chi$ of $\widetilde{GL}_2(F)$ is defined by

$$ r_\chi := \text{Ind}_{\widetilde{\text{GL}}_2(F)}^{\text{GL}_2(F)} r^\psi, $$

Then $r_\chi$ is irreducible and independent of the choice of $\psi$, and hence our notation. If $\chi(−1) = 1$, one can check that $r_\chi$ is the exceptional representation of Kazhdan-Patterson for $r = 2$ with the determinantal character $\chi^{1/2}$, which will be recalled later. If $\chi(−1) = −1$, then $r_\chi$ is described as follows: For non-archimedean $F$, it is supercuspidal ([GPS, Proposition 3.3.3]), for $F = \mathbb{R}$, it is a discrete series representation of lowest weight $3/2$ ([GPS, §6]), and finally for $F = \mathbb{C}$, it is identified with a certain induced representation ([GPS, §6]).

**Global case:**

We define the global Weil representation $r_\chi$ of $\widetilde{GL}_2(A)$ as the restricted tensor product of the local Weil representations, i.e.

$$ r_\chi = \widetilde{\otimes}_v r_{\chi_v}. $$

It is shown in [GPS, §8] that $r_\chi$ is a square integrable automorphic representation of $\widetilde{GL}_2(A)$, and moreover it is cuspidal if and only if $\chi^{1/2}$ does not exist. Also one can see that if $\chi^{1/2}$ exists, then just like the local case, $r_\chi$ is the exceptional representation of Kazhdan-Patterson for $r = 2$, which will be explained later.

### 3.2. The Weil representation of $\widetilde{M}_P$

Let us assume $r = 2q$ and $P$ is the $(2,...,2)$-parabolic $P_{2,...,2}$, so that

$$ M_P = \bigotimes_{v=1}^{q} GL_2. $$

Recall from Section 2 that we write

$$ \widetilde{M}_P = \bigotimes_{v=1}^{q} GL_2. $$

Since each element in the center $Z_{\widetilde{GL}_2q}$ is of the form $(a^2I_{2q}, \xi)$, we have $Z_{\widetilde{GL}_2q} \subseteq \widetilde{M}_P^{(2)}$. Hence the metaplectic tensor product of this Levi is unique. (In other words, there is only once choice for $\omega$.)

We extend the theory of the Weil representation both locally and globally as discussed in the previous subsection to the group $\widetilde{M}_P$ by taking the metaplectic tensor product of $q$ copies of the Weil representation of $GL_2$, and write

\begin{equation}
(3.7) \quad \Pi_\chi := (r_\chi \otimes \cdots \otimes r_\chi)_\omega,
\end{equation}

where $\omega$ is the unique choice for the character $Z_{\widetilde{GL}_r}$ which is actually given by $(a^2I_{2q}, \xi) \mapsto \xi\chi(a^2)^q$.

Also it should be mentioned that locally we have

$$ \Pi_\chi = \text{Ind}_{\widetilde{M}_P^{(2)}}^{\widetilde{M}_P} r^\psi \otimes \cdots \otimes r^\psi. $$

We call $\Pi_\chi$ the Weil representation of $\widetilde{M}_P$. 
3.3. Non-twisted exceptional representation. Let us now consider the non-twisted exceptional representation of $GL_r$ developed by Kazhdan and Patterson in [KP]. We treat both $r = 2q$ and $2q + 1$ at the same time. Also most of the time, we consider the local and global cases at the same time, and all the groups are over the local field $F$ (non-archimedean or archimedean) or the adeles $\mathbb{A}$.

For our character $\chi$, we let
$$\Omega_\chi = (\tilde{\chi} \otimes \cdots \otimes \tilde{\chi})^\omega,$$
which is a representation of the metaplectic preimage $\tilde{T}$ of maximal torus $T$. Note that $\Omega_\chi$ depends on $\omega$ if $r = 2q + 1$, but we suppress it from our notation. For each $\nu \in \Phi_F(\mathbb{C})$, let us define
$$\Omega^\nu_\chi := \Omega_\chi \otimes \exp(\nu, H_B(-))$$
where $H_B(-)$ is the Harish-Chandra homomorphism as usual. Note that $\exp(2\rho_B, H_B(-)) = \delta_B$. Then it is shown in [KP] that the induced representation $\text{Ind}_B^{GL_r} \Omega^\nu_\chi$ has its greatest singularity at $\nu = \rho_B/2$, and the quotient of $\text{Ind}_B^{GL_r} \Omega^\rho_B / 2 = \text{Ind}_B^{GL_r} \Omega_\chi \otimes \delta_B^{1/4}$ is called the exceptional representation. Namely, we have

**Proposition 3.8.** The induced representation $\text{Ind}_{TN_B}^{GL_r} \Omega_\chi \otimes \delta_B^{1/4}$ has a unique irreducible quotient, which we denote by $\theta_\chi$. For the local case, it is the image of the intertwining integral
$$\text{Ind}_{TN_B}^{GL_r} \Omega_\chi \otimes \delta_B^{1/4} \rightarrow \text{Ind}_{TN_B}^{GL_r} w_0(\Omega_\chi \otimes \delta_B^{1/4}),$$
where $w_0$ is the longest Weyl group element. For the global case, it is generated by the residues of the Eisenstein series at $\nu = \rho_B/2$ for the induced space $\text{Ind}_B^{GL_r} \Omega^\nu_\chi$, and $\theta_\chi$ is a square integrable automorphic representation of $GL_r(\mathbb{A})$. Moreover for the global $\theta_\chi$, one has the decomposition $\theta_\chi = \widetilde{\otimes}_\nu \theta_\chi^\nu$.

**Proof.** See [KP] Theorem I.2.9] for the local statement and [KP] Theorem II.2.1] for the global one. □

We call the representation $\theta_\chi$ the non-twisted exceptional representation of $\widetilde{GL}_r$ with the determinantal character $\chi$. It should be mentioned that if $r = 2$, $\theta_\chi$ is isomorphic to the Weil representation $\tau_{\chi^2}$. Note that just as $\Omega_\chi$, $\theta_\chi$ depends on $\omega$, but we suppress it from our notation.

Let us mention that a small discrepancy between the exceptional representation defined above and the one in [T1] which is defined as follows. First for the maximal torus $T \subseteq B$, we let

$$\begin{align*} T^e = \left\{ \begin{pmatrix} t_1 & \cdots & t_r \\ & \ddots & \\ & & t_r \end{pmatrix} \in T : t_1 t_2^{-1}, \ldots, t_{2q-1} t_{2q}^{-1} \text{ are squares} \right\}. \end{align*}$$

The metaplectic preimage $\tilde{T}^e$ of $T^e$ is a maximal abelian subgroup of $\tilde{T}$. Then in [T1] the non-twisted exceptional representation of $GL_r$ was defined to be the unique irreducible quotient of the induced representation $\text{Ind}_{T^e N_B}^{T^e N_B} \omega^\psi \otimes \delta_B^{1/4}$, where $\omega^\psi$ is the character on $\tilde{T}^e$ defined by

$$\begin{align*} \omega^\psi_{\chi^e}( (1, \xi) s(t) ) &= \xi \chi(\det t) \mu_\psi(t_2) \mu_\psi(t_4) \cdots \mu_\psi(t_{2q}), \end{align*}$$

where $\mu_\psi$ is the ratio of the Weil indices. (Note that even when $F$ is global, the section $s$ is defined on $T_B$ and the expression $s(t)$ makes sense.) However the exceptional representation defined this way coincides with the above $\theta_\chi$ with a certain choice of $\omega$. To see this, let us first assume that $F$ is local, and define
$$\Omega^\psi_\chi := \text{Ind}_{T^e}^{\tilde{T}^e} \omega^\psi_{\chi^e}.$$
This is irreducible ([KP p.55]). Indeed
\[ \Omega^\psi_\chi = (\tilde{\chi} \otimes \cdots \otimes \tilde{\chi})_\omega, \]
where each \( \tilde{\chi} \) is the non-genuine character on \( \tilde{\text{GL}}_1 \) defined by \( (a, \xi) \mapsto \xi \chi(a) \) and the character \( \omega \) on the center \( Z_{\text{GL}} \) is given by
\[ \omega(aI_r, \xi) = \xi \chi^{r}(a)\mu_\psi(a)^q. \]

By inducing in stages, one can see that
\[ \text{Ind}_{P=N}^G \omega^\psi \otimes \delta_B^{1/4} = \text{Ind}_B \Omega^\psi_\chi \otimes \delta_B^{1/4}, \]
which implies that the non-twisted exceptional representation in \([T1]\) is precisely our \( \theta_\chi \) with the above chosen \( \omega \). Now if \( F \) is global, we can define \( \Omega^\psi_\chi \) to be the global metaplectic tensor product \( (\tilde{\chi} \otimes \cdots \otimes \tilde{\chi})_\omega \) with \( \omega \) chosen in the same way as the local case, and hence the global exceptional representation \( \theta_\chi \) is obtained as the quotient of the global induced representation \( \text{Ind}_{B(\mathfrak{a})} \Omega^\psi_\chi \otimes \delta_B^{1/4} \), and we have \( \theta_\chi = \tilde{\Theta}'_\psi \theta_\chi \), which again coincides with the global non-twisted exceptional representation in \([T1]\).

**Remark 3.11.** It is important to note that the above discussion shows that in \([T1]\) only one particular central character \( \omega \) was used, which depends on the additive character \( \psi \) chosen. (But it is shown in \([T1]\) that after all it depends on \( \psi \) only when both \( r \) and \( q \) are odd.) In this paper, however, we always assume \( \omega \) is arbitrary. Indeed, it is crucial to do so when we compute the poles of our Eisenstein series as we will see later. Nonetheless, it should be also mentioned that to obtain the Rankin-Selberg integral of the \( L \)-function, it is necessary to choose the particular \( \omega \) as above.

### 3.4. Twisted exceptional representation

Next we consider the twisted version of the exceptional representation of \( \text{GL}_2 \) when \( r = 2q \). The local case was originally constructed by the Ph.D thesis by Banks \([B1]\) when the residue characteristic is odd, and the other cases are taken care of in \([T1]\).

Let \( P \) be the \((2, \ldots, 2)\)-parabolic whose Levi \( M_P \) is \( \text{GL}_2 \times \cdots \times \text{GL}_2 \) \((q\text{-times})\), and \( \Pi_\chi \) the Weil representation of \( \tilde{M}_P \) as in \([37]\). For each \( \nu \in \Phi_P(\mathbb{C}) \), let us define
\[ \Pi_\nu^\psi := \Pi_\chi \otimes \exp(\mu_\psi(H_P(-))) \]
where \( H_P(-) \) is the Harish-Chandra homomorphism. Analogously to the non-twisted exceptional representation of \([KP]\), the induced representation \( \text{Ind}_P^G \Pi_\nu^\psi \) has its greatest singularity at \( \nu = \rho_P/2 \), and the quotient of \( \text{Ind}_P^G \Pi_{\chi P}\nu^{P/2} = \text{Ind}_P^G \Pi_\chi \otimes \delta_{P}^{1/4} \) is called the twisted exceptional representation. Namely, we have

**Proposition 3.12.** The induced representation \( \text{Ind}_P^G \Pi_\chi \otimes \delta_{P}^{1/4} \) has a unique irreducible quotient, which we denote by \( \Theta_\chi \). For the local case, it is the image of the intertwining integral
\[ \text{Ind}_P^G \Pi_\chi \otimes \delta_{P}^{1/4} \to \text{Ind}_P^G \text{Ind}_{\rho_{P}} w_0(\Pi_\chi \otimes \delta_{P}^{1/4}), \]
where \( w_0 \) is the longest Weyl group element relative to \( P \). For the global case, it is generated by the residues of the Eisenstein series at \( \nu = \rho_P/2 \) for the induced space \( \text{Ind}_{B(\mathfrak{a})}^G \Pi_\chi \), and \( \theta_\chi \) is a square integrable automorphic representation of \( \tilde{\text{GL}}_{2q}(\mathfrak{a}) \). Moreover for the global \( \Theta_\chi \), one has the decomposition \( \theta_\chi = \tilde{\Theta}'_\psi \theta_\chi \).

**Proof.** See \([T1]\) Proposition 2.35 for the local statement and \([T1]\) Theorem 2.33 for the global statement. \( \square \)
We call \( \vartheta_\chi \) the twisted exceptional representation of \( \widetilde{GL}_{2q} \). Both locally and globally, if \( \chi^{1/2} \) exists, one can show that

\[
\vartheta_\chi = \theta_\chi^{1/2}.
\]

This is because the Weil representation \( r_\chi \) is the non-twisted exceptional representation of \( \widetilde{GL}_2 \) with the determinantal character \( \chi^{1/2} \).

**Remark 3.13.** Let us note that unlike the case \( r = 2q + 1 \), there is no choice for the central character \( \omega \) for constructing the metaplectic tensor product \( \Pi_\chi \) and hence \( \vartheta_\chi \) depends only on \( \chi \). Accordingly, there is no discrepancy between \( \vartheta_\chi \) here and the one in [T1].

### 4. Induced representations and intertwining operators

Let

\[
Q = P_{r-1,1} = (GL_{r-1} \times GL_1)N_Q
\]

be the standard \( (r - 1,1) \)-parabolic of \( GL_r \), so the Levi part is \( GL_{r-1} \times GL_1 \). The inducing data for the Eisenstein series we consider in this paper is a residual representation on the parabolic \( Q \). In this section, we first define the inducing representation, which we called the exceptional representation of \( GL_{r-1} \times \widetilde{GL}_1 \) in [T1]. This representation is the metaplectic tensor product of the exceptional representation \( \theta_\chi \) or \( \vartheta_\chi \) of \( GL_{r-1} \) and a character on \( GL_1 \). (The precise construction differs, depending on the parity of \( r \).) Then we will examine the analytic behavior of the intertwining operators on this induced representation. The main object of this section is to prove Theorem 4.3.

#### 4.1. The inducing representation for \( r = 2q \)

In this subsection we assume \( r = 2q \) and \( F \) can be both local and global, and for example the group \( GL_r \) denotes both \( GL_r(F) \) (\( F \) local) and \( GL_r(\mathbb{A}) \) (\( F \) global). Let \( \theta_\chi \) be the non-twisted exceptional representation of \( GL_{r-1} \) with the determinantal character \( \chi \). For a character \( \eta \) on \( GL_1 \), define \( \tilde{\eta} : GL_1 \to \{ \pm 1 \} \) to be the character defined by \( \tilde{\eta}(a, \xi) \mapsto \xi \eta(a) \) for \( (a, \xi) \in GL_1 \). We let

\[
\theta_{\chi, \eta} := (\theta_\chi \boxtimes \tilde{\eta})_\omega
\]

i.e. the metaplectic tensor product of \( \theta_\chi \) and \( \tilde{\eta} \). Note that since \( Z_{\widetilde{GL}_{2q}} \subseteq \widetilde{M}_Q^{(2)} \), there is no actual choice for the character \( \omega \).

It should be mentioned that even when \( r = 2q + 1 \), one can define \( \theta_\chi \), (which is equal to \( \vartheta_\chi^{1/2} \)) and hence can define \( \theta_{\chi, \eta} \), though most of the time we use the representation \( \theta_{\chi, \eta} \) for the case \( r = 2q \).

Let us mention that what we denoted by \( \theta_{\chi, \eta} \) in [T1] corresponds to what we mean by \( \theta_{\chi, \chi, \eta} \) in this paper. The reason is because at the time we wrote [T1], we did not know how to formulate the global metaplectic tensor product and as a result we constructed the representation \( \theta_{\chi, \chi, \eta} \) more directly as the unique irreducible quotient of an induced representation. But now that we have developed in the theory of global metaplectic tensor products, which includes the compatibility with parabolic inductions (Proposition 2.10), one can see that the construction in [T1] is indeed the same as the one above. Namely the representation \( \theta_{\chi, \eta} \) is, locally or globally, a unique irreducible quotient of

\[
\text{Ind}_{B^{r-1,1}}^{\widetilde{M}_Q}(\tilde{\chi} \boxtimes \cdots \boxtimes \tilde{\chi} \boxtimes \tilde{\eta})_\omega \otimes \delta_B^{1/4},
\]

where \( B^{r-1,1} \) is the Borel subgroup of \( M_Q = GL_{r-1} \times GL_1 \), namely \( B^{r-1,1} = M_Q \cap B \).
4.2. The inducing representation for $r = 2q + 1$. Next we will consider the case $r = 2q + 1$. Also keep the notation for $F$ from the previous subsection, namely $F$ is either local or global. Let $\tilde{\eta}$ be as before and $\vartheta_\chi$ the twisted exceptional representation of $GL_{2q}$, where we include the case $\chi^{1/2}$ exists. Then we define

$$\vartheta_{\chi,\eta} := (\vartheta_\chi \tilde{\eta})_\omega.$$ 

Note that if $\chi^{1/2}$ exists, we have $\vartheta_{\chi,\eta} = \vartheta_{\chi^{1/2},\eta}$.

**Remark 4.1.** Let us mention again that in \[\text{T1}\] a particular central character $\omega$ is chosen. Indeed, we used

$$\omega : (1, \xi) sQ(aI_r) \mapsto \xi \chi(a)^q \eta(a) \mu_\psi(a)^q,$$

which depends on $\psi$ if (and only if) $q$ is odd. However in this paper, $\omega$ is always arbitrary.

Just like the case for $r = 2q$, the compatibility with parabolic induction for metaplectic tensor products (Proposition 2.10) implies that $\vartheta_{\chi,\eta}$ is a unique irreducible quotient of

$$\text{Ind}_{P_{r-1,1}^r}^{M_Q} (r_\chi \otimes \cdots \otimes r_\chi \otimes \tilde{\eta})_\omega \otimes \vartheta_{1/4}^{\text{dr}} \otimes_{P_{r-1,1}^r} \vartheta_{\chi,\eta},$$

where $P_{r-1,1}^r$ is the $(2, \ldots, 2, 1)$-parabolic subgroup of $M_Q$, so the Levi part is $GL_2 \times \cdots \times GL_2 \times GL_1$.

4.3. The intertwining operator and its analytic behavior. Let $\theta = \vartheta_{\chi,\eta}$ or $\vartheta_{\chi,\eta}$ depending on the parity of $r$ and assume $F$ is global. Define

$$w_1 = \begin{cases} 
\begin{pmatrix} I_{r-2} & 1 \\
1 & 1 \\
1 & 1 \\
1 & 1 \\
\end{pmatrix}, & \text{if } r = 2q; \\
\end{cases}$$

(4.2)

In the rest of the section, we will consider the analytic behavior of the global intertwining operator

$$A(s, \theta, w_1) : \text{Ind}_{Q(\Lambda)}^{GL_{2q}(\Lambda)} \theta \otimes \delta^s_\lambda \rightarrow \text{Ind}_{w_1(M_Q(\Lambda))}^{GL_{2q}(\Lambda)} w_1^s \theta \otimes \delta^s_\lambda,$$

and will show

**Theorem 4.3.** Let us exclude the case that $r = 2$ and $\chi^2 \eta^{-2} = 1$. Then for $\text{Re}(s) \geq 0$, the above intertwining operator $A(s, \theta, w_1)$ is holomorphic except when the complete $L$-function $L(r(2s + \frac{1}{2}) - r + 1, \chi^2 \eta^{-2})$ (if $r = 2q$) or $L(r(2s + \frac{3}{2}) - r + 1, \chi^2 \eta^{-2})$ (if $r = 2q + 1$) has a pole; In other words, if $r = 2q$, it has a possible pole if and only if $\chi^2 \eta^{-2} = 1$ and $s \in \left\{ \frac{1}{4}, \frac{1}{2} - \frac{1}{2d'} \right\}$, and if $r = 2q + 1$, it has a possible pole if and only if $\chi^2 \eta^{-2} = 1$ and $s \in \left\{ \frac{1}{4}, \frac{1}{2} - \frac{1}{2d'} \right\}$.

Further if $f^* = \otimes f^*_v$ is a factorizable section and $S$ is a finite set of $v$s which contains all the archimedean places and all the non-archimedean places $v$ at which $f^*_v$ is not spherical. Then the normalized intertwining operator

$$A^*(s, \theta, w_1)f^* := \begin{cases} 
L^S(r(2s + \frac{1}{2}), \chi^2 \eta^{-2}) A(s, \theta, w_1)f^*, & \text{if } r = 2q \\
L^S(r(2s + \frac{1}{2}), \chi^2 \eta^{-2}) A(s, \theta, w_1)f^*, & \text{if } r = 2q + 1, \\
\end{cases}$$

is holomorphic for all $s \in \mathbb{C}$ except when the complete $L$-function $L(r(2s + \frac{1}{2}) - r + 1, \chi^2 \eta^{-2})$ (resp. $L(r(2s + \frac{1}{2}) - r + 1, \chi^2 \eta^{-2})$) has a pole.
The rest of the section is devoted to the proof of this theorem, which, as we will see, boils down to
determining the possible poles of the local intertwining operator.

4.4. Unramified place. To prove the above theorem, we first need the following result on the unramified place.

**Lemma 4.4.** Let \( r = 2q \) or \( 2q + 1 \). Also assume \( F \) is a non-archimedean local field of odd residue characteristic. Further assume that \( \chi, \eta \) and \( \omega \) are all unramified. Consider the intertwining operators

\[
A(s, \theta \chi, \eta, w_1) : \text{Ind}_{Q}^{GL_{2q}} \theta \chi \otimes \delta_Q \rightarrow \text{Ind}_{M_Q N_{r-1}^*}^{GL_{2q}} \omega_1(\theta \chi, \eta) \otimes \delta_Q^{-s}, \quad (r = 2q)
\]

\[
A(s, \theta \chi, \eta, w_1) : \text{Ind}_{Q}^{GL_{2q+1}} \theta \chi \otimes \delta_Q \rightarrow \text{Ind}_{M_Q N_{r-1}^*}^{GL_{2q+1}} \omega_1(\theta \chi, \eta) \otimes \delta_Q^{-s}, \quad (r = 2q + 1),
\]

where \( w_1 \) is as in (4.2).

If \( f_0^* \in \text{Ind}_{Q}^{GL_{2q}} \theta \chi \otimes \delta_Q \) (or \( \text{Ind}_{Q}^{GL_{2q+1}} \theta \chi \otimes \delta_Q \)) is the spherical section such that \( f_0^*(1) = 1 \), then

\[
A(s, \theta \chi, \eta, w_1) f_0^*(1) = \frac{L(r(2s + \frac{1}{2}) - r + 1, \chi^2 \eta^{-2})}{L(r(2s + \frac{1}{2}), \chi^2 \eta^{-2})}, \quad (r = 2q);
\]

\[
A(s, \theta \chi, \eta, w_1) f_0^*(1) = \frac{L(r(2s + \frac{1}{2}) - r + 1, \chi \eta^{-2})}{L(r(2s + \frac{1}{2}), \chi \eta^{-2})}, \quad (r = 2q + 1).
\]

**Proof.** This is [T1, Lemma 2.58]. Note that in [T1] we used \( w_0 = (1_{r-1} \ 1) \) instead of the \( w_1 \) of the lemma, but one can verify that the results are the same because we have \( w_1 = (1_{r-1} \ 1) (w'_{-1}) \), where

\[
w'_{-1} = \begin{cases} \begin{cases} I_{r-2} & \text{if } r = 2q; \\ 1 & \text{if } r = 2q + 1, \end{cases} \\ 1 & \text{if } r = 2q + 1, \end{cases}
\]

and \((w'_{-1}) \in M_Q(O_F)\). Also note that in [T1] a specific \( \omega \) was used but the proof there applies to any \( \omega \). \( \square \)

**Remark 4.7.** Note that for the case \( r = 2q + 1 \), if \( \chi \) is unramified, \( \chi^{1/2} \) exists, and hence one has \( \theta \chi, \eta = \theta \chi^{1/2}, \eta \). Then one can see that the formula for this case is actually subsumed under the formula for \( A(s, \theta \chi, \eta, w_1) f_0^*(1) \) as in the \( r = 2q \) case.

4.5. Proof of Theorem 4.3 (\( r = 2q \)). Let us consider the case \( r = 2q \), so \( \theta = \theta \chi, \eta \). This case is essentially the case treated by [BG]. However, as we pointed out in [T1], the argument in [BG] does not seem to work when they use an asymptotic formula on matrix coefficients at the archimedean place, and hence we will give an alternate argument, which follows the idea given by Jiang [Ji, 84-86] though we use many of the ideas from [BG].

First note that for a factorizable \( f^* = \otimes' f_v^* \in \text{Ind}_{Q(A)}^{GL_{2q}(A)} \theta \otimes \delta_Q \), one can, by Lemma 4.4 write

\[
A(s, \theta, w_1) f^* = \frac{L(r(2s + \frac{1}{2}) - r + 1, \chi^2 \eta^{-2})}{L(r(2s + \frac{1}{2}), \chi^2 \eta^{-2})} \left( \otimes_v \frac{L_v(r(2s + \frac{1}{2}), \chi_v^2 \eta_v^{-2})}{L_v(r(2s + \frac{1}{2}), \chi_v^2 \eta_v^{-2})} A_v(s, \theta_v, w_1) f_v^* \right),
\]
which gives

\begin{equation}
A^*(s, \theta, w_1)f^s \\
= L^S(r(2s + \frac{1}{2}), \chi^2 \eta^{-2})A(s, \theta, w_1)f^s \\
= L(r(2s + \frac{1}{2}) - r + 1, \chi^2 \eta^{-2}) \left( \bigotimes_{v \in S} \frac{1}{L_v(r(2s + \frac{1}{2}) - r + 1, \chi_v^2 \eta_v^{-2})} A_v(s, \theta_v, w_1)f^s_v \right) \\
\end{equation}

where \( S \), which depends on \( f^s \), is as in Theorem 4.3. By Lemma 4.4 and our choice of \( S \), the product

\[ \bigotimes_{v \in S} \frac{L_v(r(2s + \frac{1}{2}), \chi_v^2 \eta_v^{-2})}{L_v(r(2s + \frac{1}{2}) - r + 1, \chi_v^2 \eta_v^{-2})} A_v(s, \theta_v, w_1)f^s_v \]

is holomorphic. Also for \( \text{Re}(s) \geq 0 \), the normalizing factor \( L^S(r(2s + \frac{1}{2}), \chi^2 \eta^{-2}) \) is non-zero holomorphic, and hence in this region the poles of \( A(s, \theta, w_1)f^s \) coincide with those of \( A^*(s, \theta, w_1)f^s \).

Hence to prove Theorem 4.3 it suffices to show that the local “modified intertwining operator”

\[ \frac{1}{L_v(r(2s + \frac{1}{2}) - r + 1, \chi_v^2 \eta_v^{-2})} A_v(s, \theta_v, w_1) : \text{Ind}_{Q(F_v)}^{\tilde{G}_{L_r}(F_v)} \theta_v \otimes \delta_Q^s \rightarrow \text{Ind}_{w_1(\tilde{M}_Q(F_v))N_{1,r-1}(F_v)}^{\tilde{G}_{L_r}(F_v)} w_1 \theta_v \otimes \delta_Q^{-s} \]

is holomorphic for all \( s \in \mathbb{C} \). Thus the question is now completely local, and hence in what follows, we will omit the subscript \( v \) and assume that everything is over the local field.

Recall that the representation \( \theta_{\chi, \eta} \) is the metaplectic tensor product \( \theta_{\chi, \eta} = (\theta_\chi \otimes \tilde{\eta})_\omega \) for an appropriate \( \omega \), and further recall that the representation \( \theta_\chi \) is the exceptional representation with the determinantal character \( \chi \) which is an irreducible subrepresentation of the induced representation

\[ \text{ind}_{B^{r-1}}(\tilde{\chi} \otimes \cdots \otimes \tilde{\chi})_\omega \otimes \delta_{B^{r-1}}^{1/4} \]

(unnormalized induction) for an appropriate \( \omega \), where \( B^{r-1} \) is the Borel subgroup of \( \tilde{G}_{L_{r-1}} \). Hence by Proposition 2.10 we have

\[ \theta_{\chi, \eta} = (\theta_\chi \otimes \tilde{\eta})_\omega \subseteq \text{ind}_{\tilde{G}_{L_{r-1}}(\tilde{\chi} \otimes \cdots \otimes \tilde{\chi})_\omega}^{\tilde{G}_{L_{r-1}}(\tilde{\chi} \otimes \cdots \otimes \tilde{\chi})_\omega} \otimes \delta_{\tilde{G}_{L_{r-1}}(\tilde{\chi} \otimes \cdots \otimes \tilde{\chi})_\omega}^{1/4} \]

for an appropriate \( \omega \), where \( B^{r-1,1} \) is the Borel subgroup of \( \tilde{G}_{L_{r-1}} \times \tilde{G}_{L_1} \). By inducing in stages we have

\[ \text{ind}_{\tilde{G}_{L_{r}}(\tilde{\chi} \otimes \cdots \otimes \tilde{\chi} \otimes \tilde{\eta})_\omega}^{\tilde{G}_{L_{r}}(\tilde{\chi} \otimes \cdots \otimes \tilde{\chi} \otimes \tilde{\eta})_\omega} \otimes \delta_{\tilde{G}_{L_{r}}(\tilde{\chi} \otimes \cdots \otimes \tilde{\chi} \otimes \tilde{\eta})_\omega}^{1/4} \]

By using the normalized induction, we have

\[ \text{Ind}_{\tilde{G}_{L_{r}}(\tilde{\chi} \otimes \cdots \otimes \tilde{\chi} \otimes \tilde{\eta})_\omega}^{\tilde{G}_{L_{r}}(\tilde{\chi} \otimes \cdots \otimes \tilde{\chi} \otimes \tilde{\eta})_\omega} \otimes \delta_{\tilde{G}_{L_{r}}(\tilde{\chi} \otimes \cdots \otimes \tilde{\chi} \otimes \tilde{\eta})_\omega}^{-1/4} \delta_{\tilde{Q}} \]

Furthermore the metaplectic tensor product \( (\tilde{\chi} \otimes \cdots \otimes \tilde{\chi} \otimes \tilde{\eta})_\omega \) is a representation of the Heisenberg group \( \tilde{T} \), and hence it is induced from a representation of the maximal abelian subgroup \( \tilde{T}^e \), where \( \tilde{T}^e \) is as in (3.9). Indeed, we have

\[ (\tilde{\chi} \otimes \cdots \otimes \tilde{\chi} \otimes \tilde{\eta})_\omega = \text{Ind}_{\tilde{T}^e}^{\tilde{T}^e} \omega_{\chi, \eta} \]
for a character \( \omega_{\chi, \eta} : \mathbb{C}^1 \to \mathbb{C}^1 \) with the property that the restriction \( \omega_{\chi, \eta}|_{\mathbb{T}^{(2)}} \) to \( \mathbb{T}^{(2)} \) is \( \chi \otimes \cdots \otimes \chi \otimes \bar{\eta} \), namely
\[
\omega_{\chi, \eta}(s(\begin{pmatrix} t_1^2 \\
 \cdots \\
 t_r^2 \end{pmatrix})) = \chi(t_1^2) \cdots \chi(t_r^2) \eta(t_r^2).
\]

(OOne can write down \( \omega_{\chi, \eta} \) more explicitly but we will not need it for our purposes.) Therefore we have
\[
\text{Ind}^{GL_r}_{Q} \theta_{\chi, \eta} \otimes \delta_{Q} \subseteq \text{Ind}^{GL_r}_{T^* N^*_B} \omega_{\chi, \eta} \otimes \delta_{B^r-1, \delta_{Q}^*},
\]
and accordingly we can view each section \( f^s \in \text{Ind}^{GL_r}_{Q} \theta_{\chi, \eta} \otimes \delta_{Q}^* \) as an element in the induced representation \( \text{Ind}^{GL_r}_{T^* N^*_B} \omega_{\chi, \eta} \otimes \delta_{B^r-1, \delta_{Q}^*} \).

Now we would like to study the analytic property of the integral
\[
A(s, \theta, w) f^s(g) = \int_{N_1, r-1} f^s(s(w)n) dn.
\]

For this purpose, let
\[
w_0 = J_r = \begin{pmatrix} 1 & J_{r-2} & 1 \\
 & \ddots & \ddots \\
 & & 1 \end{pmatrix}
\]
be the longest element in the Weyl group. We use the following by-now well-known lemma, which seems to be sometimes known as the Rallis lemma.

**Lemma 4.11.** The highest pole of the intertwining operator \( A(s, \theta, w) \) is achieved by \( A(s, \theta, w_1) f^s(s(w_0)) \) as the sections \( f^s \) run through those sections with \( \text{supp}(f^s) \subseteq Qw_0 \bar{Q} \).

**Proof.** Several versions of this lemma can be found in various places such as [PSR, Lemma 4.1] and [Sh1, Lemma 4.1], and our case is the metaplectic analogue of [Ji, Lemma 2.1.1]. \( \square \)

We should also mention

**Lemma 4.12.** Let \( f^s \) be as in the above lemma, so that \( \text{supp}(f^s) \subseteq Qw_0 \bar{Q} \). Let \( N \subseteq \bar{Q} \) be a subset of \( \bar{Q} \) such that \( w_0 N w_0^{-1} \cap \bar{Q} = \{1\} \). Then for each fixed \( q \in \bar{Q} \), the map on \( N \) defined by \( n \mapsto f^s(qw_0n) \) is compactly supported.

**Proof.** Note that since \( f^s \) is in the induced space, it is compactly supported modulo \( \bar{Q} \). Now since \( w_0 N w_0^{-1} \cap \bar{Q} = \{1\} \), the natural map \( N \to \bar{Q}/Qw_0 \bar{Q} \) given by \( n \mapsto Qw_0n \) is 1-1. Hence the lemma follows. \( \square \)

Now by Lemma 4.11, we have only to show
\[
\frac{1}{L(r(2s + \frac{1}{2}) - r + 1, \chi^{2r-2})} A(s, \theta, w_1) f^s(s(w_0))
\]
is holomorphic with \( f^s \) as in the lemma, where
\[
A(s, \theta, w_1) f^s(s(w_0)) = \int_{N_1, r-1} f^s(s(w_1)n) s(w_0)) dn.
\]
Let us write each \( n \in N_1, r-1 \) as
\[
n = \begin{pmatrix} 1 & Z & y \\
 & I_{r-2} & \\
 & & 1 \end{pmatrix}.
\]
By direct computation one can verify

\[
\begin{align*}
    w_1w_0 &= \begin{pmatrix}
        1 & J_{r-2} & 0 \\
        0 & zJ_{r-2} & 1 \\
        0 & 0 & 1
    \end{pmatrix} \\
    &= \begin{pmatrix}
        -y^{-1} & -Zy^{-1} & 1 \\
        I_{r-2} & y & 0 \\
        y & 1 & 0
    \end{pmatrix} \\
    &= \begin{pmatrix}
        1 & ZJ_{r-2}y^{-1} & y^{-1} \\
        I_{r-2} & 1 & y
    \end{pmatrix}
\end{align*}
\]

provided \( y \neq 0 \). Hence

\[
    s(w_1)n(s(w_0)) = (1, \epsilon)s\left(\begin{pmatrix}
        -y^{-1} & -Zy^{-1} & 1 \\
        I_{r-2} & y & 0 \\
        y & 1 & 0
    \end{pmatrix}\right)s(w_0)\left(\begin{pmatrix}
        1 & ZJ_{r-2}y^{-1} & y^{-1} \\
        I_{r-2} & 1 & y
    \end{pmatrix}\right)
\]

for some \( \epsilon = \epsilon(y, Z) \in \{ \pm 1 \} \), which \textit{a priori} depends on \( y \) and \( Z \). (One can compute \( \epsilon \) by using the algorithm for computing the cocycle \( \sigma_r \) developed in \cite{BLS}, and can actually verify that \( \epsilon = 1 \) for any \( y \) and \( Z \). But since this computation is extremely tedious, though not so deep, and for our purposes we will not need the precise information on \( \epsilon \), we will leave \( \epsilon \) as above.)

With this computation for \( s(w_1)n(s(w_0)) \) one can write

\[
    \int_{N_{1,r-1}} f^*(s(w_1)n(s(w_0))) \, dn
    = \int_{F^x} \int_{F_{r-2}} \epsilon f^*\left(s\left(\begin{pmatrix}
        -y^{-1} & -Zy^{-1} & 1 \\
        I_{r-2} & y & 0 \\
        y & 1 & 0
    \end{pmatrix}\right)s(w_0)\left(\begin{pmatrix}
        1 & ZJ_{r-2}y^{-1} & y^{-1} \\
        I_{r-2} & 1 & y
    \end{pmatrix}\right)\right) \, dZdy
\]

\[
    = \int_{F^x} \int_{F_{r-2}} \epsilon |y|^{-r} \frac{1}{2} (\chi \otimes \cdots \otimes \tilde{\chi} \otimes \tilde{\eta})_\omega s\left(\begin{pmatrix}
        -y^{-1} \\
        I_{r-2}
    \end{pmatrix}\right) \, dZdy
\]

\[
    = \int_{F^x} \int_{F_{r-2}} \epsilon |y|^{-r} \frac{1}{2} (\chi \otimes \cdots \otimes \tilde{\chi} \otimes \tilde{\eta})_\omega s\left(\begin{pmatrix}
        -y^{-1} \\
        I_{r-2}
    \end{pmatrix}\right) \, dZdy,
\]

where we should note that \( dy \) is the additive measure and the integral over \( F^x \) is the same as the integral over \( F \) because those two sets are equal almost everywhere. By changing the variable \( ZJ_{r-2}y^{-1} \mapsto Z \), then changing the additive measure \( dy \) to the multiplicative measure \( d^*y \), and then changing the variable \( y^{-1} \mapsto y \), one can see the above integral is written as

\[
    \int_{F^x} \int_{F_{r-2}} \epsilon' |y|^{r-1} \frac{1}{2} (\chi \otimes \cdots \otimes \tilde{\chi} \otimes \tilde{\eta})_\omega s\left(\begin{pmatrix}
        -y \\
        I_{r-2}
    \end{pmatrix}\right) f^*\left(s\left(\begin{pmatrix}
        1 & ZJ_{r-2}y^{-1} & y^{-1} \\
        I_{r-2} & 1 & y
    \end{pmatrix}\right)\right) \, dZd^*y
\]

for some \( \epsilon' = \epsilon'(y, Z) \in \{ \pm 1 \} \).
Now let \( a_1, \ldots, a_l \) be a complete set of representatives of \( F^x \setminus F^x \). Then the above integral is written as

\[
\sum_{i=1}^l \int_{F^{x2}} \int_{F^{r-2}} \epsilon_i' |x_{a_i}|^{2s-\frac{2r+4}{4}} (\tilde{\chi} \cdots \tilde{\chi} \tilde{\eta}) \omega(s \left( -x_{a_i} \begin{pmatrix} I_{r-2} \\ (x_{a_i})^{-1} \end{pmatrix} \right)) \\
f^s \left( s(w_0 \begin{pmatrix} 1 & Z & x_{a_i} \\ I_{r-2} & 1 \end{pmatrix}) \right) dZ d^x x
\]

for a certain choice of the measure \( d^x x \) on \( F^{x2} \) and some \( \epsilon_i' = \epsilon_i'(x, Z) \in \{\pm 1\} \), which is further written as

\[
\sum_{i=1}^l \int_{F^{x2}} \int_{F^{r-2}} \epsilon_i' |x_{a_i}|^{2s-\frac{2r+4}{4}} (\tilde{\chi} \cdots \tilde{\chi} \tilde{\eta}) \omega(s \left( x \begin{pmatrix} I_{r-2} \\ x^{-1} \end{pmatrix} \right)) \\
f^s \left( s(-a_i \begin{pmatrix} 1 & Z & x_{a_i} \\ I_{r-2} & 1 \end{pmatrix}) s(w_0 \begin{pmatrix} 1 & Z & x_{a_i} \\ I_{r-2} & 1 \end{pmatrix}) \right) dZ d^x x.
\]

Recall from (4.13) that we can view the section \( f^s \) as an element in the induced space \( \text{Ind}_{F^{x} N_{x}^2}^{GL_r} \omega \chi \eta \otimes \delta_{B^{-1}}^{-1/4} \delta_{Q}^{s} \), and hence the above integral is written as

\[
(4.14) \quad \sum_{i=1}^l \int_{F^{x2}} \int_{F^{r-2}} \epsilon_i' |x_{a_i}|^{2s-\frac{2r+4}{4}} \tilde{\chi}(x) \eta(x)^{-1} \\
f^s \left( s(-a_i \begin{pmatrix} 1 & Z & x_{a_i} \\ I_{r-2} & 1 \end{pmatrix}) s(w_0 \begin{pmatrix} 1 & Z & x_{a_i} \\ I_{r-2} & 1 \end{pmatrix}) \right) dZ d^x x.
\]

Note that

\[
w_0 \begin{pmatrix} 1 & Z & x_{a_i} \\ I_{r-2} & 1 \end{pmatrix} w_0^{-1} = \begin{pmatrix} 1 & Z \\ x_{a_i} & 1 \end{pmatrix}
\]

and hence we can apply Lemma 1.12 to the map

\[
(x, Z) \mapsto f^s \left( s(-a_i \begin{pmatrix} 1 & Z & x_{a_i} \\ I_{r-2} & 1 \end{pmatrix}) s(w_0 \begin{pmatrix} 1 & Z & x_{a_i} \\ I_{r-2} & 1 \end{pmatrix}) \right),
\]

which implies that the one can write

\[
\epsilon_i' f^s \left( s(-a_i \begin{pmatrix} 1 & Z & x_{a_i} \\ I_{r-2} & 1 \end{pmatrix}) s(w_0 \begin{pmatrix} 1 & Z & x_{a_i} \\ I_{r-2} & 1 \end{pmatrix}) \right) = \sum_{\lambda, \phi, \phi'} \lambda(s) \phi(x) \phi'(Z)
\]

for some holomorphic functions \( \lambda \) and smooth compactly supported functions \( \phi \) and \( \phi' \) on \( F \) and \( F^{r-2} \), respectively. Hence to study the analytic behavior of (4.14) we have only to study that of

\[
\int_{F^{x2}} \int_{F^{r-2}} |x|^{2s-\frac{2r+4}{4}} \tilde{\chi}(x) \eta(x)^{-1} \phi(x) \phi'(Z) dZ d^x x,
\]

which is written as

\[
\int_{F^{x2}} |x|^{2s-\frac{2r+4}{4}} \tilde{\chi}(x) \eta(x)^{-1} \phi(x) d^x x \cdot \int_{F^{r-2}} \phi'(Z) dZ.
\]
The integral over $Z$ is independent of $s$, and hence we have only to consider the first integral. But one can see

$$
\int_{F \times Z} |x|^{r s + \frac{1}{2}} \psi(x) \omega(x) d^n x = c \int_{F \times Z} |y|^{2 r s + \frac{1}{2}} \psi(y) \omega(y) d^n y
$$

for an appropriate non-zero constant $c$. Then by Tate's thesis, one knows that this integral is $L(2(r s + \frac{1}{2}), \chi^2 \eta^{-2})$ times an entire function on $s$, where this $L$-factor is precisely the one appearing in (1.13). Therefore (1.13) is an entire function on $s$.

4.6. **Proof of Theorem 4.3** ($r = 2q + 1$). Next we consider the case $r = 2q$, so that $\theta = \theta_{\chi, \eta}$. The basic idea is the same as the case $\theta = \theta_{1, \eta}$ in which we reduce the problem to the local one and use the Rallis lemma and Tate's thesis. Namely by arguing as above, we can see that we have only to show that the local “modified intertwining operator”

$$
\frac{1}{L_v(r(2s + \frac{r}{2}) - r + 1, \chi_v \eta^{-2})} A_v(s, \theta_v, w_1) : \text{Ind}_{Q(F_v)}^{\text{GL}_r(F_v)} \theta_v \otimes \delta_Q^{s} \rightarrow \text{Ind}_{M(Q(F_v))N_1, r-1(F_v)}^{\text{GL}_r(F_v)} w_1 \theta_v \otimes \delta_Q^{-s}
$$

is holomorphic for all $s \in \mathbb{C}$. Again, we will omit the subscript $v$ and assume that everything is over the local field.

Recall that the representation $\theta_{\chi, \eta}$ is the metaplectic tensor product $\theta_{\chi, \eta} = (\theta_{\chi} \otimes \bar{\eta})_{\omega}$ for an appropriate $\omega$, and further recall that the representation $\bar{\theta}_{\chi}$ is the twisted exceptional representation on $\text{GL}_2$, which is an irreducible subrepresentation of the induced representation $\text{Ind}_{P_{2, 2, 1}}^{\text{GL}_{2q}} (r_1 \otimes \cdots \otimes r_1)_{\omega} \otimes \delta_{P_{2, 2, 1}}^{1/4}$ (unnormalized induction) for an appropriate $\omega$, where $P_{2, 2, 1}$ is the $(2, 2, 1)$-parabolic subgroup of $\text{GL}_{2q}$. Hence by Proposition 2.10 we have

$$
\theta_{\chi, \eta} = (\theta_{\chi} \otimes \bar{\eta})_{\omega} \subseteq \text{Ind}_{P_{2, 2, 1}}^{\text{GL}_r} (r_1 \otimes \cdots \otimes r_1)_{\omega} \otimes \delta_{P_{2, 2, 1}}^{1/4}
$$

for an appropriate $\omega$. By inducing in stages we have

$$
\text{ind}_{Q}^{\text{GL}_r} \theta_{\chi, \eta} \otimes \delta_Q^{s} \subseteq \text{ind}_{P_{2, 2, 1}}^{\text{GL}_r} (r_1 \otimes \cdots \otimes r_1)_{\omega} \otimes \delta_{P_{2, 2, 1}}^{1/4} \delta_Q^{s + \frac{1}{2}}.
$$

By using the normalized induction, we have

$$
\text{Ind}_{Q}^{\text{GL}_r} \theta_{\chi, \eta} \otimes \delta_Q^{s} \subseteq \text{Ind}_{P_{2, 2, 1}}^{\text{GL}_r} (r_1 \otimes \cdots \otimes r_1)_{\omega} \otimes \delta_{P_{2, 2, 1}}^{-1/4} \delta_Q^{s}.
$$

Furthermore by definition of metaplectic tensor product, we have

$$
(r_1 \otimes \cdots \otimes r_1)_{\omega} \subseteq \text{Ind}_{Z_{\text{GL}_r}(P_{2, 2, 1})}^{\text{GL}_r} \omega (r_1 \otimes \cdots \otimes r_1)_{\omega},
$$

where $P_{2, 2, 1}$ is the $(2, 2, 1)$-parabolic of $\text{GL}_r$. (One can check that this inclusion is actually equality, but since we will not need this fact, we will leave the verification to the reader.) Therefore we have

$$
(4.15) \quad \text{Ind}_{Q}^{\text{GL}_r} \theta_{\chi, \eta} \otimes \delta_Q^{s} \subseteq \text{Ind}_{Z_{\text{GL}_r}(P_{2, 2, 1})}^{\text{GL}_r} \omega (r_1 \otimes \cdots \otimes r_1)_{\omega} \otimes \delta_{P_{2, 2, 1}}^{-1/4} \delta_Q^{s},
$$

and accordingly we can view each section $f^s$ as an element in the latter induced representation. Also we should recall that the space of the representation $r_1 \otimes \cdots \otimes r_1$ is the (usual) tensor product

$$
S_\chi(F) \otimes \cdots \otimes S_\chi(F) \otimes \mathbb{C},
$$

where $S_\chi(F)$ realizes the Weil representation $r_1$. Hence for each $g \in \text{GL}_r$, we can view $f^s(g)$ as an element in this space.
Now we would like to study the analytic property of the integral
\[
A(s, \theta, w_1)f^s(g) = \int_{N_1, r^{-1}} f^s(s(w_1 n)g) \, dn.
\]

But by Lemma 4.11, we have only to show
\[\tag{4.16}
\frac{1}{L(r(2s + \frac{1}{2}) - r + 1, \chi \eta^{-2})} A(s, \theta, w_1)f^s(s(w_0))
\]
is holomorphic with \(f^s\) as in Lemma 4.11. If we write each \(n \in N_1, r^{-1}\) as
\[
n = \begin{pmatrix} 1 & Z & y \\ I_{r-2} & 1 \end{pmatrix},
\]
then, if \(y \neq 0\), we have
\[
w_1 nw_0 = \begin{pmatrix} 1 & J_{r-2}' \\ y & zJ_{r-2} \\ 1 \end{pmatrix} = \begin{pmatrix} -y^{-1} & -Zy^{-1}J_{r-2}J_{r-2}' \\ I_{r-2} & 1 \\ y \end{pmatrix} \begin{pmatrix} 1 & ZJ_{r-2}y^{-1} \\ I_{r-2} & y^{-1} \\ 1 \end{pmatrix},
\]
where
\[
w_0' = \begin{pmatrix} 1 & J_{r-2}' \end{pmatrix} \quad \text{and} \quad J_{r-2}' = \begin{pmatrix} 1 & J_{r-4} \\ 1 \end{pmatrix}.
\]
Hence
\[
s(w_1 n)s(w_0) = (1, \epsilon)s\left(\begin{pmatrix} -y^{-1} & -Zy^{-1}J_{r-2}J_{r-2}' \\ I_{r-2} & 1 \\ y \end{pmatrix}\right)s\left(\begin{pmatrix} 1 & ZJ_{r-2}y^{-1} \\ I_{r-2} & y^{-1} \\ 1 \end{pmatrix}\right)
\]
for some \(\epsilon = \epsilon(y, Z) \in \{\pm 1\}\), which a priori depends on \(y\) and \(Z\). Here \(Z\) is a \(1 \times (r - 2)\) matrix. If we write
\[
Z = (Z', z)
\]
where \(Z'\) is \(1 \times (r - 3)\) and \(z \in F\), then
\[
-Zy^{-1}J_{r-2}J_{r-2}' = -y^{-1}(z, Z'\left(I_{r-4} \ 1\right)) \quad \text{and} \quad Zy^{-1}J_{r-2}' = y^{-1}(z, Z'\left(1 \ J_{r-4}\right)),
\]
and hence

\[
\int_{N_{1, r-1}} f^\times(s(w_1 n) s(w_0)) \, dn \\
= \int_{F^\times} \int_{F_{r-3}} \int_F \epsilon f^\times(s(\begin{pmatrix} -y^{-1} & -zy^{-1} & -y^{-1}Z' & 1 \\ 1 & & & \\ I_{r-3} & & & \\ y & & & \\ \end{pmatrix})) \\
\quad \cdot s(w_0') \begin{pmatrix} 1 & y^{-1}Z' & zy^{-1} & y^{-1} \\ & I_{r-3} & & \\ I_{r-3} & & & 1 \\ & & & y \\ \end{pmatrix} \, dz \, dZ' \, dy
\]

\[
= \int_{F^\times} \int_{F_{r-3}} \int_F \epsilon |y|^{-\frac{1}{4}(r-3)+s+\frac{1}{2}|y|} (r_\chi \otimes \cdots \otimes r_\chi \otimes \eta) \omega(s(\begin{pmatrix} -y^{-1} & -zy^{-1} & 1 \\ 1 & & \\ I_{r-3} & & \\ y & & \\ \end{pmatrix})) \\
\quad \cdot f^\times(s(w_0') \begin{pmatrix} 1 & y^{-1}Z' & zy^{-1} & y^{-1} \\ & I_{r-3} & & \\ I_{r-3} & & & 1 \\ & & & y \\ \end{pmatrix}) \, dz \, dZ' \, dy
\]

By changing the variables \(zy^{-1} \mapsto z\) and \(y^{-1}Z' \mapsto Z'\), then changing the additive measure \(dy\) to the multiplicative measure \(d^x y\), and then changing the variable \(y^{-1} \mapsto y\), one can see the above integral is written as

\[
\int_{F^\times} \int_{F_{r-3}} \int_F \epsilon' |y|^{rs-\frac{1}{4} r + \frac{1}{4}} (r_\chi \otimes \cdots \otimes r_\chi \otimes \eta) \omega(s(\begin{pmatrix} y & -z & 1 \\ & I_{r-3} & \\ & & & y^{-1} \\ \end{pmatrix})) \\
\quad \cdot f^\times(s(w_0') \begin{pmatrix} 1 & Z' & z & y \\ I_{r-3} & & & 1 \\ & & & y^{-1} \\ & & & y \\ \end{pmatrix}) \, dz \, dZ' \, d^x y.
\]

for some \(\epsilon' = \epsilon'(y, Z) \in \{ \pm 1 \} \).
Now let \( a_1, \ldots, a_l \) be a complete set of representatives of \( F^\times \setminus F^\times \). Then the above integral is written as

\[
\sum_{i=1}^l \int_{F^\times} \int_{F^\times - 2} \epsilon'_i|x|a_i|^{r_1 - \frac{1}{2}}(r_1 \otimes \cdots \otimes r_1 \otimes \check{\eta}) \omega(s) \left( \begin{array}{c} -x \quad -z \\ 1 \quad 1_r \end{array} \right) \left( \begin{array}{c} (x_a)^{-1} \\ \end{array} \right) dz dZ' dx
\]

for a certain choice of the measure \( d^\times x \) on \( F^\times \) and some \( \epsilon'_i = \epsilon'_i(x, Z) \in \{ \pm 1 \} \), which is further written as

\[
\sum_{i=1}^l \int_{F^\times} \int_{F^\times - 2} \epsilon'_i|x|a_i|^{r_1 - \frac{1}{2}}(r_1 \otimes \cdots \otimes r_1 \otimes \check{\eta}) \omega(s) \left( \begin{array}{c} -x \quad -z \\ 1 \quad 1_r \end{array} \right) \left( \begin{array}{c} (x_a)^{-1} \\ \end{array} \right) dz dZ' dx.
\]

Recall that we can view the section \( f^\times \) as an element in the second induced space in (4.19), and in particular we can and do view the expression \( f^\times(\cdots) \) in the above integral as an element in the Schwartz space \( S_\chi(F) \otimes \cdots \otimes S_\chi(F) \otimes \mathbb{C} \). Therefore to show the desired holomorphy, we may consider the integral

\[
\sum_{i=1}^l \int_{F^\times} \int_{F^\times - 2} \epsilon'_i|x|a_i|^{r_1 - \frac{1}{2}}(r_1 \otimes \cdots \otimes r_1 \otimes \check{\eta}) \omega(s) \left( \begin{array}{c} -x \quad -z \\ 1 \quad 1_r \end{array} \right) \left( \begin{array}{c} (x_a)^{-1} \\ \end{array} \right) dz dZ' dx,
\]

where \( (t_1, \ldots, t_q) \in F^q \) is fixed. By (3.2), (3.5) and (3.6), one can see that the above integral is written as

\[
(4.17) \sum_{i=1}^l \int_{F^\times} \int_{F^\times - 2} \epsilon'_i|x|a_i|^{r_1 - \frac{1}{2} + \frac{1}{2}}(x^{1/2}) \eta(x)^{-1} \psi(-zt^2_1) \left( \begin{array}{c} -a_i \quad -z \\ 1 \quad 1_r \end{array} \right) \left( \begin{array}{c} (x_a)^{-1} \\ \end{array} \right) dz dZ' dx,
\]

which is independent of the choice of \( x^{1/2} \).
Recall that the support of $f^s$ is in $\widetilde{Q}w_0\widetilde{Q} = \widetilde{Q}w_0'\widetilde{Q}$. Also note that

$$w'_0 \begin{pmatrix} 1 & Z' & z & xa_i \\ I_{r-3} & 1 & 1 & 1 \end{pmatrix} w'_0^{-1} = \begin{pmatrix} 1 & I_{r-3} \\ xa_i & Z'' & z & 1 \end{pmatrix},$$

where $Z'' = Z' \begin{pmatrix} 1 & 1 \\ J_{r-4} \end{pmatrix}$. Hence by Lemma 4.12 (with $w'_0$ in place of $w_0$), the map

$$(x, Z', z) \mapsto f^s(s(-a_i \begin{pmatrix} 1 & I_{r-3} \\ a_i^{-1} \end{pmatrix}) w'_0 \begin{pmatrix} 1 & Z' & z & xa_i \\ I_{r-3} & 1 & 1 \end{pmatrix})$$

is smooth and compactly supported and hence so is the map

$$(x, Z', z) \mapsto f^s(s(-a_i \begin{pmatrix} 1 & I_{r-3} \\ a_i^{-1} \end{pmatrix}) w'_0 \begin{pmatrix} 1 & Z' & z & xa_i \\ I_{r-3} & 1 & 1 \end{pmatrix})(x^{1/2}t_1, \ldots, t_q).$$

Therefore one can write

$$\epsilon_i f^s(s(-a_i \begin{pmatrix} 1 & I_{r-3} \\ a_i^{-1} \end{pmatrix}) w'_0 \begin{pmatrix} 1 & Z' & z & xa_i \\ I_{r-3} & 1 & 1 \end{pmatrix})(x^{1/2}t_1, \ldots, t_q) = \sum_{\lambda, \phi, \phi'} \lambda(s)\phi(x)\phi'(Z', z)$$

for some holomorphic functions $\lambda$ and smooth compactly supported functions $\phi$ and $\phi'$ on $F$ and $F^{r-3} \times F$, respectively. Hence to study the analytic behavior of (4.17) we have only to study that of

$$\int_{F^{r-2}} \int_{F^{r-2}} |x|^{r_s - \frac{1}{2}r + \frac{1}{4}} \chi(x^{1/2}) \eta(x)^{-1} \phi(x)\phi'(Z) dZ d^\times x,$$

which is written as

$$\int_{F^{r-2}} |x|^{r_s - \frac{1}{2}r + \frac{1}{4}} \chi(x^{1/2}) \eta(x)^{-1} \phi(x) d^\times x \cdot \int_{F^{r-2}} \phi'(Z) dZ,$$

where recall we have put $Z = (Z', z)$. The integral over $Z$ is independent of $s$, and hence we have only to consider the first integral. But one can see

$$\int_{F^{r-2}} |x|^{r_s - \frac{1}{2}r + \frac{1}{4}} \chi(x^{1/2}) \eta(x)^{-1} \phi(x) d^\times x = c \int_{F^r} |y|^{r_s - \frac{1}{2}r + \frac{1}{4}} \chi(y) \eta(y)^{-1} \phi(y^2) d^\times y$$

for an appropriate non-zero constant $c$. By Tate’s thesis, one knows that this integral is $L(2(r_s - \frac{1}{2}r + \frac{1}{4}), \chi \eta^{-2})$ times an entire function on $s$, where this $L$-factor is precisely the one appearing in (4.16). Therefore (4.16) is an entire function on $s$. 
4.7. The case $r = 2$. In Theorem 4.3 we excluded the case that $r = 2$ and $\chi^2 \eta^{-2} = 1$. However, the argument above works even in this case except at $s = 0$; Namely Theorem 4.3 holds even when $r = 2$ and $\chi^2 \eta^{-2} = 1$ except at $s = 0$. Now at $s = 0$, since for $r = 2$ the inducing representation is cuspidal, the general theory of Eisenstein series ([MW Proposition IV. 1.11.(b)]) implies that it is actually holomorphic at $s = 0$. One can see $(\hat{\chi} \tilde{\otimes} \hat{\eta})_\omega = (\hat{\chi} \tilde{\otimes} \hat{\eta})_\omega$ by Proposition 2.12 and $w(\hat{\chi} \tilde{\otimes} \hat{\eta})_\omega = (\hat{\chi} \tilde{\otimes} \hat{\eta})_\omega$ by Proposition 2.19. Hence the map $A(0, \theta_{\chi, \eta}, w_1)$ is an endomorphism on $\text{Ind}_{\tilde{\mathbb{B}}_\omega}^{\tilde{\text{GL}}_2} = (\hat{\chi} \tilde{\otimes} \hat{\chi})_\omega$. But by the functional equation of the intertwining operator ([MW Theorem IV.1.10(b), p.141]) we must have $A(0, \theta_{\chi, \eta}, w_1)^2 = \text{Id}$. This implies that on each irreducible submodule of $\text{Ind}_{\tilde{\mathbb{B}}_\omega}^{\tilde{\text{GL}}_2} (\hat{\chi} \tilde{\otimes} \hat{\chi})_\omega$, the operator $A(0, \theta_{\chi, \eta}, w_1)$ acts as $\pm \text{Id}$. Indeed, it is shown in [BG Proposition 7.3 (iii)] that it acts as $-1$ on all of the induced space. Hence we have

**Proposition 4.18.** For $r = 2$ and $\chi^2 \eta^{-2} = 1$, the (global) intertwining operator $A(s, \theta_{\chi, \eta}, w_1)$ is holomorphic for $\text{Re}(s) \geq 0$ except a possible simple pole at $s = \frac{1}{4}$. Moreover $A(0, \theta_{\chi, \eta}, w_1)$ acts as $-1$.

5. The unnormalized Eisenstein series

Now we are ready to state the main theorem on the analytic behavior of the (unnormalized) Eisenstein series. Let $\theta = \theta_{\chi, \eta}$ or $\vartheta_{\chi, \eta}$, depending on the parity of $r$. In this paper, we consider the Eisenstein series associated to the induced representation

$$\text{Ind}_{\tilde{\mathbb{Q}}(\tilde{A})}^{\text{GL}_r(\tilde{A})} \theta \otimes \delta_{\tilde{Q}}.$$

Namely for each $f^s \in \text{Ind}_{\tilde{\mathbb{Q}}(\tilde{A})}^{\text{GL}_r(\tilde{A})} \theta \otimes \delta_{\tilde{Q}}$, we let

$$E(g, s; f^s) = \sum_{\gamma \in Q(F) \setminus \text{GL}_r(F)} f^s(s(\gamma)g; 1)$$

for $g \in \tilde{\text{GL}}_r(\tilde{A})$, where we view each section $f^s$ as a function

$$f^s : \tilde{\text{GL}}_r(\tilde{A}) \to \text{space of } \theta \otimes \delta_{\tilde{Q}}^{s + \frac{1}{2}},$$

and by $f^s(s(\gamma)g; 1)$ we mean the automorphic form $f^s(s(\gamma)g)$ on $\tilde{M}_Q(\tilde{A})$ evaluated at the identity $1$. Also for fixed $s(\gamma)g$, we often write $f^s(s(\gamma)g; -)$, which is viewed as an automorphic form in the space of $\theta \otimes \delta_{\tilde{Q}}^s$, namely the function $\tilde{m} \mapsto f^s(s(\gamma)g; \tilde{m})$ is an automorphic form on $\tilde{M}_Q(\tilde{A})$.

The main theme of this section is to prove

**Theorem 5.3.** The Eisenstein series $E(g, s; f^s)$ is holomorphic for $\text{Re}(s) \geq 0$ except that it possibly has a simple pole at $s = \frac{1}{4}$, when $\chi^2 \eta^{-2} = 1$ and $r = 2q$, or $\chi \eta^{-2} = 1$ and $r = 2q + 1$.

In what follows we will give a proof of the theorem. The basic idea seems to be standard in that we will compute a constant term of the Eisenstein series and argue inductively on $r$. Indeed, the most of the ideas (at least for the case $\theta = \theta_{\chi, \eta}$) are already present in [BG] and we will borrow many of the ideas from there. Let us note, however, that the cuspidal support of our Eisenstein series differs for the two cases $\theta = \theta_{\chi, \eta}$ and $\theta = \vartheta_{\chi, \eta}$, and hence we will compute different constant terms for those cases.
5.1. The base step of induction. The base step is \( r = 2 \) for the case \( \theta = \theta_{\chi, \eta} \) and \( r = 3 \) for the case \( \theta = \vartheta_{\chi, \eta} \) and \( \chi^{1/2} \) does not exist. (If \( r \) is odd and \( \chi^{1/2} \) exists, then we have \( \theta = \theta_{\chi^{1/2}, \eta} \) and hence the base step will be the case \( r = 2 \).)

Consider the case \( r = 2 \). Then \( \theta = \theta_{\chi, \eta} = (\tilde{\chi} \otimes \tilde{\eta})_{\omega} \). The analytic property of the Eisenstein series \( E(g, s; f^*) \) is determined by the constant term \( E_{N_B}(g, s; f^*) \) along the unipotent radical \( N_B \) of \( B \). By the standard calculation, one has

\[
E_B(g, s; f^*) = f^*(g) + \int_{N_B} f^*(s(w_1^{-1}n)g) \, dn = f^*(g) + A(s, \theta_{\chi, \eta}, w_1)f^*(g),
\]

where \( w_1 \) is as in (4.2). By Theorem 4.3 (and Proposition 4.18) above we know that \( A(s, \theta, w_1) \) is holomorphic for \( \text{Re}(s) \geq 0 \) except that if \( \chi^2 \eta^{-1} = 1 \), it has a possible pole at \( \frac{1}{4} \). Hence Theorem 5.3 holds for \( r = 2 \). (Though we do not need this fact, let us mention that at \( s = \frac{1}{4} \), the intertwining operator does have a pole and the residues generate the non-twisted exceptional representation for \( r = 2 \) of determinantal character \( \chi \), namely the Weil representation \( r_{\chi^2} \).

Next consider the case \( r = 3 \) (so \( \vartheta_{\chi, \eta} = (\tilde{\chi} \otimes \tilde{\eta})_{\omega} \) and \( \chi^{1/2} \) does not exist. Since \( \chi^{1/2} \) does not exist, necessarily \( \chi \eta^{-2} \neq 1 \). Then the inducing representation \( \vartheta_{\chi, \eta} \) is cuspidal, since \( \vartheta_{\chi} \) is cuspidal, and the metaplectic tensor product preserves cuspidality (Proposition 2.8). Moreover the Levi \( \tilde{GL}_2 \times \tilde{GL}_1 \) is maximal and non-self-conjugate, and hence the Eisenstein series \( E(g, s; f^*) \) is entire as desired.

5.2. The induction step for \( r = 2q \). Now we will prove the induction step. Let us first consider the case \( r = 2q > 2 \), namely \( \theta = \theta_{\chi, \eta} \). This basically coincides with the case treated by Bump and Ginzburg in [BG]. It seems to the author, however, that some of their arguments cannot be justified without the theory of metaplectic tensor products developed in [12]. Also in [BG], they use the induction argument for the normalized Eisenstein series. However, as we will point out later, their induction argument does not seem to work because the finite set \( S \) used to normalize the Eisenstein series depends not only on \( \chi \) and \( \eta \), but also on the choice of the section \( f^* \), which makes the induction hypothesis not applicable. (This is another error in [BG] which does not seem to have been pointed out anywhere else.) Indeed, to obtain the holomorphy for the normalized Eisenstein series, one needs to use the functional equation of the Eisenstein series as we will do in a later section. Moreover there are a quite few places in [BG] where important computations are omitted. For those reasons, we will write out the computations in detail, though our computations are quite parallel to those in [BG].

Now for the case at hand, the cuspidal support of our Eisenstein series \( E(g, s; f^*) \) is the Borel subgroup and hence the poles of the Eisenstein series are precisely the poles of the constant term along any parabolic. So in particular in this subsection we let

\[
P = P_{1, r-1} = M_P N_P \subseteq \text{GL}_r
\]
be the \((1,r-1)\)-parabolic, and will consider the constant term along the unipotent radical \(N_P\) of this parabolic. The constant term along \(N_P\) is computed as

\[
E_P(g,s;f^s) = \int_{N_P(F) \setminus N_P(k)} E(s(n)g,s;f^s) \, dn
\]

\[
= \int_{N_P(F) \setminus N_P(k)} \sum_{\gamma \in Q(F) \setminus \GL_r(F)} f^s(s(\gamma)g;1) \, dn
\]

\[
= \int_{N_P(F) \setminus N_P(k)} \sum_{\gamma \in Q(F) \setminus \GL_r(F)} \sum_{n' \in N_P(F)\gamma^{-1} \setminus N_P(F)} f^s(s(\gamma n')g;1) \, dn
\]

\[
= \sum_{\gamma \in Q(F) \setminus \GL_r(F) \setminus N_P(F)} \int_{N_P(F) \setminus N_P(k)} \sum_{n' \in N_P(F)\gamma^{-1} \setminus N_P(F)} f^s(s(\gamma n')g;1) \, dn
\]

\[
= \sum_{\gamma \in Q(F) \setminus \GL_r(F) \setminus N_P(F)} \int_{N_P(F) \setminus N_P(k)} f^s(s(\gamma n)g;1) \, dn, \tag{5.5}
\]

where \(N_P(F)\gamma^{-1} = N_P(F) \cap \gamma^{-1} Q(F) \gamma\) and also for the fourth equality we used Lemma

By the Bruhat decomposition, we have

\[
\GL_r(F) = \bigcup_{w \in \{1,w_1\}} Q(F)w^{-1}P(F),
\]

where \(w_1\) is as in \([122]\). Accordingly, we have

\[
Q/\GL_r \cap N_P = \bigcup_{w \in \{1,w_1\}} Q \cap Qw^{-1}P/\GL_r = \bigcup_{w \in \{1,w_1\}} Q \cap Qw^{-1}M_P = \bigcup_{w \in \{1,w_1\}} M_P \cap wQw^{-1} \setminus M_P,
\]

where the last equality is given by the map \(\gamma = w^{-1}m \mapsto m\) for \(m \in M_P\). Notice that

\[
M_P \cap wQw^{-1} \setminus M_P = \begin{cases} P_{r-2,1}^{r-1} \setminus \GL_{r-1}, & \text{if } w = 1; \\ 1, & \text{if } w = w_1, 
\end{cases}
\]

where \(P_{r-2,1}^{r-1}\) is viewed as a subgroup of \(\GL_r\) embedded in the lower right corner, and \(P_{r-2,1}^{r-1}\) is the \((r-2,1)\)-parabolic of \(\GL_{r-1}\). (Recall the notation from the notation section.)

Using this decomposition, one can write \([120]\) as

\[
= \sum_{\gamma \in Q(F) \setminus \GL_r(F) \setminus N_P(F)} \int_{N_P(F) \setminus N_P(k)} f^s(s(\gamma n)g;1) \, dn
\]

\[
= \sum_{m \in M_P \cap \gamma^{-1}Qw^{-1}} \int_{N_P(F) \setminus N_P(k)} f^s(s(w^{-1}mn)g;1) \, dn
\]

\[
= \left( \sum_{m \in P_{r-2,1}^{r-1}(F) \setminus \GL_{r-1}(F)} \int_{N_P(F) \setminus N_P(k)} f^s(s(mn)g;1) \, dn \right) + \int_{N_P(F)w_1 \setminus N_P(k)} f^s(s(w_1^{-1}n)g;1) \, dn
\]

\[
= E_P(g,s;f^s)_1 + E_P(g,s;f^s)_{w_1},
\]
where we have set
\[ E_P(g, s; f^s)_{\text{id}} := \sum_{m \in P^{r-1}_{-1}(F) \backslash \text{GL}_{r-1}(F)} \int_{N_P(F) \backslash N_P(\mathbb{A})} f^s(s(mn)g; 1) \, dn; \]
\[ E_P(g, s; f^s)_{\text{w}_1} := \int_{N_P(\mathbb{A})} f^s(s(w_1^{-1}n)g; 1) \, dn. \]

Note that we used \( N_P(F)_{\text{w}_1} = 1 \). To sum up, we have obtained
\[(5.6) \quad E_P(g, s; f^s) = E_P(g, s; f^s)_{\text{id}} + E_P(g, s; f^s)_{\text{w}_1}. \]

In what follows, we will show that the non-identity term \( E_P(g, s; f^s)_{\text{w}_1} \) is holomorphic for \( \text{Re}(s) \geq 0 \) except when \( \chi^2 \eta^{-2} = 1 \), in which case it has possible poles at \( s = \frac{1}{2} \) and \( s = \frac{1}{2} - \frac{1}{r} \), and then the identity term \( E_P(g, s; f^s)_{\text{id}} \) is holomorphic for \( \text{Re}(s) \geq 0 \) except when \( \chi^2 \eta^{-2} = 1 \), in which case it has a possible pole at \( s = \frac{1}{2} - \frac{1}{r} \). Then we will show that the possible poles at \( s = \frac{1}{2} - \frac{1}{r} \) (if exist at all) coming from both terms cancel each other. This will complete the induction.

**The non-identity term** \( E_P(g, s; f^s)_{\text{w}_1} \):

First consider the non-identity term. Note that the non-identity term is written as
\[ E_P(g, s; f^s)_{\text{w}_1} = \int_{N_P(\mathbb{A})} f^s(s(w_1^{-1}n)g; 1) \, dn = A(s, \theta_{\chi, \eta}, w_1) f^s, \]
where \( A(s, \theta_{\chi, \eta}, w_1) \) is the intertwining operator studied in Section 4. From Theorem 4.3, we know that \( A(s, \theta_{\chi, \eta}, w_1) \) is holomorphic for \( \text{Re}(s) \geq 0 \) except when \( \chi^2 \eta^{-2} = 1 \), in which case it has possible poles at \( s = \frac{1}{2} \) and \( s = \frac{1}{2} - \frac{1}{r} \).

**The identity term** \( E_P(g, s; f^s)_{\text{id}} \):

One can write
\[ E_P(g, s; f^s)_{\text{id}} = \sum_{m \in P^{r-1}_{-1}(F) \backslash \text{GL}_{r-1}(F)} \int_{N_P(F) \backslash N_P(\mathbb{A})} f^s(s(m)g; s(n')) \, dn', \]
where
\[ P' = P^{r-1}_{-1} = P \cap M_Q = (\text{GL}_1 \times \text{GL}_{r-2} \times \text{GL}_1) N_{P'}, \]
so \( P' \) is the parabolic subgroup of \( \text{GL}_{r-1} \times \text{GL}_1 \) whose Levi is \( \text{GL}_1 \times \text{GL}_{r-2} \times \text{GL}_1 \). If one views \( f^s(s(m)g; -) \) as an automorphic form in \( \theta \otimes \delta_Q^{0+1/2} \) on \( \tilde{M}_Q(\mathbb{A}) \) as explained at the beginning of this section, the integral in the above sum is just the constant term along \( N_{P'} \). But since \( f^s(s(m)g; -) \in \theta \otimes \delta_Q^{0+1/2} \), we need to compute the constant term of the residual representation \( \theta \otimes \delta_Q^{0+1/2} \) along \( N_{P'} \). Recall that the exceptional representation \( \theta \) is constructed as the residue of the Eisenstein series associated to the induced representation \( \text{Ind}_{B^{r-1,1}}(\tilde{\chi} \otimes \cdots \otimes \tilde{\chi} \otimes \tilde{\eta})_{\nu} \) at \( \nu = \rho_{B^{r-1,1}}/2 \), where \( B^{r-1,1} \) is the Borel subgroup of \( \text{GL}_{r-1} \times \text{GL}_1 \). Namely it is generated by
\[ \text{Res}_{\nu = \rho_{B^{r-1,1}}} E_{\tilde{M}_Q}(-, \varphi^\nu; \nu) \]
for \( \varphi^\nu \in \text{Ind}_{B^{r-1,1}}(\tilde{\chi} \otimes \cdots \otimes \tilde{\chi} \otimes \tilde{\eta})_{\nu} \). (Here the superscript for the Eisenstein series is the group on which the Eisenstein series is defined. We will use this convention in what follows as well.) But the constant term of the residue is the same as the residue of the constant term, and hence one first needs to compute the constant term \( E_{\tilde{M}_Q}(-, \varphi^\nu; \nu) \) of \( E_{\tilde{M}_Q}(-, \varphi^\nu; \nu) \) along \( N_{P'} \). For this, one can use [MW]
Proposition (ii), p.92], and obtain
\[
\tilde{E}_{\nu}^{M_p}(m', \nu'; \nu) = \sum_{\nu} E^{\tilde{M}_p}(m', M(w, \nu) \nu'),
\]
where \( m' \in \tilde{M}_p \) and \( w \) runs through all the Weyl group elements of \( \text{GL}_{r-1} \times \text{GL}_1 \) such that \( w^{-1}(\alpha) > 0 \) for all the positive roots \( \alpha \) that are in \( M_p \). Note that
\[
M(w, \nu) \nu' \in \text{Ind}_{\tilde{M}_p}^{\tilde{M}_Q} w(\bar{\chi} \otimes \cdots \otimes \bar{\chi} \otimes \bar{\eta})^w,
\]
but it is viewed as a map on \( \tilde{M}_p \) by restriction. Hence we actually have
\[
M(w, \nu) \nu' \in \text{Ind}_{\tilde{M}_p}^{\tilde{M}_Q} w(\bar{\chi} \otimes \cdots \otimes \bar{\chi} \otimes \bar{\eta})^{w/2} \delta_{B_{1,r-1,1}}^{-1/2} \delta_{B_{1,r-1,1}}^{-1/2}.
\]
(Note that since our induction is normalized, we need the modulus characters \( \delta_{B_{1,r-1,1}}^{1/2}, \delta_{B_{1,r-1,1}}^{-1/2} \).)

One can see that by using the language of permutations, \( w \) runs through all the elements of the form
\[
w = (12 \cdots k), \quad \text{for } k = 1, \ldots, r - 1.
\]
We need to compute the residue at \( \nu = \rho_{B_{1,r-1,1}}^{1/2} / 2 \) of each \( E^{\tilde{M}_p}(m', M(w, \nu) \nu') \) for such \( w \). But one can see that this Eisenstein series has a residue at \( \nu = \rho_{B_{1,r-1,1}}^{1/2} / 2 \) only for \( w = (12 \cdots r - 1) \) for the following reason: Since the cuspidal support of the Eisenstein series \( E^{\tilde{M}_p} \) is the Borel, the analytic behavior is determined by the constant term \( E_{B_{1,r-1,1}}^{\tilde{M}_p}(m', M(w, \nu) \nu') \) along the Borel \( B^{1,r-1,1} \) of \( M_p \). By using [[MW] Proposition (i), p.92], one can see
\[
E_{B_{1,r-1,1}}^{\tilde{M}_p}(m', M(w, \nu) \nu') = \sum_{w'} M(w', w) \circ M(w, \nu) \nu',
\]
where \( w' \) runs through all the elements in the Weyl group of \( M_p \). We have
\[
M(w', w) \circ M(w, \nu) = M(w'w, \nu).
\]
But we know from [KP Theorem II.1.3] that the intertwining operator \( M(w'w, \nu) \) for the exceptional representation \( (\chi \otimes \cdots \otimes \bar{\chi} \otimes \bar{\eta})^w \) has a residue at \( \nu = \rho_{B_{1,r-1,1}}^{1/2} / 2 \) only when \( w'w \) is the longest element, which implies \( w \) must be of the form \( (12 \cdots r - 1) \). Therefore we have
\[
\text{Res}_{\nu=\frac{1}{2} \rho_{B_{1,r-1,1}}} E_{B_{1,r-1,1}}^{\tilde{M}_p}(m', \nu; \nu) = \text{Res}_{\nu=\frac{1}{2} \rho_{B_{1,r-1,1}}} E_{B_{1,r-1,1}}^{\tilde{M}_p}(m', M(w, \nu) \nu'),
\]
where \( w = (12 \cdots r - 1) \).

Now as in the notation section let
\[
\nu = s_1 e_1 + \cdots + s_{r-1} e_{r-1} \in \Phi_{B_{1,r-1,1}}(\mathbb{C}) \cong \mathbb{C}^{r-2}
\]
for \( s_i \in \mathbb{C} \) with \( s_1 + \cdots + s_{r-1} = 0 \). With this notation, for \( w = (12 \cdots r - 1) \) we have
\[
w \nu = s_{r-1} e_1 + s_1 e_2 + s_2 e_3 + \cdots + s_{r-2} e_{r-1} = \frac{s_{r-1}}{r-2} (2p^r) + (s_1 + \frac{s_{r-1}}{r-2}) e_2 + (s_2 + \frac{s_{r-1}}{r-2}) e_3 + \cdots + (s_{r-2} + \frac{s_{r-1}}{r-2}) e_{r-1}.
\]
where we have set
\[ \nu' = (s_1 + \frac{s_r-1}{r-2})e_2 + (s_2 + \frac{s_r-1}{r-2})e_3 + \cdots + (s_{r-2} + \frac{s_r-1}{r-2})e_{r-1}. \]

Note that \( \nu' \in \Phi_{B^1,r-2;1}(\mathbb{C}) \cong \mathbb{C}'^{r-3} \). Now we have
\[ \frac{1}{2} \rho_{B^{r-1,1}} = \frac{1}{4}((r-2)e_1 + (r-4)e_2 + \cdots + (2-r)e_{r-1}) \]
and hence
\[ \frac{1}{2} w \rho_{B^{r-1,1}} = \frac{1}{4}((-2 \rho_{B^{r-1,1}} + (r-3)e_2 + (r-5)e_4 + \cdots + (3-r)e_{r-1}) \]
\[ = \frac{1}{2} \rho_{B^{r-1,1}} + \frac{1}{2} \rho_{B^{r-2,1}}. \]

But here \( -\frac{1}{2} \rho_{B^{r-1,1}} \) is just a character on the Levi \( M_{B^{r-1,1}} = GL_1 \times GL_{r-2} \times GL_1 \) which acts as
\[ (a, g_{r-2}, b) \mapsto |a|^{-\frac{r-2}{4}} |\det(g_{r-2})|^{\frac{1}{4}} \]
for \( (a, g_{r-2}, b) \in GL_1 \times GL_{r-2} \times GL_1 \). Hence for \( m' \in M_{B^r} = GL_1 \times GL_{r-2} \times GL_1 \), we have
\[ \Res_{\nu' = \frac{1}{2} \rho_{B^{r-1,1}}} E_{B^{r-2,1}}(m', M(w, \nu) \varphi) = \Res_{\nu' = \frac{1}{2} \rho_{B^{r-2,1}}} E_{B^{r-2,1}}(m', \varphi') \]
where
\[ \varphi' \in \Ind_{B^{r-2,1}}^{M_{B^r}}(\bar{\chi}) - |\frac{r-2}{4} \otimes (\bar{\chi} \otimes \cdots \otimes \bar{\chi})|\det_{r-2}|^{\frac{1}{4}} |\bar{\eta}|_{\omega}^{\frac{1}{2}} \delta_{B^{r-1,1}}^{1/2} \delta_{B^{r-2,1}}^{1/2}. \]

By computing \( \delta_{B^{r-1,1}}^{1/2}, \delta_{B^{r-2,1}}^{1/2} \), this induced representation is written as
\[ \Ind_{B^{r-2,1}}^{M_{B^r}}(\bar{\chi}) - |\frac{r-2}{4} \otimes (\bar{\chi} \otimes \cdots \otimes \bar{\chi})|\det_{r-2}|^{-\frac{1}{4}} |\bar{\eta}|_{\omega}. \]

The inducing representation in this induced representation is the exceptional character (with some character twits) and hence the residue at \( \nu' = \frac{1}{2} \rho_{B^{r-2,1}} \) of the Eisenstein series for this induced representation gives rise to the exceptional representation of \( \hat{M}_{B^r} \). But by the compatibility of parabolic inductions for metaplectic tensor products (Proposition 2.10), one can see that
\[ \Res_{\nu' = \frac{1}{2} \rho_{B^{r-2,1}}} E_{B^{r-2,1}}(-, \varphi') \in (\bar{\chi} - |\frac{r-2}{4} \otimes \vartheta|\det_{r-2}|^{-\frac{1}{4}} |\bar{\eta}|_{\omega}, \]
where \( \vartheta' \) is the exceptional representation of \( \hat{GL}_{r-2} \) which is the unique irreducible quotient of the induced representation
\[ \Ind_{B^{r-2}}^{GL_{r-1}}(\bar{\chi} \otimes \cdots \otimes \bar{\chi})_{\omega'} \otimes \delta_{B^{r-2}}^{1/4} \]
for an appropriate choice of \( \omega' \).

Hence for each fixed \( m \in GL_{r-1}(F) \), the function on \( \hat{M}_{B^r}(\mathbb{A}) \) given by
\[ m' \mapsto \int_{N_{B^r}(F) \cap N_{B^r}(\mathbb{A})} f^*(s(m)g(s(m')) m') \, dm' \]
is an element in \( (\bar{\chi} - |\frac{r-2}{4} \otimes \vartheta|\det_{r-2}|^{-\frac{1}{4}} |\bar{\eta}|_{\omega} \otimes \delta_{Q}^{\frac{1}{2}+1}, \delta_{Q}^{\frac{1}{2}+1} \) where \( \delta_{Q}^{\frac{1}{2}+1} \) is actually the restriction \( \delta_{Q}^{\frac{1}{2}+1} \mid_{P'} \) to \( P' \). One can compute
\[ \delta_{Q}^{\frac{1}{2}+1}(a, g_{r-2}, b) = |a|^{\frac{1}{2}+1}|\det(g_{r-2})|^{\frac{1}{2}} |b|^{-(r-1)(\frac{1}{2}+1)}. \]
Accordingly the function $f_{N_{P'(A)}}$ on $\widehat{M}_P(A)$ defined by

$$f_{N_{P'(A)}}(g) = \int_{N_{P'(A)}} f^s(g; s(n')) \, dn'$$

is in

$$\text{Ind}_{\widetilde{P}'(\mathbb{A})}^{\widetilde{P}(\mathbb{A})} \left( \chi | - \frac{i}{4} \otimes \theta' | (\text{det}_{r-2})^{-\frac{1}{2}} \otimes \tilde{\eta} \right) \omega_\delta \otimes \delta_Q^{\frac{1}{2}} \delta_{P'_r}^{-1/2} \delta_{P_{r-2,1}}^{1/2}$$

Recall we are trying to figure out the analytic behavior of the Eisenstein series

$$E_P(g, s; f^s)_{\text{id}} = \sum_{m \in P^{-1}_{r-2,1}(F) \setminus GL_{r-1}(F)} f_{N_{P'(A)}}(s(m)g; 1)$$

as $g$ runs through all elements in $\widehat{M}_P(A)$ and $f^s$ runs through all the sections. But for this purpose we may assume that $g = ((1, \xi), \xi) \in \widehat{M}_P(A)$ is such that $(1, h) \in \text{GL}_1(\mathbb{A}) \times GL_{r-1}(\mathbb{A})$, because if $g$ is not of this form, one can always translate $f^s$ by an appropriate element in $\widehat{M}_P(A)$. Namely we consider the section $f_{N_{P'(A)}}|_{GL_{r-1}(\mathbb{A})}$. Let

$$F^s := f_{N_{P'}}|_{GL_{r-1}(\mathbb{A})}$$

Then one can see, by using Proposition 2.11 on restriction of metaplectic tensor product on smaller Levi, that

$$F^s \in \bigoplus \text{Ind}_{\widetilde{P}'_{r-2,1}(\mathbb{A})}^{\widetilde{GL}_{r-1}(\mathbb{A})}(\theta' | (\text{det}_{r-2})^{-\frac{1}{2}} \otimes \tilde{\eta}) \omega_\delta \otimes \delta_Q^{\frac{1}{2}} \delta_{P'_r}^{1/2} \delta_{P_{r-2,1}}^{-1/2},$$

where $\delta$ runs through a subset of $GL_1(F)$ and $\omega_\delta$ is an appropriate character on the center of $\widetilde{GL}_{r-1}$. Hence after all, the analytic behavior of (5.7) is determined by the analytic behavior of the Eisenstein series on $GL_{r-1}(\mathbb{A})$ given by

$$F^s(s(m)g; 1) = \sum_{m \in P^{-1}_{r-2,1}(F) \setminus GL_{r-1}(F)} f^s(s(m)g; 1)$$

where $F^s \in \text{Ind}_{P_{r-2,1}(\mathbb{A})}^{GL_{r-1}(\mathbb{A})}(\theta' | (\text{det}_{r-2})^{-\frac{1}{2}} \otimes \tilde{\eta}) \omega_\delta \otimes \delta_Q^{\frac{1}{2}} \delta_{P'_r}^{1/2} \delta_{P_{r-2,1}}^{-1/2}$ and $\omega$ is some appropriately chosen character and $\delta_Q^{\frac{1}{2}} \delta_{P'_r}^{1/2} \delta_{P_{r-2,1}}^{-1/2}$ is restricted to $P_{r-2,1}^{-1}$. As a character on $GL_{r-2} \times GL_1 \subseteq P_{r-2,1}^{-1}$, one can compute

$$| \text{det}(g_{r-2}) |^{\frac{1}{4}} \delta_Q^{\frac{1}{2}} \delta_{P'_r}^{1/2} \delta_{P_{r-2,1}}^{-1/2} (g_{r-2}, b) = | \text{det}(g_{r-2}) |^{\frac{1}{4}} |b|^{-\frac{1}{2} + \frac{1}{2}(r-1) + \frac{1}{2}(r-2)}$$

$$= \delta_{P_{r-2,1}} (g_{r-1}, b) \frac{r+s}{r-1} | \text{det}(g_{r-1}) |^{\frac{1}{4} - \frac{1}{2}}$$

Thus the section $F^s$ belongs to

$$\text{Ind}_{P_{r-2,1}(\mathbb{A})}^{\widetilde{GL}_{r-1}(\mathbb{A})}(\theta' \otimes \tilde{\eta}) \omega \otimes \delta_{P_{r-2,1}}^{-1} (\text{det}_{r-1})^a,$$

where

$$a = -\frac{1}{4} r - s \quad \text{and} \quad b = \frac{rs + \frac{1}{2}}{r-1}$$

(Note that those two exponents $a$ and $b$ are precisely the ones in the middle of p. 196 of [BG], though in [BG] induction is not normalized and hence their exponents look different for ours.)
Therefore the analytic behavior of the Eisenstein series in (5.7) is determined by that of the Eisenstein series associated to the induced representation (5.8). But the twist by $|\det r|^{-s}$ does not have any affect on the analytic behavior, and hence we have to consider the Eisenstein series on $\widetilde{GL}_{r-1}(\mathbb{A})$ associated with the induced space

$$\text{Ind}_{P^{r-2,1}(\mathbb{A})}^{\widetilde{GL}_{r-1}(\mathbb{A})}(\theta' \otimes \tilde{\eta})_\omega \otimes \delta_{P^{r-2,1}}^b,$$

where $b$ is as in (5.9).

Now by the induction hypothesis, this Eisenstein series on $\widetilde{GL}_{r-1}(\mathbb{A})$ is holomorphic for $\text{Re}(b) \geq 0$, except when $\chi^2 \eta^{-2} = 1$ in which case it has a possible simple pole at $b = \frac{1}{4}$. From (5.9), $b = \frac{1}{4}$ amounts to

$$s = \frac{1}{4} - \frac{1}{2r}. \tag{5.10}$$

**Remark 5.11.** The above argument is essentially the detail of the argument outlined in [BG, p.195-196]. As we pointed out at the beginning of the section, however, in [BG] it seems the induction argument is used for the normalized Eisenstein series rather than the unnormalized one. But the normalization of the Eisenstein series depends on the choice of the set $S$ of “bad places”, which include the bad places for the section $f^*$. As one can see from the above computation, one has to apply the induction hypothesis to the new section $F^*$ on the lower rank group $\widetilde{GL}_{r-1}(\mathbb{A})$. But there is no guarantee that the same set $S$ works for $F^*$. This is why we cannot use the induction argument for the normalized Eisenstein series.

**Cancellation of the poles at $s = \frac{1}{4} - \frac{1}{2r}$:**

Finally, to complete the induction, we need to show the (possible) poles at $s = \frac{1}{4} - \frac{1}{2r}$ (if exist at all) of both the identity term $E_P(g, s; f^*)_\text{id}$ and the non-identity term $E_P(g, s; f^*)_w$ cancel out.

Set

$$s_0 := \frac{1}{4} - \frac{1}{2r}.$$

Let us note that the possible pole at $s = s_0$ will happen only when $\chi^2 \eta^{-2} = 1$, namely $\chi^2 = \eta^2$. Moreover by the uniqueness of the metaplectic tensor product (Proposition 4.12), if $\chi^2 \eta^{-2} = 1$, then

$$\left(\tilde{\chi} \otimes \cdots \otimes \tilde{\chi} \otimes \tilde{\eta}\right)_\omega = \left(\tilde{\chi} \otimes \cdots \otimes \tilde{\chi} \otimes \tilde{\eta}\right)_\omega,$$

i.e., we may (and do) assume $\eta = \chi$, although most of the time we use the notation $\left(\tilde{\chi} \otimes \cdots \otimes \tilde{\chi} \otimes \tilde{\eta}\right)_\omega$.

As we have seen above, the two terms $E_P(g, s; f^*)_\text{id}$ and $E_P(g, s; f^*)_w$ both have a possible pole at $s = \frac{1}{4} - \frac{1}{2r}$. But in what follows, we will show the Eisenstein series $E(g, s; f^*)$ does not have a pole at this point. Namely those two possible poles cancel each other or they just do not exist to begin with. This is essentially shown in [BG, p.201-203]. However many of the computations are omitted there, and hence we will give a complete proof in detail here. The basic idea is the following: First one computes the constant term of our Eisenstein series along the Borel subgroup $B$ instead of $P$.

Then one can see that all the terms in the constant term is holomorphic at $s = \frac{1}{4} - \frac{1}{2r}$ except two terms. One can then see that the treatment of those two terms can be reduced to the “$\widetilde{GL}_2$-case”, and invoke Proposition 4.13 to show the cancellation of the poles.

So let us compute the constant term $E_B(g, s; f^*)$ of the Eisenstein series along the Borel. Analogously to (5.5)

$$E_B(g, s; f^*) = \sum_{\gamma \in Q(F) \setminus \widetilde{GL}_r(F)} \int_{N_B(F)^{\gamma^{-1}} \setminus N_B(\mathbb{A})} f^*(\gamma n) g; 1) \, dn,$$
where $N_B(F)\gamma^{-1} = N_B(F) \cap \gamma^{-1} Q(F)\gamma$. For $i = 0, \ldots, r - 1$, let

$$w_i = \begin{pmatrix} I_{r-i-1} & 0 \\ 0 & I_i \end{pmatrix}.$$  

(Let us note that $w_1$ here differs from the one in (4.12), but this should not cause any confusion.) By the Bruhat decomposition, one has

$$Q(F)\setminus GL_r(F)/N_B(F) = \bigcup_{i=0}^{r-1} Q\setminus Qw_i^{-1}T_B = \bigcup_{i=0}^{r-1} T_B \cap w_iQw_i^{-1}\setminus T_B,$$

where the last equality is given by the map $\gamma = w_i^{-1}t \mapsto t$ for $t \in T_B$. But $T_B \cap w_iQw_i^{-1}\setminus T_B = 1$ for any $w_i$, and so each double coset in $Q(F)\setminus GL_r(F)/N_B(F)$ is represented by $w_i^{-1}$. Hence

$$E_B(g, s; f^s) = \sum_{i=0}^{r-1} \int _{N_B(F)w_i\setminus N_B(\mathbb{A})} f^s(s(w_i^{-1}n)g; 1) \, dn. \tag{5.12}$$

Let us put

$$c_i(g, s; f^s) = \int _{N_B(F)w_i\setminus N_B(\mathbb{A})} f^s(s(w_i^{-1}n)g; 1) \, dn,$$

so that

$$E_B(g, s; f^s) = \sum_{i=0}^{r-1} c_i(g, s; f^s). \tag{5.14}$$

Let us compute the term for $i = 0$, so $w_0 = 1$ and $N_B(F)w_0 = N_B(F)$. For $m \in \tilde{M}_Q$, define

$$\hat{f}^s(g; m) := \int _{N_{Br^{-1,1}(F)\setminus N_{Br^{-1,1}(\mathbb{A})}}} f^s(g; s(n)m) \, dn.$$  

Namely $\hat{f}^s(g)$ is the constant term of the automorphic form $f^s(g) \in \theta_{\chi, \eta} \otimes \hat{\sigma}_Q^{s+1/2}$ along the Borel subgroup $B^{r-1,1}$ of $M_Q = GL_{r-1} \times GL_1$. Then

$$c_0(g, s; f^s) = \int _{N_B(F)\setminus N_B(\mathbb{A})} f^s(s(w_0^{-1}n)g; 1) \, dn$$

$$= \int _{N_{Br^{-1,1}(F)\setminus N_{Br^{-1,1}(\mathbb{A})}}} f^s(s(n)g; 1) \, dn$$

$$= \int _{N_{Br^{-1,1}(F)\setminus N_{Br^{-1,1}(\mathbb{A})}}} f^s(g; s(n)) \, dn$$

$$= \hat{f}^s(g; 1)$$

where we used that $N_B = N_QN_{Br^{-1,1}}$ and $N_Q(\mathbb{A})$ acts trivially on $f^s$. But $\theta_{\chi, \eta}$ is generated by the residues of Eisenstein series associated with the induced representation $Ind_{Br^{-1,1}(\mathbb{A})}^{\tilde{M}_Q(\mathbb{A})}(\tilde{\chi} \otimes \cdots \otimes \tilde{\chi} \otimes \tilde{\eta})^\nu$ at $\nu = \frac{1}{2}r_{Br^{-1,1}}$. Namely

$$f^s(g; -) = \Res_{\nu = \frac{1}{2}r_{Br^{-1,1}}} E(-, \varphi^\nu) \otimes \varphi^\nu (\hat{\sigma}_Q^{s+1/2})(-)$$

for $\varphi^\nu \in Ind_{Br^{-1,1}(\mathbb{A})}^{\tilde{M}_Q(\mathbb{A})}(\tilde{\chi} \otimes \cdots \otimes \tilde{\chi} \otimes \tilde{\eta})^\nu$. Now a constant term of a residue is the residue of the constant term. Hence we need to compute the constant term $E_{Br^{-1,1}}(-, \varphi^\nu)$. By [MW, Proposition
Hence we have
\[ E_{B^{r-1}:1}(-, \varphi^\nu) = \sum_{w \in W_{GL_{r-1}}} M(w, \nu) \varphi^\nu(-) \]
where \( W_{GL_{r-1}} \) is the Weyl group of \( GL_{r-1} \) embedded into the left upper corner of \( GL_r \), and \( M(w, \nu) \) is the intertwining operator
\[ M(w, \nu) : \text{Ind}_{\tilde{\mathcal{M}}^Q(\mathbb{A})}^{\tilde{\mathcal{M}}_{B^{r-1}:1}(\mathbb{A})} (\tilde{\chi} \otimes \tilde{\chi} \otimes \tilde{\eta})^\nu \rightarrow \text{Ind}_{\tilde{\mathcal{M}}_{B^{r-1}:1}(\mathbb{A})}^{\tilde{\mathcal{M}}_Q(\mathbb{A})} w(\tilde{\chi} \otimes \tilde{\chi} \otimes \tilde{\eta})^{w\nu}. \]
But we know from [KP, Theorem II.1.3] that this intertwining operator has a residue at \( \nu = \frac{1}{2} \rho_{B^{r-1}:1} \) only for the longest element \( u = \begin{pmatrix} J_{r-1} & 0 \\ 0 & 1 \end{pmatrix} \) in \( W_{GL_{r-1}} \), where \( J_{r-1} \) is the \((r-1) \times (r-1)\) matrix with 1’s on the anti-diagonal entries and all the other entries 0. Moreover the residue is in the exceptional representation \( \theta_{X,\eta} \). Hence the function on \( B(\mathbb{A}) \) defined by \( b \mapsto \tilde{f}^*(g; b) \) for \( b \in B(\mathbb{A}) \) is an automorphic form in the space generated by the constant terms of the exceptional representation \( \theta_{X,\eta} \otimes \delta_Q^{1/4} \), which is equal to
\[(\tilde{\chi} \otimes \cdots \otimes \tilde{\chi} \otimes \tilde{\eta})^\omega \otimes \delta_{B^{r-1}:1}^{1/4} \delta_Q^{1/4} \delta_{B^{r-1}:1}^{1/4} \]
Also for each \( b, b' \in \tilde{B}(\mathbb{A}) \),
\[
\tilde{f}^*(b'g; b) = \int_{N_{B^{r-1}:1}(F) \backslash N_{B^{r-1}:1}(\mathbb{A})} f^*(b'g; s(n)b) \ dn = \int_{N_{B^{r-1}:1}(F) \backslash N_{B^{r-1}:1}(\mathbb{A})} f^*(g; s(n)b) \ dn = \tilde{f}^*(g; bb').
\]
Hence we have
\[
(5.15) \quad \tilde{f}^* \in \text{ind}_{\tilde{B}(\mathbb{A})}^{\text{GL}_{r}(\mathbb{A})} (\tilde{\chi} \otimes \cdots \otimes \tilde{\chi} \otimes \tilde{\eta})^\omega \otimes \delta_{B^{r-1}:1}^{1/4} \delta_Q^{1/4} \delta_{B^{r-1}:1}^{1/4},
\]
where the induction is not normalized.

Next consider \( i > 0 \). Let
\[ N_i := \{ \begin{pmatrix} I_{r-i-1} \\ & 1 \\ & X_i \end{pmatrix} : X_i \text{ is a } 1 \times i \text{ matrix} \}, \]
i.e. \( N_i \) is the set of the “\((r-i)^{th}\) rows” of \( N_B \). Also let
\[ U_i = \{ \begin{pmatrix} I_{r-i-1} & 0 & Y_{r-i-1} \\ 0 & I_i & 0 \\ 0 & 0 & 1 \end{pmatrix} : Y_{r-i-1} \text{ is a } (r-i-1) \times 1 \text{ matrix} \}, \]
i.e. \( U_i \) is the set of the first \( r-i-1 \) entries in the last column. Both \( N_i \) and \( U_i \) are subgroups of \( N_B \). One can see
\[ w_i U_i N_{B^{r-1}:1} w_i^{-1} N_i = N_B, \]
and
\[ N_B(F)^{w_i} = N_B(F) \cap w_i Q(F) w_i^{-1} = w_i U_i(F) N_{B^{r-1}:1}(F) w_i^{-1}. \]
Therefore we have

\[
c_i(g, s; f^s) = \int_{N_B(F)^{w_i} \setminus N_B(\mathbb{A})} f^s(s(w_i^{-1}n)g; 1) \, dn
\]

\[
= \int \int f^s(s(n'w_i^{-1}n_i)g; 1) \, dn' \, dn_i
\]

where for the last equality we used Lemma 2.5 and that \( w_i \in \text{GL}_r(F) \). Note that \( U_i(\mathbb{A}) \subseteq N_Q(\mathbb{A}) \) acts trivially on \( f^s \), and \( f^s(s(u_i)s(n_{r-1})s(w_i^{-1}n_i); 1) = f^s(s(n_{r-1})s(w_i^{-1}n_i); 1) \). Hence the innermost integral simply goes away. Furthermore, \( f^s(s(n_{r-1})s(w_i^{-1}n_i)g; 1) = f^s(s(w_i^{-1}n_i); s(n_{r-1})) \). Therefore the integral (5.16) is written as

\[
\int \int f^s(s(w_i^{-1}n_i); s(n_{r-1})) \, dn_{r-1} \, dn_i = \int f^s(s(w_i^{-1}n_i); 1) \, dn_i.
\]

Therefore

\[
(5.17) \quad c_i(g, s; f^s) = \int_{N_i(\mathbb{A})} f^s(s(w_i^{-1})s(n_i); 1) \, dn_i.
\]

This is precisely the formula stated (without a proof) at the end of p. 202 of [BG], though our \( w_i^{-1} \) is their \( w_i \).

We will show that \( c_i(g, s; f^s) \) is holomorphic at \( s = s_0 \) for all \( i < r - 2 \), and for \( i = r \) and \( r - 1 \) it does have a pole but they cancel out, and hence the constant term (5.14) has no pole at \( s = s_0 \), which implies that the Eisenstein series \( E(g, s, f^s) \) does not have a pole at \( s = s_0 \).

For this purpose, we need to reduce our situation to the “\( \text{GL}_2 \)-situation”. For this, just as is done in [BG], we need to interpret the metaplectic tensor product \( (\tilde{\chi} \otimes \cdots \tilde{\chi} \otimes \tilde{\eta})_\omega \) as follows.

**Lemma 5.18.** Let \( T^m(\mathbb{A}) \) be the subgroup of the maximum torus \( T(\mathbb{A}) \) of the form

\[
T^m(\mathbb{A}) = \{(t_1, \ldots, t_r) \in T : t_i \in F^\times \mathbb{A} \times \mathbb{F}_p(Z_{\text{GL}_r}(\mathbb{A}))\},
\]

where \( p \) is the canonical projection. Also let \( \tilde{T}^m(\mathbb{A}) \) be the metaplectic preimage of \( T^m(\mathbb{A}) \). Then \( \tilde{T}^m(\mathbb{A}) \) is a maximal abelian subgroup of \( \tilde{T}(\mathbb{A}) \). Moreover let \( \omega^m \) be any character on \( \tilde{T}^m(\mathbb{A}) \) extending the character on \( Z_{\text{GL}_r}(\mathbb{A}) \tilde{T}(\mathbb{A})^{(2)}(\mathbb{A}) \) given by

\[
\omega(\tilde{\chi}^{(2)} \otimes \cdots \tilde{\chi}^{(2)}),
\]

where \( \omega \) is chosen so that it agrees with the metaplectic tensor product \( \tilde{\chi}^{(2)} \otimes \cdots \otimes \tilde{\chi}^{(2)} \) on the overlap. Then we have

\[
(\tilde{\chi} \otimes \cdots \otimes \tilde{\chi})_\omega \cong \text{ind}_{\tilde{T}^m(\mathbb{A})}^{\tilde{T}(\mathbb{A})} \omega^m,
\]

where \( \text{ind} \) is as in [KP] p. 54.
Proof. The group \( \tilde{T}(\mathbb{A}) \) is a Heisenberg group and the group \( \tilde{T}^m(\mathbb{A}) \) is a maximal abelian subgroup by [KP II.1.1]. Hence the lemma follows from a general theory on the Heisenberg group as described in [KP p.52-56]. □

Let us note that in [BC], our \( T^m \) is denoted by \( T^i \), and our \( \omega^m \) by \( \omega^i \), but we avoid to use \( i \) in order not to confuse it with the index \( i \) we have been using.

With this lemma, one may assume

(5.19) \[ \hat{f^s} \in \text{ind}_{\tilde{T}^m(\mathbb{A})N_B(\mathbb{A})}^{\tilde{\GL}(\mathbb{A})} \omega^m \otimes \delta_{B_{T^r-1}}^1 \delta_{Q_{T^r-1}}^{1/2}, \]

where \( N_B(\mathbb{A})^* \) acts trivially as usual.

Another important property of the group \( T^m(\mathbb{A}) \) is

**Lemma 5.20.** The partial section \( s : \text{GL}_r(\mathbb{A}) \to \tilde{\GL}_r(\mathbb{A}) \) is not only defined but a homomorphism on \( T^m(\mathbb{A})N_B(\mathbb{A}) \). Also for \( t \in T^m(\mathbb{A}) \) and \( g \in \text{GL}_r(F) \), both \( s(tg) \) and \( s(gt) \) are defined and \( s(tg) = s(t)s(g) \) and \( s(gt) = s(g)s(t) \).

**Proof.** It is known that the partial section \( s \) is defined on the Borel subgroup \( B(\mathbb{A}) \) and the block-compatible cocycle is globally defined on \( B(\mathbb{A}) \times B(\mathbb{A}) \). Now if \( t, t' \in T^m(\mathbb{A}) \) and \( n, n' \in N_B(\mathbb{A}) \), one can compute

\[ \sigma_r(tn, t'n') = \sigma_r(tn^{-1}t, t'n') = \sigma_r(t, t') = 1, \]

where the last equality follows because the Hilbert symbol is trivial on \( F^{\times} \times F^{\times} \times F^{\times} \times F^{\times} \).

To show the second part of the lemma note that at every place \( v \),

\[ s_r(t,g_v) = \frac{s_r(t,v) s_r(g_v)}{\sigma_r(t,v,g_v)} \tau_r(t_v, g_v) \]

and for almost all \( v \) one can see that the right hand side is 1, and hence the product \( \prod_v s_r(t_v g_v) \) is defined, i.e. \( s(tg) \) is defined. Also this implies that globally we have

\[ s_r(tg) = \frac{s_r(t,v) s_r(g_v)}{\sigma_r(t,v,g_v)} \tau_r(t_v, g_v) = s_r(t,v) s_r(g_v) \tau_r(t_v, g_v) \]

because \( \sigma_r(t,v) = 1 \), which implies \( s(tg) = s(t) s(g) \). The same argument works for \( s(gt) \). □

Now for each \( i \), let us define the inclusion

\[ \iota_i : \text{GL}_2 \to \text{GL}_r, \quad g_2 \mapsto \begin{pmatrix} I_{r-i-1} & g_2 \\ 0 & I_{i-1} \end{pmatrix}, \]

so the first entry of \( g_2 \in \text{GL}_2 \) is in the \((r-i, r-i)\)-entry. This lifts to

\[ \tilde{i}_i : \tilde{\text{GL}}_2(\mathbb{A}) \to \tilde{\GL}_r(\mathbb{A}) \]

With this notation, define

\[ F^*_i(g_2; g) := c_i(\tilde{i}_i(g_2) g, s; f^*) = \int_{\mathbb{A}^+} \hat{f^*}(s(w_i^{-1}) s(\begin{pmatrix} I_{r-i-1} & 1 \\ 0 & I_i \end{pmatrix}) \tilde{i}_i(g_2) g) dx_i \]

\[ G^*_i(g_2; g) := c_{i-1}(\tilde{i}_i(g_2) g, s; f^*) = \int_{\mathbb{A}^+} \hat{f^*}(s(w_{i-1}^{-1}) s(\begin{pmatrix} I_{r-i} & 1 \\ 0 & I_{i-1} \end{pmatrix}) \tilde{i}_i(g_2) g) dx_{i-1}. \]

It should be noted that

(5.21) \[ G^*_i(1; g) = F^*_i(1; g). \]
Lemma 5.22. Assuming the integrals of \( F_i^s(g_2; g) \) and \( G_i^s(g_2; g) \) both converge, we have

\[
F_i^s(g_2; g) = \int_{\mathbb{A}} G_i^s(s_2 \begin{pmatrix} 1 & x \\ 1 & 1 \end{pmatrix} g_2; g) \, dx,
\]

where \( s_2 : \text{GL}_2(\mathbb{A}) \to \widetilde{\text{GL}}_2(\mathbb{A}) \) is the partial section for \( \widetilde{\text{GL}}_2(\mathbb{A}) \).

Proof. First note that \( \hat{t}_i \circ s_2 = s \circ t_i \), and hence

\[
\hat{t}_i(s_2 \begin{pmatrix} 1 & x \\ 1 & 1 \end{pmatrix} g_2) = s(t_i \begin{pmatrix} 1 & x \\ 1 & 1 \end{pmatrix}) s(\hat{t}_i(g_2)).
\]

Second note that

\[
(5.23) \quad w_i = \begin{pmatrix} I_{r-i-1} & \\ 1 & 1 \\ I_{i-1} \end{pmatrix} = \begin{pmatrix} I_{r-i-1} & \\ 1 & 1 \\ I_{i-1} \end{pmatrix} = t_i \begin{pmatrix} 1 & \\ 1 & \end{pmatrix} w_{i-1}.
\]

Then one can check the lemma by a direct computation using Lemma 2.25. \( \square \)

Now let \( B^2 = \mathcal{T}_B^2 N_{B^2} \) be the Borel subgroup of \( \text{GL}_2 \) and \( T_{2}^m \) be the analogous subgroup of \( T_{B^2} \) as defined above with \( r = 2 \). Let \( \omega^m_2 \) be the above \( \omega^m \) with \( r = 2 \), so \( \text{ind}_{T_{B^2}(\mathbb{A})}^{\mathcal{T}_B^2(\mathbb{A})} \omega^m_2 = \hat{\chi} \otimes \hat{\chi} \). (Note that for \( \widetilde{\text{GL}}_2 \) there is no choice for the central character \( \omega \) for the metaplectic tensor product because the center \( Z_{\widetilde{\text{GL}}_2} \) is already contained in \( \mathcal{T}_B^2(\mathbb{A}) \), and hence we write \( \hat{\chi} \otimes \hat{\chi} \) instead of \( (\hat{\chi} \otimes \hat{\chi})_{\omega} \).)

Let us mention

Lemma 5.24. With the above notation, we have

\[
G_i^s(-; g) \in \text{ind}_{T_{B^2}(\mathbb{A})}^{\mathcal{T}_B^2(\mathbb{A})} \omega^m_2 \otimes \delta_{B^2}^{t + \frac{1}{2}} | \det |^u,
\]

where

\[
t = \frac{1}{2} rs + \frac{1}{8} - \frac{1}{4} i, \quad u = s - \frac{1}{2} rs - \frac{3}{8} r + \frac{3}{4} i,
\]

and hence in particular at \( s = s_0 = \frac{1}{4} - \frac{1}{2} r \) we have

\[
G_i^{s_0}(-; g) \in \text{ind}_{T_{B^2}(\mathbb{A})}^{\mathcal{T}_B^2(\mathbb{A})} \omega^m_2 \otimes \delta_{B^2}^{\frac{1}{2} (r-i+1)} | \det |^{-\frac{1}{4} r + \frac{1}{2} i + \frac{1}{4} - \frac{1}{2} p},
\]

provided the integral is convergent.

Proof. Let

\[
g_2 = s(t_i(\begin{pmatrix} t_1 \\ t_2 \end{pmatrix}))
\]

where \( t_i \in F^x \mathbb{A}^{x^2} \), so \( \begin{pmatrix} t_1 \\ t_2 \end{pmatrix} \in T_{B^2}(\mathbb{A}) \). Then one can compute

\[
G_i^s(g_2; g) = \int_{\mathbb{A}^{r-i-1}} f^s(s(w_{i-1}^{-1})s(\begin{pmatrix} 1 & X_{i-1} \\ I_{i-1} \end{pmatrix}), s(\begin{pmatrix} t_1 \\ t_2 \\ I_{i-1} \end{pmatrix}) g; 1) \, dX_{i-1}
\]
Now by (5.19) and the change of variables $t$s where the last equality follows because $s$ is a group homomorphism on $\text{GL}_r(F)$. (Note that in the above computations, the global partial section $s$ is always defined.)

Next by using Lemma [5.20] and conjugating by $w_{i-1}$, we can compute

$$s(w_{i-1}^{-1}) s(\begin{pmatrix} I_r & t_1 \\ -t_2 & I_{i-1} \end{pmatrix}) s(w_{i-1}) = s(w_{i-1}^{-1}) s(w_{i-1}) s(\begin{pmatrix} I_r & t_1 \\ -t_2 & I_{i-1} \end{pmatrix})$$

$$= s(\begin{pmatrix} I_r & t_1 \\ -t_2 & I_{i-1} \end{pmatrix}) s(\begin{pmatrix} 1 \\ t_2^{-1} X_{i-1} \end{pmatrix}) g; 1) dX_{i-1},$$

where the last equality follows because $s$ is a group homomorphism on $\text{GL}_r(F)$.

Therefore we have

$$G^i_t(g_2; g) = \int_{A^{i-1}} \hat{f}^s(s(w_{i-1}^{-1}) s(\begin{pmatrix} I_r & t_1 \\ -t_2 & I_{i-1} \end{pmatrix}) s(w_{i-1}) s(\begin{pmatrix} I_r & t_1 \\ -t_2 & I_{i-1} \end{pmatrix}) g; 1) dX_{i-1}. $$

Now by (5.19) and the change of variables $t_2^{-1} X_{i-1} \mapsto X_{i-1}$, one can see

$$G^i_t(g_2; g) = |t_1|^{\frac{i}{2}(r-2(r-i)) + s + \frac{i}{2}} |t_2|^{(1-r)(s+\frac{i}{2}) + i-1} \omega^m(s(\begin{pmatrix} I_r & t_1 \\ -t_2 & I_{i-1} \end{pmatrix}))$$

$$\int_{A^{i-1}} \hat{f}^s(s(w_{i-1}^{-1}) s(\begin{pmatrix} 1 \\ X_{i-1} \end{pmatrix}) g; 1) dX_{i-1}. $$
By direct computation, one can see
\[
\omega^m(s(\begin{pmatrix} I_{r-i-2} & t_1 \\ t_1 & I_{i-1} \\ t_2 \end{pmatrix})) = \omega_2^m(s_2(\begin{pmatrix} t_1 \\ t_2 \end{pmatrix})).
\]
Then the lemma follows by simplifying the exponents for \(|t_1|\) and \(|t_2|\). \qed

Let
\[
M_2(t) : \text{ind}_{T_{n^2}(\mathbb{A})\text{N}_{n^2}(\mathbb{A})}^{\text{GL}_2(\mathbb{A})} \omega_2^m \otimes \delta_{B_2}^{t+\frac{\delta}{4}} |\det|^u \rightarrow \text{ind}_{T_{n^2}(\mathbb{A})\text{N}_{n^2}(\mathbb{A})}^{\text{GL}_2(\mathbb{A})} \omega_2^m \otimes \delta_{B_2}^{t+\frac{\delta}{4}} |\det|^u
\]
be the intertwining operator defined on the induced space in the above lemma. Then Lemma \ref{lem:5.2} says that
\[
F^s_i(g_2; g) = M_2(t)G^s_i(g_2; g),
\]
with
\[
t = \frac{1}{2}rs + \frac{1}{8} - \frac{1}{4}i,
\]
provided all the integrals are convergent, which is the case if \(\Re(s) >> 0\).

With this said, one can prove

**Proposition 5.27.** Let \(i < r - 2\). Then for each \(g\) and \(g_2\), the integral for \(F^s_i(g_2; g)\) converges absolutely at \(s = s_0\). In particular \(c_i(g, s; f^s) = F^s_i(1; g)\) converges at \(s = s_0\).

**Proof.** We prove it by induction on \(i\). For \(i = 1\), first note that \(G^s_1(g_2; g) = \hat{\delta}(\hat{\iota}_1(g_2); g, 1)\), and hence certainly \(G^s_1(g_2; g)\) converges at any \(s\). Now at \(s = s_0\) by the previous two lemmas, one can see that
\[
F^s_1(g_2; g) = M_2(\frac{1}{4}(r - i - 1))G^s_1(g_2; g),
\]
where \(M_2\) is as in \ref{lem:5.26}. But if \(i = 1\) (and \(r > 3\)), the intertwining operator \(M_2(\frac{1}{4}(r - i - 1))\) converges.

Now assume \(F^s_i(g_2; g)\) converges for all \(g_2\) and \(g\) at \(s = s_0\). Notice that
\[
G^s_{i+1}(g_2; g) = G^s_{i+1}(1; \hat{\iota}_{i+1}(g_2)g) = F^s_i(1; \hat{\iota}_{i+1}(g_2)g).
\]
Hence \(G^s_{i+1}(g_2; g)\) converges for all \(g_2\) and \(g\). Again by the previous two lemmas, one can see that
\[
F^s_{i+1}(g_2; g) = M_2(\frac{1}{4}(r - i) - \frac{1}{2})G^s_{i+1}(g_2; g),
\]
and if \(i + 1 < r - 2\), the intertwining operator here converges. \qed

Let us note that the convergence of \(c_i(g, s; f^s)\) for \(i < r - 2\) is stated without proof at the end of p.202 of \cite{BG} for the non-twisted case. The author believes that the above proof is the one they have in mind.

Finally we show the cancellation of the possible poles for \(c_{r-1}(g, s; f^s)\) and \(c_{r-2}(g, s; f^s)\) at \(s = s_0\), namely

**Proposition 5.28.** The sum
\[
c_{r-1}(g, s; f^s) + c_{r-2}(g, s; f^s)
\]
is holomorphic at \(s = s_0\).
Proof. The proof is essentially described in the first half of p.203 of [BG]. But we will repeat the argument with our notations.

First note that
\[ c_{r-2}(g, s; f^s) = F_{r-2}^s(1, g) = M_2(t)G_{r-2}^s(1; g), \]
and by the above proposition, \( G_{r-2}^s(1; g) = F_{r-3}^s(1; g) \) is convergent. With \( i = r - 2 \) and \( s = s_0 \), one can see \( t = \frac{1}{4} \). But at \( t = \frac{1}{4} \), the intertwining operator \( M_2(t) \) has a simple pole. Then we have
\[
\text{Res}_{s=s_0} c_{r-2}(g, s; f^s) = \text{Res}_{s=s_0} F_{r-2}^s(1, g) = \text{Res}_{s=s_0} G_{r-1}^s(1, g),
\]
where for the last equality we used (5.21). Second note that
\[
c_{r-1}(g, s; f^s) = F_{r-1}^s(1; g) = M(t)G_{r-1}^s(1; g),
\]
and with \( i = r - 1 \) and \( s = s_0 \), one can see \( t = 0 \), and we know by Proposition 4.18 that the intertwining operator \( M_2(0) \) is holomorphic and acts as \(-\text{Id}\). Therefore we have
\[
\text{Res}_{s=s_0} c_{r-1}(g, s; f^s) = M(0)\text{Res}_{s=s_0} G_{r-1}^s(1; g) = -\text{Res}_{s=s_0} G_{r-1}^s(1; g) = -\text{Res}_{s=s_0} c_{r-2}(g, s; f^s)
\]
Hence the residues get cancelled out. □

With those two propositions, we have proven that the constant term (5.14) is holomorphic at \( s = s_0 \), and hence the Eisenstein series \( E(g, s; f^s) \) is holomorphic at \( s = s_0 \).

5.3. The induction step for \( r = 2q + 1 \). We now consider the case \( r = 2q + 1 \), namely \( \theta = \vartheta_{\chi, \eta} \).

First of all, let us note that if \( \chi^{1/2} \) exists, then \( \vartheta_{\chi, \eta} = \vartheta_{\chi^{1/2}, \eta} \), to which case the argument for \( r = 2q \) applies. Hence we will assume that \( \chi^{1/2} \) does not exist. But even in this case, we argue similarly to the case \( r = 2q \).

This time, however, the cuspidal support of our Eisenstein series \( E(g, s; f^s) \) is the \((2, \ldots, 2, 1)\)-parabolic, and hence the poles of the Eisenstein series are precisely the poles of the constant term along any parabolic containing the \((2, \ldots, 2, 1)\)-parabolic. In particular in this subsection we let
\[
P = P_{2, r-2} = M_P N_P \subseteq \text{GL}_r
\]
be the \((2, r-2)\)-parabolic, and will consider the constant term along the unipotent radical \( N_P \) of this parabolic. Similarly to the computation for the case \( r = 2q \), the constant term along \( N_P \) is computed as follows:

\[
E_P(g, s; f^s) = \sum_{\gamma \in Q(F) \backslash \text{GL}_r(F) / N_P(F)} \int_{N_P(F)\gamma^{-1}\backslash N_P(F)} f^s(s(\gamma)n; 1) \, dn,
\]
where \( N_P(F)\gamma^{-1} = N_P(F) \cap \gamma^{-1}Q(F)\gamma \). Since, by the Bruhat decomposition, we have
\[
\text{GL}_r(F) = \bigcup_{w \in \{1, w_1\}} Q(F)w^{-1}P(F),
\]
where \( w_1 \) is as in (4.12), we have
\[
Q \backslash \text{GL}_r / N_P = \bigcup_{w \in \{1, w_1\}} M_P \cap wQw^{-1}M_P,
\]
where
\[
M_P \cap wQw^{-1}M_P = \begin{cases} P_{r-3}^{\sim-1} \backslash \text{GL}_{r-2}, & \text{if } w = 1; \\ B^{2} \backslash \text{GL}_2, & \text{if } w = w_1, \end{cases}
\]
where $\mathrm{GL}_{r-2}$ is viewed as a subgroup of $\mathrm{GL}_r$ embedded in the lower right corner, and $P_{r-3,1}^{-2}$ is the $(r - 3, 1)$-parabolic of $\mathrm{GL}_{r-2}$, and $\mathrm{GL}_2$ is embedded in the upper left corner and $\mathbb{P}^r$ is the opposite of the standard Borel subgroup of $\mathrm{GL}_2$. Using this decomposition, one can write \([5,29]\) as

$$E_P(g, s; f^s) = E_P(g, s; f^s)_{\text{Id}} + E_P(g, s; f^s)_{w_1},$$

where we have set

$$E_P(g, s; f^s)_{\text{Id}} := \sum_{m \in P_{r-3,1}^{-2}(F) \setminus \mathrm{GL}_{r-2}(F)} \int_{N_P(F) \setminus N_P(\mathfrak{A})} f^s(s(mn)g; 1) \, dn;$$

$$E_P(g, s; f^s)_{w_1} := \sum_{m \in B^r(F) \setminus \mathrm{GL}_2(F)} \int_{N_P(F) \setminus N_P(\mathfrak{A})} f^s(s(w^{-1}mn)g; 1) \, dn.$$

In what follows, we will show that the identity term $E_P(g, s; f^s)_{\text{Id}}$ is holomorphic for $\text{Re}(s) \geq 0$ and the non-identity term $E_P(g, s; f^s)_{w_1}$ vanishes, which will complete the proof of Theorem \[5.3\].

**The identity term $E_P(g, s; f^s)_{\text{Id}}$:**

The argument for the identity term is quite similar to the case $r = 2q$ in that we interpret it as the Eisenstein series on the smaller group $\mathrm{GL}_{r-2}$ and use induction. Also let us mention that, as we will see, unlike the case $r = 2q$, we will not have to show the cancellation of the pole at $s = \frac{1}{4} - \frac{1}{2r}$. First one can write

$$E_P(g, s; f^s)_{\text{Id}} = \sum_{m \in P_{r-3,1}^{-2}(F) \setminus \mathrm{GL}_{r-2}(F)} \int_{N_P(F) \setminus N_P(\mathfrak{A})} f^s(s(m)g; s(n')) \, dn',$$

where

$$P' = P_{2,r-3,1} = P \cap M_Q = (\mathrm{GL}_2 \times \mathrm{GL}_{r-3} \times \mathrm{GL}_1)N_{P},$$

so $P'$ is the parabolic subgroup of $\mathrm{GL}_{r-1} \times \mathrm{GL}_1$ whose Levi is $\mathrm{GL}_2 \times \mathrm{GL}_{r-3} \times \mathrm{GL}_1$. If one views $f^s(s(m))g; -)$ as an automorphic form in $\theta \otimes \delta_Q^{s+1/2}$ on $\widetilde{M}_Q(\mathfrak{A})$, the integral in the above sum is just the constant term along $N_{P'}$. But since $f^s(s(m))g; -) \in \theta \otimes \delta_Q^{s+1/2}$, we need to compute the constant term of the residual representation $\theta \otimes \delta_Q^{s+1/2}$ along $N_{P'}$. Recall that the exceptional representation $\theta$ is constructed as the residue of the Eisenstein series associated to the induced representation $\text{Ind}_{\overline{M}_Q,1}^{\overline{M}_Q,1,1}(r_\chi \otimes \cdots \otimes r_\chi \otimes \overline{\eta})_\nu$ at $\nu = \rho_{P_{2,r-3,1,1}}/2$, where $P_{2,r-3,1,1}^{-1,1}$ is the $(2, \ldots, 2, 1)$-parabolic subgroup of $\mathrm{GL}_{r-1} \times \mathrm{GL}_1$. Namely it is generated by

$$\text{Res}_{\nu = \frac{1}{2}P_{2,r-3,1,1}} E(-, \varphi''; \nu)$$

for $\varphi'' \in \text{Ind}_{\overline{M}_Q,1}^{\overline{M}_Q,1,1}(r_\chi \otimes \cdots \otimes r_\chi \otimes \overline{\eta})_\nu$. But the constant term of the residue is the same as the residue of the constant term, and hence one first needs to compute the constant term $E_{P'}(-, \varphi''; \nu)$ of $E(-, \varphi''; \nu)$ along $N_{P'}$. For this, one can use \[MW\] Proposition (ii), p.92, and obtain

$$E_{P'}(m', \varphi''; \nu) = \sum_w E^{P'}(m', M(w, \nu)\varphi''),$$

where $m' \in \overline{M}_{P'}$ and $w$ runs through all the Weyl group elements of $\mathrm{GL}_{r-1} \times \mathrm{GL}_1$ such that $w(\mathrm{GL}_2 \times \cdots \times \mathrm{GL}_2 \times \mathrm{GL}_1)w^{-1}$ is a standard Levi of $\mathrm{GL}_2 \times \mathrm{GL}_{r-3} \times \mathrm{GL}_1$ and $w^{-1}(\alpha) > 0$ for all
the positive roots $\alpha$ that are in $M_{r'} = \text{GL}_2 \times \text{GL}_{r-3} \times \text{GL}_1$. One can see that by using the language of permutations, $w$ runs through all the elements of the form

$$w = (12\cdots k), \quad \text{for } k = 1, \ldots, q,$$

where each permutation corresponds to a permutation of $\text{GL}_2$-blocks in the Levi $\text{GL}_2 \times \cdots \times \text{GL}_2 \times \text{GL}_1$. (Note that the last $\text{GL}_1$ is always fixed by $w$.) But by exactly the same reasoning as the case $r = 2q$, one can see that the Eisenstein series has a residue at $\nu = \rho_{P_{2r-1,2}}^{-1,1}/2$ only for $w = (12\cdots q)$ by using [T1] Proposition 2.42.

Now as in the case $r = 2q$, we can see

$$\frac{1}{2}w\rho_{P_{2r-1,2,1}} = -\frac{1}{2}\rho_{P_{2r-3,1}} + \frac{1}{2}\rho_{P_{2r-2,1}},$$

where we note that $-\frac{1}{2}\rho_{P_{2r-1,1}}$ is a character on the Levi $M_{P_{2r-1,1},2} = \text{GL}_2 \times \text{GL}_{r-3} \times \text{GL}_1$ which acts as

$$(a, g_{r-3}, b) \mapsto |\det a|^{-\frac{3}{4}|\det(g_{r-3})|^{\frac{1}{2}}},$$

for $(a, g_{r-3}, b) \in \text{GL}_2 \times \text{GL}_{r-3} \times \text{GL}_1$. Hence for $m' \in M_{r'} = \text{GL}_2 \times \text{GL}_{r-3} \times \text{GL}_1$, we have

$$\text{Res}_{\nu = \frac{3}{2}p_{2r-1,2,1}} E^{M_{r'}}(m', M(w, \nu)\varphi') = \text{Res}_{\nu' = \frac{3}{2}p_{2r-3,1}} E^{M_{r'}}(m', \varphi'^\prime),$$

where

$$\varphi'^\prime \in \text{Ind}_{F_{2r-3,2r-3,2}}^{M_{r'}(F)}(r_\chi|\det|_{2}^{-\frac{3}{4}} \otimes (r_\chi \cdots \otimes r_\chi)|\det_{r-3}|^{\frac{1}{2}} \otimes \tilde{\eta})^{\nu'}.$$ By computing $\delta_{F_{2r-3,2r-3,2}}^{1/2} \delta_{F_{2r-1,2}}^{-1/2}$, this induced representation is written as

$$\text{Ind}_{F_{2r-3,2r-3,2}}^{M_{r'}(F)}(r_\chi|\det|_{2}^{-\frac{3}{4}} \otimes (r_\chi \cdots \otimes r_\chi)|\det_{r-3}|^{\frac{1}{2}} \otimes \tilde{\eta})^{\nu'} \Delta_{F_{2r-3,2r-3,2}}^{1/4}.$$ Then as in the case of $r = 2q$, we see that

$$\text{Res}_{\nu' = \frac{3}{2}p_{2r-3,1}} E^{M_{r'}}(-, \varphi'^\prime) \in (r_\chi|\det|_{2}^{-\frac{3}{4}} \otimes \theta' |\det_{r-3}|^{\frac{1}{2}} \otimes \tilde{\eta})^\nu' \Delta_{F_{2r-3,2r-3,2}}^{1/4}.$$ where $\theta'$ is the twisted exceptional representation of $\widetilde{\text{GL}}_{r-3}$ which is the unique irreducible quotient of the induced representation

$$\text{Ind}_{Q_{2r-3,2r-3,2}}^{\widetilde{\text{GL}}_{r-3}(F)}(r_\chi \cdots \otimes r_\chi)^{\omega'} \Delta_{Q_{2r-3,2r-3,2}}^{1/4}$$

for an appropriate choice of $\omega'$.

Hence for each fixed $m \in \text{GL}_{r-2}(F)$, the function on $M_{r'}(\mathbb{A})$ given by

$$m' \mapsto \int_{N_{r'}(F) \setminus N_{r'}(\mathbb{A})} f^*(s(m)g; s(n')m') \, dn'$$

is an element in $(r_\chi|\det|_{2}^{-\frac{3}{4}} \otimes \theta' |\det_{r-3}|^{\frac{1}{2}} \otimes \tilde{\eta})^\nu' \Delta_{Q_{2r-3,2r-3,2}}^{1/4}$, where $\delta_{Q_{2r-3,2r-3,2}}^{s+\frac{1}{2}}$ is actually the restriction $\delta_{Q_{2r-3,2r-3,2}}^{s+\frac{1}{2}}|_{P'}$ to $P'$. One can compute

$$\delta_{Q_{2r-3,2r-3,2}}^{s+\frac{1}{2}}(a, g_{r-3}, b) = |\det a|^{s+\frac{1}{2}} |\det(g_{r-2})|^{s+\frac{1}{2}} |b|^{-(r-1)(s+\frac{1}{2})}.$$
Accordingly the function $f^s_{N_{\rho'}}$ on $\tilde{M}_F(A)$ defined by

$$f^s_{N_{\rho'}}(g) = \int_{N_{\rho'}(F) \backslash N_{\rho'}(A)} f^s(g; s(n')) \, dn'$$

is in

$$\text{Ind}_{\tilde{M}_F(A)}^{\tilde{M}_F(A)}(r_\chi | \det_2)^{s-\frac{1}{2}} \otimes \theta' | \det_{r-3}|^{-\frac{1}{2}} \otimes \bar{\eta})_\omega \otimes \delta^{s+\frac{1}{2}} Q \delta_{\rho'}^{-1/2}$$

and

$$\text{Ind}_{\tilde{M}_F(A)}^{\tilde{M}_F(A)}(r_\chi | \det_2)^{-\frac{1}{2}} \otimes \theta' | \det_{r-3}|^{\frac{1}{2}} \otimes \bar{\eta})_\omega \otimes \delta^{s+\frac{1}{2}} Q \delta_{\rho'}^{-1/2}.$$

Again as in the $r = 2q$ case, by restricting the section to $\tilde{\text{GL}}_{r-3}$ (Proposition 2.11) one only has to consider the analytic behavior of the $\text{GL}_{r-3}$ Eisenstein series

$$\sum_{m \in \tilde{P}_{r-3,1}(A) \backslash \tilde{P}_{r-3,1}(F)} F^s(s(m)g)$$

where

$$F^s \in \text{Ind}_{\tilde{P}_{r-3,1}(A)}^{\tilde{P}_{r-3,1}(A)}(\theta' | \det_{r-3}|^{\frac{1}{2}} \otimes \bar{\eta})_\omega \otimes \delta^{s+\frac{1}{2}} Q \delta_{\rho'}^{-1/2}$$

and $\omega$ is some appropriately chosen character and $\delta^{s+\frac{1}{2}} Q \delta_{\rho'}^{-1/2}$ is restricted to $\tilde{P}_{r-3,1}$. As a character on $\text{GL}_{r-3} \times \text{GL}_1 \subseteq \tilde{P}_{r-3,1}$, one can compute

$$| \det(g_{r-3}) |^{\frac{1}{2}} \delta^{s+\frac{1}{2}} Q \delta_{\rho'}^{-1/2} (g_{r-3}, b) = | \det(g_{r-3}) |^{s+\frac{1}{2}} |b|^{-\frac{1}{2}}$$

$$\cdot \delta_{\rho'}^{r-1} (g_{r-3}, b) \left| \det(g_{r-3}) b \right|^{-\frac{r-3}{2}}.$$

Thus the section $F^s$ belongs to

$$(5.31) \quad \text{Ind}_{\tilde{P}_{r-3,1}(A)}^{\tilde{P}_{r-3,1}(A)}(\theta' \otimes \bar{\eta})_\omega \otimes \delta_{r-3}^b | \det_{r-3}|^a,$$

where

$$(5.32) \quad a = -\frac{1}{2} r - 2 s - \frac{1}{2}, \quad b = \frac{r s + \frac{3}{2}}{r - 2}.$$

Therefore the analytic behavior of the Eisenstein series in (5.30) is determined by that of the Eisenstein series associated to the induced representation (5.31). But the twist by $| \det_{r-2}|^a$ does not have any affect on the analytic behavior, and hence we have to consider the Eisenstein series on $\text{GL}_{r-2}(A)$ associated with the induced space

$$\text{Ind}_{\tilde{P}_{r-3,1}(A)}^{\tilde{P}_{r-3,1}(A)}(\theta' \otimes \bar{\eta})_\omega \otimes \delta_{r-3}^b,$$

where $b$ is as in (5.32).

Now by the induction hypothesis, this Eisenstein series on $\tilde{\text{GL}}_{r-2}(A)$ is holomorphic for $\text{Re}(b) \geq 0$, which implies that it is holomorphic for $\text{Re}(s) \geq -\frac{1}{2}$. (Note that the induction is on $q$ and the base step is $q = 1$ i.e. $r = 3$.) Thus (5.30) is holomorphic for $\text{Re}(s) \geq 0$.

The non-identity term $E_F(g, s; f^s)_{w_1}$:
We will show that the non-identity term vanishes, i.e. $E_P(g, s; f^s)_{w_1} = 0$, which will complete our proof. First note that the non-identity term is written as

\begin{equation}
E_P(g, s; f^s)_{w_1} = \sum_{m \in \mathcal{M}^*_{r}(F) \backslash GL_r(F)} \int_{N_P(F)^{m-1} \backslash N_P(k)} f^s(s(w_1^{-1}mn)g; 1) \, dn
\end{equation}

where for the last equality we used \( (5.35) \) and the fact that \( s \) is a homomorphism on \( GL_r(F) \). Then as we will see, each integral in the sum vanishes due to the “cuspidality”. But as we just have done, to move round the section \( s \) we will frequently use Lemma 2.5 and the fact that \( s \) is a homomorphism on both \( GL_r(F) \) and \( N_B(A) \), and the Weyl group elements are in \( GL_r(F) \). The reader can verify that each manipulation on \( s \) can be justified.

By the change of variable \( mmn^{-1} \rightarrow n \), \( (5.33) \) becomes

\begin{equation}
\sum_{m \in \mathcal{M}^*_{r}(F) \backslash GL_r(F)} \int_{N_P(F)^{w_1} \backslash N_P(k)} f^s(s(w_1^{-1}n)g; 1) \, dn.
\end{equation}

Let us introduce

\[ N_1 = \left\{ \begin{pmatrix} 1 & 0 & n_1 \\ 1 & 0 & 0 \\ 1 \end{pmatrix} \right\} \quad \text{and} \quad N_2 = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 1 & n_2 & 0 \\ 1 \end{pmatrix} \right\}, \]

i.e. \( N_1 \) is the first row and \( N_2 \) is the second row of \( N_P \). Note that \( N_P = N_2N_1 \). Then by direct computation we see \( N_P^{w_1} = N_2 \). Further let us write \( N_2 = UV \), where

\[ U = \left\{ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & u & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \right\}, \quad \text{and} \quad V = \left\{ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & v & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \right\}, \]

where \( v \) is \( 1 \times (r-3) \) and \( u \) is \( 1 \times 1 \), so \( n_2 = (v, u) \). Note that \( N_2 = UV \). With those notations, each integral in the sum of \( (5.34) \) is written as

\begin{equation}
\int_{N_1(k)} \int_{N_2(F) \backslash N_2(k)} f^s(s(w_1^{-1}n_2)g; 1) \, dn_2 \, dn_1
\end{equation}

where for the last equality we used

\[ w_1^{-1}w_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & u & 1 \\ 0 & 0 & 0 \end{pmatrix} \in Q. \]

Now let

\[ w_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \]
which is an element in $M_Q(F)$. By the automorphy of the automorphic form $f^*(s(w_1^{-1}vn_1)s(m)g; -)$, the integrand of $(6.35)$ is written as

$$f^*(s(w_1^{-1}vn_1)s(m)g; s(w_1^{-1})s(u)s(w_1)) = f^*(s(w_1^{-1}vn_1)s(m)g; s(w_2^{-1}w_1^{-1}uw_1w_2^{-1}w_1^{-1})s(w_1)),$$

where we again used Lemma 2.5. Note that

$$w_2^{-1}w_1^{-1}uw_1w_2 = \begin{pmatrix} I_{r-3} & 0 & 1 & u \\ 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \in N_B.$$  

Therefore the inner most integral of $(6.35)$ is written as

$$\int_{F\setminus A} f^*(s(w_1^{-1}vn_1)s(m)g; s(\begin{pmatrix} I_{r-3} & 0 & 1 & u \\ 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix})s(w_2^{-1}w_1^{-1})s(w_1)) \, du$$

But the cuspidal support of the automorphic form $f^*(s(w_1^{-1}n_1)s(m)g; -)$ is the $(2, \ldots, 2, 1)$-parabolic, which makes this integral vanish.

6. The normalized Eisenstein series

Following [BG], we normalize the Eisenstein series by using the denominators of $(4.5)$ and $(4.6)$. Namely we set

$$(6.1) \quad E^*(g, s; f^*) = \begin{cases} L^S(r(2s + \frac{1}{2}), \chi^2\eta^{-2})E(g, s; f^*), & \text{if } \theta = \theta_{\chi, \eta}; \\ L^S(r(2s + \frac{1}{2}), \chi\eta^{-2})E(g, s; f^*), & \text{if } \theta = \theta_{\chi, \omega}, \end{cases}$$

where $S = S(\chi, \eta, \omega, f^*)$ is a finite set of places containing all the bad places with respect to $\chi, \eta, \omega$ and $f^*$; namely $S$ contains all the archimedean places, the places where $\chi, \eta$ or $\omega$ ramified, the places dividing $2$, and the places where $f^*$ is ramified. (Note that here $\omega$ is the central character used to define the metaplectic tensor product.) Let us emphasize that the normalization depends on the set $S$, which in turn depends on $f^*$.

Then we have

**Theorem 6.2.** Assume $r > 2$. The above normalized Eisenstein series $E^*(g, s; f^*)$ is holomorphic for all $s \in \mathbb{C}$, except that if $\chi^2\eta^{-2} = 1$ and $r = 2q$, or $\chi\eta^{-2} = 1$ and $r = 2q + 1$, it has a possible simple pole at $s = \frac{1}{4}$ and $-\frac{3}{4}$.

The idea of the proof is by now standard in that we use the functional equation of the Eisenstein series and the analytic behavior of the normalized intertwining operator $A^*(s, \theta, w_1)$ we obtained in Section 3

To use the functional equation, however we need to introduce the “opposite Eisenstein series” as follows. Let $Q = P_{r-1}$ be the standard $(1, r - 1)$-parabolic of $GL_r$. Define a representation $'\theta$ of the Levi part $M_Q$ by

$$'\theta := \begin{cases} (\eta \otimes \theta_{\chi})_{\omega}, & \text{if } r = 2q \\ (\eta \otimes \theta_{\chi})_{\omega}, & \text{if } r = 2q + 1, \end{cases}$$
where ω is arbitrary as long as it satisfies the requirement for the metaplectic tensor product. Then consider the global induced space Ind^{GL_r(A)}_{Q(A)} ι \cdot \theta \otimes δ^r Q. Note that this induced representation is nothing but the codomain of our intertwining operator A(s, θ, w_1). Form the corresponding “opposite Eisenstein series”

\[ \hat{E}(g,s; f^*) = \sum_{\gamma \in Q(F) \backslash GL_r(F)} f^*(\gamma g; 1) \]

for \( f^* \in \text{Ind}^{GL_r(A)}_{Q(A)} \cdot \theta \otimes δ^r Q. \) Then we have

**Theorem 6.3.** The above Eisenstein series \( \hat{E}(g,s; f^*) \) is holomorphic for \( \text{Re}(s) \geq 0 \) except that it has a possible simple pole at \( s = \frac{1}{2} \) if \( \chi^2 \eta^{-2} = 1 \) and \( r = 2q \), or \( \chi \eta^{-2} = 1 \) and \( r = 2q \).

**Proof.** The proof is completely identical to Theorem 5.3 except that we need use the parabolic subgroup \( \hat{Q} \). However when one applies the induction argument to compute the possible poles of the Eisenstein series, one needs to induct “from the bottom”. Namely, say if \( r = 2q \), one considers the constant term along the \( (r - 1,1) \)-parabolic, and then the identity term will be interpreted as the Eisenstein series on \( GL_{r-1} \) embedded in the upper left corner. To do so, one needs part (b) of Proposition 2.11 and hence one needs to assume that the metaplectic tensor product is realized as such.

Now we are ready to provide

**Proof of Theorem 6.2.** This follows from the functional equation together with the holomorphy of the two Eisenstein series (Theorem 5.3, 6.3) and the holomorphy of the normalized intertwining operator (Theorem 4.3). Though the argument seems to be standard by now, we will repeat it in what follows. We only treat the case \( r = 2q \), and the other case is identical.

Note that since we know the holomorphy for \( \text{Re}(s) \geq 0 \) by Theorem 5.3 and the partial \( L \)-function \( L^S(r(2s + \frac{1}{2}), \chi^2 \eta^{-2}) \) is holomorphic for \( \text{Re}(s) \geq 0 \), we only have to consider \( \text{Re}(s) < 0 \). Now by the functional equation of the Eisenstein series, one has

\[ \hat{E}(g,-s; A(s,\theta,w_1)f^*) = E(g,s; f^*). \]

By multiplying the normalizing factor \( L^S(r(2s + \frac{1}{2}), \chi^2 \eta^{-2}) \) to both sides, we have

(6.4) \[ \hat{E}(g,-s; A^*(s,\theta,w_1)f^*) = L^S(r(2s + \frac{1}{2}), \chi^2 \eta^{-2}) E(g,s; f^*), \]

where the right hand side is nothing but the normalized Eisenstein series \( E^*(g,s,f^*) \).

Assume \( \chi^2 \eta^{-2} \neq 1 \). Then \( A^*(s,\theta,w_1)f^* \) is holomorphic for all \( s \in \mathbb{C} \) by Theorem 4.3 and hence by Theorem 6.3, the left hand side (and hence the right hand side) is holomorphic for \( \text{Re}(s) \leq 0 \). Assume that \( \chi \eta^{-2} = 1 \). Then \( A^*(s,\theta,w_1)f^* \) is holomorphic for \( \text{Re}(s) < 0 \) again by Theorem 4.3. But by Theorem 6.3 the left hand side (and the right hand side) of (6.4) has a possible simple pole at \( s = -\frac{1}{2} \).

7. The twisted symmetric square \( L \)-function

Theorem 6.2 along with the integral representation of the symmetric square \( L \)-function of \( GL_r \) obtained in \( \text{T} \) immediately implies

**Theorem 7.1.** Let \( \pi \) be a cuspidal automorphic representation of \( GL_r(A) \) with unitary central character \( \omega_{\pi} \) and \( \chi \) a unitary Hecke character. For a sufficiently large finite set \( S \) of places, the (incomplete) twisted symmetric square \( L \)-function \( L^S(s,\pi,\text{Sym}^2 \otimes \chi) \) is holomorphic everywhere except that if \( \chi^2 \omega_{\pi}^2 = 1 \) it has a possible simple pole at \( s = 0 \) and \( s = 1 \). Indeed, the set \( S \) can be taken to be precisely the set of archimedean places, places dividing 2 and non-archimedean places at which either \( \pi \) or \( \chi \) is ramified.
Proof. Assume \( r = 2q + 1 \). Set \( \theta = \theta_{\chi^2, \chi^{-q}} = (\hat{\theta}_{\chi^2} \otimes \bar{\chi}^{-q})_{\omega} \), where \( \omega \) is chosen appropriately as in [T1] (2.57). (As one can see from there, the bad places for \( \omega \) are dependent on the choice of the additive character \( \psi \). However, one can always choose \( \psi \) so that the bad places of \( \omega \) are either dividing 2 or contained in those of \( \chi \) or \( \pi \).) Then in [T1] we defined the zeta integral

\[
Z(\phi, \theta, f^*) = \int_{Z(\mathbb{A}) \backslash GL_r(F) \backslash GL_r(\mathbb{A})} \phi(g) \Theta(\kappa(g)) E(\kappa(g), s; f^*) \, dg
\]

where \( \phi \in \pi, f^* \in \text{Ind}_{\widetilde{GL}_r(\mathbb{A})} \theta \otimes \delta_Q^r, \) and \( \Theta \) is an automorphic form in the twisted exceptional representation \( \omega_{\chi^{-q}} \) on \( GL_r(\mathbb{A}) \). Now if \( f^* = f^*_{\infty} \otimes (\otimes' f_v^*) \) is an (almost) factorizable section, we have shown in [T1] that

\[
L^S(2s + \frac{1}{2}, \pi, Sym^2 \otimes \chi)Z_S(s) = L^S(2s, \chi^r \omega^2)Z(\phi, \theta, f^*)
\]

for an appropriately chosen \( \phi \) and \( \theta \), where \( Z_S(s) \) is a product of local zeta integrals. (Note that in [T1] the induced representation for the Eisenstein series is not normalized and hence there is a shift by \( \frac{1}{2} \).) Here we may assume that the local section \( f_v^* \) is unramified if \( \nu \notin S \). Moreover one can choose the section so that \( Z_S(s) \) is non-zero holomorphic, and hence the poles of \( L^S(2s + \frac{1}{2}, \pi, Sym^2 \otimes \chi) \) are the poles of \( L^S(2s, \chi^r \omega^2)Z(\phi, \theta, f^*) \), which are among the poles of the normalized Eisenstein series \( E^*(\kappa(g), s; f^*) \), where the normalization is with respect to \( S \). Hence by using Theorem 6.2 we see that the incomplete \( L \)-function \( L^S(2s + \frac{1}{2}, \pi, Sym^2 \otimes \chi) \) is entire except that if \( \chi^r \omega^2 = 1 \) it has a possible pole at \( s = -\frac{1}{2} \) and \( s = -\frac{1}{2} \).

Assume \( r = 2q \) and \( r > 2 \). (If \( r = 2 \), the theorem is already well-known by the work of Gelbart and Jacquet ([GJ]).) Set \( \theta = \theta_{\omega_{\pi}, \chi^{-q}} = (\hat{\theta}_{\omega_{\pi}} \otimes \bar{\chi}^{-q})_{\omega} = \theta_{\omega_{\pi}} \otimes \bar{\chi}^{-q} \). (There is no actual choice for \( \omega \) here because the center of \( GL_{2q} \) is already contained in \( \widetilde{M}_{Q}^{(2)} \), and that is why we simply write \( \theta_{\omega_{\pi}} \otimes \bar{\chi}^{-q} \).) Then in [T1] we have shown that the twisted symmetric square \( L \)-function is represented by

\[
Z(\phi, \theta, f^*) = \int_{Z(\mathbb{A}) \backslash GL_r^{(2)}(F) \backslash GL_r^{(2)}(\mathbb{A})} \phi(g) \Theta(\kappa(g)) E(\kappa(g), s; f^*) \, dg
\]

where the Eisenstein series \( E(\kappa(g), s; f^*) \) is a restriction to \( GL_r^{(2)}(\mathbb{A}) \) of the Eisenstein series associated with the induced space \( \text{Ind}_{\widetilde{GL}_{2q}^{(2)}(\mathbb{A})} \theta \otimes \delta_Q^{(2)}, \) and \( \Theta \) is a restriction of an automorphic form in the exceptional representation \( \theta_{\chi^2} \). (It should be mentioned that both the restriction of the Eisenstein series and the restriction of the exceptional representation depends on the choice of \( \psi \). See [T1] for the details. But once again, one can choose \( \psi \), so that the bad places incurred by \( \psi \) are either dividing 2 or contained in the bad places of \( \chi \) or \( \pi \).) Then by arguing as above, we can derive that the poles of the twisted symmetric square \( L \)-function are among the poles of the restriction of the normalized Eisenstein series \( E^*(\kappa(g), s; f^*) \), which are among the poles of the normalized Eisenstein series itself. But by Theorem 6.2 we see that the normalized Eisenstein series is entire except that if \( \chi^r \omega_{\pi} = 1 \) it has a possible simple pole at \( s = -\frac{1}{4} \) and \( s = -\frac{1}{4} \). The proof is complete. \( \square \)

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