Resolution in Linguistic First Order Logic based on Linear Symmetrical Hedge Algebra

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Abstract. This paper focuses on resolution in linguistic first order logic with truth value taken from linear symmetrical hedge algebra. We build the basic components of linguistic first order logic, including syntax and semantics. We present a resolution principle for our logic to resolve on two clauses having contradictory linguistic truth values. Since linguistic information is uncertain, inference in our linguistic logic is approximate. Therefore, we introduce the concept of reliability in order to capture the natural approximation of the resolution inference rule.

Keywords: Linear Symmetrical Hedge Algebra; Linguistic Truth Value; Linguistic First Order Logic; Resolution; Automated Reasoning.

1 Introduction

Automated reasoning theory based on resolution rule of Robinson [15] has been research extensively in order to find efficient proof systems [1,6]. However, it is difficult to design intelligent systems based on traditional logic while most of the information we have about the real world is uncertain. Along with the development of fuzzy logic, non-classical logics became formal tools in computer science and artificial intelligence. Since then, resolution based on non-classical logic (especially multi-valued logic and fuzzy logic) has drawn the attention of many researchers.

In 1965, Zadeh introduced fuzzy set theory known as an extension of set theory and applied widely in fuzzy logic [22]. Many researchers have presented works about the fuzzy resolution in fuzzy logic [29][10][16][19][21]. In 1990, Ho and Wechler proposed an approach to linguistic logic based on the structure of natural language [11]. The authors introduced a new algebraic structure, called hedge algebra, to model linguistic truth value domain, which applied directly to semantics value in inference. There also have been many works about inference on linguistic truth value domain based on extended structures of hedge algebra such as linear hedge algebra, monotony linear hedge algebra [7][13][14]. Researchers also presented truth functions of new unary connectives (hedges) from
the set of truth values to handle fuzzy truth values in a natural way. Recently, we have presented the resolution procedure in linguistic propositional logic with truth value domain taken from linear symmetrical hedge algebra. We have constructed a linguistic logic system, in which each sentence in terms of "It is very true that Mary studies very well" is presented by $P_{\text{VeryTrue}}$, where $P$ is "Mary studies very well". Two clauses having converse linguistic truth values, such as $P_{\text{VeryTrue}}$ and $P_{\text{MoreFalse}}$, are resolved by a resolution rule. However, we cannot intervene in the structure of a proposition. For example with the knowledge base: "It is true that if a student studies hard then he will get the good marks" and "It is very true that Peter studies hard", we cannot infer to find the truth value of the sentence "Peter will get the good marks". Linguistic first order logic overcomes this drawback of linguistic propositional logic. Furthermore, knowledge in the linguistic form maybe compared in some contexts, such as when we tell about the value of linguistic variable Truth, we have LessTrue < VeryTrue or MoreFalse < LessFalse. Therefore, linear symmetrical hedge algebra is an appropriate to model linguistic truth value domain.

As a continuation of our research works on resolution in linguistic propositional logic systems, we study resolution in linguistic first order logic. We construct the syntax and semantics of linguistic first order logic with truth value domain taken from linear symmetrical hedge algebra. We also propose a resolution rule and a resolution procedure for our linguistic logic. Due to the uncertainty of linguistic information, each logical clause would be associated with a certain confidence value, called reliability. Therefore, inference in our logic is approximate. We shall build an inference procedure based on resolution rule with a reliability $\alpha$ which ensures that the reliabilities of conclusions are less than or equal to reliabilities of premises.

The paper is structured as follows: section 2 introduces basic notions of linear symmetrical hedge algebras and logical connectives. Section 3 describes the syntax and semantics of our linguistic first order logic with truth value domain based on linear symmetrical hedge algebra. Section 4 proposes a resolution rule and a resolution procedure. Section 5 concludes and draws possible future work.

2 Linear Symmetrical Hedge Algebra

We present here an appropriate mathematical structure of a linguistic domain called hedge algebra which we use to model linguistic truth domain for our linguistic logic. In this algebraic approach, values of the linguistic variable Truth such as $\{\text{True, MoreTrue, VeryPossibleTrue, PossibleFalse, LessFalse}\}$, and so on are generated from a set of generators (primary terms) $G = \{\text{False, True}\}$ using hedges from a set $H = \{\text{Very, More, Possible, Less, ...}\}$ as unary operations. There exists a natural ordering among these values, with $a \leq b$ meaning that $a$ indicates a degree of truth less than or equal to $b$, where $a < b$ iff $a \leq b$ and $a \neq b$. For example, True < VeryTrue and False < LessFalse. The relation $\leq$ is called the semantically ordering relation on the term domain, denoted by $X$. 
In general, $X$ is defined by an abstract algebra called hedge algebra $HA = (X, G, H, >)$ where $G$ is the set of generators and $H$ is the set of hedges. The set of values $X$ generated from $G$ and $H$ is defined as $X = \{ \delta | c \in G, \delta \in H \}$. $\geq$ is a partial order on $X$ such that $a \geq b$ if $a > b$ or $a = b$ ($a, b \in X$).

Each hedge $h \in H$ either strengthens or weakens the meaning of a term $x \in X$, this means $hx$ and $x$ are always comparable. For example, $\text{VeryTrue} > \text{True}$ but $\text{PossibleTrue} < \text{True}$. Therefore, the set $H$ can be decomposed into two subsets: one subset $H^+$ consists of hedges which strengthen the primary term $\text{True}$ and the other, denoted by $H^-$, consists of hedges that weaken the term $\text{True}$.

Each hedge has a strengthening or weakening degree w.r.t. linguistic terms and so the sets $H^+$ and $H^-$ maybe ordered; and they then become a poset (partially ordered set). The ordering relationship between two hedges $h$ and $k$ maybe comparable or incomparable. Thus, $H$ and $\text{Proposition 1.}$

Let $h, k$ be two hedges in the set of hedges $H$. Then $k$ is said to be positive (negative) w.r.t. $h$ if for every $x \in X$, $hx \geq x$ implies $khx \geq hx(khx \leq hx)$ or, conversely, $hx \leq x$ implies $khx \leq hx(khx \geq hx)$. $h$ and $k$ are converse if $\forall x \in X, hx \leq x$ iff $khx \geq x$, i.e. they are in the different subset. $h$ and $k$ are compatible if $\forall x \in X, x \leq hx$ iff $x \leq kx$, i.e. they are in the same subset. $h$ modifies terms stronger or equal than $k$, denoted by $h \geq k$, if $\forall x \in X, (hx \geq kx \geq x)$ or $(hx \geq kx \geq x)$.

Given a term $u$ in $X$, the expression $h_n \ldots h_1 u$ is called a representation of $x$ w.r.t. $u$ if $x = h_n \ldots h_1 u$, and it is called a canonical representation of $x$ w.r.t. $u$ if $h_n h_{n-1} \ldots h_1 u \neq h_{n-1} \ldots h_1 u$. The notation $x_{u|j}$ denotes the suffix of length $j$ of a representation of $x$ w.r.t. $u$. The following proposition shows how to compare any two terms in $X$.

Proposition 1. [[11]] Let $x = h_n h_{n-1} \ldots h_1 u$, $y = k_m k_{m-1} \ldots k_1 u$ be two canonical presentations of $x$ and $y$ w.r.t. $u \in X$, respectively. Then, there exists the largest $j \leq \min(m, n) + 1$ such that $\forall i < j, h_i = k_i$, and

i. $x = y$ iff $m = n$ and $h_j x_{u|j} = k_j x_{u|j}$ for every $j \leq n$;

ii. $x < y$ iff $h_j x_{u|j} < k_j x_{u|j}$;

iii. $x$ and $y$ are incomparable iff $h_j x_{u|j}$ and $k_j x_{u|j}$

The set of primary terms $G$ usually consists of two comparable ones, denoted by $c^- < c^+$. For the variable $\text{Truth}$, we have $c^+ = \text{True} > c^- = \text{False}$. Such HAs are called symmetric ones. For symmetric HAs, the set of hedges $H$ is decomposed into two disjoint subsets $H^+$ and $H^-$ defined as $H^+ = \{ h \in H | hc^+ > c^+ \}$ and $H^- = \{ h \in H | hc^- < c^+ \}$. Two hedges in each of the sets $H^+$ and $H^-$ maybe comparable or incomparable. Thus, $H^+$ and $H^-$ become posets.
Definition 1. A symmetric HA $AX = (X, G = \{c^-, c^+\}, H, \leq)$ is called a linear symmetric HA (lin-HA, for short) if the set of hedges $H$ is divided into two subsets $H^+$ and $H^-$, where $H^+ = \{h \in H | h \leq c^+\}$, $H^- = \{h \in H | h \geq c^+\}$, and $H^+$ and $H^-$ are linearly ordered.

Let $x$ be an element of the hedge algebra $AX$ and the canonical representation of $x$ is $x = h_1...h_1a$ where $a \in \{c^+, c^-\}$. The contradictory element of $x$ is an element $\overline{x}$ such that $\overline{x} = h_1...h_1a'$ where $a' \in \{c^+, c^-\}$ and $a' \neq a$. In lin-HA, every element $x \in X$ has an unique contradictory element in $X$.

HAs are extended by augmenting two hedges $\Phi$ and $\Sigma$ defined as $\Phi(x) = \inf \{H(x)\}$ and $\Sigma(x) = \sup \{H(x)\}$, for all $x \in X$ [4]. An HA is said to be free if $\forall x \in X$ and $\forall h \in H, hx \neq x$. It is shown that, for a free lin-HA of the variable Truth with $H \neq 0, \Phi(c^+) = \Sigma(c^-), \Sigma(c^+) = \top$ (AbsolutelyTrue), and $\Phi(c^-) = \bot$ (AbsolutelyFalse). Let us put $W = \Phi(c^+) = \Sigma(c^-)$ (called the middle truth value), we have $\bot < c^- < W < c^+ < \top$.

Definition 2. A linguistic truth domain $\overline{X}$ taken from a lin-HA $AX = (X, \{c^-, c^+\}, H, \leq)$ is defined as $\overline{X} = X \cup \{\bot, W, \top\}$, where $\bot, W, \top$ are the least, the neutral, and the greatest elements of $\overline{X}$, respectively.

Proposition 2. [4] For any lin-HA $AX = (X, G, H, \leq)$, the linguistic truth domain $\overline{X}$ is linearly ordered.

In many-valued logic, sets of connectives called Lukasiewicz, Gödel, and product logic ones are often used. Each of the sets has a pair of residual t-norm and impicator. However, we cannot use the product logic connectives when our truth values are linguistic. We showed that the logical connectives based on Gödel’s t-norm and t-conorm operators are more suitable for our linguistic logic than those based on Lukasiewicz’s [12]. Therefore, in this paper we define logical connectives using Gödel’s t-norm and t-conorm operators [17,21].

Let $K = \{n | n \in \mathbb{N}, n \leq N_0\}$. A pair of $(T, S)$ in Gödel’s logic is defined as follows:

- $T_G(m, n) = \min(m, n)$.
- $S_G(m, n) = \max(m, n)$.

It is easy to prove that $T_G, S_G$ are commutative, associate, monotonous.

Given a lin-HA $AX$, since all the values in $AX$ are linearly ordered, truth functions for conjunctions and disjunctions are Gödel’s t-norms and t-conorms, respectively.

Definition 3. Let $S$ be a linguistic truth domain, which is a lin-HA $AX = (X, G, H, \leq)$, where $G = \{\top, \text{True}, W, \text{False}, \bot\}$. The logical connectives $\land$ (respectively $\lor$) over the set $X$ are defined to be Gödel’s t-norm (respectively t-conorm), and furthermore to satisfy the following: $\neg \alpha = \overline{\alpha}$, and $\alpha \rightarrow \beta = (\neg \alpha) \lor \beta$, where $\alpha, \beta \in X$.

Proposition 3. Let $S$ be a linguistic truth domain, which is a lin-HA $AX = (X, \{\top, \text{True}, W, \text{False}, \bot\}, H, \leq)$; $\alpha, \beta, \gamma \in X$, we have:
– Double negation: \( \neg(\neg\alpha) = \alpha \)
– Commutative: \( \alpha \land \beta = \beta \land \alpha \), \( \alpha \lor \beta = \beta \lor \alpha \)
– Associative: \( (\alpha \land \beta) \land \gamma = \alpha \land (\beta \land \gamma) \), \( (\alpha \lor \beta) \lor \gamma = \alpha \lor (\beta \lor \gamma) \)
– Distributive: \( \alpha \land (\beta \lor \gamma) = (\alpha \land \beta) \lor (\alpha \land \gamma) \), \( \alpha \lor (\beta \land \gamma) = (\alpha \lor \beta) \land (\alpha \lor \gamma) \)

3 Linguistic First Order Logic based on Linear Symmetrical Hedge Algebra

In this section we define the syntax and semantics of our linguistic first-order logic.

3.1 Syntax

**Definition 4.** The alphabet of a linguistic first-order language consists of the following sets of symbols:

– constant symbols: a set of symbols \( a, b, c, \ldots \), each of 0-ary;
– logical constant symbols: MoreTrue, VeryFalse, \( \perp \), \( \top \), \( \ldots \);
– variable: \( x, y, z, \ldots \);
– predicate symbols: a set of symbols \( P, Q, R, \ldots \), each associated with a positive integer \( n \), arity. A predicate with arity \( n \) is called \( n \)-ary;
– function symbols: a set of symbols \( f, g, h, \ldots \), each associated with a positive integer \( n \), arity. A function with arity \( n \) is called \( n \)-ary;
– logical connectives: \( \lor \), \( \land \), \( \neg \), \( \rightarrow \), \( \leftrightarrow \);
– quantifies: universal quantification \( \forall \), existential quantification \( \exists \);
– auxiliary symbols: \( \Box, (,) \ldots \).

**Definition 5.** A term is defined recursively as follows:

– either every constant or every variable symbol is a term,
– if \( t_1, \ldots, t_n \) are terms and \( f \) is a \( n \)-ary function symbol, \( f(t_1, \ldots, t_n) \) is a term (functional term).

**Definition 6.** An atom is either a zero-ary predicate symbol or a \( n \)-ary predicate symbol \( P(t_1, \ldots, t_n) \), where \( t_1, \ldots, t_n \) are terms.

**Definition 7.** Let \( A \) be an atom and \( \alpha \) be a logical constant. Then \( A^\alpha \) is called a literal to represent \( A \) is \( \alpha \).

**Definition 8.** Formulae are defined recursively as follows:

– a literal is a formula,
– if \( F, G \) are formulae, then \( F \lor G \), \( F \land G \), \( F \rightarrow G \), \( F \leftrightarrow G \), \( \neg F \) are formulae, and
– if \( F \) is a formula and \( x \) is a free variable in \( F \), then \( (\forall x)F \) and \( (\exists x)F \) are formulae.
The notions of free variable, bound variable, substitution, unifier, most general unifier, ground formula, closed formula, etc. are similar to those of classical logic.

**Definition 9.** A clause is a finite disjunction of literals represented by \( L_1 \lor L_2 \lor \ldots \lor L_n \), where \( L_i (i = 1, 2, \ldots, n) \) is a literal. An empty clause is denoted by \( \Box \).

A formula is in conjunctive normal form (CNF) if it is a conjunction of clauses. It is well known that transforming a formula in first order logic into a CNF formula preserves satisfiability \([1]\). In Section 3 we shall be working with a resolution procedure which processes CNF formulae, or equivalently clause sets.

### 3.2 Semantics

**Definition 10.** An interpretation for the linguistic first order logic is a pair \( I = < D, A > \) where \( D \) is a non empty set called domain of \( I \), and \( A \) is a function that maps:

- every constant symbol \( c \) into an element \( c^A \in D \);
- every \( n \)-ary function symbol \( f \) into a function \( f^A : D^n \rightarrow X \);
- every logical constant symbol \( l \) into an element \( l^A \in X \);
- every \( n \)-ary predicate symbol \( P \) into an \( n \)-ary relation \( P^A : D^n \rightarrow X \), where \( X \) is the truth value domain taken from lin-HA;
- every variable \( x \) into a term.

Given an interpretation \( I = < D, A > \) for the linguistic first order logic, the truth value of a symbol \( S \) in the alphabet of the logic is denoted by \( I(S) \).

**Definition 11.** Given an interpretation \( I = < D, A > \), we define:

- Value of a term: \( I(t) = t^A, I(f(t_1, \ldots , t_n)) = f(I(t_1), \ldots , I(t_n)) \).
- Truth value of an atom: \( I(P(t_1, \ldots , t_n)) = P(I(t_1), \ldots , I(t_n)) \).
- Truth value of a logical constant: \( I(c) = c^A \).
- Let \( P \) be an atom such that \( I(P) = \alpha_1 \). Truth value of a literal \( P^{\alpha_2} \):

\[
I(P^{\alpha_2}) = \begin{cases} 
\alpha_1 \land \alpha_2 & \text{if } \alpha_1, \alpha_2 > W, \\
\neg(\alpha_1 \lor \alpha_2) & \text{if } \alpha_1, \alpha_2 \leq W, \\
(\neg \alpha_1) \lor \alpha_2 & \text{if } \alpha_1 > W, \alpha_2 \leq W, \\
\alpha_1 \lor (\neg \alpha_2) & \text{if } \alpha_1 \leq W, \alpha_2 > W.
\end{cases}
\]

- Let \( F \) and \( G \) be formulae. Truth value of a formula:

\[
\begin{align*}
I(\neg F) &= \neg I(F) \\
I(F \land G) &= I(F) \land I(G) \\
I(F \lor G) &= I(F) \lor I(G) \\
I(F \rightarrow G) &= I(F) \rightarrow I(G) \\
I(\forall x F) &= \min_{x \in D} I(F) \\
I(\exists x F) &= \max_{x \in D} I(F)
\end{align*}
\]

**Definition 12.** Let \( I = < D, A > \) be an interpretation and \( F \) be a formula. Then
- $F$ is true iff $I(F) \geq W$. $F$ is satisfiable iff there exists an interpretation $I$ such that $F$ is true in $I$ and we say that $I$ is a model of $F$ (write $I \models F$) or $I$ satisfies $F$.

- $F$ is false iff $I(F) < W$ and we say that $I$ falsifies $F$. $F$ is unsatisfiable iff there exists no interpretation that satisfies $F$.

- $F$ is valid iff every interpretation of $F$ satisfies $F$.

- A formula $G$ is a logical consequence of formulas $\{F_1, F_2, \ldots, F_n\}$ iff for every interpretation $I$, if $I \models F_1 \land F_2 \land \ldots \land F_n$ we have that $I \models G$.

**Definition 13.** Two formulae $F$ and $G$ are logically equivalent iff $F \models G$ and $G \models F$ and we write $F \equiv G$.

It is infeasible to consider all possible interpretations over all domains in order to prove the unsatisfiability of a clause set $S$. Instead, we could fix on one special domain such that $S$ is unsatisfiable iff $S$ is false under all the interpretations over this domain. Such a domain, which is called the Herbrand universe of $S$, defined as follows.

Let $H_0$ be the set of all constants appearing in $S$. If no constant appears in $S$, then $H_0$ is to consist of a single constant, say $H_0 = \{a\}$. For $i = 0, 1, 2, \ldots$, let $H_{i+1}$ be the union of $H_i$ and the set of all terms of the form $f^n(t_1, \ldots, t_n)$ for all $n$-place functions $f^n$ occurring in $S$, where $t_j, j = 1, \ldots, n$, are members of the set $H_i$. Then each $H_i$ is called the $i$-level constant set of $S$ and $H_\infty$ is called the Herbrand universe (or H-universe) of $S$, denoted by $H(S)$.

The set of ground atoms of the form $P^n(t_1, \ldots, t_n)$ for all n-ary predicates $P^n$ occurring in $S$, where $t_1, \ldots, t_n$ are elements of the H-universe of $S$, is called the atom set, or Herbrand base (H-base, for short) of $S$, denoted by $A(S)$.

A ground instance of a clause $C$ of a clause set $S$ is a clause obtained by replacing variables in $C$ by members of H-universe of $S$.

We now consider interpretations over the H-universe. In the following we define a special over the H-universe of $S$, called the H-interpretation of $S$.

**Definition 14.** Let $S$ be a clause set, $H$ be the H-universe of $S$, and $I=\langle D,A \rangle$ be an interpretation of $S$. $I$ is an H-interpretation of $S$ if the following holds:

- $D = H$,
- Let $c$ be a constant symbol, $c^A = c$,
- Let $f$ be a n-ary function symbol, $f^A$ maps $(h_1, \ldots, h_n) \in H^n$ to $f(h_1, \ldots, h_n) \in H$,
- Let $A = \{A_1, \ldots, A_n, \ldots\}$ be the H-base (or atom set) of $S$, H-interpretation $I = \{m_1, \ldots, m_n, \ldots\}$, where $m_j = A_j$ or $m_j = \neg A_j$.

Given $I=\langle D,A \rangle$ interpretation over $D$, an H-interpretation $I=\langle H,A \rangle$ corresponding to $I$ is an H-interpretation that satisfies the following condition:

Let $h_1, \ldots, h_n$ be elements of $H$ and let $m : H \rightarrow D$ be a mapping from $H$ to $D$ then $P^A(h_1, \ldots, h_n) = P^A(m(h_1), \ldots, m(h_n))$.

Given an Interpretation $I$, we can always find a corresponding $I$ H-interpretation.
Lemma 1. If an interpretation $I$ over some domain $D$ satisfies a clause set $S$, then any one of the $H$-interpretations $I$ corresponding to $I$ also satisfies $S$.

Proof. Assume $I$ falsifies $S$ over domain $D$. Then there must exist at least one clause $C$ in $S$ such that $I(C) < W$. Let $x_1, \ldots, x_n$ be the variables occurring in $C$. Then there exist $h_1, \ldots, h_n$ in $H(S)$ such that $I(C') < W$ where $C'$ is ground clause obtained from $C$ by replacing every $x_i$ with $h_i$. Let every $h_i$ mapped to some $d_i$ in $D$ by $I$. By the definition of $H$-interpretation of $S$ in Def. 14, if $C''$ is the ground clause obtained from $C$ by replacing every $x_i$ with $d_i$ then $I(C'') < W$. This means that $I$ falsifies $S$ which is impossible.

Theorem 1. A clause set $S$ is unsatisfiable iff $S$ is false under all the $H$-interpretations of $S$.

Proof. ($\Rightarrow$) Obviously, by definition $S$ is unsatisfiable iff $S$ is false under all the interpretations over any domain.

($\Leftarrow$) Assume that $S$ is false under all the $H$-interpretations of $S$. Suppose $S$ is satisfiable. Then there is an interpretation $I$ over some domain $D$ such that $I(S) \geq W$. Let $I$ be an $H$-interpretation corresponding to $I$. According to Lemm. 1, $I(S) \geq W$. This contradicts the assumption that $S$ is false under all the $H$-interpretations of $S$. Therefore, $S$ must be unsatisfiable.

Let $S$ be a clause set and $A(S)$ be the $H$-base of $S$. A semantic tree for $S$ is a complete binary tree constructed as follows:

- For each node $N_i$ at the $i^{th}$ level corresponds to an element $A_i$ of $A(S)$, that is, the left edge of $N_i$ is labeled $A_i < W$, the right edge of $N_i$ is labeled $A_i \geq W$.

- Conversely, each element of $A(S)$ corresponds to exactly one level in the tree, this means if $A_i \in A(S)$ appears at level $i$ then it must not be at any other levels.

Let $T$ be a semantic tree of a clause set $S$ and $N$ be a node of $T$. We denote $I(N)$ to be the union of all the sets labeled to the edges of branch of $T$ down to $N$. If there exists an $H$-interpretation $I$ in $T$ which contains $I(N)$, such that $I(N)$ falsifies some ground instance of $S$, then $S$ is said to be failed at the node $N$. A node $N$ is called a failure node of $S$ if $S$ falsifies at $N$ and $I(N')$ does not falsify any ground instance of a clause in $S$ for every ancestor node $N'$ of $N$. $N$ is called an inference node if all the immediate descendant nodes of $N$ are failure nodes. If every branch in $T$ contains a failure node, cutting off its descendants from $T$, we have $T'$ which is called a closed tree of $S$. If the number of nodes in $T'$ is finite, $T'$ is called a finite closed semantic tree.

Lemma 2. There always exists an inference node on finite closed tree.

Proof. Assume that we have a closed tree $CT$. Because $CT$ has finite level, so there exists at least one leaf node $j$ on $CT$ at the highest level. Let $i$ be parent node of $j$. By definition of closed tree, $i$ cannot be failure node. Therefore, $i$ has
another child node, named k. If k is a failure node then i is inference node, the lemma is proved. If k is not a failure node then it has two child nodes: l, m. Clearly l, m are at higher level than j. This contradicts with the assumption that j is at the highest level. Therefore k is a failure node and i is an inference node. The lemma is proved.

\[ \text{Fig. 1. Proof of inference node} \]

**Lemma 3.** Let \( S \) be a clause set. Then \( S \) is unsatisfiable iff for every semantic tree of \( S \), there exists a finite closed tree.

**Proof.** \((\Rightarrow)\) Suppose \( S \) is unsatisfiable and \( T \) is a semantic tree of \( S \). For each branch \( B \) of \( T \), let \( I_B \) be the set of all literals labeled to all edges of the branch \( B \) then \( I_B \) is an H-interpretation for \( S \). Since \( S \) is unsatisfiable, \( I_B \) must falsify a ground instance \( C' \) of a clause \( C \) in \( S \). However, since \( C' \) is finite, there must exists a failure node \( N_B \) on the branch \( B \). Since every branch of \( T \) has a failure node, there is a closed semantic tree \( T' \) for \( S \). Furthermore, since only a finite number of edges are connected to each node of \( T' \), the number of nodes in \( T' \) must be finite, for otherwise, by König Lemma, we could find an infinite branch containing no failure node. Thus, \( T' \) is a finite closed tree.

\((\Leftarrow)\) Conversely, if corresponding to every semantic tree \( T \) for \( S \) there is a finite closed semantic tree, by the definition of closed tree, every branch of \( T \) contains a failure node. This means that every interpretation falsifies \( S \). Hence \( S \) is unsatisfiable.

In the next section we present the inference based on resolution rule for our linguistic logic. Lemma 2 and Lemma 3 will be used to prove the soundness and completeness of resolution inference rule.

### 4 Resolution

In two-valued logic, when we have a set of formulae \( \{A, \neg A\} \) (written as \( \{A^\text{True}, A^\text{False}\} \) in our logic) then the set is said to be contradictory. However in
our logic, the degree of contradiction can vary because the truth domain contains more than two elements. Let us consider two sets of formulae \( \{A^{\text{VeryTrue}}, A^{\text{VeryFalse}}\} \) and \( \{A^{\text{LessTrue}}, A^{\text{LessFalse}}\} \). Then the first set of formulae is “more contradictory” than the second one. Consequently, the notion of reliability is introduced to capture the approximation of linguistic inference.

**Definition 15.** Let \( \alpha \) be an element of \( X \) such that \( \alpha > W \) and \( C \) be a clause. The clause \( C \) with a reliability \( \alpha \) is denoted by the pair \( (C, \alpha) \).

The reliability \( \alpha \) of a clause set \( S = \{C_1, C_2, \ldots, C_n\} \) is defined as follows: \( \alpha = \alpha_1 \land \alpha_2 \land \ldots \land \alpha_n \), where \( \alpha_i \) is the reliability of \( C_i \) (\( i = 1, 2, \ldots, n \)).

A clause \( (C_2, \alpha_2) \) is a variant of a clause \( (C_1, \alpha_1) \) if \( \alpha_1 \neq \alpha_2 \) or \( C_2 \) is equal to \( C_1 \) except for possibly different variable name.

### 4.1 Fuzzy linguistic resolution

The clause \( C_2 \) is a factor of clause \( C_1 \) iff \( C_2 = C_1 \sigma \), where \( \sigma \) is a most general unifier (m.g.u, for short) of some subset \( \{L_1, \ldots, L_k\} \) of \( C_1 \).

**Definition 16.** Given two clauses \((C_1, \alpha_1)\) and \((C_2, \alpha_2)\) without common variables, where \( C_1 = A^a \lor C'_1 \), \( C_2 = A^a \lor C'_2 \). Define the linguistic resolution rule as follows:

\[
\begin{align*}
(A^a \lor C'_1, \alpha_1) & \quad (B^b \lor C'_2, \alpha_2) \\
\hline \\
(C'_1 \gamma \lor C'_2 \gamma, \alpha_3)
\end{align*}
\]

where \( a, b, \) and \( \alpha_3 \) satisfy the following conditions:

\[
\begin{cases}
  a \land b < W, \\
  a \lor b \geq W,
\end{cases}
\]

\( \gamma \) is an m.g.u of \( A \) and \( B \),

\[
\alpha_3 = f(\alpha_1, \alpha_2, a, b),
\]

with \( f \) is a function ensuring that \( \alpha_3 \leq \alpha_1 \), and \( \alpha_3 \leq \alpha_2 \). \( (C'_1 \gamma \lor C'_2 \gamma, \alpha_3) \) is a binary resolvent of \((C_1, \alpha_1)\) and \((C_2, \alpha_2)\). The literals \( A^a \) and \( B^b \) are called literals resolved upon.

In Def. 16, \( \alpha_3 \) is defined so as to be smaller or equal to both \( \alpha_1 \) and \( \alpha_2 \). In fact, the obtained clause is less reliable than original clauses. The function \( f \) is defined as following:

\[
\alpha_3 = f(\alpha_1, \alpha_2, a, b) = \alpha_1 \land \alpha_2 \land (\neg (a \land b)) \land (a \lor b)
\]

(1)

Obviously, \( \alpha_1, \alpha_2 \geq W \), and \( \alpha_3 \) depends on \( a, b \). Additionally, \( a \land b < W \) implies \( \neg (a \land b) > W \). Moreover, \( (a \lor b) \geq W \). Then, by Formula (1), we have \( \alpha_3 \geq W \).

An inference is sound if its conclusion is a logical consequence of its premises. That is, for any interpretation \( I \), if the truth values of all premises are greater than \( W \), the truth value of the conclusion must be greater than \( W \).
Definition 17. A resolvent of clauses $C_1$ and $C_2$ is a binary resolvent of factors of $C_1$ and $C_2$, respectively.

Definition 18. Let $S$ be a clause set. A resolution derivation is a sequence of the form $S_0, \ldots, S_i, \ldots$, where
- $S_0 = S$, and
- $S_{i+1} = S_i \cup \{(C, \alpha)\}$, where $(C, \alpha)$ is the conclusion of a resolution inference with premises $S_i$ based on resolution rule in Def. 16, and $(C, \alpha) \not\in S_i$.

Lemma 4 (Lifting lemma). If $C'_1$ and $C'_2$ are instances of $C_1$ and $C_2$, respectively, and if $C'$ is a resolvent of $C'_1$ and $C'_2$, then there is a resolvent $C$ of $C_1$ and $C_2$ such that $C'$ is an instance of $C$.

Proof. Let $C_1 = A^a \lor C'_1$ and $C_2 = B^b \lor C'_2$.

$C'_1 = C_1 \lor T_1^\beta$, $C'_2 = C_2 \lor T_2^\beta$ (for some m.g.u of $T_1^\beta, T_2^\beta$) and $\gamma$ is a m.g.u of $\alpha, \beta$. Let $C' = C_1 \lor T_1^\gamma$. Then $C'$ is an instance of $C = \gamma \lor (C_1 \lor T_2^\beta)$.

Theorem 2. Let $S$ be a clause set, $S_0, \ldots, S_i, \ldots$ be a resolution derivation. $S$ is unsatisfiable iff there exists $S_i$ containing the empty clause $\square$.

Proof. ($\Rightarrow$) Suppose $S$ is unsatisfiable. Let $A = \{A_1, A_2, \ldots\}$ be the atom set of $S$. Let $T$ be a semantic tree for $S$. By Theo. 1, $T$ has a finite closed semantic tree $T'$.

If $T'$ consists of only one root node, then $\square$ must be in $S$ because no other clauses are falsified at the root of a semantic tree. Thus the theorem is true.

Assume $T'$ consists of more than one node, by Lemm. 2, $T'$ has at least one inference node. Let $N$ be an inference node in $T'$, and let $N_1$ and $N_2$ be the failure nodes immediately below $N$.

Since $N_1$ and $N_2$ are failure nodes but $N$ is not a failure node, there must exist two ground instances $C'_1$ and $C'_2$ of clauses $C_1$ and $C_2$ such that $C'_1$ and $C'_2$ are in $T(N_1)$ and $T(N_2)$, respectively, but both $C'_1$ and $C'_2$ are not falsified by $T(N)$. Therefore, $C'_1$ must contain a literal $A^a$ and $C'_2$ must contain a literal $B^b$ such that $T(A^a) < W$ and $T(B^b) > W$.

Let $C' = (C'_1 - A^a) \lor (C'_2 - B^b)$. $C'$ must be false in $T(N)$ because both $(C'_1 - A^a)$ and $(C'_2 - B^b)$ are false. By the Lifting Lemma we can find a resolvent $C$ of $C_1$ and $C_2$ such that $C'$ is a ground instance of $C$. 


Let $T''$ be the closed semantic tree for $(S \cup \{C\})$ obtained from $T'$ by deleting any node or edge that is below the first node where the resolvent $C'$ is falsified. Clearly, the number of nodes in $T''$ is fewer than that in $T'$. Applying the above process on $T''$, we can obtain another resolvent of clauses in $(S \cup \{C\})$. Putting this resolvent into $(S \cup \{C\})$ we can get another smaller closed semantic tree. This process is repeated until the closed semantic tree consists of only the root node. This is possible only when $\square$ is derived, therefore there is a deduction of $\square$ from $S$.

$(\Leftarrow)$ Suppose there is a deduction of $\square$ from $S$. Let $R_1, \ldots, R_k$ be the resolvents in the deduction. Assume $S$ is satisfiable then there exists $I \models S$. If a model satisfies clauses $C_u$ and $C_v$, it must also satisfy any resolvent of $C_u$ and $C_v$. Therefore $I \models (C_u \land C_v)$. Since resolution is an inference rule then if $I \models (C_u \land C_v)$ then $I \models R_i$ for all resolvents. However, one of the resolvents is $\square$ therefore $S$ must be unsatisfiable. The theorem is proved.

A resolution proof of a clause $C$ from a set of clauses $S$ consists of repeated application of the resolution rule to derive the clause $C$ from the set $S$. If $C$ is the empty clause then the proof is called a resolution refutation. We shall represent resolution proofs as resolution trees. Each tree node is labeled with a clause. There must be a single node that has no child node, labeled with the conclusion clause, we call it the root node. All nodes with no parent node are labeled with clauses from the initial set $S$. All other nodes must have two parents and are labeled with a clause $C$ such that

$$
\begin{array}{c}
\hline
C_1 & C_2 \\
\hline
\end{array}
$$

where $C_1, C_2$ are the labels of the two parent nodes. If $RT$ is a resolution tree representing the proof of a clause with reliability $(C, \alpha)$, then we say that $RT$ has the reliability $\alpha$.

Example 1. Let $AX = (X, G, H, \leq, \neg, \lor, \land, \rightarrow)$ be a lin-HA where $G = \{\perp, \text{False}, \text{W}, \text{True}, \top\}$, $\perp, \text{W}, \top$ are the smallest, neutral, biggest elements, respectively, and $\perp < \text{False} < \text{W} < \text{True} < \top$; $H^+ = \{V,M\}$ and $H^- = \{P,L\}$ ($V=$Very, $M=$More, $P=$Possible, $L=$Less); Consider the clause set after transforming into CNF as following:

1. $A(x)^{\text{MFalse}} \lor B(z)^{\text{MFalse}} \lor C(x)^{\text{PTrue}}$
2. $C(y)^{\text{MFalse}} \lor D(y)^{\text{VTrue}}$
3. $C(t)^{\text{VTrue}} \lor E(t, f(t))^{\text{MFalse}}$
4. $E(a, u)^{\text{True}}$
5. $A(a)^{\text{VTrue}}$
6. $B(a)^{\text{LTrue}}$
7. $D(a)^{\text{MFalse}}$

where $a, b$ are constant symbols; $t, x, y, u, z$ are variables. At the beginning, each clause is assigned to the highest reliability $\top$. We have two of resolution proofs as follows:
\[
\frac{(A(x)^\text{MFalse} \lor B(z)^\text{MFalse} \lor C(x)^\text{PTrue}, \top)}{(A(a)^\text{VTrue}, \top) [a/x]} \quad \frac{(B(z)^\text{MFalse} \lor C(a)^\text{PTrue}, \text{MTrue})}{(B(a)^\text{VTrue}, \top) [a/z]} \\
\frac{(C(a)^\text{PTrue}, \text{LTrue})}{(C(y)^\text{MFalse} \lor D(y)^\text{VMTrue}, \top) [a/y]} \quad \frac{(D(a)^\text{VMTrue}, \text{LTrue})}{(D(a)^\text{MFalse}, \top)}
\]

\[
\frac{(C(y)^\text{MFalse} \lor D(y)^\text{VMTrue}, \top) \quad (D(a)^\text{MFalse}, \top) [a/y]}{(C(a)^\text{MFalse}, \text{MTrue}) \quad (C(t)^\text{VTrue} \lor E(t, f(t))^\text{MFalse}, \top) [a[t]} \\
\frac{(E(a, f(a))^\text{MFalse}, \text{MTrue}) \quad (E(a, u)^\text{True}, \top) [a/u]}{(\Box, \text{LTrue})}
\]

5 Conclusion

We have presented syntax and semantics of our linguistic first order logic system. We based on linear symmetrical hedge algebra to model the truth value domain. To capture the approximate of inference in nature language, each clause in our logic is associated with a reliability. We introduced an inference rule with a reliability which ensures that the reliability of the inferred clause is less than or equal to those of the premise clauses. Based on the algebraic structure of linear symmetrical hedge algebra, resolution in linguistic first order logic will contribute to automated reasoning on linguistic information. It would be worth investigating how to extend our result to other hedge algebra structures and to other automated reasoning methods.

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