Sandpile behavior in discrete water-wave turbulence.

Sergey Nazarenko

Mathematics Institute, University of Warwick, Coventry CV4 7AL, UK

March 30, 2022

Abstract: I construct a sandpile model for evolution of the energy spectrum of the water waves in finite basins. This model takes into account loss of resonant wave interactions in discrete Fourier space and restoration of these interactions at larger nonlinearity levels. For weak forcing, the waveaction spectrum takes a critical $\omega^{-10}$ shape where the nonlinear resonance broadening overcomes the effect of the Fourier grid spacing. The energy cascade in this case takes form of rare weak avalanches on the critical slope background. For larger forcing, this regime is replaced by a continuous cascade and Zakharov-Filonenko $\omega^{-8}$ waveaction spectrum. For intermediate forcing levels, both scalings will be relevant, $\omega^{-10}$ at small and $\omega^{-8}$ at large frequencies, with a transitional region in between characterised by strong avalanches.

Keywords: Water waves, wave turbulence, sandpile models, four-wave resonance, energy and waveaction spectra, discrete Fourier space.

1 Introduction

Importance of the basin finiteness for the statistical evolution of the free water surface was recently argued in [1, 2] and [3]. Later in the present paper, we will derive an estimate (15) according to which water surface waves of steepness $\alpha$ are sensitive to the basin size $L$ if their wavelength $\lambda$ is of the order or greater than $L \alpha^4$. This means that waves with steepness $\alpha \sim 0.1$ and 1 meter wavelength will “feel” the boundaries for for lakes or gulfs up to 10 km wide.

Recent direct numerical simulations of the free water surface in finite basin in presence of gravity revealed a bursty character of the nonlinear energy cascade from small to large wavenumbers [2]. Namely, cascade strengths were measured as functions of time at two different wavenumbers within the inertial range. Intermittent bursts were observed on these graphs, initially arising at the lower wavenumber and then propagating to the higher wavenumber. This behavior reminds sandpile avalanches moving from low to high wavenumbers and their mechanism can be understood as a cycle:
• A cascade arrest due to the wavenumber discreteness leads to accumulation of energy near the forcing scale.
• This leads to widening of the nonlinear resonance.
• Sufficient resonance widening triggers the cascade thereby draining the turbulence levels and returning the system to the beginning of the cycle.

Note that presence of forcing is essential for this scenario and it should not be expected in freely decaying fields, e.g. like in [3]. Below, we present and study a simple model for such a sandpile-like evolution of the water wave spectrum.

2 Differential approximations for waves on an infinite surface.

Evolution of random weakly nonlinear gravity surface waves in basins of infinite size and depth is well described by the Hasselmann kinetic equation [4]. However, for many purposes one can use a simpler differential equation model which preserves many properties of the Hasselmann equation [5–7]:

$$\dot{n} = \frac{C_1}{g^{3/2} \omega^3} \frac{\partial}{\partial \omega} n^4 \omega^{26} \frac{\partial^2}{\partial \omega^2} \frac{1}{n},$$

(1)

where $C_1$ is a dimensionless constant. This equation preserves energy

$$E = \frac{2\pi}{g^2} \int \omega^4 n d\omega$$

(2)

and the waveaction

$$N = \frac{2\pi}{g^2} \int \omega^3 n d\omega.$$  

(3)

Even simpler differential approximation was suggested in [7],

$$\dot{n} = \frac{C_2}{g^{3/2} \omega^3} \frac{\partial}{\partial \omega} (n^2 \omega^{24}),$$

(4)

where $C_2$ is a dimensionless constant. This equation conserved both energy and the waveaction, but it does not have thermodynamic solutions corresponding to equipartition of these quantities.

If we insist that having thermodynamics equilibria is important, but ignore conservation of the waveaction (which can be done at scales smaller, but not larger, than the forcing scale) then we can use another 2nd order differential equation.

$$\dot{n} = \frac{C}{g^{3/2} \omega^4} \frac{\partial}{\partial \omega} \left( n^2 \omega^{24} \frac{\partial}{\partial \omega} (n \omega) \right),$$

(5)
where $C$ is a dimensionless constant. Similar approach in Navier-Stokes turbulence is called Leith model [8, 9]. Like in Leith model [9], we can now find a “warm cascade” solution, i.e. the general stationary solution of (5) which contains both finite flux and finite temperature components,

$$n = \frac{1}{\omega} \left( \frac{g^{3/2}P}{7C} \omega^{-2} + T^3 \right)^{1/3}$$  \hspace{1cm} (6)

where $P$ and $T$ are (dimensional) constants measuring the energy flux and the temperature respectively. For $T = 0$ we recover the pure cascade Zakharov-Filonenko state,

$$n = (P/7C)^{1/3} g^{1/2} \omega^{-8},$$  \hspace{1cm} (7)

and for $P = 0$ we get the pure thermodynamic distribution,

$$n = \frac{T}{\omega}.$$  \hspace{1cm} (8)

3  Finite-basin effects.

Let us now consider waves in a square basin with sides of length $2\pi$, so that the wavenumbers take values on a discrete lattice, $k \in \mathbb{Z}$. The main effect of the finite basin size is in loss of wavenumber resonances due to the wavenumber discreteness [1–3]. Indeed, let us consider the 4-wave resonance conditions,

$$k_1 + k_2 = k_3 + k_4,$$

$$\omega(k_1) + \omega(k_2) = \omega(k_3) + \omega(k_4).$$

(9)  \hspace{1cm} (10)

Two different classes of such solutions were found in [2]: collinear quartets (all four wavevectors are parallel to each other) and “tridents” (first two wavevectors are anti-parallel and the other two are mirror symmetric with respect to the direction of the first two). Parametrisation of the collinear quartets and quintets was done in [16]. Note that the collinear quartets are physically unimportant because of the zero nonlinear coefficient for such wavevectors, but the next order (5-wave) is nontrivial and yields an interesting kinetic equation [16]. The second class, tridents, can be parametrisated as follows [2],

$$k = (a, 0), \quad k_1 = (-b, 0), \quad k_2 = (c, d), \quad k_3 = (c, -d)$$

with

$$a = (l^2 + m^2 + lm)^2, \quad b = (l^2 + m^2 - lm)^2, \quad c = 2lm(l^2 + m^2), \quad d = l^4 - m^4,$$

where $l$ and $m$ are integers. New solutions can be obtained by further rescaling and rotating these tridents by rational angles.

Further, more resonances appear due to the nonlinear resonance broadening even when this broadening is significantly less than the wavenumber grid spacing [2]. However, the total number of both
exact and approximate resonances remain significantly depleted with respect to the continuous case and, therefore, they are inefficient for supporting the turbulent cascade unless the nonlinear resonance broadening becomes of order of the $k$-grid spacing. This allows us to formulate a simplified model for wave turbulence in finite basins as explained in the next section.

4 Wave turbulence in finite basins.

First, we need to evaluate the 4-wave resonance broadening which is of order of the characteristic nonlinear time $\tau_{NL}$. Estimate for $\tau_{NL}$ will be the same if one finds it using (1), (4), (5) or the original Hasselmann’s kinetic equation [4]. It can also be found from a simple dimensional argument and the result is

$$\tau_{NL} \sim g^{10} \omega^{-19} n^{-2}. \tag{11}$$

This corresponds to the resonance broadening in the $k$-space given by

$$\kappa_{NL} = \frac{1}{\partial \omega \tau_{NL}} \sim g^{-11} \omega^{20} n^2. \tag{12}$$

In our model, we will postulate that the wave spectrum will not evolve at $\omega$ if the resonance broadening $\kappa_{NL}$ is less than the $k$-grid spacing $\kappa$, and it will evolve as in the continuous case for $\kappa_{NL} > \kappa$. Such a “frozen turbulence” state was first observed in numerical simulations of the capillary waves [11] and it was later discussed in [2, 12]. One has to be careful, however, not to interpret literally the absence of the spectrum evolution at small amplitudes because a small number of exact resonances do survive for the gravity wave waves (see the previous section) and further resonances may re-appear at resonance broadening which is much less than the $k$-grid spacing [2]. However, the number of such resonant modes is too small to evolve the spectrum efficiently and in our simple model we just put the Heaviside step function $H(\kappa_{NL} - \kappa)$ as a pre-factor to the equation (5) in order to get a model for the spectrum evolution in discrete $k$-space,

$$\dot{n} = \frac{H(\kappa_{NL} - \kappa)}{g^{3/2} \omega^4} \frac{\partial}{\partial \omega} \left( n^2 \omega^{24} \frac{\partial}{\partial \omega} (\omega n) \right) + \gamma(\omega)n. \tag{13}$$

Here, we have added a function $\gamma(\omega)$ which models forcing at low $\omega$’s (e.g. by wind) and dissipation at high $\omega$’s (by wavebreaking) and which, in principle, can be a function of $n$. We will assume that in between of the forcing and dissipation scales there exists an inertial range where $\gamma \approx 0$. From now on, we ignore the dimensionless order-one pre-factor $C$ since our resonance broadening is given by an order-of-magnitude estimate.

5 Behavior predicted by the model.

Equation (13) with $\kappa_{NL}$ given by (12) will be our master model for the water-wave turbulence spectrum in a finite basin. Let us qualitatively consider the consequences of this model. Let us assume that
initially there is no waves in the basin and let us start forcing the system at low frequencies. Then, there will be no transfer over scales and the spectrum will grow with the growth rate $\gamma$ until it reaches the critical value where $\kappa_{NL} = \kappa$. After that the nonlinear transfer will get activated and the energy will spill into the adjacent range of frequencies. If the forcing is so weak that its characteristic $\tau_F = 1/\max\{\gamma(\omega)\}$ is much longer than the characteristic nonlinear time $\tau_{NL}$ then the level of turbulence will never greatly exceed its critical value and the critical spectrum with $\kappa_{NL} \approx \kappa$ will gradually occupy the entire inertial range. Condition $\kappa_{NL} \approx \kappa$ gives for the critical spectrum

$$n_c \sim g^{11/2} \kappa^{1/2} \omega^{-10}. \quad (14)$$

It is useful to re-write this relation in terms of the water surface angle $\alpha$ characterising the wave steepness,

$$\alpha_c \sim (\lambda/L)^{1/4}, \quad (15)$$

where $L$ is the box size and $\lambda$ is the wavelength. One can interpret this relation as an expression for the minimal steepness for which the finite box effect can be ignored. For example, for a box containing $10^4$ wavelengths the finite box effects can be ignored only for $\alpha > 0.1$.

If the forcing is stochastic then the subsequent evolution will consist of small avalanches going down the critical slope with time intervals $\Delta t$ greater than the time $\sim \tau_{NL}$ needed (according to (13)) for the avalanche to travel from the forcing to the dissipation scale. Note that this is a classical condition for sandpile models. To be specific, let us consider a type of forcing such that after each interval $\Delta t$ we add an increment $(\Delta a)e^\phi$ where phase $\phi$ is random and uniform in $(0, 2\pi]$ and $a$ is a small positive value, $\Delta a \ll \sqrt{n_c}$. If at some moment of time $n = n_c$ at the forcing scales $k \in (k_F, k_F + \Delta k)$ then with probability $1/2$ the spectrum will get greater than critical in the forcing range after time interval $\Delta t$ and approximately $n = n_c + \Delta a \sqrt{n_c}$. For $\Delta t \gg \tau_{NL}$, such disturbance will have enough time to travel/diffuse away before the next spectrum disturbance might appear at $k_F$ after another $\Delta t$ interval. Thus, the evolution of each super-critical disturbance can be treated separately. Because each of such disturbances is small, one can use the linearised version of the evolution equation (13),

$$\dot{n} = \kappa g^{19/2} \omega^{-4} \frac{\partial}{\partial \omega} \left( \omega^4 \frac{\partial}{\partial \omega} (\omega n) \right), \quad (16)$$

with initial condition

$$n|_{t=0} = \Delta a \sqrt{n_c(\omega_F)} \quad \text{for } \omega \in \omega_F + \Delta \omega, \quad \text{and } n = 0 \text{ otherwise}. \quad (17)$$

Equation (16) can be re-written as

$$\dot{\epsilon} = \kappa g^{19/2} \left( 4 \frac{\partial \epsilon}{\partial \omega} + \omega \frac{\partial^2 \epsilon}{\partial \omega^2} \right), \quad (18)$$

where $\epsilon = \omega n$ is the spectral energy density. According to this equation, the disturbance generated by forcing will propagate toward high $k$ with speed $4\kappa g^{19/2}$ while getting diffused at an increased rate (due
to moving to higher $\omega$’s). The stationary solution of (18) decays as $\epsilon \sim \omega^{-3}$ or $n \sim \omega^{-4}$, i.e. significantly slower than $n_c \sim \omega^{-10}$. Therefore, for a long enough inertial range the linear approximation will fail at some $\omega = \omega^*$ and the critical slope $n \sim \omega^{-10}$ will be replaced by the Zakharov-Filonenko slope $n \sim \omega^{-8}$ for $\omega > \omega^*$. The transitional range with $\omega \sim \omega^*$ will be characterised by strong avalanches.

For stronger forcing, transition to the Zakharov-Filonenko spectrum occurs at lower frequencies or even right at the forcing scale if $\omega_F > \omega^*$ (i.e. when $\alpha$ at the forcing scale is steeper than $\alpha_c$ at this scale). In numerical simulations, efforts are typically made to overcome “frozen turbulence” and generate the cascade. At the present level of resolution (up to $512^2$ modes) this goal can be achieved with only partial success because, according to estimate (15), the condition that turbulence is not frozen can be only be marginally reconciled with the condition for the Wave Turbulence theory to work, $\alpha < 1$. Thus, in all existing simulations (e.g. [1–3, 13–15]) turbulence, although not frozen, was still quite sensitive to the finite box effects. This state was named mesoscopic turbulence in [3]. In presence of forcing, it shows up via strong cascade avalanches coexisting with Zakharov-Filonenko state occupying about a decade long wavenumber interval.

6 Discussion

In this paper, we presented an evolution model for the spectrum of gravity water waves in finite basins. It has the following features:

- The model is given by a nonlinear second order equation in Fourier space.

- The model has the cascade Zakharov-Filonenko spectrum and the thermodynamic spectrum among its solutions. It also has a general stationary solution where both the flux and the temperature effects are present.

- The model takes into account the $k$-space discreteness by switching off the nonlinear evolution when the spectrum falls below a critical value. The critical spectrum is determined by the condition that the nonlinear resonance widening is equal to the $k$-grid spacing.

Based on this model, we established that for very weak forcing the spectrum takes the critical slope $n \sim \omega^{-10}$ with occasional weak “avalanches” running down this slope. For larger forcing the system does not feel discreteness and the spectrum takes the Zakharov-Filonenko form, $n \sim \omega^{-8}$. For intermediate levels of forcing, the spectrum may have the $-10$ exponent at low frequencies and the $-8$ at large frequencies within the inertial range. Such intermediate case is characterised by strong avalanches down the mean spectral slope, a feature observed in recent numerical simulations of the free water surface [2].

It is interesting that steeper than Kolomogorov slopes were also previously obtained for waves with narrow-band forcing [17]. The narrow band forcing also leads to quasi-discrete character of the mode excitations. To conclude, it is worth mentioning that there are plenty of other examples of dispersive waves whose resonant interaction may be affected by the finite size box and where one could expect
similar avalanche-like behaviour. Interestingly, irrespective to discreteness, the sandpile analogy have also been previously invoked in the wave turbulence context to illustrate sudden readjustments necessary to balance the turbulence sources and sinks [18].

References

[1] M. Tanaka, N. Yokoyama, Effects of discretization of the spectrum in water-wave turbulence, Fluid Dynamics Research 34 (2004) 199-216.
[2] Yu. Lvov, S.V. Nazarenko and B. Pokorni, Water-wave turbulence: statistics beyond the spectra, arXiv: math-ph/050705 (2005).
[3] V.E. Zakharov, A.O. Korotkevich, A.N. Pushkarev and A.I. Dyachenko, Mesoscopic wave turbulence, arXiv: physics/0508155 (2005).
[4] K. Hasselmann, J. Fluid Mech 12, 481 (1962).
[5] S. Hasselmann and K. Hasselmann, Computations and parametrizations of the nonlinear energy transfer in gravity wave spectrum. Part 1, J. Phys. Oceanogr., 15, 1369-1377 (1985).
[6] R.S. Iroshnikov, Possibility of a non-isotropic spectrum of wind waves by their weak interaction, Soviet Phys. Dokl, 30, 126-128 (1985).
[7] V.E. Zakharov, A.N. Pushkarev, Diffusion model of interacting gravity waves on the surface of deep fluid, Nonlin. Proc. Geophys. 6 (1) 1-10 (1999)
[8] C. Leith, Phys. Fluids 10, 1409 (1967); Phys. Fluids 11, 1612 (1968).
[9] C. Connaughton and S. Nazarenko, Warm Cascades and Anomalous Scaling in a Diffusion Model of Turbulence, Phys. Rev. Lett. 92, 044501 (2004)
[10] V.E.Zakharov and Filonenko, J. Appl. Mech. Tech. Phys. 4, 506-515 (1967).
[11] A.N. Pushkarev, On the Kolmogorov and frozen turbulence in numerical simulation of capillary waves, Eur. J. Mech. B/Fluids 18, 345-352 (1999)
[12] C.Connaughton, S.Nazarenko, A. Pushkarev, Discreteness and quasiresonances in weak turbulence of capillary waves, Physical Review E, 63, 046306, (2001).
[13] M. Onorato et al., Freely decaying weak turbulence for sea surface gravity waves, PRL 89 No.14, Sept 2002.
[14] N. Yokoyama, Statistics of Gravity Waves obtained by direct numerical simulation, JFM 501, 169-178 (2004).
[15] A.N. Pushkarev, D. Resio, V.E. Zakharov, Weak turbulent approach to the wind-generated gravity sea waves, Physica D 184 (1-4) 29-63 (2003)

[16] A.I. Dyachenko, Y.V. Lvov, V.E. Zakharov, Five-wave interaction on the surface of deep fluid, Physica D 87 (1-4) 233-261 (1995).

[17] V.E. Zakharov, V.S. L'vov, G.E. Falkovich, Kolmogorov Spectra of Turbulence, Springer, Berlin, 1992.

[18] A.C. Newell, private communication.