ON THE DLW CONJECTURES

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Abstract. In 2007, Dmytrenko, Lazebnik and Williford posed two related conjectures about polynomials over finite fields. Conjecture 1 is a claim about the uniqueness of certain monomial graphs. Conjecture 2, which implies Conjecture 1, deals with certain permutation polynomials of finite fields. Two natural strengthenings of Conjecture 2, referred to as Conjectures A and B in the present paper, were also insinuated. In a recent development, Conjecture 2 and hence Conjecture 1 have been confirmed. The present paper gives a proof of Conjecture A.

1. Introduction

Let \( F_q \) denote the finite fields with \( q \) elements. For \( f, g \in F_q[X, Y] \), \( G_q(f, g) \) is a bipartite graph with vertex partitions \( P = F_q^3 \) and \( L = F_q^3 \) and edges defined as follows: a vertex \( (p_1, p_2, p_3) \in P \) is adjacent to a vertex \( [l_1, l_2, l_3] \in L \) if and only if

\[
   p_2 + l_2 = f(p_1, l_1) \quad \text{and} \quad p_3 + l_3 = g(p_1, l_1).
\]

The graph \( G_q(f, g) \) is called a polynomial graph, and when \( f \) and \( g \) are both monomials, it is called a monomial graph. Polynomial graphs were introduced by Lazebnik, Ustimenko and Woldar in [6] to provide examples of dense graphs of high girth. In particular, the monomial graph \( G_q(XY, XY^2) \) has girth 8, and its number of edges achieves the maximum asymptotic magnitude of the function \( g_3(n) \) as \( n \to \infty \), where \( g_k(n) \) is the maximum number of edges in a graph of order \( n \) and girth \( \geq 2k + 1 \). (For surveys on the function \( g_k(n) \), see [1, 3].)

Let \( q = p^e \), where \( p \) is an odd prime and \( e \geq 0 \). It was proved in [2] that every monomial graph of girth \( \geq 8 \) is isomorphic to \( G_q(XY, X^kY^{2k}) \) for some \( 1 \leq k \leq q - 1 \), and the following conjecture was posed in [2]:

**Conjecture 1.** ([2 Conjecture 4]) Every monomial graph of girth 8 is isomorphic to \( G_q(XY, XY^2) \).

To prove Conjecture 1, it suffices to show that if \( 1 \leq k \leq q - 1 \) is such that \( G_q(XY, X^kY^{2k}) \) has girth \( \geq 8 \), then \( k \) is a power of \( p \).

A polynomial \( h \in F_q[X] \) is called a permutation polynomial (PP) of \( F_q \) if the mapping \( x \mapsto h(x) \) is a permutation of \( F_q \). For \( 1 \leq k \leq q - 1 \), let

\[
   A_k = X^k [(X + 1)^k - X^k] \in F_q[X]
\]

and

\[
   B_k = [(X + 1)^{2k} - 1]X^{q-1-k} - 2X^{q-1} \in F_q[X].
\]

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It was proved in [2] that if $1 \leq k \leq q - 1$ is such that $G_q(XY, X^kY^{2k})$ has girth $\geq 8$, then both $A_k$ and $B_k$ are PPs of $\mathbb{F}_q$. Consequently, a second conjecture was proposed:

**Conjecture 2.** ([2] Conjecture 16) If $1 \leq k \leq q - 1$ is such that both $A_k$ and $B_k$ are PPs of $\mathbb{F}_q$, then $k$ is a power of $p$.

Note that if $k$ is a power of $p$, then $A_k$ and $B_k$ are clearly PPs of $\mathbb{F}_q$. Obviously, Conjecture 2 implies Conjecture 1. Although the polynomials $A_k$ and $B_k$ are both related to the graph $G_q(XY, X^kY^{2k})$, it is not clear how they are related to each other. Therefore, it is natural to consider the polynomials $A_k$ and $B_k$ separately, giving rise to the following two stronger versions of Conjecture 2; see [4] [5] [7].

**Conjecture A.** Assume that $1 \leq k \leq q - 1$. Then $A_k$ is a PP of $\mathbb{F}_q$ if and only if $k$ is a power of $p$.

**Conjecture B.** Assume that $1 \leq k \leq q - 1$. Then $B_k$ is a PP of $\mathbb{F}_q$ if and only if $k$ is a power of $p$.

We refer to all above conjectures as the DLW conjectures (after the authors of [2]). A breakthrough on these conjectures came recently when Conjecture 2 and hence Conjecture 1 were proved in [4]. However, Conjectures A and B remained unsolved. The purpose of the present paper is to give a proof of Conjecture A.

We rely on several previous results on the polynomial $A_k$ from [4]. A summary of these results is given in Section 2. Section 3 is devoted to the proof of Conjecture A.

2. Previous Results on $A_k$

Recall that $q = p^e$, where $p$ is an odd prime and $e > 0$, and $1 \leq k \leq q - 1$. Congruence of integers modulo $p$ is denoted by $\equiv_p$. For each integer $a > 0$, let $a^* \in \{1, \ldots, q - 1\}$ be such that $a^* \equiv a \pmod{q - 1}$; in addition, we define $0^* = 0$. We will need the following known facts about the polynomial $A_k$ for the proof of Conjecture A; the proofs of these facts can be found in [4].

**Fact 2.1.** $A_k$ is a PP of $\mathbb{F}_q$ if and only if $\gcd(k, q - 1) = 1$ and

$$\sum_{1 \leq i \leq q - 2} (-1)^i \binom{s}{i} \binom{(ki)^*}{(2ks)^*} \equiv_p 0 \quad \text{for all } 1 \leq s \leq q - 2.$$  

**Fact 2.2.** Assume that $A_k$ is a PP of $\mathbb{F}_q$ and let $k' \in \{1, \ldots, q - 2\}$ be such that $k' \equiv 1 \pmod{q - 1}$. Then all the base $p$ digits of $k'$ are 0 or 1.

**Fact 2.3.** Conjecture A is true for $q = p^e$, where $e = 1$ or the greatest prime divisor of $e$ is $\leq p - 1$. In particular, Conjecture A is true for $q = p^2$ with $e \leq 2$.

3. Proof of Conjecture A

We first restate equation (2.1) in terms of $k'$ defined in Fact 2.2.

**Lemma 3.1.** $A_k$ is a PP of $\mathbb{F}_q$ if and only if $\gcd(k, q - 1) = 1$ and

$$\sum_{2 \leq i \leq q - 2} (-1)^i \binom{(k's)^*}{(ki)^*} \binom{i}{2s} \equiv_p 0, \quad 1 \leq s \leq (q - 1)/2,$$

$$\sum_{2 \leq i \leq q - 2} (-1)^i \binom{(k'(s + (q - 1)/2))^*}{(ki)^*} \binom{i}{2s} \equiv_p 0, \quad 1 \leq s < (q - 1)/2.$$
Proof. By Fact 2.1 we only have to show that (2.1) is equivalent the combination of (3.1) and (3.2). Replacing \( s \) by \((k's)^*\) and \( i \) by \((k'i)^*\) in (2.1) gives

\[
\sum_{1 \leq i \leq q - 2} (-1)^i \binom{(k's)^*}{i} \binom{i}{2s}^* \equiv_p 0, \quad 1 \leq s \leq q - 2.
\]

Note that (3.1) is (3.3) with \( 1 \) of (3.1) and (3.2). Replacing \( s \)

**Proof.** Assume that (3.2) is \((q - 1)/2 \leq \( s < q - 1 \).

For each integer \( l \geq 0 \), write \( l^* = \sum_{i=0}^{e-1} l_ip^i \), \( 0 \leq l_i \leq p - 1 \), and define

\[
d(l) = (l_0, \ldots, l_{e-1})
\]

and

\[
supp(l) = \{0 \leq i \leq e - 1 : l_i > 0\}.
\]

For \( \alpha = (\alpha_0, \ldots, \alpha_{e-1}), \beta = (\beta_0, \ldots, \beta_{e-1}) \in \mathbb{Z}^e \), the congruence \( \alpha \equiv \beta \pmod{p - 1} \) means that \( \alpha_i \equiv \beta_i \pmod{p - 1} \) for all \( 0 \leq i \leq e - 1 \).

**Lemma 3.2.** Assume that \( e \geq 3 \). Let \( 0 < t \leq e - 1 \), \( 0 \leq u, v \leq (p - 1)/2 \), \( 2s = q - 1 - 2(u + vp') \), and \( l = \sum_{i=0}^{e-1} l_ip^i \), where \( l_i \in \{0, 1\} \), and \( (l_0, \ldots, l_{e-1}) \neq (1, \ldots, 1) \). Then

\[
\sum_{2 \leq i \leq q - 2} (-1)^i \binom{(l(s + (q - 1)/2))^*}{(li)^*} \binom{i}{2s} \equiv_p \sum_{0 \leq a \leq 2u, \ 0 \leq b \leq 2v} (-1)^{(a+b+u+v)(x+y)} \binom{a}{u}^x \binom{b}{v}^y \binom{a + b}{u + v}^y \binom{2u}{a} \binom{2v}{b},
\]

where

\[
y = |supp(l) \cap supp(p'l)| = |\{0 \leq i \leq e - 1 : l_i = l_{i-1} = 1\}|,
\]

\[
x = |supp(l) \setminus supp(p'l)| = \sum_{i=0}^{e-1} l_i - y,
\]

and the subscript of \( l_{i-1} \) is taken modulo \( e \).

**Proof.** 1° We have

\[
d(2s) = (p - 1 - 2u, p - 1, \ldots, p - 1, p - 1 - 2v, p - 1, \ldots, p - 1).
\]

Hence, if \( 2 \leq i \leq q - 2 \) is such that \( \binom{i}{2s} \neq p \), we must have

\[
d(i) = (p - 1 - a, p - 1, \ldots, p - 1, p - 1 - b, p - 1, \ldots, p - 1),
\]

where \( 0 \leq a \leq 2u \) and \( 0 \leq b \leq 2v \); in this case,

\[
\binom{i}{2s} = p \binom{p - 1 - a}{p - 1 - 2u} \binom{p - 1 - b}{p - 1 - 2v} \equiv_p \binom{-(a + 1)}{2u - a} \binom{-(b + 1)}{2v - b} = (-1)^{a+b} \binom{2u}{a} \binom{2v}{b}.
\]

2° We claim that when (3.6) is satisfied,

\[
\binom{(l(s + (q - 1)/2))^*}{(li)^*} \equiv_p (-1)^{(a+b+u+v)(x+y)} \binom{a}{u}^x \binom{b}{v}^y \binom{a + b}{u + v}^y.
\]
For the sake of notational convenience, we write

\[(3.9) \quad d(l) = (1, \ldots, 1, 1, \ldots, 1, 0, \ldots, 0, 0, \ldots, 0), \]
\[(3.10) \quad d(p^i l) = (0, \ldots, 0, 1, \ldots, 1, 1, \ldots, 1, 0, \ldots, 0); \]

In doing so, we no longer maintain the alignment of the components in \((3.9)\) and \((3.10)\) with the powers of \(p\). Since \(l(s + (q - 1)/2) \equiv -(u + vp^i)l \pmod{q - 1}\), we have

\[(3.11) \quad d(l(s + (q - 1)/2)) = \underbrace{p - 1 - u, \ldots, p - 1 - u,}_{x} \underbrace{p - 1 - (u + v), \ldots, p - 1 - (u + v),}_{y} \quad \underbrace{p - 1 - v, \ldots, p - 1 - v,}_{x} \quad \underbrace{p - 1, \ldots, p - 1.}_{y}\]

We first assume that \(a + b \leq p - 1\). Since \(li \equiv -(a + bp^i) \pmod{q - 1}\), we have

\[(3.12) \quad d(li) = \underbrace{p - 1 - a, \ldots, p - 1 - a,}_{x} \underbrace{p - 1 - (a + b), \ldots, p - 1 - (a + b),}_{y} \quad \underbrace{p - 1 - b, \ldots, p - 1 - b,}_{x} \quad \underbrace{p - 1, \ldots, p - 1.}_{y}\]

Therefore

\[(3.13) \quad \left(\frac{l(s + (q - 1)/2)}{li}\right)^* \equiv_p \left(p - 1 - u\right)^x \left(p - 1 - v\right)^x \left(p - 1 - (a + b)\right)^y.\]

In the above,

\[(3.14) \quad \left(p - 1 - u\right) \equiv_p (a)^u,\]
\[(3.15) \quad \left(p - 1 - v\right) \equiv_p (b)^v,\]
\[(3.16) \quad \left(p - 1 - (a + b)\right) \equiv_p (a + b)^{u + v}.\]

Hence

\[\left(\frac{l(s + (q - 1)/2)}{li}\right)^* \equiv_p (-1)^{(a+u)x} \left(a\right)^x \left(-1\right)^{(b+v)x} \left(b\right)^x \left(-1\right)^{(a+b+u+v)y} \left(a + b\right)^y \quad \left(a + b\right)^{u + v}.\]

If \(a + b > p - 1\) but \(y = 0\), the above computation also gives \((3.8)\).

Now assume that \(a + b > p - 1\) and \(y > 0\). Then

\[\left(\frac{a + b}{u + v}\right) \equiv_p \left(a + b - p\right) = 0.\]
since \(0 \leq a + b - p \leq 2(u + v) - p < u + v\). It remains to show that the left side of (3.8) is also \(\equiv_p 0\). We have

\[
d(l) = \left( \frac{p - 1 - a, \ldots, p - 1 - a, p - 1 - (a + b), \ldots, p - 1 - (a + b)}{x}, \frac{p - 1 - b, \ldots, p - 1 - b, p - 1, \ldots, p - 1}{y} \right) \pmod{p - 1}.
\]

(3.17)

We remind the reader that for notational convenience, the components of the right side of (3.17) have not been aligned with the powers of \(p\). Now, however, it necessary to align these components with the powers of \(p\) since carries in base \(p\) will be considered.

**Case 1.** Assume that at least one component of the right side of (3.17) is \(> 0\). Since \(y > 0\), we may write

\[
d(l) = (\alpha_1, \ldots, \alpha_m) \pmod{p - 1},
\]

where each \(\alpha_j\) \((1 \leq j \leq m)\) is a block of the form

\[
\alpha_j = (p - 1 - (a + b), \ddagger, \ldots, \ddagger, \epsilon_j, *, \ldots, *),
\]

where each \(\ddagger\) is either \(p - 1 - (a + b)\) or 0, each \(*\) belongs \(\{0, \ldots, p - 1\}\), and \(0 < \epsilon_j \leq p - 1\). It follows that

\[
d(l) = (\alpha'_1, \ldots, \alpha'_m),
\]

where

\[
\alpha'_j = (2p - 1 - (a + b), *, \ldots, *, \epsilon_j - 1, *, \ldots, *).
\]

Align the components of (3.11) with those of (3.18). This gives

\[
d(l(s + (q - 1)/2)) = (p - 1 - (u + v), *, \ldots, *).
\]

Therefore

\[
((l(s + (q - 1)/2))^\ast \pmod{(l)^\ast}) \equiv_p (p - 1 - (u + v)) \cdot \ldots = 0
\]

since \(2p - 1 - (a + b) \geq 2p - 1 - 2(u + v) > p - 1 - (u + v)\).

**Case 2.** Assume that every component of the right side of (3.17) is either \(p - 1 - (a + b)\) or 0. Since \((i_0, \ldots, i_{e-1}) \neq (1, \ldots, 1)\), we have \(y < e\) and hence \(x > 0\). It follows from (3.17) that \(a = b = p - 1\). Therefore we may write

\[
d(l) = (0, \ddagger, \ldots, \ddagger) \pmod{p - 1},
\]

where each \(\ddagger\) is either 0 or \(p - (a + b)\). Hence

\[
d(l) = (p - 1, *, \ldots, *)
\]

(3.19)

Align the components of (3.11) with those of (3.19). Without loss of generality, we may write

\[
d(l(s + (q - 1)/2)) = (p - 1 - u, *, \ldots, *).
\]

Since \(a/2 \leq u \leq (p - 1)/2\), we have \(u = (p - 1)/2\). Therefore,

\[
((l(s + (q - 1)/2))^\ast \pmod{(l)^\ast}) \equiv_p (p - 1 - u) \cdot \ldots = 0
\]

3. Equation (3.3) follows from (3.7) and (3.8). (Note that \((-1)^i = (-1)^{a+b}\) by (3.6).)
Proof of Conjecture A. Assume that $A_k$ is a PP of $\mathbb{F}_q$. We show that $k$ is a power of $p$, equivalently, $k'$ is a power of $p$. By Fact 2.3 we may assume that $e \geq 3$. By Fact 2.2 $k' = \sum_{i=0}^{e-1} l_i p^i$, where $l_i \in \{0, 1\}$. Assume to the contrary that $\sum_{i=0}^{e-1} l_i > 1$. Since $\gcd(k', q - 1) = 1$, we have $(l_0, \ldots, l_{e-1}) \neq (1, \ldots, 1)$. We follow the notation of Lemma 3.2. In Lemma 3.2 let $u = v = (p - 1)/2$ and $l = k'$. Moreover, choose $0 < t \leq e - 1$ such that $y > 0$; this is possible since $l_i = 1$ for at least two $i$. Also note that $s = (q - 1)/2 - (u + vp^t) < (q - 1)/2$. Now we have (3.20)

\[
0 \equiv_p \sum_{2 \leq i \leq q - 2} (-1)^i \left(\frac{k'(s + (q - 1)/2)}{(k')^i}\right) \left(\begin{array}{c} i \\ 2s \end{array}\right) \quad \text{(by 3.2)}
\]

\[
\equiv_p \sum_{0 \leq a \leq 2b \leq 2v} (-1)^{a+b+u+v}(x+y) \left(\begin{array}{c} a \\ u \end{array}\right)^x \left(\begin{array}{c} b \\ v \end{array}\right)^y \left(\begin{array}{c} 2u \\ a \end{array}\right)^2 \left(\begin{array}{c} 2v \\ b \end{array}\right)^2 \quad \text{(by 3.4)}
\]

\[
= \sum_{(p-1)/2 \leq a, b \leq p-1} (-1)^{a+b} \left(\begin{array}{c} a \\ (p-1)/2 \end{array}\right)^x \left(\begin{array}{c} b \\ (p-1)/2 \end{array}\right)^y \left(\begin{array}{c} p-1 \\ a \end{array}\right)^x \left(\begin{array}{c} p-1 \\ b \end{array}\right)^y.
\]

(For that last step of (3.20), note that $x + y$ is odd since $\gcd(k', q - 1) = 1$.) For $(p - 1)/2 \leq a, b \leq p - 1$, $(a+b \neq p$ only if $a = b = (p - 1)/2$. Hence

the right side of (3.20)\[\equiv_p \left(\frac{p-1}{(p-1)/2}\right)^2 = 1 \neq_p 0,\]

which is a contradiction. \[\square\]

Remark. The above proof of Conjecture A uses Lemma 3.2 only for $u = v = (p - 1)/2$. We choose to present Lemma 3.2 in a more general setting in anticipation of possible applications of the result in related problems.

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