Remarks on supersymmetry of quantum systems with position-dependent effective masses

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Abstract

We apply the supersymmetry approach to one-dimensional quantum systems with spatially-dependent mass, by including their ordering ambiguities dependence. In this way we extend the results recently reported in the literature. Furthermore, we point out a connection between these systems and others with constant masses. This is done through convenient transformations in the coordinates and wavefunctions.

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Recently some of the authors of the present work did an analysis of the classification of quantum systems with position-dependent mass regarding their exact solvability \cite{1}. On a similar basis Plastino et al. \cite{2}, applied the supersymmetric quantum-mechanical approach to such systems, corresponding to effective theories related to some solid state problems. In that paper the authors considered the following kind of Shrödinger equation

\[
-\hbar^2 \nabla^2 \frac{1}{2m(\vec{r})} \nabla + V(\vec{r}) \psi(\vec{r}) = E \psi(\vec{r}),
\]

(1)

and succeed to show that some one-dimensional systems with position-dependent effective mass have a supersymmetric partner system with the same effective mass. They were also able to solve exactly some particular cases by constructing the superpotential from the form of the effective mass \(m(x)\) and generalizing the concept of shape invariance. However, this kind of physical problem is intrinsically ambiguous \cite{1, 3-4}, and consequently the above Shrödinger equation is a particular case of the most general Hamiltonian, as originally proposed by von Roos \cite{3},

\[
H_{VR} = -\frac{\hbar^2}{4} \left[ m^\delta(\vec{r}) \nabla m^\kappa(\vec{r}) \nabla m^\lambda(\vec{r}) + m^\lambda(\vec{r}) \nabla m^\kappa(\vec{r}) \nabla m^\delta(\vec{r}) \right] + V(\vec{r}),
\]

(2)

whose classical limit is identical to the first one, and the parameters are constrained by the condition \(\delta + \kappa + \lambda = -1\). Here we intend to extend the results of Plastino et al., in order to accommodate this more general situation.

In fact, the problem of ordering ambiguity is a long standing one in quantum mechanics. Some of the founders of quantum mechanics as Born, Jordan, Weyl, Dirac and von Neumann worked on this matter, see for instance the excellent critical review by Shewell \cite{4}. There are many examples of physically important systems for which such an ambiguity is quite relevant. For instance, we can cite the problem of impurities in crystals \cite{5, 6, 7}, the dependence of nuclear forces on the relative velocity of the two nucleons \cite{3, 4}, and more recently the study of semiconductor heterostructures \cite{10, 11}. In addition, taking into account the spatial variation of the semiconductor type, some effective Hamiltonians were proposed with a position-dependent mass for the carrier \cite{12-17}.

More recently, Lévy-Leblond \cite{18}, when discussing the case of discontinuous masses, argued that there is a privileged ordering, namely the one given in equation (1). This was achieved by choosing the continuity of the wave function \(\psi(x, t)\) and its derivative divided by the mass \(\frac{1}{m(x)} \frac{\partial \psi}{\partial x}\). At this point it is interesting to quote a very recent work by Dekar et al \cite{19}, where this ambiguity is still providently taken into account from the beginning and then compared with the result of Lévy-Leblond. Those authors call the attention to the fact that, even if one accept the continuity condition used by Lévy-Leblond \cite{18}, there still remains the question regarding the universality of this choice, as in the case of continuous masses. As far as we know no one has proven such universality. On the other hand, Morrow and Brownstein \cite{20}, also addressing abrupt heterojunctions, had concluded that for the Hamiltonian given in equation (2) the continuous quantities are

\[
m(x)^\delta \psi(x) \quad \text{and} \quad m(x)^{(\delta + \kappa)} \frac{\partial \psi}{\partial x},
\]

(3)
where \(2 \delta + \kappa = -1\). This shows that there exists at least a controversy about the possibility of removing the ambiguity, and that seems to be intimately linked to the choice of the continuity condition. Moreover, Henderson et al. [21] suggested some experiments, measuring the amplitude index of refraction, in order to determine the ordering parameters \((\delta, \kappa, \lambda)\) at the discontinuities, once their values seem to be dependent of the type of interfaces.

Finally, as it was seen in [18] and [1], it is possible to include all the ambiguity into an effective potential and, since the potentials play no role in the process of constructing a continuity equation (conservation of the probability), it is easy to verify that the conservation of the probability is absolutely unambiguous. Therefore, it is at least doubtful that one could use any continuity condition to get rid of the ambiguity. So it should be important to extend the supersymmetric approach in such a way that these ambiguities could be taken into account.

The extension of the supersymmetric approach devised in [2], capable of generating those cases can be done by starting from the following generalized ladder operators

\[
A = i a \left( m^\alpha \hat{p} m^\beta + b m^\beta \hat{p} m^\alpha \right) + \tilde{W}, \\
A^\dagger = -i a \left( m^\beta \hat{p} m^\alpha + b m^\alpha \hat{p} m^\beta \right) + \tilde{W},
\]

where \(\hat{p} = -i \hbar \frac{d}{dx}\) is the momentum operator acting on all factors to the right, \(\tilde{W}\) is the generalized quantum superpotential and \(\alpha, \beta, a, b\) are arbitrary parameters related by \(a = \frac{1}{\sqrt{2(b+1)}}, \alpha + \beta = -\frac{1}{2}\).

The supersymmetric partner Hamiltonians \(H_1 = A^\dagger A\) and \(H_2 = AA^\dagger\) can be written as

\[
H_i = \frac{1}{2m} \left[ \hat{p}^2 + i \hbar \left( \frac{m'}{m} \right) \hat{p} \right] + V_i
\]

where \(m' = \frac{dm}{dx}\), and \(i = 1, 2\). One can verify that the case studied by Plastino et al. is recovered by choosing \(\alpha = -\frac{1}{2}\) and \(b = 0\). In the above equations, we have defined the superpartner potentials respectively as

\[
V_1 = \tilde{W}^2 - \frac{\hbar}{\sqrt{2m(b+1)}} \left[ (\beta - \alpha)(b-1) \left( \frac{m'}{m} \right) \tilde{W} + (b+1) \tilde{W}' \right] - \frac{\hbar^2}{2m(b+1)^2} (\beta + b\alpha) \left\{ (b+1) \left( \frac{m''}{m} \right) + [\beta(2b+1) + \alpha(b+2) - (b+1)] \left( \frac{m'}{m} \right)^2 \right\},
\]

and

\[
V_2 = \tilde{W}^2 + \frac{\hbar}{\sqrt{2m(b+1)}} \left[ (\alpha - \beta)(b-1) \left( \frac{m'}{m} \right) \tilde{W} + (b+1) \tilde{W}' \right] + \frac{\hbar^2}{2m(b+1)^2} (\alpha + b\beta) \left\{ (b+1) \left( \frac{m''}{m} \right) + [\alpha(2b+1) + \beta(b+2) - (b+1)] \left( \frac{m'}{m} \right)^2 \right\}.
\]

Now, as an example, we apply the above results to the particular case of potentials with the harmonic oscillator spectrum, similarly to which was done in [2]. The shape invariance is
guaranteed by the condition $V_2(x; k) = V_1(x; k) + k$, where $k$ is a uniform energy shift. This lead us to the following equation obeyed by the superpotential $\tilde{W}$,

$$\sqrt{\frac{2\hbar}{m'}} \tilde{W}' + \frac{\hbar^2}{2m} \left( \frac{b - 1}{b + 1} \right) \left( \frac{1}{2} + 2\alpha \right) \left[ m'' + \frac{3}{2} \left( \frac{m'}{m} \right)^2 \right] = k. \quad (8)$$

Note that the shape invariance as imposed implies that the partner potentials differ only by a constant term $k$. So the expression of $V_2$ can be written through creation and annihilation operators as those written in above. Consequently, one can repeat the above procedure again and again, obtaining $V_{j+1}(x, a_j) = V_j(x, a_{j+1}) + k$,

where $a_j$ stands for the potential and ambiguity parameters, and the energy for the case of the harmonic oscillator type potential [2] is given by $E_n = n k$.

Finally, the corresponding eigenfunctions are obtained by successive applications of creation operators, as in the usual supersymmetry procedure. Another way to verify the possibility of constructing all the energy eigenspectra from the above shape invariance imposition, is to note that $H_2$ can be cast into the form $H_2 = A^\dagger A + k$.

So one can construct a Hamiltonian $H_3$, supersymmetric partner of $H_2$ given by

$$H_3 = A A^\dagger + k = \frac{1}{2m} \left[ \hat{p}^2 + i \hbar \left( \frac{m'}{m} \right) \hat{p} \right] + V_3, \quad (9)$$

where $V_3 = V_2 + k = V_1 + 2k$. In its turn, it can be rewritten as $H_3 = A^\dagger A + 2k$.

This can be done in successive steps, so that after $n$ repetitions one get

$$H_{n+1} = A^\dagger A + n k = \frac{1}{2m} \left[ \hat{p}^2 + i \hbar \left( \frac{m'}{m} \right) \hat{p} \right] + V_1 + n k = H_1 + n k. \quad (10)$$

Following this procedure, it can be proved that the energy levels of $H_2$ are the same of $H_1$, except by its ground state. In doing so, one can verify that $E_{n+1}^{(1)} = E_{n+1-1}^{(2)} = E_{n+1-2}^{(3)} = \cdots = E_{n}^{(l)}$, which lead us to the harmonic oscillator type energy, as expressed in above.

The comparison of the generalized potential $V_1(x)$ with that obtained in reference [2] can be done if one substitutes

$$m(x) = \left( \frac{\gamma + x^2}{1 + x^2} \right)^2$$

in equation(8). Then one obtains

$$\tilde{W}(x) = \frac{kx}{\sqrt{2}} + \frac{\gamma - 1}{\sqrt{2}} k \arctan x + 2 \left( \frac{b - 1}{b + 1} \right) \left( \frac{1}{2} + 2\alpha \right) \left\{ \frac{\gamma - 1}{\sqrt{2}} \frac{x}{(\gamma + x^2)^2} + 6b \left[ \frac{1}{8(\gamma - 1)\gamma^{\gamma/2}} (3\gamma^2 + 6\gamma - 1) \arctan \frac{x}{\sqrt{2}} - \frac{1}{\gamma - 1} \arctan x + 2\gamma(\gamma - 1) + (3\gamma^2 - 6\gamma + 7)(\alpha + x^2) \right] \right\}. \quad (11)$$
Figure 1: Effective Potential for two particular values of the ambiguity parameter $b$ and $\alpha = -1/2$. The case of Pastino et al: $b = 0$ (dashed line) and another possible one $b = 0.5$ (solid line).

The final expression for the potential corresponding to the above superpotential is very complicate, however one can obtain it straightforwardly. Consequently, we present in fig. 1, two particular choices of the ambiguity parameters. One of them recovers the result appearing in [2]. In fact, there exist other qualitatively different situations which we are not going to treat here, because it is out of the scope of this work.

On devising one way to implement supersymmetry in quantum mechanics for systems with position-dependent mass we have found how it is possible in some particular cases. We show through a convenient transformation of variables and redefinition of the wave function that systems with position-dependent mass can be mapped into others with a constant one, sharing the same spectrum and for which the quantum supersymmetric approach is usually well known.

One-dimensional systems with position-dependent mass are in general described by Hamiltonian (2) and consequently by the effective Schrödinger equation

$$
-\frac{\hbar^2}{2m} \left( \frac{d^2 \psi}{dx^2} - \frac{m'}{m} \frac{d \psi}{dx} \right) + V_{eff}(\delta, \kappa, \lambda; x) \psi = E \psi, \tag{12}
$$

where

$$
V_{eff}(\delta, \kappa, \lambda; x) = V(x) - \frac{\hbar^2}{4m} \left[ (\delta + \lambda) \frac{m''}{m} - 2(\delta + \lambda + \delta \lambda) \left( \frac{m'}{m} \right)^2 \right]. \tag{13}
$$

We note that the transformation of variable

$$
u = \int^x \sqrt{2m(z)dz}, \tag{14}
$$

and the redefinition of the wave function

$$
\psi(u) = [m(u)]^{1/4} \varphi(u),
$$
leaves us with the following Schrödinger equation, with mass equals to unity.

\[-\hbar^2 \frac{d^2}{du^2} \varphi(u) + U_{\text{eff}}(\delta, \lambda; u) \varphi(u) = E \varphi(u),\]  

(15)

where

\[U_{\text{eff}}(\delta, \lambda; u) = V(u) - \hbar^2 \frac{4}{m} \left\{ [1 + 2(\delta + \lambda)] \frac{d^2 m}{du^2} - \left( \frac{5}{4} + 3\delta + 3\lambda + 4\delta\lambda \right) \left( \frac{dm}{du} \right)^2 \right\}.\]  

(16)

If one can implement supersymmetry for the above potential it can naturally be done for the corresponding position-dependent mass system.

We recall that the potentials considered by Plastino et al. can be recovered by taking \(\delta = \lambda = 0\) and that is the case we consider together with \(U_{\text{eff}}(u)\) given by

\[U_{\text{eff}}(u) = \frac{\beta^2}{4} u^2 + \frac{\beta}{2},\]  

(17)

and

\[U_{\text{eff}}(u) = C \Gamma e^{\Gamma u} + (B + C e^{\Gamma u})^2,\]  

(18)

respectively for the two examples considered by Plastino et al [2]. Moreover \(\beta, B, C\) and \(\Gamma\) are constants. The first one corresponds to the harmonic oscillator and the second is the Morse potential. The supersymmetric treatment for them is well known [22].

For these cases, equation (15) can be factorized as

\[\mathcal{A}^\dagger \mathcal{A} \varphi(u) = E \varphi(u),\]

where

\[\mathcal{A} = \frac{d}{du} + \frac{\beta}{2} u \quad \text{and} \quad \mathcal{A} = \frac{d}{du} + B + C e^{\Gamma u},\]

respectively. The action of this operator on the ground state (\(\mathcal{A} \varphi_0(u) = 0\)) can be transformed into

\[\tilde{\mathcal{A}} \psi_0(u) = 0,\]

where

\[\tilde{\mathcal{A}} = \frac{d}{du} - \frac{1}{4m} \frac{dm}{du} + \frac{\beta}{2} u \quad \text{and} \quad \tilde{\mathcal{A}} = \frac{d}{du} - \frac{1}{4m} \frac{dm}{du} + B + C e^{\Gamma u}\]  

(19)

respectively, and \(\psi_0(u) = [m(u)]^{1/4} \varphi_0(u)\).

On its turn the transformation of variable (14) transforms the operator (19) into the one prescribed by Plastino et al. for systems with harmonic oscillator (HO) and Morse-like (ML) spectra and position-dependent mass, namely

\[A(x) = \frac{1}{\sqrt{2m}} \frac{d}{dx} + W(x)\]  

(20)
where
\[ W_{HO}(x) = \frac{1}{2} \frac{d}{dx} \left( \frac{1}{\sqrt{2m}} \right) + \frac{\beta}{2} \int_{x}^{0} \sqrt{2}(z) \, dz, \quad (21) \]
and
\[ W_{ML}(x) = B + \frac{1}{2} \frac{d}{dx} \left( \frac{1}{\sqrt{2m}} \right) + Ce^{-\Gamma} \int_{x}^{0} \sqrt{2m(z)} \, dz. \]

Then we have shown how systems with position-dependent mass can be mapped into isospectral ones with constant mass for which the supersymmetric approach can be implemented. This analysis is very important because in the course of demonstration one can verify why the superpartner potentials \( V_1(x) \) and \( V_2(x) \) in equations (6) and (7) should depend on the position-dependent mass in such a way that the corresponding Schrödinger equations could be exactly solvable, allowing one to understand which class of solvability it belongs \([1]\). Finally, it is important to remark that such an analysis can easily be extended to the case where some discontinuities are present. This can be achieved by performing the transformation (14) for each continuous interval under consideration and imposing the continuity conditions (8) at the discontinuities.

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