Magnetic Flux Periodicity in Second Order Topological Superconductors

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The magnetic flux periodicity of $\frac{\phi}{2\pi}$ is a well known manifestation of Cooper pairing in typical s-wave superconductors. In this paper we theoretically show that the flux periodicity of a two-dimensional second-order topological superconductor, which features zero-energy Majorana modes localized at the corners of the sample, is $\frac{\phi}{2\pi}$ instead. We further show that the periodicity changes back to $\frac{\phi}{2\pi}$ at the transition to a topologically trivial superconductor, where the Majorana modes hybridize with the bulk states, demonstrating that the doubling of periodicity is a manifestation of the non-trivial topology of the state.

INTRODUCTION

Topological insulators and superconductors are examples of symmetry-protected topological phases (SPTs) which feature a gapped bulk spectrum with gapless modes localized at the boundaries [1–3]. Free fermion SPTs with internal symmetries, such as charge-conjugation or time-reversal, can be completely classified in any dimension on the basis of which of them are present in the system [4–6]. Recent years have seen the advent of new classes of SPTs with spatial or crystalline symmetries. These systems have a much richer connection between the topological properties of the bulk and the states at the boundary. While crystalline topological insulators [7, 8] are analogous to the standard SPTs with a gapped bulk and gapless boundaries, higher-order topological phases [9–25] have a gapped bulk with boundaries which are themselves topologically non-trivial. The $n$th order topological phase in $d$ dimensions has gapless modes at its $d − n$-dimensional boundary.

Higher order topological insulators are best understood in the framework of the dipole moment theory of SPTs [10, 11]. In this theory, the dipole moment of a crystal is defined in terms of a Berry’s phase and the quantization (in presence of certain symmetries) of this moment leads to topological insulators with boundary charges. This idea can be generalized to higher multipole moments, such as the quadrupole and octopole moments defined in terms of nested Wilson loops [20]. Again in presence of certain crystalline symmetries, these moments are quantized and lead to higher-order topological insulators with boundary charges at the hinges or the corners. A topological invariant characterizing the higher-order TIs can be obtained from the Wilson loops in the same way as one obtains the topological invariant from the Floquet operator, familiar in the context of periodically driven systems [10, 11, 27].

Second order topological superconductors, in analogy with higher order topological insulators, can be written in terms of a mean field Boguliobov-de-Gennes Hamiltonian describing the bulk gapped $d$ dimensional superconductor with gapless $d − 2$ edge states, instead of $d − 1$ dimensional edge states as for usual topological superconductors.

The standard one dimensional topological superconductors [28–30] host edge Majorana modes, which are expected to obey non-abelian statistics, and hence be relevant for quantum computation. They have been shown [31, 32] to exhibit the fractional Josephson effect, where the current-phase relation has a $4\pi$, rather than a $2\pi$ periodicity. Other attempts to probe the topological order includes using non-linear Coulomb blockade using a superconducting nanoring [33], tunneling spectroscopy [34, 35] and transport experiments [36]. There have also been proposals to measure the flux periodicity in a ring geometry with either a single [37] or multiple [38] Majorana modes. An alternate system is the chiral $p$-wave superconductor, which is predicted to occur when the chiral edge modes of a quantum anomalous Hall insulator turn superconducting via the proximity effect [38] and lead to chiral Majorana states and there has been some experimental evidence for these modes. However, it has proved remarkably difficult to unambiguously prove the existence of these Majorana modes.

In this paper, we focus on a two-dimensional second order topological superconductor which hosts zero energy Majorana modes localized at the corners of the sample. We study the flux periodicity of the superconducting state after introducing a vortex in the center of the sample. The vortex makes the geometry multiply connected and thus the superconducting phase winds around the vortex in a non-trivial way. To take this into account, we compute the ground state of the mean-field BdG Hamiltonian self-consistently at each value of the flux. This self-consistent calculation shows us that the flux periodicity of the second order topological superconductor is $\frac{\phi}{2\pi}$ instead of $\frac{\phi}{\pi}$ as expected for a superconductor. To probe the origin of this period doubling, we compute the flux periodicity while varying a parameter in the Hamiltonian which drives the system into a topologically trivial phase. Interestingly we find that the flux periodicity changes to $\frac{\phi}{2\pi}$ across this transition, proving that the change in flux periodicity is related to the topologically nontrivial nature of the state.
The plan of the paper is as follows. In section II, we introduce the model, originally studied in Ref. [44], with $p_x + ip_y$ pairing in a doped Dirac semimetal with two mirror symmetries - i.e., four mirror symmetric Dirac nodes. We will then show using a concrete pairing mechanism that a second order topological superconductor TSC$_2$ can be self-consistently realised in such a model, with four Majorana corner modes. We will then introduce a vortex through the centre in an annulus geometry (a square annulus in the lattice model) and obtain the self-consistent solutions of the superconducting order parameter, as a function of the parameters of the theory. In Sec III, we will study the energy levels and the circulating current due to the insertion of the vortex and show that the flux periodicity changes from $\hbar c/e$ to $\hbar c/2e$ as a tunable parameter in the model is changed. Further tuning of the parameter to bring the system into the metallic regime, changes the flux periodicity back to $\hbar c/e$ as expected for an Aharonov-Bohm ring. Finally, in Sec. IV, we end with discussions and conclusions.

MODEL AND VORTEX INTRODUCTION

A. Second order topological superconductor (TSC$_2$)

We start with the four band Boguliobov-de Gennes model introduced in Ref. [44] with $H = \int d\mathbf{k} \Psi_{\mathbf{k}}^\dagger H(\mathbf{k}) \Psi_{\mathbf{k}}$, \begin{equation}
H(\mathbf{k}) = (b_x + \lambda \cos(k_x))\tau_z\sigma_x + \lambda \cos(k_y)\tau_y\sigma_y + \Delta \sin(k_x)\tau_y\sigma_x + \Delta \sin(k_y)\tau_x\sigma_z,
\end{equation}
and $\Psi_{\mathbf{k}} = (c_{\mathbf{k}+1}^\dagger, c_{\mathbf{k}1}^\dagger, c_{-\mathbf{k}+1}, c_{-\mathbf{k}1})$. Here, $\sigma(\tau)$ denote the operators acting on spin (Nambu) space respectively and $1_x$ represents the identity in the Nambu space. $\lambda$ denotes the hopping. This model can be shown [44] to describe a higher order topological $p_x + ip_y$ superconductor phase for a fixed $\Delta$, when $|b_x/\lambda| < 1$, with four Majorana modes localized at the four corners of the sample. This Hamiltonian has a particle-hole symmetry, with $\tau_x$ being the charge conjugation operator such that $\tau_x H(\mathbf{k})^T \tau_x^{-1} = -H(-\mathbf{k})$. Provided that we choose the pairing terms to have $p_x + ip_y$ symmetry, the model also has two mirror symmetries $M_x = \sigma_y \tau_y$ and $M_z = \sigma_y \tau_x$ such that $M_{x,y} H(\mathbf{k}) M_{x,y}^{-1} = H(-\mathbf{k}, -\mathbf{k})$ with the two mirror symmetries anti-commuting with each other. More specifically, $M_z H(k_x, k_y) M_z^{-1} = H(-k_x, k_y)$ and similarly for $M_y$.

This model has a gapped spectrum, but as shown in Ref. [44], for $|b_x|<\lambda$, the model denotes a second order topological superconductor TSC$_2$ - i.e., the edge states themselves are topological and have gapless corner states. Analogous to what was done for the model of higher order topological insulators in Ref. [45], we can plot the spectrum for open boundary conditions in both the $x$ and $y$ directions, parametrically, as a function of $b_x$, as shown in Fig. 1. The spectrum in Fig. 1(a) clearly shows the existence of zero modes for $|b_x|/\lambda < 1$, and in Fig. 1(b), the densities clearly show four localised modes at the four corners of the lattice.

A two dimensional quadrupole insulator can be characterized by a quantized quadrupole moment ($Q_{xy}$) as argued in Refs. [10], [46] and [47] and the quadrupole moment is defined as a ground state $|\Phi_0\rangle$ expectation value of a

\begin{equation}
\langle \Phi_0 | Q_{xy} | \Phi_0 \rangle = \int d\mathbf{k} \Psi_{\mathbf{k}}^\dagger Q_{xy} \Psi_{\mathbf{k}}.
\end{equation}

FIG. 1: (a) Plot of the spectrum of the Hamiltonian in Eq[1] for a 20 x 20 lattice, for open boundary conditions in both directions, with respect to the field $b_x$, which clearly shows the zero modes for $|b_x| < \lambda$. Parameter values are $\lambda = 1$, $\Delta/\lambda = 0.8$ and lattice spacing $a = 1$. (b) Plot of the electron densities which shows the four localized Majorana modes at the four corners. The inset shows four zero energy modes clearly distinguishable from the bulk spectrum at $b_x = 0$.

FIG. 2: The quadrupole moment($Q_{xy}$) as a function of $b_x/\lambda$ for the Hamiltonian in Eq[1] with $\lambda = 1$, $\Delta/\lambda = 0.8$ and lattice spacing $a = 1$. (a) For open boundary conditions in both $x$ and $y$ directions and (b) for open boundary conditions in the $y$ direction and periodic boundary conditions in the $x$ direction. This shows that the second order topological superconductor phase has $Q_{xy}/\epsilon = 0.5$ (modulo 1) and the topologically trivial phase has $Q_{xy}/\epsilon = 0.0$ (modulo 1). The phase transition from a topological to a non-topological superconducting phase occurs at $b_x/\lambda = 1.0$ in (b) showing that the minor deviation from unity in (a) here as well as in Fig[1(a)] is a finite size effect.
many-body operator as follows -
\[
Q_{xy} = \frac{e}{2\pi} \text{Im}[\ln(\langle \Phi_0 | \hat{O} | \Phi_0 \rangle)] \, (\text{modulo } 1),
\]
\[
\hat{O} = \exp \left( i\frac{2\pi xy \hat{n}(x,y)}{L_x L_y} \right),
\]
where \((x,y)\) is the lattice site index, \(\hat{n}(x,y)\) is the quasiparticle density at the site \((x,y)\) and \(L_x, L_y\) are the lengths of the 2d system. By analogy, we can define a similar quadrupole moment for the two dimensional topological superconductor, which has been plotted in Fig.3. The quadrupole moment shows a sharp transition from \(Q_{xy} = 0.5\) to \(Q_{xy} = 0\) at the value of \(b_x\) where the model transitions from a topological superconductor into a normal superconductor. This transition occurs close to \(b_x/\lambda = 1\). This is also consistent with the disappearance of the corner Majorana modes at \(b_x = 1\) as seen in Fig.1, b. Here, and in Fig.2 a), the minor deviation from unity is a finite size effect. Note, however that the density plotted in the figure is that of the Bogoliubons, linear combinations of the particle and hole operators obtained by diagonalising the Hamiltonian in Eq.1 with the pairing term \(\Delta\). This is discussed further in the next section, where we compute the quadrupole moment with a self-consistent pairing term.

The normal state of this Hamiltonian (when \(\Delta = 0\)) has four gapless mirror symmetric Dirac points, and it can be shown that at finite chemical potential, regions of the Fermi surface with opposite momenta always have the same spin texture. Hence, it is natural\(^4\) for a spin triplet superconducting gap to be induced by electronic interactions. The pairing potential \(\Delta_{ij}^\uparrow = \langle c_i^\uparrow c_j^\downarrow \rangle\) with the appropriate \(p_x + ip_y\) symmetry can be derived from a mean field treatment of the pairing interaction \(H_{int} = \frac{V}{2} \sum_{(i,j)} (n_{i\uparrow} n_{j\downarrow} + n_{i\downarrow} n_{j\uparrow})\)
where the \(\langle i, j \rangle\) denotes nearest neighbour sites. We will show in a later section that \(\Delta_{ij}^\uparrow\) can be obtained self-consistently for our model on a square lattice.

### B. Vortex insertion in an annulus geometry

The basic idea is that in a superconducting ring, the order parameter responds to a flux or vortex inserted through the ring. Just as current though a metallic ring is modulated by a \(hc/e\) periodicity due to the Aharonov-Bohm effect, the current through a superconducting ring is expected to be modulated as \(hc/2e\)\(^4\)\(^5\)\(^6\). Although naively explained in terms of the Cooper pair condensates having a charge of \(2e\), the theoretical explanation is more subtle and comes from the degeneracy between two different classes of superconducting wave-functions at \(\phi_0/2\). The first class are those wave-functions with pairing between the angular momentum states \(hk\) and \(-hk\) leading to Cooper pairs with \(hq = 0\). All even values of \(q\) can be obtained from these wave-functions by gauge transformations. The second class are those wave-functions with pairing between \(hk = \pm \hbar(k + 1)\) leading to Cooper pairs with \(hq = 1\), with again, all odd integer values of \(q\) being related to these wave-functions by gauge transformations. Both these classes of wave-functions turn out to have the same energy for the flux \(\phi = \phi_0/2 = hc/2e\). But more recently, the question of the flux periodicity has resurfaced in the context of high \(T_c\) \(d\)-wave superconductors\(^5\)\(^6\) where it was seen that the condensate reconstructs for half-integer flux quanta, and breaks the degeneracy between the state at zero flux and the state at half-integer flux. Thus, the periodicity changes back to \(hc/e\) as for normal metals. Even for \(s\)-wave superconductors, it has been shown\(^5\)\(^6\) that for superconducting rings with diameter smaller than the coherence length, the response is generally modulated as \(hc/e\) instead of \(hc/2e\).

Here we study the response in a \(p_x + ip_y\) higher order topological superconductor ‘ring’. We imagine adding an infinitely long solenoid (of infinitesimal radius) at the origin of a 2d sample so that there is no magnetic field crossing any of the sites, but a closed loop around the origin encloses a flux - thereby mimicking an annulus with flux through the hole. More specifically, we have a square geometry and assume that the lattice sites are located at \(r = (m + 1/2, n + 1/2)\) where \(m, n\) are integers from \(-L\) to \(L - 1\). This ensures that the lattice sites are symmetrically located about the origin at \(r = (0,0)\).

In the absence of superconductivity, a vortex can be added to \(H\) through the standard Peierls substitution\(^5\)\(^2\). Under this transformation, the kinetic terms change as...
follows -

\[
\psi_r^{\dagger} \psi_r \rightarrow \psi_{r+\delta}^{\dagger} \psi_{r+\delta} \exp \left( i \frac{e}{\hbar c} \int_{r}^{r+\delta} dr' \cdot \vec{A}(r') \right) = \\
\psi_{r+\delta}^{\dagger} \psi_{r+\delta} \exp \left( i \phi \int_{r}^{r+\delta} dr' \cdot \frac{1}{|r'|} \right),
\]

where \( \phi_0 = \hbar c/e \), \( \psi_r = (c_{r\uparrow}, c_{r\downarrow}, c_{rT}, c_{rL}) \) and \( \delta \) is constrained only up to nearest neighbour in both \( \hat{x} \) and \( \hat{y} \) direction. Every bond of the lattice will clearly pick up a different phase due to the \( \frac{\theta}{|r'|} \) in the integral, but at \( \phi = n\phi_0 \) (where \( n \) is an integer), the total phase accumulated by an electron going through each plaquette around the origin is \( 2\pi n \). Therefore the system behaves as if there is no magnetic field at all. For illustration, the lowest three positive and negative eigenvalues are shown as a function of the flux in Fig.4 which show that the spectrum is \( \phi_0 = \hbar c/e \) periodic as expected.

However, in the presence of pairing terms \( \Delta \neq 0 \), adding a flux or creating a vortex at the center of system makes \( \Delta \) position dependent and keeping it constant is no longer viable. We need to solve for \( \Delta \) self-consistently, which is done in the next section.

**SELF-CONSISTENT CALCULATION AND RESULTS**

We work with the Hamiltonian in Eq.1 in real space given by

\[
\mathcal{H} = \sum_{(i,j)} \left( \mathcal{H}_{ij}^0 c_{i\uparrow}^\dagger c_{j\downarrow} + h.c. \right) + \sum_{(i,j)} \left( \Delta_{ij} c_{i\uparrow}^\dagger c_{j\downarrow} + \Delta_{ij}^* c_{i\downarrow} c_{j\uparrow} \right),
\]

where \( \mathcal{H}_{ij}^0 = t_h \exp(i\phi_{ij}) + 2b_x \delta_{ij} \), with \( t_h^x = \lambda \) and \( t_h^y = -i\lambda \), are the hopping matrix elements between the nearest neighbour sites. Also \( \phi_{ij} = \int_{r_i}^{r_j} \frac{e}{\hbar c} (A(r') \cdot d\vec{r}') \) is the Peierls phase factor coming from \( \vec{A}(r) \), which is the vector potential due to flux through the origin. The order parameter of the superconducting state \( \Delta_{ij} \) is defined on the links between neighbouring sites with the appropriate \( p \)-wave symmetry, i.e, we have

\[
\Delta_{ij} = \frac{V}{2} \left[ (c_{i\uparrow} c_{j\downarrow}) + (c_{i\downarrow} c_{j\uparrow}) \right].
\]

which is symmetric under the exchange of spins and anti-symmetric under the exchange of spatial indices. Here, \( V \) is the nearest-neighbour pairing interaction strength. Now following a Boguliobov rotation using new fermionic operators (the Boguliobons) \( \gamma_n \)

\[
c_{i\sigma} = \sum_n \left( u^n_{i\sigma} \gamma_n - (v^n_{i\sigma})^* \gamma_n^\dagger \right)
\]

we find that the Hamiltonian in Eq.5 is given by

\[
\mathcal{H} = E_g + \sum_n (E_n \gamma_n^\dagger \gamma_n), \quad E_n > 0
\]

with \( E_g = -\sum_{i\sigma,n} E_n |v^n_{i\sigma}|^2 \) and with the coefficients \( (u^n) \) and \( (v^n) \) satisfying the equation

\[
\begin{pmatrix}
\mathcal{H}^0 \\
-\Delta^\dagger \\
-\Delta
\end{pmatrix}
\begin{pmatrix}
u^n \\
v^n
\end{pmatrix} = E_n \begin{pmatrix}
u^n \\
v^n
\end{pmatrix}.
\]

The order parameter \( \Delta_{ij}(\hat{b}, b_x) \) is then calculated self-consistently from the following equation

\[
\Delta_{ij} = V 4 \sum_n \left[ u^n_{i\uparrow}(v^n_{j\uparrow})^* + u^n_{i\downarrow}(v^n_{j\downarrow})^* - u^n_{i\uparrow}(v^n_{j\downarrow})^* - u^n_{i\downarrow}(v^n_{j\uparrow})^* \right] \tanh \left( \frac{E_n}{2kB_T} \right), \quad E_n > 0.
\]

and used to compute the energy eigenvalues and the total energy.

In Fig.4 we show the self-consistent energy eigenvalues in the presence of a vortex for a finite pairing interaction. We can compare these energy eigenvalues with those in Fig.3 without the pairing term and note that there are four zero energy modes well separated from the bulk states, clearly showing that the system is a second order topological superconductor. Moreover, the spectrum surprisingly shows a magnetic flux periodicity of \( \frac{2\pi}{e} \) as opposed to a magnetic flux periodicity of \( \frac{b_x}{2e} \) in a typical \( s \)-wave superconductor.

The total self-consistent energy has been shown in Fig.5 for four different values of \( b_x/\lambda \) as a function of the flux \( \phi/\phi_0 \) through the centre of the lattice. The total energy initially has a maximum at \( \phi = \phi_0/2 \) at \( b_x = 0 \) which is in the topological superconductor regime.


FIG. 5: Self-consistent total energy \((\langle E(\phi/\phi_0) - E(0) \rangle * 10^3)/|E(0)|\) as a function of the flux \(\phi/\phi_0\) through the center of a 20 \times 20 square lattice having lattice spacing \(a = 1\) for the Hamiltonian in Eq. 5. The parameters chosen are \(\lambda = 1, V = 3\lambda\) with (a) \(b_x/\lambda = 0.0\) (b) \(b_x/\lambda = 0.3\) (c) \(b_x/\lambda = 0.8\) and (d) \(b_x/\lambda = 1.5\). Note that the flux-periodicity changes from \(\phi_0\) to \(\phi_0/2\) and back again to \(\phi_0\), as \(b_x\) is tuned.

as shown in Fig. 5(a). But as \(b_x\) increases, we note that the width of the maximum reduces (as in Fig. 5(b)), and then as shown in Fig. 5(c), the total energy develops a minimum at \(\phi = \phi_0/2\). Finally, in Fig. 5(d), we note that at even higher values of \(b_x\) where the model is no longer in the topological superconductor regime, the minimum at \(\phi = \phi_0/2\) is again replaced by a maximum. So as one increase \(b_x\) from \(b_x/\lambda = 0\), the magnetic flux periodicity of total energy which was \(\phi_0\) to start with goes to \(\phi_0/2\) at intermediate \(b_x\) and then again goes back to \(\phi_0\) as one further increase \(b_x\).

The lattice current density \(\vec{J}_{ij}\) from lattice site \(i\) to \(j\) is then obtained as

\[
\vec{J}_{ij} = \frac{-2e}{\hbar} \text{Im} \left\{ t_h \langle c_{ij}^\dagger c_{ij} \rangle \exp(i\phi_{ij}) - t_h \langle c_{ij}^\dagger c_{ij} \rangle \exp(i\phi_{ji}) \right\}. \tag{11}
\]

using the continuity equation which is then used to compute the total circulating current in the system, which is shown in Fig. 6. Here again, we note that as \(b_x\) increases, there is a tendency towards period doubling i.e., the flux periodicity changes from \(\phi_0\) to \(\phi_0/2\). Further increase in \(b_x\) brings the periodicity again back to \(\phi_0\), as shown in Fig. 6(c).

Now to see why the magnetic flux periodicity of the system changes from \(\phi_0\) to \(\phi_0/2\) and goes back again to \(\phi_0\), we have calculated the self-consistent spectrum in the absence of the vortex but as a function of \(b_x/\lambda\) as illustrated in Fig. 5. The spectrum clearly shows that for small enough \(b_x/\lambda\), there are zero energy states well separated from the bulk states. Here, the system is in a second order topological superconductor phase and the magnetic flux periodicity of this topological phase is \(\phi_0/2\) as seen in the total energy in Fig. 5(a) and in the total circulating current in Fig. 6(a). Now close to \(b_x/\lambda \simeq \pm 0.6\), the bulk energy gap closes and the zero energy states mix with bulk states giving rise to a continuum of en-
These show that at close to zero energy states and signifying a change in the topology of the system.

The full spectrum, (b) spectrum close to zero energy. These show that at close to $b_x/\lambda \simeq 0.6$ zero energy is gapped out by mixing with the bulk energy, giving an indication that there is a phase transition at that point.

FIG. 7: Self-consistent spectrum for the Hamiltonian in Eq.5 on $20 \times 20$ square lattice having lattice spacing $a = 1$, without any flux as function of field $b_x/\lambda$ for $\lambda = 1, V = 3\lambda$. (a) The full spectrum, (b) spectrum close to zero energy. This causes the magnetic flux periodicity change from $\frac{\hbar c}{e}$ to $\frac{\hbar c}{2e}$ which is is seen both in the total energy and in the total circulating current in the system. So the magnetic flux periodicity change from $\frac{\hbar c}{e}$ to $\frac{\hbar c}{2e}$ is associated with the change of topology of the system, i.e., 2nd-order topological superconductor has flux-periodicity of $\frac{\hbar c}{e}$ in contrast to a non-topological superconductor which has flux-periodicity of $\frac{\hbar c}{2e}$. When $b_x$ is increased beyond $b_x/\lambda \simeq 1.0$, the pairing term goes to zero and the system is in the metallic phase. This again explains the switch back to the magnetic flux periodicity of $\frac{\hbar c}{e}$ as seen both in the total energy by Fig.5(d) and in the total persistent current in Fig.6(c). This is just the expected periodicity due to the Aharonov-Bohm effect when a flux is inserted through a metallic ring.

FIG. 8: Site average of the pairing term $\langle |\Delta| \rangle$ as a function of $b_x/\lambda$, calculated self-consistently without flux for the Hamiltonian in Eq.5 on a $20 \times 20$ square lattice for $\lambda = 1, V = 3\lambda$ and lattice spacing $a = 1$. This clearly shows that the system is in a superconducting phase until $b_x \sim 1.0$. Further increase of $b_x$ brings the system to the non-superconducting phase having zero pairing.

We have also calculated the site average of the pairing term as a function of $b_x/\lambda$ as illustrated in Fig.8 to show that the system has finite pairing term up to $b_x/\lambda \simeq 1.0$. So the system remains in a superconducting phase up to $b_x/\lambda \simeq 1.0$; however, close to $b_x/\lambda \simeq 0.6$ there is a phase transition from a topological to a non-topological superconductor, which can be seen from the vanishing of the zero energy states. This causes the magnetic flux periodicity to change from $\frac{\hbar c}{e}$ to $\frac{\hbar c}{2e}$ which is is seen both in the total energy and in the total circulating current in the system. So the magnetic flux periodicity change from $\frac{\hbar c}{e}$ to $\frac{\hbar c}{2e}$ is associated with the change of topology of the system.

We also confirm the phase transition from a topological superconductor to a normal superconductor at $b_x/\lambda \sim 0.6$ by calculating the the quadrupole moment $Q_{xy}/e$ as a function of the field $b_x/\lambda$ for (a) $\Delta/\lambda = 0.4$ (without self-consistency) for the Hamiltonian in Eq.4 and for (b) $V = 3\lambda$ (using self-consistency) for the Hamiltonian in Eq.5 for $\lambda = 1$, flux $\phi = 0$ and for $20 \times 20$ square lattice having lattice spacing $a = 1$ with open boundary condition in both $x$ and $y$ direction. In (b), the transition to a non-topological phase occurs at $b_x/\lambda \simeq 0.6$, contrast to (a) where the transition happens at $b_x/\lambda \simeq 1.0$.

FIG. 9: The quadrupole moment($Q_{xy}/e$) as a function of the field $b_x/\lambda$ for (a) $\Delta/\lambda = 0.4$ (without self-consistency) for the Hamiltonian in Eq.4 and for (b) $V = 3\lambda$ (using self-consistency) for the Hamiltonian in Eq.5. (a) $\Delta/\lambda \simeq 0.4$, (b) $V = 3\lambda$.

In (b), the transition to a non-topological phase occurs at $b_x/\lambda \simeq 0.6$, contrast to (a) where the transition happens at $b_x/\lambda \simeq 1.0$. This is consistent with the spectrum which also shows a gap closing at the same point. This confirms that the change in the magnetic flux periodicity from $\frac{\hbar c}{e}$ to $\frac{\hbar c}{2e}$ close to $b_x/\lambda \simeq 0.7$ for the system shown in Fig.5(c) and Fig.6(b) is associated with the change in the topology of the system.
DISCUSSIONS AND CONCLUSIONS

In this work, we have focussed on the flux periodicity of a second order topological superconductor in two dimensions with four Majorana modes at the corners. By implementing a ring geometry via a flux at the origin, we show that the flux periodicity changes from $\frac{hc}{e}$ to $\frac{hc}{2e}$ when the topological superconductor transitions to an ordinary superconductor.

This model hosts four Majorana modes at its corners, which is sufficient to exploit their non-abelian nature in braiding since they can be paired in different ways. By insertion of fluxes and implementing the ring geometry in generalisations of this model, one can expect to be able to move the Majorana modes and design non-abelian quantum computational protocols. Thus designing and discovering higher order topological insulators could be a pathway to topological quantum computation.

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