Orbifold instantons, moment maps
and Yang-Mills theory with sources

Tatiana A. Ivanova\textsuperscript{1}, Olaf Lechtenfeld\textsuperscript{2}, Alexander D. Popov\textsuperscript{2} and Richard J. Szabo\textsuperscript{3}

\textsuperscript{1}Bogoliubov Laboratory of Theoretical Physics, JINR
141980 Dubna, Moscow Region, Russia
Email: ita@theor.jinr.ru

\textsuperscript{2}Institut für Theoretische Physik and Riemann Center for Geometry and Physics
Leibniz Universität Hannover
Appelstraße 2, 30167 Hannover, Germany
Email: lechtenf@itp.uni-hannover.de, popov@itp.uni-hannover.de

\textsuperscript{3}Department of Mathematics, Heriot-Watt University
Colin Maclaurin Building, Riccarton, Edinburgh EH14 4AS, U.K.
and Maxwell Institute for Mathematical Sciences, Edinburgh, U.K.
and The Tait Institute, Edinburgh, U.K.
Email: R.J.Szabo@hw.ac.uk

Abstract

We revisit the problem of constructing instantons on ADE orbifolds $\mathbb{R}^4/\Gamma$ and point out some subtle relations with the complex structure on the orbifold. We consider generalized instanton equations on $\mathbb{R}^4/\Gamma$ which are BPS equations for the Yang-Mills equations with an external current. The relation between level sets of the moment maps in the hyper-Kähler quotient construction of the instanton moduli space and sources in the Yang-Mills equations is discussed. We describe two types of spherically-symmetric $\Gamma$-equivariant connections on complex $V$-bundles over $\mathbb{R}^4/\Gamma$ which are tailored to the way in which the orbifold group acts on the fibres. Some explicit abelian and nonabelian SU(2)-invariant solutions to the instanton equations on the orbifold are worked out.
1 Introduction and summary

Instantons in Yang-Mills theory [1] and gravity [2, 3] play an important role in modern field theory [4, 5, 6]. They are nonperturbative configurations which solve first order (anti-)self-duality equations for the gauge field and the Riemann curvature tensor, respectively. The construction of gauge instantons can be described systematically in the framework of twistor theory [7, 8] and by the ADHM construction [9]. There are also many methods for constructing gravitational instantons including twistor theory [8] and the hyper-Kähler quotient construction [10] based on the hyper-Kähler moment map introduced in [11].

In this paper we revisit the problem of constructing instantons on the ADE orbifolds $\mathbb{R}^4/\Gamma$. The corresponding instanton moduli spaces are of special interest in type II string theory, where they can be realized as Higgs branches of certain quiver gauge theories which appear as worldvolume field theories on $Dp$-branes in a $Dp$-$D(p+4)$ system with the $D(p+4)$-branes located at the fixed point of the orbifold [12]. The ADHM equations can be identified with the vacuum equations of the supersymmetric gauge theory, and the structure of the vacuum moduli space provides an important example of resolution of spacetime singularities by stringy effects in the form of D-brane probes. We point out in particular some salient relations between the construction of instantons and complex structures on $\mathbb{R}^4/\Gamma$.

Kronheimer [10] considers $\Gamma$-equivariant solutions of the matrix equations

$$
\begin{align*}
[W_2, W_3] + [W_1, W_4] &= \Xi_1, \\
[W_3, W_1] + [W_2, W_4] &= \Xi_2, \\
[W_1, W_2] + [W_3, W_4] &= \Xi_3,
\end{align*}
$$

(1.1)

where $\Gamma$ is a finite subgroup of the Lie group $\text{SU}(2)$ acting on the fundamental representation $\mathbb{C}^2 \cong \mathbb{R}^4$, $W_\mu$ with $\mu = 1, 2, 3, 4$ are matrices taking values e.g. in the Lie algebra $\mathfrak{u}(N)$, and $\Xi_a$ with $a = 1, 2, 3$ are matrices in the center $\mathfrak{h}$ of a subalgebra $\mathfrak{g}$ of $\mathfrak{u}(N)$. For $\Xi_a = 0$ the equations (1.1) are the anti-self-dual Yang-Mills equations on the orbifold $\mathbb{C}^2/\Gamma$ reduced by translations. Their solutions satisfy the full Yang-Mills equations. In the general case, the equations (1.1) are interpreted as hyper-Kähler moment map quotient equations, and Hitchin shows [13] that one can similarly interpret the Bogomolny monopole equations and vortex equations. Kronheimer shows that the moduli space of solutions to (1.1) in the Coulomb branch is a hyper-Kähler ALE space $M_\xi$, which is the minimal resolution

$$
M_\xi \longrightarrow M_0
$$

(1.2)

of the orbifold $M_0 = \mathbb{C}^2/\Gamma$. Here $\xi$ are parameters in the matrices $\Xi_a$ of (1.1). Similar results were obtained in [14, 15] for $\text{SU}(2)$-invariant Yang-Mills instantons on $\mathbb{R}^4$ (see also [16]). Moreover, it was shown by Kronheimer and Nakajima [17] that there exists a bundle $\mathcal{E} \rightarrow M_\xi$ with Chern classes $c_1(\mathcal{E}) = 0$ and $c_2(\mathcal{E}) = (#\Gamma - 1)/#\Gamma$ such that the moduli space of framed instantons on $\mathcal{E}$ satisfying the anti-self-dual Yang-Mills equations coincides with the base manifold $M_\xi$ itself. In the limit $\xi = 0$ one obtains $\mathbb{C}^2/\Gamma$ as the moduli space of minimal fractional instantons on the V-bundle $\mathcal{E}$ over the orbifold $M_0 = \mathbb{C}^2/\Gamma$.

In this paper we consider gauge instanton equations with matrices $\Xi_a$ on the orbifold $\mathbb{C}^2/\Gamma \cong \mathbb{R}^4/\Gamma$ and show that the choices of $\Xi_a \neq 0$ correspond to sources in the Yang-Mills equations. For gauge potentials on $\mathbb{R}^4/\Gamma$ with $\Gamma = \mathbb{Z}_{k+1}$ we analyse solutions of $\Gamma$-equivariance conditions in two different $\text{SU}(2)$-invariant bases adapted to the spherical symmetry. Recall that one can write a

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1In type II string theory in the presence of orientifold $O(p+4)$ planes one should use instead the Lie algebras of orthogonal or symplectic Lie groups.
realization of the Lie algebra $\mathfrak{so}(4) \cong \mathfrak{su}(2) \oplus \mathfrak{su}(2)$ in terms of vector fields on $\mathbb{R}^4/\Gamma$ as

$$E^a = -\eta^a_{\mu\nu} y^\mu \frac{\partial}{\partial y^\nu} \quad \text{and} \quad \tilde{E}^a = -\tilde{\eta}^a_{\mu\nu} y^\mu \frac{\partial}{\partial y^\nu},$$

(1.3)

where $\eta^a_{\mu\nu}$ and $\tilde{\eta}^a_{\mu\nu}$ are components of the self-dual and anti-self-dual 't Hooft tensors [18] and $y^\mu$ are local coordinates on $\mathbb{R}^4/\Gamma$. The commutation relations between these vector fields are

$$[E^a, E^b] = 2\varepsilon^{abc} E^c, \quad [\tilde{E}^a, \tilde{E}^b] = 2\varepsilon^{abc} \tilde{E}^c \quad \text{and} \quad [E^a, \tilde{E}^b] = 0.$$  

(1.4)

Introducing complex coordinates $z^1 = y^1 + iy^2$ and $z^2 = y^3 + iy^4$ on $\mathbb{R}^4/\Gamma \cong \mathbb{C}^2/\Gamma$, one finds that the vector fields $\tilde{E}^a$ preserve this complex structure but the vector fields $E^a$ do not, i.e. the group SU(2) acting on $\mathbb{C}^2$ is generated by $(\tilde{E}^a)$. Furthermore, the actions of the corresponding Lie derivatives are given by

$$\mathcal{L}_{\tilde{E}^a} E^a = 0 \quad \text{and} \quad \mathcal{L}_{\tilde{E}^a} \tilde{E}^a = 2\varepsilon^{abc} \tilde{c}^c,$$

(1.5)

where $e^a = e^a_\mu dy^\mu$ and $\tilde{e}^a = \tilde{e}^a_\mu dy^\mu$ are one-forms dual to the vector fields $E^a$ and $\tilde{E}^a$, respectively.

We show that both bases of one-forms $(e^a, dr)$ and $(\tilde{e}^a, dr)$ with $r^2 = \delta_{\mu\nu} y^\mu y^\nu$ can be used for describing spherically-symmetric instanton configurations, but due to (1.5) the basis $(e^a, dr)$ is more suitable for connections on V-bundles $\mathcal{E}$ with trivial action of the finite group $\Gamma \subset SU(2)$, while the basis $(\tilde{e}^a, dr)$ is more suitable for connections on V-bundles $\mathcal{E}$ with non-trivial $\Gamma$-action on the fibres of $\mathcal{E}$. Explicit examples of abelian and nonabelian SU(2)-invariant instanton solutions on $\mathbb{R}^4/\mathbb{Z}_{k+1}$ are worked out below.

The structure of the remainder of this paper is as follows. In Section 2 we consider generalized instanton equations on $\mathbb{R}^4$ which reduce to (1.1) and show that they correspond to BPS-type equations for Yang-Mills theory with sources. In Section 3 we extend these equations to the ADE quotient singularities $\mathbb{R}^4/\Gamma$, focusing on the special case $\Gamma = \mathbb{Z}_{k+1}$. In Section 4 we study the moduli spaces of translationally-invariant instantons on $\mathbb{R}^4/\Gamma$ via the hyper-Kähler quotient construction. In Section 5 we consider the construction of spherically-symmetric instanton solutions on $\mathbb{R}^4/\Gamma$ and make some preliminary comments concerning the structure of the instanton moduli spaces, though a detailed description of these moduli spaces is beyond the scope of the present work.

2 Instanton equations on $\mathbb{R}^4$

Euclidean space $\mathbb{R}^4$. Consider the two-forms

$$\omega^a := \frac{1}{2} \eta^a_{\mu\nu} dy^\mu \wedge dy^\nu,$$

(2.1)

where $y^\mu$ are coordinates on $\mathbb{R}^4$ and $\omega^a_{\mu\nu} := \eta^a_{\mu\nu}$ are components of the 't Hooft tensors given by the formulas

$$\eta^a_{bc} = \varepsilon^a_{bc} \quad \text{and} \quad \eta^a_{b4} = -\eta^a_{4b} = \delta^a_b.$$  

(2.2)

Here $\varepsilon^1_{23} = 1, \mu, \nu, \ldots = 1, 2, 3, 4$ and $a, b, \ldots = 1, 2, 3$. The forms $\omega^a$ are symplectic and self-dual,

$$d\omega^a = 0 \quad \text{and} \quad * \omega^a = \omega^a,$$

(2.3)

where $*$ is the Hodge duality operator for the flat metric

$$g = \delta_{\mu\nu} dy^\mu \otimes dy^\nu$$  

(2.4)

on $\mathbb{R}^4$.

Using the metric (2.4) we introduce three complex structures $J^a = \omega^a \circ g^{-1}$ on $\mathbb{R}^4$ with components

$$(J^a)^b_c = \omega^a_{\nu\lambda} \delta^\nu_b \delta^\lambda_c,$$

(2.5)
so that \((\mathbb{R}^4, J^a) \cong \mathbb{C}^2\). The space \(\mathbb{R}^4\) is hyper-Kähler, i.e. it is Kähler with respect to each of the complex structures (2.5). We choose one of them, \(J^3 =: J\), to identify \(\mathbb{R}^4\) and \(\mathbb{C}^2 \cong (\mathbb{R}^4, J)\). With respect to \(J\) the complex two-form
\[
\omega_C = \omega^1 + i \omega^2
\]
is closed and holomorphic, i.e. \(\omega_C\) is a \((2,0)\)-form.

**Instanton equations.** Let \(\mathcal{E}\) be a rank \(N\) complex vector bundle over \(\mathbb{R}^4 \cong \mathbb{C}^2\). We endow this bundle with a connection \(A = A_\mu \, dy^\mu\) of curvature \(\mathcal{F} = dA + A \wedge A = \frac{1}{2} F_{\mu \nu} \, dy^\mu \wedge dy^\nu\) taking values in the Lie algebra \(u(N)\). Let us constrain the curvature \(\mathcal{F}\) by the equations
\[
* \mathcal{F} + \mathcal{F} - * \mathcal{F} \wedge A = 2 \omega^a \wedge \left( d \Xi_a + [A_\mu, \Xi_a] \right),
\]
where the functions \(\Xi_a\) belong to \(u(N)\). Solutions to this equation of finite topological charge are called (generalized) instantons. If \(\Xi_a\) belong to the center \(u(1)\) of \(u(N)\) and \(d \Xi_a = 0\), then solutions to the equations (2.7) satisfy the Yang-Mills equations on \(\mathbb{R}^4\). If \(\Xi_a\) do not belong to this center, then (2.7) are BPS-type equations for Yang-Mills theory with sources which vanish only if \(\Xi_a\) are constant and \(\Xi_a \in u(1) \subset u(N)\) for \(a = 1, 2, 3\). Indeed, from (2.7) we get
\[
d * \mathcal{F} + A \wedge * \mathcal{F} - * \mathcal{F} \wedge A = 2 \omega^a \wedge \left( d \Xi_a + [A_\mu, \Xi_a] \right),
\]
which after taking the Hodge dual can be rewritten as
\[
\partial_\mu F_{\mu \nu} + [A_\mu, F_{\mu \nu}] = 4 \omega^a_{\mu \nu} \left( \partial_\mu \Xi_a + [A_\mu, \Xi_a] \right).
\]
The current
\[
j_\mu := 4 \omega^a_{\mu \nu} D_\nu \Xi_a \quad \text{with} \quad D_\mu \Xi_a := \partial_\mu \Xi_a + [A_\mu, \Xi_a]
\]
satisfies the covariant continuity equation
\[
D_\mu j_\mu = 0,
\]
as required for minimal coupling of an external current in the Yang-Mills equations.

**Variational equations.** To formulate the generalized instanton equations (2.7) as absolute minima of Euler-Lagrange equations derived from an action principle, we note that the presence of the current (2.10) in the Yang-Mills equations (2.9) requires the addition of the term
\[
\frac{1}{2} \text{tr} \, j_\mu \, A_\mu
\]
in the standard Yang-Mills lagrangian
\[
\mathcal{L}_{YM} = -\frac{1}{8} \text{tr} \, F_{\mu \nu} \, F_{\mu \nu}.
\]
Up to a total derivative the term (2.12) is equivalent to the term
\[
\frac{1}{2} \omega^a_{\mu \nu} \text{tr} \, F_{\mu \nu} \, \Xi_a.
\]
After adding the term (2.14), together with the non-dynamical term
\[
-3 \text{tr} \, \Xi_a \, \Xi_a
\]
and the topological density
\[
-\frac{1}{2} \varepsilon_{\mu \nu \lambda \sigma} \text{tr} \, F_{\mu \nu} \, F_{\lambda \sigma},
\]
later on we will consider an important example of such non-central elements \(\Xi_a\).
we obtain the lagrangian
\[ \mathcal{L} = -\frac{1}{4} \text{tr} \left( F^+_{\mu\nu} - \omega^a_{\mu\nu} \Xi_a \right) \left( F^+_{\mu\nu} - \omega^a_{\mu\nu} \Xi_a \right), \tag{2.17} \]
where
\[ F^+ = \frac{1}{2} (\ast F + F) \tag{2.18} \]
is the self-dual part of the curvature two-form \( F \). In the following we will consider constant matrices \( \Xi_a \) for which (2.15) becomes constant and the term (2.14) is topological. Constant matrices of the form \( \Xi_a = i \xi_a 1_N \) correspond to D3-branes in a non-zero \( B \)-field in string theory and can be described in terms of a noncommutative deformation of Yang-Mills theory on the space \( \mathbb{R}^4 \) (see e.g. [19, 20]).

### 3 Instanton equations on \( \mathbb{R}^4/\Gamma \)

**Orbifold \( \mathbb{R}^4/\Gamma \).** The complex structure \( J = J^3 \), introduced in (2.5), defines the complex coordinates
\[ z^1 = y^1 + i y^2 \quad \text{and} \quad z^2 = y^3 + i y^4 \tag{3.1} \]
on \( \mathbb{R}^4 \cong \mathbb{C}^2 \), where \( y^\mu \) are real coordinates. The Lie group SU(2) naturally acts on the vector space \( \mathbb{C}^2 \) with the coordinates (3.1). We are interested in the Kleinian orbifolds \( \mathbb{C}^2/\Gamma \) where \( \Gamma \) is a finite subgroup of SU(2). They have an ADE classification in which \( \Gamma \) is associated with the extended Dynkin diagram of a simply-laced simple Lie algebra. For the \( A_k \)-type simple singularities, corresponding to the cyclic group \( \Gamma = \mathbb{Z}_{k+1} \) of order \( k+1 \), explicit descriptions of instantons will be readily available. However, most of our results can be generalized to the other ADE groups \( \Gamma \) corresponding to nonabelian orbifolds \( \mathbb{C}^2/\Gamma \).

The action of \( \Gamma = \mathbb{Z}_{k+1} \) on \( \mathbb{C}^2 \) is given by
\[ (z^1, z^2) \, \mapsto \, (\zeta z^1, \zeta^{-1} z^2), \tag{3.2} \]
where
\[ \zeta = \exp \left( \frac{2\pi i}{k+1} \right) \quad \text{with} \quad \zeta^{k+1} = 1 \tag{3.3} \]
is a primitive \((k+1)\)-th root of unity. This action has a single isolated fixed point at the origin \((z^1, z^2) = (0,0)\). The orbifold \( \mathbb{C}^2/\Gamma \) is defined as the set of equivalence classes on \( \mathbb{C}^2 \) with respect to the equivalence relation
\[ (\zeta z^1, \zeta^{-1} z^2) \equiv (z^1, z^2), \tag{3.4} \]
and it has a singularity at the origin. The metric on \( \mathbb{C}^2/\Gamma \) is
\[ g = dz^1 \otimes d\bar{z}^1 + dz^2 \otimes d\bar{z}^2, \tag{3.5} \]
where the coordinates \( z^1, \bar{z}^2 \) are complex conjugated to \( z^1, z^2 \).

**V-bundles on \( \mathbb{C}^2/\Gamma \).** A V-bundle on \( \mathbb{C}^2/\Gamma \) is a \( \Gamma \)-equivariant bundle over \( \mathbb{C}^2 \), i.e. a vector bundle on \( \mathbb{C}^2 \) with a \( \Gamma \)-action on the fibres which is compatible with the action of \( \Gamma \) on \( \mathbb{C}^2 \). The orbifold group \( \Gamma = \mathbb{Z}_{k+1} \) has \( k+1 \) one-dimensional irreducible representations such that the generator of \( \mathbb{Z}_{k+1} \) acts on the \( \ell \)-th \( \Gamma \)-module as multiplication by \( \zeta^\ell \) for \( \ell = 0, 1, \ldots, k \). Let us denote by \( \mathcal{E}_\ell \) complex V-bundles over \( \mathbb{C}^2/\Gamma \) of rank \( N_\ell \) on which \( \Gamma \) acts in the \( \ell \)-th irreducible representation as
\[ v_\ell \, \mapsto \, \zeta^\ell v_\ell \quad \text{for} \quad v_\ell \in \mathbb{C}^{N_\ell}. \tag{3.6} \]
on a generic fibre $\mathbb{C}^{N_\ell}$ of $\mathcal{E}_\ell$. Then any complex $V$-bundle $\mathcal{E}$ over $\mathbb{C}^2/\Gamma$ of rank $N$ can be decomposed into isotopical components as a Whitney sum

$$\mathcal{E} = \bigoplus_{\ell=0}^{k} \mathcal{E}_\ell ,$$

(3.7)

and its structure group is of the form

$$\prod_{\ell=0}^{k} U(N_\ell) \quad \text{with} \quad \sum_{\ell=0}^{k} N_\ell = N .$$

(3.8)

From (3.6) it follows that the action of the point group on the $V$-bundle (3.7) is given by the unitary matrices

$$v \mapsto \gamma_\ell (v) \quad \text{with} \quad \gamma_\ell = \bigoplus_{\ell=0}^{k} \zeta_\ell \mathbf{1}_{N_\ell}$$

(3.9)

on vectors $v = (v_\ell)_{\ell=0}^{k}$ in the generic fibre $\mathbb{C}^N = \bigoplus_{\ell=0}^{k} \mathbb{C}^{N_\ell}$ of $\mathcal{E}$.

Simplifying the situation discussed in the previous section, we choose matrices $\Xi_a$ in the form

$$\Xi_a = \bigoplus_{\ell=0}^{k} i \xi_\ell \mathbf{1}_{N_\ell} ,$$

(3.10)

where $\xi_\ell \in \mathbb{R}$ are constants. The matrices (3.10) belong to the center of the Lie algebra of the gauge group (3.8). The diagonal $U(1)$ subgroup of scalars in (3.8) acts trivially on $(\mathcal{E}, \mathcal{A})$, so we can factor the gauge group (3.8) by this $U(1)$ subgroup to get the quotient group

$$G := \left( \prod_{\ell=0}^{k} U(N_\ell) \right) / U(1) .$$

(3.11)

Then the Lie algebra $\mathfrak{g}$ of $G$ is the traceless part of the Lie algebra of (3.8), and one should impose on $\xi_\ell$ in (3.10) the tracelessness condition

$$\sum_{\ell=0}^{k} \xi_\ell N_\ell = 0 ,$$

(3.12)

which defines the center $\mathfrak{h}$ of $\mathfrak{g}$.

**Γ-equivariant connections.** Consider a one-form

$$W = W_\mu \, dy^\mu = W_{z_1} \, dz_1 + W_{z_2} \, dz_2 + W_{\bar{z}_1} \, d\bar{z}_1 + W_{\bar{z}_2} \, d\bar{z}_2$$

(3.13)

on $\mathbb{R}^4 \cong \mathbb{C}^2$ which is invariant under the action of $\Gamma \subset SU(2) \subset SO(4)$ defined by (3.2). Then on the components

$$W_{z_1} = \frac{1}{2} (W_1 - i W_2) \quad \text{and} \quad W_{z_2} = \frac{1}{2} (W_3 - i W_4)$$

(3.14)

the action of $\Gamma$ is given by

$$W_{z_1} \mapsto \zeta^{-1} W_{z_1} \quad \text{and} \quad W_{z_2} \mapsto \zeta W_{z_2} .$$

(3.15)

The action of $\Gamma$ on the components $\mathcal{A}_\mu$ of any unitary connection $\mathcal{A} = \mathcal{A}_\mu \, dy^\mu$ on a hermitian $V$-bundle (3.7) is given by a combination of the spacetime action (3.15) and the adjoint action generated by (3.9) as

$$\mathcal{A}_{z_1} \mapsto \zeta^{-1} \gamma_\ell \mathcal{A}_{z_1} \gamma_\ell^{-1} \quad \text{and} \quad \mathcal{A}_{z_2} \mapsto \zeta \gamma_\ell \mathcal{A}_{z_2} \gamma_\ell^{-1} .$$

(3.16)
The corresponding $\Gamma$-equivariance conditions require that the connection defines a covariant representation of the orbifold group, in the sense that
\[
\gamma^\Gamma \ A_{a_1} \ 
\gamma^{-1} = \zeta \ A_{a_2} \quad \text{and} \quad \gamma^\Gamma \ A_{a_2} \ 
\gamma^{-1} = \zeta^{-1} \ A_{a_2} .
\] (3.17)
It is easy to see that the solutions to the constraint equations (3.17) are given by block off-diagonal matrices
\[
A_{a_1} = \begin{pmatrix} 0 & 0 & \cdots & 0 & \psi_{k+1} \\ \psi_1 & 0 & \cdots & 0 & 0 \\ 0 & \psi_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & 0 & 0 \\ 0 & 0 & \cdots & 0 & \psi_k \end{pmatrix}
\quad \text{and} \quad
A_{a_2} = \begin{pmatrix} 0 & \phi_1 & 0 & \cdots & 0 \\ \phi_1 & 0 & \phi_2 & \cdots & 0 \\ 0 & 0 & \phi_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & \phi_k \end{pmatrix}
\] (3.18)
together with $A_{a_1} = -A_{a_1}^\dagger$ and $A_{a_2} = -A_{a_2}^\dagger$. Here the bundle morphisms $\psi_{\ell+1} : \mathcal{E}_\ell \rightarrow \mathcal{E}_{\ell+1}$ and $\phi_{\ell+1} : \mathcal{E}_{\ell+1} \rightarrow \mathcal{E}_\ell$ are bifundamental scalar fields given fibrewise by matrices
\[
\psi_{\ell+1} \in \text{Hom}(\mathbb{C}^{N_\ell}, \mathbb{C}^{N_{\ell+1}}) \quad \text{and} \quad \phi_{\ell+1} \in \text{Hom}(\mathbb{C}^{N_{\ell+1}}, \mathbb{C}^{N_\ell})
\] (3.19)
for $\ell = 0, 1, \ldots, k$ (with indices read modulo $k+1$). Substitution of (3.18) in (2.7) then yields the generalized instanton equations on the orbifold $\mathbb{C}^2/\Gamma$. The transformations (3.15) are defined for the holonomic basis $d\eta^\mu$ of one-forms on $\mathbb{R}^4/\Gamma$ and can differ for other bases of one-forms, leading to modifications of the formulas (3.16)–(3.18).

4 Translationally-invariant instantons

Matrix equations. Consider translationally-invariant connections $\mathcal{A}$ on the V-bundle (3.7) over $\mathbb{C}^2/\Gamma$ satisfying the equations (2.7) with $\Xi_a$ given in (3.10), i.e. we assume that $\mathcal{A}_\mu$ are independent of the coordinates $y^\mu$, which reduces (2.7) to the matrix equations (1.1) with $W_\mu := \mathcal{A}_\mu$. Denoting $B_1 := \mathcal{A}_1$ and $B_2 := \mathcal{A}_2$ for $\mathcal{A}$ given by (3.18) with constant matrices $\psi_{\ell+1}$ and $\phi_{\ell+1}$ for $\ell = 0, 1, \ldots, k$, we obtain the equations
\[
\frac{1}{2} \eta^{a\mu}_{\nu} [W_\mu, W_\nu] = \Xi_a ,
\] (4.1)
which can be rewritten as
\[
[B_1, B_2] = -\frac{i}{2} (\Xi_1 - i \Xi_2) =: \Xi_\mathbb{C} ,
\] (4.2)
\[
[B_1, B_1^\dagger] + [B_2, B_2^\dagger] = -\frac{i}{2} \Xi_3 =: \Xi_\mathbb{R} .
\] (4.3)
Solutions to these equations satisfy the reduced Yang-Mills equations (2.9) with the external source
\[
j_\mu = -4 \eta^{a\mu}_{\nu} [W_\nu, \Xi_a] ,
\] (4.4)
where $W_\mu$ is given by (3.18) and $\Xi_a$ by (3.10).

Hyper-Kähler quotients. The reduced equations (4.1) (and also the instanton equations (2.7)) can be interpreted as hyper-Kähler moment map equations. For this, recall that if $(M, g, \omega^a)$ is a hyper-Kähler manifold with an action of a Lie group $G$ which preserves the metric $g$ and the three Kähler forms $^3\omega^a$, then one can define three moment maps
\[
\mu^a : M \rightarrow g^a
\] (4.5)
$^3$They are Kähler with respect to the three complex structures $J^a = \omega^a \circ g^{-1}$. With respect to the complex structure $J^1$, the two-form $\omega_\mathbb{R} = \omega^3$ is Kähler and $\omega_\mathbb{C} = \omega^1 + i \omega^2$ is holomorphic.
taking values in the dual $\mathfrak{g}^*$ of the Lie algebra $\mathfrak{g}$ of $G$ such that, for each $\xi \in \mathfrak{g}$ with triholomorphic Killing vector field $L_\xi$ generated by the $G$-action on $M$, the functions (4.5) satisfy the equations

$$\langle d\mu^a, \xi \rangle = L_\xi \lrcorner \omega^a,$$

(4.6)

where $\langle - , - \rangle$ is the dual pairing between elements of $\mathfrak{g}^*$ and $\mathfrak{g}$, and $\lrcorner$ denotes contraction of vector fields and differential forms. Denoting by $\mu = (\mu^1, \mu^2, \mu^3)$ the vector-valued moment map

$$\mu : M \longrightarrow \mathbb{R}^3 \otimes \mathfrak{g}^*,$$

(4.7)

we can consider the $G$-invariant level set

$$\mu^{-1}(\Xi)$$

(4.8)

which defines a submanifold of the manifold $M$, where $\Xi = (\Xi_1, \Xi_2, \Xi_3) \in \mathbb{R}^3 \otimes \mathfrak{h}^*$ and $\mathfrak{h}$ is the center of $\mathfrak{g}$. Then one can define the hyper-Kähler quotient as (see e.g. [10, 11, 13])

$$M_\xi = \mu^{-1}(\Xi) \sslash G,$$

(4.9)

where $\xi = (\xi^\ell_a)$ are parameters defining $\Xi = (\Xi_a) \in \mathbb{R}^3 \otimes \mathfrak{h}^*$. The hyper-Kähler metric on $M$ descends to a hyper-Kähler metric on the quotient $M_\xi$. When the group action is free, the reduced space $M_\xi$ is a hyper-Kähler manifold of dimension $\dim M_\xi = \dim M - 4 \dim G$.

In the case of the matrix model (4.1), the manifold $M$ is the flat hyper-Kähler manifold

$$M = \mathbb{R}^4 \otimes \mathfrak{u}(N),$$

(4.10)

the group $G$ is given in (3.11) and the three moment maps are

$$\mu^a(W) = \frac{1}{2} \eta^a_{\mu \nu} [W_\mu, W_\nu] \in \mathfrak{u}(N).$$

(4.11)

Solutions of the equations (4.2)–(4.3) form a submanifold $\mu^{-1}(\Xi)$ of the manifold (4.10), and by factoring with the gauge group (3.11) (which for generic parameters $\xi = (\xi^\ell_a)$ acts freely on the solutions) we obtain the moduli space (4.9). This moduli space was studied by Kronheimer [10], who showed that for $\Gamma = \mathbb{Z}_{k+1}$ and the Coulomb branch $N_0 = N_1 = \cdots = N_k = 1$ it is a smooth four-dimensional asymptotically locally euclidean (ALE) hyper-Kähler manifold $M_\xi$ with metric defined by the parameters $\xi = (\xi^\ell_a)$. The ALE condition means that at asymptotic infinity of $M_\xi$ the metric approximates the euclidean metric on the orbifold $\mathbb{C}^2/\Gamma$. Kronheimer also shows that $M_\xi$ is diffeomorphic to the minimal smooth resolution of the Kleinian singularity $M_0 = \mathbb{C}^2/\Gamma$, regarded as the affine algebraic variety $x^{k+1} + y^2 + z^2 = 0$ in $\mathbb{C}^3$. For the Hilbert-Chow map

$$\pi : M_\xi \longrightarrow M_0$$

(4.12)

the exceptional divisor of the blow-up is the set

$$\pi^{-1}(0) = \bigcup_{\ell=0}^k \Sigma_\ell,$$

(4.13)

where $\Sigma_\ell \cong \mathbb{C}P^1$ and $k = \#\Gamma - 1$. The parameters $\xi$ determine the periods of the three symplectic forms $\omega^a$ as

$$\int_{\Sigma_\ell} \omega^a = \xi_\ell^a.$$

(4.14)

---

4. We identify $\mathfrak{u}(N)^*$ and $\mathfrak{u}(N)$.

5. Recall that we consider $\Gamma = \mathbb{Z}_{k+1}$ for definiteness here, but many of these considerations generalize to the other Kleinian groups $\Gamma \subset \text{SU}(2)$. In the general case, $N_\ell$ are the dimensions of the irreducible representations of the finite group $\Gamma$ in Kronheimer’s construction.
In the general case \( \ell \geq 0 \), one can also define a map \( M_\ell \to M_0 \) which is a resolution of singularities [21].

**Hermitian Yang-Mills connections.** The matrix \( \Xi_C \) in (4.2) parametrizes deformations of the complex structure on the \( \mathcal{V} \)-bundle \( \mathcal{E} \) and it can be reabsorbed through a non-analytic change of coordinates on the space (4.10) [10, 22]. Therefore we may take \( \Xi_C = 0 \) without loss of generality; in this case the ALE space \( M_\ell \) is biholomorphic to the minimal resolution. In fact, the moduli spaces \( M_\ell \) and \( M_\ell' \) are diffeomorphic for distinct \( \xi \) and \( \xi' \) such that \( \Xi_R \neq 0 \) for both sets of parameters. For \( \Xi_C = 0 \) we have \( \Xi_1 = \Xi_2 = 0 \) and the equations (2.7) become the hermitian Yang-Mills equations \([23, 24]\)

\[
*F + F = \omega^3 \Xi_3 .
\] (4.15)

A connection \( \mathcal{A} \) on \( \mathcal{E} \) satisfying (4.15) is said to be a hermitian Yang-Mills connection. It defines a holomorphic structure on \( \mathcal{E} \) since from (4.15) it follows that the curvature \( F \) is of type \((1, 1)\) with respect to the complex structure \( J \), i.e.

\[
F^{2,0} = 0 = F^{0,2} ,
\] (4.16)

and the third equation from (4.15),

\[
\omega^3_{\mu\nu} F_{\mu\nu} = \Xi_3 ,
\] (4.17)

means that for \( \Xi_3 = i \xi \mathbf{1}_N \) the \( \mathcal{V} \)-bundle \( \mathcal{E} \) is (semi-)stable \([23, 24]\). In the special case \( \Xi_3 = 0 \) we get the standard anti-self-dual Yang-Mills equations

\[
*F = -F .
\] (4.18)

**Translationally-equivariant instantons.** Instead of constant matrices \( A_\mu \) which reduce (2.7) to the matrix equations (4.1), one can also consider the gauge potential

\[
\mathcal{A} = \frac{1}{2} \omega^a_{\mu\nu} \Xi_a y^\mu dy^\nu ,
\] (4.19)

where the commuting matrices \( \Xi_a \) are given in (3.10). The connection (4.19) is translationally-invariant up to a gauge transformation and can be extended to the orbifold \( T^4/\Gamma \), where \( T^4 \) is a four-dimensional torus. The curvature of \( \mathcal{A} \) is

\[
\mathcal{F} = d\mathcal{A} = \frac{1}{2} \omega^a_{\mu\nu} \Xi_a dy^\mu \wedge dy^\nu ,
\] (4.20)

providing in essence the three symplectic structures \( \omega^a \) from (2.1).

5 **Spherically-symmetric instantons**

**Cone \( C(S^3/\Gamma) \).** The euclidean space \( \mathbb{R}^4 \) can be regarded as a cone over the three-sphere \( S^3 \),

\[
\mathbb{R}^4 \setminus \{0\} = C(S^3)
\] (5.1)

with the metric

\[
g = \delta_{\mu\nu} dy^\mu \otimes dy^\nu = dr^2 + r^2 \delta_{ab} e^a \otimes e^b ,
\] (5.2)

where \( r^2 = \delta_{\mu\nu} y^\mu y^\nu \) and \((e^a)\) give a basis of left \( SU(2) \)-invariant one-forms on \( S^3 \). One can define \( e^a \) by the formula

\[
e^a := -\frac{1}{r^2} \eta^a_{\mu\nu} y^\mu dy^\nu ,
\] (5.3)
where the 't Hooft tensors $\eta^a_{\mu\nu}$ are defined in (2.2). The one-forms $e^a$ are dual to the vector fields $E^a$ from (1.3). By using the identities

$$e^a_{bc} \eta^b_{\mu\nu} \eta^c_{\lambda\sigma} = \delta_{\mu\lambda} \eta^a_{\nu\sigma} - \delta_{\mu\sigma} \eta^a_{\nu\lambda} - \delta_{\nu\lambda} \eta^a_{\mu\sigma} + \delta_{\nu\sigma} \eta^a_{\mu\lambda},$$  
$$\delta_{ab} \eta^a_{\mu\nu} \eta^b_{\lambda\sigma} = \delta_{\mu\lambda} \delta_{\nu\sigma} - \delta_{\mu\sigma} \delta_{\nu\lambda} + \varepsilon_{\mu\nu\lambda\sigma},$$  
one can easily verify the Maurer-Cartan equations

$$de^a + e^a_{bc} e^b \wedge e^c = 0$$  
and

$$\omega^a = \frac{1}{2} \eta^a_{\mu\nu} dy^\mu \wedge dy^\nu = \frac{1}{2} \eta^a_{\mu\nu} \hat{e}^\mu \wedge \hat{e}^\nu,$$

where

$$\hat{e}^a := e^a \quad \text{and} \quad \hat{e}^4 := dr.$$  
The relation (5.2) between the metric in cartesian and spherical coordinates can be readily checked as well.

All formulas (5.2)–(5.8) are also valid for the orbifold $\mathbb{C}^2/\Gamma$ after imposing the equivalence relation (3.4), and the orbifold is a cone over the lens space $S^3/\Gamma,^6$

$$(\mathbb{C}^2 \setminus \{0\})/\Gamma = C(S^3/\Gamma),$$  
with the metric (5.2). The one-forms (5.3) in the complex coordinates (3.1) have the form

$$e^1 + i e^2 = \frac{1}{r^2} \left( z^1 dz^2 - z^2 dz^1 \right) \quad \text{and} \quad e^3 + i e^4 = \frac{1}{r^2} \left( z^3 dz^1 + z^1 dz^3 \right),$$  
plus their complex conjugated expressions. Hence (5.10) defines two complex one-forms which are (1,0)-forms with respect to the complex structure $J = J^3$ defined in (2.5). The symplectic two-forms (5.7) and the complex structures (2.5) have the same components in the holonomic ($dy^\mu, \frac{\partial}{\partial y^\mu}$) and non-holonomic ($\hat{e}^a, \hat{E}_a$) bases, where $\hat{E}_a \hat{e}^b = \delta_a^b$. From (1.5), (3.2) and (5.10) it follows that

$$e^a \quad \text{and} \quad e^4 := \frac{dr}{r} = d\tau \quad \text{with} \quad \tau = \log r$$  
are invariant under the action of the finite group $\Gamma \subset SU(2)$.

**Nahm equations.** Consider the complex V-bundle $E$ over $\mathbb{C}^2/\Gamma$ described in Section 3. Let

$$\mathcal{A} = \hat{X}_\mu \hat{e}^\mu = \frac{1}{2} (\hat{X}_1 - i \hat{X}_2) (e^1 + i e^2) + \frac{1}{2} (\hat{X}_3 - i \hat{X}_4) (e^3 + i e^4) + \text{h.c.}$$  
be a connection on $E$ written in the basis (5.8). The corresponding $\Gamma$-equivariance conditions are

$$\gamma_\tau (\hat{X}_1 - i \hat{X}_2) \gamma_\tau^{-1} = \hat{X}_1 - i \hat{X}_2 \quad \text{and} \quad \gamma_\tau (\hat{X}_3 - i \hat{X}_4) \gamma_\tau^{-1} = \hat{X}_3 - i \hat{X}_4.$$  
Solutions to these equations are given by

$$\frac{1}{2} (\hat{X}_1 - i \hat{X}_2) = \text{diag}(\chi_0, \chi_1, \ldots, \chi_k) \quad \text{and} \quad \frac{1}{2} (\hat{X}_3 - i \hat{X}_4) = \text{diag}(\varphi_0, \varphi_1, \ldots, \varphi_k),$$  
$$\frac{1}{2} (\hat{X}_1 + i \hat{X}_2) = -\text{diag}(\chi_0^\dagger, \chi_1^\dagger, \ldots, \chi_k^\dagger) \quad \text{and} \quad \frac{1}{2} (\hat{X}_3 + i \hat{X}_4) = -\text{diag}(\varphi_0^\dagger, \varphi_1^\dagger, \ldots, \varphi_k^\dagger).$$  

---

^6The orbifolds $S^3/\Gamma$ for arbitrary ADE point groups $\Gamma$ exhaust the possible Sasaki-Einstein manifolds in three dimensions.
where $\chi_\ell$ and $\varphi_\ell$ are $N_\ell \times N_\ell$ complex matrices. Thus the $\Gamma$-equivariance conditions in the basis (5.8) forces the block-diagonal form (5.14) of the connection components $\hat{X}_\mu$, i.e. the connection $\mathcal{A}$ is reducible or else $N_\ell = 0$ for $\ell \neq 0$ if $\Gamma$ acts trivially on $E$.

The instanton equations (2.7) are conformally invariant and it is more convenient to consider them on the cylinder 
$$\mathbb{R} \times S^3 / \Gamma$$
with the metric 
$$g_{\text{cyl}} = d\tau^2 + \delta_{ab} e^a \otimes e^b = \frac{dr^2}{r^2} + \delta_{ab} e^a \otimes e^b = \frac{1}{r^2} g .$$
In the basis $(e^\mu) = (e^a, d\tau)$ the SU(2)-invariant (spherically-symmetric) connection $\mathcal{A}$ and its curvature $\mathcal{F}$ have components depending only on $r = e^\tau$ and are given by
\[
\mathcal{A} = X_\mu e^\mu \quad \text{with} \quad X_\mu = r \hat{X}_\mu ,
\]
and (2.7) reduce to a form of the generalized Nahm equations given by
\[
\frac{dX_a}{d\tau} = -[X_\tau, X_a] - 2X_a + \frac{1}{2} \varepsilon_{abc} [X_b, X_c] - \Xi_a .
\]
Introducing
\[
Y_\mu := e^{2\tau} X_\mu \quad \text{and} \quad s = e^{-2\tau} = \frac{1}{r^2} ,
\]
we obtain the equations
\[
2 \frac{dY_a}{ds} = [Y_\tau, Y_a] - \frac{1}{2} \varepsilon_{abc} [Y_b, Y_c] + \frac{1}{s^2} \Xi_a .
\]
For $\Xi_a = 0$ these equations coincide with the Nahm equations [25]. Choosing $\Xi_a = 0$ and defining
\[
\alpha := \frac{1}{2} (Y_3 + iY_4) \quad \text{and} \quad \beta := \frac{1}{2} (Y_1 + iY_2) ,
\]
we obtain the equations
\[
\frac{d}{ds} (\alpha + \alpha^\dagger) + [\alpha, \alpha^\dagger] + [\beta, \beta^\dagger] = 0 ,
\]
considered by Kronheimer [14, 15] (see also [16]) in the description of SU(2)-invariant instantons. The equations (5.21) have three obvious solutions which we now consider in turn.

**Abelian instantons with $\Xi_a \neq 0$.** For the first solution, we choose
\[
Y_a = -\frac{1}{2s} \Xi_a = -\frac{s^2}{2} \Xi_a \quad \text{and} \quad Y_4 = 0 .
\]
Then we get the solution
\[
\mathcal{A} = -\frac{1}{2} r^2 e^a \Xi_a \quad \text{and} \quad \mathcal{F} = \frac{1}{2} \eta_{\mu\nu} \Xi_a \hat{e}^\mu \wedge \hat{e}^\nu
\]
of the equations (2.7), which coincide with (4.19) and (4.20). This configuration can also be regarded as a translationally-equivariant solution of the self-dual Yang-Mills equations
\[
\ast \mathcal{F} = \mathcal{F} ,
\]
i.e. as an anti-instanton on $\mathbb{R}^4/\Gamma$ or $T^4/\Gamma$.

**Abelian instantons with poles.** For the second solution, considered in [15], we put $\Xi_a = 0$ and $\frac{dY_a}{ds} = 0$. Then the constant matrices $Y_{\mu}$ satisfy the reduced anti-self-dual Yang-Mills equations

$$[Y_a, Y_4] + \frac{1}{2} \varepsilon_{abc} [Y_b, Y_c] = 0$$  \hspace{1cm} (5.28)

considered in [10] and discussed in Section 4. Solutions to (5.28) are necessarily given by commuting matrices [10], and one can choose them in the form

$$Y_a = 2\Lambda^2 \hat{\Xi}_a \quad \text{and} \quad Y_4 = 0 ,$$  \hspace{1cm} (5.29)

where $\hat{\Xi}_a$ have the form (3.10) and $\Lambda$ is a scale parameter. For the corresponding gauge potential and its field strength, we obtain

$$A = X_a \epsilon^a = \frac{2\Lambda^2}{r^3} \hat{\Xi}_a \epsilon^a \quad \text{and} \quad F = -\frac{2\Lambda^2}{r^4} \tilde{\eta}_{a\mu
u} \hat{\Xi}_a \hat{\epsilon}^\mu \wedge \hat{\epsilon}^\nu ,$$  \hspace{1cm} (5.30)

where $\hat{\epsilon}^\mu$ are given in (5.8) and $\tilde{\eta}_{a\mu
u}$ are the anti-self-dual ’t Hooft tensors defined by

$$\tilde{\eta}_{b c} = \varepsilon_{b c} \quad \text{and} \quad \tilde{\eta}_{b 4} = -\delta_a^b .$$  \hspace{1cm} (5.31)

Thus we obtain singular abelian solutions with delta-function sources in the Maxwell equations, as discussed by [15]. The gauge potential $A$ from (5.30) can be regarded as an asymptotic approximation of a smooth solution. Note also that

$$\tilde{\omega}^a := -\frac{2\Lambda^2}{r^4} \tilde{\eta}_{a\mu
u} \hat{\epsilon}^\mu \wedge \hat{\epsilon}^\nu$$  \hspace{1cm} (5.32)

can be viewed as three additional anti-self-dual symplectic forms on the cone $(\mathbb{R}^4 \setminus \{0\})/\Gamma = C(S^3/\Gamma)$, complimentary to those given in (2.1).

**’t Hooft instantons on $C^2/\Gamma$.** For the third solution we choose $Y_4 = Y_\tau = 0 = \Xi_a$ to get

$$Y_a = \frac{2}{s + \Lambda^{-2}} I_a = \frac{2\Lambda^2 r^2}{r^2 + \Lambda^2} I_a \quad \text{with} \quad \Lambda \in \mathbb{R} \quad \text{and} \quad [I_a, I_b] = \epsilon_{a b} I_c .$$  \hspace{1cm} (5.33)

Then for the anti-self-dual connection and curvature we obtain

$$A = \frac{2\Lambda^2}{r^2 + \Lambda^2} \epsilon^a I_a \quad \text{and} \quad F = -\frac{2\Lambda^2}{(r^2 + \Lambda^2)^2} \tilde{\eta}_{a\mu
u} I_a \hat{\epsilon}^\mu \wedge \hat{\epsilon}^\nu ,$$  \hspace{1cm} (5.34)

where we used the relation $s = r^{-2}$. Here $I_a$ are the generators of the group SU(2) embedded in the broken gauge group (3.11), i.e. there are $k+1$ instanton solutions with gauge group $SU(2) \subset U(N_\ell)$ if $N_\ell \geq 2$ for all $\ell = 0, 1, \ldots, k$. From the explicit form of $\epsilon^a$ in (5.3) it follows that each of these solutions is the standard ’t Hooft instanton generalized from $\mathbb{R}^4$ to $\mathbb{R}^4/\Gamma$. For framed instantons \footnote{Framed instantons are instanton solutions modulo SU(2)-invariant gauge transformations which approach the identity at asymptotic infinity.} there are four moduli: the scale parameter $\Lambda$ and three global SU(2) rotational parameters (see e.g. [22]).

**Moduli spaces of SU(2)-invariant instantons.** In the special case where $\Gamma$ is the trivial group, we obtain SU(2)-invariant solutions of the anti-self-dual Yang-Mills equations (4.18) on $\mathbb{R}^4 \setminus \{0\} = C(S^3)$. The moduli spaces of these framed instantons (subject to appropriate boundary conditions)
are four-dimensional hyper-Kähler ALE spaces $M_\xi$ resolving $M_0 = \mathbb{C}^2/\Gamma'$ as in (4.12), where $\Gamma'$ is a finite subgroup of the group $\text{SU}(2)$ related to boundary conditions for the solutions [14]–[16]. This is the moduli space of the spherically-symmetric instanton which has the minimal topological charge $c_2(\mathcal{E}) = (\#\Gamma' - 1)/\#\Gamma'$. In our reducible case we obtain a product of hyper-Kähler moduli spaces

$$M_{\xi_0} \times M_{\xi_1} \times \cdots \times M_{\xi_k}. \quad (5.35)$$

Note that $M_{\xi_k}$ is a point if $N_\ell = 1$. For $N_\ell = 1$ one can also use the singular abelian solution from (5.30),

$$\mathcal{F}_e = -\frac{2\Lambda^2}{r^4} \tilde{\eta}_{\mu\nu} \xi^\ell \delta^\ell_{\mu} \wedge \delta^\ell_{\nu}, \quad (5.36)$$

with $\xi_a^\ell \in \mathbb{R}$.

We have seen that for constant matrices $Y_\alpha, Y_\tau$ the moduli space is the orbifold $M_0 = \mathbb{C}^2/\Gamma$. For $s$-dependent solutions $Y_\alpha, Y_\tau$, similarly to [10, 16] one can choose boundary conditions such that each block tends to a constant multiple of the identity $1_{N_\ell}$ in the limits $\tau \to \pm \infty$, while as $\tau \to 0$ the solutions define a representation of $\text{SU}(2)$. For $N_0 = N_1 = \cdots = N_k = 1$ it is natural to expect that the corresponding moduli space of solutions is a resolution of the orbifold $\mathbb{C}^2/\Gamma$.

**BPST instantons on $\mathbb{C}^2/\Gamma$.** Instead of the one-forms (5.3), one can introduce a basis of right $\text{SU}(2)$-invariant one-forms on $S^3/\Gamma$ given by

$$\tilde{e}^a := -\frac{1}{r^2} \tilde{\eta}_{\mu\nu} y^\mu dy^\nu. \quad (5.37)$$

They are dual to the vector fields $\tilde{E}^a$ given in (1.3), and they satisfy the relations

$$d\tilde{e}^a + \epsilon_{bc}^a \tilde{e}^b \wedge \tilde{e}^c = 0, \quad (5.38)$$

$$g = \delta_{\mu\nu} dy^\mu \otimes dy^\nu = dr^2 + r^2 \delta_{ab} \tilde{e}^a \otimes \tilde{e}^b \quad (5.39)$$

which are similar to those for $e^a$ and can be proven by using identities for $\tilde{\eta}_{\mu\nu}$ analogous to (5.4)–(5.5).

The complex combinations

$$\tilde{e}^1 + i \tilde{e}^2 = \frac{1}{r^2} \left( z^1 dz^2 - z^2 dz^1 \right) \quad \text{and} \quad \tilde{e}^3 + i \tilde{e}^4 = \frac{1}{r^2} \left( z^1 dz^1 + z^2 dz^2 \right) \quad (5.40)$$

are neither $(1,0)$- nor $(0,1)$-forms with respect to the complex structure $J = J^3$. One can show that the forms (5.40) are $(1,0)$-forms with respect to the complex structure

$$\tilde{J} = \tilde{J}^3 := (\tilde{\eta}_{\mu\lambda} \delta^{\lambda\nu}) \quad (5.41)$$

which is used in the consideration of self-duality equations (and anti-instantons) on $\mathbb{R}^4/\Gamma$. The one-forms (5.10) and (5.40) are related by the coordinate change $z^2 \mapsto \tilde{z}^2$ or, equivalently, by the change of orientation $x^4 \mapsto -x^4$ of $\mathbb{R}^4/\Gamma$. For a fixed orientation, this inequivalence becomes more apparent in the case of the $\mathbb{C}P^2$, $K3$ and ALE hyper-Kähler manifolds. Note that exactly $\tilde{e}^a$ (but not $e^a$) form a basis of one-forms on the Sasaki-Einstein space $S^3/\Gamma \subset \mathbb{R}^4/\Gamma$, since the complex structure on $\mathbb{C}P^1 \hookrightarrow S^3/\Gamma$ is matched with (5.39)–(5.41) but not with (2.5) or (5.10). In any case, $\tilde{e}^a$ are suitable one-forms on $\mathbb{R}^4/\Gamma$ which can be used in the ansatz for instanton solutions.

Let

$$\mathcal{A} = \tilde{X}_\mu \tilde{e}^\mu \quad (5.42)$$
be an SU(2)-invariant connection on the V-bundle $\mathcal{E}$ over $\mathbb{R}^4/\Gamma$ given in (3.7). Here $\tilde{e}^a$ are given in (5.37), $\tilde{e}^4 := d\tau = dr/r$ and $\tilde{X}_\mu$ depend only on $r = e^\tau$. The explicit form (5.40) of $\tilde{e}^\mu$ and the $\Gamma$-action (3.2) imply $\Gamma$-equivariance conditions for the components $\tilde{X}_\mu$ given by

$$
\gamma_{r} (\tilde{X}_1 + i \tilde{X}_2) \gamma_{r}^{-1} = \zeta^{-2} (\tilde{X}_1 + i \tilde{X}_2) \quad \text{and} \quad \gamma_{r} (\tilde{X}_3 + i \tilde{X}_4) \gamma_{r}^{-1} = \tilde{X}_3 + i \tilde{X}_4 . \quad (5.43)
$$

For $k \geq 2$ the non-zero blocks of $\tilde{X}_\mu$ solving (5.43) are given by the matrix elements

$$
(\tilde{X}_1 + i \tilde{X}_2)^{\ell,\ell+2} \in \text{Hom}(\mathbb{C}^{N_{\ell+2}}, \mathbb{C}^{N_{\ell}}) \quad \text{and} \quad (\tilde{X}_3 + i \tilde{X}_4)^{\ell,\ell} \in \text{End}(\mathbb{C}^{N_{\ell}}) \quad (5.44)
$$

for $\ell = 0, 1, \ldots, k$, together with corresponding non-zero blocks of $\tilde{X}_1 - i \tilde{X}_2 = -(\tilde{X}_1 + i \tilde{X}_2)^\dagger$ and $\tilde{X}_3 - i \tilde{X}_4 = -(\tilde{X}_3 + i \tilde{X}_4)^\dagger$.

In the following we consider only the case of even rank $k = 2q$, since the odd case $k = 2q+1$ can be reduced to a “doubling” of the even case. Using the property $\xi^{2q+1} = 1$, one has

$$
\text{diag}(1, \zeta^2, \ldots, \zeta^{2k}) = \text{diag}(1, \zeta^2, \ldots, \zeta^{2q}, \zeta, \zeta^3, \ldots, \zeta^{2q-1}) . \quad (5.45)
$$

Then by using the matrix

$$
\gamma_{\ell} = \text{diag}(1_{N_0}, \zeta^2 1_{N_1}, \ldots, \zeta^{2q} 1_{N_q}, \zeta 1_{N_{q+1}}, \zeta^3 1_{N_{q+2}}, \ldots, \zeta^{2q-1} 1_{N_{2q}}) \quad (5.46)
$$

in (5.43), we obtain the solution

$$
\tilde{X}_1 + i \tilde{X}_2 = \begin{pmatrix} 0 & \phi_1 & 0 & \cdots & 0 \\ 0 & 0 & \phi_2 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & \phi_k \\ \phi_{k+1} & 0 & \cdots & 0 & 0 \end{pmatrix} \quad \text{and} \quad \tilde{X}_3 + i \tilde{X}_4 = \begin{pmatrix} \rho_0 & 0 & \cdots & 0 \\ 0 & \rho_1 & \ddots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & \rho_k \end{pmatrix} \quad (5.47)
$$

where $\phi_{\ell+1} \in \text{Hom}(\mathbb{C}^{N_{\ell+1}}, \mathbb{C}^{N_{\ell}})$ and $\rho_{\ell} \in \text{End}(\mathbb{C}^{N_{\ell}})$.

In the basis $(\tilde{e}^\mu) = (\tilde{e}^a, d\tau)$ the SU(2)-invariant connection $\mathcal{A}$ and the curvature $\mathcal{F}$ are given by

$$
\mathcal{A} = \tilde{X}_\mu \tilde{e}^\mu = \tilde{X}_\mu \tilde{e}^\mu \quad \text{with} \quad \tilde{X}_\mu = \frac{1}{r} \tilde{X}_\mu \quad \text{and} \quad \tilde{e}^\mu = r \tilde{e}^\mu , \quad (5.48)
$$

$$
\mathcal{F} = \frac{1}{r^2} \left( \frac{1}{4} \varepsilon_{abc} [\tilde{X}_b, \tilde{X}_c] - \tilde{X}_a \right) \tilde{\eta}_{\mu\nu}^a dy^\mu \wedge dy^\nu + \left( \frac{d\tilde{X}_a}{d\tau} + [\tilde{X}_4, \tilde{X}_a] - 2\tilde{X}_a + \frac{1}{2} \varepsilon_{abc} [\tilde{X}_b, \tilde{X}_c] \right) \tilde{e}^4 \wedge \tilde{e}^a \quad (5.49)
$$

where we used the identity

$$
\tilde{\eta}_{\mu\nu}^a \tilde{e}^\mu \wedge \tilde{e}^\nu = \frac{1}{r^2} \tilde{\eta}_{\mu\nu}^a dy^\mu \wedge dy^\nu . \quad (5.50)
$$

From (5.49) it follows that $\mathcal{F}$ is anti-self-dual, $*\mathcal{F} = -\mathcal{F}$, if $\tilde{X}_a$ satisfy the Nahm equations

$$
\frac{d\tilde{X}_a}{d\tau} = 2\tilde{X}_a - \frac{1}{2} \varepsilon_{abc} [\tilde{X}_b, \tilde{X}_c] - [\tilde{X}_4, \tilde{X}_a] . \quad (5.51)
$$

We obtain a solution by choosing $\tilde{X}_4 = 0$ and taking

$$
\tilde{X}_a = \frac{2r^2}{r^2 + \Lambda^2} I_a \quad \text{with} \quad [I_a, I_b] = \varepsilon_{ab}^c I_c , \quad (5.52)
$$

where the $I_a$ are SU(2) generators in the irreducible representation on the space $\mathbb{C}^N$ with $N = N_0 + N_1 + \cdots + N_k$ that fits with the $\Gamma$-equivariant form (5.47). For instance, one can work in
the Coulomb branch with $N_\ell = 1$ for all $\ell = 0, 1, \ldots, k$ so that $I_a$ embed the group SU(2) into SU($k+1$). We thus obtain the configuration

$$A = -\frac{2}{r^2 + \Lambda^2} \bar{\eta}_{\mu
u} I_a y^\mu d y^\nu \quad \text{and} \quad F = -\frac{2\Lambda^2}{(r^2 + \Lambda^2)^2} \bar{\eta}_{\mu
u} I_a d y^\mu \wedge d y^\nu,$$

which is exactly the BPST instanton extended from $\mathbb{R}^4$ to $\mathbb{R}^4/\Gamma$. We again have four moduli: the scale parameter $\Lambda$ and the three parameters of global SU(2) rotations.

The 't Hooft instanton (5.34) is gauge equivalent to the BPST instanton (5.53) on the euclidean space $\mathbb{R}^4$. However, this is not so on the orbifold $\mathbb{R}^4/\Gamma$. For instance, taking $N_\ell = 1$ for $\ell = 0, 1, \ldots, k$, one can obtain only abelian solutions in the 't Hooft ansatz (5.17) while one has irreducible nonabelian BPST instantons (5.53). Of course, one can transform the solution (5.53) to a 't Hooft-type solution in a singular gauge, but this transformed solution will not be compatible with $\Gamma$-equivariance, i.e. it cannot be projected from $\mathbb{R}^4$ to $\mathbb{R}^4/\Gamma$. On the other hand, 't Hooft-type solutions are well-defined on V-bundles $\mathcal{E}$ over the orbifold $\mathbb{R}^4/\Gamma$ if the group $\Gamma$ acts trivially on the fibres of $\mathcal{E}$, i.e. if $\mathcal{E} = \mathcal{E}_0$, $N = N_0$ and $\gamma_\Gamma = 1_{N_0}$. The explicit form of such solutions for $N = N_0 = 2$ can be found e.g. in [22, 26, 27].

Acknowledgements

The work of TAI and OL was partially supported by the Heisenberg-Landau program. The work of OL and ADP was supported in part by the Deutsche Forschungsgemeinschaft under grant LE 838/13. The work of RJS was partially supported by the Consolidated Grant ST/J000310/1 from the UK Science and Technology Facilities Council, and by Grant RPG-404 from the Leverhulme Trust.

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