Lower bounds for the $A_\alpha$-spectral radius of uniform hypergraphs

Peng-Li Zhang, Xiao-Dong Zhang

School of Mathematical Sciences, MOE-LSC, SHL-MAC, Shanghai Jiao Tong University, Shanghai 200240, PR China

Abstract
For $0 \leq \alpha < 1$, the $A_\alpha$-spectral radius of a $k$-uniform hypergraph $G$ is defined to be the spectral radius of the tensor $A_\alpha(G) := \alpha D(G) + (1 - \alpha) A(G)$, where $D(G)$ and $A(G)$ are diagonal and the adjacency tensors of $G$ respectively. This paper presents several lower bounds for the difference between the $A_\alpha$-spectral radius and an average degree $km/n$ for a connected $k$-uniform hypergraph with $n$ vertices and $m$ edges, which may be considered as the measures of irregularity of $G$. Moreover, two lower bounds on the $A_\alpha$-spectral radius are obtained in terms of the maximum and minimum degrees of a hypergraph.

Keywords: uniform hypergraph, vertex degree, tensor, spectral radius
AMS Classification: 05C50, 05C65

1 Introduction
Let $G$ be a hypergraph on $n$ vertices with vertex set $V(G)$ and edge set $E(G)$. The elements of $V = V(G)$, labeled as $\{v_1, \ldots, v_n\}$, are referred to as vertices and the elements of $E = E(G)$ are called edges. If $|e| = k$ for each $e \in E(G)$, then $G$ is said to be a $k$-uniform hypergraph. For $k = 2$, it refers the ordinary graph. For a vertex $v_i \in V(G)$, we denote $E_{v_i}(G) = \{e \in E(G) | v_i \in e\}$, which is the set of edges containing the vertex $v_i$. The degree $d_G(v_i)$ (or simply $d_{v_i}$) of a vertex $v_i \in V(G)$ is defined as $d_{v_i} = |e_j : v_i \in e_j \in E(G)|$. Denote the maximum degree, the minimum degree and the average degree of $G$ by $\Delta(G)$, $\delta(G)$ and $\overline{d}(G)$, respectively. A hypergraph $G$ is $d$-regular if $\Delta(G) = \delta(G) = d$, otherwise, $G$ is irregular. A complete $k$-uniform hypergraph is defined to be a hypergraph $G = (V(G), E(G))$ with the edge set consisting of all $k$-subsets of $V(G)$. Obviously, a complete $k$-uniform hypergraph on $n$ vertices is $\binom{n}{k-1}$-regular. Here, we denote an $n$-vertex $k$-uniform complete hypergraph by $K^k_n$. The complement of a $k$-uniform hypergraph $G$ is the $k$-uniform hypergraph $\overline{G}$ with the same vertex set as $G$ and the edge set of which consists of $k$-subsets of $V(G)$ not in $E(G)$. Moreover, for different $i, j \in V(G)$, $i$ and $j$ are said to be adjacent, written $i \sim j$, if there is an edge of $G$ containing both $i$ and $j$. Two edges are said to be adjacent if their intersection is not empty. A vertex $v$ is said to be incident to an edge $e$ if $v \in e$.

*This work is supported by the National Natural Science Foundation of China (Nos. 11971311, 12026230). E-mail addresses: zpengli@sjtu.edu.cn (P.-L. Zhang), xiaodong@sjtu.edu.cn (X.-D. Zhang, corresponding author).
A walk $W$ of length $\ell$ in $G$ is a sequence of alternate vertices and edges: $v_1e_1v_2e_2\ldots v_\ell e_\ell v_{\ell+1}$, where $\{v_i, v_{i+1}\} \subseteq e_i$ for $i = 1, \ldots, \ell$. A walk of $G$ is called a path if no vertices or no edges are repeated. A hypergraph $G$ is said to be connected if every two vertices are connected by a path. Moreover, since the trivial hypergraph (i.e., $E = \emptyset$) is of less interest, we only consider hypergraph having at least one edge (i.e., nontrivial) and assume that $G$ is simple throughout this paper, which means that $e_i \neq e_j$ if $i \neq j$.

Now we give the definition of a strong independent set $[1]$ for a hypergraph.

**Definition 1.1 ([1])** A subset of vertices in a $k$-uniform hypergraph $G$ is called strong independent set if it intersects every edge of $G$ in at most one vertex.

It is easy to see that no two vertices of a strong independent set $S$ is adjacent in $G$. We denote the maximum size of a strong independent set in $G$ as $\alpha_s(G)$. A strong independent set with cardinality $\alpha_s(G)$ is called a maximum strong independent set. Also, a subset of vertices in a $k$-uniform hypergraph $G$ is called weak independent set $[1]$ if it intersects every edge of $G$ in at most $k - 1$ vertices, and any $k$ vertices of a weak independent set $S$ is not an edge in $G$. We denote the maximum size of a weak independent set in $G$ as $\alpha(G)$. A weak independent set with cardinality $\alpha(G)$ is called a maximum weak independent set. Clearly, for ordinary graphs, i.e. in the case $k = 2$, these notions coincide.

Suppose $G = (V, E)$ is a hypergraph and $f : V \to \{1, 2, \ldots, r\}$ is a vertex coloring with $r$ colors. Then $f$ is said to be strong for the hypergraph $G$, if for every edge $e \in E$, all the vertices in $e$ are colored with distinct colors, i.e.

$$|\{f(v) : v \in e\}| = |e|, \quad \text{for all } e \in E.$$  

Recall that the usual hypergraph chromatic number (i.e. weak chromatic number), $\chi(G)$, introduced by Erdős corresponds to weak colorings, i.e. colorings without monochromatic edges when $|\{f(v) : v \in e\}| \geq 2$ for all $e \in E$. For ordinary graphs, these notions coincide.

The clique $[23]$ of a $k$-uniform hypergraph $G$ is a set of vertices such that any of its vertex subsets is an edge of $G$. The largest cardinality of a clique of $G$ is called the clique number of $G$, denoted by $\omega(G)$. A clique with cardinality $\omega(G)$ is called a maximum clique.

A vertex cut $[13]$ of $G$ is a vertex subset $S \subset V(G)$ such that $G - S$ is disconnected, where $G - S$ is the graph obtained by deleting all vertices in $S$ and all incident edges. The vertex connectivity of $G$, denoted by $\nu(G)$, which is the minimum cardinality of any vertices cut $S$. A vertex cut with cardinality $\nu(G)$ is called a minimum vertex cut. The complete $k$-uniform hypergraph has no vertex cut. More notations about hypergraphs readers are referred to $[3][10]$.

For positive integers $k$ and $n$, a real tensor (also called hypermatrix) $T = (t_{i_1\ldots i_k})$ of order $k$ and dimension $n$ refers to a multidimensional array with entries $t_{i_1\ldots i_k}$ such that

$$t_{i_1\ldots i_k} \in \mathbb{R}, \quad \text{for all } i_j \in [n] = \{1, 2, \ldots, n\} \text{ and } j \in [k].$$  

The tensor $T$ is called symmetric if $t_{i_1\ldots i_k}$ is invariant under any permutation of its indices $i_1, i_2, \ldots, i_k$.

A real symmetric tensor $T$ of order $k$ dimension $n$ uniquely defines a $k$-th degree homogeneous polynomial function with real coefficient by

$$F_T(x) = T x^k = \sum_{i_1, \ldots, i_k = 1}^n t_{i_1\ldots i_k} x_{i_1} \ldots x_{i_k}.  \quad \text{2}$$
It is easy to see that $T x^k$ is a real number. Remember that $T x^{k-1}$ is a vector in $\mathbb{R}^n$, whose $i$-th component is defined as

$$(T x^{k-1})_i = \sum_{i_2, \ldots, i_k = 1}^n t_{i_2 \ldots i_k} x_{i_2} \cdots x_{i_k}. \quad (1)$$

**Definition 1.2** (20) Let $T$ be a $k$-th order $n$-dimensional real tensor and $\mathbb{C}$ be the set of all complex numbers. Then $\lambda$ is an eigenvalue of $T$ and $0 \neq x \in \mathbb{C}^n$ is an eigenvector corresponding to $\lambda$ if $(\lambda, x)$ satisfies

$$T x^{k-1} = \lambda x^{k-1},$$

where $x^{[k-1]} \in \mathbb{C}^n$ with $(x^{[k-1]})_i = (x_i)^{k-1}$.

More information on eigenvalues and eigenvectors of tensors, the readers are referred to the paper Qi [20]. Moreover, it is easy to see that

$$(T x^{k-1})_i = \lambda x_i^{k-1}, \text{ for } i = 1, \ldots, n.$$

Now we introduce the general product [21] of tensors, which is a generalization of the matrix case.

**Definition 1.3** (21) Let $A$ (and $B$) be an order $m \geq 2$ (and order $k \geq 1$), dimension $n$ tensor, respectively. Define the product $AB$ to be the following tensor $C$ of order $(m-1)(k-1)+1$ and dimension $n$ :

$$c_{i\alpha_1 \cdots \alpha_{m-1}} = \sum_{i_2, \ldots, i_m = 1}^n a_{i i_2 \cdots i_m} b_{i_2 \alpha_1 \cdots \alpha_m} \cdots a_{i i_{m-1}} \alpha_m \alpha_{m-1} \quad (i \in [n], \alpha_1, \cdots, \alpha_{m-1} \in [n]^{k-1}).$$

Note that by Definition 1.3 now $T x^{k-1}$ defined in (1) can be simply written as $T x$. The spectral radius of $T$ is defined as $\rho(T) = \max \{|\lambda| : \lambda \text{ is an eigenvalue of } T \}$.

For $k \geq 2$, let $G = (V(G), E(G))$ be a $k$-uniform hypergraph on $n$ vertices. The *adjacency tensor* [7] of $G$ is defined as the $k$-th order $n$-dimensional tensor $A(G) = (a_{i_1 \ldots i_k})$, where

$$a_{i_1 \ldots i_k} = \begin{cases} \frac{1}{(k-1)!}, & \text{if } \{i_1, \ldots, i_k\} \in E(G), \\ 0, & \text{otherwise.} \end{cases}$$

Let $D$ be a $k$-th order $n$-dimensional tensor with its diagonal element $d_{i \ldots i}$ being $d_i$, the degree of vertex $i$, for all $i \in [n]$, then $\mathcal{L} = D - A$ is the *Laplacian tensor* of the hypergraph $G$, and $Q = D + A$ is the *signless Laplacian tensor* of the hypergraph $G$. It is easy to see that both $A$ and $Q$ are always nonnegative and symmetric, and $\mathcal{L}$ is symmetric.

Inspired by the innovating work of Nikiforov [17], Lin, Guo and Zhou [14] proposed corresponding notation of the convex linear combination $A_\alpha(G)$ of $D(G)$ and $A(G)$, which is defined as

$$A_\alpha(G) = \alpha D(G) + (1 - \alpha)A(G),$$

where $0 \leq \alpha < 1$.

The spectral radius of $A_\alpha(G)$ is called the $A_\alpha$-spectral radius of $G$ and denoted by $\rho_\alpha(G)$. Then $\rho_0(G)$ is the spectral radius of $A(G)$, which is called the *adjacency spectral radius* of $G$, and denoted
by $\rho(A(G))$. Moreover, $2\rho_{1/2}(G)$ is the spectral radius of $Q(G)$, which is called the signless Laplacian spectral radius of $G$, and denoted by $\rho(Q(G))$. Also, $\rho(L(G))$ is the spectral radius of $L(G)$, which is called the Laplacian spectral radius of $G$.

For $k \geq 2$, let $G$ be a $k$-uniform hypergraph with $V(G) = [n]$, and $x$ be an $n$-dimensional column vector. Clearly, $x^T(A_\alpha(G)x) = \alpha \sum_{i \in V(G)} d_i x_i^k + (1 - \alpha) \sum_{e \in E(G)} k x^e$, or equivalently, $x^T(A_\alpha(G)x) = \sum_{e \in E(G)} \left( \alpha \sum_{i \in e} x_i^k + (1 - \alpha) k x^e \right)$, and $(A_\alpha(G)x)_{ij} = \alpha d_i x_i^{k-1} + (1 - \alpha) \sum_{e \in E_i(G)} x^e(i)$,

where $x^e = x_{i_1} x_{i_2} \cdots x_{i_k}$ for $e = \{i_1, i_2, \cdots, i_k\} \in E(G)$.

Recently, many researchers focus on the difference between the spectral radius and the average degree of hypergraphs, which is considered as a measure of irregularity. For $k = 2$, Collatz and Sinogowitz [6] stated that for any graph with $n$ vertices and $m$ edges, $\rho_0(G) \geq \frac{2m}{n}$, with equality if and only if the graph is regular. Cioabă and Gregory [5] showed that if $G$ is irregular and $n \geq 4$, then $\rho_0(G) - \frac{2m}{n} > \frac{\Delta - \delta^2}{\Delta + \delta}$. Ning, Li and Lu [19] proved that if $G$ is irregular and $n \geq 3$, then $2\rho_{1/2}(G) - \frac{4m}{n} > \frac{(\Delta - \delta)^2}{(\Delta + \delta)^2}$. For general $k \geq 2$, Si and Yuan [22] presented that for a $k$-uniform hypergraph $G$, $\rho_0(G) - \frac{2m}{n} \geq \frac{1}{2k} \left( 2^{\delta} \left( \Delta^{\frac{1}{\Delta + \delta}} + \delta^{\frac{1}{\Delta + \delta}} \right) - (\Delta + \delta) \right)$, which generalized and improved the above result of Cioabă and Gregory [5]. Besides, for a $k$-uniform hypergraph $G$, there are many lower bounds on the spectral radius of $A(G)$ and $Q(G)$ in terms of various parameters of hypergraphs, such as degrees [12, 15], co-degrees [15] and number of edges [10].

Motivated by the results described above, we consider $A_\alpha$-spectral analogues for connected $k$-uniform hypergraph. This paper is organized as follows: In Section 2, we state some basic notations of tensors and auxiliary lemmas. In Section 3, using the direct product of tensors with the same order, we present some results of Laplacian eigenvalue and $A_\alpha$-spectral radius of the direct product of hypergraphs, respectively. In Section 4, we provide several lower bounds on $A_\alpha$-spectral radius of $G$ in terms of vertex degrees. And also, we improve a result in Kang, Liu and Shan [12, Theorem 3]. Furthermore, we establish two lower bounds on the $A_\alpha$-spectral radius using maximum degree and minimum degree.

2 Preliminaries

In this section, we review some notations and helpful lemmas. For $x \in \mathbb{R}^n$, denote $\|x\|_k = x_1^k + x_2^k + \cdots + x_n^k = \sum_{i=1}^n x_i^k$. In particular, $x$ is said to be unit if $\|x\|_k = 1$. Denote the set of nonnegative (positive) real vectors of dimension $n$ by $\mathbb{R}_+^n (\mathbb{R}_+^n)$. The weak irreducibility of nonnegative tensors was defined in [8]. It was proved that if $G$ is a connected $k$-uniform hypergraph with $k \geq 2$, then $A_\alpha(G)$ is weakly irreducible (see [9]). The following result is a part of Perron-Frobenius theorem for nonnegative tensors.
Lemma 2.1 Let $T$ be a nonnegative tensor of order $k$ and dimension $n$, then we have the following statements.

1. \[ 2 \] $\rho(T)$ is an eigenvalue of $T$ with a nonnegative eigenvector corresponding to it.
2. \[ 5 \] If furthermore $T$ is symmetric and weakly irreducible, then $\rho(T)$ is the unique eigenvalue of $T$, with the unique eigenvector $x \in \mathbb{R}^n_{++}$, up to a positive scaling coefficient.

By Lemma 2.1 for a symmetric weakly irreducible nonnegative tensor $A_\alpha(G)$, $\rho_\alpha(G)$ is an eigenvalue of $A_\alpha(G)$ corresponding to a nonnegative eigenvector, which is called a Perron vector of $A_\alpha(G)$. Furthermore, if $G$ is connected, then $\rho_\alpha(G)$ is the unique eigenvalue of $A_\alpha(G)$ with the unique eigenvector $x \in \mathbb{R}^n_{++}$, up to a positive scaling coefficient. Thus, if $G$ is a connected $k$-uniform hypergraph, then there is a unique unit positive Perron vector corresponding to $\rho_\alpha(G)$.

**Lemma 2.2** (11) Let $T$ be a symmetric nonnegative tensor of order $k$ and dimension $n$. Then

\[
\rho(T) = \max \{ x^T(Tx) | x \in \mathbb{R}^n_+, \|x\|_k = 1 \}.
\]  

Furthermore, $x \in \mathbb{R}^n_+$ with $\|x\|_k = 1$ is an eigenvector of $T$ corresponding to $\rho(T)$ if and only if it is an optimal solution of the above maximization problem (2).

From Lemmas 2.1 and 2.2 for a connected $k$-uniform hypergraph $G$ and a vector $x \in \mathbb{R}^n_+$ satisfying $\|x\|_k = 1$, we have $\rho_\alpha(G) \geq x^T(A_\alpha(G)x)$ with equality if and only if $x$ is the unit positive Perron vector of $G$.

**Definition 2.3** (21) Let $A$ and $B$ be two order $k$ tensors with dimension $n$ and $m$, respectively. Define the direct product $A \otimes B$ to be the following tensor of order $k$ and dimension $nm$ (the set of subscripts is taken as $[n] \times [m]$ in the lexicographic order):

\[
(A \otimes B)_{(i_1,j_1)(i_2,j_2)\cdots(i_k,j_k)} = a_{i_1i_2\cdots i_k}b_{j_1j_2\cdots j_k}.
\]

In particular, let $u = (u_1, u_2, \ldots, u_n)^T$ and $v = (v_1, v_2, \ldots, v_m)^T$ be two column vectors with dimension $n$ and $m$, respectively. Then

\[
u \otimes v = (u_1v_1, u_2v_1, \ldots, u_nv_1, u_1v_2, u_2v_2, \ldots, u_nv_2, \ldots, u_1v_m, u_2v_m, \ldots, u_nv_m)^T.
\]

From the above definition, it is easy to have the following proposition.

**Proposition 2.4** (21) (1) $(A_1 + A_2) \otimes B = A_1 \otimes B + A_2 \otimes B$.
(2) $A \otimes (B_1 + B_2) = A \otimes B_1 + A \otimes B_2$.
(3) $(\lambda A) \otimes B = A \otimes (\lambda B) = \lambda (A \otimes B), (\lambda \in \mathbb{C})$.

The following theorem presents an important relation between the direct product of tensors and the general product of tensors in Definition 1.3.

**Theorem 2.5** (21) Let $A$ and $B$ be two order $k + 1$ tensors with dimension $n$ and $m$, respectively. Let $C$ and $D$ be two order $k + 1$ tensors with dimension $n$ and $m$, respectively. Then we have:

\[
(A \otimes B)(C \otimes D) = (AC) \otimes (BD).
\]
In 2013, Shao [21] defined the direct product of two hypergraphs.

**Definition 2.6** ([21]) (The direct product of hypergraphs). Let $G$ and $H$ be two $k$-uniform hypergraphs. Define the direct product $G \times H$ of $G$ and $H$ as $V(G \times H) = V(G) \times V(H)$, and $\{(i_1, j_1), \ldots, (i_k, j_k)\} \in E(G \times H)$ if and only if $\{i_1, \ldots, i_k\} \in E(G)$ and $\{j_1, \ldots, j_k\} \in E(H)$.

**Lemma 2.7** ([13]) Let $y_1, y_2, \ldots, y_n$ be nonnegative numbers ($n \geq 2$). Then
\[
\frac{y_1 + y_2 + \cdots + y_n}{n} - (y_1y_2\cdots y_n)^{\frac{1}{n}} \geq \frac{1}{n(n-1)} \sum_{1 \leq i < j \leq n} (\sqrt[n]{y_i} - \sqrt[n]{y_j})^2,
\]
equality holds if and only if $y_1 = y_2 = \cdots = y_n$.

**Lemma 2.8** (Jensen’s inequality). Let $y_1, y_2, \ldots, y_n$ be real numbers ($n \geq 2$). Then
\[
\frac{y_1^n + y_2^n + \cdots + y_n^n}{n} \geq \left(\frac{y_1 + y_2 + \cdots + y_n}{n}\right)^n,
\]
equality holds if and only if $y_1 = y_2 = \cdots = y_n$.

### 3 Some spectral properties of direct product of hypergraphs

In this section, we consider some spectral properties for Laplacian eigenvalue and $\mathcal{A}_n$-spectral radius of the direct product of hypergraphs.

According to Definition 2.6 for two $k$-uniform hypergraphs $G$ and $H$, Kang, Liu and Shan [12] Claim 1] proved that when $H = K_k^k$, (where $K_k^k$ is the $k$-uniform hypergraph of order $k$ consisting of a single edge), if $G$ is connected, then $G = G \times K_k^k$ is connected. Now we generalize this result to the case when $H$ is a connected $k$-uniform hypergraph.

**Lemma 3.1** Let $G$ and $H$ be two $k$-uniform hypergraphs ($k \geq 3$). If both $G$ and $H$ are connected, then $G := G \times H$ is connected.

**Proof.** It suffices to show that for any $i, s), (j, t) \in V(G)$, $i, j \in V(G), s, t \in V(H)$, there exists a walk connecting them. We distinguish the following two cases:

**Case 1.** There exists an edge $e' \in E(H)$ containing both $s$ and $t$ in $H$.

The proof is the same as the proof of Kang, Liu, and Shan [12 Claim 1]. For the sake of completeness, we write it down.

**Case 1.1.** $i \neq j, s \neq t$.

Since $G$ is connected, there exists a path $P : i = i_1e_1i_2e_2\cdots i_pe_{p+1} = j$. Since $k \geq 3$, there exists $s'$ such that $s' \neq s, s' \neq t$. From the definition of $G$, we have the following:

(a) If $p$ is odd, we have

\[
\begin{align*}
(i_h, s) &\sim (i_{h+1}, s'), & h &= 1, 3, \ldots, p-2, \\
(i_i, s') &\sim (i_{i+1}, s), & l &= 2, 4, \ldots, p-1, \\
(i_p, s) &\sim (i_{p+1}, t) = (j, t).
\end{align*}
\]
(b) If \( p \) is even, we have

\[
\begin{align*}
(i_h, s) & \sim (i_{h+1}, s'), & h = 1, 3, \ldots, p - 1, \\
(i_l, s') & \sim (i_{l+1}, s), & l = 2, 4, \ldots, p - 2, \\
(i_p, s') & \sim (i_{p+1}, t) = (j, t).
\end{align*}
\]

Hence, there exists a walk connecting \((i, s)\) and \((j, t)\).

**Case 1.2.** \( i = j, s \neq t \).

Since \( k \geq 3 \), there exist \( i' \) and \( s' \) satisfying that \( i' \neq i, s' \neq s, s' \neq t \). According to Case 1.1, we know that there exists a path connecting \((i, s)\) and \((i', s')\). Note that \( i' \neq j \) and \( s' \neq t \), there is a path connecting \((i', s')\) and \((j, t)\) by Case 1.1. So there exists a walk connecting \((i, s)\) and \((j, t)\), as desired.

**Case 2.** There does not exist an edge containing both \( s \) and \( t \) in \( H \).

**Case 2.1.** \( i \neq j, s \neq t \).

Since \( G, H \) are both connected, there exists a path \( P : i = i_1 e_1 i_2 e_2 \cdots i_p e_p i_{p+1} = j \) connecting \( i \) and \( j \), and a path \( Q : s = s_1 f_1 s_2 f_2 \cdots s_q f_q s_{q+1} = t \) connecting \( s \) and \( t \). We distinguish the following two cases:

(a) If \( p = q \), then \((i, s) = (i_1, s_1) \sim (i_2, s_2) \sim \cdots \sim (i_p, s_p) \sim (i_{p+1}, s_{p+1}) = (j, t)\), thus there exists a walk connecting \((i, s)\) and \((j, t)\), as desired.

(b) If \( p \neq q \). Without loss of generality, we assume \( p > q \). Since \( k \geq 3 \), there exists a vertex \( s' \in f_q, s' \neq s_q, s' \neq t \). Thus there exists a walk \((i, s) = (i_1, s_1) \sim (i_2, s_2) \sim \cdots \sim (i_q, s_q) \sim (i_{q+1}, s')\) connecting \((i, s)\) and \((i_{q+1}, s')\).

If \( p - q \) is odd, we have

\[
\begin{align*}
(i_h, s') & \sim (i_{h+1}, s_q), & h = q + 1, q + 3, \ldots, p - 2, \\
(i_l, s_q) & \sim (i_{l+1}, s'), & l = q + 2, q + 4, \ldots, p - 1, \\
(i_p, s') & \sim (i_{p+1}, s_{q+1}) = (j, t).
\end{align*}
\]

Thus there exists a walk connecting \((i_{q+1}, s')\) and \((j, t)\). Hence, there is a walk connecting \((i, s)\) and \((j, t)\), as desired.

If \( p - q \) is even, we have

\[
\begin{align*}
(i_h, s') & \sim (i_{h+1}, s_q), & h = q + 1, q + 3, \ldots, p - 1, \\
(i_l, s_q) & \sim (i_{l+1}, s'), & l = q + 2, q + 4, \ldots, p - 2, \\
(i_p, s_q) & \sim (i_{p+1}, s_{q+1}) = (j, t).
\end{align*}
\]

Thus there exists a walk connecting \((i_{q+1}, s')\) and \((j, t)\). Hence, there is a walk connecting \((i, s)\) and \((j, t)\), as desired.

**Case 2.2.** \( i = j, s \neq t \).

Since \( k \geq 3 \), there exist \( i' \) and \( s' \) satisfying that \( i' \neq i, s' \neq s, s' \neq t \). According to Case 2.1, we know that there exists a path connecting \((i, s)\) and \((i', s')\). Note that \( i' \neq j \) and \( s' \neq t \), there is a path connecting \((i', s')\) and \((j, t)\) by Case 2.1. So there exists a walk connecting \((i, s)\) and \((j, t)\), as desired.

Using the direct product \( \tilde{G} \) of two connected \( k \)-uniform hypergraphs \( G \) and \( H \), when \( H = K^k_2 \), Shao [2] obtained the adjacency spectral radius relationship between \( \tilde{G} := G \times K^k_2 \) and \( G : \rho_0(\tilde{G}) = \)
(k - 1)!\rho_0(G)$. Kang, Liu and Shan [12] obtained the corresponding analogues for signless Laplacian spectral radius. Here we consider the corresponding analogues for Laplacian eigenvalue, $\mathcal{A}_\alpha$-spectral radius, respectively, and generalize them to the case when $H$ is a connected $d$-regular $k$-uniform hypergraph.

**Theorem 3.2** Let $G$ be a $k$-uniform hypergraph on $n$ vertices with Perron vector $u \in \mathbb{R}^n$ corresponding to $\rho(\mathcal{L}(G))$. Let $\tilde{G} := G \times H$ be the direct product of $G$ and $H$, where $H$ is a $d$-regular $k$-uniform hypergraph on $m$ vertices. Then $(k - 1)!\rho(\mathcal{L}(\tilde{G}))$ is an eigenvalue of $\mathcal{L}(\tilde{G})$ with the corresponding eigenvector $u \otimes e$, where $e = (1, 1, \cdots, 1)^T \in \mathbb{R}^m$.

**Proof.** Since $H$ is $d$-regular, then for each $i \in V(H)$,

$$(\mathcal{L}(H)e)_i = \Delta e_i^{k-1} - de_i^{k-1} = 0,$$

then we have $\mathcal{L}(H)e = 0$. From the definition of the direct product of hypergraphs, it is obvious that $d_{\tilde{G}}((i, j)) = (k - 1)!d_i d_j$ for any $i \in V(G)$ and $j \in V(H)$. Thus $D(G \times H) = (k - 1)!D(G) \otimes D(H)$, and Shao [21] proved that $\mathcal{A}(G \times H) = (k - 1)!\mathcal{A}(G) \otimes \mathcal{A}(H)$. From the definition of Laplacian tensor, we have

$$\mathcal{L}(\tilde{G}) = \mathcal{L}(G \times H) = \mathcal{D}(G \times H) - \mathcal{A}(G \times H),$$

or equivalently,

$$\mathcal{L}(\tilde{G}) = (k - 1)!\mathcal{D}(G) \otimes \mathcal{D}(H) - (k - 1)!\mathcal{A}(G) \otimes \mathcal{A}(H).$$

Therefore, we have

$$\mathcal{L}(\tilde{G})_{(i_1, j_1)(i_2, j_2)\cdots(i_k, j_k)} = \begin{cases} 
  \frac{(k - 1)!d_i d_j}{(k - 1)!}, & \text{if } i_1 = i_2 = \cdots = i_k, \\
  \frac{1}{(k - 1)!}, & \text{if } \{i_1, i_2, \cdots, i_k\} \in E(G), \\
  0, & \text{otherwise}. 
\end{cases}$$

Also, by the definition of direct product of tensors, we have

$$\mathcal{L}(G) \otimes \mathcal{L}(H)_{(i_1, j_1)(i_2, j_2)\cdots(i_k, j_k)} = \begin{cases} 
  d_i d_j, & \text{if } i_1 = i_2 = \cdots = i_k, \\
  \frac{1}{(k - 1)!}, & \text{if } \{i_1, i_2, \cdots, i_k\} \in E(G), \\
  \frac{1}{(k - 1)!} d_j, & \text{if } \{i_1, i_2, \cdots, i_k\} \in E(H), \\
  0, & \text{otherwise}. 
\end{cases}$$

It is easy to see that

$$\mathcal{L}(\tilde{G}) = (k - 1)! (\mathcal{L}(G) \otimes \mathcal{L}(H) - 2\mathcal{A}(G) \otimes \mathcal{A}(H) + \mathcal{A}(G) \otimes \mathcal{D}(H) + \mathcal{D}(G) \otimes \mathcal{A}(H)).$$
According to Propositions 2.4 and 2.5 since $H$ is a $d$-regular $k$-uniform hypergraph, we have

$$
\mathcal{L}(\tilde{G})(u \otimes e) = (k-1)! \left( (\mathcal{L}(G) \otimes \mathcal{L}(H)) - 2\mathcal{A}(G) \otimes \mathcal{A}(H) + \mathcal{A}(G) \otimes \mathcal{D}(H) + \mathcal{D}(G) \otimes \mathcal{A}(H) \right) (u \otimes e)
$$

$$
= (k-1)!\left( (\mathcal{L}(G) \otimes \mathcal{L}(H))(u \otimes e) - 2\mathcal{A}(G) \otimes \mathcal{A}(H)(u \otimes e) + \mathcal{A}(G) \otimes \mathcal{D}(H)(u \otimes e) + \mathcal{D}(G) \otimes \mathcal{A}(H)(u \otimes e) \right)
$$

$$
= (k-1)!\left( (\mathcal{L}(G)u) \otimes (\mathcal{L}(H)e) - 2(\mathcal{A}(G)u) \otimes (\mathcal{A}(H)e) + (\mathcal{A}(G)u) \otimes (\mathcal{D}(H)e) + (\mathcal{D}(G)u) \otimes (\mathcal{A}(H)e) \right)
$$

$$
= (k-1)!\left( (\mathcal{L}(G)u) \otimes (\mathcal{A}(H)e) + (\mathcal{D}(G)u) \otimes (\mathcal{A}(H)e) \right)
$$

$$
= (k-1)!\left( (\mathcal{L}(G)u \otimes de) + (\mathcal{D}(G)u) \otimes (\mathcal{A}(H)e) \right)
$$

Thus $(k-1)!d\rho(\mathcal{L}(G))$ is an eigenvalue of $\mathcal{L}(\tilde{G})$ with the corresponding eigenvector $u \otimes e$. Hence the proof is completed. ■

**Theorem 3.3** Let $G$ be a connected $k$-uniform hypergraph on $n$ vertices with Perron vector $u \in \mathbb{R}^n_+$ corresponding to $\rho_\alpha(G)$ ($k \geq 3$). Let $\tilde{G} := G \times H$ be the product of $G$ and $H$, where $H$ is a connected $d$-regular $k$-uniform hypergraph on $m$ vertices. Then $\rho_\alpha(\tilde{G}) = (k-1)!d\rho_\alpha(G)$ and $u \otimes e$ is an eigenvector corresponding to $\rho_\alpha(\tilde{G})$, where $e = (1, 1, \cdots, 1)^T \in \mathbb{R}^m$.

**Proof.** Since $H$ is $d$-regular, then for each $i \in V(H)$,

$$
(A_\alpha(H)e)_i = \alpha de_i^{k-1} + (1 - \alpha)de_i^{k-1} = de_i^{k-1}
$$

then we have $A_\alpha(H)e = de$, by the connectedness of $H$ and Lemma 2.1 we have $\rho_\alpha(H) = d$. Since $\mathcal{D}(G \times H) = (k-1)!\mathcal{D}(G) \otimes \mathcal{D}(H)$, and Shao [21] proved that $\mathcal{A}(G \times H) = (k-1)!\mathcal{A}(G) \otimes \mathcal{A}(H)$.

From the definition of $A_{\alpha}$-tensor, we have

$$
A_\alpha(\tilde{G}) = A_\alpha(G \times H) = \alpha \mathcal{D}(G \times H) + (1 - \alpha)\mathcal{A}(G \times H),
$$

or equivalently,

$$
A_\alpha(\tilde{G}) = \alpha(k-1)!\mathcal{D}(G) \otimes \mathcal{D}(H) + (1 - \alpha)(k-1)!\mathcal{A}(G) \otimes \mathcal{A}(H).
$$

Since $A_\alpha(G) = \alpha \mathcal{D}(G) + (1 - \alpha)\mathcal{A}(G)$, where $0 \leq \alpha < 1$, we have

$$
A(G) = \frac{A_\alpha(G) - \alpha \mathcal{D}(G)}{1 - \alpha}.
$$
According to Proposition 2.4 and Proposition 2.5, since $H$ is $d$-regular, we have

$$A_\alpha(\tilde{G})(u \otimes e)$$

$$= (\alpha(k-1)!D(G) \otimes D(H) + (1-\alpha)(k-1)!A(G) \otimes A(H))(u \otimes e)$$

$$= (k-1)!(\alpha D(G) \otimes D(H) + (1-\alpha)A(G) \otimes A(H))(u \otimes e)$$

$$= (k-1)!\left(\alpha D(G) \otimes D(H) + (1-\alpha)\frac{A_\alpha(G) - \alpha D(G)}{1-\alpha} \otimes A(H)\right)(u \otimes e)$$

$$= (k-1)!(\alpha D(G) \otimes D(H) + (A_\alpha(G) - \alpha D(G)) \otimes A(H))(u \otimes e)$$

$$= (k-1)!(\alpha D(G) \otimes D(H)(u \otimes e)$$

$$+ A_\alpha(G) \otimes A(H)(u \otimes e)) - \alpha D(G) \otimes A(H)(u \otimes e))$$

$$= (k-1)!(\alpha D(G)u \otimes D(H)e + A_\alpha(G)u \otimes A(H)e - \alpha D(G)u \otimes A(H)e)$$

$$= (k-1)!\left(\alpha D(G)u \otimes de + A_\alpha(G)u \otimes A(H)e - \alpha D(G)u \otimes de\right)$$

$$= (k-1)!\left(\alpha_\alpha(G)u \otimes A(H)e\right)$$

$$= (k-1)!(\rho_\alpha(G)u \otimes de)$$

$$= (k-1)!d\rho_\alpha(G)(u \otimes e).$$

Since $G$, $H$ are both connected, then $\tilde{G}$ is connected by Lemma 3.1. Obviously, we have $A_\alpha(\tilde{G})$ is weakly irreducible. So by Lemma 2.1, we have $(k-1)!d\rho_\alpha(G)$ is the spectral radius of $A_\alpha(\tilde{G})$ with the corresponding eigenvector $u \otimes e$. Hence the proof is completed. \]

\section{4 Lower bounds for the $A_\alpha$-Spectral radius of uniform hypergraphs}

In this section, we present some lower bounds for the $A_\alpha$-spectral radius of $k$-uniform hypergraphs.

Let $G$ be a $k$-uniform hypergraph on $n$ vertices with $m$ edges. By Lemma 2.2, it is easy to see that if $x$ is a unit column vector in $\mathbb{R}^n$, then

$$\rho_\alpha(G) \geq x^T(A_\alpha(G)x) = \alpha \sum_{i=1}^{n} d_i x_i^k + (1-\alpha) \sum_{e \in E(G)} k x_e.$$

Moreover, if $G$ is connected, equality holds if and only if $x$ is an eigenvector corresponding to $\rho_\alpha(G)$.

By Lemma 2.2 and choosing $x = (\frac{1}{\sqrt{n}}, \frac{1}{\sqrt{n}}, \cdots, \frac{1}{\sqrt{n}})^T$ in the above inequality, we have the following result.

\textbf{Lemma 4.1} Let $G$ be a $k$-uniform hypergraph on $n$ vertices and $m$ edges, then

$$\rho_\alpha(G) \geq \frac{km}{n}.$$  

If $G$ is connected, equality holds if and only if $G$ is regular.

We now obtain a lower bound on $\rho_\alpha(G) - \frac{km}{n}$ in terms of the restrictions $d_S = (d_i)_{i \in S}$ of the degree sequence of $G$ to a subset $S$ of the vertex set.
Lemma 4.2 Let $G$ be a $k$-uniform hypergraph on $n$ vertices with $m$ edges, and $S$ be a strong independent set of $G$ with $|S| = s$. Then 
\[
\rho_\alpha(G) - \frac{km}{n} \geq \frac{1}{n} \left( \alpha \left( s \frac{\sum_{i \in S} d_i^{2k-1}}{\sum_{i \in S} d_i} - \sum_{i \in S} d_i \right) + (1 - \alpha)k \left( \sum_{i \in S} d_i^{\frac{k-1}{k}} \right) - \left( \sum_{i \in S} d_i \right) \right).
\]

Proof. Let $s = |S|$ denote the number of vertices in $S$. Taking $x_i = \frac{a_i}{\sqrt{n}}$ for $i \in S$ and $x_i = \frac{1}{\sqrt{n}}$ when $i \notin S$. We have $\|x\|_k = 1$ when $\sum_{i \in S} a_i^k = s$. Since $S$ is a strong independent set of $G$, we have $|e \cap S| \leq 1$ for each edge $e \in E(G)$. Then we have

\[
\rho_\alpha(G) - \frac{km}{n} \geq x^T(A_\alpha(G)x) - \frac{km}{n}
\]

\[
= \alpha \sum_{i \in V(G)} d_i x_i^k + (1 - \alpha)k \sum_{e \in E(G)} x^e - \frac{km}{n}
\]

\[
= \alpha \left( \sum_{i \in S} d_i x_i^k + \sum_{i \in V \setminus S} d_i x_i^k \right) + (1 - \alpha)k \left( \sum_{e \in E(G), e \cap S \neq \emptyset} x^e + \sum_{e \in E(G), e \cap \emptyset \neq \emptyset} x^e \right) - \frac{km}{n}
\]

\[
= \alpha \left( \sum_{i \in S} d_i a_i^k + \sum_{i \in V \setminus S} d_i \frac{1}{n} \right) + (1 - \alpha)k \left( \sum_{i \in S} d_i a_i + \sum_{i \in E(G), e \cap S = \emptyset} \frac{1}{n} \right) - \frac{km}{n}
\]

\[
= \frac{\alpha}{n} \left( \sum_{i \in S} d_i a_i^k + \left( km - \sum_{i \in S} d_i \right) \right) + (1 - \alpha)k \left( \sum_{i \in S} d_i a_i + \left( m - \sum_{i \in S} d_i \right) \right) - \frac{km}{n}
\]

\[
= \frac{\alpha}{n} \left( \sum_{i \in S} d_i a_i^k - \sum_{i \in S} d_i \right) + \frac{(1 - \alpha)k}{n} \left( \sum_{i \in S} d_i a_i - \sum_{i \in S} d_i \right)
\]

If we choose the $a_i$ so that the equality holds in the Hölder inequality, $\sum_{i \in S} d_i a_i \leq \left( \sum_{i \in S} a_i^k \right)^{\frac{1}{k}} \left( \sum_{i \in S} d_i^{\frac{k-1}{k}} \right)^{\frac{k-1}{k}}$, equality holds if and only if $a_i^k = c^k d_i^{\frac{k-1}{k}}$, i.e., $a_i = cd_i^{\frac{k-1}{k}}$, where $c$ is a constant. Therefore, we have
\[ c = \sqrt{\frac{s}{\sum_{i \in S} d_i^{k-1}}} \] by \( s = \sum_{i \in S} a_i^k = c^k \sum_{i \in S} d_i^{k-1} \). Thus we have

\[
\rho_\alpha(G) - \frac{km}{n} \\
\geq \frac{\alpha}{n} \left( \sum_{i \in S} d_i^k - \sum_{i \in S} d_i \right) + \frac{(1-\alpha)k}{n} \left( \sum_{i \in S} d_i a_i - \sum_{i \in S} d_i \right) \\
= \frac{\alpha}{n} \left( \sum_{i \in S} d_i^k \frac{sd_i^{k-1}}{\sum_{i \in S} d_i^{k-1}} - \sum_{i \in S} d_i \right) + \frac{(1-\alpha)k}{n} \left( \sum_{i \in S} d_i^k \frac{sd_i^{k-1}}{\sum_{i \in S} d_i^{k-1}} - \sum_{i \in S} d_i \right) \\
= \frac{1}{n} \left( \alpha \left( \sum_{i \in S} d_i^k \frac{s^{2k-1}}{\sum_{i \in S} d_i^{k-1}} - \sum_{i \in S} d_i \right) + (1-\alpha)k \left( \sum_{i \in S} d_i^k \frac{s^{2k-1}}{\sum_{i \in S} d_i^{k-1}} - \sum_{i \in S} d_i \right) \right).
\]

We complete the proof. \( \square \)

Taking \( S \) be a maximum strong independent set in Lemma 4.2, we obtain the following corollary.

**Corollary 4.3** Let \( G \) be a \( k \)-uniform hypergraph on \( n \) vertices with \( m \) edges, and \( S \) be a maximum strong independent set of \( G \), then

\[
\rho_\alpha(G) - \frac{km}{n} \\
\geq \frac{\alpha}{n} \left( \alpha_s(G) \frac{\sum_{i \in S} d_i^{2k-1}}{\sum_{i \in S} d_i^{k-1}} - \sum_{i \in S} d_i \right) + \frac{(1-\alpha)k}{n} \left( \alpha_s(G) \frac{\sum_{i \in S} d_i^{k-1}}{\sum_{i \in S} d_i^{k-1}} - \sum_{i \in S} d_i \right).
\]

**Theorem 4.4** Let \( G \) be a connected \( k \)-uniform hypergraph on \( n \) vertices with \( m \) edges \((k \geq 3)\), and \( S \) be a subset of \( V(G) \) with \( |S| = s \). Then

\[
\rho_\alpha(G) - \frac{km}{n} \\
\geq \frac{1}{kn} \left( \alpha \left( \sum_{i \in S} d_i^{2k-1} \frac{s^{2k-1}}{\sum_{i \in S} d_i^{k-1}} - \sum_{i \in S} d_i \right) + (1-\alpha)k \left( \sum_{i \in S} d_i^{k-1} \frac{s^{k-1}}{\sum_{i \in S} d_i^{k-1}} - \sum_{i \in S} d_i \right) \right).
\]

**Proof.** If \( S \) is a strong independent set, then by Lemma 4.2 we obtain the result. If \( S \) is not a strong independent set, then we consider the direct product \( G := G \times K_k^n \). Clearly, \( \tilde{G} \) is a \( k \)-partite \( k \)-uniform hypergraph with partition:

\[ V(\tilde{G}) = \bigcup_{j=1}^k (V(G) \times \{j\}) \]

Clearly, \( |V(\tilde{G})| = k|V(G)| \), \( |E(\tilde{G})| = k|E(G)| \), and \( d_{\tilde{G}}((i,a)) = (k-1)d_i \), where \( d_i \) is the degree of vertex \( i \) in \( G \), \( i \in V(G) \), \( a \in [k] = V(K_k^n) \).

Note that \( S \times \{a\} \) is a strong independent set in \( \tilde{G} \). Applying the previous inequality, we have

\[
\rho_\alpha(\tilde{G}) - \frac{kkm}{kn} \\
\geq \frac{(k-1)!}{kn} \left( \alpha \left( \sum_{i \in S} d_i^{2k-1} \frac{s^{2k-1}}{\sum_{i \in S} d_i^{k-1}} - \sum_{i \in S} d_i \right) + (1-\alpha)k \left( \sum_{i \in S} d_i^{k-1} \frac{s^{k-1}}{\sum_{i \in S} d_i^{k-1}} - \sum_{i \in S} d_i \right) \right).
\]
Since $K_k^3$ is 1-regular and $G$ is connected, by Theorem 3.3 we know that $\rho_\alpha(G) = (k-1)!\rho_\alpha(G)$, hence we have

$$\rho_\alpha(G) - \frac{kn}{n} \geq \frac{1}{kn} \left( \alpha \left( \sum_{i \in S} d_i^{\frac{2k-1}{k}} - \sum_{i \in S} d_i \right) + (1 - \alpha)k \left( \frac{k}{n} \left( \sum_{i \in S} d_i^{\frac{1}{k}} \right)^{\frac{k-1}{k}} - \sum_{i \in S} d_i \right) \right).$$

The proof is completed.

**Remark 4.5** For a subset $S$ of the vertex set $V(G)$ of a connected $k$-uniform hypergraph $G$ ($k \geq 3$), on the one hand, it follows from the Rearrangement inequality that

$$s \sum_{i \in S} d_i^{\frac{2k-1}{k}} = s \sum_{i \in S} d_i^{\frac{1}{k}} d_i \geq \sum_{i \in S} d_i^{\frac{k}{k}} \sum_{i \in S} d_i,$$

equality holds if and only if $d_i$ is a constant for any $i \in S$. On the other hand, when $p = k, q = \frac{k}{k-1}$, it follows from the Hölder inequality that

$$s^{\frac{k}{k}} \left( \sum_{i \in S} d_i^{\frac{1}{k}} \right)^{\frac{k-1}{k}} \geq \sum_{i \in S} d_i,$$

equality holds if and only if $d_i$ is a constant for any $i \in S$. Therefore, by $0 \leq \alpha < 1$, we have

$$\rho_\alpha(G) \geq \frac{1}{kn} \left( \alpha \left( \sum_{i \in S} d_i^{\frac{2k-1}{k}} - \sum_{i \in S} d_i \right) + (1 - \alpha)k \left( \frac{k}{n} \left( \sum_{i \in S} d_i^{\frac{1}{k}} \right)^{\frac{k-1}{k}} - \sum_{i \in S} d_i \right) \right) + \frac{km}{n},$$

which improves the corresponding result in Lemma 4.1.

Taking $S = V(G)$, we obtain the following corollary of Theorem 4.4.

**Corollary 4.6** Let $G$ be a connected $k$-uniform hypergraph on $n$ vertices with $m$ edges ($k \geq 3$). Then

$$\rho_\alpha(G) \geq \frac{\alpha \sum_{i=1}^{n} d_i^{\frac{2k-1}{k}}}{\sum_{i=1}^{n} d_i^{\frac{k}{k}}} + (1 - \alpha) \left( \frac{1}{n} \sum_{i=1}^{n} d_i^{\frac{1}{k}} \right)^{\frac{k-1}{k}} + \frac{\alpha(k-1)m}{n}. \quad (3)$$

**Remark 4.7** Let $G$ be a $k$-uniform hypergraph on $n$ vertices ($k \geq 3$) and $\bar{G} := G \times K_k^3$, Kang, Liu and Shan in [12, Claim 4] proved that $\rho_0(\bar{G}) = (k-1)!\rho_0(G)$. According to the proof of Theorem 4.4 and Lemma 2.4 we have

$$\rho(A(G)) = \rho_0(G) \geq \left( \frac{1}{n} \sum_{i=1}^{n} d_i^{\frac{1}{k}} \right)^{\frac{k-1}{k}}.$$

If $G$ is connected, equality holds if and only if $G$ is regular, which was proved in [12, Theorem 2].
Taking $S$ be a pair of vertices with distinct degrees in Theorem 4.4, we obtain the following corollary.

**Corollary 4.8** Let $G$ be a connected $k$-uniform hypergraph on $n$ vertices with $m$ edges ($k \geq 3$), suppose $i$ and $j$ are vertices of $G$ and $d_i > d_j$. Then

$$
\rho_o(G) - \frac{km}{n} \geq \frac{\alpha}{cn} \left( \frac{d_i^{2k-1} + d_j^{2k-1}}{d_i^{\frac{k-1}{k}} + d_j^{\frac{k-1}{k}}} - (d_i + d_j) \right) + \frac{(1 - \alpha)k}{cn} \left( 2^\frac{1}{k} \left( d_i^{\frac{1}{k}} + d_j^{\frac{1}{k}} \right)^{\frac{k-1}{k}} - (d_i + d_j) \right),
$$

where $c = 1$ if $i$ and $j$ are not adjacent and $c = k$ if $i$ and $j$ are adjacent.

The next result is an immediate consequence of Corollary 4.8.

**Corollary 4.9** Let $G$ be a connected $k$-uniform hypergraph on $n$ vertices with $m$ edges ($k \geq 3$), suppose $d_i = \Delta, d_j = \delta, i \neq j$, and $i, j \in V(G)$. Then

$$
\rho_o(G) - \frac{km}{n} \geq \frac{\alpha}{cn} \left( \frac{\Delta^{2k-1} + \delta^{2k-1}}{\Delta^{\frac{k-1}{k}} + \delta^{\frac{k-1}{k}}} - (\Delta + \delta) \right) + \frac{(1 - \alpha)k}{cn} \left( 2^\frac{1}{k} \left( \Delta^{\frac{1}{k}} + \delta^{\frac{1}{k}} \right)^{\frac{k-1}{k}} - (\Delta + \delta) \right)
$$

(4)

where $c = 1$ if $i$ and $j$ are not adjacent and $c = k$ if $i$ and $j$ are adjacent.

Taking $S$ be a maximum weak independent set in Theorem 4.4, we obtain the following corollary.

**Corollary 4.10** Let $G$ be a connected $k$-uniform hypergraph on $|V(G)| = n$ vertices with $|E(G)| = m$ edges ($k \geq 3$), and $S$ be a maximum weak independent set of $G$. Then

$$
\rho_o(G) - \frac{km}{n} \geq \frac{\alpha}{kn} \left( \alpha(G) \sum_{i \in S} d_i^{2k-1} - \sum_{i \in S} d_i \right) + \frac{(1 - \alpha)n}{\chi(G)} \left( \frac{n}{\chi(G)} \right)^{\frac{1}{k}} \left( \sum_{i \in S} d_i^{\frac{1}{k}} \right)^{\frac{k-1}{k}} - \sum_{i \in S} d_i.
$$

Berge [3] proved that every $k$-uniform hypergraph $G$ on $n$ vertices satisfying that $\chi(G)\alpha(G) \geq n$. According to Corollary 4.10, we have the following result.

**Corollary 4.11** Let $G$ be a connected $k$-uniform hypergraph on $n$ vertices with $m$ edges ($k \geq 3$), $\chi(G)$ is the weak chromatic number of $G$, and $S$ be a maximum weak independent set. Then

$$
\rho_o(G) - \frac{km}{n} \geq \frac{\alpha}{kn} \left( \frac{n}{\chi(G)} \sum_{i \in S} d_i^{2k-1} - \sum_{i \in S} d_i \right) + \frac{(1 - \alpha)n}{\chi(G)} \left( \frac{n}{\chi(G)} \right)^{\frac{1}{k}} \left( \sum_{i \in S} d_i^{\frac{1}{k}} \right)^{\frac{k-1}{k}} - \sum_{i \in S} d_i.
$$
For a $k$-uniform hypergraph $G$, notice that a maximum weak independence set of $G$ is a maximum clique of $\overline{G}$, and there is a relation $\alpha(G) \leq \omega(\overline{G})$ between clique number $\omega(\overline{G})$ and weak independence number $\alpha(G) : \alpha(G) = \omega(\overline{G})$.

**Corollary 4.12** Let $G$ be a connected $k$-uniform hypergraph on $n$ vertices with $m$ edges ($k \geq 3$), $\omega(\overline{G})$ is the clique number of $\overline{G}$, and $S$ be a maximum clique of $\overline{G}$. Then

$$
\rho_\alpha(G) - \frac{km}{n} \geq \frac{\alpha}{cn} \left( \nu(G) \left( \frac{\sum_{i \in S} d_i^{\frac{k}{k-1}}}{\sum_{i \in S} d_i^{\frac{k}{k-1}}} - \sum_{i \in S} d_i \right) + \frac{(1 - \alpha)k}{cn} \left( \nu(G) \left( \frac{\sum_{i \in S} d_i^{\frac{k}{k-1}}}{\sum_{i \in S} d_i^{\frac{k}{k-1}}} - \sum_{i \in S} d_i \right) \right),
$$

where $c = 1$ if $S$ is a strong independent set of $G$, and $c = k$ otherwise.

**Lemma 4.14** Let $G$ be a $k$-uniform hypergraph on $n$ vertices with $m$ edges, and $S$ be a strong independent set of $G$ with $|S| = s$. Then

$$
\rho_\alpha(G) - \frac{km}{n} \geq \frac{1}{n} \left( \alpha \sum_{i \in S} d_i \left( \frac{sd_i^{\frac{k}{k-1}}}{\sum_{i \in S} d_i^{\frac{k}{k-1}}} \right)^{\frac{1}{3}} - 1 \right)^2 + k \left( \frac{\sum_{i \in S} d_i^{\frac{k}{k-1}}}{\sum_{i \in S} d_i^{\frac{k}{k-1}}} - \sum_{i \in S} d_i \right).
$$

**Proof.** Let $s = |S|$ denote the number of vertices in $S$. Taking $x_i = \frac{1}{d_i}$ for $i \in S$ and $x_i = \frac{1}{d_i}$ when $i \notin S$. We have $\|x\|_k = 1$ when $\sum_{i \in S} d_i^k = s$. Since $S$ is a strong independent set, thus we have $|e \cap S| \leq 1$ for each edge $e \in E(G)$. Then we have

$$
\rho_\alpha(G) - \frac{km}{n} \geq \alpha \sum_{i \in V(G)} d_i x_i^k + (1 - \alpha)k \sum_{e \in E(G)} x_e - \frac{km}{n}
$$

$$
= \alpha \sum_{\{i_1, i_2, \ldots, i_k\} \in E(G)} (x_{i_1}^k + x_{i_2}^k + \cdots + x_{i_k}^k) + (1 - \alpha)k \sum_{\{i_1, i_2, \ldots, i_k\} \in E(G)} x_{i_1} x_{i_2} \cdots x_{i_k} - \frac{km}{n}
$$

$$
= \alpha \left( \sum_{\{i_1, i_2, \ldots, i_k\} \in E(G)} (x_{i_1}^k + x_{i_2}^k + \cdots + x_{i_k}^k) + k \sum_{\{i_1, i_2, \ldots, i_k\} \in E(G)} x_{i_1} x_{i_2} \cdots x_{i_k} \right) - \frac{km}{n}.
$$
Using Lemma 2.7, we have
\[
\rho_\alpha(G) - \frac{km}{n} \\
\geq \alpha k \left( \sum_{\{i_1, i_2, \ldots, i_k\} \in E(G)} \left( x_{i_1}^k + x_{i_2}^k + \cdots + x_{i_k}^k - x_{i_1} x_{i_2} \cdots x_{i_k} \right) \right) \\
+ k \sum_{\{i_1, i_2, \ldots, i_k\} \in E(G)} x_{i_1} x_{i_2} \cdots x_{i_k} - \frac{km}{n}
\]
\[
\geq \alpha k \frac{1}{k(k-1)} \sum_{e \in E(G)} \sum_{\{i, j\} \subseteq e} \left( x_i^k - x_j^k \right)^2 + k \sum_{\{i_1, i_2, \ldots, i_k\} \in E(G)} x_{i_1} x_{i_2} \cdots x_{i_k} - \frac{km}{n}
\]
\[
= \alpha \frac{1}{k(k-1)} \sum_{e \in E(G)} \sum_{\{i, j\} \subseteq e} \left( x_i^k - x_j^k \right)^2 + k \sum_{\{i_1, i_2, \ldots, i_k\} \in E(G)} x_{i_1} x_{i_2} \cdots x_{i_k} - \frac{km}{n}
\]
\[
\geq \frac{1}{n} \left( \alpha \sum_{i \in S} d_i \left( a_i^{\frac{k}{k-1}} - 1 \right)^2 + k \left( \sum_{i \in S} d_i a_i - \sum_{i \in S} d_i \right) \right).
\]

Then by the similar arguments in Lemma 4.2, we obtain the desired result. ■

**Theorem 4.15** Let $G$ be a connected $k$-uniform hypergraph on $n$ vertices with $m$ edges ($k \geq 3$), and $S$ be a subset of $V(G)$ with $|S| = s$. Then
\[
\rho_\alpha(G) - \frac{km}{n} \\
\geq \frac{1}{kn} \left( \alpha \sum_{i \in S} d_i \left( \frac{sd_i^{\frac{k}{k-1}}}{\sum_{i \in S} d_i^{\frac{k}{k-1}}} \right)^{\frac{1}{k-1}} \right) - 1 \right)^2 + k \left( s^{\frac{k}{k-1}} \left( \sum_{i \in S} d_i^{\frac{k}{k-1}} \right)^{\frac{k-1}{k}} - \sum_{i \in S} d_i \right).
\]

**Proof.** By Theorem 3.3 and Lemma 4.14 using the similar method in Theorem 4.14, we obtain the desired result. ■

Taking $S = V(G)$ in Theorem 4.15, we obtain the following result.

**Corollary 4.16** Let $G$ be a connected $k$-uniform hypergraph on $n$ vertices with $m$ edges ($k \geq 3$). Then
\[
\rho_\alpha(G) \geq \alpha \frac{1}{kn} \sum_{i=1}^{n} d_i \left( \left( \frac{nd_i^{\frac{k}{k-1}}}{\sum_{i=1}^{n} d_i^{\frac{k}{k-1}}} \right)^{\frac{1}{k-1}} - 1 \right)^2 + k \left( s^{\frac{k}{k-1}} \left( \sum_{i=1}^{n} d_i^{\frac{k}{k-1}} \right)^{\frac{k-1}{k}} - \sum_{i=1}^{n} d_i \right).
\]

**Remark 4.17** Let $G$ be a connected $k$-uniform hypergraph on $|V(G)| = n$ vertices ($k \geq 3$), taking $\alpha = \frac{1}{2}$ in Corollary 4.16. Then we have
\[
\rho(Q(G)) = 2\rho_4(G) \\
\geq \frac{1}{kn} \sum_{i=1}^{n} d_i \left( \left( \frac{nd_i^{\frac{k}{k-1}}}{\sum_{i=1}^{n} d_i^{\frac{k}{k-1}}} \right)^{\frac{1}{k-1}} - 1 \right)^2 + 2 \left( \sum_{i=1}^{n} d_i^{\frac{k}{k-1}} \right)^{\frac{k-1}{k}}
\]
\[
\geq 2 \left( \frac{1}{n} \sum_{i=1}^{n} d_i^{\frac{k}{k-1}} \right)^{\frac{k-1}{k}},
\]
(6)
which improves the result in Kang, Liu and Shan [12, Theorem 3].

**Remark 4.18** Nikiforov [18] introduced the concept of odd-colorable hypergraphs, which is a generalization of bipartite graphs. Let \( k \geq 2 \) and \( k \) be even. A \( k \)-uniform hypergraph \( G \) with \( V(G) = [n] \) is called *odd-colorable* if there exists a map \( \varphi : [n] \to [k] \) such that for any edge \( \{i_1, i_2, \cdots, i_k\} \) of \( G \), we have

\[
\varphi(i_1) + \varphi(i_2) + \cdots + \varphi(i_k) \equiv \frac{k}{2} \pmod{k}.
\]

It was proved that if \( G \) is a connected \( k \)-uniform hypergraph, then \( \rho(L(G)) = \rho(Q(G)) \) if and only if \( k \) is even and \( G \) is odd-colorable [25]. Thus by \( \rho_{2^k}(G) = \frac{1}{2} \rho(Q(G)) \), we have the following results. Let \( k \geq 4 \), and \( k \) be even, for a connected odd-colorable \( k \)-uniform hypergraph \( G \), we have

\[
\rho(L(G)) = 2\rho_{2^k}(G) \geq \frac{1}{kn} \sum_{i=1}^{n} d_i \left( \left( \frac{nd_i^{\frac{k}{2}}}{\sum_{i\in S} d_i^{\frac{k}{2}}} \right)^{\frac{1}{k}} - 1 \right)^2 + 2 \left( \frac{1}{n} \sum_{i=1}^{n} d_i^{\frac{k}{2}} \right)^{k-1}
\]

which improves the result in Kang, Liu and Shan [12, Remark 12].

**Lemma 4.19** Let \( G \) be a \( k \)-uniform hypergraph on \( n \) vertices with \( m \) edges \((k \geq 3)\), and \( S \) be a strong independent set of \( G \). Then

\[
\rho_{\alpha}(G) - \frac{km}{n} \\
\geq \frac{\alpha}{nk^{k-1}} \sum_{i \in S} d_i \left( \left( \frac{\frac{1}{k} \left( \sum_{i \in S} d_i^{\frac{k}{2}} \right)^{k-1}}{\sum_{i \in S} d_i^{\frac{k}{2}}} + k - 1 \right) - k^k \right) + \frac{(1 - \alpha)k}{n} \left( \frac{1}{\frac{1}{k} \sum_{i \in S} d_i^{\frac{k}{2}}} - \sum_{i \in S} d_i \right).
\]

**Proof.** Let \( s = |S| \) denote the number of vertices in \( S \). Taking \( x_i = \frac{n}{\varphi_i} \) for \( i \in S \) and \( x_i = \frac{1}{\varphi_i} \) when \( i \notin S \). We have \( \| x \|_k = 1 \) when \( \sum_{i \in S} a_i^k = s \). Since \( S \) is a strong independent set, thus we have \( |e \cap S| \leq 1 \) for each edge \( e \in E(G) \). Then we have

\[
\rho_{\alpha}(G) - \frac{km}{n} \\
= \alpha \sum_{i \in V(G)} d_i x_i^{\frac{k}{2}} + (1 - \alpha)k \sum_{e \in E(G)} x^e - \frac{km}{n} \\
= \alpha \sum_{\{i_1, i_2, \cdots, i_k\} \in E(G)} (x_{i_1}^{k} + x_{i_2}^{k} + \cdots + x_{i_k}^{k}) + (1 - \alpha)k \sum_{\{i_1, i_2, \cdots, i_k\} \in E(G)} x_{i_1} x_{i_2} \cdots x_{i_k} - \frac{km}{n}.
\]
By Jensen’s inequality, we have
\[
\rho_\alpha(G) - \frac{km}{n} \geq \alpha \sum_{\{i_1, i_2, \ldots, i_k\} \in E(G)} (x_{i_1}^k + x_{i_2}^k + \cdots + x_{i_k}^k) + (1 - \alpha)k \sum_{\{i_1, i_2, \ldots, i_k\} \in E(G)} x_{i_1}x_{i_2} \cdots x_{i_k} - \frac{km}{n}
\]
\[
\geq \alpha \sum_{\{i_1, i_2, \ldots, i_k\} \in E(G)} \frac{(x_{i_1} + x_{i_2} + \cdots + x_{i_k})^k}{k^{k-1}} + (1 - \alpha)k \sum_{\{i_1, i_2, \ldots, i_k\} \in E(G)} x_{i_1}x_{i_2} \cdots x_{i_k} - \frac{km}{n}
\]
\[
= \frac{\alpha}{nk^{k-1}} \sum_{i \in S} d_i ((a_i + k - 1)^k - k^k) + \frac{(1 - \alpha)k}{n} \left( \sum_{i \in S} a_i d_i - \sum_{i \in S} d_i \right).
\]
Again by the similar arguments in Lemma 4.21, we obtain the desired result. □

**Theorem 4.20** Let $G$ be a connected $k$-uniform hypergraph on $n$ vertices with $m$ edges ($k \geq 3$), and $S$ be a subset with $|S| = s$. Then
\[
\rho_\alpha(G) - \frac{km}{n} \geq \frac{\alpha}{nk^{k-1}} \sum_{i \in S} d_i \left( \left( \frac{s d_i^{\frac{k}{k-1}}}{\sum_{i \in S} d_i^{\frac{k}{k-1}}} + k - 1 \right)^k - k^k \right) + \frac{(1 - \alpha)k}{n} \left( s^+ \left( \sum_{i \in S} d_i^{\frac{k}{k-1}} \right)^{\frac{k}{k-1}} - \sum_{i \in S} d_i \right).
\]

**Proof.** By Theorem 3.3 and Lemma 4.19, using the similar method in Theorem 4.24, we obtain the desired result. □

**Remark 4.21** For a connected $k$-uniform hypergraph $G$ on $n$ vertices with $m$ edges ($k \geq 3$), let $S$ be a subset of $G$, we have
\[
\sum_{i \in S} d_i \left( \left( \frac{s d_i^{\frac{k}{k-1}}}{\sum_{i \in S} d_i^{\frac{k}{k-1}}} + k - 1 \right)^k - k^k \right) \geq \sum_{i \in S} d_i \sum_{i \in S} \left( \frac{s d_i^{\frac{k}{k-1}}}{\sum_{i \in S} d_i^{\frac{k}{k-1}}} + k - 1 \right)^k \geq k^k \sum_{i \in S} d_i,
\]
the first inequality follows from the Rearrangement inequality. At the same time, considering
\[
\sum_{i \in S} \left( \left( \frac{s d_i^{\frac{k}{k-1}}}{\sum_{i \in S} d_i^{\frac{k}{k-1}}} + k - 1 \right)^k \right),
\]
we notice that there exists an $i_0 \in S$ satisfying that \( \frac{s d_{i_0}^{\frac{k}{k-1}}}{\sum_{i \in S} d_i^{\frac{k}{k-1}}} \geq 1 \), thus the second inequality holds.

Furthermore, it follows from the Hölder inequality, we have
\[
s^+ \left( \sum_{i \in S} d_i^{\frac{k}{k-1}} \right)^{\frac{k-1}{k}} \geq \sum_{i \in S} d_i,
\]

18
equality holds if and only if \( d_i \) is a constant for any \( i \in S \). Thus we have

\[
\rho_\alpha(G) \geq \frac{\alpha}{k^{kn}} \sum_{i \in S} d_i \left( \left( \frac{\sqrt{nd_i^{\frac{k}{k-1}}} + k - 1}{\sum_{i \in S} d_i^{\frac{k}{k-1}}} \right)^k - k^k \right) + \frac{(1 - \alpha)}{n} \left( s^{\frac{k}{k-1}} \left( \sum_{i \in S} d_i^{\frac{k}{k-1}} \right)^{\frac{k-1}{k}} - \sum_{i \in S} d_i \right) + \frac{km}{n}.
\]

which improves the corresponding result in Lemma 4.1 and generalizes the result in [7, Theorem 3.8].

Taking \( S = V(G) \) in Theorem 4.20, we obtain the following result.

**Corollary 4.22** Let \( G \) be a connected \( k \)-uniform hypergraph on \( n \) vertices with \( m \) edges \((k \geq 3)\). Then

\[
\rho_\alpha(G) \geq \alpha \frac{1}{k^{kn}} \sum_{i=1}^{n} d_i \left( \left( \frac{\sqrt{nd_i^{\frac{k}{k-1}}} + k - 1}{\sum_{i=1}^{n} d_i^{\frac{k}{k-1}}} \right)^k + (1 - \alpha) \left( \frac{1}{n} \sum_{i=1}^{n} d_i^{\frac{k}{k-1}} \right)^{\frac{k-1}{k}} \right).
\]

**Remark 4.23** Let \( G \) be a connected \( k \)-uniform hypergraph on \(|V(G)| = n\) vertices and \(|E(G)| = m\) edges \((k \geq 3)\), and \( S \) be a subset of \( V(G) \), notice that the lower bound of \( \rho_\alpha(G) - \frac{km}{n} \) in Theorem 4.4 is better than the corresponding lower bound in Theorems 4.15 and 4.20.

**Remark 4.24** Let \( G \) be a \( k \)-uniform hypergraph on \( n \) vertices and \( m \) edges \((k \geq 3)\). Notice that taking \( S \) be a transversal \([10]\) (also called vertex cover or hitting set), or a dominating set \([4]\) in Theorems 4.15, 4.16 and 4.20, we deduce the corresponding results, respectively, since \( S \) is an arbitrary subset of vertex set \( V(G) \) in these three above theorems. Moreover, taking \( S \) be a maximum strong independent set, a maximum weak independent set, a minimum vertex cut in Theorems 4.15 and 4.20, we also deduce the corresponding results, respectively.

**Theorem 4.25** Suppose that \( G \) is a \( k \)-uniform hypergraph with \( n \) vertices and \( m \) edges. Let \( \Delta \) and \( \delta \) be the maximum and minimum degrees of \( G \), respectively. Then

\[
\rho_\alpha(G) - \frac{km}{n} \geq \alpha \frac{1}{2n} \left( \Delta^{\frac{k}{k+1}} - \frac{\Delta^{\frac{k}{k+1}}}{\Delta^{\frac{1}{k-1}} + \delta^{\frac{1}{k-1}}} \right) - (\Delta + \delta) \right) + \frac{(1 - \alpha)k}{2n} \left( \frac{2}{\Delta} \left( \Delta^{\frac{1}{k-1}} + \delta^{\frac{1}{k-1}} \right)^{\frac{k-1}{k}} - (\Delta + \delta) \right).
\]

**Proof.** Let \( i_0 \) and \( j_0 \) be the vertices of \( G \) with \( d_{i_0} = \Delta \) and \( d_{j_0} = \delta \). We distinguish the following two cases.

**Case 1.** \( i_0 \) and \( j_0 \) are not adjacent in \( G \).
We first define a vector \( x \in \mathbb{R}^n \) by
\[
x_i = \begin{cases} \frac{\alpha_i}{\sqrt{n}}, & i = i_0, \\ \frac{\alpha_i}{\sqrt{n}}, & i = j_0, \\ \frac{\alpha_i}{\sqrt{n}}, & \text{otherwise}, \end{cases}
\]
where
\[
a_1 = \frac{\sqrt{n} \delta^\frac{1}{1-r}}{\sqrt{\Delta + \frac{1}{1-r}}}, \quad a_2 = \frac{\sqrt{n} \delta^\frac{1}{1-r}}{\sqrt{\Delta + \frac{1}{1-r}}}
\]
It can be checked that \( \|x\|_k = 1 \). By Lemma 2.2, we have
\[
\rho_\alpha(G) - \frac{km}{n} \geq x^T (A_\alpha(G)x) - \frac{km}{n} = \alpha \sum_{i \in V(G)} d_i x_i^k + (1 - \alpha)k \sum_{e \in E(G)} x^e - \frac{km}{n} = \frac{\alpha}{n} \left( \frac{2 \left( \Delta^\frac{k-1}{k} + \delta^\frac{k-1}{k} \right)}{\Delta^\frac{1}{1-r} + \delta^\frac{1}{1-r}} - (\Delta + \delta) + km \right) + \frac{(1 - \alpha)k}{n} \left( \frac{2 \left( \Delta^\frac{k-1}{k} + \delta^\frac{k-1}{k} \right)}{\Delta^\frac{1}{1-r} + \delta^\frac{1}{1-r}} - (\Delta + \delta) \right).
\]

Case 2. \( i_0 \) and \( j_0 \) are adjacent in \( G \).

Let \( G' \) be a copy of \( G \) and \( G^* = G \cup G' \). Let \( i'_0 \) and \( j'_0 \) be the corresponding vertices of \( i_0 \) and \( j_0 \) in \( G' \), respectively. Clearly, \( i_0 \) and \( j'_0 \) are not adjacent in \( G^* \), and \( d_{G^*}(i_0) = \Delta, d_{G^*}(j'_0) = \delta \). Using the same arguments of Case 1 for \( G^* \), we have
\[
\rho_\alpha(G^*) = \frac{km}{n} - \frac{km}{n} = \rho_\alpha(G^*) - \frac{km}{n} \geq \frac{\alpha}{2n} \left( \frac{2 \left( \Delta^\frac{k-1}{k} + \delta^\frac{k-1}{k} \right)}{\Delta^\frac{1}{1-r} + \delta^\frac{1}{1-r}} - (\Delta + \delta) \right) + \frac{(1 - \alpha)k}{2n} \left( \frac{2 \left( \Delta^\frac{k-1}{k} + \delta^\frac{k-1}{k} \right)}{\Delta^\frac{1}{1-r} + \delta^\frac{1}{1-r}} - (\Delta + \delta) \right).
\]
We complete the proof. ■

**Remark 4.26** Let \( G \) be a \( k \)-uniform hypergraph on \( n \) vertices and \( m \) edges. Similar to Remark 4.5, it follows from the Rearrangement inequality and the Hölder inequality that
\[
\rho_\alpha(G) = \frac{km}{n} \geq \frac{\alpha}{2n} \left( \frac{2 \left( \Delta^\frac{k-1}{k} + \delta^\frac{k-1}{k} \right)}{\Delta^\frac{1}{1-r} + \delta^\frac{1}{1-r}} - (\Delta + \delta) \right) + \frac{(1 - \alpha)k}{2n} \left( \frac{2 \left( \Delta^\frac{k-1}{k} + \delta^\frac{k-1}{k} \right)}{\Delta^\frac{1}{1-r} + \delta^\frac{1}{1-r}} - (\Delta + \delta) \right) \geq 0,
\]

\]
which improves the result in Lemma 4.1 and generalizes the result of Si and Yuan [22, Theorem 3.2].

Furthermore, for a connected $k$-uniform hypergraph $G$ on $n$ vertices and $m$ edges ($k \geq 3$), taking $S$ be a pair of vertices $\{i, j\}$ with $d_i = \Delta, d_j = \delta, i \neq j$, and $i, j \in V(G)$. When $i, j$ are not adjacent in $G$, the first lower bound of $\rho_\alpha(G)$ in Corollary 4.9 is better than the lower bound in Theorem 4.25. When $i, j$ are adjacent in $G$, the lower bound of $\rho_\alpha(G)$ in Theorem 4.25 is better than the first lower bound in Corollary 4.9.

References

[1] A. E. Balobanov, D. A. Shabanov, On the number of independent sets in simple hypergraphs. Mat. Zametki 103 (2018), no. 1, 38-48.

[2] A. E. Balobanov, D. A. Shabanov, On the strong chromatic number of a random 3-uniform hypergraph. Discrete Math. 344 (2021), 112231, 16 pp.

[3] C. Berge, Hypergraphs: Combinatorics of finite sets, third edition, North-Holland, Amsterdam, 1973.

[4] C. Bujtás, M. A. Henning, Z. Tuza, Transversals and domination in uniform hypergraphs. European J. Combin. 33 (2012), 62-71.

[5] S. M. Cioabă, D. A. Gregory, Large matchings from eigenvalues. Linear Algebra Appl. 422 (2007), 308-317.

[6] L. Collatz, U. Sinogowitz, Spektren endlicher Grafen. Abh. Math. Sem. Univ. Hamburg 21 (1957), 63-77.

[7] J. Cooper, A. Dutle, Spectra of uniform hypergraphs. Linear Algebra Appl. 436 (2012), 3268-3292.

[8] S. Friedland, S. Gaubert, L. Han, Perron-Frobenius theorem for nonnegative multilinear forms and extensions. Linear Algebra Appl. 438 (2013), 738-749.

[9] H. Y. Guo, B. Zhou, On the $\alpha$-spectral radius of uniform hypergraphs, Discuss. Math. Graph Theory 40 (2020), 559-575.

[10] M. A. Henning, A. Yeo, Transversals in linear uniform hypergraphs. Developments in Mathematics, 63. Springer, Cham, [2020].

[11] L. Q. Qi, Symmetric nonnegative tensors and copositive tensors. Linear Algebra Appl. 439 (2013), 228-238.

[12] L. Y. Kang, L. L. Liu, E. F. Shan, Sharp lower bounds for the spectral radius of uniform hypergraphs concerning degrees. Electron. J. Combin. 25 (2018), Paper No. 2.1, 13 pp.

[13] W. Li, J. Cooper, A. Chang, Analytic connectivity of $k$-uniform hypergraphs. Linear Multilinear Algebra 65 (2017), 1247-1259.
[14] H. Y. Lin, H. Y. Guo, B. Zhou, On the $\alpha$-spectral radius of irregular uniform hypergraphs, Linear Multilinear Algebra 68 (2020), 265-277.

[15] L. L. Liu, L. Y. Kang, S. L. Bai, Bounds on the spectral radius of uniform hypergraphs. Discrete Appl. Math. 259 (2019), 160-169.

[16] L. L. Liu, L. Y. Kang, E. F. Shan, On the irregularity of uniform hypergraphs. European J. Combin. 71 (2018), 22-32.

[17] V. Nikiforov, Merging the A- and Q-spectral theories. Appl. Anal. Discrete Math. 11 (2017), 81-107.

[18] V. Nikiforov, Hypergraphs and hypermatrices with symmetric spectrum. Linear Algebra Appl. 519 (2017), 1-18.

[19] W. J. Ning, H. Li, M. Lu, On the signless Laplacian spectral radius of irregular graphs. Linear Algebra Appl. 438 (2013), 2280-2288.

[20] L. Q. Qi, Eigenvalues of a real supersymmetric tensor. J. Symbolic Comput. 40 (2005), 1302-1324.

[21] J. Y. Shao, A general product of tensors with applications. Linear Algebra Appl. 439 (2013), 2350-2366.

[22] X. L. Si, X. Y. Yuan, On the spectral radii and principal eigenvectors of uniform hypergraphs. Discrete Math. Algorithms Appl. 9 (2017), 1750048, 9 pp.

[23] J. S. Xie, L. Q. Qi, The clique and coclique numbers’ bounds based on the $H$-eigenvalues of uniform hypergraphs. Int. J. Numer. Anal. Model. 12 (2015), 318-327.

[24] Y. N. Yang, Q. Z. Yang, Further results for Perron-Frobenius theorem for nonnegative tensors. SIAM J. Matrix Anal. Appl. 31 (2010), 2517-2530.

[25] X. Y. Yuan, L. Q. Qi, J. Y. Shao, C. Ouyang, Some properties and applications of odd-colorable $r$-hypergraphs. Discrete Appl. Math. 236 (2018), 446-452.