Magnetotransport in lateral superlattices with small-angle impurity scattering: Low-field magnetoresistance

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An analytical study of the low-field magnetoresistance of a two-dimensional electron gas subject to a weak periodic modulation is presented. We assume small-angle impurity scattering characteristic for high-mobility semiconductor heterostructures. It is shown that the condition for existence of the strong low-field magnetoresistance induced by so-called channeled orbits is $\eta^{1/2} q l \gg 1$, where $\eta$ and $q$ are the strength and the wave vector of the modulation, and $l$ is the transport mean free path. Under this condition, the magnetoresistance scales as $\eta^{1/2}$.

I. INTRODUCTION

The effect of a periodic modulation (lateral superlattice) on the transport properties of a two-dimensional (2D) electron gas has been the subject of intensive research during the last decade. This interest was triggered by an experiment of Weiss et al.\textsuperscript{1} who discovered that a weak one-dimensional modulation (grating) with a wave vector $q \parallel e_x$ induces strong commensurability oscillations of the magnetoresistivity $\rho_{xx}(B)$, while showing almost no effect on $\rho_{yy}(B)$ and $\rho_{xy}(B)$. The oscillation minima have been found to satisfy the condition $2R_c/a = n - 1/4$ with integer $n$, where $R_c$ is the cyclotron radius and $a = 2\pi/q$ the grating wave length. While these findings can be explained in terms of the quantum-mechanical band structure induced by the modulation,\textsuperscript{2} the phenomenon is, in fact, of quasi-classical nature, as was demonstrated by Beenakker.\textsuperscript{3} He showed that the geometric resonance of the cyclotron motion in the grating induces an extra contribution to the drift velocity of the guiding center, whose root-mean-square amplitude oscillates as $|\cos(q R - \pi/4)|$. These arguments were corroborated by an analytical solution of the Boltzmann equation within an expansion in the relative strength $\eta$ of the modulation, the result for the modulation-induced oscillatory magnetoresistivity $\Delta \rho_{xx}$ being of the order $\eta^2$. However, although the theoretical result accounted for the above experimental features, they disagreed strongly with the experiment as far as the damping of the oscillations with decreasing magnetic field is concerned. The reason for this was an oversimplified treatment of the impurity scattering: while the theory assumed isotropic scattering, in experimentally relevant high-mobility samples the random potential is very smooth, so that the scattering is of small-angle character, with the total relaxation rate $\tau_{\text{rel}}^{-1}$ much exceeding the momentum relaxation rate $\tau_m^{-1}$. This gap in the theory was filled in by our previous paper\textsuperscript{4} where the effect of small-angle scattering on the Weiss oscillations was studied analytically. It was found that the small-angle scattering changes the dependence of the oscillation amplitude on the magnetic field $B$ completely, leading to a much stronger damping of oscillations with decreasing $B$, in very good quantitative agreement with experimental data.

This paper continues our investigation of the effect of small-angle scattering on the transport in laterally modulated structures. We will address the issue of the low-field magnetoresistivity, which has been left aside in our earlier paper\textsuperscript{5}. In combination, Ref. \textsuperscript{5} and the present paper provide a complete quasiclassical theory of magnetoresistivity of a 2D electron gas subject to a weak one-dimensional modulation and a smooth random potential.

A distinct low-field magnetoresistivity was observed, along with the commensurability oscillations, in the original experiment\textsuperscript{1}, as well as in numerous later experiments on the transport in a lateral superlattice. Specifically, in low magnetic fields $B$ a positive magnetoresistivity was found, followed by a maximum in $\rho_{xx}(B)$. For not too strong modulation, the relevant magnetic fields are much weaker than those where the Weiss oscillations are observed, so that the two effects can be easily separated. Soon after the first experimental observation it was understood\textsuperscript{6} that the low-field magnetoresistivity is related to the existence of open (channeled) orbits in the magnetic fields $B < B_c = (\eta c/2e) q m v_F$. It is worth mentioning that this effect, which is not found within the $\eta$-expansion used in Refs. \textsuperscript{6,7}, has its counterpart in the context of the sound absorption in metals in the presence of a magnetic field. There, the trapping of electrons in channeled orbits by a sound wave leads to non-linearity of the acoustic response of an electron gas, as was observed experimentally\textsuperscript{8} and analyzed theoretically\textsuperscript{9}. Though the work by Beton et al.\textsuperscript{10,11} (see also a more recent paper\textsuperscript{12}) explained qualitatively the low-field magnetoresistivity as an effect of channeled orbits, a quantitative analytical description of the problem requires that the nature of disorder be taken into account. This is done below in this paper, where we demonstrate that the small-angle character of scattering leads to a parametrically different magnitude of this effect. We will also study how the contribution of drifting orbits to $\Delta \rho_{xx}$ is modified in the low-magnetic-field range, $B \sim B_c$.\

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II. LOW-FIELD MAGNETOTRANSPORT IN SMOOTH DISORDER

A. Generalities

We consider a weak periodic potential

\[ V(x) = \eta E_F \cos qx , \quad \eta \ll 1 \]  

acting on a 2D electron system (Fermi energy \( E_F \)) at temperature \( T = 0 \). In the presence of a perpendicular magnetic field \( B \), the electron at the Fermi level will perform cyclotron motion (frequency \( \omega_c = eB/mc \), cyclotron radius \( R_c = v_F/\omega_c \), where \( v_F \) is the Fermi velocity). The periodic potential causes a slow drift motion in the \( y \)-direction with velocity \( v_d \). The classical equations of motion are (the electron charge is \( -e \))

\[ m\ddot{x} = -\frac{e}{c} \dot{y}B - \frac{dV}{dx} , \quad m\ddot{y} = \frac{e}{c} \dot{x}B . \]  

Averaging over one cyclotron revolution one finds the drift velocity

\[ v_d = -\frac{e}{cB} \frac{\omega_c}{\pi} \int_{x_{\text{min}}}^{x_{\text{max}}} dx \frac{dV}{dx} , \quad \int \]  

where \( x_{\text{min}}, x_{\text{max}} \) are the classical turning points of the periodic motion. Integration of \( \dot{y} = \omega_c(x - x_0) \),

\[ v_y \equiv \dot{y} = \omega_c(x - x_0) , \]  

where \( x_0 \) is the point on the trajectory where \( \dot{y} = 0 \) (i.e. the guiding center coordinate). The \( x \)-component of velocity is now easily found from energy conservation,

\[ v_x \equiv \dot{x} = \left[ \frac{2}{m}(E - V_{\text{eff}}(x)) \right]^{1/2} , \]  

where the effective potential of the motion in \( x \)-direction is given by

\[ V_{\text{eff}}(x) = V(x) + \frac{m}{2} \omega_c^2(x - x_0)^2 . \]  

For weak magnetic fields, \( qR_c \ll 1 \), the effective potential is given by a parabola centered at \( x_0 \), with rapid periodic oscillations superposed (Fig. 1a). Intersection of this curve with a line of constant energy yields the turning points of the classical motion. There is always a central allowed region \( x_{\text{min}} < x < x_{\text{max}} \), for which the motion consists of a drift of complete cyclotron orbits. For sufficiently strong periodic potential, there exists also additional classically allowed region, to the left (right) of the turning point \( x_{\text{min}} \left( x_{\text{max}} \right) \). These are so-called channeled orbits, winding in a snake-like fashion along \( y \). Using the fact that \( x - x_0 \approx R_c \) in the region of channeled orbits, one finds the condition for the existence of channeled orbits \( B < B_c \), where

\[ \frac{e}{c} B_c = \frac{\eta mv_F q}{2} . \]  

We will assume that \( \omega_c \tau \gg 1 \) in the relevant range of magnetic fields, which implies for \( B \sim B_c \) that \( \eta \gg 2/ql \), where \( ql \sim 300 \div 1000 \) in a typical experiment with a high mobility sample, this condition is fulfilled even for a very weak (\( \eta \) of order of few percent) modulation.

![Effective potential](image)

**FIG. 1.** (a) Effective potential \( V_{\text{eff}}(x) \), Eq. (1); (b) effective potential linearized near a turning point, Eq. (15).

B. Contribution of drifting orbits

We will first discuss the contribution of the drifting cyclotron orbits to the transport. Substituting (1), (2) into (6), we find

\[ v_d = \frac{\eta}{2\pi} v_F \int_{u_{\text{min}}}^{u_{\text{max}}} du \sin u \left[ 1 - \eta \cos u - \frac{(u - u_0)^2}{(qR_c)^2} \right]^{-1/2} , \]  

where \( u = qx, u_0 = qx_0 \), etc. In the limit \( \eta qR_c \ll 1 \) (which is equivalent to \( B \gg B_c \) the term \( \eta \cos u \) in
may be neglected. The integral may then be done, yielding
\[ v_d = \frac{1}{2} \eta v_F qR_c J_0(qR_c) \sin u_0, \]
where \( J_0(x) \) is the Bessel function. The drift velocity is oscillating with \( u_0 \), i.e. the initial conditions. In order to determine the modulation-induced correction to the diffusion coefficient, one has to evaluate the average of \( v_d^2 \):
\[
\langle v_d^2 \rangle = \frac{1}{2\pi} \int_0^{2\pi} du_0 v_d^2 = \frac{1}{8} \eta^2 v_F^2 (qR_c)^2 J_0^2(qR_c)
\]
\[
\approx \frac{1}{4\pi} \eta^2 v_F^2 qR_c \cos^2(qR_c - \pi/4)
\]
(11)
(in the second line we used the condition \( qR_c \gg 1 \)). The result (11) obtained for the first time by Beenakker induces an oscillating (with \( qR_c \)) correction \( \delta \rho_{xx} \) to the resistivity along the modulation wave vector (Weiss oscillations). The amplitude of Weiss oscillations for the case of white-noise disorder was calculated in and for the (experimentally relevant) case of a smooth random potential in. Here we are interested in the behavior of \( \Delta \rho_{xx} \) in the region of sufficiently low magnetic fields, where the oscillations are exponentially damped. This means that the position \( u_0 \) of the guiding center changes due to impurity scattering by an amount \( \delta u_0 \gg q^{-1} \) within one cyclotron revolution. Taking into account that (in view of the oscillatory factor \( \sin u \)) the main contribution to the integral (11) comes from the regions close to the turning points \( u_{min}, u_{max} \), one easily realizes that an average squared drift velocity in one cyclotron revolution is then given by non-oscillatory part of Eq. (11),
\[
\langle v_d^2 \rangle = \frac{1}{8\pi} \eta^2 v_F^2 qR_c .
\]
The diffusive process in \( y \)-direction with the velocity squared \( \langle v_d^2 \rangle \) and the time step \( \Delta t = 2\pi/\omega_c \) induces a correction to the diffusion constant
\[
\Delta D_{yy} = \langle v_d^2 \rangle \frac{\pi}{\omega_c}
\]
and thus the resistivity correction
\[
\frac{\Delta \rho_{xx}}{\rho_0} = \frac{\Delta D_{yy}}{v_F^2 \pi/2(\omega_c \tau)^2} = \frac{1}{4} \eta^2 q \Delta t ,
\]
where \( \rho_0 \) is the Drude resistivity in the absence of modulation. Eq. (14) reproduces the result of Refs. [3,4] for the saturation value of \( \Delta \rho_{xx}/\rho_0 \) at low magnetic fields.

Having demonstrated how the result (14) of \( \eta \)-expansion of the Boltzmann equation is reproduced within the present approach for \( B \gg B_c \), we are prepared to turn to the question of our main interest here, i.e. to the range of lower magnetic fields, \( B \sim B_c \). Since the main contribution to the drift velocity, Eq. (11), comes from the vicinity of the turning points, one can linearize the second (parabolic) term in the effective potential (16) near the points \( x = x_{0\pm} = x_0 \pm R_c \),
\[
V_{eff}(x) - E_F = V(x) \pm mv_F \omega_c (x - x_{0\pm})
\]
\[
= \eta E_F (\cos u + \beta(u - u_{0\pm})).
\]
We have introduced here the parameter
\[
\beta = \frac{B}{B_c}. \tag{16}
\]
The contribution of the lower limit takes then the form
\[
v_d^{min} \approx \frac{\eta v_F}{2\pi} \left( \frac{qR_c}{2} \right) \frac{1}{2} F(u_0, \beta) , \tag{17}
\]
where
\[
F(u_0, \beta) = \int_{u_{min}}^{\infty} du \sin u \left( u - u_{0\pm} - \frac{1}{\beta} \cos u \right)^{-1/2}.
\]
The upper limit contribution is found to be \( v_d^{max} = -v_d^{min} \), \( u_0 \rightarrow -u_0 \). The average value of the total drift velocity \( v_d = v_d^{max} + v_d^{min} \) is therefore equal zero, \( \langle v_d \rangle = 0 \), where, as before the averaging goes over the position of the guiding center,
\[
\langle \ldots \rangle = \frac{1}{2\pi} \int_0^{2\pi} du_0 \ldots .
\]
The average of the square of \( v_d \) is obtained as
\[
\langle v_d^2 \rangle = 2 \langle (v_d^{min})^2 \rangle - \langle v_d^{min})^2 ,
\]
where we used \( \langle (v_d^{min})^2 \rangle = \langle (v_d^{max})^2 \rangle \) and \( \langle v_d^{min})^{max} \rangle \approx \langle v_d^{min})^{max} \rangle \). The latter relation is ensured by the impurity scattering leading to a large shift \( \delta u_0 \gg q^{-1} \) within one cyclotron revolution, as has been discussed above.

Using (13), (14), we obtain thus the following contribution of the drifting orbits to the resistivity correction
\[
\frac{\Delta \rho_{xx}^{nc}}{\rho_0} = 2\pi \omega_c \tau \frac{\langle v_d^2 \rangle}{v_F} = \frac{\eta^2 q}{2\pi} F_{nc}(B/B_c) , \tag{19}
\]
where the dimensionless function \( F_{nc}(\beta) \) is defined by
\[
F_{nc}(\beta) = \langle F^2(u_0, \beta) \rangle - \langle F(u_0, \beta) \rangle^2 . \tag{20}
\]
The function \( F_{nc}(\beta) \) is shown in Fig. 1. In the limit \( \beta \rightarrow \infty \) one easily finds \( F_{nc}(\beta) \rightarrow \pi/2 \), reproducing Eq. (14).
C. Contribution of channeled orbits

We now turn to the contribution of channeled orbits. A particle will spend an average time $\tau_{\text{ch}}$ in a channeled orbit and during this time will propagate with velocity $v_F$. If the fraction of particles in channeled orbits is $P_{\text{ch}}(\beta)$, the contribution to the diffusion coefficient will be

$$\Delta D_y = \frac{1}{2} v_F^2 \tau_{\text{ch}} P_{\text{ch}}(\beta). \quad (21)$$

In order to calculate $P_{\text{ch}}(\beta)$ (see e.g. Refs. [8]), we consider again the effective potential $V_{\text{eff}}(x)$ linearized near a turning point, Eq. (15). The fraction of the phase space $(x, \phi)$ occupied by the channeled orbits is clearly given by

$$P_{\text{ch}}(\beta) = \int^{x_{\text{c}}} x_m \frac{dx}{a} \frac{2 \phi_{\text{max}}(x)}{\pi}, \quad (22)$$

where $x_m$ is the position of a maximum of $V_{\text{eff}}, x_{\text{c}}$ is the nearest point to $x_m$ satisfying $V_{\text{eff}}(x_c) = V_{\text{eff}}(x_m)$ (see Fig. 1b), and $\phi_{\text{max}}(x)$ is the limiting angle of the velocity vector with the $y$-axis. In view of $\eta \ll 1$ we have $\phi_{\text{max}} \ll 1$, so that

$$\phi_{\text{max}}(x) = \left[ \frac{V_{\text{eff}} - V_{\text{eff}}(x)}{m v_F^2 / 2} \right]^{1/2}, \quad V_m = V_{\text{eff}}(x_m). \quad (23)$$

Substitution of (23) into (22) yields

$$P_{\text{ch}}(\beta) = \frac{4}{\pi^2} (2\eta)^{1/2} \Phi(\beta), \quad (24)$$

where

$$\Phi(\beta) = \frac{1}{4\sqrt{2}} \int_{\arcsin \beta + \pi/3}^{\arcsin \beta + 2\pi} du \times \Re \left[ (1 - \beta^2 + \beta \arcsin \beta - \cos u - \beta u)^{1/2} \right]. \quad (25)$$

The function $\Phi(\beta)$ satisfies $\Phi(0) = 1$ and $\Phi(1) = 0$ and is shown in Fig. 3a.

The life time of a particle in a channeled orbit, $\tau_{\text{ch}}$, may be obtained by considering the diffusion in the space $(x, \dot{x})$ in the potential well formed by the potential $V_{\text{eff}}(x)$. Particles diffusing beyond the border line marked by the maximum of $V_{\text{eff}}(x)$ escape the confining well. The lifetime of particles may be defined from the time dependence of the total number of particles in the well, $n_w(t)$, as

$$\tau_{\text{ch}} = \int_0^\infty dt n_w(t), \quad (26)$$

where it is assumed that at $t = 0$ there is one particle in the well, $n_w(0) = 1$, spread uniformly over the volume $V$ of the corresponding phase space. In order to calculate $n_w$ we define the phase space density $f(x, \dot{x}, t)$ subject to the initial condition $f(x, \dot{x}, 0) = 1/V$ and to the boundary condition $f = 0$ at the boundary of the confinement region.

To determine $f(x, \dot{x}, t)$ analytically, we approximate the potential well by a parabola

$$V_{\text{eff}}(x) = 4\Delta E \frac{(x - (x_c + x_m)/2)^2}{(x_c - x_m)^2} + \text{const}, \quad (27)$$

where $\Delta E$ is the depth of the well. Introducing dimensionless variables

$$X = \frac{2(x - (x_c + x_m)/2)}{(x_c - x_m)}, \quad Y = \left( \frac{m}{2\Delta E} \right)^{1/2} \dot{x}, \quad (28)$$

and $\epsilon = E/\Delta E$, the energy takes the isotropic form

$$\epsilon = X^2 + Y^2, \quad \epsilon \leq 1. \quad (29)$$

The scaled phase space trajectories are thus circles. Let us now consider the effect of impurity scattering. The small-angle scattering by the smooth random potential induces a diffusive motion in the space of the velocity angle, $\langle \delta \phi^2 \rangle = 2D_\phi \delta \phi$, with the diffusion coefficient $D_\phi = 1/\tau$, where $\tau$ is the transport time. For channeled orbits we have $\dot{x} \simeq v_F$, implying a diffusion in $\dot{x}$ with diffusion coefficient $D_{\dot{x}} = v_F^2/\tau$. Provided the motion of the particles in the potential well is rapid as compared to the escape process (requiring the condition $\eta^{3/2} g l \gg 1$, see below), the effect of the anisotropic diffusion (along $Y$) is equivalent to isotropic diffusion with diffusion coefficient

$$D = \frac{1}{2} D_Y = \frac{m}{4\Delta E} \frac{v_F^2}{\tau}. \quad (30)$$

We are thus left with the diffusion equation in a circle,
supplied with the boundary condition \( n(1,t) = 0 \) and the initial condition \( n(R,0) = 1/\pi \). The solution is easily found by expanding \( n(R,t) \) in eigenfunctions \( J_0(\kappa_n R) \) of the Laplace operator,

\[
n(R,t) = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1}{\kappa_n} J_0(\kappa_n R) e^{-D\kappa_n^2 t},
\]

where \( \kappa_n \) are zeros of the Bessel function \( J_0(x) \). We find thus for the integrated staying probability \( n_w(t) = 2\pi \int_0^1 dRRn(R,t) \)

\[
n_w(t) = \sum_n \frac{4}{\kappa_n^2} e^{-D\kappa_n^2 t}.
\]

The lifetime of a particle in the channel is now found from (31) as

\[
\tau_{ch} = \frac{4}{D} \sum_n \frac{1}{\kappa_n^4} = \frac{1}{8D} = \frac{1}{2}\eta \tau \Phi_1(\beta),
\]

where

\[
\Phi_1(\beta) = \sqrt{1 - \beta^2} + \beta \arcsin \beta - \frac{\pi}{2} \beta.
\]

The dimensionless function \( \Phi_1(\beta) \) decreasing monotonically from 1 at \( \beta = 0 \) to 0 at \( \beta = 1 \) is shown in Fig. 3a.

Collecting the results for \( P_{ch}(B) \) and \( \tau_{ch} \) we find the correction to the resistivity due to transport in channeled trajectories as

\[
\frac{\Delta \rho_{xx}^{ch}}{\rho_0} = 2(\omega_c \tau)^2 P_{ch}(\beta) \tau_{ch}/\tau = \frac{\sqrt{2}}{\pi^2} \eta^2 (ql)^2 F_{ch}(\beta),
\]

where

\[
F_{ch}(\beta) = \beta^2 \Phi_1(\beta) \Phi(\beta).
\]

Therefore, the magnitude of the low-field magnetoresistance (i.e. the value of \( \Delta \rho_{xx}^{ch}/\rho_0 \) at maximum) scales as \( \eta^{7/2} \) with the modulation amplitude. The function \( F_{ch}(\beta) \) approaches zero for \( \beta \to 0 \) and \( \beta \to 1 \) and has a maximum of \( \sim 0.03 \) at \( \beta \simeq 0.4 \) (see Fig. 3b).

**D. Resistivity correction at zero magnetic field**

Formulas (19), (36) give a zero resistivity correction in the limit \( \beta \to 0 \). However, these formulas are not valid for too small \( \beta \), and in particular, at \( B = 0 \). This is clear already from the fact that we used the condition \( \omega_c \tau \gg 1 \) in the course of their derivation, which implies \( \beta \gg 1/\eta ql \). (In fact, the lower limit for \( \beta \) is even more restrictive, see Eq. (39) below). Therefore, the zero magnetic field case requires a separate treatment. Our approach, based on consideration of a disorder-induced diffusion in the space of electron orbits, can be, however, generalized to the case \( B = 0 \) as well. The calculation presented in Appendix yields the following result:

\[
\frac{\Delta \rho_{xx}(B = 0)}{\rho_0} = C \eta^{1/2},
\]

where \( C \simeq 1.24 \) is a numerical coefficient. How does Eq. (38) match the finite-\( B \) result (36)? In our derivation in Sec. 1B, 1C we assumed that the cyclotron motion away from the domain of the phase space occupied by the channeled orbits is faster than the escape process.
from a channel. Since a characteristic angle $\phi_{\text{max}}$ for the channeled orbit is $\phi_{\text{max}} \sim \eta^{1/2}$, we get the condition $\tau_{\text{ch}} \gg \eta^{1/2}/\omega_c$, which can be rewritten as

$$\beta \gg \frac{1}{\eta^{3/2} q l}. \quad (39)$$

Note that the r.h.s. of (39) is assumed to be much smaller than unity, see Sec. II C. In the opposite case, $\beta \ll 1/\eta^{3/2} q l$, the zero-field result (38) should be valid. Indeed, comparing (36) and (38), we see that they match at $\beta \sim 1/\eta^{3/2} q l$.

E. Conditions of applicability

Let us discuss conditions of validity of our consideration. First of all, the picture of the diffusion in the velocity space used above is justified if the particle undergoes many scattering events within the time $\tau_{\text{ch}} \sim \eta \tau$, which implies

$$\eta \gg \tau_s/\tau. \quad (40)$$

Assuming a value $\tau/\tau_s \sim 50$ characteristic for high mobility samples, this inequality is reasonably fulfilled for modulation strengths $\eta \gtrsim 5\%$. In the opposite case $\eta \ll \tau_s/\tau$ (i.e. for very weak modulations) one should replace $\tau_{\text{ch}}$ by $\tau_s$ in Eq. (36), yielding

$$\frac{\Delta \rho_{xx}^{\text{ch}}}{\rho_0} = \frac{2^{3/2}}{\pi^2} \eta^{3/2} q l \eta \beta^2 \Phi(\beta), \quad (41)$$

where $l_s = v_F \tau_s$, so that the $\eta^{7/2}$ behavior changes into the $\eta^{5/2}$ one. As to the zero-$B$ formula (38), we expect that it transforms at $\eta \ll \tau_s/\tau$ into $\Delta \rho/\rho_0 \sim \eta^{3/2} \tau_s/\tau_s$, matching the result $\Delta \rho/\rho_0 \sim \eta^{3/2} \tau_s$ in the limit of isotropic scattering ($\tau_{\text{ch}} = \tau$).

Another essential assumption was that the oscillatory motion within the potential well takes place on a time scale much shorter than the escape time $\tau_{\text{ch}}$. In other words, we assume that the particle performs many oscillations in the channel (with the frequency $\omega_{\text{ch}} \sim \eta^{1/2} v_F q$) before it escapes into a conventional cyclotron orbit. The condition for this is $\omega_{\text{ch}} \tau_{\text{ch}} \gg 1$, which means that

$$\eta^{3/2} q l \gg 1. \quad (42)$$

For experimentally relevant values of $q l$ this implies $\eta \gg 1 \div 2\%$, which is fulfilled in the majority of experiments. In the opposite case, $\eta^{3/2} q l \ll 1$, disorder scattering dominates over modulation-induced effects, so that a particle escapes from the channel before it “recognizes” that it is trapped there. In this limit the non-perturbative in $\eta$ effects related to existence of channeled orbits become irrelevant, and the magnetoresistivity is given by the result of the perturbative expansion (i.e. by Eq. (14) in the region of exponentially damped oscillations), without any pronounced features around $B = B_c$.

Note that in the assumed regime $\eta^{3/2} q l \gg 1$ is still violated in a narrow vicinity of $\beta = 1$, namely at $1 - \beta \lesssim (\eta^{3/2} q l)^{-4/7}$, since both $\omega_{\text{ch}}$ and $\tau_{\text{ch}}$ vanish when $\beta \rightarrow 1$, $\tau_{\text{ch}} \propto (1 - \beta)^{3/2}$ and $\omega_{\text{ch}} \propto (1 - \beta)^{1/4}$. This implies that channeled orbits get gradually destroyed by disorder as $\beta$ approaches unity and leads to a smearing of singularity in $\Delta \rho_{xx}/\rho_0$ at $\beta = 1$ [Eqs. (14), (40)] over a narrow interval $\delta \sim (\eta^{3/2} q l)^{-4/7}$.

Finally, we discuss the overall magnitude of the effect. At $B > B_c$ only the drifting orbits exist, and Eq. (40) predicts an enhancement of $\Delta \rho_{xx}/\rho_0$ [with respect to its value well above $B_c$, Eq. (14)] by the factor $F_{\text{nc}}(\beta) \sim 1$. At $B < B_c$ comparison of Eq. (36) and Eq. (40) shows that the contribution of channeled orbits dominates, yielding an enhancement of $\Delta \rho_{xx}/\rho_0$ [again with respect to (14)] by a parametrically large factor $(2^{5/2}/\pi^2) F(\beta) \eta^{3/2} q l$. Let us note, however, that in view of a rather small numerical value of $F(\beta)$ (equal to 0.03 at the maximum), this enhancement factor may be not so big, despite a large value of the parameter $\eta^{3/2} q l$.

III. SUMMARY

We have presented an analytical study of the low-field magnetoresistance of a 2D electron gas subject to a weak one-dimensional modulation. We assumed that the disorder scattering is of small-angle nature due to the long-range character of the random potential. This corresponds to the experimental situation in high-mobility semiconductor heterostructures, where the smoothness of the impurity potential is controlled by the large spacer separating the doping layer from the 2D electron gas.

We have demonstrated that a strong magnetoresistance with a pronounced maximum at $B \sim 0.4 B_c$ (with $B_c$ defined by Eq. (3)) exists for sufficiently strong modulation, $\eta^{3/2} q l \gg 1$. (In fact, due to smallness of a numerical coefficient, a rather large value of this parameter is required, $\eta^{3/2} q l \gtrsim 50$.) The amplitude of $\Delta \rho$ scales with the modulation strength $\eta$ in this regime as $\eta^{7/2}$, see Eq. (31). At zero magnetic field the grating-induced correction to the resistivity is small and scales as $\eta^{1/2}$, see Eq. (38). At $B \gg B_c$ the magnetoresistivity is described by the earlier theory using the perturbative expansion in $\eta$ (Ref. 3 for white-noise random potential and Ref. 4 for smooth disorder).

We have presented a detailed discussion of the limits of validity of the theory and, in particular, its matching with the earlier results of the $\eta$-expansion. Specifically, for a sufficiently weak modulation, $\eta^{3/2} q l \ll 1$, the result of the $\eta$-expansion (predicting $\Delta \rho \propto \eta^2$) is valid in the whole range of magnetic fields including the low-field region $B \lesssim B_c$, implying disappearance of the strong low-field magnetoresistance. Therefore, the $\eta^{7/2}$ behavior of $\Delta \rho$ at $B < B_c$, Eq. (36), does not imply a true non-analyticity of $\Delta \rho(\eta)$ at $\eta \rightarrow 0$ but rather restricts the applicability of the $\eta$-expansion at weak $B$ to the region...
of sufficiently small modulation amplitudes, \( \eta^{3/2}q l \ll 1 \).

Finally, to illustrate the magnitude of the effect, let us calculate \( \Delta \rho/\rho \) for typical experimental parameters. Specifically, we use the parameters of a recent experiment on electron density \( n = 2.84 \times 10^{11} \text{ cm}^{-2} \), modulation period \( a = 120 \text{ nm} \), transport mean free path \( l = 19 \mu\text{m} \). While the emphasis in this chapter was put on a novel type of quantum magnetoeoscillations, a pronounced magnetoresistance was observed in low magnetic fields, with a maximum at \( B_m = 0.145 \text{ T} \). Using \( B_m \approx 0.4B_c \), we infer the modulation amplitude \( \eta = 0.157 \), so that \( \eta^{3/2}ql \approx 62 \) is sufficiently large. With these values of parameters we find from (86) the value of the magnetoresistance at maximum, \( (\Delta \rho_{xx}/\rho_0)_{\text{max}} \approx 6.8 \), in good agreement with the experimentally observed magnitude of the effect, \( \rho_{xx}(B=0)/\rho_0 \approx 8 \). As concerns the resistivity correction at zero magnetic field, Eq. (88), we find \( \Delta \rho_{xx}(B=0)/\rho_0 \approx 0.49 \). Therefore, despite the parameter smallness of this correction at \( \eta \ll 1 \), its value in real experiments can be quite appreciable, due to its square-root dependence on the modulation strength \( \eta \).

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APPENDIX A: RESISTIVITY CORRECTION AT \( B = 0 \)

In zero magnetic field and in the absence of disorder equations (3), (4) trivially decouple. We denote the energies corresponding to the motion along \( x \) and \( y \) as \( E_x \) and \( E_y \), respectively (\( E_x + E_y = E_F \)). The motion along \( x \) is unbound for \( E_x > \eta E_F \); in the opposite case, \( E_x < \eta E_F \), the particle is in a channeled trajectory. The trajectories can be conveniently labeled by an angle \( \psi \) such that \( E_x = E_F \sin^2 \psi \), \( E_y = E_F \cos^2 \psi \). Transforming the uniform distribution on the Fermi surface \( n(0)(x, \phi) = 1/2\pi a \) to the variables \( (x, \psi) \) and integrating over \( x \), we find the equilibrium distribution in the \( \psi \)-space,

\[
n(0)(\psi) = \frac{1}{2\pi} r_-(\psi) ,
\]

where

\[
r_+ (\psi) = \int \frac{d\vartheta}{2\pi} R^{+1}(\vartheta, \psi) ,
\]

\[
R(\vartheta, \psi) = \left( \frac{\partial \psi}{\partial \vartheta} \right)_x = \left( 1 - \frac{\eta}{\sin^2 \psi} \cos \vartheta \right)^{1/2} ,
\]

and \( \vartheta = qx \). Eq. (A2) is valid for non-channeled orbits, \( \sin^2 \psi > \eta \) (consideration of which will be sufficient for us to determine the conductivity, as explained below).

The next step is the derivation of a kinetic equation characterizing the relaxation of \( n_\psi(\psi) \) toward equilibrium due to disorder. The starting point is the Liouville-Boltzmann equation for \( n_\psi(x, \phi) \),

\[
\partial_t n_\psi = -\psi \partial_\phi n_\psi - x \partial_x n_\psi + \frac{1}{\tau} \partial^2_\phi n_\psi ,
\]

where the last term describes the diffusion in the velocity space induced by the small-angle scattering (in the absence of this term \( \psi \) would be an integral of motion, implying \( \partial_t n_\psi = 0 \)). Now we transform Eq. (A4) from the variables \( (x, \phi) \) to \( (x, \psi) \) and use the fact that the oscillations due to the motion in \( x \)-direction can be considered as a fast process as compared to the impurity scattering [the corresponding condition is specified in Sec. (13) see eq. (12)]. This allows us to average over \( x \), which yields the following Fokker-Planck equation for \( n_\psi(\psi) \),

\[
\partial_t n_\psi = \frac{1}{\tau} \partial_\psi r_+(\psi) \partial_\psi n_\psi = \frac{n_\psi}{r_-(\psi)} ,
\]

In order to calculate the conductivity, we use the classical Kubo formula,

\[
\sigma_{xx} = e^2 \nu \int_0^\infty dt \int d\psi \int_{-\infty}^\infty d\psi' \int_0^\infty \frac{d\vartheta}{\pi} \int_0^{T(\psi')} d\tau' \times P(x, \psi; x', \psi'; t) v(x) \sin \phi v(x') \sin \phi' ,
\]

where \( v(x) = [2(E_F - V(x))/m]^{1/2} \) is the particle velocity and \( P(x, \psi; x', \psi'; t) \) is the propagator in the phase space \((x, \psi)\) (i.e. the probability density to move from a point \((x', \psi')\) to a point \((x, \psi)\) in a time \( t \)). Transforming from the variables \((x, \psi)\) to \((\tau, \psi)\), where \( \tau \) is the time along the trajectory, we get

\[
\sigma_{xx} = e^2 \nu \int_0^\infty dt \int d\psi \int_{-\infty}^\infty d\psi' \int_0^\infty \frac{d\vartheta}{\pi} \int_0^{T(\psi')} d\tau' \times v_x(\psi, \tau) v_x(\psi', \tau') v_F \sin \psi P(\tau; \psi, \psi'; t) ,
\]

where \( T(\psi) = \frac{q \vartheta}{\sin \psi} r_-(\psi) \) is the period of oscillations in \( v_x \) induced by the modulation. Averaging over the fast variable \( \tau \), we reduce Eq. (A7) to the form

\[
\sigma_{xx} = e^2 \nu v_F^2 \int_0^\infty dt \int d\psi \int \frac{d\psi' \sin \psi \sin \psi'}{2\pi r_-(\psi)} P(\psi, \psi', t) .
\]

The evolution kernel \( P(\psi, \psi', t) \) satisfies the differential equation (A8) with the initial condition \( P(\psi, \psi', 0) = \delta(\psi - \psi') \). Defining \( n(\psi) = \int_0^\infty dt \int d\psi' P(\psi, \psi', t) \sin \psi' \) and \( \bar{n}(\psi) = n(\psi)/r_-(\psi) \), we transform Eq. (A8) to the following form:

\[
\sigma_{xx} = e^2 \nu v_F^2 \int \frac{d\psi}{2\pi} \sin \psi \bar{n}(\psi) ,
\]

\[
\frac{1}{\tau} \partial_\psi r_+(\psi) \partial_\psi \bar{n}(\psi) = -\sin \psi .
\]
The range of variation of the variable $\psi$ in (A9), (A10) is restricted by the condition of non-channeled motion, $|\sin \psi| \geq \eta^{1/2}$. As soon as the particle comes into the region of channeled orbits, its velocity $v_{\psi}$ starts to oscillate rapidly around zero, so that the contribution of such trajectories to the conductivity can be neglected (the corresponding parameter is given by Eq. (38)). Therefore, Eq. (A11) should be supplemented by the condition $\tilde{n} = 0$ at the boundary of the non-channeled region. In other words, the integral $\hat{f}(\psi)/2\pi$ in (A9) is understood as $\int_{\pi}^{\pi-\eta^{1/2}} (d\psi/\pi)$ (we used $\eta \ll 1$), and the boundary condition to Eq. (A10) reads $\tilde{n}(\psi = \eta^{1/2}, \pi - \eta^{1/2}) = 0$.

The function $r_+(\psi)$ is equal to $r_+(\psi) \approx 1 - \frac{\nu}{16} \eta / \sin^2 \psi$ for $\sin^2 \psi \gg \eta$, reaching the value $2\sqrt{2}/\pi$ at the boundary, $\sin^2 \psi = \eta$. In the limit $\eta \to 0$ we have $r_+ = 1$ and $\tilde{n}(\psi) = \eta \sin \psi$, yielding the Drude conductivity $\sigma_{xx} = e^2 \nu \psi^2 \tau/2 \equiv \sigma_0$. We now want to calculate the leading correction. It will be shown below to be of order $\eta^{1/2}$, so that we will neglect all contributions of higher orders.

Let us denote by $\hat{D}$ the differential operator entering Eq. (A10), $\hat{D} = -\partial_\psi r_+(\psi) \partial_\psi$ on the interval $[\eta^{1/2}, \pi - \eta^{1/2}]$ with zero boundary conditions, so that Eq. (A9) can be rewritten as

$$\sigma_{xx} = e^2 \nu \psi^2 \tau \int_{\pi-\eta^{1/2}}^{\pi} \frac{d\psi}{\pi} \sin \psi \hat{D}^{-1} \sin \psi \quad (A11)$$

Let us further consider a complete set of normalized functions on this interval (eigenfunctions of $\partial_\psi^2$),

$$f_n(\psi) = \left(\frac{2}{\pi - 2\eta^{1/2}}\right)^{1/2} \sin \frac{n(\psi - \eta^{1/2})}{1 - (2/\pi) \eta^{1/2}} \quad (A12)$$

It is easy to see that to the first order in $\eta^{1/2}$ we can insert the projector $|f_1\rangle\langle f_1|$ in Eq. (A11),

$$\sigma_{xx} = e^2 \nu \psi^2 \tau \left(\frac{1}{\pi}\sin \psi|f_1\rangle^2 \langle f_1| \hat{D}^{-1} |f_1\rangle \right) \quad (A13)$$

where

$$\langle f|g \rangle = \int_{\eta^{1/2}}^{\pi-\eta^{1/2}} d\psi f(\psi)g(\psi).$$

With the same accuracy we have

$$\langle f_1| \hat{D}^{-1} |f_1\rangle = \langle f_1| \hat{D} |f_1\rangle^{-1} \quad (A14)$$

Expanding everything to the first order in $\eta^{1/2}$, we find

$$\langle f_1| \hat{D} |f_1\rangle = 1 + \frac{4}{\pi} (1 - C_1) \eta^{1/2} + O(\eta) \quad (A15)$$

$$\langle f_1| \sin \psi \rangle = \left(\frac{\pi}{2}\right)^{1/2} + O(\eta) \quad (A16)$$

where

$$C_1 = \int_1^\infty dx \int_0^{2\pi} \frac{d\vartheta}{2\pi} \left[ 1 - \left(1 - \frac{\cos \vartheta}{x^2}\right)^{1/2} \right] \approx 0.0241.$$

Substituting (A14), (A15), (A16) in (A13), we finally get

$$\sigma_{xx} \approx \sigma_0 [1 - C_2 \eta^{1/2}] \quad (A17)$$

with a numerical coefficient $C_2 = \frac{4}{\pi}(1 - C_1) \approx 1.24$. Since at zero magnetic field $\rho_{xx} = \sigma_{xx}^{-1}$, we arrive at the result (23) for the grating-induced correction to resistivity.

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1 D. Weiss, K. v. Klitzing, K. Ploog, and G. Weimann, Europhys. Lett. 8, 179 (1989).
2 R.R. Gerhardt, D. Weiss, and K. von Klitzing, Phys. Rev. Lett. 62, 1173 (1989); R. W. Winkler, J. P. Kotthaus, and K. Ploog, Phys. Rev. Lett. 62, 1177 (1989); P. Vasilopoulos and F.M. Peeters, Phys. Rev. Lett. 63, 2120 (1989).
3 C. W. J. Beenakker, Phys. Rev. Lett. 62, 2020 (1989).
4 A.D. Mirlin and P. Wölfle, Phys. Rev. B 58, 12986 (1998).
5 P. H. Beton, E. S. Alves, P. C. Main, L. Eaves, M. W. Delmow, M. Henini, O. H. Hughes, S. P. Beaumont, and C. D. W. Wilkinson, Phys. Rev. B 42, 9229 (1990).
6 V.D. Fil’I, V.I. Denisenko, and P.A. Bezuglyi, Fiz. Nizk. Temp. 1, 1217 (1975) [Sov. J. Low Temp. Phys. 1, 584 (1975)].
7 Yu.M. Galperin, V.L. Gurevich, and V.I. Kozub, Usp. Fiz. Nauk 128, 107 (1979) [Sov. Phys. Usp. 22, 352 (1979)] and references therein.
8 R. Menne and R. R. Gerhardt, Phys. Rev. B 57, 1707 (1998).
9 C. Albrecht, J.H. Smet, D. Weiss, K. von Klitzing, R. Hennig, M. Langenbuch, M. Suhreke, U. Rössler, V. Umansky, and H. Schweizer, Phys. Rev. Lett. 83, 2234 (1999).