Amplitudes in Noncritical Dimensions and Dimensional Regularization

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We study how the dimensional regularization works in the light-cone gauge string field theory. We show that it is not necessary to add a contact term to the string field theory action as a counter term in this regularization at least at the tree level. We also investigate the one-loop amplitudes of the bosonic theory in noncritical dimensions and show that they are modular invariant.

§1. Introduction

In the light-cone gauge NSR superstring field theory, there exist divergences caused by the colliding worldsheet supercurrents inserted at the interaction points. In order to deal with the divergences, we have proposed a dimensional regularization scheme in this theory.$^{1)–3)}$ In the conformal gauge, string theories in noncritical dimensions which appear in the process of the dimensional regularization correspond to the worldsheet theories with nonstandard longitudinal part. This is an interacting CFT that we call the $X^\pm$ CFT.$^{3)–5)}$ Using this CFT, we can rewrite the tree level amplitudes of the light-cone gauge superstring field theory into a BRST invariant form.$^{2),6)}$ In the conformal gauge formulation, the vertex operators in the Ramond sector involve the spin fields in the $X^\pm$ CFT. Since the $X^\pm$ CFT is an interacting theory, it is not straightforward to construct the spin fields. In Ref. 6), we have formulated a free field description of the $X^\pm$ CFT combined with the reparametrization ghosts, and constructed the spin fields via the free variables. Nevertheless, in the above scheme there are no states that satisfy the level matching condition in the (R,NS) and the (NS,R) sectors and thus no spacetime fermions exist in the regularized theory.$^6)$ We need therefore modify the scheme. A way of modification will be discussed elsewhere.

In this presentation, we would like to study how the dimensional regularization works in the amplitudes of the light-cone gauge superstring field theory. Among other things, we show that we obtain the results of the first-quantized formulation without adding any contact terms to the string field theory action as counter terms. It is an important question whether our regularization scheme is applicable beyond the tree level amplitudes. As a first step towards this question, we also evaluate the one-loop amplitudes in the light-cone gauge bosonic string field theory in noncritical dimensions. We show that the amplitudes are indeed modular invariant and can be recast into a BRST invariant form.$^7)$

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§2. Free field description of supersymmetric $X^\pm$ CFT

2.1. Free variables

The supersymmetric $X^\pm$ CFT is described by using the superfield variables

$$X^\pm(z, \bar{z}) \equiv X^\pm(z, \bar{z}) + i\theta \psi^\pm(z) + i\bar{\theta}\bar{\psi}^\pm(\bar{z}) + i\theta\bar{\theta}F^\pm(z, \bar{z}).$$

The Virasoro central charge of this CFT is $\hat{c} = 12 - d$. It follows that together with the super-reparametrization ghosts $B(z) \equiv \beta(z) + \theta b(z)$, $C(z) \equiv c(z) + \theta\gamma(z)$, and the transverse sector, the total system becomes a superconformal field theory of vanishing Virasoro central charge, and thus the nilpotent BRST charge $Q_B$ can be constructed.

We can define the superfields $X^+$, $X^-$ and the ghosts $B'$, $C'$, $\tilde{B}'$, $\tilde{C}'$, which satisfy the OPE’s for the free variables, as

$$B'(z) \equiv (D\Theta^+)^{2\alpha} B(z), \quad C'(z) \equiv (D\Theta^+)^{-2\alpha} C(z),$$

$$X'(z, \bar{z}) \equiv X^-(z, \bar{z}) + \alpha \left[ \partial D \left( \Sigma' + \frac{1}{2}\Phi \right) \frac{\Theta^+}{(D\Theta^+)^3} - \partial \left( \Sigma' + \frac{1}{2}\Phi \right) \left( \frac{1}{(D\Theta^+)^2} + \frac{\partial\Theta^+\Theta^+}{(D\Theta^+)^4} \right) - D \left( \Sigma' + \frac{1}{2}\Phi \right) \left( \frac{\partial\Theta^+}{(D\Theta^+)^3} + \frac{\partial D\Theta^+\Theta^+}{(D\Theta^+)^4} \right) + \text{c.c.} \right].$$

Here

$$\Theta^+(z) \equiv \frac{D X^+}{(\partial X^+)^2}(z), \quad \Phi(z, \bar{z}) \equiv \ln \left[ -4(D\Theta^+)^2(D\tilde{\Theta}^+)^2 \right],$$

$$\Sigma'(z) \equiv \sigma'(z) - \phi'(z) - \theta\beta'c'(z), \quad \alpha = \frac{d-10}{8},$$

where $\sigma'$ and $\phi'$ are defined so that $\partial\sigma' = c'b'$ and

$$\beta'(z) = e^{-\phi'} \partial\xi'(z), \quad \gamma'(z) = \eta' e^{\phi'}(z).$$

We note that $CB(z) = C'B'(z) = -D\Sigma'(z)$.

It is also possible to express $X^+$, $X^-$, $B$, $C$, $\tilde{B}$, $\tilde{C}$ in terms of the free variables.

2.2. Correlation functions in terms of free variables

Now that all the fields of the theory can be expressed in terms of the free variables, it should be possible to describe the theory using the free variables. We denote correlation functions on the complex plane in the superconformal field theory consisting of $X^\pm$ and the ghosts $B$, $C$, $\tilde{B}$, $\tilde{C}$ by $\langle \cdots \rangle_{X^\pm, B, C}$, and those of the free primed variables by $\langle \cdots \rangle_{\text{free}}$. The correlation functions that we are interested in are of the form

$$\left\langle e^{3\sigma - 2\phi}(\infty)^2 \phi_1 \phi_2 \cdots \phi_n \prod_{r=1}^N e^{-i\nu^r X^-} (Z_r, \bar{Z}_r) \right\rangle_{X^\pm, B, C},$$

(2.5)
where the ghosts are bosonized in the usual way and \(|e^{3\sigma - 2\phi}(\infty)|^2\) is inserted to soak up the ghost zero modes. We note that with the insertion of \(\prod_{r=1}^N e^{-ip_r^+ \chi^-(Z_r, \bar{Z}_r)}\), the superfield \(\chi^+(z, \bar{z})\) acquires an expectation value \(-\frac{i}{2}(\rho(z) + \rho(\bar{z}))\), where \(\rho(z) \equiv \sum_{r=1}^N \alpha_r \ln(z - Z_r)\) is the super Mandelstam mapping.

At first glance, one might expect that the correlation function (2.5) should be expressed in terms of the free variables as

\[
\left\langle e^{3\sigma - 2\phi}(\infty) \right| \left. e^{3\sigma - 2\phi}(\infty) \right\rangle_{\chi^\pm, B, C} = \left\langle e^{3\sigma - 2\phi}(\infty) \right| \left. e^{3\sigma - 2\phi}(\infty) \right\rangle_{\text{free}}^?,
\]

on the right-hand of which all the fields are considered to be expressed in terms of the free variables by using the relations (2.2). This would hold if the relations (2.2) were not singular anywhere on the complex plane. Nevertheless, the expectation values of supercovariant derivatives of \(\chi^+\) can have zeros and poles, and thus Eq. (2.6) is not true as it is. We need therefore insert operators at \(z = \tilde{z}_I, Z_r, \infty\) reflecting the singular behaviors of the expectation values of supercovariant derivatives of \(\chi^+\). Here \(\tilde{z}_I (I = 1, \ldots, N - 2)\) denotes the points determined by \(\partial \rho(\tilde{z}_I) = \partial D \rho(\tilde{z}_I) = 0\). We find that the resultant formula becomes

\[
\left\langle e^{3\sigma - 2\phi}(\infty) \right| \left. e^{3\sigma - 2\phi}(\infty) \right\rangle_{\chi^\pm, B, C} = \left\langle \mathcal{R} \phi_1(z_1, \tilde{z}_1) \phi_2(z_2, \tilde{z}_2) \cdots \phi_n(z_n, \tilde{z}_n) \prod_{I=1}^{N-2} \mathcal{O}_I \right| \left. \prod_{r=1}^N \mathcal{S}_r \right\rangle_{\text{free}},
\]

up to a numerical proportionality constant. The operators \(\mathcal{O}_I, \mathcal{R}\) and \(\mathcal{S}_r\) inserted on the right-hand side are defined as

\[\mathcal{O}_I \equiv \int_{\tilde{z}_I} dz \frac{D\Phi}{2\pi i} \left[ 1 + \frac{\alpha}{12} \frac{\partial^3 \delta X^+ \partial^2 \delta X^+}{(\partial^2 \delta X^+)^2} + \alpha \left( \frac{\alpha}{32} - \frac{1}{8} \right) \frac{\partial^3 \chi^+ \partial^2 \delta X^+ \partial X^+}{(\partial^2 \chi^+)^2} \right. \]
\[\left. - \frac{\alpha^2}{8} \frac{\partial^2 \delta X^+ \partial^2 \chi^+}{(\partial^2 \delta X^+)^2} D\Sigma' + \frac{\alpha^2}{8} \frac{\partial^2 \delta X^+ \partial X^+}{(\partial^2 \delta X^+)^2} \partial \Sigma' \right. \]
\[\left. - \frac{\alpha^2}{2} \frac{\partial^2 \chi^+ \partial \Sigma' D\Sigma'}{(\partial^2 \chi^+)^2} \right] e^{-\alpha \Sigma'} \left| \frac{2}{\partial^2 X^+} \right|^2, \tag{2.8a}\]

\[\mathcal{R} \equiv \int_{\infty} dz \frac{D\Phi}{2\pi i} \left( D\Theta^+ \right) \left[ (1 + \theta \gamma b) e^{3\sigma^+ - 2\phi}(z) \right]^2, \tag{2.8b}\]

\[\mathcal{S}_r \equiv \int_{\tilde{z}_I(r)} dz \frac{D\Phi}{2\pi i} \int_{\tilde{z}_I(r)} \frac{d\tilde{z}}{2\pi i} D\tilde{\Phi} e^{-i \frac{\alpha}{2p_r^1} \chi^+} (z, \tilde{z}) \cdot \tag{2.8c}\]
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One can show that these operators are conformal invariant.

§3. Vertex operators

3.1. Neveu-Schwarz sector

Let us consider the left-moving part of a state in the (NS,NS) sector of the light-cone gauge superstrings,

\[
\alpha_{-n_1}^i \cdots \psi_{-s_1}^{j_1} \cdots |\vec{p}\rangle_L .
\]  

(3.1)

Here \(n_i\) are positive integers and \(s_i\) are positive half odd integers. \(|\vec{p}\rangle_L\) is the state corresponding to the operator \(e^{i\vec{p} \cdot \vec{X}_L}\), where \(\vec{p}\) denotes the transverse \((d-2)\)-momentum and \(\vec{X}_L\) denotes the left-moving part of the transverse bosonic matter fields. The left-moving BRST invariant vertex operator in the conformal gauge corresponding to this state is given as

\[
V_{L}^{(-1)} \equiv e^\sigma e^{-\phi} V_{L}^{DDF} ,
\]  

(3.2)

where \(V_{L}^{DDF}\) denotes the DDF vertex operator corresponding to the state (3.1) with the momentum \(p^-\) in the \(-\) direction shifted as \(p^- + \frac{\alpha}{2p^+}\).\(^6\) The free field expression \(V_{L}^{(-1)}\) for \(V_{L}^{(-1)}\) can be obtained by using Eq. (2.7) to be

\[
V_{L}^{(-1)}' \equiv e^{(1+\alpha)\sigma'} e^{-(1+\alpha)\phi'} V_{L}^{DDF}' ,
\]  

(3.3)

where \(V_{L}^{DDF}'\) is the free field expression for \(V_{L}^{DDF}\).

3.2. Ramond sector

We turn to the left-moving part of the state in the Ramond sector of the light-cone gauge superstring,

\[
\alpha_{-n_1}^i \cdots \psi_{-m_1}^{j_1} \cdots |\vec{p},\vec{s}\rangle_L ,
\]  

(3.4)

where \(n_i\) and \(m_i\) are positive integers, and \(|\vec{p},\vec{s}\rangle_L\) is the state corresponding to the operator \(e^{i\vec{p} \cdot \vec{X}_L + i\vec{s} \cdot \vec{H}}\). Here \(\vec{H} = (H^A)\) \((A = 1, \ldots, \frac{d-2}{2})\) are defined by using the transverse matter fermions as

\[
e^{\pm iH^A} = \frac{1}{\sqrt{2}} \left( \psi^{2A-1} \pm i\psi^{2A} \right) ,
\]  

(3.5)

and \(\vec{s} = (s^A)\) with \(s^A = \frac{1}{2}\) or \(-\frac{1}{2}\). In order to express the BRST invariant vertex operator for the state (3.4) in the Ramond sector, we need use the free fields as explained in the Introduction. Mimicking \(V_{L}^{(-1)}\) in Eq. (3.3), we construct

\[
V_{L}^{(-\frac{3}{2})} \equiv e^{(1+\alpha)\sigma'} e^{-(\frac{1}{2}+\alpha)\phi'} e^{\frac{i}{2}H'} e^{\vec{s} \cdot \vec{H}} V_{L}^{DDF} ,
\]

(3.6)

\[
V_{L}^{(-\frac{1}{2})} (z) \equiv A_{-n_1}^i \cdots B_{-m_1}^{j_1} \cdots e^{(1+\alpha)\sigma'} e^{-(\frac{1}{2}+\alpha)\phi'} e^{\frac{i}{2}H'} e^{\vec{s} \cdot \vec{H}} V_{L}^{DDF} ,
\]

where \(H'\) is defined by the bosonization of the free fermions as

\[
\psi^+ = e^{iH'}, \quad \psi^- = -e^{-iH'} .
\]  

(3.7)
Let $V_L^{(-\frac{3}{2})}$ be the vertex operator in the CFT for unprimed variables which corresponds to $V_L^{(-\frac{3}{2})}$ in the free field description. Although the explicit form of $V_L^{(-\frac{3}{2})}$ is complicated, without it we can read off properties of $V_L^{(-\frac{3}{2})}$ from the free field form $V_L'(\frac{-3}{2})$. In particular, $V_L^{(-\frac{3}{2})}$ is a BRST invariant operator of $-\frac{3}{2}$ picture.6) Contrastingly, $V_L''(-\frac{3}{2})$ does not correspond to a BRST invariant vertex operator. We can obtain the left-moving BRST invariant vertex operator $V_L^{(-\frac{3}{2})}$ of $-\frac{1}{2}$ picture by applying to $V_L^{(-\frac{3}{2})}$ the picture changing operator $X(z) \equiv \{Q_B, \xi(z)\}$:

$$V_L^{(-\frac{3}{2})}(0) \equiv \lim_{z \rightarrow 0} X(z)V_L^{(-\frac{3}{2})}(0) = \lim_{z \rightarrow 0} (-e^{\phi T_F^X}) (z)V_L^{(-\frac{3}{2})}(0).$$

We define $V_L^{(-\frac{1}{2})}$ to be the free field version of $V_L^{(-\frac{3}{2})}$.

One can obtain the BRST invariant right-moving parts $V_R^{(-\frac{3}{2})}(\bar{z})$, $V_R^{(-\frac{1}{2})}(\bar{z})$ and their free field versions in the same way.

§4. Amplitudes

The tree level $N$-string amplitudes $\mathcal{A}_N$ are perturbatively computed, by starting from the string field theory action. We obtain \(^3\)

$$\mathcal{A}_N = (4ig)^{N-2} \int \left( \prod_{I=1}^{N-3} \frac{d^2 T_I}{4\pi} \right) F_N (T_I, \bar{T}_I),$$

where $T_I$ denotes the complex Schwinger parameter of the $I$th internal propagator ($I = 1, \ldots, N-3$), which are the $N-3$ complex moduli parameters of the amplitude $\mathcal{A}_N$. As was discussed in Ref. 2), on the right-hand side the integration region is taken to cover the whole moduli space and the integrand $F_N$ is described by the correlation function of the superconformal field theory for the light-cone gauge superstrings on the $z$-plane:

$$F_N (T_I, \bar{T}_I) = (2\pi)^2 \delta \left( \sum_{r=1}^N p_r^+ \right) \delta \left( \sum_{r=1}^N p_r^- \right) \text{sgn} \left( \prod_{r=1}^N \alpha_r \right) e^{-\frac{d-2}{16} \Gamma} \times \left\langle \prod_{I=1}^{N-2} \left[ (\partial^2 \rho)^{-\frac{3}{4}} T_F^{LC}(z_I) \right]^{2} \prod_{r=1}^N V_r^{LC} \right\rangle X^i.$$

Here $e^{-\frac{d-2}{16} \Gamma}$ originates from the conformal anomaly, \(^8\) and $V_r^{LC}$ denotes the vertex operators for the $r$th external string in the light-cone gauge. In the following we will consider the case in which $V_r^{LC}$ ($r = 1, \ldots, 2f$) are in the (R,R) sector and $V_r^{LC}$ ($r = 2f + 1, \ldots, N$) are in the (NS,NS) sector.

We would like to rewrite the light-cone gauge expression (4.1) of the amplitudes into a BRST invariant form, adding the longitudinal variables in the $X^\pm$ CFT and
the super-reparametrization ghosts. A key ingredient of this rewriting is to express the anomaly contribution $e^{-\frac{\pi^2}{16}\Gamma}$ in Eq. (4.2) in the form of the correlation function of the worldsheet superconformal field theory consisting of the $X^\pm$ CFT and the ghosts. It is straightforward to show that the quantity which appears on the right-hand side of Eq. (4.2) can be expressed as a correlation function in the system of the free variables defined in §2:

$$
(2\pi)^2 \delta \left( \sum_{r=1}^{N} p_r^+ \right) \delta \left( \sum_{r=1}^{N} p_r^- \right) e^{-\frac{\pi^2}{16}\Gamma} \prod_{I=1}^{N-2} \left| \partial^2 \rho \left( z_I \right) \right|^{\frac{3}{2}} \prod_{r=1}^{N} V_r^{L,C} 
$$

$$
\sim \left\langle \left| \left( \partial \rho \right)^{1+\alpha} e^{\sigma'} \left( \infty \right) \right|^{2} \prod_{I=1}^{N-2} \left| \frac{e^{-\left(1+\alpha\right)(\sigma'-\phi')}}{\left( \partial \rho \right)^{1+\alpha}} \right| \left( z_I \right) \prod_{r=1}^{N} \left( |\alpha_r|^{-\alpha} V_r^{(\frac{3}{2},-\frac{3}{2})} \left( Z_r, \bar{Z}_r \right) \right) \right\rangle_{\text{free}} 
$$

where $V_r^{(-1,-1)}$ and $V_r^{(\frac{3}{2},-\frac{3}{2})}$ are the vertex operators $V_r^{(\frac{3}{2},-\frac{3}{2})}$ and $V_r^{(-1,-1)}$ defined in Eqs. (3.3) and (3.6) for the $r$-th external string, and similarly for the right moving sector ones $V_r^{(-1)}$, $V_r^{(\frac{3}{2})}$ and $V_r^{(\frac{3}{2},-\frac{3}{2})}$.

On the right-hand side of Eq. (4.3), $X^\tau$ appears only in the form of the vertex operator $e^{i\tau X^\tau}$, and $\psi^\tau, \bar{\psi}^\tau$ do not appear. Therefore we can replace $X^\tau, \psi^\tau, \bar{\psi}^\tau$ by their expectation values $-\frac{1}{2} \left( \rho + \bar{\rho} \right)$, 0, 0 in the correlation function, and vice versa.

In this situation, without changing the value of the correlation function, we can replace $V_r^{(-\frac{1}{2},-\frac{1}{2})}$ with $V_r^{(-\frac{3}{2},-\frac{3}{2})}$ that is defined by applying the picture changing operator $X$ and its right-moving counter part $\tilde{X}$ to $V_r^{(-\frac{3}{2},-\frac{3}{2})}$ as

$$
V_r^{(-\frac{1}{2},-\frac{1}{2})} \equiv X \tilde{X} V_r^{(-\frac{3}{2},-\frac{3}{2})}. 
$$

Besides this replacement, we can further recast the right-hand side of Eq. (4.3) into

$$
\left\langle \left| \left( \sum_{r} \alpha_r Z_r \right) \lim_{z \to \infty} e^{-2(\sigma'-\phi')} \left( z \right) \right|^{2} \mathcal{R} \right\rangle 
$$

$$
\times \prod_{I=1}^{N-2} \left| \oint_{z_I} \frac{dz}{2\pi i} \frac{e^{-\sigma'}}{\left( \partial \rho \right)^{1+\alpha}} \left( z \right) \lim_{w \to z_I} \left( \left( \partial \rho \right)^{\alpha} e^{\sigma'} \right) \left( w \right) \right| O_I 
$$
where \( p_{L,r}, p_{R,r} = -\frac{1}{2}, -1, -\frac{3}{2} \) indicate the picture of the vertex operators. The choice of the picture in Eq. (4.6) is obvious from Eq. (4.3). To Eq. (4.6) we can easily apply the formula (2.7), and express it using the \( X^\pm \) CFT and the unprimed ghost fields. Substituting it into Eq. (4.2), we obtain

\[
F_N \sim \left\langle |\partial \rho c(\infty)|^2 \prod_i \int_{z_i} \frac{dz}{2\pi i} \frac{b}{\partial \rho} (z) e^{\phi T_{\mathcal{F}C}^+(z_i)} \right\rangle \prod_{r=1}^N S_r^{-1} \prod_{r=1}^N V_r^{(p_{L,r}, p_{R,r})} \langle Z_r, \bar{Z}_r \rangle. \tag{4.7}
\]

Here \( \langle \cdots \rangle \) denotes the correlation function of the CFT for the longitudinal and transverse variables and the super-reparametrization ghosts. \( V_r^{(p_{L,r}, p_{R,r})} \) is the unprimed field version of \( V_r^{(p_{L,r}, p_{R,r})} \) and it is BRST invariant. \( S_r^{-1} \) is defined as

\[
S_r^{-1} \equiv \oint_{z_i} \frac{dz}{2\pi i} \frac{b}{\partial \rho} (z) \oint_{\bar{z}_i} \frac{d\bar{z}}{2\pi i} \frac{b}{\partial \rho} (\bar{z}) e^{i\alpha \bar{\chi}^+} (z, \bar{z}), \tag{4.8}
\]

which can be shown to be the inverse of \( S_r \) in Eq. (2.8c) by replacing \( \chi^+ \) by its expectation value.

In Eq. (4.7), the right-hand side is expressed by the variables in the conformal gauge, but it is not manifestly BRST invariant. In order to get a BRST invariant form of the amplitudes, we would like to show that \( e^{\phi T_{\mathcal{F}}(z_l)} \) in Eq. (4.7) can be turned into the picture changing operator \( X(z_l) \) and thus

\[
F_N \sim \left\langle |\partial \rho c(\infty)|^2 \prod_i \int_{z_i} \frac{dz}{2\pi i} \frac{b}{\partial \rho} (z) X(z_l) \right\rangle \prod_{r=1}^N S_r^{-1} \prod_{r=1}^N V_r^{(p_{L,r}, p_{R,r})}. \tag{4.9}
\]

Let us introduce a nilpotent fermionic charge \( Q \) defined as

\[
Q \equiv \oint \frac{dz}{2\pi i} \left[ -\frac{b}{4} \frac{\partial}{\partial \rho} \left( iX_L^+ - \frac{1}{2}\rho \right) + \frac{1}{2} \frac{e^{-\phi} \partial \psi}{\partial \rho} \psi^+ \right] (z). \tag{4.10}
\]

One can show

\[
\oint_{z_i} \frac{dz}{2\pi i} \frac{b}{\partial \rho} (z) X(z_l) = - \oint_{z_i} \frac{dz}{2\pi i} \frac{b}{\partial \rho} (z) e^{\phi T_{\mathcal{F}C}^+ (z_l)} + \left[ Q, \oint \frac{dz}{2\pi i} \frac{b}{\partial \rho} (z) \oint \frac{dw}{2\pi i} \frac{A(w)}{w - z_l} e^{\phi (z_l)} \right] + \frac{1}{4} \oint \frac{dz}{2\pi i} \frac{b}{\partial \rho} (z) \oint \frac{dw}{2\pi i} \frac{\partial \psi^- (w)}{w - z_l} e^{\phi (z_l)}, \tag{4.11}
\]

where the explicit form of \( A(w) \) is given in Eq. (5.12) of Ref. 6). Substituting Eq. (4.11) into the right-hand side of Eq. (4.9) and comparing it with that of Eq. (4.7), one can see that in order to prove Eq. (4.9), one should show that the second and
third terms on the right-hand side of Eq. (4.11) do not contribute to the correlation function. One can prove the third term does not contribute to the correlation function.\(^2\),\(^6\) The second term is \(Q\)-exact. We can therefore prove that this is also irrelevant, by showing that \(Q\) (anti)commutes with all the operators in the correlation function (4.9). Thus we obtain the expression (4.9) for \(F_N\).

By deforming the contour of \(\oint z_I \frac{dz}{2\pi i b} \partial \rho(z)\) in Eq. (4.9) as was done in Ref. 2), we can obtain a manifestly BRST invariant form of the amplitude \(A_N\):

\[
A_N \sim \int \prod_{I=1}^{N-3} d^2 \mathcal{T}_I \left[ \prod_{I=1}^{N-3} \left( \oint_{C_I} \frac{dz}{2\pi i} b \partial \rho(z) \oint_{C_I} \frac{d\bar{z}}{2\pi i} \partial \bar{\rho}(\bar{z}) \right) \left| X(z_I) \right|^2 \right] \prod_{r=1}^{N} S_r^{-1} \prod_{r=1}^{N} V_r^{(p_{L,r}, p_{R,r})}.
\]

Here \(C_I\) denotes a contour which goes around the \(I\)th internal propagator. Compared with the form of the tree amplitudes in the critical case, the difference is in the insertions of \(S_r^{-1}\). These insertions are peculiar to the noncritical strings.\(^4\) It follows that in the limit \(d \to 10\) the amplitudes \(A_N\) smoothly coincide with the results of the first-quantized formalism in the critical dimensions. This implies that in our dimensional regularization scheme we need not add any contact term interactions to the string field theory action as counter terms.

§5. One-loop amplitudes of light-cone gauge bosonic string field theory in noncritical dimensions

We introduce the complex coordinate \(\rho\) on the \(N\)-string one-loop diagram in the usual way. We can map the \(\rho\)-plane to the periodic parallelogram on the complex \(u\)-plane with period 1 and \(\tau\) with \(\text{Im} \, \tau > 0\) (Fig. 1), using the Mandelstam mapping\(^8\)

\[
\rho(u) = \sum_{r=1}^{N} \alpha_r \left[ \ln \vartheta_1(u - U_r | \tau) - 2\pi i \frac{\text{Im} \, U_r}{\text{Im} \, \tau} u \right].
\]

\(U_r\) is the position of the puncture on the \(u\)-plane corresponding to the \(r\)th external string. We denote the \(N\) interaction points by \(u_I\) \((I = 1, \ldots, N)\), which are determined by \(\partial \rho(u_I) = 0\).

![Fig. 1. One-loop string diagram and the parallelogram on the \(u\)-plane.](https://example.com/fig1.png)
This leads to the modular invariance of the correlation functions of the worldsheet CFT on the $u$-plane. This takes the form

$$A_N \sim (2\pi)^2 \delta^2 \left( \sum_{r=1}^N \nu_r^\pm \right) \int d^2T \int \prod_{I=1}^{N-2} d^2T_I \int \frac{\alpha_m d\alpha_m}{4\pi} \frac{d\theta_m}{2\pi} \langle \prod_{r=1}^N V_r^{LC} \rangle e^{-\frac{d+2}{4\pi} \Gamma_{\text{loop}}}.$$  

(5.2)

Here $T$ denotes the complex Schwinger parameter for the internal propagator of the loop part, and $T_I$ ($I = 1, \ldots, N - 2$) are those for the other internal propagators. $\alpha_m$ and $\theta_m$ denote the string length and the twist angle of the internal propagator of the loop part (Fig. 1). $e^{-\frac{d-2}{4\pi} \Gamma_{\text{loop}}}$ denotes the contribution from the conformal anomaly, where

$$e^{-\Gamma_{\text{loop}}} \equiv \prod_{r=1}^N \left[ \alpha_r^{-2} e^{-2 \text{Re} \bar{N}_{00}^{rr}} \right] \prod_{l=1}^{N} \left| \partial^2 \rho(u_l) \right|^{-1},$$

$$\bar{N}_{00}^{rr} \equiv \frac{1}{\alpha_r} \left[ \sum_{s \neq r} \alpha_s \ln \vartheta_1(U_r - U_s | \tau) + i2\pi \frac{U_r}{\text{Im} \tau} \sum_{s=1}^N \alpha_s \text{Im} U_s + \rho(u_I(\tau)) \right] - \ln \vartheta_1(0|\tau).$$  

(5.3)

Let us study the behavior of the amplitude $A_N$ in Eq. (5.2) under the modular transformations

$$u \mapsto \frac{u}{ct+d}, \quad \tau \mapsto \frac{a\tau+b}{ct+d},$$

with $a, b, c, d \in \mathbb{Z}$ and $ad - bc = 1$. One can easily find that under the transformation (5.4) the Mandelstam mapping (5.1) transforms as

$$\rho(u) \mapsto \rho(u) + \frac{c}{ct+d} \sum_{r=1}^N \alpha_r U_r^2.$$  

(5.5)

This leads to the modular invariance of $A_N$ in Eq. (5.2).

Similarly to the tree level amplitudes, we can describe $e^{-\frac{d-2}{4\pi} \Gamma_{\text{loop}}}$ in terms of the correlation function of the worldsheet CFT consisting of the $X^\pm$ CFT and the $bc$-ghost system. Eventually, we can rewrite the amplitudes $A_N$ in Eq. (5.2) into a BRST invariant form as

$$A_N \sim \int d^2T \int \prod_{I=1}^{N-2} d^2T_I \int \frac{\alpha_m d\alpha_m}{4\pi} \frac{d\theta_m}{2\pi} \text{Im} \tau \times \left\langle \prod_{r=1}^N \left[ \frac{\partial^2 \rho(u)}{\partial \rho} \right] \prod_{I=1}^{N} \mathcal{S}_{(\text{bosonic}),r}^{-1} \right\rangle \times \left\langle \int_{C_m} \frac{du}{2\pi i} \partial \rho \langle u \rangle \int_{C_{m'}} \frac{du}{2\pi i} \partial \rho \langle u \rangle \right\rangle^{2 N-2} \prod_{I=\infty} \left\langle \int_{C_I} \frac{du}{2\pi i} \partial \rho \langle u \rangle \right\rangle^{2},$$

(5.6)

where $\mathcal{S}_{(\text{bosonic}),r}^{-1}$ denotes the bosonic part of $\mathcal{S}_r^{-1}$ in Eq. (4.8).
§6. Summary and discussion

We have seen that our dimensional regularization scheme works well in the light-cone gauge superstring field theory at least at the tree level. In particular, we have shown that in our scheme we can obtain the results of the first-quantized formulation without introducing any contact term interactions as counter terms.

As a first step towards the application of the dimensional regularization to the loop level, we have studied the one-loop amplitudes of the light-cone gauge bosonic string field theory in noncritical dimensions. We have shown that the amplitudes are modular invariant and can be recast into a BRST invariant form by using the $X^\pm$ CFT.

Another thing to be investigated is the Green-Schwarz formalism. As was pointed out in Ref. 9), our results seem to be useful in constructing the vertex operators in the semi-light-cone gauge Green-Schwarz string theory. In particular, the similarity transformation given in Ref. 9) looks similar to the bosonic part of Eq. (2.2). It will be interesting to apply our results to this formulation.

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