SYMPLECTIC 4-MANIFOLDS
WITH FIXED POINT FREE CIRCLE ACTIONS

JONATHAN BOWDEN

(Communicated by Daniel Ruberman)

Abstract. We show that recent results of Friedl-Vidussi and Chen imply that a symplectic 4-manifold admits a fixed point free circle action if and only if it admits a symplectic structure that is invariant under the action and we give a complete description of the symplectic cone in this case. This then completes the topological characterisation of symplectic 4-manifolds that admit non-trivial circle actions.

1. Introduction

Recently Friedl and Vidussi [7] solved the long standing Taubes Conjecture, which classifies which 4-manifolds of the form \( M \times S^1 \) admit symplectic forms. Moreover, they determined exactly which cohomology classes can be represented by symplectic forms. Using recent results of D. Wise, [13] they have extended their results to the case of non-trivial \( S^1 \)-bundles in [9]. In this note we observe that their results as well of those of Chen, who obtained partial results in the fixed point free case in [3], imply the analogue of (9, Theorem 1.3) for all fixed point free circle actions.

Before stating our main result we fix some notation and terminology. Let \( X \xrightarrow{p} M \) be an orientable 4-manifold with a fixed point free circle action and quotient space \( M = X/S^1 \). The quotient space is an orbifold whose underlying topological space \(|M|\) is a manifold since all the stabilisers of the \( S^1 \)-action are necessarily cyclic and the singular locus consists of a collection of branching circles (cf. [1], [6]).

We let \( M_{\text{reg}} \) denote the complement of an open tubular neighbourhood of the singular locus of \( M \) and \( X_{\text{reg}} = p^{-1}(M_{\text{reg}}) \), which is an honest \( S^1 \)-bundle so that the pushforward map \( p_* \) is well-defined for cohomology classes in \( H^*(X_{\text{reg}}, \mathbb{R}) \). The manifold \( M_{\text{reg}} \) has toroidal boundary and thus one may define the Thurston norm on \( H^1(M_{\text{reg}}, \mathbb{R}) \) in the usual fashion. Finally for \( \psi \in H^2(X, \mathbb{R}) \) we let \( \psi_{\text{reg}} \) denote the restriction of \( \psi \) to \( X_{\text{reg}} \).

Theorem 1. Let \( X \xrightarrow{p} M \) be an oriented manifold admitting a fixed point free \( S^1 \)-action with quotient space \( M \) and let \( \psi \in H^2(X, \mathbb{R}) \). Then the following are equivalent:

1. \( \psi \) can be represented by a symplectic form,
2. \( \psi \) can be represented by an \( S^1 \)-invariant symplectic form,
3. \( \psi^2 > 0 \) and \( p_* \psi_{\text{reg}} \in H^1(M_{\text{reg}}, \mathbb{R}) \) lies in the open cone over a fibered face of the Thurston norm ball and is the restriction of a class in \( H^1(|M|, \mathbb{R}) \).

Received by the editors May 11, 2012 and, in revised form, September 1, 2012.
2010 Mathematics Subject Classification. Primary 57R17; Secondary 57N10, 57N13.

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use
Note that if a class $\phi \in H^1(|M|, \mathbb{R})$ is integral, then $\phi$ can be represented by a fibration over $S^1$ that is transverse to the singular locus of $|M|$ if and only if its restriction to $M_{\text{reg}}$, which we denote by $\phi_{\text{reg}}$, is fibered. Recall that a fibration of a manifold with boundary is required to be transverse to the boundary. Furthermore, since $\phi_{\text{reg}}$ is the restriction of a class in $H^1(|M|, \mathbb{R})$, it automatically vanishes on the meridian classes in $\partial M_{\text{reg}}$ so that if $\phi_{\text{reg}}$ is fibered, then the induced fibration on the boundary is necessarily meridional. Thus the fibration dual to $\phi_{\text{reg}}$ extends to $|M|$ in the desired way by filling in discs near the singular locus. In particular, part (3) of Theorem 1 implies that the underlying manifold $|M|$ is fibered and we obtain a positive answer to the following conjecture, which implies [3], Conjecture 1.7, as a special case.

**Conjecture 1 (Generalised Taubes Conjecture).** Let $X$ be a symplectic 4-manifold that admits a non-trivial fixed point free circle action with quotient orbifold $M$. Then the (possibly empty) singular locus $L$ of $M$ is a meridionally fibered link.

Furthermore, as noted in [3], p. 6, Theorem 1 completes the characterisation of which symplectic manifolds admit non-trivial $S^1$-actions. For Baldridge, [1] showed that if a non-trivial $S^1$-action on a symplectic 4-manifold has fixed points, then $X$ is rational or ruled and thus admits an $S^1$-invariant symplectic form for some non-trivial $S^1$-action. In view of this we obtain the following corollary.

**Corollary 1.** Let $X$ be a symplectic 4-manifold that admits a non-trivial $S^1$-action. Then either the action is fixed point free and the quotient space fibers over $S^1$ or $X$ is rational or ruled. In either case, $X$ admits a non-trivial symplectic $S^1$-action.

2. **Proof of Theorem 1**

The proof of Theorem 1 is based on the following lemma, which provides a generalisation of [4], Theorem 5.2, to include irrational classes. For the proof we assume a certain familiarity with the basic properties of the Thurston norm (cf. [12]).

**Lemma 1.** Let $\overline{M}$ be a 3-manifold with an orientation preserving smooth action of a finite group $G$ and quotient orbifold $M = \overline{M}/G$. An element $\overline{\phi}$ in the invariant subspace $H^1(\overline{M}, \mathbb{R})^G$ admits a non-degenerate de Rham representative if and only if it admits a non-degenerate de Rham representative that is $G$-invariant.

In particular, the restriction of the associated class $\phi \in H^1(|M|, \mathbb{R}) \cong H^1(\overline{M}, \mathbb{R})^G$ to $M_{\text{reg}}$ lies in the open cone over a fibered face of the Thurston norm ball.

**Proof.** We first assume that $\overline{\phi}$ is rational. Since nothing changes after multiplying with positive constants, we may assume that $\overline{\phi}$ is in fact integral. In this case the first claim is just a restatement of [4], Theorem 5.2, which can be applied in complete generality in view of [11], Theorem 8.1. Note that the assumption $H^1(\overline{M}, \mathbb{Q})^G = \mathbb{Q}$ in [4], Theorem 5.2, can be replaced by the fact that the fibration is given by a fibered class $\overline{\phi}$ that is $G$-invariant. Moreover, the proof in [4] actually gives a fibration that is transverse to the branching locus in $\overline{M}$. The quotient map $\pi$ induces an isomorphism $H^1(\overline{M}, \mathbb{R})^G \cong H^1(|M|, \mathbb{R})$ so that there is a unique class $\phi$ with $\overline{\phi} = \pi^* \phi$ and the fibration dual to $\overline{\phi}$ descends to a fibration of $|M|$ dual to $\phi$. Finally since the fibration is transverse to the singular locus it follows that the restriction of $\phi$ to $M_{\text{reg}}$ is fibered.
We next assume that \( \overline{\phi} \) is irrational and let \( \phi \) be the unique class with \( \overline{\phi} = \pi^*\phi \). We let \( \iota_{\text{reg}} \) denote the natural inclusion \( M_{\text{reg}} \to |M| \) and set \( V = \text{Im}(\iota_{\text{reg}}^*) \). By the previous case all rational classes in \( V \) that are sufficiently close to \( \iota_{\text{reg}}^*\phi \) are fibered. If \( \phi_{\text{reg}} \) itself did not lie in the open cone over a fibered face of the Thurston unit ball, then it must lie in the closed cone over the boundary of a fibered face by the assumption that it can be approximated by fibered elements. Since the Thurston unit ball is rational, the intersection of the closed cone containing \( \phi_{\text{reg}} \) with \( V \) must contain non-fibered rational points arbitrarily close to \( \phi_{\text{reg}} \), which gives a contradiction. Thus \( \phi_{\text{reg}} \) admits a non-degenerate de Rham representative \( \eta_{\text{reg}} \). Since \( \eta_{\text{reg}} \) can be approximated by rational classes that are fibered and restrict themselves to meridional fibrations on the boundary of \( M_{\text{reg}} \), the foliation induced by \( \eta_{\text{reg}} \) on the boundary is also meridional.

We let \( (z, \theta) \in D^2 \times S^1 \) denote coordinates on a tubular neighbourhood of a component of the branching locus of \( |M| \). After applying a suitable isotopy we may assume that \( \eta_{\text{reg}} \) has the form \( f(\theta)d\theta \) near \( \partial D^2 \times S^1 \). It follows that \( \eta_{\text{reg}} \) extends to a non-degenerate closed form \( \eta \) which is transverse to the branching locus of \( |M| \). The pullback \( \overline{\eta} = \pi^*\eta \) then gives the desired non-degenerate \( G \)-equivariant representative of \( \overline{\phi} \).

\[ \square \]

**Proof of Theorem** \[ \Box \]. The implication \((2) \implies (1)\) is trivial.

\((1) \implies (3)\): Let \((X, \omega)\) be a symplectic manifold with a fixed point free \( S^1 \)-action and quotient space \( M \). By \([3]\), Proposition 1.8, there is a manifold \( \overline{M} \) and a smooth action by a finite group so that \( M = \overline{M}/G \). Furthermore, we have the following commutative diagram:

\[
\begin{array}{ccc}
\pi^*X & \xrightarrow{\overline{p}} & \overline{M} \\
\pi \downarrow & & \downarrow \pi \\
X & \xrightarrow{p} & M,
\end{array}
\]

where \( \pi \) is the quotient map, \( \overline{\pi} \) is an unramified covering and the induced \( S^1 \)-action on \( \overline{X} \) is free. Moreover, the group \( G \) acts naturally on \( \overline{X} \) as the group of deck transformations of \( \overline{\pi} \).

Thus \( \overline{\omega} = \overline{\pi^*} \omega \) is a symplectic form and by \([9]\), Theorem 1.4, its image under the pushforward map \( \overline{\pi}^*(\overline{\omega}) \in H^1(\overline{M}, \mathbb{R}) \) lies in the open cone over a fibered face of the Thurston norm ball. Since \( \overline{\omega} \) is \( G \)-invariant and the action on \( \overline{X} \) is fiber preserving, the class \( \overline{\phi} = \overline{\pi}^*(\overline{\omega}) \) is also \( G \)-invariant. We let \( \phi \) be the unique class such that \( \pi^*\phi = \overline{\phi} \). By Lemma \([\Box]\) the restriction of \( \phi_{\text{reg}} \) to \( M_{\text{reg}} \) lies in the open cone over a fibered face. Finally the naturality of the transfer homomorphism implies that the restriction of \( \phi_{\text{reg}} \) agrees with \( \overline{p}^*\omega_{\text{reg}} \).

\((3) \implies (2)\): By assumption \( \phi_{\text{reg}} = \overline{p}^*\psi_{\text{reg}} \) lies in the open cone over a fibered face of the Thurston norm ball and \( \phi_{\text{reg}} \) is the restriction of a class \( \phi \in H^1(|M|, \mathbb{R}) \). In particular, \( |M| \) fibers over \( S^1 \). We first note that \( M \) is a very good orbifold so that it is a quotient of a manifold \( \overline{M} \) by a smooth action of a finite group \( G \). For this it suffices to rule out bad \( 2 \)-suborbifolds by \([2]\), Corollary 3.28. However, a bad \( 2 \)-suborbifold is topologically a sphere that is essential in \( H_2(|M|, \mathbb{Z}) \) and as in the proof of \([3]\), Lemma 2.3, this implies that \( b^+_2(X) = b_2(|M|) - 1 \). Thus \( |M| = S^2 \times S^1 \) and \( b^+_2(X) = 0 \), contradicting the assumption that \( \psi^2 > 0 \).
Thus since $M$ is very good we can proceed as in the proof of the previous implication. In particular, $M$ is a quotient of a manifold $\overline{M}$ by a smooth action of a finite group $G$, the total space has a finite covering $X$ which is a genuine $S^1$-bundle and these bundles fit into a pullback diagram as above. Since a degree one cohomology class on $\overline{M}$ is determined by its restriction to the complement of the branching locus, we deduce that $\bar{\phi} = \pi_* \phi$ and $\bar{p}_*(\pi_* \psi)$ agree as cohomology classes. We then note that the construction of $S^1$-invariant forms in $\mathbb{R}$ and its extension to irrational classes ([8], Theorem 1.1) can be done $G$-equivariantly.

First choose a $G$-invariant representative $\gamma$ of $e(\bar{\Omega})$, which can be obtained as the curvature of a $G$-equivariant angular form. By [8], Lemma 2.1, we may write $\gamma = \bar{\phi} \wedge \beta$. After averaging over $G$ this equation still holds, so $\beta$ can be assumed to be $G$-equivariant. Let $\eta$ be a $G$-invariant angular 1-form so that $d\eta = \bar{p}^* \gamma$ and let $\bar{\Omega} \in H^2(\overline{M}, \mathbb{R})$ be the unique class such that the following holds in cohomology:

$$\pi^* \psi - \eta \wedge \bar{p}^* \bar{\phi} = \bar{p}^* \bar{\Omega}.$$ 

Such an $\bar{\Omega}$ exists in view of the Gysin sequence since the left hand lies in the kernel of $\bar{p}_*$, and since the left hand side is $G$-equivariant so is $\bar{\Omega}$. The fact that $\pi^* \psi^2 > 0$ implies that $\bar{p}^* \bar{\phi} \wedge \bar{\Omega} > 0$. Thus by [8], Lemma 2.2, there is a non-vanishing 2-form representing the class $\bar{\Omega}$ so that $\bar{\phi} \wedge \bar{\Omega} > 0$; again after averaging we may assume that $\bar{\Omega}$ is $G$-invariant. Thus the $S^1$-invariant form

$$\bar{\omega}_{inv} = \eta \wedge \bar{p}^* \bar{\phi} + \bar{p}^* \bar{\Omega}$$

represents $\pi^* \psi$ and descends to an $S^1$-invariant form $\omega_{inv}$ on $X$ which is cohomologous to $\psi$.

\[\square\]

**Remark 1.** A vital step in the proof of Theorem 1 was Chen’s observation that the base orbifold of a symplectic manifold $X$ admitting a fixed point free $S^1$-action is a quotient of a manifold by a finite group action. The main technical point in the proof of [3], Proposition 1.8, is to rule out bad 2-orbifolds in the base. This is achieved by results relating the Seiberg-Witten invariants of the base orbifold to those of the underlying manifold.

We sketch a different proof which uses more standard Seiberg-Witten vanishing results. For background on the Seiberg-Witten invariants we refer to [10] and the references therein. First observe that a bad 2-suborbifold $\Sigma$ in the quotient orbifold $M = X/S^1$ can intersect at most 2 singular curves $L_1, L_2$, each in at most one point. Taking a neighbourhood $N$ of $|\Sigma| \cup L_1$ gives a topological splitting of the base $|M| = (S^2 \times S^1) \# M'$ so that the preimage of the splitting sphere in $|M|$ induces a splitting $X = X_1 \cup_S X_2$, where $S$ is either $S^2 \times S^1$ or $S^3$ depending on whether $L_2$ is empty or not. Moreover, as in the proof of [3], Lemma 2.3, we must have $b_1(M') > 0$ by the assumption that $b_2^+(X) > 0$. If $S$ is a 3-sphere, then $b_2^+(X_2) > 0$, and by taking the covering $\overline{X}$ of $X$ induced by the natural surjection

$$\pi_1(X) \to \pi_1(S^2 \times S^1) \to \mathbb{Z}_m$$

we obtain a splitting of $X = X_1 \cup_{S^3} X_2$, where $b_2^+(X_1), b_2^+(X_2) \geq 1$. It follows that the Seiberg-Witten invariants of $\overline{X}$ are trivial.

If $S = S^2 \times S^1$, then we take the covering $\overline{X}$ of $X$ induced by a surjection

$$\pi_1(X) \to \pi_1(S^2 \times S^1) \ast \pi_1(|M'|) \to \mathbb{Z}_m \times \mathbb{Z}_n.$$
The embedded 2-sphere $S^2 \times \{pt\}$ in $S$ then becomes essential in the covering and $b_1(X)$, and hence $b_1^+ (X)$, may be assumed to be arbitrarily large. Furthermore, the sphere $S^2 \times \{pt\}$ has trivial self-intersection and consequently the Seiberg-Witten invariants of $X$ are trivial. Thus in both cases we obtain a contradiction to the non-vanishing results of Taubes for the Seiberg-Witten invariants of a symplectic 4-manifold.

Acknowledgments

The author thanks P. SuÁrez-Serrato for his stimulating questions and S. Friedl for helpful comments. The hospitality of the Max-Planck-Institut fÁr Mathematik in Bonn, where this research was carried out, is also gratefully acknowledged.

References

[1] Scott Baldridge, Seiberg-Witten vanishing theorem for $S^1$-manifolds with fixed points, Pacific J. Math. 217 (2004), no. 1, 1–10, DOI 10.2140/pjm.2004.217.1. MR2105762 (2005j:57040)
[2] Michel Boileau, Sylvain Maillot, and Joan Porti, Three-dimensional orbifolds and their geometric structures (English, with English and French summaries), Panoramas et SynthÁeses [Panoramas and Syntheses], vol. 15, SociÁtÁe MathÁematique de France, Paris, 2003. MR2060653 (2005b:57030)
[3] W. Chen, Seifert fibered four-manifolds with nonzero Seiberg-Witten invariant, Preprint: arXiv:1103.5861v3, 2011.
[4] Allan L. Edmonds and Charles Livingston, Group actions on fibered three-manifolds, Comment. Math. Helv. 58 (1983), no. 4, 529–542, DOI 10.1007/BF02564651. MR728451 (85k:57037)
[5] Marisa FernÁndez, Alfred Gray, and John W. Morgan, Compact symplectic manifolds with free circle actions, and Massey products, Michigan Math. J. 38 (1991), no. 2, 271–283, DOI 10.1307/mmj/1029004333. MR1098863 (92c:57040)
[6] Ronald Fintushel, Classification of circle actions on 4-manifolds, Trans. Amer. Math. Soc. 242 (1978), 377–390, DOI 10.2307/1997745. MR496815 (81e:57036)
[7] S. Friedl and S. Vidussi, Twisted Alexander polynomials detect fibered 3-manifolds, Ann. Math. (2) 173 (2011), no. 3, 1587–1643, DOI 10.4007/annals.2011.173.3.8. MR2800721 (2012f:57040)
[8] Stefano Vidussi and Stefano Vidussi, Twisted Alexander polynomials and fibered 3-manifolds, Low-dimensional and symplectic topology, Proc. Sympos. Pure Math., vol. 82, Amer. Math. Soc., Providence, RI, 2011, pp. 111–130. MR2768657 (2012f:57040)
[9] S. Friedl and S. Vidussi, A vanishing theorem for twisted Alexander polynomials with applications to symplectic 4-manifolds, J. Eur. Math. Soc. (JEMS) 15 (2013), no. 6, 2127–2041. MR3120739
[10] Robert E. Gompf and AndrÁÁas I. Stipsicz, 4-manifolds and Kirby calculus, Graduate Studies in Mathematics, vol. 20, American Mathematical Society, Providence, RI, 1999. MR1707327 (2000h:57038)
[11] William H. Meeks III and Peter Scott, Finite group actions on 3-manifolds, Invent. Math. 86 (1986), no. 2, 287–346, DOI 10.1007/BF01389073. MR856847 (88b:57039)
[12] William P. Thurston, A norm for the homology of 3-manifolds, Mem. Amer. Math. Soc. 59 (1986), no. 339, i–vi and 99–130. MR823443 (88k:57014)
[13] D. Wise, The structure of groups with a quasiconvex hierarchy, 187 pages, Preprint: available on the webpage for the NSF-CBMS conference.