BIFURCATION ANALYSIS OF A STAGE-STRUCTURED PREDATOR-PREY MODEL WITH PREY REFUGE

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Abstract. A stage-structured predator-prey model with prey refuge is considered. Using the geometric stability switch criteria, we establish stability of the positive equilibrium. Stability and direction of periodic solutions arising from Hopf bifurcations are obtained by using the normal form theory and center manifold argument. Numerical simulations confirm the above theoretical results.

1. Introduction. Since it takes time for species to mature, delay differential equations are widely adopted to model the corresponding stage structure. For example, a single species model with a stage structure was studied in [1, 2], while a predator-prey model with a stage-structure was considered in [19]. For persistence, stability, and bifurcations of stage-structured predator-prey models, we refer the readers to [3, 11, 14, 17, 18, 23] and their references therein.

As a result of evolution, the refuge phenomenon enriches the interaction between predators and prey. To better understand the influence of refuge on the predator-prey interaction, some mathematical models with prey refuge have been proposed and studied, see [6, 8, 7, 9, 10, 13, 15, 16, 21, 22]. In general, two types of refuge have been considered: (i) the number of prey using refuge is fixed, and (ii) a constant proportion of prey uses refuge [9].

In this paper, we assume that predators feed only on mature prey population, and we consider the following model

\[
\begin{align*}
\frac{dx_i(t)}{dt} &= \alpha x_m(t) - r_1 x_i(t) - \alpha e^{-r_1 \tau} x_m(t - \tau), \\
\frac{dx_m(t)}{dt} &= \alpha e^{-r_1 \tau} x_m(t - \tau) - \beta x^2_m(t) - r_2 x_m(t) - \frac{(x_m(t) - \xi) y(t)}{x_m(t) - \xi + y(t)}, \\
\frac{dy(t)}{dt} &= y(t)(-d + \frac{(x_m(t) - \xi) y(t)}{x_m(t) - \xi + y(t)}), \\
&= \begin{cases} \\
x_i(0) > 0, x_m(0) > \xi, y(0) > 0, x_m(t) = \phi_m(t) \geq 0, -\tau \leq t < 0, \\
\end{cases}
\end{align*}
\]

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where \( x_i(t) \) and \( x_m(t) \) represent the immature and mature prey population densities at time \( t \), respectively, and \( y(t) \) denotes the density of predator population at time \( t \).

In model (1), \( \alpha > 0 \) and \( r_1 > 0 \) represent the birth and death rates of the immature population, respectively; the death rate of the mature population is of a logistic nature, i.e., it is proportional to square of the population with a proportionality constant \( \beta > 0 \); \( r_2 > 0 \) is the harvesting rate; \( \tau \) is the time to maturity; the term \( \alpha e^{-r_1 \tau} x_m(t - \tau) \) represents the immature prey individuals who were born at time \( t - \tau \) and are alive at time \( t \); \( k > 0 \) is the rate of conversion of nutrients from the prey to the reproduction of the predator; \( d > 0 \) is the death rate of the predator population; \( \xi > 0 \) denotes the fixed quantity of mature prey protected by refuges. For continuity of initial conditions of (1), we assume that:

\[
x_i(0) = \alpha \int_{-\tau}^{0} \phi_m(s)e^{r_1 s}ds.
\]

According to [20], since the coefficients of our model are delay-dependent, the coefficients of the corresponding characteristic equation are also delay-dependent. Thus the local stability analysis becomes very complicated. In such case, the geometric stability switch criteria and center manifold argument developed in [5] can be applied to analyze the existence of Hopf bifurcation and the direction of periodic solutions branching from Hopf bifurcations.

We organized the rest of this paper as follows. In Section 2, we apply the geometric stability switch criteria to analyze the corresponding characteristic equation to derive the local stability of equilibria and establish the existence of Hopf bifurcation. In Section 3, by using the normal form theory and the center manifold theorem, we study the stability and direction of periodic solutions branching from Hopf bifurcations. In Section 4, we give some numerical simulations to support the analytic results. In the last section, we provide a conclusion to followed by some discussions.

### 2. Stability of equilibria.

Since the variable \( x_i \) does not appear in the second and the third equations of (1), we only need to consider the following system

\[
\begin{align*}
\frac{dx_m(t)}{dt} &= \alpha e^{-r_1 \tau} x_m(t - \tau) - \beta x_m^2 - r_2 x_m - \frac{(x_m - \xi)y}{x_m + \xi + y}, \\
\frac{dy(t)}{dt} &= y \left(-d + \frac{k (x_m - \xi)}{x_m + \xi + y}\right),
\end{align*}
\]

with the initial conditions

\[
x_m(t) = \phi_m(t) \geq 0, \quad -\tau \leq t < 0 \text{ and } x_m(0) > \xi, \quad y(0) > 0.
\]

Similar to Theorems 3.1 and 4.1 of [9], we can show that the solutions of (3) with (4) are positive and bounded for all \( t \geq 0 \).

For system (3), there is always the trivial equilibrium \( E_0 = (0, 0) \). There exists a boundary equilibrium \( E_1 = \left( \frac{\alpha e^{-r_1 \tau} - r_2}{\beta}, 0 \right) \), provided that \( \alpha > r_2 \) and \( \tau \in [0, \tilde{\tau}) \), where \( \tilde{\tau} = \frac{1}{r_1} (\ln \alpha - \ln r_2) \). Moreover, if the positive equilibrium \( E_2 = (x_m^*, y^*) \) exists, then we must have

\[
\begin{align*}
\beta x_m^* \frac{k}{d} - \frac{\alpha e^{-r_1 \tau}}{d} x_m^* - \xi \frac{k - d}{k} &= 0, \\
y^* &= \frac{(k - d)(x_m^* - \xi)}{d_m - \xi}.
\end{align*}
\]
If \( d < k < 1 \), then by the Descartes rule of change of sign, there exists the only positive root

\[
x^*_m = \frac{\delta + \sigma}{2\beta},
\]

where \( \delta = \alpha e^{-r_1 \tau} - r_2 - \frac{k-d}{d} \), \( \sigma = \sqrt{\delta^2 + \frac{4\xi \beta (k-d)}{k}} \). In addition, the positivity of \( y^* \) can be obtained if \( x^*_m > \xi \), i.e., \( \tau < \frac{1}{\delta}(\ln \alpha - \ln (\beta \xi + r_2)) \). In conclusion, the unique positive equilibrium \( E_2 \) exists only if

\[
(H1): d < k < 1 \text{ and } \tau \in [0, \bar{\tau}), \text{ where } \bar{\tau} = \frac{1}{r_1} \left( \ln \alpha - \ln (\beta \xi + r_2) \right)
\]

holds.

**Remark 1.** In the discussion above, \( \bar{\tau} < \bar{\tau} \). Under condition (H1), \( E_0 \) and \( E_1 \) are unstable. Details about the stability of \( E_0 \) and \( E_1 \) follow from [9, 19].

**Remark 2.** Since \( \delta = \delta(\tau) \) depends on \( \tau \), the positive root \( x^*_m \) is a function of \( \tau \). For convenience, denote \( x^*_m \) by \( x^*_m(\tau) \), \( \tau \in [0, \bar{\tau}) \). Then

\[
\frac{dx^*_m(\tau)}{d\tau} = -\frac{r_1 \alpha e^{-r_1 \tau}}{2\beta(1 + \frac{\delta(\tau)}{\sqrt{\delta^2(\tau) + \frac{4\xi \beta (k-d)}{k}}})} < 0.
\]

In other words, since \( \tau \in [0, \bar{\tau}) \), \( x^*_m(\tau) \) is decreasing in \( \tau \). Moreover, \( \xi < x^*_m(\tau) \leq \frac{\delta(0) + \sqrt{\delta(0)^2 + \frac{4\xi \beta (k-d)}{2\beta}}}{2\beta} := x^*_m(0), \tau \in [0, \bar{\tau}) \), where \( \delta(0) = \alpha - r_2 - \frac{k-d}{d} \).

In the following, we focus on the stability of the unique positive equilibrium by applying the geometric stability switch criteria developed in [5].

Linearizing system (3) at \((x^*_m, y^*)\) gives the characteristic equation

\[
G(\lambda, \tau) = P(\lambda, \tau) + Q(\lambda, \tau) e^{-\lambda \tau} = 0,
\]

where

\[
P(\lambda, \tau) = \lambda^2 + p(\tau) \lambda + q(\tau), \quad Q(\lambda, \tau) = -(s(\tau) \lambda + \gamma(\tau))
\]

and

\[
p(\tau) = 2\beta x^*_m + \frac{(k-d)^2}{k^2} + \frac{d(k-d)}{k} + r_2, \quad s(\tau) = \alpha e^{-r_1 \tau},
\]

\[
q(\tau) = \frac{d(k-d)}{k}(2\beta x^*_m + r_2 + \frac{k-d}{k}), \quad \gamma(\tau) = \frac{d(k-d)}{k} \alpha e^{-r_1 \tau}.
\]

Following the setting in [5], we have

\[
p(\tau) - s(\tau) = \sigma(\tau) - \frac{d(1-k)(k-d)}{k^2}, \quad (7)
\]

and

\[
q(\tau) - \gamma(\tau) = \frac{d(k-d)\sigma(\tau)}{k}.
\]

**Lemma 2.1.** Suppose that (H1) holds. Then \( p(\tau) - s(\tau) > 0 \) for any \( \tau \in [0, \bar{\tau}) \).

**Proof.** If \( \tau = \bar{\tau} \), then it follows from (5) and (7) that

\[
p(\bar{\tau}) - s(\bar{\tau}) = 2\beta \xi + \frac{(k-d)^2}{k^2} + \frac{d(k-d)}{k} + r_2 - r_2 - \beta \xi
\]

\[
= \beta \xi + \frac{(k-d)^2}{k^2} + \frac{d(k-d)}{k} > 0.
\]
It is easy to show that the function $\sigma(\tau)$ is decreasing in $\tau$ for $\tau \in [0, \bar{\tau})$. This completes the proof.

Now, we verify that $P(\lambda, \tau)$ and $Q(\lambda, \tau)$ satisfy the condition of the geometric stability switch criteria [5].

**Lemma 2.2.** Suppose that (H1) holds. Then

(I) $P(0, \tau) + Q(0, \tau) \neq 0$;

(II) $P(wi, \tau) + Q(wi, \tau) \neq 0$, $\forall w \in \mathbb{R}$, $\forall \tau \in \mathbb{R}_+$;

(III) $\lim_{|\lambda| \to \infty} \sup \{ \frac{|Q(\lambda, \tau)|}{|P(\lambda, \tau)|} : \text{Re} \lambda \geq 0 \} < 1$;

(IV) $F(w, \tau) = |P(wi, \tau)|^2 - |Q(wi, \tau)|^2$ for each $\tau$ has at most a finite number of real zeros;

(V) Each positive root $w(\tau)$ of $F(w, \tau) = 0$ is continuous and differentiable in $\tau$ whenever it exists.

**Proof.** Based on the above argument, we have

$$P(0, \tau) + Q(0, \tau) = q(\tau) - \gamma(\tau) > 0;$$

then (I) holds.

According to Lemma 2.1, for any $\tau \in [0, \bar{\tau})$, it follows that

$$P(wi, \tau) + Q(wi, \tau) = (wi)^2 + p(\tau)wi + q(\tau)wi - \gamma(\tau)$$

$$= q(\tau) - \gamma(\tau) - w^2 + (p(\tau) - s(\tau))wi$$

$$\neq 0,$$

and (II) is satisfied.

Since $P(\lambda, \tau)$ is a second-degree polynomial about $\lambda$ and $Q(\lambda, \tau)$ is linear in $\lambda$, we have $\lim_{|\lambda| \to \infty} \sup \{ \frac{|Q(\lambda, \tau)|}{|P(\lambda, \tau)|} : \text{Re} \lambda \geq 0 \} < 1$, and (III) holds.

Calculating $F(w, \tau) = |P(wi, \tau)|^2 - |Q(wi, \tau)|^2 = w^4 + A(\tau)w^2 + B(\tau)$, where $A(\tau) = p^2(\tau) - 2q(\tau) - \gamma^2(\tau)$, $B(\tau) = q^2(\tau) - s^2(\tau)$, we obtain (IV). $F(w, \tau)$ is continuous in $w$ and $\tau$, and differentiable in $w$, so the implicit function theorem leads to (V).

Next we discuss the stability of the positive equilibrium $E_2$ of system (3). Firstly, when $\tau = 0$, by the Routh-Hurwitz criterion, we obtain the following theorem.

**Theorem 2.3.** Suppose that (H1) holds. Then the equilibrium $E_2$ of system (3) is asymptotically stable when $\tau = 0$.

Secondly, assume that $G(wi, \tau) = 0$ for some $\tau \in (0, \bar{\tau})$ and $w > 0$, we get

$$\left\{ \begin{array}{l}
    \gamma(\tau) \cos w\tau + ws(\tau) \sin w\tau = q(\tau) - w^2, \\
    ws(\tau) \cos w\tau - \gamma(\tau) \sin w\tau = p(\tau)w.
\end{array} \right.$$

Hence, from $|Q(wi, \tau)|^2 \neq 0$, it follows that

$$\left\{ \begin{array}{l}
    \sin w\tau = \frac{(q(\tau) - w^2)ws(\tau) - p(\tau)w\gamma(\tau)}{w^2s^2(\tau) + \gamma^2(\tau)}, \\
    \cos w\tau = \frac{p(\tau)w^2s(\tau) + (q(\tau) - w^2)\gamma(\tau)}{w^2s^2(\tau) + \gamma^2(\tau)}.
\end{array} \right.$$

Let

$$F(w, \tau) = w^4 + A(\tau)w^2 + B(\tau) = 0,$$  

(9)
where \( A(\tau) = p^2(\tau) - 2q(\tau) - r^2(\tau) \) and \( B(\tau) = q(\tau)^2 - s(\tau)^2 \). The polynomial function \( F \) can be written as \( F(w, \tau) = h(w^2, \tau) \), where \( h \) is a second-degree polynomial defined by

\[
h(z, \tau) = z^2 + A(\tau)z + B(\tau).
\]  

By assumption (H1), we have \( 0 < k - d < \frac{k}{d} \), which implies that \( A(\tau) > 0 \). Therefore, Eq. (10) has positive real roots if and only if \( B(\tau) < 0 \). Due to

\[
\alpha e^{-r_1 \tau} = \frac{k\beta x_m^* - \xi(k - d)}{k x_m^*} + r_2 + \frac{k - d}{k},
\]

then inequality \( B(\tau) < 0 \) is equivalent to

\[
\beta(2(k - d) - \frac{k}{d})x_m^2 + (r_2 + \frac{k - d}{k})(k - d - \frac{k}{d})x_m + \frac{\xi(k - d)}{d} < 0.
\]

For simplicity of presentation, we write

\[
f(x) = \beta(2(k - d) - \frac{k}{d})x^2 + (r_2 + \frac{k - d}{k})(k - d - \frac{k}{d})x + \frac{\xi(k - d)}{d},
\]  

where \( x \in [\xi, x_m(0)] \). In view of \( 0 < k - d < \frac{k}{d} \) and \( k < 1 \), we have \( 0 < k - d < \frac{k}{2d} \), which implies that \( (r_2 + \frac{k-d}{k})(k - d - \frac{k}{d}) < 0 \) and \( \beta(2(k - d) - \frac{k}{d}) < 0 \). Thus if \( f(\xi) \leq 0 \), then \( f(x) < 0 \). If \( f(\xi) > 0 \), according to (11), there exists a constant \( \xi_0 > 0 \) such that \( \xi < \xi_0 \), where

\[
\xi_0 = \frac{d(k - d)^2 - (k^2 - dk(k - d))r_2}{(k^2 - 2dk(k - d))\beta}.
\]  

Similarly, there exists a constant \( \xi_1 > 0 \) such that for \( \xi = \xi_1 \), \( f(x_m(0)) = 0 \), where

\[
\xi_1 = \frac{k^2}{2\beta d(k - d)^2}((\frac{k}{2d(k - d)} - 1)\alpha + r_2 + \frac{k - d}{k}).
\]

Eventually, we have \( f(x_m(0)) \geq 0 \) if \( \xi \geq \xi_1 \), and \( f(x_m(0)) < 0 \) if \( \xi < \xi_1 \).

Since the inequality \( \xi < x_m^* \leq x_m(0) \) is equivalent to \( 0 \leq \tau < \bar{\tau} \), based on the above analysis, we have the following results.

**Lemma 2.4.** Suppose that (H1) holds and \( \xi_0, \xi_1 \) are defined by (12) and (13), respectively.

(I) When \( \xi \geq \xi_0 \), Eq. (9) has positive real roots for all \( 0 \leq \tau < \bar{\tau} \).

(II) When \( \xi < \xi_0 \),

(i) if \( \xi_1 \leq \xi < \xi_0 \), Eq. (9) has no positive real root for any \( 0 \leq \tau < \bar{\tau} \).

(ii) if \( \xi < \min\{\xi_0, \xi_1\} \), Eq. (9) has no positive real root for any \( \tau \in [\tau_0, \bar{\tau}] \). If \( \tau \in [0, \tau_0) \), Eq. (9) has only one positive real root \( w \), where \( \tau_0 \) satisfies \( f(x_m(\tau_0)) = 0 \) and \( w := w(\tau) = \sqrt{-A(\tau) + \sqrt{A^2(\tau) - 4B(\tau)}} \).

Specifically, for \( \xi < \min\{\xi_0, \xi_1\} \) and \( \tau \in [0, \tau_0) \subset [0, \bar{\tau}] \), let \( \theta(\tau) \in (0, 2\pi) \),

\[
\begin{aligned}
\sin \theta(\tau) &= \frac{q(\tau) - w^2(\tau)u(\tau)s(\tau) + p(\tau)w(\tau)\gamma(\tau)}{w^2(\tau)u(\tau)s(\tau) + p(\tau)w(\tau)\gamma(\tau)}, \\
\cos \theta(\tau) &= \frac{p(\tau)w^2(\tau)s(\tau) + q(\tau)w(\tau)\gamma(\tau)}{w^2(\tau)u(\tau)s(\tau) + p(\tau)w(\tau)\gamma(\tau)}.
\end{aligned}
\]

Let

\[
S_n(\tau) = \tau - \frac{\delta(\tau)n2\pi}{w(\tau)}, \quad n \in \mathbb{N}.
\]

We notice that \( iw(\tau^*) \) is a purely imaginary root of (9) if and only if \( \tau^* \) is the root of \( S_n(\tau) = 0 \).
To obtain our main result, we also need to analyze the stability of the equilibrium $E_2$ when $\tau = \tau_0$. In this case, the characteristic equation (9) becomes

$$w^4 + A(\tau_0)w^2 + B(\tau_0) = 0.$$  \hspace{1cm} (16)

It is not difficult to verify that $B(\tau_0) = 0$ and $A(\tau_0) = \frac{p^2(\tau_0) - 2q(\tau_0) - r^2(\tau_0)}{\tau_0}$ is unstable for all $\tau \in [0, \bar{\tau})$.

For $\tau \in [\tau_0, \bar{\tau})$, we have

$$\sign \left\{ \frac{d\Re \lambda}{dt} \bigg|_{\lambda = iw(\tau^*)} \right\} = \sign \left\{ \frac{\partial F}{\partial w}(w(\tau^*), \tau^*) \right\} \times \sign \left\{ \frac{dS_n(\tau)}{dt} \bigg|_{\tau = \tau^*} \right\},$$

and since $\frac{\partial F}{\partial w}(w, \tau) > 0$, it is clear that

$$\sign \left\{ \frac{d\Re \lambda}{dt} \bigg|_{\lambda = iw(\tau^*)} \right\} = \sign \left\{ \frac{dS_n(\tau)}{dt} \bigg|_{\tau = \tau^*} \right\}.$$  

Theorem 2.5. Suppose that (H1) holds and $\xi_0$, $\xi_1$ are defined by (12) and (13), respectively.

(I) For $\xi < \min \{\xi_0, \xi_1\}$,

(i) if $S_0(\tau)$ has no positive zeros in $[0, \tau_0)$, equilibrium point $E_2$ of system (3) is asymptotically stable for all $\tau \in [0, \bar{\tau})$.

(ii) if $S_0(\tau)$ has positive zeros in $[0, \tau_0)$, $E_2$ is asymptotically stable for all $\tau \in [0, \tau^*)$. $E_2$ becomes unstable for $\tau$ staying in some right neighborhood of $\tau^*$, where $\tau^* = \min \{\tau : \tau \in [0, \tau_0), S_0(\tau) = 0\}$. And hence system (3) undergoes Hopf bifurcation when $\tau = \tau^*$.

(iii) if $\tau \in [\tau_0, \bar{\tau})$, $E_2$ is asymptotically stable.

(II) For $\xi \geq \xi_0$, $E_2$ is unstable for $\tau \in [0, \bar{\tau})$.

(III) For $\xi_1 \leq \xi < \xi_0$, $E_2$ is asymptotically stable for all $\tau \in [0, \bar{\tau})$.

Proof. It is easy to see that $S_n(\tau) > S_{n+1}(\tau)$ for all $\tau \in [0, \tau_0)$ and $n \in \mathbb{N}$. Moreover, by (15) and (16), we have $S_n(0) < 0$ and $S_n(\tau) \to -\infty$ as $\tau \to \tau_0$. Hence, if the function $S_0(\tau)$ has no positive zeros in $[0, \tau_0)$, $S_n(\tau)$ have no positive zeros. This implies that the characteristic equation (3) has no purely imaginary roots.

3. Direction and stability of Hopf bifurcation. In this section, we obtain the conditions under which a family of periodic solutions bifurcate from the positive equilibrium point $E_2$ at the critical value of $\tau^*$. Following the ideas of Hassard et al.[12], we derive the explicit formulae which can determine the properties of the Hopf bifurcation at critical value of $\tau^*$ by using the normal form and the center manifold theory [4, 20]. Throughout this section, we always assume that system (3) undergoes a Hopf bifurcation at the positive equilibrium $E_2$ at $\tau = \tau^*$. Let $w^*i = w(\tau^*)i$ be the corresponding purely imaginary root of the characteristic equation at the positive equilibrium.

We firstly let $x_1(t) = x_m(t) - x_m^*$, $y_1(t) = y(t) - y^*$, $\tau = \tau^* + \mu$. Then $\mu = 0$ is a Hopf bifurcation value of (3). Rescale the time by $t \mapsto (t/\tau)$ to normalize the delay and denote $x_1(t\tau), y_1(t\tau)$ by $x_m(t), y(t)$. Thus, system (3) is transformed into an FDE in $C([-1, 0], \mathbb{R}^2)$ as

$$\dot{X}(t) = L_\mu X_t + f(\mu, X_t),$$  \hspace{1cm} (17)
where \( X(t) = (x_m(t), y(t))^T \in \mathbb{R}^2 \) and \( L_\mu : \mathbb{C} \to \mathbb{R}^2, \ f : \mathbb{R}^2 \times \mathbb{C} \to \mathbb{R}^2 \) are given respectively by

\[
L_\mu X_t = (\tau^* + \mu) \left[ B(\tau^* + \mu) \begin{pmatrix} x_m(t) \\ y(t) \end{pmatrix} + C(\tau^* + \mu) \begin{pmatrix} x_m(-1) \\ y(-1) \end{pmatrix} \right].
\]

Here

\[
C(\tau^* + \mu) = \begin{pmatrix} \alpha e^{-\rho_1(\tau^* + \mu)} & 0 \\ 0 & 0 \end{pmatrix},
\]

\[
B(\tau^* + \mu) = \begin{pmatrix} -r_2 - 2\beta x^*_m - \frac{y^*_2}{(x^*_m - \xi + y^*_2)^2} & \frac{(z^*_m - \xi)^2}{(x^*_m - \xi + y^*_2)^2} \\ k y^2 & \frac{k(x^*_m - \xi)^2}{(x^*_m - \xi + y^*_2)^2} - d \end{pmatrix},
\]

Moreover, \( f(\mu, X_t) = (\tau^* + \mu) \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \), where

\[
f_1 = f_{11} x^2_m(0) + f_{12} x_m(0) y(0) + f_{13} x^3_m(0) + f_{14} x^2_m(0) y(0) + O(|x_m|^4),
\]

\[
f_2 = f_{21} x^2_m(0) + f_{22} x_m(0) y(0) + f_{23} x^3_m(0) + f_{24} x^2_m(0) y(0) + O(|x_m|^4),
\]

and

\[
f_{11} = \frac{y^2}{(x_m - \xi + y^2)^2} - \beta; f_{12} = \frac{y - y^2}{(x_m - \xi + y^2)^2}; f_{13} = \frac{y^2}{(x_m - \xi + y^2)^2}; f_{14} = \frac{k y^2}{(x_m - \xi + y^2)^2};
\]

\[
f_{21} = \frac{k y^2}{(x_m - \xi + y^2)^2}; f_{22} = \frac{k y^2}{(x_m - \xi + y^2)^2}; f_{23} = \frac{k y^2}{(x_m - \xi + y^2)^2}; f_{24} = \frac{k y^2}{(x_m - \xi + y^2)^2}.
\]

Turning to the linear problem \( \dot{X}(t) = L_\mu X_t \), by the Riesz representation theorem, there exists a bounded variation matrix valued function \( \eta(\theta, \mu) : \mathbb{C} \to \mathbb{R}^2 \), such that

\[
L_\mu \phi = \int_{-1}^0 [d\eta(\theta, \mu)] \phi(\theta), \text{ for } \phi \in \mathbb{C}.
\]

In fact, we choose

\[
\eta(\theta, \mu) = (\tau^* + \mu) [B(\tau^* + \mu) \phi(\theta) - C(\tau^* + \mu) \phi(\theta + 1)],
\]

where \( \phi(\theta) \) is Dirac delta function and satisfies

\[
\left\{ \begin{array}{l}
\phi(0) = 1, \\
\phi(\theta) = 0, \quad \theta \neq 0.
\end{array} \right.
\]

For \( \phi \in \mathcal{C}^1([-1, 0], \mathbb{R}^2) \), the infinitesimal generator \( A(\mu) \) is defined by

\[
A(\mu) \phi = \left\{ \begin{array}{l}
\frac{d\phi(\theta)}{d\theta}, \quad \theta \in [-1, 0), \\
\int_{-1}^0 [d\eta(\theta, \mu)] \phi(\theta), \quad \theta = 0.
\end{array} \right.
\]

Further, let

\[
R(\mu) \phi = \left\{ \begin{array}{l}
0, \quad \theta \in [-1, 0), \\
f(\mu, \phi), \quad \theta = 0.
\end{array} \right.
\]

Then, system (17) is equivalent to

\[
\dot{X}_t = A(\mu) X_t + R(\mu) X_t,
\]

where \( X_t(\theta) = X(t + \theta) \) for \( \theta \in [-1, 0] \).

If the adjoint operator

\[
A^*(\mu) \psi(\theta) = \left\{ \begin{array}{l}
-\frac{d\psi(\theta)}{d\theta}, \quad \theta \in (0, 1], \\
\int_{-1}^0 \psi(-\theta) d\eta(\theta, \mu), \quad \theta = 0,
\end{array} \right.
\]

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and a bilinear form
\[ \langle \psi, \phi \rangle = \mathcal{B}_q(0\phi(0) - \int_{-1}^{0} \int_{0}^{\theta} \mathcal{B}_q(\zeta - \theta) d\eta(\theta)\phi(\zeta) d\zeta, \] (25)
where \( \eta(\theta) = \eta(\theta, 0) \), then \( A(0) \) and \( A^*(0) \) are adjoint operations and satisfy
\[ \langle \psi, A\phi \rangle = \langle A^*\psi, \phi \rangle. \]

If we let \( \mu = 0, \pm w^*\tau_i \) become common eigenvalues of \( A(0) \) and \( A^*(0) \). Suppose \( q_1(\theta) \) and \( q_1^*(s) \) are the eigenvectors of \( A(0) \) and \( A^*(0) \) corresponding to \( w^*\tau_i \) and \(-w^*\tau_i \), respectively. Then, we have \( A(0)q_1(\theta) = w^*\tau_iq_1(\theta) \). Based on (21), when \( \theta \in [-1, 0) \),
\[ A(0)q_1(\theta) = \frac{dq_1(\theta)}{d\theta} = iw^*\tau_iq_1(\theta). \] (26)
Therefore, we can let \( q_1(\theta) = (\varepsilon_1, \varepsilon_2) e^{iw^*\tau_i}\theta \). When \( \theta = 0 \), we have
\[ A(0)q_1(\theta) = \int_{-1}^{0} d\eta(\theta, \mu)\phi(\theta) = iw^*\tau_i \left( \begin{array}{c} \varepsilon_1 \\ \varepsilon_2 \end{array} \right). \] (27)
It follows from (26) and (27) that \( \varepsilon_1 = 1 \) and \( \varepsilon_2 = \frac{(iw^* + r_2 + 2\beta)(x_m^* - \xi + y^* + r_1 \tau^*)^2 + y^*}{(x_m^* - \xi)^2} \).

Similarly, based on (24), when \( s \in [-1, 0) \),
\[ A^*(0)q_1^*(s) = \frac{dq_1^*(s)}{ds} = -iw^*\tau_iq_1^*(s). \] (28)
Therefore, we can let \( q^*(s) = D(1, \beta_3) e^{iw^*\tau_i}s \). When \( s = 0 \), we have
\[ A^*(0)q^*(s) = \int_{-1}^{0} d\eta(0, s)q^*(-s) = D \left( \begin{array}{c} -iw^*\tau_i \\ -iw^*\beta_3 \end{array} \right)^T. \]
Comparing the above result with (28),
\[ \beta_3 = \frac{(-iw^* + r_2 + 2\beta x_m^* - \alpha e^{-r_1 \tau^* + iw^* \tau^*})(x_m^* - \xi + y^*)^2 + y^*}{ky^*}. \]
Calculating
\[ \langle q^*(s), q_1(\theta) \rangle = \bar{q}^*(0)q_1(0) - \int_{-1}^{0} \int_{0}^{\theta} \bar{q}^*(s - \theta)d\eta(\theta)q_1(s) ds \\
= D(1 + \varepsilon_2\beta_3) + D\alpha e^{-r_1 \tau^* + iw^* \tau^*}. \]
Since \( \langle q^*(s), q_1(\theta) \rangle = 1 \),
\[ \bar{D} = \frac{1}{1 + \varepsilon_2\beta_3 + \alpha e^{-r_1 \tau^* + iw^* \tau^*}}. \]
Moreover, \( \langle q^*(s), \bar{q}_1(\theta) \rangle = 0 \).

By using the idea of Hassard [12], we firstly compute the coordinates to describe the center manifold \( C_0 \) at \( \mu = 0 \), which is a local invariant attracting a two-dimensional manifold in \( C_0 \). Let \( X_t \) be the solution of (23) when \( \mu = 0 \), and define
\[ z(t) = \langle q^*, X_t \rangle. \] (29)
Then, we have \( W(t, \theta) = W(z, \bar{z}, \theta) \), where
\[ W(z, \bar{z}, \theta) = X_t(\theta) - 2Re\{z(t)q_1(\theta)\}. \] (30)
Then on the center manifold \( C_0 \), we have
\[ W(z, \bar{z}, \theta) = W_{20}(\theta)z^2 + W_{11}(\theta)z\bar{z} + W_{02}(\theta)\bar{z}^2 + \cdots, \] (31)
where $z$ and $\bar{z}$ are local coordinates for center manifold $C_0$ in the directions of $q_1$ and $\bar{q}^*$, respectively. Note that, $W$ is real if $X_t$ is real. It is easy to see that

$$z(t) = (q^*, X_t) = iw^*\tau^*z + g(z, \bar{z}),$$

(32)

where

$$g(z, \bar{z}) = g_{02}(\frac{z^2}{2} + g_{11}z\bar{z} + \frac{z^2\bar{z}}{2} + g_{21}\frac{z^2\bar{z}}{2} + \cdots).$$

(33)

Now, we determine the coefficients $W_{ij}$ in (31). It follows from (30) and (32) that

$$\begin{align*}
W & = X_t - zq_1 - z\bar{q}_1 \\
& = A(0)W + R(0)X_t - 2Re(gq_1) \\
& = \begin{cases}
A(0)W - 2Re(\overline{q^*}(0)f_0q_1(\theta)), & \theta \in [-1, 0), \\
A(0)W - 2Re(\overline{q^*}(0)f_0q_1(\theta)) + f_0, & \theta = 0,
\end{cases} \\
& := A(0)W + H(z, \bar{z}, \theta),
\end{align*}$$

(34)

and

$$H(z, \bar{z}, \theta) = H_{20}(\theta)\frac{z^2}{2} + H_{11}(\theta)z\bar{z} + H_{02}(\theta)\frac{\bar{z}^2}{2} + \cdots.$$

(35)

On the other hand, on $C_0$, we have

$$\begin{align*}
W = W_{20}(\theta)z + W_{11}(\theta)\bar{z} + W_{02}(\theta)(iW^*\tau^*z + g) + (W_{11}(\theta)z + W_{02}(\theta)\bar{z} + \cdots)(-iW^*\tau^*\bar{z} + \bar{g}).
\end{align*}$$

From (31), (32), (34), and (35), it follows that

$$A(0)W + H(z, \bar{z}, \theta) = iw^*\tau^*W_{20}(\theta)z^2 + 0z\bar{z} - iw^*\tau^*W_{02}(\theta)\bar{z}^2 + \cdots.$$
\[ M_{11} = f_{11} + f_{12}\varepsilon_2, \ M_{12} = 2f_{11} + f_{12}(\varepsilon_2 + \bar{\rho}_3), \ M_{13} = f_{11} + f_{12}\bar{\rho}_3, \]
\[ M_{14} = f_{11}\left(\frac{W_{20}^{(2)}(0)}{2} + W_{11}^{(1)}(0)\right) + f_{12}(W_{11}^{(2)}(0) + W_{11}^{(1)}(0)\varepsilon_2 + \frac{W_{20}^{(2)}(0)\bar{\rho}_3}{2}) \]
\[ - 3f_{13} + f_{14}(2\varepsilon_2 + \bar{\rho}_3), \]
\[ M_{21} = f_{21} + f_{22}\varepsilon_2, \ M_{22} = 2f_{21} + f_{22}(\varepsilon_2 + \bar{\rho}_3), \ M_{23} = f_{21} + f_{22}\bar{\rho}_3, \]
\[ M_{24} = f_{21}\left(\frac{W_{20}^{(2)}(0)}{2} + W_{11}^{(1)}(0)\right) + f_{22}(W_{11}^{(2)}(0) + W_{11}^{(1)}(0)\varepsilon_2 + \frac{W_{20}^{(2)}(0)\bar{\rho}_3}{2}) \]
\[ - 3f_{23} + f_{24}(2\varepsilon_2 + \bar{\rho}_3). \]

Hence, we have
\[ g(z, \bar{z}) = \tau^* D \{(M_{11} + M_{21}\bar{\rho}_3)z^2 + (M_{12} + M_{22}\bar{\rho}_3)z\bar{z} + (M_{13} + M_{23}\bar{\rho}_3)\bar{z}^2 \]
\[ + (M_{14} + M_{24}\bar{\rho}_3)z^2\bar{z} + \cdots \}. \tag{38} \]

Comparing the coefficients of the above equation with those in (33),
\[ g_{20} = 2\tau^* D(M_{11} + M_{21}\bar{\rho}_3), \ g_{11} = 2\tau^* D(M_{12} + M_{22}\bar{\rho}_3), \]
\[ g_{02} = 2\tau^* D(M_{13} + M_{23}\bar{\rho}_3), \ g_{21} = 2\tau^* D(M_{14} + M_{24}\bar{\rho}_3). \tag{39} \]

To get the expression of \( g_{21} \), we need to compute \( W_{20}(\theta) \) and \( W_{11}(\theta) \). From (33) and (34), for \( \theta \in [-1, 0] \), it follows that
\[ H(z, \bar{z}, \theta) = -(g_{20} \frac{z^2}{2} + g_{11}z\bar{z} + g_{02}\frac{\bar{z}^2}{2} + g_{21}\frac{z^2\bar{z}}{2} + \cdots)q_1 \]
\[ + (g_{20} \frac{\bar{z}^2}{2} + g_{11}z\bar{z} + g_{02}\frac{z^2}{2} + g_{21}\frac{z^2\bar{z}}{2} + \cdots)\bar{q}_1. \tag{40} \]

Comparing (35) with (40), we have
\[
\begin{align*}
H_{20} &= g_{20}q_1(\theta) - g_{02}\bar{q}_1(\theta), \\
H_{11} &= g_{11}q_1(\theta) - g_{11}\bar{q}_1(\theta).
\end{align*} \tag{41} \]

If \( \theta = 0 \),
\[ H(z, \bar{z}, \theta) = -2Re\{\bar{q}^*(0)f_0q_1(\theta)\} + f_0(z, \bar{z}) \]
\[ = \left[-g_{20}q_1(0) - g_{02}\bar{q}_1(0) + 2 \begin{pmatrix} M_{11} \\ M_{21} \end{pmatrix} \right]z^2 \]
\[ + \left[-g_{11}q_1(0) - g_{11}\bar{q}_1(0) + 2 \begin{pmatrix} M_{12} \\ M_{22} \end{pmatrix} \right]z\bar{z} + \cdots \tag{42} \]

From (36) and the definition of \( A(0) \), we have
\[ W_{20}(\theta) = 2i\omega^* \tau^* W_{02}(\theta) - H_{20}(\theta) \]
\[ = 2i\omega^* \tau^* W_{02}(\theta) + g_{20}q_1(0)e^{i\omega^* \tau^* \theta} + g_{02}\bar{q}_1(0)e^{-i\omega^* \tau^* \theta}. \tag{43} \]

Hence, solving the Bernoulli differential equation, we obtain
\[ W_{20}(\theta) = e^{\int 2i\omega^* \tau^* d\theta} U_1 + \int [g_{20}q_1(\theta) + g_{02}\bar{q}_1(\theta)]e^{-\int 2i\omega^* \tau^* d\theta} \]
\[ = \frac{i\bar{g}_{20}q_1(0)}{\omega^* \tau^*}e^{i\omega^* \tau^* \theta} + \frac{i\bar{g}_{02}\bar{q}_1(0)}{3\omega^* \tau^*}e^{-i\omega^* \tau^* \theta} + U_1e^{2i\omega^* \tau^* \theta}. \tag{44} \]

By a similar method, we obtain
\[ W_{11}(\theta) = -\frac{i\bar{g}_{11}q_1(0)}{\omega^* \tau^*}e^{i\omega^* \tau^* \theta} + \frac{i\bar{g}_{11}\bar{q}_1(0)}{\omega^* \tau^*}e^{-i\omega^* \tau^* \theta} + U_2, \]
where $U_1$ and $U_2$ are both two-dimensional vectors. It follows from (36) that

$$A(0)W_{20}(\theta) = 2iw^*\tau^*W_{20}(0) - H_{20}(0)$$

$$= 2iw^*\tau^*W_{20}(0) + g_{20}q_1(0) + g_{02}q_1(0) - 2\tau^*\left(\begin{array}{c} M_{11} \\ M_{21} \end{array}\right).$$

Notice that, from the definition of $A(0)$,

$$A(0)W_{20}(\theta) = \int_{-1}^{0} d\eta(0,\theta)W_{20}(\theta)$$

$$= \frac{ig_{20}}{w^*\tau^*} \int_{-1}^{0} e^{iw^*\tau^*\theta} d\eta(0,\theta)q_1(0) + \frac{ig_{02}}{3w^*\tau^*} \int_{-1}^{0} e^{-iw^*\tau^*\theta} d\eta(0,\theta)\bar{q}_1(0)$$

$$+ \int_{-1}^{0} e^{2iw^*\tau^*\theta} d\eta(0,\theta)U_1,$$

and

$$(iw^*\tau^*I - \int_{-1}^{0} e^{iw^*\tau^*\theta} d\eta(\theta))q_1(0) = iw^*\tau^*q_1(0) - A(0)q_1(0) = 0,$$

$$(-iw^*\tau^*I - \int_{-1}^{0} e^{-iw^*\tau^*\theta} d\eta(\theta))\bar{q}_1(0) = 0.$$  

Hence, from (44)-(48), it follows that

$$2iw^*\tau^*I - \int_{-1}^{0} e^{2iw^*\tau^*\theta} d\eta(\theta))U_1 = 2\tau^*\left(\begin{array}{c} M_{11} \\ M_{21} \end{array}\right).$$

Similarly, we have

$$\int_{-1}^{0} d\eta(\theta))U_2 = -\tau^*\left(\begin{array}{c} M_{12} \\ M_{22} \end{array}\right).$$

Then, we obtain

$$U_1 = 2(2iw^*I - B(\tau^*) - C(\tau^*)e^{-2iw^*\tau^*\theta})^{-1}\left(\begin{array}{c} M_{11} \\ M_{21} \end{array}\right),$$

and

$$U_2 = -(B(\tau^*) + C(\tau^*))^{-1}\left(\begin{array}{c} M_{11} \\ M_{21} \end{array}\right).$$

So we can calculate $g_{21}$ and derive the following values

$$c_1(0) = \frac{i}{2w^*\tau^*} \left(g_{11}g_{20} - 2|g_{11}|^2 - \frac{|g_{02}|^2}{3}\right) + \frac{g_{21}}{2}, \quad \nu_2 = -\frac{Re(c_1(0))}{Re(X'(\tau^*))},$$

$$\beta_2 = 2Re(c_1(0)), \quad T_2 = -\frac{Im(c_1(0)) + \mu_2Im(X'(\tau^*))}{w^*\tau^*}.$$
(III) If $T_2 > 0$ ($T_2 < 0$), then the period of the bifurcating periodic solutions increases (decreases).

4. **Numerical simulations.** In this section, we present numerical simulations of system (3) to illustrate our theoretical results.

![Graph](image_url)

**Figure 1.** For $\tau \in [0, \bar{\tau})$, the graph of $S_0(\tau)$ and $S_1(\tau)$.

We choose the set of parameters as $\alpha = 1.0$, $r_1 = 0.1$, $\beta = 0.5$, $r_2 = 0.05$, $\xi = 0.01$, $k = 0.2$ and $d = 0.1$. Then system (3) is reduced to

![Graph](image_url)

**Figure 2.** For $\tau = 0.6 < \tau^*$ and the initial value "1.0, 1.0", the positive equilibrium point of system (49) is stable.
\[
\begin{align*}
\frac{dx_m(t)}{dt} &= e^{-0.1 \tau} x_m(t - \tau) - 0.5 x_m^2 - 0.05 x_m - \frac{(x_m - 0.01)y}{x_m - 0.01 + y}, \\
\frac{dy(t)}{dt} &= y \left( -0.1 + \frac{0.2(x_m - 0.01)y}{x_m - 0.01 + y} \right).
\end{align*}
\] (49)

Substituting these parameters into (12) and (13), we have \(\xi_0 = 0.013\) and \(\xi_1 = 19.01\). Obviously, \(\xi < \min\{\xi_0, \xi_1\}\). Furthermore, \(\tau_0 = 35.9316\) and \(\bar{\tau} = 37.2970\).

Drawing the graph of \(S_0(\tau)\) and \(S_1(\tau)\) versus \(\tau\) on \([0, \bar{\tau})\) in Fig.1, one can see that \(S_n(\tau)\) has no positive zeros for \(n \geq 1\), and there exist two critical values of the delay \(\tau\), denoted by \(\tau^*\) and \(\tau^{**}\) with \(\tau^* = 1.7308\), \(\tau^{**} = 35.01\).

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure3.png}
\caption{For \(\tau = 22 > \tau^*\) and the initial value \([0.81, 0.61]\), system (49) exhibits a periodic solution.}
\end{figure}

A direct computation implies that the equilibrium is asymptotically stable for \(\tau \in [0, \tau^*) \cup (\tau^{**}, \tau_0)\) and unstable for \(\tau \in (\tau^*, \tau^{**})\). If \(\tau = \tau^*\) or \(\tau = \tau^{**}\), all the roots of the corresponding characteristic equation of system (49) have negative real parts except a pair of purely imaginary roots. When \(\tau = \tau^* = 1.7308\), Hopf bifurcation occurs. Furthermore, we can obtain the value of \(\text{Re}(c_1(0))\) by using (39). Therefore, the Hopf bifurcation of the system (49) at the positive equilibrium point is supercritical and the bifurcating periodic solutions are orbitally asymptotically stable.

5. **Conclusions.** In this work, a stage-structured predator-prey model with prey refuge has been considered. We have assumed that a fixed number of prey are protected by refuges to escape the predation. The existence and stability of the positive equilibrium \(E_2\) have been derived under some conditions. It was found that Hopf bifurcation may appear when \(\xi\) is small. If \(\xi\) is large enough, the positive...
equilibrium is unstable, while the predator population becomes smaller and can even become extinct. Using the approach of Beretta and Kuang [5], we have shown that the positive equilibrium can be destabilized through a Hopf bifurcation. Moreover, we have investigated the stability and direction of periodic solutions branching from Hopf bifurcations by using the normal form theory and the center manifold theorem. Our results show that the prey refuges play an important role in determining the dynamics of the system.

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