SDYM equations on the self-dual background

L V Bogdanov

L.D. Landau ITP RAS, Moscow, Russia

E-mail: leonid@itp.ac.ru

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Abstract
We introduce the technique combining the features of integration schemes for self-dual Yang–Mills (SDYM) equations and multidimensional dispersionless integrable equations to get SDYM equations on the conformally self-dual background. Generating differential form is defined, and the dressing scheme is developed. Some special cases and reductions are considered.

Keywords: self-dual Yang–Mills (SDYM) equations, dispersionless integrable equations, self-dual conformal structure, integrable background geometry

1. Introduction
Self-dual Yang–Mills (SDYM) (ASDYM) equations

\[ F = \pm *F \]  

represent SD (ASD) condition for the two-form \( F = dA + A \wedge A \), where the gauge field (potential) \( A \) is a one-form taking its values in some Lie algebra. The most well-known results concerning the integrability of SDYM equations are formulated in four-dimensional Euclidean space or its compactification (sometimes complexification). However, it is a well-established fact in twistor theory that SDYM equations are integrable (in terms of twistor construction) on general nontrivial geometrical background defined by self-dual conformal structure on some 4-manifold, this structure itself being integrable [1, 2]. Different reductions of SDYM equations give rise to background geometries which are themselves solutions of (dispersionless) integrable systems, the current picture of the field and many examples are provided in [3]. Thus there is an approach to consider dispersionless integrable systems as integrable background geometries for some reductions of SDYM equations. We will take the opposite direction and, starting from dispersionless integrable hierarchies, will consider an extension leading in particular to SDYM equations on the self-dual background. In the process of extension we will transfer all integrable structures of dispersionless integrable systems—Lax pairs, the hierarchy, Lax–Sato equations, the dressing scheme [4, 5]—to the case of SDYM
equations on the background. Though some of these structures have their analogues in twistor approach, there are differences in technique and setting of the problems, and comparison of these two approaches should be mutually enriching.

2. ASD conformal structures

Our starting point will be a recent result [6]

**Theorem 1 (Dunajski, Ferapontov and Kruglikov (2014)).** There exist local coordinates \((z, w, x, y)\) such that any ASD conformal structure in signature \((2, 2)\) is locally represented by a metric

\[
\frac{1}{2} g = dw dx - dz dy - F_y dz^2 - (F_x - G_y) dw dz + G_x dz^2,
\]

where the functions \(F, G : M^4 \rightarrow \mathbb{R}\) satisfy a coupled system of third-order PDEs,

\[
\begin{align*}
\partial_x (Q(F)) + \partial_y (Q(G)) &= 0, \\
(\partial_w + F_y \partial_x + G_y \partial_y) Q(G) + (\partial_z + F_x \partial_x + G_x \partial_y) Q(F) &= 0,
\end{align*}
\]

where

\[Q = \partial_w \partial_x - \partial_z \partial_y + F_y \partial_x^2 - G_x \partial_y^2 - (F_x - G_y) \partial_x \partial_y.\]

A conformal structure \([g]\) is called anti-self-dual if the self-dual part of the Weyl tensor of any \(g \in [g]\) vanishes: \(W_+ = \frac{1}{4} (W + *W) = 0\). Real case with the signature \((2, 2)\) or, generally, complex analytic case may be considered. The choice of ASD or SD case is just a convention depending on orientation, and we will follow [6] in taking ASD case (though we changed the order of variables and the sign of one variable for technical reasons).

A crucial observation made in [6] is that system (3) arises as \([X_1, X_2] = 0\) from the dispersionless Lax pair

\[
\begin{align*}
X_1 &= \partial_x - \lambda \partial_t + F_x \partial_x + G_x \partial_y + f_1 \partial \lambda, \\
X_2 &= \partial_w - \lambda \partial_x + F_y \partial_x + G_y \partial_y + f_2 \partial \lambda.
\end{align*}
\]

Due to compatibility conditions, \(f_1\) and \(f_2\) can be expressed through \(F\) and \(G\),

\[
\begin{align*}
f_1 &= -Q(G), \quad f_2 = Q(F), \\
Q &= \partial_x \partial_t - \partial_t \partial_x + F_x \partial_x^2 - G_x \partial_y^2 - (F_x - G_y) \partial_x \partial_y.
\end{align*}
\]

Theorem 1 defines an important relation between ASD conformal structures and dispersionless integrable systems. The Lax pair (4) corresponds to lowest order flows of generic four-dimensional dispersionless integrable hierarchy [4, 5], we will briefly describe main structures of this hierarchy below.

On the other hand, reductions of the Lax pair (4) correspond to important geometrical systems. A reduction to divergence-free vector fields gives the Dunajski system [7] describing null Kähler case, and a further reduction to linearly degenerate case \((f_1, f_2 = 0, \text{no derivative over } \lambda \text{ in vector fields})\) leads to the famous Plebański second heavenly equation (Einstein ASD case).
3. Extended Lax pair

Let us consider an extension of Lax pair (4) to covariant derivatives,
\[
\nabla_X^1 = \partial_z - \lambda \partial_x + F_x \partial_x + G_x \partial_y + f_1 \partial_\lambda + A_1, \\
\nabla_X^2 = \partial_w - \lambda \partial_y + F_y \partial_x + G_y \partial_y + f_2 \partial_\lambda + A_2,
\]
where gauge field components \(A_1, A_2\) do not depend on \(\lambda\) and take their values in some (matrix) Lie algebra. Lax pairs of this structure (without derivative over \(\lambda\)) were already present in the seminal work of Zakharov and Shabat [8] (1979), where it was noticed that the commutation relation splits into (scalar) vector field part, which is the same as for unextended Lax pair, and Lie algebraic part,
\[
[X^1, X^2] = [X^1, X^2] + [X^1, A^2] - [X^2, A^1] + [A^1, A^2] = 0.
\]
From the first part we get system (3) describing ASD conformal structure, and the second part gives the system for \(A_1, A_2\)
\[
\nabla_X^1 = \partial_z - \lambda \partial_x + A_1, \\
\nabla_X^2 = \partial_w - \lambda \partial_y + A_2,
\]
and the commutativity condition is
\[
\partial_\lambda A_2 = \partial_y A_1, \\
\partial_z A_2 - \partial_x A_1 + [A_1, A_2] = 0,
\]
representing a well known form of ASDYM equations (1) for constant metric \(g\) (7) in a special gauge (where two components of the gauge field are eliminated by the gauge transform).

The following statement demonstrates that general background in equations (6) has a direct geometric sense.

**Theorem 2.** Equations (6) represent ASDYM equations (1) for the background conformal structure (2) (in a special gauge).

**Proof.** First we notice that for metric (2) due to ASDYM (1) equations we have \(F_{34} = 0\), where we have used the matrix inverse to metric \(g\) defining symmetric bivector
\[
\frac{1}{2} g = \partial_\alpha \cdot \partial_\beta - \partial_\alpha \cdot \partial_\gamma + F_\gamma \partial_\alpha \partial_\beta + (G_\gamma - F_\alpha) \partial_\gamma \partial_\beta - G_\alpha \partial_\gamma \partial_\beta,
\]
\(\det g = \det q = 1\) (for this metric \(F_{12} = F_{34}\)). Then it is possible to choose a gauge such that \(A_3 = A_4 = 0\), and we have only two nontrivial gauge field components \(A_1, A_2\).

The next step is to introduce a tetrad of vector fields and dual tetrad of one-forms for which the conformal structure (2) takes especially simple form, see [6].

In terms of tetrad of one-forms...
conformal structure (2) is represented as
\[ g = 2(e_{00'} e_{11'} - e_{01'} e_{10'}) , \]
where, following [6], we use spinor notations for indices.

The dual tetrad of vector fields is
\[
\begin{align*}
e_{00'} &= \partial_w + F_y \partial_x + G_y \partial_y, \\
e_{01'} &= \partial_z + F_x \partial_x + G_x \partial_y, \\
e_{10'} &= \partial_z, \\
e_{11'} &= \partial_x,
\end{align*}
\]
and symmetric bivector (9) reads
\[ q = 2(e_{00'} e_{11'} - e_{01'} e_{10'}) . \]

ASDYM equations for this tetrad take the form
\[
\begin{align*}
F_{11'} 01' &= 0, \\
F_{00'} 10' &= 0, \\
F_{00'} 11' &= F_{10'} 01',
\end{align*}
\]
where, to calculate gauge field strength \( F \) components in the tetrad basis, we use a standard formula
\[ F(u, v) = \nabla_u \nabla_v - \nabla_v \nabla_u - \nabla_{[u,v]} \]
valid for arbitrary vector fields \( u, v \). Taking into account the structure of tetrade and the fact that for our gauge \( A_3 = A_4 = 0 \), for the curvature components we get
\[
\begin{align*}
F_{11'} 01' &= 0, \\
F_{00'} 10' &= (\partial_w + F_y \partial_x + G_y \partial_y) A_1 - (\partial_z + F_x \partial_x + G_x \partial_y) A_2 - [A_1, A_2], \\
F_{00'} 11' &= -\partial_z A_2, \\
F_{10'} 01' &= -\partial_z A_1
\end{align*}
\]
Thus ASDYM equations read
\[
\begin{align*}
(\partial_w + F_y \partial_x + G_y \partial_y) A_1 - (\partial_z + F_x \partial_x + G_x \partial_y) A_2 - [A_1, A_2] &= 0, \\
\partial_z A_2 &= \partial_z A_1,
\end{align*}
\]
that coincides with the Lie algebraic part of commutativity condition for extended Lax pair (6).

Theorem 1, theorem 2 imply the following statement:

**Corollary.** *Commutation relations \([\nabla_{X_1}, \nabla_{X_2}] = 0\) for the Lax pair*
\[
\begin{align*}
\nabla_{X_1} &= \partial_z - \lambda \partial_x + F_x \partial_x + G_x \partial_y + f_1 \partial \lambda + A_1, \\
\nabla_{X_2} &= \partial_w - \lambda \partial_x + F_x \partial_x + G_x \partial_y + f_2 \partial \lambda + A_2,
\end{align*}
\]
represent a general local form (up to coordinate and gauge transformations) of ASDYM equations (1) for ASD conformal structure (real case, signature \((2,2)\)).

Vector fields part of the commutation relation defines equations (3) for the coefficients of the metric (2) representing ASD conformal structure, and Lie-algebraic part of commutation relations (6) gives ASDYM equations for this structure (in a special gauge).

The gauge freedom can be recovered by arbitrary gauge transformation, and it is possible to write the Lax pair in gauge invariant form,
\begin{align}
\n\nabla_X t &= \partial_t + F_t \partial_t + G_t \partial_t + A_t - \lambda (\partial_t + B_t) + f_t \partial_t,
\n\nabla_X y &= \partial_y + F_y \partial_y + G_y \partial_y + A_y - \lambda (\partial_y + B_y) + f_y \partial_y
\end{align}

(12)

Vector field part of commutation relation remains the same and gives equations (3) for conformal ASD structure. Lie-algebraic part of commutation relations gives the equations

\begin{align}
\partial_t B_t - \partial_y B_y + [B_1, B_2] &= 0,
\n\partial_t A_t - \partial_y A_y + [A_1, A_2] &= \partial_y A_t - (\partial_y + F_y \partial_y + G_y \partial_y) A_y - [B_2, A_1],
\n(\partial_t + F_t \partial_t + G_t \partial_t) A_2 - (\partial_y + F_y \partial_y + G_y \partial_y) A_1
\n&+ [A_1, A_2] + Q(F) B_1 + Q(G) B_2 = 0,
\end{align}

(13)

representing gauge-invariant form of ASDYM equation (1) for ASD conformal structure (2).

Here matrices $A_1, A_2, B_1, B_2$ are defined as components of the gauge field in tetrad basis (10), (11) and can be easily expressed through coordinate components. Lax pair (12) may be considered as direct covariant extension of the vector fields pair (4) by changing partial derivatives $\partial_t, \partial_y, \partial_x, \partial_y$ to covariant derivatives $\nabla_t, \nabla_y, \nabla_x, \nabla_y$.

4. Matrix dressing on the geometric background

The dressing scheme for ASDYM equations based on matrix Riemann–Hilbert (RH) problem (see, e.g. [9]) can be extended to the case of nontrivial background. Generally, we may consider matrix RH problem

$$\Phi^+ = \Phi^- R(\psi_1, \psi_2, \psi_3),$$

(14)
defined on some oriented curve $\gamma$ in the complex plane, or matrix \(\bar{\partial}\) problem

$$\bar{\partial} \Phi = \Phi R(\psi_1, \psi_2, \psi_3),$$

(15)
defined in some region $G$, where $\psi_i(\lambda, t)$ are arbitrary wave functions of dispersionless Lax pair

$$X_1 \psi_t = (\partial_t - \lambda \partial_t + F_t \partial_t + G_t \partial_t + f_t \partial_t) \psi_t = 0,$$

$$X_2 \psi_y = (\partial_y - \lambda \partial_y + F_y \partial_y + G_y \partial_y + f_y \partial_y) \psi_y = 0,$$

defined on $\gamma$ or in $G$. Let us suggest the existence of solution $\Phi$ of RH (or \(\bar{\partial}\)) problem having no zeroes and normalized by $1$ at infinity, $\Phi^\infty = 1 + \sum_{n=1}^{\infty} \Phi_\psi(t) \lambda^{-n}$. Then $X_1 \Phi, X_2 \Phi$ satisfy the same problem $[X_1, R] = [X_2, R] = 0$, and the functions $X_1 \Phi \Phi^{-1}, (X_2 \Phi) \Phi^{-1}$ are holomorphic in the complex plane. Considering the behaviour at infinity, we get

$$X_1 \Phi \Phi^{-1} = -\partial_t \Phi_1(t),$$

$$X_2 \Phi \Phi^{-1} = -\partial_y \Phi_1(t),$$

thus $\Phi$ is a solution for the extended Lax pair (5) with the gauge field

$$A_1 = \partial_t \Phi_1(t), A_2 = \partial_y \Phi_1(t),$$

which satisfies equations (6).

Dropping the normalization condition at infinity and demanding only regularity, we will get solution for gauge-invariant extended Lax pair (12) and equations (13).
For constant metric \( g \) (7) corresponding to trivial vector fields we have wave functions 
\( \psi_1 = \lambda, \psi_2 = x + \lambda z, \psi_3 = y + \lambda w \), and RH problem (14) reduces to standard Riemann–Hilbert problem for ASDYM equations (8) (see, e.g. [9]).

Important classes of reductions of equations (3) are connected with the existence of wave functions for dispersionless Lax pair (4) with special analytic properties in \( \lambda \), e.g. polynomial wave functions 
\( \psi = P_n(\lambda) \), coefficients of the polynomial depends on times. A class of special ASDYM solutions for these background geometries is defined by the problems

\[ \Phi_+ = \Phi_- R(P^n) \quad \text{or} \quad \partial \Phi = \Phi R(P^n). \]

Another important reduction is linearly-degenerate case, for which there is no \( \partial \lambda \) in dispersionless Lax pair and \( \lambda \) is one of the wave functions,

\[ \Phi_+ = \Phi_- R(\lambda, \psi_1, \psi_2) \quad \text{or} \quad \partial \Phi = \Phi R(\lambda, \bar{\lambda}, \psi_1, \psi_2). \]

In this case ASDYM Lax pair admits rational (in \( \lambda \)) solutions with simple stationary poles (correspond to \( \delta \)-functions in the \( \partial \) problem), which can be calculated explicitly.

### 5. From the dressing scheme to the hierarchy

To introduce the hierarchy connected with extended Lax pair (5) and equations (6), we will start from extended dressing scheme and obtain generating relations for the hierarchy and Lax–Sato equations. It is also possible to consider these generating relations independently in terms of formal series, and the dressing scheme as a tool to construct solutions.

First we briefly outline the dressing scheme for multidimensional dispersionless hierarchy connected with the Lax pair (4) [4, 5]. We consider nonlinear vector Riemann–Hilbert problem on the unit circle,

\[ \begin{array}{l}
\Psi_0^0 = F_0(\Psi_0^0, \Psi_1^0, \Psi_2^0) \\
\Psi_1^0 = F_1(\Psi_0^0, \Psi_1^0, \Psi_2^0) \\
\Psi_2^0 = F_2(\Psi_0^0, \Psi_1^0, \Psi_2^0)
\end{array} \]

outside the unit circle the solutions are analytic and given by expansions of the form

\[ \begin{align*}
\Psi_0^0 &= \lambda + \sum_{n=1}^{\infty} \Psi_0^0(t^1, t^2)\lambda^{-n}, \\
\Psi_1^0 &= \sum_{n=0}^{\infty} \Psi_1^0(t^1, t^2)\lambda^{-n} \\
\Psi_2^0 &= \sum_{n=1}^{\infty} \Psi_2^0(t^1, t^2)\lambda^{-n}.
\end{align*} \]

inside the unit circle the functions are analytic. Solutions to RH problem (16) \( \Psi_0, \Psi_1, \Psi_2 \) will give wave functions for the hierarchy of commuting vector fields, defined through coefficients of expansion of these functions, \( t^1 = (t_1^1, t_2^1, \ldots), t^2 = (t_1^2, t_2^2, \ldots) \) are two infinite sets of independent variables of the hierarchy. To obtain a gauge field extension of the hierarchy, we introduce also a matrix Riemann–Hilbert problem

\[ \Phi_{in} = \Phi_{out} R(\Psi_{out}^0, \Psi_{out}^1, \Psi_{out}^2). \]

\( \Phi \) is normalized by 1 at infinity and analytic inside and outside the unit circle.
Φ_{out} = 1 + \sum_{n=1}^{\infty} \Phi_n(t^1, t^2)\lambda^{-n}

Expansions of \Psi, \Phi give coefficients for extended Lax pair, \Phi is a wave function. A general wave function is given by the expression \Phi F(\Psi^0, \Psi^1, \Psi^2), F is an arbitrary complex-analytic matrix function.

The vector fields part of the dressing scheme implies analyticity in the complex plane of the form (no discontinuity on the unit circle)

\omega = \left| \frac{D(\Psi^0, \Psi^1, \Psi^2)}{D(\lambda, x_1, x_2)} \right|^{-1} d\Psi^0 \wedge d\Psi^1 \wedge d\Psi^2,

where \( x_1 = t^0_1, x_2 = t^0_2 \) are lowest times of the hierarchy, and from matrix Riemann problem we get analyticity of the matrix-valued form

\Omega = \omega \wedge d\Phi \cdot \Phi^{-1}.

Analyticity of these forms imply the relations

\begin{align*}
(\omega_{out})_+ &= \left( \left| \frac{D(\Psi^0_{out}, \Psi^1_{out}, \Psi^2_{out})}{D(\lambda, x_1, x_2)} \right|^{-1} d\Psi^0_{out} \wedge d\Psi^1_{out} \wedge d\Psi^2_{out} \right)_+ = 0, \\
(\Omega_{out})_+ &= (\omega_{out} \wedge d\Phi_{out} \cdot \Phi^{-1}_{out})_+ = 0
\end{align*}

for the series \( \Psi^0_{out}, \Psi^1_{out}, \Psi^2_{out}, \Phi_{out} \). These relations are generating relations for the hierarchy in terms of formal series, they are equivalent to the complete set of Lax–Sato equations of the hierarchy. Though we used the dressing scheme to introduce these relations, they may be considered independently.

First relation gives Lax–Sato equations for the hierarchy of commuting polynomial in \lambda vector fields (here we drop subscript ‘out’ for the series):

\begin{align*}
\partial_k^n \Psi &= \sum_{i=0}^{2} \left( \left| \frac{D(\Psi^0, \Psi^1, \Psi^2)}{D(\lambda, x_1, x_2)} \right|^{-1} (\Psi^0)^i \right)_k \partial_i \Psi, \\
&= (V^k_n(\lambda)) \Psi, \\
\partial_k^n \Phi &= \left( V^k_n(\lambda) - (V^k_n(\lambda) \cdot \Phi^{-1})_+ \right) \Phi,
\end{align*}

where 1 \leq n < \infty, k = 1, 2, \partial_0 = \partial_\lambda, \partial_1 = \partial_{x_1}, \partial_2 = \partial_{x_2}, \Psi = (\Psi^0, \Psi^1, \Psi^2).

The second generating relation gives Lax–Sato equations for \Phi on the vector field background in terms of extended polynomial vector fields,

\begin{align*}
\partial_k^n \Psi &= \sum_{i=0}^{2} \left( \left| \frac{D(\Psi^0, \Psi^1, \Psi^2)}{D(\lambda, x_1, x_2)} \right|^{-1} (\Psi^0)^i \right)_k \partial_i \Psi, \\
\partial_k^n \Phi &= \left( V^k_n(\lambda) - (V^k_n(\lambda) \cdot \Phi^{-1})_+ \right) \Phi,
\end{align*}

where vector fields \( V^k_n(\lambda) \) are defined by formula (17). First flows give exactly extended Lax pair for ASDYM equations on ASD background (5), if we identify \( z = t^1_1, w = t^2_1, x = x_1, y = x_2. \)

5.1. Discussion, open questions

Though ASD conformal structure in Plebański–Robinson form (2), (9) can be considered in general complex case, it is not convenient to use it to obtain a real slice with Euclidean signature. Motivated by the Kähler case, we suggest to consider conformal structure defined by the symmetric bivector
\[
\frac{1}{2} q = a \partial_w \cdot \partial_w + b \partial_z \cdot \partial_w + c \partial_w \cdot \partial_z + d \partial_z \cdot \partial_z, \tag{18}
\]

and corresponding extended Lax pair
\[
\begin{align*}
\nabla X_1 &= \partial \lambda - \lambda (a \partial_w + b \partial_z) + (\lambda^2 f_1 + \lambda g_1) \partial \lambda + A_1 - \lambda B_1, \\
\nabla X_2 &= \partial \lambda + \lambda (c \partial_w + d \partial_z) + (\lambda^2 f_2 + \lambda g_2) \partial \lambda + A_2 + \lambda B_2, \tag{19}
\end{align*}
\]

Vector fields part of commutation relations gives seven equations for eight functions because of conformal freedom. To fix representative of conformal structure and close the system of equations, it is convenient to use the condition \( \det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = 1 \), in this case three independent coefficients of bivector (18) satisfy three second-order equations, and the conformal structure depends on 6 arbitrary functions of three variables. Another choice to fix conformal freedom is to put \( g_1 = g_2 = 0 \), in the scalar-flat Kähler case vector fields Lax pair of this type was considered in [10, 11].

The conjecture is that conformal structure (18) with the coefficients satisfying vector fields part of commutation relations for the Lax pair (19) gives a general local form of complex ASD conformal structure, and Lie algebraic part of commutation relations gives ASDYM equations on this background. The first part of the conjecture is connected with general recent results of the work [14]. ASD conformal structure of the form (18) can be useful for reduction to Hermitian case and real case with Euclidean signature.

Another interesting question concerns covariant extension of dispersionless integrable hierarchies in lower as well as in higher dimensions and the geometric meaning of arising systems. An important \((2 + 1)\)-dimensional example is provided by the Manakov–Santini system. Let us consider extended Lax pair
\[
\nabla X_1 = \partial_t - (\lambda - v_x) \partial_x + u_x \partial \lambda + A, \\
\nabla X_2 = \partial_t - (\lambda^2 - v_x \lambda + u - v_x) \partial_x + (u_x \lambda + u_x) \partial \lambda + \lambda A + B, \tag{20}
\]

where \( A, B \) are gauge field components. Vector field part of commutation relations gives the Manakov–Santini system \([12, 13]\)
\[
\begin{align*}
0 &= u_{xt} = u_{yy} + (uu_x)_x + v_y u_{xy} - u_{xx} v_y, \\
0 &= v_{xt} = v_{yy} + v_x u_{xx} + v_y v_{xy} - v_{xx} v_y, \tag{20}
\end{align*}
\]

describing general Einstein–Weyl geometry \([6]\), and matrix part of compatibility conditions read
\[
\begin{align*}
A_y - B_x &= 0, \\
(\partial_y + v_x \partial_x) B - (\partial_t + (v_y - u) \partial_x) A + u_t A + [A, B] &= 0
\end{align*}
\]

For the potential \( \Phi, A = \Phi_t, B = \Phi_y \), we have
\[
\begin{align*}
0 &= \Phi_{xt} - \Phi_{yy} - [\Phi_x, \Phi_y] - \partial_t (u \Phi_x) + v_y \Phi_{xx} - v_y \Phi_{xy}, \tag{21}
\end{align*}
\]

where \( u, v \) satisfy Manakov–Santini system describing Einstein–Weyl geometry.

The natural conjecture is that system (21) represents a general local form of monopole equations on Einstein–Weyl background (up to coordinate transformations and a gauge). We are planning to consider extended Manakov–Santini hierarchy in more detail in the nearest future.
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