RANGE OF BEREZIN TRANSFORM

N. V. RAO

Abstract. Let \( dA = \frac{dx dy}{\pi} \) denote the normalized Lebesgue area measure on the unit disk \( \mathbb{D} \) and \( u \), a summable function on \( \mathbb{D} \).

\[
B(u)(z) = \int_{\mathbb{D}} u(\zeta) \frac{(1 - |z|^2)^2}{|1 - \zeta z|^4} dA(\zeta)
\]

is called the Berezin transform of \( u \). Ahern [1] described all the possible triples \( \{u, f, g\} \) for which

\[
B(u)(z) = f(z)g(z)
\]

where both \( f, g \) are holomorphic in \( \mathbb{D} \). This result was crucial in solving a version of the zero product problem for Toeplitz operators on the Bergman space.

The natural next question was to describe all functions in the range of Berezin Transform which are of the form

\[
\sum_{i=1}^{N} f_i g_i
\]

where \( f_i, g_i \) are all holomorphic in \( \mathbb{D} \). We shall give a complete description of all such \( u \) and the corresponding \( f_i, g_i, 1 \leq i \leq N \). Further we give very simple proof of the result of Ahern [1] and the recent results of Ćučkovi´ć and Li [2] where they tackle the special case when \( N = 2 \) and \( g_2 = z^n \).

1. Introduction

Let \( dA = \frac{dx dy}{\pi} \) denote the normalized Lebesgue area measure on the unit disk \( \mathbb{D} \) in the complex plane. For any \( u \), a summable function on \( \mathbb{D} \),

\[
B(u)(z) = \int_{\mathbb{D}} u(\zeta) \frac{(1 - |z|^2)^2}{|1 - \zeta z|^4} dA(\zeta)
\]

is called the Berezin transform of \( u \).

**Theorem A:** (Ahern [1]) If \( u \in L^1(\mathbb{D}) \) and

\[
B(u)(z) = f(z)g(z)
\]

where both \( f, g \) are holomorphic in \( \mathbb{D} \) and not constant, then there exists an automorphism \( \phi \) of \( \mathbb{D} \) and two polynomials \( p, q \) each of degree at most 2 and the degree of \( pq \) is at most 3 such that \( f = p(\phi) \), \( g = q(\phi) \).

By the standard trick of complexification we obtain \( B(u)(z, w) \) a holomorphic function of two complex variables defined in the bidisk \( \mathbb{D} \times \mathbb{D} \) as follows:

\[
B(u)(z, w) = \int_{\mathbb{D}} u(\zeta) \frac{(1 - zw)^2}{(1 - \zeta z)^2(1 - \zeta w)^2} dA(\zeta)
\]

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and notice that (1) can be rewritten as
\[ B(u)(z) = B(u)(z, \overline{z}). \]
Further we can write
\[ B(u)(z,w) = \sum_{k=0}^{\infty} f_k(z)w^k \]
where \( f_k \) is holomorphic in \( \mathbb{D} \) for every \( k \geq 0 \). For any function \( v(z) \), let
\[ \frac{\partial v}{\partial z} = \frac{\partial}{\partial z}, \quad \Delta = \partial^2 = \partial \overline{\partial}, \text{ the Laplace Operator.} \]
We say \( B(u)(z) \) and also \( B(u)(z,w) \) is of \textbf{finite rank} if the vector space generated by \( f_k \) is finite dimensional. The following remarks are not difficult to check.

**Remark 1:** If \( B(u) \) is of finite rank, then it can be written as
\[ \sum_{i=1}^{N} f_i \overline{\theta}_i, \text{ where } f_i, g_i \text{ are holomorphic in } \mathbb{D}, \]
and conversely.

**Remark 2:**
\[ \sum_{i=1}^{N} f_i \overline{\theta}_i \]
is of rank \( N \) if and only if the set of functions \( \{ f_1, f_2, \ldots, f_N \} \) is linearly independent and so is \( \{ g_1, g_2, \ldots, g_N \} \).

This terminology could be applied to any holomorphic function of two complex variables in the bidisk. For example the function \( f(z)g(z) \) is of rank 1 because after complexification it will be \( f(z)\overline{g}(z) = \sum_{k=0}^{\infty} c_k f(z)w^k \).

Ahern’s theorem can be thought of as characterizing all functions of rank one in the range of Berezin transform on \( \mathbb{D} \).

The theorem of [2], including both cases, comes under the case of rank not exceeding 3 because any harmonic function can be written as \( f_1 \overline{\theta}_1 + f_2 \overline{\theta}_2 \) where \( f_1, g_2 \) are holomorphic and \( g_1 = f_2 = 1 \) and so case of (2) is covered under rank not exceeding 3 and case of (3) is covered under rank not exceeding 2.

2. \textbf{Main Theorem}

**Theorem 1:** If \( B(u) \) is of finite rank not exceeding \( N \), then there exist finitely many points \( a_i, 1 \leq i \leq N \) in \( \mathbb{D} \) such that
\[ u(\zeta) = h(\zeta) + \sum_{i=1}^{N} D_i \ln |\zeta - a_i| + \frac{E_i}{(\zeta - a_i)} + \frac{F_i}{(\zeta - a_i)^2} \]
where \( D_i, E_i, F_i, 1 \leq i \leq N \) are constants, many and even all of them could vanish, and \( h \) is a summable harmonic function.
Proof. We follow Ahern by applying Laplacian to $B(u)$,
\[
\Delta B(u) = \sum_{k=1}^\infty f'_k(z)kz^{k-1} = \int_\mathbb{D} u(\zeta)\Delta_z \frac{(1-|\zeta|^2)^2}{|1-\zeta z|^4} dA(\zeta)
\]
where $\Delta_z$ denotes Laplacian with respect to $z$. As noted by Ahern\cite{Ahern} there is a remarkable symmetry for the Berezin kernel
\[
\Delta_z (1-|z|^2)^2 \frac{1}{|1-\zeta z|^4} = \Delta_\zeta (1-|\zeta|^2)^2 \frac{1}{|1-\zeta z|^4}
\]
and so
\[
\sum_{k=1}^\infty f'_k(z)kz^{k-1} = \int_\mathbb{D} u(\zeta)\Delta_\zeta \frac{(1-|\zeta|^2)^2}{|1-\zeta z|^4} dA(\zeta)
\]
\[
= \int_\mathbb{D} u(\zeta)\Delta_\zeta \frac{(1-|\zeta|^2)^2}{(1-\zeta z)^2(1-\zeta z)^2} dA(\zeta)
\]
\[
= \int_\mathbb{D} u(\zeta)\Delta_\zeta \frac{(1-|\zeta|^2)^2}{(1-\zeta z)^2} \sum_{k=0}^\infty (k+1)\zeta^k dA(\zeta)
\]
\[
= \sum_{k=0}^\infty (k+1)z^k \int_\mathbb{D} u(\zeta)\Delta_\zeta \frac{(1-|\zeta|^2)^2}{(1-\zeta z)^2} \zeta^k dA(\zeta).
\]
From this we get, after complexifying both sides
\[
\sum_{k=1}^\infty f'_k(z)kw^{k-1} = \sum_{k=0}^\infty (k+1)w^k \int_\mathbb{D} u(\zeta)\Delta_\zeta \frac{(1-|\zeta|^2)^2}{(1-\zeta z)^2} \zeta^k dA(\zeta)
\]
and equating the coefficients of $w^k$,
\[
f'_{k+1}(z) = \int_\mathbb{D} u(\zeta)\Delta_\zeta \frac{(1-|\zeta|^2)^2}{(1-\zeta z)^2} \zeta^k dA(\zeta).
\]
Let us write the power series expansion for $f'_{k+1}$ as follows:
\[
f'_{k+1}(z) = \sum_{l=0}^\infty a_{k,l}z^l.
\]
From (4) we deduce
\[
f'_{k+1}(z) = \sum_{l=0}^\infty (l+1)z^l \int_\mathbb{D} u(\zeta)\Delta_\zeta (1-|\zeta|^2)^2 \zeta^k dA(\zeta)
\]
and so
\[
\frac{a_{k,l}}{l+1} = \int_\mathbb{D} u(\zeta)\Delta_\zeta (1-|\zeta|^2)^2 \zeta^k dA(\zeta) \text{ for every } k, l \geq 0.
\]
We are given that the vector space generated by $f_k$ is finite dimensional and so same goes for the vector space generated by $f'_k$ which means the matrix $\{a_{k,l}\}$ is of finite rank and therefore the matrix $\left\{\frac{a_{k,l}}{l+1}\right\}$ is also of finite rank since the column space is unaltered. Now we see the distribution $F$ with support in $\overline{\mathbb{D}}$ defined by
\[
F(\psi) = \int_\mathbb{D} u(\zeta)\Delta_\zeta (1-|\zeta|^2)^2 \psi(\zeta) dA(\zeta)
\]
for any $C^\infty$ function $\psi$ in the complex plane, satisfies the Luecking condition: The matrix $\{F(\zeta^k)\}$ is of rank, $\leq N$.

By the theorem 3.1 of Alexandrov and Rozenblum, we have that, such a distribution has only finite support, that is there exist at most $N$ points $a_i$ in $\mathbb{D}$ and linear differential operators $L_i$ such that

$$F = (1 - |\zeta|^2)^2 \Delta \zeta u(\zeta) = \sum_{i=1}^N L_i(\delta(\zeta - a_i))$$

where $\delta$ is the Dirac delta function. This implies that the distribution $\Delta u \equiv 0$ except at $a_i$ and so $u$ is harmonic in $\mathbb{D}$ with isolated singularities at $a_i$. Hence in a neighborhood of each $a_i$, it will have a Laurent series like

$$c_0 \ln |\zeta - a_i| + c_1/(\zeta - a_i) + c_2/(\zeta - a_i).$$

Higher powers will be absent since $u$ is summable. This proves the main theorem.

3. Applications

The Berezin transforms of basic functions $\ln |\zeta|$, $1/\zeta$, and $1/|\zeta|$ are calculated in Ahern’s paper. For example $B(\ln |\zeta|) = \frac{z - 1}{2}$ and $B(1/\zeta) = 2\overline{\zeta} - z\overline{\zeta}^2$ and the rest can be computed by fractional linear transformations. For example it is an easy exercise to calculate Berezin transform of $\ln |\zeta - a|$ and $1/(\zeta - a)$ because

$$B(u(\phi_a)) = B(u)(\phi_a)$$

where

$$\phi_a(\zeta) = \frac{\zeta - a}{1 - \overline{\zeta}a}$$

and if $v$ is harmonic and summable,

$$B(v) = v.$$ 

For example

$$B(\ln |\zeta - a|) = B(\ln |\phi_a(\zeta)|) + B(\ln |1 - \overline{\zeta}a|) = \frac{\phi_a \overline{\phi_a} - 1}{2} + \ln |1 - \overline{\zeta}a|,$$

$$2B(\ln |\zeta - a| - \ln |1 - \overline{\zeta}a|) = \phi_a \overline{\phi_a},$$

(6a)

and

$$(1 - a\overline{a})B\left(\frac{1}{\zeta - a}\right) = B\left(\frac{1 - \overline{\zeta}a}{\zeta - a} + \overline{\zeta}\right) = B\left(\frac{1}{\phi_a}\right) + \overline{\zeta} = 2\overline{\phi_a} - \phi_a \overline{\phi_a} + \overline{\zeta},$$

$$B\left(\overline{\zeta} + 2\overline{\phi_a} - \frac{1 - \overline{a}\overline{\zeta}}{\zeta - a}\right) = \phi_a \overline{\phi_a}.$$ 

(6b)

So we conclude

**Theorem 2.** If $u$ is summable and $B(u)$ of finite rank $N$, then

$$B(u) = h + \sum_{i=1}^N \phi_a \overline{\phi_a} (D_i + E_i \phi_a + F_i \overline{\phi_a})$$

(7)

where $h$ is a summable harmonic function and $a_i \in \mathbb{D}$ and $D_i, E_i, F_i$ are constants.
Corollary 1. (Čučković and Li, [2]) If \( u \) is summable and \( B(u) \) is harmonic, then \( u \) is harmonic.

Proof. As noted this means \( B(u) \) is of rank at most \( 2 \). So by Theorem 2 there exist at most two points \( a_1, a_2 \) in \( \mathbb{D} \) such that

\[
B(u) = h + 2 \sum_{i=1}^{2} \phi_{a_i} \phi_{a_i}(A_i + B_i \phi_{a_i} + C_i \overline{\phi}_{a_i})
\]

where \( h \) is harmonic and summable. So \( B(u) = v \) is summable and since it is given to be harmonic, we have \( B(v) = v \) and therefore \( B(u-v) = 0 \). It is well known that \( B \) is injective and hence \( u = v \) and \( u \) is harmonic. QED.

Comment. Prof. Ahern communicated to me that this fact was noted by him long ago and his proof even simpler than what is presented here, goes as follows: Main point is to prove that \( B(u) \) is summable. Using Lemma 6.23 of [5], p 148 along with the notation there, we have

\[
\tilde{\Delta}B(u) = 8(B(u) - B_1(u)). \tag{afr}
\]

But harmonicity of \( B(u) \) makes the LHS zero and so \( B(u) = B_1(u) \). Since \( B_1(u) \) is summable, \( B(u) \) is summable. The equation (afr) was already noted in [6].

Now we apply Theorem 2 to prove Theorem A:

Proof of Theorem A. So by the hypothesis of Theorem A, \( B(u) \) has rank 1 and so from Theorem 2 follows the existence of a point \( a \in \mathbb{D} \) such that

\[
f(\zeta)\overline{\phi}(\zeta) = h(\zeta) + (A\phi_a(\zeta) + B\phi_a^2(\zeta))\overline{\phi}_a(\zeta) + C\phi_a(\zeta)\overline{\phi}_a(\zeta)
\]

where \( h \) is a summable harmonic function and \( A, B, C \) are constants and \( \phi_a(\zeta) = \frac{\zeta - a}{1 - \overline{a}\zeta} \), an automorphism of \( \mathbb{D} \). By (5) and a change of variable \( z = \phi_a(\zeta) \), we get

\[
F(z)\overline{G}(z) = H(z) + (Az + Bz^2)\overline{z} + Cz\overline{z}^2 \tag{8}
\]

where \( F(z) = f(\zeta), G(z) = g(\zeta), H(z) = h(\zeta) \). Since \( H \) is harmonic in \( \mathbb{D} \), we can write it in a unique way as

\[
K(z) + L(z)
\]

where \( K, L \) are holomorphic in \( \mathbb{D} \) and \( L(0) = 0 \). Also let us write the Taylor series

\[
\overline{G}(z) = \sum_{k=0}^{\infty} g_k z^k, \overline{L}(z) = \sum_{k=1}^{\infty} l_k z^k. \tag{9}
\]

Now (8) can be written as

\[
F(z)\overline{G}(z) = \sum_{k=0}^{\infty} g_k F(z)z^k = K(z) + (l_1 + Az + Bz^2)\overline{z} + (l_2 + Cz)\overline{z}^2 + \sum_{k=3}^{\infty} l_k z^k.
\]

Comparing the coefficient of \( z^k \) on both sides, we obtain

\[
g_0 F(z) = K(z), g_1 F(z) = l_1 + Az + Bz^2, g_2 F(z) = l_2 + Cz, g_k F(z) = l_k \quad \text{for} \quad k \geq 3.
\]

So if \( g_k \neq 0 \) for some \( k > 2 \), \( F \) and so \( f \) will be constant.

Since \( f \) is not constant, we get \( g_k = 0 \) for all \( k > 2 \) and that means \( G \) is a polynomial of degree \( \leq 2 \). Now one of \( g_1, g_2 \) is different from zero for otherwise \( G \) would be constant.
Now if \( g_2 = 0, g_1 \neq 0, \) \( G \) is a polynomial of degree 1 and \( F(z) = (l_1 + Az + Bz^2)/g_1 \) is a polynomial of degree at most 2. On the other hand \( g_2 \neq 0, F(z) = (l_2 + Cz)/g_2 \) is a polynomial of degree 1 and \( G \) is of degree 2. This proves the theorem A by noting
\[
F(\phi_a(\zeta)) = f(\zeta), \quad G(\phi_a(\zeta)) = g(\zeta),
\]
\( F(z) = p(z), G(z) = q(z) \) the promised polynomials.

Let us end this section with a lemma that will be useful in the next section.

**Lemma 1:** Let \( \phi(z) = \frac{z - a}{1 - \overline{a}z}, |a| < 1 \) and \( A, B, C \) be constants, not all zero and \( \psi = (A\phi + B\phi^2)\overline{\phi} + C\phi\phi^2 \).

There are two possibilities.

a) \( BC \neq 0, \) and there exist functions \( u_1, u_2 \) both summable such that
\[
\psi = B(u_1) + B(u_2) \quad \text{and} \quad B(u_1) = f_1\overline{g_1}, B(u_2) = f_2\overline{g_2}
\]
where \( f_1, g_1, f_2, g_2 \) are holomorphic everywhere except at \( b = \frac{1}{\overline{a}} \). Further \( f_1, g_2 \) have a pole of order 2 and \( f_2, g_1 \) have a pole of order 1 at \( b \).

b) \( BC = 0 \) and there exists a summable function \( u \) such that
\[
\psi = B(u) = f\overline{g}
\]
where \( f, g \) are holomorphic everywhere except at \( b \) where each has a pole.

**Proof.** Let us assume \( BC \neq 0 \) which means \( B \neq 0 \) and \( C \neq 0 \). Now we set
\[
f_1 = A\phi + B\phi^2, g_1 = \phi, f_2 = C\phi, g_2 = \phi^2.
\]
Now from (6a) and (6b) follows the existence of summable functions \( u_1 \) and \( u_2 \) such that
\[
B(u_1) = f_1\overline{g_1}, B(u_2) = f_2\overline{g_2}.
\]
Easy to check all of a) is true.

If \( BC = 0 \), one of \( B, C \) is zero. If \( B = 0, C \neq 0 \), we set \( f = \phi, g = A\phi + C\phi^2 \) and if \( B \neq 0, C = 0 \), set \( f = A\phi + B\phi^2, g = \phi \). And if \( B = C = 0 \), we set \( f = A\phi, g = \phi \).

Now easy to check in each instance from (6a) and (6b) that there exists a summable function \( u \) such that \( B(u) = f\overline{g} \) and easy to check that all of b) is true.

4. The case of rank \( N \), any positive integer.

**Theorem 3.** If \( u \in L^1(\mathbb{D}) \) and \( B(u) \) is of rank \( N \), then either \( u = \sum_{i=1}^{N} u_i \) where each \( u_i \) is summable and each of \( B(u_i) \) is of rank one or \( u_1 + u_2 \) is summable and harmonic and for \( i > 2, u_i \) is summable and \( B(u_i) \) is of rank one.

[I think it is possible for a harmonic function to be summable but not its conjugate.]

**Proof.** Theorem 2 implies there exist at most \( N \) distinct automorphisms
\[
\phi_i = \frac{z - a_i}{1 - \overline{a}_iz}
\]
of $D$ and a harmonic function $h$ and constants $D_i, E_i, F_i, 1 \leq i \leq$, allowing the possibility for a lot of these constants to vanish, such that

$$B(u) = h + \sum_{i=1}^{N} (D_i \phi_i + E_i \phi_i^2 \overline{\phi_i} + F_i \phi_i \overline{\phi_i}^2).$$  \hspace{1cm} (10)$$

Write $h$ uniquely as $K + L$ where $K, L$ are holomorphic and $L(0) = 0$.

**Fix an $i$.** From Lemma 1, there are three possibilities for the function

$$\psi_i = (D_i \phi_i + E_i \phi_i^2 \overline{\phi_i} + F_i \phi_i \overline{\phi_i}^2),$$

either it could be zero or there exist two summable functions $u, v$ such that

$$\psi_i = B(u) + B(v), B(u) = f \overline{g}, B(v) = p \overline{q}$$

such that $f, g, p, q$ are holomorphic everywhere except at $\frac{1}{\overline{\psi_i}} = b_i$ where $f, q$ have a pole of order 2 and $g, p$ have a pole of order 1. Or there exists just one summable function $u$ such that

$$\psi_i = B(u) = f \overline{g}$$

where $f, g$ are holomorphic everywhere except at $b_i$ where each of $f, g$ has a pole of order $\leq 2$.

Now from (10), we deduce the existence of finitely many summable functions $u_j, 1 \leq j \leq M$ such that

$$B(u_j) = f_j \overline{g}_j$$

and

$$B(u) = h + \sum_{j=1}^{M} B(u_j) = h + \sum_{j=1}^{M} f_j \overline{g}_j,$$  \hspace{1cm} (11)$$

where each of the functions $f_j, g_j$ is holomorphic in the entire plane except at a single point, where each has a pole and which varies with the index $j$ and if two pairs $(f_j, g_j)$ and $(f_k, g_k)$ have poles at the same point, this can happen only for two pairs both coming from one $\psi_i$, the orders of the pole differ from $f_j$ to $f_k$ and from $g_j$ to $g_k$.

Summing up the above discussion, we have

$$B(u) = K + L + \sum_{i=1}^{M} B(u_i), \hspace{1cm} (12)$$

$$B(u_i) = f_i \overline{g}_i, \hspace{1cm} 1 \leq i \leq M,$$

where each $f_i$ and $g_i$ is meromorphic in the entire plane with a single pole, of course the pole may vary with $i$, and if two of the $f_i$ happen to have a pole at the same point, one of them will have pole of order 2 and the other of order 1. Same can be said about the $g_i$. These are the contributions from the same $\psi_j$.

Let us write the Taylor series for $g_i, L$,

$$g_i(z) = \sum_{k=0}^{\infty} \overline{T}_{i,k} z^k, L(z) = \sum_{k=1}^{\infty} \overline{T}_k z^k.$$
Let \( w_k \) denote the coefficient of \( \zeta^k \) for the function \( B(u) \). Then \( N \) is the dimension of the vector space \( W \) generated by \( w_k \). We list

\[
\begin{align*}
w_0 &= K + \sum_{i=1}^{M} \beta_{i,0} f_i \\
w_k &= l_k + \sum_{i=1}^{M} \beta_{i,k} f_i \quad \text{for } k \geq 1
\end{align*}
\]

using (10).

Evidently dimension of \( W \) space generated by \( w_k, k > 0 \) would be \( \leq N \). We claim 1, \( f_1, f_2, \ldots, f_M \) are linearly independent for if not, there would exist constants \( \xi_0, \xi_1, \ldots, \xi_M \) such that

\[
\xi_0 + \xi_1 f_1 + \cdots + \xi_m f_m = 0.
\]

As noted earlier each of \( f_i \) is meromorphic in the plane with a single pole and the poles are either different or of different orders if they happen to be the same. Hence each \( \xi_i = 0 \) for \( i > 0 \). Hence \( \xi_0 = 0 \).

Therefore \( w_k \) is represented by the row \( (l_k, \beta_{1,k}, \beta_{2,k}, \ldots, \beta_{M,k}) \). Let \( W_1 \) also denote the matrix whose rows are \( w_k, k > 0 \). Then rank of \( W_1 \) is \( \leq N \) and the matrix \( C \) obtained by dropping the first column of \( W_1 \) would have rank \( \leq N \). But rank of \( C \) is \( M \). For otherwise there would exist constants \( \xi_1, \ldots, \xi_M \), not all zero, such that

\[
\sum_{i=1}^{M} \xi_i \beta_{i,k} = 0, \forall k > 0,
\]

and so

\[
\sum_{i=1}^{M} \xi_i (g_i - g_i(0)) = 0.
\]

But again using the poles argument, we see that all \( \xi_i = 0 \). Hence the rank of \( C \) is \( M \) and \( M \leq N \). Evidently the rank of \( B(u) = K + \sum_{i=1}^{M} B(u_i) \) is less than or equal to \( 1 + 1 + M \). So

\[
N - 2 \leq M \leq N.
\]

If \( M = N - 2 \), the theorem is proved because \( B(u) = B(h) + \sum_{i=1}^{M} B(u_i) \) where \( h \) is summable and harmonic. Assume \( M = N - 1 \). There are two possibilities depending on the rank of \( W_1 \). Its rank lies between \( M \) and \( N \). If its rank is \( M = N - 1 \), the first column will depend on the rest and so we see that \( \mathcal{L} \) is a linear combination of \( \bar{g}_i - \bar{g}_i(0), 1 \leq i \leq M \). Therefore there exist constants \( \lambda_i \) such that

\[
\mathcal{L} = \sum_{i=1}^{M} \lambda_i (\bar{g}_i - \bar{g}_i(0)), \quad (13)
\]

from which and (12), we obtain

\[
B(u) = K + \sum_{i=1}^{M} \lambda_i (\bar{g}_i - \bar{g}_i(0)) + \sum_{i=1}^{M} f_i \bar{g}_i,
\]

\[
= K - \sum_{i=1}^{M} \lambda_i \bar{g}_i(0) + \sum_{i=1}^{M} (f_i + \lambda_i) \bar{g}_i,
\]

\[
= K - \sum_{i=1}^{M} \lambda_i \bar{g}_i(0) + \sum_{i=1}^{M} B(u_i + \lambda_i \bar{g}_i).
\]
Now by (13), $L$ is summable and since $K + \overline{L}$ is summable, we have $K - \sum_{i=1}^{M} \lambda_i \overline{g}_i(0) = K_1$ is summable, further since it is holomorphic, $B(K_1) = K_1$. Therefore

$$B(u) = B(K_1) + \sum_{i=1}^{M} B(u_i + \lambda_i \overline{g}_i)$$

proving the Theorem in this case.

The remaining possibility is that the rank of $W_1$ is $N$. Then $w_0$ is a linear combination of some of the rows of $W_1$, let us say

$$w_0 = \sum_{i=1}^{n} \mu_i w_i.$$  

Hence

$$w_0 = K + \sum_{j=1}^{M} \beta_{j,0} f_j = \sum_{i=1}^{n} \mu_i \left( l_i + \sum_{j=1}^{M} \beta_{j,i} f_j \right),$$

$$K = \kappa_0 + \sum_{j=1}^{M} \kappa_j f_j. \quad (14)$$

Therefore from (12), we have

$$B(u) = \overline{L} + \kappa_0 + \sum_{j=1}^{M} \kappa_j f_j + \sum_{j=1}^{M} f_j \overline{g}_j$$

$$= \overline{L} + \kappa_0 + \sum_{j=1}^{M} f_j (\overline{g}_j + \kappa_j)$$

$$= \overline{L} + \kappa_0 + \sum_{j=1}^{M} B(u_j + \kappa_j f_j).$$

Now by (14), $K$ is summable and so $\overline{L}_1 = \overline{L} + \kappa_0$ is summable and further since it is anti-holomorphic, $B(\overline{L}_1) = \overline{L}_1$. Therefore

$$B(u) = B(\overline{L}_1) + \sum_{j=1}^{M} B(u_j + \kappa_j f_j)$$

proving the Theorem in this case.

Assume $M = N$. In this case first column of $W_1$ depends on the other columns which are columns of $C$ and this means $\overline{L}$ is a linear combination of $\overline{g}_i - \overline{g}_i(0), 1 \leq i \leq M$. Following the same argument as in (13), we obtain

$$B(u) = K - \sum_{i=1}^{M} \lambda_i \overline{g}_i(0) + \sum_{i=1}^{M} (f_i + \lambda_i) \overline{g}_i.$$  

Now by introducing new functions in place of the old $K, L, f_i, g_i$, we do not change $B(u)$ and nor the new functions. They have the same properties as the old ones. So Let

$$\tilde{K} = K - \sum_{i=1}^{M} \lambda_i \overline{g}_i(0), \tilde{L} = 0, \tilde{f}_i = f_i + \lambda_i, \tilde{g}_i = g_i.$$
Here
\[ \tilde{u}_i = u_i - \lambda_i \tilde{g}_i. \]
Now the coefficient \( w_k \) of \( \tilde{f}_i \) of \( B(u) \) would look like
\[
\begin{align*}
    w_0 &= \tilde{K} + \sum_{i=1}^{M} \beta_{i,0} \tilde{f}_i \\
    w_k &= \sum_{i=1}^{M} \beta_{i,k} \tilde{f}_i \text{ for } k \geq 1.
\end{align*}
\]
Since the rank of \( B(u) \) is equal to the rank of the matrix \( (\beta_{i,k}) = C \), as argued in the above paragraph \( w_0 \) is a linear combination of \( w_k, 0 < k \leq n \) for some \( n \). Therefore we get
\[
\tilde{K} + \sum_{i=1}^{M} \beta_{i,0} \tilde{f}_i = \sum_{k=1}^{n} \mu_k \sum_{i=1}^{M} \beta_{i,k} \tilde{f}_i
\]
from which we get
\[
\tilde{K} = \sum_{i=1}^{M} \kappa_i \tilde{f}_i,
\]
and
\[
B(u) = \sum_{i=1}^{M} \kappa_i \tilde{f}_i + \sum_{i=1}^{M} \tilde{f}_i \tilde{g}_i = \sum_{i=1}^{M} \tilde{f}_i (\tilde{g}_i + \kappa_i) = \sum_{i=1}^{M} B(\tilde{u}_i + \kappa_i \tilde{f}_i).
\]
This proves the Theorem 3 completely.

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THE UNIVERSITY OF TOLEDO, COLLEGE OF ARTS AND SCIENCES,
DEPARTMENT OF MATHEMATICS, MAIL STOP 942. TOLEDO, OHIO 43606-3390, USA
E-mail address: rnegise@math.utoledo.edu