Discrete-Continual Finite Element Method for Semianalytical Analysis of Plates on Two-Parameter Elastic Foundation. Part 1: Continual Formulations of the Problem and Approximations

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Abstract. This paper is proposed discrete-continual finite element method (DCFEM) for semianalytical analysis of plates on two-parameter elastic foundation. First of all, the operational and discrete-continual formulations of the problem are described, then the approximation of unknown functions and its partial derivatives are obtained. Later the internal forces, local stiffness matrices and local load vectors are evaluated, then these local vectors and matrices are assembled to obtain resultant multipoint boundary problem and in the final step the problem is solved by the exact analytical method. This method allowed to obtain exact analytical solutions of boundary problems along the regular direction, and these solutions remain exact for arbitrary influences such as any force or moment, soil parameters, intermediate constraints and connections. Using this method, the problem remains continual in the one direction (basic), while the discrete (finite element) approximation is carried out with respect to the non-basis direction. The structural discontinuities in the analytical direction can be taken into account, by addition of new appropriate boundary conditions at the relevant section. The DCFEM increases the accuracy of the solution and significantly reduces the computational efforts, especially within analysis of extended plates such as strip foundations.

1. Introduction

Recent developments in the computational methods and applied mathematics, have resulted in emerging new branches in the field of computational mechanics, known as semianalytical or discrete-continual methods [1,2]. Semianalytical approaches allow comprehensive evaluation of influences of various local and global factors, especially in zones of edge effect [3,4] (i.e. where some components of the solution are a rapidly changing functions and their rate of change can’t always be adequately taken into account by conventional numerical methods (including the standard finite element method (FEM)). The discrete-continual finite element method (DCFEM) allows to obtain exact analytical solutions of boundary problems along the direction with constant or piecewise constant physico-geometric parameters of structure (this “regular” direction was suggested to be called “basic”), and these solutions remain correct for arbitrary influences such as any force or moment, soil parameters,
intermediate constraints and connections. This method is called discrete-continual in connection with the fact that the problem remains continuous in the basic direction, while the discrete (finite element) approximation is carried out with respect to the non-basis direction. It should be explained that an exact analytical solution is understood as the existence of an explicit formula for computing of stress distribution at an arbitrary point of the section. Besides, this formula should explicitly demonstrate the behavior of the computed factors (displacements, deformations, stresses, etc.) [5].

The technical literature contains numerous papers, conference proceedings and books about analysis of plates on elastic foundation [6-24]. Y.C. Cheung and O.C. Zinkiewicz [8] solved the problems of analysis of slabs and tanks (either isotropic or orthotropic) resting either on a semi-infinite elastic continuum or on individual springs (of the so-called Winkler’s type) by the finite element method. A.L. Yettram, J.R. Whiteman and D.J. Henwood [23] proposed the analysis of transversely loaded thin rectangular plates on Winkler foundations and used a technique, which is based on Fourier series expansions for a parametric study of plate behavior. W.T. Straughan [19] developed a three-parameter mathematical model for the analysis of plates and solved it by finite difference method. A. El-Zafrany, S. Fadhil and K. Al-Hosani [12] introduced for the first time a full derivation of a fundamental solution and boundary integral equations for thick Reissner plates resting on a Winkler elastic foundation. Moreover, a series of researches about analysis of plates on two-parameter elastic foundation, by solving same numerical examples for the comparison of proposed methods, were developed by R. Buczkowski [8], M. Celik [9], Y.C. Das [20], A.T. Daloglu [17], K. Ozgan [17], A. Saygun [9], W.T. Straughan [20], W. Torbacki [8], C.V.G. Vallabhan [20]. In the recent years, R. Li, Y. Zhong and M. Li [15] obtained analytic bending solutions of free rectangular thin plates resting on elastic foundations, based on the Winkler model and a new symplectic superposition method.

In the present paper, the discrete-continual finite element method (DCFEM) is used for the semianalytical analysis of plates on two-parameter elastic foundation. It allows construction of correct exact analytical solution in one direction and decreases the problem size to a conventional one-dimensional finite element analysis. Generally, our papers with these results are organized as follows. In the next, the discrete-continual formulation of the problem is presented. Then it is solved by an analytical solution [6]. Subsequently, the efficiency, accuracy and validity of the proposed method are demonstrated by several numerical examples. Finally, some concluding remarks are presented.

2. Continual formulations of the problem

In the present paper by using of Kirchhoff theory [25], the plate bending problem is solved by DCFEM with allowance for two-parameter elastic foundation. The considering plates have a rectangular or piecewise rectangular shape. Thus, modelling of these plates can be efficiently done by this method. Let us consider operational formulation of the problem.

Let \( x = (x_1, x_2) \) be Cartesian coordinates; \( \Omega \) be domain occupied by plate with boundary \( \Gamma_k \); \n
\[
\Omega = \bigcup_{k=1}^{n} \Omega_k ; \quad \theta_k(x) = \begin{cases} 1, & x \subset \Omega_k \\ 0, & x \subset \bar{\Omega}_k \end{cases} ; \quad \delta_{k,k} = \frac{\partial \theta_k}{\partial n} ,
\]

\( \theta \) is characteristic function of \( \Omega \); \( \delta_{k,k} \) is delta-function of \( \Gamma_k = \partial \Omega_k \); \( n \) is unit normal vector.

Let \( w \) be the vertical deflection of the plate and \( x_2 \) be the coordinate corresponding to the basic direction (\( x_1 \) is other coordinate); \( x^b_{2,k}, k = 1,...,n \) are coordinates of boundary points (including points of changes in physical and geometrical parameters). The mechanical behaviour of the soil is simulated by modulus of subgrade reaction \( k_{s,k} \) and shear deformation of the soil \( 2t_{s,k} \). It is more realistic (with respect to Winkler model) to take into account shear interactions between the soil springs. Then, the general operational formulation of problem of plate analysis has the following form:
\[ L_{4,k} \frac{\partial^4}{\partial x^4} w_k + (L_{2,k} + 2\partial_1 \theta_1 t_{1,k} \partial_1) \frac{\partial^2}{\partial x^2} w_k + (L_{0,k} + \theta_2 k_{2,k}) w_k = F_k, \quad (x_1, x_2) \in \omega_k, \]  

\[ L_{4,k} = \theta_k D_k; \quad L_{2,k} = -[\partial_1^2 \theta_1 D_1 \nu_{1,k} + 2\partial_1 \theta_1 D_1 (1 - \nu_k) \frac{\partial_1}{\partial x_1} + \theta_2 D_2 \nu_{2,k}] ; \quad L_{0,k} = -\partial_1^2 \theta_1 D_1 \frac{\partial_1}{\partial x_1} \]  

\[ F_k = \delta_k q_k - \delta_{b,k} q_k - \partial_1 (\delta_{b,k} M_{1,k}) - \partial_2 (\delta_{b,k} M_{2,k}), \]  

where \( \Omega \) is domain occupied by plate; \( D \) is bending stiffness of plate, \( \nu \) is the Poisson ratio, \( L_2 \) is symmetric differential operator; \( \partial_i = \partial/\partial x_i \), \( \partial_i^* = -\partial/\partial x_i \); \( Q_{1,k}, M_{1,k}, M_{2,k} \) are corresponding shear force and bending moments at \( \Gamma_k = \partial \Omega_k \) [26]; \( q_k \) is distributed load; \( w_k \equiv w, \quad (x_1, x_2) \in \omega_k. \)  

Let us introduce the following notations:

\[ \tilde{L}_{2,k} = L_{2,k} + 2\partial_1 \theta_1 t_{1,k} \partial_1 \partial_1 \]  

\[ \tilde{L}_{0,k} = L_{0,k} + \theta_2 k_{2,k}. \]  

The fourth order differential equation (2) can be reduced to a linear system of first order differential equations [2]. Let us use the following notation

\[ y_{1,k} = y_{1,k}(x_1, x_2) = w_k(x_1, x_2); \quad y_{i,k} = y_{i,k}(x_1, x_2) = \partial_{x_1}^{i-1} w_k(x_1, x_2), \quad i = 2, 3, 4, \]  

Using (2), (3), (5) and (6), finally the following system of equations can be obtained:

\[
\begin{bmatrix}
1 & 0 & 0 & 0 & y_{1,k} \\
0 & 1 & 0 & 0 & y_{2,k} \\
0 & 0 & 1 & 0 & y_{3,k} \\
0 & 0 & 0 & L_{4,k} & y_{4,k} \\
\end{bmatrix} \begin{bmatrix}
L_{4,k} \\
\tilde{L}_{0,k} \\
\tilde{L}_{2,k} \\
\end{bmatrix} = \begin{bmatrix}
0 \\
0 \\
0 \\
F_k \\
\end{bmatrix},
\]

Besides, we can rewrite system (7):

\[ \overline{U}_k' = \tilde{L}_k \overline{U}_k + \overline{F}_k, \]  

where \( \tilde{L}_k \) is a differential operator, which has eigenvalues of opposite signs and Jordan cells;

\[
\tilde{L}_k = 
\begin{bmatrix}
0 & I & 0 & 0 \\
0 & 0 & I & 0 \\
0 & 0 & 0 & I \\
L^{-1}_{4,k} \tilde{L}_{0,k} & 0 & L^{-1}_{4,k} \tilde{L}_{2,k} & 0 \\
\end{bmatrix} ; \quad \bar{U} = \begin{bmatrix}
y_1 \\
y_2 \\
y_3 \\
y_4 \\
\end{bmatrix} ; \quad \bar{U}' = \partial_{x_1} \bar{U} = \begin{bmatrix}
\partial_{x_1} y_1 \\
\partial_{x_1} y_2 \\
\partial_{x_1} y_3 \\
\partial_{x_1} y_4 \\
\end{bmatrix} ; \quad \overline{F}_k = \begin{bmatrix}
0 & 0 & 0 & L^{-1}_{4,k} F_k \\
0 & 0 & 0 & 0 \\
\end{bmatrix}, \]  

\( I \) is the identity matrix (identity operator).

3. Approximation of the problem

In accordance with the method of extended domain [26], the given domains \( \Omega_k, \quad k = 1, \ldots, n_k - 1 \) are bordered by extended domains \( \omega_k, \quad k = 1, \ldots, n_k - 1 \) of arbitrary shape, particularly elementary. Discrete-continual approximation model for each \( \omega_k \) is shown in Figure 1.

We have the following relations:
\[ \omega_k = \bigcup_{i=1}^{N-1} \omega_{k,i} ; \quad \omega_k = \{ (x_1, x_2, x_3) : 0 \leq x_1 \leq d_i, \quad x_{2,k}^b < x_2 < x_{2,k+1}^b, \quad 0 \leq x_3 \leq \Delta_k \} \],  \quad (10)

**Figure 1** Discrete-continual model of structure in an arbitrary interval of the basic direction

\[ \omega_{k,i} = \{ (x_1, x_2, x_3) : x_{1,i} \leq x_1 \leq x_{1,i+1}, \quad x_{2,k}^b < x_2 < x_{2,k+1}^b, \quad 0 \leq x_3 \leq \Delta_k \} , \quad (11) \]

\[ x_{1,i+1} = x_{1,i} + h_i, \quad i = 1, 2, ..., N - 1; \quad h_i = \begin{cases} x_{1,i+1} - x_{1,i}, & i < N \\ 0, & i = N \end{cases}, \quad i = 1, 2, ..., N , \quad (12) \]

where \( d_i \) is the length of structure along the \( x_i \) direction; \( N \) is the number of nodes along the \( x_i \) direction; \((x_{1,i}, x_2, x_3), \quad i = 1, 2, ..., N \) are coordinates of the nodes; \( h_i \) is the size of the \( i \)-th discrete-continual finite element (DCFEE); \( \Delta_k \) is plate thickness.

The characteristic function and material properties of a discrete-continual finite element \( \omega_{k,i} \) are determined by the formulas

\[ \theta_{k,i} = \begin{cases} 1, & \omega_{k,i} \subset \Omega_k \\ 0, & \omega_{k,i} \subset \Omega_k \end{cases} ; \quad \overline{D}_{k,i} = \theta_{k,i} D_{k,i} ; \quad \overline{t}_{s,k,i} = \theta_{k,i} t_{s,k,i} ; \quad \overline{k}_{s,k,i} = \theta_{k,i} k_{s,k,i} , \quad (13) \]

where \( D_{k,i} = E_{k,i} \Delta_k^3 / [12(1-\nu_{k,i})] \); \( E_{k,i} \) is the elastic modulus of material of plate; \( \nu_{k,i} \) is the Poisson ratio within discrete-continual finite element \( \omega_{k,i} \); \( D_{k,i} \), \( t_{s,k,i} \) and \( k_{s,k,i} \) are parameters \( D_k \), \( t_{s,k} \) and \( k_{s,k} \) within \( \omega_{k,i} \).

Let us introduce notation

\[ z_j^{(k)}(x_1, x_2) = \partial_j y_j^{(k)}(x_1, x_2), \quad j = 1, 2, 3, 4 , \quad (14) \]

Components \( y_j^{(k)}(x_1, x_2), \quad j = 1, 2, 3, 4 \) and \( z_j^{(k)}(x_1, x_2), \quad j = 1, 2, 3, 4 \), were used as the main unknowns in the nodes of the discrete-continual mesh (i.e. for the \( i \)-th node this unknowns be \( y_j^{(k,i)}(x_1, x_2), \quad j = 1, 2, 3, 4 \) and \( z_j^{(k,i)}(x_1, x_2), \quad j = 1, 2, 3, 4 \)). Then the unknown vector for each node is defined by formulas

\[ \overline{y}_{n^{(k,i)}} = \overline{y}_{n^{(k,i)}}(x_2) = [ y_1^{(k,i)} \quad z_1^{(k,i)} \quad y_2^{(k,i)} \quad z_2^{(k,i)} \quad y_3^{(k,i)} \quad z_3^{(k,i)} \quad y_4^{(k,i)} \quad z_4^{(k,i)} ]^T \], \quad (15) \]
or  
\[
\bar{y}^{(k,j)}_n = \bar{y}^{(k,j)}(x_2) = [ (\bar{y}^{(k,i)}_{n,1})^T \quad (\bar{y}^{(k,i)}_{n,2})^T \quad (\bar{y}^{(k,i)}_{n,3})^T \quad (\bar{y}^{(k,i)}_{n,4})^T ]^T, 
\]

where  
\[
\bar{y}^{(k,j)}_{n,j} = \bar{y}^{(k,j)}(x_2) = [ y^{(k,i)}_{j,n} \quad z^{(k,j)}_j ]^T, \quad j = 1, 2, 3, 4, 
\]

The vector of unknowns in all nodes of the element has the form:  
\[
\bar{y}^{(k,i)}_j = \bar{y}^{(k,i)}(x_1, x_2) = \bar{y}^{(k,i)}(t, x_2) = \alpha^{(k,i)}_{j,1} + \alpha^{(k,i)}_{j,2} t + \alpha^{(k,i)}_{j,3} t^2 + \alpha^{(k,i)}_{j,4} t^3, 
\]

or  
\[
\bar{y}^{(k,i)}_j = \bar{y}^{(k,i)}(x_2) = \bar{y}^{(k,i)}(t, x_2) = \bar{y}^{(k,i)}_j, \quad j = 1, 2, 3, 4, \quad (x_1, x_2) \in \omega_{k,i}, 
\]

where  
\[
\bar{y}^{(k,i)}_j = [ \alpha^{(k,i)}_{j,1} \quad \alpha^{(k,i)}_{j,2} \quad \alpha^{(k,i)}_{j,3} \quad \alpha^{(k,i)}_{j,4} ]^T; \quad \bar{t} = [ 1 \quad t \quad t^2 \quad t^3 ]^T. 
\]

The relation for  
\[
z^{(k,i)}_j = \partial_1 y^{(k,i)}(x_1, x_2) = h^{-1} \partial y^{(k,i)}(t, x_2) / \partial t = h^{-1} [ \alpha^{(k,i)}_{j,2} + 2 \alpha^{(k,i)}_{j,3} t + 3 \alpha^{(k,i)}_{j,4} t^2 ], 
\]

The relation (19) can be represented in the global matrix form:  
\[
y^{(k,i)}_j = \bar{t}^T C_i \bar{y}^{(k,i)}_j, \quad \bar{y}^{(k,i)}_j = C_i \bar{y}^{(k,i)}_j, \quad j = 1, 2, 3, 4, \quad (x_1, x_2) \in \omega_{k,i}, 
\]

where  
\[
C_i = \tilde{C} C; \quad \tilde{C} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1/h_i & 0 & 0 \\ 0 & 0 & 0 & 1/h_i \\ 0 & 0 & 0 & 1 \\ \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 2 & 3 \\ \end{bmatrix}. 
\]

Besides, we can rewrite (24) in the following form:  
\[
y^{(k,i)}_j = N_i(t) \bar{y}^{(k,i)}_j, \quad j = 1, 2, 3, 4, \quad (x_1, x_2) \in \omega_{k,i}, 
\]

where  
\[
N_i = N_i(t) = \tilde{t}^T C_i^{-1} = [ N_{i,1} \quad N_{i,2} \quad N_{i,3} \quad N_{i,4} ], 
\]

and the elements of the matrix  
\[N_i(t)\]  
are the shape function of DCFE
\[ N_{i,1}(t) = 1 - 3t^2 + 2t^3; \quad N_{i,2}(t) = h_i(t - 2t^2 + t^3); \quad N_{i,3}(t) = 3t^2 - 2t^3; \quad N_{i,4}(t) = h_i(-t^2 + t^3), \quad (29) \]

The partial derivatives of the unknown functions with respect to variables \( x_i \) and \( x_j \) can be obtained in the following form (where \( \xi, \eta \in (0,1) \)):

- the first order partial derivatives
  \[ \partial_j \bar{y}^{(k)}(x_i, x_j) = \bar{z}^{(k)}(x_i, x_j) = h_i^{-1} N_i'(t) \bar{y}^{(k,j)}(x_j), \quad j = 1, 2, 3, 4, \]
  \[ \partial_j \bar{y}^{(k)}(x_i, x_j) = N_i(t) \partial_j \bar{y}^{(k,i)}(x_j) = N_i(t) \bar{y}^{(k,i)}(x_j), \quad j = 1, 2, 3, \]

- the second order partial derivatives
  \[ \partial_i \partial_j \bar{y}^{(k)}(x_i, x_j) = \partial_i \bar{z}^{(k)}(x_i, x_j) = h_i^{-2} N_i''(t) \bar{y}^{(k,j)}(x_j), \quad j = 1, 2, 3, 4, \]
  \[ \partial_i \partial_j \bar{y}^{(k)}(x_i, x_j) = N_i(t) \partial_j \bar{y}^{(k,j)}(x_j) = N_i(t) \bar{y}^{(k,j)}(x_j), \quad j = 1, 2, \]

- the third order partial derivatives
  \[ \partial_i \partial_j \bar{y}^{(k)}(x_i, x_j) = \partial_i \bar{z}^{(k)}(x_i, x_j) = h_i^{-3} N_i'''(t) \bar{y}^{(k,j)}(x_j), \quad j = 1, 2, 3, 4, \]
  \[ \partial_i \partial_j \bar{y}^{(k)}(x_i, x_j) = N_i(t) \partial_j \bar{y}^{(k,j)}(x_j) = N_i(t) \bar{y}^{(k,j)}(x_j), \quad j = 1,2, \]

where

\[ N_i'(t) = \frac{d}{dt} N_i(t) = [N_{i,1}' \quad N_{i,2}' \quad N_{i,3}' \quad N_{i,4}'], \]
\[ N_i''(t) = \frac{d^2}{dt^2} N_i(t) = [N_{i,1}'' \quad N_{i,2}'' \quad N_{i,3}'' \quad N_{i,4}''], \]
\[ N_i'''(t) = \frac{d^3}{dt^3} N_i(t) = [N_{i,1}''' \quad N_{i,2}''' \quad N_{i,3}''' \quad N_{i,4}''']. \]

The corresponding nodal derivatives of (40)-(48) can be obtained by averaging as follows:

- the first order partial derivatives
  \[ \partial_i \bar{y}^{(k)}(x_i, x_j) = \bar{z}^{(k,i)}, \quad i = 1, 2, ..., N, \quad j = 1, 2, 3, 4, \]
  \[ \partial_j \bar{y}^{(k)}(x_j, x_i) = \bar{y}^{(k,i)}, \quad i = 1, 2, ..., N, \quad j = 1, 2, 3, \]

- the second order partial derivatives
where

\[ \partial^2 N_{ij}(x_1, x_2) = (\theta_{k,j} + \theta_{l,j})^{-1}[h_{j}^{-3} \theta_{k,j} N''_{ij}(1) \vec{y}^{(k,j)}_{j}(x_1, x_2) + h_{j}^{-3} \theta_{l,j} N''_{ij}(0) \vec{y}^{(k,j)}_{j}(x_1, x_2)], \]

\[ i = 2, ..., N-1, j = 1, 2, 3, 4; \]

\[ \partial^2 N_{ij}(x_1, x_2) = (\theta_{k,j} + \theta_{l,j})^{-1}[h_{j}^{-3} \theta_{k,j} N''_{ij}(1) \vec{y}^{(k,j)}_{j}(x_1, x_2) + h_{j}^{-3} \theta_{l,j} N''_{ij}(0) \vec{y}^{(k,j)}_{j}(x_1, x_2)], \]

\[ i = 2, ..., N-1, j = 1, 2, 3, 4; \]

\[ \partial^2 N_{ij}(x_1, x_2) = (\theta_{k,j} + \theta_{l,j})^{-1}[h_{j}^{-3} \theta_{k,j} N''_{ij}(1) \vec{y}^{(k,j)}_{j}(x_1, x_2) + h_{j}^{-3} \theta_{l,j} N''_{ij}(0) \vec{y}^{(k,j)}_{j}(x_1, x_2)], \]

\[ i = 2, ..., N-1, j = 1, 2, 3; \]

\[ \partial^2 N_{ij}(x_1, x_2) = (\theta_{k,j} + \theta_{l,j})^{-1}[h_{j}^{-3} \theta_{k,j} N''_{ij}(1) \vec{y}^{(k,j)}_{j}(x_1, x_2) + h_{j}^{-3} \theta_{l,j} N''_{ij}(0) \vec{y}^{(k,j)}_{j}(x_1, x_2)], \]

\[ i = 2, ..., N-1, j = 1, 2, 3, 4; \]

\[ \partial^2 N_{ij}(x_1, x_2) = (\theta_{k,j} + \theta_{l,j})^{-1}[h_{j}^{-3} \theta_{k,j} N''_{ij}(1) \vec{y}^{(k,j)}_{j}(x_1, x_2) + h_{j}^{-3} \theta_{l,j} N''_{ij}(0) \vec{y}^{(k,j)}_{j}(x_1, x_2)], \]

\[ i = 2, ..., N-1, j = 1, 2, 3; \]

\[ \partial^2 N_{ij}(x_1, x_2) = (\theta_{k,j} + \theta_{l,j})^{-1}[h_{j}^{-3} \theta_{k,j} N''_{ij}(1) \vec{y}^{(k,j)}_{j}(x_1, x_2) + h_{j}^{-3} \theta_{l,j} N''_{ij}(0) \vec{y}^{(k,j)}_{j}(x_1, x_2)], \]

\[ i = 2, ..., N-1, j = 1, 2, 3; \]

\[ \partial^2 N_{ij}(x_1, x_2) = (\theta_{k,j} + \theta_{l,j})^{-1}[h_{j}^{-3} \theta_{k,j} N''_{ij}(1) \vec{y}^{(k,j)}_{j}(x_1, x_2) + h_{j}^{-3} \theta_{l,j} N''_{ij}(0) \vec{y}^{(k,j)}_{j}(x_1, x_2)], \]

\[ i = 2, ..., N-1, j = 1, 2, 3; \]

where \( \theta_{k,j} = 0, \theta_{l,j} = 1, i = 2, 3, ..., N-1 \) for the nodes which have no materials in the left-hand side (left boundary); \( \theta_{k,j} = 1, \theta_{l,j} = 0, i = 2, 3, ..., N-1 \) for the nodes which have no materials in the right-hand side (right boundary); \( \theta_{k,j} = \theta_{l,j} = 1, i = 2, 3, ..., N-1 \) for the intermediates nodes.

The internal forces of the plate can be determined with the use of (29)-(37) by formulas:

- bending moments

\[ M_1^{(k)}(x_1, x_2) = -D_{k,j}[\partial^2 y_{1,j}^{(k)}(x_1, x_2) + v_{k,j} \partial_2 y_{2,j}^{(k)}(x_1, x_2)] = -D_{k,j}[\partial^2 y_{1,j}^{(k)}(x_1, x_2) + v_{k,j} y_{1,j}^{(k)}(x_1, x_2)]; \]

\[ M_2^{(k)}(x_1, x_2) = -D_{k,j}[v_{k,j} \partial_1 y_{1,j}^{(k)}(x_1, x_2) + \partial_2 y_{2,j}^{(k)}(x_1, x_2)] = -D_{k,j}[v_{k,j} \partial_1 y_{1,j}^{(k)}(x_1, x_2) + y_{1,j}^{(k)}(x_1, x_2)]; \]

- shear forces and torque

\[ Q_1^{(k)}(x_1, x_2) = -D_{k,j}[\partial_1 y_{1,j}^{(k)}(x_1, x_2) + \partial_2 y_{2,j}^{(k)}(x_1, x_2)] = -D_{k,j}[\partial_1 y_{1,j}^{(k)}(x_1, x_2) + z_{1,j}^{(k)}(x_1, x_2)]; \]

\[ Q_2^{(k)}(x_1, x_2) = -D_{k,j}[v_{k,j} \partial_1 y_{1,j}^{(k)}(x_1, x_2) + \partial_2 y_{2,j}^{(k)}(x_1, x_2)] = -D_{k,j}[v_{k,j} \partial_1 y_{1,j}^{(k)}(x_1, x_2) + y_{1,j}^{(k)}(x_1, x_2)]; \]

\[ Q_3^{(k)}(x_1, x_2) = -D_{k,j}[\partial_2 y_{2,j}^{(k)}(x_1, x_2) + \partial_1 y_{1,j}^{(k)}(x_1, x_2)] = -D_{k,j}[\partial_2 y_{2,j}^{(k)}(x_1, x_2) + z_{1,j}^{(k)}(x_1, x_2)]; \]

\[ Q_4^{(k)}(x_1, x_2) = -D_{k,j}[v_{k,j} \partial_2 y_{2,j}^{(k)}(x_1, x_2) + \partial_1 y_{1,j}^{(k)}(x_1, x_2)] = -D_{k,j}[v_{k,j} \partial_2 y_{2,j}^{(k)}(x_1, x_2) + y_{1,j}^{(k)}(x_1, x_2)]; \]
\[ Q^{(k)}_2(x_1, x_2) = -D_{x,j} [\partial^2_2 \partial^1_1 y^{(k)}_1(x_1, x_2) + \partial^3_1 y^{(k)}_1(x_1, x_2)] = -D_{x,j} [\partial^2_2 y^{(k)}_2(x_1, x_2) + y^{(k)}_4(x_1, x_2)]; \]

\[ M^{(k)}_{1,2}(x_1, x_2) = M^{(k)}_{2,1}(x_1, x_2) = -D_{x,j} (1 - \nu) \partial^1_1 \partial^2_2 y^{(k)}_1(x_1, x_2) = -D_{x,j} (1 - \nu) \partial^2_2 y^{(k)}_2(x_1, x_2), \]

The energy functional of the structure can be represented as a sum of corresponding functionals, defined in discrete-continual finite elements. Considering the correspondence between continual operators (3) and discrete-continual analogy for the arbitrary DCFE we have:

\[
L_{k,4} = \overline{D}_k \implies K_{k,4}^{(i)}; \quad \overline{L}_{k,0} = -\partial^2_1 \overline{D}_k \partial^2_1 + \theta_k k_{s,k} \implies \overline{K}_{k,0}^{(i)} = K_{k,0}^{(i)} + K_{k,0,2}^{(i)},
\]

\[
\overline{L}_{k,2} = -[\partial^1_1 \overline{D}_k v_k + 2 \partial_1 \overline{D}_k (1 - \nu) \partial^1_1 + \overline{D}_k v_k \partial^2_2] + 2 \partial_1 \theta_k t_{s,k} \partial^1_1 \implies \overline{K}_{k,2}^{(i)} = K_{k,2,1}^{(i)} + K_{k,2,2}^{(i)},
\]

where

\[
K_{k,4}^{(i)} = \overline{D}_k, h_{k,i} (N^0_i) \left[ \int_0^T \tau^T dt \right] N^0_i; \quad K_{k,2,1}^{(i)} = -\frac{\overline{D}_k, v_{k,i}}{h_i} (N^0_i) \left[ \int_0^T \tau^T dt \right] N^0_i,
\]

\[
K_{k,2,2}^{(i)} = 2 \frac{\overline{D}_k, (1 - \nu) v_{k,i} + \tau_{s,k,i}}{h_i} (N^0_i) \left[ \int_0^T \tau (\tau^*)^T dt \right] N^0_i; \quad K_{k,2,3}^{(i)} = \frac{\overline{D}_k, v_{k,i}}{h_i} (N^0_i) \left[ \int_0^T \tau (\tau^*)^T dt \right] N^0_i,
\]

\[
K_{k,0,1}^{(i)} = -\frac{\overline{D}_k, v_{k,i}}{h_i} (N^0_i) \left[ \int_0^T \tau (\tau^*)^T dt \right] N^0_i; \quad K_{k,0,2}^{(i)} = k_{s,k} h_{k,i} (N^0_i) \left[ \int_0^T \tau^T dt \right] N^0_i,
\]

\[
\overline{\tau} = \partial \overline{\tau} / \partial t; \quad \overline{\tau}^* = \partial^2 \overline{\tau} / \partial t^2.
\]

Finally the following formulas for element stiffness matrices were obtained after calculation of integrals in (62)-(64):

\[
K_{k,4}^{(i)} = \frac{\overline{D}_k, h_i}{420} \left[ \begin{array}{cccc}
156 & 22h_i & 54 & -13h_i \\
22h_i & 4h_i^2 & 13h_i & -3h_i^2 \\
54 & 13h_i & 156 & -22h_i \\
-13h_i & -3h_i^2 & -22h_i & 4h_i^2 \\
\end{array} \right],
\]

\[
K_{k,2,1}^{(i)} = \frac{\overline{D}_k, v_{k,i}}{30h_i} \left[ \begin{array}{cccc}
36 & 3h_i & -36 & 3h_i \\
3h_i & 4h_i^2 & -3h_i & -h_i^2 \\
33h_i & 4h_i^2 & -3h_i & -h_i^2 \\
-36 & -3h_i & 36 & -3h_i \\
\end{array} \right],
\]

\[
K_{k,2,2}^{(i)} = \frac{\overline{D}_k, (1 - \nu) v_{k,i} + \tau_{s,k,i}}{15h_i} \left[ \begin{array}{cccc}
36 & 3h_i & -36 & 3h_i \\
3h_i & 4h_i^2 & -3h_i & -h_i^2 \\
-36 & -33h_i & 36 & -33h_i \\
3h_i & -h_i^2 & -33h_i & 4h_i^2 \\
\end{array} \right],
\]

\[
K_{k,2,3}^{(i)} = \frac{\overline{D}_k, v_{k,i}}{30h_i} \left[ \begin{array}{cccc}
36 & 33h_i & -36 & 3h_i \\
3h_i & 4h_i^2 & -3h_i & -h_i^2 \\
-36 & -3h_i & 36 & -33h_i \\
3h_i & -h_i^2 & -3h_i & 4h_i^2 \\
\end{array} \right],
\]
Let the load applied to the structure be a set of concentrated forces (or this load is reduced to such set) \[27\]. Then, the corresponding vector of loads in the \(i\)-th node has the form:

\[
K_{k,0}^{(i)} = -2 \frac{D_{k,i}}{h_i^3} \begin{bmatrix}
6 & 3h_i & -6 & 3h_i \\
3h_i & 2h_i^2 & -3h_i & h_i^2 \\
-6 & -3h_i & 6 & -3h_i \\
3h_i & h_i^2 & -3h_i & 2h_i^2
\end{bmatrix}, \quad (69)
\]

\[
K_{k,0}^{(j)} = \frac{\theta_{k,j} k_{k,k,i} h_i}{420} \begin{bmatrix}
156 & 22h_i & 54 & -13h_i \\
22h_i & 4h_i^2 & 13h_i & -3h_i^2 \\
54 & 13h_i & 156 & -22h_i \\
-13h_i & -3h_i^2 & -22h_i & 4h_i^2
\end{bmatrix}, \quad (70)
\]

Therefore, the load vector in all nodes of a DCFE is defined by formula:

\[
\bar{R}_n^{(k,i)} = \bar{R}_n^{(k,i)}(x_2) = [ (\bar{R}_{n,1}^{(k,i)})^T (\bar{R}_{n,2}^{(k,i)})^T (\bar{R}_{n,3}^{(k,i)})^T (\bar{R}_{n,4}^{(k,i)})^T ]^T, \quad (71)
\]

where

\[
\bar{R}_{n,i}^{(k,i)} = \bar{R}_{n,i}^{(k,i)}(x_1) = [ \bar{R}_{n,i}^{(k,i)} ]^T, \quad j = 1, 2, 3, 4, \quad (72)
\]

Therefore, the load vector in all nodes of a DCFE is defined by formula:

\[
\bar{R}_j^{(k,i)} = \bar{R}_j^{(k,i)}(x_2) = [ (\bar{R}_{j,1}^{(k,i)})^T (\bar{R}_{j,2}^{(k,i)})^T (\bar{R}_{j,3}^{(k,i)})^T (\bar{R}_{j,4}^{(k,i)})^T ]^T, \quad (73)
\]

where

\[
\bar{R}_{j,i}^{(k,i)} = \bar{R}_{j,i}^{(k,i)}(x_1) = [ (\bar{R}_{j,i}^{(k,i)}) ]^T, \quad j = 1, 2, 3, 4, \quad (74)
\]

4. Results and discussions

Thus, initial continual formulation of the considering problem of plate analysis within the method of extended domain is presented. Basic formulas of approximation of deflection of the plate, bending moments, shear forces and torque are obtained. Corresponding element stiffness matrices and load vectors are constructed. It should be noted that along the direction of piecewise constancy of physico-geometric parameters of plate (“basic” direction) problem remains continual, we use finite element approximation only along other direction.

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