Hamiltonians with Riesz Bases of Generalised Eigenvectors and Riccati Equations

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Abstract. An algebraic Riccati equation for linear operators is studied, which arises in systems theory. For the case that all involved operators are unbounded, the existence of infinitely many selfadjoint solutions is shown. To this end, invariant graph subspaces of the associated Hamiltonian operator matrix are constructed by means of a Riesz basis with parentheses of generalised eigenvectors and two indefinite inner products. Under additional assumptions, the existence and a representation of all bounded solutions is obtained. The theory is applied to Riccati equations of differential operators.

Keywords. Riccati equation, Hamiltonian operator matrix, Riesz basis of generalised eigenvectors, invariant subspace, indefinite inner product.

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1 Introduction

We consider the algebraic Riccati equation

$$A^*X + XA + XBX - C = 0$$  \hfill (1)

for linear operators on a Hilbert space $H$ where $B, C$ are selfadjoint and nonnegative. In particular, we study the case where $B$ and $C$ are unbounded. Riccati equations of type (1) are a key tool in systems theory, see e.g. [8] and the references therein. Unbounded $B$ and $C$ appear e.g. in [20, 25, 30].

It is well known that solutions $X$ of (1) are in one-to-one correspondence with graph subspaces which are invariant under the operator matrix

$$T = \begin{pmatrix} A & B \\ C & -A^* \end{pmatrix},$$

the so-called Hamiltonian. This correspondence was extensively studied in the finite-dimensional setting and led to a complete description of all solutions of the Riccati equation, see e.g. [14, 21] and [24]. In the infinite-dimensional setting with $B, C$ bounded, the invariant subspace approach was used by Kuiper and Zwart [16] for Riesz-spectral $T$ and by Langer, Ran and van de Rotten [18] for dichotomous $T$ (see also [6]).
We extend these results to the case where $B$ and $C$ are unbounded: For Hamiltonians with a Riesz basis with parentheses of generalised eigenvectors we show the existence of infinitely many selfadjoint solutions of (1). Note that the concept of a Riesz basis with parentheses of generalised eigenvectors includes Riesz-spectral operators, and it also allows for operators which are not dichotomous.

In systems theory, solutions of (1) which are bounded and nonnegative are of particular importance. For the case that $T$ has a Riesz basis of generalised eigenvectors, that its spectrum is contained in a strip around the imaginary axis, and that $B$ and $C$ are uniformly positive, we prove that there are infinitely many bounded, selfadjoint, boundedly invertible solutions, among them a nonnegative one $X_+$ and a nonpositive one $X_-$. Moreover, for every bounded selfadjoint solution $X$ we prove the relations

$$X_- \leq X \leq X_+ \quad \text{and} \quad X = X_+ P + X_- (I - P), \quad \text{(2)}$$

where $P$ is an appropriate projection.

Bounded nonnegative solutions of (1) were obtained in [16, 18] without the assumption of uniform positivity of $B, C$. However, in [18] the spectrum $\sigma(A)$ of $A$ was restricted to a sector in the open left half-plane while here $\sigma(A)$ may also contain points in the closed right half-plane. In [16] conditions for the existence of solutions were formulated in terms of the eigenvectors of $T$ while we impose conditions on the operators $A, B, C$ only. In the system theoretic setting, the relations (2) were derived in [7, 23], yet under the explicit assumption of the existence of $X_-$. For general block operator matrices, invariant graph subspaces are connected to solutions of a corresponding Riccati equation too. This was exploited in [19, 26] for certain dichotomous operator matrices and in [15] for selfadjoint ones. We also mention that, in systems theory, nonnegative solutions of (1) are constructed by minimising a quadratic functional, see e.g. [8].

The structure of this article is as follows: In Sections 2 and 3 we study the concept of a Riesz basis of subspaces which is finitely spectral for a linear operator on a Hilbert space. Such a Riesz basis consists of finite-dimensional invariant subspaces, and it yields many non-trivial infinite-dimensional invariant subspaces, which we call compatible, see Corollary 3.9. Up to certain technical details, a finitely spectral Riesz basis of subspaces is equivalent to a Riesz basis with parentheses of generalised eigenvectors, see Remark 3.10. Here we use the basis of subspaces notion since it is more convenient for our purposes. For the relation to dichotomous operators, see Remark 3.10.

In Theorem 3.7 we use perturbation theory to prove a general existence result for finitely spectral Riesz bases of subspaces and apply it to Hamiltonians in Theorem 4.4, Theorem 4.5 even yields a Riesz basis of eigenvectors and finitely many generalised eigenvectors. On the other hand, there is a huge literature on Riesz bases (with or without parentheses) of eigenvectors for various types of operators, e.g. [13, 33, 34]; all these provide examples for finitely spectral Riesz bases of subspaces.

In Section 4 we use ideas from [13] and consider two indefinite inner products with fundamental symmetries $J_1$ and $J_2$ which are associated with the Hamiltonian: $T$ is $J_1$-skew-symmetric and $J_2$-accretive. This implies the symmetry of the spectrum of $T$ with respect to the imaginary axis and also yields a characterisation of the purely imaginary eigenvalues. In Section 5 we then construct hypermaximal
$J_1$-neutral as well as $J_2$-nonnegative and -nonpositive compatible subspaces; see Theorem 5.2 and Proposition 5.6.

The main existence theorems for solutions of (1) are presented in Sections 6 and 7: In Theorem 6.3 we establish conditions on $T$ such that every hypermaximal $J_1$-neutral compatible subspace is the graph of a selfadjoint solution of (1). $J_2$-nonnegative and -nonpositive subspaces yield nonnegative and nonpositive solutions. Corollary 6.6 provides a sufficient condition for the existence of infinitely many selfadjoint solutions. All these solutions are unbounded in general, and therefore the Riccati equation takes a slightly different form, see also Proposition 6.1 and Example 8.1. The existence of bounded solutions and the relations (2) are proved in Theorem 7.7. Finally note that the graph of an arbitrary solution of (1) is $T$-invariant, but not necessarily a compatible subspace, compare Theorem 7.4 and Example 8.3.

2 Riesz bases of subspaces

We recall the closely related concepts of Riesz bases, Riesz bases with parentheses, and Riesz bases of subspaces, see [29, §1], [12, Chapter VI], [27, §15] and [31, §2] for more details.

Let $V$ be a separable Hilbert space. We denote the subspace generated by a family $(V_\lambda)_{\lambda \in \Lambda}$ of subspaces $V_\lambda \subset V$ by

$$
\sum_{\lambda \in \Lambda} V_\lambda = \{ x_{\lambda_1} + \cdots + x_{\lambda_n} \mid x_{\lambda_j} \in V_{\lambda_j}, \lambda_j \in \Lambda, n \in \mathbb{N} \}.
$$

The family is said to be complete if $\sum_{\lambda \in \Lambda} V_\lambda \subset V$ is dense.

**Definition 2.1** Let $V$ be a separable Hilbert space.

(i) A sequence $(v_k)_{k \in \mathbb{N}}$ in $V$ is called a Riesz basis of $V$ if there is an isomorphism $\Phi : V \to V$ such that $(\Phi v_k)_{k \in \mathbb{N}}$ is an orthonormal basis of $V$.

(ii) A sequence of closed subspaces $(V_k)_{k \in \mathbb{N}}$ of $V$ is called a Riesz basis of subspaces of $V$ if there is an isomorphism $\Phi : V \to V$ such that $(\Phi(V_k))_{k \in \mathbb{N}}$ is a complete system of pairwise orthogonal subspaces.

The sequence $(v_k)_{k \in \mathbb{N}}$ is a Riesz basis if and only if span$\{v_k\} \subset V$ is dense and there are constants $m, M > 0$ such that

$$
m \sum_{k=0}^{n} |\alpha_k|^2 \leq \left\| \sum_{k=0}^{n} \alpha_k v_k \right\|^2 \leq M \sum_{k=0}^{n} |\alpha_k|^2, \quad \alpha_k \in \mathbb{C}, \ n \in \mathbb{N}.
$$

(3)

In this case every $x \in V$ has a unique representation $x = \sum_{k=0}^{\infty} \alpha_k v_k$, $\alpha_k \in \mathbb{C}$, where the convergence of the series is unconditional. The sequence of closed subspaces $(V_k)_{k \in \mathbb{N}}$ is a Riesz basis of subspaces of $V$ if and only if $(V_k)_{k \in \mathbb{N}}$ is complete and there exists a constant $c \geq 1$ such that

$$
c^{-1} \sum_{k \in F} \|x_k\|^2 \leq \left\| \sum_{k \in F} x_k \right\|^2 \leq c \sum_{k \in F} \|x_k\|^2
$$

(4)

for all finite subsets $F \subset \mathbb{N}$ and $x_k \in V_k$. 

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Proposition 2.2 A Riesz basis of subspaces \( (V_k)_{k \in \mathbb{N}} \) has the following properties:

(i) There are projections \( P_k \in L(V) \) onto \( V_k \) satisfying \( P_j P_k = 0 \) for \( j \neq k \) and a constant \( c \geq 1 \) such that

\[
c^{-1} \sum_{k=0}^{\infty} \|P_k x\|^2 \leq \|x\|^2 \leq c \sum_{k=0}^{\infty} \|P_k x\|^2 \quad \text{for all } x \in V. \tag{5}
\]

(ii) If \( x_k \in V_k \) with \( \sum_{k=0}^{\infty} \|x_k\|^2 < \infty \), then the series \( \sum_{k=0}^{\infty} x_k \) converges unconditionally.

(iii) Every \( x \in V \) has a unique expansion \( x = \sum_{k=0}^{\infty} x_k \) with \( x_k \in V_k \), and we have \( x_k = P_k x \).

Proof. The proof is immediate since all assertions hold (with \( c = 1 \)) if the \( V_k \) are pairwise orthogonal, and they continue to hold (with some \( c \geq 1 \) now) if we apply the isomorphism \( \Phi \) from Definition 2.1. \( \square \)

For a Riesz basis of subspaces \( (V_k)_{k \in \mathbb{N}} \), the unique expansion from (iii) yields a decomposition of the space \( V \) into the subspaces \( V_k \), which we denote by

\[
V = \bigoplus_{k \in \mathbb{N}}^2 V_k. \tag{6}
\]

Here, the superscript 2 indicates that, due to (5), the original norm on \( V \) is equivalent to the \( l^2 \)-type norm \( (\sum_{k \in \mathbb{N}} \|P_k x\|^2)^{1/2} \).

Consider now closed subspaces \( U_k \subset V_k \). Then evidently \( (U_k)_{k \in \mathbb{N}} \) is a Riesz basis of subspaces of the closed subspace generated by the \( U_k \), i.e.

\[
\sum_{k \in \mathbb{N}} U_k = \bigoplus_{k \in \mathbb{N}}^2 U_k.
\]

Analogously, for every \( J \subset \mathbb{N} \) we have that \( (V_k)_{k \in J} \) is a Riesz basis of subspaces of \( \bigoplus_{k \in J}^2 V_k \).

**Definition 2.3** Let \( (V_k)_{k \in \mathbb{N}} \) be a Riesz basis of subspaces of \( V \). We say that a subspace \( U \subset V \) is compatible with \( (V_k)_{k \in \mathbb{N}} \) if

\[
U = \bigoplus_{k \in \mathbb{N}}^2 U_k \quad \text{with closed subspaces } U_k \subset V_k.
\]

It is easy to see that, with \( P_k \) as above, \( U \) is compatible with \( (V_k) \) if and only if \( P_k(U) \subset U \); in this case \( U = \bigoplus_{k \in \mathbb{N}}^2 P_k(U) \).

If \( U \) and \( W \) are two subspaces of \( V \) satisfying \( U \cap W = \{0\} \), we say that their sum is algebraic direct, denoted by \( U \oplus W \). We say that the sum is topological direct and write \( U \oplus W \) if the associated projection from \( U + W \) onto \( U \) is bounded. By the closed graph theorem, if \( U \cap W = \{0\} \) and \( U \), \( W \) and \( U \oplus W \) are closed, then in fact \( U \oplus W \) is topological direct.

**Proposition 2.4** Let \( (V_k)_{k \in \mathbb{N}} \) be a Riesz basis of subspaces.

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1 Note here that Definition 2.1 implicitly covers the case of families with arbitrary index set \( J \subset \mathbb{N} \) since \( V_k = \{0\} \) is possible.
(i) If \( V_k = U_k \oplus W_k \) for all \( k \), then the sum

\[
\bigoplus_{k \in \mathbb{N}}^2 U_k + \bigoplus_{k \in \mathbb{N}}^2 W_k \subset V
\]

is algebraic direct and dense.

(ii) For \( J \subset \mathbb{N} \) we have the topological direct sum

\[
V = \bigoplus_{k \in J}^2 V_k \oplus \bigoplus_{k \in \mathbb{N} \setminus J}^2 V_k.
\]

The associated projection onto the first component is given by

\[
P_J : \sum_{k \in \mathbb{N}} x_k \mapsto \sum_{k \in J} x_k, \quad x_k \in V_k, \quad (7)
\]

and satisfies \( \|P_J\| \leq c \), where \( c \) is the constant from (5).

Proof. (i): Let \( U = \bigoplus_{k \in \mathbb{N}}^2 U_k, \ W = \bigoplus_{k \in \mathbb{N}}^2 W_k \), and \( x \in U \cap W \). We expand \( x \) in the Riesz bases \( (U_k) \) of \( U \) and \( (W_k) \) of \( W \): \( x = \sum_k u_k = \sum_k w_k \) with \( u_k \in U_k, \ w_k \in W_k \). As these are also expansions of \( x \) in the Riesz basis \( (V_k) \), we obtain \( u_k = w_k \) and thus \( u_k = 0 \) and \( x = 0 \). The sum \( U + W \) is dense in \( V \) since it contains \( \sum_{k \in \mathbb{N}} V_k \).

(ii): From (5) we have the estimate

\[
\left\| \sum_{k \in J} x_k \right\|^2 \leq c \sum_{k \in \mathbb{N}} \|x_k\|^2 \leq c \sum_{k \in \mathbb{N}} \|x_k\|^2 \leq c^2 \sum_{k \in \mathbb{N}} \|x_k\|^2.
\]

This shows that \( P_J \) defined by (7) satisfies \( \|P_J\| \leq c \). Obviously

\[
\mathcal{R}(P_J) = \bigoplus_{k \in J}^2 V_k, \quad \ker P_J = \bigoplus_{k \in \mathbb{N} \setminus J}^2 V_k
\]

and hence the topological direct sum. \( \square \)

Remark 2.5 If \( (V_k)_{k \in \mathbb{N}} \) is a Riesz basis of finite-dimensional subspaces, then we may choose a basis \( (v_{k_1}, \ldots, v_{k_n}) \) in each \( V_k \). The resulting system \( (v_{k,j})_{k,j} \) is called a Riesz basis with parentheses: Every \( x \in V \) has a unique representation

\[
x = \sum_{k=0}^{\infty} \left( \sum_{j=1}^{k_n} \alpha_{k,j} v_{k,j} \right), \quad \alpha_{k,j} \in \mathbb{C},
\]

where the series over \( k \) converges unconditionally.

3 Finitely spectral Riesz bases of subspaces

We recall some concepts for a linear operator \( T \) on a Banach space \( V \), see also [2, 14]. A point \( z \in \mathbb{C} \) is called a point of regular type if \( T - z \) is injective and the inverse \( (T - z)^{-1} \) (defined on \( \mathcal{R}(T - z) \)) is bounded. The set of all points of regular type is denoted by \( r(T) \); it is open and satisfies \( \mathcal{R}(T) \subset r(T) \) and \( \sigma_p(T) \cap r(T) = \emptyset \).
Let $T$ be a closed operator. A subspace $A \subset V$ is called a core for $T$ if for every $x \in \mathcal{D}(T)$ there is a sequence $(x_n)$ in $A$ such that $\lim x_n = x$ and $\lim Tx_n = Tx$.

Finally we denote by $\mathcal{L}(\lambda)$ the space of generalised eigenvectors or root subspace of $T$ corresponding to the eigenvalue $\lambda \in \sigma_p(T)$, i.e.

$$\mathcal{L}(\lambda) = \bigcup_{k \in \mathbb{N}} \ker(T - \lambda)^k.$$

For $\lambda \notin \sigma_p(T)$ we set $\mathcal{L}(\lambda) = \{0\}$. A sequence $x_1, \ldots, x_n \in \mathcal{L}(\lambda)$ is called a Jordan chain if $(T - \lambda)x_k = x_{k-1}$ for $k \geq 2$ and $(T - \lambda)x_1 = 0$.

**Definition 3.1** Let $T$ be a closed operator on a separable Hilbert space $V$. We say that a Riesz basis of subspaces $(V_k)_{k \in \mathbb{N}}$ of $V$ is finitely spectral for $T$ if each $V_k$ is finite-dimensional, $T$-invariant, $V_k \subset \mathcal{D}(T)$, the sets $\sigma(T|_{V_k})$ are pairwise disjoint, and $\sum_{k \in \mathbb{N}} V_k$ is a core for $T$.

**Proposition 3.2** Let $T$ be a closed operator with a finitely spectral Riesz basis of subspaces $(V_k)_{k \in \mathbb{N}}$. Then

$$\mathcal{D}(T) = \left\{ x = \sum_{k \in \mathbb{N}} x_k \big| x_k \in V_k, \sum_{k \in \mathbb{N}} \|Tx_k\|^2 < \infty \right\},$$

$$Tx = \sum_{k \in \mathbb{N}} Tx_k \text{ for } x = \sum_{k \in \mathbb{N}} x_k \in \mathcal{D}(T), x_k \in V_k.$$ 

$T$ is bounded if and only if the restrictions $T|_{V_k}$ are uniformly bounded and in this case (with $c$ from (5))

$$\|T\| \leq c \sup_{k \in \mathbb{N}} \|T|_{V_k}\|.$$ 

**Proof.** Let $P_k$ be the projections onto the $V_k$ corresponding to the Riesz basis.

(i): We derive (8) and (9). First note that for $u \in \sum_k V_k$ we have $P_kTu = TP_ku$ for all $k$ since $u$ is a finite sum of elements from the $T$-invariant subspaces $V_k$. Let now $y \in \mathcal{D}(T)$. Since $\sum_k V_k$ is a core for $T$, there is a sequence $y_n \in \sum_k V_k$ with $y_n \rightarrow y$, $Ty_n \rightarrow Ty$. Since the restriction $T|_{V_k}$ is bounded, we obtain

$$P_kTy = \lim_{n \rightarrow \infty} P_kTy_n = \lim_{n \rightarrow \infty} T|_{V_k} P_ky_n = T|_{V_k} \lim_{n \rightarrow \infty} P_ky_n = TP_ky.$$ 

Hence $\sum_k \|TP_ky\|^2 = \sum_k \|P_kTy\|^2 \leq c\|Ty\|^2 < \infty$ and

$$y = \sum_k P_ky \in \left\{ x = \sum_k x_k \big| x_k \in V_k, \sum_k \|Tx_k\|^2 < \infty \right\} \text{ with}$$

$$Ty = \sum_k P_kTy = \sum_k TP_ky.$$ 

If on the other hand $x = \sum_k x_k$ with $x_k \in V_k$, $\sum_k \|Tx_k\|^2 < \infty$, then

$$\mathcal{D}(T) \ni \sum_{k=0}^n x_k \rightarrow x \text{ and } T \sum_{k=0}^n x_k = \sum_{k=0}^n Tx_k \rightarrow \sum_{k=0}^\infty Tx_k.$$ 

Hence $x \in \mathcal{D}(T)$ since $T$ is closed.
(ii): Suppose that \( L = \sup_k \| T|_{V_k} \| < \infty \). Then for \( x = \sum_k x_k \in D(T) \):
\[
\|Tx\|^2 = \| \sum_k T|_{V_k} x_k \|^2 \leq c \sum_k \| T|_{V_k} x_k \|^2 \leq cL^2 \sum_k \| x_k \|^2 \leq c^2L^2 \|x\|^2;
\]
thus \( T \) is bounded with norm \( \leq cL \).

For the case that the \( V_k \) are pairwise orthogonal and possibly infinite-di-mensional, the spectrum of an operator defined by (S). (S) was calculated by Davies [9, Theorem 8.1.12]. We obtain:

**Corollary 3.3** Let \( T \) be a closed operator with a finitely spectral Riesz basis of subspaces \( (V_k)_{k \in \mathbb{N}} \). Then
\[
\sigma_p(T) = \bigcup_{k \in \mathbb{N}} \sigma(T|_{V_k}), \quad (10)
\]
\[
V_k = \sum_{\lambda \in \sigma(T|_{V_k})} \mathcal{L}(\lambda), \quad (11)
\]
\[
\varrho(T) = r(T) = \left\{ z \in \mathbb{C} \setminus \sigma_p(T) \bigg| \sup_{k \in \mathbb{N}} \| (T|_{V_k} - z)^{-1} \| < \infty \right\}. \quad (12)
\]

**Proof.** For the identities (10) and (11), note that if \( \lambda \in \sigma_p(T) \) and \( x = \sum_{j \in \mathbb{N}} x_j \in \mathcal{L}(\lambda) \setminus \{0\}, x_j \in V_j \), then by (10)
\[
0 = (T - \lambda)^n x = \sum_{j \in \mathbb{N}} (T|_{V_j} - \lambda)^n x_j
\]
for some \( n \in \mathbb{N} \), which implies \( (T|_{V_j} - \lambda)^n x_j = 0 \) for all \( j \). Since \( x_k \neq 0 \) for some \( k \), we obtain \( \lambda \in \sigma(T|_{V_k}) \). As the \( \sigma(T|_{V_j}) \) are disjoint, we have \( \lambda \notin \sigma(T|_{V_j}) \) and hence \( x_j = 0 \) for \( j \neq k \), i.e. \( x \in V_k \).

To show (12), first note that if \( z \in r(T) \), then for every \( k \in \mathbb{N} \), \( (T|_{V_k} - z)^{-1} \) exists and is a restriction of \( (T - z)^{-1} \), thus \( \sup_k \| (T|_{V_k} - z)^{-1} \| \leq \| (T - z)^{-1} \| < \infty \). Furthermore, if \( z \in \mathbb{C} \setminus \sigma_p(T) \) with \( \sup_k \| (T|_{V_k} - z)^{-1} \| < \infty \), then
\[
S : \sum_{k \in \mathbb{N}} x_k \mapsto \sum_{k \in \mathbb{N}} (T|_{V_k} - z)^{-1} x_k
\]
defines a bounded operator \( S : V \to V \) satisfying \( (T - z)Sx = x \) for all \( x \in V \). Consequently \( z \in \varrho(T) \) with \( (T - z)^{-1} = S \). \( \square \)

In some situations, the conditions on the closedness and the core in Definition 3.1 are automatically fulfilled:

**Proposition 3.4** Let \( T \) be an operator on \( V \), \( (V_k)_{k \in \mathbb{N}} \) a Riesz basis of finite-dimensional, \( T \)-invariant subspaces of \( V \), \( V_k \subset D(T) \) for all \( k \), and \( \sigma(T|_{V_k}) \) pairwise disjoint. Then:
(i) \( T_0 = T|_{\sum_k V_k} \) is closable and \( (V_k)_{k \in \mathbb{N}} \) is finitely spectral for \( T_0 \).
(ii) If \( r(T) \neq \emptyset \), then \( T \) is closable and \( (V_k)_{k \in \mathbb{N}} \) is finitely spectral for \( T \).
Proof. (i): Let $x_n \in D(T_0) = \sum_k V_k$ with $\lim x_n = 0$ and $\lim T_0 x_n = y$. As in the proof of Proposition 3.2 we have

$$P_k y = \lim_{n \to \infty} P_k T_0 x_n = \lim_{n \to \infty} T|_{V_k} P_k x_n = T|_{V_k} P_k \lim_{n \to \infty} x_n = 0$$

for every $k \in \mathbb{N}$ and hence $y = 0$; $T_0$ is closable. The other assertion is now immediate.

(ii): In view of (i) it suffices to show $T \subseteq T_0$; for $T$ is closable then, and from $T_0 \subseteq T$ we conclude $T_0 = T$. Let $x \in D(T)$ and $z \in r(T)$. Using the Riesz basis $(V_k)$, we have the expansion $(T - z)x = \sum_{k=0}^{\infty} y_k$ with $y_k \in V_k$. Since $T - z$ is injective and $V_k$ is finite-dimensional and $T$-invariant, $T - z$ maps $V_k$ onto $V_k$. We can thus set $x_k = (T - z)^{-1} y_k \in V_k$ and obtain $x = \sum_{k=0}^{\infty} x_k$ by the boundedness of $(T - z)^{-1}$. Consequently

$$D(T_0) \ni \sum_{k=0}^{n} x_k \to x \quad \text{and} \quad (T_0 - z) \sum_{k=0}^{n} x_k = \sum_{k=0}^{n} y_k \to (T - z)x$$

as $n \to \infty$, i.e., $x \in D(T_0)$ and $T_0 x = Tx$. □

The notion of a finitely spectral Riesz basis of subspaces contains many other types of bases related to eigenvectors and the spectrum as special cases:

**Proposition 3.5** Let $T$ be closed with $r(T) \neq \emptyset$ and $\dim L(\lambda) < \infty$ for all $\lambda \in \sigma_p(T)$. Then for the assertions

(i) $T$ has a finitely spectral Riesz basis of subspaces,

(ii) the root subspaces $L(\lambda)$ of $T$ form a Riesz basis,

(iii) $T$ has a Riesz basis of Jordan chains,

we have $(iii) \Rightarrow (ii) \Rightarrow (i)$. 

**Proof.** (ii)⇒(i) is trivial. For (iii)⇒(ii) consider for each eigenvalue $\lambda \in \sigma_p(T)$ the subspace $V_\lambda$ generated by all Jordan chains from the basis which correspond to $\lambda$. Then $(V_\lambda)_{\lambda \in \sigma_p(T)}$ is a Riesz basis of subspaces and $V_\lambda = L(\lambda)$. □

**Remark 3.6** In the situation of the previous proposition, assertion (i) is equivalent to the existence of a *Riesz basis with parentheses of Jordan chains* with the additional property that Jordan chains corresponding to the same eigenvalue lie inside the same parenthesis.

If $T$ has a compact resolvent, then (ii) holds if and only if $T$ is a *spectral operator* in the sense of Dunford, see [11] [31].

A closed operator $T$ is called *Riesz-spectral* [8] [16] if all its eigenvalues are simple, $T$ has a Riesz basis of eigenvectors, and $\sigma_p(T)$ is totally disconnected. So if $T$ is Riesz-spectral then (iii) holds.

For an operator $G$ let $N(r, G)$ be the sum of the algebraic multiplicities $\dim L(\lambda)$ for all $\lambda \in \sigma_p(G)$ with $|\lambda| \leq r$. An operator $S$ is called *p-subordinate* to $G$ with $0 \leq p \leq 1$ if $D(G) \subseteq D(S)$ and there exists $b \geq 0$ such that

$$\|Sx\| \leq b\|x\|^{1-p}\|Gx\|^p$$

for $x \in D(G)$. 8
Theorem 3.7 Let $G$ be a normal operator with compact resolvent whose eigenvalues lie on a finite number of rays $e^{i\theta j} \mathbb{R}_{\geq 0}$, $0 \leq \theta j < 2\pi$, from the origin. Let $S$ be $p$-subordinate to $G$ with $0 \leq p < 1$. If
\[
\liminf_{r \to \infty} N(r, G) \frac{1}{r^{1-p}} < \infty,
\]
then $T = G + S$ has a compact resolvent and a finitely spectral Riesz basis of subspaces $(V_k)_{k \in \mathbb{N}}$.

Proof. See Theorems 4.5 and 6.1 in [32]. In particular, note that the $V_k$ were constructed as the ranges of Riesz projections associated with disjoint parts of $\sigma(T)$, and hence the $\sigma(T|V_k)$ are disjoint. □

Now we study invariant subspaces with respect to a finitely spectral Riesz basis of subspaces.

Lemma 3.8 Let $T$ be a closed operator with a finitely spectral Riesz basis of subspaces $(V_k)_{k \in \mathbb{N}}$. For a compatible subspace $U = \bigoplus_{k \in \mathbb{N}} U_k$, $U_k \subset V_k$, the following assertions are equivalent:

(i) $U$ is $T$-invariant;

(ii) all $U_k$ are $T$-invariant.

For $z \in \varrho(T)$, (i) and (ii) are equivalent to

(iii) $U$ is $(T - z)^{-1}$-invariant.

Proof. The claim is immediate from Proposition 3.2 in particular (9). For $z \in \varrho(T)$ note that $\dim U_k < \infty$ and $U_k \subset \mathcal{D}(T)$ imply that $U_k$ is $T$-invariant if and only if $U_k$ is $(T - z)^{-1}$-invariant. □

Corollary 3.9 The subspace $U$ is $T$-invariant and compatible with $(V_k)_{k \in \mathbb{N}}$ if and only if
\[
U = \sum_{\lambda \in \sigma_p(T)} W_\lambda
\]
with $T$-invariant subspaces $W_\lambda \subset \mathcal{L}(\lambda)$. In particular, for $\sigma \subset \sigma_p(T)$ we obtain the compatible subspace
\[
U_\sigma = \sum_{\lambda \in \sigma} \mathcal{L}(\lambda)
\]
associated with $\sigma$.

Proof. If $U = \bigoplus_k U_k$ with $U_k \subset V_k$ $T$-invariant, then, since $\dim U_k < \infty$,
\[
U_k = \sum_{\lambda \in \sigma(T|V_k)} W_\lambda
\]
with $W_\lambda \subset \mathcal{L}(\lambda)$ $T$-invariant; consequently (13). On the other hand, if $U$ is given by (13), and we define $U_k$ by (15), then $U_k$ is $T$-invariant, $U_k \subset V_k$, and we obtain $U = \bigoplus_k U_k$. □
In the following, we will use the notation $\sigma^+_p(T)$, $\sigma^-_p(T)$ and $\sigma^0_p(T)$ for the set of eigenvalues of $T$ on the imaginary axis and in the open right and left half-plane, respectively.

**Remark 3.10** Let $T$ be a closed operator with a finitely spectral Riesz basis of subspaces $(V_k)_{k \in \mathbb{N}}$, $\sigma_p^+(T) = \emptyset$, and consider the invariant compatible subspaces $U_\pm$ associated with $\sigma_p^-(T)$. We have $U_\pm = \bigoplus_k V_k^\pm$ where $V_k^+ = V_k^+ \oplus V_k^-$ and $V_k^\pm$ are the spectral subspaces of $T|_{V_k}$ corresponding to the right and left half-plane. Hence $U_+ + U_- \subset V$ algebraic direct and dense by Proposition 2.4. For the operator $T$ in Example 8.1, the sum is in fact not topological direct; in particular $U_+ + U_- \not\subset V$.

On the other hand, if an operator $T$ is dichotomous (see [18]), then a strip around the imaginary axis belongs to $\varrho(T)$, and there is a topological direct decomposition $V = V_+ \oplus V_-$ such that $V_\pm$ is $T$-invariant and $\sigma(T|_{V_\pm})$ is contained in the right and left half-plane, respectively. In particular $U_\pm \subset V_\pm$. Consequently the operator in Example 8.1 is not dichotomous.

**Lemma 3.11** Let $T$ be an operator on $V$, $z_0 \in \varrho(T)$ and $U \subset V$ a closed $(T - z_0)^{-1}$-invariant subspace. Then $U$ is $(T - z)^{-1}$-invariant for all $z$ in the connected component of $z_0$ in $\varrho(T)$.

**Proof.** It suffices to show that the set

$$A = \{z \in \varrho(T) \mid U \text{ is } (T - z)^{-1}\text{-invariant}\}$$

is relatively open and closed in $\varrho(T)$. Let $z \in A$. For small $|w - z|$ a Neumann series argument shows that

$$(T - w)^{-1} = (T - z)^{-1}(I - (w - z)(T - z)^{-1})^{-1} = \sum_{k=0}^{\infty} (w - z)^k (T - z)^{-k-1}$$

If $x \in U$, then $(T - z)^{-k-1}x \in U$ for all $k \geq 0$. Hence also $(T - w)^{-1}x \in U$, i.e. $w \in A$; $A$ is an open set.

Now let $w \in \varrho(T)$ with $w = \lim_{n \to \infty} z_n$, $z_n \in A$. For $x \in U$ we then have

$$U \ni (T - z_n)^{-1}x \to (T - w)^{-1}x \in U \quad \text{as} \quad n \to \infty$$

since the resolvent $(T - z)^{-1}$ is continuous in $z$. Hence $w \in A$, i.e., $A$ is relatively closed. \hfill $\square$

**Proposition 3.12** Let $T$ be an operator with compact resolvent and a finitely spectral Riesz basis of subspaces $(V_k)_{k \in \mathbb{N}}$. If $U$ is a closed subspace which is $(T - z)^{-1}$-invariant for some $z \in \varrho(T)$, then $U$ is $T$-invariant and compatible with $(V_k)_{k \in \mathbb{N}}$.

**Proof.** Since $T$ has a compact resolvent, $\sigma(T)$ consists of isolated eigenvalues only and $\varrho(T)$ is connected. The previous lemma thus implies that $U$ is $(T - z)^{-1}$-invariant for all $z \in \varrho(T)$. Let $P_k$ be the projections corresponding to the Riesz basis. Since $\sigma_k = \sigma(T|_{V_k})$ is an isolated part of the spectrum, $P_k$ is the Riesz projection associated with $\sigma_k$, i.e.

$$P_k = \frac{i}{2\pi} \int_{\Gamma_k} (T - z)^{-1}dz$$ (16)
where $\Gamma_k$ is a simply closed, positively oriented integration contour with $\sigma_k$ in its interior and $\sigma(T) \setminus \sigma_k$ in its exterior, see e.g. [14, Theorem III.6.17]. Consequently $P_k(U) \subset U$, and $U$ is thus compatible with $(V_k)$. $T$-invariance is now a consequence of Lemma 3.8. \qed

4 Hamiltonian operator matrices

We use the following definition of a Hamiltonian operator matrix, see also [3].

**Definition 4.1** Let $H$ be a Hilbert space. A Hamiltonian operator matrix is a block operator matrix

$$T = \begin{pmatrix} A & B \\ C & -A^* \end{pmatrix}, \quad \mathcal{D}(T) = (\mathcal{D}(A) \cap \mathcal{D}(C)) \times (\mathcal{D}(A^*) \cap \mathcal{D}(B))$$

acting on $H \times H$ with densely defined linear operators $A, B, C$ on $H$ such that $B$ and $C$ are symmetric and $T$ is densely defined.

If $B$ and $C$ are both nonnegative (positive, uniformly positive), then $T$ is called a nonnegative (positive, uniformly positive, respectively) Hamiltonian operator matrix.

Hamiltonian operator matrices are connected to two indefinite inner products on $H \times H$. We recall some corresponding notions, see [4, 5] for more details: A vector space $V$ together with an inner product $\langle \cdot | \cdot \rangle$ is called a Krein space if $V$ is also a Hilbert space with scalar product $\langle \cdot | \cdot \rangle$ and there is a selfadjoint involution $J : V \to V$ such that $\langle x|y \rangle = (Jx|y)$ for all $x, y \in V$.

A subspace $U \subset V$ is called neutral if $\langle x|x \rangle = 0$ for all $x \in U$. The orthogonal complement of $U$ is defined by

$$U^{(\perp)} = \{ x \in V \mid \langle x|y \rangle = 0 \text{ for all } y \in U \}.$$

Two subspaces $U, W \subset V$ are said to be orthogonal, $U^{(\perp)}W$, if $W \subset U^{(\perp)}$. $U$ is neutral if and only if $U \subset U^{(\perp)}$. The subspace $U$ is called non-degenerate if $U \cap U^{(\perp)} = \{0\}$.

Let $T$ be a densely defined operator on $V$. It is called symmetric if $\langle Tx|y \rangle = \langle x|Ty \rangle$ for all $x, y \in \mathcal{D}(T)$. The adjoint of $T$ is defined as the maximal operator $T^{(*)}$ such that

$$\langle Tx|y \rangle = \langle x|T^{(*)}y \rangle \quad \text{for all } x \in \mathcal{D}(T), y \in \mathcal{D}(T^{(*)}).$$

$T$ is called selfadjoint if $T = T^{(*)}$, and in this case its spectrum $\sigma(T)$ is symmetric with respect to the real axis.

Consider the Krein space inner products on $H \times H$ given by

$$\langle x|y \rangle = (J_1x|y) \quad \text{with} \quad J_1 = \begin{pmatrix} 0 & -iI \\ iI & 0 \end{pmatrix}$$

and

$$\langle x|y \rangle = (J_2x|y) \quad \text{with} \quad J_2 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}.$$

2 Note that the sign convention $T = \begin{pmatrix} A & -B \\ -C & A \end{pmatrix}$, in particular with nonnegative $B, C$, is also used in the literature, e.g. in [16, 18].
Here \(\langle \cdot | \cdot \rangle\) denotes the usual scalar product on \(H \times H\). The straightforward computation
\[
\left\langle \begin{pmatrix} A & B \\ C & -A^* \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} | \begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix} \right\rangle = i(Au + Bv|\tilde{v}) - i(Cu - A^*v|\tilde{u}) = i(u|A^*\tilde{v} - C\tilde{u}) - i(v|B\tilde{v} - A\tilde{u}) = \left\langle \begin{pmatrix} u \\ v \end{pmatrix} | \begin{pmatrix} -A & -B \\ -C & A^* \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} \right\rangle
\]
shows that \(T\) is \(J_1\)-skew-symmetric, i.e.,
\[
\langle Tx|y\rangle = -\langle x|Ty\rangle \quad \text{for all } x, y \in \mathcal{D}(T).
\]
As a consequence, \(T\) is always closable. In the following, additional assumptions on \(T\) such as in Theorem 4.4 or the \(r_0\)-diagonally dominance in Section 7 will often imply that \(T\) is already closed. From
\[
\text{Re} \left[ \begin{pmatrix} A & B \\ C & -A^* \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} | \begin{pmatrix} u \\ v \end{pmatrix} \right] = \text{Re}(Au + Bv|v) + (Cu - A^*v|u) = (Bv|v) + (Cu|u)
\]
we obtain that \(T\) is nonnegative if and only if it is \(J_2\)-accretive, i.e., \(\text{Re}[Tx|x] \geq 0\) for all \(x \in \mathcal{D}(T)\).

Recall that we denote by \(\sigma_p^+(T), \sigma_p^-(T)\) and \(\sigma_p(T)\) the set of eigenvalues of \(T\) on the imaginary axis and in the open right and left half-plane, respectively. As a consequence of the \(J_1\)-skew-symmetry of \(T\) we obtain:

**Proposition 4.2** Let \(T\) be a Hamiltonian operator matrix.

(i) If \(\lambda, \mu \in \sigma_p(T)\) with \(\lambda \neq -\overline{\mu}\), then the root subspaces \(\mathcal{L}(\lambda)\) and \(\mathcal{L}(\mu)\) are \(J_1\)-orthogonal. In particular \(\mathcal{L}(\lambda)\) is \(J_1\)-neutral for \(\lambda \notin \sigma_p^+(T)\).

(ii) If \(T\) has a complete system of root subspaces, then \(\sigma_p(T)\) is symmetric with respect to the imaginary axis, and \(\mathcal{L}(\lambda) + \mathcal{L}(-\overline{\lambda})\) is \(J_1\)-non-degenerate with \(\dim \mathcal{L}(\lambda) = \dim \mathcal{L}(-\overline{\lambda})\) for every \(\lambda \in \sigma_p(T)\).

(iii) If there exists \(z\) such that \(z, -\overline{z} \in \sigma(T)\), then \(T\) is \(J_1\)-skew-selfadjoint, i.e., \(T = T^{(\ast)}\), and \(\sigma(T)\) is symmetric with respect to the imaginary axis.

In particular, the point spectrum of a Hamiltonian with a finitely spectral Riesz basis of subspaces is symmetric with respect to the imaginary axis.

**Proof of the proposition.** (i): Since \(iT\) is \(J_1\)-symmetric, this is an immediate consequence of [5, Theorem II.3.3].

(ii): Let
\[
\sigma_0 = \sigma_p^+(T) \cup \sigma_p^-(T) \cup \{-\overline{\lambda} | \lambda \in \sigma_p^-(T)\}
\]
and define \(U_\lambda = \mathcal{L}(\lambda) + \mathcal{L}(-\overline{\lambda})\) for \(\lambda \in \sigma_0\). From (i) it follows that the \(U_\lambda\) are pairwise \(J_1\)-orthogonal. For \(x \in U_\lambda \cap U_\mu^{(\perp)}\) this implies that \(\langle x|y\rangle = 0\) for all \(y \in \sum_\mu U_\mu\). Since \(\sum_\mu U_\mu \subset H \times H\) is dense by assumption, we obtain \(\langle x|y\rangle = 0\) for all \(y \in H \times H\) and thus \(x = 0\); \(U_\lambda\) is \(J_1\)-non-degenerate. For \(\lambda \in \sigma_0\) with \(\text{Re} \lambda > 0\), the subspaces \(\mathcal{L}(\lambda)\) and \(\mathcal{L}(-\overline{\lambda})\) are neutral and their sum is non-degenerate. This
implies that \( \dim \mathcal{L}(\lambda) = \dim \mathcal{L}(-\lambda) \), see [3] §I.10. In particular \( \lambda, -\lambda \in \sigma_p(T) \) and hence the symmetry of \( \sigma_p(T) \).

(iii): We have that \( iT \) is \( J_1 \)-symmetric and \( w, \bar{w} \in \varrho(iT) \) where \( w = iz \). As in the Hilbert space situation this implies that \( iT \) is \( J_1 \)-selfadjoint. Consequently, \( T \) is \( J_1 \)-skew-selfadjoint.

The \( J_2 \)-accretivity of a nonnegative Hamiltonian leads to characterisations of the spectrum at the imaginary axis:

**Proposition 4.3** Let \( T \) be a nonnegative Hamiltonian operator matrix.

(i) We have \( \sigma_p'(T) = \emptyset \) if and only if

\[
\ker(A - it) \cap \ker C = \ker(A^* + it) \cap \ker B = \{0\} \quad \text{for all} \quad t \in \mathbb{R}.
\]

(ii) If \( T \) is uniformly positive with \( B, C \geq \gamma \), then

\[
\{ z \in \mathbb{C} \mid |\text{Re}(z)| < \gamma \} \subset r(T).
\]

**Proof.** (i): We show that \((T - it)x = 0 \) for \( x = (u, v) \in \mathcal{D}(T) \) if and only if

\[
u \in \ker(A - it) \cap \ker C \quad \text{and} \quad v \in \ker(A^* + it) \cap \ker B.
\]

Indeed if \((T - it)x = 0\), then

\[
(A - it)u + Bv = 0, \quad Cu - (A^* + it)v = 0 \quad \text{and} \quad 0 = \text{Re}(it|x|) = \text{Re}[Tx|x|] = (Bv|v| + (Cu|u|).
\]

Since \( B, C \) are nonnegative, this yields \((Bv|v|) = (Cu|u|) = 0\). Now \( B \) admits a nonnegative selfadjoint extension \( \hat{B} \). We obtain \(||\hat{B}^{1/2}v||^2 = \hat{B}v|v| = 0\) and hence \( Bv = (\hat{B}^{1/2})^2v = 0\). Similarly \( Cu = 0 \) and thus also \((A - it)u = (A^* + it)v = 0\). The other implication is immediate.

(ii): For \( x = (u, v) \in \mathcal{D}(T) \) we have \( \text{Re}[Tx|x|] = (Bv|v| + (Cu|u|) \geq \gamma ||x||^2\).

Let \( z \in \mathbb{C} \setminus r(T) \). Then there exists a sequence \( x_n \in \mathcal{D}(T) \) with \(||x_n|| = 1\) and \((T - z)x_n \to 0 \) as \( n \to \infty \). For \( \alpha_n = \text{Re}[(T - z)x_n|x_n] \) this implies \( \alpha_n \to 0 \). We obtain

\[
\gamma ||x_n||^2 \leq \text{Re}[Tx_n|x_n] = \alpha_n + \text{Re}z \cdot [x_n|x_n]
\]

\[
\leq |\alpha_n| + |\text{Re}z|(|J_2x_n|x_n|) \leq |\alpha_n| + |\text{Re}z||x_n||^2 \to |\text{Re}z|
\]

as \( n \to \infty \), i.e. \( \gamma \leq |\text{Re}z|\). \( \square \)

We end this section with two perturbation theorems which ensure the existence of finitely spectral Riesz bases of subspaces for \( T \).

**Theorem 4.4** Let \( T \) be a Hamiltonian operator matrix where \( A \) is normal with compact resolvent and \( B, C \) are \( p \)-subordinate to \( A \) with \( 0 \leq p < 1 \). If \( \sigma(A) \) lies on finitely many rays from the origin and

\[
\lim_{r \to \infty} \inf \frac{N(r, A)}{r^{1-p}} < \infty,
\]

then \( T \) has a compact resolvent, is \( J_1 \)-skew-selfadjoint, and there exists a finitely spectral Riesz basis of subspaces \( (V_k)_{k \in \mathbb{N}} \) for \( T \).
Proof. This is an application of Theorem 3.7 to the decomposition
\[ T = G + S \] with
\[ G = \begin{pmatrix} A & 0 \\ 0 & -A^* \end{pmatrix}, \quad S = \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix}, \]
see [32, Theorem 7.2] for details. The skew-selfadjointness then follows by Proposition 4.2. \(\square\)

Theorem 4.5 Let \( T \) be a uniformly positive Hamiltonian such that \( A \) is skew-selfadjoint with compact resolvent, \( B, C \) are bounded and satisfy \( B, C \geq \gamma \). Let \( \lambda_k \) be the eigenvalues of \( A \) where \((\lambda_k)_{k \in \Lambda} \) is increasing and \( \Lambda \in \{ \mathbb{Z}+, \mathbb{Z}^-, \mathbb{Z} \} \). Suppose that almost all eigenvalues \( \lambda_k \) are simple and that for some \( l > b = \max\{\|B\|, \|C\|\} \) we have
\[ \lambda_{k+1} - \lambda_k \geq 2l \] for almost all \( k \in \Lambda \).

Then \( T \) has a compact resolvent, almost all of its eigenvalues are simple,
\[ \sigma(T) \subset \{ z \in \mathbb{C} \mid \gamma \leq |\Re z| \leq b \}, \]
and \( T \) admits a Riesz basis of eigenvectors and finitely many Jordan chains.

Proof. See [32, Theorem 7.3]. \(\square\)

Remark 4.6 Due to [32, Remark 6.7], Theorem 4.4 continues to hold if \( A \) is an operator with compact resolvent and a Riesz basis of Jordan chains, \( B \) is \( p \)-subordinate to \( A^* \), \( C \) is \( p \)-subordinate to \( A \), \( 0 \leq p < 1 \), almost all eigenvalues of \( A \) lie inside sets \( \{e^{i\theta}(x + iy) \mid x > 0, |y| \leq \alpha x^p \} \) with \( \alpha \geq 0 \), \( -\pi \leq \theta_j < \pi \), \( j = 1, \ldots, n \), and \( (18) \) is satisfied. Theorem 4.5 also holds if \( A \) has a compact resolvent, a Riesz basis of eigenvectors and finitely many Jordan chains, and almost all eigenvalues of \( A \) are simple and contained in a strip around the imaginary axis; the constant \( b \) has to be adjusted then.

5 Invariant subspaces of Hamiltonians

Now we investigate properties of certain invariant subspaces of the Hamiltonian with respect to the two indefinite inner products defined in the previous section.

Let \( V \) be a Krein space. Recall that a subspace \( U \subset V \) is neutral if and only if \( U \subset U^{\perp} \). It is called hypermaximal neutral if \( U = U^{\perp} \), see [1, 5]. It is not hard to see that if \( U, W \) are neutral subspaces with \( V = U \oplus W \), then \( U \) and \( W \) are hypermaximal neutral. For \( \dim V < \infty \), this is even an equivalence:

Lemma 5.1 Let \( V \) be a finite-dimensional Krein space. If \( U \subset V \) is hypermaximal neutral, then there exists a neutral subspace \( W \) such that \( V = U \oplus W \).

Proof. By induction on \( n = \dim U \) we show that there exist systems \((e_1, \ldots, e_n)\) in \( U \) and \((f_1, \ldots, f_n)\) in \( V \) which form a dual pair, i.e. \( \langle e_j | f_i \rangle = \delta_{ij} \), and are such that \( W = \text{span}\{f_1, \ldots, f_n\} \) is neutral. Indeed, if \( \dim U = n + 1 \) and \( e \in U \setminus \text{span}\{e_1, \ldots, e_n\} \), we can set
\[ e_{n+1} = e - \sum_{j=1}^{n} \langle e | f_j \rangle e_j. \]
Since \( V \) is non-degenerate, there exists \( f \in V \) with \( \langle e_n+1 | f \rangle = 1 \). Then
\[
\tilde{f} = f - \sum_{j=1}^{n} (\langle f | e_j \rangle f_j - \sum_{j=1}^{n} (\langle f | f_j \rangle e_j) \quad \text{and} \quad f_{n+1} = \tilde{f} - \frac{\langle \tilde{f} | f \rangle}{2} e_{n+1}
\]
yields the desired properties.

If \( \sum_{j=1}^{n} \alpha_j e_j + \beta f_j = 0 \), then we can take the inner product of this equation with the elements \( e_j, f_j \) and find \( \alpha_j = \beta_j = 0 \) for all \( j \); \((e_1, \ldots, e_n, f_1, \ldots, f_n)\) is linearly independent. In particular \((e_1, \ldots, e_n)\) is a basis of \( U \) and \( U \cap W = \{0\} \).

To show \( V = U \oplus W \), let \( x \in V \) and set \( u = x - w \) where \( w = \sum_{j=1}^{n} \langle x | e_j \rangle f_j \in W \). Then \( \langle u | e_j \rangle = 0 \) for all \( j \), i.e. \( u \in U^{\perp} = U \).

For an operator whose point spectrum \( \sigma_p(T) \) is symmetric with respect to the imaginary axis, we say that a subset \( \sigma \subset \sigma_p(T) \setminus i\mathbb{R} \) is an \( sc \)-set (\( sc \) for skew-conjugate) if

1. \( \lambda \in \sigma \Rightarrow -\overline{\lambda} \not\in \sigma \) and
2. \( \lambda \in \sigma_p(T) \setminus i\mathbb{R} \Rightarrow \lambda \in \sigma \) or \( -\overline{\lambda} \in \sigma \).

In other words, \( \sigma \) contains one eigenvalue from each skew-conjugate pair \((\lambda, -\overline{\lambda})\) in \( \sigma_p(T) \setminus i\mathbb{R} \).

**Theorem 5.2** Let \( T \) be a closed Hamiltonian operator matrix with a finitely spectral Riesz basis of subspaces. Then \( T \) admits a hypermaximal \( J_1 \)-neutral, \( T \)-invariant, compatible subspace if and only if for all \( \lambda \in \sigma_p(T) \) we have
\[
\mathcal{L}(it) = M_{it} \oplus N_{it} \quad \text{with} \quad M_{it}, N_{it} \ J_1 \text{-neutral and } M_{it} \ T \text{-invariant.} \tag{19}
\]

In this case, for every \( sc \)-set \( \sigma \subset \sigma_p(T) \setminus i\mathbb{R} \) the \( T \)-invariant compatible subspace
\[
U = \sum_{\lambda \in \sigma} \mathcal{L}(\lambda) + \sum_{it \in \sigma_p(T)} M_{it} \tag{20}
\]
is hypermaximal \( J_1 \)-neutral.

**Proof.** Let \((V_k)_{k \in \mathbb{N}}\) be a finitely spectral Riesz basis of subspaces for \( T \) and write \( \sigma_k = \sigma(T|V_k) \). Suppose first that \( U \) is hypermaximal \( J_1 \)-neutral, \( T \)-invariant, and compatible with \((V_k)\). So \( U \) is of the form
\[
U = \bigoplus_{k \in \mathbb{N}}^2 U_k = \sum_{\lambda \in \sigma_p(T)} M_{\lambda}
\]
where the subspaces \( U_k \subset V_k \) and \( M_{\lambda} \subset \mathcal{L}(\lambda) \) are all \( T \)-invariant, compare Corollary B. By Proposition 5.2 each \( \mathcal{L}(it) \), \( it \in \sigma'_p(T) \), is \( J_1 \)-non-degenerate and thus itself a Krein space. In view of the previous lemma it suffices to show that \( M_{it} \) is hypermaximal neutral with respect to \( \mathcal{L}(it) \), i.e., \( M_{it}^{\perp} \cap \mathcal{L}(it) = M_{it} \).

Since \( M_{it} \subset U \) we have that \( M_{it} \) is neutral and hence \( M_{it} \subset M_{it}^{\perp} \cap \mathcal{L}(it) \). Let \( x \in M_{it}^{\perp} \cap \mathcal{L}(it) \). Since \( \mathcal{L}(it) \) is \( J_1 \)-orthogonal to \( \mathcal{L}(\lambda) \) for every \( \lambda \neq it \), we see that
\( x(\perp) M_\lambda \) for all \( \lambda \) and hence \( x \in U(\perp) = U \). On the other hand \( x \in L(it) \subset V_k \) with \( k_0 \) such that \( it \in \sigma_{k_0} \). Consequently \( x \in U \cap V_k = U_{k_0} \). Now the decomposition
\[
U_{k_0} = \bigoplus_{\lambda \in \sigma_{k_0}} M_\lambda
\]
implies that \( x \in U_{k_0} \cap L(it) = M_{it} \).

For the other implication, suppose now that for every \( it \in \sigma_R(T) \) there is a decomposition \( L(it) = M_{it} \oplus N_{it} \) into neutral subspaces where \( M_{it} \) is \( T \)-invariant, let \( \sigma \subset \sigma_R(T) \setminus i\mathbb{R} \) be an sc-set, and let \( U \) be given by \( \{20\} \). Since \( U \) is the closure of the sum of neutral, pairwise orthogonal subspaces, \( U \) is neutral. Moreover, \( U \) is \( T \)-invariant and compatible with \( (V_k) \) with decomposition
\[
U = \bigoplus_{k \in \mathbb{N}^2} U_k, \quad U_k = \sum_{\lambda \in \sigma_k} L(\lambda) + \sum_{it \in \sigma^*_k} M_{it},
\]
where \( \sigma^*_k = \sigma_R(T|V_k) \). It remains to show that \( U(\perp) \subset U \). We have \( V_k = U_k \oplus W_k \) with
\[
W_k = \sum_{\lambda \in \tau_k} L(\lambda) + \sum_{it \in \sigma^*_k} N_{it}, \quad \tau_k = \sigma_k \setminus (\sigma \cup \sigma^*_k).
\]
Let \( x \in U(\perp) \). We expand \( x \) in the Riesz basis \( (V_k) \) as \( x = \sum_k (u_k + w_k) \) with \( u_k \in U_k, \ w_k \in W_k \). To show that all \( w_k \) are zero, we consider now the subspaces
\[
\tilde{U}_k = \sum_{\lambda \in \tau_k} L(-\lambda) + \sum_{it \in \sigma^*_k} M_{it}.
\]
The fact that \( \sigma \) is an sc-set yields \( \lambda \in \tau_k \Rightarrow -\lambda \in \sigma \), and therefore \( \tilde{U}_k \subset U \). Moreover \( \tilde{U}_k \) is \( J_1 \)-orthogonal to \( W_j \) for \( j \neq k \), and \( W_k \) is neutral. For \( \tilde{u} \in \tilde{U}_k \), \( \tilde{w} \in W_k \) we thus compute
\[
0 = \langle x|\tilde{u} \rangle = \sum_{j \in \mathbb{N}} \langle u_j + w_j|\tilde{u} \rangle = \langle w_k|\tilde{u} \rangle = \langle w_k|\tilde{u} + \tilde{w} \rangle.
\]
In view of Proposition \( \{12\} \), \( \tilde{U}_k + W_k \) is non-degenerate since it is the orthogonal sum of subspaces \( L(\lambda) + L(-\lambda), \ \lambda \in \tau_k \cup \sigma^*_k \). Consequently \( w_k = 0 \) for all \( k \) and hence \( x = \sum_k u_k \in U \). \( \Box \)

**Remark 5.3** Since all root subspaces of \( T \) are finite-dimensional, results about the Jordan structure of \( J \)-symmetric matrices (e.g. \( \{17\} \) Theorem 2.3.2) may be used to reformulate condition \( \{19\} \). It turns out that \( \{19\} \) holds if and only if \( L(it) = M_{it}' \oplus N_{it}' \) with neutral subspaces \( M_{it}', N_{it}' \).

Now we consider the subspaces associated with \( \sigma^\pm_R(T) \), the point spectrum of \( T \) in the right and left half-plane, respectively.

**Lemma 5.4** Let \( T \) be an operator on a Banach space with \( \sigma^\pm_R(T) = \emptyset \). Consider the algebraic direct decomposition
\[
\sum_{\lambda \in \sigma_R(T)} L(\lambda) = W_+ + W_-,
\]
where
\[
W_\pm = \sum_{\lambda \in \sigma^\pm_R(T)} L(\lambda),
\]
for
and the associated algebraic projections \( P_{\pm} \) onto \( W_{\pm} \). Then

\[
\frac{1}{i\pi} \int_{\mathbb{R}} (T - z)^{-1} x \, dz = P_+ x - P_- x \quad \text{for all} \quad x \in \sum_{\lambda \in \sigma_p(T)} \mathcal{L}(\lambda), \tag{21}
\]

where the prime denotes the Cauchy principal value at infinity, that is \( \int_{\mathbb{R}} f \, dz = \lim_{r \to \infty} \int_{-ir}^{ir} f \, dz \).

Note that the integrand in (21) is well-defined since \((T - z)^{-1}\) acts, for each \( x \), on a finite sum of finite-dimensional subspaces generated by Jordan chains; \((T - z)^{-1} x\) is thus continuous in \( z \).

**Proof of the lemma.** By linearity it suffices to consider \( x \in \mathcal{L}(\lambda) \) and the Jordan chain generated by \( x \). With respect to this Jordan chain, \( T \) is represented by the matrix

\[
E_\lambda = \begin{pmatrix} \lambda & 1 \\ & \ddots \\ & & \lambda \end{pmatrix},
\]

and it suffices to show that

\[
\int_{\mathbb{R}} (E_\lambda - z)^{-1} \, dz = \pm i\pi I
\]

for \( \text{Re} \lambda \geq 0 \). This is a straightforward calculation. \( \square \)

**Lemma 5.5** Let \( T \) be an operator with a Riesz basis \((x_k)_{k \in \mathbb{N}}\) consisting of Jordan chains. If \( \sigma'_r(T) = \emptyset \) and \( \sigma_p(T) \) is contained in a strip around the imaginary axis, then

\[
\int_{-\infty}^{\infty} \|(T - it)^{-1} x\|^2 \, dt \geq c \|x\|^2 \quad \text{for} \quad x \in \text{span}\{x_k \mid k \in \mathbb{N}\}
\]

with some constant \( c > 0 \).

**Proof.** Let \( x \in \text{span}\{x_k \mid k \in \mathbb{N}\} \). Then there is a finite system \( F = (y_1, \ldots, y_n) \subset (x_k)_{k \in \mathbb{N}} \) consisting of Jordan chains such that \( x = \alpha_1 y_1 + \ldots + \alpha_n y_n \). \text{span} \( F \) is a \( T \)-invariant subspace with basis \( F \). With respect to \( F \), \((T - it)^{-1}\) is represented by a block diagonal matrix \( D \) with blocks of the form \((E_\lambda - it)^{-1}, E_\lambda\) as in (22). Hence

\[
(T - it)^{-1} x = \sum_{k=1}^{n} \alpha_k (T - it)^{-1} y_k = \sum_{j,k=1}^{n} \alpha_k D_{jk} y_j.
\]

Let \( m, M > 0 \) be the constants from (3) for the Riesz basis \((x_k)\). Putting \( \xi = (\alpha_1, \ldots, \alpha_n) \) and using the Euclidean norm on \( \mathbb{C}^n \), we find

\[
\|(T - it)^{-1} x\|^2 \geq \sum_{j=1}^{n} \left| \sum_{k=1}^{n} \alpha_k D_{jk} \right|^2 = m \|D\xi\|^2.
\]

Now \( \|D\xi\|^2 \) is the sum of terms of the form \( \|(E_\lambda - it)^{-1} \nu\|^2 \), one for each Jordan chain in \( F \) with \( \nu \) the part of \( \xi \) corresponding to that Jordan chain. From

\[
\|E_\lambda - it\| \leq |\lambda - it| + \left\| \begin{pmatrix} 0 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix} \right\| \leq |\lambda - it| + 1
\]

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Thus \( \mathcal{W} \) let \( B, C \) is uniformly negative. It shows that \( \langle x \| x \rangle \)

imaginary axis, then eigenvalue has finite multiplicity, and \( \sigma \)

Then \( U \)

\[ U \]

\[ \mathcal{E}(\lambda) \]

Proof. Let \( W \) is \( J_{2} \)-nonnegative and \( U \) is \( J_{2} \)-nonpositive.

If in addition \( T \) is uniformly positive, has a Riesz basis of Jordan chains, each eigenvalue has finite multiplicity, and \( \sigma_{p}(T) \) is contained in a strip around the imaginary axis, then \( U \) is uniformly \( J_{2} \)-positive/-negative.

Proof. Let \( W_{\pm} = \mathcal{R}(P_{\pm}) \) as in Lemma 5.4. So \( U \) for \( x \in W_{+} \), using the \( J_{2} \)-accretivity of \( T \), we obtain

\[ \| x \| = \mathcal{R}(P_{+} x - P_{-} x) = \mathcal{R}(T - it)^{-1} x \]

\[ \frac{1}{\pi} \int_{\mathbb{R}} \mathcal{R}(T - it)^{-1} x \mathcal{R}(T - it)^{-1} x dt \geq 0. \]

Thus \( W_{+} \) and hence also \( U_{+} \) are nonnegative. For \( x \in W_{-} \) a similar calculation shows that \( \| x \| \leq 0 \) and hence \( U_{-} \) is nonpositive.

Now suppose that the additional assumptions on \( T \) are satisfied. In particular, let \( B, C \geq \gamma > 0 \). For \( x \in W_{+} \), using Lemma 5.5 we then obtain

\[ \| x \| = \gamma \pi \int_{\mathbb{R}} \mathcal{R}(T - it)^{-1} x \mathcal{R}(T - it)^{-1} x dt \geq \frac{\gamma}{\pi} \int_{\mathbb{R}} \| (T - it)^{-1} x \|^{2} dt \geq \frac{\gamma c}{\pi} \| x \|^{2}. \]

Consequently \( U_{+} \) is uniformly positive. Again, a similar reasoning yields that \( U_{-} \) is uniformly negative.
Remark 5.7 Proposition 4.2 and Theorem 5.2 also hold for arbitrary (skew-)symmetric operators on Krein spaces since in the proofs the particular structure of the Hamiltonian as a block operator matrix was not used. Similarly, Proposition 5.6 holds for arbitrary (uniformly) accretive operators.

Lemma 5.8 Let $T$ be a nonnegative Hamiltonian operator matrix with $C > 0$ and $\ker(A^* - \lambda) \cap \ker B = \{0\}$ for all $\lambda \in \mathbb{C}$. (23)

Then the root subspaces $L(\lambda)$ of $T$ are $J_2$-positive for $\Re \lambda > 0$ and $J_2$-negative for $\Re \lambda < 0$.

Proof. Suppose that $\Re \lambda > 0$; the proof for $\Re \lambda < 0$ is analogous. From Proposition 5.6 we know that $L(\lambda)$ is $J_2$-nonnegative. Let $x = (u, v) \in L(\lambda) \setminus \{0\}$ and $n \in \mathbb{N}$ minimal such that $(T - \lambda)^n x = 0$. We use induction on $n$ to show that $[x|x] \neq 0$ and thus $[x|x] > 0$.

For $n = 1$ we have

\[
\Re \lambda \cdot [x|x] = \Re[Tx|x] = (Bv|v) + (Cu|u).
\]

If $[x|x] = 0$, then $u = 0$ since $B$ is nonnegative and $C$ positive. Hence

\[
Tx = \begin{pmatrix}
Bv \\
-A^*v
\end{pmatrix} = \lambda \begin{pmatrix} 0 \\
v
\end{pmatrix},
\]

and (23) yields $v = 0$, a contradiction.

For $n > 1$ we set $y = (T - \lambda)x$; so $[y|y] > 0$ by the induction hypothesis. If $[x|x] = 0$, then

\[
0 = \Re \lambda \cdot [x|x] = \Re[Tx|x] - \Re[y|x],
\]

i.e.,

\[
\Re[y|x] = (Bv|v) + (Cu|u) \geq 0.
\]

For $r \in \mathbb{R}$ let $w = rx + y$. Then $[w|w] = 2r \Re[y|x] + [y|y]$. Since $w \in L(\lambda)$ is $J_2$-nonnegative and $r$ is arbitrary, this implies $\Re[y|x] = 0$, i.e. $(Bv|v) + (Cu|u) = 0$. So again $u = 0$ and $(Bv|v) = 0$. The reasoning from the proof of Proposition 4.3 then yields $Bv = 0$. Consequently, the first component of $y$ is zero and hence $[y|y] = 0$, again a contradiction. \qed

6 Solutions of the Riccati equation

In this section we consider Hamiltonian operator matrices which are \textit{diagonally dominant}, i.e., $B$ and $C$ are relatively bounded to $A^*$ and $A$ respectively, see [28]; in particular

\[
\mathcal{D}(A) \subset \mathcal{D}(C), \quad \mathcal{D}(A^*) \subset \mathcal{D}(B).
\]

(24)

Recall that, e.g., $C$ is \textit{relatively bounded} to $A$ if $\mathcal{D}(A) \subset \mathcal{D}(C)$ and there are constants $a, b$ such that $||Cu|| \leq a||u|| + b||Au||$ for all $u \in \mathcal{D}(A)$. The infimum of all such $b$ is called the $A$-bound of $C$. Since for a Hamiltonian $T$ the operators $B$ and $C$ are symmetric and hence closable, $T$ is diagonally dominant if $A$ is closed and (24) holds, see [25] Remark 2.2.2. In particular, the Hamiltonians from Theorem 4.4 and 4.5 are diagonally dominant.
For an operator $X$ on the Hilbert space $H$ we consider the graph subspace
\[ \Gamma(X) = \left\{ \begin{pmatrix} u \\ Xu \end{pmatrix} \mid u \in \mathcal{D}(X) \right\}. \]

It is well known that invariant graph subspaces of block operator matrices are connected to Riccati equations. Here we have the following relations, see also \[31\] Section 4.3:

**Proposition 6.1** Let $T$ be a diagonally dominant Hamiltonian and $X$ an operator on $H$.

(i) $\Gamma(X)$ is $T$-invariant if and only if $X$ satisfies the Riccati equation
\[ X(Au + BXu) = Cu - A^*Xu \quad \text{for all } u \in \mathcal{D}(A) \cap X^{-1}\mathcal{D}(A^*). \quad (25) \]
(In particular $Au + BXu \in \mathcal{D}(X)$ for $u \in \mathcal{D}(A) \cap X^{-1}\mathcal{D}(A^*)$.)

(ii) If $T$ has a finitely spectral Riesz basis of subspaces $(V_k)_{k \in \mathbb{N}}$ and $\Gamma(X)$ is $T$-invariant and compatible with $(V_k)_{k \in \mathbb{N}}$, then $\mathcal{D}(A) \cap X^{-1}\mathcal{D}(A^*)$ is a core for $X$.

(iii) If $X$ is selfadjoint and $\mathcal{D}(A) \cap X^{-1}\mathcal{D}(A^*)$ is a core for $X$, then (25) holds if and only if
\[ (Xu|Av) + (Au|Xv) + (BXu|Xv) - (Cu|v) = 0 \quad (26) \]
for all $u, v \in \mathcal{D}(A) \cap X^{-1}\mathcal{D}(A^*)$.

**Proof.** (i): $\Gamma(X)$ is $T$-invariant if and only if for all $u \in \mathcal{D}(A) \cap \mathcal{D}(X)$ with $Xu \in \mathcal{D}(A^*)$ there exists $v \in \mathcal{D}(X)$ such that
\[ T \begin{pmatrix} u \\ Xu \end{pmatrix} = \begin{pmatrix} Au + BXu \\ Cu - A^*Xu \end{pmatrix} = \begin{pmatrix} v \\ Xv \end{pmatrix}, \]
and this is obviously equivalent to (25).

(ii): By assumption, we have $\Gamma(X) = \bigoplus_k U_k$ with $U_k \in \mathcal{D}(T)$. Then $\sum_k U_k$ is dense in $\Gamma(X)$, and hence the subspace $D \subset H$ obtained by projecting $\sum_k U_k$ onto the first component is a core for $X$. Moreover $D \subset \mathcal{D}(A) \cap X^{-1}\mathcal{D}(A^*)$ since $\sum_k U_k \in \mathcal{D}(T)$; hence $\mathcal{D}(A) \cap X^{-1}\mathcal{D}(A^*)$ is a core for $X$.

(iii): Taking the scalar product of (25) with $v \in \mathcal{D}(A) \cap X^{-1}\mathcal{D}(A^*)$, we immediately get (26). On the other hand, (26) can be rewritten as
\[ (Au + BXu|Xv) = (Cu - A^*Xu|v). \]
Since $\mathcal{D}(A) \cap X^{-1}\mathcal{D}(A^*)$ is a core for $X$, this equation holds for all $v \in \mathcal{D}(X)$. Consequently $Au + BXu \in \mathcal{D}(X^*) = \mathcal{D}(X)$ and (25) follows. $\square$

Graph subspaces are also naturally connected to the Krein space inner products considered in Section 4.3 see also [10].

**Lemma 6.2** Consider an operator $X$ on the Hilbert space $H$.
(i) \( X \) is Hermitian, i.e. \((Xu | v) = (u | Xv)\) for all \(u, v \in \mathcal{D}(X)\), if and only if \( \Gamma(X) \) is \( J_1 \)-neutral.

(ii) \( X \) is selfadjoint if and only if \( \Gamma(X) \) is hypermaximal \( J_1 \)-neutral.

If \( X \) is Hermitian, then

(iii) \( X \) is nonnegative and nonpositive if and only if \( \Gamma(X) \) is \( J_2 \)-nonnegative and \( J_2 \)-nonpositive, respectively;

(iv) \( X \) is bounded and uniformly positive (negative) if and only if \( \Gamma(X) \) is uniformly \( J_2 \)-negative (positive).

Proof. The assertions (i) and (iii) are immediate. For (ii) suppose \( \Gamma(X) \) is hypermaximal \( J_1 \)-neutral. If \( w \in \mathcal{D}(X) \perp \) then

\[
\left\langle \begin{pmatrix} u \\ Xu \end{pmatrix}, \begin{pmatrix} 0 \\ w \end{pmatrix} \right\rangle = i(u | w) = 0 \quad \text{for all} \quad u \in \mathcal{D}(X).
\]

Hence \((0, w) \in \Gamma(X) = \Gamma(X) \perp \) and so \( w = 0 \); \( X \) is densely defined. Since \( X \) is also Hermitian, it is thus symmetric, \( X \subset X^* \). If now \( v \in \mathcal{D}(X^*) \), then

\[
\left\langle \begin{pmatrix} u \\ Xu \end{pmatrix}, \begin{pmatrix} v \\ X^*v \end{pmatrix} \right\rangle = i(u | X^*v) - i(Xu | v) = 0 \quad \text{for all} \quad u \in \mathcal{D}(X),
\]

which implies \((v, X^*v) \in \Gamma(X) \) and so \( v \in \mathcal{D}(X) \) and \( X^*v = Xv \). \( X \) is thus selfadjoint. The converse implication in (ii) is proved similarly.

(iv): Let \( X \) be Hermitian and \( \Gamma(X) \) uniformly \( J_2 \)-positive. Then

\[
2\|Xu\|\|u\| \geq 2(Xu | u) = \left[ \begin{pmatrix} u \\ Xu \end{pmatrix} \right] \geq \alpha \left\| \begin{pmatrix} u \\ Xu \end{pmatrix} \right\|^2 = \alpha \|u\|^2 + \alpha \|Xu\|^2,
\]

implies that \((Xu | u) \geq \frac{\alpha}{2}\|u\|^2 \) and \( \|Xu\| \leq \frac{\alpha}{2}\|u\| \). The proof of the other assertions is similar. \( \square \)

**Theorem 6.3** Let \( T \) be a diagonally dominant, nonnegative Hamiltonian operator matrix with \( \varrho(T) \cap \mathbb{R} \neq \emptyset \) and a finitely spectral Riesz basis of subspaces \((V_k)_{k \in \mathbb{N}} \).

Suppose that

(a) \( B \) is positive, or

(b) there is a connected component \( M \) of \( \varrho(A) \) such that \( M \cap \varrho(T) \cap \mathbb{R} \neq \emptyset \) and

\[
\text{span}\{(A - z)^{-1}B^*u \mid z \in M, u \in \mathcal{D}(B^*)\} \subset H \quad \text{is dense.}
\]

Then every hypermaximal \( J_1 \)-neutral, \( T \)-invariant, compatible subspace \( U \) is the graph \( U = \Gamma(X) \) of a selfadjoint operator \( X \) satisfying the Riccati equation

\[
X(Au + BXu) = Cu - A^*Xu, \quad u \in \mathcal{D}(A) \cap X^{-1}\mathcal{D}(A^*),
\]

and \( \mathcal{D}(A) \cap X^{-1}\mathcal{D}(A^*) \) is a core for \( X \).
Proof. In view of Proposition 6.1 and Lemma 6.2, we only need to show that $U$ is a graph subspace. For this it is sufficient that $(0, w) \in U$ implies $w = 0$. Suppose (a) holds and let $it \in g(T)$, $t \in \mathbb{R}$. Let $(0, w) \in U$ and set $(u, v) = (T - it)^{-1}(0, w)$. Then

$$(A - it)u + Bv = 0, \quad Cu - (A^* + it)v = w.$$ 

Since $U$ is $J_1$-neutral and invariant under $(T - it)^{-1}$, this implies

$$0 = \left\langle \begin{pmatrix} 0 \\ w \end{pmatrix}, \begin{pmatrix} u \\ v \end{pmatrix} \right\rangle = -i(w|u)$$

and thus

$$0 = (w|u) = (Cu|u) - (v|(A - it)u) = (Cu|u) + (Bv|v).$$

Since $B$ is positive and $C$ nonnegative, this implies $v = 0$, and the reasoning from the proof of Proposition 4.3 also yields $Cu = 0$. Hence $w = 0$.

In the case of (b), for $it \in M \cap g(T) \cap \mathbb{i}\mathbb{R}$ we consider $u, v$ as above and obtain now $Cu = Bv = 0$. Since $it \in g(A)$, we have $-it \in g(A^*)$. For $\tilde{u} \in D(B^*)$ we get

$$(A^* + it)^{-1}w|B^*\tilde{u}) = -(v|B^*\tilde{u}) = -(Bv|\tilde{u}) = 0.$$ 

Consequently, the function $f(z) = ((A^* - z)^{-1}w|B^*\tilde{u})$, which is holomorphic on $M$, vanishes on $M \cap g(T) \cap \mathbb{i}\mathbb{R}$. From the identity theorem we thus obtain

$$0 = ((A^* - z)^{-1}w|B^*\tilde{u}) = (w|(A - z)^{-1}B^*\tilde{u}) \quad \text{for all } z \in M,$$

and (27) now implies $w = 0$. □

Remark 6.4 Applying the previous theorem to the Hamiltonian

$$\tilde{T} = \begin{pmatrix} -A^* & C \\ B & A \end{pmatrix} = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \begin{pmatrix} A & B \\ C & -A^* \end{pmatrix} \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \tag{29}$$

we immediately get the following symmetric statement: If $C$ is positive or there is a connected component $M$ of $g(A)$ such that $M \cap g(T) \cap \mathbb{i}\mathbb{R} \neq \emptyset$ and

$$\text{span}\{((A^* - z)^{-1}C^*v \mid z \in M, v \in D(C^*))\} \subset H \quad \text{is dense}, \tag{30}$$

then a hypermaximal $J_1$-neutral, $T$-invariant, compatible subspace $U$ is the “inverse” graph

$$U = \Gamma_{inv}(Y) = \left\{ \begin{pmatrix} Yv \\ v \end{pmatrix} \mid v \in D(Y) \right\}$$

of a selfadjoint operator $Y$ such that

$$Y(CYv - A^*v) = AYv + Bv, \quad v \in D(A^*) \cap Y^{-1}D(A),$$

and $D(A^*) \cap Y^{-1}D(A)$ is a core for $Y$. In particular, if simultaneously $U = \Gamma(X) = \Gamma_{inv}(Y)$, then $X$ is injective and $X^{-1} = Y$.

For bounded $B, C$, conditions analogous to (27) and (30) have been used in [18]. In that setting, they are equivalent to the approximate controllability of the pair $(A, B)$ and the approximate observability of $(A, C)$, respectively. Here we have the following relation:
Proposition 6.5 Let $A, B$ be densely defined operators on a Hilbert space $H$ and $M \subset \sigma(A)$. Then for the assertions

(i) $\text{span}\{(A - z)^{-1}B^*v \mid z \in M, v \in \mathcal{D}(B^*)\} \subset H$ dense,

(ii) $\ker(A^* - \lambda) \cap \ker B = \{0\}$ for all $\lambda \in \mathbb{C}$,

we have the implication (i) $\Rightarrow$ (ii). If $A$ is normal with compact resolvent, $\mathcal{D}(A) \subset \mathcal{D}(B)$, and $M$ has an accumulation point in $\sigma(A)$, then (i) $\Leftrightarrow$ (ii).

Proof. For (i) $\Rightarrow$ (ii) consider $A^* u = \lambda u, B u = 0$. Then

$$((A - z)^{-1}B^*v|u) = (v|B(A^* - \bar{z})^{-1}u) = (v|(\lambda - \bar{z})^{-1}Bu) = 0$$

for every $z \in M, v \in \mathcal{D}(B^*)$ and (i) implies $u = 0$.

Now let $A$ be normal with compact resolvent. Let $(\lambda_k)_{k \in \mathbb{N}}$ be the eigenvalues of $A$ and $P_k$ the corresponding orthogonal projections onto the eigenspaces. To prove (i), let $u \in H$ be such that $((A - z)^{-1}B^*v|u) = 0$ for all $z \in M, v \in \mathcal{D}(B^*)$; we aim to show $u = 0$. The function

$$f(z) = ((A - z)^{-1}B^*v|u) = \sum_{k=0}^{\infty} \frac{1}{\lambda_k - z} (P_k B^*v|u)$$

is holomorphic on $\sigma(A)$ and vanishes on $M$; hence $f = 0$ by the identity theorem. If we integrate the series along a circle in $\sigma(A)$ enclosing exactly one $\lambda_k$, we obtain

$$0 = (P_k B^*v|u) = (B^*v|P_k u)$$

for all $v \in \mathcal{D}(B^*)$.

$i.e.$ $P_k u \in \mathcal{R}(B^*)^\perp = \ker \overline{B}$. Since $P_k u \in \mathcal{D}(A) \subset \mathcal{D}(B)$, we have in fact $P_k u \in \ker B$. Since the eigenspaces of $A$ and $A^*$ coincide, (ii) now implies $P_k u = 0$ for all $k \in \mathbb{N}$ and thus $u = 0$.

Corollary 6.6 In the situation of Theorem 6.5 we have $\sigma_p^+(T) = \emptyset$ if and only if $\ker (A - it) \cap \ker C = \{0\}$ for all $t \in \mathbb{R}$.

In this case, for every sc-set $\sigma \subset \sigma_p(T)$ the associated compatible subspace $U_\sigma$ is hypermaximal $J_1$-neutral and thus $U_\sigma = \Gamma(X_\sigma)$ with a selfadjoint solution $X_\sigma$ of (28). The solutions $X_\pm$ corresponding to $\sigma = \sigma_p^+(T)$ are nonnegative/nonpositive.

If $C$ is even positive, then every $X_\sigma$ is injective. In addition, $X_\pm$ is the uniquely determined nonnegative/nonpositive selfadjoint solution of (28) whose graph is compatible with $(V_k)_{k \in \mathbb{N}}$.

Proof. The characterisation of $\sigma_p^+(T) = \emptyset$ is immediate from Proposition 4.3 and 6.3. If $\sigma_p^+(T) = \emptyset$, then condition (19) in Theorem 5.2 is trivially satisfied and hence $U_\sigma$ is hypermaximal $J_1$-neutral. The subspace $U_\pm$ associated with $\sigma_p^+(T)$ is $J_2$-nonnegative/$J_2$-nonpositive by Proposition 5.6 and hence $X_\pm$ is nonnegative/nonpositive by Lemma 6.2.

Now suppose that $C > 0$. Then $X_\sigma$ is injective by Remark 6.4. Let $X$ be nonnegative selfadjoint and $\Gamma(X) = \bigoplus_{k=-\infty}^{\infty} U_k$ with $U_k \subset V_k$ $T$-invariant. Then each $U_k$ is $J_2$-nonnegative and the span of certain root vectors of $T$. By Proposition 6.5 and Lemma 5.8, $\Gamma(X) \subset U_+$ can be applied and yields that $U_k$ is the span of root vectors corresponding to eigenvalues in the right half-plane. Therefore $U_k \subset U_+$ and hence $\Gamma(X) \subset U_+$. Consequently $X \subset X_+$ and thus $X = X_+$ since both operators are selfadjoint. The proof of the uniqueness of $X_-$ is analogous. □
7 Bounded solutions

Consider a diagonally dominant Hamiltonian $T$ and the decomposition

$$T = G + S, \quad G = \begin{pmatrix} A & 0 \\ 0 & -A^* \end{pmatrix}, \quad S = \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix}. \quad (31)$$

**Definition 7.1** We say that $T$ is $r_0$-diagonally dominant ($r$ stands for resolvent) if there is a sequence $(z_k)$ in $\varrho(G)$ such that

$$\lim_{k \to \infty} \|S(G - z_k)^{-1}\| = 0.$$

**Lemma 7.2** (i) If $T$ is $r_0$-diagonally dominant, then $S$ is relatively bounded to $G$ with $G$-bound 0. Moreover

$$z_k \in \varrho(T) \quad \text{with} \quad (T - z_k)^{-1} = (G - z_k)^{-1}(I + S(G - z_k)^{-1})^{-1}$$

whenever $\|S(G - z_k)^{-1}\| < 1$; in particular $\varrho(T) \neq \emptyset$.

(ii) If $S$ is relatively bounded to $G$ with $G$-bound 0, and there is a sequence $(z_k)$ in $\varrho(G)$ and a constant $c > 0$ such that

$$\lim_{k \to \infty} |z_k| = \infty \quad \text{and} \quad \|(G - z_k)^{-1}\| \leq \frac{c}{|z_k|},$$

then $T$ is $r_0$-diagonally dominant.

**Proof.** (i) is a consequence of the estimate

$$\|Sx\| \leq \|S(G - z_k)^{-1}\|\|(G - z_k)x\| \leq \|S(G - z_k)^{-1}\|(\|Gx\| + |z_k|\|x\|)$$

and a Neumann series argument. (ii) follows from

$$\|G(G - z_k)^{-1}\| = \|I + z_k(G - z_k)^{-1}\| \leq 1 + c$$

and

$$\|S(G - z_k)^{-1}\| \leq a\|(G - z_k)^{-1}\| + b\|G(G - z_k)^{-1}\| \leq \frac{ac}{|z_k|} + b(1 + c),$$

where $b > 0$ can be chosen arbitrarily small. \qed

Since $p$-subordination with $p < 1$ implies relative boundedness with relative bound 0, see e.g. [31, Section 3.2], the previous lemma yields that the Hamiltonians from Theorem 4.4 and 4.5 are $r_0$-diagonally dominant.

**Proposition 7.3** Let $T$ be an $r_0$-diagonally dominant Hamiltonian and $X : H \to H$ bounded such that $\Gamma(X)$ is $T$- and $(T - z)^{-1}$-invariant for all $z \in \varrho(T)$. Then $X\mathcal{D}(A) \subset \mathcal{D}(A^*)$ and

$$A^* Xu + XAu + XB Xu - Cu = 0, \quad u \in \mathcal{D}(A). \quad (32)$$

Moreover

$$\sigma(A + BX) = \sigma(T|\Gamma(X)), \quad \sigma_p(A + BX) = \sigma_p(T|\Gamma(X)),$$

and for every $\lambda \in \sigma_p(A + BX)$ the root subspace of $A + BX$ corresponding to $\lambda$ is the projection onto the first component of the root subspace of $T|\Gamma(X)$ corresponding to $\lambda$. 

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Proof. We consider the isomorphism \( \varphi \) and the projection \( \text{pr}_1 \) given by

\[
\varphi : H \to \Gamma(X), \quad u \mapsto (u, Xu), \quad \text{and} \quad \text{pr}_1 : H \times H \to H, \quad (u, v) \mapsto u.
\]

Hence \( \varphi^{-1} = \text{pr}_1|_{\Gamma(X)} \). Using the decomposition [31] and writing \( E = \varphi^{-1}T|_{\Gamma(X)}\varphi \) and \( F = \text{pr}_1S\varphi \), we have

\[
E - F = \text{pr}_1T\varphi - \text{pr}_1S\varphi = \text{pr}_1G\varphi = A|_{\mathcal{D}(A) \cap X^{-1}\mathcal{D}(A^*)}.
\]

The \((T - z)^{-1}\)-invariance of \( \Gamma(X) \) implies that \( \varphi^{-1}(T - z)^{-1}\varphi = (E - z)^{-1} \) for \( z \in \varrho(T) \). Since \( T \) is r0-diagonally dominant, we can now find \( z \in \varrho(G) \cap \varrho(T) \) such that

\[
F(E - z)^{-1} = \text{pr}_1S\varphi \circ \varphi^{-1}(T - z)^{-1}\varphi = \text{pr}_1S(T - z)^{-1}\varphi
\]

and \( \|F(E - z)^{-1}\| < 1 \). Consequently \( z \in \varrho(E - F) = \varrho(A|_{\mathcal{D}(A) \cap X^{-1}\mathcal{D}(A^*)}) \). Since also \( z \in \varrho(A) \), we obtain \( \mathcal{D}(A) \cap X^{-1}\mathcal{D}(A^*) = \mathcal{D}(A) \), i.e. \( X\mathcal{D}(A) \subset \mathcal{D}(A^*) \). The Riccati equation (32) then follows from (25). Moreover, we have

\[
\varphi^{-1}T|_{\Gamma(X)}\varphi = A + BX,
\]

which immediately implies the equality of the spectra and point spectra of \( T|_{\Gamma(X)} \) and \( A + BX \), and that \( \varphi \) maps the root subspaces of \( A + BX \) bijectively onto the corresponding ones of \( T|_{\Gamma(X)} \). \( \square \)

Theorem 7.4 Let \( T \) be an r0-diagonally dominant Hamiltonian with compact resolvent and a finitely spectral Riesz basis of subspaces \((V_k)_{k \in \mathbb{N}} \). Let \( X : H \to H \) be bounded. Then \( \Gamma(X) \) is \( T \)-invariant and compatible with \((V_k)_{k \in \mathbb{N}} \) if and only if \( X\mathcal{D}(A) \subset \mathcal{D}(A^*) \) and \( X \) is a solution of the Riccati equation

\[
A^*Xu + XAu + XBXu - Cu = 0, \quad u \in \mathcal{D}(A). \quad (33)
\]

Proof. If \( \Gamma(X) \) is invariant and compatible, then the assertion follows from Proposition 7.3. So suppose that \( X\mathcal{D}(A) \subset \mathcal{D}(A^*) \) and that \( 33 \) holds. In view of Proposition 5.12, it suffices to find \( z \in \varrho(T) \) such that \( \Gamma(X) \) is \((T - z)^{-1}\)-invariant. Let \( \varphi \) and \( \text{pr}_1 \) be as above. Let \( z \in \varrho(G) \), in particular \( z \in \varrho(A) \). Since \( X\mathcal{D}(A) \subset \mathcal{D}(A^*) \), we have

\[
A - z = \text{pr}_1(G - z)\varphi.
\]

Set \( W = (G - z)\varphi(\mathcal{D}(A)) \). Then \( \text{pr}_1 \) maps \( W \) bijectively onto \( H \) and we have

\[
(A - z)^{-1} = \varphi^{-1}(G - z)^{-1}((\text{pr}_1|_W))^{-1}.
\]

We want to show that \( W \) is closed. Let \( x_n \in W \) with \( x_n \to x \) as \( n \to \infty \) and set \( y_n = (G - z)^{-1}x_n \). Then \( y_n \to (G - z)^{-1}x \) as well as

\[
y_n = \varphi(A - z)^{-1}\text{pr}_1x_n \to \varphi(A - z)^{-1}\text{pr}_1x.
\]

Consequently \((G - z)^{-1}x = \varphi(A - z)^{-1}\text{pr}_1x \) and hence \( x \in W \). The open mapping theorem now implies that \((\text{pr}_1|_W)^{-1} \) is bounded. Since

\[
BX(A - z)^{-1} = \text{pr}_1S\varphi \circ \varphi^{-1}(G - z)^{-1}(\text{pr}_1|_W)^{-1} = \text{pr}_1S(G - z)^{-1}(\text{pr}_1|_W)^{-1}
\]

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and due to the r0-diagonally dominance of $T$, we can find $z \in \varrho(G) \cap \varrho(T)$ such that $\|BX(A-z)^{-1}\| < 1$, which in turn yields $z \in \varrho(A+BX)$. Since $\Gamma(X)$ is $T$-invariant and $\varphi^{-1}T|_{\Gamma(X)} \varphi = A + BX$; in particular $\varrho(T|_{\Gamma(X)}) = \varrho(A + BX)$. We end up with $z \in \varrho(T) \cap \varrho(T|_{\Gamma(X)})$, which implies that $\Gamma(X)$ is $(T-z)^{-1}$-invariant. □

**Remark 7.5** Let $X$ be bounded and selfadjoint. Then $XD(A) \subset D(A^*)$ and

$$A^*Xu + XAu + BXu - Cu = 0, \quad u \in D(A),$$

if and only if $XD(A) \subset D(B)$ and

$$(Xu|Av) + (Au|Xv) + (BXu|Xv) - (Cu|v) = 0, \quad u, v \in D(A).$$

Indeed, the second equation implies that $(Xu|Av)$ is bounded in $v$; hence $Xu \in D(A^*)$ and the first equation follows.

**Lemma 7.6** Let $X_+, X_-$ be bounded selfadjoint operators on a Hilbert space $H$ with $X_+$ uniformly positive and $X_-$ nonpositive. If $X$ is a Hermitian operator on $H$ satisfying $D(X) = D_+ + D_-, X|_{D_+} = X|_{D_-}$, then $X$ is bounded.

**Proof.** First consider $u \in D_+, v \in D_-$ with $\|u\| = \|v\| = 1$. Then

$$\text{Re}(u - v|X_+u + X_-v) = \text{Re}((u|X_+u) - (v|X_+u) + (u|X_-v) - (v|X_-v))$$

$$= (u|X_+u) - (v|X_-v) \geq \gamma$$

where $X_+ \geq \gamma > 0$ and hence

$$\gamma \leq |(u - v|X_+u + X_-v)| \leq \|u - v\| \cdot (\|X_+\| + \|X_-\|).$$

This implies

$$1 - \text{Re}(u|v) = \frac{1}{2}\|u - v\|^2 \geq \delta \quad \text{with} \quad \delta = \frac{1}{2} \left(\frac{\gamma}{\|X_+\| + \|X_-\|}\right)^2 > 0.$$

Consequently

$$|(u|v)| \leq 1 - \delta \quad \text{for all} \quad u \in D_+, v \in D_- \text{ with } \|u\| = \|v\| = 1.$$  

Now for arbitrary $u \in D_+, v \in D_-$ we have the estimates

$$\|X(u + v)\| = \|X_+u + X_-v\| \leq \max\{\|X_+\|, \|X_-\|\}(\|u\| + \|v\|),$$

$$(\|u\| + \|v\|)^2 \leq 2(\|u\|^2 + \|v\|^2),$$

$$\|u + v\|^2 \geq \|u\|^2 + \|v\|^2 - 2(u|v|) \geq \|u\|^2 + \|v\|^2 - 2(1 - \delta)\|u\||v|$$

$$\geq \|u\|^2 + \|v\|^2 - (1 - \delta)(\|u\|^2 + \|v\|^2) = \delta(\|u\|^2 + \|v\|^2).$$

Therefore

$$\|X(u + v)\| \leq \sqrt{\delta} \max\{\|X_+\|, \|X_-\|\}\|u + v\|,$$

$X$ is bounded. □
Recall from Proposition 4.3 and 12 that a closed uniformly positive Hamiltonian with a finitely spectral Riesz basis of subspaces satisfies \( \{ z \in \mathbb{C} \mid |\text{Re} \, z| < \gamma \} \subset \rho(T) \) for some \( \gamma > 0 \).

**Theorem 7.7** Let \( T \) be a uniformly positive, \( r_0 \)-diagonally dominant Hamiltonian with a Riesz basis of Jordan chains, where each eigenvalue has finite multiplicity and \( \sigma_p(T) \) is contained in a strip around \( i\mathbb{R} \).

(i) If \( U \) is a hypermaximal \( J_1 \)-neutral, \( T \)-invariant, compatible subspace, then \( U = \Gamma(X) \) where \( X \) is bounded, selfadjoint, boundedly invertible, \( XD(A) = \mathcal{D}(A^*) \), and \( X \) is a solution of the Riccati equation

\[
A^*Xu + XAu + XBXu - Cu = 0, \quad u \in \mathcal{D}(A). \tag{34}
\]

Moreover, the solutions \( X_{\pm} \) corresponding to the compatible subspaces \( U_{\pm} \) associated with \( \sigma^\pm_p(T) \) are uniformly positive/negative and

\[
X_- \leq X \leq X_+, \quad X_-^{-1} \leq X^{-1} \leq X_+^{-1}. \tag{35}
\]

(ii) If \( X \) is a closed symmetric operator satisfying \( \mathcal{D}(A) \subset \mathcal{D}(X), XD(A) \subset \mathcal{D}(B) \), and

\[
(Xu|Av) + (Au|Xv) + (BXu|Xv) - (Cu|v) = 0, \quad u, v \in \mathcal{D}(A), \tag{36}
\]

then \( X \) is bounded, \( XD(A) \subset \mathcal{D}(A^*) \) and (34) and the first inequality in (35) hold. If in addition \( T \) has a compact resolvent, then \( \Gamma(X) \) is hypermaximal \( J_1 \)-neutral, \( T \)-invariant and compatible, and hence all conclusions of (i) hold.

(iii) If \( X \) is bounded and \( \Gamma(X) \) is \( T \)-invariant and compatible, then there exists a projection \( P \) such that

\[
X = X_+P + X_-(I - P).
\]

**Proof.** (i): Theorem 6.3 and Remark 6.4 yield that \( U \) is a graph \( U = \Gamma(X) \) with \( X \) selfadjoint and injective. In particular \( U_{\pm} = \Gamma(X_{\pm}) \) where \( X_{\pm} \) is also bounded and uniformly positive/negative by Proposition 5.6 and Lemma 6.2. Let \( (\lambda_k)_{k \in \mathbb{N}} \) be the eigenvalues of \( T \). Since the root subspaces \( \mathcal{L}(\lambda_k) \) of \( T \) form a Riesz basis, we have \( \Gamma(X) = \bigoplus_{k \in \mathbb{N}} U_k \) with \( T \)-invariant subspaces \( U_k \subset \mathcal{L}(\lambda_k) \). Hence

\[
\Gamma(X) = W_+ \oplus W_-, \quad W_+ = \bigoplus_{\text{Re} \, \lambda_k > 0} U_k, \quad W_- = \bigoplus_{\text{Re} \, \lambda_k < 0} U_k, \tag{37}
\]

and \( W_{\pm} \subset \Gamma(X_{\pm}) \). If \( D_{\pm} = \text{pr}_1(W_{\pm}) \) where \( \text{pr}_1 \) is the projection onto the first component, then \( \mathcal{D}(X) = D_+ + D_- \), \( X|_{D_{\pm}} = X_{\pm}|_{D_{\pm}} \), and Lemma 6.4 implies that \( X \) is bounded. From Proposition 7.3 we thus obtain \( XD(A) \subset \mathcal{D}(A^*) \) and (34). Then also (36), and the first inequality in (35) will be a consequence of (ii). As \( \Gamma(X_{\pm}) = \Gamma_{\text{inv}}(X_{\pm})^{-1} \), the above reasoning applied to the Hamiltonian \( \tilde{T} \) from (29) yields the boundedness of \( X^{-1}, X^{-1}\mathcal{D}(A^*) \subset \mathcal{D}(A) \) (hence \( XD(A) = \mathcal{D}(A^*) \)), and the second inequality in (35).
(ii): Since equation (36) holds for $X_+$, we have
\[
0 = (Au|(X_+ - X)u) + ((X_+ - X)u|Au) + (BX_+ u|X_+ u) - (B(X_+ - X)u|X_+ u)
\]
\[
= ((A + BX_+)u|(X_+ - X)u) + ((X_+ - X)u|(A + BX_+)u) - (B(X_+ - X)u|X_+ - X)u)
\]
for $u \in \mathcal{D}(A)$. With $\Delta = X_+ - X$ and $t \in \mathbb{R}$ we obtain
\[
2 \text{Re}((A + BX_+ - it)u|\Delta u) = (B\Delta u|\Delta u) \geq 0.
\]
As a consequence of Proposition 5.2 we have that $\mathbb{R} \subset \sigma(A + BX_+)$, that $\sigma_p(A + BX_+)$ is contained in the right half-plane, and that the system of root subspaces $(L_\lambda)$ of $A + BX_+$ is complete in $H$. Then
\[
\text{Re}(v|\Delta(A + BX_+ - it)^{-1}v) \geq 0 \quad \text{for} \quad v \in H,
\]
and Lemma 5.3 yields
\[
(\Delta v|v) = \frac{1}{2} \int_{\mathbb{R}} \text{Re}(\Delta v|(A + BX_+ - it)^{-1}v) \, dt \geq 0 \quad \text{for} \quad v \in \sum_{\lambda \in \sigma_p(A + BX_+)} L_\lambda.
\]
Hence $X \leq X_+$ on $\sum_{\lambda} L_\lambda$. Analogously we find $X_- \leq X$ on $\sum_{\lambda} L_\lambda$. Since $X_+$ and $X_-$ are bounded, this implies that $X$ is bounded on $\sum_{\lambda} L_\lambda$ and hence on $H$ since $X$ is closed. Consequently $X_- \leq X \leq X_+$ holds on $H$, and $XD(A) \subset \mathcal{D}(A^*)$ and (35) follow by Remark 5.3.

Let now $T$ have a compact resolvent. Theorem 7.4 implies that $\Gamma(X)$ is a compatible subspace. It is also hypermaximal $J_1$-neutral since $X$ is selfadjoint.

(iii): We have again the decomposition (37). In particular, $(U_k)$ is a Riesz basis of $\Gamma(X)$. Let $D_k = \text{pr}_1(U_k)$. Then $(D_k)$ is complete in $H$. Moreover, if $c$ is the constant from 3. for the basis $(U_k)$ and $u_k \in D_k$, then
\[
c^{-1} \sum_{k=0}^{n} \|u_k\|^2 \leq c^{-1} \sum_{k=0}^{n} \left\| \frac{u_k}{Xu_k} \right\|^2 \leq \left\| \sum_{k=0}^{n} \left( \frac{u_k}{Xu_k} \right) \right\|^2 \leq (1 + \|X\|^2) \left\| \sum_{k=0}^{n} u_k \right\|^2,
\]
\[
\left\| \sum_{k=0}^{n} u_k \right\|^2 \leq c \sum_{k=0}^{n} \left\| \frac{u_k}{Xu_k} \right\|^2 \leq c \sum_{k=0}^{n} \left\| \frac{u_k}{Xu_k} \right\|^2 \leq c(1 + \|X\|^2) \sum_{k=0}^{n} \|u_k\|^2.
\]
So $(D_k)_{k \in \mathbb{N}}$ is a Riesz basis of subspaces of $H$. Consequently, we have the decomposition
\[
H = \bigoplus_{\text{Re} \lambda_+ > 0}^2 D_k \oplus \bigoplus_{\text{Re} \lambda_- < 0}^2 D_k.
\]
Let $P : H \to H$ be the corresponding projection onto $\bigoplus_{\text{Re} \lambda_+ > 0}^2 D_k$. Since $X|_{D_k} = X_\pm|_{D_k}$ for $\text{Re} \lambda_+ \geq 0$, we obtain $X = X_+ P + X_-(I - P)$. \qed

8 Examples

In the first example we consider a Hamiltonian for which the Riccati equation has unbounded solutions which can be explicitly calculated. In the other examples we apply our theory to non-trivial Riccati equations involving differential operators.
Example 8.1 Let $T$ be a nonnegative Hamiltonian such that $A$ is normal, $B = I$, $C$ is selfadjoint, and $A$ and $C$ admit an orthonormal basis $(e_k)_{k \geq 1}$ of common eigenvectors, $Ae_k = ik^2 e_k$ and $Ce_k = ke_k$ for $k \geq 1$. Then $C$ is 1/2-subordinate to $A$ and Theorem 4.4 can be applied. The subspaces $V_k = \mathbb{C}e_k \times \mathbb{C}e_k$ constitute an orthogonal decomposition $H \times H = \bigoplus_k V_k$, which is obviously finitely spectral for $T$ with

$$T|_{V_k} \cong \begin{pmatrix} ik^2 & 1 \\ k & ik^2 \end{pmatrix}.$$ 

The eigenvalues and corresponding normalised eigenvectors of $T|_{V_k}$ are

$$\lambda_k^\pm = ik^2 \pm \sqrt{k}, \quad v_k^\pm = \frac{1}{\sqrt{1 + k}} \begin{pmatrix} e_k^- \\ \pm \sqrt{k} e_k^+ \end{pmatrix}.$$ 

The hypermaximal $J_1$-neutral compatible subspace corresponding to an sc-set $\sigma \subset \sigma(T)$ is given by

$$U_\sigma = \bigoplus_{k \geq 1}^2 U_k \quad \text{with} \quad U_k = \begin{cases} \mathbb{C}v_k^+ & \text{if } \lambda_k^+ \in \sigma, \\ \mathbb{C}v_k^- & \text{if } \lambda_k^- \in \sigma, \end{cases}$$

and it is the graph $U_\sigma = \Gamma(X_\sigma)$ of a selfadjoint solution $X_\sigma$ of (28),

$$X_\sigma e_k = \begin{cases} \sqrt{k} e_k & \text{if } \lambda_k^+ \in \sigma, \\ -\sqrt{k} e_k & \text{if } \lambda_k^- \in \sigma. \end{cases}$$

In particular, $X_\sigma$ is unbounded and boundedly invertible. Consider now the sequences $(x_k)_{k \in \mathbb{N}}, (x_k^+)_{k \in \mathbb{N}}$ and $(x_k^-)_{k \in \mathbb{N}}$ given by

$$x_k = \begin{pmatrix} \sqrt{k} e_k \\ 0 \end{pmatrix}, \quad x_k^\pm = \sqrt{\frac{1 + k}{k}} v_k^\pm.$$

Then $x_k = x_k^+ + x_k^-$ with $x_k^\pm \in \mathbb{C}v_k^\pm$, the sequence $(x_k)$ converges to zero, while the sequences $(x_k^\pm)$ do not. Consequently, the algebraic direct sum

$$\bigoplus_{k \geq 1} \mathbb{C}v_k^+ + \bigoplus_{k \geq 1} \mathbb{C}v_k^-$$

is not topological direct, the system of eigenvectors $(v_k^\pm)_{k \geq 1}$ is not a Riesz basis, and the operator $T$ is neither Riesz-spectral nor dichotomous, see also Remark 3.10.

By choosing different eigenvalues for the operators $A$ and $C$ in the previous example, it is easy to construct solutions $X_\sigma$ with different properties, for example solutions which are unbounded and not boundedly invertible.

Example 8.2 Let $H = L^2([a, b])$ and consider the operators $A, B, C$ on $H$ given by

$$Au = u''', \quad Bu = -(g_1 u')' + h_1 u, \quad Cu = -(g_2 u')' + h_2 u,$$

$$\mathcal{D}(A) = \{ u \in W^{3,2}([a, b]) \mid u(a) = u(b) = 0, u'(a) = u'(b) \},$$

$$\mathcal{D}(B) = \mathcal{D}(C) = \{ u \in C^2([a, b]) \mid u(a) = u(b) = 0 \}$$

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where $g_1, g_2 \in C^1([a,b])$, $h_1, h_2 \in L^2([a,b])$, $g_1, g_2, h_1, h_2 \geq 0$, and $W^{k,2}([a,b])$ denotes the Sobolev space of $k$ times weakly differentiable, square integrable functions. Then $A$ is skew-selfadjoint with compact resolvent, $0 \in \varrho(A)$, and $\sigma(A)$ consists of at most two sequences of eigenvalues

$$\lambda_{jk} = c_{jk}k^3, \quad k \geq k_0, \quad j = 1, 2,$$

with converging sequences $(c_{jk})$, see [22]. Since the multiplicity of every eigenvalue is at most three, this implies that

$$\sup_{r \geq 1} \frac{N(r, A)}{r^{1/3}} < \infty.$$ 

The operators $B$ and $C$ are symmetric and nonnegative. Using Sobolev and interpolation inequalities, see [1], we can find constants $b_1, b_2, b_3 \geq 0$ such that

$$\|Bu\|_{L^2} \leq \|g_1\|_\infty\|u''\|_{L^2} + \|g'_2\|_{L^2}\|u'\|_{\infty} + \|h_1\|_{L^2}\|u\|_{\infty} \leq b_1\|u\|_{W^{2,2}}$$

$$\leq b_2\|u\|_{L^2}^{1/3}\|\varphi\|_{W^{3,2}}^{2/3} \leq b_3\|u\|_{L^2}^{1/3}(\|u\|_{L^2} + \|u''\|_{L^2})^{2/3}$$

$$\leq b_3\|A^{-1}\| + 1\|u\|_{L^2}^{1/3}\|Au\|_{L^2}^{2/3}$$

for $u \in \mathcal{D}(A)$. Hence $B$, and similarly $C$, are $2/3$-subordinate to $A$. By Theorem [1,3] the Hamiltonian corresponding to $A, B, C$ thus has a finitely spectral Riesz basis of subspaces. If $g_1 > 0$ or $h_1 > 0$, and if $g_2 > 0$ or $h_2 > 0$, then both $B$ and $C$ are positive, and Corollary [6,4] yields an injective selfadjoint solution $X_\sigma$ of (28) for every sc-set $\sigma \subset \sigma(T)$.

The example above immediately generalises to normal differential operators $A$ on $[a,b]$ of order $n$ and nonnegative symmetric differential operators $B, C$ of order at most $n - 1$.

**Example 8.3** Let $H = L^2([-1,1])$ and consider the operators

$$Au = u', \quad \mathcal{D}(A) = \{ u \in W^{1,2}([-1,1]) \mid u(-1) = u(1) \},$$

$$Bu = bu, \quad Cu = cu, \quad \mathcal{D}(B) = \mathcal{D}(C) = H$$

with $b, c \in L^\infty([-1,1])$ and $b(t), c(t) \geq \gamma > 0$ for almost all $t \in [-1,1]$. $A$ is skew-selfadjoint with compact resolvent and simple eigenvalues $\lambda_k = i\pi k$. $B$ and $C$ are bounded and uniformly positive. If now $\|b\|_\infty, \|c\|_\infty < \pi/2$, then we can apply Theorems [1,5] and [4,7] and obtain bounded, selfadjoint, boundedly invertible solutions of the Riccati equation (34).

Consider now the special case that $c = \chi^2b$ with

$$\chi(t) = \begin{cases} 1, & t < 0, \\ \alpha, & t \geq 0, \end{cases} \quad \alpha \in \mathbb{R} \setminus \{0, 1\}.$$

Let $X \in L(H)$ be the operator of multiplication with $\chi$. It is not hard to see that

$$\mathcal{D}(A) \cap X^{-1}\mathcal{D}(A) = \{ u \in W^{1,2}([-1,1]) \mid u(-1) = u(0) = u(1) \}$$

and $AXu = \chi u'$ for $u \in \mathcal{D}(A) \cap X^{-1}\mathcal{D}(A)$. Hence

$$-AXu + XAu + XBXu - Cu = -\chi u' + \chi u' + \chi^2bu - cu = 0.$$
Consequently, $X$ is a solution of the Riccati equation

$$-AXu + XAu + XBXu - Cu = 0, \quad u \in \mathcal{D}(A) \cap X^{-1}\mathcal{D}(A),$$

and $\mathcal{D}(A) \cap X^{-1}\mathcal{D}(A) \subset H$ is dense. In particular, $\Gamma(X)$ is a $T$-invariant subspace. On the other hand, since $\mathcal{D}(A) \cap X^{-1}\mathcal{D}(A) \neq \mathcal{D}(A)$ we have $X\mathcal{D}(A) \not\subset \mathcal{D}(A)$, and with Theorem 7.4 we conclude that $\Gamma(X)$ is not a compatible subspace.

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