Self-similar solutions for dyadic models of the Euler equations

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Abstract
We show existence of self-similar solutions satisfying Kolmogorov’s scaling for generalized dyadic models of the Euler equations, extending a result of Barbato, Flandoli, and Morandin [1]. The proof is based on the analysis of certain dynamical systems on the plane.

1 Introduction
In this note, we address the question of the existence of self-similar solutions in the following infinite system of ordinary differential equations:

\[
\frac{da_j(t)}{dt} = (\lambda^j a_{j-1}^2(t) - \lambda^{j+1} a_j(t) a_{j+1}(t)) + \beta (\lambda^j a_{j-1}(t) a_j(t) - \lambda^{j+1} a_{j+1}^2(t)),
\]

for \( j \geq 0 \), with the boundary condition \( a_{-1}(t) \equiv 0 \).

The factor \( \lambda \) is some fixed constant greater than 1, and the coefficient \( \beta \) is taken to be a nonnegative constant. The case \( \beta = 0 \) is sometimes called the KP equations, and have first appeared in the literature in the work of Friedlander, Katz, and Pavlovic [5, 7]. The full system (1) was suggested in a work of Kiselev and Zlatos [8] — it was characterized as an infinite system of ODEs which is quadratic, conserves the energy, and contains only the nearest neighbor interactions (see [8, Proposition 2.4]). It was shown in [6] that smooth solutions of (1) blows up in finite time when \( \beta \) is small enough, extending previous results which established blow-up in the case \( \beta = 0 \) [5, 7, 8, 9, 2]. This type of equations are suggested as toy models for the dynamics of an inviscid fluid. Roughly speaking, the square of the scalar variable \( a_j(t) \) is associated with the energy of a fluid velocity vector field restricted to a frequency shell of radii \( \sim 2^j \). The quantity \( \sum_{j \geq 0} a_j^2(t) \) is then the analogue of energy, and one may check directly from (1) that it is (formally) conserved in time. Therefore, it is quite natural to restrict to non-negative and finite-energy solutions, i.e. a sequence of functions \( a_j(t) \geq 0 \) solving (1) with \( \sum_{j \geq 0} a_j^2(t) < \infty \).

If one attempts to consider (1) as a model for turbulence, it makes sense to add a constant forcing term to the lowest mode (to sustain turbulent motion):

\[
\frac{da_j(t)}{dt} = (\lambda^j a_{j-1}^2(t) - \lambda^{j+1} a_j(t) a_{j+1}(t)) + \beta (\lambda^j a_{j-1}(t) a_j(t) - \lambda^{j+1} a_{j+1}^2(t)) + f \delta_0(j),
\]

where \( f > 0 \) is a constant and \( \delta_0(j) = 1 \) for \( j = 0 \) and 0 otherwise. In the case \( \beta = 0 \), it is elementary to check that there exists a unique (finite-energy) fixed point\(^2\) of the system (2), and it has the form

\[
a_j(t) = \text{const}(f) \cdot \lambda^{-j/3}.
\]

\(^1\)It is justified, for example, when \( \sum_{j \geq 0} \lambda^{2j/3+\delta} a_j^2(t) < \infty \) for some \( \delta > 0 \).

\(^2\)The existence of a fixed point in a forced system contradicts energy conservation – this phenomenon is called either anomalous or turbulent dissipation.
It can be argued that the scaling $\lambda^{-j/3}$ corresponds to Kolmogorov’s famous 5/3 law (see [3,4] for example). Remarkably, it was shown that this fixed point attracts all other solutions [4] when $t \to +\infty$.

Let us note that a fixed point with the same scaling $\lambda$ similar solution Theorem (Barbato-Frandoli-Morandin [1]) interested in finite-energy solutions. It is surprising that this condition uniquely selects a value of $a$ which play a similar role (at least conjecturally). We say that a solution $a(t) = \{a_j(t)\}_{j \geq 0}$ is self-similar if there exists some profile $\phi(t)$ such that

$$a_j(t) = a_j^* \cdot \phi(t) \quad \text{for all } j \geq 0,$$

with constants $a_j^*$. By plugging into (11), one readily sees that $\phi(t)$ must take the form $(t - t_0)^{-1}$ for some $t_0 \in \mathbb{R}$. In addition, when $\beta = 0$, the constants $a_j^*$ should satisfy the recurrence

$$a_{n+1}^* = \frac{1}{\lambda^{n+1}} + \frac{(a_{n-1}^*)^2}{\lambda a_n^*}.$$  \hspace{1cm} (5)

It is convenient to renormalize the variables by $\lambda^n a_n^* = a_n$. Then (11) takes the form

$$a_{n+1} = 1 + \lambda^2 \frac{a_{n-1}^2}{a_n}.$$  \hspace{1cm} (6)

In the case of $\beta > 0$, the corresponding recursion takes the form

$$a_{n+1} = -\frac{\lambda}{2\beta} a_n + \sqrt{\frac{\lambda}{2\beta}} a_n + \frac{\lambda}{\beta} (\lambda a_{n-1}^2 + \beta a_{n-1} a_n + a_n).$$  \hspace{1cm} (7)

In principle, any choice of $a_0 > 0$ would yield a non-negative self-similar solution via (6), but we are interested in finite-energy solutions. It is surprising that this condition uniquely selects a value of $a_0$:

**Theorem** (Barbato-Frandoli-Morandin [1]). There exists a unique value of $a_0 > 0$ such that the self-similar solution

$$a_j(t) = \frac{a_j^*}{t - t_0}$$

obtained from the recursion (11) and the scaling $\lambda^j a_j^* = a_j$ has finite energy. Moreover, we have asymptotics

$$a_j^* \approx \text{const} \cdot \lambda^{-j/3} \quad \text{as } j \to \infty.$$  \hspace{1cm} (8)

Our result extends the existence statement to the case of small $\beta > 0$.

**Theorem 1.** There exists $\beta_0 > 0$ such that for all $\beta \in (0, \beta_0)$, there exists a value $a_0 = a_0(\beta) > 0$ such that the sequence of points $\{a_j^*\}_{j \geq 0}$ obtained from the recursion (11) and the scaling $\lambda^j a_j^* = a_j$ satisfies

$$a_j^* \approx \text{const}(\beta) \cdot \lambda^{-j/3} \quad \text{as } j \to \infty.$$  \hspace{1cm} (9)

**Remark.** The proof of [1] was based on complex analysis and it is not clear to us whether the method can be adapted to the case $\beta > 0$.

Our arguments yield the uniqueness statement for $\beta = 0$ as in [1], and also “local” uniqueness for $\beta > 0$ small (in the sense that if we slightly perturb $a_0(\beta)$ a little bit, it does not provide a finite energy self-similar solution).

**Remark.** As it was pointed out in [1], the unique solution given in the theorem automatically generates a family of self-similar solutions parametrized by $(J, t_0) \in \mathbb{Z}_+ \times \mathbb{R}_+$, where $t_0$ is the time parameter as in [3] and $J$ is the first nonzero entry in the sequence. It is reported in [1] that at least numerically, any solution of (11) (with $\beta = 0$) selects one of the self-similar solutions and converges to it exponentially in time. We have observed a similar phenomenon for the case of small positive $\beta$.

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3Solutions exist globally in $l^2$, even though they blow up in finite time with respect to smooth topologies.

4This may be viewed as a model for freely decaying turbulence, see [1].
In the proof, we fix $\lambda = 2$ but the proof carries over to any value of $\lambda > 1$. Now let us present an outline of the proof of the Theorem [1] for the case $\beta = 0$.

1. The starting point is a reformulation of the statement in terms of a dynamical system defined on the plane. If we consider the map

$$F : (x, y) \mapsto (y, 1 + \lambda^2 \frac{x^2}{y}),$$

then the equivalent problem is to find a point $(0, \alpha_0)$ with $\alpha_0 > 0$ whose iterates under $F$ have the desired asymptotics. Note that $F$ restricts to a dynamical system in the positive quadrant $\mathbb{R}_+ \times \mathbb{R}_+ = \{(x, y) : x, y > 0\}$ and is injective. As the image of the ray

$$L = \{(x, y) : x = 0, y > 0\}$$

is already contained in $\mathbb{R}_+ \times \mathbb{R}_+$, we may consider $F$ only on the positive quadrant. We then change coordinates to “diagonalize” the map $F$, so that in the new coordinate system, $F = G + E$, where $G$ is an affine map and $E$ is an error term. The map $G$ has the form $(a, b) \mapsto (-2a, b + c_0)$ and therefore the line $\{a = 0\}$ is invariant. Our goal is to show that there is an invariant curve for $F$ as well.

2. We take a rectangle of the form $X = \{(a, b) : R_0 \leq b, -R_1 \leq a \leq R_1\}$. Assuming that the inverse $F^{-1}$ is well-defined on $X$, we may write $F^{-1} = G^{-1} + H$ and derive conditions $H$ in terms of $R_0$ and $R_1$ which guarantee existence of an invariant curve for $F^{-1}$ (hence for $F$ as well) inside $X$.

3. We then find a pair $(R_0, R_1)$ for which $F$ is invertible on $X$ and the conditions from (2) on $H$ are satisfied. At this point, we deduce the existence of an invariant curve $\gamma^{inv}$.

4. We show that some forward iterate of $L$ intersects $\gamma^{inv}$ transversally. From this, we obtain the initial value $\alpha_0$.

5. Finally, we show that one can take $R_1 \sim e^{-\text{const} \cdot R_0}$ as $R_0 \to \infty$, which implies that $\gamma^{inv}$ converges exponentially to the vertical line.

In Subsection 2.2, we give a rather detailed proof in the case $\beta = 0$, following the outline described above. Then in Subsection 2.2, we treat the case $\beta > 0$, but as the structure of the argument is similar, we mainly indicate the necessary modifications.

**Notation.** Given a continuous function $f$ on $A$, we write $\|f\|_A = \sup_{x \in A} |f(x)|$. When $f = (f_{ij})$ is a matrix-valued function, we similarly write $\|f\|_A = \max_{i,j} \sup_{x \in A} |f_{ij}(x)|$. Moreover, given two sequences $c_j$ and $d_j$, we write $c_j \approx d_j$ when $c_j/d_j \to 1$ as $j \to +\infty$.

## 2 Proof of the Main Result

### 2.1 The Case of $\beta = 0$

**Step 1.** Since we are concerned with the region $x, y > 0$, we can make a logarithmic change of variables $u = \ln x, v = \ln y$. In this coordinate system, the map $F$ takes the form

$$(u, v) \mapsto (v, 2u - v + c_0 + \ln(1 + c_1 e^{-2u+v})), $$

where $c_0 = \ln(\lambda^2)$ and $c_1 = \lambda^{-2}$. We then diagonalize the affine part of $F$ by another change of coordinates $a = u - v + c_0/3, b = 2u + v$. In this coordinate system, we have

$$F : (a, b) \mapsto (-2a, b + c_0) + (-e(a, b), e(a, b)),$$  (11)
where \( e(a, b) = \ln(1 + c_2e^{-(4a+b)/3}) \) with \( c_2 = \lambda^{-26/9} \). We define the affine part and the error part of \( F \) by

\[
G(a, b) = (-2a, b + c_0), \quad E(a, b) = (-e(a, b), e(a, b)).
\]

**Step 2.** Take two positive numbers \( R_0, R_1 \) and consider the rectangles

\[
X = [-R_1, R_1] \times [R_0, \infty), \quad X^+ = [-R_1, R_1] \times [R_0 - R_1 - c_0, \infty)
\]

in the \((a, b)\)-plane. Later we will choose \( R_0, R_1 \) so that \( F(X^+) \supset X \), which (together with injectivity of \( F \)) means that \( F^{-1} \) is a well-defined as a map \( X \to X^+ \). Then, writing \( F^{-1} = G^{-1} + H \), we obtain

\[
H \circ F = -\nabla G^{-1} \circ E.
\] (12)

We write the components of \( H \) and \( E \) as \( H(a, b) = (h_1(a, b), h_2(a, b)) \) and \( E(a, b) = (e_1(a, b), e_2(a, b)) \).

Consider a space of Lipschitz continuous curves whose images lie in the set \( X \) with lipschitz constants not exceeding 1:

\[
\Gamma = \{ \gamma : [R_0, \infty) \to [-R_1, R_1] : |\gamma'| \leq 1 \},
\]

Equipped with the metric 

\[
d(\gamma_1, \gamma_2) = \sup_{R_0 \leq x} |\gamma_1(x) - \gamma_2(x)|,
\]

\((\Gamma, d)\) becomes a complete metric space. Assume that we have the following bounds on \( H \)

\[
\|H\|_X \leq \min\{\frac{R_1}{2}, c_0\}, \quad (13)
\]

together with the bounds on \( \nabla H \)

\[
\|\nabla H\|_X \leq \frac{1}{10}. \quad (14)
\]

**Claim.** The bounds (13), (14) guarantee that \( F^{-1} \) induces a contraction mapping in \((\Gamma, d)\).

Note that in the case when \( H \) is identically zero, it is clear that \( F^{-1} \) induces a map on \( \Gamma \) by taking the image \( F^{-1}(\gamma) \) and cutting away the piece which does not belong to \( X \). Denoting this map by \( T \), we have \( d(T\gamma_1, T\gamma_2) = d(\gamma_1, \gamma_2)/2 \) in this special case. To verify the claim, we will list four conditions which together imply the statement, and then proceed to show that (13), (14) imply each condition. We begin with

**Condition 1.** Given \( \gamma \in \Gamma \), the image \( F^{-1}\gamma \) is contained in \([-R_1, R_1]\) \( \times \mathbb{R} \).

This follows directly from \( \|h_1\|_X \leq \frac{R_1}{2} \) of (13). Next,

**Condition 2.** The image \( F^{-1}\gamma \) is the graph of a function \( \beta : [R_0 - r, \infty) \to [-R_1, R_1] \) for some \( r \geq 0 \).

This time, the condition \( h_2 \leq c_0 \) on \( X \) ensures that the image \( F^{-1}\gamma \) has a part belonging to the region \( \{ a \leq R_0 \} \). Then, we only need to exclude the possibility that for \( t_1 < t_2 \) in \([R_0, \infty), F^{-1}(\gamma(t_1), t_1) \) and \( F^{-1}(\gamma(t_2), t_2) \) have the same \( a \)-coordinate, i.e. \( t_1 + h_2(\gamma(t_1), t_1) = t_2 + h_2(\gamma(t_2), t_2) \). By setting \( \Delta t = t_2 - t_1 \), we note that above equality implies

\[
\Delta t \leq (\|\partial_2 h_2\|_X + \|\partial_1 h_2\|_X \cdot |\gamma'|) \Delta t.
\]

This is a contradiction since \( \|\partial_2 h_2\|_X + \|\partial_1 h_2\|_X \leq 1/5 \) by (14). Therefore, we may cut the piece of \( F^{-1}\gamma \) not contained in \( X \) and define the resulting curve \([R_0, \infty) \to [-R_1, R_1] \) as \( T\gamma \). To show that the curve obtained in this way belongs to \( \Gamma \), we only need to check

**Condition 3.** The curve \( T\gamma \) defined above is continuous with Lipschitz constant not exceeding 1.
We show that (14) implies
\[
\frac{| - \frac{1}{2}(\gamma(t_2) - \gamma(t_1)) + h_1(\gamma(t_2), t_2) - h_1(\gamma(t_1), t_1)|}{\Delta t + (h_2(\gamma(t_2), t_2) - h_2(\gamma(t_1), t_1))} \leq 1,
\] (15)
for any \( t_2 > t_1 \geq R_0 \) and \( \gamma \in \Gamma \).

Indeed, the denominator is bounded below by
\[
\Delta t(1 - \|\partial_2 h_2\|_X - \|\partial_1 h_2\|_X) \geq \frac{4}{5} \Delta t,
\]
and the first term on the numerator satisfies
\[
\left| - \frac{1}{2}(\gamma(t_2) - \gamma(t_1)) \right| \leq \frac{1}{2} \Delta t,
\]
while the second one satisfies
\[
|h_1(\gamma(t_2), t_2) - h_1(\gamma(t_1), t_1)| \leq (\|\partial_2 h_1\|_X + \|\partial_1 h_1\|_X) \Delta t \leq \frac{1}{5} \Delta t.
\]

Combining these, we obtain (15). That is, \( T \) defines a dynamical system on the set \( \Gamma \). Finally, we require that

**Condition 4.** The map \( T \) is a contraction on \( \Gamma \).

We again verify that (14) is enough to establish it. Take two curves \( \gamma_1, \gamma_2 \in \Gamma \) and \( t \geq R_0 \). Denote \( F^{-1}(\gamma_i(t), t) = O_i \), and set \( O' \) to be the point on the image \( F^{-1}\gamma_2 \) which has the same \( b \)-coordinates with \( O_1 \). It will be enough to show that \( d(O_1, O') \leq \mu |\gamma_1(t) - \gamma_2(t)| \) with some \( \mu < 1 \). For this we will bound each of \( d(O_1, O_2) \) and \( d(O_2, O') \) in terms of \( |\gamma_1(t) - \gamma_2(t)| \).

First,
\[
d(O_1, O_2) \leq d(G^{-1}(\gamma_1(t), t) - G^{-1}(\gamma_2(t), t)) + d(H(\gamma_1(t), t) - H(\gamma_2(t), t))
\leq (\frac{1}{2} + \sqrt{\|\partial_1 h_1\|_X^2 + \|\partial_2 h_2\|_X^2}) |\gamma_1(t) - \gamma_2(t)|
\leq (\frac{1}{2} + \frac{\sqrt{5}}{10}) |\gamma_1(t) - \gamma_2(t)|.
\]

Next, we set \( t^* \) to be the point such that the image of \( (\gamma_2(t^*), t^*) \) by \( F^{-1} \) has the same \( b \)-coordinates with \( O_1 \). Then \( t^* \) is determined by the equation
\[
|t^* - t| = |h_2(\gamma_2(t^*), t^*) - h_2(\gamma_1(t), t)|,
\]
(16)

We bound the right hand side of (16) by
\[
|t^* - t| \leq |\partial_1 h_2| \cdot |\gamma_2(t^*) - \gamma_1(t)| + |\partial_2 h_2| \cdot |t^* - t|
\]
which in turn implies
\[
\frac{1}{1 - \|\partial_2 h_2\|_X} |t^* - t| \leq \|\partial_1 h_2\|_X \cdot |\gamma_2(t^*) - \gamma_1(t)|
\leq \|\partial_1 h_2\|_X \cdot (|\gamma_2(t^*) - \gamma_2(t)| + |\gamma_2(t) - \gamma_1(t)|)
\leq \|\partial_1 h_2\|_X \cdot (|t^* - t| + |\gamma_2(t) - \gamma_1(t)|),
\]
and from (14), we obtain
\[
|t^* - t| \leq \frac{1}{10} |\gamma_2(t) - \gamma_1(t)|.
\]
Now we can bound $d(O_2, O')^2 \leq I^2 + II^2$, where $I$ and $II$ denote the difference of $O_2, O'$ in $a$ and $b$ coordinates, respectively. We have

$$I \leq \frac{1}{2}|t^* - t| + |h_1(\gamma_2(t^*), t^*) - h_1(\gamma_2(t), t)|$$

$$\leq \left(\frac{1}{2} + \|\partial_1 h_1\|_X + \|\partial_2 h_1\|_X\right)|t^* - t|,$$

and similarly,

$$II \leq |t^* - t| + |h_2(\gamma_2(t^*), t^*) - h_2(\gamma_2(t), t)|$$

$$\leq (1 + \|\partial_1 h_2\|_X + \|\partial_2 h_2\|_X)|t^* - t|,$$

With (14), we get

$$d(O_2, O') \leq \sqrt{2}|t^* - t| \leq \frac{\sqrt{2}}{10}\gamma_1(t) - \gamma_2(t),$$

and finally

$$d(O_1, O') \leq \left(\frac{1}{2} + \frac{\sqrt{2}}{5}\right)\gamma_1(t) - \gamma_2(t).$$

**Step 3.** We will pick a pair $(R_0, R_1)$ so that the region $X$ satisfies $F(X^+) \supset X$ and the bounds (13), (14). It is easy to see that requiring

$$\|e_1\|_{X^+}, \|e_2\|_{X^+} \leq \min\{R_1, c_0\}$$

(17)

guarantees $F(X^+) \supset X$. Next, from the expression $H \circ F(a, b) = -\nabla G^{-1} \circ E = (e_1(a, b)/2, -e_2(a, b))$, we see that (17) is sufficient to guarantee bounds on $H$ in (13).

Before we proceed further, let us fix a convention for matrix entries: $(\nabla A)_{ij} = \partial_j A_i$. With this notation, we have

$$(\nabla H \circ F)\nabla F = \nabla (-\nabla G^{-1} \circ E) = \begin{pmatrix} -\frac{1}{2} \partial_1 e_1 & -\frac{1}{2} \partial_2 e_1 \\ \partial_1 e_2 & \partial_2 e_2 \end{pmatrix}.\tag{18}$$

From $\nabla F = \nabla G + \nabla E$, we write

$$(\nabla F)^{-1} = \nabla G^{-1} (I + (\nabla E \nabla G^{-1}))^{-1} = \nabla G^{-1} \left(\sum_{n \geq 0} (\nabla E \nabla G^{-1})^n\right).$$

Since $\|\nabla E \nabla G^{-1}\|_{X^+} \leq \|\nabla E\|_{X^+}$, we have

$$\|\nabla F\|_{X^+} \leq \frac{1}{1 - \|\nabla E\|_{X^+}}$$

and we obtain from (18) that

$$\|\nabla H\|_X \leq \frac{2}{1 - \|\nabla E\|_{X^+}} \cdot \|\begin{pmatrix} -\frac{1}{2} \partial_1 e_1 & -\frac{1}{2} \partial_2 e_1 \\ \partial_1 e_2 & \partial_2 e_2 \end{pmatrix}\|_{X^+} \leq \frac{2\|\nabla E\|_{X^+}}{1 - \|\nabla E\|_{X^+}}.\tag{19}$$

Requiring $\|\nabla E\|_{X^+} \leq \frac{1}{25}$ is sufficient to obtain the bound (14) on $\nabla H$. In conclusion, the following are sufficient conditions on $R_0$ and $R_1$:

$$\|E\|_{X^+} \leq \min\{c_0, R_1\}, \quad \|\nabla E\|_{X^+} \leq \frac{1}{25}.\tag{20}$$

We proceed with the explicit formula for the error. Since $|e_i(a, b)| = \ln(1 + c_2 e^{-4(a+b)/3})$ for $i = 1, 2$, we get the maximal value of error in $X^+$ upon substituting $a = R_0 - R_1 - c_0, b = -R_1$. Hence,

$$\|e_1\|_{X^+} \leq c_2 e^{-\frac{4}{3}(4(R_0 - R_1 - c_0) - R_1)} = \lambda^{-\frac{4}{3}} e^{-\frac{4}{3}(4R_0 - 5R_1)}.\tag{21}$$
and

$$||\partial_2 e_i||_{X^+} \leq \frac{1}{3} \lambda^{-\frac{R_0}{2}} e^{-\frac{1}{2}(4R_0-5R_1)},$$

(22)

while $$||\partial_1 e_i||_{X^+} = 4 ||\partial_2 e_i||_{X^+}$$. Therefore, if we pick $$R_1 = \min\left\{ \frac{3}{100}, c_0 \right\}$$ and then $$R_0$$ in a way that $$\lambda^{-\frac{R_0}{2}} e^{-\frac{1}{2}(4R_0-5R_1)} \leq R_1$$, all the requirements in (20) are satisfied.

Since $$\lambda = 2$$, if we fix $$R_1 = 3/100$$, any value of $$R_0$$ not less than

$$\frac{3}{4} \left( \frac{\ln 100}{3} - \frac{2}{9} \ln 2 + \frac{5}{100} \right) \approx 2.55$$

would work. In particular, we have obtained the existence of a unique $$F$$-invariant curve $$\gamma^{inv} : [2.56, \infty) \to [-0.03, 0.03]$$ whose graph lies in $$X$$. We will soon take $$R_0 \to \infty$$, but by an abuse of notation, let us denote the corresponding restriction of the curve by the same letter $$\gamma^{inv}$$.

**Step 4.** We will show in Lemma 1 that there is an $$N > 0$$ such that $$F^N(L)$$ intersects $$\gamma^{inv}$$ (see Figure 1). By the definition of $$L$$, we know that the point of intersection has the form $$(\alpha_{N-1}, \alpha_N)$$ (in the $$(x, y)$$-coordinates), where $$\alpha_{N-1}$$ and $$\alpha_N$$ are from a sequence $$\{\alpha_n\}_{n \geq 0}$$ satisfying the recurrence (5) with some $$\alpha_0 > 0$$. We have obtained the value $$\alpha_0$$.

**Step 5.** We now shrink the domain $$X$$ by taking pairs $$(R_0, R_1)$$ in a way that $$R_0 \to \infty$$ and $$R_1 \to 0$$. Note that if we take $$R_0$$ large then we can take the pair in a way that

$$R_1 \approx e^{-c'R_0}$$

(23)

for some constant $$c' > 0$$. In particular, the curve $$\gamma^{inv}$$ has the asymptotics

$$|\gamma^{inv}(t)| \leq e^{-c't},$$

(24)

as $$t \to \infty$$. From the estimate (24), it follows that $$\frac{\alpha}{x^{n+\beta}} \to \text{const}$$ as $$n \to \infty$).

**2.2 The Case of $$\beta > 0$$**

Proceeding analogously as in the case $$\beta = 0$$, we define the map

$$F_\beta : (x, y) \mapsto (y, -\frac{\lambda}{2\beta} y + \frac{\lambda}{2\beta} y \sqrt{1 + Z_\beta(x, y)^2}),$$

(25)

where

$$Z_\beta(x, y) = \frac{4\beta}{\lambda} (\sqrt{x^2 + \beta \lambda y^2 + \frac{1}{y}}),$$

(26)

for $$0 < \beta < \beta_0$$. We may take $$\beta_0 = 1$$ initially, but we will need to adjust it to be small (and unspecified) at several places from now on.

For convenience, set

$$m_\beta(x, y) = -\frac{\lambda}{2\beta} y + \frac{\lambda}{2\beta} y \sqrt{1 + Z_\beta(x, y)^2}.$$ 

(27)

It is easy to check that the map $$F_\beta$$ is well-defined as a map $$\mathbb{R}_+ \times \mathbb{R}_+$$ to itself and is injective in this region. Indeed, we will consider $$F_\beta$$ in the region $$\{(x, y) : x, y > 0, Z_\beta(x, y) \leq 1/2\}$$ so that we can take the Taylor expansion of $$(1 + Z_\beta(x, y))^{1/2}$$. This will be achieved by restricting to the values of $$(x, y) > 0$$ with $$x/y \leq r_0$$ and $$1/y \leq r_1$$. Any large values of $$r_0, r_1 > 0$$ are allowed at the cost of taking $$\beta_0$$ small.

By Taylor expanding $$(1 + Z_\beta(x, y))^{1/2}$$ and collecting terms of the same degree in $$x$$ and $$y$$, we obtain

$$m_\beta(x, y) = \sum_{n \geq 0} d_n^+(\beta) \frac{x^{n+1}}{y^n} + \sum_{n, k \geq 0} d_{n, k}^-(\beta) \frac{x^n}{y^{n+k}},$$

(28)
which is uniformly convergent for the set of pairs \((x, y)\) we consider. Notice that the term \(-\lambda y/2\beta\) gets cancelled and all the terms of (28) are \(O(1)\) as \(\beta \to 0\). With logarithmic change of coordinates \(u = \ln x\) and \(v = \ln y\), the form of \(F_{\beta}\) becomes \((u, v) \mapsto (v, m_{\beta}(u, v))\), with

\[
m_{\beta}(u, v) = 2u - v + \ln \left(\sum_{n \geq 0} d_n^+(\beta)e^{(n-1)(u-v)} + \sum_{n, k \geq 0} d_n^-(\beta)e^{(n-2)(u-v)-(k+1)v}\right)
\]

\[
= 2u - v + \ln \left(f_1(\beta)(u, v) + f_2(\beta)(u, v)\right) = 2u - v + c_0 + \ln \left(1 + (f_1(\beta)(u, v)/\lambda^2 - 1) + f_2(\beta)(u, v)\right),
\]

and one can check that the term \((f_1(\beta)(u, v)/\lambda^2 - 1)\) vanishes on the line \(\{u = v\}\). Diagonalizing the affine part by the change of coordinates \(a = u - v + \frac{1}{3}c_0\) and \(b = 2u + v\), we arrive at the form

\[
F_{\beta} : (a, b) \mapsto (-2a - e_{\beta}(a, b), b + c_0 + e_{\beta}(a, b)).
\]

From the expansion in (29), one can check that the error term \(e_{\beta}\) has the form

\[
e_{\beta}(a, b) = \ln \left(1 + g_1(\beta)(a) + g_2(\beta)(a, b)e^{-b/3}\right),
\]

with estimates

\[
|g_1(\beta)(a)| \leq \beta C|a|
\]

and

\[
|g_2(\beta)(a, b) - \lambda^{-26/9}| \leq \beta C
\]

in a region of the form \(|a| \leq \ln r_0, b \geq r_2 = r_2(r_0, r_1)\), for some constant \(C = C(r_0, r_2) > 0\), and for \(\beta \in (0, \beta_0)\). We can easily deduce bounds of the similar form for the partial derivatives of \(e_{\beta}(a, b)\).

The point is that, if we consider \(X = [-R_1, R_1] \times [R_0, \infty)\) with \(R_1 \ll 1\) and \(R_0 \gg 1\), then the expansions (28), (29) and the bounds (32), (33) are valid and we have uniform convergence \(\|e - e_{\beta}\|_X \to 0\), as \(\beta \to 0\). Therefore, Steps 1–3 from the previous section goes through literally in this case as well, with the only difference being that we may need to take \(R_1 \ll 1\) and \(R_0 \gg 1\). We conclude the existence of the invariant curve \(\gamma_{\beta}^{inv}\), for each \(0 < \beta < \beta_0\). By the uniform convergence in \(\beta\) of the error term, we deduce that the invariant curves themselves converge uniformly to \(\gamma^{inv}\). Since the intersection between \(F^N(L)\) and \(\gamma^{inv}\) was transversal for all \(N\) (whenever they intersect), possibly after taking a smaller value of \(\beta_0\), for \(0 < \beta < \beta_0\), there is an iterate \(F^N L\) of the initial line \(L\) which crosses \(\gamma_{\beta}^{inv}\). This takes care of Step 4. At this point, we have obtained the values \(\{\alpha(\beta)\}_{0 < \beta < \beta_0}\). Finally, Step 5 follows from the exponential decay of the error in \(b \to \infty\) of (31). To conclude the proof, it only remains to establish the following

**Lemma 1.** Let us denote the line segment \(\{\{t, 2t - \frac{2}{3}\ln 2^t\} : -0.4 \leq t \leq 0\}\) by \(I\). Then \(F^3(I)\) intersects \(\gamma^{inv}\) transversally.

**Proof.** We investigate each iterate of \(I\). To begin with, one sees that the image \(F(I)\) is indeed a segment of the line \(\{a + b = \ln 2^\beta\}\). An explicit computation, shows that \(F(I)\) contains the line segment

\[
J := \{(s, -s + \ln 2^s) : -0.1 \leq s \leq -0.01\}.
\]

From now on, we will always assume that the variable \(s\) takes values in \([-0.1, -0.01]\). Parametrizing the set \(F(J)\) by \(s\), we have

\[
(-2s, -s + \ln 2^s) + (-e(s), e(s)), \quad \text{with} \quad e(s) = \ln(1 + 2^{-\frac{3}{10}}e^{-2s}),
\]

8
Figure 1: A few forward $F$ iterates of the line segment $I := \{(t, 2t - 3 \ln \lambda^2) : -0.25 \leq t \leq -0.15\}$: the segments represent $I$, $F(I)$, $F^2(I)$, $F^3(I)$, and $F^4(I)$, from below to above.

with crude estimates $|e(s)| < 1/8$ and $|e'(s)| < 1/4$. Next, we parametrize the set $F^2(J)$ as follows:

$$(4s + 2e(s), -s + e(s) + \ln 2^{14/3}) + (-E(s), E(s)),$$

with $E(s) = \ln(1 + 2^{-42/9} e^{6s + 2e(s)})$, and we have $|E(s)| < 1/16$ and $|E'(s)| < 3/8$. We note that $F^2(J)$ is the graph of a strictly decreasing function defined on $-0.1 \leq s \leq -0.01$. Indeed, it is enough to check $(4s + 2e(s) - E(s))'> 0$ and $(-s + e(s) + E(s))'< 0$, which follows from the inequalities above. Moreover, from the same estimates, we see that $-s + e(s) + \ln 2^{14/3} + E(s) > 3$, that is, $F^2(J)$ lies above the line $b = 3$. Finally, if we plug in the values $s = -0.1$ and $s = -0.01$, we obtain

$$4s + 2e(s) - E(s) < -0.03, \quad 4s + 2e(s) - E(s) > +0.03$$

respectively, using $\ln(1 + x) \geq x/2$ for $0 < x < 1$. This concludes the proof.

\[
\]

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