Fair and Efficient Division among Families

Sophie Bade and Erel Segal-Halevi

November 26, 2018

Abstract

Fair division theory mostly involves individual consumption. But resources are often allocated to groups, such as families or countries, whose members consume the same bundle but have different preferences. Do fair and efficient allocations exist in such an "economy of families"?

We adapt three common notions of fairness: fair-share, no-envy and egalitarian-equivalence, to an economy of families. The stronger adaptation — individual fairness — requires that each individual in each family perceives the division as fair; the weaker one — family fairness — requires that the family as a whole, treated as a single agent with (typically) incomplete preferences, perceives the division as fair. Individual-fair-share, family-no-envy and family-egalitarian-equivalence are compatible with efficiency under broad conditions. The same holds for individual-no-envy when there are only two families. In contrast, individual-no-envy with three or more families and individual-egalitarian-equivalence with two or more families are typically incompatible with efficiency, unlike the situation in an economy of individuals. The common market equilibrium approach to fairness is of limited use in economies with families. In contrast, the leximin approach is broadly applicable: it yields an efficient, individual-fair-share, and family-egalitarian-equivalent allocation.
1 Introduction

Several siblings inherit their parents’ goods. Can they divide the goods among them in a fair and efficient way? This problem has been extensively studied in economics. A classic positive result by Varian (1974) shows that, under mild conditions, there exists a Pareto-efficient division that is also fair in the sense that no sibling envies another one. But now assume that each sibling is married and has children, so that the goods have to be divided among families. Once the goods are divided, all family-members consume the same bundle: they live in the same house, use the same furniture, and enjoy the same trees and flowers in the backyard. Family members may have different preferences: while the husband in one family may prefer his family’s bundle to bundles obtained by his siblings’ families (and thus feel no envy), his wife may prefer the bundle obtained by his sister’s family.

As another example, consider the division of disputed lands among several countries. Different citizens of the same country may have different preferences over shares. This might lead to difficulties in negotiating dispute-settlement agreements, as each citizen might have different ideas on which parts of the disputed resources are more important, and thus different ideas on the fairness of each agreement to settle the dispute.

These examples provoke two questions: (a) What divisions should be considered “fair” in an economy of families? (b) In what economies do fair and efficient allocations exist?

We investigate these questions considering three classic fairness criteria. An allocation has the fair-share guarantee (FS)\(^1\) if no agent likes the average bundle better than their own bundle; if each agent likes their bundle at least as much as any other agent’s bundle then the allocation satisfies no envy (NE)\(^2\) the allocation is egalitarian-equivalent (EE) if there exists a reference bundle that each agent believes to be equivalent to their share.

Considering either the families or the individuals in our economy as the agents, we present two different adaptations of these three notions of fairness. A division is individual-fair if all family members unanimously agree that it

\(^1\) Also known as proportionality.
\(^2\) Also known as envy-freeness.
satisfies the selected fairness notion (NE or FS or EE). A division is *family-fair* if it satisfies the selected fairness notion according to the family’s — typically incomplete — preferences, which rank one bundle above another bundle if the first bundle is preferred to the second by all its members. The notions of individual fairness are stronger than the notions of family-fairness: to be considered individual-fair, all family members have to unanimously agree on the comparison between their family’s share and the other families bundles or respectively the reference bundle.

The assumption that family members consume the same bundle has two opposing effects: For one allocation to Pareto dominate another all members of all families must prefer their family’s bundle under the first allocation. So the criterion of Pareto domination becomes harder to satisfy and there should be a larger set of Pareto optima. Conversely, the existence of families makes individual fairness harder to satisfy, since all members of each family should agree that their share is fair. Since the existence of families affects efficiency and individual fairness in opposite directions, it is a-priori not clear when Pareto optimal and individual-fair allocations exist. It is however expected that the conditions which suffice for the existence of fair Pareto optima in the standard model suffice for the existence of family-fair Pareto optima.

Our main results are:

- Individual-FS Pareto optima exist for any number of families, under mild compactness assumptions (Theorem 4.1). In particular, lexicmin-optimal allocations exist, they are efficient, and with the right selection of utility functions they are also individual-FS.

- Individual-NE Pareto optima exist for two families whose members have convex preferences (Theorem 5.1), but might not exist for three or more families even with strictly monotonic convex preferences (Theorem 5.2).

- Family-NE Pareto optima, that are also individual-FS, exist for any number of families whose members have locally-non-satiated strictly convex preferences (Theorem 5.5).

- Individual-EE Pareto optima might not exist even for two families
whose members have strictly monotonic convex preferences (Theorem 6.1).

- Family-EE Pareto optima, that are also individual-FS, exist for any number of families whose members have strictly monotonic strictly convex preferences (Theorem 6.2). Without strict convexity, it is possible to guarantee any two of the three conditions (family-EE, individual-FS and Pareto-optimality) but not all three (Theorem 6.5).

In addition, we investigate the relation between fairness and market equilibrium. In a standard economy, whenever there exists a market equilibrium from equal endowments, it is both Pareto-optimal and has no envy (Varian, 1974). We show that this relation does not hold in an economy of families.

Firstly, we show in Proposition 5.3 that even if individual-NE Pareto optima exist, it may be impossible to find them as market equilibria from equal endowments. To attain individual-NE, families with divergent preferences may need more income than families with homogeneous preferences. Say three families care about only two goods: donations to charity and the acquisition of modern art. For a family which cares about both not to envy two other families who respectively only care about charity or art, the family with members of either sort needs to spend at least as much on art and charity as do the families who only spend on only one of the two. Similarly, while we show in Theorem 5.5 that economies with strictly convex and locally non-satiated preferences always have family-NE and individually-FS Pareto optima, we show in Proposition 5.4 that market equilibria from equal endowments need not have the two fairness properties. So money and markets do not guarantee fair outcomes in economies with families. Safeguards for families with discordant preferences or for individuals within families may be required to obtain fair allocations.
2 Related work

2.1 Fairness criteria

The modern study of fairness in economics was initiated by Steinhaus (1948). He proved the existence of fair share allocations of a heterogeneous good ("cake"). Since then, fairness has been extensively studied in economics (Young, 1995; Moulin, 2004; Thomson, 2011) as well as in other disciplines such as mathematics and computer science.

The existence of Pareto-optimal no-envy allocations was initially proved as a consequence of the existence of market equilibrium. If all preferences are convex and strictly monotonic, then there exists a market equilibrium from equal incomes, and it is both Pareto-optimal and envy-free. Varian (1974) showed that the convexity of preferences can be replaced by a different condition: for each weakly-Pareto-optimal utility profile, the set of allocations with this utility profile is a singleton. Svensson (1983) showed that it is sufficient that this set of allocations be convex. Diamantaras (1992) showed that it is sufficient that this set of allocations be contractible. He showed that this is true even for economies with public goods. Svensson (1994) proved existence under a different condition which he termed "sigma-optimality". In contrast to all these general existence results, we show that individual-NE Pareto optima rarely exist in economies with families — even when all agents have well-behaved preferences and there are very few families.

The egalitarian equivalence criterion was introduced by Pazner and Schmeidler (1978). They proved that a Pareto-optimal egalitarian-equivalent allocation exists even in economies with production, in contrast to Pareto-optimal no-envy allocation (Vohra, 1992). In contrast, we show non-existence in economies with families, even when there are two families and no production.

2.2 Incomplete preferences

Formally the families in this paper are agents with incomplete preferences that can be represented by vector valued utilities. Viewed that way our paper contributes to the literature on behavioral welfare economics pioneered by
Fon and Otani (1979); Bernheim and Rangel (2007, 2009); Mandler (2014, 2017); Fleurbaey and Schokkaert (2013). Consider an economy where all agents view all options through a set of different frames, as suggested by Salant and Rubinstein (2008). The notions of individual and family fairness then respectively require that an allocation is fair according to each frame of an agent or according to the incomplete preferences of the agent. If the agent uses different frames in different points of time or to evaluate uncertainty we can apply our results to economies where agents have $\beta - \delta$-preferences following Laibson (1997); ODonoghue and Rabin (1999) or exhibit Knightian uncertainty following Bewley (2002).

2.3 Public and club goods

The existence of fair allocations when some of the goods are public has been studied e.g. by Diamantaras (1992); Diamantaras and Wilkie (1994, 1996); Guth and Kliemt (2002). There, each good is either private (consumed by a single agent) or public (consumed by all agents).

We conversely consider club goods, that is goods that are public inside a family — but private outside (i.e, all family members enjoy the same bundle, but members of other families cannot enjoy it). Club goods were popularized by Buchanan (1965) and studied in various contexts. The literature on club goods studies questions such as optimal number of members in a club, optimal quantity of club-good provision, pricing policies and exclusion mechanisms (Sandler and Tschirhart, 1980; Hillman, 1993; Sandler and Tschirhart, 1997; Loertscher and Marx, 2017; Mackenzie and Trudeau, 2017). As far as we know, fair allocation of goods among different clubs has not been considered yet.\(^3\)

2.4 Families and groups

In contrast to the literature on club goods, in this paper we do not consider congestion effects: the family sizes are fixed, and all members of a family consume the same bundle regardless of the family size.

\(^3\)In contrast to the literature on club goods, in this paper we do not consider congestion effects: the family sizes are fixed, and all members of a family consume the same bundle regardless of the family size.
problem of fairly dividing a single heterogeneous resource (“cake”) or sev-
eral indivisible goods among families. The requirements in these problems
are quite different than ours: in cake-cutting the main requirement (besides
fairness) is connectivity, and with indivisible goods fairness typically can-
not be guaranteed, so the focus is on finding appropriate approximations to
fairness. They show that, in most settings, individual-fairness (which they
call “unanimous-fairness”) is not guaranteed to exist. Therefore, they relax
the fairness notion by requiring that only a certain fraction of the mem-
bers in each family perceive the division as fair (they call this relaxation
“democratic-fairness”).

It is important to distinguish our family-based fairness notions from two
different notions of group fairness.

(a) One notion of group-fairness involves the standard resource-allocation
setting in which each individual receives an individual bundle (Berliant et al.
1992, Husseinov 2011, Dall’Aglio et al. 2009, Dall’Aglio and Di Luca 2014,
Todo et al. 2011, Mouri et al. 2012). A group-envy-free division is defined
as a division in which no coalition of individuals can take the pieces allocated
to another coalition with the same number of individuals and re-divide the
pieces among its members such that all members are weakly better-off. In
our setting the families are fixed and agents do not form coalitions on-the-fly;
the challenge arises from the fact that all individuals in each family consume
the same bundle.

(b) A second notion of group-fairness comes from an entirely different
field — artificial intelligence. Consider an AI system that automatically
detects potential criminals based on their personal traits. If such a system
reports significantly more suspects with a certain skin-color, this might be
considered a violation of group-fairness — the members of the group with
that particular skin-color are treated unfairly. There is a growing literature
on various definitions of group-fairness in this context; see, for example,
Dwork et al. (2012), Hébert-Johnson et al. (2017) and the references therein.
3 Preliminaries

3.1 Economies

Let $I$ and $F$ denote the set of individuals and families. Each individual $i \in I$ belongs to exactly one family $f \in F$, where $\phi i$ is the family $f$ with $i \in f$. A family $f$ may contain one or more individuals.

There are $G$ different homogeneous divisible goods and $\mathbb{R}^G_+$ is the set of consumption bundles (the non-negative orthant of the $G$-dimensional Euclidean space). There is a total endowment $e \in \mathbb{R}^G_+$ with $e \gg 0$. The average bundle available to each family, also called the fair share, is $\bar{e} := \frac{1}{|F|} e$. An allocation is a vector $\{x_f\}_{f \in F}$ of consumption bundles $x_f \in \mathbb{R}^G_+$ whose sum does not exceed the aggregate endowment: $\sum_{f \in F} x_f \leq e$. The set of all allocations is denoted by $X$.

The preferences of each individual $i \in I$ over bundles in $\mathbb{R}^G_+$ can be represented by a continuous utility function $u_i : \mathbb{R}^G_+ \rightarrow \mathbb{R}$. Individual $i$’s utility of an allocation $x$ depends only on the bundle $x_{\phi i}$ consumed by his family. An economy is formally defined by $(I, F, \mathbb{R}^G_+, \{u_i\}_{i \in I}, e)$.

The preference represented by $u_i$ is:

- **strictly monotonic** if $x > x'$ implies $u_i(x) > u_i(x')$ for all $x, x' \in \mathbb{R}^G_+$;
- **locally non-satiated** if for every $x \in \mathbb{R}^G_+$ and $\epsilon > 0$, there exists some $x' \in \mathbb{R}^G_+$ with $d(x, x') < \epsilon$ such that $u_i(x') > u_i(x)$;
- **strictly convex** if $u_i(x) \geq u_i(x')$ together with $x \neq x'$ implies $u_i((1 - \alpha)x + \alpha x') > u_i(x')$ for all $x, x' \in \mathbb{R}^G_+$ and $\alpha \in (0, 1)$;
- **convex** if $u_i(x) > u_i(x')$ implies $u_i((1 - \alpha)x + \alpha x') > u_i(x')$ for all $x, x' \in \mathbb{R}^G_+$ and $\alpha \in (0, 1)$.

Note that strict monotonicity implies local non-satiation and that strict convexity implies convexity.

---

4For any two vectors $x, x' \in \mathbb{R}^m$ for some integer $m$, say $x \geq x'$ if $x_i \geq x_i'$ for all $i$, $x > x'$ if $x \geq x'$ but not $x = x'$ and $x \gg x'$ if $x_i > x_i'$ for all $i$. The Euclidean distance between $x, x' \in \mathbb{R}^m$ is denoted $d(x, x')$. 

8
We derive family $f$’s — typically incomplete — preferences $\succsim_f$ from the preferences of all its members. Family $f$ weakly prefers bundle $x$ to $x'$, denoted $x \succsim_f x'$, if and only if $u_i(x) \geq u_i(x')$ holds for all $i \in f$. If $x \succsim_f x'$ as well as $x' \succsim_f x$ hold then family $f$ is indifferent between $x$ and $x'$, denoted $x \sim_f x'$. Note that $x \sim_f x'$ holds if and only if each member of $f$ is indifferent between $x$ and $x'$. If $x \succsim_f x'$ holds but $x \sim_f x'$ does not, then family $f$ strictly prefers $x$ to $x'$, denoted $x \succ_f x'$. If $u_i(x) > u_i(x')$ and $u_j(x) < u_j(x')$ hold for two family members $i, j \in f$ then the family cannot rank $x$ and $x'$, denoted $x \nhd_f x'$.

All individuals — and therefore all families — only care about their own consumption. So an individual prefers one allocation to a different allocation if he prefers his family’s bundle in the first allocation to his family’s in the second allocation. Similarly a family prefers a first allocation to a second one if it prefers its bundle in the first allocation to its bundle in the second allocation.

3.2 Fairness and Efficiency

An allocation $x$ satisfies individual-no envy (is individual-NE) if no individual envies any other individual, i.e., there are no two individuals $i, i'$ with

$$u_i(x_{\phi i'}) > u_i(x_{\phi i}).$$

An allocation $x$ satisfies family-no envy (is family-NE) if no family envies another family, i.e., there are no two families $f, f'$ with:

$$x_{f'} \succ_f x_f.$$

An allocation $x$ is individual-egalitarian-equivalent (individual-EE) if there exists a bundle $r$ (called the reference bundle) such that, for every individual $i \in I$:

$$u_i(x_{\phi i}) = u_i(r)$$

An allocation $x$ is family-egalitarian-equivalent (family-EE) if there exists a bundle $r$ such that each family $f$ who can rank $x_f$ and $r$ is indifferent between
the two bundles. I.e., for every family \( f \):

\[
x_f \sim_f r \quad \text{or} \quad x_f \not\sim_f r.
\]

An allocation \( x \) satisfies the \textit{individual fair-share guarantee} (is individual-FS) if no individual strictly prefers the fair share \( \overline{e} \) (which contains a fraction \( \frac{1}{|F|} \) of each good in the endowment) to his/her family’s bundle. So \( x \) is individual-FS if for every \( i \):

\[
u_i(x_{\phi i}) \geq u_i(\overline{e}).
\]

An allocation \( x \) satisfies the \textit{family fair-share guarantee} (is family-FS) if no family strictly prefers the fair share \( \overline{e} \) to its bundle. i.e., there is no family \( f \) with:

\[
e \succ_f x_f.
\]

Any individual-NE allocation \( x \) is family-NE: if \( u_i(x_{\phi i}) \geq u_i(x_{\phi i}') \) holds for each individual \( i \) then \( x_f \preceq_f x_f' \) holds for each family \( f \). The converse does not hold: in a family-NE allocation, some family \( f \) may be unable to rank their own bundle \( x_f \) and some other family’s bundle \( x_f' \), implying that \( u_i(x_f') > u_i(x_f) \) holds for some member \( i \) of family \( f \), so that the allocation does not satisfy individual-NE. Similarly any individual-EE allocation is family-EE and every individual-FS allocation is family-FS. The converse does not hold in either case.

The set of all feasible individual-FS allocations is \( E := \{ x : \sum_{f \in F} x_f \leq e \text{ and } u_i(x_{\phi i}) \geq u_i(\overline{e}) \text{ for all } i \in I \} \). Since \( E \) contains the allocation in which each family receives \( \overline{e} \) it is non-empty. It is compact since it is bounded and since each utility \( u_i \) is continuous.

A triplet \((p,x,x^0)\) with \( x,x^0 \in X \) and \( p \in \mathbb{R}^G \) is a \textit{market equilibrium from endowment} \( x^0 \), if for each family and each bundle \( x' \) (a) \( px_f \leq px_f^0 \), and: (b) \( x' \succ_f x_f \) implies \( px' > px_f^0 \). So \((p,x,x^0)\) is a market equilibrium from endowment \( x^0 \) if each bundle \( x_f \) is affordable for that family and if any bundle \( x' \) a family strictly prefers to \( x_f \) is unaffordable for that family. If \( x^0 = (\overline{e}, \ldots, \overline{e}) \), then \((p,x,x^0)\) is called a \textit{market equilibrium from equal endowments}.
An allocation $x$ is Pareto dominated by a different allocation $x'$ if for every individual $i$: $u_i(x_{\phi i}') \geq u_i(x_{\phi i})$, and for at least one individual $j$: $u_j(x_{\phi j}') > u_j(x_{\phi j})$. Equivalently, $x$ is Pareto-dominated by $x'$ if for every family $f$: $x'_f \succ_f x_f$, and for at least one family: $x'_f \succ_f x_f$. An allocation $x$ is Pareto-optimal in a set $S$ if it is not Pareto-dominated by any allocation in the set $S$. An allocation that is Pareto optimal in the set of all allocations $X$ is simply called Pareto-optimal.

A leximin optimal allocation (Dubins and Spanier, 1961; Moulin, 2004) lexicographically maximizes the utilities from smallest to largest. A leximin optimal allocation is always defined with respect to specific utility functions. Formally, given some subset $X' \subseteq X$, define a subset $X^1 \subseteq X'$ as the set of allocations $x$ that maximize the smallest utility $u_i(x)$ over all $i \in I$ and $x \in X'$. Supposing it exists, define $X^2 \subseteq X^1$ as the set of allocations $x$ that maximize the second smallest utility $u_i(x)$ over all $i \in I$ and $x \in X^1$. Proceeding inductively, define $X^{[l]} \subseteq X^{[l]-1}$ as the set of allocations $x$ that maximize the $|I|^\text{th}$ smallest utility (i.e., the largest utility) over all $i \in I$ and $x \in X^{[l]-1}$. The set $X^{[l]}$ (if it exists) is called the set of $\{u_i\}_{i \in I}$-leximin optimal allocations in $X'$.

### 3.3 Examples with two goods

If there are only two goods in an economy, we call these two goods $y$ and $z$.

**Marginal rate of substitution.** If a utility function $u_i : \mathbb{R}^2_+ \rightarrow \mathbb{R}$ is differentiable, we define the marginal rate of substitution between $y$ and $z$, $MRS_i(y, z)$, as:

$$MRS_i(y, z) := \frac{d(u_i(y, z))}{dy} \bigg/ \frac{d(u_i(y, z))}{dz}.$$

**Single-crossing property** Two different strictly monotonic preferences represented by $u_i$ and $u_j$ satisfy the single crossing property if any two indifference-curves defined by $u_i(y, z) = \alpha$ and $u_j(y, z) = \beta$ for some $\alpha, \beta \in \mathbb{R}$ have at most one point in common. So $u_i$ and $u_j$ have the single-crossing property if for all bundles $(y, z)$ and $(y', z')$ and either $u_i = u$, $u_j = u'$ or
If one individuals’ marginal rate of substitution is higher than the other’s at each possible bundle \((y, z)\), then \(u_i\) and \(u_j\) satisfy the single crossing property.

An agent’s preference has a Cobb-Douglas utility representation if \(u_i(y, z) = y^{\alpha_i}z^{1-\alpha_i}\) holds for some \(\alpha_i \in (0, 1)\). Any such preferences are strictly monotonic and strictly convex. Two different Cobb-Douglas utilities satisfy the single crossing property.

Our proofs and examples use the following characterization of Pareto-optimal allocations among families. This characterization is a well-known result from (behavioral) welfare economics, however, it is convenient to rephrase it here in terms of family economies.

**Theorem 3.1** Consider a two-good-economy \((I, F, \mathbb{R}^2_+, \{u_i\}_{i \in I}, e)\) with differentiable utilities. Let \(x\) be an allocation in the interior of \(X\). Then \(x\) is Pareto optimal if and only if:

\[
\bigcap_{f \in F} \left[ \min_{i \in f} MRS_i(y_f, z_f), \max_{i \in f} MRS_i(y_f, z_f) \right] \neq \emptyset
\]

I.e., for each family \(f\), we define its MRS range as the interval between the smallest and the largest MRS of its members; the allocation is Pareto optimal if the common intersection of all \(|F|\) MRS ranges is non-empty.

As a special case, two single-agent families \(f = \{s\}\) and \(f' = \{s'\}\) whose utilities \(u_s\) and \(u_{s'}\) satisfy the single-crossing property must consume different bundles in any interior Pareto optimum.

This result has been shown to hold in (much) more general environments by [Fon and Otani (1979); Mandler (2014)], so we omit its proof here. The statement can also be proven with some simple modifications of the arguments in [Mas-Colell (1974)].
4 Fair Share Guarantee

In this short section, we show that individual-FS Pareto optima exist under generic conditions.

**Theorem 4.1** Any economy has an individual-FS Pareto optimum.

**Proof** Consider the set $E$ of all feasible individual-FS allocations defined above. Since $E$ is compact, there exists an allocation $x^*$ that maximizes the sum of all individuals utilities over all $x \in X$, so $x^* \in \arg \max_{x \in E} \sum_{i \in I} u_i(x_{\phi_i})$. Since $x^*$ is not Pareto-dominated by any $x' \in E$ and since $u_i(x^*_{\phi_i}) \geq u_i(\bar{e})$ holds for each $i$, $x^*$ is not Pareto dominated by any $x' \notin E$. So $x^*$ is Pareto optimal. Since $x^* \in E$ it is individual-FS. □

**Remark 4.2** An alternative proof of Theorem 4.1 relies on $\{u_i\}_{i \in I}$-leximin optimal allocations. For this proof normalize all agents utilities such that $u_i(\bar{e}) = 0$ for all $i \in I$. Since $X$ is compact and since all $u_i$ are continuous, each subset $\{X^i\}_{i \in I}$ in the definition of $\{u_i\}_{i \in I}$-leximin optimal allocations is well-defined. So a $\{u_i\}_{i \in I}$-leximin optimal allocation $x^*$ exists. Since $(\bar{e}, \ldots, \bar{e})$ is feasible, $u_i(x^*_{\phi_i}) \geq u_i(\bar{e})$ holds for all $i$, so $x^*$ is FS. Since $x^*$ is $\{u_i\}_{i \in I}$-leximin optimal, it is Pareto optimal.

5 No Envy

In this section, we prove that individual-NE Pareto optima exist under generic conditions when there are only two families, but might not exist with more than two families. Even if individual-NE Pareto optima exist, it might be impossible to find them as market equilibria from equal endowments.

The different examples used to prove the impossibility results do not rely on extreme conditions: each of these examples has only two goods and three families, two of which are singletons. All individuals have plain vanilla preferences — strictly convex and strictly monotonic.
5.1 Individual-no envy with just two families

**Theorem 5.1** Any economy with exactly two families and convex preferences has an individual-NE and individual-FS Pareto optimum.

**Proof** By Theorem 4.1, there exists an individual-FS Pareto optimum $x$. To see that $x$ is individual-NE, suppose some individual envies the other family, so that $u_i(x_{\phi i}) < u_i(e - x_{\phi i})$ holds for some $i$. By convexity, $u_i(x_{\phi i})$ must also be smaller than $u_i(\frac{1}{2}x_{\phi i} + \frac{1}{2}(e - x_{\phi i}))$. But the latter bundle is exactly the fair share $\bar{e}$ — a contradiction to the assumption that $x$ is individual-FS.

5.2 Individual-no envy with more than two families

Theorem 5.1 does not extend to more than two families. In Proposition 5.2 we show that economies with three or more families need not have individual-NE Pareto optima. To show this we construct an example with two goods and three families, only one of which is made up of a husband and a wife. The two others are singles. All preferences satisfy the single crossing property: we assume that the two singles’ preferences are intermediate between the husband’s the wife’s preferences. This assumption implies that the family must consume the same bundle as each of the singles: otherwise, either the husband or the wife would by the single crossing property envy one of the two singles. A contradiction the arises since the two singles do not consume the same bundle in any interior Pareto optimum (see Theorem 3.1). The proof is illustrated in Figure 1.

**Proposition 5.2** Some economies with three families have no individual-NE Pareto optimum.

**Proof** Consider a two-good-economy with four agents $I: = \{h, w, s, s\}'$ belonging to three families $F: = \{\{h, w\}, \{s\}, \{s\}'\}$. Each individual $i \in I$ has a strictly monotonic preference represented by a differentiable utility $u_i: \mathbb{R}^2_+ \to \mathbb{R}$, and all preferences satisfy a single-crossing property so that
Figure 1: An illustration of Proposition 5.2. There are four individuals — wife, husband and two singles $s$ and $s'$. Say $(y_f, z_f)$ is the bundle consumed by the family consisting of the husband and wife. The single-crossing property implies that the indifference curves of the husband, wife and the two singles through $(y_f, z_f)$ only cross at that point. For $s$ to not envy the family, he must consume at or above his green indifference curve. If his consumption is at the north-west of the family’s bundle, then the husband envies; if it is at the south-east, then the wife envies; hence $s$ must consume exactly $(y_f, z_f)$. The same is true for $s'$. But then the allocation cannot be Pareto-optimal, since the two singles have different MRS-s at that bundle.
for any two bundles \((y, z), (y', z')\) if

\[
\text{If } y' > y \text{ and } z' < z \text{ then:} \\
u_h(y', z') \geq u_h(y, z) \Rightarrow u_{s'}(y', z') > u_{s'}(y, z), \\
u_{s'}(y', z') \geq u_s(y, z) \Rightarrow u_s(y', z') > u_s(y, z), \\
u_s(y', z') \geq u_s(y, z) \Rightarrow u_w(y', z') > u_w(y, z).
\]

\[
\text{If } y' < y \text{ and } z' > z \text{ then:} \\
u_w(y', z') \geq u_w(y, z) \Rightarrow u_{s'}(y', z') > u_{s'}(y, z), \\
u_s(y', z') \geq u_s(y, z) \Rightarrow u_{s'}(y', z') > u_{s'}(y, z), \\
u_{s'}(y', z') \geq u_{s'}(y, z) \Rightarrow u_h(y', z') > u_h(y, z).
\]

Consider an individual-NE allocation \(x = ((y_f, z_f), (y_s, z_s), (y_{s'}, z_{s'}))\). For \(s\) not to envy \(f\), we must have:

\[
u_s(y_s, z_s) \geq u_s(y_f, z_f)
\]

Since all preferences are strictly monotonic, and since \(h\) and \(w\) may not envy \(s\), there are only three options regarding the relation of \((y_f, z_f)\) and \((y_s, z_s)\):

- If \(y_s > y_f\) and \(z_s < z_f\) then \(u_s(y_s, z_s) \geq u_s(y_f, z_f)\) and single-crossing imply that \(u_w(y_s, z_s) > u_w(y_f, z_f)\) so the wife envies \(s\).

- If \(y_s < y_f\) and \(z_s > z_f\) then \(u_s(y_s, z_s) \geq u_s(y_f, z_f)\) and single-crossing imply that \(u_h(y_s, z_s) > u_h(y_f, z_f)\) so the husband envies \(s\).

So we must have \((y_s, z_s) = (y_f, z_f)\).

Applying the same arguments to \(s'\) we see that also \((y_{s'}, z_{s'}) = (y_f, z_f)\) holds, so \((y_{s'}, z_{s'}) = (y_s, z_s)\). But then, Theorem 3.1 implies that the allocation \(x\) cannot be Pareto-optimal. □

\[\text{5For example, we can take agents with Cobb-Douglas preferences with different coefficients, such that the coefficients of the two singles \(s, s'\) are between the coefficient of the husband \(h\) and the coefficient of the wife \(w\).}\]
5.3 Market equilibrium and individual-no envy

The classic proof of the existence of NE Pareto optima (Varian, 1974) uses market equilibrium from equal endowments: consider an environment where the existence of market equilibria is guaranteed. Equalize each individual's endowment, so that each individual is endowed with a fair share. Any market equilibrium from this endowment satisfies the fair share guarantee and no envy: first, note that the fair share is in each individual's budget set. So each agent must weakly prefer his equilibrium consumption to the fair share. Since all agents have the same endowment, they all face the same budget set. Since each individual maximizes his utility for this given budget set, no individual envies any other.

This technique does not fare well in economies with families. Proposition 5.2 already shows that individual-NE Pareto optima need not exist. But even if such individual-NE Pareto optima exist, they may not arise as a market equilibrium from equal endowments.

**Proposition 5.3** Some economies with families have individual-NE Pareto optima, where no individual-NE Pareto optimum is a market equilibrium from equal endowments.

**Proof** Consider a two-good-economy \((I,F,\mathbb{R}^G_+,\{u_i\}_{i\in I},e)\) with four agents \(I: = \{h,w,s,s'\}\) belonging to three families \(f: = \{h,w\}, \{s\} \) and \(\{s'\}\). There are 3 units of good \(y\) and of good \(z\). The husband \(h\) and the single person \(s\) have the same Cobb-Douglas preference with coefficient \(\alpha_h = \frac{1}{3}\), the wife \(w\) and the single person \(s'\) have the same Cobb-Douglas preference with coefficient \(\alpha_w = \frac{2}{3}\).

For \(((y_s,z_s),(y_{s'},z_{s'}),(y_f,z_f))\) to be an allocation \(y_s + y_{s'} + y_f = 3\) and \(z_s + z_{s'} + z_f = 3\) have to hold. For the singles and the family members not to envy each other \(y_s^\frac{1}{2}z_s^\frac{2}{3} = y_f^\frac{1}{2}z_f^\frac{2}{3}\) and \(y_s^\frac{2}{3}z_s^\frac{1}{3} = y_f^\frac{2}{3}z_f^\frac{1}{3}\) must hold. If \(((y_s,z_s),(y_{s'},z_{s'}),(y_f,z_f))\) is on the boundary of \(X\) then all members of at least one family have zero utility. Since the members of this family may not envy the other families, each agent must have utility zero in the allocation \(((y_s,z_s),(y_{s'},z_{s'}),(y_f,z_f))\) which can therefore not be Pareto optimal. So \(((y_s,z_s),(y_{s'},z_{s'}),(y_f,z_f))\) must be in the interior of \(X\) and \(\frac{1}{2}y_s = \)
\( MRS_s(y_s, z_s) = MRS_{s'}(z_{s'}, y_{s'}) = 2\frac{y_{s'}}{z_{s'}} \) must hold by Theorem 3.1. The resulting system of 5 equations in 6 unknowns has a continuum of solutions. To obtain an example we additionally impose the symmetry condition \( y_f = z_f \). The resulting system has a unique solution, illustrated in Figure 2.

Now, suppose that some individual-NE Pareto optimum \(((y_s, z_s), (y_{s'}, z_{s'}), (y_f, z_f))\) could be obtained as a market equilibrium from equal endowments. By Theorem 3.1 \((y_s, z_s) \neq (y_{s'}, z_{s'})\). Therefore, either \((y_f, z_f) \neq (y_s, z_s)\) or \((y_f, z_f) \neq (y_{s'}, z_{s'})\) (or both). W.l.o.g. suppose that the former is true. Since all individuals’ preferences are strictly convex, \((y_s, z_s)\) is according to \(u_s\) the unique optimal bundle in the budget set. Since \((y_f, z_f) \neq (y_s, z_s)\) and since \((y_f, z_f)\) is in the budget set, it is according to \(u_s\) strictly worse than \((y_s, z_s)\). Since \(u_h = u_s\), the husband envies \(s\) - a contradiction. To obtain an individual-NE Pareto optimum as a market equilibrium, the family must have a higher income than the singles. \(\square\)

Family size is not the relevant factor in the above proof, intra-family heterogeneity is: the same non-existence result holds if we replace each single with a family of one hundred individuals with identical preferences. Therefore, the disconnection between fairness and market equilibrium identified in Proposition 5.3 can not be fixed by larger families higher budgets.

---

6This solution is such that

\[
\begin{align*}
y_s &\approx 0.654 \\
y_{s'} &\approx 1.308 \\
y_f &\approx 1.0381 \\
z_s &\approx 1.308 \\
z_{s'} &\approx 0.654 \\
z_f &\approx 1.0381
\end{align*}
\]

Since \(((y_s, z_s), (y_{s'}, z_{s'}), (y_f, z_f))\) is a solution to the above system of equations, the singles and the family members do not envy each other and there are no Pareto improvements involving only the two singles. It is easy to check (numerically or graphically) that also the singles do not envy each other, so the allocation is individual-NE. Moreover, the singles’ marginal rate of substitution at \((y_s, z_s)\) and respectively \((y_{s'}, z_{s'})\) lies between the husband’s and wife’s marginal rate of substitution at \((y_f, z_f)\). So by Theorem 3.1 the allocation is Pareto optimal.
Figure 2: An illustration of Proposition 5.3. The three blue discs denote an allocation among the families. The husband and single $s$ are indifferent between the family’s bundle and single $s$’s bundle as represented by the bold indifference curve through these two bundles. The wife and single $s'$ are indifferent between the family’s bundle and single $s'$’s bundle as represented by the dashed indifference curve through these two bundles. Since each individual consumes on an indifference curve that is at or above the indifference curves of the two other bundles, the allocation is individual-NE. The allocation is Pareto optimal, since the MRS of $s$ and $s'$ at their bundles, represented by the thick dotted line lies between the husband’s and the wife’s MRSs at their family’s bundle (represented by the thin dotted lines). To support this allocation as a market equilibrium, both singles have to face the budget set with the thick dotted line as its frontier. The family’s bundle is not affordable with this budget.
5.4 Market-equilibrium and the individual fair share guarantee

While we have shown in Theorem 4.1 that any economy has an individual-FS Pareto optimum, the following proposition shows that market equilibria need not be individual-FS.

**Proposition 5.4** Some economies have market equilibria from equal endowments that are not individual-FS.

**Proof** Consider a two-good-economy \((I, F, \mathbb{R}_+^2; \{u_i\}_{i \in I}, e)\) with four agents \(I = \{h, w, h', w'\}\) belonging to two families \(f = \{h, w\}\) and \(f' = \{h', w'\}\). Each individual \(i \in I\) has a Cobb-Douglas utility with coefficients \(\alpha_h = .9, \alpha_w = .6, \alpha_{w'} = .4\), and \(\alpha_{h'} = .1\). Then \((1, ((.9, .1), (.1, .9)), (\bar{e}, \bar{e}))\) is a market equilibrium from equal endowments: At the price \(p = 1\), each husband gets his best affordable bundle. However both wives prefer \(\bar{e} = (.5, .5)\) to their respective bundles: \(.9^6 \cdot 1^4 < .5^6 \cdot 5^4 = .5\). □

5.5 Family-no envy Pareto optima that satisfy the individual fair share guarantee

We next show that family-NE individual-FS Pareto optima exist under general conditions. In the upcoming proof of Theorem 5.5 we use a modified equilibrium approach in which the families’ choice sets are restricted to bundles that each member prefers to the fair share.

**Theorem 5.5** Any economy with locally non-satiated, strictly convex preferences has a family-NE and individual-FS Pareto optimum.

**Proof** We use Shafer and Sonnenschein (1975)’s result on the existence of equilibrium in abstract economies. To this end, we define an abstract economy \(\Gamma = (X_f, A_f, P_f)_{f=1}^{n+1}\), where each agent \(\{1, \ldots, n\}\) is a family and agent \(n + 1\) is a special agent called the “market maker”. The elements of the abstract economy are defined as follows:
• **Choice sets:** for each family \( f \in F \), the choice set is \( X_f := \{ x_f \in \mathbb{R}_+^G \mid x_f \leq e \text{ and } u_i(x_f) \geq u_i(e) \text{ for all } i \in f \} \). So each family only considers feasible bundles that are weakly-preferred by all members to the fair share \( e \). For the market-maker, \( X_{n+1} = \Delta \) is the simplex of all price-vectors, normalized so that their sum equals 1.

• **Action correspondences:** \( A_f : X_1 \times \cdots \times X_{n+1} \to X_f \). For each family \( f \in F \) and each \( (x_1, \ldots, x_n, p) \in X_1 \times \cdots \times X_{n+1} \) define \( A_f(x_1, \ldots, x_n, p) = A(\cdot, p) := \{ x'_f \in X_f \mid p \cdot x'_f \leq p \cdot e \} \) as the set of bundles \( x'_f \) that, at price \( p \), cost no more than the fair share \( e \). For the market maker \( n+1 \) and each \( (x_1, \ldots, x_n, p) \in X_1 \times \cdots \times X_{n+1} \) define \( A_{n+1}(x_1, \ldots, x_n, p) = A_{n+1}(\cdot, p) : = \Delta \), the full simplex.

• **Preference correspondences:** \( P_f : X_1 \times \cdots \times X_{n+1} \to X_f \). For each family \( f \in F \) and each \( (x_1, \ldots, x_n, p) \in X_1 \times \cdots \times X_{n+1} \) define \( P_f(x_1, \ldots, x_n, p) = P_f(x, \cdot) : = \{ x'_f \in X_f \mid u_i(x'_f) > u_i(x_f) \text{ for all } i \in f \} \) as the set of all bundles \( x'_f \) strictly preferred to \( x_f \) by all members of \( f \). For the market-maker \( n+1 \) and each \( (x_1, \ldots, x_n, p) \in X_1 \times \cdots \times X_{n+1} \) define \( P_{n+1}(x_1, \ldots, x_n, p) = P_{n+1}(\cdot, p) : = \{ p' \in \Delta \mid p'(\sum_{f=1}^n x_f - e) > p(\sum_{f=1}^n x_f - e) \} \).

We next show that the abstract economy \( \Gamma \) satisfies conditions a) b'), b''), c') and c'') of Shafer and Sonnenschein’s Theorem.

a) For each family \( f \in F \) the choice-set \( X_f \) is non-empty since it contains \( e \), convex since all agents’ preferences are strictly convex, and compact since it is bounded by \( e \) and closed by the continuity of all utilities \( u_i \). Since \( X_{n+1} = \Delta \) is non-empty, convex and compact, the conditions also apply for the the market maker \( f = n + 1 \).

b') \( A_f \) is continuous since it is for \( f \in F \) a standard budget correspondence and for \( f = n + 1 \) a constant.

b'') For \( f \in F \), \( A_f \) is non-empty since \( e \in A_f(x, \cdot) \); as a standard budget correspondence it is also convex. For \( f = n + 1 \) the conditions are obvious.
c') To see that $P_f$ has for each $f \in F$ an open graph, fix any $((x, p), x'_f)$ in the graph of $P_f$, so that $u_i(x'_f) > u_i(x_f)$ holds for all $i \in f$. Since $u_i$ is for each $i \in f$ continuous, there exists a small $\epsilon > 0$ such that $u_i(y'_f) > u_i(y_f)$ holds for all $(y'_f, y_f)$ with $d(y'_f, x'_f) < \epsilon$ and $d(y_f, x_f) < \epsilon$. So $P_f$ has an open graph. Since $P_{n+1}$ can be represented by a continuous utility function, $P_{n+1}$ also has an open graph.

c") For each $x \in X$, $x_f$ is not an element of the convex hull of $P_f(x, p)$. To see that this holds fix an arbitrary $(x, p) \in X_1 \times \cdots X_n + 1$ and note that the preferred-bundle set $P_f(x, p)$ is for each $f \in F$ and for the market maker convex, so that $P_f(x, p)$ equals its convex hull. Since $P_f(x, p)$ contains only bundles strictly better than $x_f$ it does not contain $x_f$.

By Shafer and Sonnenschein’s theorem this abstract economy has an equilibrium, which is a vector $(\bar{x}_1, \ldots, \bar{x}_n, \bar{p}) = (\bar{x}, p) \in X_1 \times \cdots \times X_n + \Delta$ with for all $f \in \{1, \ldots, n+1\}$:

1. $\bar{x}_f \in A_f(\bar{x}, \bar{p})$, and:
2. $P_f(\bar{x}, p) \cap A_f(\bar{x}, \bar{p}) = \emptyset$.

We now show that the allocation $\bar{x}$ is feasible, Pareto optimal, individual-FS and family-NE.

• $\bar{x}$ is feasible. Proof: Fix any $f \in F$. By (1) we have $\bar{p} \cdot \bar{x}_f \leq \bar{p} \cdot \bar{e}$. Since all agents $i \in f$ have locally non-satiated preferences, (2) implies $\bar{p} \cdot \bar{x}_f = \bar{p} \cdot \bar{e}$. Summing over all families we obtain $\bar{p} \cdot (\sum_{f=1}^{n} \bar{x}_f - e) = 0$. By (2) again, $A_{n+1}(\bar{x}, \bar{p}) \cap P_{n+1}(\bar{x}, \bar{p})$ equals $\emptyset$, so that $p \cdot (\sum_{f=1}^{n} \bar{x}_f - e) \leq \bar{p} \cdot (\sum_{f=1}^{n} \bar{x}_f - e) = 0$ holds for all $p \in \Delta$. This can only hold if $\sum_{f=1}^{n} \bar{x}_f - e = 0$. Hence $\bar{x}$ is feasible as claimed.

• $\bar{x}$ is individual-FS. Proof: For every $f \in F$, by (1), every bundle $\bar{x}_f$ is in $X_f$. By the definition of the choice sets $X_f$, every such bundle is weakly-better than $\bar{e}$ for each agent in $f$.

• $\bar{x}$ is family-NE. The proof uses the following observation. Fix any $f \in F$. There does not exist any bundle $y_f \in A_f(\bar{x}, \bar{p}) \setminus \{\bar{x}_f\}$ with $y_f \succeq_f \bar{x}_f$. Proof: Suppose such a bundle $y_f$ did exist. Since $A_f(\bar{x}, \bar{p})$
is convex, \( \frac{1}{2} \bar{x}_f + \frac{1}{2} y_f \) is also an element of \( \mathcal{A}_f(\bar{x}, \bar{p}) \). Since each agent \( i \in f \) has strictly convex preferences, and since each agent \( i \in f \) weakly prefers \( y_f \) to the (different) bundle \( \bar{x}_f \), each agent \( i \in f \) strictly prefers \( \frac{1}{2} \bar{x}_f + \frac{1}{2} y_f \) to \( \bar{x}_f \). But then we have \( \frac{1}{2} \bar{x}_f + \frac{1}{2} y_f \in P_f(\bar{x}, \bar{p}) \) a contradiction to (2).

Now, the affordability constraints (represented by the action sets \( \mathcal{A}_f(\bar{x}, \bar{p}) \)) are the same for all families \( f \in F \), so \( \bar{x}_f' \in \mathcal{A}_f(\bar{x}, \bar{p}) \) holds for any pair \( f, f' \in F \). By the preceding observation we have have that \( \bar{x}_f' \gtrsim_f \bar{x}_f \) can only hold if \( \bar{x}_f = \bar{x}_f' \). Therefore, \( x \) in family-NE.

- \( \bar{x} \) is Pareto optimal. \textit{Proof:} Suppose by contradiction that \( \bar{x} \) was Pareto dominated by some \( y \) with \( \sum_{f=1}^{n} y_f \leq e \). Since \( y \) Pareto-dominates \( \bar{x} \) and since \( \bar{x}_f \in X_f \) for each family, we have for each agent \( i \in f \) \( u_i(y_f) \geq u_i(\bar{x}_f) \geq u_i(\bar{x}) \). Since \( \sum_{f=1}^{n} y_f \leq e \) implies \( y_f \leq e \) we in sum have \( y_f \in X_f \) for each family \( f \in \{1, \cdots, n\} \).

Since \( y \neq \bar{x} \), \( y_f^* \neq \bar{x}_f^* \) must hold for some family \( f^* \in \{1, \cdots, n\} \).

Since \( y_f^* \gtrsim_{f^*} x_f^* \), the above observation implies that \( y_f^* \notin \mathcal{A}_{f^*}(\bar{x}, \bar{p}) \) and therefore \( y_f^* \cdot \bar{p} > x_f^* \cdot \bar{p} \). Summing over all families we obtain \( \sum_{f=1}^{n} \bar{p} \cdot y_f > \sum_{f=1}^{n} \bar{p} \cdot \bar{x}_f = \bar{p} \cdot e \). A contradiction arises since the feasibility of \( y \) implies \( \sum_{f=1}^{n} \bar{p} \cdot y_f = \bar{p} \cdot e \). So \( \bar{x} \) is Pareto optimal.

\( \square \)
6 Egalitarian Equivalence

6.1 Pareto-optimality contradicts individual-egalitarian equivalence

In the preceding section we showed that it does not take much for individual-NE and Pareto optimality to clash. Individual-EE is even more restrictive: even economies with two families need not have any individual-EE Pareto optima. To understand this difference between egalitarian equivalence in the standard model and individual-EE, fix an arbitrary bundle \( r \) and a family consisting of a husband and wife with differing preferences. While there generally are many bundles \( x \) such that a single agent is indifferent between \( x \) and \( r \), it is much harder to find bundles \( x \) such that both the husband and the wife are indifferent between \( x \) and \( r \). Indeed, if the husband’s and the wife’s preferences have the single crossing property then \( x = r \) is the only bundle such that the husband and the wife are indifferent between \( x \) and \( r \). In the upcoming non-existence proof we specify an economy with two such couples which must consume the same bundle in any individual-EE allocation. We obtain a contradiction by differentiating the preferences of the two couples enough, so that they must consume different bundles in any interior Pareto optimum.

**Proposition 6.1** Some economies with two families have no individual-EE Pareto optimum.

**Proof** Consider a two-good-economy with four individuals \( I = \{h, w, h', w'\} \) belonging to two families \( f = \{h, w\} \) and \( f' = \{h', w'\} \). All four individuals have strictly monotone and strictly convex preferences, satisfying the single-crossing property such that for each bundle \((y, z)\):

\[
MRS_w(y, z) < MRS_h(y, z) < MRS_{h'}(y, z) < MRS_{w'}(y, z)
\]

Assume that \( x \) is an individual-EE Pareto optimum and let \( r \) be its reference bundle. By the single crossing property there is exactly one bundle \((y_f, z_f)\) such that \( h \) and \( w \) are indifferent between that bundle and \( r \), namely \((y_f, z_f) = r \). By the same token, \( f' \) must consume \( r \) too, so that \( x = (r, r) \).
Since \( \max_{i \in f} MRS_i(r) < \min_{i \in f'} MRS_i(r) \) the allocation is by Theorem 3.1 not Pareto optimal. The proof is illustrated in Figure 3.

6.2 Family-egalitarian equivalent Pareto optima with the individual-fair share guarantee

While individual-EE Pareto optima exist only rarely, family-EE Pareto optima that are also individual-FS exist under quite general conditions:

**Theorem 6.2** If all individuals’ preferences are strictly monotonic and strictly convex, then some Pareto optimum is family-EE and individual-FS.

If the equal split \((\bar{e}, \ldots, \bar{e})\) is Pareto-optimal in some economy, then \((\bar{e}, \ldots, \bar{e})\) is a family-EE and individual-FS Pareto optimum. Our proof of Theorem 6.2 therefore focusses on the case where \((\bar{e}, \ldots, \bar{e})\) is not Pareto optimal. We start by showing that any economy with strictly monotonic and strictly convex where \((\bar{e}, \ldots, \bar{e})\) is not Pareto optimal has an individual-strict-FS allocation \(x\) in the sense that each individual \(i\) strictly prefers \(x_{\phi i}\) to the equal split \(u_i(x_{\phi i}) > u_i(\bar{e})\) for all \(i \in I\).

Given that Theorem 6.2 assumes monotonic preferences, we can normalize utilities so that \(u_i(t \cdot e) = t - \frac{1}{|F|}\) for each \(i\) and \(t \in \mathbb{R}\), implying that \(u_i(\bar{e}) = 0\) holds for each \(i\). We use leximin-optimal allocations relative to this specific choice of utility functions.

**Lemma 6.3** In any economy with strictly monotonic preferences, all families have the same minimum utility in each \(\{u_i\}_{i \in I}\)-leximin-optimal allocation

**Proof** Suppose by contradiction that some families have different minimum utility, say \(\min_{i \in f} u_i(x^*_{\phi i}) > \min_{i \in I} u_i(x^*_{\phi i})\) holds for some \(f\). Starting with \(x^*\), create a new allocation \(x'\) by redistributing a small amount of goods from \(f\) to all other families such that still \(\min_{i \in f} u_i(x'_{\phi i}) > \min_{i \in I} u_i(x^*_{\phi i})\). By strict monotonicity, \(u_i(x'_{\phi i}) > u_i(x^*_{\phi i})\) holds for all \(i \notin f\). But this means that the new allocation \(x'\) is \(\{u_i\}_{i \in I}\)-leximin-better than \(x^*\) — a contradiction. □
Figure 3: An illustration of Proposition 6.1. Family 1 has two members with solid indifference curves; Family 2 has two members with dotted indifference curves. To be indifferent between their bundle to some $r$, they must consume exactly $r$. But this allocation cannot be Pareto optimal — a Pareto improvement can be attained by giving the solid family $r + (\epsilon, -\epsilon)$ and giving the dotted family $r + (-\epsilon, \epsilon)$, for some small $\epsilon$. The same considerations are true for any $r$. 
Lemma 6.4 In any economy with strictly monotonic and strictly convex preferences, either the equal allocation \((\overline{e}, \ldots, \overline{e})\) is Pareto optimal, or there is an individual-strict-FS Pareto-optimum.

Proof If \((\overline{e}, \ldots, \overline{e})\) is \(\{u_i\}_{i \in F}\)-leximin optimal then it is Pareto-optimal so we are done. If not, Remark 4.2 implies the existence of a \(\{u_i\}_{i \in F}\)-leximin optimal allocation \(x^* \neq (\overline{e}, \ldots, \overline{e})\) in \(X\). It is individual-FS, i.e., \(\min_{i \in F} u_i(x^*_{\phi_i}) \geq 0\); we show that it is also individual-strict-FS, i.e., \(\min_{i \in I} u_i(x^*_{\phi_i}) > 0\).

Suppose by contradiction that \(\min_{i \in F} u_i(x^*_{\phi_i}) = 0\). By the Lemma 6.3, all families have the same minimal utility in \(x^*\), so each family \(f\) has a member \(i \in f\) with \(u_i(x^*_f) = 0\). Define \(x^{**} := \frac{1}{2}x^* + \frac{1}{2}(\overline{e}, \ldots, \overline{e})\). By assumption, \(x^*\) is not the equal allocation, so there are some families \(f\) with \(x^*_f \neq \overline{e}\). By strict convexity, all members of such families must now have \(u_i(x^{**}_f) > 0\). Moreover, the members of families with \(x^*_f = \overline{e}\) are indifferent between \(x^*\) and \(x^{**}\). Therefore, \(x^{**}\) is \(\{u_i\}_{i \in F}\)-leximin-better than \(x^*\), contradicting the \(\{u_i\}_{i \in F}\)-leximin-optimality of \(x^*\). \(\square\)

Proof of Theorem 6.2 If \((\overline{e}, \ldots, \overline{e})\) is is Pareto-optimal, we are done as this allocation is individual-FS and family-EE. So suppose \((\overline{e}, \ldots, \overline{e})\) is not Pareto optimal.

We consider only allocations in the set \(E\) — the set of individual-FS allocations. In this set, the utilities of all agents are at least 0. For each family \(f\), define an aggregate utility function \(U_f : \mathbb{R}^G \rightarrow \mathbb{R}\), by:

\[
U_f(x_f) := \prod_{i \in f} u_i(x_f).
\]

Since each \(U_f\) is continuous and since \(E\) is compact, there exists a \(\{U_f\}_{f \in F}\)-leximin optimal allocation in \(E\), say \(x^*\). Note that, for all \(x \in E\), \(U_f(x_f) \geq 0\), and \(U_f(x_f) = 0\) if and only if \(u_i(x_f) = 0\) for some \(i \in f\). So \(U_f(x^*_f) \geq 0\) must hold for all \(f \in F\). By Lemma 6.4 there exists an individual-strict-FS allocation \(x'\), in which \(U_f(x'_f) > 0\) for all \(f \in F\). Since \(x^*\) is \(\{U_f\}_{f \in F}\)-leximin optimal, \(\min_{f \in F} U_f(x^*_f) \geq \min_{f \in F} U_f(x'_f) > 0\). We claim that \(x^*\) is individual-FS, family-EE and Pareto optimal.

Since \(x^* \in E\) by definition, it is individual-FS.
To see that $x^*$ is Pareto optimal, suppose that some $x'$ did Pareto dominate it. Since $x' \in E$ and since $u_i(x_{\phi i}^*) \geq u_i(x_{\phi i}^*) > 0$ holds for all $i$, with first inequality holding strictly for at least some $i'$, we have $U_{f'}(x_{f'}^*) > U_{f'}(x_{f'}^*)$ for the family $f'$ containing $i'$ and $U_f(x_f^*) \geq U_f(x_f^*)$ for all $f$. So $x' \in E$ is $\{U_f\}_{f \in F}$-leximin-better than $x^*$ — a contradiction to the definition of $x^*$.

To see that $x^*$ is family-EE, we first show that $U_f(x_f^*) = U_{f'}(x_{f'}^*)$ holds for all $f, f'$. Suppose not. So some family $f'$ is such that $U_{f'}(x_{f'}^*) > \min_{f \neq f'} U_f(x_f^*)$. Since $u_i(x_{f'}^*) > u_i(\bar{x})$ holds for each $i \in f$ we can redistribute a small amount of goods from family $f'$ to all other families to obtain $x^{**} \in E$. Since each agent’s preferences are strictly monotonic, $U_f(x_f^{**}) > U_f(x_f^*)$ holds for all $f \neq f'$ — a contradiction to $x^*$ being $\{U_f\}_{f \in F}$-leximin optimal.

Now let $V \in \mathbb{R}$ be that equal aggregate utility, $V := U_f(x_f^*)$ for all $f \in F$. Define $t \in \mathbb{R}$ and a bundle $r \in X$ by:

$$t := V \frac{1}{|F|} + \frac{1}{|F|}$$

and $r := t \cdot e$.

By normalization of all utilities $u_i$ we have, for all $i \in I$, $u_i(r) = u_i(t \cdot e) = t - \frac{1}{|F|} = V \frac{1}{|F|}$. For each family, there are two cases. If $u_i(x_f^*) = V \frac{1}{|F|}$ for all $i \in f$, then each agent $i \in f$ is indifferent between $x_f^*$ and $t \cdot e$, so that $x_f^* \sim_f t \cdot e$. If not, then $U_f(x_f^*) = V$ implies that some agent $i \in f$ strictly prefers $x_f^*$ to $t \cdot e$ while another agent $j \in f$ strictly prefers $t \cdot e$ to $x_f^*$. In this latter case we have $x_f^* \succ_f t \cdot e$. So $x^*$ is family-EE with reference bundle $t \cdot e$.

\[\square\]

Theorem 6.2 is true even without strict convexity, as long as one of the following holds: (a) the equal allocation is Pareto optimal, or (b) there exists an individual-strict-FS allocation. We require strict convexity only in the proof of Lemma 6.4 to ensure that one of these conditions holds. We show next that convexity alone is not sufficient for the conclusion of the Theorem.

**Proposition 6.5** (a) Some economies with strictly monotonic and convex preferences with only three families have no individual-FS and family-EE Pareto optimum. (b) There always exist family-EE Pareto optima, individual-FS Pareto optima, and family-EE individual-FS allocations.
Figure 4: Non-existence of Pareto-optimal FS family-EE allocations.
Proof  (a) Consider a two good economy with 3/2 units of y, 3/2 units of z. There are five agents $I := \{h, w, h', w', s\}$ belonging to three families $f := \{h, w\}, f' := \{h', w'\}$ and $\{s\}$, where:

- The single $s$ and the husbands $h$ and $h'$ have identical linear preferences: $u_h(y, z) = u_{h'}(y, z) = u_s(y, z) = y + z$.

- The wives $w$ and $w'$ have different Cobb-Douglas preferences: $u_w(y, z) = y^{1/4}z^{3/4}$ and $u_{w'}(y, z) = y^{3/4}z^{1/4}$.

For the single and both husbands to enjoy the fair-share guarantee, $y_f + z_f = y_{f'} + z_{f'} = y_s + z_s = 1$ must hold. The only Pareto-optimal allocation satisfying these equations gives family $f (1/4, 3/4)$ and family $f' (3/4, 1/4)$; see Figure 4.

But this allocation is not family-EE. Suppose by contradiction that it is, and let $r := (y_r, z_r)$ be the corresponding reference bundle. Since the single must be indifferent between his bundle and $r$, $y_r + z_r$ must equal 1. This means that both husbands are indifferent between their family’s bundle and $r$. But, regardless of which $r$ we pick, at least one wife will prefer her family’s bundle to $r$.

(b) To see that there always exists a family-EE Pareto optimum, define aggregate family utilities $U_f : \mathbb{R}^G \to \mathbb{R}$ so that $U_f(x_f) := \sum_{i \in f} u_i(x_f)$ for each $f \in F$. Define $x^*$ as a $\{U_f\}_{f \in F}$-leximin optimal allocation in $X$. To see that $U_f(x^*_f) = U_{f'}(x^*_{f'})$ holds for all families $f, f'$, suppose that $U_{f'}(x^*_{f'}) > U_{f''}(x^*_{f''})$ hold for two families $f', f''$. Define a new allocation $x'$ by redistributing a small amount of goods from family $f'$ to all other families. By monotonicity, $\min_{f \in F} U_f(x^*_f) < \min_{f \in F} U_f(x'_f)$ a contradiction to the $\{U_f\}_{f \in F}$-leximin optimality of $x^*$ in $X$. Since $U_f(x^*_f) = U_{f'}(x^*_{f'})$ holds for all families $f, f', x^*$ is by the arguments in the proof of Theorem 6.2 family-EE. As a $\{U_f\}_{f \in F}$-leximin optimal allocation in $X$, $x^*$ is Pareto optimal.

Any $\{u_i\}_{i \in I}$-leximin-optimal allocation in $X$ is individual-FS and Pareto optimal. Finally $(\overline{v}, \ldots, \overline{v})$ is individually-FS and family-EE. □

To sum, without strict convexity, we can guarantee any two properties out of {Pareto optimality, individual-FS, family-EE} but not all three.
7 Alternatives

7.1 Individual-fairness vs. family-fairness

There are always two options to adapt a preference-based property for individuals to our environment with families: a strict adaptation requires that the property holds for each individual, a weak one requires that the property holds for the families’ incomplete preferences. We indeed studied the stricter notions of individual-FS, NE, and EE as well as the weaker notions of family-FS, NE, and EE.

There are two further concepts for which we only gave one adaptation: market equilibria and Pareto optima. In each case we use the — standard — family based adaptation. The alternative more restrictive adaptations pose severe existence problems. To see this say, a triplet \((p, x, x^0)\) with \(x, x^0 \in X\) and \(p \in \mathbb{R}^G\) is an “individual”-market equilibrium from endowment \(x^0\), if for each individual and each bundle \(x'\): (a) \(px_f \leq px^0_f\), and: (b) \(u_i(x') > u_i(x_{\phi i})\) implies \(px' > px^0_{\phi i}\). So \((p, x, x^0)\) is an “individual”-market equilibrium from endowment \(x^0\) if each bundle \(x_f\) is affordable for that family and if any bundle \(x'\) an individual strictly prefers to \(x_f\) is unaffordable for that individual’s family. So for \((p, x, x^0)\) to be a “individual”-market equilibrium all individuals within the same family must agree on their optimal bundle for a given budget set. Even in the simple case where families have members with different Cobb-Douglas utilities, this case will never arise.

Similarly, “individual”-Pareto optimal allocations, in the sense that any alternative allocation \(x'\) which is not indifferent for each agent must be strictly worse for some agent, rarely exist. Even an economy with two families where there two wives have identical Cobb-Douglas preferences which differ from the two husbands identical Cobb-Douglas preferences there are no such “individual”-Pareto optima.

Comparing the “individual”-adaptations of the different principles we see that the individual-EE falls into the same class as the “individual”-adaptations of market equilibria and Pareto optima: allocations that are individual-EE only exist under the rarest circumstances. No envy and the fair share guarantee present a different picture. Our results show that the individual-FS can easily be attained, even in combination with additional
fairness properties such as family-NE or family-EE.

Individual-NE falls right in the middle between individual-EE and individual-FS. While individual-NE can easily be attained when there are only two families, the prospects for individual-NE with more than two families are bleak. There is, however, another well-defined sense in which individual-NE is easier to attain than the strict adaptations of Pareto optimality, market equilibrium and egalitarian equivalence:

Consider a sequence of two-good-economies with $| F | > 2$ families, each of which consisting of one husband and one wife. Say all agents have Cobb-Douglas preferences. Consider any fixed family: while the husband and wife in that family have different coefficients in each economy their coefficients converge to the same limit. All families coefficients converge to different limits. While no economy in this sequence has an “individual”-Pareto optimum or an “individual”-market equilibrium, there is some number $n \in \mathbb{N}$ such that each economy in the sequence after $n$ has individual-NE Pareto optima.

### 7.2 Both family-no envy and family-egalitarian equivalence

When do Pareto optima exist that are both family-NE and family-EE? Thom-son (1990) showed that, with three or more individual agents, Pareto optima with an egalitarian equivalence typically do not satisfy no envy. Since the case of single agents is a special case of our model with families, there is little hope to find Pareto optima that are both family-NE and family-EE.

### 7.3 Grouping by preferences

Some of our negative examples depend on quite divergent preferences within families. For instance, the economy without individual-NE Pareto optima (Proposition 5.2) relies on the fact that the husband is more similar to one of the singles than to his wife, who in turn is more similar to the other single than to her husband. This particular economy has individual-NE Pareto optima for a different family structure where people with more similar preferences form families. So one may wonder whether economies with assortatively
matched families always have individual-NE Pareto optima.

To see that this is not the case consider a two-good-economy with three women and three men, who have to form three (heterosexual) couples. Say that all individuals have Cobb-Douglas preferences, where $m_i$ and $w_i$ respectively denote the coefficient rates of man and woman $i$. Say these coefficients are ordered such that $m_1 < m_2 < w_1 < m_3 < w_2 < w_3$. Consider the assortative matching where man $m_i$ and woman $w_i$ get married, so $f_i = \{m_i, w_i\}$ for all $i = 1, 2, 3$. By Theorem 3.1 $f_1$ and $f_3$ must consume different bundles. Since $w_1$ can't envy $f_2$ and since $w_2$ and $m_2$ cannot envy $f_1$ families $f_1$ and $f_2$ must consume the same bundle. Mutatis mutandis we see that also $f_2$ and $f_3$ must consume the same bundle, but then by Theorem 3.1 this allocation is not Pareto optimal.

If we allow any arbitrary grouping into pairs in the above problem an individual-NE Pareto optimum exists: simply match the two agents with the highest two coefficients, with the middle coefficients and the lowest coefficients. Then find a market equilibrium from equal endowments given that each pair maximizes its average utility. The question whether (and how) this observation generalizes is open.

### 7.4 Democratic fairness

Individual-NE is a very strong requirement that often fails to exist. In contrast, family-NE is a quite weak requirement that allows allocations where in each family, all agents except one are envious. A possible middle ground between these extremes is democratic-NE. An allocation is $h$-democratic-NE, for some fraction $h \in [0, 1]$, if in each family, at least a fraction $h$ of the members perceive the allocation as fair. This criterion may be of practical application if the families represent democratic countries. For example, if an allocation is $1/2$-democratic-NE, then there is a hope that this allocation will be approved in a referendum.

In general, democratic-NE Pareto-optima are not guaranteed to exist. An example similar to the one in Proposition 5.2 shows that, for every integer $k \geq 2$, there are economies with $2k - 1$ families in which no Pareto-optimum is NE for more than $1/k$ of each family’s members. In particular, with 5
or more families there might be no Pareto-optimum that is more than 1/3-
democratic NE. This leaves open the following practical question: with 3 or 4
families, is there always a Pareto-optimum that is NE for 1/2 of the members
in each family?

8 Acknowledgments

We are grateful to the Max Planck Institute for Research on Collective Goods
for supporting this research through the ARCHES prize.

The paper started as a discussion in the economics stackexchange website.

We are grateful to Shane Auerbach, Amit Goyal and Kitsune Cavalry for
participating in the discussion.

References

Berliant, M., Thomson, W., and Dunz, K. (1992). On the fair division of a
heterogeneous commodity. *Journal of Mathematical Economics*, 21(3):201–
216.

Bernheim, B. D. and Rangel, A. (2007). Toward choice-theoretic foundations
for behavioral welfare economics. *American Economic Review*, 97:464–470.

Bernheim, B. D. and Rangel, A. (2009). Beyond revealed preference: choice
theoretic foundations for behavioral welfare economics. *Quarterly Journal
of Economics*, 124:51–104.

Bewley, T. (2002). Knightian decision theory: part i. *decisions in Economics
and Finance*, 2:79–110.

Buchanan, J. M. (1965). An economic theory of clubs. *Economica*, 32(125):1–
14.

Dall’Aglio, M., Branzei, R., and Tijs, S. (2009). Cooperation in dividing the
cake. *TOP - the official journal of the Spanish Society of Statistics and
Operations Research*, 17(2):417–432.

7https://economics.stackexchange.com/q/9916/385
Dall’Aglio, M. and Di Luca, C. (2014). Finding maxmin allocations in co-operative and competitive fair division. *Annals of Operations Research*, 223(1):121–136.

Diamantaras, D. (1992). On equity with public goods. *Social Choice and Welfare*, 9(2):141–157.

Diamantaras, D. and Wilkie, S. (1994). A Generalization of Kaneko’s Ratio Equilibrium for Economies with Private and Public Goods. *Journal of Economic Theory*, 62(2):499–512.

Diamantaras, D. and Wilkie, S. (1996). On the set of Pareto efficient allocations in economies with public goods. *Economic Theory*, 7(2):371–379.

Dubins, L. E. and Spanier, E. H. (1961). How to Cut A Cake Fairly. *The American Mathematical Monthly*, 68(1):1–17.

Dwork, C., Hardt, M., Pitassi, T., Reingold, O., and Zemel, R. (2012). Fairness through awareness. In *Proceedings of the 3rd Innovations in Theoretical Computer Science Conference*, ITCS ’12, pages 214–226, New York, NY, USA. ACM.

Fleurbaey, M. and Schokkaert, E. (2013). Behavioral welfare economics and redistribution. *American Economic Journal: Microeconomics*, 5:180–205.

Fon, V. and Otani, Y. (1979). Classical welfare theorems with non-transitive and non-complete preferences. *Journal of economic theory*, 20:409–418.

Guth, W. and Kliemt, H. (2002). Non-Discriminatory, Envy Free Provision of a Collective Good: A Note. *Public Choice*, 111(1-2):179–184.

Hébert-Johnson, Ú., Kim, M. P., Reingold, O., and Rothblum, G. N. (2017). Calibration for the (computationally-identifiable) masses. *CoRR*, abs/1711.08513.

Hillman, A. L. (1993). Socialist clubs: A perspective on the transition. *European Journal of Political Economy*, 9(3):307–319.
Hüsseinov, F. (2011). A theory of a heterogeneous divisible commodity exchange economy. *Journal of Mathematical Economics*, 47(1):54–59.

Laibson, D. (1997). Golden eggs and hyperbolic discounting. *Quarterly Journal of Economics*, 112:443–477.

Loertscher, S. and Marx, L. M. (2017). Club good intermediaries. *International Journal of Industrial Organization*, 50:430–459.

Mackenzie, A. and Trudeau, C. (2017). Club good mechanisms: from free-riders to citizen-shareholders, from impossibility to characterization. Technical report. Maastricht university school of business and economics.

Mandler, M. (2014). Indecisiveness in behavioral welfare economics. *Journal of Economic Behavior & Organization*, 97:219–235.

Mandler, M. (2017). Distributive justice for behavioral welfare economics. *working paper*.

Manurangsi, P. and Suksompong, W. (2017). Asymptotic existence of fair divisions for groups. *Mathematical Social Sciences*, 89:100–108.

Mas-Colell, A. (1974). An equilibrium existence theorem without complete or transitive preferences. *Journal of Mathematical Economics*, 1(3):237–246.

Moulin, H. (2004). *Fair Division and Collective Welfare*. The MIT Press.

Mouri, T., Li, R., Todo, T., Iwasaki, A., and Yokoo, M. (2012). Envy-Freeness for Groups of Agents: Beyond Single-Minded Domain. In *Proceedings of the 11th international conference on autonomous agents and multi-agent systems (AAMAS)*, pages 19–31. IFAAMAS.

ODonoghue, T. and Rabin, M. (1999). Doing it now or later. *American Economic Review*, 89:103–124.

Pazner, E. A. and Schmeidler, D. (1978). Egalitarian equivalent allocations: A new concept of economic equity. *The Quarterly Journal of Economics*, 92(4):671–687.
Salant, Y. and Rubinstein, A. (2008). (a; f): choice with frames. *Review of Economic Studies*, 75:1287–1296.

Sandler, T. and Tschirhart, J. (1997). Club theory: Thirty years later. *Public choice*, 93(3-4):335–355.

Sandler, T. and Tschirhart, J. T. (1980). The economic theory of clubs: An evaluative survey. *Journal of economic literature*, 18(4):1481–1521.

Segal-Halevi, E. and Nitzan, S. (2015). Fair Cake-cutting among Families. *CoRR*, abs/1510.03903.

Segal-Halevi, E. and Suksompong, W. (2018). Democratic Fair Allocation of Indivisible Goods. ArXiv preprint 1709.0256.

Shafer, W. and Sonnenschein, H. (1975). Equilibrium in abstract economies without ordered preferences. *Journal of Mathematical Economics*, 2(3):345–348.

Steinhaus, H. (1948). The problem of fair division. *Econometrica*, 16(1):101–104.

Suksompong, W. (2018). Approximate maximin shares for groups of agents. *Mathematical Social Sciences*. http://doi.org/10.1016/j.mathsocsci.2017.09.004.

Svensson, L.-G. (1983). On the existence of fair allocations. *Zeitschrift für Nationalökonomie*, 43(3):301–308.

Svensson, L.-G. (1994). σ-optimality and fairness. *International Economic Review*, pages 527–531.

Thomson, W. (1990). On the non existence of envy-free and egalitarian-equivalent allocations in economies with indivisibilities. *Economics Letters*, 34(3):227–229.

Thomson, W. (2011). *Fair Allocation Rules*, volume 2, pages 393–506. Elsevier.
Todo, T., Li, R., Hu, X., Mouri, T., Iwasaki, A., and Yokoo, M. (2011). Generalizing envy-freeness toward group of agents. In Proceedings of the 20th International Joint Conference on Artificial Intelligence (IJCAI), pages 386–.

Varian, H. R. (1974). Equity, envy, and efficiency. Journal of economic theory, 9(1):63–91.

Vohra, R. (1992). Equity and efficiency in non-convex economies. Social Choice and Welfare, 9(3):185–202.

Young, H. P. (1995). Equity: in theory and practice. Princeton University Press.