Abstract

We summarize our recently improved results for the pseudoscalar [1, 2], vector and Bc [3] meson decay constants from QCD spectral sum rules where N2LO estimate of the N3LO PT in d ≤ 6 condensates have been included in the SVZ expansion. The “optimal results” based on stability criteria with respect to the variations of the Laplace/Moments sum rule variables, QCD continuum threshold and subtraction constant µ are compared with recent sum rules and lattice calculations. To understand the “apparent tension” between some recent results, we present in Section 8 a novel extraction of for fb/fB from heavy quark effective theory (HQET) sum rules by including the normalization factor (Mb/MB)2 relating the pseudoscalar to the universal HQET correlators for finite b-quark and B-meson masses. We obtain fb/fB = 1.025(16) in good agreement with 1.016(16) from spectral sum rules (SR) in full QCD [3]. We complete the paper by including new improved estimates of the scalar, axial and Bc meson decay constants (Sections 11–13). For further phenomenological uses, we attempt to extract a Global Average of the sum rules and lattice determinations which are summarized in Tables 2–6. We do not found any deviation of these SM results from the present data.

Keywords: QCD spectral sum rules, Heavy quark effective theory (HQET), Heavy-light mesons, Meson decay constants.

1. Introduction

The decay constants fP/S, fV/A of (pseudo)scalar and (axial)vector mesons are of prime interests for understanding the realizations of chiral symmetry in QCD and for controlling the meson (semi-)leptonic decay widths, hadronic couplings and form factors. In addition to the well-known values of fπ=130.4(2) MeV and fK=156.1(9) MeV which control the light flavour chiral symmetries [9], it is also desirable to extract the ones of the heavy-light charm and bottom quark systems with high-accuracy. This improvement program has been initiated by the recent predictions of fD, fB [1, 2] and pursued for fDc, fBc and fB in [3] from QCD spectral sum rules (QSSR) [10] which are improved predictions of earlier estimates [15–27] since the pioneering work of Novikov et al. (NSVZZ) [28]. Here, these decay constants are normalized as:

\[ \langle 0 | J_{P/S}(x) | P \rangle = f_{P/S} M_{P/S}^2, \quad \langle 0 | J_{V/A}(x) | V/A \rangle = f_{V/A} M_V M_A e^{\mu^a \mu^b} \]

where: \( \epsilon_\mu \) is the vector polarization; \( J_{P/S}(x) = (m_q + i M_0) \eta_s / \sqrt{2} \)Q and \( J_{V/A}(x) = \tilde{q}(\gamma_\mu / \gamma_\nu) \tilde{s} \) are the heavy-light pseudoscalar, scalar and vector currents; \( q \equiv d, c, \bar{Q} \equiv c, b, P \equiv D, B, B_c; S \equiv D_0, B_0, V \equiv D^*, B^*, A \equiv D_1, B_1 \) and their s-quark analogue; the decay constants \( f_{P/S}, f_{V/A} \) are related to the leptonic widths \( \Gamma[P/S \rightarrow \ell^+ \ell^-] \) and \( \Gamma[V/A \rightarrow \ell^+ \ell^-] \). The associated two-point correlators are:

\[ \psi_{P/S}(q^2) = i \int d^4 x e^{-iqx} \langle 0 | T J_{P/S}(x) J_{P/S}(0) | 0 \rangle, \]

\[ \Pi_{V/A}^{\mu\nu}(q^2) = i \int d^4 x e^{-iqx} \langle 0 | T J_{V/A}(x) J_{V/A}(0) | 0 \rangle - \left( g^\nu - q^\mu q^\nu / q^2 \right) \Pi_{V/A}^{\mu\nu}(q^2) + g^{\mu\nu} \Pi_{V/A}^{\nu\mu}(q^2) \]

where \( \Pi_{V/A}^{\mu\nu}(q^2) \) has more power of \( q^2 \) than the transverse two-point function used in the current literature for \( m_q = m_Q \) in order to avoid mass singularities at \( q^2 = 0 \) when \( m_Q \) goes to zero. \( \Pi_{V/A}^{\mu\nu}(q^2) \) (vector/axial) and \( \psi_{P/S}(q^2) \) (scalar/pseudoscalar) correlators are related each other through the Ward identities:

\[ g_{\mu\nu} \Pi_{V/A}^{\mu\nu}(q^2) = \psi_{S/P}(q^2) - \psi_{S/P}(0), \]

where the (non)perturbative parts of \( \psi_{S/P}(0) \) are known (see e.g. [11]) and play an important rôle for absorbing mass singularities which appear during the evaluation of the PT two-point function. For extracting the decay constants, we shall work with the well established (inverse) Laplace sum rules [we use the terminology : (inverse) Laplace instead of Borel sum rule as it has been demonstrated in [29] that its QCD radiative cor-

---

\[ ^{a} \text{Laboratoire Particules et Univers de Montpellier, CNRS-IN2P3, Case 070, Place Eugène Bataillon, 34095 - Montpellier, France.} \]

---

\[ ^{\ast} \text{Plenary talk given at QCD15 (18th International QCD Conference and 30th Anniversary), 29 June-3 July 2015 (Montpellier-FRA) and at HEPMAD15 (7th International Conference), 17-22 September 2015 (Antananarivo-MGA).} \]

\[ ^{\ast\ast} \text{En hommage aux victimes des attentats du 13 Novembre 2015 à Paris-FRA. Email address: marison@enron.fr (Stephan Narison)} \]

\[ ^{\dagger} \text{Some applications to D and B-decays can e.g be found in [5–9].} \]

\[ ^{\ddagger} \text{For reviews with complete references, see e.g. [1, 2, 4, 11–14, 20].} \]
rections satisfy the (inverse) Laplace transform properties\(^3\):

\[
\mathcal{L}_{P/S,\mu}(\tau, \mu) = \int_{(\mu + M_0)^2}^{\infty} dt \ e^{-\frac{t}{\mu}} \ \text{Im} \left[ \phi_{P/S, \Pi(\tau, \mu)}(t, \mu) \right]. \quad (5)
\]

For improving the extraction of the decay constants \(f_{D_{G_0}}\) and \(f_{B_{G_0}}\), we shall also work with the ratio:

\[
R_{\Pi(\tau, \mu)} = \frac{\mathcal{L}_{\Pi(\tau, \mu)}(\tau, \mu)}{L_{\Pi}(\tau)}, \quad (6)
\]

in order to minimize the systematics of the approach, the effects of heavy quark masses and the continuum threshold uncertainties which are one of the main sources of errors in the determinations of the individual decay constants. \(\tau_{\Pi(\tau, \mu)}\) denotes the values of the sum rule \(\tau\)-variable at which each individual sum rule is optimized (minimum or inflexion point). In general, \(\tau_{\Pi(\tau, \mu)} \neq \tau_{P}\) as we shall see later on which requires some care for a precise determination of the ratio of decay constants. This ratio of sum rule has lead to a successful prediction of the SU(3) breaking ratio \(f_P/f_{P_{\Pi}}[32]\) such that, from it, one expects to extract precise values of the ratio \(f_{\Pi}/f_P\).

### 2. QCD expression of the two-point correlators

The QCD expression of the Laplace sum rule \(L_{P/S}(\tau, \mu)\) in the (pseudo)scalar channel can be found in [1, 2, 22] for full QCD including NLO perturbative QCD contributions and corrections of non-perturbative condensates up to the complete \(d = 6\) dimension condensates\(^4\). The one of the vector channel has been given within the same approximation in [3]. Some comments are in order:

- The expressions of NLO PT in [33, 34], of NLO PT in [35], of the non-perturbative in [16, 28] and the light quark mass corrections in [16, 33, 36] have been used. The N3LO PT contributions have been estimated by assuming the geometric growth of the series [37] which is dual to the effect of a \(1/q^2\) term [38, 39]. The PT expressions have been originally obtained using a pole quark mass which is transformed into the \(\overline{MS}\)-scheme by using the relation between the running \(\hat{m}_Q(\mu)\) and pole mass \(M_Q\) in the \(\overline{MS}\)-scheme to order \(\alpha_s^2\) [40–44].

- The LO contributions in \(\alpha_s\), up to the \(d = 4\) gluon: \(\langle \alpha_s G^2 \rangle \equiv \langle \alpha_s G_{\mu}^a G_{\nu}^a \rangle\), \(d = 5\) mixed quark-gluon: \(\langle g \bar{q} G q \rangle \equiv \langle g \bar{q} G_{\mu}^{a \nu} \lambda_{\alpha} (1/2) G_{\mu, a}^{\alpha} q \rangle = M_0^2 \langle g \bar{q} q \rangle\) and \(d = 6\) quark: \(\langle \bar{q} d \rangle \equiv \langle g \bar{g} G_{\mu}^{a \nu} \bar{d} G_{\nu}^{a \mu} \rangle = g^2 \langle \bar{d} G_{\mu}^{a \nu} \bar{d} \sum_{\alpha} \bar{q} G_{\mu}^{a \nu} \lambda_{\alpha} q \rangle = -\frac{16}{9} (\alpha_s \beta_0) \rho(\bar{d} d)^2\), (after the use of the equation of motion) condensates, have been obtained originally by NSVZZ [28]. \(\rho = 3 \pm 4\) measures the deviation from the vacuum saturation estimate of the \(d = 6\) four-quark condensates [45–47].

\(\alpha_s(M_z) = 0.12(8)\) [31, 45, 48, 49], \(\alpha_s(\mu_0) = 126(12)\) MeV average [49–51], \(\bar{m}_0(\mu) = 4177(11)\) MeV average [49, 50], \(\rho_0 = (253 \pm 6)\) MeV [11, 30, 52, 53], \(M_c^2 = (0.8 \pm 0.02)\) GeV\(^2\) [22, 46, 54], \(\langle g^3 G^2 \rangle = (7 \pm 3) \times 10^{-3}\) GeV\(^4\) [45, 47, 50, 55–56], \(\langle g^5 G^4 \rangle = (8.2 \pm 2.0)\) GeV\(^2\) [50].

Table 1: QCD input parameters: the original errors for \(\langle \alpha_s G^2 \rangle\), \(\langle g^3 G^2 \rangle\) and \(\rho(\bar{q} q)^2\) have been multiplied by 2–3 for a conservative estimate.

| Parameters | Values | Ref. |
|------------|--------|------|
| \(\alpha_s(M_z)\) | 0.125(8) | [31, 45, 48, 49] |
| \(\alpha_s(\mu_0)\) | 126(12) MeV | average [49–51] |
| \(\bar{m}_0(\mu)\) | 4177(11) MeV | average [49, 50] |
| \(\rho_0\) | (253 \pm 6) MeV | [11, 30, 52, 53] |
| \(M_c^2\) | (0.8 \pm 0.02) GeV\(^2\) | [22, 46, 54] |
| \(\langle g^3 G^2 \rangle\) | (7 \pm 3) \times 10^{-3}\) GeV\(^4\) | [45, 47, 50, 55–56] |
| \(\langle g^5 G^4 \rangle\) | (8.2 \pm 2.0)\) GeV\(^2\) [50]. | [45–47] |
| \(s_0\) | (0.114 \pm 0.006) GeV | [11, 30, 52, 53, 61] |
| \(\rho = (3s_0)/(\bar{d} d)\) | (0.74 \pm 0.12) | [11, 62] |

### 3. QCD input parameters

The QCD parameters which shall appear in the following analysis will be the charm and bottom quark masses \(m_c\) (we shall neglect the light quark masses \(q = u, d\)), the light quark condensate \(\langle \bar{q} q \rangle\), the gluon condensates \(\langle \alpha_s G^2 \rangle\) and \(\langle g^3 G^2 \rangle\) the mixed condensate \(\langle g \bar{q} G_q \rangle\) defined previously and the four-quark condensates \(\rho(\bar{q} q)^2\). Their values are given in Table 1.

We shall work with the running light quark condensates and masses. They read:

\[
\langle \bar{q} q \rangle(\tau) = -\tilde{\mu}_Q^2 (1 - \beta_1) \langle \bar{q} q \rangle(\tau),
\]

\[
\langle g \bar{q} G_q \rangle(\tau) = -M_0^2 \bar{q}_Q^2 (1 - \beta_1) \langle \bar{q} q \rangle(\tau),
\]

\[
\bar{m}_1(\tau) = \bar{m}_1(\tau) \bigg/ \left( \log \sqrt{r^2 - 1} \right)^{2\beta_1} \langle \bar{q} q \rangle(\tau),
\]

where \(\beta_1 = -(1/2)(1 - 2n_f/3)\) is the first coefficient of the \(\beta\) function for \(n_f\) flavours; \(\alpha_s = \alpha_s(\tau)/\tau; \tilde{\mu}_Q\) is the spontaneous RGI light quark condensate [64]. The QCD correction factor \(C(a_s)\) in the previous expressions is numerically [44]:

\[
C(a_s) = 1 + 1.1755a_s + 1.5008a_s^2 + \ldots ; \quad n_f = 5 ,
\]

which shows a good convergence.
4. Parametrization of the spectral function

- **Minimal Duality Ansatz (MDA)**

We shall use MDA for parametrizing the spectral function:

\[
\frac{1}{\pi} \text{Im} \psi_{PS}(t) = f_P^2 M_{PS}^2 \delta(t - M_P^2) + \text{"QCD cont."} \theta(t - t_P^c),
\]

\[
\frac{1}{\pi} \text{Im} \Pi^\ell(t) = f_{\ell P}^2 M_{\ell P}^2 \delta(t - M_{\ell P}^2) + \text{"QCD cont."} \theta(t - t_{\ell P}^c),
\]

where \( f_{\ell P} \) are the decay constants defined in Eq. (1) and the higher states contributions are smeared by the "QCD continuum" coming from the discontinuity of the QCD diagrams and starting from a constant threshold \( t_{\ell P}^c \), which is independent on the subtraction point \( \mu \) in this standard minimal model.

- **Test of the Minimal Duality Ansatz from \( J/\psi \) and \( \Upsilon \)**

The MDA presented in Eq. (9), when applied to the \( \rho \)-meson reproduces within 15% accuracy the ratio \( R_{hl}^{(n)} \) measured from the total cross-section \( e^+ e^- \to 1 \) hadrons data (Fig. 5.6 of [4]), while in the case of charmonium, \( M_0^2 \) from ratio of moments \( R_{hl}^{(n)} \) evaluated at \( Q^2 = 0 \) has been compared with the one from complete data where a remarkable agreement for higher \( n \geq 4 \) values (Fig. 9.1 of [4]) has been found.

Recent tests of MDA from the \( J/\psi \) and \( \Upsilon \) systems have been done in [1]. Taking \( t_{\ell P}^c \) \( = 0.15 \) GeV and \( t_{P \ell}^c \) \( = 0.15 \) GeV, we show (for instance) the ratio between \( \mathcal{L}_{00}^{\text{exp}} \) and \( \mathcal{L}_{00}^{\text{theor}} \) in Fig. 1 for the \( J/\pi \) and \( \Upsilon \) systems indicating that for heavy quarks systems the rôle of the QCD continuum is smaller than in the case of light quarks while the exponential weight suppresses efficiently the QCD continuum contribution and enhances the one of the lowest resonance. We have used the simplest QCD continuum for massless quarks [50]:

\[
\text{QCD cont.} = (1 + a_1 + 1.5a_2 + 12.07a_3) \theta(t - t_{\ell P}^c).
\]

One can see in Fig. 1 that the MDA, with a value of \( \sqrt{\mathcal{L}} \) around the one of the 1st radial excitation mass, describes well the complete data in the region of \( \tau \)-stability of the sum rules [50]:

\[
\tau^\phi \approx (0.8 \sim 1.4) \text{ GeV}^{-2}, \quad \tau^\Upsilon \approx (0.2 \sim 0.4) \text{ GeV}^{-2},
\]

as we shall see later on. Though it is difficult to estimate with precision the systematic error related to this simple model, this feature indicates the ability of the model for reproducing accurately the data. We expect that the same feature is reproduced for the open-charm and beauty vector meson systems where complete data are still lacking. Moreover, MDA has been also used in [66] in the context of large \( N_c \) QCD, where it provides a very good approximation to the observables one computes.

5. Optimization and Stability criteria

- **\( \tau \) and \( t_{\ell P} \)-stabilities**

In order to extract an optimal information for the lowest resonance parameters from this rather crude description of the spectral function and from the approximate QCD expression, one often applies the stability criteria at which an optimal result can be extracted. This stability is signaled by the existence of a stability plateau, an extremum or an inflexion point versus the changes of the external sum rule variables \( \tau \) and \( t_{\ell P} \) where the simultaneous requirement on the resonance dominance over the continuum contribution and on the convergence of the OPE is satisfied. This optimization criterion demonstrated in series of papers by Bell-Bertmann [60] in the case of \( \tau \) by taking the examples of harmonic oscillator and charmonium sum rules and extended to the case of \( t_{\ell P} \) in [4, 11] gives a more precise meaning of the so-called “sum rule window” originally introduced by SVZ [10] and used in the sum rules literature.

- **\( \mu \)-subtraction point stability**

We shall add to the previous well-known stability criteria, the one associated to the requirement of stability versus the variation of the arbitrary subtraction constant \( \mu \) often put by hand in the current literature and which is often the source of large errors from the PT series in the sum rule analysis. The \( \mu \)-stability
procedure has been applied recently in [1, 2, 53, 67]6 which gives a much better meaning on the choice of \( \mu \)-value at which the observable is extracted, while the errors on the results have been reduced due to a better control of the \( \mu \) region of variation which is not often the case in the literature.

6. The decay constants \( f_D \) and \( f_{D^*} \)

- **Direct determinations from \( \mathcal{L}_P (\tau, \mu) \)**

We start by showing in Fig. 3, the \( \tau \)-behaviour of the decay constants \( f_D \) and \( f_D^* \) at a given value of the subtraction point \( \mu = m_c \), for different values of the continuum threshold \( t_c \). We have assumed that \( (t_D^0)^{1/2} - (t_D^0)^{1/2} \approx M_D - M_D = 140.6 \text{ MeV} \), for the vector and pseudoscalar channels where the QCD expressions are truncated at the same order of PT and NP series.

At this value of \( \mu \), we deduce the optimal value versus \( \tau \) (minimum or inflexion point for \( \tau \approx 0.8 - 1 \text{ GeV}^{-2} \)) and \( t_c \), (5.6-8 \text{ GeV}^2). Next, we study the \( \mu \) variation of these results which we show in Fig. 4. We deduce the mean result from [1]:

\[
 f_D = 204(6) \text{ MeV},
\]

where the largest error for each data point comes from \( t_c \).7 The value of \( f_{D^*} \) at the minimum in \( \mu = (1.5 \pm 0.1) \text{ GeV} \) is [3]:

\[
 f_{D^*} = 253.5(11.5)_{t_c}(5.7)_{t_c}(13)_{x_c}(1)_{\mu} = 253.5(18.3) \text{ MeV},
\]

with:

\[
 (13)_{x_c} = (0.5)_{t_c}(12.3)_{t_c}(0.6)_{m_c}(3.8)_{dGd}, \quad (1.8)_{x_c}(1.4)_{dGd}, (0.4)_{dGd}, (0.4)_{dGd}^2.
\]

where the SVZ-OPE error is dominated by the estimate of the \( x_c^2 \) corrections (95%) and where again the error due to \( \mu \) has been multiplied by a factor 2 for a more conservative error. One should notice that the accurate value of \( f_D \) comes from the mean of different data shown in Fig. 4, while the error from \( f_{D^*} \) is taken from the one at the minimum for \( \mu = 1.45 \text{ GeV} \).

- **The ratio \( f_{D^*}/f_D \) and improved determination of \( f_{D^*} \)**

One can notice in Fig. 3 that working directly with the ratio in Eq. (6) by taking the same value \( \tau \), is inaccurate as the two sum rules \( \mathcal{L}_V(\tau) \) and \( \mathcal{L}_P(\tau) \) are not optimized at the same value of \( \tau \) (minimum for \( f_D \) and inflexion point for \( f_{D^*} \)). Therefore, for a given value of \( t_c \), we take separately the value of each sum rule at the corresponding value of \( \tau \) where they present minimum and/or inflexion point and then take their ratio. For a given \( \mu \), the optimal result corresponds to the mean obtained in range of values of \( t_c \) where one starts to have a \( \tau \)-stability (\( t_c \approx 5.6 - 5.7 \text{ GeV}^{-2} \) for \( \tau \approx 0.6 \text{ GeV}^{-2} \)) and a \( t_c \)-stability (\( t_c = 9.5 - 10.5 \text{ GeV}^{-2} \) for \( \tau \approx 0.8 \text{ GeV}^{-2} \)). In Fig. 5, we look for the \( \mu \)-stability of the previous optimal ratio \( f_{D^*}/f_D \) in the set (\( \tau, t_c \)). We obtain at the minimum \( \mu = (1.5 \pm 0.1) \text{ GeV} \):

\[
 f_{D^*}/f_D = 1.218(6)_{t_c}(27)_{t_c}(23)_{x_c}(4)_{x_c} = 1.218(36)
\]

with:

\[
 (23)_{x_c} = (7)_{t_c}(2)_{t_c}(3)_{m_c}(0)(18)_{x_c}(12)_{dGd}, (0)_{dGd}, (0)_{dGd}^2.
\]

The error from \( \mu \) has been multiplied by 2 to be conservative. The largest error comes from \( \tau \) which is due to the inaccurate localization of the inflexion point. The one due to \( t_c \) is smaller as expected in the ratio which is not the case for the direct extraction of the decay constants. The ones due to \( (x_c, G^2) \) and \( \langle dGd \rangle \) are large due to the opposite sign of their contributions in the vector and pseudoscalar channels which add when taking the ratio. Using the value \( f_D = 204(6) \text{ MeV} \) in Eq. (11) and the ratio in Eq. (14), we deduce the improved value:

\[
 f_{D^*} = 248.5(10.4) \text{ MeV},
\]

where the errors have been added quadratically. Taking the mean of the two results in Eqs. (12) and (15), one obtains:

\[
 \langle f_{D^*} \rangle = 249.7(10.5)(1.2)_{\text{fstat}} = 250(11) \text{ MeV},
\]

where the 1st error comes from the most precise determination and the 2nd one from the distance of the mean value to it.

6Some alternative approaches for optimizing the PT series are in [68].

7One should note that the extraction of the charm quark running mass by requiring that the sum rules should reproduce the \( D \)-meson mass is obtained in the same range of values of the previous set of stability parameters [1]. The same results are obtained for the other mesons. This fact increases the confidence on the predictions of the unknown decay constants.

---

**Figure 4:** a) \( f_D \) from LSR versus \( \mu \) and for \( \mu_c = 1.467 \text{ MeV} \). The filled (grey) region is the average with the corresponding averaged errors. The dashed horizontal lines are the ones where the errors come from the best determination. b) \( f_{D^*} \) for different values of \( \mu \).

**Figure 5:** a) \( f_{D^*}/f_D \) versus the subtraction point \( \mu \); b) The same as a) but for \( f_{D^*} \).
of the QCD continuum contribution to the spectral function and by taking the limit where \( t_c \to \infty \) in Eq. (5) which corresponds to a full saturation of the spectral function by the lowest ground state contribution. The result of the analysis versus the change of \( \tau \) for a given value of \( \mu = 1.5 \text{ GeV} \) is given in Fig. 6a where one can observe like in the previous analysis the presence of a \( \tau \)-inflexion point. We also show the good convergence of the PT series by comparing the result at N2LO and the one including an estimate of the N3LO term based on the geometric growth of the PT coefficients. We show in Fig. 6b the variation of the optimal bound versus the subtraction point \( \mu \) where we find a region of \( \mu \) stability from 1.5 to 2 GeV. We obtain:

![Figure 6: Upper bounds on \( f_{D*} \): a) at different values of \( \tau \) for a given value \( \mu = 1.5 \text{ GeV} \) of the subtraction point \( \mu \). One can notice a good convergence of the PT series by comparing the calculated N2LO and estimated N3LO terms; b) versus the values of the subtraction point \( \mu \).

\[
\begin{align*}
\text{f}_{D*} &\leq 267(10)\tau(14)_{svz}(0)\mu \text{MeV} , \\
\text{with :} &\quad (14)_{svz} = (1.4)_{s}(13)_{v}(1.3)_{m}(3)_{(dd)} \\
&\quad (3)_{(α, G)}(2.5)_{(dGd)}(0)_{(2G)}(0.7)_{(dd)} .
\end{align*}
\]

(17)

Alternatively, we combine the upper bound \( f_D \leq 218.4(14) \text{ MeV} \) obtained in [1, 2] with the ratio in Eq. (14) and deduce:

\[
\begin{align*}
\text{f}_{D*} &\leq 266(8) \text{ MeV} ,
\end{align*}
\]

(18)

where we have added the errors quadratically. The good agreement of the results in Eqs. (17) and (18) indicates the self-consistency of the approaches. This bound is relatively strong compared to the estimate in Eq. (16). A comparison of our results with the ones from some other sources is shown in Table 4.

7. The decay constants \( f_B \) and \( f_{B*} \) from SR in full QCD

- Direct estimate of \( f_B \) and \( f_{B*} \)

![Figure 7: a) \( \tau \)-behaviour of \( f_B \) from \( \mathcal{L}_B \) for different \( t_c \) at \( \mu = m_b \); b) the same as in a) but for \( f_{B*} \) from \( \mathcal{L}_B \).]

We extend the analysis to the case of the \( B^* \) meson. We use the set of parameters in Table 1. The \( \tau \)-behaviours of \( f_B \) and \( f_{B*} \) in Fig. 7 have a shape similar to the case of \( f_D \) and \( f_{D*} \). We estimate \( f_B \) from Fig. 7b where the \( \tau \)-stability is reached from \( t_c = 34 \text{ GeV}^2 \) while the \( t_c \)-one starts from \( t_c = (55 - 60) \text{ GeV}^2 \). We show the \( \mu \)-behaviour of the optimal result in Fig. 8. At the inflexion point \( \mu = (5.5 \pm 0.5) \text{ GeV} \) we deduce:

\[
\begin{align*}
\text{f}_{B*} &\leq 295(14)_{r}(4)_{svz}(10)_{µ} = 295(18) \text{ MeV} .
\end{align*}
\]

(23)
We consider the previous values of $f_{B^*}$ and $f_B$ as improvement of our earlier results in [4] and [18].

8. $f_{B^*}/f_B$ from HQET sum rules

This ratio is under a good control from heavy quark effective theory (HQET) sum rules. It reads in the large mass limit [25, 26, 69]:

$$R_b = \frac{f_{B^*}}{f_B} \sqrt{\frac{M_{B^*}}{M_B}} = R_{PT} + R_{NPT} \frac{M_B}{M_{B^*}},$$

$$R_{PT} = 1 - \frac{2}{3} a_s - 6.56 a_t^2 - 84.12 a_s^2 = 0.892(13),$$

$$R_{NPT} = \frac{2}{3} \tilde{A} \left( 1 + \frac{5}{3} a_s \right) - 4G_F \left( 1 + \frac{3}{2} a_t \right),$$

(24)

with $a_s = (\alpha_s/\pi)(M_b) = 0.070(1)$, $\tilde{A} = M_b - M_{B^*} = (499 \pm 60)$ MeV, where we have used the pole mass $M_{B^*}=4.810(60)$ MeV to order $a_s^2$ deduced from the running mass $\bar{m}_b(m_b) = 4.177(11)$ MeV [50] and we have retained the larger error from PDG [49]. We have estimated the error in the PT series by assuming a geometric growth of the coefficients [37].

$$G_2(M_b) \equiv \frac{G_2}{5}(\alpha_s(M_b))^{1/2} + \frac{8}{27} \tilde{A},$$

(25)

where the invariant quantity: $G_2 \approx -(0.20 \pm 0.01)$ GeV has been extracted inside the $t$-stability region of Fig. 7 from [26]. We obtain:

$$R_b = 0.938(13)_{PT}(5)_{NP}(6)_{M_b,\tilde{A}} = 0.938(15).$$

(26)

To convert the above HQET result to the one of the full theory at finite quark and meson masses, we have to include the normalization factor $(M_{B^*}/M_b)^2$ relating the pseudoscalar to the universal HQET correlators according to the definition in Eq. (1). Then, we deduce (see e.g. [2, 26, 27]):

$$\frac{f_{B^*}}{f_B} = \frac{M_{B^*}}{M_b} \sqrt{\frac{M_B}{M_{B^*}}} R_b = 1.025(13)_{M_b}(15)_{M_b},$$

(27)

in fair agreement with our result in Eq. (20) and the ones in [8, 26, 70] but higher than the ones in [71, 72]. To make a direct comparison of our results with the lattice calculations, it is desirable to have a direct lattice calculation from the pseudoscalar correlator built from the current in Eq. (1). The result of [72] is difficult to interpret as they use a non-standard threshold of the QCD continuum. However, their requirement of maximal stability for $t \leq 0.15$ GeV$^{-2}$ is outside the “standard sum rule (SR) window” obtained around 0.2-0.3 GeV$^{-2}$ (minimum or inflexion point in our figures) where the lowest resonance dominates the sum rule. Taking literally their set ($t_\tau=33$ GeV$^2$, 0.05 GeV$^{-2}$) where their “duality” is obtained (Fig. 3 of [72]) into our Fig. 9, one obtains: $f_{B^*}/f_B = 0.985$ more comparable with their result 0.944 but meaningless from the SR point of view as it comes from a region dominated by the QCD continuum.

9. SU(3) breaking for $f_{D^*}$ and $f_{D}/f_{D^*}$

We pursue the same analysis for studying the SU(3) breaking for $f_{D^*}$ and the ratio $f_{D}/f_{D^*}$. We work with the complete massive ($m_s \neq 0$) LO expression of the PT spectral function obtained in [64] and the massless ($m_\mu = 0$) expression known to NLO used in the previous sections. We include the NLO PT corrections due to linear terms in $m_\mu$ obtained in [70]. We show the $\tau$-behaviour of the results in Fig. 11a for a given $\mu = 1.5$ GeV and different $t_\tau$. We study their $\mu$-dependence in Fig. 11b where a nice $\mu$ stability is reached for $\mu = 1.4 - 1.5$ GeV. We have used: $(D_{c \bar{c}}(t))^1/2 - (D_{c \bar{c}}(t))^1/2 = M_{D^*} - M_D = 102$ MeV. Taking the conservative result ranging from the beginning of $\tau$-stability ($t_\tau \approx 5.7$ GeV$^2$) until the beginning of $t_\tau$-stability of about (9–10) GeV$^2$, we obtain at $\mu = 1.5$ GeV:

$$f_{D^*} = 272(24)_{t_\tau}(18)_{t_{\tau}}(2)_{\mu} = 272(30)\text{ MeV},$$

with:

$$(18)_{t_{\tau}}(0)_{\mu}(1)_{\alpha_s}(1)_{\alpha_s}(1.5)_{\alpha_s}(1.5)_{\alpha_s}(0)(0)(0.3)_{\mu}(0.3)_{\mu}(0.3)_{\mu}(0.3)_{\mu}.$$  

(28)

Taking the PT linear term in $m_\mu$ at lowest order and $t_\tau = 7.4$ GeV$^2$, we obtain $f_{D^*} = 291$ MeV in agreement with the one 293 MeV of [70] obtained in this way. The inclusion of the complete LO term decreases this result by about 5 MeV while the inclusion of the NLO PT SU(3) breaking terms increases the result by about the same amount. Combining the result in Eq. (28) with the one in Eq. (16), we deduce the ratio:

$$f_{D}/f_{D^*} = 1.090(70),$$

(29)

where we have added the relative errors quadratically. Alternatively, we extract directly the previous ratio using the ratio of sum rules. We show the results in Fig. 12a versus $\tau$ and for different values of $t_\tau$ at $\mu = 1.5$ GeV. $\tau$-stabilities occur from $\tau \approx 1$ to 1.5 GeV$^{-2}$. We also show in Fig 12b the $\mu$-behaviour of the results where a good stability in $\mu$ is observed for $\mu = 1.4 - 1.5$ GeV in the same way as for $f_{D^*}$. We deduce:

$$f_{D^*}/f_{D} = 1.073(1)_{\mu}(16)_{t_\mu}(50)_{t_\mu} = 1.073(52),$$

with:

$$(50)_{t_\mu} = (1)_{\alpha_s}(45)_{\alpha_s}(0)_{\mu}(2)_{\mu}(16)_{\mu}(16)_{\mu}(16)_{\mu}(16)_{\mu}(4)_{\mu}(13)_{\mu}.$$  

(30)
where, for asymmetric errors, we have taken the mean of the two extremal values. The error associated to $\tau$ take into accounts the fact that, for some values of $t_c$, the $\tau$-minima for $f_{D^*}$ and $f_{B^*}$ do not coincide. Comparing Eqs. (29) and (30), one can see the advantage of a direct extraction from the ratio of the moments due to the cancellation of systematic errors in the analysis. Taking the mean of Eqs. (29) and (30), we deduce:

$$f_{D^*}/f_{B^*} = 1.08(6)(1)_{\text{asy}} ,$$  \hspace{1cm} (31)

where the 1st error comes from the most precise determination and the 2nd one from the distance of the mean value to the central value of this precise determination. Using Eq. (31), $f_{D^*}$ in Eq. (16) and its upper bound in Eq. (18), we predict in MeV:

$$f_{D^*} = 270(19) , \quad f_{B^*} \leq 287(8.6)(16) = 287(18) . \hspace{1cm} (32)$$

Future experimental measurements of $f_{D^*}$ and $f_{B^*}$ though most probably quite difficult should provide a decisive selection of these existing theoretical predictions.

10. SU(3) breaking for $f_{B^*}$ and $f_{B^*}/f_{B^*}$

We extend the analysis done for the $D^*_1$ to the case of the $B^*_1$-meson. We show, in Fig. 13a, the $\tau$-behaviour of the ratio $f_{B^*}/f_{B^*}$ at $\mu=5$ GeV and for different values of $t_c$ where the $\tau$-stability starts from $t_c = 40$ GeV$^2$ while the $t_c$ one is reached for $t_c \simeq (60 - 65)$ GeV$^2$. Our optimal result is taken in this range of $t_c$. We study the $\mu$-behaviour in Fig. 13b where an inflexion point is obtained for $\mu = (5 \pm .5)$ GeV. At this point, we obtain:

$$f_{B^*}/f_{B^*} = 1.054(8)_{h}(0)_{s}(6)_{h}(4.6)_{s} = 1.054(11) ,$$

with : \hspace{1cm} (4.6)_{s} = (2)_{a}(2.5)_{a}(0)m_{s}(2)_{dG}(1.5)_{a}, \hspace{1cm} (33)

We show, in Fig. 14a, the $\tau$-behaviour of the result for $f_{B^*}$ at $\mu=5.5$ GeV and for different values of $t_c$. For $f_{B^*}$, $\tau$-stability starts from $t_c \simeq 34$ GeV$^2$ while $t_c$-stability is reached from $t_c \simeq (50 - 65)$ GeV$^2$. We show in Fig. 14b the $\mu$-behaviour of these optimal results. One can notice a slight inflexion point at $\mu=6$ GeV which is about the one (5.0-5.5) GeV obtained previously for the ratio $f_{B^*}/f_{B^*}$. At this value of $\mu$, we obtain:

$$f_{B^*} = 271(39)_{t}(0)_{s} = 271(40) \text{ MeV} ,$$

with : \hspace{1cm} (3)_{s} = (1)_{a}(1.5)_{a}(0)_{m}(2)_{dG}(0.5)_{a}, \hspace{1cm} (34)

Combining this result with the one of $f_{B^*}$ in Eq. (19) obtained within the same approach and conditions, we deduce the ratio:

$$f_{B^*}/f_{B^*} = 1.13(25) ,$$

where the large error is due to the determinations of each absolute value of the decay constants. We take as a final value of the ratio $f_{B^*}/f_{B^*}$ the most precise determination in Eq. (33). Combining this result with the final value of $f_{B^*}$ in Eq. (22) and with the upper bound in Eq. (23), we deduce our final estimate:

$$f_{B^*} = 220(9) \text{ MeV} , \quad f_{B^*} \leq 311(19) \text{ MeV} . \hspace{1cm} (36)$$

11. The scalar meson decay constants $f_{D^*}$ and $f_{B^*}$

- Due to large discrepancies of the $f_{D^*}$ values among different SR [73–75] and between lattice [76, 77] results (see Table 3), we update our previous estimates in [4, 18, 22]. We use the same QCD inputs (N2LO for PT within the $\overline{MS}$-scheme @ $\bar{d} \leq 6$ condensates) and the same criteria as for $f_{D,B} [1]$ for obtaining the optimal results. The difference between the scalar and pseudoscalar SR is the overall factor $(M_{Q} \pm m_{s})^2$ and the signs of the $(\eta \eta)$ and mixed condensates effects for $m_{s} = 0$. We show $f_{D^*}$ versus $\tau$ in Figs. 15a,b where the $\tau$-stability about $(0.4 - 0.6)$ GeV$^{-2}$ of the SR for $\mu = \tau^{-1/2}$ constrains $t_{c}$ inside $(7 - 9)$ GeV$^2$ (Fig. 15b)\footnote{Larger $t_{c}$ corresponds to an apparent small $\tau$-stability outside the SR window and leads to a larger $f_{D^*}$ (Fig. 15b).}. In Fig. 15b, we show the SR for a given $\mu = 2$ GeV, and $t_{c} = (7 - 11)$ GeV$^2$-stability. The effect of the truncation of the PT series is analyzed by using instead the invariant quark mass in Eq. 7 and the QCD scale $\Lambda$. We show in Fig. 15c the results versus $\mu$ and using either the running (blue open circle data points) or the invariant (red triangle data) mass. Taking their average, we deduce the value in Table 2 where 95% of the error comes from the above range of $t_{c}$ values. The error becomes 12 instead of 7 if one considers the most precise determination. Upper bound is derived in Fig. 16 using the positivity of the spectral function. We extract from the SR at $\mu = \sqrt{1/\tau}$...
the SU(3) breaking ratio $f_{D^*}/f_D$ with the result in Table 3. The error of the ratio comes mainly from the $SU(3)$ breaking terms. These results confirm and improve the previous one in [22].

- Doing similar analysis for $f_{B^*}$ and $f_{B^*}$, we notice that only the SR subtracted at $\mu = \tau^{-1/2}$ (Fig. 15) exhibits $\tau$-stability for $t_c \approx (35 - 38) \text{ GeV}^2$. We use $M_{B^*} = M_{B^*} = 5733 \text{ MeV}$ assuming that $M_{B^*} - M_B = M_{B^*} - M_D = 448 \text{ MeV}$ supported by the SR estimate $413(210) \text{ MeV}$ in [4, 18]. We deduce the results in Table 2, where like for $f_{D^*}$, the errors come mainly from $t_c$ and the $SU(3)$ breaking parameters. These results improve and confirm previous estimates in [4, 18].

- $f_{D^*}$ agrees with the (inaccurate) phenomenological value 206(120) MeV from $B \rightarrow D^*\pi$ [76]. The agreement for $f_{B^*}$ but not $f_{D^*}$ with the SR one in [75] is puzzling and could be due to a different treatment of the QCD input and evolution parameters which are sensitive at low scale. A comparison with some other determinations are given in Table 3. A future precise measurement of $f_{D^*}$ will select among the theoretical estimates.

- The analysis will be similar to the one in [18, 78, 79] but we shall use the running $M_B - M_{B^*} = M_{B^*} - M_D = 448 \text{ MeV}$ as indicated by the result $417(212) \text{ MeV}$ in [4, 18]. The optimal ratios obtained in this way are given for different $\mu$ in Figs. 17c,d. The average given in Table 4 comes from the dashed coloured regions.

13. The decay constants $f_{B^*}$ and $f_{B^*}$

We complete the analysis by the estimate and bound of the decay constant $f_{B^*}$ of the $B_{c}(6277)$ meson $bc$ bound state and of $f_{B^*}$ of its vector partner $B_{c}^*$, where the light quarks $d, s$ are replaced by the heavy quark $c$. Our analysis will be very similar to the one in [18, 78, 79] but we shall use the running $c$ and $b$ quark masses instead of the pole masses and we shall include N2LO radiative corrections.

- The dynamics of the $B_{c}$ and $B_{c}^*$ is expected to be different from the $B$ and $B^*$ one because, by using the heavy quark mass expansion, the heavy quark $\langle cc \rangle$ and quark-gluon mixed $\langle cGc \rangle$ condensates defined in Section 2 behave as [78]:

$$\langle cc \rangle = \frac{1}{12\pi m_c} (\alpha_s G^2) - \frac{\langle G^2 \rangle}{1440\pi^2 m_c^3},$$

12. The axial meson decay constants $f_{D^*}$ and $f_{B^*}$

In the zero light quark mass limit, the axial-vector correlator can be deduced from the vector one by changing the sign of the chiral condensate contributions. Using the expression known to N2LO @ $d \leq 6$ condensates, we extract directly the ratio $f_D/f_D$ and $f_{B}/f_B$ by taking each value of $f_D$, and $f_{B}$ at a $t$-minimum for a given $(\mu, t_c)$ (Figs. 17a,b) and the corresponding value of $f_D$ (Fig. 3) and $f_B$ (Fig. 7) at the same of set $(\mu, t_c)$. We use $M_{B^*} = M_{B^*} = 5733 \text{ MeV}$ as indicated by the result $417(212) \text{ MeV}$ in [4, 18]. The optimal ratios obtained in this way are given for different $\mu$ in Figs. 17c,d. The average given in Table 4 comes from the dashed coloured regions.
These behaviours are in contrast with the ones of the light quark $\bar{q}q$ and mixed quark-gluon $\bar{q}Gq$ condensates [4, 11].

- The complete expression of the perturbative NLO pseudoscalar spectral function has been obtained in [16] and explicitly written in [78], where we transform the pole to the running terms of PT series [38, 39].

- The Wilson coefficients of the non-perturbative $\langle \alpha_s G^2 \rangle$ and $\langle g^2 G^2 \rangle$ contributions are also given in [78].

- Like in the previous cases, we study the SR versus $\tau$ and $t_c$ as shown in Fig. 18. One can note that the non-perturbative contributions are small (about 1–2 MeV) indicating that the dynamics of the $B_s$ meson is dominated by the perturbative contributions. This feature might explain the success of the non-relativistic potential models for describing the $B_s$-like hadrons [78]. The optimal result is obtained from $t_c = 44$ GeV$^2$ (beginning of $\tau$-stability) until $t_c = (50 - 60)$ GeV$^2$ (beginning of $\tau$-stability).

We show in Fig. 19 the $\mu$-behaviour of the results, where there is an inflexion point for $(7.5 \pm 0.5)$ GeV for the estimate (Fig. 19a) and the upper bound (Fig. 19b). At these optimal points, we deduce:

$$ f_{B_s} = 436(38)(6)_{0.05}(1.2)(7)_{0.05}(7)_{0.05}(6)_{\mu} = 436(40) \text{ MeV}, $$

$$ f_{B_s} \leq 466(9)_{0.05}(2.5)(12)_{0.05}(8)_{\mu} = 466(16) \text{ MeV}. $$

We may consider these results as a confirmation and improvement of the earlier ones in [4, 18, 78, 79]. The results in Eq. (38) will restrict the wide range of $f_{B_s}$ given in the current literature and may be used for extracting the CKM angle $\theta_{c\bar{b}}$ from the $B_s \rightarrow \tau \nu \tau$ leptonic width.

- We do a similar analysis for the $B_s$ vector meson where we replace the strange by the charm quark mass in the $B_s$ SR expression$^9$. We use the rescaled $M_{B_s}$ = 6350(20) MeV from potential model [78] using the experimental value of $M_{B_s}$. The different steps are shown in Fig. 21. We obtain the results in Table 6 for $\tau \simeq 0.15$ GeV$^{-2}$.

### Summary and Conclusions

- We have reviewed our recent determinations of the heavy-light pseudoscalar [1, 2], vector [3] mesons decay constants

---

$^9$A complete expression of the spectral function to NLO is given in [13] but it has singularity for $m_c \rightarrow 0$ and needs to be checked.
Table 3: Scalar $D_{QCD}, B_{QCD}$ meson decay constants. Values marked with * are not included in the average. The one with + has been rescaled by 1.5 for accounting a slight tension with our estimate.

| $f_{s^*}$ [MeV] | $f_{D^*}/f_D$ | $f_{B^*}/f_B$ | Sources | Ref. |
|----------------|----------------|----------------|--------|-----|
| 122(13)−360(99) | −              | −              | SR     | [77]|
| 128(13)−373(19) | −              | −              | SR     | [75]|
| 217(24)         | 202(23)        | 0.93(2)        | SR     | [75]|
| 221(12)         | 202(19)        | 0.91(23)       | SR     | New |
| 243(8)          | ≤ 222(6)       | −              | SR     | New |
| 220(11)         | 202(15)        | 0.92(15)       | SR Average |

Table 4: Vector $D^*, B^*$ and axial $D_1, B_1$ meson decay constants. The $n_f = 2$ Lattice and Global Average errors marked with * is rescaled by 1.5 for accounting slight tensions between different estimates.

| $f_{D^*}$ [MeV] | $f_{D^*}/f_D$ | $f_{B^*}/f_B$ | Sources | Ref. |
|----------------|----------------|----------------|--------|-----|
| 263(51)        | −              | −              | SR     | [4, 18]|
| 281(14)        | 274(20)        | −              | SR     | [75]|
| 271(26)        | 233(21)        | 0.865(54)      | SR     | New |
| ≤ 302(8)       | ≤ 261(17)      | −              | SR     | New |
| 278(12)        | 255(15)        | −              | SR Average |

Table 5: $SU(3)$ breakings for the vector $D^*$ and $B^*$ decay constants.

| $f_{B^*}/f_B$ [MeV] | $f_{D^*}/f_D$ | $f_{D^*}/f_D$ | Sources | Ref. |
|---------------------|----------------|----------------|--------|-----|
| 270(29)             | 1.08(6)        | 220(14)        | SR     | [3] |
| 293(17)             | 1.21(5)        | 210(11)        | SR     | [70]|
| 308(21)             | 285(19)        | −              | SR     | [75]|
| ≤ 287(18)           | ≤ 317(17)      | −              | SR     | [3] |
| ≤ 347               | ≤ 296          | −              | SR     | [70]|
| 290(11)             | 1.16(4)        | 221(7)         | SR Average |
| 306(27)             | 1.21(6)        | 214(19)        | −      | NS2R |
| 311(14)             | 1.16(6)        | −              | Latt. $n_f = 2$ [8]|
| 274(6)              | −              | ≤ 3(7)         | Latt. $n_f = 2 \neq 1 \& NR$ [71, 90]|
| 282(5)(4.8)         | 1.17(3)        | 216(9)(4.6)    | 1.064(10) | Global Average |

Table 6: Pseudoscalar $B_s$ and vector $B_s^*$ mesons decay constants. The $n_f = 2$ Lattice error marked by * is rescaled by 1.5. The same for NS2R for accounting the tension with other estimates.

| $f_{B_s^*}/f_{B_s}$ [MeV] | $f_{B_s}/f_{B_s}$ | $f_{B_s}/f_{B_s}$ | Sources | Ref. |
|---------------------------|------------------|------------------|--------|-----|
| 436(40)                   | −                | −                | SR     | [3] |
| ≤ 466(16)                 | −                | −                | SR     | [3] |
| 305(71)                   | −                | −                | Po. Mod. [78] |
| 528(29)*                  | −                | −                | NS2R [89] |
| 427(6)                    | −                | ≤ 0.988(27)     | Latt. $n_f = 2$ [71]|
| 434(23)*                  | −                | 1.043(45)       | SR     | New |
| −                         | ≤ 505(6)        | −                | SR     | New |
| 432(6)                    | −                | 1.003(23)       | Global Average |

from the standard Laplace and Moment sum rules (SR) introduced by SVZ [10]. Motivated by different tensions in the literature, we have completed the review by presenting new determinations of the vector $f_{V} / f_{B}$, scalar $f_{D_{QCD}, B_{QCD}}$, axial $f_{D_{1}, B_{1}}$, and vector $f_{B_{s}}$ meson decay constants. Our results based on stability criteria are summarized in Tables 2–6 and compared with some other recent sum rules and lattice results. Our quoted errors are mainly due to the choice of the QCD continuum threshold and to the estimate of the $a_s^2$ PT series. This latter is not considered in some other SR results.

- Then, we have attempted to give a Global Average of sum rules and lattice results (results from some other approaches can e.g be found in [91]), which can be used for further phenomenological applications. To take into account the slight tensions among different determinations, we have rescaled some errors.
- One can see in these Tables that there are fair agreements among estimates from different standard sum rule (SR) approaches. The slight difference is mainly due to a different appreciation of the optimal results from the choice of the QCD continuum threshold, subtraction constant $\mu$ and of the set of input QCD parameters. The quoted upper bounds come from the positivity of the spectral functions.

- A slight tension exists between SR, NS2R and lattice results for $f_{B_{s}} / f_{B}$ [72]. To give insight into this problem, we have rederived this ratio in Section 8 from HQET sum rules. There is also a tension for $f_{B_{s}}$, from NS2R [89] (see comments in [3]). The NS2R results do not affect significantly the SR average and they are considered in the Global Average.

- Large tensions exist in the scalar sector among various SR and between lattice results for $f_{D_{1}}$. Our updated results given in Table 3 using the running $\overline{MS}$ charm mass and SR stability criteria confirm our previous findings [22].

- The results indicate a good realization of heavy quark symmetry for the $B$ and $B^*$ mesons ($f_{B} \approx f_{B^*}$) as expected from HQET [25] but signal large charm quark mass and radiative QCD corrections for the $D$ and $D^*$ mesons ($f_{D} \approx f_{D^*}$) which are known since a long time (see e.g among others [17, 93]).

- The $SU(3)$ breaking is typically 20% for $f_{D_{1}} / f_{D} \approx f_{B_{1}} / f_{B}$ and 10% for $f_{D_{1}} / f_{D} \approx f_{B_{1}} / f_{B}$ which are almost constant as they behave like $m_{b}/\omega_{c}$ [32] where $\omega_{c} \equiv \sqrt{M_{B_{c}} - M_{B} - M_{B_{s}}}$ is independent of $M_{b}$. The reverse effect for $f_{D_{1}} / f_{D} \approx f_{B_{1}} / f_{B} < 1$ comes mainly from the overall $(1 - m_{c}/M_{Q})$ factor not compensated by $M_{D_{1}} / M_{B_{1}}$. 

- It is informative to show the behaviour of our predictions of the pseudoscalar and vector meson decay constants versus the corresponding meson masses in Fig. 22. We use $f_{p} = (221.6 \pm 1.0)$ MeV from its electronic width [49] and $f_{D_{s}} = 207$ MeV from SR analysis [4]. One can notice similar $M_{Q}$ behaviours of these couplings where the the $1/\sqrt{M_{Q}}$ HQET relations [25, 92] are not satisfied due to large $x_{t}$ and $1/M_{O_{Q}}$ corrections as noted earlier [17, 93]. One can notice small $SU(3)$ but large $SU(4)$ breaking for $B_{s}, B_{s}^{*}$. Chiral symmetries are badly broken between the vector $D^{*}, B^{*}$ and axial $D_{1}, B_{1}$ meson couplings.

- We do not find any deviation of these Standard Model results from the present data.
