A note on Polyakov’s nonlocal form of the effective action

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Abstract

A technical point regarding the invariance of Polyakov’s nonlocal form of the effective action under uniform rescalings is addressed.
1. Introduction.
The following is a brief note on the conformal anomaly and its integration to give the change in the effective action under a conformal transformation. This is a well–trodden path, especially in two dimensions, and no attempt will be made here to give proper references. A recent review, with a personal flavour, by Duff, [1], provides a reasonable, historical perspective on the conformal anomaly.

In two dimensions, which is what concerns us here, the anomaly was integrated to give the effective action by Lüscher, Symanzik and Weiss [2], and by Polyakov [3], long ago. The point at issue is Polyakov’s conversion of the local form of the effective action to a non-local expression when there is a zero mode, as for a closed space. We can therefore ignore boundary effects.

The conclusions of this paper are elementary and probably known to workers in the field. However, the author has not been able to find a suitable published discussion.

2. Integrating the anomaly
The conformal scaling is $g_{\mu\nu} \rightarrow \bar{g}_{\mu\nu} = \lambda^2 g_{\mu\nu} = \exp(-2\omega)g_{\mu\nu}$ under an infinitesimal example of which the renormalised effective action, $W$, changes by

$$\delta W[\bar{g}] = -\zeta(0, \bar{g}; \delta \omega)$$

with

$$\zeta(0, \bar{g}; f) \equiv \int_{\mathcal{M}} \zeta(0, \bar{g}, x) f(x) (\bar{g})^{1/2} dx.$$  

Standard theory gives the local value

$$\zeta(0, g, x) = \frac{1}{24\pi} R - P_0$$

where $P_0(x)$ is the projection onto the zero mode. In the following we will set $P_0 = 1/A$ where $A$ is the area of the closed 2-manifold $\mathcal{M}$ of metric $g$. The manifold of metric $\bar{g}$ is denoted by $\mathcal{M}$.

The idea is to integrate (1) knowing the explicit dependence of the right-hand side on the scaling function $\omega$. As already said, the result is very old. It is

$$W[\bar{g}, g] = -\frac{1}{2} \ln \left( \frac{A[\bar{g}]}{A[g]} \right) - \frac{1}{24\pi} \int \omega(R + \Box) \bar{g}^{1/2} dx$$

where $\Box$ is the covariant Dalembertian.

In order to get a symmetrical formula in terms of the geometry corresponding to the two metrics, we can try to eliminate the conformal factor $\omega$ by solving the conformal relation

$$\Box \omega = \frac{1}{2g^{1/2}} \left( \bar{g}^{1/2} \bar{R} - g^{1/2} R \right)$$
for $\omega$. This is possible because the right-hand side is orthogonal in $\mathcal{M}$ to the zero mode, a uniform function, by topological invariance. The solution then reads

$$\omega(x) = \frac{1}{2} \int G(x, y) (\bar{g}^{1/2} \bar{R} - g^{1/2} R)_y \, dy + \frac{1}{A} \int \omega(y) g^{1/2} \, dy,$$

where the Green function $G$ is to be thought of as having the zero mode removed, i.e. it satisfies the standard equation

$$\Box G(x, y) = \delta(x, y) - \frac{1}{A}.$$  \hfill (7)

Note that removing the zero mode destroys the equality of $G$ and $\bar{G}$, except under uniform rescalings. $\bar{G}$ obeys

$$\Box \bar{G}(x, y) = \tilde{\delta}(x, y) - \frac{1}{A},$$

with

$$\bar{g}^{1/2} \Box = g^{1/2} \Box.$$ \hfill (9)

It should be pointed out that $\delta(x, y)$ is the covariant delta function on $\mathcal{M}$. It equals $\delta_P(x, y)/\bar{g}^{1/2}$, where $\delta_P(x, y)$ is the metric-independent, Dirac delta function periodised appropriately for a closed manifold.

Substituting for $\omega$ in (4), and using (5), we obtain (cf [4] but without the zero mode contribution)

$$W[\bar{g}, g] = -\frac{1}{2} \ln \left( \frac{A[\bar{g}]}{A[g]} \right) - \frac{1}{48\pi A} \int \ln \left( \frac{\bar{g}}{g} \right) \bar{g}^{1/2} \, dx \int R \bar{g}^{1/2} \, dx$$

$$-\frac{1}{96\pi} \int (\bar{g}^{1/2} \bar{R} + g^{1/2} R)_x G(x, y) (\bar{g}^{1/2} \bar{R} - g^{1/2} R)_y \, dx \, dy.$$ \hfill (10)

The first term on the right-hand side of (10) is the standard zero mode contribution. The second term is our (possible) novelty which goes part way to relieving the unease expressed by Duff [1] regarding the variation (or rather, non-variation) of the nonlocal action under uniform rescalings. Under such, the final term in (10) is unchanged but the second yields the required variation.

The second term on the right-hand side of (10) (and of its symmetrical form) can be removed if $\omega$ is constrained to being orthogonal to the zero modes in both $\mathcal{M}$ and $\overline{\mathcal{M}}$. Of course, this then eliminates the possibility of discussing uniform rescalings in the nonlocal formulation.

It is not obvious that one can write the right-hand side of (10) as $W[\bar{g}] - W[g]$. But, using (9) and solving (5) in terms of $\bar{G}$, we find an equivalent form for $W$ that implies the required antisymmetry, $W[\bar{g}, g] = -W[g, \bar{g}]$. The second cocycle
condition, \( W[g_1, g_2] + W[g_2, g_3] + W[g_3, g_1] = 0 \), is also true. This implies the existence of the functional \( W[g] \). It is probably easiest to show this from the local expression (4) by noting that

\[
\frac{\delta^2}{\delta \omega \delta \sigma} W[e^{-2\omega g}, e^{-2\sigma g}] = 0. \tag{11}
\]

Unlike Polyakov’s case, it does seem possible to disentangle \( \bar{g} \) from \( g \) in (10) but, formally, we could define \( W[g] \), up to an additive constant, by \( W[g] = W[g, g_0] \) where \( g_0 \) is some fiducial metric.

3. Conclusion

Our unexceptional conclusion is that one must be careful when employing Polyakov’s nonlocal form of the effective action, \( W[g] \approx \int R \Box^{-1} R \). A too cavalier introduction could fabricate difficulties such as those concerning uniform scalings.

References

[1] M.Duff *Twenty years of the Weyl anomaly* CTP-TAMU-06/93, (1993).
[2] M.Lüscher, K.Symanzik and P.Weiss *Nucl. Phys.* B173 (1980) 365.
[3] A.Polyakov *Phys. Lett.* B 103B (1981) 207.
[4] L.Bukhbinder, V.P.Gusynin and P.I.Fomin *Sov. J. Nucl. Phys.* 44 (1986) 534.