Abstract

The key step of generating the well-known Hsiao code is to construct a \( \{0,1\} \)-check-matrix in which each column contains the same odd-number of 1’s and each row contains the same number of 1’s or differs at most by one for the number of 1’s. We also require that no two columns are identical in the matrix. The author solved this problem in 1986 by introducing a type of recursively balanced matrices. However, since the paper was published in Chinese, the solution for such an important problem was not known by international researchers in coding theory. In this note, we focus on how to practically generate the check matrix of Hsiao codes. We have modified the original algorithm to be more efficient and effective. We have also corrected an error in algorithm analysis presented in the earlier paper. The result shows that the algorithm attained optimum in average cases if a divide-and-conquer technique must be involved in the algorithm.

1 Introduction

Error-detections and corrections are required for computer main memory and secondary storages. In recent years, trusted computing and computing reliability have become more and more important in theory and practice. Data recovery is at the center of concerns. Error-corrections codes, especially Hamming codes and Hsiao codes are still essential to this type of technology [10][9].

The well-known Hsiao code is the most energy saving Hamming Codes [7][9][4][5][6]. It has been widely used in memory fault tolerance for more than thirty decades. As a type of SEC-DED codes, i.e. single error correction and double-error detection codes, Hsiao codes attained the optimal in minimum odd-weight of columns. Even though there was an observation indicating that cellular automata-based codes might be better than Hsiao codes in terms of check bits for cellular automata—a future computing device [4][2]. However, Hsiao code is still the most efficient code used in industry [9][6].
After a code is designed, the most important task is to find its check matrix. The author introduced an algorithm that recursively constructed the check-matrix to Hsiao codes in 1986 [1]. Unfortunately, the published paper was written in Chinese and the title of the paper did not mention the SEC-DED code. This algorithm was not known by international researchers in coding theory.

In this paper, we first explain the algorithm described in [1] where it was addressed in an abstract and in a brief manner. Then, we modified some steps to show a more detailed algorithm. Finally, we corrected an error in the algorithm analysis of [1] and provided a more precise time analysis for general cases. Theoretically, both original algorithm and the modified algorithm are very efficient and fast in terms of complexity theory. We have proved that the modified algorithm is optimal in average cases. However, in practice, we may find an even more efficient algorithm.

2 Major Steps of Generating Check Matrix H for Hsiao Code

The definition of Hsiao code is a type of SEC-DED codes whose check matrix $H$ defined on $GF(2)$ satisfies:

(1) Every column contains an odd number of 1’s.
(2) The total number of 1’s reaches the minimum.
(3) The difference of the number of 1’s in any two rows is not greater than 1.
(4) No two columns are the same.

Therefore, Hsiao called this code an optimal minimum odd-weight-column SEC-DED code.

Now, we will discuss how to generate $H$. Assume that we want to generate the SEC-DED Code with $k$ information (data) bits, first according to the general requirement of Hsiao codes [7],

$$R \geq 1 + \log_2 (k + R)$$ (1)

From this, we can determine $R$. After $R$ is determined, we define $\Delta(R, J, m)$ a $\{0, 1\}$-type $R \times m$ matrix with column weight $J$, $0 \leq J \leq R$. No two columns are the same in $\Delta(R, J, m)$.

$$\Delta(R, 2 \cdot i + 1, C_R^{2i+1}), i = 0, 1, ..., I - 1.$$ (2)

Where, $C_R^{2i+1} = \binom{R}{2 \cdot i + 1}$ is the combinatorial number. Assume,
\[ \Sigma_{i=0}^{I-1} C_R^{2i+1} \leq (k + R) \leq \Sigma_{i=0}^{I} C_R^{2i+1}. \tag{3} \]

Let \( m = (k + R) - \Sigma_{i=0}^{I-1} C_R^{2i+1} \). We can see that the most important task is to generate

\[ \Delta(R, 2 \cdot I + 1, m) \]

because the other cases are the special cases of it.

Thus,

\[ H = [\oplus_{i=0}^{I-1} \Delta(R, 2 \cdot i + 1, C_R^{2i+1})] \oplus \Delta(R, 2 \cdot I + 1, m), \tag{4} \]

where \( \oplus \) is to union these arrays together, horizontally.

### 3 The Algorithm of Generating \( \Delta(R, J, m) \)

According to Section 2, it is not difficult to see that obtaining the matrix \( \Delta(R, J, m) \) is the key to the check matrix \( H \), especially when \( m \neq C_R^{2i+1} \).

Let’s first define some notations:

- (a) \( \Delta(R, J, m) \): a \( \{0, 1\}\)-type \( R \times m \) matrix with column weight \( J \), \( 0 \leq J \leq R \).
- (b) \( < t > (m) \): a \( \{0, 1\}\)-type \( 1 \times m \) matrix in which every component is \( t \). For example, \( < 1 > (2) = (1, 1) \).
- (c) \( \circ \): the matrix up-down union operator, i.e. \( N \circ M = \begin{bmatrix} a \\ b \end{bmatrix} \)
- (d) \( \oplus \): the matrix left-right union operator, i.e. \( N \oplus M = [NM] \)
- (e) \( \overline{\oplus} \): \( N \overline{\oplus} M \) is to place the matrix \( M \) upside down, do “\( \oplus \),” and then move the rows that contain more 1’s to the top of the matrix.
- (f) \( L - condition \): \( \Delta(R, J, m) \) is said to satisfy the condition if \( 0 \leq J \leq R \) & \( 0 \leq m \leq C_J^I \).
- (g) **Matrices with equal-weight-columns/rows**: Matrices have equal-weight columns and quasi-equal-weight rows. In other words, in these matrices, each column has the same number of 1’s and each row has almost the same number of 1’s. (The difference of the numbers of 1’s is less than 1.) We will also call such a matrix a (weighted) Balanced matrix.

We also define the following special \( \Delta(R, J, m) \) that will keep the rows with more 1’s at the top of the array:

\[ \sum_{i=0}^{I-1} C_R^{2i+1} \leq (k + R) \leq \sum_{i=0}^{I} C_R^{2i+1} \]

\[ \Delta(R, 2 \cdot I + 1, m) \]
(1) If \( m = 0 \), \( \Delta(R, J, m) = \emptyset \).

(2) If \( J = 0 \), \( \Delta(R, J, m) = (0, ..., 0)^T \).

(3) If \( J = R \), \( \Delta(R, J, m) = (1, ..., 1)^T \).

(4) If \( m = 1 \), \( \Delta(R, J, m) = (1, ..., 1, 0, ..., 0)^T \), where the number of 1’s is \( J \).

(5) If \( J = 1 \), \( \Delta(R, J, m) = \begin{pmatrix} 1 & 0 & ... & 0 \\ ... & ... & ... \\ 0 & ... & 0 & 1 \\ 0 & ... & 0 \\ ... & ... & ... \\ 0 & ... & 0 \end{pmatrix} \), there are \( m \) 1’s in the matrix.

(6) If \( J = R - 1 \), \( \Delta(R, J, m) = \begin{pmatrix} 1 & 1 & ... & 1 \\ ... & ... & ... \\ 1 & ... & 1 & 1 \\ 0 & 1 & ... & 1 \\ 1 & 0 & ... & 1 \\ ... & ... & ... \\ 1 & 1 & ... & 0 \end{pmatrix} \), there are \( m \) 1’s in the matrix.

The above matrices are called the ending-states. So we can assume that \( 2 \leq J \leq R - 2 \) for later discussion.

Let \( \Delta(R, J, m) \) satisfy the \( L \)-condition. We can represent \( \Delta(R, J, m) \) in the following form.

\[
\Delta(R, J, m) = \begin{bmatrix} 1 & \cdots & 1 \\ \Delta_1(R - 1, J - 1, m_1) & \Delta_2(R - 1, J, m - m_1) \end{bmatrix}
\]  \hspace{1cm} (5)

**Theorem 1** [1] Assume \( \Delta(R, J, m) \) satisfy the \( L \)-condition. Let \( m_1 = \lfloor \frac{m - J}{R} + \frac{R - 1}{R} \rfloor \), then \( \Delta_1(R - 1, J - 1, m_1) \) and \( \Delta_2(R - 1, J, m - m_1) \) also satisfy \( L \)-condition.

**Theorem 2** [1] Suppose that \( \Delta_1(R - 1, J - 1, m_1) \) and \( \Delta_2(R - 1, J, m - m_1) \) are Matrices with equal-weight-columns/rows, where \( m_1 = \lfloor \frac{m - J}{R} + \frac{R - 1}{R} \rfloor \). Then,

\[
\Delta'(R, J, m) = \langle 1 > (m_1) \oplus < 0 > (m - m_1) \rangle \circ [\Delta_1(R - 1, J - 1, m_1) \oplus \Delta_2(R - 1, J, m - m_1)]
\]  \hspace{1cm} (6)

is a matrix with equal-weight-columns/rows.

The above two theorems indicate a recursive process of generating a matrix with equal-weight columns and quasi-equal-weight rows: The Theorem 2 provides the union process that guarantees that each row after the union has about the same weight. \( \oplus \), to place the matrix on the right side of the operator upside down, is to avoid one row containing more than one 1’s after the merge. It is the simplest mathematical way of solving the problem. The recursive matrice in (5) is also called a recursively balanced matrix.
However, by “placing a matrix upside down” and then moving rows with more 1’s to the top of the matrix after merging takes more computational time. In [1], the author made a mistake by simply stating that this process is optimum $O(R \cdot m)$. This is because some of the parts in the array may need to be “placed upside down” many times (say $\min\{R, m\}$). So, in the worst case, the complexity of the procedure given in Theorem 2 is $O(R \cdot m \cdot \min\{R, m\})$. This is still a very fast algorithm for generating such a matrix. Other generating algorithms for these check matrices were discussed in [11][5][6]. However, these algorithms are not in polynomial time.

It is obvious that if $m \cdot J$ is divisible by $R$, the procedure of “placing a matrix upside down” can be ignored. So,

**Corollary 1** Let $m \cdot J$ be divisible by $R$. Suppose that $\Delta_1(R-1, J-1, m_1)$ and $\Delta_2(R-1, J, m - m_1)$ are matrices with equal-weight-columns and equal-weight-rows, where $m_1 = \left[ \frac{m \cdot J}{R} + \frac{R-1}{R} \right]$. Then,

$$\Delta'(R, J, m) = [\langle 1 \rangle > (m_1) \oplus \langle 0 \rangle > (m - m_1)] \circ [\Delta_1(R-1, J-1, m_1) \oplus \Delta_2(R-1, J, m - m_1)].$$  \hspace{1cm} (7)

is a matrix with equal-weight-columns and equal-weight-rows.

**Corollary 2** There is $O(m \cdot R)$ algorithm for generating $\Delta(R, J, m = C^J_R)$

Based on the above discussion, we can obtain two algorithms below:

**Algorithm A: the none-recursive algorithm**

**Step 1** Check if $\Delta(R, J, m)$ satisfies the L-condition.

**Step 2** Decompose $\Delta(R, J, m)$ into $\Delta(R_1, J_1, m_1), \Delta(R_2, J_2, m_2), ..., \Delta(R_n, J_n, m_n)$, where each $\Delta(R_i, J_i, m_i), (i=1,2,...,n)$ is a ending-state.

**Step 3** Merge $\Delta(R_i, J_i, m_i), (i = 1, 2, ..., n)$ using $\oplus$ operator.

Note: ”decomposition” and ”merge” use the same objects.

**Algorithm A’: the recursive algorithm**

**Step 1** Check if $\Delta(R, J, m)$ satisfies L-condition.

**Step 2** Decompose $\Delta(R, J, m)$ into $\Delta_1(R-1, J-1, m_1)$ and $\Delta(R-1, J, m - m_1)$.

**Step 3** Merge $\Delta_1(R-1, J-1, m_1)$ and $\Delta(R-1, J, m - m_1)$ based on (6).

### 4 A Better Algorithm for Generating $\Delta(R, J, m)$

Even though Theorem 2 did not give an $O(R \cdot m)$ algorithm, it provides a great hint to design a (virtually) optimum algorithm. The key is to keep more 1’s rows at the top of the matrix and to
The following simple calculations can be used to improve the process. We would be able to know how many rows have the extra 1’s in each of \( \Delta_1(R-1, J-1, m_1) \) and \( \Delta_2(R-1, J, m-m_1) \) before the merge. After the merge, the array will not change.

Suppose that \( \Delta_1(R-1, J-1, m_1) \) and \( \Delta_2(R-1, J, m-m_1) \) are Matrices with equal-weight-columns/rows, where \( m_1 = \left\lceil \frac{mJ}{R} + \frac{R-1}{R} \right\rceil \).

First, calculate

\[
  r_1 = (J - 1) \cdot (m_1) \pmod{R-1}, 0 \leq r_1 < (R-1),
\]

\[
  r_2 = J \cdot (m - m_1) \pmod{R-1}, 0 \leq r_2 < (R-1).
\]

If \( r_1 + r_2 > (R-1) \), let \( r' = r_1 + r_2 - (R-1) \). Because \( r_1 < (R-1) \) and \( r_2 < (R-1) \), \( r_1 > r' \) and \( r_2 > r' \). Shift \( r_2 - r' \) rows in \( \Delta_2 \) to the bottom to obtain a \( \Delta'_2 \). If \( r_1 + r_2 \leq (R-1) \), then move the first \( r_2 \) rows to \( r_1 + 1 \) to \( r_1 + r_2 \), so

\[
  \Delta'(R, J, m) = \langle < 1 > (m_1) \oplus < 0 > (m-m_1) \rangle \circ [\Delta_1(R-1, J-1, m_1) \oplus \Delta'_2(R-1, J, m-m_1)]
\]

is a matrix with equal-weight-columns/rows.

The time complexity of the above process is

\[
  T(m) = T(m_1) + T(m - m_1) + O(m \cdot R).
\]

Using the same technique to analyze the average case of Quick-Sort [3], we have

\[
  T(m) = R \cdot m \log (m \cdot R) = T(m) = R \cdot m \log (m).
\]

For the detail analysis, we could view \( T(m) \) as the average time required by the randomized \( m_1 \). That means \( m_1 \) could be any number between 1 to \( m - 1 \). So

\[
  T(m) = (T(1) + T(m-1) + T(2) + T(m-2) + \ldots + T(m-1) + T(1) + O(R \cdot m \cdot m)) / (m-1),
\]

\[
  T(m) = \frac{2}{m-1} \sum_{k=1}^{m-1} T(k) + O(R \cdot m).
\]

Using the method of the mathematical induction, we can prove that \( T(m) \leq R \cdot m \cdot \log(R \cdot m) \) [3].

We know that \( R \leq m \) is held in most of cases. Therefore, we have:

**Theorem 3** The time complexity of generating \( \Delta(R, J, m) \) is \( O(R \cdot m (\log R + \log m)) \) for average cases, or \( O(R \cdot m \cdot \log m) \) for most cases.

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5 More Efficient Algorithms for Generating $\Delta(R, J, m)$?

To avoid using extra time to build $\Delta(R, J, m)$, we have to know exactly where to put the extra 1’s if $r = J \cdot m \pmod{R}$ is not zero. Except in the case of $m = C^J_R$, even though $r = J \cdot m \pmod{R}$, we may have an unbalanced number of 1’s at the next level. For example, let $R = 6$, $J = 3$, and $m = 10$. $m_1 = [(J \cdot m)/R + (R - 1)/R] = 6$ and $(m - m_1) = 6$. $r = (J - 1) \cdot m_1 \pmod{(R - 1)} = 12 \pmod{5} = 2$. Therefore, it seems impossible to design an $O(R \cdot m)$ algorithm for generating $\Delta(R, J, m)$. In other words, it is reasonable to say that the best algorithm should use the divide-and-conquer to reach the time complexity $O(R \cdot m \log m)$. Moreover, to set up a dividing point in the middle of the length of the array might break the $L$-condition, it seems hard to design an algorithm that makes a matrix based on two equal sized arrays without the $L$-condition. We have provided an analysis of average cases in Section 4. We have the reason to believe that the algorithms discussed in this note attain the optimal in theory. However, practically one can still develop more efficient algorithms for generating such a matrix.

6 Conclusion

This note translates the major steps of the author’s earlier paper written in Chinese for optimal Hsiao codes. An algorithm analysis error was found and corrected. The original algorithm has been modified and improved. A brief analysis shows that the algorithm reached optimum in average cases if a divide-and-conquer technique was involved in the algorithm.

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Appendix

Chen, Li

An optimal generating algorithm for a matrix of equal-weight columns and quasi-equal-weight rows. (Chinese. English summary) [J] J. Nanjing Inst. Technol. 16, No.2, 33-39 (1986). [ISSN 0254-4180]

According to a recursive matrix which is introduced in this paper, an algorithm for a matrix of equal-weight columns which are inequal each other and quasi-equal-weight rows is given. The algorithm is optimal in time and space, and it can be applied to generate check matrices of optimal minimum odd-weight-column SEC-DED codes. Finally, this algorithm is extended to a set of n elements.

MSC 2000: *65F30 Other matrix algorithms

Keywords: recursive matrix; equal-weight columns; quasi-equal-weight rows; optimal minimum odd-weight-column SEC-DED codes

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