AN INVERSE PROBLEM FOR A CLASS OF DIAGONAL HAMILTONIANS

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Abstract. Hamiltonians are 2-by-2 positive semidefinite real symmetric matrix valued functions satisfying certain conditions. In this paper, we solve the inverse problem for which recovers a Hamiltonian from the solution of a first-order system consisting of ordinary differential equations parametrized by complex numbers attached to a given Hamiltonian, under certain conditions for the solutions. This inverse problem is a generalization of the inverse problem for a class of two-dimensional Hamiltonian systems.

1. Introduction

In this paper, we generalize the theory on the inverse problem for a class of two-dimensional Hamiltonian systems in [15] together with some simplifications of argument.

A 2 × 2 real symmetric matrix valued function H defined on an interval I = [t_1, t_0) (−∞ < t_1 < t_0 ≤ ∞) is called a Hamiltonian if H(t) is positive semidefinite for almost all t ∈ I, H is not identically zero on any subset of I with positive Lebesgue measure, and H belongs to L^1([t_1, c], R^{2×2}) for any t_1 < c < t_0. An open subinterval J of I is called H-indivisible, if the equality

\[ H(t) = h(t) \begin{pmatrix} \cos^2 \theta & \cos \theta \sin \theta \\ \cos \theta \sin \theta & \sin^2 \theta \end{pmatrix} \]

holds on J for some positive valued function h on J and 0 ≤ θ < π. A point t ∈ I is called regular if it does not belong to any H-indivisible interval, otherwise t is called singular. The first-order system

\[-\frac{d}{dt} \begin{pmatrix} A(t, z) \\ B(t, z) \end{pmatrix} = z \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} H(t) \begin{pmatrix} A(t, z) \\ B(t, z) \end{pmatrix}\]  \hspace{1cm} (1.1)

associated with a Hamiltonian H on an interval I parametrized by all \(z \in \mathbb{C}\) is called a canonical system on I. In [13], the sign is different from the usual definition. This is because, we want to regard the value at the right end \(t_0\) of I as the initial value for our convenience. A typical source of Hamiltonians is entire functions of the Hermite–Biehler class, which is the set \(\mathbb{H}\) of all entire functions satisfying

\[|E^2(z)| < |E(z)|\]  \hspace{1cm} (1.2)

and the subset \(\mathbb{H}_B\) of \(\mathbb{H}\) consisting of E such that \(E(z) \neq 0\) for any \(z \in \mathbb{R}\), where \(\mathbb{C}_+ = \{z \mid \Im(z) > 0\}\) is the upper half-plane and \(F^2(z) := F(\bar{z})\), the notation is often used in this paper. Every \(E \in \mathbb{H}_B\) generates a de Branges space \(\mathcal{H}(E)\) which is a reproducing kernel Hilbert space of entire functions. Every de Branges space \(\mathcal{H}(E)\) has a unique maximal chain of de Branges subspaces \(\mathcal{H}(E_t)\) parametrized by t in an interval I such that \(\mathcal{H}(E_t)\) contained isometrically in \(\mathcal{H}(E)\) for almost all \(t \in I\). For the generating functions \(E_t, A_t = (E_t + E_t^*)/2\) and \(B_t = i(E_t - E_t^*)/2\) satisfy a canonical system on the interval I associated with some Hamiltonian \(H\). Such Hamiltonian is called the structure Hamiltonian of \(\mathcal{H}(E)\). Recently, a complete characterization of structure Hamiltonians of de Branges spaces is obtained by Romanov–Woracek [13]. The inverse problem for the recovering the structure Hamiltonian from given \(E \in \mathbb{H}_B\)

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has been studied by many authors; see Winkler [18], Remling [11], Romanov [12], Suzuki [10], and references therein, for example. However, if \( E \) does not necessarily belong to the Hermite–Biehler class, nothing can be said about whether Hamiltonian can be obtained from \( E \), in general.

In this paper, we prove that if we suppose several conditions on a function \( E \), a Hamiltonian is obtained from \( E \) by an explicit way of the construction, even though \( E \) does not necessarily belong to the Hermite–Biehler class. Those conditions described below may look artificial, but they naturally arise from number theory; see the final part of the introduction. The first condition for a function \( E \) is the following.

(K1) There exists \( c > 0 \) and a discrete subset \( 0 \notin \mathbb{Z} \) (which is possibility empty or infinite) of the horizontal strip \( -c \leq \Im z \leq c \) such that it is closed under the complex conjugation \( z \mapsto \bar{z} \) and the negation \( z \mapsto -z \) and \( E \) is analytic in \( \mathbb{C} \setminus \mathbb{Z} \) and that \( E \) satisfies \( E^2(z) = E(-z) \) for \( z \in \mathbb{C} \setminus \mathbb{Z} \).

Note that this condition (K1) is more general than that in [15]. In particular, it is allowed that \( E \) has an essential singularity in \( \mathbb{Z} \). We define

\[
\Theta(z) = \frac{E^2(z)}{E(z)}
\]

under (K1). Then, \( \Theta(0) = 1 \),

\[
\Theta(z)\Theta(-z) = 1 \quad \text{for} \quad z \in \mathbb{C} \setminus \mathbb{Z} \tag{1.3}
\]

and

\[
|\Theta(u)| = 1 \quad \text{for} \quad u \in \mathbb{R} \setminus \mathbb{Z} \tag{1.4}
\]

by definition. We denote by

\[
(F f)(z) = \int_{-\infty}^{\infty} f(x) e^{izx} \, dx, \quad (F^{-1} g)(z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} g(u) e^{-izu} \, du \tag{1.5}
\]

the Fourier integral and inverse Fourier integral, respectively. Then, additional conditions are as follows.

(K2) There exists a real-valued continuous function \( K \) defined on the real line such that \( |K(x)| \ll \exp(c|x|) \) as \( |x| \to \infty \) and that \( \Theta(z) = (FK)(z) \) holds for \( \Im(z) > c \), where \( c \) is the constant in (K1).

(K3) \( K \) vanishes on the negative real line \((-\infty, 0)\).

(K4) \( K \) is continuously differentiable outside a discrete subset \( \Lambda \subset \mathbb{R} \) and the derivative \( K' \) belongs to \( L^1_{loc}(\mathbb{R}) \).

These three conditions are the same as [15]. Under conditions (K2) and (K3), the map

\[
K[t] : f(x) \mapsto 1_{\leq t}(x) \int_{-\infty}^{t} K(x + y) f(y) \, dy
\]

defines a Hilbert–Schmidt operator on \( L^2(-\infty, t) \) for every \( t \in \mathbb{R} \), where \( 1_{\leq t} \) stands for the characteristic function of \((-\infty, t] \). In fact, the Hilbert–Schmidt norm of \( K[t] \) is finite:

\[
\int_{-\infty}^{t} \int_{-\infty}^{t} |K(x + y)|^2 \, dx \, dy \leq \int_{-t}^{t} dy \int_{-2t}^{2t} |K(x)|^2 \, dx = 2t \int_{0}^{2t} |K(x)|^2 \, dx < \infty.
\]

For \( t \leq 0 \), we understand \( K[t] = 0 \) by (K3). Moreover, \( K[t] \) is self-adjoint, because the kernel \( K(x + y) \) is real-valued and symmetric. Therefore, the spectrum of \( K[t] \) consists only of real eigenvalues of finite multiplicity and 0. Finally, we consider the following conditions.

(K5) There exists \( 0 < \tau \leq \infty \) such that both \( \pm 1 \) are not eigenvalues of \( K[t] \) for every \( t < \tau \).

(K6) \( \Theta \) can not be expressed as a ratio of two entire functions of exponential type.
Condition (K5) is the same as [15], but condition (K6) is added in this paper, though it is rarely used. The requirement for eigenvalues of $K[t]$ in (K5) is trivial for $t \leq 0$, since $K[t] = 0$. The set of functions satisfying (K1)$\sim$(K6) is not empty but a large; see the final part of the introduction.

Now we assume that $E$ satisfies (K1)$\sim$(K5) and define

$$m(t) := \frac{\det(1 + K[t])}{\det(1 - K[t])}, \quad H(t) := \begin{bmatrix} 1/\gamma(t) & 0 \\ 0 & \gamma(t) \end{bmatrix}, \quad \gamma(t) := m(t)^2,$$

where we understand that $m(t) = 1$ and $H(t)$ is the identity matrix if $t \leq 0$. Then $\gamma(t)$ is a continuous positive real-valued function on $I_r = (-\infty, \tau)$ (Theorem 1.3 and Proposition 2.1). Therefore $H$ is a Hamiltonian on $I_r$ consisting of continuous functions such that it has no $H$-indivisible intervals, that is, all points of $I_r$ are regular. The solution of the associated first-order system (1.1) on $I_r$ recovers the original $E$ as follows as well as the case of the inverse problem for entire function $s$ of the Hermite–Biehler class in [15]. The solution of the first-order system is explicitly described by using the unique solutions of the integral equations

$$\Phi(t, x) + \int_{-\infty}^{t} K(x + y)\Phi(t, y) \, dy = 1,$$  

$$\Psi(t, x) - \int_{-\infty}^{t} K(x + y)\Psi(t, y) \, dy = 1.$$  

**Theorem 1.1.** Let $E$ be a function satisfying (K1)$\sim$(K5), and define $A$ and $B$ by

$$A(z) := \frac{1}{2}(E(z) + E^\ast(z)) \quad \text{and} \quad B(z) := \frac{i}{2}(E(z) - E^\ast(z)).$$

Let $H$ be the Hamiltonian on $I_r = (-\infty, \tau)$ defined by (1.6) and (1.7). Let $\Phi(t, x)$ and $\Psi(t, x)$ be the unique solutions of (1.8) and (1.9), respectively. Define $A(t, z)$ and $B(t, z)$ by

$$A(t, z) = -\frac{iz}{2}E(z) \int_{t}^{\infty} \Phi(t, x)e^{ixz} \, dx,$$

$$-iB(t, z) = -\frac{iz}{2}E(z) \int_{t}^{\infty} \Phi(t, x)e^{ixz} \, dx.$$  

Then,

1. for each $t \in I_r$, $A(t, z)$ and $B(t, z)$ are well-defined for $\Im(z) > c$ and extend to analytic functions on $\mathbb{C} \setminus Z$ satisfying $A(t, -z) = A(t, z)$, $B(t, -z) = -B(t, z)$, $A(t, z) = A^\ast(t, z)$, and $B(t, z) = B^\ast(t, z)$,

2. for each $z \in \mathbb{C} \setminus Z$, $A(t, z)$ and $B(t, z)$ are continuously differentiable with respect to $t$,

3. $A(t, z)$ and $B(t, z)$ solves the first-order system (1.1) associated with $H$ on $I_r$ for every $z \in \mathbb{C} \setminus Z$

4. $A(z) = A(0, z)$, $B(z) = B(0, z)$ and $E(z) = A(0, z) - iB(0, z)$.

For $H(t) \equiv I$, the identity matrix, the functions $A(t, z) = A(z) \cos(tz) + B(z) \sin(tz)$ and $B(t, z) = -A(z) \sin(tz) + B(z) \cos(tz)$ satisfy (1)$\sim$(4) of Theorem 1.1 for any interval of $t$ containing 0, and this is the case for the subinterval $(-\infty, 0]$ of $I_r$ in the theorem. In this trivial case, $H$ has no information about the original $E$. Different from these cases, we will describe below that $H$ in Theorem 1.1 has nontrivial information about $E$ on $(0, \tau)$.

Let $H^{\infty} = H^{\infty}(\mathbb{C}_+)$ be the space of all bounded analytic functions in $\mathbb{C}_+$. A function $\theta \in H^{\infty}$ is called an inner function in $\mathbb{C}_+$ if $\lim_{y \to 0^+} |\theta(x + iy)| = 1$ for almost all $x \in \mathbb{R}$ with respect to the Lebesgue measure.
Theorem 1.2. Assume that $E$ satisfies $(K1)$~$(K3)$, and $(K5)$ with $\tau = \infty$. Then $\Theta = E^\tau/E$ is an inner function in $\mathbb{C}_+$. 

Remark 1.1. Compare this with \cite{15} Theorem 2.4, where some additional conditions are assumed to conclude that $\Theta$ is an inner function in $\mathbb{C}_+$. 

If $E \in \mathbb{H}$, $\Theta = E^\tau/E$ is an inner function in $\mathbb{C}_+$. Theorem 1.2 shows that (K5) plays the role of the condition $E \in \mathbb{H}$ for entire functions $E$; $Z = \emptyset$. On the other hand, it is known that if $\theta$ is an inner function and meromorphic in $\mathbb{C}_+$, there exists $E \in \mathbb{H}$ such that $\Theta = E^\tau/E$ (\cite{8} Sections 2.3 and 2.4). However, the existence of $\tau > 0$ in (K5) is not obvious even if we assume that $\Theta = E^\tau/E$ is inner in $\mathbb{C}_+$. Therefore, for the converse of Theorem 1.2, we require (K6). 

Theorem 1.3. Assume that $E$ satisfies $(K1)$~$(K3)$, and $(K6)$. In addition, assume that $\Theta$ is an inner function in $\mathbb{C}_+$. Then (K5) holds for $\tau = \infty$. 

Theorems \cite{14} and \cite{12} emphasize the importance of the function $m(t)$. The following simple formula is interesting from both theoretical and computational aspects.

Theorem 1.4. Assume that $E$ satisfies $(K2)$~$(K5)$. Then, 

$$m(t) = \frac{1}{\Phi(t, t)} = \Psi(t, t)$$

holds for every $t \in \mathbb{R}$. 

See Propositions 2.2 and 2.3 for other formulas of $m(t)$. If $E$ satisfies (K1)~(K5) and $\Theta$ is an inner function in $\mathbb{C}_+$, $K = \lim_{t \to \infty} K[t]$ defines a bounded operator on $L^2(\mathbb{R})$ (Lemma 1.1), and the Fourier transform $\hat{F}(\mathcal{V}_t)$ of the space $\mathcal{V}_t = L^2(\mathbb{R}) = \{f : f(\xi) = \hat{F}(f) = 0\}$ for each $0 < t < \tau$ (Section 6.1).

Theorem 1.5. The following statements hold. 

1. Assume that $E$ satisfies $(K1)$~$(K5)$ and that $\Theta$ is an inner function in $\mathbb{C}_+$. Let $A(t, z)$ and $B(t, z)$ be as in Theorem 1.7, and let $j(t; z, w)$ be the reproducing kernel of $\mathcal{F}(\mathcal{V}_t)$ for $0 < t < \tau$. Then, 

$$j(t; z, w) = \frac{1}{E(z)E(w)} \frac{A(t, z)B(t, w) - A(t, w)B(t, z)}{\pi(w - z)},$$

and $j(t; z, w) \neq 0$ as a function of $z \in \mathbb{C}_+$ for any $0 < t < \tau$.

2. Assume that $E$ satisfies $(K1)$~$(K5)$ with $\tau = \infty$. Then, $(\Theta$ is an inner function in $\mathbb{C}_+$ and) $\lim_{t \to \infty} j(t; z, w) = 0$ for every $z, w \in \mathbb{C}_+$. 

Theorem 1.3 shows that $H$ of (1.7) provides the genuine structure Hamiltonian of the de Branges space $\mathcal{H}(E)$ if $E \in \mathbb{H}$ satisfies (K1)~(K3) and (K6) by \cite{2} Theorem 40.

The basic idea for achieving the above results originates from the work of J.-F. Burnol \cite{6} (and \cite{8} \cite{11} \cite{15}) as well as \cite{14} \cite{15}, and \cite{17}. However, in this paper, the method used in \cite{6} for $\Gamma(1 - s)/\Gamma(s)$ standing on the theory of Hankel transforms is axiomatized, reorganized, and generalized, and some arguments are simplified.

Before closing the introduction, we mention a few examples of functions $E$ satisfying conditions (K1)~(K6). Let $\zeta(s)$ be the Riemann zeta function, and let $\xi(s) = s(s - 1)\pi^{-s/2}\Gamma(s/2)\zeta(s)$. Then $\xi(s)$ is an entire function taking real-values on the critical line $\Re(s) = 1/2$ and the real line such that the zeros coincide with nontrivial zeros of $\zeta(s)$. We put 

$$E_n(z) = \xi \left( \frac{1}{2} + \frac{n}{2} - iz \right)^n, \quad E_\infty(z) = \exp \left( 2\xi' \frac{1}{\xi} \left( \frac{1}{2} - iz \right) \right)$$

for $n \in \mathbb{Z}_{>0}$. Then, $E_\infty(z) = \lim_{n \to \infty} E_n(z)/\xi(1/2 - iz)^n$, and it is proved that $E_n$ (resp. $E_\infty$) satisfies the conditions (K1)~(K6) in \cite{15} Proposition 4.1 and Lemma 4.3]
The function $\xi(s)$ has no zeros in $\Re(s) > 1 + \frac{1}{2} + \frac{\tau}{2}$ if and only if $E_n$ belongs to $\mathbb{H}_2$ for each $n \geq N$ (\cite[Theorem 9.1]{15}). In particular, the de Branges space $\mathcal{H}(E_n)$ is defined for each $n \in \mathbb{Z}_{>0}$ under the Riemann hypothesis, and its structure Hamiltonian $H_n$ is constructed in \cite[Theorem 9.1]{15}. Therefore, it is natural to ask about the limit behavior of $H_n$ as $n \to \infty$. However, $\lim_{n \to \infty} E_n$ does not make sense, and $E_\infty$ is no longer an entire function, because $E_\infty$ has an essential singularity at a zero of $\xi(1/2 - iz)$. Therefore, the method constructing $H_n$ in \cite{15} can not be applied to $E_\infty$. This is the main reason why we generalized condition (K1) as above in this paper.

The paper is organized as follows. In Section 2, we study basic properties of solutions $\Phi(t, x)$ and $\Psi(t, x)$ of integral equations (1.8) and (1.9) in preparation for the proof of Theorem 1.1 and prove Theorem 1.4. In Section 3, we prove Theorem 1.1 by using results in Section 2. In Section 3, we prove Theorems 1.2 and 1.3 by studying the behavior of the operator norm of $K[t]$ when $t$ varies. In Section 5, we review the theory of model subspaces and de Branges spaces in preparation for the proof of Theorem 1.5. In Section 6, we prove Theorem 1.5 by studying the vectors representing the point evaluation maps in a model subspace. In Section 7, we comment on a relation between our inverse problem, the Cauchy problem for certain hyperbolic systems, and damped wave equations.

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2. Solutions of related integral equations

We suppose that $E$ satisfies (K2)~(K5) throughout this section. In particular, we understand that $c$ and $\tau$ are constants in (K2) and (K5), respectively. Let $L^p(I)$ be the $L^p$-space on an interval $I$ with respect to the Lebesgue measure. If $J \subset I$, we regard $L^p(J)$ as a subspace of $L^p(I)$ by the extension by zero.

2.1. Properties of $\Phi(t, x)$ and $\Psi(t, x)$

Proposition 2.1. The integral equations (1.8) and (1.9) for $(x, t) \in \mathbb{R} \times (-\infty, \tau)$ have unique solutions $\Phi(t, x)$ and $\Psi(t, x)$, respectively. Moreover,

1. $\Phi(t, x)$ and $\Psi(t, x)$ are real-valued,
2. $\Phi(t, x)$ and $\Psi(t, x)$ are continuously differentiable functions of $x \in \mathbb{R}$,
3. $\Phi(t, x) = \Psi(t, x) = 1$ for $x < -t$ and $\Phi(t, x) \ll \exp(c'x)$ and $\Psi(t, x) \ll \exp(c'x)$ as $x \to +\infty$ for any $t < \tau$ and $c' > c$, where implied constants depend on $t$,
4. $\Phi(t, t) \neq 0$ and $\Psi(t, t) \neq 0$ for every $t < \tau$,
5. if $t \leq 0$, $\Phi(t, t) = \Psi(t, t) = 1$ and

$$\Phi(t, x) = 1 - \int_0^{x+t} K(y) \, dy, \quad \Psi(t, x) = 1 + \int_0^{x+t} K(y) \, dy.$$  \hfill (2.1)

Proof. We prove only in the case of $\Phi(t, x)$, because the case of $\Psi(t, x)$ is proved in the similar argument. First, we prove the uniqueness of $\Phi(t, x)$. If $\Phi_1(t, x)$ and $\Phi_2(t, x)$ solve (1.8), the difference $f(t, x) = \Phi_1(t, x) - \Phi_2(t, x)$ satisfies $f(t, x) + \int_{-\infty}^t K(x+y) f(t, y) \, dy = 0$. This shows that $1_{x \in J}(x) f(t, x) = 0$, since $(1+K[t])$ is invertible, and hence $f(t, x) = 0$. Then (1) is obvious, since the kernel $K$ is real-valued by (K2). To prove other statements, we first suppose that $t > 0$.

We prove (2) and (3). By considering equation (1.8) on $L^2(-t, t)$, we find that $1_{[-t,t]}(x) \Phi(t, x)$ is a continuous function of $x$ on $[-t, t]$ by the continuity of $K$ and
\[1_{[-t,0]}(x),\] where \(1_A\) stands for the characteristic function of \(A \subset \mathbb{R}\). On the other hand, \(\Phi(t, x) = 1\) for \(x < -t\), since the integral in \([L, L]\) is zero for \(x < -t\) by (K3). Therefore,

\[
\Phi(t, x) = 1 - \int_{-t}^{-x} K(x + y) dy - \int_{-x}^{t} K(x + y) \Phi(t, y) dy \tag{2.2}
\]

by \([L, L]\), where the middle integral is understood as zero if \(x < t\). This equality and (K2) shows that \(\Phi(t, x)\) is a continuous function of \(x \in \mathbb{R}\) and satisfies the estimate in (3). Moreover, differentiating \([L, L]\) with respect to \(x\),

\[
\frac{\partial}{\partial x} \Phi(t, x) + \int_{-t}^{t} K'(x + y) \Phi(t, y) dy = 0. \tag{2.3}
\]

This shows that \(\Phi(t, x)\) is differentiable for \(x\) and \((\partial/\partial x) \Phi(t, x)\) is a continuous function of \(x\) by (K4).

We prove (4) by contradiction. Differentiating \([L, L]\) with respect to \(x\), and then applying integration by parts,

\[
\frac{\partial}{\partial x} \Phi(t, x) + K(x + y) \Phi(t, y) dy = 0. \tag{2.4}
\]

Therefore, if we suppose that \(\Phi(t, t) = 0\),

\[
\frac{\partial}{\partial x} \Phi(t, x) - \int_{-t}^{t} K(x + y) \frac{\partial}{\partial y} \Phi(t, y) dy = 0. \tag{2.5}
\]

This asserts that the restriction \(1_{[-t,0]}(x)(\partial/\partial x) \Phi(t, x)\) is a solution of the homogeneous equation \((1 - K[t])f = 0\) on \(L^2(-t, t)\), and thus \((\partial/\partial x) \Phi(t, x) = 0\) by \([L, L]\) and the equality \(\Phi(t, x) = 1\) for \(x < -t\), since \(K(x + y) = 0\) if \(x < -t\) and \(y < t\). Hence, we have \(c \int_{0}^{x} K(y) dy = 1\) for arbitrary \(x\) if \(\Phi(t, x) = c\). This implies that \(K \equiv 0\) on \(\mathbb{R}\), and therefore, \(\Phi(t, x) = 1\) for all \(x \in \mathbb{R}\) by \([L, L]\). This is a contradiction.

Finally, we prove (5). If \(t \leq 0\), \(K(x + y) = 0\) for \(x < t\) and \(y < t\). Thus \(\Phi(t, x) = 1\) for \(x < t\), the first equality of (2.1) holds, and \(\Phi(t, t) = 1\). Hence \(\Phi(t, x)\) is a continuously differentiable function of \(x\) on \(\mathbb{R}\) satisfying the desired estimate. \(\square\)

For convenience of studying the solutions \(\Phi(t, x)\) and \(\Psi(t, x)\), we consider the solutions of integral equations

\[
\phi^+(t, x) + \int_{-\infty}^{t} K(x + y) \phi^+(t, y) dy = K(x + t), \tag{2.6}
\]

\[
\phi^-(t, x) - \int_{-t}^{t} K(x + y) \phi^-(t, y) dy = K(x + t). \tag{2.7}
\]

The usefulness of solutions \(\phi^+(t, x)\) and \(\phi^-(t, x)\) comes from relationships with the resolvent kernels \(R^+(t; x, y)\) and \(R^-(t; x, y)\) of \((1 + K[t])\) and \((1 - K[t])\), respectively: \(1_{\leq t}(x) \phi^+(t, x) = R^+(t; x, t), 1_{\leq t}(x) \phi^-(t, x) = R^-(t; x, t)\) \([L, L] \text{ Section 3}\).

**Proposition 2.2.** The integral equations \([L, L]\) and \([L, L]\) for \((t, x) \in \mathbb{R} \times (-\infty, \tau)\) have unique solutions \(\phi^+(t, x)\) and \(\phi^-(t, x)\), respectively. Moreover,

1. \(\phi^+(t, x)\) and \(\phi^-(t, x)\) are continuous on \(\mathbb{R}\) and continuously differentiable on \(\mathbb{R} \setminus \{\lambda - t \mid \lambda \in \Lambda\}\) as a function of \(x\), where \(\Lambda\) is the set in (K4).
2. \(\phi^+(t, x)\) and \(\phi^-(t, x)\) are continuous on \([0, \tau]\) and continuously differentiable on \((0, \tau)\) except for points in \(\{\lambda - x \mid \lambda \in \Lambda\}\) as a function of \(t\),
3. for fixed \(t \in [0, \tau]\), \(\phi^+(t, x) = 0\) for \(x < -t\) and \(\phi^+(t, x) \ll e^{ct} \) as \(x \to +\infty\) for \(c > 0\) in (K2), where the implied constants depend on \(t\),
4. if \(t \leq 0\), \(\phi^+(t, t) = \phi^-(t, t) = 0\) and

\[
\phi^+(t, x) = \phi^-(t, x) = K(x + t),
\]
The differentiability of $\Psi(t, x)$ by Proposition 2.2 (2), (3), and (4). Hence $\Phi(t, x)$ is continuously differentiable with respect to $t$ on $(-\infty, \tau)$ and equalities
\begin{align}
\phi^+(t, x) &= -\frac{1}{\Phi(t, t)} \frac{\partial}{\partial t} \Phi(t, x) = \frac{1}{\Psi(t, t)} \frac{\partial}{\partial x} \Psi(t, x), \tag{2.10} \\
\phi^-(t, x) &= -\frac{1}{\Phi(t, t)} \frac{\partial}{\partial x} \Phi(t, x) = \frac{1}{\Psi(t, t)} \frac{\partial}{\partial t} \Psi(t, x) \tag{2.11}
\end{align}
hold.

Proof. Applying integration by parts to (2.3),
\begin{equation}
\frac{\partial}{\partial x} \Phi(t, x) + K(x + t) \Phi(t, t) - \int_{-\infty}^t K(x + y) \frac{\partial}{\partial y} \Phi(t, y) dy = 0. \tag{2.12}
\end{equation}
This shows that $-(\partial/\partial x) \Phi(t, x)/\Phi(t, t)$ solves (2.7) by Proposition 2.1(4). Hence the uniqueness of the solution of (2.6) concludes the first equality of (2.11).

On the other hand, we find that $1_{[-t, t]}(x) \Phi(t, x)$ is also continuous in $t$, because the resolvent kernel $R^+(t, x, y)$ of $(1 + K(t))$ is continuous in all variables (15, Section 3). Therefore, $\Phi(t, x)$ is continuous in $t$ by (2.2). By differentiating (1.8) with respect to $t$ (in the sense of weak derivative), we find that $-(\partial/\partial t) \Phi(t, x)/\Phi(t, t)$ solves (2.6) by Proposition 2.1(4). Hence the uniqueness of the solution concludes the first equality of (2.11). Moreover, the first equality of (2.10) shows that $(\partial/\partial t) \Phi(t, x)$ is continuous with respect to $t$ by Proposition 2.2 (2), (3), and (4). Hence $\Phi(t, x)$ is differentiable with respect to $t$ in the usual sense, and the derivative with respect to $t$ is continuous in $t$.

The differentiability of $\Psi(t, x)$ with respect to $t$ and the second equalities of (2.10) and (2.11) are proved by the similar argument.

2.2 Proof of Theorem 1.4. Taking $x = t$ in equation (1.8) and then differentiating it with respect to $t$,
\begin{equation}
0 = \frac{d}{dt} (\Phi(t, t)) + 2K(t) \Phi(t, t) \tag{2.13}
\end{equation}
Using the first equalities of (2.10) and (2.11) on the right-hand side,
\begin{equation}
\frac{d}{dt} (\Phi(t, t)) + 2K(t) \Phi(t, t) - \Phi(t, t) \int_{-\infty}^t K(t + x)(\phi^+(t, x) - \phi^-(t, x)) dx = 0.
\end{equation}
On the other hand, by the proof of [15, Theorem 6.1], we have
\begin{equation}
\frac{1}{2} (\phi^+(t, x) + \phi^-(t, x)) = K(x + t) - \int_{-\infty}^t K(x + y) \frac{1}{2} (\phi^+(t, y) - \phi^-(t, y)) dy.
\end{equation}
Substituting this into (2.13) after taking $x = t$, we get
\[ \frac{d}{dt}(\Phi(t, t)) + \Phi(t, t)(\phi^+(t, t) + \phi^-(t, t))) = 0. \] (2.14)
Therefore, $\Phi(t, t) = C \exp \left( -\int_0^t (\phi^+(\tau, \tau) + \phi^-(\tau, \tau)) \, d\tau \right) = Cm(t)^{-1}$ by (2.9). To determine $C$, we take $x = t = 0$ in equation (1.8). Then $\Phi(0, 0) = 1$, since the integral on the left-hand side is zero because $K(x) = 0$ for $x < 0$, and thus $C = 1$ by $m(0) = 1$. Hence we obtain the first equality of (1.12). The second equality of (1.12) is proved by the same way.

From Theorem 1.4 and Proposition 2.1, we find that $H$ of (1.7) is a Hamiltonian on $(-\infty, \tau)$ such that it consists of continuous functions and has no $H$-indivisible intervals. These properties also obtained from Proposition 2.2.

2.3. Corollaries of Proposition 2.3
Here we state a few results that easily obtained from Proposition 2.3, but note that these are of their own interest and are not used to prove the main results.

**Proposition 2.4.** The solutions of (1.8) and (1.9) are related each other as follows
\[ \Psi(t, x) = 1 - \frac{1}{\Phi(t, t)^2} \int_{-t}^{x} \frac{\partial}{\partial t} \Phi(t, y) \, dy, \] (2.15)
\[ \Phi(t, x) = 1 - \frac{1}{\Psi(t, t)^2} \int_{-t}^{x} \frac{\partial}{\partial t} \Psi(t, x) \, dy. \] (2.16)

**Proof.** Integrating the second equalities of (2.10) and using (1.12), Proposition 2.1 (3) and Proposition 2.2 (6),
\[ \Psi(t, x) = 1 + m(t) \int_{-t}^{x} \phi^+(t, y) \, dy. \] (2.17)
Substitute the first equality of (2.10) and (1.12) into (2.17), we obtain (2.15). (2.16) is also proved in a similar way. \( \square \)

**Proposition 2.5.** We have
\[ \frac{1}{m(t)} = 1 - \int_{-t}^{t} \phi^+(t, y) \, dy, \quad m(t) = 1 + \int_{-t}^{t} \phi^-(t, y) \, dy. \] (2.18)

**Proof.** The first equality is obtained by taking $x = t$ in (2.17) and noting (1.12). The second equality is proved in a similar way. \( \square \)

**Proposition 2.6.** The following partial differential equations hold
\[ \frac{\partial}{\partial t} \phi^+_t(x) - \frac{\partial}{\partial x} \phi^-_t(x) = -\mu(t) \phi^-_t(x), \quad \frac{\partial}{\partial t} \phi^-_t(x) - \frac{\partial}{\partial x} \phi^+_t(x) = \mu(t) \phi^-_t(x). \] (2.19)

**Proof.** We have
\[ \frac{\partial}{\partial t} \phi^+_t(x) = -\frac{\Psi_t(t, t) + \Psi_x(t, t)}{\Psi(t, t)} \phi^+_t(x) + \frac{\Psi_x(t, x)}{\Psi(t, t)} \phi^+_t(x), \quad \frac{\partial}{\partial x} \phi^-_t(x) = \frac{\Psi_t(t, x)}{\Psi(t, t)} \phi^-_t(x) \]
by (2.10) and (2.11). On the other hand,
\[ \frac{\Psi_t(t, t) + \Psi_x(t, t)}{\Psi(t, t)} = \frac{d}{dt} \log m(t) = \mu(t). \]
by (1.12) and Proposition 2.2 (6). Hence we obtain the first equation. The second equation is proved in a similar way. \( \square \)
3. Proof of Theorem 1.1

We suppose that \( E \) satisfies (K1)~(K5) throughout this section. We use the same notation \((1.3)\) for the Fourier transforms on \( L^1(\mathbb{R}) \) and \( L^2(\mathbb{R}) \) if no confusion arises. If we understand the right-hand sides of \((1.3)\) in \( L^2\)-sense, they provide isometries on \( L^2(\mathbb{R}) \) up to a constant multiple: \( \|Ff\|_{L^2(\mathbb{R})}^2 = 2\pi \|f\|_{L^2(\mathbb{R})}^2 \), \( \|F^{-1}f\|_{L^2(\mathbb{R})}^2 = (2\pi)^{-1}\|f\|_{L^2(\mathbb{R})}^2 \).

3.1. Proof of (1). The right-hand side of \((1.11)\) is defined for \( \Im(z) > c \) by Proposition 2.1(3). Therefore,

\[
\Omega(t, z) = \int_{-\infty}^{\infty} \left( \Phi(t, x) - \Psi(t, x) \right) e^{izx} \, dx
\]

is defined for \( \Im(z) > c \). Subtracting \((1.9)\) from \((1.8)\),

\[
(\Phi(t, x) - \Psi(t, x)) + \int_{-\infty}^{t} K(x + y)(\Phi(t, y) + \Psi(t, y)) \, dy = 0.
\]

Using this equality, we have

\[
\begin{align*}
\Omega(t, z) &= -\int_{-\infty}^{\infty} \left( \int_{-\infty}^{t} K(x + y)(\Phi(t, y) + \Psi(t, y)) \, dy \right) e^{izx} \, dx \\
&= -\int_{-\infty}^{\infty} K(x) e^{izx} \int_{-\infty}^{t} (\Phi(t, y) + \Psi(t, y)) e^{-izy} \, dy \\
&\quad + \int_{-\infty}^{t} \int_{-\infty}^{\infty} K(x + y)(\Phi(t, y) + \Psi(t, y)) \, dy e^{izx} \, dx \\
&= -\int_{-\infty}^{\infty} K(x) e^{izx} \int_{-t}^{t} (\Phi(t, y) + \Psi(t, y)) e^{-izy} \, dy \\
&\quad - \int_{-\infty}^{\infty} K(x) e^{izx} \int_{-\infty}^{-t} (\Phi(t, y) + \Psi(t, y)) e^{-izy} \, dy \\
&\quad - \int_{-\infty}^{t} (\Phi(t, x) - \Psi(t, x)) e^{izx} \, dx
\end{align*}
\]

for \( \Im(z) > c \). Because \( \Phi(t, x) = \Psi(t, x) = 1 \) for \( x < -t \), we have

\[
\begin{align*}
\Omega(t, z) &= \Theta(z) \left( 2 \frac{e^{itz}}{iz} - \int_{-t}^{t} (\Phi(t, y) + \Psi(t, y)) e^{-izy} \, dy \right) \\
&\quad - \int_{-t}^{t} (\Phi(t, x) - \Psi(t, x)) e^{izx} \, dx
\end{align*}
\]

for \( \Im(z) > c \). The right-hand side extends \( \Omega(t, z) \) to \( \mathbb{C} \setminus \mathbb{Z} \). From this equality,

\[
\Theta(z) \Omega(t, -z) = \left( -2 \frac{e^{-itz}}{iz} - \int_{-t}^{t} (\Phi(t, y) + \Psi(t, y)) e^{izy} \, dy \right) \\
- \Theta(z) \int_{-t}^{t} (\Phi(t, x) - \Psi(t, x)) e^{-izx} \, dx
\]

for \( z \in \mathbb{C} \setminus \mathbb{Z} \) by \((1.3)\). Adding \((1.8)\) and \((1.9)\),

\[
(\Phi(t, x) + \Psi(t, x)) + \int_{-\infty}^{t} K(x + y)(\Phi(t, y) - \Psi(t, y)) \, dy = 2.
\]

Integrating this with respect to \( x \) from \(-t\) to \( \infty \) after multiplying by \( e^{izx} \) both side,

\[
\int_{-t}^{\infty} (\Phi(t, x) + \Psi(t, x)) e^{izx} \, dx + \int_{-\infty}^{\infty} \int_{-\infty}^{t} K(x + y)(\Phi(t, y) - \Psi(t, y)) \, dy e^{izx} \, dx = -2 \frac{e^{-itz}}{iz}.
\]
The integration $\int_0^\infty \int_{-t}^t$ can be replaced by $\int_0^\infty \int_{-t}^t$ since $\int_{-\infty}^t K(x+y)(\Phi(t,y) - \Psi(t,y)) dy = 0$ and $\Phi(t,x) - \Psi(t,x) = 0$ for $x < -t$. Thus,

$$\int_{-t}^t (\Phi(t,x) + \Psi(t,x)) e^{itz} dx + \Theta(z) \int_{-t}^t (\Phi(t,y) - \Psi(t,y)) e^{-iy} dy = -2e^{-iz}$$

Combining this with (3.1),

$$\Theta(z) \Omega(t, -z) = \int_{-t}^t (\Phi(t,y) + \Psi(t,y)) e^{itz} dy. \quad (3.2)$$

The left-hand side extends the right-hand side to $\mathbb{C} \setminus \mathbb{Z}$. On the other hand,

$$A(t, z) + iB(t, z) = \frac{i\pi}{2} E(z) \int_{-t}^t (\Phi(t,x) - \Psi(t,x)) e^{itz} dx$$

$$= \frac{i\pi}{2} E(z) \Omega(t, z),$$

$$A(t, z) - iB(t, z) = -\frac{i\pi}{2} E(z) \int_{-t}^t (\Phi(t,x) + \Psi(t,x)) e^{itz} dx$$

$$\quad = \frac{i}{2} E(z) \Omega(t, -z) \quad (3.3)$$

by (1.11) and (3.2). These two formula extend $A(t, z)$ and $B(t, z)$ to $\mathbb{C} \setminus \mathbb{Z}$, and show that $A(t, z)$ is even and that $B(t, z)$ is odd.

If $F(z) = F(f(x))(z)$ for $\Im(z) > 0$, $\overline{F(\bar{z})} = F(\overline{f(-x)})(z)$ and $F(-z) = F(f(-x))(z)$ for $\Im(z) < 0$. Therefore, if $f(x)$ (resp. $if(x)$) is real-valued, $F(z)$ is continued to an analytic function in $\mathbb{C} \setminus \mathbb{Z}$, and $F(-z) = F(z)$ (resp. $F(-z) = -F(z)$), then $\overline{F(\bar{z})} = F(z)$ holds for $\Im(z) < 0$. From this argument and Proposition 2.1 (3), $A(t, z) = \overline{A(t, \bar{z})}$ and $B(t, z) = B(t, \bar{z})$ hold.

3.2. Proof of (2) and (3). By Proposition 2.3 formulas (2.10), (2.11), and Proposition 2.2 (3), $A(t, z)$ and $B(t, z)$ are differentiable with respect to $t$. Therefore, it remains to show that $(\frac{\partial}{\partial t}) A(t, z) = zm(t)^2B(t, z)$ and $(\frac{\partial}{\partial t}) B(t, z) = zm(t)^{-1}A(t, z)$. Using (1.12) and (2.11), we have

$$\frac{\partial}{\partial t} A(t, z) = -\frac{i\pi}{2} E(z) \left( -\Psi(t, t)e^{itz} + \int_{-t}^t \frac{\partial}{\partial t} \Psi(t, x)e^{itz} dx \right)$$

$$= -\frac{i\pi}{2} E(z) \left( -\Psi(t, t)e^{itz} - \frac{\Psi(t, t)}{\Phi(t, t)} \int_{-t}^t \frac{\partial}{\partial x} \Phi(t, x)e^{itz} dx \right)$$

$$= -\frac{i\pi}{2} E(z) \left( \frac{\Psi(t, t)}{\Phi(t, t)} i \int_{-t}^t \Phi(t, x)e^{itz} dx \right) = \frac{z}{\Psi(t, t)} B(t, z) = zm(t)^2B(t, z).$$

The second equality is proved in a similar way.

3.3. Proof of (4). For $t \leq 0$, we have

$$A(t, z) = A(z) \cos(tz) + B(z) \sin(tz),$$

$$B(t, z) = -A(z) \sin(tz) + B(z) \cos(tz) \quad (3.4)$$

by (1.11), (1.10) and (1.11). In particular, Theorem 1.1 (4) holds.

4. Proof of Theorems 1.2 and 1.3

Lemma 4.1. Let $E$ be a function satisfying (K1)∼(K3). Define $Kf = \lim_{t \to \infty} K[t]f$ in pointwise convergence for $f$ in the space $C_c^\infty(\mathbb{R})$ of all compactly supported smooth function on $\mathbb{R}$. Then the Fourier integral formula

$$(FKf)(z) = \Theta(z) (Ff)(-z) \quad (4.1)$$

holds for $\Im(z) > c$. Suppose that $\Theta = E^2/E$ is an inner function in $\mathbb{C}_+$ in addition to (K1)∼(K3). Then $Kf$ belongs to $L^2(\mathbb{R})$ for $f \in C_c^\infty(\mathbb{R})$, and the linear map $f \mapsto Kf$
extends to the isometry $K : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ satisfying $K^2 = \text{id}$ and holds for $z \in \mathbb{R} \setminus \mathcal{Z}$. Moreover, holds for $z \in (\mathbb{C}_+ \cup \mathbb{R}) \setminus \mathcal{Z}$, if $f \in L^2(\mathbb{R})$ has a support in $(-\infty, t)$ for some $t \in \mathbb{R}$.

**Proof.** This is proved by almost the same argument as Lemma 3.2, because the difference of condition (K1) between this paper and is not essential in the proof. 

Here we recall basic properties of the Hardy spaces. The Hardy space $H^2 = H^2(\mathbb{C}_+)$ in $\mathbb{C}_+$ is defined to be the space of all analytic functions $f$ in $\mathbb{C}_+$ endowed with norm $\| f \|_{H^2} := \sup_{u>0} \int_{\mathbb{R}} | f(u+it) |^2 \, du < \infty$. The Hardy space $\overline{H^2} = H^2(\mathbb{C}_-)$ in the lower half-plane $\mathbb{C}_-$ is defined in the similar way. As usual, we identify $H^2$ and $\overline{H^2}$ with subspaces of $L^2(\mathbb{R})$ via nontangential boundary values on the real line such that $L^2(\mathbb{R}) = H^2 \oplus \overline{H^2}$, where $L^2(\mathbb{R})$ has the inner product $\langle f, g \rangle = \int_{\mathbb{R}} f(u)g'(u) \, du$. The Hardy space $H^2$ is a reproducing kernel Hilbert space, in particular the point evaluation functional $f \mapsto f(z)$ is continuous for each $z \in \mathbb{C}_+$ and it is represented by $k_z(w) = (2\pi i)^{-1}(z - w)^{-1} \in H^2$ as $\langle f, k_z \rangle = f(z)$.

**Lemma 4.2.** Let $\theta$ be an analytic function defined in $\mathbb{C}_+$. Suppose that $\theta f \in H^2$ for every $f \in H^2$. Then the pointwise multiplication operator $M_\theta : f \mapsto \theta f$ is bounded on $H^2$ and $\theta \in \overline{H^2}$.

**Proof.** We find that $M_\theta$ is bounded from the closed graph theorem and continuity of point evaluations. Let $k_z \in H^2$ be the vector representing the point evaluation at $z \in \mathbb{C}_+$. Then, we have

$$\langle M_\theta f, k_z \rangle = \langle \theta f, k_z \rangle = \theta(z)f(z) = \langle f, \theta(z)k_z \rangle,$$

for $f \in H^2$. Therefore, the adjoint $M_\theta^* \theta_z$ acts on $k_z$ as $M_\theta^* k_z = \theta(z)k_z$. This implies $k_z$ is an eigenvector of $M_\theta^*$ with eigenvalue $\theta(z)$. Hence, $|\theta(z)| \leq \| M_\theta \|_{\text{op}}$ for every $z \in \mathbb{C}_+$. This shows that $\theta(z)$ is uniformly bounded on $\mathbb{C}_+$. 

**Proposition 4.1.** Suppose that $E$ satisfies (K1)∼(K3). Then the following are equivalent:

1. $\Theta = E^1/E$ is an inner function in $\mathbb{C}_+$.
2. $K[t]$ converges as $t \to \infty$ with respect to the operator norm $\| \cdot \|_{\text{op}}$.
3. $K[t]f$ converges as $t \to \infty$ with respect to $\| \cdot \|_{L^2(\mathbb{R})}$ for all $f \in L^2(\mathbb{R})$.
4. There exists $M > 0$ such that $\| K[t] \|_{\text{op}} \leq M$ for every $t > 0$.

**Proof.** We prove the implication (2)⇒(1). Suppose that $K = \lim_{n \to \infty} K[t]$ exists with respect to the operator norm. Then $Kf \in L^2(\mathbb{R})$ for any $f \in L^2(\mathbb{R})$, where we understand as

$$Kf(x) = \int_{-\infty}^{\infty} K(x+y)f(y) \, dy := \lim_{t \to \infty} \int_{-\infty}^{\infty} K(x+y)F(y) \, dy,$$

where $\lim_{t \to \infty}$ stands for limit in mean. We have $K[t] = P_tK_P$. Let $f \in L^2(-\infty, 0)$ and take $\{ f_n \}_n \subset C^\infty_c(\mathbb{R})$ such that $\lim_{n \to \infty} f_n = f$ in $L^2(\mathbb{R})$. Then,

$$(FKf)(z) = \lim_{n \to \infty} (FKf_n)(z) = \lim_{n \to \infty} \lim_{t \to \infty} (FK[t]f_n)(z),$$

since $K$ and $F$ are bounded. On the right-hand side,

$$\lim_{t \to \infty} (FK[t]f_n)(z) = \Theta(z)(Ff_n)(z)$$

for $\Theta(z) > c$ by Lemma 3.2. Therefore, $(FKf)(z) = \Theta(z)(Ff)(z)$ for $\Theta(z) > c$. On the other hand, $Kf \in L^2(0, \infty)$ by (K3). Thus $(FKf)(z)$ defines an analytic function in $\mathbb{C}_+$. Hence $(FKf)(z) = \Theta(z)(Ff)(z)$ which conclude $\Theta H^2 \subset H^2$. Hence $\Theta \in \overline{H^2}$ by Lemma 3.2. Therefore, by 3.2, $\Theta$ is an inner function in $\mathbb{C}_+$.

We prove the implication (1)⇒(3). If $\Theta$ is an inner function in $\mathbb{C}_+$, for any $f \in L^2(\mathbb{R})$, $Lf := F^{-1}M_\Theta Ff$ is defined and belongs to $L^2(\mathbb{R})$, $Lf(x) = \int K(x+y)f(y) \, dy := \int K(x+y)f(y) \, dy.$
\[ \lim_{t \to \infty} \int_{-T}^{T} K(x + y)f(y) \, dy, \text{ and } ||Lf|| = ||f|| \text{ by the argument similar to the proof of Lemma 3.2, where } Jf(x) = f(-x). \] We have \( K[t] = P_lLP_t \). Therefore,
\[ Lf - K[t]f = L(1 - P_l)f + (1 - P_l)LP_t f \to 0 \quad (t \to \infty). \]

Hence \( K[t]f \) converges to \( Lf \), and \( L = K \).

The implication (3) \( \Rightarrow \) (2) is a consequence of the Banach–Steinhaus theorem. The implication (2) \( \Rightarrow \) (4) is trivial. Finally, we show that (4) implies (3). Let \( t > s > a > 0 \), and let \( P_l \) be the projection to \( L^2(-\infty, t) \). Then, \( K[t]f - K[s]f = P_l(K[t] - P_s)f = (P_l - P_s)(K[t] - P_s)f + (P_l - P_s)KP_t f \), and thus
\[ ||K[t]f - K[s]f|| \leq ||K[t](P_l - P_s)f|| + ||(P_l - P_s)K[t](P_l - P_s)f|| + ||(P_l - P_s)KP_t f|| \leq 2M||(P_l - P_s)f|| + ||(P_l - P_s)KP_t f||. \]

The first term of the right-hand side is smaller as \( t > s > 0 \) are larger. We show that the second term is also smaller as \( t > s > 0 \) are larger by contradiction. Suppose that there exists \( a > 0 \) such that
\[ a \leq ||(P_l - P_s)KP_t f||^2 = \int_{s}^{t} \left| \int_{-\infty}^{t} K(x + y)f(y) \, dy \right|^2 \, dx \]
holds for some \( t > s > s_0 \) for arbitrary \( s_0 \). Then we can take a strictly increasing numbers \( s_0 \leq s_1 < \ldots \) such that \( a \leq \int_{s_n}^{s_{n+1}} \left| \int_{-\infty}^{s_n} K(x + y)f(y) \, dy \right|^2 \, dx \) holds. For such numbers,
\[ a \leq \int_{s_n}^{s_{n+1}} \left| \int_{-\infty}^{s_n} K(x + y)f(y) \, dy \right|^2 \, dx \leq \int_{s_n}^{s_{n+1}} \left| \int_{-\infty}^{s_0} K(x + y)f(y) \, dy \right|^2 \, dx + \int_{s_n}^{s_{n+1}} \left| \int_{s_0}^{s_n} K(x + y)f(y) \, dy \right|^2 \, dx \leq \int_{s_n}^{s_{n+1}} \left| \int_{-\infty}^{s_0} K(x + y)f(y) \, dy \right|^2 \, dx + ||(P_{s_{n+1}} - P_{s_0})K[s_{n+1}](P_{s_{n+1}} - P_{s_0})f||^2 \leq \int_{s_n}^{s_{n+1}} \left| \int_{-\infty}^{s_0} K(x + y)f(y) \, dy \right|^2 \, dx + M||P_{s_{n+1}} - P_{s_0}f||^2. \]

The second term of the right-hand side is small for large \( s_0 \). Therefore, by replacing \( a \) if necessary,
\[ a \leq \int_{s_n}^{s_{n+1}} \left| \int_{-\infty}^{s_0} K(x + y)f(y) \, dy \right|^2 \, dx \quad (n = 0, 1, 2, \ldots). \]

This implies \( ||K[s_{n+1}](P_{s_0}f)|| \geq ||K[s_1](P_{s_0}f)|| + na \), which contradicts the uniform boundedness (4).

**Proposition 4.2.** We can arrange the eigenvalues of the family \( \{K[t]\}_t \) such that \( \lambda^+_1(t) \geq \lambda^+_2(t) \geq \ldots \) are positive eigenvalues, \( \lambda^-_1(t) \leq \lambda^-_2(t) \leq \ldots \) are negative eigenvalues, where eigenvalues are repeated as many times as multiplicities, and each \( |\lambda^\pm(t)| \) is a continuous nondecreasing function of \( t \).

**Proof.** We find that \( \lambda^+_j(t) \) is a nondecreasing function of \( t \) by the min–max principle for positive eigenvalues \( \lambda^+_j(t) = \min_{V_j} \max_{f \in V_j} \langle K[t]f, f \rangle/\langle f, f \rangle \}, where \( V_j \) runs all \((j - 1)\)-dimensional subspace of \( L^2(-\infty, t) \) and \( V_j^\perp \) stands for the orthogonal complement of \( V_j \), because the maximum part of the formula is a nondecreasing function of \( t \) by the inclusion \( L^2(-\infty, s) \subset L^2(-\infty, t) \) for \( s < t \) obtained by the extension by zero. Also \( \lambda^-_j(t) \) is a nonincreasing function of \( t \) by applying the above argument to \( -K[t] \). Since
\(K(t)\) is a Hilbert–Schmidt operator, \(\sum_j \lambda_j^+(t)^2 + \sum_j \lambda_j^-(t)^2 = \int_{-t}^t \int_{-t}^t |K(x + y)|^2 \, dx \, dy\). Therefore, for sufficiently small \(h > 0\),

\[
\sum_j (\lambda_j^+(t + h)^2 - \lambda_j^+(t)^2) + \sum_j (\lambda_j^-(t + h)^2 - \lambda_j^-(t)^2)
= \left( \int_{-t}^{t+h} \int_{-t-h}^{t+h} + \int_{-t-h}^{t-h} \int_{-t}^{t+h} + \int_{-t-h}^{t-h} \int_{-t}^{-t-h} \right) |K(x + y)|^2 \, dx \, dy
\leq 4h \int_{-3t}^{3t} |K(x)|^2 \, dx,
\]

and hence \(\lambda_j^+(t + h)^2 - \lambda_j^+(t)^2 \leq 4h \int_{-3t}^{3t} |K(x)|^2 \, dx\). This implies that each \(\lambda_j^+(t)\) is right continuous. The left continuity of each \(\lambda_j^+(t)\) is proved by the same argument. \(\square\)

**Proof of Theorems 1.2 and 1.3** Recall that the operator norm of a compact self-adjoint operator is equal to the maximum of absolute values of eigenvalues. Then, condition (K5) with \(\tau = \infty\) implies that \(\|K[t]\|_{\text{op}} < 1\) by Proposition 4.2. Hence \(\Theta\) is an inner function in \(\mathbb{C}_+\) by Proposition 4.1, and Theorem 1.3 is proved. If we suppose that \(E\) satisfies (K1)∼(K3) and (K6) and that \(\Theta\) is an inner function in \(\mathbb{C}_+\), both \(\pm 1\) are not eigenvalues of \(K[t]\) for every \(t > 0\) as proved in [15, Proposition 9.1]. Then, \(\|K[t]\|_{\text{op}} < 1\) by Proposition 4.2. Hence \(\Theta\) is an inner function in \(\mathbb{C}_+\) by Proposition 4.1 and Theorem 1.3 is proved. \(\square\)

## 5. Theory of model subspaces

In this section, we review basic notions and properties of model subspaces and de Branges spaces in preparation for the next section according to Havin–Mashreghi [8], Woracek [20] for model subspaces and de Branges [2], Romanov [12], Winkler [19], Woracek [20] for de Branges spaces.

### 5.1. Model subspaces

For an inner function \(\theta\), a **model subspace** (or coinvariant subspace) \(\mathcal{K}(\theta)\) is defined by the orthogonal complement

\[
\mathcal{K}(\theta) = H^2 \ominus \theta H^2, \tag{5.1}
\]

where \(\theta H^2 = \{\theta(z)F(z) \mid F \in H^2\}\). It has the alternative representation

\[
\mathcal{K}(\theta) = H^2 \cap \bar{H}^2. \tag{5.2}
\]

A model subspace \(\mathcal{K}(\theta)\) is a reproducing kernel Hilbert space with respect to the norm induced from \(H^2\). The reproducing kernel of \(\mathcal{K}(\theta)\) is

\[
j(z, w) = \frac{1 - \overline{\theta(z)}\theta(w)}{2\pi i (z - w)} \quad (z, w \in \mathbb{C}_+),
\]

that is, \(\langle f, j(z, \cdot) \rangle_{H^2} = f(z)\) for \(f \in \mathcal{K}(\theta)\) and \(z \in \mathbb{C}_+\).

### 5.2. De Branges spaces

For \(E \in \mathbb{H}_B\), the set

\[
\mathcal{H}(E) := \{f \mid f \text{ is entire}, f/E \text{ and } f^\dagger/E \in H^2\}
\]

forms a Hilbert space under the norm \(\|f\|_{\mathcal{H}(E)} := \|f/E\|_{H^2}\). The Hilbert space \(\mathcal{H}(E)\) is called the **de Branges space** generated by \(E\). The de Branges space \(\mathcal{H}(E)\) is a reproducing kernel Hilbert space consisting of entire functions endowed with the reproducing kernel

\[
J(z, w) = \frac{E(z)E(w) - E^2(z)E^2(w)}{2\pi i (z - w)} \quad (z, w \in \mathbb{C}_+).
\]

The reproducing formula \(\langle f, J(z, \cdot) \rangle_{\mathcal{H}(E)} = f(z)\) for \(f \in \mathcal{H}(E)\) and \(z \in \mathbb{C}_+\) remains true for \(z \in \mathbb{R}\) if \(\theta = E^2/E\) is analytic in a neighborhood of \(z\).
If an inner function \( \theta \) in \( \mathbb{C}_+ \) is meromorphic in \( \mathbb{C} \), it is called a \textit{meromorphic inner function}. It is known that every meromorphic inner function is expressed as \( \theta = E^2/E \) by using some \( E \in \mathbb{H} \). If \( \theta \) is a meromorphic inner function such that \( \theta = E^2/E \), the model subspace \( K(\theta) \) is isomorphic and isometric to the de Branges space \( \mathcal{H}(E) \) as a Hilbert space by \( K(\theta) \to \mathcal{H}(E) : f(z) \mapsto E(z)f(z) \).

As developed in [3, 4, 5, 6], the Hankel type operator \( K \) with the kernel \( K(x + y) \) is useful to study model subspaces \( K(\theta) \) or de Branges spaces \( \mathcal{H}(E) \) via Fourier analysis.

6. PROOF OF THEOREM 1.5

We suppose that \( E \) satisfies (K1)∼(K5) with \( \tau = \infty \) throughout this section. Then \( \Theta = E^2/E \) is an inner function in \( \mathbb{C}_+ \) by Theorem 1.2 (thus \( Z \subset \mathbb{R} \)) and \( f \mapsto Kf \) defines an isometry of \( L^2(\mathbb{R}) \) satisfying \( K^2 = \text{id} \) by Lemma 4.1. These properties are essential in the argument of the section. For convenience, we put

\[
A(t, z) = m(t)^{-1}A(t, z), \quad B(t, z) = m(t)B(t, z).
\]

Lemma 6.1. For \( \Im(z) > c \), we have

\[
\mathcal{A}(t, z) = \frac{1}{2}E(z) \left( e^{izt} + \int_t^\infty \phi^+(t, x)e^{ixz} \, dx \right),
\]

\[
-i\mathcal{B}(t, z) = \frac{1}{2}E(z) \left( e^{izt} - \int_t^\infty \phi^-(t, x)e^{ixz} \, dx \right).
\]

Proof. Using (2.10) and (2.11),

\[
e^{izt} + \int_t^\infty \phi^+(t, x)e^{ixz} \, dx = -iz \int_t^\infty \frac{\Psi(t, x)}{\Psi(t, t)} e^{ixz} \, dx,
\]

\[
e^{izt} - \int_t^\infty \phi^-(t, x)e^{ixz} \, dx = -iz \int_t^\infty \frac{\Phi(t, x)}{\Phi(t, t)} e^{ixz} \, dx.
\]

Hence we obtain (6.2) by definition (1.11). The convergence of integrals are justified by Proposition 2.2(3). \qed

6.1. Formulas for reproducing kernels of model subspaces. Let \( \mathcal{V}_t \) be the Hilbert space of all functions \( f \) such that both \( f \) and \( Kf \) are square integrable functions having supports in \( [t, \infty) \):

\[
\mathcal{V}_t = L^2(t, \infty) \cap K(L^2(t, \infty)).
\]

If \( t > 0 \), \( F(\mathcal{V}_t) \) is a closed subspace of \( H^2 \), because the Fourier transform provides an isometry of \( L^2(\mathbb{R}) \) up to a constant such that \( H^2 = \mathcal{F}L^2(0, \infty) \) and \( \mathcal{F}L^2 = \mathcal{F}L^2(-\infty, 0) \) by the Paley-Wiener theorem, and \( \mathcal{V}_t \) is a closed subspace of \( L^2(0, \infty) \) by definition. Therefore, \( F(\mathcal{V}_t) \) is a reproducing kernel Hilbert space, since the point evaluation map \( F \mapsto F(z) \) is continuous in \( F(\mathcal{V}_t) \) for each \( z \in \mathbb{C}_+ \) as well as \( H^2 \).

Lemma 6.2. We have \( F(\mathcal{V}_0) = K(\Theta) \).

Proof. If \( f \in \mathcal{V}_0 \), \( Ff \) and \( Kf \) belong to the Hardy space \( H^2 \). On the other hand, we have \( (FKf)(z) = \Theta(z)(Ff)(-z) \) by Lemma 4.1. This implies \( (Ff)(z) = \Theta(z)(FKf)(-z) \) by (1.3). Therefore, \( Ff \) belongs to \( K(\Theta) \) by (5.2). Conversely, if \( f \in K(\Theta) \), there exists \( f \in L^2(0, \infty) \) and \( g \in L^2(-\infty, 0) \) such that \( Ff \) = \( (Ff)(z) \). We have \( (Fg)(-z) = \Theta(z)(Ff)(-z) \) by (1.3) again. Here \( (Fg)(-z) = (Fg^-)(z) \) for \( g^- \in L^2(0, \infty) \), and \( \Theta(z)(Ff)(-z) = (FKf)(z) \) as above. Hence \( Kf \) belongs to \( L^2(0, \infty) \), and thus \( f \in \mathcal{V}_0 \). \qed

Lemma 6.3. We have \( \mathcal{V}_t \neq \{0\} \) for every \( t \in \mathbb{R} \).
Proof. The case of $t = 0$ is Lemma 5.2. The orthogonal complement $\mathcal{V}_t^\perp$ of $\mathcal{V}_t$ in $L^2(\mathbb{R})$ is equal to the closure of $L^2(-\infty, t) + K(L^2(-\infty, t))$. Therefore, if $t < 0$, $\mathcal{V}_t$ contains $L^2(t, -t)$ and thus $\mathcal{V}_t \neq \{0\}$, since $K(L^2(-\infty, t)) \subset L^2(-t, \infty)$ by (K3). We suppose that $t > 0$ and show that $L^2(-\infty, t) + K(L^2(-\infty, t))$ is closed. If $w = u + Kw$ for some $u, v \in L^2(-\infty, t)$, we have $P_tw = u + K[t]v$ and $P_t Kw = K[t]u + v$ by $K^2 = \text{id}$. It is solved as $u = (1 - K[t]^2)^{-1}(P_t - K[t]P)w$ and $v = (1 - K[t])^{-1}(P_t - K[t]P_t)w$, since both $\pm 1$ are not eigenvalues of $K[t]$. Therefore, if $w_n = u_n + Kw_n$ converges in $L^2(\mathbb{R})$, it implies that both $u_n$ and $v_n$ also converge in $L^2(\mathbb{R})$, and hence the space $L^2(-\infty, t) + K(L^2(-\infty, t))$ is closed. Hence we obtain
\[
\mathcal{V}_t^\perp = L^2(-\infty, t) + K(L^2(-\infty, t)).
\] (6.3)
To prove $\mathcal{V}_t \neq \{0\}$, it is sufficient to show that $L^2(-\infty, t) + K(L^2(-\infty, t))$ is a proper closed subspace of $L^2(\mathbb{R})$. Suppose that $f \in K(L^2(-\infty, t))$. Then the restriction of $f$ to $(t, \infty)$ is continuous on $(t, \infty)$. In fact, if $f(x) := (Kg)(x) = \int_x^t K(x + y)g(y) dy$ for some $g \in L^2(-\infty, t)$, the continuity of $K$ implies the continuity of $f$ as
\[
|f(x + \delta) - f(x)| = \left| \int_{x-\delta}^x K(x + \delta + y)g(y) dy - \int_{x-\delta}^t K(x + y)g(y) dy \right|
\leq \int_{x-\delta}^x |K(x + \delta + y)||g(y)| dy + \int_{x-\delta}^t |K(x + \delta + y) - K(x + y)||g(y)| dy
\leq \left( \int_{x-\delta}^x |K(x + \delta + y)|^2 dy + C_2 \cdot \delta \cdot |t + x| \bigg) \|g\|^2.
\]
where $0 < \delta < x$ and we used the mean value theorem and the Schwartz inequality. Hence, if $f \in L^2(-\infty, t) + K(L^2(-\infty, t))$, $f$ is continuous on $(t, \infty)$. Thus $L^2(-\infty, t) + K(L^2(-\infty, t))$ is a proper closed subspace of $L^2(\mathbb{R})$. \hfill \Box

Lemma 6.4. Let $z \in \mathbb{C}_+$. Then the integral equations
\[
a^t_z + KP_t \alpha^t_z = e_z + Ke_z,
\]
(6.4)
\[
b^t_z - KP_t \beta^t_z = e_z - Ke_z
\]
(6.5)
have unique solutions, where $e_z(x) = \exp(izx)$ and $P_t$ is the projection to $L^2(-\infty, t)$. Moreover, for $\Im(z) > c$,
\[
a^t_z(t) = \frac{2\mathfrak{A}(t, z)}{E(z)}, \quad b^t_z(t) = -2i\frac{\mathfrak{A}(t, z)}{E(z)}.
\]
(6.6)
Proof. Note that $Ke_z$ makes sense as a function, because $(Ke_z)(x) = e^{-izx} \Theta(z)$ holds if $\Im(z) > c$ by (K1), $(Ke_z)(x) = (K_{1>z}e_z)(x) + \int_{-\infty}^x K(x + y)\exp(izy) dy$ holds if $0 < \Im(z) \leq c$ by (K3), and $1_{\geq z}e_z \in L^2(\mathbb{R})$. First, we show the uniqueness of the solution. After subtracting $P_t e_z + KP_t e_z$ (resp. $P_t e_z - KP_t e_z$) from both sides of (6.4) (resp. (6.5)), multiplying by $P_t$ on both sides, we find that equations (6.4) and (6.5) have unique solutions
\[
a^t_z = e_z + K(1 - P_t)e_z - K(1 + K[t])^{-1}P_t K(1 - P_t)e_z,
\]
\[
b^t_z = e_z - K(1 - P_t)e_z - K(1 - K[t])^{-1}P_t K(1 - P_t)e_z,
\]
respectively. By multiplying both sides of $a^t_z = e_z + Ke_z - KP_t a^t_z$ and $K(x + t) - \int_{-\infty}^t K(x + y)\phi^+(t, y) dy = \phi^+(t, x)$, integrating the obtained equation with respect to $x$ from $-\infty$ to $t$, and using the symmetry of the kernel $K(x + y)$, we obtain
\[
\int_{-\infty}^t a^t_z(x)K(t + x) dx = \int_{-\infty}^t \phi^+(t, x)(\exp(izx) + (Ke_z)(x)) dx.
\]
Combining this with (6.3),
\[ a_2^t(t) = e^{izt} + (Ke_2)(t) - \int_{-\infty}^{t} \phi^+(t, x)(e^{ixx} + (Ke_2)(x))\, dx. \] (6.7)

If we suppose that \( \Im(z) > c, (Ke_2)(x) = e^{-ixx}\Theta(z) \) holds, \( F(z) := F(\phi^+(t, \cdot))(z) \) is defined by Proposition 2.2 (3), and

\[ F(z) = 2\Re(t, z) - e^{izt} + \int_{-\infty}^{t} \phi^+(t, x)e^{ixx}\, dx \] (6.8)
by (6.2). On the other hand, by taking the Fourier transform of both sides of (6.6),

\[ F(z) + \Theta(z) \int_{-\infty}^{t} \phi^+(t, x)e^{-ixx}\, dx = e^{-izt}\Theta(z). \] (6.9)

Substituting (6.8) into (6.9), and arranging, we find that the right-hand side of (6.7) equals \( 2\Re(t, z)/E(z) \). Hence the formula on the left of (6.6) holds. The the formula on the right of (6.3) is proved by a similar argument. \( \Box \)

Integral equations (6.4) and (6.5) are solved as
\[ a_2^t(x) = a_2^x(x) + \int_{t}^{x} a_2^x(s)\phi^+(s, x)\, ds, \quad b_2^t(x) = b_2^x(x) - \int_{t}^{x} b_2^x(s)\phi^-(s, x)\, ds. \]

In fact, differentiating both sides of (6.4) with respect to \( t \),
\[ \frac{\partial}{\partial t} a_2^t(x) + \int_{t}^{x} K(x + y)\frac{\partial}{\partial y} a_2^y(y)\, dy = (-a_2^x(t))K(x + t). \]

This shows that \( (-a_2^x(t))^{-1}(\partial/\partial t)a_2^t(x) \) is a solution of (2.6), hence it equals to \( \phi^+(t, x) \). Then, we obtain \( a_2^t \) by integrating \( (\partial/\partial t)a_2^x(x) = -a_2^x(t)\phi^+(t, x) \) with respect to \( t \). We obtain \( b_2^t \) by a way similar to \( a_2^t \).

6.2. Proof of Theorem 1.5 (1). Let \( z \in \mathbb{C}_+ \). Then \( f \mapsto Ff(z) \) is a continuous functional on \( V_1 \), since the point evaluation \( F \mapsto F(z) \) is continuous on \( H^2 \) and the Fourier transform \( F \) is an isometry between \( L^2(0, \infty) \) and \( H^2 \) up to a constant. Therefore, there exists an unique vector \( Y_2^t \in V_1 \) such that \( \int_{0}^{\infty} f(x)Y_2^t(x)\, dx = (Ff)(z) \) holds for all \( f \in V_1 \) by the Riesz representation theorem. Let \( j(t; z, w) \) be the reproducing kernel of \( F(V_1) \). Then,
\[ j(t; z, w) = \frac{1}{2\pi} \langle Y_2^t, Y_2^t \rangle, \] (6.10)

where \( \langle f, g \rangle = \int_{\mathbb{R}} f(u)\overline{g(u)}\, du \) as above. In fact, for \( F = Ff \in F(V_1) \),
\[ \langle F(w), \langle Y_2^t, Y_2^t \rangle \rangle_{H^2} = \int_{\mathbb{R}} F(w) \left( \int_{\mathbb{R}} Y_2^t(x)Y_2^t(x)\, dx \right) \, dw \]
\[ = \int_{\mathbb{R}} F(w) \left( \int_{\mathbb{R}} e^{-iwx}Y_2^t(x)\, dx \right) \, dw = 2\pi \int_{\mathbb{R}} f(x)e^{-ixx}Y_2^t(x)\, dx = 2\pi F(z). \]

On the other hand, \( Y_2^t \) is the orthogonal projection of \( 1_{\mathbb{R}}(x)e^{ixx} \in L^2(\mathbb{R}) \) to \( V_1 \). By (6.3), there exists unique vectors \( u_2^t \) and \( u_2^t \) in \( L^2(\infty, t) \) such that
\[ 1_{[t, \infty)}e_z = Y_2^t + u_2^t + K\bar{v}_z^t. \] (6.11)

Using the solutions \( a_2^t \) and \( b_2^t \) of (6.4) and (6.5), equation (6.11) is solved as
\[ Y_2^t = (1 - P_t)\frac{1}{2}(a_2^t + b_2^t). \] (6.12)

This equality is proved as follows. Put \( U_2^t = (a_2^t + b_2^t)/2 - e_z \) and \( V_2^t = (a_2^t - b_2^t)/2 \). Then, \( U_2^t + KP_tV_2^t = 0 \) and \( V_2^t + KP_tU_2^t = K(1 - P_t)e_z \) by (6.4). By multiplying \( K \) on
both sides of the second equation, \((1 - P_t) e_z = K(1 - P_t)V^t_z + P_tU^t_z + KP_tV^t_z\). Therefore, \(Y^t_z = K(1 - P_t)V^t_z\), \(U^t_z = P_tU^t_z\), and \(V^t_z = P_tV^t_z\). Moreover,

\[
Y^t_z = K(1 - P_t)V^t_z = KV^t_z - KP_tV^t_z = ((1 - P_t)e_z - P_tU^t_z) + U^t_z = (1 - P_t)(e_z + U^t_z).
\]

Hence (6.12) holds. By differentiating both sides of (6.4) with respect to \(t\), we obtain

\[
\frac{\partial}{\partial x} a^t_z(x) - \int_{-\infty}^t K(x+y) \frac{\partial}{\partial y} a^t_z(y) dy = -K(t) a^t_z(t) + iz(e^{izx} - Ke^{izx})
\]

Multiplying (6.5) by \(iz\) and then subtracting from this equation, we find that the function \((-a^t_z(t))^{-1}(\partial/\partial x) a^t_z(x) - izb^t_z(x))\) solves (2.7). Therefore, by the uniqueness of the solution of (2.7),

\[
\frac{\partial}{\partial x} a^t_z(x) - izb^t_z(x) = -a^t_z(t)\phi^-(t, x).
\]

We also obtain

\[
\frac{\partial}{\partial x} b^t_z(x) - izb^t_z(x) = b^t_z(t)\phi^+(t, x).
\]

by a similar argument. Adding these,

\[
\frac{\partial}{\partial x} (a^t_z(x) + b^t_z(x)) - iz(a^t_z(x) + b^t_z(x)) = b^t_z(t)\phi^+(t, x) - a^t_z(t)\phi^-(t, x).
\]

Hence, if \(\Im(z) > c\) and \(\Im(w) > c\),

\[
-i(z + w) \int_0^\infty Y^t_z(x)e^{izx} dx = -i(z + w) \int_0^\infty \frac{1}{2}(a^t_z(x) + b^t_z(x))e^{izx} dx
\]

\[
= \frac{1}{2} b^t_z(t) \left(e^{iwt} + \int_t^\infty \phi^+(t, x)e^{iwx} dx\right) + \frac{1}{2} a^t_z(t) \left(e^{iwt} - \int_t^\infty \phi^-(t, x)e^{iwx} dx\right)
\]

\[
= -2i \frac{\Re(z)}{E(z)} A(t, w) - 2i \frac{\Re(z)}{E(z)} E(w)
\]

by (6.2) and (6.9). Therefore,

\[
\frac{1}{2\pi} \langle Y^t_w, Y^t_z \rangle = \frac{1}{2\pi} \int Y^t_w(x)Y^t_z(x) dx = \frac{1}{2\pi} \int Y^t_w(x)e^{-izx} dx
\]

\[
= \frac{1}{E(-z)E(w)} \left(\Re(t, w)\Re(t, z) + \Re(t, w)\Re(t, \bar{z})\right)
\]

\[
= \frac{1}{E(z)E(w)} \left(A(t, z)B(t, w) - A(t, w)B(t, z)\right)
\]

by (6.10), (6.11), and Theorem 1.11(1). Hence we obtain (1.13) by (6.10) under the restrictions \(\Im(z) > c\) and \(\Im(w) > c\), but (1.13) holds for all \(z, w \in \mathbb{C}_+\) by analytic continuation. Hence we complete the proof. \(\square\)

6.3. Proof of Theorem 1.5 (2). Suppose that \(j(t; z, w) \equiv 0\) for some \(t > 0\). Then \(V_t = \{0\}\), since \(j(t; z, w)\) is the reproducing kernel of \(F(V_t)\). This contradicts Lemma 6.3 and thus \(j(t; z, w) \neq 0\) for every \(t > 0\). We have

\[
|j(t; z, w)| = 2\pi \langle Y^t_w, Y^t_z \rangle \leq \|Y^t_w\| \cdot \|Y^t_z\|
\]

for fixed \(z, w \in \mathbb{C}_+\) by (1.13), where \(\| \cdot \| = \| \cdot \|_{L^2(\mathbb{R})}\). Therefore, for Theorem 1.5 (2), it is sufficient to show that \(\|Y^t_z\| \to 0\) as \(t \to \infty\) for a fixed \(z \in \mathbb{C}_+\). We have \(\|Y^t_z\|^2 = \langle Y^t_z, Y^t_z \rangle = (F(Y^t_z))(z) = (Y^t_z, Y^0_z)\), since \(V_t \subset V_0\). Therefore,

\[
\langle Y^t_z, Y^0_z \rangle = \int_t^\infty Y^t_z(x)Y^0_z(x) dx \leq \|Y^t_z\| \left(\int_t^\infty |Y^0_z(x)|^2 dx\right)^{1/2}
\]

by the Cauchy–Schwarz inequality. Hence, \(\|Y^t_z\|^2 \leq \int_t^\infty |Y^0_z(x)|^2 dx\). This shows that \(\|Y^t_z\| \to 0\) as \(t \to \infty\), since \(Y^0_z \in L^2(0, \infty)\). \(\square\)
6.4. Model subspaces for general $t$.

**Theorem 6.1.** Let $A(t, z)$ and $B(t, z)$ be as in Theorem 1.1 and define

$$E(t, z) = A(t, z) - iB(t, z), \quad \Theta(t, z) = \frac{E(t, z)}{E(t, \bar{z})},$$

(6.13)

for $t \in \mathbb{R}$. Then $\Theta(t, z)$ is an inner function in $\mathbb{C}_+$.

**Proof.** Note that $Z \subset \mathbb{R}$, since $\Theta$ is an inner function in $\mathbb{C}_+$ by the assumption. From Theorem 1.1(1), we have $E(t, z) = A(t, z) + iB(t, z)$. Therefore, $|\Theta(t, z)| = 1$ for almost all $z \in \mathbb{R}$. Suppose that $t \geq 0$ and put $J(t; z, w) = \overline{E(z)E(w)}j(t; z, w)$. Then, (6.13) is transformed as

$$J(t; z, w) = \frac{E(t, z)E(t, w) - E(t, \bar{z})E(t, \bar{w})}{2\pi i(z - w)}$$

(6.14)

by using (6.13) and $E(t, \bar{z}) = A(t, z) + iB(t, z)$. On the other hand, we have

$$J(t; z, w) - J(s; z, w) = \frac{1}{\pi} \int_t^s A(t, z)A(t, w) \frac{1}{\gamma(t)} dt + \frac{1}{\pi} \int_t^s B(t, z)B(t, w) \gamma(t) dt$$

(6.15)

for any $t < s < \infty$ and $z, w \in \mathbb{C}_+$ in a way similar to the proof of Lemma 2.1 of [7]. Taking $w = z \in \mathbb{C}_+$ in (6.15) and then tending $s$ to $\infty$, we have

$$J(t; z, z) = \frac{1}{\pi} \int_\mathbb{R} |A(t, z)|^2 \frac{1}{\gamma(t)} dt + \frac{1}{\pi} \int_\mathbb{R} |B(t, z)|^2 \gamma(t) dt$$

(6.16)

by Theorem 1.1(2), where $\gamma(t) = m(t)^2 > 0$. This shows that $|\Theta(t, z)| < 1$ for any $z \in \mathbb{C}_+$, since the left-hand side equals to $(|E(t, z)|^2 - |E(t, \bar{z})|^2)/(2\pi i\Im(z))$. Hence $\Theta(t, z)$ is an inner function in $\mathbb{C}_+$. For $t \leq 0$, we have

$$E(t, z) = E(z)e^{izt}, \quad \Theta(t, z) = \Theta(z)e^{-2izt}$$

(6.17)

by (6.11). Hence $\Theta(t, z)$ is an inner function in $\mathbb{C}_+$, since the factors $\Theta(z)$ and $e^{-2izt}$ are both inner functions in $\mathbb{C}_+$. $\square$

Equality (6.16) also shows the following.

**Corollary 6.1.** $\mathfrak{A}(t, z)$ and $\mathfrak{B}(t, z)$ are square integrable at $t = \infty$ for each $z \in \mathbb{C}_+$.

Recall that $\int_\alpha^\infty \gamma(t)^{-1} dt < \infty$ is a part of the sufficient condition in [2] Theorem 41] for the existence of the solution of the canonical system for $H(t) = \text{diag}(1/\gamma(t), \gamma(t))$ on $[\alpha, \infty)$. We find that it is necessary if $E(0) = A(0) \neq 0$. In fact, we have $J(t; 0, 0) = \pi^{-1}A(0)^2 \int_\mathbb{R} \gamma(t)^{-1} dt$ by taking $z = 0$ in (6.16). Therefore, $\gamma(t)^{-1}$ is $L^1$ at $t = \infty$.

By comparing the kernels of $F(\mathfrak{V}_t)$ and $K(\Theta(t, z))$, we obtain the following relation.

**Corollary 6.2.** For $t \geq 0$,

$$F(\mathfrak{V}_t) = \frac{E(t, z)}{E(z)} K(\Theta(t, z)).$$

(6.18)

If $t < 0$, the space $F(\mathfrak{V}_t)$ is no longer a subspace of $H^2 = F(L^2(0, \infty))$, since $\mathfrak{V}_t$ contains $L^2(t, -t)$ as found in the proof of Lemma 6.3 in particular, $F(\mathfrak{V}_t)$ can not be a model subspace, but the right-hand side of (6.18) can be extended to negative $t$’s. We found above that $\Theta(t, z)$ is an inner function in $\mathbb{C}_+$. The kernel of $K(\Theta(t, z))$ is

$$J(t; z, w) = \frac{A(z)B(w) - A(w)B(z)}{\pi(w - \bar{z})} \cos(t(w - \bar{z}))$$

$$- \frac{(A(z)A(w) + B(z)B(w))\sin(t(w - \bar{z}))}{\pi(w - \bar{z})}$$

- (A(z)A(w) + B(z)B(w))\sin(t(w - \bar{z}))}{\pi(w - \bar{z})}$$

by (6.13).
connection formula for solutions of canonical systems: 

\[ E(t, z) \mathcal{K}(\Theta(t, z)) = e^{itz}(\mathcal{K}(\Theta) \oplus \Theta \Pi_{-2\lambda}), \]

where \( \Pi_{-2\lambda} = \mathcal{K}(e^{itz}) \) \((\alpha > 0)\) is the Paley–Wiener space, which consists of the entire functions of exponential type at most \( \alpha \) the restrictions of which to the real line \( \mathbb{R} \) are in \( L^2(\mathbb{R}) \). Therefore, \( F(V_t) \) is a shift of a model subspace.

7. RELATED DIFFERENTIAL EQUATIONS

From Theorem 1.4 and the second equation of (2.10) and (2.11), we find that \( \Phi(t, x) \) and \( \Psi(t, x) \) are characterized as the unique solution of the Cauchy problem:

\[
\begin{cases}
\Phi_t(t, x) + \gamma(t)\Psi_x(t, x) = 0, & \Psi_t(t, x) + \gamma(t)^{-1}\Phi_x(t, x) = 0, \\
\gamma(t) = \Psi(t, t)/\Phi(t, t) (= \Phi(t, t)^{-1} = \Psi(t, t)^2), \\
\Phi(0, x) = 1 - \int_0^\infty K(y) dy, & \Psi(0, x) = 1 + \int_0^\infty K(y) dy
\end{cases}
\tag{7.1}
\]

for \((t, x) \in [0, \tau) \times \mathbb{R}\). In this formulation, (K5) is understood as a statement about the existence of a global solution. We should remark that \( \gamma(t) \) is not a given function in (7.1) different from usual Cauchy problem for hyperbolic first-order systems. It would be interesting to study the inverse problem for Hamiltonian systems in terms of this (unusual) Cauchy problem.

On the other hand, if we note that \( \Phi(t, x) \) and \( \Psi(t, x) \) have the second derivative for both variables by Propositions 2.2 and 2.5, we find that they satisfy the following damped wave equations (or wave equations with time-dependent dissipation)

\[
\begin{cases}
\Phi_{tt}(t, x) - \Phi_{xx}(t, x) - 2\mu(t)\Phi_t(t, x) = 0, \\
\Psi_{tt}(t, x) - \Psi_{xx}(t, x) + 2\mu(t)\Psi_t(t, x) = 0
\end{cases}
\]

by the first line of (7.1), where \( 2\mu(t) = \gamma(t)'/\gamma(t) \). Moreover, definition (1.11) derives the Schrödinger equations

\[
\begin{cases}
A_{tt}(t, z) + z^2A(t, z) - 2\mu(t)A_t(t, z) = 0, \\
B_{tt}(t, z) + z^2B(t, z) + 2\mu(t)B_t(t, z) = 0
\end{cases}
\]

These are of course directly proved by Theorem 1.4(3). Taking \( z = 0 \) in Theorem 1.4(3), we have \( A(t, 0) = A(0) = E(0) \) and \( B(t, 0) = 0 \) for each \( t < \tau \). Therefore, if \( E(0) = A(0) \neq 0 \) and (K5) holds for \( \sigma = \infty \), \( j(t; 0, z) = (\pi z E(z))^{-1}B(t, z) \to 0 \) as \( t \to \infty \) by Theorem 1.5(2). Hence \( B(t, z) \to 0 \) as \( t \to \infty \), and \( \Phi(t, x) \to 0 \) as \( t \to \infty \). This fact would be interesting if we recall the following J. Wirth’s results. He studied the Cauchy problem for a damped wave equation \( u_{tt} - u_{xx} + bu_t = 0 \), \( u(0, \cdot) = u_1, u_t(0, \cdot) = u_2 \) with positive time-depending dissipation \( b = b(t) \). He showed that if \( tb(t) \to \infty \) as \( t \to \infty \), \( 1/b \in L^1(0, \infty), u_1 \in W^{s,2} \), and \( u_2 \in W^{s-1,2} \), the solution \( u(t, x) \) tends in \( W^{s,2} \) to a real analytic function \( u(\infty, x) = \lim_{t \to \infty} u(t, x) \), which is nonzero except at most one \( u_2 \) for each \( u_1 \), where \( W^{s,2} \) is the Sobolev space on \( (0, \infty) \) (22 p. 76, Result 3).

From the result of Wirth, it is naturally asked whether \( A(t, z) \) and \( \Psi(t, z) \) tend to functions as \( t \to \infty \). This problem is also interesting from the viewpoint of the so-called connection formula for solutions of canonical systems:

\[
\begin{bmatrix}
A(t, z) \\
B(t, z)
\end{bmatrix} = M(t, s; z) \begin{bmatrix}
A(s, z) \\
B(s, z)
\end{bmatrix},
\]

but nothing is known about the behavior of \( A(t, z) \) or \( \Phi(t, z) \) when \( t \to \infty \).
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