LARGE TIME BEHAVIOR OF SOLUTIONS TO THE 3D ANISOTROPIC NAVIER-STOKES EQUATION

MIKIHIRO FUJII

Abstract. We consider the large time behavior of the solution to the 3D Navier-Stokes equation with horizontal viscosity \( \Delta_h u = \partial_t^\alpha u + \Delta u \) and show that the \( L^p \) decay rate of the horizontal components of the velocity field coincides to that of the 2D heat kernel, while the vertical component decays like the 3D heat kernel. Moreover, we consider the asymptotic expansion of the solution and find that a portion of the nonlinear term affect the leading term of the horizontal components of the velocity field, whereas the leading term of the vertical component is given by only the linear solution.

1. Introduction

In this paper, we consider the initial value problem for the 3D anisotropic Navier-Stokes equation:

\[
\begin{aligned}
\partial_t u - \Delta_h u + (u \cdot \nabla)u + \nabla p &= 0, & t > 0, x \in \mathbb{R}^3, \\
\nabla \cdot u &= 0, & t \geq 0, x \in \mathbb{R}^3, \\
u(0, x) &= u_0(x), & x \in \mathbb{R}^3.
\end{aligned}
\]

Here, \( u = (u_1(t, x), u_2(t, x), u_3(t, x)) \) and \( p = p(t, x) \) denote the unknown velocity and the unknown pressure of the fluid, respectively, while the vector field \( u_0 = (u_{0,1}(x), u_{0,2}(x), u_{0,3}(x)) \) is the given divergence free initial velocity of the fluid. The operator \( \Delta_h := \partial_1^2 + \partial_2^2 \) denotes the horizontal Laplacian and \( \nabla = (\partial_1, \partial_2, \partial_3) \) represents the 3D gradient. Throughout this paper, for given 3D vector \( a = (a_1, a_2, a_3) \in \mathbb{R}^3 \), we write \( a_h = (a_1, a_2) \). We also denote by \( \nabla_h = (\partial_1, \partial_2) \) the horizontal gradient.

In geophysical fluid dynamics, meteorologists modelize the turbulent diffusion with anisotropic viscosity \(-\nu \Delta_h - \varepsilon \partial_3^2\), where the horizontal kinetic viscosity coefficient \( \nu \) and the vertical kinetic viscosity coefficient \( \varepsilon \) satisfy \( 0 < \varepsilon \ll \nu \). We refer to [19, Chapter 4] for the complete discussion for the physical background. In this article, we are concerned with the system \((\ref{system})\), which corresponds to the case \( \nu = 1 \) and \( \varepsilon = 0 \).

The aim of this paper is to reveal the anisotropic effect for the large time behavior of the solution \( u \) to \((\ref{system})\) and we derive the \( L^p \) decay rate and the asymptotic expansion of the solution. More precisely, for \( s \in \mathbb{N} \) with \( s \geq 5 \) and for sufficiently small initial data \( u_0 \in H^s(\mathbb{R}^3) \cap L^1(\mathbb{R}^2; (W^{1,1} \cap W^{1,\infty})(\mathbb{R}^3)) \) with \( \nabla \cdot u_0 = 0 \), the solution \( u \) to \((\ref{system})\) satisfies

\[
\|\nabla^\alpha u_h(t)\|_{L^p} = O(t^{-1(1-\frac{1}{p})-\frac{1}{2}|\alpha|}), \quad \|\nabla_h^\alpha u_3(t)\|_{L^p} = O(t^{-\frac{1}{2}(1-\frac{1}{p})-\frac{1}{2}|\alpha|})
\]

as \( t \to \infty \) for \( 1 \leq p \leq \infty \) and \( \alpha = (\alpha_h, \alpha_3) \in (\mathbb{N} \cup \{0\})^2 \times (\mathbb{N} \cup \{0\}) \) with \( |\alpha| \leq 1 \). This implies that \( u_h(t) \) decays like the 2D heat kernel, whereas \( u_3(t) \) decays as the 3D heat kernel. Moreover, we shall show that \( u_h(t) \) and \( u_3(t) \) behave as

\[
u_h(t, x) = G_h(t, x_h) \int_{\mathbb{R}^2} u_{0, h}(y_h, x_3) dy_h
\]

\[\quad - G_h(t, x_h) \int_0^\infty \int_{\mathbb{R}^2} \partial_3(u_3 u_h)(\tau, y_h, x_3) dy_h d\tau + o(t^{1(1-\frac{1}{p})}) \text{ in } L^p(\mathbb{R}^3) \quad (1 \leq p \leq \infty),\]

\[
u_3(t, x) = G_h(t, x_h) \int_{\mathbb{R}^2} u_{0, 3}(y_h, x_3) dy_h + o(t^{-\frac{1}{2}(1-\frac{1}{p})}) \text{ in } L^p(\mathbb{R}^3) \quad (1 \leq p < \infty),\]

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as $t \to \infty$, where $G_1(t, x_h)$ denotes the 2D Gaussian. Furthermore, we also prove that if, in addition, $s \geq 9$ and $|x_h| u_0(x) \in L^1(\mathbb{R}^2_h; (L^1 \cap L^\infty)(\mathbb{R}^3_x))$, then the remainder terms $o(t^{-\frac{s-1}{2}})$ and $o(t^{\frac{1}{2}} (1 - \frac{1}{s}))$ of the above asymptotic expansions are improved to $O(t^{-(1 - \frac{1}{2}) - \frac{\alpha}{2}} \log t)$ and $O(t^{\frac{1}{2}} (1 - \frac{1}{s}) - \frac{\alpha}{2})$ for $1 < p \leq \infty$, respectively and also we obtain the higher order expansion of the vertical component of the solution.

Before we state our main theorems, let us recall known results related to the system (1.1). For the well-posedness of (1.1), Chemin, Desjardins, Gallagher and Grenier [2] proved the existence of a local solution for large data and a global solution for small data in $H^{0*,s}(\mathbb{R}^3)$ ($s > 1/2$), where

$$H^{0*,s}(\mathbb{R}^3) := L^2(\mathbb{R}^3) \cap \dot{H}^{0*,s}(\mathbb{R}^3), \quad \dot{H}^{0*,s}(\mathbb{R}^3) := (-\Delta_h)^{\frac{1}{2}} (\partial_3^2)^{\frac{1}{2}} L^2(\mathbb{R}^3).$$

The uniqueness of solutions in $H^{0*,s}(\mathbb{R}^3)$ ($s > 1/2$) is proved by Itimie [8]. Here, the regularity condition $s > 1/2$ is caused by the Sobolev embedding $H^s(\mathbb{R}^3) \hookrightarrow L^\infty(\mathbb{R}^3)$ and $s = 1/2$ corresponds to the scaling critical exponent. Paicu [17] considered the scaling critical setting and proved the global existence of a unique solution in the $L^2$-based anisotropic Besov space $B^{0,1}_p(\mathbb{R}^3)$. Chemin and Zhang [4] and Zhang and Fang [22] extended the Paicu theorem to the $L^p$ framework and proved the global existence results in the scaling critical anisotropic Besov space $B^{-\frac{1+\frac{1}{p}}{2}}_p(\mathbb{R}^3)$ ($2 \leq p < \infty$). We refer to [11][18][21][22] for other literatures on the well-posedness for (1.1).

Next, we focus on the previous studies for the large time behavior. Ji, Wu and Yang [11] proved that for given small initial data $u_0 \in H^1(\mathbb{R}^3) \cap H^{-\sigma,1}(\mathbb{R}^3)$ ($3/4 \leq \sigma < 1$), the global solution $u$ of (1.1) satisfies

$$\|u(t)\|_{H^{4+\gamma,1}} \leq C \|u_0\|_{H^{4+\gamma,1}}, \quad \|\nabla^\alpha u(t)\|_{L^2} \leq C(1 + t)^{-\frac{\alpha_1 |\alpha|}{2}},$$

for $t \geq 0$ and $\alpha = (\alpha_h, \alpha_3) \in (\mathbb{N} \cup \{0\})^2 \times (\mathbb{N} \cup \{0\})$ with $|\alpha| \leq 1$. The decay rate of (1.2) coincides to that of the linear solution $e^{it\Delta_h}u_0$. Xu and Zhang [20] relaxed the regularity condition of (1.1) and showed that for $s > 2$ and $(1 + 3s)/\{10(s - 1)\} < \sigma < 1$, the solution $u$ of (1.1) with given small initial data $u_0 \in (\dot{H}^{0,s}, \dot{H}^{-\sigma,0} \cap \dot{H}^{-\sigma,-\frac{s}{2}} \times \dot{H}^{-\sigma,-\frac{s}{2}} \cap \dot{H}^{-\sigma,-1})(\mathbb{R}^3)$ satisfies that for $\alpha_h \in (\mathbb{N} \cup \{0\})^2$ with $|\alpha_h| \leq 1$,

$$\|\nabla_{h}^{\alpha_h} u(t)\|_{L^2} = O(t^{-\frac{\alpha_1 |\alpha_h|}{2}}), \quad \|\partial_3 u(t)\|_{L^2} = O(t^{-\frac{1}{4}}),$$

$$\|\nabla_{h}^{\alpha_h} u_3(t)\|_{L^2} = O(t^{-\frac{1}{2}(\frac{3}{4} + 1 + |\alpha_h|)})$$

as $t \to \infty$. This implies that the horizontal components decay like the 2D heat kernel and the vertical component decays as the 3D heat kernel. We refer to [3][6][7][13] for the large time behavior of isotropic Navier-Stokes equations.

The purpose of this paper is to refine the results obtained by [11][20] and to clarify the effect of the anisotropy on the large time behavior of the solution in terms of $L^p$ decay rates and asymptotic expansions.

In order to state our results precisely, we prepare some notation. For $s \in \mathbb{N}$, we define a function space $X^s(\mathbb{R}^3)$ by

$$X^s(\mathbb{R}^3) := H^s(\mathbb{R}^3) \cap L^1(\mathbb{R}^2_h; (W^{1,1} \cap W^{1,\infty})(\mathbb{R}^3_x)).$$

Let $G_1(t, x_h)$ be the 2D Gaussian:

$$G_1(t, x_h) := (4\pi t)^{-1} e^{-\frac{|x_h|^2}{4t}} , \quad (t, x_h) = (t, x_1, x_2) \in (0, \infty) \times \mathbb{R}^2.$$

Our first main result reads as follows.

**Theorem 1.1.** Let $s \in \mathbb{N}$ satisfy $s \geq 5$. Then, there exists a positive constant $\delta_1 = \delta_1(s)$ such that the following properties hold:

For any $u_0 \in X^s(\mathbb{R}^3)$ satisfying $\nabla \cdot u_0 = 0$ and $\|u_0\|_{X^s} \leq \delta_1$, there exists a unique solution $u \in C([0, \infty); X^s(\mathbb{R}^3))$ of (1.7) and there exists an absolute positive constant $C$ such that

$$\|\nabla_{h}^{\alpha} u(t)\|_{L^p} \leq C t^{-\frac{\alpha_1 |\alpha|}{2}} \|u_0\|_{X^s},$$

$$\|\nabla_{h}^{\alpha} u_3(t)\|_{L^p} \leq C t^{\frac{1}{2}(1 - \frac{s}{2}) - \frac{\alpha_1 |\alpha|}{2}} \|u_0\|_{X^s}$$

for all $1 \leq p \leq \infty$, $t > 0$ and $\alpha = (\alpha_h, \alpha_3) \in (\mathbb{N} \cup \{0\})^2 \times (\mathbb{N} \cup \{0\})$ with $|\alpha| \leq 1$. 

Moreover, there hold for \( 1 \leq p \leq \infty \),
\[
\lim_{t \to \infty} t^{\frac{1}{p} - \frac{1}{p}} \left\| u_h(t, x) - G_h(t, x_h) \int_{\mathbb{R}^2} u_{0, h}(y_h, x_3) dy_h + G_h(t, x_h) \int_0^\infty \int_{\mathbb{R}^2} \partial_3(u_3 u_h)(\tau, y_h, x_3) dy_h d\tau \right\|_{L^p_x} = 0
\] (1.6)
and for \( 1 \leq p < \infty \),
\[
\lim_{t \to \infty} t^{\frac{1}{2} \left(1 - \frac{1}{p}\right)} \left\| u_3(t, x) - G_h(t, x_h) \int_{\mathbb{R}^2} u_{0, 3}(y_h, x_3) dy_h \right\|_{L^p_x} = 0.
\] (1.7)

Remark 1.2.
1. The decay rate of (1.4) and (1.5) with \( p = 2 \) is same as that of the \( L^2 \)-norm of the 2D Gaussian, which corresponds to the limiting case \( \sigma = 1 \) in (1.2) and (1.3). Thus, our result extends the previous studies \([11, 20]\).
2. Unlike isotropic nonlinear parabolic-type equations \([5, 6, 9, 10, 12, 15, 16]\), (1.6) implies the leading term of \( u_h(t) \) cannot be given by only the linear solution, and the nonlinearity affects the leading term.
3. The asymptotic limit (1.7) fails if \( p = \infty \). Indeed, there exists an initial data such that the limit (1.7) with \( p = \infty \) does not converge to 0. These facts are precisely stated in our second main result and its corollary.

If we suppose the spatial decay assumption on the initial data, then we obtain the convergence rate for the above limits (1.6)-(1.7) and the higher order asymptotic expansion of the vertical component of the velocity field. The following theorem is our second main result.

Theorem 1.3. Let \( s \in \mathbb{N} \) satisfy \( s \geq 9 \). Then, there exists a positive constant \( \delta_2 = \delta_2(s) \leq \delta_1(s) \) such that if \( u_0 \in X^s(\mathbb{R}^3) \) satisfies \( |x|^s |u_0(x)| \in L^1(\mathbb{R}^2, (L^1 \cap L^\infty)(\mathbb{R}_x)) \), \( \nabla \cdot u_0 = 0 \) and \( \| u_0 \|_{X^s} \leq \delta_2 \), then for \( 1 < p \leq \infty \) there exists a positive constant \( C = C(p) \) such that
\[
\left\| u_h(t, x) - G_h(t, x_h) \int_{\mathbb{R}^2} u_{0, h}(y_h, x_3) dy_h + G_h(t, x_h) \int_0^\infty \int_{\mathbb{R}^2} \partial_3(u_3 u_h)(\tau, y_h, x_3) dy_h d\tau \right\|_{L^p_x} \leq C \| u_0 \|_{X^s} t^{-\left(1 - \frac{1}{p}\right) - \frac{1}{2}} \log t
\] (1.8)
for \( t \geq 2 \) and for \( 1 \leq p \leq \infty \) there exists a positive constant \( C = C(p) \) such that
\[
\left\| u_3(t, x) - G_h(t, x_h) \int_{\mathbb{R}^2} u_{0, 3}(y_h, x_3) dy_h \right\|_{L^p_x} \leq C \| u_0 \|_{X^s} t^{-\left(1 - \frac{1}{p}\right) - \frac{1}{2}} \log t \quad (1 < p \leq \infty)
\] (1.9)
for all \( t \geq 2 \), where \( \| u_0 \|_{X^s} := \| u_0 \|_{X^s} + \| x_h |u_0(x)| \|_{L^1(\mathbb{R}^2, (L^1 \cap L^\infty)(\mathbb{R}_x))} \).

Furthermore, for any \( 1 < p \leq \infty \), it holds
\[
\lim_{t \to \infty} t^{\frac{1}{2} \left(1 - \frac{1}{p}\right) + \frac{1}{2}} \left\| u_3(t, x) - G_h(t, x_h) \int_{\mathbb{R}^2} u_{0, 3}(y_h, x_3) dy_h + \nabla_h G_h(t, x_h) \cdot \int_{\mathbb{R}^2} y_h u_0 u_3(y_h, x_3) dy_h - \nabla_h G_h(t, x_h) \int_0^\infty \int_{\mathbb{R}^2} (u_3 u_h)(\tau, y_h, x_3) dy_h d\tau \right\|_{L^p_x} = 0
\] (1.10)

Remark 1.4.
1. (1) The regularity assumption \( s \geq 9 \) is necessary only for the proof of (1.8). It is possible to prove (1.9) and (1.10) under the weaker assumption \( s \geq 5 \), which is the same regularity condition as in Theorem 1.1.
2. By \( t^{-\left(1 - \frac{1}{2}\right) - \frac{1}{2}} = t^{-\left(1 - \frac{1}{2}\right) - \frac{1}{2}} \), (1.4) and (1.10), we see that the second leading term of \( u_3(t) \) decays as the 2D heat kernel with the horizontal gradient \( \nabla_h e^t \Delta_h \), while the leading term of \( u_3(t) \) decays like the 3D heat kernel \( e^t \Delta \).
3. The estimate (1.8) with \( p = 1 \) holds if we additionally assume \( \nabla^a u_0 \in L^\infty(\mathbb{R}_x) \) \(|a| \leq 1\). See Remarks 1.6 and 6.2.
4. By Proposition 2.2 and Lemma 4.2 below, the nonlinear effect in the asymptotic expansion (1.10) appears only from the first Duhamel term \( \mathcal{D}_1[u] \) of \( u_3(t) \). It follows from Lemma 4.2 below that \( L^1 \) decay rates of all nonlinear terms \( \mathcal{D}_m[u](t) \) \((m = 1, 2, 3)\) of \( u_3(t) \) are equal except for \( \log t \). Therefore, in order to accurately approximate the approximation (1.10) with \( p = 1 \), we need not only information for \( \mathcal{D}_1[u](t) \) but information for all \( \mathcal{D}_m[u](t) \) \((m = 1, 2, 3)\).
As a corollary of (1.10), we see that the limit (1.7) fails if \( p = \infty \).

**Corollary 1.5.** There exists a suitable initial data \( u_0 \in \mathcal{S}(\mathbb{R}^3) \) such that the corresponding solution \( u \) satisfies

\[
\liminf_{t \to \infty} t^{\frac{4}{3}} \left\| u_3(t, x) - G_h(t, x_h) \right\|_{L^\infty(v)} > 0.
\]

This paper is organized as follows. In Section 2, we give a decomposition of the integral equation corresponding to (1.1), which is the key ingredient of our analysis. In Section 3, we prepare linear estimates. Nonlinear decay estimates are established in Section 4. In Section 5, we present the proof of Theorem 1.1. In Section 6, we establish the additional estimates for the solution. Finally, in Section 7, we prove Theorem 1.2 and Corollary 1.3.

**Notation.** At the end of this section, we summarize notation used in this paper. For given 3D vector \( a = (a_1, a_2, a_3) \in \mathbb{R}^3 \), we denote by \( a_h = (a_1, a_2) \) the horizontal components of \( a \) and \( a_3 \) is called the vertical component. The horizontal Laplacian and \( \nabla_h = (\partial_1, \partial_2) \) represents the horizontal gradient. For \( s \in \mathbb{N} \), we define a function space \( X^s(\mathbb{R}^3) \) by

\[
X^s(\mathbb{R}^3) := H^s(\mathbb{R}^3) \cap L^1(\mathbb{R}^2_+; (W^{1,1} \cap W^{1,\infty})(\mathbb{R}^3_x)).
\]

Let \( G_h(t, x_h) \) be the 2D Gaussian:

\[
G_h(t, x_h) := (4\pi t)^{-1} e^{-\frac{|x_h|^2}{4t}}, \quad (t, x_h) = (t, x_1, x_2) \in (0, \infty) \times \mathbb{R}^2.
\]

For \( 1 \leq p, q \leq \infty \), we define the anisotropic Lebesgue space by

\[
L^p_h L^q(\mathbb{R}^3) := L^p(\mathbb{R}^2_+; L^q(\mathbb{R}^3_x)).
\]

2. **INTEGRAL EQUATIONS**

In this section, we consider the integral equation corresponding to (1.1):

\[
u(t) = e^{t\Delta_h} u_0 - \int_0^t e^{(t-\tau)\Delta_h} \mathbb{P} \nabla \cdot (u \otimes u)(\tau) d\tau,
\]

(2.1)

where \( \mathbb{P} = (\delta_{kl} + R_k R_l)_{1 \leq k, l \leq 3} \) denotes the Helmholtz projection and \( \{ R_k \}_{k=1}^3 \) represents the 3D Riesz transform. The unboundedness of the operator \( \mathbb{P} \) on \( L^1(\mathbb{R}^3) \) and \( L^\infty(\mathbb{R}^3) \) prevents us from calculating \( \| u(t) \|_{L^1} \) and \( \| u(t) \|_{L^\infty} \) via the integral equation (2.1). To overcome this, we follow the idea of \( \mathbb{P} \) and decompose the nonlinear term as follows. The \( j \)-th component of \( e^{(t-\tau)\Delta_h} \mathbb{P} \nabla \cdot (u \otimes u)(\tau) \) is given by

\[
\left( e^{(t-\tau)\Delta_h} \mathbb{P} \nabla \cdot (u \otimes u)(\tau) \right)_j
\]

\[
= \sum_{k=1}^3 e^{t-\tau)\Delta_h} \partial_k (u_k u_j)(\tau) + \sum_{k,l=1}^3 \partial_j \partial_k \partial_l \mathcal{F}^{-1}_{\mathbb{R}^3} \left[ |\xi|^2 e^{-|t-\tau| |\xi|} \right] (u_k u_l)(\tau)
\]

\[
= \sum_{k=1}^3 e^{t-\tau)\Delta_h} \partial_k (u_k u_j)(\tau) + \sum_{k,l=1}^2 \partial_j \partial_k \partial_l K(t-\tau) \cdot (u_k u_l)(\tau)
\]

\[
+ 2 \sum_{k=1}^2 \partial_j \partial_k \partial_l K(t-\tau) \cdot (u_k u_l)(\tau) + \partial_j \partial_l \partial_3 K(t-\tau) \cdot (u_3(\tau)^2)
\]

(2.2)

where \( K(t, x) := \mathcal{F}^{-1}_{\mathbb{R}^3} \left[ |\xi|^2 e^{-|t-\tau| |\xi|} \right] (x) \). Let us derive the explicit formula for the function \( K(t, x) \).

**Lemma 2.1.** The following formula holds:

\[
K(t, x) = \mathcal{F}^{-1}_{\mathbb{R}^3} \left[ \frac{1}{2|\xi_h|} e^{-|\xi_h| \cdot |\xi_h|} e^{-|\xi_h| |x|} \right] (x)
\]
where \(G_v(s, x_3)\) is the 1D Gaussian.

Proof. Using the formula
\[
\int_{\mathbb{R}} \frac{e^{ix_3 \xi_3}}{\xi_3^2 + |\xi_h|^2} d\xi_3 = \frac{\pi}{|\xi_h|} e^{-|\xi_h||x_3|},
\]
we have
\[
K(t, x) = \frac{1}{(2\pi)^3} \int_{\mathbb{R}^2} e^{ix_k \cdot \xi_k} e^{-t|\xi_k|^2} \int_{\mathbb{R}} \frac{e^{ix_3 \xi_3}}{\xi_3^2 + |\xi_h|^2} d\xi_3 d\xi_h
\]
\[
= \frac{1}{2(2\pi)^2} \int_{\mathbb{R}^2} e^{ix_k \cdot \xi_k} \frac{1}{|\xi_h|} e^{-t|\xi_h|^2} e^{-|\xi_h||x_3|} d\xi_h
\]
\[
= \mathcal{F}^{-1}_{\mathbb{R}^2} \left[ \frac{1}{2|\xi_h|} e^{-t|\xi_h|^2} e^{-|\xi_h||x_3|} \right](x_h).
\]

On the other hand, we have
\[
K(t, x) = \frac{1}{(2\pi)^3} \int_{\mathbb{R}^2} e^{ix_k \cdot \xi_k} e^{-t|\xi_k|^2} \int_{\mathbb{R}} \frac{e^{ix_3 \xi_3}}{\xi_3^2 + |\xi_h|^2} d\xi_3 d\xi_h
\]
\[
= \frac{1}{(2\pi)^3} \int_{\mathbb{R}^2} e^{ix_k \cdot \xi_k} e^{-t|\xi_k|^2} \int_{\mathbb{R}} e^{ix_3 \xi_3} \int_{0}^{\infty} e^{-s|\xi_h|^2 + |x_3|^2} ds d\xi_3 d\xi_h
\]
\[
= \int_{0}^{\infty} \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} e^{ix_k \cdot \xi_k} e^{-t|\xi_k|^2} e^{-s|\xi_h|^2} \frac{1}{2\pi} \int_{\mathbb{R}} e^{ix_3 \xi_3} e^{-s\xi_h^2} d\xi_h d\xi_3 ds
\]
\[
= \int_{0}^{\infty} \mathcal{F}^{-1}_{\mathbb{R}^2} \left[ e^{-(t+s)|\xi_k|^2} \right](x_h) \mathcal{F}^{-1}_{\mathbb{R}^2} \left[ e^{-s|\xi_h|^2} \right](x_3) ds
\]
\[
= \int_{0}^{\infty} \frac{e^{-\frac{|x_3|^2}{2t+s}}}{4\pi(t+s)(4\pi s)^{\frac{3}{2}}} ds.
\]

This completes the proof. \(\square\)

By Lemma 2.1, we obtain
\[
\partial_3 K(t, x) = \mathcal{F}^{-1}_{\mathbb{R}^2} \left[ \frac{1}{2|\xi_h|} e^{-t|\xi_k|^2} \partial_3 \left( e^{-|\xi_h||x_3|} \right) \right](x_h)
\]
\[
= -\text{sgn}(x_3) \mathcal{F}^{-1}_{\mathbb{R}^2} \left[ |\xi_h| \frac{1}{2|\xi_h|} e^{-t|\xi_k|^2} e^{-|\xi_h||x_3|} \right](x_h)
\]
\[
= -\text{sgn}(x_3)(-\Delta_h)^{\frac{1}{2}} K(t, x)
\]
and
\[
\partial_3 \partial_3 K(t, x) = -\delta(x_3) \mathcal{F}^{-1}_{\mathbb{R}^2} \left[ e^{-t|\xi_k|^2} e^{-|\xi_h||x_3|} \right](x_h) - \Delta_h K(t, x),
\]
which implies
\[
\partial_3 \partial_3 K(t) * f = -e^{t\Delta_h} f - \Delta_h K(t) * f.
\]
(2.3)

Here, \(\delta(x_3)\) is the Dirac distribution on \(\mathbb{R}_x\). It follows from (2.3) and (2.4) that
\[
\partial_3 \partial_3 K(t) * f = -e^{t\Delta_h} \partial_3 f - \Delta_h \partial_3 K(t) * f
\]
\[
= -e^{t\Delta_h} \partial_3 f - \left( \text{sgn}(x_3)'(-\Delta_h)^{\frac{3}{2}} K(t, x') \right) * f.
\]
(2.5)

Hence, by (2.2), (2.3), (2.4) and (2.5), we see that
\[
\left( e^{(t-\tau)\Delta_h} \nabla \cdot (u \otimes u) (\tau) \right)_j = \partial_3 e^{(t-\tau)\Delta_h} (u_3 u_j)(\tau) + \sum_{k=1}^{2} \partial_k e^{(t-\tau)\Delta_h} (u_k u_j)(\tau)
\]
(2.6)
Proposition 2.2. Therefore, we obtain the following proposition:

\[ j \]

for \( j = 1, 2 \) and

\[
\left( e^{(t-\tau)\Delta_h} \mathbb{P} \nabla \cdot (u \otimes u) \right) = - \sum_{k=1}^{2} \partial_{x} e^{(t-\tau)\Delta_h} (u_{k} u_{3})(\tau)
\]

\[
- 2 \sum_{k=1}^{2} \left( \text{sgn}(x') \partial_{x} \partial_{x} (- \Delta_h)^{3/2} K(t-\tau, x') \right) \ast (u_{k} u_{3})(\tau)
\]

\[
\sum_{k=1}^{2} \partial_{x} e^{(t-\tau)\Delta_h} (u_{3}(\tau)^2) - \partial_{x} \Delta_h K(t-\tau) \ast (u_{3}(\tau)^2)
\]

Therefore, we obtain the following proposition:

**Proposition 2.2.** Let \( u \) be a solution to (2.1). Then, the following formula holds:

\[
\begin{cases}
  u_h(t) = e^{t \Delta_h} u_{0,h} + \sum_{m=1}^{5} \mathcal{D}^{h}_{m}[u](t), \\
  u_3(t) = e^{t \Delta_h} u_{0,3} + \sum_{m=1}^{5} \mathcal{D}^{h}_{m}[u](t),
\end{cases}
\]

(2.6)

where

\[
\mathcal{D}^{h}_{1}[u](t) := - \int_{0}^{t} e^{(t-\tau)\Delta_h} \partial_{x} (u_{3} u_{h})(\tau) d\tau,
\]

\[
\mathcal{D}^{h}_{2}[u](t) := - \int_{0}^{t} e^{(t-\tau)\Delta_h} \nabla \cdot (u_{h} \otimes u_{h})(\tau) d\tau,
\]

\[
\mathcal{D}^{h}_{3}[u](t) := \int_{0}^{t} \nabla \partial_{x} e^{(t-\tau)\Delta_h} (u_{3}(\tau)^2) d\tau,
\]

\[
\mathcal{D}^{h}_{4}[u](t) := - \sum_{k,l=1}^{2} \int_{0}^{t} \nabla \partial_{x} \partial_{x} K(t-\tau) \ast (u_{k} u_{l})(\tau) d\tau,
\]

\[
\mathcal{D}^{h}_{5}[u](t) := 2 \sum_{k=1}^{2} \int_{0}^{t} \nabla \partial_{x} (- \Delta_h)^{3/2} \tilde{K}(t-\tau) \ast (u_{3} u_{k})(\tau) d\tau + \int_{0}^{t} \nabla \partial_{x} K(t-\tau) \ast (u_{3}(\tau)^2) d\tau,
\]

and

\[
\mathcal{D}^{h}_{6}[u](t) := \int_{0}^{t} e^{(t-\tau)\Delta_h} \nabla \cdot (u_{3} u_{h})(\tau) d\tau,
\]

\[
\mathcal{D}^{h}_{7}[u](t) := 2 \sum_{k,l=1}^{2} \int_{0}^{t} (- \Delta_h)^{3/2} \partial_{x} \partial_{x} \tilde{K}(t-\tau) \ast (u_{k} u_{l})(\tau) d\tau,
\]

\[
\mathcal{D}^{h}_{8}[u](t) := 2 \sum_{k=1}^{2} \int_{0}^{t} \partial_{x} \partial_{x} K(t-\tau) \ast (u_{3} u_{k})(\tau) d\tau + \int_{0}^{t} (- \Delta_h)^{3/2} \tilde{K}(t-\tau) \ast (u_{3}(\tau)^2) d\tau.
\]

Here, \( K(t, x) \) and \( \tilde{K}(t, x) \) are the functions with the following representaions:

\[
K(t, x) = \int_{0}^{\infty} \frac{e^{-\frac{x^2}{4(t+s)}}}{4\pi(t+s)} \frac{e^{-\frac{s^2}{4t(s)}}}{(4\pi s)^{2}} ds = \int_{0}^{\infty} G_h(t+s,x)G_v(s,x_3)ds,
\]

\[
\tilde{K}(t, x) = \text{sgn}(x_3)K(t, x).
\]
In this section, we prepare some linear estimates. We start with the classical properties of heat kernels.

**Lemma 3.1.**

1. For each $\alpha = (\alpha_h, \alpha_3) \in (\mathbb{N} \cup \{0\})^2 \times (\mathbb{N} \cup \{0\})$ and $m \in \mathbb{N} \cup \{0\}$, there exists a positive constant $C = C(\alpha, m)$ such that
   \[
   \| x h \|^m \nabla_h^{\alpha_h} G_h(t, x_h) \|_{L^p(\mathbb{R}^3_h)} \leq Ct^{-\left(1 - \frac{1}{p}\right) - \frac{|\alpha_h|}{2}}
   \]
   for all $1 \leq p \leq \infty$ and $t > 0$. In particular, we have
   \[
   \| \nabla_h e^{t\Delta_h} f \|_{L^p_h L^q_x} \leq Ct^{-\left(1 - \frac{1}{p}\right) - \frac{|\alpha_h|}{2}} \| \partial_{\alpha_h}^3 g \|_{L^p_h L^q_x}
   \]
   for all $t > 0$, $1 \leq p_1 \leq p_2 \leq \infty$, $1 \leq q \leq \infty$ and all functions $f$ satisfying $\partial_{\alpha_h}^3 g \in L^p_h L^q_x(\mathbb{R}^3)$.

2. Let $1 \leq p, q \leq \infty$ and $m = 0, 1$. Then, for any function $f$ satisfying $|x h |^m f(x) \in L^1(\mathbb{R}^2_h; L^1(\mathbb{R}^3))$, it holds
   \[
   \lim_{t \to \infty} t^{-\left(1 - \frac{1}{p}\right) + \frac{m}{q}} \left\| e^{t\Delta_h} f(x) - \sum_{|\alpha_h| \leq m} \nabla_h^{\alpha_h} G_h(t, x_h) \int_{\mathbb{R}^2} (-y_h)^{\alpha_h} f(y_h) dy_h \right\|_{L^p_h L^q_x} = 0.
   \]

3. There exists a positive constant $C$ such that
   \[
   \left\| e^{t\Delta_h} f(x) - G_h(t, x_h) \int_{\mathbb{R}^2} f(y_h, x_3) dy_h \right\|_{L^p_h L^q_x} \leq C t^{-\left(1 - \frac{1}{p}\right) - \frac{m}{2}} \| x h \|_{L^1(\mathbb{R}^3)}
   \]
   for all $1 \leq p, q \leq \infty$ and all functions $f$ satisfying $|x h |^m f(x) \in L^1(\mathbb{R}^3)$.

We omit the proof of Lemma 3.1 since it is quite standard. Next, we focus on the enhanced dissipation for $e^{t\Delta_h} u_{0,3}$.

**Lemma 3.2.**

1. There exists an absolute positive constant $C$ such that
   \[
   \| e^{t\Delta_h} u_{0,3} \|_{L^p_h L^q_x} \leq C t^{-\left(1 - \frac{1}{p}\right) - \frac{1}{2}(1 - \frac{1}{q})} \| u_0 \|_{L^1}
   \]  
   (3.1)
   for all $1 \leq p, q \leq \infty$, $t > 0$ and $u_0 = (u_{0,1}, u_{0,2}, u_{0,3}) \in L^1(\mathbb{R}^3)$ satisfying $\nabla \cdot u_0 = 0$.

   Moreover, it holds
   \[
   \| x h e^{t\Delta_h} u_{0,3} \|_{L^1_t L^\infty_x} \leq C (1 + |x h|) \| u_0 \|_{L^1}
   \]  
   (3.2)
   for all $t > 0$ and $u_0 = (u_{0,1}, u_{0,2}, u_{0,3}) \in L^1(\mathbb{R}^3)$ satisfying $\nabla \cdot u_0 = 0$ and $|x h| u_0 \in L^1(\mathbb{R}^3)$.

2. There exists an absolute positive constant $C$ such that for $1 \leq p \leq \infty$, $m = 0, 1$ and any function $u_0 = (u_{0,1}, u_{0,2}, u_{0,3}) \in L^1(\mathbb{R}^3)$ satisfying $\nabla \cdot u_0 = 0$ and $|x h| u_0 \in L^1(\mathbb{R}^3)$, there exists a nonnegative function $R_{p,m}(t)$ such that $R_{p,m}(t) \to 0$ as $t \to \infty$ and
   \[
   \left\| e^{t\Delta_h} u_{0,3} - \sum_{|\alpha_h| \leq m} \nabla_h^{\alpha_h} G_h(t, x_h) \int_{\mathbb{R}^2} (-y_h)^{\alpha_h} u_{0,3}(y_h, x_3) dy_h \right\|_{L^p} \leq C t^{-\frac{1}{2}(1 - \frac{1}{p}) - \frac{1}{2}} R_{p,m}(t)^{\frac{1}{2}} \| x h |^m u_0(x) \|_{L^1}^{1 - \frac{1}{2}}
   \]  
   (3.3)
   for all $t > 0$. 

Proof. We first prove (3.1). Using \(\partial_3 u_0 = -\nabla_h \cdot u_{0,h}\), we obtain that
\[
\|e^{t \Delta_h} u_{0,3}\|_{L^\infty_t L^1_y} \leq \|e^{t \Delta_h} u_{0,3}\|_{L^1_t L^\infty_y} \|e^{t \Delta_h} u_{0,3}\|_{L^1_t L^\infty_y}^{1-1/\beta} 
\leq C \left( t^{-(1-1/\beta)} \|u_{0,3}\|_{L^1_y} \right)^{1/\beta} \|e^{t \Delta_h} \partial_3 u_{0,3}\|_{L^1_t L^1_y}^{1-1/\beta} 
= Ct^{-1/\beta} \|u_{0,3}\|_{L^1_y} \|e^{t \Delta_h} \nabla_h \cdot u_{0.h}\|_{L^1_t L^1_y}^{1-1/\beta} 
\leq Ct^{-1/\beta} \|u_{0,3}\|_{L^1_y} \left( t^{-(1-1/\beta)-1/\beta} \|u_{0,h}\|_{L^1_y} \right)^{1-1/\beta} 
\leq Ct^{-1/\beta} \|u_{0}\|_{L^1_t}.
\]
For the proof of (3.2), we see by Lemma 3.1 that
\[
\|x_h e^{t \Delta_h} u_{0,3}(x)\|_{L^1_t L^\infty_y} \leq \|x_h e^{t \Delta_h} \partial_3 u_{0,3}(x)\|_{L^1_t} 
= \|x_h e^{t \Delta_h} \nabla_h \cdot u_{0,h}(x)\|_{L^1_t} 
\leq C \|\nabla_h \cdot u_{0,h}(x)\|_{L^1_t}.
\]
Next, we show (3.3). Let
\[
F_m(t, x) := e^{t \Delta_h} u_{0,3}(x) - \sum_{|\alpha_h| \leq m} \nabla_h^{\alpha_h} G_h(t, x_h) \int_{\mathbb{R}^2} (-y_h)^{\alpha_h} u_{0,3}(y_h, x_3) dy_h.
\]
Then, we see that
\[
\|F_m(t)\|_{L^p_t} \leq \|F_m(t)\|_{L^p_t L^1_y} \|F_m(t)\|_{L^1_t L^\infty_y}^{1-1/\beta} \leq \|F_m(t)\|_{L^p_t L^1_y} \|\partial_3 F_m(t)\|_{L^1_t L^\infty_y}^{1-1/\beta}.
\]
(3.4)
Here, by Lemma 3.1 (2), we see that
\[
\|F_m(t)\|_{L^p_t L^1_y} = t^{-1/\beta} R_{p, m}(t), \quad \lim_{t \to \infty} R_{p, m}(t) = 0.
\]
(3.5)
For the case (3.3) \(m = 0\), it follows from the divergence free condition on \(u_0\) and integration by parts that
\[
\partial_3 \int_{\mathbb{R}^2} u_{0,3}(y_h, x_3) dy_h = -\int_{\mathbb{R}^2} \nabla_h \cdot u_{0,h}(y_h, x_3) dy_h = 0,
\]
(3.6)
\[
\partial_3 e^{t \Delta_h} u_{0,3}(x) = -\int_{\mathbb{R}^2} \nabla_h G_h(t, x_h - y_h) \cdot u_{0,h}(y_h) dy_h.
\]
(3.7)
Thus, we have
\[
\partial_3 F_0(t, x) = -\int_{\mathbb{R}^2} (\nabla_h G_h)(t, x_h - y_h) \cdot u_{0,h}(y_h, x_3) dy_h,
\]
which implies
\[
\|\partial_3 F_0(t, x)\|_{L^p_t L^1_y} \leq \|\nabla_h G_h(t)\|_{L^p(\mathbb{R}^2)} \|u_{0,h}(\cdot, x_3)\|_{L^1_y} 
\leq C t^{-1/\beta} \|u_{0}\|_{L^1_t}.
\]
(3.8)
Hence, we obtain by (3.4), (3.3) and (3.8) that
\[
\|F_0(t)\|_{L^p_t} \leq C t^{-1/\beta} \|u_{0}\|_{L^1_t}.
\]
This gives (3.3) with \(m = 0\). For the case \(m = 1\), it holds by the integration by parts and the divergence free condition that
\[
\int_{\mathbb{R}^2} y_h \partial_3 u_{0,3}(y_h, x_3) dy_h = -\int_{\mathbb{R}^2} y_h \nabla_h \cdot u_{0,h}(y_h, x_3) dy_h = \int_{\mathbb{R}^2} u_{0,k}(y_h, x_3) dy_h.
\]
(3.9)
Using (3.6), (3.7) and (3.9), we have
\[
\partial_3 F_1(t, x) = -\int_{\mathbb{R}^2} \{\nabla_h G_h(t, x_h - y_h) - (\nabla_h G_h(t, x_h)) \cdot u_{0,h}(y_h, x_3) dy_h 
= \int_{\mathbb{R}^2} \int_{0}^{1} (\nabla_h^2 G_h)(t, x_h - t y_h) y_h \cdot u_{0,h}(y_h, x_3) d\theta dy_h.
\]
Taking $L^p_h L^q_v$-norm, we see that
\[ \| \partial_3 F_1(t) \|_{L^p_h L^q_v} \leq C t^{-(1 - \frac{1}{p}) - 1} \| x_h | u_0(x) \|_{L^1}. \]
Hence, we obtain
\[ \| F_1(t) \|_{LP} \leq C t^{-\frac{1}{2} (1 - \frac{1}{p}) - \frac{1}{2} R_{p,1}(t)^{-\frac{1}{2}}} \| x_h | u_0(x) \|_{L^1}^{1 - \frac{1}{p}}, \]
which completes the proof.

Next, we recall the function
\[ K(t, x) = \int_0^\infty e^{-\frac{|s|^2}{4(t + s)}} e^{-\frac{s^2}{4(t + s)^2}} ds \]
\[ = \int_0^\infty G_h(t + s, x_h) G_v(s, x_3) ds, \]
which is defined in Section 2 (see Lemma 2.1). In the following lemma, we state the decay rate for the derivative of $K(t)$.

**Lemma 3.3.** Let $1 \leq p, q \leq \infty$, $(\beta, \gamma) \in (\mathbb{N} \cup \{0\})^2 \times (\mathbb{N} \cup \{0\})$ and $m \in \mathbb{N} \cup \{0\}$ satisfy
\[ |\beta| + \gamma > 2 \frac{1}{p} + 1 + m. \]
Then, there exists a positive constant $C = C(p, q, \beta, \gamma, m)$ such that
\[ \||x_h|^m \nabla_h^\beta (-\Delta_h)^{\frac{\gamma}{2}} K(t, x) \|_{L^p_h L^q_v} \leq C t^{-(1 - \frac{1}{p}) - \frac{1}{2} (1 - \frac{1}{p}) - \frac{|\beta| + \gamma - 2}{2}} \]
for all $t > 0$.
Moreover, if $1 \leq p_1, p_2 \leq p$ and $1 \leq q_1, q_2 \leq q$ satisfy
\[ \left( \frac{1}{p_1} - \frac{1}{p} \right) + \frac{1}{2} \left( \frac{1}{q_1} - \frac{1}{q} \right) + \frac{|\beta| + \gamma - 3}{2} > 0, \quad \left( \frac{1}{p_2} - \frac{1}{p} \right) + \frac{1}{2} \left( \frac{1}{q_2} - \frac{1}{q} \right) + \frac{|\beta| + \gamma - 2}{2} > 0, \]
then there exists a positive constant $C = C(p, p_1, p_2, q, q_1, q_2, \beta, \gamma)$ such that
\[ \||x_h|^m \nabla_h^\beta (-\Delta_h)^{\frac{\gamma}{2}} K(t) * f(x) \|_{L^p_h L^q_v} \leq C t^{-(1 - \frac{1}{p}) - \frac{1}{2} (1 - \frac{1}{p}) - \frac{|\beta| + \gamma - 3}{2}} \| f \|_{L^{p_1}_h L^{q_1}_v} \]
\[ + C t^{-(1 - \frac{1}{p}) - \frac{1}{2} (1 - \frac{1}{p}) - \frac{|\beta| + \gamma - 2}{2}} \||x_h| f(x) \|_{L^{p_2}_h L^{q_2}_v} \]
for all $t > 0$ and all functions $f$ satisfying $\partial_3^{\alpha_3} f \in L^{p_1}_h L^{q_1}_v (\mathbb{R}^3)$ and $\|x_h| \partial_3^{\alpha_3} f(x) \|_{L^{p_2}_h L^{q_2}_v (\mathbb{R}^3)}$.

**Remark 3.4.** Lemma 3.3 implies that the operator $\nabla_h^\beta (-\Delta_h)^{\frac{\gamma}{2}} K(t) *$ decays as the 3D heat kernel with the horizontal gradient of $|\beta| + \gamma - 2$-th order.

**Proof of Lemma 3.3.** From Lemma 2.1, it follows that
\[ |x_h|^m \nabla_h^\beta (-\Delta_h)^{\frac{\gamma}{2}} K(t, x) = \int_0^\infty G_v(s, x_3) |x_h|^m \nabla_h^\beta (-\Delta_h)^{\frac{\gamma}{2}} G_h(t + s, x_h) ds. \]
Therefore, we have
\[ \||x_h|^m \nabla_h^\beta (-\Delta_h)^{\frac{\gamma}{2}} K(t, x) \|_{L^p_h L^q_v} \leq \int_0^\infty \| G_v(s) \|_{L^4(\mathbb{R})} \||x_h|^m \nabla_h^\beta (-\Delta_h)^{\frac{\gamma}{2}} G_h(t + s) \|_{L^p(\mathbb{R}^2)} ds \]
\[ \leq C \int_0^\infty s^{-\frac{1}{2} (1 - \frac{1}{p})} (t + s)^{-1} (-\frac{1}{2} - \frac{|\beta| + \gamma - 2}{2}) ds \]
\[ = C t^{-(1 - \frac{1}{p}) - \frac{1}{2} (1 - \frac{1}{p}) - \frac{|\beta| + \gamma - 2}{2}}, \]
which proves the first estimate. For the second estimate, we see that
\[ |x_h| \| \nabla_h^\beta (-\Delta_h)^{\frac{\gamma}{2}} K(t) * f(x) \| \leq \int_{\mathbb{R}^3} |x_h - y_h| \| \nabla_h^\beta (-\Delta_h)^{\frac{\gamma}{2}} K(t, x - y) \| |f(y)| dy \]
\[ + \int_{\mathbb{R}^3} \| \nabla_h^\beta (-\Delta_h)^{\frac{\gamma}{2}} K(t, x - y) \| |y_h| f(y) dy. \]
Hence, taking $L^p_h L^q_v$-norm and applying the Hausdorff-Young inequality, we complete the proof. \(\square\)
4. Nonlinear Analysis

In this section, we establish decay estimates and asymptotic expansions for the Duhamel terms related to the vector fields \( u = (u_1(t,x), u_2(t,x), u_3(t,x)) \), which satisfies some of the following assumptions for some \( s \in \mathbb{N}, 0 < T \leq \infty \) and \( A, B \geq 0 \):

(A1) \( u \in C([0,\infty);X^s(\mathbb{R}^3)), \nabla \cdot u = 0 \) and
\[
\|u(t)\|_{H^s} \leq A, \quad \|u(t)\|_{L^1(\mathbb{R}^3; C_t^1(\mathbb{R}^3))} \leq A(1 + t)
\]
for all \( t > 0 \).

(A2) For \( 1 \leq p \leq \infty \), there hold
\[
\|\nabla^\alpha u_1(t)\|_{L^p} \leq CA_t^{-\left(1 - \frac{3}{p}\right) - \frac{|\alpha|}{2}},
\]
\[
\|\nabla^\alpha u_3(t)\|_{L^p} \leq CA_t^{-\left(1 - \frac{3}{p}\right) - \frac{|\alpha|}{2}}
\]
for all \( 0 < t < T \) and \( \alpha \in (\mathbb{N} \cup \{0\})^3 \) with \( |\alpha| \leq 1 \).

(A3) \( \|\nabla^\alpha u(t)\|_{L^\infty_t L^1_x} \leq A t^{-\left(1 - \frac{|\alpha|}{2}\right)} \) for all \( 0 < t < T \) and \( \alpha \in (\mathbb{N} \cup \{0\})^3 \) with \( |\alpha| \leq 1 \).

(A4) There hold
\[
\|x_h|u_1(t,x)\|_{L^1_t L^\infty_x} \leq B(1 + t)^{\frac{1}{2}},
\]
\[
\|x_h|u_3(t,x)\|_{L^1_t L^\infty_x} \leq B
\]
for all \( t > 0 \).

Remark 4.1. It is easy to check that if \( u \) satisfies (A1) and (A2) for some \( s \in \mathbb{N} \) with \( s \geq 3 \), \( 0 < T \leq \infty \) and \( A \geq 0 \), then there exists an absolute positive constant \( C \) such that
\[
\|u_1(t)\|_{L^p}, \|u_3(t)\|_{L^p} \leq CA(1 + t)^{-\left(1 - \frac{3}{p}\right)},
\]
\[
\|\nabla u(t)\|_{L^p} \leq \left\{
\begin{array}{ll}
CA(1 + t)^{-\left(1 - \frac{3}{p}\right) - \frac{1}{2}} & (2 \leq p \leq \infty) \\
CA^{-\left(1 - \frac{3}{p}\right) - \frac{1}{2}} & (1 \leq p < 2)
\end{array}
\right.
\]
\[
\|u_3(t)\|_{L^p} \leq CA(1 + t)^{-\left(1 - \frac{3}{p}\right) - \frac{1}{2}},
\]
\[
\|\nabla u(t)\|_{L^p} \leq \left\{
\begin{array}{ll}
CA(1 + t)^{-\left(1 - \frac{3}{p}\right) - \frac{1}{2}} & (2 \leq p \leq \infty) \\
CA^{-\left(1 - \frac{3}{p}\right) - \frac{1}{2}} & (1 \leq p < 2)
\end{array}
\right.
\]
\[
\|u_1(t)\|_{L^1_t L^\infty_x} \leq CA,
\]
\[
\|u_3(t)\|_{L^1_t L^\infty_x} \leq CA(1 + t)^{-\frac{1}{2}}
\]
for all \( 1 \leq p \leq \infty \) and \( 0 < t < T \). In the following of this paper, we often use this fact.

Decay Estimates for the Duhamel Terms. We first focus on the decay rates for the Duhamel terms.

Lemma 4.2. Let \( u \) satisfy (A1) and (A2) for some \( s \in \mathbb{N} \) with \( s \geq 5 \), \( 0 < T \leq \infty \) and \( A \geq 0 \). Then, there exists an absolute positive constant \( C \) such that
\[
\|\nabla^\alpha D^1_t[u](t)\|_{L^p} \leq CA^2(1 + t)^{-\left(1 - \frac{3}{p}\right) - \frac{|\alpha|}{2}}
\]
\[
\|\nabla^\alpha D^2_t[u](t)\|_{L^p} \leq CA^2(1 + t)^{-\left(1 - \frac{3}{p}\right) - \frac{|\alpha|}{2}} \log(2 + t),
\]
\[
\|\nabla^\alpha D^3_t[u](t)\|_{L^p} \leq CA^2(1 + t)^{-\left(1 - \frac{3}{p}\right) - \frac{|\alpha|}{2}},
\]
\[
\|\nabla^\alpha D^4_t[u](t)\|_{L^p} \leq CA^2(1 + t)^{-\left(1 - \frac{3}{p}\right) - \frac{|\alpha|}{2}} \log(2 + t),
\]
\[
\|\nabla^\alpha D^5_t[u](t)\|_{L^p} \leq CA^2(1 + t)^{-\left(1 - \frac{3}{p}\right) - \frac{|\alpha|}{2}} \log(2 + t)
\]
and
\[
\|\nabla^\alpha D^6_t[u](t)\|_{L^p} \leq CA^2(1 + t)^{-\left(1 - \frac{3}{p}\right) - \frac{|\alpha|}{2}} = CA^2(1 + t)^{-\left(1 - \frac{3}{p}\right) - \frac{|\alpha|}{2}} - \frac{|\alpha|}{2},
\]
\[
\|\nabla^\alpha D^7_t[u](t)\|_{L^p} \leq CA^2(1 + t)^{-\left(1 - \frac{3}{p}\right) - \frac{|\alpha|}{2}} \log(2 + t),
\]
\[ \| \nabla^\alpha D_h^5[u](t) \|_{L^p} \leq C A^2 (1 + t)^{-\frac{\beta}{2(1 - \frac{1}{p})} - \frac{\| u \|_\infty}{2}} \]

for all \(1 \leq p \leq \infty\), \(0 < t < T\) and \(\alpha \in (\mathbb{N} \cup \{0\})^3\) with \(|\alpha| \leq 1\).

**Proof.** It suffices to prove

\[ \left\| \nabla^\alpha \int_0^t e^{(t - \tau)\Delta_h} \partial_3(u u_h)(\tau) d\tau \right\|_{L^p} \leq C A^2 (1 + t)^{-(-1 - \frac{1}{2})^\gamma \frac{\| u \|_\infty}{2}}, \quad (4.1) \]

\[ \left\| \nabla^\alpha \int_0^t e^{(t - \tau)\Delta_h} \nabla_h(u u_h)(\tau) d\tau \right\|_{L^p} \leq \begin{cases} C A^2 (1 + t)^{-(-1 - \frac{1}{2})^\gamma \frac{\| u \|_\infty}{2}} \log(2 + t) & (k = 1, 2) \\ C A^2 (1 + t)^{-(-1 - \frac{1}{2})^\gamma \frac{\| u \|_\infty}{2}} & (k = 3) \end{cases}, \quad (4.2) \]

\[ \left\| \nabla^\alpha \int_0^t K^{(m)}_{3, \gamma}(t - \tau) * (u u_h)(\tau) d\tau \right\|_{L^p} \leq \begin{cases} C A^2 (1 + t)^{-\frac{\beta}{2(1 - \frac{1}{p})} - \frac{\| u \|_\infty}{2}} \log(2 + t) & (k = 1, 2) \\ C A^2 (1 + t)^{-\frac{\beta}{2(1 - \frac{1}{p})} - \frac{\| u \|_\infty}{2}} & (k = 3) \end{cases}, \quad (4.3) \]

for \(l = 1, 2, 3, m = 1, 2\) and \(0 < t < T\), where

\[ K^{(1)}_{3, \gamma}(t, x) := \nabla_h^\beta (-\Delta_h)^{\frac{\gamma}{2}} K(t, x), \quad K^{(2)}_{3, \gamma}(t, x) := \text{sgn}(x_3) \nabla_h^\beta (-\Delta_h)^{\frac{\gamma}{2}} K(t, x) \quad (4.4) \]

for \((\beta, \gamma) \in (\mathbb{N} \cup \{0\})^2 \times (\mathbb{N} \cup \{0\})\) satisfying \(|\beta| + \gamma = 3\). The interpolation yields that it is enough to prove \((4.1) - (4.3)\) only for the case \(p = 1\) and \(p = \infty\).

First, we show \((4.1)\). For the case \(p = 1\), using the estimates

\[ \| \partial_3(u u_h)(\tau) \|_{L^\infty} \leq \| \partial_3 u_3(\tau) \|_{L^\infty} \| u_h(\tau) \|_{L^1} + \| u_3(\tau) \|_{L^\infty} \| \partial_3 u_h(\tau) \|_{L^1} \]

\[ \leq C A^2 (1 + \tau)^{-\frac{\beta}{2}}, \]

and

\[ \| \partial_3^2(u u_h)(\tau) \|_{L^1} \leq \| \partial_3^2 u_3(\tau) \|_{L^2} \| u_h(\tau) \|_{L^2} + 2 \| \partial_3 u_3(\tau) \|_{L^2} \| \partial_3 u_h(\tau) \|_{L^2} \]

\[ \leq C \left( \| \partial_3 u_3(\tau) \|_{L^2} \| \partial_3^2 u_3(\tau) \|_{L^2} \| u_h(\tau) \|_{L^2} + \| \partial_3 u_3(\tau) \|_{L^2} \| \partial_3 u_h(\tau) \|_{L^2} \right) \]

\[ \leq C A^2 (1 + \tau)^{-\frac{\beta}{2}}, \quad (4.5) \]

we see that

\[ \left\| \nabla^\alpha \int_0^t e^{(t - \tau)\Delta_h} \partial_3(u u_h)(\tau) d\tau \right\|_{L^1} \leq C \int_0^t \| \nabla_h^\alpha G_h(t - \tau) \|_{L^1(\mathbb{R}^2)} \| \partial_3^{\alpha_3+1}(u u_h)(\tau) \|_{L^1} d\tau \]

\[ \leq C A^2 \int_0^t (t - \tau)^{-\frac{\| u \|_\infty}{2}} (1 + \tau)^{-\frac{\beta}{2}} d\tau \]

\[ \leq C A^2 (1 + t)^{-\frac{\| u \|_\infty}{2}}. \]

For the case \(p = \infty\) with \(\alpha_3 = 0\), it is easy to see that

\[ \| \partial_3(u u_h)(\tau) \|_{L^\infty L^\infty} \leq \| \partial_3 u_3(\tau) \|_{L^\infty} \| u_h(\tau) \|_{L^\infty} + \| u_3(\tau) \|_{L^\infty} \| \partial_3 u_h(\tau) \|_{L^\infty} \]

\[ \leq C A^2 (1 + \tau)^{-\frac{\beta}{2}}, \]

and

\[ \| \partial_3^2(u u_h)(\tau) \|_{L^\infty} \leq \| \partial_3 u_3(\tau) \|_{L^\infty} \| u_h(\tau) \|_{L^\infty} + \| u_3(\tau) \|_{L^\infty} \| \partial_3 u_h(\tau) \|_{L^\infty} \]

\[ \leq C A^2 (1 + \tau)^{-\frac{\beta}{2}}, \]

which yield

\[ \| \nabla_h^\alpha \int_0^t e^{(t - \tau)\Delta_h} \partial_3(u u_h)(\tau) d\tau \|_{L^\infty} \]

\[ \leq C \int_0^t \| \nabla_h^\alpha G_h(t - \tau) \|_{L^\infty(\mathbb{R}^2)} \| \partial_3(u u_h)(\tau) \|_{L^\infty} d\tau + C \int_0^t \| \nabla_h^\alpha G_h(t - \tau) \|_{L^1(\mathbb{R}^2)} \| \partial_3(u u_h)(\tau) \|_{L^\infty} d\tau \]
for $t > 0$ and
\[ \left\| \nabla_h^{\alpha_3} \int_0^t e^{(t-\tau)\Delta_h} \partial_3(u_3u_h)(\tau) d\tau \right\|_{L^\infty} \leq C \int_0^t \left\| \nabla_h^{\alpha_3} G_h(t-\tau) \right\|_{L^1(\Omega^2)} \left\| \partial_3(u_3u_h)(\tau) \right\|_{L^\infty} d\tau \leq CA^2 \int_0^t (t-\tau)^{-\frac{\alpha_3}{2}} (1+\tau)^{-\frac{5}{4}} d\tau \]
for $0 < t \leq 1$. For the case $p = \infty$ with $\alpha_3 = 1$, we have by the Gagliardo-Nirenberg interpolation inequality that
\[ \left\| \partial_3^2(u_3u_h)(\tau) \right\|_{L^1 H^p L^\infty \Omega} \leq \left\| \partial_3^2 u_3(\tau) \right\|_{L^1 L^\infty \Omega} \left\| u_3(\tau) \right\|_{L^1 L^\infty \Omega} + 2 \left\| \partial_3 u_3(\tau) \right\|_{L^1 L^\infty \Omega} \left\| \partial_3 u_3(\tau) \right\|_{L^1 L^\infty \Omega} + C \left( \left\| \partial_3 u_3(\tau) \right\|_{L^1 L^\infty \Omega} \left\| \partial_3^2 u_3(\tau) \right\|_{L^1 L^\infty \Omega} \right) \]
and also
\[ \left\| \partial_3^2(u_3u_h)(\tau) \right\|_{L^1 L^\infty \Omega} \leq \left\| \partial_3^2 u_3(\tau) \right\|_{L^1 L^\infty \Omega} \left\| u_3(\tau) \right\|_{L^1 L^\infty \Omega} + 2 \left\| \partial_3 u_3(\tau) \right\|_{L^1 L^\infty \Omega} \left\| \partial_3 u_3(\tau) \right\|_{L^1 L^\infty \Omega} + C \left( \left\| \partial_3 u_3(\tau) \right\|_{L^1 L^\infty \Omega} \left\| \partial_3^2 u_3(\tau) \right\|_{L^1 L^\infty \Omega} \right) \]
Combining these estimates, we obtain
\[ \left\| \partial_3 \int_0^t e^{(t-\tau)\Delta_h} \partial_3(u_3u_h)(\tau) d\tau \right\|_{L^\infty} \leq C \int_0^t \left\| G_h(t-\tau) \right\|_{L^1(\Omega^2)} \left\| \partial_3^2(u_3u_h)(\tau) \right\|_{L^1 L^\infty \Omega} d\tau \leq CA^2 \int_0^t (t-\tau)^{-\frac{\alpha_3}{2}} (1+\tau)^{-\frac{5}{4}} d\tau + \frac{1}{2} \int_0^t (t-\tau)^{-\frac{3}{4}} (1+\tau)^{-\frac{5}{4}} d\tau \]
for $t > 0$
We next show (4.2). For the case $p = 1$, we easily see that
\[ \left\| \partial_3^k(u_ku)(\tau) \right\|_{L^1} \leq 2 \left\| \partial_3^k u(\tau) \right\|_{L^1} \left\| u(\tau) \right\|_{L^\infty} \leq CA^2 (1+\tau)^{-1} \]
for $k = 1, 2$
\[ \left\| \partial_3^3(u_3u)(\tau) \right\|_{L^1} \leq \left\{ \begin{array}{ll}
\left\| u_3(\tau) \right\|_{L^\infty} \left\| u(\tau) \right\|_{L^1} & (\alpha_3 = 0)
\left\| \partial_3 u_3(\tau) \right\|_{L^\infty} \left\| u(\tau) \right\|_{L^1} + \left\| u_3(\tau) \right\|_{L^\infty} \left\| \partial_3 u(\tau) \right\|_{L^1} & (\alpha_3 = 1)
\end{array} \right. \]
(4.6)
and also have
\[ ||\nabla^\alpha (u_k u_l)||_{L^1} \leq 2 ||\nabla^\alpha u(\tau)||_{L^\infty} ||u(\tau)||_{L^1} \leq CA^2 (1 + \tau)^{-1 - \frac{3k}{2}}.\]

Let \( \sigma_k = 1 \) (if \( k = 1, 2 \)) and \( \sigma_3 = 3/2 \). Then, we have
\[
\left\| \nabla^\alpha \int_0^t e^{(t-\tau)\Delta_h} \nabla_h (u_k u_l)(\tau)d\tau \right\|_{L^1} \\
\leq \int_0^\frac{t}{2} \| \nabla^\alpha_h \nabla_h G_h(t-\tau) \|_{L^\infty(\mathbb{R}^2)} \| \partial^3 \nabla^\alpha (u_k u_l)(\tau) \|_{L^1} d\tau + \int_\frac{t}{2}^t \| \nabla_h G_h(t-\tau) \|_{L^1(\mathbb{R}^2)} \| \nabla^\alpha (u_k u_l)(\tau) \|_{L^1} d\tau \\
\leq CA^2 \int_0^\frac{t}{2} (t-\tau)^{-1 - \frac{3k}{2}} (1 + \tau)^{-\sigma_k} d\tau + CA^2 \int_\frac{t}{2}^t (t-\tau)^{-\frac{3}{2}} (1 + \tau)^{-1 - \frac{3k}{2}} d\tau \\
\leq \begin{cases} 
CA^2(1 + \tau)^{-\frac{3k}{2}} & (k = 1, 2) \\
CA^2(1 + \tau)^{-\frac{3}{2}} & (k = 3) 
\end{cases}
\]

For the case \( p = \infty \), we have
\[
\| \partial^3 \nabla^\alpha (u_k u_l)(\tau) \|_{L^1 L^\infty} \leq C \| \partial^3 \nabla^\alpha u(\tau) \|_{L^\infty} \|u(\tau)\|_{L^1 L^\infty} \leq CA^2 (1 + \tau)^{-1} 
\quad \text{for} \quad k = 1, 2,
\]
\[
\| \partial^3 \nabla^\alpha (u_k u_l)(\tau) \|_{L^1 L^\infty} \leq \begin{cases} 
\| u_3(\tau) \|_{L^\infty} \| u_h(\tau) \|_{L^1 L^\infty} & (\alpha_3 = 0) \\
\| \partial_3 u_3(\tau) \|_{L^\infty} \| u_h(\tau) \|_{L^1 L^\infty} + \| u_3(\tau) \|_{L^1 L^\infty} \| \partial_3 u_h(\tau) \|_{L^\infty} & (\alpha_3 = 1) 
\end{cases}
\]
and
\[
\| \nabla^\alpha (u_k u_l)(\tau) \|_{L^\infty} \leq C \| \nabla^\alpha u(\tau) \|_{L^\infty} \|u(\tau)\|_{L^1 L^\infty} \leq CA^2 (1 + \tau)^{-2 - \frac{3k}{2}}.
\]

Then, we obtain
\[
\left\| \nabla^\alpha \int_0^t e^{(t-\tau)\Delta_h} \nabla_h (u_k u_l)(\tau)d\tau \right\|_{L^\infty} \\
\leq \int_0^\frac{t}{2} \| \nabla^\alpha_h \nabla_h G_h(t-\tau) \|_{L^\infty(\mathbb{R}^2)} \| \partial^3 \nabla^\alpha (u_k u_l)(\tau) \|_{L^1 L^\infty} d\tau + \int_\frac{t}{2}^t \| \nabla_h G_h(t-\tau) \|_{L^1(\mathbb{R}^2)} \| \nabla^\alpha (u_k u_l)(\tau) \|_{L^1} d\tau \\
\leq CA^2 \int_0^\frac{t}{2} (t-\tau)^{-1 - \frac{3k}{2}} (1 + \tau)^{-\sigma_k} d\tau + CA^2 \int_\frac{t}{2}^t (t-\tau)^{-\frac{3}{2}} (1 + \tau)^{-2 - \frac{3k}{2}} d\tau \\
\leq \begin{cases} 
CA^2 t^{-1 - \frac{3k}{2}} \log(2 + t) & (k = 1, 2) \\
CA^2 t^{-\frac{3}{2}} & (k = 3) 
\end{cases}
\]
for \( t \geq 1 \) and
\[
\left\| \nabla^\alpha \int_0^t e^{(t-\tau)\Delta_h} \nabla_h (u_k u_l)(\tau)d\tau \right\|_{L^\infty} \leq C \int_0^t (t-\tau)^{-\frac{3}{2}} ||\nabla^\alpha (u_k u_l)(\tau)||_{L^\infty} d\tau \\
\leq CA^2 \int_0^t (t-\tau)^{-\frac{3}{2}} (1 + \tau)^{-2 - \frac{3k}{2}} d\tau \\
\leq CA^2
\]
for \( 0 < t \leq 1 \).

We finally prove (4.2). For the case \( p = 1 \), it follows from Lemma 3.3 and the inequalities in the proof of (4.2) that
\[
\left\| \nabla^\alpha \int_0^t K^{(m)}_{\beta, \gamma}(t-\tau) * (u_k u_l)(\tau)d\tau \right\|_{L^1} \\
\leq C \int_0^\frac{t}{2} \| \nabla_h K^{(m)}_{\beta, \gamma}(t-\tau) \|_{L^1} \| \partial^3 \nabla^\alpha (u_k u_l)(\tau) \|_{L^1} d\tau + C \int_\frac{t}{2}^t \| K^{(m)}_{\beta, \gamma}(t-\tau) \|_{L^1} \| \nabla^\alpha (u_k u_l)(\tau) \|_{L^1} d\tau \\
\leq CA^2 \int_0^\frac{t}{2} (t-\tau)^{-1 - \frac{3k}{2}} (1 + \tau)^{-\sigma_k} d\tau + CA^2 \int_\frac{t}{2}^t (t-\tau)^{-\frac{3}{2}} (1 + \tau)^{-1 - \frac{3k}{2}} d\tau
\]
Lemma 4.3. Let \( N \) be a positive integer such that
\[
\begin{aligned}
&\left\| \nabla^{\alpha} \int_{0}^{t} K_{\beta, \gamma}^{(m)}(t - \tau) * (u_k u_l)(\tau) d\tau \right\|_{L^{\infty}} \\
&\leq C \int_{0}^{\frac{t}{2}} \left\| \nabla^{\alpha} K_{\beta, \gamma}^{(m)}(t - \tau) \right\|_{L^{\infty}} \left\| \partial_{\gamma}^{3} (u_k u_l)(\tau) \right\|_{L^{1}} d\tau + C \int_{\frac{t}{2}}^{t} \left\| \nabla^{\alpha} (u_k u_l)(\tau) \right\|_{L^{4}} d\tau \\
&\leq C \int_{0}^{\frac{t}{2}} \left( t - \tau \right)^{-\frac{3}{2} - \frac{1 + |\alpha|}{2}} \left\| \partial_{\gamma}^{3} (u_k u_l)(\tau) \right\|_{L^{1}} d\tau + C \int_{\frac{t}{2}}^{t} \left( t - \tau \right)^{-\frac{3}{2}} \left\| \nabla^{\alpha} (u_k u_l)(\tau) \right\|_{L^{4}} d\tau \\
&\leq CA^2 \int_{0}^{\frac{t}{2}} \left( t - \tau \right)^{-\frac{3}{2} - \frac{1 + |\alpha|}{2}} (1 + |\alpha|)^{-\sigma} d\tau + CA^2 \int_{\frac{t}{2}}^{t} \left( t - \tau \right)^{-\frac{3}{2}} (1 + |\alpha|)^{-\sigma} d\tau \\
&\leq \begin{cases} 
CA^2 t^{-\frac{3}{2} - \frac{1 + |\alpha|}{2}} \log(2 + t) & (k = 1, 2) \\
CA^2 t^{-\frac{3}{2} - \frac{1 + |\alpha|}{2}} & (k = 3)
\end{cases}
\end{aligned}
\]
for \( t \geq 1 \) and
\[
\begin{aligned}
\left\| \nabla^{\alpha} \int_{0}^{t} K_{\beta, \gamma}^{(m)}(t - \tau) * (u_k u_l)(\tau) d\tau \right\|_{L^{\infty}} \\
&\leq C \int_{0}^{t} \left\| K_{\beta, \gamma}^{(m)}(t - \tau) \right\|_{L^{1}} \left\| \nabla^{\alpha} (u_k u_l)(\tau) \right\|_{L^{\infty}} d\tau \\
&\leq CA^2 \int_{0}^{t} \left( t - \tau \right)^{-\frac{3}{2} - \frac{1 + |\alpha|}{2}} (1 + |\alpha|)^{-\sigma} d\tau \\
&\leq CA^2
\end{aligned}
\]
for \( 0 < t \leq 1 \). Thus, we complete the proof. \( \square \)

Lemma 4.3. Let \( u \) satisfy (A1) and (A2) for some \( s \in \mathbb{N} \) with \( s \geq 9 \), \( 0 < T \leq \infty \) and \( A \geq 0 \). Then, there exists an absolute positive constant \( C \) such that
\[
\begin{aligned}
\left\| \nabla^{\alpha} D_{m}^{h}[u](t) \right\|_{L_{x}^{\infty} L_{t}^{1}} &\leq CA^2 t^{-1 - \frac{|\alpha|}{2}}, \\
\left\| \nabla^{\alpha} D_{n}^{h}[u](t) \right\|_{L_{x}^{\infty} L_{t}^{1}} &\leq CA^2 t^{-1 - \frac{|\alpha|}{2}}
\end{aligned}
\]
for all \( m = 1, 2, 3, 4, 5 \), \( n = 1, 2, 3, 0 < t < T \) and \( \alpha \in (\mathbb{N} \cup \{0\})^{3} \) with \( |\alpha| \leq 1 \).

Proof. It suffices to show
\[
\begin{aligned}
\left\| \nabla^{\alpha} \int_{0}^{t} e^{(t-\tau)\Delta_{h}} \partial_{3}(u_{3} u_{h})(\tau) d\tau \right\|_{L_{x}^{\infty} L_{t}^{1}} &\leq CA^2 t^{-1 - \frac{|\alpha|}{2}}, \\
\left\| \nabla^{\alpha} \int_{0}^{t} e^{(t-\tau)\Delta_{h}} \nabla_{h}(u_{k} u_{l})(\tau) d\tau \right\|_{L_{x}^{\infty} L_{t}^{1}} &\leq CA^2 t^{-1 - \frac{|\alpha|}{2}}, \\
\left\| \nabla^{\alpha} \int_{0}^{t} K_{\beta, \gamma}^{(m)}(t - \tau) * (u_k u_l)(\tau) d\tau \right\|_{L_{x}^{\infty} L_{t}^{1}} &\leq CA^2 t^{-1 - \frac{|\alpha|}{2}}
\end{aligned}
\]
for \( k, l = 1, 2, 3, m = 1, 2 \) and \( 0 < t < T \), where \( K_{\beta, \gamma}^{(m)} \) are defined by (1.4) for \( (\beta, \gamma) \in (\mathbb{N} \cup \{0\})^{2} \times (\mathbb{N} \cup \{0\}) \) satisfying \( |\beta| + |\gamma| = 3 \).

We first show (4.7). For the case \( \alpha_{3} = 0 \), we have
\[
\begin{aligned}
\left\| \nabla^{\alpha} \int_{0}^{t} e^{(t-\tau)\Delta_{h}} \partial_{3}(u_{3} u_{h})(\tau) d\tau \right\|_{L_{x}^{\infty} L_{t}^{1}} \\
&\leq \int_{0}^{\frac{t}{2}} \left\| \nabla^{\alpha} G_{h}(t - \tau) \right\|_{L_{x}^{\infty} (\mathbb{R}^{2})} \left\| \partial_{3}(u_{3} u_{h})(\tau) \right\|_{L^{1}} d\tau + \int_{\frac{t}{2}}^{t} \left\| \nabla^{\alpha} G_{h}(t - \tau) \right\|_{L_{x}^{\infty} (\mathbb{R}^{2})} \left\| \partial_{3}(u_{3} u_{h})(\tau) \right\|_{L_{x}^{\infty} L_{t}^{1}} d\tau \\
&\leq C \int_{0}^{\frac{t}{2}} (t - \tau)^{-1 - \frac{|\alpha|}{2}} \left( \left\| \partial_{3} u_{3}(\tau) \right\|_{L^{\infty}} \left\| u_{h}(\tau) \right\|_{L^{1}} + \left\| u_{3}(\tau) \right\|_{L^{\infty}} \left\| \partial_{3} u_{h}(\tau) \right\|_{L^{1}} \right) d\tau \\
&+ C \int_{\frac{t}{2}}^{t} (t - \tau)^{-1 - \frac{|\alpha|}{2}} \left( \left\| \partial_{3} u_{3}(\tau) \right\|_{L^{\infty}} \left\| u_{h}(\tau) \right\|_{L_{x}^{\infty} L_{t}^{1}} + \left\| u_{3}(\tau) \right\|_{L^{\infty}} \left\| \partial_{3} u_{h}(\tau) \right\|_{L_{x}^{\infty} L_{t}^{1}} \right) d\tau
\end{aligned}
\]
\[
\begin{align*}
&\leq CA^2 \int_0^\tau (t - \tau)^{-\frac{3}{2}} \frac{(1 + \tau)^{-\frac{1}{2}}}{d\tau} + CA^2 \int_0^\tau (t - \tau)^{-\frac{3}{2}} \frac{(1 + \tau)^{-\frac{1}{2}}}{d\tau} + CA^2 \int_0^\tau (t - \tau)^{-\frac{3}{2}} \frac{(1 + \tau)^{-\frac{1}{2}}}{d\tau} \\
&\leq CA^2 t^{-1 - \frac{3}{2}}.
\end{align*}
\]

For the case \(\alpha_3 = 1\), it follows from the Gagliardo-Nirenberg interpolation inequality that
\[
\begin{align*}
\|\partial_3^2 f\|_{L^\infty} &\leq C\|\partial_3^2 f\|^{\frac{2}{5}}_{L^2} \|\partial_3 f\|^{\frac{3}{5}}_{L^\infty} \leq C\|f\|^{\frac{2}{5}}_{H^2} \|\partial_3 f\|^{\frac{3}{5}}_{L^\infty}, \\
\|\partial_3^2 f\|_{L^\infty L^2} &\leq C\|\partial_3^2 f\|^{\frac{2}{5}}_{L^2} \|\partial_3 f\|^{\frac{3}{5}}_{L^\infty} \leq C\|f\|^{\frac{2}{5}}_{H^2} \|\partial_3 f\|^{\frac{3}{5}}_{L^\infty}.
\end{align*}
\]

Using (4.10) and (4.11), we obtain
\[
\begin{align*}
\|\partial_3^2 (u_3 u_h)(\tau)\|_{L^\infty L^1} &\leq \|\partial_3^2 u_3(\tau)\|_{L^\infty} \|u_h(\tau)\|_{L^\infty L^1} + \|\partial_3 u_3(\tau)\|_{L^\infty} \|\partial_3 u_h(\tau)\|_{L^\infty L^1} \\
&\leq C \left(\|u_3(\tau)\|_{H^2} \|\partial_3 u_3(\tau)\|_{L^\infty} \|u_h(\tau)\|_{L^\infty L^1} + \|\partial_3 u_3(\tau)\|_{L^\infty} \|\partial_3 u_h(\tau)\|_{L^\infty L^1} \right) \\
&\leq CA^2 \left\{ (1 + \tau)^{-\frac{1}{2}} t^{-\frac{3}{2}} + (1 + \tau)^{-\frac{1}{4}} t^{-1} \right\}.
\end{align*}
\]

By (4.8) and (4.12), it holds
\[
\begin{align*}
&\left\|\partial_3 \int_0^t e^{(t-\tau)\Delta_h} \partial_3 (u_3 u_h)(\tau) d\tau \right\|_{L^\infty L^1} \\
&\leq \int_0^\tau \|G_h(t - \tau)\|_{L^\infty} \|\partial_3^2 (u_3 u_h)(\tau)\|_{L^1} d\tau + \int_0^t \|G_h(t - \tau)\|_{L^1} \|\partial_3^2 (u_3 u_h)(\tau)\|_{L^\infty} d\tau \\
&\leq CA^2 \int_0^\tau (t - \tau)^{-1} (1 + \tau)^{-\frac{1}{4}} d\tau + CA^2 \int_0^t (1 + \tau)^{-\frac{1}{4}} t^{-\frac{3}{2}} + (1 + \tau)^{-\frac{1}{4}} t^{-1} d\tau \\
&\leq CA^2 t^{-1},
\end{align*}
\]
which completes the proof of (4.8).

Next, we show (4.9). It is easy to see that
\[
\begin{align*}
\|\nabla^3 (u_3 u_h)(\tau)\|_{L^\infty L^1} &\leq \|u(\tau)\|_{L^\infty L^1} \|\nabla^3 u(\tau)\|_{L^\infty} \leq CA^2 t^{-1 - \frac{3}{2}}.
\end{align*}
\]

It follows from (4.13) and (4.11) that
\[
\begin{align*}
\|\partial_3^4 (u_3 u_h)(\tau)\|_{L^\infty L^1} &\leq \|\partial_3^4 (u_3 u_h)(\tau)\|_{L^\infty L^1} \leq \|\partial_3^4 (u_3 u_h)(\tau)\|_{L^\infty L^1} \\
&\leq CA^2 t^{-\frac{1}{4}} (1 + \tau)^{-\frac{1}{4}}.
\end{align*}
\]

Hence, by (4.13) and (4.14), we have
\[
\begin{align*}
&\left\|\nabla^3 \int_0^t e^{(t-\tau)\Delta_h} \nabla_h (u_3 u_h)(\tau) d\tau \right\|_{L^\infty L^1} \\
&\leq \int_0^\tau \|\nabla^3 \nabla_h G_h(t - \tau)\|_{L^2} \|\partial_3^4 (u_3 u_h)(\tau)\|_{L^1} d\tau + \int_0^t \|\nabla_h G_h(t - \tau)\|_{L^1} \|\nabla^3 (u_3 u_h)(\tau)\|_{L^\infty} d\tau \\
&\leq CA^2 \int_0^\tau (t - \tau)^{-1 - \frac{3}{2}} t^{-\frac{1}{4}} (1 + \tau)^{-\frac{1}{4}} d\tau + CA^2 \int_0^t (1 + \tau)^{-\frac{1}{4}} t^{-1 - \frac{3}{2}} d\tau \\
&\leq CA^2 t^{-1 - \frac{3}{2}}.
\end{align*}
\]

This completes the proof of (4.8).

Finally, we prove (4.9). We obtain from (4.13) and (4.14) that
\[
\begin{align*}
&\left\|\nabla^3 \int_0^t K_{\beta, \gamma} (t - \tau) * (u_3 u_h)(\tau) d\tau \right\|_{L^\infty L^1}
\end{align*}
\]
If, in addition, we first note that it holds
\[ I \leq CA^2 \int_0^t (t - \tau)^{-\frac{1}{4}} \tau^{-\frac{1}{2}} (1 + \tau)^{-\frac{3}{2}} d\tau + CA^2 \int_0^t (t - \tau)^{-\frac{1}{4}} \tau^{-1} (1 + \tau)^{-\frac{3}{2}} d\tau \]
\[ \leq CA^2 t^{-1 - \frac{|m|}{2}}. \]

The proof is completed.

Asymptotic Expansions of the Duhamel Terms. Next, we consider the asymptotic expansions of Duhamel terms $D^i_k[u]$ and $D^i_1[u]$.

Lemma 4.4. Let $u$ satisfy (A1) and (A2) for some $s \in \mathbb{N}$ with $s \geq 3$, $A > 0$ and $T = \infty$. Then, for $1 \leq p \leq \infty$, it holds
\[ \lim_{t \to \infty} t^{1 - \frac{p}{2}} \left\| \int_0^t \int_{\mathbb{R}^2} \partial_3 (u_3 u_h)(\tau, y_h, x_3) dy_h d\tau \right\|_{L^p_\tau} = 0. \]  
(4.15)

If, in addition, $u$ satisfies (A3) and (A4), then for $1 < p \leq \infty$ there exists a positive constant $C = C(p)$ such that
\[ \left\| \int_0^t \int_{\mathbb{R}^2} \partial_3 (u_3 u_h)(\tau, y_h, x_3) dy_h d\tau \right\|_{L^p_\tau} \leq CA(A + B) t^{\left(1 - \frac{1}{p}\right) - \frac{1}{2} \log(2 + t)} \]  
(4.16)

for all $t > 0$.

Proof. For the proof of (4.15), we first note that it holds
\[ \| \partial_3 (u_3 u_h)(\tau) \|_{L^p L^p_\tau} \leq \| \partial_3 u_3(\tau) \|_{L^p} \| u_h(\tau) \|_{L^p_\tau} + \| u_3(\tau) \|_{L^p L^p_\tau} \| \partial_3 u_h(\tau) \|_{L^p} \]
\[ \leq \| \partial_3 u_3(\tau) \|_{L^p} \| u_h(\tau) \|_{L^p_\tau} + \| u_3(\tau) \|_{L^p L^p_\tau} \| \partial_3 u_h(\tau) \|_{L^p} \]
\[ \leq CA^2 (1 + \tau)^{-\frac{3}{2}} \in L^1_\tau(0, \infty). \]

Following the idea of [5] Theorem 4.1, let us decompose our target as follows:
\[ D^i_k[u](t, x) + G_h(t, x_h) \int_0^t \int_{\mathbb{R}^2} \partial_3 (u_3 u_h)(\tau, y_h, x_3) dy_h d\tau = - \sum_{m=1}^4 I_m(t, x), \]

where
\[ I_1(t, x) = \int_0^t \int_{\mathbb{R}^2} \{ G_h(t - \tau, x_h - y_h) - G_h(t, x_h - y_h) \} \partial_3 (u_3 u_h)(\tau, y_h, x_3) dy_h d\tau, \]
\[ I_2(t, x) = \int_0^t \int_{\mathbb{R}^2} \{ G_h(t, x - y_h) - G_h(t, x_h) \} \partial_3 (u_3 u_h)(\tau, y_h, x_3) dy_h d\tau, \]
\[ I_3(t, x) = \int_0^t e^{(t - \tau) A_h} \partial_3 (u_3 u_h)(\tau, x) d\tau, \]
\[ I_4(t, x) = -G_h(t, x_h) \int_0^t \int_{\mathbb{R}^2} \partial_3 (u_3 u_h)(\tau, y_h, x_3) dy_h d\tau. \]

On the estimate for $I_1(t)$, since
\[ I_1(t, x) = - \int_0^t \int_{\mathbb{R}^2} \tau \partial_3 (G_h(t - \theta \tau, x_h - y_h) \partial_3 (u_3 u_h)(\tau, y_h, x_3) d\theta dy_h d\tau, \]
we have
\[ \| I_1(t) \|_{L^p_\tau} \leq \int_0^t \int_{\mathbb{R}^2} \tau \| \partial_3 (G_h(t - \theta \tau)) \|_{L^p(\mathbb{R}^2)} \| \partial_3 (u_3 u_h)(\tau) \|_{L^p L^p_\tau} d\theta dy_h d\tau \]
\[ \leq CA^2 \int_0^t \int_{\mathbb{R}^2} \tau (t - \theta \tau)^{-\frac{1}{4}} d\theta dy_h d\tau \]
\[
\|I_2(t)\|_{L^p} \leq t^{-\frac{(1-p)}{2}} 2^\frac{\frac{1}{2}}{p} \int_0^t (1 + \tau)^{-\frac{1}{2}} d\tau \\
\leq CA^2 t^{-\frac{(1-p)}{2}} (1 + \tau)^{-\frac{1}{2}}.
\]

On the estimate for \(I_2(t)\), we see that
\[
\|I_2(t)\|_{L^p} \leq t^{-\frac{(1-p)}{2}} \int_0^\infty \int_{\mathbb{R}^2} \|G_h(1, \cdot - t^{-\frac{1}{2}} y_h) - G_h(1, \cdot)\|_{L^p(\mathbb{R}^2)} \|\partial_3(u_3 u_h)(\tau, y_h, \cdot)\|_{L^p(\mathbb{R}^3)} dy_h d\tau.
\]

By virtue of \(\partial_3(u_3 u_h) \in L^1(0, \infty; L^p_{3}(\mathbb{R}^3))\), the dominated convergence theorem yields that
\[
t_{1-\frac{P}{2}}\|I_2(t)\|_{L^p} \to 0 \text{ as } t \to \infty.
\]

On the estimate for \(I_3(t)\) and \(I_4(t)\), we have
\[
\|I_3(t)\|_{L^p} + \|I_4(t)\|_{L^p} \leq \int_0^t \|G_h(t - \tau)\|_{L^p(\mathbb{R}^2)} \|\partial_3(u_3 u_h)(\tau)\|_{L^p_{2}(\mathbb{R}^2)} d\tau
\]

\[
+ \|G_h(t)\|_{L^p(\mathbb{R}^2)} \int_0^\infty \|\partial_3(u_3 u_h)(\tau)\|_{L^p_{2}(\mathbb{R}^2)} d\tau
\]

\[
\leq CA^2 \int_0^t (t - \tau)^{-\frac{(1-p)}{2}} (1 + \tau)^{-\frac{1}{2}} d\tau + CA^2 t^{-\frac{(1-p)}{2}} \int_\frac{t}{2}^\infty \tau^{-\frac{1}{2}} d\tau
\]

\[
\leq CA^2 t^{-\frac{(1-p)}{2}} (1 + \tau)^{-\frac{1}{2}}.
\]

Collecting the estimates for \(I_m(t)\) \((m = 1, 2, 3, 4)\), we obtain \(\text{(4.15)}\).

For the proof of \(\text{(4.16)}\), it suffices to improve the estimate for \(I_2(t)\). By \(\text{(A3)}\) and \(\partial_3 u_3 = -\nabla_h u_h\), we see that
\[
\|\partial_3 u_3(\tau)\|_{L^p_{3}(\mathbb{R}^3)} = \begin{cases} 
\|\partial_3 u_3(\tau)\|_{L^p_{3}(\mathbb{R}^3)} & (0 < \tau < 1) \\
\|\nabla_h \cdot u_h(\tau)\|_{L^p_{3}(\mathbb{R}^3)} & (\tau \geq 1)
\end{cases}
\]

\[
\leq \begin{cases} 
A^{-1} \tau & (0 < \tau < 1) \\
A^{-\frac{1}{2}} \frac{\tau}{\|\nabla_h \cdot u_h(\tau)\|_{L^p_{3}(\mathbb{R}^3)}} & (\tau \geq 1)
\end{cases}
\]

\[
\leq C A \tau^{-1} (1 + \tau)^{-\frac{1}{2}}.
\]
\[ \leq CABt^{-(1-\frac{1}{p})-\frac{1}{p} \log(2+t)} \]
for \(1 < p \leq \infty\). This completes the proof. \(\square\)

**Remark 4.5.** From the above proof, we see that if the assumption (A3) is modified and \(\|\partial_3 u(t)\|_{L_t^\infty X_{t}^{\sigma}}\) is assumed to be bounded around \(t = 0\), then we obtain the estimate (4.15) even for the case \(p = 1\).

**Lemma 4.6.** Let \(u\) satisfy (A1) and (A2) for some \(s \geq 3\), \(A > 0\) and \(T = \infty\). Then, for \(1 \leq p \leq \infty\), it holds
\[
\lim_{t \to \infty} t^{(1-\frac{1}{p})+\frac{1}{p}} \left\| D_x^3 u(t,x) - \nabla_h G_h(t,x_h) \cdot \int_0^\infty \int_{\mathbb{R}^2} (u_3 u_1)(\tau,y_h,x_h) dy_h d\tau \right\|_{L^p} = 0.
\]

We omit the proof of Lemma 4.6 since it is similar to that of (4.15).

5. Proofs of Theorem 3.1

In this section, we prove Theorem 3.1. First of all, we recall the global existence result for (3.1):

**Proposition 5.1 (3.1).** Let \(s \in \mathbb{N}\) satisfy \(s \geq 2\). Then, there exists a positive constant \(\delta_0 = \delta_0(s)\) such that for every \(u_0 \in H^s(\mathbb{R}^3)\) satisfying \(\nabla \cdot u_0 = 0\) and \(\|u_0\|_{H^s} \leq \delta_0\), (3.1) possesses a unique solution \(u \in C([0, \infty); X^s(\mathbb{R}^3))\). Moreover, it holds
\[
\|u(t)\|_{H^s}^2 + \int_0^t \|\nabla u(\tau)\|_{H^s}^2 d\tau \leq 2\|u_0\|_{H^s}^2, \quad \sigma = 0, 1, \ldots, s
\]
for all \(t \geq 0\).

We are ready to present the proof of Theorem 3.1.

**Proof of Theorem 3.1.** We split the proof into three steps. At the first step, we prove the solution belongs to \(C([0, \infty); X^s(\mathbb{R}^3))\) if the \(H^s\)-norm of \(u_0 \in X^s(\mathbb{R}^3)\) is sufficiently small. In the second step, we show the \(L^p\)-decay estimate (4.4) and (4.5) by virtue of Lemmas 4.1 and 4.2. Finally, in the third step, we prove the asymptotic decay estimates by combining Lemmas 3.1 and 4.2 with 4.4.

**Step 1. (Global solutions in \(X^s(\mathbb{R}^3)\))**

Let \(s \in \mathbb{N}\) satisfy \(s \geq 5\) and let \(u_0 \in X^s(\mathbb{R}^3)\) satisfy \(\nabla \cdot u_0 = 0\) and \(\|u_0\|_{H^s} \leq \delta_0\). Let \(u\) be the global solution to (3.1), which is constructed by Proposition 5.1. From the integral equation (2.3), we have
\[
\|u(t)\|_{L^1_h(W^{1,1} \cap W^{1,\infty})_{x_3}} \leq \|e^{\Delta_h} u_0\|_{L^1_h(W^{1,1} \cap W^{1,\infty})_{x_3}} + C \sum_{m=1}^5 \|D^h_{x_1} u(t)\|_{L^1_h W^{2,1}_{x_3}} + C \sum_{m=1}^3 \|D^h_{x_3} u(t)\|_{L^1_h W^{2,1}_{x_3}}
\]
\[
\leq C\|u_0\|_{X^s} + C \int_0^t \|\nabla^3 (u_3 u_1)(\tau)\|_{L^1_h W^{2,1}_{x_3}} d\tau + C \sum_{k,l=1}^3 \int_0^t (t-\tau)^{-\frac{1}{2}} \|u_k u_l(\tau)\|_{L^1_h W^{2,1}_{x_3}} d\tau
\]
\[
\leq C\|u_0\|_{X^s} + C \int_0^t \|u(\tau)\|_{H^s}^2 d\tau + C \int_0^t (t-\tau)^{-\frac{1}{2}} \|u(\tau)\|_{H^2}^2 d\tau
\]
\[
\leq C \|u_0\|_{X^s} + Ct\|u_0\|_{H^s}^2 + Ct \|u_0\|_{X^s}^2
\]
for all \(t > 0\). Here, we have used the abbreviation
\[
\| \cdot \|_{L^1_h(W^{1,1} \cap W^{1,\infty})_{x_3}} = \| \cdot \|_{L^1(R^2_h(W^{1,1} \cap W^{1,\infty})(\mathbb{R}_{x_3}))},
\]
\[
\| \cdot \|_{L^1_h W^{2,1}_{x_3}} = \| \cdot \|_{L^1(R^2_h W^{2,1}(\mathbb{R}_{x_3}))}.
\]

It is easy to check that \(t \mapsto \|u(t)\|_{L^1_h(W^{1,1} \cap W^{1,\infty})_{x_3}}\) is continuous. Therefore, we have \(u \in C([0, \infty); X^s(\mathbb{R}^3))\).

**Step 2. (\(L^p\) decay estimates)**

In this step, we show (4.4) and (4.5). The idea of the proof is the continuous argument used in (3.1). By Lemmas 3.1 and 3.2, there exists an absolute positive constant \(C_1\) such that
\[
\|\nabla^\alpha e^{\Delta_h} u_{0,h}(t)\|_{L^p} \leq C_1 t^{-(1-\frac{1}{p})-\frac{1}{p(\alpha)}} \|u_0\|_{X^s},
\]
Proposition 2.2 and Lemma 4.2 with $A$ and we shall prove for $t > 0$ and $t < T$ by Lemmas 3.1, 4.2 and 4.4, we have
\[ \|\nabla^\alpha \varphi(t)\|_{L^p} \leq C_1 t^{-\frac{1}{3}(1 - \frac{1}{p})} \|u_0\|_{X^s}, \]
and we shall prove $T = \infty$. By the similar calculation as in the proof of $u(t) \in L^1(\mathbb{R}^2_0; (W^{1,1} \cap W^{1,\infty})(\mathbb{R}^3))$, it is easy to see that $T > 0$. Suppose by contradiction that $T < \infty$. Then, by Proposition 2.2 and Lemma 4.2 with $A = 2C_1\|u_0\|_{X^s}$, we see that
\[ \|\nabla^\alpha \varphi(t)\|_{L^p} \leq C_1 t^{-\frac{1}{3}(1 - \frac{1}{p})} \|u_0\|_{X^s}, \]
\[ \|\nabla^\alpha \varphi_3(t)\|_{L^p} \leq C_1 t^{-\frac{1}{3}(1 - \frac{1}{p})} \|u_0\|_{X^s}, \]
for some $C_2 > 0$ and all $0 < t < T$. Therefore, if the initial data satisfies
\[ \|u_0\|_{X^s} \leq \delta_1 := \min\{\delta_0, C_1/(2C_2)\}, \]
then we have
\[ \|\nabla^\alpha \varphi(t)\|_{L^p} \leq \frac{3}{2} C_1 t^{-\frac{1}{3}(1 - \frac{1}{p})} \|u_0\|_{X^s}, \]
\[ \|\nabla^\alpha \varphi_3(t)\|_{L^p} \leq \frac{3}{2} C_1 t^{-\frac{1}{3}(1 - \frac{1}{p})} \|u_0\|_{X^s}, \]
for $0 < t < T$. This contradicts to the definition of $T$ and we complete the proof of (1.4) and (1.5).

**Step 3. (Asymptotic expansions)**

By Lemmas 3.1, 4.2 and 4.4, we have
\[ t^{1 - \frac{1}{p}} \left\| u_3(t, x) - G_3(t, x) \int_{\mathbb{R}^2} u_0(y) dy \right\|_{L^p} \]
\[ \leq t^{1 - \frac{1}{p}} \left\| e^{t \Delta} u_3(x) - G_3(t, x) \int_{\mathbb{R}^2} u_0(y) dy \right\|_{L^p} \]
\[ + t^{1 - \frac{1}{p}} \left\| D^{\|\nabla\|}_p[u](t) + G_3(t, x) \int_{\mathbb{R}^2} \partial_3(u_3(t, y, x, d_3) dy \right\|_{L^p} \]
\[ \leq t^{1 - \frac{1}{p}} \left\| e^{t \Delta} u_3(x) - G_3(t, x) \int_{\mathbb{R}^2} u_0(y) dy \right\|_{L^p} \]
\[ + t^{1 - \frac{1}{p}} \left\| D^{\|\nabla\|}_p[u](t) + G_3(t, x) \int_{\mathbb{R}^2} \partial_3(u_3(t, y, x, d_3) dy \right\|_{L^p} \]
\[ \rightarrow 0 \quad \text{as } t \to \infty, \]
which gives (1.6). For the proof of (1.7), we see by Lemmas 3.2 and 4.2 that
\[ t^{\frac{1}{p}(1 - \frac{1}{p})} \left\| u_3(t, x) - G_3(t, x) \int_{\mathbb{R}^2} u_0(y) dy \right\|_{L^p} \]
\[ \leq t^{\frac{1}{p}(1 - \frac{1}{p})} \left\| e^{t \Delta} u_3(x) - G_3(t, x) \int_{\mathbb{R}^2} u_0(y) dy \right\|_{L^p} \]
\[ + \sum_{m=1}^{\infty} t^{\frac{1}{p}(1 - \frac{1}{p})} \left\| D^{\|\nabla\|}_m[u](t) \right\|_{L^p} \]
\[ \leq CR_p(t)^{\frac{1}{p}} \|u_0\|_{X^s}^{\frac{1}{p}} + C\|u_0\|_{X^s} \|u_0\|_{X^s}^{\frac{1}{p}} + C\|u_0\|_{X^s} \|u_0\|_{X^s}^{\frac{1}{p}} \log(2 + t). \]
Then, the right hand side converges to 0 as $t \to \infty$ if $p < \infty$. Thus, we complete the proof. 

6. Additional Estimates for the Solution

The aim of this section is to prove that the solution $u$ of (1.1) satisfies the assumptions (A3) and (A4) for $T = \infty$, $A = C\|u_0\|_{X^s}$, and $B = C\|u_0\|_{X^s}$. 

Proposition 6.1. Let $s \in \mathbb{N}$ satisfy $s \geq 9$. Then, there exist a positive constant $\delta_3 = \delta_3(s) \leq \delta_1(s)$ and an absolute positive constant $C$ such that for solutions $u$ to (1.1) with the initial data $u_0 \in X^s(\mathbb{R}^3)$ satisfying $\nabla \cdot u_0 = 0$ and $\|u_0\|_{X^s} \leq \delta_3$, it holds

$$\|\nabla^\alpha u(t)\|_{L^\infty_t L^1_x} \leq Ct^{-1-|\alpha|/2} \|u_0\|_{X^s}$$

for all $t > 0$ and $\alpha \in (\mathbb{N} \cup \{0\})^3$ with $|\alpha| \leq 1$.

Proof. By Lemma 3.1, there exists an absolute positive constant $C$ such that

$$\|\nabla^\alpha e^{\Delta_t} u_0\|_{L^\infty_t L^1_x} \leq Ct^{-1-|\alpha|/2} \|u_0\|_{X^s}$$

for all $t > 0$ and $\alpha \in (\mathbb{N} \cup \{0\})^3$ with $|\alpha| \leq 1$. Thus, the continuous argument via Lemma 4.3 completes the proof. We omit the detail since the argument is similar to the second step of the proof of Theorem 1.1. $$\Box$$

Remark 6.2. If we assume $\nabla^\alpha u_0 \in L^\infty_t L^1_x(\mathbb{R}^3)$ for $|\alpha| \leq 1$, then by the slight modification of the proof of Lemma 4.3 and Proposition 6.1, we obtain

$$\|\nabla^\alpha u(t)\|_{L^\infty_t L^1_x} \leq C(\|u_0\|_{X^s} + \|\nabla^\alpha u_0\|_{L^\infty_t L^1_x})(1 + t)^{-1-|\alpha|/2}$$

for $t > 0$ and $|\alpha| \leq 1$. Hence by Remark 4.3, 4.10 holds even when $p = 1$, if $\nabla^\alpha u_0 \in L^\infty_t L^1_x(\mathbb{R}^3)$ for $|\alpha| \leq 1$.

Proposition 6.3. For $s \in \mathbb{N}$ with $s \geq 9$, there exist an absolute positive constant $C$ and a positive constant $\delta_2 = \delta_2(s)$ such that for any $u_0 \in X^s(\mathbb{R}^3)$ satisfying $\nabla \cdot u_0 = 0$, $|x_h|u_0(x) \in L^1((\mathbb{R}^2_x; (L^1 \cap L^\infty)(\mathbb{R})) \times \mathbb{R})$ and $\|u_0\|_{X^s} \leq \delta_2$, the solution $u$ of (1.1) satisfies

$$\|x_h|u_0(t, x)\|_{L^1_t L^\infty_x} \leq C(1 + t)^{-1} \|u_0\|_{X^s},$$

$$\|x_h|u(t, x)\|_{L^1_t L^\infty_x} \leq C \|u_0\|_{X^s}$$

for all $t > 0$, where $\|u_0\|_{X^s} := \|u_0\|_{X^s} + \|x_h|u_0(x)\|_{L^1((\mathbb{R}^2_x; (L^1 \cap L^\infty)(\mathbb{R})) \times \mathbb{R})}$.

To prove Proposition 6.3, let us consider the following integral equation:

$$\begin{cases}
  v_h(t) = e^{t \Delta_h} u_{0,h} + \sum_{m=1}^{5} D_m^h[u, v](t), \\
  v_3(t) = e^{t \Delta_h} u_{0,3} + \sum_{m=1}^{3} D_m^v[u, v](t),
\end{cases}$$

where $v = (v_h(t, x), v_3(t, x))$ is the unknown vector field and $u$ is the solution to (1.1) with the initial data $u_0$ and the Duhamel terms are defined by

$$D^h_1[u, v](t) := -\int_0^t e^{(t-\tau)\Delta_h} ((\partial_3 u_3)v_h + v_3 \partial_3 u_h)(\tau)d\tau,$$

$$D^h_2[u, v](t) := -\int_0^t e^{(t-\tau)\Delta_h} \nabla_h \cdot (u_h \otimes v_h)(\tau)d\tau,$$

$$D^h_3[u, v](t) := \int_0^t \nabla_h e^{(t-\tau)\Delta_h} (u_3 v_3)(\tau)d\tau,$$

$$D^h_4[u, v](t) := -\sum_{k,l=1}^{2} \int_0^t \nabla_h \partial_k \partial_l K(t - \tau) * (u_k v_l)(\tau)d\tau,$$

$$D^h_5[u, v](t) := 2 \sum_{k=1}^{2} \int_0^t \nabla_h \partial_k ((\Delta_h)^{1/2} v_3)(t - \tau) * (u_3 v_k)(\tau)d\tau + \int_0^t \nabla_h \Delta_h K(t - \tau) * (u_3 v_3)(\tau)d\tau$$

and

$$D^v_1[u, v](t) := \int_0^t e^{(t-\tau)\Delta} \nabla_h \cdot (v_h u_3)(\tau)d\tau,$$

$$D^v_2[u, v](t) := 2 \sum_{k,l=1}^{2} \int_0^t (\Delta_h)^{1/2} \partial_k \partial_l K(t - \tau) * (u_k v_l)(\tau)d\tau,$$
\[ D^h_1[u, v](t) := 2 \sum_{k=1}^{2} \int_0^t \partial_t \frac{\partial}{\partial t} K(t - \tau) * (u_3 v_k)(\tau) d\tau + \int_0^t (-\Delta_h)^{\frac{1}{2}} K(t - \tau) * (u_3 v_3)(\tau) d\tau. \]

We note that \( u \) is a solution to (6.1). Therefore, once we prove the existence of solutions to (6.4) in an appropriate function space equipped with some weighted norm and the uniqueness in \( L^\infty(0, \infty; L^1 \xi L^\infty(\mathbb{R}^3)) \), then we obtain the desired weighted estimates for \( u \). Thanks to the definition of the Duhamel term \( D^h_1[u, v] \), the well-posedness problem for (6.1) is much easier than that for (6.4), since no \( \partial_t \) derivative loss occurs and we can construct the solution of (6.4) simply by the contraction mapping approach.

We define the solution space \( Y \) by
\[
Y := \left\{ v = (v_1, v_2, v_3) \in L^\infty(0, \infty; L^1 \xi L^\infty(\mathbb{R}^3)) : \| v \|_Y < \infty \right\},
\]
\[
\| v \|_Y := \sup_{t \geq 0} (1 + t)^{-\frac{1}{2}} \| x_h v_1(t, x) \|_{L^1_{\xi} L^\infty} + \sup_{t \geq 0} \| x_h v_3(t, x) \|_{L^1_{\xi} L^\infty}
+ \sup_{t \geq 0} \| v_3(t) \|_{L^1(\mathbb{R}^3; L^1(\mathbb{R}^3))} + \sup_{t \geq 0} (1 + t)^{\frac{1}{2}} \| v_3(t) \|_{L^1(\mathbb{R}^3; L^1(\mathbb{R}^3))}.
\]

We prepare some lemmas for the proof of Proposition 6.3.

**Lemma 6.4.** Let \( s \in \mathbb{N} \) satisfy \( s \geq 7 \). Let \( u \) be the solution to (1.7) with the data \( u_0 \in X^s(\mathbb{R}^3) \) satisfying \( \nabla \cdot u_0 = 0 \) and \( \| u_0 \|_{X^s} \leq \delta_1 \). Then, there exists an absolute positive constant \( C \) such that
\[
\| x_h [D^h_1[u, v](t)](t, x) \|_{L^1_{\xi} L^\infty} \leq C(1 + t)^{\frac{1}{2}} \| u_0 \|_{X^s} \| v \|_Y,
\]
\[
\| D^h_1[u, v](t) \|_{L^1_{\xi} L^\infty} \leq C \| u_0 \|_{X^s} \| v \|_{L^\infty(0, \infty; L^1_{\xi} L^\infty)},
\]
\[
\| \partial_t D^h_1[u, v](t) \|_{L^1_{\xi} L^\infty} \leq C \| u_0 \|_{X^s} \| v \|_Y
\]
for all \( v \in Y \) and \( t \geq 0 \).

**Proof.** For the proof of (6.2), we have by Lemma 5.1 (1) that
\[
\| x_h [D^h_1[u, v](t)](t, x) \|_{L^1_{\xi} L^\infty}
\leq \int_0^t \| x_h |G_h(t - \tau, x_h)\|_{L^1(\mathbb{R}^3)} \| (\partial_3 u_3) v_1 + v_3 \partial_3 u_h) \|_{L^1_{\xi} L^\infty} d\tau
+ \int_0^t \| G_h(t - \tau) \|_{L^1(\mathbb{R}^3)} \| (\partial_3 u_3) v_1 + v_3 \partial_3 u_h) \|_{L^1_{\xi} L^\infty} d\tau
\leq C \int_0^t (t - \tau)^{\frac{1}{2}} \left( \| \partial_3 u_3(t) \|_{L^\infty} \| v_1(t) \|_{L^1_{\xi} L^\infty} + \| v_3(t) \|_{L^1_{\xi} L^\infty} \| \partial_3 u_h(t) \|_{L^\infty} \right) d\tau
+ C \int_0^t (t - \tau)^{\frac{1}{2}} (1 + t - \tau)^{-\frac{1}{2}} d\tau \| u_0 \|_{X^s} \| v \|_Y + C \int_0^t (1 + \tau)^{-\frac{1}{2}} d\tau \| u_0 \|_{X^s} \| v \|_Y
\leq C(1 + t)^{\frac{1}{2}} \| u_0 \|_{X^s} \| v \|_Y.
\]

On the estimate for (6.3), we have
\[
\| D^h_1[u, v](t) \|_{L^1_{\xi} L^\infty} \leq \int_0^t \| G_h(t - \tau) \|_{L^1(\mathbb{R}^3)} \| (\partial_3 u_3) v_1 + v_3 \partial_3 u_h) \|_{L^1_{\xi} L^\infty} d\tau
\leq C \int_0^t \left( \| \partial_3 u_3(t) \|_{L^\infty} \| v_1(t) \|_{L^1_{\xi} L^\infty} + \| v_3(t) \|_{L^1_{\xi} L^\infty} \| \partial_3 u_h(t) \|_{L^\infty} \right) d\tau
\leq C \int_0^t (1 + \tau)^{-\frac{1}{2}} d\tau \| u_0 \|_{X^s} \| v \|_Y
\leq C \| u_0 \|_{X^s} \| v \|_Y.
\]

Finally, we prove (6.4). It follows from (4.10) that
\[
\| \partial_t D^h_1[u, v](t) \|_{L^1_{\xi} L^\infty}
\leq \int_0^t \| G_h(t - \tau) \|_{L^1(\mathbb{R}^3)} \left( \| \partial_3^2 u_3(t) \|_{L^\infty} \| v_1(t) \|_{L^1_{\xi} L^\infty} + \| \partial_3 u_3(t) \|_{L^\infty} \| \partial_3 v_3(t) \|_{L^1_{\xi} L^\infty} \right)
\]
\begin{align*}
+ \| \partial_3 v_3(\tau) \|_{L^1_t L^\infty_x} & | \partial_3 u_h(\tau) \|_{L^\infty_t L^\infty_x} + \| v_3(\tau) \|_{L^1_t L^\infty_x} \| \partial_3^2 u_h(\tau) \|_{L^\infty_t L^\infty_x} \\
\leq C \int_0^t & \left( \| u_3(\tau) \|_{H^s}^2 | \partial_3 u_3(\tau) \|_{L^1_t L^\infty_x} + \| v_3(\tau) \|_{L^1_t L^\infty_x} \| \partial_3 v_3(\tau) \|_{L^\infty_t L^\infty_x} \\
& + \| \partial_3 v_3(\tau) \|_{L^1_t L^\infty_x} \| \partial_3 u_h(\tau) \|_{L^\infty_t L^\infty_x} + \| v_3(\tau) \|_{L^1_t L^\infty_x} \| u_h(\tau) \|_{H^s}^2 \right) \| \partial_3 u_h(\tau) \|_{L^\infty_t L^\infty_x} d\tau \\
\leq C & \int_0^t (1 + \tau)^{-\frac{3l}{2}} dt \| u_0 \|_{X^s} \| v \|_{Y} \\
\leq C & \| u_0 \|_{X^s} \| v \|_{Y}
\end{align*}

Thus, we complete the proof. \( \square \)

**Lemma 6.5.** Let \( s \in \mathbb{N} \) with \( s \geq 5 \) and let \( u \) be the solution to \( (\text{1.4}) \) with the data \( u_0 \in X^s(\mathbb{R}^3) \) satisfying \( \nabla \cdot u_0 = 0 \) and \( \| u_0 \|_{X^s} \leq \delta_1 \). Then, there exists an absolute positive constant \( C \) such that

\begin{align*}
\left\| |x_h| \int_0^t e^{(t-\tau)\Delta_h} \nabla_h(u_0 v_l)(\tau, x) d\tau \right\|_{L^1_t L^\infty_x} & \leq \begin{cases}
C(1 + t)^{\frac{l}{2}} \| u_0 \|_{X^s} \| v_l \|_{Y} & (k = 1, 2) \\
C \| u_0 \|_{X^s} \| v_l \|_{Y} & (k = 3)
\end{cases} \quad (6.5) \\
\left\| \int_0^t e^{(t-\tau)\Delta_h} \nabla_h(u_0 v_l)(\tau) d\tau \right\|_{L^1_t L^\infty_x} & \leq \begin{cases}
C \| u_0 \|_{X^s} \| v_l \|_{L^\infty_t(0, \infty; L^1_x)} & (k = 1, 2) \\
C(1 + t)^{\frac{3l}{2}} \| u_0 \|_{X^s} \| v_l \|_{L^\infty_t(0, \infty; L^1_x)} & (k = 3)
\end{cases} \quad (6.6) \\
\left\| \partial_3 \int_0^t e^{(t-\tau)\Delta_h} \nabla_h(u_0 v_l)(\tau) d\tau \right\|_{L^1_t L^\infty_x} & \leq \begin{cases}
C \| u_0 \|_{X^s} \| v_l \|_{Y} & (k = 1, 2) \\
C(1 + t)^{\frac{3l}{2}} \| u_0 \|_{X^s} \| v_l \|_{Y} & (k = 3)
\end{cases} \quad (6.7)
\end{align*}

for all \( t \geq 0 \) and \( l = 1, 2, 3 \).

**Proof.** On the estimate \( (6.5) \), we see by Lemma 3.1 (1) that

\begin{align*}
\left\| |x_h| \int_0^t e^{(t-\tau)\Delta_h} \nabla_h(u_0 v_l)(\tau, x) d\tau \right\|_{L^1_t L^\infty_x} & \leq \int_0^t \left\| |x_h| \nabla_h G_h(t - \tau, x_h) \|_{L^1(\mathbb{R}^3)} \| u_0(\tau) \|_{L^\infty_t L^\infty_x} \| v_l(\tau) \|_{L^1_t L^\infty_x} d\tau \\
& + \int_0^t \| \nabla_h G_h(t - \tau) \|_{L^1(\mathbb{R}^3)} \| u_0(\tau) \|_{L^1_t L^\infty_x} \| v_l(\tau) \|_{L^1_t L^\infty_x} d\tau \\
& \leq \begin{cases}
C \int_0^t (1 + \tau)^{-\frac{3l}{2}} d\tau \| u_0 \|_{X^s} \| v \|_{Y} & (k = 1, 2) \\
+C \int_0^t (t - \tau)^{\frac{3l}{2}} (1 + \tau)^{-\frac{3l}{2}} d\tau \| u_0 \|_{X^s} \| v \|_{Y} & (k = 1, 2) \\
+C \int_0^t (1 + \tau)^{-\frac{3l}{2}} d\tau \| u_0 \|_{X^s} \| v \|_{Y} & (k = 3)
\end{cases}
\end{align*}

We next obtain \( (6.6) \) by

\begin{align*}
\left\| \int_0^t e^{(t-\tau)\Delta_h} \nabla_h(u_0 v_l)(\tau) d\tau \right\|_{L^1_t L^\infty_x} & \leq C \int_0^t (1 + \tau)^{-\frac{3l}{2}} \| u_0(\tau) \|_{L^\infty_t L^\infty_x} \| v_l(\tau) \|_{L^1_t L^\infty_x} d\tau \\
& \leq \begin{cases}
C \int_0^t (t - \tau)^{\frac{3l}{2}} (1 + \tau)^{-\frac{3l}{2}} d\tau \| u_0 \|_{X^s} \| v \|_{L^\infty_t(0, \infty; L^1_x)} & (k = 1, 2) \\
+C \int_0^t (1 + \tau)^{-\frac{3l}{2}} d\tau \| u_0 \|_{X^s} \| v \|_{L^\infty_t(0, \infty; L^1_x)} & (k = 3)
\end{cases}
\end{align*}
Finally, for the estimate (6.1), we have

\[
\left\| \partial_t \int_0^t e^{(t-\tau)\Delta_h} \nabla_h (u_k v_l)(\tau) d\tau \right\|_{L^1_t L^\infty_x} \\
\leq \int_0^t \| \nabla_h G_h(t-\tau) \|_{L^1_t L^1_x} \left( \left\| \partial \partial_t u_k(\tau) \right\|_{L^\infty_t L^1_x} + \left\| u_k(\tau) \right\|_{L^\infty_t L^1_x} \right) d\tau \\
\leq C \int_0^t (t-\tau)^{-\frac{3}{2}} (1+\tau)^{-1} d\tau \| u_0 \|_{X^\infty} \| v \|_{Y} \quad (k = 1, 2) \\
\leq C \int_0^t (t-\tau)^{-\frac{3}{2}} (1+\tau)^{-\frac{3}{2}} d\tau \| u_0 \|_{X^\infty} \| v \|_{Y} \quad (k = 3),
\]

This completes the proof. \( \square \)

**Lemma 6.6.** Let \( s \in \mathbb{N} \) with \( s \geq 9 \) and let \( u \) be the solution to (2.1) with the data \( u_0 \in X^s(\mathbb{R}^3) \) satisfying \( \nabla \cdot u_0 = 0 \| u_0 \|_{X^\infty} \leq \delta_3 \). Then there exists an absolute positive constant \( C \) such that

\[
\left\| x_h \int_0^t K^{(m)}_{\beta,\gamma}(t-\tau) \ast (u_k v_l)(\tau, x) d\tau \right\|_{L^1_t L^\infty_x} \leq C \| u_0 \|_{X^\infty} \| v \|_{Y}, \quad (6.8)
\]

\[
\left\| \int_0^t K^{(m)}_{\beta,\gamma}(t-\tau) \ast (u_k v_l)(\tau) d\tau \right\|_{L^1_t L^\infty_x} \leq C (1 + t)^{-\frac{3}{2}} \| u_0 \|_{X^\infty} \| v \|_{L^\infty_t(0,\infty; L^1_h L^\infty_x)}, \quad (6.9)
\]

\[
\left\| \partial_t \int_0^t K^{(m)}_{\beta,\gamma}(t-\tau) \ast (u_k v_l)(\tau) d\tau \right\|_{L^1_t L^\infty_x} \leq C (1 + t)^{-\frac{3}{2}} \| u_0 \|_{X^\infty} \| v \|_{Y}, \quad (6.10)
\]

for all \( k, l = 1, 2, 3, m = 1, 2 \) and \( t \geq 0 \). where \( K^{(m)}_{\beta,\gamma} \) are defined by (4.7) for \( (\beta, \gamma) \in (\mathbb{N} \cup \{0\})^2 \times (\mathbb{N} \cup \{0\}) \) satisfying \( |\beta| + |\gamma| = 3 \).

**Proof.** For the estimate (6.8), we have by Lemma 3.3 that

\[
\left\| x_h \int_0^t K^{(m)}_{\beta,\gamma}(t-\tau) \ast (u_k v_l)(\tau, x) d\tau \right\|_{L^1_t L^\infty_x} \\
\leq \int_0^t \left\| x_h K^{(m)}_{\beta,\gamma}(t-\tau, x) \right\|_{L^1_t L^2_x} \| u_k(\tau) v_l(\tau) \|_{L^1_t L^2_x} d\tau + \int_0^t \left\| K^{(m)}_{\beta,\gamma}(t-\tau) \right\|_{L^1_t L^1_x} \| u_k(\tau) \|_{L^\infty_t L^1_x} \| x_h v_l(\tau) \|_{L^1_t L^\infty_x} d\tau \\
\leq C \int_0^t (t-\tau)^{-\frac{3}{2}} \| u_k(\tau) \|_{L^1_t L^2_x} \| v_l(\tau) \|_{L^1_t L^\infty_x} d\tau + C \int_0^t (t-\tau)^{-\frac{3}{2}} \| u_k(\tau) \|_{L^\infty_t L^1_x} \| x_h v_l(\tau) \|_{L^1_t L^\infty_x} d\tau \\
\leq C \int_0^t (t-\tau)^{-\frac{3}{2}} \left( (1 + t)^{-\frac{3}{2}} d\tau \| u_0 \|_{X^\infty} \| v \|_{Y} \right) + \int_0^t (t-\tau)^{-\frac{3}{2}} \left( (1 + t)^{-\frac{3}{2}} d\tau \| u_0 \|_{X^\infty} \| v \|_{Y} \right) \\
\leq C \| u_0 \|_{X^\infty} \| v \|_{Y}. \]

On the estimate (6.9), we see that

\[
\left\| \int_0^t K^{(m)}_{\beta,\gamma}(t-\tau) \ast (u_k v_l)(\tau) d\tau \right\|_{L^1_t L^\infty_x} \\
\leq \int_0^t \left\| K^{(m)}_{\beta,\gamma}(t-\tau) \right\|_{L^1_t L^2_x} \| u_k(\tau) v_l(\tau) \|_{L^1_t L^2_x} d\tau + \int_0^t \left\| K^{(m)}_{\beta,\gamma}(t-\tau) \right\|_{L^1_t L^1_x} \| u_k(\tau) \|_{L^\infty_t L^1_x} \| v_l(\tau) \|_{L^1_t L^\infty_x} d\tau \\
\leq C \int_0^t (t-\tau)^{-\frac{3}{2}} \| u_k(\tau) \|_{L^1_t L^2_x} \| v_l(\tau) \|_{L^1_t L^\infty_x} d\tau + C \int_0^t (t-\tau)^{-\frac{3}{2}} \| u_k(\tau) \|_{L^\infty_t L^1_x} \| v_l(\tau) \|_{L^1_t L^\infty_x} d\tau \\
\leq C \int_0^t (t-\tau)^{-\frac{3}{2}} \left( (1 + t)^{-\frac{3}{2}} d\tau \| u_0 \|_{X^\infty} \| v \|_{L^\infty_t(0,\infty; L^1_h L^\infty_x)} \right) + C \int_0^t (t-\tau)^{-\frac{3}{2}} \left( (1 + t)^{-1} d\tau \| u_0 \|_{X^\infty} \| v \|_{L^\infty_t(0,\infty; L^1_h L^\infty_x)} \right) \\
\leq C A^2 \left( (1 + t)^{-\frac{3}{2}} \right) \| u_0 \|_{X^\infty} \| v \|_{L^\infty_t(0,\infty; L^1_h L^\infty_x)} \]
Finally, we can prove (6.10). By the similar calculation as above, we see that

\[ \left\| \frac{\partial}{\partial t} \int_0^t K_{\beta,\gamma}^{(m)}(t-\tau) * (u_k v_l)(\tau) d\tau \right\|_{L^1_t L^\infty Y} \]

\[ \leq \int_0^t \left\| K_{\beta,\gamma}^{(m)}(t-\tau) \right\|_{L^1_t L^2 Y} \left( \left\| \frac{\partial}{\partial t} u_k(\tau) \right\|_{L^\infty_t L^2 Y} + \left\| u_k(\tau) \right\|_{L^\infty_t L^2 Y} \left\| \frac{\partial}{\partial t} v_l(\tau) \right\|_{L^1_t L^\infty Y} \right) d\tau \]

\[ + \int_0^t \left\| K_{\beta,\gamma}^{(m)}(t-\tau) \right\|_{L^1_t L^2 Y} \left( \left\| \frac{\partial}{\partial t} u_k(\tau) \right\|_{L^\infty_t L^2 Y} + \left\| u_k(\tau) \right\|_{L^\infty_t L^2 Y} \left\| \frac{\partial}{\partial t} v_l(\tau) \right\|_{L^1_t L^\infty Y} \right) d\tau \]

\[ \leq C(1+t)^{-\frac{\delta}{2}} \| u_0 \|_{X^*} \| v \|_{Y}. \]

Hence, we complete the proof. \( \square \)

We are ready to prove Proposition 6.3.

**Proof of Proposition 6.3** Let \( s \in \mathbb{N} \) satisfy \( s \geq 9 \) and let \( u_0 \in X^*(\mathbb{R}^3) \) satisfy \( \nabla \cdot u_0 = 0 \) and \( \| u_0 \|_{X^*} \leq \delta_3 \). Let \( u \) be the solution to (1.1) with the initial data \( u_0 \). It is easy to see that there exists an absolute positive constant \( C_4 \) such that

\[ \| e^{t \Delta} u_0 \|_Y \leq C_4 \| u_0 \|_{X^*}. \]

For each \( v \in Y \), we define \( \Psi[v] = (\Psi^m[v], \Psi^n[v]) \) by

\[
\left\{ \begin{align*}
\Psi^m[v](t) &:= e^{t \Delta} u_{0,m} + \sum_{m=1}^{5} D^m_m[u,v](t), \\
\Psi^n[v](t) &:= e^{t \Delta} u_{0,n} + \sum_{m=1}^{3} D^m_m[u,v](t).
\end{align*} \right.
\]

Then, by Lemmas 6.4, 6.3 and 6.6 there exists an absolute positive constant \( C_5 \) such that

\[
\left\| \left( \sum_{m=1}^{5} D^m_m[u,v], \sum_{m=1}^{3} D^m_m[u,v] \right) \right\|_Y \leq C_5 \| u_0 \|_{X^*} \| v \|_Y,
\]

\[
\left\| \left( \sum_{m=1}^{5} D^m_m[u,w], \sum_{m=1}^{3} D^m_m[u,w] \right) \right\|_{L^\infty(0;L^1_t L^\infty Y)} \leq C_5 \| u_0 \|_{X^*} \| w \|_{L^\infty(0;L^1_t L^\infty Y)}
\]

for all \( v \in Y \) and \( w \in L^\infty(0, \infty; L^1_t L^\infty Y) \). Thus, it holds

\[
\| \Psi[v] \|_Y \leq \| e^{t \Delta} u_0 \|_Y + \left\| \left( \sum_{m=1}^{5} D^m_m[u,v], \sum_{m=1}^{3} D^m_m[u,v] \right) \right\|_Y \leq C_4 \| u_0 \|_{X^*} + C_5 \| u_0 \|_{X^*} \| v \|_Y < \infty
\]

for \( v \in Y \), which implies \( \Psi[v] \in Y \). Let \( \delta_2 = \min\{ \delta_3, 1/(2C_5) \} \) and assume \( \| u_0 \|_{X^*} \leq \delta_2 \). Then, we have

\[
\| \Psi[v_1] - \Psi[v_2] \|_Y \leq \left\| \left( \sum_{m=1}^{5} D^m_m[u,v_1 - v_2], \sum_{m=1}^{3} D^m_m[u,v_1 - v_2] \right) \right\|_Y \leq C_5 \| u_0 \|_{X^*} \| v_1 - v_2 \|_Y \leq \frac{1}{2} \| v_1 - v_2 \|_Y.
\]

Therefore, the contraction mapping principle yields that there exists a unique \( \tilde{v} \in Y \) such that \( \tilde{v} = \Psi[\tilde{v}] \). Then by (6.11), we have

\[
\| \tilde{v} \|_Y \leq C_4 \| u_0 \|_{X^*} + \frac{1}{2} \| \tilde{v} \|_Y,
\]

which implies \( \| \tilde{v} \|_Y \leq 2C_4 \| u_0 \|_{X^*} \).
Finally, we show $\tilde{v} = u$. Since $\tilde{v}, u \in L^\infty(0, \infty; L^1_t L^\infty(\mathbb{R}^3))$ and $u$ is also a solution to (6.1), we have
\[
\|\tilde{v} - u\|_{L^\infty(0, \infty; L^1_t L^\infty(\mathbb{R}^3))} = \|\Psi[\tilde{v}] - \Psi[u]\|_{L^\infty(0, \infty; L^1_t L^\infty(\mathbb{R}^3))} \\
\leq C_5 \|u_0\|_{X^s} \|\tilde{v} - u\|_{L^\infty(0, \infty; L^1_t L^\infty(\mathbb{R}^3))} \\
\leq \frac{1}{2} \|\tilde{v} - u\|_{L^\infty(0, \infty; L^1_t L^\infty(\mathbb{R}^3))}.
\]
This implies $\tilde{v} = u$. Hence, we see that $u \in Y$ and $\|u\|_Y \leq 2C_4 \|u_0\|_{X^s}$, which complete the proof. □

7. PROOFS OF THEOREM 1.3 AND COROLLARY 1.5

Now, we give the proofs of Theorem 1.3 and Corollary 1.5.

Proof of Proposition 1.3. Let $s \in \mathbb{N}$ satisfy $s \geq 9$ and let $\delta_2$ be the constant determined in Proposition 6.3. Let $u_0 \in X^s(\mathbb{R}^3)$ satisfy $|x_0| |u(x) - x_0| \in L^1(\mathbb{R}^3; (L^1 \cap L^\infty)(\mathbb{R}^3))$, $x_0 \cdot u_0 = 0$ and $\|u_0\|_{X^s} \leq \delta_2$. Then, by Theorem 1.1, Proposition 6.1 and Proposition 6.3, the solution $u$ of (1.1) with the initial data $u_0$ satisfies the assumptions (A1)–(A4) with $T = \infty$, $A = \|u_0\|_{X^s}$ and $B = \|u_0\|_{X^s}$ for some constant $C$. Hence, Lemmas 3.1, 4.2 and 4.4 imply that
\[
\|u_3(t, x) - G_h(t, x) \int_{\mathbb{R}^2} u_0, h(y, x) dy\|_{L^p_t} \\
+ G_h(t, x) \int_{\mathbb{R}^2} \partial_3(u_3u_h)(\tau, y, x) dy d\tau \|_{L^p_t} \\
\leq \frac{e^{(1-\frac{1}{p})}}{L^p_t} \|e^{i\Delta_h u_0, h(x)} - G_h(t, x) \int_{\mathbb{R}^2} u_0, h(y, x) dy\|_{L^p_t} \\
+ \frac{\|D^h[u](t)\|_{L^p_t}}{L^p_t} + \frac{\|D^h[u](t)\|_{L^p_t}}{L^p_t} + \frac{\|D^h[u](t)\|_{L^p_t}}{L^p_t} \\
\leq C \|u_0\|_{X^s} \|u_0\|_{X^s} \|u_0\|_{X^s} \|t^{-(1-\frac{1}{p}) - \frac{1}{p}} \log(2 + t)}{L^p_t} \\
\leq C \|u_0\|_{X^s} \|t^{-(1-\frac{1}{p}) - \frac{1}{p}} \log t}{L^p_t}
\]
for $t \geq 2$. By Lemmas 3.1 and 4.2 we have
\[
\|u_3(t, x) - G_h(t, x) \int_{\mathbb{R}^2} u_0, h(y, x) dy\|_{L^p_t} \\
\leq t^{\frac{3}{2}(1-\frac{1}{p})} \|u_3(t, x) - G_h(t, x) \int_{\mathbb{R}^2} u_0, h(y, x) dy\|_{L^p_t} \\
\leq C \frac{1}{p} \|u_0\|_{X^s} \|t^{-(1-\frac{1}{p}) - \frac{1}{p}} \log t}{L^p_t} \\
\leq \frac{1}{p} \|u_0\|_{X^s} \|t^{-(1-\frac{1}{p}) - \frac{1}{p}} \log t}{L^p_t} \\
\]
for $t \geq 2$. Finally, we show (1.10). By Lemmas 3.2, 4.2 and 4.4 we obtain
\[
t^{\frac{3}{2}(1-\frac{1}{p}) + \frac{1}{p}} \|u_3(t, x) - G_h(t, x) \int_{\mathbb{R}^2} u_0, h(y, x) dy\|_{L^p_t} \\
+ \partial_3(u_3u_h)(\tau, y, x) dy d\tau \|_{L^p_t} \\
\leq t^{\frac{3}{2}(1-\frac{1}{p}) + \frac{1}{p}} \|e^{i\Delta_h u_0, h(x)} - G_h(t, x) \int_{\mathbb{R}^2} u_0, h(y, x) dy\|_{L^p_t} \\
+ \frac{\|D^h[u](t)\|_{L^p_t}}{L^p_t} + \frac{\|D^h[u](t)\|_{L^p_t}}{L^p_t} + \frac{\|D^h[u](t)\|_{L^p_t}}{L^p_t} \\
\leq C t^{\frac{3}{2}(1-\frac{1}{p}) + \frac{1}{p}} R_{\frac{1}{p}, 1} \|u_0\|_{X^s} \|t^{-(1-\frac{1}{p}) - \frac{1}{p}} \log t}{L^p_t} \\
\]
\[ + t^{(1 - \frac{1}{p}) + \frac{2}{q}} \left\| D^1_t [u](t, x) - \nabla h G_h(t, x_h) \cdot \int_0^\infty \int_{\mathbb{R}^2} (u_3 u_h)(\tau, y_h, x_3) dy_h d\tau \right\|_{L^p_{y_h}} + C t^{-\frac{1}{2} - \frac{1}{2q} + \frac{1}{2} \log(2 + t)}.
\]

Then, the right hand side converges to 0 if \( 1 < p \leq \infty \). Hence, we complete the proof. \( \square \)

**Proof of Corollary** \( \square \) Passing the limit inferior as \( t \to \infty \) in
\[
\begin{align*}
t^\frac{2}{q} \left\| u_3(t, x) - G_h(t, x_h) \int_{\mathbb{R}^2} u_0, 3(y_h, x_3) dy_h \right\|_{L^\infty_x} \\
\geq \begin{cases} t^\frac{2}{q} \left\| \nabla_h G_h(t, x_h) \cdot \int_{\mathbb{R}^2} y_h u_0, 3(y_h, x_3) dy_h - \nabla_h G_h(t, x_h) \cdot \int_0^\infty \int_{\mathbb{R}^2} (u_3 u_h)(\tau, y_h, x_3) dy_h d\tau \right\|_{L^\infty_x} \\
- t^\frac{2}{q} \left\| u_3(t, x) - G_h(t, x_h) \int_{\mathbb{R}^2} u_0, 3(y_h, x_3) dy_h + \nabla_h G_h(t, x_h) \cdot \int_{\mathbb{R}^2} y_h u_0, 3(y_h, x_3) dy_h \\
- \nabla_h G_h(t, x_h) \cdot \int_0^\infty \int_{\mathbb{R}^2} (u_3 u_h)(\tau, y_h, x_3) dy_h d\tau \right\|_{L^\infty_x},
\end{cases}
\end{align*}
\]

we have by (1.10) that
\[
\liminf_{t \to \infty} t^\frac{2}{q} \left\| u_3(t, x) - G_h(t, x_h) \int_{\mathbb{R}^2} u_0, 3(y_h, x_3) dy_h \right\|_{L^\infty_x} \geq \liminf_{t \to \infty} t^\frac{2}{q} \left\| \nabla_h G_h(t, x_h) \cdot \int_{\mathbb{R}^2} y_h u_0, 3(y_h, x_3) dy_h - \nabla_h G_h(t, x_h) \cdot \int_0^\infty \int_{\mathbb{R}^2} (u_3 u_h)(\tau, y_h, x_3) dy_h d\tau \right\|_{L^\infty_x}.
\]

Here, let us consider the following initial data:
\[
u_0(x) := \eta \phi(x), \quad \phi(x) := \left(0, -x_3 e^{-|x|^2}, x_2 e^{-|x|^2}\right),
\]
where \( \eta \in (0, \delta_2(9)/\|\phi\|_{X^9}) \) is a positive constant to be determined later. It is easy to check that this \( u_0 \) satisfies \( u_0 \in X_9(\mathbb{R}^3), |x_h| u_0(x) \in L^1(\mathbb{R}^2; (L^1 \cap L^\infty)(\mathbb{R}^3)), \nabla \cdot u_0 = 0 \) and \( \|u_0\|_{X^9} \leq \delta_2(9) \).

Then, we see that
\[
\nabla_h G_h(t, x_h) \cdot \int_{\mathbb{R}^2} y_h u_0, 3(y_h, x_3) dy_h = \eta \partial_1 G_h(t, x_h) \int_{-\infty}^\infty y_1 e^{-y_1^2} dy_1 \int_{-\infty}^\infty y_2 e^{-y_2^2} dy_2 e^{-x_3^2} \\
+ \eta \partial_2 G_h(t, x_h) \int_{-\infty}^\infty e^{-y_1^2} dy_1 \int_{-\infty}^\infty y_2^2 e^{-y_2^2} dy_2 e^{-x_3^2} \\
= C_6 \eta^2 t^{-\frac{2}{q}} \partial_2 G_h(1, t^{-\frac{2}{q}} x_h) e^{-x_3^2},
\]
which implies
\[
\left\| \nabla_h G_h(t, x_h) \cdot \int_{\mathbb{R}^2} y_h u_0, 3(y_h, x_3) dy_h \right\|_{L^\infty_x} = C_6 \eta^2 t^{-\frac{2}{q}}
\]
for some absolute positive constant \( C_6 \). On the other hand, the corresponding solution \( u \) satisfies
\[
\left\| \nabla_h G_h(t, x_h) \cdot \int_0^\infty \int_{\mathbb{R}^2} (u_3 u_h)(\tau, y_h, x_3) dy_h d\tau \right\|_{L^\infty_x} \leq C t^{-\frac{2}{q}} \int_0^\infty \|u_3(\tau)\|_{L^\infty} \|u_h(\tau)\|_{L^q_{y_h} L^\infty_{x_h}} d\tau \\
\leq C \int_0^\infty (1 + \tau)^{-\frac{2}{q}} d\tau \cdot \eta^2 \|\phi\|_{X^9}^2 t^{-\frac{2}{q}} \\
= C_7 \eta^2 t^{-\frac{2}{q}}
\]
for some absolute positive constant \( C_7 \). Hence, if we choose \( \eta \) such that \( 0 < \eta \leq \min\{\delta_2(9)/\|\phi\|_{X^9}, C_6/(2C_7)\} \), then we have
\[
\liminf_{t \to \infty} t^\frac{2}{q} \left\| u_3(t, x) - G_h(t, x_h) \int_{\mathbb{R}^2} u_0, 3(y_h, x_3) dy_h \right\|_{L^\infty_x} \geq C_6 \eta - C_7 \eta^2 \geq \frac{C_6}{2} \eta > 0.
\]
This completes the proof. \( \square \)
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Graduate School of Mathematics Kyushu University, Fukuoka 819-0395, JAPAN
Email address: 3MA20005M@s.kyushu-u.ac.jp