FINITE GROUP FACTORIZATIONS AND BRAIDING

E. J. Beggs & J. D. Gould
Department of Mathematics
University College of Swansea, Swansea
Wales SA2 8PP

&

S. Majid
Department of Applied Mathematics & Theoretical Physics
University of Cambridge, Cambridge CB3 9EW, U.K.

February 1994 – revised March 1995

ABSTRACT We compute the quantum double, braiding and other canonical Hopf algebra constructions for the bicrossproduct Hopf algebra $H$ associated to the factorization of a finite group into two subgroups. The representations of the quantum double are described by a notion of bicrossed bimodules, generalising the cross modules of Whitehead. We also show that self-duality structures for the bicrossproduct Hopf algebras are in one-one correspondence with factor-reversing group isomorphisms. The example $\mathbb{Z}_6 \times \mathbb{Z}_6$ is given in detail. We show further that the quantum double $D(H)$ is the twisting of $D(X)$ by a non-trivial quantum cocycle, where $X$ is the associated double cross product group.

1 INTRODUCTION

It is known that to every factorisation $X = GM$ of a group into two subgroups $G, M$, is associated a generally non-commutative and non-cocommutative Hopf algebra $H = kM \bowtie k(G)$. Here $k(G)$ is the Hopf algebra of functions on $G$ and $kM$ is the group Hopf algebra of $M$. Further details will be recalled in the Preliminaries. These bicrossproduct Hopf algebras arose in an algebraic approach to quantum-gravity in [1] as well as having been noted in connection

$\text{Royal Society Univ. Research Fellow and Fellow of Pembroke College, Cambridge. On leave at Dept Mathematics, Harvard University during 1995+1996.}$
with extension theory in \cite{2}. Group factorisations are very common in mathematics and this bicrossproduct construction remains one of the primary sources of true non-commutative and non-cocommutative Hopf algebras. One of their novel features, which was the motivation in \cite{1} is the self-dual nature of the construction. The dual Hopf algebra is of the same general form, namely \( k(M)\triangleleft kG \).

Since this work, there have been developed a number of more modern Hopf algebra constructions related to knot and three manifold invariants. Central to these is the Quantum Double construction of V.G. Drinfeld\cite{3} which associates to a general Hopf algebra \( H \) a quasitriangular one \( D(H) \). This quantum double Hopf algebra induces on its category of representations a braiding. When \( H = kG \) the group algebra of a group, the braiding is that of the category of crossed \( G \)-modules as studied by Whitehead\cite{5}. In the present paper we compute the quantum double and braiding for the significantly more complicated bicrossproduct Hopf algebras associated to the factorisation \( X = GM \). The result is an interesting generalisation of crossed modules to bicrossed bimodules.

We also study the question of when exactly the bicrossproducts \( kM\triangleleft k(G) \) are self-dual as Hopf algebras (rather than merely of self-dual form), and find that such self-duality structures are in one-one correspondence with factor-reversing automorphisms of \( X \). In general one might hope that many properties of the bicrossproduct Hopf algebras could be related to the factorizing group \( X \), but this is the first result that we know of this type. We follow this with a relation between the quantum double of the bicrossproduct and the double \( D(X) \) of the group algebra of \( X \).

An outline of the paper is as follows. We begin in Section 2 with the self-duality result. Section 3 computes the quantum double of the bicrossproduct Hopf algebra \( H = kM\triangleleft k(G) \). Section 4 analyses their representation theory and computes the canonical Schroedinger representation of \( D(H) \) on \( H \). The induced braiding is a Yang-Baxter operator \( V \otimes V \to V \otimes V \) where \( V \) is a vector space with basis \( X \). In the paper we compute our various constructions on one of the simplest examples where the group factorisation is \( \mathbb{Z}_6 \mathbb{Z}_6 \). Finally, we show in Section 5 that the braided category of representations of \( D(H) \) as computed in earlier sections, is monoidally equivalent to the braided category of representations of \( D(X) \) (or crossed \( X \)-modules), where \( X \) is the double cross product group. This is disappointing from the point of view of obtaining
completely new braided categories but is an important result from the algebraic point of view. It means that $D(H)$ is isomorphic as a Hopf algebra to the twisting of the Hopf algebra $D(X)$ by a Hopf algebra 2-cocycle $F$ in the terminology of [4], [6].

The general theory of twisting at the level of quasiHopf algebras was introduced by Drinfeld in [7] but works also at the Hopf algebra level [8]. It consists of leaving the algebra unchanged but conjugating the coproduct by $F$. The sense in which $F$ is a cocycle is itself an interesting one and corresponds to a kind of quantum or non-Abelian cohomology as explained in [4], [6]. Previous examples have generally been Lie or deformation-theoretic: our results in this section provide $D(H)$ as one of the first discrete examples. We show that $F$ is non-trivial (not a coboundary). It is worth noting that the concept of two cocycle and 2-cohomology here is a special case of the dual of the notion occurring in another context in [9] and elsewhere.

Acknowledgements

We would like to thank Alan Thomas for some useful comments.

Preliminaries

For bicrossproducts we use the conventions of [3]. By definition, a group $X$ factorises into $X = GM$ if the map $G \times M \to X$ given by product in $X$ is a bijection. In this case define corresponding action $\triangleright : M \times G \to G$ and reaction $\triangleleft : M \times G \to M$ by $su = (s\triangleright u)(s\triangleleft u)$ for $s \in M$ and $u \in G$. One can see that they obey for all $s,t \in M$ and $u,v \in G$:

$$
\begin{align*}
\triangleright e &= s, & (s\triangleright u)v &= s\triangleleft (uv); & \triangleleft e &= u, & (st)\triangleleft u &= (s\triangleleft (tv))(t\triangleleft u) \\
\triangleright u &= e, & \triangleright (tv)u &= (st)\triangleright uv; & \triangleright e &= e, & \triangleright (uv) &= (s\triangleright u)((s\triangleleft u)\triangleright v).
\end{align*}
$$

One says that $(G,M,\triangleright,\triangleleft)$ are a matched pair of groups acting on each other. We then use this data to define the bicrossproduct Hopf algebra $k(M) \triangleright kG$ with basis $\delta_s \otimes u$ where $s \in M$ and $u \in G$, and for its dual Hopf algebra $kM \triangleright k(G)$ with dual basis $s \otimes \delta_u$. In both cases they are semidirect products as algebras and as coalgebras. Explicit formulae are as follows. For $kM \triangleright k(G)$:

$$(s \otimes \delta_u)(t \otimes \delta_v) = \delta_{u, tv}(st \otimes \delta_v), \quad \Delta(s \otimes \delta_u) = \sum_{xy = u} s \otimes \delta_x \otimes s \triangleleft x \otimes \delta_y$$
1 = \sum_u e \otimes \delta_u, \quad \epsilon(s \otimes \delta_u) = \delta_{u,e}, \quad S(s \otimes \delta_u) = (s \triangleright u)^{-1} \otimes \delta_{(s \triangleright u)^{-1}}, \quad (s \otimes \delta_u)^* = s^{-1} \otimes \delta_{s \triangleright u}

Here \( \Delta \) denotes the coproduct, \( \epsilon \) the counit and \( S \) the antipode of the Hopf algebra. Additionally we have given a formula for the star operation. For \( k(M) \triangleright kG \):

\[(\delta_s \otimes u)(\delta_t \otimes v) = \delta_{s \triangleright u, t} \delta_s \otimes uv, \quad \Delta(\delta_s \otimes u) = \sum_{ab=s} \delta_a \otimes b \triangleright u \otimes \delta_b \otimes u \]

\[1 = \sum_s \delta_s \otimes e, \quad \epsilon(\delta_s \otimes u) = \delta_{s,e}, \quad S(\delta_s \otimes u) = \delta_{(s \triangleright u)^{-1}, s \triangleright u}^{-1}, \quad (\delta_s \otimes u)^* = \delta_{s \triangleright u} \otimes u^{-1}.\]

This is all that we need from the theory of bicrossproducts. The general theory for arbitrary Hopf algebras was developed in [1] to which we refer the interested reader, or see [6]. Finally, we note that the factorising group \( X \) can also be recovered from the matched pair of groups as a double cross product or double-semidirect product construction in which both factors act on each other simultaneously. This group \( G \triangleright \bowtie M \) is built on \( G \times M \) and the explicit formulae which we shall need are:

\[(u, s)(v, t) = (u(s \triangleright v), (s \triangleright v)t), \quad e = (e, e), \quad (u, s)^{-1} = (s^{-1} \triangleright u^{-1}, s^{-1} \triangleright u^{-1}).\]

This double cross product group is isomorphic to our original \( X \) by identifying \((u, s)\) with \(us\) in \( X \). We will also need double cross products at the level of Hopf algebras [1], which general theory we recall at the beginning of the relevant sections. For general Hopf algebra constructions we use the summation notation \( \Delta h = \sum h^{(1)} \otimes h^{(2)} \) as in [10].

2 Self-Duality of bicrossproducts

Let \( X = GM \) be a group factorisation. We define a group isom \( \theta : X \to X \) to be factor-reversing if \( \theta(G) \subset M \) and \( \theta(M) \subset G \).

**Proposition 2.1** Factor-reversing isomorphisms of \( X = GM \) give rise to Hopf algebra self-duality pairings \( \langle , \rangle : H \otimes H \to k \) on the Hopf algebra \( H = kM \triangleright k(G) \). The corresponding pairing is

\[\langle s \otimes \delta_u, t \otimes \delta_v \rangle = \delta_{s, \theta(t \triangleright v)} \delta_{u, \theta(t \triangleright v)}.\]

Further, a Hopf algebra isomorphism \( H \to H^* \) which sends basis elements to basis elements must arise from a factor-reversing isomorphism of \( X = GM \).
We define a linear map \( \tilde{\theta} : H \to H^* \) by
\[
\tilde{\theta}(s \otimes \delta_u) = \delta_{\theta(s \triangleright u)} \otimes \theta(s \triangleleft u)
\]
and verify that this is a Hopf algebra isomorphism \( kM \triangleright k(G) \to k(M) \blacktriangleright kG \) if and only if \( \theta \) is a group isomorphism.

If \( \theta \) is a group homomorphism then \( \theta(tv) = \theta(t \triangleright v) \theta(t \triangleleft v) \), which is also \( \theta(t) \theta(v) \). The condition that these two expressions are the same is that, for all \( t \) and \( v \),
\[
\theta(t) = \theta(t \triangleright v) \triangleright \theta(t \triangleleft v) , \quad \theta(v) = \theta(t \triangleright v) \triangleleft \theta(t \triangleleft v).
\]

On the assumption that \( \theta \) is a group isomorphism, we now check the conditions for \( \tilde{\theta} \) to be an algebra isomorphism.
\[
\tilde{\theta}((s \otimes \delta_u)(t \otimes \delta_v)) = \delta_{u,t \triangleright v} \delta_{\theta(st \triangleright v)} \otimes \theta(st \triangleleft v).
\]
Calculated the other way gives
\[
\tilde{\theta}(s \otimes \delta_u) \tilde{\theta}(t \otimes \delta_v) = \delta_{\theta(st \triangleright v), \theta(s \triangleright u)} \delta_{\theta(s \triangleleft u)} \otimes \theta(s \triangleleft u) \theta(t \triangleleft v)
\]
\[
= \delta_{\theta(st \triangleright v), \theta(u)} \delta_{\theta(s \triangleright u)} \otimes \theta((s \triangleleft u)(t \triangleleft v))
\]
\[
= \delta_{u,t \triangleright v} \delta_{\theta(st \triangleright v)} \otimes \theta((s \triangleleft (t \triangleright v))(t \triangleleft v))
\]
\[
= \delta_{u,t \triangleright v} \delta_{\theta(st \triangleright v)} \otimes \theta(st \triangleleft v).
\]

We now check the condition for \( \tilde{\theta} \) to be a coalgebra isomorphism, i.e. \( \Delta \tilde{\theta} = (\tilde{\theta} \otimes \tilde{\theta}) \Delta \).
\[
\Delta \tilde{\theta}(s \otimes \delta_u) = \sum_{ab = \theta(s \triangleright u)} \delta_a \otimes b \triangleright \theta(s \triangleleft u) \otimes \delta_b \otimes \theta(s \triangleleft u)
\]
\[
(\tilde{\theta} \otimes \tilde{\theta}) \Delta(s \otimes \delta_u) = \sum_{xy = u} \delta_{\theta(s \triangleright x)} \otimes \theta(s \triangleleft x) \otimes \delta_{\theta((s \triangleleft x) \triangleright y)} \otimes \theta(s \triangleleft xy).
\]
If we put \( a = \theta(s \triangleright x) \) and \( b = \theta((s \triangleleft x) \triangleright y) \), then \( ab = \theta(s \triangleright u) \), and
\[
b \triangleright \theta(s \triangleleft u) = \theta((s \triangleleft x) \triangleright y) \triangleright \theta(s \triangleleft u)
\]
\[
= \theta((s \triangleright x)^{-1} s \triangleright u) \triangleright \theta(s \triangleleft u)
\]
\[
= \theta(s \triangleright x)^{-1} \triangleright \theta(s \triangleright u) \triangleright \theta(s \triangleleft u)
\]
\[
= \theta(s \triangleright x)^{-1} \triangleright \theta(s)
\]
\[
= \theta((s \triangleright x)^{-1}) \triangleright \theta(s \triangleright x) \triangleright \theta(s \triangleleft x)
\]
\[
= \theta(s \triangleleft x),
\]
so we get a coalgebra map. Next we check the effect of \( \tilde{\theta} \) on the unit and counit.

\[
\epsilon_H \cdot \tilde{\theta}(s \otimes \delta_u) = \delta_{\theta(s \triangleright u), e} = \delta_u, e = \epsilon_H (s \otimes \delta_u).
\]

\[
\tilde{\theta}(1_H) = \tilde{\theta}(\sum_u e \otimes \delta_u) = \sum_u \delta_{\theta(u)} \otimes e = 1_{H^*}.
\]

To check that the antipode is preserved, we need to note that

\[
u^{-1}s^{-1} = (s \triangleleft u)^{-1} (s \triangleright u)^{-1} = ((s \triangleleft u)^{-1} \triangleright (s \triangleright u)^{-1}) ((s \triangleleft u)^{-1} \triangleleft (s \triangleright u)^{-1}),
\]

so that

\[
\theta(u)^{-1} = \theta((s \triangleleft u)^{-1} \triangleright (s \triangleright u)^{-1}) \quad \text{and} \quad \theta(s)^{-1} = \theta((s \triangleleft u)^{-1} \triangleleft (s \triangleright u)^{-1}).
\]

Then we have that

\[
\tilde{\theta} S(s \otimes \delta_u) = \delta_{\theta(s \triangleleft u)^{-1} \triangleright (s \triangleright u)^{-1}} \otimes \theta((s \triangleleft u)^{-1} \triangleleft (s \triangleright u)^{-1}) = \delta_{\theta(u)^{-1}} \otimes \theta(s)^{-1}
\]

\[
S \tilde{\theta}(s \otimes \delta_u) = \delta_{\theta(s \triangleright u) \triangleright \theta(s \triangleleft u)^{-1}} \otimes (\theta(s \triangleright u) \triangleright \theta(s \triangleleft u))^{-1} = \delta_{\theta(u)^{-1}} \otimes \theta(s)^{-1}
\]

Finally to see that \( \tilde{\theta} \) is invertible, put

\[
\tilde{\theta}^{-1}(\delta_s \otimes u) = \theta^{-1}(s \triangleright u) \otimes \delta_{\theta^{-1}(s \triangleleft u)},
\]

then

\[
\tilde{\theta}^{-1}(\delta_s \otimes u) = \tilde{\theta}^{-1}(\delta_{\theta(s \triangleleft u)} \otimes \theta(s \triangleleft u))
\]

\[
= \theta^{-1}(\theta(s \triangleright u) \triangleright \theta(s \triangleleft u)) \otimes \delta^{-1}(\theta(s \triangleright u) \triangleright \theta(s \triangleleft u))
\]

\[
= \theta^{-1}(\theta(s)) \otimes \delta_{\theta^{-1}(\theta(u))} = s \otimes \delta_u,
\]

\[
\tilde{\theta}^{-1}(\delta_s \otimes u) = \delta_{\theta^{-1}(s \triangleright u) \triangleright \theta^{-1}(s \triangleleft u)} \otimes \theta(\theta^{-1}(s \triangleright u) \triangleleft \theta^{-1}(s \triangleleft u)) = \delta_s \otimes u.
\]

The last line proceeds since we can swap the roles of \( \theta \) and \( \theta^{-1} \) in the group identities, since both are isomorphisms. This completes the proof that \( \tilde{\theta} \) is a Hopf algebra isomorphism.

Now we assume that \( \tilde{\theta} \) is a Hopf algebra isomorphism which sends our basis elements to basis elements for the preferred descriptions of our two Hopf algebras, and prove that we can recover a group isomorphism \( \theta \). We start with functions \( m : M \times G \to M \) and \( g : M \times G \to G \) such that

\[
\tilde{\theta}^{-1}(\delta_s \otimes u) = m(s, u) \otimes \delta_{\theta(s, u)}.
\]
Since the map $m \times g : M \times G \to M \times G$ is a 1-1 correspondence, it is possible to define a map $\theta$ by

$$(m \times g)^{-1}(s, u) = (\theta(s \triangleright u), \theta(s \triangleleft u)).$$

Now $\tilde{\theta}$ is invertible, so that

$$s \otimes \delta_u = \tilde{\theta}^{-1}(s \otimes \delta_u) \quad \text{and} \quad \delta_s \otimes u = \tilde{\theta}^{-1}(\delta_s \otimes u),$$

giving the relations

$$s = m(\theta(s \triangleright u), \theta(s \triangleleft u)) \quad (a),$$
$$u = g(\theta(s \triangleright u), \theta(s \triangleleft u)) \quad (b),$$
$$s = \theta(m(s, u) \triangleright g(s, u)) \quad (c),$$
$$u = \theta(m(s, u) \triangleleft g(s, u)) \quad (d).$$

Next we use the fact that $\tilde{\theta}^{-1}$ is an algebra homomorphism in the equation

$$\tilde{\theta}^{-1}((\delta_s \otimes u)(\delta_t \otimes v)) = \tilde{\theta}^{-1}(\delta_s \otimes u)\tilde{\theta}^{-1}(\delta_t \otimes v)$$

to get the equation

$$\delta_{s \triangleleft u, t}(m(s, uv) \otimes \delta_{g(s, uv)}) = \delta_{g(s, u), m(t, v) \triangleright g(t, v)}((m(s, u)m(t, v) \otimes \delta_{g(t, v)}).$$

Thus for all $s, t \in M$ and $u, v \in G$, the following are equivalent:

$$t = s \triangleleft u \quad (e),$$
$$g(s, u) = m(t, v) \triangleright g(t, v) \quad (f),$$
$$m(s, uv) = m(s, u)m(t, v) \quad (g),$$
$$g(s, uv) = g(t, v) \quad (h).$$

Now define $\psi : X \to X$ by

$$\psi(s) = g(s, e), \quad \psi(u) = m(e, u), \quad \psi(su) = g(s, e)m(e, u) = \psi(s)\psi(u).$$

Note that $\psi$ is well defined since $G \cap M = \{e\}$. We show that $\psi \theta = \theta \psi = \text{id}_X$, so that $\theta$ is a factor reversing isomorphism. From (e) and (f) we have

$$g(s, u) = m(s \triangleleft u, v) \triangleright g(s \triangleleft u, v).$$
and replacing $s$ by $s\triangleright u^{-1}$ we obtain
\[ g(s\triangleright u^{-1}, u) = m(s, v) \triangleright g(s, v) \quad (i) . \]

The equations (e) and (h) give $g(s, uv) = g(s\triangleright u, v)$, and replacing $s$ by $s\triangleright u^{-1}$, and setting $v = e$ gives
\[ g(s\triangleright u^{-1}, u) = g(s, e) = \psi(s) \quad (j) . \]
Then using (c), (i), and (j), we get
\[ s = \theta(m(s, u) \triangleright g(s, u)) = \theta(g(s\triangleright u^{-1}, u)) = \theta\psi(s) , \]
so that $\theta\psi|_M = \text{id}_M$. Next we note that from (b) and (i),
\[ e = g(\theta(e\triangleright e), \theta(e\triangleright e)) = g(\theta(e), \theta(e)) = g(e\triangleright u^{-1}, e) = g(e, u) . \]
Then using (d) with $s = e$, we have
\[ u = \theta(m(e, u) \lhd g(e, u)) = \theta(m(e, u) \triangleright e) = \theta(m(e, u)) = \theta\psi(u) , \]
so $\theta\psi|_G = \text{id}_G$. Putting $u = e$ in (a) gives
\[ s = m(\theta(s\triangleright e), \theta(s\triangleright e)) = m(\theta(e), \theta(s)) = m(e, \theta(s)) = \psi\theta(s) , \]
so that $\psi\theta|_M = \text{id}_M$. Next putting $s = e$ in (b) gives
\[ u = g(\theta(e\triangleright u), \theta(e\triangleright u)) = g(\theta(u), \theta(e)) = \psi\theta(u) , \]
so that $\psi\theta|_G = \text{id}_G$. Finally we have from the definition of $\psi$,
\[ \theta\psi(su) = \theta(\psi(s)\psi(u)) = \theta\psi(s)\theta\psi(u) = su , \]
and since $\theta(s\triangleright u) \in M$ and $\theta(s\triangleright u) \in G$,
\[ \psi\theta(su) = \psi\theta((s\triangleright u)(s\triangleright u)) = (s\triangleright u)(s\triangleright u) = su . \]
\[ \square \]
Example 2.2 For odd $n \geq 3$, we can express the product of dihedral groups $D_n \times D_n$ as a double cross product of cyclic groups $Z_{2n}Z_{2n}$. We give the $n = 3$ case in detail, where we note that $D_3 \equiv S_3$. Consider the group $X = S_3 \times S_3$ as the permutations of 6 objects labelled 1 to 6, where the first factor leaves the last 3 objects unchanged, and the second factor leaves the first 3 objects unchanged. We take $G$ to be the cyclic group of order 6 generated by the permutation $1_G = (123)(45)$, and $M$ to be the cyclic group of order 6 generated by the permutation $1_M = (12)(456)$. Our convention is that permutations act on objects on their right, for example $1_G$ applied to 1 gives 2. The intersection of $G$ and $M$ is just the identity permutation, and counting elements shows that $GM = MG = S_3 \times S_3$. We write each cyclic group additively, for example $G = \{0_G, 1_G, 2_G, 3_G, 4_G, 5_G\}$.

In this case we have a factor reversing isomorphism on $X$, in fact several of them. The map $\theta(x) = \Theta x \Theta^{-1}$ where $\Theta$ is the permutation $(1425)(36)$ gives such an isomorphism. Also the map $\phi(x) = \Phi x \Phi^{-1}$ where $\Phi$ is the permutation $(14)(25)(36)$ gives such an isomorphism. In fact the dihedral group $D_4$ generated by $\theta$ (order 4) and $\phi$ (order 2) is all the isomorphisms of $X$ which either reverse or preserve the factors.

Example 2.3 In more generality, take the dihedral group $D_n$ to be generated by elements $a$ of order $n$, and $b$ of order 2, with relation $bab = a^{-1}$. For $n$ odd, consider the group $G$ to be the cyclic subgroup of $D_n \times D_n$ of order $2n$ generated by $1_G = (a, b)$, and $M$ to be the cyclic subgroup of order $2n$ generated by $1_M = (b, a)$. By counting elements, we see that everything in $X = D_n \times D_n$ can be written as a product on $GM$, and as a product in $MG$. This gives
$D_n \times D_n$ as a double cross product $\mathbb{Z}_{2n} \mathbb{Z}_{2n}$. The corresponding actions are (writing the cyclic groups additively)

$$s \triangleright u = \begin{cases} u & s \text{ even} \\ -u & s \text{ odd} \end{cases} \quad \text{and} \quad s \triangleleft u = \begin{cases} s & u \text{ even} \\ -s & u \text{ odd} \end{cases}.$$ 

There is a factor reversing isomorphism of $X = D_n \times D_n$ given by $\psi(x, y) = (y, x)$.

**Example 2.4** Let $X$ be a finite group whose order is divisible by only two distinct primes $p$ and $q$, i.e. $|X| = p^n q^m$. Then Sylow’s theorems state that there is at least one subgroup $G$ with order $p^n$, and at least one subgroup $M$ with order $q^m$. Since the order of $G \cap M$ would have to divide $p^n$ and $q^m$, we must have $G \cap M = \{e\}$, giving the factorisation $X = GM = MG$. However bicrossproducts formed from factorisations of this sort can never be self-dual, as the orders of $G$ and $M$ are different.

### 3 Quantum Double of a bicrossproduct

Let $H$ be a Hopf algebra. The quantum double is a Hopf algebra double cross product $D(H) = H^{* \text{op}} \bowtie H$, not to be confused with the bicrossproducts above. It is a Hopf algebra factorising into $H^{* \text{op}}$ and $H$ and given via a double-semidirect product by mutual coadjoint actions of these two factors on each other. This formulation is from [1] although the double itself as a Hopf algebra originates with V.G. Drinfeld[3]. Explicitly, it is built on $H^* \otimes H$ as a linear space with product

$$ (a \otimes h)(b \otimes g) = \sum (Sb_{(1)}, h_{(1)})b_{(2)}a \otimes h_{(2)}g(b_{(3)}, h_{(3)}) \quad (1)$$

and tensor product unit, counit, coproduct, and antipode given by the formulae:

$$1_{D(H)} = 1_{H^*} \otimes 1_H ,$$

$$\epsilon_{D(H)} = \epsilon_{H^*} \otimes \epsilon_H ,$$

$$\Delta_{D(H)} = (\text{id} \otimes \tau \otimes \text{id}) \circ (\Delta \otimes \Delta) ,$$

$$S_{D(H)}(a \otimes h) = (1 \otimes Sh)(S^{-1}a \otimes 1) ,$$

$$(a \otimes h)^* = (1 \otimes h^*)((S^2a)^* \otimes 1) ,$$

where $\tau(a \otimes h) = h \otimes a$. 

10
One can compute this directly or compute first the mutual coadjoint actions $\triangleright, \triangleleft$. In their terms, the Hopf algebra structure takes the double cross product form

$$(a \otimes h)(b \otimes g) = \sum (h_{(1)} \triangleright h_{(2)}) a \otimes (h_{(2)} \triangleleft b_{(2)}) g \quad (2)$$

We will use this double cross product form for our calculation. The required mutual coadjoint actions are defined by

$$h \triangleleft a = \sum h_{(2)} \langle a, (Sh_{(1)})h_{(3)} \rangle, \quad h \triangleright a = \sum a_{(2)} \langle h, (Sa_{(1)})a_{(3)} \rangle, \quad a \in H^*, \; h \in H. \quad (3)$$

In all these formulae, the expressions are given in terms of the Hopf algebras $H, H^*$.

**Lemma 3.1** The coadjoint action of $H = kM \triangleright kG$ on $H^* = k(M) \triangleright kG$ and vice-versa are

$$(t \otimes \delta_v) \triangleright (\delta_s \otimes u) = \delta_{u,v}(s \triangleright u) (\delta_{v,ts^{-1}} \otimes t' \triangleright u; \quad t' = t \triangleleft (s \triangleright u)^{-1})$$

$$(t \otimes \delta_v) \triangleleft (\delta_s \otimes u) = t' \otimes \delta_{(s \triangleright u)v^{-1}} \delta_{t \triangleright t'(s \triangleright u)}.$$  

**Proof**: The action of $H$ on $H^*$. Let $a = \delta_s \otimes u \in H$, then firstly we calculate

$$(\Delta \otimes \text{id}) \Delta(\delta_s \otimes u) = \sum_{cdb = s} \delta_c \otimes db \triangleright u \otimes \delta_d \otimes b \triangleright u \otimes \delta_b \otimes u.$$  

From this we get that $a_{(1)} = \delta_c \otimes b \triangleright u$, $a_{(2)} = \delta_d \otimes b \triangleright u$, and $a_{(3)} = \delta_b \otimes u$, which with $h = t \otimes \delta_v$ we substitute into the second equation of (3) to get

$$(t \otimes \delta_v) \triangleright (\delta_s \otimes u) = \sum_{cdb = s} (\delta_d \otimes b \triangleright u)(t \otimes \delta_v, S(\delta_c \otimes db \triangleright u)\delta_b \otimes u)$$

$$= \sum_{cdb = s} (\delta_d \otimes b \triangleright u)(t \otimes \delta_v, (\delta_{(c \otimes (db \triangleright u))^{-1}} \otimes (c \otimes (db \triangleright u))^{-1})(\delta_b \otimes u))$$

$$= \sum_{cdb = s} (\delta_d \otimes b \triangleright u)(t \otimes \delta_v, (\delta_{c^{-1} \otimes (s \triangleright u)} \otimes (s \triangleright u)^{-1})(\delta_b \otimes u))$$

$$= \sum_{cdb = s} (\delta_d \otimes b \triangleright u)(t \otimes \delta_v, \delta_{c^{-1} \otimes (s \triangleright u)} \otimes (s \triangleright u)^{-1},b)(\delta_{c^{-1} \otimes (s \triangleright u)} \otimes (s \triangleright u)^{-1})u)$$

$$= \sum_{cdb = s} (\delta_d \otimes b \triangleright u)(t \otimes \delta_v, \delta_{c^{-1} \otimes (s \triangleright u)} \otimes (s \triangleright u)^{-1})u)$$

$$= \sum_{cdb = s} (\delta_d \otimes b \triangleright u)(t \otimes \delta_v, \delta_{c^{-1} \otimes (s \triangleright u)} \otimes (s \triangleright u)^{-1})u)$$

$$= \delta_{t \triangleright t'(s \triangleright u)^{-1}}u \sum_{cde^{-1} = s} (\delta_d \otimes c^{-1} \triangleright u)\delta_{t \triangleright t'(s \triangleright u)} \delta_{e^{-1} \otimes (s \triangleright u)}$$

$$(c \triangleleft (db \triangleright u))^{-1} = ((cd \triangleright b \triangleleft u))^{-1}$$

$$= (db \triangleright u)(cd \triangleright b \triangleleft u)^{-1} = c^{-1} \triangleleft (cd \triangleright b \triangleleft u) = c^{-1} \triangleleft (s \triangleright u) \cdot$$
Finally to obtain the desired action we solve for $c$ which is fixed by the delta function inside the summation. We have $t = c^{-1} d(s \triangleright u)$, so $c^{-1} = t d(s \triangleright u)^{-1} = t^\prime$. Also $d = c^{-1} s c = t^\prime s t^{-1}$, so we have

$$(t \otimes \delta_v) \triangleright (\delta_s \otimes u) = \delta_v(s \triangleright u)^{-1} u \delta_{t^\prime s t^{-1} \otimes t^\prime u}.$$  

$\square$

**Proof**: The action of $H^*$ on $H$. First we calculate $(\Delta \otimes 1)\Delta(h)$ with $h = t \otimes \delta_v \in H$:

$$(\Delta \otimes 1)\Delta(h) = \sum_{wzy=v} t \otimes \delta_w \otimes t \triangleleft w \otimes \delta_z \otimes t \triangleleft wz \otimes \delta_y$$

This gives summands $h_{(1)} = t \otimes \delta_w$, $h_{(2)} = t \triangleleft w \otimes \delta_z$, and $h_{(3)} = t \triangleleft wz \otimes \delta_y$, which we substitute into the first equation of (3) to get

$$(t \otimes \delta_v) \triangleleft (\delta_s \otimes u) = \sum_{wzy=v} (t \triangleleft w \otimes \delta_z)(S(t \otimes \delta_w)(t \triangleleft wz \otimes \delta_y), \delta_s \otimes u)$$

$$= \sum_{wzy=v} (t \triangleleft w \otimes \delta_z)((t \triangleleft w)^{-1} \otimes \delta_{(tw)^{-1}})(t \triangleleft wz \otimes \delta_y), \delta_s \otimes u)$$

$$= \sum_{wzy=v} (t \triangleleft w \otimes \delta_z)\delta_{(tw)^{-1},(twz)^{-1}}((t \triangleleft w)^{-1}(t \triangleleft wz) \otimes \delta_y, \delta_s \otimes u)$$

$$= \sum_{wzy=v} (t \triangleleft w \otimes \delta_z)\delta_{(tw)^{-1},(twz)^{-1}}(t \triangleleft w^{-1},s\delta_y,u)$$

$$= \sum_{wz=uv^{-1}} (t \triangleleft w \otimes \delta_z)\delta_{(tw)^{-1},(twu^{-1})\triangleright u}\delta_{(tw)^{-1},(twu^{-1})\triangleright s}$$

Next we solve these equations for $w$ and $z$. We need the following identities for double cross product groups:

$$(t \triangleright w)^{-1} = (t \triangleleft w)\triangleright w^{-1} \quad , \quad (t \triangleleft w)^{-1} = t^{-1} d(t \triangleright w)$$

Now if $(t \triangleright w)^{-1} = (t \triangleleft w^{-1}) \triangleright u$ and $s = (t \triangleleft w)^{-1}(t \triangleleft w u^{-1})$, then $(t \triangleleft w)\triangleright w^{-1} = (t \triangleleft w^{-1}) \triangleright u$, and so

$$w^{-1} = (t \triangleleft w)^{-1} \triangleright ((t \triangleleft w^{-1}) \triangleright u) = s \triangleright u$$

Thus $w = (s \triangleright u)^{-1}$, $z = (s \triangleright u) w^{-1}$, and $t \triangleleft w = t \triangleleft (s \triangleright u)^{-1} = t^\prime$. To simplify the second delta function in the summation we note that

$$(t \triangleleft w)^{-1}(t \triangleleft w u^{-1})^{-1} = t^{-1}(t \triangleright u)^{-1}.$$  

This means that

$$\delta_{(tw)^{-1},(twu^{-1})\triangleright s} = \delta_{s,t^{-1}(tw)\triangleright u^{-1}} = \delta_{s,t^{-1}(tw)\triangleright u} = \delta_{t(s \triangleright u),t \triangleleft w}.$$
and this proves the formula for the action. □

**Proposition 3.2** The quantum double $D(kM\rhd k(G))$ is generated by $kM\rhd k(G)$ and $k(M)\triangleright kG$ as sub-Hopf algebras with cross relations defined by the product

$$(1 \otimes t \otimes \delta_v)(\delta_s \otimes u \otimes 1) = \delta_{t' s (t\triangleright u v^{-1})-1 \otimes (t\triangleright u v^{-1})} \triangleright u \otimes t' \otimes \delta_{(s\triangleright u) v u^{-1}} ,$$

where $t' = t\triangleright (s\triangleright u)^{-1}$.

**Proof** Let $h = t \otimes \delta_v \in H$, and $b = \delta_s \otimes u \in H^*$. We calculate $(1 \otimes h)(b \otimes 1)$ using (2). We have

$$\Delta(t \otimes \delta_v) = \sum_{xy = v} t \otimes \delta_x \otimes t\triangleright x \otimes \delta_y$$
$$\Delta(\delta_s \otimes u) = \sum_{wz = s} \delta_w \otimes z\triangleright u \otimes \delta_z \otimes u ,$$

so that, by lemma 3.1,

$$h_{(1)} \triangleright b_{(1)} = (t \otimes \delta_x) \triangleright (\delta_u \otimes z\triangleright u)$$
$$= \delta_{z\triangleright u, (w\triangleright u)x}.(\delta_{t' w^{-1}} t' \otimes z\triangleright u)$$
$$= \delta_{z\triangleright u, (s\triangleright u)x}.(\delta_{t' w^{-1}} t' \otimes z\triangleright u)$$

where $t' = t\triangleright (s\triangleright u)^{-1}$, since $wz = s$. Also

$$h_{(2)} \triangleright b_{(2)} = \delta_{t\triangleright x, (t\triangleright x)(z\triangleright u)}.t'' \otimes \delta_{(z\triangleright u)y u^{-1}} = \delta_{t\triangleright v, (t\triangleright x)(z\triangleright u)}.t'' \otimes \delta_{(z\triangleright u)y u^{-1}}$$

where $t'' = (t\triangleright x)(z\triangleright u)^{-1}$. Substituting into formula (2) gives

$$(1 \otimes t \otimes \delta_v)(\delta_s \otimes u \otimes 1) = \sum_{xy = v, wz = s} \delta_{z\triangleright u, (s\triangleright u)x}. \delta_{t\triangleright v, (t\triangleright x)(z\triangleright u)}.(\delta_{t' w^{-1}} t' \otimes z\triangleright u \otimes t'' \otimes \delta_{(z\triangleright u)y u^{-1}})$$

Next we solve

$$z\triangleright u = (s\triangleright u)x \quad (a) \quad \text{and} \quad t\triangleright v = (t\triangleright x)(z\triangleright u) \quad (b)$$

to find $t'z$, $t'w^{-1}$, $t''$ and $(z\triangleright u)y$ in terms of $t$, $v$, $s$ and $u$ alone. First we calculate

$$t'z\triangleright u = (t' \triangleleft (z\triangleright u))(z\triangleright u)$$
$$= ((t\triangleleft (s\triangleright u)^{-1}) \triangleleft (z\triangleright u))(z\triangleright u)$$
$$= (t\triangleleft (s\triangleright u)^{-1})(z\triangleright u)(z\triangleright u)$$
$$= (t\triangleleft x)(z\triangleright u) = t\triangleleft v .$$
Thus \( t'z = (t<v)\triangleright u^{-1} = t<vu^{-1} \). Also using (a) and \( xy = v \) we have
\[
(z\triangleright u)y = (s\triangleright u)xy = (s\triangleright u)v, \quad \text{and} \\
t'' = (t\langle x\rangle)(z\triangleright u)^{-1} = t\langle x(z\triangleright u)^{-1} \\
= t\langle (s\triangleright u)^{-1} = t'.
\]

Finally we have, using \( wz = s \),
\[
t'wt'^{-1} = t'(wz)z^{-1}t'^{-1} = t's(t'z)^{-1} = t's(t<vu^{-1})^{-1}.
\]
\(\Box\)

We now give a relation between our order reversing group automorphisms on \( X \) and the quantum double on \( H = kM\triangleright k(G) \). Remember that we had Hopf algebra isomorphisms \( \tilde{\theta} : H \to H^* \) and \( \tilde{\theta} : H^* \to H \) given by
\[
\tilde{\theta}(s \otimes \delta_u) = \delta_{\theta(s\triangleright u)} \otimes \theta(s\triangleright u) \quad \text{and} \quad \tilde{\theta}(s \otimes \delta_u) = \theta(s\triangleright u) \otimes \delta_{\theta(s\triangleright u)}.
\]

In addition we have the order reversing map \( \tau : H \otimes H^* \to H^* \otimes H \) given by \( \tau(h \otimes b) = b \otimes h \).

**Remark 3.3** We have already proved that \( \tilde{\theta} : H \to H^* \) is a Hopf algebra isomorphism. The given map \( \tilde{\theta} : H^* \to H \) is the inverse of \( \tilde{\theta}^{-1} : H \to H^* \), and so is also a Hopf algebra isomorphism. We note that for \( a \in H^* \) and \( h \in H \),
\[
\langle \tilde{\theta}(a), \tilde{\theta}(h) \rangle = \langle h, a \rangle.
\]

**Proof**
\[
\langle \tilde{\theta}(s \otimes \delta_u), \tilde{\theta}(t \otimes \delta_v) \rangle = \langle \theta(s\triangleright u) \otimes \delta_{\theta(s\triangleright u)}, \delta_{\theta(t\langle v\rangle)} \otimes \theta(t\langle v\rangle) \rangle \\
= \delta_{\theta(s\triangleright u), \theta(t\langle v\rangle)} \cdot \delta_{\theta(s\triangleright u), \theta(t\langle v\rangle)} = \delta_{s\triangleright u, t\langle v\rangle} = \delta_{s\triangleright u, t\triangleright v} = \langle t \otimes \delta_v, \delta_{s \otimes u} \rangle.
\]
This equation has used the result that if \( s\triangleright u = t\triangleright v \) and \( s\triangleright u = t\triangleright v \), then \( su = (s\triangleright u)(s\triangleright u) = (t\triangleright v)(t\triangleright v) = tv \), which means that \( s = u \) and \( t = v \). \(\Box\)

**Proposition 3.4** We have the following equations for \( h \in H \) and \( b \in H^* \):
\[
\tilde{\theta}(h \triangleright b) = \tilde{\theta}b \triangleright \tilde{\theta}h \quad (a) \quad \text{and} \quad \tilde{\theta}(h \langle b) = \tilde{\theta}b \triangleright \tilde{\theta}h \quad (b).
\]
Proof : (a) : We let \( h = s \otimes \delta_u \) and \( b = \delta_t \otimes v \), and find

\[
\tilde{\theta}(\delta_t \otimes v) \circ \tilde{\theta}(s \otimes \delta_u) = \delta_k(t \triangleright v)u \cdot (\theta(t \triangleright v) \circ \theta(s)^{-1} \otimes \delta_{\tilde{\theta}(s \triangleright u)(s \triangleright u)^{-1}}),
\]

\[
\tilde{\theta}((s \otimes \delta_u) \circ (\delta_t \otimes v)) = \delta_k(t \triangleright v)u \cdot \tilde{\theta}(\tilde{\theta}(s \triangleright u) \otimes \delta_{\tilde{\theta}(s \triangleright u)}),
\]

where \( \tilde{s} = t' t t'^{-1} \), \( \tilde{u} = t' \triangleright v \), and \( t' = s \circ (t \triangleright v)^{-1} \). To show that these are equal we calculate

\[
\tilde{s} \circ \tilde{u} = t' t t'^{-1} \circ (t' \triangleright v) = t' t \triangleright v = t' \triangleright (t \triangleright v)
\]

\[
\theta(\tilde{s} \circ \tilde{u}) = \theta((s \circ (t \triangleright v)^{-1}) \circ (t \triangleright v)) = \theta((t \triangleright v)^{-1} \circ (t \triangleright v)) = \theta(t \triangleright v) \circ \theta(s)^{-1}.
\]

Also assuming that \( v = (t \triangleright v)u \), we have

\[
\tilde{s} \circ \tilde{u} = t' t t'^{-1} \circ (t' \triangleright v) = ((t' t t'^{-1}) \circ (t' \triangleright v))^{-1}
\]

\[
= (t' t \triangleright v)(t' \triangleright v)^{-1} = (t' t \triangleright v)((s \circ (t \triangleright v)^{-1}) \circ (t \triangleright v))^{-1}
\]

\[
= (t' t \triangleright v)((s \circ u \circ v^{-1}) \circ (t \triangleright v))^{-1} = (t' t \triangleright v)(s \circ u)\circ t \triangleright v
\]

\[
= (t' \circ (t \triangleright v))(t \circ u)(s \circ u)^{-1} = ((s \circ (t \triangleright v)^{-1}) \circ (t \triangleright v))(t \circ u)(s \circ u)^{-1}
\]

\[
= s(t \circ u)(s \circ u)^{-1}.
\]

\( \square \)

Proof : (b) : We have

\[
\tilde{\theta}(\delta_t \otimes v) \circ \tilde{\theta}(s \otimes \delta_u) = (\tilde{t} \otimes \delta_{\tilde{v}}) \circ (\tilde{\delta}_t \otimes \tilde{u}) = \delta_{\tilde{u} \circ (\tilde{s} \circ \tilde{u}) \circ \tilde{v} \circ (\tilde{t} \circ \tilde{u})^{-1} \circ (t' \circ \tilde{v} \circ \tilde{u})}
\]

where \( \tilde{t} = \theta(t \triangleright v) \), \( \tilde{v} = \theta(t \circ u) \), \( \tilde{s} = \theta(s \circ u) \), \( \tilde{u} = \theta(s \circ u) \), and \( \tilde{v} = \tilde{t} \circ (\tilde{s} \circ \tilde{v})^{-1} \). Then

\[
\tilde{s} \circ \tilde{u} = \theta(s \circ u) \circ \theta(s \circ u) = \theta(s)
\]

\[
\tilde{t}' = \tilde{t} \circ s \circ u \circ t \circ u^{-1} = \theta(t \circ v) \circ \theta(s)^{-1} = \theta((s \triangleright v)^{-1}),
\]

\[
\delta_{\tilde{u},(\tilde{s} \circ \tilde{u}) \circ \tilde{v} \circ (\tilde{t} \circ \tilde{u})^{-1}} = \delta_{\theta(s \circ u), \theta(s) \circ \theta(t \circ u)} = \delta_{s \circ u, s(t \circ u)}.
\]

Next we calculate

\[
\tilde{\theta}((s \otimes \delta_u) \circ (\delta_t \otimes v)) = \delta_{s \circ u, s(t \circ u)} \circ (\theta(t \circ v)^{-1} \otimes \delta_{(t \circ v) \circ u \circ t \circ u^{-1}})
\]

\[
= \delta_{s \circ u, s(t \circ u)} \circ (\delta_{m \circ g} \otimes \theta(m \circ g)).
\]
where, if $s \triangleright u = s(t \triangleright v)$,

\[
\begin{align*}
    m \triangleright g &= (s \lhd (t \triangleright v)^{-1} \triangleright (t \triangleright v)uv^{-1} = s \triangleright uv^{-1}, \\
    m \triangleright g &= (s \lhd (t \triangleright v)^{-1} \triangleright (t \triangleright v)uv^{-1} \\
    &= ((s \lhd (t \triangleright v)^{-1} \triangleright (t \triangleright v))((s \triangleright v)^{-1} \triangleright (t \triangleright v))uv^{-1}) \\
    &= (s \triangleright (t \triangleright v)^{-1})^{-1}((s \triangleright uv^{-1}) \\
    &= (s \triangleright (t \triangleright v)^{-1})^{-1}(s \triangleright u)((s \triangleright v)^{-1}) \\
    &= (s \triangleright (t \triangleright v)^{-1})^{-1}(s \triangleright u)((s \triangleright v)^{-1}) \\
    &= (s \triangleright (t \triangleright v)^{-1})^{-1}(s \triangleright u)((s \triangleright v)^{-1}) .
\end{align*}
\]

Now we have to show that $\tilde{t} \triangleright \tilde{u} = \theta(m \triangleright g)$ (which is now automatic), and that

\[
\begin{align*}
    \tilde{t} \triangleright \tilde{u} &= \theta((s \triangleright v)^{-1} \triangleright (s \triangleright u)) \\
    &= \theta(v)\theta((s \triangleright u)^{-1} \triangleright (s \triangleright u)) \\
    &= \theta(v)\theta((s \triangleright u)^{-1})^{-1} \\
    &= \theta((s \triangleright u)^{-1} = \theta(s \triangleright uv^{-1} = \theta(m \triangleright g)
\end{align*}
\]

Remark 3.5 For $h \in H$ and $b \in H^*$, we have $\langle S(h), S(a) \rangle = \langle h, a \rangle$. This also implies that $S^2$ is the identity on $H$ and $H^*$.

Proposition 3.6 The map $\psi = \tau(\tilde{\theta} \otimes \tilde{\theta}) : D(H) \rightarrow D(H)$ is an anti-algebra isomorphism, a coalgebra isomorphism, and preserves the unit, the counit, and the antipode.

Proof First we show that $\psi$ is an anti-algebra map. On the one hand we have, using proposition 3.4,

\[
\begin{align*}
    \psi((a \otimes h)(b \otimes g)) &= \psi\left(\sum (h_{(1)} \triangleright b_{(1)})a \otimes (h_{(2)} \lhd b_{(2)})g\right) \\
    &= \sum \tilde{\theta}(h_{(2)} \lhd b_{(2)})\tilde{\theta}(g) \otimes \tilde{\theta}(h_{(1)} \triangleright b_{(1)})\tilde{\theta}(a) \\
    &= \sum (\tilde{\theta}g \otimes 1)\tilde{\theta}(h_{(2)} \lhd b_{(2)}) \otimes \tilde{\theta}(h_{(1)} \triangleright b_{(1)})((1 \otimes \tilde{\theta}a) \\
    &= \sum (\tilde{\theta}g \otimes 1)\tilde{\theta}(h_{(2)} \triangleright b_{(2)}) \otimes \tilde{\theta}(h_{(1)} \lhd b_{(1)})\lhd(\tilde{\theta}h_{(1)})((1 \otimes \tilde{\theta}a)
\end{align*}
\]
On the other hand,
\[
\psi(b \otimes g)\psi(a \otimes h) = (\hat{\theta}g \otimes \hat{\theta}b)(\hat{\theta}h \otimes \hat{\theta}a)
\]
\[
= \sum ((\hat{\theta}b)_1 \triangleright (\hat{\theta}h)_1)\hat{\theta}g \otimes ((\hat{\theta}b)_2 \triangleleft (\hat{\theta}h)_2)\hat{\theta}a
\]
\[
= \sum (\hat{\theta}g \otimes 1)(\hat{\theta}b)_1 \triangleright (\hat{\theta}h)_1) \otimes ((\hat{\theta}b)_2 \triangleleft (\hat{\theta}h)_2)(1 \otimes \hat{\theta}a).
\]

To complete the calculation we note the identity [5, Chapter 7]
\[
\sum h(1) \triangleright a(1) \otimes h(2) \triangleleft a(2) = \sum h(2) \triangleright a(2) \otimes h(1) \triangleleft a(1).
\]

We now check the condition for \( \psi \) to be a coalgebra map, i.e. \((\psi \otimes \psi)\Delta = \Delta \psi \).
\[
\Delta \psi(a \otimes h) = \Delta(\hat{\theta}h \otimes \hat{\theta}a)
\]
\[
= \sum (\hat{\theta}h)_1 \otimes (\hat{\theta}a)_1 \otimes (\hat{\theta}h)_2 \otimes (\hat{\theta}a)_2
\]
\[
= \sum \hat{\theta}(h(1)) \otimes \hat{\theta}(a(1)) \otimes \hat{\theta}(h(2)) \otimes \hat{\theta}(a(2))
\]
\[
= \sum \psi(a(1) \otimes h(1)) \otimes \psi(a(2) \otimes h(2))
\]
\[
= (\psi \otimes \psi) \sum a(1) \otimes h(1) \otimes a(2) \otimes h(2)
\]
\[
= (\psi \otimes \psi)\Delta(a \otimes h).
\]

To check that the antipode is preserved,
\[
S\psi(a \otimes h) = S(\hat{\theta}h \otimes \hat{\theta}a)
\]
\[
= (1 \otimes S\hat{\theta}a)(S\hat{\theta}h \otimes 1)
\]
\[
= (1 \otimes \tilde{\theta}S(a))(\tilde{\theta}S(h) \otimes 1)
\]
\[
= \psi(\tilde{\theta}S(a) \otimes 1)\psi(1 \otimes \tilde{\theta}S(h))
\]
\[
= \psi((1 \otimes \tilde{\theta}S(h))(\tilde{\theta}S(a) \otimes 1))
\]
\[
= \psi S(a \otimes h).
\]

Finally we check the effect of \( \psi \) on the co-unit and unit of \( D(H) \), using the fact that \( \hat{\theta} \) is a Hopf algebra isomorphism:
\[
\epsilon_{D(H)}\psi(a \otimes h) = \epsilon_{D(H)}(\hat{\theta}h \otimes \hat{\theta}a)
\]
\[
= (\epsilon \hat{\theta}h)(\epsilon \hat{\theta}a) = (\epsilon h)(\epsilon a)
\]
\[
= \epsilon_{D(H)}(a \otimes h),
\]
\[
\psi(1 \otimes 1) = \hat{\theta}(1) \otimes \hat{\theta}(1) = 1 \otimes 1.
\]
Example 3.7. In the case of our $\mathbb{Z}_{2n} \mathbb{Z}_{2n}$ example, we can write out the actions as (we use additive operations in $\mathbb{Z}_{2n}$)

\[
(t \otimes \delta_v) \triangleright (\delta_s \otimes u) = \begin{cases} 
\delta_{0,u} \delta_s \otimes u & t \text{ even } s \text{ even} \\
\delta_{0,u} \delta_s \otimes (-u) & t \text{ odd } s \text{ even} \\
\delta_{2u,v} \delta_s \otimes u & t \text{ even } s \text{ odd} \\
\delta_{2u,v} \delta_s \otimes (-u) & t \text{ odd } s \text{ odd}
\end{cases}
\]

\[
(t \otimes \delta_v) \triangleleft (\delta_s \otimes u) = \begin{cases} 
\delta_{0,s} \delta_v & u \text{ even } v \text{ even} \\
\delta_{0,s} (-t) \otimes \delta_v & u \text{ odd } v \text{ even} \\
\delta_{-2t,v} \delta_v & u \text{ even } v \text{ odd} \\
\delta_{2t,v} (-t) \otimes \delta_v & u \text{ odd } v \text{ odd}
\end{cases}
\]

4 Representations and Braiding

The representations of the quantum double of any Hopf algebra form a braided tensor category, and for this reason the category is especially interesting. By the double cross product construction we see that we have such a braided category for every group factorisation. Many results and applications flow from this. We shall mention only one or two of them.

The representations of $D(H)$ are evident from its description as a double cross product. namely, they are left $H$ and right $H^*$-modules $W$ which are compatible in such a way that

\[
h \triangleright (w \triangleleft a) = \sum ((h(1) \triangleleft a(1)) \triangleright w) \triangleleft (h(2) \triangleright a(2))
\]

which can be further computed in terms of the mutual coadjoint actions. We freely identify a left $H^{*\text{op}}$ module as a right $H^*$-module. The braiding among such modules $V, W$ is

\[
\Psi_{V,W}(v \otimes w) = \sum_a e_a \triangleright w \otimes v \triangleleft f^a
\]

where \(\{e_a\}\) is a basis of $H$ and \(\{f^a\}\) is a dual basis. The corresponding action of $D(H)$ is $(a \otimes h) \triangleright w = (h \triangleright w) \triangleleft a$.

Finally, a canonical representation of $D(H)$ (aside from the obvious left-regular representation on itself) is on $H$. This is motivated by thinking of $D(H)$ as a semidirect product of the type arising in quantum mechanics (in some cases it is) and can be called the ‘Schroedinger representation’ of the quantum double. Explicitly, it is

\[
h \triangleright g = \sum h(1) g Sh(2), \quad h \triangleleft a = \sum \langle a, h(1) \rangle h(2), \quad \forall h, g \in H, a \in H^*
\]
which are respectively the quantum adjoint action of $H$ on itself and the right coregular action of $H^*$ on $H$ induced by the left regular coaction. The braiding in this case reduces to

$$\Psi_{H,H}(h \otimes g) = \sum h(1)gSh(2) \otimes h(3).$$

(7)

This is a canonical braiding (i.e. a solution of the celebrated quantum Yang-Baxter equations) associated to any Hopf algebra. For this last observation the Hopf algebra need not be finite-dimensional.

**Proposition 4.1** The representations of the double $D(kM\bowtie k(G))$ are in one-one correspondence with vector spaces $W$ which are

(i) $G$-graded left $M$-modules such that $|t \triangleright w| = tw|w|$ for all $t \in M$, where $| |$ denotes the $G$-degree of a homogeneous element $w \in W$.

(ii) $M$-graded right $G$-modules such that $\langle w \triangleleft u \rangle = \langle w \rangle \triangleleft u$ for all $u \in G$, where $\langle \rangle$ denotes the $M$-degree of a homogeneous element $w \in W$.

(iii) mutually ‘cross modules’ according to

$$\langle t \triangleright w \rangle = t \langle w \rangle(t \triangleleft |w|)^{-1}, \quad |w \triangleright u| = (\langle w \rangle \triangleright u)^{-1}|w|u$$
on homogeneous elements.

(iv) $G - M$-‘bimodules’ according to

$$(t \triangleleft (\langle w \rangle \triangleright u)) \triangleright (w \triangleleft u) = (t \triangleright u) \triangleleft ((t \triangleleft |w|) \triangleright u)$$

The corresponding action of the double and the induced braiding are

$$(t \otimes \delta_v) \triangleright w = t \triangleright \delta_{v,w} |w|, \quad \langle w \rangle \triangleleft (\delta_s \otimes u) = \delta_{s,\langle w \rangle} w \triangleleft u.$$

$$\Psi_{V,W}(v \otimes w) = \langle v \rangle \triangleright w \otimes v \triangleleft |w|.$$  

**Proof** (if direction) Suppose that $W$ is a vector space satisfying conditions (i) to (iv). First we show that (i) ensures that $W$ is a left $H$-module. Since $W$ is $G$-graded, we have $W = \oplus_{g \in G} W_g$, where $|w| = g$ if and only if $w \in W_g$. Also $W$ is a left $k(G)$-module by $\delta_g \triangleright w = \delta_{g,|w|} w$ for homogenous $w$ (i.e. $w \in W_{|w|}$). Additionally from (i), $W$ is a left $M$-module, and hence a left $kM$-module. Now define an action of $H = kM \bowtie k(G)$ on $W$ by

$$(t \otimes \delta_v) \triangleright w = t(v \triangleright \delta_v w) = \delta_{v,|w|} t \triangleright w, \quad \text{wherew} \in W_{|w|} \quad (a).$$

19
To see that this is a left action, calculate

\[
(s \otimes \delta_u)(t \otimes \delta_v) \triangleright w = \delta_{u, t \triangleright v}(st \otimes \delta_v) \triangleright w
\]

\[
= \delta_{u, t \triangleright v}(\delta_{v, |w|} st \triangleright w),
\]

\[
(s \otimes \delta_u) \triangleright ((t \otimes \delta_v) \triangleright w) = \delta_{v, |w|}(s \otimes \delta_u) \triangleright (t \triangleright w)
\]

\[
= \delta_{v, |w|}\delta_{u, |t \triangleright w|} st \triangleright w.
\]

These are equal since \(|t \triangleright w| = t|w|\).

Also

\[
1_H \triangleright w = \left( \sum_u e \otimes \delta_u \right) \triangleright u = \sum_u \delta_{u, |w|} e \triangleright w = \delta_{u, |w|} w = w.
\]

Next we show that \((ii)\) ensures that \(W\) is a right \(H^\ast\)-module. Since \(W\) is \(M\)-graded, \(W = \bigoplus_{s \in M} W_s\), where \(\langle w \rangle = s\) if and only if \(w \in W_s\). Also \(W\) is a right \(k(M)\)-module by \(w \circ \delta_s = \delta_{s, \langle w \rangle} w\) for homogenous \(w\) (i.e. \(w \in W_{\langle w \rangle}\)). By assumption \(W\) is also a right \(kG\)-module. Now define a right action of \(H^\ast = k(M) \triangleright \triangleleft kG\) on \(W\) by

\[
w \circ (\delta_s \otimes u) = (w \circ \delta_s) \triangleright u = \delta_{s, \langle w \rangle} w \triangleright u,
\]

where \(w \in W_{\langle w \rangle}\) (\(b\)).

To see that this is a right action, calculate

\[
w \circ ((\delta_s \otimes u)(\delta_t \otimes v)) = \delta_{s \circ u, t \triangleright v}(\delta_s \circ uv)
\]

\[
= \delta_{s \circ u, t \triangleright \delta_{s, \langle w \rangle} w \triangleright uv},
\]

\[
(w \circ (\delta_s \otimes u)) \circ (\delta_t \otimes v) = \delta_{s, \langle w \rangle}(w \circ \delta_u) \circ (\delta_t \otimes v)
\]

\[
= \delta_{s, \langle w \rangle}\delta_{t, \langle w \circ u \rangle} w \circ uv.
\]

These are equal since \(\langle w \circ u \rangle = \langle w \rangle \circ u\).

The action of \(D(H)\) on \(W\) is defined by \((a \otimes h) \triangleright w = (h \triangleright w) \triangleright a\). Now we must check that this does define a left action of \(D(H)\). If \((1 \otimes h)(a \otimes 1) = a' \otimes h'\) in \(D(H)\), then we show that \((1 \otimes h) \triangleright ((a \otimes 1) \triangleright w) = (a' \otimes h') \triangleright w\), i.e. that \(h \triangleright (w \circ a) = (h' \triangleright w) \circ a'\). Putting \(h = t \otimes \delta_v\) and \(a = \delta_s \otimes u\), assuming that \(w\) is homogenous, and using formulae (a) and (b), we get the equivalent conditions

\[
\delta_{s, \langle w \rangle} \delta_{v, |w \circ u|} t \triangleright (w \circ u) = \delta_{v', |w|} \delta_{s', \langle t' \triangleright w \rangle} (t' \triangleright w) \circ u' \quad (c),
\]

where \((1 \otimes t \otimes \delta_v)(\delta_s \otimes u \otimes 1) = \delta_{s'} \otimes u' \otimes t' \otimes \delta_{v'}\), and using Proposition 3.2,

\[
s' = (t \circ (s \circ u)^{-1}) s(t \circ u)^{-1}, \quad u' = (t \circ u)^{-1} \triangleright u, \quad t' = t \circ (s \circ u)^{-1}, \quad v' = (s \circ u) \circ w^{-1}.
\]
It will be useful to calculate
\[
t'\wedge' = t\preceq (s\triangleright u)^{-1}\preceq (s\triangleright u)v^{-1} = t\wedge v^{-1},
\]
\[
s'\triangleright u' = (t\preceq (s\triangleright u)^{-1})s(t\wedge v^{-1})^{-1}p((t\wedge v^{-1})\triangleright u)
\]
\[
= (t\preceq (s\triangleright u)^{-1})s\triangleright u = (t\preceq (s\triangleright u)^{-1})\triangleright (s\triangleright u)
\]
\[
= (t\triangleright (s\triangleright u)^{-1})^{-1},
\]
\[
t'\triangleright v' = (t\preceq (s\triangleright u)^{-1})\triangleright ((s\triangleright u)v^{-1}) = ((t\preceq (s\triangleright u)^{-1})\triangleright (s\triangleright u))( (t\preceq (s\triangleright u)^{-1})\preceq (s\triangleright u))\triangleright v^{-1}
\]
\[
= (t\preceq (s\triangleright u)^{-1})\triangleright v^{-1} = (s'\triangleright u')\triangleright v^{-1},
\]
\[
s'\wedge u' = (t\preceq (s\triangleright u)^{-1})s(t\wedge v^{-1})^{-1}q((t\wedge v^{-1})\triangleright u)
\]
\[
= ((t\preceq (s\triangleright u)^{-1})s\wedge u) ((t\wedge v^{-1})^{-1}q((t\wedge v^{-1})\triangleright u))
\]
\[
= (t\preceq (s\triangleright u)^{-1}q(s\triangleright u))(s\wedge u)(t\wedge v^{-1}\triangleright u)^{-1}
\]
\[
= t(s\wedge u)(t\wedge v)^{-1}.
\]

Now we use (iii) and (iv) to show that the condition (c) holds. First we prove that
\[
s = \prec w \succ and \quad v = \mid w\wedge u \mid \quad \text{if and only if} \quad v' = \mid w \mid and s' = \prec t'\triangleright w \succ .
\]

If \( s = \prec w \succ \) and \( v = \mid w\wedge u \mid \), then
\[
v' = (s\triangleright u)v^{-1} = (\prec w \succ \triangleright u)v^{-1}
\]
\[
= \mid w\mid w\wedge u\mid^{-1}v^{-1} \quad \text{by(iii)}
\]
\[
= \mid w\mid uv^{-1}v^{-1} = \mid w \mid ,
\]
\[
s' = t's(t\wedge v^{-1})^{-1} = t's(t'\wedge v')^{-1} = t's(t'\wedge \mid w \mid)^{-1} = t' < w > (t'\wedge \mid w \mid)^{-1}
\]
\[
= \prec t'\triangleright w \succ \quad \text{by(iii)} .
\]

Conversely, if \( v' = \mid w \mid \) and \( s' = \prec t'\triangleright w \succ \), then
\[
s = t'^{-1}s'(t\wedge v^{-1}) = t'^{-1} < t'\triangleright w > (t\wedge v^{-1})
\]
\[
= t'^{-1}t' \prec w > (t'\wedge \mid w \mid)^{-1}(t\wedge v^{-1}) \quad \text{by(iii)}
\]
\[
= \prec w > (t'\wedge v')^{-1}(t'\wedge v') = \prec w > ,
\]
\[
v = (s\triangleright u)^{-1}v' u = (\prec w > \triangleright u)^{-1}\mid w \mid u
\]
\[
= \mid w\wedge u \mid \quad \text{by(iii)} .
\]

Now assuming that \( s = \langle w \rangle \) and \( v = |w\rangle u \), we show that \( t \triangleright (w \triangleleft u) = (t' \triangleright w) \triangleleft u' \). If we define \( \bar{t} = t \langle \langle w \rangle \triangleright u \rangle^{-1} \), then

\[
\begin{align*}
t \triangleright (w \triangleleft u) & = (\bar{t} \langle \langle w \rangle \triangleright u \rangle) \triangleright (w \triangleleft u) \\
& = (\bar{t} \triangleright w) \langle (\bar{t} \langle \langle w \rangle \triangleright u \rangle) \triangleright u \rangle \quad \text{by (iv)} \\
& = (\bar{t} \triangleright w) \langle (t \langle \langle w \rangle \triangleright u \rangle^{-1}) \triangleright u \rangle \quad \text{by (iii)} \\
& = (\bar{t} \triangleright w) \langle (t \langle w \triangleleft u \rangle^{-1}) \triangleright u \rangle \quad \text{by (iii)} \\
& = (\langle \langle w \rangle \triangleright u \rangle^{-1}) \triangleright u' \quad \text{by (iii)} \\
& = (t \langle w \triangleleft u \rangle^{-1} \triangleright w) \triangleleft u' \quad (\text{d}) .
\end{align*}
\]

\( \square \)

**Proof** (only if direction): Suppose now that we have a representation of \( D(H) \) on a vector space \( W \). Then \( W \) is a left \( H \)-module. But as an algebra \( H = kM \triangleright k(G) \), the semidirect product of \( kM \) and \( k(G) \). This means that \( W \) is a left \( M \)-covariant left \( k(G) \)-module. But a \( k(G) \)-module is precisely a \( G \)-graded vector space, where the action of \( \delta_g \) is \( \delta_g \triangleright w = \delta_g,|w|u \), for \( w \) homogenous. This and \( M \)-covariance gives (i). The action of \( H \) on \( W \) is \( (t \otimes \delta_v) \triangleright w = \delta_{v,|w|} t \triangleright w \), where \( |.| \) is the \( G \)-grading on \( W \).

Similarly \( W \) is a right \( H^* \)-module, and \( H^* = k(M) \triangleright kG \). By the above reasoning, \( W \) is a right \( G \)-covariant right \( k(M) \)-module. This is condition (ii). The action of \( H^* \) on \( W \) is \( w \triangleleft (\delta_s \otimes u) = \delta_{s,\langle w \rangle} w \triangleleft u \), where \( \langle . \rangle \) is the \( M \)-grading on \( W \).

Because we have a representation of \( D(H) \) on \( W \), and have determined the corresponding actions of \( H \) and \( H^* \), we know that for homogenous \( w \),

\[
\delta_{s,\langle w \rangle} \delta_{v,|w\rangle u} t \triangleright (w \triangleleft u) = \delta_{v',|w| \delta_{s',\langle t' \triangleright w \rangle} (t' \triangleright w) \triangleleft u', \quad (c) .
\]

We show that this implies (iii) and (iv). Now assume that \( s = \langle w \rangle \) and \( v = |w\rangle u \). Then

\[
|w| = v' = (s \triangleright u) vu^{-1} = (\langle w \rangle \triangleright u) vu^{-1} = (\langle w \rangle \triangleright u)|w\rangle u|u^{-1} = (\langle w \rangle \triangleright u)|w\rangle u|u^{-1} .
\]
so that $|w\triangleleft u| = (\langle w \triangleright u \rangle)^{-1}|w|u$, 

$$<w> = s = t'^{-1}s'(t\triangleleft vu^{-1}) = t'^{-1}<t\triangleright w>(t\triangleleft vu^{-1})$$

$$= t'^{-1}<t\triangleright w>(t'\triangleleft w') = t'^{-1}<t\triangleright w>(t'\triangleleft |w|),$$

so that $<t\triangleright w> = t' <w> (t'\triangleleft |w|)^{-1}$,

which gives (iii).

To prove (iv), note that by following the calculation (d) backwards to the second step (which is legitimate, since we now know that (iii) holds) gives

$$(t'\triangleright w)\triangleleft u' = (\bar{t}\triangleright w)\triangleleft ((\bar{t}\triangleleft |w|)\triangleright u)$$

where $\bar{t} = t\triangleleft (<w>\triangleright u)^{-1}$. Then

$$(t'\triangleright w)\triangleleft u' = t\triangleright (w\triangleleft u) = (\bar{t}\triangleleft (<w>\triangleright u))\triangleright (w\triangleleft u),$$

so that we have the following equation, which in turn gives (iv):

$$(\bar{t}\triangleright w)\triangleleft ((\bar{t}\triangleleft |w|)\triangleright u) = (\bar{t}\triangleleft (<w>\triangleright u))\triangleright (w\triangleleft u).$$

Let $V$ and $W$ be representations of $D(H)$. Then the induced braiding is given by (5), i.e.

$$\Psi_{V,W}(v \otimes w) = \sum (s \otimes \delta_u)\triangleright w \otimes v\triangleleft (\delta_s \otimes u) = \sum (\delta_{|w|u}\triangleleft s \triangleright w) \otimes (\delta_s, v\triangleright u)$$

$$= (v)\triangleright w \otimes v\triangleleft |w|.$$

\[\square\]

Note that conditions (iii) are each a generalisation of Whitehead’s notion of crossed modules\[5\], but now the ‘crossed $G$-module’ and the ‘crossed $M$-module’ are coupled unless one of the factors is trivial. For example, if $M$ is trivial then we have exactly a right crossed $G$-module. The connection between quantum doubles of a single group and crossed modules was introduced in \[11\], and the last proposition extends this point of view to the quantum double of a bicrossproduct associated to a matched pair of groups. In view of this, we call modules obeying the conditions in the proposition bicrossed bimodules.

**Proposition 4.2** The Schrödinger representation of the quantum double on $W = kM\triangleright k(G)$ and the induced braiding are

$$(s \otimes \delta_u)\triangleright (t \otimes \delta_v) = st(s\triangleleft u)^{-1} \otimes \delta_{(s\triangleleft u)\triangleright v, t\triangleright v}, \quad (t \otimes \delta_v)\triangleleft (\delta_s \otimes u) = \delta_s, t\triangleleft u \otimes \delta_{u^{-1}}v.$$
\[ \Psi(s \otimes \delta_u \otimes t \otimes \delta_v) = sts'^{-1} \otimes \delta_{s' \triangleright v} \otimes s' \otimes \delta_{v(t \triangleright v)^{-1}u}; \quad s' = s \triangleleft (t \triangleright v)v^{-1}. \]

**Corollary 4.3** The gradings and \( M \times G \) actions for the Schroedinger representation \( W = kM \triangleright k(G) \) according to Proposition 4.1 are

\[
|t \otimes \delta_v| = (t \triangleright v)v^{-1}, \quad \langle t \otimes \delta_v \rangle = t
\]

\[ s \triangleright (t \otimes \delta_v) = st s'^{-1} \otimes \delta_{s' \triangleright v}, \quad (t \otimes \delta_v) \triangleright u = t \triangleright u \otimes \delta_u^{-1}v; \quad s' = s \triangleleft (t \triangleright v)v^{-1}. \]

**Proof** of proposition and corollary: First we use formula (6) to calculate the action of \( H \) on itself, taking \( h = s \otimes \delta_u, \ g = t \otimes \delta_v. \) Since \( \Delta(h) = \sum_{xy = u} s \otimes \delta_x \otimes s \triangleleft x \otimes \delta_y, \)

\[
(s \otimes \delta_u) \triangleright (t \otimes \delta_v) = \sum_{xy = u} (s \otimes \delta_x)(t \otimes \delta_v)(s \triangleleft x \otimes \delta_y) S(s \triangleright x \otimes \delta_y)
\]

\[
= \sum_{xy = u} (s \otimes \delta_x)(t \otimes \delta_v)((s \triangleleft xy)^{-1} \otimes \delta((s \triangleleft x) \triangleright y)^{-1})
\]

\[
= \sum_{xy = u} \delta_{x,t \triangleright v}(st \otimes \delta_v)((s \triangleright u)^{-1} \otimes \delta((s \triangleright u)^{-1}(s \triangleright x))
\]

\[
= (s \triangleright u)^{-1} \triangleright (s \triangleright u)^{-1}(st \triangleright v) (st(s \triangleright u)^{-1} \otimes \delta((s \triangleright u)^{-1}(st \triangleright v))
\]

where we have used

\[
(s \triangleleft x) \triangleright y = (s \triangleleft x) \triangleright x^{-1}u = (s \triangleright x)^{-1}(s \triangleright u).
\]

To simplify further, we calculate

\[
(s \triangleleft u)^{-1} \triangleright (s \triangleright u)^{-1}(st \triangleright v) = ((s \triangleright u)^{-1} \triangleright (s \triangleright u)^{-1})((s \triangleleft u)^{-1} \triangleright (s \triangleright u)^{-1}) \triangleright (st \triangleright v)
\]

\[
= u^{-1}(s^{-1} \triangleright (st \triangleright v)) = u^{-1}(t \triangleright v),
\]

so that we may assume that \( t \triangleright v = uv, \) and then

\[
(s \triangleright u)^{-1}(st \triangleright v) = (s \triangleright u)^{-1}(s \triangleright uv) = (s \triangleright u)^{-1}(s \triangleright u)(s \triangleright u)(s \triangleleft u) \triangleright v = (s \triangleright u) \triangleright v.
\]

Hence we obtain

\[
(s \otimes \delta_u) \triangleright (t \otimes \delta_v) = \delta_{uv,(s \triangleright u)^{-1}((s \triangleright u)^{-1})} \triangleright (s \triangleright u) \triangleright v.
\]
Next we calculate the right coregular action of $H^*$ on $H$, using formula (6) with $h = t \otimes \delta_v$ and $a = \delta_s \otimes u$,

$$(t \otimes \delta_v) \triangleleft (\delta_s \otimes u) = \sum_{xy=v} (\delta_s \otimes u, t \otimes \delta_x)(t \otimes x) \otimes \delta_y$$

$$= \sum_{xy=v} \delta_{s,t} \delta_{u,x}(t \otimes x) \otimes \delta_y = \delta_{s,t}(t \otimes u) \otimes \delta_{u-1,v}.$$ 

We calculate the induced braiding using the formula

$$\Psi_{H,H}(v \otimes w) = <v > \triangleright w \otimes v \triangleleft |w| ,$$

from proposition 4.1. We could also use (7). But first we must determine the $M$ and $G$ gradings, and the actions of $M$ and $G$ on $H$. From the first part of the proof, and proposition 4.1,

$$(s \otimes \delta_u) \triangleright (t \otimes \delta_v) = \delta_{uv,tv} st(s \triangleleft u)^{-1} \otimes \delta_{(s \triangleleft u) \triangleright v} ,$$

$$(s \otimes \delta_u) \triangleright (t \otimes \delta_v) = \delta_{u,t \otimes \delta_v} \triangleright (t \otimes \delta_v) ,$$

so that

$$|t \otimes \delta_v| = (t \triangleright v)^{-1} ,$$

$$s \triangleright (t \otimes \delta_v) = st(s \triangleleft u)^{-1} \otimes \delta_{(s\triangleleft u)\triangleright v}$$

$$= st s'^{-1} \otimes \delta_{s'\triangleright v} ,$$

where $s' = s\triangleleft u = s\triangleleft (t \triangleright v)^{-1}$ (in view of the delta function, we may take $u = (t \triangleright v)^{-1}$).

Similarly we showed

$$(t \otimes \delta_v) \triangleleft (\delta_s \otimes u) = \delta_{s,t} t \triangleleft u \otimes \delta_{u-1,v} ,$$

$$(t \otimes \delta_v) \triangleleft (\delta_s \otimes u) = \delta_{s,\langle t \otimes \delta_v \rangle}(t \otimes \delta_v) \triangleleft u ,$$

so that

$$\langle t \otimes \delta_v \rangle = t ,$$

$$(t \otimes \delta_v) \triangleleft u = t \triangleleft u \otimes \delta_{u-1,v} .$$

Then the braiding is given by

$$\Psi_{H,H}(s \otimes \delta_u \otimes t \otimes \delta_v) = <s \otimes \delta_u \triangleright (t \otimes \delta_v) \otimes (s \otimes \delta_u) \triangleleft |t \otimes \delta_v|$$
\[
\begin{align*}
&= s \langle t \otimes \delta_v \otimes (s \otimes \delta_u) \rangle < (t \otimes v) \nu^{-1} \\
&= s t^{-1} \otimes \delta_{s^t v} \otimes s \langle (t \otimes v) \nu^{-1} \otimes \delta_{u(t \otimes v)\nu} \rangle \\
&= s t^{-1} \otimes \delta_{s^t v} \otimes s' \otimes \delta_{u(t \otimes v)\nu} \\
&= s t^{-1} \otimes \delta_{s^t v} \otimes s' \otimes \delta_{u(t \otimes v)\nu} \\
&= s t^{-1} \otimes \delta_{s^t v} \otimes s' \otimes \delta_{u(t \otimes v)\nu} \\
&= s t^{-1} \otimes \delta_{s^t v} \otimes s' \otimes \delta_{u(t \otimes v)\nu} \\
&= s t^{-1} \otimes \delta_{s^t v} \otimes s' \otimes \delta_{u(t \otimes v)\nu}.
\end{align*}
\]

\[\square\]

Example 4.4 For our \( \mathbb{Z}_{2^n} \otimes \mathbb{Z}_{2^n} \) example the brading matrix \( \Psi \) is

\[
\Psi(s \otimes \delta_u \otimes t \otimes \delta_v) = \begin{cases} 
  t \otimes \delta_v \otimes s \otimes \delta_u & t \text{ even} \quad s \text{ even} \\
  t \otimes \delta_v \otimes s \otimes \delta_{u+2v} & t \text{ odd} \quad s \text{ even} \\
  t \otimes \delta_{-v} \otimes s \otimes \delta_u & t \text{ even} \quad s \text{ odd} \\
  t \otimes \delta_{-v} \otimes s \otimes \delta_{u+2v} & t \text{ odd} \quad s \text{ odd}
\end{cases}
\]

In the case \( \mathbb{Z}_6 \otimes \mathbb{Z}_6 \) the minimal polynomial of \( \psi \) is

\[-1 - \lambda^2 - \lambda^4 + \lambda^8 + \lambda^{10} + \lambda^{12}.
\]

Proof Remember that both actions of \( \mathbb{Z}_{2^n} \) on \( \mathbb{Z}_{2^n} \) are given by an even element having trivial action, and an odd element reversing sign. Then the gradings are given by the formulae \( |t \otimes \delta_v| = 0 \) if \( t \) is even, and \( |t \otimes \delta_v| = -2v \) if \( t \) is odd. Also \( <t \otimes \delta_v> = t \). This gives the formula for \( \Psi \).

The effect of the linear transformation is to permute the elements of the bases. If we have a vector space of dimension \( m \) and a linear transformation which cyclically permutes the basis vectors, the characteristic polynomial is \( \lambda^m - 1 \). Since all the roots are distinct, this is also the minimal polynomial. If we have a vector space which is a direct sum of subspaces where a linear transformation acts on each subspace, the minimal polynomial is the least common multiple of the minimal polynomials on each subspace.

To get the minimal polynomial of \( \Psi \), we decompose the permutation of the basis vectors into disjoint cycles. A cycle of length \( m \) contributes \( \lambda^m - 1 \), and we take the least common multiple of the \( \lambda^m - 1 \) polynomials over the cycles. The order of \( \Psi \) is the least common multiple of the lengths of the cycles. Note that \( \Psi \) preserves \( s \) and \( t \) in \( s \otimes \delta_u \otimes t \otimes \delta_v \), and we can consider the different cases even and odd for \( s \) and \( t \) separately.

The calculation of the sizes of the cycles is in general a non-trivial problem in number theory. We shall only do the calculation for the case \( n = 3 \).

If both \( s \) and \( t \) are even, these subspaces are 1 dimensional (ie. no effect). The polynomial is \( \lambda - 1 \).
If $s$ is odd and $t$ even, we get the shift $(u, v) \mapsto (-v, u)$, which has order 4, and subspaces of order 1, 2 and 4. The polynomial is $\lambda^4 - 1$.

If $s$ is even and $t$ odd, we get the shift $(u, v) \mapsto (v, u + 2v)$, which has order 8, and subspaces of order 1, 2 and 8. The polynomial is $\lambda^8 - 1$.

If both $s$ and $t$ are odd, we get the shift $(u, v) \mapsto (-v, u + 2v)$, which has order 6, and subspaces of order 1, 2, 3 and 6. The polynomial is $\lambda^6 - 1$.

Taking the least common multiple of the $\lambda^m - 1$ factors, we see that $\Psi$ has minimal polynomial $(\lambda^8 - 1)(\lambda^6 - 1)/(\lambda^2 - 1) = -1 - \lambda^2 - \lambda^4 + \lambda^8 + \lambda^{10} + \lambda^{12}$. □

5. $D(H)$ as a twisting of $D(X)$.

We recall that $X$ is the double cross product group $GM$. This group has a quantum double $D(X) = k(X)\bowtie kX$ which is an ordinary crossproduct. It has operations

$$(\delta_x \otimes y)(\delta_a \otimes b) = \delta_{y^{-1}xy,a}(\delta_x \otimes yb), \quad \Delta(\delta_x \otimes y) = \sum_{ab=x} \delta_a \otimes y \otimes \delta_b \otimes y$$

$$1 = \sum_x \delta_x \otimes e, \quad e(\delta_x \otimes y) = \delta_x.e, \quad S(\delta_x \otimes y) = \delta_{y^{-1}xy^{-1}}, \quad S(\delta_x \otimes y) = \delta_{y^{-1}xy} \otimes y^{-1}, \quad S(\delta_x \otimes y)^* = \delta_{y^{-1}xy} \otimes y^{-1},$$

$$R = \sum_{y,z} \delta_y \otimes e \otimes \delta_z \otimes y .$$

The representations of $D(X)$ are given by $X$ graded left $kX$ modules. We denote the $kX$ action by $\hat{\cdot}$, and the grading by $\|\cdot\|$. In a representation of $D(X)$, the grading and $X$ action are related by

$$\|y\hat{\cdot}v\| = y\|v\|y^{-1}, \quad y \in X, v \in V ,$$

and the action of $\delta_x \otimes y \in D(X)$ is given by

$$(\delta_x \otimes y)\hat{\cdot}v = \delta_x.\|y\hat{\cdot}v\| \cdot y\hat{\cdot}v .$$

We give an operation $\chi$ which sends representations of $D(H)$ to representations of $D(X)$ in the following manner: Let $W$ be a representation of $D(H)$, as described in the previous section. Then as vector spaces, $\chi W$ is the same as $W$. The $X$-grading $\|\cdot\|$ on $\chi W$ is defined by $\|\chi(w)\| = \langle w \rangle^{-1}|w|$, and the action of $us$ in $kX$ is given by

$$us\hat{\cdot}\chi(w) = \chi(\langle (s\hat{\cdot}|w|^{-1})\hat{\cdot}w \rangle^{-1}u^{-1}w^{-1}) , \quad s \in M, u \in G .$$
Proposition 5.1 The map $\chi$ gives a 1-1 correspondence between representations of $D(H)$ and $D(X)$.

Proof We prove that $\|\|$ and $\triangleright\triangleleft$ give a representation of $D(X)$. First we must show that $\triangleright\triangleleft$ is an action, which means that

$$vt\triangleright\triangleleft((us\triangleright\triangleleft\chi(w)) = vtus\triangleright\triangleleft\chi(w),$$

for all $s, t \in M$ and $u, v \in G$. Note that

$$vt\triangleright\triangleleft(\chi(((s\triangleright|w|^{-1})\triangleright\triangleright w)\triangleleft|u|^{-1})) = \chi(((t\triangleright\triangleright|w|^{-1})\triangleright\triangleright\triangleright w)\triangleleft|v|^{-1}),$$

where $\bar{w} = ((s\triangleright|w|^{-1})\triangleright\triangleright w)\triangleleft|u|^{-1}$. On the other hand we have $vtus = v(t\triangleright u)(t\triangleleft u)s$, where $v(t\triangleright u) \in G$ and $(t\triangleleft u)s \in M$, so

$$vtus\triangleright\triangleleft\chi(w) = \chi(((t\triangleright\triangleright|w|^{-1})\triangleright\triangleright\triangleright w)\triangleleft|t\triangleleft u|^{-1}v^{-1})).$$

We must therefore show that

$$(t\triangleright\triangleright|\bar{w}|^{-1})\triangleright\triangleright\bar{w} = (((t\triangleright u)s\triangleright|w|^{-1})\triangleright\triangleright w)\triangleleft|t\triangleleft u|^{-1}.$$ 

To do this we have to find the $G$ grade of $\bar{w}$, which we do as follows: Put $\bar{w} = w'\triangleleft|u|^{-1}$, where $w' = (s\triangleright|w|^{-1})\triangleright\triangleright w$. Then using the properties listed in 4.1,

$$|w'| = |(s\triangleright|w|^{-1})\triangleright\triangleright w| = (s\triangleright|w|^{-1})\triangleright\triangleright w = (s\triangleright|w|^{-1})^{-1},$$

$$\langle w' \rangle = (s\triangleright|w|^{-1})\langle w \rangle s^{-1},$$

$$|\bar{w}| = |w'\triangleleft|u|^{-1}| = ((w')\triangleright\triangleright|w|^{-1}|w'|\triangleleft|u|^{-1} = ((w')\triangleright\triangleright|w|^{-1}^{-1}u^{-1} = (w')\triangleright\triangleright|w|^{-1}^{-1}u^{-1}.$$ 

Putting $\bar{t} = t\triangleright u(s\triangleright|w|^{-1})$, and $\bar{v} = (t\triangleright u)^{-1}$, we have

$$\left(\left((t\triangleright u)s\triangleright|w|^{-1}\right)\triangleright\triangleright w\right)\triangleleft|t\triangleleft u|^{-1} = \left(\left((t\triangleright u)s\triangleright|\bar{w}|^{-1}\right)\triangleright\triangleright w\right)\triangleleft|t\triangleleft u|^{-1}$$

$$= (\bar{t}\triangleright\bar{w})\triangleleft|\bar{v}|^{-1}$$

$$= (\bar{t}\triangleright\langle w' \rangle(\bar{t}\triangleright|w'|^{-1}\triangleright\triangleright\bar{v}))\triangleright\triangleright (w'\triangleright\langle\bar{t}\triangleright|w'|^{-1}\triangleright\triangleright\bar{v} \rangle).$$
where the derivation of the last line uses 4.1(iv). To simplify the last line, we calculate

\[ t \triangleleft |w'| = (t \triangleleft u(s \triangleright |w|^{-1}) \triangleleft (s \triangleright |w|^{-1})^{-1} = t \triangleleft u, \]

\[ (\bar{t} \triangleleft |w'|)^{-1} \triangleright \bar{u} = (t \triangleleft u)^{-1} \triangleright (t \triangleright u)^{-1} = u^{-1}. \]

Then we have

\[
\left( (t \triangleleft u) s \triangleright |w|^{-1} \triangleright w \right) \triangleleft u^{-1} = \left( \bar{t} \triangleleft (\langle w' \triangleright \bar{u}^{-1} \rangle \right) \triangleright \left( w' \triangleleft u^{-1} \right) \\
= (\bar{t} \triangleleft (\langle w' \triangleright \bar{u}^{-1} \rangle) \triangleright \bar{w} \\
= \left( (t \triangleleft u(s \triangleright |w|^{-1}) \triangleleft (\langle w' \triangleright u^{-1} \rangle \right) \triangleright \bar{w} \\
= \left( t \triangleleft |w|^{-1} \triangleright \bar{w} \right),
\]

as required. Next we must show that \( \|y \triangleright \chi(w)\| = y\|\chi(w)\|y^{-1} \). This proceeds by a similar calculation, and is left to the reader!

Finally we show that \( \chi \) is a 1-1 correspondence by giving its inverse \( \chi^{-1} \) in the following form. Let \( V \) be a representation of \( D(X) \), with \( kX \) action \( \triangleright \), and \( X \)-grading \( \| . \| \). Define a \( D(H) \) representation as follows: As a vector space \( \chi^{-1}V \) will be the same as \( V \). There are \( G \) and \( M \) gradings given by the factorisation

\[ \|v\| = \langle \chi^{-1}v \rangle^{-1}|\chi^{-1}v|, \quad \langle \chi^{-1}v \rangle \in M, \quad |\chi^{-1}v| \in G. \]

The actions of \( s \in M \) and \( u \in G \) are given by

\[ s \triangleright \chi^{-1}v = \chi^{-1}((s \triangleleft |\chi^{-1}v|) \triangleright v) \quad \chi^{-1}v \triangleleft u = \chi^{-1}(u^{-1} \triangleright v). \]

\( \square \)

A 1-1 correspondence between the representations of \( D(X) \) and \( D(H) \) suggests an algebra isomorphism between \( D(X) \) and \( D(H) \). We show that this is in fact the case.

**Proposition 5.2** There is an algebra isomorphism \( \psi : D(H) \to D(X) \) defined by the formula

\[ \psi(\delta_s \otimes u \otimes t \otimes \delta_v) = \delta_{u^{-1}s^{-1}(t \triangleright v)u} \otimes u^{-1}(t \triangleleft v), \]

and such that \( \chi \) is induced by pull back along \( \psi \).
Proof (a) To prove that $\psi$ is an algebra isomorphism, we take $\alpha = \delta_s \otimes q \otimes t \otimes \delta_v$, and $\beta = \delta_s \otimes u \otimes r \otimes \delta_w$, and their images $\psi(\alpha) = \delta_s \otimes \bar{u}$ and $\psi(\beta) = \delta_t \otimes \bar{v}$. Then the product in $D(X)$ is

$$\psi(\alpha)\psi(\beta) = \delta_{\bar{u}-1\bar{s},\bar{t}}(\delta_s \otimes \bar{u}) \otimes \bar{v}.$$ 

Expanding this using the formula for $\psi$ gives

$$\psi(\alpha)\psi(\beta) = \delta_{(t\lhd v)^{-1}-1, tv,u^{-1}s^{-1}(r\rhd w)u}(\delta_q^{-1}x^{-1}(t\lhd v)q \otimes q^{-1}(t\lhd w)u^{-1}(r\lhd w)).$$

Now using Proposition 3.2 to calculate $\alpha\beta$ in $D(H)$,

$$\alpha\beta = \delta_{s', au', x} \delta_{v', r\rhd w} \delta_{s' u' q} \otimes t' r \otimes \delta_w,$$

where

$$t' = t < (s\rhd u)^{-1}, v' = (s\rhd u)vu^{-1}, s' = t's(t\lhd vu^{-1})^{-1}, u' = (t\lhd vu^{-1})\rhd u.$$

Applying $\psi$ to this product gives

$$\psi(\alpha\beta) = \delta_{s', au', x} \delta_{v', r\rhd w} \delta_{s' u' q} \otimes (u'q)^{-1}(t' r < w)).$$

First we establish the equality of the delta functions

$$\delta_{(t\lhd v)^{-1}-1, tv,u^{-1}s^{-1}(r\rhd w)u} = \delta_{s', au', x} \delta_{v', r\rhd w}. $$

Firstly, suppose that the left hand side is non-zero. Then the uniqueness of the factorisation $X = MG$ gives $(t\lhd v)^{-1}x^{-1}t = (s\lhd u)^{-1}$ and $v = (s\rhd u)^{-1}(r\rhd w)u$. Rearranging these yields $r\rhd w = (s\rhd u)vu^{-1}$ and $x = t(s\rhd u)(t\lhd w)^{-1}$, so the right hand side is also non-zero. Similarly for the other way around.

Next, assuming the above delta functions are not zero, we show that

$$q^{-1}x^{-1}(t\rhd v)q = (s' u' q)^{-1}(t' r\rhd w)u' q.$$ 

This reduces to showing that

$$s' u' x^{-1}(t\rhd v) = (t' r\rhd w)u'.$$

Since $r\rhd w = (s\rhd u)vu^{-1}$, we can calculate

$$(t' r\rhd w)u' = (t\rhd (s\rhd u)^{-1})^{-1}(t\lhd v).$$
and since $x = t(s\triangleright u)(t\triangleright v)^{-1}$,

$$s'u'x^{-1}(t\triangleright v) = (t\triangleright (s\triangleright u))^{-1}(t\triangleright v),$$

so we have the required identity.

Lastly, again under the assumption that the delta functions are not zero, we show that

$$q^{-1}(t\triangleright v)u^{-1}(r\triangleright w) = (u'q)^{-1}(t'r\triangleright w).$$

This reduces to showing that

$$(t\triangleright v)u^{-1} = u'^{-1}(t'r\triangleright w)(r\triangleright w)^{-1}.$$ 

On substituting for $r\triangleright w$,

$$u'^{-1}(t'r\triangleright w)(r\triangleright w)^{-1} = u'^{-1}(t'\triangleright (r\triangleright w))(r\triangleright w)^{-1} = u'^{-1}t\triangleright vu^{-1} = (t\triangleright v)u^{-1}.$$ 

This completes the proof that $\psi$ is an algebra map. To show that it is an isomorphism, we can give an explicit form for the inverse. For $s, t \in M$ and $u, v \in G$, we have

$$\psi^{-1}(\delta_{su} \otimes tv) = \delta_{(s^{-1}\triangleright v)^{-1} \otimes (t\triangleright u)^{-1} \otimes t\triangleright v^{-1} \triangleright u^{-1} \triangleright v^{-1}},$$

where $\alpha = t^{-1}u^{-1}(s^{-1}t\triangleright v).$ $\Box$

**Proof** (b): We next show that the maps $\psi$ and $\chi$ are linked by the equation $\chi((a \otimes h)\triangleright w) = \psi(a \otimes h)\delta\chi(w)$.

Begin by

$$\chi((\delta_s \otimes u \otimes t \otimes \delta_v)\triangleright w) = \delta_{s, (t\triangleright w)} \delta_{v, |w|} \chi((t\triangleright w)\triangleright u),$$

$$\psi(\delta_s \otimes u \otimes t \otimes \delta_v)\triangleright \chi w = (\delta_{u^{-1}s^{-1}(t\triangleright v)u} \otimes u^{-1}(t\triangleright w)) \triangleright \chi w$$

$$= \delta_{u^{-1}s^{-1}(t\triangleright v)u, u^{-1}(t\triangleright w)} \triangleright \chi w \otimes u^{-1}(t\triangleright w) \triangleright \chi w$$

$$= \delta_{u^{-1}s^{-1}(t\triangleright v)u, u^{-1}(t\triangleright w), u^{-1}(t\triangleright w) \triangleright \chi w} \chi((t\triangleright w)(|w|^{-1})\triangleright w).$$

Now we simplify the delta function: If the delta function is non-zero, then

$$s^{-1}tv = s^{-1}(t\triangleright v)(t\triangleright v) = (t\triangleright v)||\chi w|| = (t\triangleright v)\langle w \rangle^{-1}||w||.$$ 

31
By the uniqueness of factorisation, we find that \( s^{-1}t = (t\triangleright v)(w)^{-1} \) and \( v = |w| \). But then \( s = t(w)(t\triangleright|w|)^{-1} = (t\triangleright w) \), giving the equality we want. \( \square \)

We have now discussed the algebra structure and algebra representations. To examine the coalgebra structure we turn to tensor products of representations. Recall that for a general Hopf algebra \( H \) with representations \( V \) and \( V' \), the tensor product representation \( V \otimes V' \) is defined by the action

\[
h\triangleright (v \otimes v') = \sum h_{(1)} \triangleright v \otimes h_{(2)} \triangleright v'.
\]

In the case of representations \( V \) and \( V' \) of \( D(X) \), this formula leads to the equations

\[
y\tilde{\triangleright} (v \otimes v') = y\tilde{\triangleright} v \otimes y\tilde{\triangleright} v' \quad \text{and} \quad \|v \otimes v'| = \|v\|\|v'\|.
\]

If we take representations \( W \) and \( W' \) of \( D(H) \), we get the following actions and gradings on \( W \otimes W' \):

\[
t\triangleright (w \otimes w') = (t\triangleright w) \otimes ((t\triangleright|w|)\triangleright w') \quad t \in M,
\]

\[
|w \otimes w'| = |w||w'|,
\]

\[
(w \otimes w')\langle u = (w\langle (w')\triangleright u \rangle \otimes (w'\langle u \rangle) \quad u \in G,
\]

\[
\langle w \otimes w' \rangle = \langle w \rangle \langle w' \rangle.
\]

We find that \( \chi \) does not preserve tensor products of representations. To correct for this we introduce another map \( c : \chi W \otimes \chi W' \rightarrow \chi (W \otimes W') \), defined by

\[
c(\chi(w) \otimes \chi(w')) = \chi((w')\langle|w|^{-1}\rangle\triangleright w \otimes w').
\]

**Proposition 5.3** The map \( c \) is a \( D(X) \) module map, i.e. it preserves the grading and action in the following manner:

\[
\|c(\chi w \otimes \chi w')\| = \|\chi w \otimes \chi w'\|
\]

\[
y\tilde{\triangleright} c(\chi w \otimes \chi w') = c(y\tilde{\triangleright} (\chi w \otimes \chi w')).
\]

for all \( y \in X, w \in W \) and \( w' \in W' \).
\textbf{Proof} First we begin with the grading. We know that
\[
\|\chi w \otimes \chi w'\| = \|\chi w\|\|\chi w'\| = \langle w'^{-1}w, w'^{-1}w' \rangle.
\]
On the other hand
\[
\|c(\chi w \otimes \chi w')\| = \|\chi((\langle w'\rangle^{-1}\triangleright w) \otimes w')\|
\]
\[
\begin{aligned}
\quad &= \langle((\langle w'\rangle^{-1}\triangleright w) \otimes w')^{-1}((\langle w'\rangle^{-1}\triangleright w) \otimes w')\rangle \\
\quad &= \langle w'^{-1}((\langle w'\rangle^{-1}\triangleright w) \otimes w')^{-1}((\langle w'\rangle^{-1}\triangleright w) \otimes w')\rangle.
\end{aligned}
\]
By 4.1 we can calculate
\[
\begin{aligned}
\quad &= \langle((\langle w'\rangle^{-1}\triangleright w) \otimes w')\rangle \\
\quad &= \langle((\langle w'\rangle^{-1}\triangleright w) \otimes w')\rangle \\
\quad &= \langle w'^{-1}((\langle w'\rangle^{-1}\triangleright w) \otimes w')^{-1}((\langle w'\rangle^{-1}\triangleright w) \otimes w')\rangle.
\end{aligned}
\]
which gives the result.

Now we shall show that \(c\) preserves the action, which we do separately for the two factors \(G\) and \(M\) of \(X\).

For the \(G\) action, we must show that
\[
c(u\triangleright(\chi w \otimes \chi w')) = u\triangleright c(\chi w \otimes \chi w') .
\]
By the definitions
\[
u\triangleright(\chi w \otimes \chi w') = \chi(w\triangleright u^{-1}) \otimes \chi(w'\triangleright u^{-1}) ,
\]
\[
c(u\triangleright(\chi w \otimes \chi w')) = \chi\left(\langle(w'\triangleright u^{-1})\triangleright w \otimes (w'\triangleright u^{-1})\triangleright (w\triangleright u^{-1})\rangle \otimes (w'\triangleright u^{-1})\right).
\]
Using the properties of the \(G\) and \(M\) gradings,
\[
\langle w'\triangleright u^{-1}\rangle \triangleright w \triangleright u^{-1} = ((w')\triangleright w^{-1})\triangleright ((w)\triangleright u^{-1}) ,
\]
\[
\langle w'\triangleright u^{-1}\rangle \triangleright w \triangleright u^{-1} = ((w')\triangleright w^{-1})\triangleright ((w)\triangleright u^{-1})\triangleright (w\triangleright u^{-1})
\quad = ((w')\triangleright w^{-1})\triangleright ((w)\triangleright u^{-1}) \rangle ,
\]
so that
\[
c(u\triangleright(\chi w \otimes \chi w')) = \chi\left(\langle w'\triangleright w^{-1}\rangle \triangleright (w'\triangleright u^{-1})\rangle \otimes (w'\triangleright u^{-1})\right).
\]

33
Next we calculate

\[
\begin{align*}
\tilde{u} \tilde{c}(\chi w \otimes \chi w') &= \tilde{u} \tilde{c}(\chi ((\langle w' \rangle|w|^{-1}) \triangleright w) \otimes w') \\
&= \chi \left( (((\langle w' \rangle|w|^{-1}) \triangleright w) \otimes (w') \triangleright u^{-1} \right) \\
&= \chi \left( (((\langle w' \rangle|w|^{-1}) \triangleright w) \triangleright ((w') \triangleright u^{-1})) \otimes (w' \triangleright u^{-1}) \right),
\end{align*}
\]

which agrees with the alternative calculation.

Lastly we shall show that \( c \) preserves the \( M \) action. We have

\[
\begin{align*}
\tilde{s} \tilde{c}(\chi w \otimes \chi w') &= \chi (\langle s|w|^{-1} \rangle \triangleright w) \otimes \chi (\langle s|w'|^{-1} \rangle \triangleright w') .
\end{align*}
\]

Using the equations

\[
\begin{align*}
\langle (s|w'|^{-1}) \triangleright w' \rangle &= (s|w'|^{-1}) \langle w' \rangle \ s^{-1} \\
\langle (s|w|^{-1}) \triangleright w \rangle &= (s|w|^{-1}) \langle w \rangle = (s \triangleright |w|^{-1})^{-1},
\end{align*}
\]

we see that

\[
\begin{align*}
c(\tilde{s} \tilde{c}(\chi w \otimes \chi w')) &= \chi \left( (((\langle s|w'|^{-1} \rangle \triangleright w') \triangleright (\langle s|w|^{-1} \rangle \triangleright w|^{-1}) \triangleright (\langle s|w|^{-1} \rangle \triangleright w)) \otimes (\langle s|w'|^{-1} \rangle \triangleright w') \right) \\
&= \chi \left( (((\langle s|w'|^{-1} \rangle \langle w' \rangle \ s^{-1} \triangleright (s \triangleright |w|^{-1})) \triangleright (\langle s|w|^{-1} \rangle \triangleright w)) \otimes (\langle s|w'|^{-1} \rangle \triangleright w') \right) \\
&= \chi \left( (((\langle s|w'|^{-1} \rangle \langle w' \rangle \triangleright |w|^{-1}) \triangleright w) \otimes (\langle s|w'|^{-1} \rangle \triangleright w') \right).
\end{align*}
\]

On the other hand,

\[
\begin{align*}
\tilde{s} \tilde{c}(\chi w \otimes \chi w') &= \tilde{s} \tilde{c}(\tilde{w} \otimes w') \\
&= \chi (\langle s|\tilde{w} \otimes w'|^{-1} \rangle \triangleright (\tilde{w} \otimes w')) \\
&= \chi ((\tilde{t} \triangleright \tilde{w}) \otimes ((\tilde{t} \triangleright |\tilde{w}|) \triangleright w')) ,
\end{align*}
\]

where \( \tilde{w} = (\langle w' \rangle|w|^{-1}) \triangleright w \) and \( \tilde{t} = s|\tilde{w} \otimes w'|^{-1} \). Next we calculate

\[
\begin{align*}
|\tilde{w} \otimes w'| &= |\tilde{w}||w'| , \\
|\tilde{w}| &= (\langle w' \rangle|w|^{-1})^{-1} , \\
\tilde{t} \triangleright \tilde{w} &= (s|\tilde{w}'|^{-1}) \langle w'|^{-1} \rangle \triangleright w , \\
(\tilde{t} \triangleright |\tilde{w}|) \triangleright w' &= (s|w'|^{-1}) \triangleright w' ,
\end{align*}
\]
and substitution of these gives the answer. □

We find that \((\chi, c)\) is a monoidal functor between the categories of representations. By Tannaka-Krein reconstruction theory, we then expect that the Hopf algebras \(D(H)\) and \(D(X)\) (which are already isomorphic as algebras) are related by a cocycle \(F\) in the following manner:

If \(F \in D(X) \otimes D(X)\) is an invertible, we define a new coproduct \(\tilde{\Delta}\) for \(D(X)\) and element \(R\) by the formula 6

\[
\tilde{\Delta}(h) = F(\Delta h)F^{-1}, \quad \text{and} \quad \tilde{R} = (\tau F)RF^{-1}.
\]

This also gives a quasitriangular Hopf algebra if \(F\) is a 2-cocycle. The condition for a 2-cocycle is \((1 \otimes F)(id \otimes \Delta)F = (F \otimes 1)(\Delta \otimes id)F\). The new Hopf algebra has the same representations as \(D(X)\) with a different tensor product, i.e. is monoidally equivalent to it.

**Proposition 5.4** The following formula defines a 2-cocycle:

\[
F = \sum_{x \in X, \, t \in M, \, v \in G} \delta_x \otimes t^{-1} \otimes \delta_{tv} \otimes e.
\]

**Proof** We follow our usual convention that \(r, s\) and \(t\) are in \(M\), that \(u, v\) and \(w\) are in \(G\), and that \(x, y, a, b\) and \(z\) are in \(X\). First we calculate

\[
(id \otimes \Delta)F = \sum_{x, t, v, a, b, ab = tv} \delta_x \otimes t^{-1} \otimes \delta_a \otimes e \otimes \delta_b \otimes e,
\]

\[
(1 \otimes F)(id \otimes \Delta)F = \sum_{x, t, v, a, b, ab = tv} \sum y, s, u \delta_x \otimes t^{-1} \otimes \delta_y \otimes s^{-1} \otimes \delta_a \otimes e \otimes \delta_s \otimes \delta_u \otimes e,
\]

\[
= \sum_{x, t, v, a, b, ab = tv} \sum y, s, u \delta_{ys^{-1}, a} \delta_{su, b} \delta_x \otimes t^{-1} \otimes \delta_y \otimes s^{-1} \otimes \delta_{su} \otimes e,
\]

\[
= \sum_{x, t, v, s, u} \delta_x \otimes t^{-1} \otimes \delta_{s^{-1}tvu} \otimes s^{-1} \otimes \delta_{su} \otimes e.
\]

On the other hand,

\[
(\Delta \otimes id)F = \sum_{x, t, v, a, b, ab = tv} \delta_a \otimes \tilde{t}^{-1} \otimes \delta_b \otimes \tilde{t}^{-1} \otimes \delta_{t\tilde{b}} \otimes e,
\]

\[
(F \otimes 1)(\Delta \otimes id)F = \sum_{x, t, v, a, b, ab = tv} \sum z, r, u \delta_z \otimes r^{-1} \otimes \delta_a \otimes \tilde{t}^{-1} \otimes \delta_{r\tilde{u}} \otimes \delta_{t\tilde{b}} \otimes e,
\]

\[
= \sum_{x, t, v, a, b, ab = tv} \sum z, r, u \delta_{zr^{-1}, a} \delta_{r\tilde{u}, b} \delta_z \otimes r^{-1} \otimes \delta_{r\tilde{u}} \otimes \tilde{t}^{-1} \otimes \delta_{t\tilde{b}} \otimes e,
\]

\[
= \sum_{x, t, v, a, b, ab = tv} \delta_{r^{-1}zr^{-1}, x, z, r, u} \otimes r^{-1} \otimes \delta_{r\tilde{u}} \otimes \tilde{t}^{-1} \otimes \delta_{t\tilde{b}} \otimes e.
\]
Now if we put \( \tilde{t} = s, \tilde{v} = u, r = s^{-1}t, w = vu^{-1} \), and \( \tilde{x} = s^{-1}txv^1 \), we see that the two expressions are equal. \( \square \)

We now know that twisting the coproduct of \( D(X) \) by \( F \) gives another quasi-triangular Hopf algebra. It remains to relate this to the \( D(H) \) Hopf algebra in the following manner:

**Proposition 5.5**

\[
\tilde{\Delta}(h) = F(\Delta_{D(X)}h).F^{-1} = (\psi \otimes \psi)\Delta_{D(H)}(\psi^{-1}h)
\]

for all \( h \in D(X) \). Hence the twisting of \( D(X) \) by \( F \) is isomorphic to \( D(H) \).

**Proof** Here we use the conventions that \( x, y, a, b \in X, r, s, t, \tilde{t}, a', b' \in M, \) and \( u, v, w, \tilde{v}, x', y' \in G \). Begin by calculating

\[
\delta_x \otimes y = \psi(\delta_s \otimes u \otimes t \otimes \delta_v) = \delta_{u^{-1}a^{-1}(t \triangleright v)u} \otimes u^{-1}(t \triangleleft v).
\]

Then we have

\[
(\psi \otimes \psi)\Delta(\delta_s \otimes u \otimes t \otimes \delta_v) = \sum_{a'b'=s,x'y'=v} \psi(\delta_{a'} \otimes (b' \triangleright u) \otimes t \otimes \delta_{x'}) \otimes \psi(\delta_{y'} \otimes u \otimes (t \triangleleft x') \otimes \delta_{y'})
\]

\[
= \sum_{a'b'=s,x'y'=v} \delta_{(b' \triangleright u)^{-1}a'^{-1}(t \triangleright x')(b' \triangleright u)} \otimes (b' \triangleright u)^{-1}(t \triangleleft x') \otimes \delta_{u^{-1}a'^{-1}((t \triangleleft x')(b' \triangleright u))u} \otimes u^{-1}(t \triangleright v)
\]

\[
= \sum_{a'b'=s,x'y'=v} (b' \triangleright u)^{-1}a'^{-1}(t \triangleright x')(b' \triangleright u) \otimes (b' \triangleright u)^{-1}(t \triangleleft x') \otimes \delta_{t \triangleright x} \otimes y
\]

where we have used the relabelings

\[
\tilde{t} = (b' \triangleleft u)^{-1}, \tilde{v} = (b' \triangleright u)^{-1}(t \triangleright x')^{-1}(t \triangleright v)u.
\]

Then we can find

\[
b' \triangleright u = (t^{-1} \triangleright u)^{-1},
\]

\[
t \triangleright x' = (t \triangleright v)u\tilde{v}^{-1}(b' \triangleright u)^{-1} = (t \triangleright v)u\tilde{v}^{-1}(t^{-1} \triangleright u)^{-1}.
\]

Now use the relation \( a'b' = s \) to show that

\[
(b' \triangleright u)^{-1}a'^{-1}(t \triangleright x')(b' \triangleright u) = (\tilde{t}^{-1} \triangleright u)^{-1}a'^{-1}(t \triangleright v)u\tilde{v}^{-1}
\]

\[
= (\tilde{t}^{-1} \triangleright u)^{-1}(t^{-1} \triangleleft u)^{-1}s^{-1}(t \triangleright v)u\tilde{v}^{-1}
\]

\[
= \tilde{t}^{-1}u^{-1}s^{-1}(t \triangleright v)u\tilde{v}^{-1} = \tilde{t}^{-1}x\tilde{v}^{-1}.
\]
Also we consider
\[
(b'\triangleright u)^{-1}(t\triangleleft x') = (\bar{t}^{-1}b'^{-1})(t\triangleright x')^{-1}tx'
\]
\[
= \bar{v}u^{-1}(t\triangleright v)^{-1}tx'
\]
\[
= \bar{v}u^{-1}(t\triangleleft v)^{-1}x' = \bar{v}yv^{-1}x'.
\]

On the other hand, we calculate
\[
F.\Delta(\delta_x \otimes y) = \left( \sum_{x,t,v} \delta_x \otimes \bar{t}^{-1} \otimes \delta_{\bar{t}v} \otimes e \right) \left( \sum_{ab=x} \delta_a \otimes y \otimes \delta_b \otimes y \right)
\]
\[
= \sum_{\bar{t},\bar{v}} \delta_{\bar{t}^{-1}x\bar{v}^{-1}} \otimes \bar{t}^{-1}y \otimes \delta_{\bar{t}v} \otimes y.
\]
The next stage is to calculate
\[
F.\Delta(\delta_x \otimes y).F^{-1} = \left( \sum_{\bar{t},\bar{v}} \delta_{\bar{t}^{-1}x\bar{v}^{-1}} \otimes \bar{t}^{-1}y \otimes \delta_{\bar{t}v} \otimes y \right) \left( \sum_{z,r,w} \delta_z \otimes r \otimes \delta_{rw} \otimes e \right)
\]
\[
= \sum_{r,w,\bar{t},\bar{v}} \delta_{\bar{t}^{-1}x\bar{v}^{-1}} \otimes \bar{t}^{-1}yr \otimes \delta_{\bar{t}v} \otimes y.
\]
Then the equation \( rw = y^{-1}\bar{t}\bar{v}y \) can be rearranged to give \( \bar{t}^{-1}yr = \bar{v}yw^{-1} \), which gives the required equality of the two expressions (with \( w = x'^{-1}v \)). □

This is also true at the level of quasitriangular structures. Recall first that if \( R \) is a quasitriangular structure, then so is \( \tau R^{-1} \). In our above conventions we have:

**Proposition 5.6** The relation between the quasi-triangular structures on \( D(X) \) and \( D(H) \) is given by the formula \( (\tau F)(\tau R^{-1})F^{-1} = \psi \otimes \psi(R_{D(H)}) \).

**Proof** First we calculate
\[
(\tau F)R = \left( \sum_{x,t,\bar{v}} \delta_{\bar{t}v} \otimes e \otimes \delta_x \otimes \bar{t}^{-1} \right) \left( \sum_{y,z} \delta_y \otimes e \otimes \delta_z \otimes y \right)
\]
\[
= \sum_{x,t,\bar{v}} \delta_{\bar{t}v} \otimes e \otimes \delta_x \otimes \bar{v},
\]
\[
(\tau F)RF^{-1} = \left( \sum_{x,t,\bar{v}} \delta_{\bar{t}v} \otimes e \otimes \delta_x \otimes \bar{v} \right) \left( \sum_{y,r,w} \delta_y \otimes r \otimes \delta_{r\omega} \otimes e \right)
\]
\[
= \sum_{x,t,\bar{v}} \delta_{\bar{t}v} \otimes r \otimes \delta_x \otimes \bar{v},
\]
where \( r \in M \) is the solution to the factorisation \( rw = \bar{v}^{-1}x\bar{v} \).
On the other hand,

\[(\psi \otimes \psi) R_{D(H)} = (\psi \otimes \psi) \sum_{s' \in M, u' \in G} \delta_{s'} \otimes u' \otimes e \otimes 1 \otimes 1 \otimes e \otimes \delta_{u'} \]

\[= \sum_{s', t' \in M, u', v' \in G} \delta_{u'^{-1}s'^{-1}v'u'^{-1}} \otimes u'^{-1} \otimes \delta_{v'^{-1}(s' \triangleright u') \otimes (s' \triangleleft u')} .\]

We can reorganise this by using

\[s = (s' \triangleleft u')^{-1}, \quad u = (s' \triangleright u')^{-1}, \quad v = uv'u' \quad \text{and} \quad t = t'^{-1} \quad \text{to get}\]

\[(\psi \otimes \psi) R_{D(H)} = \sum_{s, t \in M, u, v \in G} \delta_{uv} \otimes (s \triangleright u) \otimes \delta_{tu^{-1}} \otimes s^{-1} .\]

Now if we calculate (left to the reader), we find

\[(\tau(\psi \otimes \psi) R_{D(H)})(\tau F)RF^{-1} = 1_{D(X)} \otimes 1_{D(X)} .\]

\[\square\]

The cocycle \(F\) is not a coboundary. This means that there is no inner automorphism relating the two coproduct structures on \(D(X)\), although the Hopf algebras may still be isomorphic via an outer automorphism. To see this, we note that if \(F\) was a coboundary, there would be an invertible element \(\gamma \in D(X)\) so that \(\gamma \otimes \gamma = F \Delta \gamma \) in \(D(X)\). We shall take \(\gamma\) of the form

\[\gamma = \sum_{x, y \in X} \gamma_{x, y} \delta_x \otimes y ,\]

and \(F\) of the form given previously. Then we can work out the following expressions as

\[F \Delta \gamma = \sum_{z, t, v, y} \gamma_{t_z, v, y} \delta_z \otimes t^{-1} y \otimes \delta_{tv} \otimes y ,\]

\[\gamma \otimes \gamma = \sum_{z, t, v, y, x} \gamma_{z, x} \gamma_{tv, y} \delta_z \otimes x \otimes \delta_{tv} \otimes y .\]

If these are equal, we must have

\[\gamma_{z, x} \gamma_{tv, y} = \begin{cases} 0 & x \neq t^{-1} y \\ \gamma_{t_z, v, y} & x = t^{-1} y \end{cases} .\]

To make these equations clearer, we put

\[\beta_{t^{-1} y, v, y} = \gamma_{tv, y} ,\]

and write \(z = su\) to rewrite the previous equation as

\[\beta_{s^{-1} x, u, x} = \begin{cases} 0 & x \neq t^{-1} y \\ \beta_{s^{-1} x, uv, y} & x = t^{-1} y \end{cases} .\]
Now suppose that there are $x_0$, $s_0$, and $u_0$ with $\beta_{s_0^{-1}x_0, u_0, x_0} \neq 0$. The equation then shows that $\beta_{t^{-1}y, v, y} = 0$ if $y \neq s_0^{-1}x_0$. If we consider the expression $\beta_{s_0^{-1}x_0, u_0, x_0} \beta_{s_0^{-1}x_0, u_0, x_0} \neq 0$ in the equation, we find that $s_0 = e$. The only possible non-zero $\beta$s are of the form $\beta_{x_0, u, x_0}$, which corresponds to the only possible non-zero $\gamma$s being of the form $\gamma_{u, x_0}$. But this means that $\gamma$ cannot be invertible in $D(X)$ (unless one of the groups in the factorisation of $X$ is just the identity). We see that the non-abelian cohomology $H^2(D(X))$ is non-trivial.

References

[1] S. Majid. Physics for algebraists: Non-commutative and non-cocommutative Hopf algebras by a bicrossproduct construction. *J. Algebra*, 130:17–64, 1990. From PhD Thesis, Harvard, 1988.

[2] M. Takeuchi. Matched pairs of groups and bismash products of Hopf algebras. *Comm. Alg.*, 9:841, 1981.

[3] V.G. Drinfel’d. Quantum groups. In A. Gleason, editor, *Proceedings of the ICM*, pages 798–820, Rhode Island, 1987. AMS.

[4] S. Majid. Crossproduct quantisation, non-Abelian cohomology, and twisting of Hopf algebras. In H. Doebner & V. Dobrev, eds. *Generalised Symmetries in Physics*, pages 13–41, Singapore, 1994. World Sci.

[5] J.H.C. Whitehead. Combinatorial homotopy, II. *Bull. Amer. Math. Soc.*, 55:453–496, 1949.

[6] S. Majid. *Foundations of Quantum Group Theory*. In press, C.U.P. 1995.

[7] V.G. Drinfel’d. Quasi-Hopf algebras, *Algebra i Analiz*, 1(6):2, 1989.

[8] D.I. Gurevich and S. Majid. Braided groups of Hopf algebras obtained by twisting, Pacific J. Math, 162:27-44, 1994.

[9] Y. Doi. Equivalent crossed products of a Hopf algebra. *Commum. Algebra* 17: 3053-3085.

[10] M.E. Sweedler. *Hopf Algebras*. Benjamin, 1969.
[11] S. Majid. Doubles of quasitriangular Hopf algebras. *Comm. Algebra*, 19(11):3061–3073, 1991.