GAUSSIAN FLUCTUATIONS OF EIGENVALUES IN THE GUE

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Abstract. Under certain conditions on $k$ we calculate the limit distribution of eigenvalue number $k$, $x_k$, of the Gaussian unitary ensemble (GUE). More specifically, if $n$ is the dimension of a random matrix from the GUE and $k$ is such that both $k$ and $n - k$ tend to infinity as $n \to \infty$, then $x_k$ is normally distributed in the limit.

We also consider the joint limit distribution of $(x_{k_1}, \ldots, x_{k_m})$ where we require that $k_1, n - k_m$ and $k_{i+1} - k_i$, $1 \leq i \leq m - 1$, tend to infinity with $n$. The result is an $m$-dimensional normal distribution.

1. Introduction and formulation of results

The Gaussian unitary ensemble (GUE) is a classical random matrix ensemble. It is defined by the probability distribution on the space of $n \times n$ Hermitian matrices given by

$$\mathbb{P}(dH) = C_n e^{-\text{Trace}H^2} dH.$$ 

By $dH$ we mean the Lebesgue measure on the $n^2$ essentially different members of the matrix, namely

$$\{\text{Re}H_{ij}; 1 \leq i \leq j \leq n, \text{Im}H_{ij}; 1 \leq i < j \leq n\}.$$ 

In other words this means that the entries in (1.1) are independent Gaussian random variables with zero mean and variance $\frac{1 + \delta_{ij}}{4}$. The measure on the matrices naturally induces a measure on the corresponding $n$ real eigenvalues $x_i$. This induced measure can be explicitly calculated and its density is given by

$$p_n(x_1, \ldots, x_n) = \frac{1}{Z_n^{(2)}} \prod_{1 \leq i < j \leq n} |x_i - x_j|^2 e^{-x_1^2 - \cdots - x_n^2}.$$ 

The normalization constant $Z_n^{(2)}$ is called the partition function. It is often convenient to work with the eigenvalues being ordered. Naming the eigenvalues so that $x_1 < \ldots < x_n$, gives that the probability density $\rho_{n,n}(x_1, \ldots, x_n)$ of the ordered eigenvalues defined on the space

$$\mathbb{R}_{\text{ord}}^n = \{x_1, \ldots, x_n; x_1 < \ldots < x_n\}$$

is given by

$$\rho_{n,n}(x_1, \ldots, x_n) = n! p_n(x_1, \ldots, x_n).$$
This density $\rho$ is a member of a family of functions called the correlation functions. These functions are defined by

$$\rho_{n,k}(x_1, \ldots, x_k) = \frac{n!}{(n-k)!} \int_{\mathbb{R}^{n-k}} p_n(x_1, \ldots, x_n) \, dx_{k+1} \ldots dx_n = \det(K_n(x_i, x_j))_{i,j=1}^k.$$ 

Here $K_n(x, y)$ is given by

$$K_n(x, y) = \sum_{i=0}^{n-1} h_i(x) h_i(y) e^{-\frac{1}{2}(x^2+y^2)}$$

where $\{h_i\}_{i \geq 0}$ are the orthonormalized Hermite polynomials, that is

$$\int_{-\infty}^{\infty} h_i(x) h_j(x) e^{-x^2} \, dx = \delta_{ij}.$$ 

The kernel $K_n(x, y)$ can also be represented by the so called Christoffel-Darboux identity. For $x \neq y$ it holds that

$$K_n(x, y) = \left(\frac{n}{2}\right)^{1/2} \frac{h_n(x)h_{n-1}(y) - h_n(y)h_{n-1}(x)}{x-y} e^{-\frac{1}{2}(x^2+y^2)}$$

and on the diagonal one has

$$K_n(x, x) = \left(n h_n^2(x) - \sqrt{n(n+1)}h_{n-1}(x)h_{n+1}(x)\right) e^{-x^2}.$$ 

The correlation function $\rho_{n,1}$ describes the overall density of the eigenvalues and the Wigner semi-circle law states that

$$(1.2) \quad \lim_{n \to \infty} \frac{2}{n \rho_{n,1}(\sqrt{2nx})} = \begin{cases} 
\frac{2}{\pi} \sqrt{1-x^2} & \text{if } |x| \leq 1 \\
0 & \text{if } |x| > 1.
\end{cases}$$

All the results above and more can be found in the book by Mehta, [7].

This paper deals with the distribution of eigenvalue number $k$, $x_k$, of the GUE. More specifically, we look at the distribution of $x_k$ as $n$ and $k$ both tend to infinity. For example, if

$$k = k(n) = n - \log n$$

then, as $n$ becomes large, $k$ is very close (relatively) to the right edge of the spectrum. Another example is when $k = n/2$. In this case we are in the middle of the bulk of the spectrum. In both cases one ends up with a normal distribution in the limit. The following theorems generalize and specify this statement.

**Theorem 1.1 (The bulk).** Set

$$G(t) = \frac{2}{\pi} \int_{-1}^{t} \sqrt{1-x^2} \, dx \quad -1 \leq t \leq 1$$
and $t = t(k, n) = G^{-1}(k/n)$ where $k = k(n)$ is such that $k/n \to a \in (0, 1)$ as $n \to \infty$. If $x_k$ denotes eigenvalue number $k$ in the GUE it holds that, as $n \to \infty$,
\[
\frac{x_k - t \sqrt{2n}}{\left( \frac{\log n}{4(1-t^2)n} \right)^{1/2}} \to N(0, 1)
\]
in distribution.

**Theorem 1.2** (The edge). Let $k$ be such that $k \to \infty$ but $\frac{k}{n} \to 0$ as $n \to \infty$ and let $x_{n-k}$ denote eigenvalue number $n - k$ in the GUE. Then it holds that, as $n \to \infty$,
\[
\frac{x_{n-k} - \sqrt{2n}}{\left( 1 - \left( \frac{3x_k}{4\sqrt{2n}} \right)^{2/3} \right) \left( \frac{1}{(12\pi)^{2/3}} \frac{\log k}{n^{1/3}k^{2/3}} \right)^{1/2}} \to N(0, 1)
\]
in distribution.

**Remark 1.** The theorems deal with the bulk and the right spectrum edge. One gets the equivalent for the left edge with some obvious modifications.

**Remark 2.** In [12] the distribution of the largest eigenvalue was studied.

**Remark 3.** Set $I = (-\infty, s]$. In [1] it is shown that
\(\text{(1.3)}\quad P[x_k \in (s, s + ds)] = \left( \frac{1}{(k-1)!} \int_{x_{k-1}}^{x_k} J_k(x_1, \ldots, x_{k-1}, s) \mu(dx_1) \ldots \mu(dx_{k-1}) \right) \mu(ds).\)

Here $J_k$ is the so called Janossy density and
\[\mu(dx) = C e^{-x^2/2} dx.\]

In [1] it is also proven that $J_k$ can be expressed explicitly by a determinantal formula. For $k$ and $n$ as in (1.1) or (1.2) we should thus have that (1.3) is, for large $n$, approximately equal to the probability density function for the normal distribution $N(\nu, \sigma)$. The parameters $\nu$ and $\sigma$ should of course be taken to be those indicated from the relevant theorem above.

**Remark 4.** The zero number $k$ of the Hermite polynomial of degree $n$ is close to the expected value of eigenvalue number $k$ of GUE$_n$. This can be shown directly by the following result, [4]:

\[\text{(1.4)}\quad P[x_k \in (s, s + ds)] = \left( \frac{1}{(k-1)!} \int_{x_{k-1}}^{x_k} H_k(x_1, \ldots, x_{k-1}, s) \mu(dx_1) \ldots \mu(dx_{k-1}) \right) \mu(ds).\]
There are constants $k_0$ and $C$ such that for $k_0 \leq k \leq n - k_0$ and $\alpha = k/n$ it holds that
\begin{equation}
\left| \frac{z_{k,n}}{\sqrt{2n}} - G^{-1}\left[ \frac{k}{n} - \frac{1}{2\pi n} \arcsin\left(G^{-1}(k/n)\right) + \frac{1}{2n} \right] \right| \leq \frac{C}{n^2(\alpha(1-\alpha))^{1/3}}.
\end{equation}

Here $z_{1,n} < \ldots < z_{n,n}$ are the zeros of the Hermite polynomial of degree $n$. When we are in the bulk this translates into
\begin{equation}
\left| z_{k,n} - \sqrt{2n} G^{-1}(k/n) \right| \leq \frac{C}{\sqrt{n}}.
\end{equation}

This means that one can replace $t\sqrt{2n}$ by $z_{k,n}$ in Theorem 1.1. Close to the edge, for example when $k = n - \log n$, we cannot use (1.4) to conclude that this replacement is allowed.

A motivation for this approximate equality between the locations of the zeros and eigenvalues goes as follows. Set
\begin{equation}
W = \frac{1}{2} \sum_{i=1}^{n} x_i^2 - \sum_{1 \leq i < j \leq n} \log |x_i - x_j|.
\end{equation}

and note that
\begin{equation}
\rho_{n,n}(x_1, \ldots, x_n) = \text{Const} \cdot e^{-2W}.
\end{equation}

It is a fact, see [7], that $W$ obtains its minimum exactly when $x_i = z_{i,n}$, $1 \leq i \leq n$. This configuration is hence the most 'probable' for the eigenvalues. Expanding around this minimum we see that it is reasonable that $x_k$ should have Gaussian fluctuations around $z_{k,n}$.

Remark 5. If one is interested in the distribution of the eigenvalues of some other ensemble one should in many cases be able to apply the same methodology that we use in this paper.

It is also interesting to see what happens when looking at several eigenvalues at the same time. When we write $k(n) \sim n^\theta$ below we mean that $k(n) = h(n)n^\theta$ where $h$ is a function such that, for all $\epsilon > 0$,
\begin{equation}
\frac{h(n)}{n^\epsilon} \to 0 \quad \text{and} \quad h(n)n^\epsilon \to \infty
\end{equation}
as $n \to \infty$. We have the following results:

**Theorem 1.3** (The bulk). Let $\{x_k\}_{i=1}^m$ be eigenvalues of the GUE such that $0 < k_i - k_{i+1} \sim n^{\theta_i}$, $0 < \theta_i \leq 1$, and $k_i/n \to a_i \in (0,1)$ as $n \to \infty$. Define $s_i = s_i(k_i, n) = G^{-1}(k_i/n)$ and set
\begin{equation}
X_i = \frac{x_k - s_i\sqrt{2n}}{\left(\frac{\log n}{4(1-s_i^2)n}\right)^{1/2}} \quad i = 1, \ldots, m.
\end{equation}
Then as, $n \to \infty$,
\[ P \left[ X_1 \leq x_1, \ldots, X_m \leq x_m \right] \longrightarrow \Phi_\Lambda(x_1, \ldots, x_m) \]
where $\Phi_\Lambda$ is the cdf\(^1\) for the $m$-dimensional normal distribution with covariance matrix $\Lambda_{i,j} = 1 - \max\{\theta_k | i \leq k < j < m\}$ if $i < j$ and $\Lambda_{i,i} = 1$.

**Theorem 1.4** (The edge). Let $\{x_{n-k_i}\}_{i=1}^m$ be eigenvalues of the GUE such that $k_1 \sim n^\gamma$ where $0 < \gamma < 1$ and $0 < k_{i+1} - k_i \sim n^{\theta_i}$, $0 < \theta_i < \gamma$. Set
\[ X_i = \frac{x_{n-k_i} - \sqrt{2n}}{\left(\frac{1}{12}\pi\right)^{2/3} \frac{\log n}{n^{1/3} k_i^{2/3}}} \]

Then as, $n \to \infty$,
\[ P \left[ X_1 \leq x_1, \ldots, X_m \leq x_m \right] \longrightarrow \Phi_\Lambda(x_1, \ldots, x_m) \]
where $\Phi_\Lambda$ is the cdf for the $m$-dimensional normal distribution with covariance matrix $\Lambda_{i,j} = 1 - \frac{1}{n} \max\{\theta_k | i \leq k < j < m\}$ if $i < j$ and $\Lambda_{i,i} = 1$.

**Remark 1.** As one would expect the eigenvalues near the edge of the spectrum are less correlated than the ones in the bulk.

**Remark 2.** The eigenvalues are quite correlated in the bulk. In order for $x_k$ and $x_m$ to be independent in the limit it must hold that $|k - m| \sim n$. It is interesting to compare with the following result by Mosteller\(^2\), p. 201: Let $X_i$, $i = 1, \ldots, n$, be independent random variables uniformly distributed on $(0,1)$. Consider the asymptotic joint distribution of the $m$ sample quantiles $X_{n_j}$, $j = 1, \ldots, m$, where $n_j = \lfloor \lambda_j n \rfloor + 1$ and $0 < \lambda_1 < \ldots < \lambda_m$.

**Theorem 1.5** (Mosteller). As $n \to \infty$ the joint distribution of $X_{n_1}, \ldots, X_{n_m}$ tends to the $m$-dimensional normal distribution with means $\lambda_j$, variances $n^{-1}\lambda_j(1 - \lambda_j)$ and correlations
\[ \rho(X_{n_j}, X_{n_{j'}}) = \sqrt{\frac{\lambda_j(1 - \lambda_{j'})}{\lambda_{j'}(1 - \lambda_j)}} \]

Hence, in this case $X_{n_1}, \ldots, X_{n_m}$ are globally correlated in the limit.

2. Proofs of Theorems 1.1 and 1.2

The proofs of Theorems 1.1 and 1.2 rely on a theorem due to Costin, Lebowitz and Soshnikov, see \cite{2} and \cite{10}. Before presenting it we need some notation.

Let $\{P_t\}$, $t \in \mathbb{R}_+$, be a family of random point fields, see \cite{9}, on the real line such that their correlation functions have a determinantal form\(^3\).

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1 Cumulative distribution function
2 Mosteller actually allowed for $X_i$ to come from more general distributions.
3 An example is the GUE.
Call the determinant kernels $K_t(x, y)$ and let $\{I_t\}$ be a set of intervals. We denote by $A_t$ the integral operator on $L^2(I_t)$ defined by the kernel $K_t(x, y)$, $A_t : L^2(I_t) \rightarrow L^2(I_t)$. Furthermore, $\mathbb{E}_t$ and $\text{Var}_t$ denote the expectation and variance with respect to the probability measure $P_t$. Finally, let $\# I_t$ stand for the number of particles in $I_t$.

**Theorem 2.1** (Costin-Lebowitz, Soshnikov). Let $A_t = K_t \cdot \chi_{I_t}$ be a family of trace class operators associated with the determinantal random point fields $\{P_t\}$ such that $\text{Var}_t(\# I_t) = \text{Trace}(A_t - A_t^2)$ goes to infinity as $t \rightarrow \infty$. Then

$$\frac{\# I_t - \mathbb{E}[\# I_t]}{\sqrt{\text{Var}(\# I_t)}} \rightarrow N(0, 1)$$

in distribution with respect to the random point field $P_t$.

The following lemmas will be proven in sections 4 and 5:

**Lemma 2.1.** Let $t = t(k, n)$ be the solution to the equation

$$\frac{2}{\pi} \int_{-1}^{t} \sqrt{1 - x^2} \, dx = k,$$

where $k = k(n)$ is such that $k/n \rightarrow a \in (0, 1)$ as $n \rightarrow \infty$. The expected number of eigenvalues in the interval

$$I_n = \left[ \sqrt{2n} t + x \sqrt{\frac{\log n}{2n}}, \infty \right)$$

is given by

$$\mathbb{E}[\#I_n] = n - k - \frac{x}{\pi} \sqrt{(1 - t^2) \log n} + O \left( \frac{\log n}{n} \right).$$

**Lemma 2.2.** The expected number of eigenvalues in the interval $I_n = [\sqrt{2n} t, \infty)$, where $t \rightarrow 1^-$ as $n \rightarrow \infty$, is given by

$$\mathbb{E}[\#I_n] = \frac{4\sqrt{2}}{3\pi} n (1 - t)^{3/2} + O(1).$$

**Lemma 2.3.** Let $\delta > 0$ and suppose that $t$, which may depend on $n$, is such that $-1 + \delta \leq t < 1$ and $n(1 - t)^{3/2} \rightarrow \infty$ as $n \rightarrow \infty$. Then the variance of the number of eigenvalues in the interval $I_n = [\sqrt{2n} t, \infty)$ is given by

$$\text{Var}(\#I_n) = \frac{1}{2\pi^2} \log \left[ n(1 - t)^{3/2} \right] \left( 1 + \eta(n) \right)$$

where $\lim_{n \rightarrow \infty} \eta(n) = 0$.

Using these lemmas and Theorem 2.1 we are now ready to prove Theorems 1.1 and 1.2.

**Proof of Theorem 1.1:** Set

$$I_n = \left[ \sqrt{2n} + \xi \left( \frac{\log n}{4(1 - t^2)n} \right)^{1/2}, \infty \right).$$
Using Lemma 2.1 and Lemma 2.3 we get

\[
\mathbb{P}_n \left[ x_k - t\sqrt{\frac{2n}{(\log n)^{1/2}}} \leq \xi \right] = \mathbb{P}_n \left[ x_k \leq t\sqrt{\frac{2n}{4(1-t^2)n}} + \xi \left( \frac{\log n}{4(1-t^2)n} \right)^{1/2} \right]
\]

\[
= \mathbb{P}_n \left[ \#I_n \leq n - k \right] = \mathbb{P}_n \left[ \frac{\#I_n - E_n[\#I_n]}{\text{Var}_n(\#I_n)^{1/2}} \leq \frac{n - k - E_n[\#I_n]}{(\text{Var}_n(\#I_n))^{1/2}} \right]
\]

\[
= \mathbb{P}_n \left[ \frac{\#I_n - E_n[\#I_n]}{(\text{Var}_n(\#I_n))^{1/2}} \leq \xi + \epsilon(n) \right]
\]

where \( \epsilon(n) \to 0 \) as \( n \to \infty \). By the Costin-Lebowitz-Soshnikov theorem the conclusion follows.

\[ \square \]

Proof of Theorem 1.2: Let \( g(t) \) be the expected number of eigenvalues in the interval \( I_n = [t\sqrt{2n}, \infty) \). We have that

\[
\mathbb{P}_n \left[ x_{n-k} \leq t\sqrt{2n} \right] = \mathbb{P}_n \left[ \#I_n \leq k \right]
\]

\[
= \mathbb{P}_n \left[ \frac{\#I_n - g(t)}{(\text{Var}_n(\#I_n))^{1/2}} \leq \frac{k - g(t)}{(\text{Var}_n(\#I_n))^{1/2}} \right]
\]

If we can find \( t \) such that

\[
\frac{k - g(t)}{(\text{Var}_n(\#I_n))^{1/2}} \to \xi
\]

as \( n \to \infty \) then, by the Costin-Lebowitz-Soshnikov theorem, we are done. The idea now is therefore to find a candidate for \( t \). We will then insert this \( t \) in the equation above to see if it is satisfied. Set for simplicity \( h(t) = (\text{Var}_n(\#I_n))^{1/2} \). We get from Lemma 2.2 and Lemma 2.3 that

\[
g(t) = a_1 n(1-t)^{3/2} + O(1)
\]

\[
h(t) = a_2 \log^{1/2}[n(1-t)^{3/2}] + o(\log^{1/2}[n(1-t)^{3/2}])
\]

where \( a_i \) are known constants. We need to study the equation

\[
k = g(t) + \xi h(t)
\]

or, since \( g \) is a strictly decreasing function,

\[
t = g^{-1}(k - \xi h(t)) \approx g^{-1}(k) - (g^{-1})'(k) \cdot \xi h(t).
\]

Since

\[
(g^{-1})'(k) = \frac{1}{g'(g^{-1}(k))}
\]

we need to study \( g^{-1}(k) \).

\[
k \approx a_1 n(1-t)^{3/2} \implies
\]

\[
t \approx 1 - \left( \frac{k}{a_1 n} \right)^{2/3}
\]
A reasonable guess for the derivative of $g$ is that
\[ g'(t) \approx -\frac{3a_1}{2} n \sqrt{1-t}. \]

We now get
\[ g'(g^{-1}(k)) \approx -\frac{3a_1}{2} n \left( \frac{k}{a_1 n} \right)^{2/3} \] 
and
\[ h(t) \approx h(g^{-1}(k)) \approx a_2 \log^{1/2} \left[ \frac{k}{a_1 n} \right] \approx a_2 \log^{1/2} k \]

When gluing the pieces together one gets
\[ t \approx 1 - \left( \frac{k}{a_1 n} \right)^{2/3} + \xi \frac{2a_2}{3a_1^{2/3}} \frac{\log^{1/2} k}{k^{1/3} n^{2/3}}. \]

When inserting this expression in (2.1) it turns out that it all works out. Some rearranging finally yields the result. \(\square\)

3. Proof of theorems 1.3 and 1.4

We shall use the following theorem, [11]:

**Theorem 3.1** (Soshnikov). Let \((X, \mathcal{F}, P_L)\) be a family of determinantal random point fields with Hermitian locally trace class kernels \(K_L\) and
\[ \{I^{(1)}_L, \ldots, I^{(k)}_L\}_{L \geq 0} \]
be a family of Borel subsets of \(\mathbb{R}\), disjoint for any fixed \(L\), with compact closure.

Suppose that the variance of the linear statistic \(\sum_{i=-\infty}^{\infty} f_L(x_i)\), where
\[ f_L(x) = \sum_{j=1}^{k} \alpha_j \cdot \chi_{I_L^{(j)}}(x) \quad \alpha_1, \ldots, \alpha_k \in \mathbb{R}, \]
grows to infinity with \(L\) in such a way that
\[ \text{Var}_L \left( \#I^{(j)}_L \right) = \mathcal{O} \left( \text{Var}_L \left( \sum_{i=-\infty}^{\infty} f_L(x_i) \right) \right) \]
for any \(1 \leq j \leq k\). Then the Central limit theorem holds:
\[ \frac{\sum_{j=1}^{k} \alpha^{(L)}_j \#I^{(j)}_L - \mathbb{E}_L \left[ \sum_{j=1}^{k} \alpha^{(L)}_j \#I^{(j)}_L \right]}{\sqrt{\text{Var}_L \left( \sum_{j=1}^{k} \alpha^{(L)}_j \#I^{(j)}_L \right)}} \rightarrow N(0,1) \]
in distribution.
Remark 1: The theorem in [11] is actually more general than the theorem stated here.

Remark 2: If (3.1) holds for any $\alpha_1, \ldots, \alpha_k$ then $\#I_L^{(1)}, \ldots, \#I_L^{(k)}$ are jointly normally distributed in the limit, [5].

Proof of Theorem 1.3: Take $k_i, s_i, \theta_i$ and $X_i$ as in the formulation of Theorem 1.3. Note that $k_i - k_i + 1 \sim n \theta_i$ implies that $s_i - s_i + 1 \sim n^{\theta_i - 1}$.

For any real numbers $x_i$ we have the identity (for $n$ large enough)

$$P[X_1 \leq x_1, \ldots, X_m \leq x_m] = \mathbb{P}\left[\frac{\#I_1 - \mathbb{E}[\#I_1]}{(\text{Var}(\#I_1))^{1/2}} \leq \frac{n - k_1 - \mathbb{E}[\#I_1]}{(\text{Var}(\#I_1))^{1/2}}, \frac{\#I_1 + \#I_2 - \mathbb{E}[\#I_1 + \#I_2]}{(\text{Var}(\#I_1 + \#I_2))^{1/2}} \leq \frac{n - k_2 - \mathbb{E}[\#I_1 + \#I_2]}{(\text{Var}(\#I_1 + \#I_2))^{1/2}}, \ldots, \frac{\sum_{i=1}^m \#I_i - \mathbb{E}[\sum_{i=1}^m \#I_i]}{(\text{Var}(\sum_{i=1}^m \#I_i))^{1/2}} \leq \frac{n - k_m - \mathbb{E}[\sum_{i=1}^m \#I_i]}{(\text{Var}(\sum_{i=1}^m \#I_i))^{1/2}}\right].$$

Here the intervals $I_i$ are given by

$$I_1 = \left(s_1 \sqrt{2n} + x_1 \left(\frac{\log n}{4(1 - s_1^2)n}\right)^{1/2}, \infty\right)$$

$$I_i = \left(s_i \sqrt{2n} + x_i \left(\frac{\log n}{4(1 - s_i^2)n}\right)^{1/2}, \ldots, s_{i-1} \sqrt{2n} + x_{i-1} \left(\frac{\log n}{4(1 - s_{i-1}^2)n}\right)^{1/2}\right]$$

where $2 \leq i \leq m$. We will now show that the random variables

$$\#I_1, \#I_1 + \#I_2, \ldots, \sum_{i=1}^m \#I_i$$

are jointly normal in the limit. To this end we shall use Theorem 3.1 above to show that all linear combinations of the variables are normally distributed in the limit. Since

$$\alpha_1 \#I_1 + \alpha_2 (\#I_1 + \#I_2) = (\alpha_1 + \alpha_2) \#I_1 + \alpha_2 \#I_2$$
and so forth it is clear that one can instead look at all linear combinations of \{\#I_i\}^m_1. Hence, by Theorem 3.1 we must calculate (see Appendix B)

$$\text{Var} (\alpha_1 \#I_1 + \alpha_2 \#I_2 + \ldots + \alpha_m \#I_m)$$

$$= \sum_{i=1}^{m} \alpha_i^2 \iiint_{I_i \times I_i^c} K^2_n(x, y) \, dx \, dy$$

$$- \sum_{i \neq j}^{m} \alpha_i \alpha_j \iiint_{I_i \times I_j} K^2_n(x, y) \, dx \, dy$$

to see that it is of magnitude \(\log n\). First define the set \(M\) by \(\theta_k = 1\). Hence

\[M = \{k_1, \ldots, k_j\}; \quad 1 \leq k_1 < k_2 < \ldots < k_j \leq m - 1\]

for some \(j\) such that \(0 \leq j \leq m - 1\). Suppose first that \(M\) is empty which means that \(\theta_i < 1\) for all \(i\). If \(\alpha_1 \neq 0\) then, by using the inequality

\[xy \leq \frac{1}{2}(x^2 + y^2),\]

we get

$$\text{Var} (\alpha_1 \#I_1 + \alpha_2 \#I_2 + \ldots + \alpha_m \#I_m)$$

$$\geq \sum_{i=1}^{m} \alpha_i^2 \iiint_{I_i \times I_i^c} K^2_n(x, y) \, dx \, dy$$

$$- \sum_{i \neq j}^{m} \frac{1}{2}(\alpha_i^2 + \alpha_j^2) \iiint_{I_i \times I_j} K^2_n(x, y) \, dx \, dy$$

(3.2)

$$= \sum_{i=1}^{m} \alpha_i^2 \left( \iiint_{I_i \times I_i^c} K^2_n(x, y) \, dx \, dy \right.$$ \hspace{2cm} \hfill $$- \sum_{j \neq i}^{m} \iiint_{I_i \times I_j} K^2_n(x, y) \, dx \, dy \right)$$

All the terms under the \(i\)-summation in (3.2) are non-negative and the first term can be calculated as in the proof of Lemma 2.3 which we provide in section 5. In the proof of this lemma we show that in the domain

\[\Omega = \left\{ (x, y); s \leq x \leq s + \frac{1}{\log n}, s - \frac{1}{\log n} \leq y \leq s \right\}\]

it holds that

\[2nK_n(\sqrt{2nx}, \sqrt{2ny}) = \frac{1}{2\pi^2(x - y)^2} + O\left(\frac{1}{\log n}\right)\]

The theorem by Soshnikov does not apply directly to this situation since \(I_1\) does not have compact closure. This is however easily overcome simply by chopping of the interval far out where the probability of finding any eigenvalue is exponentially small in \(n\).
It is also shown that if 
\[ \Omega' = \left\{ (x, y); \sqrt{2ns} \leq x \leq \infty, -\infty < y \leq \sqrt{2ns} \right\} / \sqrt{2n} \cdot \Omega \]
then
\[ \int_{\Omega'} K_n^2(x, y) \, dx \, dy = O(\log \log n) . \]
In what follows we shall often make use of these facts without mentioning it. The main contribution to the first term in (3.2) can now be calculated to be (disregarding \( \alpha_1^2 \))
\[ \int_{s_1}^{s_1 + \frac{1}{2\log n}} \int_{s_1}^{s_1 - n^{\theta^* - 1}} \frac{1}{(x - y)^2} \, dx \, dy = \frac{1 - \theta^*}{2\pi^2} \log n + O(\log \log n) \]
where \( \theta^* = \max_i \theta_i < 1 \). By our definition of \( \sim \) above the integration in the \( y \)-variable should have been over the interval \( (s_1 - 1/\log n, s_1 - h(n)n^{\theta^* - 1}) \) where \( h(n) \) satisfies (1.3). However, because of the logarithmic answer this \( h \) will only produce lower order terms.

Now suppose that \( j = 0 \) as before, \( \alpha_1 = \ldots = \alpha_{k-1} = 0 \) but \( \alpha_k \neq 0 \). In this case we get
\[
\text{Var} \left( \alpha_k \#I_k + \ldots + \alpha_m \#I_m \right) \\
\geq \sum_{i=k}^{m} \alpha_i^2 \left( \int I_i \times I_i^c K_n^2(x, y) \, dx \, dy \\
- \sum_{k \leq j \neq i} \int I_i \times I_j K_n^2(x, y) \, dx \, dy \right) .
\]
Using the estimates above it is straightforward to verify that the \( k \)-term is of order \( \log n \).

When \( j \geq 1 \) meaning that there is at least one \( k \) with \( \theta_k = 1 \), things are only slightly more complicated. Let \( k^* \) be the largest integer \( i \) such that \( \theta_i = 1 \). It is sufficient to consider the case when there exists \( i \geq k^* + 1 \) such that \( \alpha_i \neq 0 \). On the other hand if this is the case then we are in a situation very similar to when \( j = 0 \). Either \( \alpha_{k^*+1} \neq 0 \) or \( \alpha_{k^*+1} = \ldots = \alpha_{l-1} = 0 \) but \( \alpha_l \neq 0 \). The details are left out.

It is hence a fact that
\[ \#I_1, \#I_1 + \#I_2, \ldots, \sum_{i=1}^{m} \#I_i \]
in the limit have a joint normal distribution.

To complete the proof we need to calculate the correlations between the different \( \#I_i \)'s. If \( j < i \) we have that \( s_j - s_i \sim n^{-\gamma} \) where \( \gamma = 1 - \max_{j \leq k \leq \theta_k} \).

\[ ^5 \text{Since } K_n(x, y) = K_n(y, x) \text{ it is clear that the same estimates hold in the domains obtained from reflection with respect to the } x = y \text{-line.} \]
Set
\[ X_k = \sum_{m=1}^{k} \# I_m. \]

From a straightforward calculation (as above) we get that
\[
\text{Var}(X_i - X_j) = \text{Var}\left( \sum_{k=j+1}^{i} \# I_k \right) = \text{Var}\left( \# \bigcup_{k=j+1}^{i} I_k \right)
= \frac{1 - \gamma}{\pi^2} \log n + \mathcal{O}(\log \log n).
\]

Since
\[
\text{Var}(X_k) = \frac{1}{2\pi^2} \log n + \mathcal{O}(\log \log n)
\]
the correlation \( \rho \) is given by
\[
\rho(X_i, X_j) = \frac{1}{\sqrt{\text{Var}(X_j)} \text{Var}(X_j)} \left( \frac{\text{Var}(X_j) + \text{Var}(X_i) - \text{Var}(X_i - X_j)}{\text{Var}(X_j) \text{Var}(X_j)} \right)^{1/2} = \gamma + o(1).
\]

\[ \square \]

**Proof of Theorem 1.4.** This proof is of course very similar to the previous one so some details will be skipped.

With notation as in the formulation of Theorem 1.4 the intervals of interest (confer with the previous proof) are in this case
\[
I_1 = \left( \sqrt{2n} \left( 1 - C_1 \left( \frac{k_1}{n} \right)^{2/3} \right) + x_1 C_2 \left( \frac{\log k_1}{n^{1/3} k_1^{2/3}} \right)^{1/2}, \infty \right)
\]
\[
I_i = \left( \sqrt{2n} \left( 1 - C_1 \left( \frac{k_i}{n} \right)^{2/3} \right) + x_i C_2 \left( \frac{\log k_i}{n^{1/3} k_i^{2/3}} \right)^{1/2}, \right.
\]
\[
\left. \sqrt{2n} \left( 1 - C_1 \left( \frac{k_{i-1}}{n} \right)^{2/3} \right) + x_{i-1} C_2 \left( \frac{\log k_{i-1}}{n^{1/3} k_{i-1}^{2/3}} \right)^{1/2} \right]
\]
where \( C_1, C_2 \) are known constants and \( 2 \leq i \leq m \). Given any \( x_1, \ldots, x_m \) it is easy to see that the sets \( I_i \) are intervals if \( n \) is sufficiently large. As in the previous proof we want to show that
\[
\# I_1, \# I_2, \ldots, \# I_m
\]
are jointly normally distributed. The way to prove this is the same as before but some details are different. By Lemma 2.3 we need to show that
\[
\log n = \mathcal{O} \left( \text{Var} \left( \sum_{i=1}^{m} \alpha_i \# I_i \right) \right)
\]
for any real \( \alpha_i \)'s such that \( \alpha_i \neq 0 \) for some \( i \).
Let $t = t(n)$ be such that $n^{\epsilon - \frac{2}{3}} \leq 1 - t \leq n^{-\epsilon}$ for some $0 < \epsilon \leq 1/3$. From the proof of Lemma 2.3 below one can see that in the set

$$\Omega_t = \left\{(x, y); t \leq x \leq t + \frac{1 - t}{\log n}, t - \frac{1 - t}{\log n} \leq y \leq t\right\}$$

it holds that

$$2nK_n^2(\sqrt{2nx}, \sqrt{2ny}) = \frac{1}{2\pi^2(x - y)^2} + O\left(\frac{1}{\log n}\right).$$

Returning to the variance calculation we first assume that $\alpha_1 \neq 0$. We know from the previous proof that in this case it is sufficient to show that

$$\frac{1}{\log n}$$

is of order $\log n$. In fact since the integrand is non-negative it is enough if

$$\int \int_{I^* \times J^*} \frac{1}{(x - y)^2} \, dy \, dx$$

is of order $\log n$ where

$$I^* = \left(t_1 + r_1, t_1 + \frac{1 - t_1}{\log n}\right)$$

$$J^* = \left(t_1 - \frac{1 - t_1}{\log n}, t_1\right)$$

and

$$t_i = 1 - C_1 \left(\frac{k_i}{n}\right)^{2/3}$$

$$r_i = x_i C_2 \left(\frac{\log k_i}{n^{1/3}k_i^{2/3}}\right).$$

An elementary calculation shows that this integral is indeed of order $\log n$.

If $\alpha_1 = \ldots = \alpha_{k-1} = 0$ but $\alpha_k \neq 0$ it is sufficient that the integral

$$\int \int_{J^* \times J} \frac{1}{(x - y)^2} \, dy \, dx$$

is of order $\log n$ where

$$J^* = \left(t_{k-1} + r_{k-1}, t_{k-1} + \frac{1 - t_{k-1}}{\log n}\right)$$

$$J^* = \left(t_k, t_{k-1}\right).$$

Again we get the size $\log n$. This proves that we get a normal distribution in the limit. The calculations of the correlations are very similar to the bulk case and the details are not presented here. \qed
4. The expected number of eigenvalues in $I_n$

In this section and the next we shall need asymptotics for the Airy function and the Hermite polynomials. In [4] the asymptotics for a class containing the Hermite case was studied. It is shown that there exists a $\delta > 0$ such that the following holds:

1. $-1 + \delta \leq x \leq 1 - \delta$

\[
h_n \left( \sqrt{2nx} \right) e^{-nx^2} = \sqrt{\frac{2}{\pi \sqrt{2n(1-x^2)^{1/4}}} \left( \cos \left[ 2nF(x) - \frac{1}{2} \arcsin(x) \right] + O(n^{-1}) \right)}
\]

2. $1 - \delta \leq x < 1$

\[
h_n \left( \sqrt{2nx} \right) e^{-nx^2} = (2n)^{-1/4} \times \left\{ \left( \frac{1+x}{1-x} \right)^{1/4} [3nF(x)]^{1/6} \text{Ai} \left( -[3nF(x)]^{2/3} \right) (1 + O(n^{-1})) \right. \]
\[
- \left. \left( \frac{1-x}{1+x} \right)^{1/4} [3nF(x)]^{-1/6} \text{Ai}' \left( -[3nF(x)]^{2/3} \right) (1 + O(n^{-1})) \right\}
\]

3. $1 < x \leq 1 + \delta$

\[
h_n \left( \sqrt{2nx} \right) e^{-nx^2} = (2n)^{-1/4} \times \left\{ \left( \frac{x+1}{x-1} \right)^{1/4} [3nF(x)]^{1/6} \text{Ai} \left( [3nF(x)]^{2/3} \right) \right.
\]
\[
- \left. \left( \frac{x-1}{x+1} \right)^{1/4} [3nF(x)]^{-1/6} \text{Ai}' \left( [3nF(x)]^{2/3} \right) \right\} (1 + O(n^{-1}))
\]

4. $x > 1 + \delta$

\[
h_n \left( \sqrt{2nx} \right) e^{-nx^2} = O \left( n^{-1/4} e^{-nF(x)} \right)
\]

In these expressions $\text{Ai}$ stands for the Airy function and

(4.1) \[ F(x) = \int_x^1 \sqrt{1-y^2} \, dy. \]

There are of course also similar asymptotics for the Hermite polynomials near the point $-1$.

The Airy function is bounded on the real line. It is exponentially small in $x$ on $\mathbb{R}_+$ and for $r > 0$ it holds that, [3],

\[
\text{Ai}(-r) = \pi^{-1/2} r^{-1/4} \left\{ \cos \left[ \frac{2}{3} r^{3/2} - \frac{\pi}{4} \right] + O(r^{-3/2}) \right\}
\]
\[
\text{Ai}'(-r) = \pi^{-1/2} r^{1/4} \left\{ \sin \left[ \frac{2}{3} r^{3/2} - \frac{\pi}{4} \right] + O(r^{-3/2}) \right\}.
\]
Proof of Lemma 2.1: Set

\[ f_n(t) = t + x \frac{\log n}{2n}. \]

We have that

\[ \mathbb{E} [\# I_n] = \int_{f_n(t)}^{\infty} n \rho_n(x) \, dx \]

where \( \rho_n \) is the scaled density for the eigenvalues (the limiting density has support in \([-1, 1])\). From symmetry one gets

\[ \int_{f_n(t)}^{\infty} n \rho_n(x) \, dx = n \frac{2}{\pi} \int_{-1}^{1} \sqrt{1 - x^2} \frac{1}{x} - \frac{1}{x + 1} \cos \left( 2n \frac{1}{\pi} \int_{x}^{1} \sqrt{1 - y^2} \, dy \right) + O(n^{-1}) \]

if \( x \in [-1 + \delta, 1 - \delta] \). We get that

\[ \mathbb{E} [\# I_n] = n \frac{2}{\pi} \int_{-1}^{1} \sqrt{1 - x^2} \, dx + O(n^{-1}) \]

\[ = n - n \frac{2}{\pi} \int_{-1}^{1} \sqrt{1 - x^2} \, dx + O(n^{-1}) \]

\[ = n - n \frac{2}{\pi} \left( \int_{-1}^{1} \sqrt{1 - x^2} \, dx + \sqrt{1 - t^2} x \sqrt{\frac{n}{2n}} + O \left( \frac{\log n}{n^2} \right) \right) + O(n^{-1}) \]

\[ = n - k - x \frac{2}{\pi} \sqrt{(1 - t^2) \log n} + O \left( \frac{\log n}{n} \right). \]

□

Proof of Lemma 2.2: From the formulas (4.4) and (4.21) in [6] one gets after some minor calculations that

\[ n \rho_n(x) = \left( \frac{\Phi'(x)}{4 \Phi(x)} - \frac{\gamma'(x)}{\gamma(x)} \right) [2 \text{Ai}(\Phi(x)) \text{Ai}'(\Phi(x))] \]

\[ + \Phi'(x) \left[ (\text{Ai}'(\Phi(x)))^2 - \Phi(x)(\text{Ai}(\Phi(x)))^2 \right] + O \left( \frac{1}{n(\sqrt{1 - x})} \right) \]

in a fixed neighborhood of \([0, 1]\). As before \( \rho_n \) is the scaled density for the eigenvalues so that

\[ g(t) = \int_{t}^{\infty} n \rho_n(x) \, dx \]
and the functions $\gamma$ and $\Phi$ are given by

$$
\gamma(x) = \left(\frac{x - 1}{x + 1}\right)^{1/4}
$$

$$
\Phi(x) = \begin{cases} 
- \left(3n \int_1^x \sqrt{1 - y^2} \, dy\right)^{2/3} & \text{if } x \leq 1 \\
\left(3n \int_1^x \sqrt{y^2 - 1} \, dy\right)^{2/3} & \text{if } x > 1.
\end{cases}
$$

The function $\gamma$ is evaluated by taking the limit from the upper half plane using the principal branch.

The fact that the asymptotics only holds for $x \in [0, 1 + \delta]$ for some $\delta > 0$ (independent of $n$) is not a problem. It is not difficult to show that, for $x \geq 1 + \delta$, $\rho_n(x)$ is exponentially small in $n$ and exponentially decaying in $x$.

We shall now study the different terms in the asymptotical expression for $\rho_n$ above. When looking at the asymptotics for $\text{Ai}$ and $\text{Ai}'$ it easy to see that

$$|	ext{Ai}(x)\text{Ai}'(x)| = \mathcal{O}(1).$$

A calculation shows that

$$\left(\frac{\Phi'(x)}{4\Phi(x)} - \frac{\gamma'(x)}{\gamma(x)}\right) = \mathcal{O}(1)$$

hence

$$\int_t^{1+\delta} \left(\frac{\Phi'(x)}{4\Phi(x)} - \frac{\gamma'(x)}{\gamma(x)}\right) [2\text{Ai}(\Phi(x))\text{Ai}'(\Phi(x))] \, dx = \mathcal{O}(1).$$

The main contribution comes from the second term. In fact a primitive function can be found for this expression (see Appendix A):

$$\int_t^{1+\delta} \Phi'(x) \left[\text{Ai}^2(\Phi(x)) - \Phi(x)\text{Ai}^2(\Phi(x))\right] \, dx
= \left[\text{set } y = \Phi(x)\right]
= \int_{\Phi(t)}^{\Phi(1+\delta)} \text{Ai}^2(y) - y\text{Ai}^2(y) \, dy
= - \left[\frac{2}{3}\left(y^2\text{Ai}^2(y) - y\text{Ai}'^2(y)\right) - \frac{1}{3}\text{Ai}(y)\text{Ai}'(y)\right]_{\Phi(t)}^{\Phi(1+\delta)}
= \frac{2}{3}\left(\Phi^2(t)\text{Ai}^2(\Phi(t)) - \Phi(t)\text{Ai}^2(\Phi(t))\right)
- \frac{1}{3}\text{Ai}(\Phi(t))\text{Ai}'(\Phi(t)) + \mathcal{O}(e^{-cn})
$$

Here $c$ is a positive constant. Integrating the third term only gives a contribution of order $n^{-1}$. One can now use the asymptotics for the Airy function and its derivative to get the stated result. □
5. The variance of the number of eigenvalues in $I_n$

Proof of Lemma 2.3: The proof will be divided into two basic cases. The first case is when $1 - t > \delta$ for a fixed $\delta > 0$, that is, we are in the bulk. The second case is when $t = t(n) \to 1^-$ as $n \to \infty$, that is, we are close to the right spectrum edge.

First define $I_n = [t \sqrt{2n}, \infty)$ and $\#I_n$ as the number of eigenvalues in $I_n$. It is a fact (see Appendix B) that

$$\text{Var}(I_n) = \int_{I_n} \int_{I_n} K_n^2(x, y) dxdy$$

where $K_n$ is the Hermite kernel. The advantage with this representation is that there is only one singular point involved in the Christoffel-Darboux representation of $K_n(x, y)$:

$$K_n(x, y) = \sqrt{\frac{n}{2}} \frac{h_n(x)h_{n-1}(y) - h_{n-1}(x)h_n(y)}{x - y} e^{-\frac{1}{2}(x^2 + y^2)}$$

Case I (the bulk): After a change of variables ($x \to \sqrt{2n}x$) we get the integrand

$$\left[ \sqrt{2n} K_n(\sqrt{2n}x, \sqrt{2n}y) \right]^2.$$ 

First consider the domain where both variables are in the Bulk:

$$\Gamma = \{(x, y); t \leq x \leq 1 - \delta, -1 + \delta \leq y \leq t\}$$

Recall that in $\Gamma$, $h_n$ has asymptotics given by

$$h_n(\sqrt{2n}x)e^{-nx^2} = \left( \frac{2}{\pi \sqrt{2n}} \right)^{1/2} \frac{1}{(1 - x^2)^{1/4}} \left( \cos \left[ 2nF(x) - \frac{1}{2} \arcsin(x) \right] + O(n^{-1}) \right)$$

where

$$F(x) = \int_x^1 \sqrt{1 - z^2} \, dz = \frac{1}{2} \left( \arccos x - x \sqrt{1 - x^2} \right).$$

The asymptotics for $h_{n-1}$ becomes

$$h_{n-1}(\sqrt{2n}x)e^{-nx^2} = \left( \frac{2}{\pi \sqrt{2(n-1)}} \right)^{1/2} \frac{1}{(1 - x_n^2)^{1/4}} \left( \cos \left[ 2(n-1)F(x_n) - \frac{1}{2} \arcsin(x_n) \right] + O(n^{-1}) \right)$$

where

$$x_n = \frac{1}{\sqrt{2n}}.$$
where \( x_n = \sqrt{\frac{n}{n-1}}x \). A Taylor expansion gives
\[
F(x_n) = F(x) - \frac{x}{2(n-1)} \sqrt{1-x^2} + \mathcal{O}(n^{-2})
\]
and hence
\[
2(n-1)F(x_n) = 2nF(x) - 2F(x) - x\sqrt{1-x^2} + \mathcal{O}(n^{-1})
= 2nF(x) - \arccos x + \mathcal{O}(n^{-1}).
\]
One can now write
\[
h_n(\sqrt{2nx})h_{n-1}(\sqrt{2ny})e^{-n(x^2+y^2)}
= \frac{2}{\pi \sqrt{2n(1-x^2)^{1/4}(1-y^2)^{1/4}}}
\times \cos \left[ 2nF(x) - \frac{1}{2} \arcsin x \right] \cos \left[ 2nF(y) - \frac{1}{2} \arcsin y - \arccos y \right]
+ \mathcal{O}(n^{-3/2}).
\]
Set
\[
\alpha_x = 2nF(x) - \frac{1}{2} \arcsin x
\theta_x = \arccos x.
\]
By the Christoffel-Darboux formula
\[
\sqrt{2n} \, K_n(\sqrt{2nx}, \sqrt{2ny}) = \frac{1}{\pi \sqrt{2n(1-x^2)^{1/4}(1-y^2)^{1/4}}}
\times \frac{\cos \alpha_x \cos \theta_y - \cos [\alpha_x - \theta_x] \cos \alpha_y + \mathcal{O}(n^{-1})}{x-y}.
\]
To prepare for integration we now divide \( \Gamma \) into four disjoint sets. Set
\[
\Gamma_0 = \left\{ (x,y); t \leq x \leq t + \frac{1}{n}, t - \frac{1}{n} \leq y \leq t \right\}
\Gamma_1 = \Gamma_1 \cup \Gamma_2 = \left\{ (x,y); t \leq x \leq t + \frac{1-t}{r(n)}, t - \frac{1-t}{r(n)} \leq y \leq t - \frac{1}{n} \right\} \cup \left\{ (x,y); t + \frac{1}{n} \leq x \leq t + \frac{1-t}{r(n)}, t - \frac{1}{n} \leq y \leq t \right\}
\Gamma_2 = \Gamma \setminus (\Gamma_0 \cup \Gamma_1)
\]
where \( r(n) = \log n \) and \( \Gamma \) was defined in (5.1).
\( \Gamma_0 \): When integrating over \( \Gamma_0 \) one can use the fact that
\[
\sqrt{2n} \left| K_n(\sqrt{2nx}, \sqrt{2ny}) \right| \leq Cn \left| \frac{\sin(x-y)}{x-y} \right|
\]
where \( C > 0 \). Hence
\[
\int_{\Gamma_0} \left[ \sqrt{2n} \, K_n(\sqrt{2nx}, \sqrt{2ny}) \right]^2 \, dx \, dy = \mathcal{O}(1).
\]
\( \Gamma_1 \): For \( x \in \Gamma_1 \) it holds that
\[
\theta_x = \arccos x = \arccos t + O \left( \frac{1}{r(n)} \right)
\]
and of course also the equivalent for \( \theta_y \). Defining \( \theta = \arccos t \) we get by the use of some trigonometric identities that
\[
\cos \alpha_x \cos[\alpha_y - \theta] \cos[\alpha_x - \theta_x] \cos \alpha_y = \cos \alpha_x \cos[\alpha_y - \theta] - \cos[\alpha_x - \theta] \cos \alpha_y + O \left( \frac{1}{r(n)} \right)
\]
\[
= \sqrt{1 - t^2} \sin[\alpha_y - \alpha_x] + O \left( \frac{1}{r(n)} \right).
\]
Since
\[
\frac{\sqrt{1 - t^2}}{(1 - x^2)^{1/4}(1 - y^2)^{1/4}} = 1 + O \left( \frac{1}{r(n)} \right)
\]
and
\[
\alpha_x - \alpha_y = 2n(F(x) - F(y)) + O \left( \frac{1}{r(n)} \right)
\]
we now get
\[
\int_{\Gamma_1} \left[ \sqrt{2n} K_n(\sqrt{2nx}, \sqrt{2ny}) \right]^2 \, dx \, dy
\]
\[
= \int_{\Gamma_1} \frac{1}{\pi^2} \frac{\sin^2[2n(F(x) - F(y))] + O \left( \frac{1}{r(n)} \right)}{(x - y)^2} \, dx \, dy + \int_{\Gamma_1} \frac{O(1)}{(x - y)^2} \, dx \, dy
\]
\[
= \frac{1}{2\pi^2} \int_{\Gamma_1} 1 - \cos \left[ 4n(F(x) - F(y)) \right] \frac{1}{(x - y)^2} \, dx \, dy + O \left( \log r(n) \right)
\]
\[
= \frac{1}{2\pi^2} \log n - \frac{1}{2\pi^2} \int_{\Gamma_1} \frac{\cos \left[ 4n(F(x) - F(y)) \right]}{(x - y)^2} \, dx \, dy + O \left( \log r(n) \right).
\]
The remaining integral is not bigger than a constant as will now be shown. A partial integration in the \( y \)-variable gives
\[
\int_{\Gamma_1} \frac{\cos \left[ 4n(F(x) - F(y)) \right]}{(x - y)^2} \, dx \, dy
\]
\[
= \int_t^{t + \frac{1}{r(n)\alpha}} \left[ \frac{\sin \left[ 4n(F(x) - F(y)) \right]}{4n [F'(y)(x - y)^2]} \right]_t^{t + \frac{1}{r(n)\alpha}} dF' + \int_{t - \frac{1}{r(n)\alpha}}^{t + \frac{1}{r(n)\alpha}} \sin \left[ 4n(F(x) - F(y)) \right] \left( \frac{1}{4n [F'(y)(x - y)^2]} \right)_{x \to y} dF'(x)
\]
\[
= I_1 - I_2.
\]
Both these integrals are easy to estimate:

\[
|I_1| \leq C \int_t^{t+1/2(n)} \frac{1}{n(x - (t - n^{-1}))^2} \, dx = O(1).
\]

We have

\[
\left( \left[ F'(y)(x - y)^2 \right]^{-1} \right)' = -\frac{y}{(1 - y^2)^{3/2}(x - y)^2} - \frac{2}{\sqrt{1 - y^2}(x - y)^3}
\]

which gives

\[
|I_2| \leq C \int_{\Gamma_1} \frac{1}{n(x - y)^3} = O(1).
\]

\(\Gamma_2\): In \(\Gamma_2\) it holds that

\[
\left[ \sqrt{2n} K_n(\sqrt{2nx}, \sqrt{2ny}) \right]^2 = O\left( \frac{1}{(x - y)^2} \right)
\]

and a trivial calculation gives

\[
\int_{\Gamma_2} \frac{1}{x - y)^2} \, dx dy = O(\log r(n)).
\]

To complete case I we must also integrate over \(I_n \times I_n^c \setminus \Gamma\). By using the appropriate asymptotics for the Hermite polynomials, the Airy function and its derivative and then taking absolute values in the integrals it is straightforward to show that the contribution from this domain is \(O(1)\). We omit the details.

**Case II (the spectrum edge):** First consider the subdomain

\[
\Omega = \{(x, y); t \leq x \leq 1 - Cn^{-1}, 1 - \delta \leq y \leq t\}
\]

where \(C\) is a large positive constant. After a change of variables the contribution \(J_\Omega\) from \(\sqrt{2n} \cdot \Omega\) to the variance can be written as

\[
J_\Omega = \iint_{\Omega} \left[ \sqrt{2n} K_n(\sqrt{2nx}, \sqrt{2ny}) \right]^2 \, dx dy.
\]

In order to deal with this integral we must first study the integrand and, via Christoffel-Darboux, especially the difference

\[
D = \left[ h_n \left( \sqrt{2nx} \right) h_{n-1} \left( \sqrt{2ny} \right) - h_{n-1} \left( \sqrt{2nx} \right) h_n \left( \sqrt{2ny} \right) \right] e^{-n(x^2+y^2)}.
\]

We will show that in \(\Omega\) it holds that

\[
b(4n(n - 1))^{1/4} D = \left[ \text{Ai} \left( -[3nF(x)]^{2/3} \right) \text{Ai}' \left( -[3nF(y)]^{2/3} \right)
\]

\[
-\text{Ai}' \left( -[3nF(x)]^{2/3} \right) \text{Ai} \left( -[3nF(y)]^{2/3} \right) \right] + O \left( \frac{1}{n(1-x)} \right) + O \left( \frac{(1-y)^{3/4}}{(1-x)^{1/4}} \right).
\]
If we disregard the \( O \) written as a sum of four differences \( D_n \) stands for the Airy function and

\[
F(x) = \int_x^1 \sqrt{1 - t^2} \, dt.
\]

In \( \Omega \) \( h_n \) has the following asymptotics:

\[
h_n(\sqrt{2nx})e^{-nx^2} = (2n)^{-1/4} \left\{ \left( \frac{1 + x}{1 - x} \right)^{1/4} \left[ 3nF(x) \right]^{1/6} \left[ 3nF(y) \right]^{1/6} \times \text{Ai} \left( -\left[ 3nF(x) \right]^{2/3} \right) \text{Ai} \left( -\left[ 3nF(y) \right]^{2/3} \right) \\
- \left( \frac{1 - x}{1 + x} \right)^{1/4} \left[ 3nF(x) \right]^{-1/6} \left[ 3nF(y) \right]^{-1/6} \times \text{Ai} \left( -\left[ 3nF(x) \right]^{2/3} \right) \text{Ai} \left( -\left[ 3nF(y) \right]^{2/3} \right) \right\}
\]

If we disregard the \( O(n^{-1}) \) terms in the \( h_n \)-asymptotics, then \([5.2]\) can be written as a sum of four differences \( D_1 \) to \( D_4 \):

\[
(4n(n - 1))^{1/4} D_1
\]

\[
= \left( \frac{1 + x}{1 - x} \right)^{1/4} \left( \frac{1 + y}{1 - y} \right)^{1/4} \left[ 3nF(x) \right]^{1/6} \left[ 3nF(y) \right]^{1/6} \times \text{Ai} \left( -\left[ 3nF(x) \right]^{2/3} \right) \text{Ai} \left( -\left[ 3nF(y) \right]^{2/3} \right)
\]

\[
- \left( \frac{1 + x_n}{1 - x_n} \right)^{1/4} \left( \frac{1 + y_n}{1 - y_n} \right)^{1/4} \left[ 3nF(x_n) \right]^{1/6} \left[ 3nF(y_n) \right]^{1/6} \times \text{Ai} \left( -\left[ 3nF(x_n) \right]^{2/3} \right) \text{Ai} \left( -\left[ 3nF(y_n) \right]^{2/3} \right)
\]

\[
(4n(n - 1))^{1/4} D_2
\]

\[
= \left( \frac{1 + x}{1 - x} \right)^{1/4} \left( \frac{1 - y_n}{1 + y_n} \right)^{1/4} \left[ 3nF(x) \right]^{1/6} \left[ 3nF(y_n) \right]^{-1/6} \times \text{Ai} \left( -\left[ 3nF(x) \right]^{2/3} \right) \text{Ai} \left( -\left[ 3nF(y_n) \right]^{2/3} \right)
\]

\[
- \left( \frac{1 + x_n}{1 - x_n} \right)^{1/4} \left( \frac{1 - y}{1 + y} \right)^{1/4} \left[ 3nF(x_n) \right]^{1/6} \left[ 3nF(y) \right]^{-1/6} \times \text{Ai} \left( -\left[ 3nF(x_n) \right]^{2/3} \right) \text{Ai} \left( -\left[ 3nF(y) \right]^{2/3} \right)
\]

\[
(4n(n - 1))^{1/4} D_3
\]

\[
= \left( \frac{1 - x_n}{1 + x_n} \right)^{1/4} \left( \frac{1 + y_n}{1 - y_n} \right)^{1/4} \left[ 3nF(x_n) \right]^{-1/6} \left[ 3nF(y_n) \right]^{1/6} \times \text{Ai} \left( -\left[ 3nF(x_n) \right]^{2/3} \right) \text{Ai} \left( -\left[ 3nF(y_n) \right]^{2/3} \right)
\]
\[
\begin{align*}
&= -\left(1 - x\right)^{1/4} \left(1 + \frac{y_n}{1 - y_n}\right)^{1/4} [3nF(x)]^{-1/6} [3n'F(y_n)]^{1/6} \\
&\quad \times \text{Ai}' \left(-[3nF(x)]^{2/3}\right) \text{Ai} \left(-[3n'F(y_n)]^{2/3}\right)
\end{align*}
\]

\[
(4n(n - 1))^{1/4} D_4
\]

\[
= \left(\frac{1 - x}{1 + x}\right)^{1/4} \left(\frac{1 - y_n}{1 + y_n}\right)^{1/4} [3nF(x)]^{-1/6} [3n'F(y_n)]^{-1/6} \\
\quad \times \text{Ai}' \left(-[3nF(x)]^{2/3}\right) \text{Ai} \left(-[3n'F(y_n)]^{2/3}\right)
\]

\[
= \left(\frac{1 - x_n}{1 + x_n}\right)^{1/4} \left(\frac{1 - y}{1 + y}\right)^{1/4} [3n'F(x_n)]^{-1/6} [3nF(y)]^{-1/6} \\
\quad \times \text{Ai}' \left(-[3n'F(x_n)]^{2/3}\right) \text{Ai} \left(-[3nF(y)]^{2/3}\right)
\]

We have used the notation \(n' = n - 1\) and \(x_n = \sqrt{\frac{n}{n-1}}x\). Note that \(x_n < 1\) in \(\Omega\).

\(D_1\): A calculation using the series expansion

\[
\frac{F^{1/6}(x)}{(1 - x)^{1/4}} = a_0 + c_1(1 - x) + \ldots
\]

gives

\[
\begin{align*}
\left(\frac{1 + x_n}{1 - x_n}\right)^{1/4} \left(\frac{1 + y}{1 - y}\right)^{1/4} [3(n - 1)F(x_n)]^{1/6} [3nF(y)]^{1/6} \\
= \left(\frac{1 + x}{1 - x}\right)^{1/4} \left(\frac{1 + y}{1 - y}\right)^{1/4} [3nF(x)]^{1/6} [3nF(y)]^{1/6} + O \left(n^{1/3}(1 - x)\right)
\end{align*}
\]

\[
= a_1 n^{1/3} + O \left(n^{1/3}(1 - y)\right)
\]

where

\[
a_1 = \lim_{x \to 1^-} \sqrt{\frac{1}{1-x}} \frac{(3F(x))^{1/3}}{\sqrt{1-x}}.
\]

Since

\[
\text{Ai} \left(-[3nF(x)]^{2/3}\right) = O \left(\frac{1}{n^{1/6}(1 - x)^{1/4}}\right)
\]
it holds that
\[
\left(4n(n-1)\right)^{1/4}D_1
= a_1 n^{1/3} \left[ \text{Ai} \left( - [3nF(x)]^{2/3} \right) \text{Ai} \left( - [3n'F(y_n)]^{2/3} \right) - \text{Ai} \left( - [3n'F(x_n)]^{2/3} \right) \text{Ai} \left( - [3nF(y)]^{2/3} \right) \right]
+ \mathcal{O} \left( \frac{(1-y)^{3/4}}{(1-x)^{1/4}} \right).
\]

\[\text{D}_2 - \text{D}_4: \] The same procedure as above gives
\[
\left(4n(n-1)\right)^{1/4}D_2
= \mathcal{O}(1) \left[ \text{Ai} \left( - [3nF(x)]^{2/3} \right) \text{Ai}' \left( - [3n'F(y_n)]^{2/3} \right) - \text{Ai} \left( - [3n'F(x_n)]^{2/3} \right) \text{Ai}' \left( - [3nF(y)]^{2/3} \right) \right]
+ \mathcal{O} \left( \frac{(1-y)^{5/4}}{(1-x)^{1/4}} \right)
\]

\[
\left(4n(n-1)\right)^{1/4}D_3
= \mathcal{O}(1) \left[ \text{Ai}' \left( - [3n'F(x_n)]^{2/3} \right) \text{Ai} \left( - [3nF(y)]^{2/3} \right) - \text{Ai}' \left( - [3nF(x)]^{2/3} \right) \text{Ai} \left( - [3n'F(y_n)]^{2/3} \right) \right]
+ \mathcal{O} \left( \frac{(1-y)^{5/4}}{(1-x)^{1/4}} \right)
\]

\[
\left(4n(n-1)\right)^{1/4}D_4
= \mathcal{O}(n^{-1/3}) \left[ \text{Ai}' \left( - [3nF(x)]^{2/3} \right) \text{Ai}' \left( - [3n'F(y_n)]^{2/3} \right) - \text{Ai}' \left( - [3n'F(x_n)]^{2/3} \right) \text{Ai}' \left( - [3nF(y)]^{2/3} \right) \right]
+ \mathcal{O} \left( (1-y)^{3/2} \right)
\]

Now consider the difference still left in \(D_1\):
\[
\text{Ai} \left( - [3nF(x)]^{2/3} \right) \text{Ai} \left( - [3n'F(y_n)]^{2/3} \right)
- \text{Ai} \left( - [3n'F(x_n)]^{2/3} \right) \text{Ai} \left( - [3nF(y)]^{2/3} \right)
To deal with this expression we shall first investigate the argument
\[ [3n'F(x_n)]^{2/3} = \left[ 3(n-1)F\left(\sqrt{\frac{n}{n-1}}x\right)\right]^{2/3}. \]

A simple integration shows that
\[ F(x) = \int_x^1 \sqrt{1-t^2} \, dt = \frac{1}{2} \left( \arccos x - x\sqrt{1-x^2} \right). \]

Since
\[ x_n = \sqrt{\frac{n}{n-1}}x = x + \frac{x}{2(n-1)} + O(n^{-2}) \]
we get
\[ F(x_n) = F(x) + F'(x) \left( \frac{x}{2(n-1)} + O(n^{-2}) \right) + O(F''(x)n^{-2}) \]
\[ = F(x) - \frac{x\sqrt{1-x^2}}{2(n-1)} + O\left( \frac{1}{n^2\sqrt{1-x}} \right) \]
and hence
\[ 3n'F(x_n) = 3(n-1)F(x) - \frac{3}{2} x\sqrt{1-x^2} + O\left( \frac{1}{n\sqrt{1-x}} \right) \]
\[ = 3nF(x) - \frac{3}{2} \arccos x + O\left( \frac{1}{n\sqrt{1-x}} \right). \]

The argument can now finally be written as
\[ (5.3) - [3n'F(x_n)]^{2/3} = -[3nF(x)]^{2/3} + \frac{\arccos x}{(3nF(x))^{1/3}} + O\left( \frac{1}{n^{4/3}(1-x)} \right). \]

Note in the last expression that
\[ \frac{\arccos x}{(3nF(x))^{1/3}} \sim n^{-1/3}. \]

It is now possible to expand the difference in a Taylor series around the point \(-[3nF(x)]^{2/3}\) and the result is
\[ \frac{a_2}{n^{1/3}} \left[ \text{Ai} \left( -[3nF(x)]^{2/3} \right) \text{Ai}' \left( -[3nF(y)]^{2/3} \right) \right. \]
\[ \left. - \text{Ai}' \left( -[3nF(x)]^{2/3} \right) \text{Ai} \left( -[3nF(y)]^{2/3} \right) \right] \]
\[ + O\left( \frac{1}{n^{4/3}(1-x)} \right) + O\left( \frac{(1-y)^{3/4}}{n^{1/3}(1-x)^{1/4}} \right) \]
where \(a_2\) is defined by
\[ a_2 = \lim_{x \to 1^-} \frac{\arccos x}{(3F(x))^{1/3}}. \]
Similar computations can be done in $D_2 - D_4$ and one then ends up with

$$(4n(n-1))^{1/4}(D_2 + D_3 + D_4) = O\left((1 - y)^{1/2}\right).$$

Adding everything up we now have that

$$\frac{(4n(n-1))^{1/4}}{a_1 a_2} D = \left[\text{Ai} \left( - [3nF(x)]^{2/3} \right) \text{Ai}' \left( - [3nF(y)]^{2/3} \right) - \text{Ai}' \left( - [3nF(x)]^{2/3} \right) \text{Ai} \left( - [3nF(y)]^{2/3} \right) \right]$$

$$+ O\left(\frac{1}{n(1 - x)}\right) + O\left(\frac{(1 - y)^{3/4}}{(1 - x)^{1/4}}\right).$$

As we shall see the main contribution comes from the domain

$$\Omega_1 = \{(x, y); t \leq x \leq t + \frac{1 - t}{r(n)}, t - \frac{1 - t}{r(n)} \leq y \leq t - \epsilon\}.$$  

Here $r(n)$ is a function tending slowly to infinity as $n$ tends to infinity and $\epsilon = \frac{1}{n(1 - t)^{3/2}}$, the size of the expected distance between two eigenvalues at $t$. The reason why this $\epsilon$ is necessary lies in the asymptotics for the Hermite polynomials. The error term given there, however small, will cause problems since the integral

$$\int_t^{t+\epsilon} \int_{t-\epsilon}^t \frac{1}{(x - y)^2} \, dy \, dx$$

is divergent.

From the asymptotics of the Airy function and its derivative we get, for $(x, y) \in \Omega_1$, that

$$\text{Ai} \left( - [3nF(x)]^{2/3} \right) \text{Ai}' \left( - [3nF(y)]^{2/3} \right)$$

$$= \left( \frac{1}{\sqrt{\pi}} (nF(x))^{-1/6} \sin \left[2nF(x) + \frac{\pi}{4}\right] + O \left((nF(x))^{-7/6}\right) \right)$$

$$\times \left( \frac{1}{\sqrt{\pi}} (nF(y))^{1/6} \sin \left[2nF(y) - \frac{\pi}{4}\right] + O \left((nF(y))^{-5/6}\right) \right)$$

$$= \frac{1}{\pi} \left( \frac{F(y)}{F(x)} \right)^{1/6} \sin \left[2nF(x) + \frac{\pi}{4}\right] \sin \left[2nF(y) - \frac{\pi}{4}\right] + O \left((nF(x))^{-1}\right).$$

If we define $r(n)$ by

$$\frac{1}{r(n)} = \max \left(\sqrt{1 - t}, \frac{1}{\log[n(1 - t)^{3/2}]}\right)$$
we get that

\[
\left( \frac{F(y)}{F(x)} \right)^{1/6} = 1 + \mathcal{O}\left( \frac{1}{r(n)} \right)
\]

\[
\left( \frac{F(x)}{F(y)} \right)^{1/6} = 1 + \mathcal{O}\left( \frac{1}{r(n)} \right)
\]

\[
\frac{1}{n(1-x)} = \mathcal{O}\left( \frac{1}{nF(x)} \right) = \mathcal{O}\left( \frac{1}{r(n)} \right)
\]

\[
\frac{(1-y)^{3/4}}{(1-x)^{1/4}} = \mathcal{O}\left( \frac{1}{r(n)} \right).
\]

From this it follows that in \( \Omega \) \( D \) can be written as

\[
\frac{(4n(n-1))^{1/4}}{a_1a_2} D
\]

\[
= \frac{1}{\pi} \left( \sin \left[ 2nF(x) + \frac{\pi}{4} \right] \sin \left[ 2nF(y) - \frac{\pi}{4} \right] \right.
\]

\[
- \sin \left[ 2nF(x) - \frac{\pi}{4} \right] \sin \left[ 2nF(y) + \frac{\pi}{4} \right] \left. \right) + \mathcal{O}\left( \frac{1}{r(n)} \right)
\]

\[
= \frac{1}{\pi} \sin [2n(F(x) - F(y))] + \mathcal{O}\left( \frac{1}{r(n)} \right).
\]

The numerator in the integral of interest is

\[
\frac{n}{2\sqrt{n(n-1)}} D^2
\]

\[
= \frac{(a_1a_2)^2}{4\pi^2} \sin^2 [2n(F(x) - F(y))] + \mathcal{O}\left( \frac{1}{r(n)} \right)
\]

\[
= \frac{1}{2\pi^2} (1 - \cos [4n(F(x) - F(y))] + \mathcal{O}\left( \frac{1}{r(n)} \right).
\]

We used here that \( a_1a_2 = 2 \). A simple integration gives

\[
\int \int_{\Omega_1} \frac{1}{(x-y)^2} \, dx \, dy = \log[n(1-t)^{3/2}] + \mathcal{O}(\log r(n)).
\]

The integral

\[
I = \int \int_{\Omega_1} \frac{\cos [4n(F(x) - F(y))]}{(x-y)^2} \, dx \, dy
\]
is $O(1)$: By doing a partial integration $I$ can be split into two integrals:

$$
I = \int_t^{t+\frac{1}{r(n)}} \left( \left[ \sin [4n(F(x) - F(y))] - 4nF'(y)(x-y)^2 \right]_{t-\frac{1}{r(n)}}^{t+\frac{1}{r(n)}} + \int_{t-\frac{1}{r(n)}}^{t-\epsilon} \sin [4n(F(x) - F(y))] \left( \frac{1}{4nF'(y)(x-y)^2} \right)' dy \right) dx
$$

$$
=: I_1 + I_2
$$

We can estimate $I_1$ in the following way.

$$
|I_1| \leq 2 \int_t^{t+\frac{1}{r(n)}} \frac{1}{4n\sqrt{1-t(x-(t-\epsilon))^2}} dx
$$

$$
= \epsilon \cdot \frac{1}{2} \left[ \frac{-1}{x-t+\epsilon} \right]_t^{t+\frac{1}{r(n)}} \leq \frac{\epsilon}{2} \cdot \frac{2}{\epsilon} = 1
$$

Since

$$
\left( [F'(y)(x-y)^2]^{-1} \right)'_y = -\frac{y}{(1-y^2)^{3/2}(x-y)^2} - \frac{2}{\sqrt{1-y^2}(x-y)^3}
$$

we get

$$
|I_2| \leq C \left( \iint_{\Omega_1} \frac{1}{n(1-y)^{3/2}(x-y)^2} dxdy + \iint_{\Omega_1} \frac{1}{n\sqrt{1-y}(x-y)^3} dxdy \right).
$$

The first part is small:

$$
\iint_{\Omega_1} \frac{1}{n(1-y)^{3/2}(x-y)^2} dxdy
$$

$$
\leq \frac{1}{n(1-t)^{3/2}} \int_{\Omega_1} \frac{1}{(x-y)^2} dxdy = O \left( \frac{\log [n(1-t)^{3/2}]}{n(1-t)^{3/2}} \right)
$$

The second part is also easily estimated:

$$
\iint_{\Omega_1} \frac{1}{n\sqrt{1-y}(x-y)^3} dxdy \leq \epsilon \int_{\Omega_1} \frac{1}{(x-y)^3} dxdy = O(1)
$$

This concludes the calculations in $\Omega_1$.

The calculations made above can also be applied to the thin slice

$$
\{(x,y); t+\epsilon \leq x \leq t+\frac{1-t}{r(n)}, t-\epsilon \leq y \leq t\}
$$

and the result is $O(\log [r(n)])$.

The corner

$$
\Omega_0 = \{(x,y); t \leq x \leq t+\epsilon, t-\epsilon \leq y \leq t\}$$
requires a special technique. In this domain a different representation of $K_n$ will be used, namely

$$K_n(x, y) = \sum_{i=0}^{n-1} p_i(x)p_i(y)e^{-\frac{1}{2}(x^2+y^2)}.$$  

By use of the Cauchy-Schwartz inequality we get that

$$K_n^2(x, y) \leq K_n(x, x)K_n(y, y).$$

Having separated the variables we can now use the calculations from the proof of Lemma 2.2 to see that

$$\int_{t-\epsilon}^{t+\epsilon} \int_{t}^{t+\epsilon} \left(\sqrt{2nK_n(\sqrt{2nx}, \sqrt{2ny})}\right)^2 \, dx \, dy = O(1).$$

Note that

$$\int_{t}^{t+\epsilon} K_n(\sqrt{2nx}, \sqrt{2nx}) \, dx = g(t) - g(t + \epsilon)$$

where $g(t)$ is the expected number of eigenvalues in the interval $(t\sqrt{2n}, \infty)$.

Now we shall look at the other part still left of $\Omega$. This domain can conveniently be written as $\Omega_2 \cup \Omega_3$ where

$$\Omega_2 = \{(x, y); t \leq x \leq 1 - Cn^{-1}, 1 - \delta \leq y \leq t - \frac{1 - t}{r(n)}\}$$

and

$$\Omega_3 = \{(x, y); t + \frac{1 - t}{r(n)} \leq x \leq 1 - Cn^{-1}, t - \frac{1 - t}{r(n)} \leq y \leq t\}$$

When looking at the expression for $D$ in (5.4) above it is clear that every term is smaller than

$$n^{-1/2} Ai\left(-[3nF(x)]^{2/3}\right)Ai'\left(-[3nF(y)]^{2/3}\right) = O\left(n^{-1/2} \left(\frac{1 - y}{1 - x}\right)^{1/4}\right).$$

This means that it is sufficient to calculate the integrals

$$\int_{\Omega_i} \frac{\sqrt{1 - y}}{\sqrt{1 - x(x - y)^2}} \, dx \, dy \quad i = 2, 3.$$

The calculations are straightforward so some details will be skipped. When first integrating with respect to the x-variable one gets

$$\int_{L_1}^{H_1} \frac{\sqrt{1 - y}}{\sqrt{1 - x(x - y)^2}} \, dx$$

$$= \frac{1}{2(1 - y)} \log \left[\left(\frac{\sqrt{1 - y} + \sqrt{1 - L_1}}{\sqrt{1 - y} + \sqrt{1 - H_1}}\right) \left(\sqrt{1 - y} - \sqrt{1 - H_1}\right)\right]$$

$$+ \frac{1}{\sqrt{1 - y}} \left(\frac{1}{\sqrt{1 - y} - \sqrt{1 - L_1}} - \frac{1}{\sqrt{1 - y} + \sqrt{1 - L_1}}\right).$$
Letting $H_1 = 1$ instead of $1 - Cn^{-1}$ we get nicer expressions. This is allowed since the domain of integration becomes larger. The task is to get an upper bound for the integrals

$$A = \int_{L_2}^{H_2} \frac{1}{2(1-y)} \log \left[ \frac{\sqrt{1-y} + \sqrt{1-L_1}}{\sqrt{1-y} - \sqrt{1-L_1}} \right] \, dy$$

and

$$B = \int_{L_2}^{H_2} \frac{1}{\sqrt{1-y}} \left( \frac{1}{\sqrt{1-y} - \sqrt{1-L_1}} - \frac{1}{\sqrt{1-y} + \sqrt{1-L_1}} \right) \, dy$$

where

$$L_2 = 1 - \delta, \quad H_2 = t - \frac{1 - t}{r(n)}\quad \text{and} \quad L_1 = t.$$

When manipulating the integrand in $A$ one gets

$$\frac{1}{z} \log \left[ 1 + \frac{2\sqrt{1-L_1}}{z - \sqrt{1-L_1}} \right] = \frac{1}{z} O \left( \frac{\sqrt{1-L_1}}{z - \sqrt{1-L_1}} \right).$$

Some algebra shows that

$$\frac{\sqrt{1-L_1}}{z (z - \sqrt{1-L_1})} = \frac{1}{z - \sqrt{1-L_1}} - \frac{1}{z}$$

which can easily be integrated:

$$A \leq C \left[ \log \left( \frac{z - \sqrt{1-L_1}}{z} \right) \right]^{\sqrt{1-L_2}}_{\sqrt{1-H_2}} = O(\log r(n))$$

The integral $B$ is even easier and one gets

$$B = 2 \left[ \log \left( \frac{z - \sqrt{1-L_1}}{z + \sqrt{1-L_1}} \right) \right]^{\sqrt{1-L_2}}_{\sqrt{1-H_2}} = O(\log r(n)).$$

$\Omega_3$: The same procedure as in $\Omega_2$ gives that the contribution to the variance from this domain is $o(1)$.

We shall now consider the thin strip

$$\Omega_4 = \{ x, y; 1 - Cn^{-1} \leq x \leq 1 + Cn^{-1}, 1 - \delta \leq y \leq t \}.$$

The asymptotics here is similar to that in $\Omega$ and hence many of the calculations already done can be applied here as well. As before $D$ can be split
up in $D_1 - D_4$ which can all be treated similarly. Therefore we only look at $D_1$ here. We have that
\[
(4n(n - 1))^{1/4} D_1 = a_1 n^{1/3} \left[ \text{Ai} \left( \pm \left[ 3n F(x) \right]^{2/3} \right) \text{Ai} \left( - \left[ 3n' F(y_n) \right]^{2/3} \right) - \text{Ai} \left( \pm \left[ 3n' F(x_n) \right]^{2/3} \right) \text{Ai} \left( - \left[ 3n F(y) \right]^{2/3} \right) \right] + \mathcal{O} \left( \frac{(1 - y)^{3/4}}{(1 - x)^{1/4}} \right),
\]
where
\[
\pm \left[ 3n F(x) \right]^{2/3} = \begin{cases} 
- \left[ 3n F(x) \right]^{2/3} & \text{if } x < 1 \\
\left[ 3n F(x) \right]^{2/3} & \text{if } x \geq 1.
\end{cases}
\]
This follows from the calculations done above and the asymptotics for the Hermite polynomials when $x > 1$. In $\Omega_4$ we have
\[
\text{Ai} \left( \pm \left[ 3n F(x) \right]^{2/3} \right) = \text{Ai}(0) + \mathcal{O}(n^{-1/3})
\]
and by using equation (5.3) (for the $y$-variable) one gets
\[
(4n(n - 1))^{1/4} D_1 = \mathcal{O} \left( \frac{(1 - y)^{1/4}}{|1 - x|^{1/4}} \right).
\]
The error term here has actually already been dealt with in the estimations of the contribution coming from $\Omega_2$.

Rather than to repeat a lot of calculations we now just give ideas of how to to treat what is left of $[t, \infty) \times (-\infty, t]$.

In the domain
\[
\{x, y; 1 + C n^{-1} \leq x \leq 1 + \delta, 1 - \delta \leq y \leq t\}
\]
one can perform much the same calculations as in $\Omega$ and the contribution is $\mathcal{O}(1)$. In
\[
\{x, y; t \leq x \leq 1 + \delta, -1 - \delta \leq y \leq 1 - \delta\}
\]
one can use the fact that $x - y \geq \delta$ to show that the contribution from this domain is $\mathcal{O}(1)$. If $x \geq 1 + \delta$ or $y \leq -1 - \delta$ one easily gets from the asymptotics for the Hermite polynomials that $K_n(\sqrt{2n} x, \sqrt{2n} y)$ is exponentially small in $n$ and exponentially decaying in $x^2$ (or $y^2$). Thus the contribution from this domain is $o(1)$.

\[\square\]
**A: Some integrals.** The following equalities hold:

\[
\int_x^\infty \text{Ai}^2(y)\,dy = \text{Ai}^2(x) - x\text{Ai}^2(x)
\]

\[
\int_x^\infty y\text{Ai}^2(y)\,dy = \frac{1}{3} \left( x\text{Ai}^2(x) - x^2\text{Ai}^2(x) - \text{Ai}(x)\text{Ai}'(x) \right)
\]

\[
\int_x^\infty \text{Ai}'^2(y)\,dy = \frac{1}{3} \left( x^2\text{Ai}^2(x) - x\text{Ai}^2(x) - 2\text{Ai}(x)\text{Ai}'(x) \right)
\]

\[
\int_x^\infty y^2\text{Ai}^2(y)\,dy = \frac{1}{5} \left( x^2\text{Ai}^2(x) - x^3\text{Ai}^2(x) - 2x\text{Ai}(x)\text{Ai}'(x) + \text{Ai}^2(x) \right)
\]

\[
\int_x^\infty y\text{Ai}'^2(y)\,dy = \frac{1}{5} \left( x^3\text{Ai}^2(x) - x^2\text{Ai}'^2(x) - 3x\text{Ai}(x)\text{Ai}'(x) + \frac{3}{2}\text{Ai}^2(x) \right)
\]

The first integral is obtained by performing a partial integration while remembering that

\[ \text{Ai}''(x) = x\text{Ai}(x). \]

The integrals 3-5 can be obtained rather easily from the second which can be treated as follows:

Set \[ u_\alpha(x) = \text{Ai}(\alpha x) \quad \alpha > 0. \]

The relationship

\[ \left[ u'_\alpha u_\beta - u_\alpha u'_\beta \right]' = u''_\alpha u_\beta - u_\alpha u''_\beta = x(\alpha^3 - \beta^3)u_\alpha u_\beta \]

holds since

\[ u''_\alpha(x) = \alpha^2\text{Ai}''(\alpha x) = \alpha^3 x\text{Ai}(\alpha x) = \alpha^3 xu_\alpha(x). \]

Hence

\[
\int_a^\infty xu_\alpha(x)u_\beta(x)\,dx = \frac{1}{\alpha^3 - \beta^3} \left[ u'_\alpha u_\beta - u_\alpha u'_\beta \right]_a^\infty
\]

\[ = \frac{u_\alpha(a)u'_\beta(a) - u'_\alpha(a)u_\beta(a)}{\alpha^3 - \beta^3}. \]

The idea now is to let \( \alpha, \beta \) tend to one. Set \( \alpha = 1 + h \) and \( \beta = 1 - h \) where \( h > 0 \) and small. The left hand side tends to

\[ \int_a^\infty x\text{Ai}^2(x)\,dx \]

as \( h \to 0^+ \). Standard calculations show that at the same time the right hand side tends to

\[ \frac{1}{3} \left( -a^2\text{Ai}^2(a) - \text{Ai}(a)\text{Ai}'(a) + a\text{Ai}'^2(a) \right). \]
B: Variance calculations. Let $I_1, \ldots, I_m$ be a set of disjoint intervals and $\#I_i$ be the number of eigenvalues of the GUE in the interval $I_i$. We shall give a formula for $\text{Var} (\alpha_1 \#I_1 + \ldots + \alpha_m \#I_m)$. Clearly

$$\#I_i = \sum_{k=1}^{n} \chi_{I_i}(x_k) \quad 1 \leq i \leq n$$

where $\chi_B$ is the characteristic function for the set $B$ and $\{x_k\}_{k=1}^{n}$ are the unordered eigenvalues. The expected value is easy to compute:

$$\mathbb{E} [\#I_i] = \int_{I_i} \rho_{n,1}(x) \, dx = \int_{I_i} K_n(x,x) \, dx$$

The correlation functions $\rho_{n,k}$ were defined in the introduction. We also need to calculate $\mathbb{E}[\#I_i^2]$:

$$\mathbb{E} [\#I_i^2] = \mathbb{E} \left[ \sum_{j,k=1}^{n} \chi_{I_i}(x_k)\chi_{I_i}(x_j) \right]$$

$$= \sum_{k=1}^{n} \mathbb{E}[\chi_{I_i}(x_k)] + \sum_{j \neq k} \mathbb{E}[\chi_{I_i}(x_k)\chi_{I_i}(x_j)]$$

$$= \int_{I_i} K_n(x,x) \, dx + \iint_{I_i \times I_i} \rho_{n,2}(x,y) \, dxdy$$

$$= \int_{I_i} K_n(x,x) \, dx + \left( \int_{I_i} K_n(x,x) \, dx \right)^2 - \iint_{I_i \times I_i} K_n^2(x,y) \, dxdy$$

It now follows that

$$\text{Var}(\#I_i) = \int_{I_i} K_n(x,x) \, dx - \iint_{I_i \times I_i} K_n^2(x,y) \, dxdy.$$ 

To get a more convenient formula to work with one can now use the identities $K_n(x,y) = K_n(y,x)$ and

$$\int_{\mathbb{R}} K_n(x,y) K_n(y,z) \, dy = K(x,z)$$

to get

$$\text{Var}(\#I_i) = \int_{I_i} \left( \int_{\mathbb{R}} K_n^2(x,y) \, dy \right) \, dx - \iint_{I_i \times I_i} K_n^2(x,y) \, dxdy$$

$$= \iint_{I_i \times I_i} K_n^2(x,y) \, dxdy.$$ 

In more generality one gets

$$\mathbb{E}[\alpha_1 \#I_1 + \ldots + \alpha_m \#I_m] = \sum_{i=1}^{m} \alpha_i \int_{I_i} K_n(x,x) \, dx$$
and
\[(\alpha_1\#I_1 + \ldots + \alpha_m\#I_m)^2\]
\[= \sum_{i=1}^{m} \alpha_i^2 \left( \sum_{k=1}^{n} \chi_{I_i}(x_k) \right)^2 + \sum_{i \neq j}^{m} \alpha_i \alpha_j \left( \sum_{k=1}^{n} \chi_{I_i}(x_k) \right) \left( \sum_{k=1}^{n} \chi_{I_j}(x_k) \right) \]
\[=: S_1 + S_2.\]

From the calculations above we know that
\[E[S_1] = \sum_{i=1}^{m} \alpha_i^2 \left\{ \int_{I_i} K_n(x, x) \, dx + \left( \int_{I_i} K_n(x, x) \, dx \right)^2 \right. \]
\[\left. - \iint_{I_i \times I_i} K_n^2(x, y) \, dx \, dy \right\} \]
so it remains to calculate \(E[S_2]\). We have
\[\left( \sum_{k=1}^{n} \chi_{I_i}(x_k) \right) \left( \sum_{k=1}^{n} \chi_{I_j}(x_k) \right) = \sum_{k \neq l}^{n} \chi_{I_i}(x_k) \chi_{I_j}(x_l) \]
and hence
\[E[S_2] = \sum_{i \neq j}^{m} \alpha_i \alpha_j \iint_{I_i \times I_j} \rho_{n,2}(x, y) \, dx \, dy \]
\[= \sum_{i \neq j}^{m} \alpha_i \alpha_j \left( \int_{I_i} K_n(x, x) \, dx \int_{I_j} K_n(x, x) \, dx - \iint_{I_i \times I_j} K_n^2(x, y) \, dx \, dy \right). \]

Since
\[(E[\alpha_1\#I_1 + \ldots + \alpha_m\#I_m])^2\]
\[= \sum_{i=1}^{m} \alpha_i^2 \left( \int_{I_i} K_n(x, x) \, dx \right)^2 + \sum_{i \neq j}^{m} \alpha_i \alpha_j \int_{I_i} K_n(x, x) \, dx \int_{I_j} K_n(x, x) \, dx \]
we finally get (with manipulations as before)
\[\text{Var}(\alpha_1\#I_1 + \ldots + \alpha_m\#I_m)\]
\[= \sum_{i=1}^{m} \alpha_i^2 \iint_{I_i \times I_i} K_n^2(x, y) \, dx \, dy - \sum_{i \neq j}^{m} \alpha_i \alpha_j \iint_{I_i \times I_j} K_n^2(x, y) \, dx \, dy. \]

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