Research Article

Global Attractor for Coupled Beam Equations with Nonhomogeneous Boundary Conditions

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We study the longtime behavior for thermoelastic coupled beam equations with strong damping and past history thermal memory

\[ u_{tt} + u_{xxxx} + u_{xxxxx} + \theta_{xx} - M \left( \int_0^l u_x^2 \, dx \right) u_{xx} = h(x), \quad \text{in} \ [0, l] \times \mathbb{R}^+, \]

\[ \theta_t - \eta \theta_{xx} - \int_0^\infty k(s) \theta_{xx}(t-s) \, ds - u_{xxt} + p(\theta) = q(x), \]

with \( 1/2 \leq \eta < 1 \), where we consider the nonlinear boundary conditions

\[ u(0, t) = u_x(0, t) = u_{xx}(l, t) = 0, \]

\[ u_{xxx}(l, t) + u_{xxt}(l, t) - M \left( \int_0^l u_x^2 \, dx \right) u_x(l, t) = f(u(l, t)) + g(u_x(l, t)), \]

\[ \theta(0, t) = \theta_x(l, t) = 0 \]

for every \( t > 0 \) and the initial conditions

\[ u(x, 0) = u_0(x), \]

\[ u_t(x, 0) = u_1(x), \]

\[ \theta(x, 0) = \theta_0(x) \]

for every \( x \in [0, l] \). This problem arises from a model of the nonlinear thermoelastic coupled vibration beam with a nonlinear damping acting on \( x = l \).

1. Introduction

In this paper, we are concerned with the longtime dynamics for the 1-dimensional nonlinear thermoelastic coupled beam equations with strong damping and past history thermal memory

\[ u_{tt} + u_{xxxx} + u_{xxxxx} + \theta_{xx} - M \left( \int_0^l u_x^2 \, dx \right) u_{xx} = h(x), \quad \text{in} \ [0, l] \times \mathbb{R}^+, \]

\[ \theta_t - \eta \theta_{xx} - \int_0^\infty k(s) \theta_{xx}(t-s) \, ds - u_{xxt} + p(\theta) = q(x), \]

on which we consider the nonlinear boundary conditions

\[ u(0, t) = u_x(0, t) = u_{xx}(l, t) = 0, \]

\[ u_{xxx}(l, t) + u_{xxt}(l, t) - M \left( \int_0^l u_x^2 \, dx \right) u_x(l, t) = f(u(l, t)) + g(u_x(l, t)), \]

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for every \( t > 0 \) and the initial conditions

\[ u(x, 0) = u_0(x), \]

\[ u_t(x, 0) = u_1(x), \]

\[ \theta(x, 0) = \theta_0(x) \]

for every \( x \in [0, l] \). This problem arises from a model of the nonlinear thermoelastic coupled vibration beam with a nonlinear damping acting on the \( x = l \) end, which simultaneously considers the medium damping, the viscous effect, the nonlinear constitutive relation, and thermoelasticity based on a theory of non-Fourier heat flux.

Here the unknown functions \( u(x, t) \) and \( \theta(x, t) \) are the elevation of the surface of beam and vertical component of the temperature gradient, respectively, \( u'(x), u'(x) \) and \( \theta'(x) \) are the given initial value functions, the subscripts \( t \) and \( x \) denote the derivatives with respect to \( t \) and \( x \), respectively, \( M(\cdot) \) is the nonlinearity of the material and continuous nonnegative nonlinear real function, \( f(u(l)) \) and \( p(\theta) \) and \( g(u, l) \) are essentially \( |u(l)|^p u'(l) \) and \(|\theta|^q \theta \) and \(|u(l)|^r u'(l) \), respectively, with \( \rho, \varphi, r > 0, k : \mathbb{R}^+ \rightarrow \mathbb{R} \) is memory.
kernel and assumed to be a positive bounded convex function vanishing at infinity, and \(h(x)\) is the external heat supply. The detailed assumptions on nonlinear functions \(g(\cdot), f(\cdot), p(\cdot), \) and \(g(\cdot)\) and the external force functions \(h(x)\) and \(q(x)\) will be specified later.

It is well known that the infinite dimensional dynamical systems determined by the elastic beam are different because of the difference of boundary conditions. Bertì et al. [1] and Fastovska T. [2] and Giorgi et al. [3] and Wu [4] all considered the long-time dynamics of one-dimensional thermoelastic coupled beam equations under null boundary conditions. Barbose et al. [5] studied the longtime behavior for a class of two-dimensional thermoelastic coupled beam equations subjected to the hinged conditions

\[
\begin{align*}
\theta_t - i \triangle \theta - (1 - i) \int_0^\infty k(t - s) \triangle \theta ds - v \triangle u_t &= 0 \\
\theta(0) &= 0, \\
x &\in \Gamma.
\end{align*}
\]

Dell'Oro et al. [6] dealt with the long-term properties of the thermoelastic nonlinear string-beam system related to the well-known Lazer-McKenna suspension bridge model

\[
\begin{align*}
\triangle^2 u_M - M \left( \int_\Omega |Vu|^2 \, dx \right) \triangle u - \triangle u_t + f(u) + v \\
\theta &= h(x) \\
\theta_t - i \theta - (1 - i) \int_0^\infty k(t - s) \theta ds &= 0
\end{align*}
\]

subjected to the hinged conditions

\[
\begin{align*}
u = \Delta u &= 0, \\
\theta &= 0, \\
x &\in \Gamma.
\end{align*}
\]

We also refer the reader to [7–11] and the references therein for thermoelastic coupled beam equations with null boundary conditions. However, abovementioned longtime dynamics for the thermoelastic coupled beam equations are all subjected to null boundary conditions.

Under nonlinear boundary conditions, we refer the reader to the following works. One of the first studies in this direction was done by Pažanin and Perla Menzala [12], where stabilization of a thermoelastic extensible beam was considered. Motivated by the result, Ma proved the existence of global solutions and the existence of a global attractor for the Kirchhoff-type beam equation

\[
\begin{align*}
u_t + \Delta^2 \nu &= f(T) + \int_0^L |\nu_x|^2 \, dx + \theta_{xx} + (u - v)^+ + \theta_{xx} \\
\theta_t - \Delta \theta - (1 - i) \int_0^\infty k(t - s) \Delta \theta ds &= 0 \\
\theta(0) &= 0, \\
x &\in \Gamma.
\end{align*}
\]

2. Basic Assumptions and Basic Lemmas

Our fundamental assumptions on \(M(\cdot), f(\cdot), g(\cdot), \mu(\cdot)\) and \(h(x), q(x)\) are given as follows.

We assume that \(M(\cdot) \in C^1(\mathbb{R})\) satisfying

\[
M(z)z \geq \tilde{M}(z) \geq 0, \quad \forall z \geq 0
\]

where \(\tilde{M}(z) = \int_0^z M(s)ds\). This condition is promptly satisfied if \(M(\cdot)\) is nondecreasing with \(M(0) = 0\).

The function \(f(\cdot): \mathbb{R} \to \mathbb{R}\) is of class \(C^1(\mathbb{R})\) and satisfies

\[
f(0) = 0, \quad \text{and there exist constants } k, \rho > 0 \text{ such that}
\]

\[
|f(u) - f(v)| \leq k (1 + |u|^p + |v|^p) |u - v|, \quad \forall u, v \in \mathbb{R},
\]

where \(\tilde{f}(z) = \int_0^z f(s)ds\).

The function \(p(\cdot): \mathbb{R} \to \mathbb{R}\) is of class \(C^1(\mathbb{R})\) and satisfies

\[
p(0) = 0, \quad \text{and there exist constants } k_2, q > 0 \text{ such that}
\]

\[
(p(\theta) - p(\tilde{\theta})) (\theta - \tilde{\theta}) \geq k_2 (\theta - \tilde{\theta})^{q+2}, \quad \forall \theta, \tilde{\theta} \in \mathbb{R},
\]

The function \(g(\cdot): \mathbb{R} \to \mathbb{R}\) is of class \(C^1(\mathbb{R})\) and satisfies

\[
g(0) = 0, \quad \text{and there exist constants } k_3, k_4, \text{ and } r > q \text{ such that}
\]

\[
|g(u) - g(v)| \leq k_3 (1 + |u|^r + |v|^r) |u - v|, \quad \forall u, v \in \mathbb{R},
\]

The assumptions on \(k(\cdot)\) are as follows: \(k\) is vanishing at \(\infty\); moreover

\[
(1 - i) k'(s) = \mu(s)
\]

where \(\mu \in C^1(\mathbb{R}) \cap L^1(\mathbb{R}^+)\) and there exists a constant \(\delta_1 > 0\), \(\forall s \in \mathbb{R}^+\) such that

\[
\mu'(s) \leq -\delta_1 \mu(s), \quad s \geq 0.
\]

Finally, we assume that \(h(x), q(x) \in L^2(0, l)\).

To prove the main result, we need the following Lemmas.
Lemma 1 (see [16]). Let \( \varphi(t) \) be a nonnegative continuous function defined on \([0, T)\), \( 1 < T \leq \infty \), which satisfies \( \sup_{0 \leq t \leq 1} \varphi(t)^{1+\eta} \leq M_0 (\varphi(t) - \varphi(t+1)) + M_1, 0 \leq t \leq T - 1 \), where \( M_0, M_1, \eta \) are positive constants. Then we have

\[
\varphi(t) \leq \left( M_0^{-\eta} (t-1)^{+} + \left( \sup_{0 \leq s \leq 1} \varphi(s)^{-\eta} \right) \right)^{-1/\eta} + M_1^{1/(\eta+1)}, \quad 0 \leq t \leq T
\]

Lemma 2 (see [17]). Assume that, for any bounded positive invariant set \( B \subset H \) and for any \( \epsilon > 0 \), there exists \( T = T(\epsilon, B) \) such that \( d(S(T)x, S(T)y) \leq \epsilon + \omega_r(x, y), \forall x, y \in B \), where \( \omega_r : H \times H \rightarrow \mathbb{R} \) satisfies, for any sequence \( \{z_n\} \subset B \), \( \liminf_{n \rightarrow \infty} \inf_{t \geq 0} \omega_r(z_n, z_m) = 0 \). Then \( S(t) \) is asymptotically smooth.

Under our hypotheses on \( M(\cdot), f(\cdot), p(\cdot), g(\cdot), k(s), h(x) \), and \( q(x) \), we can derive the existence and uniqueness of global solutions and the existence of a global attractor for 1-dimensional nonlinear thermoelastic coupled equations systems (1)-(6) step by step.

3. Basic Space and Global Solutions

First we proceed as in Barbose et al. [5], Giorgi [18], and Dafermos [19] and define a new variable \( \chi = \chi(t, s) \) by

\[
\chi = \chi'(t, s) = \int_0^t \theta(x, t - \tau) d\tau,
\]

\((t, s) \in [0, \infty) \times \mathbb{R}^\ast \)

From the definition of \( \chi' \), for all \( t \geq 0 \), we have \( \chi'(t, 0) = 0 \) in \([0, l], t \in \mathbb{R}^\ast \) and \( \chi'(0, s) = \chi_0(s) \) in \([0, l], s \in \mathbb{R}^\ast \), where \( \chi_0(s) = \int_0^l \theta_0(\tau) d\tau, s \in \mathbb{R}^\ast \). Differentiate (21) with respect to \( t \) and \( s \) on both sides, respectively, and make the sum to get

\[
\chi_t + \chi_s = \theta(x, t) \quad \text{in} \quad [0, l] \times \mathbb{R}^\ast \times \mathbb{R}^\ast
\]

So

\[
\chi_{xx} + \chi_{xxx} = \theta_{xx}(t - s) \quad \text{in} \quad [0, l] \times \mathbb{R}^\ast + \mathbb{R}^\ast
\]

Thus

\[
\chi_{xxx} = \theta_{xx}(t - s)
\]

Therefore thermal memory can be rewritten to be

\[
-\int_0^\infty k(s) \theta_{xx}(t - s) ds = -\int_0^\infty k(s) d\chi_{xx}
\]

\[
= -k(s) \chi_{xx} \bigg|_{0}^{\infty} + \int_0^\infty k'(s) \chi_{xx} ds
\]

Then, from the assumption (19) of kernel \( k(s) \), problems (1)-(6) are transformed into the new system

\[
u_{xx} + u_{xxxx} + u_{xxxt} + \theta_{xx} - M \left( \int_0^l (u_x)^2 dx \right) u_{xx} = h(x)
\]

\[
\theta_t - \theta_{xx} - \int_0^\infty \mu(s) X_{xx}(s) ds - u_{xxt} + p(\theta) = q(x)
\]

\[
\chi' = \theta(t) - \chi'(0), \quad \text{in} \quad [0, l] \times \mathbb{R}^\ast
\]

with the initial conditions

\[
u(x, 0) = u_x(0) = 0,
\]

\[
u_{xxx} (l, t) + u_{xxt} (l, t) - M \left( \int_0^l (u_x)^2 dx \right) u_x (l, t)
\]

\[
= f(u(l, t)) + g(u_{l}(l, t)),
\]

\[
\theta(0, t) = \theta_0(l, t) = 0,
\]

\[
\chi'(0, s) = \chi_0(l, s) = 0.
\]

Our analysis is based on the following Sobolev spaces. Let

\[
L^2(0, l),
\]

\[
U = \{u \in H^1(0, l) : u(0) = 0\}
\]

\[
V = \{u \in H^2(0, l) : u(0) = u_x(0) = 0\}
\]

\[
W = \{u \in H^4(0, l) \cap V; u_{xx}(l) = 0\}
\]

\[
X = \{\theta \in H^1(0, l) : \theta(0) = 0\}
\]

\[
Y = \{\theta \in H^2(0, l) : \theta(0) = \theta_x(l) = 0\}
\]

and with respect to the new variable \( \chi \), we define the weighted space

\[
L^2_{\mu}(\mathbb{R}^\ast, X),
\]

\[
\mu = \left\{ \chi : \mathbb{R}^\ast \rightarrow X | \int_0^\infty \mu(s) \|\chi\|_X^2 ds \right\}
\]

which is a Hilbert space with inner product and the norm defined by

\[
\langle \chi, \bar{\chi} \rangle = \langle \chi, \bar{\chi} \rangle_{L^2_{\mu}(\mathbb{R}^\ast, X)} = \int_0^\infty \mu(s) \int_0^l \chi_x \bar{\chi}_x dx ds,
\]

\[
\|\chi\|^2_{\mu, X} = \int_0^\infty \mu(s) \|\chi\|_X^2 ds
\]
Motivated by the boundary condition (4), we assume, for regular solutions, that data $u^0$ and $u^1$ satisfy the compatibility condition

$$u^{0}_{xxx}(l) + u^{1}_{xxx}(l) - M \left( \|u_x^0\|^2 \right) u^0_x (l) = f \left( u^0 (l) \right) + g \left( u^1 (l) \right) \tag{35}$$

Then for regular solutions we consider the phase space

$$\mathbb{H}_1 = \left\{ \left( u^0, u^1, \theta, \chi \right) \in W \times W \times Y \times L^2 \left( R^+, Y \right) \text{ satisfying } (35) \right\}. \tag{36}$$

In the case of weak solutions we consider the phase space

$$\mathbb{H}_0 = V \times L^2 \left( 0, l \right) \times L^2 \left( 0, l \right) \times L^2 \left( R^+, X \right) \tag{37}$$

In $\mathbb{H}_0$ we adopt the norm defined by

$$\left\| \left( u, v, \theta, \chi \right) \right\|_{\mathbb{H}_0} = \|u_x\|^2 + \|v\|^2 + \|\theta\|^2 + \|\chi\|^2_{L^2 \left( 0, l \right)} \tag{38}.$$

Using the classical Galerkin method, we can establish the existence and uniqueness of regular solution and weak solution to problems (26)-(31) as in the work of Ma et al. [14] and Cavalcanti et al. [20].

**Theorem 3.** Under the assumptions on $M(\cdot), f(\cdot), p(\cdot), g(\cdot), k(s), h(x)$, and $q(x)$, for any initial data $\left( u^0, u^1, \theta^0, \chi^0 \right) \in \mathbb{H}_1$, there exists a unique regular solution $(u, \theta, \chi)$ of problems (26)-(31) such that

$$u \in L^\infty \left( R^+ ; V \right) \cap C^0 \left( [0, \infty) ; V \right),$$

$$\cap C^1 \left( [0, \infty) ; L^2 \left( 0, l \right) \right),$$

$$u_x \in L^\infty \left( R^+ ; L^2 \left( 0, l \right) \right),$$

$$\theta, \theta_t \in L^\infty \left( R^+ ; L^2 \left( 0, l \right) \right),$$

$$\chi, \chi_t \in L^\infty \left( R^+ ; L^2 \left( R^+ \times X \right) \right). \tag{39}$$

In addition, if the initial data $\left( u^0, u^1, \theta^0, \chi^0 \right) \in \mathbb{H}_0$, there exists a unique weak solution $(u, \theta, \chi)$ of problems (26)-(31) such that

$$\left( u, \theta, \chi \right) \in C \left( R^+ , \mathbb{H}_0 \right) \tag{40}$$

which depends continuously on initial data with respect to the norm of $\mathbb{H}_0$. What is more, in both cases, is that

$$\|u\|^2 + \|u_x\|^2 + \tilde{M} \left( \|u_x\|^2 \right) + \|\theta\|^2 + \|\chi\|^2_{L^2 \left( R^+ \times X \right)} \leq C \tag{41}$$

where $C$ is a constant and $C$ denotes different constant in different expression in this paper.

The existence of weak solutions follows from density arguments as shown in [14, 21].

Theorem 3 implies that problems (26)-(31) define a nonlinear $C_0$-semigroup $S(t)$ on $\mathbb{H}_0$. Moreover, the operator $S(t)$ defined in $\mathbb{H}_0$ meets the usual semigroup properties $S(t + \tau) = S(t)S(\tau)$, $S(0) = I, \forall t, \tau \in R$.  

**4. Global Attractor**

The main result of this paper reads as follows.

**Theorem 4.** Assume the hypotheses of Theorem 3; then the corresponding semigroup $S(t)$ of problem (26)-(31) has an absorbing set $B$ in $\mathbb{H}_0$.

**Proof.** Now we show that semigroup $S(t)$ has as absorbing set $B$ in $\mathbb{H}_0$. Firstly, we can calculate total energy functional

$$E (t) = \frac{1}{2} \left\{ \|u_t\|^2 + \|u_{xx}\|^2 + \tilde{M} \left( \|u_x\|^2 \right) + \|\theta\|^2 \right\}$$

$$+ \int_0^l \left( f (u (l)) - \int_0^l \right) \mu d x \tag{42}$$

Also, since $u(0) = u_x (0) = 0$, the following inequalities hold:

$$\|u (l)\|^2 \leq \int_0^l d x \int_0^l u_x^2 d x = \left\| u_x \right\|^2 \tag{43}$$

and

$$\|u_x (l)\|^2 \leq \int_0^l d x \int_0^l u_{xx}^2 d x = \left\| u_{xx} \right\|^2 \tag{44}$$

Let us fix an arbitrary bounded set $B \subset \mathbb{H}_0$ and consider the solutions of problems (26)-(31) given by $(u(t), u_x (t), \theta (t)) = S(t) (u^0, u^1, \theta^0)$. Our analysis is based on the modified energy function

$$\tilde{E} (t) = E (t) + a_0 + \frac{1}{\lambda_1} \|\theta\|^2 \geq 0, \tag{45}$$

where $\lambda_1 > 0$ is the first eigenvalue of the operator $\partial_{xx}$ in $H^2 (0, l)$. It is easy to see that $\tilde{E}(t)$ dominates $\|u(t)\|_{\mathbb{H}_0}$ and $\|u_{xx}(s)\|^2 \leq 4E(s)$.

By multiplying (26) by $u$ and integrating over $[0, l]$

$$\|u_{xx}\|^2 = \|u_t\|^2 - \frac{d}{d t} \left( u_t, u \right) - \int_0^l u_{xx} u_{xx} \mu d x$$

$$+ u_x (l) \theta (l) - \int_0^l \theta u_{xx} d x - \tilde{M} \left( \int_0^l \left| u_x \right|^2 d x \right) \|u_x\|^2 - f (u (l)) u (l)$$

$$- g (u_t (l)) u (l) + \int_0^l \mu d x \tag{46}$$

and inserting it into $\tilde{E}(t)$, then integrating it from $t_1$ to $t_2$, and considering (12) and (14), we obtain that

$$\int_{t_1}^{t_2} \tilde{E} (t) d s$$

$$\leq \int_{t_1}^{t_2} \|u_t\|^2 d s$$

$$- \frac{1}{2} \left( \int_0^l u_t (t_2) u (t_2) d x - \int_0^l u_t (t_1) u (t_1) d x \right)$$
\[
- \int_{t_1}^{t_2} \frac{1}{2} \left( \frac{1}{2} u_{xx} u_{xx} \right) dx ds + \int_{t_1}^{t_2} \frac{1}{2} u_{x}(l) \theta(l) ds \\
- \frac{1}{2} \int_{t_1}^{t_2} \theta u_{xx} dx ds - \frac{1}{2} \int_{t_1}^{t_2} g(u_l(l)) u_l(l) ds \\
+ (a_0 + a_1) + \frac{1}{4} \|q_l\|^2 - \frac{1}{2} \int_{t_1}^{t_2} hu dx ds \\
+ \frac{1}{2} \int_{t_1}^{t_2} \|\theta\|^2 ds
\]

(47)

where \(t_1, t_2 \in [t, t+1]\).

Now let us begin to estimate the right hand side of (47) to use the above Lemma 1 of Nakao.

First, multiplying (26) by \(u_l\) and integrating over \([0, l]\), multiplying (27) by \(\theta\) and integrating over \([0, l]\), and then taking the sum and integrating from \(t\) to \(t+1\), we get

\[
\int_{t}^{t+1} \|u_{xxl}\|^2 d\tau + t \int_{t}^{t+1} \|\theta_x\|^2 d\tau \\
+ \int_{t}^{t+1} \int_{0}^{l} p(\theta) \theta dx d\tau + \int_{t}^{t+1} \int_{0}^{l} g(u_l(l)) u_l(l) ds \\
= E(t) - E(t+1) + \int_{t}^{t+1} \int_{0}^{l} q \theta dx d\tau \\
+ \int_{t}^{t+1} u_{xl}(l) \theta(l) d\tau
\]

(48)

Using Schwarz inequality and Young inequality with \((q + 1)/(q + 2) + 1/(q + 2) = 1\) and imbedding inequality, we get

\[
\int_{t}^{t+1} \int_{0}^{l} q \theta dx d\tau \leq \int_{t}^{t+1} \|q\| \|\theta\| d\tau \\
\leq \frac{\varepsilon^{-1/(q+1)}}{4} \|q\|^{(q+2)/(q+1)} \\
+ \frac{\varepsilon}{\|q\|^{(q+2)/(q+1)}} \int_{t}^{t+1} \|\theta\|^2 d\tau \\n\leq \frac{\varepsilon^{-1/(q+1)}}{4} \|q\|^{(q+2)/(q+1)} \\
+ \int_{t}^{t+1} \|\theta\|^{q+2} d\tau
\]

(49)

Considering (44) and using Young inequality, we have

\[
\int_{t}^{t+1} u_{xl}(l) \theta(l) d\tau \leq \int_{t}^{t+1} \|u_{xxl}\| \|\theta_x\| d\tau \\
\leq \int_{t}^{t+1} \left( \frac{1}{3l} \|u_{xxl}\|^2 + \frac{3l}{4} \|\theta_x\|^2 \right) d\tau
\]

(50)

With the assumption (15) of \(f(\cdot)\), we have

\[
\frac{1}{2} \int_{t}^{t+1} \int_{0}^{l} f(\theta) \theta dx d\tau \geq \frac{k_2}{2} \int_{t}^{t+1} \int_{0}^{l} |\theta|^{r+2} dx d\tau
\]

\[
= \frac{k_2}{2} \int_{t}^{t+1} ||\theta||_{l+1}^{r+2} d\tau
\]

(51)

As \(\varepsilon \leq k_2/2\) and \(1/2 \leq r < 1\), combined with (50), from (48) we obtain

\[
\frac{1}{3} \int_{t}^{t+1} \int_{0}^{l} u_{xxl}^2 dx d\tau + t \int_{t}^{t+1} \int_{0}^{l} \|\theta_x\|^2 dx d\tau \\
+ \frac{1}{2} \int_{t}^{t+1} \int_{0}^{l} p(\theta) \theta dx d\tau \\
+ \int_{t}^{t+1} \int_{0}^{l} g(u_l(l)) u_l(l) ds \\
\leq E(t) - E(t+1) + \frac{\varepsilon^{-1/(q+1)}}{4} \|q\|^{(q+2)/(q+1)}
\]

(52)

Taking into account assumption (15) of \(p(\cdot)\) and assumption (16) of \(g(\cdot)\), we have

\[
E(t) + \frac{\varepsilon^{-1/(q+1)}}{4} \|q\|^{(q+2)/(q+1)} \geq E(t+1),
\]

(53)

and then we define an auxiliary function \(I^2(t)\) by putting

\[
I^2(t) = E(t) + \frac{\varepsilon^{-1/(q+1)}}{4} \|q\|^{(q+2)/(q+1)}
\]

(54)

And we can get

\[
\int_{t}^{t+1} \int_{0}^{l} u_{xxl}^2 dx d\tau \leq 3I^2(t), \\
\int_{t}^{t+1} \|\theta_x\|^2 dx d\tau \leq \frac{4}{l} I^2(t)
\]

(55)

Using twice holder inequality with \(r/(r+2) + 2/(r+2) = 1\) and considering the assumption (16) of \(g(\cdot)\), we obtain that

\[
\int_{t}^{t+1} \|u_l\|^2 ds - \int_{t}^{t+1} \int_{0}^{l} u_{xl}^2 dx ds \\
\leq \int_{t}^{t+1} \left( \int_{0}^{l} 1^{(r+2)/2} dx \right)^{r/(r+2)} \\
\cdot \left( \int_{0}^{l} u_{xl}^{(r+2)/2} dx \right)^{1/(r+2)} ds = |l|^{r/(r+2)}
\]
Using Schwarz inequality and Young inequality combined the second inequality of (55), we have

\[
\frac{1}{2} \int_0^{t+1} \left( \int_0^t u_{t+2} \right)^{r/(r+2)} ds \leq \|\theta\|^2 \left( \int_0^{t+1} \left( \int_0^t u_{t+2} \right)^{r/(r+2)} dx \right)^{\theta/(\theta+2)}
\]

\[
\left( \int_0^{t+1} \theta^{r+2} ds \right)^{2/(r+2)} \left( \int_0^t \left( \int_0^t u_{t+2} \right)^{r/(r+2)} dx \right)^{\theta/(\theta+2)} \leq \frac{1}{2} \|\theta\|^2 \left( \int_0^{t+1} \left( \int_0^t u_{t+2} \right)^{r/(r+2)} dx \right)^{\theta/(\theta+2)}
\]

(61)

Since (56), in view of the Mean Value Theorem for integral, there exist number \( t_1 \in [t, t + 1/4] \) and number \( t_2 \in [t + 3/4, t + 1] \) such that

\[
\|u_t(t_1)\|^2 \leq \|\theta\|^2 \left( \int_0^{t+1} \left( \int_0^t u_{t+2} \right)^{r/(r+2)} dx \right)^{\theta/(\theta+2)}
\]

and \( \|u_t(t_2)\|^2 \leq \|\theta\|^2 \left( \int_0^{t+1} \left( \int_0^t u_{t+2} \right)^{r/(r+2)} dx \right)^{\theta/(\theta+2)} \) (57)

Thus from Schwarz inequality combined with (57), noting that \( \|u_{xx}(s)\|^2 \leq 4\bar{E}(s) \), we have

\[
\frac{1}{2} \left( \int_0^t u_t(t_2) u(t_2) dx - \int_0^t u_t(t_1) u(t_1) dx \right) \leq \frac{1}{\sqrt{\lambda_1}} \left( \|u_t(t_2)\| \|u_{xx}(t_2)\| + \|u_t(t_1)\| \|u_{xx}(t_1)\| \right) \leq \frac{1}{\sqrt{\lambda_1}} \|\theta\|^{2(r+2)} \left( \int_0^{t+1} \left( \int_0^t u_{t+2} \right)^{r/(r+2)} dx \right)^{\theta/(\theta+2)}
\]

\[
\leq \frac{1}{\sqrt{\lambda_1}} \|\theta\|^{2(r+2)} \left( \int_0^{t+1} \left( \int_0^t u_{t+2} \right)^{r/(r+2)} dx \right)^{\theta/(\theta+2)} \sup_{t \in [s,t+1]} \|u_{xx}\| \leq \frac{1}{\lambda_1 \eta} \|\theta\|^{2(r+2)} \left( \int_0^{t+1} \left( \int_0^t u_{t+2} \right)^{r/(r+2)} dx \right)^{\theta/(\theta+2)} + \eta \sup_{t \in [s,t+1]} \bar{E}(s) \] \quad (58)

Using Schwarz inequality and Young inequality combined with the first inequality of (55), we get

\[
\frac{1}{2} \int_0^{t+1} \left( \int_0^t u_{xx} u_{t+2} dx dx \right) ds \leq \frac{3}{4\eta} t^2(t) + \eta \sup_{t \in [s,t+1]} \bar{E}(s) \] \quad (59)

Considering (44) and using Young inequality combined with the second inequality of (55), we have

\[
\int_0^{t+1} \int_0^t u_{xx}(t) \theta \theta(t) dx ds \leq \int_0^{t+1} \int_0^t \|u_{xx}\| \|\theta_x\| ds \leq \int_0^{t+1} \left( \lambda \|\theta_x\|^2 + \eta \|u_{xx}\|^2 \right) ds \leq \frac{4}{\eta} \lambda \int_0^{t+1} \|\theta_x\|^2 ds + \eta \sup_{t \in [s,t+1]} \bar{E}(s) \] \quad (60)

Using twice Holder inequality with \( q/(q + 2) + 2/(q + 2) = 1 \), combined with the assumption (15) on \( p(\theta) \) and the fourth inequality of (55), we have

\[
\frac{1}{2} \int_0^{t+1} \|\theta\|^2 ds \leq \frac{1}{2} \int_0^{t+1} \left( \int_0^t u_{t+2} \right)^{\theta/(\theta+2)} \left( \int_0^{t+1} \left( \int_0^t u_{t+2} \right)^{r/(r+2)} dx \right)^{\theta/(\theta+2)}
\]

\[
\left( \int_0^{t+1} \theta^{r+2} ds \right)^{2/(r+2)} \left( \int_0^t \left( \int_0^t u_{t+2} \right)^{r/(r+2)} dx \right)^{\theta/(\theta+2)} \leq \frac{1}{2} \|\theta\|^{\theta/(\theta+2)}
\]

(61)

Thus using Young inequality and considering (61), we have

\[
-\frac{1}{2} \int_0^{t+1} \int_0^t \theta u_{xx} dx ds \leq \int_0^{t+1} \frac{1}{4\eta} \|\theta\|^2 ds + \int_0^{t+1} \frac{3}{4\eta} \|u_{xx}\|^2 ds \]

\[
\leq \frac{1}{4\eta} \left( \frac{2}{k_2} \right) \|\theta\|^{2/(\theta+2)} \left( \int_0^{t+1} \left( \int_0^t u_{t+2} \right)^{r/(r+2)} dx \right)^{\theta/(\theta+2)} + \eta \sup_{t \in [s,t+1]} \bar{E}(s) \] \quad (62)

Considering the assumptions (17) of \( g(\cdot) \) combined with (43), we obtain

\[
\frac{1}{2} \int_0^{t+1} \int_0^t g(u, l) u(l) ds \leq \frac{k_4}{2} \int_0^{t+1} \sqrt{\|u(l)\|} \|u_x\| ds \]

\[
+ \frac{k_4}{2} \int_0^{t+1} \|u(l)\|^{1+1} \sqrt{\|u_x\|} ds \] \quad (63)

With

\[
|u_x|^2 = \|u(x, t) - u(x, 0, t)|^2 \leq \int_0^x \int_0^t u_{xx}^2 dx dx \leq k_4 \int_0^x \int_0^t u_{xx}^2 dx dx \] \quad (64)

we get

\[
\|u_x\|^2 \leq \frac{1}{k_4} \int_0^x \int_0^t u_{xx}^2 dx dx \leq \frac{1}{k_4} \int_0^x \int_0^t u_{xx}^2 dx dx \] \quad (65)
Using Young inequality combined with (43) and (65) and the first inequality of (55), we have

\[ \frac{k_1}{2} \int_{t_1}^{t_2} \sqrt{\| u_l(t) \| u_{xx} } \, ds \leq \frac{k_1^{3/2}}{2} \int_{t_1}^{t_2} \| u_l(t) \| u_{xx} \, ds \]

\[ \leq \frac{1}{\eta} \left( \frac{k_1^{3/2}}{2} \right)^2 \int_{t_1}^{t_2} \| u_l(t) \|^2 \, ds + \eta \int_{t_1}^{t_2} \| u_{xx} \|^2 \, ds \]

\[ \leq \frac{1}{\eta} \left( \frac{k_1^{3/2}}{2} \right)^2 \int_{t_1}^{t_2} \| u_{xx} \|^2 \, ds + \eta \sup_{t \leq s \leq t+1} \| u_{xx} \| \tag{66} \]

and using twice Holder inequality with \((r+1)/(r+2) + 1/(r+2) = 1\) combined with the embedding theorem and Young inequality,

\[ \frac{k_2}{2} \int_{t_1}^{t_2} \| u_l(t) \|^r \sqrt[n]{\| u_{xx} \|} \, ds \leq \frac{k_2}{2} \int_{t_1}^{t_2} \| u_{xx} \|^2 \, ds \]

\[ \leq \frac{1}{(k_3)^{(r+1)/(r+2)}} \frac{k_4}{2} \int_{t_1}^{t_2} \| u_l(t) \| u_{xx} \, ds \]

\[ = \frac{1}{(k_3)^{(r+1)/(r+2)}} \left( \frac{2}{(k_4)^{(r+1)/(r+2)}} \right) \int_{t_1}^{t_2} \| u_{xx} \|^2 \, ds \]

\[ \leq \frac{1}{(k_3)^{(r+1)/(r+2)}} \left( \frac{2}{(k_4)^{(r+1)/(r+2)}} \right) \int_{t_1}^{t_2} \| u_{xx} \|^2 \, ds \]

\[ \leq \frac{1}{(k_3)^{(r+1)/(r+2)}} \left( \frac{2}{(k_4)^{(r+1)/(r+2)}} \right) \sup_{t \leq s \leq t+1} \| u_{xx} \| \%

Thus we can obtain

\[ \frac{1}{2} \int_{t_1}^{t_2} g(u_l(t)) \, u(l) \, ds \]

\[ \leq \frac{3}{\eta} \left( \frac{k_3^{3/2}}{2} \right)^2 \int_{t_1}^{t_2} \| u_l(t) \|^2 \, ds \]

\[ + \frac{1}{\eta} \left( \frac{1}{(k_3)^{(r+1)/(r+2)}} \right) \frac{k_4}{2} \int_{t_1}^{t_2} \| u_{xx} \|^2 \, ds \]

\[ + 2 \eta \sup_{t \leq s \leq t+1} \| E(s) \| \]

Finally using Young inequality, we get that

\[ \frac{1}{2} \int_{t_1}^{t_2} \| u(t) \|^2 \, ds \]

\[ \leq \frac{1}{4 \eta \lambda_1} \| h \|^2 + \eta \sup_{t \leq s \leq t+1} \| E(s) \| \]

Inserting (56), (58)-(62), and (68)-(69) into (47), we obtain

\[ \int_{t_1}^{t_2} \| u(t) \|^2 \, ds \]

\[ \leq \left( \int_{t_1}^{t_2} \| u_l(t) \|^2 \, ds \right) \frac{1}{4 \eta \lambda_1} \| h \|^2 \]

\[ + \left( \int_{t_1}^{t_2} \| u_{xx} \|^2 \, ds \right) \frac{1}{4 \eta \lambda_1} \| h \|^2 \]

\[ + 7 \eta \sup_{t \leq s \leq t+1} \| E(s) \| \]

For the left hand side of (70), we use the Mean Value Theorem; then there exists number \( r \in [t_1, t_2] \) such that

\[ \int_{t_1}^{t_2} \| u_l(r) \|^2 \, ds \]

\[ \leq \frac{1}{2} \| E(t) \| \]

\[ + \frac{e^{-1/(q+1)}}{8} \| q \|^2 \| q \|^2 \]

So we conclude that

\[ \| E(t) \| \leq I(t) + 2 \int_{t_1}^{t_2} \| E(s) \| \, ds \]

\[ + \frac{e^{-1/(q+1)}}{8} \| q \|^2 \| q \|^2 \]
Inserting (70) into (72), we obtain that

\[ E(t) \leq 2 \left( \frac{|l|^{(r+2)}}{k_3} \left( \frac{1}{k_3} \right) \right) + \left[ 1 + \frac{1}{1-14\eta} \left( \frac{1}{\eta} \right)^{2/2} \right] I^2(t) \]

and then (74) can be rewritten as

\[ (73) \]

\[ E(t) \leq C_1 I (t)^{4/(r+2)} + \frac{2}{1-14\eta} (a_0 + a_1) \]

and then (74) can be rewritten as

\[ E(t)^{1+r/2} \leq C_1 \left( \frac{E(t) - E(t+1)}{1} \right) + \frac{2}{1-14\eta} (a_0 + a_1) \]

Using Nakao's Lemma 1, we conclude that

\[ E(t) \leq C_1 I (t)^{4/(r+2)} \]

and then (74) can be rewritten as

\[ (75) \]

As \( t \to \infty \), the first term of the right side of (76) goes to zero; thus with \( \tilde{E}(t) \geq (1/4)(\|u_x\|^2 + \|v_x\|^2 + \|\theta\|^2 + \|\chi\|_{H^2}^2) \), we conclude that

\[ \mathbb{B} = \left\{ (u, v, \theta, \chi) \in \mathbb{H}_0 \mid \|u_x\|^2 + \|v_x\|^2 + \|\theta\|^2 + \|\chi\|_{H^2}^2 \right\} \]

is an absorbing set for \( S(t) \) in \( \mathbb{R}_+ \). Theorem 4 is proved. \( \square \)

The main result of a global attractor reads as follows.

**Theorem 5.** Assume the hypotheses of Theorem 3; then the corresponding semigroup \( S(t) \) of problems (26)-(31) is asymptotically smooth in space \( \mathbb{H}_0 \).

**Proof.** We are going to apply Lemmas 1 and 2 to prove the asymptotic smoothness. Given initial data \((u^0, \theta^0, \chi^0) \) and \((v^0, \nu, \vartheta, \chi^0) \) \( \in \mathcal{B} \) in a bounded invariant set \( \mathbb{B} \subset \mathbb{H}_0 \), let \((u, \theta, \chi), (v, \vartheta, \chi) \) be the corresponding weak solutions of problems (27)-(31). Then the differences \( w = u - v, \theta = \theta - \vartheta, \Pi = \chi - \chi \) are the weak solutions of

\[ \begin{align*}
    w_t + w_{xxx} + w_{xxt} + \theta_{xx} - M \left( \|u_x\|^2 \right) w_{xx} &= \Delta M v_{xxx}, \\
    \theta_t - \vartheta_{xx} - \int_0^\infty \mu(s) \Pi'_{xx} (s) ds - w_{xxt} + \Delta \rho &= 0, \\
    \Pi'_t &= -\Pi'_x + \theta, \\
    w(0) &= u^0 - v^0, \\
    w_t (0) &= u^1 - v^1, \\
    \theta(0) &= \theta^0 - \vartheta_0, \\
    \Pi(0) &= \chi^0 - \chi^0, \\
    w(0) &= w_{xx} (0) = w_{xx} (l) = 0, \\
    \theta(0) &= \theta_{xx} (l) = 0, \\
    \Pi(0) &= \Pi_{xx} (l) = 0, \\
    w_{xxt} (l) + w_{xxt} (l) - M \left( \|u_x\|^2 \right) w_x (l, t) &= -\Delta M v_{xxx} (l, t) + \Delta f + \Delta g
\end{align*} \]
where

\[
\Delta M = M \left( \| u_x \|^2 \right) - M \left( \| v_r \|^2 \right),
\]
\[
\Delta p = p(\theta) - p(\bar{\theta}),
\]
\[
\Delta f = f(u(l,t)) - f(v(l,t)),
\]
\[
\Delta g = g(u(l,t)) - g(v(l,t)).
\]

(79)

Let us define

\[
E_w(t) = \| u_1 \|^2 + \| w_{xx} \|^2 + M \left( \| u_x \|^2 \right) \| w_x \|^2 + \| \theta \|^2
\]
\[
+ \| \Pi \|^2_{\mu}.
\]

(80)

We can assume formally that \( w \) is sufficiently regular by density. Then, multiplying the first equation in (78) by \( w_t \) and integrating over \([0, l]\), multiplying the second equation in (78) by \( \theta \) and integrating over \([0, l]\) and taking the inner with \( \Pi \) for the third equation in (78) in space \( L^2_\mu(R^n, X) \), and then taking the sum, we get

\[
\frac{1}{2} \frac{d}{dt} E_w(t) + \| w_{xx} \|^2 + t \| \theta \|^2
\]
\[
+ \int_0^l \mu(s) \int_0^l \Pi_x \Pi_x dx ds
\]
\[
+ \int_0^l \Delta p \theta dx
\]
\[
+ \Delta gw_t(l)
\]
\[
= -M' \left( \| u_x \|^2 \right) \| w_x \|^2 \int_0^l u_{xx} u_t dx
\]
\[
- \Delta M v_x(l) w_t(l) - w_{xt}(l) \theta(l)
\]
\[
+ \int_0^l \Delta M v_{xx} w_t dx - \Delta f w_t(l)
\]

(81)

Let us estimate the right hand side of (81). Considering the continuity of \( M'(\cdot) \) and the estimates (41), we have

\[
- \Delta M v_x(l) w_t(l)
\]
\[
\leq M' \left( \| u_x \|^2 \right) \left( \| u_x \| + \| v_r \| \right) \| v_r(l) \| \| w_t(l) \|
\]
\[
\leq C \| w_x \| \left( \| u_x \| + \| v_r \| \right) \| v_r(l) \| \| w_t(l) \|
\]
\[
\leq C \| w_{xx} \|^2 \left( r+2 \right) \frac{1}{(r+1)^2} + k_2 \| w_t(l) \|^{r+2}
\]
\[
+ \frac{k_1}{2} \left( \| \theta \| \xi \right)^{r+2} + k_2 \| \theta \| \| \Pi \|^2_{\mu}
\]
\[
\leq C \left( \| w_{xx} \|^2 + \| w_x \|^2 + \| w_{xx} \|^{(r+2)/(r+1)} \right)
\]

(82)

Applying the mean value theorem, then considering (44) and the estimates (41), we have

\[
- \Delta M v_x(l) w_t(l)
\]
\[
\leq M' \left( \| u_x \|^2 \right) \left( \| u_x \| + \| v_r \| \right) \| v_r(l) \| \| w_t(l) \|
\]
\[
\leq C \| w_x \| \left( \| u_x \| + \| v_r \| \right) \| v_r(l) \| \| w_t(l) \|
\]
\[
\leq C \| w_{xx} \|^2 \left( r+2 \right) \frac{1}{(r+1)^2} + k_2 \| w_t(l) \|^{r+2}
\]
\[
+ \frac{k_1}{2} \left( \| \theta \| \xi \right)^{r+2} + k_2 \| \theta \| \| \Pi \|^2_{\mu}
\]
\[
\leq C \left( \| w_{xx} \|^2 + \| w_x \|^2 + \| w_{xx} \|^{(r+2)/(r+1)} \right)
\]

(83)

where \( \xi \) is among \( \| u_x \|^2 \) and \( \| v_r \|^2 \). Also considering (43) and (44), then using Young inequality, we have

\[
w_{xt}(l) \theta(l) \, dt \leq \| w_{xx} \| \| \theta \| \leq \| w_{xx} \| \| \theta \| \| w_{xx} \|
\]
\[
\leq \frac{1}{3} \| w_{xx} \|^2 + \frac{3}{4} \| \theta \|^2
\]
\[
\leq \frac{1}{3} \| w_{xx} \|^2 + \frac{3}{4} \| \theta \|^2
\]
\[
\leq \frac{1}{3} \| w_{xx} \|^2 + \frac{3}{4} \| \theta \|^2
\]

(84)

Also applying the mean value theorem combined with the estimates (41), we get

\[
\Delta M \int_0^l v_{xx} w_t dx \leq C \| w_t \|.
\]

(85)

Considering that the assumption (13) of \( f(\cdot) \) combined with the estimates (41) and (43), then Young inequality \((r+1)/(r+2) + 1/(r+2) = 1\)

\[
| - \Delta f w_t(0) |
\]
\[
\leq k_1 \left( 1 + \sqrt{7} \| u_x \|^{r+2} \right) \sqrt{7} \| w_t \| \| w_t(l) \|
\]
\[
\leq 3k_1 C \| w_t \| \| w_t(l) \|
\]
\[
\leq k_1 C \| w_t \|^{(r+2)/(r+1)} + k_2 \| w_t(l) \|^{r+2}
\]

(86)

On the other hand, combining with the assumption (19) on \( \mu(s) \)

\[
\int_0^l \Delta p \theta dx \geq k_2 \| \theta \|^{r+2}_{\varepsilon_2},
\]

(88)

Also by view of the assumption (16) on \( g(\cdot) \), we have

\[
\Delta g w_t(l) \geq k_3 \| u_t(l) - v_t(l) \|^{r+2} = k_3 w_t(l) \|^{r+2}.
\]

(89)

Thus by inserting (82)-(89) into (81), with \( 1/2 \leq t \leq 1 \), we get that

\[
\int_0^l \frac{d}{dt} E_w(t) + \frac{1}{3} \| w_{xx} \|^2 + \frac{1}{4} \| \theta \|^2 + \frac{1}{2} \| \Pi \|^2_{\mu}
\]
\[
+ \frac{k_1}{2} \left( \| \theta \| \xi \right)^{r+2} + k_2 \| \theta \| \| \Pi \|^2_{\mu}
\]
\[
\leq C \left( \| w_{xx} \|^2 + \| w_x \|^2 + \| w_{xx} \|^{(r+2)/(r+1)} \right)
\]

(90)

Then integrating from \( t \) to \( t+1 \) and defining an auxiliary function \( F^2(t) \), we get

\[
\int_0^{t+1} \| w_{xx} \|^2 ds + \frac{1}{4} \int_0^{t+1} \| \theta \|^2 ds
\]
\[
+ \int_0^{t+1} \| \Pi \|^2_{\mu} ds + \frac{k_1}{2} \int_0^{t+1} |w_t(l)|^{r+2} ds
\]
\[
+ k_2 \int_0^{t+1} \| \theta \|^{r+2}_{\varepsilon_2} ds \leq \frac{1}{2} (E_w(t) - E_w(t+1))
\]
\[
+ C \left( \| w_{xx} \|^2 + \| w_x \|^2 + \| w_{xx} \|^{(r+2)/(r+1)} \right) ds
\]

(91)
Thus we can get
\[
\begin{align*}
\frac{1}{3} \int_{t}^{t+1} \|w_{xx}\|^2 \, ds & \leq F(t)^2, \\
\frac{1}{4} \int_{t}^{t+1} \|w\|^2 \, ds & \leq F(t)^2, \\
\int_{t}^{t+1} \frac{\delta_{t}}{2} \|\Delta w\|^2 \, ds & \leq F(t)^2, \\
k_{3} \int_{t}^{t+1} |w_{l}(l)|^{r+2} \, ds & \leq F(t)^2, \\
k_{2} \int_{t}^{t+1} \|\tilde{w}_{0}^{r+2}\| \, ds & \leq F(t)^2
\end{align*}
\]  
(92)

Then by multiplying first equation in (78) by \(w\) and integrating over \([0, l]\), then integrating from \(t_{1}\) to \(t_{2}\), we get that
\[
\begin{align*}
\int_{t_{1}}^{t_{2}} \left( \|w_{xx}\|^2 + \sigma \left( \|w_{x}\|^2 \right) \|w_{x}\|^2 \right) \, ds \\
= - \int_{0}^{l} w_{l}(t_{2}) w(t_{2}) \, dx + \int_{0}^{l} w_{l}(t_{1}) w(t_{1}) \, dx \\
+ \int_{t_{1}}^{t_{2}} \|w_{x}\|^2 \, dt - \int_{t_{1}}^{t_{2}} \Delta f w(l) \, ds \\
- \int_{t_{1}}^{t_{2}} \Delta g w(l) \, ds - \int_{t_{1}}^{t_{2}} \Delta M v_{x}(l) w(l) \, ds \\
+ \int_{t_{1}}^{t_{2}} \Delta M \int_{0}^{l} v_{xx} w dx \, ds + w_{x}(l) \theta(l) \\
- \int_{t_{1}}^{t_{2}} \int_{0}^{l} \theta w_{xx} dx \, ds
\end{align*}
\]  
(94)

Now let us estimate the right-hand side of (94). Firstly, from the first inequality of (92), we infer that
\[
\int_{t}^{t+1} \|w_{x}\|^2 \, ds \leq \int_{t}^{t+1} \frac{1}{\lambda_{1}} \|w_{x}\| \, ds \leq \frac{3}{\lambda_{1}} \|F\|^{2}(t)
\]  
(95)

Thus we combine with the first inequality of (92), and there exists \(t_{1} \in [t, t + 1/4]\) and \(t_{2} \in [t + 3/4, t + 1]\) such that
\[
\|w_{l}(t_{1})\|^2 \leq \frac{6}{\lambda_{1}} \|F\|^{2}(t) \quad \text{and} \quad \|w_{l}(t_{2})\|^2 \leq \frac{6}{\lambda_{1}} \|F\|^{2}(t),
\]  
(96)

then we can deduce that
\[
\begin{align*}
- \int_{0}^{l} w_{l}(t_{2}) w(t_{2}) \, dx + \int_{0}^{l} w_{l}(t_{1}) w(t_{1}) \, dx \\
\leq \frac{2 \sqrt{6}}{\lambda_{1}} F(t) \sup_{t \in [t_{1}, t_{2}]} \|w_{x}\| \\
\leq \frac{48}{\lambda_{1}} F(t)^{2} + \frac{1}{8} \sup_{t \in [t_{1}, t_{2}]} E_{w}(\sigma).
\end{align*}
\]  
(97)

The assumption (13) on \(f(\cdot)\) combined with (43) and the estimates (41) implies that
\[
\begin{align*}
\int_{t_{1}}^{t_{2}} F(l) d\mu(l) d\nu(l) \\
\leq k_{4} \int_{t_{1}}^{t_{2}} \left( 1 + |u_{x}(l)|^{r} + |\nu_{x}(l)|^{r} \right) \nu(l) \, ds
\end{align*}
\]  
(98)

By the assumption (17) on \(g(\cdot)\) combined with (43) and the estimates (41), and using the Hölder inequality and Young inequality, we have
\[
\begin{align*}
- \int_{t_{1}}^{t_{2}} \Delta g w(l) \, ds & \leq k_{4} \int_{t_{1}}^{t_{2}} \left( 1 + |u_{x}(l)|^{r} + |\nu_{x}(l)|^{r} \right) \nu(l) \, ds \\
& \leq k_{4} \int_{t_{1}}^{t_{2}} \left( 1 + \sqrt{\|u_{x}\|} + \sqrt{\|v_{x}\|} \right) \sqrt{\|w_{x}\|} \, ds \\
& \leq k_{4} \int_{t_{1}}^{t_{2}} \|u_{x}\| \, ds \leq C \int_{t_{1}}^{t_{2}} \|u_{x}\| \|w_{x}\| \, ds \leq \frac{8}{\lambda_{2}} \|u_{x}\| \, ds
\end{align*}
\]  
(99)

where \(\lambda_{2} > 0\) is the first eigenvalue of the operator \(\partial_{x}\) in \(H^{1}(0, l)\). Applying the Mean value theorem and using Schwarz inequality and Young inequality, we have
\[
\begin{align*}
\int_{t_{1}}^{t_{2}} \Delta M v_{x}(l) w(l) \, ds \\
\leq \int_{t_{1}}^{t_{2}} \sigma(\xi_{l}) \|w_{x}\| \left( \|u_{x}\| + \|v_{x}\| \right) \|v_{x}(l) \| \, ds \\
\leq \int_{t_{1}}^{t_{2}} C \|w_{x}\| \|w_{l}(l)\| \, ds \\
\leq \int_{t_{1}}^{t_{2}} C \|w_{x}\| \|w_{l}(l)\| \, ds + \int_{t_{1}}^{t_{2}} \frac{k_{3}}{4} |w_{l}(l)|^{r+2} \, ds
\end{align*}
\]  
(100)

where \(\xi_{l}\) is among \(\|u_{x}\|^{2}\) and \(\|v_{x}\|^{2}\). Also apply the Mean value theorem combined with the estimates (41) to get
\[
\begin{align*}
\int_{t_{1}}^{t_{2}} \frac{1}{t_{2}} \int_{0}^{l} v_{xx} w \, dx \, ds & \leq C \int_{t_{1}}^{t_{2}} \|w_{x}\| \, ds
\end{align*}
\]  
(101)
Finally, using twice Holder inequality with inequality of (92), and (95) and (103), we obtain that

\[
\int_{t_1}^{t_2} |w_x(t) \vartheta(t)| \, ds \leq \int_{t_1}^{t_2} \|w_{xx}\| \|\vartheta_x\| \, ds \\
\leq \int_{t_1}^{t_2} \left( \frac{8}{\eta} \|\vartheta_x\| + \frac{\eta}{32} \|w_{xx}\|^2 \right) \, ds \\
\leq \frac{32}{\eta} F^3(t) + \frac{\eta}{8} \sup_{s \leq \tau \leq t+1} E_w(s)
\]

Using Young inequality, we have

\[
\int_l^2 \frac{1}{t} \left( \frac{1}{t} \right)^{\eta/(\eta+2)} \, ds \\
\leq \frac{\eta}{32} \left( \frac{1}{k_2} \right)^{2/(\eta+2)} F(t)^{4/(\eta+2)}
\]

thus by Schwarz inequality and Young inequality, we get

\[
\int_{t_1}^{t_2} \int_0^t \delta w_{xx} \, dx \, ds \leq 8 t^\theta/(\eta+2) \left( \frac{1}{k_2} \right)^{2/(\eta+2)} F(t)^{4/(\eta+2)}
\]

By inserting (95) and (97)-(104) into (94), we obtain that

\[
\frac{1}{2} \int_{t_1}^{t_2} \left[ \|w_{xx}\|^2 + \sigma (\|w_x\|^2 \|w_x\|^2) \right] \, ds \\
\leq C \int_{t_1}^{t_2} \|w_x\|^2 \, ds \\
+ \frac{9}{2} t^{\theta/(\eta+2)} \left( \frac{1}{k_2} \right)^{2/(\eta+2)} F(t)^{4/(\eta+2)}
\]

From (91), we see that

\[
E_w(t) + 2 F^2(t)
\]

Let \(E_w(t) = \sup_{t \leq \tau \leq t+1} E_w(s)\) with \(\tau \in [t, t+1]\), then integrate (90) over \([t, \tau]\) and over \([\tau', t+1]\) to have

\[
\sup_{t \leq \tau \leq t+1} E_w(s) = E_w(\tau) \\
\leq E_w(t) + C \int_t^{t_2} \left( \|w_x\|^2 + \|w_{xx}\|^2 + \|w_{x}\|^{(2)/(r+1)} \right) \, ds
\]
From Nakao’s Lemma 1, there exists $C_B > 0$ and $C_T > 0$ such that

$$E_w (t) \leq C_B \left[ (t - 1)^+ \right]^{2/\gamma} + C_T \left\{ \sup_{0 \leq \sigma \leq T} \int_{\sigma}^{\sigma + 1} \left( \|w_x\|^2 + \|w_x\|^{(r+2)/(r+1)} \right) \right\} ds$$

(114)

From the definition of $E_w (t)$, we have

$$\left\|(w, w_x, \theta) \right\|_{\mathcal{H}_0} \leq C_B \left[ (t - 1)^+ \right]^{2/\gamma} + C_T \left\{ \sup_{0 \leq \sigma \leq T} \int_{\sigma}^{\sigma + 1} \left( \|w_x\|^2 + \|w_x\|^{(r+2)/(r+1)} \right) \right\} ds$$

(115)

Given $\varepsilon > 0$, we choose $T$ large such that

$$C_B \left[ (t - 1)^+ \right]^{2/\gamma} \leq \varepsilon$$

and define $\omega_T : \mathcal{H}_0 \times \mathcal{H}_0 \to R$ as

$$\omega_T \left( (u^0, u^1, \theta^0, \chi^0), (v^0, v^1, \theta^0, \chi^0) \right)$$

$$= C_T \left\{ \sup_{0 \leq \sigma \leq T} \int_{\sigma}^{\sigma + 1} \left( \|w_x\|^2 + \|w_x\|^{(r+2)/(r+1)} \right) \right\} ds$$

(117)

Then from (115)-(117), we get

$$\left\| S(T) \left( (u^0, u^1, \theta^0, \chi^0), (v^0, v^1, \theta^0, \chi^0) \right) - S(T) \left( (v^0, v^1, \theta^0, \chi^0) \right) \right\|_{\mathcal{H}_0}$$

$$\leq \varepsilon + \omega_T \left( (u^0, u^1, \theta^0, \chi^0), (v^0, v^1, \theta^0, \chi^0) \right)$$

(118)

for all $(u^0, u^1, \theta^0, \chi^0), (v^0, v^1, \theta^0, \chi^0) \in B$.

Let $(u^0_n, u^1_n, \theta^0_n, \chi^0_n)$ be a given sequence of initial data in $B$. Then the corresponding sequence $(u_{nk}, u_{nk}, \theta_{nk}, \chi_{nk})$ of solutions of the problems (26)-(31) is uniformly bounded in $\mathcal{H}_0$, because $B$ is bounded and positively invariant. So $(u_{nk})$ is bounded in $C([0, \infty), \mathcal{V}) \cap C^1 \left( [0, \infty), L^2 (0, I) \right)$. Since $V \hookrightarrow U$ compactly, there exists a subsequence $u_{nk}$ which converges strongly in $C([0, T], U)$. Therefore

$$\lim_{k \to \infty} \lim_{l \to \infty} \int_0^T \left( \| (u_{nk} - (u_{nk})_l \|^2 + \| (u_{nk} - (u_{nk})_l \|^{(r+2)/(r+1)} \right) ds = 0$$

(119)

and

$$\lim_{k \to \infty} \lim_{l \to \infty} \omega_T \left( (u^0_{nk}, u^1_{nk}, \theta^0_{nk}, \chi^0_{nk}), (u^0_{nl}, u^1_{nl}, \theta^0_{nl}, \chi^0_{nk}) \right) = 0.$$  

(120)

So $S(t)$ is asymptotically smooth in $\mathcal{H}_0$. That is, Lemma 2 holds. Thus Theorem 5 is proved.

**Theorem 6.** The corresponding semigroup $S(t)$ of problems (26)-(27) has a compact global attractor in the phase space $\mathcal{H}_0$.

**Proof.** In view of Theorems 3–5, we directly get Theorem 6. \qed

**Data Availability**

The data used to support the findings of this study are available from the corresponding author upon request.

**Conflicts of Interest**

The authors declare that they have no conflicts of interest.
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