On the Order of Convexity for the Shifted Hypergeometric Functions

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Abstract
In the present paper, we study the order of convexity of $z^2 F_1(a, b; c; z)$ with real parameters $a$, $b$ and $c$ where $2 F_1(a, b; c; z)$ is the Gaussian hypergeometric function. First we obtain some conditions for $z^2 F_1(a, b; c; z)$ with no finite orders of convexity by considering its asymptotic behavior around $z = 1$. Then the order of convexity of $z^2 F_1(a, b; c; z)$ is demonstrated for some ranges of real parameters $a$, $b$ and $c$. In the last section, we give some examples as applications of the main results.

Keywords Gaussian hypergeometric functions · Order of convexity

Mathematics Subject Classification Primary 30C45; Secondary 33C05

1 Introduction and Main Results

The Gaussian hypergeometric function plays an important role in special function theory and is related to many elementary functions. It is connected with conformal mappings, quasiconformal theory, differential equations, continued fractions and so on. For complex parameters $a, b, c, c \neq 0, -1, -2, \ldots$, the hypergeometric function is defined by the power series

$$2 F_1(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n n!} z^n$$
for $z \in \mathbb{D} = \{ z \in \mathbb{C} : |z| < 1 \}$, where $(a)_n$ is the Pochhammer symbol; namely, 
$(a)_0 = 1$ and $(a)_n = a(a+1) \cdots (a+n-1) = \Gamma(a+n)/\Gamma(a)$ for $n = 1, 2, \ldots$. 
If $c = a + b$, the hypergeometric function is termed as the zero-balanced one. Note 
that $z_2 F_1(a, b; c; z)$ is usually called a shifted hypergeometric function. For instance 
$z_2 F_1(1, 1; 2; z) = -\log(1-z)$ is a shifted zero-balanced hypergeometric function. In 
present paper, we only restrict to the real parameters $a, b$ and $c$. For the basic 
properties of hypergeometric functions we refer to [1,13,25].

The behavior of the hypergeometric function $2 F_1(a, b; c; z)$ near $z = 1$ varies 
according to the sign of $a + b - c$, namely,

(i) For $a + b - c < 0$

$$2 F_1(a, b; c; 1) = \frac{\Gamma(c) \Gamma(c-a-b)}{\Gamma(c-a) \Gamma(c-b)} < \infty. \quad (1.1)$$

(ii) For $a + b - c = 0$

$$2 F_1(a, b; a+b; z) \sim -\frac{\Gamma(a) \Gamma(b)}{\Gamma(a+b)} \log(1-z) \text{ as } z \to 1. \quad (1.2)$$

(iii) For $a + b - c > 0$

$$2 F_1(a, b; c; z) \sim \frac{\Gamma(c) \Gamma(a+b-c)}{\Gamma(a) \Gamma(b)} (1-z)^{c-a-b} \text{ as } z \to 1. \quad (1.3)$$

For more details see [6,13,15] and the references therein. For a refined asymptotic 
relation as $z \to 1$ for $c \leq a + b$, one may refer to [2,4,16].

For a function $f$ analytic in $\mathbb{D}$ and normalized by $f(0) = f'(0) - 1 = 0$, the order 
of convexity of $f$ is defined by

$$\kappa = \kappa(f) := 1 + \inf_{z \in \mathbb{D}} \Re \frac{zf''(z)}{f'(z)} \in [-\infty, 1].$$

It is known that $f$ is convex, i.e. $\kappa(f) \geq 0$ if and only if $f$ is univalent in $\mathbb{D}$ and 
$f(\mathbb{D})$ is a convex domain. It is also true that if $\kappa(f) \geq -1/2$, then $f$ is univalent in 
$\mathbb{D}$ and $f(\mathbb{D})$ is convex in (at least) one direction, see [24] and [19, p. 17, Thm. 2.24; 
p. 73]. We make the convention that $\kappa(f) = -\infty$ only if $f'$ has no zeros in $\mathbb{D}$ and 
$\Re [zf''(z)/f'(z)]$ is not bounded from below in $\mathbb{D}$, whereas $\kappa(f)$ is regarded to be 
defined if $f'$ has zeros in $\mathbb{D}$.

The starlikeness and other geometric properties of the shifted hypergeometric functions have been extensively researched, see [8,9,11,12,14,21,23] and so on. Via the Alexander transform and the identity

$$\frac{d}{dz} 2 F_1(a, b; c; z) = \frac{ab}{c} 2 F_1(a+1, b+1; c+1; z),$$

the starlikeness of the shifted hypergeometric function can be transformed to the convexity of $2 F_1(a, b; c; z)$ or of the normalized one $\frac{c}{ab}(2 F_1(a, b; c; z) - 1)$. Comparing

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with the fruitful researches on the convexity of $\,_{2}F_{1}(a, b; c; z)$, there are only a few studies on the convexity of the shifted one in the existing literature. For instance, Silverman constructed some sufficient conditions for convexity on the coefficients of the MacLaurin series in [22]; Sugawa and the author got a condition for the convexity of a special shifted hypergeometric function with complex parameters by Jack’s Lemma in [23]; Ponnusamy and Vuorinen in [17] gave several necessary conditions for convexity by considering the bounds of the modulus of the $n$-th coefficients of the MacLaurin series; Küstner in [9] obtained the order of convexity of $z\,_{2}F_{1}(1, b; c; z)$ and $z\,_{2}F_{1}(a, b; 2; z)$ for some special cases by transforming the convexity to the star-likeness of $z\,_{2}F_{1}(2, b; c; z)$ and $z\,_{2}F_{1}(a, b; 1; z)$ respectively. We only state one of Küstner’s results here, because it is closely related to our main results.

Theorem A (\cite[Cor. 8]{9})

(a) If $0 < a \leq b \leq 1$ then

$$\kappa(z\,_{2}F_{1}(a, b; 2; z)) = 1 - \frac{2\,_{2}F_{1}(a, b; 1; -1)}{2\,_{2}F_{1}(a, b; 1; -1)}.$$

(b) If $0 < -a \leq b \leq 1$ then

$$\kappa(z\,_{2}F_{1}(a, b; 2; z)) = 1 + \frac{2\,_{2}F_{1}(a, b; 1; 1)}{2\,_{2}F_{1}(a, b; 1; 1)} = -\infty.$$

(c) If $0 < b < -a \leq 1$ then

$$\kappa(z\,_{2}F_{1}(a, b; 2; z)) = 1 + \frac{2\,_{2}F_{1}(a, b; 1; 1)}{2\,_{2}F_{1}(a, b; 1; 1)} = 1 - \frac{ab}{a + b}.$$

(d) If $0 < a < 1 < b \leq 2 - a$ then

$$\kappa(z\,_{2}F_{1}(a, b; 2; z)) = -\infty.$$

(e) If $0 < a \leq 1 \leq b \leq 2 < a + b$ then

$$\kappa(z\,_{2}F_{1}(a, b; 2; z)) = 1 + \frac{(1-a)(1-b)}{a+b-2} + \frac{1-a-b}{2}.$$

By observing the behavior of hypergeometric function around $z = 1$ in $\mathbb{D}$, we derive some conditions for the shifted hypergeometric function to have $-\infty$ as its order of convexity.

Theorem 1.1 For real parameters $a, b$ and $c$ none of which are negative integers satisfying $c - a \notin -\mathbb{N}$ and $c - b \notin -\mathbb{N}$, if one of the following conditions holds:

(1) $0 < ab < 1$ and $a + b \leq c < 1 + a + b - ab$;
(2) $ab < 0$ and $a + b \leq c < 1 + a + b$;

then the order of convexity of the function $z\,_{2}F_{1}(a, b; c; z)$ is $-\infty$. 
Note that Theorem 1.1 generalizes the cases (b) and (d) in Theorem A and a result due to Küstner in [9, Cor. 9, case (e)]. Letting \( c = a + b \) in Theorem 1.1, we obtain a result on the non-convexity of the zero-balanced hypergeometric functions as follows.

**Corollary 1.2** If \( a \) and \( b \) are real constants satisfying \( a \neq 0, -1, -2, \ldots, b \neq 0, -1, -2, \ldots \) and \( ab < 1 \), then the shifted zero-balanced hypergeometric function \( z_2 F_1(a, b; a + b; z) \) is not convex.

The representations of the ratio of two hypergeometric functions by a continued fraction and Riemann-Stieltjes type integral play crucial roles in our proofs. Some other ratios of hypergeometric functions are also quite useful, for example, a formula of Ramanujan for the representation of the ratio has been used to solve a conjecture on certain inequalities in [5] and later in [3] to prove generalizations of Elliott’s identity. The forthcoming two theorems are about the order of convexity of the shifted hypergeometric functions.

**Theorem 1.3** Suppose \( a \) and \( c \) are real parameters with \( 0 < a \leq c \).

1. If \( c \geq 2 \), then
   \[
   \kappa(z_2 F_1(a, 1; c; z)) = \frac{4 - a - c}{2} + \frac{c - 2}{2} \frac{2 F_1(a, 1; c; -1)}{2 F_1(a, 2; c; -1)}.
   \]

2. If \( 1 \leq c < \min\{2, 1 + a\} \), then
   \[
   \kappa(z_2 F_1(a, 1; c; z)) = \frac{(c - a)(a + c - 3)}{2(1 + a - c)}.
   \]

   In particular, \( 2 F_1'(a, 1; c; z) \neq 0 \) for all \( z \in D \) in these two cases.

**Remark 1.4** Theorem 1.3 is also proved by Küstner in [9, Cor. 9, cases (a) and (f)] by the relationship between the starlikeness of \( z_2 F_1(a, 2; c; z) \) and the convexity of \( z_2 F_1(a, 1; c; z) \), although the order in the first case is given in different forms.

**Theorem 1.5** For real parameters \( a, b \) and \( c \) satisfying \( 0 < a < 1 \) and \( a \leq b \leq c \), the order of convexity of the function \( z_2 F_1(a, b; c; z) \) is given as follows:

1. If \( b < 1 \) and \( c \geq 1 + a + b - ab \), then
   \[
   \kappa(z_2 F_1(a, b; c; z)) = \frac{5 - c - a - b}{2} + \frac{c - 2 - (1 - a)(1 - b)}{2 \left(1 - a + b \frac{2 F_1(a + 1, b; c; -1)}{2 F_1(a, b; c; -1)}\right)}.
   \]

2. If \( b > 1 \) and \( c < \min\{a + b, 1 + a + b - ab\} \), then
   \[
   \kappa(z_2 F_1(a, b; c; z)) = \frac{c^2 - a^2 - b^2 + 3(a + b - c) - 2}{2(a + b - c)}.
   \]

   In particular, \( 2 F_1'(a, b; c; z) \neq 0 \) for all \( z \in D \) in these two cases.
It is worth pointing out that the order of convexity of $z_2F_1(a, b; c; z)$ with real parameters satisfying $0 < a < 1 < b \leq c$ and $a + b \leq c < 1 + a + b - ab$ is $-\infty$ which is already shown in the first case of Theorem 1.1.

**Remark 1.6** Put $c = 2$ in Theorem 1.5, then the first result is reduced to the case $(a)$ in Theorem A, although the orders are given in different forms; the second result becomes the case $(e)$ in Theorem A, since the condition $0 < a < 1 < b \leq 2$ implies $1 + a + b - ab > 2$.

A direct use of Theorems 1.3, Theorem 1.5 and Lemma 2.2 yields the result on the lower orders of convexity for some special hypergeometric functions.

**Corollary 1.7** Suppose $0 < a \leq b \leq \min\{1, c\}$, then the order of convexity for the shifted hypergeometric function $z_2F_1(a, b; c; z)$ satisfies

$$\kappa \geq \begin{cases} \frac{(4-ab)c-ab(5-a-b)}{2(2c-ab)}, & c \geq 3-a-b+ab \\ \frac{2c+(a^2-5a+2)b}{2(b+c-ab)}, & 1+a+b-ab \leq c < 3-a-b+ab \end{cases} \geq 0.$$

Furthermore, the shifted hypergeometric function $z_2F_1(a, b; c; z)$ is convex in $\mathbb{D}$ if $0 < a \leq b \leq 1$ and $c \geq 1 + a + b - ab$.

### 2 Some Lemmas

This section is devoted to introducing several lemmas for later use.

**Lemma 2.1** ([8, Thm. 1.5], [25, pp. 337–339 and Thm. 69.2]) If $-1 \leq a \leq c$ and $0 \leq b \leq c \neq 0$, the ratio of two hypergeometric functions can be written as a continued fraction and integral as

$$\frac{2F_1(a+1, b; c; z)}{2F_1(a, b; c; z)} = \frac{1}{1 - \frac{(1-g_0)z}{1-(1-g_1)z}} = \int_0^1 \frac{d\mu(t)}{1-tz}, \quad z \in \mathbb{C} \setminus [1, +\infty)$$

where

$$g_n = \begin{cases} 0 & \text{for } n = 0; \\ \frac{a+k}{c+2k-1} & \text{for } n = 2k \geq 2, \ k \geq 1; \\ \frac{b+k}{c+2k-2} & \text{for } n = 2k-1 \geq 1, \ k \geq 1 \end{cases} \quad (2.1)$$

and $\mu : [0, 1] \to [0, 1]$ is non-decreasing with $\mu(1) - \mu(0) = 1$. Thus the ratio is holomorphic in $\mathbb{C} \setminus [1, +\infty)$ and it maps the unit disc and the half plane $\{z \in \mathbb{C} : \text{Re} \ z < 1\}$ univalently onto domains that are convex in the direction of the imaginary axis.
Since the values of the hypergeometric functions at $z = \pm 1$ appear in Theorems 1.3 and 1.5, the next two lemmas deal with the estimations at $z = -1$ and the behaviors around $z = 1$ respectively.

**Lemma 2.2** If $-1 \leq a \leq c$ and $0 \leq b \leq c \neq 0$, then

$$\frac{c}{b + c} \leq \frac{2F_1(a + 1, b; c; -1)}{2F_1(a, b; c; -1)} \leq \frac{2c - b}{2c}.$$  

**Proof** Since $-1 \leq a \leq c$ and $0 \leq b \leq c \neq 0$, Lemma 2.1 implies that

$$\frac{2F_1(a + 1, b; c; -1)}{2F_1(a, b; c; -1)} = \frac{1}{1 + \frac{(1-g_0)g_1}{1+...}}.$$  

where $\{g_n\}_{n=0}^\infty$ is the non-negative sequence shown in (2.1). Thus it is obvious that the above continued fraction is not less than $\frac{c}{b + c}$. On the other hand, [25, Thm. 11.1, p. 46] demonstrates that the values of the continued fraction

$$\frac{g_1}{1 - \frac{(1-g_1)g_2}{1-(1-g_2)g_1}}$$

lie in the disc

$$\left| w - \frac{1}{2 - g_1} \right| \leq \frac{1 - g_1}{2 - g_1}.$$  

Therefore

$$\frac{1}{1 + \frac{(1-g_0)g_1}{1+...}} \leq \frac{1}{1 + \frac{g_1}{2-g_1}} = \frac{2c - b}{2c}.$$  

We verify the assertions.

Note that the approximants in Lemma 2.2 are also used in [8, Rem. 2.3], although the explicit forms are not given there.

**Lemma 2.3** Let $a, b$ and $c$ be non-zero real constants with $a, b, c \notin -\mathbb{N}$, $c - a \notin -\mathbb{N}$ and $c - b \notin -\mathbb{N}$.

(1) If $a + b < c < a + b + 1$, then

$$\frac{2F_1(a + 1, b; c; z)}{2F_1(a, b; c; z)} = \frac{A}{(1-z)^{1-a}} + O(|1-z|^{\varepsilon-1}) \quad (2.2)$$

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where

\[
A = \frac{\Gamma(a + b + 1 - c)\Gamma(c - a)\Gamma(c - b)}{\Gamma(a + 1)\Gamma(b)\Gamma(c - a - b)},
\]

\(\alpha = c - a - b \in (0, 1)\) and \(\varepsilon = \min\{2\alpha, 1\} \).

(2) If \(a + b = c\), then

\[
\frac{2F_1(a + 1, b; c; z)}{2F_1(a, b; c; z)} = \frac{1}{-a(1 - z) \log(1 - z)} + O\left(\log\frac{1}{|1 - z|}\right). \quad (2.3)
\]

(3) If \(c < a + b\), then

\[
\frac{2F_1(a + 1, b; c; z)}{2F_1(a, b; c; z)} = \frac{a + b - c}{a(1 - z)} + O\left(|1 - z|^{a+b-c}\right). \quad (2.4)
\]

Proof (1) Since \(a + b < c < a + b + 1\), by applying the formula

\[
2F_1(a, b; c; z) = \frac{\Gamma(c)\Gamma(c - a - b)}{\Gamma(c - a)\Gamma(c - b)} 2F_1(a, b; a + b - c + 1; 1 - z)
+ (1 - z)^c-a-b \frac{\Gamma(c)\Gamma(a + b - c)}{\Gamma(a)\Gamma(b)} 2F_1(c - a, c - b; c - a - b + 1; 1 - z)
\]

(2.5)

to the functions \(2F_1(a + 1, b; c; z)\) and \(2F_1(a, b; c; z)\), we have for \(z \to 1\) in \(\mathbb{D}\),

\[
\frac{2F_1(a + 1, b; c; z)}{2F_1(a, b; c; z)}
= (1 - z)^c-a-b-1 \frac{\Gamma(c)\Gamma(a + b + 1 - c)\Gamma(c - a)\Gamma(c - b)}{\Gamma(a + 1)\Gamma(b)\Gamma(c - a - b)} + O(|1 - z|^{\varepsilon-1})
= (1 - z)^c-a-b-1 \frac{\Gamma(a + b + 1 - c)\Gamma(c - a)\Gamma(c - b)}{\Gamma(a + 1)\Gamma(b)\Gamma(c - a - b)} + O(|1 - z|^{\varepsilon-1})
\]

where

\[
A = \frac{\Gamma(a + b + 1 - c)\Gamma(c - a)\Gamma(c - b)}{\Gamma(a + 1)\Gamma(b)\Gamma(c - a - b)},
\]

\(\varepsilon = \min\{2(c - a - b), 1\}\) and \(\alpha = c - a - b \in (0, 1)\) since \(a + b < c < a + b + 1\).

(2) If \(c = a + b\), by virtue of (2.5) and the following formula due to Ramanujan

\[
2F_1(a, b; a + b; z) = \frac{\Gamma(a + b)}{\Gamma(a)\Gamma(b)} \left( R(a, b) - \log(1 - z) \right) + O\left( |1 - z| \log\frac{1}{|1 - z|} \right)
\]
as $z \to 1$ in $D$, where

$$R(a, b) = 2\psi(1) - \psi(a) - \psi(b)$$

and $\psi(x) = \Gamma'(x)/\Gamma(x)$ denotes the digamma function, we have around $z = 1$,

$$\frac{2F_1(a + 1, b; c; z)}{2F_1(a, b; c; z)} = (1 - z)^{a - 1} \frac{\Gamma(a + b + 1 - c)}{\Gamma(a + 1)\Gamma(b)} \times \frac{\Gamma(a)\Gamma(b)}{-\log(1 - z)\Gamma(a + b)} + O\left(\log \frac{1}{|1 - z|}\right)$$

$$= \frac{1}{-a(1 - z)\log(1 - z)} + O\left(\log \frac{1}{|1 - z|}\right).$$

(3) Since $c < a + b$, we deduce from the transformation (2.5) that around $z = 1$,

$$\frac{2F_1(a + 1, b; c; z)}{2F_1(a, b; c; z)} = (1 - z)^{-1} \frac{\Gamma(a + b + 1 - c)}{\Gamma(a + 1)\Gamma(b)} \times \frac{\Gamma(a)\Gamma(b)}{\Gamma(a + b - c)} + O\left(|1 - z|^{a + b - c}\right)$$

$$= \frac{a + b - c}{a(1 - z)} + O\left(|1 - z|^{a + b - c}\right).$$

**Lemma 2.4** [20] Let $F(z)$ be analytic in the slit domain $\mathbb{C} \setminus [1, +\infty)$. Then

$$F(z) = \int_0^1 \frac{d\mu(t)}{1 - tz}$$

for some probability measure $\mu$ on $[0, 1]$, if and only if the following conditions are fulfilled:

(1) $F(0) = 1$;
(2) $F(x) \in \mathbb{R}$ for $x \in (-\infty, 1)$;
(3) $\text{Im } F(z) \geq 0$ for $\text{Im } z > 0$;
(4) $\lim_{n \to \infty} F(z_n)/z_n = 0$ for some sequence $z_n \in \mathbb{C}$ with $\text{Im } z_n \to +\infty$, and $\text{Im } z_n \geq \delta \text{Re } z_n$ for some positive constant $\delta$;
(5) $\limsup_{x \to +\infty} F(-x) \geq 0$.

The measure $\mu$ and the functions $F$ are in one-to-one correspondence.

It is noteworthy that Liu and Pego in [10] proved that the condition (4) in Lemma 2.4 is superfluous.

The next lemma is a direct consequence of Lemma 2.4, thus we omit its proof.

**Lemma 2.5** If

$$f(z) = \int_0^1 \frac{d\mu(t)}{1 - tz}$$
for some probability measure \( \mu \) on \([0, 1]\), then for any \( 0 < a \leq 1 \), there exists some probability measure \( \nu \) on \([0, 1]\) such that
\[
\frac{1}{(1-z)(1-a+af)} = \int_0^1 \frac{d\nu(t)}{1-tz}.
\]

3 Proofs of the Main Results

Before proceeding to prove the main results, we first prepare some materials which will be used several times in the proofs.

Let \( F(z) = _2F_1(a, b; c; z) \), \( G(z) = _2F_1(a + 1, b; c; z) \) and \( H(z) = _2F_1(a + 1, b + 1; c + 1; z) \) for simplicity. Contiguous relations of hypergeometric functions imply that
\[
G(z) - F(z) = \frac{b}{c}zH(z).
\]

In order to obtain the order of convexity of \( zF(z) \), we need to estimate the real part of \( 1 + zF''(z)/(zF)'(z) \) in \( \mathbb{D} \). By combining the derivative formula
\[
zF'(z) = \frac{ab}{c}zH = -aF(z) + aG(z)
\]
and the hypergeometric differential equation
\[
z(1 - z)F''(z) + [c - (a + b + 1)z]F'(z) - abF(z) = 0,
\]
a routine computation shows that
\[
1 + \frac{z(zF)''}{(zF)'} = 1 + \frac{2zF' + z^2F''}{F + zF'}
= 1 + \frac{z[2 - c + (a + b - 1)z]F' + abzF}{(1 - z)(F + zF')}
= \frac{a[3 - c + (a + b - 2)z](G - F) + [1 + (ab - 1)z]F}{(1 - z)[F + a(G - F)]}
= 3 - c + (a + b - 2)z + \frac{c - 2 + (1 - a)(1 - b)z}{1 - z}(1 - a + a\frac{G}{F})
= 3 - c + (1 + a + b - c)\frac{z}{1 - z} + \frac{c - 2 + (1 - a)(1 - b)z}{(1 - z)(1 - a + a\frac{G}{F})}.
\]

Denote
\[
W(z) = \left[3 - c + (1 + a + b - c)\frac{z}{1 - z}\right] + M(z),
\]

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with
\[ M(z) = \frac{c - 2 + (1 - a)(1 - b)z}{(1 - z)(1 - a + a G/F)}. \] (3.3)

Thus in order to prove Theorems 1.1, 1.3 and 1.5, it suffices to find the infimum of the real part of \( W(z) \) in the unit disc \( \mathbb{D} \).

If \( 0 \leq a \leq \min\{c, 1\} \) and \( 0 \leq b \leq c \), a combination of Lemmas 2.1 and 2.5 implies that there exists a probability measure \( \nu \) on \([0, 1]\) such that
\[
\frac{1}{(1 - z)(1 - a + a G/F)} = \int_0^1 \frac{d\nu(t)}{1 - tz}, \tag{3.4}
\]

Now we are ready for the proofs of the main results.

**Proof of Theorem 1.1** Since the first term of \( W(z) \) in (3.2) has bounded real part on the unit circle \( \partial \mathbb{D} := \{ z \in \mathbb{C} : |z| = 1 \} \) and \( 1 + a + b - c > 0 \) by the assumptions, it is sufficient to only consider the asymptotic behavior of the second term \( M(z) \) around \( z = 1 \). Rewrite
\[
M(z) = \frac{c - 2 + (1 - a)(1 - b)z}{(1 - z)(1 - a + a G/F)} = \frac{p}{(1 - z)(1 - a + a G/F)} - \frac{(1 - a)(1 - b)}{1 - a + a G/F},
\]
with \( p = c - 1 - a - b + ab \).

Both of the assumptions guarantee that \( p < 0 \) and \( a + b \leq c < a + b + 1 \). Thus the Eqs. (2.2) and (2.3) in Lemma 2.3 imply that for \( z \to 1 \) in \( \mathbb{D} \), the second term on the right hand side of \( M(z) \) is bounded. To verify \( \Re M(z) \) tends to \( -\infty \) for \( z \to 1 \) in \( \mathbb{D} \), we need only to analyze the asymptotic behavior of \((1 - z)(1 - a + a G/F)\) in the first term. Let \( z_\theta \in \mathbb{D} \) with \( z_\theta = 1 - r e^{i\theta} \), thus \( -\pi/2 < \theta < \pi/2 \) and \( 0 \leq r < 2 \cos \theta \). Since the asymptotic behavior of the ratio of two hypergeometric functions \( G/F \) around \( z = 1 \) depends on to the sign of \( c - a - b \), we divide the proof into two parts.

**Case I:** Let \( a + b < c < 1 + a + b \). The Eq. (2.2) yields
\[
\tan\left[ \arg(1 - z_\theta) \left( 1 - a + a G/F(z_\theta) \right) \right] = \frac{\Im \left[ (1 - a)re^{i\theta} + O(r^\varepsilon) + aA(re^{i\theta})^\alpha \right]}{\Re \left[ (1 - a)re^{i\theta} + O(r^\varepsilon) + aA(re^{i\theta})^\alpha \right]}
= \frac{(1 - a)r \sin \theta + \Im O(r^\varepsilon) + aA r^\alpha \sin[\alpha \theta]}{(1 - a)r \cos \theta + \Re O(r^\varepsilon) + aA r^\alpha \cos[\alpha \theta]}
= \frac{(1 - a)r^{1-\alpha} \sin \theta + \Im O(r^{\varepsilon-\alpha}) + aA \sin[\alpha \theta]}{(1 - a)r^{1-\alpha} \cos \theta + \Re O(r^{\varepsilon-\alpha}) + aA \cos[\alpha \theta]}.
\]

Since \( \alpha \in (0, 1) \) and \( \varepsilon > \alpha \), if we let \( \theta \to \pm \pi/2 \) and \( r \to 0 \) correspondingly, then
\[
\arg(1 - z_\theta) \left( 1 - a + a G/F(z_\theta) \right) \to \pm \frac{\alpha \pi}{2}.
\]

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and
\[ \left| (1 - z_\theta) \left( 1 - a + a \frac{G}{F}(z_\theta) \right) \right| \to 0. \]

Finally we proved
\[ \text{Re} \frac{1}{(1 - z_\theta)(1 - a + a \frac{G}{F}(z_\theta))} \to +\infty \quad (3.5) \]
as \( \theta \to \pm \pi / 2 \).

**Case II:** If \( a + b = c \), the Eq. (2.3) implies that
\[
\tan \arg \left[ (1 - z_\theta) \left( 1 - a + a \frac{G}{F}(z_\theta) \right) \right] = \tan \arg \left[ (1 - z_\theta)(1 - a) - \frac{1}{\log(1 - z_\theta)} + O(1 - z_\theta \log \frac{1}{1 - z_\theta}) \right] \\
= \frac{(1 - a)r \sin \theta + \frac{\theta}{\log^2 r + \theta^2} + \text{Im } O(-r \log r)}{(1 - a)r \cos \theta - \frac{\log r}{\log^2 r + \theta^2} + \text{Re } O(-r \log r)}. 
\]

After multiplying the term \( \log^2 r \) in the denominator and numerator of the above equation, if we let \( \theta \to \pm \pi / 2 \) and \( r \to 0 \) correspondingly, then
\[
\arg (1 - z_\theta) \left( 1 - a + a \frac{G}{F}(z_\theta) \right) \to 0
\]
and
\[
\left| (1 - z_\theta) \left( 1 - a + a \frac{G}{F}(z_\theta) \right) \right| \to 0,
\]
from which we also deduce (3.5) in this case.

Therefore in both of these two cases, we have
\[ \text{Re} \frac{p}{(1 - z_\theta)(1 - a + a \frac{G}{F}(z_\theta))} \to -\infty \]
as \( \theta \to \pm \pi / 2 \), since \( p < 0 \). The proof is complete.

**Proof of Theorem 1.3** First note that the hypergeometric functions are symmetric with respect to the parameters \( a \) and \( b \); namely, \( _2F_1(a, b; c; z) = _2F_1(b, a; c; z) \). In order to apply the formulas given in the beginning of this section, we exchange the roles of \( a \) and \( b \) in the following proof. So we turn to show the order of convexity of the shifted hypergeometric function \( z_2F_1(1, b; c; z) \).
Since $a = 1$, by making use of the representation (3.4), we can rewrite $W(z)$ as

$$W(z) = 3 - c + (2 + b - c) \frac{z}{1-z} + \frac{c-2}{(1-z)G(z)}$$

$$= 3 - c + (2 + b - c) \frac{z}{1-z} + (c-2) \int_0^1 \frac{d\nu(t)}{1-tz},$$

where $\nu$ is a probability measure on $[0, 1]$. Note that the function $z/(1-z)$ maps the unit disc $\mathbb{D}$ conformally onto the right half plane $\{z \in \mathbb{C} : \text{Re} z > -1/2\}$.

We first consider the case when $c - 2 \geq 0$. By observing the above form of $W(z)$, we thus have

$$\text{Re } W(z) \geq W(-1) = \frac{4-b-c}{2} + \frac{c-2\, _2F_1(1, b; c; -1)}{2\, _2F_1(2, b; c; -1)},$$

for $z$ on the unit circle. We infer from Lemma 2.2 that

$$W(-1) = \frac{4-b-c}{2} + \frac{c-2\, _2F_1(1, b; c; -1)}{2\, _2F_1(2, b; c; -1)} \leq \frac{4-b-c}{2} + \frac{(c-2)(2c-b)}{4c}$$

$$= 1 - \frac{b(3c - 2)}{4c} < 1 = W(0)$$

as $b > 0$ and $c \geq 2$. Therefore we conclude that $\inf_{z \in \mathbb{D}} \text{Re } W(z) = W(-1)$, if $0 < b \leq c$ and $c \geq 2$.

Note that if $1+b \leq c < 2$, the first result in Theorem 1.1 demonstrates that the order of convexity is $-\infty$. Thus it is sufficient to only consider the case $c < \min\{2, 1+b\}$.

For any $z \in \mathbb{D}$, we get

$$\text{Re } W(z) \geq \frac{4-b-c}{2} + (c-2) \lim_{z \to 1} \frac{\, _2F_1(1, b; c; z)}{(1-z)\, _2F_1(2, b; c; z)},$$

since $2+b-c$ is positive in this case. On the other hand, the real part of the Möbius transform $z/(1-z)$ can approach $1/2$ as $z \to 1$ in $\mathbb{D}$. For example we can choose a sequence $z_n = 1/n + (1 - 1/n)e^{i\pi/n} \in \mathbb{D}$, for $n \geq 2$. Since

$$\text{Re } \frac{z_n}{1-z_n} \to \frac{1}{2}$$

for $n \to \infty$, we get

$$\inf_{z \in \mathbb{D}} \text{Re } W(z) = \frac{4-b-c}{2} + (c-2) \lim_{z \to 1} \frac{\, _2F_1(1, b; c; z)}{(1-z)\, _2F_1(2, b; c; z)}. $$
Since \( c < \min\{2, 1 + b\} \), the equation (2.4) shows that
\[
\lim_{z \to 1} \frac{2F_1(1, b; c; z)}{(1-z)2F_1(2, b; c; z)} = \lim_{z \to 1} \frac{1}{1 + b - c + O(|1 - z|^{2b+c})} = \frac{1}{1 + b - c}.
\]

Therefore we obtain the assertion
\[
\inf_{z \in \mathbb{D}} \Re W(z) = \frac{(c - b)(c + b - 3)}{2(1 + b - c)}
\]
for \( 0 < b \leq c \) and \( 1 \leq c < \min\{2, 1 + b\} \).

**Proof of Theorem 1.5** The representation (3.4) implies that there exists a probability measure \( \nu \) on \([0, 1]\) such that
\[
M(z) = \int_0^1 \frac{c - 2 + (1 - a)(1 - b)z}{1 - tz} d\nu(t)
= c - 2 + \int_0^1 [(1 - a)(1 - b) + (c - 2)t] \frac{z}{1 - tz} d\nu(t).
\]

For fixed \( t \in [0, 1] \), the second factor of the integrand in the integral form of \( M(z) \) is a Möbius transform which maps the unit disc \( \mathbb{D} \) onto a disc or a half plane which is symmetric with respect to the real axis. Thus we arrive at the following claims.

(1) If \((1 - a)(1 - b) + (c - 2)t \geq 0\) for all \( t \in [0, 1] \), i.e. \( 1 - b \geq 0 \) and \( c - 2 + (1 - a)(1 - b) \geq 0 \), then
\[
\inf_{z \in \mathbb{D}} \Re M(z) = M(-1).
\]

By recalling the form of the function \( W(z) \), to prove that it attains the infimum of the real part in the unit disc at \( z = 1 \), it is sufficient to only consider the case \( 5 - a - b - c < 0 \). By evoking Lemma 2.2, we can compare the values of \( W(-1) \) and \( W(0) \) as follows:
\[
W(-1) = \frac{5 - a - b - c}{2} + \frac{c - 2 - (1 - a)(1 - b)}{2 \left( 1 - a + a \frac{2F_1(1, b; c; -1)}{2F_1(a, b; c; -1)} \right)}
\leq \frac{5 - a - b - c}{2} + \frac{c - 2 - (1 - a)(1 - b)}{2 \left( 1 - a + \frac{ab}{b+c} \right)}
\leq 1 = W(0)
\]
as \( 5 - a - b - c < 0 \) and \( 0 < b \leq 1 \) imply \( c - 2 - (1 - a)(1 - b) > 0 \). Therefore we conclude that
\[
\inf_{z \in \mathbb{D}} \Re W(z) = \frac{5 - a - b - c}{2} + M(-1)
\]
in this case.

(2) If \((1 - a)(1 - b) + (c - 2)t \leq 0\) for all \(t \in [0, 1]\), i.e. \(1 - b \leq 0\) and \(c - 2 + (1 - a)(1 - b) \leq 0\), then

\[
\inf_{z \in \mathbb{D}} \Re M(z) = M(1),
\]

where \(M(1)\) is regarded as the (unrestricted) limit of \(M(z)\) as \(z \to 1\) in \(\mathbb{D}\). Recall that

\[
M(z) = \frac{c - 2 + (1 - a)(1 - b)z}{(1 - z)(1 - a + aG_F(z))}.
\]

Since we assume \(c < a + b\) in addition, it follows with (2.4) that

\[
M(z) = \frac{c - 2 + (1 - a)(1 - b)z}{(1 - z)(1 - a) + a + b - c + O(|1 - z|^{1 + a + b - c})} \to \frac{c - 2 + (1 - a)(1 - b)}{a + b - c}
\]

for \(z \to 1\) and \(z \in \mathbb{D}\).

By investigating the original form of \(W(z)\), we have

\[
\Re W(z) \geq \frac{5 - a - b - c}{2} + M(1), \quad z \in \mathbb{D}
\]

as \(1 + a + b - c > 0\) by the assumption. We further obtain that

\[
\inf_{z \in \mathbb{D}} \Re W(z) = \frac{5 - a - b - c}{2} + M(1),
\]

by the same sequence technique used in the proof of Theorem 1.3 since the first term of \(W(z)\) is also a Möbius transform.

Therefore we have verified all the assertions in this theorem. The proof is completed.

**Proof of Corollary 1.7** It is easy to check that all the assumptions of Case (1) in both Theorems 1.5 and 1.3 are satisfied, therefore we obtain the order of convexity is

\[
\kappa(\,_{2}F_{1}(a, b; c; z)) = \frac{5 - c - a - b}{2} + \frac{c - 2 - (1 - a)(1 - b)}{2 [1 - a + aG_F(z)]},
\]

for both \(0 < b < 1\) and \(b = 1\). Next we divide the proof into two cases according to the sign of \(c - 2 - (1 - a)(1 - b)\).

Case I: \(c - 2 - (1 - a)(1 - b) \geq 0\), i.e. \(c \geq 3 + a + b - ab\). By substituting the upper bound of \((G/F)(-1)\) in Lemma 2.2 into the above order \(\kappa\), we get

\[
\kappa \geq \frac{5 - c - a - b}{2} + \frac{c - 2 - (1 - a)(1 - b)}{2 (1 - a + aG_F(z))} = \frac{(4 - ab)c - ab(5 - a - b)}{2(2c - ab)} := \kappa_1.
\]
In order to show the convexity of the shifted hypergeometric functions in this case, it is sufficient to verify the above lower bound $\kappa_1$ is non-negative. As $0 < ab \leq 1$, $\kappa_1 \geq 0$ if and only if $c \geq ab(5 - a - b)/(4 - ab)$. An elementary calculation yields that under the condition $0 < a \leq b \leq 1$,

$$\frac{ab(5 - a - b)}{4 - ab} < 3 + a + b - ab,$$

is valid which means the lower bound $\kappa_1$ is positive in this case. Thus $z_2 F_1 (a, b, c, z)$ is convex in $\mathbb{D}$.

Case II: $1 + a + b - ab \leq c < 3 + a + b - ab$. We deduce from Lemma 2.2 that

$$\kappa \geq \frac{5 - c - a - b}{2} + \frac{c - 2 - (1 - a)(1 - b)}{2 \left(1 - a + \frac{c}{b+c}\right)} = \frac{2c + (a^2 - 5a + 2)b}{2(b + c - ab)} := \kappa_2.$$

By a similar argument to Case I, to show the convexity of the shifted hypergeometric functions, it is sufficient to verify that

$$c \geq \frac{(a^2 - 5a + 2)b}{2},$$

under the conditions of this case. An elementary calculation also yields that under the condition $0 < a \leq b \leq 1$, the inequality

$$-\frac{(a^2 - 5a + 2)b}{2} < 1 + a + b - ab,$$

holds which means the lower bound $\kappa_2$ is positive in this case. Therefore $z_2 F_1 (a, b, c, z)$ is convex in $\mathbb{D}$.

We thus complete the proof of this theorem.

### 4 Some Examples and Remarks

In this section, we find some applications of the main results, as well as some comparisons with the previous known results.

By specifying $(b, c) = (1, 3), (b, c) = (3/2, 3), (b, c) = (1, 3/2)$ in Theorem 1.3 successively, we find the explicit order of convexity for some shifted hypergeometric functions.

**Example 4.1**

1. The order of convexity of the function

$$2 + 2 \frac{1 - z}{z} \log(1 - z) = z_2 F_1 (1, 1; 3; z)$$

is $\log 2/[2(2 \log 2 - 1)] \approx 0.8971$. 

(2) The order of convexity of the function
\[
\frac{4z}{(1 + \sqrt{1 - z})^2} = z_2 F_1(1, 3/2; 3; z)
\]
is \((1 + \sqrt{2})/4 \approx 0.6035\).

(3) The order of convexity of the function \(z_2 F_1(1, 1; 3/2; z)\) is \(-1/4\).

Remark 4.2 Putting \(c = 1 + a\) in Theorem 1.3, we arrive at the consequence that for \(a \geq 1\), the function
\[
z_2 F_1(a, 1; 1 + a; z) = \sum_{n=1}^{\infty} \frac{a}{a - 1 + n} z^n
\]
is convex of order \(1/(1 + a)\). Note that the convexity of this function is contained in Ruscheweyh [18] and Sugawa and the author [23]. In [23], the complex parameter \(a\) is considered. Although here we only deal with the real one, the explicit order of convexity is given.

In view of Theorem 1.3 and Lemmas 2.2 and 2.3, we obtain a result on the convexity of the shifted hypergeometric function.

Corollary 4.3 Assume \(a\) and \(c\) are real constants satisfying \(0 < a \leq c\), the function \(z_2 F_1(a, 1; c; z)\) is convex if one of the following conditions holds:

(1) \(0 \leq a \leq 4\) and \(c \geq 2\);

(2) \(1 \leq c < \min\{2, 1 + a\}\) and \(c \geq 3 - a\).

Proof In view of the different cases in Theorem 1.3, we divide the proof into two cases accordingly.

Case I: Let \(c \geq 2\). Theorem 1.3 together with Lemma 2.2 shows that the order of convexity of the function \(z_2 F_1(a, 1; c; z)\) satisfies
\[
\kappa \geq \frac{4 - a - c}{2} + \frac{c - 2}{2} \times \frac{2c}{2c - a} = \frac{(4 - a)c + a^2 - 4a}{2(2c - a)} \geq 0
\]
since \(0 < a \leq 4\) and \(a \leq c\).

Note that the case \(0 < a \leq 1\) has already been proved in Corollary 1.7.

Case II: Assume \(1 \leq c < \min\{2, 1 + a\}\), then we infer from Theorem 1.3 that the order of convexity is
\[
\kappa = \frac{(c - a)(a + c - 3)}{2(1 + a - c)}
\]
which is obviously non-negative if \(c \geq 3 - a\).

We verify all the cases of this corollary.
Recall that Hästö et al. in [7] proved that for non-zero real numbers $b$ and $c$, if $c \geq \max(3 - b, 3b)$, then $z_2 \text{F}_1(1, b; c; z)$ is a convex function. Therefore the above corollary partially generalizes their result.

**Example 4.4** Letting $(a, b, c) = (1/2, 1/2, 3)$ and $(a, b, c) = (1/2, 1/2, 2)$ in Corollary 1.7 and $(a, b, c) = (3/4, 3/2, 2)$ in Theorem 1.5 successively, we obtain that

1. The order of convexity of the functions $z_2 \text{F}_1(1/2, 1/2; 3; z)$ and $z_2 \text{F}_1(1/2, 1/2; 2; z)$ are at least 41/46 and 31/36 respectively.
2. The order of convexity of the function $z_2 \text{F}_1(3/4, 3/2; 2; z)$ is $-1/8$.

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