FINITENESS OF MAPPING DEGREES AND PSL(2, R)-VOLUME
ON GRAPH MANIFOLDS

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ABSTRACT. For given closed orientable 3-manifolds $M$ and $N$ let $\mathcal{D}(M, N)$ be the set of mapping degrees from $M$ to $N$. We address the problem: For which $N$, $\mathcal{D}(M, N)$ is finite for all $M$? The answer is known for prime 3-manifolds unless the target is a non-trivial graph manifold. We prove that for each closed non-trivial graph manifold $N$, $\mathcal{D}(M, N)$ is finite for all graph manifold $M$.

The proof uses a recently developed standard forms of maps between graph manifolds and the estimation of the $PSL(2, \mathbb{R})$-volume for certain class of graph manifolds.

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1. INTRODUCTION

Let $M$ and $N$ be two closed oriented 3-dimensional manifolds. Let $\mathcal{D}(M, N)$ be the set of degree of maps from $M$ to $N$, that is

$$\mathcal{D}(M, N) = \{d \in \mathbb{Z} | f: M \to N, \ deg(f) = d\}.$$

According to [CT], M. Gromov think it is a fundamental problem in topology to determine the set $\mathcal{D}(M, N)$. Indeed the supremum of absolute values of degrees in $\mathcal{D}(M, N)$ has been addressed by J. Milnor and W. Thutson in 1970’s [MT]. A basic property of $\mathcal{D}(M, N)$ is reflected in the following

Question 1. (see also [Re2 Problem A] and [W2 Question 1.3]): For which closed orientable 3-manifold $N$, the set $\mathcal{D}(M, N)$ is finite for all closed orientable 3-manifolds $M$?

This question can be interpreted as a way to detect some new rigidity properties of the geometry-topology of a manifold. More precisely, when $M$ is fixed, then one can expect that if the geometry-topology of a manifold $N$ is complicated, then the possible degree of maps $f: M \to N$ is essentially controled by the datas of $N$. For geometric 3-manifolds (i.e. 3-manifolds which admits a locally homogeneous complete Riemannian metric) the answser to this question is summarised in the following

Theorem 1.1 ([Th2], [BG1], [BG2], [W2]). Let $N$ denote a closed orientable geometric 3-manifold.

(i) If $N$ supports the hyperbolic or the $\widetilde{\text{PSL}}(2, \mathbb{R})$ geometry, then $\mathcal{D}(M, N)$ is finite for any $M$.

(ii) If $N$ admits one of the six remainder geometry, $S^3$, $S^2 \times \mathbb{R}$, Nil, $\mathbb{R}^3$, $H^2 \times \mathbb{R}$ or Sol then $\mathcal{D}(N, N)$ is infinite.

To study the set $\mathcal{D}(M, N)$ we need to introduced a special kind of 3-manifolds invariants. More precisely, we say that a non-negative 3-manifold invariant $\omega$ has degree property or simply Property D, if for any map $f: M \to N$, $\omega(M) \geq |\deg(f)|\omega(N)$. Say $\omega$ has covering property or simply Property C, if for any covering $p: M \to N$, $\omega(M) = |\deg(p)|\omega(N)$. The invariants with Property D are important to study Question 1 due to the following fact (see Lemma 3.1):

Fact (*) If $\omega$ has property $D$ and if $N$ admits a finite covering $\tilde{N}$ such that $\omega(\tilde{N}) \neq 0$ then the set $\mathcal{D}(M, N)$ is finite for all $M$.

When $N$ is hyperbolic, the finiteness of the set $\mathcal{D}(M, N)$ is essentially controled by the volume associated to the Riemannian metric with constant negative sectional curvature which satisfies Property D. When $N$ admits a $\widetilde{\text{PSL}}(2, \mathbb{R})$ geometry $\mathcal{D}(M, N)$ is essentially controled by the $\text{PSL}(2, \mathbb{R})$-volume $\text{SV}$ introduced in [BG1] and [BG2]: it satisfies Property D and it is non-zero for Seifert manifolds supporting $\text{PSL}(2, \mathbb{R})$ geometry.

To study Question 1 for more general manifolds, M. Gromov introduced in [G] the simplicial volume $\|N\|$ of a manifold $N$. This invariant always satisfies Property D. For example, using the simplicial volume and the work of Connel Farb in...
Lafont and Schmidt have generalized in \cite{LS} point (i) of Theorem 1.1 when the target manifold \( N \) is a closed locally symmetric \( n \)-manifold with nonpositive sectional curvature. However, a closed locally symmetric manifold is a special class of complete locally homogeneous manifolds and thus Question 1 is still open for non-geometric manifolds with zero simplicial volume.

In this paper we focus on closed 3-manifolds. Recall that according to the Perelman geometrization Theorem, 3-manifolds with zero Gromov simplicial volume are precisely graph manifolds. We call that a 3-manifold covered by either a torus bundle, or a Seifert manifold is a trivial graph manifold. Hence for prime 3-manifolds, Question 1 is reduced to

**Question 2.** Suppose \( N \) is a non-trivial graph manifold, is \( D(M, N) \) finite for all closed orientable 3-manifolds \( M \)?

The main difficulty to study Question 2 for a non-trivial graph manifold \( N \) is to find a 3-manifold invariant satisfying property \( D \) which does not vanish of \( N \). Based on Fact (*), it is natural to ask

**Question 3.** Let \( N \) be a closed orientable non-trivial graph manifold. Does \( SV(\tilde{N}) \neq 0 \) for some finite covering \( \tilde{N} \) of \( N \)?

The \( \tilde{PSL}(2, \mathbb{R}) \)-volume is rather strange and was very little known. It can be either zero or non-zero for hyperbolic 3-manifolds \cite{BG1}; if it has Property \( C \) is still unclear, and it was not addressed for non-geometric 3-manifolds since it was introduced more than 20 years ago.

A main result of this paper is a partial answer of Question 3 we verify that for a family of non-geometric graph manifolds \( N \), they do have finite cover \( \tilde{N} \) with \( SV(\tilde{N}) \neq 0 \) (Proposition 4.1). Such a partial answer, combined with the standard form of nonzero degree maps developed in \cite{D1}, enable us to solve Question 2 when we restrict on graph manifolds.

**Theorem 1.2.** For any given closed non-trivial prime graph manifold \( N \), \( D(M, N) \) is finite for any graph manifold \( M \).

**Remark 1.3.** Some facts related to Theorem 1.2 are known before: \( D(N, N) \) is finite for any prime non-trivial graph manifold \( N \) \cite{W1}, see also \cite{D2}). The covering degrees is uniquely determined by the graph manifolds involved \cite{YW}.

This paper is organized as follows.

In Section 2 we define the objects which will be used in the paper: For graph manifolds, we will define their coordinates and gluing matrices, canonical framings, the standard forms of nonzero degree maps, the absolute Euler number and the absolute volume. We also recall \( \tilde{PSL}(2, \mathbb{R}) \)-volume and its basic properties.

In section 3 we state and prove some results on coverings of graph manifolds which will be used in the paper.

Section 4 is devoted to the proof of Proposition 4.1. The strategy is to use a finite sequence of coverings to get a very "large" and "symmetric" covering space which allows some free action of a finite cyclic group so that the quotient can be sent onto a 3-manifold supporting \( \tilde{PSL}(2, \mathbb{R}) \) geometry via a nonzero degree map.
In Section 5 we prove Theorem 1.2. The strategy is to use the standard form of nonzero degree maps between graph manifolds to show that one can reduce the problem to the case where the target is a graph manifold satisfying the hypothesis of Proposition 4.1.

2. Notations and known results

From now on all 3-manifolds are irreducible and oriented, and all graph manifolds are non-trivial.

Suppose \( F \) (resp. \( P \)) is a properly embedded surface (resp. an embedded 3-manifold) in a 3-manifold \( M \). We use \( M \setminus F \) (resp. \( M \setminus P \)) to denote the resulting manifold obtained by splitting \( M \) along \( F \) (resp. removing \( \text{int} P \), the interior of \( P \)).

2.1. Coordinated graph manifolds and gluing matrices. Let \( N \) be a graph manifold. Denote by \( T_N \) the family of JSJ tori of \( N \), by \( N^\ast \) the set \( N \setminus T_N = \{ \Sigma_1, ..., \Sigma_n \} \) of the JSJ pieces of \( N \), by \( \tau: \partial N^\ast \to \partial N^\ast \) the associated sewing involution defined in [JS].

A dual graph of \( N \), denoted by \( \Gamma_N \), is given as follows: each vertex represents a JSJ piece of \( N \); each edge represents a JSJ torus of \( N \); an edge \( e \) connects two vertices \( v_1 \) and \( v_2 \) (may be \( v_1 = v_2 \)) if and only if the corresponding JSJ torus is shared by the corresponding JSJ pieces.

Call a dual graph \( \Gamma_N \) directed if each edge of \( \Gamma_N \) is directed, in other words, is endowed with an arrow. Once \( \Gamma_N \) is directed, the sewing involution \( \tau \) becomes a well defined map, still denoted by \( \tau: \partial N^\ast \to \partial N^\ast \).

Suppose \( N^\ast \) contains no pieces homeomorphic to \( I(K) \), the twisted \( I \)-bundle over the Klein bottle.

Let \( \Sigma \) be an oriented Seifert manifold which admits a unique Seifert fibration, up to isotopy, and \( \partial \Sigma \neq \emptyset \). Denote by \( h \) the homotopy class of the regular fiber of \( \Sigma \), by \( O \) the base 2-orbifold of \( \Sigma \) and by \( \Sigma^0 \) the space obtained from \( \Sigma \) after removing the singular fibers of \( \Sigma \). Then \( \Sigma^0 \) is a \( S^1 \)-bundle over a surface \( O^0 \) obtained from \( O \) after removing the exceptional points. Then there exists a cross section \( s: O^0 \to \Sigma^0 \). Call \( \Sigma \) is coordinated, if

1. such a a section \( s: O^0 \to \Sigma^0 \) is chosen,
2. both \( \partial O^0 \) and \( h \) are oriented so that their product orientation is matched with the orientation of \( \partial \Sigma \) induced by that of \( \Sigma \).

Once \( \Sigma \) is coordinated, then the orientation on \( \partial O^0 \) and the oriented fiber \( h \) gives a basis of \( H_1(T; \mathbb{Z}) \) for each component \( T \) of \( \partial \Sigma \). We also say that \( \Sigma \) is endowed with a \( (s, h) \)-basis.

Since \( N^\ast \) has no \( I(K) \)-components then each component \( \Sigma_i \) of \( N^\ast \) admits a unique Seifert fibration, up to isotopy. Moreover each component \( \Sigma_i \) has the orientation induced from \( N \). Call \( N \) is coordinated, if each component \( \Sigma_i \) of \( N^\ast \) is coordinated and \( \Gamma_N \) is directed.

Once \( N \) is coordinated, then each torus \( T \) in \( T_N \) is associated with a unique \( 2 \times 2 \)-matrix \( A_T \) provided by the gluing map \( \tau|: T_-(s_-, h_-) \to T_+(s_+, h_+) \): where \( T_-, T_+ \) are two torus components in \( \partial N^\ast \) provided by \( T \), with basis \( (s_-, h_-) \) and
Denote by Proposition 2.2. are proved in [D1] in a more general case. The first result is related to the standard Lemma 2.1.

For a given closed graph manifold \( L \) the family canonical framed canonical submanifolds 2.3.:

\[
\tau(s_-, h_-) = (s_+, h_+)A_T.
\]

Call \( \{A_T, T \in T\} \) the gluing matrices.

2.2. Canonical framings and canonical submanifold. Let \( \Sigma \) denote an orientable Seifert manifold with regular fiber \( h \). A framing \( \alpha \) of \( \Sigma \) is to assign a simple closed essential curve not homotopic to the regular fiber of \( \Sigma \), for each component \( T \) of \( \partial \Sigma \). Denote by \( \Sigma(\alpha) \) the closed Seifert 3-manifold obtained from \( \Sigma \) after Dehn fillings along the family \( \alpha \) and denote by \( \pi_\Sigma \colon \Sigma \to \Sigma(\alpha) \) the natural quotient map. Let \( p \colon \hat{\Sigma} \to \Sigma \) be a finite covering. Assume that \( \Sigma \) and \( \hat{\Sigma} \) are endowed with a framing \( \alpha \) and \( \hat{\alpha} \). Then we say that \( (\Sigma, \hat{\alpha}) \) covers \( (\Sigma, \alpha) \) if each component of \( \hat{\alpha} \) is a component of \( p^{-1}(\alpha) \). In this case, the map \( p \colon (\hat{\Sigma}, \hat{\alpha}) \to (\Sigma, \alpha) \) extends to a map \( \hat{p} \colon \hat{\Sigma}(\hat{\alpha}) \to \Sigma(\alpha) \) and the Euler number of \( \Sigma(\alpha) \) is nonzero iff the Euler number of \( \hat{\Sigma}(\hat{\alpha}) \) is nonzero by [LW]. When \( N \) contains no \( I(K) \)-component in its JSJ-decomposition, each Seifert piece \( \Sigma \) of \( N^* \) is endowed with a canonical framing \( \alpha_\Sigma \) given by the regular fiber of the Seifert pieces of \( N^* \) adjacent to \( \Sigma \). Denote by \( \hat{\Sigma} \) the space \( \Sigma(\alpha_\Sigma) \). By minimality of JSJ decomposition, \( \hat{\Sigma} \) admits a unique Seifert fibration extending that of \( \Sigma \).

Call a submanifold \( L \) of a graph manifold \( N \) is canonical if \( L \) is a union of some components of \( N \setminus T \), where \( T \) is subfamily of \( T_N \). Similarly call \( \alpha_L = \{t_U \subset U\} \) where \( t_U \) is the regular fiber of the Seifert piece adjacent to \( L \) along the component \( U \), when \( U \) runs over the components of \( \partial L \), the canonical framing of \( L \), and denote by \( \hat{L} \) the closed graph 3-manifold obtained from \( L \) after Dehn fillings along the family \( \alpha_L \). From the definition we have

**Lemma 2.1.** For a given closed graph manifold \( M \), there are only finitely many canonical framed canonical submanifolds \( (L, \alpha_L) \), and thus only finitely many \( \hat{L} \).

2.3. Standard forms of nonzero degree maps. We recall here two results which are proved in [D1] in a more general case. The first result is related to the standard forms of nonzero degree maps.

**Proposition 2.2.** ([D1] Lemma 3.4) For a given closed graph manifold \( M \), there is a finite set \( \mathcal{H} = \{M_1, \ldots, M_k\} \) of closed graph manifolds satisfying the following property: for any nonzero degree map \( g \colon M \to N \) into a closed non-trivial graph manifold \( N \) without \( I(K) \) piece in \( N^* \), there exists some \( M \) in \( \mathcal{H} \) and a map \( f \colon M_i \to N \) such that:

(i) \( \deg(f) = \deg(g) \),

(ii) for each piece \( Q \) in \( N^* \), \( f^{-1}(Q) \) is a canonical submanifold of \( M \).

The following technical "mapping lemma" will be also useful:

**Lemma 2.3.** ([D1] Lemma 4.3) Suppose \( f \colon M \to N \) is a map between closed graph manifolds and \( N^* \) contains no \( I(K) \) piece. Let \( S \) and \( S' \) be two components of \( M^* \) which are adjacent in \( M \) along a subfamily \( T \) of \( T_M \) and satisfy:

(i) \( f(S') \subset \text{int}(\Sigma') \) for some piece \( \Sigma' \) of \( N^* \),

(ii) \( f_*([h_S]) \neq 1 \), where \( t_S \) is the regular fiber of \( S \).
Then there exists a piece $\Sigma$ of $N^*$ and a homotopy of $f$ supported in a regular neighborhood of $S$ such that $f(S) \subset \text{int}(\Sigma)$. Moreover if $f(h_S)$ is not homotopic to a non-trivial power of the regular fiber of $\Sigma$, then one can choose $\Sigma = \Sigma'$.

[D1, Lemma 4.3] was stated for Haken manifolds. Since here we consider only non-trivial graph manifolds instead of Haken manifolds, then we can state [D1, Lemma 4.3] in term of the JSJ-pieces of $N$ instead of in term of the characteristic Seifert pair of $N$.

2.4. PSL(2, $\mathbb{R}$)-volume, absolute volume, and absolute Euler number.

PSL(2, $\mathbb{R}$)-volume $SV$ is introduced in [BG1] and [BG2]. (It is also considered as a special case of volumes of representations, see [Re1], [Re2] and [WZ]). Two basic properties of $SV$ are reflected in the following

Lemma 2.4. (i) $SV$ has Property D [BG1].

(ii) If $N$ supports PSL(2, $\mathbb{R}$) geometry, i.e., $N$ admits a Seifert fibration with nonzero Euler number $e(N)$ and whose base 2-orbifold $O_N$ has a negative Euler characteristic, then [BG2]

$$SV(N) = \left| \frac{\chi^2(O_N)}{e(N)} \right|.$$  

When $N$ is a closed graph manifold with no $I(K)$ piece in $N^*$, using the notations introduced in Section 2.2, one can define the so-called absolute volume $|SV|$ by setting

$$|SV|(N) = \sum_{\Sigma \in N^*} SV(\hat{\Sigma}).$$

In the same way one can define the absolute Euler number of $N$ by setting

$$|e|(N) = \sum_{\Sigma \in N^*} |e(\hat{\Sigma})|.$$  

In Section 3.3 we will study the relations between $|e|(N)$ and $|SV|(N)$ (see Lemma 3.6).

3. REDUCTION OF COMPLEXITY VIA COVERINGS

In this section we state some results on finite coverings of surfaces and 3-manifolds which will be used in the proofs of Proposition 4.1 and Theorem 1.2.

3.1. Two general statements. The first result says that to prove the finiteness of the set $\mathcal{D}(M, N)$ one can replace $N$ by a finite covering of it.

Lemma 3.1. (1) Let $M$ and $N$ denote two closed oriented manifolds of the same dimension and let $p : N' \rightarrow N$ be a finite covering of $N$. If $\mathcal{D}(P, N')$ is finite for any closed manifold $P$ then the set $\mathcal{D}(M, N)$ is finite.

(2) If all manifolds involved in (1) are graph manifolds, then the conclusion of (1) is still hold.
Proof: (1) For each nonzero degree map \( f : M \rightarrow N \), let \( M(f) \) be the connected covering space of \( M \) corresponding to the subgroup \( f^{-1}(p_*(\pi_1 N')) \) of \( \pi_1 M \) which we denote by \( r : M(f) \rightarrow M \). Let \( f' : M(f) \rightarrow N' \) be a lift of \( f \), then \( p \circ f' = f \circ r \). We claim that the set \( C = \{ M(f), \text{ when } f \text{ runs over the nonzero degree maps from } M \text{ to } N \} \)
is finite. To see this, first note that the index of \( f^{-1}(p_*(\pi_1 N')) \) in \( \pi_1 M \) is bounded by the index of \( p_*(\pi_1 N') \) in \( \pi_1 N \). Indeed, the homomorphism \( f_* : \pi_1 M \rightarrow \pi_1 N \)
descends through an injective map \( \tilde{f}_* : \pi_1 M \rightarrow \pi_1 N \).

Since \( \pi_1 M \) contains at most finitely many subgroup of a bounded index, it follows that \( M(f) \) has only finitely many choices which proves that the set \( C \) is finite. By the construction we have

\[
\deg(f) = \frac{\deg(p)}{\deg(r)} \deg(f').
\]

By the finiteness of the set \( C \) and assumption on \( N' \), the set of \( \{ \deg(f') | f : M' \rightarrow N', M' \in C \} \) is finite. Clearly \( \deg(r) \) have only finitely many choice, so the lemma is proved.

(2) If \( M \) and \( N \) are graph manifolds, then all manifolds \( M(f), N' \) in the proof of (1) are graph manifolds. Clearly (2) follows. \( \square \)

Lemma 3.2. Let \( N \) be a closed 3-manifold with non-trivial JSJ-decomposition. Then there exists a 2-fold covering \( \tilde{N} \) of \( N \) such that each JSJ-torus of \( \tilde{N} \) is shared by two different pieces of \( \tilde{N}^* \).

Proof: Let \( \{ T_1, ..., T_k \} \) be the union JSJ-tori of \( N \) with each \( T_i \) is shared by the same piece of \( N^* \). Let \( e_1, ..., e_k \) be the corresponding edges in \( \Gamma_N \). Then \( e_1, ..., e_k \) are the edges of \( \Gamma_N \) with the two ends of each \( e_i \) being at the same vertex. Clearly \( H_1(\Gamma_N; \mathbb{Z}) = \langle e_1 \rangle \oplus ... \oplus \langle e_k \rangle \oplus G \).

Let \( r : N \rightarrow \Gamma_N \) be the retraction. Consider the following epimorphism

\[
\phi : H_1(N; \mathbb{Z}) \overset{r_*}{\rightarrow} H_1(\Gamma_N; \mathbb{Z}) \overset{q}{\rightarrow} \langle e_1 \rangle \oplus ... \oplus \langle e_k \rangle \overset{\lambda}{\rightarrow} \mathbb{Z}/2\mathbb{Z}
\]
where $r_*$ is induced by $r$, $q$ is the projection, and $\lambda$ is defined by $\lambda([e_i]) = T$ for $i = 1, \ldots, k$. Then the double covering $\tilde{N}$ of $N$ corresponding to $\phi$ satisfies the conclusion of the lemma, since the double covering of $\Gamma_N$ corresponding to $\lambda \circ q$, which is the dual graph of $\tilde{N}$, contains no edge with two ends being at the same vertex. See Figure 1 for the local picture.

3.2. Separable and characteristic coverings. Let $N$ be a closed graph manifold without $I(K)$ JSJ-piece. Let $T$ be a union of tori and let $m$ be a positive integer. Call a covering $p : \tilde{T} \to T$ $m$-characteristic if for each component $T$ of $T$ and for each component $\tilde{T}$ of $\tilde{T}$ over $T$, the restriction $p| : \tilde{T} \to T$ is the covering map associated to the characteristic subgroup of index $m \times m$ in $\pi_1T$. Call a finite covering $\tilde{N} \to N$ of a graph manifold $N$ $m$-characteristic if its restriction to $\tilde{T}_N \to T_N$ is $m$-characteristic.

Next we define the separable coverings. Let $\Sigma$ be a component of $N^*$ with base 2-orbifold $O$. Let $\Sigma^0$, $O^0$, and $s : O^0 \to \Sigma^0$ are be given as in Section 2.1. Let $p : \tilde{\Sigma} \to \Sigma$ denote a finite covering. Recall that $p$ is a fiber preserving map.

Recall that the vertical degree of $p$ is the integer $d_v$ such that $p_*(\bar{h}) = h^{d_v}$, where $h$ and $\bar{h}$ denote the homotopy class of the regular fiber in $\Sigma$ and $\tilde{\Sigma}$, and the horizontal degree $d_h$ is the degree of the induced branched covering $\bar{p} : \tilde{O} \to O$, where $\tilde{O}$ denotes the base of the bundle $\tilde{\Sigma}$. We have $\deg(p) = d_v \times d_h$.

On the other hand, $p$ induces a finite covering $p| : \tilde{\Sigma}^0 = p^{-1}(\Sigma^0) \to \Sigma^0$ and a covering $p|_b : \tilde{\mathcal{F}}^0 \to O^0$, with $\tilde{\mathcal{F}}^0$ connected. More precisely, $p|_b$ corresponds to the subgroup $s_*^{-1}((p|)_*(\pi_1\Sigma^0))$. Note that $p$ and $p|$ have the same degree, same vertical degree and same horizontal degree. If $\deg(p|_b) = d_h$, then we say that the covering $p$ is separable. The following result provides two classes of separable coverings which will be used later.

**Lemma 3.3.** Let $p : \tilde{\Sigma} \to \Sigma$ be an oriented Seifert manifold finite covering.

(i) If $p$ has fiber degree one, then $p$ is a separable covering.

(ii) If $\Sigma = F \times S^1$ and $p$ is a regular covering corresponding to an epimorphism $\phi : \pi_1N = \pi_1F \times \mathbb{Z} \to G = G_1 \times G_2$ satisfying $\phi(\pi_1F \times \{1\}) = G_1$ and $\phi(\{1\} \times \mathbb{Z}) = G_2$ then $p$ is separable.

**Proof.** Using the same notations as above it is easy to see that the map $p|_b : \tilde{\mathcal{F}}^0 \to O^0$ factors through covering maps $q : \tilde{\mathcal{F}}^0 \to \tilde{\mathcal{O}}^0$ and $\bar{p} : \tilde{\mathcal{O}}^0 \to \mathcal{O}^0$ where $\tilde{\mathcal{O}}^0$ denote the base of the bundle $\tilde{\Sigma}^0$. Then we get

$$\deg(p|_b) = d_h \times \deg(q) = \deg(p) \times \deg(q)$$

since $p$ has vertical degree one. On the other hand, since $\deg(p|_b) \leq \deg(p)$ then $\deg(q) = 1$. This proves (i).

If $\Sigma$ is homeomorphic to a product $F \times S^1$ then we have the following commutative diagram

$$
\begin{array}{ccc}
\tilde{F} & \xrightarrow{s} & \tilde{\Sigma} \\
\downarrow{p_b} & & \downarrow{p} \\
F & \xrightarrow{s} & \Sigma
\end{array}
$$
where \( \tilde{F} \) is connected. Since \( \phi(\pi_1 F \times \{1\}) = G_1 \) then \( p^{-1} (s(F)) \) has \( |G_2| \) components and thus \( \text{deg}(p) = \text{deg}(p_b) \times |G_2| \). Since \( \text{deg}(p) = |G_1| \times |G_2| \) then \( \text{deg}(p_b) = |G_1| = d_h \). This proves (ii). \( \square \)

3.3. **Lifting of coordinates and gluing matrices.** From now on we assume the graph manifold \( N \) is coordinated. Let \( p : \tilde{N} \rightarrow N \) be a finite covering of graph manifolds. Then obviously \( \Gamma_{\tilde{N}} \) can be directly in a unique way such that the induced map \( p^\# : \Gamma_{\tilde{N}} \rightarrow \Gamma_N \) preserves the directions of the edges. Below we also assume that \( \Gamma_{\tilde{N}} \) is directed in such a way.

Let \( p : \tilde{N} \rightarrow N \) be a finite covering of graph manifolds. Call \( p \) is separable if the restriction \( p| : \tilde{\Sigma} \rightarrow \Sigma \) on connected Seifert pieces is separable for all possible \( \tilde{\Sigma} \) and \( \Sigma \). Call a coordinate on \( \tilde{N} \) is a lift of the coordinate of \( N \), if for each possible covering \( p| : \tilde{\Sigma} \rightarrow \Sigma \) on connected Seifert pieces, the \((s, h)\)-basis of \( \tilde{\Sigma} \) is lifted from the \((s, h)\)-basis of \( \Sigma \).

**Lemma 3.4.** (i) Let \( p : \tilde{N} \rightarrow N \) be a separable finite covering of graph manifolds. Then the coordinate of \( N \) can be lifted on \( \tilde{N} \).

(ii) Moreover, if the covering \( p \) is characteristic, then for each component \( T \) of \( T_N \) and for each component \( \tilde{T} \) over \( T \) we have \( A_T = \tilde{A}_T \), where the coordinate of \( \tilde{N} \) is lifted from \( N \).

**Proof.** To prove (i), one need only to show that for a separable finite covering \( p : \tilde{\Sigma} \rightarrow \Sigma \) of connected Seifert piece, then any \((s, h)\)-basis of \( \tilde{\Sigma} \) lifts to a \((s, h)\)-basis of \( \Sigma \).

Using the same notation as in the proof of Lemma 3.3 we have

\[
\text{deg}(p|_b) = d_h \times \text{deg}(q)
\]

Since we assume \( \text{deg}(p|_b) = d_h \), then \( \text{deg}(q) = 1 \) and thus \( \tilde{s} \) is a cross section. This proves (i).

Once \( N \) is coordinated, then each torus \( T \) in \( T_N \) is associated with a unique \( 2 \times 2 \)-matrix \( A_T \) provided by the gluing map \( \tau : T_- (s_-, h_-) \rightarrow T_+ (s_+, h_+) \) such that \( \tau(s_-, h_-) = (s_+, h_+) \tilde{A}_T \).

Similarly with lifted coordinate on \( \tilde{N} \) we have \( \tilde{\tau} : \tilde{T}_- (\tilde{s}_-, \tilde{h}_-) \rightarrow \tilde{T}_+ (\tilde{s}_+, \tilde{h}_+) \) and \( \tau(\tilde{s}_-, \tilde{h}_-) = (\tilde{s}_+, \tilde{h}_+) \tilde{A}_T \).

Since the coordinate of \( \tilde{N} \) are lifted from \( N \), and \( p \) is \( m \)-characteristic for some \( m \), we have the following commutative diagram

\[
\begin{array}{ccc}
(\tilde{s}_-, \tilde{h}_-) & \xrightarrow{\tilde{A}_T} & (\tilde{s}_+, \tilde{h}_+) \\
\times m & & \times m \\
(s_-, h_-) & \xrightarrow{A_T} & (s_+, h_+)
\end{array}
\]

Then one verifies directly that \( \tilde{A}_T = A_T \). This proves (ii). \( \square \)
3.4. The absolute volume and the absolute Euler number. We end this section with a result (see Lemma [3.6]) which states the relation between the absolute volume and the absolute Euler number of a graph manifold. First we begin with a technical result.

Lemma 3.5. Suppose $N$ is a closed graph manifolds without $I(K)$ JSJ-piece.

(i) For any finite covering $\tilde{N} \to N$, $|e|(\tilde{N}) = 0$ if and only if $|e|(N) = 0$.

(ii) There is a finite covering $p: \tilde{N} \to N$ which is separable and characteristic, and each Seifert piece of $\tilde{N}$ is the product of a surface of genus at least 2 and the circle. Moreover $\tilde{N}$ may be chosen so that $\Gamma_{\tilde{N}}$ has two vertices if $\Gamma_N$ has two vertices.

Proof. (i) follows from the definition and [LW, Proposition 2.3].

(ii) It has been proved in [LW, Proposition 4.4], that there is a characteristic finite covering $p: \tilde{N} \to N$ whose each piece is the product of a surface and the circle. By checking the proof, it is easy to see that the condition ”genus at least 2” can be satisfied; moreover the restriction $p|: \tilde{\Sigma} \to \Sigma$ on connected JSJ-pieces is a composition of separable coverings described in Lemma 3.3 which is still separable.

If moreover $\Gamma_N$ has exactly two vertices $\Sigma_1$ and $\Sigma_2$, then for $i = 1, 2$, denote by $p_i: \tilde{\Sigma}_i \to \Sigma_i$ the $m$-characteristic separable finite covering such that $\tilde{\Sigma}_i$ is the product of a surface of genus at least 2 and the circle. There exists a 1-characteristic finite covering $q_i: \tilde{\Sigma}_i \to \Sigma_i$ such that $\partial \tilde{\Sigma}_1$ and $\partial \tilde{\Sigma}_2$ have the same number of components. Next one can glue $\tilde{\Sigma}_1$ and $\tilde{\Sigma}_2$ by the lift of the sewing involution of $N$ to get a characteristic and separable finite covering $p: \tilde{N} \to N$ whose dual graph has two vertices. This completes the proof of the lemma.

Lemma 3.6. Let $N$ be a closed graph manifold without $I(K)$ JSJ-pieces.

(i) If $|e|(N) \neq 0$ then $N$ admits a finite covering $\tilde{N}$ with $|SV|(\tilde{N}) \neq 0$.

(ii) If $|e|(N) = 0$ then $N$ admits a finite covering $\tilde{N}$ which can be coordinated such that each gluing matrix is in the form $\pm \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

Proof. By Lemma 3.5(ii) let $p: \tilde{N} \to N$ be a finite covering which is separable and characteristic and each piece of $\tilde{N}^*$ is a product $F \times S^1$ with $g(F) \geq 2$.

By Lemma 3.5(i) $|e|(N) \neq 0$ implies $|e|(\tilde{N}) \neq 0$. By definition of $|e|$, $e(\tilde{\Sigma}) \neq 0$ for some $\Sigma = F \times S^1 \in \tilde{N}^*$. Since $g(F) \geq 2$, $SV(\Sigma) \neq 0$, and hence $|SV|(\tilde{N}) \neq 0$ by definition in Section 2.3. This proves (i).

Denote by $\Sigma_1, ..., \Sigma_n$ the components of $N^*$. For each $i = 1, ..., n$, denote by $(\Sigma_i, \alpha_i)$ the Seifert piece $\Sigma_i$ of $N^*$ endowed with its canonical framing. Since $e(N) = 0$ then $e(\Sigma_i(\alpha_i)) = e(\tilde{\Sigma}_i) = 0$ and thus there exists a finite covering of $\Sigma_i$, with fiber degree one, homeomorphic to a product. By pulling back this covering via the quotient map $\pi: \Sigma_i \to \tilde{\Sigma}_i$ we get a covering $\tilde{\Sigma}_i$ of $\Sigma_i$ such that the framing $(\Sigma_i, \tilde{\alpha}_i)$ satisfies the following condition: there exists a properly embedded incompressible surface $F_i$ in $\tilde{\Sigma}_i$ such that $\tilde{\Sigma}_i \simeq F_i \times S^1$ and $\partial F_i = \tilde{\alpha}_i$. 

Suppose $T$ is a component of $\partial \Sigma_i$ and $T'$ is a component of $\partial \Sigma_j$ such that $T$ is identified to $T'$ then the sewing involution $\tau|T : T \to T'$ lifts to a sewing involution $\tilde{\tau} : \tilde{T} \to \tilde{T}'$, where $\tilde{T}$, resp. $\tilde{T}'$, denotes a component of $\partial \tilde{\Sigma}_i$, resp. a component of $\partial \tilde{\Sigma}_j$, over $T$, resp. $T'$. Indeed by our construction the induced coverings $\tilde{T} \to T$ and $\tilde{T}' \to T'$ correspond exactly to the subgroup of $\pi_1 T$, resp. of $\pi_1 T'$, generated by $h$ and $h'$, where $h$ is the fiber of $\Sigma_i$ represented in $T$ and $h'$ is the fiber of $\Sigma_j$ represented in $T'$, hence the gluing map lifts by the lifting criterion.

Denote by $\eta_i$ the degree of the covering map $\tilde{\Sigma}_i \to \Sigma_i$. Let

$$\eta = \text{l.c.m.}\{\eta_1, ..., \eta_n\}$$

For each $i = 1, ..., n$, take $t_i = \frac{n}{\eta_i}$ copies of $\tilde{\Sigma}_i$ and glue the components of

$$\prod_{i=1}^{m} \left(t_i\text{ copies of } \tilde{\Sigma}_i\right)$$

together via lifts of the sewing involution $\tau$ of $N$ to get a separable finite covering $p : \tilde{N} \to N$. By coordinating each piece $\tilde{\Sigma}_i$ of $\tilde{N}^*$ with such a section $F_i$ and its regular fiber, $\tilde{N}$ is coordinated. Clearly each component of $\partial F_i$ is identified with the regular fiber of its adjacent piece and vice versa. Therefore each gluing matrix should be in the form of

$$\begin{pmatrix} 0 & \pm 1 \\ \pm 1 & 0 \end{pmatrix}.$$  

Since the determinant should be $-1$, therefore the gluing matrix is in the form of $\pm \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. This proves point (ii). □

4. $\widetilde{PSL(2,\mathbb{R})}$-VOLUME OF GRAPH MANIFOLDS

Let $N$ be a closed graph manifold which consists of two JSJ-pieces $\Sigma_1$ and $\Sigma_2$ and $n$ JSJ-tori $\{T_1, ..., T_n\}$, moreover no $\Sigma_i$ is $I(K)$ and each $T_i$ is shared by both $\Sigma_1$ and $\Sigma_2$. Call such a manifold $n$-multiple edges graph manifold, whose dual graph $\Gamma_N$ is shown in Figure 2. We assume $\Gamma_N$ is also directed as in Figure 2. In this section we use $A_i$ for $A_{T_i}$ for short.

![Figure 2: Multiple edges graph manifold](image)

\[ T_1 \to T_2 \to \cdots \to T_n \]

\[ \Sigma_1 \to \Sigma_2 \]
Proposition 4.1. Let $N$ be a $n$-multiple edges graph manifold which is coor-
inated. Assume that the gluing matrices of $N$ satisfy the condition $A_1 = \pm A_2 = \ldots = \pm A_n$. Then $N$ admits a finite covering space $\tilde{N}$ such that $SV(\tilde{N}) \neq 0$.

Corollary 4.2. Suppose $N$ is a closed graph manifold whose dual graph has two vertices and one edge. Then $|D(M, N)|$ is finite for any 3-manifold $M$.

Proof. We may suppose that $N$ contains no $I(K)$ piece. Otherwise $N$ is doubly covered by a non-trivial graph manifold which contains no $I(K)$ piece and whose dual graph still has two vertices and one edge (since we assume that $N$ is non-trivial graph manifold). In any case $N$ has a finite cover $\tilde{N}$ such that $SV(\tilde{N}) \neq 0$ by Proposition 4.1. Then by Lemmas 2.4 and 3.1 $|D(M, N)|$ is finite for any 3-manifold $M$. □

The proof of Proposition 4.1 follows from the a sequence of lemmas below.

Let $N$ be a $n$-multiple edges coordinated graph manifold. We say that $N$ satisfies the Property $I$ if

1. The JSJ-piece $\Sigma_i$ is homeomorphic to a product $F_i \times S^1$ where $F_i$ is an oriented surface with genus $\geq 2$, for $i = 1, 2$;
2. $A_1 = A_2 = \ldots = A_n$.

Lemma 4.3. Let $N$ be a $n$-multiple edges graph manifold satisfying the assumption of Proposition 4.1. Then there exist separable and characteristic finite coverings $p_1: N_1 \to N$ and $p_2: N_1 \to N_2$ such that $N_2$ satisfies Property $I$.

Proof. By Lemma 3.5 (ii) and Lemma 3.4, we may assume that $N$ is a $n$-multiple edges graph manifold satisfying the assumption of Proposition 4.1 and moreover $\Sigma_i$ is homeomorphic to a product $F_i \times S^1$ where $F_i$ is an oriented surface with genus $\geq 2$. May assume that $A_1 = \ldots = A_k = -A$ and $A_{k+1} = \ldots = A_n = A$, $0 < k < n$, shown as in the left of Figure 3.

Denote by $c_{i,j}$ the loops of $\Gamma_N$ corresponding to the "composition" $T_i, (-T_j)$, note that here $T_i$ represents an oriented edge. Then $c_{i,k+1}$ for $i = 1, \ldots, k$ and $c_{j,n}$ for $j = k+1, \ldots, n-1$ form a basis of $H_1(\Gamma_N)$ and we have

$$H_1(\Gamma_N) = (\oplus_{i=1}^{k} \langle c_{i,k+1} \rangle) \oplus (\oplus_{j=k+1}^{n-1} \langle c_{j,n} \rangle)$$

Next we define an epimorphism

$$\phi: H_1(N, Z) \overset{r_*}{\to} H_1(\Gamma_N; Z) \overset{q}{\to} \oplus_{i=1}^{k} \langle c_{i,k+1} \rangle \overset{\lambda}{\to} Z/2Z$$

where $r_*$ is induced by the retraction $r: N \to \Gamma_N$, $q$ is the projection and $\lambda$ is defined by $\lambda(c_{i,k+1}) = 1$ for any $i \in \{1, \ldots, k\}$. Denote by $p_1: N_1 \to N$ the 2-fold covering corresponding to $\phi$, and by $\mu$ the deck transformation of this covering.

It is easy to see this covering is separable and 1-characteristic. Moreover with the lifted coordinates of $N$, the directed graph $\Gamma_{N_1}$ with gluing matrices $\pm A$, as well as the two lifts $\Sigma^1_i$ and $\Sigma^2_i$ of $\Sigma_i$, $i = 1, 2$, are shown in the right of Figure 3.
Let \( \Sigma_i^j = F_i^j \times S^1 \). It is not difficult to see that there is an orientation preserving involution \( \eta^j_i \) on \( \Sigma_i^j \) satisfying the following

1. for \( \eta^j_i \) reverses both the orientation of \( F_i^j \) and \( S^1 \),
2. for each coordinated component \((T, (s, h))\) of \( \partial \Sigma_i^j \), \( \eta^j_i((T, (s, h))) = (T, (-s, -h)) \).

Then all those \( \eta^j_i \), \( i, j = 1, 2 \) match together to get an involution \( \eta \) on \( N_1 \).

Keep the coordinate of \( \Sigma_i^1 \) for \( i = 1, 2 \), and re-coordinate \( \Sigma_i^2 \) for \( i = 1, 2 \) by \((T, (-s, -h))\) for each component of \( \partial \Sigma_i^2 \) for \( i = 1, 2 \), and denoted new coordinated graph manifold denoted by \( N'_1 \) (\( N'_1 \) is \( N_1 \) if we forget their coordinates). Then all gluing matrices of \( N'_1 \) are \( A \).

Now consider the composition \( \eta \circ \mu \), it is easy to see that

1. \( \eta \circ \mu \) is free involution on \( N'_1 \),
2. for each JSJ piece \( \Sigma \) of \( N'_1 \), \( \eta \circ \mu \) send the coordinate systems of \( \Sigma \) to the the coordinate systems of \( \eta \circ \mu (\Sigma) \).

Now consider the double covering \( p_2 : N_1 = N'_1 \to N_2 = N'_1 / \eta \circ \mu \). Since the coordinates of \( N'_1 \) are invariant under \( \eta \circ \mu \), and all gluing matrices of \( N'_1 \) are \( A \), we conclude that \( N_2 \) has Property 1.

**Lemma 4.4.** Let \( N \) be a \( d \)-multiple edges graph manifold satisfying the property 1. Then there exists a finite separable \( d \)-characteristic covering \( p : N_1 \to N \) such that \( N_1 \) is \( d \)-multiple edges graph manifold satisfying the property 1 and each JSJ-piece \( \Sigma_i^1 \) is the product \( F_i^1 \times S^1 \) with \( g(F_i^1) = a_i d + b_i \geq 2 \) for some positive integers \( a_i, b_i \).

**Proof.** Denote by \( F_i \) the orbit space of \( \Sigma_i \), by \( h_i \) its fiber, and by \( c_1^i, ... , c_d^i \) the components of \( \partial F_i \) and consider the homomorphism

\[
\varepsilon_i : \pi_1 \Sigma_i = \pi_1 F_i \times \langle [h_i] \rangle \to \mathbb{Z}^d \mathbb{Z} \times \mathbb{Z}^d \mathbb{Z}
\]

defined by \( \varepsilon_i(a_l) = (0, \varepsilon_l) \) for \( l \geq 1 \), \( \varepsilon_i(b_j) = (0, 0) \) for \( j \geq 1 \), \( \varepsilon_i(c_1^i) = ... = \varepsilon_i(c_{d-1}^i) = (1, 0) \) and \( \varepsilon_i(h_i) = (0, 1) \), where \( \pi_1 F_i \) has a presentation:

\[
\langle a_1, b_1, ... , a_{g_i}, b_{g_i}, c_1, ... , c_d^i \mid [a_1, b_1]...[a_{g_i}, b_{g_i}]c_1^i...c_d^i = 1 \rangle,
\]
where \( g_i = g(F_i) \). Since \( c_1^i + \ldots + c_{d-1}^i + c_d^i = 0 \) in \( H_1(F_i; \mathbb{Z}) \) and since \( d - 1 \) is invertible in \( \mathbb{Z}/d\mathbb{Z} \) then \( \varepsilon_i(c_d^i) \) is of order \( d \) in \( \mathbb{Z}/d\mathbb{Z} \) for \( l = 1, \ldots, d \). Denote by \( p_1^i : \Sigma_1^i \to \Sigma_i \) the associated covering, then the number of components of \( \partial \Sigma_1^i \) is \( d \) by the construction. Denote by \( p : N_1 \to N \) the \( d^2 \)-fold covering of \( N \) obtained by gluing \( \Sigma_1^i \) with \( \Sigma_2^i \). This is possible since the \( p_1^i \) induce the \( d \)-characteristic covering on the boundary for \( i = 1, 2 \). This defines a finite separable \( d \)-characteristic covering by construction. Since \( p_1^i \) has horizontal degree \( d \) then \( \chi(F_1^i) = d\chi(F_i) \), where \( F_1^i \) denotes the orbit space of \( \Sigma_1^i \). This implies that

\[
2g(F_1^i) + d - 2 = d(2g(F_i) + d - 2)
\]

Hence we get

\[
g(F_1^i) = d(g(F_i) - 1) + \left( \frac{d(d - 1)}{2} + 1 \right)
\]

This proves the lemma. \( \square \)

Call a proper degree one map \( p : F' \to F \) between compact surfaces is a **pinch** if there is a disc \( D \) in \( \text{int}F \) such that \( p|_{p^{-1}(V)} : V \to \text{int}(D) \) is a homeomorphism, where \( V = F - \text{int}(D) \). We call a proper degree one map \( f : F' \times S^1 \to F \times S^1 \) a **vertical pinch** if \( f = p \times \text{id} \), where \( p \) is a pinch.

**Lemma 4.5.** Let \( N \) be a \( d \)-multiple edges graph manifold whose gluing matrices satisfy the condition I and assume that \( g(F_i) = a_id + b_i \) for some positive integers \( a_i, b_i \) and for \( i = 1, 2 \). Then \( N \) dominates a \( \text{PSL}(2, \mathbb{R}) \)-manifold.

**Proof.** First note that after performing a vertical pinch on \( \Sigma_1 = F_1 \times \langle h_1 \rangle \) and on \( \Sigma_2 = F_2 \times \langle h_2 \rangle \) we may assume that \( g_1 = g_2 = ad + 1 \) for some \( a \in \mathbb{Z}_+ \). Note there is a cyclic \( d \)-fold covering \( p'_1 : F_i \to F_i' \) with \( g(F_i') = a + 1, \partial F_i' \) connected, and the restriction of \( p'_1 \) is trivial on each component of \( \partial F_i \). This covering is given by a rotation of angle \( 2\pi/d \) on \( F_i \) whose axis does not meet \( F_i \) (see Figure 4).
The coverings $p'_i: F_i \to F'_i$ trivially extend to coverings $p'_i: \Sigma_i = F_i \times \langle h_i \rangle \to \Sigma'_i = F'_i \times \langle h'_i \rangle$ by setting $p'_i(h_i) = h'_i$ for $i = 1, 2$. Since all the gluing matrices of $N$ are $A$ by Property I, then the coverings $p'_i: \Sigma_i \to \Sigma'_i$ extend to a covering $p': N \to N'$, where the graph manifold $N'$ consists of the Seifert pieces $\Sigma'_1$ and $\Sigma'_2$ and the gluing matrix $A$ under obvious basis.

We fix some notations. For $i = 1, 2$, denote by $\partial F'_i = s_i$ and $\tau': \partial \Sigma'_1 = s_1 \times h'_1 \to \partial \Sigma'_2 = s_2 \times h'_2$ the induced sewing map satisfying $\tau'(s_1, h'_1) = (s_2, h'_2)A$, where $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}(2, \mathbb{Z})$ with $ad - bc = -1$. Moreover $b \neq 0$ by the basic properties of JSJ-decomposition.

Note that $\tau'_1(s_1) = as_2 + ch'_2$ and $\tau'^{-1}_1(s_2) = -ds_1 + ch'_1$. If $ac \neq 0$, then first pinch $\Sigma'_1 = F'_1 \times h'_1$ into a solid torus $V_1 = D^2 \times h'_1$ by killing $s_1$. This pinch provides a degree one map $\pi: N' \to \hat{\Sigma}'_2$ where $\hat{\Sigma}'_2$ is the closed 3-manifold obtained from $\Sigma'_2$ by Dehn filling along the curve $as_2 + ch'_2$. Since $ac \neq 0$ then $\Sigma'_2$ is
a $\text{PSL}(2, \mathbb{R})$-manifold. If $dc \neq 0$ similarly one can perform the same construction with $\Sigma'_1$. This proves the lemma when $ad \neq 0$ or $dc \neq 0$.

Let us assume now that $ac = dc = 0$. Then either $c = 0$ or $a = d = 0$. Since $ad - bc = -1$, then either $A = \pm \begin{pmatrix} 1 & b \\ 0 & -1 \end{pmatrix}$ with $b \neq 0$ or $A = \pm \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

Keeping the coordinate on $\Sigma'$ and re-coordinating $\Sigma'_2$ by $(-s_2, -h_2')$ if needed, we may assume that $A = \begin{pmatrix} 1 & b \\ 0 & -1 \end{pmatrix}$ or $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

Suppose first $A = \begin{pmatrix} 1 & b \\ 0 & -1 \end{pmatrix}$. Denote $F'_1 \simeq F'_2$ by $F$. Let denote by

$$\pi: F'_1 \times h'_1 \coprod F'_2 \times h'_2 \rightarrow \Sigma = F \times h$$

the trivial 2-fold covering map, where $\pi(h'_i) = h$ and $\pi(s_i) = s$. Denote by $\rho: \Sigma \rightarrow \hat{\Sigma}$ the quotient map associated with the Dehn filling on $\partial \Sigma$ along the curve $\frac{b}{(2,b)} s - \frac{2}{(2,b)} h$, where $(2, b)$ denotes the greatest common divisor of 2 and $b$. Note that, since $b \neq 0$, then $\hat{\Sigma}$ is a $\text{PSL}(2, \mathbb{R})$-manifold.

One can verify routinely that in the $\pi_1$ level the relations provided by gluing $\Sigma'_1$ and $\Sigma'_2$ via $\tau'$ are sent to the relation provided by Dehn filling on $\Sigma$ via $\frac{b}{(2,b)} s - \frac{2}{(2,b)} h$ under $\rho$, hence the map $\rho \circ \pi: \Sigma'_1 \coprod \Sigma'_2 \rightarrow \hat{\Sigma}$ factors through $\Sigma'_1 \coprod \Sigma'_2 \simeq N'$ which is sent into $\hat{\Sigma}$ by a degree 2 map, since the sewing involution $\tau'$ is orientation reversing so that $N'$ inherits compatible orientations from the pieces $\Sigma'_1$ and $\Sigma'_2$.

In the case of $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, we can perform the same construction as above, just replace the filling curve $\frac{b}{(2,b)} s - \frac{2}{(2,b)} h$ by the curve $s - h$. This proves Lemma 4.5.

By Lemmas 4.3, 4.4 and 4.5 and their proofs, we have the following diagram:

$$\begin{array}{c}
N_1 \\
p_1 \downarrow \\
N \\
p_2 \\
N_2 \\
p_3 \\
N_3 \\
p_4 \\
N_4 \\
N \\
N_2 \\
N_4 \\
N \\
N_2 \\
N_4 \\
\end{array}$$

where $p_1$ and $p_2$ are coverings provided by Lemma 4.3, $p_3$ is the coverings provided by Lemma 4.4 and the non-zero degree map $p_4$ is provided by Lemma 4.5, where $SV(N_4) \neq 0$. Since $SV$ has property $D$, $SV(N_3) \neq 0$.

Consider the covering $\hat{N}$ corresponding to the finite index subgroup $p_2 \ast (\pi_1 N_1) \cap p_3 \ast (\pi_1 N_3)$ in $\pi_1 N_2$. Then $\hat{N}$ covers both $N_1$ (and thus $N$) and $N_3$, and $SV(\hat{N}) \neq 0$. Then the proof of Proposition 4.1 is complete.

5. PROOF OF THEOREM 1.2

5.1. SIMPLIFICATIONS. Let $\hat{N}$ be a closed non-trivial graph manifold. We are going to show that $|D(M, N)|$ is finite for any given graph manifold $M$. 

(1) First we simplify \( N \): By Lemma 3.2 and Lemma 3.5, there is a finite covering \( \tilde{N} \) of \( N \) satisfying the condition (*): each JSJ piece of \( \tilde{N} \) is a product of an oriented surface with genus \( \geq 2 \) and the circle, and each JSJ torus is shared by two different JSJ pieces.

By Lemma 3.1 if \( |D(M, N)| \) is not finite for some graph manifold \( M \), then \( |D(P, \tilde{N})| \) is not finite for some graph manifold \( P \). So we may assume \( N \) already satisfies the condition (*).

(2) Then we simplify \( M \): For given \( M \), let \( H = \{ M_1, \ldots, M_k \} \) be the finite set of graph manifolds provided by Proposition 2.2. By Proposition 2.2 (i), if \( |D(M, N)| \) is not finite, then \( |D(M_i, N)| \) is not finite for some \( M_i \in H \). So may assume that (** \( M = M_i \in H \) for some \( i \in \{ 1, \ldots, k \} \).

5.2. Proof of Theorem 1.2 when \( |e|(N) \neq 0 \). Suppose \( |e|(N) \neq 0 \). By Lemma 3.6 and (*) in 5.1, we may assume that \( |SV(\tilde{N})| \neq 0 \). Then there exists a Seifert piece \( Q \) of \( N^* \) such that \( SV(\tilde{Q}) \neq 0 \). By (** in 5.1 and Proposition 2.2 (ii), we may assume that \( L_Q(f) = f^{-1}(Q) \) is a canonical submanifold of \( M \). Below we denote \( L_Q(f) \) as \( L_Q \) for short.

Lemma 5.1. \( L_Q \) can be chosen so that any component \( T \) of \( \partial L_Q \) is shared by two distinct Seifert pieces of \( M \): one in \( L_Q \) and another in \( M \setminus L_Q \).

Proof: Indeed if not, then there exists two distinct components \( T \) and \( T' \) of \( \partial L_Q \) which are identified by the sewing involution \( \tau_M \) of \( M \) and such that \( T \) and \( T' \) are sent by \( f \) into the same component of \( \partial Q \). Denote by \( \tilde{L}_Q \) the canonical submanifold of \( M \) obtained by identifying \( T \) and \( T' \) via \( \tau_M \). Since each component of \( \partial Q \) is shared by two distinct Seifert pieces of \( N \) by the assumption in 5.1, \( f \) induces a proper map \( \tilde{f}: \tilde{L}_Q \to Q \). After finitely many such operations, we reach a new \( L_Q \) satisfying the requirement of Lemma 5.1 \( \square \)

Below we assume that \( L_Q \) satisfies the requirement of Lemma 5.1. Now we choose \( L_Q \) to be maximal in the sense that for any Seifert piece \( S \) in \( M \setminus L_Q \) adjacent to \( L_Q \), \( S \) is not able to be added into \( L_Q \) by homotopy on \( f \). Then \( f(S) \subset B_S \), where \( B_S \) is a Seifert piece of \( N \) distinct from \( Q \) and adjacent to \( Q \).

Since \( L_Q \) is maximal, by Lemma 2.3 we deduce that for any Seifert piece \( S \) adjacent to \( L_Q \) along a component of \( \partial L_Q \), \( f|S: S \to B_S \) is fiber preserving. Hence the proper map \( f|L_Q : L_Q \to Q \) preserves the canonical framings, and it induces a map \( \hat{f}: \hat{L}_Q \to \hat{Q} \) between the closed manifolds obtained after Dehn filling along the canonical framings. By Lemma 2.4 we have

\[
SV(\hat{L}_Q) \geq |\deg(\hat{f})|SV(\hat{Q}).
\]

Since \( \deg(f) = \deg(f|L_Q) = \deg(\hat{f}) \), we get

\[
|\deg(f)| \leq \frac{SV(\hat{L}_Q)}{SV(\hat{Q})}.
\]

Therefore
By Lemma 2.1 there are only finitely many \( \hat{Q} \) and only finitely many \( \hat{L} \). So the right side of the above inequality is finite. This completes the proof of Theorem 1.2 when \( |e|(N) \neq 0 \).

5.3. Proof of Theorem 1.2 when \( |e|(N) = 0 \). By Lemma 3.1 and Lemma 3.6 we can assume each gluing matrix of \( N \) is equal to \( \pm \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \).

Choose two distinct adjacent Seifert pieces \( S_1 \) and \( S_2 \) in \( N \), denote by \( T = \partial S_1 \cap \partial S_2 \) and by \( Q \) the connected graph manifold \( S_1 \cup_T S_2 \) (such Seifert pieces exist by 5.1). By Proposition 2.2 we may assume that \( f^{-1}(Q) = L_Q \) is a canonical submanifold of \( M \).

Since each JSJ-torus of \( N \) is shared by two different JSJ-pieces, by the same arguments as in 5.2, we may assume that each component of \( \partial L_Q \) is shared by two distinct Seifert pieces of \( M \) one in \( L_Q \) and another in \( M \setminus L_Q \). Furthermore we can arrange \( L_Q \) to be maximal in the sense of 5.2, then by Lemma 2.3 we deduce that any Seifert piece \( S' \) of \( M \) adjacent to \( L_Q \) is sent by \( f \) to a Seifert piece \( B' \) adjacent to \( Q \) such that \( f|S': S' \to B' \) is fiber preserving.

As in 5.2, it follows that the proper map \( f|L_Q: L_Q \to Q \) induces a map \( \hat{f}: \hat{L}_Q \to \hat{Q} \) between closed graph manifolds obtained by Dehn filling along the canonical framings. Moreover, as in 5.2 we have \( \deg(f) = \deg(f|L_Q) = \deg(\hat{f}) \) and thus

\[
|D(M, N)| \leq \max \left\{ |D(\hat{L}, \hat{Q})| \mid L \text{ is canonical in } M \right\}
\]

Note that \( \hat{Q} = \hat{S}_1 \cup_T \hat{S}_2 \), where \( \hat{S}_i \) is obtained by Dehn filling along the canonical framings on \( \partial S_i \setminus T \), \( i = 1, 2 \). It follows that \( \hat{Q} \) satisfies the hypothesis of Proposition 4.1. Then \( \hat{Q} \) has a finite covering \( \bar{Q} \) with \( SV(\bar{Q}) \neq 0 \) by Proposition 4.1. Hence by Lemma 5.1 the set \( |D(\hat{L}, \hat{Q})| \) is finite for any \( \hat{L} \). Since by Lemma 2.1 there are only finitely many \( \hat{L} \), this completes the proof of Theorem 1.2 when \( |e|(N) \neq 0 \). Hence Theorem 1.2 is proved.

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