HOPF-CYCLIC HOMOLOGY
OF QUANTUM INSTANTONS

TOMAZS MASZCZYK AND SERKAN SÜTLÜ

Abstract. For noncommutative principal bundles corresponding to Hopf-Galois extensions Jara and Ştefan established an isomorphism between the relative cyclic homology and a cyclic dual Hopf-cyclic homology with appropriate stable anti-Yetter-Drinfeld coefficients. However, known noncommutative deformations of principal bundles, such as the quantum circle bundles over Podleś spheres and the quantum instanton bundles, go beyond the Hopf-Galois context. The case of quantum circle bundles over Podleś spheres were discussed by the authors in a previous paper, and an analogical isomorphism for homogeneous quotient coalgebra-Galois extensions were constructed. In the present paper, the isomorphism for arbitrary module coalgebra-Galois comodule algebra extensions, covering the case of quantum instanton bundles, is treated.

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1. Introduction

Hopf-cyclic homology arose from studying the index theory of transversally elliptic operators [15] to become an invariant of a noncommutative symmetry given in terms of a Hopf algebra. As such, it is a generalization of group and Lie algebra homology. Its importance stems from the existence of characteristic maps relating it with cyclic homology, establishing a relation between noncommutative symmetry and noncommutative topology.

Presented as a homology theory associated to a Hopf algebra and a pair of elements, consisting of a group-like and a character, called a modular pair in involution (MPI) in [15], Hopf-cyclic homology was axiomatized in [18, 17] as a homology theory associated to a Hopf algebra and a coefficient space, called a stable anti-Yetter-Drinfeld (SAYD) module. In the language of [17], the MPI of [15] represented a trivial (one dimensional) coefficient space. In [18, 17] some more general examples of SAYD’s were given, all somehow related to the Hopf-Galois context.

However, a concrete nontrivial application of a Hopf-cyclic coefficient space had to wait until the paper [21] of Jara and Ştefan, in which a class of SAYD modules, a generalization of one of examples from [18, 17], arose in a canonical way within the context of the Hopf-Galois extensions. The application consisted there in establishing an isomorphism between the relative cyclic homology of the extension, and the Hopf-cyclic homology, with coefficients in the SAYD module associated to the Hopf-Galois symmetry of the extension.

Next, in [29], further examples of MPI’s were constructed for bicrossproduct Hopf algebras associated to Lie algebras. These MPI’s were then upgraded into nontrivial SAYD modules in [28]. More precisely, a 4-dimensional SAYD module over the Schwarzian quotient $H_{1,S}$ of the Connes-Moscovici Hopf algebra $H_1$ was illustrated in [28]. On the other hand, SAYD modules over the universal enveloping algebras were studied in [27], and a SAYD module was constructed for any given dimension; namely the truncated Weil algebras.

In all the above examples and applications it is assumed that the symmetry of the problem is described in terms of a Hopf algebra. However, Galois symmetry in many important noncommutative deformations of principal bundles, such as the quantum circle
bundles over Podleś spheres and the quantum instanton bundles [5, 4] go beyond the Hopf-Galois context. Instead of a Hopf algebra coaction they need coaction of a quotient module coalgebra of a Hopf algebra.

The aim of the present paper is to generalize the Jara-Ștefan isomorphism from [21] to encompass such examples. It is worth mentioning that the comparison between our construction and that one from [21] is quite nontrivial. The main difference consists in the construction of an appropriate SAYD module, quite simple in [21], and more complicated in our case, as shown by the following Theorem.

**Theorem 3.1.** For any right $H$-comodule algebra $A$ the quotient vector space

$$M := (H \otimes A)/\langle ha'(1) \otimes aa'(0) - h \otimes a'a \mid h \in H, a, a' \in A \rangle.$$

is a stable anti-Yetter Drinfeld $H$-module with the left $H$-module structure

$$H \otimes M \rightarrow M,$$

$$h' \otimes (h \otimes a) \mapsto h'(h \otimes a) := h'h \otimes a,$$

and the right $H$-comodule structure given by

$$M \rightarrow M \otimes H,$$

$$(h \otimes a) \mapsto (h \otimes a)<_{0}> \otimes (h \otimes a)<_{1}> := (h_{(2)} \otimes a_{(0)}) \otimes h_{(3)}a_{(1)}S(h_{(1)}).$$

As we show, in the commutative case our construction produces the affine Brylinski $G$-scheme, whose action groupoid is the inertia groupoid of the action groupoid corresponding to a given $H$-comodule algebra. This explains the classical meaning of our construction.

We show also, constructing an explicit isomorphism, that in the Hopf-Galois case, when the module coalgebra is the Hopf-algebra itself, our apparently different construction produces the same SAYD-module as the one of Jara and Ștefan [21].

In the previous paper [23] the authors generalized a special case of the Jara-Ștefan isomorphism from homogeneous Hopf-Galois to homogeneous quotient coalgebra-Galois extensions, covering i.a. the case of quantum circle bundles over Podleś spheres. In this generality, the thus obtained isomorphism is a cyclic homological enhancement of the Takeuchi-Galois correspondence between the left coideal subalgebras and the quotient right module coalgebras of a Hopf algebra [23]. The latter generalizes the classical bijective correspondence between orbits and stabilizers.

In the present paper we generalize the full Jara-Ștefan isomorphism from Hopf-Galois to quotient coalgebra-Galois extensions, covering i.a. the case of a quantum instanton bundle. More precisely, we do it in two steps. First, we prove the following technical result.
Theorem 3.3. The map

\[ \Phi : ((A \otimes C) \otimes_A \ldots \otimes_A (A \otimes C)) \otimes_H A \rightarrow (C \otimes \ldots \otimes C) \otimes_H M \]

\[ \left( (a^0 \otimes c^0) \otimes_A \ldots \otimes_A (a^n \otimes c^n) \right) \otimes_H 1 \mapsto \left( c^0 \cdot a^1(1) \ldots a^n(1) \otimes c^1 \cdot a^2(2) \ldots a^n(2) \otimes \ldots \otimes c^{n-2} \cdot a^{n-1}(n-1) \otimes c^{n-1} \cdot a^n(n) \otimes c^n \right) \otimes_H [1 \otimes a^0 a^1(0) \ldots a^n(0)] \]

is an isomorphism of cyclic objects.

Next, we combine Theorem 3.3 with the bijectivity of the canonical map (the Galois condition). The latter can be viewed as an isomorphism of corings with a group-like element. One coring encodes descend along the algebra extension, and the other one passing from equivariant to invariant with respect to symmetry given in terms of a free coaction.

The cyclic-dual Hopf-cyclic homology of the first coring turns out to be isomorphic to the relative cyclic homology of the extension. At least at the level of relative periodic cyclic homology and in the classical commutative case, it depends only on the relative topology of the corresponding principal fibration.

The Hopf-cyclic homology of the second coring, on the other hand, turns out to be isomorphic to the Hopf-cyclic homology of the module-coalgebra coacting freely on a comodule algebra. At least at the level of the classical commutative case, it depends only on the Galois symmetry given as a free action of a subgroup and the inertia of the corresponding action of the overgroup. In the case of an instanton bundle, the action of an overgroup is transitive.

The main application of the resulting isomorphism of cyclic objects is subsumed in the following theorem.

Theorem 5.1. For any \( H \)-module coalgebra \( C \) and any \( C \)-Galois \( H \)-comodule algebra \( A \) extension of an algebra \( B \) the relative periodic cyclic homology of the extension \( B \subseteq A \) is isomorphic to the Hopf-cyclic homology of the \( H \)-module coalgebra \( C \) with SAYD-coefficients in \( M \)

\[ HP_\bullet(A|B) \cong HP^H_\bullet(C, M). \]

The main point here is that in the classical situation the left hand side, when equipped with the Gauss-Manin connection \([16, 32, 25]\), is an invariant of the relative topology of a fibration, while the right hand side is an invariant of the action of a group, a subgroup of which acting freely.

2. Extensions and Cyclic Homology

In this section we recall the material we shall use in the sequel. We begin with the notion of coalgebra-Galois extension. This is followed by the Hopf-cyclic homology
of corings, with coefficients, and finally we shall recall the cyclic homology of algebra extensions.

2.1. **Coalgebra-Galois extensions.** We recall the definition and basic properties of the coalgebra-Galois extensions from [11].

Let $C$ be a coalgebra, and $A$ be an algebra. Let $A$ also be a right $C$-comodule via $\rho : A \to A \otimes C$, which we denote by $\rho(a) = a_{<0>} \otimes a_{<1>}$. Then the coaction invariants of $A$ is defined to be the subset

$$A^\co_c := \{ b \in A \mid \rho(ba) = b\rho(a) \}.$$  

It follows, from this definition, that $A^\co_c \subseteq A$ is a subalgebra.

An algebra extension $B \subseteq A$ is called a $C$-extension if $B = A^\co_c$. Finally, a $C$-extension $B \subseteq A$ is said to be Galois if the left $A$-linear right $C$-colinear map

$$\text{can} : A \otimes_B A \to A \otimes C, \quad a \otimes_B a' \mapsto aa'_{<0>} \otimes a'_{<1>}$$

is bijective. The map (2.2) is called the canonical map of the extension.

An interesting, due to the geometric examples it covers, class of coalgebra-Galois extensions is called a quotient coalgebra-Galois extension. In this setting, one lets $H$ to be a Hopf algebra, $I \subseteq H$ a coideal right ideal, and $A$ a right $H$-comodule algebra via $\nabla : A \to A \otimes H$. Then the composition

$$\rho : A \xrightarrow{\nabla} A \otimes H \xrightarrow{\Id \otimes \pi} A \otimes H/I$$

determines a right $H/I$-coaction on $A$. Finally, a Galois $H/I$-extension $A^\co_{H/I} \subseteq A$ is called a quotient coalgebra-Galois extension.

For $I = 0$, the quotient coalgebra-Galois extensions recover the Hopf-Galois extensions, and in case of $A = H$ they are called homogeneous coalgebra-Galois extensions. In particular, viewing $H$ as a right $H$-comodule algebra via its comultiplication, one obtains the homogeneous $H/I$-Galois extensions

$$\rho : H \xrightarrow{\Delta} H \otimes H \xrightarrow{\Id \otimes \pi} H \otimes H/I, \quad h \mapsto h_{(1)} \otimes h_{(2)}.$$  

Such Galois extensions correspond to the quantum homogeneous spaces. Concrete examples include Manin’s plane and Podlés sphere [8], as well as a quantum spherical fibration of $SU_q(2)$, [10].

The quantum instanton bundle introduced in [5] is also a quotient coalgebra-Galois extension, [4]. This examples uses the full generality of the quotient coalgebra-Galois extensions: $A \neq H$ and $I \neq 0$.

In fact, the cyclic object on the right hand side of Theorem 3.3 makes sense in even more general case of extensions, which we define as follows.
If $A$ is a right $H$-comodule algebra with the coaction

$$A \rightarrow A \otimes H, \quad a \mapsto a(0) \otimes a(1)$$

and $C$ be a right $H$-module coalgebra with a group-like element $e \in C$, such that, the induced coaction

$$A \rightarrow A \otimes C, \quad a \mapsto a_{<0>} \otimes a_{<1>} := a(0) \otimes ea(1)$$

has the subalgebra of invariants $B = A^\co C$, then we say that $A$ is a module coalgebra-comodule algebra extension of $B$.

The following lemma shows however that in the Galois context, this reduces to the quotient coalgebra-Galois case.

**Lemma 2.1.** Let $H$ be a Hopf algebra and $B \subseteq A$ be a module coalgebra-comodule-Galois algebra extension for an $H$-module coalgebra $C$ coaugmented with a group-like $e \in C$. Then the right $H$-module coalgebra map

$$\pi : H \rightarrow C, \quad h \mapsto eh$$

is surjective, hence makes $C$ a quotient coaugmented right $H$-module coalgebra and $B \subseteq A$ a quotient coalgebra-Galois extension.

**Proof.** Consider a diagram commuting by virtue of the definition of a module coalgebra-comodule algebra extension

$$\begin{array}{ccc}
A \otimes A & \xrightarrow{\beta} & A \otimes H \\
\downarrow & & \downarrow A \otimes \pi \\
A \otimes_B A & \xrightarrow{\text{can}} & A \otimes C.
\end{array}$$

Since the left vertical arrow (the balancing the tensor product over $B$) is surjective and the canonical map is bijective, the map $A \otimes \pi$ is surjective as well. Since $A$ is (faithfully) flat over the base field we have $\ker(A \otimes \pi) = A \otimes \ker(\pi)$ and hence by exactness of the short sequence

$$0 \rightarrow A \otimes \ker(\pi) \rightarrow A \otimes H \xrightarrow{A \otimes \pi} A \otimes C \rightarrow 0$$

and faithfull flatness of $A$ the map $\pi$ itself has to be surjective. \qed

Given a $C$-Galois extension $B \subseteq A$, there is also an $A$-coring structure on $A \otimes C$, called the Galois-coring of the extension, whose underlying $A$-bimodule structure reads as

$$a'(a \otimes c) = a'a \otimes c, \quad (a \otimes c)a' = aa'(0) \otimes ca'(1).$$

This can be understood as the coincidence of the canonical entwining

$$c \otimes a \mapsto \text{can}(\text{can}^{-1}(1 \otimes c)a)$$

of the $C$-Galois extension $A$, with the standard entwining

$$c \otimes a \mapsto a(0) \otimes ca(1)$$

for a right $H$-module coalgebra and a right $H$-comodule algebra $A$, [11, 12].
The fact that the canonical map (2.2) is an isomorphism of corings will play a crucial role later on. Note that \( A \otimes_B A \), called the Sweedler coring, encodes the relative geometry of a noncommutative fibration in terms of the descent data, while \( A \otimes C \), called the Galois coring, encodes the action of the symmetry given in terms of the coaction of the coalgebra [13].

The main aim of the present paper is to enhance this isomorphism to an isomorphism of cyclic objects providing invariants of the relative topology of a fibration (the topological side) on one hand, and invariants of the action of symmetry (the dynamical side) on the other.

2.2. Hopf-cyclic homology of corings. We now recall the cyclic homology of a module coring, with coefficients in a stable-anti-Yetter-Drinfeld (SAYD) module, under the symmetry of a right \( \times \)-Hopf algebra from [3, 19].

Let \( A \) and \( B \) be two algebras. Let also \( s : A \to B \) and \( t : A^\text{op} \to B \) be two algebra maps with commuting ranges. Then \( B \) is equipped with an \( A \)-bimodule structure by

\[
(2.3) \quad a \cdot b \cdot a' = bs(a')t(a)
\]

for any \( a, a' \in A \) and any \( b \in B \). Let, in addition, \( B \) be an \( A \)-coring with the comultiplication \( \Delta : B \to B \otimes_A B \), and the counit \( \varepsilon : B \to A \). Then \((B, s, t, \Delta, \varepsilon)\) is called a right \( A \)-bialgebroid if for \( b, b' \in B \) and \( a \in A \)

(i) \( b_{(1)} \otimes_A t(a) b_{(2)} = s(a) b_{(1)} \otimes_A b_{(2)}, \)
(ii) \( \Delta(1_B) = 1_B \otimes_A 1_B, \quad \Delta(bb') = b_{(1)} b'_{(1)} \otimes_A b_{(2)} b'_{(2)}, \)
(iii) \( \varepsilon(1_B) = 1_A, \quad \varepsilon(bb') = \varepsilon(s(\varepsilon(b))b'). \)

Finally, a right \( A \)-bialgebroid \( B = (B, s, t, \Delta, \varepsilon) \) is said to be a right \( \times_A \)-Hopf algebra provided the map

\[
\nu : B \otimes_{A^\text{op}} B \to B \otimes_A B, \quad b \otimes_{A^\text{op}} b' \mapsto bb'_{(1)} \otimes_A b'_{(2)}
\]

is bijective. As for the inverse, we use the notation

\[
\nu^{-1}(1 \otimes_{A^\text{op}} b) = b^- \otimes_A b^+
\]

for any \( b \in B \). For further information on bialgebroids and \( \times \)-Hopf algebras we refer the reader to [2, 12, 30, 31], and we recall here [19, Ex. 2.3].

Example 2.2. For an algebra \( A \), its enveloping algebra \( A^e := A \otimes A^\text{op} \) is a right \( \times_A \)-Hopf algebra with the structure maps

\[
s : A \to A^e, \quad a \mapsto a \otimes 1; \quad t : A^\text{op} \to A^e, \quad a \mapsto 1 \otimes a,
\]
\[
\Delta^e : A^e \to A^e \otimes_A A^e, \quad (a, a') \mapsto (1, a') \otimes_A (a, 1),
\]
\[
\varepsilon^e : A^e \to A, \quad \varepsilon^e(a, a') = a'a,
\]
\[
\nu(((a', a') \otimes_{A^\text{op}} (b, b'))) = (a, b'a') \otimes_A (b, 1),
\]
\[
\nu^{-1}((a, a') \otimes_A (b, b')) = (ab', a') \otimes_{A^\text{op}} (b, 1).
\]
Now we recall SAYD modules over a right $\times_A$-Hopf algebras from [3, 19]. Let $B$ be a right $\times_A$-Hopf algebra, and $M$ be a left $B$-module and a right $B$-comodule via $\nabla: M \rightarrow M \otimes_A B$ defined by $m \mapsto m_{(0)} \otimes_A b_{(1)}$. Then $M$ is called an AYD module over $B$ if the $A$-bimodule structures on $M$, being a left module over a right $\times_A$-Hopf algebra and a left comodule over an $A$-coring, coincide, i.e.

\begin{equation}
(2.4) \quad m \cdot a = t(a) \triangleright m, \quad a \cdot m = m_{(0)} \cdot \varepsilon(s(a)m_{(1)}) = s(a) \triangleright m,
\end{equation}

and

\begin{equation}
(2.5) \quad \nabla(b \triangleright m) = b_{(1)}^+ \triangleright m_{(0)} \otimes_A b_{(2)}m_{(1)}b_{(1)}^-.
\end{equation}

Furthermore, $M$ is considered stable if $m_{(1)} \triangleright m_{(0)} = m$. The following example will be used later.

**Example 2.3.** Let $A$ be an algebra and $A^e$ be the enveloping algebra of $A$ as the right $\times_A$-Hopf algebra introduced in Example 2.2. Then $A$ is a left $A^e$-module via

\begin{equation}
(2.6) \quad A^e \otimes A \rightarrow A, \quad (a', a'') \triangleright a := a'aa'',
\end{equation}

for all $a, a', a'' \in A$. Moreover, $A$ is a left $A^e$-comodule via

\begin{equation}
(2.7) \quad \nabla^e: A \rightarrow A \otimes_A A^e, \quad a \mapsto a_{(0)} \otimes_A a_{(1)} := 1 \otimes_A (a, 1).
\end{equation}

For any $a \in A$ we have

\begin{equation}
(2.8) \quad a_{(1)} \triangleright a_{(0)} = (a, 1) \triangleright 1 = a,
\end{equation}

hence $A$ is stable. As for the AYD compatibility, for any $a, b \in A$ we have

\begin{equation}
(2.9) \quad ab = (1, b) \triangleright a = t(b) \triangleright a,
\end{equation}

as well as

\begin{equation}
(2.10) \quad a_{(0)} \varepsilon^e(s(b)a_{(1)}) = \varepsilon^e((b, 1)(a, 1)) = ba = (b, 1) \triangleright a = s(b) \triangleright a.
\end{equation}

As a result, (2.4) is satisfied. Finally, for any $a, b', b'' \in A^e$,

\[
(b', b'')_{(1)}^+ \triangleright a_{(0)} \otimes_A (b', b'')_{(2)}a_{(1)} = (1, b'')^+ \triangleright 1 \otimes_A (b', 1)(a, 1)(1, b'')^- = (1, 1) \triangleright 1 \otimes_A (b', 1)(a, 1)(b'', 1) = 1 \otimes_A (b'ab'', 1) = \nabla^e((b', b'') \triangleright a).
\]

Therefore (2.5) is also satisfied, and we conclude that $A$ is a left-right SAYD module over the right $\times_A$-Hopf algebra $A^e$.

Let us next recall the cyclic homology of module corings with SAYD coefficients from [3, 19]. Let $B$ be a right $\times_A$-Hopf algebra, $\mathcal{C}$ an $A$-coring as well as a right $B$-module coring, i.e. $\mathcal{C}$ is a right $B$-module and for any $c \in \mathcal{C}$ and $b \in B$,

\begin{enumerate}
  \item $\varepsilon_\mathcal{C}(c \cdot b) = \varepsilon_\mathcal{C}(c) \cdot b = \varepsilon_\mathcal{B}(s(\varepsilon_\mathcal{C}(c))b),$
  \item $\Delta_\mathcal{C}(c \cdot b) = \Delta_\mathcal{C}(c) \cdot b = c_{(1)} \cdot b_{(1)} \otimes_A c_{(2)} \cdot b_{(2)}$.
\end{enumerate}

Let also $M$ be a left-right SAYD module over $B$. Then,

\begin{equation}
(2.11) \quad \mathfrak{Y}_n(\mathcal{C}, B, M) := \mathcal{C}^{\otimes_A n+1} \otimes_B M, \quad n \geq 0
\end{equation}
As it is remarked in [21], for any $B$-bimodules $M, B$ of algebras from [21]. To this end, we first recall the cyclic tensor product from [26]. The symmetry of the right $HC$-module, denoted by $\mathcal{M}$, is equipped with the morphisms

\begin{equation}
\delta_i : \mathcal{M}_{n+1}(C, B, M) \rightarrow \mathcal{M}_n(C, B, M), \quad 0 \leq i \leq n + 1
\end{equation}

\begin{equation}
\delta_i(c^0 \otimes_A \cdots \otimes_A c^{n+1} \otimes_B m) = c^0 \otimes_A \cdots \otimes_A (c^i) \otimes_A \cdots \otimes_A c^{n+1} \otimes_B m,
\end{equation}

\begin{equation}
\varepsilon_i : \mathcal{M}_{n-1}(C, B, M) \rightarrow \mathcal{M}_n(C, B, M), \quad 0 \leq i \leq n - 1
\end{equation}

\begin{equation}
\varepsilon_i(c^0 \otimes_A \cdots \otimes_A c^{n-1} \otimes_B m) = c^0 \otimes_A \cdots \otimes_A c^{(i)} \otimes_A \cdots \otimes_A c^{n-1} \otimes_B m,
\end{equation}

and

\begin{equation}
t : \mathcal{M}_n(C, B, M) \rightarrow \mathcal{M}_n(C, B, M),
\end{equation}

\begin{equation}
t(c^0 \otimes_A \cdots \otimes_A c^n \otimes_B m) = c^n \otimes m_{<1>} \otimes_A c^0 \otimes_A \cdots \otimes_A c^{n-1} \otimes_B m_{<0>}.
\end{equation}

The cyclic homology of this cyclic module, denoted by $HC_B(C, M)$, is called the cyclic homology of the $B$-module coring $C$ with coefficients in the SAYD module $M$ under the symmetry of the right $\times_A$-Hopf algebra $B$.

### 2.3. Cyclic homology of algebra extensions.

We recall the relative cyclic homology of algebras from [21]. To this end, we first recall the cyclic tensor product from [26]. The cyclic tensor product of $B$-bimodules $M_1, \ldots, M_n$ is defined by

\[ M_1 \hat{\otimes}_B \cdots \hat{\otimes}_B M_n := (M_1 \otimes_B \cdots \otimes_B M_n) \otimes_{B^e} B. \]

As it is remarked in [21], for any $B$-bimodule $M$ one has $M \otimes_{B^e} B \cong M_B$, where $M_B = M/[M, B]$ and $[M, B]$ is the subspace of $M$ generated by the commutators, that is, the elements of the form $[m, b] := m \cdot b - b \cdot m$.

The relative cyclic homology of an algebra extension $B \subseteq A$ is computed by the cyclic object

\begin{equation}
Z(A|B, A) = \bigoplus_{n \geq 0} Z_n(A|B, A), \quad Z_n(A|B, A) := A \hat{\otimes}_B^{n+1}
\end{equation}

equipped with the morphisms

\begin{equation}
\delta_i : Z_n(A|B, A) \rightarrow Z_{n-1}(A|B, A), \quad 0 \leq j \leq n,
\end{equation}

\begin{equation}
\delta_i(a^0 \otimes_B a^1 \hat{\otimes}_B \cdots \hat{\otimes}_B a^n) =
\begin{cases}
a^0 a^1 \hat{\otimes}_B \cdots \hat{\otimes}_B a^n, & i = 0, \\
a^0 a^1 \hat{\otimes}_B \cdots \hat{\otimes}_B a^i a^{i+1} \hat{\otimes}_B \cdots \hat{\otimes}_B a^n, & 1 \leq i \leq n - 1, \\
a^n a^0 \hat{\otimes}_B a^1 \hat{\otimes}_B \cdots \hat{\otimes}_B a^{n-1}, & i = n,
\end{cases}
\end{equation}

\begin{equation}
\sigma_j : Z_n(A|B, A) \rightarrow Z_{n+1}(A|B, A), \quad 0 \leq j \leq n,
\end{equation}

\begin{equation}
\sigma_j(a^0 \hat{\otimes}_B \cdots \hat{\otimes}_B a^n) = a^0 \hat{\otimes}_B a^1 \hat{\otimes}_B \cdots \hat{\otimes}_B a^i \hat{\otimes}_B a^j a^{j+1} \hat{\otimes}_B \cdots \hat{\otimes}_B a^n,
\end{equation}

and

\begin{equation}
\tau_n : Z_n(A|B, A) \rightarrow Z_n(A|B, A),
\end{equation}

\begin{equation}
\tau_n(a^0 \hat{\otimes}_B \cdots \hat{\otimes}_B a^n) = a^n \hat{\otimes}_B a^0 \hat{\otimes}_B \cdots \hat{\otimes}_B a^{n-1}.
\end{equation}
The cyclic (resp. periodic cyclic, negative cyclic, Hochschild) homology of the complex (2.15) is called the relative cyclic homology of the extension $B \subseteq A$, and it is denoted by $HC(A|B)$ (resp. $H_P(A|B)$, $HN(A|B)$, $HH(A|B)$).

2.4. Cyclic homology of corings associated to the extension. The cyclic module (2.15) can also be interpreted in terms of the cyclic module (2.11) of the Sweedler coring $A \otimes_B A$ of the extension with coefficients in the left-right SAYD module $A$ over the right $\times_A$-Hopf algebra $A^e$, see Example 2.3, via the isomorphism

$$
\begin{align*}
\mathcal{Y}_n(A \otimes_B A, A^e, A) &\longrightarrow Z_n(A/B, A), \\
(a^0 \otimes_B 1) \otimes_A \cdots \otimes_A (a^n \otimes_B 1) \otimes_{A^e} 1 &\mapsto a^0 \otimes_B \cdots \otimes_B a^n.
\end{align*}
$$

The cyclic structure of the former is then given explicitly by

$$
\begin{align*}
\delta_i &: \mathcal{Y}_n(A \otimes_B A, A^e, A) \longrightarrow \mathcal{Y}_{n-1}(A \otimes_B A, A^e, A), \quad 0 \leq i \leq n, \\
\delta_i((a^0 \otimes_B 1) \otimes_A \cdots \otimes_A (a^n \otimes_B 1) \otimes_{A^e} 1) &\mapsto (a^0 \otimes_B 1) \otimes_A \cdots \otimes_A (a^i \otimes_B 1) \otimes_A \cdots \otimes_A (a^n \otimes_B 1) \otimes_{A^e} 1, \\
\sigma_j &: \mathcal{Y}_n(A \otimes_B A, A^e, A) \longrightarrow \mathcal{Y}_{n+1}(A \otimes_B A, A^e, A), \quad 0 \leq j \leq n, \\
\sigma_j((a^0 \otimes_B 1) \otimes_A \cdots \otimes_A (a^n \otimes_B 1) \otimes_{A^e} 1) &\mapsto (a^0 \otimes_B 1) \otimes_A \cdots \otimes_A (a^j \otimes_B 1) \otimes_A \cdots \otimes_A (a^n \otimes_B 1) \otimes_{A^e} 1, \\
\tau_n &: \mathcal{Y}_n(A \otimes_B A, A^e, A) \longrightarrow \mathcal{Y}_n(A \otimes_B A, A^e, A), \\
\tau_n((a^0 \otimes_B 1) \otimes_A \cdots \otimes_A (a^n \otimes_B 1) \otimes_{A^e} 1) &\mapsto (a^n \otimes_B 1) \otimes_A \cdots \otimes_A (a^0 \otimes_B 1) \otimes_A \cdots \otimes_A (a^{n-1} \otimes_B 1) \otimes_{A^e} 1.
\end{align*}
$$

Given a $C$-Galois extension $B \subseteq A$, there is also an $A$-coring structure on $A \otimes C$, called the Galois-coring of the extension, whose underlying $A$-bimodule structure reads as

$$
a'(a \otimes c) = a'a \otimes c, \quad (a \otimes c)a' = a_{<0>}a' \otimes ca'_{<1>}.
$$

This can be understood as the coincidence of the canonical entwining $c \otimes a \mapsto \text{can(can}^{-1}(1 \otimes c)a)$ of the $C$-Galois extension $A$, with the standard entwining $c \otimes a \mapsto a_{<0>} \otimes ca_{<1>}$ for a right $H$-module coalgebra and a right $H$-comodule algebra $A$, [11, 12].

Iterating the canonical map (2.2), we may further identify the relative cyclic homology of a $C$-Galois extension $B \subseteq A$ with the Hopf-cyclic homology of the Galois-coring $A \otimes C$, with coefficients in the SAYD module $A$ over the $\times_A$-Hopf algebra $A^e$ via

$$
\begin{align*}
\mathcal{Y}_n&(A \otimes_B A, A^e, A) \longrightarrow \mathcal{Y}_n(A \otimes C, A^e, A), \\
((a^0 \otimes_B 1) \otimes_A \cdots \otimes_A (a^n \otimes_B 1)) \otimes_{A^e} 1 &\mapsto ((a^0 1_{<0>} \otimes 1_{<1>}) \otimes_A \cdots \otimes_A (a^n 1_{<0>} \otimes 1_{<1>})) \otimes_{A^e} 1,
\end{align*}
$$
whose inverse is given by

\[(2.21)\]
\[\mathcal{Y}_n(A \otimes C, A^e, A) \rightarrow \mathcal{Y}_n(A \otimes_B C, A^e, A),\]
\[(\langle a^0 \otimes c^0 \rangle \otimes_A \ldots \otimes_A \langle a^n \otimes c^n \rangle ) \otimes_{A^e} 1 \mapsto ((\langle a^0 c^0 \rangle \otimes_B c^0) \otimes \ldots \otimes_A \langle a^n c^n \rangle \otimes_B c^n) \otimes_{A^e} 1,\]

where \(\tau : C \rightarrow A \otimes_B C, c \mapsto c_1 \otimes_B c_2\) is the translation map, i.e. \(\tau(c) = \text{can}^{-1}(1 \otimes c)\), for any \(c \in C\). The cyclic structure associated to the Galois coring is then given explicitly by

\[\delta_i : \mathcal{Y}_n(A \otimes C, A^e, A) \rightarrow \mathcal{Y}_{n-1}(A \otimes C, A^e, A), \quad 0 \leq i \leq n,\]
\[\delta_i((\langle a^0 \otimes c^0 \rangle \otimes_A \ldots \otimes_A \langle a^n \otimes c^n \rangle) \otimes_{A^e} 1) = (\langle a^0 \otimes c^0 \rangle \otimes_A \ldots \otimes_A (a^i \otimes c^i) \otimes_A \ldots \otimes_A (a^n \otimes c^n) \otimes_{A^e} 1,\]

\[\sigma_j : \mathcal{Y}_n(A \otimes C, A^e, A) \rightarrow \mathcal{Y}_{n+1}(A \otimes C, A^e, A), \quad 0 \leq j \leq n,\]
\[\sigma_j((\langle a^0 \otimes c^0 \rangle \otimes_A \ldots \otimes_A (a^n \otimes c^n) \otimes_{A^e} 1) = (\langle a^0 \otimes c^0 \rangle \otimes_A \ldots \otimes_A \Delta(a^j \otimes c^j) \otimes_A \ldots \otimes_A (a^n \otimes c^n) \otimes_{A^e} 1,\]

\[\tau_n : \mathcal{Y}_n(A \otimes C, A^e, A) \rightarrow \mathcal{Y}_n(A \otimes C, A^e, A),\]
\[\tau_n((\langle a^0 \otimes c^0 \rangle \otimes_A \ldots \otimes_A (a^n \otimes c^n) \otimes_{A^e} 1) = (\langle a^n \otimes c^n \rangle \otimes_A (a^0 \otimes c^0) \otimes_A \ldots \otimes_A (a^{n-1} \otimes c^{n-1}) \otimes_{A^e} 1.\]

3. Hopf-cyclic homology for quotient coalgebra-Galois extensions

Let us assume that \(B \subseteq A\) is a quotient coalgebra-Galois extension, for given a quotient right \(H\)-module coalgebra \(C\) and a right \(H\)-comodule algebra \(A\).

3.1. The SAYD module associated to the extension. In the present subsection we construct an SAYD module over the Hopf algebra \(H\), involved in the \(C\)-Galois extension \(B \subseteq A\). In fact, this construction depends only on a given right \(H\)-comodule algebra \(A\).

**Theorem 3.1 (The SAYD module).** For any right \(H\)-comodule algebra \(A\) the quotient vector space

\[M := (H \otimes A)/\langle ha'(a) \otimes aa'(a) - h \otimes a'a \mid h \in H, a, a' \in A\rangle.\]

is a stable anti-Yetter Drinfeld \(H\)-module with the left \(H\)-module structure

\[H \otimes M \rightarrow M,\]
\[h' \otimes [h \otimes a] \mapsto h' \cdot [h \otimes a] := [h'h \otimes a],\]

and the right \(H\)-comodule structure given by

\[M \rightarrow M \otimes H,\]
\[[h \otimes a] \mapsto [h \otimes a]_{<\alpha>} \otimes [h \otimes a]_{<\beta>} := [h_{(2)} \otimes a_{(0)}] \otimes h_{(3)} a_{(1)} S(h_{(1)}).\]
Proof. It is obvious that the left $H$-module structure on $H \otimes A$ is well defined by the formula (3.1). The left module structure on its quotient $M$ is well defined since

$$h'(ha'_{(1)} \otimes aa'_{(0)} - h \otimes a'a) = (h'h)a'_{(1)} \otimes aa'_{(0)} - (h'h) \otimes a'a$$

which means that elements $ha'_{(1)} \otimes aa'_{(0)} - h \otimes a'a$ span a left $H$-submodule in the left $H$-module $H \otimes A$.

The formula (3.2) defines a right $H$-comodule structure on $H \otimes A$ since

$$[h \otimes a]_{<0> <1>} \otimes [h \otimes a]_{<0> <1>} \otimes [h \otimes a]_{<1>} = [h_{(2)} \otimes a_{(0)}]_{<0> <1>} \otimes [h_{(2)} \otimes a_{(0)}]_{<1>} \otimes h_{(3)} a_{(1)} S(h_{(1)})$$

$$= [h_{(2)} \otimes a_{(0)(0)}] \otimes h_{(2)}(a_{(0)(1)} S(h_{(2)(1)}) \otimes h_{(3)} a_{(1)} S(h_{(1)})$$

$$= [h_{(2)} \otimes a_{(0)}] \otimes h_{(4)} a_{(1)} S(h_{(2)}) \otimes h_{(5)} a_{(2)} S(h_{(1)})$$

$$= [h_{(2)} \otimes a_{(0)}] \otimes h_{(3)} a_{(1)} S(h_{(2)(1)}) \otimes h_{(3)} a_{(2)} S(h_{(1)(1)})$$

$$= [h_{(2)} \otimes a_{(0)}] \otimes (h_{(3)} a_{(1)} S(h_{(1)}))_{(1)} \otimes (h_{(3)} a_{(1)} S(h_{(1)}))_{(2)}$$

$$= [h \otimes a]_{<0> <1>} \otimes [h \otimes a]_{<1> (1)} \otimes [h \otimes a]_{<1> (2)}$$

and

$$[h \otimes a]_{<0>} \varepsilon([h \otimes a]_{<1>}) = [h_{(2)} \otimes a_{(0)}] \varepsilon(h_{(3)} a_{(1)} S(h_{(1)}))$$

$$= [h_{(2)} \otimes a_{(0)}] \varepsilon(h_{(3)}) \varepsilon(a_{(1)}) \varepsilon(S(h_{(1)})) = \varepsilon(h_{(1)}) h_{(2)} \varepsilon(h_{(3)}) \otimes a_{(0)} \varepsilon(a_{(1)})$$

$$= [h \otimes a].$$

As for the well-definedness of the comodule structure on $M$, let us first compute

$$(ha'_{(1)} \otimes aa'_{(0)})_{<0> \otimes (ha'_{(1)} \otimes aa'_{(0)})_{<1>}}$$

$$= ((ha'_{(1)})_{(2)} \otimes (aa'_{(0)})_{(0)}) \otimes (ha'_{(1)})_{(3)} (aa'_{(0)})_{(0)} S((ha'_{(1)})_{(1)})$$

$$= (h_{(2)} a'_{(1)(2)} \otimes a_{(0)} a'_{(0)(0)}) \otimes h_{(3)} a'_{(1)(3)} a_{(1)} a'_{(0)(0)} S(h_{(1)} a'_{(1)(1)})$$

$$= (h_{(2)} a'_{(3)(2)} \otimes a_{(0)} a'_{(0)(0)}) \otimes h_{(3)} a'_{(4)(3)} a_{(1)} a'_{(2)} S(a'_{(3)}) S(h_{(1)})$$

$$= (h_{(2)} a'_{(2)(2)} \otimes a_{(0)} a'_{(0)(0)}) \otimes h_{(3)} a'_{(3)(3)} a_{(1)} \varepsilon(a'_{(3)}) S(h_{(1)})$$

$$= (h_{(2)} a'_{(1)(1)} \otimes a_{(0)} a'_{(0)(0)}) \otimes h_{(3)} a'_{(2)(2)} a_{(1)} S(h_{(1)})$$

$$= (h_{(2)} a'_{(0)(1)} \otimes a_{(0)} a'_{(0)(0)}) \otimes h_{(3)} a'_{(1)(3)} a_{(1)} S(h_{(1)})$$

$$= (h_{(2)} a'_{(0)(1)} \otimes a_{(0)} a'_{(0)(0)}) - h_{(2)} \otimes a'_{(0)(0)} a_{(0)} \otimes h_{(3)} a'_{(1)(3)} a_{(1)} S(h_{(1)})$$

$$+ (h_{(2)} \otimes a'_{(0)(0)}) \otimes h_{(3)} a'_{(1)(1)} a_{(1)} S(h_{(1)})$$

and compare it with

$$(h \otimes a'a)_{<0> \otimes (h \otimes a'a)_{<1>}}$$

$$= (h_{(2)} \otimes (a'a)_{(0)}) \otimes h_{(3)} (a'a)_{(1)} S(h_{(1)})$$

$$= (h_{(2)} \otimes a'_{(0)(0)}) \otimes h_{(3)} a'_{(1)(1)} a_{(1)} S(h_{(1)}).$$
The difference is

\[(ha'(1) \otimes a a'(0) - h \otimes a'a)_{<0>} \otimes (ha'(1) \otimes a a'(0) - h \otimes a'a)_{<1>}\]

\[= (h(2) a'(0)(1) \otimes a(0) a'(0)(0) - h(2) \otimes a'(0) a(0)) \otimes h(3) a'(1) a(1) S(h(1))\]

which means that elements \(ha'(1) \otimes a a'(0) - h \otimes a'a\) span a right \(H\)-subcomodule in a the right \(H\)-comodule \(H \otimes A\).

Now we check the stability condition. We have,

\[(h \otimes a)_{<1>} (h \otimes a)_{<0>} = h(3) a(1) S(h(1))(h(2) \otimes a(0))\]

\[= h(3) a(1) S(h(1)) h(2) \otimes a(0) = \varepsilon(h(1)) h(2) a(1) \otimes a(0)\]

\[= ha(1) \otimes a(0) = (ha(1) \otimes 1 \cdot a(0) - h \otimes a \cdot 1) + h \otimes a,\]

as such, we see that the stability holds only in the level of the quotient \(M\), not on the level of \(H \otimes A\).

Finally, we check the AYD-condition. We have,

\[(h' \cdot [h \otimes a])_{<0>} \otimes (h' \cdot [h \otimes a])_{<1>}\]

\[= [h'h \otimes a]_{<0>} \otimes [h'h \otimes a]_{<1>}\]

\[= ([h'h(2) \otimes a(0)] \otimes (h'h)(3) a(1)) S((h'h)(1))\]

\[= [h'(2) h(2) \otimes a(0)] \otimes h'(3) h(3) a(1) S(h(1)) S(h'(1))\]

\[= h'(2) \cdot [h(2) \otimes a(0)] \otimes h'(3) (h(3) a(1)) S(h(1)) S(h'(1))\]

\[= h'(2) \cdot [h \otimes a]_{<0>} \otimes h'(3) [h \otimes a]_{<1>} S(h'(1)).\]

\[\square\]

3.2. The identification of the associated cyclic objects. Let \(B \subseteq A\) be a \(C\)-Galois extension detailed at the introduction of this section. We shall now identify the Hopf-cyclic cohomology of the Galois-coring of the extension, and hence in view of (2.19) and (2.20) the relative cyclic homology of the extension, with the Hopf-cyclic homology of the coalgebra \(C\), with coefficients in the SAYD module of Theorem 3.1.

To this end, let us first note the following lemma.

**Lemma 3.2.** Given a \(C\)-Galois extension \(B \subseteq A\) as above,

\[((1 \otimes c^0) \otimes_A \cdots \otimes_A (1 \otimes c^0)) \otimes_A e' a'\]

\[= ((1 \otimes c^0 a'(1)) \otimes_A (1 \otimes c^0 a'(2)) \otimes_A \cdots \otimes_A (1 \otimes c^0 a'(n+1))) \otimes_A e' a a'(0).\]

for any \(a, a' \in A\), and any \(c^0, \ldots, c^n \in C\).
Proof. The claim follows at once from the observation
\[
((1 \otimes c^0) \otimes_A \cdots \otimes_A (1 \otimes c^n)) \otimes_{A^e} a' a
\]
\[
= ((1 \otimes c^0) \otimes_A \cdots \otimes_A (1 \otimes c^n))a' \otimes_{A^e} a
\]
\[
= a'(0) ((1 \otimes c^0 a'(1)) \otimes_A (1 \otimes c^1 a'(2)) \otimes \cdots \otimes_A (1 \otimes c^n a'(n+1))) \otimes_{A^e} a
\]
\[
= ((1 \otimes c^0 a'(1)) \otimes_A (1 \otimes c^1 a'(2)) \otimes \cdots \otimes_A (1 \otimes c^n a'(n+1))) \otimes_{A^e} aa'(0).
\]
\[\square\]

We are now ready to prove our main result.

Theorem 3.3. The map
\[
\Phi : ((A \otimes C) \otimes_A \cdots \otimes_A (A \otimes C)) \otimes_{A^e} A \longrightarrow (C \otimes \cdots \otimes C) \otimes_H M
\]
\[
(\Phi) (a^0 \otimes \cdots \otimes a^n) \otimes_{A^e} 1 \mapsto \left( e^0 \cdot a^1(1) \otimes a^{n-1}(n-1) a^n(n) \otimes c^n \otimes e \right) \otimes_H [1 \otimes a^0 a^1(0) \cdots a^n(0)]
\]
is an isomorphism of cyclic objects.

Proof. Let us first consider the mapping
\[
\Phi : ((A \otimes C) \otimes A \cdots \otimes A (A \otimes C)) \otimes A \longrightarrow (C \otimes \cdots \otimes C) \otimes_H M
\]
\[
((a_0 \otimes c^0) \otimes \cdots \otimes (a_n \otimes c^n)) \otimes a' \mapsto
\]
\[
(c^0 \cdot a^1(1) \otimes a^{n-1}(n-1) a^n(n) \otimes c^n) \otimes_H [1 \otimes a^0 a^1(0) \cdots a^n(0)].
\]
We note that for \(1 \leq k \leq n,\)
\[
\Phi \left( ((a_0 \otimes c^0) \otimes \cdots \otimes (a^{k-1} \otimes c^{k-1}) \otimes (a^k \otimes c^k) \otimes \cdots \otimes (a_n \otimes c^n)) \otimes a' \right)
\]
\[
= \Phi \left( ((a_0 \otimes c^0) \otimes \cdots \otimes (a^{k-1} a^k(0) \otimes c^{k-1} a^k(1)) \otimes (1 \otimes c^k) \otimes \cdots \otimes (a_n \otimes c^n)) \otimes a' \right),
\]
and that
\[
\Phi \left( ((a' a^0 c^0) \otimes \cdots \otimes (a^n c^n)) \otimes 1 \right)
\]
\[
= (c^0 \cdot a^1(1) \otimes a^2(2) \cdots a^n(n) \otimes c^n) \otimes_H [1 \otimes a' a^0 a^1(0) \cdots a^n(0)]
\]
\[
= (c^0 \cdot a^1(1) \otimes a^2(2) \cdots a^n(n) \otimes c^n)
\]
\[
\otimes_H [a'(0) \otimes a^0 a^1(0) \cdots a^n(0) a'(0)]
\]
\[
= (c^0 \cdot a^1(1) \otimes a^2(2) \cdots a^n(n) a'(n) c^n) \otimes_H [1 \otimes a^0 a^1(0) \cdots a^n(0) a'(0)]
\]
\[
= \Phi \left( ((a^0 \otimes c^0) \otimes \cdots \otimes (a^n a'(0) \otimes c^n a'(1))) \otimes 1 \right),
\]
using Lemma 3.2. As a result, (3.7) induces the map (3.6).
Let us next define the mapping

\[
\tilde{\Psi} : (C \otimes \cdots \otimes C) \otimes H \otimes A \longrightarrow ((A \otimes C) \otimes_A \cdots \otimes_A (A \otimes C)) \otimes_{A^e} A
\]

\[
c^0 \otimes \cdots \otimes c^n \otimes (h \otimes a) \mapsto \\
\left( (1 \otimes c^0 \cdot h_{(1)} S^{-1}(a_{(n+1)})) \otimes_A (1 \otimes c^1 \cdot h_{(2)} S^{-1}(a_{(n)})) \otimes_A \cdots \otimes_A (1 \otimes c^n \cdot h_{(n+1)} S^{-1}(a_{(1)})) \right) \otimes_{A^e} a_{(0)}.
\]

We first note that for any \( a, a' \in A, \)

\[
\tilde{\Psi}(c^0 \otimes \cdots \otimes c^n \otimes (ha'_{(1)} \otimes aa'_{(0)}))
\]

\[
= \left( (1 \otimes c^0 \cdot h_{(1)} a'_{(1)} S^{-1}(a_{(n+1)})) \otimes_A (1 \otimes c^1 \cdot h_{(2)} a'_{(2)} S^{-1}(a_{(n)})) \otimes_A \cdots \right.
\]

\[
\left. \otimes_A (1 \otimes c^n \cdot h_{(n+1)} a'_{(n+1)} S^{-1}(a_{(1)})) \right) \otimes_{A^e} a_{(0)} a'
\]

\[
= \left( (1 \otimes c^0 \cdot h_{(1)} S^{-1}(a_{(n+1)})) \otimes_A (1 \otimes c^1 \cdot h_{(2)} S^{-1}(a_{(n)})) \otimes_A \cdots \right.
\]

\[
\left. \otimes_A (1 \otimes c^n \cdot h_{(n+1)} S^{-1}(a_{(1)})) \right) \otimes_{A^e} (a_{(0)}, 1) \cdot a'
\]

\[
= \left( (1 \otimes c^0 \cdot h_{(1)} S^{-1}(a_{(n+1)})) \otimes_A (1 \otimes c^1 \cdot h_{(2)} S^{-1}(a_{(n)})) \otimes_A \cdots \right.
\]

\[
\left. \otimes_A (1 \otimes c^n \cdot h_{(n+1)} S^{-1}(a_{(1)})) \right) \cdot a_{(0)} \otimes_{A^e} a'
\]

\[
= \left( (a \otimes c^0 \cdot h_{(1)}) \otimes_A (1 \otimes c^1 \cdot h_{(2)}) \otimes_A \cdots \otimes_A (1 \otimes c^n \cdot h_{(n+1)}) \right) \otimes_{A^e} a'
\]

\[
= \left( (a \otimes c^0 \cdot h_{(1)}) \otimes_A (1 \otimes c^1 \cdot h_{(2)}) \otimes_A \cdots \otimes_A (1 \otimes c^n \cdot h_{(n+1)}) \right) \otimes_{A^e} (a', 1) \cdot 1
\]

\[
= (a', 1) \cdot \left( (a \otimes c^0 \cdot h_{(1)}) \otimes_A (1 \otimes c^1 \cdot h_{(2)}) \otimes_A \cdots \otimes_A (1 \otimes c^n \cdot h_{(n+1)}) \right) \otimes_{A^e} a'
\]

\[
= \left( (a' a \otimes c^0 \cdot h_{(1)}) \otimes_A (1 \otimes c^1 \cdot h_{(2)}) \otimes_A \cdots \otimes_A (1 \otimes c^n \cdot h_{(n+1)}) \right) \otimes_{A^e} 1
\]

\[
= (a' a, 1) \cdot \left( (1 \otimes c^0 \cdot h_{(1)}) \otimes_A (1 \otimes c^1 \cdot h_{(2)}) \otimes_A \cdots \otimes_A (1 \otimes c^n \cdot h_{(n+1)}) \right) \otimes_{A^e} 1
\]

\[
= \left( (1 \otimes c^0 \cdot h_{(1)}) \otimes_A (1 \otimes c^1 \cdot h_{(2)}) \otimes_A \cdots \otimes_A (1 \otimes c^n \cdot h_{(n+1)}) \right) \otimes_{A^e} a' a
\]

\[
= \left( (1 \otimes c^0 \cdot h_{(1)} S^{-1}(a'_{(n+1)} a_{(n+1)}) \otimes_A (1 \otimes c^1 \cdot h_{(2)} S^{-1}(a'_{(n)} a_{(n)})) \otimes_A \cdots \right.
\]

\[
\left. \otimes_A (1 \otimes c^n \cdot h_{(n+1)} S^{-1}(a'_{(1)} a_{(1)})) \right) \otimes_{A^e} a'_{(0)} a_{(0)}
\]

\[
= \tilde{\Psi}(c^0 \otimes \cdots \otimes c^n \otimes (h \otimes a')).
\]

where we used Lemma 3.2 once more at the eleventh equality. Moreover, it is evident that

\[
\tilde{\Psi}(c^0 \otimes \cdots \otimes c^n \otimes (h \otimes a)) = \tilde{\Psi}(c^0 \cdot h_{(1)} \otimes \cdots \otimes c^n \cdot h_{(n+1)} \otimes (1 \otimes a)).
\]
As a result, (3.8) induces a map

\[(3.9)\]
\[
\Psi : (C \otimes \cdots \otimes C) \otimes_H \otimes M \longrightarrow ((A \otimes C) \otimes_A \cdots \otimes_A (A \otimes C)) \otimes_{A^e} A
\]
\[
c^0 \otimes \cdots \otimes c^n \otimes_H [1 \otimes a] \mapsto
\left( (1 \otimes c^0 \cdot S^{-1}(a_{(n+1)})) \otimes_A (1 \otimes c^1 \cdot S^{-1}(a_{(n)})) \otimes_A \cdots \otimes_A (1 \otimes c^n \cdot S^{-1}(a_{(1)})) \right) \otimes_{A^e} a_{(0)}.
\]

Finally we see that
\[
\Phi \circ \Psi((c^0 \otimes \cdots \otimes c^n) \otimes_H [1 \otimes a])
\]
\[
= \Phi\left( (1 \otimes c^0 \cdot S^{-1}(a_{(n+1)})) \otimes_A (1 \otimes c^1 \cdot S^{-1}(a_{(n)})) \otimes_A \cdots \otimes_A (1 \otimes c^n \cdot S^{-1}(a_{(1)})) \right) \otimes_{A^e} a_{(0)}
\]
\[
= \Phi\left( (a \otimes c^0) \otimes_A (1 \otimes c^1) \otimes_A \cdots \otimes_A (1 \otimes c^n) \right) \otimes_{A^e} 1
\]
\[
= (c^0 \otimes \cdots \otimes c^n) \otimes_H [1 \otimes a],
\]
and that
\[
\Psi \circ \Phi((1 \otimes c^0) \otimes_A (1 \otimes c^1) \otimes_A \cdots \otimes_A (1 \otimes c^n)) \otimes_{A^e} a
\]
\[
= \Psi((c^0 \otimes \cdots \otimes c^n) \otimes_H [1 \otimes a])
\]
\[
= ((a \otimes c^0) \otimes_A (1 \otimes c^1) \otimes_A \cdots \otimes_A (1 \otimes c^n)) \otimes_{A^e} 1
\]
\[
= (a, 1) \cdot ((1 \otimes c^0) \otimes_A (1 \otimes c^1) \otimes_A \cdots \otimes_A (1 \otimes c^n)) \otimes_{A^e} 1
\]
\[
= ((1 \otimes c^0) \otimes_A (1 \otimes c^1) \otimes_A \cdots \otimes_A (1 \otimes c^n)) \otimes_{A^e} (a, 1) \cdot 1
\]
\[
= ((1 \otimes c^0) \otimes_A (1 \otimes c^1) \otimes_A \cdots \otimes_A (1 \otimes c^n)) \otimes_{A^e} a
\]

which proves that (3.6) and (3.9) are inverse to each other.

Let us finally show that the mapping (3.6) commutes with the cyclic structure maps. To this end, we start with the commutation with the faces. Namely,

\[
\Phi(\delta_i((a^0 \otimes c^0) \otimes_A \cdots \otimes_A (a^n \otimes c^n) \otimes_{A^e} 1))
\]
\[
= \Phi((a^0 \otimes c^0) \otimes_A \cdots \otimes_A \varepsilon(a^i \otimes c^i) \otimes_A \cdots \otimes_A (a^n \otimes c^n) \otimes_{A^e} 1)
\]
\[
= \varepsilon(c^i) \Phi((a^0 \otimes c^0) \otimes_A \cdots \otimes_A (a^i \otimes c^{i-1}) \otimes_A (a^i a^{i+1} \otimes c^{i+1}) \otimes_A (a^n \otimes c^n) \otimes_{A^e} 1)
\]
\[
= \varepsilon(c^i) \left( c^0 \cdot a^1_{(1)} \cdot a^1_{(2)} \cdots a^n_{(n+1)} \otimes (c^0 \otimes \cdots \otimes c^n) \otimes_H [1 \otimes a^0 a^1_{(0)} \cdots a^n_{(0)}] \right)
\]
\[
= \left( c^0 \cdot a^1_{(1)} \otimes \cdots \otimes a^n_{(n)} \otimes c^1 \cdot a^2_{(2)} \cdots a^n_{(n)} \otimes \cdots \otimes c^0 \cdot a^n_{(0)} \otimes c^n \otimes (c^0 \otimes \cdots \otimes c^n) \otimes_H [1 \otimes a^0 a^1_{(0)} \cdots a^n_{(0)}] \right)
\]
\[
= d_i(\Phi((a^0 \otimes c^0) \otimes_A \cdots \otimes_A (a^n \otimes c^n) \otimes_{A^e} 1)),
\]
for $0 \leq i \leq n$. On the next round, we consider the degeneracy operators. We have,

\[
\Phi(\sigma_j((a^0 \otimes c^0) \otimes_A \ldots \otimes_A (a^n \otimes c^n) \otimes_{A^e} \otimes 1))
\]

\[
= \Phi((a^0 \otimes c^0) \otimes_A \ldots \otimes_A \Delta(a^j \otimes c^j) \otimes_A \ldots \otimes_A (a^n \otimes c^n) \otimes_{A^e} \otimes 1)
\]

\[
= (c^0 \cdot a^1(1) \ldots a^n(1) \otimes c^1 \cdot a^2(2) \ldots a^n(2) \otimes \ldots
\]

\[
\ldots \otimes c^{j-1} \cdot a^j(j) \cdot a^{j+1}(j) \ldots a^n(j) \otimes c^j(1) \cdot a^{j+1}(j+1) \ldots a^n(j+1) \otimes c^j(2) \cdot a^{j+2}(j+2) \ldots a^n(j+2) \otimes \ldots
\]

\[
\ldots \otimes c^{n-1} \cdot a^{n+1}(n+1) \otimes c^n) \otimes_H [1 \otimes a^0 a^1(0) \ldots a^n(0)]
\]

\[
= s_j(\Phi((a^0 \otimes c^0) \otimes_A \ldots \otimes_A (a^n \otimes c^n) \otimes_{A^e} \otimes 1)),
\]

for $0 \leq j \leq n$. As for the cyclic operator, we have

\[
\Phi(\tau_n((a^0 \otimes c^0) \otimes_A \ldots \otimes_A (a^n \otimes c^n) \otimes_{A^e} \otimes 1))
\]

\[
= \Phi((a^n \otimes c^n) \otimes_A (a^0 \otimes c^0) \otimes_A \ldots \otimes_A (a^{n-1} \otimes c^{n-1}) \otimes_{A^e} \otimes 1)
\]

\[
= (c^n \cdot a^0(1) \ldots a^{n-1}(1) \otimes c^0 \cdot a^1(2) \ldots a^{n-1}(2) \otimes \ldots
\]

\[
\ldots \otimes c^{n-2} \cdot a^{n-1}(n) \otimes c^{n-1}) \otimes_H [1 \otimes a^0 a^0(0) \ldots a^{n-1}(0)]
\]

\[
= (c^n \cdot a^0(1) \ldots a^{n-1}(1) \otimes c^0 \cdot a^1(2) \ldots a^{n-1}(2) \otimes \ldots
\]

\[
\ldots \otimes c^{n-2} \cdot a^{n-1}(n) \otimes c^{n-1}) \otimes_H [a^n(1) \otimes a^0(0) \ldots a^{n-1}(0)a^n(0)]
\]

\[
= (c^n \cdot a^0(1) \ldots a^{n-1}(1) \cdot a^n(1) \otimes c^0 \cdot a^1(2) \ldots a^n(2) \otimes \ldots
\]

\[
\ldots \otimes c^{n-2} \cdot a^{n-1}(n) \otimes c^{n-1}) \cdot a^n(n) \otimes H [1 \otimes a^0(0) \ldots a^{n-1}(0)a^n(0)]
\]

\[
= t_n(\Phi((a^0 \otimes c^0) \otimes_A \ldots \otimes_A (a^n \otimes c^n) \otimes_{A^e} \otimes 1)),
\]

using Lemma 3.2 one last time at the third equality.

\[\square\]

4. Special cases

In the present section we explain that the SAYD module constructed in Theorem 3.1 is a noncommutative counterpart of the Brylinski $G$-scheme. Next, we specialise our construction to the case of Hopf-Galois extensions, to recover the SAYD module constructed by Jara and Ţeanu [21].

4.1. The Brylinski $G$-scheme. The next proposition claims that beyond the Hopf-Galois context our SAYD module is a noncommutative counterpart of the Brylinski $G$-scheme,
The inertia groupoid \([24]\) of a given \(G\)-action groupoid for a given \(G\)-scheme \(X\) is nothing but the action groupoid of the Brylinski \(G\)-scheme. The fundamental role of the Brylinski scheme relies on the fact that its category of equivariant sheaves of \(O\)-modules is equivalent to the Drinfeld double of the monoidal category of equivariant sheaves of \(O\)-modules on the \(G\)-scheme \(X\) itself, at least for a finite group \(G\), conjecturally in much wider generality \([20]\). The inertia groupoid, through so called extended quotient construction, plays a fundamental role in local Langlands duality \([1]\) and the orbifold and stringy cohomology \([14, 22, 7]\).

The relation between the resulting cyclic object computing the Hopf-cyclic homology and inertia phenomena playing a role in constructing the orbifold cohomology, had been noticed in \([23]\) in a special case of the homogeneous quotient-coalgebra-Galois extensions. The following proposition allows us to extends this relation from the Jara-Ştefan case of arbitrary Hopf-Galois extensions to the case of arbitrary quotient module coalgebra-Galois extensions, relating it to the Brylinski scheme in the commutative case.

**Proposition 4.1.** If the Hopf algebra \(H\) and the right \(H\)-comodule algebra are commutative, the SAYD module \(M\) becomes a commutative right \(H\)-comodule algebra. If \(G = \text{Spec}(H)\) is a corresponding affine group scheme and \(X = \text{Spec}(A)\) a corresponding affine \(G\)-scheme, \(\text{Spec}(M)\) is the Brylinski \(G\)-scheme, on points defined as

\[
\{(g, x) \in Ad(G) \times X \mid xg = x\},
\]

with the diagonal right \(G\)-action by right conjugations on \(Ad(G) = G\) and a given right \(G\)-action on \(X\), equipped with a \(G\)-equivariant morphism into \(Ad(G)\).

**Proof.** The algebra structure comes from the fact that in the coproduct commutative algebra \(H \otimes A\) the \(H\)-submodule of relations defining \(M\) is an ideal. This ideal is generated by elements of the form

\[
a'(1) \otimes a'(0) - h \otimes a'
\]

defining a closed \(G\)-invariant subscheme of the \(G\)-scheme \(G \times X\) with the diagonal \(G\)-action by conjugations on \(G\), which results from evaluation the \(H\)-comodule structure on points. By evaluating on points the \(H\)-module structure gives a map into \(Ad(G)\) induced by the restriction of the projection \(Ad(G) \times X \to Ad(G)\). By the AYD-property the map into \(Ad(G)\) is \(G\)-equivariant. The stability property of the SAYD module is equivalent to the fact that in the pair \((g, x)\) the group element \(g\) stabilizes \(x\). \(\square\)

In the case of a transitive right \(G\)-action on \(X\) with a distinguished point \(x_0 \in X\), covering the instanton case, we can assume that \(X = K\backslash G\), the left quotient by the stabilizer \(K\) of \(x_0\). Then the Brylinski scheme is a right \(G\)-orbit of \(K = K \times \{Ke\}\) in \(Ad(G) \times (K\backslash G)\) and in the inertia groupoid the stabilizer of a point \((k, Ke)g\) of the Brylinski \(G\)-scheme is equal to \(g^{-1}Z_K(k)g\), the conjugated centralizer \(Z_K(k)\) of \(k\) in \(K\).

**4.2. The Jara-Ştefan SAYD module.** In this subsection we shall prove that the SAYD module of Theorem 3.1 generalizes the one introduced by Jara and Ştefan in \([21]\). For that
Lemma 4.2. For any right Hopf-Galois extension \( A \) of an algebra \( B \), with the Galois Hopf algebra \( H \) and the translation map

\[
\tau : H \to A \otimes_B A, \quad h \mapsto h^1 \otimes_B h^2 \tag{4.3}
\]

one has

\[
(hh')^1 \otimes_B (hh')^2 = h'^1 h^1 \otimes_B h^2 h'^2. \tag{4.4}
\]

Proof. By the definition of the translation map as the restriction of the inverse \( \text{can}^{-1} \) to the canonical map \( \text{can} \) to \( 1 \otimes H \subseteq A \otimes H \) one has

\[
1 \otimes h = \text{can}(\text{can}^{-1}(1 \otimes h)) = \text{can}(h^1 \otimes_B h^2) = h^1 h^2 \otimes 1 \otimes h^2(1) . \tag{4.5}
\]

Applying (4.5) to \( 1 \otimes hh' \) one gets

\[
1 \otimes hh' = (hh')^1(hh')^2 \otimes (hh')^2(1) . \tag{4.6}
\]

Applying (4.5) to \( 1 \otimes h' \) in \( 1 \otimes hh' = (1 \otimes h')(1 \otimes h') \), inscribing 1 between the two left-most tensorands, applying (4.5) to \( 1 \otimes h \) and finally using the definition of a comodule algebra one gets

\[
1 \otimes hh' = h'^1 h^1 h^2 (0) \otimes hh^2(1)
\]

\[
= h'^1 h^1 h^2 (0) \otimes (h^2 h'^2)(1) . \tag{4.7}
\]

Now, applying \( \text{can}^{-1} \) to the equality

\[
(hh')^1(hh')^2 \otimes (hh')^2(1) = h'^1 h^1 h^2 h'^2 \otimes (h^2 h'^2)(1) , \tag{4.8}
\]

resulting from comparing the right hand sides of (4.6) and (4.7), one obtains the desired identity.

Lemma 4.3. The image of the translation map belongs to the \( B \)-centralizer \( (A \otimes_B A)^B \) of the \( A \)-bimodule \( A \otimes_B A \), namely,

\[
\tau(H) \subseteq (A \otimes_B A)^B . \tag{4.9}
\]

Proof. Let us multiply the element \( \tau(h) = h^1 \otimes h^2 \) from both sides by \( b \in B = A^{op}H \subseteq A \).

\[
br(h) = bh^1 \otimes h^2 = \text{can}^{-1}(1 \otimes h) = \text{can}^{-1}(b \otimes h),
\]

\[
\tau(h)b = h^1 \otimes h^2 b = \text{can}^{-1}(1 \otimes h)b = \text{can}^{-1}((1 \otimes h)b) = \text{can}^{-1}(b(0) \otimes hb^2(1)) = \text{can}^{-1}(b \otimes h), \tag{4.10}
\]

hence \( b\tau(h) = \tau(h)b \).
Lemma 4.4. Given a Hopf-Galois extension $B \subseteq A$, and $a \in A$, the identity
\begin{equation}
 a \otimes 1 = a_{(2)}^{[1]} \otimes a_{(0)} a_{(1)}^{[1]}
\end{equation}
holds in $(A \otimes A)_B$.

Proof. It suffices to prove the flipped version in $A \otimes_B A$, that is,
\begin{equation}
 1 \otimes_B a = \text{can}^{-1}(\text{can}(1 \otimes_B a)) = \text{can}^{-1}(a_{(0)} \otimes a_{(1)}) = a_{(0)} a_{(1)}^{[1]} \otimes_B a_{(2)}^{[2]}.
\end{equation}
\hfill \square

In the following proposition, for any $B$-bimodule $N$ we use the symbol $N_B$ to denote the quotient space $N/[B,N]$.

Proposition 4.5. For any right Hopf-Galois extension $A$ of an algebra $B$, with the Galois Hopf algebra $H$ the SAYD module $M$ is isomorphic to the quotient space $A_B$ with its canonical structure of an SAYD module.

Proof. Applying the bijective linear map (the flipped $\text{can}^{-1}$)
\begin{equation}
 \alpha : H \otimes A \rightarrow (A \otimes A)_B, \quad h \otimes a \mapsto h^{[2]} \otimes ah^{[1]}
\end{equation}
to $h a_{(1)}' \otimes a a_{(0)}'$ in $H \otimes A$ and next the flipped version of the identity (4.4), we obtain
\begin{equation}
 \alpha(h a_{(1)}' \otimes a a_{(0)}') = (h a_{(1)}')^{[2]} \otimes a a_{(0)}'(h a_{(1)})^{[1]}
 = h^{[2]} a_{(1)}'^{[2]} \otimes a a_{(0)}' a_{(1)}'^{[1]} h^{[1]} = h^{[2]} a' \otimes ah^{[1]}.
\end{equation}
in view of (4.11). Note that the application of (4.11) to prove the latter equality is legitimate by the flipped version of (4.9).

We have also
\begin{equation}
 \alpha(h \otimes a') = h^{[2]} \otimes a'ah^{[1]},
\end{equation}
hence finally
\begin{equation}
 \alpha(h a_{(1)} \otimes a a_{(0)} - h \otimes a'a) = h^{[2]} a' \otimes ah^{[1]} - h^{[2]} \otimes a'ah^{[1]}.
\end{equation}
Since by (4.13) the elements $h^{[2]} \otimes ah^{[1]}$ span $(A \otimes A)_B$, the latter means that (4.13) induces the isomorphism
\begin{equation}
 M \rightarrow (A \otimes_A A)_B = A_B, \quad h \otimes a \mapsto h^{[2]} ah^{[1]}.
\end{equation}
Note that by the flipped version of (4.4), the latter isomorphism is left $H$-linear. Its right $H$-colinearity, on the other hand, follows from the fact that the translation map is a morphism of $H$-bicomodules [9, Prop. 3.6] which is equivalent to the identity
\begin{equation}
 h_{(1)} \otimes h_{(2)}^{[1]} \otimes_B h_{(2)}^{[2]} \otimes h_{(3)} = S^{-1}(h_{(1)}^{[1]}) \otimes h_{(0)}^{[1]} \otimes_B h_{(0)}^{[2]} \otimes h_{(1)}^{[2]}.
\end{equation}
in $H \otimes A \otimes_B A \otimes H$. Applying the antipode $S : H \to H$ to the left most tensorand, next moving the first tensorand to the last position, and finally flipping the first two tensorands of both sides of the resulting identity, we arrive at the identity

\[(4.19) \quad h_{(2)}^{[2]} \otimes h_{(2)}^{[1]} \otimes h_{(3)} \otimes S(h_{(1)}) = h_{(0)}^{[2]} \otimes h_{(0)}^{[1]} \otimes h_{(1)}^{[2]} \otimes h_{(1)}^{[1]} .\]

in $(A \otimes A)_B \otimes H \otimes H$. The latter implies

\[(4.20) \quad h_{(2)}^{[2]} a_{(0)} h_{(2)}^{[1]} \otimes h_{(3)} a_{(1)} S(h_{(1)}) = h_{(0)}^{[2]} a_{(0)} h_{(0)}^{[1]} \otimes h_{(1)}^{[2]} a_{(1)} h_{(1)}^{[1]} \]

in $A_B \otimes H$, which amounts to the fact that the map (4.17) of SAYD modules is also $H$-colinear. \qed

5. The application to noncommutative principal fibrations

5.1. Relative topology of a fibration. Assume that the extension of $\mathbb{C}$-algebras $B \subseteq A$ of finite type describes a smooth morphism $f : X \to Y$ of affine algebraic manifolds.

Then the relative periodic cyclic homology $HP_*(A|B)$ is a finitely generated projective $B$-module which describes the vector bundle of the de Rham cohomology of the fibers of the fibration $f : X \to Y$. Equipped with the Gauss-Manin connection it describes the $\mathbb{Z}/2$-graded system of local coefficients $R^\bullet f_* \mathbb{C}_X$ in euclidean topology.

The Gauss-Manin connection makes sense also for noncommutative fibrations over commutative bases [16, 32, 25].

5.2. The main application. The main application of arguments of previous sections is that identifying the Hopf-cyclic homologies of the Sweedler and Galois corings, in view of the Galois condition, we can identify an invariant of the relative topology of a principal fibration with an invariant of the subgroup-Galois symmetry and the inertia of the overgroup, as subsumed in the following theorem.

**Theorem 5.1.** For any $H$-module coalgebra $C$ and any $C$-Galois $H$-comodule algebra $A$ extension of an algebra $B$ the relative periodic cyclic homology of the extension $B \subseteq A$ is isomorphic to the Hopf-cyclic homology of the $H$-module coalgebra $C$ with SAYD-coefficients in $M$

\[ HP_*(A|B) \cong HP^H_*(C, M). \]

Finally, we note that it would require further investigation to study what happens to the Gauss-Manin connection under this isomorphism.

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References

[1] A.-M. Aubert, P. Baum, R. Plymen, and M. Solleveld. Geometric structure in smooth dual and local Langlands conjecture. *Jpn. J. Math.*, 9(2):99–136, 2014.

[2] G. Böhm. Hopf algebroids. In *Handbook of algebra. Vol. 6*, volume 6 of *Handb. Algebr.*, pages 173–235. Elsevier/North-Holland, Amsterdam, 2009.

[3] G. Böhm and D. Ţeanâ. (Co)cyclic (co)homology of bialgebroids: an approach via (co)monads. *Comm. Math. Phys.*, 282(1):239–286, 2008.

[4] F. Bonechi, N. Ciccoli, L. Dąbrowski, and M. Tarlini. Bijectivity of the canonical map for the noncommutative instanton bundle. *J. Geom. Phys.*, 51(1):71–81, 2004.

[5] F. Bonechi, N. Ciccoli, and M. Tarlini. Noncommutative instantons on the 4-sphere from quantum groups. *Comm. Math. Phys.*, 226(2):419–432, 2002.

[6] J.-L. Brylinski. Cyclic homology and equivariant theories. *Ann. Inst. Fourier (Grenoble)*, 37(4):15–28, 1987.

[7] J.-L. Brylinski and V. Nistor. Cyclic cohomology of étale groupoids. *K-Theory*, 8(4):341–365, 1994.

[8] T. Brzeziński. Quantum homogeneous spaces as quantum quotient spaces. *J. Math. Phys.*, 37(5):2388–2399, 1996.

[9] T. Brzeziński. Translation map in quantum principal bundles. *J. Geom. Phys.*, 20(4):349–370, 1996.

[10] T. Brzeziński. Quantum homogeneous spaces and coalgebra bundles. *Rep. Math. Phys.*, 40(2):179–185, 1997. Quantizations, deformations and coherent states (Białowieża, 1996).

[11] T. Brzeziński and P. M. Hajac. Galois-type extensions and equivariant projectivity, arXiv:0901.0141, (2009).

[12] T. Brzeziński and R. Wisbauer. *Corings and comodules*, volume 309 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 2003.

[13] S. Caenepeel. Galois corings from the descent theory point of view. In *Galois theory, Hopf algebras, and semiabelian categories*, volume 43 of *Fields Inst. Commun.*, pages 163–186. Amer. Math. Soc., Providence, RI, 2004.

[14] W. Chen and Y. Ruan. A new cohomology theory of orbifold. *Comm. Math. Phys.*, 248(1):1–31, 2004.

[15] A. Connes and H. Moscovici. Hopf algebras, cyclic cohomology and the transverse index theorem. *Comm. Math. Phys.*, 198(1):199–246, 1998.

[16] E. Getzler. Cartan homotopy formulas and the Gauss-Manin connection in cyclic homology. In *Quantum deformations of algebras and their representations (Ramat-Gan, 1991/1992; Rehovot, 1991/1992)*, volume 7 of *Israel Math. Conf. Proc.*, pages 65–78. Bar-Ilan Univ., Ramat Gan, 1993.

[17] P. M. Hajac, M. Khalkhali, B. Rangipour, and Y. Sommerhäuser. Hopf-cyclic homology and cohomology with coefficients. *C. R. Math. Acad. Sci. Paris*, 338(9):667–672, 2004.

[18] P. M. Hajac, M. Khalkhali, B. Rangipour, and Y. Sommerhäuser. Stable anti-Yetter-Drinfeld modules. *C. R. Math. Acad. Sci. Paris*, 338(8):587–590, 2004.

[19] M. Hassanzadeh and B. Rangipour. Equivariant Hopf Galois extensions and Hopf cyclic cohomology. *J. Noncommut. Geom.*, 7(1):105–133, 2013.

[20] V. Hinich. Drinfeld double for orbifolds. In *Quantum groups*, volume 433 of *Contemp. Math.*, pages 251–265. Amer. Math. Soc., Providence, RI, 2007.

[21] P. Jara and D. Ţeanâ. Hopf-cyclic homology and relative cyclic homology of Hopf-Galois extensions. *Proc. London Math. Soc. (3)*, 93(1):138–174, 2006.

[22] E. Lupercio and B. Uribe. Inertia orbifolds, configuration spaces and the ghost loop space. *Q. J. Math.*, 55(2):185–201, 2004.

[23] T. Maszczyk and S. Sütlü. Cyclic homology and quantum orbits. *SIGMA Symmetry Integrability Geom. Methods Appl.*, 11:041, 27 pages, 2015.

[24] I. Moerdijk. Orbifolds as groupoids: an introduction. In *Orbifolds in mathematics and physics (Madison, WI, 2001)*, volume 310 of *Contemp. Math.*, pages 205–222. Amer. Math. Soc., Providence, RI, 2002.

[25] A. Petrov, D. Vaintrob, and V. Vologodsky. The Gauss-Manin connection on the periodic cyclic homology. *Selecta Math. (N.S.)*, 24(1):531–561, 2018.
[26] D. Quillen. Cyclic cohomology and algebra extensions. *K-Theory*, 3(3):205–246, 1989.
[27] B. Rangipour and S. Sütlü. Cyclic cohomology of Lie algebras. *Documenta Math.*, 17:483–515, 2012.
[28] B. Rangipour and S. Sütlü. SAYD Modules over Lie-Hopf Algebras. *Comm. Math. Phys.*, 316(1):199–236, 2012.
[29] B. Rangipour and S. Sütlü. A van Est isomorphism for bicrossed product Hopf algebras. *Comm. Math. Phys.*, 311(2):491–511, 2012.
[30] P. Schauenburg. Bialgebras over noncommutative rings and a structure theorem for Hopf bimodules. *Appl. Categ. Structures*, 6(2):193–222, 1998.
[31] P. Schauenburg. Duals and doubles of quantum groupoids (×_H-Hopf algebras). In *New trends in Hopf algebra theory (La Falda, 1999)*, volume 267 of *Contemp. Math.*, pages 273–299. Amer. Math. Soc., Providence, RI, 2000.
[32] B. Tsygan. On the Gauss-Manin connection in cyclic homology. *Methods Funct. Anal. Topology*, 13(1):83–94, 2007.

Instytut Matematyki, Uniwersytet Warszawski, ul. Banacha 2, 02–097 Warszawa, Poland

E-mail address: t.maszczyk@uw.edu.pl

İşik University, Department of Mathematics, 34980, Şile, İstanbul, Turkey

E-mail address: serkan.sutlu@isikun.edu.tr