Tunable Aharonov-Bohm-like cages for quantum walks

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(Dated: October 3, 2019)

Aharonov-Bohm cages correspond to an extreme localization phenomenon for two-dimensional tight-binding electrons in a transverse magnetic field. If the flux per plaquette is settled to a half quantum, a destructive interference forbids the particle to diffuse away from a small cluster. Another signature is the pinching of the energy levels into a set of highly degenerate discrete levels as the flux approaches the critical value. We show here that cages also occur for discrete-time quantum walks on either the diamond chain or the \( T_3 \) tiling. The corresponding quasi-energies when plotted as a function of the flux result in a Hofstadter-like butterfly, that displays pinching for a critical flux. The associated quantum walk cages have larger sizes than their tight-binding counterparts, and the critical flux may be tuned away from half a quantum.

The behavior of electrons in two-dimensional (2d) structures under a magnetic field has been of special interest in condensed matter physics, leading to various subtle effects ranging from the (integer and fractional) quantum Hall effects, very well documented at the experimental levels [1, 2], to the predicted self-similar Hofstadter butterfly pattern [3] describing the complex structure of energy levels for tight-binding (TB) electrons with respect to the magnetic flux. In that case, the magnetic field enters the Hamiltonian through vector potential derived additional phases multiplying the hopping terms (known as Peierls substitution [4]). This pattern is difficult to observe in real atomic 2d structures, due both to the lack of simple experiments able to probe the energy level organization “inside” the butterfly, and to the rather high magnetic fields at which these effects should be seen. The possibility of generating “artificial” magnetic fields acting on cold atoms assemblies opens the way to different types of experiments in that direction [5].

About 20 years ago, an extreme localization effect was proposed, which occurs for tight-binding particles in certain periodic lattices, like the 2d \( T_3 \) rhombus tiling, at half a magnetic flux quantum per plaquette [6]. This is due to a complete destructive interference preventing the particle to escape from finite clusters, known as Aharonov-Bohm (A-B) cages. Generic structures, like the infinite square lattice studied in [3], lead to energy bands for rational fluxes (measured in units of the flux quantum), and more complex density of states for irrational fluxes. In contrast, the \( T_3 \) butterfly displays the surprising feature that the density of states pinches near a half flux quantum, leading to an energy spectrum consisting of three highly degenerate energy levels. The one at zero energy is present at any flux and is due to a chiral (bipartite) symmetry [7]. The other two are the result of destructive interferences tuned by the magnetic field. It was demonstrated that a particle, placed at initial time on one of the three different sites of the lattice, displays a quantum diffusion limited to a small cluster of sites, and eventually periodically bounces back and forth to its original position. This effect disappears if the flux is tuned away from half a flux quantum [8] or if interactions between particles [5] or disorder [9] are introduced.

This predicted phenomenon inspired several experimental implementations (or propositions), among which mesoscopic systems, like superconducting wire networks [10], Josephson junction arrays [11], cold atomic gases [12], photonic lattices [13], ions micro traps [14], etc...

Here, we show that, beside systems modelled on the original TB Hamiltonian [6], a related caging phenomenon can be obtained in the a priori different case of unitary quantum walks (QW).

Quantum walks - QW were introduced in 1993 [15] as a quantum generalization of classical random walks, and their main interest arises in the quantum information framework [16]. We focus here on discrete-time QWs usually described as follows: a particle, equipped with a finite dimensional internal state (with associated Hilbert space \( \mathcal{H}_c \) known as the “coin space”), is subject to unitary shifts along edges on a graph \( \Lambda \) (with Hilbert space \( \mathcal{H}_\Lambda = \text{span}\{\ket{i}, i \in \Lambda\} \)), in a direction guided by the internal state. At each step, a unitary transformation, called a quantum coin operator \( C \), is uniformly applied on the internal states, eventually affecting the next unitary shift \( S \) and leading to the QW operator \( W = \sqrt{S} C \). After \( N \) time steps, the initial state \( \ket{\psi(0)} \) becomes \( \ket{\psi(N)} = W^N \ket{\psi(0)} \).

QW protocols have already led to several experimental realizations in 1d with trapped atoms [17], single photons [18] or Bose-Einstein condensates in momentum space [19]. In 2d, a generalization has been done with light [20], and the tentative addition of a magnetic field has also been discussed [21–23].

Here we consider a particle performing a QW subject to a transverse magnetic field, and look for cage effects on periodic graphs: (i) the diamond chain (DC) with four-fold coordinated sites \( a \) and two-fold sites \( b, c \) (Figure 1) (ii) the \( T_3 \) tiling with six-fold sites \( a \) and three-fold sites...
(b, c) (Figure 4). In each unit cell, the internal space size is 8 for DC and 12 for \( T_3 \). In a standard QW, the internal space codifies for the direction of the walk (in 1d, a pseudo-spin 1/2 internal state is used to specify along which direction, left or right, to move). Applied to cases with sites of unequal connectivity would lead to internal spaces of different dimension on the different types of sites. Following ref. [24] (and references therein), it is much simpler to consider that a Hilbert space basis state is associated with a pair composed of a vertex and the directed edge incident on that vertex (see top of Figure 1). The shift operator \( S \) is then a sum of simple hopping terms between the two states associated with each edge (one per opposite sites of this edge) and is therefore unitary and hermitian. The unitary coin operator \( C_s \) shuffles the states associated with site \( s = a, b, c \) and its different incident edges. The total coin operator \( C \) is the direct sum of \( C_a, C_b \) and \( C_c \). It may also be hermitian, but generically does not commute with \( S \), leading to a non trivial unitary \( W \) operator.

We define here four types of \( n \times n \) unitary coins, with \( n = 2, 3, 4, 6 \) associated respectively with sites of coordination number \( n \). The generic space of parameters for such matrices is a priori quite large. Previous works on QW have focused mainly on three types of coins [16]: Hadamard \( \text{H}_n \), easy to construct for sizes \( n = 2^m \), Grover \( \text{G}_n \) and DFT (Discrete Fourier Transform) \( \text{D}_n \) coins, which can be constructed for any matrix size \( n \). We found that \( D_n \) coins do not lead to caging effects and therefore will not discuss them in the following. Groover coins read \( \text{G}_n = (2/n) \mathbf{1}_n - \mathbb{I}_n \), where \( \mathbf{1}_n \) and \( \mathbb{I}_n \) are (respectively) the \( n \times n \) full (of 1) and identity matrices.

For the two-fold sites, we use generic \( U(2) \) unitary coins:

\[
U_2(\theta, \varphi, \omega, \beta) = \begin{pmatrix} \cos \theta e^{i\beta} - \sin \theta e^{i(\varphi + \omega)} \\ \sin \theta e^{-i\omega} \cos \theta e^{i(\varphi - \beta)} \end{pmatrix}.
\]

\( U_2(\pi/4, \pi, 0, 0) \) gives the standard Hadamard coin \( \text{H}_2 \), and \( U_2(0, 0, 0, 0) \) corresponds to \( \text{SO}(2) \) rotations. For \( n = 3 \), we use \( \text{SO}(3) \) rotations \( R_3(\alpha, \gamma) \), of angle \( \alpha \) around a restricted set of unit 3d vectors \( \vec{\nu} \) having at least two coordinates (say \( x \) and \( z \)), in the form \( \vec{\nu} = \left( \frac{\cos \gamma}{\sqrt{2}}, \sin \gamma, \frac{\cos \gamma}{\sqrt{2}} \right) \). Such a matrix reduces to a Grover \( \text{G}_3 \) whenever \( (\alpha, \gamma) = (\pi, \sin^{-1}(1/\sqrt{3})) \). For \( n = 4 \), we use either \( \text{H}_2 \) or \( \text{H}_2 \otimes \text{H}_2 \), or the Grover \( \text{G}_4 \) coin, and finally \( \text{G}_6 \). Grover coins are used for \( n = 6 \). The basis conventions are explicit in appendix A.

The shift operator reads

\[
S = \sum_{\langle (i,j), (i',j') \rangle} |i,j \rangle \langle i',j'| + hc,
\]

where \( i \) and \( i' \) are neighbouring sites, and \( j \) and \( j' \) two opposite directed edges between these two sites. The magnetic field \( B \) enters as a Peierls substitution on the hopping term of the shift \( S \): with \( A \) the vector potential \( (B = |\nabla \times A|) \), the hopping term gets multiplied by a phase factor \( e^{iA} f_j \mathbf{d} A \), \( \phi_0 = h/e \) denoting the flux quantum.

The QW amounts to apply \( W \) at each time step, its effect being therefore encoded in the \( W \) eigenspectrum. Being unitary, \( W \) can be written as the exponential \( e^{-iH_{\text{eff}}} \) of an effective Hamiltonian \( H_{\text{eff}} \), whose evolution is only considered at discrete times. Its eigenvalues are pure phases, called quasi-energies and defined modolo 2\( \pi \).

\[ FIG. 1. \text{Bottom: a piece of a diamond chain, with sites of type } a, b \text{ and } c. \text{Coloured sites indicate the maximal extension of a cage (at } f_j \text{), for an initial state localized at a (circled) } a \text{ site. Top: the QW Hilbert space is schematized, with four (resp. two) basis states for } a \text{ (resp. } b, c \text{) sites, shown here as circles. Coins operate on states inside a circle, and shifts along edges.} \]
graph [23]) and two mirrors, $f \leftrightarrow -f + \omega/\pi$ (with $k \leftrightarrow -k - \pi + \phi$) and $\epsilon \leftrightarrow -\epsilon + \phi/2$ (with $k \leftrightarrow k + \pi$). Figure 2-a,b shows the resulting $W$ quasi-energies, only displayed between 0 and $\pi$ owing to the above translation symmetry, for $C_b = G_4$ and symmetric coins ($C_b = C_c$, meaning $\omega = 0$). They display two striking features: (i) the existence of flat bands (versus $k$ and $f$, which are even $\delta$-independent whenever $\beta - \phi/2 = \pm \pi/2$); (ii) a pinching phenomenon for the dispersive bands, recalling that found for A-B cages [6].

Note however that the critical value $f_c$ at which the pinching occurs can be tuned at will, if $\omega \neq 0$ (therefore $C_b \neq C_c$), as $f_c = 1/2 + \omega/2\pi$, and even made to vanish (not shown in the Figure).

These two features are independent: indeed, using a Hadamard $H_4$ coin operator on $a$ sites (instead of Grover $G_4$) still leads to a pinching phenomenon, but without flat bands (Figure 2-c,d). Here $f_c = \omega/2\pi$ and pinching occurs at vanishing magnetic field whenever identical coin operators are applied on sites $b$ and $c$ (i.e. $\omega = 0$).

At $f = f_c$ the 8 quasi-energies read:

$$\epsilon = \frac{\pi}{4} \pm \frac{\pi}{4} + \frac{\omega}{4} \pm \frac{\omega}{4} + \frac{\beta}{2} \sqrt{\frac{1}{2}} \cos^{-1}\left(\frac{\sin(\beta - \frac{\pi}{4}) \cos \theta \pm \sqrt{2 \sin^2 \theta + \cos^2 \theta \sin^2(\beta - \frac{\pi}{4})}}{2}\right)$$

The observed pinching is associated with an A-B-like caging effect. In the original tight-binding A-B cages case [6], the caging effect was proved by analyzing the local density of states, with a Lanczos tridiagonalization showing a vanishing recursion coefficient for $f_c = 1/2$, signing a non-propagating quantum evolution. Here we use a related approach, adapted to the unitary transformation $W$, which amounts to put $W$ into a Hessenberg form, an almost triangular form (with an added subdiagonal), using the so-called Arnoldi iteration [25] (described in appendix C). The latter consists, starting from a given state, called $|0\rangle$, to iteratively apply the $W$ operator, each obtained new state being brought orthogonal to all previous ones: $b_{n+1}|n+1\rangle = W|n\rangle - \sum_{m<\omega} (m|W|n\rangle |m\rangle$.

A vanishing coefficient $b_j$ terminates the iteration and indicates a caging effect. Indeed, starting from a localized state, this proves that under the $W$ operator action, only a finite number of states can be reached, leading to a (possibly complicated) quantum evolution inside the subspace spanned by the states $\{|n\rangle\}$, with $n < j$. A vanishing coefficient (here $b_k$) is indeed found for all above described DC cases, with an initial state localized at any state on an $a$ site, see Figure 3-a. This leads to a cage of maximal radius twice the size of the DC unit cell (see Figure 1).

A last point concerns the time evolution inside a cage (for $f = f_c$). Analysis of the above quasi-energies shows that they are incommensurate for generic ($\theta, \phi, \beta$) values, leading to a quasiperiodic time behaviour. A closer inspection (see appendix D) shows that a periodic behaviour can nevertheless arise, as displayed in Table 1.

A sketch of a period 8 cage is shown in appendix E.

**QW on a $T_3$ lattice under magnetic field** - With the above diamond chain QW results, it is tempting to address the $T_3$ case (see Figure 4). Using the Grover $G_6$ coin on a sites, we indeed found suitable $R_3(\alpha, \gamma)$ coins for $(b,c)$ sites that lead to a caging effect. As in the DC case, we allow for a different $R_3$ coin operating on the $c$ sites, with a new angle $\omega$ entering $R_3(\alpha, \gamma, \omega)_{i,j} = R_3(\alpha, \gamma)_{i,j} e^{-i\omega \sum_{k} \epsilon_{ijk}}$ with $\epsilon_{ijk}$ the skew-symmetric Levi-Civita tensor. Note that this coin, although unitary, is no more a 3d rotation, and that the introduction of $\omega \neq 0$ breaks time-reversal symmetry. Using a Landau gauge, periodicity is present in the (say) $y$ direction (with a magnetic unit cell containing $12q$ states), and $W$ can be block-diagonalized for rational $f = p/q$ into $12q \times 12q$ blocks, leading to the butterfly-like pattern shown in Figure 5 for two values of $(\alpha, \gamma)$. It displays many interesting features, like flat bands, apparent
TABLE I. Examples of caging effects for QW on DC and $\mathcal{T}_3$, for different coins. $f_c$ is the critical flux for the spectral pinching, FB tells whether flat bands are present, and the last column specifies the period of the time evolution when it exists (QP meaning a quasiperiodic case).

| Graph | $C_a$ | $C_e$ | FB | $f_c$ | Period |
|-------|-------|-------|-----|-------|--------|
| DC $G_1$ | $U_2(\theta, \varphi, \omega, \beta)$ | generic | Yes | $\frac{\pi}{2} + \frac{\pi}{72}$ | QP |
| DC $G_4$ | $U_2(\theta, \varphi, \omega, \frac{\pi}{3} + \varphi)$ | Yes | $\frac{\pi}{2} + \frac{\pi}{72}$ | 8 |
| DC $G_0$ | $U_2(\frac{2\pi}{3}, \varphi, \omega, -\frac{\pi}{2})$ | Yes | $\frac{\pi}{2} + \frac{\pi}{72}$ | LCM(4, q) |
| DC $H_1$ | $U_2(\theta, \varphi, \omega, \beta)$ | generic | No | $\frac{\pi}{2}$ | QP |
| DC $H_4$ | $U_2(\frac{\pi}{2}, \varphi, \omega, 0)$ | No | $\frac{\pi}{2}$ | 24 |
| DC $H_4$ | $U_2(\frac{\pi}{2}, \frac{\pi}{2}, \omega, 0)$ | No | $\frac{\pi}{2}$ | 10 |
| DC $H_4$ | $U_2(\frac{\pi}{2}, 0, \omega, 0)$ | No | $\frac{\pi}{2}$ | 12 |
| $T_3$ $G_0$ | $R_3(\alpha, \gamma)$ | generic | Yes | $\frac{\pi}{2}$ | QP |
| $T_3$ $G_0$ | $R_3(\alpha, \gamma, \frac{2\pi}{3})$ | Yes | $\frac{\pi}{2}$ | QP |
| $T_3$ $G_0$ | $R_3(\frac{2\pi}{3}, \gamma)$ | Yes | $\frac{\pi}{2}$ | 12 |

self-similar sub-patterns and pseudo-Landau levels. But we shall only focus here on the spectral pinching that occurs near $f_c = 1/2$. Here the quasi-energies, doubly degenerate and $k$-independent, read:

$$\epsilon = 0, \pm \frac{\pi}{2}, \pm \frac{\alpha}{2}, \pm \frac{\alpha}{2} - \frac{\pi}{2}, \pm \frac{\alpha}{2},$$

$$\pm \frac{\pi}{2} \pm \frac{1}{2} \cos^{-1}\left(\frac{2 + \cos \alpha}{3}\right), \pm \frac{\pi}{2} \pm \frac{1}{2} \cos^{-1}\left(\frac{2 + \cos \alpha}{3}\right).$$

From these expressions, it is possible to determine whether the dynamics of the QW cages is periodic (when quasi-energies differences are commensurate) or not (see Appendix D).

FIG. 4. Left: a piece of $\mathcal{T}_3$ tiling, with sites of type $a$, $b$ and $c$. Coloured sites form a cage (at $f_c$), centered on the initial (circled) $a$ site. Right: the QW Hilbert space is schematized, with six (resp. three) basis states for $a$ (resp. $b$, $c$) sites, shown here as circles. Coins operate on states inside a circle, and shifts along edges.

Using Arnoldi iteration, one numerically finds that the $b_{12}$ coefficient vanishes at $f_c = 1/2$, as shown in Figure 3-b. The associated A-B cage depends on the precise chosen initial state, the largest one, displayed in Figure 4, being larger than in the TB case.

The flux values $\pm 1/3$ are particular: we can define a different gauge, which now shares the tiling periodicity, as in the DC case (see appendix F). As a consequence, $W$ can be diagonalized into $12 \times 12$ blocks $W(k_x, k_y)$. Going further, we ask whether asymmetric $R_3$ coins may then change the critical $f_c$ value. It indeed works whenever $\omega = -2\pi/3$, in which case $f_c$ can be shifted from $1/2$ to $1/6$, according to $f_c = 1/2 + \omega/2\pi$. But, in contrast to the DC case, here $f_c$ cannot be tuned continuously.

FIG. 5. Quasi energy spectra of QW on $\mathcal{T}_3$ for three values of the $R_3$ coin operator: top: $\alpha = 2\pi/3, \gamma = \sin^{-1}(1/\sqrt{3}), \omega = 0$, middle: $\alpha = 2\pi/3, \gamma = 0, \omega = 0$, bottom: $\alpha = \pi/2, \gamma = \sin^{-1}(1/\sqrt{3}), \omega = -2\pi/3$. In this third case, the quasi-energies pinching occurs at $f_c = 1/6$.

Discussion – The A-B cages result from a destructive interference effect. In the TB case, this occurs on the higher coordinated sites where edges meet and partial wave-function amplitude adds destructively. In contrast, in the QW case and because of unitarity, the shift part only carries the Peierls phases, and the destructive interference effect occurs through the coin operator action on the internal degrees of freedom. Amplitudes vanish on those states that would allow the particle, upon further shift, to escape the cage. As a result, the A-B cages in the QW case have a larger size than their TB counterparts. In addition, the quantum evolution is generically quasiperiodic in the QW case, while it was found to be periodic in the TB case. Also, QW A-B cages may occur at critical fluxes others than $1/2$. Necessary conditions for this tuning of the critical flux to happen are the existence both of a gauge respecting the translation symmetries of the...
structure, and of a coin that breaks time-reversal symmetry (whenever $\omega \neq 0$), and partially compensates the effect of the applied magnetic field.

The QW A-B cages should be readily observable. Recently, a concrete experimental implementation of a magnetic QW in 2d has already been proposed [23]. The extra challenge in order to observe A-B cages is to implement such a QW on a lattice with varying coordination number. QW on such lattices have been theoretically studied for many years [24], but we are not aware of an experimental realization to date. A striking confirmation of the above predictions would be to test the tunability of the critical flux at which caging occurs.

### Appendix A: Basis conventions

The different coin operators, acting at a given site, are given in the main text. We add here our chosen basis ordering convention in the four cases, relative to the action of different coins.

FIG. 6. Basis states ordering convention for coordinated sites. (Top) diamond chain with two 2-fold and one 4-fold sites. (Bottom) $T_5$ tiling with two 3-fold and one 6-fold sites.

### Appendix B: Isospectrality of $W^2$ sub-blocks

In the two examples treated here, the underlying graph is bi-partite. The Hilbert space basis is the union of two parts $B_a$ and $B_{b,c}$, and the action of the unitary operator $W$ is bi-partite: it sends components in $B_a$ onto components in $B_{b,c}$ and vice versa. As a consequence $W^2$ sends each sub-basis onto itself, and appears therefore block-diagonal.

Now, suppose that $|\psi\rangle = |\psi_a\rangle + |\psi_{b,c}\rangle$ is an eigenvector of $W$, with eigenvalue $e^{i\epsilon}$, with $|\psi_a\rangle$ (resp. $|\psi_{b,c}\rangle$) being the subpart of $|\psi\rangle$ in $B_a$ (resp. $B_{b,c}$). $|\psi\rangle$ is an eigenvector of $W^2$, with eigenvalue $e^{2i\epsilon}$. In each diagonal sub-blocks of $W^2$, spanned by $B_a$ and $B_{b,c}$, we therefore have that $|\psi_a\rangle$ and $|\psi_{b,c}\rangle$ are separately eigenvectors of $W^2$ with the same eigenvalue $e^{2i\epsilon}$, the two blocks being therefore isospectral.
Appendix C: Arnoldi iteration

The Arnoldi algorithm [25] amounts to transform a generic square matrix into a so-called “Hessenberg form”, which is an almost triangular form. More precisely, it results to a (say upper) triangular form plus the lower sub-diagonal.

Here we apply the Arnoldi iteration to reduce the unitary QW operator. We start with a localized state \(|0\rangle\) (the initial condition stating where the wavefunction is concentrated at \(t = 0\)) and successively apply the QW operator \(W\). At each step, a new state is obtained from which the part which belongs to states already explored is removed:

\[
    b_{n+1} |n + 1\rangle = \hat{W} |n\rangle - \sum_{m \leq n} \langle m | W | n\rangle |m\rangle.
\]

As a consequence, a new orthonormal basis \(|n\rangle\) is generated in which \(\hat{W}\) has an Hessenberg form. Note that, when the operator is hermitian, the Arnoldi algorithm reduces to the Lanczos tridiagonalization.

The \(b_n\) are defined positive, and the algorithm ends whenever a new generated state is null (\(b_j = 0\)). As a consequence, starting from the (localized) state \(|0\rangle\), the quantum walk explores only a finite number of states, leading to a cage trapping.

Appendix D: Criteria for periodic dynamics at the caging critical value

At the critical caging value \(f_c\), an interesting question is to analyse the confined dynamics. It is periodic whenever the differences between quasi-energies are commensurate. In the tight binding case the dynamics were found to be periodic in all cases (DC and \(T_3\)). QW caging effects are found here with a larger set of parameters and are generically quasiperiodic. However, analysis of the quasi-energies in the expressions given in the main text allows one to find some conditions for periodicity.

- **DC case** The fact that caging effects are found with a generic \(U(2)\) coin \(U_2(\theta, \varphi, \omega, \beta)\) leads to a large spectrum of parameters. In the case \(C_4 = G_4\), one finds for instance that the dynamics is periodic whenever \(\cos^{-1}(\cos(\beta - \varphi/2) \cos \theta)\) is commensurate with \(\pi\). For instance, if \(\beta - \varphi/2 = 0\), we find a periodic dynamics, with period \(\text{LCM}(4, q)\), whenever \(\theta = 2\pi p/q\) with \((p, q) \in \mathbb{Z}\). On the other hand, if \(\beta - \varphi/2 = \pm \pi/2\), the resulting dynamics is periodic for any \(\theta\).

- **\(T_3\) case**

  Substracting the quasi-energies at the critical \(f_c\) value, we have that \(\alpha\) must be commensurate with \(\pi\), say \(\alpha = \pi p_1/q_1\) and satisfy an additional more complicate equation: \(\cos^{-1}\left(\frac{2 + \cos \frac{\pi p_1}{q_1}}{3}\right) = \frac{\pi p_2}{q_2}\). If this is verified, the quantum walk is periodic, with period \(4 \times \text{LCM}(q_1, q_2)\). We find that for \(\alpha = 2\pi/3\), the dynamics has period 12, and numerically that no other solution exists for \(q_1, q_2 \leq 100\).

Appendix E: Sketch of the cage effect in a simple DC case

We show here a sketch of a periodic quantum walk on the diamond chain, at the critical value corresponding to a caging effect. We take the simple example of a period 8 quantum walk, with parameters given in the caption.

Appendix F: Example of a periodic gauge at flux -1/3

Figure 8 shows an example of a gauge at flux -1/3 sharing the tiling periodicity (whose unit vectors are displayed in red).
FIG. 7. Sketch of a period 8 quantum walk on DC. The presence probability on each eigenbasis state is shown, with $C_a = G_4, C_b = C_c = U_2(\pi/4, 0, 0, -\pi/2)$ and $f_c = 1/2$.

FIG. 8. An example of a periodic gauge at flux $-1/3$