Equivariant quantization of spin systems

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Abstract. We investigate the geometric and conformally equivariant quantizations of the supercotangent bundle of a pseudo-Riemannian manifold \((M, g)\), which is a model for the phase space of a classical spin particle. This is a short review of our previous works [10, 11].

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INTRODUCTION

Quantization is born with quantum mechanics, as the fundamental attempt to establish a correspondence between the mathematical structures of classical and quantum mechanics, presented in the following table.

| Phase space | classical | quantum |
|-------------|-----------|---------|
| symplectic manifold \((\mathcal{M}, \omega)\) | | Hilbert space \(\mathcal{H}\) |
| Observables | Poisson algebra \(A \subset C^\infty(\mathcal{M})\) | associative algebra \(A \subset \mathcal{L}(\mathcal{H})\) |
| Symmetries | Lie subalgebra \(\mathfrak{g} \subset \text{ham}(\mathcal{M}, \omega)\) | Lie subalgebra \(\mathfrak{g} \subset \text{u}(\mathcal{H})\). |

One of the most celebrated quantization procedure is the geometric quantization [8, 15], whose main drawback is its too small set of quantizable observables. Equivariant quantization [9, 3] aims to overcome this issue for systems admitting a configuration space \(M\) with a large enough group \(G\) of (local) symmetries, as the projective or conformal group. More precisely, the inverse of the obtained quantization map is a \(G\)-equivariant symbol map on \(M\), from differential operators to symmetric tensors.

We present here the equivariant quantization of spin systems whose configuration space is a spin manifold \(M\) endowed with a metric \(g\) of signature \((p,q)\). This suppose to introduce a framework for classical mechanics of spin systems, namely the supercotan-

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gent bundle of $(M,g)$ [1, 4] endowed with its canonical symplectic structure [14]. We recover some of the main objects of spin geometry via its geometric quantization, and the privileged status of conformal transformations of $(M,g)$ is highlighted. Restricting us to a conformally flat manifold, we describe explicitly the action of $\mathfrak{o}(p+1,q+1)$ on the space of classical and quantum observables, plus on a related space of tensors. We state then our main results on the existence and uniqueness of conformally equivariant quantization and superization, which are isomorphisms between these three $\mathfrak{o}(p+1,q+1)$-modules. Some applications are given. We refer to [10, 11] for more details and proofs.

**FROM CLASSICAL TO QUANTUM SPIN SYSTEMS**

The quantum framework for spin systems with configuration space $(M,g)$ is well-known: the state space $\mathcal{H}$ is obtained by completion of the space of sections of the spinor bundle $S \to M$, and we choose the algebra $\mathcal{D}(M,S)$ of spinor differential operators as space of quantum observables. Its usual algebra of symbols is $\text{Pol}(T^*M) \otimes \Gamma(\mathbb{C}l(M,g))$, the tensor product of the space of functions on $T^*M$, which are polynomial in the fiber variables, with the space of sections of the complex Clifford bundle of $(M,g)$. Replacing $\Gamma(\mathbb{C}l(M,g))$ by its graded counterpart, namely the algebra of complex differential forms $\Omega(M)$, we end up with a superalgebra of functions on the supercotangent bundle of $M$. That provides us with the algebra of symbols for $\mathcal{D}(M,S)$ w.r.t. its bifiltration [4], as well as with the classical setting for a spin system on $M$.

**Supercotangent bundle and pseudomechanics**

The supercotangent bundle of the manifold $M$ is $\mathcal{M} = T^*M \oplus \Pi TM$, i.e. the direct sum of the cotangent bundle and the tangent bundle with reverse parity. Thus, its superalgebra of functions is $\mathcal{C}^\infty(T^*M) \otimes \Omega(M)$, generated locally by coordinates $(x^i, p_i, \xi^i)$, where $\xi^i$ identifies with $dx^i$. The general study of symplectic supermanifolds by Rothstein [14] proves that a symplectic structure on $M$ is equivalent to the data of a metric and a compatible connexion on $M$. As a consequence, to any pseudo-Riemannian manifold $(M,g)$ corresponds a canonical symplectic form $\omega$ on $\mathcal{M}$, given by

$$\omega = d\alpha \quad \text{and} \quad \alpha = p_i dx^i + \frac{\hbar}{2i} g_{ij} \xi^i d^V \xi^j,$$

where $d^V$ is the covariant differential w.r.t. the Levi-Civita connexion of $g$. The spin components are defined as $S^{ij} = \frac{\hbar}{i} \xi^i \xi^j$ so that, together with the Poisson bracket associ-
ated to $\omega$, they generate a Lie algebra isomorphic to $\mathfrak{o}(p,q)$. The equations of motion of a spin particle in an exterior electromagnetic field can be easily recovered in that framework [1, 10], and also the coupling of the spin with the gravitation [13, 10]. Thus, the Hamiltonian flows of the kinetic energy $g^{ij} p_i p_j$ leads to the Papapetrou’s equations [12],
\[
\dot{x}^i \nabla_j x^j = -\frac{1}{2} g^{ik} R(S)_{jk} \dot{x}^j, \\
\dot{x}^k \nabla_k S^{ij} = 0,
\]
where the spin happens to be coupled with the curvature via $R(S)_{jk} = g_{im} R^i_{jk} S^{jk}$, with $(R^i_{jk})$ the components of the Riemann tensor.

**Spin geometry via geometric quantization**

Thanks to geometric quantization, we can built the main objects of spin geometry from $(\mathcal{M}, \omega)$ endowed with a polarization, i.e. a Lagrangian distribution. Upon topological restrictions on $M$, the vertical polarization of the cotangent bundle of $M$ can be completed by a maximal isotropic complex distribution for $g$ on $\Pi \mathcal{T} M$, to give a polarization on $(\mathcal{M}, \omega)$. In the simplest case of a Riemannian metric, the geometric quantization leads then to the construction of Hitchin [6] for the spinor bundle $S$ of a pseudo-Hermitian manifold $(M, g)$, where spinor fields identify with antiholomorphic differential forms tensorized with square root of the volume form of $(M, g)$.

Besides, for Darboux coordinates $(x^i, \tilde{p}_i, \tilde{\xi}^i)$ of $(\mathcal{M}, \omega)$, the quantum map $^2$ satisfies
\[
\mathcal{D}(x^i) = x^i, \quad \mathcal{D}(\tilde{p}_i) = \frac{\hbar}{i} \partial_i \quad \text{and} \quad \mathcal{D}(\tilde{\xi}^i) = \frac{\tilde{g}^i}{\sqrt{2}}, \tag{2}
\]
where $\tilde{g}^i$ is a Clifford matrix for the flat metric given by $(\eta_{ij}) = I_p \oplus -I_q$. Let us remind that the vector fields on $M$ can be lifted to Hamiltonian vector fields on $T^*M$, giving rise to a tautological momentum map $J$, such that $J_X = p_i X^i$ for $X = X^i \partial_i \in \text{Vect}(M)$. This map can be lifted to $\mathcal{M}$ and quantized,
\[
\mathcal{D}(J_X) = \frac{\hbar}{i} \nabla_X, \tag{3}
\]
giving rise to an essential object of the spin geometry: the covariant derivative of spinors.

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$^2$ we consider here that $\mathcal{D}$ takes its values in usual spinor fields rather than spinor half-densities.
Conformal geometry of the supercotangent and spinor bundles

The symplectic structure of the supercotangent bundle $\mathcal{M}$ depends on a metric $g$ on $M$, as a consequence the natural lift to $\mathcal{M}$ of $X \in \text{Vect}(M)$ does not preserve the potential 1-form $\alpha$. Moreover, the condition $L_\tilde{X} \alpha = 0$ does not uniquely determine the lift $\tilde{X}$ of $X$. One way out is to impose the further condition $L_\tilde{X} \beta \propto \beta$, where $d \beta = g_{ij} d^\nabla \xi^i \wedge dx^j$ is the odd symplectic form on $\Pi T M$. The lift is then unique but exists only if $X$ is a conformal Killing vector field. We refer to it later as $\tilde{X}$. Let us notice that, as expected, the momentum of an infinitesimal rotation $X_{ij}$ is,

$$J_{X_{ij}} = p_i x_j - x_j p_i + S_{ij}.$$  \hfill (4)

Remarkably, the quantization of the momentum map $J$ leads to

$$\mathcal{D}(J_X) = \frac{\hbar}{i} L_X,$$  \hfill (5)

the Lie derivative of spinors introduced by Kosmann [7], well-defined precisely for conformal Killing vector fields. Via geometric quantization, we get thus the conformal geometry of the spinor bundle of $(M, g)$ out of the one of its supercotangent bundle.

CONFORMALLY EQUIVARIANT QUANTIZATION

We suppose from now on that $(M, g)$ is a conformally flat spin manifold, i.e. $g_{ij} = F \eta_{ij}$ for some positive function $F$. The conformal Killing vector fields on $(M, g)$ generate then a Lie algebra isomorphic to $\mathfrak{o}(p + 1, q + 1)$, which is the one of infinitesimal conformal transformations of $(\mathbb{R}^n, \eta)$. The aim is to compare the action of those vector fields on $D(M, S)$, the algebra of spinor differential operators, with those on its algebras of symbols, i.e. to compare their $\mathfrak{o}(p + 1, q + 1)$-module structures.

Conformal geometry of spinor differential operators and of their symbols

Let us define the space of $\lambda$-densities by $\mathcal{F}^\lambda = \Gamma(\wedge^n T^* M|^{\otimes \lambda})$, with $\lambda \in \mathbb{R}$ and $n = \dim M$. Instead of $D(M, S)$, we will rather study the two parameters family of $\mathfrak{o}(p + 1, q + 1)$-modules $(D^{\lambda, \mu})$. Each of those modules is defined as the space of differential operators $D : \Gamma(S) \otimes \mathcal{F}^\lambda \to \Gamma(S) \otimes \mathcal{F}^\mu$, endowed with the adjoint action

$$\mathcal{L}_X^{\lambda, \mu} D = L^\lambda_X D - L^\mu_X D,$$  \hfill (6)
where $L^\lambda_X = L_X + \lambda \text{Div}(X)$ is the action of $X$ on $\Gamma(S) \otimes F^\lambda$. The corresponding $o(p+1,q+1)$-module of classical observables is naturally the space $\mathcal{S}^\delta[\xi] = \text{Pol}(T^*M) \otimes \Omega_C(M) \otimes F^\delta$, with $\delta = \mu - \lambda$, endowed with the Hamiltonian action

$$L^\delta_X = \tilde{X} + \delta \text{Div}(X). \tag{7}$$

The explicit expressions of these both classical and quantum actions have been computed in [11], showing that $D^\lambda,\mu$ and $\mathcal{S}^\delta[\xi]$ are filtered modules, by the order of differential operators and the degree in the $p$ variables respectively. Their common associated graded module is the module of tensorial symbols $T^\delta[\xi] = \bigoplus_{n=0}^\infty \text{Pol}(T^*M) \otimes \Omega^K C \otimes F^\delta - n$, endowed with the natural action on weighted tensors. It identifies to $\text{Pol}(T^*M) \otimes \Omega_C(M)$ as an algebra and to the usual space of symbols $\text{Pol}(T^*M) \otimes \Gamma(\text{Cl}(M,g))$ as a module.

**Main results**

Let us begin with few definitions. A map is called conformally equivariant if it is an isomorphism of $o(p+1,q+1)$-modules. Besides, as $\text{Pol}(T^*M)$ is a submodule of $\mathcal{S}^0[\xi]$, a linear isomorphism $\mathcal{S}^\delta[\xi] \to \mathcal{S}^\delta[\xi]$ which preserves the principal symbol is named a superization, whereas a linear isomorphism $\mathcal{S}^\delta[\xi] \to D^\lambda,\mu$ preserving the principal symbol is a quantization, since it relates classical and quantum observables.

**Theorem 1** There exists $I^\delta_S \subset I_S \subset \mathbb{Q}^*_+$ such that the conformally equivariant superization $S^\delta : \mathcal{S}^\delta[\xi] \to \mathcal{S}^\delta[\xi]$, exists if $\delta \not\in I^\delta_S$ and is unique if $\delta \not\in I_S$.

**Theorem 2** Let $\delta = \mu - \lambda \in \mathbb{R}$. There exists $I^\delta_Q \subset I_Q \subset \mathbb{Q}^*_+$ such that the conformally equivariant quantization $Q^\lambda,\mu : \mathcal{S}^\delta[\xi] \to D^\lambda,\mu$, exists and is unique if $\delta \not\in I_Q$, and exists for at least one value of $\lambda \in \mathbb{R}$ if $\delta \not\in I^\delta_Q$.

The values of $\delta$ for which existence or uniqueness of $Q^\lambda,\mu$ is lost are called resonances. The fact that $\delta = 0$ is not a resonance is crucial, thus the conformally equivariant quantization $Q^{1,1} : \mathcal{S}^\delta[\xi] \to D^\lambda,\mu$ extends uniquely the quantization map provided by geometric quantization. Let us remark that $Q^\lambda,\mu \circ S^\delta_Q$ is a particular case of AHS-equivariant quantization [2], but each single map deserves interest, at least from a physical point of view.

The idea of the proofs is the same than in the spinless case [3], and relies on the use of the Casimir operators of each module. To be concrete, we name $C^\mathcal{S}$ and $C^\mathcal{S}$

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3 that is the higher order term of an element w.r.t. a filtration, here it identifies to a tensor over $M$. 


those of $\mathcal{T}^\delta[\xi]$ and $\mathcal{S}^\delta[\xi]$. If the conformally equivariant superization exists, then $C_S^\delta S_T^\delta = S_T^\delta C_S^\delta$, and in particular every eigenvectors of $C_S^\delta$ is send to an eigenvector of $C_S^\delta$ with the same eigenvalue and the same principal symbol. The main point is to prove that the eigenvectors of $C_S^\delta$ are uniquely determined by their eigenvalue and their principal symbol. Then, if this is the case for those of $C_S^\delta$ too, we get the uniqueness of the superization. As $C_S^\delta$ is equal to $C_T^\delta$ plus an operator lowering the degree in the $p$ variable, this is simply checked by the resolution of triangular systems, and the resonant values of $\delta$ are precisely those leading to degenerated systems. If unique, the map constructed in this way is easily proved to be conformally equivariant.

**Some applications**

We give now two applications that we hope to investigate further in forthcoming papers. The first one relies on the explicit formulas for the conformally equivariant superization that we determine in [10] for symbols of degree 1 in $p$. Let us recall that a (conformal) Killing-Yano tensor on $M$ is a skew-symmetric tensor describing higher symmetries of $(M,g)$. As Killing tensors, it generates constant of motion but for spin particles [5]. In fact, a correspondence has been obtained in [16] between Killing-Yano tensors and classical supercharges, which happens to be generalized by the conformally equivariant superization.

**Theorem 3** Let $f$ be a skew-symmetric tensor and $P_f = f^{i_1 \ldots i_{k-1} j_1 \ldots j_{k-1} \xi} p_i$ the associated tensor symbol. Denoting by $\Delta = p_i \xi^i$, we have

$$\{\Delta, S^0_T(P_f)\} = 0 \propto \Delta \iff f \text{ is a (conformal) Killing-Yano tensor.} \quad (8)$$

The second application deals with the conformal invariants of the $o(p + 1, q + 1)$-modules that we have introduced. Let $R = g^{ij} p_i p_j$. The Weyl theory of invariants together with the explicit actions of $o(p + 1, q + 1)$ on each module leads to the following Theorem.

**Theorem 4** The conformal invariants of each family of modules are

1. $\Delta^a R^s \in \mathcal{T}^{\frac{2s+a}{n}}[\xi]$, where $s \in \mathbb{N}$ and $a = 0, 1$.
2. $\Delta R^s \in \mathcal{S}^{\frac{2s}{n}}[\xi]$, where $s \in \mathbb{N}$.
3. $\mathcal{D}^\lambda \mu(\Delta R^s) \in \mathcal{D}^{\frac{n-2s-1}{2n} - \frac{n+2s+1}{2n}},$ where $s \in \mathbb{N}$. 
In particular, we get the Dirac operator as the lower order conformal invariant of \((D^\lambda,\mu)\). Since the maps \(S_\delta^\frac{2s}{n}\) and \(\mathcal{Q}^\lambda,\mu\) preserve conformal invariance, we deduce from the last Theorem that \(S_\delta^\frac{2s}{n}\) does not exist and that the module \(D^{\frac{n-2s+1}{2n}}\) is exceptional: this the only one in the family \((D^\lambda,\lambda+\frac{2s+1}{n})\) to be isomorphic to its space of symbols \(\mathcal{Q}^{\frac{2s+1}{n}}[\xi]\). As a consequence, \(\delta = \frac{2s+1}{n}\) are resonances. In fact, every resonances correspond to some conformal invariants, that will be the matter of our next paper.

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