Poincaré cycle of a multibox Ehrenfest urn model with directed transport

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(March 22, 2022)

We propose a generalized Ehrenfest urn model of many urns arranged periodically along a circle. The evolution of the urn model system is governed by a directed stochastic operation. Method for solving an $N$-ball, $M$-urn problem of this model is presented. The evolution of the system is studied in detail. We find that the average number of balls in a certain urn oscillates several times before it reaches a stationary value. This behavior seems to be a peculiar feature of this directed urn model. We also calculate the Poincaré cycle, i.e., the average time interval required for the system to return to its initial configuration. The result can be easily understood by counting the total number of all possible microstates of the system.

PACS numbers: 05.20.-y, 02.50.Ey, 02.50.-r, 64.60.Cn

I. INTRODUCTION

Physical laws governing the microscopic processes are mostly reversible in time. In macroscopic world, however, people often experience time-irreversible phenomena in their daily life. To understand why the reversible microscopic processes lead to irreversible macroscopic manifestations one refers to the Poincaré Theorem, which states that a system having a finite energy and confined to a finite volume will, after a sufficient long time – the so called Poincaré cycle, return to an arbitrarily small neighborhood of almost any given initial state [1]. The key point is to note that the typical value of a Poincaré cycle for even a moderate-sized system is far beyond the meaningful time scale one can measure or experience, thus the irreversibility is realized.

Usually to describe a macroscopic system one has to know only a few parameters, such as volume, pressure, and temperature. However, to describe the same system in terms of its microscopic constituents, one has to deal with a large number of parameters, such as the momenta and positions of a huge amount of particles, which are impossible to calculate in practice. Based on this reason together with the fact that the macroscopic laws are insensitive to the microscopic details (of system history), it is natural for people to adopt the probability (ensemble) description in statistical mechanics, which deals with the equilibrium state (a macroscopic state that has stationary value of state parameters) of a macroscopic system. In this kind of description the macroscopic quantities are defined as the ensemble average of their microscopic correspondences. This definition connects the microscopic and macroscopic worlds.

To study how a system approaches its equilibrium state one also uses probability description, where the evolution of the system is treated as a stochastic process. One famous model for simulating such a process was proposed by Ehrenfest one century ago [2], which is an $N$-ball, $2$-urn problem. In the beginning $N$ numbered balls are distributed arbitrarily in either urn $A$ or urn $B$. At each time step one ball is picked out at random and then put into the other urn. This simple model can be exactly solved to give an explicit Poincaré cycle. This model was then generalized by several authors to mimic more complicated situations encountered in real physical phenomena [3–5]. An attractive feature of these urn model problems is that they are easy to formulate but not always easy to solve. The solutions obtained have, therefore, sometimes led to new mathematical techniques and insights [6–9]. Recently, some new urn models were proposed and solved analytically or numerically. Their results provide very good descriptions on granular and glass systems [10–14].

In this paper, we obtain the exact solution of a generalized urn model. Hereafter we call it “periodic urn model”. In this model, one considers $N$ distinguishable balls which are distributed in $M$ urns. These $M$ urns are arranged along a circle and numbered one by one to form a cycle, that is, we define the $(M + 1)$th urn as the 1st urn (See Fig. 1). To begin with, the initial distribution of the $N$ balls in the $M$ urns is given by $|m_{1,0}, m_{2,0}, \cdots, m_{M,0} \rangle \equiv |\mathbf{m}_0 \rangle$, where $m_{i,0}$ is the number of balls in the $i$th urn at the start. At each time step one ball is picked out of the $N$ balls such that every ball has an equal probability of being picked up. The ball is then placed into the next numbered urn. The state that the $i$th urn contains $m_i$ balls is represented by $|m_1, m_2, \cdots, m_M \rangle \equiv |\mathbf{m} \rangle$, which we name it state vector. Hereafter we call a distribution string $\mathbf{m}$ (without knowing the numbering of the balls) a configuration of the system. Otherwise, if we also know the location of each numbered ball, we call such a distribution a microstate.
of the system.

\[ \langle m_i \rangle = \sum_{\{m\}} m_i \langle m | P^s | m_0 \rangle, \]  

where \( \{m\} \) include all the configurations satisfying the constraint (2).

Let \( A(m) = m_i \), then from Eq. (1) and (3) we have

\[ \langle m_i \rangle_s = \sum_{\{m\}} m_i \langle m | P^s | m_0 \rangle, \]

\[ = \sum_{\{m\}} \sum_{j=1}^M m_i j \frac{1}{N} \]

- \( \cdots m_j + 1, m_j + 1 - 1, \cdots | P^{s-1} | m_0 \),

\[ = \sum_{j=1}^M \left( \frac{\langle m_j m_j \rangle_{s-1}}{N} - \frac{\langle m_j \rangle_{s-1}}{N} \delta_{j,i} + \frac{\langle m_j \rangle_{s-1} - \delta_{j+1,i}}{N} \right), \]

\[ = \sum_{j=1}^M \frac{\langle m_j m_j \rangle_{s-1}}{N} - \frac{\langle m_j \rangle_{s-1}}{N} + \frac{\langle m_{j-1} \rangle_{s-1}}{N}, \]

\[ = \left( 1 - \frac{1}{N} \right) \langle m_j \rangle_{s-1} + \frac{\langle m_{j-1} \rangle_{s-1}}{N}, \]  

here we have used the constraint (2).

Now we are ready to solve \( \langle m_i \rangle_s \). The recurrence relation (4) can be written as

\[ M_s = P_{ave} M_{s-1}, \]

where \( M_s \) is a \( M \times 1 \) column vector defined by

\[ M_s = \begin{bmatrix} \langle m_1 \rangle_s \\ \langle m_2 \rangle_s \\ \vdots \\ \langle m_M \rangle_s \end{bmatrix}, \]

and \( P_{ave} \) is a \( M \times M \) matrix written as

\[ P_{ave} = \begin{bmatrix} 1 - \frac{1}{N} & 0 & \cdots & \frac{1}{N} \\ \frac{1}{N} & 1 - \frac{1}{N} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 - \frac{1}{N} \end{bmatrix}. \]

By means of the recurrence relation (5), \( M_s \) can be deduced:

\[ M_s = P_{ave}^s M_0, \]

where

\[ M_0 = \begin{bmatrix} m_{1,0} \\ m_{2,0} \\ \vdots \\ m_{M,0} \end{bmatrix} \]
represents the initial state. $P^s_{ave}$ can be calculated if one knows the eigenvalues $\lambda_m$ and eigenvectors $Q_m$ of $P_{ave}$. They are given by (See Fig. 2)

$$\lambda_m = 1 - \frac{1}{N} + \frac{1}{N}q^*_m,$$

$$Q_m = \frac{1}{\sqrt{M}} \begin{bmatrix} q_m \\ q_m^2 \\ \vdots \\ q_m^M \end{bmatrix},$$  \hspace{1cm} (10)

where

$$q_m = \exp \left( \frac{2m\pi i}{M} \right), \quad m = 1, 2, \cdots, M. \hspace{1cm} (11)$$

where we have used the following properties of $R$

$$R = R^t, \quad (R^{-1}) = R^t, \quad R_{mn} = q^m_n = q^{*n}_m. \hspace{1cm} (15)$$

Now the average number of balls in the $i$th urn after $s$ steps can be determined:

$$\langle m_i \rangle_s = \frac{1}{M} \sum_{j=1}^M \sum_{k=1}^M q_j^i \lambda^*_k \delta_{jk} q_i^{-k} \langle m_i \rangle_0,$$

$$= \frac{1}{M} \sum_{j=1}^M q_j^i \langle m_i \rangle_0, \hspace{1cm} (16)$$

where $\langle m_i \rangle_0 = m_{i,0}$ is the initial number of balls in the $i$th urn.

Denote $R$ as the $M \times M$ matrix of the eigenvectors $Q_m$

$$R = [Q_1, Q_2, \cdots, Q_M]$$

$$= \frac{1}{\sqrt{M}} \begin{bmatrix} q_1^1 & q_2^1 & \cdots & q_M^1 \\ q_1^2 & q_2^2 & \cdots & q_M^2 \\ \vdots & \vdots & \ddots & \vdots \\ q_1^M & q_2^M & \cdots & q_M^M \end{bmatrix}, \hspace{1cm} (12)$$

and $\Lambda$ as the diagonal matrix of $P_{ave}$‘s eigenvalues $\lambda_m$

$$\Lambda = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_M \end{bmatrix} \hspace{1cm} (13)$$

then we obtain

$$P^s_{ave} = R\Lambda^s R^{-1} = R\Lambda^s R^t = R\Lambda^s R^*, \hspace{1cm} (14)$$

FIG. 2. Eigenvalues $\{\lambda_m\}$ of the matrix $P_{ave}$ (represented by the tiny circles). Here $R = 1$ and $r = 1/N$ are the radii of two reference circles, and $O_A$ and $O_B$ are their centers, respectively. The eigenvalues of $P_{ave}$ are distributed uniformly on the small reference circle centered at $O_B = (1 - 1/N, 0)$.

Let us now consider a simple example. Suppose initially all the $N$ balls are in the first urn, that is,

$$m_{1,0} = N, \quad m_{2,0} = m_{3,0} = \cdots = m_{M,0} = 0, \hspace{1cm} (17)$$

then according to Eq. (16), we have

$$\langle m_1 \rangle_s = \frac{N}{M} \sum_{j=1}^M \lambda_j^s \hspace{1cm} (18)$$

Fig. 3 shows the results for $N = 50$ at $M = 2, 5, 10, 25$. The $M = 2$ case is the original Ehrenfest model, in which the average number of balls in the first urn decays to $N/2$ in a period of steps of order $N$. For any $M > 2$ case, however, we observe that before the system arrives its true equilibrium (here we mean the value of $\langle m_i \rangle_s$ for each $i$ does not change anymore), $\langle m_1 \rangle_s$ undergoes
several oscillations, which seems to be a unique feature of this model and have never been found in other kinds of urn models—to our knowledge. Furthermore, in Fig. 3 the \( M = 25 \) case shows that before the appearance of the first peak of \( \langle m_1 \rangle_s \), there is a period during which \( \langle m_1 \rangle_s \) is almost zero. This phenomenon together with the oscillations mentioned before seem to be typical results when both \( M \) and \( N \) are large.

\[ \langle m_1 \rangle_s = \frac{N}{M} \sum_{j=1}^{M} \exp \left[ \frac{s(q_j^{-1} - 1)}{N} \right], \quad (19) \]

hence \( \langle m_1 \rangle_s/N \) becomes a universal function of \( s/N \) for a fixed \( M \). In Fig. 4 we plot the \( \langle m_1 \rangle_s/N \) curves for \( M = 30 \) (which is large enough in practice) and \( N = 5, 15, 30, 60 \). The \( N = 5 \) case is shown here too to compare with those cases with large \( N \). As one can see, except for the \( N = 5 \) case, the \( \langle m_1 \rangle_s/N \) curves corresponding to different \( N \)'s merge and become one universal function of \( s/N \). In the beginning this function decays from 1 exponentially like that in the \( M = 2 \) case. However, this curve does not decay to its stationary value \( 1/M \) directly, instead, before it growing up again during a period it becomes too tiny to be visible. In practice this tiny value can be ignored thus hereafter we treat this period as the “empty-urn” period. The values of \( \langle m_1 \rangle_s/N \) in this period for our chosen cases are of the order \( 10^{-4} \). The precise duration of the empty-urn period depends on how accurate we treat the urn as empty. After this period the curve grows up again and oscillates several times with evanescent amplitudes and finally approaches the stationary mean value \( 1/M \).

Further exploration shows that there can have more than one empty-urn period. And the total number of these periods, their durations, and the local maxima and minima of each \( \langle m_1 \rangle_s/N \) curve are all determined by \( M \). Fig. 5 shows the results for \( N = 60 \) at \( M = 30, 60, 90 \). These three \( M \)'s are chosen intentionally to have a simple integer proportion \( 1:2:3 \) in order to help us inspect relevant results without much effort. The first maxima of \( \langle m_1 \rangle_s/N \) for these three \( M \)'s are located at \( s/N = 30, 60, 90 \), which are exactly these \( M \)'s. In addition, the locations of the first maxima for \( M = 60 \) and 90 are just the same locations of the second and third maxima for \( M = 30 \), respectively. Moreover, for the \( M = 60 \) and \( M = 90 \) cases the total number of empty-urn periods are more than one.

Starting from Eq. (19), we now derive two useful approximations of \( \langle m_1 \rangle_s/N \) to help us understand these observations. First, define

\[ \tau \equiv \frac{s}{N}, \quad (20) \]

and expand the exponential functions in Eq. (19) as power series of \( \tau \), we have

\[ \frac{\langle m_1 \rangle_s}{N} = \frac{e^{-\tau}}{M} \sum_{j=1}^{M} \left( 1 + \tau q_j^{-1} + \frac{1}{2} \tau^2 q_j^{-2} + \cdots \right) = e^{-\tau} \left[ 1 + \frac{\tau^M}{M!} + \frac{\tau^{2M}}{(2M)!} + \cdots \right]. \quad (21) \]

In deriving Eq. (21), we have used the fact that \( \sum_{j=1}^{M} q_j^{-k} = \sum_{j=1}^{M} q_1^k = 0 \) except for \( k = rM \) with...
where we have defined

\[ \theta = \frac{2\pi j}{M}, \quad \theta = \frac{2\pi}{M}. \]

Remember that \( M \) is a large number, so

\[ \sin \theta_1 \approx \theta, \quad 1 - \cos \theta_1 \approx \frac{\theta^2}{2}. \]

We thus have the approximation

\[ \frac{\langle m_1 \rangle_s}{N} \approx 1 + 2e^{-\tau \theta^2/2} \cos (\tau \theta) + 2e^{-2\tau \theta^2} \cos (2\tau \theta) \frac{M}{M} \]

for describing the behaviors of \( \langle m_1 \rangle_s/N \) curve in the region \( \tau >> M^2/2\pi^2 \), that is, \( \tau (1 - \cos \theta_1) \approx \tau \theta^2/2 >> 1 \). From this expression we also get the result

\[ \lim_{s \to \infty} \langle m_1 \rangle_s = \frac{N}{M}, \]

which must be true after the system has reached its stationary state.

FIG. 6. Plot of the trace of the center of mass (COM) as a function of time step \( s \), assuming initially all the balls are in the first urn. Here the curves are plotted for \( s = 0 \) to \( s = 2MN \) with \( M = 2, 5, 15, 60 \). Each curve with large enough \( M \) (\( M \geq 10 \)) has circulated the origin (the “x” symbol) of the complex plane twice after evolving \( 2MN \) steps.
Up to now we have been focusing our attention on only one single urn. In fact, we can also understand the behaviors of the system in a global manner. First, let's define the "phase angle" of the kth urn (see Fig. 1) as
\[ \phi_k = -(k-1) \theta. \] (32)
Now we define the "center-of-mass" (COM) of the N-ball, M-urn system as
\[ \text{COM} = \sum_{k=1}^{M} e^{i \phi_k} \langle m_k \rangle_s. \] (33)

![FIG. 7. These COM curves are plotted for s = 0 to s = M^2 N/π^2 with M = 2, 5, 15, 60. The norm of a COM in the complex plane becomes \( e^{-2} \). \]

Here we see that the COM/N curve is approximately described by a spiral circulating inside a unit circle. The angular frequency of the circulating motion of the COM with respect to \( s \) is \( -\theta = -2\pi/M \) (clockwise), consistent with both the definition of our game and the oscillation behaviors of the COM/N curve discussed before. Furthermore, when \( \tau > M^2/2\pi^2 \) we have \( r < e^{-1} \), which indicates that the system has a wide extent of the ball-distribution, also consistent with the previous discussions. Some examples are illustrated in Fig 6 and Fig. 7.

**III. STATE MATRIX AND GENERATING FUNCTION**

Now we calculate \( \langle m | P^s | m_0 \rangle \) – the transition probability from \( |m_0\rangle \) to \( |m\rangle \) after \( s \) steps. Once one knows the exact solution of \( \langle m | P^s | m_0 \rangle \), any quantity can be calculated explicitly.

Define
\[ S_{mm'} = \langle m | P | m' \rangle, \] (36)
then we have:
\[ \langle m | P^s | m_0 \rangle = (S^s)_{mm_0}. \] (37)
Here \( S \) is a \( H^M_N \times H^M_N \) matrix, we name it state matrix, and \( |m\rangle \) is a \( H^M_N \) column vector, here
\[ H^M_N = C^{N+M-1}_M = \frac{(N+M-1)!}{N!(M-1)!}. \] (38)
Like before, \( S^s \) can be calculated by means of its eigenvalues and eigenstates. According to Eq. (1), the matrix \( S \) has components
\[ S_{mm'} = \sum_{i=1}^{M} \frac{m_i + 1}{N} \delta_{m_1,m'_1} \delta_{m_2,m'_2} \cdots \times \delta_{m_{M+1},m'_{M+1}} \delta_{m_{M+1},m'_{M+1}} \cdots \delta_{m_M,m'_M}, \] (39)
where \( m_{M+1} = m_1 \) have been assumed. The eigenvalue equation can be written as
\[ \sum_{\{m'\}} S_{mm'} \phi_{m'} = \gamma \phi_{m}, \] (40)
or more explicitly
\[ \sum_{i=1}^{M} \frac{m_i + 1}{N} \phi_{m_1,m_2,\cdots,m_i+1,m_{i+1},\cdots,m_M} = \gamma \phi_{m_1,m_2,\cdots,m_i,m_{i+1},\cdots,m_M}. \] (41)
In the above expression we have set \( \phi_{m_1,m_2,\cdots,m_M} = 0 \) for those \( |m\rangle = |m_1, m_2, \cdots, m_M \rangle \) that do not satisfy the constraint (2).
It is not an easy task to diagonalize \( S \) directly. Thus we adopt another strategy. We first construct a generating function for \( \phi_{m_1,m_2,...,m_M} \) and then transform the matrix eigenvalue equation (40) to its differential equation form. We find that the differential equation can be solved analytically.

By introducing variables \( x_1, x_2, \cdots, x_M \), the generating function can be defined as

\[
f(x_1, x_2, \cdots, x_M) \equiv \sum_{\{m\}} \phi_{m_1,m_2,...,m_M} x_1^{m_1} x_2^{m_2} \cdots x_M^{m_M}.
\]

(42)

Hereafter we also use the following expression

\[
f(X) = \sum_{\{m\}} \phi_m X^m,
\]

(43)

where \( X \) and \( X^m \) are defined by

\[
X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_M \end{bmatrix}, \quad X^m = x_1^{m_1} x_2^{m_2} \cdots x_M^{m_M}.
\]

(44)

To proceed further, note that \( f(X) \) satisfies the following two relations:

\[
\partial_{x_i} f(X) = \sum_{\{m\}} (m_i + 1) \phi_{m_1,m_2,...,m_i+1,...,m_M} X^m,
\]

\[
x_i f(X) = \sum_{\{m\}} \phi_{m_1,m_2,...,m_i-1,...,m_M} X^m,
\]

as can be easily checked. Multiplying \( X^m \) on both sides of Eq. (41), summing over all \( \{m\} \), and using the results of (45) , we get

\[
\sum_{i=1}^{M} \frac{x_{i+1}}{N} \partial_{x_i} f(X) = \gamma f(X),
\]

(46)

or equivalently

\[
\sum_{i=1}^{M} x_{i+1} \partial_{x_i} \ln[f(X)] = N \gamma,
\]

(47)

which is the desired differential equation form of the eigenvalue equation (41). Define

\[
x_{q_j} = x_1 q_j + x_2 q_j^2 + \cdots + x_M q_j^M,
\]

(48)

we find

\[
\sum_{i=1}^{M} x_{i+1} \partial_{x_i} \ln(x_{q_j}) = q_j^{-1} = q_j^*.
\]

(49)

This implies that the complete solution of \( \ln[f(X)] \) can be written as

\[
\ln[f_n(X)] = \sum_{j=1}^{M} n_j \ln(x_{q_j}),
\]

(50)

which gives us

\[
f_n(X) = \prod_{j=1}^{M} x_{q_j}^{n_j} = X_q^n.
\]

(51)

Here \( f_n(X) \) (a homogeneous Nth power function) and the eigenvalue \( \gamma_n \) are characterized by \( n = [n_1,n_2,\cdots,n_M] \) and \( q = [q_1,q_2,\cdots,q_M] \), satisfying

\[
N = \sum_{j=1}^{M} n_j, \quad \gamma_n = \frac{1}{N} \sum_{j=1}^{M} n_j q_j^* = \frac{n \cdot q^*}{N}.
\]

(52)

Denoting the \( n \)-th eigenvector of \( S \) as \( \phi(n) \), Eq. (40) and (43) now become

\[
\sum_{\{m\}} S_{nm} \phi_m(n) = \gamma_n \phi_m(n)
\]

(53)

and

\[
f_n(X) = \sum_{\{m\}} \phi_m(n) X^m = X_q^n.
\]

(54)

To diagonalize \( S \) we first define an orthogonal transformation matrix \( U \):

\[
U_{mn} = \phi_m(n),
\]

(55)

where \( \phi_m(n) \) according to Eq. (54) is the coefficient of \( X^m \) that appears in the expansion of \( f_n(X) = X_q^n \).

We now are ready to solve the matrix \( U^{-1} \). Multiplying \( X^m = x_1^{m_1} x_2^{m_2} \cdots x_M^{m_M} \) on both sides of

\[
\sum_{\{n\}} \phi_m(n) U^{-1}_{nl} = \sum_{\{n\}} U_{mn} U^{-1}_{nl} = \delta_{ml}
\]

(56)

and summing over all possible \( \{m\} \), we get

\[
\sum_{\{n\}} f_n(X) U^{-1}_{nm} = \sum_{\{n\}} X_q^n U^{-1}_{nm} = X_m^n.
\]

(57)

Furthermore, define two vectors \( Y \) and \( X_q \) as

\[
Y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_M \end{bmatrix} \equiv \begin{bmatrix} q_1 & q_2 & \cdots & q_m \\ q_1^2 & q_2^2 & \cdots & q_m^2 \\ \vdots & \vdots & \ddots & \vdots \\ q_1^M & q_2^M & \cdots & q_m^M \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_M \end{bmatrix},
\]

(58)

and

\[
X_q = \begin{bmatrix} x_{q_1} \\ x_{q_2} \\ \vdots \\ x_{q_M} \end{bmatrix}.
\]

(59)
we have

\[ Y = \sqrt{MR}X. \]  

(60)

Using the same notation and remember that \( R^{-1} = R^* \), we find

\[ X = \frac{1}{\sqrt{M}}R^*Y = \frac{1}{M}Y_{q^*}. \]  

(61)

These results further lead to

\[
X^m = \frac{1}{MN} (Y_{q^*})^m = \frac{1}{MN} Y^m_q = \frac{1}{MN} \sum_{(n)} \phi_n(m) Y^m_q = \frac{1}{MN} \sum_{(n)} f_n(X) \phi_n(m).
\]  

(62)

Comparing Eq. (62) with (57), we get:

\[ U^{-1}_{mn} = \frac{1}{MN} \phi_n(m), \]  

(63)

where we have used the relations:

\[
Y^m_q = y^{m_1}_{q_1} y^{m_2}_{q_2} y^{m_3}_{q_3} \cdots y^{m_{M-1}}_{q_{M-1}} y^{m_{M}}_{q_{M}},
\]

\[
= y^{m_{M-1}}_{g_{M-2}} y^{m_{M-3}}_{g_{M-3}} \cdots y^{m_{1}}_{g_{1}} y^{m_{0}}_{g_{0}},
\]

\[
= \tilde{Y}^m_q = f_m(Y),
\]

(64)

and defined \( \tilde{m} \) as

\[ \tilde{m} \equiv [m_{M-1}, m_{M-2}, \ldots, m_{1}, m_{0}]. \]  

(65)

Finally we obtain the desired solution of \( \langle m | P^s | m_0 \rangle \):

\[
\langle m | P^s | m_0 \rangle = (S^s)_{m_0 m},
\]

\[
= (U^s U^{-1})_{m_0 m},
\]

\[
= \frac{1}{MN} \sum_{m'} \gamma^s_{m'} \phi_m(m') \phi_m(m_0),
\]  

(66)

where \( \Gamma \) is the eigenvalue matrix of \( S \), which has components \( \Gamma_{mm'} = \gamma^s_{m'} \delta_{mm'} \).

IV. POINCARÉ CYCLE

In this section, we study the Poincaré cycle of our periodic urn model. For simplicity we first consider the situation that initially all the \( N \) balls are stayed in the last urn, i.e.,

\[ m_0 = [0, 0, \ldots, N]. \]  

(67)

From (67) and (65) we have

\[ m_0 = \tilde{m}_0. \]  

(68)

Now we want to know how many time steps on average are required for all of the \( N \) balls to return to the last urn (the initial state). We thus have to calculate

\[ \langle m_0 | P^s | m_0 \rangle = \frac{1}{MN} \sum_{m} \gamma^s_{m_0} \phi_m(m_0) \phi_m(m_0). \]  

(69)

Recall that \( \phi_m(m) \) is nothing but the coefficient of \( X^m = x_1^{m_1} \cdots x_M^{m_M} \) appearing in the expansion of \( f_n(X) = X^m_q = x_1^{m_1} \cdots x_M^{m_M} \). From Eq. (54) and (67), we have

\[ \phi_m(m_0) = 1, \]  

(70)

\[ \phi_m(m_0) = \frac{N!}{m_1! m_2! m_3! \cdots m_M!} \equiv \binom{N}{m}, \]  

(71)

and hence

\[ \langle m_0 | P^s | m_0 \rangle = \frac{1}{MN} \sum_{m} \binom{N}{m} \gamma^s_{m_0} \equiv \mathcal{P}(s). \]  

(72)

Here \( \mathcal{P}(s) \) represents the transition probability for the system to return to the initial state after \( s \) steps. It does not preclude the possibility that the initial state has already been re-arrived before.

Since Poincaré cycle is defined as the time interval required for the event of first return to happen, so we have to do more calculations to extract what we really want. We define a function \( Q(s) \) as the probability for the event of first return to happen at the \( s \)th step. The Poincaré cycle can thus be defined as

\[ \text{P. C.} = \sum_{s=0}^{\infty} s Q(s). \]  

(73)

By definition \( Q(s) \) relates to \( \mathcal{P}(s) \) via the relation

\[ \mathcal{P}(s) = Q(s) + \sum_{k=1}^{s-1} Q(k) \mathcal{P}(s-k), \]  

(74)

and hence \( Q(s) \) can be calculated from \( \mathcal{P}(s) \). To ease the calculation we now use again the generating function method. We first define two generating functions:

\[ h(z) \equiv \sum_{s=1}^{\infty} \mathcal{P}(s) z^s, \quad g(z) \equiv \sum_{s=1}^{\infty} Q(s) z^s, \]  

(75)

and then we find from Eq. (74) that

\[ g(z) = \frac{h(z)}{h(z) + 1}. \]  

(76)

These two generating functions also lead to

\[ \sum_{s=0}^{\infty} s Q(s) = \left( \frac{dg}{dz} \right)_{z=1}, \quad g' = \frac{h'}{(1 + h)^2}, \]  

(77)
which can determine the Poincaré cycle. We now calculate $h(z)$. From Eq. (72) and (85) we obtain
\[
h(z) = \frac{1}{M^N} \sum_{\{m\}} \left( \sum_{s=1}^{\infty} (\gamma_m z)^s \right),
\]
\[
= \frac{1}{M^N} \sum_{\{m\}} \left( \sum_{s=1}^{\infty} \frac{(\gamma_m z)^s}{1 - \gamma_m z} \right).
\]
(78)
Since we know from Eq. (52) that $\gamma_{m_0} = 1$, thus when $z \to 1^-$, $h(z)$ becomes singular:
\[
\lim_{z \to 1^-} h(z) = \frac{1}{M^N} \frac{z}{1 - z} + \text{regular function}.
\]
(79)
In this limit, we obtain
\[
\lim_{z \to 1^-} g' = \lim_{z \to 1^-} \left( \frac{h'}{(1 + h)^2} \right) = M^N,
\]
(80)
which gives us the desired Poincaré cycle:
\[
P.C. = \sum_{s=0}^{\infty} sQ(s) = M^N.
\]
(81)

To understand this result, we refer to the ergodic theorem [1], which says that if one waits a sufficiently long time, the locus of the representative point of a system will cover the entire accessible phase space. For our periodic urn model, the “representative point” corresponds to the microstate of the arrangement of balls, the “accessible phase space” is the set of total microstates, and the “locus” means the evolution history of the system (See Fig. 1). Since in our periodic model every microstate has the same probability to appear (a fundamental assumption of statistical mechanics), and the set of total microstates contains $M^N$ microstates, therefore on average one has to wait $M^N$ time steps to see a microstate reappearing again.

What will be the Poincaré cycle if the initial state is different from the $m_0$ in Eq. (67)? Let’s denote the initial state by $\tilde{d}$:
\[
|\tilde{d}\rangle = |d_1, d_2, \ldots, d_M\rangle.
\]
(82)
Now we have
\[
\mathcal{P}(s) = \langle d|P^s|d \rangle = \frac{1}{M^N} \sum_{\{m\}} \gamma_m^s \phi_d(\{m\}) \phi_{\tilde{d}}(\{m\})
\]
and
\[
h(z) = \frac{1}{M^N} \sum_{\{m\}} \phi_d(\{m\}) \phi_{\tilde{d}}(\{m\}) \left( \frac{\lambda_m z}{1 - \lambda_m z} \right).
\]
(84)
In general it is difficult to calculate $\phi_d(\{m\})$ and $\phi_{\tilde{d}}(\{m\})$. However, we do not need to calculate them all. Remember that to determine the Poincaré cycle the the knowledge of the asymptotic form of $h(z)$ near $z = 1$ is enough. This is given by
\[
\lim_{z \to 1^-} h(z) = \frac{\phi_d(m_0)\phi_{\tilde{d}}(d_0)}{M^N} \frac{z}{1 - z} + \text{regular function}.
\]
(85)
Substituting
\[
\phi_d(m_0) = \left( \frac{N}{d_0} \right), \quad \phi_{\tilde{d}}(d) = 1
\]
into (85), we find
\[
P.C. = \lim_{z \to 1^-} \frac{h'}{(1 + h)^2} = \frac{M^N}{\left( \frac{N}{d_0} \right)}
\]
(87)
This result conforms with our intuition. Consider the difference between the concepts of configuration and microstate, one can easily understand this result because the degeneracy of the configuration $d$ is given by
\[
\left( \frac{N}{d_0} \right) = \frac{N!}{d_1!d_2!\cdots d_M!}.
\]

V. SUMMARY

In this work we propose a generalized Ehrenfest urn model of many urns arranged periodically along a circle. We solve an $N$-ball, $M$-urn problem explicitly. The evolution of the system is studied, and the average number of balls in a certain urn at any time step is calculated. We find that this mean value oscillates several times before it arrives the stationary value. We also obtained the Poincaré cycle. The result is consistent with the consideration of the total number of possible microstates of the system.

ACKNOWLEDGMENTS

The authors would like to acknowledge helpful discussions with professor T. F. Jiang. This work received support from the National Science Council, Republic of China through Grant No. NSC 90-2811-M-009-019.

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