Some Estimates for Martingale Representation under $G$-Expectation

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Abstract

We provide some useful estimates for solving martingale representation problem under $G$-expectations. We also study the corresponding conditions for the existence and uniqueness.

1 Introduction

Many important progresses in the domain of $G$-expectation and related $G$-Brownian motion have been made in recent years since the introduction of this theory. But some very important questions still remain open. An interesting and challenging one is the so called $G$-martingale representation problem proposed in [8, Peng2007] of the following form:

$$M_t = \mathbb{E}_G[X|\Omega_t] = \mathbb{E}_G[X] + \int_0^t z_s dB_s + \int_0^t \eta_s d\langle B \rangle_s - \int_0^t 2G(\eta_s)ds,$$

for some given element $X \in L^2_G(\Omega_T)$. As Peng pointed out in many of his lectures and discussions, this formulation permits us to treat the following version of BSDE driven by $G$-Brownian motion of the form

$$y_t = X + \int_t^T f(s,y_s,z_s,\eta_s)ds - \int_t^T z_s dB_s - \int_t^T \eta_s d\langle B \rangle_s + \int_t^T 2G(\eta_s)ds.$$

[8, 10, Peng2007, Peng2010] had only treated the above form of $G$-martingale for the situation where $X \in L_{lip}(\Omega_T)$ and $G$ is non-degenerate.

The first step towards a proof to (1) for a more general $X \in L^2_G(\Omega_T)$ was given by [13] for the case where $X$ satisfies $\mathbb{E}_G[X] = -\mathbb{E}_G[-X]$. In this case

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the $G$-martingale $M_t = \mathbb{E}_G[X|\Omega_t], t \geq 0$, is symmetric, namely $-M$ is also a $G$-martingale. They obtained a representation of the form $X = \mathbb{E}_G[X] + \int_0^T z_s dB_s$, corresponding to the case $\eta \equiv 0$.

Observe that in the formulation of (1) $M_t = M^*_t - A_t$, where $M^*_t = \mathbb{E}_G[X] + \int_0^t z_s dB_s$ is a symmetric martingale and

$$A_t = \int_0^t 2G(\eta_s)ds - \int_0^t \eta_s d\langle B\rangle_s, \quad t \in [0, T]$$

is a nondecreasing process with $A_0 = 0$ such that $-A_t$ is a $G$-martingale. A very interesting problem is whether $M$ has a unique decomposition $M = M^*_t - A_t$. An important progress of this problem was obtained by Soner, Touzi and Zhang [11]. They have proved that, under the condition

$$\|X\|_{L^2} := \mathbb{E}_G[\sup_{t \in [0,T]} |M_t|^2] < \infty$$

there exists a unique decomposition $M = M^*_t - A_t$ such that

$$\mathbb{E}_G[A_T^2] + \mathbb{E}_G[\int_0^T |z_s|^2 ds] \leq C^* \|X\|_{L^2}$$

where $C^*$ is a universal constant. More recently, [12, Song, 2010] has significantly improved their result by only assuming that $X \in L^p_G(\Omega_T)$ for $p > 1$. His result will be used in this paper (see the next section).

In this paper we give an a priori estimate of $\eta$ in the representation of (1): if for a fixed $\varepsilon > 0$, $G_\varepsilon(a) = G(a) - \varepsilon/2|a|$ is a sublinear function of $a$, then we have

$$\mathbb{E}_G[\int_0^T |\eta_t|^2 dt] \leq \varepsilon^{-1}(\mathbb{E}_G[X] + \mathbb{E}_G[\eta^2])$$

This estimate indicates clearly that the norm of $\eta$ can be dominated by $\mathbb{E}_G[X] + \mathbb{E}_G[\eta^2]$ and thus provides a new proof of [8, 10] for the existence of the representation in the case $X \in L^p_{\text{lip}}(\Omega_T)$. We will also give a proof of uniqueness of $\eta$. We then show that, if the increasing part $A$ of $M$ satisfies a type of bounded variation condition, then we can also prove the existence of the representation.

This paper is organized as follows: after given some basic settings in the next section, we give the a priori estimate and then a proof of the representation for the case where $X \in L^p_{\text{lip}}(\Omega_T)$, in Section 3. In Section 4 we study the uniqueness of representation theorem for $X \in L^p_G(\Omega_T)$. In Section 5 we study the existence of the representation of $G$-martingales.

## 2 Preliminaries

We present some preliminaries in the theory of sublinear expectations and the related $G$-Brownian motions. More details can be found in Peng [6], [7] and [8].
Definition 1 Let $\Omega$ be a given set and let $\mathcal{H}$ be a linear space of real valued functions defined on $\Omega$ with $c \in \mathcal{H}$ for all constants $c$, and $|X| \in \mathcal{H}$, if $X \in \mathcal{H}$. $\mathcal{H}$ is considered as the space of our “random variables”. A sublinear expectation $\hat{E}$ on $\mathcal{H}$ is a functional $\hat{E} : \mathcal{H} \to \mathbb{R}$ satisfying the following properties: for all $X, Y \in \mathcal{H}$, we have

(a) **Monotonicity:** If $X \geq Y$ then $\hat{E}[X] \geq \hat{E}[Y]$.
(b) **Constant preserving:** $\hat{E}[c] = c$.
(c) **Sub-additivity:** $\hat{E}[X] - \hat{E}[Y] \leq \hat{E}[X - Y]$.
(d) **Positive homogeneity:** $\hat{E}[\lambda X] = \lambda \hat{E}[X]$, $\forall \lambda \geq 0$.

The triple $(\Omega, \mathcal{H}, \hat{E})$ is called a sublinear expectation space. $X \in \mathcal{H}$ is called a random variable in $(\Omega, \mathcal{H})$. We often call $Y = (Y_1, \cdots, Y_d)$, $Y_i \in \mathcal{H}$ a $d$-dimensional random vector in $(\Omega, \mathcal{H})$. Let us consider a space of random variables $\mathcal{H}$ satisfying: if $X_i \in \mathcal{H}$, $i = 1, \cdots, d$, then

$$\varphi(X_1, \cdots, X_d) \in \mathcal{H}, \text{ for all } \varphi \in C_{b,Lip}(\mathbb{R}^d),$$

where $C_{b,Lip}(\mathbb{R}^d)$ is the space of all bounded and Lipschitz continuous functions on $\mathbb{R}^d$. An $m$-dimensional random vector $X = (X_1, \cdots, X_m)$ is said to be independent of another $n$-dimensional random vector $Y = (Y_1, \cdots, Y_n)$ if

$$\hat{E}[\varphi(X, Y)] = \hat{E}[\varphi(X, y)]_{y=Y}, \text{ for } \varphi \in C_{b,Lip}(\mathbb{R}^m \times \mathbb{R}^n).$$

Let $X_1$ and $X_2$ be two $n$-dimensional random vectors defined respectively in sublinear expectation spaces $(\Omega_1, \mathcal{H}_1, \hat{E}_1)$ and $(\Omega_2, \mathcal{H}_2, \hat{E}_2)$. They are called identically distributed, denoted by $X_1 \sim X_2$, if

$$\hat{E}_1[\varphi(X_1)] = \hat{E}_2[\varphi(X_2)], \forall \varphi \in C_{b,Lip}(\mathbb{R}^n).$$

If $X, \bar{X}$ are two $m$-dimensional random vectors in $(\Omega, \mathcal{H}, \hat{E})$ and $\bar{X}$ is identically distributed with $X$ and independent of $X$, then $\bar{X}$ is said to be an independent copy of $X$.

Definition 2 (**G-normal distribution**) A $d$-dimensional random vector $X = (X_1, \cdots, X_d)$ in a sublinear expectation space $(\Omega, \mathcal{H}, \hat{E})$ is called $G$-normal distributed if for each $a, b \geq 0$ we have

$$aX + b\bar{X} \sim \sqrt{a^2 + b^2}X, \quad (2)$$

where $\bar{X}$ is an independent copy of $X$. Here the letter $G$ denotes the function

$$G(A) := \frac{1}{2}\hat{E}[(AX, X)] : \mathbb{S}_d \to \mathbb{R}.$$

It is also proved in Peng [1, 2] that, for each $a \in \mathbb{R}^d$ and $p \in [1, \infty)$

$$\hat{E}[|a, X|^p] = \frac{1}{\sqrt{2\pi\sigma_{aa}^2}} \int_{-\infty}^{\infty} |x|^p \exp \left( \frac{-x^2}{2\sigma_{aa}^2} \right) dx,$$

where $\sigma_{aa}^2 = 2G(aa^T)$. 

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Definition 3. A d-dimensional stochastic process \( \xi_t(\omega) = (\xi^1_t, \cdots, \xi^d_t)(\omega) \) defined in a sublinear expectation space \((\Omega, \mathcal{H}, \mathbb{E})\) is a family of d-dimensional random vectors \( \xi_t \) parameterized by \( t \in [0, \infty) \) such that \( \xi^i_t \in \mathcal{H} \), for each \( i = 1, \cdots, d \) and \( t \in [0, \infty) \).

The most typical stochastic process in a sublinear expectation space is the so-called G-Brownian motion.

Definition 4. (\cite{G} and \cite{S}) Let \( G: \mathbb{S}_d \mapsto \mathbb{R} \) be a given monotonic and sublinear function. A process \( \{B_t(\omega)\}_{t \geq 0} \) in a sublinear expectation space \((\Omega, \mathcal{H}, \mathbb{E})\) is called a G-Brownian motion if for each \( n \in \mathbb{N} \) and \( 0 \leq t_1, \cdots, t_n < \infty \), \( B_{t_1}, \cdots, B_{t_n} \in \mathcal{H} \) and the following properties are satisfied:

(i) \( B_0(\omega) = 0 \);
(ii) For each \( t, s \geq 0 \), the increment \( B_{t+s} - B_t \) is independent of \( (B_{t_1}, B_{t_2}, \cdots, B_{t_n}) \), for each \( n \in \mathbb{N} \) and \( 0 \leq t_1 \leq \cdots \leq t_n \leq t \);
(iii) \( B_{t+s} - B_s \sim B_t \), for \( s, t \geq 0 \) and \( \mathbb{E}[|B_t|^3]/t \to 0 \), as \( t \to 0 \).
(iv) \( \mathbb{E}[B_t] = -\mathbb{E}[-B_t] = 0 \), for \( t \geq 0 \).

It was proved that, for each \( t > 0 \), \( B_t/\sqrt{t} \) is \( G \)-normal distributed with \( G(A) = \mathbb{E}[\{AB_1, B_1\}] \). In many cases \( B \) is also called G-Brownian motion when it only satisfies (i)-(iii), and a G-Brownian motion satisfying (i)-(iv) is called symmetric G-Brownian motion. In this paper we only discuss symmetric G-Brownian motions.

We denote:
\( \Omega = C^d_0(\mathbb{R}^+) \) the space of all \( \mathbb{R}^d \)-valued continuous functions \( (\omega_t)_{t \in \mathbb{R}^+} \), with \( \omega_0 = 0 \), equipped with the distance
\[
\rho(\omega^1, \omega^2) := \sum_{i=1}^{\infty} 2^{-i} [\max_{t \in [0, i]} |\omega^1_t - \omega^2_t|] \wedge 1.
\]
We denote by \( \mathcal{B}(\Omega) \) the Borel \( \sigma \)-algebra of \( \Omega \) and by \( \mathcal{M} \) the collection of all probability measure on \((\Omega, \mathcal{B}(\Omega))\).

We also denote, for each \( t \in [0, \infty) \):

- \( \Omega_t := \{\omega_{\Lambda t} : \omega \in \Omega\} \),
- \( \mathcal{F}_t := \mathcal{B}(\Omega_t) \),
- \( L^0(\Omega) \): the space of all \( \mathcal{B}(\Omega) \)-measurable real functions,
- \( L^0(\Omega_t) \): the space of all \( \mathcal{B}(\Omega_t) \)-measurable real functions,
- \( B_b(\Omega) \): all bounded elements in \( L^0(\Omega) \), \( B_b(\Omega_t) := B_b(\Omega) \cap L^0(\Omega_t) \),
- \( C_b(\Omega) \): all continuous elements in \( B_b(\Omega) \), \( C_b(\Omega_t) := B_b(\Omega) \cap L^0(\Omega_t) \).

In \cite{G} \cite{S}, a G-Brownian motion is constructed on a sublinear expectation space \((\Omega, L^p_0(\Omega), \mathbb{E})\) for \( p \geq 1 \), with \( L^p_0(\Omega) \) such that \( L^p_0(\Omega) \) is a Banach space
under the natural norm \( \|X\|_p := \hat{E}[|X|^p]^{1/p} \). In this space the corresponding canonical process \( B_t(\omega) = \omega_t, t \in [0, \infty), \) for \( \omega \in \Omega \) is a \( G \)-Brownian motion.

Moreover, the notion of \( G \)-conditional expectation was also introduced \( \hat{E}[\cdot|\Omega_t] : L^p_G(\Omega) \to L^p_G(\Omega_t), \) for each \( t \geq 0 \). It satisfies: \( \hat{E}[\hat{E}[X|\Omega_t]|\Omega_s] = \hat{E}[X|\Omega_t \wedge \Omega_s] \) and

(a) Monotonicity: if \( X \geq Y \) then \( \hat{E}[X|\Omega_t] \geq \hat{E}[Y|\Omega_t] \),

(b) Constant preserving: \( \hat{E}[\eta|\Omega_t] = \eta, \eta \in L^p_G(\Omega_t), \)

(c) Sub-additivity: \( \hat{E}[X|\Omega_t] - \hat{E}[Y|\Omega_t] \leq \hat{E}[X - Y|\Omega_t] \).

(d) Positive homogeneity: \( \hat{E}[\eta X|\Omega_t] = \eta \hat{E}[X|\Omega_t], \) for bounded and non-negative \( \eta \in L^p_G(\Omega_t) \).

Furthermore, it is proved in [3] (see also [4] for a simple proof) that \( L^0(\Omega) \supset L^p_G(\Omega) \supset C_b(\Omega) \), and there exists a weakly compact family \( \mathcal{P} \) of probability measures defined on \( (\Omega, \mathcal{B}(\Omega)) \) such that

\[ \hat{E}[X] = \sup_{P \in \mathcal{P}} E_P[X], \text{ for } X \in C_b(\Omega). \]

We introduce the natural Choquet capacity (see [4]):

\[ \hat{c}(A) := \sup_{P \in \mathcal{P}} P(A), \text{ for } A \in \mathcal{B}(\Omega). \]

The space \( L^2_G(\Omega) \) was also introduced independently in [2] in a quite different framework.

**Definition 5** A set \( A \subset \Omega \) is polar if \( \hat{c}(A) = 0 \). A property holds “quasi-surely” (q.s.) if it holds outside a polar set.

\( L^p_G(\Omega) \) can be characterized as follows:

\[ L^p_G(\Omega) = \{ X \in L^0(\Omega) | \sup_{P \in \mathcal{P}} E_P[|X|^p] < \infty, \text{ and } X \text{ is } \hat{c} \text{-quasi surely continuous} \}. \]

We also denote, for \( p > 0 \),

- \( \mathcal{L}^p := \{ X \in L^0(\Omega) : \hat{E}[|X|^p] = \sup_{P \in \mathcal{P}} E_P[|X|^p] < \infty \}; \)
- \( \mathcal{N}^p := \{ X \in L^0(\Omega) : \hat{E}[|X|^p] = 0 \}; \)
- \( \mathcal{N} := \{ X \in L^0(\Omega) : X = 0, \hat{c} \text{-quasi surely (q.s.)} \}. \)

It is easy to see that \( \mathcal{L}^p \) and \( \mathcal{N}^p \) are linear spaces and \( \mathcal{N}^p = \mathcal{N} \), for each \( p > 0 \). We denote by \( \mathcal{L}^p : = \mathcal{L}^p/\mathcal{N} \). As usual, we do not make the distinction between classes and their representatives.

Now, we give the following two propositions which can be found in [3].

**Proposition 6** For each \( \{X_n\}_{n=1}^\infty \) in \( C_b(\Omega) \) such that \( X_n \downarrow 0 \) on \( \Omega \), we have \( \hat{E}[X_n] \downarrow 0 \).

**Proposition 7** We have
1. For each $p \geq 1$, $L^p$ is a Banach space under the norm $\|X\|_p := \left(\mathbb{E}[|X|^p]\right)^{\frac{1}{p}}$.

2. $L^p_*$ is the completion of $B_b(\Omega)$ under the Banach norm $\mathbb{E}[|X|^p]^{1/p}$.

3. $L^p_G$ is the completion of $C_b(\Omega)$.

The following proposition is obvious.

**Proposition 8** We have

1. $L^p_* \subset L^p \subset L^q_*$, $0 < p \leq q \leq \infty$;
2. $\|X\|_p \uparrow \|X\|_\infty$, for each $X \in L^\infty$;
3. $p, q > 1$, $\frac{1}{p} + \frac{1}{q} = 1$. Then $X \in L^p$ and $Y \in L^q$ implies $XY \in L^1$ and $\mathbb{E}[|XY|] \leq \left(\mathbb{E}[|X|^p]\right)^{\frac{1}{p}} \left(\mathbb{E}[|Y|^q]\right)^{\frac{1}{q}}$.

**Proposition 9** For a given $p \in (0, +\infty]$, let $\{X_n\}_{n=1}^\infty$ be a sequence in $L^p$ which converges to $X$ in $L^p$. Then there exists a subsequence $(X_{n_k})$ which converges to $X$ quasi-surely in the sense that it converges to $X$ outside a polar set.

We also have

**Proposition 10** For each $p > 0$, 

$$L^p_0 = \{X \in L^p : \lim_{n \to \infty} \mathbb{E}[|X|^p 1_{\{|X| > n\}}] = 0\}.$$

We introduce the following properties. They are important in this paper:

**Proposition 11** For each $0 \leq t < T$, $\xi \in L^2(\Omega_t)$, we have 

$$\mathbb{E}[\xi(B_T - B_t)] = 0.$$

**Proof.** Let $P \in \mathcal{P}$ be given. If $\xi \in C_0(\Omega_t)$, then we have 

$$0 = -\mathbb{E}[-\xi(B_T - B_t)] \leq E_P[\xi(B_T - B_t)] \leq \mathbb{E}[\xi(B_T - B_t)] = 0.$$

In the case when $\xi \in L^2(\Omega_t)$, we have $E_P[|\xi|^2] \leq \mathbb{E}[|\xi|^2] < \infty$.

Since it is known that $C_0(\Omega_t)$ is dense in $L^2_p(\Omega_t)$, we then can choose a sequence $\{\xi_n\}_{n=1}^\infty$ in $C_0(\Omega_t)$ such that $E_P[|\xi - \xi_n|^2] \to 0$. Thus

$$E_P[\xi(B_T - B_t)] = \lim_{n \to \infty} E_P[\xi_n(B_T - B_t)] = 0.$$

The proof is complete. ■

From now on and throughout this paper we restrict ourselves to the situation of 1-dimensional $G$-Brownian motion case. In this case $G(a)$ becomes a given sublinear and monotonic real valued function defined on $\mathbb{R}$. $G$ can be written as

$$G(a) = \frac{1}{2}(\sigma^2a^+ - \sigma^2a^-).$$
Proposition 12 For each $0 \leq t \leq T$, $\xi \in B_b(\Omega_t)$, we have

$$\hat{E}[\xi^2(B_T - B_t)^2 - \sigma^2 \xi^2(T - t)] \leq 0. \quad (3)$$

Proof. If $\xi \in C_b(\Omega_t)$, then by [6], we have the following Itô formula:

$$\xi^2[(B_T - B_t)^2 - (\langle B_T \rangle - \langle B_t \rangle)] = 2 \int_t^T \xi^2 B_s dB_s.$$  

It follows that $\hat{E}[\xi^2(B_T - B_t)^2 - \xi^2(\langle B_T \rangle - \langle B_t \rangle)] = 0$. On the other hand, we have $\langle B_T \rangle - \langle B_t \rangle \leq \sigma^2(T - t)$, quasi surely. Thus (3) holds for $\xi \in C_b(\Omega_t)$. It follows that, for each fixed $P \in \mathcal{P}$, we have

$$E_P[\xi^2(B_T - B_t)^2 - \xi^2(\langle B_T \rangle - \langle B_t \rangle)] \leq 0. \quad (4)$$

In the case when $\xi \in B_b(\Omega_t)$, we can find a sequence $\{\xi_n\}_{n=1}^\infty$ in $C_b(\Omega_t)$, such that $\xi_n \to \xi$ in $L^p(\Omega, F_t, P)$, for some $p > 2$. Thus we have

$$E_P[\xi_n^2(B_T - B_t)^2 - \xi_n^2(\langle B_T \rangle - \langle B_t \rangle)] \leq 0,$$

and then, by letting $n \to \infty$, we obtain (3) for $\xi \in B_b(\Omega_t)$. Thus (3) follows immediately for $\xi \in B_b(\Omega_t)$.  

The space $L^p_G(\Omega)$ is a Banach space under the norm $\| \cdot \|_p := E_G[| \cdot |^p]^{1/p}$. We have also introduced a space of ‘adapted processes’ $M^p_G(0, T)$ which is also a Banach space under the following norm:

$$\| \eta \|_{M^p_G(0, T)} = \left( \int_0^T E_G[| \eta_t |^p] dt \right)^{1/p}, \quad \eta \in M^p_G(0, T).$$

Exactly following Itô’s original idea, for each $\eta \in M^2_G(0, T)$, we have introduced Itô’s integral

$$\int_0^T \eta_t dB_t.$$  

We have

$$E_G[\int_0^T \eta_t dB_t] = 0, \quad E_G\left[\left( \int_0^T \eta_t dB_t \right)^2 \right] = E_G\left[\int_0^T |\eta_t|^2 d\langle B \rangle_t \right].$$

Moreover for each $\zeta \in M_G^1(0, T)$ we have the following estimate. For each $\eta \in M^2_G(0, T)$ we have

$$\sigma^2 E_G[\int_0^T |\zeta_s| ds] \geq E_G[\int_0^T |\zeta_s d\langle B \rangle_s|] \geq 2\sigma^2 E_G[\int_0^T |\zeta_s| ds]. \quad (5)$$

The following relations play an essentially important role in this paper (see [10]): for each $\zeta \in M^1_G(0, T)$,

$$\int_0^T \zeta_t d\langle B \rangle_t - \int_0^T 2G(\zeta_t) dt \leq 0, \quad E_G[\int_0^T \zeta_t d\langle B \rangle_s] - \int_0^T 2G(\zeta_s) ds = 0. \quad (6)$$

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In this paper we introduce a norm \( \| \cdot \|_{M^p_G(0,T)} \) which is weaker than \( \| \cdot \|_{M^p_G(0,T)} \):

\[
\| \eta \|_{M^p_G(0,T)} = \left( \mathbb{E}_G \left[ \int_0^T |\eta_t|^p dt \right] \right)^{1/p}.
\]

The completion of \( M^p_G(0,T) \) under this norm is denoted by \( M^p_G(0,T) \). It is easy to check that the definition of Itô’s integral can be extended to the case \( \eta \in M^2_G(0,T) \) and the integral \( \int_0^T \xi d\langle B \rangle_t \) can be defined also for \( \zeta \in M^1_G(0,T) \). The above relations still hold true.

We list the result of Song of \( G \)-martingale decomposition which generalizes that of [11]. Let \( H^0_G(0,T) \) be the family of simple processes of form \( z = \sum_{i=0}^{N-1} z_i 1_{[t_i, t_{i+1})} \), \( t \in [0,T] \), \( z_i \in L^p(\Omega_{t_i}) \). For each \( z \) in this space, we define the following norm

\[
\| z \|_{H^p_G(0,T)} = \left( \int_0^T |z_s|^2 ds \right)^{p/2} \right)^{1/p}
\]

and denote by \( H^p_G(0,T) \) the completion of \( H^0_G(0,T) \) under this norm.

**Theorem 13** ([12], Theorem 4.5) For any given \( p > 1 \) and \( X \in L^p_G(\Omega_T) \) the \( G \)-martingale \( M_t = \mathbb{E}_G[X|\Omega_t], t \in [0,T] \), has the following decomposition:

\[
M_t = \mathbb{E}_G[X] + \int_0^t z_s dB_s - A_t, \ t \in [0,T],
\]

where \( z \in H^1_G(0,T) \) and \( A \) is a continuous increasing process with \( A_0 = 0 \) such that \( (-A_t)_{0 \leq t \leq T} \) is a \( G \)-martingale. Furthermore the above decomposition is unique and \( z \in H^p_G(0,T) \), \( K_T \in L^2_G(\Omega_T) \) for any \( 1 \leq \alpha < p \).

### 3 A priori estimates and representation theorem for \( L^p_G(\Omega_T) \)

Let the function \( G \) be given as the above. Then for a fixed \( \varepsilon \in (0, (\sqrt{2} - \sqrt{2})^2)/2 \), we set \( G_\varepsilon(a) = G(a) - \varepsilon |a| \).

**Theorem 14** We assume that \( \xi \in L^2_G(\Omega_T) \) has the following representation: there exists a pair of processes \( (z, \eta) \in H^p_G(0,T) \times M^1_{G_\varepsilon}(0,T), \) with \( p \in [1,2], \) such that

\[
\xi = \mathbb{E}_G[\xi] + \int_0^T z_s dB_s + \int_0^T \eta_s d\langle B \rangle_s - \int_0^T 2G(\eta_s^2) ds. \tag{7}
\]

Then we have

\[
\varepsilon \mathbb{E}_{G_\varepsilon} \left[ \int_0^T |\eta_s|^2 ds \right] \leq \mathbb{E}_G[\xi] + \mathbb{E}_{G_\varepsilon}[-\xi]. \tag{8}
\]
Proof. From
\[ 0 = \mathbb{E}_{\xi} \left[ \int_0^T z_s^\delta dB_s + \int_0^T \eta_s^\delta d\langle B \rangle_s - \int_0^T 2G_{\xi}(\eta_s^\delta)ds \right] \]
\[ = \mathbb{E}_{\xi} \left[ \xi - \mathbb{E}_{\xi}[\xi] + \int_0^T 2(G(\eta_s^\delta) - G(\eta_t^\delta))ds \right] \]
\[ = \mathbb{E}_{\xi} \left[ \xi - \mathbb{E}_{\xi}[\xi] + \int_0^T \varepsilon|\eta_s^\delta|ds \right] \]
\[ \geq \mathbb{E}_{\xi} \left[ \int_0^T \varepsilon|\eta_s^\delta|ds - \mathbb{E}_{\xi}[\xi] - \mathbb{E}_{\xi}[\xi] \right] \]
we immediately have (8). ■

For the rest of this paper we assume that $\xi \geq \underline{\xi} > 0$.

We set
\[ L_{ip}(\Omega_T) = \{ \xi = \varphi(B_{t_1}, B_{t_2}, \ldots, B_{t_n}), \varphi \in C_{Lip}(\mathbb{R}^n), t_i \in [0, \infty), i = 1, 2, \cdots, n \geq 1 \} \]

Theorem 15 For each $\xi \in L_{ip}(\Omega_T)$, we have a unique representation (7).

Proof. It suffices to prove the existence of $(z, \eta)$ for such type of $\xi$.

We first consider the case where $\xi = \varphi(B_{t_2} - B_{t_1})$ with $t_1 < t_2 \leq T$ and $\varphi$ is a Lipschitz and bounded function on $\mathbb{R}$. Let $V$ be the unique viscosity solution of
\[ \partial_t V + G(D^2 V) = 0, \ (t, x) \in [t_1, t_2) \times \mathbb{R}^d, \]
with terminal condition
\[ V(t_2, x) = \varphi(x). \]

Since (9) is a uniform parabolic PDE and $G$ is a convex and Lipschitz function thus, by the regularity of $V$ (see Krylov [5], Example 6.1.8 and Theorem 6.4.3, see also Appendix of [10]), we have
\[ ||V||_{C^{1+\alpha/2,2+\alpha}(\{t_1, t_2\} \times \mathbb{R}^d)} < \infty, \text{ for some } \alpha \in (0, 1). \]

Then, for each $\delta \in (0, t_2 - t_1)$, we set
\[ \eta_s^\delta = 1_{(t_1, t_2-\delta]}(s) \frac{1}{2} V_{xx}(s, B_s), \ z_s^\delta = 1_{(t_1, t_2-\delta]}(s) V_x(s, B_s), \ s \in [0, T], \]
and
\[ \eta_s = 1_{(t_1, t_2]}(s) \frac{1}{2} V_{xx}(s, B_s), \ z_s = 1_{(t_1, t_2]}(s) V_x(s, B_s), \ s \in [0, T]. \]

It is clear that $(z^\delta, \eta^\delta) \in M^2_G(0, T)$ for each fixed $\delta \in (0, t_2 - t_1)$. By Itô’s formula we have
\[ V(t_2 - \delta, B_{t_2-\delta}) - V(t_1, 0) \]
\[ = \int_0^{t_2-\delta} z_s^\delta dB_s + \int_0^{t_2-\delta} \eta_s^\delta d\langle B \rangle_s - \int_0^{t_2-\delta} 2G(\eta_s^\delta)ds. \]
On the other hand, for each \( t, t' \in [t_1, t_2] \) and \( x, x' \in \mathbb{R}^d \), we have
\[
|V(t, x) - V(t', x')| \leq C(\sqrt{|t - t'|} + |x - x'|).
\]
Thus we have
\[
|\mathbb{E}[V(t_2 - \delta', B_{t_2 - \delta'}) - V(t_2 - \delta, B_{t_2 - \delta})]|^2
\leq C(\sqrt{\delta - \delta'} + |\delta - \delta'|) + C\mathbb{E}[|B_{t_2 - \delta} - B_{t_2 - \delta'}|^2]
\leq C(\sqrt{\delta - \delta'} + |\delta - \delta'|).
\]
Now for each \( t_2 - t_1 > \delta > \delta' > 0 \),
\[
V(t_2 - \delta, B_{t_2 - \delta}) - V(t_2 - \delta', B_{t_2 - \delta'})
\begin{align*}
&= \int_{t_1}^{t_2} (z_s^\delta - z_s^{\delta'})dB_s + \int_{t_1}^{t_2} (\eta_s^\delta - \eta_s^{\delta'})d\langle B \rangle_s - 2\int_{t_1}^{t_2} (G(\eta_s^\delta) - G(\eta_s^{\delta'}))ds \\
&= \int_{t_1}^{t_2} (z_s^\delta - z_s^{\delta'})dB_s + \int_{t_1}^{t_2} (\eta_s^\delta - \eta_s^{\delta'})d\langle B \rangle_s - 2\int_{t_1}^{t_2} G(\eta_s^\delta - \eta_s^{\delta'})ds.
\end{align*}
\]
We then can apply (8) to prove that
\[
\varepsilon\mathbb{E}_{G_\varepsilon}[\int_{t_1}^{t_2} |\eta_s^\delta - \eta_s^{\delta'}|ds] \leq \mathbb{E}_G[V(t_2 - \delta, B_{t_2 - \delta}) - V(t_2 - \delta', B_{t_2 - \delta'})]
+ \mathbb{E}_G[-V(t_2 - \delta, B_{t_2 - \delta}) + V(t_2 - \delta', B_{t_2 - \delta'})]
\leq C(\sqrt{\delta - \delta'} + |\delta - \delta'|).
\]
Thus, when \( \delta_i \downarrow 0 \), \( \{\eta_t^{\delta_i}\}_{i=1}^\infty \) forms a Cauchy sequence in \( M^1_{G_\varepsilon}(0, T) \), and the representation of \( \xi = \varphi(B_{t_2} - B_{t_1}) \) is uniquely given by
\[
\xi = V(t_1, 0) + \int_0^T z_sdB_s + \int_0^T \eta_s d\langle B \rangle_s - \int_0^T 2G(\eta_s)ds.
\]
Moreover we have \( V(t_1, 0) = \mathbb{E}_G[X] = \mathbb{E}_G[\varphi(B_T)] \).
We now consider the case where \( \xi = \varphi(B_{t_1}, B_{t_2} - B_{t_1}) \), where \( \varphi \) is a Lipschitz and bounded function on \( \mathbb{R}^2 \). For this random variable we first solve the following PDE for a fixed parameter \( y \in \mathbb{R} \):
\[
\partial_t V^y + G(V^y_x) = 0, \quad s \in [t_1, t_2], \quad V^y(t_2, x) = \varphi(y, x).
\]
Then, setting
\[
\eta_s^\delta = 1_{[t_1, t_2-\delta]}(s)\frac{1}{2}V^{B_{t_1}}(s, B_s - B_{t_1}), \quad z_s^\delta = 1_{[t_1, t_2-\delta]}(s)V^{B_{t_1}}(s, B_s - B_{t_1}), \quad \eta_s = 1_{[t_1, t_2]}(s)\frac{1}{2}V^{B_{t_1}}(s, B_s - B_{t_1}), \quad z_s = 1_{[t_1, t_2]}(s)V^{B_{t_1}}(s, B_s - B_{t_1}), \quad s \in [t_1, T].
\]
Exactly as in the first case, we can prove that
\[ \xi = \mathbb{E}G[\xi|\Omega_{t_1}] + \int_{t_1}^{T} z_s dB_s + \int_{t_1}^{T} \eta_s d\langle B\rangle_s - \int_{t_1}^{T} 2G(\eta_s)ds \] (11)
and, moreover \( \mathbb{E}G[X|\Omega_{t_1}] = V^{B_{t_1}}(t_1, 0) \). It is easy to check that \( V^y(t, 0) \) is a bounded and Lipschitz function of \( y \in \mathbb{R} \). We then can further solve, backwardly,
\[ \partial_t \bar{V} + G(\bar{V}_{xx}) = 0, \quad t \in [0, t_1], \quad V(t, x) = V_x(t, 0). \]

Setting
\[ \eta^\delta_s = 1_{[0,t_1-\delta]}(s) \frac{1}{2} \bar{V}_{xx}(s, B_s), \quad z^\delta_s = 1_{[0,t_1-\delta]}(s) \bar{V}_x(s, B_s), \quad s \in [0, t), \]
and
\[ \eta_s = 1_{[0,t_1]}(s) \frac{1}{2} \bar{V}_{xx}(s, B_s), \quad z_s = 1_{[0,t_1]}(s) \bar{V}_x(s, B_s), \quad s \in [0, t_1), \]
and then using again the same approach, we arrive at
\[ \mathbb{E}G[\xi|\Omega_{t_1}] = \mathbb{E}G[\xi] + \int_{t_1}^{T} z_s dB_s + \int_{t_1}^{T} \eta_s d\langle B\rangle_s - \int_{t_1}^{T} 2G(\eta_s)ds. \] (12)
This with (11) yields that
\[ \xi = \mathbb{E}G[\xi] + \int_{0}^{T} z_s dB_s + \int_{0}^{T} \eta_s d\langle B\rangle_s - \int_{0}^{T} 2G(\eta_s)ds. \]

We can use exactly the same approach to find, for \( X = \varphi(B_{t_1}, B_{t_2} - B_{t_1}, \ldots, B_{t_n} - B_{t_{n-1}}) \), for any given bounded and Lipschitz function \( \varphi \) defined on \( \mathbb{R}^n \) and for any \( 0 \leq t_1 < t_2 < \cdots < t_n \leq T \). The proof is complete. □

## 4 Uniqueness of representation theorem

**Lemma 16** We assume \( \xi \in L^2_G(\Omega_T) \), with \( G = G \sigma \) where
\[ \sigma_\varepsilon(a) = \frac{1}{2}[(a^2 - \varepsilon)a^+ - \sigma_a^2 a^-], \quad 0 < \sigma_a^2 \leq (\sigma^2 - \varepsilon) \]
\[ G_\varepsilon(a) = \frac{1}{2}[\sigma_x^2 a^+ - (\sigma^2 + \varepsilon)a^-], \quad (\sigma^2 + \varepsilon) \leq \sigma_x^2. \]

Then there exists at most one \((z, \eta) \in \mathcal{M}^2_G(0, T) \times \mathcal{M}^1_G(0, T)\) satisfying
\[ \xi = \mathbb{E}G[\xi] + \int_{0}^{T} z_s dB_s + \int_{0}^{T} \eta_s d\langle B\rangle_s - \int_{0}^{T} 2G(\eta_s)ds. \]
Proof. Let \((z^i, \eta^i) \in M^2_G(0, T) \times M^1_G(0, T), i = 1, 2\) satisfy the above representation. Then
\[
\int_0^t \dot{z}_s d B_s + \int_0^t \dot{\eta}_s d \langle B \rangle_s - \int_0^t 2[G(\eta^1_s) - G(\eta^2_s)] ds \equiv 0,
\]
where we denote \(\dot{z} = z^1 - z^2, \dot{\eta} = \eta^1 - \eta^2\). For each bounded process \(\mu \in M^2_G(0, T)\), we have
\[
\int_0^T \mu_s \dot{z}_s d B_s + \int_0^T \mu_s \dot{\eta}_s d \langle B \rangle_s - \int_0^T 2 \mu_s [G(\eta^1_s) - G(\eta^2_s)] ds \equiv 0.
\]
Fix \(\delta > 0\), we put
\[
\mu_s = \frac{\dot{\eta}_s}{\delta + |\dot{\eta}_s|}.
\]
Since \((a - b)(G(a) - G(b)) = |a - b| \cdot |G(a) - G(b)|\),
\[
I_\varepsilon = \int_0^T \mu_s \dot{z}_s d B_s + \int_0^T \frac{|\dot{\eta}_s|^2}{\delta + |\dot{\eta}_s|} d \langle B \rangle_s - \int_0^T 2 \varepsilon_G(\frac{|\dot{\eta}_s|^2}{\delta + |\dot{\eta}_s|}) ds
\]
\[
= \int_0^T \frac{2[G(\eta^1_s) - G(\eta^2_s)] \cdot |\dot{\eta}_s|}{\delta + |\dot{\eta}_s|} ds - \int_0^T 2 \varepsilon_G(\frac{|\dot{\eta}_s|^2}{\delta + |\dot{\eta}_s|}) ds
\]
\[
\geq \int_0^T \frac{2 \varepsilon |\dot{\eta}_s|^2}{\delta + |\dot{\eta}_s|} ds - (2 \varepsilon - \varepsilon) \frac{|\dot{\eta}_s|^2}{\delta + |\dot{\eta}_s|} ds
\]
\[
= \varepsilon \int_0^T \frac{|\dot{\eta}_s|^2}{\delta + |\dot{\eta}_s|} ds.
\]
Since \(E_{\varepsilon} I_\varepsilon = 0\), we then have
\[
E_{\varepsilon} \left\{ \int_0^T \frac{|\dot{\eta}_s|^2}{\delta + |\dot{\eta}_s|} ds \right\} = 0.
\]
Sending \(\delta\) to 0 we deduce that \(E_{\varepsilon} \left\{ \int_0^T |\dot{\eta}_s| ds \right\} = 0\). We thus have \(\dot{\eta}_s \equiv 0\) in \(M^1_G(0, T)\). Thus
\[
E_{\varepsilon} \left\{ \int_0^T |z^1_s - z^2_s|^2 ds \right\} \leq \frac{1}{\alpha^2} E_{\varepsilon} \left\{ \left( \int_0^T (z^1_s - z^2_s) d B_s \right)^2 \right\} = 0.
\]
From which we have \(z^1 = z^2\).  

Remark 17 **We can also take**

\[
\mu_s = \frac{-\dot{\eta}_s}{\delta + |\dot{\eta}_s|}
\]

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In this case we have

\[
I = \int_0^T \mu_s \hat{z}_s dB_s + \int_0^T |\hat{\eta}_s|^2 \delta \, d\langle B \rangle_s - \int_0^T 2G^2(\frac{|\hat{\eta}_s|^2}{\delta + |\eta_s|}) ds \\
= \int_0^T \left[ -2(G(\eta^1_s) - G(\eta^2_s)) \cdot |\hat{\eta}_s| ds - 2G^2(\frac{|\hat{\eta}_s|^2}{\delta + |\eta_s|}) \right] ds \\
\geq \int_0^T \left[ \frac{\sigma^2 |\hat{\eta}_s|^2}{\delta + |\eta_s|} + (\sigma^2 + \varepsilon) \frac{|\hat{\eta}_s|^2}{\delta + |\eta_s|} \right] ds \\
= \varepsilon \int_0^T \frac{|\hat{\eta}_s|^2}{\delta + |\eta_s|} ds.
\]

Since \( E_G[I] = 0 \), we then have

\[
\varepsilon E_G[\int_0^T |\hat{\eta}_s|^2 \delta + |\eta_s| ds] = 0.
\]

Sending \( \delta \) to 0 we deduce that \( \varepsilon E_G[\int_0^T |\hat{\eta}_s| ds] = 0 \).

5 Existence of the representation

We will use Theorem 13. For each \( \xi \in L^2_G(\Omega_T) \), there exists a unique decomposition

\[
\xi = E_G[\xi] + \int_0^T z_s dB_s - A_T^\xi
\]

with \( A_0^\xi = 0 \), \( A_t^\xi - A_s^\xi \geq 0 \) and \( E_G[-A_T^\xi] = -A_s^\xi \), for \( s \leq t \).

Definition 18 We define the following distance in \( L^2_G(\Omega_T) \): given \( \alpha \in (1, 2) \),

\[
\rho(\xi_1, \xi_2) = E_G[|\xi_1 - \xi_2|^2]^{1/2} + E_G[\sup_{\pi_N[0,T]} \left| \sum_{i=0}^N (A_{i+1}^\xi - A_i^\xi) - (A_{i+1}^{\xi_2} - A_i^{\xi_2}) \right|^{\alpha}]^{1/\alpha} \\
< \infty, \ \xi_1, \xi_2 \in L^2_G(\Omega_T).
\]

where \( \pi_N[0,T] \) is the collection of all (deterministic) finite partitions of \( [0,T] \).

Since \( (L_{ip}(\Omega_T), \rho) \) forms a metric space, we denote the completion of this space under \( \rho \) by \( L^*_{ip}(\Omega_T) \). For each \( \xi \in L^*_{ip}(\Omega_T) \), there exists a sequence \( \{\xi_n\}_{n=1}^\infty \) in \( L_{ip}(\Omega_T) \) such \( \lim_{n \to \infty} \rho(\xi, \xi_n) = 0 \). We have

\[
\xi = E_G[\xi] + \int_0^T z_s dB_s - A_T^\xi.
\]

The following Lemma is easy.
Lemma 19 For each \( \xi \in L^2_{\text{loc}}(\Omega_T) \) and \( \mu \in M^1_{\text{loc}}(0,T) \) such that \( |\mu| \leq c \), we have
\[
\sup_{\mu \in M^1_{\text{loc}}(0,T), |\mu| \leq c} \mathbb{E}(\int_0^T \mu_s dA^\xi_t - \int_0^T \mu_s dA^\xi_t) \leq c \rho(\xi, \xi').
\]

Theorem 20 For each \( \xi \in L^{2,\alpha}_T(\Omega) \) and for each \( \alpha \in (1,2) \), there exists a \( z^\xi \in H^\alpha_{\text{loc}}(0,T) \) and \( \eta^\xi \in M^1_{\text{loc}}(0,T) \) such that
\[
\xi = \mathbb{E}[\xi] + \int_0^T z^\xi_s dB_s + \int_0^T \eta^\xi_s d\langle B \rangle_s - \int_0^T \int 2G(\eta^\xi_s)ds.
\]

Proof. Let \( \{\xi_n\}_{n=1}^\infty \) be a Cauchy sequence in the metric space \( (L^{2,\alpha}_T(\Omega_T), \rho) \) such that \( \rho(\xi_n, \xi) \to 0 \) as \( n \to \infty \). We have the following unique representation:
\[
\xi_n = \mathbb{E}[\xi_n] + \int_0^T z^{\xi_n}_s dB_s + \int_0^T \eta^{\xi_n}_s d\langle B \rangle_s - \int_0^T \int 2G(\eta^{\xi_n}_s)ds,
\]
with \( (z^{\xi_n}, \eta^{\xi_n}) \in H^\alpha_{\text{loc}}(0,T) \times M^1_{\text{loc}}(0,T), i = 1, 2 \) be have the above representation. Then
\[
\mathbb{E}[\xi_m | \Omega_T] - \mathbb{E}[\xi_n | \Omega_T] = \mathbb{E}[\xi_m] - \mathbb{E}[\xi_n] + \int \hat{z}^{m,n}_s dB_s + \int \hat{\eta}^{m,n}_s d\langle B \rangle_s - \int 2G(\hat{\eta}^{m,n}_s)ds,
\]
where we denote \( \hat{z}^{m,n} = z^m - z^n, \hat{\eta} = \eta^m - \eta^n \). Fix \( \delta > 0 \), we set
\[
\mu^{m,n}_s = \frac{\hat{\eta}^{m,n}_s}{\delta + |\hat{\eta}^{m,n}_s|}.
\]
Since \( (a - b)(G(a) - G(b)) = |a - b| \cdot |G(a) - G(b)| \),
\[
I^m_n = \int_0^T \frac{\hat{\eta}^{m,n}_s}{\delta + |\hat{\eta}^{m,n}_s|} d\langle B \rangle_s - \int_0^T 2G_s(\frac{\hat{\eta}^{m,n}_s}{\delta + |\hat{\eta}^{m,n}_s|})ds
\]
\[
= \int_0^T \frac{\hat{\eta}^{m,n}_s}{\delta + |\hat{\eta}^{m,n}_s|} d\langle B \rangle_s - \int_0^T 2G_s(\frac{\hat{\eta}^{m,n}_s}{\delta + |\hat{\eta}^{m,n}_s|})ds - \int_0^T \mu^{m,n}_s d(A^\xi_t - dA^\xi_t)
\]
\[
> \int_0^T \frac{\varepsilon |\hat{\eta}^{m,n}_s|^2}{\delta + |\hat{\eta}^{m,n}_s|} ds - \int_0^T \mu^{m,n}_s d(A^\xi_t - dA^\xi_t)
\]
\[
= \int_0^T \varepsilon |\hat{\eta}^{m,n}_s|^2 ds - \int_0^T \mu^{m,n}_s d(A^\xi_t - dA^\xi_t)
\]
Since \( \mathbb{E}[I^m_n] = 0 \), we then have
\[
0 \geq \mathbb{E}[\int_0^T \frac{\varepsilon |\hat{\eta}^{m,n}_s|^2}{\delta + |\hat{\eta}^{m,n}_s|} ds - \int_0^T \mu^{m,n}_s d(A^\xi_t - dA^\xi_t)]
\]
\[
\geq \mathbb{E}[\int_0^T \frac{\varepsilon |\hat{\eta}^{m,n}_s|^2}{\delta + |\hat{\eta}^{m,n}_s|} ds - \mathbb{E} \int_0^T \mu^{m,n}_s d(A^\xi_t - dA^\xi_t)].
\]
Thus
\[
\mathbb{E}_{\mathcal{G}_t}\left[\int_0^T \frac{e^{m,n}}{\delta + |\tilde{\eta}_{m,n}|^2} ds \right] \leq \mathbb{E}_{\mathcal{G}_t}\left[\int_0^T \rho_s m,n d(A_{t}^{x_m} - dA_{t}^{x_n})\right] \\
\leq \rho(\xi_m, \xi_n).
\]

It then follows that \( \{\eta^n\}_{n=1}^\infty \) is a Cauchy sequence in \( M_{\mathcal{G}_t}^2(0, T) \). We then can pass limit on the both sides of (14) to obtain (13). By using the same argument as Song [12] Theorem 4.5, we can also prove that \( \{z^n\}_{n=1}^\infty \) is a Cauchy sequence in \( H_{\alpha, \mathcal{G}_t}^2(0, T) \). The proof is complete. ■

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