Positivity of LCK potential

Liviu Ornea\textsuperscript{1} and Misha Verbitsky\textsuperscript{2}

Abstract
Let $M$ be a complex manifold and $L$ an oriented real line bundle on $M$ equipped with a flat connection. An LCK ("locally conformally Kähler") form is a closed, positive (1,1)-form taking values in $L$, and an LCK manifold is one which admits an LCK form. Locally, any LCK form is expressed as an $L$-valued pluri-Laplacian of a function called LCK potential. We consider a manifold $M$ with an LCK form admitting an LCK potential (globally on $M$), and prove that $M$ admits a positive LCK potential. Then $M$ admits a holomorphic embedding to a Hopf manifold, as shown in [OV1].

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1 Introduction

There are several equivalent ways to define locally conformally Kähler (LCK) manifolds, but the most appropriate for the present paper is the following. Let \( M \) be a complex manifold and \( L \) an oriented real line bundle on \( M \) equipped with a flat connection. An LCK form is a closed, positive \((1,1)\)-form taking values in \( L \), and an LCK manifold is one which admits an LCK form. For a more explicit and detailed exposition, see Subsection 2.1.

Locally, a Kähler form is always equal to \( d d^c \varphi \) for some function \( \varphi \) which is called Kähler potential. This statement has an analogue for LCK manifolds. Denote by \( d_\theta \) the de Rham differential on \( L \)-valued differential forms (Remark 2.6) and let \( d_\theta^c := I d_\theta I^{-1} \) be its complex conjugate. Then locally any LCK form is expressed as \( d_\theta d_\theta^c \varphi \) where \( \varphi \) is called LCK potential.

On a compact complex manifold, any plurisubharmonic function is constant, and this means that such a manifold cannot have a global Kähler potential. However, an LCK manifold might have a global LCK potential – this was first observed in [Ve] and [OV1] and much used since then ([G], [OV2], [OV3], [OV4], [O] and so on). In the first papers on this subject, the LCK potential was assumed to be positive, but then we realized that the existence of a potential is a cohomological notion, see Remark 2.10, and the focus was shifted on the study of the corresponding cohomology group.

LCK manifolds with potential enjoy several properties which make this notion quite useful. First, unlike the LCK manifolds (and like the Kähler manifolds) the class of LCK manifolds with potential is “deformationally stable”: a small deformation of an LCK manifold with potential is again LCK with potential. Also, any LCK manifold \( M \) with potential, \( \dim \mathbb{C} M > 2 \), can be holomorphically embedded in a Hopf manifold \( \mathbb{C}^n \backslash \{0\}/\mathbb{Z} \), and, conversely, any complex submanifold of a Hopf manifold is LCK with potential.

To reconcile the cohomological and geometrical approach, we need to prove that any LCK manifold with the LCK-form \( \omega \) in the image of \( d_\theta d_\theta^c \varphi \) admits another LCK potential which is positive, and this is the aim of the present paper.

The obvious solution which one would use in Kähler case (adding a constant) does not work, because the operator \( d_\theta d_\theta^c \varphi \) does not vanish on constants. However, we were able to find a positive function \( h \) such that \( d_\theta d_\theta^c(h) \) is non-negative; adding \( C d_\theta d_\theta^c(h) \) to \( \omega \), \( C \gg 0 \), gives us an LCK form with positive potential in the same cohomology class.

For a while we tried to prove that any LCK potential for compact LCK manifolds is positive, but this claim was wrong; we are grateful to Victor Vuletescu who disbusted us of this fallacy.

Vuletescu’s example is the following. Take a Hopf manifold \( \mathbb{C}^n \backslash \{0\}/\mathbb{Z} \) where \( \mathbb{Z} \) acts by multiplication by \( \lambda > 1 \) and let \( L \) be the local system with the same
monodromy. Then the usual flat Kähler form on $\mathbb{C}^n$ can be considered as a closed Hermitian form with values in the bundle $L^2$. Its LCK potential is a function $m(z) := |z|^2$. Any quadratic polynomial on $\mathbb{C}^n$ gives a holomorphic section of $L^2 \otimes \mathbb{R} \mathbb{C}$; let $\nu$ be its real part. Then $d\theta d^c\theta(\nu) = 0$, because $\nu$ is the real part of a holomorphic section of $L^2$, and $d\theta d^c\theta(m + Ah) = d\theta d^c\theta(m)$ is the LCK form on $M$, for any real constant $A$. However, for $A$ large enough, the LCK potential $m + Ah$ is non-positive.

Subsections 2.1 and 2.2 are devoted to presenting the precise definitions and results that will be further used, and to stating the main result of the paper. Section 3 recalls a classical Remmert Reduction Theorem. Section 4 extends Demailly’s technique of gluing Kähler metrics to the LCK setting. In Section 5 we begin the proof of our main result, showing that an LCK potential cannot be strictly negative, while in Section 6 we prove that an LCK potential $\varphi$ which is positive somewhere can be glued with another one which is positive on the set where $\varphi < 0$ to obtain an LCK potential which is strictly positive.

2 Introduction to LCK geometry

2.1 LCK manifolds

Definition 2.1: A complex manifold $(M, I)$ of complex dimension at least 2, is called locally conformally Kähler (LCK) if it admits a Hermitian metric $g$ and a closed 1-form $\theta$, called the Lee form, such that the fundamental 2-form $\omega(\cdot, \cdot) := g(\cdot, I\cdot)$ satisfies the integrability condition

$$d\omega = \theta \wedge \omega, \quad d\theta = 0. \quad (2.1)$$

Remark 2.2: As shown by Vaisman (see [DO, Theorem 2.1]), a compact locally conformally Kähler manifold with $\theta$ non-exact cannot admit any Kähler form. On the other hand, an LCK form can be made into a Kähler one whenever $\theta$ is exact; indeed, if $\theta = df$, the form $e^{-f}\omega$ is Kähler. Such a manifold is sometimes called “globally conformally Kähler”. Throughout this paper, we shall tacitly exclude this case and assume that $\theta$ is non-exact for any closed LCK manifold we consider.

An equivalent definition is given as follows (see e.g. [DO]):

Proposition 2.3: A complex manifold $(M, I)$, of complex dimension at least 2, is LCK if and only if it admits a covering $\pi : \tilde{M} \to M$ endowed with a Kähler form $\tilde{\omega}$ with the deck group $\text{Aut}_M(\tilde{M})$ acting on $(\tilde{M}, \tilde{\omega})$ by holomorphic homotheties.
Hence, if $\tau \in \text{Aut}_M(\tilde{M})$, then $\tau^* \tilde{\omega} = c_\tau \cdot \tilde{\omega}$, where $c_\tau \in \mathbb{R}^{>0}$ is the scale factor. This defines a character

$$\chi : \text{Aut}_M(\tilde{M}) \rightarrow \mathbb{R}^{>0}, \quad \chi(\tau) = c_\tau. \quad (2.2)$$

**Definition 2.4:** Differential forms $\tilde{\eta}$ on $\tilde{M}$ which satisfy $\gamma^* \eta = c_\gamma \tilde{\eta}$ for any deck transform map $\gamma \in \text{Aut}_M(\tilde{M})$, where $c_\gamma := \chi(\gamma) \neq 1$, are called *automorphic*. In particular, $\tilde{\omega}$ is automorphic.

We shall denote by $L$ the flat line bundle on $M$ associated to this character (it is precisely the bundle of densities of weight 1 from conformal geometry). We fix a trivialization $\nu$ of $L$ such that $\theta$ is a connection form in $L$. The complexification of $L$, considered as a holomorphic line bundle, will be denoted with the same letter (it will be clear from the context which one we refer to). This holomorphic line bundle is equipped with a natural Hermitian metric which is constant on $\nu$. We shall call $L$ simply the *weight bundle*, which is a standard term in conformal geometry.

**Definition 2.5:** The rank of the abelian subgroup $\text{im}(\chi)$ in $\mathbb{R}^{>0}$ is called the *LCK rank* of $M$. It equals the rank of the monodromy group of the bundle $(L, \theta)$.

**Remark 2.6:** The cohomology $H^*(M, L)$ of the local system $L$ is isomorphic with the cohomology of the complex $(\Omega^*(M), d_\theta)$, where $d_\theta := d - \theta \wedge$, and is called *Morse Novikov cohomology*.

**Example 2.7:** The known examples of LCK manifolds include: Hopf manifolds (e.g. [OV7]), Oeljeklaus-Toma manifolds with $t = 1$ ([OT]), almost all compact complex surfaces ([GO, Be, Bru]). Such examples fully justify the interest in LCK geometry. At the moment, there exists only one example of non-Kähler compact surfaces which do not admit LCK metric, which is one of the three Inoue surfaces. If the famous spherical shell conjecture is true, the rest of non-Kähler surfaces are LCK ([OV9]).

### 2.2 LCK manifolds with potential

In [OV1] we introduced the following subclass of LCK manifolds, which, as we proved, share essential features of Kähler manifolds: stability at small deformations and the Kodaira-type embedding theorem, providing a holomorphic embedding into a Hopf manifold.

**Definition 2.8:** An LCK manifold with potential is a manifold which admits a Kähler covering $(\tilde{M}, \tilde{\omega})$ and a smooth function $\tilde{\varphi} : \tilde{M} \rightarrow \mathbb{R}^{>0}$ (the LCK potential)
satisfying the following conditions:

(i) $\tilde{\varphi}$ is proper, i.e. its level sets are compact;

(ii) $\tilde{\varphi}$ is automorphic, i.e. $\tau^*(\varphi) = \chi(\tau)\varphi$, for all $\tau \in \text{Aut}_M(\tilde{M})$.

(iii) $\tilde{\varphi}$ is a Kähler potential, i.e. $dd^c\tilde{\varphi} = \tilde{\omega}$.

For the geometric interpretation of these conditions, see Section 1.

**Example 2.9: Vaisman manifolds** are LCK manifolds $(M, g, \theta)$ with parallel Lee form (with respect to the Levi Civita connection of the LCK metric). The $\pi^*g$-squared norm of $\pi^*\theta$ is a positive, automorphic potential on the universal cover $\pi: \tilde{M} \to M$ ([Ve]). In particular, diagonal Hopf manifolds are LCK with potential and, by stability at small deformations, non-diagonal Hopf manifolds (which are non-Vaisman) are LCK with potential too, [OV1].

On the other hand, Inoue surfaces ([O]), and the LCK Oeljeklaus-Toma manifolds ([OV8]) are not LCK with potential, and hence this subclass is strict.

**Remark 2.10**: The existence of a LCK potential immediately implies the vanishing of the class $[\omega]$ in the Bott-Chern cohomology of $M$ with values in $L$, see [OV2].

The meaning of the properness condition (i) in the definition is explained by the following equivalence:

**Proposition 2.11**: ([OV6]) Let $M$ be a compact manifold, $\tilde{M}$ a covering, and $\tilde{\varphi}: \tilde{M} \to \mathbb{R}^0$ an automorphic function. Then $\tilde{\varphi}$ is proper if and only if the deck transform group $\text{Aut}_M(\tilde{M})$ of $\tilde{M}$ is virtually cyclic (i.e. it contains $\mathbb{Z}$ as a finite index subgroup).

In particular, a compact LCK manifold with potential has LCK rank 1 if and only if the automorphic potential is proper.

**Remark 2.12**: Examining the proof, one can see that it equally works for $\varphi < 0$: what is important is that $\varphi$ does not pass through zero.

**Definition 2.8** can be reformulated to avoid the Kähler cover.

**Proposition 2.13**: ([OV6]) Let $(M, \omega, \theta)$ be an LCK manifold. Then $M$ is LCK with potential if and only if there exists a positive function $\varphi \in C^\infty(M)$ satisfying

$$dd^c\varphi = \omega,$$  \hspace{1cm} (2.3)
where $d^c = I d_I I^{-1}$.

Explicitly, if $\pi^* \theta = d \varphi$ on $\bar{M}$, then the Kähler potential on $M$ is given by $e^{-f} \cdot \pi^* \varphi$.

**Remark 2.14:** Note that the LCK potential $\varphi$ above is defined on the manifold $M$, which is often compact. It is not a Kähler potential in the usual sense.

Automorphic potentials can be approximated by proper ones, and hence the properness condition in the definition is not essential as long as one is only interested in complex and differential properties (and not in metric ones).

** Proposition 2.15:** ([OV6]) Let $(M, \omega, \theta)$ be an LCK manifold, and $\varphi \in C^\infty(M)$ a function satisfying $d^c d^\omega (\varphi) = \omega$. Then $M$ admits an LCK structure $(\omega', \theta')$ of LCK rank 1, approximating $(\omega, \theta)$ in $C^\infty$-topology.

This paper is instead concerned with the positivity of the potential. The main result of this paper is the following theorem.

**Theorem 2.16:** Let $M$ be an LCK manifold with a Kähler covering admitting an automorphic Kähler potential. Then $M$ also admits an LCK metric with a positive automorphic potential.

This theorem, which is proven in Section 6 has the following useful corollary ([OV1]).

**Theorem 2.17:** Let $M$ be an LCK manifold manifold with a Kähler covering admitting an automorphic Kähler potential. Then $M$ admits a holomorphic embedding to a Hopf manifold.

The main tool in the proof will be the gluing of LCK metrics (see Section 4) which is based on Demailly’s regularized maximum of two functions.

**Remark 2.18:** Theorem 2.16 fills a gap in the proofs of the following previous results of ours:

- [OV2] Theorem 1.4, where we claimed the existence of an LCK potential and, in fact, we only proved the vanishing of $[\bar{\omega}]$ in the Bott-Chern group $H^{1,1}_{BC}(M, L)$, i.e. the existence of an automorphic potential which was not necessarily positive;

- [OV4] Theorem 2.3, where this result was used to embed an LCK manifold admitting a holomorphic circle action which is not conformal to isometry.
to a Hopf manifold. It is proven by taking an average of the LCK form with respect to the circle action and noticing that its Bott-Chern class vanishes.

Now the results are true as stated.

3 Remmert Reduction Theorem

For the sake of completion, we quote here, without proof, a classical result we shall use further on.

**Theorem 3.1:** (Remmert reduction, [Re])

Let $X$ be a holomorphically convex space. Then there exist a Stein space $Y$ and a proper, surjective, holomorphic map $f : X \to Y$ such that

(i) $f_* O_X = O_Y$.

Moreover, the fact that $Y$ is Stein and (i) imply:

(ii) $f$ has connected fibers.

(iii) The map $f^* : O_Y(Y) \to O_X(X)$ is an isomorphism.

(iv) The pair $(f, Y)$ is unique up to biholomorphism, i.e. for any other pair $(f_0, Y_0)$ with $Y_0$ Stein and property 1., there exists a biholomorphism $g : Y \to Y_0$ such that $f_0 = g \circ f$.

4 Gluing Kähler forms and LCK forms

4.1 Regularized maximum of $\partial_0 \partial_0^c$-plurisubharmonic functions

In [D1], the notion of a *regularized maximum* of two functions was defined as follows.

**Definition 4.1:** ([D1]) Choose $\varepsilon > 0$, and let $\max_\varepsilon : \mathbb{R}^2 \to \mathbb{R}$ be a smooth, convex function, monotonous in both variables, which satisfies $\max_\varepsilon(x, y) = \max(x, y)$ whenever $|x - y| > \varepsilon$. Then $\max_\varepsilon$ is called a *regularized maximum*.

**Theorem 4.2:** ([D1]) The regularized maximum of two plurisubharmonic functions is again plurisubharmonic.

**Claim 4.3:** Let $\theta$ be a closed form on a complex manifold, and $\varphi, \psi$ two $\partial_0 \partial_0^c$-plurisubharmonic functions. Then $\max_\varepsilon(\varphi, \psi)$ is also $\partial_0 \partial_0^c$-plurisubharmonic; it is strictly $\partial_0 \partial_0^c$-plurisubharmonic if $\varphi$ and $\psi$ are strictly $\partial_0 \partial_0^c$-plurisubharmonic.
Proof: Since this result is local, we may always assume that $\theta = d\rho$ for some positive function $\rho$. Then $d_\theta(\eta) = e^{-\rho} d(e^\rho \eta)$ and
\[
d_\theta d_\theta^c(\max(\varphi, \psi)) = e^{-\rho} d d^c(\max(e^\rho \varphi, e^\rho \psi)).
\]
Since $e^\rho \varphi$, $e^\rho \psi$ are plurisubharmonic, the form $d d^c(\max(e^\rho \varphi, e^\rho \psi))$ is positive.

4.2 Gluing of LCK potentials

The following procedure is well known; it was much used by J.-P. Demailly (see e.g. [DP]), and, in LCK context, in our paper [OV5]. We call it “Gluing of Kähler metrics”.

Proposition 4.4: (gluing of Kähler metrics)

Let $(M, \omega)$ be a Kähler manifold, and $D \subset M$ a submanifold of the same dimension with smooth compact boundary such that $\omega = dd^c \varphi$ in a smooth neighbourhood of $D$, with $\varphi$ a plurisubharmonic function. Let $\psi$ be another plurisubharmonic function with $\psi = \varphi$ on $\partial D$. Consider a vector field $X \in TM|_{\partial D}$, which is normal and outward-pointing everywhere in $\partial D$, and let $\text{Lie}_X$ denote the derivative of a function along $X$. Assume that $\text{Lie}_X \psi < \text{Lie}_X \varphi$ everywhere on $\partial D$. Let $D_-$ be an open subset of $D$ which does not intersect a neighbourhood $U$ of $\partial D$, and $D_+$ an open subset of $M \setminus D$ which does not intersect $U$. Then there exists a Kähler form $\omega_1$ which is equal to $\omega$ on $D_+$ and to $dd^c \psi$ on $D_-$. 

Proof: Consider the function $\max_c(\psi, \varphi)$ defined as in [Definition 4.1] where $\psi, \varphi$ are defined as above. Since $\text{Lie}_X \psi < \text{Lie}_X \varphi$, the maximum $\max(\psi, \varphi)$ is equal to $\psi$ in $D_-$ near $\partial D$ and equal to $\varphi$ in $D_+$ near $\partial D$. We choose $D_+, D_-$ sufficiently big in such a way that $\psi > \varphi$ in a neighbourhood of the boundary $\partial D_-$ and $\psi < \varphi$ in a neighbourhood of the boundary $\partial D_+$. This gives $\max(\varphi, \psi) = \varphi$ on $D_+$ and $\max(\psi, \varphi) = \psi$ on $D_-$. 

Choosing $\varepsilon$ sufficiently small, the same would hold for for the regularized maximum $\max_c(\psi, \varphi)$. Now we can extend $dd^c \max_c(\psi, \varphi)$ to $D_-$ as $dd^c \psi$ and to $D_+$ as $\omega_1$. 

Replacing $d, dd^c$ by $d_\theta$ and $d_\theta^c$ and using the regularized maximum of $d_\theta d_\theta^c$-plurisubharmonic functions as in [Claim 4.3] we obtain the following LCK-version of this result; the proof is the same after we replace $d, dd^c$ by $d_\theta$ and $d_\theta^c$ (note that below $\varphi, \psi$ denote functions on $M$, and not on its Kähler covering).
Proposition 4.5: (gluing of LCK metrics)
Let \((M, \theta, \omega)\) be an LCK manifold, and \(D \subset M\) a submanifold of the same dimension with smooth compact boundary such that \(\omega = d_\theta d_\theta^c \phi\) in a smooth neighbourhood of \(D\), with \(\phi\) a \(d_\theta d_\theta^c\)-plurisubharmonic function. Let \(\psi\) be another \(d_\theta d_\theta^c\)-plurisubharmonic function with \(\psi = \phi\) on \(\partial D\). Consider a vector field \(X \in T_M|_{\partial D}\) which is normal and outward-pointing everywhere in \(\partial D\), and let \(\text{Lie}_X\) denote the derivative of a function along \(X\). Assume that \(\text{Lie}_X \psi < \text{Lie}_X \phi\) everywhere on \(\partial D\). Let \(D_+\) be an open subset of \(D\) which does not intersect a neighbourhood \(U\) of \(\partial D\), and \(D_-\) an open subset of \(M \setminus D\) which does not intersect \(U\). Then there exists an LCK form \(\omega_1\) which is equal to \(\omega\) on \(D_+\) and to \(dd^c \psi\) on \(D_-\).

Proof: We use the same proof as for Proposition 4.4, and note that the regularized maximum of \(d_\theta d_\theta^c\)-plurisubharmonic functions is \(d_\theta d_\theta^c\)-plurisubharmonic by Claim 4.3.

Remark 4.6: Proposition 4.5 is true also if \(\psi\) and \(\phi\) are not strictly \(d_\theta d_\theta^c\)-plurisubharmonic. In this case the gluing construction works, but it gives a function which is \(d_\theta d_\theta^c\)-plurisubharmonic, but not strictly \(d_\theta d_\theta^c\)-plurisubharmonic.

5 Negative automorphic potentials for LCK metrics

Theorem 5.1: Let \((M, \theta, \omega)\) be an LCK manifold which is not Kähler, and \(\omega = d_\theta d_\theta^c(\varphi)\) for some smooth \(d_\theta d_\theta^c\)-plurisubharmonic function \(\varphi\). Then \(\varphi > 0\) at some point of \(M\).

Proof: Suppose, by absurd, that \(\varphi \leq 0\) everywhere on \(\hat{M}\). Since the LCK potential is stable under \(C^2\)-small deformations of \(\varphi\), \(\varphi - \epsilon\) is also an LCK potential. Therefore, we may assume that \(\varphi < 0\) everywhere. Define
\[
\psi := -\log(-\varphi).
\]
Since \(x \rightarrow -\log(-x)\) is strictly monotonous and convex, the function \(\psi\) is strictly plurisubharmonic. Moreover, for every \(\gamma \in \Gamma\), we have
\[
\gamma^* \psi = -\log((-\varphi \circ \gamma)) = -\log(\chi(\gamma)) - \log(-\varphi) = \text{const} + \psi.
\]
Therefore, the Kähler form \(dd^c \psi\) is \(\Gamma\)-invariant and descends to \(M\).
6 LCK potentials on Stein manifolds

6.1 Submanifolds with strictly pseudoconvex boundary and positivity of LCK potentials

Let \((M, \theta, \omega)\) be an LCK manifold, with \(\omega = d_\theta d_\theta^c(\varphi)\) for some smooth \(d_\theta d_\theta^c\)-plurisubharmonic function \(\varphi\). The condition \(\omega = d_\theta d_\theta^c(\varphi) > 0\) is open in \(C^2\)-topology on the set of all functions \(\varphi\) on \(M\). Adding a \(C^2\)-small function to \(\varphi\) if necessary, we may assume that 0 is a regular value of \(\varphi\). The pullback of \(\varphi^{-1}(0)\) to \(\tilde{M}\) is the set of zeros of a Kähler potential. Therefore, it is strictly pseudoconvex, and the same is true about \(\varphi^{-1}(c)\) for small \(c\), these sets are also strictly pseudoconvex.

Choose a regular value \(c > 0\) of \(\varphi\) such that \(\varphi^{-1}(c)\) is non-empty and pseudoconvex. Then \(\varphi^{-1}(c)\) is a strictly pseudoconvex CR-submanifold in \(M\), and \(D := \varphi^{-1}(]-\infty, c[)\) is a strictly pseudoconvex set with boundary. Note that the interior of \(D\) is an open submanifold in \(M\), and hence it is LCK.

Then our main result (Theorem 2.16) follows from the gluing theorem (Proposition 4.5) and the following result about LCK manifolds with pseudoconvex boundary.

**Theorem 6.1:** Let \((M, \theta, \omega)\) be a compact LCK manifold of LCK rank 1 with smooth boundary which is strictly pseudoconvex. Assume that \(\omega = d_\theta d_\theta^c(\varphi)\) for some smooth \(d_\theta d_\theta^c\)-plurisubharmonic function \(\varphi\). Then \(M\) admits a positive \(d_\theta d_\theta^c\)-plurisubharmonic function \(\psi\) such that \(\psi\) is constant on the boundary \(\partial M\).

We prove Theorem 6.1 later in this section. Let us deduce Theorem 2.16 from Theorem 6.1 and gluing.

**Theorem 6.2:** Let \((M, \omega, \theta)\) be an LCK manifold manifold with a Kähler covering admitting an automorphic Kähler potential \(\tilde{\varphi}\). Then \(M\) also admits an LCK metric with a positive automorphic potential and the same Bott-Chern class of the fundamental form.

**Proof:** Let \(\varphi\) be the corresponding potential on \(M\), \(d_\theta d_\theta^c\varphi = \omega\). Then \(\varphi > 0\) somewhere on \(M\) (Theorem 5.1). As above, choose a regular value \(c > 0\) of \(\varphi\) such that \(\varphi^{-1}(c)\) is non-empty, and let \(D := \varphi^{-1}(]-\infty, c[)\) be the corresponding strictly pseudoconvex set with boundary. By Theorem 6.1 \(D\) admits a positive \(d_\theta d_\theta^c\)-plurisubharmonic function \(\psi\) such that \(\psi\) is constant on the boundary \(\partial M\). Choosing \(\varepsilon > 0\) sufficiently small, and modifying \(\psi\) by adding a \(C^2\)-small function for transversality, we may assume that the set \(S := \{m \in M \mid \varepsilon \psi(m) = \varphi(m) > 0\}\) is smooth and compact in \(\varphi^{-1}(]0, c[)\), and \(\text{Lie}_X \varphi > \varepsilon \text{Lie}_X \varphi\) on \(S\) as...
Then we may glue $\varphi$ and $\varepsilon \psi$ (Proposition 4.5, Remark 4.6). We obtain an everywhere positive $d_\theta d_\theta^c$-plurisubharmonic function $\varphi_1$. Adding $\delta \varphi$, for $\delta > 0$ sufficiently small, we make sure that $\varphi_1 + \delta \varphi$ is everywhere positive and strictly $d_\theta d_\theta^c$-plurisubharmonic.

### 6.2 LCK potentials on submanifolds with strictly pseudoconvex boundary

To finish the proof of the main theorem, it remains to construct positive LCK potentials on LCK manifolds with pseudoconvex boundary (Theorem 6.1).

**Theorem 6.3**: Let $(D, \theta, \omega)$ be a compact LCK manifold of LCK rank 1 with smooth boundary which is strictly pseudoconvex. Assume that $\omega = d_\theta d_\theta^c(\varphi)$ for some smooth $d_\theta d_\theta^c$-plurisubharmonic function $\varphi$, such that for some $\varepsilon > 0$ the function $\varphi - \varepsilon$ is also $d_\theta d_\theta^c$-plurisubharmonic, vanishes on the boundary of $D$, and is strictly negative on $D$. Then $D$ admits a positive $d_\theta d_\theta^c$-plurisubharmonic function $\psi$ which is constant on the boundary $\partial D$.

**Proof**: Note that a manifold $D$ with smooth, strictly pseudoconvex boundary is holomorphically convex. Then the Remmert reduction (Theorem 3.1) implies that $D$ admits a proper, surjective and holomorphic map $\pi : D \rightarrow D_0$ with connected fibres to a Stein variety $D_0$ with isolated singularities.

Let $\Phi$ be a negative Kähler potential on the covering $\tilde{D}$ of $D$, obtained from $\varphi - \varepsilon$. Then the function $\Psi := -\log(-\Phi)$ is strictly plurisubharmonic on $\tilde{D}$ (see Theorem 5.1), hence the 1-form $\Theta := d^c \Psi$ is defined on $D$. On the other hand, $d\Theta$ is a Kähler form. This implies that $D$ has no compact subvarieties (without boundary), and the map $\pi : D \rightarrow D_0$ is bijective. This implies that $D$ is Stein.

Now, let $L$ be the weight bundle on $D$, associated to the character (2.2). Denote by $\tilde{D}$ the smallest Kähler covering of $D$. Then $\tilde{D}$ is a $\mathbb{Z}$-covering of $D$, and $L$ is trivial on $\tilde{D}$. We call a function $f$ on $\tilde{D}$ *automorphic* if for each $\gamma \in \pi_1(D)$, we have $\gamma^* (f) = \chi(\gamma)^A f$.

Clearly, 1-automorphic holomorphic functions on $\tilde{D}$ correspond to holomorphic sections of $L$. Since $D$ is Stein, the space of sections of a holomorphic bundle is globally generated; this assures the existence of sufficiently many holomorphic sections of $L$. Then, for a sufficiently big collection $f_i$ of sections of $L$, the sum $\sum |f_i|$ is positive everywhere on $D$. This gives an 1-automorphic plurisubharmonic function on $\tilde{D}$.

We have proven that $D$ admits a positive LCK potential. To obtain a potential which is constant on $\partial D$, we perform the following trick.

Let $\{f_1, \ldots, f_n\}$ be the set of holomorphic sections of $L$ without common zeros on $\partial D$. Such a set exists and is nonempty because the pushforward of $L$ to a
Stein variety $D_0$ is globally generated. The following claim finishes the proof of Theorem 6.3 because $|f_i| \cdot |b|$ is the length of the section $f_i b$ of $L$ for any holomorphic function $b$ on $D$.

**Claim 6.4:** Let $\{a_1, \ldots, a_n\}$ be a collection of non-negative functions on a complex manifold $D$ with a smooth strictly pseudoconvex boundary. Assume that $a_i$ have no common zeroes on the boundary. Then for any positive function $A$ on the boundary there exists a collection $\{b_1, \ldots, b_n\}$ of positive functions such that $A = \sum a_i b_i$ and each $b_i$ can be obtained as the limit of a sum of absolute values of holomorphic functions.

**Proof. Step 1:** By a theorem of Bremmermann ([Bre1, Theorem 2]), every positive plurisubharmonic function on a pseudoconvex manifold is Hartogs, that is, it belongs to the closure of the cone generated by absolute values of holomorphic functions. Therefore, it would suffice to find a sum $A = \sum a_i b_i$ with $b_i$ positive, continuous and plurisubharmonic.

**Step 2:** By another theorem of Bremmermann ([Bre2, Theorem 7.2]), any function on the boundary of a bounded holomorphically convex domain can be extended inside to a plurisubharmonic function $b$. Applying this result to $\log A$ and then taking the exponential, we can make sure that $b$ is positive. To prove Claim 6.4 it remains to find a collection of positive continuous functions $\{b_1, \ldots, b_n\}$ on the boundary of $D$ such that $A = \sum a_i b_i$.

**Step 3:** Since $a_i$ are non-negative and have no common zeros, their sum $B$ is positive. Then $\sum_i AB^{-1} a_i = AB^{-1} \sum_i a_i = A$. $\blacksquare$

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LIVIU ORNEA
University of Bucharest, Faculty of Mathematics,
14 Academiei str., 70109 Bucharest, Romania, and:
Institute of Mathematics "Simion Stoilow" of the Romanian Academy,
21, Calea Grivitei Str. 010702-Bucharest, Romania
lornea@fmi.unibuc.ro, Liviu.Ornea@imar.ro

MISHA VERBITSKY
Laboratory of Algebraic Geometry,
Faculty of Mathematics, National Research University HSE,
7 Vavilova Str. Moscow, Russia, also:
Université Libre de Bruxelles, Département de Mathématique
Campus de la Plaine, C.P. 218/01, Boulevard du Triomphe
B-1050 Brussels, Belgium
verbit@verbit.ru, mverbits@ulb.ac.be