Online Learning of Caching and Recommendation Policies in a Multi-BS Cellular Networks

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Abstract

Mobile edge computing is a key technology for the future wireless networks and hence, the efficiency of a cache-placement algorithm is important to seek the cache content which satisfies the maximum user demands. Since recommendations personalizes an individual’s choices, it is responsible for a significant percentage of user requests, and hence recommendation can be utilized to maximize the overall cache hit rate. Hence, in this work, joint optimization of both recommendation and caching is proposed. The influence of recommendation on the popularity of a file is modelled using a conditional probability distribution. To this end, the concept of probability matrix is introduced and a Bayesian based model, specifically Dirichlet distribution is used to predict and estimate the content request probability and hence the average cache hit is derived. Joint recommendation and caching algorithm is presented to maximize the average cache hits. Subsequently, theoretical guarantees are provided on the performance of the algorithm. Also, a heterogeneous network consisting of $M$ small base stations and one macro base station is also presented. Finally, simulation results confirm the efficiency of the proposed algorithms in terms of average cache hit rate, delay and throughput.

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Index Terms

Cache placement, content delivery, recommendation, bayesian learning.

I. INTRODUCTION

Due to the fast development of communication based applications, it is expected that there will be 5.3 billion total Internet users (66 percent of global population) by 2023, up from 3.9 billion (51 percent of global population) in 2018 \[2\]. In such a scenario, content delivery networks (CDNs) emerged as one of the promising technologies. To further enhance the user’s quality of experience, mobile edge computing (MEC) has been proposed which stores the popular contents close to the edge devices beforehand, and hence reduces the delay and alleviates the backhaul congestion \[3\]–\[5\]. While, on the other hand rapidly growing file sizes, reducing cache sizes (compared to the traditional content delivery networks), and unpredictable user demands make the task of caching algorithms even more difficult. For example, the total data generated by Google per day is in the order of PBs, while installing 1TB in every small cell in the heterogeneous network will only shift less than 1 % of the data for even one content provider. To overcome these issues, it has been observed that user demands are increasingly driven by recommendation based systems. Recommendation based on an individual’s preference have become an integral part of e-commerce, entertainment and other applications. The success of recommender systems in Netflix and Youtube shows that 80% of hours streamed at Netflix and 30% of the overall videos viewed owes to recommender systems \[6\], \[7\]. With recommendation the user’s request can be nudged towards locally cached contents, and hence resulting in lower access cost and latency.

The content placement in the wireless network largely depends on the user behaviour and file popularity. The recent success of integration of artificial intelligence in the wireless communications has further led to better understanding of user behaviors and the characteristics of the
network [8]. Especially the edge networks can now predict the content popularity profile hence increasing the average cache hit. The high accuracy in prediction by the neural networks has resulted in many of the content popularity prediction models, such as, collaborative filtering with recurrent neural networks [9], the stack auto encoder [10], deep neural networks [11] and others. However, the local content popularity profile need not match the global prediction by the central server. Many of the recent works have proposed the edge caching strategies by learning the user preferences and content popularity [12]. Context awareness helps in classifying the environment, hence enabling the intelligent decisions at the edge to select the appropriate contents, for instance, Chen et al. [13] presented the edge cooperative strategy based on neural collaborative filtering. Jiang et al. [14] used the offline user preferences data and statistical traffic patterns and proposed an online content popularity tracking algorithm. However, the offline data will not be available always. These works assume that users have identical preferences and no correlation amongst the data, which may not be the case in a practical situation.

Recommendation and caching can be individually approximated, however the joint optimization is NP hard without an optimal decomposition [15]. Hence, in this paper, two estimation procedures namely Point estimation and Bayesian estimation have been proposed for optimization. Further, considering a real-time system, the estimation has been applied to a more practical heterogeneous network. A typical heterogeneous network consists of a macro base station (MBS) and various low power nodes. Similarly, in this work we have considered a heterogeneous system for the estimation model.

The main contributions of the paper are summarized as follows:

- For the first time two estimation procedures, Point and Bayesian estimation are provided. A probabilistic model using Bayesian inference based on Dirichlet distribution is proposed. Specifically, the influence of recommendation on the popularity profile is modelled
using a conditional probability distribution. A high probability guarantee on the estimated caching and recommendation strategies is provided. Irrespective of the estimation method, it is shown that with a probability of $1 - \delta$ the proposed caching and recommendation strategy is $\epsilon$ close to the optimal solution.

- A high probability bound on the regret for Bayesian estimation method is provided. To compare and contrast the obtained regret bound, we also derive a regret bound on the genie aided scenario using the Point estimation method. First, we consider the Point estimation case and provide lower bound on the waiting time that is required to achieve an error $\epsilon$ optimal solution with high probability. Assuming a genie aided scenario, we prove a regret bound of $O\left(\frac{T^{2/3}}{\sqrt{\log T}}\right)$. Using the Martingale difference technique, we prove a high probability bound on the regret achieved by the Bayesian estimation method. In particular, we show that $O(\sqrt{T})$ regret is achievable, which is better than the genie aided scenario.

- The proposed Point estimation and Bayesian estimation are further extended to a heterogeneous network consisting of $M$ SBSs with a central MBS. The MBS computes an estimate of the probability transition matrix (PTM) and gives an update to each of the SBSs. For computing guarantees, first $M$ is taken as two for the sake of simplicity and a lower bound on the waiting time is provided. The same is generalized to a heterogeneous network consisting of $M$ SBSs, the estimation of the probability matrix has been derived and useful insights are drawn.

- Numerical results are presented and it is shown that the proposed algorithm outperforms the existing least recently/frequently used (LRFU), least frequently used (LFU) and least recently used (LRU) in terms of average cache hit.

**Notation:** Bold uppercase letter denotes matrices. $\mathbb{E}(\cdot)$ denotes the statistical expectation
operator. \( f(\cdot) \) represents the probability density function (PDF). Superscript \((\cdot)^T\) represents transposition. \( \| \cdot \|_F, \| \cdot \|_{op} \) and \( \text{vec} \) indicates the Frobenius norm, operator norm and vector respectively. \( I_d \) represents the \( d \times d \) identity matrix. Further, \( \text{Dirich}(\alpha_1, \alpha_2, \ldots, \alpha_K) \) is the Dirichlet distribution with parameters \( \alpha_1, \alpha_2, \ldots, \alpha_K \).

II. SYSTEM MODEL AND PROBLEM STATEMENT

The system model consists of a wireless distributed content storage network with \( M \) SBSs serving multiple users and one central MBS, as shown in Fig. 1. Each SBS can store up to \( F \) contents/files of equal sizes from a catalog of contents denoted by \( \mathcal{C} := \{1, 2, \ldots, F\} \). The requests are assumed to be independent and identically (iid) distributed across time.\(^1\) As we know, recommending a file influences the users request process, and hence recommendation can provide “side information” about the future requests. Therefore, we consider the problem of jointly optimizing recommendation and caching policies in a cellular network. We model the influence of recommendation on the request via a conditional probability distribution denoted \( p_{ij,k} \), which represents the probability that a user requested a file \( i \) to the SBS \( k \) given the content \( j \) was recommended \(^{[16]}\). We assume that the time is slotted, and the PTM matrix for the \( k \)-th SBS denoted by \( (P_k)_{ij} := p_{ij,k}, i,j = 1, 2, \ldots, F \) is assumed to be fixed across time slots. For the sake of simplicity, it is assumed that at least one file is requested in every slot by each user \( N \) in the network.\(^2\) Let us use \( u_i \) and \( v_j \) to represent the probabilities with which a file \( i \) is cached and file \( j \) is recommended at any SBS, respectively. This induces a set of caching and recommendation strategies denoted by

\[
\mathcal{C}_{c,r} := \{ (u, v) \in [0, 1]^{2	imes F} : u^T 1 \leq c, v^T 1 \leq r \},
\]

\(^1\)A more general model of non-stationary requests can be handled based on the insights provided in the later part of our paper.

\(^2\)This can be ensured if the slot duration is chosen to be large enough.
where \( r \) and \( c \) are recommendation and cache constraints, respectively. In the sequel, the strategy is defined by the pair \((u, v)\). For a given strategy \((u, v) \in C_{c,r}\), the average cache hit at the SBS \(k\) is given by \(u^T P_k v\). If the matrix \(P_k\) is known apriori at the SBS \(k\), the optimal strategy can be found by solving \(\max_{(u, v) \in C_{c,r}} u^T P_k v\). However, the matrix \(P_k\) is unknown, and therefore it needs to be estimated from the demands. Let the variable \(d_{k,t}^{(t)}\) denotes the demand at the SBS \(k\), and is defined as the total number of requests in the time slot \(t\) for the file \(i\). Since the demands arrive in a sequential manner, the PTMs need to be estimated and updated in an online fashion.

The performance of such algorithms is measured in terms of regret. As apposed to adversarial setting of online learning, here we have assumed that there is an underlying distribution from which the requests are generated, namely the PTM. Accordingly, the following provides the definition of the regret, which depends on the PTM:

**Definition 1:** (Regret) The regret at the SBS \(k\) after \(T\) time slots with respect to any sequence of strategy \((u_{k,t}, v_{k,t}), t = 1, 2, \ldots, T\) is defined as

\[
\text{Reg}_k,T := T u_{k,*}^T P_k v_{k,*} - \sum_{t=1}^{T} u_{t}^T P_k v_t,
\]

(2)

where \((u_{k,*}, v_{k,*}) := \arg \max_{(u, v) \in C_{c,r}} u^T P_k v\) is the optimal strategy at the SBS \(k\).
The goal of the paper is to come up with a strategy at each SBS that results in a minimum regret. Any caching and recommendation algorithm, either directly or indirectly estimates the PTM $P_k$. Therefore, the above goal translates to finding better estimates of the PTMs. In this paper, we consider two approaches to finding the estimates of the PTMs in an online fashion, namely (i) Point estimation and (ii) Bayesian estimation methods. Further, when there are multiple SBSs, any given SBS can potentially improve its estimate of PTM by fusing the estimates of the other SBSs. The question of how to fuse the estimates that results in a good regret is another question to which we will shed some lights. In the following section, we provide caching and recommendation algorithms for single SBS scenario, and provide theoretical guarantees for them.

III. JOINT CACHING AND RECOMMENDATION FOR SINGLE SBS SCENARIO

In this section, we assume single SBS, and therefore, $M = 1$ as shown in Fig. 2. As mentioned above, using the demands obtained at the SBS, an estimate of the PTM matrix is computed using either Point estimation or Bayesian estimation method. Given an estimate $\hat{P}_k^{(t)}$, the caching and recommendation strategies will be found by solving the following problem.\(^3\)

$$\left(\hat{u}_{o,t}^*, \hat{v}_{o,t}^*\right) = \arg \max_{(u,v) \in C_c,r} u^T \hat{P}_k^{(t)} v. \quad (3)$$

\(^3\)For theoretical analysis, we assume that the problem can be solved exactly.
Now, we present the following two estimation procedures used in this paper.

- **Point estimation**: Given any SBS $k$, in this method, the demands until $t$ time slots is used to compute an estimate of the matrix $P_k$. During the first $t$ time slots, recommendation and caching are done with probabilities $q$ and $p$, respectively. Let $v_{jk}^t = 1$ if file $j$ was recommended in slot $t-1$, and zero otherwise. The recommendation and caching constraints in (1) are satisfied by choosing $q := r/F$ and $p := c/F$. We can see that as the value of $t$ increases, the estimate becomes better, and hence results in better performance. The estimate of the $ij$-th entry for SBS $k$ of the $P_k$ matrix is given by

$$
\hat{p}_{ij,k}^{(t)} := \frac{\sum_{s=0}^{t-1} d_{ik}^{(s)} v_{jk}^{s-1}}{N \sum_{s=0}^{t-1} v_{jk}^{s-1}}.
$$

(4)

The above is a naive estimate of the probabilities by using a simple counting of events. The corresponding estimate of the matrix $P_k$ be denoted by $\hat{P}_k^{(t)}$. Since $\mathbb{E}\{\hat{P}_{ij,k}^{(t+1)} | \sum_{s=0}^{t-1} v_{jk}^{s} > 0\} = p_{ij,k}$, the point estimator is an unbiased estimator. After every time slot $t$, the recommendation and caching probabilities are selected by solving the optimization problem in (3) with $\hat{P}_k^{(t)}$ obtained in (4). A procedure to find a strategy is given in Algorithm 1.

- **Bayesian estimation**: In this method, for a given time slot, the rows of the matrix $P_k^{(t)}$ is sampled using a prior distribution, which is updated based on the past demands. This may tradeoff the exploration versus exploitation while solving for the optimal recommendation and caching strategies. Here, Dirichlet distribution is chosen as a prior. The Dirichlet pdf is a multivariate generalization of the Beta distribution, and is given by

$$
f(x_1, \ldots, x_M, \alpha_1, \ldots, \alpha_M) = \frac{\Gamma(\sum_{j=1}^{M} \alpha_j) \prod_{j=1}^{M} x_j^{\alpha_j-1}}{\prod_{j=1}^{M} \Gamma(\alpha_j)},
$$

(5)

$\alpha_j \geq 0 \ \forall \ j$. The Dirichlet distribution is used as a conjugate pair in bayesian analysis and the shape of the distribution is determined by the parameter $\alpha_j$. If $\alpha_j = 1$ for all $j$, then it leads to a uniform distribution. The higher the value of $\alpha_j$, the greater the probability
of occurrence of $x_j$. The notation $(x_1, x_2, \ldots, x_M) \sim \text{Dirch}(\alpha_1, \alpha_2, \ldots, \alpha_M)$ indicates that
$(x_1, x_2, \ldots, x_M)$ is sampled from a Dirichlet distribution in (5). An estimate in the beginning of the time slot $t$ of the $i$-th row of the matrix $\hat{P}_k^{(t)}$ is given by

$$(\hat{P}_k^{(t)})_{i} \sim \text{Dirch} \left( \sum_{s=1}^{t-1} d_{ik}^{(s)} v_{jk}^{s-1}, \sum_{i=1}^{t-1} d_{jk}^{(s)} v_{jk}^{s-1}, \sum_{s=1}^{t-1} d_{sk}^{(s)} v_{jk}^{s-1} \right),$$

(6)

where $v_{jk}^{s-1}$ is as defined earlier with $v_{jk}^{0}$ sampled from $\{0, 1\}$ with probability $q := r/F$.

After every time slot $t$, the recommendation and caching probabilities are selected by solving

\begin{algorithm}
\begin{algorithmic}[1]
\State \textbf{procedure} \text{POINT ESTIMATION/BAYESIAN ESTIMATION}
\State $\hat{u}_{b,0}^{*}$ \text{i.i.d.} $\sim \{0, 1\}$ from $p = c/F$, and $\hat{v}_{b,0}^{*}$ \text{i.i.d.} $\sim \{0, 1\}$ from $q = r/F$.
\State Recommend and cache according to $\hat{v}_{b,0}^{*}$ and $\hat{u}_{b,0}^{*}$.
\For{$t = 0, 1, \ldots, T$}
\State Observe demands $d_{ik}^{(t)}$ in slot $t$.
\State Compute $\hat{P}_{k}^{(t)}$ from (4) for point estimation
\State Compute $\hat{P}_{k}^{(t)}$ from (6) for Bayesian estimation
\State Solve (3)
\State Use $(\hat{v}_{b,t}^{*}, \hat{u}_{b,t}^{*})$ to recommend and cache.
\EndFor
\State \textbf{end procedure}
\end{algorithmic}
\end{algorithm}

the optimization problem in (3) with $\hat{P}_{k}^{(t)}$ obtained in (6). A procedure to find a strategy is given in Algorithm 1.

In the following subsection, we provide theoretical guarantees of the above algorithm.

A. Theoretical Guarantees

In this section, we provide a high probability bound on the regret for both Point estimation and Bayesian estimation. For the Point estimation case, we start by providing an lower bound on the waiting time which is $\epsilon$ close to the optimal caching strategies. The result will be of the following form: With a probability of at least $1 - \delta$, the following holds provided $t \geq \text{constant}$
\( u_t^T P_k v_t \geq \sup_{(u,v) \in C_{c,r}} u^T P_k v - \epsilon, \)  \( (7) \)

where \((u_t, v_t)\) is the caching strategy obtained by using any algorithm. The constant \(\epsilon\) depends on various parameters, as explained next. Towards stating theoretical guarantees, the following definition is useful.

**Definition 2:** (Covering number) A set \(N_\epsilon := \{(x_1, y_1), (x_2, y_2), \ldots, (x_{N_\epsilon}, y_{N_\epsilon})\}\) is said to be an \(\epsilon\)-cover of \(C_{c,r}\) if for any \((u, v) \in C_{c,r}\), there exists \((x_j, y_j) \in N_\epsilon\) for some \(j\) such that \(\|u - x_j\| \leq \frac{\epsilon}{8}\) and \(\|v - y_j\| \leq \frac{\epsilon}{8}\).

The following theorem provides a bound that is useful to provide the final result.

**Theorem 3.1:** For a given estimate of the PTM denoted \(\hat{P}^{(t)}_k\) using Point estimation or Bayesian estimation, the following holds good

\[
\Pr\left\{ \sup_{(u,v) \in C_{c,r}} u^T P_k v - u_t^T P_k v_t \geq \epsilon \right\} \leq |N_\epsilon| \Pr\left\{ \|\hat{\Delta}^{(t)}_P\|_F \geq \frac{\epsilon}{4\kappa F} \right\}, \tag{8}
\]

where \((u_t, v_t)\) is the output of the Algorithm 1 at time \(t\), and \(\hat{\Delta}^{(t)}_P := P_k - \hat{P}^{(t)}_k\). Further, \(\kappa > 0\) is some constant.

**Proof:** See Appendix A.

Using the above result, in the following, we provide our first main result on the performance of the Point estimation scheme.

**Theorem 3.2:** Using \((3)\) for caching and recommendation in slot \(t\), for any \(\epsilon > 0\), with a probability of at least \(1 - \delta\), \(\delta > 0\), \((u^*_{o,t})^T P_k v^*_{o,t} \geq \sup_{(u,v) \in C_{c,r}} u^T P_k v - \epsilon\) provided

\[
t \geq \frac{1}{q \left(1 - \exp\left(-\frac{N^2}{8\kappa^2 F^2 c^2 r^2}\right)\right)} \log \frac{2N_\epsilon F^2}{\delta}. \tag{9}
\]

**Proof:** See Appendix B.

As we know, the regret achieved by the Point estimation method is \(O(T)\) as it incurs non-zero constant average error for all the slots \(t\) satisfying \(9\). In this method, the estimation of PTM is done using the samples obtained from the first \(t\) slots, and the caching strategy is
decided based on this estimate. However, an improvement over this is to continuously update the estimates, and the caching/recommendation strategies. Instead of analyzing the regret for this, we assume that at any time slot $t$, a genie provides an estimate of the PTM as in (4) to compute the caching/recommendation strategies, and provide the corresponding approximate regret bound. In particular, in Appendix C, we show that a regret of $O(T^{2/3}\sqrt{\log T})$ can be achieved through the genie aided point estimation method. As opposed to point estimation method, here ( genie aided) the caching/recommendation decisions can be made using an improved estimate of the PTM in every time slot leading to a better regret. We use this as a benchmark to compare the regret obtained from the Bayesian estimation method. In the next section, we extend the result to two SBS scenario, and use the insights to extend it further in the later part of the paper to any number of SBSs.

B. Bayesian Estimation: Single SBS Scenario

Note that unlike the analysis for point estimation, in this case, the strategies are correlated across time. This makes the analysis non-trivial. The approach we take is to convert a sequence of random variables (function of caching and recommendation across time) into a Martingale difference. This enables us to use the Azuma’s inequality, which can be used to provide high probability result on the regret. In the following, we provide the result.

Theorem 3.3: For the Bayesian estimation in Algorithm 1 for any $\epsilon > 0$, with a probability of at least $1 - \delta$, $\delta > 0$, the following bound on the regret holds

$$\text{Reg}_T \leq 2rc \max_{ij} p_{ij}|N_i| \sum_t \exp \left\{ - \frac{8\psi_t^2}{cr|N_i|^2\sigma_t^2(t)} \right\} + 2 \sum_t \psi_t + \sqrt{128r^2c^2T\log(1/\delta)},$$

(10)

where $\alpha_{ij}^{(t)} = \sum_{q=1}^{t-1} d_i^{(q)} v_j^{(q-1)}$, $\bar{\sigma}_t^2 := \left[ \sum_{j=1}^{F} \frac{1}{(\sum_{i} \alpha_{ij}^{(t)} + 1)^2} \right]$, and $\psi_t$ is any non-negative number.

Proof: See Appendix E.
Remark: Note that the above result is an algorithm dependent bound as it depends on the recommendation strategy, which is determined by the algorithm. In order to provide more insights into the result, we will make certain assumptions about the demands. In particular, if the demands \( d_i^{(q)} > 0 \) almost surely for all \( i \) and \( q \), then, \( \alpha_{ij}^{(t)} \) will be 0 when \( v_j^{(q-1)} = 0 \) for all \( q \leq t - 1 \) or \( O(t) \) in case of \( \sum_{q=1}^{t} v_j^{(q-1)} = O(\sqrt{t}) \). Note that this depends on the algorithm output, which we presume that at least files of higher probability transition values with recommendation will be sampled multiple times in a time frame. This leads to \( \tilde{\sigma}_t^2 = O(\frac{1}{T}) \). Thus, assuming \( \Psi_t = O(\sqrt{t}) \), the summation in the first term of the regret is \( O(1) \). Overall, this results in \( O(\sqrt{T}) \) regret. Recall that an approximate regret of \( O(T^{2/3} \sqrt{\log T}) \) is shown for the genie aided case while the Bayesian estimation method achieves a regret of the order \( \sqrt{T} \). The genie aided regret is worse by an order of \( T^{1/6} \), which is partly due to the fact that the genie estimation of PTM is less accurate than the point estimation method. In the next section, we extend our results to two SBS scenario.

IV. Proposed Caching and Recommendation Strategies With Multiple SBSs

In this section, we present caching and recommendation algorithms when there are multiple SBSs. In particular, we provide insights on how to use the neighboring SBSs estimates to further improve the overall caching and recommendation performance of the network. First, we present the results for two SBS scenario, and similar analysis will be used to extend the results to multiple SBSs.

A. Two Small Base Station Scenario

In this subsection, we consider a two SBSs scenario, as shown in Fig. 3. As described in Section II, \( P_1 \) and \( P_2 \) represents PTM for SBS-1 and SBS-2, respectively. The central MBS sends the global update of the recommendation and caching decisions to each SBS. Assume that the request across SBSs are independent. Let each SBS use one of the estimation methods
in Algorithm 1. Let $\hat{P}_1^{(t)}$ and $\hat{P}_2^{(t)}$ be the corresponding estimates (either point or Bayesian estimate) of $P_1$ and $P_2$, respectively. The two SBSs convey their respective PTM to the central MBS. The central MBS computes an estimate $\hat{Q}_k^{(t)}$, $k = 1, 2$ for SBS 1 and SBS 2 as a linear combination of the two estimates as given below

$$\hat{Q}_k^{(t)} = \lambda_k \hat{P}_1^{(t)} + (1 - \lambda_k) \hat{P}_2^{(t)},$$

(11)

where $\lambda_k \in [0, 1]$, $k = 1, 2$ strikes a balance between the two estimates. The above estimate is used to compute the respective caching and recommendation strategies for the two SBSs and will be communicated to the respective SBSs. The above results in a better estimate, for example, when $P_1 = P_2$ or when the two matrices are close to each other. The corresponding algorithm is shown below. First, we prove the following guarantee for the Point estimation method.

**Theorem 4.1:** For Algorithm 2 with point estimation, for any SBS $k$ and for any $\epsilon > 0$, with a probability of at least $1 - \delta$, $\delta > 0$, the regret $\text{Reg}_{k,T} < \epsilon$, i.e., $\Pr\left\{ \left( u_{k,t}^* \right)^T P_k v_{k,t}^* \geq \sup_{(u,v) \in C_{c,r}} u^T \hat{Q}_k^{(t)} v - \epsilon_k \right\} > 1 - \delta$ provided

$$t \geq \max \left\{ \tau \left( \frac{\epsilon_k}{\lambda_k}, \frac{\delta}{2} \right), \tau \left( \frac{\epsilon_k}{1 - \lambda_k}, \frac{\delta}{2} \right) \right\},$$

(12)

where

$$\tau(\epsilon, \delta) := \frac{1}{q \left( 1 - \exp\{-\frac{N_\epsilon^2 F^2 c^2}{\delta} \} \right)} \log \frac{2|N_\epsilon| F^2}{\delta}.$$

(13)

Further, $\epsilon_k := \epsilon/2 - (1 - \lambda_k) \sup_{(u,v) \in C_{c,r}} |u^T (P_2 - P_1)v|$
Proof: See Appendix [F]

Algorithm 2 Caching and recommendation algorithm (two SBS case)

1: procedure POINT ESTIMATION/BAYESIAN ESTIMATION
2:  
3:  
4:  
5:  
6:  
7:  
8:  
9:  
10:  
11:  
12:  
13:  
14: end procedure

As in the single SBS case, to benchmark the performance of Bayesian estimation method, we consider a genie aided scenario, and in Appendix [D] we show that it achieves an approximate regret of

\[
\text{Reg}_{k,T} \lesssim \max \left\{ \Theta \lambda_k, \Theta(1 - \lambda_k) \right\} T^{2/3} + 2T(1 - \lambda_k)\mathcal{V}_{12},
\]

where \(\mathcal{V}_{12} := \sup_{(u,v) \in C_{c,r}} |u^T(P_2 - P_1)v|\) and \(\Theta = \sqrt[3]{8\kappa^2 c^2 r^2 (\log 4F^2T^2 + F)} \frac{qN}{qN}\). The above clearly shows the trade-off between the two terms. The first term scales as \(T^{2/3}\) while the second term scales with \(T\) linearly. This can be balanced by using \(\lambda_k = 1 - \frac{1}{\sqrt{T}}\), which results in \(O(\sqrt{T})\) scaling of regret. Note that the choice \(\lambda_k = 1 - \frac{1}{\sqrt{T}}\) reveals that as time progresses, i.e., as the BS \(k\) collects more samples, the weights allocated to the neighboring BS should go down to zero, as expected. Furthermore, by appropriately choosing \(\lambda_k\) as above, the regret obtained is of the order \(T^{2/3}\). Next, we present the guarantees for Algorithm [2]

**Theorem 4.2:** For Algorithm [2] with Bayesian estimation, for

\[
(14) \quad \epsilon > 2\max_{k=1,2}(1 - \lambda_k) T \sup_{(u,v) \in C_{c,r}} |u^T(P_1 - P_2)v|,
\]
with probability of at least $1 - \delta$, $\delta > 0$, for any BS $k \in \{1, 2\}$, the regret can be bounded as

$$\text{Reg}_{k,T} \leq \max \left\{ R_k \left( \frac{2\epsilon_k}{\lambda_k}, \frac{\delta}{2} \right), R_k \left( \frac{2\epsilon_k}{(1 - \lambda_k)}, \frac{\delta}{2} \right) \right\}. \quad (15)$$

In the above,

$$R_k(\epsilon, \delta) := 2rc \max_{ijk} \left| \mathcal{N}_c \right| \sum_t \exp \left\{ -\frac{8\psi_t^2}{cr \left| \mathcal{N}_c \right|^2 \bar{\sigma}_k^2(t)} \right\} + 2 \sum_t \psi_t + \sqrt{128r^2c^2 T \log(1/\delta)}, \quad (16)$$

$$\alpha_{ijk}^{(t)} = \sum_{q=1}^{t-1} d_{ik}^{(q)} v_{jk}^{(q-1)}, \quad \epsilon_k := \epsilon/2 - (1 - \lambda_k) T \sup_{(u,v) \in \mathcal{C}_{c,r}} \left| u^T (P_1 - P_2) v \right|,$$

$$\bar{\sigma}_k^2(t) := \left[ \sum_{j=1}^F \frac{1}{(\sum_i \alpha_{ijk}^{(t)})^2} \right].$$

**Proof:** See Appendix G.

**Remark:** The result shows the tradeoff exhibited by $\lambda_k$. In particular, larger $\lambda_k$ makes the first regret inside the max term in (16) larger, and smaller $\lambda_k$ ensures that the second term inside the max above dominates. It is essential to balance the two depending on the number of samples received. We further elaborate this in the simulation results. Further, the above result is an algorithm dependent bound as the bound depends on the recommendation strategy, which is determined by the algorithm. Following the single SBS analysis, we make similar assumptions in two SBS scenario. In particular, if the demands $d_{ik}^{(q)} > 0$ almost surely for all $i$ and $q$, then, $\alpha_{ijk}^{(t)}$ will be 0 when $v_{jk}^{(q-1)} = 0$ for all $q \leq t - 1$ or $O(t)$ in case of $\sum_{q=1}^t v_{jk}^{(q-1)} = O(\sqrt{t})$. Note that this depends on the algorithm output, which we presume that at least files of higher probability transition values with recommendation will be sampled multiple times in a time frame. This leads to $\bar{\sigma}_k^2(t) = O(\frac{1}{T^2})$. Thus, assuming $\Psi_t = O(\sqrt{t})$, the summation in the first term of the regret is $O(1)$. Overall, this results in $O(\sqrt{T})$ regret. Furthermore, the regret obtained by the genie aided method is of the order $T^{2/3}$, which is higher than the one achieved by the Bayesian estimation method. This along with the experimental results establishes the superiority of the proposed Bayesian estimation method.
B. Multiple Small Base Station Scenario

In this section, we consider a heterogeneous network with \( M \) SBSs connected to a central MBS. The requests at each SBS are assumed to be i.i.d. with PTM \( P_1, P_2, \ldots, P_M \) as described in Section II. Similar to the two SBS model, each SBS computes an estimate of the PTM as follows

\[
\hat{Q}_k^{(t)} = \lambda_1^{(k)} \hat{P}_1^{(t)} + \lambda_2^{(k)} \hat{P}_2^{(t)} + \ldots + \lambda_M^{(k)} \hat{P}_M^{(t)},
\]

where \( \lambda_1^{(k)}, \lambda_2^{(k)}, \ldots, \lambda_M^{(k)}, k = 1, 2, \ldots, M \) are coefficients to be determined later. Further, it satisfies \( \sum_{j=1}^{M} \lambda_j^{(k)} = 1 \).

The following theorem is a generalization of two BS model which provides a guarantee on the minimum time required to achieve a certain level of accuracy with high probability.

**Theorem 4.3:** Using (17) for any \( \epsilon > M^2 \max_k \{(1 - \lambda_1^{(k)})D_1 - \lambda_2^{(k)}D_2 - \ldots - \lambda_M^{(k)}D_M\} \), for point estimation, with a probability of at least \( 1 - \delta \), \( \delta > 0 \), for any BS \( k \), the regret \( (\text{Reg}_{k,T}) \) is less than \( \epsilon \) \( \text{i.e.} \) \( (u_{o,t}^*)^T P_1 v_{o,t}^* \geq \sup_{(u,v) \in C_{c,r}} u^T P_1 v - \epsilon \) provided

\[
t \geq \max \left\{ \tau \left( \frac{\epsilon_1}{\lambda_1^{(k)}}, \frac{\delta}{M} \right), \tau \left( \frac{\epsilon_2}{\lambda_2^{(k)}}, \frac{\delta}{M} \right), \ldots, \tau \left( \frac{\epsilon_M}{\lambda_M^{(k)}}, \frac{\delta}{M} \right) \right\},
\]

where

\[
\tau(\epsilon, \delta) := \frac{1}{q \left( 1 - \exp \left\{ -\frac{N_{c}^{2}}{8n^{2}F_{c}^{2}c_{r}^{2}} \right\} \right)} \log \frac{2N_{c}F_{c}^{2}}{\delta},
\]

and \( \epsilon_k := \epsilon/M^2 - (1 - \lambda_1^{(k)})D_1 + \lambda_2^{(k)}D_2 + \ldots + \lambda_M^{(k)}D_M \), and \( D_k := \sup_{(u,v) \in C_{c,r}} |u^T P_k v| \) \( \forall k = 1, 2, \ldots, M \).

**Proof:** See Appendix H.

The regret for the genie aided case after appropriate choice for \( \lambda_k \) turns out to be of the order of \( \sqrt{T} \). The genie aided regret analysis is relegated to Appendix D-A. Next we present the regret bound for the Bayesian estimation method.
Theorem 4.4: Using (17) for any
\[ \epsilon > M^2 \max_k \{ (1 - \lambda_1^{(k)}) I_1 - \lambda_2^{(k)} I_2 - \ldots, -\lambda_M^{(k)} I_M \}, \]
with probability of at least \( 1 - \delta \), \( \delta > 0 \), for any BS \( k, k \in \{1, 2, \ldots, M\} \) \( \text{Reg}_{k,T} \), the regret of the Bayesian estimation method satisfies the following bound
\[
\text{Reg}_{k,T} \leq \max \left \{ R_k \left( \frac{\epsilon_1}{\lambda_1^{(k)}}, \frac{\delta}{M} \right), R_k \left( \frac{\epsilon_2}{\lambda_2^{(k)}}, \frac{\delta}{M} \right), \ldots, R_k \left( \frac{\epsilon_M}{\lambda_M^{(k)}}, \frac{\delta}{M} \right) \right \}. \tag{20}
\]

In the above
\[
R_k(\epsilon, \delta) := 2r c \max_{ij} p_{ijk} |N_t| \sum_t \exp \left\{ - \frac{8 \psi_t^2}{cr |N_t|^2 \sigma_k^2(t)} \right\} + 2 \sum_t \psi_t + \sqrt{128 r^2 c^2 T \log \frac{1}{\delta}},
\]
\[
\alpha_{ijk}^{(t)} = \sum_{q=1}^{t-1} d_{ik}^{(q)} v_{jk}^{(q-1)}, \quad \epsilon_k := \epsilon / M^2 - (1 - \lambda_1^{(k)}) I_1 + \lambda_2^{(k)} I_2 + \ldots, + \lambda_M^{(k)} I_M,
\]
\[
I_k := \sup_{(u,v) \in C_{c,r}} |u^T P_k v| \forall k = 1, 2, \ldots, M \text{ and } \sigma_k^2(t) := \left[ \sum_{j=1}^F 1 \left( \sum_{i} \alpha_{ijk}^{(t)} \right) \right]^2.
\]

Proof: See Appendix I.

Remark: Note that the above result is an algorithm dependent bound as it depends on the recommendation strategy, which is determined by the algorithm. Following the single SBS analysis, we make similar assumptions in multiple SBS scenario. In particular, if the demands \( d_{ik}^{(q)} > 0 \) almost surely for all \( i \) and \( q \), then, \( \alpha_{ijk}^{(t)} \) will be 0 when \( v_{jk}^{(q-1)} = 0 \) for all \( q \leq t - 1 \) or \( O(t) \) in case of \( \sum_{q=1}^{t} v_{jk}^{(q-1)} = O(\sqrt{t}) \). Note that this depends on the algorithm output, which we presume that at least files of higher probability transition values with recommendation will be sampled multiple times in a time frame. This leads to \( \sigma_k^2(t) = O(\frac{1}{T}) \). Thus, assuming \( \Psi_t = O(\sqrt{t}) \), the summation in the first term of the regret is \( O(1) \). Overall, this results in \( O(\sqrt{T}) \) regret. As in the case of two SBS, the regret obtained is better than the genie aided scenario.

In the next section, we present experimental results that corroborates some of our theoretical observations.

V. Simulation Results

In this section, simulation results are presented to highlight performance of the proposed caching and recommendation model. The simulation setup consists of one SBS model, two-SBS
model and a heterogeneous model with multiple users. We assume a time-slotted system in the simulation setup. For the heterogeneous model, the simulation consists of two scenarios as follows:

- Fixed Link Scenario: In this, the links between SBS and users are uniformly and independently distributed in \{0, 1\} with probability 1/2.

- \textbf{SINR} Based Scenario: In this, the SBS and users are assumed to be distributed uniformly in a geographical area of radius 500m. It is assumed that a SBS and user can communicate only if the corresponding \textbf{SINR} is greater than a threshold. This \textbf{SINR} takes into account the fading channel, the path loss, power used, and the distance between the user and the SBS. The minimum rate at which a file can be transferred from the SBS to a user is given by the threshold, and hence the reciprocal of the rate indicates the delay. In the simulation, we have used $\tau := \frac{1}{\log(1+\text{SINR})}$ as a measure of the delay between a user and a SBS. However, when the requested file is absent, a backhaul fetching delay of $\alpha \times \tau$ is counted in addition to the down link delay of $\tau$, i.e., the overall delay when the file is absent is $(\alpha + 1)\tau$, with $\alpha = 10$. Also, if the threshold is $R$, then at least $R$ bits can be sent in a time duration of at most $1/\log(1 + \text{SINR})$ seconds, and hence the throughput is roughly $R\log(1 + \text{SINR})$ bits/second.

Fig. 4 shows the plot for a heterogeneous system the metric used for comparison is throughput. In Fig. 4 the number of SBSs, users, the total number of files and threshold value for \textbf{SINR} are 5, 30, 100, and 12dB, respectively. The throughput for the proposed algorithm with recommendation is 225 bits/s for a cache size of 24, while LRFU, LRU and LFU algorithm has a throughput of 100 bits/s, 85 bits/s and 70 bits/s respectively for the same cache size. Thus from Fig. 4 we can see that the proposed algorithm has higher throughput as compared to the existing algorithms.

Fig. 5 corresponds to the \textbf{SINR} scenario for a two SBS model. Fig. 5 shows the average
delay versus cache size plot for (a) cache placement algorithm with recommendation (b) cache placement algorithm without recommendation, (c) LRFU algorithm, (d) LRU algorithm and (e) LFU algorithm. In Fig. 5, SBSs, the number of users, the total number of files and threshold value for $\text{SINR}$ are 2, 25, 100 and 12dB, respectively. From Fig. 5 we can observe that the delay of both the proposed algorithms is less as compared to the other benchmark algorithms, since pre-fetching files according to the estimated methods results in lower fetching costs from the backhaul and hence less delay.
Fig. 6 shows the plot for two BS model. The value of $\lambda_1$ is varied between 0.1 and 1. From the Fig. 6 we can observe that as the value of $\lambda_1$ approaches 0.5, the average cache hit increases, this is because for $\lambda_1 = 0.5$ and $P_1 = P_2$, the $Q$ popularity profile matrix of BS will have maximum similarity to the individual SBS popularity profile matrix and hence the cache hit will be maximum for $\lambda_1 = 0.5$ and it will gradually decrease as we further increase the value of $\lambda_1$. 

Fig. 7 shows the plot for average cache hit versus $\lambda$ for 2 SBS when $P_1 \neq P_2$. From the Fig. 7 we can observe that for larger $T$, the optimal lambda value is close to 1. Also, for smaller value of $T$, depending on the value of $\Theta$, the optimal value of $\lambda$ is less than 1. Thus, the simulation results prove that the recommendation helps in increasing the average cache hit when compared to the algorithm without recommendation and it also performs better than the existing popular LRFU, LRU and LFU algorithms.

VI. CONCLUSION

The deployment of MEC in the current wireless heterogeneous networks is an important application for the smooth transition to the distributed cloud based platform. In this paper, a model that captures the caching decisions along with recommendation has been introduced. Implications of recommendation on user requests has been studied and it has been observed that the recommendation does influence the demands from the users. Bayesian and point estimation methods are used to determine the user request pattern. An algorithm is then proposed to jointly optimize caching and recommendation. A multi-tier heterogeneous model consisting of MBS and SBSs is also presented and an upper bound on the estimation accuracy of popularity profile is provided. Finally, simulation results and theoretical proofs support the superior performance of the proposed method over the existing algorithms.
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From [17], it follows that
\[ \sup_{(u,v) \in \mathcal{C}_{c,r}} u^* P_k v^* - u^T P_k v \leq \frac{1}{2} \sup_{(u,v) \in \mathcal{C}_{c,r}} \left| u^T \hat{\Delta} P^{(t)} v \right|. \] (21)

Let \( x^* \) and \( y^* \) be solutions to \( \sup_{(u,v) \in \mathcal{C}_{c,r}} \left| u^T \hat{\Delta} P^{(t)} v \right| \). Since \( x^* \) and \( y^* \) belong to \( \mathcal{C}_{c,r} \), for some \( i = 1, 2, \ldots, N \), there exist \( x_i \) and \( y_i \) in \( \mathcal{A} \) such that \( \| x^* - x_i \|_2 \leq \epsilon/8 \), and \( \| y^* - y_i \|_2 \leq \epsilon/8 \). Further, by adding and subtracting \( x_i \hat{\Delta} P^{(t)} y_i \) and \( x_i \hat{\Delta} P^{(t)} y_i \), we get
\[ \left| (u^*)^T \hat{\Delta} P^{(t)} v^* \right| \leq \left| x_i \hat{\Delta} P^{(t)} y_i \right| + \frac{\epsilon \| \hat{\Delta} P^{(t)} \|_{op}}{4}. \] (22)

From (21) and (22)
\[ \Pr \left\{ \sup_{(u,v) \in \mathcal{C}_{c,r}} u^T P_k v - u^* P_k v^* \geq \epsilon \right\} \leq \Pr \left\{ \bigcup_{i=1}^{N} B_i \geq g_c \right\} \leq \sum_{i=1}^{N} \Pr \left\{ B_i \geq \frac{\epsilon}{4} \right\}, \]
where \( g_c := \frac{\epsilon}{2} - \epsilon \| \hat{\Delta} P^{(t)} \|_{op}/4 \leq \frac{\epsilon}{4} \), using \( \| \hat{\Delta} P^{(t)} \|_{op} \leq 1 \), and \( B_i := \left| x_i \hat{\Delta} P^{(t)} y_i \right| \). Using the fact that \( \left| x_i \hat{\Delta} P^{(t)} y_i \right| \leq \kappa \max_j \| x_j \|_1 \max_i \| y_i \|_1 \| \hat{\Delta} P^{(t)} \|_F \leq \kappa r_c \| \hat{\Delta} P^{(t)} \|_F \), the above can be further bounded to get \( N \Pr \left\{ \| \hat{\Delta} P^{(t)} \|_F \geq \frac{\epsilon}{4 r c} \right\} \). This completes the proof.

APPENDIX B

PROOF OF THEOREM 3.2

Consider the following
\[ \Pr \left\{ \| \hat{\Delta} P^{(t)} \|_F^2 \geq \gamma \right\} \leq \Pr \left\{ \max_{k,l} (p_{kl} - \hat{p}_{kl}^{(t)})^2 \geq \frac{\gamma}{F^2} \right\} \leq F^2 \Pr \left\{ (p_{kl} - \hat{p}_{kl}^{(t)})^2 \geq \frac{\gamma}{F^2} \right\}, \] (23)
where $\gamma := \frac{\epsilon}{4F^2c}$, and the second inequality above follows from the union bound. Conditioning on $V_{ls} := \sum_{s=1}^{t} v_l^{(s-1)} = m$, there are $Nm$ i.i.d. samples available to estimate $p_{kl}$. Using Hoeffding’s inequality
\[
\mathbb{E} \Pr \left\{ \left( p_{kl} - \hat{p}_{kl}^{(t)} \right)^2 \geq \frac{\gamma}{F^2} \mid V_{ls} = m \right\} \leq 2 \mathbb{E} \exp \left\{ -\frac{2Nm\gamma}{F^2} \right\}.
\]
Since $V_{ls}$ is a binomial random variable with parameter $q$, the above average with respect to $V_{ls}$ becomes
\[
2 \left( 1 - q \left( 1 - \exp \left\{ -\frac{2N\gamma}{F^2} \right\} \right) \right)^t.
\]
The following bound on the left hand side of (8) can be obtained using the above in (23), and substituting it in (8)
\[
2N\epsilon F^2 \left( 1 - q \left( 1 - \exp \left\{ -\frac{2N\gamma}{F^2} \right\} \right) \right)^t.
\]
An upper bound on the above can be obtained by using $1 - x \leq e^{-x}$. Using the resulting bound,
\[
\Pr \left\{ \sup_{(u,v) \in C,c,r} u^T P_k v - u^* P_k v^* \geq \epsilon \right\} < \delta \text{ provided } t \text{ satisfies the bound in the theorem.}
\]

**APPENDIX C**

**GENIE AIDED REGRET ANALYSIS: HEURISTICS FOR TWO SBSs CASE**

Consider the instantaneous regret given by $\text{Reg}_k(t) := (u^*_{k,t})^T P_k v^*_{k,t} \geq \sup_{(u,v) \in C,c,r} u^T P_k v - u^* P_k v^*$ at time $t$. Using the union bound, we can write
\[
\Pr \left\{ \frac{1}{T} \sum_{t=1}^{T} \text{Reg}_k(t) \geq \frac{1}{T} \sum_{t=1}^{T} \epsilon_t \right\} \leq \sum_{t=1}^{T} \Pr \left\{ \text{Reg}_k(t) \geq \epsilon_t \right\},
\]
where $\epsilon_t > 0$. From Theorem 3.2 it follows that for any $\epsilon_t > 0$, we have $\Pr \{ \text{Reg}_k(t) \leq \epsilon_t \} \geq 1 - \delta$ provided (9). By choosing $\delta = \frac{1}{T^2}$, the approximation $e^{-x} \approx 1 - x$ for small $x$, and $|N_e| \approx 1/\epsilon_t$ in Theorem 3.2, we get $\Pr \{ \text{Reg}_k(t) \leq \epsilon_t \} \geq 1 - \frac{1}{T^2}$ provided
\[
t \geq \frac{8F^2c^2}{qN\epsilon_t^4} \log \frac{2F^2T^2}{\epsilon_t F},
\]
where $\gtrapprox$ is used to denote “approximately greater than or equal to”. Assuming $\epsilon_t < 1$ and using
\[
\log x \approx x \text{ for small } x, \text{ we have } \log \frac{2F^2T^2}{\epsilon_t} = \log 2F^2T^2 + F \log \frac{1}{\epsilon_t} \leq \frac{(\log 2F^2 + T^2)F}{\epsilon_t}.
\]
Now, we can

\[\text{The case of } \epsilon_t > 1 \text{ can be handled in a similar fashion, and hence ignored.}\]
use (27) to write $\epsilon_t$ in terms of $t$ to get

$$\epsilon_t \gtrapprox \sqrt{\frac{3k^2F^2c^2r^2(\log 2F^2T^2 + F)}{qNt}}.$$  

(28)

In other words, with a probability of at least $1 - \frac{1}{T^2}$, $\text{Reg}_k(t) \leq \sqrt{\frac{3k^2F^2c^2r^2(\log 2F^2T^2 + F)}{qN}}$. Using this result in (26), we get the following result. With a probability of at least $1 - \frac{1}{T}$,

$$\text{Reg}_{k,T} \gtrapprox \sqrt{\frac{3k^2F^2c^2r^2(\log 2F^2T^2 + F)}{qN}} \sum_{t=1}^{T} \frac{1}{t^{1/3}} = O(T^{2/3} \sqrt{\log T}).$$  

(29)

Thus, the above shows that the regret achieved grows sub-linearly with time, and hence (genie aided) achieves a zero asymptotic average regret.

**APPENDIX D**

**REGRET ANALYSIS FOR TWO SBS: HEURISTICS**

The analysis here is very similar to the analysis of single BS case. We repeat some of the analysis for the sake of clarity and completeness. Let the instantaneous regret at the BS $k$ at time $t$ is given by

$$\text{Reg}_k(t) := (u^*_k)^T P_k v^*_k \geq \sup_{(u,v) \in C_{c,r}} u^T_k P_k v_k,$$

at time $t$. Using the union bound, we can write

$$\Pr \left\{ \frac{1}{T} \sum_{t=1}^{T} \text{Reg}_k(t) \geq \frac{1}{T} \sum_{t=1}^{T} \epsilon_{k,t} \right\} \leq \sum_{t=1}^{T} \Pr \{ \text{Reg}_k(t) \geq \epsilon_{k,T} \},$$

(30)

where $\epsilon_{k,t} > 0$ and $\epsilon_{k,T} = \frac{1}{T} \sum_{t=1}^{T} \epsilon_{k,t}$. From Theorem 3.2, it follows that for any $\epsilon_{k,t} > 0$, we have $\Pr \{ \text{Reg}_k(t) \leq \epsilon_{k,t} \} \geq 1 - \delta$ provided (12) is satisfied. By choosing $\delta = \frac{1}{T^2}$, assuming $\epsilon_{k,t} < 1$, using the approximations $e^{-x} \approx 1 - x$ for small $x$, and $|N_k| \lesssim 1/\epsilon^F$, we have

$$\log \frac{2F^2T^2}{\epsilon_{k,t}} \leq \frac{\log 4F^2T^2 + F}{\epsilon_{k,t}}.$$  

Using this in Theorem 3.2, we get

$$\Pr \{ \text{Reg}_k(t) \leq \epsilon_{k,t} \} \geq 1 - \frac{1}{T^2}$$

provided

$$T \left( \frac{\epsilon_{k,t}}{\lambda_k} \frac{\delta}{2} \right) \gtrapprox \sqrt{\frac{8k^2F^2c^2r^2\lambda_k^3(\log 4F^2T^2 + F)}{qN\epsilon_{k,t}^2}},$$

(31)

where $\gtrapprox$ is used to denote “approximately greater than or equal to”. Using the above in (12), we get
\[
t \geq \max \left\{ \frac{8\kappa^2F^2c^2r^2\lambda_k^3(\log 4F^2T^2 + F)}{qN\epsilon_{k,t}^3}, \frac{8\kappa^2F^2c^2r^2(1 - \lambda_k)^3(\log 4F^2T^2 + F)}{qN\epsilon_{k,t}^3} \right\}.
\]

By rearranging and summing over \( t \), the error can be written as follows
\[
\epsilon_{k,t} \leq \frac{1}{\sqrt{t}} \max \left\{ \Theta \lambda_k, \Theta(1 - \lambda_k) \right\},
\]
where \( \Theta = \frac{3\sqrt{8\kappa^2F^2c^2r^2(\log 4F^2T^2 + F)}}{qN} \). Using this in the place of \( \epsilon_{k,t} \) in the above theorem, and summing over \( t \), we get with a probability of at least \( 1 - \frac{1}{T^2} \), the following holds for BS \( k \)
\[
\text{Reg}_{k,T} \leq \max \left\{ \Theta \lambda_k, \Theta(1 - \lambda_k) \right\} \frac{1}{\sqrt{T}} \sum_{t=1}^{T} 1 + 2T(1 - \lambda_k)\nu_{12},
\]
where \( \nu_{12} := \sup_{(u,v) \in C_{c,r}} |u^T(P_2 - P_1)v| \) and \( \Theta = \frac{3\sqrt{8\kappa^2F^2c^2r^2(\log 4F^2T^2 + F)}}{qN} \). This completes the approximate analysis.

A. Regret Analysis for Multiple SBS: Heuristics

The analysis here is again very similar to the analysis of single BS case. From Theorem 3.2, it follows that for any \( \epsilon_{k,t} > 0 \), we have \( \text{Pr}\{\text{Reg}_{k}(t) \leq \epsilon_{k,t}\} \geq 1 - \delta \), where \( \epsilon_{k,t} > 0 \) and \( \epsilon_k = \frac{1}{T} \sum_{t=1}^{T} \epsilon_{k,t} \). By choosing \( \delta = \frac{1}{T^2} \), assuming \( \epsilon_{k,t} < 1 \), using the approximation \( e^{-x} \approx 1 - x \) for small \( x \), and \( |N_\epsilon| \approx 1/\epsilon^F \), we have \( \log \frac{2F^2T^2}{\epsilon_{k,t}} \leq \frac{(\log 2F^2T^2 + F)}{\epsilon_{k,t}} \). Using this in Theorem 3.2, we get \( \text{Pr}\{\text{Reg}_{k}(t) \leq \epsilon_{k,t}\} \geq 1 - \frac{1}{T^2} \) provided
\[
\tau \left( \frac{\epsilon_1}{\lambda_1^{(k)}}, \frac{\delta}{M} \right) \geq \frac{8\kappa^2F^2c^2r^2\lambda_k^3(\log 4F^2T^2 + F)}{qN\epsilon_{k,t}^3},
\]
where \( \geq \) is used to denote “approximately greater than or equal to”. Using the above in (18), we get
\[
t \geq \max \left\{ \Theta^3 \epsilon_{k,t}^3, \Theta \lambda_1^{(k)}, \ldots, \Theta \lambda_M^{(k)} \right\},
\]
where, \( \Theta = \frac{3\sqrt{8\kappa^2F^2c^2r^2(\log 4F^2T^2 + F)}}{qN} \). From the above, it is clear that the waiting time \( t \) scales as the square of \( F, c \) and \( r \), and is inversely proportional to the error \( \epsilon_{k,t}^3 \). By rearranging and summing over \( t \), the error can be written as \( \epsilon_{k,t} \leq \frac{1}{\sqrt{t}} \max \left\{ \Theta \lambda_1^{(k)}, \ldots, \Theta \lambda_M^{(k)} \right\} \). Using this in
the place of $\epsilon_{k,t}$ in the above theorem, and summing over $t$, we get with a probability of at least $1 - \frac{1}{T}$ the following result on the regret for BS $k$ holds

$$\text{Reg}_{k,T} \lesssim \frac{\Theta M^2}{\sqrt{T}} \max \left\{ \lambda^{(k)}_1, \ldots, \lambda^{(k)}_M \right\} \sum_{t=1}^T 1 + M^2 T \left\{ (1 - \lambda^{(k)}_1) D_1 - \lambda^{(k)}_2 D_2 - \ldots, -\lambda^{(k)}_M D_M \right\}$$

**Remark:** Note that the value of regret depends on the values of $\lambda^{(k)}_1, \ldots, \lambda^{(k)}_M$ and the term $D_k$.

The first term scales as $T^{2/3}$ while the second term scales with $T$ linearly. This can be balanced by using $\lambda_1 = 1 - \frac{1}{\sqrt{T}}$ and $\lambda_k = \frac{1}{(M-1)\sqrt{T}}$, which results in $O(\sqrt{T})$ scaling of regret. Similar to the single SBS case, the choice $\lambda_k = \frac{1}{(M-1)\sqrt{T}}$ reveals that as time progresses, i.e., as the BS $k$ collects more samples, the weights allocated to the neighboring BS should go down to zero, as expected. Otherwise, one can optimize the above regret with respect to $\lambda_k$'s, and find the optimal choice.

**APPENDIX E**

**PROOF OF THEOREM 3.3**

Similar to the proof of Theorem 3.1 from [17], it follows that at time $t$, the performance gap of the proposed algorithm with respect to the optimal is given by

$$u^* P_k v^* - u^*_t P_k v_t \leq 2 \sup_{(u,v) \in \mathcal{C}_{c,r}} \left| u^T \hat{\Delta} P^{(t)} v \right|.$$

Summing the above over all $t$, we get

$$Tu^* P_k v^* - \sum_t u^*_t P_k v_t \leq 2 \sum_t \sup_{(u,v) \in \mathcal{C}_{c,r}} \left| u^T \hat{\Delta} P^{(t)} v \right|.$$

For a given $\epsilon$, the above implies that

$$\Pr \left\{ Tu^* P_k v^* - \sum_t u^*_t P_k v_t \geq \frac{\epsilon}{2} \right\} \leq \Pr \left\{ \sum_t \sup_{(u,v) \in \mathcal{C}_{c,r}} \left| u^T \hat{\Delta} P^{(t)} v \right| \geq \frac{\epsilon}{2} \right\}$$

$$= \Pr \left\{ \sum Y_t \geq \frac{\epsilon}{2} - \sum_t \mathbb{E} \left[ \sup_{(u,v) \in \mathcal{C}_{c,r}} \left| u^T \hat{\Delta} P^{(t)} v \right| \right] \right\}$$

(33)

where $Y_t = \sup_{(u,v) \in \mathcal{N}_e} \left| u^T \hat{\Delta} P^{(t)} v \right| - \mathbb{E} \left[ \sup_{(u,v) \in \mathcal{N}_e} \left| u^T \hat{\Delta} P^{(t)} v \right| \right]$ is a martingale difference, i.e., $\mathbb{E}\{Y_t\} = 0$, and $\mathcal{N}_e$ is the covering set of $\mathcal{C}_{c,r}$ as in Definition I. By Azuma’s inequality, we have
Since 

\[ \Pr \left\{ \sum_t Y_t > \frac{\epsilon}{2} \right\} \leq \exp \left\{ -\frac{\epsilon^2}{2(4rc)^2} \right\}. \]  

(34)

The above follows due to the fact that \( |Y_t| \leq 4rc \), which is explained below:

\[
Y_t \leq \sup_{(u,v)\in\mathcal{N}_c} |u^T \Delta P^{(t)} v| + \mathbb{E} \left[ \sup_{(u,v)\in\mathcal{N}_c} |u^T \Delta P^{(t)} v| \right] \\
\leq \sup_{(u,v)\in\mathcal{N}_c} |u^T P v| + \sup_{(u,v)\in\mathcal{N}_c} |u^T P^{(t)} v| + \mathbb{E} \left[ \sup_{(u,v)\in\mathcal{N}_c} |u^T P v| \right] + \mathbb{E} \left[ \sup_{(u,v)\in\mathcal{N}_c} |u^T P^{(t)} v| \right] \\
\leq 4rc, 
\]

(35)

which follows from \( |Y_t| \leq 4rc \) and \( \sup_{(u,v)\in\mathcal{C}_{c,r}} |u^T P v| \leq rc \). Thus it follows from (34), \( \Pr \left\{ \sum_t Y_t > \frac{\epsilon}{2} \right\} \leq \delta \) if \( \epsilon \geq 32r^2c^2T \log(1/\delta) \). Using this definition of \( |Y_t| \), it follows that with a probability of at least \( 1 - \delta \), we have

\[
\sum_t \sup_{(u,v)\in\mathcal{N}_c} |\bar{u}^{\Delta P^{(t)}} v| \geq 2 \sum_t \mathbb{E} \left[ \sup_{(u,v)\in\mathcal{C}_{c,r}} |u^T \Delta P^{(t)} v| \right] + \sqrt{128r^2c^2T \log(1/\delta)}.
\]

Choosing \( \epsilon = 32r^2c^2T \log(1/\delta) \) in (33), the following bound for regret is satisfied with a probability of at least \( 1 - \delta \):

\[
u^* \hat{P}_k v^* - \frac{1}{T} \sum_{t=1}^T u_t^T \hat{P}_k v_t < 2 \sum_t \mathbb{E} \left[ \sup_{(u,v)\in\mathcal{C}_{c,r}} |u^T \Delta P^{(t)} v| \right] + \sqrt{128r^2c^2 \log(1/\delta)}.
\]

(36)

Now, it remains to bound the first term on the right hand side above. For a given \( \psi_t > 0 \) (to be chosen later), using total expectation rule, we get

\[
\sum_t \mathbb{E} \left[ \sup_{(u,v)\in\mathcal{N}_c} |u^T \Delta P^{(t)} v| \right] \leq \sum_t \mathbb{E} \left[ \sup_{(u,v)\in\mathcal{N}_c} |u^T \Delta P^{(t)} v|, \sup_{(u,v)\in\mathcal{N}_c} |u^T \Delta P^{(t)} v| > \psi_t \right] \\
\times \Pr \left\{ \sup_{(u,v)\in\mathcal{N}_c} |u^T \Delta P^{(t)} v| > \psi_t \right\} + \sum_t \psi_t \\
\leq \sum_{t} rc \max_{kl} \mathbb{P}_{kl} \Pr \left\{ \sup_{(u,v)\in\mathcal{N}_c} |u^T \Delta P^{(t)} v| > \psi_t \right\} + \sum_t \psi_t,
\]

where the first inequality above follows by using the bound \( \sup_{(u,v)\in\mathcal{C}_{c,r}} |u^T \Delta P^{(t)} v| \leq \psi_t \). Since \( \sup_{(u,v)\in\mathcal{C}_{c,r}} |u^T \Delta P^{(t)} v| > \sup_{(u,v)\in\mathcal{N}_c} |u^T \Delta P^{(t)} v| \), we get

\[
\sum_t \mathbb{E} \left[ \sup_{(u,v)\in\mathcal{N}_c} |u^T \Delta P^{(t)} v| \right] \leq rc|\mathcal{N}_c| \max_{kl} \mathbb{P}_{kl} \sum_t \Pr \left\{ \sup_{(u,v)\in\mathcal{N}_c} |u^T \Delta P^{(t)} v| > \psi_t \right\} + \sum_t \psi_t.
\]

(37)
Now, consider
\[
\sum_t \Pr \left\{ \sup_{(u,v) \in \mathcal{N}_t} \left| u^T \Delta \tilde{P}^{(t)} v \right| > \psi_t \right\} \leq \sum_t \Pr \left\{ \sum_{i,j=1}^{\lvert \mathcal{N}_t \rvert} u_i^T \tilde{P}^{(t)} v_j > \psi_t \right\}
\]
where \( X_j := \sum_{l=1}^{\lvert \mathcal{N}_t \rvert} v_l \Delta \tilde{P}^{(t)}_{ij} \). In the above, (a) follows from the covering argument, and (b) follows from the Chernoff bound. From [18], using the optimal proxy variance, we get the following bound
\[
\sum_t \Pr \left\{ \sup_{(u,v) \in \mathcal{N}_t} \left| u^T \Delta \tilde{P}^{(t)} v \right| > \psi_t \right\} \leq \sum_t \exp \left\{ -s\psi_t + s^2 \sum_{i,j,l=1}^{\lvert \mathcal{N}_t \rvert} \frac{\lVert u_i \rVert^2 \lVert v_j \rVert^2 \sigma_j^2}{2} \right\},
\]
where an upper bound on \( \sigma_t \) (see [18]) is given by \( \sigma_j \leq \frac{1}{4(\sum_i a_{ij}^{(q)} + 1)} \) and \( \alpha_{ij}^{(t)} = \sum_{q=1}^{t-1} d_i^{(q)} v_{ij}^{(q-1)} \).

Optimizing the exponent in (39), the optimal \( s^* = \frac{\psi_t}{\sum_{i,j,l=1}^{\lvert \mathcal{N}_t \rvert} \lVert u_i \rVert^2 \lVert v_j \rVert^2 \sigma_j^2} \). Further, \( \lVert u_i \rVert^2 \leq \lVert u_{it} \rVert \leq c \)
and \( \lVert v \rVert^2_t \leq r \lvert \mathcal{N}_t \rvert^2 \). Using these bounds, and the bound on \( \sigma_t \) above, (39) can be written as follows
\[
\sum_t \Pr \left\{ \sup_{(u,v) \in \mathcal{N}_t} \left| u^T \Delta \tilde{P}^{(t)} v \right| > \psi_t \right\} \leq \sum_t \exp \left\{ - \frac{8\psi_t^2}{cr |\mathcal{N}_t|^2 \hat{\sigma}^2(t)} \right\},
\]
where \( \hat{\sigma}^2(t) := \left[ \frac{1}{\sum_j \left( \frac{1}{\sum_i a_{ij}^{(q)} + 1} \right)^2} \right] \). Substituting the above in (37) results in
\[
\sum_t \mathbb{E} \left[ \sup_{(u,v) \in \mathcal{N}_t} \left| u^T \Delta \tilde{P}^{(t)} v \right| \right] \leq rc \max_{ij} p_{ij} |\mathcal{N}_t| \sum_t \exp \left\{ - \frac{8\psi_t^2}{cr |\mathcal{N}_t|^2 \hat{\sigma}^2(t)} \right\} + \sum_t \psi_t.
\]
Thus, using the above, the regret can be written as
\[
\text{Reg}_T \leq 2rc \max_{ij} p_{ij} |\mathcal{N}_t| \sum_t \exp \left\{ - \frac{8\psi_t^2}{cr |\mathcal{N}_t|^2 \hat{\sigma}^2(t)} \right\} + 2 \sum_t \psi_t + \sqrt{128r^2c^2T \log(1/\delta)}.
\]
APPENDIX F

PROOF OF THEOREM 4.1

The analysis is done only for the first SBS as the analysis for the second SBS is similar. As in \(^{(21)}\), since \(\Pr\left\{ (u_{k,t}^*)^T P_k v_{k,t}^* \geq \sup_{(u,v) \in C_{c,r}} u^T \hat{\Delta} P^{(t)} v - \epsilon \right\} \leq \Pr\left\{ 2 \sup_{(u,v) \in C_{c,r}} \left| u^T \hat{\Delta} P^{(t)} v \right| > \epsilon \right\}\), it is sufficient to consider the following

\[
\sup_{(u,v) \in C_{c,r}} \left| u^T \hat{\Delta} P^{(t)} v \right| = \sup_{(u,v) \in C_{c,r}} \left| u^T (P_1 - \hat{Q}_1^{(t)}) v \right| = \lambda_1 \sup_{(u,v) \in C_{c,r}} \left| u^T \Delta P_1^{(t)} v \right| + (1 - \lambda_1) \sup_{(u,v) \in C_{c,r}} \left| u^T \Delta P_2^{(t)} v \right| + (1 - \lambda_1) u^T (P_1 - P_2) v \leq \lambda_1 \mathcal{V}_1 + (1 - \lambda_1) \mathcal{V}_2 + (1 - \lambda_1) \mathcal{V}_{12},
\]

(43)

where \(\mathcal{V}_1 := \sup_{(u,v) \in C_{c,r}} \left| u^T \Delta P_1^{(t)} v \right|\), \(\mathcal{V}_2 := \sup_{(u,v) \in C_{c,r}} \left| u^T \Delta P_2^{(t)} v \right|\), and \(\mathcal{V}_{12} := \sup_{(u,v) \in C_{c,r}} \left| u^T (P_2 - P_1) v \right|\). Here, \(\Delta P^{(t)} := P_1 - \hat{Q}_1^{(t)}\), \(\Delta P_k^{(t)} := P_k - \hat{P}_k^{(t)}\), \(k = 1, 2\).

Using the union bound, we get the following

\[
\Pr\{\lambda_1 \mathcal{V}_1 + (1 - \lambda_1) \mathcal{V}_2 > \epsilon_1\} \leq \Pr\left\{ \mathcal{V}_1 > \frac{\epsilon_1}{\lambda_1} \right\} + \Pr\left\{ \mathcal{V}_2 > \frac{\epsilon_1}{1 - \lambda_1} \right\},
\]

(44)

where \(\epsilon_1 := \epsilon/2 - (1 - \lambda_1) \mathcal{V}_{12}\). Using results from Theorem 3.2 to each of the above term with \(\epsilon\) replaced by \(\epsilon_1/\lambda_1\) and \(\epsilon_1/(1 - \lambda_1)\) with \(\delta\) replaced by \(\delta/2\) proves the theorem.

APPENDIX G

PROOF OF THEOREM 4.2

Similar to the proof provided of Theorem 4.1, the analysis is done only for the first SBS using Bayesian estimate.

\[
Tu^* P_1 v^* - \sum_t u_i^T \hat{Q}_k^{(t)} v_t \leq \lambda_1 \mathcal{U}_1 + (1 - \lambda_1) \mathcal{U}_2 + (1 - \lambda_1) \mathcal{U}_{12},
\]
where $U_1 := \sum_{t} \sup_{(u,v) \in C_{c,r}} |u^T \Delta P_1^{(t)} v|$, $U_2 := \sum_{t} \sup_{(u,v) \in C_{c,r}} |u^T \Delta P_2^{(t)} v|$, and

$U_{12} := \sum_{t} \sup_{(u,v) \in C_{c,r}} |u^T (P_1 - P_2) v|$. Here, $\Delta P_k^{(t)} := P_k - \hat{P}_k^{(t)}$, $k = 1, 2$. Consider the following

$$
\Pr\{Tu^* P_1 v^* - \sum_t u_t^T \hat{Q}_1^{(t)} v_t \geq \epsilon\} \leq \Pr\{\lambda_1 U_1 + (1 - \lambda_1) U_2 + (1 - \lambda_1) U_{12} \geq \epsilon\}
$$

\leq \Pr\left\{U_1 > \frac{\epsilon_1}{\lambda_1}\right\} + \Pr\left\{U_2 > \frac{\epsilon_1}{1 - \lambda_1}\right\},

where $\epsilon_1 := \epsilon/2 - (1 - \lambda_1) U_{12}$. Using results from Theorem 3.3 to each of the above term with $\epsilon$ replaced by $2\epsilon_1/\lambda_1$ and $\delta$ replaced by $\delta/2$, we get the desired result.

**APPENDIX H**

**PROOF OF THEOREM 4.3**

The proof for multiple SBSs is a generalization of two SBSs and the analysis is done only for the first SBS, the rest of the SBSs are similar.

$$
\sup_{(u,v) \in C_{c,r}} |u^T \Delta P_1^{(t)} v| = \sup_{(u,v) \in C_{c,r}} |u^T (P_1 - \lambda_1^{(k)} \hat{P}_1^{(t)} - \lambda_2^{(k)} \hat{P}_2^{(t)} + \ldots - \lambda_M^{(k)} \hat{P}_M^{(t)}) v|
$$

$$
= \sup_{(u,v) \in C_{c,r}} |\lambda_1^{(k)} u^T (P_1 - \hat{P}_1^{(t)}) v + \lambda_2^{(k)} u^T (P_2 - \hat{P}_2^{(t)}) v + \ldots + \lambda_M^{(k)} u^T (P_M - \hat{P}_M^{(t)}) v|
$$

$$
+ (1 - \lambda_1^{(k)}) \left\{ \sup_{(u,v) \in C_{c,r}} |u^T P_1 v| \right\} \ldots - \lambda_M^{(k)} \left\{ \sup_{(u,v) \in C_{c,r}} |u^T P_M v| \right\}
$$

\leq \lambda_1^{(k)} V_1 + \lambda_2^{(k)} V_2 + \ldots, + \lambda_M^{(k)} V_M + (1 - \lambda_1^{(k)}) D_1 - \lambda_2^{(k)} D_2 - \ldots, - \lambda_M^{(k)} D_M|

where $V_l := \sup_{(u,v) \in C_{c,r}} |u^T \Delta P_l^{(t)} v|$, $l = 1, 2, \ldots, M$, $D_1 := \sup_{(u,v) \in C_{c,r}} |u^T P_1 v|$, $D_2 := \sup_{(u,v) \in C_{c,r}} |u^T P_2 v|$, $\ldots$, $D_M := \sup_{(u,v) \in C_{c,r}} |u^T P_M v|$. Here, $\Delta P_k^{(t)} := P_k - \hat{P}_k^{(t)}$, $\Delta P_k^{(t)} := P_k - \hat{P}_k^{(t)}$, $k = 1, 2, \ldots, M$. Thus we can write the following:

$$
\Pr\{\lambda_1^{(k)} V_1 + \lambda_2^{(k)} V_2 + \ldots, + \lambda_M^{(k)} V_M > \epsilon\} \leq \sum_{l=1}^{M} \Pr\left\{V_l > \frac{\epsilon_l}{\lambda_l^{(k)}}\right\},
$$

where $\epsilon' := \epsilon/M - (1 - \lambda_1^{(k)}) D_1 + \lambda_2^{(k)} D_2 + \ldots, + \lambda_M^{(k)} D_M$, and $\epsilon_k := \epsilon/M^2 - (1 - \lambda_1^{(k)}) D_1 + \lambda_2^{(k)} D_2 + \ldots, + \lambda_M^{(k)} D_M$. Using results from Theorem 3.2 to each of the above term with $\epsilon$ replaced by $\epsilon'$ and $\epsilon_k$ and $\delta$ replaced by $\delta/M$ proves the theorem.
APPENDIX I

PROOF OF THEOREM 4.4

The proof for multiple SBSs is a generalization of two SBSs and the analysis is done only for the first SBS for Bayesian estimate, the rest of the SBSs are similar. First consider the following

\[ T u^* P_1 v^* - \sum_t u_t^T P_1 v_t \leq \sum_{j=1}^M \lambda_j^{(k)} W_j + (1 - \lambda_1^{(k)}) I_1 - \sum_{j=2}^M \lambda_j^{(k)} I_j, \quad (38) \]

where \( W_j := \sum_t \sup_{(u,v) \in C_{c,r}} \left| u^T \hat{\Delta} P_j^{(t)} v \right|, I_j := \sum_t \sup_{(u,v) \in C_{c,r}} \left| u^T P_j v \right|, j = 1, 2, \ldots, M. \) Here, \( \hat{\Delta} P_k^{(t)} := P_k - \hat{P}_k^{(t)}, k = 1, 2, \ldots, M. \) Thus we can write the following:

\[ \Pr\{\lambda_1^{(k)} W_1 + \ldots + \lambda_M^{(k)} W_M \geq \epsilon'\} \leq \sum_{l=1}^M \Pr\left\{ W_l > \frac{\epsilon_l}{\lambda_l^{(k)}} \right\}, \quad (39) \]

where \( \epsilon' := \epsilon/M - (1 - \lambda_1^{(k)}) I_1 + \sum_{l=2}^M \lambda_l^{(k)} I_k, \) and \( \epsilon_k := \epsilon/M^2 - (1 - \lambda_1^{(k)}) I_1 + \sum_{l=2}^M \lambda_l^{(k)} I_k, \) for all \( k = \{1, 2, \ldots, M\}. \) Using results from Theorem 3.3 to each of the above term with \( \epsilon \) replaced by \( \epsilon' \) and \( \epsilon_k \) and \( \delta \) replaced by \( \delta/M, \) we get the regret bound described in the theorem.