In this paper, we demonstrate that in frustrated magnets when several conventional (i.e., symmetry-breaking) orders compete, and are “intertwined” by a Wess-Zumino-Witten (WZW) term, the possibility of spin liquid arises. The resulting spin liquid could have excitations which carry fractional spins and obey non-trivial self/mutual statistics. As a concrete example, we consider the case where the competing orders are the Néel and valence-bond solid (VBS) order on square lattice. Examining different scenarios of vortex condensation from the VBS side, we show that the intermediate phases, including spin liquids, between the Néel and VBS order always break certain symmetry. Remarkably, our starting theory, without fractionalized particles (partons) and gauge field, predicts results agreeing with those derived from a parton theory. This suggests that the missing link between the Ginzberg-Landau-Wilson action of competing order and the physics of spin liquid is the WZW term.

I. INTRODUCTION

In the Landau paradigm, the phases of matter are characterized by broken symmetries. In the last two decades, it is recognized that the phases of quantum material might not follow this paradigm. Consequently a frontier of condensed matter physics is the study of orders without broken symmetry. This includes symmetry-protected topological (SPT) order, such as topological insulator and superconductor [1–3], and “intrinsic topological order” such as that exhibited by spin liquids [4].

Generally speaking, in 2+1 dimensions, theories of non-chiral spin liquids (for early examples, see Ref.[5, 6]) have two key ingredients. (i) Spins fractionalize into partons. These partons usually couple to a gauge field, which tends to glue the partons back into the physical spins; (ii) A Higgs phenomenon, in which Higgs bosons carrying multiples of the fundamental gauge charge condense. After (ii) the gauge field is (partially) gapped and the partons are deconfined [5]. Moreover, the ungapped gauge flux has non-trivial mutual statistics with the particle carrying the fundamental gauge charge. In fact, (ii) can happen without (i). For example, the condensation of double vortices in a theory of U(1) bosons gaps the gauge field (which is dual to original boson phase), so that the gauge group becomes $\mathbb{Z}_2$. The $\mathbb{Z}_2$ flux has mutual $-1$ Berry’s phase with the single vortex.

In this paper we take a different perspective, where the starting point is a Ginzburg-Landau-Wilson action describing the competition of conventional orders (hence no partons and no gauge field). However, the action contains a WZW term that “intertwines” different order parameters. In the concrete example we study, the competing orders are the Néel and valence-bond-solid (VBS) order of a frustrated magnet. The fractional spin is carried by the VBS vortex [7–9], the phase of the VBS order parameter acts as a gauge field, and spin liquid is triggered by the condensation of double vortices.

The example described above is relevant to the frustrated spin 1/2 Heisenberg model. For the square lattice $J_1-J_2$ Heisenberg model, numerical evidence suggests the ground state evolves from Néel to VBS order as a function of increasing $J_2/J_1$. Interestingly, for a narrow interval of $J_2/J_1$ neither order is present. It is widely felt that some sort of quantum spin liquid could be present in this $J_2/J_1$ interval. This includes the critical $U(1)$ spin liquid associated with the “deconfined quantum critical point” [10], as proposed in Ref.[11], and gapped [12] or gapless spin liquid [13–17] phases.

This paper begins by deriving the theory of Ref.[8], namely, the VBS-Néel $O(5)$ non-linear sigma model with a WZW term (Eq.(15)). We start with the famous parton theory (the $\pi$-flux mean-field theory) of Affleck and Marston [18]. Here the partons are massless Dirac fermions. These partons couple to a $SU(2)$ gauge field which, as we shall show, is confining. In Ref.[8], it is shown that a WZW term exists among the five mass terms of these massless fermions. However, this is done without worrying about the $SU(2)$ gauge field. In this paper, we begin by reviewing a derivation of the theory in Ref.[8] in the presence of confining $SU(2)$ gauge field fluctuations [19].

In a nutshell, the derivation amounts to the following.
Without worrying about the $SU(2)$ gauge field, there are 36 possible mass terms, each corresponds to a parton-antiparton bound state. Borrowing the language of QCD, the partons are quarks and the mass terms are mesons. After the $SU(2)$ confinement, only 5 mesons survive low energies. They correspond to the Néel and VBS order parameters in Ref.[8]. Importantly, it is shown that the WZW term survives the confinement, leading to a five-component non-linear sigma model with a WZW term (Eq.(15)).

The WZW term predicts the the vortex of the VBS order possesses fractional spins ($S = 1/2$) [7–9]. We shall present a microscopic understanding of this spin fractionalization via the fermion zero modes, and show that it manifests the WZW term. Moreover, when the same WZW term is evaluated for the space-time process involving the exchange of vortices, it predicts the statistics of the vortex is bosonic. This is also confirmed by the fermion integration. Motivated by the double-vortex condensation in the $U(1)$ boson theory, we consider different sequences of VBS vortex condensation. Here we show the condensation of double VBS vortices triggers spin liquids. However, as we shall see, the spin liquids so obtained always break certain symmetry.

Our results are summarized by Fig. 1, Fig. 2 and Fig. 3. In each figure there are two alternative paths (marked as green and yellow) leading from VBS to Néel order. The green path realizes the direct transition between VBS and Néel order. It passes through the “deconfined critical point” (marked by the red cross) of Ref.[10]. The yellow paths, on the other hand, involve intermediate phases. These include different types of $Z_2$ spin liquids which break either the lattice, time reversal or the spin rotation symmetry. They are accompanied by phases that break spin rotation symmetry. Here the red crosses mark the respective phase transitions. Remarkably, the scenario in Fig. 1 was predicted by the Schwinger-boson parton theory of Sachdev and Read [6].

It is important to point out that while the scenarios presented in Fig. 1 and Fig. 2 require the existence of further-neighbor, same-sublattice, spin interaction which favors spin singlet. In contrast scenario in Fig. 3 requires the interaction to favor spin triplet. Therefore, Fig. 1 and Fig. 2 are more relevant for the frustrated Heisenberg model.

The organization of this paper is as follows. In section II we briefly review the derivation of the Néel-VBS non-linear sigma model with a WZW term. Here we start from partons which are 4-flavor Dirac fermions (it turns out that it is more convenient to view them as 8-flavor Majorana fermions) representing the quasiparticles of the $\pi$-flux phase [18]. These fermions couple to a confining charge-$SU(2)$ gauge field [20]. We will treat such confinement. In section III, we derive the spin (with $S = 1/2$) in the core of a single VBS vortex. Here we take two alternative approaches. In subsection III A, we determine the Berry phase of the vortex core from the WZW term, and identify it with the Berry phase of $S = 1/2$ in 0+1 dimension. In subsection III B, we solve for the fermion zero modes in the vortex core, and show that the occupation of these zero modes leads to $S = 1/2$ which is a charge-$SU(2)$ singlet, hence survives the confinement. The consistent results
of subsections IIIA and IIIB imply that these two approaches are equivalent, i.e., the fermion zero mode is the manifestation of the WZW term. In section IV we determine the statistics of the VBS vortex by computing the Berry phase associated with exchanging two vortices. In IV A we compute this phase using the WZW term, and in IV B we compute it by fermion integration. Again, the consistency of the results in subsections IV A and IV B implies the equivalence of the two approaches. In section V we determine the statistics of the VBS vortex by using Majorana fermions

$$S_i^a = \frac{1}{2} \xi_{iab} \xi_{iab} f_i^a f_i^b.$$ (1)

The Hilbert space of spins is recovered after imposing the single occupation constraints

$$f_i^a f_i^b + f_i^b f_i^a = 1$$
$$f_i^a f_i^a = 0$$
$$f_i^a f_i^a = 0.$$ (2)

Because the last two lines of Eq.(2) break the spinon number conservation, it’s convenient to rewrite the spinon operators using Majorana fermions

$$f_i^a := F_{i,1a} + iF_{i,2a}.$$ (3)

In terms of the Majorana operators, the spin operators are represented as

$$S_i^a = \frac{1}{2} F_i^\dagger \Sigma^a F_i,$$ where

$$\Sigma^a = (-YX, IY, -YZ),$$ (4)

where $F_i$ has four components. In the last line of Eq.(3), $X,Y,Z$ stand for Pauli matrices $\sigma_0, \sigma_x, \sigma_y, \sigma_z$, and two Pauli matrices standing next to each other denotes tensor product. In Eq.(3), the first and second Pauli matrices carry the Majorana and spin indices respectively. Using $F_i$ the occupation constraints read

$$F_i^T (XY) F_i := F_i^T T_i F_i = 0$$
$$F_i^T (ZY) F_i := F_i^T T_i F_i = 0$$
$$F_i^T (-YI) F_i := F_i^T T_i F_i = 0.$$ (5)

In the Mott insulating phase, the low energy Hamiltonian is a function of spin operators. Expressed in terms of the parton (Majorana fermion) operators, even the simplest bilinear spin-spin interaction involves four fermion operators. By (i) introducing the auxiliary fields $\chi_{ij}$ and $\Delta_{ij}$ that decouple such four fermion operators into fermion bilinear, and (ii) impose the constraint in Eq.(4) by space-time dependent Lagrange multipliers $a_{10}^\dagger, a_{20}^\dagger, a_{30}^\dagger$, we obtain the following spinon path integral [21]

$$Z = \int D[F] D[\chi] D[\Delta] D[a_0] \exp (-S).$$
\[
S = \int_0^\beta d\tau \left\{ \sum_i F_i^T \partial_\tau F_i + \sum_{\langle ij \rangle} \frac{3}{8} J_{ij} \left[ F_i^T \left( Re[\chi_{ij}] Y I \right) + i Im[\chi_{ij}] I I + Re[\Delta_{ij}] X Y - Im[\Delta_{ij}] Y Z \right] F_j \right. \\
+ |\chi_{ij}|^2 + |\eta_{ij}|^2 \left. \right] - i \sum_i a_i^0 \left( F_i^T T^b F_i \right) \right\}. \tag{5}
\]

In Eq. (5) \( J_{ij} \) is the exchange constant between spins on site \( i \) and \( j \).

### A. The \( \pi \)-flux saddle point

The Affleck-Marston \( \pi \)-flux phase is a saddle point solution of Eq. (5) where \( \{ J_{ij} \} \) only extends to the nearest neighbors\(^1\). It corresponds to

\[
\Delta_{ij} = 0, \quad \hat{\chi}_{i+\hat{x},i} = i\chi, \quad \hat{\chi}_{i+\hat{y},i} = i(-1)^{i\pi} \chi, \quad a_i^{1,2,3} = 0. \tag{6}
\]

where \( \chi \) is a real number. The hopping pattern in Eq. (6) is shown in Fig. 4, where the arrows point in the direction of \(+i\chi \) hopping.

Choosing a 4-site unit cell and define \( \psi^T = (F_1, F_2, F_3, F_4)^T \), the momentum-space mean-field Hamiltonian read

\[
\hat{H}_{\text{MF}} = -\frac{3}{8} J\chi \sum_k \psi_k^T \left[ [I I \otimes h(k)] \psi_k \right], \tag{7}
\]

where

\[
h(k) = \begin{bmatrix}
0 & -i(1 - e^{ik_1}) & -i(1 - e^{ik_2}) & 0 \\
i(1 - e^{-ik_1}) & 0 & 0 & i(1 - e^{ik_2}) \\
i(1 - e^{-ik_2}) & 0 & 0 & -i(1 - e^{ik_1}) \\
0 & -i(1 - e^{-ik_2}) & i(1 - e^{-ik_1}) & 0
\end{bmatrix}
\]

The fermion dispersion is given by

\[
E(\mathbf{k}) = \pm \frac{3J}{8} \chi \sqrt{4 - 2(\cos k_1 + \cos k_2)},
\]

where each branch has 8-fold degeneracy.

The fermion low energy modes occur near the node \( \mathbf{k}_0 = 0 \). Writing \( \mathbf{k} = \mathbf{k}_0 + \mathbf{q} \) and expanding the mean-field Hamiltonian to linear order in \( \mathbf{q} \) we obtain

\[
\hat{H}_{\text{MF}} = \frac{3J}{4} \chi \sum_q \psi^T(-\mathbf{q})(q_1 \Gamma_1 + q_2 \Gamma_2) \psi(\mathbf{q}), \tag{8}
\]

\[\Gamma_1 = IIIX, \quad \Gamma_2 = IIXZ. \tag{9}\]

In Eq. (9) the first two Pauli matrices correspond to the Majorana and spin degrees of freedom. The indices of the tensor product of the last two Pauli matrices label the \( 2 \times 2 \) unit cell in Fig. 4, namely,

- \( ZI = +1 \), \( IZ = +1 \) : site 1
- \( ZI = +1 \), \( IZ = -1 \) : site 2
- \( ZI = -1 \), \( IZ = +1 \) : site 3
- \( ZI = -1 \), \( IZ = -1 \) : site 4

Fourier transforming the Hamiltonian back to the real space, we obtain

\[
\hat{H}_{\text{MF}} = \frac{3J}{16} \chi^2 \int d^2 r \psi^T(r) (-i \Gamma_2 \partial_j) \psi(r), \tag{10}
\]

where \( \psi(r) = \sum_q \psi(q)e^{iqr} \). Eq. (10) is the point of departure for the following discussions.

### B. The \( SU(2) \) gauge field and the confinement

Because the operators in Eq. (3) are invariant under the following local “charge-\( SU(2) \)” transformation

\[
F_j \rightarrow e^{i \delta \theta_j} F_j, \quad F_b = (X Y, Z Y, - Y I) \text{ or }
\psi(r) \rightarrow e^{i \delta \theta(r) T^b} \psi(r), \quad T^b = (X Y I, Z Y I, - Y I I I), \tag{11}
\]

we expect that the spinon field theory should involve a charge-\( SU(2) \) gauge field. We stress that this \( SU(2) \) is different from the spin \( SU(2) \), which is generated by

\[
\Sigma^a = (-Y X I I, I Y I I, - Y Z I I).
\]

\(^1\) In the following we shall start with the nearest-neighbor interaction and derive Eq. (15). The further neighbor interactions will be incorporated later on, in the non-linear sigma model as the \( \cdots \) in Eq. (29). This approach is useful so long as the Néel and VBS remain to be the most prominent competing order parameters.
The spatial components of the charge $SU(2)$ gauge field are identified as

$$U_{ij} = \tilde{U}_{ij} e^{ia_{ij}},$$

where $a_{ij} = a_{ij}^b T^b$, while $a_{0i}^b$ act as the time component of the charge-$SU(2)$ gauge field [21]. Thus, the continuum version of the low energy theory is given by

$$S = \int d^d x d^d y \left\{ \psi^T \left[ \left( \partial_0 - ia_0 \right) - \frac{3J}{16} \right] \Gamma_i \left( \partial_i - ia_i \right) \psi \right\} + \frac{1}{2g^2} \text{Tr} \left\{ f_{\mu\nu} f_{\mu\nu} \right\}$$

(12)

where $\Gamma_i$ is given by Eq.(9), and $a_\mu = a_{\mu,b} T^b$ with $T^b$ given by integrating out the high energy fermion degrees of freedom.

Via non-abelian bosonization [19], it is shown in the absence the SU(2) gauge field, there are 35 mass (gap-opening) terms for Eq.(8) relevant to the discussion here. Each of these mass terms has the form

$$\psi^T M_4 \psi$$

and corresponds to a spinon-antispinon bilinear. Borrowing the language of QCD, the order parameters correspond to these meson fields. Under the constraint that the total gap is a constant, the meson fields form the manifold $\Omega$, and the $\Omega(8)$ non-linear sigma model with a WZW term governs the dynamics of the mesons. After integrating out the charge-$SU(2)$ gauge field, the spinons are confined into five $SU(2)$ singlet mesons with the $M_i$ in Eq.(13) given by

$$M_i = Y X Z Z, I Y Z Z, Y Z Z Z, I I I Y, I I Y Z.$$  

(14)

These mesons form an $S^4$ sub-manifold in $\Omega$. Substituting these meson fields into the $\Omega$, non-linear sigma model, we obtain Eq.(15),

$$W[\tilde{\Omega}] = \frac{1}{2g^2} \int d^2 x \left( \partial_\mu \Omega_i \right)^2 + W_{\text{WZW}}[\tilde{\Omega}].$$

(15)

Here the order parameter $\tilde{\Omega} = (n_1, n_2, n_3, v_1, v_2)$ is a five-component unit vector associated with the masses $(M_1, M_2, M_3, M_4, M_5)$ in Eq.(14). Because $M_{1,2,3}$ are the masses that correspond to the Néel order, and $M_{4,5}$ are to the VBS order, (see Fig. 5), we identify Eq.(15) as the non-linear sigma model governing the Néel-VBS competition. Due to the charge-$SU(2)$-singlet nature of the masses, Eq.(15) is immune to the $SU(2)$ gauge fluctuations!

We stress that aside from the WZW term, Eq.(16) is the “fixed-length” version of the Ginzberg-Landau-Wilson action of five competing order parameters. $W_{\text{WZW}}$, given by Eq.(16), is pure imaginary with quantum mechanical origin. As we shall see, it is the missing link between classical competing order and spin liquid physics.

**FIG. 5:** Spin configurations associated with the VBS and Néel order. The green box marks a unit cell. The orange ellipse denotes spin-singlets. The first four figures represent the $SU(2)$ singlet order in specific $(n_1, n_2, n_3, v_1, v_2)$. Figure (e) is the Néel order in specific $(n_1, n_2, n_3)$ direction.

**C. The Wess-Zumino-Witten term**

The WZW term is a Berry’s phase, namely,

$$W_{\text{WZW}}[\tilde{\Omega}] = -\frac{2\pi i}{64\pi^2} \int \epsilon^{ijklm} \tilde{\Omega}_i \tilde{d}\tilde{\Omega}_j \tilde{d}\tilde{\Omega}_k \tilde{d}\tilde{\Omega}_l \tilde{d}\tilde{\Omega}_m.$$  

(16)

In Eq.(16) $\tilde{\Omega}(\tau, x, y, u)$ represents a one-parameter-family extension of the space-time configuration $\tilde{\Omega}(x, y, \tau)$. At $u = 0$, the configuration is trivial, say, $\tilde{\Omega}(\tau, x, y, 0) = (0, 0, 0, 0, 1)$. At $u = 1$ the $\tilde{\Omega}$ realizes the physical configuration. It can be shown that $\exp \left( -W_{\text{WZW}} \right)$ is independent of the specific choice of the one-parameter family as long as the coefficient in Eq.(16) is an integer multiple of $\frac{2\pi i}{64\pi^2}$.  

2 In order for the interpolation between the trivial and non-trivial
III. THE SPIN OF A VBS VORTEX

A. The WZW term restricted to the vortex core

In this subsection, we show that the WZW term in Eq.(16) predicts that the core of a VBS vortex harbors a spin with $S = 1/2$.

In a VBS vortex configuration, $\hat{\Omega} = (0, 0, 0, v_1, v_2)$ with $v_1^2 + v_2^2 = 1$ everywhere except close to the vortex core, where $v_1^2 + v_2^2 \to 0$ while $n_3^2 + n_3^2 \to 1$. Let’s compute the Berry phase associated with a single vortex loop, which is a one-dimensional closed curve parametrized by $\tau_v$ (the blue dashed line in Fig. 6). The core region, drawn as the gray tube in Fig. 6, has radius $r_c$. Thus the tube region forms $S^1 \times D^2$ where $S^1$ is the vortex loop and $D^2$ is the 2D disk with radius $r_c$. Within the tube, $(n_1, n_2, n_3)$ has non-zero magnitude.

Let’s use $r$ and $\phi$ to parametrize the local radial distance from the vortex loop and the angle around it (see Fig. 6). The configuration of $\hat{\Omega}$ is given by

$$\hat{\Omega}(r, \phi, \tau_v) = \left( \sqrt{1 - f^2(r)} \hat{n}(\tau_v), f(r) \cos \phi, f(r) \sin \phi \right)$$

(17)

where $f(r)$ is a smooth function satisfying $f(r) \to 1$ for $r > r_c$, and $f(r) \to 0$ as $r \to 0$. Since the vortex core is microscopic in size, in Eq.(17) we have assumed that there is no spatial (i.e., $r, \phi$) variation of $\hat{n}$. Nonetheless, the direction of $\hat{n}$ can vary along the vortex loop (i.e., depends on $\tau_v$). To evaluate the WZW term, we smoothly deform Eq.(17) to a configuration where $\hat{n}$ is constant along the vortex loop (due to $\pi_3(S^2) = 0$ such a deformation exists). Let $u \in [0, 1]$ parametrizes the deformation, i.e., $\hat{n}(\tau_v, u = 0) = \hat{n}(\tau_v)$ while $\hat{n}(\tau_v, u = 1) = \hat{\tau}_v$. The corresponding deformed $\Omega$ is then given by

$$\tilde{\hat{\Omega}}(r, \phi, \tau_v, u) = \left( \sqrt{1 - f^2(r)} \hat{n}(\tau_v, u), f(r) \cos \phi, f(r) \sin \phi \right)$$

(18)

Plugging Eq.(18) into Eq.(16), we obtain,

$$W_{\text{WZW}} = -2\pi i \left( \frac{1}{4\pi} \int d^2 r d\tau_v \epsilon^{abc} \hat{n}_a \partial_a \hat{n}_b \partial_{\tau_v} \hat{n}_c \right).$$

Under this change of basis

$$h_v(r) \to (II \otimes R) \cdot h_v(r) \cdot (II \otimes R)^\dagger$$

(22)

where

$$Q = -i [X \partial_1 + Z \partial_2 + v_1(r)E + v_2(r)I].$$

(23)

The fermion zero modes are the normalizable zero energy eigenvectors satisfying

$$Q^\dagger \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = 0, u_3 = u_4 = 0 \text{ or } \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}.$$
\[ Q \begin{pmatrix} u_3 \\ u_4 \end{pmatrix} = 0, u_1 = u_2 = 0. \]  
\[ (24) \]

Given \( v_1(r) \) and \( v_2(r) \) forming a vortex pattern, Eq.(23) and Eq.(24) were solved by Jackiw and Rossi [22].

Because the sites in each unit cell have been reordered, namely,
\[ 1 \leftrightarrow \text{lower} - \text{left}, 2 \leftrightarrow \text{upper} - \text{right}, \]
\[ 3 \leftrightarrow \text{upper} - \text{left}, 4 \leftrightarrow \text{lower} - \text{right}. \]  
\[ (25) \]
the zero mode localizes on the lower-left and upper-right lattice sites if there is a normalizable solution for
\[ Q \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = 0. \]  
\[ (26) \]
Similarly, it is localized on the upper-left and lower-right lattice sites if there is a normalizable solution for
\[ Q \begin{pmatrix} u_3 \\ u_4 \end{pmatrix} = 0. \]  
\[ (27) \]
Ref.[22] explicitly solved the differential equation. Their solution shows that, depending on the vorticity, the zero mode splits the zero modes and produces a \( S \) = 1/2 magnetic moment.

In the following, we present an argument which suggests that under the space-time configuration discussed above, the Berry phase arising from the WZW term in Eq.(16) is zero. To argue that the Berry phase is zero involves two steps. (i) Using the result in Ref.[23], one can show that when any one of the five components of \( \hat{\Omega} \) is zero, the WZW term reduces to the topological \( \theta \)-term (with \( \theta = \pi \)) for the remaining four components. This topological term is non-zero only when the wrapping number associated with the mapping from the space-time to the order parameter manifold, in this case \( S^3 \), is non-zero. (ii) We note that in the present situation only three components of \( \hat{\Omega} \) are non-zero. By counting the dimension of the space-time image, we conclude that any such \( \hat{\Omega} \) cannot produce a non-zero wrapping number in \( S^3 \), hence the Berry phase vanishes.

**IV. THE STATISTICS OF THE VBS VORTEX**

**A. The WZW term restricted to the vortex exchange process**

We can determine the statistics of the VBS vortex by computing the Berry phase associated with the space-time configuration \( \hat{\Omega} \) in which two vortices are exchanged. Technically, because our space-time is \( S^3 \), we need to consider the process in which two pairs of vortex-anti-vortex are created out of vacuum, exchange the two vortices, and then annihilate the vortices with the anti-vortices at the end. Such Berry’s phase contains two contributions: 1) the Berry phase due to exchange of the vortices and 2) the Berry phase associated with the spin 1/2 in the vortex/antivortex cores. To isolate the Berry phase due to the exchange, we lock the core spins in, say, the positive \( n_3 \)-direction.

In the following, we present an argument which suggests that under the space-time configuration discussed above, the Berry phase arising from the WZW term in Eq.(16) is zero. To argue that the Berry phase is zero involves two steps. (i) Using the result in Ref.[23], one can show that when any one of the five components of \( \hat{\Omega} \) is zero, the WZW term reduces to the topological \( \theta \)-term (with \( \theta = \pi \)) for the remaining four components. This topological term is non-zero only when the wrapping number associated with the mapping from the space-time to the order parameter manifold, in this case \( S^3 \), is non-zero. (ii) We note that in the present situation only three components of \( \hat{\Omega} \) are non-zero. By counting the dimension of the space-time image, we conclude that any such \( \hat{\Omega} \) cannot produce a non-zero wrapping number in \( S^3 \), hence the Berry phase vanishes.

**B. Fermion integration**

The \( \hat{\Omega} \) configuration discussed in the preceding subsection represents a map from the space-time \( (S^3) \) to the order parameter space \( S^2 \) formed by \( (n_3, v_1, v_2) \). In general, such a mapping allows for a Hopf term.

Let’s go back to the Majorana fermion representation. Integrating out the fermion gives rise to the WZW term in Eq.(16). The difference now is that only \( (n_3, v_1, v_2) \) are non-zero and the corresponding spinon masses are given by \( (M_3, M_4, M_5) \) in Eq.(14). Because both the gamma matrices in Eq.(9) and all the mass terms in Eq.(14) conserve the fermion number, (the associated \( U(1) \) symmetry is generated by \( Q = YIII \), it’s convenient to complexify the Majorana fermion so that
\[ \Gamma_1 \rightarrow IXZ, \Gamma_2 \rightarrow IXX \]
\[ M_3 \rightarrow ZII, M_4 \rightarrow IXY, M_5 \rightarrow IYY. \]
These matrices commute with $ZII$, and hence can be block-diagonalized into the $ZIII = \pm 1$ sectors. Within these each sector, the fermion integration leads to a coefficient $\pm \pi$ Hopf term (see [19, 23]). Thus, the net Hopf terms cancel, resulting in a vanishing Berry’s phase.

Therefore both subsection IV A and subsection IV B points to the fact that the $S = 1/2$ VBS vortex is a boson.

V. TRANSLATING THE VORTEX

Due to the hopping phase in Fig. 4, translation by one lattice constant (of the square lattice with the original $1 \times 1$ unit cell) is realized projectively. This is because in order to restore the mean-field Hamiltonian, translation must be accompanied by a gauge transformation. Restoring the original site labeling in Fig. 4, it can be shown that the combined transformations are given by

\[
\begin{pmatrix}
\psi(r)_{\alpha 1} \\
\psi(r)_{\alpha 2} \\
\psi(r)_{\alpha 3} \\
\psi(r)_{\alpha 4}
\end{pmatrix}
\overset{\hat{T} x}{\rightarrow}
\begin{pmatrix}
\psi(r + \hat{x})_{\alpha 1} \\
\psi(r + \hat{x})_{\alpha 2} \\
\psi(r + \hat{x})_{\alpha 3} \\
\psi(r + \hat{x})_{\alpha 4}
\end{pmatrix}
\]

where the subscripts of $\psi$ are $\alpha = $ Majorana index, $\sigma = $ the spin index, and $1, 2, 3, 4$ are the site indices in the unit cell (Fig. 4). This gives the following transformation matrices for the low energy spinon modes,

\[
T^x = IIIXZ, T^y = IIXXI.
\]

As before, the Pauli matrices are ordered according to

\[
\text{Majorana} \otimes \text{spin} \otimes 4 \times 4 \text{ unit-cell sites}
\]

When acted on the masses of Eq.(14), this leads to the following transformations of the VBS and Néel order parameters

\[
(n_1, n_2, n_3, v_1, v_2) \rightarrow (n_1, n_2, n_3, -v_1, v_2)
\]

The same conclusion can be reached without worrying about the fermion projective transformations by directly inspecting Fig. 5.

Because reversing the sign of $v_1$ or $v_2$ changes the vorticity of a VBS vortex, and due to the discussion in section III B, we conclude that translation results in particle-antiparticle conjugation on the spin label of (the relativistic) $\phi_\sigma$, namely,

\[
\phi_\sigma \overset{\hat{T} x}{\rightarrow} \hat{T} x \phi^*_\sigma.
\]

This transformation law will be useful later when we discuss the symmetries of the vortex field theory.

We note that many of the results in this section, e.g., vortex spin, the effect of translation on vortex field, are readily obtained by Levin and Senthil[7] using picturesque arguments.

VI. THE FIELD THEORY OF SINGLE VBS VORTEXES

Eq.(15) possesses an emergent symmetry (the $O(5)$ symmetry) which is not shared by the microscopic model. Therefore generically we should supplement Eq.(15) with additional $O(5)$ symmetry breaking terms, i.e,

\[
W[\hat{\Omega}] = \frac{1}{2g} \int_M d^3x \left( \partial_\mu \hat{\Omega}_\mu \right)^2 + W_{\text{WZW}}[\hat{\Omega}]
\]

These additional terms can be generated by, e.g., integrating out the high energy fermion modes, or by integrating out the charge-$SU(2)$ gauge fluctuations. A simple example of such term is

\[
\mathcal{D} \int d^3x \left( \sum_{i=1}^3 v_i^2 - \sum_{i=1}^2 v_i^2 \right),
\]

which tips the balance between the Néel and VBS order.

In the rest of the paper we shall start from the VBS phase and look at the possible vortex proliferations that can eliminate the VBS order [7].

As seen in the previous section, the VBS vortex is a spin-1/2 boson. In addition, the core-spin resides on opposite sublattice for vortex and anti-vortex. Hence conservation of vorticity implies that the vortex hops among the same sublattice. In terms of the vortex annihilation operator $\phi_\sigma$, the action for a relativistic boson field $\phi_\sigma$ has the form

\[
S_\phi = \int d^3x \left[ \left( \partial_\mu - ia_\mu \right) \phi_\sigma \phi^*_\sigma + \frac{1}{2} \left( \phi_\sigma \right)^4 \right.
\]

\[
+ \frac{1}{2g} \left( \epsilon_{\mu \nu \rho} \partial_\mu a_\rho \right)^2 \right],
\]

where $\sigma = \uparrow, \downarrow,$ and $a_\mu$ is the $U(1)$ gauge field (not to be confused with the charge-$SU(2)$ gauge field, which had been integrated out) whose fluctuation is the dual representation of the phase fluctuation of the VBS order parameter $\psi_{\text{VBS}} = v_1 + iv_2$. The theory in Eq.(30) can be obtained by the usual $\hat{U}(1)$ boson duality of the

\[3\] Here we emphasize that the spin is a flavor degree of freedom. It is not the “spin” in the Dirac sense, otherwise, the spin-statistics theorem would be violated.
Vorticity conservation: \( \psi_{VBS} \), except that the vortex core is decorated with \( S = 1/2 \).

Eq.(30) has the following symmetries,

- **Vorticity conservation**: \( \psi_{VBS} \)
- **U(1) gauge symmetry**: \( \phi_{\sigma} \to e^{i\eta} \phi_{\sigma} \), \( a_{\mu} \to a_{\mu} + \partial_{\mu} \eta(x) \)
- **Flavor (spin) SU(2)**: \( \phi_{\sigma} \to u_{\sigma\sigma'} \phi_{\sigma'} \), where \( u_{\sigma\sigma'} \in SU(2) \)

Flavor (spin) SU(2) symmetry implies the double vortex can be either ±double vortex (with vorticity \( \pm 2 \)) or ±spin-singlet double vortex. Since preceding the single vortex condensation, the antiferromagnetic order parameter given by \( |\langle \sigma \rangle| \neq 0 \) such single vortex condensation leads to the Néel state with long-range order.

The preferred angles of \( \psi_{VBS} \) will depend on whether we have the columnar or plaquette VBS order.

Note that unlike ordinary translations, \( T^2_y = T^2_y = -1 \).

As it stands the gauge field in Eq.(30) is non-compact. However, because of the underlying square lattice, the VBS order parameter is subjected to a 4-fold anisotropy. This means that \( a_{\mu} \) is in fact a compact gauge field, and in carrying out the path integral we need to include the space-time events where degree 4 monopole in \( a_{\mu} \) can pop in/out. In the phase where such monopole condenses, the vortices are confined and the VBS long-range order prevails. However, we shall be interested in condensing various types of VBS vortices in the following. In those phases \( a_{\mu} \) is necessarily deconfining. Therefore to address the properties of these vortex condensed phases, we shall ignore the monopoles and start with Eq.(30).

### VII. THE CONDENSATION OF SINGLE VORTICES – THE DIRECT VBS TO NÉEL TRANSITION

In Ref.[7] it is proposed that the condensation of \( \phi_{\sigma} \) leads to the Néel order. Moreover, it is argued that the de-confined quantum critical point (DCQCP) between VBS and Néel order is realized when the \( s \) in Eq.(30) is tuned from positive to negative so that \( \langle |\phi_{\sigma}|^2 \rangle \neq 0 \). Such single vortex condensation leads to the Néel state with the antiferromagnetic order parameter given by

\[
\vec{n} = \phi_{\sigma}^* \bar{\sigma}_{\alpha\beta} \phi_{\beta}.
\]

This direct VBS to Néel transition is marked by the red cross on the green paths in Fig. 1-Fig. 3.

In the next three sections we shall study the scenarios in which preceding the single vortex condensation, the double vortex (with vorticity ±2) condenses first. Since a double vortex is the bound state of two single vortices, and the single vortex is a spin 1/2 boson, exchange symmetry implies the double vortex can be either 1) spin-singlet and odd-parity, or 2) spin-triplet and even-parity. The fact that spin-singlet double vortex has to be odd in parity is responsible for the breaking of the point group or time reversal symmetry in the coming sections.

Note that such broken symmetry depends on the geometry of the underlying lattice. In Ref.[24] it is pointed out that on the honeycomb lattice it is possible for spin-singlet odd parity double vortex to preserve the point group symmetry.

### VIII. THE CONDENSATION OF SPIN-SINGLET ODD-PARITY DOUBLE VORTEX

In terms of \( \phi_{\sigma} \), the double vortex annihilation operator is given by

\[
\hat{\Phi}_{\alpha}(x) = \frac{1}{\sqrt{2}} \phi_{\sigma}^*(x) E_{ab} \partial_\alpha \phi_b(x).
\]

Here \( \alpha = 1, 2 \) label the spatial directions, \( a, b \) labels the spins, and \( E_{ab} \) is the two-index anti-symmetric tensor. Recall that on bipartite lattice, vortices with the same vorticity resides on the same sublattice, hence for the square lattice \( \alpha = 1 \) means \( \sigma = \frac{1}{2} \) and \( \alpha = 2 \) means \( \sigma = \frac{-1}{2} \).

The action which involves both the single and double vortex fields is given by

\[
S_{\phi, \Phi} = \int d^3x \left\{ \frac{1}{4g} (\epsilon_{\mu\nu\rho} \partial_\mu a_\rho)^2 + |(\partial_\mu - ia_\mu) \phi_{\sigma}|^2 \\
+ |(\partial_\mu - i2a_\mu) \Phi_\alpha|^2 + V_{\phi_{\sigma}} |\phi_{\sigma}| + V_{\Phi_\alpha} |\Phi_\alpha| \\
- \lambda \left[ \Phi_\alpha^* \left( \frac{1}{\sqrt{2}} \phi^T E \partial_\alpha \phi \right) + \text{c.c.} \right] \right\}.
\]

In Eq.(33)

\[
V_{\phi_{\sigma}} |\phi_{\sigma}| = s |\phi_{\sigma}|^2 + \frac{v}{2} |\phi_{\sigma}|^4
\]

\[
V_{\Phi_\alpha} |\Phi_\alpha| = \tilde{s} |\Phi_\alpha|^2 + \frac{V_1}{4} (|\Phi_1|^4 + |\Phi_2|^4) + V_2 (|\Phi_1|^2 |\Phi_2|^2)
\]

\[
+ W \left( \Phi_1^* \Phi_2^2 + \text{c.c.} \right),
\]

and the \( \lambda \) term enforces Eq.(32). Eq.(33) is consistent with the result of Ref.[6, 25].

On square lattice, the quartic terms in \( V_{\Phi_\alpha} |\Phi_\alpha| \) are constructed to be invariant under the spatial 90 degree rotation, inversion, and time-reversal transformations

\[
\begin{align*}
(\hat{\Phi}_1, \hat{\Phi}_2) \quad &\text{90 deg rotation} \quad \rightarrow (\hat{\Phi}_2, -\hat{\Phi}_1) \\
(\hat{\Phi}_1, \hat{\Phi}_2) \quad &\text{Inversion} \quad \rightarrow (-\hat{\Phi}_1, -\hat{\Phi}_2) \\
(\hat{\Phi}_1, \hat{\Phi}_2) \quad &\text{time-reversal} \quad \rightarrow (\hat{\Phi}_1, \hat{\Phi}_2).
\end{align*}
\]
Let’s first focus on $V_{\tilde{\phi}_\alpha}[\tilde{\Phi}_\alpha]$. Assume
\[ \tilde{\Phi}_\alpha = \rho_\alpha e^{i \Theta_\alpha}, \]
the potential energy is given by
\[
V_{\tilde{\phi}_\alpha}[\tilde{\Phi}_\alpha] = \tilde{S}(\rho_1^2 + \rho_2^2) + \frac{V_1}{4}(\rho_1^4 + \rho_2^4) + V_2 \rho_1 \rho_2^2 + W \rho_1^2 \rho_2^2 \cos(2(\Theta_1 - \Theta_2)).
\]
Due to the last term, $V_{\tilde{\phi}_\alpha}[\tilde{\Phi}_\alpha]$ is minimized when
\[
\Theta_1 = \Theta_2 \text{ or } \Theta_1 = \Theta_2 + \pi \quad \text{for } W < 0
\]
\[
\Theta_1 = \Theta_2 \pm \frac{\pi}{2} \quad \text{for } W > 0.
\]
(36)

Setting $\Theta_1$ and $\Theta_2$ to satisfy Eq.(36) we obtain, for either sign of $W$,
\[
V_{\tilde{\phi}_\alpha}[\tilde{\Phi}_\alpha] = \tilde{S}(\rho_1^2 + \rho_2^2) + \frac{V_1}{4}(\rho_1^4 + \rho_2^4) + V_2 \rho_1 \rho_2^2 - |W| \rho_1^2 \rho_2^2.
\]
It is straightforward to show the extrema of the above equation occur at
\[
\rho_1^2 = \rho_2^2 = \frac{\tilde{S}(\frac{V_1}{2} + V_2 - |W|)}{2} \text{ or } 0.
\]
The non-zero solution would require
\[
\tilde{S} \left( \frac{V_1}{2} + V_2 - |W| \right) < 0.
\]

Moreover, for the $V_{\tilde{\phi}_\alpha}[\tilde{\Phi}_\alpha]$ to be bounded from below we must have $\frac{V_1}{2} + V_2 - |W| > 0$. Therefore

For $\tilde{S} > 0$ : $\tilde{\phi}_1 = \tilde{\phi}_2 = 0$

For $\tilde{S} < 0$ : \[
\begin{cases}
\tilde{\phi}_1 = \pm i \tilde{\phi}_2 := \rho e^{i \Theta} & \text{for } W < 0 \\
\tilde{\phi}_1 = \pm \tilde{\phi}_2 := \rho e^{i \Theta} & \text{for } W > 0
\end{cases}
\]
(37)

Note that $\tilde{\phi}_1 = \pm i \tilde{\phi}_2$ preserves time-reversal but breaks 90 degree rotation symmetry, while $\tilde{\phi}_1 = \pm \tilde{\phi}_2$ breaks time-reversal but preserves 90 degree rotation symmetry.

A. The condensation of p-wave double vortices – a nematic, symmetry-enriched, $\mathbb{Z}_2$ spin liquid

The sign of $W$ relevant to this subsection is $W < 0$. Let us consider the case where $\tilde{S} < 0$ (this is the mean-field value, in reality there will be fluctuation correction). Let’s take, e.g.,
\[
\tilde{\phi}_1 = \tilde{\phi}_2 = \rho e^{i \Theta}.
\]
Plug this into Eq.(33) we obtain
\[
S_{\phi, \Phi} = \int d^3x \left\{ \frac{1}{2g} (\epsilon_{\mu \nu \rho} \partial_\mu a_\rho)^2 + |(\partial_\mu - i a_\mu) \phi_\sigma|^2 + 2 \rho^2 (|\partial_\mu \Theta - 2 a_\mu|^2 + V_{\phi_\sigma}[\phi_\sigma] - \lambda \rho \left\{ e^{i \Theta} \left( \frac{1}{\sqrt{2}} \phi^T E \partial_x \phi \right) + c.c. \right\} \right\}.
\]
(38)

In passing to the last line we have used the fact that $\partial_1 + \partial_2 = \partial_x$.

It is important to note that under inversion, $\Theta \rightarrow \Theta + \pi$. Since $\Theta$ can be absorbed by $a_\mu$, Eq.(38) is invariant under inversion. Therefore, although naively p-wave paired double vortex breaks the inversion symmetry, only the 90 degree rotation is broken because the sign change can be absorbed into the phase of $\Phi_{1,2}$. The resulting phase is nematic.

Isolating the parts of Eq.(38) that depends on $\phi_\sigma$ we obtain
\[
\int d^3x \left\{ |(\partial_\mu - i a_\mu) \phi_\sigma|^2 + V_{\phi_\sigma}[\phi_\sigma] - \lambda \rho \left\{ e^{-i \Theta} \left( \frac{1}{\sqrt{2}} \phi^T E \partial_x \phi \right) + c.c. \right\} \right\} = \int d^3x \left\{ \left| (\partial_\mu - i a_\mu) \phi_\sigma \right|^2 + V_{\phi_\sigma}[\phi_\sigma] - \frac{1}{\sqrt{2}} \lambda \rho \left[ e^{-i \Theta} \phi^T E \partial_x \phi + c.c. \right] \right\}.
\]
(39)

In passing to the last line we have used the fact that $\partial_1 + \partial_2 = \partial_x$.

Next, we perform the gauge transformation
\[
\phi_\sigma \rightarrow e^{i \frac{\Theta}{2}} \phi_\sigma,
\]
under which the action becomes
\[
S_\phi = \int d^3x \left\{ |(\partial_\mu - i A_\mu) \phi_\sigma|^2 + V_{\phi_\sigma}[\phi_\sigma] - \lambda \sum_{i=1}^2 \left[ \rho \left( \frac{1}{\sqrt{2}} \phi^T E \partial_x \phi \right) + c.c. \right] \right\}.
\]
(40)

where
\[
A_\mu := a_\mu - \frac{1}{2} \partial_\mu \Theta.
\]

Note that $\phi_\sigma$ is a spin 1/2 field under the global spin $SU(2)$ transformation. Moreover, because of the double vortex condensation, the gauge charge (with respect to $a_\mu$) of $\phi_\sigma$ becomes conserved mod 2, i.e., equal to either 0 or 1.

Let’s now consider the case where $\phi_\sigma$ remains massive and ask what is momentum location where the mass gap is minimum. For this purpose, we temporarily turn off the coupling to $A_\mu$ in Eq.(40). Decomposing $\phi_\sigma$ into the real and imaginary parts
\[
\phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = \begin{pmatrix} u_1 + i u_3 \\ u_2 + i u_4 \end{pmatrix},
\]
(41)
the action that is quadratic in $u$ can be written as

$$\int d^3x \left\{ -u^T (\partial^2 + sII) u - \sqrt{2} \lambda \rho u^T ZE \partial_z u \right\},$$

(42)

where

$$u^T = (u_1, u_2, u_3, u_4).$$

Upon Fourier transformation Eq. (42) turns into

$$\int \frac{d^3q}{(2\pi)^3} u^T_{-q} h(q) u_q$$

where

$$h(q_0, q_x, q_y) = (q^2 + s) II + \sqrt{2} \lambda \rho ZY q_x.$$

The eigenvalues of $h(q_0, q_x, q_y)$ are

$$q_0^2 + q_x^2 + q_y^2 + s \pm \sqrt{2} \lambda \rho q_x$$

$$= \left( q_x \pm \frac{1}{\sqrt{2}} \lambda \rho \right)^2 + q_y^2 + \left[ s - \frac{1}{2} (\lambda \rho)^2 \right].$$

For $s > \frac{1}{2} (\lambda \rho)^2$, $\phi_\sigma$ is gapped, and the minimum of the gap occurs at $\pm q_0$ where

$$q_0 = \left( \frac{1}{\sqrt{2}} \lambda \rho, 0 \right).$$

(43)

Now let’s turn the coupling to $A_\mu$ back on, and integrate out the massive $\phi_\sigma$ to yield an effective theory for $A_\mu$. (For details see appendix A.) To the leading order (in $\lambda \rho/\sqrt{s}$) the result is

$$\frac{\sqrt{s}}{24\pi} \left( \frac{\lambda \rho}{\sqrt{s}} \right)^2 \int d^3x A_\mu^2 = \frac{\sqrt{s}}{96\pi} \left( \frac{\lambda \rho}{\sqrt{s}} \right)^2 \int (\partial_\mu \Theta - 2a_\mu)^2$$

This spatial anisotropic Higgs term should be added to the

$$\int d^3x |(\partial_\mu - 2a_\mu)\Phi_1|^2 + |(\partial_\mu - 2a_\mu)\Phi_2|^2$$

$$= 2\rho^2 \int d^3x (\partial_\mu \Theta - 2a_\mu)^2$$

term in Eq. (33) to yield

$$\int d^3x \left\{ (2\rho^2) (\partial_\mu \Theta - 2a_\mu)^2 + \frac{\sqrt{s}}{96\pi} \left( \frac{\lambda \rho}{\sqrt{s}} \right)^2 (\partial_\mu \Theta - 2a_\mu)^2 \right\}$$

$$= \int d^3x \frac{g_\mu}{2} (\partial_\mu \Theta - 2a_\mu)^2$$

(44)

where $\mu$ is summed over 0, 1, 2.

Now we perform the standard duality transformation [26] on the Boltzmann weight associated with Eq. (44). The first step includes a Hubbard-Stratonovich transformation followed by decomposing $e^{i\Theta}$ into a smooth and vortex part, namely, $e^{i\Theta} = e^{i\Theta_\ast} \times \chi_{\nu}$. This leads to

$$\exp \left\{ -\int d^3x \frac{g_\mu}{2} (\partial_\mu \Theta - 2a_\mu)^2 \right\}$$

$$= \int D[J^\mu] \exp \left\{ -\int d^3x \left[ \frac{1}{g_\mu} (J^\mu)^2 - iJ^\mu (\partial_\mu \Theta - 2a_\mu) \right] \right\}.$$  

(45)

In the second step, we perform the integration over $\Theta_\ast$ and write the path integral over $\chi_{\nu}$ as the sum over the vortex world-lines (vw)

$$\sum_{vw} \int D(\Theta_\ast) D[J^\mu] \exp \left\{ -\int \left[ \frac{1}{2g_\mu} (J^\mu)^2 \right. \right.$$  

$$- iJ^\mu \left( \partial_\mu \Theta_\ast + \frac{1}{i} \chi_{\nu} \partial_\nu \chi_{\nu} - 2a_\mu \right) \left. \right\}$$

$$= \sum_{vw} \int D[J^\mu] \exp \left\{ -\int \left[ \frac{1}{2g_\mu} (J^\mu)^2 \right. \right.$$  

$$- iJ^\mu \left( \frac{1}{i} \chi_{\nu} \partial_\nu \chi_{\nu} - 2a_\mu \right) \left. \right\}.$$  

(46)

Third, we solve the constraint $\partial_\mu J^\mu = 0$ by introducing a gauge field

$$J^\mu = \frac{1}{2\pi} e^{\mu \nu \rho} \partial_\nu b_\rho,$$

Eq. (46) becomes

$$\sum_{vw} \int D[b_\mu] \exp \left\{ -\int \left[ \frac{1}{2g_\mu} (e^{\mu \nu \rho} \partial_\nu b_\rho)^2 \right. \right.$$  

$$- \frac{i}{2\pi} e^{\mu \nu \rho} \partial_\nu b_\rho \left( \frac{1}{i} \chi_{\nu} \partial_\nu \chi_{\nu} - 2a_\mu \right) \left. \right\}$$

$$= \sum_{vw} \int D[b_\mu] \exp \left\{ -\int \left[ \frac{1}{8\pi^2 g_\mu} (e^{\mu \nu \rho} \partial_\nu b_\rho)^2 \right. \right.$$  

$$+ \frac{i}{\pi} e^{\mu \nu \rho} \partial_\nu b_\rho \partial_\nu a_\rho \right. \right.$$  

$$\left. + i b_\mu K_{\nu}^\mu \right\}.$$  

(47)

where $K_\nu^\mu$ is defined by

$$K_\nu^\mu = \frac{1}{2\pi} e^{\mu \nu \rho} \partial_\rho \left( \frac{1}{i} \chi_{\nu} \partial_\nu \chi_{\nu} \right).$$

Physically $K_\nu^\mu$ is the vortex current associated with $\chi_{\nu}$. To summarize, after including the Maxwell term for $a_\mu$ in Eq. (33), the total effective action read

$$S_{\text{eff}} = \int d^3x \left[ \frac{1}{2g} (e_{\mu \nu \rho} \partial_\nu a_\rho)^2 + \frac{1}{8\pi^2 g_\mu} (e^{\mu \nu \rho} \partial_\nu b_\rho)^2 \right. \right.$$  

$$+ \frac{i}{\pi} e^{\mu \nu \rho} \partial_\nu b_\rho \partial_\nu a_\rho \right. \right.$$  

$$+ i b_\mu K_{\nu}^\mu \right].$$  

(48)

Eq. (48) can be rewritten as

$$S_{\text{eff}} = \int d^3x \left[ \frac{1}{2g} (e_{\mu \nu \rho} \partial_\nu a_\rho)^2 + \frac{1}{8\pi^2 g_\mu} (e^{\mu \nu \rho} \partial_\nu b_\rho)^2 \right. \right.$$  

$$+ \frac{i}{4\pi} \int d^3x \left[ e_{\mu \nu \lambda} \sum_{l} a_\mu^l K_{l, l} \partial_\nu a_\lambda^l + i \sum_{a} J_a^l (a_{\lambda}, a_\lambda^l) \right].$$  

(49)
Here \( I, J = 1, 2 \) with \( a^1_\mu = a_\mu \) and \( a^2_\mu = b_\mu, \alpha = \Upsilon_v, \phi_\sigma \). In addition,

\[
\begin{align*}
    l_{\phi_\sigma} &= \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \\
    l_{K_{\Upsilon_v}} &= \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \\
    K &= \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix} \Rightarrow K^{-1} = \begin{pmatrix} 0 & 1/2 \\ 1/2 & 0 \end{pmatrix} \\
    j^\mu_\sigma(x, \tau) &= \delta^2(x - x_\sigma(\tau))(1, \dot{x}_\sigma).
\end{align*}
\]

In Eq.(50) \( x_\sigma(\tau) \) is the world line of the \( \alpha \)-type quasiparticle\(^5\). It is important to point out that in Eq.(49) we have put back the current of a \( \phi_\sigma \) test-particle in order to keep track of the statistics. At low energies and long wavelengths the first two Maxwell terms in Eq.(49) are irrelevant, hence can be omitted. The remaining parts of Eq.(49) determines the topological order. Eq.(49) is consistent with the matter-double-Chern-Simons action of Ref.[27].

This \( K \) matrix in Eq.(50) contains the information of the ground state degeneracy on Riemann surfaces and the self and mutual statistics of quasiparticles [28]. On a genus \( g \) Riemann surface, the degeneracy \( = |\text{det}(K)|^g = 4^g \). The exchange statistics of the \( \alpha \)-type quasiparticle is given by the Berry phase

\[
\pi \left( l^T_\alpha \cdot K^{-1} \cdot l_\alpha \right).
\]

The mutual statistics is given by the Berry phase

\[
2\pi \left( l^T_\alpha \cdot K^{-1} \cdot l_\beta \right)
\]

arising from particle of type \( \alpha \) circling around particle of type \( \beta \). From the above formula, it’s easy to check that both particles are bosons. Circulating \( \phi_\sigma \) around \( K_{\Upsilon_v} \) gets a phase of \( e^{i\pi} = -1 \). This last fact implies that the bound state of \( \phi_\sigma \) and \( K_{\Upsilon_v} \) is a fermion, with

\[
l_{\phi_\sigma \otimes K_{\Upsilon_v}} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.
\]

Topologically, the statistics of the \( \phi_\sigma \) particle, the \( \Upsilon_v \) vortex, and the bound state of \( \phi_\sigma \) and \( \Upsilon_v \) vortex are the same as the anyons \( (e, m, e) \) in a \( \mathbb{Z}_2 \) spin-liquid, namely,

\[
\phi_\sigma \text{ particle } \rightarrow e \\
\Upsilon_v \text{ vortex } \rightarrow m
\]

Bound state of \( \Upsilon_v \) vortex and \( \phi_\sigma \) particle \( \rightarrow \epsilon = e \cdot m. \)

(51)

It is important to point out that in addition to the \( \mathbb{Z}_2 \) topological order in Eq.(51), this spin liquid breaks spatial rotation symmetry as shown in Eq.(43) and Eq.(44). Moreover, since \( \phi_\sigma \) carries spin 1/2 quantum number, this anyon is symmetry-enriched. Therefore the spin liquid we obtain is a nematic, symmetry-enriched, \( \mathbb{Z}_2 \) spin liquid.

\section{The subsequent condensation of single vortex – an uni-directional spiral AF phase}

The subsequent condensation of \( \phi_\sigma \) in the nematic symmetry-enriched \( \mathbb{Z}_2 \) spin liquid occurs when \( s < \frac{1}{2}(\lambda \rho)^2 \) (mean-field value, in reality there will be fluctuation correction). Under such condition the momentum modes \( q_\mu = \pm \frac{1}{\sqrt{2}} \lambda \rho (0, 1, 0) \) of \( \phi_\sigma \) condenses. Plugging into \( \vec{n} = \phi^\dagger \vec{\sigma} \phi \) and taking into account of Eq.(28) and Eq.(43), this gives incommensurate spiral with wave vector

\[
(\pi, \pi) \pm 2q_0.
\]

The phase diagram is shown in Fig. 1, which also appears in Ref.[6, 25, 27]

\section{The condensation of \( p_x + ip_y \) double vortices – a time-reversal breaking symmetry-enriched \( \mathbb{Z}_2 \) spin liquid}

The sign of \( W \) in Eq.(37) relevant to this subsection is \( W > 0 \). Let’s consider \( \vec{S} < 0 \) (mean-field value, in reality there should be fluctuation correction) and choose, e.g.,

\[\tilde{\Phi}_1 = -i \tilde{\Phi}_2 = \rho e^{i\theta}.\]

Here we consider the situation that \( \phi_\sigma \) remains massive. We determine the momentum location where the mass gap of \( \phi_\sigma \) is minimized by turning off the coupling to \( A_\mu \) in

\[
S_\phi = \int d^3x \left\{ (|\partial_\mu - iA_\mu| \phi_\sigma|^2 + s|\phi_\sigma|^2 + \frac{\rho}{2}|\phi_\sigma|^4 - \lambda \sum_{i=1}^2 \left[ \rho \left( \frac{1}{\sqrt{2}} \phi^T E(\partial_\mu + i \partial_\mu) \phi \right) + c.c. \right) \right\}. \quad (52)
\]

After the decomposition in Eq.(41), the action that is quadratic in \( u \) reads

\[
\int d^3x \left\{ - u^T (\partial^2 + sII) u \\
- \sqrt{2} \lambda \rho \ u^T \left( ZE \partial_1 + XE \partial_2 \right) u \right\}. \quad (53)
\]

Upon Fourier transformation, Eq.(53) turns into

\[
\int \frac{d^3q}{(2\pi)^3} \ u^T_q h(q) \ u_q \]
where
\[ h(q_0, q_1, q_2) = (q^2 + s) II + \sqrt{2} \lambda \rho \ (ZY \ q_1 + XY \ q_2). \]  
(54)

The eigenvalues of \( h(q_0, q_1, q_2) \) are
\[ (q_0^2 + |q|^2) + s \pm \sqrt{2} \lambda \rho \ |q| \]
\[ = q_0^2 + \left( |q| \pm \frac{1}{\sqrt{2}} \lambda \rho \right)^2 + \left[ s - \frac{1}{2} (\lambda \rho)^2 \right]. \]
For \( s > \frac{1}{2} (\lambda \rho)^2 \), the minimum gap of \( \phi_\sigma \) occurs at
\[ |q| = \frac{1}{\sqrt{2}} \lambda \rho, \]
which is a ring in momentum space.

Because the eigenvector of Eq.(54) is not invariant under the time reversal transformation
\[ h(q_0, q) \rightarrow (ZE)^{-1} \cdot h(q_0, -q) \cdot ZE, \]  
(55)
the \( \phi_\sigma \) mode at momentum \( q \) and spin \( \sigma \) does not have the same energy as that at momentum \(-q\) and spin \(-\sigma\), i.e.,
\[ E_\sigma(q) \neq E_{-\sigma}(-q). \]  
(56)
Eq.(56) is a manifestation of the breaking of time reversal symmetry.

Since \( \phi_\sigma \) is gapped we can integrate it out to yield an effective action for \( A_\mu \). To the leading order in \( \frac{\lambda}{\sqrt{s}} \) the answer is (the details are given in appendix A)
\[ \frac{\sqrt{s}}{24\pi} \left( \frac{\lambda \rho}{\sqrt{s}} \right)^2 \int d^3x \ (A_1^2 + A_2^2) \]
\[ = \frac{\sqrt{s}}{96\pi} \left( \frac{\lambda \rho}{\sqrt{s}} \right)^2 \int d^3x \sum_{i=1}^2 \left( \partial_i \Theta - 2a_i \right)^2. \]
This Higgs term should be added to
\[ \int d^3x \left\{ |(\partial_\mu - 2a_\mu) \bar{\Phi}_1|^2 + |(\partial_\mu - 2a_\mu) \bar{\Phi}_2|^2 \right\} \]
\[ = 2\rho^2 \int d^3x (\partial_\mu \Theta - 2a_\mu)^2 \]
of Eq.(33) to yield
\[ \int d^3x \left\{ (2\rho^2) (\partial_\mu \Theta - 2a_\mu)^2 \right. \]
\[ + \frac{\sqrt{s}}{96\pi} \frac{\rho^2 \lambda^2}{s} \left[ \sum_{i=1}^2 (\partial_i \Theta_i - 2a_i)^2 \right] \right\} \]
(57)
This time, the additional contribution to the Higgs term breaks the space-time rotation but preserves the spatial rotational symmetry. In the subsequent duality transformation, all the derivations in subsection VIII A follow, except that the Maxwell term for \( b_\mu \) becomes space direction-independent. The K-matrix which determines the self and mutual statistics of quasiparticles remains unchanged. Here the spin liquid is a time-reversal breaking, symmetry-enriched, \( \mathbb{Z}_2 \) spin liquid.

D. The subsequent condensation of single vortex – a “double spiral” AF phase

For \( s > \frac{1}{2} (\lambda \rho)^2 \) (mean-field value, in reality there should be fluctuation correction), a whole ring of momentum modes of \( \phi_\sigma \), with \( q_0 = 0 \) and \( |q| = \frac{1}{\sqrt{2}} \lambda \rho \), become unstable. This preserves the spatial rotational symmetry but breaks the time-reversal symmetry (Eq.(56)).

We have checked that up to the fourth-order terms in \( a \), all the modes lying on the momentum ring remain degenerate. Plugging into \( \bar{n} = \phi^\dagger \bar{\sigma} \phi \), this leads to a ring of momentum center around \((\pi, \pi)\). Following Ref.[29] we refer to this phase as the “double spiral” AF phase. The phase diagram corresponds to \( W > 0 \) is given in Fig. 2.

IX. THE SPIN TRIPLET DOUBLE VORTEX

Here we consider the scenario that the double vortex is a spin-triplet, even-parity, bound state of single vortices. In terms of \( \phi_\sigma \), the double vortex field is given by
\[ \Phi_n(x) = \frac{1}{\sqrt{2}} \phi_\sigma(x)(E \sigma_n)_{ab} \phi_b(x), \quad n = x, y, z. \]  
(58)
Under the action of the symmetries in Eq.(31), \( \Phi_n \) transforms according to
Vorticity conservation U(1): \( \Phi_n \rightarrow e^{i2\eta} \Phi_n \)
Flavor SU(2): \( \Phi_m \rightarrow R_{mn} \Phi_n \), \( R_{mn} \in \) spin-1 rep of SU(2)
\( T_x, T_y: \Phi_n \rightarrow -\Phi_n^* \)
\( T: \Phi_n \rightarrow -\Phi_n \)  
(59)
The action involving \( \phi_\sigma \) and \( \Phi_n \), and is invariant under Eq.(31) and Eq.(59), is given by
\[ S_{\phi, \Phi} = \int d^3x \left\{ |(\partial_\mu - ia_\mu) \phi_\sigma|^2 + |(\partial_\mu - ia_\mu) \Phi_n|^2 \right\} + \frac{s}{2} |\phi_\sigma|^2 + \frac{\sqrt{s}}{4} |\Phi_n|^2 \]
\[ + \frac{V_1}{2} (\Phi_n^* \Phi_n) (\Phi_m^* \Phi_m) + \frac{V_2}{2} (\Phi_n^* \Phi_n^*) (\Phi_m \Phi_m) \]
\[ - \lambda \left[ \Phi_n^* \left( \frac{1}{\sqrt{2}} \phi_\sigma^T (E \sigma_n)_{ab} \phi_b \right) + \text{h.c.} \right] + \frac{1}{2g} (\epsilon_{\mu\nu\rho} \partial_\mu a_\rho)^2 \]
where the $\lambda$ term enforces Eq.(58). In the following, we consider the scenario where the double-vortex $\Phi_n$ condenses while the single-vortex $\phi_\sigma$ remain gapped.

### A. The condensation of spin triplet double vortices – the AF* phase

For $S < 0$ (mean-field value, in reality there should be fluctuation correction), the magnitude of $\Phi_n$ acquires a non-zero expectation value and the fluctuation of $|\Phi_n|$ is massive. Under such condition it is convenient to separate the massive and massless degrees of freedom by writing

$$\Phi_n = \rho_\Phi e^{i\Theta}Z_n(\tilde{\Omega}),$$  

(61)

Here $Z(\tilde{\Omega})$ is the spin-1 coherent state, which is related to the spin 1/2 coherent state,

$$z(\tilde{\Omega}) := \begin{pmatrix} \cos(\theta/2) \\ \sin(\theta/2) e^{i\gamma} \end{pmatrix} := \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

$$\tilde{\Omega} = (\sin\theta \cos\gamma, \sin\theta \sin\gamma, \cos\theta)$$

by

$$Z_n(\tilde{\Omega}) = \frac{1}{\sqrt{2}} z(\tilde{\Omega})^T E\sigma_n z(\tilde{\Omega}) = \begin{pmatrix} \frac{1}{\sqrt{2}}(\alpha^2 - \beta^2) \\ \frac{1}{\sqrt{2}}(\alpha^2 + \beta^2) \\ -\sqrt{2} \alpha \beta \end{pmatrix}.$$  

(62)

Plugging Eq.(61) and Eq.(62) into the stiffness terms for $\Phi_n$ in Eq.(60), we get

$$\int d^3x [(\partial_\mu - i2a_\mu)\Phi_n]^2$$

$$= \rho_\Phi^2 \int d^3x \left\{ \left[ \left( \frac{1}{i} \gamma \frac{\partial_\mu}{i} \gamma - 2a_\mu + \frac{1}{i} Z^i \partial_\mu Z^i \right)^2 \right. 

+ (\partial_\mu Z^i)(\partial_\mu Z^i) + (Z^i \partial_\mu Z^i)(Z^i \partial_\mu Z^i) \right\}$$

$$= \rho_\Phi^2 \int d^3x \left\{ \left[ \left( \frac{1}{i} \gamma \frac{\partial_\mu}{i} \gamma - 2a_\mu + \frac{1}{i} Z^i \partial_\mu Z^i \right)^2 \right. 

\left. + \frac{1}{2} (\partial_\mu \tilde{\Omega})^2 \right\},$$

(63)

where

$$\gamma := e^{i\Theta}.$$

Note that

$$\left( \frac{1}{i} Z^i dZ^i \right) = 2 \left( \frac{1}{i} z^i dz^i \right) = 2 \sin^2 \frac{\theta}{2} d\gamma$$

In the phase where $\phi_\sigma$ is massive, we can integrate out $\phi_\sigma$. The parts of action involving the single vortex field $\phi_\sigma$ read

$$S_\phi = \int d^3x \left\{ (\partial_\mu - i2a_\mu)\phi_\sigma)^2 + s|\phi_\sigma|^2 + \psi^4 \phi_\sigma \right\}$$

$$= - \lambda_{\rho\phi} \left[ e^{-i\Theta} Z_n \left( \frac{1}{\sqrt{2}} \phi^T E\sigma_n \phi \right) + h.c. \right].$$

To leading order in $\lambda_{\rho\phi}/s$, the result of $\phi_\sigma$ integration (see appendix A for details) is

$$- \frac{\sqrt{s}}{2\pi} \left( \frac{\lambda_{\rho\phi}}{s} \right)^2 \int d^3x \left\{ 7(\partial_\mu \tilde{\Omega})^2 \right. 

\left. + 3 \left( \gamma \frac{\partial_\mu}{i} \gamma - 2a_\mu + \frac{1}{i} Z^i \partial_\mu Z^i \right)^2 \right\}$$

Adding the above result to Eq.(63), we have

$$\int d^3x \left\{ \left( \frac{1}{2\lambda_{\Theta}} \left( \gamma \frac{\partial_\mu}{i} \gamma - 2a_\mu + \frac{1}{i} Z^i \partial_\mu Z^i \right)^2 

\left. + \frac{1}{2\lambda_{\Omega}} (\partial_\mu \tilde{\Omega})^2 \right\} \right.$$

where

$$\frac{1}{2\lambda_{\Theta}} = \rho_\Phi^2 \left( 1 - \frac{3\lambda^2}{32\pi s^{3/2}} \right)$$

$$\frac{1}{2\lambda_{\Omega}} = \rho_\Phi^2 \left( 1 - \frac{7\lambda^2}{16\pi s^{3/2}} \right).$$

Now we generalize the duality transformation for $S = 1/2$ bosons in Ref.[30] to $S = 1$ bosons. First we introduce the Hubbard-Stratonovich field $J^\mu$ and split

$$\gamma = e^{i\Theta} \cdot \gamma_v$$

to obtain

$$\exp \left\{ - \int d^3x \left( \frac{1}{2\lambda_{\Theta}} (\partial_\mu \gamma_v)^2 \right. 

\left. + \frac{1}{2\lambda_{\Theta}} \left( \frac{1}{i} \gamma_v \frac{\partial_\mu}{i} \gamma_v - 2a_\mu + \frac{2}{i} z^i \partial_\mu z^i \right)^2 \right\} \right.$$}

$$= \int D[J^\mu] \exp \left\{ - \int d^3x \left( \frac{1}{2\lambda_{\Theta}} (\partial_\mu \gamma_v)^2 \right. 

\left. + \frac{\lambda_{\theta}}{2} J^\mu \left( \partial_\mu \Theta_s + \frac{1}{i} \gamma_v \frac{\partial_\mu}{i} \gamma_v - 2a_\mu + \frac{2}{i} z^i \partial_\mu z^i \right) \right\}.$$

Second, we perform the integration over $\Theta_s$ and write the path integral over $\gamma_v$ as the sum over the vortex world lines,

$$\sum_{vw} \int D[\Theta_s] D[J^\mu] \exp \left\{ - \int \left[ \frac{1}{2\lambda_{\Theta}} (\partial_\mu \gamma_v)^2 + \frac{\lambda_{\theta}}{2} J^\mu \right. \right.$$}

$$\left. + \frac{\lambda_{\theta}}{2} J^\mu \left( \partial_\mu \Theta_s + \frac{1}{i} \gamma_v \frac{\partial_\mu}{i} \gamma_v - 2a_\mu + \frac{2}{i} z^i \partial_\mu z^i \right) \right\} \right.$$}

$$= \sum_{vw} \int D[J^\mu] \exp \left\{ - \int \left[ \frac{1}{2\lambda_{\Theta}} (\partial_\mu \gamma_v)^2 \right. \right.$$}

$$\left. + \frac{\lambda_{\theta}}{2} J^\mu \left( \partial_\mu \gamma_v - 2a_\mu + \frac{2}{i} z^i \partial_\mu z^i \right) \right.$$}

(65)
Third, we solve the constraint $\partial_{\mu} J^\mu = 0$ by introducing a gauge field

$$J^\mu = \frac{1}{2\pi} \epsilon^{\mu\nu\rho} \partial_{\nu} b_{\rho},$$

after which Eq. (65) becomes

$$\sum_{vw} \int D[b_{\nu}] \exp \left\{ -\left[ \frac{1}{2\lambda_\Omega} (\partial_{\mu} \hat{\Omega})^2 + \frac{\lambda_\Theta}{8\pi^2} (\epsilon^{\mu\nu\rho} \partial_{\nu} b_{\rho})^2 ight] - \frac{i}{2\pi} \epsilon^{\mu\nu\rho} \partial_{\nu} b_{\rho} \left( \frac{1}{i} \gamma^{\mu}_v \frac{\partial_{\mu} \hat{\Omega}}{i} \gamma^{\nu}_v - 2a_{\mu} + \frac{2}{i} z^{\dagger} \gamma^{\mu}_v z \right) \right\}$$

$$= \sum_{vw} \int D[b_{\nu}] \exp \left\{ -\left[ \frac{1}{2\lambda_\Omega} (\partial_{\mu} \hat{\Omega})^2 + \frac{\lambda_\Theta}{8\pi^2} (\epsilon^{\mu\nu\rho} \partial_{\nu} b_{\rho})^2 ight] - \frac{i}{2\pi} \epsilon^{\mu\nu\rho} b_{\nu} \partial_{\rho} \left( \frac{1}{i} \gamma^{\mu}_v \frac{\partial_{\mu} \hat{\Omega}}{i} \gamma^{\nu}_v - 2a_{\rho} + \frac{2}{i} z^{\dagger} \gamma^{\rho}_v z \right) \right\}$$

$$= \sum_{vw} \int D[b_{\nu}] \exp \left\{ -\left[ \frac{1}{2\lambda_\Omega} (\partial_{\mu} \hat{\Omega})^2 + \frac{\lambda_\Theta}{8\pi^2} (\epsilon^{\mu\nu\rho} \partial_{\nu} b_{\rho})^2 ight] + \frac{i}{\pi} \epsilon^{\mu\nu\rho} b_{\nu} \partial_{\rho} a_{\rho} + ib_{\mu} \left( K^\mu_{\gamma_v} + 2K^\mu_{\hat{\Omega}} \right) \right\} (66)$$

where $K^\mu_{\gamma_v}$ and $K^\mu_{\hat{\Omega}}$ are defined by

$$K^\mu_{\gamma_v} = \frac{1}{2\pi} \epsilon^{\mu\nu\rho} \partial_{\nu} \left( \frac{1}{i} \gamma^{\mu}_v \frac{\partial_{\mu} \gamma^{\nu}_v}{i} \right)$$

$$K^\mu_{\hat{\Omega}} = \frac{1}{2\pi} \epsilon^{\mu\nu\rho} \partial_{\nu} \left( \frac{1}{i} z^{\dagger} \partial_{\rho} z \right) = \frac{1}{4\pi} \epsilon^{\mu\nu\rho} \sin \theta \partial_{\nu} \theta \partial_{\rho} \gamma$$

$$= \frac{1}{8\pi} \epsilon^{\mu\nu\rho} e^{abc}\hat{\Omega}^a \cdot \partial_{\nu} \hat{\Omega}^b \partial_{\rho} \hat{\Omega}^c$$

Physically $K^\mu_{\gamma_v}$ is the vortex current in $\gamma_v$, and $K^\mu_{\hat{\Omega}}$ is the skyrmion current in $\hat{\Omega}$. To summarize, after including the Maxwell term for $a_{\mu}$ in Eq. (60), the total effective action reads

$$S_{\text{eff}} = \int d^3x \left[ \frac{1}{2\lambda_\Omega} (\epsilon_{\nu\rho\mu} \partial_{\nu} a_{\rho})^2 + \frac{\lambda_\Theta}{8\pi^2} (\epsilon^{\mu\nu\rho} \partial_{\nu} b_{\rho})^2 \right] + \frac{i}{\pi} \epsilon^{\mu\nu\rho} b_{\nu} \partial_{\rho} a_{\rho} + ib_{\mu} \left( K^\mu_{\gamma_v} + 2K^\mu_{\hat{\Omega}} \right) (67)$$

The parts of Eq. (67) that determine the topological order is the same as Eq. (50) except now there are three quasiparticles, namely, $\alpha = \phi_{\sigma}$, vortex in $\gamma_v$, and skyrmion in $\hat{\Omega}$. The current of these quasiparticles couples to the gauge fields via Eq. (50) where

$$l_{\phi_{\sigma}} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad l_{K_{\gamma_v}} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad l_{K_{\hat{\Omega}}} = \begin{pmatrix} 0 \\ 2 \end{pmatrix}. \quad \text{(68)}$$

Follow the same procedure as in subsection VIII.A, it’s easy to check that all these three quasiparticle are bosons. $K_{\hat{\Omega}}$ has mutual boson statistics with all other particles, while the mutual Berry phase $\phi_{\sigma}$ between and $K_{\gamma_v}$ is $e^{\pi \tau} = -1$. This last fact implies that the bound state of $\phi_{\sigma}$ and $K_{\gamma_v}$ is a fermion. Topologically, the self/mutual statistics of the $\phi_{\sigma}$ particle, the $\gamma_v$ vortex, and the bound state of $\phi_{\sigma}$ and $\gamma_v$ vortex are the same as in the anyon quasiparticles $(e, m, \epsilon)$ in a $Z_2$ spin-liquid, namely,

$$\hat{\Omega} \text{ skyrmion} \rightarrow 1$$

$$\phi_{\sigma} \text{ particle} \rightarrow e$$

$$\gamma_v \text{ vortex} \rightarrow m$$

Bound state of $\gamma_v$ vortex and $\phi_{\sigma}$ particle $\rightarrow \epsilon = e \cdot m$.

In this spin liquid there is a coexisting Néel long-range order. The Goldstone mode of such order is described by the term proportional to $\int d^3x (\partial_{\mu} \hat{\Omega})^2$ in Eq. (67). This coexisting symmetry-enriched $Z_2$ topological order and Néel order is referred to as “AF*” in the literature.

B. The subsequent condensation of single vortex – back to the Néel state

From the AF* phase discussed in the preceding subsection, the condensation of $\phi_{\sigma}$ will Higgs the gauge field $a_{\mu}$, and consequently the $Z_2$ topological order is lost. The resulting phase is the same as the Néel state obtained directly from condensing the single VBS vortex (see Fig. 3). The reason there is no more phase transition in the lower half of the phase diagram in Fig. 3 is the following. Once the single vortex $\phi_{\sigma}$ condenses, the $\lambda$ term in Eq. (60) acts as an "magnetic field term" for $\Phi_n$. Hence $\Phi_n$ will automatically acquire a non-zero expectation value. Consequently, there is no transition when one tunes $S$ from positive to negative.

As mentioned earlier, the scenario depicted in Fig. 3 requires two $S = 1/2$ vortices on the same sublattice (hence the same vorticity) to form a triplet bound state. It is unclear to us what kind of spin interaction will favor that.

X. FINAL DISCUSSION

In this section, we will not talk about the AF* phase in section IX due to the irrelevancy to frustrated antiferromagnets.

As mentioned earlier, it is quite remarkable that the result of subsection VIII.A and VIII.B agree with the result of Schwinger boson parton approach in Ref. [6]. It turns out that the spin-1/2 VBS vortex in the current work plays the role of the Schwinger boson in Ref. [6]. The $U(1)$ gauge field, which is dual to the phase of the VBS order parameter in the current work, plays the role of the gauge field arising from the fractionalization of the physical, spin, into bilinear in Schwinger bosons. Since the spin liquid state possesses $\pi$-flux anyon, the
bosonic spinon can form a bound state with it and become a fermion. The quantum number of such excitation is consistent with the fermionic spinon obtained in the fermionic parton approach of $Z_2$ spin liquid [5, 25].

The nematicity in the spin liquid discussed in subsection VIII A is due to a spontaneous symmetry breaking. It is possible for quantum fluctuations to suppress the nematic order and restore the 90-degree rotation symmetry. In that case, the spin liquid discussed in subsection VIII A will become a true symmetry-enriched $Z_2$ spin liquid (i.e., without symmetry breaking). Similarly, quantum fluctuations can also restore the time-reversal symmetry and render the spin liquid discussed in subsection VIII C a true symmetry-enriched $Z_2$ spin liquid.

Finally, we would like to say a few words about hole doping. Motivated by the results of subsection III B, we expect doping to remove the $S = 1/2$ in the VBS vortex core. Moreover, we can show that after such spin removal, the vortex becomes a spinless fermion. Since the vorticity remains unchanged by such a removal process, naively one expects these fermions to inherit the vorticity quantum number, namely, $f_+$ and $f_-$, and live on opposite sublattices. However, in the spin liquid phase, the double vortex has condensed. This makes the vorticity only defined mod 2. As a result, $f_+$ and $f_-$ are no longer distinct. The resulting single fermion species live on both sublattices. This is consistent with the bosonic spinon approach where the doped holes are spinless fermions. When such a fermion bounds with a neutral bosonic vortex, the vorticity cancels. (Note that in the approach where the doped holes are spinless fermions.

\[ \text{APPENDICES} \]

A. INTEGRATING OUT THE MASSIVE $\phi_\sigma$

A. The general formalism

In integrating out $\phi_\sigma$, we face the following functional integral

\[ Z[a_\mu, \Theta] = e^{-\int Du(x)} e^{-\int S[u(x), a_\mu, \Theta]} \]

where \( S[u(x), a_\mu, \Theta] = \int d^3 x u^T \mathcal{D}[a_\mu, \Theta] u. \) (69)

Here we have adapted Eq.(41) so that $u(x)$ is a real boson field and $\mathcal{D}[a_\mu, \Theta]$ is a symmetric space-time (differential) operator. A convenient trick for doing such integration is to perform the corresponding complex boson integration and divide the resulting effective action by two.

To see this, consider two copies of real boson fields $u_\alpha$ and $u_\beta$ coupled identically to $a_\mu$ and $\Theta$. After the boson integration, the result should be the square of that in Eq.(69), namely,

\[ \int Du_\alpha D u_\beta \ e^{-\int S[u_\alpha(x), a_\mu, \Theta] + S[u_\beta(x), a_\mu, \Theta]} \]

\[ = [Z[a_\mu, \Theta]]^2 = e^{-2W[a_\mu, \Theta]} \]

On the other hand, we can combine $u_{\alpha, \beta}$ into a complex fermion field

\[ \varphi = u_\alpha + i u_\beta, \]
Consequently if \( \tilde{W}[a_\mu, \Theta] \) is the effective action due to the complex boson integration, we have

\[
W[a_\mu, \Theta] = \frac{1}{2} \tilde{W}[a_\mu, \Theta]
\]

Note that the cross terms cancel out, due to the commutation relation between \( u_\alpha \) and \( u_\beta \), and the fact that

\[
[D[a_\mu, \Theta]]^T = D[a_\mu, \Theta].
\]

Consequently if \( \tilde{W}[a_\mu, \Theta] \) is the effective action due to the complex boson integration, we have

\[
W[a_\mu, \Theta] = \frac{1}{2} \ln \left[ \frac{\det(D[a_\mu, \Theta])}{\det(D_0)} \right] = \frac{1}{2} \text{Tr} \ln(D[a_\mu, \Theta])]. \tag{70}
\]

Due to Eq.(70), we shall focus on the complex boson integration.

To proceed we write

\[
D[a_\mu, \Theta] = D_0 + V[a_\mu, \Theta] = (-\partial^2 + s) + V[a_\mu, \Theta], \tag{71}
\]

where \( V[a_\mu, \Theta] \) includes the terms describing the coupling of \( \phi_\sigma \) to the gauge field and \( \Theta \). By the usual perturbative expansion we have

\[
W[a_\mu, \Theta] = \frac{1}{2} \text{Tr} \ln(D[a_\mu, \Theta]) \approx \frac{1}{2} \text{Tr} \ln(D_0) + \sum_{n=1}^{\infty} \left( \frac{-1}{n} \text{Tr} (D_0^{-1} V)^n \right).
\]

The control parameter of this expansion will be discussed separately for the spin-triplet parity-even and spin-singlet odd parity double vortex cases. Finally, the \( \text{Tr} \ln(D_0) \) term is a constant independent of the external field so that we can neglect it.

B. Spin-singlet odd-parity double vortex condensation

For spin-triplet-parity-even double vortex the part of action that involves \( \phi_\sigma \) is given by

\[
S_\phi = \int d^3x \left\{ |(\partial_\mu - iA_\mu)\phi_\sigma|^2 + s|\phi_\sigma|^2 - \lambda \sum_{i=1}^2 \left[ \rho_i e^{-\Theta} \left( \frac{1}{\sqrt{2}} \phi^T E \partial_\mu \phi \right) + \text{h.c.} \right] \right\}.
\]

Here we do the transformation

\[
\phi \to e^{i \frac{\Theta}{2}} \phi,
\]

under which the action becomes

\[
S_\phi = \int d^3x \left\{ |(\partial_\mu - iA_\mu)\phi_\sigma|^2 + s|\phi_\sigma|^2 - \lambda \sum_{i=1}^2 \left[ \rho_i \left( \frac{1}{\sqrt{2}} \phi^T E \partial_\mu \phi \right) + \text{h.c.} \right] \right\}
\]

where

\[
A_\mu := a_\mu - \frac{i}{2} \partial_\mu \Theta \tag{72}
\]

is a U(1) gauge field.

In terms of the real boson field in Eq.(69), the potential \( V[a_\mu, \Theta] \) in Eq.(71) is

\[
V[a_\mu, \Theta] = \int d^3x \left\{ -i\sqrt{2} \lambda \sum_{i=1}^2 \rho_i \tilde{M}_i \partial_i \times \right. \]

\[
+ i(\partial_\mu A_\mu + A_\mu \partial_\mu) \Sigma_0 + A_\mu A_\nu \right\}
\]

where \( A_\mu \) is given by Eq.(72), and \( \rho_{1i} \) and \( \rho_{2i} \) are the real and imaginary parts of \( \rho_i \), and

\[
\Sigma_0 = -Y I, \quad \tilde{M}_1 = ZY, \quad \tilde{M}_2 = XY.
\]

The remaining real boson integration is straightforward. The leading order (in \( \lambda \rho_i / \sqrt{s} \)) contribution gives

\[
W = \frac{\sqrt{s}}{24\pi} \sum_{i=1}^2 \int A_i [\rho_{ai} \rho_{ai} \delta_{ij} - \rho_{ai} \rho_{aj}] A_j
\]

For the nematic spin liquid in Eq.(39), we have \( \rho_i = \delta_{ix} \rho \). This gives us

\[
W = \frac{\sqrt{s}}{24\pi} \left( \frac{\lambda \rho}{\sqrt{s}} \right)^2 \int d^3x A_y^2
\]

\[
= \frac{\sqrt{s}}{96\pi} \left( \frac{\lambda \rho}{\sqrt{s}} \right)^2 \int d^3x (\partial_y \Theta - 2a_y)^2
\]

In the case of the time-reversal breaking spin liquid we have \( \rho_1 = \rho \) and \( \rho_2 = ip \). This results in

\[
W = \frac{\sqrt{s}}{24\pi} \left( \frac{\lambda \rho}{\sqrt{s}} \right)^2 \int d^3x (A_i^2 + A_j^2)
\]

\[
= \frac{\sqrt{s}}{96\pi} \left( \frac{\lambda \rho}{\sqrt{s}} \right)^2 \int d^3x \sum_{i=1}^2 (\partial_i \Theta - 2a_i)^2.
\]

Again the presence of the prefactor \( \sqrt{s} \) can be determined from dimension counting.
C. Spin-triplet even-parity double vortex condensation

For spin-triplet-parity-even double vortex, the term in the action which involves $\phi_\sigma$ is given by

$$S_\phi = \int d^3 x \left\{ |(\partial_\mu - i a_\mu) \phi_\sigma|^2 + s|\phi_\sigma|^2 \right. $$

$$- \lambda \rho_\phi \left[ e^{-i \Theta} Z_n^* (\hat{\Omega}) \left( \frac{1}{\sqrt{2}} \phi^T E \sigma_n \phi \right) + h.c. \right] \right\}$$

Before continuing, note that we can do the transformation

$$\phi \rightarrow e^{i \frac{\pi}{6}} u^\dagger \phi,$$

where

$$u(\hat{\Omega}) = \left( \begin{array}{c} \cos \frac{\phi}{2} \\ \sin \frac{\phi}{2} e^{-i \phi} \end{array} \right),$$

and action becomes

$$S_\phi = \int d^3 x \left\{ |(\partial_\mu - i A_\mu) \phi_\sigma|^2 + s|\phi_\sigma|^2 \right.$$

$$- \lambda \rho_\phi \left[ Z_n^* (\hat{\Omega}) \left( \frac{1}{\sqrt{2}} \phi^T E \sigma_n \phi \right) + h.c. \right] \right\}$$

Here

$$Z_n (\hat{\Omega}) = \frac{1}{\sqrt{2}} (1, i, 0),$$

and $A_\mu$ contains both the $U(1)$ and the $SU(2)$ parts, namely,

$$A_\mu := a_\mu - \frac{1}{2} \partial_\mu \Theta - \frac{1}{2} i u \partial_\mu u^\dagger$$

$$= \sum_{a=0}^3 A_\mu^a \sigma_a$$

where

$$A_\mu^0 = a_\mu - \frac{1}{2} \partial_\mu \Theta$$

$$A_\mu^1 = \frac{1}{2} (\sin \gamma \partial_\mu \theta + \sin \theta \cos \gamma \partial_\mu \gamma)$$

$$A_\mu^2 = \frac{1}{2} (\cos \gamma \partial_\mu \theta + \sin \theta \sin \gamma \partial_\mu \gamma)$$

$$A_\mu^3 = -\frac{1}{2} (1 - \cos \theta) \partial_\mu \gamma = -\frac{1}{i} z^\dagger \partial_\mu z.$$

In terms of the real boson field in 69, the coupling potential is

$$V[a_\mu, \Theta, \hat{\Omega}] = -\sqrt{2} \alpha Z_{an} (\hat{\Omega}) M_{an}$$

$$+ i (\partial_\mu A_\mu^a + A_\mu^a \partial_\mu) \Sigma_a + A_\mu^a A_\mu^b \Sigma_a \Sigma_b$$

where $\alpha = 1, 2$ and $a, b, n = 1, 2, 3$. The $Z_{1n}(\hat{\Omega})$ and $Z_{2n}(\hat{\Omega})$ are the real and imaginary parts of $Z_n(\hat{\Omega})$, and

$$M_{1n} = (ZZ, -XI, -ZX)$$

$$M_{2n} = (-XZ, -ZI, XX)$$

$$\Sigma_a = (-YI, -YX, XY, -YZ)$$

The remaining perturbative integration of the boson is straightforward. The leading order (in $\lambda \rho_\phi / s$) contribution gives

$$W[\Theta, \Omega, a_\mu]$$

$$= -\frac{\sqrt{2}}{32 \pi} \left( \frac{\lambda \rho_\phi}{s} \right)^2 \int \left\{ 28 [(A_\mu^0)^2 + (A_\mu^2)^2] + 12 (A_\mu^0 + A_\mu^2)^2 \right\}$$

$$= -\frac{\sqrt{2}}{32 \pi} \left( \frac{\lambda \rho_\phi}{s} \right)^2 \int \left\{ 7 (\partial_\mu \Theta)^2 + 3 (2 a_\mu - \partial_\mu \Theta - \frac{2}{i} z^\dagger \partial_\mu z)^2 \right\}$$

The presence of the prefactor $\sqrt{s}$ is can be determined by dimension counting $^6$

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$^6$ The reason the dimensionless parameter is $\lambda \rho_\phi / s$ instead of $\lambda \rho_\phi / \sqrt{s}$ is that in the absence of derivative, the dimension of the coupling $\lambda$ changes.
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