The strong amalgamation property into union

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Abstract. We consider the situation in which some class of structures has the Strong Amalgamation Property (SAP) with the further requirement that the amalgamating structure can be taken over the set theoretical union of (the images of) the domains of the structures to be amalgamated. We call this property SAPU.

The main advantage of SAPU over SAP is that there are many preservation theorems showing that we can merge different theories with SAPU still obtaining a theory with SAPU, hence with SAP. In particular, we get SAPU for various theories with many binary relations, each relation satisfying any set of properties chosen among transitivity, reflexivity, symmetry, antireflexivity, antisymmetry. We may also add unary operations, possibly satisfying some coarseness, isotonicity and closure conditions.

SAPU is not limited to relational theories: the varieties defining the most usual Maltsev conditions in universal algebra have SAPU. Other examples include bounded directoids, order algebras and various generalizations.

1. Introduction.

The Amalgamation Property (AP) has found many important applications in algebra, logic, category theory and, recently, computer science. See, e. g., [BGR, H, GM, GG, J, KMPT, Mac, Mad, MMT].

We study theories with the amalgamation property and with the further requirement that the amalgamating structure can be taken over the set-theoretical union of the (images of the) structures to be amalgamated. This applies to any universal theory in a purely relational language and with AP, in particular, this is the case for partially ordered sets (henceforth posets, for short), more generally, for structures with a binary relation satisfying any (fixed in advance) number of the following properties: transitivity, reflexivity, symmetry, antireflexivity, antisymmetry. See Proposition 3.2.

The advantage of AP over Union (APU) is that we can frequently merge different theories with APU, still obtaining a theory with APU, hence with AP. For example, this holds for theories with a pair of binary relations as above, possibly with the condition asserting that one relation is coarser than (i. e.,...
contains) another. See Theorem 3.4. In general, the result fails when three relations are taken into account.

On the other hand, we do have APU when all the relations under consideration are assumed to be transitive; this applies to an arbitrary number of relations. In particular, APU holds for any class of multiposets with any prescribed set of coarseness relations. With a few exceptions, APU is maintained when unary operations preserving one or more relations are added; in particular, all multiposets with operators have APU. See Theorems 4.2 and 4.10. Many applications of APU are relatively simple, however, most of the results are really fine-tuned, in that, just weakening some assumption, counterexamples can be found, e. g., Theorem 3.6(7), Examples 3.7, 4.5(d), 7.7, 7.10, 7.11, 7.12, and Propositions 3.5, 6.4, 7.5, 7.6.

There are results special to APU which generally do not hold for AP. If $T$ is a theory with APU and we add to $T$ universal-existential sentences in which only one variable is bounded by the universal quantifier, then the resulting theory has still APU. In particular, if some class of partially ordered sets with a unary operation has APU, we still have APU if we ask that the operation is an involution, or that the operation, if order preserving, is a closure operation. In general, there are plenty of conditions $H$ such that if $K$ is a class with APU, then the subclass of those structures in $K$ satisfying $H$ maintains APU. See Section 4. Counterexamples are provided for theories with AP not into Union.

In another direction, studying APU is useful for discovering results which hold in general for AP. Dealing with the Strong APU (see below for the definition), it is almost immediate to show that the union of theories in disjoint languages with SAPU has still SAPU. This fact turns out to be true for SAP, as well, but with a not entirely trivial argument. Counterexamples exist showing that it is necessary to deal with the strong variants.

At first sight, the reader might expect that APU is a phenomenon almost exclusively limited to relational languages. This is not the case. On one hand, we can consider unary functions, usually getting APU “almost for free”. On the other hand, there is a bunch of examples of theories with APU in languages with $n$-ary functions, for $n \geq 2$. See Section 5 but also Propositions 4.1 and 4.3. Not only APU has many interesting and useful consequences, but it is applicable to a number of nontrivial examples.

2. The Strong Amalgamation Property into Union.

We work with classes of structures with finitary relations and functions. Structure and model are synonymous. As usual in model theory, equality is considered as a logical symbol, namely, it can be interpreted in every structure, and it is actually interpreted as identity. Under a frequent terminology, this means that we work with normal models. In particular, we do not include equality in the symbols belonging to some language $\mathcal{L}$, so that when $\mathcal{L} = \emptyset$, then $\mathcal{L}$ is the pure language of identity.
We do not take explicit position on the admissibility or not of structures with empty domain. Generally, our results hold in both settings; otherwise, we shall mention the assumptions explicitly.

An embedding $\iota$ from some structure $A$ into a structure $B$ for the same language is an injective function $\iota: A \to B$ such that, for every $a_1, a_2, \ldots \in A$, the following hold:

$$\iota(f_A(a_1, a_2, \ldots)) = f_B(\iota(a_1), \iota(a_2), \ldots),$$

for every function symbol $f$ in the language, and

$$R_A(a_1, a_2, \ldots) \text{ if and only if } R_B(\iota(a_1), \iota(a_2), \ldots),$$

for every relation symbol $R$ in the language.

If we drop the requirement of injectivity and weaken condition (2.1) to

$$R_A(a_1, a_2, \ldots) \text{ implies } R_B(\iota(a_1), \iota(a_2), \ldots),$$

we get the weaker notion of a homomorphism. Here we shall consider amalgamation properties with respect to embeddings. Were we considering injective homomorphisms, instead, we would get completely different results. See Remark 7.1 below.

We shall possibly deal also with constants (= selected elements). Embeddings and homomorphisms are assumed to satisfy $\iota(c_A) = c_B$, for every constant symbol $c$. Subscripts will be dropped when no risk of confusion might arise. Full formal details about the above notions can be found in any textbook on model theory, e. g., [H].

If $A$ and $B$ are structures for the same language and $A \subseteq B$ (as sets), then we say that $A$ is a substructure of $B$ if the inclusion map from $A$ to $B$ is an embedding of $A$ into $B$. In the above situation we shall write $A \subseteq B$.

**Definition 2.1.** (a) A class $\mathcal{K}$ of structures for the same language has the amalgamation property (AP) if, whenever $A, B, C \in \mathcal{K}$, $\iota_{C,A}: C \to A$ and $\iota_{C,B}: C \to B$ are embeddings, then there is a structure $D \in \mathcal{K}$ and embeddings $\iota_{A,D}: A \to D$ and $\iota_{B,D}: B \to D$ such that $\iota_{C,A} \circ \iota_{A,D} = \iota_{C,B} \circ \iota_{B,D}$. Namely, the following diagram can be commutatively completed as requested.

(b) A class of structures has the strong amalgamation property (SAP) if, under the assumptions in (a), the conclusion can be strengthened to the effect that the intersection of the images of $\iota_{A,D}$ and $\iota_{B,D}$ is equal to the image of $\iota_{C,A} \circ \iota_{A,D}$ (hence also of $\iota_{C,B} \circ \iota_{B,D}$).

(c) A class of structures has the (strong) amalgamation property into union (SAPU) if, in addition, $D$ can be chosen in such a way that its domain is the union of the images of $\iota_{A,D}$ and $\iota_{B,D}$. 

\[ \begin{array}{ccc}
A & \to & D \\
\downarrow & & \downarrow \\
B & \to & \text{completes to}
\end{array} \]

\[ \begin{array}{ccc}
A & \to & B \\
\downarrow & & \downarrow \\
C & \to & C
\end{array} \]
Here “union” is meant in the set theoretical sense, not in the model theoretical sense: we are considering the union of the domains, not of the structures. See below for more details.

If $T$ is a first-order theory, we say that $T$ has AP, SAP, APU, SAPU if the class of models of $T$ has the property.

Formally, we assume that an empty class of structures shares all the above properties. This is consistent with standard conventions about universal quantification over empty domains.

**Remark 2.2.** In our present context, if $\mathcal{K}$ is closed under isomorphism, then the above definitions (b) and (c) can be simplified.

(d) A class $\mathcal{K}$ closed under isomorphism has SAP (resp., SAPU) if and only if, whenever $A, B, C \in \mathcal{K}$, $C \subseteq A$, $C \subseteq B$ and $C = A \cap B$, then there is a structure $D \in \mathcal{K}$ such that $A \subseteq D$, $B \subseteq D$ (and, resp., $A \cup B = D$).

We shall sometimes informally refer to a triple $A, B, C$ as above as a *triple to be amalgamated*, for short, a *TBA triple*.

For simplicity, we shall generally work in the simplified setting described in the previous paragraph, namely, we shall deal with inclusions as above, rather than with arbitrary embeddings as in Definition 2.1. In particular, we shall always assume that classes of structures are closed under isomorphism. In any case, the setting in which we work shall always be clear from the context.

Results about AP and SAP appear scattered in the literature, sometimes in different settings or terminology. A survey of results about AP and related properties appears in [KMPT], where the notions are also inserted in a general categorical framework. A survey of various applications of AP to model theory can be found in [H].

For relational languages, the special case of SAPU when $D$ can be taken as the model-theoretical union of $A$ and $B$ has been considered by various authors, generally under the name *free amalgamation*. This is the particular case when $R_D = R_A \cup R_B$, for every relation symbol $R$ in the language. For example, see [Bo, F2, Mac] and further references there.

To the best of our knowledge, the explicit definitions of APU and SAPU in the general case are new when considered for a whole class of structures. The first implicit appearance of SAPU possibly occurs in Fraïssé argument [F1, Section 9] showing that the class of linear orders and the class of posets have SAP. Compare also Jónsson [J, Lemma 2.3]. In another direction, particular situations in which the amalgamating structure can be taken over $A \cup B$ have been considered in lattice theory. See [G, IV, Section 2.3 and VI, Exercise 4.11] and further references there.

The main interest of SAPU comes from the fact that there are various methods to join or modify some theories with SAPU in order to obtain other theories with SAPU. See Section [I] below. In particular, if we merge distinct theories in disjoint languages and having SAPU, we still obtain a theory with SAPU. See Proposition [I] below. The result is true also for SAP (Proposition
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8.1), however, in the case of SAPU the proof is simpler and, as the main point in the present note, in certain cases the argument applies also to theories which are not in disjoint languages. See Theorems 4.2 and 4.10.

Notice that there are theories with both SAP and APU but without SAPU. See Proposition 6.4 and Example 7.2. Compare also Example 7.9.

If $R$ is a binary relation on some set, we frequently write $a \mathrel{R} b$ in place of $R(a, b)$ or $(a, b) \in R$. Moreover, $a \mathrel{R} b \mathrel{S} c$ is a shorthand for $a \mathrel{R} b$ and $b \mathrel{S} c$.

In most cases, we shall prove a property somewhat stronger than SAPU.

**Definition 2.3.** If $K$ is a class of structures closed under isomorphism and in a language with a binary relation symbol $R$, we say that $K$ has superSAPU (resp., superSAP) with respect to $R$ if, whenever $A, B, C \in K$ is a triple to be amalgamated as in Remark 2.2, then there exists an amalgamating structure $D \in K$ witnessing SAPU (resp., SAP) and such that, for every $a \in A \setminus B$ and $b \in B \setminus A$,

(i) if $a \mathrel{R_D} b$ then there is $c \in C$ such that $a \mathrel{R_A} c \mathrel{R_B} b$, and
(ii) if $b \mathrel{R_D} a$ then there is $c \in C$ such that $b \mathrel{R_B} c \mathrel{R_A} a$.

Whenever we speak of superSAP(U), we always assume that at least one relation $R$ is specified, and superSAP(U) is meant with respect to the specified relation(s). We shall omit the reference to the relations, when they are understood.

Usually, the superamalgamation property (not necessarily into union) is considered with respect to some ordering relation, see [GM], but many applications to algebraic logic are known even in the case of an arbitrary binary relation, see [Mad]. The assumption that $a \in A \setminus B$ and $b \in B \setminus A$ in the hypothesis of superamalgamation properties is frequently weakened to $a \in A$ and $b \in B$. We shall use the modified version as in Definition 2.3 in order to simplify statements. Of course, the two definitions are equivalent if $R$ is a reflexive relation (since if, say, $a \in A \cap B$ and $a \mathrel{R} b$, then we can take $c = a$ in (i), and similarly for (ii)).

Another property related to AP has proven very important in model theory.

**Definition 2.4.** The joint embedding property is the special instance of AP when the structure $C$ is empty. In detail, a class $K$ of structures for the same language has the joint embedding property (JEP) if, for every $A, B \in K$, there are a structure $D \in K$ and embeddings $\iota : A \rightarrow D$ and $\kappa : B \rightarrow D$.

In Section 6 we shall present and study variations on JEP in the same spirit of Definition 2.1.

**Remark 2.5.** As mentioned, when we allow the empty structure, JEP is a special instance of AP. However, there are situations in which it is not convenient to consider the empty structure or, plainly, such a structure does not exist, e. g., when the language has some constant.

Nevertheless, even assuming AP only for nonempty structures, we can always reduce ourselves to a situation in which JEP holds. As well-known,
assuming AP for nonempty structures, if we set $A \sim B$ when $A$ and $B$ can be joint embedded into some $D$ as in Definition 2.4, then $\sim$ turns out to be an equivalence relation, hence JEP holds when restricted to each equivalence class. Clearly, AP is maintained relative to each equivalence class, so that we have both AP and JEP on each equivalence class.

If $\mathcal{K}$ is a class of finitely generated structures in a countable language, a Fraïssé limit of $\mathcal{K}$ is a countable ultrahomogeneous structure of age $\mathcal{K}$. See [H, Section 7.1] for details. A countable (up to isomorphism) hereditary class $\mathcal{K}$ with AP and JEP has a Fraïssé limit [H, Theorem 7.1.2].

Throughout, we suppose that all classes of models under consideration are closed under taking isomorphism. Compare Remark 2.2.

3. Orders, binary relations, adding operators.

For relational languages the next observation is folklore. Here and in similar situations below we point out the lesser known fact that the argument carries over when considering also unary function symbols.

Observation 3.1. (a) A universal theory in a language without function symbols of arity $\geq 2$ has APU (resp., SAPU, superSAPU) if and only if it has AP (resp., SAP, superSAP).

(b) More generally, suppose that $\mathcal{K}$ is a class of structures for a language without function symbols of arity $\geq 2$ and suppose further that $\mathcal{K}$ is preserved under taking substructures. Then $\mathcal{K}$ has APU (resp., SAPU, superSAPU) if and only if $\mathcal{K}$ has AP (resp., SAP, superSAP).

(c) The class of all models for some given language—in other words, any theory without nonlogical axioms—has SAPU and, in the presence of a binary relation, superSAPU.

Proof. (a) To prove the non trivial implication, under the assumptions of (super)(S)APU, take some $D$ witnessing (super)(S)AP and consider the union $D_1$ of the images of $\iota_{A,D}$ and of $\iota_{B,D}$. Since $T$ is universal, then the restriction $D_1$ of $D$ to $D_1$ is a model of $T$, thus $D_1$ witnesses (super)(S)APU. Notice that $D_1$ is actually a structure, since it is the union of two structures and then unary functions do not send elements of $D_1$ outside. As far as superSAP(U) is concerned, notice that the defining property for superSAP(U) speaks only of elements in the images of $\iota_{A,D}$ and $\iota_{B,D}$, of elements of $A \cup B$, in the simplified setting of Remark 2.2.

(b) is proved in a similar way.

(c) The argument generalizes [F1] 9.1. Relations cause no trouble and operations can be extended in an arbitrary way in the union, since no axiom is prescribed. □
Most results in the present section are taken from \[L1, L3\], where they are stated without the specification “into Union”. However, the “U” follows from Observation 3.1 (and, in any case, directly from the original proofs).

See \[L3\] for historical comments concerning the next proposition.

**Proposition 3.2.** \[L3\] Consider the following properties of a binary relation R.

1. R is transitive;
2. R is reflexive;
3. R is symmetric;
4. R is antireflexive, that is, \(x R x\) never holds;
5. R is antisymmetric.

Then the following statements hold.

(A) For every \(P \subseteq \{1, 2, 3, 4, 5\}\), the class \(K_P\) of the structures with a binary relation R satisfying the corresponding properties has superSAPU.

(B) For each \(P \subseteq \{1, 2, 3, 4, 5\}\), let \(K^f_P\) be the class of structures with an added unary operation \(f\) which is R-preserving, that is,

\[
x R y \implies f(x) R f(y). \tag{3.1}
\]

Then \(K^f_P\) has superSAPU.

(C) For each \(P \subseteq \{1, 2, 3, 4, 5\}\), let \(K^g_P\) be the class of structures with an added unary operation \(g\) which is R-reversing, that is,

\[
x R y \implies g(y) R g(x). \tag{3.2}
\]

Then \(K^g_P\) has superSAPU.

(D) For every \(P \subseteq \{1, 2, 3, 4, 5\}\) and for every pair \(F, G\) of sets, let \(K^F,G_P\) be the class of models obtained from members of \(K_P\) by adding an \(F\)-indexed set of unary operations satisfying (3.1) and a \(G\)-indexed set of unary operations satisfying (3.2).

Then \(K^F,G_P\) has superSAPU.

**Sketch of proof.** Assume that A, B, C is a TBA triple (a Triple to Be Amal-gamated). The proof of (A) is divided into two cases.

(A, case a) Suppose that \(1 \notin P\). Then let R on \(A \cup B\) be defined by R = \(R_A \cup R_B\). Namely, \(d R e\) if and only if

either \(d, e \in A\) and \(d R_A e\), or \(d, e \in B\) and \(d R_B e\). \(\tag{3.3}\)

Let \(D = (A \cup B, R)\).

(A, case b) Suppose that \(1 \in P\). Let \(d R e\) if either (3.3), or

\(d \in A, e \in B\) and there is \(c \in C\) such that \(d R_A c R_B e\), or

\(d \in B, e \in A\) and there is \(c \in C\) such that \(d R_B c R_A e\). \(\tag{3.4}\)
Thus in this case we set \( R = R_A \cup R_B \cup (R_A \circ R_B) \cup (R_B \circ R_A) \). Again, let \( D = (A \cup B, R) \).

In both cases it can be checked that \( D \) superamalgamates \( A \) and \( B \) over \( C \) and \( R \) on \( D \) satisfies the required conditions. This proves (A).

Given the proof of (A), clauses (B), (C) and (D) are proved by observing that the operations \( f \) and \( g \) are uniquely defined on \( D \), since \( D = A \cup B \), and then by checking that \( f \) satisfies the required properties. \( \square \)

In order to get APU, operations should be unary in \( 3.2 \) (B)-(D). See Proposition 7.4

**Corollary 3.3.** The following classes have superSAPU and JEP:

(a) partially ordered sets, (b) preordered sets, (c) undirected graphs (sets with a symmetric antireflexive binary relation), (d) directed graphs (sets with an antireflexive binary relation), (e) sets with an equivalence relation, (f) sets with a binary transitive relation, (g) sets with a binary symmetric and reflexive relation (a tolerance).

SuperSAPU and JEP are maintained when an \( F \)-indexed family of relation-preserving \( 3.1 \) unary operations and a \( G \)-indexed family of relation-reversing \( 3.2 \) unary operations are added.

In Section 4 below we shall deal with AP for structures with many binary relations. We now present a theorem (from \( L3 \)) which does not seem to follow from the results in the subsequent sections.

Given two binary relations \( R \) and \( S \) on the same domain, we say that \( S \) is coarser than \( R \) if \( R \subseteq S \), more explicitly, if \( a R b \) implies \( a S b \), for all \( a \) and \( b \) in the domain. If this is the case, we shall also say that \( R \) is finer than \( S \). Notice that we shall always use the expression “coarser” in the sense of “coarser than or equal to”.

**Theorem 3.4.** \( L3 \)

(A) For every pair \( P, Q \subseteq \{1, 2, 3, 4, 5\} \), the class \( K_{P,Q} \) of structures with

(a) a binary relation \( R \) satisfying the properties from \( P \) and
(b) a coarser relation \( S \) satisfying the properties from \( Q \)

has SAPU. Actually, superSAPU holds both with respect to \( R \) and \( S \).

(B) SuperSAPU is maintained if we add families of

(i) unary operations which are both \( R \)- and \( S \)-preserving;
(ii) unary operations which are both \( R \)- and \( S \)-reversing;
(iii) unary operations which are \( R \)-preserving;
(iv) unary operations which are \( R \)-reversing;

SuperSAPU is maintained if we consider the subclass consisting of those structures satisfying any set of properties among those we shall list in Propositions 4.5 4.7 and Theorem 4.10 below.

(C) On the other hand, the class of structures with a transitive relation \( R \), a coarser binary relation \( S \) and an \( S \)-preserving function \( f \) has not AP (here we do not include the condition that \( f \) is \( R \)-preserving).
(D) Similarly, the class of structures with a partial order \( \leq \), a coarser symmetric and reflexive relation \( S_1 \) and an \( S_1 \)-preserving function \( f \) has not AP.

**Proposition 3.5.** \[L3\] The following theories do not have AP.

(a) The theory of an antisymmetric relation \( S \) with two partial orders \( \leq \) and \( \leq' \) both finer than \( S \).

(b) The theory of an antisymmetric relation \( S \) with two transitive relations both finer than \( S \).

(c) The theory of a partial order with a coarser antisymmetric relation \( S \) and a bijective \( S \)-preserving unary operation.

Notice that either (a) or (b) in Proposition 3.5 shows that a universal Horn theory in a pure relational language does not necessarily have AP.

We now recall some results from \[L1\]. A generalization of Corollary 3.3(2) to linear orders holds, but, rather unexpectedly, it holds only in the case of just one additional operation.

If \( f \) is an unary operation on a poset, we say that an element \( c \) is a center, or a fixed point for \( f \) if \( f(c) = c \). In general, when we refer to a center \( c \) without further specifications, we shall mean that \( c \) is a center for all the operations under consideration.

**Theorem 3.6.** \[L1\] The following classes have SAPU.

1. The class of linearly ordered sets with one order preserving unary operation.
2. The class of linearly ordered sets with one order reversing unary operation with a center.
3. For every set \( F \), the class of linearly ordered sets with an \( F \)-indexed family of order automorphisms.
4. For every pair \( F \) and \( G \) of sets, the class of linearly ordered sets with an \( F \)-indexed family of order automorphisms and a \( G \)-indexed family of bijective order reversing unary operations, all operations from both families with a common center.

The following classes have APU but not SAP.

5. The class of linearly ordered sets with one strict order preserving unary operation.
6. The classes of linearly ordered sets with one order reversing, resp., one strict order reversing unary operation.

The following classes have not AP.

7. The classes of linearly ordered sets with two order preserving, resp., two strict order preserving unary operations.
8. More generally, the classes of linearly ordered sets with two unary operations and with each operation either order preserving, or strict order preserving, or order reversing, or strict order reversing.
Of course, when dealing with linearly ordered sets, we cannot have super-
SAP. Given $A$, $B$ and $C$ linearly ordered sets to be amalgamated, if $a \in A \setminus C$, $b \in B \setminus C$ and $\{c \in C \mid c <_A a\} = \{c \in C \mid c <_B b\}$, then in any strong amalgamating linear order we have either $a < b$ or $b < a$, but no such relation is witnessed by means of some $c \in C$.

Theorem 3.6(1) does not hold for binary operations which are order pre-
serving on each component, as we are going to show in the next example.

Example 3.7. The class of linearly ordered sets with a binary operation which is order preserving on each component has not AP.

Take $C = \{c\}$, $A = \{a, c\}$, with $c < a$ and a binary function $f$ defined by $f(a, c) = c$, $f(c, a) = f(a, a) = a$, and $B = \{b, c\}$, with $c < b$ and $f$ defined by $f(c, b) = c$, $f(b, c) = f(b, b) = b$.

In any amalgamating algebra we cannot have $b \leq a$, since then $b = f(b, c) \leq f(a, c) = c$, as $f$ is requested to be order preserving on the first component, Symmetrically, we cannot have $a \leq b$, hence $A$, $B$ and $C$ cannot be amalga-
mated to a linear order.

4. Preservation conditions.

We now show how to construct new theories with SAPU starting from some theories with the property. For theories in a purely relational language, Part (a) in the following proposition is folklore. Then Part (b) is an immediate consequence, but it seems to have not received due attention in the literature.

Proposition 4.1. (a) If $(T_i)_{i \in I}$ is a sequence of theories in pairwise disjoint languages and each $T_i$ has SAPU, then $T = \bigcup_{i \in I} T_i$ has SAPU.

(b) If $T_1$ is a theory in some language $\mathcal{L}$ and $T_1$ has (super)SAPU, then $T_1$ has (super)SAPU even when considered as a theory in some language $\mathcal{L}' \supseteq \mathcal{L}$.

Proof. (a) As in Remark 2.2 suppose that $A$, $B$, $C$ is a TBA triple consisting of models of $T$. For each $i \in I$, the reducts to the language of $T_i$ can be amalgamated to a model over $A \cup B$. Since the languages are pairwise disjoint, we get a model of the whole $T$ over $A \cup B$.

To prove (b), let $T_2$ be the empty theory in the language $\mathcal{L}' \setminus \mathcal{L}$. By Observation 3.1(c), $T_2$ has SAPU. Then apply the first statement. If $T_1$ has superSAPU, then superSAPU is maintained by construction. \qed

See Proposition 8.2 below for a result analogue to Proposition 4.1(a) for the superamalgamation property.

We shall see in Proposition 8.1 that the analogue of Proposition 4.1 holds when we replace SAPU with SAP. However, the proof of Proposition 4.1 is much simpler and the method of proof can be applied to more situations, see Theorems 4.2 and 4.10. Moreover, there are results holding for SAPU but not for (S)AP: compare Proposition 4.3 with Example 7.10 below. See also Remark 7.3 and Example 7.9. The assumption that the $T_i$’s have SAPU in
Proposition 4.1 cannot be weakened to APU. See Example 7.7 below. Compare also Example 7.10(b). A slightly more general version of Proposition 4.1 is stated as Proposition 7.8 below.

In order to present the following results in due generality, we need to introduce some terminology and notation. An \((I\)-indexed) multiposet is a set endowed with a family \((\leq_i)_{i \in I}\) of partial orders. It is immediate from Proposition 4.1 and Corollary 3.3 that, for every set \(I\), the class of all the \(I\)-indexed multiposets has SAPU. We are going to prove a more general fact about multiposets on which some coarseness conditions are assumed. In contrast with Theorem 3.4, here we impose no bound on the cardinality of \(I\).

A coarseness condition on an \((I\)-indexed) multiposet is a condition of the form “\(\leq_i\) is coarser than \(\leq_j\)”, for some pair \((i, j)\), with \(i, j \in I\). Thus a family of coarseness conditions is (represented by) a subset \(F\) of \(I \times I\): we are asking that \(\leq_i\) is coarser than \(\leq_j\) for all pairs \((i, j)\) \(\in F\).

If \(K\) is a class of structures, let us denote by \(K_{\text{fin}}\) the class of the finite members of \(K\).

**Theorem 4.2.** (1) For any index set \(I\), the class of all \(I\)-indexed multiposets satisfying any given family of coarseness conditions on \(I\) has SAPU, actually, superSAPU with respect to each \(\leq_i\).

(2) Suppose that \(J \subseteq I\), \(F\) is a family of coarseness conditions on \(I\) and let \(K\) be the class of all \(I\)-indexed multiposets which satisfy the coarseness conditions in \(F\) and such that all orders \(\leq_j\) with \(j \in J\) are linear. Then \(K\) has SAPU.

(3) In particular, the theory of a partial order \(\leq\) together with a linearization of \(\leq\) has SAPU.

If \(I\) is finite, then in each case the class of finite structures has a Fraïssé limit \(M\) and the first-order theory \(\text{Th}(M)\) is \(\omega\)-categorical and has quantifier elimination; moreover, \(\text{Th}(M)\) is the model completion of the first-order theory axiomatizing the class under consideration.

**Proof.** (1) Formally, the theorem is not a consequence of Proposition 4.1. However, if we apply the proof of Proposition 3.2 simultaneously for all the relations involved, we get a structure for the appropriate language. Since the order relations are all transitive, we are always in case b, hence coarseness is preserved.

(2) If \(\leq_j\) is a linear order on \(A\), \(B\) and \(C\), then the proof of Proposition 3.2 generally provides only a partial order \(\leq_{j, D}\); however, any linearization of \(\leq_{j, D}\) works so as to get an amalgamating structure with a linear order. If some coarseness condition asserts that \(\leq_j\) is coarser than \(\leq_i\), and \(i \in I \setminus J\), that is, \(\leq_i\) is assumed to be partial, then, as in (1), \(\leq_{j, D}\) is coarser than \(\leq_{i, D}\) and thus any linearization of \(\leq_{j, D}\) is coarser than \(\leq_{i, D}\). The other case is trivial: if some coarseness condition asserts that \(\leq_i\) is coarser than some linear order \(\leq_j\), then necessarily \(\leq_i = \leq_j\), hence we can take \(\leq_i = \leq_j\) in \(D\).

To prove the last statement, first notice that, for every class \(K\) under consideration, the class \(K_{\text{fin}}\) has AP, since we can amalgamate into union. JEP
follows since here we are allowed to consider an empty $C$. Then use \[\text{Theorems 7.1.2 and 7.4.1}\]. The finiteness assumption is necessary in order to have a countable number of structures under isomorphism.

We now turn to another method which produces theories with SAPU, starting from theories satisfying the property. The proof is trivial, but the method is useful. For short, SAPU is preserved if we only consider those models which satisfy some given set of universal-existential sentences in which only one variable is bounded by the universal quantifier.

**Proposition 4.3.** Fix some language for all the sentences and the models under consideration.

Suppose that $\Sigma = \{\sigma_i \mid i \in I\}$ is a set of universal-existential sentences in which at most one variable is bounded by the universal quantifier, namely, sentences of the form

$$\forall x \exists y_1 y_2 \ldots \varphi_i \quad \text{or} \quad \forall x \varphi_i \quad \text{or} \quad \exists y_1 y_2 \ldots \varphi_i \quad (4.1)$$

where in each case $\varphi_i$ is quantifier-free.

(a) If $K$ is a class of structures with (super)(S)APU, then the class $K'$ of all structures in $K$ which satisfy $\Sigma$ has (super)(S)APU.

(b) In particular, if $T$ is a theory with (super)(S)APU then $T \cup \Sigma$ has (super)(S)APU.

(c) Any theory with only axioms of the form (4.1) has SAPU, superSAPU, in the presence of a binary relation.

**Proof.** If $D = A \cup B$ and both $A$ and $B$ satisfy some sentence of the form (4.1), then $D$ satisfies such a sentence, if both $A$ and $B$ are substructures of $D$. Notice that at most one variable is bounded by $\forall$.

The last statement follows from Observation 3.1(c). \hfill $\square$

For example, by a formula of the form (4.1) we can express the condition that some unary operation $f$ satisfies identically $f(f(x)) = x$, or $f(x) \geq x$. We can say that two unary operations $f$ and $g$ are comparable, $f(x) \geq g(x)$. We can say that some function is surjective, or even that some relation is surjective with respect to some component, e. g., $\forall x_1 \exists x_2, \ldots, x_n R(x_1, x_2, \ldots, x_n)$, etc.

**Example 4.4.** A. Kisielewicz [Ki] presents examples of varieties without nontrivial finite algebras. A simple example is the variety $V$ with three unary operations $f$, $g$ and $h$ satisfying $fgh(x) = x$ and $fh(x) = f h(y)$ identically. Indeed, the first identity implies that $f$ is surjective and $h$ is injective. The second identity implies that either $h$ is not surjective or $f$ is constant.

As an application of Proposition 4.3 we show that $V$ has SAPU (for nonempty algebras). Formally, Proposition 4.3 does not apply to the the second identity; however, if we introduce a new constant $c$ and we replace the second identity by $fh(x) = c$, then, for nonempty algebras, we get exactly the same morphisms and embeddings, hence Proposition 4.3 can be applied.
The “U” in SAPU is necessary in Proposition 4.3; see Example 7.10 below. In Proposition 4.3 it is necessary to assume that in (4.1) at most one variable is bounded by $\forall$. See Remark 4.5(d) or Example 7.11 below.

Remarks 4.5. (a) We do not need the sentences in Proposition 4.3 to be finitary, they might possibly be infinitary. We might have infinitely many variables $y_j$, as far as at most one variable is bounded by $\forall$ and the $\varphi_i$’s are quantifier-free.

(b) There are first-order sentences for which the statement of Proposition 4.3 holds (limited to SAPU), but which do not have the form (4.1). For example, if some class $\mathcal{K}$ in the language with a unary operation $f$ has SAPU, then the subclass of those structures in $\mathcal{K}$ in which $f$ is bijective has still SAPU. We shall show in Example 7.12 that APU is not preserved by adding a sentence saying that some function is bijective; actually, AP might be destroyed.

(c) The assertion that $f$ is bijective cannot be expressed by a sentence of the form (4.1), however, it can be expressed by a particularly simple second-order sentence, since $f$ is bijective if and only if $f$ has an inverse. Hence the solution to the problem of finding the most general form of Proposition 4.3 (see Problem 4.6 below) might involve second-order sentences.

(d) Notice that the assertion that $f$ is surjective can be actually expressed by a sentence of the form (4.1). On the other hand, SAPU for some class is not preserved by adding the condition that some operation is injective. Adding such a condition might even destroy AP:

For example, consider the class $\mathcal{K}$ of all structures with a unary operation $f$ and a unary predicate $V$. The class $\mathcal{K}$ has SAPU by Observation 3.1(c). Let $\mathcal{C}$ be $\mathbb{N}$ with $f$ the successor operation. Let $A = \mathbb{N} \cup \{a\}$ with $f(a) = 0$ and $B = \mathbb{N} \cup \{b\}$ with $f(b) = 0$. If $f$ is still to be injective, we cannot have SAP. If we further set $V(a)$ and $\neg V(b)$, even AP fails.

Hence SAPU is not preserved by adding the condition that some operation is injective.

Problem 4.6. (a) Characterize those sets of sentences $\Sigma$ for which the analogue of Proposition 4.3 holds, either for SAPU or for APU. Notice that the two cases are distinct, by Remark 4.5(b) and Example 7.12.

(b) Are there even more sentences for which Proposition 4.3(c) holds? Some affirmative answers are provided in [L2].

(c) More generally, characterize those sets of (not necessarily first-order) properties $H$ such that, whenever $\mathcal{K}$ is a class with (S)APU, then the subclass of $\mathcal{K}$ consisting of those structures satisfying the properties in $H$ has still (S)APU. See Proposition 4.9 for more examples of such properties.

(d) Solve the above problems for (S)AP in place of (S)APU. From Example 7.10 below, we see that Proposition 4.3 as stated, fails for AP in place of SAPU. However, it might happen that some version of 4.3 holds for SAP, when a more restricted set of sentences is taken into account. There are trivial cases, for example, Proposition 4.3 holds for AP, when we restrict (4.1) to existential sentences.
As a test case, is it true that if $T$ has SAP, then the theory which further asserts that some unary function is bijective has still SAP?

Recall that a closure operation on some poset $A$ is an order preserving unary operation $f$ such that $x \leq f(x) = f(f(x))$, for every $x$. See [E] for further information and pictures. In the presence of a semilattice operation, some authors include an additivity requirement in the definitions of closure. We shall adopt the more general convention [E] according to which no additivity assumption is made.

An (antitone) involution is an (order-reversing) unary operation $'$ such that $x'' = x$.

**Corollary 4.7.** (A) For every pair $F$, $G$ of sets, the class of posets with an $F$-indexed family of closure operations and a $G$-indexed family of antitone involutions has superSAPU.

(B) The class of linearly ordered sets with one closure operation has SAPU.

(C) The class of linearly ordered sets with one antitone involution has APU.

(D) The class of linearly ordered sets with two closure operations has not AP.

(E) The class of linearly ordered sets with two antitone involutions has not AP.

(F) The class of linearly ordered sets with a family of antitone involutions with a common fixed point has SAPU.

**Proof.** (A) follows from Proposition 3.2(D) and Proposition 4.3, considering sentences of the form $\forall x \ x \leq f(x) = f(f(x))$ and $\forall x \ x'' = x$.

(B) and (C) follow from Proposition 4.3 and [L1, Theorems 3.1(a) and 4.3(a)], which have been recalled in Theorem 3.6.

(D) appears in [L1, Remarks 3.2].

(E) Case (b)(iii) in the proof of [L1, Theorem 4.3] provides a counterexample, though involutions are not explicitly mentioned in [L1].

(F) follows from [L1, Theorem 4.5], using again Proposition 4.3. Notice that the assumption that two involutions $'$ and $^*$ have a common fixed point can be expressed by the sentence $\exists x \ (x' = x \& x^* = x)$, which has the form (4.1).

□

There are also many non first-order properties which preserve SAPU. We present some examples.

If $R$ is a binary relation on some set $A$, an $R$-antichain is a subset $X$ of $A$ such that not $a R b$, for every pair of distinct elements $a, b \in X$.

It is trivial that if $A$, $B$, $C$ is a TBA triple of connected graphs, $C$ is nonempty and $D$ is an amalgamating structure, then $D$ is connected, too. The notion of connectedness can be generalized in various ways in model theory. We present a quite general version.

**Definition 4.8.** If $A$ is a model, two elements $a, b \in A$ are adjacent if $R(a_1, a_2, \ldots, a, \ldots, b, \ldots)$ holds, for some relation $R$ in the language of $A$. 

and some \( a_1, a_2, \ldots \in A \). We are not assuming that \( a \) occurs before \( b \) in the expression \( R(a_1, a_2, \ldots, a, \ldots, b, \ldots) \). If the above holds, we also say that \( a, b \in A \) are \( R \)-adjacent and also that there is an \( R \)-\( m \)-\( n \)-directed edge from \( a \) to \( b \), where \( a \) occurs in the \( m \)th position and \( b \) occurs in the \( n \)th position in \( R(a_1, a_2, \ldots, a, \ldots, b, \ldots) \).

A structure \( A \) is connected if every pair of elements of \( A \) can be connected by a path consisting of adjacent elements. In other words, \( A \) is connected if the transitive closure of the adjacency relation on \( A \) is the largest relation \( A \times A \). The structure \( A \) is \( R \)-connected if every pair of elements of \( A \) can be connected by a path consisting of \( R \)-adjacent elements.

Still more generally, let \( \mathcal{R} \) be a family of triples of the form \( (R, m, n) \), where \( R \) varies among the relations in the language and \( m, n \leq \) the number of arguments of \( R \). An \( \mathcal{R} \)-path is a sequence \( a_1, \ldots, a_h \) such that, for every \( i < h \), there are some \( (R, m, n) \in \mathcal{R} \) and an \( R \)-\( m \)-\( n \)-directed edge from \( a_i \) to \( a_{i+1} \). A model \( A \) is \( \mathcal{R} \)-connected if, for every \( a, b \in A \), there is an \( \mathcal{R} \)-path with initial point \( a \) and final point \( b \). See [C] for related notions.

For the purpose of the above definitions of connectedness, we can take into account also function symbols: think of an \( n \)-ary function \( f \) as an \( n+1 \)-ary relation given by \( R(a_1, a_2, \ldots, a_n, a_{n+1}) \) if \( f(a_1, a_2, \ldots, a_n) = a_{n+1} \).

Proposition 4.9. If \( K \) is a class of structures with \( (S)APU \), then, for any set of properties chosen from the list below, the subclass of \( K \) consisting of those structures in \( K \) which satisfy the chosen properties has \( (S)APU \).

(\( A_1 \)) The domain is finite (or has cardinality \( < \lambda \)).
(\( A_2 \)) The domain is finitely generated (generated by a set of cardinality \( < \lambda \)).
(\( A_3 \)) For some binary relation \( R \) assumed to be a partial order: \( R \) is well-founded (has no strict descending chain of length \( \geq \lambda \)) (has no strict ascending chain of length \( \geq \lambda \)).
(\( A_4 \)) For some binary relation \( R \), there is no infinite \( R \)-antichain (there is no \( R \)-antichain of cardinality \( < \lambda \))
(\( A_5 \)) (only for \( SAPU \)) For some natural numbers \( n, m \) fixed in advance, the domain has cardinality \( kn + m \), for some \( k \).

As far as the following properties are concerned, we consider the version of \( (S)APU \) in which the bottom structure \( C \) is assumed to be nonempty (compare the second paragraph in Section 2).

(\( C \)) The structure is connected (\( R \)-connected, for some relation symbol \( R \)) (\( \mathcal{R} \)-connected, for some family of triples \( \mathcal{R} \) as in Definition 4.8).

More generally, if \( V \) is a unary predicate, we can consider anyone of the above properties when restricted to the domain \( \{ x \mid V(x) \} \) of \( V \). When applying condition (\( C \)) we should assume that the domain of \( V \) is nonempty.

The proof of Proposition 4.9 is immediate; in fact, if some of these properties hold in \( A, B \) and \( C \), then the property holds in some amalgamating structure,
since we can construct it on $A \cup B$. However, the corresponding statements generally fail when SAPU is weakened to SAP. For instance, see Example 7.2(c).

Recall that a well partial order, or wpo is a well-founded partial order without infinite antichains. Hence, because of Proposition 4.9(A3)(A4), if some class with (S)APU has a partial order relation $\leq$, then the subclass of those structures in which $\leq$ is a wpo has still (S)APU.

Some assumption on the relation $R$ in Proposition 4.9(A3) is necessary: see Example 7.13.

Theorem 4.2 and Corollaries 3.3 and 4.7 have not the most general form, rather, they are just exemplifications. We are now going to state a very general result which can be obtained by the present methods.

By the proof of Proposition 7.2, we can deal with structures with a family of transitive relations, each relation satisfying any set of properties chosen from 2.- 5. Structures with partial orders are just an instance of this more general case. In particular, we can deal with any class of structures with many preordered sets, many equivalence relations, or even simultaneously posets, preordered sets and equivalence relations.

Even in this general setting we can define a coarseness condition, namely, a condition of the form $R_j \subseteq R_i$. That is, if $x R_j y$, then $x R_i y$.

Due to Propositions 3.2(D) and 4.1(b), we can add a family of unary operations, as well as conditions asking that some operation is $R$-preserving or $R$-reversing. We can add sentences of the form (4.1), in view of Proposition 4.3. In conclusion, here is a general result we have got.

**Theorem 4.10.** All the classes $\mathcal{K}$ described below have SAPU; actually, superSAPU with respect to all the binary relations $R_i$ involved.

We assume that $\mathcal{K}$ is the class of models for some theory $T$ in a language with a sequence $(R_i)_{i \in I}$ of binary relation symbols and a sequence $(f_h)_{h \in H}$ of unary function symbols. We require that $T$ asserts that each $R_i$ is a transitive relation. Moreover, $T$ is allowed to contain some axioms, possibly none, from the list below (each axiom might appear for as many indices as wanted).

1. Some relation $R_i$ is reflexive;
2. Some relation $R_i$ is symmetric;
3. Some relation $R_i$ is antireflexive;
4. Some relation $R_i$ is antisymmetric;
5. Some $f_h$ preserves some $R_i$;
6. Some $f_h$ is $R_i$-reversing;
7. Some $R_i$ is coarser than some $R_j$;
8. Any sentence of the form (4.1), in particular, the universal closures of sentences of the form
   a) $f_h(x) R_i f_k(x)$,
   b) $x R_i f_h(x)$,
   c) $f_h(f_h(x)) = f_h(x)$,
(d) \( f_h(f_k(x)) = f_k(f_h(x)) \), etc.

(9) Some \( f_h \) strictly preserves some \( R_i \), provided \( T \) asserts that \( R_i \) is a partial order;

(10) Some \( f_h \) is strictly \( R_i \)-reversing, provided \( T \) asserts that \( R_i \) is a partial order.

Furthermore, we are allowed to expand the language by adding any number of symbols of any kind, as far as the axioms involving such new symbols are only of the form (4.1).

Still more generally, for each class \( \mathcal{K} \) as above, the subclass of those substructures satisfying any given set of conditions taken from Proposition 4.7 has SAPU.

Proof. As in the proof of 4.2, apply the proof of Proposition 3.2 individually for each relation and function, joining everything in the model \( D \). By the proof of Proposition 3.2 if any sentence equivalent to some condition from (1) - (6) is included in \( T \), then \( D \) satisfies this sentence. By assumption, all the relations are transitive, hence we are always in case b in the proof of 3.2 this implies that coarseness is preserved. This argument takes care of (7). Clause (8) follows from Proposition 4.3. Clauses (9) and (10) follow from Remark 2.2 in [L3], where we check that the construction in case b in the proof of Proposition 3.3 commutes in passing from some partial order to the corresponding strict order.

As far as the penultimate statement is concerned, use Proposition 4.1(b) and again Proposition 4.3. The last statement is immediate from Proposition 4.9. □

Remark 4.11. In most cases, the class \( \mathcal{K}^{fin} \) has a Fraïssé limit, for a class \( \mathcal{K} \) as considered in Theorem 4.10. Compare Theorems 4.2.

However, there are some limitations. First, we generally need the language to be finite, in order to get only a countable number of finite models modulo isomorphism. Second, only universal sentences of the form (4.1) can be considered, if we want hereditariness [H 7 (1.1)] to be preserved. Finally, some of the conditions mentioned in Theorem 4.10 might destroy JEP, for example, adding constants to the language, or using Clause (C) from Proposition 4.9. However, we retain JEP when restricted to any equivalence class as described in Remark 2.5, hence we have a Fraïssé limit for each equivalence class.

We leave the details to the interested reader.

Remark 4.12. If we merge transitive relations with relations which are not supposed to be transitive, then coarseness is not always preserved when trying to amalgamate structures; see Proposition 3.5.

In particular, Theorem 4.10 does not necessarily hold when transitive relations are merged with nontransitive relations, if we ask for coarseness conditions as in (7). The case of just two relations, as given by Theorem 3.4, is a notable exception. Of course, as in the first lines of the proof of 3.4 if we
compare any number of relations which are not required to be transitive, then
closeness is preserved. Similarly, we can ask that some relation is finer than an-
other transitive relation, since the definition of $R$ in the proof of Proposition
3.2 case b, always produces a coarser relation, in comparison with $R$ as
defined in case a.

Quite unexpectedly, a common generalization of Theorems 3.4 and 4.10(1)-(4),
(7)-(8) holds even when dealing with relations supposed to be coarser than
other transitive relations. We shall show that the only obstacle to amalgama-
tion are antisymmetric relations supposed to be coarser than a pair of incompa-
table transitive relations; essentially, the only counterexamples to amalgama-
tion are given by the examples we have described in Proposition 3.5. However,
in the general case which unifies Theorems 3.4 and 4.10, the possibility of
adding relation-preserving operations is not as neat as in Proposition 3.2(D)
or Theorem 4.10(6), (7), (9), (10). Samples witnessing this difficulty are given
here in Theorem 3.4(C)(D) and Proposition 3.5(c). We shall present further
details elsewhere.

Remark 4.13. Theorem 4.10 is quite powerful. We present a simple example.

A bounded poset is a poset with a maximum and a minimum element, both
elements interpreted as constants. We could repeat all the above arguments
getting corresponding theorems for bounded posets.

However, the results follow automatically from Proposition 4.3 and Theorem
4.10 Indeed, by Propositions 3.2 and 4.10(b), the class of posets in the language
with two constant symbols added has SAPU. Then the assertion that, say the
constant 1 is interpreted as a maximum can be expressed by the sentence
$\forall x x \leq 1$, having the form (4.1) from Proposition 4.3.

The above arguments apply in the same way in order to show that bounded
posets with families of (strict) order preserving (or reversing) unary operations
have SAPU. Of course, we can also assume only the existence of a maximum,
or only the existence of a minimum (provided, as above, each one is interpreted
as a constant).

5. Binary and $n$-ary operations.

Reading the previous sections, the reader might expect that (S)APU is a
phenomenon typical of relational structures only, with possibly unary opera-
tions added. By Observation 3.1(c) and Proposition 4.1(b), we may have
SAPU when no axiom involves binary or $n$-ary operations, but, in a sense,
this is a trivial case. However, there are many examples of classes with $n$-ary
operations, for $n \geq 2$, and sharing SAPU.

Some varieties of groupoids with SAPU are obtained in [K, Proposition
5.19].

Varieties defined by linear equations. We now present a general result from
[L2] dealing with varieties. Recall that a variety $\mathcal{V}$ is a class of nonempty
structures for a language without relation symbols, such that \( \mathcal{V} \) can be defined by equations, i.e., universal closures of atomic formulae. An equation is linear if there is at most one occurrence of an operation symbol on each side (constants are not counted as operations, here). Notice that the terminology is not uniform in the literature. Examples of linear equations are \( f(x, x, y) = y, f(x, x, y) = g(x, x, y), h(x, c) = x \) or \( f(c, y, y, z) = g(x, x, d, y) \).

Linear equations are important because they are almost invariably encountered in the definition of Maltsev conditions, for examples, the conditions characterizing congruence permutability, distributivity, modularity... See, e.g., \([B]\) for details. See \([L2]\) for further comments, examples and related results.

**Theorem 5.1.** \([L2]\) Any variety which can be defined by a set of linear equations has SAPU.

**Proof.** (Sketch) Fix some arbitrary element \( \bar{d} \) of \( D = A \cup B \). For every operation \( f \) and \( d_1, d_2, \ldots \in D \), set \( f(d_1, d_2, \ldots) = \bar{d} \), unless the value of \( f(d_1, d_2, \ldots) \) is forced by some identity to be satisfied, or by the requirement that both \( A \) and \( B \) should embed into \( D \). Check that such conditions do not clash and that if \( f(d_1, d_2, \ldots) = g(e_1, e_2, \ldots) \) is (the evaluation of) an identity to be satisfied, then the value of \( f(d_1, d_2, \ldots) \) is not forced if and only if the value of \( g(e_1, e_2, \ldots) \) is not forced (this might be cumbersome, in general). If this is the case, both values are equal to \( \bar{d} \), hence the identity is satisfied. Full details appear in \([L2]\). \( \square \)

We now present other examples of structures with a binary operation and with SAPU.

**Directoids and related structures.** A directoid is a set with a binary operation \( \sqcup \) such that the identities \( x \sqcup x = x \), \( x \sqcup y = y \sqcup x \) and \( x \sqcup ((x \sqcup y) \sqcup z) = (x \sqcup y) \sqcup z \) hold. Directoids are an algebraization of directed sets, see \([CL, CGKLP]\) for details. The above directoids are the “commutative” ones; there is also a noncommutative version, but notice that the terminology in the literature is not uniform. Most of the following results hold also in the noncommutative case.

A bidirectoid is a set with two directoid operations satisfying the absorption laws \( a \sqcap (a \sqcup b) = a \) and \( a \sqcup (a \sqcap b) = a \). Bidirectoids correspond to posets which are both upward and downward directed.

A maximum for a directoid is an element 1 such that \( x \sqcup 1 = 1 \). A minimum is an element 0 such that \( x \sqcap 0 = x \). When we speak of a directoid with a maximum (minimum), we simply assert that such a maximum (minimum) exists, but we do not assume that it is interpreted by a constant. In particular, embeddings of directoids with a maximum need not preserve maxima.

On the other hand, an upper bounded directoid (bounded directoid) is a directoid with a maximum (and a minimum) interpreted as constant(s). In this case embeddings are supposed to preserve maxima (and minima).
The following theorem is immediate from the proof of [CGKLP, Theorems 10 and 11]. The case of bounded bidirectoids is proved in essentially the same way. See [CGKLP] for the definition of an involutive directoid.

**Theorem 5.2.** [CGKLP] The classes of upper bounded directoids, bounded directoids, bounded bidirectoids and of bounded involutive directoids have SAPU.

In [CGKLP] it is also proved that the classes of (not bounded) directoids and bidirectoids have SAP. The proof does not give SAPU, however, since a new element is added to the union of the amalgamating structures. We shall prove in [L4] that this new element is necessary in general, but we can do without adding new elements in the finite case.

As above, when we speak of a posets with a maximum, the maximum is not interpreted as a constant; in particular, embeddings need not preserve maxima. The case when maxima are interpreted as constants has been dealt with in Remark 4.13. Similarly, a finite directoid has necessarily a maximum, but we do not require that embeddings preserve maxima. As we mentioned, maxima are required to be preserved only in the class of upper bounded directoids. See [L4] for the proof of the next proposition.

**Proposition 5.3.** [L4] The following classes have SAPU.

(a) The class of posets with a maximum.
(b) The class of directoids with a maximum.
(c) The class of bidirectoids with a maximum and a minimum.
(d) The class of finite directoids.
(e) The class of finite bidirectoids.

The following classes have SAP but not APU.

(f) The classes of upward directed (downward directed, both upward and downward directed) posets.
(g) The class of directoids.
(h) The class of bidirectoids.

Using the methods of the present paper, Proposition 5.3 can be generalized so that structures with many directoid operations, with possibly many unary operations can be taken into account. We refer again to [L4] for more details.

**Order algebras.** We now outline a general method to obtain classes of algebraic structures with SAPU from corresponding classes of relational structures. We first recall the original motivating example.

If \((P, \leq)\) is a poset, define a binary operation \(\cdot\) (henceforth denoted by juxtaposition) on \(P\) by

\[
ab = \begin{cases} 
a & \text{if } a \leq b, 
b & \text{otherwise.}
\end{cases}
\]

The structures \((P, \cdot)\) which can be obtained in this way have been described in [N] under the name pogroupoids, but are called order algebras in the more
recent literature, e.g., [FJM]. Clearly, we can retrieve the order \( \leq \) from \( \cdot \) by setting \( a \leq b \) if \( ab = a \).

Neggers [N] has showed, among other, that if \( P, Q \) are posets and \( \iota: P \to Q \) is a function, then \( \iota \) is a homomorphism of the corresponding order algebras if and only if \( \iota \) is an order-morphism with the further property that any pair of incomparable elements are sent either to the same element, or to another pair of incomparable elements. It follows immediately that if \( \iota \) is injective, then \( \iota \) is an embedding of order algebras if and only if \( \iota \) is an order-embedding, since order-embeddings are exactly injective ordermorphisms which send incomparable pairs to incomparable pairs.

**Corollary 5.4.** The class of order algebras has SAPU.

**Proof.** Immediate from the above observations and the fact, generalized here in Proposition 3.2, that the class of posets has SAPU. \( \square \)

**Algebraizing relations.** The above argument has a more general flavor. Let \( \mathcal{K} \) be a class of structures such that \( R(x, x, x, \ldots) \) holds in every structure in \( \mathcal{K} \), for every relation symbol \( R \) in the language. The language of \( \mathcal{K} \) is allowed to contain relation, constant and function symbols. Let us associate to \( \mathcal{K} \) a class \( \mathcal{K}_a \) defined in the following way. To every model \( A \) in \( \mathcal{K} \) one associates a model \( A_a \) obtained from \( A \) by replacing every \( n \)-ary relation \( R \) by an \( n \)-ary function \( f_R \) defined by

\[
f_R(a_1, a_2, a_3, \ldots) = \begin{cases} 
a_1 & \text{if } R(a_1, a_2, a_3, \ldots) \text{ holds in } A, 
a_i & \text{otherwise, where } i \text{ is the smallest index such that } a_i \neq a_1.\end{cases}
\]

Notice that in the second clause such an \( i \) exists, since \( R(a, a, a, \ldots) \) holds in \( A \). The class \( \mathcal{K}_a \) is the class of models which can be obtained in this way. As in the case of order algebras, from \( A_a \) we can retrieve the structure of \( A \). It follows that the (class) function which sends \( A \) to \( A_a \) is bijective from \( \mathcal{K} \) to \( \mathcal{K}_a \). Again as in the case of order algebras, the notion of homomorphism in \( \mathcal{K}_a \) is stronger than the notion of homomorphism in \( \mathcal{K} \); however, the notions of embedding coincide. This is proved just arguing as in the last lines of [N] proof of Theorem 1. Henceforth we get the following proposition.

**Proposition 5.5.** Under the above notations and conventions, a class \( \mathcal{K} \) has AP (APU, SAP, SAPU) if and only if \( \mathcal{K}_a \) has AP (APU, SAP, SAPU).

In particular, \( \mathcal{K}_a \) has AP (and frequently SAPU) for all the classes \( \mathcal{K} \) described in Propositions 3.2, 3.3, 6.1(a)-(b), Corollaries 3.3, 4.7(A)/(B)/(C)/(F) and Theorems 3.4(A)/(B), 3.6(1)-(6), 4.2, 4.10, 5.2

APU for order algebras and “algebraized” structures can be seen from a more general perspective.

**Proposition 5.6.** Suppose that \( \mathcal{K} \) is a class of structures for the same language.
(1) Suppose further that, for every \( A \in K \), every \( n \)-ary function \( f_A \) and \( a_1, \ldots, a_n \in A \), we have \( f_A(a_1, \ldots, a_n) \in \{ a_1, \ldots, a_n \} \).

If \( K \) has (S)AP and is closed under taking substructures, then \( K \) has (S)APU.

(2) More generally, suppose that, for every \( A \in K \), every \( n \)-ary function \( f_A \) and \( a_1, \ldots, a_n \in A \), there are a unary term \( t(x) \) and \( i \leq n \) such that \( f_A(a_1, \ldots, a_n) = t(a_i) \).

If \( K \) has (S)AP and is closed under taking substructures, then \( K \) has (S)APU.

Proof. Any TBA triple \( A, B, C \in K \) can be amalgamated to some \( D \in K \). Under the assumptions, \( A \cup B \) is a substructure of \( D \).

### 6. The joint embedding property into union.

The classical joint embedding property (JEP) \([\mathbb{H}]\) admits variations in the same spirit of Definition 2.1. For ordered sets with operators, the corresponding theory is quite simple, in the sense that the resulting disjoint embedding property into union turns out to be generally either trivially true, or trivially false. In any case, JEP plays a fundamental role in model theory, hence we shall explicitly mention the JEP-related properties.

**Definition 6.1.** (a) Recall from Definition 2.4 that a class \( K \) has the joint embedding property (JEP) if, for every \( A, B \in K \), there are a structure \( D \in K \) and embeddings \( \iota : A \rightarrow D \) and \( \kappa : B \rightarrow D \).

(b) \( K \) has the disjoint embedding property (DJEP) if, under the assumptions from (a), \( D, \iota \) and \( \kappa \) can be chosen in such a way that the images of \( \iota \) and \( \kappa \) are disjoint.

Clearly, this is impossible if some constant is present in the language. This is the main difference with respect to SAP.

(c) We say that \( K \) has the joint embedding property into union (JEPU) if, under the assumptions from (a), \( D, \iota \) and \( \kappa \) can be chosen in such a way that \( D \) is the union of the images of \( \iota \) and \( \kappa \).

(d) We say that \( K \) has the disjoint embedding property into union (DJEPU) if (b) and (c) can always be accomplished simultaneously, namely, \( D \) can be chosen to be the disjoint union of the images of \( \iota \) and \( \kappa \).

If \( K \) is closed under isomorphism, then a remark parallel to 2.2 applies, namely, \( K \) has DJE if and only if, whenever \( A, B \in K \) and \( A \cap B = \emptyset \), then there is a structure \( D \in K \) such that \( A \subseteq D \) and \( B \subseteq D \). Thus \( K \) has DJEPU if and only if \( D \) as above can be chosen in such a way that \( D = A \cup B \).

The classical joint embedding property dates back at least to \([\mathbb{F}1]\). The disjoint embedding property has been sometimes used in the literature, at least from the '80's in the last century, e. g., Pouzet \([\mathbb{P}]\). For certain classes, e. g., linearly or partially ordered sets, DJEPU is trivially satisfied but, again, we do not know of a study of such properties for their own sake.
If we work in a language without constant symbols, then in most arguments from the preceding sections we may allow $C$ to be empty. As we mentioned, this special instance of AP (and variations) turns out to be exactly JEP (and variations). This is the reason why remarks parallel to Observation 3.1 and Propositions 4.1, 4.3, 4.9 hold, possibly with a difference for languages with constant symbols.

**Observation 6.2.** If some language $\mathcal{L}$ has no constant symbol, then the class $K$ of all models for $\mathcal{L}$ has DJEPU and any theory for $\mathcal{L}$ with only axioms of the form (4.1) from Proposition 4.3 has DJEPU.

For arbitrary languages, Observation 3.1(a)(b) and Propositions 4.1, 4.3(a)(b), 4.9(A1)-(A5) hold when (S)AP(U) is replaced everywhere by (D)JEP(U). However, in Propositions 4.1(b) we need to assume that $\mathcal{L}'$ has no constant symbol.

Notice that constants caused no trouble in getting strong AP in Observation 3.1(c), since constants are always already interpreted in $C$. On the other hand, as we have mentioned in Definition 6.1(b), the existence of some constant in the language always forbids disjoint JEP (and frequently forbids JEP).

**Theorem 6.3.** All the classes and theories considered in Propositions 3.2, 5.3(a)-(e), Corollaries 3.3, 4.7(A)(B) and Theorems 3.4(A)(B), 5.2(1)-(4), 4.3, 4.10 have DJEPU, with the provision that no new constant is added in the last two statements in Theorem 4.10.

The classes considered in Corollary 4.7(C)(F) have JEPU and the classes considered in Proposition 5.3(f)-(h) and Theorem 3.6(5)-(6) have DJEP.

In particular, all the above classes have JEP.

**Proof.** In most of the above arguments we have not forbidden the possibility that $C$ is empty, hence, as remarked above, this special instance of (S)AP(U) provides (D)JEP(U). In fact, in this special case proofs turn out to be generally much simpler: for example, in the case of posets, just consider $R = R_A \cup R_B$ on $A \cup B$. As another example, in the case of linearly ordered sets, just let all the elements of $A$ to be $<$ than all the elements of $B$. Adding operators presents no significant trouble, too.

Some care is needed in the presence of constants. However, in the case of bounded directoids, and variants, the constants always generate isomorphic subalgebras, hence we can consider, as $C$, any copy of this “prime” subalgebra, and then apply (S)AP(U).

On the other hand, in comparison with Theorem 5.1, only varieties defined by a special kind of linear identities have JEP, possibly, DJEPU. See [L2], in particular, Theorem 3.1, Remark 4.1 and Corollary 4.6 there.

Dense linear orders provide interesting examples concerning the properties dealt with in the present section.
Proposition 6.4. (a) The theory of dense linear orderings has SAP, DJEP, JEPU, APU but neither DJEPU nor SAPU.

(a') The theory of dense linear orderings without endpoints has SAP, DJEPU, APU but not SAPU.

(b) The theory of dense linear orderings with a closure operation has SAP, DJEP, JEPU but neither DJEPU, nor APU.

(c) The theory of dense linear orderings with two closure operations has not AP.

Proof. (a) SAP follows immediately from the facts that the class of linear orders has SAP (actually, SAPU) and that every linear order can be embedded into some dense linear order. DJEP is the special case when $C$ is the empty structure.

We now disprove SAPU. Let $C$ be $\mathbb{Q}$ with the standard order, consider two distinct copies $r$ and $r'$ of the same real (not rational) number and let $A = \mathbb{Q} \cup \{r\}$, $B = \mathbb{Q} \cup \{r'\}$. However we linearly order $A \cup B$, either $r'$ is the immediate predecessor of $r$, or conversely. The counterexample works for dense linear orders without endpoints, too.

Then we disprove DJEPU. Let $A$ and $B$ be two disjoint copies of the real interval $[0, 1]$ and suppose by contradiction that $D = A \cup B$ can be densely linearly ordered extending the orders on $A$ and $B$. Define the following equivalence relation on $A$: $r \sim s$ if $\{t' \in B \mid t' <_{D} r\} = \{t' \in B \mid t' <_{D} s\}$. Thus the $\sim$-equivalence classes partition $A$ and each class is a convex subset of $A$, hence an interval. It is easy to see that if $[0, 1]$ is partitioned into intervals, then at least one interval is a closed interval of the form $[r, s]$, possibly with $r = s$.

Indeed, if the class of 0 has the form $[0, v]$, we are done. Otherwise, let $r$ be the largest real such that, for every $t < r$, the equivalence class of $t$ has the form $[u, v)$. Namely, $r$ is the supremum of those $v$ such that some equivalence class has the form $[u, v)$ and also all preceding classes have that form. Then the class of $r$ has necessarily the form $[r, s]$, since the form $[r, s)$ would contradict the definition of $r$.

So let $[r, s]$ be a $\sim$-equivalence class and let $T' = \{t' \in B \mid t' <_{D} r\}$. If $T'$ is empty, then $0'$ in $B$ is the immediate successor of $s$ in $D$. If $T'$ has a maximum $u'$ in $B$, then $r$ is the immediate successor of $u'$ in $D$. On the other hand, if the supremum $u'$ of $T'$ does not belong to $T'$, then $u'$ is the immediate successor of $s$ in $D$. In any case, we have found two elements without intermediate elements, hence the order in $D$ is not dense, a contradiction.

Next, we prove JEPU. Actually, we show that DJEPU fails “for just one element”, namely we get JEPU by identifying at most one element from $A$ with at most one element from $B$.

So let $A$ and $B$ be two dense linear orderings. If either $A$ has no maximum or $B$ has no minimum, simply put all the elements of $A$ before all the elements of $B$. Otherwise, $A$ has a maximum $\bar{a}$ and $B$ has a minimum $\bar{b}$. Identify $\bar{a}$
and \( \bar{b} \) in \( D \) and, again, put all the other elements of \( A \) before all the elements of \( B \).

Having proved JEPU, it is rather easy to prove APU.

Let \( A_1, A_2, C \) be a TBA triple of dense linear orderings, with embeddings \( \iota_i : C \to A_i \), for \( i = 1, 2 \). Recall that if \( C \) is a linearly ordered set, a cut of \( C \) is a pair \((C', C'')\) such that \( C' \cup C'' = C \) and \( c' < c'' \), for every \( c' \in C' \) and \( c'' \in C'' \). We allow \( C' \) or \( C'' \) to be empty. To any cut \((C', C'')\) of \( C \) one associates on \( A_i \setminus \iota(i) \) the components \( \{ a \in A_i \setminus \iota(i) \mid \iota_i(c') <_{A_i} a <_{A_i} \iota_i(c'') \}, \) for all \( c' \in C' \) and \( c'' \in C'' \), for \( i = 1, 2 \). Conversely, to each \( a \in A_i \setminus \iota(i) \) one can associate the cut formed by \( C' = \{ c \in C \mid \iota_i(c) <_{A_i} a \} \) and \( C'' = \{ c \in C \mid a <_{A_i} \iota_i(c) \} \). The nonempty components partition both \( A_1 \setminus \iota(i) \) and \( A_2 \setminus \iota(i) \). Moreover, if \((C', C'')\) is associated to \( a \in A_i \) and \((C'_e, C''_e)\) is associated to \( a_e \in A_j \) with, say, \( C'_e \subseteq C''_e \), then, for any possible amalgamating structure \( D \) through embeddings \( \kappa_i : A_i \to C \) (\( i = 1, 2 \)), we should have \( \kappa_i(a) <_D \kappa_j(a_e) \). This implies that, in order to construct \( D \) and the \( \kappa_i \)'s, it is enough, for every cut, to set the relative order between the elements of the components on \( A_1 \setminus \iota(i) \) and \( A_2 \setminus \iota(i) \) associated to the cut. See the proof of [L1 Thorem 3.1(a)] for more details.

We are almost done. If \( E_i \), for \( i = 1, 2 \), are the components on \( A_1 \setminus \iota(C) \) associated to some cut, then it is enough to embed the two \( E_i \)'s into some dense linear order using JEPU. It is easy to see that, letting the cut vary among all cuts of \( C \) and putting together all the structures as above, we get a dense linear order.

(a’) As in (a), we have SAP since linear orders have SAP and every linear order can be embedded into some dense linear order without endpoints. The failure of SAPU has already been taken care of. As far as APU is concerned, the argument in (a) works in the present case, too, since if \( A_1, A_2 \) and \( C \) have no endpoint, then the model \( D \) we have constructed has no endpoint. DJEPU is trivial, just let every element of \( A \) be \( \leq \) than every element of \( B \).

(b) By Corollary [L7, B]), the theory of linearly ordered sets with a closure operation \( f \) has SAP. Hence if \( A, B, C \) is a TBA triple of dense linear orders with a closure operation, then there is an amalgamating linear order \( E \) with a closure operation. As an order, \( E \) can be embedded into a complete dense linear order \( D \) in such a way that, for every \( d \in D \), there is \( e \in E \) such that \( d \leq_D e \). Now define \( f \) on \( D \) by \( f(d) = \inf \{ f_E(e) \mid e \in E, \ d \leq_D f_E(e) \} \) and it is easy to see that, with \( f \) as defined, \( E \) embeds in \( D \) and \( f \) is a closure operation, thus \( D \) amalgamates the original triple. We have proved SAP.

DJEPU is the special case of SAP when \( C \) is the empty structure.

We now check that JEPU holds. As in (a), if either \( A \) has no maximum or \( B \) has no minimum, put all the elements of \( A \) before all the elements of \( B \). Otherwise, \( A \) has a maximum \( \bar{a} \) and \( B \) has a minimum \( \bar{b} \). Then identify \( \bar{a} \) and \( f(\bar{b}) \) and put all the other elements of \( A \) before all the other elements of \( B \). Thus all the (images of the) elements of \( A \setminus \{ \bar{a} \} \) precede all the elements \( b_1 \in B \) such that \( b_1 < f(\bar{b}) \) and all such elements are bounded by \( \bar{a} = f(\bar{b}) \). All
the other elements of \(B\) are larger. Notice that \(f(\bar{a}) \geq \bar{a}\) in \(A\), hence \(f(\bar{a}) = \bar{a}\),
since \(\bar{a}\) is the maximum of \(A\). Moreover, \(f(f(\bar{b})) = f(\bar{b})\), since \(f\) is a closure
operation, hence the identification of \(\bar{a}\) and \(f(\bar{b})\) is compatible.

Since dense linear orderings have not DJEPU, then dense linear orderings
with a closure operation have not DJEPU: just consider the same counterex-
ample with constant functions added as operations.

In order to disprove APU, we shall modify the counterexample to SAPU
given in (a). Fix \(r \in \mathbb{R} \setminus \mathbb{Q}\) and \(q \in \mathbb{Q}\) with \(r < q\). Let \(C = \mathbb{Q}\) with the
standard order and define \(f\) on \(C\) by

\[
f(c) = \begin{cases} 
  c & \text{if } c < r, \\
  q & \text{if } r < c \leq q, \\
  c & \text{if } q < c,
\end{cases}
\]

thus \(f\) is a closure operation on \(C\).

Let \(A = C \cup \{r\}\) with \(f(r) = r\). Let \(r'\) be a copy of \(r\) and define \(B\) by
\(B = C \cup \{r'\}\) with \(f(r') = q\). As in (a), if amalgamation into union holds,
then \(r\) and \(r'\) should be identified, but this is impossible because of \(f\).

(c) Consider the last example and add another operation \(g\) defined as \(f\)
on \(\mathbb{Q}\) and such that \(g(r) = q\) in \(A\) and \(g(r') = r'\) in \(B\). As above, both \(f\) and
\(g\) forbids the identification of \(r\) and \(r'\). Hence in any amalgamating structure
we have either \(r < r'\) or \(r' < r\). If \(r' < r\), then \(q = f(r') \leq f(r) = r\), a
contradiction. Symmetrically, \(r' < r\) cannot hold, thus AP fails.

\[\square\]

7. More examples and counterexamples.

Remark 7.1. If in Definition 2.1 we replace embeddings with injective homomorphisms, the results in the present note do not necessarily hold.

(a) The class of posets has not AP with respect to injective homomorphisms. Indeed, let \(C\) be the poset with just two incomparable elements \(a\) and \(b\). If
\(A = \{a,b\}\) with \(a < b\) in \(A\), then the identity is an ordermorphism (but not an embedding!) \(\iota_{C,A}\) from \(C\) to \(A\). Similarly, let \(B = \{a,b\}\) with \(b < a\) in \(B\)
and \(\iota_{C,B}\) be the identity map. In any amalgamating poset \(D\) we should have
\(\iota_{A,D}(a) \leq \iota_{A,D}(b)\) and \(\iota_{B,D}(b) \leq \iota_{B,D}(a)\). Since we require \(\iota_{C,A} \circ \iota_{A,D} = \iota_{C,B} \circ \iota_{B,D}\), then by antisymmetry \(\iota_{A,D}(a) = \iota_{A,D}(b)\), hence it is not possible
to have \(\iota_{A,D}\) an injective homomorphism.

(b) The main obstacle to AP for injective homomorphisms in (a) is antisymmetry. In fact, the arguments in (a) show that the class of sets with an
antisymmetric binary relation has not AP with respect to injective homomorph-
isms.

(c) In contrast with (a) and confirming (b), the class of preorders has SAPU
with respect to injective homomorphisms.

Parallel to Remark 2.2, we can assume that \(C = A \cap B\) and that the
inclusions from \(C\) to \(A\) and from \(C\) to \(B\) are homomorphisms. Then it is
enough to endow $A \cup B$ with the transitive closure of $\leq_A \cup \leq_B$. Of course, this is not the only possibility, we could even have done with the \textit{discrete} preorder (all pairs of elements are connected).

(d) It is probably an interesting possibility to mix the two approaches, namely, to consider embeddings (condition (2.1) is required) with respect to a certain set of relations, and homomorphisms (condition (2.2) is required) with respect to another set of relations. Notice that there is no distinction between embeddings and injective homomorphisms, when constants or functions are taken into account.

In the above proposal we intend to strictly remain within the realm of model theory. The amalgamation property can be defined in a categorical setting and, of course, this abstract setting encompasses all the above possibilities \cite{KMPT}.

\textbf{Example 7.2.} (a) The theory (in the empty language) asserting that the universe has not cardinality 3 has APU and SAP, but not SAPU. Just let $|C| = 1$ and $|A| = |B| = 2$.

(b) As above, the following theory $T$ in the empty language has APU and SAP, but not SAPU. The theory $T$ has sentences asserting

\begin{enumerate}
\item[(♦)] If there are at least 3 distinct elements, then there are at least $n$ distinct elements,
\end{enumerate}

for every $n \geq 3$.

The class of finite models of $T$ has APU but not SAP.

(c) If we consider the theory from (b) in the language with a unary predicate $U$, then $T$ has SAP, but the class of finite models of $T$ has not even AP. Let $C$ have one element $c$, let $A$ have one more element $a$ such that $U(a)$ and $B$ have another element $b$ such that not $U(b)$. Any amalgamating structure has at least 3 elements, hence is infinite.

\textbf{Remark 7.3.} (a) If we allow the empty model in the definition of SAPU and some class $\mathcal{K}$ has SAPU and both an empty model and a model of cardinality 1, then $\mathcal{K}$ has models of any finite cardinality.

(b) If $\mathcal{K}$ has SAPU and has a model of cardinality 1 which embeds into some model of cardinality 2, then $\mathcal{K}$ has models of any finite nonzero cardinality.

(c) More generally, if $\mathcal{K}$ has SAPU and has a model of cardinality $n$ which embeds into some model of cardinality $m > n$, then $\mathcal{K}$ has models of cardinality $m + k(m - n)$, for every $k$.

\textbf{Remark 7.4.} (a) In most cases, the theories we have considered in this note are universal Horn, hence they have pushouts.

In general, the homomorphisms given by a pushout are not embeddings, but if some class $\mathcal{K}$ has both AP and pushouts, then, for every TBA triple in $\mathcal{K}$, the pushout is an amalgamating structure. In fact, in most cases, here we have proved AP just by constructing the pushout and showing that the homomorphisms towards the pushout are embeddings.
(b) In the class $\mathcal{K}_0$ of sets (models for the empty language $\mathcal{L}_0$) the pushout of a TBA triple $A_0, B_0$ and $C_0$ is the model $D_0$ over $D_0 = A_0 \cup B_0$ without structure. Hence some class $\mathcal{K}$ in some language $\mathcal{L}$ has SAP(U) if and only if, for every TBA triple $A, B, C$ in $\mathcal{K}$, there is an amalgamating structure $D$ such that if $D^*$ is the pushout in $\mathcal{K}_0$ of $A \upharpoonright \mathcal{L}_0, B \upharpoonright \mathcal{L}_0$ and $C \upharpoonright \mathcal{L}_0$, then $D^*$ is a subreduct (a reduct) of $D$.

(c) For transitive relations, a similar rephrasing of the superamalgamation property is possible.

Let $\mathcal{L}_0 = \{R\}$ and $\mathcal{L} \supseteq \mathcal{L}_0$. If $\mathcal{K}$ is a class of models for $\mathcal{L}$ and $R$ is transitive in any member of $\mathcal{K}$, then $\mathcal{K}$ has the superamalgamation property (into union) with respect to $R$ if and only if, as above,

(*) for every TBA triple $A, B, C$ in $\mathcal{K}$, there is an amalgamating structure $D$ such that if $D^*$ is the pushout in $\mathcal{K}_0$ of $A \upharpoonright \mathcal{L}_0, B \upharpoonright \mathcal{L}_0$ and $C \upharpoonright \mathcal{L}_0$, then $D^*$ is a subreduct (a reduct) of $D$.

(d) On the other hand, (*) form (c) above is not necessarily equivalent to the superamalgamation property, for relations which are not supposed to be transitive. Take $\mathcal{L}_0 = \{S\}$ and $\mathcal{L} = \{R, S\}$. In the terminology of Theorem 3.4(A), if $1 \in P$ and $1 \notin Q$, then $\mathcal{K}_{P,Q}$ has the superamalgamation property with respect to $S$, but (*) is not satisfied, since, as shown by the proof, we necessarily should add $S$-related pairs which are not related in the $\mathcal{L}_0$ pushout.

We now show that, in order to get APU, we need to consider only unary operations in Proposition 3.2(B)-(D).

**Proposition 7.5.** The theory $T$ of posets with a binary operation $f$ which is order preserving on each component has not APU.

*Proof. Let $C = \{c\}$ with the only possible structure. Let $A = \{a, c\}$ with $c < a$ and $f$ the projection onto the first component. Let $B = \{b, c\}$ with $c < b \neq a$ and $f$ the projection onto the second component.

In any amalgamating structure we should have $f(a, b) \geq f(a, c) = a$ and $f(a, b) \geq f(c, b) = b$. If amalgamation is into union, then either $f(a, b) = a$, or $f(a, b) = b$. Suppose the former, hence necessarily $b \leq a$, since $b \leq f(a, b)$. Then $c = f(c, a) \geq f(c, b) = b$, impossible. □

Not all possible variations on Proposition 3.2 hold.

**Proposition 7.6.** Let $T$ be the theory with a binary reflexive and transitive relation $R$ and a unary function $f$ which strictly preserves $R$, namely,

\[ d R e \text{ and } d \neq e \implies \text{ both } f(d) R f(e) \text{ and } f(d) \neq f(e). \]

Then $T$ has not AP.

On the other hand, the theory $T$ with a binary reflexive and transitive relation $R$ and a bijective function $f$ which preserves $R$ has superSAPU. This is immediate from the proof of Proposition 3.2 and also a special case of Theorem 4.10. Obviously, $R$-preserving bijective functions are also strict $R$-preserving.
The strong amalgamation property into union

Example 7.7. (a) We now show that the assumption that the \( T_i \)'s have SAPU in Proposition 4.1(a) cannot be weakened to APU, even for just one among the \( T_i \)'s.

Let \( T_1 \) be the theory in the pure language of identity asserting that the universe has cardinality \( < 3 \). Clearly, \( T_1 \) has APU. On the other hand, SAP fails: just let \( |C| = 1 \) and \( |A| = |B| = 2 \).

Let \( T_2 \) be the theory of partially ordered sets. The classical proof that \( T_2 \) has SAP actually provides SAPU, as we noticed in Proposition 3.2. Let \( C = \{c\} \) and \( A = \{c, a\} \), \( B = \{b, c\} \), with \( c < a \) in \( A \) and \( b < c \) in \( B \). The structures \( A, B \) and \( C \) are also models of \( T_1 \), but any amalgamating structure must be of cardinality \( \geq 3 \), hence is not a model of \( T_1 \).

(b) APU is not sufficient in Proposition 4.1(b), either. Let \( T_1 \) be as above, and \( L' = \{U\} \), where \( U \) is a unary predicate. As above, let \( |C| = 1 \) and \( |A| = |B| = 2 \). Let \( U(a) \) hold in \( A \), for \( a \in A \setminus C \) and \( U(b) \) fail in \( B \), for \( b \in B \setminus C \). Thus in any amalgamating structure \( D \) we have \( a \neq b \), hence \( |D| \geq 3 \) and \( D \) is not a model of \( T_1 \).

In other words, \( T_1 \) has APU in the pure language of identity, but has not even AP in the language \( L' \).

A slightly more general version of Proposition 4.1 holds with the same proof.

Proposition 7.8. Suppose that \( \mathcal{L} = \bigcup_{i \in I} \mathcal{L}_i \) and the \( \mathcal{L}_i \)'s are pairwise disjoint languages. Suppose that, for each \( i \in I \), \( K_i \) is a class of structures for \( \mathcal{L}_i \) and \( K_i \) has SAPU. Then \( K = \{A \mid A \text{ is an } \mathcal{L}-\text{structure and } A|_{\mathcal{L}_i} \in K_i, \text{ for all } i \in I\} \) has SAPU.

The next example is rather tricky, but it explains quite clearly why the “S” and the “U” in SAPU are necessary in Proposition 7.8.

Example 7.9. Take \( \mathcal{L}_1 = \mathcal{L}_2 = \emptyset \) and let \( K_1 \) be the class of models of either odd finite cardinality or of cardinality \( \leq 2 \). Let \( K_2 \) be the class of models of either even finite cardinality or of cardinality \( \leq 2 \). Both \( K_1 \) and \( K_2 \) have SAP and APU but not SAPU. If \( K \) is defined as in Proposition 7.8 then \( K \) is the class of the models of cardinality \( \leq 2 \), thus \( K \) has APU but not SAP.

If \( \mathcal{L}_3 = \{U\} \) and \( K_3 \) is the class of all models for \( \mathcal{L}_3 \), then, as in the proof in Example 7.7(b), \( K \) has not even AP.

Example 7.10. We now provide counterexamples showing that the version of Proposition 4.1 fails when (S)APU is weakened to (S)AP.

(a) Let \( T \) be the theory of abelian groups in the language with sum, opposite and a constant for the neutral element, with a further unary predicate \( U \) and axioms stating, for every \( n \in \mathbb{N} \):

(i) if there are at least \( n \) distinct elements such that \( U(x) \), then there are at least \( n \) distinct elements such that not \( U(x) \).
Clearly, (i) is expressible as a set of first-order sentences. The theory $T$ has SAP. Indeed, given a TBA triple $A$, $B$, $C$ of models of $T$, there is obviously an amalgamating abelian group $G$. We have to interpret $U$ on $G$ in such a way that the expansion of $G$ provides a model $D$ of $T$. The interpretation of $U$ on $A\cup B$ is forced by the request that $A \subseteq D$ and $B \subseteq D$. We can suppose that $G$ properly extends both $A$ and $B$, since the other cases are trivial. Then, considering laterals, we have $|G \setminus (A \cup B)| \geq |B \setminus A|$. We are allowed to interpret $U$ in an arbitrary way over $G \setminus (A \cup B)$, hence if we let $U(x)$ always fail on $G \setminus (A \cup B)$, then $U(x)$ fails for at least half the elements of $G \setminus A$. Since $A$ is a model of $T$, then $U(x)$ fails for at least half the elements of $A$. In conclusion, with the above interpretation, $U(x)$ fails for at least half the elements of $G$, thus $D$ is a model of $T$.

Hence $T$ has SAP. However, if $\sigma$ is the sentence $\forall x \ (x = 0 \lor U(x))$, which has the form (4.1), then $T \cup \{\sigma\}$ has not SAP. Indeed, let $C$ be a trivial group in which $U(0)$ fails and extend $C$ to $A$ and $B$, two disjoint copies of $\mathbb{Z}_2$ in which $U(1)$ holds for both copies of 1. Any strong amalgamating group has cardinality $\geq 4$, hence, if we interpret $U$ in such a way that (i) holds, we have at least one element $d$ distinct from 0 and such that not $U(d)$. But then $\sigma$ fails.

(b) The theory $T \cup \{\sigma\}$ in the above counterexample has not SAP, but $T \cup \{\sigma\}$ has obviously AP. Indeed, modulo isomorphism, the only models of $T \cup \{\sigma\}$ are the trivial group and the two elements group, with $U$ interpreted as above.

However, the example can be modified in order to get a theory $T$ with SAP such that $T \cup \{\sigma\}$ has not even AP, for the same sentence $\sigma$ above. Simply consider the theory $T$ introduced in (a), but in a language with a further unary operation $f$ and no axiom mentioning $f$. The theory $T$ has SAP even in the extended language: just amalgamate the structures without considering $f$ and then interpret $f$ in an arbitrary compatible way in the amalgamating structure (this argument is the SAP-analogue of Proposition 4.1(b)). However, $T \cup \{\sigma\}$, for $\sigma$ as in (a), has not AP: consider the same counterexample as in (a), letting $f(1) = 1$ in $A$ and $f(1) = 0$ in $B$. By the considerations in (a), the copies of 1 in $A$ and $B$ should be identified, but this is prevented by the behavior of $f$.

In fact, the above considerations are an example of a general phenomenon: classes with SAP and classes with AP but not SAP are distinguished by their behavior with respect to expansions: the former classes are exactly those classes with AP such that AP is preserved by expanding the language. We shall present details elsewhere.

(c) In the above examples the sentence $\sigma$ is universal positive. We can modify the examples in such a way that the sentence is universal Horn. Let $T'$ be the theory of abelian groups with a further unary predicate $V$ and axioms stating, for every $n \in \mathbb{N}$:
(ii) if there are at least \( n \) distinct elements such that not \( V(x) \), then there are at least \( n \) distinct elements such that \( V(x) \).

By the same arguments as in (a), \( T' \) has SAP. Let \( \sigma' \) be \( \forall x \ (V(x) \Rightarrow x = 0) \). Then \( T' \cup \sigma' \) has not SAP. Let \( C \) be a trivial group in which \( V(0) \) holds, and let \( A \) and \( B \) be two disjoint copies of \( \mathbb{Z}_2 \) in which \( V(1) \) fails for both copies of \( 1 \). Then argue as above.

If we add a dummy unary function in the language, as in (b), then \( T' \) has still SAP in the expanded language, while \( T' \cup \sigma' \) has not AP.

(d) The above example can be refined in order to obtain a finitely axiomatizable universal theory \( T'' \) with SAP and a universal sentence \( \sigma'' \) such that \( T'' \cup \sigma'' \) has not AP and \( \sigma'' \) has the following properties: \( \sigma'' \) is the universal closure of an atomic formula, no constant appears in \( \sigma'' \) and only one variable appears in \( \sigma'' \).

The language of \( T'' \) consists of a binary operation for addition, a unary operation for opposite, two more unary operations \( f \) and \( g \) and a unary predicate symbol \( V \). Axioms of \( T'' \) contain axioms for abelian groups; notice that we can do without a constant for the neutral element by asking \( (x + (-x)) + y = y \), for all \( x \) and \( y \). A further axiom of \( T'' \) asserts that \( g \) is injective from \( \{ x \mid \neg V(x) \} \) to \( \{ x \mid V(x) \} \), namely,

(iii) for every \( x, y \), if \( x \neq y \), \( \neg V(x) \) and \( \neg V(y) \), then \( g(x) \neq g(y) \), \( V(g(x)) \) and \( V(g(y)) \).

Since \( g \) can be defined in an arbitrary way on \( \{ x \mid V(x) \} \), the arguments in (a)-(c) above show that \( T'' \) has SAP. Let \( \sigma'' \) be \( \forall x \ (x + g(x) = x) \). We show that \( T'' \cup \sigma'' \) has not AP. As in (c), let \( C \) be a trivial group in which (necessarily, because of (iii)) \( V(0) \) holds. Let \( A \) and \( B \) be two copies of \( \mathbb{Z}_2 \) in which \( V(0) \) holds and \( V(1) \) fails, for both copies of \( 1 \), hence necessarily \( g(1) = 0 \). As in (b), let \( f(1) = 1 \) in \( A \) and \( f(1) = 0 \) in \( B \), thus the two copies of \( 1 \) cannot be identified in any amalgamating structure. Hence APU fails and any amalgamating structure has at least one new element \( d \), hence at least one such element.

**Example 7.11.** In Proposition 4.3 it is necessary to assume that in (4.1) only one variable is bounded by \( \forall \). An example has been provided in Remark 4.5(d): here is another example.

The theory \( T \) without axioms in the language with two unary relations \( R \) and \( S \) has SAPU, by Observation 3.1(c).

If we add to \( T \) the axiom
\[
\forall xy \ (R(x) \Rightarrow S(y)), \tag{7.1}
\]
then AP fails for the extended theory.

Take \( C = \{ c \} \) and let \( \neg R(c) \) and \( S(c) \) hold in \( C \).

Let \( A \) over \( A = \{ a, c \} \) extend \( C \) with \( R(a) \) and \( S(a) \). Let \( B \) over \( B = \{ b, c \} \) extend \( C \) with \( \neg R(b) \) and \( \neg S(b) \).

Then \( A \) and \( B \) cannot be amalgamated over \( C \) if we want that (7.1) is satisfied.
Example 7.12. As we mentioned in Remark 4.5(b), if \( K \) is a class with SAPU in a language with a unary function symbol \( f \), then the subclass of \( K \) consisting of those structures in which \( f \) is bijective has SAPU.

We show that the corresponding statement is not true when SAPU is weakened to APU. Let \( T \) be the following theory in a language with two unary predicates \( U \) and \( V \) and a unary function \( f \). The theory \( T \) asserts that

1. there is at most one element \( x \) such that \( V(x) \), and
2. for every \( n \in \mathbb{N} \), if \( V(x) \), then the elements \( x, f(x), \ldots, f^n(x) \) are pairwise distinct and do not lie in \( U \).

Arguing as in Observation 3.1(c) we get that \( T \) has APU, since, once there is some element \( c \) such that \( V(c) \), the theory describes completely the set \( \{c, f(c), \ldots, f^n(c), \ldots\} \) and tells nothing about all the other potential elements, except that \( V \) never holds there. In more detail, if \( V(c) \), for some \( c \in C \), then we have an amalgamating structure on \( A \cup B \), as usual. This is the case also when \( V(a) \), for some \( a \in A \), but not \( V(b) \), for every \( b \in B \), and conversely. Similarly, we can amalgamate on the union when not \( V(x) \), for \( x \in A \cup B \).

On the other hand, SAP fails, since if in \( C \) there is no element \( x \) such that \( V(x) \) but such elements exist both in \( A \) and \( B \), then they should be identified. We still retain APU, since if \( V(a) \) and \( V(b) \), for some \( a \in A \) and \( b \in B \), then we get an amalgamating structure by identifying \( a \) with \( b \), \( f(a) \) with \( f(b) \), . . . , with no further identification on \( D \). The identification can be made coherently because of clause (2).

Let \( T' = T \cup \{\sigma\} \), where \( \sigma \) says that \( f \) is bijective. Then \( T' \) has not AP. Indeed, let \( C = \{c\} \) with \( f(c) = c \), not \( U(c) \) and not \( V(c) \). Extend \( C \) to models \( A \) and \( B \) by adding in each case a copy of \( \mathbb{Z} \), with \( f(z) = z + 1 \), for \( z \in \mathbb{Z} \), and \( V(1) \), in both cases, but with \( U(0) \) in \( A \) and not \( U(0) \) in \( B \). By (1) the two copies of 1 should be identified in any amalgamating structure, hence, if \( f \) is injective, then the two copies of 0 should be identified, but this is impossible because of \( U \).

The next example contrasts case (A3) in Proposition 7.10.

Example 7.13. Consider the theory \( T \) in the language with two binary relations \( R \) and \( S \) and asserting that \( x R y \) and \( y R z \) imply \( x S z \).

(a) The theory \( T \) has superSAPU with respect to \( R \).

Indeed, given a TBA triple \( A, B, C \), let \( R = R_A \cup R_B \) on \( D = A \cup B \). Let \( S \) be defined on \( D \) by \( d S e \) if

- either \( d, e \in A \) and \( d S_A e \), or \( d, e \in B \) and \( d S_B e \), or
- \( d R f R e \), for some \( f \in D \).

Then \( D \) is a model of \( T \) by construction. It remains to show that \( D \) extends \( A \) and \( B \). Again, this holds by construction, as far as \( R \) is concerned; moreover, the inclusions from \( A \) and \( B \) to \( D \) are homomorphisms.
Suppose that $d S e$ in $D$ and, say, $d, e \in A$. Then either $d S A e$, or $d R f R e$, for some $f \in D$. By the definition of $R$ on $D$, we necessarily have $f \in A$ and $d R_A f R_A e$, thus $d S_A e$, since $A$ is a model of $T$. From $d, e \in A$ and $d S e$ in $D$ we have got $d S_A e$ in each case, and this means that the inclusion of $A$ into $D$ is an embedding. The argument for the inclusion of $B$ into $D$ is symmetrical.

We have showed that $T$ has SAP. SuperSAPU with respect to $R$ follows from the definitions.

(b) There is a TBA triple $A, B, C$ of models of $T$ such that no infinite chain of elements such that $a_1 S a_2 S a_3 S a_4 \ldots$ exists in $A, B$ or $C$, but such a chain exists in any amalgamating structure.

Let $C = \{c_1, c_2, c_3, \ldots \}$ be a countably infinite set with no pair of elements $R$-related and no pair of elements $S$-related. Let $A = C \cup \{a_1, a_2, a_3, \ldots \}$ with $a_i R c_i S a_{i+1}$, for all indices $i$ and no other pair of elements $R$- or $S$-related. Let $B = C \cup \{b_1, b_2, b_3, \ldots \}$ with $c_i R b_i S a_i$, for all indices $i$, and no other pair of elements $R$- or $S$-related. In any amalgamating structure which is a model of $T$ we must have $a_1 S b_1 S c_1 S a_2 S b_2 \ldots$

Thus, in contrast with the case of partially ordered sets, Proposition 4.9(A3), SAPU does not prevent the creation of infinite chains of elements related by the same relation.

8. Further remarks.

**Proposition 8.1.** Suppose that $(T_i)_{i \in I}$ is a sequence of theories in disjoint languages $\mathcal{L}_i$. If each $T_i$ has SAP, then $T = \bigcup_{i \in I} T_i$ has SAP.

**Proof.** (Sketch) First observe that if $\mathcal{K}$ is a class of structures closed under isomorphism with SAP and $A \in \mathcal{K}$ has no proper extension in $\mathcal{K}$, then $A$ has no proper substructure in $\mathcal{K}$. Indeed, were $C$ a proper substructure of $A$, we could amalgamate $A$ with an isomorphic copy of $A$ intersecting $A$ in $C$, getting a proper extension of $A$. Notice that here it is fundamental to assume the strong version of AP.

Thus structures without proper extensions for some $T_i$ give no trouble. Otherwise, if some model $A$ of $T_i$ has a proper extension satisfying $T_i$, then, for every infinite cardinal $\lambda \geq |\mathcal{L}_i|$, $A$ has a proper extension of cardinality $\lambda$ satisfying $T$. This is immediate from the Löwenheim-Skolem-Tarski Theorem if $A$ is infinite; otherwise, by the preceding paragraph, $T_i \cup \text{Diag}(A)$ has models of cardinality $\geq n$, for arbitrary $n \in \mathbb{N}$, hence an infinite model by compactness.

Thus, given a triple to be amalgamated, we can amalgamate their reducts to the language of $T_i$ to models $D_i$ such that $D_i \setminus (A \cup B)$ has the same cardinality for each $i$. Since the languages are disjoint and each $T_i$ has SAP, we can arrange things in such a way that the $D_i$’s have the same domain. \qed
See [BGR] for a detailed proof of a slightly different statement, connections with quantifier-free interpolation and application to verification and automated reasoning.

For theories with superAPU Propositions 4.1 and 8.1 allow a generalization to non-disjoint languages.

**Proposition 8.2.** Suppose that \((T_i)_{i \in I}\) is a sequence of theories in languages \(L_i\) and suppose that \(L_i \cap L_j = \{R\}\), for \(i \neq j \in I\), where \(R\) is a binary relation symbol.

If each \(T_i\) has superSAPU and asserts that \(R\) is transitive, then \(T = \bigcup_{i \in I} T_i\) has superSAPU.

**Proof.** Suppose that \(A, B, C\) is a TBA-triple of models of \(T\) and, as usual, let \(D = A \cup B\). Since \(R\) is assumed to be transitive, if \(A\) and \(B\) embed in a model \(D\) over \(D\), then the interpretation of \(R\) is uniquely determined by superSAPU. By assumption, each \(L_i\)-reduct of the triple can be superamalgamated in some model over \(D\). Then \(R\) is interpreted in the same way, for each \(i \in I\). Since the languages pairwise intersect in \(\{R\}\), we can join all the interpretations in a model for the full language of \(T\). \(\square\)

**Remark 8.3.** The assumption that the common relation is transitive is necessary in Proposition 8.2. Let \(T_1\), resp., \(T_2\) be the theories of an antisymmetric relation \(S\) with a finer partial order \(\leq_1\), resp, a finer partial order \(\leq_2\). Then both \(T_1\) and \(T_2\) have superSAPU with respect to \(S\), by Theorem 3.4(A).

On the other hand, \(T_1 \cup T_2\) has not AP, by Proposition 3.5(a).

**Problem 8.4.** Find other ways, besides Propositions 4.1, 4.3, 4.9, 7.8, 8.1 and Theorems 4.2 and 4.10, to merge classes with (S)AP (not necessarily into union) in such a way that a class with (S)AP is obtained.

**Problem 8.5.** If \(T\) is a first-order theory, let \(T_1\) be the set of all consequences of \(T\) of the form (4.1). By Proposition 4.3(c), \(T_1\) has SAPU, hence SAP.

Due to the importance of SAP, it is probably interesting to study the relationships between \(T, T_1\) and their models.

Under suitable assumptions, if the class \(\mathcal{K}\) of finite substructures of models of \(T_1\) has JEP (this happens, for example, if the language of \(T\) has no constant) then \(\mathcal{K}\) has a Fraïssé limit \(M\). See [H, Section 7.1].

Study the relationships between \(T\) and \(M\). Are there some other ways to extract a subtheory of \(T\) having SAP and JEP?

As another proposal for further research, it seems that SAPU fits well with models enriched with topological structures. Moreover, SAPU can be used in order to prove that certain theories have SAP, though not necessarily SAPU. An example appears in the proof of [H, Theorem 3.5], where SAPU for posets is implicitly used in order to prove SAP for lattices. Hence it is likely that the present methods can be extended in order to prove SAP for many more theories.
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