After the discovery of quasicrystals [1], quasiperiodic tilings [2] as well as their randomized counterpart, random rhombus tilings [3], rapidly appeared to be suitable paradigmatic models for quasicrystalline alloys [4]. Meanwhile, these systems also became an active topic in discrete mathematics (see [5] or [6] for examples). Fig. 1 displays a random tiling, which belongs to the class of plane tilings with octagonal symmetry. Beyond this case, plane tilings with larger symmetries including Penrose tilings [2] and space tilings with icosahedral symmetries were proposed to model every kind of quasicrystal. The present letter is devoted to dynamical properties of random rhombus tilings in terms of local dynamical rules, the so-called phason-flips, which consist of local rearrangement of tiles (Fig. 2). These dynamics are of interest for several reasons. On the one hand, it is more and more clear that phason-flips exist in real quasicrystals [7] and can be modeled in large approximations by tile-flips; they are a new source of atomic mobility, as compared to usual crystalline materials. In particular, they could carry their own contribution to self-diffusion [8], even if the efficiency of such processes remains controversial [9]. They are also involved in some specific mechanical properties of quasicrystals, such as plasticity via dislocation mobility [10]. Therefore a complete understanding of flip dynamics is essential in quasicrystal physics. The present work is a first step in this direction. On the other hand, a lot of numerical work has been carried out to characterize statistical properties of tiling sets, a part of which was based on Monte Carlo techniques which rely on a faithful sampling of tiling sets (see [11,12]). So far, no systematic study of the relaxation times between two independent numerical measures has been accomplished, whereas it is an essential ingredient for a suitable control of error bars. However, there exist exact results in the simplest case of random rhombus tilings with hexagonal symmetry [13,14] and several estimates of relaxation times in larger symmetries, either numerical or in the approximate frame of Langevin dynamics [11,12].

Random rhombus tilings are made of rhombi of unitary side length. They are classified according to their global symmetries [3]. The simplest class of hexagonal tilings – made of 60° rhombi with 3 possible orientations – has been widely explored [13,14]. Tilings with octagonal symmetry are made of 6 different tiles (characterized by their shape and orientation): two squares and four 45° rhombi (Fig. 1). Beyond these two cases, one can define tilings with higher symmetries (e.g. Penrose tilings [2] or of higher dimensions [3]). For sake of technical simplicity, we focus on tilings filling a centrally symmetric polygon with integral side lengths (Fig. 1). We are interested in the large size limit where the polygon becomes large keeping a fixed shape. Such tilings can schematically be seen as frozen near their boundary, and strain-free in their central region [17]. Dynamics of fixed boundary tilings should be the manifestation of dynamics of their strain-free center, thus relating both boundary conditions. Here we suppose that all tilings have the same probability; we work at infinite temperature.

FIG. 1. Examples of octagonal fixed boundary tiling and of elementary flip. We have also displayed (in gray) the de Bruijn lines of a family, among 4 families. They are lines of adjacent tiles sharing an edge with a fixed orientation.

The set of all the tilings of such a region together with the flip dynamical rule define a discrete time Markov chain: at each step, a vertex of the tiling is uniformly chosen at random and if this vertex is surrounded by 3 tiles in flippable configuration, then we flip it. Since sets of plane rhombus tilings are connected via elementary flips [15], this process can reach any tiling. It converges toward the uniform equilibrium distribution, since it satisfies detailed balance. All the difficulty is to characterize how many flips one needs to get close to equilibrium. Generally speaking, let us consider a Markov chain on a finite configuration space L, which converges toward a
stationary distribution \( \pi \). Let \( x_0 \) be any initial configuration and \( P(x, t|x_0, 0) \) be the probability that the process has reached the configuration \( x \) after \( t \) steps. Then

\[
\Delta(t, x_0) = 1/2 \sum_{x \in L} |P(x, t|x_0, 0) - \pi(x)|
\]

usually measures the distance between both distributions \([13]\). Given \( \varepsilon > 0 \) we define the ergodic or mixing time \( \tau(\varepsilon) \) so that whatever \( x_0 \), after \( \tau(\varepsilon) \) steps, one is sure to stay within distance \( \varepsilon \) of equilibrium:

\[
\tau(\varepsilon) = \max_{x_0} \min_{t_0} \{t_0/\forall t \geq t_0, \Delta(t, x_0) \leq \varepsilon\}.
\]

In this letter, we prove that for a tiling of \( N_T \) tiles

\[
\tau(\varepsilon) \leq Cst. (N_T)^\nu \ln N_T \ln(1/\varepsilon).
\]

More precisely, we establish, with the help of reduced numerical work, that such a bound holds for \( \nu = 3 \), then we argue that \( \nu = 2 \) should be the correct exponent. Whatever this exponent, this proves that flip dynamics are rapidly mixing at infinite temperature. As for the \( \ln(N_T) \) correction in \([3]\), as discussed in \([13]\), it is a feature of our choice of distance \( \Delta(t, x_0) \): an Euclidean norm would not display this correction but it is less natural in the context of measure of probability distributions convergence.

The coupling technique \([19]\) has been successfully applied to estimate mixing times of several systems, such as hexagonal tilings \([4,13]\). It relies on the surprising idea that following the dynamics of couples of configurations instead of a single one might provide the properties of the original dynamics on single configurations. A coupling is a Markov chain on \( L \times L \); couples of configurations are updated simultaneously and are strongly correlated, but each configuration, viewed in isolation, performs transitions of the original Markov chain. Moreover, the coupled process is designed so that when both configurations happen to be identical, then they follow the same evolution and remain identical forever. Then the central idea of the technique is that the average time the two configurations need to couple (or to coalesce) provides a good upper bound on the original mixing time \( \tau(\varepsilon) \): given an initial couple \( (x_0, y_0) \) at time \( t = 0 \), define the coalescence time \( T(x_0, y_0) \) as the minimum time that both configurations need to coalesce, and the coupling time as

\[
T = \max_{(x_0, y_0) \in L \times L} < T(x_0, y_0) >,
\]

where the last mean is taken over realizations of the coupled Markov chain. The following theorem, central in the coupling approach, provides an upper bound for the ergodic times of the original \( (not\ coupled) \) process \([3]\):

\[
\tau(\varepsilon) \leq T e \ln(1/\varepsilon) + 1 \simeq T e \ln(1/\varepsilon),
\]

where \( e = \exp(1) \). If the configuration set \( L \) can be endowed with a partial order relation \( \succeq \) with unique minimum and maximum elements \( 0 \) and \( 1 \), the implementation of the technique is highly facilitated, provided the coupled dynamics is monotous, i.e. if \( x(t) \succeq y(t) \), then \( x(t+1) \succeq y(t+1) \): let \( (x_0, y_0) \) be any initial couple such that \( 1 \succeq x_0 \succeq y_0 \succeq 0 \). After any number of steps, the 4 configurations remain in this order. When the iterates of \( 0 \) and \( 1 \) have coalesced, the iterates of \( x_0 \) and \( y_0 \) also have. Thus \( T(x_0, y_0) \leq T(0, 1) \) and \( T = < T(0, 1) > \).

A convenient representation of random rhombus tilings was introduced by de Bruijn \([20]\). It consists of following in a tiling lines made of adjacent tiles sharing an edge with a fixed orientation (Fig. 1). The set of lines associated with an orientation is called a de Bruijn family. In an octagonal tiling, there are 4 families. When removing a family from an octagonal tiling, one gets an hexagonal tiling. Conversely, this remark enables one to propose a convenient construction of tilings \([21,12,18]\): directed paths are chosen on an hexagonal tiling, called the base tiling. They are represented by dark lines in Fig. 2. They go from left to right without crossing (but they can have contacts). When they are “opened” following a new edge orientation, they generate de Bruijn lines of the fourth family. In this one-to-one representation, a tiling flip involving tiles of the fourth family becomes a path flip: the path jumps from one side of a tile to the opposite side.

![FIG. 2. Example of 2-line coupling in the directed path representation. Dark lines represent a configuration and dotted lines the second one. Both configurations form a couple.](image-url)
where \( \text{Id} \) is the identity. Indeed, each coefficient \( \mathcal{M}(x, y) \) appears in all four matrices \( \mathcal{M}_i \), but one, since the corresponding flip involves 3 de Bruijn lines.

Now we implement the above coupling technique on each fiber. To begin with, we suppose that there is only one line in the flipping family, denoted by \( \ell \). As in reference \([13]\), in order to have a monotonous coupling, we slightly modify the Markov chain: at each step, we choose uniformly at random an internal vertex of \( \ell \), the \( n \)-th one starting form the left, and a number \( r \in \{0, 1\} \). If this vertex is flippable upward (resp. downward) and \( r = 0 \) (resp. \( r = 1 \)), then we flip it. Note that this Markov chain has a time unit different from the original one.

We now define the order relation \((\succeq)\): given two lines \( \ell_1 \) and \( \ell_2 \), \( \ell_1 \succeq \ell_2 \) if \( \ell_1 \) is entirely above \( \ell_2 \). The maximum (resp. minimum) configuration clearly lies on the top (resp. bottom) boundary of the hexagonal domain. If the two flips on \( \ell_1 \) and \( \ell_2 \) occur with same \( n \) and \( r \) and if \( \ell_1 \succeq \ell_2 \) then their images satisfy the same relation. Indeed, as in reference \([13]\), thanks to the introduction of \( r \), if a flip could bring \( \ell_1 \) below \( \ell_2 \), then the same flip would also apply to \( \ell_2 \), thus preserving the order between lines: the coupling is monotonous.

In the general case with \( p \) non-intersecting lines in each configuration (Fig. 3), let us denote by \( \ell^{(j)}_i \), \( j = 1, \ldots, p \), the \( p \) lines of each configuration \( \gamma_i \). Then \( \gamma_1 \succeq \gamma_2 \) if for any \( j \), \( \ell^{(j)}_1 \succeq \ell^{(j)}_2 \). The configuration \( \gamma \) is maximum (resp. minimum) when each of its lines is maximum (resp. minimum). At each step, the index \( j \) of the line to be flipped is chosen between 1 and \( p \), the same \( j \) for both \( \gamma_i \).

To begin with, we numerically study the diagonal case, where the 4 sides of the octagonal tilings are equal to \( k \). For a given base tiling \( T_0 \), we run a number \( m \) of couplings until they coalesce, and then estimate the coupling time \( T(T_0) \). We then make a second average on \( M \) different tilings, in order to get the time \( \mathbf{T} \) averaged on tilings \( T_a \). We also keep track of the standard deviation \( \Delta T \). From our numerical data (see Fig. 3), we draw the following conclusions: \( \Delta T/\mathbf{T} \) decreases toward a constant \((\simeq 0.07)\) as \( k \rightarrow \infty \), which means that the average coupling time \( T(T_0) \) goes on depending on the base tiling \( T_0 \) at the large size limit. However, most \( T(T_a) \) are of order \( \mathbf{T} \), and the mixing times \( \tau(\varepsilon) \) on most fibers are controlled by \( \mathbf{T} \). Nevertheless, the effect of few “slower” fibers will deserve a detailed discussion below. Moreover, the measures of \( \mathbf{T} \) are compatible with a \( k^4 \ln k \) behavior (Fig. 3, inset). In particular, this fit with logarithmic corrections is much better than a simple power-law fit. This result is consistent with known results in the case of hexagonal tilings \([13,14]\), where \( T \) also grows like \( k^4 \ln k \).

We also have explored coupling times on fibers in non-diagonal cases and our conclusions remain identical. As a consequence, couplings in fibers behave like couplings in hexagonal tiling problems, up to different numerical prefactors: the dynamics on each fiber is rapidly mixing.

![Fig. 3](image-url). A distribution of coupling times \( T \) in the case of a diagonal base tiling \((k = 10)\) and \( p = 1 \), fitted by a Gumbel distribution (continuous curve; see eq. \([10]\)). Inset: in the diagonal case \((k = p)\), numerical estimates of \( T \) in function of \( k^4 \ln k \) (circles), up to \( k = 15 \), and linear fit. Error bars are smaller than the size of symbols. The slope is 25.51 ± 0.05.

![Fig. 4](image-url). The lattice of tilings filling an octagon of sides 1,1,1 and 2. The edges represent possible flips. Two fibrations among four are represented (continuous and dotted lines).

Now we return to the dynamics on the whole set of tilings. Examining Fig. 4, one remarks that two fibrations are to a certain extent “transverse”: in octagonal tiling sets, one can connect any two tilings using flips of only two fibers \([22]\): if the dynamics is rapidly mixing in each fiber, the combination of dynamics on two (and even four) fibrations will certainly also be rapidly mixing.

We now establish properly this point: it is common in the field of Markov processes to relate rates of convergence to spectra of transition matrices. Generally speaking, given a transition matrix \( M \), 1 is always the largest eigenvalue in modulus, and the difference \( g(M) \) between 1 and the second largest eigenvalue is called the first gap of \( M \). Then \( \tau(\varepsilon) \simeq \ln(1/\varepsilon)/g(M) \) for small \( \varepsilon \) \([14]\). By simple arguments from linear algebra and Euclidean geometry (since matrices are symmetric), the following central gap relation can be established \([24]\), based upon \([8]\) and the smallness of the intersection of two fibers:

\[
g(M) \geq \inf_{i} (g(M_i)) . \tag{7}
\]

This result restores the symmetry lost in the fibration...
process. It implies that the mixing time $\tau(\varepsilon)$ on $L$ is smaller than the mixing time on the slowest fibration derived from $\mathcal{T}$. Hence $\tau(\varepsilon)$ is smaller than the mixing time on the slowest fiber. We have seen that the average coupling time $T$ grows like $k^4 \ln k$ in diagonal cases. Since the number of tiles is $N_T = 6k^2$, $T \simeq \kappa_4 N_T^2 \ln(N_T)$, where $\kappa_4 = 1.189 \pm 0.003$ in the original time unit. However, the coupling time $T$ depends on the base tiling $T_0$ and the distribution of times $T$ in a given fibration have a certain width around $T$ (Fig. 3). And even if the typical values of coupling times are of the order of magnitude of the previous average value, this does not exclude the existence of rare slow fibers in the upper tails of these distributions. However, we recall that coupling times $T$ are maxima of coalescence time distributions. Therefore the expected shape of the distribution of $T(T_0)$ ought to be sought in the specific class of extreme-value distributions, namely Gumbel distributions: consider $N$ independent identical random variables $T_0$, whose probability densities decay rapidly at large $T$:

$$p(T) \simeq \frac{C_1}{T^\beta} \exp(-C_2 T^\beta),$$

(8)

where $C_1, C_2, \beta > 0$. If $T_{\text{max}} = \max_0 T_0$, then at large $N$, the probability density of $T_{\text{max}}$ satisfies

$$p(u) = \exp(-u - \exp(-u)),$$

(9)

where $u = (T_{\text{max}} - T_0)/\delta T$ is a suitably rescaled variable.

Now, even if coalescence times are not strictly speaking independent variables, our numerical distributions appear to be well fitted by this kind of distribution (Fig. 3). This result provides the large $T$ behavior of coupling time distributions: $p(T_{\text{max}}) \sim \exp(-u) \sim \exp(-T_{\text{max}}/\delta T)$, and therefore an estimation of the largest coupling time. Let us focus on diagonal cases: $T_0$ as well as $\delta T$ behave like $k^4 \ln k$. But for a tiling $i$, there are $N_i$ base tilings $T_0$. Therefore if $T^*$ is the largest coupling time on all tilings $T_0$, it is estimated by $N_i \exp(-T^*/\delta T) \approx 1$. Now $N_i$ grows exponentially with the number of tiles [3]: $\ln N_i \approx Cst_3 k^2$. Thus

$$T^* \approx Cst_2 k^6 \ln k \approx Cst_3 N_T^3 \ln N_T$$

(10)

and $\tau(\varepsilon) \leq Cst_4 N_T^3 \ln N_T / (1/\varepsilon)$. However these extreme values should not be relevant: because of the exponential decay of $p(u)$, slow fibers are rare and can be bypassed via rapid ones. More precisely, perturbation theory arguments suggest that rare slow fibers can be seen as a small perturbation of an otherwise rapid transition matrix and have a vanishing influence on its spectral gap [22]: they do not significantly slow rapid dynamics and the typical coupling time $T$ drives the dynamics on fibers, leading to $\nu = 2$ in [3] and $Cst_4 \leq e \kappa_4$.

What does this analysis become in the case of larger symmetry tilings or of higher dimensional tilings, such as icosahedral ones? Plane rhombus tilings with 2D-fold symmetry could be addressed by our approach without significant technical complication, leading to laws similar to (3), up to different prefactors $Cst_2$. As for higher dimensional tilings, the fibration process remains valid, but the connectivity of fibers is not established [8], making impossible a naive generalization of this approach.

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