World volume Supermembrane Instantons

in the light-cone frame.*

E.G. Floratos\textsuperscript{a,b} and G.K. Leontaris\textsuperscript{c}

\textsuperscript{a}Institute of Nuclear Physics, NRCS Demokritos, Athens, Greece
\textsuperscript{b}Physics Department, University of Iraklion, Crete, Greece
\textsuperscript{c}Theoretical Physics Division, Ioannina University, GR-45110 Ioannina, Greece

Abstract

In this review we present the octonionic duality for membranes. We start with a discussion on the relation of the Yang Mills theories and the supermembrane Hamiltonian in the light-cone gauge. We further derive the self-duality equations for the membranes and discuss the integrability of the system in 7 and 3 dimensions. Finally, we present classical Euclidean time solutions of these equations and examine the supersymmetries left intact by the self-duality equations.

IOA-TH/99-05
May 1999

* Talk presented by E.G. Floratos in the workshop “Symmetries and Conservation Laws in Intermediate and High Energy Physics”, Ioannina, 1-5 October 1998
1. Introduction

One of the basic ingredients of $M$-theory[1] is the eleven dimensional ($11 - d$) supermembrane for which some years ago[2] a consistent action has been written in a general background of the $11 - d$ supergravity. The supermembrane has a uniquely defined self-interaction which, in contrast to the superstring, comes from an infinite dimensional gauge symmetry apparent in the light-cone gauge as the area-preserving diffeomorphisms ($SDiff\{M\}$) on the surface of the membrane. There is no topological expansion over all possible three manifolds analogous to the topological expansion of the string case, because of the absence of the dilaton field for the supermembrane. The supermembrane, due to its unique self-interaction, is possible to break into other supermembranes so in a sense is already a second quantized theory but up to now there is no consistent perturbative expansion around free harmonically oscillating supermembrane configurations in topologically trivial space-time backgrounds. In the light-cone gauge, and flat space-time, there are two classes of membrane vacua, points and tensionless strings, so a low energy effective field theory of supermembrane massless excitations would be either eleven-dimensional supergravity or a field theory for tensionless strings.

In a different approach, one may exploit the non-perturbative structure of the supermembrane vacuum in flat space-time studying the corresponding Euclidean time equations of motion which describe quantum tunneling between classical membrane configurations[3, 4]. The structure of quantum mechanical vacua of supermembrane in flat backgrounds can be studied through a new kind of self-duality[4] which is apparent in the light-cone gauge as an analogue of electric-magnetic non-abelian duality. Indeed, explicit construction of the self-duality occurs in three[4] and seven[5, 6] space dimensions due to the existence of unique vector cross product defined by the quaternionic and octonionic division algebras correspondingly. As in the case of non-abelian duality, the corresponding (Euclidean space-time) instantons have a topological charge which has a space-time interpretation for
the membrane and which satisfies a Bogomol’nyi bound giving rise to supersymmetric invariance for the instantons. In recent works, we have found explicit instanton solutions in three and seven dimensions for the $S^2$ and $T^2$-membranes and we have shown that the corresponding surviving supersymmetries are 8 and 1. The $3-d$ instantons in principle can be constructed using Lax pair techniques due to the integrability of this system. In seven dimensions, there is an integrable sector but due to the non-associativity of the octonionic algebra it seems that the complete system is not integrable. In the following sections we present a review on the derivation of the self-duality equations and present some solutions. We will also discuss the supersymmetries left unbroken by particular classes of solutions. We note that analogous self-duality equations were written for Yang-Mills (YM)-theories in [7] and described a subclass of solutions with interesting properties[8]. Other generalizations of such equations have also been intensively explored recently [9, 10, 11].

2. Matrix model and Membranes.

It has been known since sometime that the supermembrane Hamiltonian in the light-cone gauge is a very close relative of YM-theories in the gauge $A_0 = 0$ and in one space dimension less [12]. To describe in some more detail this relationship, we restrict our discussion to the bosonic part of the Hamiltonian of the supermembrane in the light-cone gauge and to spherical topology for the membrane[12, 13, 14]. In reference[14] it was pointed out that in the large $N$-limit, $SU(N)$ YM theories have, at the classical level, a simple geometrical structure with the $SU(N)$ matrix potentials $A_\mu(X)$ replaced by $c$-number functions of two additional coordinates $\theta, \phi$ of an internal sphere $S^2$ at every space-time point, while the $SU(N)$ symmetry is replaced by the infinite dimensional algebra of area preserving diffeomorphisms of the sphere $S^2$ called SDiff$(S^2)$. The $SU(N)$ fields ($N \times N$ matrices)

$$A_\mu(X) = A_\mu^\alpha(X) \tau^\alpha,$$
\( t^\alpha \in SU(N), \)

\[ \alpha = 1, 2, \ldots, N^2 - 1, \quad \mu = 0, 1, \ldots, d - 1 \quad (1) \]

in the large \( N \)-limit become \( c \)-number functions of an internal sphere \( S^2 \),

\[ A_\mu(X, \theta, \phi) = \sum_{l=1}^{\infty} \sum_{m=-l}^{l} A_{\mu}^{lm}(X)Y_{lm}(\theta, \phi), \quad (2) \]

where \( Y_{lm}(\theta, \phi) \) are the spherical harmonics on \( S^2 \). The local gauge transformations

\[ \delta A_\mu = \partial_\mu \omega + [A_\mu, \omega], \quad \omega = \omega^\alpha t^\alpha, \quad (3) \]

and

\[ \delta F_{\mu\nu} = [F_{\mu\nu}, \omega], \quad (4) \]

\[ F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu] \quad (5) \]

are replaced by

\[ \delta A_\mu(X, \theta, \phi) = \partial_\mu \omega(X, \theta, \phi) + \{A_\mu, \omega\}, \quad (6) \]

\[ \delta F_{\mu\nu}(X, \theta, \phi) = \{F_{\mu\nu}, \omega\}, \quad (7) \]

where

\[ F_{\mu\nu}(X, \theta, \phi) = \partial_\mu A_\nu - \partial_\nu A_\mu + \{A_\mu, A_\nu\}, \quad (8) \]

and the Poisson bracket on \( S^2 \) is defined as follows:

\[ \{f, g\} = \frac{\partial f}{\partial \phi} \frac{\partial g}{\partial \cos \theta} - \frac{\partial g}{\partial \phi} \frac{\partial f}{\partial \cos \theta} \quad (9) \]

So the commutators are replaced by Poisson brackets according to

\[ \lim_{N \to \infty} N[A_\mu, A_\nu] = \{A_\mu, A_\nu\} \quad (10) \]

Then the YM action in the large \( N \)-limit becomes \[14\]

\[ S_\infty = \frac{1}{16\pi g^2} \int_{S^2} d\Omega \int d^4XF_{\mu\nu}(X, \theta, \phi)F^{\mu\nu}(X, \theta, \phi), \quad (11) \]
where

\[ g = \lim_{N \to \infty} \frac{g_N}{N^{3/2}} \]  

This large \( N \)-limit of SU(\( N \)) YM-theories was found by making use of the relation between the SU(\( N = 2s + 1 \)) algebra in a particular basis (up to spin \( s \) SU(2)–tensor \( N \times N \)-matrices) and SDiff(S\(^2\)) in the basis of the spherical harmonics \( Y_{lm}(\theta, \phi) \). In the present day language, this SDiff(S\(^2\)) YM-theory corresponds to the effective theory of infinite number, \( N \to \infty \), \( d - 1 \) dimensional Dirichlet branes[15]. Here we note that the recently proposed matrix theory which is claimed to be the long sought formulation of M theory, is nothing but the SU(N) supersymmetric YM-mechanics which was used as a consistent truncation of the supermembrane[12, 14].

The above considered large \( N \)-limit, is a very specific one which depends on the appropriate basis of SU(\( N \)) generators convenient for the topology of the membrane and it has nothing to do, at least in a direct way, with the planar approximation of YM-theories. Also, it is different from the large \( N \)-limit used in Matrix theory. In the case of the spherical membranes the SDiff(S\(^2\)) YM-theory describes the dynamics of an infinite number of D0-branes forming a topological 2-sphere. In the light-cone gauge the transverse coordinates \( X_i \), \( (i, 1, ..., 9) \) of the \( 11 \) – \( d \) bosonic part of the supermembrane satisfy the following equations

\[ \ddot{X}_i = \{X_k, \{X_k, X_i\}\}; \quad i, k = 1, \ldots 9 \]  

where summation over repeated indices is implied. The corresponding Gauss law which is the generator of the SDiff(S\(^2\)) group is given by the constraint

\[ \{X_i, \dot{X}_i\} = 0 \]  

From the point of view of Superstring theories the D0-branes are point like solutions of low energy effective supergravity description and the string excitations have to live in these backgrounds. The superposition on the same point of \( N \) D0-branes still is a
classical solution with its own space-time metric. The supersymmetric $SU(N)$ matrix model appears to be the low energy effective description of the excitations of this bound state and the $SU(N)$ matrices which depend only on time, represent the non-commutative position coordinates of these excitations. The link with the old work of ref [17] about the discretization of the membrane is the following [18]. The question was posed in ref [17] about the geometrical meaning of the $SU(N)$ truncation of the area preserving diffeomorphism symmetry of the light-cone membrane Hamiltonian. The interpretation found was that it is the algebra of canonical commutation Weyl-Heisenberg relations for the momentum and position operators $P$ and $Q$ satisfying $QP = wPQ$ with $w = Exp(2\pi i/N)$ which represent the non-commuting coordinates of the discrete membrane surface viewed as a discrete and finite $N \times N$ rectangular and periodic phase space (toroidal membranes). The $N^2$ points of this lattice represent due to non-commutativity $N$ degrees of freedom. These are the $N$ D0-branes which make up the membrane in this truncation. Finally the finite quantum mechanics developed in [17, 19] could be interpreted as single D0-excitations with linear dynamics on the discrete membrane. According to this interpretation the large $N$-limit of the $SU(N)$ supersymmetric YM-quantum mechanics should provide the quantum membrane or an infinite number of interacting strings Up to now only the infinite string picture has been discussed in the literature with some convincing success [20]. Perhaps the sector of finite number of interacting strings provides another cut for the membrane which could be seen as a field theory of interacting strings, still in first quantized form.

3. Self-dual membranes

In this section we present a heuristic derivation of the self-duality equations for the supermembrane and will show that it can be formulated only in three and seven space dimensions. The solutions of these equations, the supermembrane instantons are characterized by a topological charge which is the degree of the map from 3-parameter space of the
membrane to the $3-d$ world volume and which provides a lower bound for the Euclidean action, in the same way as the instanton Bogomol’nyi bound for Yang-Mills (YM) theories. We shall need only the bosonic part of the light-cone Hamiltonian in flat space-time and the corresponding constraint \[13\].

In order to explain the appearance of non-abelian electric-magnetic type of duality in the membrane theory, we recall that for YM-potentials independent of space coordinates the self-duality equation in the gauge $A_0 = 0$, is \[21, 22\]

$$\dot{A}_i = \frac{1}{2} \varepsilon_{ijk} [A_j, A_k], \quad i,j,k = 1,2,3 \quad (15)$$

According to ref \[23\] the only non-trivial higher dimensional YM self-duality equations exist in 8 space-time dimensions which for the $7$-space coordinate independent potentials can be written (in the $A_0 = 0$ gauge) as

$$\dot{A}_i = \frac{1}{2} \Psi_{ijk} [A_j, A_k], \quad i,j,k = 1, \cdots 7 \quad (16)$$

where $\Psi_{ijk}$ is the multiplication table of the seven imaginary octonionic units \[24\].

It is now tempting to take the large $N$-limit and replace the commutators by Poisson brackets to obtain non-abelian (area preserving diffeomorphisms) self-duality equations for membranes in the light cone gauge for $3$ and $7$ space dimensions. In this limit we replace the gauge potentials $A_i$ by the membrane coordinates $X_i$. Then, the $3-d$ system is

$$\dot{X}_i = \frac{1}{2} \varepsilon_{ijk} \{X_j, X_k\}, \quad i,j,k = 1,2,3 \quad (17)$$

while in seven space dimensions

$$\dot{X}_i = \frac{1}{2} \Psi_{ijk} \{X_j, X_k\}, \quad i,j,k = 1, \cdots, 7 \quad (18)$$

It is easy to see that the self-duality membrane equations, imply the second order Euclidean-time, equations of motion in the light-cone gauge as well as the Gauss law. Indeed, the Gauss law results automatically by making use of the $\Psi_{ijk}$ cyclic symmetry

$$\{\dot{X}_i, X_i\} = 0 \quad (19)$$
The Euclidean equations of motion are obtained as follows

\[ \ddot{X}_i = \frac{1}{2} \Psi_{ijk} \left( \{\dot{X}_j, X_k\} + \{X_j, \dot{X}_k\} \right) \]

\[ = \{X_k, \{X_i, X_k\}\} \quad (20) \]

where use has been made of the identity

\[ \Psi_{ijk} \Psi_{lmk} = \delta_{il} \delta_{jm} - \delta_{im} \delta_{jl} + \phi_{ijlm} \]

\[ (22) \]

and of the cyclic property of the symbol \( \phi_{ijlm} \).

It has been noticed that the seven dimensional self-duality equations can be appropriately presented using the octonionic or Cauley algebra. The octonionic units \( o_i \) satisfy the algebra

\[ o_i o_j = -\delta_{ij} + \Psi_{ijk} o_k. \quad (23) \]

where \( o_i, i = 1, \ldots, 7 \) are the 7 octonionic imaginary units with the property

\[ \{o_i, o_j\} = -2\delta_{ij}, \quad (24) \]

We choose the multiplication table

\[ \Psi_{ijk} = \begin{cases} 1 & 2 & 4 & 3 & 6 & 5 & 7 \\ 2 & 4 & 3 & 6 & 5 & 7 & 1 \\ 3 & 6 & 5 & 7 & 1 & 2 & 4 \end{cases} \]

\[ (25) \]

For later use, we note that the automorphism group of the octonionic multiplication table is the exceptional group \( G_2 \). In terms of these units an octonion can be written as follows

\[ X = x_0 o_0 + \sum_{i=1}^{7} x_i o_i \]

\[ (26) \]

with \( o_0 \) the identity element. The conjugate octonion is

\[ \bar{X} = x_0 o_0 - \sum_{i=1}^{7} x_i o_i \]

\[ (27) \]
4. Integrability

In this section we study the integrability of the self-dual equations in 3 and 7 dimensions. Before doing this, we want to factorize the time dependence of the membrane coordinates in order to show the similarity of the system with the matrix Nahm equations which are known to be integrable and provide a general method for the construction of the spherically symmetric BPS YM monopoles[24]. Recent work of Fairlie and Ueno[26] has shown that the 7-d generalization of the Euler-Top (a special case of Nahm equations) provide an integrable system which has been solved completely in terms of hyperelliptic functions[27]. So, the time factorization of the self-dual membrane equations provide integrable systems.

As in the case of the 3-d system[4] we may try to factorize the time dependence in seven dimensions. We assume the following factorization:

\[ X_i = Z_{ij}(t)f_j(\xi) \]  

Then, from eq.(18), we obtain

\[ \dot{Z}_{im}f_m = \frac{1}{2} \Psi_{ijk}Z_{jl}Z_{kn}\{f_i, f_n\} \]  

We observe that if we make the ansatz for the \(7 \times 7\) matrix

\[ \dot{Z}_{im}(t)\Psi_{mln} = \Psi_{ijk}Z_{jl}(t)Z_{kn}(t) \]  

then the equation

\[ f_i = \frac{1}{2} \Psi_{ijk}\{f_j, f_k\} \]  

is automatically satisfied, while at the same time we have succeeded in disentangling the time dependence from the self-duality equation. Therefore, the problem is reduced to finding solutions for \(f_i(\xi)\) and \(Z_{kl}\) equations separately.

Another equivalent form of the previous equation for the matrices \(Z_{ij}\) is

\[ \dot{Z}_{ij} = \frac{1}{6} \Psi_{ikl}\Psi_{jmn}Z_{km}Z_{ln} \]
In the case of diagonal matrices $Z_{ij} = \delta_{ij} R_j(t)$, we have

$$\dot{R}_i = \frac{1}{6} \Psi_{ikl}^2 R_k R_l$$ (33)

We now make some observations about the symmetries of eqs(31). If $X_i$ is a solution of (31), then for every matrix $R$ of the group $G_2$, which is a subgroup of $SO(7)$,

$$Y_i = R_{ij} X_j$$ (34)

is automatically a solution of the same equation, because the elements of $G_2$ preserve the structure constants $\Psi_{ijk}$. In components,

$$\Psi_{ijk} R_{kl} = \Psi_{imn} R_{mj} R_{nl}$$ (35)

The above relation shows the way to defining $G_2$ group elements starting from two orthonormal seven-vectors. The equation is obviously covariant under $SDiff(S^2)$ transformations. One can define combined $G_2$ and $SDiff(S^2)$ transformations to get $SO(3)$ spherically symmetric solutions since $SO(3)$ can be realized as a subalgebra of $SDiff(S^2)$.

We note that in principle it is possible to look for non-linear symmetries of the self-duality equations, generalizing (34)

$$Y_i = f_i(X)$$ (36)

where $f_i(X)$ must satisfy the equation

$$\Psi_{ijk} \frac{\partial f_k}{\partial X_l} = \Psi_{imn} \frac{\partial f_m}{\partial X_j} \frac{\partial f_n}{\partial X_l}. $$ (37)

In the following we examine the self-consistency of eq.(31). Multiplying by $\Psi_{ilm}$, we get

$$\Psi_{ilm} f_i = \{f_i, f_m\} + \frac{1}{2} \phi_{lmjk} \{f_j, f_k\}$$ (38)

Then, since the Poisson brackets satisfy the Jacobi identity, the above equation is constrained to satisfy the identity

$$\frac{1}{3} \phi_{ijkl} f_l = \Psi_{ijm} \{f_m, f_k\} + \text{cyclic perm. of } (ijk).$$ (39)
This system of equations is exactly the same as in (31). Another check for the self-consistency of $f_i$ equations can be found as follows. Define the tensors

$$X_{ij}^{kl}(u) = \Delta_{ij}^{kl} + \frac{u}{4} \phi_{ij}^{kl}, \quad (40)$$

$$X_{ij}^{kl}(u) = \Delta_{ij}^{kl} + \frac{u}{4} \phi_{ij}^{kl}, \quad (41)$$

where $\Delta_{ij}^{kl} = \frac{1}{2} (\delta_i^k \delta_j^l - \delta_i^l \delta_j^k)$ and the symbol $\phi_{ij}^{kl} \equiv \phi_{ijkl}$. Then, eqs(31) can be written as follows

$$\Psi_{ijk} f_k = X_{ij}^{lm}(2) \{f_l, f_m\}. \quad (42)$$

Using now the algebra of the $X_{ij}^{kl}(u)$ tensors discussed in detail in the Appendix of ref[6], we can prove that both the identities

$$\Psi_{ijk} X_{jk}^{lm} (-1) = 0 \quad (43)$$

and

$$X_{ij}^{mn} (-1) X_{mn}^{kl} (2) = 0 \quad (44)$$

hold, and this terminates the second consistency check.

The 3–d self-duality system has a Lax pair and infinite number of conservation laws[4]. In order to see this, first we rewrite eqs.(17) in the form

$$\dot{X}_+ = i \{X_3, X_+\}, \quad \dot{X}_- = i \{X_3, X_-\}, \quad \dot{X}_3 = \frac{1}{2} i \{X_+, X_-\}, \quad (45)$$

where

$$X_\pm = X_1 \pm i X_2 \quad (46)$$

The Lax pair equations can be written as

$$\dot{\psi} = L_{X_+} \lambda X_- \psi, \quad \dot{\psi} = L_{X_-} \lambda X_+ \psi, \quad (47)$$

where the differential operators $L_f$ are defined as

$$L_f \equiv i (\frac{\partial f}{\partial \phi} \frac{\partial}{\partial \cos \theta} - \frac{\partial f}{\partial \cos \theta} \frac{\partial}{\partial \phi}), \quad (48)$$
The compatibility condition of (47) is

\[
[ \partial_t - L_{X_3 + \lambda X}, \partial_t - L_{X_3 - X_3} ] = 0,
\]

from which, comparing the two sides for the coefficients of the powers \( \lambda^0, \lambda^1 \) of the spectral parameter \( \lambda \), we find (43). From the linear system (47) using the inverse scattering method, one could in principle construct all solutions of the self-duality equations.

The infinite number of conservation laws, are derived as follows [28]: From the Cartesian formulation

\[
\frac{dX_i}{dt} = \frac{1}{2} \varepsilon_{ijk} \{ X_j, X_k \} \tag{50}
\]

contracting with a complex 3-vector \( u_i \), such that

\[
u_i = \varepsilon_{ijk} u_j v_k \tag{51}
\]

where \( u_i u_i = 0 \), and \( v \) another complex vector with \( v_i v_i = -1 \) and \( u_i v_i = 0 \), we find,

\[
\frac{d(u \cdot X)}{dt} = \{ u \cdot X, v \cdot X \} \tag{52}
\]

The latter is a Lax pair type equation which implies

\[
\frac{d}{dt} \int d^2 \xi (u \cdot X)^n = 0 \tag{53}
\]

Application of the same method after determining two complex 7-vectors \( u_i, v_i \) such that \( u_i u_i = 0, v_i v_i = -1 \) and \( u_i v_i = 0 \) leads to the equation

\[
\frac{d(u \cdot X)}{dt} = \{ u \cdot X, v \cdot X \} + \frac{1}{2} \phi_{jklm} u_j v_k \{ X_l, X_m \} \tag{54}
\]

The curvature tensor \( \phi_{jklm} \) is defined as the dual of \( \Psi_{ijk} \) in seven dimensions. When the previous equation is restricted to three dimensions we recover (52).
5. Multistring solutions

We have seen in the previous section that the self-duality equations of the membrane are first order differential equations which imply the second order -Euclidean time- equations of motion. These equations are non-linear and a general solution is rather a hard task. It is possible however to work out systematically membrane solutions which exhibit particular symmetries. Such cases are those with spherical or axial symmetry, string like configurations etc. We will present here few of the classical solutions of refs. We start in this section with the string-like solution of the self-duality equation (18) in 7 dimensions. We assume the form

\[ X_i(\sigma_1, \sigma_2, t) = A_i \sigma_1 + B_i \sigma_2 + P_i t + f_i(\sigma_1, \sigma_2, t) \]  

(55)

with \( i = 1, ..., 7 \), and \( f \) being a periodic function of \( \sigma_1, \sigma_2 \) and \( A, B \) integer vectors. Then we obtain

\[ P_i = \Psi_{ijk} A_j B_k \]  

(56)

\[ \dot{f}_i = \Psi_{ijk} \left( \frac{A_j}{\partial \sigma_2} - \frac{B_j}{\partial \sigma_1} \right) f_k + \frac{1}{2} \Psi_{ijk} \{ f_j, f_k \} \]  

(57)

Since \( f \) is a periodic function with respect to \( \sigma_1, \sigma_2 \), we can expand it in terms of an infinite number of strings, depending on the coordinate \( \sigma_1 \):

\[ f_i(\sigma_1, \sigma_2, t) = \sum_n X_i^n(\sigma_1, t) e^{in\sigma_2}. \]  

(58)

Then, from the self-duality equations (56,57) we find that the winding number of the membrane is related to the center-of-mass momentum, which is transverse to the compactification directions \( A \) and \( B \). Also, the infinite number of strings are coupled through the following equations

\[ \dot{X}_i^n(\sigma_1, t) = \Psi_{ijk} \left( m A_j - B_j \frac{\partial}{\partial \sigma_1} \right) X_k^n + \frac{1}{2} \Psi_{ijk} \sum_{n_1 + n_2 = n} \left( n_2 \frac{\partial X_{n_1}^n}{\partial \sigma_1} X_k^{n_2} - n_1 X_{n_1}^n \frac{\partial X_{n_2}^n}{\partial \sigma_1} \right) \]  

(59)
The string-like solution corresponds to the particular case \( \partial f_i / \partial \sigma_2 = 0 \), where we obtain
\[
X_i^0 = X_i(\sigma_1, t) \rightarrow \dot{X}_i = \Psi_{ijk}B_k \frac{\partial X_j}{\partial \sigma_1}. \tag{60}
\]
This equation is formally solved in vector form by
\[
X(\sigma_1, t) = e^{tM \frac{\partial}{\partial \sigma_1}} X(\sigma_1, 0) \tag{61}
\]
where we defined the \( 7 \times 7 \) matrix \( M_{ij} = \Psi_{ijk}B_k \).

Relation (56) is now simply written \( P_i = M_{ij}A_j \). On the other hand, explicit solutions of (61) are found by expanding \( X_i \) in terms of the eigenvectors of \( M \). In fact, since \( M \) is real and antisymmetric, the real 7-dimensional vector space decomposes into three orthogonal two-dimensional subspaces, each corresponding to a pair of imaginary eigenvalues \( \pm i\lambda \), and a one-dimensional subspace, in the direction of \( B_i \), corresponding to the zero eigenvalue. Since, in addition, \( (M^2)_{ij} = -B^2\delta_{ij} + B_iB_j \) (as can be checked), we see that the imaginary eigenvalue pairs are all \( \pm |B| \). Therefore the problem decomposes into three 3-dimensional problems (one for each subspace) of the kind we solved before. The general solution is then
\[
X_1^{(n)} + iX_2^{(n)} = F_n(\sigma_1 - iBt) , \quad n = 1, 2, 3
\]
\[
X^{(0)} = |B|t
\]
where \( (X_1^{(n)}, X_2^{(n)}) \) are the projections of the membrane coordinates on the \( n \)-th two-dimensional eigenspace and \( X^{(0)} = X_iB_i/|B| \) is the projection on \( B_i \). As an example, if we choose \( B_i \) in the third direction, \( B_i = B\delta_{i3} \), we have
\[
X_1 + iX_2 = F_1(\sigma_1 - iBt), \quad X_5 + iX_4 = F_2(\sigma_1 - iBt)
\]
\[
X_6 + iX_7 = F_3(\sigma_1 - iBt), \quad \text{and} \quad X_3 = Bt.
\]

In 3 dimensions there is a variety of solutions. Restricting to the axi-symmetric class of instanton like solutions, the self-duality equations result to the continuous Toda equation
Indeed, the Ansatz
\[
A_1 = R(\sigma_1, t) \cos \sigma_2, \quad A_2 = R(\sigma_1, t) \sin \sigma_2, \quad A_3 = z(\sigma_1, t)
\] 
(62)
leads to the axially symmetric continuous Toda equation
\[
\frac{d^2 \Psi}{dt^2} + \frac{d^2 e^\Psi}{d\sigma_1^2} = 0,
\]
(63)
where \( R^2 = e^\Psi \). Solutions of this equation have been discussed in the literature in connection with the self-dual 4d Einstein metrics with rotational and axial Killing vectors \[30\]. Here though, we note that \( \sigma_1 \) runs in a compact interval \((0, 2\pi)\) for torus and \((-1, 1)\) for the sphere.

A specific example of a solution with separation of variables of the Toda equation, in the case of spherical topology \((\sigma_1 = \cos \theta, \sigma_2 = \phi)\) is
\[
R(\theta, t) = \kappa \frac{\sin \theta}{\sinh [\kappa(t_0 - t)]},
\]
\[
z(\theta, t) = \kappa \coth [\kappa(t_0 - t)] \cos \theta,
\]
where \( \kappa \) is a constant. A second solution can be obtained if we change the hyperbolic functions with ordinary trigonometric ones. A variation of the method of ref. \[30\] where by inversion of the non-linear system \((17)\) is finally presented in ref\[29\].

6. **Supersymmetries**

In our analysis up to now we dealt only with the bosonic part of the Hamiltonian. In this section we will explore the number of supersymmetries preserved by \((3+1)\)- and \((7+1)\)-dimensional solutions. We will see that 3-d solutions preserve as many as eight out of the sixteen supersymmetries while the 7-d self-duality equations preserve only one supersymmetry \[31\].
There is a connection of the octonionic and Clifford algebra in $d = 8$ dimensions which may be used to study the octonionic self-duality equations under supersymmetry transformations. The relation with the octonions has been noticed in studying compactifications of 11-d supergravity on $S^7$ as well as in the case of $N = 8$ gauged supergravities\cite{33, 34}. The embedding of YM-instantons in 10-d has been constructed in\cite{35}.

The supersymmetry transformation is defined \cite{2}

$$\delta \theta = \frac{1}{2} \left( \Gamma_+(\Gamma_I \dot{X}^I + \Gamma_-) + \frac{1}{2} \Gamma_+ \Gamma^{IJ} \{X_I, X_J\} \right) \begin{pmatrix} \nu \epsilon_A \\ \epsilon_B \end{pmatrix}$$

In terms of the $16 \times 16$ $\gamma$-matrices, the above is written

$$\delta \theta = \begin{pmatrix} 0 & 0 \\ i\sqrt{2} \left( \gamma^I \dot{X}_I + \frac{1}{2} \gamma^{IJ} \{X_I, X_J\} \right) & -2 \cdot 1_{16} \end{pmatrix} \begin{pmatrix} \nu \epsilon_A \\ \epsilon_B \end{pmatrix}$$

which implies that

$$\sqrt{2} \left( \gamma^I \dot{X}_I + \frac{1}{2} \gamma^{IJ} \{X_I, X_J\} \right) \epsilon_A + 2 \cdot 1_{16} \epsilon_B = 0$$

where $\epsilon_A, \epsilon_B$ are 16-dimensional spinors. From the form of eq.\eqref{65}, we observe that if self-duality equations are going to play a role in the preservation of a number of supersymmetries, we should necessarily impose the condition $\epsilon_B = 0$. Thus, at least half of the supersymmetries are broken. Now, the last term in \eqref{65} is zero and eq.\eqref{65} simply becomes

$$\left( \gamma^I \dot{X}_I + \frac{1}{2} \gamma^{IJ} \{X_I, X_J\} \right) \epsilon_A = 0$$

Under the assumption that $\dot{X}_{8,9} = 0$, it can be shown that the above reduces to a simpler $-8 \times 8$ matrix equation. In order to find a convenient explicit form, we first express the $16 \times 16$ matrices in terms of the octonionic structure constants $\Psi_{ijk}$ as follows: let the index $n$ run from 1 to 7; then we define

$$\gamma_8 = \begin{pmatrix} 0 & 1_8 \\ -1_8 & 0 \end{pmatrix}, \quad \gamma_n = \begin{pmatrix} 0 & \beta_n \\ -\beta_n & 0 \end{pmatrix}$$

\cite{67}
where $1_8$ is the $8 \times 8$-identity matrix and $\beta_n$ are seven $8 \times 8 \gamma$-matrices with elements 

$$
(\beta_n)_j^i = \Psi_{ij}, \quad (\beta_n)_8^i = \delta_j^i, \quad (\beta_n)_j^8 = -\delta_j^i
$$

while it can be easily checked that $\beta_1 \cdots \beta_7 = -1_8$ and

$$
\gamma_9 = \begin{pmatrix} 1_8 & 0 \\ 0 & -1_8 \end{pmatrix}
$$

The commutation relations of $\beta_m$ give:

$$
([\beta_m, \beta_n])_j^8 = +2\Psi_{nmj}
$$

$$
([\beta_m, \beta_n])^i_j = -2\Psi_{nmj}
$$

$$
([\beta_m, \beta_n])_j^8 = -2\mathcal{X}^{mn}_{ij}(-4)
$$

where the tensors $\mathcal{X}^{mn}_{ij}(u)$ are defined as follows

$$
\mathcal{X}^{ij}_{kl}(u) = \Delta^{ij}_{kl} + \frac{u}{4} \phi^{ij}_{kl}
$$

where $\Delta^{ij}_{kl} = \frac{1}{2} (\delta^i_k \delta^j_l - \delta^i_l \delta^j_k)$.

Next, we impose the following condition on the components of the 16-spinor $\epsilon_A$

$$
\epsilon_A = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \otimes \varepsilon
$$

where $\otimes$ stands for the direct product and $\varepsilon$ is an eight-component spinor whose components are left unspecified. Clearly, condition (74) reduces further the sixteen supersymmetry charges to eight. Separating the eight components of $\varepsilon = (\varepsilon_7, \varepsilon_1)$ where $\varepsilon_7(1)$ is a seven-(one-) dimensional vector, we find that eq.(69) reduces to the matrix equation

$$
\mathcal{O} \varepsilon = \begin{pmatrix} \Psi_{ij} \dot{X}_m + \frac{i}{2} \mathcal{X}^{mn}_{ij}(-4)\{X_m, X_n\} & \dot{X}_i + \frac{i}{2} \Psi_{imn}\{X_m, X_n\} \\ -\{\dot{X}_i + \frac{i}{2} \Psi_{imn}\{X_m, X_n\}, 0 \end{pmatrix} \begin{pmatrix} \varepsilon_7 \\ \varepsilon_1 \end{pmatrix} = 0
$$

The rather interesting fact here is that the matrix elements $\mathcal{O}_{8j}$ and $\mathcal{O}_{j8}$, $(j = 1, \ldots, 7)$ multiplying the $\varepsilon_1$-component are the self-duality equations (??) in eight dimensions when the
Euclidean time-parameter $t$ is replaced with $it$ (Minkowski). Thus, $\varepsilon_1$-component remains unspecified and there is always one supersymmetry unbroken for any eight-dimensional solution of the self-duality equations.

Let us now turn our discussion to the upper $7 \times 7$ part of the matrix equation (75). In general, the quantity specifying these elements, namely

$$\Psi_{imj} \dot{X}_m + \frac{i}{2} \chi^m_{nnij} (-4) \{ X_m, X_n \}$$

is not automatically zero. However, there is a particular case –which turns out to be the most interesting one– where the above quantity is the self-duality equation itself. In fact, if we consider only three-dimensional solutions of the equations, the ‘curvature’ factor $\phi^{ij}_{kl}$ is automatically zero while the tensor $\chi^{ij}_{kl}$ simply becomes

$$\chi^{ij}_{kl} = \Delta^{ij}_{kl} = \frac{1}{2} (\delta^i_k \delta^j_l - \delta^i_l \delta^j_k) \text{ for } \phi^{ij}_{kl} = 0.\quad (77)$$

In this case, it can be easily seen that (76) reduces to the self-duality equations in three-dimensions. In this latter case, all eight supersymmetries survive.

We conclude this short review with a few general remarks. The octonionic algebra gives a useful formulation of the self-duality equations, which includes in a natural way the three-dimensional system and the corresponding generalized Nahm’s equations for $SDiffS_2$. By introducing in the place of the $SU(2)$ algebra of functions on the sphere, a quadratic algebra of seven functions with $G_2$ symmetry, we succeeded to factorize the time in a simple way, which may facilitate the study of solutions of the self-duality equations. Although the general system of self-duality equations in seven dimensions does not seem to have a Lax pair, at least in a direct way, due to the non-associativity of the octonionic algebra, it may happen that there is a generalization of the zero-curvature condition under which this system is integrable. We also explored the supersymmetric self-duality configurations in three and seven dimensions where we showed that in the three-dimensional integrable self-dual sector where eight supersymmetries survive.
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