Tempered Distribution Version of the Tumarkin Result

Mejdin Saliji\textsuperscript{a}\textsuperscript{*}, Bedrije Bedjeti\textsuperscript{b} and Vesna Manova Erakovik\textsuperscript{c}

\textsuperscript{a} Ukshin Hoti University, Faculty of Education, Mehmet Kasi bb, 20000, Prizren, Republic of Kosovo.
\textsuperscript{b} Univerziteti of Tetovo, Tetovo, Republic of North Macedonia.
\textsuperscript{c} Cyril and Methodius University, Faculty of Mathematics and Natural Sciences, Arhimedova bb, Gazi baba, 1000, Skopje, Republic of North Macedonia.

Authors’ contributions

This work was carried out in collaboration among all authors. All authors read and approved the final manuscript.

Article Information

DOI: 10.9734/JAMCS/2022/v37i130428

Abstract

We transfer a classical result of Tumarkin, about approximation of $L^p(d\sigma, R)$ functions with certain rational functions, in distributional setting. This is achieved by embedding the functions from classical spaces into the space of tempered distributions.

Keywords: Boundary values of distributions; distributions; Tumarkin.

1 Introduction

This section is devoted to known notions and results in the literature that we use in the article. Let $z_j, j \in N$, be a sequence of complex numbers from $\Pi^+ = \{ z \in \mathbb{C}| \text{Im } z > 0 \}$. In addition, let the following condition

$$\sum_{j=1}^{\infty} \frac{y_j}{1+|z_j|} < \infty, \text{where } y_j = \text{Im } \{z_j\},$$

is satisfied. The holomorphic function on $\Pi^+$ defined with

\textsuperscript{*}Corresponding author: Email: mejdins@googlemail.com, mejdins@gmail.com;
where \( w \in \mathbb{H}^+ \), for arbitrary \( w \in \mathbb{H}^+ \), is called Blaschke product on the upper half space with zeroes in \( z_j, j \in \mathbb{N} \).

For the definition of the desired rational functions, again one considers finite or infinite sequences of complex numbers with imaginary part not equal to 1, moreover such that some of the elements of the sequence may be the same and some of them may be \( \infty \) (in the last case we impose its’ imaginary part to be 0). For that kind of sequence, denoted again with \((z_j)_{j \in \mathbb{N}}\), consider the rational functions

\[
R(z) = \frac{c_0 z^{p_0} + c_1 z^{p_1} + \cdots + c_p}{(z-z_1)(z-z_2)\cdots(z-z_p)}, z \in \mathbb{H}^+, \tag{3}
\]

with poles some of the complex numbers of \((z_j)_{j \in \mathbb{N}}\), and \( c_j \in \mathbb{C}, j \in \mathbb{N} \), is another sequence. Additionally, in order equation (3) to make sense in the case when some of the elements of the sequence equal \( \infty \), one use the convention: if \( z_k = \infty \), then one sets \( z - z_k = 1 \) in (3).

Denote by \( z_j, j \in \mathbb{N} \), all of the elements from \((z_j)_{j \in \mathbb{N}}\) such that \( \text{Im} \, z_j > 0 \), and respectively \( z_j^*, j \in \mathbb{N} \), the elements from \((z_j)_{j \in \mathbb{N}}\) such that \( \text{Im} \, z_j < 0 \). One associate the following sums for the previously defined couple of sequences

\[
S_k = \sum_{j=1}^{k} \frac{\text{Im} \, z_j}{1 + |z_j|^2}, \quad S_k^* = \sum_{j=1}^{k} \frac{-\text{Im} \, z_j^*}{1 + |z_j^*|^2}.
\]

On these sequences we impose the conditions

\[
\lim_{k \to \infty} S_k < \infty, \quad \lim_{k \to \infty} S_k^* = \infty. \tag{4}
\]

Denote by \( B_k \) the Blaschke product with zeros \( z_1, z_2, \ldots, z_k \) from the sequence numbers \((z_j)_{j \in \mathbb{N}}\) for \( k \in \mathbb{N} \). If the condition (4) is satisfied, then the function \( \lim_{k \to \infty} \log |B_k(z)| \) is a subharmonic function on the upper half space and not equal to \( \infty \). If \( \mu(z) = \lim_{k \to \infty} \log |B_k(z)| \) and \( u(z) \) is harmonic majorant of \( \mu(z) \) on upper half space one consider another function \( \phi(z) = e^{u(z)+v(z)} \), where \( v(z) \) stands for the harmonic conjugate of \( u(z) \).

2 Known Results

The previous section is actually preparation for the classical results that follow. We state them without proof. We use the notation introduced in previous section.

**Theorem 1.** [4] We assume that condition (4) is satisfied. For an arbitrary continuous function \( F \) on \( \mathbb{R} \) there exist a sequence \((R_j)_{j \in \mathbb{N}}\) of rational functions given in (3) which converges to \( F \), uniformly on \( \mathbb{R} \) if and only if \( F \) coincides on \( \mathbb{R} \) with the boundary value of the holomorphic function \( \tilde{F} \) on \( \mathbb{H}^+ \) given with

\[
\tilde{F}(z) = \frac{\psi(z)}{B(z)\psi(z)}, z \in \mathbb{H}^+, \tag{5}
\]

\( \psi \) is any bounded analytic function on \( \mathbb{H}^+ \).
Let $\sigma$ be a nondecreasing function of bounded variation on $\mathbb{R}$. The space $L^p(d\sigma, \mathbb{R}), p > 0$ denotes the space of set of all complex valued functions $F$, such that $\int_{\mathbb{R}} |F(x)|^p d\sigma(x) < \infty$. Here $\sigma$ stands for a nondecreasing, function of bounded variation on $\mathbb{R}$ (note that the integral is Lebesgue–Stieltjes integral).

In order to formulate the next result, one needs to impose another condition concerning the growth of $\sigma$. Indeed one needs

$$-\infty < \int_{\mathbb{R}} \frac{\log^{\sigma^8(x)}}{1+x^2}. \tag{6}$$

Finally we are able to formulate the main result which we transfer in tempered distribution setting. We state the classical result again without proof. The notation is the same as in the previous section [6].

**Theorem 2.** We assume that conditions (4) and (6) are fulfilled and let $F \in L^p(d\sigma, \mathbb{R}), p > 0$. Then there exist a sequence $(R_j)_{j \in \mathbb{N}}$ of rational functions given in the form (3) such that

$$\lim_{j \to \infty} \int_{\mathbb{R}} |F(x) - R_j(x)|^p d\sigma(x) = 0$$

if and only if the function $F$ coincide with the boundary value of a holomorphic function $\bar{F}$ on $\Pi^+$, almost everywhere on $\mathbb{R}$, where $\bar{F}$ is given in form (5). The functions $B$ and $\varphi$ are as in Theorem 1, $\psi$ is analytic function on the upper half space and in $\Pi^+$.

**Distributions.** The space $S = S(\mathbb{R}^n)$ is the space of infinitely differentiable complex valued function $\varphi$ on $\mathbb{R}^n$ which satisfy the conditions

$$\sup_{t \in \mathbb{R}^n} \left| t^{\beta} D^n \varphi(t) \right| < \infty$$

for every $\alpha = (\alpha_1, ..., \alpha_n), \beta = (\beta_1, ..., \beta_n), \alpha_i, \beta_i \in N, i = 1, 2, ..., n$. The space $S = S(\mathbb{R}^n)$ is a Frechet space which implies that the convergence can be considered in the following way: $\varphi_{\lambda} \to \varphi_{\lambda_0}$, where $\varphi_{\lambda} \in S$, in $S$ as $\lambda \to \lambda_0$ if and only if

$$\lim_{\lambda \to \lambda_0} \sup_{t \in \mathbb{R}^n} \left| t^{\beta} D^n (\varphi_{\lambda}(t) - \varphi_{\lambda_0}(t)) \right| = 0$$

for every $\alpha = (\alpha_1, ..., \alpha_n), \beta = (\beta_1, ..., \beta_n), \alpha_i, \beta_i \in N, i = 1, 2, ..., n$.

The space $S'(\mathbb{R}^n)$ is the space of all continuous, lineal functionals on $S(\mathbb{R}^n)$. The space $S'(\mathbb{R}^n)$ is called the space of tempered distributions. Unless stated differently, it is imposed that the space $S'(\mathbb{R}^n)$ is endowed with the strong topology. Also, we use the following standard conventions about distributions: $(T, \varphi) = T(\varphi)$ for the value of the functional $T$ acting on the function $\varphi$, and standard embedding of locally integrable functions: for $f(x) \in L^1_{\text{loc}}(\mathbb{R}^n)$ one can define the linear and continuous functional $T_f$ on $S(\mathbb{R}^n)$ with

$$T_f(\varphi) = \int_{\mathbb{R}^n} f(t) \varphi(t) dt, \varphi \in S.$$

By the definition, $T_f \in S'(\mathbb{R}^n)$ and that distribution is called tempered (regular) associated to the function $f$.

Now we state the main result of the article.

**Theorem 3.** Let $(z_j)_{j \in \mathbb{N}}$ is a sequence of complex numbers which satisfy (4) and $\bar{F}$ be of the form (5) from Theorem 2. Denote with $T_{\bar{F}}, \bar{F} \in L^p(\mathbb{R})$, the tempered distribution in associated with the boundary value $F(x)$ of $\bar{F}(z)$ on $\Pi^+$.

Then there exists a sequence, $(R_j)_{j \in \mathbb{N}}$ of rational functions of the form (3) and, respectively, a sequence $(T_{R_j})_{j \in \mathbb{N}}, T_{R_j} \in S$ generated by $R_j$, satisfying
Proof.

(i) The general idea is to prove the convergence in $S'$ in weak sense and to use Banach Steinhaus theorem to obtain the strong convergence (note that the space $S$ is Montel space) \cite{5}.

We start with applying Theorem 2. For the function $F$ one obtains a sequence $(R_j)_{j \in \mathbb{N}}$ of rational functions of the form (3) for which the following holds

$$\lim_{j \to \infty} \int_R |F(x) - R_j(x)|^p \, dx = 0.$$ 

Triangle inequality implies

$$\|R_j\|_{L^p(d\sigma(x))} = \left( \int_R |R_j(x)|^p d\sigma(x) \right)^{\frac{1}{p}} = \left( \int_R |R_j(x) - F(x) + F(x)|^p d\sigma(x) \right)^{\frac{1}{p}}$$

$$\leq \left( \int_R |R_j(x) - F(x)|^p d\sigma(x) \right)^{\frac{1}{p}} + \left( \int_R |F(x)|^p d\sigma(x) \right)^{\frac{1}{p}} < \mathcal{C},$$

hence $R_j(x) \in L^p(d\sigma,R)$.

Now choose arbitrary $\phi \in S$ and fix it. We denote with $q$ the Hölder conjugate of $p$, i.e.

$$\frac{1}{p} + \frac{1}{q} = 1.$$ 

We can estimate as follows

$$\left| \int_R f(x) \phi(x) \, dx \right| \leq \int_R |f(x) \phi(x)| \, dx \leq \left( \int_R |f(x)|^p \, dx \right)^{\frac{1}{p}} \left( \int_R |\phi(x)|^q \, dx \right)^{\frac{1}{q}},$$

for arbitrary function $f \in L^p, \phi \in S$.

Using the previous estimate one obtains

$$\left| (T_{R_j}, \phi) - (T_F, \phi) \right| = \left| \int_{-\infty}^{\infty} R_j(x) \phi(x) \, dx - \int_{-\infty}^{\infty} F(x) \phi(x) \, dx \right|$$

$$= \left| \int_{-\infty}^{\infty} [R_j(x) - F(x)] \phi(x) \, dx \right| \leq \int_{-\infty}^{\infty} |R_j(x) - F(x)| |\phi(x)| \, dx$$

$$\leq \left( \int_R |R_j(x) - F(x)|^p \, dx \right)^{\frac{1}{p}} \left( \int_R |\phi(x)|^q \, dx \right)^{\frac{1}{q}}$$

$$\leq M \left( \int_R |R_j(x) - F(x)|^p \, dx \right)^{\frac{1}{p}} \to 0,$$

when $j \to \infty$.

In the previous calculations we used $M = \sup \{ |\phi(x)| | x \in R \}$ which is obviously finite since the inclusion $S \subset L^q$ is continuous and dense for arbitrary $1 \leq q < \infty$. The discussion on the start of the proof implies the claim or $T_{R_j} \to T_F, j \to \infty$ in $S'$ in the strong topology.
(ii) Let $\varphi \in S$ be arbitrary and fixed. The Minkowski inequality implies

$$\left( \int_{\mathbb{R}} |R_1(x)|^p \varphi(x) \, dx \right)^{\frac{1}{p}} \leq M_1^p \left( \int_{\mathbb{R}} |R_1(x) - F(x)|^p \, dx \right)^{\frac{1}{p}}$$

$$\leq M_1^p \left( \int_{\mathbb{R}} |R_1(x) - F(x)|^p \, dx \right)^{\frac{1}{p}} + M_2^p \left( \int_{\mathbb{R}} |F(x)|^p \, dx \right)^{\frac{1}{p}} = I_1 + I_2.$$  

The integral $I_1$ tends to 0 when $j \to \infty$ which implies that $\int_{\mathbb{R}} |R_j(x)|^p \varphi(x) \, dx \leq M_1^p \|F\|_p + C$. for arbitrary $j \in \mathbb{N}$. The latter implies (ii).

### 3 Conclusion

We were able to consider (tempered) distribution variant of Tumarkin result concerning approximation with rational functions. We embed $L^p$ functions continuously into $S'$ in a natural way and consider analog result, but now for their representation in $S'$.

### Competing Interests

Authors have declared that no competing interests exist.

### References

[1] Duren PL, Theory of $H^p$ Spaces, Acad. Press, New York; 1970.

[2] Bremermann H, Distribution, Complex Variables and Fourier Transforms, Addison –Wesley Publishing Co., Inc., Reading, Massachusetts-London; 1965.

[3] Beltrami EJ, Wohlers MR, Distributions and the boundary values of analytic functions, Academic Press, New York; 1966.

[4] Carmichael R, Mitrovic D, Distributions and analytic functions, Pitman Research Notes in Mathematics Series, 206, Longman Scientific & Technical, Harlow; John Wiley & Sons, Inc., New York; 1989.

[5] Treves F, Topological Vector Spaces, Distributions and Kernels, Academic Press New York-London; 1967.

[6] Tumarkin GT, Approximation with respect to various metrics of functions defined on the circumference by sequences of rational fractions with fixed poles, Akad. Nauk SSSR Ser. Mat.1966,30(4):721–766.

[7] Jantcher L, Distributionen, Walter de Gruyter, Berlin, New York; 1971.

[8] Manova E V, Bounded subsets of distributions in $D'$ generated with boundary values of functions of the space $H^p$, $1 \leq p < \infty$, Godisben zbornik na Institutot za matematika, Annuaire, ISSN 0351-7241. 2001; 31-40.

[9] Manova EV, Pandevski N, Nastovski L, Distribution analogue of the Tumarkin result, Bulletin T. CXXIII de l’Akademie serbedes sciences et des arts, Classe des Sciences mathematiqes et naturelles Sciences mathematiques. 2006;31.

[10] Reckoski N, One proof for the analytic representation of distributions, Matematicki Bilten 28 (LIV), Skopje. 2004;19-30.
[11] Reckovski N, Manova EV, Bedjeti B, Iseni E, For some boundary value problems in distributions, Journal of Advances in Mathematics. 2019;16:8331-8339.

[12] Rudin W. Real and complex analysis. McGraw-Hill Inc., New York; 1966.

[13] Rudin W. Functional analysis. North-Holland publishing company, Amsterdam; 1974.

© 2022 Saliji et al.; This is an Open Access article distributed under the terms of the Creative Commons Attribution License (http://creativecommons.org/licenses/by/4.0), which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

Peer-review history:
The peer review history for this paper can be accessed here (Please copy paste the total link in your browser address bar)
https://www.sdiarticle5.com/review-history/84338