Computing twisted KLV polynomials

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This is an attempt to convert [5] to the atlas setting, and write down explicit recursion formulas for the Kazhdan-Lusztig-Vogan polynomials which arise.

1 The Setup

The starting point is: a group $G$, a Cartan involution $\theta$, and another involution $\sigma$ of finite order, commuting with $\theta$.

It is natural to consider the coset $\sigma K = \{\sigma \circ \text{int}(k) \mid k \in K\} \subset \text{Aut}(G)$. Every element of this coset commutes with $\theta$.

We're mainly interested when $\sigma$ is an involution, especially the case $\sigma = \theta$.

Now fix a pinning $P = (H, B, \{X_\alpha\})$, and write $\delta, \epsilon \in \text{Aut}(G)$ for the images via the embedding $\text{Out}(G) \hookrightarrow \text{Aut}(G)$ (the image consists of $P$-distinguished automorphisms). Then $\delta, \epsilon \in \text{Aut}(G)$ commute.

We now introduce the usual atlas structure. See [2] for details. Let $G = G \rtimes \langle \delta \rangle$ be the usual extended group; $\sigma$ acts on it, trivially on $\delta$.

Recall $\mathcal{X} = \{\xi \in \text{Norm}_{G\delta}(H) \mid \xi^2 \in Z(G)\}/H$, and $\tilde{\mathcal{X}}$ is the numerator. We'll write $x$ for elements of $\mathcal{X}$, $\xi$ for elements of $\tilde{\mathcal{X}}$, and $p : \tilde{\mathcal{X}} \to \mathcal{X}$. It is important to distinguish between elements of $\mathcal{X}$ and $\tilde{\mathcal{X}}$.

For $\xi \in \tilde{\mathcal{X}}$, let $r_\xi = \text{int}(\xi) \in \text{Aut}(G)$, $K_\xi = G^{r_\xi}$. The restriction of $r_\xi$ to $H$ only depends on $p(\xi) \in \mathcal{X}$, and is denoted $r_x$. It is important to remember $r_x$ is only an involution of $H$, not of $G$, and $K_x$ is not well defined.

It is immediate that $\sigma(\mathcal{X}) = \mathcal{X}$.

After conjugating we can assume $\theta = \text{int}(\xi_0)$ for some $\xi_0 \in \tilde{\mathcal{X}}$. Let $x_0 = p(\xi_0)$, and define $\mathcal{X}_0 = \{x \in \mathcal{X} \mid x \text{ is } G\text{-conjugate to } x_0\}$ (G-conjugacy
of elements of $X$ is well defined). Let $K = G^\theta = K_{\xi_0}$. Then there is a canonical bijection $X_\theta \leftrightarrow K\backslash G/B$.

Since $\{\sigma, \theta\} = 1$, $\sigma$ acts on $K\backslash G/B$. Write $x \to \sigma^\dagger(x)$ for the automorphism of $X_\theta$, corresponding to the action of $\sigma$ on $K\backslash G/B$. We need to compute $\sigma^\dagger$.

The condition $\{\sigma, \theta\} = 1$ holds if and only if $\sigma(\xi_0) \in \xi_0 Z$. It is convenient to define $z_0 = \sigma(\xi_0^{-1})\xi_0 \in Z^{-\sigma}$, so

$$\sigma(\xi_0) = \xi_0 z_0^{-1}.$$ (The choice of inverse on $z_0$ is so that it goes away later.) It makes sense to write:

(1.1) \hspace{1cm} \sigma(x_0) = x_0 z_0^{-1}.

(both sides being defined up to conjugacy by $H$).

**Proposition 1.2** After replacing $\sigma$ by another element of $\sigma K$, we may assume $\sigma$ normalizes $H$. Define $v \in \text{Norm}_G(H)$ by $\sigma(B) = vBv^{-1}$. Then

(1.3)(a) \hspace{1cm} \sigma^\dagger(x) = v^{-1} \sigma(x) vz_0 \hspace{1cm} (x \in X_\theta).

If sigma preserves the base orbit, i.e. $\sigma(K \cdot B) = K \cdot B$, then after replacing $\sigma$ with another element of $\sigma K$ we may assume $\sigma$ normalizes $(B, H)$. In this case

(1.3)(b) \hspace{1cm} \sigma^\dagger(x) = \sigma(x) z_0.

The coset $vH$ of $v$ in $W = \text{Norm}_G(H)/H$ is well defined, and it makes no difference if we view $v$ as an element of $\text{Norm}_G(H)$ or $W$.

**Remark 1.4** After replacing $\sigma$ with another element of $\sigma K$ we may assume $\sigma(H) = H$, and $\sigma(B \cap K) = B \cap K$. Having done this, we conclude $v$ normalizes $B \cap K$. The normalizer in $W$ of $B \cap K$, equivalently $\rho_K$, is very small; in particular it is a product of $A_1$ factors.

Assume $\sigma(K \cdot B) = K \cdot B$, i.e. $v = 1$. Then we are in the following setting. We have an involution $\theta$ and an automorphism $\sigma$, preserving $(B, H)$, and commuting with $\theta$. The induced automorphism of $(W, S)$ also written $\sigma$, has finite order ($S$ is the set of simple reflections). Finally the corresponding automorphism $\sigma^\dagger$ of $X_\theta$ is $\sigma^\dagger(x) = \sigma(x) z_0$.

The main case of interest is:
Corollary 1.5  Suppose $\sigma = \theta$. After replacing $\theta$ with a $G$-conjugate, we have $\sigma^\dagger(x) = \delta(x)$, and $\theta, \delta$ commute.

**Proof.** After replacing $\theta$ with a $G$-conjugate we may assume $\xi_0 \in H \delta$. Then $\sigma(B) = \theta(B) = B$, also $z_0 = 1$, so $\sigma^\dagger(x) = \sigma(x) = \theta(x)$. But $\theta = \text{int}(\xi_0)$ and $\delta$ differ by an element of $H$, so $\theta(x) = \delta(x)$ (since $X$ has conjugation by $H$ built in).

**Proof of the Proposition.**

We recall a few details of the bijection $X_\theta \leftrightarrow K \backslash G/B$ [2, Sections 8 and 9].

Let $\hat{P}^\sigma = \{(x, B') \mid x \in X_\theta, B' \in B\}/G$. There are bijections:

\[
X_\theta \leftrightarrow \hat{P}^\sigma \leftrightarrow K \backslash B
\]

(1.6)

\[
x \rightarrow (x, B) \\
(x_0, B') \leftrightarrow K \cdot B'
\]

Since $H$ is a fundamental Cartan subgroup with respect to $\theta$, and all such Cartan subgroups are $K$-conjugate, we can modify $\sigma$ by $k$ so that $\sigma$ normalizes $H$.

Here is the computation.

\[
X_\theta \ni x \rightarrow (x, B) \in \hat{P}^\sigma
\]

\[
= (gx_0g^{-1}, B) \quad (x = gx_0g^{-1})
\]

\[
= (x_0, g^{-1}Bg)
\]

\[
\rightarrow g^{-1}Bg \in K \backslash B
\]

\[
\rightarrow \sigma(g^{-1}Bg) \text{ by the action of } \sigma \text{ on } K \backslash B
\]

\[
= \sigma(g^{-1})vBu^{-1}\sigma(g) \text{ where } \sigma(B) = vBu^{-1}
\]

\[
\rightarrow (x_0, \sigma(g^{-1})vBu^{-1}\sigma(g)) \in \hat{P}^\sigma
\]

\[
= (v^{-1}\sigma(g)x_0\sigma(g^{-1})v, B)
\]

\[
\rightarrow (v^{-1}\sigma(g)x_0\sigma(g^{-1})v) \in X_\theta
\]

\[
= v^{-1}\sigma(g\sigma(x_0)g^{-1})v,
\]

\[
= v^{-1}\sigma(gx_0z_0^{-1}g^{-1})v \quad (\text{by (1.1)})
\]

\[
= v^{-1}\sigma(gx_0g^{-1})\sigma(z_0^{-1})v
\]

\[
= v^{-1}\sigma(x)\sigma(z_0^{-1})v \quad (gx_0g^{-1} = x)
\]

\[
= v^{-1}\sigma(x)\nu z_0 \quad (\sigma(z_0^{-1}) = z_0)
\]

□
2 The group $W^\sigma$ and the twisted Hecke algebra $H$

We continue in the setting of Section 1, and we now assume $\sigma$ is an involution. Let $K = G^\theta$.

For simplicity let’s assume $\sigma(K \cdot B) = K \cdot B$, so $v = 1$ (see Proposition 1.2). Then, after replacing $\sigma$ with an element of $\sigma K$, we may assume $\sigma$ commutes with $\theta$, and satisfies $\sigma(B, H) = (B, H)$. Although $\sigma$ may not have finite order, $\sigma^2 = \text{int}(h)$ for some $h \in H$, so $\sigma$ induces an involution, also denoted $\sigma$, of $(W, S)$. Let $\overline{S}$ be the set of orbits of the action of $\sigma$ on $S$.

We are primarily interested in the case $\sigma = \theta$. In this case, after conjugating by $G$ we may assume the induced automorphism of $(W, S)$ is $\delta$ (Corollary 1.5), and $\{ \theta, \delta \} = 1$.

If $\kappa \in \overline{S}$ let $W(\kappa)$ be the subgroup of $G$ generated by $\kappa$. Write $\kappa = \{ s_{\alpha} \}$ or $\{ s_{\alpha}, s_{\beta} \}$, with $\alpha, \beta$ simple. In each case there is a unique long element $w_{\kappa} \in W(\kappa)$. Define $\ell(\kappa) = \ell(w_{\kappa})$.

\[
W(\kappa) = \begin{cases} 
\ell(\kappa) = 1 & S_2 \\
\ell(\kappa) = 2 & S_2 \times S_2 \\
\ell(\kappa) = 3 & S_3 
\end{cases}
\quad \sigma(s_{\alpha}) = s_{\alpha} \quad \sigma(s_{\beta}, \langle \alpha, \beta^\vee \rangle) = 0 \quad \sigma(s_{\beta}, \langle \alpha, \beta^\vee \rangle) = -1
\]

Lusztig and Vogan define a Hecke algebra $H$ over $\mathbb{Z}[u, u^{-1}]$ ($u$ is an indeterminate). See the end of [5, Section 3.1]. It has generators $T_w$ ($w \in W^\sigma$) and relations

\[
T_w T_{w'} = T_{ww'} \quad w, w' \in W^\sigma, \quad \ell(ww') = \ell(w) + \ell(w')
\]

\[
(T_{w_{\kappa}} + 1)(T_{w_{\kappa}} - u^{\ell(w_{\kappa})}) = 0 \quad (\kappa \in \overline{S})
\]

The quotient of the root system by $\sigma$ is itself a root system (nonreduced if length 3 occurs), with simple roots parametrized by $\overline{S}$, and $W^\sigma$ is the Weyl group of this root system. In particular $W^\sigma$ is generated by $\{ w_{\kappa} \mid \kappa \in \overline{S} \}$, and $H$ is generated by $\{ T_{w_{\kappa}} \mid \kappa \in \overline{S} \}$. So in fact $H$ has generators and relations

\[
T_{w_{\kappa}} T_{w'} = T_{w_{\kappa} w'} \quad \kappa \in \overline{S}, \quad \ell(w_{\kappa} w') = \ell(w_{\kappa}) + \ell(w')
\]

\[
(T_{w_{\kappa}} + 1)(T_{w_{\kappa}} - u^{\ell(w_{\kappa})}) = 0 \quad (\kappa \in \overline{S})
\]
This makes $H$ a quasisplit Hecke algebra \cite[§4.7]{5}.

Let $D = \mathcal{Z}[x]$ be the subset of $\mathcal{Z}$ having to do with $x$. That is $\mathcal{Z}[x] \subset \mathcal{X}[x] \times \mathcal{X}^\vee$ ($\mathcal{X}^\vee$ is the dual KGB space), where $\mathcal{Z}[x] \simeq K_\xi \backslash G/B$, and $D$ is parametrized by the $K_\xi$-invariant local systems on $K_\xi \backslash G/B$. Then that $\sigma$ acts on $D$, and let $D^\sigma$ be the fixed points.

Lusztig and Vogan define a $H$ module $M$, with basis $\{a_\gamma \mid \gamma \in D^\sigma\}$. We are going to write down formulas for the action of $H$ on $M$.

### 3 Extended Cartans and Parameters

If $\gamma \in D^\sigma$ there is an isomorphism between the representations parametrized by $\gamma$ and $\sigma(\gamma)$. It is possible to normalize this isomorphism to have “square 1”, i.e. in such a way that there are two choices, $\pm \alpha_\gamma$. This leads to extended parameters: for each $\gamma \in D^\sigma$ there are two extended parameters corresponding to the two choices of $\alpha_\gamma$. Write $\hat{\gamma}$ for an extended parameter corresponding to $\gamma$.

Strictly speaking, the module $M$ is spanned by vectors $a_\hat{\gamma}$ as $\hat{\gamma}$ runs over extended parameters, and the Hecke algebra action is naturally defined in these terms. If $\hat{\gamma}^\pm$ are the two choices of extension, in the module $M$ we have $a_{\hat{\gamma}^-} = -a_{\hat{\gamma}^+}$, and the dimension of $M$ is $|D^\sigma|$.

**Desideratum 3.1** For each parameter $\gamma \in D^\sigma$, it is possible to choose one extended parameter, denoted $\hat{\gamma}_+$ so that the formulas of \cite{5} hold with $\hat{\gamma}_+$ and $a_{\hat{\gamma}_+}$ everywhere.

### 3.1 Extended Cartans

We probably don’t need this subsection and the next one. They are vestiges of a version in which we worked in terms of extended parameters. But it might be helpful to include a few basic facts.

In this section and the next we assume $\sigma = \delta$. Probably this isn’t serious, but in any event in the rest of the paper we only assume $\sigma$ is an involution.

We are interested in KGB elements $x \in \mathcal{X}$ which are fixed by $\delta$. A key point is that if $\delta(x) = x$, and $\xi \in p^{-1}(x) \in \hat{\mathcal{X}}$, then $\delta(\xi) = h\xi h^{-1}$ for some $h \in H$. We cannot assume we can choose $\xi$ so that $\delta(\xi) = \xi$.

**Lemma 3.1.1** Suppose $\xi \in \hat{\mathcal{X}}$, and let $x = p(\xi) \in \mathcal{X}$. The following conditions are equivalent.
1. $\theta_\xi$ normalizes $\delta H$, and $(\delta H)^{\theta_\xi}$ meets both components of $\delta H$;

2. $\delta(x) = x$.

**Proof.** Suppose (1) holds. The second part of (1) says that $\xi(t\delta)\xi^{-1} = t\delta$ for some $t \in H$, i.e.

\[(3.1.2)\text{(a)} \quad \theta_x(t)(\xi\delta\xi^{-1}) = t\delta.\]

The first part of (1) says $\xi\delta\xi^{-1} = h\delta$ for some $h \in H$, i.e.

\[(3.1.2)\text{(b)} \quad \delta(\xi) = h^{-1}\xi.\]

Plug in $\xi\delta\xi^{-1} = h\delta$ to (a): $\theta_x(t)h\delta = t\delta$, so $h^{-1} = t^{-1}\theta_x(t)$. Then by (b):

\[(3.1.2)\text{(c)} \quad \delta(\xi) = h^{-1}\xi = t^{-1}\theta_x(t)\xi = t^{-1}\xi t.\]

Projecting to $\mathcal{X}$ this says $\delta(x) = x$.

Conversely, suppose $\delta(x) = x$. By definition this means $\delta(\xi) = h^{-1}\xi h$ for some $h \in H$. Note that

\[(3.1.3) \quad \delta(\xi) = h^{-1}\xi h \iff \xi(h\delta)\xi^{-1} = h\delta.\]

In other words the second condition of (1) holds. Also the first condition holds: the right hand side gives $\theta_x(h)\xi\delta\xi^{-1} = h\delta$, i.e. $\xi\delta\xi^{-1} \in H\delta$. □

From now on we will usually assume $\delta(x) = x$.

**Definition 3.1.4** Suppose $\xi \in \tilde{\mathcal{X}}$. Let $x = p(\xi) \in \mathcal{X}$, and assume $\delta(x) = x$. The extended Cartan defined by $\xi$ is $1^H \xi = (\delta H)^{\theta_\xi}$. It contains $H^{\theta_\xi}$ as a subgroup of index 2, and meets both components of $\delta H$.

In other words

\[(3.1.5) \quad 1^H \xi = \langle H^{\theta_\xi}, h\delta \rangle\]

where $\xi(h\delta)\xi^{-1} = h\delta$, equivalently $h^{-1}\xi h = \delta(\xi)$.

This is related to [1, Definition 13.5]. Note that $h\delta$ normalizes $\Delta^+$.  

6
3.2 Extended Parameters

We work only at a fixed regular infinitesimal character, so we fix \( \lambda \in \mathfrak{h}^* \), dominant for \( \Delta^+ \).

By a character we mean a pair \((x, \Lambda)\), where \( \Lambda \) is an \((\mathfrak{h}, H^\theta x)\)-module. We’re ignoring the \( \rho \)-cover; this isn’t hard to fix. The differential of \( \Lambda \) (the \( \mathfrak{h} \) part) is \( \lambda \in \mathfrak{h}^* \), so we usually identify \( \Lambda \) with a character of \( H^\theta x \).

**Definition 3.2.1** Given \( \xi \in \tilde{X} \), an extended character is a pair \((\xi, ^1\Lambda)\) where \(^1\Lambda \) is an \((\mathfrak{h},^1H^\xi)\) module. Equivalence of extended characters is by conjugation by \( H \).

Recall given a parameter \((x, y)\) as usual, compatible with \( \lambda \) we obtain a character \((x, \Lambda)\), as above we think of \( \Lambda \) as a character of \( H^\theta x \).

**Definition 3.2.2** An extended parameter, at infinitesimal character \( \lambda \), is a quadruple \((\xi, y, h\delta, z)\) satisfying the following conditions. Set \( x = p(\xi) \in X \), and we assume \( \delta(x) = x \).

1. \((x, y)\) is a parameter at \( \lambda \), defining a character \( \Lambda \) of \( H^\theta x \).
2. \( \Lambda^\delta = \Lambda \) (i.e. \( \Lambda \) is fixed by \( \delta \)),
3. \( h\delta \) is in the extended group \((^\delta H)^\xi\), i.e. \( h\delta \) commutes with \( \xi \),
4. \( z \in \mathbb{C}^* \), \( z^2 = \Lambda(h\delta(h)) \)

Equivalence of extended parameters is generated by conjugation by \( H \), and

\[
(\xi, y, h\delta, z) \simeq (\xi, y, th\delta, \Lambda(t)z) \quad (t \in H^\theta x).
\]

**Remark 3.2.4** Condition (2) implies (but is not equivalent to): \( \delta^t(y) = y \). So we may as well assume this holds as well as \( \delta(x) = x \).

**Proposition 3.2.5** There is a bijection between equivalence classes of extended characters and equivalence classes of extended parameters.

The bijection is

\[
(\xi, y, h\delta, z) \leftrightarrow (\xi, ^1\Lambda)
\]

From left to right, take \(^1\Lambda|_{H^\theta x}\) to be the character defined by \((x = p(\xi), y)\), and \(^1\Lambda(h\delta) = z\). Conversely, given \(^1\Lambda\), choose \( y \) so that \((x, y)\) corresponds
Choose any $h_\delta \in (\delta H)^{\theta_\xi}$, and let $z = 1^A(h_\delta)$. There are a few straightforward checks that this works. One of the main points is that if we choose $h_\delta \in (\delta H)^{\theta_\xi}$ ($i = 1, 2$), then $h_2 = h_1 t$ for $t \in H^{\theta_x}$, which is taken care of by the equivalence.

4 Cayley transforms and cross actions

Cayley transforms and cross actions can naturally be defined in terms of extended parameters. (This was done in an earlier version of these notes.) As discussed at the beginning of Section 3.2, implicit in [5] is the assertion that, for each parameter $\gamma$, there is a choice of extended parameter, which we'll label $\hat{\gamma}^+$, so that the following formulas hold with $\hat{\gamma}^+$ in place of $\gamma$ everywhere except in cases $2i12/2r21$. For these see Section [5].

Some of these new “Cayley transforms” are iterated Cayley transforms, but some involve a combination of cross actions and Cayley transforms.

4.1 Length 1

In length 1, these are essentially the usual definitions, except in the $1i2s$ case, when the Cayley transform is not defined.

Suppose $\ell(\kappa) = 1$, so $\kappa = \{s_\alpha\}$ and $w_\kappa = s_\alpha$, where $\sigma(\alpha) = \alpha$.

In the classical case $\alpha$ has type $C^+, C^-, i, i2, ic, r, r2$ or $rn$. We write these $1C^+, \ldots, 1rn$ to emphasize the length of $\kappa$.

Suppose $\alpha$ is of type $1i2$, so the Cayley transform is double valued: $\gamma_1^\alpha, \gamma_2^\alpha$. Then $\sigma(\alpha) = \alpha$ implies $\sigma$ preserves the set $\{\gamma_1^\alpha, \gamma_2^\alpha\}$. This yields two subcases in the new setting: denote these $1i2f$ (“fixed”) or $1i2s$ (“switched”), depending on whether $\sigma$ acts trivially on this set, or interchanges the two members.

Type $1r1$ is similar; the double-valued Cayley transform is written $\{\gamma_1^\alpha, \gamma_2^\alpha\}$.
| type       | definition                        | Cayley transform                        |
|------------|-----------------------------------|-----------------------------------------|
| 1C+        | $\alpha$ complex, $\theta \alpha > 0$ | $\gamma^\kappa = \gamma^\alpha$         |
| 1C-        | $\alpha$ complex, $\theta \alpha < 0$ |                                          |
| 1i1        | $\alpha$ imaginary, noncompact, type 1 | $\gamma^\kappa = \gamma^\alpha$         |
| 1i2f       | $\alpha$ imaginary, noncompact, type 2 | $\gamma^\kappa = \gamma^\alpha = \{\gamma_1^\kappa, \gamma_2^\kappa\}$ |
| 1i2s       | $\alpha$ imaginary, noncompact, type 2 | $\gamma^\kappa = \gamma^\alpha = \{\gamma_1^\kappa, \gamma_2^\kappa\}$ |
| 1c         | $\alpha$ compact imaginary        |                                          |
| 1r1f       | $\alpha$ real, parity, type 1     | $\gamma^\kappa = \gamma^\alpha = \{\gamma_1^\kappa, \gamma_2^\kappa\}$ |
| 1r1s       | $\alpha$ real, parity, type 1     |                                          |
| 1r2        | $\alpha$ real, parity, type 2     | $\gamma^\kappa = \gamma^\alpha$         |
| 1rn        | $\alpha$ real, non-parity         |                                          |

### 4.2 Length 2

Suppose $\alpha \in S, \beta = \sigma(\alpha) \in S$, and $\langle \alpha, \beta^\vee \rangle = 0$. Let $\kappa = \{s_\alpha, s_\beta\}$, so $w_\kappa = s_\alpha s_\beta \in W^\sigma$. It is easy to see that $\alpha, \beta$ have the same type with respect to $\theta$. Here are the twelve cases as listed in [5, Section 7.5].

In the length 2 and 3 cases we include the terminology from [5] in a separate column.
| type      | LV terminology       | definition                           | Cayley transform |
|-----------|----------------------|--------------------------------------|------------------|
| 2C+       | two-complex ascent   | \( \alpha, \beta \) complex \( \theta \alpha > 0 \) \( \theta \alpha \neq \beta \) |                  |
| 2C−       | two-complex ascent   | \( \alpha, \beta \) complex \( \theta \alpha < 0 \) \( \theta \alpha \neq \beta \) |                  |
| 2Ci*      | two-semiimaginary ascent | \( \alpha, \beta \) complex \( \theta \alpha = \beta \) | \( \gamma^\kappa = s_\alpha \times \gamma = s_\beta \times \gamma \) |
| 2Cr*      | two-semi-real descent | \( \alpha, \beta \) complex \( \theta \alpha = -\beta \) | \( \gamma^\kappa = s_\alpha \times \gamma = s_\beta \times \gamma \) |
| 2i11      | two-imaginary noncpt  | \( \alpha, \beta \) noncpt imaginary, type 1 \( (\gamma^\alpha)^\beta \) single valued | \( \gamma^\kappa = (\gamma^\alpha)^\beta \) |
|           | type I-I ascent      |                                      |                  |
| 2i12†     | two-imaginary noncpt  | \( \alpha, \beta \) noncpt imaginary, type 1 \( (\gamma^\alpha)^\beta \) double valued | \( \gamma^\kappa = \{\gamma_1^\kappa, \gamma_2^\kappa\} = (\gamma^\alpha)^\beta \) |
|           | type I-II ascent      |                                      |                  |
| 2i22      | two-imaginary noncpt  | \( \alpha, \beta \) noncpt imaginary, type 1 \( (\gamma^\alpha)^\beta \) has 4 values | \( \gamma^\kappa = \{\gamma_1^\kappa, \gamma_2^\kappa\} = \{\gamma^\alpha, \beta\}^\sigma \) |
|           | type II-II ascent     |                                      |                  |
| 2r22      | two-real              | \( \alpha, \beta \) real, parity, type 2 \( (\gamma_\alpha)^\beta \) single valued | \( \gamma^\kappa = (\gamma_\alpha)^\beta \) |
|           | type II-II descent    |                                      |                  |
| 2r21      | two-real              | \( \alpha, \beta \) real, parity, type 2 \( (\gamma_\alpha)^\beta \) double valued | \( \gamma^\kappa = \{\gamma_1^\kappa, \gamma_2^\kappa\} = (\gamma_\alpha)^\beta \) |
|           | type II-I descent     |                                      |                  |
| 2r11      | two-real              | \( \alpha, \beta \) real, parity, type 2 \( (\gamma_\alpha)^\beta \) has 4 values | \( \gamma^\kappa = \{\gamma_1^\kappa, \gamma_2^\kappa\} = \{\gamma_\alpha, \beta\}^\sigma \) |
|           | type I-I descent      |                                      |                  |
| 2rn       | two-real nonparity    | \( \alpha, \beta \) real, nonparity |                  |
| 2ic       | two-imaginary compact descent | \( \alpha, \beta \) compact imaginary |                  |

*: defect = 1 (see Definition 9.1.4).
†: See Section 5

### 4.3 Length 3

Suppose \( \alpha \in S, \beta = \sigma(\alpha) \in S \), and \( \langle \alpha, \beta^\gamma \rangle \neq 0 \) (equivalently \( \pm 1 \)). In this case \( w_\kappa = s_\alpha s_\beta s_\alpha = s_\beta s_\alpha s_\beta \in W^\sigma \).

Again it is easy to see that \( \alpha, \beta \) have the same type with respect to \( \theta \).

Here are the cases.

**Length 3**

10
The type of $\kappa$ depends on a parameter $\gamma \in D^\sigma$. We say $\kappa$ is of a given type with respect to $\gamma$.

**Definition 4.3.1** If $\kappa \in \overline{S}$ and $\gamma \in D^\sigma$ write $t_\gamma(\kappa)$ for the type of $\kappa$ with respect to $\gamma$.

For the notion of ascent/descent in these tables see Lemma 9.3.1.

**Definition 4.3.2** The $\tau$-invariant of $\gamma \in D^\sigma$ is

$$\tau(\gamma) = \{ \kappa \in \overline{S} \mid \kappa \text{ is a descent for } \gamma \}.$$  

Here is a list of the $10 + 12 + 8 = 30$ types:

**Table 4.3.3**

| $\ell(\kappa)$ | ascent ($\kappa \not\in \tau(\gamma)$) | descent ($\kappa \in \tau(\gamma)$) |
|-----------------|-------------------------------------|-------------------------------------|
| 1               | 1C+, 1i1, 1i2f, 1i2s, 1rn           | 1C-, 1r1f, 1r1s, 1r2, 1ic          |
| 2               | 2C+, 2Ci, 2i11, 2i12, 2i22, 2rn     | 2C-, 2Cr, 2r11, 2r21, 2r22, 2ic    |
| 3               | 3C+, 3Ci, 3i, 3rn                  | 3C-, 3Cr, 3r, 3ic                  |

*: defect=1 (see Definition 9.1.4).
5 Cases 2i12 and 2r21

Recall we need to address the issue, discussed at the beginning of Section 4, of types 2i12/2r21.

Suppose $\gamma \in D^\sigma$, and $\kappa = \{\alpha, \beta\}$ is of type 2i12 with respect to $\gamma$. Then $s_\alpha \times \gamma = s_\beta \times \gamma$ is also of type 2i12. Label these two parameters $\{\gamma_1, \gamma_2\}$. Then, on the level of non-extended parameters, $(\gamma_1)^\kappa = (\gamma_2)^\kappa$ is double-valued, label these two parameters $\lambda_1, \lambda_2$.

Thus we are given two unordered pairs $\{\gamma_1, \gamma_2\}$ and $\{\lambda_1, \lambda_2\}$; $\kappa$ is of type 2i12 and 2r21 respectively.

We want to define the extensions inductively, starting on the fundamental Cartan. Assume that we have already chosen an extension $\hat{\gamma}_1^+$ of $\gamma_1$.

The Cayley transform of $\hat{\gamma}_1^+$ by $\kappa$ is a well defined pair of extended parameters (see the old version of these notes). Use this to define the + labelling on the extended parameters for $\lambda$: 

\[(5.1)(a) \quad (\hat{\gamma}_1^+)^\kappa = \{\hat{\lambda}_1^+, \hat{\lambda}_2^+\}\]

Now fix an extension $\hat{\gamma}_2^+$ of $\gamma_2$. Then $(\hat{\gamma}_2^+)^\kappa$ is either $\{\hat{\lambda}_1^+, \hat{\lambda}_2^-\}$ or $\{\hat{\lambda}_1^-, \hat{\lambda}_2^+\}$. (This computation was done in the old notes, in $SL(4, \mathbb{R})$). After switching $\lambda_1, \lambda_2$ if necessary, we can assume

\[(5.1)(b) \quad (\hat{\gamma}_2^+)^\kappa = \{\hat{\lambda}_1^+, \hat{\lambda}_2^-\}\]

Alternatively, define $\hat{\gamma}_2^+$ by the requirement: $(\hat{\gamma}_2^+)^\kappa = \{\hat{\lambda}_1^+, \hat{\lambda}_2^-\}$ (then $(\hat{\gamma}_2^-)^\kappa = \{\hat{\lambda}_1^-, \hat{\lambda}_2^+\}$).

Clearly the chosen extensions of $\lambda_1, \lambda_2$ depend on the extensions of $\gamma_1, \gamma_2$, and also the fact that we’ve chosen an order of each pair $(\gamma_1, \gamma_2)$ and $(\lambda_1, \lambda_2)$. For example, suppose we switch $\gamma_1, \gamma_2$, but keep the same extensions of these two parameters. This would induce new definitions of $\hat{\lambda}_1^+, \hat{\lambda}_2^+$: $\hat{\lambda}_2^-$ wouldn’t change, but what we labelled $\hat{\lambda}_1^+$ before would now be labelled $\hat{\lambda}_2^-$. See the table at the end of this section.

Conclusion: some additional information is needed to determine unique preferred extensions for $\lambda_1, \lambda_2$.

What was here before was incorrect, and I don’t know how to fix it at the moment. So I’m leaving this as a conjecture.

**Conjecture 5.2** Assume $\kappa = \{\alpha, \beta\}$, where $\alpha, \beta$ are orthogonal and interchanged by $\sigma$. Suppose $\kappa$ is of type 2i12 or 2r21 for parameters $\gamma$ and
\( \gamma' = s_{\alpha} \times \gamma = s_{\beta} \times \gamma \). There is a canonical way to distinguish \( \gamma, \gamma' \), and so write them as an ordered pair \((\gamma_1, \gamma_2)\).

Assuming this, start with the ordered pair \((\gamma_1, \gamma_2)\), and assume we have chosen \(\hat{\gamma}_1^+\). Then the Cayley transform \((\gamma_1)^\kappa = (\gamma_2)^\kappa\) is an ordered pair \((\lambda_1, \lambda_2)\). Define \(\hat{\lambda}_1^+, \hat{\lambda}_2^+\) by (a): \((\hat{\gamma}_1^+)^\kappa = \{\hat{\lambda}_1^+, \hat{\lambda}_2^+\}\). Furthermore define \(\hat{\gamma}_2^\pm\) by the requirement: \((\hat{\gamma}_2^\pm)^\kappa = \{\lambda_1^+, \lambda_2^\pm\}\) (exactly one of the two extensions of \(\hat{\gamma}_2\) satisfy this).

Clearly the choice of these extensions depends on the fact that \((\gamma_1, \gamma_2)\) and \((\lambda_1, \lambda_2)\) are ordered pairs.

There might be an issue of consistency here: if we’ve already chosen \(\hat{\gamma}_2^+\), it may conflict with the one just made. (Similar issues possibly could arise elsewhere.) Let’s ignore this issue for now, and hope it works. If so, we have chosen a preferred extension of each parameter, and all formulas are in terms of this extension. See the Desideratum 3.1.

### 5.1 A Table

It is possible that the ordering we’ve chosen in the previous section, while natural, isn’t the right one. Hopefully this table will never be needed, but it shows the affect of different choices.

Assume we’ve decided on an ordering of \(\gamma_1, \gamma_2\), and extensions of these two parameters, labelled +. This uniquely determines an ordering of \(\lambda_1, \lambda_2\), and extensions of these. This is the first row of the table.

The subsequent rows show the affect of the choices. For example, suppose we keep the same order of \(\gamma_1, \gamma_2\), but choose the other extension of \(\gamma_2\). Then since

\[
\hat{\gamma}_1^+ \rightarrow \hat{\lambda}_1^+, \hat{\lambda}_2^+ \\
\hat{\gamma}_2^+ \rightarrow \hat{\lambda}_1^+, \hat{\lambda}_2^-
\]

with our new choices we have

\[
\hat{\gamma}_1^+ \rightarrow \hat{\lambda}_1^+, \hat{\lambda}_2^+ \\
\hat{\gamma}_2^- \rightarrow \hat{\lambda}_1^+, \hat{\lambda}_2^+
\]

meaning the sign has changed on the first member of the target pair. So we should change their order:

\[
\hat{\gamma}_1^+ \rightarrow \hat{\lambda}_2^+, \hat{\lambda}_1^+ \\
\hat{\gamma}_2^- \rightarrow \hat{\lambda}_2^+, \hat{\lambda}_1^-
\]
This amounts to switching $\lambda_1, \lambda_2$, giving the second row of the table.

| $\gamma_1^+$ | $\gamma_1^-$ | $\lambda_1^+$ | $\lambda_1^-$ |
|--------------|-------------|--------------|--------------|
| $\gamma_2^+$ | $\gamma_2^-$ | $\lambda_2^+$ | $\lambda_2^-$ |

5.2 The sign $\epsilon(\gamma, \lambda)$

The sign which arises in the $2i12/2r21$ cases appears frequently, so we need some notation for it. We need to refer forward to the definition of $\gamma \rightarrow \lambda$ (Definition 6.3.2).

**Definition 5.2.1** Suppose $\gamma, \lambda \in D^\sigma$, $\kappa \in \overline{S}$, and $\gamma \rightarrow \lambda$.

If $t_\gamma(\kappa) \neq 2r21$ define $\epsilon(\gamma, \lambda) = 1$.

Assume $t_\gamma(\kappa) = 2r21$, so $t_\lambda(\kappa) = 2i12$. By the discussion at the beginning of this section, $\gamma$ and $\lambda$ are members of ordered pairs $(\gamma_1, \gamma_2)$ and $(\lambda_1, \lambda_2)$, respectively. Define:

$$
(5.2.2) \quad \epsilon(\gamma_i, \lambda_j) = \begin{cases} 
-1 & i = j = 2 \\
1 & \text{otherwise}
\end{cases}
$$

6 Formulas for the $H$ action on $M$

Implicit in the following formulas is the fact that we have chosen an extension of each parameter as discussed in Section 3.2. For each $\gamma \in D^\sigma$ we have chosen an extension $\hat{\gamma}^+$; in the following formulas each $a_\gamma$ is really $a_{\hat{\gamma}^+}$. 
6.1 Length 1

Suppose $\sigma(\alpha) = \alpha$, and $\gamma \in \mathcal{D}^\sigma$. Then $T_{w_\alpha}(\gamma)$ is given by the usual formulas, taking the quotients in types $1i2s, 1r1s$ into account. The first column is $t_\gamma(\kappa)$, the type of $\kappa$ with respect to $\gamma$.

1C+: $T_{w_\alpha}(a_\gamma) = a_{w_\alpha \times \gamma}$

1C-: $T_{w_\alpha}(a_\gamma) = (u - 1)a_\gamma + ua_{w_\alpha \times \gamma}$

1i1: $T_{w_\alpha}(a_\gamma) = a_{w_\alpha \times \gamma} + a_\gamma^\alpha$

1i2f: $T_{w_\alpha}(a_\gamma) = a_\gamma + (a_{\gamma_1} + a_{\gamma_2}^\gamma)$

1i2s: $T_{w_\alpha}(a_\gamma) = -a_\gamma$

1ic: $T_{w_\alpha}(a_\gamma) = ua_\gamma$

1r1f: $T_{w_\alpha}(a_\gamma) = (u - 2)a_\gamma + (u - 1)(a_{\gamma_1} + a_{\gamma_2}^\gamma)$

1r1s: $T_{w_\alpha}(a_\gamma) = ua_\gamma$

1r2: $T_{w_\alpha}(a_\gamma) = (u - 1)a_{\gamma - a_{w_\alpha \times \gamma}} + (u - 1)a_\gamma^\alpha$

1rn: $T_{w_\alpha}(a_\gamma) = -a_\gamma$

6.2 Length 2

2C+: $T_{w_\alpha}(a_\gamma) = a_{w_\alpha \times \gamma}$

2C-: $T_{w_\alpha}(a_\gamma) = (u^2 - 1)a_\gamma + u^2a_{w_\alpha \times \gamma}$

2Ci: $T_{w_\alpha}(a_\gamma) = ua_\gamma + (u + 1)a_\gamma^\alpha$

2Cr: $T_{w_\alpha}(a_\gamma) = (u^2 - u - 1)a_\gamma + (u^2 - u)a_\gamma^\alpha$

2i11: $T_{w_\alpha}(a_\gamma) = a_{w_\alpha \times \gamma} + a_\gamma^\alpha$

2i12: $T_{w_\alpha}(a_\gamma) = a_\gamma + \sum_{\lambda, \lambda \leq \gamma} \epsilon(\lambda, \gamma) a_\lambda$

2i22: $T_{w_\alpha}(a_\gamma) = a_\gamma + (a_{\gamma_1}^\alpha + a_{\gamma_2}^\gamma)$
\[ T_{w_n}(a_\gamma) = (u^2 - 1)a_\gamma - a_{w_n \times \gamma} + (u^2 - 1)a_{\gamma n} \]

\[ T_{w_n}(a_\gamma) = (u^2 - 1)a_\gamma + (u^2 - 1) \sum_{\lambda | \gamma \to \lambda} \epsilon(\gamma, \lambda)a_\lambda \]

\[ T_{w_n}(a_\gamma) = (u^2 - 2)a_\gamma + (u^2 - 1)(a_{\gamma 1} + a_{\gamma 2}) \]

\[ T_{w_n}(a_\gamma) = -a_\gamma \]

\[ T_{w_n}(a_\gamma) = u^2a_\gamma \]

**Remark 6.2.1** In the 2i12 case, if \( \lambda \overset{\kappa}{\to} \gamma \) (Definition 6.3.2) recall \( \lambda, \gamma \) occur in ordered pairs \((\lambda_1, \lambda_2)\) and \((\gamma_1, \gamma_2)\) (Section 5). With \( \epsilon(\lambda, \gamma) \) given by Definition 5.2.1 the stated formula in this case is shorthand for:

\[ T_{w_n}(a_{\gamma 1}) = a_{\gamma 1} + (a_{\lambda 1} + a_{\lambda 2}) \]
\[ T_{w_n}(a_{\gamma 2}) = a_{\gamma 2} + (a_{\lambda 1} - a_{\lambda 2}), \]

We could state the other formulas using \( \epsilon(\gamma, \lambda) \) as well, for example if \( t_{\gamma}(\kappa) = 2r22 \). But this doesn’t seem worth it.

### 6.3 Length 3

\[ T_{w_n}(a_\gamma) = w_\kappa \times a_\gamma \]

\[ T_{w_n}(a_\gamma) = (u^3 - 1)a_\gamma + u^3(a_{w_n \times \gamma}) \]

\[ T_{w_n}(a_\gamma) = ua_\gamma + (u + 1)a_{\gamma n} \]

\[ T_{w_n}(a_\gamma) = (u^3 - u - 1)a_\gamma + (u^3 - u)a_{\gamma n} \]

\[ T_{w_n}(a_\gamma) = ua_\gamma + (u + 1)a_{\gamma ^n} \]

\[ T_{w_n}(a_\gamma) = (u^3 - u - 1)a_\gamma + (u^3 - u)a_{\gamma n} \]

\[ T_{w_n}(a_\gamma) = -a_\gamma \]

\[ T_{w_n}(a_\gamma) = u^3a_\gamma \]
Remark 6.3.1 In terms of extended parameters, the $2i_{12}/2r_{21}$ cases are simpler. Suppose $\kappa$ is of type $2i_{12}$ with respect to an ordinary parameter $\gamma$.

Suppose $\hat{\gamma}$ is an extension of $\gamma$. On the level of extended parameters $\hat{\gamma}$ has a well defined Cayley transform $(\hat{\gamma})^\kappa$, which is an unordered pair of extended parameters. Then:

$$T_{w_\kappa}(a_{\hat{\gamma}}) = a_{\hat{\gamma}} + \sum_{\hat{\lambda} \in (\hat{\gamma})^\kappa} a_{\hat{\lambda}}$$

If we write $-\hat{\gamma}$ for the opposite extension then $(-\hat{\gamma})^\kappa$ is the same set of two elements, with the sign changed on one of them. There are no choices involved here.

We have the usual notion of the W-graph associated to this Hecke algebra action.

Definition 6.3.2 Suppose $\kappa \in \overline{S}$, $\gamma, \lambda \in D^\sigma$, and $\kappa \in \tau(\gamma)$. Then we say $\gamma \leadsto \lambda$ if $\kappa \not\in \tau(\lambda)$, and $a_\lambda$ appears in $T_{w_\kappa}(a_\gamma)$.

7 Kazhdan-Lusztig-Vogan algorithm

We use a hybrid notation combining Fokko’s notes Implementation of Kazhdan-Lusztig Algorithm, and [5].

Recall $H$ is an algebra over $\mathbb{Z}[u, u^{-1}]$, with generators parametrized by $\overline{S}$. Also $M$ is a $H$-module, with $\mathbb{Z}[u, u^{-1}]$-basis $\{a_\gamma | \gamma \in D^\sigma\}$.

Write $D$ for the canonical involution of $M$. It satisfies

$$D(um) = u^{-1}D(m).$$

The order on $D^\sigma$ is defined in [5, 5.1], and length $\ell(\gamma)$ is inherited from $D$.

Theorem 7.2 ([5], Theorem 5.2) There is a unique basis $\{C_\delta | \delta \in D^\sigma\}$ of $M$ satisfying the following conditions. There are polynomials $P^\sigma(\gamma, \delta) \in \mathbb{Z}[u]$ such that

$$C_\delta = \sum_{\gamma} P^\sigma(\gamma, \delta) a_\gamma,$$

and:

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1. $D(C_\gamma) = u^{-\ell(\gamma)}C_\gamma$;

2. $P^\sigma(\gamma, \delta) \neq 0$ implies $\gamma \leq \delta$;

3. $P^\sigma(\gamma, \gamma) = 1$

4. $\deg(P^\sigma(\gamma, \delta)) \leq \frac{1}{2}(\ell(\delta) - \ell(\gamma) - 1)$.

Introduce a new variable $v$ satisfying $v^2 = u$, and tensor everything with $\mathbb{Z}[v, v^{-1}]$, so $H$ becomes an algebra over $\mathbb{Z}[v, v^{-1}]$. Define

\[(7.4)(a) \quad \hat{a}_\gamma = v^{-\ell(\gamma)}a_\gamma\]

and

\[(7.4)(b) \quad \hat{C}_\delta = v^{-\ell(\delta)}C_\delta\]

With this notation Theorem 7.2 can be written as

**Theorem 7.5** There is a unique basis $\{\hat{C}_\delta \mid \delta \in D^\sigma\}$ of $M$ satisfying the following conditions. There are polynomials $\hat{P}^\sigma(\gamma, \delta)(v) \in \mathbb{Z}[v, v^{-1}]$ such that

\[(7.6) \quad \hat{C}_\delta = \sum_\gamma \hat{P}^\sigma(\gamma, \delta)(v)\hat{a}_\gamma,\]

and:

1. $D(\hat{C}_\gamma) = \hat{C}_\gamma$;

2. $\hat{P}^\sigma(\gamma, \delta) \neq 0$ implies $\gamma \leq \delta$;

3. $\hat{P}^\sigma(\gamma, \gamma) = 1$;

4. if $\gamma \neq \delta$, then $\hat{P}^\sigma(\gamma, \delta) \in v^{-1}\mathbb{Z}[v^{-1}]$; and

5. $\deg(\hat{P}^\sigma(\gamma, \delta))(v^{-1}) \leq \ell(\gamma) - \ell(\delta)$

It is easy to see that

\[(7.7) \quad \hat{P}^\sigma(\gamma, \delta)(v) = v^{\ell(\gamma) - \ell(\delta)}P^\sigma(\gamma, \delta)(v^2).\]

Fix $\gamma < \delta$, and suppose

\[(7.8)(a) \quad P^\sigma(\gamma, \delta) = c_0 + c_1u + \cdots + c_nu^n\]
with

\[ n = \begin{cases} 
(\ell(\delta) - \ell(\gamma) - 1)/2 & \ell(\delta) - \ell(\gamma) \text{ odd} \\
(\ell(\delta) - \ell(\gamma) - 2)/2 & \ell(\delta) - \ell(\gamma) \text{ even}
\end{cases} \]

Then

\[ \hat{P}^\sigma(\gamma, \delta) = \begin{cases} 
\hat{c}_n v^{-1} + \hat{c}_{n-1} v^{-3} + \cdots + \hat{c}_0 v^{\ell(\gamma) - \ell(\delta) = -(2n+1)} & \ell(\delta) - \ell(\gamma) \text{ odd} \\
\hat{c}_n v^{-2} + \hat{c}_{n-1} v^{-4} + \cdots + \hat{c}_0 v^{\ell(\gamma) - \ell(\delta) = -(2n+2)} & \ell(\delta) - \ell(\gamma) \text{ even}
\end{cases} \]

or alternatively

\[ \hat{P}^\sigma(\gamma, \delta) = \begin{cases} 
v^{-1}[\hat{c}_n + \hat{c}_{n-1} v^{-2} + \cdots + \hat{c}_0 v^{\ell(\gamma) - \ell(\delta) + 1}] & \ell(\gamma) - \ell(\delta) \text{ odd} \\
v^{-1}[\hat{c}_n v^{-1} + \hat{c}_{n-1} v^{-3} + \cdots + \hat{c}_0 v^{\ell(\gamma) - \ell(\delta) + 1}] & \ell(\gamma) - \ell(\delta) \text{ even}
\end{cases} \]

8 Action of $T_{w_\kappa} + 1$

An important role is played by the operator $T_{w_\kappa} + 1$, which we renormalize. For $\kappa \in \mathcal{S}$ define

\[ \hat{T}_\kappa = v^{-\ell(\kappa)}(T_{w_\kappa} + 1) \]

Note that Fokko has both $T_s$ and $t_s$. One can deduce they are related by $t_s = v^{-1}T_s$. These correspond to our $T_{w_\kappa}$ and $v^{-\ell(w_\kappa)}T_{w_\kappa}$, respectively. Also Fokko has an operator $c_s$, which can be seen to be $t_s + v^{-1} = v^{-1}(T_s + 1)$, which is our $\hat{T}_\kappa$.

8.1 Formulas for $\hat{T}_\kappa$

Here are 30 formulas, for $\hat{T}_\kappa(\hat{a}_\gamma)$, depending on the type of $\kappa$ with respect to $\gamma$.

Type 1: $\hat{T}_\kappa = v^{-1}(T_{w_\kappa} + 1) = v^{-1}(T_{s_\kappa} + 1)$

These are copied from [3] Section 1.
\[
t_{\gamma}(\kappa) \quad \hat{T}_\kappa(\hat{a}_\gamma)
\]

**Type 2:** \(\hat{T}_\kappa = v^{-2}(T_{\omega \kappa} + 1) = v^{-2}(T_{s\alpha s\beta} + 1)\)

| \(t_{\gamma}(\kappa)\) | \(\hat{T}_\kappa(\hat{a}_\gamma)\) |
|----------------------|------------------|
| 1C+                  | \(v^{-1}\hat{a}_\gamma + \hat{a}_{\omega \kappa}\) |
| 1C-                  | \(v\hat{a}_\gamma + \hat{a}_{\omega \kappa}\) |
| 1i1                  | \(v^{-1}(\hat{a}_\gamma + \hat{a}_{\omega \kappa}) + \hat{a}_{\gamma^*}\) |
| 1i2f                 | \(2v^{-1}\hat{a}_\gamma + (\hat{a}_{\gamma^*} + \hat{a}_{\gamma^*}^2)\) |
| 1i2s                 | 0 |
| 1r1f                 | \((v - v^{-1})\hat{a}_\gamma + (1 - v^{-2})(\hat{a}_{\gamma^1} + \hat{a}_{\gamma^2})\) |
| 1r1s                 | \((v + v^{-1})\hat{a}_\gamma\) |
| 1r2                  | \(v\hat{a}_\gamma - v^{-1}\hat{a}_{\omega \kappa} + (1 - v^{-2})\hat{a}_{\gamma^*}\) |
| 1rn                  | 0 |
| 1ic                  | \((v + v^{-1})\hat{a}_\gamma\) |

**Type 3:** \(\hat{T}_\kappa = v^{-3}(T_{\omega \kappa} + 1) = v^{-3}(T_{s\alpha s\beta} + 1)\)

| \(t_{\gamma}(\kappa)\) | \(\hat{T}_\kappa(\hat{a}_\gamma)\) |
|----------------------|------------------|
| 2C+                  | \(v^{-2}\hat{a}_\gamma + \hat{a}_{\omega \kappa}\) |
| 2C-                  | \(v^2\hat{a}_\gamma + \hat{a}_{\omega \kappa}\) |
| 2Ci                  | \((v + v^{-1})[v^{-1}\hat{a}_\gamma + \hat{a}_{\gamma^*}]\) |
| 2Cr                  | \((v^2 - 1)\hat{a}_\gamma + (v - v^{-1})\hat{a}_{\gamma^*}\) |
| 2i1f                 | \(v^{-2}(\hat{a}_\gamma + \hat{a}_{\omega \kappa}) + \hat{a}_{\gamma^*}\) |
| 2i12                 | \(2v^{-2}\hat{a}_\gamma + \sum_{\gamma'\gamma^*,\gamma^*} \epsilon(\gamma',\gamma)\hat{a}_{\gamma'}\) |
| 2i22                 | \(2v^{-2}\hat{a}_\gamma + \hat{a}_{\gamma^1} + \hat{a}_{\gamma^2}\) |
| 2r22                 | \(v^2\hat{a}_\gamma - v^{-2}\hat{a}_{\omega \kappa} + (1 - v^{-4})\hat{a}_{\gamma^*}\) |
| 2r21                 | \((v^2 - v^{-2})\hat{a}_\gamma + (1 - v^4)\sum_{\gamma'\gamma^*,\gamma^*} \epsilon(\gamma,\gamma')\hat{a}_{\gamma'}\) |
| 2r11                 | \((v^2 - v^{-2})\hat{a}_\gamma + (1 - v^{-4})(\hat{a}_{\gamma^1} + \hat{a}_{\gamma^2})\) |
| 2rn                  | 0 |
| 2ic                  | \((v^2 + v^{-2})\hat{a}_\gamma\) |
\[ t_\gamma(\kappa) = \tilde{T}_\kappa(\hat{a}_\gamma) \]

| \( t_\gamma(\kappa) \) | \( \tilde{T}_\kappa(\hat{a}_\gamma) \) |
|-------------------------|-----------------------------|
| 3C+                    | \( v^{-3}\hat{a}_\gamma + \hat{a}_{w_\kappa \times \gamma} \) |
| 3C-                    | \( v^3\hat{a}_\gamma + \hat{a}_{w_\kappa \times \gamma} \) |
| 3Ci                    | \( (v + v^{-1})v^{-2}\hat{a}_\gamma + (v + v^{-1})\hat{a}_\gamma \) |
| 3Cr                    | \( (v^2 - v^{-2})v\hat{a}_\gamma + (v^2 - v^{-2})v^{-1}\hat{a}_\gamma \) |
| 3i                      | \( (v + v^{-1})v^{-2}\hat{a}_\gamma + (v + v^{-1})\hat{a}_\gamma \) |
| 3r                      | \( (v^2 - v^{-2})v\hat{a}_\gamma + (v^2 - v^{-2})v^{-1}\hat{a}_\gamma \) |
| 3rn                    | 0 |
| 3ic                    | \( (v^3 + v^{-3})\hat{a}_\gamma \) |

### 8.2 Summary

We write some of these formulas in a slightly different form in the following table of all \( \tilde{T}_\kappa\hat{a}_\gamma \).
### Table 8.2.1

| $t_\gamma(\kappa)$ | $\hat{T}_\kappa \hat{a}_\gamma$ | $t_\gamma(\kappa)$ | $T_\kappa \hat{a}_\gamma$ |
|-------------------|-------------------------------|-------------------|---------------------|
| 1C+               | $[\hat{a}_{\nu_x \times \gamma} + v^{-1}a_{\gamma}]$ | 1C-               | $v[\hat{a}_\gamma + v^{-1}\hat{a}_{\nu_x \times \gamma}]$ |
| 1i1               | $[\hat{a}_{\gamma \times} + v^{-1}(\hat{a}_\gamma + \hat{a}_{\nu_x \times \gamma})]$ | 1r1f              | $(v - v^{-1})[\hat{a}_\gamma + v^{-1}(\hat{a}_{\gamma_1} + \hat{a}_{\gamma_2})]$ |
| 1i2f              | $[\hat{a}_{\gamma_1} + v^{-1}a_{\gamma}] + [\hat{a}_{\gamma_2} + v^{-1}a_{\gamma}]$ | 1r2               | $v[\hat{a}_\gamma + v^{-1}\hat{a}_{\gamma_1}]$ $-v^{-1}[\hat{a}_{\nu_x \times \gamma} + v^{-1}a_{\gamma}]$ |
| 1i2s              | 0                              | 1r1s              | $(v + v^{-1})[\hat{a}_\gamma]$ |
| 1rn               | 0                              | 1ic               | $(v + v^{-1})[\hat{a}_\gamma]$ |
| 2C+               | $[\hat{a}_{\nu_x \times \gamma} + v^{-2}a_{\gamma}]$ | 2C-               | $v^2[\hat{a}_\gamma + v^{-2}\hat{a}_{\nu_x \times \gamma}]$ |
| 2Ci               | $(v + v^{-1})[\hat{a}_{\gamma \times} + v^{-1}a_{\gamma}]$ | 2Cr               | $v(v - v^{-1})[\hat{a}_\gamma + v^{-1}\hat{a}_{\gamma_1}]$ |
| 2i11              | $[\hat{a}_{\gamma \times} + v^{-2}(\hat{a}_\gamma + \hat{a}_{\nu_x \times \gamma})]$ | 2r11              | $(v^2 - v^{-2})[\hat{a}_\gamma + v^{-2}(\hat{a}_{\gamma_1} + \hat{a}_{\gamma_2})]$ |
| 2i12              | $\sum_{\gamma' \times \gamma \times \mu} \epsilon(\gamma', \gamma)[\hat{a}_{\gamma'} + v^{-2} \sum_{\mu' \times \gamma \times \mu} \epsilon(\gamma', \mu)\hat{a}_{\mu}]$ | 2r21              | $(v^2 - v^{-2})[\hat{a}_\gamma + v^{-2} \sum_{\gamma' \times \gamma \times \mu} \epsilon(\gamma', \gamma)\hat{a}_{\gamma'}$ |
| 2i22              | $[\hat{a}_{\gamma_1} + v^{-2}a_{\gamma}] + [\hat{a}_{\gamma_2} + v^{-2}a_{\gamma}]$ | 2r22              | $v^2[\hat{a}_\gamma + v^{-2}a_{\gamma}]$ $-v^{-2}[\hat{a}_{\nu_x \times \gamma} + v^{-2}a_{\gamma}]$ |
| 2rn               | 0                              | 2ic               | $(v^2 + v^{-2})[\hat{a}_\gamma]$ |
| 3C+               | $[\hat{a}_{\nu_x \times \gamma} + v^{-3}a_{\gamma}]$ | 3C-               | $v^3[\hat{a}_\gamma + v^{-3}\hat{a}_{\nu_x \times \gamma}]$ |
| 3Ci, 3i           | $(v + v^{-1})[\hat{a}_{\gamma \times} + v^{-2}a_{\gamma}]$ | 3Cr, 3r           | $v(v^2 - v^{-2})[\hat{a}_\gamma + v^{-2}a_{\gamma}]$ |
| 3rn               | 0                              | 3ic               | $(v^3 + v^{-3})[\hat{a}_\gamma]$ |

We’re going to simplify this table - see Table 9.1.3

**Remark 8.2.2** The identity in the 2i12 case is tricky, let’s write it out. Suppose $t_\gamma(\kappa) = 2i12$, and $\gamma$ is one member of the ordered pair $(\gamma_1, \gamma_2)$. Similarly $\gamma'$ is an ordered pair $(\gamma'_1, \gamma'_2)$.

The formula from Section 8.1 is:

\[(8.2.3)(a) \quad 2v^{-2}a_{\gamma} + \epsilon(\gamma', \gamma)a_{\gamma_1} + \epsilon(\gamma'_2, \gamma)a_{\gamma_2} \]

whereas Table 8.2.1 gives:

\[
\epsilon(\gamma'_1, \gamma)[a_{\gamma_1} + v^{-2}(\epsilon(\gamma'_1, \gamma_1)a_{\gamma_1} + \epsilon(\gamma'_1, \gamma_2)a_{\gamma_2})] + \\
\epsilon(\gamma'_2, \gamma)[a_{\gamma_2} + v^{-2}(\epsilon(\gamma'_2, \gamma_1)a_{\gamma_1} + \epsilon(\gamma'_2, \gamma_2)a_{\gamma_2})].
\]

Using the definition of $\epsilon$ this equals:

\[(8.2.3)(b) \quad [a_{\gamma_1} + v^{-2}(a_{\gamma_1} + a_{\gamma_2})] + \epsilon(\gamma'_2, \gamma)[a_{\gamma_2} + v^{-2}(a_{\gamma_1} - a_{\gamma_2})].\]
Plugging $\gamma = \gamma_1$ or $\gamma_2$ in to (a) and (b) and comparing confirms the identity.

9 Image of $\hat{T}_\kappa$

9.1 $\hat{T}_\kappa(\hat{a}_\gamma)$

Lemma 9.1.1 Fix $\kappa \in \mathcal{S}$.

1. The image of $\hat{T}_\kappa$ is equal to the $(v^{\ell(\kappa)} + v^{-\ell(\kappa)})$ eigenspace of $\hat{T}_\kappa$. This is also equal to the kernel of $T_\kappa - v^{\ell(\kappa)}$.

2. Suppose $\kappa \in \tau(\lambda)$. For each $\lambda'$ satisfying $\lambda \overset{\kappa}{\rightarrow} \lambda'$, the sign $\epsilon(\lambda, \lambda') = \pm 1$ (Definition 5.2.1) is the unique integer such that

\begin{equation}
  \hat{a}_\lambda^\kappa = \hat{a}_\lambda + v^{\ell(\lambda') - \ell(\lambda)} \sum_{\lambda' \overset{\kappa}{\rightarrow} \lambda'} \epsilon(\lambda, \lambda') \hat{a}_{\lambda'}
\end{equation}

belongs to the image of $\hat{T}_\kappa$.

3. The elements

\{ $\hat{a}_\lambda^\kappa$ | $\gamma \in D^s, \kappa \in \tau(\gamma)$ \}

form a basis of the image of $\hat{T}_\kappa$.

Part (1) follows from the quadratic relation (2.3). Statements (2) and (3) follow from an examination of Table 8.2.1. In each entry of the table the terms in square brackets are the $\hat{a}_\delta$ (not including the $v^{\text{def}_\delta(\kappa)}$ term). This amounts to the fact that we can rewrite Table 8.2.1 as in Table 9.1.3.

Recall $\epsilon(\delta, \delta') = 1$ except in cases $2i12/2r21$.

Table 8.2.1 now simplifies.

Table 9.1.3

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| $t_\gamma(\kappa)$ | $\hat{T}_\kappa(\hat{a}_\gamma)$ | $\tau_\gamma(\kappa)$ | $\hat{T}_\kappa(\hat{a}_\gamma)$ |
|-----------------|---------------------------------|------------------------|---------------------------------|
| 1C+             | $\hat{a}_{w_n}^{\kappa \times \gamma}$ | 1C-                    | $v \hat{a}_\gamma^{\kappa}$    |
| 111             | $\hat{a}_{\gamma_r}^{\kappa}$        | 1r1f                   | $(v - v^{-1}) \hat{a}_\gamma^{\kappa}$ |
| 1i2f            | $\hat{a}_{\gamma_1}^{\kappa} + \hat{a}_{\gamma_2}^{\kappa}$ | 1r2                    | $v \hat{a}_\gamma^{\kappa} - v^{-1} \hat{a}_{w_n}^{\kappa \times \gamma}$ |
| 1i2s            | 0                                 | 1r1s                   | $(v + v^{-1}) \hat{a}_\gamma^{\kappa}$ |
| 1rn             | 0                                 | 1ic                    | $(v + v^{-1}) \hat{a}_\gamma^{\kappa}$ |
| 2C+             | $\hat{a}_{w_n}^{\kappa \times \gamma}$ | 2C-                    | $v^2 \hat{a}_\gamma^{\kappa}$   |
| 2Ci             | $(v + v^{-1}) \hat{a}_{\gamma_{\kappa}}^{\kappa}$ | 2Cr                    | $v(v - v^{-1}) \hat{a}_\gamma^{\kappa}$ |
| 2i11            | $\hat{a}_{\gamma_{\kappa}}^{\kappa}$ | 2r22                   | $v^2 \hat{a}_\gamma^{\kappa} - v^{-2} \hat{a}_{w_n}^{\kappa \times \gamma}$ |
| 2i12            | $\sum_{\gamma' \rightarrow \gamma} \epsilon(\gamma', \gamma) \hat{a}_\gamma^{\kappa}$ | 2r21                   | $(v^2 - v^{-2}) \hat{a}_\gamma^{\kappa}$ |
| 2r22            | $\hat{a}_{\gamma_{\kappa}}^{\kappa} + \hat{a}_{\gamma_2}^{\kappa}$ | 2r11                   | $(v^2 - v^{-2}) \hat{a}_\gamma^{\kappa}$ |
| 2rn             | 0                                 | 2ic                    | $(v^2 + v^{-2}) \hat{a}_\gamma^{\kappa}$ |
| 3C+             | $\hat{a}_{w_n}^{\kappa \times \gamma}$ | 3C-                    | $v^3 \hat{a}_\gamma^{\kappa}$   |
| 3Ci, 3i         | $(v + v^{-1}) \hat{a}_{\gamma_{\kappa}}^{\kappa}$ | 3Cr, 3r                | $v(v^2 - v^{-2}) \hat{a}_\gamma^{\kappa}$ |
| 3rn             | 0                                 | 3ic                    | $(v^3 + v^{-3}) \hat{a}_\gamma^{\kappa}$ |

Those extra powers of $v$ in cases 2Cr, 3Cr, 3r are important. Suppose $\kappa \in \tau(\lambda)$. Then $\ell(\lambda')$ is the same for all $\lambda \rightarrow^\kappa \lambda'$; typically (always in the classical case) $\ell(\lambda) - \ell(\lambda') = \ell(\kappa)$. In general $\ell(\lambda) - \ell(\lambda') \leq \ell(\kappa)$.

**Definition 9.1.4** Suppose $\lambda \rightarrow^\kappa \lambda'$. Define the $\kappa$-defect of $\lambda$ and $\lambda'$ to be

$$
\text{def}_\lambda(\kappa) = \text{def}_\kappa(\lambda') = \ell(\kappa) - \ell(\lambda) + \ell(\lambda').
$$

(If $\{\lambda' \mid \lambda \rightarrow^\kappa \lambda'\} = \emptyset$ define $\text{def}(\kappa, \lambda) = 0$).

Checking the cases gives:

**Lemma 9.1.6**

$$
\text{def}_\lambda(\kappa) = \begin{cases} 
1 & t_\lambda(\kappa) = 2\text{Ci}, 3\text{Ci}, 3\text{i}; 2\text{Cr}, 3\text{Cr}, 3\text{r} \\
0 & \text{else}
\end{cases}
$$

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We can now write Table 9.1.3 more concisely. For $\kappa \in \tau(\gamma)$ define:

$$
\zeta_\kappa(\gamma) = \begin{cases} 
1 & t_\gamma(\kappa) = 1c, 2ic, 3ic, 1r1s \\
0 & t_\gamma(\kappa) = 1c-, 2c-, 3c- \\
-1 & \text{otherwise}
\end{cases}
$$

Lemma 9.1.7 Fix $\kappa \in \mathcal{S}$, $\gamma \in \mathcal{D}^s$, and set $d = \text{def}_\gamma(\kappa)$. Then

$$(9.1.8) \quad \widehat{T}_\kappa(\hat{a}_\gamma) = \begin{cases} 
(v + v^{-1})^d \sum_{\gamma' \gamma} \epsilon(\gamma', \gamma) \hat{a}_{\gamma'}^\kappa & \kappa \notin \tau(\gamma) \\
v^d [v^{\ell(\kappa) - d} \hat{a}_\gamma^\kappa + \zeta_\kappa(\gamma)v^{-\ell(\kappa) + d} \hat{a}_{w_k \times \gamma}^\kappa] & \kappa \in \tau(\gamma)
\end{cases}$$

In the second case:

1. if $t_\gamma(\kappa) = 1r2, 2r22$ then $w_\kappa \times \gamma \neq \gamma$, and $\kappa \in \tau(w_\kappa \times \gamma)$ - there are two terms;
2. if $t_\gamma(\kappa) = 1c-, 2c-, 3c-$ then $w_\kappa \times \gamma \neq \gamma$, but $\kappa \notin \tau(w_\kappa \times \gamma)$ - since $\zeta = 0$ there is only one term;
3. in all other cases $w_\kappa \times \gamma = \gamma$ (there is one term with a coefficient of $v^d (v^{\ell(\kappa) - d} \pm v^{-\ell(\kappa) + d})$).

9.2 $\widehat{T}_\kappa(\hat{C}_\lambda$) in terms of $\hat{a}_\gamma^\kappa$ We can now compute $\widehat{T}_\kappa(\hat{C}_\lambda)$ in the basis of $\hat{a}_\gamma^\kappa$ (and the unknown $\hat{P}^s(\gamma, \lambda)$).

Write $\hat{C}_\lambda = \sum_{\gamma \leq \lambda} \hat{P}^s(\gamma, \lambda) \hat{a}_\gamma$, $\widehat{T}_\kappa(\hat{C}_\lambda) = \sum_{\gamma \leq \lambda} \hat{P}^s(\gamma, \lambda) \widehat{T}_\kappa(\hat{a}_\gamma)$. The condition $\gamma \leq \lambda$ is superfluous because of the $\hat{P}^s(\gamma, \lambda)$ term. Apply Lemma 9.1.7
\[ \hat{T}_\kappa(\hat{C}_\lambda) = \sum_{\gamma} P(\gamma, \lambda) \hat{T}_\kappa(\hat{a}_\gamma) \]
\[ = \sum_{\gamma' : \kappa \not\in \tau(\gamma')} P(\gamma', \lambda) \hat{T}_\kappa(\hat{a}_{\gamma'}) + \sum_{\gamma' : \kappa \in \tau(\gamma')} P(\gamma', \lambda) \hat{T}_\kappa(\hat{a}_{\gamma'}) \]
\[ = \sum_{\gamma' : \kappa \not\in \tau(\gamma')} \left[ P(\gamma, \lambda)(v + v^{-1})^{\text{def}_{\gamma}(\kappa)} \sum_{\gamma' : \gamma' \not\in \gamma} \varepsilon(\gamma', \gamma) \hat{a}_{\gamma'} \right] + \]
\[ \sum_{\gamma' : \kappa \in \tau(\gamma')} v^{\text{def}_{\gamma}(\kappa)} [P(\gamma, \lambda)(v^\ell(\kappa) - \text{def}_{\gamma}(\kappa)) \hat{a}_{\gamma} + \zeta_\kappa(\gamma) v^{-\ell(\kappa)} + \text{def}_{\gamma}(\kappa) \hat{a}_{w_\kappa \times \gamma}] \]
\[ = \sum_{\gamma' : \kappa \not\in \tau(\gamma')} \left[ (v + v^{-1})^{\text{def}_{\gamma}(\kappa)} \sum_{\gamma' : \gamma' \not\in \gamma} \text{def}_{\gamma'} \epsilon(\gamma', \gamma') \right] \hat{a}_{\gamma'} + \]
\[ \sum_{\gamma' : \kappa \in \tau(\gamma')} v^{\text{def}_{\gamma}(\kappa)} [P(\gamma, \lambda)(v^\ell(\kappa) - \text{def}_{\gamma}(\kappa)) \hat{a}_{\gamma} + P(\gamma, \lambda) \zeta_\kappa(\gamma) v^{-\ell(\kappa)} + \text{def}_{\gamma}(\kappa) \hat{a}_{w_\kappa \times \gamma}] \]

Interchange \( \gamma, \gamma' \) in the first sum to conclude:
\[ \hat{T}_\kappa(\hat{C}_\lambda) = \sum_{\gamma' : \kappa \not\in \tau(\gamma')} \left[ (v + v^{-1})^{\text{def}_{\gamma}(\kappa)} \sum_{\gamma' : \gamma' \not\in \gamma} \text{def}_{\gamma'} \epsilon(\gamma', \gamma') \right] \hat{a}_{\gamma'} + \]
\[ \sum_{\gamma': \kappa \in \tau(\gamma')} v^{\text{def}_{\gamma}(\kappa)} [v^\ell(\kappa) - \text{def}_{\gamma}(\kappa) P(\gamma, \lambda) + \zeta_\kappa(\gamma) v^{-\ell(\kappa)} + \text{def}_{\gamma}(\kappa) P(w_\kappa \times \gamma, \lambda)] \hat{a}_{\gamma} \]

**Lemma 9.2.1** Fix \( \gamma \) with \( \kappa \in \tau(\gamma) \). The coefficient of \( \hat{a}_{\gamma} \) in \( \hat{T}_\kappa(\hat{C}_\lambda) \) is
\[ v^{\text{def}_{\gamma}(\kappa)} [v^\ell(\kappa) - \text{def}_{\gamma}(\kappa) \hat{P}\sigma(\gamma, \lambda) + \]
\[ \zeta_\kappa(\gamma) v^{-\ell(\kappa)} + \text{def}_{\gamma}(\kappa) \hat{P}\sigma(w_\kappa \times \gamma, \lambda)] + \]
\[ (v + v^{-1})^{\text{def}_{\gamma}(\kappa)} \sum_{\gamma' : \gamma' \not\in \gamma} \hat{P}\sigma(\gamma', \lambda) \epsilon(\gamma', \gamma) \]

Here is the information needed to make this explicit. Assume \( \kappa \in \tau(\gamma) \).
If \( t_\gamma(\kappa) = 1r2, 2r22, 1c-, 2c-, 3c- \) then \( w_\kappa \times \gamma \neq \gamma \). In all other cases \( w_\kappa \times \gamma = \gamma \).
We need \( \{ \gamma' : \gamma \not\rightarrow \gamma' \} \):

1. \( t_\gamma(\kappa) = 1c-, 2c-, 3c- : w_\kappa \times \gamma; \)

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2. \( t_\gamma(\kappa) = 1r2, 2Cr, 2r22, 3Cr, 3r \): \( \gamma_\kappa \) (single valued);
3. \( t_\gamma(\kappa) = 1r1f, 2r21, 2r11 \): \{\( \gamma_\kappa^1, \gamma_\kappa^2 \)\} (double valued);
4. \( t_\gamma(\kappa) = 1r1s, 1ic, 2ic, 3ic \): none

The defect \( \text{def}_\gamma(\kappa) \) is 1 if \( t_\gamma(\kappa) = 2Cr, 3Cr, 3r \), and 0 otherwise.

\[
\zeta_\kappa(\gamma) = \begin{cases} 
1 & t_\gamma(\kappa) = 1ic, 2ic, 3ic \\
0 & t_\gamma(\kappa) = 1C-, 2C-, 3C- \\
-1 & \text{otherwise}
\end{cases}
\]

Table 9.2.3

| \( t_\gamma(\kappa) \) | first term on the RHS of (9.2.2) | second term on RHS of (9.2.2) |
|--------------------------|---------------------------------|---------------------------------|
| 1C-                      | \( v\hat{P}^\sigma(\gamma, \lambda) \) | \( \hat{P}^\sigma(w_\kappa \times \gamma, \lambda) \) |
| 1r1f                    | \( (v - v^{-1})\hat{P}^\sigma(\gamma, \lambda) \) | \( \hat{P}^\sigma(\gamma_\kappa^1, \lambda) + \hat{P}^\sigma(\gamma_\kappa^2, \lambda) \) |
| 1r1s                    | \( (v + v^{-1})\hat{P}^\sigma(\gamma, \lambda) \) | | |
| 1r2                     | \( v\hat{P}^\sigma(\gamma, \lambda) - v^{-1}\hat{P}^\sigma(w_\kappa \times \gamma, \lambda) \) | \( \hat{P}^\sigma(\gamma_\kappa, \lambda) \) |
| 1ic                     | \( (v + v^{-1})\hat{P}^\sigma(\gamma, \lambda) \) | | |
| 2C-                      | \( v^2\hat{P}^\sigma(\gamma, \lambda) \) | \( \hat{P}^\sigma(w_\kappa \times \gamma, \lambda) \) |
| 2Cr                     | \( v(v - v^{-1})\hat{P}^\sigma(\gamma, \lambda) \) | \( (v + v^{-1})\hat{P}^\sigma(\gamma_\kappa, \lambda) \) |
| 2r22                    | \( v^2\hat{P}^\sigma(\gamma, \lambda) - v^{-2}\hat{P}^\sigma(w_\kappa \times \gamma, \lambda) \) | \( \hat{P}^\sigma(\gamma_\kappa, \lambda) \) |
| 2r21                    | \( (v^2 - v^{-2})\hat{P}^\sigma(\gamma, \lambda) \) | \( \sum_{\gamma' | \gamma \to \gamma'} \epsilon(\gamma, \gamma')\hat{P}^\sigma(\gamma', \lambda) \) |
| 2r11                    | \( (v^2 - v^{-2})\hat{P}^\sigma(\gamma, \lambda) \) | \( \hat{P}^\sigma(\gamma_\kappa^1, \lambda) + \hat{P}^\sigma(\gamma_\kappa^2, \lambda) \) |
| 2ic                     | \( (v^2 + v^{-2})\hat{P}^\sigma(\gamma, \lambda) \) | | |
| 3C-                      | \( v^3\hat{P}^\sigma(\gamma, \lambda) \) | \( \hat{P}^\sigma(w_\kappa \times \gamma, \lambda) \) |
| 3Cr                     | \( v(v^2 - v^{-2})\hat{P}^\sigma(\gamma, \lambda) \) | \( (v + v^{-1})\hat{P}^\sigma(\gamma_\kappa, \lambda) \) |
| 3r                       | \( v(v^2 - v^{-2})\hat{P}^\sigma(\gamma, \lambda) \) | \( (v + v^{-1})\hat{P}^\sigma(\gamma_\kappa, \lambda) \) |
| 3ic                     | \( (v^3 + v^{-3})\hat{P}^\sigma(\gamma, \lambda) \) | | |
Here is a condensed version of this table. Let \( k = \ell(\kappa) \).

### Table 9.2.4

| \( t_\gamma(\kappa) \) | first term on the RHS of (9.2.2) | second term on RHS of (9.2.2) |
|------------------------|----------------------------------|----------------------------------|
| 1C-, 2C-, 3C-          | \( v^k \hat{P}_\sigma(\gamma, \lambda) \) | \( \hat{P}_\sigma(w_\kappa \times \gamma, \lambda) \) |
| 1ic, 2ic, 3ic, 1rls    | \((v^k + v^{-k}) \hat{P}_\sigma(\gamma, \lambda)\) | \( \hat{P}_\sigma(w_\kappa \times \gamma, \lambda) \) |
| 2Cr, 3Cr, 3r           | \( v(v^k - v^{-k+1}) \hat{P}_\sigma(\gamma, \lambda) \) | \( (v + v^{-1}) \hat{P}_\sigma(\gamma_\kappa, \lambda) \) |
| 1r1f, 2r11             | \( (v^k - v^{-k}) \hat{P}_\sigma(\gamma, \lambda) \) | \( \hat{P}_\sigma(\gamma_1, \lambda) + \hat{P}_\sigma(\gamma_2, \lambda) \) |
| 1r2, 2r22              | \( v^k \hat{P}_\sigma(\gamma, \lambda) - v^{-k} \hat{P}_\sigma(w_\kappa \times \gamma, \lambda) \) | \( \hat{P}_\sigma(\gamma_\kappa, \lambda) \) |
| 2r21                   | \( (v^2 - v^{-2}) \hat{P}_\sigma(\gamma, \lambda) \) | \( \sum_{\gamma' | \gamma \rightarrow \gamma'} \epsilon(\gamma, \gamma') \hat{P}_\sigma(\gamma', \lambda) \) |

#### 9.3 \( \hat{T}_\kappa(\hat{C}_\mu) \) in terms of \( \hat{C}_\gamma \)

**Lemma 9.3.1** Suppose \( \gamma \in D^\sigma, \kappa \in \overline{S} \). Then \( \kappa \in \tau(\lambda) \) iff

\[
\hat{T}_\kappa \hat{C}_\lambda = (v^{\ell(\kappa)} + v^{-\ell(\kappa)}) \hat{C}_\lambda.
\]

This is the way “descent” is defined. In the geometric language of [3], the condition means that the corresponding perverse sheaf is pulled back from the partial flag variety of type \( \kappa \). Compare [4, Theorem 4.4(c)] and [6, Lemma 6.7].

Recall the image of \( \hat{T}_\kappa \) has \( \{ \hat{a}_\gamma^\kappa \mid \gamma \in D^\sigma, \kappa \in \tau(\gamma) \} \) as a basis. We can use \( \{ \hat{C}_\gamma \mid \gamma \in D^\sigma, \kappa \in \tau(\gamma) \} \) instead.

**Lemma 9.3.2** Fix \( \kappa \in \overline{S} \).

1. Suppose \( \mu \in D^\sigma, \) and \( \kappa \in \tau(\mu) \). Then

\[
(9.3.3) \quad \hat{C}_\mu = \sum_{\gamma | \kappa \in \tau(\gamma)} \hat{P}_\sigma(\gamma, \mu) \hat{a}_\gamma^\kappa;
\]

The coefficient polynomials are exactly the ones from Theorem 7.3. In particular \( \hat{C}_\mu \) is in the image of \( \hat{T}_\kappa \).
2. The elements \( \{ \hat{C}_\mu \mid \mu \in \mathcal{D}^\sigma, \kappa \in \tau(\mu) \} \)
form a basis of the image of \( \hat{T}_\kappa \).

**Proof.** For (1), write

\[(9.3.4)(a) \quad \hat{C}_\mu = \sum_{\gamma \mid \kappa \in \tau(\gamma)} \hat{P}^\sigma(\gamma, \mu) \hat{a}_\gamma + \sum_{\gamma \mid \kappa \notin \tau(\gamma)} \hat{P}^\sigma(\gamma, \mu) \hat{a}_\gamma.\]

By Lemmas 9.3.1 and 9.1.1(3) we can also write \( \hat{C}_\mu = \sum_{\gamma \mid \kappa \in \tau(\gamma)} \hat{R}^\sigma(\gamma, \mu) \hat{a}_\gamma \) for some \( \hat{R}^\sigma(\gamma, \mu) \in \mathbb{Z}[v, v^{-1}] \). Plugging in the definition of \( \hat{a}_\gamma \) (Lemma 9.1.1(2)) gives

\[(9.3.4)(b) \quad \hat{C}_\mu = \sum_{\gamma \mid \kappa \in \tau(\gamma)} \hat{R}^\sigma(\gamma, \mu) \left[ \hat{a}_\gamma + v^{\ell(\gamma')-\ell(\gamma)} \sum_{\gamma' \mid \gamma' \rightarrow \gamma'} \epsilon(\gamma, \gamma') \hat{a}_{\gamma'} \right].\]

Since \( \kappa \notin \tau(\gamma') \) for each term in the last sum, comparing coefficients of \( \hat{a}_\gamma \) \((\kappa \in \tau(\gamma))\) in (a) and (b) gives \( \hat{R}(\gamma, \mu) = \hat{P}^\sigma(\gamma, \mu) \). This gives (1), and (2) follows. \( \square \)

Comparing the coefficients of \( \hat{a}_\gamma \) with \( \kappa \notin \tau(\gamma) \) gives the “easy” recurrence relations for the \( \hat{P}^\sigma(\gamma, \lambda) \). See Section 10.

Next we want to compute \( \hat{T}_\kappa \hat{C}_\lambda \) in the basis \( \hat{C}_\gamma \). When \( \kappa \in \tau(\lambda) \) this is given in Lemma 9.3.1. We turn now to the case \( \kappa \notin \tau(\lambda) \).

**Lemma 9.3.5 (compare \[4\] Theorem 4.4(a,b))** Suppose \( \kappa \notin \tau(\lambda) \). Then

\[(9.3.6) \quad \hat{T}_\kappa \hat{C}_\lambda = \sum_{\gamma \mid \kappa \in \tau(\gamma)} m_\kappa(\gamma, \lambda) \hat{C}_\gamma \]

for some \( m_\kappa(\gamma, \lambda) \in \mathbb{Z}[v, v^{-1}] \). Each \( m_\kappa(\gamma, \lambda) \) is self-dual, and is of the form

\[
m_\kappa(\gamma, \lambda) = \begin{cases} 
m_{\kappa,0}(\gamma, \lambda) & \ell(\kappa) = 1 \\
m_{\kappa,0}(\gamma, \lambda) + m_{\kappa,1}(\gamma, \lambda)(v + v^{-1}) & \ell(\kappa) = 2 \\
m_{\kappa,0}(\gamma, \lambda) + m_{\kappa,1}(\gamma, \lambda)(v + v^{-1}) + m_{\kappa,2}(\gamma, \lambda)(v^2 + v^{-2}) & \ell(\kappa) = 3 \end{cases}
\]

for some integers \( m_{\kappa,i}(\gamma, \lambda) \).
Proof. The existence of \( m_\kappa(\gamma, \delta) \) is (2) of Lemma 9.3.2. That the left side of (9.3.6) is self-dual is \([4. 4.8(e)]\), and since \( \widehat{C}_\delta \) is self-dual this implies \( m_\kappa(\gamma, \delta) \) is self-dual.

The highest order term of \( m_\kappa(\gamma, \lambda) \) is \( v^{\ell(\kappa)-1} \). This follows by downward induction on \( \ell(\gamma) \). See \([4. \text{pg. 17 (Proof of Theorem 4.4)}]\). This gives the remaining assertion. □

Remark 9.3.7 It is easy to see \( m_\kappa(\gamma, \lambda) \neq 0 \) implies \( \gamma \xrightarrow{\kappa} \lambda \), or \( \gamma < \lambda \), or \( \gamma \xrightarrow{\kappa} \gamma' \) for some \( \gamma' < \lambda \). We make this more precise in Lemmas 9.4.1 and 9.4.5.

In the classical setting \( \mu(\gamma, \lambda) \) is defined to be the coefficient of the top degree term in \( P(\gamma, \lambda) \), i.e. \( q_{\frac{1}{2}(\ell(\lambda)-\ell(\gamma)-1)} \). Furthermore if \( \gamma < \lambda \) then \( m_\kappa(\gamma, \lambda) = \mu(\gamma, \lambda) \).

With our normalization the top degree term in \( \widehat{P}_\sigma(\gamma, \lambda) \) is \( v^{-1} \), which is zero unless \( \ell(\lambda) - \ell(\gamma) \) is odd. We need a generalization to take \( \kappa \) of length 2, 3 into account.

Definition 9.3.8 For \( i = -1, -2, -3 \) let \( \widehat{\mu}^\sigma_i(\gamma, \lambda) \) be the coefficient of \( v^i \) in \( \widehat{P}_\sigma(\gamma, \lambda) \).

So

\[
\widehat{P}_\sigma(\gamma, \lambda) = \widehat{\mu}^{-3}_3(\gamma, \lambda)v^{-3} + \widehat{\mu}^{-2}_2(\gamma, \lambda)v^{-2} + \widehat{\mu}^{-1}_1(\gamma, \lambda)v^{-1} \pmod{v^{-4}}.
\]

It is clear that

\[
(9.3.9) \quad \widehat{\mu}^{-k}_{-k}(\gamma, \lambda) = 0 \quad \text{unless} \quad \ell(\gamma) - \ell(\lambda) = k \pmod{2}.
\]

We can now state the main result of this section.

Theorem 9.3.10 Suppose \( \kappa \notin \tau(\lambda) \). Then

\[
\widehat{T}_\kappa \widehat{C}_\lambda = \sum_{\gamma: \kappa \in \tau(\gamma)} m_\kappa(\gamma, \lambda) \widehat{C}_\gamma.
\]

for coefficients \( m_\kappa(\gamma, \lambda) \) given as follows.
1. If $\gamma \xrightarrow{\kappa} \lambda$ then

\[(9.3.11) \quad m_\kappa(\gamma, \lambda) = (v + v^{-1})^{\text{def}_\kappa(\gamma)}\epsilon(\gamma, \lambda) = \begin{cases} \epsilon(\gamma, \lambda) & \text{def}_\lambda(\kappa) = 0 \\ (v + v^{-1}) & \text{def}_\lambda(\kappa) = 1 \end{cases}\]

(recall $\epsilon(\lambda, \gamma) = \pm 1$, and is 1 unless $t_\gamma(\kappa) = 2r21$, cf. Definition 5.2.1).

2. Assume $\gamma \not\xrightarrow{\kappa} \lambda$, and $\ell(\kappa) = 1$. Then

$m_\kappa(\gamma, \lambda) = \hat{\mu}_{-1}^{\sigma}(\gamma, \lambda)$.

3. Assume $\gamma \not\xrightarrow{\kappa} \lambda$, and $\ell(\kappa) = 2$.

(a) If $\ell(\gamma) \not\equiv \ell(\lambda) (\text{mod } 2)$ then

$m_\kappa(\gamma, \lambda) = \hat{\mu}_{-1}^{\sigma}(\gamma, \lambda)(v + v^{-1})$.

(b) If $\ell(\gamma) \equiv \ell(\lambda) (\text{mod } 2)$ then

$m_\kappa(\gamma, \lambda) = \hat{\mu}_{-2}^{\sigma}(\gamma, \lambda) - \sum_{\kappa \in \tau(\delta) \atop \gamma < \delta < \lambda} \hat{\mu}_{-1}^{\sigma}(\gamma, \delta)\hat{\mu}_{-1}^{\sigma}(\delta, \lambda) - \begin{cases} \hat{\mu}_{-1}^{\sigma}(\gamma, \lambda^\kappa) & t_\lambda(\kappa) = 2Ci \\ 0 & \text{else} \end{cases} + \begin{cases} \hat{\mu}_{-1}^{\sigma}(\lambda^\kappa, \lambda) & t_\gamma(\kappa) = 2Cr \\ 0 & \text{else} \end{cases}\]

4. Assume $\gamma \not\xrightarrow{\kappa} \lambda$, and $\ell(\kappa) = 3$.

(a) If $\ell(\gamma) \equiv \ell(\lambda) (\text{mod } 2)$ then

$m_\kappa(\gamma, \lambda) = [\hat{\mu}_{-2}^{\sigma}(\gamma, \lambda) - \sum_{\kappa \in \tau(\delta) \atop \gamma < \delta < \lambda} \hat{\mu}_{-1}^{\sigma}(\gamma, \delta)\hat{\mu}_{-1}^{\sigma}(\delta, \lambda)](v + v^{-1})$
(b) If $\ell(\gamma) \not\equiv \ell(\lambda) \pmod{2}$ then

$$m_\kappa(\gamma, \lambda) = \tilde{\mu}_1^\sigma(\gamma, \lambda)(v^2 + v^{-2}) + \tilde{\mu}_3^\sigma(\gamma, \lambda) + \sum_{\delta, \phi \in \tau(\delta), \kappa \in \tau(\phi)} \tilde{\mu}_1(\gamma, \delta)\tilde{\mu}_1(\delta, \phi)\tilde{\mu}_1(\phi, \lambda) +$$

$$- \sum_{\kappa \in \tau(\delta)} \tilde{\mu}_2(\gamma, \delta)\tilde{\mu}_2(\delta, \lambda) + \tilde{\mu}_2(\gamma, \delta)\tilde{\mu}_1(\delta, \lambda)$$

$$- \begin{cases} 
\mu^\sigma(\gamma, \lambda^\kappa) & t_\lambda(\kappa) = 3Ci \text{ or } 3i \\
0 & \text{else} 
\end{cases}$$

$$+ \begin{cases} 
\mu^\sigma(\gamma, \lambda) & t_\gamma(\kappa) = 3Cr \text{ or } 3r \\
0 & \text{else} 
\end{cases}$$

I’ve written the proof in great length for the sake of finding errors. See the Appendix.

Theorem 9.3.10 gives us a basic identity which we use repeatedly. Suppose $\kappa \not\in \tau(\lambda)$. By Theorem 9.3.10 and (7.6):

$$\tilde{T}_\kappa\tilde{C}_\lambda = \sum_{\delta | \kappa \in \tau(\delta)} m_\kappa(\delta, \lambda)\tilde{C}_\delta$$

$$= \sum_{\delta | \kappa \in \tau(\delta)} m_\kappa(\delta, \lambda) \sum_{\gamma} \tilde{P}^\sigma(\gamma, \lambda)\hat{a}_\gamma$$

$$= \sum_{\gamma} \left[ \sum_{\delta | \kappa \in \tau(\delta)} \tilde{P}^\sigma(\gamma, \delta)m_\kappa(\delta, \lambda) \right]\hat{a}_\gamma$$

Proposition 9.3.13 Fix $\kappa \in \bar{S}$, $\gamma, \lambda \in D^\sigma$, with $\kappa \not\in \tau(\lambda)$. Then

$$\sum_{\delta | \kappa \in \tau(\delta)} \tilde{P}^\sigma(\gamma, \delta)m_\kappa(\delta, \lambda) = \text{multiplicity of } \hat{a}_\gamma \text{ in } \tilde{T}_\kappa(\tilde{C}_\lambda)$$

If $\kappa \in \tau(\gamma)$ the same equality holds with $\hat{a}_\kappa$ on the right hand side.
9.4 Nonvanishing of $m_\kappa(\gamma, \lambda)$

It is important to know when $m_\kappa(\gamma, \lambda)$ can be nonzero.

**Lemma 9.4.1** Assume $\kappa \in \tau(\gamma), \kappa \notin \tau(\lambda)$, and $m_\kappa(\gamma, \lambda) \neq 0$. Then one of the following conditions holds:

(a) $\gamma \xrightarrow{\kappa} \lambda$

(b) $\gamma < \lambda$

(c) $\text{def}_\gamma(\kappa) = 1$, $\gamma \neq \lambda$, and $\tilde{\mu}_{-1}(\gamma, \lambda) \neq 0$.

(c’) $\text{def}_\lambda(\kappa) = 1$, $\gamma < \lambda$, and $\tilde{\mu}_{-1}(\gamma, \lambda^\kappa) \neq 0$.

See Definition 9.1.4 for $\text{def}_\gamma(\kappa)$. Compare [6, Lemma 6.7], and [3, Section 3,II].

**Remark 9.4.2** In the classical case either $\gamma \xrightarrow{\kappa} \lambda$, or $m_\kappa(\gamma, \lambda) = \mu(\gamma, \lambda)$ (the top degree term of $P(\gamma, \lambda)$), which is nonzero only if $\gamma < \lambda$. So cases (c), (c’) don’t occur. Since they allow $m_\kappa(\gamma, \lambda) \neq 0$ for some $\gamma < \lambda$, these cause some trouble.

**Proof.** Consulting the cases in the Theorem, if $m_\kappa(\gamma, \lambda) \neq 0$ then either:

1. $\gamma \xrightarrow{\kappa} \lambda$ (Case (1) of the Theorem)

2. Some $\tilde{\mu}_{-k}(\gamma, \delta) \neq 0$ with $\delta \leq \lambda$. This implies $\gamma < \lambda$ ($\gamma \neq \lambda$ since they have opposite $\tau$-invariants).

3. One of the terms in braces in Cases (3b) or (4b) is nonzero.

The cases $2\text{Cr}$, $3\text{Cr}$, $3\text{r}$, $2\text{Ci}$, $3\text{Ci}$, $3i$ are exactly the ones in which the defect is 1, and since $\tilde{\mu}_{-1}(\gamma, \lambda^\kappa) \neq 0$ or $\tilde{\mu}_{-1}(\gamma^\kappa, \lambda) \neq 0$ this gives the result.

**Definition 9.4.3** Suppose $\kappa \in \tau(\gamma), \kappa \notin \tau(\lambda)$. We say $\gamma \xrightarrow{\kappa} \lambda$ if one of conditions (b,c,c’) of the Lemma hold:

(b) $\gamma < \lambda$

(c) $\text{def}_\gamma(\kappa) = 1$, $\gamma \neq \lambda$, and $\tilde{\mu}_{-1}(\gamma, \lambda) \neq 0$.
(c') \( \text{def}_\lambda(\kappa) = 1, \gamma \not\preceq \lambda, \text{ and } \hat{\mu}^-_{\gamma}(\gamma, \lambda^\kappa) \neq 0. \)

Thus (9.3.6) becomes

\[
\hat{T}_\kappa C_\lambda = \sum_{\gamma: |\kappa \in \tau(\gamma), \gamma \succeq \lambda} m_\kappa(\gamma, \lambda) C_\gamma + \sum_{\gamma: |\kappa \in \tau(\gamma), \gamma \prec \lambda} m_\kappa(\gamma, \lambda) \hat{C}_\gamma
\]

We want to replace (b),(c),(c') with (weaker) conditions in terms of length. Obviously (b) implies \( \ell(\gamma) < \ell(\lambda) \). Suppose \( \ell(\gamma) \geq \ell(\lambda) \), and (c) or (c') holds. This is quite rare.

Consider Case (c). We're assuming \( \ell(\gamma_\kappa) < \ell(\lambda) \leq \ell(\gamma). \) It is hard to satisfy this. Subtract \( \ell(\gamma_\kappa) \) from each term, and use \( \ell(\gamma_\kappa) = \ell(\gamma) - \ell(\kappa) + 1 \), to see

\[
0 < \ell(\lambda) - \ell(\gamma_\kappa) \leq \ell(\kappa) - 1 \in \{1, 2\}
\]

But \( \hat{\mu}^-_{\gamma_\kappa}(\gamma, \lambda) \neq 0 \) implies \( \ell(\lambda) - \ell(\gamma_\kappa) \) is odd, so it equals 1, and

\[
\ell(\gamma) = \ell(\lambda) + \ell(\kappa) - 2, \ell(\gamma_\kappa) = \ell(\lambda) - 1.
\]

Case (c') is similar: \( t_\lambda(\kappa) = 2cr, 3cr, 3r, \ell(\gamma) = \ell(\lambda) + \ell(\kappa) - 2, \) and \( \ell(\lambda^\kappa) = \ell(\gamma) + 1. \)

These are illustrated by the following pictures. An arrow with a label: \( \alpha \rightarrow^k \beta \) indicates \( k = \ell(\alpha) - \ell(\beta). \)

The preceding argument shows that \( j = 1 \) in both cases.

This gives a nonvanishing criterion in terms of length.

**Lemma 9.4.5** Assume \( \kappa \in \tau(\gamma), \kappa \not\in \tau(\lambda), \) and \( \gamma \kappa \not\preceq \lambda. \) Then one of the following conditions holds:

(b) \( \ell(\gamma) < \ell(\lambda) \)

(c) \( \ell(\gamma) = \ell(\lambda) + \ell(\kappa) - 2, \) \( \text{def}_\gamma(\kappa) = 1 \)

(c') \( \ell(\gamma) = \ell(\lambda) + \ell(\kappa) - 2, \) \( \text{def}_\lambda(\kappa) = 1 \)
Remark 9.4.6 Explicitly, Cases (c) and (c') of the Lemma are:
1. \( \ell(\kappa) = 2, t_\gamma(\kappa) = 2Cr \text{ or } t_\lambda(\kappa) = 2Ci, \) and \( \ell(\gamma) = \ell(\lambda); \)
2. \( \ell(\kappa) = 3, t_\gamma(\kappa) = 3Cr, 3r \text{ or } t_\lambda(\kappa) = 3Ci, 3i, \) and \( \ell(\gamma) = \ell(\lambda) + 1. \)

Cases (a,b,c,c') in Lemmas 9.4.1 and 9.4.5 don’t precisely line up. There can be \( \gamma \) in case (c) or (c') of Lemma 9.4.1, so \( \gamma \not< \lambda, \) but \( \ell(\gamma) < \ell(\lambda), \) putting it in case (b) of Lemma 9.4.5.

10 Computing \( \hat{\mathcal{P}}^\sigma(\gamma, \mu) \)

10.1 Easy recurrence relations
Recall (9.3.4)(a) and (b)
\[
\hat{C}_\mu = \sum_{\gamma | \kappa \in \tau(\gamma)} \hat{\mathcal{P}}^\sigma(\gamma, \mu) \hat{a}_\gamma + \sum_{\gamma | \kappa \not\in \tau(\gamma)} \hat{\mathcal{P}}^\sigma(\gamma, \mu) \hat{a}_\gamma.
\]

and
\[
\hat{C}_\mu = \sum_{\gamma | \kappa \in \tau(\gamma)} \hat{\mathcal{P}}^\sigma(\gamma, \mu) \hat{a}_\gamma + \sum_{\gamma | \kappa \not\in \tau(\gamma)} \sum_{\gamma' | \gamma \not\rightarrow \gamma'} v^{\ell(\gamma') - \ell(\gamma)} \hat{\mathcal{P}}^\sigma(\gamma, \mu) \epsilon(\gamma, \gamma') \hat{a}_\gamma'.
\]

Equate the coefficients of \( \hat{a}_\gamma (\kappa \not\in \tau(\gamma)), \) and use (9.1.5) to conclude the “easy” relations:

Lemma 10.1.1 Suppose \( \kappa \not\in \tau(\gamma), \kappa \in \tau(\mu). \) Then
\[
(10.1.2) \quad \hat{\mathcal{P}}^\sigma(\gamma, \mu) = v^{-\ell(\kappa) + \text{def}_*(\kappa)} \sum_{\gamma' | \gamma \not\rightarrow \gamma'} \epsilon(\gamma', \gamma) \hat{\mathcal{P}}^\sigma(\gamma', \mu).
\]

We will compute \( \hat{\mathcal{P}}^\sigma(\gamma, \mu) \) (and \( m_\alpha(\gamma, \mu) \)) by induction on length as follows. To compute \( \hat{\mathcal{P}}^\sigma(\gamma, \mu) \) we may assume we know:
\[
(10.1.3) \quad \hat{\mathcal{P}}^\sigma(\gamma, \mu) \text{ if } \ell(\mu') < \ell(\mu) \]
\[
\hat{\mathcal{P}}^\sigma(\gamma', \mu) \text{ if } \ell(\gamma') > \ell(\gamma).
\]

We know the right hand side of (10.1.2) by the inductive assumption. If there is only one term on the right hand side \( \hat{\mathcal{P}}^\sigma(\gamma, \mu) \) is equal (up to a power of \( v \)) to a polynomial we have already computed. Otherwise \( \hat{\mathcal{P}}^\sigma(\gamma, \mu) \) is the sum of two terms.
Definition 10.1.4 Suppose $\gamma < \mu$. Then $(\gamma, \mu)$ is:

- **extremal** if $\kappa \in \tau(\mu) \Rightarrow \kappa \in \tau(\gamma)$
- **primitive** if $\kappa \in \tau(\mu) \Rightarrow \kappa \in \tau(\gamma)$ or $\kappa \notin \tau(\gamma)$, $|\{\gamma' \mid \gamma' \xrightarrow{\kappa} \gamma\}| = 2$.

I find the converses more natural:

Definition 10.1.5 Suppose $\gamma < \mu$. Then $(\gamma, \mu)$ is:

- **non-extremal** if there exists $\kappa \in \tau(\mu), \kappa \notin \tau(\gamma)$.
- **non-primitive** if there exists $\kappa \in \tau(\mu), \kappa \notin \tau(\gamma)$ and $|\{\gamma' \mid \gamma' \xrightarrow{\kappa} \gamma\}| < 2$.

Explicitly $(\gamma, \mu)$ is:

- **non-primitive** if there exists $\kappa \in \tau(\mu), \kappa \notin \tau(\gamma)$ and $t_\gamma(\kappa) \neq 112f, 2i12$.

Thus extremal $\subset$ primitive and non-primitive $\subset$ non-extremal.

If $(\gamma, \mu)$ is non-primitive, (10.1.2) writes $P(\gamma, \mu) = v^k P(\gamma', \mu)$.

### 10.2 Direct Recursion Relations

Recall Proposition 9.3.13:

\[(10.2.1)(a) \sum_{\delta \mid \kappa \in \tau(\delta)} \hat{P}^\sigma(\gamma, \delta)m_\kappa(\delta, \lambda) = \text{multiplicity of } \hat{a}_\gamma \text{ in } \hat{T}_\kappa(\hat{C}_\lambda),\]

and if $\kappa \in \tau(\gamma)$ the same equality holds with $\hat{a}_\kappa$ on the right hand side.

By (9.4.4) the left hand side is:

\[(10.2.1)(b) \sum_{\delta \mid \kappa \in \tau(\delta)} \hat{P}^\sigma(\gamma, \delta)m_\kappa(\delta, \lambda) + \sum_{\delta \mid \kappa \in \tau(\delta)} \hat{P}^\sigma(\gamma, \delta)m_\kappa(\delta, \lambda)\]

We introduce some notation for the final sum. See [6, after Lemma 6.7].

**Definition 10.2.2** For $\kappa \notin \tau(\lambda)$ define:

\[\hat{U}_\kappa(\gamma, \lambda) = \sum_{\delta \mid \kappa \in \tau(\delta)} \hat{P}^\sigma(\gamma, \delta)m_\kappa(\delta, \lambda).\]
Lemma 10.2.3  Fix $\gamma, \lambda$, with $\kappa \notin \tau(\lambda)$. Then
\begin{equation}
\sum_{\delta | \delta \xrightarrow{\kappa} \lambda} \hat{P}^{\sigma}(\gamma, \delta)m_{\kappa}(\delta, \lambda) = [\text{coefficient of } \hat{a}_{\gamma} \text{ in } \hat{T}_{\kappa}\hat{C}_{\lambda}] - \hat{U}_{\kappa}(\gamma, \lambda).
\end{equation}

If $\kappa \in \tau(\gamma)$ we can replace the term in brackets with

\[\text{coefficient of } \hat{a}_{\gamma}^\kappa \text{ in } \hat{T}_{\kappa}\hat{C}_{\lambda}\]

We first dispense with a case which won’t be used until Section 11. In the setting of the Lemma, if $t_{\lambda}(\kappa) = 112s, 1ic, 2ic, 3ic$ then, even though $\kappa \in \tau(\lambda)$, there are no $\delta$ occurring in the sum on the left hand side.

Lemma 10.2.5  Assume $t_{\lambda}(\kappa) = 112s, 1rn, 2rn, 3rn$. Equivalently $\kappa \notin \tau(\lambda)$ but there does not exist $\delta$ satisfying $\delta \xrightarrow{\kappa} \lambda$. Then

\[\text{coefficient of } \hat{a}_{\gamma}^\kappa \text{ in } \hat{T}_{\kappa}\hat{C}_{\lambda} = \hat{U}_{\kappa}(\gamma, \lambda)\]

We turn now to the main case, in which the left hand side of (10.2.4) is nonempty. In this case this sum has 1 or 2 terms. We’re mainly interested when it has 1 term, in which case it gives a formula for $\hat{P}^{\sigma}(\gamma, \mu)$. For this reason it is convenient to change variables. This gives the main result.

Proposition 10.2.6  Suppose $\kappa \in \tau(\gamma)$, $\kappa \in \tau(\mu)$, and $t_{\mu}(\kappa) \neq 1r1s, 1ic, 2ic, 3ic$. Choose $\lambda$ satisfying $\mu \xrightarrow{\kappa} \lambda$. Then
\begin{equation}
\sum_{\mu' | \mu' \xrightarrow{\kappa} \lambda} \hat{P}^{\sigma}(\gamma, \mu')m_{\kappa}(\mu', \lambda) = [\text{coefficient of } \hat{a}_{\gamma}^\kappa \text{ in } \hat{T}_{\kappa}\hat{C}_{\lambda}] - \hat{U}_{\kappa}(\gamma, \lambda)
\end{equation}

The sum on the left hand side is over $\{\mu, w_{\kappa} \times \mu\}$ if $t_{\mu}(\kappa) = 1r2, 2r22, 2r21$, and just $\mu$ otherwise.

This is our main recursion relation. We analyse its effectiveness in the next section.

Since we are assuming $\kappa \in \tau(\gamma)$ we have replaced $\hat{a}_{\gamma}$ with $\hat{a}_{\gamma}^\kappa$ as in Lemma 10.2.3.

Note that the term $m_{\kappa}(\mu, \lambda)$ is computed by the first case of Theorem 9.3.10, i.e. $\mu \xrightarrow{\kappa} \lambda$, and either equals $\epsilon(\mu, \lambda) = \pm 1$ (see Definition 5.2.1) or $(v + v^{-1})$. 37
Consulting Theorem 9.3.10 we make the left side of the Proposition explicit.

Table 10.2.8

| $t_\mu(\kappa)$ | LHS of (10.2.7) |
|-----------------|-----------------|
| 1C-,1r1f,2C-,2r11,3C- | $\hat{P}(\gamma, \mu)$ |
| 1r2,2r22 | $\hat{P}(\gamma, \mu) + \hat{P}(\gamma, w_\kappa \times \mu)$ |
| 2r21 | $\sum_{\mu' \mu \rightarrow \mu'} \epsilon(\mu, \mu') \hat{P}(\gamma, \mu')$ |
| 2Cr,3Cr,3r | $(v + v^{-1}) \hat{P}(\gamma, \mu)$ |

In the 2r21 case, recall $\mu$ comes in an ordered pair $(\mu_1, \mu_2)$. Then $\mu \rightarrow \lambda$ says that $\lambda$ is one member of the ordered pair $\mu_\kappa = (\lambda_1, \lambda_2)$.

We turn to the right hand side of (10.2.7). The first term is given by Lemma 9.2.1. Let $k = \ell(\kappa)$. This is copied from Table 9.2.4, which is a condensed version of Table 9.2.3.

Table 10.2.9

| $t_\gamma(\kappa)$ | first term on RHS of (10.2.7) |
|-----------------|-----------------|
| 1C-,2C-,3C- | $v^k \hat{P}(\gamma, \lambda) + \hat{P}(w_\kappa \times \gamma, \lambda)$ |
| 1ic,2ic,3ic | $(v^k + v^{-k}) \hat{P}(\gamma, \lambda)$ |
| 2Cr,3Cr,3r | $v(v^{k-1} - v^{-k+1}) \hat{P}(\gamma, \lambda) + (v + v^{-1}) \hat{P}(\gamma_\kappa, \lambda)$ |
| 1r1f,2r11 | $(v^k - v^{-k}) \hat{P}(\gamma, \lambda) + \hat{P}(\gamma_1^\kappa, \lambda) + \hat{P}(\gamma_2^\kappa, \lambda)$ |
| 1r1s | $(v - v^{-1}) \hat{P}(\gamma, \lambda)$ |
| 1r2,2r22 | $v^k \hat{P}(\gamma, \lambda) - v^{-k} \hat{P}(w_\kappa \times \gamma, \lambda) + \hat{P}(\gamma_\kappa, \lambda)$ |
| 2r21 | $(v^2 - v^{-2}) \hat{P}(\gamma, \lambda) + \sum_{\gamma' \gamma \rightarrow \gamma'} \epsilon(\gamma, \gamma') \hat{P}(\gamma', \lambda)$ |
10.3 Analysis of the Recursion

Fix $\gamma, \mu$ with $\kappa \in \tau(\gamma)$ and $t_\mu(\kappa) = 1C-, 1r1f, 2C-, 2r11, 3C-, 2Cr, 3Cr, 3r$. Choose $\lambda$ with $\mu \nrightarrow \lambda$. Then Proposition 10.2.6 says

\[(10.3.1) \hat{P}^\sigma(\gamma, \mu) = \text{coefficient of } \hat{a}^\kappa_\gamma \text{ in } \hat{T}_\kappa \hat{C}_\lambda - \hat{U}_\kappa(\gamma, \lambda).\]

Recall $\hat{U}_\kappa(\gamma, \lambda)$ is given in Definition 10.2.2. We analyse how this fits in our recursive scheme.

Lemma 10.3.2 Fix $\kappa$. For all $\gamma, \lambda$ with $\kappa \in \tau(\gamma), \kappa \n.coord {\tau(\lambda)}$: 

\[
\hat{U}_\kappa(\gamma, \lambda) = \sum_{\delta | \kappa \in \tau(\delta), \ell(\gamma) \leq \ell(\delta) < \ell(\lambda)} \hat{P}^\sigma(\gamma, \delta) m_\kappa(\delta, \lambda) + \sum_{\delta | \ell(\kappa) = 2Cr, 3Cr, 3r, \ell(\delta) = \ell(\lambda) + \ell(\kappa) - 2} \hat{P}^\sigma(\gamma, \delta) \hat{P}^\sigma_{-1}(\delta_\kappa, \lambda) + \sum_{\delta | \delta \kappa \in \tau(\delta), \ell(\delta) = \ell(\lambda) + \ell(\kappa) - 2} \hat{P}^\sigma(\gamma, \delta) \hat{P}^\sigma_{-1}(\delta, \lambda^\kappa). \]

Proof. Recall $\hat{U}_\kappa(\gamma, \lambda)$ is defined by the sum over $\delta \n.coord {\kappa} \lambda$. Of course $\hat{P}^\sigma(\gamma, \delta) \neq 0$ implies $\gamma \leq \delta$. Therefore, by the definition of $\n.coord {\kappa}$, $\hat{U}_\kappa(\gamma, \lambda)$ is the sum over $\delta$ satisfying $\kappa \in \tau(\delta)$, and one of:

1. $\gamma \leq \delta < \lambda$
2. $\text{def}_\delta(\kappa) = 1, \delta \n.coord {\kappa} \lambda, \delta_\kappa \leq \lambda$
3. $\text{def}_\lambda(\kappa) = 1, \delta \n.coord {\kappa} \lambda, \delta < \lambda^\kappa$

Every such term appears in Lemma 10.3.2. Conversely any nonzero term in Lemma 10.3.2 appears in one of these cases. \hfill \Box

Recall to compute $P(\gamma, \mu)$ we may assume we know:

\[(10.3.3)(a) \quad \hat{P}^\sigma(\ast, \mu') \text{ if } \ell(\mu') < \ell(\mu)\]
\[(10.3.3)(b) \quad \hat{P}^\sigma(\gamma', \mu) \text{ if } \ell(\gamma') > \ell(\gamma).\]
Lemma 10.3.4 Suppose $\kappa \in \tau(\gamma), \tau(\mu)$, and $\mu \rightarrow^\kappa \lambda$, and assume we know the terms (10.3.3)(a) and (b). Then

$$\hat{U}_\kappa(\gamma, \lambda) = (*) + \text{def}_\lambda(\kappa)(-1)^{\ell(\gamma)<\ell(\lambda)}\hat{\mu}_{-1}^\sigma(\gamma, \mu)$$

where $(*)$ is known.

To be explicit: the extra term is 0 unless $\text{def}_\lambda(\kappa) = 1$ in which case it equals:

\[
(10.3.5)(a) \begin{cases} 
\hat{\mu}_{-1}^\sigma(\gamma, \mu) & \ell(\gamma) = \ell(\mu) - 1 \\
-\hat{\mu}_{-1}^\sigma(\gamma, \mu) & \ell(\gamma) < \ell(\mu) - 1 
\end{cases}
\]

Equivalently if $\text{def}_\lambda(\kappa) = 1$ it is equal to

\[
(10.3.5)(b) \begin{cases} 
\hat{\mu}_{-1}^\sigma(\gamma, \mu) & \ell(\gamma) = \ell(\mu) - 1 \\
-\hat{\mu}_{-1}^\sigma(\gamma, \mu) & \ell(\gamma) = \ell(\mu) - 3, 5, 7 \ldots 
\end{cases}
\]

and is 0 otherwise.

Note that

$$\text{def}_\lambda(\kappa) = 1 \Leftrightarrow \text{def}_\mu(\kappa) = 1 \Leftrightarrow t_\lambda(\kappa) = 2C_i, 3C_i, 3i \Leftrightarrow t_\mu(\kappa) = 2C_r, 3C_r, 3r.$$

**Proof.** According to Lemma 10.3.2 $\hat{U}_\kappa(\gamma, \lambda)$ is equal to:

\[
(10.3.6)(a) \sum_{\delta | \kappa \in \tau(\delta), \ell(\gamma)<\ell(\delta)<\ell(\lambda)} \hat{P}_\sigma(\gamma, \delta)m_\kappa(\delta, \lambda) + \\
(10.3.6)(b) \sum_{\delta | t_\delta(\kappa)=2C_r, 3C_r, 3r, \ell(\delta)=\ell(\mu)-2+\text{def}_\lambda(\kappa)} \hat{P}_\sigma(\gamma, \delta)\hat{\mu}_{-1}^\sigma(\delta, \lambda) + \\
(10.3.6)(c) \sum_{\delta | t_\delta(\kappa)=2C_r, 3C_r, 3r, \ell(\delta)=\ell(\mu)-2+\text{def}_\lambda(\kappa)} \hat{P}_\sigma(\gamma, \delta)\hat{\mu}_{-1}^\sigma(\delta, \lambda) + \\
(10.3.6)(d) \text{def}_\lambda(\kappa) \sum_{\delta | t_\delta(\kappa)=2C_r, 3C_r, 3r, \ell(\delta)=\ell(\mu)-2+\text{def}_\lambda(\kappa)} \hat{P}_\sigma(\gamma, \delta)\hat{\mu}_{-1}^\sigma(\delta, \lambda)
\]

For (a) we have separated out the term $\ell(\gamma) = \ell(\delta)$ from the first sum in Lemma 10.3.2. Since then $\hat{P}_\sigma(\gamma, \delta) = 0$ unless $\gamma = \delta$ this gives (a) (the
bracket is due to the summation condition). Also, we have replaced $\ell(\lambda) + \ell(\kappa)$ with $\ell(\mu) + \text{def}_\lambda(\kappa)$ (Definition 9.1.4) and have used $\text{def}_\lambda(\kappa) = 1$ in the final summand.

We analyse each of the terms appearing, starting with the $\hat{P}^\sigma$ and $\mu_{-1}$ terms.

- In (b,c,d) we know $\hat{P}^\sigma(\gamma, \delta)$ since $\ell(\delta) < \ell(\mu)$ (by (10.3.3)(a))
- In (c) we know $\hat{\mu}_{-1}^\sigma(\delta, \lambda)$ since $\ell(\lambda) < \ell(\mu)$ ((10.3.3)(a))
- In (d) we know $\hat{P}^\sigma(\gamma, \delta)$ if $\ell(\delta) > \ell(\gamma)$ (by (10.3.3)(b))
- In (d) suppose $\ell(\delta) < \ell(\gamma)$. Then $\hat{P}^\sigma(\gamma, \delta) = 0$.
- In (d) suppose $\ell(\delta) = \ell(\gamma)$. Then $\hat{P}^\sigma(\gamma, \delta) = 0$ unless $\gamma = \delta$.

So the only additional contribution to (d) is

\[(10.3.7) \quad \text{def}_\lambda(\kappa)[\ell(\gamma) = \ell(\mu) - 1] \hat{\mu}_{-1}^\sigma(\gamma, \mu)\]

Now consider $m_\kappa(\delta, \lambda)$ in (b). By Theorem 9.3.10 this requires various $\mu_{-k}(\ast, \lambda)$ ($k = 1, 2$), and also $\mu_{-k}(\ast, \lambda')$ with $\ell(\lambda') < \ell(\lambda)$, all of which we know. The only potential problems are the terms $\hat{\mu}_{-1}^\sigma(\delta, \lambda^\kappa)$ in Theorem 9.3.10 (3)(a) and (4)(a). But $\lambda^\kappa = \mu$, and $\ell(\delta) > \ell(\gamma)$ (from the summation condition in (10.3.6)(b)), so we know these terms by (10.3.3)(b).

The only remaining term is $[\ell(\gamma) < \ell(\lambda)] m_\kappa(\gamma, \lambda)$ in (10.3.6)(a). This is the same as the previous case $m_\kappa(\delta, \lambda)$, except for the very last case: there may be a term $\mu_{-1}(\gamma, \mu)$ from Theorem 9.3.10 (3)(a) or (4)(a). These arise when $\text{def}_\lambda(\kappa) = 1$ and also certain length conditions hold. There are two cases.

- $\ell(\kappa) = 2$. By (9.3.10)(3)(b) we get an extra term $-\hat{\mu}_{-1}^\sigma(\gamma, \mu)$ if $\ell(\gamma) = \ell(\lambda)$ (mod 2) and $\ell(\gamma) < \ell(\lambda)$. On the other hand $\ell(\kappa) = 2$ implies $\ell(\mu) - \ell(\lambda) = 1$, so (10.3.7) is nonzero only if $\ell(\gamma) = \ell(\lambda)$. So the combined extra contribution is ($\text{def}_\lambda(\kappa)$ times)

\[
\begin{cases}
\hat{\mu}_{-1}^\sigma(\gamma, \mu) & \ell(\gamma) = \ell(\lambda) \\
-\hat{\mu}_{-1}^\sigma(\gamma, \mu) & \ell(\gamma) = \ell(\lambda) - 2k \quad (k = 1, 2, 3, \ldots)
\end{cases}
\]
If \( \ell(\lambda) - \ell(\gamma) \) is odd then \( \ell(\mu) - \ell(\lambda) \) is even, so \( \widehat{\mu}^{-1}(\gamma, \mu) = 0 \). Therefore this can be written

\[
(10.3.8a) \quad \begin{cases} 
\widehat{\mu}^{-1}(\gamma, \mu) & \ell(\gamma) = \ell(\lambda) \\
-\widehat{\mu}^{-1}(\gamma, \mu) & \ell(\gamma) < \ell(\lambda).
\end{cases}
\]

\( \ell(\kappa) = 3 \). By \( 9.3.10(4)(b) \) we get an extra term if \( \ell(\lambda) - \ell(\gamma) = 1 \text{(mod 2)} \) and \( \ell(\gamma) < \ell(\lambda) \). On the other hand \( \ell(\kappa) = 2 \) implies \( \ell(\mu) - 1 = \ell(\lambda) + 1 \), so \( (10.3.7) \) is nonzero only if \( \ell(\gamma) = \ell(\lambda) + 1 \). So the extra contribution is \( (\text{def}_{\lambda}(\kappa) \text{ times}) \)

\[
(10.3.8)(b) \quad \begin{cases} 
\widehat{\mu}^{-1}(\gamma, \mu) & \ell(\gamma) = \ell(\lambda) + 1 \\
-\widehat{\mu}^{-1}(\gamma, \mu) & \ell(\gamma) = \ell(\lambda) + 1 - 2k \quad (k = 1, 2, \ldots)
\end{cases}
\]

If \( \ell(\lambda) - \ell(\gamma) \) is even then so is \( \ell(\mu) - \ell(\lambda) \) so \( \widehat{\mu}^{-1}(\gamma, \mu) = 0 \). Therefore this term is

\[
(10.3.8)(c) \quad \begin{cases} 
\widehat{\mu}^{-1}(\gamma, \mu) & \ell(\gamma) = \ell(\lambda) + 1 \\
-\widehat{\mu}^{-1}(\gamma, \mu) & \ell(\gamma) < \ell(\lambda)
\end{cases}
\]

We can combine the two cases:

\[
\text{def}_{\lambda}(\kappa) (-1)^{\ell(\gamma) < \ell(\lambda)} \widehat{\mu}^{-1}(\gamma, \mu)
\]

\[\square\]

**Proposition 10.3.9 (Direct Recursion)** Suppose \( \kappa \in \tau(\gamma) \), \( \kappa \in \tau(\mu) \), and \( t(\mu) = 1C-, 1r1f, 2C-, 2r11, 3C-, 2Cr, 3Cr, 3r \).

(a) We know all terms necessary to compute \( \widehat{P}^\sigma(\gamma, \mu) \) unless \( \text{def}_{\mu}(\kappa) = 1 \) and \( \ell(\mu) - \ell(\gamma) \) is odd (and positive).

(b) Suppose \( \text{def}_{\mu}(\kappa) = 1 \), i.e. \( t(\kappa) = 2Cr, 3Cr, 3r \), and \( \ell(\mu) - \ell(\gamma) \) is odd. Then

\[
(v + v^{-1}) \widehat{P}^\sigma(\gamma, \mu) = c \widehat{\mu}^{-1}(\gamma, \mu) + (*)
\]

where all terms in \( (*) \), and \( c = 1, 2 \), are known. This can be solved for \( \widehat{P}^\sigma(\gamma, \mu) \).
Proof. This follows from the preceding Lemma, with the final case provided by the next Lemma. □

Lemma 10.3.10 Suppose \( f(v) \in v^{-1}Z[v^{-2}] \), and we know all terms of \( f(v)(v + v^{-1}) \) except the constant term. Then we can compute \( f(v) \).

Proof. Write \( f(v) = c_nv^{-1} + c_{n-1}v^{-3} + \cdots + c_0v^{-(2n+1)} \), and

\[
(v + v^{-1})f(v) = b_{n+1} + b_nv^{-2} + v_{n-1}v^{-4} + \cdots + b_0v^{-2n-2}
\]

and we know \( b_0, \ldots, b_n \). Starting at \( v^{-2n-2} \) we see \( c_0 = b_0, (c_1 + c_0) = b_1, (c_2 + c_1) = b_2, \ldots \). This is easy to solve for \( c_i \): \( c_0 = b_0, c_1 = b_0 - b_1, c_2 = b_2 - b_1 + b_0, \ldots \). That is:

\[
c_k = (-1)^k \sum_{j=0}^k (-1)^j b_j \quad (0 \leq k \leq n).
\]

□

Remark 10.3.11 It might be easier to think about this if we multiply by \( v^{2n+2} \) and replace \( v^2 \) with \( q \). This gives

\[
(1 + q)(c_0 + c_1q^2 + \cdots + c_nq^n) = b_0 + b_1q + \cdots + b_{n+1}q^{n+1}
\]

If we know all terms on the right hand side except for \( b_{n+1} \), we can find all \( c_i \).

Remark 10.3.12 This computes \( \hat{P}^\sigma(\gamma, \mu) \) \( (\kappa \in \tau(\gamma), \tau(\mu)) \) unless:

1. there is no \( \lambda \) with \( \mu \xrightarrow{\kappa} \lambda \): 1r1s,1ic,2ic,3ic
2. \( \mu \xrightarrow{\kappa} \lambda \) for some \( \lambda \), but \( |\{\mu' | \mu' \xrightarrow{\kappa} \lambda\}| = 2 \): 1r2,2r22,2r21.

See Section [12]

11 New Recursion Relations

We return now to the setting of Lemma 10.2.3 and the case we skipped earlier.
Lemma 11.1 Assume $t_\chi(\kappa) = 1i2s,1rn,2rn,3rn$. Equivalently $\kappa \not\in \tau(\lambda)$ but there does not exist $\lambda'$ satisfying $\lambda' \xrightarrow{\kappa} \lambda$. Then for any $\gamma$:

$$\sum_\mu \widehat{P}^\sigma(\mu, \lambda)\text{(multiplicity of }\hat{a}_\gamma\text{ in }\widehat{T}_\kappa(\hat{a}_\mu)) = \widehat{U}_\kappa(\gamma, \lambda) \tag{11.2}$$

This is an immediate consequence of Lemma 10.2.5 which says that under this assumption the coefficient of $\hat{a}_\gamma$ in $\widehat{T}_\kappa \widehat{C}_\lambda = \widehat{U}_\kappa(\gamma, \lambda)$.

The left hand side has at most 3 terms, which can be read off from the tables in Section 8.1 or Table 8.2.1. One of the terms is a polynomial times $\widehat{P}^\sigma(\gamma, \lambda)$, and we wish to solve for $\widehat{P}^\sigma(\gamma, \lambda)$.

If $\kappa \not\in \tau(\gamma)$ then the only possibilities for $\mu$ are $\mu = \gamma, \mu = w_\kappa \times \gamma$, or $\mu \xrightarrow{\kappa} \gamma$. These are all well suited to using induction to computing $\widehat{P}^\sigma(\gamma, \lambda)$, unless $t_\gamma(\kappa) = 1i2s,1rn,2rn,3rn$, in which case the coefficient of $\widehat{P}^\sigma(\gamma, \lambda)$ is 0.

If $\kappa \in \tau(\gamma)$ then $\mu = \gamma, \mu = w_\kappa \times \gamma$ or $\gamma \xrightarrow{\kappa} \mu$. If $\gamma \xrightarrow{\kappa} \mu$ then $\gamma > \mu$ and this is not well suited to our inductive hypothesis. So this case is only effective if there is no such $\mu$, i.e. $t_\gamma(\kappa) = 1ris,1ic,2ic,3ic$.

Here are the resulting formulas.

Table 11.3
Formula (11.2): $\kappa \notin \tau(\gamma)$

| $t_\gamma(\kappa)$ | LHS                                                                 |
|---------------------|----------------------------------------------------------------------|
| 1C+                 | $v^{-1} \tilde{P}\sigma(\gamma, \lambda) + \tilde{P}\sigma(w_\kappa \times \gamma, \lambda)$ |
| 1i1                 | $v^{-1} \tilde{P}\sigma(\gamma, \lambda) + v^{-1} \tilde{P}\sigma(w_\kappa \times \gamma, \lambda) + (1 - v^{-2}) \tilde{P}\sigma(\gamma^\kappa, \lambda)$ |
| 1i2f                | $2v^{-1} \tilde{P}\sigma(\gamma, \lambda) + v^{-1}(v - v^{-1})(\tilde{P}\sigma(\gamma^\kappa_1, \lambda) + \tilde{P}\sigma(\gamma^\kappa_2, \lambda))$ |
| 1i2s                | 0                                                                   |
| 1rn                 | 0                                                                   |
| 2C+                 | $v^{-2} \tilde{P}\sigma(\gamma, \lambda) + \tilde{P}\sigma(w_\kappa \times \gamma, \lambda)$ |
| 2Ci                 | $v^{-1}(v + v^{-1}) \tilde{P}\sigma(\gamma, \lambda) + (v - v^{-1}) \tilde{P}\sigma(\gamma^\kappa, \lambda)$ |
| 2i11                | $v^{-2}(\tilde{P}\sigma(\gamma, \lambda) + \tilde{P}\sigma(w_\kappa \times \gamma, \lambda)) + (1 - v^{-4}) \tilde{P}\sigma(\gamma^\kappa, \lambda)$ |
| 2i12                | $2v^{-2} \tilde{P}\sigma(\gamma, \lambda) + (1 - v^{-4}) \sum_{\gamma'|\gamma' \subseteq \gamma} \epsilon(\gamma', \gamma) \tilde{P}\sigma(\gamma', \lambda)$ |
| 2i22                | $2v^{-2} \tilde{P}\sigma(\gamma, \lambda) + (1 - v^{-4})(\tilde{P}\sigma(\gamma^\kappa_1, \lambda) + \tilde{P}\sigma(\gamma^\kappa_2, \lambda))$ |
| 2rn                 | 0                                                                   |
| 3C+                 | $v^{-3} \tilde{P}\sigma(\gamma, \lambda) + \tilde{P}\sigma(w_\kappa \times \gamma, \lambda)$ |
| 3Ci,3i              | $v^{-2}(v + v^{-1}) \tilde{P}\sigma(\gamma, \lambda) + (v^2 - v^{-2})v^{-1} \tilde{P}\sigma(\gamma^\kappa, \lambda)$ |
| 3rn                 | 0                                                                   |

Table 11.4

Formula (11.2): $\kappa \in \tau(\gamma)$

| $t_\gamma(\kappa)$ | LHS                                                                 |
|---------------------|----------------------------------------------------------------------|
| 1r1s                | $(v + v^{-1}) \tilde{P}\sigma(\gamma, \lambda)$                   |
| 1ic                 | $(v + v^{-1}) \tilde{P}\sigma(\gamma, \lambda)$                   |
| 2ic                 | $(v^2 + v^{-2}) \tilde{P}\sigma(\gamma, \lambda)$                  |
| 3ic                 | $(v^3 + v^{-3}) \tilde{P}\sigma(\gamma, \lambda)$                  |

Solve these for $\tilde{P}\sigma(\gamma, \lambda)$.

Lemma 11.5 Assume $t_\lambda(\kappa) = 1i2s, 1rn, 2rn, 3rn$. Then:
Table 11.6

| $t_\gamma(\kappa)$ | Formula for $\hat{P}^\sigma(\gamma, \lambda)$ |
|---------------------|-----------------------------------------------|
| $\kappa \not\in \tau(\gamma)$ | none |
| 1C+                 | $\hat{P}^\sigma(\gamma, \lambda) = -v\hat{P}^\sigma(w_\kappa \times \gamma, \lambda) + v\hat{U}_\kappa(\gamma, \lambda)$ |
| 1i1                 | $\hat{P}^\sigma(\gamma, \lambda) + \hat{P}^\sigma(w_\kappa \times \gamma, \lambda) = -(v - v^{-1})\hat{P}^\sigma(\gamma^\kappa, \lambda) + v\hat{U}_\kappa(\gamma, \lambda)$ |
| 1i2f                | $\hat{P}^\sigma(\gamma, \lambda) = -2(v - v^{-1})(\hat{P}^\sigma(\gamma^\kappa, \lambda) + \hat{P}^\sigma(\gamma^\kappa_2, \lambda)) + 2v\hat{U}_\kappa(\gamma, \lambda)$ |
| 1i2s                | none |
| 1rn                 | none |
| 2C+                 | $\hat{P}^\sigma(\gamma, \lambda) = -v^2\hat{P}^\sigma(w_\kappa \times \gamma, \lambda) + v^2\hat{U}_\kappa(\gamma, \lambda)$ |
| 2Ci                 | $(v + v^{-1})\hat{P}^\sigma(\gamma, \lambda) = -v(v - v^{-1})\hat{P}^\sigma(\gamma^\kappa, \lambda) + v\hat{U}_\kappa(\gamma, \lambda)$ |
| 2i11                | $\hat{P}^\sigma(\gamma, \lambda) + \hat{P}^\sigma(w_\kappa \times \gamma, \lambda) = -(v^2 - v^{-2})\hat{P}^\sigma(\gamma^\kappa, \lambda) + v^2\hat{U}_\kappa(\gamma, \lambda)$ |
| 2i12                | $2v^{-2}\hat{P}^\sigma(\gamma, \lambda) = -\frac{1}{2}(v^2 - v^{-2}) \sum_{\gamma' \neq \gamma} \epsilon(\gamma', \gamma) \hat{P}^\sigma(\gamma', \lambda) + \frac{1}{2}v^2\hat{U}_\kappa(\gamma, \lambda)$ |
| 2i22                | $\hat{P}^\sigma(\gamma, \lambda) = -\frac{1}{2}(v^2 - v^{-2})(\hat{P}^\sigma(\gamma^\kappa_1, \lambda) + \hat{P}^\sigma(\gamma^\kappa_2, \lambda)) + \frac{1}{2}v^2\hat{U}_\kappa(\gamma, \delta)$ |
| 3rn                 | none |
| 3C+                 | $\hat{P}^\sigma(\gamma, \lambda) = -v^3\hat{P}^\sigma(w_\kappa \times \gamma, \lambda) + v^3\hat{U}_\kappa(\gamma, \delta)$ |
| 3Ci, 3i             | $(v + v^{-1})\hat{P}^\sigma(\gamma, \lambda) = -v(v^2 - v^{-2})\hat{P}^\sigma(\gamma^\kappa, \lambda) + v^2\hat{U}_\kappa(\gamma, \lambda)$ |
| 3rn                 | none |
| $\kappa \in \tau(\gamma)$ | none |
| 1r1s                | $(v + v^{-1})\hat{P}^\sigma(\gamma, \lambda) = \hat{U}_\kappa(\gamma, \lambda)$ |
| 1ic                 | $(v + v^{-1})\hat{P}^\sigma(\gamma, \lambda) = \hat{U}_\kappa(\gamma, \lambda)$ |
| 2ic                 | $(v^2 + v^{-2})\hat{P}^\sigma(\gamma, \lambda) = \hat{U}_\kappa(\gamma, \lambda)$ |
| 3ic                 | $(v^3 + v^{-3})\hat{P}^\sigma(\gamma, \lambda) = \hat{U}_\kappa(\gamma, \lambda)$ |

In every case we know all terms on the RHS by induction, with the possible exception of $\hat{U}_\kappa(\gamma, \lambda)$. By Lemma [10.3.2], since $t_\lambda(\kappa) \neq 2C_1, 3C_i, 3i$, (11.7)

\[
\hat{U}_\kappa(\gamma, \lambda) = \sum_{\delta|\kappa \in \tau(\delta)} \hat{P}^\sigma(\gamma, \delta)m_\kappa(\delta, \lambda) + \sum_{\delta|t_\delta(\kappa) = 2C_i, 3C_i, 3i} \hat{P}^\sigma(\gamma, \delta)\hat{P}^\sigma(\delta, \lambda)
\]

\[
\hat{U}_\kappa(\gamma, \lambda) = \sum_{\ell(\gamma) \leq \ell(\delta) < \ell(\lambda)} \hat{P}^\sigma(\gamma, \delta)m_\kappa(\delta, \lambda) + \sum_{\ell(\delta) = \ell(\lambda) + \ell(\kappa) - 2} \hat{P}^\sigma(\gamma, \delta)\hat{P}^\sigma(\delta, \lambda)
\]
The second sum is problematic, so we give it a name.

**Definition 11.8** Suppose $\kappa \not\in \tau(\lambda)$. Define:

\begin{equation}
\hat{U}_{\kappa}^\dagger(\gamma, \lambda) = \sum_{\substack{\delta | t_3(\kappa) = 2Cr, 3Cr, 3r \\
\ell(\delta) = \ell(\lambda) + \ell(\kappa) - 2}} \hat{P}^\sigma(\gamma, \delta) \hat{\mu}_{-1}^\sigma(\delta_\kappa, \lambda)
\end{equation}

This term doesn’t fit well in our recursion scheme, since $\ell(\delta) > \ell(\lambda)$. (In the direct recursion section this wasn’t an issue, since we were computing $\hat{P}^\sigma(\gamma, \mu)$, and $\ell(\mu) > \ell(\delta)$.) We’re hoping this term is usually 0.

**Definition 11.10** Fix $\kappa \in \overline{S}$, $\gamma, \lambda \in \mathcal{D}^\sigma$, with $\kappa \not\in \tau(\lambda)$.

We say condition A holds for $(\kappa, \gamma, \lambda)$ if

(A) $\kappa \in \tau(\delta)$, $\text{def}_\delta(\kappa) = 1$, $\ell(\delta) = \ell(\lambda) + \ell(\kappa) + 2 \Rightarrow \hat{P}^\sigma(\gamma, \delta) \hat{\mu}_{-1}^\sigma(\delta_\kappa, \lambda) = 0$

This is automatic if $\ell(\kappa) = 1$.

We say condition (†) holds for $(\kappa, \gamma, \lambda)$ if

(†) $\hat{U}_{\kappa}^\dagger(\gamma, \lambda) = 0$

Obviously (A) $\Rightarrow$ (†), although (A) is easier to check. It often holds simply because the set in (A) is empty.

So now we need to compute:

\begin{equation}
\hat{U}_{\kappa}(\gamma, \lambda) = m_\kappa(\gamma, \lambda) + \sum_{\substack{\delta | \kappa \in \tau(\delta) \\
\ell(\gamma) < \ell(\delta) < \ell(\lambda) \\ell(\delta) = \ell(\lambda) + \ell(\kappa) - 2}} \hat{P}^\sigma(\gamma, \delta) m_\kappa(\delta, \lambda) + \hat{U}_{\kappa}^\dagger(\gamma, \lambda)
\end{equation}

with the first term present only if $\kappa \in \tau(\gamma)$.

As in the discussion in Section [10.3] we know all of the terms in the first sum. There is one crucial difference with the direct recursion of Proposition [10.3.9] In that lemma we were computing $\hat{P}^\sigma(\gamma, \mu)$, and we needed $\hat{P}^\sigma(\gamma, \lambda)$, where $\mu \rightarrow \lambda$. This gave us a little extra room when applying the inductive hypothesis that we know $\hat{P}^\sigma(\ast, \mu')$ with $\mu' < \mu$.

We consider the terms appearing in the sum in (11.11).

If $\kappa \in \tau(\gamma)$ we don’t know the term $m_\kappa(\gamma, \lambda)$.

By the inductive hypothesis we know all terms $\hat{P}^\sigma(\gamma, \delta)$ occuring in (11.11). Consider $m_\kappa(\delta, \lambda)$. This requires various $\hat{P}^\sigma(\ast, \lambda')$ with $\ell(\lambda') < \ell(\lambda)$, and
\( \hat{P}^\sigma (\delta', \lambda) \) with \( \ell(\delta') \geq \ell(\delta) > \ell(\gamma) \), all of which we know. The only potential problems are the terms

\[
\hat{P}^\sigma (\gamma, \delta) \mu_{\sigma - 1}^\sigma (\delta_\kappa, \lambda) \text{ if } \text{def}_\gamma (\kappa) = 1
\]

which occur in the formulas for \( m_\kappa (\delta, \lambda) \) (the opposite case \( \hat{P}^\sigma (\delta, \lambda_\kappa) \) does not occur since \( t_\lambda (\kappa) \neq 2Ci, 3Ci, 3i \)). If \( \ell(\delta_\kappa) > \ell(\gamma) \) we know this by induction. This leaves:

\[
\hat{P}^\sigma (\gamma, \delta) \mu_{\sigma - 1}^\sigma (\delta_\kappa, \lambda) \quad \ell(\delta_\kappa) \leq \ell(\gamma) < \ell(\delta), \text{def}_\gamma (\kappa) = 1.
\]

This might be an issue. (In the Direct Recursion this was taken care of by the fact that \( \lambda < \mu \)). As in the discussion after Definition 9.4.3 this term is nonzero only if:

\[
\ell(\delta) = \ell(\gamma) + \ell(\kappa) - 2, \quad \ell(\gamma_\kappa) = \ell(\delta) - 1
\]

**Definition 11.12** Fix \( \kappa \in S, \gamma, \lambda \in D^\sigma \), with \( \kappa \notin \tau(\lambda) \). We say condition (B) holds for \((\kappa, \gamma, \lambda)\) if

(B) \( \kappa \in \tau(\delta), \text{def}_\delta (\kappa) = 1, \ell(\delta) = \ell(\gamma) + \ell(\kappa) + 2 \Rightarrow \hat{P}^\sigma (\gamma, \delta) \mu_{\sigma - 1}^\sigma (\delta_\kappa, \lambda) = 0 \)

This is automatic if \( \ell(\kappa) = 1 \).

Here is the conclusion.
Proposition 11.13  Fix $\kappa \in S, \gamma, \lambda \in D^\sigma$, satisfying:

1. $t_\gamma(\kappa) = 1C^+, 1i2f, 2C^+, 2Ci, 2i12, 2i22, 3C^+, 3Ci, 3i, 1r1s, 1ic, 2ic, 3ic$.

2. $t_\lambda(\kappa) = 1i2s, 1rn, 2rn, 3rn$.

Assume conditions (A) and (B) hold. Then the formulas of Table 11.6 give an effective recursion relation for $\hat{P}(\gamma, \lambda)$. It is sufficient to assume Conditions $(†)$ and (B).

If $t_\gamma(\kappa) = 1i1, 2i11$ the same result holds, except that we get a formula for $\hat{P}(\gamma, \lambda) + \hat{P}(w_\kappa \times \gamma, \lambda)$.

Remark 11.14

1. We allow $\lambda$ if $\kappa \notin \tau(\lambda)$ but there is no $\lambda'$ with $\lambda' \rightarrow_1 \lambda$ (see Lemma 11.1): $1i2s, 1rn, 2rn, 3rn$. This gives the $\lambda$ of the Proposition.

2. If $\kappa \notin \tau(\gamma)$ we exclude $\gamma$ if the formula has two terms on the LHS: $1i1, 2i11$.

3. If $\kappa \notin \tau(\gamma)$ we exclude $\gamma$ if there is no $\gamma'$ with $\gamma' \rightarrow_1 \gamma$: type $1i2s, 1rn, 2rn, 3rn$. Together (2) and (3) leave: $1C^+, 1i2f, 2C^+, 2Ci, 2i12, 2i22, 3C^+, 3Ci, 3i$.

4. If $\kappa \in \tau(\gamma)$ we include $\gamma$ if there is no $\gamma'$ with $\gamma \rightarrow_1 \gamma'$: $1r1s, 1ic, 2ic, 3ic$. (2-4) give the $\gamma$ of the Proposition.

If $\kappa \notin \tau(\gamma)$ the term $m_\kappa(\gamma, \lambda)$ does not appear in Lemma 10.3.2 and the Lemma is evident from the preceding discussion. If $\kappa \in \tau(\gamma)$ we need a generalization of Lemma 10.3.10. Here are the cases.

In the column $m_\kappa(\gamma, \lambda)$ $\alpha$ is an unknown constant. We have written $\hat{P}(\gamma, \lambda) = v^{-1}f(v^{-2})$ or $v^{-2}f(v^{-2})$, depending on the parity of $\ell(\lambda) - \ell(\gamma)$, where $f$ is a polynomial. In the last column $g$ is a polynomial which is known, and $\alpha, \beta, \gamma$ are unknown constants. We want to solve for $f$. 
Lemma 11.15 In each case in the table we can solve for $f$.

After multiplying by the appropriate power of $v$, these all come down to:

Lemma 11.16 Suppose $(1 \pm q^k)f(q) = g(q)$ where $g(q)$ is a polynomial. If we know all but the top $k$ coefficients of $g$, then we can solve for $f$.

12 Guide

In the following tables, we’ve indicated which formulas to use in various cases.

1. *: not primitive, easy recursion
2. NE: not extremal (but primitive): easy recursion, but has a sum on the right hand side
3. *0: not primitive, necessarily 0
4. DR: direct recursion (Proposition 10.3.9)
5. DR+: new type of direct recursion (Proposition 11.13, $\ell(\kappa) = 1$)
6. DR+: new type of direct recursion, but the recursion may not work.
   See Proposition 11.13, $\ell(\kappa) = 2, 3$; we need conditions (A) and (B).

In (2), $\{\text{non-primitive}\} \subset \{\text{non-extremal}\}$; the pairs marked NE are in the second set, but not the first (they are not non-primitive).
### Type 1

|      | 1C- | 1r1f | 1r1s | 1r2 | 1ic | 1C+ | 1i1 | 1i2f | 1i2s | 1rn |
|------|-----|------|------|-----|-----|-----|-----|------|------|-----|
| 1C-  | DR  | DR   |      |     |     |     |     |      |      |     |
| 1r1f | DR  | DR   |      |     |     |     |     |      |      |     |
| 1r1s | DR  | DR   |      |     |     |     |     |      |      | DR+ |
| 1r2  | DR  | DR   |      |     |     |     |     |      |      |     |
| 1ic  | DR  | DR   |      |     |     |     |     |      |      | DR+ |
| 1C+  | *   | *    | *    | *   | *   |     |     |      |      | DR+ |
| 1i1  | *   | *    | *    | *   | *   |     |     |      |      |     |
| 1i2f | NE  | NE   | NE   | NE  | NE  |     |     |      |      | DR+ |
| 1i2s | *0  | *0   | *0   | *0  | *0  |     |     |      |      |     |
| 1rn  | *0  | *0   | *0   | *0  | *0  |     |     |      |      |     |

### Type 2

|      | 2C- | 2Cr  | 2r22 | 2r21 | 2r11 | 2ic | 2C+ | 2Cif | 1i11 | 1i12 | 2i22 | 2rn |
|------|-----|------|------|------|------|-----|-----|------|------|------|------|-----|
| 2C-  | DR  | DR   | DR   |      |      |     |     |      |      |      |      |     |
| 2Cr  | DR  | DR   | DR   |      |      |     |     |      |      |      |      |     |
| 2r22 | DR  | DR   | DR   |      |      |     |     |      |      |      |      |     |
| 2r21 | DR  | DR   | DR   |      |      |     |     |      |      |      |      |     |
| 2r11 | DR  | DR   | DR   |      |      |     |     |      |      |      |      |     |
| 2ic  | DR  | DR   | DR   |      |      |     |     |      |      |      |      | DR+ |
| 2C+  | *   | *    | *    | *    | *    |     |     |      |      |      |      |     |
| 2Ci  | *0  | *0   | *0   | *0   | *0   |     |     |      |      |      |      |     |
| 2i11 | *   | *    | *    | *    | *    |     |     |      |      |      |      |     |
| 2i12 | NE  | NE   | NE   | NE   | NE   | NE  |     |      |      |      |      |     |
| 2i22 | *   | *    | *    | *    | *    |     |     |      |      |      |      |     |
| 2rn  | *0  | *0   | *0   | *0   | *0   | *0  |     |      |      |      |      |     |
### Type 3

|       | 3C- | 3Cr | 3r  | 3ic | 3C+ | 3Ci | 3i  | 3rn |
|-------|-----|-----|-----|-----|-----|-----|-----|-----|
| 3C-   | DR  | DR  | DR  |     |     |     |     |     |
| 3Cr   | DR  | DR  | DR  |     |     |     |     |     |
| 3r    | DR  | DR  | DR  |     |     |     |     |     |
| 3ic   | DR  | DR  | DR  |     |     |     |     | DR+|
| 3C+   | *   | *   | *   | *   |     |     |     |     |
| 3Ci   | *   | *   | *   | *   |     |     |     |     |
| 3i    | *   | *   | *   | *   |     |     |     |     |
| 3rn   | *0  | *0  | *0  | *0  |     |     |     |     |
13 Appendix I: Proof of Theorem 9.3.10

Throughout this section we assume $\kappa \in \tau(\gamma), \kappa \not\in \tau(\lambda)$. Recall we are trying to find $m_\kappa(\gamma, \lambda)$ such that:

$$\hat{T}_\kappa \hat{C}_\lambda = \sum_{\gamma | \kappa \in \tau(\gamma)} m_\kappa(\gamma, \lambda) \hat{C}_\gamma$$

The main tool is the identity 9.3.14 (for $\kappa \not\in \tau(\lambda)$):

(13.1) $$\sum_{\mu | \kappa \in \tau(\mu)} \hat{P}_\sigma(\gamma, \mu) m_\kappa(\mu, \lambda) = \text{multiplicity of } \hat{a}_\gamma \text{ in } \hat{T}_\kappa(\hat{C}_\lambda)$$

The right hand side is given by Table 10.2.9 which we reproduce here.

Table 13.2

| $t_\gamma(\kappa)$ | RHS of (13.1) |
|---------------------|----------------|
| 1C-, 2C-, 3C-       | $v^k \hat{P}_\sigma(\gamma, \lambda) + \hat{P}_\sigma(w_\kappa \times \gamma, \lambda)$ |
| 1ic, 2ic, 3ic, 1r1s | $(v^k + v^{-k}) \hat{P}_\sigma(\gamma, \lambda)$ |
| 2Cr, 3Cr, 3r        | $v(v^{k-1} - v^{-k+1}) \hat{P}_\sigma(\gamma, \lambda) + (v + v^{-1}) \hat{P}_\sigma(\gamma_\kappa, \lambda)$ |
| 1r1f, 2r11          | $(v^k - v^{-k}) \hat{P}_\sigma(\gamma, \lambda) + \hat{P}_\sigma(\gamma_\kappa^1, \lambda) + \hat{P}_\sigma(\gamma_\kappa^2, \lambda)$ |
| 1r2, 2r22           | $v^k \hat{P}_\sigma(\gamma, \lambda) - v^{-k} \hat{P}_\sigma(w_\kappa \times \gamma, \lambda) + \hat{P}_\sigma(\gamma_\kappa, \lambda)$ |
| 2r21                | $(v^2 - v^{-2}) \hat{P}_\sigma(\gamma, \lambda) + \sum_{\gamma' | \gamma \rightarrow \gamma'} \epsilon(\gamma, \gamma') \hat{P}_\sigma(\gamma', \lambda)$ |

We are going to look at the top degree terms of both sides.
Write any element of $\mathbb{Z}[v, v^{-1}]$ as $f = f^+ + f^-$ where $f^+ \in \mathbb{Z}[v]$ and $f^- \in v^{-1}\mathbb{Z}[v^{-1}]$. We make frequent use of:

**Lemma 13.3** If $\gamma = \mu$ then

$$[\hat{P}^\sigma(\gamma, \mu)m_\kappa(\mu, \lambda)]^+ = m_{\kappa,0}(\mu, \lambda) + m_{\kappa,1}(\mu, \lambda)v + m_{\kappa,2}(\mu, \lambda)v^2$$

If $\gamma < \mu$ then

$$\hat{P}^\sigma(\gamma, \mu)m_\kappa(\mu, \lambda) = [\hat{\mu}_{-2}^\sigma(\gamma, \mu)m_{\kappa,2}(\mu, \lambda), \hat{\mu}_{-1}^\sigma(\gamma, \mu)m_{\kappa,1}(\mu, \lambda)$$

$$+ \hat{\mu}_{-1}^\sigma(\gamma, \mu)m_{\kappa,2}(\mu, \lambda)v$$

**Lemma 13.4** If $\ell(\kappa) = 1$ then

\[
(13.5) \quad m_\kappa(\gamma, \lambda) = m_{\kappa,0}(\gamma, \lambda) = \begin{cases} 
1 & \gamma \overset{\kappa}{\rightarrow} \lambda \\
\hat{\mu}_{-1}^\sigma(\gamma, \lambda) & \gamma < \lambda \\
0 & \text{else}
\end{cases}
\]

**Proof.** Since $\ell(\kappa) = 1$, $m_\kappa(\mu, \lambda) = m_{\kappa,0}(\mu, \lambda)$ for all $\mu$, and on the left hand side of (13.1) the maximal degree is 0. In each term on the right hand side, $[v\hat{P}^\sigma(\gamma, \lambda)]^+ = \hat{\mu}_{-1}^\sigma(\gamma, \lambda)$ and $[v^{-1}\hat{P}^\sigma(\gamma, \lambda)]^+ = 0$. On the other hand a term $\hat{P}^\sigma(\mu, w_\kappa \times \gamma), \hat{P}^\sigma(\mu, \gamma_\kappa)$ or $\hat{P}^\sigma(\mu, \gamma_i^\kappa)$ contributes 1 if and only if the two arguments are equal. So, $[\ ]^+$ of both sides gives (the last column is $t_\gamma(\kappa)$):

\[
(13.6) \quad m_\kappa(\gamma, \lambda) = \hat{\mu}_{-1}^\sigma(\gamma, \lambda) + \begin{cases} 
\delta_{w_\kappa \times \gamma, \lambda} & 1\text{C} \\
\delta_{\gamma, \lambda} + \delta_{\gamma^2, \lambda} & 1\text{r1f} \\
0 & 1\text{r1s} \\
\delta_{\gamma, \lambda} & 1\text{r2} \\
0 & 1\text{i1c}
\end{cases}
\]

Each Kronecker $\delta$ after the brace is non-zero precisely when $\gamma \overset{\kappa}{\rightarrow} \lambda$, in which case $\gamma > \lambda$, so $\hat{\mu}_{-1}^\sigma(\gamma, \lambda) = 0$.\qed

This proves Cases (1) ($\ell(\kappa) = 1$) and (2) of Theorem [9.3.10]. Note that $m_\kappa(\gamma, \lambda) = 0$ unless $\gamma \overset{\kappa}{\rightarrow} \lambda$ or $\gamma < \lambda$.\[54\]
Take the + part of both sides of (13.1). The left hand side is

\[(13.1.1)(a) \quad [m_{\kappa,0}(\gamma, \lambda) + \sum_{\mu: \kappa \in \tau(\mu)} \hat{\mu}_{-1}^\sigma(\gamma, \mu)m_{\kappa,1}(\mu, \lambda)] + m_{\kappa,1}(\gamma, \lambda)v\]

The first and last terms are from \(\mu = \gamma\), and the second sum is from all other terms \(\mu \neq \gamma\). (Note that the summand is 0 if \(\mu = \gamma\).)

The right hand side is

\[(13.1.1)(b) \quad \hat{\mu}_{-2}(\gamma, \lambda) + \hat{\mu}_{-1}^\sigma(\gamma, \lambda)v + \left\{ \begin{array}{ll}
\delta_{w_{\kappa} \times \gamma, \lambda} & 2C- \\
\hat{\mu}_{-1}^\sigma(\gamma, \lambda) + \delta_{\gamma, \lambda}v & 2Cr \\
\delta_{\gamma, \lambda} & 2r22 \\
\sum_{\gamma' | \gamma \rightarrow \gamma'} \epsilon(\gamma, \gamma')\delta_{\gamma', \lambda} & 2r21 \\
\delta_{\gamma, \lambda} + \delta_{\gamma, \lambda} & 2r11 \\
0 & 2ic
\end{array} \right.\]

Equating the coefficient of \(v\) in (a) and (b) gives

\[(13.1.1)(c) \quad m_{\kappa,1}(\gamma, \lambda) = \left\{ \begin{array}{ll}
\hat{\mu}_{-1}^\sigma(\gamma, \lambda) & t_{\gamma}(\kappa) \neq 2Cr \\
\hat{\mu}_{-1}^\sigma(\gamma, \lambda) & t_{\gamma}(\kappa) = 2Cr, \text{ and } \gamma < \lambda \\
1 & t_{\gamma}(\kappa) = 2Cr, \text{ and } \gamma \rightarrow \lambda \\
0 & t_{\gamma}(\kappa) = 2Cr, \text{ otherwise}
\end{array} \right.\]

We can rewrite this

\[(13.1.1)(d) \quad m_{\kappa,1}(\gamma, \lambda) = \hat{\mu}_{-1}^\sigma(\gamma, \lambda) + \left\{ \begin{array}{ll}
\delta_{\gamma, \lambda} & t_{\gamma}(\kappa) = 2Cr \\
0 & \text{otherwise}
\end{array} \right.\]

In particular \(m_{\kappa,1}(\gamma, \lambda) = 0\) unless \(\gamma < \lambda\) or \(\gamma \rightarrow \lambda\).

Now (d) holds for all \(\gamma\) with \(\kappa \in \tau(\gamma)\), so we can apply it to all \(\gamma\) occurring the sum in (13.1).
So plug this back in to (a), keep only the constant term, and set this equal to the constant term of (b) (13.1.1)(e)

\[ m_{\kappa,0}(\gamma, \lambda) + \sum_{\mu|\kappa \in \tau(\mu)} \tilde{\mu}_{-1}^\sigma(\gamma, \mu) \left[ \tilde{\mu}_{-1}^\sigma(\mu, \lambda) + \begin{cases} \delta_{\mu,\lambda} & t_\mu(\kappa) = 2Cr \\ 0 & \text{else} \end{cases} \right] = \]

(13.1.2)

Note that each Kronecker \( \delta \) after the brace is 1 iff \( \gamma \xrightarrow{\kappa} \lambda \). Also we can put \( \epsilon(\gamma, \lambda) \) in front of each such term without changing anything (these are 1 unless \( t_\gamma(\kappa) = 2r21 \)). Therefore

\[ m_{\kappa,0}(\gamma, \lambda) = \tilde{\mu}_{-2}^\sigma(\gamma, \lambda) - \sum_{\mu} \tilde{\mu}_{-1}^\sigma(\gamma, \mu) \tilde{\mu}_{-1}^\sigma(\mu, \lambda) \]

Then
\[ \sum_{\mu} \hat{\mu}_{-1}^\sigma(\gamma, \mu) \times \begin{cases} \delta_{\mu_\kappa, \lambda} & t_\mu(\kappa) = 2Cr \\ 0 & \text{else} \end{cases} \]

is equal to

\[ \begin{cases} \hat{\mu}_{-1}^\sigma(\gamma, \lambda^\kappa) & t_\lambda(\kappa) = 2Ci \\ 0 & \text{else} \end{cases} \]

**Lemma 13.1.5** Assume \( \kappa \in \tau(\gamma), \kappa \notin \tau(\lambda), \ell(\kappa) = 2 \). Then

\[
m_{\kappa,0}(\gamma, \lambda) = \hat{\mu}_{-2}^\sigma(\gamma, \lambda) - \sum_{\mu | \kappa \in \tau(\mu)} \hat{\mu}_{-1}^\sigma(\gamma, \mu) \hat{\mu}_{-1}^\sigma(\mu, \lambda)
\]

\[
\begin{cases} \hat{\mu}_{-1}^\sigma(\gamma, \lambda^\kappa) & t_\lambda(\kappa) = 2Ci \\ 0 & \text{else} \end{cases}
\]

\[ + \begin{cases} \hat{\mu}_{-1}^\sigma(\gamma^\kappa, \lambda) & t_\gamma(\kappa) = 2Cr \\ \epsilon(\gamma, \lambda) & \gamma \xrightarrow{\kappa} \lambda, t_\gamma(\kappa) \neq 2Cr \\ 0 & \text{else} \end{cases} \]

and

\[ m_{\kappa,1}(\gamma, \lambda) = \hat{\mu}_{-1}^\sigma(\gamma, \lambda) + \begin{cases} \delta_{x_\kappa, \lambda} & t_\gamma(\kappa) = 2Cr \\ 0 & \text{otherwise} \end{cases} \]

Let’s look at some cases. First assume \( \gamma \not\xrightarrow{\kappa} \lambda \). In particular \( \gamma > \delta \), so the first two terms are 0. A little checking gives

\[ m_{\kappa,0}(\gamma, \lambda) = \begin{cases} \epsilon(\lambda, \gamma) & t_\lambda(\kappa) \neq 2Ci \\ 0 & t_\lambda(\kappa) = 2Ci \end{cases} \]

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Putting this together with formula (13.1.1)(d) for $m_{κ,1}$ we get:

\[
(13.1.8) \quad \gamma \xrightarrow{κ} \lambda \Rightarrow m_κ(γ, λ) = \begin{cases} 
  v + v^{-1} & t_λ(κ) = 2Cr \\
  \epsilon(γ, λ) & \text{else}
\end{cases}
\]

Assume $γ \xrightarrow{κ} λ$. We see:

\[
m_κ(γ, λ) = \hat{μ}_{-2}(γ, λ) - \sum_{μ | κ \in τ(μ)} \hat{μ}_{-1}(γ, μ)\hat{μ}_{-1}(μ, λ) - \begin{cases} 
  \hat{μ}_{-1}(γ, λ^κ) & t_λ(κ) = 2Ci \\
  0 & \text{else}
\end{cases} + \begin{cases} 
  \hat{μ}_{-1}(γ, λ) & t_γ(κ) = 2Cr \\
  0 & \text{else}
\end{cases} + \hat{μ}_{-1}(γ, λ)(v + v^{-1})
\]

If $ℓ(λ) \not\equiv ℓ(γ)(\text{mod } 2)$ all terms but the last are 0, so

\[
(13.1.9) \quad γ \xrightarrow{κ} λ, ℓ(γ) \not\equiv ℓ(λ) \ (\text{mod } 2) \Rightarrow m_κ(γ, λ) = \hat{μ}_{-1}(γ, λ)(v + v^{-1})
\]

On the other hand $ℓ(γ) = ℓ(λ)(\text{mod } 2)$ implies the last term is 0, and

\[
(13.1.10) \quad m_κ(γ, λ) = \hat{μ}_{-2}(γ, λ) - \sum_{μ | κ \in τ(μ)} \hat{μ}_{-1}(γ, μ)\hat{μ}_{-1}(μ, λ) - \begin{cases} 
  \hat{μ}_{-1}(γ, λ^κ) & t_λ(κ) = 2Ci \\
  0 & \text{else}
\end{cases} + \begin{cases} 
  \hat{μ}_{-1}(γ, λ) & t_γ(κ) = 2Cr \\
  0 & \text{else}
\end{cases}
\]

I believe this agrees with [4, Theorem 4.4].

Note that all terms of $m_κ(γ, δ)$ are 0 unless $γ \xrightarrow{κ} δ$ or $γ < δ$, except possibly the last two.

\textbf{13.2} $ℓ(κ) = 3$

We continue to assume $κ \in τ(γ), κ \notin τ(λ)$. In particular $γ \not= λ$.  

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Take the + part of both sides of (13.1). The left hand side is:

\[(13.2.1)(a)\]
\[m_{\kappa,0}(\gamma, \lambda) + \sum_{\mu|\kappa \in \tau(\mu)} \tilde{\mu}^\sigma_{-1}(\gamma, \mu)m_{\kappa,1}(\mu, \lambda) + \sum_{\mu|\kappa \in \tau(\mu)} \tilde{\mu}^\sigma_{-2}(\gamma, \mu)m_{\kappa,2}(\mu, \lambda)]
\[+ [m_{\kappa,1}(\gamma, \lambda) + \sum_{\mu|\kappa \in \tau(\mu)} \tilde{\mu}^\sigma_{-1}(\gamma, \mu)m_{\kappa,2}(\mu, \lambda)]v + m_{\kappa,2}(\gamma, \lambda)v^2\]

The right hand side is

\[(13.2.1)(b)\]
\[\tilde{\mu}^\sigma_{-3}(\gamma, \lambda) + \tilde{\mu}^\sigma_{-2}(\gamma, \lambda)v + \tilde{\mu}^\sigma_{-1}(\gamma, \lambda)v^2 + \begin{cases} 
\delta_{w,\times \gamma,\lambda} & 3C- \\
\tilde{\mu}^\sigma_{-1}(\gamma, \gamma, \lambda) + \delta_{\gamma,\lambda}v & 3Cr \\
\tilde{\mu}^\sigma_{-1}(\gamma, \gamma, \lambda) + \delta_{\gamma,\lambda}v & 3r \\
0 & 3ic 
\end{cases}\]

Comparing the coefficient of \(v^2\) gives

\[(13.2.1)(c)\]
\[m_{\kappa,2}(\gamma, \lambda) = \tilde{\mu}^\sigma_{-1}(\gamma, \lambda)\]

Plugging this in to (13.2.1), the coefficient of \(v\) gives

\[(13.2.1)(d)\]
\[m_{\kappa,1}(\gamma, \lambda) + \sum_{\mu|\kappa \in \tau(\mu)} \tilde{\mu}^\sigma_{-1}(\gamma, \mu)\tilde{\mu}^\sigma_{-1}(\mu, \lambda) = \tilde{\mu}^\sigma_{-2}(\gamma, \lambda) + \begin{cases} 
0 & 3C- \\
\delta_{\gamma,\lambda} & 3Cr \\
\delta_{\gamma,\lambda} & 3r \\
0 & 3ic 
\end{cases}\]

i.e.,

\[(13.2.1)(e)\]
\[m_{\kappa,1}(\gamma, \lambda) = \tilde{\mu}^\sigma_{-2}(\gamma, \lambda) - \sum_{\mu|\kappa \in \tau(\mu)} \tilde{\mu}^\sigma_{-1}(\gamma, \mu)\tilde{\mu}^\sigma_{-1}(\mu, \lambda) + \begin{cases} 
0 & t_{\gamma}(\kappa) = 3C-, 3ic \\
\delta_{\gamma,\lambda} & t_{\gamma}(\kappa) = 3Cr, 3r 
\end{cases}\]

Turn the crank one more time, plugging this in, to compute the constant
term: 
(13.2.1)(f) 
\[ m_{\kappa, 0}(\gamma, \lambda) = - \sum_{\mu | \kappa \in \tau(\mu)} \hat{\mu}_{-1}^\sigma(\gamma, \mu)m_{\kappa, 1}(\mu, \lambda) - \sum_{\mu | \kappa \in \tau(\mu)} \hat{\mu}_{-2}^\sigma(\gamma, \mu)m_{\kappa, 2}(\mu, \lambda) + \]
\[ \hat{\mu}_{-3}^\sigma(\gamma, \lambda) + \begin{cases} 
\delta_{w_\kappa \times \gamma, \lambda} & 3c- \\
\hat{\mu}_{-1}^\sigma(\gamma_\kappa, \lambda) & 3Cr \\
\hat{\mu}_{-1}^\sigma(\gamma_\kappa, \lambda) & 3r \\
0 & 3ic 
\end{cases} \]

Plug in (c) and (e):
(13.2.1)(g) 
\[ m_{\kappa, 0}(\gamma, \lambda) = \]
\[ - \sum_{\mu | \kappa \in \tau(\mu)} \hat{\mu}_{-1}^\sigma(\gamma, \mu) \left[ \hat{\mu}_{-2}^\sigma(\mu, \lambda) - \sum_{\phi | \kappa \in \tau(\phi)} \hat{\mu}_{-1}^\sigma(\mu, \phi) \hat{\mu}_{-1}^\sigma(\phi, \lambda) \right] + \begin{cases} 
0 & t_\lambda(\kappa) = 3C-, 3ic \\
\delta_{\lambda_\kappa, \lambda} & t_\lambda(\kappa) = 3Cr, 3r 
\end{cases} \]
\[ - \left[ \sum_{\mu | \kappa \in \tau(\mu)} \hat{\mu}_{-2}^\sigma(\gamma, \mu) \hat{\mu}_{-1}^\sigma(\mu, \lambda) \right] + \hat{\mu}_{-3}^\sigma(\gamma, \lambda) + \begin{cases} 
\delta_{w_\kappa \times \gamma, \lambda} & t_\gamma(\kappa) = 3c- \\
\hat{\mu}_{-1}^\sigma(\gamma_\kappa, \lambda) & t_\gamma(\kappa) = 3Cr \\
\hat{\mu}_{-1}^\sigma(\gamma_\kappa, \lambda) & t_\gamma(\kappa) = 3r \\
0 & t_\gamma(\kappa) = 3ic 
\end{cases} \]

Note that if \( t_\gamma(\kappa) = 3C- \), \( \delta_{w_\kappa \times \gamma, \lambda} = 1 \) if \( \gamma \rightarrow \lambda \), and 0 otherwise. Also, evaluating

(13.2.1)(h) 
\[ \sum_{\mu} \hat{\mu}_{-1}^\sigma(\gamma, \mu) \ast \begin{cases} 
0 & t_\lambda(\kappa) = 3C-, 3ic \\
\delta_{\lambda_\kappa, \lambda} & t_\lambda(\kappa) = 3Cr, 3r 
\end{cases} \]
as in the length 2 case gives

(13.2.1)(i) 
\[ \begin{cases} 
\hat{\mu}_{-1}^\sigma(\gamma_\kappa) & t_\lambda(\kappa) = 3Ci \text{ or } 3i \\
0 & \text{else} 
\end{cases} \]
Inserting this information, moving a few terms around, and taking \( \lambda < \lambda \) in all sums as in the previous cases, gives

\[
m_{\kappa,0}(\gamma, \lambda) = \tilde{\mu}_{-3}^\sigma(\gamma, \lambda) + \sum_{\substack{\mu|\kappa \in \tau(\mu) \\ \phi|\kappa \in \tau(\phi)}} \tilde{\mu}_{-1}^\sigma(\gamma, \mu)\tilde{\mu}_{-1}^\sigma(\mu, \phi)\tilde{\mu}_{-1}^\sigma(\phi, \lambda) - \sum_{\mu|\kappa \in \tau(\mu)} \left[ \tilde{\mu}_{-1}^\sigma(\gamma, \mu)\tilde{\mu}_{-2}^\sigma(\mu, \lambda) + \tilde{\mu}_{-2}^\sigma(\gamma, \mu)\tilde{\mu}_{-1}^\sigma(\mu, \lambda) \right]
\]

(13.2.1)(j)

\[
- \begin{cases} 
\tilde{\mu}_{-1}^\sigma(\gamma, \lambda) & t_\lambda(\kappa) = 3Ci \text{ or } 3i \\
0 & \text{else} \\
1 & t_\gamma(\kappa) = 3C- \text{, } \gamma \overset{\kappa}{\rightarrow} \lambda \\
0 & t_\gamma(\kappa) = 3C- \text{, } \gamma \not\rightarrow \lambda \\
\tilde{\mu}_{-1}^\sigma(\gamma, \lambda) & t_\gamma(\kappa) = 3Cr \\
\tilde{\mu}_{-1}^\sigma(\gamma, \lambda) & t_\gamma(\kappa) = 3r \\
0 & t_\gamma(\kappa) = 3ic 
\end{cases}
\]

Summarizing the length 3 case:

**Lemma 13.2.2** Assume \( \kappa \in \tau(\gamma), \kappa \not\in \tau(\lambda), \ell(\kappa) = 3 \). Then \( m_{\kappa,2}, m_{\kappa,1}, m_{\kappa,0} \) are given by (13.2.1)(c),(e), and (j), respectively.
Let's look at some cases.
Suppose $\gamma \rightarrow^\kappa \lambda$. All $\hat{\mu}_{i-1}$ terms are 0, and

$$
(13.2.3) \quad \gamma \rightarrow^\kappa \lambda \Rightarrow m_\kappa(\gamma, \lambda) = \begin{cases} 
1 & t_\gamma(\kappa) = 3C - \\
(v + v^{-1}) & t_\gamma(\kappa) = 3Cr, 3r \\
0 & t_\gamma(\kappa) = 3ic
\end{cases}
$$

Now assume $\gamma \not\rightarrow^\kappa \lambda$, and $\ell(\lambda) \equiv \ell(\gamma) \pmod{2}$. Then $m_{\kappa,2}(\gamma, \lambda) = m_{\kappa,0}(\gamma, \lambda) = 0$. The formula for $m_{\kappa,1}$ doesn't simplify, except that last term is 0 since $\gamma \not\rightarrow^\kappa \lambda$, so

$$
(13.2.4) \quad m_\kappa(\gamma, \lambda) = \left[\hat{\mu}_{-2}^\sigma(\gamma, \lambda) - \sum_{\mu | \kappa \in \tau(\mu)} \hat{\mu}_{-1}^\sigma(\gamma, \mu)\hat{\mu}_{-1}^\sigma(\mu, \lambda)\right](v + v^{-1})
$$

Finally assume $\gamma \not\rightarrow^\kappa \lambda$, and $\ell(\lambda) \not\equiv \ell(\gamma) \pmod{2}$. Then $m_{\kappa,1}(\gamma, \lambda) = 0$, and $m_{\kappa,0}$ simplifies a little, to give:

$$
m_\kappa(\gamma, \lambda) = \hat{\mu}_{-1}^\sigma(\gamma, \lambda)(v^2 + v^{-2}) + \hat{\mu}_{-1}^\sigma(\gamma, \lambda)
+ \sum_{\mu | \kappa \in \tau(\mu)} \hat{\mu}_{-1}^\sigma(\gamma, \mu)\hat{\mu}_{-1}^\sigma(\mu, \lambda)
+ \hat{\mu}_{-1}^\sigma(\gamma, \lambda)\hat{\mu}_{-1}^\sigma(\mu, \lambda)
- \sum_{\mu | \kappa \in \tau(\mu)} \left[\hat{\mu}_{-1}^\sigma(\gamma, \mu)\hat{\mu}_{-2}^\sigma(\mu, \lambda) + \hat{\mu}_{-2}^\sigma(\gamma, \mu)\hat{\mu}_{-1}^\sigma(\mu, \lambda)\right]
$$

$$
(13.2.5) \quad m_\kappa(\gamma, \lambda) = \begin{cases} 
\hat{\mu}_{-1}^\sigma(\gamma, \lambda) & t_\gamma(\kappa) = 3Ci \text{ or } 3i \\
0 & \text{else}
\end{cases}
+ \begin{cases} 
\hat{\mu}_{-1}^\sigma(\gamma, \lambda) & t_\gamma(\kappa) = 3Cr \text{ or } 3r \\
0 & \text{else}
\end{cases}
$$

As in the length 2 case, all terms are zero unless $\gamma \rightarrow^\kappa \lambda$ or $\gamma < \lambda$, except possibly the last two.
14 Appendix II: Some supplementary material

14.1 Explanation of the 1i2s/1r1s cases

This was originally a separate note on these cases.

We recall some notation from \([5]\). There is a space \(D\) of parameters (Langlands parameters for \(G\)) with an action of \(\sigma\). There is a space of extended parameter \(\tilde{D}\) for the extended group \(\tilde{G}\). Each \(\gamma \in D\) gives rise to two parameters \((\gamma, \pm)\) in \(\tilde{D}\) (a \(\sigma\)-fixed representation extends in two ways to the extended group). If \(\sigma\gamma = \gamma' \neq \gamma\) then both \(\gamma, \gamma'\) give a single parameter \((\gamma') = (\gamma) \in \tilde{D}\). See \([5\) Section 2.3]; these are the elements \((\mathcal{L}, \pm q^k \beta \mathcal{L})\) and \((\mathcal{L}, q^{2k} t \mathcal{L})^0\).

We have a Hecke algebra \(H\), and an \(H\)-module \(M\) with basis \(\{a_\mu | \mu \in \tilde{D}\}\). This is the module \(\mathfrak{M}(C)\) of \([5\) Section 2.3]. (It isn’t entirely clear from \([5\) that \(M\) carries an action of \(H\), but David assures me this is so.)

This is not the main module \(M\) of loc. cit., which is a quotient of \(M\): \(M\) has basis \(\{a_\mu | \mu \in \tilde{D}\}\) modulo relations:

(14.1.1)(a) \(a_{(\gamma, \mp)} = -a_{(\gamma, \pm)} \) \((\gamma \in D)\)

(14.1.1)(b) \(a_{(\gamma)} = 0 \) \((\gamma \in D - D)\)

See the discussion of the image of the homomorphism \(\theta\), and the definition of \(M\), in \([5\) Section 2.3].

Now suppose \(\kappa = \{\alpha\}\) where \(\alpha\) is a \(\sigma\)-fixed root, \(\gamma \in D\), and \(\alpha\) is imaginary or real with respect to \(\gamma\). Associated to \(\kappa\) is a Hecke operator \(T_\kappa\). The problem is to compute the action of \(T_\kappa\). There are two easy cases and two hard cases.

(a) 1i2f/1r1. This means \(\alpha\) is imaginary for \(\gamma\), \(c^\alpha(\gamma) = \{\gamma', \gamma''\}\) is double valued, and \(\gamma', \gamma'' \in D\) (one discrete series and two principal series). The \(f\) refers to the fact that \(\gamma', \gamma''\) are \(\sigma\)-fixed.

There is a 6-dimensional subspace of \(M\) on which the Hecke operator \(T_\kappa\) acts, with basis \(a_{(\gamma, \pm)}, a_{(\gamma', \pm)}, a_{(\gamma'', \pm)}\) In the quotient \(M\) this becomes 3-
dimensional by (14.1.1)(a). The matrix of $T_\kappa$ on this space is

\[
\begin{pmatrix}
1 & q - 1 & q - 1 \\
1 & q - 1 & -1 \\
1 & -1 & q - 1
\end{pmatrix}
\]

with eigenvalues $u, u, -1$, and corresponding eigenvectors $(1, 1, 0), (1, 0, 1)$ (eigenvalue $u$) and $(u - 1, -1, -1)$ (eigenvalue $-1$).

(b) 1i1/1r1f. This means $\alpha$ is imaginary for $\gamma, \gamma' = s_\alpha \gamma \neq \gamma$ and $c^\alpha(\gamma) = c^\alpha(\gamma') = \gamma''$ is single valued (two discrete series and one principal series) (also $\sigma(\gamma) = \gamma$). Again there is 6-dimensional subspace of $\mathcal{M}$ on which the Hecke operator $T_\kappa$ acts, in the quotient $M$ this becomes 3-dimensional, and the matrix of $T_\kappa$ on this space is

\[
\begin{pmatrix}
0 & 1 & q - 1 \\
1 & 0 & q - 1 \\
1 & 1 & q - 2
\end{pmatrix}
\]

with eigenvalues $q, -1, -1$, and corresponding eigenvectors $(1, 1, 1)$ (eigenvalue $q$) and $(q - 1, 0, -1), (0, q - 1, -1)$ (eigenvalue $-1$).

Now the hard cases.

(c) 1i2s: Just as in the 1i2f case we have three parameters $\gamma$ and $c^\alpha(\gamma) = \{\gamma', \gamma''\}$, except now $\sigma$ switches $\gamma', \gamma''$ (hence the $s$).

Now we need to be careful. There is a 3 dimensional space $V$ invariant by $T_\kappa$, with basis

$$a_{(\gamma, +)}, a_{(\gamma, -)}, a_{(\gamma')}$$

Note that $(\gamma') = (\gamma'')$. In the quotient $M$ we have

$$a_{(\gamma, -)} = -a_{(\gamma, +)} \quad \text{by (14.1.1)(a)}$$

and

$$a_{(\gamma')} = 0 \quad \text{by (14.1.1)(b)}.$$ 

The subspace we are modding out by is spanned by $a_{(\gamma, +)} + a_{(\gamma, -)}$ and $a_{(\gamma')}$, i.e. $(1, 1, 0)$ and $(0, 0, 1)$ in the given basis. The quotient $V$ is 1 dimensional.

So to calculate the action of $T_\kappa$ on the one-dimensional space $V$, calculate it on the 3-dimensional space $\mathcal{V}$, and mod out by the span of $(1, 1, 0), (0, 0, 1)$ (which better be $T_\kappa$ invariant).
Now comes some guesswork. Recall we started with 1 discrete series $\gamma$, and two principal series $\gamma', \gamma''$. However on the extended group this becomes two discrete series $(\gamma, \pm)$ and one principal series $(\gamma')$. This suggests that the action of the Hecke operator $T_\kappa$ on $M$ is not (14.1.8) (the 1i2f case) but rather (14.1.3) (from the 1i1 case).

**Conjecture 14.1.4** In the 1i2s case $T_\kappa$ acts on $M$, with basis $a_{(\gamma, +)}, a_{(\gamma, -)}, a_{(\gamma')}$ with matrix

\[
\begin{pmatrix}
0 & 1 & q - 1 \\
1 & 0 & q - 1 \\
1 & 1 & q - 2
\end{pmatrix}
\]

(14.1.5)

Recall this matrix has eigenvalues $q, -1, -1$.

Assuming this, the subspace spanned by $(1, 1, 0)$ and $(0, 0, 1)$ is $T_\kappa$-invariant: $(1, 1, 1)$ has eigenvalue $q$ and $(q - 1, q - 1, -2)$ has eigenvalue $-1$. Therefore there is one remaining eigenvalue $-1$, and we conclude

**Lemma 14.1.6** Assuming the conjecture, in the 1i2s case $T_\kappa$ acts with eigenvalue $-1$ on the one-dimensional space $V$.

(d) 1r1s As in the 1r1f case there are two discrete series $\gamma, \gamma'$, one principal series $\gamma''$, except that now $\sigma(\gamma) = \gamma'$. There is a 3-dimensional space $V$ spanned by

$a_{(\gamma)}, a_{(\gamma'', +)}, a_{(\gamma'', -)}$

(recall $(\gamma) = (\gamma')$). For the quotient we have relations

$a_{(\gamma)} = 0$

and

$a_{(\gamma'', -)} = -a_{(\gamma'', +)}$

so the subspace is spanned by $(1, 0, 0)$ and $(0, 1, 1)$.

**Conjecture 14.1.7** In the 1r1s case the matrix of $T_\kappa$, in the basis $a_{(\gamma)}, a_{(\gamma'', +)}, a_{(\gamma'', -)}$, is

\[
\begin{pmatrix}
1 & q - 1 & q - 1 \\
1 & q - 1 & -1 \\
1 & -1 & q - 1
\end{pmatrix}
\]

(14.1.8)
Recall this matrix has eigenvalues \( q, q, -1 \).

Assuming the conjecture, the subspace spanned by \((1, 0, 0)\) and \((0, 1, 1)\) is \( T_\kappa \)-invariant: \((2, 1, 1)\) has eigenvalue \( q \) and \((u - 1, -1, -1)\) has eigenvalue \(-1\).

**Lemma 14.1.9** Assuming the conjecture, in the \( \text{ir1s} \) case \( T_\kappa \) acts on the one-dimensional space \( V \) with eigenvalue \( q \).

**Conclusion**

Assume the conjectures.

In case \( \text{ii2s} \) there is a single fixed discrete series parameter \( \gamma \), with two extensions \( (\gamma, \pm) \). Recall \( a_{(\gamma,+)} = -a_{(\gamma,-)} \). Then (in \( M \)):

\[
\text{(14.1.10)(a)} \quad T_\kappa a_{(\gamma,+)} = -a_{(\gamma,+)}
\]

In case \( \text{ir1s} \) there is a single fixed principal series parameter \( \gamma \), with two extensions \( (\gamma, \pm) \). Recall \( a_{(\gamma,+)} = -a_{(\gamma,-)} \). Then (in \( M \)):

\[
\text{(14.1.10)(b)} \quad T_\kappa a_{(\gamma,+)} = qa_{(\gamma,+)}
\]

In the notation of my notes *Computing Twisted KLV polynomials*, Section 6, these would be written simply

\[
\begin{align*}
T_\kappa(a_\gamma) &= -a_\gamma \quad (\text{ii2s case}) \\
T_\kappa(a_\gamma) &= qa_\gamma \quad (\text{ir1s case})
\end{align*}
\]

And here is an email from Marc completing the argument.

Date: Fri, 13 Dec 2013 17:05:09 +0100
From: Marc van Leeuwen <Marc.van-Leeuwen@math.univ-poitiers.fr>
To: Jeffrey Adams <jda@math.umd.edu>
CC: David Vogan <dav@math.mit.edu>,
Subject: Confirmation: \( \text{ii2s} \) must be an ascent (Re: \( \text{ii2s/ir1s} \))

On 06/12/13 04:18, Jeffrey Adams wrote:

> Marc raised a (yet another) valid objection to my formulas. After
> talking to David I arrived at the resolution explained in the attached
file. I’m quite confident in the whole picture. However I’m not confident in my ability to calculate the 3x3 matrices in the two conjectures. I give a plausibility argument for them, and hope that David is able to confirm or fix them.

I am not able to see how to compute those 3x3 matrices by sheer brain power either. However, what I can do is compute braid relations. I needed a case where one of these types occur, and fortunately there is an easy one:

> empty: type
> Lie type: A3 sc s
> main: extblock
> (weak) real forms are:
> 0: sl(2,H)
> 1: sl(4,R)
> enter your choice: 1
> possible (weak) dual real forms are:
> 0: su(4)
> 1: su(3,1)
> 2: su(2,2)
> enter your choice: 1
> Name an output file (return for stdout, ? to abandon):
> 0 1 [2C+ ,1rn ] 4 0 (\*,\*) (\*,\*) 2\-e
> 4 3 [2C- ,1i2s] 0 4 (\*,\*) (\*,\*) 1x2\-e

Now there is the first Hecke generator $T_{\{1,3\}}$, which acts by the companion matrix $C$ of the quadratic relation $(X-q^2)(X+1)=X^2-(q^2-1)X-q^2$, and the second Hecke generator $T_{\{2\}}$ which acts by a diagonal matrix $D$ with diagonal coefficients $x=-1$ and $y\in\{-1,q\}$, the latter depending on whether type 1i2s is an ascent ($y=-1$) or a descent ($y=q$). If it is an ascent, then the two matrices obviously commute, and the required braid relation $CD\-CDC=DCCD\$ is satisfied. If however $y=q$, then the matrices do not commute, and a fairly easy computation shows that the off-diagonal coefficients of $CD\-CDC$ and $DCDC$ would not match up. Therefore 1i2s must be an ascent. I suppose that by duality (more or less) 1r1s must be a descent, but I did not really check.

This confirms what Jeff wrote. Cheers,
14.2 email from David Vogan regarding the outline

Date: Thu, 07 Nov 2013 14:31:17 -0500
From: David Vogan <dav@math.mit.edu>
To: Marc.van-Leeuwen@math.univ-poitiers.fr, jeffreydavidadams@gmail.com
Subject: Re: twisted KLV

Dear Marc,

You're absolutely correct that none of the references gives a very clear picture of how the recursions work. The point of Jeff's notes "Computing twisted KLV polynomials" ([CTKLP]) was to do that, and I think he does a good job of writing down all the details properly; but my understanding of your objection is that you want to know where the details come from in order to be able to implement them reliably.

So here is the picture. We have the Coxeter group \((W,S)\) with involutive automorphism \(\sigma\) (defining an involutive automorphism of \(S\)). Therefore \(W^\sigma\) is a Coxeter group with one generator \(\kappa\) for each orbit (also called \(\kappa\)) of \(\sigma\) on \(S\). The orbit \(\kappa\) defines a Levi subgroup of \(W\) of type \(A_1\) or \(A_1 \times A_1\) or \(A_2\), which has a long element \(w_\kappa\) of length 1, 2, or 3 accordingly.

Accordingly we get an unequal parameter Hecke algebra with one generator \(T_\kappa\) for each orbit. The new-looking relation is (2.2) in [CTKLP]:

\[(T_\kappa + 1)(T_\kappa - u^{\ell(w_\kappa)}) = 0.\]

Of course this means that in any module, \(T_\kappa\) has eigenvalues \(-1\) and \(u^{\ell(w_\kappa)}\). It's convenient to introduce \(v = u^{1/2}\) and work instead with
\( \widehat{T}_{\kappa} = v^{-\ell(w_{\kappa})}(T_{\kappa} + 1) \) (Fokko's "c_s"). This element has eigenvalues 0 and \((v^{\ell(w_{\kappa})} + v^{-\ell(w_{\kappa})})\). Obviously
\[
\ker(\widehat{T}_{\kappa}) = \text{zero eigenspace}
\]
\[
\im(\widehat{T}_{\kappa}) = v^{\ell(w_{\kappa})} + v^{-\ell(w_{\kappa})} \text{ eigenspace}.
\]

As you know, the module for the Hecke algebra has a "standard" basis of various \( \widehat{a}_{\gamma} \), and a KL basis of \( \widehat{C}_{\gamma} \), both indexed by the same "parameters" \( \gamma \); and of course the (twisted) KL polynomials are the transition matrix for expressing the \( \widehat{C}_{\gamma} \) in terms of the \( \widehat{a}_{\gamma} \). Here are the key facts.

1. The action of the \( \widehat{T}_{\kappa} \) on the \( \widehat{a}_{\gamma} \) is known, (involving more or less just Cayleys and crosses on \( \gamma \) by \( \kappa \)).

2. For each \( \kappa \), the parameters divide more or less evenly into those for which \( \kappa \) is a DESCENT and those for which it is an ASCENT. These terms are defined by (3) below.

3. \( \kappa \) is a DESCENT for \( \gamma \) if and only if
\[
\widehat{T}_{\kappa} \widehat{C}_\gamma = (v^{\ell(w_{\kappa})} + v^{-\ell(w_{\kappa})}) \widehat{C}_\gamma.
\]
The \( \widehat{C}_\gamma \) with \( \gamma \) a descent are a basis of this eigenspace of \( \widehat{T}_{\kappa} \).

4. If \( \kappa \) is an ASCENT for \( \lambda \), then
\[
\widehat{T}_{\kappa} \widehat{C}_\lambda = \text{combination of } \widehat{C}_\gamma \text{ as in 3.}
\]
Statement 4. is a consequence of the last assertion in 3. and the (obvious) statements about eigenspaces, ker, and im above. It’s Lemma 69.
9.3.5 in [CTKLP], made more explicit in Theorem 9.3.10.

Statement 3. leads to the "easy recursions," because it relates the KL polynomials for \( \gamma \) and \( \delta \) to those for \( \gamma \) and \( \kappa \)-Cayleys and crosses of \( \delta \).

The way to use Statement 4. is in constructing \( \widehat{C}_\lambda \) by induction on \( \lambda \). Given a new and unknown \( \lambda' \), try to write it as cross or Cayley of a shorter \( \lambda \). If you can do this, then \( \widehat{C}_{\lambda'} \) will appear on the right side of the formula in 4., probably with some very simple coefficient. The left side of 4. is known. Try to see that all the other \( C_\gamma \) on the right in this formula are already known, and their coefficients are already known; then you can solve 4. for the unknown \( C_{\lambda'} \).

I’ve written too many words, I’m afraid, but at least it’s shorter than the references. One last point: it might or might not help to list my Park City paper as one of the references for the classical KLV case. But my copy of that has a reasonable number of typos marked on it, so don’t read it too closely.

Take care,
David

14.3 Further explanation of the algorithm

Date: Thu, 19 Dec 2013 21:08:13 +0100
From: Marc van Leeuwen <Marc.van-Leeuwen@math.univ-poitiers.fr>
To: Jeffrey Adams <jeffreydavidadams@gmail.com>
Subject: Re: modules

... 
I’ve not stumbled on anything too difficult to understand, but I do admit being puzzled about where this is going. I’ve checked table 9.1.3, which is OK except for a minus sign at 1r1s that (given the mentioned resolution) should become ‘+’, but I’m not entirely convinced of the utility of the simplification (at the expense of the new notation $\hat{a}_{\lambda \kappa}$); even more so about the notations ‘def._\lambda’ and \( \zeta_{\kappa} \) that follow. It looks like

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there are (a lot) more rewritings of the same stuff before one comes to actual recursion relations; maybe you could give an idea of the big picture for these recursions that would help understanding why these kind of reformulations are useful/necessary. It might help me speed up my reading; I am not particularly good at digesting long and numerous formulas.

Some explanation...

The normalization $\hat{a}_\gamma = v^{-\ell(\gamma)}a_\gamma$ (which I got from Fokko) is just for convenience. On the other hand the terminology $\hat{a}_\kappa^\gamma$ is more serious: this is designed to make Lemma 9.1.1(3) hold, i.e. $\hat{a}_\kappa^\gamma$ for $\kappa \in \tau(\gamma)$ are a basis of the image of $\hat{T}_\kappa$.

The def_\lambda and $\zeta_\kappa$ terminology are intended partly to make coding easier: I assume it is easier to code (and debug) a smaller table like 9.2.4 than the bigger one 9.2.3. But whatever works best is fine.

As far as the algorithm goes, on the one hand if $\kappa \not\in \tau(\lambda)$ then Section 9.2 gives

(14.3.1)(a) $\hat{T}_\kappa(\hat{C}_\lambda) = \sum_{\gamma | \kappa \in \tau(\gamma)} c(\gamma, \lambda)\hat{a}_\gamma^\kappa$

where $c(\gamma, \lambda)$ can be computed provided we know various $\hat{P}_\sigma(\gamma, \mu)$ , see Lemma 9.2.1 which we will.

On the other hand Theorem 9.3.10 says that

(14.3.1)(b) $\hat{T}_\kappa(\hat{C}_\lambda) = \sum_{\gamma | \kappa \in \tau(\gamma)} m_\kappa(\gamma, \lambda)\hat{C}_\gamma$

for certain coefficients $m_\kappa(\gamma, \lambda)$.

Suppose $\kappa \in \tau(\gamma), \tau(\mu)$, and $\mu \rightarrow_\kappa \lambda$, and compare the coefficients of $\hat{a}_\gamma^\kappa$ in (a) and (b). We hope to get a formula for $\hat{P}_\sigma(\gamma, \mu)$.

To compute $P(\gamma, \mu)$ we may assume we know:

$\hat{P}_\sigma(\gamma, \mu)$ if $\ell(\mu) < \ell(\mu)$

$\hat{P}_\sigma(\gamma, \mu)$ if $\ell(\gamma) > \ell(\gamma)$.

So, since $\ell(\lambda) < \ell(\mu)$ we know $c(\gamma, \lambda)$ in (a). On the other hand the coefficient of $\hat{a}_\gamma^\kappa$ in (b) is roughly speaking

(14.3.1)(c) $\hat{P}_\sigma(\gamma, \mu) + \sum_{\delta | \kappa \in \tau(\delta), \ell(\delta) < \ell(\lambda)} \hat{P}_\sigma(\gamma, \delta)m_\kappa(\delta, \lambda)$
Setting this equal to \( c(\gamma, \lambda) \) we conclude

\[
(14.3.1)(d) \quad \hat{P}_\sigma(\gamma, \mu) = c(\gamma, \lambda) - \sum_{\substack{\delta | \kappa \in \tau(\delta) \\ \ell(\delta) < \ell(\lambda)}} \hat{P}_\sigma(\gamma, \delta) m_\kappa(\delta, \lambda)
\]

Since \( \ell(\delta) < \ell(\lambda) < \ell(\mu) \) we know \( \hat{P}_\sigma(\gamma, \delta) \). Again, \textit{roughly speaking}, \( m_\kappa(\delta, \lambda) \) is in terms of various \( \hat{P}_\sigma(\gamma, \lambda') \) with \( \ell(\lambda') \leq \ell(\lambda) < \ell(\mu) \), and \( \hat{P}_\sigma(\delta, \mu) \) with \( \ell(\delta) > \ell(\gamma) \), which we also know. So by induction we can compute \( \hat{P}_\sigma(\gamma, \mu) \).

This argument works precisely as stated in some cases. However it can run into trouble in one or both \textit{roughly speaking} clauses. For one thing the left hand side of (d) may have two terms. (Actually the left hand side of (d) is multiplied by \( \pm 1 \) or \( (v + v^{-1}) \), from Theorem 9.3.10(1), but this isn’t serious). For another it isn’t true that \( \ell(\delta) < \ell(\lambda) \) in (c), only that this holds in most cases, and for most terms. See Section 10.3 for details.

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