Complements and Improvements Regarding Distributivity of the Product for $\sigma$-Algebras with Respect to the Intersection

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Abstract

We present a variety of refined conditions for $\sigma$-algebras $A$ (on a set $X$), $\mathcal{F}, \mathcal{G}$ (on a set $U$) such that the distributivity equation

$$(A \otimes \mathcal{F}) \cap (A \otimes \mathcal{G}) = A \otimes (\mathcal{F} \cap \mathcal{G}),$$

holds – or is violated.

The article generalizes the results in [16] and includes a positive result for $\sigma$-algebras generated by at most countable partitions, was not covered before. We also present a proof that counterexamples may be constructed whenever $X$ is uncountable and there exist two $\sigma$-algebras on $X$ which are both countably separated, but their intersection is not. We present examples of such structures. In the last section, we extend [16, Theorem 2] from analytic to the setting of Blackwell spaces.

Keywords: sigma algebra; intersection of sigma algebras; product sigma algebras; counterexample for sigma algebras

1 Introduction

The question of distributivity of $\sigma$-algebras with respect to the intersection has recently been studied in [16], motivated by questions from stochastic analysis (therein and in [17]), but also coming from a question about sequences of probability spaces in general stochastics [13, 15]$^3$. While the problem can be formulated quite simply using only well known basic constructions, its answer still is nontrivial. The question, apart from the investigation in [16] has not been addressed in various remarkable sources for results on $\sigma$-algebras (sometimes referred to as Borel structures) exceeding standard literature, such as Aumann [1], Basu [2], Bhaskara Rao and Bhaskara Rao [3], Bhaskara Rao and Rao [4], Bhaskara Rao and Shortt [6], Blackwell [7], Georgiou [8], Grzegorek [9], Rao [14] to mention an (incomplete) list of contributions to the theory. A precise description of the problem is the following.

Let $\mathcal{A}$ be a $\sigma$-algebra on a (nonempty) set $X$ and $\mathcal{F}, \mathcal{G}$ be two $\sigma$-algebras on a (nonempty) set $U$. The product of the $\sigma$-algebras $\mathcal{C}$ on a set $Y$ and $\mathcal{D}$ on a set $Z$ is denoted by $\mathcal{C} \otimes \mathcal{D}$ on $Y \times Z$ and is defined

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$^3$There was an erroneous statement in [16] regarding the discussions in the references [13, 15]: The assertion that $(\mu \otimes \nu)(K) = 0$ for every $K \in \bigcap n \in \mathbb{N}(An \otimes U) \setminus \left( \bigcap n \in \mathbb{N}An \right) \otimes U$ has to be replaced with: For all $K \in \bigcap n \in \mathbb{N}(An \otimes U)$ there is an $L \in \left( \bigcap n \in \mathbb{N}An \right) \otimes U$ such that $(\mu \otimes \nu)(L \triangle K) = 0$. 

as the smallest $\sigma$-algebra containing all Cartesian products (or rectangles) $\{C \times D : C \in \mathcal{C}, D \in \mathcal{D}\}$.

We ask, for which $\sigma$-algebras $\mathcal{A}, \mathcal{F}, \mathcal{G}$ is

$$(\mathcal{A} \otimes \mathcal{F}) \cap (\mathcal{A} \otimes \mathcal{G}) = \mathcal{A} \otimes (\mathcal{F} \cap \mathcal{G})$$

which means that '$\otimes$' is distributive with respect to '$\cap$'.

The first trivial observation is the inclusion

$$(\mathcal{A} \otimes \mathcal{F}) \cap (\mathcal{A} \otimes \mathcal{G}) \supseteq \mathcal{A} \otimes (\mathcal{F} \cap \mathcal{G}) .$$

Also rather easy to show is the relation

$$(\mathcal{A} \otimes \mathcal{F}) \lor (\mathcal{A} \otimes \mathcal{G}) = \mathcal{A} \otimes (\mathcal{F} \lor \mathcal{G}) ,$$

see e.g. [17, Proof of Lemma 3.2, Step 2]. The nontrivial results in [16] so far pointed out a counterexample to (1) (Theorem 1) and showed that in case of analytic measurable spaces $(X, \mathcal{M}), (U, \mathcal{U})$ with $\mathcal{A} \subseteq \mathcal{M}, \mathcal{F}, \mathcal{G} \subseteq \mathcal{U}$, and countably generated $\sigma$-Algebras $\mathcal{F} \cap \mathcal{G}, \mathcal{A}, (\mathcal{A} \otimes \mathcal{F}) \cap (\mathcal{A} \otimes \mathcal{G})$, the equation (1) is equivalent to all atoms of $(\mathcal{A} \otimes \mathcal{F}) \cap (\mathcal{A} \otimes \mathcal{G})$ being products (Theorem 2).

Here, we generalize these results in the following directions:

- We show that the distributivity equation (1) holds if either one of the three $\sigma$-algebras are given by a countable partition or if $\mathcal{F} \cap \mathcal{G}$ is given by a countable partition. This is not covered in [16].

- We show that counterexamples are not limited to the special one from [16, Theorem 1] but can be constructed whenever $X$ is not countable and $\mathcal{F}, \mathcal{G}$ are countably separated, but $\mathcal{F} \cap \mathcal{G}$ is not.

- We obtain a characterization of the distributivity equation (1) in terms of its atoms, similar to [16, Theorem 2], but now for strongly Blackwell spaces instead of analytic ones.

### 2 Distributivity for $\sigma$-algebras given by countable partitions

In this section we will treat the countable partition case, as the finite case basically works in a similar, but easier way. We will write $\sigma(\{C_1, C_2, \ldots\})$ for the $\sigma$-algebra generated by $\{C_1, C_2, \ldots\}$. Let $\mathbb{N}$ stand for the set of positive integers. We will say that a $\sigma$-algebra $\mathcal{C}$ is given by a countable partition if there are countably many (or finitely many) pairwise disjoint nonempty sets $\{C_1, C_2, \ldots\}$ in $\mathcal{C}$ such that $\{\bigcup_{i \in I} C_i : I \subseteq \mathbb{N}\} = \mathcal{C}$. Note that on a countable set, every $\sigma$-algebra is given by a countable partition. Observe also that if $\mathcal{C}$ is a $\sigma$-algebra given by a countable partition $\{C_1, C_2, \ldots\}$ and $\mathcal{D}$ is another $\sigma$-algebra then every set in $\mathcal{C} \otimes \mathcal{D}$ is of the form $\bigcup_{i \in \mathbb{N}} (C_i \times D_i)$ for some sequence $(D_i)_{i \in \mathbb{N}}$ such that $D_i \in \mathcal{D}$ for all $i \geq 1$. This representation is also unique. We will be using this result repeatedly in some of the proofs below.

**Theorem 2.1.** Let $\mathcal{A}$ be a $\sigma$-algebra given by a countable partition of $X$. Let $\mathcal{F}, \mathcal{G}$ be $\sigma$-algebras on $U$. Then equation (1) is true.

**Proof.** Let $\mathcal{A}$ be given by a countable partition $\{A_1, A_2, \ldots\}$.

We will take a general set $B \in (\mathcal{A} \otimes \mathcal{F}) \cap (\mathcal{A} \otimes \mathcal{G})$ and show that $B \in \mathcal{A} \otimes (\mathcal{F} \cap \mathcal{G})$. By the above mentioned result, $B$ can be written as $\bigcup_{i \in \mathbb{N}} (A_i \times F_i)$ for some $F_i \in \mathcal{F}$ for every $i$. Since $B \in \mathcal{A} \otimes \mathcal{G}$, by taking an $x$ in $A_i$ and considering the section $B_x = F_i$, we see that $F_i \in \mathcal{G}$ for every $i$. This shows that $B \in \mathcal{A} \otimes (\mathcal{F} \cap \mathcal{G})$. \[\square\]
The following result concerns the right factors.

**Theorem 2.2.** If at least one of $\mathcal{F}$ and $\mathcal{G}$ is given by a countable partition, equation (1) is true. More generally, if $\mathcal{F} \cap \mathcal{G}$ is given by a countable partition, equation (1) is true.

*Proof.* Clearly, if $\mathcal{F}$ (or $\mathcal{G}$) is given by a countable partition, $\mathcal{F} \cap \mathcal{G}$ is also given by a countable partition. Hence it is sufficient to prove the second part of the statement of the theorem. Let $\mathcal{F} \cap \mathcal{G}$ be given by a countable partition $\{H_1, H_2, \ldots \}$.

We will take a general set $B \in (\mathcal{A} \otimes \mathcal{F}) \cap (\mathcal{A} \otimes \mathcal{G})$ and show that $B \in \mathcal{A} \otimes (\mathcal{F} \cap \mathcal{G})$. Let $x$ be a point in $X$. Consider the $x$-section $B_x = \{u : (x, u) \in B\}$. Then, $B_x \in \mathcal{F} \cap \mathcal{G}$. Hence $B_x \supseteq H_i$ for some $i$ if $B_x \neq \emptyset$. Now, for a particular $i \in \mathbb{N}$, if we define the set $A_i = \{x : B_x \supseteq H_i\}$ then, $A_i \in \mathcal{A}$ and $B \supseteq A_i \times H_i$. It easily follows that $B = \bigcup_{i \in \mathbb{N}}(A_i \times H_i)$. Thus $B \in \mathcal{A} \otimes (\mathcal{F} \cap \mathcal{G})$.

Thus we have seen that if at least one of the three $\sigma$-algebras $\mathcal{A}, \mathcal{F}, \mathcal{G}$ in equation (1) is given by a countable partition, (1) is true. We will extend this result to the case of $\sigma$-algebras with the property that every countably generated sub-$\sigma$-algebra is given by a countable partition. The $\sigma$-algebra of countable and co-countable sets on any infinite set is an example of such a $\sigma$-algebra (which itself is not even countably generated). This $\sigma$-algebra is atomic. There are examples of atomless $\sigma$-algebras (on any uncountable set) with this property. See [5, Remark 5, p. 108] for examples of such $\sigma$-algebras.

**Theorem 2.3.** If one of the three $\sigma$-algebras $\mathcal{A}, \mathcal{F}$ and $\mathcal{G}$ in (1) is such that every countably generated sub-$\sigma$-algebra is given by a countable partition, then (1) is true. However, if $\mathcal{F} \cap \mathcal{G}$ has the property that every countably generated sub-$\sigma$-algebra is given by a countable partition, but $\mathcal{F} \cap \mathcal{G}$ itself is not given by a countable partition, (1) need not be true.

*Proof.* We will treat the case of $\mathcal{F}$ having the property that every countably generated sub-$\sigma$-algebra is given by a countable partition $\{F_1, F_2, \ldots \}$. Other cases can be covered using a similar argument. We will take a general set $B \in (\mathcal{A} \otimes \mathcal{F}) \cap (\mathcal{A} \otimes \mathcal{G})$ and show that $B \in \mathcal{A} \otimes (\mathcal{F} \cap \mathcal{G})$.

We know that if a set $B \in \mathcal{C} \otimes \mathcal{D}$ there exists $\{C_1, C_2, \ldots \}$ in $\mathcal{C}$ and $\{D_1, D_2, \ldots \}$ in $\mathcal{D}$ such that $B \in C_0 \otimes D_0$ where $C_0$ is the sub-$\sigma$-algebra generated by $\{C_1, C_2, \ldots \}$ and $D_0$ is the sub-$\sigma$-algebra generated by $\{D_1, D_2, \ldots \}$. We will apply this result to $\mathcal{A} \otimes \mathcal{F}$.

Since $B \in \mathcal{A} \otimes \mathcal{F}$ there exists $\{A_1, A_2, \ldots \}$ in $\mathcal{A}$ and $\{F_1, F_2, \ldots \}$ in $\mathcal{F}$ such that $B \in A_0 \otimes F_0$ where $A_0$ is the sub-$\sigma$-algebra generated by $\{A_1, A_2, \ldots \}$ and $F_0$ is the sub-$\sigma$-algebra generated by $\{F_1, F_2, \ldots \}$. Since $F_0$ is a countably generated sub-$\sigma$-algebra it is given by a countable partition. The previous theorem gives us that $B \in A_0 \otimes (F_0 \cap G_0)$. Hence $B \in \mathcal{A} \otimes (\mathcal{F} \cap \mathcal{G})$.

The example in [16, Theorem 1] has the property that $\mathcal{F} \cap \mathcal{G}$ is the countable co-countable $\sigma$-algebra and this $\sigma$-algebra has the property that every countably generated sub-$\sigma$-algebra is given by a countable partition. Thus we have concluded the second statement of the theorem.

In Theorem 3.6 below, we will show that the second part of the above theorem is true in general. We remark that, as a consequence of Theorem 2.2, if $\mathcal{F} \cap \mathcal{G}$ is the trivial $\sigma$-algebra $\{\emptyset, U\}$ or a finite $\sigma$-algebra, then also (1) is true.

### 3 Counterexamples for Uncountable Sets

In this section here, we first point out, that for $\sigma$-algebras $\mathcal{F}$ and $\mathcal{G}$ on a set $U$, which are not given by countable partitions, such that $\mathcal{F} \cap \mathcal{G}$ is infinite and neither $\mathcal{F} \subseteq \mathcal{G}$ nor $\mathcal{G} \subseteq \mathcal{F}$ hold, and for a
σ-algebra $A$, there are examples such that (1) does not hold. We already saw classes of examples of σ-algebras such that (1) holds.

Let us recall a basic definition first.

**Definition 3.1.** Let $(Y, C)$ be a measurable space.

(i) The σ-algebra $C$ is called **separated** if there is a set $I$, together with sets $\{A_i : i \in I\} \subset C$, that separate the points of $Y$, that is, for any two points $x, y \in Y$ there is an $i \in I$ such that $x \in A_i$ and $y \notin A_i$. We call such a system of sets separator.

(ii) The σ-algebra $C$ is called **countably separated** if $C$ contains a countable separator.

**Remark 3.2.** (i) Note that if a countable separator exists for $(Y, C)$ then $C$ contains all singletons: just write for $x \in Y$,

$$
\{x\} = \bigcap_{i \geq 1} A_i \cap \bigcap_{x \notin A_i} A_i^c.
$$

(ii) If $C$ is separated and countably generated by the generator $\{A_i : i \geq 1\}$, then this generator features as countable separator as well: Assume the contrary, that there are no sets separating the points $x$ and $y \in Y$. That means that all sets $A_i$ contain either both, $x$ and $y$, or none of them. The smallest σ-algebra that contains all the $A_i$ and which does not separate $x$ and $y$ is the σ-algebra generated by $\{A_i : i \geq 1\}$ and must thus be $C$. But $C$ is separable. Hence $\{A_i : i \geq 1\}$ is a separator.

**Example 3.3.** (a) The Borel σ-algebra of any separable metric space is countably generated and countably separated.

(b) The σ-algebra of Borel sets of $\mathbb{R}$ invariant of translation by 1 (or any other nonzero number) is countably generated but not countably separated (any set containing 1 also contains $\mathbb{Z}$).

(c) The σ-algebra generated by the analytic sets of $[0, 1]$ is not countably generated [4, p. 15] but countably separated (just take the separator of the Borel sets).

(d) The σ-algebra of countable and co-countable sets on an uncountable set $Y$ is neither countably generated nor countably separated.

The next theorem is a (substantial) extension of [16, Theorem 1] in the sense that we provide classes of examples for which (1) does not hold.

**Theorem 3.4.** Let $B$ and $D$ be any countably separated σ-algebras on an uncountable set $X$, such that $B \cap D =: C$ is not countably separated. Then,

$$
((D \vee B) \otimes B) \cap ((D \vee B) \otimes D) \neq (D \vee B) \otimes (D \cap B). \tag{2}
$$

**Proof.** We will first show that the diagonal $\Delta = \{(x, x) : x \in X\}$ is in in the left hand side of (2), Let $\{A_i : i \geq 1\}$ be a countable separator of $B$. $\Delta$ can be expressed as

$$
\Delta = \bigcap_{i \geq 1} ((A_i \times A_i) \cup (A_i^c \times A_i^c))
$$
This can be seen, as for all \( x \in X \) and \( i \geq 1 \), \((x, x)\) is either contained in \( A_i \times A_i \) or \( A_i^c \times A_i^c \). To exclude any \((x, y)\) with \( x \neq y\), take an \( A_i \) separating \( x \) and \( y\), s.t. say \( x \in A_i \) and \( y \in A_i^c \). Now \((x, y) \notin (A_i \times A_i) \cup (A_i^c \times A_i^c)\). It follows that \( \Delta \in \mathcal{B} \otimes \mathcal{B} \). In the same way, \( \Delta \in \mathcal{D} \otimes \mathcal{D} \), showing that \( \Delta \) is contained in the left hand side of (2).

If for \( \sigma \)-algebras \( \mathcal{H}, \mathcal{I} \) on \( X \), the diagonal \( \Delta \) is contained in \( \mathcal{H} \otimes \mathcal{I} \), then \( \mathcal{I} \) needs to be countably separated. This is a special case e.g. of [12, Proposition 2.1] taking \( f \) equal to the identity function.

Since \( \mathcal{C} \) is not countably separated, it follows that \( \Delta \) cannot be contained in the right hand side of (2).

Two examples of pairs of \( \sigma \)-algebras satisfying the conditions of the above theorem can be found in [4]. The first such example is from [4, p. 16] and the second from [4, Proposition 57, p. 55].

**Example 3.5.** (a) This example was first published in [1], following an idea from P. R. Halmos, and was also used as a counterexample for (1) in [16].

The \( \sigma \)-algebra \( \mathcal{D} \) is the preimage of the Borel sets \( \mathcal{B} \) on \( X = [0, 1] \) under a certain function \( f : [0, 1] \to [0, 1] \) constructed as follows (see also [1], [14]):

Let \( \omega_c \) be the first ordinal corresponding to the cardinal \( c \) of the continuum. Let \( (M_\alpha)_{1 \leq \alpha < \omega_c} \) be an enumeration of all uncountable Borel subsets of \([0, 1]\) with uncountable complement. Since all uncountable Borel sets have cardinality \( c \) (see e.g. [10, Theorem 13.6]) we can associate to each ordinal \( \alpha < \omega_c \) a triplet \((x_\alpha, y_\alpha, z_\alpha)\) such that \( x_\alpha, y_\alpha \in M_\alpha, z_\alpha \in [0, 1] \setminus M_\alpha \) and \( \{x_\alpha, y_\alpha, z_\alpha\} \cap \bigcup_{\beta < \alpha} \{x_\beta, y_\beta, z_\beta\} = \emptyset \) (as the cardinality of \( \bigcup_{\beta < \alpha} \{x_\beta, y_\beta, z_\beta\} \) is strictly smaller than \( c \)). Define \( f \) as the function that for each \( \alpha < \omega_c \) maps \( x_\alpha \mapsto z_\alpha, z_\alpha \mapsto x_\alpha \) and keeps \( y_\alpha \) and all points outside \( \bigcup_{\beta < \omega_c} \{x_\beta, y_\beta, z_\beta\} \) fixed.

Finally, set

\[
\mathcal{D} := \sigma(f) = \{f^{-1}(B) : B \in \mathcal{B}\}.
\]

The intersection \( \mathcal{B} \cap \mathcal{D} \) is the \( \sigma \)-algebra of countable and co-countable sets.

(b) [4, Proposition 57] Let \( f \) be a bijection of \([0, 1]\) to some analytic, non-Borel set \( A \subseteq [0, 1] \). Let \( \mathcal{B} = \mathcal{B}_f([0, 1]) \), and let \( \mathcal{B}_A \) be the trace \( \sigma \)-algebra on \( A \). Set \( \mathcal{D} = f^{-1}(\mathcal{B}_A) \). Assume that \( \mathcal{B} \cap \mathcal{D} \) were countably separated. Then it contains a separator \( \{A_i : i \geq 1\} \) and \( \sigma(A_i : i \geq 1) \) is countably generated and countably separated. It follows through [4, Proposition 5] (the Blackwell nature of \(([0, 1], \mathcal{B})\) and \(([0, 1], \mathcal{B})\)) that \( \sigma(A_i : i \geq 1) = \mathcal{B} = \mathcal{D} \) and that \( f \) were an isomorphism \(([0, 1], \mathcal{B}([0, 1])) \leftrightarrow (A, \mathcal{B}_A)\). This would imply that \( A \) is a standard Borel set, which it is not.

Hence \( \mathcal{B} \cap \mathcal{D} \) is not countably separated.

A consequence of Theorem 3.4 is that the second part of Theorem 2.3 is in fact true in general if \( \mathcal{F} \) and \( \mathcal{G} \) are countably separated.

**Theorem 3.6.** If \( \mathcal{F} \) and \( \mathcal{G} \) are countably separated and if \( \mathcal{F} \cap \mathcal{G} \) has the property that every countably generated sub-\( \sigma \)-algebra is given by a countable partition, but \( \mathcal{F} \cap \mathcal{G} \) itself is not given by a countable partition, (1) is not true.

**Proof.** If \( \mathcal{F} \cap \mathcal{G} \) has the mentioned property, then, we show that \( \mathcal{F} \cap \mathcal{G} \) is not countably separated. If it is, let \( \{A_1, A_2, \ldots\} \) be a countable separator. Then \( \sigma(\{A_1, A_2, \ldots\}) \) being a countably generated sub-\( \sigma \)-algebra is given by a countable partition. If \( \{B_1, B_2, \ldots\} \) is the countable partition, all the \( B_i \)s need to be singleton sets. This implies that \( \sigma(\{B_1, B_2, \ldots\}) = \sigma(\{A_1, A_2, \ldots\}) = \mathcal{F} \cap \mathcal{G} \) and that \( \mathcal{F} \cap \mathcal{G} \) is given by a countable partition and this is a contradiction. Thus, \( \mathcal{F} \cap \mathcal{G} \) is not countably separated. From Theorem 3.4 the result follows.

\( \square \)
4 Distributivity and Blackwell spaces

Definition 4.1. Let \((Y, C)\) be a measurable space.

(i) An atom \(K \in C\) is a nonempty set, such that no proper nonempty subset is contained in \(C\). In other words, they are the minimal elements with respect to ‘\(\subseteq\)’ in \(C\).

(ii) A \(\sigma\)-algebra \(C\) is called atomic if \(Y\) is the union of the atoms of \(C\).

This definition of an atom does not coincide with the similar notion from measure theory, where it is a set of positive measure, such that all of its proper subsets in the \(\sigma\)-algebra have zero measure. There are \(\sigma\)-algebras containing atoms and atomless \(\sigma\)-algebras [4, Chapter 3]. Note that if two atoms of \(C\), when they exist, are disjoint.

Definition 4.2. Let \((Y, C)\) be a measurable space.

(i) A subset of \([0, 1]\) is called analytic if it is the image of a Polish space under a continuous function.

(ii) We call \((Y, C)\) analytic if \(Y\) is isomorphic (i.e. there is a bijective, bimeasurable function) to an analytic subset of the unit interval and \(C\) is countably generated and contains all singletons of \(Y\).

We cite the following result of Blackwell [7] and Mackey [11], see also [4, Chapter 2].

Lemma 4.3 ([7, Section 4], [11, Section 4], [4, Proposition 6]).
If \((Y, C)\) is an analytic space and \(W, V\) are countably generated sub-\(\sigma\)-algebras of \(C\) with the same atoms, then \(W = V\).

To generalize the situation of Lemma 4.3, we recall the definition of a Blackwell space:

Definition 4.4. Let \((Y, C)\) be a countably generated and separated (and thus countably separated) measurable space.

(i) The space \((Y, C)\) is called Blackwell if the only separated sub-\(\sigma\)-algebra of \(C\) is itself.

(ii) The space \((Y, C)\) is called strongly Blackwell if any two countably generated sub-\(\sigma\)-algebras with the same atoms coincide.

For a thorough account on Blackwell spaces see [6]. Therein it is also shown that under the continuum hypothesis (CH) there are Blackwell spaces which are not strongly Blackwell. The same holds if Martin’s axiom and \((\neg\text{CH})\) is assumed.

To obtain a distributivity characterization in the sense of [16, Theorem 2] in the case of Blackwell instead of analytic spaces, we formulate the following Lemma (which has been shown in several places in the literature during proofs, such as in [4] or [16]).

Lemma 4.5. If \(B\) on \(X\) and \(C\) on \(Y\) are atomic \(\sigma\)-algebras, then \(B \otimes C\) is atomic and atoms of \(B \otimes C\) are cartesian products.

Proof. If \(B \in B\) and \(C \in C\) are atoms, to show that \(B \times C\) is an atom, observe that every rectangle in \(B \otimes C\) either contains \(B \times C\) or is disjoint with \(B \times C\). On the other hand, \(\{ A \in B \otimes C : A\) either contains \(B \times C\) or is disjoint with \(B \times C\}\) is a \(\sigma\)-algebra. Hence every set \(B \times C\) either contains \(B \times C\) or is disjoint with \(B \times C\). But then the union of all such sets (atoms) is equal to \(X \times Y\). Thus \(B \otimes C\) is atomic.
Theorem 4.6. Let \((X, \mathcal{A}), (\mathcal{U}, \mathcal{F})\) and \((U, \mathcal{G})\) be measure spaces. In the following, \([(i) \Rightarrow (ii)]\) is always true. The assertion \([(ii) \Rightarrow (iii)]\) is true if \(\mathcal{A}\) and \(\mathcal{F} \cap \mathcal{G}\) are atomic.

(i) \((\mathcal{A} \otimes \mathcal{F}) \cap (\mathcal{A} \otimes \mathcal{G}) = \mathcal{A} \otimes (\mathcal{F} \cap \mathcal{G})\)

(ii) \((\mathcal{A} \otimes \mathcal{F}) \cap (\mathcal{A} \otimes \mathcal{G})\) and \(\mathcal{A} \otimes (\mathcal{F} \cap \mathcal{G})\) have the same atoms

(iii) All atoms of \((\mathcal{A} \otimes \mathcal{F}) \cap (\mathcal{A} \otimes \mathcal{G})\) are cartesian products

If \((\mathcal{A} \otimes \mathcal{F}) \cap (\mathcal{A} \otimes \mathcal{G})\) is strongly Blackwell (or a sub \(\sigma\)-algebra of a strongly Blackwell space \((X, \mathcal{M})\)), \((\mathcal{A} \otimes \mathcal{F}) \cap (\mathcal{A} \otimes \mathcal{G})\) is countably generated and \(\mathcal{A} \otimes (\mathcal{F} \cap \mathcal{G})\) is countably generated, then (iii) implies (i).

Proof. The proof is a direct consequence of Lemma 4.5 and the definition of a strongly Blackwell space.

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