Abstract
We study optimal quadrature formulas for arbitrary weighted integrals and integrands from the Sobolev space $H^1([0,1])$. We obtain general formulas for the worst case error depending on the nodes $x_j$. A particular case is the computation of Fourier coefficients, where the oscillatory weight is given by $\varrho_k(x) = \exp(-2\pi i k x)$. Here we study the question whether equidistant nodes are optimal or not. We prove that this depends on $n$ and $k$: equidistant nodes are optimal if $n \geq 2|k| + 1$ but might be suboptimal for small $n$. In particular, the equidistant nodes $x_j = j/|k|$ for $j = 0, 1, \ldots, |k| = n + 1$ are the worst possible nodes and do not give any useful information.

Keywords Oscillatory integrals · Quadrature formulas · Sobolev space

Mathematics Subject Classification 65Y20 · 42B20 · 65D30 · 68Q25

1 Introduction

We know many results about optimal quadrature formulas, see Brass and Petras [2] for a recent monograph. This book also contains important results for the approximation of Fourier coefficients of periodic functions, mainly for equidistant nodes. As a general survey for the computation of oscillatory integrals we recommend Huybrechs and Olver [4].
It follows from results of [8,9] that equidistant nodes lead to quadrature formulas that are asymptotically optimal for the standard Sobolev spaces $H^s([0, 1])$ of periodic functions and also for $C^s$ functions.

We want to know whether equidistant nodes are optimal or not. Žensykbaev [12] proved that for the classical (unweighted) integrals of periodic functions from the Sobolev space $W^p_0([0, 1])$, the best quadrature formula of the form $A_n(f) = \sum_j a_j f(x_j)$ is the rectangular formula with equidistant nodes. Algorithms with equidistant nodes were also studied by Boltæev et al. [1] for the numerical calculation of Fourier coefficients. In their paper not only the rectangular formula was studied but all (and optimal) formulas based on equidistant nodes. It is not clear however, whether equidistant nodes are optimal or not.

We did not find a computation of the worst case error of optimal quadrature formulas for general $x_j$ in the literature. Related to this, we did not find a discussion about whether equidistant nodes are optimal for oscillatory integrals or not. We find it interesting that the results very much depend on the frequency of oscillations and the number of nodes.

This paper has two parts. In the first part, we present general formulas for the worst case error of arbitrary weighted integrals in the Sobolev space $H^1$ for arbitrary nodes. In the second part we consider oscillatory integrals and prove that equidistant nodes are optimal for relatively large $n$, but can be very bad for small $n$.

We now describe our results in more detail. We study optimal algorithms for the computation of integrals

$$I_\varrho(f) = \int_0^1 f(x)\varrho(x) \, dx,$$

where the density $\varrho$ can be an arbitrary integrable function. We assume that the integrands are from the Sobolev space $H^1$ and, for simplicity, often assume zero boundary values, i.e., $f \in H^1_0$. Here $H^1_0 = H^1_0([0, 1])$ is the space of all absolutely continuous functions with values in $\mathbb{C}$ such that $f' \in L^2$ and $f(0) = f(1) = 0$. The norm in $H^1_0$ (and semi-norm in $H^1$) is given by $\|f\| := \|f'\|_{L^2}$. We simply write $\| \cdot \|_2$ instead of $\| \cdot \|_{L^2}$.

We study algorithms that use a “finite information” $N : H^1 \to \mathbb{C}^n$ given by

$$N(f) = (f(x_1), \ldots, f(x_n)).$$

We may assume that

$$0 \leq x_1 < x_2 < \cdots < x_n \leq 1.$$

Let $A_n$ be an algorithm that uses the information $N$, such as

$$A_n(f) = \sum_{j=1}^n a_j f(x_j)$$

and let $F$ be a set of integrands. Then the worst case error of $A_n$ is given by

$$e(A_n, F) := \sup_{f \in F : \|f\|_F \leq 1} |I_\varrho(f) - A_n(f)|.$$

The radius of the information $N$ is

$$r(N) := \inf_{A_n} e(A_n, F),$$

it is the (worst case) error of the optimal $A_n$ that uses the information $N$.

We prove general results for the worst case error for arbitrary $\varrho$ and nodes $(x_j)_j$ and then study in more detail integrals with the density function $\varrho_k(x) = \exp(-2\pi i k x)$. Here we
want to know whether equidistant nodes are optimal or not. We shall see that this depends on $n$ and $k$: equidistant nodes are optimal if $n \geq 2.7|k| + 1$ but might be suboptimal for small $n$. In particular, the equidistant nodes $x_j = j/|k|$ for $j = 0, 1, \ldots, |k| = n + 1$ are the worst possible nodes and do not give any useful information.

The aim of this paper is to prove some exact formulas on the $n$th minimal (worst case) errors, for $F \in \{H_0^1, H^1\}$,

$$e(n, I_\varrho, F) := \inf_{A_n} \sup_{f \in F : \|f\|_F \leq 1} |I_\varrho(f) - A_n(f)|.$$ 

This number is the worst case error on the unit ball of $F$ of an optimal algorithm $A_n$ that uses at most $n$ function values for the approximation of the functional $I_\varrho$. Hence an algorithm $A_n$ is called optimal (for the class $F$) if $e(A_n, F) = e(n, I_\varrho, F)$. Such an algorithm makes optimal use of $n$ function values.

The initial error is given for $n = 0$ when we do not sample the functions. In this case the best we can do is to take the zero algorithm $A_0(f) = 0$, and

$$e(0, I_\varrho, F) := \sup_{f \in F : \|f\|_F \leq 1} |I_\varrho(f)| = \|I_\varrho\|_F.$$ 

Let us collect the main results of this paper:

(i) For general (integrable) weight functions $\varrho : [0, 1] \to \mathbb{C}$, we derive formulas for the initial error (Proposition 2) and for the radius of information (worst case error of the optimal algorithm) for arbitrary nodes (Theorem 3).

(ii) We study oscillatory integrals with the weight function $\varrho_k(x) = \exp(-2\pi i k x)$ for the space $H_0^1([0, 1])$. In Proposition 5 we compute the initial error for $k \in \mathbb{Z}\{0\}$ and the main result is Theorem 7 for $k \in \mathbb{R}\{0\}$, where we prove that equidistant nodes are optimal if $n \geq 2.7|k| - 1$.

(iii) Then we study the full space $H^1([0, 1])$ and again prove that equidistant nodes are optimal for $k \in \mathbb{R}\{0\}$ and large $n$. See Theorem 11 for the details. We could prove very similar results also for the subspace of $H^1([0, 1])$ of periodic functions or for functions with a boundary value (such as $f(0) = 0$). Since the results and also the proofs are similar, we skip the details.

(iv) In Sect. 4 we discuss results for equidistant nodes $x_j = j/n$, for $j = 0, 1, \ldots, n$, and prove certain asymptotic results (which are the same for equidistant and optimal nodes). In particular we obtain

$$\lim_{|k| \to \infty} e(n, I_{\varrho_k}, H^1) \cdot |k| = \frac{1}{2\pi}$$

for each fixed $n$ and

$$\lim_{n \to \infty} e(n, I_{\varrho_k}, H^1) \cdot n = \frac{1}{2\sqrt{3}}$$

for each fixed $k \in \mathbb{R}\{0\}$.

\section{2 Arbitrary Density Functions}

We start with

$I_\varrho(f) = \int_a^b f(x) \varrho(x) \, dx$
for \( f \in H^1_0([a, b]) \) and want to compute the so called initial error

\[
e_0 := \sup_{\|f\| \leq 1} |I_0(f)|.
\]

Since the complex valued case is considered here, the inner product in the spaces \( H^1_0([a, b]) \) is given by

\[
(f, g) = \int_a^b f'(x)g'(x) \, dx.
\]

Using the integration by parts formula we see that the initial error is given by

\[
e_0 = \sup_{\|f\| \leq 1} \left| \int_a^b f'(x) \cdot R(x) \, dx \right|,
\]

where \( R(t) = \int_a^t \varrho(x) \, dx \) for \( t \in [a, b] \). To solve the extremal problem

\[
\sup_{\|g\| \leq 1} \left| \int_a^b g(x)R(x) \, dx \right|,
\]

we decompose \( R \) into a constant \( c \) and an orthogonal function \( \widetilde{R} \), \( R = \widetilde{R} + c \), hence \( c = \frac{1}{b-a} \int_a^b R(x) \, dx \), \( \widetilde{R} = R - c \) and \( \int_a^b \widetilde{R}(x) \, dx = 0 \). It then follows from the Cauchy-Schwarz inequality that every \( g^* = \gamma \frac{\widetilde{R}}{\|\widetilde{R}\|_2} \) with \( |\gamma| = 1 \) solves the extremal problem and the respective maximum is \( \|R - c\|_2 = \|R\|_2 \).

We define \( f^* \) by

\[
f^*(t) = -\int_a^t \frac{\widetilde{R}(x) - c}{\|R - c\|_2} \, dx.
\]

Then \( f^*(a) = f^*(b) = 0 \) and \( f^* \in H^1_0([a, b]) \). Further,

\[
\int_a^b f^*(x)\varrho(x) \, dx = \int_a^b f^*(x) \, dR(x)
\]

\[
= f^*(x)R(x)|^b_a - \int_a^b (f^*)'(x)R(x) \, dx
\]

\[
= - \int_a^b (f^*)'(x)(R(x) - c) \, dx
\]

\[
= \int_a^b \frac{\widetilde{R}(x) - c}{\|R - c\|_2} (R(x) - c) \, dx
\]

\[
= \|R - c\|_2.
\]

**Remark 1** It is easy to check that the property \( R(a) = 0 \) is not used in the above computations. Therefore it is not important what is chosen as the lower limit of the integral in the definition of \( R \).

Hence we have proved the following proposition.
Proposition 2 Consider $I_\phi : H^1_0([a, b]) \to \mathbb{C}$ with an integrable density function $\phi$. Then

$$e_0 = \sup_{\|f\| \leq 1} |I_\phi(f)| = \|R - c\|_2,$$

where $R(t) = \int_a^t \phi(x) \, dx$ for $t \in [a, b]$ and $c = \frac{1}{b-a} \int_a^b R(x) \, dx$. Moreover the maximum is assumed for $f^* \in H^1_0([a, b])$, given by

$$f^*(t) = -\int_a^t \frac{R(x) - c}{\|R - c\|_2} \, dx,$

i.e., $I_\phi(f^*) = \|R - c\|_2$ and $\|f^*\| = 1$ with $f^*(a) = f^*(b) = 0$. □

The initial error $e_0$ clearly depends on $a$, $b$ and $\phi$ and later we will write $e_0(a, b, \phi)$ for it.

We are in a Hilbert space setting (with the two Hilbert spaces $H = H^1([0, 1])$ and $H_0^1([0, 1])$) and the structure of optimal algorithms $A = \phi \circ N$, for a given information $N : H \to \mathbb{C}^n$, is known: the spline algorithm is optimal and the spline $\sigma$ is continuous and piecewise linear, see [10, Cor. 5.7.1] and [11, p. 110].

More exactly, if $N(f) = y \in \mathbb{C}^n$ are the function values at $(x_1, \ldots, x_n)$, then $A(f) = \phi(y) = I_\phi(\sigma)$. In the case $H = H_0^1([0, 1])$ the spline $\sigma$ is given by $\sigma(0) = \sigma(1) = 0$ and $\sigma(x_i) = f(x_i) = y_i$ and piecewise linear. In the case $H = H^1([0, 1])$ the spline is constant in $[0, x_1]$ and $[x_n, 1]$, otherwise it is the same function as in the case $H = H_0^1([0, 1])$.

Moreover, we have the general formula for the worst case error of optimal algorithms $A$

$$\sup_{\|f\| \leq 1} |I_\phi(f) - A(f)| = \sup_{\|f\| \leq 1, N(f)=0} |I_\phi(f)|.$$

This number is also called the radius $r(N)$ of the information $N$ and to distinguish the two cases, we also write $r(N, H^1)$ and $r(N, H_0^1)$, respectively, see [10, Thm. 5.5.1 and Cor. 5.7.1] and [11, Thm. 2.3 of Chap. 1].

We are ready to present a general formula for $r(N, H_0^1)$ and afterwards solve another extremal problem to present the formula for $r(N, H^1)$.

We put $x_0 = 0$ and $x_{n+1} = 1$ and then have $n + 1$ intervals $I_j = [x_j, x_{j+1}]$, where $j = 0, 1, \ldots, n$. For the norm $\|f\| := \|f^\prime\|_2$, the worst case function $f^*_j$ is, on any interval $I_j$, as in Proposition 2. The norm of $f^*_j$ is one and the integral is $e_0(x_j, x_{j+1}, \phi) = c_j$. Then the radius of information of the information $N$ is given by

$$r(N) = \max_{a_j \geq 0} \sum_{j} \alpha_j e_0(x_j, x_{j+1}, \phi) \sum_{\alpha_j^2 + 1}$$

and it is easy to solve this extremal problem. The maximum is taken for $\alpha_j = (\sum_j c_j^2)^{-1/2} c_j$ and then the total error is the radius of information, $r(N) = \sum_j \alpha_j c_j = (\sum_j c_j^2)^{1/2}$. As a result we obtain the following assertion.

Theorem 3 In the case of $H_0^1([0, 1])$ the radius of information is given by

$$r(N) = \left( \sum_{j=0}^n e_0(x_j, x_{j+1}, \phi)^2 \right)^{1/2}.$$
Moreover, the worst case function $f^*$ is given by
\[ f^*|_{I_j} = \left( \sum_{j=0}^{n} c_j^2 \right)^{-1/2} \cdot c_j \cdot f_j^*, \]
where $c_j = e_0(x_j, x_{j+1}, \varrho)$. In particular we have $f^* \in H^1_0([0, 1])$ with norm 1 and $N(f^*) = 0$ with $I_0(f^*) = r(N)$. \hfill \square

Now we turn to the space $H^1([0, 1])$. In this case, we need a small modification for the intervals $[0, x_1]$ and $[x_n, 1]$ since the value of $f(0)$ is unknown if $x_1 > 0$ and $f(1)$ is unknown if $x_n < 1$.

For those functions $f \in H^1([a, b])$ satisfying $f(a) = 0$, we take $R(t) = \int_a^b \varrho(x) \, dx$, $t \in [a, b]$. Then $R(b) = 0$ and the respective maximum is $\|R\|_2$. We define $f^* \in H^1([a, b])$ by
\[ f^*(t) = \int_a^t \frac{R(x)}{\|R\|_2} \, dx. \]
Then $f^*(a) = 0$, $(f^*)'(b) = 0$ and $\|f^*\| = 1$. Afterwards,
\[ \int_a^b f^*(x) \varrho(x) \, dx = - \int_a^b f^*(x) \, dR(x) \]
\[ = - f^*(x) R(x)|_a^b + \int_a^b (f^*)'(x) R(x) \, dx \]
\[ = \int_a^b (f^*)'(x) R(x) \, dx \]
\[ = \int_a^b \frac{R(x)}{\|R\|_2} R(x) \, dx \]
\[ = \|R\|_2. \tag{1} \]

Similarly, for the functions $f \in H^1([a, b])$ satisfying $f(b) = 0$, we take $R(t) = \int_t^b \varrho(x) \, dx$, $t \in [a, b]$. Then $R(a) = 0$ and the respective maximum is $\|R\|_2$. We define $f^*$ by
\[ f^*(t) = - \int_t^b \frac{R(x)}{\|R\|_2} \, dx. \]
Then $f^*(b) = 0$, $(f^*)'(a) = 0$ and $\|f^*\| = 1$. Also, $I_0(f^*) = \|R\|_2$.

Hence, we obtain almost the same assertion for the full space $H^1([0, 1])$ as in Theorem 3. Here, $c_0 = e_0(0, x_1, \varrho) = \|R\|_2$ on $[0, x_1]$ if $x_1 > 0$ and $c_n = e_0(x_n, 1, \varrho) = \|R\|_2$ on $[x_n, 1]$ if $x_n < 1$, instead of so-called $\|R - c\|_2$. Accordingly, $f_0^*$ and $f_n^*$ should be changed.

Observe that the initial error is infinite if $I(\varrho) \neq 0$ since all constant functions have a semi-norm zero. Therefore we now assume that $I(\varrho) = 0$. Then for the full space $H^1([0, 1])$, the initial error of the problem $I_0$ is, as in (1),
\[ e_0(H^1, \varrho) := \sup_{\|f\|_{H^1} \leq 1} |I_0(f)| = \|R\|_2. \tag{2} \]
where $R(t) = \int_t^1 \varrho(x) \, dx$ for $t \in [0, 1]$. \hfill \square
Remark 4 In the case \( \varrho_k(x) = \exp(-2\pi ikx) \), the worst case function is, in each interval \( I_j = [x_j, x_{j+1}] \), of the form
\[
f(x) = c_j \exp(-2\pi ikx) + a_j x + b_j,
\]
with \( f(x_j) = 0 \) for \( j = 1, \ldots, n \) and \( f'(0) = 0 \) if \( x_1 > 0 \) and \( f'(1) = 0 \) if \( x_n < 1 \).

3 Oscillatory Integrals: Optimal Nodes

In this section we consider optimal nodes for integrals with the density function
\[
\varrho_k(x) = \exp(-2\pi ikx), \quad k \in \mathbb{R} \setminus \{0\}, \quad x \in [0, 1].
\]
The integrands are from the spaces \( H^1_0([0, 1]) \) or \( H^1([0, 1]) \), respectively.

3.1 The Case with Zero Boundary Values

We want to know whether in this case equidistant nodes, i.e.,
\[
x_j = \frac{j}{n+1}, \quad j = 1, \ldots, n,
\]
are optimal for the space \( H^1_0([0, 1]) \) or not. We will see that they are optimal for large \( n \), but not for small \( n \).

Following Sect. 2, in this case we can consider a general interval \( [a, b] \) and compute \( R(x) \), constant \( c \), and the initial error \( \| R - c \|_2 \). Then we obtain that the initial error depends only on \( k \) and the length \( L = b - a \) of the interval, it is nondecreasing with \( L \). We establish that equidistant \( x_j = \frac{j}{n+1} \) are optimal for large \( n \) compared with \( |k| \).

According to Remark 1, we modify the lower limit of the integral for \( R(x) \) and define simply
\[
R(x) := \int_0^x \varrho_k(t) \, dt = \int_0^x e^{-2\pi ikt} \, dt = \frac{e^{-2\pi ikx} - 1}{-2\pi ik},
\]
and
\[
c := \frac{1}{b-a} \int_a^b R(x) \, dx = -\frac{e^{-2\pi ikb} - e^{-2\pi ika}}{4\pi^2 k^2 L} + \frac{1}{2\pi ik}.
\]
Then on the interval \( [a, b] \),
\[
\| R - c \|_2^2 = \int_a^b (R(x) - c)(R(x) - c) \, dx
\]
\[
= \int_a^b R(x)\overline{R(x)} \, dx - Lc\overline{c}
\]
\[
= \int_a^b e^{-2\pi ikx} - 1 \frac{e^{2\pi ikx} - 1}{2\pi ik} \, dx
\]
\[
-L \left( \frac{e^{-2\pi ikb} - e^{-2\pi ika}}{4\pi^2 k^2 L} + \frac{1}{2\pi ik} \right) \cdot \left( \frac{e^{2\pi ikb} - e^{2\pi ika}}{4\pi^2 k^2 L} - \frac{1}{2\pi ik} \right)
\]
\[\square\]
\[ I(x) := \int_a^b 2(1 - \cos(2\pi kx)) \, dx - \frac{1 - \cos(2\pi kL)}{8\pi^4 k^4 L} - \frac{\sin(2\pi kb) + \sin(2\pi ka)}{4\pi^3 k^3} - \frac{L}{4\pi^2 k^2} \]

which is independent of \(a\) and \(b\) and stays the same even if \(R(x) := \int_a^x e^{-2\pi ikt} \, dt\).

From Proposition 2, we easily obtain the following assertion concerning the initial error.

Proposition 5. Consider the oscillatory integral \(I_{\Omega_k} : H^1_0([0, 1]) \to \mathbb{C}\) with \(k \in \mathbb{Z}\setminus\{0\}\). Then the initial error is given by

\[ e_0 = \sup_{\|f\| \leq 1} |I_{\Omega_k}(f)| = \frac{1}{2\pi |k|}. \]

Moreover the maximum is assumed for \(f^* \in H^1_0([0, 1])\), given by

\[ f^*(t) = \frac{1}{2\pi |k|} \left( e^{2\pi ikt} - 1 \right), \]

i.e., \(I_{\Omega_k}(f^*) = e_0\) and \(\|f^*\| = 1\) with \(f^*(0) = f^*(1) = 0\). \(\square\)

Following Theorem 3, denote \(L_j = |I_j|\), then \(\sum_{j=0}^n L_j = 1\) and \(c_j = \|R - \beta_j\|_2\) with \(\beta_j = \frac{1}{L_j} \int I_j R(x) \, dx\). The radius of information \(r(N) = \left( \sum_{j=0}^n c_j^2 \right)^{1/2}\) is

\[ r(N) = \frac{1}{2\pi |k|} \left( 1 - \frac{1}{\pi^2 k^2} \sum_{j=0}^n \frac{1 - \cos(2\pi kL_j)}{L_j} \right)^{1/2} \]

\[ = \frac{1}{2\pi |k|} \left( 1 - \frac{1}{\pi^2 k^2} \sum_{j=0}^n \frac{\sin^2(\pi k L_j)}{L_j} \right)^{1/2}. \]

To make the worst case error as small as possible, we want to find the optimal distribution of information nodes \((x_j)_{j=1}^n\), in particular for large \(n\). That is,

\[ \inf_{L_j \geq 0, \sum_{j=0}^n L_j = 1} \left( 1 - \frac{1}{\pi^2 k^2} \sum_{j=0}^n \frac{\sin^2(\pi k L_j)}{L_j} \right)^{1/2}. \]

For this, we prove the following lemma, where \(t^* > 0\) is given by \(\tan t^* = 2t^*\), hence \(t^* \approx 1.165561\).

Lemma 6. Let \(k \in \mathbb{R}\setminus\{0\}\), \(0 = x_0 < x_1 < x_2 < \cdots < x_n < x_{n+1} = 1\) and \(L_j = x_{j+1} - x_j\), \(j = 0, 1, \ldots, n\). Suppose that \(n + 1 \geq \pi |k|/t^*\). Then

\[ \sup_{L_j \geq 0, \sum_{j=0}^n L_j = 1} \left( \sum_{j=0}^n \frac{\sin^2(\pi k L_j)}{L_j} \right)^{1/2} = \left( n + 1 \right)^2 \sin^2 \left( \frac{\pi k}{n + 1} \right), \quad (4) \]

\(\square\)
i.e., equidistant $x_j$ with $L_j = \frac{1}{n+1}$ for all $j = 0, 1, \ldots, n$ are optimal.

**Proof** Let $t^*$ be the first positive argument where the function $t \mapsto \sin^2 t$ has a local maximum. It is easy to check that this is also the global maximum and that the function is concave on $[0, t^*]$. The number $t^*$ is given by $\tan t^* = 2 t^*$ and the function

$$f(t) = \begin{cases} \sin^2 t, & \text{for } 0 \leq t \leq t^* \\ \frac{\sin^2 t^*}{t^*}, & \text{for } t > t^* \end{cases}$$

is concave for $t \geq 0$. Therefore we obtain

$$\frac{1}{n+1} \sum_{j=0}^{n} \frac{\sin^2 t_j}{t_j} \leq \frac{\sin^2 s}{s}$$

for all $0 < s \leq t^*$ and $t_j > 0$ such that $\frac{1}{n+1} \sum_{j=0}^{n} t_j = s$. Application of this fact to $t_j = \pi |k| L_j$ gives the statement of the lemma. \(\square\)

We are now ready to give sharp estimates on the worst case error.

**Theorem 7** In the case of $H_0^1([0, 1])$ with $\varrho_k(x) = \exp(-2\pi i k x)$ and $k \in \mathbb{R}\setminus\{0\}$, the radius of information is given by

$$r(N) = \frac{1}{2\pi|k|} \left( 1 - \frac{1}{k^2 \pi^2} \sum_{j} \sin^2(\pi k L_j) \right)^{1/2}$$

where $L_j = x_{j+1} - x_j$, $j = 0, 1, \ldots, n$, with $0 = x_0 < x_1 < x_2 < \cdots < x_n < x_{n+1} = 1$.

Moreover, if $n \geq 2.7|k| - 1$, then equidistant nodes ($L_j = \frac{1}{n+1}$, $j = 0, 1, \ldots, n$) are optimal and the worst case error is

$$e(n, I_{\varrho_k}, H_0^1) = \frac{1}{2\pi|k|} \left( 1 - \frac{(n+1)^2}{k^2 \pi^2} \sin^2 \left( \frac{k\pi}{n+1} \right) \right)^{1/2}$$

\(\square\)

Furthermore, we establish a few nice asymptotic properties for the $n$th minimal errors as follows.

**Corollary 8** Under the same assumption of Theorem 7, the following statements hold:

(i) For fixed $k \in \mathbb{R}\setminus\{0\}$ and (optimal) equidistant nodes, we have

$$\lim_{n \to \infty} e(n, I_{\varrho_k}, H_0^1) \cdot n = \frac{1}{2\sqrt{3}}.$$

(ii) For fixed $n \in \mathbb{N}$ and arbitrary nodes, we have

$$\lim_{|k| \to \infty} e(n, I_{\varrho_k}, H_0^1) \cdot |k| = \frac{1}{2\pi}.$$

(iii) Suppose in addition that $k \in \mathbb{Z}\setminus\{0\}$. Then for fixed $n \in \mathbb{N}$ and arbitrary nodes,

$$\lim_{|k| \to \infty} e(n, I_{\varrho_k}, H_0^1) = 1.$$

\(\square\)
Proof Point (i) can be proved via Taylor’s expansion in the same manner as in Theorem 15. Point (ii) is known from the result for the radius of information in Theorem 7. This implies point (iii) by Proposition 5. □

Remark 9 The optimal distribution of information nodes is much more complicated if \( n + 1 < 2.6954|k| \). In the case of \( n = |k| - 1 \) equidistant nodes are the worst nodes, in this case these \( n \) function values are useless: the radius of information is the same as the initial error of the problem. This follows from Theorem 7 since, in this case, \( \sin(\pi k L_j) = 0 \).

Related to the field of digital signal processing, a famous assertion, the Nyquist Sampling Theorem states that, see [5,7]: If a time-varying signal is periodically sampled at a rate of \( \text{at least twice} \) the frequency of the highest-frequency sinusoidal component contained within the signal, then the original time-varying signal can be exactly recovered from the periodic samples. It seeks in essence for the reconstruction of continuous periodic functions. In contrast, for oscillatory integrals of periodic functions from \( H^1 \), the multiple number 2.7 assures that equidistant nodes achieve the optimal quadrature.

3.2 The General Case

We want to find optimal nodes,

\[ 0 \leq x_1 < \cdots < x_n \leq 1, \]

for the oscillatory integrals and integrands from the full space \( H^1([0, 1]) \) with \( k \in \mathbb{R}\setminus\{0\} \). We will prove some nice formulas for large \( n \), but not for small \( n \). For convenience we take \( x_0 = 0 \). \( I_j = [x_j, x_{j+1}] \) and \( L_j = |I_j| \).

To compute the number \( r(N) \) as in Theorem 3 with arbitrary nodes mentioned above, firstly we consider the initial errors for all intervals under the assumption \( N(x_1, \ldots, x_n) = 0 \).

On the intervals \( I_j \), \( j = 1, \ldots, n-1 \), we know from (3) that the initial error is \( \|R - c\|_{2,j} \) with

\[
\|R - c\|_{2,j}^2 := \int_{x_j}^{x_{j+1}} (R(x) - c) \left( \bar{R}(x) - c \right) \, dx = \frac{L_j}{4\pi^2 k^2} - \frac{1}{8\pi^4 k^4 L_j} \left( 1 - \cos(2\pi k L_j) \right).
\]

On the interval \( I_0 = [0, x_1] \), we obtain from (1) that the initial error is \( \|R_0\|_{2,0} \) with \( R_0(t) = \int_0^t \varrho_k(x) \, dx, \ t \in [0, x_1] \), and for \( k \in \mathbb{R}\setminus\{0\} \),

\[
\|R_0\|_{2,0}^2 := \int_0^{x_1} R_0(x) \cdot \bar{R_0}(x) \, dx = \frac{1}{4\pi^2 k^2} \left( 2L_0 - \frac{\sin(2\pi k L_0)}{\pi k} \right)
\]

Similarly on the interval \( I_n = [x_n, 1] \), the initial error is \( \|R_n\|_{2,n} \) with \( R_n(t) = \int_t^1 \varrho_k(x) \, dx, \ t \in [x_n, 1] \), and for \( k \in \mathbb{R}\setminus\{0\} \),

\[
\|R_n\|_{2,n}^2 := \int_{x_n}^{x_n} R_n(x) \cdot \bar{R_n}(x) \, dx = \frac{1}{4\pi^2 k^2} \left( 2L_n - \frac{\sin(2\pi k L_n)}{\pi k} \right)
\]

As usual, the initial error is given by taking the zero algorithm \( A_0(f) = 0 \). If \( k \in \mathbb{Z}\setminus\{0\} \), we have \( I(\varrho_k) = 0 \), and by (2),

\[
e(0, I_{\varrho_k}, H^1) = \sup_{f \in H^1 : \|f\| \leq 1} |I_{\varrho_k}(f)| = \sup_{f \in H^1 : \|f\| \leq 1} |I_{\varrho_k}(f - f(0))| = \frac{\sqrt{2}}{2\pi |k|}.
\]

\( \square \)
Following the same lines as in Sect. 3.1, the radius of information is,
\[
\left( \sum_{j=1}^{n-1} \| R - c \|_{2,j}^2 + \| R_0 \|_{2,0}^2 + \| R_n \|_{2,n}^2 \right)^{1/2}
\]
\[
= \frac{1}{2\pi |k|} \left( L_0 - \frac{\sin(2\pi k L_0)}{\pi k} + L_n - \frac{\sin(2\pi k L_n)}{\pi k} + 1 - \frac{1}{\pi^2 k^2} \sum_{j=1}^{n-1} \sin^2(\pi k L_j) \right)^{1/2}.
\]

For this, we prove the following lemma.

**Lemma 10** Let \( k \in \mathbb{R} \setminus \{0\} \) and \( n - 1 \geq 2.7|k| \). Then for
\[
\inf_{x,y_1,...,y_{n-1},z \geq 0, \ x+y_1+...+y_{n-1}+z=1} \left( 1 + x - \frac{\sin(2\pi k x)}{\pi k} + z - \frac{\sin(2\pi k z)}{\pi k} - \frac{1}{\pi^2 k^2} \sum_{j=1}^{n-1} \sin^2(\pi k y_j) \right)^{1/2}, \tag{5}
\]
the unique solution of the minimum \((x^*, y_1^*, \ldots, y_{n-1}^*, z^*)\) satisfies \( x^* = z^*, y_1^* = \ldots = y_{n-1}^* \) and \( x^* \) is the stationary point of the function,
\[
S(x) = 2x - \frac{2 \sin(2\pi k x)}{\pi k} - \frac{(n-1)^2 \sin^2 \left( \frac{\pi k \cdot \frac{1-2x}{n-1}}{\pi k} \right)}{1-2x},
\]
in the interval \( \left( \frac{1}{2}, \min \left( \frac{1}{k}, \frac{1}{6|k|} \right) \right) \).

**Proof** Without loss of generality, we assume that \( k \in \mathbb{R}^+ \) and \( z \leq 1/2 \) since \( x + y_1 + \ldots + y_{n-1} + z = 1 \).

Step 1: Following Lemma 6, we obtain that for any fixed \( x \) and \( z \), equal \( y_j = \frac{1-x-z}{n-1} \) for all \( j = 1, \ldots, n-1 \) are optimal. Afterwards, we have to find the optimal values, \( x \) and \( z \), for
\[
\inf_{x,y,z \geq 0, \ x+(n-1)y+z=1} \frac{1}{2\pi |k|} \left( x - \frac{\sin(2\pi k x)}{\pi k} + z - \frac{\sin(2\pi k z)}{\pi k} + 1 - \frac{(n-1) \sin^2(\pi k y)}{\pi^2 k^2} \right)^{1/2}.
\]

Step 2: For the above extremal problem, we state that the minimum values, \( x \) and \( z \), should be relatively small, at least in \((0, 1/(6k))\).

Firstly, for any fixed \( z = z_0 \), we know \( x + (n-1)y = 1 - z_0 \geq 1/2 \). That is to consider
\[
\inf_{x,y \geq 0, \ x+(n-1)y=1-z_0} \left( x - \frac{\sin(2\pi k x)}{\pi k} - \frac{(n-1) \sin^2(\pi k y)}{\pi^2 k^2} \right).
\]

We define \( S_1(x) := g_1(x) - F_1(x), \ x \in [0, 1-z_0] \), where
\[
g_1(x) = x - \frac{\sin(2\pi k x)}{\pi k}, \ x \in [0, 1-z_0].
\]
and
\[
F_1(x) = \frac{(n-1)}{\pi^2 k^2} f(y) \quad \text{with} \quad y = \frac{1-z_0-x}{n-1} \in \left[ 0, \frac{1-z_0}{n-1} \right],
\]
with \( f(y) = \sin^2(\pi k y)/y, \ y \in (0, 1], \ f(0) = 0 \) as in Lemma 6.
One can decompose the function $S_1$ into two parts, $x - F_1(x)$ and $-\sin(2\pi k x)/(\pi k)$. One part, $x - F_1(x)$, is monotone increasing on $[0, 1 - z_0]$. The other, $-\sin(2\pi k x)/(\pi k)$, is $1/k$-periodic. This implies that the unique minimum point $x = \tilde{x}$ for $S_1$ appears in $[0, 1/(4k)]$.

By the intermediate value theorem, there is one point $\tilde{x} < 1/(6k)$ such that $S_1'(\tilde{x}) = 0$, and the monotonicity of $S_1'$ assures the uniqueness of $\tilde{x}$ in $(0, \min\{1 - z_0, \frac{1}{6k}\})$.

Secondly, the same statement holds for finding the optimal value of $z$. That is, by fixing $x = \tilde{x}$, we obtain a minimum point for $z = \tilde{z} \in (0, \min\{1 - \tilde{x}, \frac{1}{6k}\})$.

Step 3: Iterate the process by fixing $y$ above. One knows easily $x + z = 1 - (n - 1)y < 1/(3k)$ and considers

$$g_1(x) = x - \frac{\sin(2\pi k x)}{\pi k}, \quad x \in \left[0, \frac{1}{3k}\right].$$

The convexity of this function implies that $x = z = p/2$ is the unique solution of the problem,

$$\inf_{x, z \geq 0, x + z = p} \left(x - \frac{\sin(2\pi k x)}{\pi k} + z - \frac{\sin(2\pi k z)}{\pi k}\right) \quad \text{for any fixed} \quad p \in \left[0, \frac{1}{3k}\right].$$

Step 4: The above three steps shift the extremal problem (5) to the simpler case below,

$$r(x^*) := \inf_{x, y \geq 0, 2x + (n-1)y = 1} \left(2x - \frac{2}{\pi k} \sin(2\pi k x) - \frac{(n-1) \sin^2(\pi k y)}{y^2}\right)^{1/2},$$

which depends only on some point $x = x^*$ since this number determines all $n + 1$ values ($N_{x^*}$ for short) in the minimum case.

We follow Step 2 with a few small modifications. Here we define $S(x) := g(x) - F(x)$, $x \in [0, \frac{1}{2}]$, where

$$g(x) = 2g_1(x) = 2x - 2\frac{\sin(2\pi k x)}{\pi k}, \quad x \in \left[0, \frac{1}{2}\right],$$

and

$$F(x) = \frac{(n-1)}{\pi^2 k^2} f(y) \quad \text{with} \quad y = \frac{1 - 2x}{n-1} \in \left[0, \frac{n}{n-1}\right].$$

Together with the intermediate value theorem, similar decomposition of the function $S$ into two parts implies that the unique minimum point for $S$ appears in $(0, 1/(6k))$.

Again, there exists one point $x^* \in (0, \min\{\frac{1}{2}, \frac{1}{6k}\})$ such that $S'(x^*) = 0$. The monotonicity of $S'$ on this interval shows that $S(x)$ is decreasing for $x < x^*$, and then is increasing for $x > x^*$. This confirms the unique solution of the minimum point for the problem (5).

In the case of $k \in \mathbb{R}\{0\}$, we use $|k|$ instead of $k$. Hence the proof is finished. \hfill \Box

This enables us to give sharp estimates on the worst case error for the full space $H^1([0, 1])$.

**Theorem 11** In the case of $H^1([0, 1])$ with $k \in \mathbb{R}\{0\}$, the radius of information is given by

$$r(N) = \frac{1}{2\pi |k|} \left(L_0 - \frac{\sin(2\pi k L_0)}{\pi k} + L_n - \frac{\sin(2\pi k L_n)}{\pi k} + 1 - \frac{1}{\pi^2 k^2} \sum_{j=1}^{n-1} \sin^2(\pi k L_j) L_j\right)^{1/2},$$

where $L_0 = x_1$, $L_j = x_{j+1} - x_j$, $j = 1, \ldots, n - 1$ and $L_n = 1 - x_n$, with $0 \leq x_1 < \cdots < x_n \leq 1$. 

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Moreover, if \( n - 1 \geq 2.7|k| \), then \( x_1 = 1 - x_n = x^* \) with \( x^* \) from Lemma 10, and equidistant \( x_j = \frac{j-1}{n-1} \cdot (x_n - x_1) + x_1, \ j = 2, \ldots, n-1 \), are optimal in the worst case. \( \square \)

Remark 12 Although we do not give an explicit formula for the point \( x^* \) above, it is easy to obtain the numerical solution for \( x^* \) when \( k \) and \( n \) are known. We want also to ask whether equidistant nodes, \( x_j = \frac{j}{n+1}, \ j = 1, \ldots, n \), are optimal for some \( k \) and \( n \). The answer is negative. Firstly, it can only happen if \( n+1 > 6|k| \). We take \( t = \frac{|k|}{n+1} \in (0, 1/6) \) and find that, from Lemma 10,

\[
S' \left( \frac{1}{n+1} \right) = 2 - 4 \cos(2\pi t) + \frac{2}{\pi^2} \frac{\sin(\pi t)}{t^2} (2\pi t \cos(\pi t) - \sin(\pi t)) > S'(x^*) = 0, \quad t \in \left( 0, \frac{1}{6} \right).
\]

This tells us that, \( x^* < \frac{1}{n+1} \) if \( n-1 > 2.7|k| \). Even we have \( x^* < \frac{1}{2n} \), since for the midpoint rule, i.e., \( x_j = \frac{2j-1}{2n}, \ j = 1, \ldots, n \),

\[
S' \left( \frac{1}{2n} \right) = 2 - 4 \cos(\pi t) + \frac{2}{\pi^2} \frac{\sin(\pi t)}{t^2} (2\pi t \cos(\pi t) - \sin(\pi t)) > 0, \quad t = \frac{|k|}{n} \in \left( 0, \frac{1}{3} \right).
\]

That is, the endpoints nearby are much closer to the optimal \( x_1 \) and \( x_n \) than \( x_2 \) and \( x_{n-1} \), respectively, with the distance \( x^* < \frac{1}{2n} < \frac{1}{n+1} < \frac{1}{n} < \frac{1-2x^*}{n-1} = x_{j+1} - x_j < \frac{1}{n-1}, \ j = 1, \ldots, n-1 \). \( \square \)

4 Oscillatory Integrals: Equidistant Nodes

In this section, we want to discuss the case of equidistant nodes for the Sobolev space \( H^1([0, 1]) \) of non-periodic functions. Throughout this section, we assume that one uses equidistant nodes

\[
x_j = \frac{j}{n}, \quad j = 0, 1, \ldots, n. \tag{6}
\]

This case was already studied by Boltaev et al. [1], using the S. L. Sobolev’s method.

Then the oscillatory integral \( I_{\omega^k} \) of the piecewise linear function \( \sigma \) (the spline algorithm) is given by

\[
A_{n+1}^k(f) = I_{\omega^k}(\sigma) = \sum_{j=0}^{n} a_j f(x_j), \tag{7}
\]

where the coefficients \( a_j \)'s are given as follows. We skip the proof since the result is known, see [1, Theorem 8].

**Proposition 13** Let \( k \in \mathbb{Z} \setminus \{0\}, \ n \in \mathbb{N}, \) and \( x_j = j/n, \ j = 0, 1, \ldots, n \). Assume that \( f : [0, 1] \to \mathbb{C} \) is an integrable function with \( f(x_0), f(x_1), \ldots, f(x_n) \) given, and \( \sigma \) is the piecewise linear function of \( f \) at \( n+1 \) equidistant nodes \( \{x_j\}_{j=0}^{n} \). Then \( I_{\omega^k}(\sigma) = \)
\[ \sum_{j=0}^{n} a_j f(x_j), \]

where

\[ a_0 = \frac{n}{4k^2 \pi^2} \left( 1 - \frac{2\pi i k}{n} - e^{-2\pi i k/n} \right), \]

\[ a_j = \frac{n}{k^2 \pi^2} \sin^2 \left( \frac{\pi k}{n} \right) e^{-2\pi i kj/n}, \quad j = 1, \ldots, n-1, \]

\[ a_n = \frac{n}{4k^2 \pi^2} \left( 1 + \frac{2\pi i k}{n} - e^{2\pi i k/n} \right), \]

and \( \sum_{j=0}^{n} a_j = 0. \]

\[ \sum_{j=0}^{n} a_j f(x_j), \]

\[ a_0 = \frac{n}{4k^2 \pi^2} \left( 1 - \frac{2\pi i k}{n} - e^{-2\pi i k/n} \right), \]

\[ a_j = \frac{n}{k^2 \pi^2} \sin^2 \left( \frac{\pi k}{n} \right) e^{-2\pi i kj/n}, \quad j = 1, \ldots, n-1, \]

\[ a_n = \frac{n}{4k^2 \pi^2} \left( 1 + \frac{2\pi i k}{n} - e^{2\pi i k/n} \right), \]

Remark 14 We comment on the weights \( a_j \) in Proposition 13. Obviously, for every \( j = 1, \ldots, n-1 \), we have

\[ \lim_{n \to \infty} a_j e^{2\pi i kj/n} \cdot n = 1 \quad \text{and} \quad \lim_{n \to \infty} a_0 \cdot n = \lim_{n \to \infty} a_n \cdot n = \frac{1}{2}. \]

Therefore, we conclude that for sufficiently large \( n \), the linear algorithm is almost a QMC (quasi Monte Carlo) algorithm with equidistant nodes, which is used in [8].

Clearly, from Theorem 7, the algorithm \( A_{n+1}^k \) with equidistant nodes is optimal for the space \( H^1_0 \) in the worst case if \( n \geq 2|k| \). Here, \( n \) stands for the number of the intervals. Boundary values are fixed for \( f \in H^1_0([0, 1]) \), i.e., \( f(0) = f(1) = 0 \).

Furthermore, we have the following assertion for the space \( H^1 \), in which the point (i) is already proved in [1, Theorem 9].

Theorem 15 Consider the integration problem \( I_{\varrho_k} \) defined for functions from the space \( H^1_0([0, 1]) \). Suppose \( k \in \mathbb{Z} \) and \( k \neq 0 \).

(i) The worst case error of \( A_{n+1}^k \), \( n \in \mathbb{N} \), is

\[ e(A_{n+1}^k, I_{\varrho_k}, H^1) = \frac{1}{2\pi |k|} \left( 1 - \frac{n^2}{k^2 \pi^2} \sin^2 \left( \frac{k\pi}{n} \right) \right)^{1/2}. \]

(ii) For \( n \in \mathbb{N} \), we have

\[ e(A_{n+1}^k, I_{\varrho_k}, H^1) < e(0, I_{\varrho_k}, H^1_0) = \frac{1}{2\pi |k|}, \quad \text{if} \quad k \neq 0 \mod n. \]

(iii) For fixed \( n \in \mathbb{N} \), we have

\[ \lim_{|k| \to \infty} e(A_{n+1}^k, I_{\varrho_k}, H^1) \cdot |k| = \frac{1}{2\pi}. \]

(iv) For any \( k \in \mathbb{Z}\{0\}, n \in \mathbb{N} \), we have

\[ e(A_{n+1}^k, I_{\varrho_k}, H^1) \leq \frac{1}{2\sqrt{3} \cdot n}. \]

(v) For fixed \( k \in \mathbb{Z}\{0\} \), we have the sharp constant of asymptotic equivalence \( \frac{1}{2\sqrt{3}} \), i.e.,

\[ \lim_{n \to \infty} e(A_{n+1}^k, I_{\varrho_k}, H^1) \cdot n = \frac{1}{2\sqrt{3}}. \]
Proof  The point (i) follows from Theorem 7 directly since \( N(f) = 0 \) tells us that \( f(0) = f(1) = 0 \) and \( f \in H^1_0 \). Then points (ii) and (iii) follow clearly. We use Taylor’s expansion of the cosine function at zero. For any \( k \in \mathbb{Z} \setminus \{0\}, n \in \mathbb{N}, \)

\[
\sin^2 \left( \frac{k \pi}{n} \right) = \frac{1 - \cos \left( \frac{2k \pi}{n} \right)}{2} = \frac{k^2 \pi^2}{2 n^2} - \frac{1}{2} R_3 \left( \frac{2k \pi}{n} \right).
\]

Here, the third Lagrange’s remainder term satisfies, for some \( \theta = \theta \left( \frac{2k \pi}{n} \right) \in (0, 1), \)

\[
\left| R_3 \left( \frac{2k \pi}{n} \right) \right| = \left| \cos^{(4)} \left( \theta \cdot \frac{2k \pi}{n} \right) \right| \cdot \frac{(2k \pi)^4}{4! \cdot n^4} \leq \frac{2}{3} \left( \frac{k \pi}{n} \right)^4.
\]

This implies that, for any \( k \in \mathbb{Z} \setminus \{0\}, n \in \mathbb{N}, \)

\[
0 < 1 - \frac{n^2}{k^2 \pi^2} \sin^2 \left( \frac{k \pi}{n} \right) = \frac{n^2}{2k^2 \pi^2} \cdot \left| R_3 \left( \frac{2k \pi}{n} \right) \right| \leq \frac{1}{3} \left( \frac{k \pi}{n} \right)^2.
\]

Hence, for any \( k \in \mathbb{Z} \setminus \{0\}, n \in \mathbb{N}, \)

\[
\epsilon(A_{n+1}^k, I_{\omega_k}, H^1) \leq \frac{1}{2 \sqrt{3}} \frac{1}{n}.
\]

This proves (iv).

Moreover, if \( k \) is fixed and nonzero, we have that for any \( \theta \in (0, 1), \)

\[
\lim_{n \to \infty} \cos^{(4)} \left( \theta \cdot \frac{2k \pi}{n} \right) = 1.
\]

This leads to

\[
\lim_{n \to \infty} \epsilon(A_{n+1}^k, I_{\omega_k}, H^1) \cdot n = \frac{1}{2 \sqrt{3}},
\]

as claimed in (v). □

We comment on Theorems 7 and 15. Theorem 15 deals with \( k \in \mathbb{Z} \setminus \{0\} \) and equidistant nodes, while Theorem 7 works even for \( k \in \mathbb{R} \setminus \{0\} \). However, Theorem 7 studies only the space \( H^1_0 \) instead of \( H^1 \).

For \( k \in \mathbb{R} \setminus \{0\} \), the same statements, as in Theorem 15, hold true for the space \( H^1_0 \), since the spline algorithm is optimal. Due to the zero boundary values, the number of information is \( n - 1 \) for \( H^1_0 \), instead of \( n + 1 \). This is indeed a special case of Theorem 7.

Moreover, thanks to the equidistant nodes including endpoints, the formula in point (i) of Theorem 15 remains valid for \( k \in \mathbb{R} \setminus \{0\} \) (and \( H^1 \)), as well as points (iii)-(v). In the computation of \( r(N, H^1) \), we usually work with

\[
N(f) = \left( f(0), f \left( \frac{1}{n} \right), \ldots, f \left( \frac{n-1}{n} \right), f(1) \right) = 0 \quad \text{for} \quad f \in H^1([0, 1]).
\]

This is equivalent to the computation of \( r(N_1, H^1_0) \) in Theorem 7 with

\[
N_1(f) = \left( f \left( \frac{1}{n} \right), \ldots, f \left( \frac{n-1}{n} \right) \right) = 0 \quad \text{for} \quad f \in H^1_0([0, 1]).
\]
That is shortly, for \( k \in \mathbb{R} \setminus \{0\} \),
\[
e^{(A_{n+1}^k I_{\varrho_k} H^1)} = r(N, H^1) = \sup_{f \in H^1 : \|f\| \leq 1} |I_\varrho(f)| = \sup_{f \in H_0^1 : \|f\| \leq 1} |I_\varrho(f)| = r(N_1, H_0^1) = \frac{1}{2\pi |k|} \left( 1 - \frac{n^2}{k^2\pi^2} \sin^2 \left( \frac{k\pi}{n} \right) \right)^{1/2}.
\]

**Remark 16** It is easy to prove that these asymptotic statements (iii) and (v) also hold for optimal nodes, i.e., for the numbers \( e(n, I_{\varrho_k} H^1) \) with \( k \in \mathbb{R} \setminus \{0\} \). More precisely, for fixed \( n \) and \( k \to \infty \), one can take \( L_0 = L_n = 0 \) in Theorem 11 to get the asymptotic property of \( e(n, I_{\varrho_k} H^1) \). For fixed \( k \in \mathbb{R} \setminus \{0\} \) and \( n \to \infty \), Theorem 11 gives by Taylor’s expansions the same asymptotic constant for \( e(n, I_{\varrho_k} H^1) \) since \( x^* < \frac{1}{2n} \) and \( \frac{1}{n} < \frac{1-2x^*}{n-1} < \frac{1}{n-1} \). Finally, together with Corollary 8, we find out the same asymptotic constants, \( 1/(2\pi) \) and \( 1/(2\sqrt{3}) \), for both the spaces \( H_0^1 \) and \( H^1 \).

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**References**

1. Boltav, N.D., Hayotov, A.R., Shadimetov, KhM: Construction of optimal quadrature formula for Fourier coefficients in Sobolev space \( L_2^{(m)}(0, 1) \). Numer. Algorithms 74(2), 307–336 (2017)
2. Brass, H., Petras, K.: Quadrature Theory: The Theory of Numerical Integration on a Compact Interval, p. 363. AMS Mathematical Surveys and Monographs, Rhode Island (2011)
3. Ciarlet, P.G.: The finite element method for elliptic problems. Society for Industrial and Applied Mathematics, Philadelphia. Class. Appl. Math. 40, 1–511 (2002)
4. Huybrechs, D., Olver, S.: Highly oscillatory quadrature, Chapter 2 in: Highly Oscillatory Problems, London Math. Soc. Lecture Note Ser. 366, Cambridge, pp. 25–50 (2009)
5. Landau, H.J.: Necessary density conditions for sampling and interpolation of certain entire functions. Acta Math. 117(1), 37–52 (1967)
6. Lax, P.D., Milgram, A.N.: Parabolic equations. Ann. Math. 33, 167–190 (1954)
7. Mishali, M., Eldar, Y.C.: Blind multiband signal reconstruction: compressed sensing for analog signals. IEEE Trans. Signal Proces. 57(3), 993–1009 (2009)
8. Novak, E., Ullrich, M., Woźniakowski, H.: Complexity of oscillatory integration for univariate Sobolev spaces. J. Complex. 31(1), 15–41 (2015)
9. Novak, E., Ullrich, M., Woźniakowski, H., Zhang, S.: Complexity of oscillatory integrals on the real line. Adv. Comput. Math. 43(3), 537–553 (2017)
10. Traub, J.F., Wasilkowski, G.W., Woźniakowski, H.: Information-Based Complexity. Academic Press, Cambridge (1988)
11. Traub, J.F., Woźniakowski, H.: A General Theory of Optimal Algorithms. Academic Press, Cambridge (1980)
12. Žensykaev, A.A.: Best quadrature formula for some classes of periodic differentiable functions, Izv. Akad. Nauk SSSR Ser. Mat. 41(5), 1110–1124, 1977 (in Russian); [English transl., Math. USSR Izv. 41(5), 1055–1071, (1977)]