Einstein-Hilbert action nontrivially coupled with topological characteristic classes: generating Torsion and Contortion

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Abstract

In this work we consider the effects of coupling characteristic classes to gravity by introducing appropriate operators in the Einstein–Hilbert action. As it is well known, this approach strays from the framework of General Relativity since it results in theories in which torsion can be present. An important point of our approach is that the solutions obtained explicitly carry topological information of the considered space-time manifold. We consider here all the characteristic classes that are consistent with a space-time manifold leading to the definition of an “effective Cosmological Constant” that inherits topological information. We present analytical solutions for the contortion 1-form that can be obtained under general conditions in various cases of interest. We show how to use these solutions to study cosmological scenarios that are obtained mainly by selecting a metric and an ideal fluid. We also discuss some of the consequences of these cosmological models over the topological and differential structure of the space-time manifold considered.

Keywords: torsion two-form; contortion one-form; contortion connection; general relativity; differential geometry; topological invariants; space-time topology

1. Introduction

This paper studies the consequences of adding characteristic classes into the classical Einstein-Hilbert action using the method of Lagrange multipliers. This procedure is known to change the bulk and boundary conditions resulting in gravitational theories with Torsion. Here we only consider characteristic classes which are consistent with a space-time four manifold configuration, i.e. those constructed from the rational cohomology ring such as the Euler and the Pontryagin classes \textsuperscript{1} and the Chern type Nieh-Yan class which is the only density immediately null in the absence of torsion \textsuperscript{2}. These classes can be cast as exact forms, moreover they only show dependence on the spin connection and its derivatives. The effects of adding the aforementioned classes can be acknowledged for by introducing a new term into the connection 1-form called contortion, which in turn is responsible for the torsion \textsuperscript{3}.

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Preprint submitted to Elsevier

November 3, 2014
A somewhat similar approach to the one here presented has been used in [4, 5] were the focus was to study the effects of these couplings over the topology and differential structure of the associated symplectic manifold. This was driven by having in mind the later canonical quantization of the theory. Quantization is beyond the scope of the present work, rather we focus in obtaining closed analytical expressions then study their cosmological, topological and differential compatibility.

In the procedure we follow, we allow the characteristic classes to be non-zero from the beginning, as can be explicitly seen in the way the corresponding terms are inserted into the action, thus taking into account non trivial topologies. A very interesting feature of this approach is the fact that the cosmological constant can be interpreted as a quantity that inherits topological information. In fact, an ”effective topological constant” appears even without introducing an explicit one at the level of the action. This opens the question whether the cosmological constant suggested by experiments could actually be due to a topological effect, and henceforth whether by measuring it, it would be possible to deduce the topological structure of the space-time.

In addition, in recent years the interest over theories of gravity that encompass torsion has steadily grown, as is expressed by the appearance of such theories as the Poincaré Gauge Theory of Gravity, Teleparallel Gravity, \( f(T) \)-Gravity and \( f(R, T) \)-Gravity among others. The source of torsion is the spin connection and hence the curvature and torsion become independent degrees of freedom of the gravitational field. It is known that in 4 dimensions the torsion form is null in the absence of sources, as it happens in the Einstein-Cartan theory. However, this can be overpassed by considering additional fields or adding higher order corrections to the curvature, as it will be evident in the following sections. The theory considered here has no spin densities but there is a propagating torsion due to the coupling 0-forms. This situation is different from theories like Teleparallel Equivalent of General Relativity (TEGR) where the curvature is zero.

Regarding the physical applications, the introduction of torsion induces new physical effects and modifies the local degrees of freedom of the theory. For instance, the Einstein-Hilbert theory can be interpreted as a reduction of a higher dimensional model of the Chern-Simons type in which a propagating non null torsion solution naturally appears [6, 7]. The addition of torsion also enhances the possibility of an accelerated expansion of the universe among others scenarios [8, 9, 10, 11]. It has also been proposed that the appearance of torsion would induce observable effects over the neutrino oscillation [12].

The structure of the paper is as follows, section 2 introduces some general considerations and the notions of differential geometry required in the body of the work. In section 3 four cases for different choices of the available characteristic classes are studied with non local formalism (vierbein). We present then in section 4 several examples in which we show step by step how to use the analytical expressions of the previous section to build cosmological solutions. We close with some remarks and discussion of the examples we worked out.

2. General Considerations

We begin summarizing the standard vierbein formalism and the basics of differential geometry that will be needed in the next sections [13, 14, 15, 16, 17]. On a four dimensional space-time manifold \( \mathcal{M} \) with a cosmological constant \( \Lambda \) the Einstein-Hilbert action can be written as

\[
S_{EH}[e^a, \bar{\omega}^a] = \int_{\mathcal{M}} \frac{1}{k^2} \left( e^{abcd} e_a \wedge e_b \wedge \bar{R}_{cd} + \frac{\Lambda}{6} e^{abcd} e_a \wedge e_b \wedge e_c \wedge e_d \right). \tag{1}
\]
where $e^a$ denotes the 1-form frame fields or vierbein, $\kappa$ is the Newton constant and $\bar{\omega}^{ab}$ is the connection 1-form (Levi-Civita), the latter defines the curvature to be

$$R^{ab} \equiv d\bar{\omega}^{ab} + \bar{\omega}^{ac} \wedge \bar{\omega}^{cb}. \quad (2)$$

The Palatini variation of the action (1) gives the Einstein’s Field equations and also, as an independent constraint, the null torsion condition

$$T^a \equiv de^a + \bar{\omega}^{ab} \wedge e^b = 0. \quad (3)$$

The latter allows to write the action as well as the connection entirely in terms of the vierbein (or ultimately in terms of the metric) as

$$\bar{\omega}^{ab} = \frac{1}{2} \left[ \bar{d} (de^b) - \bar{i}^b (de^a) + \bar{i}^{ab} (de_c) e^c \right], \quad (4)$$

where $d(\cdot)$ is the exterior derivative, $\bar{i}^c (\cdot) \equiv \bar{i} (\cdot)$ is the interior product with respect to the vierbein. We have also denoted $i^{ab} (\cdot) = -i^{ba} (\cdot) = i^a (i^b (\cdot))$. Equation (4) defines the connection one-form for GR, also known as the Levi-Civita connection. We can think of $M$ as being a $SO(4)$-bundle or a $SO(3,1)$-bundle space-time, depending on the signature of the metric.

A common practice to obtain invariants of smooth manifolds, beyond the homology groups, is to construct characteristic classes of the tangent bundle. For an oriented real four-manifold $M$ the characteristic classes available comprise the Stiefel-Whitney classes $w_i (TM) \in H^i (M; \mathbb{Z}/2)$ and the Euler and Pontryagin classes $e(M), p_1 (TM) \in H^4 (M; \mathbb{Z}) = \mathbb{Z}$ [1]. We will allow the manifold $M$ to be complex, hence we can use the generalizations of the Stiefel-Whitney classes, namely the Chern classes. Specifically the Pontryagin and Euler densities, as well as the Chern type Nieh-Yan density which has been proven to be important to study the spin structure of the theory in a quantum context [4, 5, 18]. These have locally the following representations as 4-forms, respectively.

$$C_P \equiv d (\tilde{C}_P) = d \left( \omega^a_b \wedge \left[ R^b_a - \frac{1}{3} \omega^b_c \wedge \omega^c_a \right] \right) = R^b_a \wedge \bar{R}^b_a, \quad (5)$$

$$C_E \equiv d (\tilde{C}_E) = e_{abcd} d \left( \omega^{ab} \wedge \left[ R^{cd} - \frac{1}{3} \omega^f_j \wedge \omega^j_f \right] \right) = e_{abcd} R^{ab} \wedge \bar{R}^{cd}, \quad (6)$$

$$C_{NY} \equiv d (\tilde{C}_{NY}) = d (T^a \wedge e_a) = T^a \wedge T_a - R_{ab} \wedge e^a \wedge e^b, \quad (7)$$

where we have denoted by a tilde the corresponding 3-form associated with each exact 4-form presented above in an obvious way.

We consider now the torsion $T^a$ by generalizing the spin connection to the following one-form [19, 20, 21, 22, 23, 24, 25, 19] as

$$\omega^{ab} = \tilde{\omega}^{ab} + K^{ab}, \quad (8)$$

$$T^a = d_a e^a, \quad (9)$$

where $d(\cdot)$ is the exterior derivative, $\bar{i}^c (\cdot) \equiv \bar{i} (\cdot)$ is the interior product with respect to the vierbein. We have also denoted $i^{ab} (\cdot) = -i^{ba} (\cdot) = i^a (i^b (\cdot))$. Equation (4) defines the connection one-form for GR, also known as the Levi-Civita connection. We can think of $M$ as being a $SO(4)$-bundle or a $SO(3,1)$-bundle space-time, depending on the signature of the metric.
where \( K^{ab} \) is called the \textit{contortion connection} and \( d_\omega \) is the covariant derivative with respect to the connection \( \omega^{ab} \). Equation (9) is known as the structure equation and when combined with (8) leads to

\[
T^a = K^{ab} \land e_b,
\]

hence, the contortion \( K^{ab} \) is the object responsible for the presence of torsion \( T^a \) as anticipated. Similarly, the curvature two-form can thus be written in the following way

\[
\bar{R}^{ab} = \frac{d_\omega}{d}/\epsilon_{ab} + \omega^{ac} \land K^b_c - K^{ac} \land K^b_c,
\]

where equations (2) and (8) have been used. There is a further restriction that comes from the Bianchi identities and reads

\[
d_\omega T^a = R^a_b \land e_b,
\]

\[
d_\omega R^b_a = 0,
\]

which, when combined with the relations above, leads to the restriction \( d_\omega \epsilon_{abc} = 0 \) in terms of the connection. Note also that the latter implies the following special cases \( d_\omega \epsilon_{abc} \land \bar{\omega}_{ab} = dK^{ac} \land K_{cb} = 0 \) leading to the compatibility condition between the connections

\[
d_\omega \epsilon_{abc} \land \bar{\omega}_{ab} \land e_c = 0,
\]

with the two forms being equivalent. This condition can be imposed over the solutions obtained from the equations of motion.

2.1. The full action

It should be clear following the discussion in the previous section that knowing the curvature 2-form \( \bar{R}^{ab} \), the spin connection \( \bar{\omega}^{ab} \) and the contortion \( K^{ab} \) 1-forms is sufficient to describe all the physical observables of the theory. And, since we know how to obtain the first two from the vierbein, the problem narrows down to find an expression for \( K^{ab} \). As Cambiaso [21] showed though, this is not an easy task and only some perturbative solutions are known.

Let us address the problem by considering the following family of actions

\[
S \left[ e^a, \omega^a_{\, b}, \tau, \varphi, \chi \right] = S_{EH} \left[ e^a, \omega^a_{\, b} \right] + \int_M \frac{1}{\kappa} \left\{ \left( \tau C_p - \frac{n_p}{V_M} \right) + \varphi \left( C_{NY} - \frac{n_{NY} \epsilon}{V_M} \right) + \chi \left( C_E - \frac{n_E \epsilon}{V_M} \right) \right\},
\]

where we have introduced the parameters \( \tau, \varphi \) and \( \chi \), that couple to the characteristic classes [5], [7] and [6], respectively. We have denoted the volume form by \( \epsilon = \frac{1}{4!} \epsilon_{abcde} e^a \land e^b \land e^c \land e^d \) such that \( V_M = \int_M \epsilon \). The additional parameters \( n_{NY}, n_p, n_E \in \mathbb{R} \) are the Nieh-Yan number, Pontryagin number and the Euler number, respectively. These quantities satisfy

\[
\int_M C_i = n_i, \quad i = NY, P, E,
\]
where the $C_i$s have been defined in (5 - 7). The study of differential topology [1, 26] shows that for certain classes of maps $\tilde{\varsigma}: M \to N$ each of the term $n_i$ above is invariant, i.e. $\tilde{\varsigma}_i^*(n_i) = n_i$. Let us then restrict our analysis to the subclass $\tilde{\varsigma} = \bigcap_i \tilde{\varsigma}_i$, so the results will be valid for all manifolds $M$ and $N$ that are $\tilde{\varsigma}$-related. Note that we can define an effective Einstein-Hilbert action by collecting these terms depending on the characteristic densities over (16) and adding them into (1), yielding
\[ S_{\text{eff}}^{\text{EH}}[e^\ell, \omega^a_b, \tau, \phi, \chi] = \int_M \frac{1}{\kappa} \left( e^{abcd} e_d \wedge e_b \wedge R_{cd} + \frac{\Lambda_{\text{eff}}}{6} \right), \] (17)
where
\[ \Lambda_{\text{eff}} = \Lambda + \frac{1}{4} \left( n_p \frac{\tau}{V_M} + n_{NY} \frac{\phi}{V_M} + n_E \frac{\chi}{V_M} \right). \] (18)

On the other hand, even though restricted to $\tilde{\varsigma}$-related $M$ and $N$ manifolds that leaves all the topological numbers $n_i$ invariant, equation (18) behaves non-trivially under maps. In fact, the relevant terms above pullback according to
\[ \tilde{\varsigma}^* \left( \frac{n_i}{V_M} \right) = \tilde{\varsigma}^* \left( \frac{n_i}{V_N} \right) = \tilde{\varsigma}^* \left( \frac{n_i}{V_N} \right), \]
this shows that if we want to interpret (18) to be an effective CC we need to consider the couplings as either being constants or as slow varying 0-forms. In this direction, the appearance of the volume $V_N$ could be used to ensure a series expansion of the couplings with a rapid convergence in some weak sense.

Coming back to the action (16), we can perform an integration by parts on the terms that were not used to define (17) to obtain
\[ S \left[ e^\ell, \omega^a_b, \tau, \phi, \chi \right] = S_{\text{eff}}^{\text{EH}} \left[ e^\ell, \omega^a_b, \tau, \phi, \chi \right] + \int_M \left( d\tau \wedge \tilde{C}_P + d\phi \wedge \tilde{C}_{NY} + d\chi \wedge \tilde{C}_E \right) + \int_{\partial M} \left( \tau \tilde{C}_P + \phi \tilde{C}_{NY} + \chi \tilde{C}_E \right), \] (19)
where the first term on the right hand side was defined in (17), and where we also used Stokes theorem to obtain the last term. At this point we note that there are several cases we need to analyze, and choose carefully among them if we want to discard the last term in (19) thus avoiding extra complications when extremizing the action
- a) $\tau, \phi, \chi \in \mathbb{R}$ and $M$ a compact manifold
  Since the manifold has no boundary, the last term is immediately null, but it also follows that all the topological numbers are trivially zero. This kills the topological degrees of freedom we want to incorporate into the model.
- b) $\tau, \phi, \chi: M \to \mathbb{R}$ and $M$ a compact manifold
  Since the manifold has no boundary the last term is null in this case. The couplings immediately become bounded functions, however, all the topological numbers are zero. As before, this kills the topological degrees of freedom.
• c) \( \tau, \varphi, \chi \in \mathbb{R} \) and \( M \) a non-compact manifold

It can be immediately seen that for this setup the second term is null and by the fact that the last term is an exact form it will not have contributions over the field equations. However, there are strong non-linear high derivative contributions coming from the characteristic densities over the boundary. If this is considered leads to nontrivial results in the context of AdS/CFT correspondence which are beyond the scope of the present work that were studied in the following references [27, 28].

• d) \( \tau, \varphi, \chi : M \to \mathbb{R} \) and \( M \) a non-compact manifold

This option maintains the topological degrees of freedom over all. The last term is not contributing to the field equations because of being an exact differential. However, as it was mentioned earlier, it has nontrivial contributions over the boundary. By imposing that the couplings behave as cutoffs over the boundary of the space-time manifold, i.e. \( \tau|_{\partial M} = \varphi|_{\partial M} = \chi|_{\partial M} = 0 \) we are just considering the bulk contributions and setting boundary conditions for the couplings.

Since option d) is the most suitable for our search, we will refer almost exclusively to it from now on. Note that by considering the setup here presented, the effects of the CC can be absorbed by the topological terms at the level of the action. An important consequence of introducing in the action the characteristic classes is therefore that it may be possible to “naturally” adjust the available parameters to obtain a prediction for the effective cosmological constant that matches current observations. The interpretation of the current acceleration of the expansion of the universe would then take a topological explanation, hopefully allowing to avoid the well known fine tuning issue present in the standard interpretation of the \( \Lambda \)CDM model.

3. Analytical solutions for the Contorsion and Torsion in different topological setups

Before tackling the problem described by the action introduced in section 2.1 (equation (19)), we propose few simplified cases in which we analyze the separated effect of the topological terms. We first take a look at the two simplest scenarios, where we consider the contributions from the Nieh-Yan and Pontryagin terms, and Pontryagin and Euler terms respectively; for these two cases in fact we can obtain analytical solutions. Strong of the experience with the first two cases, we present two more that we approach in a more heuristic way. The latter cases describe the contribution of the Nieh-Yan and Euler terms and the general case in which all three terms are allowed to be present. The case by case approach is justified by the fact that the information on the topological and differential structure is different in each case. In what follows we drop the “\( M \)” from the volume \( V_M \) to make the notation clearer. As stated previously, we start by varying the action (19) considering the setup d) of section 2.1, i.e. couplings as 0-forms and \( M \) a non compact manifold.

3.1. Case 1: Coupling Nieh-Yan and Pontryagin densities to the Einstein-Hilbert action

This first case can be carried out very straightforwardly. The field equations are

\[
\begin{align*}
\delta e & : 0 = -\epsilon_{abcd} e^b \wedge R^a \wedge + \frac{\Lambda_{\text{eff}}}{2} \epsilon_{abcd} e^b \wedge e^d \wedge + d\varphi \wedge T^a, \\
\delta \omega & : 0 = -2 \epsilon^{abcd} T_c \wedge e_d - 2 d\tau \wedge R_{ab} - d\varphi \wedge e^d \wedge e^b, \\
\delta \varphi & : \frac{m_{\text{eff}}}{\epsilon} = T^a \wedge T_a - R_{ab} \wedge e^d \wedge e^b, \\
\delta \tau & : \frac{m_{\text{eff}}}{\varphi} \epsilon = R^a_b \wedge R_a^b .
\end{align*}
\]
Let us define for later convenience the skew symmetric bilinear operator \( \mathcal{L}_{ab} (f, g) \) to be
\[
\mathcal{L}_{ab} (f, g) \triangleq i_{f a b} (df \wedge dg), \quad \mathcal{L}_{ab} (f) = \mathcal{L}_a (f) \mathcal{L}_b (g), \quad (24)
\]
where \( f : M \to \mathbb{R} \) and \( g : M \to \mathbb{R} \) be two zero-forms. Then, combining equations (20) and (21) and using (10) we obtain
\[
e^a \wedge e^b \wedge (4K_{ab} - \left[ \mathcal{L}_{ac} (\tau, \varphi) K^c_b - \mathcal{L}_{bc} (\tau, \varphi) K^c_a \right] + \left( -\mathcal{L}_{ab} (\tau, \varphi) \xi^d (K_{de}) e^e + \epsilon_{abcd} \epsilon^d \left( d\varphi + \frac{2\Lambda_{eff}}{3} d\tau \right) \right) = 0,
\]
where \( \mathcal{L}^d (\xi) \equiv \mathcal{L}^d (\xi) \) is the Lie derivative of the zero-form \( \xi \) along \( e_d \). After some algebraic manipulations we get
\[
4K_{ab} - \left[ \mathcal{L}_{ac} (\tau, \varphi) K^c_b - \mathcal{L}_{bc} (\tau, \varphi) K^c_a \right] - \mathcal{L}_{ab} (\tau, \varphi) \xi^d (K_{de}) e^e + \epsilon_{abcd} \epsilon^d \left( d\varphi + \frac{2\Lambda_{eff}}{3} d\tau \right) = A_{ab}, \quad (25)
\]
where \( A_{ab} \) is a skew symmetric one-form subjected to the condition \( A_{ab} \wedge e^a \wedge e^b = 0 \), and is yet to be determined. Note that
\[
A_{ab} \wedge e^a \wedge e^b = 0 \quad \Rightarrow \quad A_{ab} = -[i_a (A_{bc}) - i_b (A_{ac})] \wedge e^c. \quad (26)
\]
Combining (26) and (25) an expression depending on \( K_{ab} \) and its interior products exclusively can be derived, the explicit form of which is not illustrative and so we do not report it in here. On the other hand, imposing a relation between the coupling through a pullback \( \tau = \phi^* \varphi \) allows us to obtain the following
\[
K_{ab}^{(i)} = -\frac{1}{4} \left( \frac{n_N}{6} \varphi + \frac{n_P}{6} \varphi + \frac{2}{3} \Lambda + \frac{\partial \varphi}{\partial \tau} \right) \epsilon_{abcd} e^c \mathcal{L}^d (\tau) + \left[ \mathcal{L}_{a} (\theta_1) e_b - \mathcal{L}_{b} (\theta_1) e_a \right], \quad (27)
\]
where \( \theta_1 : M \to \mathbb{R} \) is a zero form that needs to be adjusted in order to fulfill the rest of the restrictions imposed by the field equations. Note that the topological information is shown explicitly in the contortion through the presence of the topological parameters.

The torsion is obtained using (10) to yield
\[
T_a^{(i)} = \frac{1}{4} \left( \frac{\partial \varphi}{\partial \tau} + \frac{n_N}{6} \varphi + \frac{n_P}{6} \varphi + \frac{2}{3} \Lambda \right) \epsilon_{abcd} e^b \wedge e^c \wedge \mathcal{L}^d (\tau) + \mathcal{L}_{b} (\theta_1) e^d \wedge e_a, \quad (28)
\]
while the curvature can be obtained after some lengthy algebra using (11), it is necessary to take account of the topological restrictions (22) and (23), as well as to choose a metric. We postpone then this part of the analysis until the next section.

3.2. Case 2: Coupling Nieh-Yan and Euler characteristic invariants to the Einstein-Hilbert action

The second case we consider is also quite straightforward. By imposing \( \tau = 0 \) in the action (19) the following field equations are obtained
\[
\delta e : 0 = -\epsilon_{abcd} e^b \wedge R^{cd} + \frac{\Lambda_{eff}}{3} \epsilon_{abcd} e^b \wedge e^d + d\varphi \wedge T_d, \quad (29)
\]
\[
\delta \omega : 0 = -2\epsilon_{abcd} T^c \wedge e^d - d\varphi \wedge e_a \wedge e_b - 2\epsilon_{abcd} d\chi \wedge R^{cd}, \quad (30)
\]
\[
\delta \chi : \frac{n}{4} \epsilon = \epsilon_{abcd} R^{ab} \wedge R^{cd}, \quad (31)
\]
\[
\delta \varphi : \frac{n}{2} \epsilon = T^a \wedge T_a - R_{ab} \wedge e^a \wedge e^b, \quad (32)
\]
with the last equation coinciding with equation (23), as expected.

Here, the approach goes along the lines of what presented for the previous case. Equations (29) and (30) when combined lead to

\[
\frac{\Lambda_{\text{eff}}}{3} e_{abcd} d\chi \wedge e^b \wedge e^c \wedge e^d + d\chi \wedge d\varphi \wedge T_a + e_{abcd} T^b \wedge e^c \wedge e^d = 0,
\]

and, by using the differential operator defined in (24), we get

\[
- \epsilon_{abcd} \int \left( \frac{\Lambda_{\text{eff}}}{3} d\chi \right) e^d - \frac{1}{2} \left[ \mathcal{L}_{\text{ab}} (\chi, \varphi) K^e_b - \mathcal{L}_{\text{bc}} (\chi, \varphi) K^e_a \right] + \\
+ \frac{1}{2} \mathcal{L}_{\text{ab}} (\chi, \varphi) \int (K_{ed}) e^d + \epsilon_{abcd} \int (K_{ed}) e^d + \epsilon_{abcd} K^e = B_{ab},
\]

which is an explicit skew-symmetric one-form subjected to the condition

\[
B_{ab} \wedge e^a \wedge e^b = 0 \quad \Rightarrow \quad B_{ab} = - \left[ i_a (B_{bc}) - i_b (B_{ac}) \right] \wedge e^c.
\]

As in the previous case, by combining (34) and (35) we get an equation in terms of the contortion and its interior products only. As before, we will assume that \( \chi = \phi^\ast \varphi \) yielding

\[
\left\{ \dot{\rho} \left[ \frac{\Lambda}{3} + \frac{n NY}{12 V} \varphi + \frac{n E}{12 V} \chi d\chi \right] e^b - \dot{\rho} \left[ \frac{\Lambda}{3} + \frac{n NY}{12 V} \varphi + \frac{n E}{12 V} \chi d\chi \right] e^d + \\
+ 2 \left\{ \dot{\rho} (K^e c) - \dot{\rho} (K^e c) \right\} e^e - \left\{ i_e (K^e) e^b - i_e (K^e c) e^d \right\} = -4 K^{ab},
\]

which eventually leads to

\[
K_{ab}^{(2)} = \left[ \frac{\Lambda}{3} + \frac{n NY}{12 V} \varphi + \frac{n E}{12 V} \chi \right] \left[ \mathcal{L}_{\text{ab}} (\chi, \varphi) e^b - \mathcal{L}_{\text{bc}} (\chi, \varphi) e_a \right] + \epsilon_{abcd} e^a \mathcal{L}^d (\theta_2),
\]

where \( \theta_2 : \mathcal{M} \to \mathbb{R} \) is a zero form which must be adjusted using the field equations. Finally, with the aid of (10) and (37) we can find the following expression for the torsion

\[
T_{a}^{(2)} = \left[ \frac{\Lambda}{3} + \frac{n P}{12 V} \varphi + \frac{n E}{12 V} \chi \right] d\chi \wedge e_a - \epsilon_{abcd} e^b \wedge e^d \mathcal{L}^d (\theta_2). 
\]

Like before, we will delay the calculation to find the curvature from (11) until later, when we will take into consideration the restrictions (32) and (31) and we will make an explicit choice for the metric.

3.3. Case 3: Pontryagin and Euler characteristic invariants added to the Einstein-Hilbert action

Setting \( \varphi = 0 \) in (19) yields the following field equations

\[
\delta e : 0 = - \epsilon_{abcd} e^b \wedge R^d + \frac{\Lambda_{\text{eff}}}{3} \epsilon_{abcd} e^b \wedge e^c \wedge e^d,
\]

\[
\delta \omega : 0 = - \epsilon_{abcd} \tau^c \wedge e^d - d\tau \wedge R_{ab} - \epsilon_{abcd} d\chi \wedge R^{cd},
\]

\[
\delta \tau : \frac{n}{n} \varepsilon = R^a_b \wedge R^b_a,
\]

\[
\delta \chi : \frac{n}{n} \varepsilon = \epsilon_{abcd} R^{ab} \wedge R^{cd}.
\]
Note the presence of the field equations (41) and (42) which we have already encountered in the previous cases. The standard approach we used before can’t be applied here directly, in fact, combining equations (39) and (40) will lead to expressions that can’t be cast in terms of the contortion and its interior products only. Hence, we take a more heuristic approach to finding a solution for this third case.

We proceed as follows, equation (39) by itself gives

\[ R^{(3)}_{ab} = \Lambda_{e f f} e^a \wedge e_b + \delta e_{abcd} e^c \wedge e^d, \tag{43} \]

where \( \delta : M \to \mathbb{R} \) is a zero form that must depend on the parameters and the couplings. By combining it with equation (40) the torsion is found to be

\[ T^{(3)}_a = -\left( \delta d\tau + \frac{\Lambda_{e f f}}{3} d\chi \right) \wedge e_a - \frac{1}{4\sigma} e_{abcd} e^c \wedge e^d \left( 8\sigma \delta d\chi + \frac{2\Lambda_{e f f}}{3} d\tau \right), \]

where \( \sigma = +1 \) for an Euclidean like metric and \( \sigma = -1 \) for a Lorentzian like metric. Using the identity

\[ K_{ab} = -\frac{1}{2} \left( i_a (T_b) - i_b (T_a) - i_{ab} (T_c) \wedge e^c \right), \]

and equation (18) with \( \phi = 0 \), after some algebra we get the expression

\[ K^{(3)}_{ab} = i_a \left( \delta d\tau + \frac{\Lambda_{e f f}}{3} d\chi \right) \wedge e_b - i_b \left( \delta d\tau + \frac{\Lambda_{e f f}}{3} d\chi \right) \wedge e_a + \]

\[ + \frac{1}{4\sigma} e_{abcd} e^c \wedge e^d \left( 8\sigma \delta d\chi + \frac{2\Lambda_{e f f}}{3} d\tau \right), \tag{44} \]

Equation (44) has been rearranged in order to highlight the similarities with (27) and (17). By comparison we can see that the heuristic approach paid off. From here it is straightforward to get the final expression for the torsion

\[ T^{(3)}_a = \left( \frac{\delta}{\partial \chi} + \frac{\Lambda}{3} + \frac{np}{12V} \delta + \frac{ne}{12V} \right) d\chi \wedge e_a + \]

\[ - \frac{1}{4\sigma} \left( \frac{np}{6V} \delta + \frac{ne}{6V} + 8\sigma \frac{\delta}{\partial \tau} + \frac{2\Lambda}{3} \right) e_{abcd} e^c \wedge e^d \wedge \mathcal{L}^d (\tau). \tag{45} \]

where again we have used a condition of the type \( \chi = \phi^* \tau \) to simplify the expression.

The 0-form \( \delta \) introduced as a device of calculation fulfills the two branched equation

\[ \delta = \pm \sqrt{\frac{ne}{4\sigma V} - \frac{1}{4\sigma} \left( \frac{\Lambda_{e f f}}{3} \right)^2}, \tag{46} \]

obtained by combining (43) and (31). A similar situation will appear on the next case.

Finally, the curvature can be obtained by using equation (11).
3.4. Case 4: Nieh-Yan, Pontryagin and Euler characteristic invariants added to the Einstein-Hilbert action

The more heuristic approach used on the previous case can also be applied here with some minor differences. Let us now consider the entire action \( \text{[19]} \), its variation yields

\[
\delta e : 0 = -\varepsilon_{abcd} e^b \wedge R^{cd} + \frac{\Lambda_{\text{eff}}}{3} e^c \wedge e^d + d\varphi \wedge T_a, \quad (47)
\]

\[
\delta \omega : 0 = -2\varepsilon_{abcd} T^c \wedge e^d - 2d\tau \wedge R_{ab} - d\varphi \wedge e_a \wedge e_b - 2\varepsilon_{abcd} d\chi \wedge R^{cd}, \quad (48)
\]

\[
\delta \varphi : \frac{\eta}{T} \varepsilon = T^a \wedge T_a - R_{ab} \wedge e^b, \quad (49)
\]

\[
\delta \chi : \frac{\eta}{\varphi} \varepsilon = \varepsilon_{abcd} R^{ab} \wedge R^{cd}, \quad (50)
\]

\[
\delta \tau : \frac{\eta}{\varphi} \varepsilon = R^a_b \wedge R^b_a. \quad (51)
\]

We also consider the couplings to be some pullback of a common zero-form \( \eta: \mathcal{M} \rightarrow \mathbb{R} \) such that \( \varphi = \phi^a \eta \), \( \tau = \phi^c \eta \) and \( \chi = \phi^d \eta \).

Multiplying equation \( (47) \) by \( d\varphi \) leads to

\[
\varepsilon_{abcd} \left( R^{cd} - \frac{\Lambda_{\text{eff}}}{3} e^c \wedge e^d \right) \wedge e^b \wedge d\varphi = 0, \quad (52)
\]

which in turn leads to the following skew symmetric curvature 2-form

\[
R_{ab}^{(4)} = \frac{\Lambda_{\text{eff}}}{3} e_a \wedge e_b + (\gamma_1 d\tau + \gamma_2 d\varphi + \gamma_3 d\chi) \wedge (e_a - e_b) + \gamma \varepsilon_{abcd} e^c \wedge e^d, \quad (53)
\]

where \( \gamma_1, \gamma_2, \gamma_3 : \mathcal{M} \rightarrow \mathbb{R} \). Inserting this back in \( (52) \) yields

\[
T_{a}^{(4)} = -\left( \gamma d\tau + \frac{\Lambda_{\text{eff}}}{3} d\chi \right) \wedge e_a - \frac{1}{4\sigma} \varepsilon_{abcd} e^b \wedge e^c \wedge d\varphi + 8\sigma r d\varphi dyd\chi, \quad (54)
\]

Note that this last expression is completely independent of the auxiliary zero-forms \( \gamma_1, \gamma_2 \) and \( \gamma_3 \). However, these are not the final expressions we are searching for the curvature and torsion forms.

After some algebra the previous equations lead to the following expression for the contortion

\[
K_{ab}^{(4)} = i_a \left( \frac{\Lambda_{\text{eff}}}{3} d\chi + \gamma d\tau \right) \wedge e_b - i_b \left( \frac{\Lambda_{\text{eff}}}{3} d\chi + \gamma d\tau \right) \wedge e_a + \frac{1}{4\sigma} \varepsilon_{abcd} e^c \wedge d\varphi + 8\sigma r d\varphi d\chi dyd\varphi + 2\Lambda_{\text{eff}} d\tau d\varphi + 8\sigma r d\varphi d\chi dyd\varphi,
\]

\[
\frac{1}{4\sigma} \varepsilon_{abcd} \left( \frac{\eta}{\varphi} d\tau + \frac{n_p}{6\varphi} \tau + \frac{n_{NY}}{12\varphi} \tau + \frac{n_p}{12\chi} \chi \right) \left( L_{ab}(\chi) e_b - L_b(\chi) e_a \right) + \frac{1}{4\sigma} \varepsilon_{abcd} \left( \frac{n_p}{6\varphi} \tau + \frac{n_{NY}}{6\varphi} \tau + \frac{n_p}{6\varphi} \chi \right) \left( \frac{\delta}{\delta \tau} + \frac{2\Lambda_{\text{eff}}}{3} \frac{\delta}{\delta \tau} + 8\sigma r \frac{\delta}{\delta \tau} \right), \quad (55)
\]

where in the last step we have written the contortion in a way suitable for comparison with the previous cases. The solution for the torsion is then

\[
T_{a}^{(4)} = -\left( \gamma \frac{\delta}{\delta \chi} + \frac{\Lambda_{\text{eff}}}{3} \frac{\delta}{\delta \chi} + \frac{n_{NY}}{12\tau} \frac{\delta}{\delta \tau} + \frac{n_p}{12\tau} \frac{\delta}{\delta \tau} + \frac{n_p}{12\chi} \frac{\delta}{\delta \tau} \right) d\chi \wedge e_a + \frac{1}{4\sigma} \varepsilon_{abcd} \left( \frac{n_p}{6\varphi} \tau + \frac{n_{NY}}{6\varphi} \tau + \frac{n_p}{6\varphi} \chi \right)
\]

\[
\frac{1}{4\sigma} \varepsilon_{abcd} \left( \frac{n_p}{12\varphi} \tau + \frac{n_{NY}}{12\varphi} \tau + \frac{n_p}{12\chi} \chi \right) \left( \frac{\delta}{\delta \tau} + \frac{2\Lambda_{\text{eff}}}{3} \frac{\delta}{\delta \tau} + 8\sigma r \frac{\delta}{\delta \tau} \right), \quad (55)
\]
Finally, we have that Restrictions (23) and (31), combined with (52) lead to the equation
\[ \gamma^3 + \left[ 3 \frac{\Lambda_{\text{eff}}}{4\sigma} \right]^2 \gamma - \frac{n_E}{4\sigma V} \frac{\Lambda_{\text{eff}}}{3} = 0, \] (56)
which is a monic trinomial with three solutions similarly to the case of the \( \delta \) 0-form from the previous case. The curvature is obtained from (11). The extra 0-forms \( \gamma_1, \gamma_2 \) and \( \gamma_3 \) are unimportant, while \( \gamma \) explicitly appears in the solutions.

3.5. On the topological restrictions as field equations

A brief discussion on the restrictions coming from equations (22), (23) and (31) is needed before analyzing some explicit examples in the next section. After an integration using Stokes theorem they all take the form
\[ \int_{\partial M} \tilde{C}_i = n_i, \] (57)
where \( i = NY, P, E \) and the three-forms \( \tilde{C}_i \) have been defined in (5), (6) and (7).

Since the manifold \( M \) is non-compact, the topological numbers are generally different from zero and explicitly depend on the couplings as well as on the connection and the vierbein.

It is best to treat the topological restrictions in the integral form (57) rather than (22), (23) and (31) since we already set the boundary conditions for the couplings. The equations obtained will then give boundary conditions for the derivatives of the couplings.

To be more concrete, let us consider restriction (22) in the setup of case 4. Using (55), it reads
\[ \int_{\partial M} \left( \frac{\partial \varphi}{\partial \tau} + 8\sigma \gamma \frac{\partial \chi}{\partial \tau} + \frac{2}{3} \Lambda \right) \epsilon_{abcd} a^a \wedge b^b \wedge c^c \mathcal{L}^d (\tau) = 4\sigma n_{NY}, \] (58)
which is in the form of (57), and where we used for the boundary conditions of the couplings \( \varphi|_{\partial M} = \tau|_{\partial M} = \chi|_{\partial M} = 0 \). These restrictions are consistent with a non-compact manifold.

Note that equation (58) depends on the other characteristic numbers through the appearance of \( \gamma \), defined in equation (56). Note also that if we set \( n_{NY} = 0 \) we obtain the following boundary condition for the derivative of the couplings
\[ \left( \frac{\partial \varphi}{\partial \tau} \right)|_{\partial M} + 8\sigma \gamma|_{\partial M} \frac{\partial \chi}{\partial \tau}|_{\partial M} = -\frac{2}{3} \Lambda. \] (59)
These boundary conditions are difficult to analyze since they are not strictly Cauchy’s or Dirichlet’s, rather they are mixed.

A typical choice for the set of parameters could be \( n_{NY} = n_P = n_E = 0 \), since it is known that it reproduces some well known model [29]. However, setting \( n_{NY} = 0 \) means that the sum of the winding numbers of the manifold \( M \) is zero [2], which in turn has strong consequences in the differential structure of the manifold. Setting \( n_P = 0 \) instead defines a cobordism class of manifolds for which the results obtained are valid [1, 26]. And finally, \( n_E = 0 \) restricts to an oriented closed manifold with genus 1 [1, 26].

As it can be seen a case by case study must be done for each example.
4. Cosmological Solutions

In the previous section we presented analytical solutions of the system described by the Einstein-Hilbert action together with the addition of topological terms via the introduction of characteristic classes. The final form of the four proposed solutions, each describing a particular class of possible physical manifolds (as was discussed at the end of the previous section) is summarized in equations (27), (37), (44), and (54) that give the contortion for each combination of topological terms considered.

We now want to use the results we found to build cosmological solutions. We start then by choosing the following form for the metric

\[ ds^2 = dt^2 + \sigma(t)^2(dx^2 + dy^2 + dz^2), \]

where \( \sigma = -1 \) for a Lorentzian metric (FLRW metric with flat spatial curvature \( k = 0 \)), and \( \sigma = +1 \) for a Euclidean metric. We allow for the freedom in signature choice in order to recover interesting solutions that have been reported (torsion vortex metric \([22]\)). The tetrad is easily obtained to be

\[ e^0 = dt, \quad e^1 = a(t)dx, \quad e^2 = a(t)dy, \quad e^3 = a(t)dz, \]

and from (3) we obtain the following expressions for the spin connection \( \bar{\omega}^{ab} \)

\[ \bar{\omega}^0 = He^t, \quad \bar{\omega}^i_j = 0, \quad i, j = 1, 2, 3 \]

where we have defined the Hubble parameter \( H = \dot{a} / a \) in the usual way, and where \( a' \) denotes the time derivative of the scale factor.

In order to study a realistic cosmological scenario we also add in the action (19) a term taking care of the matter source

\[ S_m = \int M L_m, \]

which when varied with respect to the vierbein yields the energy-momentum form \( \tau_a = \delta L_m / \delta e^a \) and therefore the term \(-2T_{ab} \wedge \star e^b\), where the \( \star \) represents the Hodge dual mapping that must be included in the left side of the field equation (47) or its equivalent in the other cases examined. We focus our attention on a pressure-less dust scenario described by a perfect fluid with energy-momentum tensor \( T_{ab} = (\rho, 0, 0, 0) \), with density \( \rho \). Since we are eventually interested in describing realistic solutions, and we know isotropy and homogeneity are characteristics of our Universe at large scales, we can consider from the start all the couplings we introduced in the action to be dependent on the time coordinate only.

Equations (61) and (62) are then used to construct the corresponding contortion \( K^{(i)}_{ab} \), torsion \( T^{(i)}_a \) and curvature \( R^{(i)}_{ab} \) forms with \( i = 1, 2, 3, 4 \) labeling each of the cases considered in the previous section. We have to recall though, that the expressions \( K_{ab}, T_a^{(i)} \) and \( R^{(i)}_{ab} \) we calculated have been obtained in the absence of matter and thus need to be corrected for the introduction of it. In this case, having in mind equation (47) with a dust type of matter, it is easy to show that equation (52) behaves as follows: \( R^i \) acquires the term \( \rho/3e^i \wedge e^i \) and \( R^0 \) acquires the term \(-\rho/6e^0 \wedge e^i \) for \( i = 1, 2, 3 \). By also using equation (48) the expression obtained is the equation (55) with the substitution \( \Lambda_{\text{eff}} \rightarrow \rho + \Lambda_{\text{eff}} \). Summarizing, the effective changes in our equations for a dust type of matter eventually reduce to applying the replacement \( \Lambda_{\text{eff}} \rightarrow \rho + \Lambda_{\text{eff}} \).
We take a look now at the system of differential equations related to the case 4 since it is the one with all the couplings turned on. Collecting all the field equations (47 - 51), and also using (15) we get the following system of equations

\[-4(\mathcal{H} - x) + \sigma r'xz + 4\chi'\left(x^2 + \frac{\sigma z^2}{16}\right) = 0,\]

\[z + 2\tau'\left(x^2 + \frac{\sigma z^2}{16}\right) - \varphi' + 2\chi'xz = 0,\]

\[3\left(x^2 + \frac{\sigma z^2}{16}\right) - \Lambda = \rho,\]

\[-2\sigma\left(x^2 + \frac{\sigma z^2}{16}\right) - \frac{4\sigma}{a(t)} \frac{d}{dt} (xa(t)) + 2\sigma \Lambda + \frac{1}{2}\varphi z = 0,\]

\[xz\frac{d}{dt} (za(t)) + 8\sigma\left(x^2 + \frac{\sigma z^2}{16}\right) \frac{d}{dt} (xa(t)) = 0,\]

\[\frac{2xz}{a(t)} \frac{d}{dt} (xa(t)) - \frac{1}{a(t)} \left(x^2 + \frac{\sigma z^2}{16}\right) \frac{d}{dt} (za(t)) = 0,\]

\[z' + 3Hz = 0,\]

where we have set

\[x = \mathcal{H} - \frac{1}{3}(\Lambda + \rho) \chi' + \gamma r',\]

\[z = -\frac{2}{3}(\Lambda + \rho) \tau' + \varphi' + 8\sigma \gamma \chi',\]

for convenience. We note that equations (68), (69) and (70) come from the variation with respect to the Euler, Pontryagin and Nieh-Yan terms respectively, and so are present in the case at hand while may not be part of the set of equations when one or more of the characteristic classes are "switched off" from the start.

In fact, all the equations needed to describe the cases we considered in the previous section can be obtained from the system above, recalling that when we switched off one or more couplings, we did so at the level of the action. This leads to a change in the number of differential equations we then obtain from the action principle, beside of course the vanishing of the corresponding couplings in the remaining equations. We will also cover some cases not treated in section 3 but that can be obtained from these solutions and have a cosmological interest.

4.1. Setting $z = 0$ and $\chi = 0$

For this class of solutions we obtain $\gamma \tau' = 0$, so either $\gamma = 0$ or $\tau = constant$. Having $\tau = constant$ implies $\varphi = constant$, i.e. we recover General Relativity. If we consider $\gamma = 0$, the torsion becomes null, and we recover the equations and solutions of General Relativity, too

\[3H^2 = \rho + \Lambda,\]

\[\rho' + 3H\rho = 0.\]

However, the extra structure departing from General Relativity shows itself in the presence of the Pontryagin and Nieh-Yan couplings that satisfy the following equation

\[2\tau' H^2 - \varphi' = 0.\]

Note also that if $\tau$ is constant then Equation (75) implies that (with $H \neq 0$) $\varphi$ is a constant too, and vice versa. Since the solutions for the scale factor $a(t)$ for General Relativity are known we are not considering them here.
4.2. Setting $\tau = 0$ and $\chi = 0$

Although in principle this is not a case that was analyzed in section 3, it is clear from the equations obtained that the appearance of just one of the density classes is enough to have torsion. The set (64 - 69) thus allows us to study the present configuration which is interesting because of reproducing some models already found in literature. Note that in this case the system (64 - 69) looses the equations coming from the couplings $\tau$ and $\chi$ leading to

$$\begin{align*}
x &= H, \\
z &= \varphi' = \frac{A}{a^i},
\end{align*}$$

where $A$ is an integration constant.

It is worth to notice that considering $\rho \neq 0$ combined with the energy-momentum conservation equation, which is contained in the above system of differential equations, we arrive at

$$\rho' + 3H\rho = 0, \quad \Rightarrow \quad \rho = \frac{\rho_0}{a^3},$$

which in principle represents a cosmological solution. We now present several sub-cases derived from the above

- for $\rho = 0$, $\sigma = +1$ and $\Lambda > 0$

$$\begin{align*}
H(t) &= \sqrt{\frac{\Lambda}{3}} \tanh \left( \sqrt{3\Lambda} (t - t_0) \right), \\
a(t) &= a_0 \left( \cosh \left( \sqrt{3\Lambda} (t - t_0) \right) \right)^{1/3}, \\
\varphi(t) &= \frac{2A}{a_0^2 \sqrt{3\Lambda}} \arctan \left( \frac{\cosh \left( \sqrt{3\Lambda} t \right)}{2} \sinh \left( \frac{\sqrt{3\Lambda}}{2} (t - 2t_0) \right) \right),
\end{align*}$$

where $t_0$ is an integration constant. Note that equation (80) is exactly the solution known as the torsion vortex [22]. In fact, this solution exactly corresponds to the one given in Eq. (33) of [22], with $\varphi(0) = a(t)$. Note also that in this paper and for an Euclidean metric ($\sigma = +1$), $\Lambda > 0$ corresponds to the anti-de Sitter case, because we have written the cosmological constant term in the action (1) with a plus sign. This choice was done in order that for a Lorentzian metric ($\sigma = -1$) with signature (+,−,−,−) $\Lambda < 0$ corresponds to the anti-de Sitter case.

- for $\rho \neq 0$, $\sigma = -1$ and $\Lambda = 0$

$$\begin{align*}
a(t) &= \frac{1}{2^{4/3}} \left( \frac{A^2}{B} - 36B(t - t_0)^2 \right)^{1/3}, \\
\varphi(t) &= \frac{8}{3} \arctanh \left( \frac{6B}{A} (t - t_0) \right), \\
\rho(t) &= -\frac{3B}{a^3(t)},
\end{align*}$$

with $t_0$ and $B < 0$ integration constants.
The deceleration parameter \( q \), defined by

\[
q = -\frac{a(t)a'(t)}{a''(t)},
\]

is given by

\[
q = \frac{1}{2} + \frac{A^2}{24B^2(t - t_0)^2},
\]

which is always positive, therefore the solution can not describe an accelerating universe. Also, if \( a(t) \) is real then \( \phi(t) \) must be imaginary, necessarily. Figure 1 shows the behavior of \( a(t), q(t) \) and \( \phi(t) \) for \( A = 10, B = -0.1 \) and \( t_0 = 0 \).

![Figure 1: Behavior of \( a(t), q(t) \) and \( \phi(t) \) for \( A = 10, B = -0.1 \) and \( t_0 = 0 \)](image)

- for \( \rho \neq 0, \sigma = -1 \) and \( \Lambda > 0 \), we obtain the solutions

\[
a(t) = \frac{1}{2\Lambda^{1/3}} e^{-\sqrt{\Lambda}(t-t_0)} \left( 3\Lambda \right)^{2/3} \left( e^{\sqrt{3\Lambda}(t-t_0)} - 6B \right)^{1/3},
\]

\[
\varphi(t) = \frac{8}{3} \text{arctanh} \left( \frac{e^{\sqrt{3\Lambda}(t-t_0)} - 6B}{A\sqrt{3\Lambda}} \right),
\]

\[
\rho(t) = -\frac{3B}{a^3(t)},
\]

who’s behavior is shown in Figure 2 for \( A = 100, B = -1, t_0 = 0 \) and \( \Lambda = 0.1 \) and

\[
a(t) = \frac{1}{2\Lambda^{1/3}} e^{-\sqrt{\Lambda}(t-t_0)} \left( 3\Lambda \right)^{2/3} \left( e^{\sqrt{3\Lambda}(t-t_0)} - 6B \right)^{1/3},
\]

\[
\varphi(t) = -\frac{8}{3} \text{arctanh} \left( \sqrt{3} \left( \Lambda^{2/3} e^{\sqrt{3\Lambda}(t-t_0)} + 2B - 12B^2 e^{\sqrt{3\Lambda}(t-t_0)} \right)/A \right),
\]

\[
\rho(t) = -\frac{3B}{a^3(t)}.
\]
with $t_0$ and $B < 0$ integration constants. In Figure 2 we show this set of solutions of $a(t)$, $q(t)$ and $\varphi(t)$ for $A = 100$, $B = -1$, $t_0 = 0$ and $\Lambda = 0.1$

Figure 2: Behavior of $a(t)$, $q(t)$ and $\varphi(t)$ for $A = 100$, $B = -1$, $t_0 = 0$ and $\Lambda = 0.1$

for $\rho \neq 0$, $\sigma = -1$ and $\Lambda < 0$, we obtain the solutions

$$a(t) = \frac{1}{24^{1/3} (\Lambda)^{1/2}} e^{-\sqrt{-\Lambda} t_0}\left(-12\Lambda A^2 - \left(e^{\Lambda_3 A(t-t_0)} \sqrt{-\Lambda - 12B}\right)^2\right)^{1/3}, \quad (93)$$

$$\varphi(t) = \frac{8}{3} \arccoth\left(\frac{2\Lambda A \sqrt{3}}{\Lambda e^{\Lambda_3 A(t-t_0)} + 12B \sqrt{-\Lambda}}\right), \quad (94)$$

Figure 3: Behavior of $a(t)$, $q(t)$ and $\varphi(t)$ for $A = 10$, $B = -1$, $t_0 = 0$ and $\Lambda = 1$

• for $\rho \neq 0$, $\sigma = -1$ and $\Lambda < 0$, we obtain the solutions

$$a(t) = \frac{1}{24^{1/3} (\Lambda)^{1/2}} e^{-\sqrt{-\Lambda} t_0}\left(-12\Lambda A^2 - \left(e^{\Lambda_3 A(t-t_0)} \sqrt{-\Lambda - 12B}\right)^2\right)^{1/3}, \quad (93)$$

$$\varphi(t) = \frac{8}{3} \arccoth\left(\frac{2\Lambda A \sqrt{3}}{\Lambda e^{\Lambda_3 A(t-t_0)} + 12B \sqrt{-\Lambda}}\right), \quad (94)$$

Figure 3: Behavior of $a(t)$, $q(t)$ and $\varphi(t)$ for $A = 100$, $B = -1$, $t_0 = 0$ and $\Lambda = 0.1$
\[ \rho (t) = -\frac{3B}{a^2(t)}, \]

where in Figure 4 we show the previous set of \(a(t), q(t)\) and \(\varphi(t)\) for \(A = 100, B = -1, t_0 = 0\) and \(\Lambda = -0.1\). and

\[ a(t) = \frac{1}{2^{4/3} (-\Lambda)^{1/2}} e^{\frac{\sqrt{-3} (t-t_0)}{2}} \left(-12\Lambda A^2 \left( e^{\sqrt{-3} (t-t_0)} \sqrt{-\Lambda + 12B} \right)^2 \right)^{1/3}, \]

\[ \varphi (t) = -\frac{8}{3} \text{arccoth} \left( \frac{2A \sqrt{3}}{\Lambda e^{-\sqrt{-3} (t-t_0)} - 12B \sqrt{-\Lambda}} \right), \]

\[ \rho (t) = -\frac{3B}{a^2(t)}, \]

with \(t_0\) and \(B < 0\) integration constants. In Figure 5 we show this set of solutions of \(a(t), q(t)\) and \(\varphi(t)\) for \(A = 10, B = -1, t_0 = 0\) and \(\Lambda = -1\).

- for \(\rho = 0, \sigma = -1\) and \(\Lambda > 0\) the cosmological solutions coincide with those of (79)-(81).
- for \(\rho = 0, \sigma = -1\) and \(\Lambda < 0\) we have
  \[ a(t) = a_0 \left( \cos \left( \sqrt{-3} \Lambda (t-t_0) \right) \right)^{1/3}, \]
  \[ \varphi (t) = \frac{2A}{\sqrt{-3} A_0} \text{arctanh} \left( \sec \left( \frac{\sqrt{-3} \Lambda}{2} \right) \sin \left( \frac{\sqrt{-3} \Lambda}{2} (t - 2t_0) \right) \right). \]
- for \(\rho = 0, \sigma = -1\) and \(\Lambda = 0\) we obtain
  \[ a(t) = a_0^{1/3}, \]
  \[ \varphi (t) = \frac{A}{a_0^3} \log (t), \]
4.3. Setting $\chi = 0$ and $\varphi = 0$

As before, since only one coupling is needed to obtain Torsion, we go to analyze the case in which only the Pontryagin class is present. When discarding equations coming from the Nieh-Yan invariant constraint and from the Euler invariant constraint it can be shown that in the remaining system (64 - 69) not all equations are independent. There are three equations remaining which do not contain the energy-momentum conservation equation (78). Several solutions are presented in what follows

- for $\rho = 0$ and $\Lambda > 0$

\[ H(t) = \sqrt[3]{\Lambda} \]  \hspace{1cm} (103)
\[ a(t) = a_0 e^{\sqrt[3]{\Lambda} t} \]  \hspace{1cm} (104)

and $\tau$ constant, i.e. we recover general relativity.

- $\rho \neq 0$

If additionally we impose the condition (78), then $\tau$ must be a constant, i.e. we recover General Relativity. The solutions are

\[ \text{-- } \Lambda < 0 \]
\[ H(t) = \sqrt[3]{-\Lambda} \tan\left(-\frac{\sqrt{-3\Lambda}}{2}(t-t_0)\right) \]  \hspace{1cm} (105)
\[ a(t) = a_0 \left(\cos\left(\frac{\sqrt{-3\Lambda}}{2}(t-t_0)\right)\right)^{2/3} \]  \hspace{1cm} (106)
– \( \Lambda > 0 \)

\[
H(t) = \sqrt{\frac{\Lambda}{3}} \tanh\left(\frac{\sqrt{3} \Lambda}{2} (t-t_0)\right),
\]

\[
a(t) = a_0 \left( \cosh\left(\frac{\sqrt{3} \Lambda}{2} (t-t_0)\right) \right)^{2/3},
\]

– \( \Lambda = 0 \)

\[
H(t) = \frac{2}{3t-t_0},
\]

\[
a(t) = (3t-t_0)^{2/3}.
\]

4.4. Setting \( \varphi = 0 \) and \( \tau = 0 \)

As before, this is a case that can be studied beginning from the system (64 - 69). This was not a case considered in section 3, however, it shows some interesting features. When discarding those equations coming from Nieh-Yan invariant constraint and from the Pontryagin invariant constraint the remaining system contains the conservation equation (78). After solving the remaining system the following solutions are derived

- for \( \rho = 0 \) and \( \Lambda < 0 \)

\[
x(t) = \sqrt{-\frac{\Lambda}{2}} \tan\left(-\sqrt{-\frac{\Lambda}{2}} (t-t_0)\right),
\]

\[
\chi(t) = -\frac{1}{\Lambda} \log\left(\sin\left(\sqrt{-\frac{\Lambda}{2}} (t-t_0)\right)\right),
\]

\[
a(t) = a_0 \frac{\cos\left(\sqrt{-\frac{\Lambda}{2}} (t-t_0)\right)}{\left(\sin\left(\sqrt{-\frac{\Lambda}{2}} (t-t_0)\right)\right)^{1/3}}.
\]

- for \( \rho = 0 \) and \( \Lambda > 0 \)

\[
x(t) = \sqrt{\frac{\Lambda}{2}} \tanh\left(\sqrt{\frac{\Lambda}{2}} (t-t_0)\right),
\]

\[
\chi(t) = -\frac{1}{\Lambda} \log\left(\sinh\left(\sqrt{\frac{\Lambda}{2}} (t-t_0)\right)\right),
\]

\[
a(t) = a_0 \frac{\cosh\left(\sqrt{\frac{\Lambda}{2}} (t-t_0)\right)}{\left(\sinh\left(\sqrt{\frac{\Lambda}{2}} (t-t_0)\right)\right)^{1/3}}.
\]
• for $\rho = 0$ and $\Lambda = 0$

\[
\begin{align*}
    x(t) &= \frac{1}{t - t_0}, \\
    \chi(t) &= -\frac{t(t - 2t_0)}{4}, \\
    a(t) &= t - t_0.
\end{align*}
\] (117)

Note that an Euclidean metric with $\sigma = 1$ only satisfies the condition $\Lambda > 0$.

We don’t show the behavior of the solutions for a dust type of matter since they were studied in [29] and [30].

4.5. Setting $\tau = 0, \chi = 0$ and $n_{NY} \neq 0$

As it was stressed in section 2.1, the possibility of discarding the CC, and emulating its effect using topological constants is briefly studied in this example. This case presents an exact solution in which only the Nieh-Yan density is added to the Einstein-Hilbert action contrasting with the previous cases. Here we study a setup in which $n_{NY} \neq 0$, noting that this allows to set to zero the cosmological constant and, as discussed in section 3, still have its effects played by the topological number $n_{NY}$ to find

\[
\Lambda_{\text{eff}} = \frac{n_{NY}}{4V} \rho,
\] (120)

where equation (18) was used. From here, the Friedmann equations in the standard form follows

\[
\begin{align*}
    3H^2 &= \rho + \tilde{\rho}, \\
    3H^2 + 2H' &= -\bar{P},
\end{align*}
\] (121)

where

\[
\begin{align*}
    \tilde{\rho} &= \frac{n_{NY}}{4V} \phi + \frac{3}{16} \phi', \\
    \bar{P} &= -\frac{n_{NY}}{4V} \phi + \frac{3}{16} \phi',
\end{align*}
\] (123)

represent the density and pressure due to the torsion. The remaining equation reads

\[
\phi'' + 3H \phi' = \frac{2n_{NY}}{3V}.
\] (125)

The total energy $\rho + \tilde{\rho}$ satisfy the conservation equation (18). However, they do not satisfy this equation separately. Being so, a solution to the system reads

\[
\begin{align*}
    \phi(t) &= a t^2, \\
    H(t) &= \frac{1}{9a} \left( \frac{n_{NY}}{V} - 3a \right) \frac{1}{t}, \\
    a(t) &= a_0 e^{\left( \frac{n_{NY}}{3V} - 3a \right)} t,
\end{align*}
\] (126)

which depending on the values of the parameters we can have different cosmological scenarios. We let the discussion of these solutions to further works since the idea was to stress the existence of solutions with cosmological constant replaced by a topological parameter.
5. Final Remarks

As we have stressed before, the closed solutions for the Contortion 1-form are suitable for constructing the curvature 2-form in each case, and hence sufficient for studying the cosmological implications of a gravitational theory with Torsion. By taking a metric such as the one showed in we were able to reproduce some well known models in a more or less general framework, for instance the torsion vortex solution reported in [22]. Also we obtained a class of new solutions that have interest for cosmology.

Since the cases of section were studied in a coordinate free fashion, another alternative to encompass the task of study the material presented here is to consider different metrics than the one used here. This program will allow to find some known solutions as well as possibly new ones and will be considered in future works.

As we pointed out before, by allowing the topological parameters to be different from zero we can consider the cosmological constant to be some reminiscent of the topology of the space-time manifold. The fact that these solutions allows for an effective CC is a feature of the model that has very interesting consequences. We found one solution that presents an interaction between different densities but it is by no means unique. This possibility will be further explored in future works.

6. Acknowledgements

The authors want to thank Gastón Giribet for his comments and valuable suggestions for starting this work.

The authors also want to thank Michele Fontanini for his valuable corrections, comments and suggestions in the final steps of this work.

This work was funded by Comisión Nacional de Ciencias y Tecnología through FONDECYT Grant 11121148 (Y.V., J.L.).

J.L. acknowledge the hospitality of the Universidad de La Serena where part of this work was carried out.

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