Interlace polynomials and Tutte polynomials

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Abstract

Let $G$ be a graph with adjacency matrix $A(G)$. Consider the matrix $IA(G) = (I | A(G))$, where $I$ is the identity matrix, and let $M(IA(G))$ be the binary matroid represented by $IA(G)$. Then suitable parametrized versions of the Tutte polynomial of $M(IA(G))$ yield the interlace polynomials of $G$, introduced by Arratia, Bollobás and Sorkin [J. Combin. Theory Ser. B 92 (2004) 199-233; Combinatorica 24 (2004) 567-584]. Interlace polynomials subsequently introduced by other authors may be obtained from parametrized Tutte polynomials of the binary matroid represented by $(I | A(G) | I + A(G))$.

Keywords. interlace polynomial, matroid, multimatroid, Tutte polynomial

Mathematics Subject Classification. 05C50

1 Introduction

Motivated by problems that arise in the study of DNA sequencing, Arratia, Bollobás and Sorkin introduced a one-variable graph polynomial, the vertex-nullity interlace polynomial, in [2]. In subsequent work [3, 4] they observed that this one-variable polynomial may be obtained from the Tutte-Martin polynomial of isotropic systems studied by Bouchet [5], introduced an extended two-variable version of the interlace polynomial, and observed that the interlace polynomials are given by formulas that involve the nullities of matrices over the two-element field, $GF(2)$. Inspired by these ideas, Aigner and van der Holst [11], Courcelle [13] and the author [25, 26] introduced several different variations on the interlace polynomial theme.

All these references share the underlying presumption that although the theory of the interlace polynomials is connected to that of the Tutte polynomial in some ways, the two theories are largely separate in general. In this short note we point out that in fact, the interlace polynomials of graphs can be derived from parametrized Tutte polynomials of binary matroids associated with adjacency matrices. We presume the reader is familiar with the standard terminology of graph theory and matroid theory; see [18, 22, 28, 29] for instance.
2 The identity-adjacency matroid

We restrict our attention to looped simple graphs. That is, a graph $G$ is given by specifying a finite set $V(G)$ of vertices, declaring that certain vertices are looped and the others are not, and declaring that certain pairs of distinct vertices are neighbors and the other pairs are not. The adjacency matrix of $G$ is the $V(G) \times V(G)$ matrix $A(G)$ with entries in $GF(2)$ given by: a diagonal entry is 1 if and only if the corresponding vertex is looped, and an off-diagonal entry is 1 if and only if the corresponding vertices are neighbors.

If $I$ is the $|V(G)| \times |V(G)|$ identity matrix then $IA(G)$ is the $|V(G)| \times 2^{|V(G)|}$ matrix $IA(G) = (I \mid A(G))$.

For convenience of notation and in order to indicate the relationship with our previous work on the interlace polynomials, we use the Greek letters $\phi$ and $\chi$ to refer to the columns of the indicated submatrices of $IA(G)$: the column of $I$ corresponding to $v$ is denoted $v_\phi$, and the column of $A(G)$ corresponding to $v$ is denoted $v_\chi$.

**Definition 1** The identity-adjacency matroid $M(IA(G))$ is the binary matroid represented by $IA(G)$.

That is, $M(IA(G))$ is a matroid on the ground set $W(G) = \{v_\phi, v_\chi \mid v \in V(G)\}$, and if $T \subseteq W(G)$ then the rank $r^G(T)$ of $T$ in $M(IA(G))$ equals the dimension of the $GF(2)$-vector space spanned by the columns of $IA(G)$ corresponding to elements of $T$.

One way to define the Tutte polynomial of $M(IA(G))$ is a polynomial in the variables $s$ and $z$, given by the subset expansion

$$t(M(IA(G))) = \sum_{T \subseteq W(G)} s^{r^G(W(G)) - r^G(T)} z^{|T|} - r^G(T).$$

We do not give a general account of this famous invariant of graphs and matroids here; thorough introductions may be found in [6, 12, 14, 18].

Tutte polynomials of graphs and matroids are remarkable both for the amount of structural information they contain and for the range of applications in which they appear. Some applications (electrical circuits, knot theory, network reliability, and statistical mechanics, for instance) involve graphs or networks whose vertices or edges have special attributes of some kind – impedances and resistances in circuits, crossing types in knot diagrams, probabilities of failure and successful operation in reliability, bond strengths in statistical mechanics. A natural way to think of these attributes is to allow each element to carry two parameters, $a$ and $b$ say, with $a$ contributing to the terms of the Tutte polynomial corresponding to subsets that include the given element, and $b$ contributing to the terms of the Tutte polynomial corresponding to subsets that do not. Zaslavsky [30] calls the resulting polynomial

$$\sum_{T \subseteq W(G)} \left( \prod_{t \in T} a(t) \right) \left( \prod_{w \notin T} b(w) \right) s^{r^G(W(G)) - r^G(T)} z^{|T|} - r^G(T)$$

(1)
the parametrized rank polynomial of \(M(IA(G))\); we denote it \(\tau(M(IA(G)))\).

We do not give a general account of the theory of parametrized Tutte polynomials here; the interested reader is referred to the literature, for instance [7, 16, 23, 24, 30]. However it is worth taking a moment to observe that parametrized polynomials are very flexible, and the same information can be formulated in many ways. For instance if \(s\) and the parameter values \(b(w)\) are all invertible then formula (1) is equivalent to

\[
s_{\tau(G(W))} \cdot \left( \prod_{w \in W(G)} b(w) \right) \cdot \sum_{T \subseteq W(G)} \left( \prod_{t \in T} \frac{a(t)}{b(t)s} \right) (sz)^{|T| - r^G(T)},
\]

which expresses \(\tau(M(IA(G)))\) as the product of a prefactor and a sum that is essentially a parametrized rank polynomial with only \(a\) parameters and one variable, \(sz\). We prefer formula (1), though, because we do not want to assume invertibility of the \(b\) parameters.

Suppose that the various parameter values \(a(w)\) and \(b(w)\) are independent indeterminates, and let \(P\) denote the ring of polynomials with integer coefficients in the set of \(2 + 4 |V(G)|\) independent indeterminates \(\{s, z\} \cup \{a(w), b(w) \mid w \in W(G)\}\). Let \(J\) be the ideal of \(P\) generated by the set of \(2 |V(G)|\) products \(\{a(v_\phi)a(v_\chi), b(v_\phi)b(v_\chi) \mid v \in V(G)\}\), and let \(\pi : P \to P/J\) be the canonical map onto the quotient. Then the only summands of (1) that make nonzero contributions to \(\pi \tau(M(IA(G)))\) correspond to subsets \(T \subseteq W(G)\) with the property that \(|T \cap \{v_\phi, v_\chi\}| = 1 \forall v \in V(G)\). Each such \(T\) is a transversal of the partition of \(W(G)\) into 2-element subsets \(\{v_\phi, v_\chi\}\) corresponding to vertices of \(G\); we denote the collection of all such transversals \(\mathcal{T}(W(G))\). Each \(T \in \mathcal{T}(W(G))\) has \(|T| = |V(G)| = r^G(W(G))\), so \(s\) and \(z\) have the same exponent in the corresponding term of \(\pi \tau(M(IA(G)))\):

\[
\pi \tau(M(IA(G))) = \pi \left( \sum_{T \in \mathcal{T}(W(G))} \left( \prod_{t \in T} a(t) \right) \left( \prod_{w \notin T} b(w) \right) (sz)^{|V(G)| - r^G(T)} \right).
\]

Every generator of \(J\) involves the product of two different parameters \(a(v_i), b(v_i)\) corresponding to a single vertex \(v\) of \(G\). It follows that \(\pi\) is injective when restricted to the additive subgroup \(A\) of \(P\) generated by products

\[
\left( \prod_{t \in T} a(t) \right) \left( \prod_{w \notin T} b(w) \right) (sz)^k
\]

where \(k \geq 0\) and \(T \in \mathcal{T}(W(G))\). Consequently there is a well-defined isomorphism of abelian groups \(\pi^{-1} : \pi(A) \to A\), and we have

\[
\pi^{-1} \pi \tau(M(IA(G))) = \sum_{T \in \mathcal{T}(W(G))} \left( \prod_{t \in T} a(t) \right) \left( \prod_{w \notin T} b(w) \right) (sz)^{|V(G)| - r^G(T)}. \quad (2)
\]

Note that \(\pi^{-1} \pi \tau(M(IA(G)))\), the image of the parametrized Tutte polynomial \(\tau(M(IA(G)))\) under the mappings \(\pi\) and \(\pi^{-1}\), might also be described as
the section of $\tau(M(IA(G))$ corresponding to $T(W(G))$. Either way, formula (2) describes an element of $P$, where $s$, $z$ and the various parameter values $a(w)$, $b(w)$ are all independent indeterminates.

Arratia, Bollobás and Sorkin define the two-variable interlace polynomial $q(G)$ by the formula

$$q(G) = \sum_{S \subseteq V(G)} (x - 1)^{r(A(G)[S])} (y - 1)^{|S| - r(A(G)[S])}$$

$$= \sum_{S \subseteq V(G)} \left( \frac{y - 1}{x - 1} \right)^{|S| - r(A(G)[S])} (x - 1)^{|S|}.$$  

Here $r(A(G)[S])$ denotes the $GF(2)$-rank of the principal submatrix of $A(G)$ involving rows and columns corresponding to vertices from $S$.

For $T \in T(W(G))$ let $S(T) = \{v \in V(G) \mid v \chi \in T\}$; then $T \mapsto S(T)$ defines a bijection from $T(W(G))$ onto the power-set of $V(G)$. As $r^G(T)$ is the $GF(2)$-rank of the matrix

$$(\text{columns } v_\phi \text{ with } v \notin S(T) \mid \text{columns } v_\chi \text{ with } v \in S(T))$$

and the columns $v_\phi$ are columns of the identity matrix,

$$r^G(T) = |V(G)| - |S(T)| + r(A(G)[S(T)]).$$

It follows that $q(G)$ may be obtained from $\pi^{-1} \pi \tau(M(IA(G))$ by setting $a(v_\phi) \equiv 1$, $a(v_\chi) \equiv x - 1$, $b(v_\phi) \equiv 1$, $b(v_\chi) \equiv 1$, $s = y - 1$ and $z = 1/(x - 1)$. These assignments are not unique; for instance the values of $s$ and $z$ may be replaced by $s = (y - 1)/\sigma$ and $z = \sigma/(x - 1)$ for any invertible $\sigma$.

## 3 Recursive formulas

In this section we show how a recursive description of the parametrized rank polynomial yields the recursive description of the interlace polynomial given by Arratia, Bollobás and Sorkin. Recall that a coloop of a matroid is an element that is included in every basis, and a loop is an element that is excluded from every basis. Suppose $r^M$ is the rank function of a matroid $M$ on the ground set $W$. If $w \in W$ then $M - w$ is the matroid on $W - \{w\}$ whose rank function is given by $r^M(T)$, for $T \subseteq W - \{w\}$; $M/w$ is the matroid on $W - \{w\}$ whose rank function is given by $r^M(T \cup \{w\}) - r^M(\{w\})$.

Parametrized rank polynomials may be calculated recursively as follows:

1. If $\emptyset$ is the empty matroid then $\tau(\emptyset) = 1$.
2. If $M$ is a matroid on $W$ and $w$ is a coloop of $M$ then $\tau(M) = (a(w) + sb(w)) \cdot \tau(M/w)$.
3. If $w$ is a loop of $M$ then $\tau(M) = (b(w) + za(w)) \cdot \tau(M - w)$. 


4. If \( w \) is neither a coloop nor a loop of \( M \) then \( \tau(M) = b(w)\tau(M - w) + a(w)\tau(M/w) \).

Suppose \( G \) is a graph and \( v \in V(G) \) is not isolated; let \( w \) be a neighbor of \( v \). If \( v \) is looped then \( \{x_\phi \mid x \in V(G)\} \) and \( \{v_\phi \} \cup \{x_\phi \mid x \neq v\} \) are both bases of \( M(I\!A(G)) \), because \( I \) and
\[
\begin{pmatrix}
1 & 0 \\
\ast & 1'
\end{pmatrix}
\]
are both of rank \( |V(G)| \), where \( 1' \) is the identity matrix of order \( |V(G)| - 1 \). If \( v \) is unlooped then \( \{x_\phi \mid x \in V(G)\} \) and \( \{v_\chi, w_\chi \} \cup \{x_\phi \mid v \neq x \neq w\} \) are both bases of \( M(I\!A(G)) \), because \( I \) and
\[
\begin{pmatrix}
0 & 1 & 0 \\
1 & \ast & 0 \\
\ast & \ast & 1''
\end{pmatrix}
\]
are both of rank \( |V(G)| \), where \( 1'' \) is the identity matrix of order \( |V(G)| - 2 \). In either case, we see that \( v_\phi \) is not a coloop or a loop of \( M(I\!A(G)) \).

If \( v \) is looped then \( \{v_\chi \} \cup \{x_\phi \mid x \neq v\} \) and \( \{w_\chi \} \cup \{x_\phi \mid x \neq v\} \) are both bases of \( M(I\!A(G)) - v_\phi \). If \( v \) is not looped then \( \{w_\chi \} \cup \{x_\phi \mid x \neq v\} \) and \( \{v_\chi, w_\chi \} \cup \{x_\phi \mid v \neq x \neq w\} \) are both bases of \( M(I\!A(G)) - v_\phi \). In either case, we see that \( v_\chi \) is not a coloop or a loop of \( M(I\!A(G)) - v_\phi \).

As \( v_\phi \) has only one nonzero entry, the definition of matroid contraction mentioned at the beginning of this section tells us that the rank function of \( M(I\!A(G))/v_\phi \) is the function on \( W(G) - \{v_\phi\} \) defined using the columns of the matrix \( I\!A(G)' \) obtained from \( I\!A(G) \) by removing both the column \( v_\phi \) and the row corresponding to \( v \). In particular, the rank of the whole matroid is \( |V(G)| - 1 \). As \( \{x_\phi \mid x \neq v\} \) and \( \{v_\chi \} \cup \{x_\phi \mid v \neq x \neq w\} \) are both bases, \( v_\chi \) is not a coloop or a loop of \( M(I\!A(G))/v_\phi \).

Let \( G - v \) be the graph obtained from \( G \) by removing \( v \) and all edges incident on it. Then \( I\!A(G - v) \) is the matrix obtained by removing the column \( v_\chi \) from \( I\!A(G)' \), so
\[
M(I\!A(G - v)) = (M(I\!A(G))/v_\phi) - v_\chi.
\]
Using step 4 of the recursion to remove \( v_\phi \) and then \( v_\chi \), we see that
\[
\pi\tau(M(I\!A(G))) = 
\pi a(v_\phi) b(v_\chi) \tau((M(I\!A(G))/v_\phi) - v_\chi) + \pi b(v_\phi) a(v_\chi) \tau((M(I\!A(G)) - v_\phi)/v_\chi)
= \pi a(v_\phi) b(v_\chi) \tau(M(I\!A(G - v))) + \pi b(v_\phi) a(v_\chi) \tau((M(I\!A(G)) - v_\phi)/v_\chi).
\]
With the parameter values given at the end of Section 2, this yields
\[
\pi^{-1}\pi\tau(M(I\!A(G))) = \pi^{-1}\pi\tau(M(I\!A(G - v)) + (x-1)\pi^{-1}\pi\tau((M(I\!A(G)) - v_\phi)/v_\chi)
\]
or equivalently,
\[
q(G) = q(G - v) + (x - 1)\pi^{-1}\pi\tau((M(I\!A(G)) - v_\phi)/v_\chi).
\]
Suppose \( v \) is looped, and let \( G' \) be the graph obtained from \( G \) by toggling all adjacencies between neighbors of \( v \), and also toggling the loop status of every neighbor of \( v \). Consider two matrices

\[
\begin{pmatrix}
1 & 1 & 0 \\
1 & B & C \\
0 & D & E
\end{pmatrix}
\quad \text{and} \quad
\begin{pmatrix}
1 & 0 & 0 \\
1 & \overline{B} & C \\
0 & D & E
\end{pmatrix}
\]

where bold numerals indicate row and column vectors and the overbar indicates a submatrix in which all entries have been toggled (reversed). Elementary column operations tell us that the two matrices have the same \( GF(2) \)-rank. Consequently, if \( v_\chi \in T \in \mathcal{T}(W(G)) \) then \( r^G(T) = 1 + r^{G'\setminus v}(T - \{v_\chi\}) \). The definition of matroid contraction tells us that if \( v_\phi \in T \in \mathcal{T}(W(G)) \) then the rank of \( T - \{v_\phi\} \in (M_{IA(G)} - v_\phi)/v_\chi \) is \( r^G(T) - 1 \). It follows that the ranks of \( T - \{v_\phi\} \in (M_{IA(G)} - v_\phi)/v_\chi \) and \( M_{IA(G')} \) are equal. Combining this equality with \( (3) \), we see that if \( v \) is looped then

\[
q(G) = q(G - v) + (x - 1)\pi^{-1}\pi'(M_{IA(G')} - v)) = q(G - v) + (x - 1)q(G' - v).
\]

Formula \( (4) \) is one of the two fundamental recursive formulas for \( q(G) \) \( (4) \).

Deriving the other fundamental recursive formula takes a little more work, because the term \( \pi^{-1}\pi'((M_{IA(G)} - v_\phi)/v_\chi) \) in formula \( (3) \) does not correspond to a single interlace polynomial.

Suppose that \( v \) is unlooped and has an unlooped neighbor \( w \). As in \( (4) \), say that two vertices \( x, y \not\in \{v, w\} \) are distinguished by \( \{v, w\} \) if they have different, nonempty neighborhoods in \( \{v, w\} \), and let \( G^{vw} \) denote the graph obtained from \( G \) by toggling all adjacencies between vertices distinguished by \( \{v, w\} \).

The matroid \( (M_{IA(G)} - v_\phi)/v_\chi \) has \( W(G) - \{v_\phi, v_\chi\} = W(G - v) \) as its ground set, and \( \pi^{-1}\pi'((M_{IA(G)} - v_\phi)/v_\chi) \) includes nonzero contributions from elements of \( \mathcal{T}(G - v) \). Split \( \pi^{-1}\pi'((M_{IA(G)} - v_\phi)/v_\chi) \) into two parts, \( S_\phi \) and \( S_\chi \), with \( S_\phi \) including the contributions from elements of \( \mathcal{T}(G - v) \) that include \( w_\phi \) and \( S_\chi \) including the contributions from elements of \( \mathcal{T}(G - v) \) that include \( w_\chi \).

Consider three matrices

\[
\begin{pmatrix}
0 & 0 & 1 & 1 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 \\
1 & 0 & B_{11} & B_{12} & B_{13} & B_{14} \\
1 & 0 & B_{21} & B_{22} & B_{23} & B_{24} \\
0 & 0 & B_{31} & B_{32} & B_{33} & B_{34} \\
0 & 0 & B_{41} & B_{42} & B_{43} & B_{44}
\end{pmatrix}
= \begin{pmatrix}
0 & 0 & 1 & 1 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 \\
1 & 0 & B_{11} & B_{12} & B_{13} & B_{14} \\
1 & 0 & B_{21} & B_{22} & B_{23} & B_{24} \\
0 & 0 & B_{31} & B_{32} & B_{33} & B_{34} \\
0 & 0 & B_{41} & B_{42} & B_{43} & B_{44}
\end{pmatrix}
\]

and

\[
\begin{pmatrix}
0 & 0 & 1 & 1 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 \\
1 & 0 & B_{11} & B_{12} & B_{13} & B_{14} \\
1 & 0 & B_{21} & B_{22} & B_{23} & B_{24} \\
0 & 0 & B_{31} & B_{32} & B_{33} & B_{34} \\
0 & 0 & B_{41} & B_{42} & B_{43} & B_{44}
\end{pmatrix}
\]
Elementary column operations tell us that the first two matrices have the same $GF(2)$-rank, and elementary row operations tell us that the third also has the same $GF(2)$-rank. Removing the second row and column from the third matrix reduces the $GF(2)$-rank by 1, clearly. It follows that if $v_\chi, w_\phi \in T \in T(W(G))$ then

$$r^G(T) - 1 = r^{G^{vw} - w}(T - \{w_\phi\}).$$

Recall that $r^G(T) - 1$ is the rank of $T - \{v_\chi\}$ in $(M(IA(G)) - v_\chi)/v_\chi$, and the parameter values given in Section 2 include $a(v_\chi) = x - 1$ and $a(w_\phi) = 1$. It follows that the contribution of $T - \{w_\phi\}$ to $\pi^{-1}\pi\tau(M(IA(G^{vw} - w)))$ is the product of $x - 1$ and the contribution of $T - \{v_\chi\}$ to $\pi^{-1}\pi\tau((M(IA(G)) - v_\chi)/v_\chi)$.

Consequently, if we split $q(G^{vw} - w)$ into two parts, $q_\phi$ and $q_\chi$, with $q_\phi$ including the contributions from elements of $T(G^{vw} - w)$ that include $v_\phi$ and $q_\chi$ including the contributions from elements of $T(G^{vw} - w)$ that include $v_\chi$, then

$$(x - 1)S_\phi = q_\chi.$$ 

If $v_\phi \in T \in T(G^{vw} - w)$, then the corresponding column has only one nonzero entry; elementary column operations show that the rank of $T$ in $M(IA(G^{vw} - w))$ is 1 more than the rank of $T - \{v_\phi\} in M(IA(G^{vw} - v - w))$. It follows that

$$q_\phi = q(G^{vw} - v - w).$$

Formula (3) now tells us that

$$q(G) = q(G - v) + (x - 1)(S_\phi + S_\chi) = q(G - v) + q_\chi + (x - 1)S_\chi \quad (5)$$

$$= q(G - v) + q(G^{vw} - w) - q_\phi + (x - 1)S_\chi$$

$$= q(G - v) + q(G^{vw} - w) - q(G^{vw} - v - w) + (x - 1)S_\chi.$$

It remains only to discuss $S_\chi$. Consider two matrices

$$\begin{pmatrix} 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & B_{11} & B_{12} & B_{13} & B_{14} \\ 1 & 0 & B_{21} & B_{22} & B_{23} & B_{24} \\ 0 & 1 & B_{31} & B_{32} & B_{33} & B_{34} \\ 0 & 0 & B_{41} & B_{42} & B_{43} & B_{44} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & \overline{B}_{11} & \overline{B}_{12} & \overline{B}_{13} & \overline{B}_{14} \\ 1 & 0 & \overline{B}_{21} & \overline{B}_{22} & \overline{B}_{23} & \overline{B}_{24} \\ 0 & 1 & \overline{B}_{31} & \overline{B}_{32} & \overline{B}_{33} & \overline{B}_{34} \\ 0 & 0 & \overline{B}_{41} & \overline{B}_{42} & \overline{B}_{43} & \overline{B}_{44} \end{pmatrix}.$$ 

Elementary column operations tell us that the two matrices have the same $GF(2)$-rank. It follows that if $v_\chi, w_\chi \in T \in T(W(G))$ then

$$r^G(T) = 2 + r^{G^{vw} - v - w}(T - \{v_\chi, w_\chi\}).$$

The rank of $T - \{v_\chi\}$ in $(M(IA(G)) - v_\phi)/v_\chi$ is

$$r^G(T) - 1 = 1 + r^{G^{vw} - v - w}(T - \{v_\chi, w_\chi\}).$$
so considering the parameter values given at the end of Section 2, we see that the contribution of \( T - \{ v_\chi \} \) to \( S_\chi \) is the product of \( x - 1 \) and the contribution of \( T - \{ v_\chi, w_\chi \} \) to \( \pi^{-1} \pi \tau(M \{ IA(G^{vw} - v - w) \}) \). It follows that

\[
S_\chi = (x - 1)q(G^{vw} - v - w),
\]

so formula (5) tells us that

\[
q(G) = q(G - v) + q(G^{vw} - w) - q(G^{vw} - v - w) + (x - 1)^2 q(G^{vw} - v - w).
\]

This is the second fundamental recursive formula for \( q(G) \) given in [4].

4 The identity-adjacency-sum matroid

We call the binary matroid \( M(\text{IAS}(G)) \) represented by the matrix

\[
\text{IAS}(G) = (I \mid A(G) \mid A(G) + I)
\]

the identity-adjacency-sum matroid of the graph \( G \). The column of \( A(G) + I \) corresponding to a vertex \( v \) is denoted \( v_\phi \), and the other columns are denoted \( v_\psi, v_\chi \) as in Section 2; then \( W'(G) = \{ v_\phi, v_\chi, v_\psi \mid v \in V(G) \} \) is the ground set of \( M(\text{IAS}(G)) \). Let \( P' \) be the ring of polynomials with integer coefficients in the set of \( 2 + 6 |V(G)| \) independent indeterminates \( \{ s, z \} \cup \{ a(w), b(w) \mid w \in W'(G) \} \), let \( J' \) be the ideal of \( P' \) generated by the set of \( 4 |V(G)| \) products \( \{ a(v_\phi)a(v_\chi), a(v_\phi)a(v_\psi), a(v_\chi)a(v_\psi), b(v_\phi)b(v_\chi)b(v_\psi) \mid v \in V(G) \} \), and let \( \pi' : P' \to P'/J' \) be the canonical map onto the quotient. Then the discussion of Section 2 is readily modified to show that \( (\pi')^{-1} \pi \tau(M(\text{IAS}(G))) \) consists only of terms associated with

\[
T'(W'(G)) = \{ T \subseteq W'(G) \mid |T \cap \{ v_\phi, v_\chi, v_\psi \}| = 1 \ \forall v \in V(G) \}.
\]

The reader familiar with the interlace polynomials introduced by Aigner and van der Holst [1], Courcelle [13], and the author [26] will have no trouble modifying the discussion of Section 2 to show that appropriate values for \( s, z \) and the \( a \) and \( b \) parameters yield all of these interlace polynomials from the parametrized rank polynomial \( \tau(M(\text{IAS}(G))) \). Notice also that \( M(IA(G)) \) is a sub-matroid of \( M(\text{IAS}(G)) \), and \( (\pi')^{-1} \pi \tau(M(\text{IAS}(G))) \) yields \( \pi^{-1} \pi \tau(M(IA(G))) \) by assigning \( a(v_\phi) \equiv 0 \) and \( b(v_\psi) \equiv 1 \); consequently the theory given in Section 2 is contained in the one described here. (It is for Expository convenience, not generality, that we detail the theory of \( M(IA(G)) \) rather than that of \( M(\text{IAS}(G)).\))

5 Comments

1. As explained by Ellis-Monaghan and Sarmiento [15], results of Las Vergnas [19, 20] and Martin [21] on circuit partitions of planar 4-regular graphs imply that if \( G \) happens to be a circle graph obtained from a planar 4-regular graph,
then the vertex-nullity interlace polynomial of $G$ may be obtained from the “diagonal” Tutte polynomial of an associated checkerboard graph. (The diagonal Tutte polynomial is obtained by setting the two variables of the ordinary (non-parametrized) Tutte polynomial equal to each other.) This connection cannot extend directly to non-planar graphs, as Martin [21, p.76] pointed out, because the complete graph $K_5$ has too many Euler circuits to be represented in a 5-element matroid.

Also, Aigner and van der Holst [1] observed that the vertex-nullity interlace polynomial of a bipartite graph may be obtained from the diagonal Tutte polynomial of an associated binary matroid. This result is connected to the preceding paragraph through de Fraysseix’s theorem connecting bipartite circle graphs to planar graphs [17]. More recently, Brijder and Hoogeboom [11] have introduced interlace polynomials for delta-matroids, and in particular for matroids. They use them to extend the connection between vertex-nullity interlace polynomials and diagonal Tutte polynomials to arbitrary matroids.

One way to summarize the content of the present note is this: Using parameters has the effect of algebraically restricting $\tau(M(IA(G)))$ to $T(W(G))$ and $\tau(M(IAS(G)))$ to $T'(W'(G))$, and unlike restriction to the diagonal Tutte polynomial, these restrictions are effective for all graphs and all interlace polynomials.

2. The matrix $IA(G)$ appears in [1] and [3], together with the observation that the vertex-nullity interlace polynomial of $G$ is equal to Bouchet’s Tutte-Martin polynomial of the isotropic system associated with the row space of $IA(G)$ [8, 10].

The matrix $IAS(G)$ also appears in [1], where Aigner and van der Holst showed that their interlace polynomial $Q$ may be obtained by summing over submatrices of $IAS(G)$ associated with elements of $T'(W'(G))$. The content of this note came to mind after R. Brijder pointed out the appearance of the same submatrices in our work with nonsymmetric modified interlacement matrices [27], when we read Bouchet’s comment [9] that Eulerian multimatroids are “sheltered” by matroids and wondered whether $M(IAS(G))$ is in general an appropriate “sheltering” matroid for the 3-matroid associated with an isotropic system with fundamental graph $G$. Indeed it is!

3. Some known properties of interlace polynomials can be readily explained using known properties of Tutte polynomials. For instance, the analogy between pendant-twin reductions for the interlace polynomial and series-parallel reductions for the Tutte polynomial noted by Bläser and Hoffman [5], Ellis-Monaghan and Sarmiento [15] and the author [25, 26] is more than an analogy: when $v$ is pendant on $w$, $v_\chi$ and $w_\beta$ are parallel in $M(IA(G))$; and when $v$ and $w$ are twins, $v_\chi$ and $w_\chi$ are parallel in $M(IA(G))/\{v_\beta, w_\beta\}$. Known properties of Tutte polynomials also provide new insights into interlace polynomials; for instance, the interlace polynomials of $G$ have activities expansions with respect to bases of $M(IA(G))$ or $M(IAS(G))$.

4. In closing: understanding more about the graph-theoretic significance of the matroids $M(IA(G))$ and $M(IAS(G))$ would help in understanding the significance of the interlace polynomials.
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