FURTHER RESULTS AND EXAMPLES FOR
FORMAL MATHEMATICAL SYSTEMS
WITH STRUCTURAL INDUCTION

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Abstract. In the former article “Formal mathematical systems including a structural induction principle” we have presented a unified theory for formal mathematical systems including recursive systems closely related to formal grammars, including the predicate calculus as well as a formal induction principle. In this paper we present some further results and examples in order to illustrate how this theory works.

Keywords: Formal mathematical systems, elementary proof theory, languages and formal grammars, structural induction, \(\omega\)-consistency of Peano arithmetic.

Mathematics Subject Classification: 03F03, 03B70, 03D03, 03D05

1. Introduction

In this article I refer to my former work [2], which is inspired by Hofstadter’s book [1] as well as by Smullyan’s “Theory of formal systems” in [3].

The recursive systems introduced in [2, Section 1] may be regarded as variants of formal grammars, but they are better adapted for use in mathematical logic and enable us to generate in a simple way the recursively enumerable relations between lists of terms over a finite alphabet, using the R-axioms and the R-rules of inference. The R-axioms of a recursive system are special quantifier-free positive horn formulas. In addition, the recursive system contains R-axioms for the
use of equations. Three R-rules of inference provide the use of R-axioms, the Modus Ponens Rule and a simple substitution mechanism in order to obtain conclusions from the given R-axioms.

In Section 2 of the paper on hand we present an example of a recursive system which represents the natural numbers in two different ways. The recursive system generates a specific relation between the dual representation of any natural number \( n \) and its representation as a tally \( a^n = a \ldots a \) of length \( n \) with the single symbol \( a \).

In [2, Section 3] a general recursive system \( S \) is embedded into a formal mathematical system \([M; \mathcal{L}]\) based on the predicate calculus and a formal induction principle. The set \( \mathcal{L} \) of restricted argument lists contains the variables and is closed with respect to substitutions. The embedding is consistent in the sense that the R-axioms of \( S \) with argument lists in \( \mathcal{L} \) will become special axioms of \([M; \mathcal{L}]\) and that the R-rules of inference with substitutions of lists restricted to \( \mathcal{L} \) will be special rules of inference in \([M; \mathcal{L}]\). The formal structural induction in a mathematical system is performed with respect to the axioms of the underlying recursive system \( S \), and the formal induction principle for the natural numbers is a special case.

The three examples in Section 3.1 make use of the axioms and rules for managing formulas with quantifiers in the formal mathematical systems, and namely the first example is needed in Section 3.6.

In Section 3.2 we will present some technical results concerning the substitution of variables in formulas, because substitutions in formulas with quantifiers need special care.

An example with formal induction involving equations will be given in Section 3.3. The underlying recursive system of \([M; \mathcal{L}]\) is simple and similar to that in Section 2, but its R-axioms contain equations, and the formulas which we will deduce in \([M; \mathcal{L}]\) need more effort than it seems at a first glance.

In Section 3.4 we develop a simple procedure in order to eliminate certain prime formulas from formal proofs which do not occur with a given arity in the basis axioms of the mathematical system.
In [2, Section 5] we have stated Conjecture (5.4) which characterizes the provability of variable-free prime formulas in special axiomatized mathematical systems \([M; \mathcal{L}]\) whose basis-axioms coincide with the basis R-axioms of their underlying recursive systems. In Section 3.5 of the paper on hand we present a proof of this conjecture via Theorem 3.5. At least under a natural interpretation of the formulas the theorem shows that the axioms and rules of inference including the Induction Rule (e) from [2, (3.13)(e)] correspond to correct methods of deduction. As a further application of Theorem 3.5 we will give a proof for the \(\omega\)-consistency of the Peano arithmetic, see Theorem 3.8.

In Section 3.6 we will come back to the recursive system \(S\) from Section 2 and will present another instructive example for the use of the Induction Rule (e).

2. Recursive systems

For the preparation of this section we need [2, (1.1)-(1.12)]. There recursively enumerable relations are defined. These are special relations between lists of symbols, and they are generated in a very simple way by three rules of inference, namely Rules (a), (b) and (c) given in (1.11). We start with the following example:

2.1. Dual representation of natural numbers.

We consider the recursive system \(S = [A; P; B]\) with \(A = [a; 0; 1]\), \(P = [D]\), with distinct variables \(x, y \in X\) and with \(B\) consisting of the following six basis R-axioms:

\[
\begin{align*}
(\alpha) & \quad D 1 \\
(\beta) & \quad \rightarrow D x D x0 \\
(\gamma) & \quad \rightarrow D x D x1 \\
(\delta) & \quad D 1, a \\
(\varepsilon) & \quad \rightarrow D x, y D x0, yy \\
(\zeta) & \quad \rightarrow D x, y D x1, yya
\end{align*}
\]

The 1-ary predicate \(D x\) represents natural numbers \(x\) in dual form. The 2-ary predicate \(D x, y\) gives the dual representation \(x\) of a natural number \(y = a^n\), represented as a tally with the single symbol “a”. Note
that the predicate symbol “$D$” is used 1-ary as well as 2-ary within the same recursive system $S$, which has to be mentioned separately in each case. There results another recursive system $\tilde{S}$ if we replace the first three R-axioms by a single one $\rightarrow D x, y D x$. The elementary prime R-formulas derivable in $S$ and $\tilde{S}$ are the same. In Section 3.6 we will come back to the recursive system $S$ and will present an instructive example for a mathematical system with formal induction.

Now we present an R-derivation of the formula $D_{101}$, $aaaaaa$ in the recursive system $S$. It means that 5 (represented by $a^5 = aaaaa$) has the dual representation 101:

(1) $D_{1}, a$ Rule (a) and ($\delta$).
(2) $\rightarrow D x, y D x0, yy$ Rule (a) and ($\varepsilon$).
(3) $\rightarrow D x, y D x1, yya$ Rule (a) and ($\zeta$).
(4) $\rightarrow D_{1}, y D 10, yy$ Rule (c), (2) with $x = 1$.
(5) $\rightarrow D_{1}, a D 10, aa$ Rule (c), (4) with $y = a$.
(6) $D_{10}, aa$ Rule (b), (1) and (5).
(7) $\rightarrow D_{10}, y D 101, yya$ Rule (c), (3) with $x = 10$.
(8) $\rightarrow D_{10}, aa D 101, aaaaa$ Rule (c), (7) with $y = aa$.
(9) $D_{101}, aaaaaa$ Rule (b), (6) and (8).

3. Formal mathematical systems

For the preparation of this section we need [2, (3.1)-(3.15)]. In [2, Section 3] a recursive system $S$ is embedded into a formal mathematical system $M$. This embedding is consistent in the sense that the R-axioms of $S$ will become special axioms of $M$ and that the R-rules of inference will be special rules of inference in $M$. In [2, (3.13)] we use five rules of inference, namely Rules (a)-(e). Rule (e) enables formal induction with respect to the recursively enumerable relations generated by the underlying recursive system $S$.

In [2, (3.15)] formal mathematical systems $[M; L]$ with restrictions in the argument lists of the formulas are introduced. The set of restricted argument lists $L$ contains the variables and is closed with respect to substitutions.
3.1. Generally valid formulas with quantifiers.

Example 1: This is needed in Section 3.6. Let $F$ be a formula of a mathematical system $[M; L]$ and let $x \in X$ be a variable. Then we obtain the following proof of the generally valid formula $\rightarrow F \exists x F$ in $[M; L]$, using the rules in [2, (3.13)].

(1) $\rightarrow \forall x \neg F \rightarrow F$ Rule (a), quantifier axiom (3.11)(a).
(2) $\rightarrow \rightarrow \forall x \neg F \rightarrow F \rightarrow \forall x \neg F$
   Rule (a) with the identically true propositional function
   $\rightarrow \rightarrow \xi_1 \rightarrow \xi_2 \rightarrow \xi_3 \rightarrow \xi_1$.
(3) $\rightarrow F \rightarrow \forall x \neg F$ Rule (b), (1), (2).
(4) $\leftrightarrow \rightarrow \forall x \neg F \exists x F$ Rule (a), quantifier axiom (3.11)(a).
(5) $\rightarrow \rightarrow \forall x \neg F \rightarrow \leftrightarrow \rightarrow \leftrightarrow \rightarrow \rightarrow \leftrightarrow \rightarrow \rightarrow \leftrightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow 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and $\xi_1 = \forall x F; \xi_2 = F; \xi_3 = \forall x \to FG; \xi_4 = G.$

(4)

$$\rightarrow \rightarrow \forall x \to FG \to FG$$

$$\rightarrow \forall x \to FG \to \forall x FG$$

Rule (b), (1), (3).

(5) $\rightarrow \forall x \to FG \to \forall x FG$ Rule (b), (2), (4).

(6) $\forall x \to \forall x \to FG \to \forall xFG$ Rule (d), (5).

(7) Next we apply Rule (a), quantifier axiom (3.11)(b):

$$\rightarrow \forall x \to \forall x \to FG \to \forall xFG$$

$$\rightarrow \forall x \to FG \forall x \to \forall xFG$$

(8) $\rightarrow \forall x \to FG \forall x \to \forall xFG$ Rule (b), (6), (7).

(9) $\rightarrow \forall x \to \forall x FG \to \forall x F \forall x G$

Rule (a), quantifier axiom (3.11)(b).

(10)

$$\rightarrow \rightarrow \forall x \to FG \forall x \to \forall x FG$$

$$\rightarrow \rightarrow \forall x \to \forall x FG \to \forall x F \forall x G$$

$$\rightarrow \forall x \to FG \to \forall x F \forall x G$$

Rule (a) with the identically true propositional function

$$\rightarrow \xi_1 \xi_2$$

$$\rightarrow \xi_2 \rightarrow \xi_3 \xi_4$$

$$\rightarrow \xi_1 \rightarrow \xi_3 \xi_4$$

and $\xi_1 = \forall x \to FG; \xi_2 = \forall x \to \forall x FG; \xi_3 = \forall x F; \xi_4 = \forall x G.$

(11)

$$\rightarrow \rightarrow \forall x \to \forall x FG \to \forall x F \forall x G$$

$$\rightarrow \forall x \to FG \to \forall x F \forall x G$$

Rule (b), (8), (10).

(12) $\rightarrow \forall x \to FG \to \forall x F \forall x G$ Rule (b), (9), (11).

The formulas in Example 1 and Example 2 are generally valid because we have only used non-basis axioms and the rules of inference.
Example 3: We show that \( x \notin \text{free}(F) \) is an essential restriction for the quantifier axiom \( \forall x \rightarrow FG \rightarrow F \forall xG \) in [2] (3.11)(b)]. Let \( a \in A \) be a constant and put \( F = G = \sim x, a \) with \( x \in \text{free}(F) \). Ignoring the condition \( x \notin \text{free}(F) \) would give the invalid “proof”

\[
\begin{align*}
(1) \quad & \rightarrow \forall x \rightarrow \sim x, a \sim x, a \rightarrow \sim x, a \forall x \sim x, a \\
& \text{incorrect use of [2] (3.11)(b)].}
\end{align*}
\]

\[
\begin{align*}
(2) \quad & \rightarrow \sim x, a \sim x, a \quad \text{Rule (a), since } \rightarrow \xi_1 \xi_1 \text{ is identically true.}
\end{align*}
\]

\[
\begin{align*}
(3) \quad & \forall x \rightarrow \sim x, a \sim x, a \quad \text{Rule (d), (2).}
\end{align*}
\]

\[
\begin{align*}
(4) \quad & \rightarrow \sim x, a \forall x \sim x, a \quad \text{Rule (b), (1), (3).}
\end{align*}
\]

\[
\begin{align*}
(5) \quad & \rightarrow \sim x, a \forall x \sim x, a \quad \text{Rule (c), (4) with } x = a.
\end{align*}
\]

\[
\begin{align*}
(6) \quad & \sim x, x \quad \text{Rule (a), axiom of equality.}
\end{align*}
\]

\[
\begin{align*}
(7) \quad & \sim a, a \quad \text{Rule (c), (6) with } x = a.
\end{align*}
\]

\[
\begin{align*}
(8) \quad & \forall x \sim x, a \quad \text{Rule (b), (5), (7).}
\end{align*}
\]

In general the result \( \forall x \sim x, a \) is false.

3.2. Collision-free substitutions in formulas.
Let \([M; L]\) be a mathematical system with restricted argument lists in \( L \), let \( F, G \) be formulas in \([M; L]\) and \( x, y, z \in X \). We make especially use of [2 (3.1)-(3.7)] and want to present the whole proof of [2 Lemma (3.16)(a)] with technical details. This is necessary because substitutions in formulas with quantifiers need special care. The lemma states that we have for all \( x, z \in X \) with \( z \notin \text{var}(F) \):

\[
\begin{align*}
\text{(i) } & \text{CF}(F; z; x) \text{ and} \\
\text{(ii) } & \text{CF}(F_{z/x}^z; x; z) \text{ and} \\
\text{(iii) } & F_{z/x}^z = F.
\end{align*}
\]

**Proof.** We say that a formula \( F \) in \([M; L]\) satisfies the condition (⋆) if we have for all \( x \in X \setminus \text{free}(F) \) and for all \( \mu \in L \):

\[
\text{CF}(F; \mu; x) \quad \text{and} \quad F_{x/x}^\mu = F.
\]

We first use induction on well-formed formulas to show that condition (⋆) is satisfied for all formulas \( F \) in \([M; L]\). Afterwards we prove [2 Lemma (3.16)(a)].
(a) From [2 (3.7)(a)] and [2 (3.5)], [2 (3.6)(a)] we see that condition (⋆) is satisfied for all prime formulas $F$.

(b) If (⋆) is satisfied for a formula $F$, then also for the formula $\neg F$ due to [2 (3.7)(b)] and [2 (3.6)(b)].

(c) Next we assume that $F$ and $G$ both satisfy condition (⋆) (induction hypothesis) and put $H = JFG$ for $J \in \{\to; \leftrightarrow; \&; \lor\}$. Let $x \in X \setminus \text{free}(H)$, $\mu \in L$. From \text{free}(H) = \text{free}(F) \cup \text{free}(G) we obtain that $x \in X \setminus \text{free}(F)$ as well as $x \in X \setminus \text{free}(G)$.

We conclude from our induction hypothesis and [2 (3.7)(c)] that $\text{CF}(F; \mu; x), \text{CF}(G; \mu; x)$ and $\text{CF}(H; \mu; x)$. Next we obtain from our induction hypothesis and [2 (3.6)(c)] that

\[
\text{SbF}(H; \mu; x) = \text{SbF}(JFG; \mu; x) \\
= J \text{SbF}(F; \mu; x) \text{SbF}(G; \mu; x) \\
= JFG = H.
\]

(d) Assume that $F$ satisfies condition (⋆) (induction hypothesis) and put $H = QyF$ for $Q \in \{\forall; \exists\}$. Let $x \in X \setminus \text{free}(H)$, $\mu \in L$ and note that \text{free}(H) = \text{free}(F) \setminus \{y\}.

We have $\text{CF}(H; \mu; x)$ immediately from [2 (3.7)(d)] and obtain that $x \in X \setminus \text{free}(F)$ or $x = y$. For the substitution we distinguish two cases according to [2 (3.6)(d)]:

Case 1: $x = y$. Then

\[
\text{SbF}(H; \mu; x) = \text{SbF}(QyF; \mu; x) = QyF = H.
\]

Case 2: $x \neq y$ and $x \in X \setminus \text{free}(F)$. Then we obtain from our induction hypothesis that

\[
\text{SbF}(H; \mu; x) = \text{SbF}(QyF; \mu; x) = Qy \text{SbF}(F; \mu; x) = QyF = H.
\]

We have shown that condition (⋆) is valid for all formulas $F$ in $[M; L]$.

We say that a formula $F$ in $[M; L]$ satisfies the condition (⋆⋆) if it satisfies (i), (ii) and (iii) for all $x, z \in X$ with $z \notin \text{var}(F)$. For the proof of [2 Lemma (3.16)(a)] we use induction on well-formed formulas to show that condition (⋆⋆) is satisfied for all formulas $F$ in $[M; L]$. 
(a) From [2] (3.7)(a)] and [2] (3.5), [2] (3.6)(a) we see that condition \( \star \star \) is satisfied for all prime formulas \( F \).

(b) If \( \star \star \) is satisfied for a formula \( F \), then also for the formula \( \neg F \) due to [2] (3.7)(b) and [2] (3.6)(b)].

(c) Assume that \( F \) and \( G \) both satisfy condition \( \star \star \) (induction hypothesis) and put \( H = JFG \) for \( J \in \{\to; \leftrightarrow; \&; \vee\} \). Let \( x, z \in X \) with \( z \notin \text{var}(H) = \text{var}(F) \cup \text{var}(G) \). We have

\[
z \notin \text{var}(F), \quad z \notin \text{var}(G)
\]

and conclude from our induction hypothesis and [2] (3.7)(c)], [2] (3.6)(c)] that

(i)’ \( \text{CF}(F; z; x), \text{CF}(G; z; x) \) and hence \( \text{CF}(H; z; x) \),
(ii)’ \( \text{CF}(F^z_x; x; z), \text{CF}(G^z_x; x; z) \) and hence \( \text{CF}(H^z_x; x; z) \),
(iii)’ \( H^z_x = J \text{SbF}(F^z_x; x; z) \text{SbF}(G^z_x; x; z) = JFG = H \).

(d) Assume that \( F \) satisfies condition \( \star \star \) (induction hypothesis) and put \( H = QyF \) for \( Q \in \{\forall; \exists\} \). Let \( x, z \in X \), \( z \notin \text{var}(H) \) and note that \( z \notin \text{var}(F) \cup \{y\} \), especially \( z \neq y \).

Case 1: We suppose that \( x \notin \text{free}(H) \). Since \( H \) satisfies the former condition \( \star \) we obtain that \( \text{CF}(H; z; x), H^z_x = H \), and from \( z \notin \text{var}(H) \) that \( \text{CF}(H^z_x; x; z) \) as well as \( H^z_x = H = H^z_x \).

Hence \( H \) satisfies condition \( \star \star \) in case 1.

Case 2: We suppose that \( x \in \text{free}(H) = \text{free}(F) \setminus \{y\} \). We have \( \text{CF}(F; z; x) \) from our induction hypothesis, recall that \( y \neq z \) and conclude \( \text{CF}(H; z; x) \) from [2] (3.7)(d)ii)]. Next we use that \( y \neq x \) and have from the induction hypothesis

\[
H^z_x = Qy F^z_x \quad \text{and} \quad \text{CF}(F^z_x; x; z).
\]

We obtain \( \text{CF}(H^z_x; x; z) \) from [2] (3.7)(d)ii)] and see

\[
\text{SbF}(H^z_x; x; z) = Qy \text{SbF}(F^z_x; x; z) = Qy F = H
\]

from \( z \neq y \) and the induction hypothesis. Hence \( H \) satisfies condition \( \star \star \) in case 2.

We have shown that condition \( \star \star \) is valid for all formulas \( F \) in \([M; \mathcal{L}]\). \( \square \)
3.3. An example with formal induction and equations.

With $A_S := [a; b; f]$ and $P_S := [W]$ we define a recursive system $S = [A_S; P_S; B_S]$ by the following list $B_S$ of basis R-axioms, where $x, y, s, t, u, v \in X$ are distinct variables:

(1) $W a$
(2) $W b$
(3) $\rightarrow W x \rightarrow W y W xy$
(4) $\sim f(a), a$
(5) $\sim f(b), b$
(6) $\rightarrow W x \rightarrow W y \sim f(xy), f(y)f(x)$.

The strings consisting of the symbols $a$ and $b$ are generated by the R-axioms (1)-(3). They are indicated by the predicate symbol $W$, which is used only 1-ary here, whereas $f$ denotes the operation which reverses the order of such a string. For example, $\sim f(abaab), baaba$ is R-derivable, and equations like $\sim f(abaab), f(aab)ba$ and R-formulas like $W f(aab)ba$ are also R-derivable. The R-formula

$$(\star) \quad \rightarrow W x \sim f(f(x)), x$$

is not R-derivable in $S$. But we will show that the latter formula is provable in the mathematical system $[M; L]$ with $M = [S; A_S; P_S; B_S]$ and the set $L$ generated by the following rules:

(i) $x \in L$ for all $x \in X$,
(ii) $a \in L$ and $b \in L$,
(iii) If $\lambda, \mu \in L$ then $\lambda \mu \in L$,
(iv) If $\lambda \in L$ then $f(\lambda) \in L$.

The R-axioms (1)-(6) also form a proof in the mathematical system $[M; L]$ which is extended to the following proof in $[M; L]$:

(7) $\sim x, x$ Rule (a), axiom of equality.
(8) $\sim f(s), f(s)$ Rule (c), (7) with $x = f(s)$.
(9) $\rightarrow \sim f(s), f(s) \rightarrow \sim s, t \sim f(s), f(t)$ Rule (a), axiom of equality.
(10) $\rightarrow \sim s, t \sim f(s), f(t)$ Rule (b), (8), (9).
(11) $\rightarrow \sim f(a), t \sim f(f(a)), f(t)$ Rule (c), (10) with $s = f(a)$.
(12) $\rightarrow \sim f(a), a \sim f(f(a)), f(a)$ Rule (c), (11) with $t = a$. 
(13) $\sim f(f(a)), f(a)$ Rule (b), (4), (12).

(14) $\rightarrow \sim s, t \rightarrow \sim t, u \sim s, u$
   Rule (a), axiom of equality.

(15) $\rightarrow \sim f(f(a)), t \rightarrow \sim t, u \sim f(f(a)), u$
   Rule (c), (14) with $s = f(f(a))$.

(16) $\rightarrow \sim f(f(a)), f(a) \rightarrow \sim f(a), u \sim f(f(a)), u$
   Rule (c), (15) with $t = f(a)$.

(17) $\rightarrow \sim f(f(a)), f(a) \rightarrow \sim f(a), a \sim f(f(a)), a$
   Rule (c), (16) with $u = a$.

(18) $\rightarrow \sim f(a), a \sim f(f(a)), a$ Rule (b), (13), (17).

(19) $\sim f(f(a)), a$ Rule (b), (18).

(20) $\rightarrow \sim f(b), t \sim f(f(b)), f(t)$ Rule (c), (10) with $s = f(b)$.

(21) $\rightarrow \sim f(b), b \sim f(f(b)), f(b)$ Rule (c), (20) with $t = b$.

(22) $\sim f(f(b)), f(b)$ Rule (b), (5), (21).

(23) $\rightarrow \sim f(f(b)), t \rightarrow \sim t, u \sim f(f(b)), u$
   Rule (c), (14) with $s = f(f(b))$.

(24) $\rightarrow \sim f(f(b)), f(b) \rightarrow \sim f(b), u \sim f(f(b)), u$
   Rule (c), (23) with $t = f(b)$.

(25) $\rightarrow \sim f(f(b)), f(b) \rightarrow \sim f(b), b \sim f(f(b)), b$
   Rule (c), (24) with $u = b$.

(26) $\rightarrow \sim f(b), b \sim f(f(b)), b$ Rule (b), (22), (25).

(27) $\sim f(f(b)), b$ Rule (b), (5), (26).

(28) $\rightarrow \sim s, s \rightarrow \sim s, t \sim t, s$ Rule (a), axiom of equality.

(29) $\sim s, s$ Rule (a), axiom of equality.

(30) $\rightarrow \sim s, t \sim t, s$ Rule (b), (28), (29).

(31) $\rightarrow \sim f(a), t \sim t, f(a)$ Rule (c), (30) with $s = f(a)$.

(32) $\rightarrow \sim f(a), a \sim a, f(a)$ Rule (c), (31) with $t = a$.

(33) $\sim a, f(a)$ Rule (b), (4), (32).

(34) $\rightarrow \sim f(b), t \sim t, f(b)$ Rule (c), (30) with $s = f(b)$.

(35) $\rightarrow \sim f(b), b \sim b, f(b)$ Rule (c), (34) with $t = b$.

(36) $\sim b, f(b)$ Rule (b), (5), (35).

(37) $\rightarrow \sim s, t \rightarrow Ws Wt$ Rule (a), axiom of equality.

(38) $\rightarrow \sim a, t \rightarrow Wa Wt$ Rule (c), (37) with $s = a$.

(39) $\rightarrow \sim a, f(a) \rightarrow Wa Wf(a)$ Rule (c), (38) with $t = f(a)$.
(40) $\to Wa Wf(a)$ Rule (b), (33), (39).
(41) $Wf(a)$ Rule (b), (1), (40).
(42) $\to \sim b, t \to Wb Wt$ Rule (c), (37) with $s = b$.
(43) $\to \sim b, f(b) \to Wb Wf(b)$ Rule (c), (42) with $t = f(b)$.
(44) $Wb Wf(b)$ Rule (b), (36), (43).
(45) $Wf(b)$ Rule (b), (2), (44).
(46) $\to Wb \to Wf(b) \to \sim f(f(b)), a \& \& Wa Wf(a) \sim f(f(a)), a$
    Rule (a), axiom of the propositional calculus.
(47) $Wf(a) \to \sim f(f(a)), a \& \& Wa Wf(a) \sim f(f(a)), a$
    Rule (b), (1), (46).
(48) $\to \sim f(f(a)), a \& \& Wa Wf(a) \sim f(f(a)), a$
    Rule (b), (41), (47).
(49) $\& \& Wa Wf(a) \sim f(f(a)), a$ Rule (b), (19), (48).
(50) $Wb \to Wf(b) \to \sim f(f(b)), b \& \& Wb Wf(b) \sim f(f(b)), b$
    Rule (a), axiom of the propositional calculus.
(51) $Wf(b) \to \sim f(f(b)), b \& \& Wb Wf(b) \sim f(f(b)), b$
    Rule (b), (2), (50).
(52) $\to \sim f(f(b)), b \& \& Wb Wf(b) \sim f(f(b)), b$
    Rule (b), (45), (51).
(53) $\& \& Wb Wf(b) \sim f(f(b)), b$ Rule (b), (27), (52).

At this place we stop the proof in the mathematical system $[M; L]$, introduce two different and new constant symbols $c, d$ not occurring in $[M; L]$ and define the extension $A := [a; b; f; c; d]$ of the alphabet $A_S$. With $M_A := [S; A; P_S; B_S]$ and

$$L' := \{ \frac{\lambda t_1}{x_1} \ldots \frac{t_m}{x_m} : \lambda \in L, x_1, \ldots, x_m \in X, t_1, \ldots, t_m \in \{c, d\}, m \geq 0 \}$$

there results the mathematical system $[M_A; L']$ due to [2] Definition (4.2)(d) and [2] Corollary (4.9)(a)]. Next we make use of the abbreviation

$$G(\lambda) := \& \& W \lambda Wf(\lambda) \sim f(f(\lambda)), \lambda \text{ with } \lambda \in L'$$

and adjoin to $[M_A; L']$ the two statements

$$** \quad \varphi_1 := G(c), \quad \varphi_2 := G(d).$$
There results the extended mathematical system \([M'; \mathcal{L}']\) with

\[
M' := M_A(\{\varphi_1, \varphi_2\}) = [S; A; P_S; B_S \cup \{\varphi_1, \varphi_2\}]
\]
due to \(\text{[2]}\) Definition (4.2)(b)]. Note that the abbreviations \(G(\lambda), \varphi_1, \varphi_2\) are not part of the formal system. We keep in mind that any proof in \([M; \mathcal{L}]\) is also a proof in \([M'; \mathcal{L}']\) and that the mathematical systems \([M; \mathcal{L}], [M_A; \mathcal{L}'], [M'; \mathcal{L}']\) all have the same underlying recursive system \(S\). Hence (1)-(53) also constitutes a proof in \([M'; \mathcal{L}']\), and we extend it to the following proof of the formula \(G(cd)\) in \([M'; \mathcal{L}']\):

\[
\begin{align*}
(54) & \quad G(c) \quad \text{Rule (a) with axiom } \varphi_1 = G(c). \\
(55) & \quad G(d) \quad \text{Rule (a) with axiom } \varphi_2 = G(d). \\
(56) & \quad \rightarrow G(c) \ W c \\
& \quad \text{Rule (a), axiom of the propositional calculus.} \\
(57) & \quad \rightarrow G(c) \ W f(c) \\
& \quad \text{Rule (a), axiom of the propositional calculus.} \\
(58) & \quad \rightarrow G(c) \sim f(f(c)), c \\
& \quad \text{Rule (a), axiom of the propositional calculus.} \\
(59) & \quad \rightarrow G(d) \ W d \\
& \quad \text{Rule (a), axiom of the propositional calculus.} \\
(60) & \quad \rightarrow G(d) \ W f(d) \\
& \quad \text{Rule (a), axiom of the propositional calculus.} \\
(61) & \quad \rightarrow G(d) \sim f(f(d)), d \\
& \quad \text{Rule (a), axiom of the propositional calculus.} \\
(62) & \quad W c \quad \text{Rule (b), (54), (56).} \\
(63) & \quad W d \quad \text{Rule (b), (55), (59).} \\
(64) & \quad W f(c) \quad \text{Rule (b), (54), (57).} \\
(65) & \quad W f(d) \quad \text{Rule (b), (55), (60).} \\
(66) & \quad \sim f(f(c)), c \quad \text{Rule (b), (54), (58).} \\
(67) & \quad \sim f(f(d)), d \quad \text{Rule (b), (55), (61).} \\
(68) & \quad \rightarrow W c \rightarrow W y W cy \quad \text{Rule (c), (3) with } x = c. \\
(69) & \quad \rightarrow W c \rightarrow W d W cd \quad \text{Rule (c), (68) with } y = d. \\
(70) & \quad \rightarrow W d W cd \quad \text{Rule (b), (62), (69).} \\
(71) & \quad W cd \quad \text{Rule (b), (63), (70).} \\
(72) & \quad \rightarrow W f(d) \rightarrow W y W f(d)y \quad \text{Rule (c), (3) with } x = f(d). 
\end{align*}
\]
(73) \[ \rightarrow W f(d) \rightarrow W f(c) W f(d) f(c) \]
Rule (c), (72) with \( y = f(c) \).

(74) \[ \rightarrow W f(c) W f(d) f(c) \] Rule (b), (65), (73).

(75) \[ W f(d) f(c) \] Rule (b), (64), (74).

(76) \[ \rightarrow W f(d) W f(c) f(y) f(c) \]
Rule (c), (6) with \( x = c \).

(77) \[ \rightarrow W c \rightarrow W d \sim f(cd), f(d) f(c) \]
Rule (b), (62), (77).

(78) \[ \sim f(d) f(c), f(d) f(c) \] Rule (c), (76) with \( y = d \).

(79) \[ \sim f(cd), f(d) f(c) \] Rule (b), (63), (78).

(80) \[ \rightarrow \sim f(cd), t \sim t, f(cd) \]
Rule (c), (30) with \( s = f(cd) \).

(81) \[ \rightarrow \sim f(cd), f(d) f(c) \sim f(d) f(c), f(cd) \]
Rule (c), (80) with \( t = f(d) f(c) \).

(82) \[ \sim f(d) f(c), f(cd) \] Rule (b), (79), (81).

(83) \[ \rightarrow \sim f(d) f(c), t \rightarrow W f(d) f(c) W t \]
Rule (c), (37) with \( s = f(d) f(c) \).

(84) \[ \rightarrow \sim f(d) f(c), f(cd) \rightarrow W f(d) f(c) W f(cd) \]
Rule (c), (83) with \( t = f(cd) \).

(85) \[ \rightarrow W f(d) f(c) W f(cd) \] Rule (b), (82), (84).

(86) \[ W f(cd) \] Rule (b), (75), (85).

(87) \[ \rightarrow \sim s, t, v \sim s, t, v \sim st, sv \] Rule (a), axiom of equality.

(88) \[ \sim s, t, v \sim s, t, v \sim st, sv \] Rule (c), (7) with \( x = st \).

(89) \[ \rightarrow \sim s, t, v \sim st, sv \] Rule (b), (87), (88).

(90) \[ \rightarrow \sim s, t, v \sim st, sv \sim s, u \sim st, uv \] Rule (a), axiom of equality.

(91)

\[ \rightarrow \rightarrow \sim t, v \sim st, sv \]
\[ \rightarrow \rightarrow \sim s, u \sim st, uv \]
\[ \rightarrow \sim s, u \rightarrow \sim t, v \sim st, uv \]

Rule (a) with the identically true propositional function

\[ \rightarrow \rightarrow \xi_1 \xi_2 \]
\[ \rightarrow \rightarrow \xi_2 \rightarrow \xi_3 \xi_4 \]
\[ \rightarrow \xi_3 \rightarrow \xi_1 \xi_4 \]
and $\xi_1 = \sim t, v; \xi_2 = \sim st, sv; \xi_3 = \sim s, u; \xi_4 = \sim st, uv$.

(92)

\[ \rightarrow \rightarrow \sim st, sv \rightarrow \sim s, u \sim st, uv \]

\[ \rightarrow \sim s, u \rightarrow \sim t, v \sim st, uv \]

Rule (b), (89), (91).

(93) $\rightarrow \sim s, u \rightarrow \sim t, v \sim st, uv$ Rule (b), (90), (92).

(94) $\rightarrow W f(d) \rightarrow W y \sim f(f(d)y), f(y)f(f(d))$

Rule (c), (6) with $x = f(d)$.

(95) $\rightarrow W f(d) \rightarrow W f(c) \sim f(f(d)f(c)), f(f(c))f(f(d))$

Rule (c), (94) with $y = f(c)$.

(96) $\rightarrow W f(c) \sim f(f(d)f(c)), f(f(c))f(f(d))$

Rule (b), (65), (95).

(97) $\sim f(f(d)f(c)), f(f(c))f(f(d))$

Rule (b), (64), (96).

(98) $\rightarrow \sim f(f(c)), u \rightarrow \sim t, v \sim f(f(d))t, uv$

Rule (c), (93) with $s = f(f(c))$.

(99) $\rightarrow \sim f(f(c)), c \rightarrow \sim t, v \sim f(f(c))t, cv$

Rule (c), (98) with $u = c$.

(100) $\rightarrow \sim f(f(c)), c \rightarrow \sim f(f(d)), v \sim f(f(c))f(f(d)), cv$

Rule (c), (99) with $t = f(f(d))$.

(101) $\rightarrow \sim f(f(c)), c \rightarrow \sim f(f(d)), d \sim f(f(c))f(f(d)), cd$

Rule (c), (100) with $v = d$.

(102) $\rightarrow \sim f(f(d)), d \sim f(f(c))f(f(d)), cd$

Rule (b), (66), (101).

(103) $\sim f(f(c))f(f(d)), cd$

Rule (b), (67), (102).

(104) $\rightarrow \sim f(f(d)f(c)), t \rightarrow \sim t, u \sim f(f(d)f(c)), u$

Rule (c), (14) with $s = f(f(d)f(c))$.

(105) $\rightarrow \sim f(f(d)f(c)), f(f(c))f(f(d)) \rightarrow \sim f(f(c))f(f(d)), u$

$\sim f(f(d)f(c)), u$ Rule (c), (104) with $t = f(f(c))f(f(d))$.

(106) $\rightarrow \sim f(f(d)f(c)), f(f(c))f(f(d)) \rightarrow \sim f(f(c))f(f(d)), cd$

$\sim f(f(d)f(c)), cd$ Rule (c), (105) with $u = cd$.

(107) $\rightarrow \sim f(f(c))f(f(d)), cd \sim f(f(d)f(c)), cd$

Rule (b), (97), (106).
(108) \( \sim f(f(d)f(c)), cd \) Rule (b), (103), (107).

(109) \( \rightarrow \sim f(cd), t \sim f(f(cd)), f(t) \)

Rule (c), (10) with \( s = f(cd) \).

(110) \( \sim f(cd), f(d)f(c) \sim f(f(cd)), f(f(d)f(c)) \)

Rule (c), (109) with \( t = f(d)f(c) \).

(111) \( \sim f(f(cd)), f(f(d)f(c)) \) Rule (b), (79), (110).

(112) \( \rightarrow \sim f(f(cd)), t \rightarrow \sim t, u \sim f(f(cd)), u \)

Rule (c), (14) with \( s = f(f(cd)) \).

(113) \( \rightarrow \sim f(f(cd)), f(f(d)f(c)) \rightarrow \sim f(f(d)f(c)), u \sim f(f(cd)), u \)

Rule (c), (112) with \( t = f(d)f(c) \).

(114) \( \rightarrow \sim f(f(cd)), f(f(d)f(c)) \rightarrow \sim f(f(d)f(c)), cd \sim f(f(cd)), cd \)

Rule (c), (113) with \( u = cd \).

(115) \( \rightarrow \sim f(f(d)f(c)), cd \sim f(f(cd)), cd \)

Rule (b), (111), (114).

(116) \( \sim f(f(cd)), cd \) Rule (b), (108), (115).

(117) \( \rightarrow W cd \rightarrow W f(cd) \rightarrow \sim f(f(cd)), cd G(cd) \)

Rule (a), axiom of the propositional calculus.

(118) \( \rightarrow W f(cd) \rightarrow \sim f(f(cd)), cd G(cd) \) Rule (b), (71), (117).

(119) \( \rightarrow \sim f(f(cd)), cd G(cd) \)

Rule (b), (86), (118).

(120) \( G(cd) \) Rule (b), (116), (119).

We have deduced \( G(cd) \) in \([M'; L']\). It follows from the Deduction Theorem [2 (4.5)] that the formula \( \rightarrow G(c) \rightarrow G(d) G(cd) \) is provable in \([M_A; L']\). From the generalization of the constant symbols \( c, d \) according to [2 Corollary (4.9)(b)] we see that

\[ \rightarrow G(x) \rightarrow G(y) G(xy) \]

is provable in the original mathematical system \([M; L]\). Moreover, the formulas \( G(a) \) in (49) and \( G(b) \) in (53) are also provable in \([M; L]\). We apply Rule (e) in \([M; L]\) on the last three formulas and finally conclude that the formulas \( \rightarrow W u G(u) \) and hence

\[ \rightarrow W x W f(x) \quad \text{and} \quad \rightarrow W x \sim f(f(x)), x \]

are provable in \([M; L]\).
3.4. On prime formulas not occurring in the basis axioms.

In this note we determine a simple procedure in order to eliminate prime formulas from formal proofs which do not occur with a given arity in the basis axioms of a mathematical system.

Let \([M; \mathcal{L}]\) with \(M = [S; A_M; P_M; B_M]\) be a mathematical system with restricted argument lists in \(\mathcal{L}\). Assume that \(q \in P_M\) does not occur \(j\)-ary in the basis axioms \(B_M\), where \(j \geq 0\) is a given integer number. Let \([\Lambda] = [F_1; \ldots; F_l]\) be a proof in \([M; \mathcal{L}]\) with the steps \(F_1; \ldots; F_l\). For a variable \(z \in X\) not involved in \(B_M\) we put as abbreviation the contradiction

\[ C = \& \forall z \sim z, z \sim z. \]

If replace in each formula \(F\) of \([M; \mathcal{L}]\) all subformulas of the form \(q\lambda_1, \ldots, \lambda_j\) with \(\lambda_1, \ldots, \lambda_j \in \mathcal{L}\) by the contradiction \(C\) then we obtain the formula \(C(F)\) with argument lists in \(\mathcal{L}\). We will show that

\[ [C(\Lambda)] = [C(F_1); \ldots; C(F_l)] \]

is again a proof in \([M; \mathcal{L}]\), where \(q\) does not occur \(j\)-ary in \([C(\Lambda)]\).

We can subsequently apply this procedure and obtain the following result: Apart from the equations we can replace all prime formulas in the original proof \([\Lambda]\) by \(C\) which do not appear as subformulas with a given arity in the basis axioms of \([M; \mathcal{L}]\). All other prime formulas which occur as subformulas in the steps of \([\Lambda]\) are not affected by this procedure.

In the sequel we fix the quantities \(q \in P_M\) and \(j \in \mathbb{N}_0\) in the definition of \(C\) and \(C(\cdot)\).

**Lemma:** Let \(F\) be a formula in \([M; \mathcal{L}]\) Then for every list \(\mu \in \mathcal{L}\) and for all variables \(x \in X\) with \(CF(F; \mu; x)\) there holds the condition \(CF(C(F); \mu; x)\) and the equation

\[ C(SbF(F; \mu; x)) = SbF(C(F); \mu; x). \]

**Proof.** We use induction with respect to the rules for generating formulas in \([M; \mathcal{L}]\) and fix a variable \(x \in X\) as well as a list \(\mu \in \mathcal{L}\).

We say that a formula \(F\) in \([M; \mathcal{L}]\) satisfies Condition (*) if the condition \(CF(F; \mu; x)\) implies the condition \(CF(C(F); \mu; x)\) and the equation \(C(SbF(F; \mu; x)) = SbF(C(F); \mu; x)\).
We prove that Condition (\(\ast\)) is satisfied for all formulas \(F\) in \([M; \mathcal{L}]\). We use the definitions [2, (3.6) and (3.7)] and the notations occurring there by treating the corresponding cases (a)-(d) in these definitions.

(a) If \(F\) is a prime formula in \([M; \mathcal{L}]\) of the form \(q\lambda_1, \ldots, \lambda_j\) then 
\[
\mathcal{C}(F) = C \quad \text{with} \quad \mathcal{C}(F; \mu; x) \quad \text{and} \quad \mathcal{C}(C; \mu; x),
\]
and we have 
\[
\mathcal{C}(\text{SbF}(F; \mu; x)) = C = \text{SbF}(\mathcal{C}(F); \mu; x)
\]
Otherwise \(F\) is a prime formula in \([M; \mathcal{L}]\) different from \(q\lambda_1, \ldots, \lambda_j\) with \(\mathcal{C}(F) = F\). In both cases we have confirmed Condition (\(\ast\)) for the prime formulas.

(b) We assume that Condition (\(\ast\)) is satisfied for a formula \(F\) in \([M; \mathcal{L}]\) and that \(\mathcal{C}(\neg F; \mu; x)\). Then there holds the condition \(\mathcal{C}(F; \mu; x)\), and we have \(\mathcal{C}(- F) = \neg \mathcal{C}(F)\). Since \(F\) satisfies Condition (\(\ast\)), we conclude that \(\mathcal{C}(\mathcal{C}(F); \mu; x)\) and \(\mathcal{C}(\mathcal{C}(\neg F); \mu; x)\) are valid and that the equations
\[
\mathcal{C}(\text{SbF}(\neg F; \mu; x)) = \neg \mathcal{C}(\text{SbF}(F; \mu; x)) = \text{SbF}(\mathcal{C}(\neg F); \mu; x)
\]
are satisfied. Thus we have confirmed Condition (\(\ast\)) for \(\neg F\).

(c) We assume that Condition (\(\ast\)) is satisfied for the \([M; \mathcal{L}]\)-formulas \(F, G\) and that \(\mathcal{C}(J \ F \ G; \mu; x)\) holds. We obtain \(\mathcal{C}(F; \mu; x)\) and \(\mathcal{C}(G; \mu; x)\). Since \(F\) and \(G\) satisfy Condition (\(\ast\)), we conclude that \(\mathcal{C}(\mathcal{C}(F); \mu; x)\) and \(\mathcal{C}(\mathcal{C}(G); \mu; x)\) are both valid. Therefore \(\mathcal{C}(J \mathcal{C}(F) \mathcal{C}(G); \mu; x)\) and \(\mathcal{C}(\mathcal{C}(J \ F \ G); \mu; x)\) are satisfied. Since \(F\) and \(G\) satisfy Condition (\(\ast\)), we obtain
\[
\mathcal{C}(\text{SbF}(J \ F \ G; \mu; x)) = \mathcal{C}(J \ F \mathcal{\mu}_x \ G \mathcal{\mu}_x) = J \mathcal{C}(F \mathcal{\mu}_x) \mathcal{C}(G \mathcal{\mu}_x)
\]
\[
= J \mathcal{C}(F) \mathcal{\mu}_x \mathcal{C}(G) \mathcal{\mu}_x = \text{SbF}(\mathcal{C}(J \ F \ G); \mu; x),
\]
i.e. Condition (\(\ast\)) is satisfied for \(J \ F \ G\).

(d) We assume that Condition (\(\ast\)) is satisfied for a formula \(F\) in \([M; \mathcal{L}]\) and that there holds \(\mathcal{C}(Q \ y \ F; \mu; x)\). We further keep in mind that \(\text{free}(\mathcal{C}(F)) \subseteq \text{free}(F)\) and that \(\mathcal{C}(Q \ y \ F) = Q \ y \mathcal{C}(F)\).
In the case \(x \notin \text{free}(F) \setminus \{y\}\) we have \(x \notin \text{free}(\mathcal{C}(F)) \setminus \{y\}\) and conclude that \(\mathcal{C}(\mathcal{C}(Q \ y \ F); \mu; x)\) as well as 
\[
\mathcal{C}(\text{SbF}(Q \ y \ F; \mu; x)) = \mathcal{C}(Q \ y \ F) = \text{SbF}(\mathcal{C}(Q \ y \ F); \mu; x).
\]
Otherwise we use that $\text{CF}(Q y F; \mu; x)$ is satisfied with $x \neq y$ and conclude that $y \notin \text{var}(\mu)$ and $\text{CF}(F; \mu; x)$. Recall that $F$ satisfies the Condition (\*) which implies $\text{CF}(C(F); \mu; x)$. From $y \notin \text{var}(\mu)$ and $\text{CF}(C(F); \mu; x)$ we conclude $\text{CF}(Q y C(F); \mu; x)$, i.e. $\text{CF}(Q y F; \mu; x)$ is again satisfied. Since $F$ satisfies the Condition (\*), we finally conclude due to $x \neq y$ that
\[
C(SbF(Q y F; \mu; x)) = Q y C(SbF(F; \mu; x))
= Q y C(F; \mu; x)
= SbF(C(Q y F; \mu; x)),
\]
i.e. Condition (\*) is satisfied for $Q y F$.

Thus we have proved the lemma. □

**Theorem:** With the assumptions of this subsection we obtain that
\[
[C(\Lambda)] = [C(F_1); \ldots; C(F_l)]
\]
is again a proof in $[M; \mathcal{L}]$.

**Proof.** We employ induction with respect to the rules of inference. First we note that for the “initial proof” $[\Lambda] = []$ we can choose $[C(\Lambda)] = []$.

In the sequel we assume that $[\Lambda]$ as well as $[C(\Lambda)] = [C(F_1); \ldots; C(F_l)]$ are both proofs in $[M; \mathcal{L}]$.

(a) Let $H$ be an axiom in $[M; \mathcal{L}]$. Then $[\Lambda_*] = [\Lambda ; H]$ is also a proof in $[M; \mathcal{L}]$ due to Rule (a). We show that $C(H)$ is again an axiom. Then $[C(\Lambda_*)] = [C(\Lambda) ; C(H)]$ is a proof in $[M; \mathcal{L}]$ due to Rule (a). For this purpose we distinguish four cases.

1.) Let $\alpha = \alpha(\xi_1, \ldots, \xi_m)$ be an identically true propositional function of the distinct propositional variables $\xi_1, \ldots, \xi_m$, $m \geq 1$. We suppose without loss of generality that all $m$ propositional variables occur in $\alpha$. If $H_1, \ldots, H_m$ are formulas in $[M; \mathcal{L}]$ with $H = \alpha(H_1, \ldots, H_m)$, then $C(H) = \alpha(C(H_1), \ldots, C(H_m))$ is again an axiom of the propositional calculus in $[M; \mathcal{L}]$.

2.) If $H$ is an axiom of equality $[2] (3.10)(a),(b)$ in $[M; \mathcal{L}]$ then $C(H) = H$. For $[2] (3.10)(c)$, $p \neq q$ or $n \neq j$ we have again $C(H) = H$. For $[2] (3.10)(c)$, $p = q$ and $n = j$ we obtain that $C(H)$ is an axiom of the propositional calculus.
3.) If \( H \) is a quantifier axiom \( [3.11] \) then \( \mathcal{C}(H) \) is again a quantifier axiom. For the quantifier axioms (3.11)(b) we further have to note that \( x \notin \text{free}(F) \) implies \( x \notin \text{free}(\mathcal{C}(F)) \).

4.) For \( H \in B_M \) we obtain again \( \mathcal{C}(H) = H \).

(b) Let \( F, G \) be two formulas in \([M; \mathcal{L}]\) and \( F, \rightarrow F G \) both steps of the proof \([\Lambda]\). Then \( [\Lambda_*] = [\Lambda; G] \) is also a proof in \([M; \mathcal{L}]\) due to Rule (b). It follows that \( \mathcal{C}(F) \) and \( \mathcal{C}(\rightarrow F G) = \rightarrow \mathcal{C}(F) \mathcal{C}(G) \) are both steps of the proof \([\mathcal{C}(\Lambda)]\) due to our assumptions, and due to Rule (b) we put \( [\mathcal{C}(\Lambda_*)] = [\mathcal{C}(\Lambda); \mathcal{C}(G)] \) for the required proof in \([M; \mathcal{L}]\).

(c) Let \( F \in [\Lambda] \), \( x \in X \) and \( \lambda \in \mathcal{L} \). Suppose that there holds the condition \( \text{CF}(F; \lambda; x) \). Then \( [\Lambda_*] = [\Lambda; F \, \lambda \, x] \) is also a proof in \([M; \mathcal{L}]\) due to Rule (c). We obtain from the previous lemma that there holds the condition \( \text{CF}(\mathcal{C}(F); \lambda; x) \) and the equation \( \mathcal{C}(F \, \lambda \, x) = \mathcal{C}(F) \, \lambda \, x \). Since \( \mathcal{C}(F) \in [\mathcal{C}(\Lambda)] \) we conclude that \( [\mathcal{C}(\Lambda_*)] = [\mathcal{C}(\Lambda); \mathcal{C}(F \, \lambda \, x)] \) is a proof in \([M; \mathcal{L}]\) due to Rule (c).

(d) Let \( F \in [\Lambda] \) and \( x \in X \). Then \( [\Lambda_*] = [\Lambda; \forall x F] \) is also a proof in \([M; \mathcal{L}]\) due to Rule (d). Since \( F \in [\Lambda] \) implies \( \mathcal{C}(F) \in [\mathcal{C}(\Lambda)] \) and since \( \mathcal{C}(\forall x F) = \forall x \mathcal{C}(F) \), we can apply Rule (d) on \([\mathcal{C}(\Lambda)]\), \( \mathcal{C}(F) \) in order to conclude that \( [\mathcal{C}(\Lambda_*)] = [\mathcal{C}(\Lambda); \forall x \mathcal{C}(F)] \) is a proof in \([M; \mathcal{L}]\).

(e) In the following we fix a predicate symbol \( p \in P_S \), a list \( x_1, \ldots, x_i \) of \( i \geq 0 \) distinct variables and a formula \( G \) in \([M; \mathcal{L}]\). We suppose that \( x_1, \ldots, x_i \) and the variables of \( G \) are not involved in \( B_S \).

Then to every R-formula \( F \) of \( B_S \) there corresponds exactly one formula \( F' \) of the mathematical system, which is obtained if we replace in \( F \) each \( i \)-ary subformula \( p \lambda_1, \ldots, \lambda_i \), where \( \lambda_1, \ldots, \lambda_i \) are lists, by the formula \( G \, \lambda_1 \, x_1 \ldots \lambda_i \, x_i \). Note that in this case \( \lambda_1, \ldots, \lambda_i \in \mathcal{L} \).

Suppose that \( F' \) is a step of \([\Lambda]\) for all R-formulas \( F \in B_S \) for which \( p \) occurs \( i \)-ary in the R-conclusion of \( F \). Then

\[ [\Lambda_*] = [\Lambda; \rightarrow p x_1, \ldots, x_i \, G] \]

is also a proof in \([M; \mathcal{L}]\) due to Rule (e).
We distinguish two cases: In the first case we assume that $p = q$ and $i = j$. Then we can apply Rule (a) on the formula

$$C(\rightarrow px_1, ..., x_i G) = \rightarrow C C(G),$$

which is an axiom of the propositional calculus, and conclude that

$$[C(\Lambda_\ast)] = [C(\Lambda); C(\rightarrow px_1, ..., x_i G)]$$

is a proof in $[M; \mathcal{L}]$. See also [2, (3.14), Example 2].

In the second case we assume that $p \neq q$ or $i \neq j$. For every $R$-formula $F$ of $B_S$ we define the formula $F''$ of $[M; \mathcal{L}]$ which is obtained if we replace in $F$ each $i$-ary subformula $p \lambda_1, ..., \lambda_i$ with $\lambda_1, ..., \lambda_i \in \mathcal{L}$ by the formula $C(G) \frac{\lambda_1}{x_1} ... \frac{\lambda_i}{x_i}$ and note that the variables in $C(G)$ are not involved in $B_S$, because we have assumed that the bound variable $z$ in the contradiction $C$ does not occur in $B_S$. We have assumed that $q$ does not occur $j$-ary in $B_M$, hence in the formula $F'$ the symbol $q$ can only occur $j$-ary within the subformulas $G \frac{\lambda_1}{x_1} ... \frac{\lambda_i}{x_i}$. From the previous lemma we obtain

$$C(G) \frac{\lambda_1}{x_1} ... \frac{\lambda_i}{x_i} = C(G) \frac{\lambda_1}{x_1} ... \frac{\lambda_i}{x_i}.$$

We see that $F'' = C(F')$ and recall that $[\Lambda]$ as well as $[C(\Lambda)] = [C(F_1); ...; C(F_l)]$ are both proofs in $[M; \mathcal{L}]$. Hence $F''$ is a step of the proof $[C(\Lambda)]$ for all $R$-formulas $F \in B_S$ for which $p$ occurs $i$-ary in the $R$-conclusion of $F$. Therefore we can apply Rule (e) on $[C(\Lambda)]$ and conclude that

$$[C(\Lambda_\ast)] = [C(\Lambda); C(\rightarrow px_1, ..., x_i G)]$$

with $C(\rightarrow px_1, ..., x_i G) = \rightarrow px_1, ..., x_i C(G)$ is a proof in $[M; \mathcal{L}]$.

Thus we have proved the theorem. $\square$

The lemma and theorem of this subsection have a strong resemblance to [2] (4.7) Lemma, (4.8) Theorem], and the proofs are very similar. It arises the question whether there is a more general result which is relevant in elementary proof theory.
3.5. A general theorem concerning formal induction and its application to PA.

We consider a mathematical system \( M = [S; A_M; P_M; B_M] \) with an underlying recursive system \( S = [A_S; P_S; B_S] \) such that \( A_M = A_S \), \( P_M = P_S \) and \( B_M = B_S \) and assume that \( [M; \mathcal{L}] \) is a mathematical system with restricted argument lists in \( \mathcal{L} \). We suppose that \( \mathcal{L} \) is enumerable, for more details see the text introducing [2, (5.4) Conjecture]. We will study this mathematical system \( [M; \mathcal{L}] \) until we discuss its application to Peano arithmetic PA.

**Definition 3.1.** An R-derivation \([\Lambda]\) in \([S; \mathcal{L}]\) is defined as an R-derivation in \(S\) with the following restrictions: The R-formulas in \([\Lambda]\) and the R-formulas \(F\), \(G\) in [2 (1.11)] have only argument lists in \(\mathcal{L}\), and the use of the Substitution Rule [2 (1.11)(c)] is restricted to \(\lambda \in \mathcal{L}\). Then the R-formulas in \([\Lambda]\) are called R-derivable in \([S; \mathcal{L}]\). By \(\Pi_R(S; \mathcal{L})\) we denote the set of all R-derivable R-formulas in \([S; \mathcal{L}]\), by \(\Pi(M; \mathcal{L})\) the set of all provable formulas in \([M; \mathcal{L}]\).

From [2 Section 3, Example 2] we know that the formula \(\neg q x_1, \ldots, x_j\) with \(x_1, \ldots, x_j \in X\) is provable in \([M; \mathcal{L}]\) whenever \(q \in P_S\) does not occur \(j\)-ary in \(B_S\). On the other hand we have shown the consistency of \([M; \mathcal{L}]\) in [2, (5.1) Proposition].

We will first simplify the syntax of the formulas \(F\) in \([M; \mathcal{L}]\) by removing the quantifier \(\exists\) and the symbols \(\lor\), \& and \(\leftrightarrow\). By \(\text{Form}(M; \mathcal{L})\) we denote the set of all formulas in \([M; \mathcal{L}]\). Let \(\mathcal{F}\) be the set of all formulas in \(\text{Form}(M; \mathcal{L})\) without the symbols \(\exists, \lor, \&\) and \(\leftrightarrow\) and define the mapping \(\Theta : \text{Form}(M; \mathcal{L}) \mapsto \mathcal{F}\) as follows:

1. \(\Theta(F) = F\) if \(F\) is a prime-formula in \([M; \mathcal{L}]\).
2. \(\Theta(\neg F) = \neg \Theta(F)\) for all formulas \(F\) in \([M; \mathcal{L}]\).
3. For all \(F, G \in \text{Form}(M; \mathcal{L})\) we have
   i. \(\Theta(\rightarrow FG) = \rightarrow \Theta(F)\Theta(G)\).
   ii. \(\Theta(\lor FG) = \neg \Theta(F)\Theta(G)\).
   iii. \(\Theta(\& FG) = \neg \rightarrow \Theta(F)\neg \Theta(G)\).
   iv. \(\Theta(\leftrightarrow FG) = \neg \rightarrow \Theta(F)\Theta(G)\rightarrow \Theta(G)\Theta(F)\).
4. i. \(\Theta(\forall x F) = \forall x \Theta(F)\) for all \(x \in X\) and \(F \in \text{Form}(M; \mathcal{L})\).
   ii. \(\Theta(\exists x F) = \neg \forall x \neg \Theta(F)\) for all \(x \in X\), \(F \in \text{Form}(M; \mathcal{L})\).
Theorem 3.2. Let $[\Lambda] = [F_1; \ldots; F_l]$ be a proof in $[M; \mathcal{L}]$ with the steps $F_1; \ldots; F_l$. Then

$$[\Theta(\Lambda)] = [\Theta(F_1); \ldots; \Theta(F_l)]$$

is again a proof in $[M; \mathcal{L}]$. For all $k = 1, \ldots, l$ the formula $\Theta(F_k)$ can be derived with the same rule of inference that was used for the derivation of $F_k$ in the proof $[\Lambda]$.

Proof. We employ induction with respect to the rules of inference. First we note that for the “initial proof” $[\Lambda] = []$ we can choose $[\Theta(\Lambda)] = []$. In the sequel we assume that $[\Lambda]$ as well as $[\Theta(\Lambda)] = [\Theta(F_1); \ldots; \Theta(F_l)]$ are both proofs in $[M; \mathcal{L}]$.

(a) The basis axioms in $[M; \mathcal{L}]$ are exactly the basis R-axioms of the underlying recursive system, hence $\Theta(F) = F$ for all formulas $F \in B_M = B_S$. If $F$ is an axiom of equality then we have again $\Theta(F) = F$. If $F$ is an axiom of the propositional calculus, then also $\Theta(F)$. If $F$ is an quantifier axiom (3.11)(a),(b), then also $\Theta(F)$. If $F$ is an quantifier axiom (3.11)(c), then $\Theta(F)$ is an axiom of the propositional calculus. We conclude that $\Theta$ maps axioms into axioms.

(b) Let $F, G$ be two formulas in $[M; \mathcal{L}]$ and $F, \to FG$ both steps of the proof $[\Lambda]$. Then $[\Lambda_*] = [\Lambda ; G]$ is also a proof in $[M; \mathcal{L}]$ due to Rule (b). It follows that $\Theta(F)$ and $\Theta(\to FG) = \to \Theta(F) \Theta(G)$ are both steps of the proof $[\Theta(\Lambda)]$ due to our assumptions, and due to Rule (b) we put $[\Theta(\Lambda_*)] = [\Theta(\Lambda) ; \Theta(G)]$ for the required proof in $[M; \mathcal{L}]$.

(c) Let $F \in [\Lambda]$, $x \in X$ and $\lambda \in \mathcal{L}$. Suppose that there holds the condition $CF(F; \lambda; x)$. Then $[\Lambda_*] = [\Lambda ; F \lambda x]$ is also a proof in $[M; \mathcal{L}]$ due to Rule (c). We obtain that there holds the condition $CF(\Theta(F); \lambda; x)$ and the equation $\Theta(F \lambda x) = \Theta(F) \lambda x$. Since $\Theta(F) \in [\Theta(\Lambda)]$ we conclude that $[\Theta(\Lambda_*)] = [\Theta(\Lambda) ; \Theta(F \lambda x)]$ is a proof in $[M; \mathcal{L}]$ due to Rule (c).

(d) Let $F \in [\Lambda]$ and $x \in X$. Then $[\Lambda_*] = [\Lambda ; \forall x F]$ is also a proof in $[M; \mathcal{L}]$ due to Rule (d). Since $F \in [\Lambda]$ implies
\( \Theta(F) \in [\Theta(\Lambda)] \) and since \( \Theta(\forall x F) = \forall x \Theta(F) \), we can apply Rule (d) on \([\Theta(\Lambda)]\), \( \Theta(F) \) in order to conclude that \([\Theta(\Lambda_\ast)] = [\Theta(\Lambda) ; \Theta(\forall x F)] \) is a proof in \([M; \mathcal{L}]\).

(e) In the following we fix a predicate symbol \( p \in P_S \), a list \( x_1, \ldots, x_i \) of \( i \geq 0 \) distinct variables and a formula \( G \) in \([M; \mathcal{L}]\). We suppose that \( x_1, \ldots, x_i \) and the variables of \( G \) are not involved in \( B_S \).

Then to every R-formula \( F \) of \( B_S \) there corresponds exactly one formula \( F' \) of the mathematical system, which is obtained if we replace in \( F \) each \( i \)-ary subformula \( p \lambda_1, \ldots, \lambda_i \), where \( \lambda_1, \ldots, \lambda_i \) are lists, by the formula \( G \frac{\lambda_1}{x_1} \ldots \frac{\lambda_i}{x_i} \). Note that in this case \( \lambda_1, \ldots, \lambda_i \in \mathcal{L} \).

Suppose that \( F' \) is a step of \([\Lambda]\) for all R-formulas \( F \in B_S \) for which \( p \) occurs \( i \)-ary in the R-conclusion of \( F \). Then

\[ [\Lambda_\ast] = [\Lambda; \rightarrow p x_1, \ldots, x_i \ G] \]

is also a proof in \([M; \mathcal{L}]\) due to Rule (e).

For every R-formula \( F \) of \( B_S \) we define the formula \( F'' \) of \([M; \mathcal{L}]\) which is obtained if we replace in \( F \) each \( i \)-ary subformula \( p \lambda_1, \ldots, \lambda_i \) with \( \lambda_1, \ldots, \lambda_i \in \mathcal{L} \) by the formula \( \Theta(G) \frac{\lambda_1}{x_1} \ldots \frac{\lambda_i}{x_i} \) and note that the variables in \( \Theta(G) \) are not involved in \( B_S \), because \( \Theta(G) \) and \( G \) both have the same variables. We see that \( F'' = \Theta(F') \) and recall that \([\Lambda]\) and \([\Theta(\Lambda)] = [\Theta(F_1); \ldots; \Theta(F_l)] \) are both proofs in \([M; \mathcal{L}]\). Hence \( F'' \) is a step of the proof \([\Theta(\Lambda)] \) for all R-formulas \( F \in B_S \) for which \( p \) occurs \( i \)-ary in the R-conclusion of \( F \). Therefore we can apply Rule (e) on \([\Theta(\Lambda)] \) and conclude that

\[ [\Theta(\Lambda_\ast)] = [\Theta(\Lambda) ; \Theta( \rightarrow p x_1, \ldots, x_i \ G)] \]

with \( \Theta( \rightarrow p x_1, \ldots, x_i \ G) = \rightarrow p x_1, \ldots, x_i \Theta(G) \) is a proof in \([M; \mathcal{L}]\).

Thus we have proved the theorem. \( \square \)

Now we will roughly divide the formulas in \( \mathcal{F} \) into equivalence classes. This is used in order to present a well defined interpretation of the formulas \( F \in \mathcal{F} \) in the mathematical system.
Definition 3.3. Equivalence classes $\langle F \rangle$ of formulas $F \in \mathcal{F}$.

1. By $\mathcal{P}$ we denote the set of all prime formulas in $[M; \mathcal{L}]$. For any prime formula $F \in \mathcal{P}$ we have $F \in \mathcal{F}$ and put $\langle F \rangle = \mathcal{P}$.
2. For $F \in \mathcal{F}$ we also have $\neg F \in \mathcal{F}$ and put $\langle \neg F \rangle = \neg \langle F \rangle = \{ \neg F' : F' \in \langle F \rangle \}$.
3. For $F, G \in \mathcal{F}$ we also have $\rightarrow FG \in \mathcal{F}$ and put $\langle \rightarrow FG \rangle = \rightarrow \langle F \rangle \langle G \rangle = \{ \rightarrow F'G' : F' \in \langle F \rangle, G' \in \langle G \rangle \}$.
4. For $x \in X, F \in \mathcal{F}$ we also have $\forall x F \in \mathcal{F}$ and put $\langle \forall x F \rangle = \forall \langle F \rangle = \{ \forall x'F' : x' \in X, F' \in \langle F \rangle \}$.

The sets $\langle F \rangle$ with $F \in \mathcal{F}$ give a well-defined partition of $\mathcal{F}$; two formulas $F$ and $F'$ in $\mathcal{F}$ are equivalent if and only if $\langle F \rangle = \langle F' \rangle$ for their equivalence classes $\langle F \rangle$ and $\langle F' \rangle$, respectively. The construction of each class starts with $\mathcal{P}$ and terminates in a finite number of steps. It is purely syntactic, for example $\neg \forall \rightarrow \mathcal{P}\mathcal{P}$ and $\forall \rightarrow \mathcal{P}\mathcal{P}$ are disjoint.

By $\mathcal{F}_*$ we denote the set of all formulas in $\mathcal{F}$ without free variables, also called statements. Let $\mathcal{L}_*$ be the set of all lists in $\mathcal{L}$ without variables.

We suppose that $\mathcal{L}_*$ is not empty. Now we will give an interpretation of all statements $F \in \mathcal{F}_*$ in the mathematical system $[M; \mathcal{L}]$.

Using the verum $\top$, the empty set $\emptyset$ and formulas $F, G \in \mathcal{F}$ we define the following function $V : \mathcal{F}_* \mapsto \{ \emptyset, \{ \top \} \}$:

1. If $\lambda, \mu \in \mathcal{L}_*$ then
   
   $$V(\sim \lambda, \mu) = \begin{cases} \{ \top \} & \text{if } \sim \lambda, \mu \in \Pi_R(S; \mathcal{L}), \\ \emptyset & \text{otherwise} \end{cases}.$$ 

   Let $p \in P_S$ and $\lambda_1, ..., \lambda_i \in \mathcal{L}_*$ for $i \geq 0$ be elementary $A_S$-lists in $\mathcal{L}_*$. Then we evaluate
   
   $$V(p \lambda_1, ..., \lambda_i) = \begin{cases} \{ \top \} & \text{if } p \lambda_1, ..., \lambda_i \in \Pi_R(S; \mathcal{L}), \\ \emptyset & \text{otherwise} \end{cases}.$$ 

2. For $\neg F \in \mathcal{F}_*$ we also have $F \in \mathcal{F}_*$ and require
   
   $$V(\neg F) = \{ \top \} \setminus V(F).$$
3. For \( \rightarrow FG \in \mathcal{F}_* \) we also have \( F, G \in \mathcal{F}_* \) and require
\[
V(\rightarrow FG) = (\{\top\} \setminus V(F)) \cup V(G).
\]

4. For \( x \in X, \forall x F \in \mathcal{F}_* \) and \( \lambda \in \mathcal{L}_* \) we have \( F^\lambda_x \in \mathcal{F}_* \), recall \( \mathcal{L}_* \neq \emptyset \) and require
\[
V(\forall x F) = \bigcap_{\lambda \in \mathcal{L}_*} V\left(F^\lambda_x\right).
\]

We say that a formula \( F \in \mathcal{F}_* \) is true if and only if \( \top \in V(F) \). The sets \( \langle F \rangle_* = \langle F \rangle \cap \mathcal{F}_* \) with \( F \in \mathcal{F}_* \) form a partition of \( \mathcal{F}_* \), and induction on the equivalence classes \( \langle F \rangle_* \) in \( \mathcal{F}_* \) shows that the function \( V \) is well-defined. Of course, in general the evaluation of \( V(F) \) must be highly non-constructive.

**Definition 3.4.** Let \( F \) be a formula in \([M; \mathcal{L}]\). Let \( x_1 = x_{j_1}, \ldots, x_m = x_{j_m} \) be the list of all free variables in \( F \), ordered with increasing indices \( j_1 < \ldots < j_m \) of the variables. We define
\[
\text{Free}(F) = [x_1; \ldots; x_m], \quad \text{Gen}(F) = \forall x_1 \ldots \forall x_m F,
\]

namely the list \( \text{Free}(F) \) of free variables in \( F \) and the generalization of the formula \( F \), respectively. Especially for statements \( F \) we have \( \text{Free}(F) = [\] and \( \text{Gen}(F) = F \).

Now we make use of Theorem 3.2 which guarantees that proofs with formulas in \( \mathcal{F} \) are not a real restriction and present our main result, namely

**Theorem 3.5.** Let \( F \in \mathcal{F} \) be a formula which is provable in \([M; \mathcal{L}]\). Suppose that the set \( \mathcal{L}_* \) of all lists in \( \mathcal{L} \) without variables is not empty and that \( \mathcal{L} \) is enumerable. Then \( \top \in V(\text{Gen}(F)) \).

**Proof.** Let \( F \in \mathcal{F} \) be a formula in \([M; \mathcal{L}]\) with \( \text{Free}(F) = [x_1; \ldots; x_m] \). Then \( \top \in V(\text{Gen}(F)) \) iff \( \top \in V(F^\lambda_{x_1} \ldots ^\lambda_{x_m}) \) for all \( \lambda_1, \ldots, \lambda_m \in \mathcal{L}_* \). This will be used throughout the whole proof.

We want to show for each proof \([\Lambda] \) in \([M; \mathcal{L}]\) with steps only in \( \mathcal{F} \) that \( \top \in V(\text{Gen}(F)) \) for all \( F \in [\Lambda] \). We employ induction with respect to the rules of inference. First we note that the statement is true for the “initial proof” \([\Lambda] = [\].\)
Let $\Lambda = [F_1; \ldots; F_l]$ be a proof in $[M; \mathcal{L}]$ with the steps $F_1; \ldots; F_l \in \mathcal{F}$. Our induction hypothesis is $\top \in V(\text{Gen}(F_k))$ for all $k = 1, \ldots, l$.

(a) Here we show that $\top \in V(\text{Gen}(F))$ for all axioms $F \in \mathcal{F}$. Then the extended proof $[\Lambda_*] = [\Lambda; F]$ will also satisfy the statement.

- The basis axioms and the axioms of equality in $[M; \mathcal{L}]$ are R-axioms of the underlying recursive system. Assume that $F$ is such an axiom with $\text{Free}(F) = [x_1; \ldots; x_m]$ and that $\lambda_1, \ldots, \lambda_m \in \mathcal{L}_*$. Then $\tilde{F} = F^{\lambda_1}_{x_1} \cdots \lambda_m_{x_m}$ is an elementary R-formula in $\mathcal{F}$. We have $\top \in V(\tilde{F})$ iff there is an R-premise in $\tilde{F}$ which is not R-derivable in $[S; \mathcal{L}]$ or if the R-conclusion of $\tilde{F}$ is R-derivable in $[S; \mathcal{L}]$. But due to the Modus Ponens Rule the R-conclusion of $\tilde{F}$ is R-derivable in $[S; \mathcal{L}]$ if all R-premises in $\tilde{F}$ are R-derivable in $[S; \mathcal{L}]$. We see $\top \in V(\tilde{F})$ and hence $\top \in V(\text{Gen}(F))$ in the case that $F$ is a basis axiom or an axiom of equality in the mathematical system $[M; \mathcal{L}]$.

- Suppose that $\alpha = \alpha(\xi_1, \ldots, \xi_j)$ is an identically true propositional function defined in [2, (3.8)] which is only constructed with the negation symbol “¬” and the implication arrow “→” and that $F_1, \ldots, F_j \in \mathcal{F}$ are formulas in $[M; \mathcal{L}]$. Then the formula $F = \alpha(F_1, \ldots, F_j) \in \mathcal{F}$ is an axiom of the propositional calculus. Prescribe $\lambda_1, \ldots, \lambda_m \in \mathcal{L}_*$ and put

$$
\tilde{F} = F^{\lambda_1}_{x_1} \cdots \lambda_m_{x_m}, \quad \tilde{F}_k = F_k^{\lambda_1}_{x_1} \cdots \lambda_m_{x_m}
$$

for $k = 1, \ldots, j$ and $\text{Free}(F) = [x_1; \ldots; x_m]$. For any two formulas $F', F'' \in \mathcal{F}_*$ we have $\top \in V(\neg F')$ iff $\top \notin V(F')$ and $\top \in V(F' \rightarrow F'')$ iff $\top \in V(F')$ implies $\top \in V(F'')$, respectively. We see that $\tilde{F} = \alpha(\tilde{F}_1, \ldots, \tilde{F}_j) \in \mathcal{F}_*$ is an axiom of the propositional calculus with $\top \in V(\tilde{F})$.

- Suppose that $x \in X$, that $F \in \mathcal{F}$ and

$\text{Free}(\forall x F) = [x_1; \ldots; x_m]$.
We put $H = \rightarrow \forall x F$ and note that $x \notin \{x_1; \ldots; x_m\}$. We see $\top \in V(Gen(H))$ iff 

$$T \in V(SbF(\tilde{H}; \mu; x)) = V(\rightarrow \forall x \tilde{F} \ SbF(\tilde{F}; \mu; x))$$

for all $\mu, \lambda_1, \ldots \lambda_m \in L_*$, using $\tilde{F} = F_{\lambda_1}^{x_1} \ldots_{x_m}^{\lambda_m}$ and

$$\tilde{H} = H_{\lambda_1}^{\frac{x_1}{1}} \ldots_{x_m}^{\lambda_m} = \rightarrow \forall x \tilde{F}$$

as abbreviations.

Now $\top \in V(\forall \tilde{F})$ implies indeed $\top \in V(SbF(\tilde{F}; \mu; x))$ for all $\mu, \lambda_1, \ldots \lambda_m \in L^*$, independent of $x \in Free(F)$ or $x \notin Free(F)$.

- Suppose that $x \in X$, that $F, G \in F$ and that $x \notin Free(F)$, $Free(\forall x \rightarrow FG) = Free(\rightarrow F \forall x G) = \{x_1; \ldots; x_m\}$. We put

$$H = \rightarrow \forall x \rightarrow FG \rightarrow F \forall x G,$$

fix arbitrary lists $\lambda_1, \ldots \lambda_m \in L_*$ and make use of the abbreviations $\tilde{F} = F_{\lambda_1}^{x_1} \ldots_{x_m}^{\lambda_m}$ and $\tilde{G} = G_{\lambda_1}^{x_1} \ldots_{x_m}^{\lambda_m}$. We have

$$\tilde{H} = H_{\lambda_1}^{\frac{x_1}{1}} \ldots_{x_m}^{\lambda_m} = \rightarrow \forall x \rightarrow \tilde{F} \tilde{G} \rightarrow \tilde{F} \forall x \tilde{G}$$

with $\tilde{H} \in F_*$. In order to show $\top \in V(\tilde{H})$ we assume $\top \in V(\forall x \rightarrow \tilde{F} \tilde{G})$ and note that $x \notin Free(\tilde{F})$. Then

$$T \in V(\forall x \rightarrow \tilde{F} \tilde{G}) \iff T \in V(\rightarrow \tilde{F} SbF(\tilde{G}; \lambda; x))$$

for all $\lambda \in L_*$. Hence $T \in V(\tilde{F})$ implies $T \in V(SbF(\tilde{G}; \lambda; x))$ for all $\lambda \in L_*$, i.e. $T \in V(\tilde{F})$ implies $T \in V(\forall x \tilde{G})$, and we have shown $T \in V(\rightarrow \tilde{F} \forall x \tilde{G})$ and $T \in V(\tilde{H})$.

- Recall that the quantifier axiom (3.11)(c) is replaced by an axiom of the propositional calculus due to Theorem 3.2.

(b) Suppose that $F$ and $H = \rightarrow FG$ are both steps of the proof $[\Lambda] = [F_1; \ldots; F_l]$ with $Free(\rightarrow FG) = \{x_1; \ldots; x_m\}$. Then $T \in V(Gen(F))$ and $T \in V(Gen(H))$ from our induction hypothesis. Fix $\lambda_1, \ldots \lambda_m \in L_*$ and put $\tilde{F} = F_{\lambda_1}^{x_1} \ldots_{x_m}^{\lambda_m}$, $\tilde{G} = G_{\lambda_1}^{x_1} \ldots_{x_m}^{\lambda_m}$.
For
\[ \tilde{H} = H \frac{\lambda_1 \ldots \lambda_m}{x_1 \ldots x_m} = \to \tilde{F} \tilde{G} \]
we have \( \tilde{F}, \tilde{G}, \tilde{H} \in \mathcal{F}_s, \top \in V(\tilde{F}), \top \in V(\tilde{H}) \) and \( \top \in V(\tilde{G}) \).

Note that substitutions of variables in \([x_1; \ldots; x_m]\) not occurring in \( F \) or \( G \) are allowed, because they do not have any effect. We obtain that the extended proof \([\Lambda_\ast] = [\Lambda; G] \) also satisfies our statement.

(c) Let \( x \in X \) and suppose that \( F \in \mathcal{F} \) is a step of the proof \([\Lambda] = [F_1; \ldots; F_l] \). Let \( \lambda \in \mathcal{L} \) and suppose that there holds the condition \( \text{CF}(F; \lambda; x) \). Note that \( \top \in V(\text{Gen}(F)) \) from our induction hypothesis.

Without loss of generality we may assume that \( x \in \text{free}(F) \), where we use the set \( \text{free}(F) = \{x, x_1, \ldots, x_m\} \) (instead of ordered lists) with distinct variables \( x, x_1, \ldots, x_m \in X \), and put
\[
\Phi(F) = \{ F \frac{\lambda_0 \lambda_1 \ldots \lambda_m}{x_1 \ldots x_m} : \lambda_0, \lambda_1, \ldots, \lambda_m \in \mathcal{L}_s \}.
\]
We write \( \text{var}(\lambda) = \{y_1, \ldots, y_k\} \) and \( \lambda = \lambda(y_1, \ldots, y_k) \). From \( x \in \text{free}(F) \) and \( \text{CF}(F; \lambda; x) \) we see that
\[
\text{var}(\lambda) \subseteq \text{free} \left( F \frac{\lambda}{x} \right).
\]
Hence we can write \( \text{free}(F \frac{\lambda}{x}) = \{y_1, \ldots, y_n\} \) with \( n \geq k \) distinct variables \( y_1, \ldots, y_n \in X \) and define the new set
\[
\Phi(F; \lambda; x) = \{ F \frac{\lambda \mu_1 \ldots \mu_n}{x y_1 \ldots y_n} : \mu_1, \ldots, \mu_n \in \mathcal{L}_s \}.
\]
Again from \( \text{CF}(F; \lambda; x) \) we conclude that
\[
\Phi(F; \lambda; x) = \left\{ F \frac{\lambda(\mu_1, \ldots, \mu_k)}{x y_1 \ldots y_n} : \mu_1, \ldots, \mu_n \in \mathcal{L}_s \right\},
\]
hence \( \Phi(F; \lambda; x) \subseteq \Phi(F) \) and
\[
\text{V}(\text{Gen}(F)) = \bigcap_{G \in \Phi(F)} \text{V}(G)
\]
\[
\subseteq \bigcap_{G \in \Phi(F; \lambda; x)} \text{V}(G) = \text{V} \left( \text{Gen} \left( F \frac{\lambda}{x} \right) \right).
\]
We obtain from our induction hypothesis $\top \in V(\text{Gen}(F))$ that $\top \in V(\text{Gen}(\overline{F}_x))$. Now the extended proof $[\Lambda_*] = [\Lambda; F_\lambda]$ satisfies our statement.

(d) Let $F$ be a step of the proof $[\Lambda] = [F_1; \ldots; F_l]$. If $x \in X$ is not a free variable of $F$ then $F_\lambda = F$ for all $\lambda \in \mathcal{L}_*$ and

$$V(\text{Gen}(\forall x F)) = V(\text{Gen}(F)).$$

Then $\top \in V(\text{Gen}(\forall x F))$ from our induction hypothesis. Now we suppose that $x \in \text{free}(F) = \{x_1, \ldots, x_m\}$ with distinct variables $x_1, \ldots, x_m$. In this case we see $\top \in V(\text{Gen}(\forall x F))$ iff for all $\lambda_1, \ldots, \lambda_m \in \mathcal{L}_*$

$$\top \in V\left(\overline{F}_{x_1}^{\lambda_1} \ldots \overline{F}_{x_m}^{\lambda_m}\right),$$

i.e. $V(\text{Gen}(\forall x F)) = V(\text{Gen}(F))$, and obtain $\top \in V(\text{Gen}(\forall x F))$ again from our induction hypothesis. In any case the extended proof $[\Lambda_*] = [\Lambda; \forall x F]$ satisfies our statement.

(e) In the following we fix a predicate symbol $p \in P_S$, a list $x_1, \ldots, x_i$ of $i \geq 0$ distinct variables and a formula $G \in \mathcal{F}$. We suppose that $x_1, \ldots, x_i$ and the variables of $G$ are not involved in $B_S$. Then to every $R$-formula $F$ of $B_S$ there corresponds exactly one formula $F' \in \mathcal{F}$ of the mathematical system, which is obtained if we replace in $F$ each $i$-ary subformula $p \lambda_1, \ldots, \lambda_i$, where $\lambda_1, \ldots, \lambda_i$ are lists, by the formula $G \overline{\lambda_1} \ldots \overline{\lambda_i}$. We suppose that $F'$ is a step of a proof $[\Lambda]$ for all $R$-formulas $F \in B_S$ for which $p$ occurs $i$-ary in the $R$-conclusion of $F$. Now $[\Lambda_*] = [\Lambda; \rightarrow px_1, \ldots, x_i G]$ is also a proof in $[M; \mathcal{L}]$ with formulas in $\mathcal{F}$. To finish the proof of the main theorem it remains to show that $\top \in V(\text{Gen}(\rightarrow px_1, \ldots, x_i G))$. We may write free($\rightarrow px_1, \ldots, x_i G) = \{x_1, \ldots, x_m\}$ with $m \geq i$ distinct variables $x_1, \ldots, x_m$. For $m > i$ we choose $\tilde{\lambda}_{i+1}, \ldots, \tilde{\lambda}_m \in \mathcal{L}_*$ arbitrary but fixed and put

$$\tilde{G} = G \overline{\lambda_{i+1}}_{x_{i+1}} \ldots \overline{\lambda_m}_{x_m},$$
and otherwise we put $\tilde{G} = G$. It is sufficient to show that $\top \in V(\text{Gen}(\to px_1, \ldots, x_i \tilde{G}))$ with the formula $\tilde{G} \in \mathcal{F}$ satisfying free($\tilde{G}$) $\subseteq \{x_1, \ldots, x_i\}$. Note that the variables of $\tilde{G}$ are not involved in $B_S$. For $\lambda_1, \ldots, \lambda_i \in \mathcal{L}$ we can also write

$$\tilde{G}(\lambda_1, \ldots, \lambda_i) = \tilde{G} \frac{\lambda_1}{x_1} \cdots \frac{\lambda_i}{x_i},$$

provided that $x_1, \ldots, x_i$ and the variables of $G$ are not involved in $\lambda_1, \ldots, \lambda_i$. Especially for $i = 0$ we put $\tilde{G}(\lambda_1, \ldots, \lambda_i) = \tilde{G}$.

We have to show that $p\lambda_1, \ldots, \lambda_i \in \Pi_R(S; \mathcal{L}) \Rightarrow \top \in V(\tilde{G}(\lambda_1, \ldots, \lambda_i))$ for all $\lambda_1, \ldots, \lambda_i \in \mathcal{L}$, see Definition 3.1.

We will show that $\tilde{G}(\lambda_1, \ldots, \lambda_i)$ can be derived in $[M; \mathcal{L}]$ from the given proof $[\Lambda] = [F_1; \ldots; F_l]$ by using only Rules (a)-(d) whenever $p\lambda_1, \ldots, \lambda_i \in \Pi_R(S; \mathcal{L})$ for $\lambda_1, \ldots, \lambda_i \in \mathcal{L}$. Then we can first apply Theorem 3.2 in order to obtain an extension of the proof $[\Lambda]$ which consists only on formulas in $\mathcal{F}$ and which contains the formula $\tilde{G}(\lambda_1, \ldots, \lambda_i)$ as a final step. This will conclude the proof of the theorem because $[\Lambda]$ satisfies the induction hypothesis and Rules (a)-(d) applied step by step on the extensions of $[\Lambda]$ with formulas in $\mathcal{F}$ can only produce further new formulas $F$ satisfying $\top \in V(\text{Gen}(F))$.

Following our strategy we can construct an algorithm $\mathcal{A}$ with the following properties:

- $\mathcal{A}$ generates an infinite sequence $R_1; R_2; R_3; \ldots$ of $R$-formulas such that each finite part $[R_1; \ldots; R_n]$ with $n \in \mathbb{N}$ is an $R$-derivation in $[S; \mathcal{L}]$. Note that $\mathcal{A}$ makes only use of the rules of inference (1.11)(a),(b),(c) in [2].
- All elementary prime $R$-formulas in $\Pi_R(S; \mathcal{L})$ occur at least one time in the sequence $R_1; R_2; R_3; \ldots$.
- We suppose that $x_1, \ldots, x_i$ and the variables of $G$ are not involved in $R_1; R_2; R_3; \ldots$, which is not a real restriction.

Depending on $\mathcal{A}$ we define a second algorithm $\mathcal{B}$ with the following properties:
• Algorithm $B$ generates a (finite or infinite) sequence of formulas $F_1; F_2; F_3; \ldots$ in $[M; L]$. Each finite part $[F_1; \ldots; F_n]$ of the sequence is a proof in $[M; L]$. For $n > l$ algorithm $B$ makes only use of the rules of inference (3.13)(a)-(d) in $[L]$ in order to derive $F_n$.

• First of all we start with algorithm $B$ and prescribe the formulas $F_1; F_2; \ldots; F_l$ in the proof $[\Lambda]$. Next we extend $[\Lambda]$ to a proof $[\Lambda_0]$ by applying only Rule (c) a finite number of times in order to substitute all variables $x_{i+1}, \ldots, x_m$ by $\tilde{\lambda}_{i+1}, \ldots, \tilde{\lambda}_m$ in the formulas $F' \in [\Lambda]$ for all R-formulas $F \in B_S$ for which $p$ occurs $i$-ary in the R-conclusion of $F$. After the construction of $[\Lambda_0]$ we pause $B$ and start $A$.

• Each time when $A$ has generated a prime R-formula $R_\kappa$ (including equations and with or without variables) we pause algorithm $A$ and activate algorithm $B$ to generate $R_\kappa$ as well in the sequence of formulas $F_1; F_2; F_3; \ldots$. Moreover, if $R_\kappa = p \lambda_1, \ldots, \lambda_i$ with lists $\lambda_1, \ldots, \lambda_i \in L$, then $B$ will also generate the formula $\tilde{G}(\lambda_1, \ldots, \lambda_i)$ in a finite number of steps. Afterwards we pause algorithm $B$ and activate algorithm $A$ again, and so on.

It is clear that any R-derivation in $[S; L]$ can also be performed in $[M; L]$. To prove that algorithm $B$ is well defined we have to show that it is able to generate the formulas $\tilde{G}(\lambda_1, \ldots, \lambda_i)$ once algorithm $A$ has produced the next prime formula of the form $p \lambda_1, \ldots, \lambda_i$. This will be explained now.

Let $F$ be any R-formula in $[S; L]$ and suppose that $x_1, \ldots, x_i$ and the variables of $G$ are not involved in $F$. To $F$ there corresponds exactly one formula $\hat{F} \in \mathcal{F}$ of $[M; L]$ which results if we replace in $F$ each $i$-ary subformula $p \lambda_1, \ldots, \lambda_i$ with $\lambda_1, \ldots, \lambda_i \in L$ by the formula $\tilde{G}(\lambda_1, \ldots, \lambda_i)$. We have assumed that the variables of $G$ are not involved in $B_S$. Then we obtain that $\hat{F}$ is a step of the extended proof $[\Lambda_0]$ for all R-formulas $F \in B_S$ for which $p$ occurs $i$-ary in the R-conclusion of $F$.

\footnote{We have $\hat{F} = F$ if $p$ does not occur $i$-ary in $F$.}
Beside the axioms \( F \) in \( B_S \) for which \( p \) occurs \( i \)-ary in the R-conclusion of \( F \) algorithm \( A \) can also make use of the R-axioms of equality (1.9)(c) with \( n = i \) in order to deduce prime R-formulas \( p \lambda_1, \ldots, \lambda_i \) in \([S; \mathcal{L}]\) from equations and these R-axioms. Let

\[ F = \rightarrow \sim y_1, y'_1 \ldots \rightarrow \sim y_i, y'_i \rightarrow p y_1, \ldots, y_i p y'_1, \ldots, y'_i \]

be such an R-axiom of equality with variables \( y_k, y'_k \in X \). We suppose that \( x_1, \ldots, x_i \) and the variables of \( G \) are not involved in \( F \). Then we infer in \([M; \mathcal{L}]\) the formula

\[ \hat{F} = \rightarrow \sim y_1, y'_1 \ldots \rightarrow \sim y_i, y'_i \rightarrow \tilde{G}(y_1, \ldots, y_i) \tilde{G}(y'_1, \ldots, y'_i). \]

That this is possible can be seen by using [2, (4.9) Corollary] combined with the Deduction Theorem [2, (4.5)], by using the axioms of equality and the Equivalence Theorem [2, Theorem (3.17)(a)]. This will only require the use of the Rules (a)-(d).

Let \( R_\lambda \) be any R-axiom generated by algorithm \( A \) and assume that \( p \) occurs \( i \)-ary in the R-conclusion of \( R_\lambda \). Then we summarize and keep in mind that we can derive the corresponding formula \( \hat{R}_\lambda \) in \([M; \mathcal{L}]\) from \([\Lambda]\) by using only Rules (a)-(d).

Initially \( B \) generates \([\Lambda_0]\). We consider a finite part \( R_1; \ldots; R_\alpha \) of the R-formulas from \( \Pi_R(S; \mathcal{L}) \) generated by the algorithm \( A \), and we assume that \( R_\alpha \) is a prime R-formula. Then we activate algorithm \( B \) and proceed with a further expansion \( F_1; \ldots; F_\beta \) of the list of formulas from \( \Pi(M; \mathcal{L}) \) until we have derived \( R_\alpha \) and \( F_\beta = \hat{R}_\alpha \). This can be achieved if \( B \) mimics the R-derivation \( R_1; \ldots; R_\alpha \) in the following way:

- For any R-axiom \( R_\lambda \) with \( \lambda \leq \alpha \) algorithm \( B \) generates the formula \( R_\lambda \) as well. If \( p \) occurs \( i \)-ary in the R-conclusion of \( R_\lambda \) then \( B \) generates \( \hat{R}_\lambda \).
- Suppose that the R-formula \( R_\lambda \) was derived from the prime R-formula \( R_\kappa \) and the R-formula \( \rightarrow R_\kappa R_\lambda \) with Rule (b), \( \kappa < \lambda \leq \alpha \). Then algorithm \( B \) derives the formula \( R_\lambda \) as well. Suppose that \( \hat{R}_\kappa \) and \( \rightarrow \hat{R}_\kappa \hat{R}_\lambda \) were already derived by \( B \), but not \( \hat{R}_\lambda \). Then \( B \) generates \( \hat{R}_\lambda \) from Rule (b).
• Assume that the R-formula $R_\lambda = R_\kappa \nu \, x$ with $x \in X$ and $\nu \in \mathcal{L}$ was derived from the R-formula $R_\kappa$ with Rule (c), $\kappa < \lambda \leq \alpha$. Then algorithm $B$ derives the formula $R_\lambda$ as well. Suppose that $\hat{R}_\kappa$ was already derived by $B$, but not $\hat{R}_\lambda$. Then $B$ generates $\hat{R}_\lambda = \hat{R}_\kappa \nu \, x$ from Rule (c).

\[\square\]

Under the mild additional condition $\mathcal{L}_* \neq \emptyset$ we have proved a slightly more general version of [2, (5.4) Conjecture], namely the following theorem, which makes use of Definition 3.1:

**Theorem 3.6.** Let $M = [S; A_M; P_M; B_M]$ be a mathematical system with an underlying recursive system $S = [A_S; P_S; B_S]$ such that $A_M = A_S$, $P_M = P_S$, $B_M = B_S$. Suppose that $[M; \mathcal{L}]$ is a mathematical system with restricted argument lists in $\mathcal{L}$ and that $\mathcal{L}$ is enumerable. Let $\mathcal{L}_* \neq \emptyset$ be the set of all $A_S$-lists in $\mathcal{L}$ without variables.

i. Let be $\lambda, \mu \in \mathcal{L}_*$. Then

$$\sim \lambda, \mu \in \Pi(M; \mathcal{L}) \iff \sim \lambda, \mu \in \Pi_R(S; \mathcal{L}).$$

ii. Let $p \in P_S$ and $\lambda_1, \ldots, \lambda_i \in \mathcal{L}_*$ for $i \geq 0$ be elementary $A_S$-lists. Then

$$p \lambda_1, \ldots, \lambda_i \in \Pi(M; \mathcal{L}) \iff p \lambda_1, \ldots, \lambda_i \in \Pi_R(S; \mathcal{L}).$$

**Remark 3.7.** We have assumed that $\mathcal{L}_* \neq \emptyset$ in order to avoid trouble with the definition of the interpretation $V$ of the formulas $F \in \mathcal{F}_*$.

From [2, Section 5.3] and Theorem 3.6 we obtain a consistency proof for the following Peano arithmetic PA: Let $\bar{S}$ be the recursive system $\bar{S} = [\bar{A}; \bar{P}; \bar{B}]$ where $\bar{A}$, $\bar{P}$ and $\bar{B}$ are empty, and introduce the alphabets $A_{PA} = [0; s ; + ; *]$, $P_{PA} = [\,]$. We define the set $\mathcal{L}$ of numeral terms by the recursive definition

(i) 0 and $x$ are numeral terms for any $x \in X$.

(ii) If $\vartheta$ is a numeral term, then also $s(\vartheta)$.

(iii) If $\vartheta_1$, $\vartheta_2$ are numeral terms, then also $+(\vartheta_1 \vartheta_2)$ and $*(\vartheta_1 \vartheta_2)$.

We define the mathematical system $M' = [\bar{S}; A_{PA}; P_{PA}; B_{PA}]$ by giving the following basis axioms for $B_{PA}$ with distinct variables $x, y$
Moreover, for all formulas \( F \) (with respect to \( A_{PA} \) and \( P_{PA} \)) which have only numeral argument lists, the following formulas belong to \( B_{PA} \) according to the Induction Scheme (IS):

\[
\forall x \sim (0.x), x
\]
\[
\forall x \forall y \sim +(s(x)y), s((xy))
\]
\[
\forall x \sim *(0.x), 0
\]
\[
\forall x \forall y \sim *(s(x)y), +(s(xy)y)
\]
\[
\forall x \forall y \rightarrow \sim s(x), s(y) \sim x, y
\]
\[
\forall x \sim s(x), 0.
\]

Using the results in [2, Chapter (5.3)] and Theorem 3.6 the inconsistency of PA would imply that there is an elementary numeral term \( \lambda \), i.e. a numeral term without variables, such that \( N_0 \lambda \) as well as \( \sim s(\lambda), 0 \) are \( R \)-derivable in \( S \), which is impossible.

Instead of explaining these earlier results in detail again we will now derive a stronger result from Theorem 3.5, namely

**Theorem 3.8. The \( \omega \)-consistency of the Peano arithmetic PA**

Let \( F \) be a formula in PA with free(\( F \)) = \{\( x \)\}, i.e. let \( x \in X \) be the only free variable of \( F \). Suppose that \( \neg F_x^\lambda \) is provable in PA for all elementary numeral terms \( \lambda \). Then \( \exists xF \) is not provable in PA.
Proof. We assume that $\exists x F$ is provable in PA, i.e.

\begin{equation}
\exists x F \in \Pi(PA),
\end{equation}

and will show that this leads to a contradiction.

Step 1. We make use of the recursive system $S = [A_S; P_S; B_S]$ with the basis R-axioms (1)-(7) given above and define the mathematical system $M = [S; A_S; P_S; B_S]$. Recall the set $\mathcal{L}$ of numeral terms. Now $[M; \mathcal{L}]$ satisfies the conditions of Theorem (3.3) mentioned at the beginning of this section. We will show for all formulas $H$ in $[M; \mathcal{L}]$ that

\begin{equation}
\rightarrow \forall x \rightarrow N_0 x \& \frac{H^0}{x} \rightarrow H \frac{H^{s(x)}}{x} \quad \forall x \rightarrow N_0 x H
\end{equation}

is provable in $[M; \mathcal{L}]$, which is the induction principle for $[M; \mathcal{L}]$. Without loss of generality we may assume that free($H$) = $\{x, x_1, \ldots, x_m\}$ with disjoint variables $x, x_1, \ldots, x_m$ and $m \geq 0$. Corresponding to the variables $x_1, \ldots, x_m$ we choose new and different constant symbols $c_1, \ldots, c_m$ and put

$$\tilde{H} = H \frac{c_1}{x_1} \cdots \frac{c_m}{x_m}.$$ 

We define $A = A_S \cup \{c_1, \ldots, c_m\}$,

$$\mathcal{L}_A = \{ \lambda \frac{c_1}{y_1} \ldots \frac{c_m}{y_m} : \lambda \in \mathcal{L}, y_1, \ldots, y_m \in X \}$$

and the extension $M_A = [S; A; P_S; B_S]$. Due to [2] Definition (4.2)(d)] and [2] Corollary (4.9)(a)] we obtain a mathematical system $[M_A; \mathcal{L}_A]$ with argument lists restricted to $\mathcal{L}_A$. We adjoin the statement

$$\varphi = \forall x \rightarrow N_0 x \& \frac{\tilde{H}^0}{x} \rightarrow \tilde{H} \frac{\tilde{H}^{s(x)}}{x}$$

to $[M_A; \mathcal{L}_A]$ and obtain the extended system $[M_A(\varphi); \mathcal{L}_A]$, see [2] Definition (4.2)(a)]. Now $\varphi$ is provable in $[M_A(\varphi); \mathcal{L}_A]$, and we obtain from the quantifier axiom (3.11)(a) and the Modus Ponens Rule in [2] that

\begin{equation}
\rightarrow N_0 x \& \frac{\tilde{H}^0}{x} \rightarrow \tilde{H} \frac{\tilde{H}^{s(x)}}{x} \in \Pi(M_A(\varphi); \mathcal{L}_A).
\end{equation}

Let $u \in X$ be any variable which is neither involved in $H$ nor in $B_S$ and put

\begin{equation}
\tilde{G} = \& N_0 u \frac{\tilde{H}^u}{x}.
\end{equation}
In \([M_A(\varphi); \mathcal{L}_A]\) we obtain a proof containing the formula in (3.3):

\[
\ldots \rightarrow N_0 x \& \overline{H}^0 x \rightarrow \overline{H} \overline{H}^{s(x)} x; \\
\rightarrow \rightarrow N_0 x \& \overline{H}^0 x \rightarrow \overline{H} \overline{H}^{s(x)} x \rightarrow N_0 x \overline{H}^0 x; \\
\rightarrow N_0 x \overline{H}^0 x; \rightarrow N_0 0 \overline{H}^0 x; N_0 0; \overline{H}^0 x; \\
\rightarrow N_0 0 \rightarrow \overline{H}^0 x \& N_0 0 \overline{H}^0 x; \\
\rightarrow \overline{H}^0 x \& N_0 0 \overline{H}^0 x; \\
\& N_0 0 \overline{H}^0 x; \\
\rightarrow N_0 x N_0 s(x); \\
\rightarrow \rightarrow N_0 x \& \overline{H}^0 x \rightarrow \overline{H} \overline{H}^{s(x)} x \\
\rightarrow \rightarrow N_0 x N_0 s(x) \\
\rightarrow \& N_0 x \overline{H} \& N_0 s(x) \overline{H}^{s(x)} x; \\
\rightarrow \rightarrow N_0 x N_0 s(x) \\
\rightarrow \& N_0 x \overline{H} \& N_0 s(x) \overline{H}^{s(x)} x; \\
\rightarrow \& N_0 x \overline{H} \& N_0 s(x) \overline{H}^{s(x)} x; \\
\rightarrow N_0 u \overline{G}].
\]

The last step results from the Induction Rule (e), using the abbreviation \(\overline{G}\) in (3.4). We see that the two formulas

\[
\rightarrow N_0 x \overline{H} \quad \text{and} \quad \forall x \rightarrow N_0 x \overline{H}
\]

are also provable in \([M_A(\varphi); \mathcal{L}_A]\). It follows from the Deduction Theorem [2 (4.3)] that the formula

\[
\rightarrow \forall x \rightarrow N_0 x \& \overline{H}^0 x \rightarrow \overline{H} \overline{H}^{s(x)} x \forall x \rightarrow N_0 x \overline{H}
\]

is provable in \([M_A; \mathcal{L}_A]\). From the generalization of the constant symbols \(c_1, \ldots, c_m\) according to [2, Corollary (4.9)(b)] we see that the formula (3.2) is provable in the original mathematical system \([M; \mathcal{L}]\).
Step 2: Following [2, Section 5] we construct from PA a related mathematical system \( P_{AN_0} = [M_{P_{AN_0}}; \mathcal{L}] \) with argument lists restricted to the numerals \( \mathcal{L} \) as follows: We put \( M_{P_{AN_0}} = [\tilde{S}; A_S; P_S; B_{P_{AN_0}}] \) with the underlying recursive system \( \tilde{S} = [[]; []; []], \) and recall that \( A_S = [0; s; +; *], \) \( P_S = [N_0]. \) The basis axioms \( B_{P_{AN_0}} \) of \( P_{AN_0} \) are given by the two formulas \( N_0 0 \) and \( \rightarrow N_0 x N_0 s(x) \) with \( x \in X \) and by all the formulas \( \Gamma_{N_0}(G) \Psi_{N_0}(G), \) where \( G \) is any basis axiom of PA (including the formulas from the induction scheme). Here \( \Gamma_{N_0}(G) \) and \( \Psi_{N_0}(G) \) are defined in [2, Section 5] for every PA-formula \( G \) as follows:

- We put \( \Gamma_{N_0}(G) = \rightarrow N_0 x_1 ... \rightarrow N_0 x_n \) for the block of \( N_0 \)-premises with respect to \( \text{free}(G) = \{x_1; ...; x_n\}, \) \( x_1, \ldots, x_n \) ordered according to their first occurrence in \( G. \) For \( n = 0 \) the string \( \Gamma_{N_0}(G) \) is defined to be empty.
- \( \Psi_{N_0}(G) \) results from \( G \) if we replace simultaneously in every subformula \( \forall z G' \) of \( G \) the part \( \forall z \rightarrow N_0 z, \) and in every subformula \( \exists z G' \) of \( G \) the part \( \exists z \) by \( \exists z \& N_0 z, \) with \( z \in X. \)

Lemma (5.2)(iii) in [2, Section 5] states that \( \Gamma_{N_0}(G) \Psi_{N_0}(G) \) is provable in \( P_{AN_0} \) for every formula \( G \) which is provable in PA.

From \( \neg F^\lambda x \in \Pi(PA) \) for all \( \lambda \in \mathcal{L}_* \) and from (3.1) we see that

\[
(3.5) \quad \neg \Psi_{N_0}( \frac{F^\lambda}{x} ) = \neg \Psi_{N_0}(F) \frac{\lambda}{x} \in \Pi( P_{AN_0} ),
\]

\[
(3.6) \quad \exists x & N_0 x \Psi_{N_0}(F) \in \Pi( P_{AN_0} ).
\]

Step 3. To the mathematical system \( [M; \mathcal{L}] \) we adjoin the single statement

\( \forall x \rightarrow N_0 x \neg \sim s(x), 0 \)

and obtain the mathematical system \( M_{PA} = [M((*)); \mathcal{L}]. \) We see from the first step that every formula which is provable in \( P_{AN_0} \) is also provable in \( M_{PA}. \) It follows from (3.5), (3.6) and the Deduction Theorem [2, (4.3)] for all \( \lambda \in \mathcal{L}_* \) that

\[
(3.7) \quad \rightarrow (* \neg \Psi_{N_0}(F) \frac{\lambda}{x} \in \Pi(M; \mathcal{L}),
\]

\[
(3.8) \quad \rightarrow (* \exists x & N_0 x \Psi_{N_0}(F) \in \Pi(M; \mathcal{L}).
\]
Step 4. We extend the function $V$ to the set of all statements in $[M; L]$. Let $F_R$ be an elementary prime formula in $[M; L]$. Then $F_R$ is also an elementary prime $R$-formula in $[S; L]$. In this case we recall that $V(F_R) = \{ \top \}$ iff $F_R$ is $R$-derivable in $[S; L]$, i.e. iff $F_R \in \Pi_R(S; L)$, and otherwise we have $V(F_R) = \emptyset$. We put in addition for all statements $G, H$ of $[M; L]$:

$$V(\neg G) = \{ \top \} \setminus V(G),$$

$$V(\rightarrow GH) = (\{ \top \} \setminus V(G)) \cup V(H),$$

$$V(\lor GH) = V(G) \cup V(H),$$

$$V(\& GH) = V(G) \cap V(H),$$

$$V(\leftrightarrow GH) = V(\rightarrow GH) \cap V(\rightarrow HG).$$

Recall the set $\mathcal{L}_*$ of all elementary numeral terms (without variables) and let $G$ be a formula of $[M; L]$ with $\text{free}(G) \subseteq \{ z \}$. Then $G^z_\lambda$ is a statement for all $\lambda \in \mathcal{L}_*$, and we put

$$V(\forall z G) = \bigcap_{\lambda \in \mathcal{L}_*} V\left( G^\lambda_z \right),$$

$$V(\exists z G) = \bigcup_{\lambda \in \mathcal{L}_*} V\left( G^\lambda_z \right).$$

Recalling the definition of the function $\Theta$ we can use induction over well formed formulas and the Equivalence Theorem \cite[3.17]{2}(a) Theorem] and obtain for all formulas $G$ of $[M; L]$:

$$(3.9) \quad \leftrightarrow G \Theta(G) \in \Pi(M; L).$$

Next we define a degree for all formulas in $[M; L]$. We put $\deg(G) = 0$ for all prime formulas $G$. For general formulas $G, H$ in $[M; L]$ we put $\deg(\neg G) = \deg(G) + 1$,

$$\deg(J GH) = \max(\deg(G), \deg(H)) + 1 \text{ for } J \in \{ \rightarrow; \lor; \&; \leftrightarrow \},$$

and for $z \in X$ we put

$$\deg(\forall z G) = \deg(\exists z G) = \deg(G) + 1.$$

Induction over $n \in \mathbb{N}_0$ with respect to $\deg(G) \leq n$ gives for all statements $G$ that

$$(3.10) \quad V(\Theta(G)) = V(G).$$
Step 5. In the final step we apply Theorem 3.5, (3.9), (3.10) on the mathematical system \([M; L]\) with the underlying recursive system \(S\) and on the two statements in (3.7), (3.8). We obtain for all \(\lambda \in L_*\) that

\[(3.11) \quad V \left( \rightarrow (\star) \neg \Psi_{N_0}(F) \frac{x}{x} \right) = \{ \top \},\]

\[(3.12) \quad V \left( \rightarrow (\star) \exists x \& N_0 x \Psi_{N_0}(F) \right) = \{ \top \} .\]

Using \(N_0 \lambda \in \Pi_R(S; L), \sim s(\lambda), 0 \notin \Pi_R(S; L)\) for all \(\lambda \in L_*\) we see \(V ((\star)) = \{ \top \} \). We obtain from (3.12)

\[\{ \top \} = V \left( \exists x \& N_0 x \Psi_{N_0}(F) \right) = \bigcup_{\lambda \in L_*} V \left( \Psi_{N_0}(F) \frac{x}{x} \right) \cdot\]

But (3.11) gives \(\emptyset = V \left( \Psi_{N_0}(F) \frac{x}{x} \right)\) for all \(\lambda \in L_*\), a contradiction. \(\Box\)

3.6. A further example with formal induction.

Finally we go back to the recursive system \(S = [A; P; B]\) introduced in Section 2.1 with \(A = [a; 0; 1]\), \(P = [D]\) and the set \(B\) consisting of the six basis R-axioms \((\alpha)-(\zeta)\). Let \(L\) be the set generated by the rules

(i) \(x \in L\) for all \(x \in X\),
(ii) \(0 \in L\), \(1 \in L\) and \(a \in L\),
(iii) If \(\lambda, \mu \in L\) then \(\lambda \mu \in L\).

We define the mathematical system \([M; L]\) with \(M = [S; A; P; B]\) and will show that the formula \(\forall x \iff D x \exists y D x, y\) is provable in \([M; L]\). We will present a short semi-formal proof. Due to Rule (d) it is sufficient to show that \(\iff D x \exists y D x, y\) is provable in \([M; L]\). For this purpose we will apply the Induction Rule (e) twice to deduce \(\rightarrow \exists y D x, y D x\) as well \(\rightarrow D x \exists y D x, y\) in \([M; L]\). Let \(x, y, u, v \in X\) be distinct. Due to Rule (a) the \(R\)-axioms in \(B\) are provable in \([M; L]\):

1. \(D 1\)
2. \(\rightarrow D x D x 0\)
3. \(\rightarrow D x D x 1\)
4. \(D 1, a\)
5. \(\rightarrow D x, y D x 0, yy\)
6. \(\rightarrow D x, y D x 1, yya\)
For the first application of Rule (e) we put $p = D$, $i = 2$, $x_1 = u$, $x_2 = v$ and $G = Du$. In axioms 4.-6. we replace the prime subformulas $D \lambda_1, \lambda_2$ by $D \lambda_1$ and obtain axioms 1.-3. for the 1-ary predicate “$D$”. Due to Rule (e)

\begin{align*}
7. & \rightarrow D u, v D u \\
\end{align*}

is provable in $[M; L]$, and also the formulas

\begin{align*}
8. & \rightarrow \neg D x \neg D x, y \\
9. & \forall y \rightarrow \neg D x \neg D x, y \quad \text{from 8. & Rule (d)} \\
10. & \rightarrow \neg D x \forall y \neg D x, y \quad \text{with 9. & quantifier axiom (3.11)(b)} \\
11. & \rightarrow \neg \forall y \neg D x, y D x \\
12. & \rightarrow \exists y D x, y D x \quad \text{with 11. & quantifier axiom (3.11)(c)} \\
\end{align*}

This is the first implication.

For the second one we deduce the following formulas in $[M; L]$: 

\begin{align*}
13. & \rightarrow D x, y \exists y D x, y \quad \text{from example 1 in Section 3.1} \\
14. & \rightarrow D 1, a \exists y D 1, y \quad \text{from 13. and two times Rule (c)} \\
15. & \exists y D 1, y \quad \text{with 4. and 14.} \\
16. & \rightarrow D x0, y \exists y D x0, y \quad \text{from example 1 in Section 3.1} \\
17. & \rightarrow D x0, yy \exists y D x0, y \quad \text{from 16. and Rule (c)} \\
18. & \rightarrow D x, y \exists y D x0, y \quad \text{with 5. and 17.} \\
19. & \rightarrow \neg \exists y D x0, y \neg D x, y \\
20. & \forall y \rightarrow \neg \exists y D x0, y \neg D x, y \quad \text{from 19. & Rule (d)} \\
21. & \rightarrow \neg \exists y D x0, y \forall y \neg D x, y \quad \text{with 20. & quantifier axiom (3.11)(b)} \\
22. & \rightarrow \neg \forall y \neg D x, y \exists y D x0, y \\
23. & \rightarrow \exists y D x, y \exists y D x0, y \quad \text{with 22. & quantifier axiom (3.11)(c)} \\
24. & \rightarrow D x1, y \exists y D x1, y \quad \text{from example 1 in Section 3.1} \\
25. & \rightarrow D x1, yya \exists y D x1, y \quad \text{from 24. and Rule (c)} \\
26. & \rightarrow D x, y \exists y D x1, y \quad \text{with 6. and 25.} \\
27. & \rightarrow \neg \exists y D x1, y \neg D x, y \\
28. & \forall y \rightarrow \neg \exists y D x1, y \neg D x, y \quad \text{from 27. & Rule (d)} \\
29. & \rightarrow \neg \exists y D x1, y \forall y \neg D x, y \quad \text{with 28. & quantifier axiom (3.11)(b)} \\
30. & \rightarrow \neg \forall y \neg D x, y \exists y D x1, y \\
31. & \rightarrow \exists y D x, y \exists y D x1, y \quad \text{with 30. & quantifier axiom (3.11)(c)} \\
\end{align*}
From formulas 15, 23, 31. and [2, Theorem (3.17)(b)] we obtain that the formulas

\begin{align*}
32. & \exists \lambda D 1, v \\
33. & \rightarrow \exists \lambda D x, v \exists \lambda D x0, v \\
34. & \rightarrow \exists \lambda D x, v \exists \lambda D x1, v \\
\end{align*}

are provable in $[M; \mathcal{L}]$.

For the second application of Rule (e) we put $p = D, i = 1, x_1 = u$ and $G = \exists \lambda D u, v$. We replace the prime subformulas $D \lambda_1$ in axioms 1.-3. by $\exists \lambda D \lambda_1, v$ and obtain formulas 32.-34., respectively.

Due to Rule (e) we see that $\rightarrow D u \exists \lambda D u, v$ and hence $\rightarrow D x \exists \lambda D x, y$ are both provable in $[M; \mathcal{L}]$.

4. CONCLUSIONS AND OUTLOOK

We have presented contributions to elementary proof theory. Especially in Section 3.4 we have determined a simple procedure in order to eliminate prime formulas from formal proofs which do not occur with a given arity in the basis axioms of a mathematical system. We also hope to develop a method in order to eliminate equations from formal proofs if there are no equations in the basis axioms.

Our most important contribution is Theorem 3.5, which is a general result of mathematical logic concerning formal induction. We have presented two applications of this theorem in Section 3.5, namely the proof of [2, (5.4) Conjecture], see Theorem 3.6 and the $\omega$-consistency of the Peano arithmetic PA in Theorem 3.8.

It would be very interesting to create a computer program which is able to check semiformal proofs like in Section 3.6. First a machine should be able to check fully formalized proofs with certain restrictions. For example, the number of propositional variables in the axioms [2, (3.9)] must be small enough for an efficient calculation. In a next step the program should be extended to analyze the use of the axioms and rules in order to develop further composed rules of inference, especially for the propositional calculus and for the treatment of equations.

An advanced program should also make use of [2, Theorem (3.17), Propositions (3.18),(3.19), Theorems (4.5),(4.8), Corollaries (4.9),(4.10)].
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