IDEALS IN SOME RINGS OF NEVANLINNA–SMIRNOV TYPE

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Abstract. Let $N^p (1 < p < \infty)$ denote the algebra of holomorphic functions in the open unit disk, introduced by I. I. Privalov with the notation $A_q$ in [8]. Since $N^p$ becomes a ring of Nevanlinna–Smirnov type in the sense of Mortini [7], the results from [7] can be applied to the ideal structure of the ring $N^p$. In particular, we observe that $N^p$ has the Corona Property. Finally, we prove the $N^p$-analogue of the Theorem 6 in [7], which gives sufficient conditions for an ideal in $N^p$, generated by a finite number of inner functions, to be equal to the whole algebra $N^p$.

1. Introduction and Preliminaries

Let $D$ denote the open unit disk in the complex plane and let $T$ denote the boundary of $D$. Let $L^p(T) (0 < p \leq \infty)$ be the familiar Lebesgue spaces on $T$. The Nevanlinna class $N$ is the set of all functions $f$ holomorphic on $D$ such that

$$\sup_{0 < r < 1} \int_0^{2\pi} \log^+ |f(re^{i\theta})| \frac{d\theta}{2\pi} < \infty,$$

where $\log^+ |x| = \max(\log |x|, 0)$.

The Smirnov class $N^+$ consists of those functions $f \in N$ for which

$$\lim_{r \to 1} \int_0^{2\pi} \log^+ |f(re^{i\theta})| \frac{d\theta}{2\pi} = \int_0^{2\pi} \log^+ |f^*(e^{i\theta})| \frac{d\theta}{2\pi} < \infty,$$

where $f^*$ is the boundary function of $f$ on $T$, i.e.,

$$f^*(e^{i\theta}) = \lim_{r \to 1} f(re^{i\theta})$$

1991 Mathematics Subject Classification. Primary 30H05, 46J15. Secondary 46J20.

Key words and phrases. Rings of Nevanlinna–Smirnov type, Classes $N^p$, Trace of an ideal, Corona Property, Interpolating Blaschke product.
is the radial limit of \( f \) which exists for almost every \( e^{i\theta} \).

Recall that the Hardy space \( H^p \) (\( 0 < p \leq \infty \)) consists of all functions \( f \) holomorphic in \( D \), which satisfy

\[
\sup_{0 < r < 1} \int_0^{2\pi} \left| f(re^{i\theta}) \right|^p \frac{d\theta}{2\pi} < \infty.
\]

If \( 0 < p < \infty \), and which are bounded when \( p = \infty \);

\[
\sup_{z \in D} |f(z)| < \infty.
\]

Following R. Mortini [7], a ring \( R \) satisfying \( H^\infty \subset R \subset N \) is said to be of Nevanlinna–Smirnov type if every function \( f \in R \) can be written in the form \( g/h \), where \( g \) and \( h \) belong to \( H^\infty \) and \( h \) is invertible element in \( R \). This is true of \( N \) itself and the Smirnov class \( N^+ \); hence the name (see [1, Chapter 2]). Further, Mortini noted that by a result of M. Stoll [10], the space \( F^+ \), the containing Fréchet envelope for \( N^+ \), consists of those functions \( f \) holomorphic in \( D \) satisfying

\[
\limsup_{r \to 1} (1 - r) \log M(r, f) = 0
\]

with \( M(r, f) = \max_{|z| = r} |f(z)| \) (see Yanagihara [11]).

Namely, Stoll [10] proved that \( F^+ \cap N = \{ f/S_\mu : f \in N^+, S_\mu \text{ is a singular inner function with } \mu \text{ a nonnegative continuous singular measure} \} \).

The class \( N^p \) (\( 1 < p < \infty \)) consists of all holomorphic functions \( f \) on \( D \) for which

\[
\sup_{0 < r < 1} \int_0^{2\pi} \left( \log^+ |f(re^{i\theta})| \right)^p \frac{d\theta}{2\pi} < \infty.
\]

These classes were introduced in the first edition of Privalov’s book [8, p. 93], where \( N^p \) is denoted as \( A_q \). It is known [6] that

\[
N^q \subset N^p \quad (q > p), \quad \bigcup_{p > 0} H^p \subset \bigcap_{p > 1} N^p, \quad \text{and} \quad \bigcup_{p > 1} N^p \subset N^+,
\]

where the above containment relations are proper.

**Theorem A** ([8, p. 98]). A function \( f \in N^p \setminus \{0\} \) has a unique factorization of the form

\[
f(z) = B(z)S(z)F(z)
\]

where \( B(z) \) is the Blaschke product with respect to zeros \( \{z_k\} \subset D \) of \( f(z) \), \( S(z) \) is a singular inner function, \( F(z) \) is an outer function in
$N^p$, i.e.,

\[ B(z) = z^m \prod_{k=1}^{\infty} \frac{|z_k|}{z_k} \frac{z_k - z}{1 - \overline{z_k} z} \]

with $\sum_{k=1}^{\infty} (1 - |z_k|) < \infty$, $m$ a nonnegative integer,

\[ S(z) = \exp \left( - \int_{0}^{2\pi} \frac{e^{it} + z}{e^{it} - z} d\mu(t) \right) \]

with positive singular measure $d\mu$, and

\[ F(z) = \lambda \exp \left( \frac{1}{2\pi} \int_{0}^{2\pi} \frac{e^{it} + z}{e^{it} - z} \log |f^*(e^{it})| \, dt \right) , \]

where $|\lambda| = 1$, $\log |f^*|$ and $(\log^+ |f^*|)^p$ belong to $L^1(T)$.

Conversely, every such product $B(z)S(z)F(z)$ belongs to $N^p$.

Remark. If we exclude only the condition $(\log^+ |f^*|)^p \in L^1(T)$ from Theorem A, we obtain the well known canonical factorization theorem for the class $N^+$ (see [1, p. 26] or [8, p. 89]). Recall that a function of the form $\varphi(z) = B(z)S(z)$ is called an inner function, while the above function $F(z)$ for which $\log |f^*| \in L^p(T)$ is called an outer function.

By Theorem A, it is easy to show (see [2], where $N^p$ is denoted as $N^+_0$) that a function $f$ is in $N^p$ if and only if it can be expressed as the ratio $g/h$, where $g$ and $h$ are in $H^\infty$, and $h$ is an outer function with $\log |h^*| \in L^p(T)$. Clearly, such function $h$ is an invertible element of $N^p$, and hence we have the following result.

**Theorem B.** $N^p (1 < p < \infty)$ is a ring of Nevanlinna–Smirnov type.

In Section 2, the ideal structure of subrings $N^p$ of $N$ is described as consequences of the results in [7, Sections 1 and 3] given for an arbitrary ring of Nevanlinna–Smirnov type. In the next section, we note that the algebra $N^p$ has the Corona Property. Finally, we prove two theorems which generalize the results from [7, Satz 5 and Satz 6] obtained for the classes $H^1$ and $N^+$.

2. Ideals in $N^p$

In this Section, as an application of Theorems A and B and the results of Mortini in [7], we obtain some facts about the ideal structure of the algebra $N^p$. We say that an ideal $I$ in $H^\infty$ is the trace of an
ideal $J$ in $N^p$ if $I = J \cap H^\infty$. The following result is an immediate consequence of Theorems A, B and [7, Satz 1, Satz 2].

**Theorem 1.** An ideal $I$ in $H^\infty$ is the trace of an ideal $J$ in $N^p$ if and only if the following condition is satisfied: If $f \in I$, $F$ is an outer function with $\log |F^*| \in L^p(T)$, and if $fF \in H^\infty$, then $fF \in I$. In this case, $J$ is a unique ideal in $N^p$ with $I = J \cap H^\infty$, and there holds $J = IN^p$.

Further, by the above theorem, it follows immediately the following theorem.

**Theorem 2.** Suppose that $I$ is an ideal in $H^\infty$ such that $f \in I$ implies that the inner factor of $f$ also belongs to $I$. Then $I$ is the trace of an ideal $J$, in $N^p$ and there holds $J = IN^p$.

**Remark.** It remains an open question is it true the converse of Theorem 2. While this is true for the Nevanlinna class and the Smirnov class [7, Korrolar 1 and Korrolar 2, resp.], here the corresponding problem is complicated by the fact that there exist outer functions which are not invertible in $N^p$. For example, the converse of Theorem 2 holds if we suppose in addition that $I$ is a closed subset of $N^p$. Namely, by Theorem 2 of [6] and the condition from Theorem 1, it follows that whenever $f$ is in $I$, then necessarily the inner factor of $f$ is also in $I$.

Recall that an ideal $P$ in a ring $R$ is prime if whenever $fg \in P$, $f, g \in R$, then either $f$ or $g$ is in $P$. Using the characterization of the invertible elements in $N^p$, by [7, Satz 3], we obtain the following.

**Theorem 3.** A prime ideal $P$ in $H^\infty$ is the trace of some prime ideal $Q$ in $N^p$ if and only if $P$ contains no outer functions $F$ for which $\log |F^*| \in L^p(T)$. When this is the case, $Q$ is a unique prime ideal in $N^p$ with this property, and there holds $Q = PN^p$.

**Remark.** By [6, Theorem 3], every prime ideal of $N^p$ which is not dense in $N^p$ is equal to the set of functions in $N^p$ vanishing at a specific point of $D$. The analogous result for the class $N^+$ is proved in [9, Theorem 1].

An ideal $J$ in the ring $R$, $H^\infty \subset R \subset N$, is called finitely generated if there exist elements $f_1, \ldots, f_n \in R$ such that

$$J = (f_1, \ldots, f_n) = \left\{ \sum_{i=1}^n g_i f_i : g_i \in R \right\}.$$
If \( n \) can be chosen to be one, then \( J \) is a principal ideal. A ring \( R \) is said to be coherent if the intersection of two finitely generated ideals is finitely generated. Using the result in [5] that \( H^\infty \) is a coherent ring, it is shown in [7, Satz 7] that this is true of all rings of Nevanlinna–Smirnov type. In particular, we have the following

**Theorem 4.** \( N^p \) is a coherent ring for all \( p > 1 \).

### 3. The Corona Property

We say that a commutative ring \( R \) with unit of holomorphic functions on \( D \) has the **Corona Property** if the ideal generated by \( f_1, \ldots, f_n \in R \) is equal to \( R \) if and only if there is an invertible element \( f \) of \( R \) such that

\[
|f(z)| \leq \sum_{i=1}^{n} |f_i(z)| \quad (z \in D).
\]

This definition is motivated by the famous Corona Theorem of Carleson (for example see [3, p. 324], or [1, p. 202]), which states that the algebra \( H^\infty \) of all bounded holomorphic functions on \( D \) has the Corona Property. Mortini noted [7, Satz 4] that by a result of Wolff [3, p. 329], it is easy to show that every ring of Nevanlinna–Smirnov type has the Corona Property. Hence, we have the following theorem.

**Theorem 5.** The algebra \( N^p \) has the Corona Property.

**Remark.** It is proved in [4, Theorem 7] that there exists a subalgebra of \( N \) containing \( N^+ \) without the Corona Property.

A sequence \( \{z_k\} \subset D \) is called an **interpolating sequence** (for \( H^\infty \)) if for every bounded sequence \( \{\omega_k\} \) of complex numbers there exists a function \( f \) in \( H^\infty \) such that \( f(z_k) = \omega_k \) for every \( k \). An **interpolating Blaschke product** is a Blaschke product whose (simple) zeros form an interpolating sequence.

The following two theorems generalize Theorems 5 and 6 in [7], respectively. We follow [7] for the proofs of theorems below.

**Theorem 6.** Let \( 0 < q < \infty \) and let \( I = (f_1, \ldots, f_n) \) be a finitely generated ideal in \( H^\infty \). Assume that the ideal \( I \) contains a zero-free holomorphic function \( F \) on \( D \) such that \( \log F \in H^q \). Then

\[
\sum_{k=1}^{\infty} (1 - |z_k|^2) \left| \log (|f_1(z_k)| + \cdots + |f_n(z_k)|) \right|^q < \infty.
\]
Proof. Let \( g_1, \ldots, g_n \) be functions in \( H^\infty \) such that \( F = \sum_{k=1}^n f_k g_k \). Then there exists a positive constant \( C_1 \) such that

\[
|F(z)| \leq C_1 \sum_{k=1}^n |f_k(z)| \quad \text{for all} \quad z \in D.
\]

Put \( S(z) = \sum_{k=1}^n |f_k(z)| \), and suppose that \( S(z) \leq C_2 \) for all \( z \in D \), with a positive constant \( C_2 \). Using the fact that \( \sum_{k=1}^\infty (1 - |z_k|) = C_3 < \infty \), applying the inequality \((a + b)^q \leq C_4 (a^q + b^q)\) with \( C_4 = 2^{\max(q,1) - 1} \), \( a, b \geq 0 \), and the main interpolation theorem for the class \( H^q \) [1, p. 149], we obtain

\[
\sum_{k=1}^\infty (1 - |z_k|^2) \left| \log S(z_k) \right|^q
\]

\[
= \sum_{k=1}^\infty (1 - |z_k|^2) \left( \log^+ S(z_k) + \log^+ \frac{1}{S(z_k)} \right)^q
\]

\[
\leq \sum_{k=1}^\infty (1 - |z_k|^2) C_4 \left( \left( \log^+ S(z_k) \right)^q + \left( \log^+ \frac{1}{S(z_k)} \right)^q \right)
\]

\[
\leq C_4 (C_2)^q \sum_{k=1}^\infty (1 - |z_k|^2) + C_4 \sum_{k=1}^\infty (1 - |z_k|^2) \left( \log^+ \frac{C_1}{|F(z_k)|} \right)^q
\]

\[
\leq 2C_3 C_4 (C_2)^q + C_4 \sum_{k=1}^\infty (1 - |z_k|^2) \left| \log \frac{F(z_k)}{C_1} \right|^q < \infty.
\]

This gives the desired result. \( \Box \)

**Theorem 7.** Assume that \( I \) is an ideal in \( N^p \) generated by inner functions \( \varphi_1, \ldots, \varphi_n \), and suppose that \( I \) contains an interpolating Blaschke product \( B \) with zeros \( \{z_k\} \) such that

\[
\sum_{k=1}^\infty (1 - |z_k|^2) \left| \log (|\varphi_1(z_k)| + \cdots + |\varphi_n(z_k)|) \right|^p < \infty.
\]

Then \( I = N^p \).

**Proof.** Put \( c_k = \sum_{i=1}^n |\varphi_i(z_k)|^2 \) for all \( k \). Using the inequalities

\[
\left( \sum_{i=1}^n |\varphi_i(z_k)| \right)^2 \leq c_k \leq \left( \sum_{i=1}^n |\varphi_i(z_k)| \right)^2.
\]
it is routine to estimate that
\[ |\log c_k| \leq 2 \left| \log \left( \sum_{i=1}^{n} |\varphi_i(z_k)| \right) \right| + \log n, \]
whence by the inequality \((a + b)^p \leq 2^{p-1}(a^p + b^p), \ a, b \geq 0\), using the assumption of the theorem, we have
\[
\sum_{k=1}^{\infty} (1 - |z_k|^2) |\log c_k|^p \\
\leq 2^{p-1} \sum_{k=1}^{\infty} (1 - |z_k|^2) \left| \log \left| \sum_{i=1}^{n} |\varphi_i(z_k)| \right| \right|^p \\
+ 2^{p-1} \log^p n \sum_{k=1}^{\infty} (1 - |z_k|^2) < \infty.
\]
Hence, by the theorem of Shapiro and Shields [1, p. 149, Theorem 9.1], there exists a function \(g \in H^p\) with \(g(z_k) = \log c_k\) for every \(k\). It is easy to verify that the function \(F = \exp g\) is invertible in \(N^p\), and there holds \(F(z_k) = c_k\).

The rest of the proof is the same as that of [7, Satz 6]. To complete the proof, we write this part.

Since \(\{z_k\}\) is an interpolating sequence, by [1, p. 149], we know that there exist functions \(f_i \in H^\infty\) \((i = 1, \ldots, n)\) such that for every \(i = 1, \ldots, n\) there holds
\[
f_i(z_k) = \overline{\varphi_i(z_k)} \quad (k = 1, 2, \ldots)
\]
Observe that the function \(F - \sum_{i=1}^{n} f_i \varphi_i\) is in \(N^p\), and that there holds
\[
F(z_k) - \sum_{i=1}^{n} f_i(z_k) \varphi_i(z_k) = c_k - \sum_{i=1}^{n} |\varphi_i(z_k)|^2 = 0 \quad (k = 1, 2, \ldots)
\]
Hence, by Theorem A, there exists a function \(h \in N^p\) such that
\[
F - \sum_{i=1}^{n} f_i \varphi_i = Bh.
\]
This shows that \(F\) belongs to the ideal \((\varphi_1, \ldots, \varphi_n, B) = I\). Since \(F\) is an invertible element in \(N^p\), it follows that
\[
I = (\varphi_1, \ldots, \varphi_n) = N^p.
\]
This completes the proof of the theorem. \(\square\)
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