What can Koopmanism do for attractors in dynamical systems?

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Abstract
We characterize the longterm behavior of a semiflow on a compact space $K$ by asymptotic properties of the corresponding Koopman semigroup. In particular, we compare different concepts of attractors, such as asymptotically stable attractors, Milnor attractors and centers of attraction. Furthermore, we give a characterization for the minimal attractor for each mentioned property. The main aspect is that we only need techniques and results for linear operator semigroups, since the Koopman semigroup permits a global linearization for a possibly non-linear semiflow.

Keywords Attractors · Dynamical systems · Koopman semigroups

Mathematics Subject Classification 37B25 · 47D03

1 Introduction

The concept of an attractor of a dynamical system has been of great interest in the last 50 years. After the term first occurred in 1964 in [1, p.55], cf. [17, p.177], many variations and modifications have been defined, each of them yielding different examples. The survey article “On the Concept of Attractor” from 1985 by Milnor, cf. [17], treats the history of its many definitions and even adds an additional concept. He justifies this by describing some of the stability properties in the

To Prof. S. H. Kulkarni on the occasion of his 65th birthday.

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previous definitions as “awkward”, [17, p.178]. In his opinion prior definitions seem to be too restrictive omitting some interesting examples, [17, p.178].

The scope of this article is to establish a systematic hierarchy of the attractors mentioned in [17] and, additionally, treat so-called minimal centers of attraction. We will do so by “translating” these concepts into operator theoretic terms by linearizing the non-linear dynamical system globally. We call this process “Koopmanism”. Indeed, this idea first appeared around 1930 in the papers by Koopman [13] and by von Neumann [19] and provided the precise mathematical terms to treat the so-called ergodic hypothesis from Boltzmann formulated in [5]. It is based on the distinction between the state space $K$ of a (physical) system and the associated observable space $\mathcal{O}$ being a (vector) space of real or complex valued functions on $K$. If the non-linear semiflow

$$\varphi_t : K \to K, \ t \in \mathbb{R}_+,$$

describes the dynamics on the state space, the maps

$$f \mapsto T(t)f := f \circ \varphi_t, \ f \in \mathcal{O},$$

become linear operators and, if $\mathcal{O}$ remains invariant, $(T(t))_{t \geq 0}$ is a one-parameter semigroup of linear operators on $\mathcal{O}$.

This idea led to the proof of the classical ergodic theorems of von Neumann [22] and Birkhoff [3] and even gave rise to ergodic theory as a mathematical discipline.

The recent state of the art of this operator theoretic approach to ergodic theory is presented in the monograph [8]. In this paper we show how this can be used to classify and discuss attractors in topological dynamical systems.

For this purpose we consider a pair $(K,(\varphi_t)_{t \geq 0})$ consisting of a compact topological Hausdorff space $K$ and a continuous semiflow $(\varphi_t)_{t \geq 0}$ on $K$, cf. [18, Def.3.1]. On the Banach space $C(K)$ of all real-valued continuous functions on $K$ we obtain the corresponding $C_0$-semigroup $(T(t))_{t \geq 0}$, called Koopman semigroup, given by the operators

$$T(t)f := f \circ \varphi_t \text{ for } f \in C(K), \ t \geq 0.$$ 

An attractor is a closed and $(\varphi_t)_{t \geq 0}$-invariant subset $\emptyset \neq M \subseteq K$ possessing a certain asymptotic property. See [17] for a survey of various concepts. In our perspective, every such subset of $K$ corresponds to the closed and $(T(t))_{t \geq 0}$-invariant ideal $I_M := \{ f \in C(K) \mid f|_M \equiv 0 \}$ in the Banach algebra $C(K)$. Essential to this matter is that all closed ideals in $C(K)$ are of the form $I_M$ where $M$ is a closed subset of $K$ (cf. [8, Theorem 4.8]).

$$M \subseteq K \text{ closed subset } \iff I_M \subseteq C(K) \text{ closed ideal}$$

$$M(\varphi_t)_{t \geq 0} \text{-invariant } \iff I_M (T(t))_{t \geq 0} \text{-invariant}.$$ 

Given an attractor $M$, our idea is to restrict the Koopman semigroup to the corresponding closed ideal $I_M$ and characterize the long-term behavior of $(\varphi_t)_{t \geq 0}$ around $M$ by asymptotic properties of $(T(t))_{t \geq 0}$ restricted to $I_M$. So ur Leitmotiv can be visualized as
"φ_t \to M" ⇔ "T(t)|_{I_M} \to 0".

The idea to characterize attractivity properties of invariant sets of a flow by stability properties of the associated Koopman operators restricted to functions vanishing on the attractor is due to Mauroy and Mezić, see [16, II.Prop.1]. Their stability corresponds to what we call weak stability later in this article.

In Sect. 1 we overview the stability theory for strongly continuous operator semigroups with more details on almost weak stability. In the following sections we then apply this theory to attractors in dynamical systems. In Sect. 2 we characterize absorbing sets and in Sect. 3 treat well-known attractivity and stability properties of dynamical systems by “translating” them into stability properties of the restricted Koopman semigroup. We then, in Sect. 4, prove the existence of minimal attractors and characterize these for each possible asymptotic behavior.

Now we recall some basic facts and fix the notation. Let \( K \) be a compact Hausdorff space. A family of self-mappings \( (φ_t)_{t \geq 0} \) on \( K \) is called semiflow if \( φ_0 \equiv id_K \) and \( φ_{t+s} = φ_t \circ φ_s \) for all \( s, t \geq 0 \). We call \( (φ_t)_{t \geq 0} \) a continuous semiflow if
\[
[0, \infty) \times K \to K,
(t, x) \mapsto φ_t(x)
\]
is continuous with respect to the product topology. Thus, a dynamical system is a pair \((K, (φ_t)_{t \geq 0})\) consisting of a compact Hausdorff space \( K \) and a continuous semiflow \( (φ_t)_{t \geq 0} \). We also call \( K \) the state space and the elements \( x \in K \) states. The induced Koopman semigroup \((T(t))_{t \geq 0} \) on \((C(K), \|\cdot\|_∞)\), as defined above, is strongly continuous if and only if \((φ_t)_{t \geq 0}\) is continuous, cf. [18, B-II, Lem. 3.2].

We recall that \((C(K), \|\cdot\|_∞)\) is a \( C^∞ \)-algebra and a Banach lattice for the usual pointwise operations. Given a function \( f : K \to \mathbb{R} \) and \( a \in \mathbb{R} \) we use the notation
\[
[f < a] := f^{-1}((-∞, a)), \quad [f \leq a] := f^{-1}((-∞, a]), \quad [f = a] := f^{-1}\{a\}
\]
and, analogously, \([f > a]\) and \([f \geq a]\). The sets \([|f| > 0], f \in C(K)\), form a basis for the topology on \( K \) since \( K \) is completely regular, [8, Appendix A.2] and [8, Proof of Lem. 4.12]. This is equivalent to the fact that the zero sets \([f = 0], f \in C(K)\), form a basis of the closed subsets of \( K \) or that the topology on \( K \) coincides with the initial topology induced by \( C(K) \). Combining these facts, given a closed subset \( M \subseteq K \) and \( U \) an open neighborhood of \( M \) there exists \( f \in C(K) \) with \( M \subseteq [f = 0] \) and \( \varepsilon > 0 \) such that \([|f| < \varepsilon] \subseteq U\), i.e. the sets of the form \( U_{ε,f} := [|f| < \varepsilon], f \in C(K), f(M) = 0 \) and \( \varepsilon > 0 \) form a basis for the system of neighborhoods of \( M \) which we will denote by \( U(M) \).

As already mentioned above, for every closed ideal \( I \subseteq C(K) \) there exists a closed subset \( M \subseteq K \) such that
\[
I = I_M = \{ f \in C(K) \mid f|_M \equiv 0 \} \quad \text{and} \quad M = \bigcap_{f \in I} [f = 0],
\]
see [8, Thm. 4.8]. A closed subset \( M \subseteq K \) is said to be \((φ_t)_{t \geq 0}\)-invariant if \( φ_t(M) \subseteq M \) for all \( t \geq 0 \). Remark that a subset \( M \) is \((φ_t)_{t \geq 0}\) invariant if and only if the corresponding ideal \( I_M \) is \((T(t))_{t \geq 0}\)-invariant, [8, Lem. 4.18]. Furthermore, for \( M \subseteq
$K$ closed, $I_{M} \cong C_{0}(K \setminus M)$ by $f \mapsto f|_{K \setminus M}$, where $C_{0}(K \setminus M)$ is the space of all real-valued continuous functions on $K \setminus M$ that vanish at infinity, cf., [20, Sect. 1.7.6]. By the Riesz’ representation theorem (cf., [8, Thm. 5.7 & Rem. 5.8]) we identify the dual spaces $C(K)'$ and $I_{M} = C_{0}(K \setminus M)'$ of $C(K)$ and $I_{M}$ with the finite regular Borel measures on $K$ and $K \setminus M$ respectively. For a subset $M \subseteq K$ we either write $M^{c}$ or $K \setminus M$ for its complement in $K$.

2 Stability of $C_{0}$-semigroups

In this section we recall various stability properties of $C_{0}$-semigroups on a Banach space $X$. The concept of almost weak stability is treated in more detail.

**Definition 1.1** Let $(T(t))_{t \geq 0}$ be a strongly continuous semigroup on a Banach space $X$. Then $(T(t))_{t \geq 0}$ is said to be

(a) **nilpotent** if there exists $t_{0} > 0$ such that

$$\|T(t_{0})\| = 0,$$

(b) **uniformly exponentially stable** if there exists $\delta > 0$ such that

$$\lim_{t \to \infty} e^{\delta t}\|T(t)\| = 0,$$

(c) **uniformly stable** if

$$\lim_{t \to \infty} \|T(t)\| = 0,$$

(d) **strongly stable** if

$$\lim_{t \to \infty} \|T(t)x\| = 0 \text{ for all } x \in X,$$

(e) **weakly stable** if

$$\lim_{t \to \infty} \langle T(t)x, x' \rangle = 0 \text{ for all } x \in X, \ x' \in X' \text{ and }$$

(f) **almost weakly stable** if for all pairs $(x, x') \in X \times X'$ there exists a subset $R \subseteq \mathbb{R}_{+}$ with density\footnote{The density of a subset $R \subseteq \mathbb{R}_{+}$ is $d(R) := \lim_{t \to \infty} \frac{1}{t} \cdot \lambda([0, t] \cap R)$, $\lambda$ Lebesgue measure, if the limit exists.} 1 such that

$$\lim_{t \to \infty; t \in R} \langle T(t)x, x' \rangle = 0.$$

In the above definition the following chain of implications holds
All implications are strict except b) \(\iff\) c) which can be found in [9, Chapter V, Section 1]. For examples we refer to [7, Chapter III] and [9, Chapter V, Section 1].

For our later study of stability of Koopman semigroups on \(C(K)\)-spaces we need an additional definition.

**Definition 1.2** Let \((K,(\varphi_t)_{t \geq 0})\) be a dynamical system and \(\mu\) a quasi invariant regular Borel measure\(^2\) on \(K\), \((T(t))_{t \geq 0}\) the corresponding Koopman semigroup on \(C(K)\). Then \((T(t))_{t \geq 0}\) is said to be **almost everywhere pointwise stable** (for \(\mu\)) if for every \(f \in C(K)\)

\[ T(t)f(x) \to 0 \text{ as } t \to \infty \text{ for } \mu\text{-almost all } x \in K. \]

We also recall that the **growth bound** \(\omega_0\) of a strongly continuous semigroup \((T(t))_{t \geq 0}\) on a Banach space \(X\) is

\[ \omega_0 := \inf \{ \omega \in \mathbb{R} \mid \|T(t)\| \leq Me^{\omega t} \text{ for suitable } M > 0 \text{ and all } t \geq 0 \}. \]

### 2.1 Almost weak stability

For a complete treatment of almost weak stability for \(C_0\)-semigroups on Banach spaces with relatively weakly compact orbits we refer to [7, Chapter III, Section 5] and, for a time-discrete variant to [8, Chapter 9]. The tools and ideas used in this subsection are based on [8, Chapter 9].

The following proposition can already be found in a more general setting in [10, Thm. 2.2].

**Proposition 1.3** Let \((T(t))_{t \geq 0}\) be a bounded \(C_0\)-semigroup on some Banach space \(X\). Then the following are equivalent

(a) \((T(t))_{t \geq 0}\) is almost weakly stable,
(b) \[
\lim_{T \to \infty} \frac{1}{T} \int_0^T |\langle T(t)x, x' \rangle| \, dt = 0
\]

for all \(x \in X, x' \in X'\),
(c) \[
\lim_{T \to \infty} \sup_{x' \in X', \|x'\| \leq 1} \frac{1}{T} \int_0^T |\langle T(t)x, x' \rangle| \, dt = 0
\]

for all \(x \in X\).

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\(^2\) A measure \(\mu\) on \(K\) is called **quasi invariant** (with respect to \((\varphi_t)_{t \geq 0}\)) if \(\mu(\varphi_t^{-1}(N)) = 0\) for all \(t \geq 0\) if and only if \(\mu(N) = 0\).
The equivalence (a) \(\iff\) (b) follows from the so-called Koopman-von Neumann Lemma, see for example [7, Chapter III, Lemma 5.2]. The implication (b) \(\implies\) (c) in the time discrete analogue is due to Jones and Lin, cf. [12]. We adapt the proof given in [8, Prop. 9.17]. Every operator \(T(t)\) as its adjoint \(T(t)'\) is a contraction and the dual unit ball \(B'\) is compact with respect to the weak-* topology. Due to these facts we can define the Koopman system

\[
(C(B'), (\tilde{T}(t))_{t \geq 0})
\]

where

\[
\tilde{T}(t)f(x') := f(T(t)'x')
\]

for \(t \geq 0, f \in C(B'), x' \in B'.\) Fix \(x \in X\) and define \(g_x \in C(B')\) by \(g_x(x') := |\langle x, x' \rangle|\). By (b)

\[
\lim_{T \to \infty} \frac{1}{T} \int_0^T \tilde{T}(t)g_x(x') \, dt = 0
\]

pointwise in \(x'\) and by Lebesgue’s theorem of dominated convergence also weakly and thus in the norm of \(C(B')\), cf. [7, Chapter I, Theorem 2.25] or [8, Proposition 8.18].

**Remark 1.4** Let \((T(t))_{t \geq 0}\) be a lattice semigroup on \(C(K)\), i.e., \(T(t)|f| = |T(t)f|\) for all \(t \geq 0, f \in C(K)\). Then \((T(t))_{t \geq 0}\) is almost weakly stable if and only if

\[
\lim_{T \to \infty} \frac{1}{T} \int_0^T T(t)|f|(x) \, dt = 0
\]

for all \(x \in K\) and \(f \in C(K)\).

**Proof** This is a direct consequence of Lebesgue’s theorem of dominated convergence. The implication “\(\Rightarrow\)” follows from Proposition 1.3. For the other implication let \((t_n)_{n \in \mathbb{N}}\) be a sequence in \([0, \infty)\) with \(t_n \xrightarrow{n \to \infty} f \in C(K)\) and \(x \in K\). Remark that

\[
\frac{1}{t_n} \int_0^{t_n} |T(t)f(x)| \, dt = \frac{1}{t_n} \int_0^{t_n} T(t)|f|(x) \, dt = \left\langle \frac{1}{t_n} \int_0^{t_n} T(t)|f| \, dt, \delta_x \right\rangle.
\]

Since \(|T(t)f(x)| \leq \|f\|_\infty \mathbb{1}_K(x)\) for all \(t \in [0, \infty), x \in K\), Lebesgue’s theorem of dominated convergence applies and thus for \(\mu \in C(K)'\)
\[
\frac{1}{t_n} \int_0^{t_n} \|T(t)f, \mu\| \, dt \leq \frac{1}{t_n} \int_0^{t_n} \|T(t)\| \, dt
\]

\[
= \lim_{n \to \infty} \frac{1}{t_n} \int_0^{t_n} \|T(t)\| \, dt, \quad t \to \infty
\]

This implies the assertion. \(\square\)

**Remark 1.5** Let \((T(t))_{t \geq 0}\) be a bounded \(C_0\)-semigroup on some Banach space \(X\). The subset

\[
I_{\text{aws}} := \{ x \in X \mid \lim_{T \to \infty} \sup_{x' \in X', \|x'\| \leq 1} \frac{1}{T} \int_0^T |\langle T(t)x, x' \rangle| \, dt = 0 \}
\]

is a closed, \((T(t))_{t \geq 0}\)-invariant subspace of \(X\).

**Proof** Let \((x_n)_{n \in \mathbb{N}}\) be a convergent sequence in \(I_{\text{aws}}\) with limit \(x \in X\) and take \(\varepsilon > 0\). Then there exists \(n \in \mathbb{N}\) such that \(\|x_n - x\| < \frac{\varepsilon}{2}\). By Proposition 1.3 (c) there exists \(t(n) \geq 0\) such that

\[
\sup_{x' \in X', \|x'\| \leq 1} \frac{1}{T} \int_0^T |\langle T(t)x_n, x' \rangle| \, dt < \frac{\varepsilon}{2}
\]

for all \(T > t(n)\). This implies

\[
\sup_{x' \in X', \|x'\| \leq 1} \frac{1}{T} \int_0^T |\langle T(t)x, x' \rangle| \, dt
\]

\[
\leq \sup_{x' \in X', \|x'\| \leq 1} \frac{1}{T} \int_0^T |\langle T(t)(x_n - x), x' \rangle| \, dt + \sup_{x' \in X', \|x'\| \leq 1} \frac{1}{T} \int_0^T |\langle T(t)x_n, x' \rangle| \, dt
\]

\[
\leq \|x - x_n\| + \sup_{x' \in X', \|x'\| \leq 1} \frac{1}{T} \int_0^T |\langle T(t)x_n, x' \rangle| \, dt < \varepsilon \quad \text{for all } T \geq \max\{n, t(n)\}.
\]

\(\square\)

**Remark 1.6** Let \((T(t))_{t \geq 0}\) be a bounded \(C_0\)-semigroup on some Banach space \(X\). In analogy to the previous remark, we define

\[
I_{\text{ss}} := \{ x \in X \mid \|T(t)x\| \to 0 \text{ as } t \to \infty \} \quad \text{and}
\]

\[
I_{\text{ws}} := \{ x \in X \mid \langle T(t)x, x' \rangle \to 0 \text{ as } t \to \infty \text{ for all } x' \in X' \}.
\]

Both are clearly closed subspaces of \(X\).
Remark 1.7 Let \((K, (\varphi_t)_{t \geq 0})\) be a dynamical system, \(\mu\) a quasi invariant regular Borel measure on \(K\) and \((T(t))_{t \geq 0}\) the corresponding Koopman semigroup on \(C(K)\). We define a closed subspace of \(C(K)\) as
\[
I_{\text{aes}} := \{ f \in C(K) \mid T(t)f(x) \to 0 \text{ as } t \to \infty \text{ for } \mu\text{-almost all } x \in K \}.
\]

Proposition 1.8 Let \((T(t))_{t \geq 0}\) be a bounded \(C_0\)-semigroup on a Banach space \(X\). If for all \(x \in X\) there exists a sequence \((t_n)_{n \in \mathbb{N}}\) in \([0, \infty)\) with \(t_n \to \infty\) as \(n \to \infty\) such that for all \(x' \in X'\)
\[
\lim_{n \to \infty} \langle T(t_n)x, x' \rangle = 0,
\]
then \((T(t))_{t \geq 0}\) is almost weakly stable.

Proof Take \(x \in X\) and \((t_n)_{n \in \mathbb{N}}, t_n \to \infty\) such that
\[
\lim_{n \to \infty} \langle T(t_n)x, x' \rangle = 0
\]
for all \(x' \in X'\). As in the proof of Proposition 1.3 we consider the induced Koopman system \((C(B'), (\tilde{T}(t))_{t \geq 0})\) and the function
\[
g_x(x') := |\langle x, x' \rangle|.
\]
If \(\mu \in C(B')\) vanishes on \(\bigcup_{t \geq 0} (\text{rg}(\text{Id} - \tilde{T}(t)))\), then
\[
\langle g_x, \mu \rangle = \langle \tilde{T}(t_n)g_x, \mu \rangle
\]
for all \(n \in \mathbb{N}\). We observe that
\[
\langle \tilde{T}(t_n)g_x, \mu \rangle = \int_{B'} \tilde{T}(t_n)g_x(x') \, d\mu(x')
= \int_{B'} |\langle x, T(t_n)x' \rangle| \, d\mu(x')
= \int_{B'} |\langle T(t_n)x, x' \rangle| \, d\mu(x').
\]
The functions \(|\langle T(t_n)x, x' \rangle|\) converge to 0 pointwise in \(x'\) by assumption and by Lebesgue’s Theorem the integral \(\int_{B'} |\langle T(t_n)x, x' \rangle| \, d\mu(x')\) goes to 0 as well implying \(\langle g_x, \mu \rangle = 0\). This implies \(g_x \in \overline{\text{lin}} \bigcup_{t \geq 0} (\text{rg}(\text{Id} - \tilde{T}(t)))\) by the theorem of Hahn-Banach. Thus,
\[
\frac{1}{T} \int_0^T |\langle T(t)x, x' \rangle| \, dt
= \frac{1}{T} \int_0^T (\tilde{T}(t)g_x)(x') \, dt \xrightarrow{T \to \infty} 0
\]
for all $x' \in X'$, cf., [9, Chapt. V, Sect. 4] and in particular [9, Chapt. V, Lem. 4.4].
Since $x$ was arbitrary, $(T(t))_{t \geq 0}$ is almost weakly stable. 

\textbf{Remark 1.9} For bounded $C_0$-semigroups with relatively weakly compact orbits the assertions in Proposition 1.8 are equivalent, see [7, Chapter III, Section 5]. Here we are able to prove one implication without assuming relatively weakly compact orbits. It still remains open whether the opposite implication also holds true in this case.

From now on $(K, (\varphi_t)_{t \geq 0})$ is a topological dynamical system with compact state space $K$ if not otherwise stated and $(T(t))_{t \geq 0}$ denotes the induced Koopman semigroup on $C(K)$.

\section{Absorbing sets}

The following section is dedicated to \textit{absorbing sets}, these are compact invariant subsets of the state space that eventually contain every initial state. We follow the definition in [6, Def. 2.1.1] and differ between two types of such sets.

\textbf{Definition 2.1} A closed invariant set $M \subseteq K$ is called
(a) \textit{absorbing} if there exists $t_0 > 0$ such that 
\[ \varphi_{t_0}(K) \subseteq M, \]
(b) \textit{pointwise absorbing} if for all $x \in K$ there exists $t_0 > 0$ such that \[ \varphi_{t_0}(x) \in M. \]

This gives rise to the notion of \textit{dissipative systems}, cf. [6, Def. 2.1.1].

\textbf{Definition 2.2} A dynamical system $(K, (\varphi_t)_{t \geq 0})$ is called (point) \textit{dissipative} if it contains a (point) absorbing set.

\textbf{Proposition 2.3} Let $M \subseteq K$ be a closed invariant set and $(S(t))_{t \geq 0}$ the restricted Koopman semigroup, i.e. $S(t) := T(t)|_{B_M}$ for $t \geq 0$. Then all the assertions in (I) and all the assertions in (II) are equivalent.

(I)
(a) $(S(t))_{t \geq 0}$ is nilpotent.
(b) $(S(t))_{t \geq 0}$ is uniformly stable.
(c) $\omega_0 = -\infty$.
(d) $M$ is absorbing.

(II)
(a) For all Dirac measures $\delta_x \in C(K)^\prime$ there exists $t_0 > 0$ such that
\[ S(t_0)^\prime \delta_x = 0. \]

(b) $M$ is pointwise absorbing.

**Proof** We begin with the proof of (I). Clearly, (a)$\Rightarrow$(b). For the implication (b)$\Rightarrow$d) assume $M$ not to be absorbing and fix $t_0 > 0$, thus there is $x_0 \in K \setminus M$ with $\varphi_{t_0}(x_0) \in K \setminus M$. Since $K$ is completely regular there exists $f \in I_M$ with $\|f\| = 1$ and $f(\varphi_{t_0}(x_0)) = 1$. Therefore,
\[ \|S(t_0)\| \geq \|S(t_0)f\| \geq S(t_0)f(x_0) = 1. \]
Since $t_0$ was arbitrary $\|S(t)\| = 1$ for all $t \geq 0$ which contradicts (b). The implication (d)$\Rightarrow$(a) can be seen as follows. Let $t_0 > 0$ be such that $\varphi_{t_0}(K) \subseteq M$, thus $S(t_0)f(x) = f(\varphi_{t_0}(x)) = 0$ for every $f \in I_M$ and $x \in K$. This implies $\|S(t_0)\| = \sup_{\|f\| \leq 1} \|S(t_0)f\| = 0$. Additionally, clearly (a) implies (c) and (c) implies (b).

Proof of (II): These equivalences are quite clear since (a) implies that for all $x \in K$ there exists $t_0 > 0$ such that
\[ \varphi_{t_0}(x) \in \bigcap_{f \in I_M} [f = 0] = M. \]

Next we give a condition under which the two concepts coincide. We recall that a compact space $K$ is a Baire space, thus for a sequence of closed subsets $K_n$, $n \in \mathbb{N}$, with
\[ K = \bigcup_{n \in \mathbb{N}} K_n \]
there exists $n \in \mathbb{N}$ such that $K_n$ has non-empty interior.

**Remark 2.4** Let $M \subseteq K$ be closed and invariant. If $M$ is pointwise absorbing the sets $\varphi_n^{-1}(M) =: K_n$ form a closed cover of $K$.

**Proposition 2.5** Let $M \subseteq K$ be closed and invariant and $K_n$ as defined in Remark 2.4. The set $M$ is absorbing if and only if it is pointwise absorbing and $M \subseteq K_n$ for some $n \in \mathbb{N}$.

**Proof** Clearly, if $M$ is absorbing it is pointwise absorbing and there exists $n \in \mathbb{N}$ such that $M \subseteq K = \overset{\circ}{K_n}$ in above construction. For the other implication consider the following. By assumption for every $x \in K$ there exists $t_x \geq 0$ such that
\[ \varphi_{t_x}(x) \in \overset{\circ}{K_n} \subset K_n. \]
By continuity $\varphi^{-1}_{t_0}(\overset{\circ}{K_n})$ is open for every $x \in K$ and

$$K \subseteq \bigcup_{x \in K} \varphi^{-1}_{t_0}(\overset{\circ}{K_n}).$$

Since $K$ is compact there exist finitely many $x_1, \ldots, x_j$ for some $j \in \mathbb{N}$ such that

$$K \subseteq \bigcup_{k=1}^j \varphi^{-1}_{t_k}(\overset{\circ}{K_n}).$$

This implies for $y \in K$ that

$$\varphi_{t_k}(y) \in \overset{\circ}{K_n} \subset K_n$$

for some $k \in \{1, \ldots, j\}$ and therefore

$$\varphi_{t_k+n}(y) \in M.$$

Define $T := \max\{t_{x_k} \mid k \in \{1, \ldots, j\}\}$, then

$$\varphi_{T+n}(y) \in M$$

by invariance of $M$. 

\[\square\]

## 4 Asymptotics of dynamical systems

In this section we consider asymptotic properties of semiflows around closed invariant sets and give an operator theoretic characterization of each such property. We also discuss relations between them.

**Definition 3.1** A closed invariant set $\emptyset \neq M \subseteq K$ is called

(a) **uniformly attractive** if for all $U \in \mathcal{U}(M)$ there exists $t_0 > 0$ such that

$$\varphi_t(K) \subseteq U \text{ for all } t \geq t_0,$$

(b) **(pointwise ) attractive** if for all $x \in K$ and $U \in \mathcal{U}(M)$ there exists $t_0 > 0$ such that

$$\varphi_t(x) \in U \text{ for all } t \geq t_0,$$

(c) **likely limit set (or Milnor attractor) for $\mu$, where $\mu$ is a quasi invariant Borel measure on $K$, if for all $U \in \mathcal{U}(M)$ and $\mu$-almost every $x \in K$ there exists $t_0 > 0$ with

$$\varphi_t(x) \in U \text{ for all } t \geq t_0,$$

(d) **center of attraction** if for all $U \in \mathcal{U}(M)$
\[
\lim_{t \to \infty} \frac{1}{t} \lambda(\{s \in [0, t] \mid \varphi_s(x) \in U\}) = 1
\]
for all \(x \in K\), where \(\lambda\) denotes the Lebesgue measure on \([0, \infty)\),

(e) stable in the sense of Lyapunov if for all \(U \in \mathcal{U}(M)\) there exists \(V \in \mathcal{U}(M)\), \(V \subseteq U\) such that

\[
\varphi_t(V) \subseteq U \text{ for all } t \geq 0.
\]

The concepts (a), (b) and (e) in Definition 3.1 have been established by Lyapunov in his dissertation ([15]) in 1892 and have since been broadly applied and investigated for dynamics on locally compact metric spaces, cf., [2, Chapt. II]. The property d) in Definition 3.1 appears in Birkhoff’s monograph “Dynamical Systems” [4, Chapt. VII] as “central motion” and has been further investigated by Hilmy [11], see for example, Sigmund [21] and by Kreidler [14, Sect. 4] to name a few. Definition (c) for semiflows on smooth compact manifolds is due to Milnor and can be found in [17, Section 2].

Remark 3.2 If \((K, (\varphi_t)_{t \geq 0})\) is a dynamical system with metric \(K\) then there exists one \(\mu\)-null set satisfying the assumptions in Definition 3.1 (c) that does not depend on \(U \in \mathcal{U}(M)\) since there is a countable neighborhood basis and the countable union of null sets is again a null set.

Remark 3.3 For the concepts defined in Definition 3.1 the following implications hold.

\[
\begin{array}{cccc}
  a) & \implies & b) & \implies & c) \\
  \Downarrow & & \Downarrow & & \Downarrow \\
  e) & & d) \\
\end{array}
\quad \text{and} \quad
\begin{array}{cccc}
  a) & \iff & b) \quad \text{and} \quad e) \\
\end{array}
\]

The opposite implications do not hold true in general which can be seen in the following examples. The equivalence of Definition 3.1 (a) \(\iff\) (b) + (e) will be proven in below Proposition 3.8 and Proposition 3.9. We quickly remark that these invariant subsets of dynamical systems that are attractive and stable in the sense of Lyapunov or equivalently uniformly attractive are often referred to as “asymptotically stable” in standard literature.

Example 3.4

(a) Consider \(K := \mathbb{R} \cup \{\infty\}\) the one-point compactification of \(\mathbb{R}\) and the semiflow \((\varphi_t)_{t \geq 0}\) defined by

\[
\varphi_t(x) := \begin{cases} 
  x + t & x \in \mathbb{R} \\
  \infty & x = \infty.
\end{cases}
\]

Then \(M := \{\infty\}\) is attractive but not uniformly attractive.
(b) Take $K := [0, \infty]$ the one-point compactification of $[0, \infty)$ and the semiflow $(\phi_t)_{t \geq 0}$ on $K$, with

$$\phi_t(x) := \begin{cases} e^{-t}x & x \in [0, \infty) \\ \infty & x = \infty \end{cases}.$$ 

Consider the standard Gaussian measure $\gamma$ on $[0, \infty]$ which is a regular Borel measure on $K$ that is quasi-invariant with respect to $(\phi_t)_{t \geq 0}$ since it is equivalent to the Lebesgue measure $\lambda$. In particular, $\gamma(\{\infty\}) = 0$. Then $M := \{0\}$ is a likely limit set for $\gamma$ since $\gamma([0, \infty)) = 1$ and $\phi_t(x) \to 0$ for all $x \in [0, \infty)$ but it is neither attractive nor a center of attraction since $\phi_t(\infty) = \infty$ for all $t \geq 0$.

(c) In [11, p.287], Hilmy gave a concrete example for a center of attraction that is not attractive. We give a simplified version of this example. Take the following differential equation

$$\begin{cases} \dot{r} = -r \log(r) \left( (1 - r)^2 + \sin^2(\theta) \right) \\ \dot{\theta} = (1 - r)^2 + \sin^2(\theta) \end{cases}.$$

given in polar coordinates on $K := \{z \in \mathbb{C} \mid 1 \leq |z| \leq 2\}$. The solutions of above differential equation exist for all times and all initial values in $K$ and form a semiflow $(\phi_t)_{t \geq 0}$ thereon.

Since $\dot{r}(t) \leq 0$ for all $t \geq 0$ the radius $r$ is monotonically decreasing and since $\dot{\theta}(t) \geq 0$ for all $t \geq 0$ the angle $\theta$ is monotonically increasing, these facts clearly imply that the orbit of an initial state with radius $r$ \([1 \leq r < 2\)$ forms a spiral towards the unit circle. On the unit circle the radius is constant and the rate of change of $\theta$ is given by the differential equation

$$\dot{\theta} = \sin^2(\theta).$$

This implies that the unit circle is $(\phi_t)_{t \geq 0}$-invariant and attractive. Furthermore, $z_1 = 1 = e^0$ und $z_2 = -1 = e^{\pi i}$ are fixed points, because $\sin^2(0) = \sin^2(\pi) = 0$. Thus, points on the unit circle converge to either $z_1$ or $z_2$. Remark that $M := \{z_1\} \cup \{z_2\}$ is not attractive. Take $z \in K$ with $|z| > 1$ and a neighborhood $U$ of $M$. Since $\theta$ is monotonically increasing the set \(\{t \in [0, \infty) \mid \phi_t(z) \notin U\}\) is unbounded and hence $M$ is not attractive. We claim that the set $M$ is a center of attraction for $(K, (\phi_t)_{t \geq 0})$ and is even minimal with this property. We prove these facts in Example 4.14.

(d) Stability in the sense of Lyapunov does not imply any of the other properties in Definition 3.1. Consider $\phi_t := \text{id}_K$ for all $t \geq 0$. Then every closed subset of $K$ is stable in the sense of Lyapunov but none of the other properties in Definition 3.1 apply.

The next proposition characterizes all above mentioned attractivity properties by means of the corresponding Koopman semigroup.
Proposition 3.5 Let \((K, (\varphi_t)_{t \geq 0})\) be a dynamical system, \(\emptyset \neq M \subseteq K\) a closed invariant set, \(\mu\) a quasi-invariant Borel measure on \(K\) and \((S(t))_{t \geq 0}\) the restricted Koopman semigroup, i.e. \(S(t) := T(t)|_M\) for \(t \geq 0\). Then the assertions in (I), (II), (III) and (IV), respectively, are equivalent.

(I)

(a) \((S(t))_{t \geq 0}\) is strongly stable.
(b) \(M\) is uniformly attractive.

(II)

(a) \((S(t))_{t \geq 0}\) is weakly stable.
(b) \(M\) is attractive.

(III)

(a) \((S(t))_{t \geq 0}\) is almost everywhere pointwise stable (for \(\mu\)).
(b) \(M\) is a likely limit set (for \(\mu\)).

(IV)

(a) \((S(t))_{t \geq 0}\) is almost weakly stable.
(b) \(M\) is a center of attraction.

Proof Proof of (I): First we show (a) \(\Rightarrow\) (b). Take \(U \in \mathcal{U}(M)\). Since \(K\) is completely regular there is \(f \in I_M\) and \(\varepsilon > 0\) such that \(U_{\varepsilon,f} \subseteq U\). Recall that \(U_{\varepsilon,f} := \{|f| < \varepsilon\}\). By assertion a) there is \(t_0 > 0\) such that \(\|S(t)f\| < \varepsilon\) for all \(t \geq t_0\). This implies
\[
|S(t)f(x)| = |f(\varphi_t(x))| < \varepsilon \quad \text{for all } x \in K, \ t \geq t_0.
\]
Therefore, \(\varphi_t(K) \subseteq U_{\varepsilon,f} \subseteq U\) for all \(t \geq t_0\). Also (b) \(\Rightarrow\) (a) because for every \(\varepsilon > 0\) and \(f \in I_M\) there is \(t_0 > 0\) such that \(\varphi_t(K) \subseteq U_{\varepsilon,f}\) for all \(t \geq t_0\). This implies \(|S(t)f(x)| < \varepsilon\) for all \(t \geq t_0\) and \(x \in K\) and therefore \(\|S(t)f\| = \sup_{x \in K} |S(t)f(x)| < \varepsilon\) for all \(t \geq t_0\).

Proof of (II): To prove (a) \(\Rightarrow\) (b) take \(U \in \mathcal{U}(M)\) and \(x \in K\). Then there exist \(\varepsilon > 0\) and \(f \in I_M\) such that \(U_{\varepsilon,f} \subseteq U\) and since \((S(t))_{t \geq 0}\) is weakly stable there exists \(t_0 > 0\) such that
\[
\langle S(t)f, \delta_x \rangle = f(\varphi_t(x)) < \varepsilon \quad \text{for all } t \geq t_0
\]
which implies \(\varphi_t(x) \in U_{\varepsilon,f} \subseteq U\) for all \(t \geq t_0\). For the opposite implication let \(\varepsilon > 0\), \(f \in I_M\) and \(x \in K\). By b) there exists \(t_0 > 0\) such that \(\langle S(t)f, \delta_x \rangle < \varepsilon\) for all \(t \geq t_0\) and thus
\[
\langle S(t)f, \delta_x \rangle \to 0 \quad \text{as } t \to \infty
\]
for all Dirac measures $\delta_x$ and by Lebesgue’s theorem of dominated convergence
$$\langle S(t)f, \mu \rangle \to 0 \text{ as } t \to \infty$$
for all $\mu \in I'_M$.

Proof of (III): We first prove (a) implies (b). First take a neighborhood $U \in \mathcal{U}(M)$, then there exist $f \in I_M$ and $\varepsilon > 0$ with $U_{\varepsilon f} \subseteq U$. By assumption there is a $\mu$-null set $N_f$ depending on $f$ such that for every $x \in N_f$ there is $t_0 > 0$ such that
$$S(t)f(x) < \varepsilon$$
for all $t \geq t_0$. Clearly, this implies $\varphi_t(x) \in U_{\varepsilon f} \subseteq U$ for all $t \geq t_0$. The other implication follows similarly.

Proof of (IV): We prove first the implication (b) $\Rightarrow$ (a). Recall that $I_M \cong C_0(K \setminus M)$ by $f \mapsto f|_{K \setminus M}$. If $M$ is a center of attraction and $U \in \mathcal{U}(M)$ open, then
$$\lim_{t \to \infty} \frac{1}{t} \lambda\left(\{s \in [0, t] \mid \varphi_s(x) \in U^c\}\right) = 0$$
for all $x \in K$. Now take $f \in C_c(K \setminus M)$ and denote its support by $L := \text{supp}(f)$. By assumption
$$\frac{1}{t} \int_0^t |S(s)f(x)| \, ds \leq \frac{1}{t} \int_0^t \|f\|_\infty \|L(\varphi_s(x))\|_1 \, ds \to 0$$
since the complement $L^c$ in $K$ is a neighborhood of $M$. This proves the assertion by the fact that $\overline{C_c(K \setminus M)}_\| \| = C_0(K \setminus M) \cong I_M$ and Remark 1.4.

For the other implication take $x \in K$, $f \in I_M$, $f \geq 0$ and $\varepsilon > 0$. By assumption there exists a subset $R \subseteq [0, \infty)$ with density 1 and $t_0 > 0$ such that
$$\langle S(t)f, \delta_x \rangle < \varepsilon \quad \text{for all } t \geq t_0, \ t \in R.$$ 
Since $R \cap [t_0, \infty)$ still has density 1 we obtain
$$\frac{1}{t} \lambda\left(\{s \in [0, t] \mid \varphi_s(x) \in U_{\varepsilon f}\}\right) \to 1.$$ 
This implies the assertion since the neighborhoods of the form $U_{\varepsilon f}$ form a basis for the neighborhoods of $M$. $\square$

To conclude this section we show that the concepts of uniform attractivity and pointwise attractivity coincide if and only if $M$ is stable in the sense of Lyapunov. To do so we first characterize stability in the sense of Lyapunov further.

**Proposition 3.6** Let $\emptyset \neq M \subseteq K$ be closed and invariant. Then the following are equivalent.

(a) The set $M$ is stable in the sense of Lyapunov.
(b) Every $U \in \mathcal{U}(M)$ contains an invariant $V \in \mathcal{U}(M)$. If $U$ is closed, $V$ can be chosen closed as well.
Proof For the implication (a)⇒(b) take $U \in \mathcal{U}(M)$ closed and $V \in \mathcal{U}(M)$, $V \subseteq U$ such that $x \in V$ implies $\varphi_t(x) \in U$ for all $t \geq 0$. Consider $W := \bigcup_{t \geq 0} \varphi_t(V)$. Then $V \subseteq W \subseteq U$. Therefore, $W$ is still a closed neighborhood of $M$ which is invariant. The implication (b)⇒(a) is trivial.

Remark 3.7 Furthermore, a closed invariant set $\emptyset \neq M \subseteq K$ is Lyapunov stable if and only if

$$M = \bigcap_{V \in \mathcal{U}(M)\text{ inv.}} V = \bigcap_{W \in \mathcal{U}(M)\text{ closed}&\text{inv.}} W.$$ 

We prove that the assertion implies (b) in Proposition 3.6. The converse is clear. Let $U$ be an open neighborhood of $M$ and assume there is no invariant neighborhood $V$ of $M$ with $V \subseteq U$. Then for all invariant neighborhoods $V$ there exists $x_V \in V$ with $x_V \in U^c$. This defines a net $(x_V)_{V \in \mathcal{U}(M)\text{ inv.}}$ which has a convergent subnet since $U^c$ is compact. We will denote said convergent subnet by $(x_V)_i \in I$ and its limit by $\bar{x}$. Now fix $W \in \mathcal{U}(M)$ closed and invariant. By cofinality of the the index set $I$ there exists $i_0 \in I$ such that $x_{V_i} \in W$ for all $i \geq i_0$. This implies $\bar{x} \in W$ and since $W$ was arbitrary $\bar{x} \in \bigcap_{W \in \mathcal{U}(M)\text{ closed}&\text{inv.}} W = M$ which is a contradiction to $\bar{x} \in U^c$. This implies (b). \qed

Proposition 3.8 Let $(K, (\varphi_t)_{t \geq 0})$ be a dynamical system and $\emptyset \neq M \subseteq K$ a closed invariant subset which is uniformly attractive, then $M$ is stable in the sense of Lyapunov.

Proof Let $M$ be uniformly attractive and assume that it is not stable in the sense of Lyapunov. Take $U \in \mathcal{U}(M)$ open. Then for every $V \in \mathcal{U}(M)$, $V \subseteq U$, there is $x_V \in V$ and $t_V > 0$ such that $\varphi_{t_V}(x_V) \in U^c$.

Since $U^c$ is compact, the net $(\varphi_{t_V}(x_V))_{V \subseteq U}$ has a convergent subnet $(\varphi_{t_{V_i}}(x_{V_i}))_{i \in I}$ with limit $y \in U^c$. Since there exists $t_0 > 0$ such that $\varphi_t(U^c) \subseteq U$ for all $t \geq t_0$ by assumption, the net $(t_{V_i})_{i \in I}$ is bounded by $t_0$ and therefore also has a convergent subnet, which we again denote by $(t_{V_i})_{i \in I}$, with limit $0 \leq t^* \leq t_0$. Furthermore, the net $(x_{V_i})_{i \in I}$ has a convergent subnet, again denoted by $(x_{V_i})_{i \in I}$, with limit $x \in M$. By continuity and invariance of $M$ it follows that

$$\varphi_{t_{V_i}}(x_{V_i}) \to \varphi_{t^*}(x) \in M.$$ 

Then, $\varphi_{t^*}(x) = y \in U^c$ which is a contradiction. \qed

On the other hand stability in the sense of Lyapunov and attractivity imply uniform attractivity.

Proposition 3.9 Let $(K, (\varphi_t)_{t \geq 0})$ be a dynamical system and $\emptyset \neq M \subseteq K$ attractive and stable in the sense of Lyapunov. Then $M$ is uniformly attractive.

Proof Let $M$ be pointwise attractive and stable in the sense of Lyapunov. Take $U \in \mathcal{U}(M)$ open and invariant. Then for every $x \in K$ there exists $t_0 = t_0(x, U)$ such that

$\varphi_{t_0}(x) = y \in U^c$ which is a contradiction.
\[ \varphi_t(x) \in U \text{ for all } t \geq t_0. \]

Since \( \varphi_{t_0} \) is continuous there exists an open neighborhood \( U_x \) of \( x \) with

\[ \varphi_t(U_x) \subseteq U \text{ for all } t \geq t_0 \]

since \( U \) is invariant. Now since \( K \) is compact there exist \( x_1, \ldots, x_n \in K \) for some \( n \in \mathbb{N} \) such that \( K \subseteq \bigcup_{i=1}^{n} U_{x_i} \) and for \( t \geq \max_{i=1,\ldots,n} t_0(x_i, U) \)

\[ \varphi_t(K) \subseteq \varphi_t \left( \bigcup_{i=1}^{n} U_{x_i} \right) \subseteq \bigcup_{i=1}^{n} \varphi_t(U_{x_i}) \subseteq U. \]

This implies the assertion. \( \square \)

5 Existence and characterization of minimal attractors

Given a dynamical system \( (K, (\varphi_t)_{t \geq 0}) \) there always exist attractors in the sense of above Definition 3.1 a-d) since the subspaces \( I_{ss}, I_{ws}, I_{aeps} \) and \( I_{aws} \) are all closed ideals of \( C(K) \) and are maximal with this property. We thus obtain a corresponding closed invariant set \( \emptyset \neq M \subseteq K \) that is uniformly attractive, attractive, a likely limit set or a center of attraction and it is minimal with this property by construction. In this subsection we will discuss what the corresponding minimal attractor \( M \) looks like. First, we clarify that \( I_{ss}, I_{ws}, I_{aws} \) and \( I_{aeps} \) are in fact not only closed subspaces but ideals in \( C(K) \).

**Remark 4.1** The closed subspaces \( I_{ss}, I_{ws}, I_{aeps} \) and \( I_{aws} \subseteq C(K) \) are lattice or equivalently algebra ideals in \( C(K) \).

**Proof** We only compute this for \( I_{aws} \), because \( I_{ws} \) and \( I_{aeps} \) follow analogously and \( I_{ss} \) is clearly a lattice ideal.

By Remark 1.5, \( I_{aws} \) is a closed subspace of \( C(K) \). It remains to show that it is an algebra or equivalently a lattice ideal. Take \( f \in I_{aws} \) we first show that \( |f| \in I_{aws} \).

Take \( x \in K \) and recall that for every \( t \geq 0 \), \( |T(t)f, \delta_x| = |T(t)f(x)| = T(t)|f|(x) \). We recall that

\[
0 \xrightarrow{T \to \infty} \frac{1}{T} \int_{0}^{T} \langle T(t)f, \delta_x \rangle \, dt
\]

\[ = \frac{1}{T} \int_{0}^{T} \langle T(t)|f|, \delta_x \rangle \, dt \]

\[ = \left( \frac{1}{T} \int_{0}^{T} T(t)|f| \, dt, \delta_x \right), \]

therefore the assertion follows by Remark 1.4. Additionally, if for \( g \in C(K), |g| \leq f \) for some \( f \in I_{aws} \) it follows that \( g \in I_{aws} \). \( \square \)
5.1 Uniform attractivity

The following proposition gives a characterization of uniform attractivity.

**Proposition 4.2** Let \((K, (\varphi_t)_{t \geq 0})\) be a dynamical system and \(0 \neq M \subseteq K\) closed and invariant. Then the following are equivalent.

(a) The set \(M\) is uniformly attractive,

(b) \(\bigcap_{t \geq 0} \varphi_t(K) \subseteq M\).

**Proof** If (a) is true then for every \(U \in \mathcal{U}(M)\) there is a \(t_0 > 0\) such that

\[
\bigcap_{t \geq 0} \varphi_t(K) \subseteq \bigcap_{t \geq t_0} \varphi_t(K) \subseteq U.
\]

This implies

\[
\bigcap_{t \geq 0} \varphi_t(K) \subseteq \bigcap_{U \in \mathcal{U}(M)} U = M.
\]

The opposite implication is true since \(\bigcap_{t \geq 0} \varphi_t(K)\) is itself uniformly attractive because \(\varphi_r(K) \subseteq \varphi_s(K)\) for \(r \geq s \geq 0\) and for \(V \in \mathcal{U}(\bigcap_{t \geq 0} \varphi_t(K))\) there exists \(t_0 > 0\) such that

\[
\bigcap_{t \geq 0} \varphi_t(K) \subseteq \varphi_{t_0}(K) \subseteq V.
\]

As an immediate result we obtain the following.

**Proposition 4.3** Let \((K, (\varphi_t)_{t \geq 0})\) be a dynamical system. Then there exists a unique minimal uniformly attractive subset of \(K\) given by

\[
\bigcap_{t \geq 0} \varphi_t(K).
\]

**Proof** The set \(\bigcap_{t \geq 0} \varphi_t(K)\) is closed as an intersection of compact sets, nonempty by the finite intersection property of \(K, (\varphi_t)_{t \geq 0}\)-invariant and is uniformly attractive by Proposition 4.2 (b) and is minimal with this property by construction.

**Remark 4.4** Combining Proposition 4.2 and Proposition 4.3 one obtains

\[
I_{ss} = I_{\bigcap_{t \geq 0} \varphi_t(K)}.
\]
5.2 Attractivity and $\omega$-limit sets

It is useful to introduce $\omega$-limit sets to study asymptotic properties of dynamical systems. Similar concepts have already been used by Poincaré, but Birkhoff first introduced the term $\omega$-limit set in [4, Chapt. VII, p.198]. The characterization of attractors via $\omega$-limit sets is due to Bhatia and Szegö and can be found in [2, Chapt. II].

**Definition 4.5** For $x \in K$ we define the $\omega$-limit set of $x$ as

$$\omega(x) := \bigcap_{T \geq 0} \{ \varphi_t(x) \mid t \geq T \}.$$  

**Proposition 4.6** For every $x \in K$

$$\omega(x) = \{ y \in K \mid \exists \text{ a net } (t_i)_{i \in I} \text{ in } [0, \infty), \ t_i \to \infty \text{ such that } \varphi_{t_i}(x) \to y, i \in I \}.$$  

**Proof** Take $x \in K$ and $y \in \omega(x)$, i.e., $y \in \{ \varphi_t(x) \mid t \geq T \}$ for all $T \geq 0$. In particular, $y \in \text{orb}(x)$. Thus, by definition of the closure there exists a net in $\text{orb}(x)$ converging to $y$. For the other implication let $(t_i)_{i \in I}$ be a net with $t_i \to \infty$ such that $\varphi_{t_i}(x)$ converges to $y$. For fixed $T \geq 0$ there exists $i_0 \in I$ such that $t_i \geq T$ for all $i \geq i_0$. Then $(\varphi_{t_i}(x))_{i \in I, i \geq i_0}$ is still a net converging to $y$. Therefore, $y \in \{ \varphi_t(x) \mid t \geq T \}$ for all $T \geq 0$. Thus the assertion follows. 

Next, we discuss some properties of $\omega$-limit sets.

**Proposition 4.7** The set $\omega(x)$ is non-empty, closed and invariant under $$(\varphi_t)_{t \geq 0}$$ for all $x \in K$.

**Proof** Take $x \in K$. The set $\omega(x)$ is closed by definition as an intersection of closed sets and non-empty by the finite intersection property of $K$. For the invariance take $r > 0$ and $y \in \omega(x)$. Then there exists a net $(\varphi_{t_i}(x))_{i \in I}$ converging to $y$ for $i \in I$, since $\varphi_r$ is continuous, $(\varphi_r(\varphi_{t_i}(x)))_{i \in I}$ converges to $\varphi_r(y)$, thus $\varphi_r(y) \in \omega(x)$. 

**Proposition 4.8** Let $(K, (\varphi_t)_{t \geq 0})$ be a dynamical system and $\emptyset \neq M \subseteq K$ closed and invariant. Then the following are equivalent.

(a) The set $M$ is attractive,

(b) $\omega(x) \subseteq M$ for all $x \in K$.

**Proof** To prove (a)$\Rightarrow$(b) take $x \in K$. By (a)

$$\omega(x) \subseteq \bigcap_{U \in \mathcal{U}(M)} U = M.$$  

Consider $U \in \mathcal{U}(M)$ open and assume that a) does not hold, i.e., there exists $x \in K \setminus M$ with $\varphi_{t_i}(x) \in U^c$ for infinitely many $t > 0$. Since $U^c$ is closed and hence
compact there exists a convergent subnet \((t_i)_{i \in I}, t_i \to \infty\) such that \(\varphi_{t_i}(x) \to z \in U^c\) which is a contradiction to (b) by Proposition 4.6.

**Proposition 4.9** Let \((K, (\varphi_t)_{t \geq 0})\) be a dynamical system. Then there exists a unique minimal attractive subset of \(K\) given by

\[ \bigcup_{x \in K} \omega(x). \]

**Proof** In Proposition 4.8 (b) we have seen that \(\omega(x)\) is contained in every closed, \((\varphi_t)_{t \geq 0}\)-invariant and attractive subset \(\emptyset \neq M \subseteq K\) therefore also

\[ \bigcup_{x \in K} \omega(x) \subseteq M. \]

Also the closure \(\bigcup_{x \in K} \overline{\omega(x)}\) is contained in every such \(M\) and \((\varphi_t)_{t \geq 0}\)-invariant, attractive itself and minimal with this property by construction.

**Remark 4.10** Combining Proposition 4.8 and Proposition 4.9 one obtains

\[ I_{\omega} = I_{\bigcup_{x \in K} \omega(x)}. \]

**Proposition 4.11** Let \((K, (\varphi_t)_{t \geq 0})\) be a dynamical system with \(K\) metric, \(\mu\) a quasi invariant regular Borel measure on \(K\) and \(\emptyset \neq M \subseteq K\) closed and invariant. Then the following are equivalent.

(a) The set \(M\) is a likely limit set,

(B) \(\omega(x) \subseteq M\) for \(\mu\)-almost every \(x \in K\).

**Proof** We prove this similarly to Proposition 4.8. Let \(M\) be a likely limit set for \(\mu\). Then there exists a \(\mu\)-null set \(N\) such that for all \(U \in \mathcal{U}(M)\) and \(x \in N^c\) there exists \(t_0 > 0\) such that \(\varphi_{t_0}(x) \in U\) for all \(t \geq t_0\). Remark that \(N\) can be chosen independently from \(U\) since \(K\) is metric. Hence, \(\omega(x) \subseteq \bigcap_{U \in \mathcal{U}(M)} U = M\) for all \(x \in N^c\).

Now assume there exists a \(\mu\)-null set \(N\) such that \(\omega(x) \subseteq M\) for all \(x \in N^c\). Take \(U \in \mathcal{U}(M)\) open. If a) does not hold there exists \(x \in N^c\) such that \(\varphi_t(x) \in U^c\) for infinitely many \(t > 0\). Since \(U^c\) is compact there exists a convergent subnet of \((\varphi_t(x))_{t \geq 0}\) with limit in \(U^c\) which is a contradiction to \(\omega(x) \subseteq M\) by Proposition 4.6.

**Remark 4.12** Let \((K, (\varphi_t)_{t \geq 0})\) be a dynamical system with \(K\) metric and \(\mu\) a quasi invariant regular Borel measure on \(K\). By Proposition 4.11 there exists a \(\mu\)-null set \(N\) such that

\[ I_{\omega} = I_{\bigcup_{x \in N^c} \omega(x)}. \]
5.3 Minimal centers of attraction and ergodic measures

An interesting fact is that the minimal center of attraction is characterized by the ergodic measures on $K$. We recall that a regular Borel measure $\mu$ on $K$ is called invariant if $\mu(\varphi_t^{-1}(A)) = \mu(A)$ for all Borel measurable sets $A$ and $t \geq 0$. An invariant probability measure is called ergodic if the corresponding measure-preserving system $(K, (\varphi_t)_{t \geq 0}, \mu)$ is ergodic, i.e., if $A \subseteq K$ is Borel measurable and invariant then $\mu(A) \in \{0, 1\}$. In the following we write $M^1(K)$ for the set of all regular Borel probability measures on $K$.

**Proposition 4.13**  The minimal center of attraction is given by the union of supports of ergodic measures, i.e.,

$$I_{\text{aws}} = \bigcup_{\mu \in M^1(K)} \text{supp}(\mu).$$

**Proof**  By [8, p.193, (10.1)] it suffices to show that $I_{\text{aws}} = I_{\text{inv}}$ where $M_{\text{inv}} := \bigcup_{\mu \in M^1(K)} \text{supp}(\mu)$. First we show “$\subseteq$”. Let $\mu \in M^1(K)$ be invariant. For $f \in I_{\text{aws}}$

$$\langle |f|, \mu \rangle = \frac{1}{t} \int_0^t \langle |f|, \mu \rangle \, ds$$

$$\Rightarrow \mu \text{ inv. } \frac{1}{t} \int_0^t \langle T(s)|f|, \mu \rangle \, ds \to 0$$

by Remark 1.4 and Remark 4.1. Therefore, $f \big|_{\text{supp}(\mu)} \equiv 0$ for all invariant $\mu \in M^1(K)$. For the implication “$\supseteq$” let $x \in K$ and $\delta_x$ the corresponding Dirac measure and $f \in I_{\text{inv}}$. We observe that

$$\frac{1}{t} \int_0^t \langle T(s)f, \delta_x \rangle \, ds = \frac{1}{t} \int_0^t \langle |f|, T(s)'\delta_x \rangle \, ds$$

$$= \langle |f|, \frac{1}{t} \int_0^t T(s)'\delta_x \, ds \rangle.$$ 

Since the dual unit ball $B'$ of $C(K)$ is compact in the weak-*-topology and $\frac{1}{T} \int_0^T T(t)'\delta_x \, dt$ is bounded, every subnet of $\left(\frac{1}{T} \int_0^T T(t)'\delta_x \, dt\right)_{t \geq 0}$ has a convergent subnet in $B'$, i.e.,

$$\langle |f|, \frac{1}{t} \int_0^t T(s)'\delta_x \, ds \rangle \to \langle |f|, \mu \rangle$$

with $\mu$ an invariant probability measure. Also, $\mu \neq 0$ since $\langle 1, \mu \rangle = 1$. 

\[ \square \]
Example 4.14  Continuation of Example 3.4 (c). As we have seen, the set $M = \{z_1\} \cup \{z_2\}$ is a closed and $(\varphi_t)_{t \geq 0}$-invariant subset of the unit circle $T$ which is the minimal attractive subset. Furthermore, the point evaluations $\delta_{z_1}$ and $\delta_{z_2}$ are invariant measures, hence

$$M \subseteq M_{\text{inv}} \subseteq T.$$  

However, by [8, Lem. 10.7] for every invariant measure $\mu$, $\varphi_t(\text{supp}(\mu)) = \text{supp}(\mu)$ for all $t \geq 0$. Since $\theta$ is strongly increasing on $T \setminus M$, $\varphi_t(L) \neq L$ for all sets $M \subseteq L \subseteq T$ and $t > 0$. Hence, there cannot be an invariant measure $\mu$ with $M \subseteq \text{supp}(\mu) \subseteq T$ and therefore $M$ is the minimal center of attraction by Proposition 4.13.

Compliance with ethical standards

Conflicts of interest  The author declares that she has no conflict of interest.

Ethical approval  This article does not contain any studies with human participants or animals performed by the author.

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