A Superconformal Index for HyperKähler Cones

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Abstract

We define an index for $\mathfrak{osp}(4^*|4)$ superconformal quantum mechanics on a hyperKähler cone. The index is defined on an equivariant symplectic resolution of the cone, which acts as a regulator. We present evidence that the index does not depend on the choice of resolution parameters and encodes information about the spectrum of (semi-) short representations of the superconformal algebra of the unresolved space. In particular, there are two types of multiplet which can be counted exactly using the index. These correspond to holomorphic functions on the cone and to the generators of the Borel-Moore homology on the resolved space respectively. We calculate the resulting index by localisation for a large class of examples.

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1 Introduction

Conformal quantum mechanics refers to a quantum mechanical system in which the Hamiltonian is accompanied by generators for dilatations and special conformal transformations which together form an SO(2,1) symmetry group, [13]. Such models are interesting in the context of holographic duality where they may provide a boundary description of geometries containing an AdS$_2$ factor such as the near horizon region of a black hole. Conformal quantum mechanics also arises as the discrete light-cone quantisation (DLCQ) of higher-dimensional CFTs, [112].

The maximally supersymmetric case we study here, arises for a quantum mechanical $\sigma$-model with a hyperKähler target manifold. As usual the resulting theory has $\mathcal{N} = (4,4)$ supersymmetry, [3]. The target space also satisfies the criterion for $\mathfrak{so}(2,1)$ conformal invariance if it admits a closed homothety in the sense of [20]. Such manifolds are known as hyperKähler cones. Provided the homothety is also triholomorphic, the theory acquires a larger $\mathfrak{osp}(4^*|4)$ superconformal symmetry, whose bosonic subalgebra includes $\mathfrak{so}(2,1)$ conformal transformations together with an $\mathfrak{su}(2) \oplus \mathfrak{usp}(4)$ algebra of R-symmetry transformations, [43]. Large classes of hyperKähler cones obeying this condition arise as the Higgs branches of supersymmetric gauge theories in higher dimensions. This provides a physical context for the corresponding $\sigma$-models; compactifying the spatial dimensions of the gauge theory on a torus, the resulting theory flows to superconformal quantum mechanics in the IR. In mathematical terms the Higgs branch of a supersymmetric gauge theory with eight supercharges corresponds to a particular hyperKähler quotient of flat space. For quiver gauge theories, the resulting spaces also have an algebro-geometric description and are known as Nakajima quiver varieties. As we discuss below, these are singular spaces and a resolution of the singularities is needed in order to properly define supersymmetric quantum mechanics.

One particularly interesting model in this class is the ADHM quiver corresponding to the moduli space of Yang-Mills instantons on $\mathbb{R}^4$. This model is believed to provide a DLCQ description of the (2,0) theory in six dimensions. This identification is consistent with the fact that $\mathfrak{osp}(4^*|4)$ is a factor in the subalgebra of the (2,0) superconformal algebra in six dimensions which is left

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1For brevity, this condition is implied whenever we use the term ‘hyperKähler cone’ in the following.
unbroken by compactification of a null direction. It further suggests the existence of a \textit{an \textit{osp}(4*|4)} invariant quantum mechanics on the instanton moduli space with a spectrum consisting of lowest-weight unitary representations of this algebra. These states should be the ones arising from the branching of unitary representations of the full (2,0) superconformal algebra onto those of the light-cone subalgebra. In \cite{2}, Aharony et al regulated the ADHM \textit{σ}-model by suitably resolving the singularities of the instanton moduli space. Any such deformation inevitably breaks superconformal invariance. However, the authors of \cite{2} argued that the resolution corresponds to a UV regulator for six-dimensional spacetime theory and also suggested a limiting procedure to extract the observables of conformal quantum mechanics from those of the regulated theory.

In this paper we will proceed in a similar way for more general models in the same class and test the hypothesis that \textit{every} hyperKähler cone defines an \textit{osp}(4*|4) invariant superconformal quantum mechanics with a spectrum consisting of lowest-weight unitary representations. For a large class of such models there is a natural choice of regulator; we replace the hyperKähler cone by its \textit{equivariant symplectic resolution}\footnote{The resolution is in fact a projective symplectic resolution, which necessarily has the property of being equivariant.}. For cones which arise as Higgs branches of quiver gauge theories, such resolutions are provided by turning on generic values of the real Fayet-Illiopoulos parameters in the gauge theory. In the case of the ADHM quiver this coincides with the resolution considered in \cite{2}.

As for superconformal algebras in higher dimensions, representations of \textit{osp}(4*|4) are naturally classified as (semi-) short or long depending on whether the dimensions of the primary state saturate a BPS bound. In recent work, \cite{42} (see also \cite{15}) Singleton defined a superconformal index which counts the (semi-) short multiplets modulo the possibility of recombination into long ones. As well as encoding the dimensions and R-charges of superconformal primaries, the index also grades states according to their quantum numbers under the global symmetry group of the theory. In the case of flat \(\mathbb{C}^{2n}\), the states contributing to the index are in one-to-one correspondence with polynomial-valued holomorphic forms on the target space.

In the general case we replace the singular cone by its equivariant symplectic resolution. Although the full superconformal algebra is broken by the resolution, the Cartan generators of the little-group associated with the superconformal index all remain unbroken as do the triholomorphic isometries of the hyperKähler cone. In the following, we will argue that the superconformal index can be identified with an appropriate index of the resolved space, namely a certain equivariant Euler character of sheaf cohomology. We will show that the index defined in this way correctly captures the expected features of superconformal
quantum mechanics on the underlying singular space and present evidence that it does not depend on the choice of symplectic resolution.

To define the index we need to pass to the algebraic description of the hyperKähler cone $X$ and its equivariant symplectic resolution $\tilde{X}$ as a complex variety. In this context we consider the sheaf $\mathcal{A}^p(\tilde{X})$ of $p$-forms on $\tilde{X}$ and the associated sheaf cohomology $H^*$ in the Zariski topology. In section 2, we argue that the superconformal index can be identified with an equivariant analog of the Hirzebruch genus in this complex. The resulting index $Z(\mathcal{X})$ for a hyperKähler cone $X$ of quaternionic dimension $d_H$, is a function of two fugacities $\tau$ and $y$ which can be identified with the Cartan generators of the $SU(1|2)$ little group preserved by the BPS shortening condition of the superconformal algebra as well as additional fugacities $Z = \{z_i\}$, corresponding to global symmetries.

$$Z(\mathcal{X}; \tau, y, Z) := \sum_{p,q=0}^{2d_H} (-)^{p+q} \left( \frac{y}{\tau} \right)^{p-d_H} \text{tr}_{H^*(\tilde{X}, \mathcal{A}^p(\tilde{X}))} \left( \tau^R \prod_i z_i^{\mathcal{J}_i} \right).$$

(1.1)

Geometrically, the powers of $y/\tau$ appearing in the index correspond to holomorphic degree, $p$, in sheaf cohomology while those of $\tau$ record the grade $R$ under the $\mathbb{C}^\times$ action preserved by the symplectic resolution. The global symmetry fugacities $z_i$, grade the cohomology classes according to their charge under the triholomorphic isometries of $\tilde{X}$.

For all the cases we study in this paper, the index can be localised at fixed points of a group action on the manifold and calculated exactly. Applying the relevant localisation theorems we find,

$$Z(\mathcal{X}) = \left( \frac{\tau}{y} \right)^{d_H} \sum_{x \in \mathcal{X}_T} \text{PE} \left[ \left( 1 - \frac{y}{\tau} \right) \text{ch}_T(T^*_x \tilde{X}; \tau, Z) \right].$$

(1.2)

Here the sum is over fixed points $x$ of $\tilde{X}$ under the torus $T$ generated by the isometries $\mathcal{J}_i$ and $\mathbb{C}^\times$. In the above formula $T^*_x \tilde{X}$ is the tangent space to the manifold at the fixed point $x$ considered as a $T$-module and $\text{ch}_T$ denotes the corresponding character. Finally, $\text{PE}$ denotes the plethystic exponential. For the full definitions see Section 2.1.

By construction, the index has no continuous dependence on the resolution parameters. However, it is known that hyperKähler cones can have inequivalent symplectic resolutions corresponding to different chambers in the parameter space separated by walls of codimension one, [32]. One might therefore worry that the index we define depends on the choice of chamber. In the following, we will present evidence that the index is in fact independent of this choice. The evidence consists of several different limits and specialisations of the index for which this can be proven. Finally, we perform some explicit calculations for a generic quiver of low dimension.
As in higher dimensions, the expansion of the index in characters of the little group provides information about the spectrum of $\mathfrak{osp}(4^*|4)$ multiplets. Although the index typically provides only a lower bound on the spectrum, there are two types of multiplets which we can count precisely. Both are related to the geometry of the target space. The first are the 1/2-BPS short multiplets which we show are in one-to-one correspondence with the generators of Borel-Moore homology of the resolved space. These states cannot be lifted and the index counts them without sign. The second are a particular type of semi-short multiplet of $\mathfrak{osp}(4^*|4)$ which are in one-to-one correspondence with holomorphic functions on the target space. The partition function for these two classes of protected states are related to the Hilbert series and Poincare polynomial of $\tilde{X}$ respectively. In fact, both these partition functions arise as particular limits of the super conformal index. As we explain in Section 3, the Hilbert series arises in the $y \to 0$ limit of $Z(\tilde{X})$ while the Poincare polynomial in Borel-Moore homology appears in the $\tau \to \infty$ limit with $y/\tau$ held fixed. The full index interpolates between these two quantities and encodes a lower bound on the degeneracies of the other (semi-)short multiplets of $\mathfrak{osp}(4^*|4)$ in each irreducible representation of the global symmetry.

In the special case of the ADHM quiver, the index (1.1) coincides with the corresponding instanton contribution to the Nekrasov partition function of a certain five-dimensional supersymmetric gauge theory, and agrees with an earlier proposal for a super conformal index for ADHM quantum mechanics, [24]. In this case, we confirm the identification of 1/2-BPS multiplets with compactly supported cohomology classes which was anticipated in [2, 24]. This identification reproduces the known spectrum of chiral primaries of the $(2,0)$ theory corresponding to the Kaluza-Klein modes of eleven dimensional supergravity on $AdS_7 \times S^4$. The semi-short multiplets of $\mathfrak{osp}(4^*|4)$ should be related to the spectrum of 1/8-BPS states of the $(2,0)$ theory. The resulting bound from the index should be relevant for counting the microstates of supersymmetric black holes in $AdS_7$.

The superconformal index defined in this paper has a number of other interesting properties. HyperKähler cones typically contain many cones of lower dimension, which appear as fixed points of symmetries. The superconformal index is stable under this reduction in the following sense; the index of the fixed point submanifold is obtained simply by taking the fugacity of the corresponding symmetry to zero in the index of the original space. This connection provides an efficient way of generating the superconformal indices of all quivers varieties of type $A_n$ and $\hat{A}_n$, as these all appear as fixed points in the instanton moduli space [31]. The connection to the Poincare polynomial, allows us to write an explicit formula for the partition function of 1/2-BPS states for superconformal quantum mechanics on any Nakajima quiver variety. There are interesting connections with the work of Nakajima, who constructed an action of a quantum affine algebra on the Borel-Moore homology of these spaces [32, 33]. Indeed the
resulting generating function can be identified with graded character of a specific module of this algebra.

The paper is organised as follows. In Section 2, we will review $\mathfrak{osp}(4^*|4)$ superconformal quantum mechanics on a hyperKähler cone. We also review the symplectic resolution of the cone and define a regulated superconformal index. Finally, we give a general formula for the index via localisation to the fixed points of a group action. In Section 3, we discuss various properties of the index including the limiting behaviour mentioned above. To show that our interpretation of $\mathcal{Z}$ as a superconformal index is consistent we compare our formula (1.2) with the index evaluated on a general spectrum of unitary lowest weight representations of $\mathfrak{osp}(4^*|4)$. This comparison provides several non-trivial tests of our proposal. Finally, in Section 4, we apply the fixed point formula to compute the index in numerous cases.

2 The superconformal index

Following [42, 43], we will study supersymmetric quantum mechanics on a hyperKähler manifold. As usual, the Hamiltonian $H$ of the system is identified with the Laplacian acting on forms and the hyperKähler condition gives an $\mathcal{N} = (4, 4)$ supersymmetry algebra. We specialize further to the case of a hyperKähler cone (see [14]).

Definition 1. For $\mathcal{X}$ a hyperKähler manifold, it is a hyperKähler cone if it has a homothetic conformal Killing vector field $V_D$,

$$\mathcal{L}_{V_D}g = 2g.$$  \hspace{1cm} (2.1)

We define $d_H := \frac{1}{4}\dim_{\mathbb{R}}\mathcal{X} \in \mathbb{Z}_{\geq 0}$. The homothety $D$ yields, a dilatation operator $\mathbb{D}$ and also implies the existence of a scalar function $K$. The corresponding operator $\mathbb{K}$ together with $\mathbb{H}$ and $\mathbb{D}$ generates an $\mathfrak{so}(2,1)$ conformal algebra. As the space $\mathcal{X}$ is necessarily non-compact, these operators act on the Hilbert space of $L^2$ normalisable forms. In the simplest case of flat space one finds a discrete spectrum\footnote{In conventional quantum mechanics on $X$, one would instead diagonalise $\mathbb{H}$ in the slightly larger Hilbert space of plane-wave normalisable forms leading to a continuous spectrum of scattering states.} for the dilatation operator $\mathbb{D}$ corresponding to a set of unitary representations of $\mathfrak{so}(2,1)$. More generally, a standard argument shows that $\mathbb{D}$ is isospectral to $L_0 = \mu^{-1} \mathbb{H} + \mu \mathbb{K}$ for arbitrary $\mu$. The addition of $\mathbb{K}$ to the Hamiltonian provides a harmonic potential on $\mathcal{X}$ which should lead to a discrete spectrum for $L_0$ and thus for $\mathbb{D}$.

If the homothety $D$ is triholomorphic then the $\mathfrak{so}(2,1)$ conformal algebra combines with $\mathcal{N} = (4,4)$ supersymmetry give a larger $\mathfrak{osp}(4^*|4)$ superconformal algebra [43]. In particular, in this reference, Singleton gave an explicit presentation of the generators of $\mathfrak{osp}(4^*|4)$ acting on the the space of forms on $\mathcal{X}$. If $\mathcal{X}$
is smooth, this algebra is therefore realised in the spectrum of supersymmetric quantum mechanics on the manifold. In the case of a flat target space \( \mathbb{R}^{4n} \), the model can be solved exactly and one finds a spectrum consisting of positive energy, unitary irreducible representations of the superconformal algebra. Unfortunately, this is the only non-singular hyperKähler manifold which admits a triholomorphic homothety. In particular, the hyperKähler cones we consider here are singular spaces and additional input is required to define a sensible model.

In this paper, we will consider the hypothesis that there is an \( \mathfrak{osp}(4^*|4) \) invariant theory associated with each hyperKähler cone \( X \) which in particular has a spectrum consisting of a countable set of positive-energy, irreducible unitary representations of this algebra. We will now review the classification of these representations and the corresponding superconformal index.

The bosonic subalgebra of the superconformal algebra is
\[
\mathfrak{g}_B = \mathfrak{so}(2,1) \oplus \mathfrak{su}(2) \oplus \mathfrak{usp}(4).
\] (2.2)

The Cartan subalgebra of \( \mathfrak{g}_B \) is generated by \( \mathbb{D}, \mathbb{J}_3, \mathbb{M} \) and \( \mathbb{N} \), with \( \mathbb{J}_3 \) the Cartan generator of \( \mathfrak{su}(2) \), and \( \mathbb{M} \) and \( \mathbb{N} \) the Cartan generators of \( \mathfrak{usp}(4) \). The weight lattice is generated in the orthogonal basis as \( \epsilon_1 \mathbb{Z} \oplus \epsilon_2 \mathbb{Z} \oplus \delta_1 \mathbb{Z} \oplus \delta_2 \mathbb{Z} \), defined such that if \( v_\lambda \) has eigenvalues \((\Delta, -2j, -m, -n)\) under \((\mathbb{D}, 2\mathbb{J}_3, \mathbb{M}, \mathbb{N})\), then \( v_\lambda \) has weight
\[
\lambda = \frac{\Delta}{2} (\epsilon_1 + \epsilon_2) - j(\epsilon_1 - \epsilon_2) - m\delta_1 - n\delta_2.
\] (2.3)

In order for \( v_\lambda \) to be a lowest weight vector for a unitary irreducible representation of \( \mathfrak{osp}(4^*|4) \), it is necessary that \( \Delta \geq 0, (2j, m, n) \in \mathbb{Z}_{\geq 0}^3 \) and \( m \geq n \). On a generic state, \( |\psi\rangle \), corresponding to a \((p, q)\)-form on \( X \), we have
\[
\mathbb{M}|\psi\rangle = (q - d_H)|\psi\rangle, \quad \mathbb{N}|\psi\rangle = (p - d_H)|\psi\rangle.
\] (2.4)

As for superconformal algebras in higher dimension, the unitary irreducible lowest weight representations are classified by the value of the lowest weight. The lowest weight state is annihilated by the lowering operators of \( \mathfrak{osp}(4^*|4) \). In addition there are BPS bounds on the dimension \( \Delta \). If \( \Delta \) saturates the bound, then the lowest weight vector is annihilated by one or more additional generator. In the work [42], Singleton found a full classification of unitary irreducible lowest weight representations,

**Theorem 1.** Unitary, irreducible, lowest weight representations of \( \mathfrak{osp}(4^*|4) \) are obtained from the Verma module generated by the action of \( \mathfrak{osp}(4^*|4) \) on \(|\Delta, j, m, n\rangle\), by quotienting out null states. They come in the following types:

- **Generic ‘long’ representations** \( L(\Delta, j, m, n) \) with \( \Delta > 2(j + m + 1) \).
- ‘Semishort’ representations \( SS(j, m, n) \) with \( \Delta = 2(j + m + 1) \).
- ‘Short’ representations \( S(m, n) \) with \( \Delta = 2m \) and \( j = 0 \). These split into \( 1/2 \)-BPS representations with \( m = n \) and \( 1/4 \)-BPS otherwise.
In [42], Singleton, defined a superconformal index which receives contributions solely from the short and semi-short representations. The index can be used to count the (semi-)short representations up to the possibility of recombination into long multiplets. To define the index we pick a supercharge \( q \) and its conjugate, \( s = q^\dagger \) such that,

\[
\{ q, s \} = \mathcal{H} = \frac{1}{2}L_0 + J_3 + M.
\]  

(2.5)

Thus \( \mathcal{H} \) has eigenvalues,

\[
E := \frac{1}{2}\Delta - j - m.
\]  

(2.6)

Each (semi)-short multiplet contains states which are annihilated by \( \mathcal{H} \). Assuming a discrete spectrum, these states are in one-to-one correspondence with the cohomology classes of the supercharge \( s \). The choice of BPS bound breaks the full superconformal algebra down to the \( \mathfrak{su}(1|2) \) subalgebra commuting with \( q \) and \( s \). This subalgebra has Cartan generators \( T = -(M + 2J_3) \) and \( N \). The superconformal index counts states saturating the bound, graded by their charges under \( T \) and \( N \) and any additional mutually commuting global symmetry generators \( \{ J_i \} \). The fermion number is given by \( F = M + N \). The resulting index is given as,

\[
\mathcal{I}(t, y, Z) = \text{Tr} \left[ (-1)^F e^{-\beta \mathcal{H}} \tau^T y^N \prod_i z_i^{J_i} \right].
\]  

(2.7)

Assuming a discrete spectrum, the superconformal index is invariant under all deformations of the system which preserve the supercharges \( q \) and \( s \). In particular, the index is independent of the parameter \( \beta \).

The states with \( E = 0 \) in each (semi-)short multiplet of \( \mathfrak{osp}(4^*|4) \) transform in representations of the “little group” \( SU(1|2) \) their contribution to the index is the corresponding character. The 1/2- and 1/4-BPS short representations \( S(m, m) \) and \( S(m, n) \) with \( m > n \geq 0 \) have characters,

\[
I_{m, m} (\tau, y) = \tau^m \left[ \chi_m(y) - \tau \chi_{m-1}(y) \right],
\]

\[
I_{m, n} (\tau, y) = \tau^m \left[ (1 + \tau^2)\chi_n(y) - \tau (\chi_{n+1}(y) + \chi_{n-1}(y)) \right],
\]  

(2.8)

where \( \chi_n(y) \) is the character of the spin \( n/2 \) representation of \( \mathfrak{su}(2) \);

\[
\chi_n(y) = y^n + y^{n-2} \ldots + y^{-n}.
\]  

(2.9)

\( \chi_0(y) = 1 \) and we adopt the convention that \( \chi_{-1}(y) = 0 \). Similarly, the semi-short representation \( SS(j, m, n) \) yields the character,

\[
I_{j, m, n} (\tau, y) = \tau^{m+2j+2} \left[ (1 + \tau^2)\chi_n(y) - \tau (\chi_{n+1}(y) + \chi_{n-1}(y)) \right]
\]

\[= \tau^{2j+2}I_{m, n} (\tau, y).\]  

(2.10)
For a generic $\mathfrak{osp}(4^*|4)$ invariant theory, with (semi-) short spectrum,\
\[ S = \left( \bigoplus_{m=0}^{d_H} N^{(m,m)} S(m,m) \right) \oplus \left( \bigoplus_{m=n+1}^{d_H} \bigoplus_{n=0}^{d_H-1} N^{(m,n)} S(m,n) \right) \oplus \left( \bigoplus_{m=n}^{d_H-1} \bigoplus_{n=0}^{d_H-1} N^{(j,m,n)} SS(j,m,n) \right), \tag{2.11} \]

The upper bounds in the direct sums in equation (2.11) come from the geometric constraint that the holomorphic and antiholomorphic degrees $p$ and $q$ of any form are both bounded by $2d_H$.

The superconformal index can be expressed in terms of $N^{(m,m)}, N^{(m,n)}, N^{(j,m,n)} \in \mathbb{Z}_{\geq 0}[Z, Z^{-1}]$ as,
\[ Z(\tau, y; Z) = \sum_{m=0}^{d_H} N^{(m,m)} I_{m,m}(\tau, y) - \sum_{m=1}^{d_H} N^{(m,m-1)}(Z) I_{m,m-1}(\tau, y) \]
\[ + \sum_{m=n+2}^{\infty} \sum_{n=0}^{d_H-1} (-1)^{m-n} \tilde{N}^{(m,n)}(Z) I_{m,n}(\tau, y), \tag{2.12} \]

where for $m \geq n + 2$, we have,
\[ \tilde{N}^{(m,n)}(Z) = N^{(m,n)}(Z) + \sum_{k=\max\{0, m-1-d_H\}}^{m-n-2} (-1)^{k+1} N^{(k/2, m-2-k,n)}(Z). \tag{2.13} \]

Given the value of the index $I$ as a function of $\tau$ and $y$ it is possible to read off the numbers $N^{(m,m)}, N^{(m+1,m)}$ and $\tilde{N}^{(m,n)}$. The alternating signs in the expression for $\tilde{N}^{m,n}$ correspond to potential cancellations between different (semi-)short multiplets contributing to the index. In some special cases these cancellations are absent, and we can therefore uniquely determine the degeneracies of the corresponding multiplets. Specifically, we can uniquely determine the numbers of 1/2-BPS short multiplets $S(m,m)$, 1/4-BPS short multiplets $S(n+1,n)$, and also the semishort representations $SS(j,d_H-1,d_H-1)$. However, for the other short and semi-short multiplets of $\mathfrak{osp}(4^*|4)$ the index instead provides a lower bound for the degeneracies of these states.

As mentioned above, Singleton constructed a geometric action of $\mathfrak{osp}(4^*|4)$ on the space of differential forms on a hyperKähler cone $\mathcal{X}$. After the change of basis from $\mathbb{D}$ to $\mathbb{L}_0$, the inner product on the space of forms becomes,
\[ (\alpha, \beta) = \int_{\mathcal{X}} d^{4d_H} x \sqrt{g} \alpha \wedge \overline{\beta} e^{-\mu K}, \tag{2.14} \]

The polynomials $N^{(m,m)}$ and $N^{(j,m,n)}$ are characters of the global symmetry. When the global symmetry is non-Abelian, the polynomials will be invariant under the corresponding Weyl group. We will see that the $N^{(m,m)}$ have no $Z$ dependence for all cases we investigate.
where $4d_H = \dim \mathcal{X}$, and $K$ is the function on $\mathcal{X}$ corresponding to the special conformal generator $K$. In the case of flat affine space $\mathcal{X} = \mathbb{R}^{4n} = \mathbb{C}^{2n}$ the index can be calculated easily. In this case, the representatives in each superconformal multiplet which contribute to the index are in one to one correspondence with the holomorphic forms on the target space. The following analysis of the index is only rigorously correct for the flat space case, but will suggest a regularised definition of the index in the general case. The resulting index is a trace over the space of forms with vanishing $H$-eigenvalue on $\mathcal{X}$ graded by the triholomorphic isometries of the manifold, $G_H$, a Lie group whose Cartan subalgebra is generated by $J_i$, and two Cartan elements of the little group $su(1|2)$,

$$Z_{\mathcal{X}} = \text{tr}_{\Omega^*(\mathcal{X})} \left( (-)^{M+N} e^{-\beta H} y^N \tau^{-M-2J_3} \prod_i z_i^J_i \right). \quad (2.15)$$

More precisely, the trace is over the space of forms which are normalisable with respect to the inner product (2.14).

If we choose a particular complex structure, the hyperKähler space $\mathcal{X}$ becomes a holomorphic symplectic manifold. The $R$-symmetry on $\mathcal{X}$ yields a holomorphic $\mathbb{C} \times$-action on $\mathcal{X}$ under which the holomorphic symplectic form has charge 2. It is generated by a vector field $V_R$. On flat space, for $a \in \mathbb{C} \times$ it acts as $z \mapsto az$ and $\bar{z} \mapsto a^{-1} \bar{z}$. In general, the induced action on forms is given by

$$L_{V_R} = -M + N - 2J_3. \quad (2.16)$$

From this, we see that we can write

$$Z_{\mathcal{X}} = \text{tr}_{\Omega^*(\mathcal{X})} \left( (-)^{M+N} e^{-\beta H} \left( \frac{y}{\tau} \right)^N \tau^R \prod_i z_i^J_i \right). \quad (2.17)$$

As mentioned above, the ground states contributing to the index can be identified with cohomology classes of the supercharge $s$. From [42] chapter 7, we know that if $\beta$ is any form on $\mathcal{X}$ and $\alpha = \beta e^{-\mu K}$, then

$$s\alpha = \frac{1}{\sqrt{\mu}} \partial \beta e^{-\mu K}. \quad (2.18)$$

Hence, $s$ acts as a Dolbeault operator, up to the overall exponential factor and $Z_{\mathcal{X}}$ formally coincides with an index of the corresponding Dolbeault complex.

The condition of finite norm under (2.14) determines the space of forms we should consider. In the case of affine space $\mathcal{X} = \mathbb{C}^{2n}$ with complex coordinates $(q^i, \bar{q}_i), i = 1, \ldots, n$, the inner product is given by (2.14) with $K = \sum_i (|q^i|^2 + |\bar{q}_i|^2)$. In this case one finds that the Hilbert space is

$$\{ \text{states with } E = 0 \} \cong \mathbb{C}[q, \bar{q}, dq, d\bar{q}], \quad (2.19)$$

where the right hand side is the space of polynomials in the Grassmann even $q$ and $\bar{q}$ and the Grassmann odd $dq$ and $d\bar{q}$. In otherwords, the states annihilated

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by \( H \) are precisely the polynomial-valued holomorphic forms on \( X = \mathbb{C}^{2n} \) and the supercharge \( s \) can be identified with the Dolbeault operator \( \bar{\partial} \) acting on these forms. Each monomial has a definite \( \mathbb{C}^\times \)-grade corresponding to its degree which is essentially the dimension of the corresponding state. Thus, it is natural to work in the basis of homogeneous polynomials. (See [42] section 7.3 for further details of the flat space case).

The flat space result reviewed above can also be described in a slightly different way: the cohomology of the Dolbeault operator on polynomial-valued holomorphic forms also coincides with the sheaf cohomology of the affine variety \( X = \mathbb{C}^{2n} \) in the Zariski topology. Thus, the analytic Dolbeault cohomology of \( \mathbb{C}^{2n} \) with finite norm under the inner product is equal to the sheaf cohomology in the Zariski topology, provided that we restrict our attention to forms of finite \( \mathbb{C}^\times \)-grade. We will assume that this identification also holds for a general hyperKähler cone and hence we think of \( X \) as a variety from now on and assume that the space of \( E = 0 \) states is given by Dolbeault cohomology in the Zariski topology giving,

\[
Z_X = \sum_{p,q=0}^{2d_H} (-)^{p+q} \left( \frac{y}{\tau} \right)^{p-d_H} \text{tr}_{H^{p,q}(X)} \left( \tau^R \prod_i z_i^{J_i} \right). \tag{2.20}
\]

With this, we can use Dolbeault’s theorem to write

\[
Z_X = \sum_{p,q=0}^{2d_H} (-)^{p+q} \left( \frac{y}{\tau} \right)^{p-d_H} \text{tr}_{H^q(X;A^p(X))} \left( \tau^R \prod_i z_i^{J_i} \right). \tag{2.21}
\]

Dolbeault’s theorem (see [17]) states that for \( M \) a complex manifold

\[
H^q(M;A^p(M)) = H^{p,q}(M), \tag{2.22}
\]

where the right hand side is the \( \bar{\partial} \)-Dolbeault cohomology, and the left hand side is the sheaf cohomology of \( A^p(M) \), the sheaf of holomorphic \( p \)-forms on \( M \).

The problem with this definition of the superconformal index is that, in most examples, hyperKähler cones are not smooth. They have singularities, notably at the origin of the cone, but also along subspaces that intersect the origin. The space of forms is not defined at the singularities. Only for \( p = 0 \) is the summand well-defined. In order to define our index, it is necessary that we introduce some form of regularisation. In this work, we propose that the Dolbeault cohomology of the projective symplectic resolution of \( X \) is the appropriate regularisation. As above, we specifically mean the Dolbeault cohomology with respect to the Zariski topology, where this restriction from the analytic topology to the Zariski topology is a consequence of the finite norm restriction under the inner product (2.14).

We define a projective symplectic resolution. For the definition of words such as proper, projective etc. see [18]. A symplectic variety \( X \), is a variety, with an open set of smooth points \( X^{\text{reg}} \) on which is defined a holomorphic symplectic 2-form.
Definition 2. For $X$ a symplectic variety, a resolution of $X$ is a proper surjective morphism, $\pi : \tilde{X} \to X$, such that $\tilde{X}$ is smooth, and $\pi^{-1}(X^{\text{reg}}) \to X^{\text{reg}}$ is an isomorphism. If $\pi$ is a projective morphism, then this a projective resolution.

A symplectic resolution is one where $\pi^*\omega$, the pullback of the symplectic form on $X^{\text{reg}}$, the open set of smooth points in $X$, can be extended to a symplectic form on all of $\tilde{X}$.

We will assume that a projective symplectic resolution of $X$, $\tilde{X}$, exists. This is certainly the case for a large class of hyperKähler cones $X$ corresponding to Nakajima quiver varieties. We briefly recall their definition.

A quiver $\Gamma = (V, \Omega)$ is a set $V$ of vertices and $\Omega$ a set of arrows, $h = (i, j) \in \Omega$ corresponds to $i \to j$ for $i, j \in V$, we write $\text{in}(h) = i$, $\text{out}(h) = j$. We then provide the data $k \in \mathbb{Z}^V_{>0}$, $N \in \mathbb{Z}^V_{\geq 0}$; and $\zeta \in \mathbb{R}^{3V}$. With this, we define the affine space of complex matrices

$$M \equiv M(k, N) := \bigoplus_{(i,j) \in \Omega} \text{Hom}(\mathbb{C}^k_i, \mathbb{C}^k_j) \oplus \text{Hom}(\mathbb{C}^{k_i}, \mathbb{C}^{k_j}) \oplus \bigoplus_{i \in V} \text{Hom}(\mathbb{C}^N_i, \mathbb{C}^{k_i}) \oplus \text{Hom}(\mathbb{C}^{k_i}, \mathbb{C}^{N_i}) \quad (2.23)$$

This space is hyperKähler. Elements $(X, \tilde{X}, q, \tilde{q}) \equiv (X_{ij}, \tilde{X}_{ij}, q_i, \tilde{q}_i)_{i,j} \in M$ transform under $g \in G \equiv G_k = \prod_i \text{GL}(\mathbb{C}^{k_i})$ as

$$(X_{ij}, \tilde{X}_{ij}, q_i, \tilde{q}_i) \mapsto (g_j X_{ij} g_i^{-1}, g_i \tilde{X}_{ij} g_j^{-1}, g_i q_i, \tilde{q}_i g_i^{-1}). \quad (2.24)$$

This action is smooth (except for the zeroes), Hamiltonian, isometric and triholomorphic. So, we have three moment maps

$$\mu_{\mathbb{R}} := [X, X^\dagger] + [\tilde{X}, \tilde{X}^\dagger] + q q^\dagger - \tilde{q} \tilde{q} \in \prod_{a \in V} u(k_a),$$

$$\mu_{\mathbb{C}} := [X, \tilde{X}] + q \tilde{q} \in \prod_{a \in V} \text{gl}(k_a). \quad (2.25)$$

We define the Nakajima quiver variety as

$$\mathcal{M}_{\zeta_{\mathbb{R}}, \zeta_{\mathbb{C}}} \equiv \mathcal{M}_{\zeta_{\mathbb{R}}, \zeta_{\mathbb{C}}}^0(k, N) := \mu_{\mathbb{R}}^{-1}(\zeta_{\mathbb{R}}) \cap \mu_{\mathbb{C}}^{-1}(\zeta_{\mathbb{C}})/G. \quad (2.26)$$

A Nakajima quiver variety is hyperKähler by virtue of the hyperKähler quotient construction.

Such varieties arise as the Higgs branch moduli space of eight supercharge quiver gauge theories, where $\tilde{\zeta}$ correspond to the Fayet-Iliopoulos parameters in the field theory Lagrangian. Setting these parameters to zero, the singular or unresolved Nakajima quiver $X = \mathcal{M}_0 := \bar{\mu}^{-1}(0)/G$ is a hyperKähler cone with a triholomorphic homothety and thus gives rise to $\text{osp}(4^*|4)$ superconformal quantum mechanics.
Often, there are values of the level set \( \vec{\zeta} \in \mathbb{R}^3 \otimes \pi_1(G)^\vee \) such that \( \mathcal{M}_{\vec{\zeta}} := \vec{\mu}^{-1}(\vec{\zeta})/G \) is smooth, thus providing a resolution of the singularity. The values of \( \zeta \) such that \( \mathcal{M}_{\vec{\zeta}} \) is smooth are known as generic values. Either there are no generic values, or they form a subset of \( \mathbb{R}^3 \otimes \pi_1(G)^\vee \) whose complement is codimension 3. The \( \mathbb{C}^\times \)-action generated by \( R \) that we grade by is only defined if \( \zeta_C = 0 \). In this case, for \( (\zeta_R, 0) \) is generic, the resulting manifold, \( \mathcal{M}_{\vec{\zeta}_R, 0} \), is a projective symplectic resolution of the singular space \( \mathcal{M}_0 \). From \([16]\), we know that all holomorphic Hamiltonian vector field actions on \( \mathcal{M}_0 \) lift to an action on the projective symplectic resolution \( \mathcal{M}_{\vec{\zeta}_R} \), thus the resolution is equivariant.

In the following we will often specialize to the case of a singular Nakajima quiver variety \( \mathcal{X} := \mathcal{M}_0 \) and its resolution \( \tilde{\mathcal{X}} := \mathcal{M}_{\vec{\zeta}_R, 0} \), where \( (\zeta_R, 0) \) is generic. In the more general case of a hyperKähler cone that is not a Nakajima quiver variety, we restrict to cones such that the projective symplectic resolution exists, and that it is equivariant with respect to the \( \mathbb{C}^\times \times G_H \) action. We will abuse notation by using \( J_i \) and \( R \) to denote the corresponding actions on \( \tilde{\mathcal{X}} \).

We are now ready to define our regularised superconformal index, \( \mathcal{Z} \):

\[
\mathcal{Z}(\tilde{\mathcal{X}}; \tau, y, Z) := \sum_{p,q=0}^{2d_H} (-)^{p+q} \left( \frac{y}{\tau} \right)^{p-d_H} \text{tr}_{H^p(\tilde{\mathcal{X}}, \mathbb{A}^p(\tilde{\mathcal{X}}))} \left( \tau R \prod_i z_i^{J_i} \right). \tag{2.27}
\]

Note that \( \mathcal{Z} \) is an analog of the Hirzebruch \( \chi_-y \)-genus of \( \tilde{\mathcal{X}} \).

For a hyperKähler cone \( \mathcal{X} \) with a \( \mathbb{C}^\times \)-action such that the space of polynomial-valued holomorphic functions on \( \mathcal{X} \) is non-negatively graded under \( \mathbb{C}^\times \), the zeroth graded component being solely the constant functions, and the holomorphic symplectic form is homogeneous with respect to this grading, Namikawa in \([36]\) showed that there are only finitely many non-isomorphic projective symplectic resolutions of \( \mathcal{X} = \mathcal{M}_0 \). We may ask, given two non-isomorphic equivariant projective symplectic resolutions of \( \mathcal{X} \), \( \tilde{\mathcal{X}} \) and \( \tilde{\mathcal{X}}' \), do they have the same superconformal index? If this is the case, then the index computed on \( \tilde{\mathcal{X}} \) can be regarded as an invariant of \( \mathcal{X} \), and our regularisation by working on the resolution makes sense. We conjecture that this is indeed the case for all such hyperKähler cones, and will write

\[
\mathcal{Z} \equiv \mathcal{Z}(\mathcal{X}). \tag{2.28}
\]

We will present various pieces of evidence for this in the following. In particular we will prove that this property holds in various limits and specialisations of the index. We also perform some explicit calculations to verify this property for quivers of low dimension.

First, we note that that any choice of projective symplectic resolution of a Nakajima quiver variety will give the same index if we set \( \tau \) to 1. This is because of the following theorem:
Theorem 2. (A simple generalisation of 3.4 in [39]) If \( \vec{\zeta} \) and \( \vec{\zeta}' \) are both generic, then \( M_{\vec{\zeta}} \) and \( M_{\vec{\zeta}'} \) are \( G_H \)-equivariant diffeomorphic.

Essentially this theorem holds because provided we forget about the \( \mathbb{C}^\times \)-grading by setting \( \tau = 1 \) we are free to turn on \( \zeta_C \), and so can form a homotopically unique path through regular values from one resolution to the other. This ceases to be true if we turn on \( \tau \), which grades with respect to a preferred complex structure. To preserve the \( \mathbb{C}^\times \)-grading we are required to set \( \zeta_C = 0 \) and typically the space of generic values on this slice of the parameter space is disconnected by walls of codimension one. So we are claiming that there is no wall crossing for the superconformal index of a resolved Nakajima quiver variety. We discuss this in more detail in Appendix B with examples.

Two further pieces of evidence for our conjecture are that two limits of the superconformal index discussed in section 3.1, the Poincaré polynomial and the Hilbert series, are known to be the same for different resolutions. The Poincaré polynomial does not depend on the choice of resolution, because all resolutions are diffeomorphic. The Hilbert series does not depend on the choice of resolution, because it is equal to the count of holomorphic functions on the unresolved space, [23].

2.1 Localisation

Now that we have defined the index, it remains to compute it. We show that in many cases the superconformal index can be computed exactly. We write \( T \) for the maximal torus of \( \mathbb{C}^\times \times G_H \), and will assume that \( \mathcal{X} \) has a unique fixed point under \( T \) and that \( \tilde{\mathcal{X}} \) has finitely many fixed points under \( T \). This is indeed the case for all Nakajima quiver varieties that we investigate in this paper.

We will express \( Z \) as a sum of equivariant Euler characters of equivariant coherent sheaves. This means that we are able to use localisation theorems from equivariant K-theory to exactly compute \( Z \) as a sum over fixed points. Writing \( \chi_T \) for the \( T \)-equivariant Euler character, it is defined for any holomorphic \( T \)-equivariant coherent sheaf \( \mathcal{V} \) as

\[
\chi_T(\mathcal{V}) := \sum_{i=0}^{2d_H} (-)^i \text{ch}_T H^i(\tilde{\mathcal{X}}, \mathcal{V}).
\]  

It is immediate from equation (2.27) that

\[
Z(\mathcal{X}; \tau, y, Z) := \sum_{p=0}^{2d_H} (-)^p \left( \frac{y}{\tau} \right)^{p-d_H} \chi_T(A^p(\tilde{\mathcal{X}})).
\]  

We pause to note the importance of the factor of \( \tau^{d_H-p} \). If we had not taken this factor out, then we would have been grading with respect to \( -M - 2J_3 \). This is an action on the space of forms, but not an action on the base manifold and hence we cannot form an equivariant action on the sheaves with this action.
However, $R$ and $J_i$ are actions on $\mathcal{X}$, and hence we have written $Z$ as a sum of equivariant Euler characters.

The localisation theorem for equivariant K-theory is due to [44], though a preliminary version is in [5]. This theorem tells us that the natural inclusion map, $\iota : \tilde{X}^T \to \tilde{X}$, of the $T$-fixed points of $\tilde{X}$ into $\tilde{X}$ induces a homomorphism, $\iota^*$, that is an isomorphism after localisation.

**Proposition 1.** (from [35]) Let $V$ be a $T$-equivariant locally free sheaf on $\tilde{X}$. Then we have that
\[
\chi_T(V) = \sum_{x \in \tilde{X}^T} \iota_x^*(V) \text{PE}[\chi_T(T_x^*\tilde{X}; \tau, Z)].
\] (2.31)

Here $\tilde{X}^T$ is the space of $T$-fixed points of $\tilde{X}$, $\iota_x : \{x\} \hookrightarrow \tilde{X}$ is the inclusion of the fixed point $x$ in $\tilde{X}$, and $\chi$ is the $\mathbb{C}^\times \times G_H$-equivariant Euler character.

PE is the plethystic exponential. It is defined as
\[
\text{PE}[f(t_1, \ldots, t_n)] := \exp \left( \sum_{r=1}^{\infty} \frac{f(t_1^r, \ldots, t_n^r)}{r} \right).
\] (2.32)

The plethystic exponential of a polynomial is as follows
\[
\text{PE} \left[ \sum_i t_i - \sum_j s_j \right] = \prod_i (1 - s_j) \prod_i (1 - t_i),
\] (2.33)

where the $t_i$ and $s_j$ are monomials.

We can now compute our superconformal index using localisation.
\[
Z(\mathcal{X}) = \sum_{p=0}^{2d_H} (-)^p \left( \frac{y}{\tau} \right)^{p-d_H} \chi(A^p(\tilde{X}); \tau, Z)
\]
\[
= \sum_{p=0}^{2d_H} (-)^p \left( \frac{y}{\tau} \right)^{p-d_H} \sum_{x \in \tilde{X}^T} \text{ch}_T(A^p(T_x^*\tilde{X}); \tau, Z) \text{PE}[\chi_T(T_x^*\tilde{X}; \tau, Z)]
\] (2.34)
\[
= \left( \frac{\tau}{y} \right)^{d_H} \sum_{x \in \tilde{X}^T} \text{PE} \left[ (1 - \frac{y}{\tau}) \sum_{x \in \tilde{X}^T} \text{ch}_T(T_x^*\tilde{X}; \tau, Z) \right].
\]

We introduce the following notation for the contribution at each fixed point,
\[
Z \equiv \left( \frac{\tau}{y} \right)^{d_H} \sum_{x \in \tilde{X}^T} Z_x
\]
\[
\equiv \left( \frac{\tau}{y} \right)^{d_H} \sum_{x \in \tilde{X}^T} \text{PE} \left[ (1 - \frac{y}{\tau}) \sum_{x \in \tilde{X}^T} m_\alpha(\tau, Z) \right],
\] (2.35)

where $\alpha = (\alpha_0, \gamma)$, $\alpha_0 \in \mathbb{Z}$ and $\gamma$ is a weight of $G_H$; $m_\alpha(\tau, Z) = \tau^{\alpha_0} Z^\gamma$ is a monomial; and $J_x$ is the collection of $T$-weights of the module $T_x^*(\tilde{X})$. 
An alternative procedure to evaluating these quantities is via the use of the Jeffrey-Kirwan localisation theorem, [22]. This theorem is for the evaluation of the integral of a form over a symplectic quotient. In the case that $\tilde{X}$ is a hyperKähler quotient, we can reduce the evaluation of $Z(X)$ to this problem via the use of Grothendieck-Riemann-Roch theorem. We can then use the different choices of chambers in the Jeffrey-Kirwan residue, corresponding to the different choices of resolutions, to explicitly see that the superconformal index does not depend on the choice of resolution.

3 Properties of the superconformal index

In order for $Z(X)$ defined above to be a superconformal index of $\mathfrak{osp}(4^*|4)$ representations, it is necessary that it coincides with the form predicted in (2.12). Notably, there must be a $Z[Z^\pm 1]$-expansion in 1/2- and 1/4-BPS $\mathfrak{su}(1|2)$ characters, and the coefficients of all 1/2-BPS representations are positive integers with no $Z$-dependence\textsuperscript{5}. We shall show that this is true, at least for all examples we investigate.

We use the result in appendix A.2 this tells us that if $Z$ obeys four equations then it obeys the necessary property to be a $\mathfrak{osp}(4^*|4)$ superconformal index listed above. There is a further positivity condition imposed on the coefficient of the semishort representations $SS(j/2,d_H-1,d_H-1)$. We show this is indeed true by investigating what we call the Hilbert series limit of the superconformal index.

The four equations are

$$Z(\tau, y, Z) = Z(\tau, 1/y, Z).$$

(3.1)

Writing

$$Z(\tau, y, Z) = \sum_{a=0}^{\infty} \sum_{b=-d_H}^{d_H} \alpha_{a,b}(Z) \tau^a y^b = \sum_{a=0}^{\infty} \sum_{b=-m}^{m} \alpha_{a,b}(Z) \tau^{a+b} \left( \frac{y}{\tau} \right)^b,$$

(3.2)

we have

$$\alpha_{a,b} = 0 \text{ for } a < |b|,$$

(3.3)

$$Z(\tau, \tau, Z) \in \mathbb{Z}_{\geq 0}, \text{ so that } \frac{d}{d\tau} Z(\tau, \tau, Z) = \frac{d}{dz_i} Z(\tau, \tau, Z) = 0, \text{ and}$$

(3.4)

$$\lim_{y/\tau \to 0} Z(\tau, y, Z) = \sum_{a=0}^{d_H} \alpha_{a,-a}(Z) \left( \frac{\tau}{y} \right)^a \in \mathbb{Z}_{\geq 0} \left[ \frac{\tau}{y} \right].$$

(3.5)

\textsuperscript{5}Note that there is a condition on the coefficients of the 1/4-BPS states $S(n+1,n)$. We do not investigate this condition in this paper.
Note that equation (3.1) is manifest\(^6\). This follows because the fugacity \(y\) corresponds to the Cartan generator of the \(SU(2)\) Lefschetz action, whose raising operator is wedging with \(\omega_C\). The resulting \(y \mapsto 1/y\) symmetry is known as Serre duality. Equation (3.2) is true for all examples where the resolution has isolated fixed points and equation (3.3) is true for all Nakajima quiver varieties.

We show that equation (3.3) holds in all examples we investigate. In all examples we investigate we find that the \(y \mapsto 1/y\) symmetry is preserved at the level of the fixed point contributions to \(Z\). We shall use to show that in a Taylor expansion in powers of \(\tau\), for all monomials of the form \(\tau^a y^b\) we have \(a \geq |b|\).

In the superconformal index, the contribution from each fixed points has a factor of \(1 - \tau/y\) in the plethystic exponential. Assuming a \(y \mapsto 1/y\) symmetry at each fixed point, there must also be a factor of \(1 - y\tau\). Furthermore, since the solution is a finite polynomial in \(y\), we expect to be able to write the superconformal index as

\[
Z = \sum_{x \in \mathcal{X}} \text{PE}[p_x(\tau, Z)(1 - \tau y)(1 - \tau/y)] ,
\]

where \(p_x\) is a Laurent polynomial in \(\tau\) and \(Z\) with positive integer coefficients, \(p_x(\tau, Z) \in \mathbb{Z}_{\geq 0}[\tau^{\pm 1}, Z^{\pm 1}]\). One can then directly calculate that equation (3.3) does indeed hold for every fixed point.

Note that in order for equation (3.6), it is necessary that, for \(x\) a \(T\)-fixed point of \(\mathcal{X}\),

\[
\text{ch}_T(T^*_x(\mathcal{X}); \tau, Z) = p_x(\tau^{-1}, Z^{-1}) + \tau^2 p_x(\tau, Z) .
\]

Equation (3.4), that the superconformal index at \(\tau = y\) is a positive integer, is an immediate consequence of equation (2.34).

In order to conclude that \(Z(\mathcal{X})\) is in the form of a \(\mathfrak{osp}(4^*|4)\) superconformal index it is necessary that equation (3.5) holds, namely

\[
\lim_{\substack{\tau \to 0 \\ y/\tau \text{ finite}}} Z_{\mathcal{X}}(\tau, y, Z) \in \mathbb{Z}_{\geq 0} \left[ \frac{y}{\tau} \right] .
\]

We shall show this in the next subsection for all Nakajima quiver varieties. It follows from the fact that the \(\tau \to 0\) limit with \(y/\tau\) fixed of the superconformal index is the Hirzebruch \(\chi_{-y}\)-genus of the \(\mathbb{C}_x\)-fixed point submanifold\(^7\). From \([29]\), it is known that this submanifold is compact, and hence the superconformal index is the Poincaré polynomial, and moreover it is known the space’s odd homology vanishes. Hence the superconformal index is a positive polynomial, the fact that \(Z_{\mathcal{X}}(\tau, y, Z) \in \mathbb{Z}_{\geq 0}\) means that the \(\tau \to 0\) limit cannot depend on \(Z\).

\(^6\)If we taken \(\tau\) to be the fugacity for the scaling symmetry, \(R\), this would be equivalent to using a fugacity \(\tilde{y} := y\tau\) to count the \(p\)-grading of a \((p,q)\)-form. We would have had less trouble using equivariant localisation theorems, but at the cost of losing the manifest \(y \mapsto 1/y\) symmetry.

\(^7\)It is not a superconformal index, as this space is not the resolution of a cone.
3.1 Limits of the superconformal index

In this subsection, we shall consider three limits of the superconformal index which exhibit interesting behaviour. In particular, we consider limits where the superconformal index reduces to the Poincaré polynomial of $\mathcal{X}$ and to its Hilbert series. Finally, we show that if one hyperKähler cone $\mathcal{Y}$ is contained inside another $\mathcal{X}$ as a fixed point subspace of a triholomorphic isometry then there is a limit in which the superconformal index of $\mathcal{X}$ reduces to that of $\mathcal{Y}$.

We start by considering the generating series for Borel-Moore homology. For $M$ a manifold (possibly non-compact), the Borel-Moore homology of $M$ is defined as the relative singular homology of the one point compactification of $M$, $\overline{M}$, with respect to the point at infinity, and so for compact manifolds the Borel-Moore homology is identical to the singular homology.

$$H_{BM}^*(M) := H_*(\overline{M}, \{\infty\}). \quad (3.9)$$

Pick a $\mathbb{C}^\times$-action on $\tilde{\mathcal{X}}$ defined by some

$$\lambda: \mathbb{C}^\times \hookrightarrow \mathbb{C}^\times \times T_H = T \cong (\mathbb{C}^\times)^{r+1}. \quad (3.10)$$

We assume that this is a generic action, this means that it has isolated fixed points. We further assume that $\lim_{t_1 \to 0} \lambda(t_1) = \infty$. We have defined $r := \text{rk}(G_H)$.

Theorem 3.7 (3) and (4) of [34] easily lift to any Nakajima quiver variety with isolated fixed points. It tells us that the homology vanishes at odd degree; is freely generated at even degree; and that each fixed point contributes a single generator, whose homology degree is given by the dimension of the (+)-attracting set at that point. That is, for $x \in \tilde{\mathcal{X}}^T$ a fixed point, the (+)-attracting set is

$$S_x = \{ p \in \tilde{\mathcal{X}} | \lim_{t \to 0} \lambda(t) \cdot p = x \}. \quad (3.11)$$

We then define the Poincaré polynomial as the generating function of equivariant Borel-Moore homology:

$$P_{\tilde{\mathcal{X}}}(q) := \sum_{i=0}^{2d_H} \dim \left( H_{BM}^{2i}(\tilde{\mathcal{X}}) \right) q^i = \sum_{x \in \tilde{\mathcal{X}}^T} q^{\dim C S_x}. \quad (3.12)$$

We now show that the contribution to the superconformal index at each fixed point of $\tilde{\mathcal{X}}$, contains information about the dimension of the (+)-attracting set. A generic choice of $\lambda$ is given by

$$\lambda(t) = (t^{n_1}, t^{n_2}, \ldots, t^{n_r}) \quad (3.13)$$
for some
\[ 0 > n_1 > \cdots > n_r \gg m. \] (3.14)

We write our superconformal index as a function of the fugacities \( \tilde{y}, \tau, Z \), where \( \tilde{y} = y/\tau \). Then one makes the following replacements for the fugacities appearing in the formula (2.34) for \( Z^X \):
\[ \tau \mapsto s^m, \quad z_i \mapsto s^{n_i}, \] (3.15)
for \( s \) a non-zero complex number. Finally, one takes the limit \( s \to 0 \). Because the powers of \( s \) appearing in the numerator and denominator of each factor of \( Z_x \) agree, the limit of the index is a Laurent polynomial in \( \tilde{y} \) with positive integer coefficients.

For a particular fixed point, \( x \in \tilde{X}^T \), one has a product of the form
\[
Z_x(\tilde{y}, s) = \tilde{y}^{-d_H} \prod_{\alpha \in J_x} \frac{1 - \tilde{y} s^{\alpha_0 m + \sum_i \gamma_i n_i}}{1 - s^{\alpha_0 m + \sum_i \gamma_i n_i}}.
\] (3.16)
where we have defined \( \ell_\alpha := \alpha_0 m + \sum_i \gamma_i n_i \). Note that the \( \mathbb{C}^X \)-action being generic necessarily means that \( \ell_\alpha \neq 0 \) for all \( \alpha \in J_x \) for all \( x \in \tilde{X}^T \). We then take the limit \( s \to 0 \) and obtain
\[
\lim_{s \to 0} Z_x = \tilde{y}^{-d_H + |\{ \alpha \mid \ell_\alpha < 0 \}|} \] (3.17)
Since the sign of \( \ell_\alpha \) tells us about whether the tangent direction \( \alpha \) at the fixed point is an attracting or repelling one, we have that \( |\{ \alpha \mid \ell_\alpha < 0 \}| = \dim_{\mathbb{C}} S_x \).
Thus, the Poincaré polynomial is given by
\[
P_{\tilde{X}}(\tilde{y}) = \tilde{y}^{d_H} \lim_{s \to 0} Z_x(\tilde{y}, \tau = s^m, z_i = s^{n_i}). \] (3.18)

Since the superconformal index is a grading under \( \mathfrak{su}(1|2) \) and global symmetries \( G_H \), it has a character expansion as
\[
Z = \sum_{R_1, R_2} \chi_{\mathfrak{su}(1|2)}(R_1; \tau, y) \chi_{G_H}(R_2; Z), \] (3.19)
where the sum is over finite dimensional irreducible representations of \( \mathfrak{su}(1|2) \) and \( G_H \); and \( \chi_G(R; W) \) is the character of the representation \( R \) of \( G \), with fugacities \( w_1, \ldots, w_{\text{rk}(G)} \).

In equation (2.28), we find the characters of all finite dimensional irreducible representations of \( \mathfrak{su}(1|2) \). Importantly, if we set \( \tau = y \), then we see that the only non-zero contributions to the index are from the 1/2-BPS short multiplets \( S(m, m) \), which each contribute unity to the superconformal index. If we look in the formula (2.31) for the superconformal index, we see that we get contribution of 1 for each fixed point, so we may identify each fixed point with a 1/2-BPS multiplet.
If we keep $\tau/y$ fixed while sending $\tau$ as well as the fugacities $Z$ to zero we compute the Poincaré polynomial with grading by homological degree. Since the \(1/2\)-BPS states are necessarily invariant under $G_H$ (see [2]), one can see that the only terms that survive in $Z$ are the $1/2$-BPS multiplets. Their contribution is $(\tau/y)^m$, where $m$ is the highest power of $y$ that was in the $\mathfrak{su}(1|2)$ character.

This means that if $P = \sum_{n=0}^{d_H} b_{2n} q^n$ is the Poincaré polynomial\(^8\) of $\tilde{X}$, then we can reconstruct the $1/2$-BPS state spectrum as

$$
\mathcal{H}_{1/2\text{-BPS}} = \bigoplus_{n=0}^{d_H} b_{2(d_H-n)} S(n, n).
$$

(3.20)

Note that this is nothing more than the statement that the $1/2$-BPS multiplets of the superconformal algebra are in one-to-one correspondence with the compactly supported cohomology classes of $\tilde{X}$, and Poincaré duality relates this to the Borel-Moore homology. In the case where $X$ is the moduli space of Yang-Mills instantons, this correspondence was anticipated in [2].

Equation (3.20) clearly shows that the multiplicities of $1/2$-BPS states are non-negative integers independent of the flavour fugacities. Together with the earlier results in this section, this means that $Z$ satisfies all the criteria imposed by the condition that it is a superconformal index of an $\mathfrak{osp}(4^*|4)$ representation.

In the works [20, 21], Hausel gives the generating function for the Poincaré polynomial of any Nakajima quiver variety. Assuming that the variety has isolated fixed points, we can then use this to give the full $1/2$-BPS spectrum of the theory.

While further results such as [27], mean that for $A$-type quivers, as well as the Coulomb branch of $DE$-type quivers, the $1/2$-BPS states highest states are given by the fusion product of fundamental Kirillov-Reshetikhin modules of $ADE$-type, and can be given by Hatayama’s fermionic form [19].

Now we discuss the Hilbert series limit of the superconformal index. In the works [3-12], the Hilbert series, $\text{HS}$, is the count of polynomial valued holomorphic functions on $\mathcal{X}$ graded by global symmetries and the $\mathbb{C}^\times$-action. Note that for any variety $Y$, $\Gamma(Y, \mathcal{O}_Y)$ is defined as the space of polynomial valued global holomorphic functions on $Y$. The Hilbert series is

$$
\text{HS}(\mathcal{X}) = \text{tr}_{\Gamma(\mathcal{X}, \mathcal{O}_\mathcal{X})} \left( \tau^R \prod_i z_i^2 \right).
$$

(3.21)

\(^8\)The reason that this sum goes up to $d_H$ and not to $2d_H$ as one might naively expect, is because for Nakajima quiver varieties $\mathcal{X}$ is homotopic to one of its Lagrangian subvarieties, for example see [16].
One can easily see from the definition of $Z$ that the coefficient of $y^{d_H}$ divided by $\tau^{d_H}$ is $\chi(O_{\tilde{X}})$, the equivariant Euler character of the structure sheaf. Explicitly,

$$
\lim_{y \to \infty} y^{-d_H} \tau^{-d_H} Z_X = \sum_{q=0}^{2d_H} (-)^q \text{tr} \ H^q(\tilde{X}) \left( \tau^R \prod_i z_i^{J_i} \right).
$$

(3.22)

For Nakajima quiver varieties, we have that the space of holomorphic functions on $\tilde{X}$ and $X$ are isomorphic as graded Poisson algebras. Namely,

$$
\pi^* : \Gamma(X, O_X) \to \Gamma(\tilde{X}, O_{\tilde{X}})
$$

(3.23)

is an isomorphism of graded Poisson algebras, see [23].

We further have (from [16]) that for all $q > 0$,

$$
H^q(\tilde{X}, O_{\tilde{X}}) = 0.
$$

(3.24)

This means that for Nakajima quiver varieties the coefficient of $y^d$ in $Z$ divided by $\tau^d$ is the Hilbert series,

$$
\text{HS}(X) = \lim_{y \to \infty} y^{-d_H} \tau^{-d_H} Z_X.
$$

(3.25)

Like the limit corresponding to the Poincare polynomial, the Hilbert series limit also yields a precise counting of certain superconformal multiplets. In particular, a holomorphic function of charge $r \in \mathbb{N}_0$ under the $\mathbb{C}^\times$-action generated by $R$ corresponds to a (semi)-short multiplet,

$$
S(d_H, d_H) \quad \text{for } r = 0,
$$

$$
S(d_H, d_H - 1) \quad \text{for } r = 1,
$$

$$
SS(R/2 - 1, d_H - 1, d_H - 1) \quad \text{for } r \geq 2.
$$

(3.26)

This means that

$$
\text{HS}(X) = N^{(d_H, d_H)} + N^{(d_H, d_H - 1)} \tau + \sum_{r=2}^{\infty} N^{(r/2 - 1, d_H - 1, d_H - 1)} \tau^r.
$$

Finally, we discuss the limit of the superconformal index that gives the superconformal index of a fixed point subspace. Let $X$ be a hyperKähler cone, $\tilde{X}$ its projective symplectic resolution with isolated fixed points under $T = \mathbb{C}^\times \times T_H$. If $N \subset \tilde{X}$ is the fixed point subspace (generally not connected) under a closed Lie subgroup of isometries $T' \subset T_H$, then we have that $\tilde{N}$ will be the projective symplectic resolution of a disjoint union of hyperKähler cones $N$. This follows from the fact that the resolution is $T_H$-equivariant.

---

9Note because of the $y \to 1/y$ symmetry, one can swap the limit of $y$ to $\infty$ for a limit of $y$ to 0, at the cost of multiplying by $y^{d_H}$ instead of $y^{-d_H}$. 
The superconformal index of \( \mathcal{N} \) will be graded by \( \mathbb{C}^\times \times T_H/T' \). Suppose without loss of generality that \( T' = U(1) \) (without loss of generality as we can do this \( \text{rk} \ T' \) times). We write \( \bar{z}_i \) for \( i = 1, \ldots, \text{rk} \ T_H - 1 \) as the fugacities of \( T_H/T' \). The inclusion of \( T' \) into \( T_H \) defines a relabelling of fugacities \( z_i \mapsto s^{f_i} \bar{z}'_i(i) \), where \( f_i \in \mathbb{Z} \) and \( s \) is the fugacity for \( T' = U(1) \). The superconformal index of \( \mathcal{N} \) will be the superconformal index of \( \mathcal{X} \), but with all the cotangent directions at a fixed point in \( \tilde{\mathcal{X}} \) that have any charge under \( s \) thrown away. One can achieve this by sending \( s \to 0 \):

\[
Z_{\mathcal{N}}(\tau, y; \bar{Z}) = \lim_{s \to 0} Z_{\mathcal{X}}(\tau, y; s^{f_1} \bar{z}'_1(1), \ldots, s^{f_{\text{rk}(T_H)}} \bar{z}'_{(\text{rk}(T_H))}).
\]  

(3.27)

A similar limit exists where we send \( \tau \to 0 \), restricting to the fixed point submanifold of the \( \mathbb{C}^\times \)-action, \( \tilde{\mathcal{X}}^{\mathbb{C}^\times} \). This necessarily breaks the hyperKähler structure, but the superconformal index, \( Z(\tilde{\mathcal{X}}^{\mathbb{C}^\times}) \), is still defined. The space \( \tilde{\mathcal{X}}^{\mathbb{C}^\times} \) is projective and connected (lemma 7.3.3 and proposition 7.3.4 of [29]) for all Nakajima quiver varieties. This means that the de Rham cohomology and the Borel-Moore homology coincide. One can easily compute that

\[
Z(\tilde{\mathcal{X}}^{\mathbb{C}^\times}) = \left( \frac{\tau}{y} \right)^{d_{H^-}} \sum_{x \in \tilde{\mathcal{X}}^{\mathbb{C}^\times}} \prod_{(\alpha,\gamma) \in \mathcal{J}_x, \alpha_0 = 0} \frac{1 - \frac{y}{Z} Z^{\gamma}}{1 - Z^{\gamma}}
\]

\[
= \sum_{r} \text{tr}_{H^r(\tilde{\mathcal{X}}^{\mathbb{C}^\times})} \left( (-)^{r} y^{N} \prod_{i} z_i^{f_i} \right). \tag{3.28}
\]

It is known that the odd cohomology vanishes. So, we have that namely that \( Z(\tilde{\mathcal{X}}^{\mathbb{C}^\times}) \in \mathbb{Z}_{\geq 0}((Z))[\tau/y^{\pm 1}] \). However, we also have that \( Z_X(\tau, \tau) \in \mathbb{Z}_{\geq 0} \). From this we may conclude that

\[
\lim_{\gamma \to 0} \frac{Z(\tilde{\mathcal{X}}^{\mathbb{C}^\times})}{\gamma \text{ fixed}} \in \mathbb{Z}_{\geq 0} \left[ \frac{\tau}{y} \right]. \tag{3.29}
\]

This confirms that equation (3.3) does indeed hold, and hence the superconformal index does obey the necessary properties to be a superconformal index of \( \mathfrak{osp}(4^*|4) \).

We consider a simple example to illustrate each of the different limits. Take \( \mathbb{C}^4 \), with coordinate ring \( \mathbb{C}[X_1, \bar{X}_1, X_2, \bar{X}_2] \). \( X_i \) is charged\(^{10} \) as \( \tau s_i \) and \( \bar{X}_i \) is charged as \( \tau/s_i \) for \( i = 1, 2 \). \( \tau/s_i \) and \( \tau s_i \) are fugacities for \( \mathbb{C}_{1,1}^\times \times \mathbb{C}_{1,2}^\times \) rotating the target space \( \mathbb{C}_{3}^\times = \mathbb{C}_{1,1} \times \mathbb{C}_{1,2} \). The diagonal subgroup of \( \mathbb{C}_{1,1}^\times \times \mathbb{C}_{1,2}^\times \) is counted with the same fugacity as the diagonal subgroup of \( \mathbb{C}_{2,1}^\times \times \mathbb{C}_{2,2}^\times \), this is the \( \mathbb{C}^\times \) generated by \( R \). \( dX_i \) and \( d\bar{X}_i \) are charged as \( y s_i \) and \( y/s_i \) respectively, where the fugacity \( y \) is for a \( \mathbb{C}^\times \) that rotates the cotangent fibres. The superconformal index is,

\[
Z_{\mathbb{C}^4}(\tau, y; s_1, s_2) = \left( \frac{\tau}{y} \right)^2 \prod_{i=1}^{2} \frac{(1 - y s_i)(1 - y/s_i)}{(1 - \tau s_i)(1 - \tau/s_i)}. \tag{3.30}
\]

\(^{10}\)One should think of this as the charge of the operator given by multiplication by \( X_i \), and similarly for the other variables.
The Hilbert series is the coefficient of the highest power of $y$ divided by $\tau^2$,
\[ \text{HS}(\mathbb{C}^4) = \prod_{i=1}^{2} \frac{1}{(1 - \tau s_i)(1 - \tau/s_i)}. \quad (3.31) \]

The Borel-Moore homology of $\mathbb{C}^4$ has only one generator, the fundamental class $[\mathbb{C}^4]$. Hence
\[ H^\text{BM}_i(\mathbb{C}^4) = \begin{cases} \mathbb{Z} & \text{if } i = 8, \\ 0 & \text{otherwise}. \end{cases} \quad (3.32) \]

To form the Poincaré polynomial we rewrite $\mathbb{Z}$ in terms of $\tilde{\tau} = s - 5$, $s_1 \mapsto s - 2$, and $s_2 \mapsto s - 1$. This gives
\[ \mathbb{Z}_{\mathbb{C}^4}(s^{-5}, s^{-5} \tilde{\tau}; s^{-2}, s^{-1}) = \tilde{\tau}^{-2}(1 - \tilde{\tau}s^{-7})(1 - \tilde{\tau}s^{-3})(1 - \tilde{\tau}s^{-6})(1 - \tilde{\tau}s^{-4}). \quad (3.33) \]

From this we see
\[ P_{\mathbb{C}^4}(\tilde{\tau}) = \tilde{\tau}^4 = \lim_{s \to 0} \mathbb{Z}_{\mathbb{C}^4}(s^{-5}, s^{-5} \tilde{\tau}; s^{-2}, s^{-1}). \quad (3.34) \]

We consider restricting to the hyperKähler submanifold invariant under the subgroup $\{(x, 1/x) | x \in \mathbb{C}^\times\} \subset \mathbb{C}^\times_{1,1} \times \mathbb{C}^\times_{1,2}$, namely $\mathbb{C}^\times_{2,1} \times \mathbb{C}^\times_{2,2}$. We do this by discarding all generators with non-zero power of $s_1$. To take this limit, we take the limit $s_1 \to 0$ in the index. This gives
\[ \mathbb{Z}_{\mathbb{C}^2}(\tau, y; s_2) = \frac{\tau (1 - y s_2)(1 - y/s_2)}{y (1 - \tau s_2)(1 - \tau/s_2)}. \quad (3.35) \]

We then get the Hilbert series by taking the highest power of $y$ divided by $\tau$,
\[ \text{HS}(\mathbb{C}^2) = \frac{1}{(1 - \tau s_2)(1 - \tau/s_2)}, \quad (3.36) \]

while the Poincaré polynomial is
\[ P_{\mathbb{C}^2}(\tilde{\tau}) = \tilde{\tau}^2 = \lim_{s \to 0} \mathbb{Z}_{\mathbb{C}^4}(s^{-5}, s^{-5} \tilde{\tau}; s^{-2}). \quad (3.37) \]

## 4 Examples

### 4.1 Instanton moduli space

\[ \begin{array}{c} \text{k} \\ \text{N} \end{array} \]

**Figure 1:** The ADHM quiver.

The Nakajima quiver variety associated to the ADHM quiver, in figure 1, has known fixed points, with the associated tangent space at the fixed point calculated in [28, 37]. The global symmetry is
\[ G_H = \text{SU}(N) \times \text{SU}(2), \quad (4.1) \]
where the $SU(N)$ is the flavour symmetry associated to the box and the $SU(2)$ is a flavour symmetry associated to the adjoint of $U(k)$. The fields are $X, \tilde{X} \in \text{End}(\mathbb{C}^k)$, $Q \in \text{Hom}(\mathbb{C}^N, \mathbb{C}^k)$ and $\tilde{Q} \in \text{Hom}(\mathbb{C}^k, \mathbb{C}^N)$. The moment map equations defining the quiver are known as the ADHM equation, and are

\[
\begin{align*}
\mu_R &= [X, X^\dagger] + [\tilde{X}, \tilde{X}^\dagger] + QQ^\dagger - \tilde{Q}^\dagger \tilde{Q}, \\
\mu_C &= [X, \tilde{X}] + QQ.
\end{align*}
\] (4.2)

We use fugacities $z_1, \ldots, z_N$ for $SU(N)$ and $x$ for $SU(2)$.

The ADHM quiver has known fixed points under the action of the Cartan torus of $G_H \times \mathbb{C}^\times$, with the associated character of tangent space at the fixed point calculated in [28, 37]. The fixed points are given by $N$-coloured Young tableaux of $k$. We define the functions of a box, $s = (a, b)$, at row $a$ and column $b$ in the $i$th partition $Y_i$ of a coloured Young tableau $\tilde{Y}$

\[
f_{ij}(s) := -a_i(s) - l_j(s) - 1, \quad g_{ij}(s) := -a_i(s) + l_j(s),
\] (4.3)

where $a_i(s) := Y_{ia} - b$ the arm length and $l_j(s) := (Y_j^\vee)_b - a$ the leg length relative to $Y_j$. We can write the superconformal index of the ADHM quiver as

\[
Z_{k,N} = \sum_{\{Y_i\}} \prod_{i=1}^N \prod_{s \in Y_i} \text{PE} \left[ \tau^{g_{ij}(s)} - 1 \cdot \frac{z_i}{z_j} (1 - \tau/y)(1 - \tau y) \right],
\] (4.4)

where $\sum_{\{Y_i\}}$ is the sum over the $N$-coloured Young tableaux corresponding to all $N$-coloured partitions of $k$.

The superconformal index is therefore equal to the $k$ instanton contribution to the Nekrasov partition function of $\mathcal{N} = 1^*$ five dimensional supersymmetric Yang-Mills theory with gauge group $SU(N)$ compactified to four dimensions on a circle. This agrees with an earlier proposal for a superconformal index of the ADHM moduli space quantum mechanics [24].

The dictionary between the parameters of the superconformal index and those of the Nekrasov partition is as follows: the parameters $\tau$ and $x$ are related to the deformation parameters for the $\Omega$-background via

\[
\tau = e^{\frac{c_1 + c_2}{2}}, \quad x = e^{\frac{c_1 - c_2}{2}};
\] (4.5)

if $m$ is the mass of the adjoint hypermultiplet, then

\[
y = e^m;
\] (4.6)

if $a_i$ for $i = 1, \ldots, N$ are the Coulomb branch parameters, then we have

\[
z_i = e^{a_i}.
\] (4.7)
Here the five dimensional parameters are measured in units of the radius of the compactification circle.

The Poincaré polynomial limit of the superconformal index reproduces the result

\[
P(\tilde{y}) = \sum_{\{Y_i\}} \prod_{i=1}^{N} \tilde{y}^{2N|Y_i|-2i\ell(Y_i)}.
\] (4.8)

4.2 \(\mathbb{C}^2/\mathbb{Z}_n\)

In this subsection we work through an example where the orbifold cohomology of [7] is the same as the cohomology of the symplectic resolution.

If \(\mathcal{X}\) is a hyperKähler orbifold, then the cohomological hyperKähler resolution conjecture states that if \(\tilde{\mathcal{X}}\) is a hyperKähler resolution of \(\mathcal{X}\), then the cohomology on \(\tilde{\mathcal{X}}\) is the orbifold cohomology on \(\mathcal{X}\). See conjecture 6.3 of [40] for the first statement of this conjecture, and [41] for a slightly more sophisticated wording of it. The orbifold cohomology was first defined in [7], the key point is that it depends solely on \(\mathcal{X}\), and so is independent of the choice of resolution. It is known to be true for the following example of \(\mathbb{C}^2/\mathbb{Z}_n\). Due to how the orbifold cohomology is constructed, when \(\mathcal{X}\) is a hyperKähler orbifold, the cohomology of \(\tilde{\mathcal{X}}\) contains, as a subring, the cohomology of \(\mathcal{X}\).

Using the orbifold cohomology, we shall calculate the superconformal index, and compare it to the localisation formulae. The quiver gauge theory we look at can be found in figure 2. The unresolved hyperKähler cone is \(\mathbb{C}^2/\mathbb{Z}_n\), which has an orbifold singularity at the origin.

![Figure 2: The quiver whose corresponding unresolved variety is \(\mathbb{C}^2/\mathbb{Z}_n\). There are \(n - 1\) nodes, \(k = (1^{n-1})\) and \(N = (1, 0^{n-3}, 0)\). When \(n = 2\), \(N = (2)\), and the quiver is known as \(T(SU(2))\).](image)

Chen Ruan cohomology involves taking the cohomology of the smooth part of the manifold, as well as the addition of twisted sectors, which live at the orbifold singularities. In the case of \(\mathbb{C}^2/\mathbb{Z}_n\), there are \(n - 1\) twisted sectors, corresponding to all non-identity elements of \(\mathbb{Z}_n\). Each twisted sector is the point set, \(\{\ast\}\). So we have

\[
H^{p,q}_{\text{orb}}(\mathbb{C}^2/\mathbb{Z}_n) = H^{p,q}(\mathbb{C}^2/\mathbb{Z}_n) \oplus H^{p-1,q-1}(\{\ast\}).
\] (4.9)

\[\text{The twisted sectors correspond to conjugacy classes in general, but this group is Abelian, so they correspond to elements.}\]
The ordinary cohomology bigraded-ring, $H^{p,q}(\mathbb{C}^2/\mathbb{Z}_n)$, is simply given by taking $\mathbb{Z}_n$-invariant holomorphic forms. $H^{0,0}(\{\ast\}) = 1$ and $H^{p,q}(\{\ast\}) = 0$ for $(p,q) \neq (0,0)$.

Hence, we have that the superconformal index defined by the Chen Ruan cohomology is

$$Z_{\text{orb}}(\mathbb{C}^2/\mathbb{Z}_n) = \frac{1}{n} \sum_{i=1}^{n} \frac{\tau}{1-y(-z^2)^{i/n}} \left( 1 + \frac{y}{1 - \tau(-z^2)^{i/n}} + n - 1 \right).$$ \tag{4.10}

One can then test this against localisation formulae. Using the analysis of section 4.3, we know there are $n$ fixed points. Their contribution to the index is

$$Z(\mathbb{C}^2/\mathbb{Z}_n) = \sum_{a=1}^{n} \text{PE} \left[ z^{2k-2a} \left( 1 - \frac{\tau}{y} \right) (1 - \tau y) \right].$$ \tag{4.11}

One can check that equations (4.10) and (4.11) are indeed the same.

If we set $\tau = y$ to count the 1/2-BPS states, note that there is a contribution of $n - 1$ from the twisted sectors.

### 4.3 $A$-type quivers

In this section, we compute the superconformal index for the case when $X$ is a Nakajima quiver variety of either $A$-type or $\hat{A}$-type. We do this by using the known answer for instanton moduli space, an observation by Nakajima in [31], that $A$-type and $\hat{A}$-type Nakajima quiver varieties are $\mathbb{C}^\times$-fixed point submanifolds of instanton moduli space.

We explain the construction in [31]. We take $\mathcal{M}_{k,0}$ to be the resolved moduli space of $k$ $SU(N)$ instantons on $\mathbb{C}^2$ of section 4.1. The construction takes a certain $\mathbb{C}^\times$-subgroup of $T := \mathbb{C}^\times \times G_H$, and restricts $\mathcal{M}_{k,0}$ to the fixed point submanifold. This submanifold is a disjoint union of linear quivers.

We want the fixed points of the $\mathbb{C}^\times \ni t_1$ action on the set $\mu_{\mathbb{C}^1}^{-1}(0)$ given by

$$(X, \tilde{X}, Q, \tilde{Q}) \mapsto (t_1 X, t_1^{-1} \tilde{X}, Q \rho_W(t_1)^{-1}, \rho_W(t_1) \tilde{Q}).$$ \tag{4.12}

This corresponds to a choice of homomorphism $\rho_V : \mathbb{C}^\times \rightarrow GL(\mathbb{C}^k)$, such that

$$(t_1 X, t_1^{-1} \tilde{X}, Q \rho_W(t_1)^{-1}, \rho_W(t_1) \tilde{Q}) = (\rho_V(t_1)^{-1} X \rho_V(t_1), \rho_V(t_1)^{-1} \tilde{X} \rho_V(t_1), \rho_V(t_1)^{-1} Q, \rho_V(t_1)).$$ \tag{4.13}

$\rho_V$ is a homomorphism, because the action of $GL(\mathbb{C}^k)$ is free on the space of $(X, \tilde{X}, Q, \tilde{Q})$ that obey $\mu = \zeta_\mathbb{R}$ for $\zeta_\mathbb{R}$ generic. A choice of conjugacy class of $\rho_V$ and $\rho_W$ defines a particular linear quiver. The conjugacy class of $\rho_V$ is determined by an $\vec{n} \in \mathbb{Z}^k/S_k$, such that

$$t_1 \mapsto \begin{pmatrix} t_1^{n_1} \\ t_1^{n_2} \\ \vdots \\ t_1^{n_k} \end{pmatrix}.$$ \tag{4.14}

26
Similarly, \( \rho_W \)'s conjugacy class is determined by \( \bar{m} \in \mathbb{Z}^N / S_N \).

We order these integers from smallest to largest. Let \( p := \min(m_1, n_1) \) and \( q := \max(n_k, m_N). \) Define \( n := q - p + 1. \) For \( a = 1, \ldots, n \), we define the spaces

\[
\begin{align*}
V_a &= \text{Eigenspace of } \mathbb{C}^k \text{ with eigenvalue } t_1^{q+1-a}, \\
W_a &= \text{Eigenspace of } \mathbb{C}^N \text{ with eigenvalue } t_1^{q+1-a}.
\end{align*}
\]  

(4.15)

Note that unless \( m_1 \geq n_1 \) and \( m_N \leq n_k \), the fixed point set will be empty, so we may as well take \( n = n_k - n_1 + 1. \)

The \( \mathbb{C}^\times \)-fixed points respect the eigenspace structure of \( \rho_V \) and \( \rho_W \): we see from equation (4.13) that for \( v \in V_i \),

\[
t_1 X v = \rho_V(t_1)^{-1} X \rho_V(t_1)v = t_1^{q+1-a} \rho_V(t_1)^{-1} X v, \\
\implies \rho_V(t_1) X v = t_1^{q-a} X v.
\]  

(4.16)

This implies that \( X : V_a \to V_{a+1} \). Similarly, \( t_1^{-1} \tilde{X} = \rho_V(t_1)^{-1} \tilde{X} \rho_V(t_1), \) \( Q \rho_W(t_1)^{-1} = \rho_V(t_1)^{-1} Q \) and \( \rho_W(t_1) \tilde{Q} = \tilde{Q} \rho_V(t_1) \) means

\[
\begin{align*}
X : V_a &\to V_{a+1}, \\
\tilde{X} : V_a &\to V_{a-1}, \\
Q : W_a &\to V_a, \\
\tilde{Q} : V_a &\to W_a.
\end{align*}
\]  

(4.17)

So we exactly have the \( A_n \) linear quiver. We define

\[
k_a := \dim V_a, \quad N_a := \dim W_a.
\]  

(4.18)

Calling the Nakajima quiver variety associated to the linear quiver \( M(\rho_V, \rho_W) \), we have found that

\[
\prod_{\rho_V} M(\rho_V, \rho_W) = \{ \mathbb{C}^\times \text{ fixed points of } \mathfrak{M}_{k,0} \}.
\]  

(4.19)

In the evaluation of the superconformal index of the linear quiver, the analysis of section 2.1 means that one need only consider the fixed points of the action of \( T \). As discussed in section 4.1 on instanton moduli space the fixed points correspond to \( N \)-coloured Young tableaux of total size \( k \). For a particular choice of \( \rho_W \), each fixed point will lie inside an individual linear quiver, corresponding to some \( \rho_V \). We explain here how to work out which \( \rho_V \), and hence which linear quiver, the fixed point is an element of. Note that since the fixed points are invariant under the whole of \( (\mathbb{C}^\times)^{N+2} \), they are invariant under the particular \( \mathbb{C}^\times \) we used to restrict to the linear quivers, and hence must lie in some linear quiver.
The Higgs branch of a linear quiver is non-empty if and only if it has a fixed point under $T$. The only if is trivial, as the fixed point is an element of the Higgs branch, while the other way is true because it must be closed under the action of $(\mathbb{C}^\times)^{N+2}$, lie within instanton moduli space, and every point on instanton moduli space flows under the action of $(\mathbb{C}^\times)^{N+2}$ to a fixed point, [34].

The fixed points are the maps $X, \tilde{X}, Q, \tilde{Q}$ such that $\mu_R = \zeta_R$ and $\mu_C = 0$, and

$$\begin{align*}
(\phi_l - \phi_m + \epsilon_1)X_{lm} &= 0, \\
(\phi_l - \phi_m + \epsilon_2)\tilde{X}_{lm} &= 0, \\
(\phi_l - \epsilon_1 + \epsilon_2 - a_i)Q_{li} &= 0, \\
(\phi_l + \epsilon_1 + \epsilon_2 - a_i)\tilde{Q}_{li} &= 0,
\end{align*}$$

(4.20)

where $l, m = 1, \ldots, k$ and $i = 1, \ldots, N$, and the $(\phi_l)_l$ are diagonalised gauge transformations. The coloured Young tableaux give us a way of reindexing the numbers $l = 1, \ldots, k$ as $(i, (\alpha, \beta))$ for $(\alpha, \beta) \in Y_i$ and $i = 1, \ldots, N$. Exactly $k$ components of the $2kN + 2k^2$ components of $(X, \tilde{X}, Q, \tilde{Q})$ are non-zero. They are

$$X_{i(\alpha, \beta),i(\alpha+1,\beta)}, \tilde{X}_{i(\alpha, \beta),i(\alpha, \beta+1)}, \tilde{Q}_{i(1,1),i} \neq 0.$$

(4.21)

Suppose $e_{ais}$ is a basis for $\mathbb{C}^k$ for $a = 1, \ldots, n$, $i = 1, \ldots, N_a$ and $s \in Y_{ai}$, and $f_{ai}$ a basis for $\mathbb{C}^{N_a}$ for $a = 1, \ldots, n$, $i = 1, \ldots, N_a$. Then we have that $e_{ai(1,1)} \in V_a$, because $\tilde{Q}_{ai(1,1)ai} f_{ai} \propto e_{ai(1,1)}$ and $\tilde{Q}_{ai(1,1)ai} \neq 0$. Now we see that if $(2,1) \in Y_{ai}$, then $X e_{ai(2,1)} \propto e_{ai(1,1)}$, and so $e_{ai(2,1)} \in V_{a-1}$. Through this, we see that

$$e_{ai(\alpha, \beta)} \in V_{a-\alpha+\beta}.$$

(4.22)

This fully determines the value of the $k_a$’s. Note that there can be values of $a$ where $N_a = 0$ and $k_a \neq 0$.

A special class of linear quivers are known as $T^\rho_{\sigma}(SU(M))$ quivers. $\sigma$ and $\rho$ are partitions of $M$ determined by $\rho^W$ and $\rho^V$. The $\rho$ and $\sigma$ are defined by $\rho_i^V - \rho_i^W = N_i$ and $k_i = \rho_1^V + \cdots + \rho_i^V - \sigma_1 - \cdots - \sigma_i$. The quiver will not be a $T^\rho_{\sigma}$ if the gauge group ranks do not define a partition through the equation for $k_i$ before. However, if this is the case, the quiver is Seiberg dual to a $T^\rho_{\omega}$, see appendix [13]. The number of fixed points on the resolved space for a $T^\rho_{\sigma}(SU(M))$ is

$$\text{# of fixed points} = \sum_{\rho \lessdot \nu \lessdot \sigma^\vee} K_{\nu \rho} K_{\rho \vee \sigma},$$

(4.23)

where $K_{\alpha \beta}$ are the Kostka numbers. This is due to the expression for the Poincaré polynomial of $A$-type Nakajima quiver varieties in [38], and the fact that each fixed point contributes a generator to the homology.
Figure 3: An example of a linear quiver. It is a connected component of the resolved moduli space of $k = l^2 \, SU(N)$ instantons on $\mathbb{C}^2$.

We look at an example, to show how the Young tableaux are chosen. Take the linear quiver in figure 3

The coloured Young tableau that we restrict to are exactly the ones such that

\[\exists \text{ exactly } l \, s \in Y_i \text{ s.t. } s = (b, b) \text{ for some } b \in \mathbb{N},\]
\[\exists \text{ exactly } l - 1 \, s \in Y_i \text{ s.t. } s = (b - 1, b) \text{ for some } b \in \mathbb{N},\]
\[\exists \text{ exactly } l - 1 \, s \in Y_i \text{ s.t. } s = (b, b - 1) \text{ for some } b \in \mathbb{N},\]
\[\exists \text{ exactly } l - 2 \, s \in Y_i \text{ s.t. } s = (b - 2, b) \text{ for some } b \in \mathbb{N},\]

\[\text{etc.}\]  

(4.24)

So if $N = 1$, the only pole is given by a single square Young tableaux of height and width $l$.

We write the superconformal index for the linear quiver defined by the conjugacy classes of $\rho_V$ and $\rho_W$. We restrict to the fixed points corresponding to the linear quiver, scale the fugacities according to the $\mathbb{C}^\times$-action and take the limit $x \to 0$. This gives

\[Z_{\rho_V, \rho_W} = \sum_{\{Y_{a,i}\}_{a,b=1}^n} \prod_{a,b=1}^n \prod_{i,j=1}^{N_a} \prod_{s \in Y_{a,i}} \prod_{f(a,i)(b,j)(s)=a-b} \text{PE} \left[ \frac{z_{a,i}}{z_{b,j}} \tau^{g(a,i)(b,j)(s)-1} \right] (1 - \tau/y)(1 - \tau y).\]  

(4.25)

In this expression, the $\{Y_{a,i}\}_{a,b=1}^n$ means restricting the sum to all fixed points corresponding to the linear quiver $M(\rho_V, \rho_W)$. Note that unlike the instanton moduli space’s superconformal index, a generic box from a coloured Young tableaux, associated to a fixed point within the manifold, need not contribute an individual term to the index. Indeed, if this were so then the highest power of $y$ in the index would be $kN$, which is strictly greater than the quaternionic dimension of $M(\rho_V, \rho_W)$.

Since the manifold is connected, and the contribution at each fixed point corresponds to the tangent space at that point, we would expect that the highest power of $y$ at each point would be the quaternionic dimension of the manifold $\dim \mathbb{H} M(\rho_V, \rho_W) = \sum_a \left( k_a k_{a+1} + k_a N_a - k_a^2 \right)$. This is a non-trivial combinatorial condition on the coloured Young tableaux that appears to be true.
Figure 4: A quiver whose resolved space is a cotangent bundle to a flag variety.

From this we can conclude that

\[
\lim_{\rho_W, x \to 0} Z_{k,N} = \sum_{\rho_V} Z_{\rho_V, \rho_W}.
\]

(4.26)

This sum has multiplicity one for each \( \rho_V \), but we might find that \( Z_{\rho_V, \rho_W} = 0 \), and we may also have two equivalent linear quivers for different \( \rho_V \)'s, for example (1)-(1)-[1] and [1]-[1]-[1]. Furthermore, we may not have a connected quiver for a specific \( \rho_V \) and \( \rho_W \).

4.3.1 An example: cotangent bundles of flag varieties

We look at a special class of linear quivers, the ones in figure 4.

The quiver ranks obey \( N \geq k_1 \geq k_2 \geq \cdots \geq k_n > 0 \), otherwise the Higgs branch is empty. The unresolved space is a nilpotent orbit, while the resolved space is the cotangent bundle to the flag variety \( \mathbb{C}^{k_1} \hookrightarrow \mathbb{C}^{k_2} \hookrightarrow \cdots \hookrightarrow \mathbb{C}^{k_n} \).

The calculation of the Hilbert series of this quiver was done via Lefschetz fixed point theorem directly in [9], we find that our analysis exactly reproduces their results for a choice of \( k \) and \( N \) such that it is a \( T_\rho(SU(N)) \) theory.

Define the composition of \( N \)

\[
l_1 = k_n, \quad l_2 = k_{n-1} - k_n, \quad l_3 = k_{n-2} - k_{n-1}, \ldots, l_n = k_1 - k_2, \quad l_{n+1} = N - k_1.
\]

(4.27)

Since there is only one flavour node, and it is on the far left node, the fixed points are coloured Young tableaux of length 1. They are given by \( l_1 \) lots of \( (n) \), \( l_2 \) lots of \( (n-1) \), \ldots, \( l_n \) lots of \( (1) \) and \( l_{n+1} \) lots of \( \varnothing \). One then needs to sum over the Weyl group \( S_N \) modulo the Weyl group of the Levi subgroup,

\[
\hat{W} := \prod_{a=1}^{n+1} S_{l_a}.
\]

(4.28)

This is precisely the same parameterisation of the fixed points found in [9]. The quaternionic dimension of the manifold is

\[
d_H := \sum_{a>b} l_a l_b.
\]

(4.29)

Suppose \( Y_i = (a) \) and \( Y_j = (b) \), then we have that \( f_{ij}(s) \) is zero if and only if \( b \leq a - 1 \) and \( s = (1,a) \) (the last box). For this box we have that \( g_{ij}(s) = -1 \). Define the function on indices \( h : \{1, \ldots, N\} \to \{0,1,\ldots,n\} \) via

\[
i = l_1 + l_2 + \cdots + l_{h(i)} + j, \quad \text{for} \quad j = 1, \ldots, l_{h(i)+1}.
\]

(4.30)
The superconformal index is
\[
Z = \sum_{w \in S_{\tilde{N}} / \tilde{W}} \prod_{h(i) > h(j)} w \left( \frac{\tau}{y} \left( \frac{1 - \frac{z_i}{z_j}}{1 - \frac{\tilde{y}_i}{\tilde{y}_j}} \right) \left( 1 - \frac{\tilde{y}_i + \tau y}{\tilde{y}_j} \right) \right) .
\]

(4.31)

The orbit under Seiberg duality of the quiver in figure 4 is a family of the same structure, but with varying gauge ranks. There is an element of this orbit where \((l)\), the corresponding composition of \(N\), is in fact a decreasing sequence, and so defines a partition \(\rho = (l_{n+1}, l_n, \ldots, l_1)\). This quiver is the \(T_{\rho}(SU(N))\)-quiver.

Note that the resolved space is the cotangent bundle to a flag variety, with \(\tau\) the grading for the \(C^\times\)-action rotating the cotangent fibre. This means that if we set \(\tau \to 0\) then we restrict to the flag variety itself. The flag variety is projective and is a deformation retract of its cotangent bundle. This means that the \(\tau \to 0\) limit of the superconformal index is the Poincaré polynomial of the cotangent bundle to the flag variety, giving
\[
P_{T_{\rho}(SU(N))}(\tilde{y}) = \sum_{\sigma \in S_{\tilde{N}} / \tilde{W}} \sigma \left( \prod_{h(i) > h(j)} \frac{1 - \frac{\tilde{y}_i}{\tilde{y}_j}}{1 - \frac{z_i}{z_j}} \right) .
\]

(4.32)

Using our own limit for the Poincaré polynomial we obtain
\[
P_{T_{\rho}(SU(N))}(\tilde{y}) = \sum_{\sigma \in S_{\tilde{N}} / \tilde{W}} \tilde{y}^{\ell(\sigma)} ,
\]

(4.33)

where \(\ell(\sigma)\) is the length of the shortest element of the coset \(\sigma \tilde{W}\).

Finally, it is also known classically that the Poincaré polynomial of the flag variety is
\[
P_{T_{\rho}(SU(N))}(\tilde{y}) = \sum_{\nu \geq \rho} K_{\nu^\vee \langle 1^N \rangle} K_{\nu \rho}(\tilde{y})
\]
\[
= \frac{\prod_{i=1}^{N}(1 - \tilde{y}^i)}{\prod_{j=1}^{\ell}\prod_{i=1}^{\rho_j}(1 - \tilde{y}^i)} .
\]

(4.34)

It is proven in [6] that any linear quiver’s superconformal index can be reached by taking a certain limit of the flavour fugacities of a linear quiver whose resolved space is the cotangent bundle to a flag variety. This means we can write the superconformal index of any linear quiver as a single Weyl group sum.

### 4.4 \(\hat{A}\)-type quivers

The construction of \(A_n\)-type quivers can be easily adapted to give us the fixed points of generic \(\hat{A}_n\)-type quivers. The associated variety to a \(\hat{A}_n\) quiver is the moduli space of instantons on \(\mathbb{C}^2/\mathbb{Z}_n\), [25].

We restrict \(t_1\) in equation (4.13) to lie in a finite cyclic group, \(t_1 \in \mathbb{Z}_n \subset \mathbb{C}^\times\). If we do this, then we have the same argument as for the linear quiver. \(\mathbb{C}^k\) is split
into \(n\) pieces, \(V_0, \ldots, V_{n-1}\), with \(V_a\) having weight \(t_a^1\), and similarly \(\mathbb{C}^N\) splits into \(n\) pieces \(W_0, \ldots, W_{n-1}\). As before we define

\[
k_a := \dim V_a, \quad N_a := \dim W_a.
\]

The difference now is the periodicity, namely

\[
X : V_{n-1} \to V_0, \quad \tilde{X} : V_0 \to V_{n-1}.
\]

This periodicity is important for identifying which \(\hat{A}_n\)-type quiver a particular fixed point lies in given the choice of \(\rho_W\).

The superconformal index for \(\hat{A}_n\) is

\[
Z_{\rho_V, \rho_W}(\hat{A}_n) = \sum_{\{Y_{a,i}\}} \prod_{a,b=1}^n \prod_{i=1}^{N_a} \prod_{j=1}^{N_b} \prod_{s \in Y_{a,i}} \text{PE} \left[ \frac{z^{a_i}}{z^{b_j}} \tau^{g_{(a,i),(b,j)}(s)} (1 - \tau/y)(1 - \tau y) \right].
\]

In this expression, the \(\{Y_{a,i}\}\) means restricting the sum to all fixed points corresponding to the affine quiver fixed by \(\rho_V\) and \(\rho_W\).

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**A Some superalgebra details**

**A.1 Flat space quantum mechanics**

We consider the problem of free quantum mechanics on \(\mathbb{C}^{2n}\), with complex coordinates \((q_i, \bar{q}_i)_{i=1}^n\). Consider the action of \(-M - 2J_3 + N\) on the forms

\[
\alpha = \prod_{i=1}^n q_i^{a_i} \bar{q}_i^{\bar{a}_i} q_i^{b_i} \bar{q}_i^{\bar{b}_i} dq_i^{\delta_i} \wedge d\bar{q}_i^{\bar{\delta}_i} \wedge dq_i^{\bar{\epsilon}_i} \wedge d\bar{q}_i^{\epsilon_i},
\]

(A.1)
with \( a, \bar{a}, b, \bar{b} \in \mathbb{Z}_{\geq 0} \) and \( \delta, \bar{\delta}, \epsilon, \bar{\epsilon} \in \{0, 1\}^n \), then we have that\(^{12}\)

\[
2\mathcal{J}_3 \alpha = \sum_{i=1}^{n} (a_i + b_i - \bar{a}_i - \bar{b}_i) \alpha, \\
\mathcal{M} \alpha = \sum_{i=1}^{n} (\bar{\delta}_i + \bar{\epsilon}_i - n) \alpha, \\
\mathcal{N} \alpha = \sum_{i=1}^{n} (\delta_i + \epsilon_i - n) \alpha.
\]  

(A.2)

So, \(-\mathcal{M} - 2\mathcal{J}_3 + \mathcal{N}\) acts as the \( \mathbb{C}^\times\)-scaling\(^{13}\) which we call \( \mathcal{R} \):

\[
\mathcal{R} \alpha = (-\mathcal{M} - 2\mathcal{J}_3 + \mathcal{N}) \alpha = \sum_{i=1}^{n} (a_i + b_i - \bar{a}_i - \bar{b}_i + \delta_i + \epsilon_i - \bar{\delta}_i - \bar{\epsilon}_i) \alpha.
\]  

(A.3)

### A.2 \( \mathfrak{su}(1|2) \) character

Suppose that \( p(\tau, y) \in \mathbb{Z}((Z))[[\tau]][1/y] \) is such that

\[
p(\tau, y, Z) = p(\tau, 1/y, Z),
\]

(A.4)

a formal Laurent series in \( Z \), a formal power series in \( \tau \) and a finite Laurent expansion in \( y \), writing it as

\[
p(\tau, y, Z) = \sum_{a=0}^{\infty} \sum_{b=-m}^{m} \alpha_{a,b}(Z) \tau^a y^b = \sum_{a=0}^{\infty} \sum_{b=-m}^{m} \alpha_{a,b}(Z) \tau^{a+b} \left( \frac{y}{\tau} \right)^b,
\]

(A.5)

we further have that

\[
\alpha_{a,b} = 0 \text{ for } a < |b|,
\]

(A.6)

that

\[
p(\tau, \tau, Z) \in \mathbb{Z}_{\geq 0}, \text{ so that } \frac{d}{d\tau} p(\tau, \tau, Z) = \frac{d}{dz_i} p(\tau, \tau, Z) = 0,
\]

(A.7)

and

\[
\lim_{y/\tau \text{ finite}} p(\tau, y, Z) = \sum_{a=0}^{m} \alpha_{a,-a}(Z) \left( \frac{\tau}{y} \right)^a \in \mathbb{Z}_{\geq 0} \left[ \frac{\tau}{y} \right].
\]

(A.8)

We have that in the text equation (A.4) corresponds to (3.1), equation (A.6) corresponds to (3.3), equation (A.7) corresponds to (3.4) and equation (A.8) corresponds to (3.5).

\(^{12}\)Explicit expressions for the \( \mathfrak{osp}(4^*|4) \) generators on flat space can be found in appendix E.5 of \(^{42}\).  

\(^{13}\)On flat space, this is the Lie derivative with respect to the Hamiltonian vector field \( \sum_{i=1}^{n} \left( q_i \frac{\partial}{\partial p_i} + \bar{q}_i \frac{\partial}{\partial \bar{p}_i} - \bar{q}_i \frac{\partial}{\partial q_i} - \bar{\bar{q}}_i \frac{\partial}{\partial \bar{p}_i} \right) \).
We show in this subsection that if \( p \) obeys all these properties, then it can be written as

\[
p(\tau, y) = \sum_{a=0}^{m} N_{a,a} I_{a,a} + \sum_{a>b} N_{a,b} I_{a,b},
\]

(A.9)

with \( N_{a,a} \in \mathbb{Z}_{\geq 0} \) and \( \tilde{N}_{a,b} \in \mathbb{Z}[\mathbb{Z}, \mathbb{Z}^{-1}] \).

We prove this via induction on \( m \). If \( m = 0 \), then since \( p(\tau, \tau, Z) = p(\tau, y, Z) = \alpha_{0,0} \in \mathbb{Z}_{\geq 0} \) and \( I_{0,0} = 1 \), we have

\[
p(\tau, y) = \alpha_{0,0} I_{0,0}.
\]

(A.10)

Now suppose it is true up to the highest power of \( y \) in \( p \) being \( m - 1 \). We take

\[
p(\tau, y, Z) = q(\tau, Z)(y^m + 1/y^m) + \sum_{a=0}^{\infty} m \sum_{b=1-m}^{m-1} \alpha_{a,b}(Z) \tau^a y^b,
\]

(A.11)

for some \( q(\tau, Z) \in \tau^m \mathbb{Z}((Z))[\tau] \), \( q(\tau, Z) = \sum_{a=m}^{\infty} q_a(Z) \tau^a \). Equation (A.8) necessarily means that \( q_m(Z) \in \mathbb{Z}_{\geq 0} \). We see that we can write

\[
p(\tau, y, Z) = q_m I_{m,m} - \sum_{a=m+1}^{\infty} q_a(Z) I_{a-1,m-1}(\tau, y) + \tilde{p}(\tau, y, Z),
\]

(A.12)

where the highest power of \( y \) in \( \tilde{p}(\tau, y) \) is \( m - 1 \) and it clearly obeys all the necessary properties (A.4), (A.6), (A.7) and (A.8).

\section*{B Wall crossing}

We conjecture that the superconformal index does not depend on the choice of projective symplectic resolution. We discuss this further in this appendix for certain Nakajima quiver varieties, providing some of the evidence for this conjecture.

Define \( \Gamma = (V, \Omega) \) to be a quiver of ADE-type. If \( \omega_i \) are the fundamental weights and \( \alpha_i \) are the simple roots of \( \Gamma \). Then for a given choice of gauge ranks \( k \in \mathbb{Z}_{>0}^V \), flavour ranks \( N \in \mathbb{Z}_{>0}^V \), choice of FI parameter \( \zeta = \zeta_{\mathbb{R}} \in \mathbb{R}^V \) and background baryonic charge \( B \in \mathbb{Z}^V \), we define

\[
\lambda := \sum_{i \in V} N_i \omega_i, \quad \alpha := \sum_{i \in V} k_i \alpha_i, \\
\zeta := \sum_{i \in V} \zeta_i \omega_i.
\]

(B.1)

There is an action of the Weyl group of the quiver, \( W \equiv W_\Gamma \). For \( w \in W \):

\[
\zeta \mapsto w(\zeta), \quad B \mapsto w(B), \\
\alpha \mapsto w* \alpha := \lambda - w(\lambda - \alpha).
\]

(B.2)
From the work [30], we know that
\[
\mathcal{M}(\lambda, \alpha, \zeta) \cong \mathcal{M}(\lambda, w \alpha, w(\zeta)),
\]
where the isomorphism is a hyperKähler isometry, an isometry preserving all three complex structures. This transformation is known in the mathematical literature as reflection functors, while in the physical literature it is known as three dimensional Seiberg duality.

The Hilbert series is known to be independent of the choice of \(\zeta\), [16]. However, in general the fixed point cotangent space \(T\)-module structure does depend on the resolution.

\[\begin{array}{cc}
2 & 1 \\
\downarrow & \uparrow \\
2 & 1
\end{array}\]  \[\begin{array}{cc}
2 & 1 \\
\uparrow & \downarrow \\
1 & 1
\end{array}\]  \[\begin{array}{cc}
2 & 1 \\
\uparrow & \downarrow \\
2 & 2
\end{array}\]

**Figure 5**: We investigate the quiver on the left. For this quiver \(\lambda = 2\omega_1 + \omega_2\) and \(\lambda - \alpha = \omega_3 + \omega_2 - \omega_1\). The quiver in the middle is the image under the Weyl group transformation (12), and has \(\lambda - (12) \ast \alpha = \omega_3 + \omega_1\). The quiver on the right is the image under (23) and has \(\lambda - (23) \ast \alpha = 2\omega_3 - \omega_2\). The middle quiver is the \(T_{(1,1)}^{(3,1)}(SU(4))\) quiver.

For example, the polynomial \(p_x(\tau, Z) \in \mathbb{Z}_{\geq 0}[\tau^{\pm 1}, Z^{\pm 1}]\) from equation (3.7) for the quiver on the left in figure 5 has the following values for the five fixed points of the two resolutions:

\[
p_x \text{ for } \zeta^{(1)} = (1, 1) : \\
\frac{z_{21,1}}{z_{21,2}} + \tau \frac{z_{21,1}}{z_{22,1}}, \\
\frac{z_{21,2}}{z_{21,1}} + \tau \frac{z_{21,2}}{z_{22,1}}, \\
\tau^{-1} \frac{z_{21,1}}{z_{22,1}} + \tau^{-1} \frac{z_{21,2}}{z_{22,1}}, \\
\frac{z_{21,1}}{z_{21,1}} + \tau \frac{z_{22,1}}{z_{21,2}}, \\
\frac{z_{21,2}}{z_{21,1}} + \tau \frac{z_{22,1}}{z_{21,2}},
\]  \(\text{(B.4)}\)

\[
p_x \text{ for } \zeta^{(2)} = (2, -1) : \\
\frac{z_{21,2}}{z_{21,1}} + \tau \frac{z_{21,1}}{z_{22,1}}, \\
\frac{z_{21,2}}{z_{22,1}} + \tau \frac{z_{21,1}}{z_{22,1}}, \\
\frac{z_{21,1}}{z_{21,1}} + \tau \frac{z_{22,1}}{z_{22,1}}, \\
\frac{z_{21,2}}{z_{21,1}} + \tau \frac{z_{22,1}}{z_{22,1}}, \\
\frac{z_{21,1}}{z_{21,1}} + \tau \frac{z_{22,1}}{z_{22,1}}.
\]

We see that the fixed point structure is different for these two different choices of resolution. Nonetheless, both the Hilbert series and the superconformal index are the same. We have tested this for multiple length two quivers and some length three quivers and the same structure is persistent.

Note that \(\zeta^{(1)} = (23)(\zeta^{(2)})\), while the subgroup of the Weyl group of the quiver, \(W_{A_2} = S_3\), for which \(\mu\) is invariant, namely the group \(\{1, (13)\} \cong \mathbb{Z}_2\), the fixed point structure is invariant. This holds more generally, as the resolutions are Seiberg dual to each other (equivalently one is given by acting with a reflection functor on the other). This further means that for the quiver on the right in figure 5, the fixed point structure for the left quiver with choice of resolution...
$\zeta_{\text{left}}$ is equal to the fixed point structure for the right quiver with resolution

$\zeta_{\text{right}} := (23)(\zeta_{\text{left}})$. 
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