Inverse Problems

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Wavefield reconstruction inversion: an example

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Abstract

Nonlinear least squares data-fitting driven by physical process simulation is a classic and widely successful technique for the solution of inverse problems in science and engineering. Known as ‘full waveform inversion (FWI)’ in application to seismology, it can extract detailed maps of earth structure from near-surface seismic observations, but also suffers from a defect not always encountered in other applications: the least squares error function at the heart of this method tends to develop a high degree of nonconvexity, so that local optimization methods (the only numerical methods feasible for field-scale problems) may fail to produce geophysically useful final estimates of earth structure, unless provided with initial estimates of a quality not always available. A number of alternative optimization principles have been advanced that promise some degree of release from the multimodality of FWI, amongst them wavefield reconstruction inversion (WRI), the focus of this paper. Applied to a simple 1D acoustic transmission problem, both full waveform and WRI methods reduce to minimization of explicitly computable functions, in an asymptotic sense. The analysis presented here shows explicitly how multiple local minima arise in FWI, and that WRI can be vulnerable to the same ‘cycle-skipping’ failure mode.

Keywords: seismic inversion, optimization, nonlinear least squares

(Some figures may appear in colour only in the online journal)

1. Introduction

Full waveform inversion (FWI) is the current nomenclature in the seismology literature for data-fitting earth structure estimation driven by wavefield modeling. The earth properties to be estimated (material densities, stiffnesses, attenuation rates, . . .) form a vector \( c \) of spatially varying fields that appear as coefficients in systems of hyperbolic partial differential equations, modeling seismic wave propagation. The wavefields used in structure estimation (‘imaging’)

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are small motion disturbances of the earth’s equilibrium state, so the equations of motion are typically linear(ized). Right-hand side vectors $f$ in these systems model energy input that initiates waves (earthquakes, man-made sources such as explosives or mechanical vibrators). Data vectors $d$ are simulated by sampling the solution fields at the locations of measurement devices (accelerometers, microphones, . . .) over appropriate time intervals. The relation between the energy source $f$ and the simulated data is linear in $f$, but nonlinear in the coefficient vector $c$, so is naturally represented by a family of linear operators $S[c]$ parametrized by the coefficient vector $c$.

The objective of FWI is to find $c$ and $f$, given $d$, so that $S[c]f \approx d$. A typical method for achieving this goal is the minimization of an objective misfit measure (objective, for short), the most common choice being the square norm of a Hilbert space in which the data is presumed to reside:

$$\text{Given } d, \text{ find } c \text{ and } f \text{ to minimize } ||d - S[c]f||^2. \quad (1)$$

In equation (1), $|| \cdot ||$ is the norm in a suitable Hilbert space.

This approach was first suggested in the 1980’s (Bamberger et al. (1979), Tarantola (1984), Kolb et al. (1986), Crase et al. (1990), and many other papers since then). Usually some form of regularization is applied, to compensate for poorly determined aspects of $c$ and/or $f$, as is explained in Tarantola’s influential book (Tarantola 2005). Also, $f$ may be constrained in one way or another to embody characteristics of field energy sources, or even regarded as known (an example of this is given below).

Within a few years of its introduction into quantitative seismology, FWI was understood to suffer from a severe limitation. Because of the typical dimensions of earth models, and consequent cost of accurate computation of $S$, iterative local optimization provides the only feasible route to estimation of $c$ via solution of the optimization problem (1). However the objective function of this optimization problem (the mean-square residual appearing in display (1)) has many local minima in general, most having nothing to do with a usable estimate of earth structure. (An explicit example of this multi-modal behaviour appears below.) Reliable estimation of $c$ via iterative local optimization requires that the initial estimate predict the correct arrival time of waves, as they appears in the data, within a wavelength (in fact, conventionally a half wavelength) at dominant frequencies (Gauthier et al. 1986 and Virieux and Operto 2009).

Despite this severe constraint, FWI has shown enough promise as a tool for both industrial and academic seismology that it is now a mainstream research topic, and to some extent a commercial product. Estimation of sufficiently accurate initial models via non-FWI methods is common practice, though what ‘sufficiently accurate’ means may be difficult to discern (Plessix et al. 2010). The wavelength criterion mentioned in the last paragraph may be made less onerous by collection of relatively low-frequency data (Dellinger et al. 2016). Finally, many alternatives to straightforward least-squares data fitting have been suggested, some of which appear to exhibit less tendency to develop local minima than does the problem described in (1). For example, other metrics than the $L^2$ distance employed in definition (1) may be used to characterize the size of the data residual $S[c]f - d$, and some of these result in larger regions of convexity about global minima than possessed by the $L^2$ based objective (1). A particularly promising class of alternative distance measures derives from optimal transport theory (Yang et al. 2018 and Métiévier et al. 2018). Of course, practical seismology has estimated earth mechanical parameters, including wave velocities, for decades. Some of these practical methods are based on model extension, that is, the addition of extra degrees of freedom beyond those required by basic physics. More recently, the extension concept has been used to reformulate the FWI problem. Symes (2008) describes several variants of extension-based inversion and its origins in seismic data processing.
This topic of this paper is one of these extension-based inversion approaches, wavefield reconstruction inversion (‘WRI’). WRI was introduced by van Leeuwen and Herrmann (2013), and further developed by van Leeuwen and Herrmann (2016), Wang et al (2016), Li et al (2018), van Leeuwen (2019), Rizzuti et al (2019), Aghamiry et al (2019), Fang et al (2018), Aghamiry et al (2020), Louboutin et al (2020), and other authors. It is based on the presumption that the correct source (right-hand side in the equations of motion) \( q \) is known, and combines a penalty for data misfit with a penalty for failing to solve the equations of motion with the correct right-hand side:

Given \( d \) and \( q \), find \( c \) and \( f \) to minimize \( \| d - S[c]f \|^2 + \alpha^2 \| f - q \|^2 \). \hspace{1cm} (2)

The formulation of WRI given by van Leeuwen and Herrmann (2013) appears to differ from the alternative formulation (2), introduced by Wang et al (2016) but in fact the two are completely equivalent, in the sense of having corresponding local minimizers. The original form and its equivalence with the problem (2) are explained in the fourth section of this paper. As noted by van Leeuwen (2019), either formulation is available not just for seismic inverse problems, but for any inverse problem based on a separable (partly linear) modeling operator.

Numerical examples provided in the cited literature on WRI suggest that the problem defined in display (2) is less likely to develop uninformative local minima (that is, ‘less nonlinear’) than is the least squares problem (1) in application to seismic inversion. The most-often cited justification for this point of view is that the WRI problem (2) involves a search for an optimum over many more degrees of freedom than does the FWI problem (1), since the trial source \( f \) is also to be determined. More concretely, van Leeuwen and Herrmann (2016) show that if \((c^*, f^*)\) is a stationary point of the objective defined in problem (2), then \( c^* \) is within (roughly) the data residual scaled by the reciprocal penalty parameter (that is, \( O(\alpha^{-1} \| d - S[c^*]f^* \|) \)) of a global FWI minimizer. [The precise statement of this result, which follows from theorem 4.3 in (van Leeuwen and Herrmann 2016), depends on analysis of the FWI problem (1) as an equality-constrained optimization; we refer the reader to the cited reference for details.] This is however a local characterization of WRI stationary points: it shows that these are close to solutions of the FWI problem only for those that produce a small data residual, and does not address the global question of whether large-residual stationary points exist, as appears to be the case for FWI.

The main result of this paper is that in one simple case, in which all of the necessary computations can be carried out by hand, the WRI objective does indeed possess stationary points with large data residual. In fact, in this case, minimization of the WRI objective is just as likely to be trapped in a spurious local minimizer as is the FWI objective. The context of this conclusion is a simple transmission inverse problem for the 1D acoustic wave system, which models pulse transmission along a 1D continuum from a source point to a receiver point. The pulses used in this thought experiment are short, so the main information content of the data is the time of transit from source to receiver. The predominant information about the material model, in this case reduced to the wave velocity (a scalar function of position), is just this transit time, so I constrain both the target wave velocity \( c^* \) generating the data and the trial wave velocity \( c_t \) to be constant, that is, independent of position along the 1D continuum. I introduce a family of inverse problems, depending on a parameter \( \lambda \) playing the role of wavelength. For sufficiently small \( \lambda \), it is possible to show explicitly that the FWI problem possesses local minimizers far from the global minimizer at the target \( c^* \), and that initiating a local iterative optimization, such as steepest descent or Newton’s method, at a distance from \( c^* \), bounded below by a multiple of the ‘wavelength’ \( \lambda \) will result in convergence to these spurious local minima. That is, FWI behaves in exactly the manner described in much of the literature on this topic. However, an analysis of WRI applied to the same context yields the same result: spurious local
minima exist for sufficiently small $\lambda$, and will be found by local optimization unless the starting point is within $O(\lambda)$ of the target. That is, WRI behaves in a manner qualitatively indistinguishable from FWI. In particular, its ability to allow good fit to data for small $\alpha$ does not safeguard it from failure to converge globally to a ‘good’ local minimum. In fact, the $\alpha \to 0$ limit of the WRI objective is well-defined (after scaling by $1/\alpha^2$) and also behaves in qualitatively the same way as the FWI objective. So the apparent ability to maintain better data fit via reduction of $\alpha$ does not lead to global behaviour asymptotically more amenable to local optimization.

I emphasize that this conclusion holds a priori for the 1D transmission inverse problem described in the next section, and only for this problem. The results presented here imply nothing about the behaviour of any other instance of WRI. In particular, the analysis takes advantage of the diagonal (in fact, scalar) nature of the operator $S[c]S[c]^T$ (the central role played by this operator will become clear in the fourth section below). For other problems, for instance the simple example presented in van Leeuwen (2019), this operator is not diagonal, so the short cuts used in the analysis given here are not available. However, it should be understood that the various heuristic justifications found in the literature for expecting WRI to avoid cycle-skipping, and the theoretical analysis in van Leeuwen and Herrmann (2016), all apply to the 1D transmission problem as well, and therefore in themselves are insufficient to imply avoidance of cycle-skipping.

This paper begins with a description of the 1D inverse transmission problem and explicit computation of various components of the FWI approach, based on explicit solution of the 1D acoustic system presented in appendix A. In the third section I introduce the $\lambda$-dependent family of problems, and establish the asymptotic properties of FWI as $\lambda \to 0$. The fourth section develops the algebraic structure of WRI, culminating in a remarkable identity revealing WRI to be equivalent to minimization of a weighted norm of the data residual, with a weight operator depending on the coefficient vector $c$. This identity has also been derived by van Leeuwen (2019), using a different argument. The result is quite general, applying to essentially any realization of WRI, and already shows that its behaviour must be closely related to that of FWI. In the fifth section I return to 1D acoustics problem and the $\lambda$-dependent family of inverse problems, compute that weight operator explicitly, and deduce the global behaviour of WRI in this instance. This behaviour is illustrated by a very simple numerical example in the sixth section. The paper ends with a discussion of the relation of the analysis presented here to a previous analysis of optimization formulations of wave inversion problems based on parameter-dependent quadratic forms (Stolk and Symes 2003), and a brief discussion of an extension-based inversion approach using a different penalty than does WRI, which in application to the 1D transmission problem provably avoids cycle-skipping.

2. FWI for 1D acoustics

The example of FWI to be explored in this paper is one of the simplest possible, based on the 1D acoustics system connecting excess pressure $p$, particle velocity $v$, constitutive law defect (‘source’) $f$, density $\rho$, and wave velocity $c$:

$$
\begin{align*}
\frac{\partial p}{\partial t} + \rho c^2 \frac{\partial v}{\partial z} &= f \\
\rho \frac{\partial v}{\partial t} + \frac{\partial p}{\partial z} &= 0 \\
p, v &= 0, \ t \ll 0.
\end{align*}
$$

(3)
The fields \( p, v, f \) are functions of spatial position \( z \in \mathbb{R} \) and time \( t \in \mathbb{R} \), whereas \( c, \rho \) are functions of \( z \) alone, so that the system (3) is autonomous.

The system (3) has classical (smooth) solutions \((p, v)\) when \( c, \rho, \) and \( f \) are smooth, and \( \log c \) and \( \log \rho \) are bounded on \( \mathbb{R} \), as is well-established (Lax 2006). In this paper, for reasons to be discussed below, \( c \) and \( \rho \) are constrained to be constant \((z\text{-independent})\) in which case solutions may be constructed by elementary methods (appendix A).

In fact, I shall assume \( \rho > 0 \) to be fixed for the remainder of this paper, that is, not updated in the inversion process. The wave velocity \( c \) will range over an interval: it is the parameter to be inverted. To be specific, choose \( c_{\max} > c_{\min} > 0 \), and require that \( c \) satisfy \( c_{\min} \leq c \leq c_{\max} \).

Limit observations to the time interval \([0, T]\), at the spatial (‘receiver’) location \( z_r \). The modeling operator outputs the pressure trace \( p(z_r, t) \) over the time interval \([0, T]\):

\[
S[c]f = p|_{(z_r) \times [0, T]}.
\]

This trace is well-defined if \( f \) is smooth, since then the solution \((p, v)\) is smooth. For homogeneous \((z\text{-independent})\) \( c \), appendix A provides an explicit expression:

\[
S[c]f(t) = \frac{1}{2c} \int_{c_{\min}}^{c_{\max}} zf \left( z, t - \frac{|z_r - z|}{c} \right) \, dz.
\]

From this expression it clear that \( S[c] \) extends a bounded operator

\[
S[c] : L^2([z_{\min}, z_{\max}] \times \mathbb{R}) \to L^2[0, T].
\]

That is, the support of \( f \) will be assumed to lie in the strip \([z_{\min}, z_{\max}] \times \mathbb{R}\). The choice of spatial interval \([z_{\min}, z_{\max}]\) is arbitrary, so long as it has positive length. The role of compact support in \( z \) in ensuring boundedness of \( S[c] \) is also evident from equation (5).

It should be mentioned that the triviality of the conclusion (6) is a happy accident of one space dimension: for analogous hyperbolic initial-boundary value problems in more than one space dimension, simple integral representations such as (5) are not to be expected, and the regularity of traces becomes a much more involved matter—see for example Symes and Payne (1983), Lasiecka and Trigianni (1989), and Bao and Symes (1991).

The formulation of the inverse problem via least-squares requires a choice of Hilbert space structure for the domain and range of \( S[c] \), given for this problem in display (6). The data \( d \) is presumed to lie in the range space of \( S[c] \): \( d \in L^2[0, T] \). It is simple to verify from the identity (5) that \( S[c] \) is also surjective. That is, any data at all can be fit by \( S[c] \) with an appropriate choice of input. This observation is important in the development of WRI, to be explained below.

The version of FWI discussed here presumes that the source field corresponding to the data \( d \) is supported at a point \( z_s \in \mathbb{R} \), and is known. The source position \( z_s \) must satisfy \( z_{\min} \leq z_s \leq z_{\max} \) and \( z_s \neq z_r \), but is otherwise arbitrary.

This point source field depends on a function of time (‘wavelet’) \( w \in L^2(\mathbb{R}) \). Formally, the resulting acoustic field satisfies the system (3) with \( f(z, t) = w(t) \delta(z - z_s) \). Inserting this expression in the explicit expression (5), obtain

\[
f(z, t) = w(t) \delta(z - z_s),
\]

\[
(S[c]f)(t) = \frac{1}{2c} w \left( t - \frac{|z_r - z_s|}{c} \right)
\]

\[
= (S[p]c)w(t).
\]
\(S_p[c]\) is a bounded operator with domain \(L^2(\mathbb{R})\) and range \(L^2([0, T])\).

For this restricted class of source fields, the FWI problem (1) can be redefined as minimization of the objective

\[
J_{\text{FWI}}[c; d, w] = \frac{1}{2} \|S_p[c]w - d\|^2,
\]

over \(c\), with \(d\) and \(w\) treated as parameters.

To end this section, it is necessary to address an irritating technical point: the point source defined above is not a member of the domain of \(S[c]\), as it was defined in display (6), so the left-hand side of equation (7) does not actually make sense. The fix for this incompatibility actually elucidates the relation between \(S[c]\) and \(S_p[c]\). Appendix B describes the construction of a family of bounded injective operators \(E[c]: L^2(\mathbb{R}) \rightarrow L^2([z_{\min}, z_{\max}] \times \mathbb{R})\) for which

\[
S_p[c] = S[c] \circ E[c].
\]

This relation exhibits \(S[c]\) as an extension of \(S_p[c]\) as described by Symes (2009). WRI for 1D acoustic transmission is based on \(S[c]\), and so is identified as an extended version of FWI.

The construction described in appendix B requires that \(z_s \in (z_{\min}, z_{\max})\), as mentioned above.

### 3. Global asymptotics of 1D transmission FWI

The FWI objective is well-known to exhibit non-convexity unless data frequency content is limited to a small range near 0 Hz, how small being determined by other scales and by the extent to which the initial wave velocity differs from the target.

To understand the non-convexity phenomenon and the relation of the various scales, it is advantageous to introduce a family of source wavelets, depending on a parameter \(\lambda\), having dimensions of time and playing the role of wavelength:

\[
w_\lambda(t) = \frac{1}{\sqrt{\lambda}} w_1 \left( \frac{t}{\lambda} \right).
\]

In the definition (10), the ‘mother wavelet’ \(w_1 \in C_0^\infty(0, 1)\) has dimensionless argument, and the scaling is chosen so that \(\|w_\lambda\|_{L^2(\mathbb{R})}\) is independent of \(\lambda > 0\).

Evidently there is no control of \(c\) at all in the data if the time interval of the observation, namely \([0, T]\), is so short that no signal arrives within it. Accordingly, add to the other assumptions made so far the requirement that the transit time between source and receiver at the slowest permitted velocity is less than \(T\):

\[
\frac{|z_s - z_r|}{c_{\min}} < T.
\]

Choose a target wave velocity \(c_* \in [c_{\min}, c_{\max}]\). Introduce a family of consistent data \(d_\lambda\), generated by \(c_*\) and the wavelet family \(w_\lambda\):

\[
d_\lambda = S_p[c_*]w_\lambda
\]

and a corresponding family of FWI objectives:
\[
J_{FWI}[c; d_{\lambda}, w_{\lambda}] = \frac{1}{2} \|d_{\lambda} - S_p[c]w_{\lambda}\|^2
\]
\[
= \frac{1}{2} \int_0^T dt \left[ \frac{1}{2c_*}w_{\lambda} \left( t - \frac{|z_s - z_r|}{c_*} \right) - \frac{1}{2c}w_{\lambda} \left( t - \frac{|z_s - z_r|}{c} \right) \right]^2.
\]
(13)

Note that \(\text{supp } w_{\lambda} \subset [0, \lambda]\), so
\[
\text{supp } S_p[c]w_{\lambda} \subset \left[ \frac{|z_s - z_r|}{c}, \lambda + \frac{|z_s - z_r|}{c} \right] \cap [0, T].
\]
(14)

The transit time condition (11) implies that there exists \(\lambda_0 > 0\) so that for \(\lambda < \lambda_0\),
\[
\lambda + \frac{|z_s - z_r|}{c} < T
\]
for all admissible \(c\). That is, for \(\lambda < \lambda_0\), \(\text{supp } S_p[c]w_{\lambda} \subset (0, T)\) for \(c \in [c_{\min}, c_{\max}]\), and
\[
\|S_p[c]w_{\lambda}\|^2 = \frac{1}{4c^2} \int_{-\infty}^{\infty} dt \left| w_{\lambda} \left( t - \frac{|z_s - z_r|}{c} \right) \right|^2 = \frac{\|w_1\|^2}{4c^2}.
\]
(15)

Recall that the object of this study is the global behaviour of objective functions for velocity estimation: in this context, that means the behaviour for \(c\) far from \(c_*\). Define
\[
L = \frac{2c_{\max}^2}{|z_s - z_r|}.
\]
(16)

Then if \(|c_* - c| > L\lambda\),
\[
\left| \frac{|z_s - z_r|}{c} - \frac{|z_s - z_r|}{c_*} \right| = \frac{|c - c_*||z_s - z_r|}{cc_*}
\]
\[
\geq \frac{|c - c_*||z_s - z_r|}{c_{\max}}
\]
\[
> L\lambda \frac{|z_s - z_r|}{c_{\max}}
\]
\[
= 2\lambda.
\]
(17)

That is, equation (17) shows that when the condition (16) is satisfied, the infima of the supports of \(S_p[c]w_{\lambda}, d_{\lambda} = S_p[c_*]w_{\lambda}\) are further apart than the lengths of these supports. Then necessarily \(\text{supp } S_p[c]w_{\lambda} \cap \text{supp } S_p[c_*]w_{\lambda} = \emptyset\), so \(S_p[c]w_{\lambda}\) and \(S_p[c_*]w_{\lambda}\) are orthogonal in \(L^2[0, T]\), and
\[
J_{FWI}[c; d_{\lambda}, w_{\lambda}] = \frac{1}{2} \left( \frac{1}{4c_*^2} + \frac{1}{4c^2} \right) \|w_1\|^2.
\]
(18)

Amongst other consequences, one immediately deduces from the expression 18 the non-convexity result:

**Theorem 1** For \(L > 0\) given by equation (16) and \(\lambda < \lambda_0\), the minimizer of \(J_{FWI}[c; d_{\lambda}, w_{\lambda}]\) on the complement of \([c_* - L\lambda, c_* + L\lambda]\) is \(c = c_{\max}\).
That is, outside of a neighborhood of width proportional to a wavelength, minimization of $J_{FWI}$ yields a local minimizer far from the target velocity $c^*$ that generates the (noise-free) data.

For this 1D problem, a happy 1D accident occurs: a descent minimization starting at $c_0 < c^*$ will at least initially proceed in the right direction. With sufficiently small steps, it is possible that an interaction might land in the (small) domain of attraction around $c^*$. However neither this nor various other accidental advantages stemming from the very special form of this problem should be regarded as of any importance.

4. Wavefield reconstruction inversion

This section will describe WRI and develop some of its formal algebraic properties. I will follow the development of the basic concept and its variable projection refinement from the papers of van Leeuwen and Herrmann (van Leeuwen and Herrmann 2013 and van Leeuwen and Herrmann 2016), then show how WRI may be cast as an extended source inversion. In this section, consistent with the literature on this topic, I will abjure precise definitions of spaces and operators involved in the problem formulations. I introduce just enough mathematical structure to conclude that these three variants of WRI are strictly equivalent: all three are optimization formulations, and a local minimizer of any one of the corresponds one-to-one with local minimizers of the others by well-defined transformations.

These conclusions will be applied to 1D acoustics in the following section, with $c$ specialized to a scalar (wave velocity) and all of the missing definitions filled in.

In the formulation of van Leeuwen and Herrmann (2013), WRI involves a dynamical wavefield $u$, a data vector $d$, a data sampling operator $P$, a vector of material parameters $c$, a wave operator $L[c]$ depending on $c$, a penalty weight $\alpha$, and a target source $q$. The domains and ranges of $P$ and $L[c]$ are presumed to be Hilbert spaces, in which the various vectors sit, and the symbol $\| \cdot \|$ will represent the appropriate Hilbert norm in each place where it appears.

The WRI objective function is defined as

$$\phi_\alpha[c, u; d, q] = \frac{1}{2}(\|Pu - d\|^2 + \alpha^2\|L[c]u - q\|^2).$$

(19)

Note that I have changed the notation of van Leeuwen and Herrmann (2013) slightly to conform more closely to that used in the rest of this paper, however the meaning is exactly the same as their equation (5).

Note also that $d$ and $q$ appear as parameters in $\phi_\alpha$, which is to be optimized over $c, u$. In particular, the target source $q$ must be known. Fang et al (2018) describe an algorithm for incorporating estimation of $q$ into WRI.

The objective $\phi_\alpha[c, u; d, q]$ is quadratic in $u$, so given the other three fields, estimation of $u$ results from solution of a linear least squares problem. van Leeuwen and Herrmann (2013) assume that $L[c]$ is the Helmholtz operator and $c$ is the square slowness. In that case (and some others), $\phi_\alpha[c, u; d, q]$ is quadratic in $c$, so minimization over $c$ with fixed $u$ is also a linear least squares problem (in fact, a very simple one, in the Helmholtz case, as it may be solved by pointwise division). Based on these observations, van Leeuwen and Herrmann (2013) minimize $\phi_\alpha$ via coordinate search, alternating minimization over $u$ with minimization over $c$.

van Leeuwen and Herrmann (2016) used the variable projection method (VPM) (Golub and Pereyra 2003), rather than coordinate search, to reduce the optimization of $\phi_\alpha[c, u; d, q]$ to an optimization in $c$ alone. Define

$$u_\alpha[c; d, q] = \arg \min_u \phi_\alpha[c, u; d, q].$$

(20)
Then assuming (as do van Leeuwen and Herrmann (2016)) that $L[c]$ is invertible for any admissible $c$, $L[c]^T L[c]$ is symmetric positive definite, and therefore $u_0$ is uniquely determined as the solution of a normal equation [equation after display (20) in (van Leeuwen and Herrmann 2016)]

$$u_0[c; d, q] = (L[c]^T L[c] + \alpha^{-2} P^T P)^{-1}(L[c]^T q + \alpha^{-2} P^T d).$$

(21)

The VPM reduction of the objective function is

$$\phi_{\alpha}^{\text{red}}[c; d, q] = \phi_{\alpha}[c, u_0[c; d, q]; d, q].$$

(22)

Note that $u_0$ is defined by solution of an unconstrained convex quadratic minimization (20). Therefore $c, u$ is a local minimizer of $\phi_{\alpha}[\cdot; d, q]$ if and only if $c$ is a local minimizer of $\phi_{\alpha}^{\text{red}}[\cdot; d, q]$ and $u = u_0[c; d, q]$. That is, local minimizers of the reduced objective $\phi_{\alpha}^{\text{red}}[\cdot; d, q]$ are in one-to-one correspondence with local minimizers of the basic WRI objective $\phi_{\alpha}[\cdot; d, q]$. In that sense, the original and reduced WRI problems are completely equivalent.

The assumption that $L[c]$ is invertible is essential to the arguments of van Leeuwen and Herrmann (2013) and van Leeuwen and Herrmann (2016). That is, the equation $L[c] u = f$ has a unique solution $u = L[c]^{-1} f$ for any $f$ in the range space of $L[c]$ (as in the cited papers, I will leave the actual identity of both domain and range of $L[c]$ a bit vague). With $u_0$ defined as in equation (21), compute

$$f_0[c; d, q] \equiv L[c] u_0[c; d, q]$$

$$= L[c](L[c]^T L[c] + \alpha^{-2} P^T P)^{-1} L[c]^T (L[c]^T q + \alpha^{-2} P^T d)$$

$$= (I + \alpha^{-2} P L[c]^{-1} P^T)(P L[c]^{-1})^{-1} (q + \alpha^{-2} (P L[c]^{-1})^T d).$$

Define $S[c] = P L[c]^{-1}$. Then with a little rearranging,

$$f_0[c; d, q] = (S[c]^T S[c] + \alpha^2 I)^{-1} (\alpha^2 q + S[c]^T d).$$

(23)

Inspection shows that $f_0[c; d, q]$ is the solution of a least squares problem:

$$f_0[c; d, q] = \arg \min_{f} \frac{1}{2} \|d - S[c] f - d\|^2 + \alpha^2 \|f - q\|^2.$$ 

(24)

VPM (elimination of $f$) applied to the objective on the right-hand side of equation (24) yields the reduced objective

$$J_{\text{WRI}}^{\alpha}[c; d, q] = \min_{f} \frac{1}{2} \|d - S[c] f - d\|^2 + \alpha^2 \|f - q\|^2$$

$$= \frac{1}{2} \|d - S[c] f_0[c; d, q] - q\|^2 + \alpha^2 \|f_0[c; d, q] - q\|^2.$$ 

(25)

The main point to be made here is that the objectives $\phi_{\alpha}^{\text{red}}$ and $J_{\text{WRI}}^{\alpha}$ are the same:

$$\phi_{\alpha}^{\text{red}}[c; d, q] = \frac{1}{2} \|L[c] u_0[c; d, q] - q\|^2 + \alpha^2 \|d - P u_0[c; d, q]\|^2$$

$$= \frac{1}{2} \|f_0[c; d, q] - q\|^2 + \alpha^2 \|d - P L[c]^{-1} f_0[c; d, q]\|^2$$

$$= J_{\text{WRI}}^{\alpha}[c; d, q].$$ 

(26)
Since these are the same functions, they have the same local minimizers. As noted earlier, $c$ is a local minimizer of $\phi^\text{non-radiating}_{\text{WRI}}(c; d, q)$ if and only if $(c, u_c[c; d, q])$ is a local minimizer of $\phi_c(c, u; d, q)$. Since $c \mapsto u_c[c; d, q]$ is injective, conclude that the sets of local minimizers of the three optimization problems defined so far in this section are in one-to-one correspondence.

Define the residual with the target source $q$ as $r[c] = d - S[c]q$, and set $g = f - q$. Then the definition (25) can be rewritten as

$$J^\text{WR}_{\text{WRI}}[c; d, q] = \min_{g} \frac{1}{2} \left( \|r[c] - S[c]g\|^2 + \alpha^2 \|g\|^2 \right).$$

This formulation was introduced by Wang et al (2016).

‘Most’ source fields $f$ are non-radiating, that is, $S[c]f = 0$, and such sources contribute nothing to the data fit term in the definition of $J^\text{WR}_{\text{WRI}}$. If the domain and range of $S$ were finite dimensional (which of course they are, after discretization), then the fundamental theorem of linear algebra identifies the null space of $S[c]$ (the non-radiating sources) as the orthocomplement of the range of the transpose $S[c]^T$ (Strang 1993). In the infinite-dimensional setting of this paper, the closed range theorem (Yosida 1996) states that the same is true if the range of $S[c]$ is closed. There are various ways to ensure this property, but the simplest is relevant here:

For all admissible models $c$, $S[c]$ is surjective.

That is, any data can be fit exactly using the extended model space (the domain of $S$). This property is a characteristic of extended modeling methods (Symes 2008): the ability to fit any data appears to be essential for such methods to produce objectives without spurious local minima. It holds for the problems for which WRI has been advocated. For the simple model problem considered here, surjectivity follows from the explicit expression for $S[c]f$, as was noted in the discussion following equation (5).

Assuming that $S[c]$ is surjective for any admissible $c$, the orthocomplement of the subspace of non-radiating sources is the range of the adjoint operator $S[c]^T$. In the definition (27), decompose $g = S[c]^Te + n$, in which $e$ is the same type of object as $d$ and $S[c]n = 0$ (that is, $n$ is a non-radiating source), and note that the decomposition is orthogonal. Then

$$J^\text{WR}_{\text{WRI}}[c; d, q] = \min_{r, n} \frac{1}{2} \left( \|d - S[c](S[c]^Te + q)\|^2 + \alpha^2 (\|S[c]^Te\|^2 + \|n\|^2) \right)$$

$$= \min_{r, n} \frac{1}{2} \left( \|r[c] - S[c]S[c]^Te\|^2 + \alpha^2 \|S[c]^Te\|^2 \right).$$

(28)

This reformulation has some computational advantages (Wang et al 2016 and Rizzuti et al 2019), but also leads to a useful analytic transformation of the WRI problem. The minimizer on the RHS of equation (28) is the solution $e = e_n[c]$ of the normal equation

$$(S[c]S[c]^T)^{-1} + \alpha^2 S[c](S[c]^T) = S[c]S[c]^T r[c],$$

whence

$$S[c]S[c]^T e_n[c] = S[c]S[c]^T(S[c]S[c]^T + \alpha^2 I)^{-1} r[c].$$

Since the null space of $S[c]$ is orthogonal to the range of $S[c]^T$ under the surjectivity assumption, $S[c]S[c]^T$ is injective, whence

$$e_n[c] = (S[c]S[c]^T + \alpha^2 I)^{-1} r[c].$$

(29)
Consequently

\[
J_{\text{WRI}}^\alpha[c; d, q] = \frac{1}{2} (\|r[c] - S[c]S[c]^T e_o[c]\|^2 + \alpha^2 \|S[c]^T e_o[c]\|^2) \\
= \frac{1}{2} (\|r[c] - S[c]S[c]^T (S[c]S[c]^T + \alpha^2 I)^{-1} r[c]\|^2 \\
+ \alpha^2 \langle (S[c]S[c]^T + \alpha^2 I)^{-1} r[c], S[c]S[c]^T (S[c]S[c]^T + \alpha^2 I)^{-1} r[c]\rangle) \\
= \frac{1}{2} (\|\alpha^2 (S[c]S[c]^T + \alpha^2 I)^{-1} r[c]\|^2 \\
+ \alpha^2 \langle (S[c]S[c]^T + \alpha^2 I)^{-1} r[c], S[c]S[c]^T (S[c]S[c]^T + \alpha^2 I)^{-1} r[c]\rangle) \\
= \frac{\alpha^2}{2} \langle (S[c]S[c]^T + \alpha^2 I)^{-1} r[c], r[c]\rangle. 
\]

(30)

Rearranging the RHS of equation (30), obtain

\[
J_{\text{WRI}}^\alpha[c; d, q] = \frac{1}{2} \langle r[c], W_\alpha[c] r[c]\rangle, 
\]

(31)

with

\[
W_\alpha[c] = \frac{\alpha^2}{2} (S[c]S[c]^T + \alpha^2 I)^{-1}. 
\]

(32)

This remarkable identity shows that the WRI objective function is a weighted norm of the data residual \(r[c]\).

van Leeuwen (2019) gives a different derivation of an identity equivalent to equations (31) and (32).

Note that \(\alpha^2\) must have the same units as \(S[c]S[c]^T\) (or \(S[c]^T S[c]\)), hence \(W_\alpha[c]\) is dimensionless. Of course this also follows from the definition of \(J_{\text{WRI}}^\alpha\), which must have the same units as \(J_{\text{FWI}}\).

To end this discussion, it must stipulated that in any penalty method, control of the penalty parameter has a large influence on the speed of convergence. Aghamiry et al (2019) use an augmented Lagrangian algorithm to minimize the influence of the penalty weight choice. Alternatively, one can use a version of the discrepancy principle to adjust \(\alpha\) dynamically (Fu and Symes 2017), as the WRI problem has the necessary features described in that paper. Since the properties of WRI to be established in the next section are actually independent of \(\alpha > 0\), I leave the matter of \(\alpha\) control at that.

5. Global asymptotics of 1D transmission WRI

The preceding section provides the necessary ingredients for an assessment of the relation between WRI and FWI. While the conclusion reached below applies to many wave propagation settings, the 1D acoustic setting is particularly simple and yet illustrates clearly the nature of this relation.
The first task is to give an explicit expression for the operator $S[c]S[c]^T$ appearing repeatedly in the expression (32). From the definition (5), it follows immediately that

$$S[c]^T e(z, t) = \begin{cases} 
\frac{1}{2c} e \left( t + \frac{|z - \bar{z}|}{c} \right), & z_{\min} \leq z \leq z_{\max}; \\
0, & \text{else,}
\end{cases} \quad (33)$$

whence

$$S[c]S[c]^T e(t) = \frac{z_{\max} - z_{\min}}{4c^2} e(t),$$

that is,

$$S[c]S[c]^T = \frac{z_{\max} - z_{\min}}{4c^2} I. \quad (34)$$

Thus the weight operator $W[c]$ appearing in (31) takes the form

$$W_{\alpha}[c] = u_{\alpha}[c] I,$$

$$u_{\alpha}[c] = \frac{\alpha^2}{2} \left( \frac{z_{\max} - z_{\min}}{4c^2} + \alpha^2 \right)^{-1}. \quad (35)$$

Next, suppose that $q = w\delta(z - z_s)$, that is, the target source is a point source, so that $SI[c]q = S_p[c]w$ in the notation used in the discussion of FWI. Thus (31) can be re-written as

$$J_{WR}[c; d, q] = u_{\alpha}[c] J_{FWI}[c; d, w]. \quad (36)$$

Recall the wavelength-dependent family of problems introduced in the derivation and statement of theorem 1: target wave velocity $c^*$, wavelength parameter $\lambda$, parametrized family of wavelets $w_\lambda$ and corresponding data $d_\lambda$.

Define

$$\beta = \frac{z_{\max} - z_{\min}}{c^2} - 4\alpha^2. \quad (37)$$

**Theorem 2** For $L$ as defined in (16), and $\lambda < \lambda_0$, the minimizer of $J_{WR}[c; d_\lambda, w_\lambda \delta(c - z_s)]$ on the complement of $[c^* - L\lambda, c^* + L\lambda]$ is

- $c = c_{\max}$ if $\beta < 0$;
- $c = c_{\min}$ if $\beta > 0$;
- any $c < c^* - L\lambda$ or $> c^* + L\lambda$ if $\beta = 0$.

**Proof.** From (18), (35), and (36), if $|c - c^*| > L\lambda$,

$$J_{WR}[c; d_\lambda, w_\lambda \delta(c - z_s)] = \frac{\alpha^2}{2} \left( \frac{z_{\max} - z_{\min}}{4c^2} + \alpha^2 \right)^{-1} \frac{1}{2} \left( \frac{1}{4c^2} + \frac{1}{4c^2} \right) \|w_1\|^2$$

$$= \frac{\beta^2}{4} \frac{1 + \frac{\beta^2}{4c^4}}{z_{\max} - z_{\min} + 4c^2\alpha^2} \|w_1\|^2. \quad (38)$$

The linear fractional function of $c^2$ on the RHS of equation (38) is increasing, decreasing, or constant if $\beta > 0$, $\beta < 0$ or $\beta = 0$, respectively. \qed
In other words, $J_{\alpha}^{WRI}$ has local minima far from the target velocity $c_\ast$, in the same way as does $J_{FWI}$. One of the local minima will be the result of a local optimization almost surely, unless the initial estimate of $c$ is "within a wavelength" of the target velocity.

Note that $L$ is independent of $\alpha$ (definition (16)), and for small enough $\alpha, \beta > 0$ (definition (37)). Conclude that the region $\{c \in [c_{\min}, c_{\max}] : |c - c_\ast| > L\lambda \}$ is independent of $\alpha$, and the minimizer of $J_{WRI}[c; d, w_\lambda(\cdot - z_\ast)]$ in this region (away from $c_\ast$) is $c = c_{\min}$ for small enough $\alpha$. Therefore taking $\alpha$ small does not change the multimodal nature of $J_{WRI}$: there remain multiple far-apart local minima, no matter how small $\alpha$ may be.

6. Numerical illustration

The Ricker wavelet (normalized second derivative of a Gaussian) is a commonly used synthetic seismic signal, as it closely resembles short segments of field data traces. Figure 1 displays a Ricker wavelet of 20 Hz peak frequency and zero phase (symmetric about $t = 0$). The Ricker wavelet of 1 Hz peak frequency decreases by almost three orders of magnitude at $t = \pm 1$, relative to its peak at $t = 0$, so could reasonably be truncated to $[-1, 1]$ with a bit of smoothing, hence could be taken as the 'mother wavelet' $w_1$ of the discussion in section 2. Then the signal displayed in figure 1 is $w_\lambda$ for $\lambda = 0.05$.

Figure 2 shows the simulated data $d = S[c_\ast]w_\lambda$ for $c_\ast = 2.5$ km s$^{-1}$ and source-receiver offset $|z_r - z_s| = 1$ km. The time shift is calculated via piecewise linear interpolation on a grid with time interval $\Delta t = 0.001$ s.

Figures 3 and 4 show FWI and VPM-reduced WRI objectives ($J_{FWI}$ and $J_{\alpha}^{WRI}$ respectively) as functions of slowness (reciprocal velocity). The FWI (mean square data residual) values are computed directly from equation (1), using the trapezoidal rule to approximate the integral. The VPM-reduced WRI values are calculated on the basis of the weighted-residual formulas (31) and (32), using the result (35) which holds for the 1D transmission problem (that is, the relation (36)).

The figures 3 and 4 differ in choice of penalty weight $\alpha$ for the WRI objective. For the 1D transmission problem, the unit of $\alpha$ is (time)/$\sqrt{\text{length}}$, so $s (\sqrt{\text{km}})^{-1}$ in the units used here.
Figure 2. Predicted data for 1D transmission problem, source wavelet = 20 Hz Ricker wavelet of figure 1, source–receiver offset = 1 km, velocity = 2.5 km s$^{-1}$.

Figure 3. Blue curve = FWI objective, red curve = VPM–WRI objective, plotted as functions of slowness, $\alpha = 0.1$ s ($\sqrt{\text{km}}$)$^{-1}$.

For figure 3, $\alpha = 0.1$ s ($\sqrt{\text{km}}$)$^{-1}$, whereas for figure 4, $\alpha = 1.0$ s ($\sqrt{\text{km}}$)$^{-1}$. For $J_{\text{FWI}}$, $c = c_{\text{min}}$ is a local minimizer, consistent with theorem 1, and corresponding to the largest admissible slowness. For the parameters of figure 3, the quantity $\beta$ defined in equation (37) is positive, so $c = c_{\text{min}}$ is a local minimizer of $J_{\text{WRI}}$, corresponding to the largest admissible slowness, per theorem 2. For the parameters pertaining to figure 4, $\beta$ is negative, so $c = c_{\text{max}}$ is a local minimizer of $J_{\text{WRI}}$, corresponding to the smallest admissible slowness.
Figure 4. Blue curve = FWI objective, red curve = VPM–WRI objective, plotted as functions of slowness, $\alpha = 1.0 \text{ s (\sqrt{km})}^{-1}$.

In all cases, the global minimizer occurs at the target velocity $c^* = 2.5 \text{ km s}^{-1}$ (slowness $= 0.4 \text{ s km}^{-1}$). It is ‘defended’ by close-in local minimizers roughly half a wavelength on either side. Our analysis would be just as valid for non-oscillatory wavelets, so cannot predict this smaller-scale structure.

7. Discussion

Theorems 1 and 2 call out the chief conclusions of this work: that at least for the 1D acoustic transmission inverse problem, both $J_{\text{FWI}}[c; d, w]$ and $J_{\text{WRI}}[c; d, q]$ exhibit local minima (in $c$) far from the global minimum for consistent data, the domain of attraction of the global minimizer can be arbitrarily small, and these properties persist as $\alpha \to 0$. These are striking conclusions, but the 1D acoustic transmission problem is very special and lacks fidelity to field practice. I shall show how these approaches to solving this special problem share properties with a much larger family of inversion methods. The theory developed to explain these properties suggests methods that may not suffer the non-convexity of FWI and WRI, and in fact do not in several cases that I will mention.

First, a consequence of the results proven here: $J_{\text{FWI}}[c; d, w]$ and $J_{\text{WRI}}[c; d, q]$ are not smooth as joint functions of model ($c$) and data ($d$) vectors. If they were, their derivatives would be bounded uniformly over bounded sets in $c, d$, but the two main results show that this is not the case. As $\lambda \to 0$, $d = d_\lambda$ varies within a ball $B \subset L^2([0, T])$ of radius $\|d_1\|$ (since the $\|d_\lambda\|$ is independent of $\lambda$), but the value of either objective changes from a positive value (bounded away from zero independently of $\lambda$) to zero over an interval of $c$ of length $O(\lambda)$. Therefore the derivatives with respect to $c$ of both objective functions are not bounded over $[c_{\text{min}}, c_{\text{max}}] \times B$.

While lack of smoothness is not in itself the most important property established in the preceding sections, it is a necessary condition for stable and reliable parameter recovery via local optimization. Moreover, necessary conditions for smoothness are known for a much wider class of quadratic form objectives for inverse problems.
These results concern optimization problems of the general form

\[ \text{Given } d, \text{ find } c \text{ to extremize } J[c, d] = \langle G[c]d, A[c]G[c]d \rangle. \] \tag{39} 

In this prescription, \( d \) is a data vector, as in the examples above, \( c \) is a vector of material parameters to be estimated, \( G[c] \) is a \( c \)-dependent family of operators, whose common range is a Hilbert space with inner product \( \langle \cdot, \cdot \rangle \). \( A[c] \) is an operator-valued function of \( c \), with domain and the range equal to the range of \( G[c] \).

Stolk and Symes (2003) assume that the operator-valued function \( G[c] \) is of a class typical of modeling operators for wave equation inverse problems, or their inverses or adjoints. The precise characterization of these so-called microlocally elliptic Fourier integral operators is quite technical (Duistermaat 1996). Roughly speaking, such operators map high-frequency localized wave packets to other such packets with well-defined changes of position and direction of oscillation. The simulation operators \( S[c] \) and \( S_p[c] \) figuring in the preceding discussion are particularly simple examples of this type.

If the dependence of such an operator \( G[c] \) on \( c \) is of sufficiently full rank, in the sense that destination packets can be shifted in any direction by changing \( c \) appropriately, along with a couple of other technical assumptions, one can conclude that \( J \) as defined in (39) is smooth in \( c \) and \( d \) jointly if and only if the operator \( A[c] \) is a pseudodifferential operator—again, a class of operators whose precise definition is quite technical (Duistermaat 1996 and Taylor 1981). However these operators also have a rough characterization: they do not change the location of oscillatory wave packets or alter their direction of oscillation, but only scale such packets by smooth functions, to good approximation.

With a bit of fiddling, the FWI problem (8) for 1D acoustic transmission inversion can be rewritten in the form (39). Note that \( S_p[c] \) is invertible (more specifically, has a right inverse; if \([0, T]\) were extended to \((−\infty, \infty)\) it would have a left inverse too). From the definition (12) of the data family \( d_\lambda \), one sees that \( w_\lambda = S_p[c_\lambda]^{-1}d_\lambda \). Therefore

\[
J_{\text{FWI}}[c; d_\lambda, w_\lambda] = \frac{1}{2} \| (I - S_p[c]S_p[c]^{-1})d_\lambda \|^2
= \frac{1}{2} (\| d_\lambda \|^2 + \| S_p[c]S_p[c]^{-1}d_\lambda \|^2) + \langle d_\lambda, S_p[c]S_p[c]^{-1}d_\lambda \rangle.
\tag{40}
\]

For the operator family \( S_p[c] \) defined above, it is easy to see that the second term in the right-hand side of equation (40) is smooth in \( c \), and the first is constant. The third can be rewritten as

\[ \langle d_\lambda, S_p[c]S_p[c_\lambda]^{-1}d_\lambda \rangle = \langle S[c]^T d_\lambda, (S_p[c]^T S_p[c_\lambda])^{-1}S_p[c]^T d \rangle. \tag{41} \]

The RHS of equation (41) has the form (39) with the choices \( G[c] = S_p[c]^T \), \( A[c] = (S_p[c]^T S_p[c_\lambda])^{-1} \). A similar manipulation exhibits \( J_{\text{WRI}} \) as the sum of harmless terms and a quadratic form of the form (39).

Given the rough understanding of the results of Stolk and Symes (2003) sketched above, one would conclude that neither \( J_{\text{FWI}} \) nor \( J_{\text{WRI}} \) are likely to be smooth jointly in \( c \) and \( d \). Indeed, apart from scale, \( S_p[c] \) is composition with a shift (translation) by \((\zeta_{\text{max}} - \zeta_{\text{min}})c^{-1}\), so \( A[c] = (S_p[c]^T S_p[c_\lambda])^{-1} \) is composition with a shift by \((\zeta_{\text{max}} - \zeta_{\text{min}})(c^{-1} - c_\lambda^{-1}) \). Thus application of \( A[c] \) does not leave the position of a wave packet fixed, unless \( c = c_\lambda \)—and indeed \( A[c] \) is not a pseudodifferential operator unless \( c = c_\lambda \). On the other hand, \( G[c] \) is a shift operator, the simplest prototype of an elliptic Fourier integral operator. Therefore the conclusion, derived...
directly from theorem 1, that $J_{FWI}$ is not smooth jointly in $c$ and $d$ would also appear to follow from the main result of (Stolk and Symes 2003). This conclusion can be made precise by proper attention to detail, and the same is true of $J_{WRI}$.

In the context of the 1D acoustic transmission problem as formulated here, the question immediately arises: do quadratic forms (39) exist that are smooth jointly in $c$ and $d$, and whose global minimizer is the correct velocity $c = c_*$? An affirmative answer is provided for precisely this example problem in (Symes 2020). The operator $G[c] = S_p[c]^T$ is precisely the same as appeared in the reformulation of $J_{FWI}$. Ignoring a $c$-dependent multiplier,

$$A[c]u(t) = tu(t).$$

Applied to the wavelength-dependent family of data $d_\lambda$ and source wavelets $w_\lambda$ used repeatedy throughout this paper, $A[c]$ yields a vanishing result as $\lambda \to 0$ for the correct velocity $c = c_*$ and a stably non-zero result otherwise. For these choices, it can be established that

$$\frac{d}{dc}J[c, d_\lambda] \begin{cases} > 0 & \text{if } c < c_* + O(\lambda), \\ < 0 & \text{if } c > c_* + O(\lambda) \end{cases}.$$ 

That is, all local minima of $J[c, d_\lambda]$ lie within $O(\lambda)$ (‘a wavelength’) of the target velocity $c_*$. Not surprisingly the wavelength parameter also regulates the accuracy of the inversion.

The reader is directed to (Huang et al 2019) for an extensive discussion of other similar source extension methods for various wave inversion problems, and for references to earlier work on this topic.

8. Conclusion

The tendency of iterative FWI to be trapped in uninformative local minima has been much discussed and still drives a substantial worldwide research program, almost 35 years after the phenomenon was first identified. WRI is amongst the many remedies proposed for this pathology, and numerical experiments have appeared to suggest that it may succeed. The example investigated in this report is simple enough to allow for rigorous mathematical conclusions regarding the behaviour of both FWI and WRI. The complete explanation for the behaviour of FWI is no surprise. As it turns out, the same conclusion may be reached for WRI: in this example at least, iterative minimization of the WRI objective it is no more likely to produce a useful estimate of wave velocity than is FWI, and for the same reason—indeed, the two are very closely linked (equations (31) and (32)).

While these specific conclusions are of course tied to the extremely simple homogeneous acoustic 1D transmission inverse problem studied here, the relations (31) and (32) are straightforward algebraic properties of WRI and appear to link it closely to FWI in any wave propagation setting. Since the commonly cited arguments for reduced nonlinearity of WRI over FWI also apply in the context of the 1D transmission problem, these arguments cannot be regarded as having any actual force. As explained in the discussion section, even mere smoothness of a quadratic form objective function in both the model and data parameters may impose restrictions on the operators involved in the construction of the form. These restrictions are generally not met by any version of FWI. An examination of other versions of WRI from this point of view may prove informative.

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Appendix A. 1D radiation problem

Begin with the 1D acoustics point source system.

\[
\begin{align*}
\frac{\partial p}{\partial t} + \rho c^2 \frac{\partial v}{\partial z} &= w(t)\delta(z - z_s) \\
\rho \frac{\partial v}{\partial t} + \frac{\partial p}{\partial z} &= 0 \\
p, v &= 0, \ t \ll 0.
\end{align*}
\]  

(A-1)

Since the right-hand side is singular, so is the solution, so it must be a solution in the weak sense. It follows from the weak solution conditions that the pressure is continuous at \( z = z_s \), whence \( v \) must have a discontinuity.

In \( z \neq z_s \), the right-hand side vanishes, so the solution must be locally a combination of plane waves; causality implies that

\[
p(z, t) = a \left( t - \frac{|z - z_s|}{c} \right), \quad v(z, t) = \text{sgn}(z - z_s) b \left( t - \frac{|z - z_s|}{c} \right).
\]

From the second dynamical equation (Newton’s law) it follows that \( b = a/(\rho c) \). The singularity on the LHS of the first dynamical equation (constitutive law) is

\[
\rho c^2 [v]_{z = z_s} \delta(z - z_s) = 2 \rho c^2 b \delta(z - z_s) = 2ca \delta(z - z_s).
\]

This must in turn equal the RHS of the constitutive law, whence \( a = w/(2c) \). Thus

\[
p(z, t) = \frac{1}{2c} w \left( t - \frac{|z - z_s|}{c} \right), \quad v(z, t) = \text{sgn}(z - z_s) \frac{1}{2 \rho c^2} w \left( t - \frac{|z - z_s|}{c} \right).
\]  

(A-2)

This result (computation of the Green’s function for the acoustic system) permits an explicit expression for the system with a space–time source:

\[
\begin{align*}
\frac{\partial p}{\partial t} + \rho c^2 \frac{\partial v}{\partial z} &= f(z, t) \\
\rho \frac{\partial v}{\partial t} + \frac{\partial p}{\partial z} &= 0 \\
p, v &= 0, \ t \ll 0.
\end{align*}
\]  

(A-3)

Since

\[
f(z, t) = \int dz_1 f(z_1, t) \delta(z - z_1),
\]
vanish near
obtain

\begin{align}
p(z, t) &= \frac{1}{2c} \int dz_1 f \left( z_1, t - \frac{|z - z_1|}{c} \right), \quad (A-4) \\
v(z, t) &= \frac{1}{2\rho c^2} \int dz_1 \text{sgn}(z - z_1) f \left( z_1, t - \frac{|z - z_1|}{c} \right). \quad (A-5)
\end{align}

**Appendix B. Equivalence of point and non-point sources**

As noted in the text, the point source \(w(t)\delta(z - z_0)\) is not a member of the domain of the simulation operator \(S[c]\), as it is not square-integrable. The object of this appendix is to construct a square-integrable right-hand side in the system (3) for which the pressure field \(p\) is the same as that of the weak solution to the point source problem (A-1) constructed in the last section, near the receiver point \(z = z_r\), and to exhibit this square-integrable replacement for the point source as the image of the point source wavelet under a bounded extension map, as in equation (9).

One step in this construction involves building a constitutive defect (pressure) source that is equivalent to a force (velocity) source, in the sense of generating the same solution outside of the source support. This construction is presented here in the context of the 1D acoustic problem, but is a special case of a much more general construction of considerable interest in its own right (Burridge and Knopoff 2003).

Let \(\epsilon\) be any positive number \(<|z_s - z_c|\). Denote by \((p, v)\) the (weak) solution (A-2) of the point source problem constructed in the last section. Pick \(\phi \in C^\infty_0(\mathbb{R})\) so that \(\phi = 1\) if \(|z - z_0| \leq \epsilon/2\) and \(\phi(z) = 0\) if \(|z - z_0| \geq \epsilon\). Set \(p_0 = p(1 - \phi), \, v_0 = v(1 - \phi)\). Then

\begin{align}
\frac{\partial p_0}{\partial t} + \rho c^2 \frac{\partial v_0}{\partial z} &= f_0, \\
\rho \frac{\partial v_0}{\partial t} + \frac{\partial p_0}{\partial z} &= g_0,
\end{align}

in which

\begin{align}
f_0(z, t) &= -\rho c^2 v(z, t) \frac{\partial}{\partial z} (1 - \phi(z)), \\
g_0(z, t) &= -p(z, t) \frac{\partial}{\partial z} (1 - \phi(z)).
\end{align}

vanish near \(z = z_c\). If \(w \in L^2(\mathbb{R})\) and vanishes for large negative \(t\) (as it must, for the system (A-1) to be compatible), then from expressions (A-2) the distributions \(p, v\) are locally square-integrable in \( \{z : |z - z_s| \geq \epsilon/2\} \times \mathbb{R} \) and vanish for large negative \(t\), whence the same is true of \(f_0, g_0\).

Assume for the moment that \(w \in C^\infty_0(\mathbb{R})\), so that \(p, v\) are smooth away from \(z = z_s\) and \(p_0, v_0, f_0, g_0\) are smooth. Then \(p_0\) is also the solution of the second-order initial value problem

\begin{align}
\frac{1}{\rho c^2} \frac{\partial^2 p_0}{\partial t^2} - \frac{1}{\rho} \frac{\partial^2 p_0}{\partial z^2} &= F, \\
\rho \frac{\partial v_0}{\partial t} + \frac{\partial p_0}{\partial z} &= 0, \quad t \ll 0,
\end{align}

with the right-hand side \(F\) given by

\begin{align}
F &= \frac{1}{\rho c^2} \frac{\partial f_0}{\partial t} - \frac{1}{\rho} \frac{\partial g_0}{\partial z}.
\end{align}
Define $f$ by

$$
 f(z, t) = \rho c^2 \int_{-\infty}^{t} ds \, F(z, t) = f_0(z, t) - c^2 \int_{-\infty}^{t} ds \, \frac{\partial g_0}{\partial z}(z, s).
$$

Then setting $p_1 = p_0$, $v_1 = \frac{1}{\rho} \int_{-\infty}^{t} \frac{\partial p_0}{\partial z}$, it follows from (B-3) and (B-5) that $p_1, v_1$ solves (3) with $f$ as given above. Since $p_1 = p_0$, and $p_0 = p$ in a neighborhood of $z = z_r$, it follows that

$$
 S[c] f = S[p][w],
$$

that is, that using RHS $f$ in (A-3) produces the same pressure field near $z = z_r$ as does the point source in (A-1). Also

$$
 f(z, t) = -\rho c^2 v(z, t) \frac{\partial}{\partial z} (1 - \phi(z)) - c^2 \int_{-\infty}^{t} ds \, \frac{\partial}{\partial z}\left( -p(z, s) \frac{\partial}{\partial z} (1 - \phi(z)) \right)
$$

$$
 = -c^2 \left( \int_{-\infty}^{t} ds \left( \rho v - \frac{\partial p}{\partial z} \right)(z, s) \frac{\partial}{\partial z} (1 - \phi(z)) - p(z, v) \frac{\partial^2}{\partial z^2} (1 - \phi(z)) \right)
$$

$$
 = -2\rho c^2 (v(z, t)) \frac{\partial}{\partial z} (1 - \phi(z)) + c^2 \frac{\partial^2}{\partial z^2} (1 - \phi(z)) \int_{-\infty}^{t} ds p(z, s),
$$

using the second equation (momentum balance) in the system (A-1). Use (A-2) to replace $p, v$ by explicit expressions in $w$:

$$
 = -\text{sgn}(z - z_s) w \left( t - \frac{|z - z_s|}{c} \right) \frac{\partial}{\partial z} (1 - \phi(z)) + \frac{\partial^2}{\partial z^2} (1 - \phi(z)) c^2 \left( \int_{-\infty}^{t} w \right) \left( t - \frac{|z - z_s|}{c} \right)
$$

$$
 = E[c] w(z, t),
$$

whence the image of $w$ under $E[c]$ is square-integrable, $E[c]$ extends to a bounded operator $L^2(\mathbb{R}) \rightarrow L^2([z_{min}, z_{max}] \times \mathbb{R})$, and from equation (B-6)

$$
 S[c] \circ E[c] = S[p][w],
$$

as asserted in equation (9).

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