Intersection numbers of twisted cycles and the correlation functions of the conformal field theory
II

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October 23, 2018

Abstract: We evaluate the intersection numbers of loaded cycles and twisted forms associated with an $n$-fold Selberg-type integral. The result is deeply related with the geometry of the configuration space of $n + 3$ points in the projective line. The polyhedral geometrical method developed in [KY] and [Yo2] is used in full. This is a continuation of [MiY].

Keywords: twisted (co)homology, loaded cycles, intersection numbers, hypergeometric integrals, Selberg-type integrals, conformal field theory, correlation functions, Terada-3.

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Introduction

We evaluate the intersection numbers of loaded cycles associated with the $n$-fold integral – a variant of the Selberg integral –

$$
\int u(t) \frac{dt_1 \wedge \cdots \wedge dt_n}{\prod_{i=1}^{n} t_i(1-t_i)(z-t_i)}, \quad u = \prod_{i=1}^{n} t_i^{a_i} (1-t_i)^{\beta} (z-t_i)^{\delta} \prod_{1 \leq i < j \leq n} (t_i - t_j)^{2\gamma},
$$

where $z \neq 0, 1$ is a complex parameter. Let $S = S_u$ be the local system defined by the integrand $u$ on $X = \{ t = (t_1, \ldots, t_n) \in \mathbb{C}^n \mid t_i \neq 0, 1, z, t_j (j \neq i) 1 \leq i \leq n \}$, $S^-$ its dual, i.e. the local system on $X$ defined by $u^{-1}$. When $z$ is real, the real locus $X^R$ of $X$ breaks into disjoint $n$-cells. We load each cell in a standard way, say $\Delta$ with $u^\pm$ as is explained in §6, and make it loaded cycles $\Delta^\pm$, that is, elements of the locally finite $n$-dimensional homology group $H^f_n(X, S^\pm)$ with coefficients in $S$.

There is a natural dual pairing

$$
H^f_n(X, S) \times H_n(X, S^-) \longrightarrow \mathbb{C}
$$

called the intersection pairing. In this paper we assume that the exponents $\alpha, \beta, \gamma, \delta$ are sufficiently generic so that the natural map

$$
H_n(X, S) \longrightarrow H^f_n(X, S)
$$

is an isomorphism; the inverse map is called the regularization and is denoted by reg. We evaluate the intersection numbers of these cycles. The intersection number of the cycles $\Delta^+_n$ and reg $\Delta^-_n$ will be denoted by $\Delta^+_1 \bullet \text{reg} \Delta^-_2$ or simply $\Delta^+_1 \bullet \Delta^-_2$; $\Delta \bullet \Delta$ will often be called the self-intersection number of $\Delta$.

The symmetric group $S_n$ acts on $X$ as permutations of the coordinates $t_j$. Since the local system $S$ is invariant under this action, $S_n$ acts also on the homology groups. We are specially interested in the symmetric part, which is by definition the subspace consisting of $S_n$-invariant elements. It is well known that the symmetric part of $H^f_n(X, S)$ is $(n+1)$-dimensional, and that (just for simplicity, we assume $1 < z$)

$$
\Delta^+_n := \sum_{\sigma \in S_n} \{ t \mid 0 < t_{\sigma(1)} < \ldots < t_{\sigma(k)} < 1, z < t_{\sigma(k+1)} < \cdots < t_{\sigma(n)} \}, \quad k = 0, \ldots, n
$$

form a basis. Since these are disjoint each other, among the intersection numbers $\Delta^+_1 \bullet \Delta^+_n$, only the self-intersection numbers only are non-zero. As is easily known from the expression above, $\Delta^+_k$ is the disjoint union of $\binom{n}{k}$ copies of the direct product $\Delta^+_k \times \Delta^+_{n-k}$. The self-intersection number of a direct product is the product of the self-intersections of the factors. Thus the computation in question boils down
to that for $\Delta_n^m$ and $\Delta_n^n$, which are isomorphic. So in this paper we evaluate the self-intersection number of $\Delta^n = \Delta_n^n$.

In the complex $(x_1, \ldots, x_n)$-space, we consider the multi-valued function

$$v := \prod_{i=1}^{n} t_i^\alpha (1 - t_i)^\beta \prod_{1 \leq i < j \leq n} (t_i - t_j)^{2\gamma}.$$ 

Let $S_v$ be the local system defined by $v$ on

$$Y = \{(t_1, \ldots, t_n) \in \mathbb{C}^n \mid t_i \neq 0, 1, t_j (j \neq i) 1 \leq i \leq n\}.$$

**Theorem 1** Let $D^\sigma$ be the loaded $n$-cycle loaded standardly with $v$ supported by

$$\{t = (t_1, \ldots, t_n) \in Y_R \mid 0 < t_{\sigma(1)} < \ldots < t_{\sigma(n)} < 1\}, \quad \sigma \in S_n$$

and put $\Delta^n := \sum_{\sigma \in S_n} D^n_\sigma$. Then the self-intersection number $J_n := \Delta^n \bullet \Delta^n$ is equal to

$$n! \prod_{j=1}^{n} \frac{1 - abg^{n+j-2}}{(1-ag^{j-1})(1-bg^{j-1})} \frac{1-g}{1-g^j},$$

where $a = \exp 2\pi i\alpha$, $b = \exp 2\pi i\beta$, $g = \exp 2\pi i\gamma$.

**Theorem 2** Let

$$\omega := \frac{dt_1 \wedge \cdots \wedge dt_n}{\prod_{i=1}^{n} t_i (1 - t_i)}$$

represent an element of $n$-th cohomology group $H^n(Y, S_v)$ with coefficients in $S_v$. The self-intersection number $\omega \bullet \omega$ is equal to

$$(2\pi i)^n \prod_{j=1}^{n} \frac{\alpha + \beta + (n + j - 2)\gamma}{(\alpha + (j - 1)\gamma)(\beta + (j - 1)\gamma)}.$$ 

In this paper, we give a direct geometric proof of Theorems 1 and 2. In the complex $t = (t_1, \ldots, t_n)$-space, we consider the hyperplanes

$$t_i = 0, 1, t_j (j \neq i), \quad i = 1, \ldots, n,$$

and blow-up the $t$-space along the non-normally crossing loci of the union of these hyperplanes. Since the hyperplanes are defined over the reals, combinatorial information (e.g. intersection pattern) of the exceptional divisors together with these hyperplanes can be best understood if we study the real locus of the blow-up space: The real $t$-space is cut by these hyperplanes into $(n + 2)!/2!$ (non-empty) pieces defined by

$$a_1 < \cdots < a_{n+2}, \quad \{a_1, \ldots, a_{n+2}\} = \{0, t_1, \ldots, t_n, 1\},$$

among which are $n!$ bounded chambers (simplices) $\sigma(1) \cdots \sigma(n)$ defined by

$$0 < t_{\sigma(1)} < \cdots < t_{\sigma(n)} < 1, \quad \sigma \in S_n.$$
Blowing-up along the non-normal crossing loci in the complex space corresponds to the truncation of chambers in the real space. Each bounded chambers are truncated to be a $n$-polyhedron called $‘$Terada-$n$‘. This polyhedron is used firstly to describe homotopy $(n + 2)$-associativity (in this context Terada-$n$ is called the Stasheff $n$-cell), and then to describe the $S_{n+3}$-equivariant minimal smooth compactification of the configuration space of the colored $n+3$ distinct points in the projective line (cf. \cite{Yo1}). Terada-1 is a segment, Terada-2 is a pentagon, Terada-3 is a polyhedron with six pentagonal faces and three rectangular faces, and Terada-$n$ is described in §2.1.

Let $T$ be the Terada-$n$ coded by $12\cdots n$ and $T^\sigma$ ($\sigma \in S_n$) the Terada-$n$ coded by $\sigma(1)\cdots \sigma(n)$. Once we know the adjacency of the $n!$ Terada-$n$‘s, we can express the intersection number $T \cdot T^\sigma$ of $T$ and $T^\sigma$ loaded standardly with $v$ as a rational form in the exponents $a, b$ and $g$; it is the sum of the local contributions along all faces of $T \cap T^\sigma$. The sum

$$\frac{J_n}{n!} = \sum_{\sigma \in S_n} T \cdot T^\sigma$$

factorizes: The factors of the denominators correspond bijectively to the hyperfaces (with the exponents attached) of $T$, and the factors of the numerators correspond bijectively to the divisors (with the exponents attached) at infinity.

The self-intersection number of the $n$-form $\omega$ is the sum of the local contribution of all vertices of $T - \sum_{\sigma \in S_n} T^\sigma$. Recall that evaluation of the intersection forms of cohomology is much simpler than that of homology, in general. Though it is also the case here, our geometric interpretation confirms that the cohomology intersection is the counterpart of the homology intersection.

**Relation with the Selberg integral.** The reader may notice the resemblance between our Theorems and the Selberg integral; indeed they are deeply related. Consider the Selberg function

$$\text{Sel}_n(\alpha, \beta, \gamma) := \int_{(0,1)^n} t_1^\alpha (1 - t_1)^\beta \prod_{1 \leq i < j \leq n} |t_i - t_j|^{2\gamma} \prod_{i=1}^n \frac{dt_i}{t_i(1 - t_i)}.$$ 

If we admit the well-known fact that the symmetric part ($S_n$-invariant part) of $H_n(Y, S_v)$ and $H^\alpha(Y, S_v)$ are 1-dimensional, the quadratic relation of the hypergeometric integral (see [MaY]) implies the reciprocity relation

$$\text{Sel}_n(\alpha, \beta, \gamma) \cdot \text{Sel}_n(-\alpha, -\beta, -\gamma) = \Delta_n \cdot \Delta_n \cdot \omega \cdot \omega.$$ 

The Selberg formula

$$\text{Sel}_n(\alpha, \beta, \gamma) = \prod_{j=1}^n \frac{\Gamma(\alpha + (j - 1)\gamma) \Gamma(\beta + (j - 1)\gamma) \Gamma(j\gamma + 1)}{\Gamma(\alpha + \beta + (n + j - 2)\gamma) \Gamma(\gamma + 1)}$$

and the reciprocity relation of the Gamma function

$$\Gamma(\alpha)\Gamma(1 - \alpha) = \frac{\pi}{\sin \pi \alpha} \quad \text{or} \quad \Gamma(\alpha)\Gamma(-\alpha) = \frac{a}{1 - a} \cdot \frac{2\pi i}{\alpha} \quad (a := \exp 2\pi i \alpha)$$

tell that Theorem 1 implies Theorem 2 and vice versa. The situation is exactly the same as the simplest case $n = 1$: The reciprocity relation

$$B(\alpha, \beta) \cdot B(-\alpha, -\beta) = \frac{1 - ab}{(1 - a)(1 - b)} \cdot \frac{2\pi i (\alpha + \beta)}{\alpha \beta}$$
of the Beta function

\[ B(\alpha, \beta) := \int_0^1 t^\alpha (1-t)^\beta \frac{dt}{t(1-t)} \]

follows directly from the formula

\[ B(\alpha, \beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha + \beta)} \]

and the reciprocity relation of the Gamma function.

1 Observation of 2D and 3D cases

1.1 2D: Geometry of a pentagon ([MiY])

In the \((x, y)\)-space, we consider the lines

\[ x = 0, \quad y = 0, \quad x = 1, \quad y = 1, \quad x = y, \]

with exponents \(a, a, b, b, g^2\), respectively. The non-normally crossing points are the two points

\[ x = y = 0, \quad x = y = 1. \]

These lines cut the real \((x, y)\)-plane into \(4!/2! = 12\) non-empty chambers

\[ t_0 < t_1 < t_2 < t_3 < t_4 \quad \{t_0, t_1, t_2, t_3\} \subset \{0, x, y, 1\}, \]

among which are \(2!\) bounded chambers defined by \(0 < x < y < 1\) and \(0 < y < x < 1\).

We blow-up the \((x, y)\)-plane at the two singular points. Let us consider the triangle \(0xy1\) bounded by the three edges

\[ 0 = x, \quad x = y, \quad y = 1, \]

with the three vertices

\[ 0 = x = y, \quad x = y = 1, \quad \{0 = x, \ y = 1\}. \]

By the blowing-up, the triangle \(0xy1\) is truncated to be a pentagon bounded by the five segments

\[ 0 = x (a), \quad 0 = x = y (a^2g^2), \quad x = y (g^2), \quad x = y = 1 (g^2b^2), \quad y = 1 (b), \]

where the corresponding exponents are given in the parentheses. Here \(0 = x = y\) is regarded as the exceptional curve coming from the singular point \(0 = x = y\), through which three lines \(0 = x, \ 0 = y, \ x = y\) with exponents \(a, a, g^2\) pass; the exponent along this exceptional curve is just the product of these three exponents. We will also call this pentagon \(0xy1\). As we explained in [KY] the intersection number \(0xy1 \cdot 0xy1\) is the sum of the local contributions at the baricenters of all the possible faces of the pentagonal cell \(0xy1\), where the contribution of the 2-cell is 1, that of the 1-face \(x = 0\) is \(\frac{1}{a-1}\), that of the 1-face \(y = 1\) is \(\frac{1}{e-1}\), that of the vertex \(\{x = 0\} \cap \{y = 1\}\) is \(\frac{1}{(a-1)(e-1)}\), and so on. The intersection number \(0xy1 \cdot 0yx1\) of the two pentagons is
\( \frac{g^2}{g^2-1} \) times the intersection number of the common edge. For notational simplicity, we introduce some functions:

\[
F(x) := \frac{1}{x-1}, \quad S(x, y) := 1 + F(x) + F(y) \quad \left( = \frac{xy - 1}{(x-1)(y-1)} \right),
\]

\[
P(x_1, \ldots, x_5) := 1 + F(x_1) + \cdots + F(x_5) + F(x_1)F(x_2) + \cdots + F(x_4)F(x_5) + F(x_5)F(x_1).
\]

(Note that \( S \) and \( P \) are initials of segment and pentagon, while \( F \) means nothing special.) Then we have

\[
\frac{J_2}{2} = 0xy1 \cdot (0yx1 + 0xy1) = -gF(g^2)S(a^2g^2, g^2b^2) + P(a, a^2g^2, a^2, g^2b^2, b).
\]

We evaluate this. Before directly computing

\[
\frac{J_2}{2} = -\frac{g}{g^2-1} \left( \frac{1}{a^2g^2 - 1} + \frac{1}{g^2b^2 - 1} \right) + 1 + \frac{1}{a-1} + \frac{1}{a^2g^2 - 1} + \frac{1}{g^2 - 1} + \frac{1}{g^2b^2 - 1} + \frac{1}{b-1}
\]

\[
+ \frac{1}{(a-1)(a^2g^2 - 1)} + \frac{1}{(a^2g^2 - 1)(g^2 - 1)} + \frac{1}{(g^2 - 1)(g^2b^2 - 1)} + \frac{1}{(g^2b^2 - 1)(b-1)} + \frac{1}{(b-1)(a-1)},
\]

let us reflect a little: The summands above tell that the denominator of \( J_2 \) divides

\[
(a - 1)(a^2g^2 - 1)(g^2 - 1)(b - 1)(b^2g^2 - 1).
\]

On the other hand, the exponents along the lines at infinity (after blowing-up to make the singular lines cross normally) tell that the numerator divides

\[
(a - 1)(a^2g^2 - 1)(g^2 - 1)(b - 1)(b^2g^2 - 1).
\]

The numerator and the denominator have the same degree with respect to each of \( a, b \) and to \( g \); they are symmetric with respect to \( a \) and \( b \). These conditions are
not enough to determine $J_2$; we do not know a priori which factor is needed among
$\{(g-1), (g+1)\}$, among $\{(ag-1), (ag+1)\}$, among $\{(gb-1), (gb+1)\}$ and among
$\{(agb-1), (agb+1)\}$.

Let us add the terms above. First the terms free of $a$ and $b$:

$$1 + \frac{1}{g^2-1} - \frac{g}{g^2-1} = \frac{g}{g+1}.$$ 

Next the terms with the factor $a^2g^2-1$ as denominators:

$$\frac{1}{a^2g^2-1} \left(1 + \frac{1}{a-1} + \frac{1}{g^2-1} - \frac{g}{g^2-1}\right) = \frac{1}{(a-1)(ag-1)(g+1)}.$$

Finally,

$$J_2 = \frac{g}{g+1} + \frac{1}{a-1} + \frac{1}{b-1} + \frac{1}{(a-1)(b-1)(g+1)}$$

$$+ \frac{1}{(a-1)(ag-1)(g+1)} + \frac{1}{(b-1)(gb-1)(g+1)};$$

we know the denominator already. The numerator is a symmetric polynomial in $a$
and $b$; writing this as a polynomial in $ab$ and $a+b$, we can easily check that the
coefficients of the power of $a+b$ are 0. The result is

$$J_2 = \frac{(agb-1)(ag^2b-1)}{(a-1)(ag-1)(b-1)(gb-1)(g+1)}.$$

1.2 3D: Geometry of Terada-3

In the $(x, y, z)$-space, we consider the planes

$$x, y, z = 0; \quad x, y, z = 1; \quad x = y, y = z, z = x;$$

with exponents $a, b, g^2$, respectively. The non-normally crossing loci are the lines

$$x = y = 0, 1; \quad y = z = 0, 1; \quad z = y = 0, 1; \quad x = y = z$$

and the points

$$x = y = z = 0, 1.$$

These planes cut the real $(x, y, z)$-space into $5!/2!$ chambers defined by

$$t_0 < t_1 < t_2 < t_3 < t_4 \quad \{t_0, t_1, t_2, t_3, t_4\} \in \{0, x, y, z, 1\};$$

among which are $3!$ bounded chambers defined by

$$0 < t_1 < t_2 < t_3 < 1 \quad \{t_1, t_2, t_3\} \in \{x, y, z\};$$

these six tetrahedra fill the cube $0 < x, y, z < 1$. These chambers are encoded as $t_0 \cdots t_4$; and $0t_1t_2t_31$ is often abbreviated as $t_1t_2t_3$. Note that two such chambers
are adjacent if and only if these codes can be changed from one to the other by a permutation of adjacent two letters, and that two chambers $0xyz1$ and $01zyx$ are antipodal with center $(1,1,1)$.

We blow-up the $(x, y, z)$-space at the singular points and then along the singular lines. We describe the resulting object by the truncation of the chambers. Let us consider the tetrahedron $0xyz1$ bounded by the four planes

$$0 = x, x = y, y = z, z = 1,$$

with the six edges

$$0 = x = y, x = y = z, y = z = 1; \{0 = x, y = z\}, \{0 = x, z = 1\}, \{x = y, z = 1\},$$

and with the four vertices

$$0 = x = y = z; \{0 = x, y = z = 1\}, \{0 = x = y, z = 1\}.$$

The first three edges and the first two vertices above are in the singular loci. Such an edge is truncated to be a rectangle, and such a vertex is truncated to be a pentagon. As a result, the tetrahedron $xyz1$ is truncated to be a Terada-3, which is bounded by six pentagons and three rectangles. This Terada-3 is adjacent to $0xyz1$ through the pentagon $x = y$, to $0zyx1$ through the (exceptional) rectangle born from the edge $x = y = z$, and to $01zyx$ through the (exceptional) pentagon born from the vertex $x = y = z = 1$. These faces will be denoted by

$$\{xy\}, \{xyz\}, \{xyz1\},$$

respectively. In this way, we can easily know the codes of the adjacent (blown-up) chambers.

Once the faces of the truncated chambers and the adjacency through the faces are known, we can compute the intersection numbers following the recipe shown in [KY].

Let the exponents be given as follows:

$$x, y, z = 0 \cdots a, \quad x, y, z = 1 \cdots b, \quad x = y \cdots f^2, \quad y = z \cdots g^2, \quad z = x \cdots h^2.$$
The 2-faces of the Terada-3 $0xyz1$ are shown with their exponents. The three 2-faces
\[ \{xy\}, \{yz\}, \{xyz\}, \]
and the two edges
\[ \{xy\} \cap \{xyz\}, \{yz\} \cap \{xyz\} \]
touching five other Terada-3's are painted in gray, while the four vertices not touching any of these are marked.

Figure 3: The faces of Terada-3 $0xyz1$ with different information

The Terada-3 obtained from the tetrahedron $0xyz1$ has nine faces: the six pe-
natagons
\[ \{0x\}, \{xy\}, \{yz\}, \{z1\}, \{0xyz\}, \{xyz1\}, \]
and the three rectangles
\[ \{0xy\}, \{xyz\}, \{yz1\}, \]
and 21 edges and 14 vertices (see Figure 3).

Let us evaluate the intersection numbers of the $S_3$ (acting as permutations of $\{x, y, z\}$) orbits of the simplex $0xyz1$. The simplex $0xyz1$ (for simplicity we write $xyz$ for the time being) is truncated to be a Terada-3, which is also denoted by $xyz$. The Terada-3 touches other Terada-3s in the $S_3$-orbit along the following faces:
\[
\begin{align*}
xyz \cap yxz & = \text{pentagon } \{xy\}, \\
xyz \cap xyz & = \text{pentagon } \{yz\}, \\
xyz \cap zyx & = \text{rectangle } \{xyz\}, \\
xyz \cap zxy & = \text{segment } \{xy\} \cap \{xyz\}, \\
xyz \cap yzx & = \text{segment } \{yz\} \cap \{xyz\}.
\end{align*}
\]

We compute the intersection number
\[
\frac{J_3}{3!} = xyz \cdot (y x z + x z y + z y x + z x y + y x z + y z x + z x z + y x z).
\]

Notation for Terada-3: $Q(p_1, q_1, r_1; p_2, q_2, r_2; p_3, q_3, r_3)$

\[
Q = \begin{pmatrix} p_1 & q_1 \\ r_1 \\ p_2 & q_2 \\ r_2 \\ p_3 & q_3 \\ r_3 \end{pmatrix} \begin{pmatrix} 1 + \sum F(r_i)S(p_i, q_{i+1})S(p_{i+1}, q_i) \\ + \sum F(p_i) + F(q_i) + F(p_i)F(p_{i+1}) + F(q_i)F(q_{i+1}) \\ + F(p_1)F(p_2) + F(p_1)F(q_2)F(q_3) + \sum F(p_i)F(q_i) \end{pmatrix}.
\]
The nine faces of a Terada-3 are shown with exponents $p_i, q_i, r_i$. The self-intersection number of this Terada-3 is given by

$$Q(p_1, q_1; p_2, q_2; p_3, q_3).$$

Figure 4: A stereographic image of a Terada-3.
Next the terms with the factor $a^3g^6 - 1$ as denominators: $(a^3g^6 - 1)^{-1}$ times

$$F(a^2g^2)[gF(g^2) - S(g^2, a)] + F(g^6)[ \text{as above }] + F(a)(gF(g^2) - F(g^2) - 1) + 2gF(g^2) - 2F(g^2) - 1 = \frac{a^2g^4 + ag^2 + 1}{(a - 1)(ag - 1)(g + 1)(g^2 + g + 1)}.$$

Among the remaining terms, the terms with the factor $a^2g^2 - 1$ as denominators: $(a^2g^2 - 1)^{-1}$ times

$$\left(1 + \frac{1}{b - 1}\right) \frac{ag + 1}{(a - 1)(g + 1)}.$$

Finally left are

$$F(a)(gF(g^2) - F(g^2) - 1) + F(b)(gF(g^2) - F(g^2) - 1) - F(b)F(a).$$

So we know the denominator of $J_3$; the numerator can be computed without difficulty and we get

$$J_3 = -\frac{3!(g^2ba - 1)(g^3ba - 1)(g^4ba - 1)}{(a - 1)(ag - 1)(ag^2 - 1)(b - 1)(gb - 1)(g^2b - 1)(g + 1)(g^2 + g + 1)}.$$

### 2. nD case

In the real $(x_1, \ldots, x_n)$-space, we consider the hyperplanes

$$x_i = 0, 1, \ x_j \ (j \neq i), \ i = 1, \ldots, n.$$

They cut the space into $(n + 2)!/2!$ chambers (simplices) defined by

$$a_1 < \cdots < a_{n + 2}, \ \{a_1, \ldots, a_{n+2}\} = \{0, x_1, \ldots, x_n, 1\},$$

among which are $n!$ bounded chambers defined by

$$0 < x_{\sigma(1)} < \cdots < x_{\sigma(n)} < 1, \ \sigma \in S_n.$$

Now we blow-up along the non-normally crossing loci of the union of these hyperplanes. Each simplex is transformed into an $n$-polygon called Terada-$n$. (A Terada-2 is a pentagon, and a Terada-3 appeared in the previous subsection.) We encode the Terada-$n$ coming from the simplex $0 < x_{\sigma(1)} < \cdots < x_{\sigma(n)} < 1$ by the word

$$0\sigma(1)\cdots\sigma(n)(n + 1)(n + 2),$$

which will be called an $(n + 3)$-juzu; note that point 1 is coded by $n + 1$.

### 2.1 Shape of Terada-$n$ and Juzus

A $k$-juzu is a sequence of $k$ numerals coding a Terada-$(k - 3)$; two juzus are identified if one is obtained from the other by a cyclic permutation and/or inversion. For example, a 3-juzu

$$012 = 120 = \cdots = 210$$
is a point, a 4-juzu
\[ 0123 = 1230 = \cdots = 3210 \]
is a segment, and a 5-juzu is a pentagon. To make the formulae looks nicer, we often abbreviate \( n \pm 1 \) and \( n \pm 2 \) as
\[ n - 2 = "n", \quad n - 1 = 'n', \quad n + 1 = n', \quad n + 2 = n''. \]
Let us describe the boundary of the Terada-\( n \) \( T := 01 \cdots nn'n'' \) coming from the \( n \)-simplex
\[ 0 < x_1 < \cdots < x_n < 1. \]
The Terada-(\( n - k \)) coming from the \((n - k)\)-simplex defined by
\[ 0 < x_1 < \cdots < x_{i-1} < x_i = \cdots = x_{i+k} < x_{i+k+1} < \cdots < x_n < 1, \]
which is part of the boundary of the \( n \)-simplex above, will be coded by the \((n-k+3)\)-juzu
\[ 0 \, 1 \, \cdots \, 'i \, (i \, \cdots \, i + k) \, (i + k)' \, \cdots \, n \, n' \, n'', \]
regarding the \( k + 1 \) numerals in the parentheses as a single numeral. The \((n - 1)\)-faces of the Terada-\( n \) \( T = 01 \cdots nn'n'' \) are direct products of Terada-(\( n - i \)) and Terada-(\( i - 1 \)):
\[ (0 \cdots i) \, i' \cdots n'' \times 0 \cdots i'(i' \cdots n''0), \quad (1 \cdots i') \, i'' \cdots n''0 \times 1 \cdots i''(i'' \cdots n''0), \cdots \]
\[ (n''0 \cdots 'i) \, i' \cdots n' \times n''0 \cdots 'i \, (i' \cdots n'). \quad i = 1, \ldots, \left[ \frac{n+3}{2} \right]; \]
they will be abbreviated as
\[ (0 \cdots i) \text{ or } (i' \cdots n''), \quad (1 \cdots i') \text{ or } (i'' \cdots n''0), \quad \ldots, \quad (n''0 \cdots 'i) \text{ or } (i' \cdots n'), \]
respectively. The \((n - k)\)-faces are intersections of \( k \) of these \((n - 1)\)-faces, say
\[ (a_1 a'_1 \cdots z_1), \quad (a_2 a'_2 \cdots z_2), \quad \ldots, \quad (a_k a'_k \cdots z_k) \]
such that for any \( i \) and \( j \), \( \{a_i, a'_i, \ldots, z_i\} \neq \{a_j, a'_j, \ldots, z_j\} \) and either
\[ \{a_i, a'_i, \ldots, z_i\} \cap \{a_j, a'_j, \ldots, z_j\} = \emptyset \quad \text{or} \]
\[ \{a_i, a'_i, \ldots, z_i\} \subset \{a_j, a'_j, \ldots, z_j\} \text{ or } \{a_j, a'_j, \ldots, z_j\} \subset \{a_i, a'_i, \ldots, z_i\}. \]
**Terminology:** In the sequel, in place of saying ‘for any \( i \) and \( j \), …’, we say ‘the sets
\[ \{a_1, a'_1, \ldots, z_1\}, \quad \{a_2, a'_2, \ldots, z_2\}, \quad \ldots, \quad \{a_k, a'_k, \ldots, z_k\} \]
have the disjoint/include property.’

Note that this definition fits the two alternative expression of an \((n - 1)\)-face above.

- When \( n = 1 \), Terada-1 0123 is a segment. Its boundary consists of two points:
\[ (01) = (23), \quad (12) = (30). \]
• When \( n = 2 \), Terada-2 012345 is a pentagon. Its boundary consists of
  
  - five edges:
    \[(01), (12), (34), (45), (50)\]
  
  - and five vertices
    \[(01) \cap (34), (34) \cap (51), (50) \cap (12), (12) \cap (45), (45) \cap (01).\]

• When \( n = 3 \), the boundary of the Terada-3 012345 consists of
  
  - 2D faces of 2 kinds
    \[(01), \ldots, (50); (012) = (345), (123) = (456), (234) = (501),\]
  
  - 1D faces of 2 kinds
    \[(01) \cap (23); (01) \cap (012),\]
  
  - 0D faces of 2 kinds
    \[(01) \cap (23) \cap (45), (12) \cap (34) \cap (50); (01) \cap (12) \cap (012).\]

Let us see how this Terada-\( n \) \( T \) touches other Terada-\( n \)'s. Through every face

\[(a_1a'_1 \cdots z_1) \cap (a_2a'_2 \cdots z_2) \cap \cdots \cap (a_ka'_k \cdots z_k)\]

with

\[a_1, a'_1, \ldots, z_1; a_2, a'_2, \ldots, z_2; \ldots; a_k, a'_k, \ldots, z_k \in \{1, \ldots, n\}\]

touches \( T \) the Terada-\( n \)

\[0\sigma(1) \cdots \sigma(n)n'n'',\]

where \( \sigma \in S_n \) is the product of the commutative \( k \) cyclic permutaion of elements

\[\{a_1, a'_1, \ldots, z_1\}, \{a_2, a'_2, \ldots, z_2\}, \ldots, \{a_k, a'_k, \ldots, z_k\}.\]

Note that not all of the \( n! \) Terada-\( n \)'s touches the Terada-\( n \) \( T \).

• When \( n = 2 \), the Terada-2 01234 touches the Terada-2 02134 through the edge (12). Note that if you perform the cyclic permutaion of \( \{3, 4, 0\} \) you get also the juzu 02134.

• When \( n = 3 \), the Terada-3 012345 touches the five others
  
  - through 2D faces: (12), (23), (123),
  
  - through 1D faces: (12) \( \cap \) (123), (23) \( \cap \) (123).

• When \( n = 4 \) already, among the 4! Terada-4's, there are Terada-4's which do not touch 0123456; they are 0241356 and 0314256.
2.2 Exponents along the hyperfaces

Since
\[ u = \prod_{i=1}^{n} t_i^{\alpha_i} (1 - t_i)^{\beta_i} \prod_{1 \leq i < j \leq n} (t_i - t_j)^{2\gamma}, \]
the exponents along the hyperplanes in the \( t \)-space are given as follows
\[ t_i = 0 \cdots a, \quad t_i = 1 \cdots b, \quad x_i = x_j \cdots g^2, \]
where \( a = \exp 2\pi i \alpha, \ b = \exp 2\pi i \beta, \ g = \exp 2\pi i \gamma \). Through the \((n-k)\)-simplex
\[ 0 = x_1 = \cdots = x_k < x_{k+1} < \cdots < x_n < 1, \]
pass the hyperplanes
\[ 0 = x_i, \quad 1 \leq i \leq k \quad \text{and} \quad x_i = x_j, \quad 1 \leq i < j \leq k; \]
through the \((n-k)\)-simplex
\[ 0 < x_1 < \cdots < x_{p-1} < x_p = \cdots = x_{p+k} < x_{p+k+1} < \cdots < x_n < 1, \]
pass the hyperplanes
\[ x_i = x_j, \quad p \leq i < j \leq p+k; \]
through the \((n-k)\)-simplex
\[ 0 < x_1 < \cdots < x_{n-k} < x_{n-k+1} = \cdots = x_n = 1, \]
pass the hyperplanes
\[ x_i = x_j, \quad n-k+1 \leq i < j \leq n \quad \text{and} \quad x_i = 1, \quad n-k+1 \leq i \leq n. \]

Thus, after necessary blowing-up, the exponents along the hyperfaces \(((n-1)\)-faces) of the Terada-\(n\) \(T = 01 \cdots mn'n''\) are give as
\[ (01 \cdots k) \cdots a^k g^{k(k-1)}, \quad (p \cdots p + k) \cdots g^{k(k+1)}, \quad (n - k + 1 \cdots n') \cdots b^k g^{k(k-1)}, \]
where \( 1 \leq k, p, k + p \leq n. \)

2.3 Intersection numbers

The self-intersection number of an \(n\)-polygon bounded by \textit{normally crossing} hyperplanes is the sum of the local contributions of all the possible faces, where the contribution of the \(n\)-face (the polygon itself) is 1, that of a hyperface with exponent \(e\) is \(\frac{1}{e^2 - 1}\), and that of a \(p\)-codimensional face (which is the intersection of exactly \(p\) hyperfaces) is the product of the contributions of the \(p\) intersecting hyperfaces.

\textbf{Notation:} For a hyperface \((qq' \cdots r)\) \(0 \leq q < r \leq n'\) with exponent \(e\), we set
\[ [q \cdots r] := \frac{1}{e^2 - 1}. \]

Thus, taking the Terada-\(n\) as a polygon, we have
**Proposition 1** The self-intersection number $T \bullet T$ of the Terada-$n$ $T$ is the sum of all the possible product

$$[a_1a'_1 \cdots z_1][a_2a'_2 \cdots z_2] \cdots [a_k a'_k \cdots z_k],$$

$0 \leq a_1, \ldots, z_1, a_2, \ldots, z_2, \ldots, a_k, \ldots, z_k \leq n' = n + 1$, $0 \leq k \leq n'$, such that $[a_i a'_i \cdots z_i] \neq [01 \cdots n']$ and,

$$\{a_i, a'_i, \ldots, z_i\} \ i = 1, \ldots, k$$

have the disjoint/include property. When $k = 0$ the product is regarded as 1.

**Proposition 2** Let a Terada-$n$ $T'$ touches $T$ along the hyperface $H := (p, p', \ldots, p + q)$ $(1 \leq p, p + q \leq n)$. The intersection number of the Terada-$n$’s $T$ and $T'$ is given by

$$T \bullet T' = (-)^q g^{q(q+1)/2}[pp' \cdots p + q] H \bullet H,$$

where the self-intersection number $H \bullet H$ of $H$ is the sum of all the possible product

$$[a_1a'_1 \cdots z_1][a_2a'_2 \cdots z_2] \cdots [a_k a'_k \cdots z_k],$$

$0 \leq a_1, \ldots, z_1, a_2, \ldots, z_2, \ldots, a_k, \ldots, z_k \leq n' = n + 1$, $0 \leq k \leq n'$, such that $[a_i a'_i \cdots z_i] \neq [01 \cdots n']$ and,

$$\{p, p', \ldots, p + q\}, \ {a_i, a'_i, \ldots, z_i} \ i = 1, \ldots, k$$

have the disjoint/include property.

**Corollary 1** Let a Terada-$n$ $T'$ touches $T$ along a face $F$ which is the intersection of the hyperfaces $H_t := (p_t, p'_t, \ldots, p_t + q_t)$ $(1 \leq p_t, p_t + q_t \leq n)$. The intersection number of the Terada-$n$’s $T$ and $T'$ is given by

$$T \bullet T' = \prod_t (-)^{q_t} g^{q_t(q_t+1)/2}[p_t p'_t \cdots p_t + q_t] F \bullet F,$$

where the self-intersection number $F \bullet F$ of $F$ is the sum of all the possible product

$$[a_1a'_1 \cdots z_1][a_2a'_2 \cdots z_2] \cdots [a_k a'_k \cdots z_k],$$

$0 \leq a_1, \ldots, z_1, a_2, \ldots, z_2, \ldots, a_k, \ldots, z_k \leq n' = n + 1$, $0 \leq k \leq n'$, such that

$$\{p_t, p'_t, \ldots, p_t + q_t\}, \ {a_i, a'_i, \ldots, z_i}$$

have the disjoint/include property.
3 Evaluation

For $\sigma \in S_n$, let $T^\sigma$ denote the Terada-$n$ $0\sigma(1)\ldots\sigma(n)n''$. We would like to evaluate the sum

$$J_n = \frac{1}{n!} \sum_{\sigma \in S_n} T \cdot T^\sigma;$$

when $T$ and $T^\sigma$ do not touch, their intersection number is 0, of course. Proposition 1 and Corollary 1 imply that $(-)^n J_n/n!$ is the sum of all the possible product

$$\prod_i (-\alpha^q g^{\alpha(q+1)/2}[p_ip_i'\ldots p_i+q_i]\cdot \prod_j [a_j a_j' \ldots z_j]),$$

where $1 \leq p_i, p_i+q_i \leq n$, $0 \leq a_j, \ldots, z_j \leq n+1$, and $[a_j a_j' \ldots, z_j] \neq [01\ldots n']$ and,

$$\{p_i, p_i', \ldots, p_i+q_i\}, \{a_j, a_j', \ldots, z_j\}$$

have disjoint/include property. Empty products are regarded as 1.

Notation: For $1 \leq p, p+q \leq n$, put

$$\langle pp'\ldots p+q \rangle := [pp'\ldots p+q](1+(-\alpha^q g^{(q+1)/2})$$

$$= \frac{1+(-\alpha^q g^{(q+1)/2})}{g^{2(q+1)/2}-1} = \frac{-1}{1-(-\alpha^q g^{(q+1)/2})}.$$ 

Terminology: $\langle \cdots \rangle$ and $[\cdots]$ are called sequences. A monomial in $0, 1, \ldots, n, n+1$ is a product of sequences

$$\prod_i \langle pp'\ldots p_i+q_i \rangle \cdot \prod_j [a_j a_j' \ldots z_j],$$

where

$$\{p_i, p_i', \ldots, p_i+q_i\}, \quad 1 \leq p_i, p_i+q_i \leq n,$$

and

$$\{a_j, a_j', \ldots, z_j\}, \quad 0 \leq a_j, \ldots \leq n+1, \quad 0 = a_j \text{ or } z_j = n+1$$

have disjoint/include property. The length of a monomial is the length of the longest sequence.

For example, when $n = 4$,

$$[0123] \langle 23 \rangle \langle 123 \rangle [45], \quad \langle 12 \rangle \langle 34 \rangle \langle 1234 \rangle [12345]$$

are monomials of length 4 and 5.

Let $U$ be a family of subsets of $\{0, 1, \ldots, n'\}$ with disjoint/include property, and $J(U)$ the sum of possible products in the form (1), where

$$U = \cup \{p_i, \ldots, p_i+q_i\} \cup \cup \{a_j, \ldots, z_j\}.$$

The sum $J(U)$ can be computed as follows: Divide $U$ into three subsets:

$$1U_n := \{\{p_i, \ldots, p_i+q_i\} \in U \mid 1 \leq p_i, p_i+q_i \leq n\},$$

$$0U := \{\{0, \ldots, z_i\} \in U \mid z_i \leq n\},$$

$$U_{n'} := \{\{a_i, \ldots, n'\} \in U \mid 1 \leq a_i\}.$$
Then the sum $J(U)$ can be factorized as

$$J(U) = \prod_{i} [p_{i} \cdots p_{i} + q_{i}] \prod_{o} [0 \cdots z_{i}] \prod_{U_{n'}} [a_{i} \cdots n'] .$$

We hope that the reader readily guess the reason of the above claim by the following example for $n = 3$ and $U = \{\{1, 2\}, \{1, 2, 3\}\}$ (cf. §4.2):

$$[12][123] + g[12] \cdot [123] - g^{3}[123] \cdot [12] + g[12](-g^{3}[123])$$

$$= (1 - g)[12] \cdot (1 + g^{3})[123] = \langle 12 \rangle \langle 123 \rangle .$$

Letting $U$ vary all the families of subsets of $\{0, 1, \ldots, n'\}$ with disjoint/include property, we have

**Proposition 3** The quantity $(-)^{n}J_{n}/n!$ is the sum of the possible monomials in $0, 1, \ldots, n, n + 1$ of length at most $n + 1$.

For each monomial, notice the longest sequence including 0, and the longest sequence including $n + 1$. Then we are led to

**Proposition 4** For $k = 1, \ldots, n$, put

$$X_{k} := \text{sum of the possible monomials in } 1, 2, \ldots, k,$$

$$A_{k} := [01 \cdots k] \times \{ \text{sum of the possible monomials in } 0, 1, \ldots, k \text{ of length at most } k \},$$

$$B_{k} := [n - k + 1 \cdots n n + 1] \times \{ \text{sum of the possible monomials in } n - k + 1, \ldots, n + 1 \text{ of length at most } k \} .$$

Then we have

$$(-)^{n}J_{n}/n! = \sum_{i,j \geq 0, \ i+j \leq n} A_{i} \ X_{n-i-j} \ B_{j} .$$

### 3.1 Evaluation of $X_{n}$

We found that the sum $X_{n}$ of the terms free of $a$ and $c$ is the sum of monomials in $1, \ldots, n$. Indeed we have

$$X_{2} = 1 + \langle 12 \rangle ,$$

$$X_{3} = (1 + \langle 123 \rangle)(1 + \langle 12 \rangle + \langle 23 \rangle) ,$$

$$X_{4} = (1 + \langle 1234 \rangle)(1 + \langle 12 \rangle + \langle 23 \rangle + \langle 34 \rangle + \langle 12 \rangle \langle 34 \rangle$$

$$+ (1 + \langle 12 \rangle + \langle 23 \rangle) \langle 123 \rangle + (1 + \langle 23 \rangle + \langle 34 \rangle) \langle 234 \rangle ) ,$$

$$\ldots$$

These can be evaluated as

$$X_{2} = \frac{g}{g + 1} ,$$

$$X_{3} = \frac{g^{3}}{(g + 1)(g^{2} + g + 1)} ,$$

$$X_{4} = \frac{g^{6}}{(g + 1)(g^{2} + g + 1)(g^{4} + g^{2} + g + 1)} ,$$

$$\ldots$$
Let $Y(k, n)$ be the sum of the monomials in $1, \ldots, n$ of length exactly $k$, and $X(k, n)$ be the sum of the monomials in $1, \ldots, n$ of length at most $k$. Then we have

$$X(k, n) = \sum_{j=1}^{k} Y(j, n), \quad X_n = X(n, n),$$

where we put $Y(1, 2) = Y(1, 3) = \cdots = 1$, so we have $X(1, 2) = X(1, 3) = \cdots = 1$.

**Notation:**

$$[n] := [n]_g = 1 + g + g^2 + \cdots + g^{n-1},$$

$$[n]! := [n]_g! = [n][n-1]\cdots[1],$$

$$\begin{bmatrix} n \\ m \end{bmatrix} := \begin{bmatrix} n \\ m \end{bmatrix}_g = \frac{[n]!}{[m]! \cdot [n-m]!}, \quad 0 \leq m \leq n,$$

$$(a)_n := (a; g)_n = (1 - a)(1 - ag)(1 - ag^2) \cdots (1 - ag^{n-1}).$$

Since we have, by definition,

$$Y(n, n) = \langle 1 \cdots n \rangle X(n-1, n), \quad X(n, n) = Y(n, n) + X(n-1, n)$$

and

$$\langle 1 \cdots n \rangle = \frac{-1}{1 + (-)^ng^{(n)}};$$

**Lemma 1**

$$X(n-1, n) = \frac{g^{(n)}}{[n]!} + (-)^n, \quad n = 2, 3, \ldots$$

implies

**Proposition 5**

$$Y(n, n) = \frac{(-)^{n+1}}{[n]!}, \quad X(n, n) = \frac{g^{(n)}}{[n]!}.$$ 

**Proof.** We prove this Lemma (as well as this Proposition) by induction on $n$. For each monomial of length at most $n-1$ in $\{1, 2, \ldots, n\}$, we notice the longest sequence including 1, and divide the sequences in two parts: those which are part of the longest one, and those which are disjoint with the longest one. In this way, we have

$$X(n-1, n) = \langle 1 \cdots n-1 \rangle X(n-2, n-1) + \langle 1 \cdots n-2 \rangle X(n-3, n-2) X(2, 2)$$

$$\vdots$$

$$+ \langle 1 \cdots n-k \rangle X(k-1, k) X(n-k, n-k)$$

$$\vdots$$

$$+ \langle 12 \rangle X(12) X(n-2, n-2) + X(n-1, n-1).$$

By the induction hypothesis of the Lemma and the Proposition, we have

$$\langle 1 \cdots n-k \rangle X(k-1, k) X(n-k, n-k) = \frac{(-)^k}{[k]!} \cdot \frac{g^{(n-k)}}{[n-k]!}.$$
and so
\[ X(n-1, n) = \sum_{k=1}^{n-1} \frac{(-1)^{k+1}}{[k]!} \frac{g^{(n-k)}}{[n-k]!} = \frac{1}{[n]!} \sum_{k=1}^{n-1} (-1)^{k+1} \binom{n}{k} g^{(n-k)}. \]

On the other hand, the \(g\)-binomial theorem (cf. [GR])
\[(x)_n = \sum_{k=0}^{n} \binom{n}{k} g^{1+\cdots+(k-1)}(-x)^k\]
with \(x = 1\) yields
\[ \sum_{k=0}^{n} \binom{n}{k} g^{(k)}(-)^k = 0 \quad \text{or} \quad \sum_{k=0}^{n} \binom{n}{k} g^{(n-k)}(-)^{n-k} = 0. \]

This leads to
\[ X(n-1, n) = \frac{g^{(n)} + (-)^n}{[n]!}, \]
proving the Lemma, and so the Proposition.

### 3.2 Evaluation of \(A_k\)

**Proposition 6**
\[ A_k = \{01\cdots k\} \times \{\text{sum of the monomials in } 0, 1, \ldots, k \text{ of length at most } k\} \]
\[ = \{01\cdots k\} \sum_{p=0}^{k-1} A_p \ X_{k-p}, \]
where \(A_0 = 1\).

For example
\[ A_1 = [01], \quad A_2 = [012](X_2 + [01]), \quad A_3 = [0123](X_3 + [01]X_2 + A_2), \ldots \]

**Proposition 7**
\[ A_k = \frac{(-)^k (1 - g)^k}{(a)_k (g)_k}, \quad k = 0, 1, \ldots \]

\(B_k\) is given from this formula by replacing \(a\) by \(b\).

**Proof.** We compute
\[ \sum_{k=0}^{n-1} A_k X_{n-k} = X_n \left[ 1 + A_1 \frac{X_{n-1}}{X_n} + \cdots + A_k \frac{X_{n-k}}{X_n} + \cdots + A_{n-1} \frac{X_1}{X_n} \right]. \]

Since we have
\[ X_k = \frac{(1-g)^k g^{(k)}}{(g)_k} = \frac{1}{(1+g^{-1})\cdots(1+g^{-1}+\cdots+g^{-k+1})} = \frac{(1-g^{-1})^k}{(1-g^{-1})(1-g^{-2})\cdots(1-g^{-k})} = \frac{(1-g^{-1})^k}{(g^{-k})_k}, \]
and so
\[
\frac{X_{n-k}}{X_n} = (1 - g^{-1})^{-k} \frac{(g^{-n})_k}{(g^{-n-k})_{n-k}}
\]
\[
= (1 - g^{-1})^{-k}(1 - g^{-n}) \cdots (1 - g^{-n+k-1}) = \frac{(g^{-n})_k}{(1 - g^{-1})^k} = g^k \frac{(g^{-n})_k}{(g - 1)^k},
\]

by the induction hypothesis, we have
\[
A_k \frac{X_{n-k}}{X_n} = \frac{(g^{-n})_k}{(a)_k (g)_k} g^k.
\]

On the other hand, the reversed \( q \)-Chu-Vandermonde formula (cf. §6.2) on the basic hypergeometric function \( \varphi \):
\[
\sum_{k=0}^{n} \frac{(q^{-n})_k (b)_k}{(q)_k (c)_k} = 2 \varphi_1 \left( \begin{array}{c} q^{-n}, b \\ c; q \end{array} \right) \frac{(c/b; q)_n}{(c; q)_n} b^n
\]
with \( q \to g, c \to a, \) and \( b \to 0 \) yields
\[
\sum_{k=0}^{n} \frac{(g^{-n})_k}{(a)_k (g)_k} g^k = \frac{(-a)^n g^{1+\cdots+(n+1)}}{(a)_n} = \frac{(-a)^n (g(2))}{(a)_n}.
\]

Thus we have
\[
1 + A_1 \frac{X_{n-1}}{X_n} + \cdots + A_k \frac{X_{n-k}}{X_n} + \cdots + A_{n-1} \frac{X_1}{X_n} = \frac{(-a)^n (g(2))}{(a)_n} - \frac{(g^{-n})_n g^n}{(a)_n (g)_n}
\]
\[
= a^n g^{(2)} - g^{-n(2)} = (-)^n,
\]

and so
\[
\sum_{k=0}^{n-1} A_k X_{n-k} = X_n \left[ 1 + \cdots + A_n \frac{X_1}{X_n} \right]
\]
\[
= \frac{(1 - g)^n g^{(2)}}{(g)_n} \cdot \frac{a^n g^{(2)} - g^{-n(2)}}{(a)_n (-)^n} = \frac{(g - 1)^n (a^n g^{(2)} - 1)}{(a)_n (g)_n}.
\]

Therefore we have
\[
A_n = [01 \cdots n] \sum_{k=0}^{n-1} A_k X_{n-k} = \frac{(g - 1)^n}{(a)_n (g)_n},
\]
ending the proof.

### 3.3 Evaluation of \( J_n \), end of the proof of Theorem 1

Proposition 8
\[
\sum_{i,j \geq 0, i+j \leq n} A_i X_{n-i-j} B_j = \frac{(g - 1)^n (abq^{n-1})}{(a)_n (b)_n (g)_n}.
\]
Proof. In the course of the proof of the previous Proposition, we proved
\[
\sum_{i=0}^{m} A_i X_{m-i} = \frac{(g - 1)^m a^m g^{2n}}{(a)_m (g)_m} = C_m.
\]
We compute
\[
\sum_{j=0}^{n} \sum_{i=0}^{n-j} A_i X_{n-i-j} B_j = \sum_{j=0}^{n} C_{n-j} B_j = C_n \sum_{j=0}^{n} B_j \frac{C_{n-j}}{C_n}.
\]
Since we have
\[
(a)_n = (1 - a)(1 - ag) \cdots (1 - ag^{n-1}) = (-a)^n g^{1+\cdots+(n-1)}(1 - a^{-1})(1 - a^{-1}g^{-1}) \cdots (1 - a^{-1}g^{1-n}),
\]
and so
\[
\frac{(a)_n}{(a)_{n-j}} = (-a)^j \frac{g^{1+\cdots+(n-1)}}{g^{1+\cdots+(n-j-1)}} (a^{-1}g^{1-n})_j,
\]
\[
\frac{(g)_n}{(g)_{n-j}} = (-g)^j \frac{g^{1+\cdots+(n-1)}}{g^{1+\cdots+(n-j-1)}} (g^{-n})_j,
\]
we have
\[
\frac{C_{n-j}}{C_n} = (g - 1)^{-j} a^{-j} \frac{(a)_n (g)_n}{(a)_{n-j} (g)_{n-j}} g^{(n-j)(n-j-1)g^{-n(n-1)}}
\]
\[
= (a^{-1}g^{1-n})_j (g^{-n})_j (g - 1)^{-j} g^j,
\]
and so
\[
B_j \frac{C_{n-j}}{C_n} = \frac{(g^{-n})_j}{(b)_j} \frac{(a^{-1}g^{1-n})_j}{(g)_j} g^j.
\]
On the other hand the reversed \(q\)-Chu-Vandermonde formula above yields
\[
\sum_{j=0}^{n} \frac{(g^{-n})_j}{(b)_j} \frac{(a^{-1}g^{1-n})_j}{(g)_j} g^j = \frac{(abg^{n-1})_n}{(b)_n} (a^{-1}g^{1-n})^n.
\]
Now we finish the proof:
\[
\sum_{j=0}^{n} \sum_{i=0}^{n-j} A_i X_{n-i-j} B_j = C_n \sum_{j=0}^{n} B_j \frac{C_{n-j}}{C_n} = C_n \frac{(abg^{n-1})_n}{(b)_n} (a^{-1}g^{1-n})^n
\]
\[
= \frac{(g - 1)^n a^m g^{n(n-1)}}{(a)_m (g)_m} \cdot \frac{(abg^{n-1})_n}{(b)_n} (a^{-1}g^{1-n})^n
\]
\[
= \frac{(g - 1)^n (abg^{n-1})_n}{(a)_m (b)_n (g)_m}.
\]

4 Review of 2D and 3D cases

4.1 2D case

The Pentagon \( T := xy = 0xy1 \) is a Terada-2 with code 01234. It has 1D faces
\[
(01), (12), (23), (34) = (012), (40) = (123),
\]
0D faces

\((01) \cap (23), (01) \cap (34), (12) \cap (34), (12) \cap (40), (23) \cap (40)\).

Exponents of the 1D faces are

\[
(01) \cdots a, \quad (12) \cdots g^2, \quad (23) \cdots b,
\]

\[
(34) = (012) \cdots a^2 g^2, \quad (40) = (123) \cdots g^2 b^2;
\]

put

\[
[01] = \frac{1}{a - 1}, \quad [12] = \frac{1}{g^2 - 1}, \quad [34] = [012] = \frac{1}{a^2 g^2 - 1},
\]

and so on. Then we have the self-intersection number of \(T = xy = 01234\):

\[
x y \cdot x y = 1 + [01] + [12] + [23] + [34] + [40] + [01][23] + [01][34] + [12][34] + [12][40] + [23][40].
\]

Since \(yx = 02134\) is adjacent to \(T\) along the edge \((12)\), computing the self-intersection number of the edge \((12)\), we have

\[
x y \cdot y x = -g[12]\{1 + [34] + [40]\}.
\]

Let us add these:

\[
1 + (1 - g)[12] + [01] + [012](1 + (1 - g)[12] + [01]) + [23] + [123](1 + (1 - g)[12] + [23]) + [01][23].
\]

Note that the sum of the terms free of \(a\) and \(b\) is \(1 + (1 - g)[12]\), and the sum of the terms with the factor \(a^2 g^2 - 1\) is \([012](1 + (1 - g)[12] + [01])\).

### 4.2 3D case

The Terada-3 \(T := 0xyz1\) is now coded by 012345. It has

2D faces of 2 kinds

\((01), \ldots, (50); \quad (012) = (345), \quad (123) = (456), \quad (234) = (501)\),

1D faces of 2 kinds

\((01) \cap (23); \quad (01) \cap (012)\),

0D faces of 2 kinds

\((01) \cap (23) \cap (45), \quad (12) \cap (34) \cap (50); \quad (01) \cap (12) \cap (012)\).

Exponents of the 2D faces are

\[
(01) \cdots a, \quad (12). (23) \cdots g^2, \quad (34) \cdots b,
\]

\[
(45) = (0123) \cdots a^3 g^6, \quad (50) = (1234) \cdots g^6 b^3,
\]
Recall the notation:

\[ [01] := F(a), \ [12] := F(g^2), \ldots, \ [50] := F(g^6b^3), \ldots, \ [234] := F(g^2b^2). \]

The faces touching the 3! T3's are \( xyz = 123 \) itself and through 2D faces:

\( (12), \ (23), \ (123), \)

through 1D faces:

\( (12) \cap (123), \ (23) \cap (123). \)

Then counting all the faces of 012345, we have the self-intersection number of \( T \):

\[
xyz \bullet xyz = -\{ 1 + [01] + \cdots + [50] + [012] + [123] + [234] + [01][23] + \cdots \}.
\]

Through the face \( (12) \), \( T \) is adjacent to \( yxz = 021345 \); evaluating the self-intersection number of the face \( (01) \), we have

\[
xyz \bullet yxz = g[12] \{ 1 + [34] + [45] + [50] + [012] + [34][50] + [34][012] + \cdots \}
\]

Through the face \( (123) \), \( T \) is adjacent to \( zyx = 032145 \); evaluating the self-intersection number of the face \( (123) \), we have

\[
xyz \bullet zyx = -g^3 [123] \{ 1 + [12] + [23] + [45] + [50] + [12][45] + \cdots \}
\]

Along the face \( (12) \cap (123) \), \( T \) touches \( zxy = 031245 \); evaluating the self-intersection number of the face \( (12) \cap (123) \), we have

\[
xyz \bullet zxy = g[12]g^3[123] \{ 1 + [45] + [50] \}.
\]

Let us add these terms. First the sum \( X_3 \) of the terms free of \( a \) and \( b \):

\[
X_3 := 1 + [12] + [23] + [123] + [12][123] + [23][123] - g[12] \{ 1 + [123] \} - g[23] \{ 1 + [123] \} + g^3 [123] \{ 1 + [12] + [23] \}
\]

\[
= (1 - [12](g - 1) - [23](g - 1))(1 + [123](1 + g^3))
\]

\[
= \frac{g^3}{(g + 1)(g^3 + g + 1)}.
\]

Next sum of the terms with the factor \([0123] = (a^3g^6 - 1)^{-1} \) is \([0123]\) times

\[
-X_3 - ([01] + [01][23] + [012](1 + [01] + [12]) + g[12][012] + g[23][01] = -X_3 - [01](1 - [23](g - 1)) - [012](1 - [12](g - 1) + [01])
\]

\[
= -X_3 - \frac{1}{a - 1} g + \frac{1}{a^2g^2 - 1} (a - 1)(a + 1)
\]

\[
= \frac{(a - 1)(ag - 1)(ag^2 - 1)(g + 1)(g^2 + g + 1)}{(a^3g^6 - 1)}.
\]
5 Proof of Theorem 2

Evaluation of intersection of cohomology is simpler than that of homology. Especially the evaluation of the self-intersection number of the form corresponding to a chamber is the sum of the contribution of every vertices. Let us see for example the $n$-beta function

$$B(\alpha_0, \ldots, \alpha_n) := \int_{t_j > 0, \, t_1 + \cdots + t_n < 1} t_1^{\alpha_1 - 1} \cdots t_n^{\alpha_n - 1} (1 - t_1 - \cdots - t_n)^{\alpha_0 - 1} dt_1 \cdots dt_n.$$ 

The quadratic relation reads

$$B(\alpha_0, \ldots, \alpha_n) \cdot B(-\alpha_0, \ldots, -\alpha_n) = \frac{1 - \prod a_i}{\prod (1 - a_i)} \cdot (2\pi i)^n \frac{\alpha_0 + \cdots + \alpha_n}{\alpha_0 \cdots \alpha_n},$$

where the first factor of the right hand-side is the self-intersection of the loaded simplex

$$\Delta : \quad t_j > 0, \, t_1 + \cdots + t_n < 1$$

and the second factor is the self-intersection of the twisted form

$$\frac{dt_1 \cdots dt_n}{t_1 \cdots t_n (1 - t_1 - \cdots - t_n)}.$$

The former is the sum of the contributions of all faces of $\Delta$:

$$1 + \sum_i \frac{1}{a_i - 1} + \sum_{i < j} \frac{1}{(a_i - 1)(a_j - 1)} + \cdots + \sum \frac{1}{\prod_{j \neq i} (a_j - 1)},$$

and the latter is the sum of the contribution of just the vertices of $\Delta$:

$$\sum \frac{1}{\prod_{j \neq i} \alpha_j} \quad \text{at the point } x_j = 0 \, (j \neq i).$$

Now we consider Terada-$n$ $T$ as a polyhedron. The self-intersection number

$$\omega := \frac{dt_1 \wedge \cdots \wedge dt_n}{\prod_{i=1}^n t_i (1 - t_i)}$$

is $(2\pi i)^n n!$ times the sum of the local contribution at every vertex of $T$ not touching another Terada-$n$, because $\omega$ has no poles of type

$$\frac{dt_1 \wedge \cdots \wedge dt_n}{t_1 \wedge \cdots \wedge t_n}$$

at vertices touching another Terada-$n$. Such a vertex will be said to be admissible. Let us work in this line.

- 2D case: admissible vertices are (in Figure 1, they are marked)

$$(012) \cap (01), \, (01) \cap (34), \, (34) \cap (234);$$

so the sum of the contributions are

$$\frac{1}{2\alpha + 2\gamma} \frac{1}{\alpha} + \frac{1}{\alpha} \frac{1}{\beta} + \frac{1}{\beta} \frac{1}{2\beta + 2\gamma} = \frac{1}{2} \frac{(\alpha + \beta + \gamma)(\alpha + \beta + 2\gamma)}{\alpha(\alpha + \gamma)\beta(\beta + \gamma)}.$$
• 3D case: admissible vertices are (in Figure 3, they are marked)

\[(0123) \cap (012) \cap (01), (012) \cap (01) \cap (45), (01) \cap (45) \cap (345), (45) \cap (345) \cap (2345);\]

so the sum of the contributions are

\[
\frac{1}{3\alpha + 6\gamma} \frac{1}{2\alpha + 2\gamma} \frac{1}{2\alpha + 2\gamma} \frac{1}{\alpha} \frac{1}{\alpha \beta} \frac{1}{\alpha \beta} \frac{1}{2b + 2g} \frac{1}{\beta} \frac{1}{2\beta + 2\gamma} \frac{1}{3\beta + 6\gamma} = \frac{1}{3!} \frac{(\alpha + \beta + 2\gamma)(\alpha + \beta + 3\gamma)(\alpha + \beta + 4\gamma)}{\alpha(\alpha + \gamma)(\alpha + 2\gamma)(\beta + \gamma)(\beta + 2\gamma)}.
\]

• nD case: admissible vertices are, for \(k = 0, \ldots, n,\)

\[(01) \cap (012) \cap \cdots \cap (01 \cdots k) \cap (k' \cdots n') \cap \cdots \cap (\'nnn') \cap (nn');\]

so the sum of the contributions are

\[
\sum_{k=0}^{n} \frac{1}{\prod_{i=1}^{k}(i\alpha + i(i-1)\gamma)} \frac{1}{\prod_{j=1}^{n-k}(j\beta + j(j-1)\gamma)}.
\]

Theorem 2 claims that this is equal to

\[
= \frac{1}{n!} \frac{(\alpha + \beta + (n - 1)\gamma)}{\alpha(\alpha + \gamma) \cdots (\alpha + (n - 1)\gamma)} \frac{\beta(\beta + \gamma) \cdots (\beta + (n - 1)\gamma)}{\alpha(\alpha + \gamma)(\alpha + 2\gamma)(\beta + \gamma)(\beta + 2\gamma)}.
\]

Proof. In this proof \((\alpha)_n := \alpha(\alpha + 1) \cdots (\alpha + n - 1).\) The sum in question equals

\[
= \frac{1}{n!} \sum_{k=0}^{n} \left( \binom{n}{k} \right) \frac{1}{\prod_{i=1}^{k}(\alpha + (i-1)\gamma)} \frac{1}{\prod_{j=1}^{n-k}(\beta + (j-1)\gamma)}
\]

\[
= \frac{1}{n!} \frac{1}{\prod_{j=1}^{n}(\beta + (j-1)\gamma)} \sum_{k=0}^{n} \left( \binom{n}{k} \right) \frac{1}{\prod_{j=1}^{n-k}(\alpha + (i-1)\gamma)} \frac{1}{\prod_{j=1}^{n-k}(\beta + (j-1)\gamma)}
\]

\[
= \frac{1}{n!} \frac{1}{\prod_{j=1}^{n}(\beta + (j-1)\gamma)} \sum_{k=0}^{n} \frac{(-n)_k}{k!} \left( \frac{\beta - n + 1}{\gamma} \right)_k
\]

On the other hand, the Chu-Vandermonde formula (a special case of Gauss’s summation formula) for the value of the hypergeometric function \(2F1\) at 1:

\[
_{2}F_{1} \left( \begin{array}{c} -n, \beta \\ \gamma \end{array} ; 1 \right) = \frac{(\gamma - \beta)_n}{(\gamma)_n}
\]

implies that the sum above equals

\[
\sum_{k=0}^{n} \frac{(-n)_k}{k!} \left( \frac{\beta - n + 1}{\gamma} \right)_k = \frac{1}{\prod_{j=1}^{n}(\alpha + \beta + (n - 1 + j - 1)\gamma)} \frac{1}{\prod_{j=1}^{n}(\alpha + (j - 1)\gamma)}
\]

which ends the proof.
6 Appendix

6.1 Standard loading

For a set of hyperplanes \( f_j(t) = 0 \) in the complex affine \( n \)-space and complex numbers (called exponents) \( \alpha_j \), we consider a (multi-valued) function

\[
  u = \prod_j f_j(t)^{\alpha_j}
\]

on \( X := \mathbb{C}^n - \bigcup_j \{ f_j = 0 \} \). The function \( u \) determines the local system \( S = S_u \) on \( X \). When every \( f_j \) is defined over \( \mathbb{R} \), the real locus \( X_{\mathbb{R}} \) breaks into simply connected chambers. For each chamber \( D \) we load

\[
  \prod_j (\varepsilon_j \cdot f_j(t))^{\alpha_j}, \quad \arg \varepsilon_j \cdot f_j(t) = 0,
\]

where \( \varepsilon_j = \pm \) is so determined that \( \varepsilon_j \cdot f_j(t) \) is positive on \( D \). This way of loading is said to be standard. This loading is already used in [MiY], though the terminology ‘standard’ is not used.

In all the other publications of the authors, chambers are loaded in a different way: each chamber is loaded with a branch (result of analytic continuation along a fixed path) of a fixed function element of \( u \) at a specific base point. This loading is often convenient in practical computation. But it has a fatal disadvantage: it depends on the choice of the base point and the paths. As a result, the intersection matrices are not symmetric.

On each chamber, the old loading and the standard loading differ only by a multiplicative constant, and their global monodromy representation (of the fundamental group) coincides, of course. So the difference might look minor, but this standard loading has apparent advantages: it does not require any base point nor fixed paths, and the intersection matrices are symmetric.

Here we present the simplest example. In the \( x \)-space, we consider

\[
  u = x^{\alpha}(1 - x)^\beta(2 - x)^\delta.
\]

Let us load the interval \( I_0 := (0, 1) \) and \( I_1 := (1, 2) \) in a standard way, that is, we load \( I_0 \) with \( u_0 := x^{\alpha}(1 - x)^\beta(2 - x)^\delta \), and \( I_1 \) with \( u_1 := x^{\alpha}(x - 1)^\beta(2 - x)^\delta \), where the argument of positive numbers are supposed to be 0. By making regularizations (by putting kintamas), we evaluate the intersection numbers as:

\[
  (\text{reg } I_1) \bullet \tilde{I}_0 = (\text{reg } I_0) \bullet \tilde{I}_1 = \frac{e^{\pi i \beta}}{e^{2\pi i \beta - 1}}.
\]

If we choose a base point in the lower half \( x \)-space, and load these intervals with the result of analytic continuations of \( u \) along paths in the lower half-space connecting the base point and the intervals, and if we write the loads on the intervals \( I_0 \) and \( I_1 \) by \( u'_0 \) and \( u'_1 \), respectively, they are related as

\[
  u'_0 = Cu_0, \quad u'_1 = Ce^{\pi i \beta}u_1,
\]
where $C$ is a constant depending on the choice of the branch of $u$ at the base point. Their intersection numbers are now evaluated as
\[
(\text{reg } I_1) \cdot \mathcal{I}_0 = \frac{e^{2\pi i\beta}}{e^{2\pi i\beta} - 1}, \quad (\text{reg } I_0) \cdot \mathcal{I}_1 = \frac{1}{e^{2\pi i\beta} - 1}.
\]

These non-symmetric formulae are commonly presented, for instance in \[Yo1\] (Chap 4 §7).

### 6.2 $q$-Chu-Vandermonde formulae

We give a proof of the reversed $q$-Chu-Vandermonde formula
\[
2\varphi_1 \left( \frac{q^{-n}}{c} : q, q \right) = (c/b)_n b^n.
\]

Reversing the order of the finite sum
\[
2\varphi_1 \left( \frac{q^{-n}}{c} : q, x \right) = \sum_{i=0}^n A_i, \quad A_i = \frac{(q^{-n})_i (b)_i}{(q)_i (c)_i} x^n,
\]
as
\[
A_n \left( 1 + \frac{A_{n-1}}{A_n} + \ldots + \frac{A_1}{A_n} + \ldots + \frac{A_n}{A_n} \right),
\]
we have
\[
2\varphi_1 \left( \frac{q^{-n}}{c} : q, x \right) = (-)^n q^{-(n+1)} \frac{(b)_n}{(c)_n} x^n \quad 2\varphi_1 \left( \frac{q^{-n}, c^{-1} q^{1-n}}{b^{-1} q^{1-n}} : q, \frac{c}{b} q^{n+1} \right). \tag{3}
\]

Indeed since
\[
\frac{(q^{-n})_{n-i}}{(q^{-n})_n} \cdot \frac{(q)_n}{(q)_{n-i}} = q^{(n+1)} \frac{(q^{-i})_i}{(q)_i} \quad \text{and} \quad \frac{(c)_n}{(c)_{n-i}} \cdot \frac{(b)_{n-i}}{(b)_n} = \left( \frac{c}{b} \right)^i \frac{(c^{-1} q^{1-n})_i}{(b^{-1} q^{1-n})_i}
\]
hold, we have
\[
\frac{A_{n-i}}{A_n} = \frac{(q^{-n})_{n-i} (q)_n}{(q^{-n})_n (q^{-i})_i} \cdot \frac{(q)_{n-i} (c)_n}{(c)_{n-i} (b)_n} x^{-i} = \frac{(c^{-1} q^{1-n})_i (q^{-n})_i}{(b^{-1} q^{1-n})_i (q)_i} \left( \frac{c}{b} q^{n+1} \right)^i,
\]
which proves the above formula \[(3).\] Putting $x = cq^n/b$ in this formula, we have
\[
2\varphi_1 \left( \frac{q^{-n}}{c} : q, \frac{c}{b} q^n \right) = (-)^n \left( \frac{c}{b} \right)^n q^{\frac{n(n-1)}{2}} \frac{(b)_n}{(c)_n} \quad 2\varphi_1 \left( \frac{q^{-n}, c^{-1} q^{1-n}}{b^{-1} q^{1-n}} : q, q \right). \tag{4}
\]

On the other hand the $q$-analogue of the Gauss’s sum due to Heine (cf. \[GR\]) reads
\[
2\varphi_1 \left( \frac{a}{c}, \frac{b}{c} ; q, \frac{c}{ab} \right) = \frac{(c/b)_\infty (c/b)_\infty}{(c)_\infty (c/ab)_\infty}.
\]
This yields $(a = q^{-n})$ the $q$-Chu-Vandermonde formula
\[
2\varphi_1 \left( \frac{q^{-n}}{c} : q, \frac{c}{b} q^n \right) = \frac{(c/b)_n}{(c)_n}. \tag{5}
\]
Since the left hand-sides of (4) and (5) coincide, we have

\[ 2\varphi_1\left(\frac{q^{-n}, c^{-1}q^{1-n}}{b^{-1}q^{1-n}}; q, q\right) = \frac{(c/b)_n}{(b)_n} (-)^n \left(\frac{b}{c}\right)^n q^{-\frac{1}{2}n(n-1)}. \]

If we put \( c^{-1}q^{1-n} \rightarrow b \) and \( b^{-1}q^{1-n} \rightarrow c \), then this leads to (2).

Acknowledgement. The second author is grateful to Joichi Kaneko for his encouragement.

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