Analytical Solution for Space-Charge Waves in a Two-Stream Cylindrical Electron Beam

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Abstract—We present an analytical method to compute the wavenumbers and electric fields of the space-charge-wave eigenmodes supported by a two-stream electron beam, consisting of a solid inner cylindrical stream and a coaxial outer annular stream, both contained within a cylindrical metallic tunnel. We extend the analytical model developed by Ramo to the case of two streams. The method accounts for the interaction between the two streams with the presence of the beam-tunnel wall; it can be used to model the complex wavenumbers associated with the two-stream instability and the plasma frequency reduction effects in vacuum electronic amplifiers and other vacuum electronic devices.

Index Terms—Bifurcation points, double-stream, eigenmode solution, electron beam devices, exceptional points of degeneracy (EPDs), transition points, two-stream instability.

I. INTRODUCTION

Vacuum electronic devices with high power and broad bandwidth have a competitive edge in various applications, such as electronic countermeasures, satellite communications, plasma diagnostics, and high resolution radars [1], [2]. Lately, the designers of microwave tubes have faced many difficult design challenges, such as reducing operating voltages and minimizing the weight and dimensions of the devices and their power supplies. In addition, with the high demand for vacuum electronics applications that operate at high frequencies, the dimensions of these devices are being reduced, and at the same time, electron beams with high current density are also required to obtain high output power [3]. There are some technical limitations to increasing both the output power and the operating frequency. The power-frequency-bandwidth product $P f_0\Delta f$ ($P$ is the average output power, $f_0$ is the center frequency, and $\Delta f$ is the bandwidth) for state-of-the-art vacuum electronics is a figure of merit (FOM) that tends to follow a growing linear trend with time [4], [5]. However, it is not certain how long vacuum electronic devices will continue to follow this trend. A promising engineering solution to continue improving this FOM for vacuum electronic devices is to use multiple electron beams [6], [7].

The interaction of multistream electron beams has been studied theoretically for many years. Many authors have also proposed the multiple beam concept since the 1940s for use in electron beam devices. As a pioneer in this field, Pierce [8] predicted the gain of a double-stream amplifier having thin concentric electron streams of different velocities that are modulated by input and output cavities. Then, Swift-Hook [9] analyzed the validity of the theory of the double stream amplification model proposed by Pierce. As an early work on this topic, beam–beam interaction in concentric-beam dual-mode traveling-wave tubes (TWTs) is presented in [10]. Chen [11] analyzed the conversion mechanism from the kinetic energy of electron beams to electromagnetic wave energy in the two-stream amplifier and how the efficiency of a two-stream instability amplifier increases with relativistic beam velocities. Wave coupling in multiple beam TWTs to increase the power level of vacuum electronic devices has also been studied in [12]. On the other hand, many works have begun exploring and showing realistic structures for multibeam generation. Zavadil [13] proposed a dual-cathode electron gun incorporating an annular hollow beam cathode, concentric, and co-planar with a solid beam cathode. Some work has used multiple cathode sources to produce multiple electron beams in low-power microwave sources where two separate power supplies power each cathode at different voltages [14], [15]. Also, some work has been published in the past that uses conventional vacuum electron beam device design concepts to generate multiple electron beams and extract power from them [16], [17], [18], [19]. Multiple electron beam generation with comparable currents and different energies from a single cathode--anode voltage for high-power applications has been studied recently in [20] and [21]. Another significant motivation for our work has been the analytic theory developed for multistream electron beam devices in [22] and [23] and the theoretical work involving modal degeneracies in linear beam tubes [24], [25], [26], [27], [28], [29].

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Modern communications’ increasing range and data handling requirements have given rise to a need for microwave tubes with power output and bandwidth capabilities that greatly exceed those of present-day state-of-the-art single-stream electron beam devices. The multiple-beam concept was developed to address this need and was applied to a resonant klystron [30], which was demonstrated to be capable of an order of magnitude higher power output than single-beam devices using the same electron beam. Then, the development of multistream klystron to provide low operating voltages, high power, low noise, and the possibility of larger operating bandwidth is further studied in many papers, such as [6], [31], and [32]. Recently, several research papers have focused on electron beam devices that utilize multiple electron streams, namely, multistream folded waveguide structures [33], [34], [35], [36], two-stream gyrotron TWT amplifiers [37], staggered dual-beam waveguides [38], [39], [40], [41], dual-beam sines waveguide TWTs [42], [43], two-stream backward wave oscillators (BWOs) [44], and other unique TWT configurations [45], [46], [47], [48], [49]. The aforementioned devices can employ the advantages of two-stream beams to improve the output power significantly and/or increase the bandwidth for various applications, such as telecommunication and high resolution radar.

The problem of space-charge waves in an electron beam is a topic of interest, since such waves are excited and utilized in a variety of electron tubes. These tubes may be used to generate, amplify, and detect signals. Furthermore, such tubes utilizing multistream electron beams may either be designed to utilize or avoid strong coupling between electron streams that leads to the two-stream instability under certain conditions. Two-stream instability conditions depend on the velocity differences between electron streams, their respective current densities, operating frequency, and geometry [8]. In particular, Pierce uses an ad hoc separation parameter $S$ to model how strongly electron streams are coupled when they are close together in a multistream beam, which affects the growth rate of space-charge waves under the two-stream instability regime [8]. However, no simple models have yet been developed to analytically determine the growth rate and conditions for two-stream instability as a function of stream geometry, voltage, and current. Ramo [50], [51] studied the propagation of space-charge waves in the case of a single electron beam propagating within a metallic beam tunnel. The theory of Ramo was extended in [52], where they defined the plasma frequency reduction factor and considered the case of an annular electron beam within a cylindrical metallic tunnel. Here, we extend the work of Ramo to the case of two concentric electron streams within a metallic tunnel, which allows us to analytically determine the conditions for two-stream instability and its growth rates without using ad hoc parameters. We consider an electron beam composed of a solid stream inside a hollow coaxial stream as in Fig. 1. Knowledge of the complex propagation constants of space-charge waves supported by the two-stream system may be useful for designing and analyzing two-stream instability amplifiers and two-stream TWTs, which depend strongly on the geometric configuration of the two-stream electron beam. This work can also lead to future definitions of the effective plasma frequency reduction factor for each individual stream in the two-stream system. However, the two-stream reduction factor definition will be more complicated than what was done in [52] due to the fact that the two streams interact, leading to four modes. Furthermore, some of the modes can have complex propagation constants when two-stream instability occurs.

II. FORMULATION

A. Problem Setup

The electron beam is assumed to be made of two concentric streams: stream 1 (the inner stream) is solid with a circular cross section, and it exists for $0 \leq r \leq R_1$; stream 2 (the outer stream, coaxial with stream 1) has an annular cross section, and it exists for $R_2 \leq r \leq R_{o2}$. The beam tunnel is assumed to be cylindrical with radius $R_1$ and made of a perfect electric conductor (PEC), as shown in Fig. 1. For convenience, we use a cylindrical coordinate system $(r, \theta, z)$ in this article to represent both the beam and the electromagnetic fields.

The two streams are assumed to possess uniform dc charge densities of $\rho_{o1}$ and $\rho_{o2}$ in both transverse and longitudinal directions. The uniform axial dc magnetic field is assumed to be strong enough to confine each of the two streams, such that all charges travel in the axial direction only (a common simplifying assumption seen in other linear beam tube work, such as [8], [50], [53], [54], and [55]) with dc...
velocities \(u_{0,1}\) and \(u_{0,2}\) for stream 1 and stream 2, respectively. Analogously, the existence of a strong axial dc magnetic field leads also to the assumption that the ac modulation in the velocity of electrons is only in the axial direction. Therefore, the radial and the azimuthal components of electron velocities are assumed to be vanishing \([50], [51], [53], [8], [54], [55]\). It is our goal to find the eigenmodes that represent the space-charge waves in this configuration. The propagating space-charge wave consists of a modulation in the beam’s volumetric charge density and axial velocity, as well as its associated electromagnetic fields, that, in the phasor domain, are all proportional to the wave function \(e^{i(o-t-\phi)}\). As another simplifying assumption, we only consider modes with azimuthal symmetry; therefore, we assume that \(\partial/\partial\theta = 0\) for all beam and field quantities. However, the presented formalism could be extended to find modes that do not possess azimuthal symmetry. Nevertheless, the case of azimuthal symmetry is the most significant one, in practice, for the operation of TWTs. Therefore, the instantaneous total (both dc and ac components) axial velocity and volumetric charge density for each stream are written as follows:

\[
\begin{align*}
    u_1(r, z, t) &= u_{0,1} + \mathfrak{Re}(u_{m,1}(r)e^{i(o-t-\phi)}) \\
    u_2(r, z, t) &= u_{0,2} + \mathfrak{Re}(u_{m,2}(r)e^{i(o-t-\phi)}) \\
    \rho_1(r, z, t) &= \rho_{0,1} + \mathfrak{Re}(\rho_{m,1}(r)e^{i(o-t-\phi)}) \\
    \rho_2(r, z, t) &= \rho_{0,2} + \mathfrak{Re}(\rho_{m,2}(r)e^{i(o-t-\phi)})
\end{align*}
\]

(1)

where \(u_{m,1}(r), u_{m,2}(r), \rho_{m,1}(r), \rho_{m,2}(r)\) are the radial distributions of the stream velocities and charge densities expressed in phasor domain; the subscript “0” denotes the dc component; “m” denotes ac modulation component; and “1” and “2” denote stream 1 and stream 2, respectively. The total radial-dependent volumetric charge density inside the tunnel is expressed as a piecewise function as follows:

\[
\rho(r, z, t) = \begin{cases} 
    \rho_1(r, z, t), & 0 \leq r \leq R_1 \\
    \rho_2(r, z, t), & R_1 \leq r \leq R_{o,2} \\
    0, & \text{otherwise}
\end{cases}
\]

(2)

We assume that the ac modulation of each electron-beam stream is small compared with the corresponding dc part. Therefore, under this small-signal approximation, the electron beam streams have current densities in the axial direction (\(J_1 = J_1\hat{z}\) and \(J_2 = J_2\hat{z}\)) in the form of

\[
\begin{align*}
    J_1(r, z, t) &= u_{0,1}\rho_1 \approx J_{0,1} + \mathfrak{Re}(J_{m,1}(r)e^{i(o-t-\phi)}) \\
    J_2(r, z, t) &= u_{0,2}\rho_2 \approx J_{0,2} + \mathfrak{Re}(J_{m,2}(r)e^{i(o-t-\phi)})
\end{align*}
\]

(3)

where the dc current densities of stream 1 and stream 2 are \(J_{0,1} = \rho_{0,1}u_{0,1}\) and \(J_{0,2} = \rho_{0,2}u_{0,2}\), and the linearized ac current densities of stream 1 and stream 2 are \(J_{m,1}(r) = \rho_{0,1}u_{m,1}(r) + u_{0,1}\rho_{m,1}(r)\) and \(J_{m,2}(r) = \rho_{0,2}u_{m,2}(r) + u_{0,2}\rho_{m,2}(r)\), respectively. The total currents for stream 1 and stream 2 are found using integration over each of the transverse cross sections of the stream regions as

\[
\begin{align*}
    j_1(z, t) &= \int_{A_1} J_1(r, z, t) \, dA \quad \text{and} \quad j_2(z, t) = \int_{A_2} J_2(r, z, t) \, dA,
\end{align*}
\]

where \(A_1 = \pi R_1^2\) and \(A_2 = \pi(R_{o,2}^2 - R_{k,2}^2)\) are the cross-sectional areas of regions 1 and 2, respectively. This integration yields

\[
\begin{align*}
    j_1(z, t) &= -I_{0,1} + \mathfrak{Re}(I_{m,1}e^{i(o-t-\phi)}) \\
    j_2(z, t) &= -I_{0,2} + \mathfrak{Re}(I_{m,2}e^{i(o-t-\phi)})
\end{align*}
\]

(5)

where \(I_{0,1} = A_1\rho_{0,1}u_{0,1}\) and \(I_{0,2} = A_2\rho_{0,2}u_{0,2}\) are the dc currents of stream 1 and stream 2, respectively, and \(I_{m,1}\) and \(I_{m,2}\) are the ac currents of stream 1 and stream 2 in phasor domain, respectively.

The electromagnetic fields associated with the two-stream electron beam are represented using the electric scalar potential and the magnetic vector potential, which are expressed as follows:

\[
\begin{align*}
    \phi(r, z, t) &= \mathfrak{Re}(f_\phi(r)e^{i(o-t-\phi)}) \\
    \mathbf{A}(r, z, t) &= \mathfrak{Re}(f_A(r)e^{i(o-t-\phi)}) \hat{z}
\end{align*}
\]

(6)

(7)

where \(\hat{z}\) is the unit vector in the \(z\)-direction. The chosen magnetic vector potential has only an axial component (in the \(z\)-direction), because we assume that only the longitudinal component of current modulation is present (we neglect current directions that are not longitudinal, because we assume to have a very high, confining, axial dc magnetic field). We use the International System of Units (SI) in the following analysis, whereas centimeter-gram-second (CGS) units were used in \([50]\) and \([51]\). The electric and magnetic fields in the structure are expressed in terms of the scalar electric potential and vector magnetic potential as \(E = -\nabla\phi - d\mathbf{A}/dt\) and \(\mathbf{H} = \nabla \times \mathbf{A}/\mu_0\). Here, we use the Lorentz gauge \(\nabla \cdot \mathbf{A} = -\mu_0\varepsilon_0\partial\phi/\partial t\) \([56]\), which leads to the relation \(f_A(r) = (\omega_0\varepsilon_0/k) f_\phi(r)\), as shown in Appendix A. Because the tunnel region is not homogeneously filled, we represent the radially dependent electric scalar potential function \(f_\phi(r)\) in (6) as follows:

\[
\begin{align*}
    f_\phi(r) &= \begin{cases} 
        f_{\phi,1}(r), & 0 \leq r \leq R_1 \\
        f_{\phi,2}(r), & R_1 \leq r < R_{o,2} \\
        f_{\phi,3}(r), & R_{o,2} \leq r \leq R_o \\
        0, & r > R_o
    \end{cases}
\end{align*}
\]

(8)

The time-domain electric and magnetic fields (which do not depend on \(\theta\) due to the assumption of azimuthal symmetry) are then given by

\[
\begin{align*}
    E_z(r, z, t) &= \mathfrak{Re}\left((-f_\phi'(r)e^{i(o-t-\phi)})\right) \\
    E_z(r, z, t) &= \mathfrak{Re}\left(-\frac{\varepsilon_0}{k} f_\phi(r)e^{i(o-t-\phi)}\right) \\
    H_\theta(r, z, t) &= \mathfrak{Re}\left(-\frac{\varepsilon_0}{k} f_\phi(r)e^{i(o-t-\phi)}\right)
\end{align*}
\]

(9)

whereas the rest of the time-varying field components are vanishing, i.e., \(E_\theta = H_r = H_z = 0\).

B. Governing Equations

We start by writing Newton’s second law (without relativistic corrections for the electron mass), which describes the equations of motion for each stream individually. The basic equations that govern the charges’ longitudinal motion are

\[
\begin{align*}
    m_0 \frac{du_{1,t}}{dt} &= -eE_{z,1} \\
    m_0 \frac{du_{2,t}}{dt} &= -eE_{z,2}
\end{align*}
\]

(10)

(11)
where $E_{z,1}$ and $E_{z,2}$ are the longitudinal electric fields that stream 1 and stream 2 experience in each of the regions denoted by indices “1” and “2,” respectively (see Fig. 1). $m_0 = 9.109 \times 10^{-31}$ kg is the rest mass of an electron, and $e = +1.602 \times 10^{-19}$ C is the elementary charge. The longitudinal electric fields $E_{z,1}$ and $E_{z,2}$ are determined from (9) in region 1 and region 2, as we will discuss later. Each electron flow should be continuous, and there should be no leakage or accumulation of charges. Therefore, the continuity equation for each stream is written as $\nabla \cdot \mathbf{J}_i = -\partial \rho_i/\partial t$ and $\nabla \cdot \mathbf{J}_2 = -\partial \rho_2/\partial t$, which are simplified as follows:

$$\frac{\partial (\rho_1 u_1)}{\partial z} = -\frac{\partial \rho_1}{\partial t} \quad (12)$$

$$\frac{\partial (\rho_2 u_2)}{\partial z} = -\frac{\partial \rho_2}{\partial t}. \quad (13)$$

As explained in Appendix A, these charge continuity and force equations lead to the velocity and charge modulations of each stream, expressed in terms of the scalar potential as follows:

$$u_{m,i}(r) = \frac{\eta}{u_{0,i} k (k - \beta_{0,i})} f_{\phi,i}(r) \quad (14)$$

$$\rho_{m,i}(r) = -\frac{\eta \rho_{0,i}}{u_{0,i}^2 (k - \beta_{0,i})} f_{\phi,i}(r) \quad (15)$$

depends on which stream is referred to as 1 or 2. Furthermore, $\beta_{0,i}$ is the electronic phase constant of the $i$th electron stream, $\eta = e/m_0 = 1.758829 \times 10^{11}$ C/kg is the charge to mass ratio of an electron, and $k_0 = \omega/(\mu_0 \epsilon_0)$. 1/2. 2. 1/2.

Following what was done in [50] and [51] for a single-stream electron beam inside a concentric metallic tunnel, the governing equation in each region of our problem is found by substituting the definition $\mathbf{E} = -\nabla \phi - dA/\partial t$ into Gauss’s law $\nabla \cdot \mathbf{E} = \rho/\epsilon_0$, and by using the Lorentz gauge (see Appendix A), leading to

$$\left( \nabla^2 - \mu_0 \epsilon_0 \frac{\partial^2}{\partial t^2} \right) \phi = -\frac{\rho}{\epsilon_0}. \quad (16)$$

The charge density term in (16) is either $\rho = 0$, in the two vacuum regions, or is given by (15) in the two stream regions. Substituting these values for charge density into (16), we arrive at the Bessel differential equations for the electric potential in both the vacuum and stream regions, as explained further in Appendixes A and B. As a result, the two stream regions (i.e., regions 1 and 2), the potential solution is expressed in terms of Bessel functions of the first and second kinds, and order zero. For the vacuum regions between the electron streams and near the metallic wall (i.e., regions 3 and 4), the potential solution is written in terms of modified Bessel functions of the first and second kinds, and order zero. An alternative formulation is based on taking the electric field expressions in (9) and the charge density expression in (15) into Gauss’ law $\nabla \cdot \mathbf{E} = \rho/\epsilon_0$, and the Bessel equations are determined by expressing everything in terms of $f_{\phi}(r)$.

C. Boundary Conditions

Aside from the fact that the potential function $f_{\phi}(r)$ is finite at $r = 0$, building on what was done in [50] and [51] for a single stream, we enforce that the potential function $f_{\phi}(r)$ and its derivative are continuous across the three boundaries between the concentric regions illustrated in Fig. 1 as follows:

$$f_{\phi,1}(R_1) = f_{\phi,3}(R_1), \quad f'_{\phi,1}(R_1) = f'_{\phi,3}(R_1)$$

$$f_{\phi,3}(R_2) = f_{\phi,2}(R_2), \quad f'_{\phi,3}(R_2) = f'_{\phi,2}(R_2)$$

$$f_{\phi,2}(R_3) = f_{\phi,4}(R_3), \quad f'_{\phi,2}(R_3) = f'_{\phi,4}(R_3). \quad (17)$$

Due to the assumption that a tunnel is made of PEC, we also enforce that the potential function vanishes at the tunnel wall

$$f_{\phi,4}(R_i) = 0. \quad (18)$$

In the following section, we chose a representation for the solution of the potential functions in each region. Then, we enforce the aforementioned boundary conditions to find a linear system whose solution provides the eigenmodes of charge waves of the two-stream electron beam.

III. Modal Dispersion Equation

The general solutions of the scalar electric potential as a function of radius in each region shown in Fig. 1 are found based on the derivation in Appendixes A and B. We write the radially dependent scalar potential function in (8) as follows:

$$f_{\phi,1}(r) = c_{1} I_0(T_1 r)$$

$$f_{\phi,3}(r) = c_{2} K_0(\tau r) + c_{3} I_0(\tau r) \quad (19)$$

$$f_{\phi,2}(r) = c_{4} I_0(T_2 r) + c_{5} Y_0(T_2 r) \quad (20)$$

$$f_{\phi,4}(r) = c_{6} K_0(\tau r) + c_{7} I_0(\tau r) \quad (21)$$

where the values of $c_n (n = 1, \ldots, 7)$ are arbitrary constants that are determined by imposing the boundary conditions in Section II-C. The parameters $T_1$ and $T_2$ in the arguments of the above Bessel functions are related to regions 1 and 2, i.e., in the electron streams, and the parameter $\tau$ is related to the vacuum regions outside of the electron streams (regions 3 and 4), defined as follows:

$$T_1^2 = \frac{1}{\tau^2} \left( \frac{(\beta_{p,1})^2 - (k - \beta_{0,1})^2}{(k - \beta_{0,1})^2} \right) \quad (23)$$

$$T_2^2 = \frac{1}{\tau^2} \left( \frac{(\beta_{p,2})^2 - (k - \beta_{0,2})^2}{(k - \beta_{0,2})^2} \right) \quad (24)$$

$$\tau^2 = (k^2 - k_0^2). \quad (25)$$

In these equations, $\beta_{p,1} = \omega_{p,1}/u_{0,1}$ and $\beta_{p,2} = \omega_{p,2}/u_{0,2}$ are the plasma phase constants related to stream 1 and stream 2, respectively, related to the two plasma frequencies $\omega_{p,1} = (\eta \rho_{0,1}/\epsilon_0)^{1/2}$ and $\omega_{p,2} = (\eta \rho_{0,2}/\epsilon_0)^{1/2}$. Note that we did not consider the Bessel’s function of the second kind (Neumann’s function) $Y_0(T r)$ in $f_{\phi,1}(r)$, because the scalar potential should be finite at $r = 0$, and $Y_0(T r)$ has a singularity at $r = 0$. When the six boundary conditions in (17) are enforced, together with the PEC condition at the tunnel wall in (18), the resulting set of seven equations are cast in matrix form as $\mathbf{M} \mathbf{c} = \mathbf{0}$, where $\mathbf{c} = [c_1, c_2, c_3, c_4, c_5, c_6, c_7]^T$ (T denotes the transpose operation), and the matrix $\mathbf{M}$ is defined as in (26), shown at the bottom of the next page.
A solution exists when one finds \( k \), such that \( \det(\mathbf{M}) = 0 \). Here, we look for complex \( k \) solutions, though they may also be purely real. The matrix \( \mathbf{M} \) may become ill conditioned (i.e., the condition number of the matrix becomes large [37]) when using an imaginary part of the wavenumber \( k \) that makes the Bessel functions extremely large or small in value. This also occurs when \( k \) is nearly equal to \( \beta_{0,1} \) or \( \beta_{0,2} \), which are the poles of \( T_1 \) and \( T_2 \), respectively. To overcome this issue, we follow the same procedure as in [50] and [51]: i.e., we reduce the number of equations describing the boundary conditions until we obtain only one characteristic equation, \( C_e = 0 \), that is described in terms of the space-charge wavenumber \( k \) of the system. Numerical solutions for \( k \), which make \( |C_e| = 0 \), are eigenmode solutions of the two-stream system.

IV. ILLUSTRATIVE EXAMPLES

As an illustrative example, we consider an electron beam consisting of two streams that have equivalent kinetic dc voltages of \( V_{0,1} = 7 \) kV and \( V_{0,2} = 6 \) kV, corresponding to average electron speeds of \( u_{0,1} = 0.164c \) and \( u_{0,2} = 0.152c \), respectively, from the relativistic relation \( V_o = ((1 - (u_0/c)^2)^{1/2} - 1)c^2/\eta \). Stream 1 is a solid cylinder with a circular cross section and has an outer radius \( R_1 = 0.1 \) mm. Stream 2 is annular in cross section, with inner and outer radii of \( R_{o,2} = 1 \) mm and \( R_{o,2} = 1.1 \) mm, respectively. The metallic tunnel is made of a PEC and has an inner radius of \( R_i = 2 \) mm, as illustrated in Fig. 1. The dc currents of stream 1 and stream 2 are \( I_{0,1} = A_1\rho_{0,1}u_{0,1} = 50 \) mA and \( I_{0,2} = A_2\rho_{0,2}u_{0,2} = 50 \) mA, respectively, corresponding to dc volume charge densities of \( \rho_{0,1} = 0.0325 \) C/m\(^3\) and \( \rho_{0,2} = 0.00166 \) C/m\(^3\) for streams 1 and 2, respectively.

We show in Fig. 2(a) the magnitude of the characteristic equation, \( C_e \), defined in Appendix C, in log scale when the real and imaginary parts of \( k \) are swept at a fixed frequency of \( f = 5 \) GHz. The roots of the characteristic equation correspond to locations in Fig. 2(a) where \( C_e \) tends to zero (shown by the dark blue regions). We label the four modes in Fig. 2(a) that correspond to the dominant modes of the system. We also verify that, for the four \( k \) solutions, the scalar potential function \( f_\theta(r) \) is continuous at the radii, where we have the boundaries between the electron stream and vacuum regions. All the points in the complex \( k \) plane of Fig. 2(a) where \( C_e \) is not vanishing (i.e., everywhere, except for the labeled four dominant modal solutions and the bright-yellow spots corresponding to higher-order modes near \( \Re(k) \approx 640 \) m\(^{-1}\)) correspond to scalar potential functions \( f_\theta(r) \) that have discontinuities at the radii corresponding to the boundaries between regions; thus, these points are not valid solutions. The bright yellow spots that are not labeled in Fig. 2(a) correspond to valid solutions of the characteristic equation. However, the potential functions corresponding to such modal solutions were found to have one or more zero crossings within the stream regions, unlike the potential functions of the four dominant modes shown in Fig. 2. Thus, we classify them as higher-order modes. Higher-order modes occur also for single-stream configurations, as indicated in [50] and [51], but they are generally ignored as well, because they do not strongly interact with the conventional longitudinal electric field of the electromagnetic modes utilized by vacuum electronic devices, such as klystrons or TWTs. The average phase constant for stream 1, \( \beta_{0,1} \approx 640 \) m\(^{-1}\), which is a pole of \( T_1^2 \), lies exactly between the aforementioned higher-order solutions on the \( \Re(k) \) axis. The radial profiles of the scalar electric potential for the four dominant modes labeled in Fig. 2(a) are shown in Fig. 2(b)–(e).

In Fig. 3, we show the four modes of the two-stream system when the dc current of stream 2 is swept, while the dc current of stream 1 is held constant at \( I_{0,1} = 50 \) mA. The frequency is fixed at \( f = 5 \) GHz, and the equivalent kinetic stream voltages are held constant at \( V_{0,1} = 7 \) kV and \( V_{0,2} = 6 \) kV. Also, the study can be used to verify the validity of the mode-finding method that we use. When the dc current of stream 2 approaches zero, this is equivalent to a case where stream 1 only exists in the tunnel. Therefore, when \( I_{0,2} \to 0 \), one sees only two solutions that coincide with the two conventional plasma modes that have wavenumbers described as \( k = \beta_{0,1} \pm \beta_{0,1}^- \), where \( \beta_{0,1}^- = R_{\infty}\omega_p/\mu_{0,1} \), and \( R_{\infty} \) is the plasma frequency reduction factor, calculated using the method shown in [52]. Fig. 3 shows that there exist two transition points (bifurcations) close to \( I_{0,2} = 2 \) mA and \( I_{0,2} = 62.2 \) mA, between which exponentially growing space-charge waves occur due to the two-stream instability effect, which happens when there is a sufficient velocity difference between electron streams and sufficient stream currents, as predicted in [8] and [58] using an abstract theoretical model. The bifurcation points in Fig. 3 are exceptional points of degeneracy (EPDs) [26], [28], [59], which are conditions where two or more eigenmodes coalesce in their wavenumbers and eigenvectors. In Appendix D, we show the effect of swapping the two stream velocities on the resulting space-charge wavenumbers. We find that the dispersion diagram does not change significantly compared with the case studied in this section.

To understand the conditions resulting in the beam instability, we have swept both dc currents of stream 1 and stream 2.
at a fixed frequency of $f = 5$ GHz and fixed equivalent kinetic stream voltages of $V_{0,1} = 7$ kV and $V_{0,2} = 6$ kV, and we monitored the imaginary part of the wavenumber for the modes that exhibit instability. We show in Fig. 4 the absolute value of the imaginary part of the wavenumber for the mode that exhibits a growing instability. Note that instability occurs when either stream 1 has a higher dc voltage than stream 2 or vice versa. Similar to the previous figure, the white dashed lines represent the boundary between the stability and instability.
regions. For the studied range of stream 1 and stream 2 dc voltages in Fig. 5, one finds that instability occurs when the difference between the equivalent kinetic dc voltages of the two streams is approximately $1 \text{kV} \lesssim |V_{0,1} - V_{0,2}| \lesssim 1.5 \text{kV}$.

Similar to [60] and [61], where the modal wavenumber-frequency dispersion relations for space-charge waves in single-stream systems are found, we show the modal wavenumber-frequency dispersion relation for the two-stream system in Fig. 6 when the operating frequency is swept while the two beam currents are held constant at $I_{0,1} = 50 \text{ mA}$ and $I_{0,2} = 50 \text{ mA}$, and the equivalent stream voltages are held constant at $V_{0,1} = 7 \text{ kV}$ and $V_{0,2} = 6 \text{ kV}$ (as in the first and second examples above). The figure shows that the amplification resulting from the two-stream instability occurs from dc up to a threshold frequency, which is $15 \text{ GHz}$ in this case (note that the cutoff frequency of the lowest transverse electric (TE) mode ($\text{TE}_{11}$) in a metallic circular waveguide of radius $R_i = 2 \text{ mm}$ is approximately $44 \text{ GHz}$). For the case of a two-stream instability amplifier, which has been experimentally investigated in works, such as [14] and [17], it may be potentially beneficial to have a growing instability up to a threshold frequency that is below the lowest cutoff frequency of a circular waveguide, since the device will be less susceptible to regenerative oscillations or backward-wave oscillations that would otherwise exist on a slow wave structure in a conventional TWT amplifier, as explained in [11]. However, in [62], it was stated that backward-wave oscillations may still be an issue if long slow wave structures are used to extract amplified waves from a two-stream amplifier.

Next, observing the modal dispersion of the wavenumbers as the distance between the two streams varies, Fig. 7, gives us a better understanding of how the two streams are coupled. The coupling is controlled in this example by sweeping the inner radius $R_{i,2}$ of stream 2 while keeping its width, $R_{o,2} - R_{i,2} = 0.1 \text{ mm}$, constant. We also keep the radius of stream 1 constant at $R_i = 0.1 \text{ mm}$. Therefore, what varies is the distance between the two streams, $R_{o,2} - R_1$. In Fig. 7, we show the four complex wavenumbers of the two charge-wave eigenmodes versus $R_{i,2}$. The operating frequency is still kept at $f = 5 \text{ GHz}$, the currents of the streams are $I_{0,1} = 50 \text{ mA}$ and $I_{0,1} = 50 \text{ mA}$, and the equivalent kinetic voltages of the streams are still kept at $V_{0,1} = 7 \text{ kV}$ and $V_{0,2} = 6 \text{ kV}$. Note that the radius $R_{i,2} = 1 \text{ mm}$ was considered in all the previous examples, and that for the given values of $I_{0,1}, I_{0,2}, V_{0,1}$, and $V_{0,2}$ considered here, the two-stream electron beam exhibits instability, as was shown in Fig. 3. The plots in Fig. 7 reveal that when the distance between the two beams $R_{o,2} - R_1$ gets smaller, $\text{Im}(k)$ of the unstable eigenmode gets larger. Conversely, when the distance between the streams gets larger, $\text{Im}(k)$ gets smaller. Furthermore, for all values of the distance $R_{o,2} - R_1$, there are always two eigenmodes with purely real $k$, represented by the red and blue curves, and the difference between their $k$ values remains more or less constant for all considered distances $R_{o,2} - R_1$.

Finally, to demonstrate a more significant gain per unit length that can be utilized for a two-stream instability amplifier with the same stream configuration shown in Fig. 1, we use higher stream voltages of $V_{0,1} = 18 \text{ kV}$ and $V_{0,2} = 10 \text{ kV}$ for streams 1 and 2, respectively. In this example, the outer radius of stream 1 is $R_1 = 1.296 \text{ mm}$, with stream 2 having dimensions of $R_{i,2} = R_1 + 0.1 \text{ mm}$ (i.e., 0.1 mm is the radial gap between streams) and $R_{o,2} = 1.905 \text{ mm}$ to give both streams equal areas with a small spacing between them to have
strong coupling between the streams without mixing them. Furthermore, the wall radius is $R_t = 4R_{o,2}$. Stream currents for this example are $I_{o,1} = I_{o,2} = 4.22$ A to achieve equal current densities of $80$ A/cm$^2$ in both streams, as was done in [62] for copropagating streams of equal radius.

In Fig. 8, we show the complex modal dispersion diagram for the frequency of interest, as was done for the previous example, but now, we achieve a much more significant peak gain per unit length of approximately 37 Nb/m (321 dB/m). A two-stream linear beam amplifier with such a growth rate would only need to be approximately 9.3 cm in length in order to have a small-signal gain of 30 dB, which can, of course, be made longer to compensate for launching losses or imperfect beam focusing.

V. CONCLUSION

We have provided an analytical method to determine the wavenumbers and associated electric fields of space-charge waves supported by an electron beam made of two coaxial streams. Our analytical model has determined the two-stream instability regions and the bifurcation points in the wavenumber-dispersion diagrams for a coaxial two-stream system operating in the linear regime with non-relativistic stream voltages. Specific combinations of dc current and dc equivalent kinetic voltages cause instability. In addition, this instability growth rate is enhanced when the two streams are closer together, as was predicted in [8]. The analytical model developed here may serve as a useful design tool for modeling electron beam systems consisting of two coaxial streams, such as those generated by multistream electron guns described in [15], [21], and [20].

APPENDIX A

GENERAL SOLUTION OF THE SCALAR ELECTRIC POTENTIAL FUNCTION IN A REGION WITH MOVING ELECTRONS (STREAM REGION)

We provide the basic steps to find the general solution that satisfies the differential equation governing the electron beam and the electromagnetic fields. For a given region containing a single electron stream [i.e., either region 1 or 2 in Fig. 1(b)], the electrons in that stream are assumed to be traveling only in the axial direction with uniform dc velocity $u_0$ and uniform dc charge density $\rho_0$ (e.g., region 1 in Fig. 1(b) has $u_0 = u_{0,1}$ and $\rho_0 = \rho_{0,1}$). We assume that the electron beam and the electromagnetic dynamics follow a time-harmonic wave function $e^{j\omega t - jkz}$ that involves modulation in the charges’ axial speed and charge density, which in cylindrical coordinates are written as follows:

\[
\begin{align*}
  u(r, z, t) &= u_0 + \Re[(u_m(r)e^{j\omega t - jkz})] \\
  \rho(r, z, t) &= \rho_0 + \Re[(\rho_m(r)e^{j\omega t - jkz})] \\
  \phi(r, z, t) &= \Im[(f_\phi(r)e^{j\omega t - jkz})] \\
  A(r, z, t) &= \Im[(f_A(r)e^{j\omega t - jkz})] \hat{\mathbf{z}}.
\end{align*}
\]

Because of the strong external axial magnetic field that confines the beam, we have assumed that the beam modulation occurs only along the axial direction. Starting from the divergence relation of the magnetic vector potential \( \nabla \cdot \mathbf{A} = -\mu_0 \epsilon_0 \partial \phi / \partial t \), one finds that

\[
f_A(r) = \frac{\omega \mu_0 \epsilon_0}{k} f_\phi(r). \tag{31}
\]

The axial electric field component is found using the relation $\mathbf{E} = -\nabla \phi - \partial \mathbf{A} / \partial t$, which yields

\[
\begin{align*}
  E_z(r, z, t) &= -\partial \phi / \partial z - \partial (A_z / \partial t) \\
  &= \Re((jkf_\phi(r) - j\omega f_A(r))e^{j\omega t - jkz}). \tag{32}
\end{align*}
\]

We use the relation in (31) to simplify (32) as follows:

\[
E_z(r, z, t) = \Re\left((j\frac{k^2 - \omega^2 \mu_0 \epsilon_0}{k}) f_\phi(r)e^{j\omega t - jkz}\right). \tag{33}
\]

Newton’s second law that describes the equation of motion for electrons (for a strongly confined beam of electrons, i.e., with no radial or azimuthal motion) is written as $mdu/dt = -eE_z$. First, we express the total derivative of the velocity of the electrons in the phasor domain as follows:

\[
\frac{du}{dt} = \frac{\partial u}{\partial t} + u_0 \frac{\partial u}{\partial z}
= \Re((j\omega - jku_0)u_m(r)e^{j\omega t - jkz}). \tag{34}
\]

When this is inserted into Newton’s second law applied to the electrons, we find the relation between the velocity and electric potential functions as follows:

\[
u_m(r) = \frac{\eta}{u_0(k - \beta_0)} k_0^2 - k^2 f_\phi(r) \tag{35}
\]

where $k_0 = \omega \sqrt{\mu_0 \epsilon_0}$ is the free space wavenumber, and $\beta_0 = \omega / u_0$ for the stream-containing region of interest. Moreover,
we consider the continuity equation or conservation of charge

$$\frac{\partial (\rho u)}{\partial z} = -\frac{\partial \rho}{\partial t}$$

(36)

which is simplified (using a small-signal approximation) to

$$\rho_m(r) = \frac{k_0}{u_0(\beta_0 - k)} u_m(r).$$

(37)

By substituting (35) into (37), the latter equation yields the relation between the charge density and the potential function as follows:

$$\rho_m(r) = -\frac{\eta_0}{u_0^2} \frac{k_0^2 - k^2}{(k - \beta_0)^2} f_\phi(r).$$

(38)

The final equation that is used to find the potential function is obtained from Gauss’ law, \(\nabla \cdot \mathbf{A} = \rho/\epsilon_0\), where \(\mathbf{E} = -\nabla \phi - d\mathbf{A}/dt\), leading to

$$\nabla^2 \phi - \frac{\partial}{\partial t}(\nabla \cdot \mathbf{A}) = -\frac{\rho}{\epsilon_0}.$$  

(39)

Then, using the Lorentz gauge \(\nabla \cdot \mathbf{A} = -\mu_0 \varepsilon_0 \partial \phi/\partial t\), we obtain the inhomogeneous wave equation that governs the scalar electric potential \((\nabla^2 - \mu_0 \varepsilon_0 (d^2/dt^2)) \phi = -\rho/\epsilon_0\), where the charge density is cast in terms of the scalar potential function \(f_\phi(r)\) using (38). Due to the azimuthal symmetry of our system in cylindrical coordinates, we have \(\nabla^2 \phi = (1/r)(\partial/\partial r)(r(\partial \phi/\partial r)) + (\partial^2 \phi/\partial z^2)\). After taking the time and \(z\) derivatives, the final equation to be solved is

$$\frac{d^2 f_\phi(r)}{dr^2} + \frac{1}{r} \frac{df_\phi(r)}{dr} + T^2 f_\phi(r) = 0$$

(40)

where

$$T^2 = \left(k^2 - k_0^2\right) \left(\beta_p^2 - (k - \beta_0)^2\right).$$

(41)

Here, \(\beta_p = \omega_p/u_0\), and \(\omega_p = (\eta_0/\varepsilon_0)^{1/2}\) is the plasma frequency of the electron stream in the region considered (either stream 1 or stream 2). The general solution of the differential equation in (40) is written in terms of Bessel’s functions as follows:

$$f_\phi(r) = a_1 I_0(T_r) + a_2 Y_0(T_r)$$

(42)

where \(a_1\) and \(a_2\) are arbitrary constants, \(I_0\) is the Bessel function of the first kind and order zero, and \(Y_0\) is the Bessel function of the second kind (Neumann’s function) and order zero.

One may also find the general solution in terms of modified Bessel’s functions, as follows. By transforming the differential equation in (40) using \(\kappa = jr\), as was done in [64], we find

$$\frac{d^2 g(\kappa)}{d\kappa^2} + \frac{1}{\kappa} \frac{dg(\kappa)}{d\kappa} - T^2 g(\kappa) = 0$$

(43)

which has a general solution \(g(\kappa) = d_1 I_0(T \kappa) + d_2 K_0(T \kappa)\). Thus, one may rewrite the general solution of (40) in terms of modified Bessel functions as \(f_\phi(r) = d_1 I_0(j T r) + d_2 K_0(j T r)\).

APPENDIX B

GENERAL SOLUTION OF THE SCALAR ELECTRIC POTENTIAL FUNCTION IN AN EMPTY REGION (VACUUM)

We consider the case where the studied region is empty; i.e., it does not contain charges. Starting with (16), the steps
are exactly as the previous case in Section A, except that in the vacuum region \( \rho(r) = 0 \). This leads to the homogeneous (i.e., source-free) wave equation that governs the scalar electric potential

\[
\frac{d^2 f_\phi(r)}{dr^2} + \frac{1}{r} \frac{df_\phi(r)}{dr} - \tau^2 f_\phi(r) = 0 \quad (44)
\]

where

\[
\tau^2 = k^2 - k_0^2.
\]

The general solution of the differential equation in (44) is a linear combination of modified Bessel functions

\[
f_\phi(r) = b_1 I_0(\tau r) + b_2 K_0(\tau r)
\]

where \( b_1 \) and \( b_2 \) are arbitrary constants, \( I_0 \) is the modified Bessel function of the first kind and order zero, and \( K_0 \) is the modified Bessel function of the second kind and order zero.

**APPENDIX C**

**CHARACTERISTIC EQUATION DEFINITION AND MODE PROFILE**

We show the steps we used to find solutions to the system of equations in \( \mathbf{M} \mathbf{c} = 0 \), where \( \mathbf{c} = [ c_1, c_2, c_3, c_4, c_5, c_6, c_7 ]^{\top} \), and the matrix \( \mathbf{M} \) is given in (26). Once we assume the stream parameters and the radii of the structure, the only unknowns we are left with are the seven constants in \( \mathbf{c} \) and the wavenumber \( k \). First, we find \( c_1 \) by assuming that the potential function at the center of the stream region 1 (given by (19)) is normalized, such that \( f_{\phi,1}(r = 0) = c_1 = 1 \, \text{V} \).

Then, for any given wavenumber \( k \), one finds the other seven constants of \( \mathbf{c} \) by solving the last six equations in \( \mathbf{M} \mathbf{c} = 0 \), which yields (47), as shown at the top of the page.

Reaching this stage, we have found the seven constants in \( \mathbf{c} \) that satisfy six out of the seven boundary conditions. In the next step, we solve for \( k \) that will satisfy the first boundary condition at \( r = R_1 \) [that was eliminated when we considered (47) instead of (26)] as follows:

\[
C_e = J_0(T_1 R_1) - c_2 K_0(\tau R_1) - c_3 I_0(\tau R_1) = 0 \quad (48)
\]

where \( c_2 \) and \( c_3 \) are known. We find solutions of \( k \) by searching for complex wavenumbers that guarantee that (48) is satisfied, which implicitly guarantees that the remaining equations are also satisfied, since \( c_2 \) and \( c_3 \) are found based on satisfying the rest of the boundary conditions. The modes’ profile in Fig. 2(a) is found based the constants calculated from (47) with \( c_1 = 1 \, \text{V} \) to have a normalized electric potential at the center of stream 1 as \( f_\phi(r = 0) = 1 \, \text{V} \).

**APPENDIX D**

**EFFECT OF SWAPPING STREAM VELOCITIES**

We show in Fig. 9(a) the magnitude of the value of the characteristic equation in log scale for the same case as the results shown in Fig. 2(a), except that the beam dc voltages of the two streams are swapped, i.e., \( V_{0,1} = 6 \, \text{kV} \) and \( V_{0,2} = 7 \, \text{kV} \) instead of \( V_{0,1} = 7 \, \text{kV} \) and \( V_{0,2} = 6 \, \text{kV} \). We label the four modes in Fig. 9(a) that correspond to the dominant modes of the system. Compared with the previous case, we still see that the interaction between the two streams results in instability.

In Fig. 10, we show the four modes of the two-stream system when the dc current of stream 2 is swept similar to case shown in Fig. 3, except that the beam dc voltages of the two streams are swapped, i.e., now \( V_{0,1} = 6 \, \text{kV} \) and \( V_{0,2} = 7 \, \text{kV} \) instead of \( V_{0,1} = 7 \, \text{kV} \) and \( V_{0,2} = 6 \, \text{kV} \) as in Fig. 3. In Fig. 3, there exist two transition points (bifurcations) close to \( I_{0,2} = 1 \, \text{mA} \) and \( I_{0,2} = 54.7 \, \text{mA} \). Between the bifurcation points, exponentially growing space-charge waves occur due to the two-stream instability effect. This is similar to the other case in Fig. 3, except that the transition points in Fig. 3 occurred at approximately \( I_{0,2} = 2 \, \text{mA} \) and \( I_{0,2} = 62.2 \, \text{mA} \).

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