A MEAN VALUE INEQUALITY FOR THE GENERALIZED SELF-EXPANDER TYPE SUBMANIFOLDS AND ITS APPLICATION

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Abstract. In this paper we get a version of mean value inequality for generalized self-expander type submanifolds in Euclidean space. As the application, we prove that if mean curvature flow $M(t)$ on the self-expander in Euclidean space subconverges to an $n$-rectifiable varifold $T$ in weak sense for $t$ goes to the singular time, then $T$ must be the cone.

1. Introduction

Let $x_0 : M^n \to \mathbb{R}^{n+m}$ be a complete smooth immersed submanifold in Euclidean space. Consider the mean curvature flow

$$\frac{\partial x}{\partial t} = \vec{H},$$

with the initial data $x_0$, where $\vec{H} = -H\nu$ is the mean curvature vector and $\nu$ is the outer unit normal vector. The self-similar solutions, including self-shrinkers, translators and self-expanders, are one of the important subject in the study of mean curvature flow. For other works for studying the self-similar solutions of the mean curvature flow, one may see [3], [6], [15], and [15]. Recall

Definition 1.1. The immersed submanifold $x : M^n \to \mathbb{R}^{n+m}$ is called self-expanders of mean curvature flow if it satisfies

$$\vec{H} = \mu(x - p_0)^2,$$

for some fixed vector $p_0 \in \mathbb{R}^{n+m}$ and nonnegative constant $\mu \geq 0$.

The mean curvature flow $x(\cdot, t)$ on the self-expander (1.7) satisfying

$$x(p, t) = \sqrt{1 + 2\mu(t - \frac{1}{2\mu})(x(\phi^\mu_t(p)) - p_0)},$$

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with $\phi_t$ being the tangent diffeomorphisms on $M^n$ with
\[
x(p, \frac{1}{2\mu}) = x(\phi_t^*(p)),
\]
and
\[Dx\left(\frac{\partial}{\partial t}\phi_t^*(p)\right) = -\frac{\mu}{1+2\mu(\frac{t}{\sqrt{\mu}})}(x(\phi_t^*(p)) - p_0)^T.
\]

Notice that the mean curvature flow always blows up at finite time. For noncompact hypersurfaces, solution to the mean curvature flow may exist for all times. For example, Ecker and Huisken [8] showed that the mean curvature flow on locally Lipschitz continuous entire graph in Euclidean space exists for all time. The self-expanders appear as the singularity model of the mean curvature flow which exists for long time. For the entire graphs have the bounded gradient

\[
(\langle x_0, \nu \rangle)^2 \leq c(1 + |x_0|^2)^{1-\delta}
\]

at time $t = 0$, where $c < \infty$ and $\delta > 0$, Ecker and Huisken [7] proved the normalized mean curvature flow
\[
\frac{\partial \tilde{x}}{\partial s} = \tilde{H} - \tilde{x},
\]

with initial data $x_0$, obtained under the rescaling
\[
\tilde{x}(\cdot, s) = \frac{1}{\sqrt{2t+1}}x(\cdot, t), \quad s = \frac{1}{2}\log(2t+1),
\]

converges as $s \to \infty$ to a self-expander.

In this paper, we study the following generalized version of self-expanders for the mean curvature flow:

**Definition 1.2.** The immersed submanifold $x : M^n \to \mathbb{R}^{n+m}$ is called generalized self-expander type submanifold if it satisfies
\[
\tilde{H} \geq \mu(x - p_0)^2,
\]

We first prove the following mean value inequality.

**Theorem 1.3.** Let $x : M^n \to \mathbb{R}^{n+m}$ be the submanifold in the Euclidean space satisfying $\tilde{H} \cdot (x - p_0) \geq \mu |(x - p_0)^{\perp}|^2$ for some fixed vector $p_0 \in \mathbb{R}^{n+m}$ and nonnegative constant $\mu \geq 0$. Set $M = x(M^n)$. Then
\[
\frac{d}{dR} \left( \frac{1}{R^n} \int_{M \cap B_R(p_0)} f \right) \geq \frac{d}{dR} \int_{M \cap B_R(p_0)} \frac{|(x - p_0)^{\perp}|^2}{|x - x_0|^2} f + \frac{1}{2R^{n+1}} \int_{M \cap B_R(p_0)} (R^2 - |x - p_0|^2) \Delta f
\]
\[
+ \frac{\mu}{R^{n+1}} \int_{M \cap B_R(p_0)} |(x - p_0)^{\perp}|^2 f,
\]

for any smooth nonnegative function $f$ on $M$. Moreover, the equality of (1.3) holds if and only if $x$ satisfies $\tilde{H} \cdot (x - p_0) = \mu |(x - p_0)^{\perp}|^2$. 
Next we give an application to the mean value inequality (1.8). Recall that if \( x_0 \) is the graphical cone, then the solution to the mean curvature flow (1.1) must be the self-expander. The argument is this (see [14]): the rescaled flow \( x_j(t, t) = \sqrt{\lambda_j} x(\cdot, t_j \lambda_j t) \) solves (1.1) with the same initial condition \( x_0 \) since \( x_0 \) is a cone, so it must be equal to \( x(t) \) by the uniqueness of the solution of graphically initial data (if the uniqueness fails, then one may not have the mean curvature flow coming out of cone is the self-expander, see [1]). On the contrary, a natural question is that on what conditions the corresponding mean curvature flow (1.3) for the self-expanders converges to the cone as \( t \to 0 \)?

**Theorem 1.4.** Let \( x : M^n \to \mathbb{R}^{n+m} \) be the self-expander (1.7) in the Euclidean space. Let \( x(t) \) be the corresponding solution (1.3) to mean curvature flow for the self-expander \( x \). Set \( M(t) = x(M^n, t) \). If \( M(t) \) subconverges to an n-rectifiable varifold \( T \) in weak sense for some sequence \( t_j \to 0 \), then \( T \) must be a cone.

The structure of this paper is as follows. In section 2, we give the proof of Theorem 1.3. Then we give two corollaries of Theorem 1.3 as the direct applications to Theorem 1.3. In section 3, we give the proof of Theorem 1.4.

2. **Monotonicity Formula and Mean Value Inequality**

First we give the proof of Theorem 1.3.

**Proof of Theorem 1.3.** With losing of generality, we may assume \( p_0 = 0 \) and \( B_R(p_0) = B_R(0) \). We have

\[
\int_{M \cap B_R(p_0)} \text{div} \left( \frac{x}{R} \right) f = -\int_{M \cap B_R(p_0)} \tilde{H} \cdot \frac{x}{R} f + \int_{M \cap \partial B_R(p_0)} \frac{x}{R} f \cdot \nu \\
\leq -\mu \int_{M \cap B_R(p_0)} \frac{|x|^2}{R} f + \int_{M \cap \partial B_R(p_0)} f |\nabla|x||, 
\]

where we use \( \nu = \frac{x}{|x|} \) and \( \frac{x}{R} \cdot \nu = |\nabla|x|| \). Then by coarea formula we have

\[
\frac{d}{dR} \int_{M \cap B_R(p_0)} f = \int_{M \cap \partial B_R} \frac{f}{|\nabla|x||} \\
\geq \int_{M \cap \partial B_R} \frac{f}{|\nabla|x||} \left[ \frac{1}{R} - \frac{|\nabla|x|^2}{|\nabla|x||} \right] + \int_{M \cap B_R(p_0)} \text{div} \left( \frac{x}{R} \right) f + \mu \int_{M \cap B_R(p_0)} \frac{|x|^2}{R} f \\
= \frac{d}{dR} \int_{M \cap B_R} f \frac{|x|^2}{|x|^2} + \frac{n}{R} \int_{M \cap B_R(p_0)} f + \int_{M \cap B_R(p_0)} \frac{x}{R} \cdot \nabla f \\
+ \mu \int_{M \cap B_R(p_0)} \frac{|x|^2}{R} f, 
\]
where we use \( 1 - |\nabla x|^2 = \frac{1}{|p|^2} \). Since \( x^F = -\frac{1}{2} \nabla (R^2 - |x|^2) \), we conclude that
\[
\frac{d}{dR} \int_{M \cap B_R(p_0)} f \geq \frac{d}{dR} \int_{M \cap B_R} f \frac{|x|^2}{|x|^2} + \frac{n}{R} \int_{M \cap B_R(p_0)} f + \frac{1}{2R} \int_{M \cap B_R(p_0)} (R^2 - |x|^2)\Delta f
\]
\[
+ \mu \int_{M \cap B_R(p_0)} \frac{|x|^2}{R} f.
\]

It follows that
\[
\frac{d}{dR} \left( \frac{1}{R^n} \int_{M \cap B_R(p_0)} f \right) \geq \frac{d}{dR} \int_{M \cap B_R} f \frac{|x|^2}{|x|^2} + \frac{1}{2R^{n+1}} \int_{M \cap B_R(p_0)} (R^2 - |x|^2)\Delta f
\]
\[
+ \frac{\mu}{R^{n+1}} \int_{M \cap B_R(p_0)} |x|^2 f.
\]

In view of (2.1), the equality of (1.8) holds if and only if \( \int_{M \cap B_R(p_0)} \tilde{H} \cdot \frac{|x|^2}{R} f = \mu \int_{M \cap B_R(p_0)} \frac{|x|^2}{R} f \) for any smooth function \( f \) on \( M \). It follows that (1.8) holds if and only if \( x \) satisfying \( \tilde{H} \cdot (x - p_0) = \mu |(x - p_0)|^2 \).

As the corollary to Theorem 1.3, we have the following mean value inequality.

**Corollary 2.1.** Let \( x : M^n \to \mathbb{R}^{n+m} \) be the submanifold in the Euclidean space satisfying and \( \tilde{H} \cdot (x - p_0) \geq \mu |(x - p_0)|^2 \) for some fixed vector \( p_0 \in \mathbb{R}^{n+m} \) and nonnegative constant \( \mu \geq 0 \). Assuming that \( B_{R_0}(p_0) \cap M \neq \emptyset \) with \( M = x(M^n) \). If \( f \) is a nonnegative function with \( \Delta f \geq -(\lambda + 2\mu |x - p_0|^2)R_0^{-2} f \), then the function
\[
g(R) = e^{\frac{\lambda R^2}{2R_0^2}} \int_{M \cap B_R(p_0)} f
\]

is monotone non-decreasing for any \( 0 \leq R \leq R_0 \). In particular, if \( p_0 \in M \), then
\[
f(p_0) \leq e^{\frac{\lambda}{\text{Vol}(B_1 \subset \mathbb{R}^n)}R_0^{-2} f}
\]

**Proof.** It follows from (1.8) that
\[
g'(R) \geq -\frac{\lambda}{2} R_0^{-2} R^{1-n} \int_{M \cap B_R(p_0)} f = -\frac{\lambda}{2} R_0^{-2} R g(R).
\]

We have \( g'(R) \geq -\frac{\lambda}{2} R_0^{-2} R \geq -\frac{\lambda}{2R_0} g(R) \). Then (2.2) and (2.3) follow immediately. \( \square \)

For the another corollary to Theorem 1.3, by taking \( f = 1 \) in (1.8), we have
Corollary 2.2. Let $x: M^n \to \mathbb{R}^{n+m}$ be the submanifold in the Euclidean space satisfying $\tilde{H} \cdot (x - p_0) \geq \mu |(x - p_0)^{+}|^2$ for some fixed vector $p_0 \in \mathbb{R}^{n+m}$ and nonnegative constant $\mu \geq 0$. Then
\[
\frac{d}{dR} \left( \frac{H''(M \cap B_R(p_0))}{R^n} \right) \geq \frac{d}{dR} \int_{M \cap B_R(p_0)} \frac{|(x - p_0)^{+}|^2}{|x - p_0|^2} + \frac{\mu}{R^{n+1}} \int_{M \cap B_R(p_0)} |(x - p_0)^{+}|^2,
\]
with the equality holds if and only if $x$ satisfies $\tilde{H} \cdot (x - p_0) = \mu |(x - p_0)^{+}|^2$.

3. PROOF OF THEOREM 1.4

Before the proof of Theorem 1.4, we need the following lemma.

Lemma 3.1. Let $x: M^n \to \mathbb{R}^{n+m}$ be the submanifold in the Euclidean space satisfying $\tilde{H} \cdot (x - p_0) \geq 0$ for some fixed vector $p_0 \in \mathbb{R}^{n+m}$. If $\lambda_j(M - p_0)$ subconverges an $n$-rectifiable $T$ in weak sense for some sequence $\lambda_j \to 0$, then $T$ is an $n$-rectifiable cone.

Proof. With losing of generality, we may assume $p_0 = 0$ and $B_R(p_0) = B_R(0)$. We first prove that the varifold $T$ satisfies the following monotonicity formula
\[
t^{-n} \mu_T(B_t(0)) - s^{-n} \mu_T(B_s(0)) \geq \int_{(B_t(0) \setminus B_s(0)) \times G(n,n+m)} r^{-n} |\nabla \omega^N|^2 dT(x, \omega),
\]
for any $0 < s < t$, where $r = |x|$ and $\omega^N$ denote the orthogonal $m$-plane to $\omega$. The proof of (3.1) is similar to the case of stationary varifolds (see Proposition 3.7 in).

Let $\phi$ be a nonnegative cutoff function with $\phi'(s) \leq 0$ which is one on $[0, \frac{1}{2}]$ and supported on $[0, 1]$. Denote $\eta(r) = \phi(\frac{r}{s})$ so that $r\eta'(r) = -s \frac{d}{dr} (\phi(\frac{r}{s}))$. Since $\lambda_j M$ subconverges to $T$ in weak sense and $\tilde{H} \cdot x \geq 0$ on $M$, we have
\[
\int \text{div}_\omega(\eta(r)x) dT(x, \omega) = \lim_{j \to \infty} \int \text{div}_\omega(\eta(r)x) dT_j(x, \omega) = -\lim_{j \to \infty} \int \tilde{H}_j \cdot \eta(r)x d\mu_j \leq 0,
\]
where $\tilde{H}_T$ is the the mean curvature vector with respect to the varifold $T$ and $\tilde{H}_j$ is the the mean curvature vector with respect to the varifold $\lambda_j M$. Then
\[
0 \geq \int \text{div}_\omega(\eta(r)x) dT(x, \omega)
 = \int (m\eta(r) + r\eta'(r)) dT(x, \omega) - \int r\eta'(r) |\nabla \omega^N r|^2 dT(x, \omega).
\]
It follows that
\[ \int n\phi(r) - s \frac{d}{ds}(\phi(r))dT(x, \omega) \leq -s \int \frac{d}{ds}(\phi(r))|\nabla \omega| r^2 dT(x, \omega). \]
Hence
\[ \frac{d}{ds}(s^n - s \int \phi(r) dT(x, \omega)) \geq s^{-n} \frac{d}{ds} \int \phi(r)|\nabla \omega| r^2 dT(x, \omega). \]
Let \( \phi \) increase to the characteristic function of \([0, 1]\), we conclude (3.1) holds. From Corollary 2.2 and taking \( \mu = 0 \), we get \( t^{-n} \mu_T(B_t(0)) \equiv C \). Then \( T \) is the cone in view of (3.1).

Now we give the proof of Theorem 1.4.

**Proof of Theorem 1.4.** Since \( \phi_t \) are the tangent diffeomorphisms on \( M^n \), we
\[ M(t) = x(M^n, t) = \sqrt{1 + 2\mu(t - \frac{1}{2\mu})(x(\phi_t^*((M^n)) - p_0)) = \sqrt{1 + 2\mu(t - \frac{1}{2\mu})(x(M^n) - p_0)}. \]
Then Theorem 1.4 holds by Lemma 3.1.

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**References**

[1] S.B. Angenent, D.L. Chopp, and T. Ilmanen. *A computed example of nonuniqueness of mean curvature flow in R3*. Comm. Partial Differential Equations, 20 (1995), no. 11-12, 1937C1958
[2] J. Bode; *Mean Curvature Flow of Cylindrical Graphs*. Ph.D. thesis (2007), Freie Universität Berlin, Universitätbibliothek
[3] Cao H. D., Li H. *A gap theorem for self-shrinkers of the mean curvature flow in arbitrary codimension*. Calculus of Variations and Partial Differential Equations, 2013, 46(3-4): 879-889.
[4] Cheng, L., Sesum N. *Asymptotic behavior of Type III mean curvature flow on noncompact hypersurfaces*. arXiv preprint [arXiv:1403.0235], 2014.
[5] Cheng, L.; Zhu, A *On the weighted forward reduced volume of Ricci flow*. Proc. Amer. Math. Soc. 141 (2013), no. 8, 2859-2868.
[6] Colding T. H, Minicozzi II W.P. *Generic mean curvature flow I; generic singularities*. Annals of Mathematics, 2012, 175(2): 755-833.
[7] K. Ecker, G. Huisken, *mean curvature evolution of entire graphs*. Ann. Math. 130 (1989), 453-471
[8] K. Ecker, G. Huisken, *Interior estimates for hypersurfaces moving by mean curvature*. Invent. Math. 105 (1991), 547-569
[9] G. Huisken, *Asymptotic behavior for singularities of the mean curvature flow*. J. Differential Geom. 31 (1990), no. 1, 285-299.
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[10] G. Huisken and C. Sinestrari, *Mean curvature flow singularities for mean convex surface*, Calc. Var. PDE, 8(1999), 1-14.

[11] G. Huisken and C. Sinestrari, *Convexity estimates for mean curvature flow and singularities of mean convex surfaces*, Acta Math., 183(1999), 47-70.

[12] Tom Ilmanen, *Singularities of mean curvature flow of surfaces*, preliminary version, available under [http://www.math.ethz.ch/~ilmanen/papers/sing.ps](http://www.math.ethz.ch/~ilmanen/papers/sing.ps).

[13] Tom Ilmanen, *Elliptic regularization and partial regularity for motion by mean curvature*, Mem. Amer. Math. Soc. 108 (1994), no. 520, x90.

[14] Tom Ilmanen, *Lectures on mean curvature flow and related equations*, lecture notes, ICTP, Trieste, 1995, [http://www.math.ethz.ch/Ilmanen/papers/pub.htm](http://www.math.ethz.ch/Ilmanen/papers/pub.htm).

[15] Ma L, Vicente M. *Bernstein theorem for translating solitons of hypersurfaces*. arXiv preprint arXiv:1405.3042, 2014.

[16] Martin F, Savas-Halilaj A, Smoczyk K. *On the topology of translating solitons of the mean curvature flow*. arXiv preprint arXiv:1404.6703, 2014.

[17] B. White, *Stratification of minimal surfaces, mean curvature flows, and harmonic maps*. J. Reine Angew. Math. 488 (1997), 1-35.