THREE SPHERES THEOREM FOR p-HARMONIC FUNCTIONS

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Abstract. Three spheres theorem type result is proved for the p-harmonic functions defined on the complement of k-balls in the Euclidean n-dimensional space.

1. Introduction

A classical theorem by J. Hadamard gives the following relation between the maximum absolute values of an analytic function on three concentric circles.

1.1. Theorem. Let \( R_1 < r_1 < r_2 < r_3 < R_2 \) and let \( f \) be an analytic function in the annulus \( \{ z \in \mathbb{C} : R_1 < |z| < R_2 \} \). Denote the maximum of \( |f(z)| \) on the circle \( |z| = r \) by \( M(r) \). Then

\[
M(r_2)^{\log(r_3/r_1)} \leq M(r_1)^{\log(r_3/r_2)} M(r_3)^{\log(r_2/r_1)}.
\]

This result, known as the three circles theorem, was given by Hadamard without proof in 1896 [3]. For a discussion of the history of this result, see e.g. [8] and [5, pp. 323–325]. It is a natural question, what results of this type can be proved for other classes of functions. For example, a version of Hadamard’s theorem can be proved for subharmonic functions in \( \mathbb{R}^n, n \geq 2 \), see [7, pp. 128–131].

Some generalizations of the three circles theorem will be studied here. For the formulation of our main result, Theorem 2.1, we recall some standard notation and definitions from the book [4]. We will consider solutions \( v : \Omega \to \mathbb{R} \) of the \( p \)-Laplace equation

\[
\text{div} \left( |\nabla v|^{p-2} \nabla v \right) = 0, \quad 1 < p < \infty,
\]
on an open set \( \Omega \subset \mathbb{R}^n \) in the sense that will be described shortly. When \( p = 2 \) equation (1.2) reduces to the Laplace equation \( \Delta u = 0 \), whose solutions, harmonic functions, are studied in the classical potential theory. When \( p \neq 2 \) equation (1.2) is nonlinear and degenerates at the zeros of the gradient of \( v \). It follows that the solutions, \( p \)-harmonic functions, need not be in \( C^2(\Omega) \) and the equation must be understood in the weak sense. A weak solution of (1.2) is a function \( v \) in the Sobolev space \( W^{1,p}_{\text{loc}}(\Omega) \) such that

\[
\int_{\Omega} \langle |\nabla v|^{p-2} \nabla v, \nabla \varphi \rangle \, dm = 0
\]
for all \( \varphi \in C_0^\infty(\Omega) \), where \( \langle \cdot, \cdot \rangle \) denotes the scalar product of vectors in \( \mathbb{R}^n \), and \( m \) is the Lebesgue measure in \( \mathbb{R}^n \).

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Let 0 < α < β < ∞ be fixed and let
\[ D_{α, β} = \{ x \in \mathbb{R}^n : α < d_k(x) < β \}. \]

For k = 1 the set \( D_{α, β} \) is the union of the two layers between two parallel hyperplanes. For 1 < k < n the boundary of the domain \( D_{α, β} \) consists of two coaxial cylindrical surfaces.

Let \( v \in C^0(D_{r,R}) \), and let \( M(r) = \limsup_{z \to \Sigma_k(r)} v(z) \). Suppose that \( M(R) > M(r) \). Consider the function
\[ v_{r,R}(x) = \frac{v(x) - M(r)}{M(R) - M(r)}, \]
for \( r < R \). Clearly, \( \limsup_{z \to \Sigma_k(r)} v_{r,R}(z) \leq 0 \) and \( \limsup_{z \to \Sigma_k(R)} v_{r,R}(z) \leq 1 \). Let \( \xi(r, t) = \int_r^t s^{(1-k)/(p-1)} ds \), and \( u_0^{k,p}(t) = \frac{\xi(r, t)}{\xi(r, R)} \).

Let \( u(x) = u_0^{k,p}(d_k(x)) \) for \( x \in D_{r,R} \). It is clear (see Lemma 3.5) that \( u \) is a \( C^2 \)-solution to (1.2). We have
\[ u(x)|_{\Sigma_k(r)} \equiv 0, \quad u(x)|_{\Sigma_k(R)} \equiv 1, \]
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and

\( u(x) \geq v_{r,R}(x) \) if \( x \in \Sigma_k(r) \) or \( x \in \Sigma_k(R) \).

2. Main results

We will prove the following Hadamard type theorem for the $p$-harmonic functions defined on the complement of a $k$-ball. We use the method of proof from [6].

2.1. Theorem. Let \( 1 < p < \infty \), \( R > r > 0 \) and let \( v(x) \in W^{1,p}_{\text{loc}}(D_{r,\infty}) \) be a continuous weak solution of (1.2) such that

\[
\int_0^\infty dt \left( \int_{\Sigma_k(t)} |v_{r,R} - u|^2 \left( |\nabla v_{r,R}|^{p-2} + |\nabla u|^{p-2} \right) d\mathcal{H}^{n-1} \right)^{-1} = \infty,
\]

where \( \mathcal{H}^{n-1} \) is the \((n-1)\)-dimensional Hausdorff measure. Then for all \( t \in (r,R) \),

\[
M(t) \leq (M(R) - M(r)) u_0^{k,p}(t) + M(r).
\]

Note that for \( k = n \) (2.3) follows immediately from the comparison principle, see [4, p. 133].

2.4. Corollary. Let \( 1 < p < \infty \), \( R > r > 0 \) and let \( v(x) \in W^{1,p}_{\text{loc}}(D_{r,\infty}) \), be a continuous weak solution of (1.2) such that

\[
\lim_{S \to \infty} \frac{1}{S^2} \int_{D_{r,S}} |v_{r,R} - u|^2 \left( |\nabla v_{r,R}|^{p-2} + |\nabla u|^{p-2} \right) dm = 0.
\]

Then for all \( t \in (r,R) \) the inequality (2.3) holds.

2.6. Corollary. Let \( 1 < p < \infty \), \( R > r > 0 \) and let \( v(x) \in W^{1,p}_{\text{loc}}(D_{r,\infty}) \) be a continuous weak solution of (1.3) such that

\[
\int_{D_{r,\infty}} |v_{r,R} - u|^2 \left( |\nabla v_{r,R}|^{p-2} + |\nabla u|^{p-2} \right) dm \leq M < \infty.
\]

Then for all \( t \in (r,R) \) the inequality (2.3) holds.

For the formulation of a result of S. Granlund [2], Theorem 2.7 below, we introduce some notation and terminology. Let \( p > 1 \), \( \Omega \subset \mathbb{R}^n \) be a bounded domain and let \( F: \Omega \times \mathbb{R}^n \to \mathbb{R} \) be such that the following conditions hold.

1. There are constants \( \beta > \alpha > 0 \) such that for a.e. \( x \in \Omega \)

\[
\alpha |z|^p \leq F(x,z) \leq \beta |z|^p.
\]

2. For a.e. \( x \in \Omega \) the function \( z \mapsto F(x,z) \) is convex.

3. The function \( x \mapsto F(x,\nabla u(x)) \) is measurable for all \( u \in W^{1,p}(\Omega) \).

Let

\[
I(u) = \int_\Omega F(x,\nabla u(x)) \, dm.
\]

A function \( u \in W^{1,p}(\Omega) \) is a subminimum in \( \Omega \) if \( I(u) \leq I(u - \eta) \) for all non-negative \( \eta \in W^{1,p}_0(\Omega) \). Let

\[
M(r) = \text{ess sup}_{x \in \overline{B}^\Omega(r)} u(x), \quad \overline{B}^\Omega(r) \subset \Omega.
\]

The following Hadamard type theorem was proved by S. Granlund in [2].
2.7. Theorem. Let $u$ be a subminimum of
\[ I(u) = \int_{\Omega} F(x, \nabla u(x)) \, dm, \]
r_1 < r < r_2, and $\overline{B}^n(r_2) \subset \Omega$. Then $u$ is bounded from above, and there is a constant
\[ \lambda = \lambda(n, p, r/r_1, r_2/r, \alpha/\beta), \]
$0 < \lambda < 1$ such that
\[ M(r) \leq \lambda M(r_1) + (1 - \lambda) M(r_2). \]

Since $p$-harmonic functions minimize (see e.g. [4, p. 59]) the integral
\[ I(u) = \int_{\Omega} |\nabla u|^p \, dm, \]
Theorem 2.7 is related to Theorem 2.1 with $k = n$.

3. Preliminaries

We start by recalling some basic properties of the Sobolev spaces from [4]. Let $\Omega$ be a nonempty open set in $\mathbb{R}^n$.

3.1. Lemma. [4, Theorem 1.24] Let $u \in W^{1,p}_0(\Omega)$ and $v \in W^{1,p}(\Omega)$ be bounded. Then $uv \in W^{1,p}_0(\Omega)$.

3.2. Lemma. [4, Lemma 3.11] If $v \in W^{1,p}(\Omega)$ is a weak solution of (1.2) in $\Omega$, then
\[ \int_{\Omega} \langle |\nabla v|^{p-2}\nabla v, \nabla \varphi \rangle \, dm = 0 \]
for all $\varphi \in W^{1,p}_0(\Omega)$.

3.3. Theorem. [1, p. 99] Let $f : \mathbb{R}^n \to \mathbb{R}$ be a locally Lipschitz mapping. Let $E \subset \mathbb{R}^n$ be an $n$-measurable set and $g : E \to \mathbb{R}$ be a nonnegative measurable function. Then
\[ \int_{E} g(x)|\nabla f(x)| \, dx_1 \cdots dx_n = \int_{\mathbb{R}} \left( \sum_{x \in f^{-1}(y)} g(x) \right) d\mathcal{H}^n(y). \]

3.5. Lemma. Let $1 < p < \infty$, $0 < r < d_k(x)$ and fix an integer $1 \leq k \leq n$. Then
\[ u(x) = \int_{r}^{d_k(x)} \frac{1}{s^{n-k}} \, ds \]
is a solution of (1.2), i.e.
\[ \sum_{i=1}^{n} \left\{ \frac{\partial}{\partial x_i} \left( u_{x_i} u_{x_1}^2 + \ldots + u_{x_n}^2 \frac{x_i}{x_1} \right) \right\} = 0. \]

Proof. We note that
\[ \frac{\partial}{\partial x_i} d_k(x) = \frac{x_i}{d_k(x)}, \]
and hence \( u_{x_i} = x_i d_k(x)^{\frac{1-k}{p-1}-1} \). Then
\[
\left( u_{x_1}^2 + \ldots + u_{x_k}^2 \right)^{\frac{p-2}{2}} = x_i d_k(x)^{\frac{1-k}{p-1}-1} \left[ d_k(x)^{\frac{2(1-p)}{p-1}} - 2 \left( \sum_{j=1}^k x_j^2 \right) \right]^{\frac{p-2}{2}} = x_i d_k(x)^{\frac{1-k}{p-1}-1} d_k(x) \frac{(1-k)(p-2)}{p-1} = x_i d_k(x)^{-k}.
\]
It follows that
\[
\sum_{i=1}^k \partial \frac{d_k(x)^{-k}}{d \partial x_i} = \sum_{i=1}^k d_k(x)^{-k} - k \sum_{i=1}^k x_i^2 d_k(x)^{-k-2} = kd_k(x)^{-k} - kd_k(x)^{-k-2} \left( \sum_{i=1}^k x_i^2 \right) = 0.
\]

Next we will prove two lemmas which are used later in the proof of Theorem 2.1.

3.6. Lemma. Let \( a > b > 0, p > 1 \). Then
\[
C_1 \frac{a^{p-1} - b^{p-1}}{a - b} \leq \frac{a^{p-1} + b^{p-1}}{a + b} \leq C_2 \frac{a^{p-1} - b^{p-1}}{a - b},
\]
with some constants \( C_1, C_2 > 0 \).

Proof. We examine the function
\[
g_1(x) = \frac{(x^{p-1} + 1)(x - 1)}{(x^{p-1} - 1)(x + 1)}, \quad x > 1.
\]
It is clear that
\[
\lim_{x \to 1} g_1(x) = \frac{1}{p-1}, \quad \lim_{x \to \infty} g_1(x) = 1.
\]
It is sufficient to find positive bounds for \( g_1(x) \) for \( x > 1 \). We will prove that the bounds are in fact given by (3). First we note that
\[
\begin{cases}
(p - 2)(x^p - 1) + p(x - x^{p-1}) < 0, & \text{for } p \in (1, 2), \\
(p - 2)(x^p - 1) + p(x - x^{p-1}) = 0, & \text{for } p = 2, \\
(p - 2)(x^p - 1) + p(x - x^{p-1}) > 0, & \text{for } p > 2,
\end{cases}
\]
and
\[
\begin{cases}
x - x^{p-1} < 0, & \text{for } p \in (1, 2), \\
x - x^{p-1} = 0, & \text{for } p = 2, \\
x - x^{p-1} > 0, & \text{for } p > 2.
\end{cases}
\]
Hence
\[
\begin{cases}
g_1(x) \in (1, 1/(p-1)), & \text{for } p \in (1, 2), \\
g_1(x) = 1, & \text{for } p = 2, \\
g_1(x) \in (1/(p-1), 1), & \text{for } p > 2.
\end{cases}
\]

\( \square \)
3.9. Lemma. Let $a > b > 0$. Then

\[ C_3(a^{p-2} + b^{p-2}) \leq \frac{a^{p-1} - b^{p-1}}{a-b} \leq C_4(a^{p-2} + b^{p-2}), \]

for $p \geq 2$, and

\[ C_3(a^{2-p} + b^{2-p})^{-1} \leq \frac{a^{p-1} - b^{p-1}}{a-b} \leq C_4(a^{2-p} + b^{2-p})^{-1}, \]

for $p \in (1,2]$ with some constants $C_3, C_4 > 0$.

Proof. The proof is similar to that of Lemma 3.6. First we study the function

\[ g_2(x) = \frac{x^{p-1} - 1}{(x-1)(x^{p-2} + 1)}. \]

As in Lemma 3.6 it is sufficient for (3.10) to find positive bounds for $g_2(x)$ for $x > 0$.

We note that $\lim_{x \to 1} g_2(x) = (p-1)/2$ and $\lim_{x \to \infty} g_2(x) = 1$. We obtain

\[
\begin{cases}
(p-3)(1 - x^{p-1}) + (p-1)x(1 - x^{p-3}) < 0, & \text{for } p \in (1,3), \\
(p-3)(1 - x^{p-1}) + (p-1)x(1 - x^{p-3}) = 0, & \text{for } p = 3, \\
(p-3)(1 - x^{p-1}) + (p-1)x(1 - x^{p-3}) > 0, & \text{for } p > 3,
\end{cases}
\]

and

\[
\begin{cases}
x(x^{p-3} - 1) < 0, & \text{for } p \in (1,3), \\
x(x^{p-3} - 1) = 0, & \text{for } p = 3, \\
x(x^{p-3} - 1) > 0, & \text{for } p > 3.
\end{cases}
\]

It follows that

\[
\begin{cases}
g_2(x) \in ((p-1)/2,1), & \text{for } p \in (1,3), \\
g_2(x) = 1, & \text{for } p = 3, \\
g_2(x) \in (1,(p-1)/2), & \text{for } p > 3.
\end{cases}
\]

To prove (3.11) we study the function

\[ g_3(x) = \frac{(x^{p-1} - 1)(x^{2-p} + 1)}{x-1}. \]

Now $\lim_{x \to 1} g_3(x) = 2(p-1)$ and $\lim_{x \to \infty} g_3(x) = 1$. Again, we have

\[
\begin{cases}
(-2p + 3)(x - 1) + (x^{p-1} - x^{2-p}) < 0, & \text{for } p \in (1,3/2), \\
(-2p + 3)(x - 1) + (x^{p-1} - x^{2-p}) = 0, & \text{for } p = 3/2, \\
(-2p + 3)(x - 1) + (x^{p-1} - x^{2-p}) > 0, & \text{for } p > 3/2,
\end{cases}
\]

and

\[
\begin{cases}
x^{p-1} - x^{2-p} < 0, & \text{for } p \in (1,3/2), \\
x^{p-1} - x^{2-p} = 0, & \text{for } p = 3/2, \\
x^{p-1} - x^{2-p} > 0, & \text{for } p > 3/2,
\end{cases}
\]

and thus

\[
\begin{cases}
g_3(x) \in (2(p-1),1), & \text{for } p \in (1,3/2), \\
g_3(x) = 1, & \text{for } p = 3/2, \\
g_3(x) \in (1,2(p-1)), & \text{for } p > 3/2.
\end{cases}
\]
4. Proof of Theorem 2.1

Suppose the contrary, that is, there exists \( x_0 \in D_{r,R} \) such that

\[
(4.1) \quad v(x_0) > (M(R) - M(r))u(x_0) + M(r),
\]

or

\[
v_{r,R}(x_0) > u(x_0).
\]

Fix some \( \varepsilon_0 > 0 \), for which

\[
v_{r,R}(x_0) - u(x_0) > \varepsilon_0.
\]

Consider the set

\[
U = \{ x \in D_{r,R} : v_{r,R}(x) - u(x) > \varepsilon_0 \} \neq \emptyset.
\]

Choose a component \( O \) of \( U \) such that \( x_0 \in O \). It is clear that \( \overline{O} \cap \partial D_{r,R} = \emptyset \) and \( (v_{r,R}(x) - u(x))|_{\partial O} = 0 \). Fix \( \varepsilon_2 > \varepsilon_1 > 0 \) and the balls \( O_1 = B_k(x_0, \varepsilon_1), \)
\( O_2 = B_k(x_0, \varepsilon_2) \). Let \( \varphi(x) = \eta(d_k(x)) \) be a locally Lipschitz function with the properties:

\[
(4.2) \begin{cases} 
\varphi \equiv 1 & \text{for all } x \in O_1, \\
\varphi \equiv 0 & \text{for all } x \in D_{r,R} \setminus O_2.
\end{cases}
\]

Then the function \( \psi = (v_{r,R}(x) - u(x)) \varphi^2 \) has a support \( \text{supp} \psi \subset \overline{O}_2 \) and by Lemma 3.1 \( \psi \in W^{1,p}_0(\Omega) \) for all \( \Omega \) such that \( \text{supp} \psi \subset \Omega \). Since \( v_{r,R} \) and \( u \) are generalized solutions of (1.2) we have by Lemma 3.2

\[
0 = \int_{\text{supp} \psi} \langle \nabla \psi, |\nabla v_{r,R}|^{p-2}\nabla v_{r,R} - |\nabla u|^{p-2}\nabla u \rangle \, dm
\]

\[
= \int_{\text{supp} \psi} \langle \nabla \psi, |\nabla v_{r,R}|^{p-2}\nabla v_{r,R} \rangle \, dm - \int_{\text{supp} \psi} \langle \nabla \psi, |\nabla u|^{p-2}\nabla u \rangle \, dm = 0.
\]

Next, we note that

\[
\nabla \psi = \varphi^2(\nabla v_{r,R} - \nabla u) + 2\varphi(v_{r,R} - u)\nabla \varphi.
\]

Thus, we may write

\[
0 = \int_{\text{supp} \psi} \langle \nabla \psi, |\nabla v_{r,R}|^{p-2}\nabla v_{r,R} - |\nabla u|^{p-2}\nabla u \rangle \, dm
\]

\[
= \int_{O \cap O_2} \langle \nabla \psi, |\nabla v_{r,R}|^{p-2}\nabla v_{r,R} - |\nabla u|^{p-2}\nabla u \rangle \, dm
\]

\[
= \int_{O \cap O_2} \varphi^2(\nabla v_{r,R} - \nabla u, |\nabla v_{r,R}|^{p-2}\nabla v_{r,R} - |\nabla u|^{p-2}\nabla u) \, dm
\]

\[
+ 2\int_{O \cap O_2} \varphi(v_{r,R} - u)\langle \nabla \varphi, |\nabla v_{r,R}|^{p-2}\nabla v_{r,R} - |\nabla u|^{p-2}\nabla u \rangle \, dm
\]
or
\[
\int_{O \cap O_2} \varphi^2 (\nabla v_{r,R} - \nabla u, |\nabla v_{r,R}|^{p-2} \nabla v_{r,R} - |\nabla u|^{p-2} \nabla u) \, dm
\]
\[
= -2 \int_{O \cap O_2} \varphi (v_{r,R} - u) (\nabla \varphi, |\nabla v_{r,R}|^{p-2} \nabla v_{r,R} - |\nabla u|^{p-2} \nabla u) \, dm
\]
or
\[
(4.3) \quad \left| \int_{O \cap O_2} \varphi^2 (\nabla v_{r,R} - \nabla u, |\nabla v_{r,R}|^{p-2} \nabla v_{r,R} - |\nabla u|^{p-2} \nabla u) \, dm \right|
\]
\[
\leq 2 \int_{O \cap O_2} |\varphi| (v_{r,R} - u) |\nabla \varphi| |\nabla v_{r,R}|^{p-2} \nabla v_{r,R} - |\nabla u|^{p-2} \nabla u | \, dm.
\]
Let
\[
\Phi(\lambda) = |\nabla (\lambda v_{r,R} + (1 - \lambda)u)|^{p-2} \nabla (\lambda v_{r,R} + (1 - \lambda)u)
\]
for \( \lambda \in [0, 1] \), and note that
\[
\Phi(0) = |\nabla u|^{p-2} \nabla u \quad \text{and} \quad \Phi(1) = |\nabla v_{r,R}|^{p-2} \nabla v_{r,R}.
\]
Now we write
\[
(4.4) \quad |\nabla v_{r,R}|^{p-2} \nabla v_{r,R} - |\nabla u|^{p-2} \nabla u = \Phi(1) - \Phi(0) = \int_0^1 \Phi'(\lambda) \, d\lambda
\]
\[
= \int_0^1 \left[ (\nabla v_{r,R} - \nabla u) |\nabla (\lambda v_{r,R} + (1 - \lambda)u)|^{p-2} + (p - 2) \nabla (\lambda v_{r,R} + (1 - \lambda)u) \right.
\]
\[
\left. \cdot |\nabla (\lambda v_{r,R} + (1 - \lambda)u)|^{p-4} (\nabla v_{r,R} - \nabla u, \nabla (\lambda v_{r,R} + (1 - \lambda)u)) \right] \, d\lambda,
\]
and obtain
\[
(4.5) \quad \langle \nabla v_{r,R} - \nabla u, |\nabla v_{r,R}|^{p-2} \nabla v_{r,R} - |\nabla u|^{p-2} \nabla u \rangle
\]
\[
= |\nabla v_{r,R} - \nabla u|^2 \int_0^1 |\nabla (\lambda v_{r,R} + (1 - \lambda)u)|^{p-2} \, d\lambda
\]
\[
+ (p - 2) \int_0^1 |\nabla (\lambda v_{r,R} + (1 - \lambda)u)|^{p-4} (\nabla v_{r,R} - \nabla u, \nabla (\lambda v_{r,R} + (1 - \lambda)u))^2 \, d\lambda.
\]
If \( p \geq 2 \) then
\[
(4.6) \quad \langle \nabla v_{r,R} - \nabla u, |\nabla v_{r,R}|^{p-2} \nabla v_{r,R} - |\nabla u|^{p-2} \nabla u \rangle
\]
\[
\geq |\nabla v_{r,R} - \nabla u|^2 \int_0^1 |\nabla (\lambda v_{r,R} + (1 - \lambda)u)|^{p-2} \, d\lambda.
\]
If \( p < 2 \), we have
\[
|\nabla v_{r,R} - \nabla u|^2 \int_0^1 |\nabla (\lambda v_{r,R} + (1 - \lambda)u)|^{p-2} \, d\lambda
\]
\[+ (p-2) \int_0^1 |\nabla (\lambda v_{r,R} + (1 - \lambda)u)|^{p-4} \langle \nabla v_{r,R} - \nabla u, \nabla (\lambda v_{r,R} + (1 - \lambda)u) \rangle^2 \, d\lambda \]
\[\geq (p-1)|\nabla v_{r,R} - \nabla u|^2 \int_0^1 |\nabla (\lambda v_{r,R} + (1 - \lambda)u)|^{p-2} \, d\lambda.
\]
This together with (4.5) gives
\[
\langle \nabla v_{r,R} - \nabla u, |\nabla v_{r,R}|^{p-2} \nabla v_{r,R} - |\nabla u|^{p-2} \nabla u \rangle
\[\geq (p-1)|\nabla v_{r,R} - \nabla u|^2 \int_0^1 |\nabla (\lambda v_{r,R} + (1 - \lambda)u)|^{p-2} \, d\lambda, \quad 1 < p \leq 2.
\]
It follows from (4.4) that for every \( p > 1 \),
\[
|\nabla v_{r,R}|^{p-2} |\nabla v_{r,R} - |\nabla u|^{p-2} |\nabla u| \leq C_5 |\nabla v_{r,R} - \nabla u| \int_0^1 |\nabla (\lambda v_{r,R} + (1 - \lambda)u)|^{p-2} \, d\lambda,
\]
at every point where \( v_{r,R} \) has differential. Here \( C_5 = 1 + |p-2| \). Setting
\[
I(p) = \int_0^1 |\nabla (\lambda v_{r,R} + (1 - \lambda)u)|^{p-2} \, d\lambda
\]
and using (4.3), (4.6), (4.7) and (4.8) we obtain
\[
\int_{O \cap O_2} \varphi^2 I(p)|\nabla v_{r,R} - \nabla u|^2 \, dm \leq C_6 \int_{O \cap O_2} I(p)||\nabla \varphi|||\nabla v_{r,R} - \nabla u|| \, dm,
\]
where \( C_6 = 2C_5/\min\{1, p-1\} \).

We note that
\[
|\nabla (\lambda v_{r,R} + (1 - \lambda)u)|^2 = \lambda^2 |\nabla v_{r,R}|^2 + 2\lambda(1-\lambda)\langle \nabla v_{r,R}, \nabla u \rangle + (1 - \lambda)^2 |\nabla u|^2,
\]
and therefore
\[
|\lambda| |v_{r,R}| - (1 - \lambda) |\nabla u| \leq |\nabla (\lambda v_{r,R} + (1 - \lambda)u)| \leq |\nabla v_{r,R}| + (1 - \lambda)|\nabla u|
\]
for an arbitrary \( \lambda \in [0,1] \). Let \( p \geq 2 \). We suppose that \( |\nabla v_{r,R}| > |\nabla u| \). Then by (4.10),
\[
I(p) \leq \int_0^1 \left( \lambda(|\nabla v_{r,R}| - |\nabla u|) + |\nabla u| \right)^{p-2} \, d\lambda
\]
\[= \frac{1}{|\nabla v_{r,R}| - |\nabla u|} \int_{|\nabla u|}^{|
abla v_{r,R}|} s^{p-2} \, ds = \frac{1}{p-1} \frac{|\nabla v_{r,R}|^{p-1} - |\nabla u|^{p-1}}{|\nabla v_{r,R}| - |\nabla u|}.
\]
Next by (4.10),

\[ I(p) \geq \int_0^1 \left| \lambda |\nabla v_{r,R}| - (1 - \lambda) |\nabla u| \right|^{p-2} d\lambda \]

\[ = \int_0^1 \left| \lambda (|\nabla v_{r,R}| + |\nabla u|) - |\nabla u| \right|^{p-2} d\lambda \]

\[ = \int_0^1 \left( \lambda (|\nabla v_{r,R}| + |\nabla u|) - |\nabla u| \right)^{p-2} d\lambda \]

\[ + \int_0^s \left( |\nabla u| - \lambda (|\nabla v_{r,R}| + |\nabla u|) \right)^{p-2} d\lambda, \]

where

\[ s = \frac{|\nabla u|}{|\nabla v_{r,R}| + |\nabla u|}. \]

By computing both of the last two integrals, we obtain

\[ I(p) \geq \frac{1}{(p-1)} \frac{|\nabla v_{r,R}|^{p-1} + |\nabla u|^{p-1}}{|\nabla v_{r,R}| + |\nabla u|}. \]

Let \( 1 < p < 2 \). As above, we assume \( |\nabla v_{r,R}| > |\nabla u| \). Then by (4.10),

\[ I(p) \leq \int_0^1 \left| \lambda |\nabla v_{r,R}| - (1 - \lambda) |\nabla u| \right|^{2-p} d\lambda \]

\[ = \int_0^1 \left| \lambda (|\nabla v_{r,R}| + |\nabla u|) - |\nabla u| \right|^{2-p} d\lambda \]

\[ = \int_0^s \left( |\nabla u| - \lambda (|\nabla v_{r,R}| + |\nabla u|) \right)^{2-p} d\lambda \]

\[ + \int_0^1 \left( \lambda (|\nabla v_{r,R}| + |\nabla u|) - |\nabla u| \right)^{2-p} d\lambda, \]

where \( s \) is defined in (4.12), and hence

\[ I(p) \leq \frac{1}{(p-1)} \frac{|\nabla v_{r,R}|^{p-1} + |\nabla u|^{p-1}}{|\nabla v_{r,R}| + |\nabla u|}. \]

By (4.11), it follows that

\[ I(p) \geq \frac{1}{(p-1)} \frac{|\nabla v_{r,R}|^{p-1} - |\nabla u|^{p-1}}{|\nabla v_{r,R}| - |\nabla u|}. \]
Setting \(a = |\nabla v_{r,R}|\) and \(b = |\nabla u|\) in (3.7), (3.10) and (3.11), we can obtain by (4.11), (4.13), (4.14) and (4.15), for \(p \geq 2\)

\[
(4.16) \quad C_7 \left( |\nabla v_{r,R}|^{p-2} + |\nabla u|^{p-2} \right) \leq I(p) \leq C_8 \left( |\nabla v_{r,R}|^{p-2} + |\nabla u|^{p-2} \right),
\]

or

\[
(4.17) \quad C_7 \left( |\nabla v_{r,R}|^{2-p} + |\nabla u|^{2-p} \right)^{-1} \leq I(p) \leq C_8 \left( |\nabla v_{r,R}|^{2-p} + |\nabla u|^{2-p} \right)^{-1},
\]

where \(1 < p \leq 2\), with some constants \(C_7, C_8 > 0\). The case \(|\nabla v_{r,R}| < |\nabla u|\) is analogous.

Thus by (4.9), (4.18) we find,

\[
(4.19) \quad \int_{O_2} \phi^2 |\nabla v_{r,R} - \nabla u|^2 \left( |\nabla v_{r,R}|^{p-2} + |\nabla u|^{p-2} \right) \, dm
\]

\[
\leq C_{11} \int_{O_2} |\phi| |v_{r,R} - u| |\nabla \phi| |\nabla v_{r,R} - \nabla u| \left( |\nabla v_{r,R}|^{p-2} + |\nabla u|^{p-2} \right) \, dm
\]

\[
\leq C_{11} \left( \int_{O_2} |\nabla \phi|^2 |v_{r,R} - u|^2 \left( |\nabla v_{r,R}|^{p-2} + |\nabla u|^{p-2} \right) \, dm \right)^{1/2}
\]

\[
\cdot \left( \int_{O_2} \phi^2 |\nabla v_{r,R} - \nabla u|^2 \left( |\nabla v_{r,R}|^{p-2} + |\nabla u|^{p-2} \right) \, dm \right)^{1/2}
\]

and

\[
\int_{O_2} \phi^2 |\nabla v_{r,R} - \nabla u|^2 \left( |\nabla v_{r,R}|^{p-2} + |\nabla u|^{p-2} \right) \, dm
\]

\[
\leq C_{11}^2 \int_{O_2} |\nabla \phi|^2 |v_{r,R} - u|^2 \left( |\nabla v_{r,R}|^{p-2} + |\nabla u|^{p-2} \right) \, dm.
\]

Remembering (4.2) we have

\[
\int_{O_1} |\nabla v_{r,R} - \nabla u|^2 \left( |\nabla v_{r,R}|^{p-2} + |\nabla u|^{p-2} \right) \, dm
\]

\[
\leq C_{11}^2 \int_{D_{r,R}\cap (O_2 \setminus O_1)} |\nabla \phi|^2 |v_{r,R} - u|^2 \left( |\nabla v_{r,R}|^{p-2} + |\nabla u|^{p-2} \right) \, dm.
\]

Because \(\phi\) is constant on \(\Sigma_k(t)\) and \(|\nabla d_k| \equiv 1\), we have by Theorem 3.3

\[
\int_{D_{r,R}\cap (O_2 \setminus O_1)} |\nabla \phi|^2 |v_{r,R} - u|^2 \left( |\nabla v_{r,R}|^{p-2} + |\nabla u|^{p-2} \right) \, dm
\]

\[
\leq \int_{\{x : \varepsilon < d_k(x) < 2\varepsilon\}} |\nabla \phi|^2 |v_{r,R} - u|^2 \left( |\nabla v_{r,R}|^{p-2} + |\nabla u|^{p-2} \right) \, dm
\]

\[
= \int_{\varepsilon_1}^{2\varepsilon} \eta^2 H(t) \, dt,
\]
where

\begin{equation}
H(t) = \int_{\Sigma_k(t)} |v_{r,R} - u|^2 (|\nabla v_{r,R}|^{p-2} + |\nabla u|^{p-2}) \, d\mathcal{H}^{n-1}.
\end{equation}

By Hölder’s inequality

\[ 1 \leq \int_{\varepsilon_1}^{\varepsilon_2} \eta' \frac{H(t)^{1/2}}{H(t)^{-1/2}} \, dt \leq \left( \int_{\varepsilon_1}^{\varepsilon_2} \eta'^2 H(t) \, dt \right)^{1/2} \left( \int_{\varepsilon_1}^{\varepsilon_2} H^{-1}(t) \, dt \right)^{1/2}. \]

It follows that

\begin{equation}
\left( \int_{\varepsilon_1}^{\varepsilon_2} H^{-1}(t) \, dt \right)^{-1} \leq \int_{\varepsilon_1}^{\varepsilon_2} \eta'^2 H(t) \, dt,
\end{equation}

for all \( \varphi(x) = \eta(d_k(x)) \) satisfying (4.2). We define a function \( \hat{\eta} \) by the formula

\[ \hat{\eta}(s) = \left( \int_{\varepsilon_1}^{s} H^{-1}(t) \, dt \right) \left( \int_{\varepsilon_1}^{\varepsilon_2} H^{-1}(t) \, dt \right)^{-1}. \]

Now \( \hat{\eta}(\varepsilon_1) = 0 \) and \( \hat{\eta}(\varepsilon_2) = 1. \) Because

\[ \hat{\eta}'(s) = \frac{1}{H(s)} \left( \int_{\varepsilon_1}^{\varepsilon_2} \frac{dt}{H(t)} \right)^{-1}, \]

we have by (4.21)

\[ \left( \int_{\varepsilon_1}^{\varepsilon_2} H^{-1}(t) \, dt \right)^{-1} \leq \inf_{\varphi} \int_{\varepsilon_1}^{\varepsilon_2} \eta'^2 H(t) \, dt \leq \int_{\varepsilon_1}^{\varepsilon_2} \eta'^2 H(t) \, dt = \left( \int_{\varepsilon_1}^{\varepsilon_2} H^{-1}(t) \, dt \right)^{-1}. \]

Because

\[ \int_{\varepsilon_1}^{\varepsilon_2} H^{-1}(t) \, dt = \int_{\varepsilon_1}^{\varepsilon_2} \frac{dt}{H(t)} \int_{\Sigma_k(t)} |v_{r,R} - u|^2 (|\nabla v_{r,R}|^{p-2} + |\nabla u|^{p-2}) \, d\mathcal{H}^{n-1} \rightarrow \infty, \]

as \( \varepsilon_2 \rightarrow \infty, \) the claim follows. \( \square \)

5. Proofs of the corollaries

Proof of Corollary 2.4. Let

\begin{equation}
H(t) = \int_{\Sigma_k(t)} |v_{r,R} - u|^2 (|\nabla v_{r,R}|^{p-2} + |\nabla u|^{p-2}) \, d\mathcal{H}^{n-1}.
\end{equation}

By Hölder’s inequality

\[ (S - r)^2 = \left( \int_r^S \frac{dt}{H^{-1/2}(t)} \right)^2 \leq \left( \int_r^S H^{-1/2}(t) \, dt \right)^2 \leq \left( \int_r^S H^{-1}(t) \, dt \right) \left( \int_r^S H(t) \, dt \right). \]
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Hence

\[(S - r)^2 \left( \int_r^S H^{-1}(t) \, dt \right)^{-1} \leq \left( \int_r^S H(t) \, dt \right).\]

Now by (5.2) and Theorem 3.3

\[
\left[ \int_r^S dt \left( \int_{\Sigma_k(t)} |v_{r,R} - u|^2 \left( |\nabla v_{r,R}|^{p-2} + |\nabla u|^{p-2} \right) \, dH^{n-1} \right)^{-1} \right]^{-1}
\leq \frac{1}{(S - r)^2} \int_{D_{r,S}} |v_{r,R} - u|^2 \left( |\nabla v_{r,R}|^{p-2} + |\nabla u|^{p-2} \right) \, dm \to 0,
\]

as \( S \to \infty \), proving the claim. \( \square \)

Proof of Corollary 2.6. Since

\[
\int_{D_{r,\infty}} |v_{r,R} - u|^2 \left( |\nabla v_{r,R}|^{p-2} + |\nabla u|^{p-2} \right) \, dm \leq M < \infty,
\]

we have for \( S > r \),

\[
\frac{1}{S^2} \int_{D_{r,S}} |v_{r,R} - u|^2 \left( |\nabla v_{r,R}|^{p-2} + |\nabla u|^{p-2} \right) \, dm \leq \frac{M}{S^2} \to 0,
\]

as \( S \to \infty \). \( \square \)

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