ON HARDY-SOBOLEV EMBEDDING

WILLIAM BECKNER

Abstract. Linear interpolation inequalities that combine Hardy’s inequality with sharp Sobolev embedding are obtained using classical arguments of Hardy and Littlewood (Bliss lemma). Such results are equivalent to Caffarelli-Kohn-Nirenberg inequalities with sharp constants. A one-dimensional convolution inequality for the exponential density is derived as an application of these methods.

1. Interpolation inequalities.

A classical problem in analysis is to understand how “smoothness” controls norms that measure the “size” of functions. Maz’ya recognized in his classic text on Sobolev spaces the intrinsic importance of inequalities that would refine both Hardy’s inequality and Sobolev embedding. Dilation invariance and group symmetry play an essential role in determining sharp constants. Recent interest has focused on how to add “error terms” to the classical estimates. The objective here is the following theorem drawn as a novel consequence of this effort to extend Sobolev embedding.

Theorem 1. For \( f \in S(\mathbb{R}^n) \), \( n \geq 3 \) and \( 2 < q \leq q^*_n = \frac{2n}{n-2} \)

\[
\int_{\mathbb{R}^n} |\nabla f|^2 \, dx \geq \frac{(n-1)(n-3)}{4} \int_{\mathbb{R}^n} \frac{1}{|x|^2} |f|^2 \, dx + C_q \left[ \int_{\mathbb{R}^n} |x|^{-n(q^*-q)/q} |f|^q \, dx \right]^{2/q} \tag{1}
\]

\[
C_q = \left( \frac{2\pi^{n/2}}{\Gamma(n/2)} \right)^{1/\alpha} \left( \frac{q}{2} \right)^{q/2} \left[ \frac{\Gamma(\alpha)\Gamma(\alpha+1)}{\Gamma(2\alpha)} \right]^{1/\alpha}, \quad \alpha = \frac{q}{q - 2}.
\]

It is important to emphasize that this is a global estimate on \( \mathbb{R}^n \), and that it improves over a convex linear combination of the two terms on the right-hand side with their respective sharp constants. A comparison can be made with Hebey’s AB program (see [10], chapter 7) where for this setting, optimal pairs of constants \( A, B \) would be determined for

\[
\int_{\mathbb{R}^n} |\nabla f|^2 \, dx \geq A \int_{\mathbb{R}^n} \frac{1}{|x|^2} |f|^2 \, dx + B \left[ \int_{\mathbb{R}^n} |f|^{q^*} \, dx \right]^{2/q^*}
\]

and additionally the treatment of forms for elliptic differential operators on hyperbolic space (see [1]). This framework suggests that it would be interesting to study this inequality for values of \( A \) less than the sharp value of \( (n-2)^2/4 \). The sharp value of \( B \) (with \( A = 0 \)) is the Sobolev embedding constant \( \pi n(n-2)\Gamma(n/2)\Gamma(n) / \Gamma(n/2) \Gamma(n) \)^{2/n}. Moreover, as this is an \textit{a priori} inequality, the limiting extremal functions may be singular at the origin. A further unexpected feature is that such inequalities are equivalent to three-dimensional Sobolev embedding estimates.

Proof of Theorem 1. Under radial decreasing rearrangement, the gradient term decreases and the terms on the right-hand side increase so it suffices to prove this inequality for radial decreasing functions. By setting \( f(x) = u(x)|x|^{-(n-1)/2} \) with \( u(0) = 0 \), then inequality (1) is equivalent to

\[
\int_0^\infty \left| \frac{\partial u}{\partial r} \right|^2 \, dr \geq (\omega_{n-1})^{2/q} - 1 \cdot C_q \left[ \int_0^\infty |u|^q r^{-\frac{n-2}{2}} \, dr \right]^{2/q} \tag{2}
\]
where \( \omega_{n-1} \) is the surface area of the unit sphere \( S^{n-1} \). Since \( |f(x)| \leq \sigma(x)|x|^{-\frac{n}{2}+1} \) with \( \lim_{|x| \to 0} \sigma(x) = 0 \), observe that \( |u(x)|^2|x|^{-1} \leq \sigma^2(x) \to 0 \) as \( |x| \to 0 \), and there is no difficulty with the accompanying integration by parts in the functional transformation of the inequality. Further, for \( w = |x|^{-1} u \)

\[
\int_0^\infty \left( \frac{\partial u}{\partial r} \right)^2 dr = \int_0^\infty r^2 \left( \frac{\partial w}{\partial r} \right)^2 dr \geq (\omega_{n-1})^{2/q - 1} C_q \left[ \int_0^\infty |w|^q r^{\frac{n}{2} - 1} dr \right]^{2/q}.
\]

Then set \( h(r) = \partial w/\partial r \) so that \( w(r) = -\int_r^\infty h(t) dt \), and

\[
\int_0^\infty r^2|h|^2 dr \geq (\omega_{n-1})^{2/q - 1} C_q \left[ \int_0^\infty \int_r^\infty h(t) dt \left| \int_0^r |g(s)| ds \right|^q r^{\frac{n}{2} - 1} dr \right]^{2/q}.
\]

Now set \( g(r) = r^{-2}h(1/r) \), and change variables \( r \to 1/r \)

\[
\int_0^\infty |g|^2 dr \geq (\omega_{n-1})^{2/q - 1} C_q \left[ \int_0^\infty \int_0^r |g(s)| ds \left| \int_0^s \frac{r\Gamma(q/r)}{\Gamma((q-1)/r)} \right|^{rp/q} dr \right]^{2/q}.
\]

Now to calculate \( C_q \), apply the Bliss lemma (\[5\]):

**Bliss Lemma.** For \( s \geq 0, q > p > 1, r = \frac{q}{p} - 1 \)

\[
\left[ \int_0^\infty \int_0^s g(t) dt \left| \int_0^s t^{r-q} ds \right|^p \right]^{1/p} \leq K \int_0^\infty |g|^p ds
\]

\( K = (q - r - 1)^{-p/q} \left[ \frac{r\Gamma(q/r)}{\Gamma((q-1)/r)} \right]^{rp/q} \).

Equality is attained for functions of the form

\( g(s) = A(cs^r + 1)^{-(r+1)/r}, \quad c > 0 \).

Then \( C_q \) in equation (\[1\]) is given by

\[
C_q = (\omega_{n-1})^{1/\alpha} \left( \frac{q}{2} \right)^{2/q} \left[ \frac{\Gamma(\alpha)\Gamma(\alpha + 1)}{\Gamma(2\alpha)} \right]^{1/\alpha}
\]

for \( \alpha = q/(q - 2) \) and \( \omega_{n-1} = 2\pi^{n/2}/\Gamma(n/2) \). Tracing back the functional transformations, an extremal function for inequality (\[1\]) is given by

\( f(x) = |x|^{-(n-3)/2} \left( 1 + |x|^{\beta} \right)^{-1/\beta}, \quad \beta = \frac{q}{2} - 1 \).

Perhaps conceptually it is surprising that since extremals do not exist for Hardy’s inequality, this linear combination with a Sobolev embedding term suffices to determine an extremal function where equality is attained. The connection with Caffarelli-Kohn-Nirenberg inequalities becomes more explicit with the following extension of Theorem (\[1\])

**Theorem 2.** For \( f \in S(\mathbb{R}^n), n \geq 3, 2 < q \leq q_* = \frac{2n}{n-2} \) and \( 0 < a < (n-2)/2 \)

\[
\int_{\mathbb{R}^n} |\nabla f|^2 dx \geq a(n-2-a) \int_{\mathbb{R}^n} \frac{1}{|x|^2} |f|^2 dx + D_{q,a} \left[ \int_{\mathbb{R}^n} |x|^{-n(q-2)/q_*}|f|^q dx \right]^{2/q}
\]

\[
D_{q,a} = (n-2-2a)^{2/q} + 1 C_q
\]
Proof. Under radial decreasing rearrangement, the gradient term decreases and the terms on the right-hand side increase so it suffices to prove this inequality for radial decreasing functions. By setting \( f(x) = u(x)|x|^{-a} \) with \( u(0) = 0 \) and \( 0 < a < (n-2)/2 \), then inequality (4) is equivalent to

\[
\int_0^\infty r^{n-2-2a} \left( \frac{\partial u}{\partial r} \right)^2 dr \geq (\omega_{n-1})^{2/q - 1} D_{q,a} \left[ \int_0^\infty |u|^{q} r^2 |x|^{n-2a-2} dr \right]^{2/q}.
\]  

(6)

Since \( |f(x)| \leq \sigma(x)|x|^{-n/2 + 1} \) with \( \lim_{|x| \to 0} \sigma(x) = 0 \), observe that \( |u(x)|^2 |x|^{n-2a} \leq \sigma^2(x) \to 0 \) as \( |x| \to 0 \), and there is no difficulty with the accompanying integration by parts in the functional transformation of the inequality. Note that for values of \( a(n - 2 - a) \) less than the maximum \( (n - 2)^2/4 \), there are two roots and the restriction \( 0 < a < (n-2)/2 \) means that the smaller root is selected. This contrasts with the method used in the proof of Theorem 1. Now make the change of variables \( s = r^{n-2-2a} \), then

\[
\int_0^\infty s^2 \left( \frac{\partial u}{\partial s} \right)^2 ds \geq (n - 2 - 2a)^{-2/q - 1} (\omega_{n-1})^{2/q - 1} D_{q,a} \left[ \int_0^\infty |u|^{q} s^{q/2 - 1} ds \right]^{2/q}
\]  

(7)

and using the previous calculation from equation (3)

\[
D_{q,a} = (n - 2 - 2a)^{(2/q + 1) C_q}
\]

Surprisingly the dependence on the parameter \( a \) is simple and allows an immediate recovery of Hardy’s inequality by setting \( a = (n - 2)/2 \). Further, setting \( a = (n - 3)/2 \) which corresponds to the special case of Theorem 1 gives \( D_{q,a} = C_q \). Again as in the proof of Theorem 1 an extremal function for inequality (5) is given by

\[
f(x) = |x|^{-a} (1 + |x|^3(n-2-2a))^{-1/3}, \quad \beta = \frac{q}{2} - 1.
\]

And a close look at inequality (7) shows an equivalent three-dimensional inequality by using radial decreasing symmetrization:

\[
\int_{\mathbb{R}^3} |\nabla u|^2 dx \geq E_q \left[ \int_{\mathbb{R}^3} |u|^q |x|^{q/2 - 3/2} dx \right]^{2/q} \quad (8)
\]

\[
E_q = (4\pi)^{1 - 2/q} \left( \frac{q}{2} \right)^{2/q} \left[ \frac{\Gamma(\alpha) \Gamma(\alpha + 1)}{\Gamma(2\alpha)} \right]^{1/\alpha}, \quad \alpha = \frac{q}{q - 2}
\]

Symmetrization can be applied here since

\[
\frac{q}{2} - 3 \leq \frac{q}{2} - 3 = \frac{n}{n - 2} - 3 \leq 0 \quad \text{for} \quad n \geq 3.
\]

This is simply the \( a = 0, n = 3 \) case of Theorem 2 so in fact the full framework of that theorem can be bootstrapped from this one case! This inequality was first obtained by Glaser, Martin, Grosse and Thirring [8], and then extended to the \( n \)-dimensional setting by Lieb [11]. □

Remarks. Several recent papers provide context for the development of the main theorems here. Musina’s short paper [13] shows the connection between error estimates for Hardy’s inequality and the Caffarelli-Kohn-Nirenberg inequalities. From the nature of the proof of his Proposition 1.4 which holds only for radial functions on the unit ball in \( \mathbb{R}^n \), one can obtain an equivalence with the global estimates in Theorem 1. Various forms of embedding estimates for fractional smoothness are given in the author’s papers [2], [3]. With respect to reduction to one-dimensional estimates, see especially section 5 in [3]. After formulating the substance of this paper, the author became aware of the article by Dolbeault, Esteban, Loss and Tarantello [7] which gives new results on existence and symmetry of extremals for the Caffarelli-Kohn-Nirenberg inequalities.
2. Applications — Young’s inequality.

The weighted Hardy inequality (4) has varied applications, including the calculation of the sharp Sobolev embedding constant. In general, inequalities that occur for the “line of duality”, \( L^p \rightarrow L^{p'} \) with \( 1 < p < 2 \) and \( 1/p + 1/p' = 1 \), are especially interesting; examples include the Hausdorff-Young theorem and the Hardy-Littlewood-Sobolev inequality. Here the Bliss lemma for \( q = 2 \) is applied to calculate a one-dimensional convolution inequality on the line of duality.

**Theorem 3** (Young’s inequality for an exponential density). For \( f \in L^p(\mathbb{R}) \), \( 1 < p < 2 \), \( 1/p + 1/p' = 1 \) and \( \varphi(x) = e^{-|x|} \)

\[
\| \varphi \ast f \|_{L^{p'}(\mathbb{R})} \leq A_p \| f \|_{L^p(\mathbb{R})} \tag{9}
\]

\[
A_p = \left( \frac{1}{2} \right)^{2/p} \frac{1}{\Gamma(\frac{2p}{2-p})} \left[ \frac{\Gamma(\frac{2p}{2-p})}{\Gamma(\frac{2}{2-p})} \Gamma(\frac{p}{2-p}) \right]^{2/p - 1}
\]

**Proof.** From the Bliss lemma for \( q = 2 \):

\[
\int_0^\infty \left| \int_0^s g(t) dt \right|^2 s^{-1} - 2/p' ds \leq K^{2/p} \left[ \int_0^\infty |g|^p ds \right]^{2/p}
\]

First, set \( g(t) = h(t)t^{-1/p} \), and then set \( t = e^x \)

\[
\int_0^\infty \left| \int_0^s h(t) (t/s)^{1/p'} \frac{1}{t} dt \right|^2 s^{-1} ds \leq K^{2/p} \left[ \int_0^\infty |h|^p \frac{1}{s} ds \right]^{2/p}
\]

\[
\int_{-\infty}^{\infty} \left| \int_{-\infty}^x h(y) e^{-(x-y)/p'} dy \right|^2 dx \leq K^{2/p} \left[ \int_{-\infty}^\infty |h|^p dx \right]^{2/p}
\]

\[
\left| \int_{\mathbb{R} \times \mathbb{R}} h(u)e^{-2/p'|u-v|} h(v) du dv \right| \leq \frac{2}{p'} K^{2/p} \left[ \int_{\mathbb{R}} |h|^p dx \right]^{2/p}
\]

\[
A_p = \left( \frac{1}{2} \right)^{2/p - 1} K^{2/p} = \left( \frac{1}{2} \right)^{2/p} \frac{\Gamma(\frac{2p}{2-p})}{\Gamma(\frac{2}{2-p}) \Gamma(\frac{p}{2-p})} \left[ \frac{\Gamma(\frac{2p}{2-p})}{\Gamma(\frac{2}{2-p})} \Gamma(\frac{p}{2-p}) \right]^{2/p - 1}
\]

An extremal function for inequality (9) is given by

\[
f(x) = \cosh[p' x/(4p \delta)]^{-\delta}, \quad \delta = \frac{2}{2 - p}.
\]

\[\square\]

3. Stein-Weiss potentials.

The pure Sobolev embedding portion of Theorem 2 (with \( a = 0 \)) gives

\[
\int_{\mathbb{R}^n} |\nabla f|^2 dx \geq (n - 2)^{2/q} + C_q \left[ \int_{\mathbb{R}^n} |x|^{-\alpha_0} |f|^q dx \right]^{2/q}, \quad \alpha = \frac{n}{q} - \frac{n}{q_*} \tag{10}
\]

which determines a Stein-Weiss potential map on the line of duality through application of the fundamental solution for the Laplacian:
Theorem 4. For $h \in L^p(\mathbb{R}^n)$, $n > 2$, $\frac{2n}{n+2} < p < 2$, $(2n/p') - 2\alpha = n - 2$

$$\left| \int_{\mathbb{R}^n \times \mathbb{R}^n} h(x)|x|^{-\alpha}|x - y|^{-(n-2)}|y|^{-\alpha}h(y) \, dx \, dy \right| \leq A_n(\|h\|_{L^p(\mathbb{R}^n)})^2$$  \hspace{1cm} (11)

with $q = p'$ and $\delta = p/(2 - p)$.

Observe that $0 < \alpha < 1$. By applying symmetrization and using the dilation invariance, study of this inequality can be reduced to radial functions with an inversion symmetry: $h(|x|) = |x|^{-2n/p}h(1/|x|)$. Lieb [11] shows that extremal functions exist for general Stein-Weiss maps from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ where $p \neq q$. In general, one would like to calculate sharp constants for this inequality in the “line of duality”:

$$\left| \int_{\mathbb{R}^n \times \mathbb{R}^n} h(x)|x|^{-\alpha}|x - y|^{-\lambda}|y|^{-\alpha}h(y) \, dx \, dy \right| \leq A_{\lambda,\alpha}(\|h\|_{L^p(\mathbb{R}^n)})^2$$  \hspace{1cm} (12)

$1 < p < 2$, $\lambda = (2n/p') - 2\alpha$ and $0 < \alpha < n/p'$. The case $\alpha = 0$ corresponds to the Hardy-Littlewood-Sobolev inequality, and the case $\lambda = n - 2$ corresponds to Theorem 4. Sharp constants for such maps from $L^p(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ are calculated in [2].

4. Caffarelli-Kohn-Nirenberg inequalities.

One form of the Caffarelli-Kohn-Nirenberg inequalities is given by

$$\int_{\mathbb{R}^n} |x|^{-2\alpha} \nabla u|^2 \, dx \geq D_{q,a} \left( \int_{\mathbb{R}^n} |u|^q |x|^{-\frac{n}{q} - a} \, dx \right)^{2/q} \hspace{1cm} (13)$$

where $n \geq 3$, $2 < q < q_* = 2n/(n - 2)$ and $a < \frac{n^2}{2}$. Observe that for radial functions this is exactly inequality (11) above. In contrast to Theorem 4 there is no reason here in the application of the Bliss lemma why the parameter $a$ can not assume negative values. Hence the computation follows the same argument as above with

$$D_{q,a} = (n - 2 - 2a)^{(2/q + 1)} C_q$$

and the corresponding radial extremal

$$u(|x|) = (1 + |x|^\beta(n - 2 - 2a))^{-1/\beta} \hspace{1cm} \beta = \frac{q}{2} - 1 .$$

More generally, this result extends from $L^2(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ in the case of radial functions.

Theorem 5. For $u \in \mathcal{S}(\mathbb{R}^n)$, $1 < p < q < \infty$, $n > p$ and $0 < a < (n - p)/p$

$$\int_{\mathbb{R}^n} |x|^{-p\alpha} \nabla u|^p \, dx \geq D_{p,q,a} \left( \int_{\mathbb{R}^n} |u|^q |x|^{-q\alpha + a} \, dx \right)^{p/q}$$  \hspace{1cm} (14)

with $u$ begin a radial function and $q^* = pn/(n - p)$. If $a = 0$, then in addition the result holds for non-radial functions.

$$D_{p,q,a} = \left[ \frac{n - p(a + 1)}{p - 1} \right]^{(p-1) + \frac{p}{q}} (\omega_{n-1})^{1 - \frac{p}{q}} \left( \frac{\Gamma[qp/(q - p)]}{\Gamma(q/(q - p))\Gamma((q - 1)p/(q - p))} \right)^{1 - \frac{p}{q}} .$$
Proof. For radial functions, this inequality can be written as
\[
\int_0^\infty r^{n-pa-1} \frac{\partial u}{\partial r}^p dr \geq (\omega_{n-1})^{p/q} - 1 D_{p,q,a} \left( \int_0^\infty r^{q(\frac{n}{a}-a)-1} |u|^q dr \right)^{p/q} .
\]
Now make the change of variables \( s = r^{(n-p(a+1))/(p-1)} \); then
\[
\int_0^\infty s^{2p-1} \left( \frac{\partial u}{\partial s} \right)^p ds \geq \left[ \frac{n-p(a+1)}{p-1} \right]^{-(p-1)\frac{p}{q}} (\omega_{n-1})^{p/q} - 1 D_{p,q,a} \left( \int_0^\infty |u|^q s^{\frac{2q}{p-1}-(p-1)} ds \right)^{p/q} .
\]
Set \( h = \frac{\partial u}{\partial s} \) so that \( u(s) = -\int_s^\infty h(t) dt \), and then let \( g(r) = r^{-2} h(1/r) \), and change variables \( r \to 1/r \)
\[
\int_0^\infty |g|^p dr \geq \left[ \frac{n-p(a+1)}{p-1} \right]^{-(p-1)\frac{p}{q}} (\omega_{n-1})^{p/q} - 1 D_{p,q,a} \left( \int_0^\infty \int_0^r |g(s)| ds |r^{-\frac{2q}{p(1-1)}} dr \right)^{p/q}.
\]
Applying the Bliss lemma, one finds
\[
\left[ \frac{n-p(a+1)}{p-1} \right]^{-(p-1)\frac{p}{q}} (\omega_{n-1})^{p/q} - 1 D_{p,q,a} \frac{\Gamma(qp/(q-p))}{\Gamma(q/(q-p)) \Gamma((q-1)p/(q-p))} \right]^{1-p/q} .
\]
An extremal function for inequality (14) is given by
\[
u(x) = \left[ 1 + x^{\beta(n-p(a+1))/(p-1)} \right]^{-1/\beta} , \quad \beta = \frac{q}{p} - 1 .
\]
In the case \( a = 0 \), apply radial decreasing symmetrization for reduction to radial functions. □

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