An elementary formula for entanglement entropies of fermionic systems

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An elementary formula for the von Neumann and Rényi entropies describing quantum correlations in two-fermionic systems having four single particle states is presented. An interesting geometric structure of fermionic entanglement is revealed. A connection with the generalized Pauli principle is established.

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I. INTRODUCTION

During the past decade the characterization of inseparable quantum correlations, or entanglement has become one of the most active research fields. The reason for this flurry of activity is two-fold. First the attitude of regarding entanglement like energy as a resource paved the way to the appearance of quantum information science including such exciting applications like, teleportation [1], quantum cryptography [2] and more importantly quantum computing [3]. Second, entanglement as ”the characteristic trait of quantum mechanics” [4] is of fundamental importance for a deeper understanding of the conceptual foundations of quantum theory.

The main problem is how to quantify entanglement. In this respect as far as entanglement of distinguishable particles is concerned a large number of useful results exists. Entanglement measures for bipartite [5] and multipartite [6] pure states have been defined and used in a wide variety of interesting physical applications. However, the challenging problem of quantifying also mixed state entanglement is still at its infancy. Although the development in this field is apparent, apart from systems [7], [8], [9] of two qubits, and a qubit and a qutrit no simple sufficient and necessary conditions are known for deciding whether a state is entangled or not.

Quantifying quantum correlations for systems of indistinguishable particles is a relatively new topic. As a first step in this direction Schliemann et al. [10] characterized and classified quantum correlations in two-fermion systems having $2K$ single-particle states. For pure states they introduced in analogy to the Schmidt decomposition, a decomposition in terms of Slater determinants. States with Slater rank (i.e., the number of Slater determinants occurring in the canonical form) greater than one are called entangled. A sufficient and necessary condition for a state being entangled was established for $K = 2$ in [10], and for arbitrary $K$ later in [11]. For $K = 2$ a measure $0 \leq \eta \leq 1$ was introduced, Slater rank one (nonentangled) states correspond to $\eta = 0$, Slater rank two states with maximal entanglement correspond to $\eta = 1$. This quantity in many respect behaves similarly to the well-known concurrence [12] $0 \leq C \leq 1$ quantifying two qubit entanglement for distinguishable particles. In a special case they can in fact be related [13].

Some problems arise when we calculate the reduced (single particle) density matrix. Regarding the von Neumann entropy $S$ as a good correlation measure for fermions [14], raises the following puzzling issue. $S$ attains its minimum value $S_{\text{min}} = 1$ corresponding to Slater rank one i.e., nonentangled states. This situation is to be contrasted with the case familiar for two distinguishable particles where for nonentangled Schmidt rank one states one has $S_{\text{min}} = 0$. However, as was shown in [15] this contradiction is arising from the fact that the correlations of the Slater rank one state with $S_{\text{min}} = 1$ are related merely to the exchange properties of the indistinguishable fermions. Since these correlations cannot be used to implement a teleportation process or to violate Bell’s inequality they cannot be regarded as manifestations of entanglement.

The aim of the present paper is to study these issues by explicitly working out the example of two correlated fermions having four single particle states. Motivated by geometric considerations after employing a special representation for the complex amplitudes of our fermionic wave function we show that the von Neumann and Rényi entropies can be expressed in terms of the measure $\eta$ via a simple formula. Our elementary formula is entirely analogous to the one known for distinguishable particles using the concurrence $C$. Moreover, our construction yields the canonical form (Slater decomposition) explicitly. Unlike however, the canonical form of [11] where the expansion coefficients are complex in our form they are nonnegative real numbers. Having real expansion coefficients this decomposition is closer to the spirit of the Schmidt decomposition for distinguishable particles than the one presented in Refs. [11] and [14]. As a next step by calculating the von Neumann and Rényi entropies it is also shown that in this picture the residual entropy $S_{\text{min}} = 1$ reflecting the exchange properties of the fermions can be reinterpreted as a manifestation of the generalized Pauli exclusion principle. Finally it is shown that the residual entropy can be given a nice geometric interpretation in terms of a nonseparable quadric surface in the five dimensional complex projective space.

The organization of this paper is as follows. In Sec. II, using a convenient representation the structure of the density matrix is elucidated and the canonical form with real expansion coefficients is achieved. In Sec. III the
von Neumann and Rényi entropies are calculated and the limiting cases are discussed. Here the connection with the generalized Pauli exclusion principle is established. In Sec. IV the geometric background underlying our construction is illuminated. Some comments and the conclusions are left for Section V.

II. THE DENSITY MATRIX

As a starting point let us assume that the Hilbert space $\mathcal{H}$ describing the quantum correlations of two fermionic systems with four single particle states is of the form $\mathcal{H} \equiv \mathcal{A}(\mathbb{C}^3 \otimes \mathbb{C}^4)$ where $\mathcal{A}$ refers to antisymmetrization. An arbitrary element $|\Psi\rangle$ of $\mathcal{H}$ has the form

$$|\Psi\rangle \equiv \sum_{\mu, \nu=0}^{3} P_{\mu \nu} c_{\mu}^\dagger c_{\nu} |0\rangle \in \mathcal{H},$$

where $c_{\mu}^\dagger$ and $c_{\mu}$, $\mu = 0, 1, 2, 3$ are fermionic creation and annihilation operators satisfying the usual anticommutation relations

$$\{c_{\mu}, c_{\nu}\} = \delta_{\mu \nu}, \quad \{c_{\mu}, c_{\nu}^\dagger\} = 0, \quad \{c_{\mu}^\dagger, c_{\nu}\} = 0,$$

and $|0\rangle$ is the fermionic vacuum. Due to anticommutation the $4 \times 4$ matrix $P$ with complex elements is an antisymmetric one i.e., we have $P^{T} = -P$. Using these relations it can be shown that the normalization condition $\langle \Psi | \Psi \rangle = 1$ implies

$$2 \text{Tr} PP^{\dagger} = 1.$$  

It will be instructive in the following to stress the similarity with the structure of invariants characterizing fermionic entanglement and the ones arising from electrodynamics hence we parametrize our matrix $P$ as

$$P_{\mu \nu} \equiv \begin{pmatrix}
0 & E_{1} & E_{2} & E_{3} \\
-E_{1} & 0 & -B_{3} & B_{2} \\
-E_{2} & B_{3} & 0 & -B_{1} \\
-E_{3} & -B_{2} & B_{1} & 0
\end{pmatrix},$$

i.e., $P_{0j} = E_{j}$, $P_{j k} = -e_{j k l} B_{l}$, $j, k, l = 1, 2, 3$. It is important to emphasize at this point that unlike in electrodynamics here $E$ and $B$ are merely complex three vectors, i.e., $E, B \in \mathbb{C}^{3}$.

As was demonstrated in [10] local unitary transformations $U \otimes U$ with $U \in U(4)$ acting on $\mathbb{C}^4 \otimes \mathbb{C}^4$ do not change the fermionic correlations we are intending to study. Under such transformations $P$ transforms as

$$P \rightarrow UPU^{T},$$

A representation convenient for our purposes can be obtained by choosing the unitary matrix $U \in U(4)$ as

$$U_{\mu \nu} \equiv \frac{1}{\sqrt{2}} \begin{pmatrix}
1 & 0 & 0 & 1 \\
0 & 1 & -i & 0 \\
0 & 1 & i & 0 \\
1 & 0 & 0 & -1
\end{pmatrix},$$

where $i$ is the imaginary unit. The geometric meaning of the representation (hereafter to be called the "$U$-representation") arising from using this unitary transformation will be explained later. Straightforward calculation shows that the transformed matrix $P' \equiv UPU^{T}$ has the form

$$P' = UPU^{T} = \frac{1}{2} (\varepsilon \otimes \varepsilon) (I \otimes a\sigma + b\sigma \otimes I),$$

where

$$a = E + iB, \quad b = E - iB, \quad \varepsilon \equiv i\sigma_{2} = \begin{pmatrix}
0 & 1 \\
-1 & 0
\end{pmatrix},$$

and according to the normalization condition [4]

$$||a||^{2} + ||b||^{2} = \frac{1}{2}, \quad \text{with} \quad ||a||^{2} \equiv \bar{a}a, \quad ||b||^{2} \equiv \bar{b}b.$$  

Since the fermions are indistinguishable the reduced one-particle density matrices are equal and have the form [13]

$$\rho = 2PP^{\dagger}.$$  

Then a calculation using [14] shows that

$$2U\rho U^{T} = (I \otimes a\sigma + b\sigma \otimes I) (I \otimes \bar{a}\sigma + \bar{b}\sigma \otimes I).$$

Using the relation $(u\sigma)(v\sigma) = (uv)I + i(u \times v)\sigma$ and the normalization condition [15], we obtain the result

$$U\rho U^{\dagger} = \frac{1}{4} (1 + \Lambda),$$

where

$$\Lambda = 2 (I \otimes x\sigma + y\sigma \otimes I + b\sigma \otimes \bar{a}\sigma + \bar{b}\sigma \otimes a\sigma),$$

and

$$x \equiv -ia \times \bar{a}, \quad y \equiv ib \times \bar{b}, \quad 1 \equiv I \otimes I.$$  

Notice that the vectors $x$ and $y$ are real ones i.e., elements of $\mathbb{R}^{3}$.

In order to obtain the eigenvalues of $\rho$ we calculate $\Lambda^{2}$. The calculation is easily performed after noticing that due to the relations $x\sigma = x\bar{a} = y b = y\bar{b} = 0$ the first two and the last two terms in $\Lambda$ anticommute with each
other. A straightforward calculation using the definitions \[ ||x||^2 = ||a||^4 - a^2a^2, \] and the relations \[ ||x||^2 = ||a||^4 - a^2a^2, \] \[ ||y||^2 = ||b||^4 - b^2B^2 \] shows that
\[
A^2 = (1 - 64||EB||^2)(I \otimes I) = (1 - \eta^2)1. \quad (16)
\]
The quantity
\[
0 \leq \eta \equiv 8|P_{01}P_{23} - P_{02}P_{13} + P_{03}P_{12}| = 8||EB|| \leq 1 \quad (17)
\]
is the measure for fermionic correlations introduced in [10]. Since \[\text{Det} \, P = (EB)^2\] we see that \[\eta\] is invariant under local unitary transformations of the form \[\rho\].

Now Eq. (16) implies that the eigenvalues of \[\Lambda\] are
\[
\lambda_{\pm} = \frac{1}{4} \left( 1 \pm \sqrt{1 - \eta^2} \right), \quad (18)
\]
each of them doubly degenerate.

In [10] a fermionic analogue of the usual Schmidt decomposition for distinguishable particles was introduced. Adapted to our situation the theorem of [10] states that there exists a unitary matrix \[U \in U(4)\] (not to be confused with our \[U\] of expression (6)) such that
\[
Z = UPU^T, \quad \text{where} \quad Z = \begin{pmatrix}
0 & z_1 & 0 & 0 \\
-z_1 & 0 & 0 & 0 \\
0 & 0 & z_2 & 0 \\
0 & 0 & -z_2 & 0
\end{pmatrix}, \quad (19)
\]
where \[z_1\] and \[z_2\] are complex numbers. When one of the complex numbers \[z_i, i = 1, 2\] is zero we have Slater rank one, for both \[z_i\] being nonzero Slater rank two states. According to [10], a fermionic state is called entangled if and only if its Slater number is strictly greater than one.

However, according to a theorem of Zumino [16] even more can be said.

**Theorem.** If \(P\) is a complex \(2K \times 2K\) skew symmetric matrix, then there exist a unitary transformation \(V \in U(2K)\) such that
\[
R = VPY^T, \quad \text{where} \quad R = \text{diag}[R_1, R_2, \ldots R_K], \quad (20)
\]
with
\[
R_i = \begin{pmatrix}
0 & r_i \\
-r_i & 0
\end{pmatrix}, \quad (21)
\]
where \[r_i, i = 1, 2, \ldots K\] are nonnegative real numbers. Notice that unlike the decomposition of [10] the one presented in [20] is closer to the spirit of the usual Schmidt decomposition where the expansion coefficients are nonnegative real numbers.

Using the theorem above we see that the canonical form of our \(P\) in Eq. (1) is given by (20) with \(K = 2\) and
\[
r_1 = \sqrt{\frac{\lambda_1}{2}}, \quad n_2 = \sqrt{\frac{\lambda_2}{2}}. \quad (22)
\]
With this notation
\[
|\Psi\rangle = \sqrt{2\lambda_1} c'_0 c_0 |0\rangle + \sqrt{2\lambda_2} c'_2 c_2 |0\rangle, \quad (23)
\]
where
\[
c'_\mu = \sum_{\nu=0}^3 v_{\nu\mu} c^{(3)}_\nu. \quad (24)
\]
It is clear that for the calculation of the Slater states (the analogues of the Schmidt states) appearing in (20) we have to determine the unitary \(V\) along the lines as presented in (16). Notice that in our case this \(V\) as a function of the complex numbers \(E\) and \(B\) can be obtained explicitly. In order to give some hints notice that according to (16) the \(4 \times 4\) matrices
\[
\Pi_{\pm} \equiv \frac{1}{2} \left( 1 \pm \frac{1}{r} \Lambda \right), \quad r \equiv \sqrt{1 - \eta^2} \quad (25)
\]
are orthogonal projectors of rank two, i.e., they satisfy \[\Pi^2_{\pm} = \Pi_{\pm}\], and \[\Pi_+ \Pi_- = 0\]. Let us define the vectors
\[
v_0 = N_0 \Pi_+ e_0, \quad v_1 = N_1 \Pi_+ e_1. \quad (26)
\]
\[
v_2 = N_2 \Pi_- e_2, \quad v_3 = N_3 \Pi_- e_3, \quad (27)
\]
where \[e_{\mu} \mu = 0, 1, 2, 3\] are unit vectors corresponding to the columns of the unitary in Eq. (16), \(N_{\mu}\) are normalization constants. Then the \(v_{\mu}\) are normalized eigenvectors of \(\rho\). With the help of these eigenvectors we can build up the unitary diagonalizing the density matrix with the dependence on \(E\) and \(B\) explicitly displayed, the first step needed for the determination of \(V\) [16].

**III. ENTROPY**

Having the eigenvalues and the canonical form at our disposal we can now write down the explicit form of entropies used in quantum information theory. These are the von Neumann and the quantum counterpart of Rényi's \(\alpha (\alpha = 2, 3, \ldots)\) entropies [17] defined as
\[
S_1 \equiv -\text{Tr} \rho \log_2 \rho, \quad S_\alpha \equiv \frac{1}{1 - \alpha} \log_2 \text{Tr} \rho^\alpha, \quad \alpha > 1. \quad (28)
\]
Notice that for convenience we have chosen 2 for the base of the logarithm, and the von Neumann entropy can be regarded as the \( \alpha \) tends to 1 decreasingly limit of \( S_\alpha \).

Using (18) we obtain the explicit formula

\[
S_1(\eta) = 1 - x \log_2 x - (1 - x) \log_2 (1 - x), \tag{29}
\]

\[
S_\alpha(\eta) = 1 + \frac{1}{1 - \alpha} \log_2 (x^\alpha + (1 - x)^\alpha), \quad \alpha > 1 \tag{30}
\]

where

\[
x = \frac{1}{2} (1 + \sqrt{1 - \eta^2}), \tag{31}
\]

with \( \eta \) given by (17). All of our entropies satisfy the inequalities

\[
1 \leq S_\alpha(\eta) \leq 2, \quad \alpha \geq 1. \tag{32}
\]

Note, that the left hand side inequality is a consequence of the antisymmetry property of the two-particle state \(| \Psi \rangle \) as it was shown in (17). This statement, however, is a special case of a more general result obtained from the so-called Pauli principle for density matrices, related to the \( N \)-representability problem of \( \rho \). For \( N \)-particle fermionic systems the following question is of physical relevance. Given a one-particle density matrix \( \rho \), does there exist an \( N \)-particle density matrix \( \rho_N \) with the usual properties and which is antisymmetric with respect to the exchange of particles, satisfying the relation \( \rho = \text{Tr}_{2,\ldots,N} \rho_N \)? Operation \( \text{Tr}_{2,\ldots,N} \) is a partial trace on the particle indices 2, \ldots, \( N \). Clearly, a physical reduced density matrix should satisfy this requirement, in this case it is called \( N \)-representable. If, furthermore, \( \rho_N = | \Psi \rangle \langle \Psi | \), \( \rho \) is called pure state \( N \)-representable.

It is a result of Coleman [19] that a necessary and sufficient condition for \( N \)-representability can be formulated using the eigenvalues \( \lambda_0, \ldots, \lambda_{M-1} \) of \( \rho \) (here \( M \) stands for the dimension of the basis set of the one-particle Hilbert space). The reduced density operator \( \rho \) is \( N \)-representable iff

\[
0 \leq \lambda_\mu \leq 1/N \quad \text{for any } \mu = 0, \ldots, M - 1. \tag{33}
\]

The above condition is known as the generalized Pauli principle in the literature and is obviously satisfied by the eigenvalues (18) with \( N = 2, M = 2K = 4 \).

Considering now the entropy expressions (28) Jensen’s inequality results in the standard relations

\[
0 \leq S_1 \leq \log_2 M, \quad 0 \leq S_2 \leq \log_2 M, \tag{34}
\]

moreover, it can be shown [19] that

\[
S_2 \leq S_1 \tag{35}
\]

also holds. Applying (33)

\[
-S_2 = \log_2 \sum_{\mu=0}^{M-1} \lambda_\mu^2 \leq \log_2 \sum_{\mu=0}^{M-1} \lambda_\mu \frac{1}{N} = -\log_2 N \tag{36}
\]

and using (35) finally leads to

\[
\log_2 N \leq S_1 \leq \log_2 M, \quad \log_2 N \leq S_2 \leq \log_2 M, \tag{37}
\]

which is clearly a generalization of (32) for an arbitrary particle number \( N \) with \( \alpha = 1, 2 \).

Notice that \( S_\alpha = 1 \) iff \( \eta = 0 \). These are the states having Slater rank one, i.e., \(| \Psi \rangle \) in this case can be transformed via local unitaries \( U \otimes U \) with \( U \in U(4) \) to a single Slater determinant. Mathematically this means that \( P_{\mu\nu} \) of (11) is a separable bivector, i.e., there exist four-vectors \( u_\mu \) and \( v_\nu, \mu, \nu = 0, 1, 2, 3 \) such that \( P_{\mu\nu} = u_\mu v_\nu - u_\nu v_\mu \).

In order to study in our formalism the \( S_\alpha \) with both indices \( \alpha \geq 1 \), one has to consider the special case corresponding to Slater rank two states satisfying an additional requirement we introduce some terminology. Let us define the matrix

\[
g \equiv \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & -1
\end{pmatrix}. \tag{38}
\]

Then a short calculation using (18) shows that

\[
U g U^T = \varepsilon \otimes \varepsilon. \tag{39}
\]

We use \( g \) to raise and lower indices in the usual way hence for example we have \( P^{\mu\nu} \equiv g^{\mu\kappa} g^{\nu\rho} P_{\kappa\rho} \), in short a quantity like \( g P g \) corresponds to \( P \) with both indices raised.

Now let us define the dual \( P \) of \( P \) as

\[
*P_{\mu\nu} = \frac{1}{2} \epsilon_{\mu\nu\kappa\rho} P^{\kappa\rho}. \tag{40}
\]

Here \( \epsilon_{\mu\nu\kappa\rho} \) is the fourth order totally antisymmetric tensor defined by the condition \( \epsilon_{0123} = 1 \). Then we see that \( *E = -B \) and \( *B = E \). Moreover for the \( U(4) \) invariant \( \eta \) occurring in our formulae for the entropy we have

\[
\eta = 2 | *P_{\mu\nu} P^{\mu\nu} | = 2 | \text{Tr} ( \ast (P g P g) ) |. \tag{41}
\]

Comparing this with (33) we see that \( \eta = 1 \) iff

\[
P = e^{i \theta} ( g P g ) g, \tag{42}
\]

where \( e^{i \theta} \) is an arbitrary complex phase factor. In terms of \( E \) and \( B \) this means that \( \eta = 1 \) iff

\[
E = e^{i \theta} B. \tag{43}
\]
Transforming this equation with the unitary \( U \) we get

\[
P' = e^{i\theta} (\varepsilon \otimes \varepsilon P e^{-i\theta} (\varepsilon \otimes \varepsilon). \tag{44}\]

With an abuse of notation we can omit the prime and we can say that in the \( U \)-representation of Eq. \( \eta = 1 \) if and only if

\[
^*P = e^{i\theta} \tilde{P}, \tag{45}\]

where we have introduced the spin flip operation of Wooters \( \| \) playing a crucial role in the definition of the entanglement of formation for two-qubit systems, (recall that \( i\sigma_2 = \varepsilon \))

\[
\tilde{P} = \sigma_2 \otimes \sigma_2 ^* \sigma_2 \otimes \sigma_2. \tag{46}\]

Hence in the \( U \)-representation for states with maximal fermionic entanglement their duals are equal to their spin-flipped transforms (up to a phase). This result has to be compared with the similar one obtained in \( \| \). Here we also uncovered the instructive connection of dualization and its connection with the spin flip operation (i.e., time reversal) of quantum information theory. Notice also that in the original \( \| \) representation taking the spin-flip transform amounts to complex conjugation followed by raising both indices with the matrix \( g \) known from special relativity. The roots of this correspondence will be revealed in the next section.

\section*{IV. THE GEOMETRY OF FERMIONIC ENTANGLEMENT}

In this section we clarify the geometric meaning of our \( U \)-representation, and the residual entropy \( S_{\text{min}} = 1 \). To begin with, notice that our \( U \)-representation is a variant of the method of expressing quantities instead of the computational base in the so called "magic base" of Hill and Wootters \( \| \). The use of this base has its roots in the group theoretical correspondences \( (SL(2,\mathbb{C}) \times SL(2,\mathbb{C}))/\mathbb{Z}_2 \simeq SO(4,\mathbb{C}) \), \( (SU(2) \times SU(2))/\mathbb{Z}_2 \simeq SO(4) \). These correspondences have been used successfully for establishing exact results for the behavior of the entanglement of formation \( \| \). Here we would like to provide a different insight on the effectiveness of this base provided by geometry.

Let us consider the quantities (Infeld–van der Waerden symbols)

\[
\sigma_0^{AB'} = \sigma_0^{A'B'} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{47}\]

\[
\sigma_1^{AB'} = \sigma_1^{A'B'} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{48}\]

\[
\sigma_2^{A'B'} = -\sigma_2^{AB'} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \tag{49}\]

\[
\sigma_3^{A'B'} = \sigma_3^{AB'} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{50}\]

Here \( A, B' = 0, 1 \) are the matrix (spinor) indices of the Pauli matrices. The quantities \( \sigma_\mu^{A'B'} \) and \( \sigma_\mu^{A'B'} \mu = 0, 1, 2, 3 \) can be used to convert vector and spinor indices back and forth. For example for a four vector \( a_\mu \) we can form the four component spinorial object \( a_{A'B'} = \sigma_\mu^{A'B'} a_\mu \) where summation over \( \mu \) is understood. Writing out this relation explicitly we have

\[
\begin{pmatrix} a_{00'} \\ a_{01'} \\ a_{10'} \\ a_{11'} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & -i & 0 \\ 0 & 1 & i & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{pmatrix}. \tag{51}\]

Comparing this with Eq. \( \| \) we see that our use of the \( U \)-representation amounts to reverting to the spinorial analogue of our tensorial quantities. In particular the transformation of Eq. \( \| \) in this formalism takes the form

\[
P_{\mu\nu} \mapsto P_{AB'A'B'} = \sigma_\mu^{A'A'} \sigma_\nu^{B'B'} P_{\mu\nu}. \tag{52}\]

Moreover, Eq. \( \| \) becomes one of the basic identities of the spinorial formalism

\[
\varepsilon_{AB} \varepsilon_{A'B'} = \sigma_\mu^{A'A'} \sigma_\nu^{B'B'} g_{\mu\nu}. \tag{53}\]

Our decomposition \( \| \) in this picture corresponds to the well-known one in the spinor formalism of Penrose and Rindler \( \| \)

\[
P_{AB'A'B'} = \varepsilon_{AB} \psi_{A'B'} + \varphi_{AB} \varepsilon_{A'B'}. \tag{54}\]

Notice that the symmetric spinors \( \psi \) and \( \varphi \) correspond to \( \frac{1}{2} \varepsilon(\mathbf{a}\sigma) \) and \( \frac{1}{2} \varepsilon(\mathbf{b}\sigma) \), respectively. It is straightforward to check that \( ^*\mathbf{a} = i\mathbf{a} \) and \( ^*\mathbf{b} = -i\mathbf{b} \). Tensors satisfying \( ^*T = \pm iT \) are called \( \| \) self dual and anti self-dual, respectively. Hence our decomposition \( \| \) is in terms of the self-dual and anti-self dual parts of our matrix \( P \). As it is well-known spinorial methods proved to be of basic importance for a Petrov type of classification of curvature tensors in general relativity \( \| \). It is interesting to note that these methods proved to be of relevance for the classification of three-qubit (and possibly \( n \)-qubit) entanglement, too \( \| \). The basic idea behind this approach to \( n \)-qubit entanglement is to convert spinorial indices reflecting transformation properties under the group of \( n \)-fold tensor products of \( SL(2,\mathbb{C}) \) representing stochastic local operations and classical communication of the
entangled parties to vectorial ones (or vice versa) and then use the techniques as developed in twistor theory.

Finally, let us discuss the geometric meaning of $S_{\text{min}} = 1$ characterizing nonentangled fermionic states. From Eq. 1 it is clear that an unnormalized fermionic state $|\Psi\rangle$ can be characterized by six complex numbers, i.e., elements of $C^6$. However, the space of states of a quantum system is the space of rays $\mathcal{P}$ a space obtained by identifying states $|\Psi\rangle$ and $|\Phi\rangle$ if they are related by $|\Psi\rangle = c|\Phi\rangle$ where $0 \neq c \in C$. In our case $\mathcal{P}$ is the five dimensional complex projective space i.e., $\mathcal{P} \simeq CP^5$. Alternatively, one can consider the space of normalized states which is the 11 dimensional sphere $S^{11} \subset C^6 \simeq R^{12}$. In this case $CP^5$ can also be regarded as the space of equivalence classes of normalized states defined up to a complex phase. ($S^{11}$ has eleven real dimensions, a complex phase of unit magnitude is the circle $S^3$ which has one real dimension, and $CP^5$ has $11 - 1 = 10$ real dimensions.)

Let us now consider the constraint $\eta = 0$ which gives rise according to Eqs. 29-31 to $S_{\text{min}} = 1$ for all of our entropies. This condition is

$$P_{01}P_{23} - P_{02}P_{13} + P_{03}P_{12} = 0,$$  \hspace{1cm} (55)

which is the Plücker relation among the six complex coordinates characterizing separable bivectors. As it is well-known, a bivector (an antisymmetric $4 \times 4$ matrix) is separable if and only if condition (55) holds. This result dates back to the work of Plücker and Klein in the middle of the 19th century and was rediscovered in the context of fermionic entanglement in [11]. The separability conditions for an arbitrary dimensional bivector can be found in Penrose and Rindler [21], or in connection with fermionic entanglement in [11].

Using the coordinates

$$(z_0, z_1, z_2, z_3, z_4, z_5) \equiv (a_1, a_2, a_3, ib_1, ib_2, ib_3)$$ \hspace{1cm} (56)

where the components of $a$ and $b$ are related to the Plücker coordinates $P_{\mu \nu}$ by Eqs. 1 and 31, the Plücker relations can be written as

$$z_0^2 + z_1^2 + z_2^2 + z_3^2 + z_4^2 + z_5^2 = 0.$$ \hspace{1cm} (57)

This equation is homogeneous of degree two and defines a quadric surface $Q_4(C)$ (the so called Klein quadric) in $CP^5$. As far as we know the $Q_4(C)$ quadric has made its debut to physics as early as 1936 in the seminal work of Dirac [22] of conformal geometry and wave equations. Here the Klein quadric is the eight real (four complex) dimensional manifold characterizing nonentangled fermionic states. $Q_4(C)$ is a submanifold of $CP^5$. States of $CP^5$ not lying in $Q_4(C)$ are exhibiting nontrivial fermionic correlations hence they are entangled.

In order to gain more insight on the nature of fermionic entanglement let us compare these results with the corresponding ones known for two distinguishable qubits. Two unnormalized qubits are characterized by four complex numbers, hence the relevant space of rays in this case is the three dimensional complex projective space $CP^3$. Nonentangled states are the ones for which the concurrence $C$ is zero. It can be shown [12] (again by using the magic base) that four complex coordinates $w_0, w_1, w_2, w_3$ can be introduced such that for vanishing $C$ they satisfy the relation

$$w_0^2 + w_1^2 + w_2^2 + w_3^2 = 0.$$ \hspace{1cm} (58)

This equation defines the four real dimensional quadric $Q_2(C)$ in $CP^3$. Now as in the fermionic case nonentangled states are parametrized by the points of $Q_2(C)$, and entangled ones belong to its complement in $CP^3$.

It was shown in [21] that for distinguishable qubits a measure of entanglement can be defined as follows. The space of rays $CP^3$ can be equipped with the Fubini-Study metric [25], which is induced by the standard Hermitian scalar product on $C^4$. Let us fix an entangled state off the $Q_2(C)$ quadric. Then the measure of entanglement for this state is related to the length of the shortest arc of the geodesic with respect to the Fubini-Study metric, connecting the state in question with the $Q_2(C)$ quadric. More precisely we have

$$\cos^2 s = \frac{1}{2}(1 + \sqrt{1 - C^2})$$ \hspace{1cm} (59)

where $0 \leq C \leq 1$ is the concurrence, and $s$ is the geodesic distance. The separable states corresponding to the two points of intersection of this geodesic with $Q_2(C)$ are just the ones occurring in the Schmidt decomposition [21]. The proof of this theorem in [21] can be trivially generalized for an arbitrary quadric $Q_{n-1}(C)$ in $CP^n$. In particular for $n = 5$ we get the result

$$\cos^2 s = \frac{1}{2}(1 + \sqrt{1 - \eta^2}).$$ \hspace{1cm} (60)

Hence the measure of nontrivial fermionic correlations is related to the geodesic distance between the fermionic state in question and the Klein quadric of nonentangled states by Eq. 60. Notice also that the Slater decomposition of Eq. 28 can be reexpressed as

$$|\Psi \rangle = \cos s \frac{C_1^*}{\sqrt{2}} C_0^* 1|10\rangle + \sin s \frac{C_1^*}{\sqrt{2}} C_3^* 0|0\rangle,$$ \hspace{1cm} (61)

which for variable $s$ describes a family of entangled states lying on a horizontal geodesic. The normalized separable Slater states $C_0^* 1|10\rangle$ and $C_3^* 0|0\rangle$ are on the quadric $Q_2(C)$. They can also be calculated by a Lagrange multiplier technique as in [21] giving a geometric meaning to the Slater decomposition of a fermionic state having four single particle states.

Now the question arises: is there any basic difference between the quadrics $Q_2(C)$ and $Q_4(C)$ that can
account for the different physical situations as reflected by the different minimum values of their respective entropies? The answer to this question is surprisingly yes. It is a theorem in differential geometry that the quadrics $Q_{n-1}(C)$ in $\mathbb{C}P^n$ parametrized by homogeneous coordinates $Z_0, Z_1, \ldots, Z_n$ satisfying the additional constraint $\sum_{j=0}^{n} Z_j^2 = 0$ are symmetric spaces that can be represented in the form $[25]$

$$Q_{n-1}(C) \simeq SO(n+1)/SO(2) \times SO(n-1). \quad (62)$$

For the very special value $n = 3$, $SO(4) \sim SU(2) \times SU(2)$ i.e., this group exhibits a product structure. Since $SO(2) \simeq U(1)$ and $SU(2)/U(1) \simeq S^3$ one can show that $Q_2(C) \simeq S^2 \times S^2$, i.e., the direct product of two-spheres. These spheres are just the Bloch spheres corresponding to the distinguishable qubits in a separable state. The embedding of $Q_2(C)$ in the form $S^2 \times S^2 \hookrightarrow \mathbb{C}P^3$ is the special case of the so called Segre embedding having already been used in geometric descriptions of separability for distinguishable particles $[25]$.

For $n \geq 4$ the corresponding symmetric spaces are irreducible $[25]$, hence they cannot be represented in a product form of two manifolds. Hence we can conclude that the manifold of nonentangled states representing quantum systems of distinguishable or indistinguishable constituents exhibits different topological structure. For nonentangled distinguishable particles ($S_{\min} = 0$) we have a product structure of the state space $Q_2(C)$ which conforms with our expectations coming from classical mechanics. However, for nonentangled indistinguishable fermions ($S_{\min} = 1$) no product structure of the state space $Q_4(C)$ can be observed due to correlations reflecting the exchange properties of the fermions. These correlations are of intrinsically quantum in nature. However, they are not to be confused with the correlations that can be regarded as true manifestations of entanglement. Representative states of this kind belonging to the complement of $Q_4(C)$ can be used to implement quantum information processing tasks.

V. CONCLUSIONS

In this paper we studied the nature of quantum correlations for fermionic systems having four single particle states. Though these are the simplest systems among the fermionic ones exhibiting such correlations, but they clearly show some of the basic differences between entanglement properties of quantum systems with distinguishable and indistinguishable constituents.

As a starting point, we employed a comfortable, the so-called $U$-representation for the $4 \times 4$ antisymmetric matrix $P$ containing the six complex amplitudes representing our fermionic system. This representation enabled an explicit construction of the reduced density matrix, its eigenvalues and eigenstates. We have shown that these eigenvalues can be expressed in terms of the invariant $\eta$ of $[10]$ via an elementary formula analogous to the one well-known for distinguishable qubits. In this way we managed to represent our entangled state in a canonical form (the so called Slater decomposition) with real nonnegative expansion coefficients. This decomposition is closer to the spirit of the standard Schmidt decomposition than the one presented in $[10]$, and $[11]$ with complex expansion coefficients.

Using these results we have computed the von Neumann and Rényi entropies that can also be used to characterize fermionic correlations. These entropies satisfy the bound $1 \leq S_{\alpha}(\eta)$, $\alpha = 1, 2, \ldots$. This inequality is to be contrasted with the corresponding one $0 \leq S_{\alpha}(C)$ known for distinguishable qubits, where $C$ is the concurrence. We have shown that the difference in the bounds can be traced back to fact that the so-called Pauli principle for density matrices has to hold. We related the special values for the entropies satisfying the lower or upper bounds to the algebraic properties of the matrix $P$.

We have also clarified the geometric meaning of the $U$-representation. An interesting and useful connection with the spinor formalism of Penrose and Rindler hitherto used merely within the rather exotic realm of twistor theory was pointed out. Next we initiated the study of quadrics $Q_{n-1}(C)$ embedded in the space of rays $\mathbb{C}P^n$ for revealing the geometric aspects of entanglement. The cases $n = 3$ and $n = 5$ correspond to the simplest cases of entanglement for systems with distinguishable and indistinguishable constituents. We noticed that previous results connecting the measure of entanglement with the geodesic distance between states and the quadric can be generalized for the fermionic case as well. Finally we have proved that the different physical situations showing up as the occurrence of different minimum values for the entanglement entropies, are also reflected in the different topological properties of the quadrics $Q_2(C)$ and $Q_4(C)$. $Q_2(C)$ exhibits a product structure $S^2 \times S^2$ of two Bloch-spheres which conforms with our expectations based in classical physics. However, for $Q_4(C)$ i.e., the Klein quadric, as the manifold of nonentangled states no product representation is available. This fact can be regarded as a geometric manifestation of the Pauli principle, showing the existence of correlations related entirely to the exchange properties of fermions. The main concern of quantum information science can be the use of states off the Klein quadric. These are the ones that can be used to implement quantum information processing tasks.

VI. ACKNOWLEDGEMENTS

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