Breakdown Criteria for Nonvacuum Einstein Equations

Arick Shao

University of Toronto

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The Breakdown Problem

- General question: *Under what conditions can an existing local solution of an evolution equation on a finite interval* \([0, T)\) *be further extended past* \(T\) ?
- Why is this useful?
  1. Characterize breakdown of solutions.
  2. Global existence problem.
Nonlinear Wave Equations

- Equations of the form

\[ \square \phi = (\partial \phi)^2, \quad \phi|_{t=0} = \phi_0, \quad \partial_t \phi|_{t=0} = \phi_1. \]

- Local existence for \( H^s \)-spaces.
- If local solution on \([0, T)\) satisfies

\[ \| \partial \phi \|_{L^\infty} < \infty, \quad (1) \]

then solution can be extended past \( T \).
- Time of existence controlled by \( H^s \)-norms, which can be uniformly controlled on \([0, T)\) using (1).
Incompressible 3-d Euler equations

\[ u : \mathbb{R}^{1+3} \rightarrow \mathbb{R}^3, \quad p : \mathbb{R}^{1+3} \rightarrow \mathbb{R}. \]

\[ \partial_t u + u \cdot \nabla u + \nabla p = 0, \quad \nabla \cdot u = 0. \]

Vorticity: \( \omega = \nabla \times u. \)

Beale, Kato, Majda (1984): If a local solution has \( \omega \) bounded in \( L^1_t L^\infty_x \), then it can be extended.

- Need not bound all of \( \nabla u. \)
Yang-Mills Equations

- Eardley, Moncrief (1982): global existence in $\mathbb{R}^{1+3}$.
  - Continuation criterion: $\|F\|_{L^\infty} < \infty$
  - $F$ - Yang-Mills “curvature”.
  - $\|F\|_{L^\infty}$ controlled using wave equations and fundamental solutions.

- Chruściel, Shatah (1997): generalized to globally hyperbolic $(1 + 3)$-dim. Lorentz manifolds.
Results for Vacuum Equations

- Einstein vacuum: \((1 + 3)\)-dim. spacetimes \((M, g)\),
  \[ \text{Ric}_g = 0. \]

- Anderson (2001): \(\| R_g \|_{L^\infty} < \infty \Rightarrow \) solution can be extended.
  - Geometric, requires two derivatives of \(g\).

- Other continuation criteria:
  \[ \| \partial g \|_{L^\infty} < \infty, \text{ or } \| \partial g \|_{L^1_t L^\infty_x} < \infty. \]
  - Not geometric, depends on choice of coordinates.
Improved Results

- Klainerman, Rodnianski (2008): improved breakdown criterion for vacuum:

\[ \| k \|_{L^\infty} + \| \nabla (\log n) \|_{L^\infty} < \infty \]

- CMC foliation, compact time slices.
- \( k, n \) - second fundamental form, lapse of time slices.
- Geometric, do not need full coordinate system.
- \( k \) and \( \nabla (\log n) \) at level of \( \partial g \), but do not cover all components of \( \partial g \).

- D. Parlongue (2008): vacuum, maximal foliation, asymptotically flat time slices, replaced \( L^\infty \) by \( L_t^2 L_x^\infty \).

- Q. Wang (2010): vacuum, CMC, compact time slices, replaced \( L^\infty \) by \( L_t^1 L_x^\infty \).
General Einstein Equations

- Spacetime \((M, g, \Phi)\), \(\Phi\) - matter fields.

- Einstein equations:

\[ R_{\alpha \beta} - \frac{1}{2} R g_{\alpha \beta} = Q_{\alpha \beta}. \]

\(Q\) - energy-momentum tensor.

- Einstein-scalar \((\Phi = \phi\) - scalar):

\[ \Box g \phi = 0, \quad Q_{\alpha \beta} = \partial_\alpha \phi \partial_\beta \phi - \frac{1}{2} g_{\alpha \beta} \partial^\mu \phi \partial_\mu \phi. \]

- Einstein-Maxwell \((\Phi = F\) - 2-form):

\[ D^\alpha F_{\alpha \beta} = 0, \quad D_{[\alpha} F_{\beta \gamma]} = 0, \]

\[ Q_{\alpha \beta} = F_{\alpha \mu} F_{\beta}^{\mu} - \frac{1}{4} g_{\alpha \beta} F_{\mu \nu} F^{\mu \nu}. \]
The Main Questions

- Does there exist a “breakdown criterion” similar to K-R for Einstein-scalar and Einstein-Maxwell spacetimes.
- Other nonvacuum settings?
The Basic Setting

- Same setting as K-R, but with E-S or E-M spacetime \((M, g, \Phi)\) rather than E-V.
  - Time foliation:
    \[ M = \bigcup_{t_0 < \tau < t_1} \Sigma_\tau, \quad t_0 < t_1 < 0. \]
  - \(\Sigma_\tau\)'s are compact.
  - CMC foliation: \(\text{tr} \ k = \tau < 0\) on \(\Sigma_\tau\).
The Main Theorem

Theorem

Assume an Einstein-scalar or Einstein-Maxwell spacetime \((M, g, \Phi)\) in the setting of the previous slide. If

\[
\sup_{t_0 \leq \tau < t_1} \left( \| k(\tau) \|_{L^\infty} + \| \nabla (\log n)(\tau) \|_{L^\infty} \right) < \infty,
\]

(2)

and the following bounds hold for the matter field,

\[
(E-S) \quad \sup_{t_0 \leq \tau < t_1} \| D\Phi(\tau) \|_{L^\infty} < \infty,
\]

(3)

\[
(E-M) \quad \sup_{t_0 \leq \tau < t_1} \| F(\tau) \|_{L^\infty} < \infty,
\]

(4)

then \((M, g, \Phi)\) can be extended as a CMC foliation beyond time \(t_1\).
Additional Remarks

- Strategy of proof analogous to K-R.
- We focus on E-M setting, since E-S is easier.
- The theorem extends to Einstein-Klein-Gordon and Einstein-Yang-Mills spacetimes (nontrivial).
- Result can likely be extended to $L^2_t L_x^\infty$ and $L^1_t L_x^\infty$ breakdown criteria.
Main Issues

- Presence of nontrivial Ricci curvature.
- Coupling between curvature and matter fields.
- E-M: New types of nonlinearities in wave equations for $DF$ and curvature $R$.
  - Cannot be treated using methods of K-R.
The Cauchy Problem

- Given $(\Sigma_0, \gamma_0, k_0, \Phi_0)$, where
  - $\Sigma_0$ - Riemannian 3-manifold.
  - $\gamma_0$ - metric on $\Sigma$.
  - $k_0$ - symmetric 2-tensor (“second fundamental form”).
  - $\Phi_0$ - initial values for matter fields.

- Assume initial data satisfies \textit{constraint equations}.

- Solve for spacetime $(M, g, \Phi)$, where $M \cong I \times \Sigma_0$:
  - $(\Sigma_0, \gamma_0)$ imbedded as “initial” time slice of $M$, with second fundamental form $k_0$.
  - $\Phi_0$ corresponds to value of $\Phi$ on $\Sigma_0$. 
Local Well-Posedness

- Local existence: time of existence depends on
  \[ \mathcal{E}_0 \sim \| k_0 \|_{H^3} + \| R_0 \|_{H^2} + \| \Phi_0 \|_{H^3}, \]
  and geometric properties of \( \Sigma_0 \).
  - \( R_0 \) - Ricci curvature of \( \Sigma_0 \).
  - E-M: \( \Phi_0 = (E_0, H_0) \) - electromagnetic decomposition
- Main goal: uniformly control analogous quantities \( \mathcal{E}(\tau) \) for each \( \Sigma_\tau \) for all \( t_0 < \tau < t_1 \).
  - Apply local existence theorem to each \( \Sigma_\tau \).
- Elliptic estimates: suffices to uniformly bound spacetime quantities
  \[ \mathcal{E}(\tau) \sim \| R(\tau) \|_{H^2} + \| F(\tau) \|_{H^3}. \]
Important Preliminaries

- Breakdown criterion $\Rightarrow$ the deformation tensor
  \[ T_{\pi} = \mathcal{L}_T g \]
  is uniformly bounded (i.e. $T$ "almost Killing").
  - $T$ - future unit normal to $\Sigma_{\tau}$’s.
- Construct “energy-momentum tensors” similar to $Q$ for scalar and Maxwell fields.
  - Generalized (tensorial) wave equations.
  - Generalized Maxwell-type equations.
- The above two ideas imply energy inequalities.
A Priori Energy Estimates

Define
\[ \mathcal{E}_0 (\tau) = \| R (\tau) \|_{L^2} + \| DF (\tau) \|_{L^2}. \]

Using generalized EMT’s from $R$ and $F$, we obtain
\[ \mathcal{E}_0 (\tau) \lesssim \mathcal{E}_0 (t_0). \]

Due to coupling, $R$ and $DF$ must be handled concurrently.
Higher Order Energy Quantities

- Define higher order energy quantities

\[
\mathcal{E}_1 (\tau) = \| DR (\tau) \|_{L^2} + \| D^2 F (\tau) \|_{L^2}, \\
\mathcal{E}_2 (\tau) = \| D^2 R (\tau) \|_{L^2} + \| D^3 F (\tau) \|_{L^2}.
\]

- \( R, DR, DF, D^2 F \) satisfy covariant wave equations.

- Goal: show uniformly in \( \tau \),

\[
\mathcal{E}_1 (\tau) + \mathcal{E}_2 (\tau) \leq C. \tag{5}
\]

- Main difficulty: must also bound

\[
\| R (\tau) \|_{L^\infty} + \| DF (\tau) \|_{L^\infty}.
\]
Null Cones

- For $p \in M$, we can define past null cone $N^-(p)$ about $p$.
  - Near $p$, $N^-(p)$ is smooth and parametrized by $s \in (0, \infty)$ and $\omega \in S^2$; call this portion $\mathcal{N}^-(p)$.
  - $L$ - geodesic null tangent vector field.
  - Null frames $L, L, e_1, e_2$ - locally defined w.r.t. spherical foliation of $\mathcal{N}^-(p)$. 

Null Cones
A priori $L^2$ flux bounds for $R$ and $DF$ on $\mathcal{N}^-(p)$, again using EMT's.

- Flux does not control all components of $R$ and $DF$.
  - $R$: excludes $R_{\underline{L}e_a\underline{L}e_b}$.
  - $DF$ excludes $D_{\underline{L}}F_{\underline{L}e_a}$.

- Also need higher-order flux estimates for $DR$ and $D^2F$ on $\mathcal{N}^-(p)$.
  - Cannot control all components of $DR$ and $D^2F$.
  - Also need uniform bounds for $R$ and $DF$. 
Revisiting the Uniform Bound

Recall: need uniform bounds for

$$\| R(\tau) \|_{L^\infty} + \| DF(\tau) \|_{L^\infty}.$$ 

Main idea: $R, DF$ satisfy system of wave equations:

$$\Box_g R \cong F \cdot D^2 F + (R + DF)^2 + \text{l.o.}, \quad (6)$$

$$\Box_g DF \cong F \cdot DR + (R + DF)^2 + \text{l.o.}.$$ 

$(R + DF)^2$ - quadratic terms.

$F \cdot D^2 F, F \cdot DR$ - first-order terms,

Compare to vacuum case (K-R):

$$\Box_g R \cong R \cdot R.$$
The Kirchhoff-Sobolev Parametrix

In $\mathbb{R}^{1+3}$, Kirchhoff’s formula for scalar wave equations:

$$\Box \phi = \psi, \quad \phi (p) \approx \int_{\mathcal{N}^{-}(p)} \frac{1}{d(q, p)} \psi (q) \, d\sigma(q) + i.v.$$

“Kirchhoff-Sobolev parametrix” (K-R): first-order generalization to curved spacetimes.

- Valid on regular past null cones on a Lorentzian manifold (i.e., within null radius of injectivity).
- Valid for covariant tensorial wave equations.
- Supported entirely on past null cone.
- Generalizable to covariant wave equations on arbitrary vector bundles (application: Y-M equations).
The Explicit Formula, Abridged

- Covariant wave equation $\Box g \Phi = \Psi$.
- Transport equation on $N^{-}(p)$:
  
  $$D_{L}A = -\frac{1}{2} (\text{tr} \chi) A, \quad sA|_{p} = J_{p},$$

- $A$ - tensor field on $N^{-}(p)$, of same rank as $\Phi, \Psi$ - corresponds to $r^{-1}$ in $\mathbb{R}^{1+3}$.
- $s$ - affine parameter (or another foliating function).
- $\text{tr} \chi$ - expansion of $N^{-}(p)$.

- Kirchhoff-Sobolev parametrix given by
  
  $$4\pi \cdot g \left( \Phi|_{p}, J_{p} \right) = \int_{N^{-}(p)} [g (A, \Psi) + \text{Error}] + i.v..$$
Consider vacuum case, $\Box g R \simeq R^2$.

To bound $\|R\|_{L^\infty}$, we must control principal term

$$\int_{\mathcal{N}^-(p)} |A| |R \cdot R|.$$ 

Main trick: the “Eardley-Moncrief” observation - one of the $R$'s must be a flux component.

Must also bound “error terms” and $A$. 
Uniform Bounds in E-M

- Wave equations for $R$ and $DF$.
- Quadratic terms $(R + DF)^2$ handled as in vacuum.
- However, cannot handle first-order terms this way.
Generalizing the Parametrix

- Alter the K-S parametrix to handle systems of covariant wave equations, with first-order terms.
- General form: for all $1 \leq m \leq n$,

\[
\Box_g (m \Phi)_I + \sum_{c=1}^{n} (mc P)_{\mu I}^J \cdot D^\mu (c \Phi)_J = (m \Psi)_I.
\]

- Main idea: handle the $mc P$'s through $A$, by altering the transport equation for $A$.
- Solve a coupled system of transport equations:

\[
D_L (m A)_I = -\frac{1}{2} (\text{tr} \chi) (m A)_I + \frac{1}{2} \sum_{c=1}^{n} (cm P)_{IJ} L^I (c A)^J.
\]
The Generalized Formula, Abridged

Generalized formula given by:

\[ 4\pi \cdot \sum_{m=1}^{n} g \left( (^m\Phi)|_p, (^mJ_p) \right) \]

\[ = \int_{\mathcal{N}^{-}(p)} \left[ \sum_{m=1}^{n} g ((^mA), (^m\Psi)) + \text{Error} \right] + \text{i.v..} \]

Used different proof than in K-R.

- Avoids distributions.
- Discretionary integration by parts - “never leaves the null cone.”
- Avoids the optical function - weakens assumptions needed in K-R.
- Gives initial value terms explicitly.
- Again, can generalize to arbitrary vector bundles.
Uniform Bounds in E-M, Revisited

- In E-M case, $n = 2$, $(^1\Phi) = R$, $(^2\Phi) = DF$.
- $(^1P)$ and $(^2P)$ vanish, while $(^1^2P)$, $(^2^1P) \cong F$.
- $L^\infty$-bounds for $F$, flux bounds for $R$ and $DF \Rightarrow$ bounds for $A$.
- Remark: Generalization to vector bundles $\Rightarrow$ similar uniform bounds for Einstein-Yang-Mills.
More Issues

- To apply the K-S parametrix (E-V and E-M), we need:
  - Control for null injectivity radius.
  - Bounds for Ricci coefficients $\text{tr} \chi, \hat{\chi}, \zeta, \eta, \text{tr} \chi, \hat{\chi}$ on $\mathcal{N}^-(p)$, and their first derivatives.
- This is hard!
- Difficulty: we must control everything by $L^2$-quantities for $R$ and $DF$, and by the breakdown criterion.
- Remark: We cannot similarly bound causal inj. radius. Thus, it is essential that the K-S parametrix depends only on null inj. radius.
A Basic Outline

- E-V: series of papers by K-R.
- Main task: extend to E-S and E-M settings.

1. Gigantic bootstrap: assume conditional bounds for Ricci coeff.
2. Assumptions for \( \text{tr} \chi \Rightarrow \) control null conj. radius
3. “Regularity” of time foliation \( \Rightarrow \) control null inj. radius
4. Prove improved bounds for Ricci coeff.

- Remark: Must assume null injectivity radius to make full sense of \( \text{tr} \chi \), etc. on \( \mathcal{N}^-(p) \).
Notes on Steps 2 and 3

- **Step 2:** Finiteness of $\text{tr} \, \chi \Rightarrow$ null exponential map remains nonsingular.
- **Step 3:** Must control cut locus points.
  - Main tool: Existence of “almost Minkowski” coordinate systems $\Rightarrow N^-(p)$ comparable to Minkowski cones.
  - At first cut locus point, show that distinct null geodesics intersect at angle $\pi$. 
Discussion of Step 4

- Past results:
  - Klainerman, Rodnianski (2005): geodesic foliation, truncated null cones.
  - Q. Wang (2006): geodesic foliation, null cones.
  - D. Parlongue (2008): time foliation, truncated null cones.
  - Assume unit interval and small curvature flux, control Ricci coeff. by curvature flux (and time foliation).

- The nonvacuum analogue:
  - Time foliation, null cones.
  - Matter fields: control by both curvature and matter flux.
  - Assume small time interval and only bounded flux.
A Sample of the Results

- Main estimates:

\[
\left\| \text{tr} \chi - \frac{2}{t} \right\|_{L^\infty_t L^2_\omega} + \left\| \hat{\chi} \right\|_{L^\infty_t L^2_\omega} + \left\| \zeta \right\|_{L^\infty_t L^2_\omega} \lesssim 1,
\]

\[
\left\| \text{tr} \chi - \frac{2}{t} \right\|_{\mathcal{H}^1} + \left\| \hat{\chi} \right\|_{\mathcal{H}^1} + \left\| \zeta \right\|_{\mathcal{H}^1} \lesssim 1,
\]

- On sufficiently small segment \( \mathcal{N} \) of \( \mathcal{N}^-(p) \).

- Constant depends on flux and time foliation quantities.

- We also have the following:
  - \( \left\| \text{tr} \chi - 2t^{-1} \right\|_{L^\infty} \lesssim 1 \).
  - Improved \( \mathcal{H}^1 \)-estimates for \( \text{tr} \chi, \hat{\chi}, \eta \) (uses recent results of Q. Wang: \( k \) satisfies tensor wave equation).
The Bootstrap Argument

- Assume main estimates hold on $\mathcal{N}$ of “small” length $\delta_0$ with right-hand side replaced by “large” constant $\Delta_0$.
- Conditional assumption: only when $\mathcal{N}$ remains regular, e.g., within the null injectivity radius.
- Show everything is $\lesssim \delta_0^{\frac{1}{2}} \Delta_0^2 + 1 \leq \Delta_0/2$.
- Main steps:
  - Integrate Raychaudhuri equation for $\text{tr} \chi - 2t^{-1}$.
  - Integrate evolution equations for special derivative components $\nabla \text{tr} \chi, \mu$.
  - Elliptic estimates for $\nabla (\text{tr} \chi), \nabla \hat{\chi}, \nabla \zeta$.
  - Sharp trace estimates for $\hat{\chi}, \zeta$. 
Motivation for Sharp Trace Estimates

- In \( I \times \mathbb{R}^2 \), with \( I \) an interval, we have trace estimates

\[
\| \partial_t f \|_{L^\infty_t L^2_x} \lesssim \| f \|_{H^2},
\]

\[
\left\| \int_I \partial_t f \cdot g_{(t, \cdot)} \ dt \right\|_{B^0_{2,1}(\mathbb{R}^2)} \lesssim \| f \|_{H^1} \| g \|_{H^1},
\]

and other similar estimates.

- Goal: Prove similar tensorial estimates on \( \mathcal{N}^-(p) \).

- Problems:
  - Cannot use classical Littlewood-Paley theory - not enough metric regularity.
  - Validity of estimates relies on bootstrap assumptions, i.e., \textit{derivation of sharp trace estimates must be a part of the gigantic bootstrap argument for Ricci coefficients!}
The Main Estimates, Abridged

- K-R (2006): geometric L-P theory on manifolds.
  - Based on heat flow.
  - Can construct Besov spaces.
  - Can derive product estimates.

- Using L-P theory, derive sharp trace estimates:
  \[
  \left\| \int_0^t \nabla_L F \cdot G ds \right\|_{L^\infty_{0,0}B_{2,1}^0} \lesssim \|F\|_{H^1} \left( \|G\|_{H^1} + \|G\|_{L^\infty_{0,0}L^2_t} \right).
  \]

  Other similar estimates hold.

  - Major difficulty: Commutator estimates involving $P_k$. 

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  \]

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Trace Estimates for $\hat{\chi}$, $\zeta$

- Trace estimate: if $\nabla F = \nabla L P + E$, then

$$\| F \|_{L^\infty_t L^2} \lesssim \| F \|_{\mathcal{H}^1} + \| P \|_{\mathcal{H}^1} + \| E \|_{L^2_t B^0_{2,1}}.$$ 

- Goal: Apply to $\hat{\chi}$, $\zeta$.

- Problem: Not clear $\nabla \hat{\chi}$, $\nabla \zeta$ is of the above forms.

- Remark: Also need similar decompositions of $D$ Ric.

- To show this, we must use the following:
  - “Inverses” $D^{-1}$ of elliptic Hodge operators.
  - Null Bianchi identities.
  - Commutators involving $D^{-1}$.
  - Elaborate Besov estimates.
The End

Thank you!