A new approach to the Darboux-Bäcklund transformation versus the standard dressing method

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Abstract

We present a new approach to the construction of the Darboux matrix. This is a generalization of the recently formulated method based on the assumption that the square of the Darboux matrix vanishes for some values of the spectral parameter. We consider the multisoliton case, the reduction problem and the discrete case. The relationship between our approach and the standard dressing method is discussed in detail.

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1 Introduction

There are several methods to construct the Darboux matrix (which generates soliton solutions) \[1, 2, 3, 4, 6, 5, 7, 8\]). However, these methods are technically difficult when applied to the matrix versions of the spectral problems which are naturally represented in Clifford algebras \[9, 10, 12\]. Some of these problems are avoided in our recent paper \[13\]. In the present paper we develop the ideas of \[13\] in the matrix case. We extend our approach on the multisoliton case and consider the reduction problem and the discrete case. We also show that our approach, although different, is to some extent equivalent to the standard dressing method. We compare our method with the Zakharov-Shabat approach \[1, 14\] and the Neugebauer-Meinel approach \[3, 15\].

We consider the spectral problem
\[
\Psi_{\mu} = U_{\mu} \Psi, \quad (\mu = 1, \ldots, m) \tag{1}
\]
(with no assumptions on \(U_{\mu}\) except rational dependence on \(\lambda\)) and the Darboux transformation
\[
\tilde{\Psi} = D \Psi, \tag{2}
\]
which means that
\[
\tilde{\Psi}_{\mu} = \tilde{U}_{\mu} \tilde{\Psi}, \tag{3}
\]
where \(\tilde{U}_{\mu}\) and \(U_{\mu}\) have the same rational dependence on \(\lambda\) (\(U_{\mu}\) and \(\Psi\) are \(n \times n\) matrices but our approach works well also in the Clifford numbers case \[13\]).

The construction of the Darboux transformation is well known (especially in the matrix case) \[7, 14\]. The first step is the equation for \(D\) resulting from \[1, 2\] and \[3\]:
\[
D_{\mu} + D U_{\mu} = \tilde{U}_{\mu} D . \tag{4}
\]

In our earlier paper \[13\] we proposed the following procedure. We assume that there exist two different values of \(\lambda\), say \(\lambda_{+}\) and \(\lambda_{-}\), satisfying
\[
D^2(\lambda_{\pm}) = 0 . \tag{5}
\]
Denoting \(\Psi(\lambda_{\pm}) = \Psi_{\pm}\), \(D(\lambda_{\pm}) = D_{\pm}\), evaluating \[4\] at \(\lambda = \lambda_{\pm}\) and multiplying \[4\] by \(D_{\pm}\) from the right, we get:
\[
D_{\mu}, D_{\pm} + D_{\pm} U_{\mu}(\lambda_{\pm}) D_{\pm} = 0 . \tag{6}
\]
We assume that $\Psi(\lambda_\pm)$ are invertible (which is obviously true in the generic case). It is not difficult to check that $D_\pm$ given by
\[
D_\pm = \varphi_\pm \Psi_\pm d_\pm \Psi_\pm^{-1}, \quad d_\pm^2 = 0 ,
\]
(7)
(where $d_\pm = \text{const}$ and $\varphi_\pm$ are scalar functions) satisfy equations (5), (6). Assuming that $D$ is linear in $\lambda$, i.e.,
\[
D(\lambda) = A_0 + A_1 \lambda ,
\]
(8)
we can easily express $A_0, A_1$ by $D_\pm$ to get
\[
D(\lambda) = \frac{\lambda - \lambda_-}{\lambda_+ - \lambda_-} \varphi_+ \Psi_+ d_+ \Psi_+^{-1} + \frac{\lambda - \lambda_+}{\lambda_- - \lambda_+} \varphi_- \Psi_- d_- \Psi_-^{-1} .
\]
(9)

2 One-soliton case and the Zakharov-Shabat approach

We confine ourselves to the case linear in $\lambda$ (see (8)). The condition (5) can be easily realized if
\[
D_+^2(\lambda) = \sigma (\lambda - \lambda_+)(\lambda - \lambda_-)I
\]
(10)
where $\sigma \neq 0$ is a constant, $\lambda_+ \neq \lambda_-$ and $I$ is the identity matrix. The identity matrix will be sometimes omitted (i.e., for $a \in \mathbb{C}$ we write $aI = a$).

In the case (10) from (5) and (9) it follows that
\[
D_+D_- + D_-D_+ = -\sigma (\lambda_+ - \lambda_-)^2 .
\]
(11)

Lemma 1 $D$ of the form (8) satisfies (10) if and only if $n$ is even and
\[
D = \mathcal{N} (\lambda - \lambda_+ + (\lambda_+ - \lambda_-)P)
\]
(12)
where the matrices $\mathcal{N}$ and $P$ satisfy
\[
P^2 = P , \quad \mathcal{N}^2 = \sigma , \quad \mathcal{N}P\mathcal{N}^{-1} = I - P .
\]
(13)
In this case the Darboux matrices (4) and (12) are equivalent.

Proof: We denote $\mathcal{N} := A_1$. From (8) we get
\[
D^2(\lambda) = A_0^2 + (A_0 \mathcal{N} + \mathcal{N} A_0)\lambda + \mathcal{N}^2 \lambda^2 ,
\]
i.e., $D^2(\lambda)$ is a quadratic polynomial. It is proportional to the identity matrix $I$ (compare (10)) iff

$$N^2 = \sigma, \quad A_0 N + N A_0 = -\sigma (\lambda_+ + \lambda_-), \quad A_0^2 = \sigma \lambda_+ \lambda_- . \quad (14)$$

Multiplying the second equation by $N A_0$ we get

$$\sigma^2 \lambda_+ \lambda_- + (N A_0)^2 + \sigma (\lambda_+ + \lambda_-) N A_0 = 0 .$$

Hence $(N A_0 + \sigma \lambda_+)(N A_0 + \sigma \lambda_-) = 0$, and, denoting $Q := N A_0 + \sigma \lambda_+$, we have

$$Q^2 = (\lambda_+ - \lambda_-) \sigma Q$$

which means that $Q = (\lambda_+ - \lambda_-) \sigma P$, where $P^2 = P$. Therefore, taking into account $N^2 = \sigma$, we get (12). Now, we take into account the third equation of (14). First, $A_0^3 P = \sigma \lambda_+ \lambda_- P$ yields $\lambda_- (\lambda_+ - \lambda_-) N P N P = 0$. Then the equation $A_0^3 = \sigma \lambda_+ \lambda_- P$ is equivalent to $\lambda_- (\lambda_+ - \lambda_-)(\sigma (I - P) - N P N) = 0$. Therefore $N P N^{-1} = I - P$. This equality means that ker $P = N^{-1} \text{Im} P$ which implies dim ker $P = \text{dim \ Im} P$. Thus $n$ is even which completes the proof.

The case $\lambda_+ = \lambda_-$ can be treated in a similar way and it leads to the nilpotent case [7]:

$$D = N (\lambda - \lambda_+ + M), \quad M^2 = 0, \quad N^2 = \sigma, \quad M = -N M N^{-1} .$$

Our method is closely related to the standard dressing transformation [1, 7, 14]. The Darboux matrix (12) can be rewritten as

$$D = (\lambda - \lambda_+) N \left( I + \frac{\lambda_+ - \lambda_-}{\lambda - \lambda_+} P \right) . \quad (15)$$

We recognize the standard one-soliton Darboux matrix in the Zakharov-Shabat form [7, 14]. We point out that usually one considers the Darboux matrix $D = (\lambda - \lambda_+)^{-1} D$ which is equivalent to $D$ given by (15) because the multiplication of $D$ by a constant factor leaves the equation (11) invariant [16]. $N$ is known as the normalization matrix and $P$ is a projector expressed by the background wave function:

$$\ker P = \Psi(\lambda_+) V_{ker}, \quad \text{im} P = \Psi(\lambda_-) V_{im} , \quad (16)$$

$V_{ker}$ and $V_{im}$ are some constant vector spaces, $\lambda_+$ and $\lambda_-$ are constant complex parameters. The last constraint of (13) has the following interpretation. Let $N P N^{-1} = I - P$. Then

$$v \in \text{im} P \iff (I - P)v = 0 \iff PN^{-1}v = 0 \iff N^{-1}v \in \ker P$$

$$v \in \ker P \iff Pv = 0 \iff PN^{-1}v = N^{-1}v \iff N^{-1}v \in \text{im} P$$
Hence, \( \dim \text{im} P = \dim \ker P = d \equiv n/2 \), which implies \( \dim V_{im} = \dim V_{ker} \).

In this case, given a projector \( P \), one can always find a corresponding \( N \).

Indeed, let \( v_1, \ldots, v_d \) be a basis in \( \text{im} P \) and \( w_k := N^{-1} v_k \) \((k = 1, \ldots, d)\) an associated basis in \( \ker P \). By virtue of \( N^2 = \sigma \) we have \( N^{-1} w_k = \sigma^{-1} v_k \).

Therefore
\[
N^{-1}(v_1, \ldots, v_d, w_1, \ldots, w_d) = (w_1, \ldots, w_d, v_1/\sigma, \ldots, v_d/\sigma)
\]
(where \( (v_1, v_2, \ldots) \) denotes the matrix with columns \( v_1, v_2, \ldots \)) and, finally,
\[
N = (v_2, \ldots, v_d, w_1, \ldots, w_d)(w_1, \ldots, w_d, v_1/\sigma, \ldots, v_d/\sigma)^{-1}.
\]  

The \( N \) obtained in this way depends on the choice of the bases \( v_1, \ldots, v_d \) and \( w_1, \ldots, w_d \) (we can put \( Av_k, \det A \neq 0, \) in the place of \( v_k \) and \( Bw_j, \det B \neq 0, \) in the place of \( w_j \)). In other words, \( N \) is given up to nondegenerate \( d \times d \) matrices \( A \) and \( B \).

The formulas (9) and (12) coincide after appropriate identification of the parameters. Indeed, comparing coefficients by powers of \( \lambda \) we have:
\[
N = \frac{\varphi_+ \Psi_+ d_+ \Psi_+^{-1} - \varphi_- \Psi_- d_- \Psi_-^{-1}}{\lambda_+ - \lambda_-},
\]
\[
N(-\lambda_+ + (\lambda_+ - \lambda_-)P) = \frac{\lambda_+ \varphi_- \Psi_- d_- \Psi_-^{-1} - \lambda_- \varphi_+ \Psi_+ d_+ \Psi_+^{-1}}{\lambda_+ - \lambda_-},
\]
and after straightforward computation we get
\[
P = (\varphi_+ \Psi_+ d_+ \Psi_+^{-1} - \varphi_- \Psi_- d_- \Psi_-^{-1})^{-1} \varphi_+ \Psi_+ d_+ \Psi_+^{-1},
\]
\[
I - P = (\varphi_- \Psi_- d_- \Psi_-^{-1} - \varphi_+ \Psi_+ d_+ \Psi_+^{-1})^{-1} \varphi_- \Psi_- d_- \Psi_-^{-1}.
\]

Taking into account the assumption (11) we have:
\[
P = \frac{D_- D_+}{D_+ D_- + D_- D_+} = \frac{-D_- D_+}{\sigma(\lambda_+ - \lambda_-)^2}.
\]

The above results are valid for \( n \times n \) matrix linear problems. Now, we focus on the \( 2 \times 2 \) case. Because the elements \( d_+ \), \( d_- \) are nilpotent \((d_\pm = 0)\), then there exist vectors \( v_+, v_- \) such that
\[
d_+ v_+ = 0, \quad d_- v_- = 0.
\]

Then from (19) it follows immediately \( P \Psi_+ v_+ = 0 \) and \((I - P) \Psi_- v_- = 0\), i.e., \( \Psi_+ v_+ \) span \( \ker P \) and \( \Psi_- v_- \) span \( \text{im} P \). Hence, \( v_+ \in V_{ker} \) and \( v_- \in V_{im} \).
It is not difficult to check that the general form of $2 \times 2$ matrices $d_\pm$ such that $d_\pm^2 = 0$ is given by

$$d_\pm = \begin{pmatrix} -a_\pm b_\pm & b_\pm^2 \\ -a_\pm^2 & a_\pm b_\pm \end{pmatrix} = \begin{pmatrix} b_\pm \\ -a_\pm \end{pmatrix} \begin{pmatrix} -a_\pm & b_\pm \end{pmatrix},$$

(22)

where $a_\pm, b_\pm$ are complex numbers. Therefore, to satisfy (21), we can take

$$v_+ = \begin{pmatrix} b_+ \\ a_+ \end{pmatrix}, \quad v_- = \begin{pmatrix} b_- \\ a_- \end{pmatrix}.$$  

(23)

We have almost unique correspondence (i.e., up to a scalar factor) between $v_+$ and $d_+$ and between $v_-$ and $d_-$. 

Denoting

$$\Psi_+ v_+ \equiv \begin{pmatrix} B_+ \\ A_+ \end{pmatrix}, \quad \Psi_- v_- \equiv \begin{pmatrix} B_- \\ A_- \end{pmatrix},$$

we get the explicit formula for $P$

$$P = \begin{pmatrix} 0 & B_- \\ 0 & A_- \end{pmatrix} \begin{pmatrix} B_+ & B_- \\ A_+ & A_- \end{pmatrix}^{-1} = \begin{pmatrix} -A_+ B_- & B_+ B_- \\ -A_+ A_- & B_+ A_- \end{pmatrix} = \begin{pmatrix} \frac{A_+ B_- - A_- B_+}{A_+ B_+ - A_- B_-} & \frac{B_+ B_-}{A_+ B_+ - A_- B_-} \\ \frac{B_+ A_- - A_- B_+}{A_+ B_+ - A_- B_-} & \frac{B_+ B_-}{A_+ B_+ - A_- B_-} \end{pmatrix}$$

(24)

The corresponding $N$ reads (compare (17)):

$$N = \frac{1}{A_+ B_+ - A_- B_-} \begin{pmatrix} \sigma A_+ B_- - A_+ B_+ & B_+ B_- - \sigma B_+^2 \\ \sigma A_- B_- - A_- B_+ & A_+ B_+ - \sigma A_- B_- \end{pmatrix}$$

(25)

Although we can reduce our approach to the explicit formulas (24) and (25) the main advantage of our method consists in expressing the Darboux transformation in terms of $\Psi_\pm d_\pm \Psi_\pm^{-1}$ and avoiding difficulties with parameterizing kernel and image of the projector $P$ which is especially troublesome in the Clifford algebras case.

## 3 Reductions

Let us consider the unitary reduction

$$U_\mu^1(\lambda) = -U_\mu(\lambda).$$  

(26)

If $U_\mu$ is a polynom in $\lambda$, then the condition (26) means that the coefficients of this polynom by powers of $\lambda$ are $u(n)$-valued.
One can easily prove that (26) implies \( \Psi^\dagger(\bar{\lambda})\Psi(\lambda) = C(\lambda) \), where \( C(\lambda) \) is a constant matrix \( (C_{\nu\nu} = 0) \). The matrix \( C \) can be fixed by a choice of the initial conditions. Usually we confine ourselves to the case

\[
\Psi^\dagger(\bar{\lambda})\Psi(\lambda) = k(\lambda)I ,
\]

(27)

where \( k(\lambda) \) is analytic in \( \lambda \). From (27) we can derive \( \bar{k(\lambda)} = k(\lambda) \). By virtue of (2), the Darboux matrix have to satisfy the analogical constraint:

\[
D^\dagger(\bar{\lambda})D(\lambda) = p(\lambda)I .
\]

(28)

Assuming that \( D \) is a polynomial with respect to \( \lambda \), compare (8), we get that \( p(\lambda) \) is a polynomial with constant real coefficients, i.e., \( p(\lambda) = p(\lambda) \) and \( p,\nu = 0 \).

**Lemma 2** If \( D \) is linear in \( \lambda \) and (28) holds, then roots of the equation \( \det D(\lambda) = 0 \) satisfy the quadratic equation \( p(\lambda) = 0 \).

**Proof:** Let \( p(\lambda) = \alpha \lambda^2 + \beta \lambda + \gamma \). From (8), (28) it follows

\[
\begin{align*}
A_0^\dagger A_0 &= \gamma , & A_1^\dagger A_1 &= \alpha , & A_0^\dagger A_1 + A_1^\dagger A_0 &= \beta \\
\end{align*}
\]

(29)

which can be easily reduced to a single equation for \( S := -A_0 A_1^{-1} \). Namely,

\[
\alpha S^2 + \beta S + \gamma = 0 .
\]

(30)

Therefore, the eigenvalues of \( S \) have to satisfy the equation \( p(\lambda) = 0 \). Indeed, if \( S\vec{v} = \mu \vec{v} \), then \( (\alpha \mu^2 + \beta \mu + \gamma)\vec{v} = 0 \). On the other hand, the equation \( \det D(\lambda) = 0 \) can be rewritten as

\[
0 = \det(\lambda I - S) \det A_1,
\]

(31)

which means that the roots of \( \det D(\lambda) = 0 \) coincide with eigenvalues of \( S \). \( \Box \)

**Lemma 3** We assume (10). Then the reduction (27) imposes the following constraints on the Darboux matrix (3):

\[
\lambda_\pm = \lambda^\dagger_\pm , \quad d^\dagger_- d_+ = 0 ,
\]

(32)

and (for \( n = 2 \)) \( \langle v_+ | v_- \rangle = 0 \).
In particular, by virtue of (5), we can take \( d_- = f d_+^\dagger \), where \( f \) is a scalar function.

**Proof:** Let us denote zeros of the polynom \( p(\lambda) \) by \( \lambda_1, \lambda_2 \). Because \( p(\lambda) = \overline{p(\lambda)} \) there are two possibilities: either \( \lambda_2 = \lambda_1 \) or \( \lambda_1, \lambda_2 \) are real. From (10) we have

\[
(\det D(\lambda))^2 = \sigma^n(\lambda - \lambda_+)^n(\lambda - \lambda_-)^n.
\] (33)

Therefore, in the case (10), Lemma 2 means that \( \lambda_+ = \lambda_- \) coincide with \( \lambda_1, \lambda_2 \).

Suppose that \( \lambda_+ \in \mathbb{R} \). Then from (28) we have \( (D(\lambda_+))^{\dagger} D(\lambda_+) = 0 \) which implies \( D_+ \equiv D(\lambda_+) = 0 \) (because for any vector \( v \in \mathbb{C}^n \) the scalar product \( \langle v | D_+^\dagger D_+ v \rangle = 0 \), hence \( (D_+ v | D_+ v) = 0 \), and, finally \( D_+ v = 0 \)). Therefore \( \lambda_+ \) (and, similarly, \( \lambda_- \)) cannot be real. Thus \( \lambda_- = \lambda_+^\dagger \). In this case (28) reads

\[
(D(\lambda_-))^{\dagger} D(\lambda_+) = 0.
\] (34)

Using (7) and (27) (assuming \( k(\lambda_\pm) \neq 0 \)) we get

\[
(D(\lambda_-))^{\dagger} = \bar{\varphi}_- (\Psi_\dagger_-)^{-1} \bar{d}_- \Psi_\dagger_- = \bar{\varphi}_- \Psi_+ \bar{d}_+ \Psi^{-1}
\]

and (34) assumes the form \( \varphi_+ \bar{\varphi}_- \Psi_+ \bar{d}_- \Psi^{-1} = 0 \). Hence \( \bar{d}_- d_+ = 0 \).

Finally, in the case \( n = 2 \), we use (29). Then the condition \( \bar{d}_- d_+ = 0 \) is equivalent to \( a_+ \bar{a}_- + b_+ \bar{b}_- = 0 \), i.e., \( \langle v_+ | v_- \rangle = 0 \).

Another very popular reduction is given by

\[
U_\mu(-\lambda) = J U_\mu(\lambda) J^{-1}, \quad J^2 = c_0 I,
\] (35)

then one can prove that \( \Psi(-\lambda) = J \Psi(\lambda) C(\lambda) \), and we choose such initial conditions that \( C(\lambda) = J^{-1} \), i.e.,

\[
\Psi(-\lambda) = J \Psi(\lambda) J^{-1}, \quad D(-\lambda) = J D(\lambda) J^{-1}.
\] (36)

Such choice of \( C(\lambda) \) is motivated by a natural requirement that \( \Psi, \bar{\Psi}, D \) are elements of the same loop group (by the way, the formula (27) has the same motivation).

**Lemma 4** *We assume (10).* Then the reduction (36) imposes the following constraints on the Darboux matrix (9):

\[
\lambda_- = -\lambda_+, \quad \varphi_+ = \varphi_- , \quad d_+ = J^{-1} d_- J ,
\] (37)

and (for \( n = 2 \)) \( v_- = J v_+ \).
Proof: From (36) it follows that \( \det D(\lambda) = \det D(-\lambda) \) which means that the set of roots of the equation \( \det D(\lambda) = 0 \) is invariant under the transformation \( \lambda \rightarrow -\lambda \). Therefore \( \lambda_\pm = -\lambda_\mp \). Then, using once more (36) we get \( D_- = JD_+J^{-1} \) and \( \Psi_- = J\Psi_+J^{-1} \). Hence \( \varphi_+d_+ = \varphi_-J^{-1}d_-J \). Thus \( \varphi_+ = c_0\varphi_- \), where \( c_0 \) is a constant. Without loss of the generality we can take \( c_0 = 1 \) (redefining \( d_\pm \) if necessary). In the case \( n = 2 \) the kernels of \( d_\pm \) are 1-dimensional. Therefore \( 0 = d_+v_+ = J^{-1}d_-Jv_+ \) implies \( v_- = c_1Jv_+ \), where \( c_1 = \text{const} \). We can take \( v_+ = Jv_- \). \( \blacksquare \)

Other types of reductions (compare [2, 7]) can be treated in a similar way.

4 The multi-soliton Darboux matrix

In this section we generalize the approach of [13]. First, we relax the assumption [5]. Second, we consider the \( N \)-soliton case (the Darboux matrix is a polynomial of order \( N \)):

\[
D(\lambda) = A_0 + A_1\lambda + \ldots + A_N\lambda^N.
\]

The condition [5] will be replaced by:

\[
D(\lambda_k)T(\lambda_k) = 0
\]

We denote \( D_k \equiv D(\lambda_k) \), \( T_k \equiv T(\lambda_k) \), \( \Psi_k \equiv \Psi(\lambda_k) \) and \( U_{k\mu} \equiv U_{\mu}(\lambda_k) \). Evaluating (31) at \( \lambda = \lambda_k \) and multiplying the resulting equation by \( T_k \) from the right we get:

\[
D_{k;\mu}T_k + D_kU_{k\mu}T_k = 0
\]

To solve the equation (40) we define \( d_k \) and \( h_k \) by

\[
D_k = \Psi_kd_k\Psi_k^{-1}, \quad T_k = \Psi_kh_k\Psi_k^{-1}
\]

\[
D_{k;\mu} = \Psi_kd_{k;\mu}d_k\Psi_k^{-1} + \Psi_kd_{k;\mu}\Psi_k^{-1} - \Psi_kd_k\Psi_k^{-1}\Psi_{k;\mu}\Psi_k^{-1}.
\]

Therefore

\[
D_{k;\mu} = U_{k\mu}D_k + \Psi_kd_{k;\mu}\Psi_k^{-1} - D_kU_{k\mu},
\]

and, taking into account (39) and (41), we rewrite (40) as follows

\[
\Psi_kd_{k;\mu}h_k\Psi_k^{-1} = 0.
\]
Finally, as a straightforward consequence of (39) and (42) we get the following constraints on $d_k$ and $h_k$:

$$d_k h_k = 0, \quad d_k h_{k,\mu} = 0. \quad (43)$$

In [13] we confined ourselves to the case $T(\lambda) = D(\lambda)$, i.e., $d_k = \varphi_k d_{0k}$, ($\varphi_k$ scalar functions, $d_{0k}$ constant elements satisfying $d_{0k}^2 = 0$), $h_k = d_k$. Now we are going to obtain the general solution of (43) in the case of $2 \times 2$ matrices.

**Lemma 5** Let $d$ and $h$ are $2 \times 2$ matrices depending on $x^1, \ldots, x^n$ such that $dh = 0$, $dh_{,\mu} = 0$ and $d \neq 0$, $h \neq 0$. Then there exist constants $c^1, c^2$ and scalar functions $q^1, q^2, p^1, p^2$ (depending on $x^1, \ldots, x^n$) such that

$$d = \begin{pmatrix} q^1 c^2 & -q^1 c^1 \\ q^2 c^2 & -q^2 c^1 \end{pmatrix}, \quad h = \begin{pmatrix} c^1 p^1 & c^1 p^2 \\ c^2 p^1 & c^2 p^2 \end{pmatrix} \equiv cp^T \quad (44)$$

**Proof:** The columns of $h$ are orthogonal to the rows of $d$. If $\det(d) \neq 0$, then, obviously, $h = 0$ in contrary to our assumptions. Therefore $\det(d) = 0$ which means that the rows of $d$ are linearly dependent. Similarly, the columns of $h$ are linearly dependent as well. We denote them by $p^1 c$ and $p^2 c$ (where $c$ is a column vector). Thus $h = cp^T$, where $p^T := (p^1, p^2)$.

$dh = 0$ means that the columns of $h$ are orthogonal to the rows of $d$. Therefore these rows are of the form $q^1 c^\perp$, $q^2 c^\perp$, where $c^\perp$ is a vector orthogonal to $c$, and, finally $d = qc^\perp$. Thus we obtained (44).

Taking into account the condition $dh_{,\mu} = 0$ we get

$$0 = qc^\perp (c_{,\mu} p^T + cp^T_{,\mu}) = qc^\perp c_{,\mu} p^T \quad \Rightarrow \quad c^\perp c_{,\mu} = 0$$

This means that $c^2 c^1_{,\mu} = c^1 c^2_{,\mu},$ or $c^2 / c^1$ is a constant. In other words, $c^1 = f x^{10}$, $c^2 = f x^{20}$ ($f$ is a function, $x^{10}, x^{20}$ are constants. To complete the proof we redefine $p \rightarrow fp$, $q \rightarrow fqq$, and $c^{10} \rightarrow c^k$. $\square$

Therefore,

$$D(\lambda_k) = \Psi(\lambda_k) q_k c^k_1 \Psi^{-1}(\lambda_k), \quad (45)$$

where $c_k$ are given constant column unit vectors, $c^k_1$ is a row vector orthogonal to $c_k$ and $q_k$ are some vector-valued functions (column vectors). We keep the notation $q_k c^k_1 \equiv d_k$, but now in general $d^2_k \neq 0$.

We notice that the freedom concerning the choice of $q_k$ corresponds to the arbitrariness of the normalization matrix. In particular, the condition
imposes strong constraints on $N$. The condition (5) can be rewritten as
$q_k = φ_kc_k$

The constraint (33) implies $\det D(λ_k) = 0$. In the case of $2 \times 2$ matrices the equation $\det D(λ) = 0$ (where $D$ is given by (38)) has $2N$ roots (at most): $λ_1, \ldots, λ_{2N}$.

Taking any $N + 1$ pairwise different roots (say $λ_1, \ldots, λ_{N+1}$) and using Lagrange’s interpolation formula for polynomials, we get the generalization of the formula (9):

$$ D(λ) = \sum_{k=1}^{N+1} \left( \prod_{j=1}^{N+1} \frac{λ - λ_j}{λ_k - λ_j} \right) ψ(λ_k)q_kc_k^⊥ψ^{-1}(λ_k). $$

(46)

We have also $N - 1$ matrix constraints which result from evaluating the formula (46) at $λ_{N+2}, \ldots, λ_{2N}$:

$$ \sum_{k=0}^{N+1} \frac{ψ(λ_k)q_kc_k^⊥ψ^{-1}(λ_k)}{(λ_k - λ_0) \ldots (λ_k - λ_{k-1})(λ_k - λ_{k+1}) \ldots (λ_k - λ_{N+1})} = 0, \quad (47) $$

where $λ_0 = λ_{N+2}, \ldots, λ_{2N}$.

We denote

$$ Q_k := ψ(λ_k)q_k, \quad C_k^⊥ := c_k^⊥ψ^{-1}(λ_k) $$

(48)

The Darboux matrix is parameterized by $2N$ constants $λ_k$, $2N$ vector functions $q_k$ and $2N$ constant vectors $c_k$ subject to the constraints (47).

The crucial point consists in solving the system (47) in order to get parameterization of the Darboux matrix by a set of independent quantities. We plan to express $2N - 2$ functions from among $Q_1, \ldots, Q_{2N}$ by other data. For instance, we choose $Q_1, Q_2$ as independent functions (they correspond to the normalization matrix $N$).

We rewrite the system (47) as

$$ σ_{ν0}Q_νC_ν^⊥ + \sum_{k=1}^{N+1} σ_{νk}Q_kC_k^⊥ = 0, \quad (ν = N + 2, \ldots, 2N), \quad (49) $$

where

$$ σ_{νk} = \frac{1}{(λ_k - λ_ν)(λ_k - λ_1) \ldots (λ_k - λ_{k-1})(λ_k - λ_{k+1}) \ldots (λ_k - λ_{N+1})} $$

$$ σ_{ν0} = \frac{1}{(λ_ν - λ_1) \ldots (λ_ν - λ_N)(λ_ν - λ_{N+1})}. $$

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Thus we have a system linear with respect to $Q_k$. We are going to express $2N - 2$ vector functions $Q_3, \ldots, Q_{2N}$ by $Q_1, Q_2$ and the other parameters: $C_k, \lambda_k$. Then, using (48), we could get $q_3, \ldots, q_{2N}$, etc. However, it is better to write (46) in terms of $Q_k$:

$$D(\lambda) = \sum_{k=1}^{N+1} \left( \prod_{j=1}^{N+1} \frac{\lambda_j - \lambda_k}{\lambda_j - \lambda_{j'}} \right) Q_k c_k^\perp \Psi^{-1}(\lambda_k).$$

(50)

Taking the scalar product of (49) by $C_1$ we get

$$Q_1 = -\sum_{k=2}^{N+1} \frac{\sigma_{\nu k} \langle C^\perp_k | C_1 \rangle}{\sigma_{\nu 0} \langle C^\perp_\nu | C_1 \rangle} Q_k,$$

(51)

and the scalar product of $\nu$th equation of (49) by $C_\mu$ yields

$$\sum_{k=1}^{N+1} \sigma_{\nu k} \langle C^\perp_k | C_\nu \rangle Q_k = 0,$$

(52)

This is a system of $N - 1$ linear equations with respect to $Q_1, \ldots, Q_{N+1}$. Therefore, we can (for instance) express $Q_3, \ldots, Q_{N+1}$ in terms of $Q_1, Q_2$. Then, using (51), we have $Q_{N+2}, \ldots, Q_{2N}$ expressed in the similar way.

Our method is closely related to the Neugebauer-Meinel approach [3].

Let $D$ is given by (38). We denote by $F(D(\lambda))$ the adjugate (or adjoint) matrix of $D$ which is, obviously, a polynom in $\lambda$. Thus

$$D(\lambda) F(D(\lambda)) = w(\lambda) I$$

(53)

where $w(\lambda) = \det(D(\lambda))$ is a scalar polynom and $I$ is the identity matrix.

Therefore, we can put $T(\lambda) = F(D(\lambda))$ in the formula (39) and identify $\lambda_k$ with zeros of $\det D(\lambda)$.

In the Neugebauer approach the matrix coefficients $A_k$ of the Darboux matrix are obtained by solving the following system

$$D(\lambda_k) \Psi(\lambda_k) c_k = 0, (k = 1, \ldots, nN)$$

(54)

where $\lambda_k$ and constant vectors $c_k$ are treated as given parameters. Thus one has $n^2N$ scalar equations for $(N+1)n^2$ scalar variables. One of the matrices $A_k$, say $A_N$, is considered as undetermined normalization matrix.

We point out that $D(\lambda_k)$ given by the formula (45) satisfy (54).
5 The discrete case

The discrete analogue of (1) is the following system of linear difference equation

\[ T_\mu \Psi = U_\mu \Psi, \quad (\mu = 1, \ldots, m), \]

(55)

where \( T_\nu \) denotes the shift in \( \nu \)-th variable, i.e., \( (T_\nu \Psi)(x^1, \ldots, x^\nu, \ldots, x^m) := \Psi(x^1, \ldots, x^\nu + 1, \ldots, x^m) \). The Darboux transformation is defined in the standard way:

\[ \tilde{\Psi} = D\Psi, \quad T_\mu \tilde{\Psi} = \tilde{U}_\mu \tilde{\Psi}. \]

(56)

Therefore \((T_\mu D)(T_\mu \Psi) = \tilde{U}_\mu D\Psi\), and, finally

\[ (T_\mu D)U_\mu = \tilde{U}_\mu D \]

(57)

If \( D^2(\lambda_1) = 0 \), then multiplying (57) by \( D(\lambda) \) from the right, and evaluating the obtained equation at \( \lambda = \lambda_1 \) we see that the right hand side vanishes and we get:

\[ (T_\mu D_1)U_\mu(\lambda_1)D_1 = 0 \]

(58)

where we denote \( D_1 := D(\lambda_1) \). In order to solve (58) we put

\[ D_1 = \varphi_1 \Psi_1 d_1 \Psi_1^{-1} \]

where \( \Psi_1 := \Psi(\lambda_1) \). Then (58) takes the form:

\[ \varphi_1 T_\mu(\varphi_1)(T_\mu \Psi_1)(T_\mu d_1)d_1 \Psi_1^{-1} = 0. \]

Therefore, if

\[ (T_\mu d_1)d_1 = 0 \]

(59)

then the equation (58) is satisfied. The condition (59) can be rewritten (at least in the matrix case) as

\[ \text{Im}d_1 \subset \ker(T_\mu d_1) \]

In other words, the sequence of linear operators

\[ \ldots \rightarrow T_\mu^{-1}d_1 \rightarrow d_1 \rightarrow T_\mu d_1 \rightarrow T_\mu^2 d_1 \rightarrow \ldots \]

is an exact sequence [17].

Similarly as in the smooth case we mostly confine ourselves to the simplest solution of (59), i.e., \( d_1 = \text{const} \) which implies \( d_1^2 = 0 \). The Darboux matrix has the same form [19] as in the continuum case.

Summary. In this paper we developed the approach of [13] considering explicitly the most important reductions, extending our results on the \( N \)-soliton case, and showing that the discrete case is, as usual, very similar to the continuous one.
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