W-algebras arising as chiral algebras of conformal field theory

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It is argued that chiral algebras of conformal field theory possess a W-algebra structure. A survey of explicitly known W-algebras and their constructions is given.

Introduction

The presence of infinite dimensional chiral symmetries is central to the study of two dimensional conformal field theories. These symmetries, called chiral algebras, are formed by the purely holomorphic and purely anti-holomorphic fields, respectively. How the field content of a conformal field theory (CFT) is organised into representations of the chiral algebras is encoded in the fusion ring. The task of solving and classifying general conformal field theories can be split into two separate problems: to find chiral algebras of conformal field theory and to determine possible fusion rings of chiral algebras.

This programme has been most successful for the Virasoro minimal series [1], where the fields fall into finitely many representations of the Virasoro algebra. This has subsequently enabled the complete characterisation of the field content of such theories [2]. But considerable progress has also been made in constructing other chiral algebras of CFT. I present here a survey of these results.

Chiral algebras

The (left) chiral algebra of a two dimensional conformal field theory (CFT) is the symmetry algebra formed by all fields $\phi(z)$ which depend analytically on the coordinates. Locality implies that $[\phi(z), \psi(\zeta)] = 0$, for $z \neq \zeta$ and that all correlation functions are meromorphic in the arguments with poles only at coinciding arguments. From this one can derive the short distance behaviour of chiral fields which is given by the operator product expansion (OPE),

$$
\phi(z)\psi(\zeta) = \sum_n (z-\zeta)^n \chi_n(\zeta),
$$

where the $\chi_n(\zeta)$ are again local fields and only finitely many singular terms appear. Fields which transform homogeneously under projective transformations $\gamma: z \mapsto (az+b)/(cz+d)$ are

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called quasi-primary (or non-derivative). They correspond to highest weights of the \( su(1,1) \) algebra generated by the modes \( L_0, L_{\pm 1} \) of the stress tensor. All other fields can be obtained by taking derivatives. We will thus restrict attention to quasi-primary fields and choose a basis \( \{ \chi^i \} \) with conformal weights \( \{ h_i \} \). Using \( su(1,1) \)-covariance and the usual contour deformation argument we find from the OPE that the commutator of two quasi-primary fields has the form

\[
[\chi^i_m, \chi^j_n] = \sum_k C^{ij}_k p^{h_i h_j}(m, n) \chi^k_{m+n} + D^{ij}(m+h_i-1)_2 \delta_{m+n}.
\]  

(2)

The contributions from a particular \( su(1,1) \) representation are collected in the \( su(1,1) \)-Clebsch-Gordon coefficients \( p^{h_i h_j}(m, n) \) which are polynomials in \( m \) and \( n \) depending only on the conformal weights of the fields involved [3]. The \( D^{ij} = \langle \chi^i | \chi^j \rangle \) define a metric on the space of quasi-primary fields which can be used to raise and lower indices. The field content of a particular chiral algebra is encoded in the structure constants \( D^{ij} \) and \( C^{ijk} = \langle \chi^i | \chi^j(1) | \chi^k \rangle \).

Locality of the chiral algebra is equivalent to the Jacobi identity for the commutator algebra which takes the form

\[
\sum_{N=1}^{h_k+h_l-1} \left( \sum_{q,d} C^{ijq} C^{kld'} D_{qq'} \right) p^{h_q h_l}(m, n + p) p^{h_k h_l}(n, p) + \text{cyclic} = 0.
\]  

(3)

The study of chiral algebras is thus equivalent to the study of the Lie algebra of quasi-primary fields. However, we still have to deal with an infinite number of quasi-primary fields.

In addition to the Lie bracket structure, which is determined solely by the singular part of the OPE, chiral algebras admit a second structure encoded in the regular part of the OPE, the so-called normal ordered products (NOPs). They are usually defined as the the constant term of the OPE, \( (\phi \psi)(z) = \oint \frac{d\zeta}{2\pi i} (\zeta - z)^{-1} \phi(\zeta) \psi(z) \). However, \( (\phi \psi)(z) \) is not quasi-primary. We therefore define a new NOP, \( \phi \diamond \psi \), by projecting \( (\phi \psi)(z) \) onto quasi-primary fields, i.e. \( \phi \diamond \psi \) is the new quasi-primary field entering the OPE of \( \phi \) and \( \psi \) at the constant term. The \( \diamond \)-product is commutative, but still non-associative, and differs from the usual NOP only by a finite number of correction terms which are known explicitly [3].

We call fields which can be obtained from the \( \diamond \)-product composite. Fields which are orthogonal to all composite fields are called simple. The commutators of simple fields yields fields which are in general not simple but involve NOPs of simple fields. The algebraic structure given by commutators and NOPs of simple fields is called a W-algebra.

The simple fields generate the complete chiral algebra through NOPs and derivatives and all structure constants involving composite fields are determined by the couplings between simple fields. This means that the chiral algebra is completely determined by the W-algebra. Typically, a W-algebra has only a finite number of simple fields reducing the problem of constructing a chiral algebra to the finite problem of finding solutions to the associativity constraints (3).

One starts be specifying the weights of simple fields, \( (h_1, \ldots, h_r) \) and then attempts to construct the W-algebra \( W_{(h_1, \ldots, h_r)} \) as follows.
• Construct all composite fields of weight less than twice the maximum of the $h_i$. These are the only fields which can appear in the commutator of two simple fields.

• Express the structure constants for composite fields in terms of the structure constants for simple fields only.

• Impose the Jacobi identity for simple fields.

This yields a system of polynomial equations for the simple structure constants which one now has to solve. There are three possibilities:

• $W(h_1,...,h_r)$ is consistent for generic values of $c$ (generic $W$-algebras),

• $W(h_1,...,h_r)$ is consistent for a finite set of $c$-values (exceptional $W$-algebras),

• $W(h_1,...,h_r)$ is inconsistent for all values of $c$.

This approach is systematic enough to be amenable to algebraic computing. The results obtained so far [3–5] fall into the following classes.

**Generic $W$-algebras**

**Current algebras:** If the simple fields are all of weight one they form a current algebra. Their commutator algebra is a Kac-Moody algebra and the $W$-algebra is given by all derivatives and NOPs of the currents.

**Toda theories:** $W$-algebras also appear as the chiral symmetry algebras of Toda field theory. The $W$-algebra fields are normal ordered differential polynomials of the Toda fields which commute with each of the exponential terms in the Toda Lagrangian [6]. The $W$-algebra $Wg$ obtained from Toda field theory based on a Lie algebra $g$ has rank $g$ simple fields. Their weights are equal to the orders of the independent Casimir operators of $g$ [7]. The following quantum $W$-algebras have been constructed so far:

| $Wg$              | weights |
|-------------------|---------|
| $WA_1$            | 2       |
| $W(A_1 \times U(1))$ | 2, 1    |
| $W(A_1 \times A_1)$ | 2, 2    |
| $WA_2$            | 2, 3    |
| $WC_2$            | 2, 4    |
| $WG_2$            | 2, 6    |
| $WA_3$            | 2, 3, 4 |
| $WB_3$            | 2, 4, 6 |
| $WC_3$            | 2, 4, 6 |
| $W(A_1^n)$        | $2^n$   |

It has been realised recently that Toda field theories can be viewed as Hamiltonian reductions of constrained WZNW theories [8]. In this context it is possible to associate (at least classically) a $W$-algebra to every integral $sl(2)$ embedding into a simple Lie algebra. In this case the weights of the simple fields are one higher than the exponents of $g$ with respect to the $sl(2)$ embedding. However, no quantum version of such a generalised algebra has been constructed as yet.
**Orbifold models**: Whenever a W-algebra has a (discrete) automorphism \( \sigma \) the subspace of fields invariant under \( \sigma \) forms again a W-algebra. An example of this is given by one of the algebras \( W_{(2,4,6)} \), which turns out to be the bosonic projection of the super-Virasoro algebra [9].

**Exceptional W-algebras**

**Extensions**: If for a specific \( c \)-value a generic W-algebra possesses representations of integral weight with appropriate fusion rules then it is possible to enlarge the generic W-algebra by these representations to obtain an exceptional W-algebra. This is indicated by the appearance of a non-diagonal modular invariant [3]. The simplest example is given by the \((A_{p-1}, D_{q/2+1})\) modular invariant of the Virasoro algebra. Here, the field \( \phi_{p-1,q-1} \) has weight \( h = (p-2)(q-2)/4 \) and fusion rule \( \phi_{p-1,q-1} \times \phi_{p-1,q-1} = 1 \). Whenever \( p \) or \( q \) is even, this yields a W-algebra \( W_{(2,h)} \). Exceptional W-algebras coming from such an extension of the Virasoro algebra are listed below. Extensions of other generic W-algebras have not yet been constructed.

| weights | \( c \)-values | weights | \( c \)-values |
|---------|---------------|---------|---------------|
| 2, 2    | \(-\frac{44}{5}, -\frac{39}{10}, -\frac{494}{35}\) | 2, 7    | \(-\frac{25}{2}\) |
| 2, 3    | \(-\frac{114}{7}, -2, \frac{4}{5}\)   | 2, 8    | \(-\frac{944}{17}, \frac{21}{22}, -\frac{224}{65}\) |
| 2, 4    | \(-\frac{11}{14}\)                   | 2, 3, 3 | \(-2\)        |
| 2, 5    | \(-\frac{350}{11}, -7, \frac{6}{7}\) | 2, 5, 5 | \(-7\)        |
| 2, 6    | \(-\frac{516}{13}, -\frac{306}{35}\) | 2, 7, 7 | \(-\frac{25}{2}\) |

**Decoupling**: It can happen that at specific values of \( c \) some of the simple fields of a generic W-algebra decouple and one is left with an exceptional W-algebra. The simplest example is the Ising-model \( c = \frac{1}{2} \) which is in the \( WE_8 \) minimal series. In this case all simple fields apart from the stress tensor decouple. Known exceptional W-algebras arising from this decoupling procedure are:

| \( Wg \) | weights | \( c \)-values | weights | \( c \)-values |
|---------|---------|---------------|---------|---------------|
| \( WE_6 \) | 2, 5, 6, 8, 9, 12 | \( -\frac{350}{11}, -7, \frac{6}{7} \) | 2, 5 | \( -\frac{944}{17} \) |
| \( WE_8 \) | 2, 8, 12, 14, 18, 20, 24, 30 | \( -\frac{944}{17} \) | 2, 8 | \( -\frac{224}{65} \) |
| \( WD_4 \) | 2, 4, 4, 6 | \( 1, -\frac{656}{11} \) | 2, 4, 4 | \( 1, -\frac{656}{11} \) |

These results support the conjecture that all generic W-algebras are all given by Hamiltonian reduction of constrained WZWN models and their orbifolds. The picture for exceptional W-algebras is much more complicated. While the majority of cases fit the two procedures of extending and decoupling there are a large number of cases which do not fall into these patterns [3, 10].
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