QUANTUM CURVES

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Abstract
One says that a pair \((P, Q)\) of ordinary differential operators specify a quantum curve if \([P, Q] = ℏ\). If a pair of difference operators \((K, L)\) obey the relation \(KL = λLK\) where \(λ = e^ℏ\) we say that they specify a discrete quantum curve.

This terminology is prompted by well known results about commuting differential and difference operators, relating pairs of such operators with pairs of meromorphic functions on algebraic curves obeying some conditions.

The goal of this paper is to study the moduli spaces of quantum curves. We will relate the moduli spaces for different \(ℏ\). We will show how to quantize a pair of commuting differential or difference operators (i.e. to construct the corresponding quantum curve or discrete quantum curve).

1. Introduction
One says that a pair \((P, Q)\) of ordinary differential operators specify a quantum curve if \([P, Q] = ℏ\) \([10], [13], [3]\). If a pair of difference operators \((K, L)\) obey the relation \(KL = λLK\) where \(λ = e^ℏ\) we say that they specify a discrete quantum curve.

This terminology is prompted by well known results about commuting differential and difference operators \([8], [12]\), relating pairs of such operators with pairs of meromorphic functions on algebraic curves obeying some conditions.

The goal of this paper is to study the moduli spaces of quantum curves. We will relate the moduli spaces for different \(ℏ\). We will show how to quantize a pair of commuting differential or difference operators (i.e. to construct the corresponding quantum curve or discrete quantum curve). This construction generalizes the considerations of \([13]\).

The KP-hierarchy acts on the moduli space of quantum curves; we prove that similarly the discrete KP-hierarchy acts on the moduli space of discrete quantum curves.

We consider also matrix differential and difference operators and obtain similar results. (The generalization of \([13]\) to matrix differential operators was given in \([7]\).)

Eynard-Orantin topological recursion \([4]\) gives a construction of free energy and correlation functions corresponding to an algebraic curves and two meromorphic functions on it. We construct a quantum curve starting with the same data (but the conditions that we impose on meromorphic functions are different). It seems that the paper \([5]\) can be considered as a bridge between topological recursion and our constructions. The modification of topological recursion that was required to ”remodel B-model” \([2]\) is related to discrete quantum curves in our sense.

We will discuss the relation of our constructions to the results of \([3]\).

2. Differential operators. Quantum curves.
Let us define a pseudodifferential operator as a formal series
\[
L = \sum_{k} a_k(x) D^k
\]
where \( D = \frac{d}{dx} \) and \( a_k(x) \) stands for a formal power series: 
\[
a_k(x) = \sum a_k x^i.
\]
We assume that \( k \in \mathbb{Z} \) and \( a_k(x) = 0 \) for \( k >> 0 \). The operator has order \( q \) if its leading term (the non-zero term with greatest \( k \)) is equal to \( a_q(x)D^q \); the operator is monic if \( a_q(x) = 1 \), a monic operator is normalized if \( a_{q-1}(x) = 0 \). Monic pseudodifferential operators of order 0 form a group denoted by \( \mathcal{G} \).

We denote by \( \mathcal{H} \) the space of Laurent polynomials \( \sum_{k<<\infty} c_k z^k \), by \( \mathcal{H}_+ \) its subspace consisting of polynomials and by \( \mathcal{H}_- \) the subspace spanned by \( z^n \) where \( n < 0 \). Pseudodifferential operators act on \( \mathcal{H} \); the differentiation \( D \) acts as multiplication by \( z \) and multiplication by \( x \) acts as \( -\frac{d}{dx} \). Differential operators can be characterized as pseudodifferential operators preserving \( \mathcal{H}_+ \). Every pseudodifferential operator \( L \) can be represented as a sum of differential operator \( L_+ = \sum_{k>0} a_k(x)D^k \) and “integral” operator \( L_- = \sum_{k<0} a_k(x)D^k \).

We will denote by \( \mathcal{G}_r \) the space of all subspaces \( V \subset \mathcal{H} \) such that the natural projection \( \pi_+ : \mathcal{H} \rightarrow \mathcal{H}_+ \) induces an isomorphism between \( V \) and \( \mathcal{H}_+ \). In other words the subspace \( V \in \mathcal{G}_r \) if it has a basis of the form \( v_n = z^n + r_n \) where \( n \geq 0 \) and \( r_n \in \mathcal{H}_- \). The space \( \mathcal{G}_r \) is called Sato Grassmannian.

The following theorems belongs to Sato (see [11] for the proof):

**Theorem 2.1.** There exists one-to-one correspondence between the elements of the group \( \mathcal{G} \) of monic zeroth order differential operators and points of \( \mathcal{G}_r \). Namely, every subspace \( V \in \mathcal{G}_r \) has a unique representation in the form \( V = S \mathcal{H}_+ \) where \( S \in \mathcal{G} \).

The commutative Lie algebra \( \gamma_+ \) of polynomials \( \sum_{k\geq0} t_k z^k \) acts on \( \mathcal{G}_r \) in natural way. (This action comes from the remark that we can multiply the elements of \( \mathcal{H} \) by \( g(t) = \exp(\sum_{k\geq0} t_k z^k) \) where \( t_k \) are nilpotent parameters.)

It is clear from Theorem 2.1 that \( \gamma_+ \) acts also on \( \mathcal{G} \):

\[
\frac{\partial S}{\partial t_n} = (SD^n S^{-1})_- S.
\]

**Theorem 2.2.** Every normalized pseudodifferential operator \( Q \) of order \( q \) can be represented in the form \( S^{-1} D^n S \) where \( S \in \mathcal{G} \); this representation is unique up to multiplication by an operator with constant coefficients.

Using this statement we can construct the action of Lie algebra \( \gamma_+ \) on the space of normalized pseudodifferential operators differentiating the relation \( Q(t) = S^{-1}(t)D^n S(t) \) with respect to \( t_n \).

The action on this space can be written in the form of differential equation

\[
\frac{\partial Q}{\partial t_n} = [Q_+^{\frac{n}{2}}, Q]
\]

Notice that this formula determines also the action of Lie algebra \( \gamma_+ \) on the space of normalized differential operators.

All actions we described can be considered as different forms of KP-hierarchy.

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1Pseudodifferential operators constitute an associative algebra \( \mathcal{A} \) for appropriate definition of multiplication. We will not give this definition; see, for example, [11]. Let us notice, however, that the multiplicative structure in \( \mathcal{A} \) can be recovered from the representation of \( \mathcal{A} \) by the operators in \( \mathcal{H} \) that is described in the next paragraph.

2It is the big cell of the index zero part of the infinite-dimensional Grassmannian; we do not need the description of this Grassmannian.

3More generally, one can consider the action of the Lie algebra \( \gamma \) of polynomials \( \sum_{-\infty<<k<<\infty} t_k z^k \); this action will be important in the next section.

4The fractional powers entering this formula can be defined using the representation \( Q = S^{-1} D^n S \).
We would like to solve the equation $[P, Q] = h$. We assume that $P$ is a differential operator of order $p$ and $Q$ is a normalized differential operator of order $q$. Using Theorem 2.2 we construct the operator $S \in \mathcal{G}$ such that $SQS^{-1} = D^q$. Introducing the notation $V = SH_+$ we obtain a subspace $V \in Gr$ invariant with respect to multiplication by $z^q$ and with respect to the action of the operator $\tilde{P} = h \frac{d}{dz^q} + b(z)$ where $b(z)$ stands for the multiplication by a Laurent series denoted by the same letter. (We use the fact that the action of $D^q$ can be interpreted as multiplication by $z^q$ and the fact that $H_+$ is invariant with respect to multiplication by differential operators. The form of the operator $\tilde{P} = SQS^{-1}$ follows from the relation $[\tilde{P}, \tilde{Q}] = h$ where $\tilde{Q} = SQS^{-1} = z^q$.) We can invert this consideration to obtain the following statement [13]:

**Theorem 2.3.** If $V \in Gr$ is invariant with respect to the operator of multiplication by $z^q$ and with respect to the operator $h \frac{d}{dz^q} + b(z)$ we can construct a differential operator $P$ and a normalized differential operator $Q$ obeying $[P, Q] = h$. The leading term of the operator $P$ is determined by the leading term of the Laurent series $b(z)$.

The construction is based on the Sato theorem: We represent $V$ in the form $V = SH_+$ where $S \in \mathcal{G}$ and transform the operators acting on $V$ by means of the operator $S$. We obtain pseudodifferential operators acting in $H_+$, i.e. differential operators.

It follows from Theorem 2.3 that the Lie algebra $\Gamma_+$ acts on the space of pairs $(P, Q)$ of differential operators obeying $[P, Q] = h$. (We assume that $Q$ is normalized.) The proof is based on the remark that for $V \in Gr$ satisfying the conditions of the theorem the subspace $V(t) = g(t)V$ also obeys the same conditions with $\tilde{P} = h \frac{d}{dz^q} + b(z)$ replaced with

$$g^{-1}(t)\tilde{P}g(t) = h \frac{d}{dz^q} + b(z) - \sum_i \frac{i}{q} t_i z^{i-1}.$$ 

In other words, we can say that the KP-flows [2] are defined on the space of pairs we are interested in.

We say that the vectors $v_0, ..., v_{q-1}$ form a $g$-basis in $V$ if the vectors $g^m v_i$ where $0 \leq i < q, 0 \leq m$, form a basis of $V$. We will use this definition in the case $g = z^q$.

To construct an example of $z^q$-basis we recall that $V \in Gr$ has a basis $v_n = z^n + r_n$ where $r_n \in H_-$. The first $q$ vectors of this basis (vectors $v_0, ..., v_{q-1}$) form a $z^q$-basis of $V$.

If $SQS^{-1} = D^q, V = SH_+, v_0, ..., v_{q-1}$ is a $z^q$-basis of $V$ we represent the operator $\tilde{P} = SQS^{-1}$ in this basis. We obtain

$$(3) \quad \tilde{P} v_j = M^j_i (z^q) v_j$$

where the entries of the matrix $M$ are polynomials with respect to $z^q$. We say that the matrix $M$ is the companion matrix of the pair $(P, Q)$. (Alternatively one can define the companion matrix as a matrix of $P$ in a $Q$-basis of $H_+$.) The companion matrix depends on the choice of $z^q$-basis. More precisely, two $z^q$-bases are related by the formula $\tilde{v}_i = A^j_i (z^q) v_j$ where $A^j_i (z^q)$ is an invertible polynomial matrix (its entries are polynomials with respect to $z^q$). We will choose the basis $v_i$ in such a way that $v_i = c_i z^i$+lower order terms, $c_i \neq 0$. (Notice that this condition does not specify the vectors $v_i$ uniquely; we can replace $v_i$ with $\tilde{v}_i = t_i^j v_j$ where $t$ is a constant invertible triangular matrix: $t_i^j = 0$ for $i \leq j$.)

Let us prove the following theorem:

**Theorem 2.4.** If the leading coefficient (the coefficient of the leading term) of the matrix $z^{j-i} M^j_i (z^q)$ has $q$ distinct eigenvalues then there exist $q$ pairs of differential operators obeying $[P, Q] = h$ and having $M$ as the companion matrix.
To prove this theorem we should find $q$ Laurent series $b(z)$ and corresponding $v_0, \ldots, v_{q-1}$ in such a way that

$$ (\hbar \frac{d}{dz}^q + b(z))v_i = M^j_i(z^q)v_j $$

and $z^{mq}v_i$ where $0 \leq i < q$, $0 \leq m$ form a basis of a subspace $V \in Gr$. Then we can apply Theorem \ref{2.3}.

We change the variables in the equation (4) substituting $v_i = z^i u_i$ where $u_i = c_i + \text{lower order terms}$. We obtain the equation

$$ (\hbar \frac{d}{dz}^q + b(z))u_i = B^j_i(z)u_j $$

where

$$ B(z) = (B^j_i(z)) = \left( M^j_i(z^q)z^{j-i} - \frac{i\hbar}{qz^q}\delta^j_i \right) $$

Let us consider first the case when $\hbar = 0$. Then $b(z)$ is one of eigenvalues $\lambda_k(z)$ of the matrix $B^j_i(z)$ and $u_i$ are components of the eigenvector. The existence of $b$ and of the vector $u(z) = (u_0, \ldots, u_{q-1})$ obeying the conditions we need follows immediately from the perturbation theory. (If $B$ is replaced by its leading term this statement follows from our assumptions. All other terms of $B$ can be considered as a perturbation of the leading term. The leading term of the matrix $B^j_i(z)$ coincides with the leading term of the matrix $z^{j-i}M^j_i(z^q)$, therefore its eigenvalues are distinct. This allows us to construct $b$ as a Laurent series and $u(z)$ as a power series with respect to $z$.)

If $\hbar \neq 0$ we consider the auxiliary equation

$$ \hbar \frac{d}{dz}^q w_i = B^j_i(z)w_j, $$

or, equivalently,

$$ \hbar \frac{d}{dz} w_i = qz^{q-1}B^j_i(z)w_j. $$

As we have noticed the eigenvalues of the leading term of $B(z)$ are distinct. This allows us to say that the equation can be diagonalized by means of the formal change of variables $w(z) = R(z)t(z)$ where $R(z) = 1 + \sum_{k \geq 1} R_k z^{-k}$; see \ref{14}. This means that the equation for the components of the vector $t(z)$ looks as follows:

$$ \hbar \frac{d}{dz} t_i = \Lambda_i t_i. $$

Let us consider $q$ solutions of the equation (9) having the form

$$ t_k = \exp\left( \int \hbar^{-1} \Lambda_k(z) qz^{q-1} dz \right), $$

$t_i = 0$ for $i \neq k$.

The corresponding solutions of the equation (6) have the form

$$ w_i = \exp\left( \int \hbar^{-1} \Lambda_k(z) qz^{q-1} dz \right) r_{ik}(z), $$

where $r_{ik}(z) = c_{ik} + \text{lower order terms}$. Now for every $k$ it is easy to find $b(z)$ in such a way that $r_{ik}(z)$ becomes a solution to the equation (5). Namely, we should take

$$ b_k(z) = \Lambda_k(z). $$
(We use the fact that the equation (5) can be reduced to the equation (6) by means of the substitution \( w = \rho u \).) This gives the proof of the theorem.

Notice that \( \Lambda_k(z) = \lambda_k(z) + O(\hbar) \) where \( \lambda_k(z) \) stands for the eigenvalue of the matrix \( B^j_2(z) \).

(This follows, for example, from the comparison with the case \( \hbar = 0 \).)

Let us give another proof of the theorem that can be applied in more general situations. We would like to find solutions of the equation (5) as power series:

\[
    u(z) = \sum_{k \geq 0} k u z^{-k},
\]

\[
    b(z) = \sum_{k \geq 0} k b z^{p-k}.
\]

We introduce the notation

\[
    B = \sum_{0 \leq k \leq p} k B z^{p-k}.
\]

Here \( u(z) \) and the corresponding coefficients \( k u \) are considered as \( k \)-dimensional vectors; \( B(z) \) and \( k B \) are \( q \times q \) dimensional matrices. We can solve the equation (5). The recursion formula looks as follows:

\[
(0 b - 0 B)(k u) = (k B - k b)(0 u) + \text{known terms}.
\]

In particular,

\[
(0 b - 0 B)(0 u) = 0,
\]

i.e. \( 0 b \) is an eigenvalue of \( 0 B \). It follows from our assumptions that all eigenvalues of \( 0 B \) (that coincides with the leading coefficient of \( z^{-q} M_1^j(z^q) \)) are simple. This means that the image of the operator \( 0 b - 0 B \) has codimension 1. We denote by \( \rho \) a non-zero linear functional vanishing on this image. (It can be interpreted as an eigenvector the matrix transposed to \( 0 B \) with eigenvalue \( 0 b \).)

Applying \( \rho \) to both parts of recursion formula and noticing that \( \rho(0 u) \neq 0 \) we can calculate \( k b \).

Then the recursion formula gives us \( k u \). Noticing that we can take any eigenvalue of \( 0 B \) as \( 0 b \) we obtain the proof of the theorem.

The group \( C_q \) of \( q \)-th roots of unity acts on solutions of the equation (6) (if \( (u_0(z), \ldots, u_{q-1}(z)) \) is a solution and \( \epsilon^j = 1 \) then \( (u_0(\epsilon z), \ldots, \epsilon^{-j} u_1(\epsilon z), \ldots) \) is again a solution). One can use this fact to check that the group \( C_q \) acts also on the set of solutions constructed above. If \( p \) and \( q \) are coprime then this action is transitive therefore for appropriate labeling \( \Lambda_{k+1}(z) = \Lambda_k(\epsilon z) \). (Here \( \epsilon \) stands for a primitive root of unity.) Similarly, we can assume that \( \lambda_{k+1}(z) = \lambda_k(\epsilon z) \). From this equation one can derive the properties of eigenvalues of the leading term of the matrix \( B^j_1(z) \). If this leading term has degree \( p \) then the leading terms of eigenvalues have the same degree: \( \lambda_k(z) = \alpha_k z^p + \ldots \) and we obtain that \( \alpha_{k+1} = \epsilon^q \alpha_k \), hence \( \alpha_k = \epsilon^{kp} \alpha_1 \). Therefore in the case when \( p \) and \( q \) are coprime the numbers \( \alpha_k \) are distinct.

Let us consider pairs \( (P,Q) \) where \( P \) is a differential operator of order \( p \), \( Q \) is a normalized differential operator of order \( q \), the orders \( p \) and \( q \) are coprime and \( [P,Q] = \hbar \). It follows from the above considerations that the order of \( b(z) \) is equal to \( p \), therefore the leading coefficient of the matrix \( B^j_1(z) \) (coinciding with the leading coefficient of the matrix \( z^{q-j} M_1^j(z^q) \)) has distinct eigenvalues. This allows us to describe the moduli space of such pairs and to prove that this moduli space does not depend on \( \hbar \) (see [13]). We can formulate this description in the following way.

Let us say that the polynomial \( q \times q \) matrix \( M_1^j(z^q) \) is regular if the leading coefficient of the matrix \( z^{q-j} M_1^j(z^q) \) has \( q \) distinct eigenvalues. If this leading coefficient has degree \( p \) we say that...
a regular matrix $M$ belongs to the space $\mathcal{M}_{p,q}$. The solution of the equation $[P,Q] = \hbar$ where $P$ is a differential operator of order $p$, $Q$ is a normalized differential operator of order $q$ is regular if the companion matrix $M$ is regular. It follows from the above consideration that the moduli space of regular solutions does not depend on $\hbar$. It can be considered as a $q$-fold covering of the space $\mathcal{M}_{p,q}/T$ where $T$ denotes the group of triangular matrices. If $p$ and $q$ are coprime all solutions of the equation $[P,Q] = \hbar$ are regular.

In the above statements we have used the choice of $z^q$-basis specified by the condition $v_i = c_i z^i + \text{lower order terms}$. We say that a companion matrix in general $z^q$-basis is regular, if the corresponding matrix in the preferred basis is regular. The condition of regularity for other choices of $z^q$-basis is not so simple. Notice, however, that the eigenvalues of the companion matrix do not depend on the choice of the basis. Using this remark one can give a necessary condition of regularity that is valid in any basis. Namely, we should consider the characteristic polynomial
\[
\det(M - \lambda \cdot 1) = \sum A_i(z^q)\lambda^i.
\]

Let us suppose that $\det M = A_0(z^q)$ is a polynomial of degree $p$ with respect to $z^q$. If $M$ is regular then the degree of $A_1$ with respect to $z^q$ is less or equal than $\left\lfloor \frac{p(q-1)}{q} \right\rfloor$.

The results we have obtained for scalar differential operators can be generalized to the case of matrix differential operators. Instead of Grassmannian $Gr$ we should consider the vector Sato Grassmannian $Gr_s$. It consists of subspaces $V$ of $\mathcal{H}^s = \mathcal{H} \otimes \mathbb{C}^s$ (of direct sum of $s$ copies of $\mathcal{H}$) such that the natural projection of $V$ to $\mathcal{H}^s_+ = \mathcal{H}_+ \otimes \mathbb{C}^s$ is an isomorphism.

It is easy to generalize Theorem 2.1 and Theorem 2.3 to this case; see [9], Th.6.2, [7], Prop 2.1.

Theorem 2.5. If $V \in Gr_s$ is invariant with respect to the operator of multiplication by $z^q$ and with respect to the operator $\tilde{P} = \hbar \frac{d}{dz} + b(z)$ we can construct a matrix differential operator $P$ and a normalized matrix differential operator $Q$ of order $q$ obeying $[P,Q] = \hbar$. Here $b(z)$ is a Laurent series having $s \times s$ matrices as coefficients; if its leading term has degree $p$ then the operator $\tilde{P}$ is of order $p$.

Again we can define the companion matrix of the pair $(P,Q)$ as a matrix of $\tilde{P}$ in a $z^q$-basis of $V$. It is convenient to use a $z^q$-basis obeying
\[
v_i = c_{i\alpha} z^i c_{\alpha} + \text{lower order terms}
\]

Here $0 \leq i < q, 1 \leq \alpha \leq s$, $c_{\alpha}$ denotes the standard basis of $\mathbb{C}^s$ and $c_{i\alpha}$ are non-vanishing constants.

The companion matrix in this basis can be regarded as a $q \times q$ matrix with entries that are $s \times s$ matrices depending polynomially on $z^q$. We denote this matrix as $M^q_1$. It is defined by the equation (1) where $v_i$ denotes now a $q$-dimensional vector having $s$-dimensional vectors as components. Introducing the notation $u_{i\alpha} = z^{-i} v_{i\alpha}$ we obtain the equation (6) where the matrix $B$ is defined by the same formula as in scalar case.

We will prove the following theorem:

Theorem 2.6. Let us suppose that the entries of the leading coefficient of the matrix $z^{q-i} M^q_1(z^q)$ are scalar matrices. In other words we assume that $0 B$ (the leading coefficient of the matrix $B$) has the form $0 B = \sigma \otimes I_s$ where $\sigma$ is a $q \times q$ matrix with complex entries and $I_s$ stands for unit $s \times s$ matrix; we assume that $\sigma$ has $q$ distinct eigenvalues. Then there exist $q$ pairs of matrix differential operators obeying $[P,Q] = \hbar$ and having $M$ as the companion matrix.

We derive this statement from Theorem 2.1 generalizing the second proof of Theorem 2.4. It is sufficient to check that the equation (5) has $q$ solutions obeying
$u_{i\alpha} = c_{i\alpha} + \text{lower order terms}$.

Then we can define $V$ as a subspace spanned by $z^m v_{i\alpha}$ where $v_{i\alpha} = z^i u_{i\alpha}$.

The recursion formula (9) for the solution of (5) can be written in more detail in the form

$$(0 \cdot b_{i\alpha}^\beta_{i\alpha} - 0 \cdot B_{i\alpha}^j)(U_{i,j}) = (k \cdot B_{i\alpha}^j - k \cdot b_{i\alpha}^\beta_{i\alpha})(0 \cdot u_{i,j}) + \text{known terms}.$$

In particular,

$$(0 \cdot b_{i\alpha}^\beta_{i\alpha} - 0 \cdot B_{i\alpha}^j)(0 \cdot u_{i,j}) = 0.$$

Recall that $0 \cdot B_{i\alpha}^j = \sigma_j^\alpha \delta_{i\alpha}^\beta$ if $\sigma_j^\alpha \delta_{i\alpha}^\beta$ is a diagonal operator. If the sum is finite then $L$ is a diagonal operator. If the sum is infinite we assume that the natural projection $\pi : V \to H$ is an isomorphism. Let us denote by $u_{i\alpha}$ the eigenvector of the transposed matrix $\sigma$ having eigenvalue $\lambda$; it follows from our assumptions that $< \rho, s > \neq 0$. The inner product of LHS of the recursion relation with $\rho$ vanishes; this allows us to calculate $k \cdot b_{i\alpha}^\beta$. Then the recursion formula gives us $k \cdot u_{i\alpha}$.

The conditions of Theorem 2.6 are satisfied if the degree $p$ of the leading term of $B$ is coprime with $q$; the proof is the same as in scalar case.

3. Difference operators. Discrete quantum curves.

We would like to solve the equation $PQ = \lambda QP$ where $P$ and $Q$ are difference operators. Our consideration will be based on the notion of pseudodifference operator.

Let us consider linear operators acting on the space of Laurent series $H$. We say that a (doubly infinite) sequence $a = (a_k)$ specifies a diagonal operator transforming a sequence $c_k$ into a sequence $a_k c_k$ (or equivalently a series $\sum_{k \leq 0} a_k c_k z^k$ into the series $\sum_{k \leq 0} a_k c_k z^k$); we denote this operator by the same letter $a$. The shift operator $\Lambda$ transforms a sequence $c_k$ into the sequence $c_{k-1}$ (equivalently the series $c(z) = \sum_{k \leq 0} c_k z^k$ goes to the series $z c(z) = \sum_{k \leq 0} c_{k-1} z^k$).

Notice, that one can consider $k$ as a continuous parameter $k \in \mathbb{R}$. Then we can modify the definition of the shift operator considering the operator $\Lambda_h$ that transforms $c(k)$ into $c(k - h)$. Of course, for fixed $h$ this makes no difference. However, in applications we should consider $h$ as a small parameter and work with power series with respect to $h$.

One defines pseudodifference operators by the formula

$$L = \sum_{-\infty < s < \infty} a(s) \Lambda^s$$

where $a(s)$ are diagonal operators. If the sum is finite then $L$ is a difference operator. Restricting the summation to negative $s$ we obtain the operator $L_-$. Taking the sum over $k \geq 0$ we obtain the operator $L_+$; notice that $L_+$ is a difference operator.

An operator of the form

$$\Lambda^n + \sum_{-\infty < s < n} a(s) \Lambda^s$$

is a monic pseudodifference operator of order $n$. Monic pseudodifference operators of order zero form a group denoted by $S$.

The space $H$ has natural decreasing filtration $H_n = z^n H_+ = \text{span}(z^n, z^{n+1}, \ldots)$. One can characterize difference operators as pseudodifference operators compatible with this filtration.

Let us say that a flag $V$ in $H$ is a decreasing filtration $V_n$ such that $z^{-n} V_n \in Gr$. In other words we assume that the natural projection $\pi_n : V_n \to H_n$ is an isomorphism. Let us denote by $w_n(z)$

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5 We say that an operator $A$ and a decreasing filtration $F_n$ are compatible if for some integer $a$ we have $AF_n \subset F_{n-a}$ for all $n$. 
the point of $V_n$ obeying $\pi_n(w_n) = z^n$. A flag is specified by an arbitrary sequence of series $w_n(z)$ obeying the condition

$$w_n(z) = z^n + \sum_{k<n} w_{n,k} z^k,$$

hence $w_{n,k}$ can be considered as coordinates in the space $F$ of flags. (We should take $V_n = \text{span}(w_n, w_{n+1}, \cdots)$.)

The notion of flag was essentially used in [1], the theorems below are closely related to the results of this paper. However, they admit simple independent proofs.

**Theorem 3.1.** For every monic pseudodifference operator of order zero $S \in S$ we can construct a flag $V$ in $H$ taking $V_n = S H_n$. This construction gives a one-to-one correspondence between $S$ and the space of flags $F$.

To prove this fact we notice that the operator $S = 1 + \sum_{r>0} s(r) \Lambda^{-r}$ where $s(r)$ are diagonal operators transforms $z^n$ into

$$S(z^n) = z^n + \sum_{r>0} s_{n-r}(r) z^{n-r}.$$ We see immediately that $SH_n = \text{span}(S(z^n), S(z^{n+1}), \cdots)$ is a flag in $H$. The map $S \to F$ is bijective (we can identify the coordinates in these spaces using the formula $s_{n-r}(r) = w_{n-n-r}$).

**Corollary 3.1.** The representation $V_n = SH_n$ induces a correspondence between pseudodifference operators compatible with the flag $V = (V_n)$ and difference operators (=pseudodifference operators compatible with the flag $H_n$).

**Theorem 3.2.** Every monic pseudodifference operator $Q$ of order $q$ can be transformed into operator $\Lambda^q$ by means of monic pseudodifference operator of order zero:

$$\Lambda^q = SQS^{-1},$$

where $S \in S$.

The commutative Lie algebra $\gamma$ of polynomials $\sum_{-\infty < k < \infty} t_iz^i$ acts on $H$, hence on $Gr$ and on the space of flags $F$. As in Section 2 this follows from the remark that for $g(t) = \exp(\sum_{k \geq 0} t_iz^i)$ where $t_i$ are nilpotent parameters and a flag $V = (V_n)$ we can consider a flag $g(t)V = (g(t)V_n)$.

We can use Theorem 3.1 to define the action of the Lie algebra $\gamma$ on $G$ and Theorem 3.2 to define the action on monic pseudodifference operators. These actions can be written in the form

$$\frac{\partial S}{\partial t_n} = (S \Lambda^n S^{-1})_n S$$

$$\frac{\partial Q}{\partial t_n} = [Q_{n+1}, Q]$$

These formulas are called discrete KP equations (or Toda equations) [1]. They are very similar to formulas of Section 2. However, here $n \in \mathbb{Z}$, all operators are pseudodifference operators, $D$ is replaced by $\Lambda$. Notice, that formula (12) can be considered also as an action of $\gamma$ on the space of monic difference operators.

To solve the equation $PQ = \lambda QP$ where $Q$ is a monic difference operator we consider the flag $V_n = SH_n$ where $S$ is defined by the formula (10). If $P$ is a difference operators then the flag $V_n = SH_n$ is compatible with operators $P = SPS^{-1}$ and $\Lambda^q = SQS^{-1}$. Using Theorem 3.1 we obtain
Theorem 3.3. For every pair \((P, Q)\) of difference operators obeying \(PQ = \lambda QP\) where \(Q\) is monic of order \(q\) one can construct a flag \(V = (V_n)\) in \(\mathcal{H}\) such that it is compatible with operator \(\Lambda^q\) and with pseudodifference operator \(\tilde{P}\) obeying \(\tilde{P}\Lambda^q = \lambda \Lambda^q \tilde{P}\).

Conversely, if we have a flag \(V = (V_n)\) in \(\mathcal{H}\) such that it is compatible with operator \(\Lambda^q\) and with pseudodifference operator \(\tilde{P}\) obeying \(\tilde{P}\Lambda^q = \lambda \Lambda^q \tilde{P}\) we can construct a pair \((P, Q)\) of difference operators obeying \(PQ = \lambda QP\).

Recall that the operator \(\Lambda^q\) acts as multiplication by \(z^q\). The operator \(\tilde{P}\) can be represented in the form
\[
\tilde{P} = (0)b + (1)b \Lambda^{-1} + \cdots
\]
where \(n_b\) are diagonal operators obeying \(n_b = \lambda (n_b - q)\).

It follows from Theorem 5.3 that the Lie algebra \(\gamma\) acts on the moduli space \(\mathcal{P}\) of pairs \((P, Q)\) of difference operators obeying \(PQ = \lambda QP\) (we assume that \(Q\) is monic). In other words, the moduli space we consider is invariant with respect to the discrete KP hierarchy.

If the flag \(V_n = \text{span}(w_n, w_{n+1}, \cdots)\) is compatible with the operator \(\Lambda^q\) then the vectors \(w_0, \ldots, w_{q-1}\) form a \(z^{\pm q}\)-basis of \(\bigcup \mathbb{V}_n\) (i.e. the vectors \(z^m w_i\) where \(m \in \mathbb{Z}, 0 \leq i < q\) form a basis of this space). In the conditions of Theorem 3.3 the operator \(\tilde{P}\) acts in \(\bigcup \mathbb{V}_n\), hence we can consider the matrix \(M^q_i\) of \(\tilde{P}\) in this basis. By definition this matrix is the companion matrix of the pair \((P, Q)\). (The entries of this matrix are polynomials of \(z^q\) and \(z^{-q}\).)

It follows from this definition that
\[
\sum (n) \Lambda^{p-n} w_i = M^q_i w_j.
\]
Our goal is to find a pair of difference operators having companion matrix \(M\). It will be more convenient to work with \(u_i = z^{-i} w_i\) and with the matrix \(B^q_i = z^{i} M^q_i (z^q)\). Then we should consider the equation
\[
\sum (n) \Lambda^{p-n} (z^i u_i) = z^i B^q_i u_j
\]
that can be written as
\[
\sum_{n \geq 0, k \geq 0} n_b p - n + k u_i z^{p-n-k} = B^q_i u_j.
\]
Here
\[
u_i = \sum_{k \geq 0} u_i z^{-k}
\]
and
\[
(n) b_k = \lambda (n_b - q).
\]
We assume that the matrix
\[
B^q_i = \sum_{m \geq 0} m B^q_i z^{s-m}
\]
is known. We should find \(n_b\) and \(n_{u_i}\) by induction. First of all looking at the leading terms we see that \(p = s\) and
\[
(0) b_{p+1} (0) u_i = (0) B^q_i (0) u_j.
\]
The recursion formula formula is similar to the formula in Section 2:
\[
(0) b_{p+n+1} (n) u_i - (0) B^q_i (n) u_j = (n) B^q_i (0) u_j - (n) b_{p-n+1} (0) u_i + \text{known terms}.
\]
To guarantee the existence of solutions we assume that $^0b_k$ is an eigenvalue of $^0B^i_j$ for all $k$ and that $^0B^i_j$ has $q$ distinct eigenvalues. As in Section 2 knowing that $\alpha$ is an eigenvalue of $^0B^i_j$ and assuming that $p$ and $q$ are coprime we can say that all eigenvalues have the form $\epsilon r \alpha$ where $\epsilon^q = 1$. Combining this statement with equality (15) we obtain that the solution can exist only in the case when $\lambda^q = 1$. From the other side if $\lambda^q = 1$ we can take as $^0 \lambda$ for $0 \leq i < q$ arbitrary eigenvalues of $^0B^i_j$; then the formula $^0 \lambda_k = \lambda^0 (^0 \lambda_{k-q})$ specifies all other $^0 \lambda_i$ as eigenvalues of $^0B^i_j$. We obtain the following

**Theorem 3.4.** Let us suppose that the leading term matrix $^0B^i_j$ of the matrix $B^i_j = z^{j-i}M^j_i(z^q)$ has $q$ distinct eigenvalues and for every $k$ the number $^0b_k$ is equal to one of these eigenvalues. Assume that $^0b_k$ obey $^0b_k = \lambda^0 (^0b_{k-q})$ and $^0u_i$ obey (17). Then we can construct a pair of difference operators with companion matrix $M^j_i(z^q)$ solving the recursion formula (18).

The proof repeats the second proof of Theorem 2.4. Notice that the arguments used in this proof give us the numbers $^n b_k$ only for $k = p - n + i$ where $0 \leq i < q$. However, knowing these numbers we can find all $^n b_k$ from (19).

One can modify the above consideration to study the solutions to the equation $\lambda = \epsilon \mathbf{h}$ in the limit $\epsilon \to 0$ as power series with respect to $\epsilon$. In this situation the equation (14) can be written as

$$
\sum_{n \geq 0, k \geq 0} ^n b_{p-n-k+i} k \epsilon^k \epsilon^{m-k} = B^j_i u_j,
$$

where

$$
\epsilon^k = \sum_{k \geq 0, r \geq 0} k^r \epsilon^k \epsilon^{m-k} h^r,
$$

$$
^n b_k = \sum_{r \geq 0} ^n \lambda^r b_k
$$

and

$$
^n \lambda^r b_k = ^n \lambda^r b_{k-q} + \sum_{s \geq 0} ^n \lambda^{r-s} b_{k-s} \frac{1}{s!}.
$$

(The last equation follows from (16).)

We can find the coefficients $^n \lambda^r u_i$ and $^n \lambda^r b_k$ using double recursion. We are writing (19) as a system of equations for these coefficients. The equations for $r = 0$ coincide with the equations coming from (15) for $\lambda = 1$; we have solved them by means of recursion with respect to $n$. Now we assume that we have found all coefficients with $r < s$. Then we can find the coefficients $^n \epsilon^r u_i$ and $^n \epsilon^r b_k$ using the recursion formula with respect to $n$.

We obtain

**Theorem 3.5.** Let us suppose that the leading term matrix $^0B^i_j$ of the matrix $B^i_j = z^{j-i}M^j_i(z^q)$ has $q$ distinct eigenvalues. Then for every pair $(P, Q)$ of commuting difference operators with companion matrix $M^j_i(z^q)$ we can find a formal deformation $(P_h, Q_h)$ having the same companion matrix and obeying $P_hQ_h = \epsilon \mathbf{h} Q_h P_h$.

(Saying that $(P_h, Q_h)$ is a formal deformation of $(P, Q)$ we have in mind that $P_h$ and $Q_h$ are power series with respect to $\epsilon \mathbf{h}$ giving $P, Q$ for $\epsilon \mathbf{h} = 0$.)
4. Quantization

Let us consider a pair \((P, Q)\) of commuting differential operators, or a pair \((K, L)\) of commuting difference operators. We say that a pair \((P_\hbar, Q_\hbar)\) of differential operators obeying \([P_\hbar, Q_\hbar] = \hbar\) (quantum curve) is obtained by quantization of the pair \((P, Q)\) if it has the same companion matrix. Similarly, a pair \((K_\lambda, L_\lambda)\) of difference operators obeying \(K_\lambda L_\lambda = \lambda L_\lambda K_\lambda\) (discrete quantum curve) is obtained by quantization of the pair \((K, L)\) if it has the same companion matrix. Notice that in these definitions we can work with matrix differential or difference operators.\(^6\)

A pair \((P, Q)\) of commuting differential operators satisfies an algebraic equation \(A(P, Q) = 0\). This means that \(P\) and \(Q\) can be considered as meromorphic functions \(f, g\) on an algebraic curve \(A(x, y) = 0\). Let us describe a procedure \(^5\) that permits us to construct commuting differential operators starting with two meromorphic functions on an algebraic curve \(C\). We start for simplicity with the case when the functions \(f, g\) have only one pole at a smooth point \(a\) (the function \(f\) has a pole of order \(p\), the function \(g\) has a pole of order \(q\)). Let us suppose that we have found a subspace \(\mathcal{E}\) of the space of meromorphic functions on \(C\) having the following properties a) the space \(\mathcal{E}\) contains precisely one function (up to a constant factor) that has a pole of order \(n\) at the point \(a\) and these functions span \(\mathcal{E}\) (here \(n \in \mathbb{N}\) or \(n = 0\); saying that a function has a pole of order 0 we have in mind that it it is holomorphic and does not vanish at \(a\)), b) this space is invariant with respect to multiplication by \(f\) and \(g\). We introduce a coordinate \(z\) in the neighborhood of the point \(a\) in such a way \(g = z^q\) in this neighborhood (hence \(z = \infty\) at \(a\)). Considering the Laurent series (with respect to \(z\)) of functions belonging to \(\mathcal{E}\) we obtain a subspace \(V \subset \mathcal{H}\); it is easy to check that we can apply Theorem 2.3 for \(\hbar = 0\) to operators of multiplication acting on \(V \in \mathcal{G}\) to get commuting differential operators.

Now we should describe the construction of the space \(\mathcal{E}\). The first idea is to take as \(\mathcal{E}\) the space of all functions having a pole only at \(a\). However, this does not work -the projection of the corresponding subspace of \(\mathcal{H}\) on \(\mathcal{H}_a\) has the index \(g - 1\) where \(g\) denotes the genus of the curve \(C\) (this follows from well known fact that the sequence of orders of poles has \(g\) gaps). We should extend this space allowing functions with poles at the points of some divisor (the extended space is still invariant with respect to multiplication by \(f\) and \(g\)). Generic divisor of degree \(g - 1\) (non-special divisor) will give the subspace we need ( any divisor of degree \(g - 1\) gives a subspace of index 0, generically it belongs to the big cell).

A pair \(f, g\) of meromorphic functions on the curve \(C\) embeds this curve into \(\mathbb{C} \times \mathbb{C}\). If the points of embedded curve satisfy the equation \(A(x, y) = 0\) the pair \((P, Q)\) of commuting differential operators we constructed obeys the same equation. We can quantize this pair constructing differential operators obeying the equation \([P_\hbar, Q_\hbar] = \hbar\) and having the same companion matrix. It is important to notice that the explicit construction of the pair \((P, Q)\) is not necessary in the quantization procedure. We can define the companion matrix directly in terms of functions \(f, g\). Namely, we should find a \(g\)-basis of the space \(\mathcal{E}\) and calculate the matrix of multiplication by \(f\) in this basis.

The construction we have described can be generalized to the case when functions \(f\) and \(g\) have multiple poles. Let us suppose, for example, that the function \(g\) has \(s\) poles, all of order \(q\), at the points \(a_1, \cdots, a_s\). Let us introduce the coordinate \(z_i\) in the neighborhood of \(a_i\) in such a way that \(g = z_i^q\) in this neighborhood (hence \(z_i = \infty\) at \(a_i\)). Then we should find a subspace \(\mathcal{E}\) of the space of meromorphic functions on \(C\) having the following properties a) for arbitrary integers

\(^6\)Recall that the companion matrix is not unique. It is defined up to polynomial change of basis if we are working with general definition. Hence it would be more precise to say that the quantization does not change the set of companion matrices.
Consider the space $K$ we obtain two commuting difference operators $a, b$ at points order at $a$ neighborhood (we assume that $s$ by functions subspace $C$): this space is invariant with respect to multiplication by $f$ and $g$. To every meromorphic function on $C$ we should assign $s$ Laurent series representing this function in the neighborhoods of points $a_i$. This construction sends $E$ to a subspace $V \subset \mathcal{H}^a$; it follows from the condition a) that $V \in \text{Gr}_a$. Applying Theorem 3.3 for $h = 0$ we obtain a pair of commuting differential matrix operators.

To construct the space $E$ obeying the above conditions we start with with the space of meromorphic functions having poles only at the points $a_1, \ldots, a_s$. If $f$ has poles only at $a_1, \ldots, a_s$ this space is invariant with respect to multiplication by $f$ and $g$ (property b). This property is preserved if we allow additional poles of order 1 at the points of some divisor. Choosing the divisor appropriately we can satisfy the property a).

Very similar construction works for difference operators. Let us consider an algebraic curve $C$ and two meromorphic functions $f, g$ that are holomorphic everywhere except two smooth points $a$ and $b$. We introduce a coordinate $z$ in the neighborhood of $a$ in such a way that $g = z^q$ in this neighborhood (we assume that $z = \infty$ at $a$ hence $g$ has a pole of order $q$). Let us suppose that a subspace $E$ of the space of meromorphic functions on $C$ has the following properties a) it is spanned by functions $s_n, n \in \mathbb{Z}$ such that $s_n$ has a pole of order $n$ at $a$ for non-negative $n$ and zero of order $-n$ for negative $n$, b) it is invariant with respect to multiplication by $f$ and $g$, c) $f s_n$ and $g s_n$ are linear combinations of $s_r$ with $n - \alpha < r < n + \beta$ for some $\alpha, \beta$. The functions $s_n$ specify a flag in $\mathcal{H}_a$ (To define this flag we consider Laurent series of $s_n$ at the point $a$ with respect to the coordinate $z$). This flag is compatible with multiplication by $f$ and $g$. Applying Theorem 3.3 in the case $\lambda = 1$ we obtain two commuting difference operators.

Now we should explain how to find the subspace $E$ satisfying the conditions a), b) and c). Let us consider the space $K$ consisting of meromorphic sections of some line bundle having poles only at points $a, b$ (equivalently we can talk about meromorphic functions on $C$ having poles of any order at $a$ and $b$, that can also have simple poles at the points of some divisor). Let us denote by $K(na)$ the subspace $K$ consisting of functions of the form $z^n f(z)$ where $f$ is finite at $a$; the space $K(na)$ is defined in similar way. One can prove that for appropriate choice of $K$ the space $K(na) \cap K((-n + 1)b)$ is one-dimensional; the function $s_n$ can be defined as non-zero element of this space.

5. D-modules

In [3] quantum curves were studied from the viewpoint of $D$-modules. The approach of this paper is closely related to the approach of [13] and present paper. The construction of the point of Grassmannian used in [13] plays fundamental role also in [3]. The companion matrix specifies a meromorphic connection

\[ \nabla = h \frac{d}{dz^q} - B_i^j(z), \]

the flat sections of this connection can be identified with solutions to the equation (6). The flat connection $\nabla$ can be identified with $D$-module studied in [3] (better to speak about the family of connections and a family of $D$-modules parametrized by $h$ or about $D_h$-module). We consider connections in the neighborhood of $z = \infty$.

Notice that an equivalent $D$-module can be defined by means of the matrix $M_j^i$.

It is well known that every $D$-module defined by meromorphic connection in dimension 1 has rank 1; it can be represented in the form $D/D \cdot P$ where $D$ denotes the algebra of differential
operators with meromorphic coefficients (i.e. polynomials with respect to $\partial z$ with coefficients that are meromorphic with respect to $z$) and $P \in D$. This is the form mostly used in [3].

The above statement means that the system of equations for flat sections $\nabla u = 0$ is equivalent to a single differential equation. One can use this statement to rewrite the system (6) or (4) as a single equation

$$\hat{A}w_0 = 0,$$

where $\hat{A}$ is a differential operator with meromorphic coefficients. Namely, we should consider $w = (w_0, \cdots, w_{q-1})$ as an element of $F^q$ where $F$ denotes the field of meromorphic functions. Then $w_0 = <e_0, w>$ where $e_0 = (1, 0, \cdots, 0)$ and $<a, b> = \sum a_i b_i \in F$. Defining $\nabla_s$ by the formula

$$\nabla_s = \hbar \frac{d}{dz^q} + B_i(z),$$

and using $\nabla w = 0$ we obtain that

$$(\hbar \frac{d}{dz^q})^s w_0 = <\nabla_s e_0, w>.$$

To find $\hat{A}$ we notice that the vectors $\nabla_s e_0$ with $s = 0, \cdots q$ are linearly dependent in $q$-dimensional vector space over $F$. If

$$\sum_{0 \leq s \leq q} a_s(z, \hbar) \nabla_s e_0 = 0$$

we can take

$$(22) \quad \hat{A} = \sum_{0 \leq s \leq q} a_s(z, \hbar)(\hbar \frac{d}{dz^q})^s.$$

Notice that knowing an operator annihilating $w_0$ it is easy to find an operator annihilating $u_0 = v_0$:

$$(\hat{A} - s_h(z))u_0 = 0.$$

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