Reflecting rough differential equations

Shigeki Aida*
Mathematical Institute
Tohoku University, Sendai, 980-8578, JAPAN
e-mail: aida@math.tohoku.ac.jp

Abstract

In this paper, we study the reflecting differential equations driven by continuous $p$-variation paths ($1 \leq p < 2$) and $p$-rough paths ($2 \leq p < 3$) on domains in Euclidean spaces whose boundary may not be smooth. We define reflecting rough differential equations and prove the existence of the solutions. Also we discuss the relation between the solution of SDE and reflecting rough differential equation when the driving process is a Brownian motion.

1 Introduction

In [2], we proved the strong convergence of the Wong-Zakai approximations of the solutions of reflecting stochastic differential equations defined on domains in Euclidean spaces whose boundary may not be smooth. The driving stochastic process in the equation is a Brownian motion. Recently, many researchers have been studying differential equations driven by more general stochastic processes and irregular paths. Of course this is due to the development of rough path theory which gives new meaning of stochastic integrals. In view of this, it is natural to study reflecting differential equations driven by irregular paths or rough paths rather than semi-martingales. The aim of this paper is to study such equations and prove the existence of solutions. We use the Euler approximation of the differential equations by modifying the idea of Davie [4]. When the equation has reflection term, the Euler approximation becomes the implicit Skorohod equation and it is not trivial to see the existence of the solutions. Hence, we need stronger assumptions than those given in [2] on the boundary of the domain to prove the existence of solutions. At the moment, we neither have uniqueness of solutions nor continuity theorem with respect to driving paths.

The paper is organized as follows. In Section 2, we recall conditions of the boundary under which reflecting rough differential equations are studied and prepare necessary lemmas. In Section 3, we study the reflecting differential equations driven by $p$-variation paths with $1 \leq p < 2$. The meaning of the integral in this equation is justified by the Young integrals. We prove the existence of solutions by using Davie’s approach [4]. This problem was already studied when $D$ is a half space in [8]. Our existence theorem is valid for more general domains. In Section 4, we study the case where the driving path is $p$-rough path with $2 \leq p < 3$. In this case, we consider stronger assumptions than that in previous sections. First, we give the meaning

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the reflecting rough differential equations and prove the existence of solutions and estimates on solutions. Also we explain the reason of the difficulty to prove the uniqueness of solutions and continuity theorems with respect to driving rough paths. In Section 5, we go back to reflecting SDEs driven by Brownian motion. We explain relations between the solutions of them and the solutions of reflecting rough differential equations driven by Brownian rough paths.

2 Preliminary

First, we prepare necessary definitions and results for our purposes. The following conditions on the connected domain $D \subset \mathbb{R}^d$ are standard assumptions for reflecting SDE and can be found in [11, 17, 20] and we will study our equations on domains which satisfy these conditions. We will introduce other conditions later. For other references of reflecting SDEs related with this paper, we refer the readers to [2, 23, 5, 6, 7, 14, 15, 16, 18, 19]. In a forthcoming paper [1], we study Wong-Zakai approximations in the two cases, (i) the domain is convex, (ii) the conditions (A) and (B) are satisfied which are not contained in the result in [2].

Recall that the set $\mathcal{N}_x$ of inward unit normal vectors at the boundary point $x \in \partial D$ is defined by

$$\mathcal{N}_x = \cup_{r>0} \mathcal{N}_{x,r},$$

$$\mathcal{N}_{x,r} = \{ n \in \mathbb{R}^d \mid |n| = 1, B(x-rn, r) \cap D = \emptyset \},$$

where $B(z, r) = \{ y \in \mathbb{R}^d \mid |y-z| < r \}$, $z \in \mathbb{R}^d$, $r > 0$.

**Definition 2.1.** (A) There exists a constant $r_0 > 0$ such that

$$\mathcal{N}_x = \mathcal{N}_{x,r_0} \neq \emptyset \quad \text{for any } x \in \partial D.$$

(B) There exist constants $\delta > 0$ and $\beta \geq 1$ satisfying:

for any $x \in \partial D$ there exists a unit vector $l_x$ such that

$$(l_x, n) \geq \frac{1}{\beta} \quad \text{for any } n \in \cup_{y \in B(x,\delta) \cap \partial D} \mathcal{N}_y.$$

(C) There exists a $C^2_b$ function $f$ on $\mathbb{R}^d$ and a positive constant $\gamma$ such that for any $x \in \partial D$, $y \in \bar{D}$, $n \in \mathcal{N}_x$ it holds that

$$(y-x, n) + \frac{1}{\gamma} ((Df)(x), n) |y-x|^2 \geq 0.$$

We use the following quantities of paths $w_t$ as in [2].

$$\|w\|_{\infty,[s,t]} = \max_{s \leq u \leq v \leq t} |w_u - w_v|, \quad (2.1)$$

$$\|w\|_{[s,t]} = \sup_{\Delta} \sum_{k=1}^N |w_{t_k} - w_{t_{k-1}}|, \quad (2.2)$$
where $\Delta = \{s = t_0 < \cdots < t_N = t\}$ is a partition of the interval $[s, t]$. When the domain $D$ satisfies the conditions (A) and (B), the Skorohod problem associated with a continuous path $w \in C([0, T] \to \mathbb{R}^d)$:

$$
\xi_t = w_t + \phi_t, \quad \xi_t \in \bar{D} \quad 0 \leq t \leq T,
$$

$$
\phi_t = \int_0^t 1_{\partial D}(\xi_s)n(s)d\|\phi\|_{[0, s]}, \quad n(s) \in N_{\xi_s} \text{ if } \xi_s \in \partial D
$$

can be uniquely solved. See [17]. When the mapping $w \mapsto \xi$ is unique, we write $\Gamma(w)_t = \xi_t$ and $L(w)_t = \phi_t$. The following lemma can be proved by a similar proof to that of Lemma 2.3 in [2].

**Lemma 2.2.** Assume conditions (A) and (B) hold. Let $w_t$ be a $p$-variation continuous path such that

$$
|w_t - w_s| \leq \omega(s, t)^{1/p} \quad 0 \leq s \leq t \leq T,
$$

where $p \geq 1$ and $\omega(s, t)$ is the control function of $w_t$. Then the local time $\phi$ of the solution to the Skorohod problem associated with $w$ has the following estimate.

$$
\|\phi\|_{[s, t]} \leq \beta \left( \left\{ \delta^{-1}G(\|w\|_{\infty, [s, t]}) + 1 \right\}^p \omega(s, t) + 1 \right) \left( G(\|w\|_{\infty, [s, t]}) + 2 \right) \|w\|_{\infty, [s, t]}, \quad (2.3)
$$

where

$$
G(a) = 4 \left\{ 1 + \beta \exp \left\{ \beta \frac{(2\delta + a)}{(2r_0)} \right\} \right\} \exp \left\{ \beta \frac{(2\delta + a)}{(2r_0)} \right\}, \quad a \in \mathbb{R}.
$$

3 Reflecting differential equations driven by $p$-variation path with $1 \leq p < 2$

Let $x_t$ ($0 \leq t \leq T$) be a continuous $p$-variation path with the control function $\omega(s, t)$, where $1 \leq p < 2$. We prove the existence of a solution $y_t$ which is also a continuous $p$-variation path to the reflecting differential equation driven by $x$:

$$
y_t = y_0 + \int_0^t \sigma(y_s)dx_s + \Phi(t), \quad y_0 \in \bar{D}. \quad (3.1)
$$

The integral in this equation is a Young integral [22]. The following is a main result in this section. See Remark 4.6.

**Theorem 3.1.** Assume that (A) and (B) hold. Then there exists a solution $(y, \Phi)$ to (3.1) and satisfies

$$
|y_t - y_s| \leq C\omega(s, t)^{1/p} \quad (3.2)
$$

$$
\|\Phi\|_{[s, t]} \leq C\omega(s, t)^{1/p}. \quad (3.3)
$$

Here $C$ is a constant which depends on $\omega(0, T)$ and $\sigma$ and $r_0, \beta, \delta$. 
We solve this equation by using the Euler approximation. Let $\Delta : 0 = t_0 < t_1 < \cdots < t_N = T$ be a partition of $[0,T]$. We define $y^\Delta$ by the solution to the Skorohod equation:

$$y^\Delta_t = y^\Delta_{t_{k-1}} + \sigma(y^\Delta_{t_{k-1}})(x_t - x_{t_{k-1}}) + \Phi^\Delta(t) - \Phi^\Delta(t_{k-1}) \quad t_{k-1} \leq t \leq t_k.$$  

Let

$$I^\Delta_s(t) = y^\Delta_s - y^\Delta_s - \sigma(y^\Delta_s)(x_t - x_s) - (\Phi^\Delta(t) - \Phi^\Delta(s)) \quad s \leq t. \quad (3.4)$$

By the definition, we have $I^\Delta_{t_k}(t) = 0$ for all $t_k \leq t \leq t_{k+1}$ and for any $s \leq t \leq u$,

$$I^\Delta_s(u) - I^\Delta_s(t) - I^\Delta_t(u) = (\sigma(y^\Delta_s) - \sigma(y^\Delta_t))(x_u - x_t).$$

For simplicity we may omit the notation $\Delta$. Also we write $\pi^\Delta(t) = \max\{t_k \mid t_k \leq t\}$.

In the following lemma, we use a constant in the estimate (2.3). Let $C_0$ be a positive constant for which

$$\|\phi\|_{[s,t]} \leq C_0 (\omega(s,t) + 1) \left( e^{C_0 \omega(s,t)^{1/p}} + 1 \right) \omega(s,t)^{1/p} \quad (3.5)$$

holds. Hence for any small positive $\varepsilon$, if $\omega(s,t)$ is sufficiently small, $\|\phi\|_{[s,t]} \leq (2 + \varepsilon)C_0 \omega(s,t)^{1/p}$ holds.

**Lemma 3.2.** Let $1 \leq p < \gamma \leq 2$. Let $C_1 = 9C_0(1 + \|\sigma\|_\infty)$, $C_2 = 1 + \|\sigma\|_\infty + 3C_0 \|\sigma\|_\infty$ and $M = \frac{C_2(1 + 2\|D\sigma\|_\infty)}{1 - 2^{1/(\gamma/p)}}$. For sufficiently small $\varepsilon(\leq 1)$ which depends only on $\sigma$ and $C_0$ such that for any $t$ with $\omega(t_k,t) \leq \varepsilon$,

$$|I^\Delta_{t_k}(t)| \leq M \omega(t_k,t)^{\gamma/p},$$

$$\|\Phi^\Delta\|_{[t_k,t]} \leq C_1 \omega(t_k,t)^{1/p}. \quad (3.6)$$

**Proof.** Note that if (3.6) and (3.7) hold, then by taking $\varepsilon$ to be sufficiently small, we have

$$|y^\Delta(t) - y^\Delta(t_k)| \leq \left( M\varepsilon^{(\gamma-1)/p} + \|\sigma\|_\infty + 3C_0 \|\sigma\|_\infty \right) \omega(t_k,t)^{1/p} \leq C_2 \omega(t_k,t)^{1/p}.$$ 

Let $K$ be a positive integer. Consider a claim which depends on $K$: The estimates (3.6) and (3.7) hold for all $t_k$ and $t$, where $t_k \leq t \leq t_{k+K}$ and $0 \leq k \leq N - 1$. We prove this claim by an induction on $K$. Let $K = 1$. Then $I^\Delta_{t_k}(t) = 0$ for all $t_k \leq t \leq t_{k+1}$. Also by taking $\varepsilon$ to be sufficiently small,

$$\|\Phi^\Delta\|_{[t_k,t]} \leq 3C_0 \|\sigma\|_\infty \omega(t_k,t)^{1/p}.$$ 

Suppose the claim holds for all $K$ which is smaller than or equal to $K' - 1$. We prove the case $K = K'$. Let $t_l$ be the largest partition point such that $t_k \leq t_l < t \leq t_{k+K'}$ and $\omega(t_k,t_l) \leq \frac{1}{2} \omega(t_k,t)$. There are two cases, (a) $t_l < \pi^\Delta(t)$ and (b) $t_l = \pi^\Delta(t)$. We consider the case (a). In this case, $t_l < t_{l+1} \leq \pi^\Delta(t)$. By the definition, we have $\omega(t_k,t_{l+1}) > \frac{1}{2} \omega(t_k,t)$. By the superadditivity of $\omega$, we have

$$\omega(t_{l+1},t) \leq \frac{1}{2} \omega(t_k,t).$$
We have

\[ |I_{t_k}^{\Delta}(t)| \leq |I_{t_k}^{\Delta}(t_i)| + |I_{t_i}^{\Delta}(t_{i+1})| + |I_{t_{i+1}}^{\Delta}(t)| + |\sigma(y_{t_{i+1}}) - \sigma(y_t)| |x_t - x_{t_{i+1}}| \\
+ |\sigma(y_t) - \sigma(y_{t_k})||x_t - x_{t_i}| \]

By the assumption of the induction, we have

\[ |I_{t_k}^{\Delta}(t_i)| \leq M\omega(t_k, t_i)^{1/p}, \quad |I_{t_{i+1}}^{\Delta}(t)| \leq M\omega(t_{i+1}, t)^{1/p} \]
\[ |\sigma(y_{t_{i+1}}) - \sigma(y_t)||x_t - x_{t_{i+1}}| \leq C_2||D\sigma||_{\infty} \omega(t_i, t_{i+1})^{1/p}\omega(t_{i+1}, t)^{1/p} \]
\[ |\sigma(y_t) - \sigma(y_{t_k})||x_t - x_{t_i}| \leq C_2||D\sigma||_{\infty} \omega(t_k, t)^{1/p}\omega(t, t)^{1/p} \]

Therefore

\[ |I_{t_k}^{\Delta}(t)| \leq M\left(2^{1-(\gamma/p)} + 2C_2M^{-1||D\sigma||_{\infty}^{(2-\gamma)/p}}\right)\omega(t_k, t)^{\gamma/p} \leq M\omega(t_k, t)^{\gamma/p}. \]

In the case of (b), by using the assumption of the induction, we obtain

\[ |I_{t_k}^{\Delta}(t)| \leq |I_{t_k}^{\Delta}(t_i)| + |I_{t_i}^{\Delta}(t)| + |\sigma(y_t) - \sigma(y_{t_k})||x_t - x_{t_i}| \]
\[ \leq M\omega(t_k, t_i)^{1/p} + C_2\omega(t_i, t)^{1/p}\omega(t, t)^{1/p} \]
\[ \leq M\left(2^{-\gamma/p} + 2C_2M^{-1||D\sigma||_{\infty}^{(2-\gamma)/p}}\right)\omega(t_k, t)^{\gamma/p} \]
\[ \leq M\omega(t_k, t)^{\gamma/p}. \]

Next we show \(||\Phi^{\Delta}||_{[t_k, t]} \leq C_1\omega(t_k, t)^{1/p}\) for \(t_k, t\) with \(t_k \leq t \leq t_{k+1}/2\). To this end, we note that \(\Phi^{\Delta}(s) - \Phi^{\Delta}(t) = L(z^{\Delta}(s))(t)\), where \(z^{\Delta}(t) = I_{t_k}^{\Delta}(t) + y_{t_k}^{\Delta} + \sigma(y_{t_k}^{\Delta})(x_t - x_{t_k})\). By Lemma 2.2, it suffices to estimate \(z^{\Delta}(t)\). Take \(s, t\) such that \(t_k \leq s \leq t \leq t_{k+1}/2\) and \(\omega(t_k, s) \leq \varepsilon, \omega(t_k, t) \leq \varepsilon\). We estimate \(I_{t_k}^{\Delta}(t) - I_{t_k}^{\Delta}(s)\) by using

\[ I_{t_k}^{\Delta}(t) - I_{t_k}^{\Delta}(s) = I_{t_k}^{\Delta}(s) + (\sigma(y_s) - \sigma(y_{t_k}^{\Delta}))(x_t - x_s). \] (3.8)

Let \(t_m\) be the largest number such that \(t_m \leq s\). Then we have two cases, (a) \(t_k \leq t_m \leq s < t_{m+1} \leq t\) and (b) \(t_k \leq t_m \leq s < t < t_{m+1}\). First we consider the case (a). We have

\[ I_{t_k}^{\Delta}(t) = I_{t_k}^{\Delta}(t_{m+1}) + I_{t_{m+1}}^{\Delta}(t) + (\sigma(y_{t_{m+1}}) - \sigma(y_s))(x_t - x_{t_{m+1}}). \]

Since \(I_{t_k}^{\Delta}(t_{m+1}) = -(\sigma(y_s) - \sigma(y_{t_m}^{\Delta}))(x_{t_{m+1}} - x_s)\), we have

\[ |I_{t_k}^{\Delta}(t_{m+1})| \leq C_2||D\sigma||_{\infty} \omega(t_m, s)^{1/p}\omega(s, t_{m+1})^{1/p} \leq C_2||D\sigma||_{\infty} \varepsilon^{1/p}\omega(s, t)^{1/p}. \]

By the hypothesis of the induction, \(|I_{t_{m+1}}^{\Delta}(t)| \leq M\omega(t_{m+1}, t)^{1/p} \leq M\varepsilon^{(\gamma-1)/p}\omega(s, t)^{1/p}\). Also,

\[ |(\sigma(y_{t_{m+1}}) - \sigma(y_s))(x_t - x_{t_{m+1}})| \leq 2||D\sigma||_{\infty} \omega(t_m, t_{m+1})^{1/p}\omega(t_{m+1}, t)^{1/p} \]
\[ \leq 2\varepsilon^{1/p}||D\sigma||_{\infty} \omega(s, t)^{1/p}. \]

Hence

\[ |I_{t_k}^{\Delta}(t)| \leq \left(3C_2||D\sigma||_{\infty} \varepsilon^{1/p} + M\varepsilon^{(\gamma-1)/p}\right)\omega(s, t)^{1/p} \]
\[ |I_{t_k}^{\Delta}(t) - I_{t_k}^{\Delta}(s)| \leq \left(4C_2||D\sigma||_{\infty} \varepsilon^{1/p} + M\varepsilon^{(\gamma-1)/p}\right)\omega(s, t)^{1/p}. \]
Consequently, for sufficiently small $\varepsilon$,
\[ \|\Phi^\Delta\|_{[t_k, t]} \leq 3C_0 (1 + \|\sigma\|_{\infty}) \omega(t_k, t)^{1/p}. \]
We consider the case (b). In this case, $I^\Delta_s(t) = - (\sigma(y_s) - \sigma(y_{t_m})) (x_t - x_s)$. So, we have
\[ |z^\Delta(t) - z^\Delta(s)| \leq C_2 \|D\sigma\|_{\infty} \varepsilon^{1/p} \omega(s, t)^{1/p} + 3 \|\sigma\|_{\infty} \omega(s, t)^{1/p} \]
which completes the proof. \hfill \Box

By Lemma 3.2 we can prove the following.

**Lemma 3.3.** Let $\varepsilon$ be a positive number in Lemma 3.2. Let $\Delta = \{t_k\}_{k=0}^N$ be a partition of $[0, T]$ such that $\omega(t_k, t_{k+1}) \leq \varepsilon$ for all $k$. Then there exists $C > 0$ such that for any $0 \leq s \leq t \leq T$ the following estimates hold. The constant $C$ depends only on $\sigma$, $p$ and $D$.

1. $|y^\Delta(t) - y^\Delta(s)| \leq C (1 + \omega(0, T)) \omega(s, t)^{1/p}$
2. $\|\Phi^\Delta\|_{[s, t]} \leq C (1 + \omega(0, T)) \omega(s, t)^{1/p}$.

**Proof of Lemma 3.3.** We note that the statement is true if $t_k \leq s \leq t \leq t_{k+1}$ for some $k$ by Lemma 2.2. Let us consider general cases. We define a subsequence $\{s_k\}_{k=0}^{N'}$ of $\{t_k\}$ in the following way. Let $s_0 = t_0 = 0$. When $s_k$ is defined, we define $s_{k+1}$ as the smallest $t_i$ such that $\omega(s_k, t_i) \leq \varepsilon$ and $t_i > s_k$. Then by the superadditivity of $\omega$, we have $N'\varepsilon \leq \omega(0, T)$ and so $N' \leq (\omega(0, T)/\varepsilon)$. For $0 \leq s \leq t \leq T$, let us choose the numbers $l, m$ so that $s_l \leq s < s_{l+1} \leq s_m \leq t < s_{m+1}$. Then
\[ |y^\Delta(t) - y^\Delta(s)| \leq |y^\Delta(t) - y^\Delta(s_m)| + \sum_{k=l+1}^{m-1} |y^\Delta(s_{k+1}) - y^\Delta(s_k)| + |y^\Delta(s_{l+1}) - y^\Delta(s)| \]
\[ \leq C_2 ((\omega(0, T)/\varepsilon) + 2) \omega(s, t)^{1/p}. \]
A similar estimate for $\Phi^\Delta$ holds and this completes the proof. \hfill \Box

**Proof of Theorem 3.1.** Let us consider a sequence of partitions $\Delta(n) = \{t(n)_k\}$ such that
(a) $\omega(t(n)_{k}, t(n)_{k+1}) \leq \varepsilon$ for all $k$,
(b) $\Delta(n + 1)$ is a subdivision of $\Delta(n)$,
(c) $\lim_{n \to \infty} \max_{k \geq 0} |t(n)_{k+1} - t(n)_k| = 0$.

Then by Lemma 3.3 there exists a subsequence $y^{\Delta(n_k)}$ and $\Phi^{\Delta(n_k)}$ converge uniformly to continuous paths $y^\infty$ and $\Phi^\infty$ respectively which satisfy (3.2) and (3.3). Then these subsequences converge in $p'$-variation norm. The solution $y^{\Delta(n_k)}$ satisfies
\[ y^{\Delta(n_k)}_t = y_0 + \int_0^t \sigma(y^{\Delta(n_k)}(\pi^\Delta(u))) du + \Phi^{\Delta(n_k)}(t). \]
By taking the limit $n_k \to \infty$ and by the continuity theorem of Young integral, we see that $(y^\infty, \Phi^\infty)$ is a solution of the equation. \hfill \Box
4 Reflecting differential equations driven by \( p \)-rough path with \( 2 \leq p < 3 \)

In this section, we prove the existence of solutions to reflecting differential equations driven by rough path. We mainly follow the formulation of rough path in [12, 13, 4]. See also [9, 10]. First, we define reflecting differential equation driven by rough path.

**Definition 4.1.** Let \( D \) be a connected domain in \( \mathbb{R}^d \) for which the condition (A) holds. Let \( 2 \leq p < 3 \). Let \( X_{s,t} = (1, X^1_{s,t}, X^2_{s,t}) \in \Omega_p(\mathbb{R}^n) \) \((0 \leq s \leq t \leq T)\) be a \( p \)-rough path. Let \( Y_{s,t} = (1, Y^1_{s,t}, Y^2_{s,t}) \in \Omega_p(\mathbb{R}^d) \) be a \( p \)-rough path and \( \Phi_t \) \((0 \leq t \leq T)\) be a continuous bounded variation path on \( \mathbb{R}^d \). The pair \((Y, \Phi)\) is called a solution to a rough differential equation on \( D \) with normal reflection with the starting point \( y_0 \in \hat{D} \) if the following hold.

1. Let \( Y_t = y_0 + Y^1_{0,t} \). Then \( Y_t \in \hat{D} \) \((0 \leq t \leq T)\) and it holds that there exists a Borel measurable map \( s(\in [0, T]) \to n(s) \in \mathbb{R}^d \) such that \( n(s) \in N_{Y_s} \) if \( Y_s \in \partial D \) and

\[
\Phi_t = \int_0^t 1_{\partial D}(Y_s)n(s)d||\Phi||_{[0,s]} \quad 0 \leq t \leq T. \tag{4.1}
\]

2. \( Y_{s,t} \) is a solution to the following rough differential equation.

\[
dY_t = \sigma(Y_t)dX_t + d\Phi_t \quad 0 \leq t \leq T, \quad Y_0 = y_0. \tag{4.2}
\]

Precisely, the driving rough path \( \hat{X} \in \Omega_p(\mathbb{R}^n \oplus \mathbb{R}^d) \) of this equation is given below.

\[
\hat{X}^1_{s,t} = (X^1_{s,t}, \Phi_t - \Phi_s)
\]

\[
\hat{X}^2_{s,t} = (X^2_{s,t}, \int_s^t X^1_{s,u} \otimes d\Phi_u, \int_s^t \Phi_u - \Phi_s) \otimes dX^1_{s,u}, \int_s^t (\Phi_u - \Phi_s) \otimes d\Phi_u).
\]

To solve this equation, we consider the Euler approximation modifying the Davies’ approximation for rough differential equations without reflection terms. Let \( \Delta : 0 = t_0 < t_1 < \cdots < t_N = T \) be a partition of \([0, T]\). Let us consider a Skorohod problem:

\[
y^\Delta_t = y^\Delta_{t_{k-1}} + \sigma(y^\Delta_{t_{k-1}})(x_t - x_{t_{k-1}}) + (D\sigma)(y^\Delta_{t_{k-1}})(\sigma(y^\Delta_{t_{k-1}})X^2_{t_{k-1},t})
\]

\[
+ (D\sigma)(y^\Delta_{t_{k-1}}) \left( \int_{t_{k-1}}^t (\Phi^\Delta(r) - \Phi^\Delta_{(t_{k-1})}) \otimes dx_r \right) + \Phi^\Delta_{(t)} - \Phi^\Delta(t_{k-1})
\]

\[
y^\Delta_{t_{k-1}} \in \hat{D}, \quad y^\Delta_0 = y_0, \quad t_{k-1} \leq t \leq t_k, \quad 1 \leq k \leq N, \tag{4.3}
\]

where \( x_t = X^1_{0,t} \). That is, the pair \((y^\Delta_t, \Phi^\Delta_t - \Phi^\Delta_{t_{k-1}})\) is the solution of the Skorohod problem associated with the continuous path

\[
y^\Delta_{t_{k-1}} + \sigma(y^\Delta_{t_{k-1}})(x_t - x_{t_{k-1}}) + (D\sigma)(y^\Delta_{t_{k-1}})(\sigma(y^\Delta_{t_{k-1}})X^2_{t_{k-1},t})
\]

\[
+ (D\sigma)(y^\Delta_{t_{k-1}}) \left( \int_{t_{k-1}}^t (\Phi^\Delta(r) - \Phi^\Delta_{(t_{k-1})}) \otimes dx_r \right) \quad t_{k-1} \leq t \leq t_k.
\]

Since this is an implicit Skorohod problem, the existence of the solution is not trivial. In view of this, we consider the following condition (D) and assumptions (H1) and (H2) on \( D \).
Assumption 4.2. (D) Condition (A) is satisfied and there exist constants $K_1 \geq 0$ and $0 < K_2 < r_0$ such that

$$|\bar{x} - \bar{y}| \leq (1 + K_1\varepsilon)|x - y|$$

holds for any $x, y \in \mathbb{R}^d$ with $|x - \bar{x}| \leq K_2$, $|y - \bar{y}| \leq K_2$, where $\varepsilon = \max\{|x - \bar{x}, |y - \bar{y}|\}$. Here $\bar{x}$ denotes the nearest point of $x$ in $\bar{D}$.

(H1) The condition (A) holds and the Skorohod problem is uniquely solved for any $w$. Moreover, there exists a positive constant $C_D$ such that for all continuous paths $w$ on $\mathbb{R}^d$

$$\|L(w)\|_{[s,t]} \leq C_D\|w\|_{\infty,[s,t]} \quad 0 \leq s \leq t \leq T.$$

(H2) The condition (A) holds and the Skorohod problem is uniquely solved for any $w$. Moreover, there exists a positive constant $C_D'$ such that for all continuous paths $w, w'$ on $\mathbb{R}^d$

$$\|L(w) - L(w')\|_{\infty,[0,t]} \leq C_D' \{\|w - w'\|_{[0,t]} + |w(0) - w'(0)|\}.$$

Remark 4.3. It is proved in [20] that the condition (H1) holds if $D$ is convex and there exists a unit vector $l \in \mathbb{R}^d$ such that

$$\inf\{(l, n(x)) \mid n(x) \in N_x, x \in \partial D\} > 0.$$

The condition (H2) holds if the conditions (B) and (D) are satisfied. This is due to [17].

About the existence and uniqueness of solutions to (4.3), we have the following.

Lemma 4.4. Let $\eta_t$ be a continuous path on $\mathbb{R}^n$ with $\eta_0 = 0$ and $x_t$ be a continuous $p$-variation path on $\mathbb{R}^d$ for some $p \geq 1$. Let $F$ be a linear mapping from $\mathbb{R}^d \otimes \mathbb{R}^n$ to $\mathbb{R}^d$. We consider the following implicit Skorohod equation:

$$y_t = y_0 + \eta_t + F\left(\int_0^t \Phi(r) \otimes dx_r\right) + \Phi(t) \quad 0 \leq t \leq T, \quad (4.4)$$

where $y_0 \in \bar{D}$ ($0 \leq t \leq T$) and $\Phi(t)$ is a continuous bounded variation path which satisfies

$$L\left(y_t + \eta_t + F\left(\int_0^t \Phi(r) \otimes dx_r\right)\right)_t = \Phi_t \quad 0 \leq t \leq T.$$

1. Assume (H2) are satisfied and $x_t$ is bounded variation. Then there exists a unique solution $(y_t, \Phi(t))$ to (4.3).

2. Assume (H1) holds. There exists a solution $(y_t, \Phi(t))$ to (4.3).

Proof. (1) By (H2), we see the unique existence of $\Phi$, by a standard iteration procedure, considering the equation in the small interval, if necessary. This arguments produce the solution for the whole interval $[0, T]$.

(2) First we prove the existence of a solution on a small interval $[0, T']$, where $T' < T$. We specify $T'$ later. Let $\Delta = \{t_k\}_{k=0}^N$ be a partition of $[0, T']$. We consider the Euler approximation of $y$.

$$y^\Delta_t = y^\Delta_{t_k} + \eta_t - \eta_{t_k} + F(\Phi^\Delta(t_k) \otimes (x_t - x_{t_k})) + \Phi^\Delta(t) - \Phi^\Delta(t_k) \quad t_k \leq t \leq t_{k+1}.$$
Hence, again by applying (H1), we obtain
\[ y^\Delta_t = y_0 + \eta_t + F \left( \int_0^t \Phi^\Delta (\pi^\Delta(r)) \otimes dx_r \right) + \Phi^\Delta (t) \quad 0 \leq t \leq T'. \]

Let \( 0 \leq s < t \leq T' \). If \( t_{k-1} \leq s < t \leq t_k \) for some \( k \), then
\[ \int_s^t \Phi^\Delta (\pi^\Delta(r)) \otimes dx_r = \Phi^\Delta (t_{k-1}) \otimes (x_t - x_s). \]  

(4.5)

We consider the case where \( 0 \leq t_{k-1} \leq s < t_k < \cdots < t_l \leq t < t_{l+1} \leq T' \). Then
\[
\begin{align*}
\int_s^t \Phi^\Delta (\pi^\Delta(r)) \otimes dx_r &= \Phi^\Delta (t_{k-1}) \otimes (x_t - x_s) + \Phi^\Delta (t_l) \otimes (x_l - x_t) \\
&\quad + \Phi^\Delta (t_l) \otimes x_l - \Phi^\Delta (t_k) \otimes x_k \\
&\quad + \sum_{m=k}^{l-1} (\Phi^\Delta (t_m) - \Phi^\Delta (t_{m+1})) \otimes x_{t_{m+1}}.
\end{align*}
\]

(4.6)

Therefore we have for all \( 0 \leq s < t \leq T' \),
\[
\left| \int_s^t \Phi^\Delta (\pi^\Delta(r)) \otimes dx_r \right| \leq 5\| \Phi^\Delta \|_{[0,T']} \| x \|_{[0,T']}. 
\]

Hence by (H1),
\[
\| \Phi^\Delta \|_{[0,T']} \leq C_D \left( \| \eta \|_{[0,T']} + 5\| F \| \| x \|_{[0,T']} \| \Phi^\Delta \|_{[0,T']} \right).
\]

(4.7)

Therefore if \( \| x \|_{[0,T']} \leq 1/(10C_D\| F \|) \),
\[
\| \Phi^\Delta \|_{[0,T']} \leq 2C_D \| \eta \|_{[0,T']}. 
\]

Combining (4.7), (4.5) and (4.6) we have for any \( 0 \leq s \leq t \leq T' \),
\[
\left| \int_s^t \Phi^\Delta (\pi^\Delta(r)) \otimes dx_r \right| \leq 6C_D \| \eta \|_{[0,T']} \| x \|_{[0,T']}, \| x \|_{[0,T']} + 2C_D \| \Phi \|_{[s,t]} \| x \|_{[0,T']}.
\]

(4.8)

Hence, again by applying (H1), we obtain
\[
\| \Phi^\Delta \|_{[s,t]} \leq C_D \| \eta \|_{[s,t]} + 2C_D^2 \| F \| \left( 3\| \eta \|_{[0,T']} \| x \|_{[0,T']} + \| \Phi \|_{[s,t]} \| x \|_{[0,T']} \right). 
\]

(4.9)

Consequently, if
\[
\| x \|_{[0,T']} \leq \min \left( (10C_D\| F \|)^{-1}, (4C_D^2\| F \|)^{-1} \right)
\]

(4.10)

then
\[
\| \Phi^\Delta \|_{[s,t]} \leq C_D \left( \| \eta \|_{[s,t]} + 6C_D\| F \| \| \eta \|_{[0,T']} \| x \|_{[s,t]} \right) \quad 0 \leq s \leq t \leq T'.
\]

Now we choose \( T' \) so that (4.10) holds. Then \( \{ \Phi^\Delta_n \} \) is a family of equicontinuous and bounded functions on \([0,T']\) and so there exists a sequence \( n \to 0 \) such that \( \Phi^\Delta_n \) converges to a
certain $\Phi$ uniformly on $[0, T']$. By the estimate (4.7), this convergence takes place for all $p$-variation norm ($p > 1$) on $[0, T']$. Therefore $F \left( \int_0^t \Phi \left( \pi^{\Delta_n}(r) \right) \otimes dx_r \right)$ converges uniformly to $F \left( \int_0^t \Phi(r) \otimes dx_r \right)$. Here we use the property of Young integrals. Also $y^{\Delta_n}_t$ converges uniformly.

We denote the limit by $y$. Then $(y, \Phi(t))$ $(0 \leq t \leq T')$ is a solution of (4.4). Next, we need to construct a solution after time $T'$. For $t \geq T'$, (4.4) reads

$$y_t = y_{T'} + (\eta_t - \eta_{T'}) + F \left( \Phi_{T'} \otimes (x_t - x_{T'}) \right) + F \left( \int_{T'}^t (\Phi(r) - \Phi(T')) \otimes dx_r \right) + \Phi_t - \Phi_{T'}.$$ (4.11)

Since $T'$ depends only on $C_D$ and $\|F\|$, by iterating the above procedure, we can get a solution defined on $[0, T']$.

By the above lemma, we see that there exist a solution $y^\Delta, \Phi^\Delta$ to the implicit Skorohod equation (4.3). Using this approximation solution, we can prove the existence of a solution of reflecting rough differential equations. Now we state our main theorem in this section.

**Theorem 4.5.** Assume (H1) holds. Let $\omega$ be the control function of $X_{s,t}$, i.e., it holds that

$$|X_{s,t}^i| \leq \omega(s,t)^{i/p} \quad 0 \leq s \leq t \leq T, \quad i = 1, 2.$$ (4.12)

Then there exists a solution $(Y, \Phi)$ to the reflecting rough differential equation (4.2) such that for all $0 \leq s \leq t \leq T$,

$$|Y_{s,t}^i| \leq C (1 + \omega(0,T))^3 \omega(s,t)^{i/p}, \quad i = 1, 2,$$ (4.12)

$$\|\Phi\|_{s,t} \leq C (1 + \omega(0,T))\omega(s,t)^{1/p},$$ (4.13)

where the positive constant $C$ depends only on $\sigma, C_D, p$.

Here we make remarks on this theorem together with Theorem 3.1

**Remark 4.6.** (1) At the moment, I do not prove the uniqueness yet and it is not clear to see whether the functionals $X \mapsto \Phi$, $X \mapsto Y$ are continuous or not. Actually, at the moment, I do not know the existence of Borel measurable selection of the mapping. If there are no boundary terms, the functional $X \mapsto Y$ is continuous and this is known as Lyons’ continuity theorem and universal limit theorem. If the continuity theorem would hold, then by applying it to the case of Brownian rough path, it would imply the strong convergence of Wong-Zakai approximation which was proved in [2] under general conditions on the boundary. We discuss the relation between the solution of reflecting rough differential equation driven by Brownian rough path and the solution of reflecting SDE driven by Brownian motion later.

(2) We consider the case where $D$ is a half space. In this simplest case too, we have difficulties to prove the uniqueness of solutions and continuity theorems with respect to driving paths (rough paths) in the equations (3.1) and (4.2). We explain the reason. When $D$ is a half space, the Skorohod mapping $\Gamma$ is given explicitly and it is globally Lipschitz continuous in the set of continuous path spaces with the sup-norm. This nice result is used in the studies [3, 5]. However, it is not Lipschitz continuous in the $\lambda$-Hölder continuous path spaces $C^\lambda_D$. This is pointed out by Ferrante and Rovira [8] who studied reflecting differential equations driven by
Hölder continuous paths on half spaces. This implies the difficulty of the study of the uniqueness of solutions of reflecting differential equations as pointed out in their paper. In the usual rough differential equations, we have locally Lipschitz continuities of the solutions with respect to the driving rough paths. On the other hand, it is not difficult to show that \( \Gamma \) is Hölder continuous restricted sets of rough paths and Hölder continuity of the solution mapping in such a set.

To prove this theorem, we argue similarly to the case \( 1 \leq p < 2 \). When \( \Phi^\Delta(t) \) is defined, let

\[
J^\Delta_{k}(t) = I^\Delta_{k}(t) - D\sigma(y^\Delta_{s})(\sigma(y^\Delta_{s}))(X^2_{s,t}) - (D\sigma)(y^\Delta_{s})\left(\int_{s}^{t}(\Phi^\Delta(r) - \Phi^\Delta(s)) \otimes dx_r \right),
\]

The definition of \( I^\Delta_{k}(t) \) is similar to (3.4) just replacing \( \Phi^\Delta \) by a solution of (4.3). By the definition of \( y^\Delta \), we have \( J^\Delta_{k}(t) = 0 \) for \( t_k \leq t \leq t_{k+1} \). We define \( J^\Delta(s,t,u) = J^\Delta_{k}(u) - J^\Delta_{k}(t) - J^\Delta_{k}(u) \). By an easy calculation, we have for \( s \leq t \leq u \),

\[
J^\Delta(s,t,u) = \left(\sigma(y^\Delta_{s}) - \sigma(y^\Delta_{t}) - (D\sigma)(y^\Delta_{s})(y^\Delta_{t} - y^\Delta_{s}) - (D\sigma)(y^\Delta_{s})(I^\Delta_{k}(t))\right)(x_t - x_u)
\]

\[
+ \left(\left(D\sigma(y^\Delta_{s})(\sigma(y^\Delta_{s})) - (D\sigma)(y^\Delta_{s})(\sigma(y^\Delta_{s}))\right)(X^2_{s,t})
\]

\[
+ \left(\left(D\sigma(y^\Delta_{s}) - (D\sigma)(y^\Delta_{s})\right)\left(\int_{t}^{u}(\Phi^\Delta - \Phi^\Delta_{s}) \otimes dx_r \right)\right).
\]

This relation plays important role as in [4] and the proof in Lemma 3.2 in the calculation below.

**Lemma 4.7.** Suppose (H1) hold. Let \( 2 \leq p < \gamma \leq 3 \). There exist positive constants \( M \) and \( \varepsilon \) which depend only on \( \sigma \) and \( C_D \) such that if \( \omega(t_k,t) \leq \varepsilon \), then

\[
|J^\Delta_{k}(t)| \leq M\omega(t_k,t)^{\gamma/p}
\]

\[(4.14)\]

\[
\|\Phi^\Delta\|_{[t_k,t]} \leq C_3\omega(t_k,t)^{1/p},
\]

\[(4.15)\]

where \( C_3 = 2C_D\|\sigma\|_\infty \). The constant \( M \) is specified in (4.17).

**Proof.** If (4.14) and (4.15) hold, then

\[
|y^\Delta(t) - y^\Delta(t_k)| \leq \left(M\omega(t_k,t)^{\gamma-1/p} + \|\sigma\|_\infty + C_3 + \|D\sigma\|\|\sigma\|\varepsilon^{1/p} + \|D\sigma\|\varepsilon^{1/p}\right)\omega(t_k,t)^{1/p}
\]

\[
\leq C_4\omega(t_k,t)^{1/p},
\]

where \( C_4 = 1 + C_3 + \|\sigma\|_\infty \). Also

\[
|I^\Delta_{k}(t)| \leq M\omega(t_k,t)^{\gamma/p} + \|D\sigma\|\|\sigma\|\omega(t_k,t)^{2/p} + 2C_3\|D\sigma\|\|\sigma\|\omega(t_{k-1},t)^{2/p}
\]

\[
\leq C_5\omega(t_k,t)^{2/p},
\]

where \( C_5 = 1 + 2C_3\|D\sigma\|_\infty + \|D\sigma\|_\infty \|\sigma\|_\infty \). Let

\[
z^\Delta_t = y^\Delta_{t_k} + \sigma(y^\Delta_{t_k})(x_t - x_{t_k}) + (D\sigma)(y^\Delta_{t_k})(\sigma(y^\Delta_{t_k})X^2_{t_k,t})
\]

\[
+ (D\sigma)(y^\Delta_{t_k})\left(\int_{t_k}^{t}(\Phi^\Delta(r) - \Phi^\Delta(t_k)) \otimes dx_r \right) + J^\Delta_{k}(t),
\]

\( t \geq t_k \).
Similarly, \( \Phi^\Delta(t) - \Phi^\Delta(t_k) = L(z^\Delta)_t \) for \( t \geq t_k \). We use this relation to estimate \( \Phi^\Delta \). Let \( K \) be a positive integer. Consider a claim which depends on \( K \): The estimates (4.14) and (4.15) hold for all \( t_k \) and \( t \), where \( t_k \leq t \leq t_{k+K} \) and \( 0 \leq k \leq N - 1 \). We prove this claim by an induction on \( K \). Let \( K = 1 \). By the definition, \( J_{t_k}^\Delta(t) = 0 \) for any \( t_k \leq t \leq t_{k+1} \). We estimate the bounded variation norm of \( \Phi^\Delta \). By (H1),

\[
\| \Phi^\Delta \|_{[t_k,t]} \leq C_D \left( \| \sigma \|_\infty + \| D\sigma \|_\infty \| \sigma \|_\infty \varepsilon^{1/p} \right) \omega(t_k, t)^{1/p} + 2C_D \varepsilon^{1/p} \| D\sigma \|_\infty \| \Phi^\Delta \|_{[t_k,t]}
\]

which implies for sufficiently small \( \varepsilon \),

\[
\| \Phi^\Delta \|_{[t_k,t]} \leq 2C_D \| \sigma \|_\infty \omega(t_k, t)^{1/p}.
\]

Suppose the claim holds for all \( K \) which is smaller than or equal to \( K' - 1 \). We prove the case \( K = K' \). Let \( t_i \) be the largest partition point such that \( t_k \leq t_i < t \leq t_{k+K} \) and \( \omega(t_k, t_i) \leq \frac{1}{2} \omega(t_k, t) \). There are two cases, (a) \( t_i < \pi^\Delta(t) \) and (b) \( t_i = \pi^\Delta(t) \). We consider the case (a). In this case, \( t_i < t_{i+1} \leq \pi^\Delta(t) \). By the definition, we have \( \omega(t_k, t_{i+1}) > \frac{1}{2} \omega(t_k, t) \). By the superadditivity of \( \omega \), we have

\[
\omega(t_{i+1}, t) \leq \frac{1}{2} \omega(t_k, t).
\]

We have

\[
|J_{t_k}^\Delta(t)| \leq |J_{t_k}^\Delta(t_i)| + |J_{t_i}^\Delta(t_{i+1})| + |J_{t_{i+1}}^\Delta(t)|
\]

\[
+ |J^\Delta(t_k, t_i)| + |J^\Delta(t_i, t_{i+1})|
\]

By the assumption of the induction and the choice of \( t_i \),

\[
|J_{t_k}^\Delta(t_i)| \leq 2^{-\gamma/p} M \omega(t_k, t_i)^{\gamma/p}, \quad |J_{t_{i+1}}^\Delta(t)| \leq 2^{-\gamma/p} M \omega(t_k, t)^{\gamma/p}.
\]

By the assumption of the induction, we have

\[
|J^\Delta(t_k, t_i, t)| \leq (C_4/2) \| D^2 \sigma \|_\infty \omega(t_k, t)^{2/p} \omega(t_i, t)^{1/p} + C_5 \| D\sigma \|_\infty \omega(t_k, t)^{2/p} \omega(t_i, t)^{1/p}
\]

\[
+ C_4 \left( \| D^2 \sigma \|_\infty \| \sigma \|_\infty + \| D\sigma \|_\infty^2 \right) \omega(t_k, t)^{1/p} \omega(t_i, t)^{2/p}
\]

\[
+ 2C_4C_3 \| D^2 \sigma \|_\infty \omega(t_k, t)^{1/p} \omega(t_i, t)^{2/p}.
\]

Here we have used that if \( t_k < t_i \),

\[
\int_{t_i}^t (\Phi^\Delta(r) - \Phi^\Delta(t_i)) \otimes dx_r = (\Phi^\Delta(t) - \Phi^\Delta(t_i)) \otimes (x_t - x_{t_i}) - \int_{t_i}^t d\Phi^\Delta(r) \otimes (x_r - x_{t_i})
\]

\[
= 2\| \Phi^\Delta \|_{[t_i,t]} \omega(t_i, t)^{1/p}
\]

\[
\leq 2C_3 \omega(t_i, t)^{2/p}.
\]

Similarly,

\[
|J^\Delta(t_i, t_{i+1}, t)| \leq (C_4/2) \| D^2 \sigma \|_\infty \omega(t_i, t_{i+1})^{2/p} \omega(t_{i+1}, t)^{1/p} + C_5 \| D\sigma \|_\infty \omega(t_i, t_{i+1})^{2/p} \omega(t_{i+1}, t)^{1/p}
\]

\[
+ C_4 \left( \| D^2 \sigma \|_\infty \| \sigma \|_\infty + \| D\sigma \|_\infty^2 \right) \omega(t_i, t_{i+1})^{1/p} \omega(t_{i+1}, t)^{2/p}
\]

\[
+ 2C_3C_4 \| \sigma \|_\infty \| D^2 \sigma \|_\infty \omega(t_i, t_{i+1})^{1/p} \omega(t_{i+1}, t)^{2/p}.
\]
Consequently,
\[
|J^\Delta_{t_k}(t)| \leq 2^{1-(\gamma/p)} M \omega(t_k, t)^{\gamma/p} + \varepsilon^{(3-\gamma)/p} C_6 \omega(t_k, t)^{\gamma/p},
\]
where
\[
C_6 = C_4 \|D^2 \sigma\|_\infty + 2 C_5 \|D \sigma\|_\infty + 2 C_4 \left( \|D^2 \sigma\|_\infty \|\sigma\|_\infty + \|D \sigma\|^2_\infty \right) + 4 C_3 C_4 \|D^2 \sigma\|_\infty.
\]
Therefore, if \( M \) satisfies
\[
M \geq \frac{C_6}{1 - 2^{1-(\gamma/p)}},
\]
then the desired estimate for \( J^\Delta_{t_k}(t) \) holds. In the case of (b), by using the assumption of the induction and noting \( J^\Delta_{t_k}(t) = 0 \), we obtain
\[
|J^\Delta_{t_k}(t)| \leq |J^\Delta_{t_k}(t)| + |J^\Delta_{t_k}(t_k, t)| \leq M \omega(t_k, t)^{\gamma/p} + |J^\Delta_{t_k}(t, t_k)| \leq 2^{-\gamma/p} M \omega(t_k, t)^{\gamma/p} + \left( \varepsilon^{(3-\gamma)/p}/2 \right) C_6 \omega(t_k, t)^{\gamma/p}.
\]
Hence, under the condition \( [1.17] \), the desired estimate for \( J^\Delta_{t_k}(t) \) holds. We show \( \| \Phi^\Delta \|_{[t_k, t]} \leq C_3 \omega(t_k, t)^{1/p} \) for \( t_k, t \) with \( t_k \leq t \leq t_k+k' \). Take \( s, t \) such that \( t_k \leq s \leq t \leq t_k+k' \). We have
\[
J^\Delta_{t_k}(t) - J^\Delta_{t_k}(s) = J^\Delta_{t_k}(s) + J^\Delta_{t_k}(t_k, s, t).
\]
Let \( t_m \) be the largest number such that \( t_m \leq s \). Then we have two cases, (a) \( t_k \leq t_m \leq s < t_m+1 < t \) and (b) \( t_k \leq t_m \leq s < t \leq t_m+1 \). We consider the case (a). We can apply the assumption of the induction to \( t_k, s \) and we obtain,
\[
|J^\Delta_{t_k}(s, t)| \leq 2^{-1} \|D^2 \sigma\|_\infty C_4 \omega(t_k, s)^{2/p} \omega(s, t)^{1/p} + C_5 \|D \sigma\|_\infty \omega(t_k, s)^{2/p} \omega(s, t)^{1/p} + C_4 \left( \|D^2 \sigma\|_\infty \|\sigma\|_\infty + \|D \sigma\|^2_\infty \right) \omega(t_k, s)^{1/p} \omega(s, t)^{2/p} + 2 C_4 \|D^2 \sigma\|_\infty \omega(t_k, s)^{1/p} \|\Phi^\Delta\|_{[s, t]} \omega(s, t)^{1/p}.
\]
We have
\[
J^\Delta_{t_k}(t) = J^\Delta_{t_k}(t_m+1) + J^\Delta_{t_{m+1}}(t) + J^\Delta_{t_k}(s, t_{m+1}, t).
\]
Since \( J^\Delta_{t_k}(t_{m+1}) = - J^\Delta_{t_k}(t_m, s, t_{m+1}) \),
\[
|J^\Delta_{t_k}(t_{m+1})| \leq 2^{-1} \|D^2 \sigma\|_\infty \omega(t_m, t)^{2/p} \omega(t_m, t_{m+1})^{1/p} + C_5 \|D \sigma\|_\infty \omega(t_m, t)^{2/p} \omega(t_m, t_{m+1})^{1/p} + C_4 \left( \|D^2 \sigma\|_\infty \|\sigma\|_\infty + \|D \sigma\|^2_\infty \right) \omega(t_m, t)^{1/p} \omega(s, t_{m+1})^{2/p} + 2 C_4 \|D^2 \sigma\|_\infty \omega(t_m, t)^{1/p} \|\Phi^\Delta\|_{[s, t_{m+1}]} \omega(s, t_{m+1})^{1/p}.
\]
Note that
\[
|I^\Delta_{t_k}(t_{m+1})| \leq |I^\Delta_{t_m}(t_{m+1})| + |I^\Delta_{t_m}(s)| + |\sigma(y_s) - \sigma(y_{t_m})| : \|x_{t_{m+1}} - x_s\| \leq C_5 \omega(t_m, t_{m+1})^{2/p} + C_5 \omega(t_m, s)^{2/p} + \|D \sigma\|_\infty C_4 \omega(t_m, s)^{1/p} \omega(s, t_{m+1})^{1/p} \leq (2 C_5 + C_4 \|D \sigma\|_\infty) \varepsilon^{1/p} \omega(s, t)^{1/p} \leq 2 C_4 \omega(t_m, t_{m+1})^{1/p}.
\]

Therefore, for sufficiently small $\varepsilon$, the induction, we have the following estimates hold. The constant 

$$
|J^\Delta(s, t_{m+1}, t)| \leq 2C_d^2\|D^2\sigma\|_{\infty}\omega(t_m, t_{m+1})^{2/p}\omega(t_{m+1}, t)^{1/p} \\
+ C_5\|D\sigma\|_{\infty}(2C_5 + C_4\|D\sigma\|_{\infty})\varepsilon^{1/p}\omega(s, t)^{1/p}\omega(t_{m+1}, t)^{1/p} \\
+ 2C_1\|D^2\sigma\|_{\infty}\|\sigma\|_{\infty} + \|D\sigma\|_{\infty}^2 \omega(t_m, t_{m+1})^{1/p}\omega(t_{m+1}, t)^{2/p} \\
+ \varepsilon^2 \omega(t_m, t_{m+1})^{1/p}\|\Phi^\Delta\|_{[t_{m+1}, t]}\omega(t_{m+1}, t)^{1/p}.
$$

By the assumption of induction,

$$
|J^\Delta_{t_{m+1}}(t)| \leq M\omega(t_{m+1}, t)^{1/p}.
$$

Because

$$
\int_s^t (\Phi^\Delta(r) - \Phi^\Delta(t_k)) \otimes dx_r = (\Phi^\Delta(t) - \Phi^\Delta(t_k)) \otimes (x_t - x_s) - \int_s^t d\Phi^\Delta(r) \otimes (x_r - x_s),
$$

we have

$$
\left| \int_s^t (\Phi^\Delta(r) - \Phi^\Delta(t_k)) \otimes dx_r \right| \leq \|\Phi^\Delta\|_{[t_k, t]}\omega(s, t)^{1/p} + \|\Phi^\Delta\|_{[s, t]}\omega(s, t)^{1/p}.
$$

Putting the estimates above together, by (H1), for sufficiently small $\varepsilon$, we have

$$
\|\Phi^\Delta\|_{[t_k, t]} \leq C_D \|\Delta\|_{\infty, [t_k, t]} \\
\leq C_D \|D\sigma\|_{\infty}(C_5\omega^{2/p} + 2\varepsilon^{1/p})\|\Phi^\Delta\|_{[t_k, t]} + C_D\|\sigma\|_{\infty}\omega(t_k, t)^{1/p} \\
+ 2C_D\|D\sigma\|_{\infty}\|\sigma\|_{\infty}\varepsilon\omega(t_k, t)^{1/p}.
$$

Therefore, for sufficiently small $\varepsilon$, we obtain $\|\Phi^\Delta\|_{[t_k, t]} \leq C_3\omega(t_k, t)^{1/p}$. We consider the case (b). Since $I^\Delta_{t_k}(s) = I^\Delta_{t_k}(t_m) + I^\Delta_{t_m}(s) + (\sigma(y_{t_m}) - \sigma(y_{t_k}))(x_s - x_{t_m})$, by using the assumption of the induction, we have

$$
|I^\Delta_{t_k}(s)| \leq C_5\omega(t_k, t_m)^{2/p} + C_5\omega(t_m, s)^{2/p} + C_4\|D\sigma\|_{\infty}\omega(t_k, t_m)^{1/p}\omega(t_m, s)^{1/p} \\
\leq (2C_5 + C_4\|D\sigma\|_{\infty})\varepsilon^2.
$$

Since $J^\Delta_{s}(t) = -J^\Delta_{t}(t, s, t)$, we have

$$
|J^\Delta_{s}(t)| \leq 2^{-1}C_d^2\|D^2\sigma\|_{\infty}\omega(t_m, s)^{2/p}\omega(s, t)^{1/p} + C_5\|D\sigma\|_{\infty}\omega(t_m, s)^{2/p}\omega(s, t)^{1/p} \\
+ C_4\|D^2\sigma\|_{\infty}\|\sigma\|_{\infty} + \|D\sigma\|_{\infty}^2 \omega(t_m, s)^{1/p}\omega(s, t)^{2/p} \\
+ 2C_4\|D^2\sigma\|_{\infty}\omega(s, t_m)^{1/p}\omega(s, t)^{1/p}\|\Phi^\Delta\|_{[s, t]}.
$$

Therefore, by the same argument as the case (a), we complete the proof of the case (b) and the proof of the lemma is finished. 

**Lemma 4.8.** Let $\varepsilon$ be a positive number in Lemma [4.7]. Let $\Delta = \{t_k\}_{k=1}^N$ be a partition of $[0, T]$ with $\omega(t_k, t_{k+1}) \leq \varepsilon$ for all $k$. Then there exists $C > 0$ such that for any $0 \leq s \leq t \leq T$ the following estimates hold. The constant $C$ depends only on $\sigma$, $p$ and $D$.

1. $|y^\Delta(t) - y^\Delta(s)| \leq C(1 + \omega(0, T))\omega(s, t)^{1/p}$
(2) $\|\Phi_\Delta\|_{[s,t]} \leq C (1 + \omega(0, T)) \omega(s, t)^{1/p}$.

**Proof.** The proof of this lemma is similar to that of Lemma 3.3. 

**Proof of Theorem 4.5** Let $\hat{X}_\Delta$ be the naturally defined $p$-rough path whose 1-st level path is $(X, \hat{\Phi}_\Delta)$. Thanks to the above lemma, this family of $p$-rough path has a common control function $C \omega$ for some positive constant $C$ which is independent of $\Delta$. Let $p' > p$. Since the two-parameter functions $(s, t) \mapsto \hat{X}_{\Delta s,t} y^\Delta(t)$ are equicontinuous (we need Chen’s identity to prove the equicontinuity of the former), there exist subsequences $\hat{X}_{\Delta_n s,t} y^\Delta_n$, where $\Delta_{n+1}$ is a subdivision of $\Delta_n$ and $|\Delta_n| \to 0$, a $p'$-rough path $\hat{X} \in \Omega_p(\mathbb{R}^n \oplus \mathbb{R}^d)$, a continuous path $y$ and a positive decreasing sequence $\delta_n \downarrow 0$ such that

$$\left| \hat{X}_{\Delta_n s,t} - \hat{X}_{s,t} \right| \leq \delta_n \omega(s, t)^{1/p'} \quad 0 \leq s \leq t \leq T,$$

$$\lim_{n \to \infty} \max_{0 \leq t \leq T} |y^\Delta_n(t) - y(t)| = 0.$$

We denote the limit of $\Phi_{\Delta_n}(t)$ by $\Phi(t)$. Clearly, the estimate (4.13) holds for this $\Phi$ and we have for all $0 \leq s \leq t \leq T$,

$$\left| y_t - y_s - \sigma(y_s)(x_t - x_s) - (\Phi(t) - \Phi(s)) - (D\sigma)(y_s)(\sigma(y_s)X^2_{s,t}) - (D\sigma)(y_s) \left( \int_s^t (\Phi(r) - \Phi(s)) \otimes dx_r \right) \right| \leq C \omega(s, t)^{\gamma/p}.$$

This shows $y_t$ is a solution of

$$dy_t = \sigma(y_t) dX_t + d\Phi(t)$$

in the sense of Davie [4]. Also we can find a $p$-rough path $Y_{s,t} \in \Omega_p(\mathbb{R}^d)$ so that $y_t = y_0 + Y^1_{0,t}$ and the equation (4.12) is satisfied. We write $Y_t = y_0 + Y^1_{0,t}$. We refer the reader for this to [4]. Since $\hat{X}_{s,t}$ has the control function $C(1 + \omega(0, T)) \omega(s, t)$, the estimate on the rough differential equations implies the estimate (4.12). We have to show $Y_t$ and $\Phi(t)$ is the solution of the Skorohod problem associated with the first level path $y_0 + \int_0^t \sigma(Y_s) dX^1_s$. To this end, we consider the solution $Y^\Delta_{s,t}$ associated with $\hat{X}^\Delta$. Let $Y^\Delta_t = y_0 + (Y^\Delta)^1_{0,t}$. Since $\hat{X}^\Delta$ has the same common control function $C \omega(s, t)$ independent of $\Delta$, by the estimate in [4],

$$\lim_{n \to \infty} ||Y^\Delta_n - y^\Delta_n||_{[0,T]} = 0.$$

By Lyons’ continuity theorem for the integrals of $p'$-rough path,

$$\lim_{n \to \infty} \left\| \int_0^t \sigma(Y^\Delta_s) dX^1_s - \int_0^t \sigma(Y_s) dX^1_s \right\|_{[0,T]} = 0.$$

Let $z^\Delta_t = y^\Delta_t - \Phi^\Delta_t$. Then $(y^\Delta_t, \Phi^\Delta_t)$ is the solution of the Skorohod problem associated with $z^\Delta_{s,t}$. Because $Y^\Delta_t = y_0 + \int_0^t \sigma(Y^\Delta_s) dX^1_s + \Phi^\Delta(t)$, $z^\Delta_{s,t}$ converges to $y_0 + \int_0^t \sigma(Y_s) dX^1_s$ uniformly. By the continuity of the Skorohod map, this shows the desired result. 

□
5 Back to reflecting SDE driven by Brownian motion

In this section, we consider the case where $X_{s,t}$ is the Brownian rough path $B_{s,t} \in G\Omega_\mu(\mathbb{R}^d)$, where $2 < p < 3$. The set of geometric rough paths, $G\Omega_\mu(\mathbb{R}^d)$, is the closure of the set of smooth rough paths defined by continuous bounded variation paths with respect to the distance $d_{\mu}$ below and consists $X_{s,t} = (1, X^1_{s,t}, X^2_{s,t})$ where $X^1_{s,t}, X^2_{s,t}$ are $\mathbb{R}^d$ and $\mathbb{R}^d \times \mathbb{R}^d$-valued continuous maps satisfying Chen’s identity and

\[
\sup_{0 \leq s < t \leq T} \frac{|X^i_{s,t}|}{|t - s|^{1/p}} < \infty. \tag{5.1}
\]

The distance is given by

\[
d_{\mu}(X, X') = \sum_{i=1}^{2} \sup_{0 \leq s < t \leq T} \frac{|X^i_{s,t} - (X')^i_{s,t}|}{|t - s|^{1/p}}, \quad X, X' \in G\Omega_\mu(\mathbb{R}^d).
\]

$(G\Omega_\mu(\mathbb{R}^d), d_{\mu})$ is a complete separable metric space. Let $W^d = C([0, T] \to \mathbb{R}^d \mid B(0) = 0)$ be a Wiener space. That is, $W^d$ is a probability space with the Wiener measure $\mu$. The coordinate process $t \mapsto B(t)$ is a realization of Brownian motion. Let

\[
B^N(t) = B(t^N) + \frac{B(t^N_k) - B(t^N_{k-1})}{\Delta N_k} (t - t^N_k) \quad t^N_k \leq t \leq t^N_{k+1},
\]

where $t^N_k = kT/(2^N)$ $(1 \leq k \leq 2^N)$, $\Delta N = 2^{-N}T$ and $\Delta N_k B^N = B(t^N_k) - B(t^N_{k-1})$. We may omit superscript $N$ in the notation $t^N_k$. Consider a smooth rough path $B^N_{s,t}$ over $B^N$. Then we can see that there exists a subset $\Omega \subset W^d$ such that $\mu(\Omega) = 1$ and any $B \in \Omega$ satisfies $i = 1, 2$

\[
|B^i_{s,t} - (B^N)^i_{s,t}| \leq \varepsilon_{p,N}(B)(t - s)^{i/p}, \quad i = 1, 2 \tag{5.2}
\]

\[
\max \{|B^i_{s,t}|, (B^N)^i_{s,t}| \leq (C_p(B)(t - s))^{i/p}, \quad i = 1, 2 \tag{5.3}
\]

hold, where $\lim_{N \to \infty} \varepsilon_{p,N}(B) = 0$ and $2 < p < 3$. We can take $\omega(s, t) = C_p(B)(t - s)$ as a control function for $B^N_{s,t}$ and $B_{s,t}$. Let $Y^N$ be the solution to reflecting ODE:

\[
dY^N(t) = \sigma(Y^N(t))dB^N(t) + d\Phi^N(t), \quad Y^N(0) = x.
\]

By estimates (5.2) and (5.3) and our main theorem in the Section 4, we have

\[
|(Y^N(B))^i_{s,t}| \leq (C_p(B')(t - s))^{i/p}, \quad i = 1, 2 \tag{5.4}
\]

\[
||\Phi^N(B)||[s,t] \leq (C_p(B')(t - s))^{1/p}. \tag{5.5}
\]

Therefore, for any $B \in \Omega$, there exists a subsequence $N_k(B) \uparrow +\infty$ such that $Y^{N_k(B)}_{s,t}(B)_{s,t}$ and $\Phi^{N_k(B)}(B)(t)$ converges in $p'$-rough path sense and $p'$-variation path respectively. The limit is a solution of reflecting rough differential equation driven by $B_{s,t}$. However, we cannot conclude that the limit and the solution is unique by this argument. However, on the other hand, the solution $Y^N$ is the Wong-Zakai approximation of $Y^S(t)$ which is the solution to the reflecting SDE driven by Brownian motion:

\[
dY^S(t) = \sigma(Y^S(t)) \circ dB(t) + d\Phi^S(t), \quad Y^S(0) = x,
\]

\[
\]
where \( \circ dB(t) \) denotes the Stratonovich integral and \( \Phi^S \) is the local time term. We use the notation \( Y^S \) to distinguish the solution in the sense of Itô calculus from the solution in the sense of rough path. Note that in [2], we used the notation \( X^N(t) \) for the Wong-Zakai approximation. If SDE does not contain reflection term, then Lyon’s continuity theorem and the uniqueness of solution imply that the Wong-Zakai approximation of the solution converges to the true solution uniformly. However, we cannot do such a thing here. In [2], we proved that \( Y^N(t) \) converges to \( Y^S(t) \) uniformly on \([0,T]\) for almost all \( B \). By the results in [2], we can prove the following lemma.

**Proposition 5.1.** Assume conditions (A), (B), (C) are satisfied for \( D \). Then for any \( \varepsilon > 0 \), there exists a positive constant \( C_\varepsilon \) independent of \( N \) such that

\[
E \left[ \max_{0 \leq t \leq T} \left| \int_0^t \sigma(Y^N(s))dB^N(s) - \int_0^t \sigma(Y^S(s)) \circ dB(s) \right|^2 \right] \leq C_\varepsilon \cdot 2^{-(1-\varepsilon)N/6}.
\]

Thanks to the lemma above, applying the Borel-Cantelli lemma, we see that there exists a full measure subset \( \Omega' \subset \Omega \) such that \( \int_0^T \sigma(Y^N(s))dB^N(s) \) converges to \( \int_0^T \sigma(Y^S(s)) \circ dB(s) \) for all \( B \in \Omega' \). Hence by the continuity property of the Skorohod mapping, \( \Phi^N(t) \) also converges to \( \Phi^S(t) \) uniformly for all \( B \in \Omega' \). Therefore, \( Y^N(B)_{s,t} \) converges to a certain \( p \)-rough path \( Y(B)_{s,t} \) for all \( B \in \Omega' \cap \Omega \), without taking subsequences, and \( Y(B)_{s,t} \) is a solution of rough differential equation.

**Proof of Proposition 5.1.** In this proof, we use the estimate obtained in [2]. Note that some notation there are different from those in this paper. Take points such that \( t_l < t \leq t_{l+1} \). We have

\[
\left| \int_0^t \sigma(Y^N(s))dB^N(s) - \int_0^t \sigma(Y^N(s))dB^N(s) \right| \leq C|\Delta_lB^N|.
\]

Hence

\[
E \left[ \max_{0 \leq s \leq t} \left| \int_0^s \sigma(Y^N(u))dB^N(u) - \int_0^s \sigma(Y^S(u)) \circ dB(u) \right|^2 \right]
\]

\[
\leq 3E \left[ \max_{0 \leq k \leq l} \left| \int_0^{t_k} \sigma(Y^N(s))dB^N(s) - \int_0^{t_k} \sigma(Y^S(s)) \circ dB(s) \right|^2 \right]
\]

\[
+ 3CE \left[ \max_k |\Delta_kB^N|^2 \right] + 3E \left[ \max_{u-v \leq T/N, 0 \leq u \leq v \leq T} \left| \int_u^v \sigma(Y^S(s)) \circ dB(s) \right|^2 \right]
\]

\[
\leq 3E \left[ \max_{0 \leq k \leq l} \left| \int_0^{t_k} \sigma(Y^N(s))dB^N(s) - \int_0^{t_k} \sigma(Y^S(s)) \circ dB(s) \right|^2 \right] + C_\varepsilon \left( 2^{-N}T \right)^{1-\varepsilon},
\]

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where $\epsilon$ is any positive number. Let $\pi^N(t) = \max \{ t_k^N \mid t_k^N \leq t \}$.

\[
\int_0^{t_t} \sigma(Y^N(s)) dB^N(s) - \int_0^{t_t} \sigma(Y^S(s)) \circ dB(s)
= \int_0^{t_t} (\sigma(Y^N(\pi^N(s))) - \sigma(Y^S(s))) dB(s)
+ \sum_{k=1}^l \int_{t_{k-1}}^{t_k} \left( \int_{t_{k-1}}^{s} (D\sigma)(Y^N(u)) \left( \sigma(Y^N(u)) \frac{\Delta_k B^N}{\Delta_N} \right) du \right) \left( \frac{\Delta_k B^N}{\Delta_N} \right) ds
- \int_0^{t_t} \frac{1}{2} \text{tr} (D\sigma)(Y^S(s)) (\sigma(Y^S(s))) ds
+ \sum_{k=1}^l \int_{t_{k-1}}^{t_k} \int_{t_{k-1}}^{s} (D\sigma)(Y^N(u)) \left( \sigma(Y^N(u)) d\Phi^N(u) \right) \left( \frac{\Delta_k B^N}{\Delta_N} \right) ds
=: I^N_1(t_t) + I^N_2(t_t) + I^N_3(t_t).
\]

Noting

\[
I^N_1(t_t) = \int_0^{t_t} (\sigma(Y^N(\pi^N(s))) - \sigma(\pi^N(s))) dB(s)
+ \int_0^{t_t} (\sigma(\pi^N(s)) - \sigma(Y^S(s))) dB(s),
\]

and by using Burkholder-Davis-Gundy’s inequality and estimates in Theorem 2.9 and Lemma 4.5 in [2], we obtain

\[
E \left[ \max_{0 \leq s \leq t} |I^N_1(s)|^2 \right] \leq C \epsilon^4 \left( 2^{-N} T \right)^{(1-\epsilon)/6} + C \cdot 2^{-N} tT.
\]

\[
I^N_2(t_t) = \frac{1}{2} \int_0^{t_t} \left( \text{tr} (D\sigma)(Y^N(\pi^N(s)))(\sigma(Y^N(\pi^N(s)))) - \text{tr} (D\sigma)(Y^S(s))(\sigma(Y^S(s))) \right) ds
+ \sum_{k=1}^l \int_{t_{k-1}}^{t_k} \int_{t_{k-1}}^{s} \left\{ (D\sigma)(Y^N(u)) \left( \sigma(Y^N(u)) \frac{\Delta_k B^N}{\Delta_N} \right) \right\} du \left( \frac{\Delta_k B^N}{\Delta_N} \right) ds
+ \frac{1}{2} \sum_{k=1}^l \left\{ (D\sigma)(Y^N(t_{k-1})) \left( \sigma(Y^N(t_{k-1})) \frac{\Delta_k B^N}{\Delta_N} \right) \right\} \left( \frac{\Delta_k B^N}{\Delta_N} \right) ds
- \sum_{i=1}^d (D\sigma)(Y^N(t_{k-1})) \left( \sigma(Y^N(t_{k-1})) e_i \right) (e_i) 2^{-N} T, \right\}
=: I^N_{2,1}(t_t) + I^N_{2,2}(t_t) + I^N_{2,3}(t_t)
\]
where \(e_i\) is a unit vector in \(\mathbb{R}^d\) whose \(i\)-th element is equal to 1. We have
\[
|I_{2,1}^N(t_l)| \leq C \int_0^{t_l} |Y^N(s) - Y^S(s)| ds \\
+ Ct_l \max_{0 \leq u \leq v \leq T, |v-u| \leq T/N} \left( |Y^N(v) - Y^N(u)| + |Y^S(v) - Y^S(u)| \right),
\]
\[
|I_{2,2}^N(t_l)| \leq C \sum_{k=1}^l \max_{0 \leq u \leq v \leq T, |v-u| \leq T/N} |Y^N(v) - Y^N(u)| \cdot |\Delta_k B^N|^2.
\]
By the Burkholder-Davis-Gundy’s inequality, we have
\[
E \left[ \max_{1 \leq k \leq l} |I_{2,3}^N(t_k)|^2 \right] \leq CE \left[ \sum_{k=1}^l \eta_k \right],
\]
where
\[
\eta_k = \sum_{i=1}^d \left( (\xi^i_k)^2 - 2^{-N}T \right)^2 + \sum_{1 \leq i < j \leq d} (\xi^i_k)^2 (\xi^j_k)^2.
\]
Here \(\xi^i_k = B^i(t_k) - B^i(t_{k-1}) (1 \leq i \leq d)\) which is the increment of the \(i\)-th element of the Brownian motion. By the estimates in Lemma 2.8, Theorem 2.9 and Lemma 4.5 in [2] and arguing similarly to pages 3813 and 3814 in [2], we have
\[
E \left[ \max_{1 \leq k \leq l} |I_{2,1}^N(t_k)|^2 \right] \leq Ct_l^2 \left( 2^{-N}T \right)^{(1-\varepsilon)/6} + Ct_l^2 \left( 2^{-N}T \right)^{1-\varepsilon}
\]
\[
E \left[ \max_{1 \leq k \leq l} |I_{2,2}^N(t_k)|^2 \right] \leq C \left( 2^{-N}T \right)^{1-\varepsilon}
\]
\[
E \left[ \max_{1 \leq k \leq l} |I_{2,3}^N(t_k)|^2 \right] \leq C \cdot 2^{-N}T.
\]
Finally, since \(\max_{0 \leq t \leq T} |I_3^N(t)| \leq C\|\Phi^N\|_{[0,T]} \max_k |\Delta_k B^N|\) we have
\[
E \left[ \max_{0 \leq t \leq T} |I_3^N(t)|^2 \right] \leq (2^{-N}T)^{1-\varepsilon}
\]
which completes the proof. \(\square\)

Finally, we discuss the relation between the solution of reflecting rough differential equation which is obtained as a limit of the Euler approximation defined in (4.3) and \(Y^S\). For each \(B_{s,t}\), we see the existence of the solution \(y^\Delta(B, t)\). However, it is not trivial to see that a certain version of \(y^\Delta(B, t)\) is a semimartingale. Therefore we need the following proposition.

**Proposition 5.2.** Assume \(D\) satisfies (A), (B), (C), (H1). Let \(\{B_t(\omega)\}\) be an \(\mathcal{F}_t\)-Brownian motion and \(\eta_t(\omega)\) be a continuous \(\mathcal{F}_t\)-semimartingale with \(E[\|\eta\|_{[0,T]}^q] \leq C_q(t-s)^{q/2}\) for all \(q \geq 1\) and \(0 \leq s \leq t \leq T\). We consider the following equation.
\[
Y_t(\omega) = y_0 + \eta_t(\omega) + F \left( \int_0^t \Phi(r, \omega) \otimes dB_r(\omega) \right) + \Phi(t, \omega) \quad 0 \leq t \leq T, \tag{5.6}
\]
where $Y_t(\omega)$ is an $\mathcal{F}_t$-adapted continuous process and $\Phi(t, \omega)$ is an $\mathcal{F}_t$-adapted continuous bounded variation process, and $(Y_t(\omega), \Phi(t, \omega))$ is the solution of Skorohod problem associated with $y_0 + \eta_t(\omega) + \int_0^t \Phi(r, \omega) \otimes dB_r(\omega)$. For this problem, there exists a unique solution.

Proof. We consider again an Euler approximation. Let $\Delta = \{t_k\}$ be a partition of $[0, T]$. We write $|\Delta| = \max_k (t_k - t_{k-1})$ and $\pi(\Delta) = \max \{t_k \mid t_k \leq t\}$. Let $Y^\Delta_t$ be the solution to the Skorohod equation:

$$Y^\Delta_t = Y^\Delta_{t_k} + \eta_t - \eta_{t_k} + F(\Phi^\Delta_{t_k} \otimes (B_t - B_{t_k-1})) + \Phi^\Delta(t) - \Phi^\Delta(t_{k-1}) \quad t_{k-1} \leq t \leq t_k.$$  

Then $Y^\Delta, \Phi^\Delta$ satisfy

$$Y^\Delta_t = y_0 + \eta_t + \int_0^t F(\Phi^\Delta(\pi^\Delta(t))) \otimes dB(t) + \Phi^\Delta(t).$$

Let $q \geq 2$. By the assumption (H1), we have

$$E \left[ \left\| \Phi^\Delta \right\|_{[0,t]}^q \right] \leq C_q (t - s)^{q/2} + C_q(t - s)^{(q-1)/2} \int_s^t E \left( |\Phi^\Delta(\pi^\Delta(\omega))|^q \right) du.$$ 

Hence by considering the case where $s = 0$, we have

$$E \left[ \left\| \Phi^\Delta \right\|_{[0,t]}^q \right] \leq C_q t^{q/2} + C_q(t - s)^{(q-1)/2} \int_0^t E \left( \left\| \Phi^\Delta \right\|_{[0,u]}^q \right) du$$

and by the Gronwall inequality, we get $E \left[ \left\| \Phi^\Delta \right\|_{[0,t]}^q \right] \leq C_q T^{q/2} \exp \left( T^{(q+1)/2} \right)$. Thus, we obtain

$$E \left[ \left\| \Phi^\Delta \right\|_{[0,t]}^q \right] \leq C_q \left( 1 + T^{q/2} \exp \left( T^{(q+1)/2} \right) \right) (t - s)^{q/2} \quad 0 \leq s \leq t \leq T. \quad (5.7)$$

Let $\Delta'$ be another partition of $[0, T]$. Define

$$Z(t) = Y^{\Delta}(t) - Y^{\Delta'}(t),$$

$$\mu(t) = e^{-\frac{2}{\gamma} \int (y^\Delta(t) + f(y^\Delta(t))) dt},$$

$$k(t) = \mu(t)|Z(t)|^2,$$

where $f$ is the function in the condition (C). By the Ito formula, we have

$$dk(t) = \mu(t) \left\{ 2 \left( Z(t), F\left( \Phi^\Delta(\pi^\Delta(t)) - \Phi^\Delta\prime(\pi^\Delta'(t)) \right) \otimes dB(t) \right) \\
+ \sum_{i=1}^d \left| F\left( \Phi^\Delta(\pi^\Delta(t)) - \Phi^\Delta\prime(\pi^\Delta'(t)), e_i \right) \right|^2 dt \\
+ 2\mu(t) \left( Z(t), d\Phi^\Delta(t) - d\Phi^\Delta\prime(t) \right) \\
- \frac{2\mu(t)}{\gamma} |Z(t)|^2 \left\{ \left( (Df)(Y^\Delta(t)), d\Phi^\Delta(t) \right) + \left( (Df)(Y^\Delta\prime(t)), d\Phi^\Delta\prime(t) \right) \right\} \\
- \frac{2\mu(t)}{\gamma} |Z(t)|^2 \left\{ \left( (Df)(Y^\Delta(t)), F\left( \Phi^\Delta(\pi^\Delta(t)) \otimes dB(t) \right) \right) + \left( (Df)(Y^\Delta\prime(t)), F\left( \Phi^\Delta\prime(\pi^\Delta'(t)) \otimes dB(t) \right) \right. \right\} \right\}. \quad (5.8)$$
By the condition (C), we obtain
\[ E\left[|Y_t^\Delta - Y_t^{\Delta'}|^2\right] \leq C_F \int_0^t E\left[|\Phi^\Delta(u) - \Phi^{\Delta'}(u)|^2\right] du + C(|\Delta| + |\Delta'|)t, \]
where we have used the estimate \((5.7)\) and the positive constant \(C_F\) depends on the (Hilbert-Schmidt) norm of \(F\). Combining the above inequality and the identity
\[ \Phi^\Delta(t) - \Phi^{\Delta'}(t) = Y^\Delta(t) - Y^{\Delta'}(t) - \int_0^t F\left(\Phi^\Delta(u) - \Phi^{\Delta'}(u) \otimes dB(u)\right), \quad (5.9) \]
we obtain
\[ E\left[|Y_t^\Delta - Y_t^{\Delta'}|^2\right] \leq C(|\Delta| + |\Delta'|)t + 2C_F \int_0^t E\left[|Y_u^\Delta - Y_u^{\Delta'}|^2\right] du + 2C_F^2 \int_0^t \int_u^t E\left[|\Phi^\Delta(r) - \Phi^{\Delta'}(r)|^2\right] dr du. \]
Iterating this procedure, we have
\[ E\left[|Y_t^\Delta - Y_t^{\Delta'}|^2\right] \leq C(|\Delta| + |\Delta'|)t + C \int_0^t E\left[|Y_s^\Delta - Y_s^{\Delta'}|^2\right] ds. \]
By the Gronwall inequality, we obtain
\[ E|Y_t^\Delta - Y_t^{\Delta'}|^2 \leq C(|\Delta| + |\Delta'|)e^{Ct}. \]
Therefore, by \((5.9)\),
\[ E\left[|\Phi^\Delta(t) - \Phi^{\Delta'}(t)|^2\right] \leq 2C(|\Delta| + |\Delta'|)e^{Ct} + C_F \int_0^t E\left[|\Phi^\Delta(s) - \Phi^{\Delta'}(s)|^2\right] ds \]
and
\[ E\left[|\Phi^\Delta(t) - \Phi^{\Delta'}(t)|^2\right] \leq 2C(|\Delta| + |\Delta'|)e^{(C+C_F)t}. \]
Therefore \(L^2\)-limit \(Y_t := \lim_{|\Delta| \to 0} Y_t^\Delta\) and \(\Phi(t) := \lim_{|\Delta| \to 0} \Phi^\Delta(t)\) exist. Moreover there exists a subsequence \(\Delta\) such that \(\int_0^t F\left(\Phi^\Delta(\sigma(s)) \otimes dB(s)\right)\) converges to \(\int_0^t F\left(\Phi(s) \otimes dB(s)\right)\) \(0 \leq t \leq T\) uniformly \(P\)-a.s. \(\omega\). Thus, by the continuity of the Skorohod mapping, we see that the pair \((Y, \Phi)\) is a solution. We prove the uniqueness. Let \((Y, \Phi)\) and \((Y', \Phi')\) be solutions to \((5.6)\). Then by a similar calculation to \((5.8)\), we have
\[ E\left[|Y(t) - Y'(t)|^2\right] \leq C_F \int_0^t |\Phi(s) - \Phi'(s)|^2 ds. \]
By arguing similarly to the above, we complete the proof. \(\square\)

We consider solutions to \((1.3)\) when \(X_{s,t} = B_{s,t}\). By applying the above proposition, we see that the solution \((y^\Delta(B), \Phi^\Delta(B))\) is unique in the set of semimartingales. Hence, we obtain the following.
**Theorem 5.3.** Assume the conditions (A), (B), (C) and (H1) are satisfied. Let $\Delta_N = \{2^{-NkT}\}_{k=0}^{2N}$. There exists a full measure set $\Omega' \subset \Omega$ such that the following hold for any $B \in \Omega'$.

1. $y^\Delta_N(B)$, $\Phi^\Delta_N(B)$ and $\hat{X}^\Delta_{N,s,t}$ converge as in the proof of Theorem 4.5.
2. $y^\Delta_N$ and $\Phi^\Delta_N$ converge to $Y^S$ and $\Phi^S$ uniformly respectively.
3. We have

$$|Y^S(t) - Y^S(s)| \leq C(1 + \omega_B(0,T))^3\omega_B(s,t)^{1/p},$$

$$\|\Phi^S\|_{[s,t]} \leq C(1 + \omega_B(0,T))\omega_B(s,t)^{1/p},$$

where $\omega_B$ is the control function of $B_{s,t}$ and the constant $C$ depends only on $\sigma, D, p$.

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