Logic of Dynamics & Dynamics of Logic; Some Paradigm Examples

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Abstract
The development of “operational quantum logic” points out that classical boolean structures are too rigid to describe the actual and potential properties of quantum systems. Operational quantum logic bears upon basic axioms which are motivated by empirical facts and as such supports the dynamic shift from classical to non-classical logic resulting into a dynamics of logic.

On the other hand, an intuitionistic perspective on operational quantum logic, guides us in the direction of incorporating dynamics logically by reconsidering the primitive propositions required to describe the behavior of a quantum system, in particular in view of the emergent disjunctivity due to the non-determinism of quantum measurements.

A further elaboration on “intuitionistic quantum logic” emerges into a “dynamic operational quantum logic”, which allows us to express dynamic reasoning in the sense that we can capture how actual properties propagate, including their temporal causal structure. It is in this sense that passing from static operational quantum logic to
dynamic operational quantum logic results in a true logic of dynamics that provides a unified logical description of systems which evolve or which are submitted to measurements. This setting reveals that even static operational quantum logic bears a hidden dynamic ingredient in terms of what is called “the orthomodularity” of the lattice-structure.

Focusing on the quantale semantics for dynamic operational quantum logic, we delineate some points of difference with the existing quantale semantics for (non)-commutative linear logic. Linear logic is here to be conceived of as a resource-sensitive logic capable of dealing with actions or in other words, it is a logic of dynamics.

We take this opportunity to dedicate this paper to Constantin Piron at the occasion of his retirement.

1. INTRODUCTION

As a starting point for our discussion on the dynamics of logic we quote G. Birkhoff and J. von Neumann, confronting the then ongoing tendencies towards intuitionistic logic with their observation of the “logical” structure encoded in the lattice of closed subspaces of a Hilbert space, the “semantics” of quantum theory (Birkhoff and von Neumann 1936):

“The models for propositional calculi [of physically significant statements in quantum mechanics] are also interesting from the standpoint of pure logic. Their nature is determined by quasi-physical and technical reasoning, different from the introspective and philosophical considerations which have to guide logicians hitherto […]. whereas logicians have usually assumed that [the orthocomplementation] properties L71-L73 of negation were the ones least able to withstand a critical analysis, the study of mechanics points to the distributive identities L6 as the weakest link in the algebra of logic.” (p.839)

They point at a fundamental difference between Heyting algebras (the semantics of intuitionistic propositional logic) and orthomodular lattices (the “usual” semantics of quantum logic) when viewed as generalizations of Boolean algebras (the semantics of classical propositional logic). A new intuitionistic perspective on operational quantum logic (see below) provides a way of blending these seemingly contradicting directions in which logic propagated during the previous century (Coecke 2002). In this paper we focus on two
new logical structures, namely (intuitionistic) linear logic (Girard 1987, Abrusci 1990) which emerged from the traditional branch of logic, and dynamic operational quantum logic (Coecke (nd), Coecke and Smets 2001), emerging from an elaboration on the above mentioned blend. We also briefly consider the “general dynamic logic” proposed in van Benthem (1994). We do mention epistemic action logic (e.g., Baltag 1999) and computation and information flow (e.g., Abramsky 1993, Milner 1999) as other examples of dynamic aspects in logic, which we unfortunately will not be able to consider in this paper. We also won’t discuss the “geometry of interaction” paradigm which provides a (promising) different perspective on linear logic (Abramsky and Jagadeesan 1994), but is still in full development. Concretely we start in section 2 with an outline of static operational quantum logic. In section 3 we survey dynamic operational quantum logic, and demonstrate that the emerging structures do not fit in the logical system proposed in van Benthem (1994). In section 4 we compare dynamic operational quantum logic with linear logic while we focus on formal and methodological differences.

2. STATIC “OLD-STYLE” OPERATIONAL QUANTUM LOGIC

The Geneva School approach to the logical foundations of physics originated with the work in Jauch and Piron (1963), Piron (1964), Jauch (1968), Jauch and Piron (1969), Piron (1976) and Aerts (1981) as an incarnation of the part of the research domain “foundations of physics” that is nowadays called “Operational quantum logic” (OQL) — see Coecke et al (2000) for a recent overview of its general aspects. With respect to the intuitionistic perspective and the dynamic aspects which we put forward below, we will further refer in this paper to OQL as “sOQL”, emphasizing its static nature. More concretely, sOQL as a theory aims to characterize physical systems, ranging from classical to quantum, by means of their actual and potential properties, in particular by taking an ontological rather than an empirical perspective, but, still providing a truly operational alternative to the standard approaches on the logical status of quantum theory. Since Moore (1999), Coecke et al ((nd)a,b) and Smets (2001) provide recent and detailed discussions of sOQL, we intend in this paper to give only a concise overview, focusing on its basic concepts and underlying epistemology. We are aware of the strong conceptual restrictions imposed by the rigid foundation of sOQL, necessary in order to obtain a framework with fully well-defined primitive notions. Clearly, in view of still existing conceptual incompatibilities at the foundational level between quantum theory and relativistic space-time and field theoretic considerations,
the development of an essentially “towards dynamics directed”-formalism for quantum logicality should range beyond the rigid concepts of sOQL. Therefore, we conceive of the notions inherited from sOQL as a stepping-stone for further development rather than something necessarily “to carry all the way”. This is the reason why we lowercase ‘o’ in our notation DoQL referring to “dynamic operational quantum logic” and IoQL, when referring to “intuitionistic operational quantum logic”. In particular the sOQL assumptions (see below) of “a particular physical system which is considered as distinct from its surroundings”, “the specification of states as a (pre-defined) set” and “the a priori specification of a particular physical system itself” need too be reconsidered when crossing the edges of sOQL (which itself was designed to clarify the structural description of quantum systems and justify an ontological perspective for non-relativistic quantum theory).

First we want to clarify that the operationalism which forms the core of sOQL points to a pragmatic attitude and not to any specific doctrine one can encounter in last century’s philosophy of science (Coecke et al (nd)a). Linked to the fact that we defend the position of critical scientific realism in relation of sOQL, operationality points to the underlying assumption that with every property of a physical system we can associate experimental procedures that can be performed on the system, each such experimental procedure including the specification of a well-defined positive result for which certainty is exactly guaranteed by the actual existence of the corresponding property. In particular, while on the epistemological level our knowledge of what exists is based on what we could measure or observe, on the ontological level “physical” properties have an extension in reality and are not reducible to sets of procedures.3 Focusing on the stance of critical scientific realism, we first adopt an ontological realistic position and a correspondence theory of truth.4 Though, contrary to naive realists we do adopt the thesis of fallibilism by which truth in relation to scientific theories has to be pursued but can only be approached. As such we agree with Niiniluoto (1999) that scientific progress can be characterized in terms of increasing truthlikeness. Furthermore, we believe that reality can indeed be captured in conceptual frameworks, though contrary to Niiniluoto we like to link this position to a “weak” form of conceptual idealism. Concentrating for instance on Rescher’s conceptual idealism as presented in Rescher (1973, 1987 §11) and revised in Rescher (1995 §8), this position maintains that a description of physical reality involves reference to mental operations, it doesn’t deny ontological realism and also adopts the
thesis of fallibilism. This position is opposed to an ontological idealism in which the mind produces the “real” objects. As one might expect, we want to stress that we are not inclined to adopt Rescher’s strict Kantian distinction between reality out there and reality as we perceive it, though we are attracted by the idea of capturing ontological reality in mind-correlative conceptual frameworks. Once we succeed in giving such a description of reality, it approaches according to us the ontological world close enough to omit a Kantian distinction between the realms of noumena and phenomena. Why it is of importance for us to reconcile critical realism with a weak form of conceptual idealism becomes clear when we focus on sOQL. Firstly, we cannot escape the fact that in our theory we focus on parts of reality considered as well-defined and distinct which we can then characterize as physical systems. Secondly we have to identify the properties of those systems which is a mind-involving activity. To be more explicit, a physicist can believe that a system ontologically has specific actual and potential properties, but to give the right characterization of the physical system he has to consider the definite experimental projects which can test those properties so that, depending on the results he would obtain, he can be reinforced in his beliefs or has to revise them. How sOQL formally is built up, using the notions of actual properties, potential properties and definite experimental projects, will be clarified below. To recapitulate we finish this paragraph by stressing that we focus in our scientific activity on parts of the external world, reality is mind-independent, though the process of its description “involves” some mind-dependent characterizations.

Let us introduce the primitive concrete notions on which sOQL relies, explicitly following Coecke et al (nd):

- We take a particular physical system to be a part of the ostensively external world which is considered as distinct from its surroundings — see Moore (1999) for a discussion on this matter;

- A singular realization of the given particular physical system is a conceivable manner of being of that system within a circumscribed experiential context;

- States $\mathcal{E} \in \Sigma$ of a given particular physical system are construed as abstract names encoding its possible singular realizations;

- A definite experimental project $\alpha \in \mathcal{Q}$ on the given particular physi-
cal system is a real experimental procedure which may be effectuated on that system where we have defined in advance what would be the positive response should it be performed.

- Properties $a \in \mathcal{L}$ of a given particular physical system are construed as candidate elements of reality corresponding to the definite experimental projects defined for that system. We as such obtain a mapping of definite experimental projects $Q$ on properties $\mathcal{L}$.

The notion of an element of reality was first introduced in Einstein et al (1935) as follows:

“If, without in any way disturbing a system, we can predict with certainty [...] the value of a physical quantity, then there exists an element of physical reality corresponding to this physical quantity” (p.777).

While this formulation is explicitly the starting point for Piron’s early work, we insist that Piron’s operational concept of element of reality is both more precise and allows theoretical deduction, being based on an empirically accessible notion of counterfactual performance rather than a metaphorical notion of non-perturbation. The basic ingredient that we inherit from this setting is the agreement that actual properties exist. Before we give a more rigid characterization of how an actual property is conceived within sOQL, we want to remark that Piron adopted the Aristotelian concepts of actuality and potentiality and placed them in a new framework (see e.g. Piron (1983)). For an analysis of how these notions, used within sOQL, are still related to the old Aristotelian ones, we refer to Smets (2001). Let us just briefly mention here that with regard to actuality, an actual property is within sOQL conceived as an attribute which “exists”; it is some realization in reality or in other words; an element of reality. A potential property on the other hand, does not exist in the same way as an actual one, it is conceived merely as a capability with respect to an actualization since there is always a chance — i.e. except for the absurd property — that it could be realized after the system has been changed without destroying it. It will become clear that a property can be actual or potential depending on the state in which we consider the particular physical system. Similarly we can say that certainty of obtaining a positive answer when performing a definite experimental project depends on the state of the system. In order to construct our theory further we need to
introduce the following relationship between definite experimental projects, states and properties:

- A definite experimental project $\alpha$ is called certain for a given singular realization of the given particular physical system if it is sure that the positive response would be obtained should $\alpha$ be effectuated;

- A property $a$ is called actual for a given state if any, and so all, of the definite experimental projects corresponding to the property $a$ are certain for any, and so all, of the singular realizations encoded by that state. A property is called potential when it is not actual.

In particular is one of the essential achievements of sOQL that it gives a consistent and coherent ontological account of physical properties contra certain ‘overextrapolations’ claimed to be inherent in quantum theory. The quantum mechanical formalism itself indeed allows a characterization of the properties of a quantum system as being in correspondence with the closed subspaces of the Hilbert space describing the system in the above sense: Definite experimental projects $\alpha$ expressible in quantum theory are of the form, “the value of an observable is in region $E \subset \sigma(H)$”, where $\sigma(H)$ is the spectrum of the self-adjoint operator $H$ describing the observable, thus we can write $\alpha(H, E)$; more explicitly, the definite experimental project $\alpha(H, E)$ consists of measuring the observable $H$ and obtaining an outcome in $E$; the corresponding property $a$ is then represented by the closed subspace of fixpoints of the projector $P_E$ that arises via decomposition of $H$ according to von Neumann’s spectral decomposition theorem since only the states that imply the actuality of that property, that is, the states represented by a ray included in that subspace, will yield a positive outcome with certainty when “we would perform $\alpha(H, E)$”.

Given the above notions, it becomes possible to introduce an operation on the collection $\mathcal{Q}$ of definite experimental projects. What we have in mind is the product of a family of definite experimental projects $\Pi A$ which obtains its operational meaning in the following way:

- The product $\Pi A$ of a family $A$ of definite experimental projects is the definite experimental project “choose arbitrarily one $\alpha$ in $A$ and effectuate it and attribute the obtained answer to $\Pi A$.”
More explicitly, given a particular realization of the system, $\Pi A$ is a certain definite experimental project if and only if each member of $A$ is a certain definite experimental project. Formally it becomes possible to pre-order definite experimental projects by means of their certainty:

- $\alpha \prec \beta := \beta$ is certain whenever $\alpha$ is certain.

The notions of a trivial definite experimental project, which is always certain and an absurd definite experimental project, which is never certain, can be introduced and play the role of respectively maximal and minimal element of the collection $Q$. The trivial and absurd definite experimental projects give rise on the level of properties to a trivial and absurd property, the first is always actual while the latter is always potential. Through the correspondence between definite experimental projects and properties, the mentioned pre-order relation induces a partial order relation on $L$:

- $a \leq b := b$ is actual whenever $a$ is actual.

The product of a collection of definite experimental projects with corresponding collection of properties $A$ provides a greatest lower bound or meet for any $A \subseteq L$, and as such the set of properties $L$ forms a complete lattice. Indeed, once we have the “meet” of any collection of properties, we can construct the operation of “join” or least upper bound via Birkhoff’s theorem stating that for a given $A \subseteq L$:

$$\bigvee A = \bigwedge \{x \in L \mid (\forall a \in A)a \leq x\}.$$ 

Note however that contrary to the meet, the join admits of no direct operational meaning in sOQL. Even more, we see that the join of a collection of properties need not, and generically does not, correspond to a classical disjunction since the following implication is only secured in one direction,

- $(\exists a \in A: a$ is actual) $\Rightarrow \bigvee A$ is actual.

With respect to the meet we do have

- $(\forall a \in A: a$ is actual) $\iff \bigwedge A$ is actual,

following directly from the identification of the meet with the product of definite experimental projects.
The property lattice description of a physical system in sOQL allows a dual description by means of the system’s states (Moore 1995, 1999) in terms of maximal state sets \( \mu(a) \subseteq \Sigma \) for which a common property \( a \in \mathcal{L} \) is actual whenever the system is in a state in \( \mathcal{E} \in \mu(a) \), these sets being ordered by inclusion. More explicitly, the state-property duality may be straightforwardly characterized once we introduce the forcing relation \( \triangleright \) defined by \( \mathcal{E} \triangleright a \) if and only if the property \( a \) is actual in the state \( \mathcal{E} \), by the fact that we can associate to each property \( a \) the set \( \mu(a) = \{ \mathcal{E} \in \Sigma \mid \mathcal{E} \triangleright a \} \) of states in which it is actual, and to each state \( \mathcal{E} \) the set \( S(\mathcal{E}) = \{ a \in \mathcal{L} \mid \mathcal{E} \triangleright a \} \) of its actual properties. Formally \( \mu : \mathcal{L} \to P(\Sigma) : a \mapsto \mu(a) \) is injective, satisfies the condition \( \mu(\bigwedge A) = \bigcap \mu[A] \) and is called the Cartan map. In particular we now see that actuality can be studied at the level of either states or properties since we have

\[ \mathcal{E} \in \mu(a) \iff \mathcal{E} \triangleright a \iff a \in S(\mathcal{E}). \]

We will be a bit more explicit about the dual description of a physical system by means of its states and operationally motivate the introduction of a symmetric and antireflexive orthogonality relation on the set \( \Sigma \):

- Two states \( \mathcal{E} \) and \( \mathcal{E}' \) are called orthogonal, written \( \mathcal{E} \perp \mathcal{E}' \), if there exists a definite experimental project which is certain for the first and impossible for the second.

If we equip \( \Sigma \) with \( \perp \), we can consider \( \mathcal{L}_\Sigma \) as the set of biorthogonal subsets, i.e., those \( A \subseteq \Sigma \) with \( A^\perp = A \) for \( A^\perp = \{ \mathcal{E}' \mid (\forall \mathcal{E} \in A) \mathcal{E} \perp \mathcal{E}' \} \). Whenever \( \perp \) is separating (Coecke et al (nd)a), the codomain restriction of the Cartan map \( \mu \) to the set of biorthogonals gives us an isomorphism of complete atomistic orthocomplemented lattices (see below) extending the dual description of a physical system by its state space \( \Sigma \) and its property lattice \( \mathcal{L} \) — for details we refer to Moore (1999) and Coecke et al ((nd)a). It thus follows that \( \mathcal{L}_\Sigma \) is an atomistic lattice where the singletons \( \{ \mathcal{E} \} = \{ \mathcal{E} \}^{\perp\perp} \) are the atoms.

Let us first introduce the notion of an atomistic lattice more explicitly:

- A complete lattice \( \mathcal{L} \) is called atomistic if each element \( a \in \mathcal{L} \) is generated by its subordinate atoms,

\[ a = \bigvee \{ p \in \Sigma_\mathcal{L} \mid p \leq a \}, \]

where the \( p \in \Sigma_\mathcal{L} \) are by definition the minimal nonzero elements of \( \mathcal{L} \).
Under the assumption that states are in bijective correspondence with atoms, \( \Sigma = \Sigma_L \subseteq L \), each property lattice is atomistic in the sense that 

\[
a = \bigvee\{ \bigwedge S(\mathcal{E}) \mid \mathcal{E} \in \mu(a) \}
\]

for each \( a \in L \). Note that \( \bigwedge S(\mathcal{E}) \) is the strongest actual property of the collection \( S(\mathcal{E}) \) which as such “represents” the state.

While we will only work with complete atomistic lattices in the following, we want to finish off this section with an example explaining that a property lattice description for a quantum system will not lead to a boolean algebra. Under the assumption that properties have opposite properties and even more that each property \( a \in L \) is the opposite of another one, we can formally introduce an orthocomplementation, i.e.:

- A surjective antitone involution \( \cdot : L \to L \) satisfying \( a \land a' = 0 \), \( a \leq b \Rightarrow b' \leq a' \) and \( a'' = a \). A lattice equipped with an orthocomplementation is usually abbreviated as an ortholattice.

Consider now the property lattice description of a photon as presented in Piron (1978). Take as a physical system a propagating photon which is linearly polarized. A definite experimental project \( \alpha \phi \) is then defined by:

- i. The apparatus: A polarizer oriented with angle \( \phi \) and a counter placed behind it; ii. The manual: Place the polarizer and counter within the passage of the photon; iii. If one registers the passage of the photon through the polarizer, the result is “yes” and otherwise “no”.

Clearly, a property \( a \) corresponding to \( \alpha \phi \) is called actual if it is certain that \( \alpha \phi \) would lead to the response “yes”, should we perform the experiment. In the diagram (Fig 1) of this photon, we consider some of its properties explicitly: \( a \) corresponds to \( \alpha \phi \), \( b \) to \( \alpha' \phi \), \( a' \) to \( \alpha \phi + \pi/2 \) and \( b' \) to \( \alpha' \phi + \pi/2 \). One immediately sees that this lattice is not boolean since distributivity is violated in the sense that \( a \land (b \lor a') = a \) while \( (a \land b) \lor (a \land a') = 0 \).
3. DYNAMIC OPERATIONAL QUANTUM LOGIC

Contrary to the static approach outlined above we will now analyze how an actual property before alteration will induce a property to be actual afterwards and, conversely, we characterize causes for actuality. The obtained result will then give rise to DoQL, when passing via IoQL. We stress that both these approaches are still under full development. These developments were preceded by a representation theorem for deterministic evolutions of quantum systems as given in Faure et al (1995) and for which DoQL provides an extension to non-deterministic cases. The new primitive concept (as compared to sOQL) is the notion of induction, defined in Amira et al (1998) — see also Coecke et al (2001) and Smets (2001):

- An induction \( e \in \epsilon_s \) is a physical procedure that can be effectuated on a particular physical system \( s \). This procedure, when carried out, might change \( s \), modify the collection of its actual properties and thus its state, or even destroy \( s \).

On the collection \( \epsilon_s \) of all inductions performable on a physical system \( s \) we can consider two operations, one corresponding to the arbitrary choice of inductions and one corresponding to a finite concatenation of inductions. Following Amira et al (1998) and Coecke et al (2001), the finite concatenation of inductions \( e_1, e_2, ..., e_n \in \epsilon_s \) is the induction \( e_1 \& e_2 \& ... \& e_n \) which consists of first performing \( e_1 \) then \( e_2 \), then ... until \( e_n \). The arbitrary choice of inductions in \( \{ e_i \mid i \in I \} \subseteq \epsilon_s \) is the induction \( \bigvee_i e_i \) consisting of performing one of the \( e_i \), chosen in any possible way. Defined as such, an act of induction can for example be the assurance of a free evolution (Faure et al 1995), indeterministic evolution (Coecke and Stubbe 1999), e.g., a measurement (Coecke and Smets 2000, 2001), or the action of one subsystem in a compound system on another one (Coecke 2000). For reasons of formal simplicity we will only focus on inductions which cannot lead to the destruction of the physical system under consideration. As such we presume that an induction cannot alter the nature of a physical system, whereby we mean that what can happen is that a system’s state is shifted within the given initial state space, i.e., that the actuality and potentiality of the properties in the initial property lattice is changed. This implies that the description of a physical system by means of its state space or property lattice encompasses those states in which the system may be after performing an induction. More explicitly, we can point to the particular type of properties which are actual or potential before
the system is altered and which contain information about the actuality or potentiality of particular properties after the system is altered. We introduce this particular type of properties formally (Coecke et al 2001):

\[(e, a) : \epsilon \times L^{op} \rightarrow L^{op} : (e, a) \mapsto e.a \]  

where the reason for reversal of the lattice order (this is what “\(^{op}\)” in \(L^{op}\) stands for) will be discussed below. The property \(e.a\) stands for “guaranteeing the actuality of \(a\)”. Existence of a property \(e.a\) is indeed operationally assured in the following way: given that \(a\) corresponds to \(\alpha\), \(e.a\) corresponds to a definite experimental project “\(e.\alpha\)” of the form “first execute the induction \(e\) and then perform the definite experimental project \(\alpha\), and, attribute the outcome of \(\alpha\) to \(e.\alpha\)” (Faure et al 1995, Coecke 2000, Coecke et al 2001). In terms of actuality, following Smets (2001):

“\(e.a\) is an actual property for a system in a certain realization if it is sure that \(a\) would be an actual property of the system should we perform induction \(e\).”

This explains that if \(e.a\) is actual it indeed “guarantees” the actuality of \(a\) with respect to \(e\), while if \(e.a\) is potential it does not. This expression crystallizes into the idea of introducing a causal relation:

\[\sim_{\epsilon} \subseteq L_1 \times L_2,\]

where subscript 1 points to the lattice before \(e\) and 2 to the lattice after \(e\), as follows (Coecke et al 2001):

- \(a \sim_{\epsilon} b := \text{the actuality of } a \text{ before } e \text{ induces (or, guarantees) the actuality of } b \text{ after } e.\)

Against the background of our characterization of \(e.a\) we now see that \(e.a \sim_{\epsilon} a\) will always be valid and that

\[a \sim_{\epsilon} b \iff a \leq e.b,\]

so \(\sim_{\epsilon}\) fully characterizes the action of \(e.- : L_2 \rightarrow L_1\). In case \(e\) stands for the induction “freeze” (with obvious significance, given a referential), conceived as timeless, then \(\sim_{\epsilon}\) reduces to the partial ordering \(\leq\) of \(L_1 = L_2\).

To link the physical-operational level to a mathematical level, we associate with every induction \(e \in \epsilon_s\) a map called property propagation and a map called property causation (Coecke et al 2001):
1) Property Causation:
\[ \bar{c}_* : \mathcal{L}_2 \rightarrow \mathcal{L}_1 : a_2 \mapsto e.a_2 = \bigvee \{ a_1 \in \mathcal{L}_1 \mid a_1 \looparrowleft a_2 \}; \]

2) Property Propagation:
\[ \bar{c}^* : \mathcal{L}_1 \rightarrow \mathcal{L}_2 : a_1 \mapsto \bigwedge \{ a_2 \in \mathcal{L}_2 \mid a_1 \looparrowright a_2 \}. \]

Given those mappings, clearly \( \bar{c}_*(a_2) \) is the weakest property whose actuality guarantees the actuality of \( a_2 \) and \( \bar{c}^*(a_1) \) is the strongest property whose actuality is induced by that of \( a_1 \). Further we immediately obtain the Galois adjunction \( \bar{c}^*(\bar{c}_*(a_2)) \leq a_2 \) and \( a_1 \leq \bar{c}_*(\bar{c}^*(a_1)) \), denoted as \( \bar{c}^* \dashv \bar{c}_* \), since
\[ a \leq \bar{c}_*(b) \iff a \looparrowright b \iff \bar{c}^*(a) \leq b. \]

In Coecke et al (2001) this adjunction is referred to as “causal duality”, since it expresses the dual expressibility of dynamic behavior for physical systems respectively in terms of propagation of properties and causal assignment. We also recall here that this argument towards causal duality suffices to establish evolution for quantum systems, i.e., systems with the lattice of closed subspaces of a Hilbert space as property lattice, in terms of linear or anti-linear maps (Faure et al 1995) and compoundness in terms of the tensor product of the corresponding Hilbert spaces (Coecke 2000). It also follows from the above that the action defined in eq. defines a quantale module action (Coecke et al 2001) — quantales will be discussed below. Crucial here is the fact that \( (\bigvee_i e_i) \cdot \alpha \) and \( \Pi_i (e_i \cdot \alpha) \) clearly define the same property \( (\bigvee_i e_i) \cdot a = \bigwedge_i (e_i \cdot a) \), since both \( \Pi \) for definite experimental projects and \( \bigvee \) for inductions express choice. Accordingly, the opposite ordering in eq. then matches \( \mathcal{L} \)-meets with \( \epsilon \)-joins.

In the above we discussed the strongest property \( \bar{c}^*(a_1) \) of which actuality is induced by an induction due to actuality of \( a_1 \), but only for maximally deterministic evolutions this fully describes the system’s behavior. In other cases it makes sense to consider how (true logical) disjunctions of properties propagate, as such allowing accurate representation of for example the emergence of disjunction in a perfect quantum measurement (Coecke 2002, Coecke and Smets 2000, Smets 2001) due to the uncertainty on the measurement outcome whenever the system is not in an eigenstate of this measurement. First we introduce the notion of an actuality set as a set of properties of which at
least one element is actual, clearly encoding logical disjunction in terms of actuality. As before we want to express propagation and causal assignment of these actuality sets. While we shift from the level of properties to sets of properties, we will have to take care that we don’t loose particular information on the structure of \( \mathcal{L} \), in particular its operationally derived order. The solution to this problem consists in considering a certain kind of ideal. More specifically we work with the set \( DI(\mathcal{L}) \subseteq P(\mathcal{L}) \), \( P(\mathcal{L}) \) being the powerset of \( \mathcal{L} \), of which the elements are called property sets and which formally are the so-called distributive ideals of \( \mathcal{L} \), introduced in a purely mathematical setting in Bruns and Lakser (1970):

- A distributive ideal is an order ideal, i.e. if \( a \leq b \in I \) then \( a \in I \) and \( I \neq \emptyset \), and is closed under distributive joins, i.e. if \( A \subseteq I \in DI(\mathcal{L}) \) then \( \bigvee A \in I \) whenever we have

\[
\forall b \in \mathcal{L} : b \wedge \bigvee A = \bigvee \{ b \wedge a \mid a \in A \}.
\]

Intuitively, this choice can be motivated as follows (a much more rigid argumentation does exists):¹² i. a first choice for encoding disjunctions would be the powerset itself, however, if \( a \leq b \) we don’t have \( \{a\} \subseteq \{b\} \) so we do not preserve order; otherwise stated, if \( a < b \) then the “propositions” \( \{a\} \) and \( \{a,b\} \) (read: either \( a \) or \( b \) is actual) mean the same thing, since actuality of \( b \) is implied by that of \( a \); ii. we can clearly overcome this problem by considering order ideals

\[
I(\mathcal{L}) := \{ \downarrow [A] \mid A \subseteq \mathcal{L} \} \subset P(\mathcal{L});
\]

however, in case the property lattice would be a complete Heyting algebra in which all joins encode disjunctions, then \( A \) and \( \bigvee A \) again mean the same thing; this redundancy is then exactly eliminated by considering distributive ideals (Coecke 2002, Coecke and Smets 2001). For \( \mathcal{L} \) atomistic and \( \Sigma \subseteq \mathcal{L} \), \( DI(\mathcal{L}) \cong P(\Sigma) \) which implies that \( DI(\mathcal{L}) \) is a complete atomistic boolean algebra (Coecke 2002).

Similarly as for properties, for property sets we can operationally motivate the introduction of a causal relation \( \tau \subseteq DI(\mathcal{L})_1 \times DI(\mathcal{L})_2 \):

- \( A \xleftarrow{\tau} B := \) if property set \( A \) is an actuality set before \( e \), \( A \) induces that property set \( B \) is an actuality set after \( e \).
To every induction we associate a map called property set causation and a map called property set propagation:

1) Property set causation

\[ \hat{e}_*: DI(L)_1 \rightarrow DI(L)_2 : A_1 \mapsto \bigcup \{ A_2 \in DI(L)_1 \mid A_1 \sim_e A_2 \} \]

2) Property set propagation

\[ \hat{e}^* : DI(L)_1 \rightarrow DI(L)_2 : A_1 \mapsto \bigcap \{ A_2 \in DI(L)_2 \mid A_1 \sim_e A_2 \} \]

where

\[ C : P(L) \rightarrow P(L) : A \mapsto \bigcap \{ B \in DI(L) \mid A \subseteq B \} . \]

Similar as above we obtain an adjunction \( \hat{e}^* \dashv \hat{e}_* \) following from:

\[ \hat{e}^*(A)_1 \subseteq A_2 \iff A_1 \sim_e A_2 \iff A_1 \subseteq \hat{e}_*(A)_2 \]

In case the induction \( e \) stands for “freeze” we obtain \( A \sim_e B = A \subseteq B \). Note that the join preservation that follows from the adjunction \( \hat{e}^* \dashv \hat{e}_* \) expresses the physically obvious preservation of disjunction for temporal processes.

It is however also important to stress here that not all maps \( \hat{e}^* : DI(L)_1 \rightarrow DI(L)_2 \) are physically meaningful. Indeed, any physical induction admits mutually adjoint maps \( \tilde{e}^* \) and \( \tilde{e}_* \) with the significance discussed above, the existence of such a map \( \tilde{e}^* : L_1 \rightarrow L_2 \) forcing \( \hat{e}^* \) to satisfy a join continuity condition (Coecke and Stubbe 1999, Coecke et al 2001, Coecke 2002), namely:

\[ \bigvee A = \bigvee B \implies \bigvee \hat{e}^*(A) = \bigvee \hat{e}^*(B) , \tag{2} \]

which indeed expresses well-definedness of a corresponding \( \tilde{e}^* \) since given an actuality set \( A \), the strongest property that is actual with certainty is \( \bigvee A \). It is exactly in the existence of a non-trivial condition as in eq.(2) that the non-classicality of quantum theory comes in. As such, both causal dualities, the one on the level of properties and the other on the level of property sets, provide a physical law on transitions, respectively condition eq.(2) and preservation of \( DI(L) \)-joins. We also want to stress a manifest difference here with the setting in van Benthem (1994):

“The most general model of dynamics is simply this: some system moves through a space of possibilities. Thus there is to be some
set $\Sigma$ of relevant states (cognitive, physical, etc.) and a family $\{R_e | e \in \epsilon\}$ of binary transition relations among them, corresponding to actions that could be performed to change from one state to another. 

Let us briefly consider a number of dynamic ‘genres’, 

- Real action in the world changes actual physical states. 
- What are most general operations on actions? Ubiquitous examples are sequential composition and choice.” (p.109, 110, 112)

Thus it seems to us that the author aims to cover our study of dynamic behavior of physical systems. He moreover states (van Benthem 1994):

“The main claim of this paper is that the above systems of relational algebra and dynamic logic provide a convenient architecture for bringing out essential logical features of action and cognition.” (p.130)

We claim here that relational structures are inappropriate for modeling physical dynamics, even classically! Let us motivate this claim. It follows from the above that transitions of properties of physical systems are internally structured by the causal duality, which in the particular case of non-classical systems restricts possible transitions. Recalling that for atomistic property lattices we have $P(\Sigma) \cong DI(\mathcal{L})$ it is definitely true that any join preserving map $\hat{e}^* : DI(\mathcal{L})_1 \rightarrow DI(\mathcal{L})_2$ defines a unique relation $R_e \subseteq \Sigma_1 \times \Sigma_2$, and conversely, any relation $R \subseteq \Sigma_1 \times \Sigma_2$ defines a unique (join preserving) map $f_R : DI(\mathcal{L})_1 \rightarrow DI(\mathcal{L})_2$. Next, any relation $R \subseteq \Sigma_1 \times \Sigma_2$ has an inverse $R^{-1} \subseteq \Sigma_2 \times \Sigma_1$ and this inverse plays a major role in van Benthem (1991, 1994) as converse action. However, nothing assures that when $\hat{e}^*$ satisfies eq.(2) that the map $f_{R^{-1}} : DI(\mathcal{L})_2 \rightarrow DI(\mathcal{L})_1$ encoding the relational inverse satisfies eq.(4), and as such, has any physical significance at all.\footnote{Obviously, this argument applies only to non-classical systems. More generally, however, since it is the duality between causation and propagation at the $DI(\mathcal{L})$-level that guarantees preservation of disjunction we feel that it should be present in any modelization, and although relations and union preserving maps between powersets are in bijective correspondence, they have fundamentally different dual realizations: relations have inverses, and union preserving maps between powersets have adjoints, and these two do not correspond at all, respectively being encoded as (in terms of maps between powersets):

$$f_{R_e^{-1}}(A) = \{p \in \Sigma | \exists q \in A : q \in \hat{e}^*(\{p\})\}$$}
\[ \hat{e}^*(A) = \{ p \in \Sigma | \forall q \in A : q \in \hat{e}^*(\{p\}) \} \]

if it was even only by the fact that one preserves unions and the other one intersections. As such, the seemingly innocent choice of representation in terms of relations or union preserving maps between powersets does have some manifest consequence in terms of the implementation of causal duality.

In the remaining part of this section we concentrate on the logic of actuality sets as initiated in Coecke (2002). Introduce the following primitive connectives:

\[
\begin{align*}
\bigwedge_{DI(\mathcal{L})} & : P(DI(\mathcal{L})) \rightarrow DI(\mathcal{L}) : A \mapsto \bigcap A; \\
\bigvee_{DI(\mathcal{L})} & : P(DI(\mathcal{L})) \rightarrow DI(\mathcal{L}) : A \mapsto \mathcal{C}(\bigcup A); \\
\rightarrow_{DI(\mathcal{L})} & : DI(\mathcal{L}) \times DI(\mathcal{L}) : (B, C) \mapsto \bigvee_{DI(\mathcal{L})} \{ A \in DI(\mathcal{L}) | A \cap B \subseteq C \} \\
& = \{ a \in \mathcal{L} | \forall b \in B : a \land b \in C \} ; \\
\mathcal{R}_{DI(\mathcal{L})} & : DI(\mathcal{L}) \rightarrow DI(\mathcal{L}) : A \mapsto \downarrow (\bigvee_{\mathcal{L}} A) .
\end{align*}
\]

While the first three connectives are standard in intuitionistic logic, \( \mathcal{R}_{DI(\mathcal{L})} \) should be conceived as a “resolution-connective” allowing us to recuperate the logical structure of properties on the level of property sets (Coecke 2002). In particular, for classical systems we have \( \mathcal{R}_{DI(\mathcal{L})} = id_{DI(\mathcal{L})} \). Note that the condition in eq.(2) now becomes:

\[
\mathcal{R}_{DI(\mathcal{L})}(A) = \mathcal{R}_{DI(\mathcal{L})}(B) \implies \mathcal{R}_{DI(\mathcal{L})}(\hat{e}^*(A)) = \mathcal{R}_{DI(\mathcal{L})}(\hat{e}^*(B)) , \quad (3)
\]

restricting the physically admissible transitions. Clearly, this condition is trivially satisfied for classical systems.

When concentrating on the material implication, we want to stress that on the level of \( \mathcal{L} \) only for the properties in a distributive sublattice we can say the following:

\[
p \models (a \rightarrow_{\mathcal{L}} b) \iff \{p\} \cap \mu(a) \subseteq \mu(b) \\
\iff p \in \mu(a) \implies p \in \mu(b) \\
\iff p \models a \implies p \models b .
\]

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In that case, this implication satisfies the strengthened law of entailment:

\[ \mu(a \rightarrow_L b) = \Sigma \iff \mu(a) \subseteq \mu(b) \iff a \leq b. \]

Note that in the non-distributive case for \( a \rightarrow_L b = a' \lor b \in \mathcal{L} \) we can only say that \( (p \models a \rightarrow_L b) \iff (p \models a \implies p \models b) \), which goes together with the fact that there are examples of orthomodular lattices for which \( a' \lor b = 1 \) while \( a \not\leq b \). Hence in general, for \( T = \{ p \in \Sigma \mid p \models a \implies p \models b \} \) there will not be an element \( x \in \mathcal{L} \) for which \( \mu(x) = T \). There are of course examples of other implications which do satisfy the strengthened law of entailment in the orthomodular case — see for instance Kalmbach (1983). It is now our aim to focus on the implication for elements in \( DI(\mathcal{L}) \). First we lift the Cartan map to the level of property sets \( \mu(A) := \bigcup \mu[A] \subseteq \Sigma_L \), then we obtain the following semantical interpretation:

\[
\begin{align*}
p \models (A \rightarrow_{DI(\mathcal{L})} B) & \iff \{p\} \cap \mu(A) \subseteq \mu(B) \\
& \iff p \in \mu(A) \implies p \in \mu(B) \\
& \iff p \models A \implies p \models B.
\end{align*}
\]

From this it follows that

\[ \mu(A \rightarrow_{DI(\mathcal{L})} B) = \{ p \in \Sigma_L \mid (p \models A) \implies (p \models B) \}, \]

which allows us to reformulate the given static implication \((- \rightarrow_{DI(\mathcal{L})} -)\) as follows (Coecke (nd), Coecke and Smets 2001):

\[
(A \rightarrow_{DI(L)} B) = \bigvee_{DI(L)} \{ C \in DI(L) \mid \forall D \models C : (D \models A \implies D \models B) \}
\]

\[
= \{ c \in L \mid \forall d \leq c : (d \in A \implies d \in B) \},
\]

where \( D \models A \iff \forall p \in \mu(D) : p \in \mu(A) \). Again this implication satisfies the strengthened law of entailment in the sense that

\[ (A \rightarrow_{DI(\mathcal{L})} B) = \mathcal{L} \iff A \subseteq B \]

and is as operation the right adjoint with respect to the conjunction \( \bigwedge_{DI(\mathcal{L})} \), as it is always the case for the implication connective of a complete Heyting algebra — see for example Borceux (1994) or Johnstone (1982):

\[ (A \land_{DI(\mathcal{L})} -) \vdash (A \rightarrow_{DI(\mathcal{L})} -). \]
As for the static material implication defined above, we now want to look for a dynamic propagation-implication satisfying the following:

\[(A \rightarrow B) = \mathcal{L} \iff A \xrightarrow{e} B.\]

The candidate which naturally arises is (Coecke (nd), Coecke and Smets 2001):

\[(A \xrightarrow{e} B) := \{c \in L \mid \forall d \leq c : (d \in A \Rightarrow \hat{e}^*(\downarrow d) \subseteq B)\}.

indicated by a semantical interpretation as

\[\mu(A \rightarrow_{DI(L)} B) = \{p \in \Sigma_L \mid (p \models A) \Rightarrow (\hat{e}^*(\{p\}) \models B)\}.

In case \(e\) stands for the induction “freeze” we see that \(\xrightarrow{e}\) reduces to \(\rightarrow_{DI(L)}\).

Similar as in the static case, we can find an induction-labeled operation as left adjoint to the dynamic propagation-implication, i.e.,

\[(A \otimes_e -) \dashv (A \xrightarrow{e} -) \quad \text{with} \quad A \otimes_e B := \hat{e}^*(A \land_{DI(L)} B).

It is important here to notice that this dynamic conjunction is a commutative operation. Since \(\hat{e}^*\) preserves joins and since in \(DI(L)\) binary meets distribute over arbitrary joins (being a complete Heyting algebra) we moreover have (Coecke (nd)):

\[A \otimes_e \left( \bigvee_{DI(L)} B \right) = \bigvee_{DI(L)} \{A \otimes_e B \mid B \in B\},

so \(DI(L)\) is equipped with operations \(\xrightarrow{e}\) and \(\otimes_e\), for every \(e \in \epsilon_s\) the latter yielding “commutative quantales”. Let us give the definition of a quantale (Rosenthal 1990, 1996, Paseka and Rosicky 2000):

- A **quantale** is a complete lattice \(Q\) together with an associative binary operation \(\circ\) that satisfies \(a \circ (\bigvee_i b_i) = \bigvee_i (a \circ b_i)\) and \((\bigvee_i b_i) \circ a = \bigvee_i (b_i \circ a)\) for all \(a, b_i \in Q\).

Thus, for each induction \(e\) we obtain that \((DI(L), \bigvee_{DI(L)}, \otimes_e)\) is a commutative quantale since \(\otimes_e\) is a commutative operation. In case \(e\) stands for “freeze”, the mentioned quantale becomes a locale \((DI(L), \bigvee_{DI(L)}, \land_{DI(L)}\), i.e., a complete Heyting algebra. Recall here that a locale is a quantale with
as quantale product the meet-operation of the complete lattice, and one verifies that this definition exactly coincides with that of a complete Heyting algebra — for details we refer again to Borceux (1994) or Johnstone (1982).

As for the propagation-implications which, when valid, express a forward causal relation between property sets, we can introduce causation-implications. The relation to which these causation-implications match will be a backward relation introduced as follows (Coecke and Smets 2001):

- \( A \xleftarrow{e} B := \text{If property set } B \text{ is necessarily an actuality set after } e \text{ then property set } A \text{ was an actuality set before } e. \)

Formally we see

\[
A \xleftarrow{e} B \iff \hat{e}(B) \subseteq A.
\]

The causation-implications we want to work with now have to satisfy:

\[
A \xleftarrow{e} B = \mathcal{L} \iff A \xleftarrow{e} B.
\]

The candidate which satisfies this condition is:

\[
(A \xleftarrow{e} B) := \{ c \in \mathcal{L} \mid \forall d \leq c : (d \in A \iff \hat{e}(\downarrow d) \subseteq B) \}.
\]

indicated by a semantical interpretation as

\[
\mu(A \xleftarrow{DI(L)} B) = \{ p \in \Sigma_{\mathcal{L}} \mid (p \models A) \iff (\hat{e}(\{p\}) \models B) \}.
\]

As such we see that when \( A \xleftarrow{e} B \) is valid (i.e. equal to \( \mathcal{L} \)) then it expresses that if property set \( B \) is an actuality set after \( e \) then property set \( A \) was an actuality set before \( e \). Again we have a left adjoint for the causation-implication:

\[
(\_ \otimes e B) \dashv (\_ \xleftarrow{e} B) \quad \text{with} \quad A \otimes e B := A \wedge_{DI(L)} \hat{e}(B).
\]

Thus we can additionally equip \( DI(\mathcal{L}) \) with \( \xleftarrow{e} \) and \( \otimes e \), for each \( e \in \epsilon_s \) the latter yielding non-commutative co-quantales \( (DI(\mathcal{L}), \wedge_{DI(L)}, e \otimes) \), i.e., with a distributive property with respect to meets. Note that the preservation of joins for propagation versus that of meets for causation reflects here in a two-sided distributivity respectively with respect to joins and meets. Indeed, since we have:

\[
(\mathcal{L} \otimes e -) = \hat{e}(-) \quad \text{and} \quad (\mathcal{L} \otimes -) = \hat{e}_s(-)
\]

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this distributivity truly encodes the respective join and meet preservation and consequently, also the causal duality.

It is important to remark that the semantics we obtain is the complete Heyting algebra of actuality sets $DI(\mathcal{L})$ equipped with additional dynamic connectives to express causation and propagation:

$$\left(\left(DI(\mathcal{L}), R_{DI(\mathcal{L})}, \bigvee_{\mathcal{L}}, \bigwedge_{\mathcal{L}}, \neg, \{\otimes, \epsilon, \epsilon, \rightarrow, \leftarrow, \neg, e \in \epsilon\}\right)\right).$$

It turns out that we have a forward negation $\neg$ which does not depend on $e$, and thus coincides with that of “freeze”, i.e., the static one, and a backward negation $\neg$ which by contrast does depend on $e$ — for a discussion of these negations we refer to Coecke (nd) and Smets (2001). Note here that $DI(\mathcal{L})$ is also a left quantale module for $\hat{\epsilon} = \{\hat{e} \mid e \in \epsilon_s\}$ when considering the pointwise action of $(e, -)$. This then intertwines the two quantale structures that emerge in our setting. We end this paragraph by mentioning that it is possible to implement other kinds of implications on $DI(\mathcal{L})$ that extend the causal relation. As an example, it is possible to extend $DI(\mathcal{L})$ with bi-labeled families of non-commutative quantales rendering bi-labeled implications, some of them extending the ones presented in this paper.

In Coecke and Smets (2001) it is argued that the Sasaki adjunction is an incarnation of causal duality for the particular case of a quantum measurement $\varphi_a$ with a projector (on $a$) as corresponding self-adjoint operator. This has as a striking consequence, since validity of the Sasaki adjunction is equivalent to “orthomodularity”, sOQL embodies a hidden dynamical ingredient which is algebraically identifiable as orthomodularity. One could as such argue that the necessity of the passage from sOQL to DoQL was already announced within sOQL itself, it was just waiting to be revealed. As a more radical statement one could say that due to this hidden dynamical ingredient, it is impossible to give a full sense to quantum theory in logical terms within an essentially static setting. Following Coecke and Smets (2001), this fact can be deduced from DoQL by eliminating the emergent disjunctivity when introducing modalities with respect to actuality and conditioning. We can in that case derive that the labeled dynamic hooks that encode quantum measurements act on properties as

$$(a_1 \rightarrow a_2) := (a_1 \rightarrow_L (a \rightarrow_S a_2)) \quad \text{and} \quad (a_1 \rightarrow a_2) := ((a \rightarrow_S a_2) \rightarrow_L a_1)$$
where \((- \mathcal{S} -)\) is the well-known Sasaki hook and we identify \(a\) and \(\{a\}\). One could say that the transition from either classical or intuitionistic logicality to “true” quantum logicality entails besides the introduction of an additional unary connective “operational resolution” the shift from a binary implication connective to a ternary connective where two of the arguments have an ontological connotation and the third, the new one, an empirical.

4. COMPARISON WITH LINEAR LOGIC

We will analyze another logic of dynamics, namely linear logic, while focusing on the differences with DoQL especially with respect to the quantale structures mentioned above. We intend to give a brief overview of the basic ideas behind “Linear logic” as introduced in Girard (1987, 1989). It’s categorical semantics in terms of a \(*\)-autonomous category appeared in Barr (1979) and it is fair to say that already in Lambek (1958) a non-commutative fragment of linear logic was present. We follow the discussion of Smets (2001).

The main advantage linear logic has with respect to classical/intuitionistic logic is that it allows us to deal with actions versus situations in the sense of stable truths (Girard 1989). This should be understood in the sense that linear logic is often called a resource sensitive logic. The linear logical formulas can be conceived as expressing finite resources, the classical formulas then being interpretable as corresponding to unlimited or eternal resources. Allowing ourselves to be a bit more formal on this matter, resource sensitivity is linked to the explicit control of the weakening and contraction rules. As structural rules, weakening and contraction will be discarded in the general linear logical framework. Note that in non-commutative linear logic, to which we will come back later, the exchange-rule will also be dropped.

We use \(\mathcal{A}, \mathcal{B}\) for sequences of well formed formulas and \(a, b\) for well formed formulas. Sequents are conceived as usual:

\[
\begin{align*}
\text{(weakeningL)} & \quad \frac{\mathcal{A} \rightarrow \mathcal{B}}{\mathcal{A}, a \rightarrow \mathcal{B}} & \text{(weakeningR)} & \quad \frac{\mathcal{A} \rightarrow \mathcal{B}}{\mathcal{A} \rightarrow \mathcal{B}, a} \\
\text{(contractionL)} & \quad \frac{\mathcal{A}, a, a \rightarrow \mathcal{B}}{\mathcal{A}, a \rightarrow \mathcal{B}} & \text{(contractionR)} & \quad \frac{\mathcal{A} \rightarrow a, a, \mathcal{B}}{\mathcal{A} \rightarrow a, \mathcal{B}} \\
\text{(exchangeL)} & \quad \frac{\mathcal{A}_1, a, b, a_2 \rightarrow \mathcal{B}}{\mathcal{A}_1, b, a, a_2 \rightarrow \mathcal{B}} & \text{(exchangeR)} & \quad \frac{\mathcal{A} \rightarrow \mathcal{B}_1, a, b, \mathcal{B}_2}{\mathcal{A} \rightarrow \mathcal{B}_1, b, a, \mathcal{B}_2}
\end{align*}
\]

Dropping weakening and contraction implies that linear formulas cannot be duplicated or contracted at random, in other words, our resources are restricted. An important consequence of dropping these two structural rules
is the existence of two kinds of “disjunctions” and “conjunctions”. We will obtain a so-called additive disjunction $\oplus$ and additive conjunction $\sqcap$ and a so-called multiplicative disjunction $\wp$ and multiplicative conjunction $\otimes$. The following left and right rules will make their differences clear.

$$(\sqcap R) \quad \frac{A \rightarrow a, B}{A, a \sqcap b, B} \quad (\oplus L) \quad \frac{A, a \rightarrow B, B}{A, a \oplus b \rightarrow B}$$

$$(\sqcap L) \quad \frac{A, a \rightarrow B, B}{A, a \sqcap b \rightarrow B, B} \quad (\oplus R) \quad \frac{A, a \rightarrow b, B}{A, a \oplus b \rightarrow B}$$

$$(\otimes R) \quad \frac{A, a \rightarrow b, B}{A, a \otimes b, B, B} \quad (\wp L) \quad \frac{A, a \rightarrow b_1, B_1, B_2}{A, a, a \wp b \rightarrow b_1, B_1, B_2}$$

$$(\otimes L) \quad \frac{A, a \otimes b \rightarrow B}{A, a, b \rightarrow B} \quad (\wp R) \quad \frac{A, a \rightarrow b, B}{A, a, a \wp b \rightarrow B}$$

By allowing the structural rules and by using $(\otimes)$ we can express the $(\sqcap)$-rules and vice versa. A similar result can be obtained for the $(\oplus)$ and $(\wp)$-rules.

To understand linear logic, it is necessary to take a look at the intuitive meaning of the above additives and multiplicatives derived from their use by the above rules. First it is important to note that to obtain $\otimes$ in a conclusion, no sharing of resources is allowed, while the contrary is the case for $\sqcap$. Similarly, there is a difference between $\wp$ and $\oplus$. In the line of thought exposed in Girard (1989), the meanings to be attached to the connectives are the following:

- $a \otimes b$ means that both resources, $a$ and $b$ are given simultaneously;
- $a \sqcap b$ means that one may choose between $a$ and $b$;
- $a \oplus b$ means that one of both resources, $a$ or $b$, is given though we have lack of knowledge concerning the exact one;
- $a \wp b$ expresses a constructive disjunction.

The meaning of $\wp$ becomes clearer when we follow J.Y. Girard in his construction that every atomic formula of his linear logical language has by definition a negation $(-)^\perp$. Running a bit ahead of our story, the meaning of $a \wp b$ will now come down to the situation where “if” not $a$ is given “then” $b$ is given and “if” not $b$ is given “then” $a$ is given. Of course this explanation is linked to the commutative case where a linear logical implication is defined as $a^\perp \wp b := a \rightarrow b$ which by transposition equals $b^\perp \rightarrow a^\perp$. In the same
sense, only in the commutative case where \( a \otimes b \) equals \( b \otimes a \), does it make sense to say that \( a \otimes b \) comes down to \textit{simultaneous} given resources.

We will be more specific on the linear logical implication and analyze the underlying philosophical ideas as presented in Girard (1989) where of course \( \rightarrow \) is defined by means of \( \wp \). What is important is that the linear implication should mimic exactly what happens when a non-iteratable action is being performed, where we conceive of a non-iteratable action to be such that after its performance the initial resources are not available any more as initial resources. The linear implication should as such express the consumption of initial resources and simultaneously the production of final resources. Indeed, as stated in Girard (1995), the linear implication \( \rightarrow \) expresses a form of causality: \( a \rightarrow b \) is to be conceived as “from \( a \) get \( b \)”. More explicitly (Girard 1989):

“\( A \) causal implication cannot be iterated since the conditions are modified after its use; this process of modification of the premises (conditions) is known in physics as \textit{reaction}.” (p.72)

or in Girard (2000):

“C’est donc une vision causale de la déduction logique, qui s’oppose à la pérennité de la vérité traditionnelle en philosophie et en mathématiques. On a ici des vérités fugaces, contingentes, dominées par l’idée de ressource et d’action.” (p.532)

If we understand this correctly, the act of consumption and production is called a non-iteratable action while the process of modification of initial conditions, the deprivation of resources, is called reaction. The idea of relating action and reaction is in a sense metaphorically based on Newton’s action-reaction principle in physics. Girard uses this metaphor also when he explains why every formula has by definition a linear negation which expresses a duality or change of standpoint (Girard 1989):

“action of type \( A \) = reaction of type \( A^\perp \).” (p.77)

or in Girard (2000):

“Concrètement la négation correspond à la dualité “action/réaction” et pas du tout à l’idée de ne pas effectuer une action: typiquement lire/écrire, envoyer/recevoir, sont justicibles de la négation linéaire.” (p.532)
In terms of functional programming or categorical semantics, the negation represents an input-output duality. In game terms, it is an opponent-proponent duality. In Girard (1989) the notion of a reaction of type \( A \perp \), as dual to an action of type \( A \), is quite mysterious and not further elaborated. The only thing Girard mentions is that it should come down to an “inversion of causality, i.e. of the sense of time” (Girard 1989). It is also not clear to us what this duality would mean in a commutative linear logic context when we consider the case of an action of type \( a \multimap b \) (or \( b \multimap a \perp \)) and a reaction of type \( a \otimes b \perp \) (or \( b \perp \otimes a \)) — where \( a \multimap b = a \perp \wp b \) and \((a \perp \wp b)^\perp = a \otimes b \perp \). Thus we tend to agree with Girard (1989) where he says that this discussion involves not standard but non-commutative linear logic. Indeed, against the background of the causation-propagation duality elaborated upon above, if Girard has something similar to causation in mind, we know that what is necessary is an implication and “non-commutative” conjunction which allow us to express causation. As we will show further on, switching from standard to non-commutative linear logic effects the meanings of the linear implication and linear negation.

Leaving this action-reaction debate aside, we can now explain why our given interpretation of non-iterability is quite subtle. First note that in Girard’s standard linear logic, \( \vdash a \multimap a \) is provable from an empty set of premises. As such \( \multimap \) represents in \( a \multimap a \) the identity-action which does not really change resources, but only translates initial ones into final ones. And although we could perform the action twice in the following sense: \( a_1 \multimap a_2 \multimap a_3 \) — for convenience we labeled the resources — this still does not count as an iterable action since \( a_1 \) is to be conceived as an initial resource, different from \( a_2 \), the final resource of the first action. But we can go further in this discussion and follow Girard in stipulating the fact that we may still encounter situations in which the picture of “consuming all initial resources” does not hold. Linear logic henceforth allows also the expression of those actions which deal with stable situations and which are iterable. In the latter case the use of exponentials is necessary where for instance the exponential ! gives !a the meaning that a’s use as a resource is unlimited.

In Girard (1989), Girard discusses the link between states, transitions and the linear implication. In particular, for us it is the following statement which places the linear implication in an interesting context, thinking of course about the above discussed DoQL, (Girard 1989):

“In fact, we would like to represent states by formulas, and tran-
sitions by means of implications of states, in such a way that the
state \( S' \) is accessible from \( S \) exactly when \( S \rightarrow S' \) is provable
from the transitions, taken as axioms.” (p.74)

In Girard (1989) this statement applies to for instance systems such as Petri
nets, Turing machines, chessboard games, etc. Focusing “in this sense” on
physical systems, and using \( \rightarrow \) for transitions of states, it becomes interesting
to investigate how \( \rightarrow \) can be conceived in the context of our logic of actuality
sets. While a formal comparison on the semantical level will be given in the
next paragraph we have to stress here that there is on the methodological level
the following point of difference between DoQL and linear logic: contrary to
DoQL, it should be well understood that linear logic is not a temporal logic,
no preconceptions of time or processes is built into it. More explicitly (Girard
1989):

> “Linear logic is eventually about time, space and communication,
but is not a temporal logic, or a kind of parallel language: such
approaches try to develop preexisting conceptions about time,
processes, etc. In those matters, the general understanding is so
low that one has good chances to produce systems whose aim is
to avoid the study of their objects [...] The main methodologi-
ical commitment is to refuse any *a priori* intuition about these
objects of study, and to assume that (at least part of) the tempo-
ral, the parallel features of computation are already in Gentzen’s
approach, but are simply hidden by taxonomy.” (p.104)

As such DoQL started out with a different methodology. The objects of
study are well-known “scientific objects” such as physical systems and their
properties and the inductions performable on physical systems. In a sense
this information has been encrypted in the formulas we used. All dynamic
propagation- and causation-implications have been labeled by inductions,
and this is different from the linear logical implications which are used to ex-
press any (non-specified) transition. In view of quantum theory it is indeed
the case that one cannot speak about observed quantities without specifica-
tion of the particular measurement one performs, and as such, the cor-
responding induction that encodes von Neumann’s projection postulate, or
in more fashionable terms, state-update. Exactly this could form an argu-
ment against applying the linear logical implications in a context of physical
processes. Thus, we are not tempted to agree with the proposal in Pratt
(1993) of adding linear logical connectives as an extension to quantum logic, but rather focus on the development of a new logical syntax which will however have some definite similarities with linear logic, in particular with its quantale semantical fragment.

In order to get a grasp on the quantale semantics of linear logic we have to say something about its non-commutative variants. In the literature on non-commutative linear logic, two main directions emerge. In a first direction one introduces non-commutativity of the multiplicatives by restricting the exchange-rule to circular permutations while in a second direction one completely drops all structural rules. Concentrating on the first direction, here linear logic with a cyclic exchange rule is called cyclic linear logic and has mainly been developed by D.N. Yetter in Yetter (1990), though we have to note that Girard already makes some remarks on cyclic exchanges in Girard (1989). More explicitly we see that the restriction to cyclic permutations means that we consider the sequents as written on a circle (Girard 1989). This then should come down to the fact that $\mathbf{a}_1 \otimes \ldots \otimes \mathbf{a}_{n-1} \otimes \mathbf{a}_n \vdash \mathbf{a}_n \otimes \mathbf{a}_1 \otimes \ldots \otimes \mathbf{a}_{n-1}$ is provable in cyclic linear logic. Of course the meaning of $\otimes$ with respect to the standard case changes in the sense that it expresses now “and then” (Yetter 1990) or when following Girard (1989) it means that “in the product $b \otimes a$, the second component is done before the first one”. As we will explain in the next paragraph on Girard quantales, it is exactly the difference between $-\otimes a$ and $a \otimes -$ which leads to the introduction of two different implication-connectives in cyclic linear logic: $\rightarrow a$ and $a \rightarrow$.

Given a unital quantale $(Q, \bigvee, \otimes)$, with 1 as the multiplicative neutral element with respect to $\otimes$, it then follows that the endomorphisms $a \otimes -$, $- \otimes a : Q \rightarrow Q$ have right adjoints, $a \rightarrow -$ and $- \rightarrow a$ respectively:

$$a \otimes c \leq b \iff c \leq a \rightarrow b$$
$$c \otimes a \leq b \iff c \leq b \rightarrow a$$

$$a \rightarrow b = \bigvee \{c \in Q : a \otimes c \leq b\} \quad \quad b \rightarrow a = \bigvee \{c \in Q : c \otimes a \leq b\}.$$  

We know that in the standard linear logic as presented by Girard, the following holds $a \rightarrow b = a^\perp \varphi b = (a \otimes b^\perp)^\perp = (b^\perp \otimes a)^\perp = b^\perp \varphi a^\perp = b^\perp \rightarrow a^\perp$. In cyclic linear logic where we have now two implications, things change in the sense that we obtain:

$$(a \otimes b)^\perp = b^\perp \varphi a^\perp \quad \quad (a \varphi b)^\perp = b^\perp \otimes a^\perp$$
\[
\begin{align*}
  a \multimap b &= a^\perp \multimap b \\
  b \multimap a &= b \multimap a^\perp
\end{align*}
\]

To be more explicit, the linear negation can be interpreted in the unital quantale \((Q, \top, \otimes, 1)\) by means of a \textit{cyclic dualizing element}, which can be defined as follows (Rosenthal 1990, Yetter 1990):

- An element \(\perp \in Q\) is dualizing iff
  \(\perp \multimap (a \multimap \perp) = a = (\perp \multimap a) \multimap \perp\)
  for all \(a \in Q\). It is cyclic iff
  \(a \multimap \perp = \perp \multimap a\), for every \(a \in Q\).

Here the operation \(\multimap \multimap \perp\) or equivalently \(\perp \multimap \perp\) is called the linear negation and can be written as \((-)^\perp\). Note that a unital quantale with a cyclic dualizing element \(\perp\) is called a \textit{Girard quantale}, a notion having been introduced in Yetter (1990). These Girard quantales can be equipped with modal operators to interpret the linear logical exponentials and form as such a straightforward semantics for the cyclic as well as standard linear logical syntax. In the latter case \(a \multimap b = b \multimap a\). The disadvantage suffered by cyclic linear logic is that it is still not “non-commutative enough to properly express time’s arrow” (Yetter 1990). Indeed, if \(\multimap\) is to be conceived as a causal implication then it would have been nice to conceive \(\multimap\) as expressing past causality, though this interpretation is too misleading according to (Girard 1989). In a way we agree with him since in cyclic linear logic \(a \multimap b\) and \(a^\perp \multimap b^\perp\) are the same — in the sense that they are both equal to \(a^\perp \multimap b\).

Focusing on the second direction in the non-commutative linear logical literature, we first have to mention the work of J. Lambek. Lambek’s \textit{syntactic calculus} originated in Lambek (1958) and as Girard admits, is the non-commutative ancestor of linear logic. However it has to be mentioned that Lambek’s syntactic calculus, as originally developed against a linguistic background, is essentially multiplicative and intuitionistic. Later on Lambek extended his syntactic calculus with additives and recently renamed his formal calculus \textit{bilinear logic}. In the same direction we can place the work of V.M. Abrusci who developed a non-commutative version of the intuitionistic linear propositional logic in Abrusci (1990) and of the classical linear propositional logic in Abrusci (1991). Specific to Abrusci’s work, however, is the fact that a full removal of the exchange rule requires the introduction of two different negations and two different implications. To explain this in detail we switch to the semantical level of quantales; where we want to note that Abrusci works in Abrusci (1990, 1991) with the more specific structure of phase spaces which as proved in Rosenthal (1990) are examples of quantales. This then leads to the fact that for \(a \in Q\): \(\perp \multimap (a \multimap \perp) \neq (\perp \multimap a) \multimap a\). Here we can
follow Abrusci and define $\bot \circ (a \circ \bot) = (\bot a)^{\bot}$ and $(\bot \circ a) \circ \bot = (\bot a)^{\bot}$, where on the syntactical level $a^{\bot}$ is called the linear postnegation, $\bot a$ the linear retronegation, $a \rightarrow b$ the linear postimplication and $b \circ a$ the linear retroimplication. Further on the syntactical level we obviously have $a^{\bot} \varphi b = a \rightarrow b$ while $b \varphi a = b \circ a$. Let us finish of this paragraph with a note on the fact that there is of course much more to say about (non)-commutative linear logic, indeed contemporary research is in full development and heads in the direction of combining cyclic linear logic with commutative linear logic, we however limit ourselves for the time being to the overview given.

Given the above expositions, it follows that $(DI(\mathcal{L}), \bigvee_{DI(\mathcal{L})}, \otimes_{e})$, the quantale fragment emerging for propagation for a specific induction $e$, provides an example of the quantale obtained for commutative linear logic. Note that the difference of course lies in the fact that in commutative linear logic no retro-implication different from $\circ$ is present, not even as an additional structure. In this respect, to obtain a retro-implication in linear logic it is necessary to move to a non-commutative linear logic providing a single quantale in which to interpret $\rightarrow$ and $\circ$, in sharp contrast to our constructions allowing the interpretation of $\varphi$ and $\circ$ for each specific $e$. Finally, let us note that while the implication $\rightarrow$, when focusing on it as an implication expressing simultaneous consumption of initial resources and production of final resources, is much stronger than $\circ$, it can nevertheless be reconstructed within the framework of DoQL, for which we refer to Smets (2001).

5. CONCLUSION

It seems to us that an actual attitude towards the logic of dynamics ought to be pluralistic, as it follows from our two main paradigmatic examples, dynamic operational quantum logic and Girard’s linear logic.

The mentioned attempts that aim to integrate the logic of dynamics as it emerges from for example physical and proof theoretic considerations fail either on formal grounds or due to conceptual inconsistency. We indeed provided a non-classical physical counterexample to van Benthem’s general dynamic logic, and argued that even for classical systems, from a physical perspective adjoints rather than relational inverses should be the formal bases for the logic of dynamics.

We end by mentioning two promising recent alternative approaches in relating quantum features and linear logic, in Blute et al (2001) in terms of
polycategories and deduction systems and in Abramsky and Coecke (2002) in terms of geometry of interaction in categorical format, that is, in terms of traced monoidal categories. It is however too soon to obtain conclusions from these lines of thought.

6. NOTES

1. From now on OQL will only refer to Geneva School operational quantum logic.

2. It is exactly the particular operational foundation of ontological concepts that has caused a lot of confusion with respect to this approach, including some attacks on it due to misunderstandings stemming from identification of “what is”, “what is observed”, “what will be observed”, “what would be observed”, “what could be observed”, etc. We don’t refer to these papers but cite one that refutes them in a more than convincing way, namely Foulis and Randall (1984). We recall here that it were D.J. Foulis and C.H. Randall who developed an empirical counterpart to C. Piron’s ontological approach (Foulis and Randall 1972, Foulis et al 1983, Randall and Foulis 1983). We also quote the review of R. Piziak in Mathematical Reviews of one of these attacks of Piron’s approach exposing their somewhat doubtful aims (MR86i:81012):

“... in fact, they confused the very essence of Piron’s system of questions and propositions, the sharp distinction between properties of a physical system and operationally testable propositions about the system. [...] In their reply to Foulis and Randall, HT ignore the list of mathematical errors, confusions and blunders in their papers except for one (a minor one at that). HT simply reissue their challenge and dismiss the work of Foulis and Randall as well as Piron as being “useless from the physicist’s point of view”. However, when one finds an argument in HTM (the main theorem of their 1981 paper) to the effect that the failure to prove the negation of a theorem constitutes a proof of the theorem, one might form a different opinion as to whose work is ‘useless’. It is right and proper for any scientific work to be scrutinized and criticized according to its merits. Indeed, this is a main impetus to progress. But if mathematics is to be used as a tool of criticism, let it be used properly.”

See also Smets (2001) for an overview of most of the criticism and its refutation on sOQL. We refer to Coecke (nd) for a formulation of sOQL where a conceptually somewhat less rigid, but more general perspective is proposed, avoiding the notion of test or definite experimental project in the definition of a physical
property as an ontological quality of a system. One of the motivations for this reformulation is exactly the confusion that the current formulation seems to cause — although there is definitely nothing wrong with it as the truly careful reader knows, on the contrary in fact.

3. We are not implying that our scientific theories are to be based on obtained measurement-results. The oft-drawn conclusion from this stating that “ontological existence is independent from any measurement or observation” also holds in our view. It is however specific for our position, which may not be share by every scientific realist, that our knowledge of what exists is linked to what we could measure, stated counterfactually. As such we adopt an “endo perspective”, measurements are not a priori part of our universe of discourse but incorporated in a conditional way, it is in this sense that e.g. two not simultaneously observable properties which ontologically exist, can both unproblematically be incorporated in our description. We refer to Coecke and Smets (2001) for more details on the “endo versus exo perspective”. A somewhat related view we want to draw to your attention is put forward in Ghins (2000), where he argues that from the affirmation of some “existence”, under a criterion of existence based on the conditions of “presence” and “invariance”, certain counterfactuals should reasonably also be affirmed.

4. We are well aware of the fact that several correspondence theories of truth have been put forward and have also been criticised. Adhering to some form of scientific realism does not necessarily imply that one accepts a correspondence theory, even more it has been suggested that the debate on the notion of truth can be cut loose from the debate on realism versus anti-realism — see for instance Horwich (1997), Tarski (1944). Still, against the background of sOQL, we are sympathetic towards a contemporary account of a Tarskian-style semantic correspondence theory — see for example Niiniluoto (1999, §3.4).

5. In the line of C. Piron (1981) we note that in general, the actual performance of a definite experimental project can at most serve to prove the falsity of a physicists’ assumptions, it cannot hand out a prove for them to be true.

6. Note here the similarity with the generation of a quantale structure within the context of process semantics for computational systems sensu Abramsky and Vickers (1993) and Resende (2000).

7. Different approaches however do exist for considering potentially destructive measurements, see for example Faure et al (1995), Amira et al (1998), Coecke and Stubbe (1999), Coecke et al (2001) and in particular Sourbron (2000).

8. From this point on we will identify those inductions which have the same action on properties, i.e., we abstract from the physical procedure to its transitional effect.

9. A pair of maps $f^* : L \to M$ and $f_* : M \to L$ between posets $L$ and $M$ are
Galois adjoint, denoted by \( f^* \vdash f_* \), if and only if \( f^*(a) \leq b \iff a \leq f_*(b) \) if and only if \( \forall a \in M : f^*(f_*(a)) \leq a \) and \( \forall a \in L : a \leq f_*(f^*(a)) \). One could somewhat abusively say that Galois adjointness generalizes the notion of inverse maps to non-isomorphic objects: in the case that \( f^* \) and \( f_* \) are inverse, and thus \( L \) and \( M \) isomorphic, the above inequalities saturate in equalities. Whenever \( f^* \vdash f_* \), \( f^* \) preserves all existing joins and \( f_* \) all existing meets. This means that for a Galois adjoint pair between complete lattices, one of these maps preserves all meets and the other preserves all joins. Conversely, for \( L \) and \( M \) complete lattices, any meet preserving map \( f_* : M \to L \) has a unique join preserving left Galois adjoint \( f^* : a \mapsto \bigwedge \{ b \in M | a \leq f_*(b) \} \) and any join preserving map \( f^* : L \to M \) a unique meet preserving right adjoint \( f_* : b \mapsto \bigvee \{ a \in L | f^*(a) \leq b \} \).

10. This dual representation is included in the duality between the categories \textbf{Inf} of complete lattices and meet-preserving maps and \textbf{Sup} of complete lattices and join preserving maps, since this duality is exactly established in terms of Galois adjunction at the morphism level — see Coecke et al (2001) for details.

11. See also Amira et al (1998) and Coecke and Stubbe (1999) for a similar development in terms of a so-called operational resolution on the state set.

12. Let us briefly describe the more rigid argumentation. Consider the following definitions for \( A \subseteq \mathcal{L} \): i. \( \bigvee A \) is called disjunctive iff \( \bigvee A \) actual \( \iff \exists a \in A : a \) actual); ii. Superposition states for \( \bigvee A \) are states for which \( \bigvee A \) is actual while no \( a \in A \) is actual; iii. Superposition properties for \( \bigvee A \) are properties \( c < \bigvee A \) whose actuality doesn’t imply that at least one \( a \in A \) is actual. We then have that \( \bigvee A \) disjunctive \( \iff \bigvee A \) distributive) provided that existence of superposition states implies existence of superposition properties (Coecke 2002). Moreover, any complete lattice \( L \) has the complete Heyting algebra \( DI(L) \) of distributive ideals as its distributive hull (Bruns and Lakser 1970), providing it with a universal property. The inclusion preserves all meets and existing distributive joins. Thus, \( DI(\mathcal{L}) \) encodes all possible disjunctions of properties, and moreover, it turns out that all \( DI(\mathcal{L})\)-meets are conjunctive and all \( DI(\mathcal{L})\)-joins are disjunctive — note that this is definitely not the case in the powerset \( P(\mathcal{L}) \) of a property lattice, nor in the order ideals \( I(\mathcal{L}) \) ordered by inclusion. It follows from this that the object equivalence between: i. complete lattices, and, ii. complete Heyting algebras equipped with a distributive closure (i.e., it preserves distributive sets), encodes an intuitionistic representation for operational quantum logic — see Coecke (2002) for details.

13. The following map \( \hat{e}^* : P(\Sigma)_1 \to P(\Sigma)_2 \) provides a counterexample: Let \( \Sigma_1 = \Sigma_2 := \{ p, q, r, s \} \) with as closed subsets \( \emptyset, \{ p \}, \{ q \}, \{ r \}, \{ s \}, \{ q, r, s \}, \Sigma \subset P(\Sigma)_1 = P(\Sigma)_2 \), and set \( \hat{e}^*(\{ p \}) \mapsto \{ p, q \}, \hat{e}^*(\{ q \}) \mapsto \{ q \}, \hat{e}^*(\{ r \}) \mapsto \{ r \}, \hat{e}^*(\{ s \}) \mapsto \{ s \} \); one verifies that although \( \bigvee \{ q, r \} = \bigvee \{ r, s \} \) we have

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\[ \bigvee f_{R^{-1}}(\{q, r\}) = \bigvee \{p, q, r\} \neq \bigvee \{r, s\} = \bigvee f_{R^{-1}}(\{r, s\}) \] since \( \bigvee \{p, q, r\} \) yields the top element of the property lattice and \( \bigvee \{r, s\} \) doesn’t.

14. Note here that in accordance to the literature on linear logic, contra section 2 and section 3 of this paper, we do use the terms conjunction and disjunction beyond their strict intuitionistic significance. This however should not cause any confusion.

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