Critical Behaviour of Random Bond Potts Models: A Transfer Matrix Study

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We study the two-dimensional Potts model on the square lattice in the presence of quenched random bond impurities. For \( q > 4 \) the first-order transitions of the pure model are softened due to the impurities, and we determine the resulting universality classes by combining transfer matrix data with conformal invariance. The magnetic exponent \( \beta/\nu \) varies continuously with \( q \), assuming non-Ising values for \( q > 4 \), whereas the correlation length exponent \( \nu \) is numerically consistent with unity. We present evidence for the correctness of a formerly proposed phase diagram, unifying pure, percolative and non-trivial random behaviour.

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I. INTRODUCTION

The effect of quenched bond randomness on a classical statistical mechanics system whose pure version undergoes a second-order phase transition is well understood. Namely, the so-called Harris criterion states that if the critical exponent \( \alpha^{\text{pure}} \) governing the divergence of the specific heat at the transition point of the pure system is negative, weak bond randomness is irrelevant in the renormalisation group (RG) sense and the pure fixed point (FP) is stable \([1]\). On the other hand, if \( \alpha^{\text{pure}} > 0 \) the randomness is relevant and causes a cross-over to critical behaviour governed by a new random FP nearby, at least if the cross-over exponent \( \alpha^{\text{pure}} \) is small.

This should be contrasted with the more dramatic effects of randomness in the field conjugate to the local magnetisation. Such randomness can eliminate low-dimensional phase transitions altogether, and at least it always changes the values of the critical exponents \([3]\). For this reason most early research was concentrated on field randomness. In this context a particularly popular model is the random field Ising model (RFIM) for which a classical argument due to Imry and Ma \([4]\) from a simple comparison of the field fluctuations with the stabilising effect caused by the formation of a domain wall concluded, that the lower critical dimension of the RFIM is \( d_l=2 \).

The issue of quenched bond randomness imposed on a system that undergoes a thermal first-order phase transition is less studied. An adaptation of the Imry-Ma argument can be established by noting that the bond randomness couples to the local energy density, which differs for the two phases that co-exist at the critical point of the pure system, in exactly the same way that the random field couples to the local magnetisation in the RFIM. Consequently the existence of a non-vanishing latent heat for \( d < 2 \) can be ruled out. Early work by Imry and Wortis \([5]\) furnished a heuristic argument, reminiscent of that of the Harris criterion, that the bond randomness indeed softens any such phase transition in \( d = 2 \) to a continuous one. A subsequent phenomenological RG argument by Hui and Berker \([6]\) confirmed that the lower critical dimension for random bond tricriticality and end-point criticality is \( d_t=2 \). As the dimensionality increases, tricritical points and critical end points emerge from \( T=0 \). Finally, a mathematically rigorous theorem by Aizenman and Wehr \([7]\) stated that quite generally for \( d \leq 2 \) an arbitrarily weak amount of quenched bond randomness leads to the elimination of any discontinuity in the density of the variable conjugate to the fluctuating parameter.

The question then emerges whether this softening of the phase transition can be verified for specific models and, if so, what are the universality classes of these novel second-order phase transitions. An investigation along these lines has recently been initiated by one of us \([8]\), by considering a system of \( N \) two-dimensional Ising models coupled by their energy operators which, according to mean-field theory (MFT), is supposed to display a second-order phase transition. For \( N > 2 \), however, the RG flow of the model exhibits a runaway behaviour, which is characteristic of a fluctuation driven first-order transition \([8]\). In this sense the transition is only weakly first-order and hence amenable to perturbative calculations. On adding weak bond randomness it was found that the RG trajectories curl back towards the pure decoupled Ising FP, and consequently Ising exponents are expected, up to possible logarithmic corrections. This study was extended by Pujol \([9]\) to the case of \( N \) coupled \( q \)-state random bond Potts models for \( 2 \leq q \leq 4 \), but here the universality class of the impurity softened transition was found to depend on the coupling between the models.

A more interesting model for studying the effect of quenched bond impurities on a first-order transition is the \( q \)-state random bond Potts model (RBPM). For \( q > 4 \)
the phase transition of the pure system is first order with a latent heat that is an increasing function of $q$. In fact, since the transition is first-order already in MFT, on the RG level it is controlled by a zero-temperature discontinuity FP with the eigenvalue of the relevant scaling operator being $y = \frac{d}{2}$. Quenched randomness coupling to the local energy density thus has the eigenvalue $d - 2(d - y) = d$ and is strongly relevant, whence an RG treatment appears to be problematic.

The work undertaken until now has therefore mainly been numerical. Extensive Monte Carlo (MC) simulations have been carried out for $q = 8$ by Chen, Ferrenberg and Landau [11] confirming the transition softening scenario outlined above, and finding critical exponents numerically consistent with those of the pure Ising model. Similar conclusions were reached by Domany and Wiseman [2] for $q = 4$ and also for the Ashkin-Teller model. It thus appears that in a variety of situations the universality class of the bond disordered models is that of the Ising model, irrespective of the symmetry underlying the original model.

To explain these findings Kardar et al. [13] have proposed an interface model for the RBPM which, after several approximations, is amenable to an RG treatment that is exact on the hierarchical lattice. In the pure model the interface exhibits a branching structure with fractal dimension at criticality, but when randomness is present the critical interface is asymptotically linear. Assuming that the vanishing of the interfacial free energy is governed by a zero-temperature FP, the Widom exponent $\mu$ turns out to be independent of $q$ for all sufficiently large $q$, taking the Ising value $\mu = 1$.

This is in contrast to the perturbative expansion in powers of $(q - 2)$ investigated by Ludwig and Cardy [14], Ludwig [15] and Dotsenko et al. [17]. Using the RG approach for the perturbation series around the conformal field theories representing the pure models, these authors find the critical behaviour of the RBPM to be controlled by a new random FP which merges with the pure FP as $q \to 2$. Critical exponents are found to depend continuously on $q$, at least for $(q - 2)$ small, and in the case of the magnetic exponent $\chi_H$ a calculation to three loop order yields a prediction which is supposed to be very precise even up to $q = 3$. Unfortunately, extending these results beyond $q = 4$ is impossible, even in principle, since this is the limiting case in the range of minimal conformal theories around which the perturbative calculations take place. Another interesting implication of this line of research is that the local operators exhibit multiscaling [16], meaning that correlation functions of different moments of such operators decay with powers that are, in general, independent.

It has been suggested by Kardar et al. [13] and one of us [1] that these contrasting theories describe very different FPs. Indeed, it can be argued that the interface model pertains to the case of strong non-self-dual randomness, whilst the $(q - 2)$-expansion is relevant for weak self-dual randomness. Also, even though it may turn out that the critical exponents do not depend on $q$, the central charge $c$ evidently must, since even when the critical behaviour is controlled by a decoupled Ising FP there is generally not just one Ising model but several.

To resolve this controversy we have undertaken an extensive study of the $d = 2$ RBPM where finite-size data obtained from transfer matrix (TM) calculations were combined with the powerful techniques of conformal invariance. We have extended the random cluster model TMs of Blöte and Nightingale [18] to the case of bond randomness whilst taking into account that in the impure case such TMs do not commute and hence must be discussed in terms of their Lyapunov (rather than the eigenvalue) spectra. Because of the lack of self-averaging the relation between the Lyapunov spectra and the critical exponents is inferred indirectly through a cumulant expansion, which has the advantage of illustrating the multiscaling properties of the correlation functions [16] explicitly. The number of Potts states $q$ enters our TMs only as a continuous parameter, both facilitating the comparison with analytical results within the $(q - 2)$ expansion and making the interesting regime $q > 4$ readily accessible.

Although the cumulant expansion yields very appealing results in the case of the magnetic exponent it works poorly for the thermal one. For reasons yet not fully understood such results, when taken at face value, seem to hint at a conformal field theory violating the bound $\nu \geq 2/d$. On the other hand, using phenomenological RG techniques we find results consistent both with the bound and with the $(q - 2)$-expansion explicitly.

Some of our results have been reported in a Letter [23], where we also described a mapping between the interface models of the RBPM for large $q$ and the RFIM. This mapping, which is asymptotically exact in the limit $q \to \infty$, allowed us to establish a schematic phase diagram for the RBPM unifying pure, non-trivial random, and percolative behaviour. In the present paper the evidence for this phase diagram will be collected and discussed.

The outline of this paper is as follows. In Sect. II we define the model and discuss the principles of extracting physical information from the Lyapunov spectrum of the TMs. The proposed phase diagram is reviewed along with the translation of the renormalisation group equations from the RFIM to the problem at hand. Then, in Sect. III, the TM formalism of Ref. [18] is generalised to the random case. This relies on the mapping of the RBPM to the random cluster model and on two complementary representations of the connectivity of a row of spin in the latter model. By decomposing the TM into sparse single-bond TMs we arrive at a highly efficient algorithm, the implementation of which is considered in detail. The magnetic properties can be accessed
by adding a ghost site, but we also describe an alternative route in which the two-point correlator is related to a disorder operator under duality. The corresponding implementation of the TMs has a seam spanning the length of the cylinder. In the percolation limit the TMs take on a particularly simple form, allowing us to obtain very accurate results.

Sect. V is dedicated to the presentation of our numerical results. From the scaling of the free energy we find evidence that the first-order phase transition is indeed softened due to the randomness. The effective central charge is determined both at the random FP and in the percolation limit. In the latter case we obtain excellent agreement with our analytical result. It is shown how a cumulant expansion leads to very accurate values of $x_H$. These depend continuously on $q$, and are in perfect agreement with the $(q-2)$-expansion at $q=3$. For larger $q$ the values stay far away from the Ising value, in sharp contrast to the results of Ref. [11]. For $q > 8$ the expansion begins to break down, and we give an argument why this must be so in terms of a model of coupled replicas. The problems encountered when trying to extract $x_T$ in a similar fashion then lead us to discuss the method of extracting physical observables from the Lyapunov spectrum in more physical terms. We then consider the constraints put on the multiscaling exponents by a conformal sum rule. A reliable determination of $x_T$ is furnished by a variant of the phenomenological RG scheme, in which the shape of the self-dual surface is explicitly taken into account. The criticism of Ref. [19] recently raised by Pázmándi et al. [23] is shown not to apply to the RBPM. We conclude the section with a discussion of the higher Lyapunov spectrum and its possible relation to the (presently unknown) conformal field theory underlying the model.

Finally, Sect. V contains a discussion of our findings. We seek to explain the discrepancy with Ref. [11], and we discuss other types of randomness relevant to the question whether a first-order phase transition is softened due to impurities. A list of unsettled questions relevant for future research is also given.

II. THE MODEL AND ITS PHASE DIAGRAM

A. The random bond Potts model

The $q$-state Potts model [25] is defined by the reduced Hamiltonian

$$\mathcal{H} = -\sum_{\langle ij \rangle} K_{ij} \delta_{\sigma_i, \sigma_j}, \quad (1)$$

where the spins, defined on the vertices of the square lattice, can take the values $\sigma_i = 1, 2, \ldots, q$, and the summation is over all nearest neighbour bonds in the lattice. We shall specialise to the ferromagnetic case, where the reduced couplings $K_{ij} \geq 0$ measure the strength of the aligning tendency of nearest neighbour spins.

Although the free energy of the pure model ($K_{ij} \equiv K$) is not known in closed form for general $q$, a wide range of exact results is nevertheless available [29]. In particular it is well-known that the model exhibits a second-order phase transition for $q \leq 4$ and a first-order one for $q > 4$ [10].

However, in this paper we are mainly concerned with the random bond Potts model (RBPM) for which much less is known. Here the couplings $K_{ij}$ are quenched random variables, typically drawn from the symmetric binary distribution

$$P(K) = \frac{1}{2} \delta(K - K_1) + \delta(K - K_2), \quad (2)$$

where the ratio between strong and weak bonds $R = K_2/K_1$ measures the strength of the randomness. For the special choice

$$(e^{K_1} - 1)(e^{K_2} - 1) = q \quad (3)$$

the model is on average self-dual, as discussed in more detail in Sect. III below. Assuming that the phase transition is unique the model is therefore at its critical point [23].

Other self-dual distributions of the random bonds than that of Eq. (2) have also been investigated in order to check our results. In particular, we have found the trinary distribution introduced in Sect. V of useful, since it gives us a clearer idea about the length scale associated with the random impurities.

B. Lyapunov spectrum of the transfer matrix

The construction of the transfer matrices (TMs) for the RBPM is described in detail in Sect. III. It is well-known that in the pure case ($R = 1$) the operator content of the conformal field theory (CFT) underlying the model is related to the eigenvalue spectrum $\{\lambda_i(L)\}$, $i = 0, 1, 2, \ldots$, of the TM for a strip of width $L$ through

$$f_i(L) - f_0(L) = \frac{2\pi x_i}{L^2} + \cdots, \quad (4)$$

where $f_i(L) = -\frac{1}{T} \ln \lambda_i(L)$ are the generalised free energies per site (in units of $k_B T$) and $x_i$ the scaling dimensions of the corresponding operators. Similarly the central charge $c$, measuring the number of bosonic degrees of freedom of the CFT, is related to the finite-size corrections to the customary free energy through

$$f_0(L) = f_0(\infty) - \frac{\pi c}{6L^2} + \cdots. \quad (5)$$

In the random case the TMs are no longer constant but depend on the particular realisation of the random
bonds within each row of strip. Accordingly the concept of eigenvalues generalises to that of Lyapunov exponents. Starting with some suitable initial vector of unit norm \( |v_0 \rangle \), the leading Lyapunov exponent can be found by the Furstenberg method \([31]\).

\[
\Lambda_0(L) = \lim_{m \to \infty} \frac{1}{m} \ln \left\| \left( \prod_{j=1}^{m} T_j \right) |v_0 \rangle \right\|, \tag{6}
\]

where \( T_j \) is the TM acting between rows \( j-1 \) and \( j \). The average free energy per site is given as before by \( f_0(L) = -\frac{1}{L} \Lambda_0(L) \). Higher exponents are found by iterating a set of \( n \) vectors \( \{|v_k \rangle \}_{k=0}^{n-1} \), where a given \( |v_k \rangle \) is orthogonalised to the set \( \{|v_l \rangle \}_{l=1}^{k} \) after each multiplication by \( T_j \) \([31]\). Surprisingly, this method works even for a non-hermitian TM, and it is numerically shown to be independent of the choice of the initial vectors.

When some symmetry (e.g., spin reversal or duality) is manifest in \( T_j \) the orthogonalisation can be circumvented by iterating vectors which belong to definite irreducible components of that symmetry, but the \( S_q \) permutation symmetry inherent in the Potts model has been lost through the mapping to the random cluster model which forms the backbone of our TMs; see Sect. 2.3.

As to the extraction of physical information from the spectra, Eq. (6) is supposed to retain its validity provided that \( c \) is replaced by the effective central charge \( c' \), that in the standard replica formalism is the derivative of \( c(n) \) with respect to the number of replicas at \( n = 0 \) \([4]\). The question to which extent Eq. (6) also remains valid is by no means trivial and we shall dedicate a fair part of the subsequent discussion to it.

C. Phase diagram

In the limit \( q \to \infty \) the behaviour of the pure model is readily understood \([23]\). At the self-dual point the partition function is dominated by two contributions, namely those corresponding to the \( q \) completely ordered states and the completely disordered state respectively. All other configurations are down by powers of \( 1/\sqrt{q} \) and have recently been enumerated to 10th order in this small parameter \([22]\). The dominating states have identical free energy but different internal energy densities of \( K \) and 0 for the ordered and the disordered phase respectively, so the transition is, as expected, first order.

The bond randomness is then included through the parametrisation \( e^{K_{ij}} - 1 = q^{2^{w_{ij}}} \), where \( w_{ij} = \pm w \) and \( w > 0 \) measures the strength of the randomness. It can now be shown \([23]\) that as \( q \to \infty \) the model for an interface between these two phases of the RBPM is exactly the same as that of an interface between the spin-up and spin-down phases of the RFIM.

\[
\mathcal{H}_{\text{RFIM}} = -J \sum_{\langle ij \rangle} s_i s_j - \sum_i h_i^{\text{RF}} s_i - h \sum_i s_i \tag{7}
\]

with \( h_i^{\text{RF}} = \pm h^{\text{RF}} \), provided that one translates quantities between the two models using the “dictionary”

\[
\begin{align*}
& h^{\text{RF}} \leftrightarrow \frac{1}{2} w \ln q, \\
& J \leftrightarrow \frac{1}{8} \ln q, \\
& h \leftrightarrow \frac{1}{4} t \ln q.
\end{align*}
\]

Here \( t = (T - T_c)/T_c \) is the reduced temperature.

The infinitesimal RG equations can now be inferred from the similar results for the RFIM \([33]\). Near \( d = 2 \) they read

\[
\begin{align*}
dw/dl &= -(d/2 - 1)w + Aw^3 + \cdots, \tag{9} \\
d(ln q)^{-1}/dl &= -(\ln q)^{-1}((d - 1) - Aw^2 + \cdots), \tag{10} \\
dt/dl &= t(1 + Aw^2 + \cdots), \tag{11}
\end{align*}
\]

where \( A > 0 \) is a non-universal constant. The RG flows for \( d > 2 \) and the proposed phase diagram are shown in Fig. 1 (see Ref. \([23]\) for details). The shaded region of non-vanishing latent heat is bounded by the line \( Rq_2 \) of tricritical points. This line, controlled by the fixed point \( R \) at infinite \( q \), merges with the abscissa as \( d \to 2 \). At \( q = \infty \) the interfacial mapping is exact so that the flows along \( R P_1 \) must extend all the way to \( w = \infty \), and since \( w^{-1} \) is known to be a relevant scaling variable at the percolation limit \([34]\), Ref. \([23]\) concluded that \( Rq_2 \) must be separated from the percolative behaviour along \( P_1 P_2 \) by another line of stable FPs emerging from \( P_1 \). It was then conjectured that this connects on to the line of random FPs found in the \((q-q_1)\)-expansion \([4]\). In this paper we shall present the evidence for this conjecture for \( d = 2 \), when \( q_1 = 2 \) and \( q_2 = 4 \).

III. THE TRANSFER MATRICES

In spite of the large amount of high-precision results obtained by combining transfer matrix (TM) techniques with finite-size scaling for almost any conceivable type of pure statistical mechanics system (see, e.g., Ref. \([35]\) for a review) the use of TMs in the study of disordered systems seems to have attracted rather little interest as compared with the complementary approach of Monte Carlo simulations.

A straightforward way of setting up the TMs for the \( q \)-state Potts model is to use the traditional spin basis where the state of a row of \( L \) spins is labelled by the \( q^L \) basis states \( \{\sigma_1, \sigma_2, \ldots, \sigma_L\} \), \( \sigma_i = 1, \ldots, q \). Whilst this approach is highly efficient for \( q = 2, 3 \) it has two major shortcomings in the general case. First, the dimension
of the matrices becomes forbiddingly for large $q$, in particular making unaccessible the regime of $q > 4$ which is our main concern. Second, the restriction to integer values of $q$ is unnecessary and in fact makes it difficult to compare numerical results with analytical calculations in the $(q-2)$-expansion.

Both these shortcomings can be remedied by writing the TMs in the connectivity basis introduced by Blöte and Nightingale. In this representation the dimension of the TMs is independent of $q$ which enters only as a continuous parameter. In fact, the number of basis states is asymptotically $\sim 4^L$ (or $\sim 5^L$ upon imposition of a magnetic field) with a rather small coefficient of proportionality, in practice making this basis the preferred choice for all but the Ising model ($q = 2$).

We have generalised these TMs to include quenched bond randomness, and also devised an alternative method of accessing the magnetic properties through the introduction of a seam along the strip. Furthermore, in the percolation limit the TMs are found to simplify in a manner that makes calculations for rather large $L$ feasible. For convenience these results are presented along with a review of the relevant parts of Ref. [8] thus making our description of the Potts model TMs self-contained.

A. Mapping to the random cluster model

Introducing an imaginary ‘ghost site’ with fixed spin $\sigma_0 = 1$ the partition function for the Potts model can be written as

$$Z = \sum_\sigma \left( \prod_{(ij)} \exp(K_{ij}\delta_{\sigma_i\sigma_j}) \right) \left( \prod_{(0)} \exp(H_i\delta_{\sigma_i\sigma_0}) \right),$$

(12)

where $\prod_{(ij)}$ is the usual product over pairs of nearest neighbour sites and each site $i$ has been connected to the ghost site 0 with a similar notation. The reduced magnetic field $H_i$, here taken to be site dependent, now enters at the same footing as the reduced exchange couplings $K_{ij}$. It should be pointed out, however, that a random coupling to the ghost site is not a true random field, since the latter would try to force different sites into different Potts states and not just into the particular state of the ghost site with a site-dependent probability. To avoid any confusion we shall therefore specialise to the case of a homogeneous field $H_0 \equiv H$.

The site variables can now be traded for bond variables through the mapping to the random cluster model introduced by Kasteleyn and Fortuin. In terms of the variables $u_{ij} = e^{K_{ij}} - 1$ and $v = e^H - 1$ we arrive at

$$Z = q^N \sum_{G \subseteq L} \sum_{G_0 \subset L_0} \left( \prod_{(ij) \in G} \frac{u_{ij}}{q} \right) \left( \prod_{(0) \in G_0} \frac{v}{q} \right) q^{l(G \cup G_0)},$$

(13)

where $\mathcal{L}$ denotes the set of all nearest neighbour bonds, $\mathcal{L}_0$ the bonds from each of the $N$ Potts spin to the ghost site, and $l(G \cup G_0)$ is the number of independent loops on the combined graph $G \cup G_0$.

The usual construction of the transfer matrix $\mathcal{T}$ for a strip of width $L$ seems to be obstructed by the non-local factor $l(G \cup G_0)$, but this can be taken into account by choosing a basis containing information about which sites of a given row are interconnected through the part of the lattice below that row (including connections via the ghost site). This leads us to the concept of connectivity states, which we consider next.

B. The connectivity states

In order to determine the number of loop closures induced by appending a new row of $L$ sites along with the corresponding $L$ connections to the ghost site to the top of $G \cup G_0$, we need information about how the sites in the top row of $G \cup G_0$ were previously interconnected. This information is comprised in the connectivity state $(i_1 i_2 \ldots i_L)$, where $i_t = 0$ if site $t$ is connected to 0 within the combined graph $G \cup G_0$ and, otherwise, $i_t = i_s$ is a (non-unique) positive integer if and only if sites $r$ and $s$ are connected within $G$.

Whilst this ‘index representation’ is useful for determining whether a newly appended bond does or does not close a loop, and thus will allow us to explicitly construct the single-bond TMs in the next subsection, a one-to-one mapping to the set of consecutive integers $\{1, 2, \ldots\}$ is clearly needed to define a ‘number representation’ which will enable us to label the entries of the TM and thus to perform actual computations. These representations and the mapping were supplied by Ref. [8] as were the determination of the number of connectivity states ($d_L$ with and $c_L$ without a magnetic field). We shall review the necessary details and also give details on the construction of the inverse of the mapping just mentioned.

Consider first the case of $H = 0$ where all ghost bonds carry zero weight ($v = 0$). The connectivity states then have all $i_t > 0$ and can be recursively ordered by noting that the index representation is well-nested, i.e., for $r < s < t < u$

$$i_t = i_r \land i_s = i_u \implies i_s = i_t.$$  

(14)

It follows that if we define the cut function $\rho(i_1 i_2 \ldots i_L)$ to be the smallest $t > 1$ such that $i_1 = i_t$, if such a $t$ exists, and $L + 1$ otherwise, the left $(i_2 i_3 \ldots i_{p-1})$ and right $(i_p i_{p+1} \ldots i_L)$ parts of the index representation are both well-nested. A complete ordering of the well-nested
sequences is now induced by applying the cut function first to the whole sequence, then recursively to its right and finally to its left part.

More precisely, the mapping from the index to the number representation is effected by

$$
\sigma(i_1 i_2 \ldots i_L) = \begin{cases} 1 & \text{if } L \leq 1 \\ c_{L,k-1} + [\sigma(i_k \ldots i_L) - 1]c_{k-2} + \sigma(i_2 \ldots k-1) & \text{otherwise}, \end{cases}
$$

(15)

where $k = \rho(i_1 \ldots i_L)$ and $c_{n,l} = \sum_{i=2}^l c_{i-2}c_{n-i+1}$ with

$$
c_n \equiv c_{n,n+1} = \frac{(2n)!}{n!(n+1)!}
$$

(16)

giving the number of well-nested $n$-point connectivities. Explicit values are shown in Table II.

To consider the general case of $v \neq 0$ we remark that the subsequence of non-zero indices is still well-nested. A complete ordering of an index representation $(i_1 i_2 \ldots i_L)$ with precisely $s$ zero indices is then induced by first ordering according to the value of $s$, then lexicographically ordering the zeros, and finally using the ordering of the well-nested subsequence $(i_{p_1} i_{p_2} \ldots i_{p_L})$ given by Eq. (13). The lexicographic ordering is carried out by

$$
\psi(i_1 i_2 \ldots i_L) = \begin{cases} 1 & \text{if } L = 1 \text{ or } s = L \\ \psi(i_2 i_3 \ldots i_L) & \text{if } i_1 \neq 0 \\ (L-s)^{-1} + \psi(i_2 i_3 \ldots i_L) & \text{if } i_1 = 0, \end{cases}
$$

(17)

and the mapping to the number representation is finally

$$
\tau(i_1 i_2 \ldots i_L) = d_{L,s-1} + [\psi(i_2 i_3 \ldots i_L) - 1]c_{L-s} + \sigma(i_{p_1} i_{p_2} \ldots i_{p_{L-s}}),
$$

(18)

where $d_{n,l} = \sum_{i=0}^l \binom{n}{i} c_{n-i}$ with

$$
d_n \equiv d_{n,n} = \sum_{i=0}^n \binom{n}{i} c_{n-i}
$$

(19)

giving the number of general $n$-point connectivities. Again, explicit values are presented in Table II.

To construct the inverse mapping, i.e., the one taking us from the number to the index representation, we solve $\tau = \tau(i_1 i_2 \ldots i_L)$ for the indices $(i_1 i_2 \ldots i_L)$ by performing the following steps. First, the number of zero indices is found as $s = \max\{s|d_{L,s-1} < \tau\}$. Second, perform a slightly modified integer division by writing $\tau - d_{L,s-1} = Qc_{L-s} + R$, where the remainder $R$ is restricted to take its values in the interval $[1,c_{L-s}]$. From Eq. (13) we infer that $\psi = Q + 1$ and $\sigma = R$. Third, the position of the first (leftmost) zero index is given by $i_L-k = 0$, where $k = \max\{|k|\} < \psi$. This procedure of finding the zero indices is then iterated with $\psi \to \psi(1) \equiv \psi - (k)$ until $\psi(s') = 1$, and the remaining $s-s'$ zeros are filled in from the right: $i_{s'+1} = \cdots = i_{s-1} = i_s = 0$.

It remains to deduce the subsequence of non-zero indices by inverting $\sigma = \sigma(i_{p_1} i_{p_2} \ldots i_{p_s})$ with $L = l - s$. After initialising $i_{p_1} = p_1$ we proceed by recursion as follows. First, choose $k = \min\{k|c_{L-k} + c_{k-2}i_{k-1} \geq \sigma\}$. If $k \leq l$ we have then found a connection: $i_{p_1} = i_{p_k}$. This procedure of finding the connections is now iterated on the left $(i_{p_2}, \ldots, i_{p_{k-1}})$ and the right $(i_{p_{k+1}}, \ldots, i_{p_L})$ parts of the remaining sequence. If $k \geq 2$ the assignment $i_{p_2} = p_2$ is performed. By (modified) integer division we then write $\sigma - c_{L-k} = Qc_{k-2} + R$ with $R \in [1,c_{k-1}]$, and pass over the left part of the sequence with $\sigma \to \sigma(1) = R$ and $l \to l(1) = k - 2$, and the right part with $\sigma \to \sigma(1) = Q + 1$ and $l \to l(1) = l - k + 1$. The recursion stops when for any sequence $l(m) \leq 2$. If then $l(m) = 2$ and the sequence is $(i_{p_a}, i_{p_{a+1}})$ we perform the assignment $i_{p_{a+1}} = i_{p_a}$ if $\sigma(m) = 1$ and $i_{p_{a+1}} = p_{a+1}$ if $\sigma(m) = 2$.

Any way of constructing the index representation $(i_1 i_2 \ldots i_L)$ will of course reflect the above-mentioned arbitrariness as to the actual values of the non-zero indices, but the particular procedure just outlined is easily seen to ensure that all indices are $\leq L$. This invariant is useful since then any given site $t$ can be disconnected from the rest by assigning $i_t = L + 1$.

C. The single-bond transfer matrices

The amount of computer time necessary for building up a long strip by repeated application of the transfer matrix $T$ can be enormously reduced by decomposing the latter as a product of sparse matrices, each corresponding to the addition of a single bond to $L$.

Specifically we write $T = T^0 T^h T^v$, where $T^v = T_{L-1}^v \cdots T_2^v T_1^v$ is connecting each of the $L$ spin sites in the uppermost row of the strip to a new spin site situated vertically above it, and $T^h = T_{L}^h \cdots T_{L-2}^h T_{L-1}^h$ is finishing the new row of $L$ by appending horizontal bonds between each of the nearest-neighbour dangling ends created by $T^v$. The matrix $T_{L}^h$ imposes periodic boundary conditions by interconnecting the newly added spins at sites $L$ and 1. Finally $T^0 = T_{L}^0 \cdots T_{2}^0 T_{1}^0$ furnishes the bonds of $L_0$ from each of the new spin sites to the ghost site. Each of these single-bond TMs is implicitly understood to depend on the particular realisation of the bond and, in the case of $T_1^0$, the field randomness pertaining to the bond in question.

Upon addition of one single bond the summation over graphs in Eq. (13) is augmented by a sum over the two possible states of this new degree of freedom, viz. the bond added to $L$ ($L_0$) can be either present or absent in $G (G_0)$. Correspondingly each column of the TM has at most two distinct non-zero entries.

Consider first adding a vertical bond by action of $T^v_i$, $l \in \{1, \ldots, L\}$. If the bond is ‘present’ any given con-
nectivity state \((i_1i_2\ldots i_L)\) of the \(L\) uppermost spin sites will be left unchanged. In case of an ‘absent’ bond site \(l\) will be disconnected, and the number representation of the new connectivity state can be found by assigning \(i_l = L + 1\) and using Eq. (18). Interpreting the factor of \(q^N\) in Eq. (13) as an extra factor of \(q\) going with each vertical bond we see that the non-zero entries in \(T^T\) corresponding to a column with a given connectivity number are a diagonal contribution of \(u_{ij}\) and a possibly off-diagonal contribution of \(q\). In particular the vertical bonds do not induce any loop closures.

Similarly the TM of a horizontal bond \(T^h_{i,l+1}\) has a diagonal entry of 1 for each column, corresponding to the bond being absent. The other non-zero entry corresponds to a present bond, and its value depends on whether a loop is being closed or not. Given the connectivity state \((i_1i_2\ldots i_{l+1}\ldots i_L)\) of some column in the TM this is determined by comparing \(i_l\) and \(i_{l+1}\): if they are equal we get an additional diagonal contribution of \(u_{i,l+1}\) corresponding to a loop closure, whereas if they are different there is an off-diagonal entry with value \(u_{i,l+1}/q\). In the latter case the connectivity number is found by assigning the value \(\min\{i_l, i_{l+1}\}\) to all indices that were formerly equal to either \(i_l\) or \(i_{l+1}\) and applying Eq. (18). (The reason why we copy the \(\min\) index is to ensure the proper handling of spins connected to the ghost site.)

Finally, the TM of a ghost bond \(T^g_l\) has the same form as in the case of a horizontal bond if we make the substitutions \(u_{i_l,l+1} \rightarrow v_l\) and \(i_{l+1} \rightarrow 0\).

**D. Magnetic properties**

It is well known, at least in the case of a pure system, that physically interesting quantities like the central charge \(c\) as well as the thermal \((x_T)\) and the magnetic \((x_H)\) scaling dimensions can be extracted from the transfer matrix spectrum. Consider for the moment the case of vanishing magnetic field, \(H = 0\). Since connections to the ghost site are then generated with zero weight \((v = 0)\) such connections can only be present in any row if they were already there in the preceding row. In particular, noting that in the numbering of connectivities induced by Eq. (13) the non-ghost connectivities precede the others, we see that the TM assumes the following block form

\[
T = \begin{bmatrix} T^{11} & T^{12} \\ 0 & T^{22} \end{bmatrix}.
\]

where superscript 2 (1) refers to the (non-)ghost connectivities.

The largest and the next-largest eigenvalues of \(T\) turn out to be the largest eigenvalue of block \(T^{11}\) and \(T^{22}\) respectively, and from the corresponding (reduced) free energies per site \(f_0^i(L) = -\frac{1}{L} \lambda_0^i (i = 1, 2)\) for a strip of width \(L\) the magnetic scaling dimension can be found from the CFT formula

\[
f_0^{22}(L) - f_0^{11}(L) = \frac{2\pi x_H}{L^2} + \cdots.
\]

Physically this relation to \(x_H\) can be understood by noting that by acting repeatedly with \(T^{22}\) on some initial (row) state \(|\psi_0\rangle \neq 0\) one measures the decay of clusters extending back to row 0. This must have the same spatial dependence as the spin-spin correlation function and hence be related to \(x_H\). Analogously \(T^{11}\) measures the decay of two-point correlations between pairs of spins being interconnected within the random cluster model. This is nothing but the energy-energy correlation in the strip geometry, and accordingly we expect that

\[
f_0^{11}(L) - f_0^{11}(L) = \frac{2\pi x_T}{L^2} + \cdots.
\]

We have checked the results for \(x_H\) by constructing a realisation of the TM in the presence of a seam spanning the length of the cylinder. Our algorithm also merits attention on its own right since it improves the asymptotic number of basis states necessary for finding \(f_0^{22}(L)\) from \(d_L - c_L \sim 5^L\) (the dimension of \(T^{22}\)) to \(Lc_L \sim L^2\). In practice, however, with the strip widths \(L\) accessible using present-day computers the two algorithms perform more or less equally fast (see Table 1 for a comparison).

The well-known duality relation for the Ising model partition function without a magnetic field is easily extended to the case of the Potts model on a cylinder. For \(v = 0\) the partition function of the random cluster model, Eq. (13), can be rewritten as

\[
Z = \sum_{G \subseteq \mathcal{L}} \prod_{\langle ij \rangle \in G} u_{ij}^{C(G)} q^{C(G)},
\]

where \(C(G)\) is the number of independent clusters on \(G\). We first stipulate the duality between two very special graphs. Namely, the full graph \(G = \mathcal{L}\) with partition function \(Z_{\text{full}} \{ \{ u_{ij} \} \} = q \prod_{\langle ij \rangle \in \mathcal{L}} u_{ij}\) is taken to be dual to the empty graph \(G^* = 0\) with \(Z_{\text{empty}} \{ \{ u_{ij}^* \} \} = q^{N^*}\), where the number of dual sites \(N^*\) is fixed by the Euler relation.

Establishing the duality then amounts to ascertaining that all other graphs have the same weight relative to this reference state as is the case in the dual model. In the terminology introduced above, duality states that a graph configuration \(G\) on the original lattice \(\mathcal{L}\) is dual to a configuration \(G^*\) on the dual lattice \(\mathcal{L}^*\) in which every bond of strength \(u_{ij}\) being ‘present’ in \(G\) corresponds to the dual bond of strength \(u_{ij}^*\) being ‘absent’ from \(G^*\).

In particular, removing one bond from the full graph (relative weight: \(1/u_{ij}\)) must correspond to adding the corresponding dual bond to the empty dual graph (relative weight: \(u_{ij}^*/q\)), meaning that the bond strengths and their duals must obey the relation...
\[ u_{ij} u_{ij}^* = q. \] (24)

When removing further bonds from \( G \) it may happen that a new cluster is separated from the rest of the graph, yielding an additional factor of \( q \). But such a cluster formation corresponds precisely to a loop closure on the dual lattice, also giving an extra factor of \( q \)!

Since all graph configurations \( G \) can be constructed by successive removals of bonds from the full reference state we have thus proven the fundamental duality relation

\[ Z\{\{u_{ij}\}\} = q C Z^\ast\{\{u_{ij}\}\}, \] (25)

where \( C = q^{-N^\ast} \prod_{(ij)} u_{ij} \) is some constant.

A similar duality relation can be established for the spin-spin correlation function. As usual we define the local order parameter as \( s \)

\[ M_a(r) = \left( \delta_{(r),a} - \frac{1}{q} \right), \quad a = 1, \ldots, q. \] (26)

In the high temperature phase all components of the order parameter vanish, whilst in the ordered (low temperature) phase the \( Z_q \) symmetry is spontaneously broken and one of the components, say \( a = 1 \), has a positive expectation value. A simple calculation now shows that the correlation function \( G_{aa}(r_1, r_2) = \langle M_a(r_1) M_a(r_2) \rangle \) is proportional to the probability that the points \( r_1 \) and \( r_2 \) belong to the same cluster.

In a cylindrical geometry the graphs with \( r_1 \) and \( r_2 \), taken to be at opposite ends of the cylinder, connected correspond to dual graphs where clusters are forbidden to wrap around the cylinder. This is equivalent to computing the dual partition function with twisted boundary conditions \( \sigma \rightarrow (\sigma + 1) \mod q \) across a seam running from \( r_1 \) to \( r_2 \). By permuting the Potts spin states the shape of this seam can be deformed at will as long as it connects \( r_1 \) and \( r_2 \). Duality thus maps the correlation function onto a disorder operator

\[ \langle M_a(r_1) M_a(r_2) \rangle = \left( \prod_{\text{seam}} \exp(-K^\ast \delta_{(r_1, r_2)}) \right) Z^\ast, \] (27)

where \( Z^\ast = Z^\ast\{\{K^\ast\}\} \) is the dual partition function with periodic boundary conditions.

The construction of the TM in the presence of a seam is facilitated by the following observation: If no cluster is allowed to wrap the cylinder, each graph contributing to the partition function can be associated with a function \( s(j) \) of the row number \( j \), such that \( s(j) = k \in \{1, \ldots, L\} \) means that in row \( j \) no horizontal bond connecting sites \( k \) and \( k+1 \mod L \) is present. For obvious reasons we shall refer to \( s \) as the virtual seam. We can then write the TM in a basis which is the direct product of the \( L \) possible values of the virtual seam and the costumary \( c_L \) non-ghost connectivities. The virtual seam is initialised by assigning to it a definite value in row 0, viz. \( s(0) = L \) for all graph configurations of that row.

The single-bond TM of a vertical bond is diagonal in \( s \), but a present horizontal bond not inducing a loop closure may alter the value of the virtual seam. Let us recall from Sect. [III C] that to find the connectivity state \( (i_1 \ldots i_{l+1} \ldots i_L) \) giving the row label of \( T_r^L \) corresponds to the off-diagonal entry with value \( u_{l+1} / q \) we would join the two distinct clusters formerly labelled by either \( i_1 \) or \( i_{l+1} \). But such a merger would ruin the invariant stated above, unless we move the virtual seam at the same time. On the other hand, if \( i_1 = i_{l+1} \) and \( s(j) = l \) we must explicitly prevent a cluster from wrapping the cylinder by leaving out that extra diagonal contribution which would otherwise be implied by the condition \( i_l = i_{l+1} \). In this case the virtual seam is not moved.

To conclude this section we remark that in the case of a planar geometry any \( n \)-point Potts correlation function can be mapped to a generalised surface tension by duality [III E].

### E. The percolation limit

In the random bond Potts model the couplings \( u_{ij} \geq 0 \) are quenched random variables, and the critical point can be accessed by drawing them from the symmetric binary distribution \( P(u) = \frac{1}{2} [\delta(u - u_1) + \delta(u - u_2)] \), where \( u_1 u_2 = q \). For details, see Sect. [IV]. Bond percolation can be studied in the limit \( u_1 \rightarrow 0, u_2 \rightarrow \infty \) of infinitely weak and strong bonds respectively. In this limit considerable simplifications occur in the TM, rendering computations with rather large strip widths feasible.

In the percolation limit all single-bond TM have only one non-zero entry per column. Recall from Sect. [III C] that in the general case there are two such entries of which one is diagonal and the other is ‘non-trivial’. In the case of the strong vertical bonds and the weak horizontal bonds only the diagonal entries survive, so that the matrices \( T^\ast_{\text{strong}} = u_2 \mathbf{1} \) and \( T^\ast_{\text{weak}} = \mathbf{1} \) both become trivial. On the other hand, a weak vertical bond corresponds to a TM having one non-trivial entry of \( q \) per column, whilst a strong horizontal bond is represented by a TM that is \( u_2 \) times a non-trivial matrix with entries of 1’s and \( 1/q \)’s.

The factors of \( u_2 \) multiplying both \( T^\ast_{\text{strong}} \) and \( T^h_{\text{strong}} \) are innocuous albeit infinite, since of the \( 2L \) single-bond matrices constituting the entire \( T \) there will on average be \( L \) strong ones, hence \( L \) factors of \( u_2 \). On the level of the specific free energy this amounts to an infinite additive constant

\[ f^{11}_0(L) = - \ln u_2 + f^{11}_0(L) \] (28)

independent of the strip width \( L \). In particular, the central charge \( c \) can be extracted from the finite quantity
\[ f_0^{11}(L). \]

As we shall see in Sect. [IV] this quantity can be found by measuring the asymptotic growth of the norm of
\[ \left( \prod_{j=1}^{n} T_j \right) |v_0\rangle, \]
where \( |v_0\rangle \) is some largely arbitrary initial vector. In the percolation limit the TM turn out to be so sparse that after a very few iterations the resulting vector has only one non-zero component. Computationally this means that it is sufficient to store the row index of that non-zero component as well as its norm. Both time and memory requirements are thus enormously reduced, allowing us to access larger system sizes.

The disadvantage of this projective quality of the percolation point TM is that neither the thermal nor the magnetic scaling dimensions can be found from the Lyapunov spectrum. In the case of \( x_H \) an initial vector in the \( T^{22} \) sector will rapidly decay to zero, thus invalidating the procedure for finding \( f_0^{22}(L) \), and the alternative of using a seam is obstructed by the fact that disallowing the entry in the horizontal bond TM that corresponds to a cluster wrapping the cylinder is incompatible with the argument of pulling out an overall factor of \( u_2 \) from the TM.

**IV. RESULTS**

**A. Softening of the transition**

Before attempting to determine the universality classes of the RBPM it is essential to make sure that quenched bond randomness indeed renders the phase transitions second order. For \( q > 4 \) the pure system has a first-order transition for which the free energy per site is expected to scale like [13]
\[ f_0(L) = f_0(\infty) + aL^{-d}\exp(-L/\xi), \]
where \( \xi \) is the bulk correlation length and \( a \) is an amplitude depending on \( q \). In Fig. [3] we show plots of the function
\[ \lambda(L) \equiv \ln[f_0(L) - f_0(\infty)] + d\ln L \]
\[ \sim \text{const} - L/\xi \]
for various values of \( q \) and the randomness strength \( R \). These plots are rather sensitive to the value of \( f_0(\infty) \), but although this is only known exactly for the pure model [11] it can nevertheless be determined with sufficient accuracy from the parabolic fits described in Sect. [IV.B] below.

For \( q = 8 \) the finite correlation length of the pure system (\( \xi \sim 70 \)) is seen to be rendered effectively infinite (\( \xi \sim 10^3 \)) upon imposition of the randomness, whilst the transition of the Ising model (\( q = 2 \)) simply stays second order. Despite the simplicity of these plots we also find a fair agreement with the recently found analytical values of \( \xi \) for the pure systems [11]; near \( q = 4 \) these assume the simple form
\[ \xi \approx \sqrt{2\over 8} \exp\left(\frac{\pi^2}{\sqrt{q-4}}\right). \]

Another criterion for distinguishing between first and second-order phase transitions is the values of the (effective) exponents \( x_H \) and \( x_T \) as found from Eq. (21) and (23) respectively. Generally speaking, for pure systems with \( q > 4 \) these equations give rise to rather poor fits which however have extrapolated values of the effective exponents that are in the vicinity of, and slightly below, zero, whereas when randomness is added the fits are much better and yield exponents in the interval \([0, 2]\]. In view of the problems justifying such fits in the random case (see below) this evidence for a softening of the transition is however not to be taken too seriously.

**B. Central charge at the random FP**

The free energy per site \( f_0^{11}(L) \) for the RBPM on long strips of width \( L \) is readily found from Eq. (4) applied to the \( T^{11} \) sector of the TM. We have performed extensive simulations for various values of \( q \) and the randomness strength \( R \), though in most cases \( R = 2 \) was found to describe the random FP adequately.

Representative samples of our data are shown in Table [4]. For each run a normalised initial vector \( |v_0\rangle \) was prepared by choosing its components randomly, and after discarding the results of the first 2,000 multiplications by \( T_j^{11} \) in order to eliminate transients, data collection was made for each 200 iterations until a strip of a total length of \( m = 10^3 \) had been built up. For \( q > 2 \) a total of 100 independent runs were made for \( 1 \leq L \leq 8 \), and 3 runs for \( 9 \leq L \leq 12 \), whilst for the Ising model (\( q = 2 \)) we were able to make 100 runs for \( 1 \leq L \leq 13 \) by using the conventional spin basis. Final results and error bars were extracted by computing the mean and the standard deviation for the totality of patches of length 200.

It is not a priori obvious that the Lyapunov exponents found from Eq. (4) are independent of the norm used. The standard norm in both the spin basis and the connectivity basis is given by the square root of the sum of the squared components, and these two are of course not identical. To impose the spin basis norm on the connectivity basis each term in the sum must be weighted by a factor \( q^C \), where \( C \) is the number of clusters in the relevant connectivity state. We have checked the consistency of our results by comparing the first few Lyapunov exponents obtained from imposing the two different norms on the connectivity basis, and we find that not only are the results identical but there is even a complete agreement of the first three significant digits of the error bars. For \( q = 2 \) we found that the results using the spin basis and
the connectivity basis were consistent, but that the error bars obtained using the spin basis were slightly smaller.

Our results for the free energies of the random-bond Ising model agree with, and are more precise than, those of de Queiroz\textsuperscript{\ref{12}}.

Values of the effective central charge $c'$ can be extracted from Eq. \textsuperscript{[8]} by employing various fitting procedures. In spite of the relatively slow convergence of both two-point fits ($L, L + 1$) and straight-line least-squares fits against $1/L^2$ \textsuperscript{[12]}, iterating such fits yields quite good results in the pure model. When randomness is added this is no longer so, since rather substantial error bars on the first estimates prevent us from efficiently iterating the fits.

A better scheme is to include the leading correction to the scaling of Eq. \textsuperscript{[8]}, which in the pure case has been shown numerically to take the form \textsuperscript{[13]}

$$f_0^{11}(L) = f_0(\infty) - \frac{\pi c'}{6L^2} + \frac{A}{L^4} + \cdots \quad (32)$$

One then performs either three-point fits ($L, L + 1, L + 2$) or parabolic least-squares fits against $1/L^2$, and because of the much faster convergence no iteration is needed \textsuperscript{[42]}. Although a correction proportional to $1/L^4$, due to the operator $T \mathcal{T}$, must necessarily be present in every system that is conformally invariant \textsuperscript{[13]} it can of course not be guaranteed to be the dominant one in general.

In Table \textsuperscript{[1]} the results of parabolic fits including the data points for $L_0 \leq L \leq L_{\text{max}}$ have been shown as a function of $L_0$. It is seen that $L_0$ must be chosen large enough to justify the omission of higher terms in the series \textsuperscript{[12]}, and small enough to minimise error bars. From the special cases of the Ising model and of the percolation point (see Sect. \textsuperscript{IV C} below) we concluded that the choice $L_0 = 3$ is optimal.

Apart from the results shown in Table \textsuperscript{[1]} we have also performed some runs for $q = 1.5$, finding, as expected from the Harris criterion \textsuperscript{[1]}, no difference between the results for the pure and the random model.

In the intermediate regime $2 \leq q \leq 4$ our results compare favourably to those of the $(q-2)$-expansion, at least up to $q = 3$. On the other hand, it is evident from Fig. \textsuperscript{[3]} that the difference between $c$ for the pure model and $c'$ for the random one is of the same order of magnitude as our error bars, and only near $q = 4$, where the expansion is expected to break down anyway, are our results able to distinguish between the two different behaviours. Exactly at $q = 2$ the randomness is marginal and logarithmic corrections to the finite-size scaling forms, Eqs. \textsuperscript{[8]} and \textsuperscript{[1]}, are expected. Whilst this issue has recently attracted considerable interest in the case of the critical exponents \textsuperscript{[1]}, the corrections to the central charge are much weaker \textsuperscript{[17]} and accordingly our result is consistent with that of the pure Ising model.

In Fig. \textsuperscript{[10]} we have displayed our results for $c'$ as a function of $\log_{10} q$ for selected values of $q \in [1.5, 64]$. We have juxtaposed the results for two strengths of the randomness, namely weak randomness ($R = 2$, closed circles on Fig. \textsuperscript{[10]} and strong randomness ($R = 10$, open circles). For small values of $q$ both randomness strengths give rise to the same $c'$, as witnessed by the overlap of the $q = 4$ data points. However, for larger $q$ the $R = 2$ curve flattens out and grows slower than logarithmically. Sample runs show that the same is true for larger values of $R$, the difference being that the range of $q$-values for which the growth is logarithmic is extended as $R$ is increased. This is illustrated by the $R = 10$ curve’s staying above, but very close to, the percolative result $\sim \log q$ (see Sect. \textsuperscript{IV C} below) for the whole range of $q$-values shown on the plot.

Another way to state this is that for fixed $q$ and varying $R$, the quantity $c'$ is an increasing function of $R$ that eventually reaches a plateau as $R$ becomes large enough. It then appears from Fig. \textsuperscript{[10]} that for $q \leq 64$ the randomness strength $R = 10$ is sufficient to reach this plateau.

These findings are interpreted as follows. According to the $(q-2)$-expansion the randomness strength $R^*$ corresponding to the random FP is an increasing function of $q$. Assuming this FP to persist as we enter the regime $q > 4$ (see Fig. \textsuperscript{[10]}) we now claim that the monotonicity of $R^*(q)$ also holds true when the $(q-2)$-expansion breaks down. From the RG flows given in Fig. \textsuperscript{[3]} we see that any initial value of $R \in [1, \infty]$ will eventually flow to the random FP as the system is viewed on larger and larger length scales. However, if we start out very far from $R^*$ the onset of the asymptotic scaling given by Eq. \textsuperscript{[8]} may be deferred to much larger length scales than the strip widths $L$ numerically accessible for our TMs. We therefore expect poor scaling for strip widths $L \leq L_{\text{max}}$. Conversely, if we choose the strength of the randomness as $R \sim R^*$ the resulting value of $c'$ is expected to be more or less independent of the precise choice of $R$ and equal to the true value of the central charge. But in our simulations we find that this is precisely accomplished by choosing $R$ as an increasing function of $q$. Further justification for this interpretation is found from the phenomenological RG treatment in Sect. \textsuperscript{IV F} below.

A heuristic argument explaining that the “effective” $c'(R, q)$ obtained for small values of $R$ is less than the

\textsuperscript{[1]} Actually they differ by a constant, since Ref. \textsuperscript{[12]} defines the Hamiltonian as $-\sum_{(ij)} K_{ij} \sigma_i \sigma_j$ as opposed to our Eq. \textsuperscript{[8]}. Since $s_i s_j = 2 \delta_{s_i \sigma_j} - 1$ there is a free energy difference of $2 K_{ij}$, which for $R = 2$ equals 0.91407.
“correct” value \( c'(R^*, q) \) associated with the random FP is readily furnished, at least for large \( q \). Namely, from Zamolodchikov’s \( e \)-theorem we know that there exists a function \( c(\{K\}) \) of the couplings that decreases along the RG flow and equals the central charge at the fixed points. As a corollary the curves of constant \( c \) are orthogonal to the RG flow. In particular, for large \( q \) where the RG flow is known from the mapping to the RFIM (see Eqs. (10) and (11)), it is evident from Fig. 1 that \( c'(R, q) \) is equal to \( c'(R^*, q^*) \) for some \( q^* < q \). Since our numerical results indicate that \( c'(R^*, q) \) is an increasing function of \( q \) the proposition follows.

To check our results for \( c' \) we have also made 100 independent runs for each of the strip widths \( 1 \leq L \leq 8 \) where the random bonds were drawn from the trinary distribution

\[
P(K) = p[\delta(K - K_1) + \delta(K - K_2)] + (1 - 2p)\delta(K - K^*),
\]

where \( K_1 \) and \( K_2 = 1000K_1 \) satisfy the criterion (8) and \( (\exp K^* - 1)^2 = q \). Here \( p \ll 1 \) is the strength of the randomness. Of course this realisation of the randomness also preserves self-duality, and hence the model is again at its critical point (2).

Numerical results for \( c' \) using trinary randomness are shown in Table V, and they are consistent with the binary results given above, again provided that \( p \) is increased as we go larger and larger \( q \). In particular it is reassuring to verify that we seem to probe the true random behaviour when \( 2/p \) (the length scale associated with this randomness) is comparable to the correlation length of the pure system (31).

An interesting question is whether the asymptotic value of \( c' \) is approached from above or below when the system is viewed on larger and larger length scales. For models exhibiting reflection positivity Zamolodchikov’s \( e \)-theorem ensures that the convergence is from above. In particular the condition of positivity holds true for unitary models, whereas for a random model it may well fail to be fulfilled. Indeed, in the case of the RBPM a perturbative calculation suggests that the convergence may be from below in some cases.

In order to discuss this point the parabolic fits versus \( 1/L^2 \) employed above are no longer appropriate. Apart from speeding up the rate of convergence to a point where information about its direction becomes obliterated due to error bars the inclusion of higher-order corrections to the finite-size scaling form (3) may have the effect of reversing this direction. E.g., in the case of the pure Ising model it is found that the estimators obtained from parabolic fits converge from below, whereas the corresponding linear fits (i.e., without the \( 1/L^4 \) correction) yield estimators that converge from above in accordance with the theoretical prediction.

In Table V we show the results of such linear least-squares fits for several values of \( q \). The randomness strength \( R \) was chosen in accordance with the considerations given above. It appears that in all cases the finite-size estimators converge towards the asymptotic values of Table \( \mathrm{II} \) from above.

We remark that values of \( c' \) similar to ours have recently been reported by Picco (17). For \( q = 8 \) this author found \( c' = 1.45 \pm 0.06 \) which agrees with our result of respectively \( c' = 1.52 \pm 0.02 \) for binary randomness of strength \( R = 10 \), and \( c' = 1.51 \pm 0.04 \) for trinary randomness of strength \( p = 0.10 \). Our observation that \( c' \) appears to be an increasing function of \( R \), eventually reaching a plateau as \( R \) becomes large enough, was confirmed by Ref. (17) that used binary randomness of strength \( R = 10 \) throughout. Strong evidence was also given that \( c'(q) \) grows roughly logarithmically with \( q \) in the regime \( q \in [5, 256] \), but a further discussion of what this implies will be deferred to Sect. \( \mathrm{V} \) below.

It is worthwhile to compare the TM algorithm used in Ref. (17) to ours. It was found that the number of distinct entries in the pure model TM in the spin basis is

\[
b_L = \sum_{i_2+1} \sum_{i_3=1} \sum_{i_4=1} \cdots \sum_{i_L=1} 1,
\]

where \( m_i = \max(i_2, i_3, \ldots, i_{L-1}) \), and \( L \) designates the strip width as usual. Further taking into account the \( 2^L \) different realisations of the binary randomness in each strip, recursion relations between the different elements of the TM were found by computing a total of \( (b_L)^{2^L} \) polynomials. Since this number of polynomials increases rapidly with \( L \) high-precision computations could only be performed up to \( L_{\text{max}} = 6 \). The number of iterations used to determine \( f_0(L = 6) \) was similar to ours, whereas more iterations were used for the smaller strip widths.

Evidently this algorithm also has the advantage that \( q \) enters only as a parameter, thus making accessible any value of \( q \) for the simulations. However, for large \( L \) it performs inefficiently, as we will now show. The numbers \( b_L \) of Eq. (15) are by no means unfamiliar. Indeed, they are nothing but the total number of \( L \)-point connectivities, including the non-well-nested ones (39). Alternatively they can be viewed as the number of ways that \( L \) objects can be partitioned into indistinguishable parts (40). With \( m_\nu \), parts of \( \nu \) objects each \( (\nu = 1, 2, \ldots) \) this can be rewritten as

\[
b_L = \sum_{m_\nu=0}^{\infty} \prod_{\nu=1}^{L} \frac{L!}{(\nu!)^{m_\nu} m_\nu!},
\]

where the primed summation is constrained by the condition \( \sum_{\nu=1}^{L} \nu m_\nu = L \). From this representation the generating function can be immediately inferred

\[
\exp(e^t - 1) = \sum_{n=0}^{\infty} \frac{b_n t^n}{n!}.
\]
Explicit values, found by Taylor expansion of the left-hand side, are shown in Table 3. Asymptotically the $b_L$ are seen to grow faster than $L^2$, whereas the well-nested connectivities only grow as $\sim 4^L$.

C. The percolation limit

In the case of the binary randomness (2) the percolation limit is reached by letting $(e^{K_1} - 1) \to 0$ and $(e^{K_2} - 1) \to \infty$ whilst maintaining the self-duality criterion (3). The partition function of the random cluster model is then dominated by one graph only, viz. the one that covers all of the strong bonds and none of the weak ones. (Note in particular that the limits $R \to \infty$ and $q \to \infty$ do not commute.) Expressed in terms of the free energy per site this reads

$$f_0^{\text{perc}} = -\frac{B}{N} \ln(e^{K_2} - 1) - \frac{C}{N} \ln q,$$

where $B$ is the number of strong bonds and $C$ is the number of clusters in the dominant graph.

The quenched average over the randomness must be taken on the level of the free energy. Evidently, with the chosen distribution of the randomness, $B = N$ whence the first term is simply a trivial, albeit infinite, constant. (Incidentally this is the same constant that was pulled out in Eq. (23).) On the other hand, the average number of percolation clusters is related to a derivative in the pure $Q$-state Potts model

$$C = \left. \frac{\partial}{\partial Q} \ln Z(Q) \right|_{Q=1},$$

thus determining the effective central charge $c'(q)$ at percolation as

$$c'(q) = \ln q \left. \frac{\partial c(Q)}{\partial Q} \right|_{Q=1}.\quad (39)$$

An alternative argument for this relation is furnished by the observation that the replicated model is simply the Potts model with $q^n$ states; differentiating $c(q^n)$ with respect to the number of replicas $n$ and taking the limit $n \to 0$ one recovers the result (39). The central charge of the pure model is given by an expression due to Kadanoff

$$c = \frac{(2 - 3y)(1 + y)}{(2 - y)},\quad (40)$$

where $\sqrt{q} = 2 \cos(\pi y/2)$ and $0 \leq y \leq 1$, and taking the appropriate derivative of this we finally arrive at

$$c'(q) = \frac{5\sqrt{3}}{4\pi} \ln q.\quad (41)$$

As described in Sect. (11), the single-bond TMs in the percolation limit have only one non-zero entry per column, equal to either $q$, 1 or $1/q$. Taken together with their projective quality and Eq. (11) for the largest Lyapunov exponent it is clear that the free energy, and hence the central charge, must be explicitly proportional to $\ln q$. So it suffices to do the numerics for one value of $q \neq 1$. Because of the simple form of these TMs we were able to average $f_0^{11}(L)$ of Eq. (28) over 100 strips of length $m = 10^5$ for the range $1 \leq L \leq 19$. Consequently the factor of proportionality could be determined quite accurately as $0.688 \pm 0.003$, in excellent agreement with $\frac{5\sqrt{3}}{4\pi} \simeq 0.689$.

It is evident from the mapping between bond percolation and the pure $Q = 1$ Potts model that the critical exponents of the two models are identical: $x_T = \frac{2}{3}$ and $x_F = \frac{\pi}{2\sqrt{3}}$ [13]. Since all correlation functions at percolation can only take the values 0 and 1, it is also clear that different moments of a given correlation function all have the same scaling dimension. Thus, in the notation of Ludwig [13], $x_n = x_1$ for all $n > 1$. The pure model represents the other trivial extreme case of multiscaling behaviour: $x_n = n x_1$.

D. The cumulant expansion

The concept of multiscaling is best understood in terms of a simple example [17]. The random-bond Ising chain. From the reduced Hamiltonian $H = -\sum_{i=1}^N K_i s_i s_{i+1}$ the partition function is easily found as $Z = \prod_{i=1}^N c_i$, where $c_i = 2 \cosh K_i$. In particular the quenched average $\bar{Z} = \exp[N \log c_1]$ does not coincide with the most probable value $Z_{\text{m.p.}} = \exp[N \log c_1]$. On the other hand, the reduced free energy is $F = -\sum_{i=1}^N \ln c_i$ so that $\overline{F} = F_{\text{m.p.}} = -N \log c_1$. The free energy is thus self-averaging, i.e., it takes on its sample averaged value with probability unity in the thermodynamic limit. Indeed, by the central limit theorem, $F$ is normally distributed, it being a sum of random numbers, whereas $Z$ is a product of random numbers and therefore log-normally distributed. Similarly the correlation function $\langle s_1 s_R \rangle = \prod_{i=1}^{R-1} \tanh K_i$ is non-self-averaging. In particular $\langle s_1 s_R \rangle^2 \neq \langle s_1 s_R \rangle$. In Sect. (11) we related the spin-spin correlation function $G(m)$ on a strip of the RBPM to the free energy in the presence of a seam of frustrated bonds (or with a ghost site). Taking the logarithm of Eq. (27) and exploiting the self-duality of the lattice we have

$$\Delta f(L) \equiv f_0^{22}(L) - f_0^{11}(L) = \frac{1}{mL} \ln G(m),$$

and in the pure system, according to conformal symmetry [28], this decays along the strip as $2\pi x_H/L^2$, cf. Eq. (22).
When randomness is present $\Delta f(L)$ is a fluctuating quantity, and since free energies are supposed to be normally distributed these fluctuations are $O(1/\sqrt{m})$. Consequently $\ln G$ is a self-averaging quantity and $G$ is not \(\text{E} \), exactly as in the simple example given above.

In the multiscaling scenario of Ludwig [10] different moments $G(m)^n$ scale with dimensions $x_n$ which, as opposed to what is the case in the pure model, are not necessarily linear in $n$. (In this notation $x_H \equiv x_1$.) For $n_1 > n_2$ we have $x_{n_1} \geq x_{n_2}$ and $x_{n_1/n_1} \leq x_{n_2/n_2}$ (convexity); pure and percolative behaviour are thus realisations of the two possible extremes of multiscaling.

Since translational invariance is one of the basic assumptions of conformal symmetry \([43]\), the latter only refers to the averaged quantities $G(m)^n$ and not to the $G(m)^n$ themselves. These averages cannot be computed directly in a numerical experiment because of the lack of self-averaging; this can however be circumvented by performing a cumulant expansion

$$\ln G^n = n \ln G + \frac{1}{2} n^2 (\ln G - \ln G)^2 + \cdots , \tag{43}$$

where each term on the right-hand side is self-averaging and can be directly extracted from the statistical fluctuations in $\Delta f(L)$ between the patches of length 200 into which we have divided our strip.

Quite generally for a stochastic variable $X$ we have

$$\langle \exp X \rangle = \exp \sum_{j=1}^{\infty} \frac{1}{j!} k_j, \tag{44}$$

where explicit expressions for the six first cumulants $k_i$ in terms of the moments $m_i$ of $X$ are given by \([40]\)

\begin{align*}
k_1 &= m_1 \\
k_2 &= m_2 - m_1^2 \\
k_3 &= m_3 - 3m_2 m_1 + 2m_1^3 \\
k_4 &= m_4 - 4m_3 m_1 - 3m_2^2 + 12m_2 m_1^2 - 6m_1^4 \\
k_5 &= m_5 - 5m_4 m_1 - 10m_3 m_2 + 20m_3 m_1^2 + 30m_2^2 m_1 - 60m_2 m_1^3 + 24m_1^5 \\
k_6 &= m_6 - 6m_5 m_1 - 15m_4 m_2 + 30m_3 m_1^2 - 10m_2^3 + 120m_3 m_2 m_1 - 120m_3 m_1^3 + 30m_2^3 + 270m_2 m_1^4 + 360m_2 m_1^6 - 120m_1^6.
\end{align*}

We have computed these six cumulants of $\Delta f(L)$ for various values of $R$ and $q$, based on 100 independent strips of length $m = 10^5$ and width $1 \leq L \leq 7$. Sample results for $R = 2$ and $q = 3, 8$ are shown in Table \([V]\).

For $q = 3$ the cumulant expansion converges well. The magnitude of the higher cumulants decreases very rapidly, especially for $L > 3$, and reliable estimates for the left-hand side of Eq. \((33)\) can be obtained simply by summing the first 3 or 4 cumulants, at least when $n$ is not too large. Performing parabolic least-squares fits using Eq. \((21)\) with an $1/L^4$ correction we thus expect to extract quite accurate values of $x_n$ at the random FP.

As $q$ increases the convergence is slower. This is witnessed by the $q = 8$ results of Table \([VI]\) decreasing noticeably slower, both for a definite cumulant as a function of $L$ (vertically) and for a definite $L$ as a function of the cumulant number (horizontally). The approximation of leaving out the higher cumulants in the sum \((43)\) thus becomes increasingly difficult to justify, and eventually the cumulant expansion breaks down. This problem is enhanced by the fact that for $q > 8$ we expect a randomness strength of $R = 2$ to be insufficient in order to access the true behaviour at the random FP. We are thus forced to increase $R$, whence the fluctuations become even more violent and the cumulant expansion accordingly ill-behaved.

Our results for $x_1$ are shown in Fig. \([I]\). Since error bars on the individual cumulants are related to the magnitude of the higher cumulants the question of how to assign a final error bar to $x_1$ becomes a delicate one. We have addressed this issue by averaging the estimates for $x_1$ obtained from various parabolic least-squares fits. More precisely, the average is calculated from 4 values, namely fits with $L_0 = 3$ or 4 and including the first 3 or 4 cumulants on the right-hand side of Eq. \((43)\). The consistency of these 4 values is regarded as a check of the validity of the expansion.

In particular, for $q = 3$ we find $x_1(3) = 0.13467 \pm 0.00013$ which is 10 standard deviations above the value $x_1^{\text{pure}}(3) = \frac{2}{9} \approx 0.13333$ of the pure three-state Potts model \([43]\) and at the same time in perfect agreement with the result $x_1(3) = 0.13465 + O(c^4)$ of the $(q = 2)$-expansion \([17]\). The Monte Carlo result $x_1(3) = 0.1337 \pm 0.0007$ of Picco \([22]\) was not able to distinguish convincingly between pure and random behaviour.

For $q = 4$ our result is $x_1(4) = 0.1396 \pm 0.0005$, in nice agreement with Picco’s preliminary result $x_1(4) \sim 0.139 \pm 0.0002$ and decidedly different from the corresponding pure value of $x_1^{\text{pure}}(4) = \frac{2}{9}$.

As discussed at length in the Introduction a major motivation for this work was to determine whether the impurity soften transitions for $q > 4$ do or do not have the critical exponents of the pure Ising model. The data of Fig. \([I]\) clearly show a smooth continuation of the perturbative results \([14, 17]\) exhibiting no singularity whatsoever at $q = 4$. Our result $x_1(8) = 0.1415 \pm 0.0036$ is comfortably away from the pure Ising value and provides a striking piece of evidence for both our phase diagram and the FP structure of the $(q - 2)$-expansion.

All the results quoted for $x_1$ were computed using $R = 2$. We have checked that other values of $R$ yield results consistent herewith, provided that $R$ is chosen neither too small, in which case the cross-over length $\xi_X \sim \exp(1/2Aw^2)$ found from Eq. \((3)\) becomes too large for the random FP to be reached, nor too large, in which
case the cumulant expansion breaks down. The same holds true when the random bonds are drawn from the trinary distribution \[ 33 \] with various values for the dilution parameter \( p \).

Because of the positive sign of the second cumulant the values of \( x_1 \) are invariably smaller than those one would have obtained without the cumulant expansion (i.e., using only the first cumulant). The latter, however, determines a universal exponent \( \alpha_0 \) that describes the asymptotic decay of the spin-spin correlation function in a fixed sample at criticality. In terms of the multiscaling exponents this reads

\[
\alpha_0 = \left. \frac{dx_n}{dn} \right|_{n=0} \tag{45}
\]

Near \( q = 2 \) Ludwig obtained the expansion \[ 16 \]

\[
\alpha_0 = x_1^{\text{pure}} + \frac{1}{16} n(n-1) y + \mathcal{O}(y^2), \tag{46}
\]

where \( y \) is the RG eigenvalue of the energy operator coupling to the bond randomness. Our results for small fractional values of \((q - 2)\) are in good agreement with this expression, and if one takes into account the logarithmic corrections expected exactly at \( q = 2 \) it seems that the theoretical prediction and the numerical results have a common tangent at \( q = 2 \). For the physically interesting case of \( q = 3 \) the agreement is not so good. We believe that this apparent discrepancy would be resolved if the expansion \[ 16 \] could be carried through to three-loop order as in the case of \( x_1 \).

Similar remarks can be made about the higher moments of the spin-spin correlation function for which we are unable to verify Ludwig’s expansion \[ 16 \]

\[
x_n = n x_1^{\text{pure}} - \frac{1}{16} n(n-1) y + \mathcal{O}(y^2). \tag{47}
\]

Nevertheless it should be remarked that the fact that the higher cumulants do not vanish in itself implies multiscaling.

Before concluding this section we should like to give a heuristic argument that the cumulant expansion breaks down for large \( q \). In a replica formulation we can imagine the central charge \( c(n) \) as a function of the number of replicas \( n \). In this notation the central charge of the pure and the random systems are \( c(1) \) and \( c'(0) \) respectively, where the prime denotes a differentiation with respect to \( n \). The partition function of the replicated strip is then

\[
\bar{Z}^n = \int \exp(-nmL f) P(f) df = \exp \left( -mL \mathfrak{f} + \frac{nmc(n)}{6L} \frac{\pi^2}{8} \right), \tag{48}
\]

where \( P(f) \) is the probability distribution of the free energy. Differentiating this expression twice with respect to \( n \) and taking the replica limit \( n \to 0 \) we infer that the second cumulant of \( f \) contains a term that is proportional to \( c''(0) \). The cumulant expansion is thus expected to break down if \( c(n) \) has a large curvature at \( n = 0 \).

For \( 2 \leq q \leq 4 \) the replicas are weakly coupled, since \( c(1) \approx c'(0) \) \[ 16 \]. Hence \( c''(0) \ll 1 \). But when \( q = 4 + \epsilon \) the transition of the pure system goes first order so that the function \( c(n) \) starts out with slope \( c'(0) = 1 \) and somehow curves down to assume the value \( c(1) = 0 \). Consequently \( c''(0) = \mathcal{O}(1) \) and the higher cumulants begin to contribute significantly to the sum \[ 13 \]. Finally, for \( q \gg 4 \) we are in the strong coupling regime. We still have \( c(1) = 0 \) and as our numerical data indicate that \( c'(0) \sim \ln q \) it follows that \( c''(0) \gg 1 \). This means that the cumulant expansion must break down.

One may speculate whether the transition actually becomes first-order whenever \( q^n > 4 \). Clearly this is the case for the pure Potts model \[ 10 \], but a similar statement is true when \( N \) Ising models are coupled by their local energy density. Namely, in this case an RG analysis \[ 3 \] implies a fluctuation-driven first-order transition whenever \( N > 2 \), that is to say for \( 2^N > 4 \). If this conjecture is correct one would then suppose the function \( c(n) \) to vanish for \( n \geq n_0 \), where \( q^{n_0} = 4 \). Evidently such a scenario is in accordance with our observation that \( c''(0) \gg 1 \) for \( q \gg 4 \).

E. The thermal exponent

Because of the rather striking success of the cumulant expansion for \( x_1 \) one would now expect the thermal exponent \( x_T \) to be similarly related to the fluctuations of \( \Delta f_T(L) = f_{11}(L) - f_{01}(L) \). Surprisingly, this seems not to be the case. Computing the equivalent of \( \alpha_0 \), i.e., using only the first cumulant, we find the following results for different values of \( q \): \( \alpha_0^T(2) = 1.028 \pm 0.001 \), \( \alpha_0^T(3) = 0.91 \pm 0.01 \), \( \alpha_0^T(4) = 0.81 \pm 0.02 \) and \( \alpha_0^T(8) = 0.65 \pm 0.01 \). As remarked above the results using more cumulants can only be lower.

This is bad news since the quenched correlation length exponent \( \nu \) can be shown quite rigorously to satisfy the bound \[ 19,20 \]

\[
\nu \geq \frac{2}{d}, \tag{49}
\]

or, in our notation, \( x_T \geq 1 \). Though the proof of Ref. \[ 14 \] refers to the divergence of the correlation length as the critical point is approached, and hence strictly speaking does not apply to the system under consideration since we work exactly at the critical point, the RBPM is among the simplest physical systems for which Eq. \[ 19 \] is believed to be valid \[ 20 \]. The point is strengthened by noting that the \( (q-2)\)-expansion yields \( x_T = 1.02 + \mathcal{O}(c^3) \) at \( q = 3 \) \[ 13 \]. It is therefore difficult to have confidence
in the cumulant expansion for the thermal exponent, and independent methods of assessing \( x_T \) must be devised.

At this point we note that although the RG equation \( \Gamma \) seems to warrant an effective exponent of \( x_T^{\text{eff}} = 1 - Aw^2 \) for \( q \) large, this argument is only superficially true. Indeed, near \( q = \infty \) the RG flows must extend to infinite \( w \) before reaching the random FP, and consequently an expansion valid for weak randomness is not to be trusted.

The alternative method for finding \( x_T \) that comes closest to the spirit of Refs. 13,14 is that of finite-size scaling off the critical point. This is discussed at length in the next subsection, and for the moment we concentrate on less “obvious” possibilities.

One of the key points in the construction of the cumulant expansion was the realisation that the spin-spin correlation function was mapped onto a surface tension under duality, and hence could be expressed in terms of the largest Lyapunov exponent of a TM with twisted boundary conditions. Reinterpreting the latter as a free energy the self-averaging property was evident, and the cumulant expansion correspondingly behaved quite well if the fluctuations were not too large. It has recently been shown \( 39,40 \) that under duality four-point correlation functions are similarly mapped onto (generalised) surface tensions. Presently these duality relations have only been worked out for planar graphs, but there is some hope that they may be extended to the case of cylindrical boundary conditions as well. Taking two of the points as nearest neighbours on either end of the cylinder we would then recover the energy-energy correlator, and if the corresponding boundary conditions can be implemented in the TM \( x_T \) follows from a cumulant expansion.

Next, we discuss the method of iterating orthogonal vectors in order to extract the second Lyapunov exponent \( 31 \) in more physical terms. The energy-energy correlator (Green’s function) can be written as

\[
\langle E(r_1)E(r_2) \rangle = \frac{\text{Tr} \, E(r_1)E(r_2) \exp(-\mathcal{H})}{\text{Tr} \, \exp(-\mathcal{H})}. \tag{50}
\]

Now imagine building up the strip by repeated action by the random TMs on some initial state situated at \( r = -\infty \). When we reach \( r_1 \) the system is in a state \( |a_0\rangle \) on which we act with the energy operator to define \( |b_0\rangle = E(r_1)|a_0\rangle \). After \( n \) further iterations these states become

\[
|a_n\rangle = T_n \cdots T_2 T_1 |a_0\rangle \quad |b_n\rangle = T_n \cdots T_2 T_1 |b_0\rangle. \tag{51}
\]

Defining a new state \( |\tilde{b}_n\rangle \) by orthogonalising \( |b_n\rangle \) with respect to \( |a_n\rangle \)

\[
|\tilde{b}_n\rangle = |b_n\rangle - \frac{\langle a_n |b_n\rangle}{\langle a_n |a_n\rangle} |a_n\rangle \tag{52}
\]

we find that

\[
\frac{\langle \tilde{b}_n |\tilde{b}_n\rangle}{\langle a_n |a_n\rangle} = \frac{\langle b_n |b_n\rangle}{\langle a_n |a_n\rangle} - \frac{\langle a_n |b_n\rangle \langle b_n |a_n\rangle}{\langle a_n |a_n\rangle^2} = \langle E(r_1)E(r_2) \rangle - \langle E(r_1) \rangle \langle E(r_2) \rangle \tag{53}
\]

Thus the process of orthogonalisation corresponds precisely to subtracting off the disconnected part of the correlation function.

When \( n \gg 1 \) the states \( |b_n\rangle \) and \( |a_n\rangle \) are almost identical due to contamination and have a huge norm \( \sim A_0^n \). The idea of orthogonalising them is therefore numerically extremely unsound. Fortunately a simple calculation shows that orthogonalising after \( n_1 \) iterations and then again after \( n - n_1 \) further iterations is equivalent to orthogonalising only once, as above. Hence, by induction, we are allowed to orthogonalise after each iteration, leaving us with the method of Benettin et al. 31. Similar observations hold true for the higher Lyapunov spectrum.

At this point an objection may be raised. Since

\[
\langle a_n |a_n\rangle = \langle a_0 |T_1^\dagger T_2^\dagger \cdots T_n^\dagger T_n \cdots T_2 T_1 |a_0\rangle, \tag{54}
\]

where the dagger denotes transposition, the correlation function \( 53 \) corresponds to a realisation of the randomness that is always symmetric around the midpoint of \( r_1 \) and \( r_2 \). From the above physical picture leading to Eq. \( 53 \) it seems that what we really ought to compute is

\[
\frac{\langle \tilde{b}_n |\tilde{b}_n\rangle}{\langle a_n' |a_n\rangle}, \tag{55}
\]

where the (transposed) TMs used to obtain the states on the left implement a different realisation of the randomness than that used to obtain the states on the right.

Numerically we are now facing the problem of computing the average of huge numbers that are no longer necessarily positive. As discussed in connection with Eq. \( 43 \) we do not expect the correlation function to be self-averaging, and because of possible negative values of Eq. \( 55 \) the subterfuge of averaging its logarithm will not help us out. Trial runs seem to indicate that for sufficiently small values of \( q \) and \( R \) (such as \( q = 3 \) and \( R = 2 \)) the matrix elements appearing in Eq. \( 55 \) computed for the usual samples of length 200 may have either sign, but that their quotient is invariably positive. The corresponding result for \( x_T \) is roughly equal to that obtained from the cumulant expansion. Unfortunately, for larger values of \( q \) or \( R \) rare events of negative quotients begin to occur, and any attempt of averaging Eq. \( 55 \) without resorting to logarithms is hampered by such large fluctuations as to render the results insufficiently accurate at the very best. Computations along these lines, though physically appealing, must therefore be abandoned on numerical grounds.
Yet another possibility of determining at least an approximate value of \( x_T \) is through the conformal sum rule that for an \( n \)-fold replicated system reads
\[
\frac{c(n)}{12} = \frac{\sum_i d_i(n) x_i e^{-2\pi x_i}}{1 + \sum_i d_i(n) e^{-2\pi x_i}},
\]
where the sum runs over all operators in the theory, including the descendants of the Verma module with their appropriate multiplicities, and \( d_i(n) \) are the multiplicities pertaining to the permutational symmetry of the \( n \) replicas and the \( q \) Potts states. For the magnetic operator \( d_i(n) = n(q - 1) \), since there are \((q - 1)\) independent order parameters, and in the case of the energy operator \( d_i(n) = n \). In the pure system this yields quite accurate estimates for \( x_T \) if the exact values of \( c(1) \) and \( x_H \) are inserted along with the first descendant of the latter. Differentiating and going to the replica limit we find that for a random system
\[
\frac{c'(0)}{12} = x_1(q - 1)e^{-2\pi x_1} - \frac{x_2}{2}(q - 1)^2e^{-2\pi x_2} + \cdots,
\]
so that for values of \( x_1, x_2 \) and \( x_T \) near those of the Ising model the term with \( x_T \) enters only as a small correction. Consequently, at the very best only \( x_2 \) can be determined with some confidence from our previous results for \( c'(0) \) and \( x_1 \). Its value appears to be consistent with that obtained from the cumulant expansion.

Finally, we should like to mention that preliminary studies of exact partition function zeros for small \( L \times L \) lattices with quenched bond randomness hints at an interesting new method of estimating \( x_T \). Although the different realisations of the quenched bond randomness in general lead to a considerable scatter in the positions of such zeros, it turns out that the zeros that are closest to the real axis only exhibit a very weak dependence on the realisation. But these zeros are precisely those that fix \( x_T \) through their finite-size scaling. Results along these lines, both for zeros of the Lee-Yang and of the Fisher type, will be published elsewhere.

### F. Phenomenological renormalisation

In view of the difficulties encountered in our attempts to extract \( x_T \) directly at the critical point we turn our attention to the method of phenomenological renormalisation [21], which is closer in spirit to the ideas that lead to the bound [13].

The magnetic correlation length can be found from the TM spectra through
\[
\xi(L, T)^{-1} = \ln \left( \frac{\Lambda_{11}^0}{\Lambda_0^2} \right) = L(f_0^{22} - f_0^{11}),
\]
and we note that this quantity would be self-averaging in the random case. Motivated by the form \( \xi \sim (T - T_c)^{-\nu} \) of the divergence of the correlation length in the infinite system we make the finite-size scaling ansatz
\[
\xi(L, t) = L\phi((T - T_c)L^{1/\nu}).
\]
For pure systems, then, one traditionally scans through the vicinity of \( T_c \) to find an effective \( T_c(L) \) as the solution of
\[
\frac{\xi(L, T_c(L))}{L} = \frac{\xi(L - 1, T_c(L))}{L - 1},
\]
and computes an approximant \( \nu(L) \) from
\[
1 + \frac{1}{\nu(L)} = \frac{\ln \left( \frac{\mu(L, T)/\mu(L - 1, T)}{\ln \left( \frac{L}{(L - 1)} \right)} \right)}{T = T_c(L)} ,
\]
where the derivatives
\[
\mu(L, T) = \frac{\partial \xi(L, T)}{\partial T} = L^{1+\nu}\phi'((T - T_c)L^{1/\nu})
\]
are found by numerical differentiation. As \( L \to \infty \) we have \( T_c(L) \to T_c \) and \( \nu(L) \to \nu \).

In the random case the extracted values of \( \xi(L, T) \) are hampered by substential error bars, and the method just outlined becomes by far too inefficient. Fortunately the very costly idea of scanning for \( T_c(L) \) can be discarded, since the exact \( T_c \) of the RBPM is known from Eq. [3]. Consequently the derivatives [22] and the approximants [14] are evaluated at the exact \( T_c \), whence the only remaining source of errors is that of statistical fluctuations over the different realisations of the randomness.

Naively one would now find the derivative [22] by subtraction of the free energies evaluated at \( T = T_c(1 \pm \epsilon) \), where \( \epsilon \ll 1 \). Although this method yields reasonable results for \( \epsilon \sim 10^{-2} \) it is way too inaccurate, since it involves the subtraction of almost identical quantities (with error bars). A superior strategy is to divide the strip into patches of length 200, calculate exact\(^2\) values of \( \mu(L) \)

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\(^2\)Since we are now faced with differentiating a quantity that is known with full machine precision \((10^{-16})\) we can concentrate on minimising the rounding and truncation errors. This is accomplished by choosing \( \epsilon = 10^{-5} \). See Sect. 5.7 of: W. H. Press \textit{et al.}, \textit{Numerical Recipes in C}, second edition (Cambridge University Press, Cambridge, 1992).
for each of those, and finally average over the totality of such patches. In this way one exploits the fact that \( \xi(L, T_c(1 - \epsilon)) \) and \( \xi(L, T_c(1 + \epsilon)) \) are strongly correlated when the realisation of the randomness is kept fixed. In practice we found that this trick lead to a reduction of the error bars with a factor \( \sim 120 \).

Sample results obtained by using these prescriptions are shown in Table VII. It is seen that although the convergence is still too slow for reliable extrapolations to the limit of an infinite system to be made, the conflict with the bound (19) appears to be resolved.

We have found that the convergence of the estimators \( \nu(L) \) can be significantly sped up if one performs the numerical differentiation (22) by going perpendicularly off the self-duality criterion in \( (K_1, K_2) \) space instead of maintaining the condition \( K_2 = R K_1 \). Indeed, one may imagine that there is another exponent associated with motion along the critical surface, and maintaining \( K_2 = R K_1 \) one would then measure an admixture of this spurious exponent, in particular for large \( R \). A simple calculation using Eq. (2) shows that one should then evaluate \( \xi(L, K_1', K_2') \) at

\[
K_1' = K_1 \left( 1 + \epsilon \frac{R e^{K_1}}{q e^{K_2}} (e^{K_2} - 1)^2 \right)
\]

\[
K_2' = K_2 (1 \pm \epsilon).
\]

The sample results shown in Table VIII exhibit a conspicuous improvement over those of Table VII. Not only is the convergence faster, but it is even seen that the estimators \( \nu(L) \) form a monotonically increasing sequence for low values of \( R \) and a monotonically decreasing one for high \( R \). The extrapolated \( \nu \) is pinched between those two sequences and consequently quite accurately determined.

Plots of the estimators \( \nu(L) \) for \( 3 \leq L \leq 7 \) and several values of \( R \) are shown in Figs. 6 and 7 for \( q = 8 \) and \( q = 64 \) respectively. It is seen that the curves for \( \nu(L) \) and \( \nu(L - 1) \) intersect at a unique value of \( L \), seemingly converging quite fast to a definite value \( R^* \) as \( L \) increases. We interpret \( R^* \) as the randomness strength at the critical FP and the corresponding value of \( \nu \) as the correct critical exponent. It is tempting to conjecture that the curves \( \nu(L) \) approach \( \nu \) on the entire interval \( R \in ]1, \infty[ \) as \( L \to \infty \). From the graphs it seems that the convergence is faster for large \( q \).

The values of \( \nu \) and \( R^* \) corresponding to this scenario are shown in Table IX. In accordance with the phase diagram (Fig. 1) \( R^* \) is a slowly, supposedly logarithmically\(^3\), increasingly function of \( q \). For \( q = 2 \) the deviation from \( \nu = 1 \) can be ascribed to logarithmic corrections (44), and for \( q = 3 \) our result for \( \nu \) is in agreement with the \((q - 2)\)-expansion (17) though the possibility of replica symmetry breaking cannot be ruled out (22). Also for \( q > 4 \) are our values for \( \nu \) numerically consistent with unity, indicating that, unlike what is the case for the magnetic exponent, the thermal one displays only a weak \( q \)-dependence.

From Figs. 6 and 7 a remarkable feature about the pure system \( (R = 1) \) is apparent. For \( q = 8 \) the estimators \( \nu(L) \) seem to converge to \( \nu = \frac{1}{3} \), whilst for \( q = 64 \) the extrapolated value is \( \nu \approx 0 \). The former value is hardly surprising since, as we also remarked above, a first-order transition is expected to exhibit scaling with trivial effective exponents (in this case: \( \nu = 0 \)). On the other hand, \( \nu \approx 0 \) for \( q = 64 \) has to do with the length scales of the system. Namely, from the asymptotic behaviour of the correlation length of the pure system (11)

\[
\xi \sim \frac{2}{\ln q} \quad \text{as} \quad q \to \infty
\]

we infer that \( \xi \approx 1 \ll L \) at the transition point of the \( q = 64 \) system. But away from the transition point we also expect \( \xi = O(1) \), since the lattice spacing (unity) is the least length scale in the system. After all there is a ferromagnetic interaction between nearest neighbour spins. We thus conclude that \( \xi \) is roughly temperature independent in this case. In order for this to be consistent with the asymptotic behaviour of the finite-size scaling ansatz (59)

\[
\phi(x) \sim x^{-\nu} \quad \text{for} \quad x \ll 1
\]

we must then have \( \nu \approx 0 \). This is to be contrasted to the case of \( q = 8 \) where \( \xi \gg L \) so that we clearly “see” the phase transitions in our strips of width \( L \).

Very recently the bound (19) was challenged by Pázmáni et al. (24). These authors claimed that the standard method of averaging over the disorder in finite-size (FS) systems introduces a new diverging length scale into the problem, whence the resulting \( \nu_{FS} \) may be unrelated to the true exponent \( \nu \) governing the divergence of the correlation length in the infinite system. In particular \( \nu \) can be less than \( \frac{1}{3} \), and if this is the case the standard method is liable to yield exactly \( \nu_{FS} = \frac{2}{3} \). Ref. (24) then went on to suggest a noise reduction procedure that professedly would allow one to access the true \( \nu \). For each realisation of the binary randomness (4) used in the disorder average there will be a fraction \( \mathcal{P} \) of weak bonds \( (K_1) \). The noise due to the fluctuations of \( p \) around its average value \( \bar{p} = \frac{1}{2} \) can then be reduced by adjusting the

\(^3\)This supposition constitutes the simplest possibility allowed by the phase diagram of Fig. 1 in which \( R^* \to \infty \) as \( \ln q \to \infty \).
couplings \((K_1, K_2)\) for that particular realisation to the values they would assume at the critical point of an infinite system with a (mean) fraction of weak bonds equal to \(p\).

To implement this we are faced with the task of finding the two-dimensional critical surface in the space \((K_1, K_2, p)\) using our knowledge of its one-dimensional intersection with the plane \(p = \frac{1}{2}\), viz. Eq. (53). Let the fraction of weak bonds in a particular realisation be

\[
p = \frac{1}{2}(1 + \epsilon_p),
\]

where \(\epsilon_p \ll 1\). The symmetry \(p \leftrightarrow 1 - p\) ensures that, to first order in \(\epsilon_p\), we must still go perpendicularly off the self-dual curve in the \((K_1, K_2)\) subspace, as in Eq. (63). Since an increase in the number of weak bonds must be offset by an increase of the \(K\)'s in order to keep the coupling to the energy density constant, the correct prescription is

\[
K_1 \rightarrow K_1 \left(1 + \epsilon_K \frac{\text{Re}K_1}{q e^{K_2}}(e^{K_2} - 1)^2\right)
\]

\[
K_2 \rightarrow K_2(1 + \epsilon_K)
\]

for some \(\epsilon_K > 0\). Demanding that the combined change in \(p\) and in \((K_1, K_2)\) must leave the coupling to the energy density invariant furnishes a relation between \(\epsilon_p\) and \(\epsilon_K\)

\[
\epsilon_K = \epsilon_p \frac{K_2 - K_1}{K_1 \frac{\text{Re}K_1}{q e^{K_2}}(e^{K_2} - 1)^2 + K_2}
\]

and the derivatives (62) are now evaluated at these values of the parameters by going perpendicularly off the critical surface. To first order, of course, Eq. (63) still gives the correct way of doing so.

Our confidence in the results of Table X is increased by observing that the implementation of this novel averaging procedure does not alter our results. Indeed, trial runs for \(q = 8\), where the discrepancy between the \(x_T\) extracted from the Lyapunov spectrum and phenomenological RG respectively is large, render the values of the estimators \(\mu(L)\) unchanged within (small) error bars. It is thus concluded that even though our results for \(\nu\) are conspicuously close to satisfying the bound (49) with equality, this is not due to an artifact in the averaging procedure.

G. The higher Lyapunov spectrum

Although the second Lyapunov exponent of \(T^{11}\) fails to yield the thermal scaling dimension \(x_T\) in the standard way it is hard to believe that the Lyapunov spectrum is not in some way related to the operator content of the CFT underlying the RBPM. In the case of the pure three-state Potts model, for example, it is well known [43] that the first five gaps of the \(Z_2\)-even sector are related to the energy density \(\epsilon\), its first descendants \(L_{-1} \epsilon\) and \(T_{-1} \epsilon\), the stress tensor \(T\) and its conjugate \(\bar{T}\). To wit, the scaling dimensions of these operators can be found from the gaps through Eq. (4), and we have verified this using our connectivity basis TMs.

In view of the bound (43) it is problematic to associate the first gap with the energy density in the random case, but it is nevertheless a beguiling question whether such concepts as descendants and the stress tensor are preserved by the randomness. To investigate this issue we have computed the first few Lyapunov exponents of \(T^{11}\) for \(1 \leq L \leq 8\), averaging over 100 runs as usual. The scaling dimensions corresponding to the first five gaps for \(q = 3, 4, 5\) and \(R = 2\) are shown in Table X. Self-averaging was ensured by utilising the cumulant expansion, and parabolic least-squares fits included the first three cumulants.

It is quite remarkable that even if the scaling dimension corresponding to the first gap may not be equal to \(x_T\) our data give strong reasons to believe that it has a descendant, and that this descendant has the expected degeneracy of two. And even though the scaling dimensions in general depend on \(q\) those corresponding to the fourth and the fifth gaps are constant within error bars and very close to 2, as is expected for the stress tensor of a conformally invariant system [13]. Preliminary data for even higher Lyapunov exponents seem to hint at descendants at level two, but since we have found that in the pure system higher and higher eigenvalues require larger and larger system sizes before the asymptotic scaling form (4) is valid, massive computations are needed to establish reliable results for all but the first few scaling dimensions.

Another interesting feature of our data for the higher Lyapunov exponents is that the Harris criterion seems to be valid in a very complete sense. Namely, trial runs for \(q = 1\) seem to indicate that although individual cumulants exhibit a pronounced dependence of \(R\), their sum is virtually independent of the strength of the randomness in the whole range \(R \in [1, 2]\). It thus appears that all exponents \(x_i\) that we can extract numerically from the Lyapunov spectrum, using Eq. (42) and the cumulant expansion, obey the Harris criterion. Since the connection between these exponents and the scaling dimensions of the underlying CFT is not completely known (witness \(x_T\)) this may well turn out to be a non-trivial observation.

V. DISCUSSION AND OUTLOOK

In a recent paper by Picco [14] it has been suggested that for \(q = 2^p\) the effective central charge at the random FP is \(c' = \frac{6}{7}\), and that this class of models thus
behaves as $p$ decoupled Ising models. Even without refer-
ing to our values of the magnetic exponent we should like to point out that all the data show is that $\epsilon' \propto \ln q$ with a constant of proportionality that is very close to $\frac{1}{2\ln 2} \approx 0.721$. But this constant is also very close to that of the percolation point, viz. $\frac{1}{2\ln 2} \approx 0.689$. Indeed, these two numbers differ by less than 5%, and since our error bars and those of Picco are in the 2% and the 4% range respectively, there is no irrefutable way of numerically distinguishing between percolative, Ising-like or indeed some other, presently unknown, behaviour of the central charge. A similar observation is valid for $2 \leq q \leq 4$ where our numerical data as displayed in Fig. 3 are compatible, within error bars, with both the values at the pure and the random FP (but not, in this case, with those at the percolation point).

On the other hand, our results for the magnetic exponent should leave no doubt that the correct CFT describing the RBPM cannot be that of a number of decoupled Ising models. In particular, the non-Ising value at $q = 8$ is in sharp disagreement with the Monte Carlo results of Ref. [11]. One possible explanation of this discrepancy is that these authors define a non-standard order parameter through

$$ \rho = L^{-d} \langle \text{max}(N_1, N_2, \ldots, N_q) \rangle, $$

where $N_i$ is the number of spins in state $i$, which is related to our local order parameter defined in Eq. (69) by $N_i = \sum_r (M_i(r) + q^{-1})$. The site label $r$ runs over a hypercube of side $L'$ with $24 \leq L' \leq 84$. Note that $\rho$ may also be written as

$$ \rho = L^{-d} \lim_{k \to \infty} \left( \sum_{i=1}^{q} \langle N_i^k \rangle \right)^{1/k}. $$

Expressed in terms of the local order parameter, $\langle N_i^k \rangle$ gives a sum of terms each of the form

$$ \sum_{r_j} \langle M_i(r_1)^{k_1} M_i(r_2)^{k_2} \ldots M_i(r_n)^{k_n} \rangle, $$

where $k_1 + k_2 + \cdots = k$. As $k \to \infty$ at fixed $L'$, it is clear that at least some of the $k_j$ must grow large. In the pure system, this should not matter, since each term will scale in the same manner. But when multiscaling is present, the scaling behaviour of the $k_j$ power of the local order parameter may be different. Indeed, in the limit of $k \to \infty$ one would expect $\rho$ to scale with dimension $\lim_{k \to \infty} x_k / k$, which is less than $x_1$ by convexity.

Another criticism of Ref. [11] is that the realisations of the binary randomness considered were confined to those for which the number of strong and weak bonds in each of the two lattice directions were equal. Though this restriction is clearly innocuous in the limit $L' \to \infty$ this may not be so as far as the finite-size scaling is concerned. From trial runs where similar restrictions were imposed to the bond distributions of the TMs we found that seemingly harmless noise reductions schemes can influence the output substantially.

Finally, the mapping to the RFIM [23] illustrated that for large $q$ typical configurations consist of large clusters of spins in the same Potts state, separated by domain walls. Whilst our very long strips are guaranteed to accomodate large regions in which all $q$ values of the order parameters are realised, it is not clear that this should be the case in the much smaller square geometries of Ref. [11]. Indeed it seems likely that one would find Ising exponents if the geometry under consideration typically can accomodate at most two different large clusters.

We now turn our attention to the thermal exponent. If the phenomenological RG scheme is to be trusted the values of $x_T$ only exhibit a weak dependence on $q$, although the $(q-2)$-expansion gives us reason to believe that there is some variation [15]. It is interesting that $x_T$ stays so close to unity even at very high $q$, but presently we do not have any arguments to explain this finding. Unfortunately the method employed was unable to resolve the slight deviations from unity, and it is indeed a challenge to future research to find more accurate ways of assessing $x_T$ for disordered systems. Our results on the higher Lyapunov spectrum are nothing if not intriguing, and we believe that a great effort must be made to understand why the first gap in the spectrum fails to be related to $x_T$ in the standard way, even though the higher gaps show clear indications of a conformal field theory underlying the RBPM.

A very interesting issue to be addressed by future research is that of the dynamical universality class of the RBPM. In particular it would be interesting to see whether the dynamic critical exponent $z$ does or does not agree with the Ising value of $z \approx 2$, or whether, in analogy with the RFIM, there is logarithmically slow dynamics due to the pinning of domain walls by impurities.

Other types of randomness are also of interest to the question whether a first-order phase transition is softened due to impurities. In this paper we have studied the effect of quenched bond randomness in a flat, regular lattice. A somewhat different scenario is obtained by investigating the pure $q$-state Potts model on lattices with quenched connectivity disorder. In Ref. [25] MC simulations of the $q = 8$ model on two-dimensional Poissonian random lattices (Voronoi tessellations) with toroidal topology unambiguously showed that the first-order nature of the transition was not modified.

An argument why this must be so, at least for large $q$, can readily be given. For simplicity consider the model on the dual Delaunay random lattice, which per construction is a triangulation of space [26]. As on the regular lattice, at large $q$ there are only two important states in the equivalent random cluster model: the empty
lattice, which contributes a term $q^{N_{\text{vertices}}}$ to the partition function, and the full lattice, contributing a factor $u^{N_{\text{edges}}}$, where $u = e^K - 1$. Since for any triangulation $2N_{\text{edges}} = 3N_{\text{vertices}}$, the transition occurs when $u \sim q^{2/3}$ for any geometry. If we now consider that part of the lattice within a large hypercube of side $L_u \sim N_{\text{lattice}}$, which contributes a term $q^{2N_u}$, these are very much smaller than the analogous fluctuations which are present when random bonds are added: these are quite different from ours, and neither our arguments nor the results of Ref. [55] works for any triangulation, the fluctuations in the difference of the energies of these two states inside this region will come solely from the edges which penetrate the boundary. On average, the difference in the energies of these two states will still be zero, but there will be fluctuations of the order of the square root of the number of bonds which penetrate the boundary, which will therefore be $O(L^{(d-1)/2})$. These are very much smaller than the analogous fluctuations which are present when random bonds are added: these are $O(L^{d/2})$, which leads to the Imry-Ma type of argument [3,5,6]. For $d = 2$ Voronoi tessellations the fluctuations are thus always smaller than the domain wall energy $O(L^{d-1})$, and we conclude that such randomness is strongly irrelevant (at least for large $q$), in agreement with the results of Ref. [56].

Yet another kind of randomness is obtained by studying the Potts model on quenched random gravity graphs, for which MC simulations for $q = 10$ have provided strong evidence for a softening scenario similar to ours [57]. However, in this case the curvature is random and hence when the lattice is embedded in the plane, it is fractal. Although the argument about compensation of the bulk energies when $u \sim q^{2/3}$ works for any triangulation, the number of boundary edges may well scale in a different manner for these lattices. Whilst it would be interesting to study this in detail, it is clear that this problem is quite different from ours, and neither our arguments nor those of Refs. [38,39] can be directly applied.

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FIG. 1. Schematic phase diagram in the critical surface for \( d > 2 \). \( q \) increases to the left and \( w \) is the disorder strength, with \( P_1 P_2 \) being the percolation limit. RG flows are indicated. The latent heat is non-vanishing within the shaded region, and elsewhere the transition is continuous, controlled by the line of fixed points \( P_1 q_1 \). As \( d \to 2 \) the shaded region collapses to a line \( q_2 O \) of first-order transitions in the pure system.

FIG. 2. Plots of \( \lambda(L) \), normalised to \( \lambda(1) = 1 \), showing that bond randomness renders the phase transition second order. The random systems have \( R \equiv K_2/K_1 = 2 \).

FIG. 3. The effective central charge \( c' \) as a function of \( q \) for \( 2 \leq q \leq 4 \). The perturbative results by Ludwig and Cardy [14] do not differ appreciably within the range of \( q \)-values where the expansion is supposed to be valid. Accordingly the numerical data are unable to distinguish between pure and non-trivial random behaviour. They are also quite close to, but distinguishable from, the percolation point values.

FIG. 4. Effective central charge as a function of \( \log_{10} q \) for \( 1.5 \leq q \leq 64 \). For large \( q \) the data for \( R = 10 \) are supposed to represent the true behaviour at the random fixed point, as argued in the text.

FIG. 5. Magnetic exponent \( x_1 = \beta/\nu \) for \( R = 2 \) as obtained from the cumulant expansion. \( x_1 \) is an increasing function of \( q \), continuously connecting onto the perturbative results near \( q = 2 \). For \( q > 8 \) the cumulant expansion begins to break down.

FIG. 6. Estimators \( \nu(L) \) for the thermal exponent at \( q = 8 \) as obtained from phenomenological renormalisation applied to strips of width \( L \) and \( L - 1 \). In the pure system (\( R \to 1 \), see rightmost inset) the estimators converges to \( \nu(\infty) = \frac{1}{2} \). Curves for neighbouring system sizes intersect at values of \( \nu \) and \( R \) that converge to those at the random fixed point as \( L \to \infty \). In this case \( \nu = 1.01 \pm 0.02 \) and \( R^* \approx 9 \) (see left inset). Error bars are no larger than the size of the symbols.

FIG. 7. Phenomenological renormalisation at \( q = 64 \). The curves intersect at larger angles than before, allowing for a rather accurate determination \( \nu = 1.02 \pm 0.03 \) in spite of the large fluctuations. Error bars are comparable to the size of the symbols. From the rightmost inset it is seen that \( \nu \to 0 \) in the pure systems, as explained in the text. The left inset is a magnification of the region around \( R^* \approx 10 \).
| \( L \) | \( c_L \) | \( d_L \) | \( d_L - c_L \) | \( Lc_L \) | \( b_L \) |
|---|---|---|---|---|---|
| 1 | 1 | 2 | 1 | 1 | 1 |
| 2 | 2 | 5 | 3 | 4 | 2 |
| 3 | 5 | 15 | 10 | 15 | 5 |
| 4 | 14 | 51 | 37 | 56 | 15 |
| 5 | 42 | 188 | 146 | 210 | 52 |
| 6 | 132 | 731 | 599 | 792 | 203 |
| 7 | 429 | 2,950 | 2,521 | 3,003 | 877 |
| 8 | 1,430 | 12,235 | 10,805 | 11,440 | 4,140 |
| 9 | 4,862 | 51,822 | 46,960 | 43,758 | 21,147 |
| 10 | 16,796 | 223,191 | 206,395 | 167,960 | 115,975 |
| 11 | 58,786 | 974,427 | 915,641 | 646,646 | 678,570 |
| 12 | 208,012 | 4,302,645 | 4,094,633 | 2,496,144 | 4,213,597 |

**TABLE I.** The number of connectivity states for a Potts model transfer matrix of width \( L \) with \((d_L)\) and without \((c_L)\) an external magnetic field. Also shown is the size of the magnetic sector when using a ghost site \((d_L - c_L)\) and a seam \((Lc_L)\). For large strip widths the seam is advantageous. The number \(c_L\) of well-nested \(L\)-point connectivities should be compared to the total number of \(L\)-point connectivities \(b_L\) which increases faster than exponentially as a function of \(L\).

| \( L \) | \( q = 2 \) | \( q = 3 \) | \( q = 4 \) | \( q = 8 \) | \( q = 16 \) | \( q = 64 \) |
|---|---|---|---|---|---|---|
| 1 | 2.17460 (12) | 2.62881 (13) | 2.96193 (13) | 3.80035 (16) | 4.68198 (18) | 6.54635 (24) |
| 2 | 1.95329 (8) | 2.26650 (9) | 2.49558 (9) | 3.06980 (10) | 3.67393 (11) | 4.95619 (14) |
| 3 | 1.90971 (7) | 2.19534 (7) | 2.40405 (7) | 2.92819 (8) | 3.48241 (9) | 4.59557 (10) |
| 4 | 1.89550 (6) | 2.17203 (6) | 2.37431 (6) | 2.88328 (7) | 3.37498 (7) | 4.53797 (8) |
| 5 | 1.88895 (5) | 2.16182 (5) | 2.36126 (5) | 2.86392 (6) | 3.39934 (7) | 4.56442 (9) |
| 6 | 1.88568 (5) | 2.15649 (5) | 2.35449 (5) | 2.85395 (6) | 3.38683 (6) | 4.54893 (8) |
| 7 | 1.88377 (4) | 2.15328 (4) | 2.35040 (4) | 2.84798 (5) | 3.37948 (6) | 4.53429 (7) |
| 8 | 1.88250 (4) | 2.15113 (4) | 2.34782 (4) | 2.84424 (5) | 3.37479 (5) | 4.53394 (7) |
| 9 | 1.88164 (4) | 2.14993 (4) | 2.34624 (4) | 2.84172 (5) | 3.37186 (5) | 4.53017 (5) |
| 10 | 1.88098 (4) | 2.14858 (3) | 2.34504 (3) | 2.84011 (3) | 3.36918 (4) | 4.52653 (5) |
| 11 | 1.88048 (4) | 2.14804 (2) | 2.34386 (2) | 2.83851 (2) | 3.36768 (2) | 4.52465 (3) |
| 12 | 1.88017 (3) | 2.14744 (2) | 2.34314 (2) | 2.83765 (2) | 3.36639 (2) | 4.52316 (3) |

**TABLE II.** Critical free energies per site, \(-\beta f_1^1(L)\), for \( R = 2 \) and various values of \( q \). The figures in parentheses indicate the error bar on the last quoted digits.

| \( L_0 \) | \( q = 2 \) | \( q = 3 \) | \( q = 4 \) | \( q = 8 \) | \( q = 16 \) | \( q = 64 \) |
|---|---|---|---|---|---|---|
| 1 | 0.563 (1) | 0.927 (1) | 1.184 (1) | 1.787 (1) | 1.76 (1) | 2.33 (1) | 2.322 (4) | 3.120 (1) | 3.476 (5) |
| 2 | 0.508 (2) | 0.825 (3) | 1.042 (2) | 1.515 (3) | 1.586 (10) | 1.864 (3) | 2.101 (10) | 2.194 (4) | 3.150 (13) |
| 3 | **0.499 (3)** | **0.800 (6)** | **1.003 (6)** | **1.441 (7)** | **1.521 (23)** | **1.752 (8)** | **2.052 (25)** | **2.065 (10)** | **3.034 (30)** |
| 4 | 0.500 (6) | 0.813 (14) | 0.994 (14) | 1.424 (15) | 1.548 (52) | 1.750 (17) | 2.089 (57) | 2.157 (22) | 3.079 (68) |
| 5 | 0.505 (11) | 0.842 (30) | 1.005 (31) | 1.426 (33) | 1.622 (113) | 1.785 (38) | 2.203 (125) | 2.305 (47) | 3.209 (148) |
| 6 | 0.500 (20) | 0.818 (62) | 0.963 (63) | 1.360 (67) | 1.587 (228) | 1.794 (78) | 2.196 (251) | 2.384 (93) | 3.213 (300) |

**TABLE III.** Effective central charge \( c' \) extracted from parabolic fits with \( L_0 \leq L \leq L_{\text{max}} \), as described in the text. Error bars on the last quoted digit are shown in parentheses. The choice \( L_0 = 3 \) appears to be optimal, provided that the strength of the randomness \( R \) is large enough (see text), and the corresponding values of \( c' \), shown in bold face, should be regarded as our results.
TABLE IV. Effective central charge $c'$ obtained using a trinary distribution of the random bonds. There is a fraction $p$ of very weak and very strong bonds respectively, the remaining fraction $1 - 2p$ being critical.

| $L_0$ | $q = 2$ | $q = 3$ | $q = 4$ | $q = 8$ | $q = 16$ | $q = 64$ |
|-------|--------|--------|--------|--------|--------|--------|
|       |        |        |        |        |        |        |
| 2     | 0.5662 (6) | 0.9300 (7) | 1.1903 (7) | 1.723 (3) | 2.313 (3) | 3.469 (4) |
| 3     | 0.535 (1) | 0.878 (1) | 1.117 (1) | 1.656 (5) | 2.207 (6) | 3.311 (7) |
| 4     | 0.521 (2) | 0.848 (3) | 1.075 (3) | 1.605 (10) | 2.148 (11) | 3.206 (13) |
| 5     | 0.514 (3) | 0.836 (5) | 1.051 (5) | 1.585 (19) | 2.125 (20) | 3.163 (24) |
| 6     | 0.512 (4) | 0.839 (9) | 1.043 (9) | 1.595 (34) | 2.144 (38) | 3.174 (45) |
| 7     | 0.510 (6) | 0.840 (18) | 1.022 (18) |        |        |        |
| 8     | 0.509 (9) | 0.812 (33) | 1.011 (34) |        |        |        |
| 9     | 0.503 (14) |        |        |        |        |        |
| 10    | 0.500 (23) |        |        |        |        |        |

TABLE V. Effective central charge $c'$ extracted from linear fits of $f_{11}^{L}(L) - f_{11}^{L}(\infty)$ versus $1/L^2$, with $L_0 \leq L \leq L_{\text{max}}$. In all cases the approach towards the asymptotic values of Table III is from above. Error bars on the last quoted digit are shown in parentheses.

| $L$ | 1. cumulant | 2. cumulant | 3. cumulant | 4. cumulant | 5. cumulant | 6. cumulant |
|-----|-------------|-------------|-------------|-------------|-------------|-------------|
| $q = 3$ |           |     |     |     |     |     |
| 1   | -1.039786 (242) | 0.060716 | -0.00791 | 0.000830 | -0.004583 | -0.000778 |
| 2   | -0.253209 (146) | 0.012391 | -0.00347 | 0.000279 | 0.000059 | 0.000386 |
| 3   | -0.106163 (113) | 0.004963 | -0.00246 | 0.000063 | -0.000102 | 0.000046 |
| 4   | -0.057901 (95)  | 0.002784 | -0.00143 | 0.000006 | 0.000034 | 0.000003 |
| 5   | -0.036521 (84)  | 0.001810 | -0.00105 | 0.00001 | -0.00002 | -0.000003 |
| 6   | -0.025172 (76)  | 0.001289 | -0.00075 | 0.000008 | -0.00001 | -0.000002 |
| 7   | -0.018426 (69)  | 0.000968 | -0.00069 | 0.000002 | 0.000002 | -0.000001 |
| $q = 8$ |           |     |     |     |     |     |
| 1   | -1.380171 (289) | 0.104382 | 0.004069 | 0.014001 | 0.019452 | -0.013889 |
| 2   | -0.326484 (177) | 0.028366 | -0.001683 | -0.000432 | 0.000157 | -0.003145 |
| 3   | -0.132560 (138) | 0.014908 | -0.001822 | 0.000356 | 0.000221 | -0.00083 |
| 4   | -0.071296 (115) | 0.010129 | -0.001610 | 0.000319 | -0.000959 | 0.002323 |
| 5   | -0.044886 (102) | 0.007880 | -0.001619 | 0.000252 | -0.000082 | -0.000456 |
| 6   | -0.031045 (92)  | 0.006450 | -0.001538 | 0.000607 | -0.000184 | -0.001196 |
| 7   | -0.028851 (84)  | 0.005401 | -0.001505 | 0.000237 | 0.000234 | -0.000280 |

TABLE VI. The first six cumulants of $-\Delta f(L)$ for $1 \leq L \leq 7$ and $R = 2$. The error bar on the first cumulant (shown in parentheses) is related to the second cumulant; error bars on the higher cumulants are not shown.
TABLE VII. Phenomenological renormalisation for the thermal scaling dimension $x_T = 2 - 1/\nu$ at $q = 8$ and $R = 10$. For each strip width $L$ the 100 independent strips of length $m = 10^5$ are divided into patches of length 200. Within each patch the exact $\mu(L, T_c)$ is computed, based on evaluations of $\xi(L, K'_1, K'_2)$ at $K'_1 = K_1(1 \pm \epsilon_K)$ and $K'_2 = RK'_1$, where $K_1$ is found from Eq. (3). Final results and error bars are obtained as the mean value and the standard deviation over the totality of patches.

| L  | $\mu(L)$ | $\nu(L)$ |
|----|----------|----------|
| 1  | 1.087 (1)| -        |
| 2  | 4.229 (3)| 1.041 (1)|
| 3  | 10.426 (8)| 0.816 (2)|
| 4  | 19.682 (18)| 0.827 (3)|
| 5  | 31.867 (33)| 0.863 (5)|
| 6  | 46.994 (53)| 0.885 (7)|
| 7  | 65.020 (79)| 0.904 (9)|

TABLE VIII. Phenomenological renormalisation going perpendicularly off the critical surface for $q = 8$ and $R = 6$ and 10 respectively. The data collection was done as before.

| q  | $\nu$ | $R^*$ |
|----|-------|------|
| 2  | 1.12 (3)| 7 (1) |
| 3  | 1.04 (4)| 8 (1) |
| 8  | 1.01 (2)| 9 (1) |
| 64 | 1.02 (3)| 10 (1)|

TABLE IX. Values of the critical exponent $\nu$ and the randomness strength $R^*$ at the random fixed point as obtained from phenomenological renormalisation.

| q  | 1. gap | 2. gap | 3. gap | 4. gap | 5. gap |
|----|--------|--------|--------|--------|--------|
| 3  | 0.899 (4)| 1.877 (13)| 1.885 (12)| 2.045 (24)| 2.050 (23)|
| 4  | 0.817 (5)| 1.811 (9)| 1.818 (8)| 2.043 (23)| 2.049 (24)|
| 5  | 0.754 (6)| 1.771 (6)| 1.779 (6)| 2.058 (24)| 2.065 (25)|

TABLE X. Scaling dimensions corresponding to the first five gaps in the Lyapunov spectrum of $T^{11}$ for $R = 2$. The parabolic least-squares fits included the first three cumulants of the probability distribution, and error bars were extracted based on the fits with $L_0 = 4, 5$ and 6.
