METRIC INDEPENDENT ANALYSIS OF
SECOND ORDER STRESSES

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ABSTRACT. A metric independent geometric analysis of second order stresses in continuum mechanics is presented. For a vector bundle $W$ over the $n$-dimensional space manifold, the value of a second order stress at a point $x$ in space is represented mathematically by a linear mapping between the second jet space of $W$ at $x$ and the space of $n$-alternating tensors at $x$. While only limited analysis can be performed on second order stresses as such, they may be represented by non-holonomic stresses, whose values are linear mapping defined on the iterated jet bundle, $f^1(f^1W)$, and for which an iterated analysis for first order stresses may be performed. As expected, we obtain the surface interactions on the boundaries of regions in space.

1. INTRODUCTION

A metric independent geometric analysis of second order stresses in continuum mechanics is presented. A vector bundle $W$ over the $n$-dimensional space manifold $S$ is considered whose sections are interpreted as virtual generalized velocity fields. In the standard case, $W = TS$, the tangent bundle of $S$. A second order stress $S$ is defined to be a tensor field over space such that the value $S(x)$ is a linear mapping between the second jet $j^2W_x$ space of $W$ at $x$ and the space $\wedge^nT^*_xS$ of $n$-alternating tensors at $x$. Thus, for a smooth $n$-dimensional submanifold $B$ in space, representing the image of a body, and a virtual velocity field $w$,

$$\int_B S(j^2w),$$

where $j^2w$ denotes the second jet of $w$, represents the virtual power performed by the stress field.

Only limited analysis can be performed on second order stresses. However, second order stresses may be represented, non-uniquely, by non-holonomic stress. By a non-holonomic stress we refer to a tensor field whose value at a point $x \in S$ is a linear mapping from the iterated jet bundle, $f^1(f^1W)_x$, to $\wedge^2T^*_xS$. Non-holonomic stresses may be treated by repeating the metric

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independent analysis for first order stresses as in \cite{Seg02, Seg13}. As expected, we obtain the surface interactions on the boundaries of regions in space.

2. Notation and Preliminaries

We will use the same scheme of notation as in \cite{Seg13} and we will offer use the same notation for a mapping and variables in the co-domains thereof. Thus, Let $\xi : W \to S$ be a vector bundle. We recall that a section $w : S \to W$ of $\xi$ is represented locally in the form

$$
(x^1, \ldots, x^n) \mapsto (x^1, \ldots, x^n, w^1(x^i), \ldots, w^d(x^i))
$$

(2.1)

where $(x^1, \ldots, x^n)$ is a local coordinate system and a local basis $\{g_1, \ldots, g_d\}$ was used for the fibers of $W$. Let $I = (i_1, \ldots, i_n)$, for non-negative integers $i_j$, be a multi-index and let $|I| = \sum_{j=1}^n i_j$. We use the notation

$$
\frac{\partial^{|I|} w^\alpha}{\partial x^I}(x^i_0) = \frac{\partial^{|I|} w'^\alpha}{\partial x^I}(x^i_0)
$$

(2.2)

Two sections $w$ and $w'$ have the same $k$-jet at $x_0 \in S$ if

$$
\frac{\partial^{|I|} w^\alpha}{\partial x^I}(x^i_0) = \frac{\partial^{|I|} w'^\alpha}{\partial x^I}(x^i_0)
$$

(2.3)

for all $I$ such that $|I| \leq k$ and all $\alpha = 1, \ldots, d$. Clearly, if this condition holds in one vector bundle chart in a neighborhood of $x_0$, it will hold using any other chart and it induces an equivalence relation on the vector space $C^k(\xi)$ of $C^k$-sections of the vector bundle. An equivalence class for this relation is a $k$-jet at $x_0$. Given a section $w$, the jet it induces at $x_0$—the jet of $w$ at $x_0$—will be denoted as $j^k(w)(x_0)$. Given a chart in a neighborhood of $x_0$, $j^k(w)(x_0)$ is represented by

$$
\left\{ w^\alpha_I(x_0) := \frac{\partial^{|I|} w^\alpha}{\partial x^I}(x_0) \mid |I| \leq k, \ \alpha = 1, \ldots, d \right\}.
$$

(2.4)

The collection of all $k$-jets of sections at $x_0 \in S$ is the $k$-jet space of the vector bundle at $x$ and is denoted as $J^k_x W$. Given a point $x_0 \in S$ and an element $w_0 \in W_{x_0}$, the collection of all $k$-jets at $x_0$ such that each jet is represented by a section $w$ with $w(x_0) = w_0$ will be referred to as the $k$-jet space at $w_0$ and will be denoted by $J^k_{w_0} W$. Evidently,

$$
J^k_{x_0} W = \bigcup_{w_0 \in W_{x_0}} J^k_{w_0} W.
$$

(2.5)

The $k$-jet bundle $J^k W$ is the collection of all $k$-jets at the various points in $S$ so that

$$
J^k W = \bigcup_{x \in S} J^k_x W = \bigcup_{w \in W} J^k_w W.
$$

(2.6)
A natural vector bundle structure
\[ \pi^k : J^k W \rightarrow S, \] (2.7)
is available on the jet bundle by which \( \pi^k(A) = x \) if \( A \in J^k_x W \). The linear structure on the fibers is given by \( a_1 A_1 + a_2 A_2 = j^k(a_1 w_1 + a_2 w_2)(x) \), for \( A_1, A_2 \) in \( J^k_x W \), \( a_1, a_2 \in \mathbb{R} \), and representing sections \( w_1 \) and \( w_2 \). Evidently, the result is independent of the choice of representative sections. The fiber \( J^k_x W \) of this vector bundle over \( x \in S \) is isomorphic with \( W_x \oplus \cdots \oplus L^p(T_x S, W_x) \oplus \cdots \oplus L^k(S_x, W_x) \), (2.8)
where \( L^p(S_x, W_x) \) denotes the vector space of \( p \)-multilinear symmetric mappings from \( T_x S \) to \( W_x \). Thus, an element in \( J^k_x W \) is represented locally in the form
\[
(A^{a_0}_{\alpha_0}, A^{a_1}_{\alpha_1}, \ldots, A^{a_p}_{\alpha_p}, \ldots, A^{a_k}_{\alpha_k}) = (A^p_{\alpha_k}),
\] (2.9)
where \( p = 0, \ldots, k \), \( |I_p| = p \), \( \alpha = (a_0, \ldots, a_k) \), \( \alpha_p = 1, \ldots, d \), and evidently, \( A^{a_0}_0 \) represents an element of \( W_x \). Evidently, each section \( w \) of \( W \) induces the section \( j^k w \) of the \( k \)-th jet bundle and if fact we have a linear mapping
\[
j^k : C^p(U) \rightarrow C^0(J^k W),
\] (2.10)
where \( C^p(U) \) represents the vector space of sections of the vector bundle \( U \) of class \( p \). For additional information on jet bundles, some of which will be used in the following sections see [Sau89].

The jet bundle has also a natural projection
\[
\pi^k_p : J^k W \rightarrow J^p W, \quad 0 \leq p \leq k,
\] (2.11)
characterized by \( \pi^k_p(A) = j^p(w)(x) \) where \( x = \pi^k(A) \) and \( w \) is any section of \( W \) that represents \( A \). The mapping \( \pi^k_p \) is a vector bundle morphism over the identity of \( S \).

### 3. Simple Stresses

As a primitive mathematical object pertaining to stress theory for continuum mechanics of order 1 we take the variational stress, a smooth section \( S \) of the vector bundle \( L(J^1 W, \wedge^n T^* S) \) for some vector bundle \( W \rightarrow S \), where \( \wedge^n T^* S \) is the vector bundle of \( n \)-alternating covariant tensors over \( S \). For motivation, see [Seg86, Seg02, Seg13]. In particular, for an \( n \)-dimensional submanifold with boundary \( B \subset S \), one is interested in the linear functional
\[
w \mapsto \int_B S(j^1 w)
\] (3.1)
which is interpreted as the virtual power performed by the variational stress \( S \) for the virtual generalized velocity field \( w \) inside the region \( B \).
Locally, $S$ is represented locally in the form $(x^i, R_{1...dp}, S^k_{1...dp})$ or equivalently

$$
\left( \sum_{\alpha} R_{1...na} (dx^1 \wedge \cdots \wedge dx^n) \otimes g^\alpha, \sum_{i, \alpha} S^i_{1...na} (dx^1 \wedge \cdots \wedge dx^n) \otimes \frac{\partial}{\partial x^i} \otimes g^\alpha \right)
$$

so that $S(j(w))$ is represented locally by

$$
\left( \sum_{\alpha} R_{1...na} w^\alpha + \sum_{i, \alpha} S^i_{1...na} w^\alpha \right) dx^1 \wedge \cdots \wedge dx^n. \tag{3.2}
$$

For a vector bundle $V \to S$, let $\wedge^p(T^*S, V)$ denote the vector bundle over $S$ whose fiber at $x$ is the vector space of $p$-alternating multilinear mappings from $T_xS$ to $V_x$. Consider the isomorphism

$$
tr : \wedge^p(T^*S, V^*) \to L(V, \wedge^n T^*S) \tag{3.3}
$$
defined as follows. For $T \in \wedge^p(T^*S, V^*)$, $T^{tr} = tr(T)$ is given by

$$
T^{tr}(v)(u_1, \ldots, u_p) = T(u_1, \ldots, u_p)(v). \tag{3.4}
$$

Thus, for a variational stress $S$ one may consider $S^* = tr^{-1}(S)$, an $n$-form on $S$ valued in the dual of the jet bundle.

Using the vector bundle morphism $\pi^0 : JW \to W$, consider the vertical sub-bundle $VJ^1W = \text{Kernel } \pi^0$. Since $\pi^0$ is represented locally by $(x^i, u^p, A^q_k)$ $\mapsto (x^i, u^p)$, an element of $VJ^1W$ is represented locally in the form $(x^i, 0, A^q_k)$. In other words, the fiber $VJ^1_{x_0} W$ contains jets of sections of $W$ that vanish at $x_0$. It is noted that there is a natural vector bundle isomorphism $VJ^1W \cong L(TS, W)$ by which an element of $VJ^1W$ is represented in the form $(x^i, A^q_k)$. We use $\iota_V : VJ^1W \hookrightarrow J^1W$ to denote the inclusion of the vertical sub-bundle—a vector bundle morphism over $S$ represented by $(x^i, A^q_k) \mapsto (x^i, 0, A^q_k)$. Then, the dual vector bundle morphism $\iota_{V^*} : (J^1W)^* \to (VJ^1W)^* \cong L(W, TS)$ is a projection represented locally in the form $(x^i, \xi_p, \Xi^q_k) \mapsto (x^i, \Xi^q_k)$—the restriction of $\xi \in (J^1W)^*$ to vertical elements of the jet bundle. Thus, $\iota_{V^*}(\xi)(A), A \in VJ^1W$, is represented by $\sum_{i,q} \Xi^q_k A^q_k$. Similarly, for a section $S$ of $L(J^1W, T^*S)$, $\iota_S := \iota_{V^*} \circ S$, a section of $L(VJ^1W, \wedge^n T^*S)$, is given by $\iota_S(S)(x)(A) = S(x)(\iota_{V^*}(A)) \in \wedge^n T^*_x S$.

The evaluation $\iota_{V^*}(S)(x)(A)$ is represented by $\sum_{k,\alpha} S^k_{1...na} (x) A^k_{\alpha} dx^1 \wedge \cdots \wedge dx^n$ and so $\iota_{V^*}(S)$ is represented in the form

$$
\sum_{k,\alpha} S^k_{1...na} (dx^1 \wedge \cdots \wedge dx^n) \otimes \frac{\partial}{\partial x^k} \otimes g^\alpha. \tag{3.6}
$$

The object $\iota_{V^*}(S)$ is the symbol of the linear differential operator $S$ as defined in [Pal68].
Using the isomorphism $VJ^1W \cong L(TS, W)$, we regard $i_V^*(S)$ as a section of
\[
L(L(TS, W), \wedge^n T^*S) \cong (\wedge^n T^*S) \otimes L(TS, W)^*,
\]
\[
\cong (\wedge^n T^*S) \otimes L(W, TS), \quad (3.7)
\]
\[
\cong (\wedge^n T^*S) \otimes TS \otimes W^*.
\]

It follows that a section of $\wedge^n(T^*S, L(W, TS))$ may be represented locally in the form $\sum_\theta \otimes v_\alpha \otimes \varphi^a$ for an $n$-form $\theta$ and pairs $v_\alpha, \varphi^a$ of sections of $TS$ and $W^*$, respectively. We can use the contraction of the second and first factors in the product to obtain $\sum_\theta(v_\alpha, \varphi^a) \otimes \varphi^a$. Thus, we have a natural mapping
\[
C : L(L(TS, W), \wedge^n T^*S) \rightarrow \wedge^{n-1}(T^*S) \otimes W^*
\]
\[
\cong L(W, \wedge^{n-1} T^*S). \quad (3.8)
\]

The mapping $C$ is represented locally by
\[
\sum_{k,\alpha} S^k_{1...n\alpha} (dx^1 \wedge \cdots \wedge dx^n) \otimes \frac{\partial}{\partial x^k} \otimes g^\alpha \rightarrow \sum_{k,\alpha} S^k_{1...n\alpha} \frac{\partial}{\partial x^k} (dx^1 \wedge \cdots \wedge dx^n) \otimes g^\alpha,
\]
\[
= \sum_{k,\alpha} (-1)^{k-1} S^k_{1...n\alpha} (dx^1 \wedge \cdots \wedge \hat{dx}^k \wedge \cdots \wedge dx^n) \otimes g^\alpha, \quad (3.9)
\]
where a superimposed “hat” indicates the omission of the associated term.

The mapping
\[
p_{\sigma} := C \circ i_V^* : L(L(TS, W), \wedge^n T^*S) \rightarrow L(W, \wedge^{n-1} T^*S)
\]
associates a section $\sigma = p_{\sigma} \circ S$ of $L(W, \wedge^{n-1} T^*S)$ with any variational stress $S$. We refer to a section of $L(W, \wedge^{n-1} T^*S)$ as a traction stress. Such a section is represented locally in the form $(x^1, \sigma_1\ldots\sigma_n(x^1))$ or
\[
\sum_{k,\alpha} \sigma_{1...\hat{\alpha}\ldots n\alpha} (dx^1 \wedge \cdots \wedge \hat{dx}^k \wedge \cdots \wedge dx^n) \otimes g^\alpha. \quad (3.11)
\]

The transposed, $\sigma^T$, is represented by
\[
\sum_{k,\alpha} \sigma_{1...\hat{\alpha}\ldots n\alpha} g^\alpha \otimes (dx^1 \wedge \cdots \wedge \hat{dx}^k \wedge \cdots \wedge dx^n)
\]
\[
= \sum_{k,\alpha} \sigma_{1...\hat{\alpha}\ldots n\alpha} w^a dx^1 \wedge \cdots \wedge \hat{dx}^k \wedge \cdots \wedge dx^n. \quad (3.12)
\]
and $\sigma(w)$ is represented locally by
\[
\sum_{k,\alpha} \sigma_{1...\hat{\alpha}\ldots n\alpha} w^a dx^1 \wedge \cdots \wedge \hat{dx}^k \wedge \cdots \wedge dx^n. \quad (3.13)
\]

We conclude that in case $\sigma = p_{\sigma}(S)$, then,
\[
\sigma_{1...\hat{\alpha}\ldots n\alpha} = (-1)^{k-1} S^k_{1...n\alpha} \quad (3.14)
\].
For each \((n - 1)\)-dimensional oriented submanifold \(P \subset S\), in particular, the boundary \(\partial B\) of submanifold with boundary \(B \subset S\), one may integrate \(\sigma(\omega)\) over \(P\), and evaluate

\[
\int_P \iota_P^*(\sigma(\omega)).
\]  

(3.15)

Here, \(\iota_P : P \to S\) is the natural inclusion so that \(\iota_P^*\) is the restriction of forms. We conclude that \(t_P = \iota_P^* \circ \sigma\) is the surface force induced by \(\sigma\) and the integral above represents the power produced by the traction.

The divergence, \(\text{div} S\), of the variational stress field \(S\) is a section of \(L(W, \Lambda^n(T^* S))\) which is defined invariantly by

\[
\text{div} S(\omega) = d(p_\tau(S)(\omega)) - S(j^1(\omega)),
\]  

(3.16)

for every differentiable vector field \(\omega\). To present the local expression for \(\text{div} S\) we first note that if \(\sigma = p_\tau(S)\), then \(d(\sigma(\omega))\) is represented locally by

\[
\sum_{k,a} d(\sigma_{1...\hat{k}...n}^a \omega^a) \wedge dx^1 \wedge \cdots \wedge \widehat{dx^k} \wedge \cdots \wedge dx^n
\]

\[
= \sum_{k,a} (\sigma_{1...\hat{k}...n}^a \omega^a)_k dx^k \wedge dx^1 \wedge \cdots \wedge \widehat{dx^k} \wedge \cdots \wedge dx^n,
\]

\[
= \sum_{k,a} (\sigma_{1...\hat{k}...n}^a \omega^a)_k (-1)^{k-1} dx^1 \wedge \cdots \wedge dx^n,
\]

(3.17)

\[
= \sum_{k,a} (S_{1...\hat{n}a}^k \omega^a)_k dx^1 \wedge \cdots \wedge dx^n.
\]

Using Equation (3.17), the local expression for \(\text{div} S(\omega)\) is therefore

\[
\sum_{k,p} \left[ (S_{1...n}^a \omega^a)_k - \left( \sum_a R_{1...n}^a \omega^a + \sum_{k,a} S_{1...\hat{n}a}^k \omega^a \right)_k \right] dx^1 \wedge \cdots \wedge dx^n
\]

\[
= \sum_{k,a} (S_{1...\hat{n}a,k} - R_{1...n}^a) \omega^a dx^1 \wedge \cdots \wedge dx^n,
\]  

(3.18)

so that \(\text{div} S\) is represented locally by

\[
\sum_{k,p} (S_{1...\hat{n}a,k} - R_{1...n}^a) \omega^a dx^1 \wedge \cdots \wedge dx^n \otimes g^a.
\]  

(3.19)

It is noted that in the case where \(R_{1...n}^a = 0\) locally, the expression for the divergence reduces to the traditional expression for the divergence of a tensor field in an Euclidean space.

Given a variational stress \(S\), and setting

\[
\mathbf{b} = - \text{div} S,
\]  

(3.20)

for every \(n\)-dimensional submanifold with boundary \(B \subset S\), one has

\[
\int_B \mathbf{b}(\omega) + \int_{\partial B} t_{\partial B}(\omega) = \int_B S(j(\omega))
\]  

(3.21)
which is our generalization of the principle of virtual work.

4. HIGH ORDER STRESSES

For continuum mechanics of order greater than one, the fundamental object we consider is the $k$-th order variational stress which is a smooth section of the vector bundle $L(j^k U, \bigwedge^n T^* S) \cong \bigwedge^n T^* S \otimes (j^k U)^*$, for some vector bundle $U \to S$ whose sections are interpreted as virtual generalized velocities (see [Seg86] and [SD91] for motivation). Thus, the virtual power performed by the $k$-th order variational stress $S$ for the virtual generalized velocity $u$ in a body $B \subset S$ is given by the action of the functional

$$u \mapsto \int_B S(j^k u). \quad (4.1)$$

Observing (2.8), it follows that the fiber, $L(j^k U, \bigwedge^n T^* S)_x$ of $L(j^k U, \bigwedge^n T^* S)$ at $x \in S$ is isomorphic with

$$\bigwedge^n T^* S \otimes \left( U_x^* \oplus L(T_x S, U_x)^* \oplus \cdots \oplus L^n_S(T_x S, U_x)^* \oplus \cdots \oplus L^n_S(T_x S, U_x)^* \right).$$

(4.2)

Let $S^p$ denote the component of $S$ in $\bigwedge^n T^* S \otimes L^n_S(T_x S, W_x)^*$. It follows that the stress may be represented locally in the form $(S^0, S^1, \ldots, S^k)$ and $S^p$ is represented locally by an array in the form $S^p_{1\ldots n\alpha}$, the $I$ is a multi-index with $|I| = p$. The action $S(A)$ for an element $A \in j^k U$ is given by

$$\sum_{|I| \leq k \alpha} S^p_{1\ldots n\alpha} A^p_{I} \, dx^1 \wedge \cdots \wedge dx^n, \quad (4.3)$$

It is our objective to represent the virtual power for high order stresses (4.1) in a form analogous to (3.21). We will concentrate on the case $k = 2$.

5. THE ITERATED JET BUNDLE

Since for any vector bundle $U, W = j^1 U \to S$ is a vector bundle, also, one may consider the vector bundle $j^1 W = j^1(j^1 U) \to S$. As any section $A$ of $j^1 U$ is locally of the form $(u^a(x), A^I_i (x)) = (A_{0\alpha_0}^I(x), A_{1\alpha_1}^I(x))$, an element of the iterated jet bundle is of the form $(x^i, B_{0\alpha_0}^I, B_{1\alpha_1}^I, B_{2\alpha_2}^I, B_{3\alpha_3}^I)$. It is observed that no “compatibility”, or “holonomicity”, is imposed, and for example, $B_{3\alpha_3}^I$ need not be symmetric in the $i_3, i_4$ indices.

It follows that an element $X$ of the dual, $j^1(j^1 U)^*$, of the iterated jet bundle is represented in the form

$$(x^i, X^0_{\alpha_0}, X^1_{\alpha_1}, X^2_{\alpha_2}, X^3_{\alpha_3}) \quad (5.1)$$

whose action on $A \in j^1(j^1 U)_x$ is given by

$$\sum_{a, i, j} X^0_{\alpha_0} B^a_{0\alpha_0} + X^1_{\alpha_1} B^a_{1\alpha_1} + X^2_{\alpha_2} B^a_{2\alpha_2} + X^3_{\alpha_3} B^a_{3\alpha_3} \quad (5.2)$$
It is noted that there is a natural vector bundle inclusion \( \iota : J^2 U \rightarrow J^1(J^1 U) \) such that \( \iota_x : J^2 U_x \rightarrow J^1(J^1 U)_x \) is given as follows. Let \( u \) be a section of \( U \) the represents and element \( B \in J^2 U \). Then, \( w = j^1 u \) is a section of \( J^1 U \) whose jet is the target element in \( J^1(J^1 U) \). Thus,

\[
\iota_x : (u^0, u^1, u^2) \longmapsto (u^0, u^1, u^2, u^3). \tag{5.3}
\]

Evidently, the result is independent of the section chosen and locally the inclusion is in the form

\[
(A^{0\alpha}, A^{1\alpha_1}_i, A^{2\alpha_2}_{ij}) \longmapsto (A^{0\alpha}, A^{1\alpha_1}_i, A^{1\alpha_2}_i, A^{2\alpha_2}_{ij}). \tag{5.4}
\]

Thus, the image of \( \iota \) contains elements for which the second and third groups of components are identical and the components in the fourth group are symmetric.

Since the inclusion \( \iota \) is injective, the dual \( \iota^* : J^1(J^1 U)^* \rightarrow (J^2 U)^* \) is therefore surjective and it satisfies

\[
\iota^*(X)(A) = X(\iota A)
\]

\[
= \sum \alpha X^0_\alpha A^{0\alpha} + \sum_{\alpha,i} (X^{1i}_\alpha + X^{2i}_\alpha) A^{1\alpha}_i + \sum_{\alpha} X^{3ij}_\alpha A^{2\alpha}_{ij}, \tag{5.5}
\]

\[
= \sum \alpha X^0_\alpha A^{0\alpha} + \sum_{\alpha,i} (X^{1i}_\alpha + X^{2i}_\alpha) A^{1\alpha}_i + \sum_{\alpha} \frac{1}{2}(X^{3ij}_\alpha + X^{3ji}_\alpha) A^{2\alpha}_{ij}.
\]

The dual is therefore a restriction represented locally by

\[
(x^i, X^0_\alpha X^{1i}_\alpha, X^{2i}_\alpha, X^{3ij}_\alpha) \longmapsto (x^i, X^0_\alpha X^{1i}_\alpha, X^{2i}_\alpha, \frac{1}{2}(X^{3ij}_\alpha + X^{3ji}_\alpha)) \tag{5.6}
\]

and being surjective, every second order stress \( S \) is of the form \( S = \iota^* X \) for some non-unique section \( X \) of \( J^1(J^1 U) \). Thus, whatever properties we deduce for elements of \( J^1(J^1 U) \) will hold for their restriction to Image \( \iota \).

It is observed finally (see [Sau89, p. 169]) that there is no natural inverse to \( \iota \), i.e., a projection \( J^1(J^1 U) \rightarrow J^2 U \).

### 6. Non-Holonomic Stresses

As elements of \((J^1(J^1 U))^*\) represent elements of \((J^2 U)^*\) using the surjective mapping \( \iota^* \), every second order stress \( S \), a section of \( L(J^2 U, \wedge^n T^* S) \cong \wedge^n T^* S \otimes (J^2 U)^* \), may be represented by a section \( Y \) of \( L(J^1(J^1 U), \wedge^n T^* S) \cong \wedge^n T^* S \otimes (J^1(J^1 U))^* \) in the form \( S(j^2 u) = Y(\iota \circ j^2 u) \). We will refer to such a section \( Y \) as a non-holonomic stress. The second order stress \( S \) induced by the non-holonomic stress \( Y \) may therefore be written as

\[
S = \iota^* Y. \tag{6.1}
\]

We thus concentrate our attention in this section to analysis of the action of non-holonomic stresses in the form

\[
u \longmapsto \int_B Y(j^1(A)) \tag{6.2}
\]
for a section $A$ of $J^1U$, and in particular, the compatible case where $A = J^1u$, for a section $u$ of $U$.

Since $J^1U \to S$ is a vector bundle, we may apply to it all the analysis described in Section 3 by substituting $W = J^1U$. Using the notation of Section 5, a non-holonomic stress may be represented locally in the form

$$
(dx^1 \wedge \cdots \wedge dx^n) \otimes \left( \sum_{a} Y^0_{1...na} g^a, \sum_{a,i} Y^1_{1...na} \frac{\partial}{\partial x^i} \otimes g^a, \sum_{a,i} Y^2_{1...na} \frac{\partial}{\partial x^i} \otimes g^a, \ldots \right),
$$

where we have omitted the indication of the dependence of the various fields on $x \in S$. Thus, the action $Y(B), B \in J^1(J^1U)$ is given by

$$
\left( \sum_{a,i,j} (Y^0_{1...na} B^0_{ia} + Y^1_{1...na} B^1_{ia} + Y^2_{1...na} B^2_{ia} + Y^3_{1...na} B^3_{ia}) \right) dx^1 \wedge \cdots \wedge dx^n
$$

and for the jet $j^1A$ of a section $A$ of $J^1U$,

$$
\left( \sum_{a,i,j} (Y^0_{1...na} A^0_{ia} + Y^1_{1...na} A^1_{ia} + Y^2_{1...na} A^2_{ia} + Y^3_{1...na} A^3_{ia}) \right) dx^1 \wedge \cdots \wedge dx^n.
$$

It is observed that in the last expressions the components $Y^0_{1...na}$ and $Y^1_{1...na}$ assume the roles of the components $R_{1...na}$ in (3.3) and the components $Y^2_{1...na}$ together with $Y^3_{1...na}$ assume the roles of $S^i_{1...na}$.

Using the operator $p_c$ as in (3.10), one can extract a section $Z = p_c Y$ of the vector bundle $L(J^1U, \Lambda^{n-1}T^*S)$. The local representation of $Z$ is of the form

$$
\sum_{k,\alpha} (dx^1 \wedge \cdots \wedge dx^k \wedge \cdots \wedge dx^n) \otimes \left( Z^0_{1...k...na} \otimes g^a, Z^1_{1...k...na} \frac{\partial}{\partial x^i} \otimes g^a, \ldots \right)
$$

From (3.14) it follows that

$$
Z^0_{1...k...na} = (-1)^{k-1} Y^2_{1...na}, \quad Z^1_{1...k...na} = (-1)^{k-1} Y^3_{1...na}.
$$

Equation (3.16) assumes the form

$$
\text{div } Y(A) = d (p_c(Y)(A)) - Y(j^1A),
$$

in which $\text{div } Y$, a section of $L(J^1U, \Lambda^n(T^*S))$ is represented locally by

$$
(\omega^1 \wedge \cdots \wedge \omega^n) \otimes \sum_{a,i,j} \left( Y^2_{1...na,j} - Y^0_{1...na} + (Y^3_{1...na,j} - Y^1_{1...na}) \frac{\partial}{\partial x^i} \right) \otimes g^a.
$$

One conclude that for $Z = p_c(Y),

$$
\int_B Y(j^1A) = \int_B d(Z(A)) - \int_B \text{div } Y(A) = \int_{\partial B} Z(A) - \int_B \text{div } Y(A).
$$
For the case where \( A = j^1 u \) for a section \( u \) of \( U \),
\[
Y(j^1(j^1 u)) = d \left( p_\sigma(Y) (j^1 u) \right) - \text{div} \ Y(j^1 u), \tag{6.11}
\]
and
\[
\int_B Y(j^1(j^1 u)) = \int_B d(Z(j^1 u)) - \int_B \text{div} \ Y(j^1 u) = \int_{\partial B} Z(j^1 u) - \int_B \text{div} \ Y(j^1 u). \tag{6.12}
\]

Similarly to a simple variational stress, \( \text{div} \ Y \) is a section of \( L(j^1U, \wedge^n (T^* S)) \), we may apply the definition of the generalized divergence (3.16) to it (substituting \( \text{div} \ Y \) for \( S \)) and so
\[
\text{div} \ Y(j^1 u) = d(p_\sigma(\text{div} \ Y)(u)) - \text{div} \ (\text{div} \ Y)(u). \tag{6.13}
\]
Here, similarly to a traction stress \( p_\sigma(\text{div} \ Y) \) is a section of \( L(U, \wedge^{n-1} T^* S) \) and \( \text{div} \ (\text{div} \ Y) \) is a section of \( L(U, \wedge^n (T^* S)) \), similarly to a body force. Thus,
\[
\int_B Y(j^1(j^1 u)) = \int_{\partial B} Z(j^1 u) - \int_B d(p_\sigma(\text{div} \ Y)(u)) + \int_B \text{div} \ (\text{div} \ Y)(u),
\]
\[
= \int_{\partial B} Z(j^1 u) - \int_{\partial B} p_\sigma(\text{div} \ Y)(u) + \int_B \text{div} \ (\text{div} \ Y)(u). \tag{6.14}
\]

We note that in the first integral \( Z \) is a section of \( L(j^1U, \wedge^{n-1} T^* S) \), and so it plays the role of a variational stress on the \((n-1)\)-dimensional manifold \( \partial B \). One may therefore use the definition of the generalized divergence to obtain
\[
Z(j^1 u) = d(p_\sigma(Z)(u)) - \text{div} \ Z(u) \tag{6.15}
\]
where \( p_\sigma(Z) \) is a section of \( L(U, \wedge^{n-2} T^* \partial B) \) and \( \text{div} \ Z \) is a section of \( L(U, \wedge^n (\partial B)) \).

In other words, \( p_\sigma(Z) \) is a surface stress as one would expect in second order continuum mechanics. We conclude that
\[
\int_B Y(j^1(j^1 u)) = \int_{\partial B} d(p_\sigma(Z)(u)) - \int_{\partial B} \text{div} \ Z(u)
\]
\[
- \int_{\partial B} p_\sigma(\text{div} \ Y)(u) + \int_B \text{div} \ (\text{div} \ Y)(u),
\tag{6.16}
\]
\[
= \int_{\partial B} p_\sigma(Z)(u) - \int_{\partial B} \text{div} \ Z(u)
\]
\[
- \int_{\partial B} p_\sigma(\text{div} \ Y)(u) + \int_B \text{div} \ (\text{div} \ Y)(u),
\]
As \( \partial (\partial B) = 0 \), it follows that
\[
\int_B Y(j^1(j^1 u)) = + \int_B \text{div} \ (\text{div} \ Y)(u) - \int_{\partial B} (\text{div} p_\sigma(Y))(u) - \int_{\partial B} p_\sigma(\text{div} Y)(u). \tag{6.17}
\]
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