Discrete and continuous approaches for the failure analysis of masonry structures subjected to settlements

Marco Pepe\textsuperscript{1,*}, Marialuigia Sangirardi\textsuperscript{2}, Emanuele Reccia\textsuperscript{3}, Marco Pingaro\textsuperscript{1}, Patrizia Trovalusci\textsuperscript{1} and Gianmarco de Felice\textsuperscript{2}

\textsuperscript{1} Department of Structural and Geotechnical Engineering, Sapienza University of Rome, Rome, Italy  
\textsuperscript{2} Department of Engineering, Roma Tre University, Rome, Italy  
\textsuperscript{3} Department of Civil and Environmental Engineering and Architecture, University of Cagliari, Cagliari, Italy

Correspondence*:
Corresponding Author
marco.pepe@uniroma1.it

ABSTRACT
Numerical modelling of masonry structures is nowadays still an active research field, given a number of open issues related to preservation and restoration of historical constructions and the availability of computational tools that have become more and more refined. This work focuses on the analysis of settlement-induced failure patterns characterizing the in plane response of two-dimensional dry-joints masonry panels, which differ in terms of texture, geometry and settlement configuration. Brick-block masonry, interpreted as a jointed assembly of prismatic particles in dry contact, can be modelled as a discrete system of rigid blocks interacting through contact surfaces with no tensile strength and finite friction, modelled as zero thickness elasto-plastic Mohr-Coulomb interfaces. Different approaches and numerical models are adopted herein: Limit Analysis (LA), a discrete model DEM and a continuous Finite Element Model (FEM). Limit Analysis is able to provide fast and reliable results in terms of collapse multiplier and relative kinematics. In this work, a standard LA procedure is coded through Linearised Mathematical Programming to take into account sliding mechanisms through dilatant joints. Discrete models are particularly suitable to study historical masonry materials, where rigid bodies interacts between contact and friction. Here, a combined Finite/Discrete Element approach (FEM/DEM) is adopted. Finally, analyses are conducted through the Finite Element approach, resorting to a continuum anisotropic elastic-perfectly plastic constitutive model. Some selected case-studies have been investigated adopting the above mentioned models and numerical results have been interpreted to highlight the capability of the approaches to predict failure patterns for various geometrical features of the structure and settlement configurations.

Keywords: Limit Analysis, FEM/DEM, Finite Element, Masonry Structures, Rigid Blocks, Settlements.

1 INTRODUCTION
Masonry is one of the most ancient structural material and constitutes a vast majority of the World’s architectural heritage. It is a composite and heterogeneous medium, resulting from the assemblage of natural or artificial blocks by means of mortar layers or dry joints. Being characterized by an internal
structure, which reflects in a complex mechanical response, masonry and its constitutive behaviour still represent a challenging research field. Thanks to the availability of more and more powerful computational resources, during last decades a large number of numerical applications have been developed, resorting to different constitutive assumptions and solution algorithms. Nonetheless, it is not possible to state that each of these models suits any structural problem, but their applicability needs to be evaluated case by case, on the basis of geometrical features, extent of the structure and boundary conditions. Among the available numerical modelling techniques for masonry structures, a broad distinction can be made between micromechanical, macromechanical and multiscale models.

According to micromodeling strategy, the constituents, that is units, mortar (if present) and unit/mortar interfaces, are separately modelled and each part is assigned a properly calibrated constitutive law. This approach is particularly suitable if the response of the assemblage needs to be accurately described (Lotfi and Shing (1994); Lourenço and Rots (1997); Oliveira and Lourenço (2004); Alfano and Sacco (2006); Serpieri et al. (2017)), but a major hindrance is still represented by its high computational cost, consequence of the large number of degrees of freedom needed to describe the structural configuration (Clementi et al., 2019, 2020). In addition, the adopted constitutive assumptions often imply the calibration of a large set of parameters which are not always easily determined.

Following a macromechanical approach, the heterogeneous medium is modelled as a continuum and the constitutive behaviour is usually described through phenomenologically based mathematical relations, in which degrading phenomena are accounted through damage or friction variables. In this case, macroscopic mechanical properties are more easily derived from standard experimental tests on small masonry specimens. These models are, if compared to micromechanical ones, more efficient from a computational point of view (Del Piero, 1989; Gambarotta and Lagomarsino, 1997; Roca et al., 2005; Sangirardi et al., 2019a) and are widely used for real-scale applications in which, depending on the complexity of the geometry of the structure, different discretization strategies might result in the adoption of monodimensional, bidimensional or three-dimensional finite elements.

Multiscale, i.e. micro-macro, continuum models represent a very promising approach for the analysis of masonry structures since they can accurately keep track of the the mechanical and geometrical properties of the material at the microstructure with a reduced computational cost if compared to a fully micromechanical model (Masiani and Trovalusci, 1996; Trovalusci et al., 2010; Leonetti et al., 2018; Reccia et al., 2018). These models are often derived by considering two material scales: a microscale where, having deduced the mechanical properties of the components through experimental tests, a material representative volume element (RVE) is defined and a macroscale structural level, where a homogeneous continuum is obtained by performing a homogenization procedure based on the solution of boundary conditions problems for the RVE (Addessi et al., 2018, 2016; Greco et al., 2016, 2017). Other multiscale strategies make use of different homogenization techniques based on the so-called Cauchy rule and its generalizations (Capecchi et al., 2011) allowing the derivation of generalized continua such as micropolar continua able to properly represent scale effects, that in masonry materials are significant (Masiani and Trovalusci, 1996; Trovalusci and Masiani, 2003; Pau and Trovalusci, 2012; Fantuzzi et al., 2019; Leonetti et al., 2019).

Due to their characteristics, masonry constructions have proven to be particularly vulnerable not only to earthquakes but also to structural settlements. Cracking and damage, in fact, often occur as a consequence of ground movements. In urban areas, this phenomenon is related to the realization of underground infrastructures or, more in general, to anthropic triggering factors, while natural hazards (as for example slow-moving landslides, liquefaction or consolidation processes) are more likely to interfere with masonry constructions in rural areas. In both cases the understanding of the phenomenon ad its description are
fundamental to identify the causes as well as to prevent the effects with appropriate protection measures, for modern and, moreover, for historical constructions. Several approaches for the prediction of settlement-induced damage may be found in the literature, that consider masonry either as an assembly of discrete blocks (De Jong et al., 2016; Portioli and Cascini, 2016) or as a continuum medium (Burd et al., 2000; Amorosi et al., 2014). The comparison between discrete and continuous models is also a widespread topic (Casalegno et al., 2013; Zampieri et al., 2019; Landolfo et al., 2020).

In this work, three modelling techniques are adopted to describe settlement induced crack patterns in masonry panels characterized by different geometrical configurations and boundary conditions, reproducing the ground movement as a downward moving rigid block. Limit Analysis, largely recognized as a very effective tool to estimate collapse load and collapse mechanisms for masonry structures (Baggio and Trovalusci, 2000; Milani, 2011; Portioli et al., 2014; Pepe et al., 2019b; Milani and Taliercio, 2016; Pavlovic et al., 2016; Cascini et al., 2018), is used to determine the failure configuration of dry joints masonry panels subjected to settlements, modelling the walls (according to the microscale approach) as an assemblage of rigid blocks in contact through frictional interfaces.

The results of the analyses, performed on panels, walls with openings and facades, are compared with the ones obtained through a FEM/DEM approach (Baraldi et al., 2018) and with those obtained by means of a continuum (macroscale) Finite Element approach, adopting an elasto-plastic anisotropic constitutive model (Lasciarrea et al., 2019; Sangirardi et al., 2019b). In the following, the main features of the models are recalled in Section 2, the selected case-studies and the results of the analyses are then presented in Section 3 and critically compared in order to highlight the influence of walls aspect ratio, width of the settling area and presence of openings. Finally in Section 4 some remarks and future developments are reported.

2 ADOPTEO MICROMODELS

2.1 Rigid block model for limit analysis

The first adopted model is framed within the Limit Analysis (LA) theory, taking into account the presence of friction. The model considers as a system of $n$ rigid blocks directly interacting through $m$ contact surfaces unable to carry tension and resistant to sliding by friction. The blocks can translate and rotate about the edges of the contact surfaces (hinging) as well as sliding along the joints.

In order to provide the mechanical details of the model, let consider two simple blocks represented in Figure 1, introducing $e_i = \{e_1, e_2, e_3\}^T$ the orthonormal basis in the three-dimensional space. Loads are applied to the centroid of each rigid block $i-$th: static ‘dead’ loads are collected in vector $f^i_0 = \{f^i_{01}, f^i_{02}, m^i_{03}\}^T$, live loads are collected in the vector $f^i_L = \{f^i_{L1}, f^i_{L2}, m^i_{L3}\}^T$. For the whole structure it results $f_0 = \{f^i_0\}$ and $f_L = \{f^i_L\}$, with $i = 1, \ldots, n$. The vector of the load over the whole system is $f = f_0 + \alpha f_L$, where live loads are proportional to the dead loads through a non-negative coefficient, $\alpha$, called collapse multiplier. Let $u^i = \{u^i_1, u^i_2, \theta^i_3\}$ denote the vector of generalized displacement of the centroid of each $i-$th block. The vector $u = \{u^i\}$, with $i = 1, \ldots, n$, collects the displacement for the whole structure which correspond in a virtual work sense to loads $f$.

The static variables are the internal forces acting at each $j-$th contact surface between blocks, that is the normal force $N^j$, the shear force $T^j$ and the moment $M^j$. For each joint they are collected in vector $\sigma^j = \{N^j, T^j, M^j\}^T$. The vector $\sigma = \{\sigma^j\}$, with $j = 1, \ldots, m$, refers to the whole structure.

The kinematic variables, or generalized strain, are the relative displacement rates at joints, that is normal displacement $\xi^j$, tangential displacement $\gamma^j$ and rotation $\chi^j$. For each joint $j = 1, \ldots, m$ they are collected
in the vector $\varepsilon^j = \{\xi^j, \gamma^j, \chi^j\}^T$. The vector $\varepsilon = \{\varepsilon^j\}$ refers to the whole structure and corresponds in a virtual work sense to the vector of static variables $\sigma$.

Figure 1. a) Simple two-blocks structure b) Displacement components for a rigid block c) Model for settlement: panel with fictitious block

The kinematic compatibility for the whole system is expressed by equation

$$\varepsilon = Bu,$$

(1)

where $B$ represents the compatibility matrix defined in [Baggio and Trovalusci, 2000].

The equilibrium of the whole structure is defined by the equation

$$B^T \sigma + f = 0.$$

(2)

The generalized yield domain of the system can be written as

$$y = N^T \sigma \leq 0,$$

(3)

where $N$ is the block-diagonal gradient matrix referred to the adopted failure surface.

The flow rule expresses the vector $\varepsilon$ as a linear combination of non-negative coefficients $\lambda$, called inelastic multiplier, and it can be written as

$$\varepsilon = M \lambda.$$

(4)

The plastic behavior of contact surfaces is defined through the complementarity condition

$$\lambda^T y = 0.$$

(5)
Furthermore, the non-negative work of the live loads which cause the collapse mechanism is defined by the following equation

\[ f_L^T u = 1. \]  

(6)

In (Baggio and Trovalusci 2000), the authors presented a home-made code, ALMA (Analisi Limite Murature Attritive) based on a two-step procedure, to deal with the non-linear and non-convex programming problem (NLNCP) related to the presence of frictional interfaces (non-standard LA) of bidimensional and threedimensional block masonry structures.

As reported in (Pepe et al., 2019a,b), following the approach in (Baggio and Trovalusci, 2000), a new version of the code, ALMA 2.0, was implemented using MATLAB® for linear optimization and a Python™ interface for pre and post processing operations. In particular, this new version is based on the kinematic approach of Limit Analysis and considers for sliding a linearized behaviour of joints. This aspect is significant because the computational effort due to the solution of the NLNCP is avoided. Indeed, the advantage of a linear mathematical programming technique, deriving by the presence of the dilatant behaviour of the contact surfaces, can provide a unique and quite fast solution of the problem.

Here, the optimization problem presented in (Pepe et al., 2019b) has been modified to include the presence of settlements into the model (Pepe, 2020). A preliminary modification of the model has been the introduction of the possibility to add kinematic constraints to the block of the structure. The displacement components of every points of a generic \( i \)th block have been expressed as function of the displacement components of its center of gravity. Let consider a generic point \( A \) of the \( i \)th block shown in Figure 1b.

Equation of rigid motion, describing its horizontal, vertical and rotation movement are reported as follows

\[ \hat{u}_A^1 = u_G^1 - \theta_G^3 h_{v,i}, \]
\[ \hat{u}_A^2 = u_G^2 + \theta_G^3 h_{o,i}, \]
\[ \hat{\theta}_A^3 = \theta_G^3, \]

(7)

assumed in a compact form as

\[ \hat{\mathbf{u}} = \mathbf{V} \mathbf{u}, \]

(8)

where, for the whole structure, the matrix \( \mathbf{V} \) contains the geometrical information of the points where the kinematical constraints are inserted. In particular, the components of displacement \( \hat{\mathbf{u}} \) became identically null depending on the typology of the external constraint considered. The modified programming problem, considering the expression of \( \mathbf{u} \) wrote as function of the inelastic multiplier \( \lambda \), is represented by

\[ \min \alpha = -\lambda^T (A_0N_1)^T f_0 \text{ subjected to } \]
\[ (AN_1 - N_2)\lambda = 0, \]
\[ \lambda^T (A_0N_1)^T f_L - 1 = 0, \]
\[ \mathbf{V}A_0N_1\lambda = 0, \]
\[ \lambda \geq 0, \]

(9)

where introducing \( B_1 \), that is the kinematical submatrix of maximum rank and \( B_2 \) the rest of the kinematical matrix, the matrix \( A_0 \) is the inverse of \( B_1 \). The matrix is defined as \( A = B_2B_1^{-1} \) and \( N \) is the transpose of block-diagonal gradient matrix (\( N = [N_1, N_2] \)).
Following the idea of (Portioli and Cascini, 2016), in order to introduce a local foundation settlement into the model, the mathematical and geometrical formulation of the model has been modified with the addition of a fictitious rigid block, with degrees of freedom associated to the imposed movement. In Figure 1c: an example of the modified geometric model for a simple masonry panel is shown, indicating in yellow the fictitious rigid block.

Another difference introduced into the model concerns the definition of the loads applied to the blocks. Indeed, in that case, every block of the structure is subjected only to its dead load, while live load is applied only on the block that simulate the settlement. In details, for that block, dead load is an upward force assumed proportional to an admissible base reaction without foundation settlement, considering a uniform distribution of vertical loads in the structure and denoted as \( f_{(0,r)} \) while live load \( f_L \) is a downward force equal to dead load \( f_{(0,r)} \) and proportional to the collapse multiplier \( \alpha \). Figure 1c shows the loading condition for the support block. The optimization problem has been changed taking advantage from the previously modification developed to include kinematic constraints. Indeed the displacement components of any point of the fictitious block \( \hat{u}_s \) are related to displacement components of the centroid \( u_s \) by means a matrix \( S \) that plays the same role of matrix \( V \) introduced for kinematic constraints

\[
\hat{u}_s = S u_s. \tag{10}
\]

It is consequently possible to particularize the movement of the support block \( u_s \) using the components described by the vector \( \hat{u}_s \). Indeed, depending on the desired typology of settlement, some of its components could be imposed identically null. If for example the fictitious block have to move vertically, user have to put equal to zero only the horizontal and rotation components. The modified programming problem, considering the expression of \( u_s \) wrote as function of the inelastic multiplier \( \lambda \), is represented by Eq. 9

\[
\min \alpha = -\lambda^T (A_0 N_1)^T f_0 \quad \text{subjected to} \\
(A N_1 - N_2) \lambda = 0, \\
\lambda^T (A_0 N_1)^T f_L - 1 = 0, \\
S A_0 N_1 \lambda = 0, \\
\lambda \geq 0. \tag{11}
\]

2.2 Combined Finite/Discrete Element Model

A discrete model, made by means of a combined Finite-Discrete Element Model (FEM/DEM) approach, is here adopted.

Discrete Element Models (DEM) (Cundall, 1988) are a specific class of discrete models in which distinct elements can move independently, can come in or loose contact with other elements (the contact detection is governed by a molecular algorithm) and large displacements are considered. (Cundall and Hart, 1992). DEM are suitable to study non-linear problems characterized by the mutual movement of rigid bodies interacting by means of both contact and friction, such as jointed rock and granular assemblies (Cundall and Strack, 1979), and recently have been adopted for masonry modelling (Lemos, 2007; Cecchi and Sab, 2004, 2009; Baraldi et al., 2015a).

In order to describe the deformability of the elements, simple FE discretizations have been proposed since the beginning of the development of DE Method (Cundall et al., 1985). Here, the combined FEM/DEM approach proposed by Munjiza (Munjiza, 2004) and developed by the Toronto Geo Group (Mahabadi...
et al., 2010) is adopted. The approach relies in a combination of FEM and DEM: DEs are meshed into FEs with embedded crack elements that activate whenever the peak strength is reached. In this way, elastic deformation in the continuum is accounted by FEs, while interaction, fracture and fragmentation processes are modelled by DEs.

The FEM/DEM approach here adopted has been successfully adopted for masonry structures by some of the authors (Reccia et al., 2012; Baraldi et al., 2013, 2015b; Reccia et al., 2016; Baraldi et al., 2018; Reccia et al., 2018; Baraldi et al., 2019; Pepe et al., 2019a,b) and by other research groups (Smoljanović et al., 2013, 2015, 2017).

Numerical analyses are performed through the open source computer codes Y2D/Y-GUI (Mahabadi et al., 2010) and Y-Geo (Mahabadi et al., 2012), while input and results have been processed by means of CAD, ad-hoc MATLAB® scripts and spreadsheets.

A mesh of triangular CST FEs is made to model the specimens, under the hypothesis of plane stress. Masonry is modelled as an assemblage of rigid bricks, adopting a very high value of Young’s Modulus and avoiding cracks inside the. Cracking may occur only in the joints between the bricks, modelled as zero-thickness Mohr-Coulomb interfaces, with only friction and no cohesion.

2.3 FEM

The third approach adopted to analyse failure mechanisms characterizing the response to settlements of masonry constructions is a continuum finite element (FE) one. The constitutive model, implemented in the FE code PLAXIS 3D®, is a three-dimensional anisotropic elastic-perfectly plastic one. It stems from the Jointed Rock Model, and it has been enriched considering block aspect ratio and staggering joints effects. The Jointed Masonry Model (Lasciarrea et al., 2019) is characterized by isotropic elasticity and anisotropic yielding and can be included in the class of multi-laminates models (Pietruszczak and Niu, 1992). Macroscopic elastic properties of the continuum are derived from the joints and blocks ones through a homogenization procedure (De Buhan and de Felice, 1997; de Felice et al., 2010) and an equivalent isotropic behaviour can also be assumed assigning the material an average elastic modulus $E$ as in the presented case-studies. A set of (maximum) three sliding directions, on which failure is meant to occur, is defined in the $xyz$ space and described by means of dip ($\alpha_1$) and strike ($\alpha_2$). These parameters represent, for each plane, the positive rotation along the $x$-axis and the negative rotation along the $z$-axis respectively. In case of masonry panels with regular texture, these angles can be easily defined according to Figure 2. In the proposed examples, only two planes (head and bed joints) are activated, while a third plane might be considered in case of walls with double facing. Yield functions are defined, for each orientation, in terms of local stress components according to Coulomb’s and tensile criterion as follows:

$$f_i^C = |\tau_i| + \sigma_{n,i} \tan \phi_i - c_i$$
$$f_i^T = \sigma_{n,i} - \sigma_{t,i}$$

where $i = 1,2,3$ is the plane id, $\sigma_{n,i}$ and $\tau_i$ are the normal and the shear stress along each orientation, $\phi_i$ is the friction angle, $c_i$ is the cohesion and $\sigma_{t,i}$ is the tensile strength along the joints. The interlocking effect is accounted by modifying the strength parameters on the head-joints plane, stemming from equilibrium conditions and considering the aspect ratio of the blocks through the parameter $\beta$, which also depends on the friction angle of the bed joints:

$$\beta = \tan \phi_2 \frac{b}{2a}$$
Figure 2. JMM bed and head joint plane orientation.

Tensile strength and cohesion on the head joints are hence:

\[
\sigma_{t,1} = \sigma_{t0,1} - \beta \sigma_{n,2} + c_0,2 \frac{\beta}{\tan \phi_2}
\]

\[
c_1 = c_{0,1} - \left( \beta \sigma_{n,2} - c_{0,2} \frac{\beta}{\tan \phi_2} \right) \tan \phi_1
\]

and the modified strength criterion is reported in Figure 3. Table 1 reports the parameters adopted in the analyses.

Figure 3. Modified Mohr-Coulomb criterion.

3 NUMERICAL ANALYSES

The models have been validated using as benchmark the experimental and numerical results obtained by Portioli and Cascini (2016) who studied the collapse of different wall masonry panels made of dry jointed tuff blocks subjected to settlements. The test set-up has been designed ad-hoc, allowing a downward displacement of a portion of the structure, being the rest of the wall simply supported. Block dimensions...
Table 1. Model parameters, 2D dry masonry walls subjected to settlements.

| Test-id          | $G$       | $\nu$ | $\beta$ | $c_{0,i}$ | $\phi_i$ | $\psi_i$ | $\sigma_{t,i}$ | $\gamma$ |
|------------------|-----------|-------|---------|-----------|---------|---------|----------------|--------|
| Test 8C-12C      | 477 MPa   | 0.12  | 0.8     | 0 MPa     | 21.8°   | 21.8°   | 0 MPa          | 12 kN/m³ |
| Panel 1-5, Facade 1-3 | 511 MPa   | 0.12  | 0.4     | 0 MPa     | 21.8°   | 21.8°   | 0 MPa          | 12 kN/m³ |

are equal to 100×200×50 mm³, the two walls are 1100 mm wide, 100 mm thick and differ in terms of height, since Test 8C is made of eight courses resulting in 400 mm height, while Test 12C is made of 12 courses (600 mm height).

Figure 4. a) Experimental and b) numerical results for Test 8C. Modified from (Portioli and Cascini, 2016).

Figure 5. a) LA, b) FEM/DEM and c) FEM results for Test 8C.

Figure 4 shows the experimental and numerical benchmark results producing a mechanism characterized by three macro-blocks: a first one, which behaves as a rigid block, at the left bottom side of the panel; a central one that rotates around the outer right vertex of the first macro-block, in which the blocks are subjected to sliding and rocking; a third macro-block that is separated by the others through a ‘stair-stepped’ path and lies on the moving support. Figure 5 presents the collapse mechanism obtained with the different models referring to Test 8C. Figure 5a) and Figure 5b) present the mechanism of collapse obtained with LA and FEM/DEM. Both mechanisms agree with experimental and numerical results obtained by Portioli, producing a collapse where the three macro-blocks previously described can be easily identified. Figure 5c) reports the distribution of the plastic points (blue points indicate tensile failure while grey points localize shear failure points). According to the typically observed crack patterns in case of long settlement (Mastrodicasa, 1958), tensile failure points are concentrated on the top part of the wall, while shear failure point are located along a 45° oriented direction going through the panel.
Figure 6. a) Experimental and b) numerical results for Test 12C. Modified from (Portioli and Cascini, 2016).

Figure 7. a) LA, b) FEM/DEM and c) FEM results for Test 12C.

Figure 6 shows the experimental and numerical benchmark results producing a mechanism characterized by two macro-blocks separated by a ‘stair-stepped’ crack: the first macro-block, which is supported by the fixed base, and the second one, translating downwards on the movable support. Figure 7 presents the failure mechanisms obtained with the different models for Test 12C. Figure 7a) and 7b) present the mechanism obtained with LA and FEM/DEM which result in perfect accordance with experimental as well as numerical results obtained by the author. Indeed also in this case, the two macro-blocks previously described are easily distinguishable. Figure 7c) refers to continuum FEM modelling. The plastic point distribution is the one at failure, i.e. once the vertical reaction at the base of the moving support is constant, and if compared to the Test 8C case, less tensile plastic points can be observed. Nonetheless it has to be remarked that the adopted continuum approach allows to follow the entire settlement process and its evolution. In case of shorter settlement, shear failure first appears at the lower right corner of the wall, and the upper part of the panel is involved only in the last stages of the analysis. Conversely in case of Panel 8C the first plastic points to appear were the tensile ones in the upper portion of the specimen.

3.1 Square panels with opening

After this first validation, some geometries of masonry walls with openings or façades are analysed under settlement conditions involving a portion of the structure.

Two identical dry joints masonry walls, named Panel 1 and Panel 2, characterized by the presence of an opening are considered. The panels are constituted by 55 blocks having dimensions $50 \times 100 \times 50$ mm$^3$ and are 600 mm wide, 50 mm thick and 550 mm high; opening dimensions are $300 \times 250$ mm$^2$. They differ in terms of length of the settling area, namely $200 \times 50$ mm$^2$ for Panel 1 and $300 \times 50$ mm$^2$ for Panel 2; the effect of this feature is here investigated with the three approaches. Figure 8 presents the collapse mechanism obtained with the different models, referring to Panel 1. LA, Figure 8a), and FEM/DEM, 8b), show a fragile behavior, with the macro-block (1) cracking into several parts some rotating other sliding,
the portion upon the architrave (2) slides while the macro-block upon the fictitious block (3) follows its  
downward movement cracking into several portions, which rotate and slide. Figure 8c) reports the plastic points  
distribution at collapse obtained with the FEM approach. Tensile failure points are located at the top of the  
panel and at the lower right corner of the architrave, which is here modelled as an elastic element. The  
configuration described by this plastic point distribution is in good agreement with the results of the LA  
and FEM/DEM analysis.

Figure 9 presents the collapse mechanism of Panel 2, obtained with the different models.  
LA and FEM/DEM, Figure 9a) and 9b), exhibit the formation of three distinct macro-blocks divided by ‘stair- 
stepped’ cracks: a first one remains stable (1), a second one represented by the portion upon the architrave  
rotates around the upper right corner of the first block and a third portion of the wall (3) follows, without  
crashing, the downward movement of the fictitious block.  
FEM results, presented in terms of plastic points distribution (Figure 9c), show that the overall collapse  
mechanism is affected by the length of the settlement only in the supported part of the wall, while in the  
top part, as in the other two approaches, cracking is located at the corners.
3.2 Slender panels with opening

In the following analyses the effect of settlement configuration is investigated together with the influence of wall height and presence of openings. Three slender panels, characterized by different height and number of openings, named Panel 3, Panel 4 and Panel 5, have been studied. Dimensions of blocks are $50 \times 100 \times 50$ mm$^3$ for all the specimens and the openings are $100 \times 250$ mm$^2$ wide. Panel 3 and Panel 4 are both made of 73 blocks and have the same dimensions, that is 400 mm width, 50 mm thickness and 900 mm height, but they differ because of the extension of the portion involved in the settlement. The fictitious downward moving block has dimension of $100 \times 150 \times 50$ mm$^3$ for Panel 3 and dimension $100 \times 250 \times 50$ mm$^3$ for Panel 4. Panel 5 is taller and characterized by the presence of three openings. It is 400 mm wide, 100 mm thick and 1300 mm high and is made of 105 blocks. The dimensions of the fictitious block are the same adopted in the Panel 4 case.

For Panel 3 and Panel 4, as expected, results show that collapse mechanism is affected by the portion of structure involved into settlement.

Figure 10 presents the mechanism obtained using the different models, referring to Panel 3. Results of LA and FEM, Figure 10a) and b), indicate that the masonry wall is almost stable with the separation of only one little macro-block which follows the downward movement of the settlement. The FEM simulation (Figure 9c) is in partial agreement with the results of the two previous methods, since the portion affected by the settlement is fairly larger. Nonetheless, apart from a slight localization of shear failure point on the first floor spandrel, it can be observed that the continuum model is able to reproduce the collapse configuration in which the settlement practically involves only the supported portion of the wall.

Figure 11 refers to Panel 4. The collapse mechanism obtained with LA, Figure 11a), caused the generation of three distinct macro-blocks separated by ‘stair-stepped’ cracks: the first one (1), positioned at ground level, remains stable, the second portion of the panels (2) follows, without crashing, the downward movement of the fictitious block and the third one (3), which corresponds to the portion of the panel upon the architrave of first opening, rotates without cracking around the upper left corner of the first macro-block. FEM/DEM, Figure 11b), produces a similar collapse but the portion of the panel at first floor, unlike LA results, splits in two macro-block (3) and (4) with a almost symmetric mechanism if compared with that of ground level. FEM simulation results (Figure 9c) show a localization of the plastic points that reflects the
kinematics described above. In fact, failure involves the bottom spandrel at the end of the supported part, the first floor spandrel with prevalence of shear failure mechanisms and it is possible to see the formation of tensile failure points at the top left part of the wall, according to the mechanism obtained through a FEM/DEM approach.

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**Figure 11.** a) LA, b) FEM/DEM and c) FEM results for Panel 4 case.

**Figure 12.** a) LA, b) FEM/DEM and c) FEM results for Panel 5 case.

Figure 12 refers to Panel 5. The different height of the panel does not influence the collapse mechanism obtained with LA and FEM, Figure 12a) and 12b), which results are similar to those obtained for Panel 4. In particular, it may be observed the formation of three macro-blocks: portion (1) is stable, portion (2) follows the downward movement of the fictitious block and the portion (3) acts as a unique rigid body which rotates around the lower right corner of the macro-block (2). Furthermore the presence of two
openings seems that does not affects the collapse mechanism of this portion of the wall. Apart from the slight cracking occurring in the top spandrel, FEM results (9c) are in fair agreement with the observations reported above and the failure mechanism described by the plastic points is very similar to Panel 4 one.

3.3 Facades with opening

The collapse mechanism of two-storey and three-storey masonry façades characterized by different geometries is investigated. Façade 1 and Façade 2 have the same geometry and are characterized by the presence of four openings, two doors at the ground level and two windows at the first floor which have the same dimensions (200×350 mm²). Each façade is 1500 mm wide, 50 mm thick and 1200 mm high, made of 320 blocks with dimensions 50×100×50 mm³. The dimension of the fictitious block is 100×300×50 mm³ for both the analysed cases. Façade 1 is affected by a lateral settlement, while Façade 2 to a central settlement.

Figure 13. a)LA, b)FEM/DEM and c)FEM results for Façade 1 case.
Figure 14. a) LA, b) FEM/DEM and c) FEM results for Façade 2 case.

The downward movement of the central pier causes the shear failure of the spandrels while tensile failure is mainly concentrated at the lintel ends. Concluding, a three-storey structure, named Façade 3, shown in Figure 15 and subjected to a foundation settlement at the right pier is analysed. The structure is characterized by the presence of six openings, two doors at ground level and four windows, two for each storey, all having 200 mm width and 350 mm height. The façade is 1500 mm wide, 50 mm thick and 1800 mm high. It is made of 480 blocks with dimensions $50 \times 100 \times 50$ mm$^3$. The dimension of the fictitious block is $100 \times 300 \times 50$ mm$^3$. Figure 15 presents the collapse obtained with the different models. As for previously analysed cases, the mechanisms obtained with LA, Figure 15a), and FEM, Figure 15b), are in good agreement, being possible to identify the formation of similar macro-blocks: part (1), positioned upon the fictitious block, follows the downward movement of the settlement without cracking. The part (2) similarly follows the movement of the portion below with a slight sliding behavior and a more evident rotation around its lower right corner; part (3), corresponding to the portion of masonry upon the architrave of the right door at ground level, rotates rigidly as well as part (4) which includes a portion of masonry upon the architrave of the window at first floor and a portion of the lateral side of the façade at the second storey. Macro-block (5), corresponding to the final portion of masonry upon the architrave of the window at second floor, rotates around the lower left corner of the architrave. Interesting to notice, as for other façades, the presence of diagonal cracks passing through the central portion of the
façade, that originate in correspondence of the upper right corner of door and windows. From the analysis of the collapse mechanism it is possible to notice also a little rigid rotation of the macro-block (6) and (7) identified by those diagonal cracks. The left side of structure as well as the central wall at ground level remain stable. FEM analysis results (Figure 13c) show a concentration of tensile failure points in the top part of the wall, which is compatible with the formation of block 5 reported in the LA and FEM/DEM results and its clock-wise rotation. As in the two previous cases, at collapse, only slight damage occurs in the supported part of the structure, while tensile and shear cracking affects the right portion, above the downward moving pier.

4 CONCLUDING REMARKS

This work presents the comparison of failure patterns characterizing the response of dry joints masonry walls subjected to settlements. Three numerical formulations have been adopted: a Limit Analysis, a FEM/DEM and FEM approach. The models have been briefly described and preliminary validated referring to experimental and numerical literature results. A comparative study has then been performed, varying the main factors affecting the response of masonry structures in case of settlements, that is wall dimensions, presence of openings, extension of the settling area. All the models have proven to be very efficient from a computational point of view and able to reproduce the collapse mechanisms, and can thus be considered a useful tool to back-analyse real-scale problems in order to identify the causes of observed crack pattern or to predict the damage distribution when a settlement is expected to occur, as in the case of underground excavation or in case of natural triggering factors. The adoption of micro-models, i.e. Limit Analysis and FEM/DEM, implies that the structure is described with its real texture, taking into account block dimensions and the internal structure of the wall. While in most continuum approach this aspect is only marginally considered, the FEM model adopted in this study is founded on a constitutive model in which both joint orientations and block proportions are taken into account. A specific advantage of Limit Analysis approach is the low number of parameters required to perform analysis, being friction and self-weight per unit volume the only mechanical information needed. The same holds for the FEM model, but the elastic parameters adopted need to be determined through a homogeneization procedure in order to be assigned to the continuum. The comparisons have been made on a total set of eight walls (if the benchmark cases are excluded) and results have been compared in terms of crack pattern at collapse (LA and FEM/DEM) and failure (plastic) points (FEM) in which either tensile or shear criterion is reached. In all the analysed cases the models are in fair agreement, showing the strong influence of the extension of the settling area and of the opening distribution. The cases of Panel 4 and Panel 5 have shown an almost negligible effect of the panel height (over a certain slenderness of the structure) in case of localized settlement.

AUTHOR CONTRIBUTIONS

MPE and MPI developed a novel feature of ALMA 2.0 to take into account the effect of foundation settlement and performed the analysis with the LA approach. ER and MS carried out the analysis with FEM/DEM and FEM approach respectively. All authors equally contributed to the common sections and took care of the description of the adopted models. PT and GdF supervised the whole work during the entire process.

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