Arguments for the Unsuitability of Convolutional Neural Networks for Non–Local Tasks

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Abstract
Convolutional neural networks have established themselves over the past years as the state of the art method for image classification, and for many datasets, they even surpass humans in categorizing images. Unfortunately, the same architectures perform much worse when they have to compare parts of an image to each other to correctly classify this image.

Until now, no well-formed theoretical argument has been presented to explain this deficiency. In this paper, we will argue that convolutional layers are of little use for such problems, since comparison tasks are global by nature, but convolutional layers are local by design. We will use this insight to reformulate a comparison task into a sorting task and use findings on sorting networks to propose a lower bound for the number of parameters a neural network needs to solve comparison tasks in a generalizable way. We will use this lower bound to argue that attention, as well as iterative/recurrent processing, is needed to prevent a combinatorial explosion.

Keywords: , convolutional neural networks, sorting networks, relational reasoning, attention, locality

1. Introduction

Being able to compare objects in a scene and making decisions based on that information is an essential skill for humans, who can compare completely novel shapes and objects without being familiar with them, something which for example Fleuret et al. (2011) were able to show using the SVRT dataset.

Since 2012, deep learning and Convolutional Neural Networks (CNNs) have emerged as state-of-the-art methods in computer vision. It is therefore natural to extend the use of such networks to tasks involving judgments about similarity and identity.

Unfortunately, experiments by us (Stabinger et al. (2016)) have shown that CNNs perform very poorly on classification tasks that require comparison of

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shapes. Although multiple authors have confirmed this problem of CNNs by now (e.g. [Ricci et al. (2018) and [Kim et al.] (2018)), to our knowledge, no convincing theoretical arguments on why comparison tasks are so difficult for these architectures have been proposed in detail.

In this paper, we will try to shed some light on this class of comparison problems by analyzing one specific task in detail. We will try to convince the reader of three important aspects regarding comparison tasks: 1) They are inherently difficult for CNNs, 2) Attention drastically reduces the size of a network needed to solve such tasks and 3) Iterative processing further reduces the complexity of the task considerably. In Section 2, we will present the current research on solving comparison tasks using CNNs. The identity task, which we will concentrate our analysis on, will be presented in Section 3. The same section will also present a theoretical framework to analyze such tasks. Section 4 will use this framework to introduce the concept of the locality of a task, which we will use in Section 5 to argue that the convolutional layers are of limited usefulness for solving the identity task. We will also show, that the identity task can, in the optimal case, be reduced to sorting a list of numbers in the fully connected part of a network. In Section 6, we will use research on sorting networks to propose a way to generate neural networks to sort numbers and offer a lower bound on the number of parameters such a neural network needs. Experiments in Section 7 will fortify some of the theoretical arguments with practical results. In Section 8, we will present conclusions and discuss our findings.

2. Related Work

The SVRT dataset by [Fleuret et al. (2011)] consists of multiple problems, each built around the classification of abstract images containing shape outlines. For some problems, it is necessary to compare the shapes to each other to be able to classify the image correctly. For example, in problem 1 of the SVRT dataset, two shapes are visible in each image. The image belongs to class 1 if both shapes are different, and to class 2 if they are identical (see Figure 1 for a few example images).

![Figure 1: Examples of the two classes of problem 1 from the SVRT dataset by Fleuret et al. (2011). For class 1 the two shapes are different, for class 2 they are identical.](image)
Figure 2: Examples from the PSVRT dataset by Kim et al. (2018) that offer two classification tasks with the same images either classifying identity of the patches or their orientation to each other.

Using this SVRT dataset, we could show in Stabinger et al. (2016) showed that CNNs struggle with solving tasks that require the comparison of shapes. Ricci et al. (2018) later came to the same conclusion and Kim et al. (2018) extended this work by introducing the PSVRT dataset (see Figure 2 for examples) using randomly generated checkerboards instead of shape outlines.

Further research by Messina et al. (2019) showed that ResNet by He et al. (2016) as well as CorNet-S by Kubilius et al. (2018) can solve some of the previously unsolved problems of the SVRT dataset, but needed to use 400,000 training images to do so. Funke et al. (2020) finally were able to achieve accuracies above 90% on all problems of the SVRT dataset using a ResNet architecture with just 28,000 images.

The fact that the SVRT dataset has been solved by Funke et al. (2020) probably has more to do with shortcomings of the dataset and less with neural networks being able to solve the underlying comparison problem. Our experiments with a seemingly easier dataset in Section 7.3 seem to support this hypothesis. In our opinion, the SVRT dataset has two shortcomings:

First, the images for the different problems also have different complexity (e.g. the number, size and distribution of the shapes). Therefore, it is hard to judge whether a system struggles because the actual problem is harder to solve, or because the images have higher variability than images for other problems (e.g. there are more shapes, the size of the shapes is different, the positions of the shapes are more variable, ...). Kim et al. (2018) created the PSVRT (see Figure 2) dataset to take image complexity out of the equation by using the same images for two different tasks (spatial orientation or identity). The authors used PSVRT to confirm that tasks involving comparison are truly more difficult to learn than those involving other relations (like spatial relations in the PSVRT case).

Second, even an approximate comparison of the shapes of the SVRT dataset is sufficient to claim identity. Since the shapes are generated entirely at random,
different shapes, in most cases, do not resemble each other even on a coarse level. Thus, two shapes that roughly look-alike will be identical with quite a high probability. Because of this, even a small number of training images might be sufficient to have seen all approximate shapes and the separation between training and testing set can not be ensured anymore. This means that it is hard to judge whether a system has learned a task or has just memorized the training samples.

Humans, as well as more classical machine learning approaches like support vector machines with handcrafted features, do not show this systematically lower performance on tasks involving comparison, which was shown by Fleuret et al. in their original paper.

3. The Identity Task

Because of these shortcomings, we will use a simplified task in this paper that was influenced by the identity task of the PSVRT dataset by Kim et al. (2018). The task consists of images $I$ with dimension $N \times N$ where each pixel can have one of two possible states $\{0, 1\}$. Each image contains a randomly chosen amount $c \geq 3$ of non-overlapping, randomly positioned patches $P$ of size $n \times n$, where $n \ll N$. The patches are filled with random pixels, according to a Bernoulli distribution with $p = 0.5$ and $P_{xy} \in \{0, 1\}$. Each image is categorized into one of two classes. An image is of class one if at least half of the patches are identical, and class two otherwise. Two patches $P^i$ and $P^j$ are identical iff $\forall x, y \in \{1, ..., n\} : P^i_{xy} = P^j_{xy}$. The goal of this task is to decide which class a given image belongs to, given $M$ image/class pairs for training. We call this classification task and the accompanying dataset the identity task. Example images from this dataset can be seen in Figure 3.

3.1. Determining the Usefulness of Convolutional Layers

In this section, we will present arguments why convolutional layers are of limited usefulness for solving the identity task. As an abstraction of our actual
task, let us assume that some system has to compare two image patches and
decide whether they are identical or not.

We assume that all patches that should be considered identical are assigned
one unique symbol by a function $E(p)$ where $p$ is the patch to be mapped to a
symbol. We will call this function the encoder. The set of all possible symbols
for a task will be named $S$ so that $\forall p E(p) \in S$. One symbol therefore encodes
$\log_2 |S|$ bit of information, where $|S|$ signifies the number of symbols in $S$.

The task of comparing two patches for identity can now be solved by a
decision function $D(s_1, s_2)$ taking in two symbols. This function detects whether
$s_1$ and $s_2$ are identical or not (see Equation 1), and we will call it the decision–
maker.

$$D(s_1, s_2) = \begin{cases} \text{same iff } s_1 = s_2 \\ \text{different otherwise} \end{cases}$$  (1)

Deciding whether the patches $p_1$ and $p_2$ are identical therefore is reduced to the
question of whether the result from $D(E(p_1), E(p_2))$ is "same" or "different".

In our identity task, the goal is not to compare two patches for identity,
but to compare $c$ patches. A decision–maker, processing all patches in one
step, therefore, would need $c$ symbols as input and therefore $c \log_2 |S|$ bit of
information from the encoders. We will now investigate the following question:
Can we somehow reduce the amount of information the decision–maker needs
to solve a given task by some form of hierarchical preprocessing?

3.1.1. Hierarchical Preprocessing

We define a preprocessor $P_i(s_1, s_2)$ as a function that takes two symbols from
a set of symbols $S_{i-1}$ as arguments and returns a single symbol from a new set
of symbols $S_i$. A preprocessor, therefore, performs lossless compression of the
received information with respect to the given task. How much a preprocessor
can compress the incoming information depends on the task that has to be
solved. The ratio between the amount of information going into a preprocessor
and the amount of information being passed along to higher-level preprocessors
will be called the compression factor, defined as

$$C(P_i) = \frac{2 \log_2 |S_{i-1}|}{\log_2 |S_i|}$$

This compression factor directly correlates to the amount of processing that
can be performed in the preprocessor $P_i$. A high compression factor indicates
that a lot of the processing can be done in $P_i$, and a compression factor of 1
indicates that no processing can occur in $P_i$ and all the incoming information has
to be passed along unchanged. Preprocessors can be stacked in a hierarchical
arrangement (see Figure 4).
Figure 4: Example of a hierarchical arrangement of encoders $E$, preprocessors $P_1$ and $P_2$, and a decision–maker $D$ for the presented identity task

4. Locality

Given a preprocessor $P_i$, we define the \textit{receptive field size} $\text{rfs}(P_i)$ as the number of inputs its output, directly or indirectly, depends on. E.g. for the hierarchical structure from Figure 4, the first layer of preprocessors ($P_1$) will have a receptive field size of 2, the next layer will have a receptive field size of 4 and so forth (i.e. $\text{rfs}(P_i) = 2^i$).

We will now investigate how the minimum amount of information that is needed to fully represent the content of a specific receptive field changes with respect to its size. This minimum amount of information is equivalent to the number of output symbols a preprocessor in our hierarchical processing scheme needs.

If the minimal amount of information needed to represent a receptive field does not depend on the receptive field size, we have \textit{perfect locality} (i.e. all the processing can be done on a local scope). If the amount of information grows directly proportional to the receptive field size we have \textit{no locality} at all (i.e. all processing has to be done on a global scope by the decision-maker).

Therefore, we define the locality $L$ of a task as the limit of the compression factor of our preprocessors while going up the hierarchy and therefore simultaneously increasing the receptive field size of the preprocessors (we assume that the receptive field size and the depth of the hierarchy are not limited):

$$L = \lim_{i \to \infty} \frac{C(P_i)}{C(E)} - 1$$

Informally, taking the identity task as an example, this can be seen as having an infinite number of patches to analyze. Therefore, the hierarchy of Figure 4 would have to have infinite depth, and the locality is the compression factor the preprocessors approach while going up this infinite hierarchy, minus one.

A locality of 0 indicates that no local processing can be performed if the receptive field becomes very big (i.e. the preprocessor is not able to compress anything). A locality $L < 0$ can not occur since it is always possible to pass
along the full incoming information. A locality of 1 means that all processing can be done on a local level, i.e. the preprocessor can compress two symbols from $S_{i-1}$ to one symbol from $S_i$ and the amount of information needed to represent one symbol from $S_{i-1}$ and $S_i$ is the same ($|S_{i-1}| = |S_i|$). In other words, the amount of information needed to represent a receptive field does not depend on its size. A locality $L$ between 0 and 1 indicates that a varying degree of processing can be performed on a local level.

**Hypothesis:** Problems exhibiting low locality are ill fitted to be solved by Convolutional Neural Networks (CNNs)

Intuitively, CNNs are ill suited to solve non-local tasks, since the convolutional part of the network is local by design.

### 4.0.1. The Identity Task as an Example

As previously described, we assume that $c$ patches of size $n \times n$, consisting of black and white pixels, are given, and the task is to decide whether at least half of the patches are identical or not. Since two patches are only considered equal if they contain the same pattern, the encoder needs to use $2^{n^2}$ symbols in $S_0$ to preserve comparability. Thus, one such symbol encodes $n^2$ bit of information.

The preprocessors of the first layer $P_1$ each get $2^{n^2}$ bit of information as input, assuming that each possible pattern can occur with the same probability. No matter whether the two incoming symbols are the same or different, the output has to encode all possible combinations of incoming symbols since no decision about which patch has to be available for comparison at a later stage can be made at this point. Since the order of symbols is not relevant (i.e. $P_1(s_1, s_2) = P_1(s_2, s_1)$), the preprocessor has to use $|S_0|^2 - \left(\frac{|S_0|}{2}\right)$ symbols for $S_1$.

Intuitively, one would assume that at least some information can be left out in cases where a preprocessor receives the same symbol twice (meaning that the two patches are identical), but this is not the case. The information that the same patch was received twice needs one symbol as a representation, the same way that any other combination of received patches needs exactly one symbol to represent.

This gives us the following compression factor for $P_1$:

$$C(P_1) = \frac{2 \log_2 |S_0|}{\log_2 \left(|S_0|^2 - \left(\frac{|S_0|}{2}\right)\right)}$$  \hspace{1cm} (2)

The preprocessors of the next layer $P_2$ again have to use one symbol in $S_2$ for all possible pairs of symbols from $S_1$, ignoring the order of symbols. This procedure does not change for any of the following layers of preprocessors, and we can define a general formula for the compression factor of any preprocessor as

$$C(P_i) = \frac{2 \log_2 |S_{i-1}|}{\log_2 \left(|S_{i-1}|^2 - \left(\frac{|S_{i-1}|}{2}\right)\right)}$$  \hspace{1cm} (3)
For better readability we define \( s = |S_{i-1}| \). Since the number of symbols in \( S_i \) monotonically increases while going up the hierarchy, we can rewrite the locality of this task as follows:

\[
L = \lim_{s \to \infty} \frac{2 \log_2 s}{\log_2 \left( s^2 - \left( \frac{s}{2} \right)^2 \right)} - 1 = \lim_{s \to \infty} \frac{2 \log_2 s}{\log_2 \left( s^2 - \frac{s^2 - s}{2} \right)} - 1 = (4)
\]

\[
2 \lim_{s \to \infty} \frac{d}{ds} \log_2 \left( s^2 - \frac{s^2 - s}{2} \right) - 1 = 2 \lim_{s \to \infty} \frac{s + 1}{2s + 1} - 1 = 0 \quad (5)
\]

With a locality of 0, we can see that processing can only happen once the system has reached a global receptive field.

5. Application to the Identity Task

We will now see how the theoretical architecture of encoder, preprocessor and decision–maker can be translated to the different parts of a Convolutional Neural Network (CNN). Commonly, a CNN for classification is constructed from two main parts: A varying amount of convolutional and pooling layers extracting feature descriptors with increasingly global scope and a part of fully connected layers generally considered to use those features for the actual classification.

The symbols the encoder is extracting can be interpreted as the activations of the neurons of some of the first layers in the CNN where the receptive fields of the neurons are still relatively small. The following convolutional and pooling layers can be interpreted as the preprocessors that compress and extract information as the receptive field becomes more extensive, either through pooling or through the stride of the convolutions. The following fully connected layers have a global receptive field and can be seen as the decision–maker, taking the extracted information from the convolutional layers and calculating the class probabilities.

As we have shown in Section 4, the only operation that can be performed on a local scope is extracting the information contained in the patches, and forwarding this information to the decision–maker. Therefore, the best the convolution and pooling part of the network can do is to find an efficient, lossless encoding of the information contained in each patch. If we allow for arbitrary precision real numbers, a convolution can in principle completely encode the content of a patch in a single real-valued output. Those real-valued outputs can be interpreted as the symbols forwarded to the decision–maker.

A kernel of size \( n \times n \) that is capable of capturing all information of a patch with binary pixels (i.e. black or white) in a single real number can be constructed by setting the \( n^2 \) weights of the kernel to \( \forall i \in 1, \ldots, n^2 : w_i = \frac{1}{2} \). This ensures that each bit of the incoming patch is represented by exactly one bit of the fraction bits of the resulting floating-point number. See Figure 5 for an example of a \( 3 \times 3 \) kernel constructed in this manner.
There are two possibilities when extracting the patches in the convolutional layers. If some attentional mechanism allows the convolutional layers to extract exactly one number for each patch, we end up with \( c \) (the number of patches) numbers that need to be compared. Since we have to design our network in a way so that the worst case can also be handled, we have to assume the maximal number of patches that are possible. This means that we have to allow for \( (N/n)^2 \) numbers to be compared where \( N \) is the width and height of the image in pixels and \( n \) is the width and height of the patch in pixels. If no attentional mechanism is present, the convolutional layers will have to generate one number for each possible patch position, resulting in \( (N - 1)^2 \) numbers that have to be processed.

In reality, the situation is even worse since the precision of floating-point numbers is limited. Using a 32-bit floating-point number (the most common size used for neural networks) only 24 bit can be reliably encoded with the presented method. Patches above a size of \( 4 \times 4 \) would therefore have to be encoded by multiple numbers.

After the convolution and pooling part of the network, in the most favourable case, we would have several inputs to our fully connected layers, and each input encodes the information for each patch (in the case with attention) or each possible patch position (in the case without attention). The most efficient way of detecting the amount of identical numbers in a list is to sort the list first. Once the list is sorted, identical numbers can be identified by merely checking neighbouring numbers for equality. In the experimental section, we will show that this also holds in practice using neural networks.

The question now is, how a neural network can sort numbers and how big it has to be to do so. The next section will start with sorting networks to estimate a lower bound on the number of parameters needed to achieve sorting using a neural network under the assumption that it is constructed from fully connected layers.

6. Sorting Networks

A sorting network by O’connor & Nelson (1962) consists of wires and comparators. Wires "transport" comparable values (e.g. real numbers in our case).
Figure 6: Example of a sorting network that is able to sort four real numbers in descending order. The horizontal lines represent the wires, the vertical lines and black dots represent the comparators. Values flow along the wires from left to right. Swapped values are highlighted and the three layers of the network are labeled $L_1$ through $L_3$.

Pairs of wires can be connected by comparators that swap the values transported on the wires if they are not already in the correct order. Multiple comparators can swap values in parallel as long as each wire is only connected to one comparator. We will call such a parallel evaluation of comparators a layer. A sorting network is a fixed arrangement of comparators and wires so that any combination of possible values sent along the wires is sorted after passing all comparators. Figure 6 shows a sorting network in operation. Sorting networks take the task of sorting - which is usually perceived as an iterative process - and converts it into a highly parallel, purely feed-forward problem.

The number of layers of a sorting network is also called its depth. Extensive research exists on the theoretical properties of such sorting networks, and lower bounds on their depth have been published. There exists an information-theoretical lower bound of $\log_2 n$ regarding the depth of a sorting network to sort $n$ numbers and Kahale et al. [Kahale et al. 1995] were able to tighten that lower bound to $(c - o(1)) \log_2 n$ with $c \approx 3.27$.

Since we want to transfer the knowledge on how to construct a sorting network to neural networks, we first have to find a way to implement a comparator using a neural network. We assume the neural network has two inputs $x_1$ and $x_2$ in the range $[0, 1]$ and two outputs $y_1$ and $y_2$. We expect the following behavior: $y_1 = x_1$ and $y_2 = x_2$ in case $x_1 \geq x_2$ and $y_1 = x_2$, $y_2 = x_1$ in case of $x_1 < x_2$. Figure 7 shows a minimal implementation of such a comparator neural network. The implementation assumes $\max(0, x)$ (a rectified linear unit) as the activation function of the neurons and biases are not needed. The following equations describe the hidden neurons:

$$z_1 = \max(0, x_1) \rightarrow z_1 = x_1$$
$$z_2 = \max(0, x_2) \rightarrow z_2 = x_2$$
$$z_3 = \max(0, x_2 - x_1)$$
Figure 7: One possible, minimal implementation of a comparator as a neural network.

So if $x_1 \geq x_2$ then $z_3 = 0$ otherwise $z_3 = x_2 - x_1$. This means we can calculate $y_1$ and $y_2$ by

$$y_1 = z_1 + z_3 = x_1 + \max(0, x_2 - x_1)$$
$$y_2 = z_1 - z_3 = x_2 - \max(0, x_2 - x_1)$$

In case of $x_1 \geq x_2$

$$y_1 = x_1 + 0 = x_1$$
$$y_2 = x_2 - 0 = x_2$$

In case of $x_1 < x_2$

$$y_1 = x_1 + x_2 - x_1 = x_2$$
$$y_2 = x_2 - x_2 + x_1 = x_1$$

Which is the expected behavior of a comparator.

6.1. Constructing Neural Networks from Sorting Networks

A sorting network can trivially be used to create a neural network for sorting numbers. For each layer of the sorting network, two hidden layers in the neural network of appropriate width are created. The wires are implemented by passing values through neurons with a weight of 1 (i.e. the values are unchanged). A comparator in the sorting network is replaced by a comparator neural network after which the two numbers passed through will be sorted. Figure 8 shows a minimal sorting network to sort three numbers and Figure 9 the corresponding neural network.

As we have shown above, one comparator, implemented using a neural network, needs three layers. However, since the input and output layer of comparators following each other can be combined, we need two fully connected layers in our neural network for each layer in the sorting network. The first layer needs
Figure 8: Example of an optimal sorting network for three numbers.

Figure 9: Example of a neural network with 72 parameters, capable of sorting three numbers, constructed using the sorting network from Figure 8. Weights with a value of 0.0 are omitted for clarity.

\[
\lceil 1.5x \rceil \text{ neurons and the second needs } x \text{ neurons to implement one layer of a sorting network which sorts } x \text{ numbers. The number of parameters } p \text{ needed for such a neural network, that implements a sorting network with depth } d, \text{ sorting } x \text{ numbers are:}
\]

\[
p = 2d\lceil 1.5x \rceil x \tag{6}
\]

Slightly fewer parameters can be achieved, depending on the exact structure of the sorting network. More specifically, if a layer of the sorting network passes through values without processing (i.e. in cases where a line is not connected to any comparator), some neurons can be saved. See Figure 9 as an example: One neuron can be saved for each second neural network layer because every layer of the sorting network passes through one value unchanged. If we assume that the neural sorting network has to be learned from data, we can not assume an architecture that is so closely fitted to the problem, and the parameter count of Equation 6 is more realistic than the version with neurons removed from individual layers.

Experimental evidence suggests that neural networks constructed in this manner are minimal solutions to the sorting problem when using fully connected layers with rectified linear units as activation functions and the number of parameters from Equation 6 is close to the minimum. The smallest network we could train to sort three numbers needs 97 parameters, which is higher than our proposed lower bound from Equation 6 of 90 parameters.
6.2. Predicted Number of Parameters for the Identity Task

Using a typical input image size of 224 × 224 pixels, the amount of numbers to process depends, as previously mentioned, on whether there is some form of attentional mechanism or not. With attention, the system will be able to detect which part of the image is a patch and which one is not. Therefore, we have to be able to process the maximal amount of patches that could fit into an image without overlapping. Assuming a patch size of 8 × 8, we need to process a maximum of 784 patches, and therefore numbers. Without attention, the system has to be able to process all 46,656 possible patch positions.

Assuming the smaller, information-theoretic, lower bound on the number of layers needed in a sorting network, we need ten layers and 16 layers with and without attention respectively. Following Equation 6, the neural sorting networks need ≈ 18.4 million parameters with and ≈ 65.3 billion parameters without an attention mechanism. This difference in parameters shows how important attention is in such cases to prevent a combinatorial explosion. This need for attention, to bring the complexity of vision tasks down to practical levels, was already shown by Tsotsos (1988).

6.3. Advantages of Iterative Processing

We have previously mentioned, that we suspect that iterative processing should, similarly to attention, considerably reduce the complexity of the identity task. This advantage of iterative processing is connected to the problem that purely feed forward networks have with sorting numbers.

If we are allowed to send a list of numbers through a sorting network and apply the same sorting network to the resulting list of numbers repeatedly, we can sort any list of numbers with a network with only two layers. Such a network can be implemented by connecting every second pair of wires with a comparator, starting with the first wire in layer one, and starting with the second wire for layer two (see Figure 10 for such a sorting network for six numbers).

Sending numbers through this sorting network ensures that each number will move at least one step in the direction of its correctly sorted position. Therefore, a list of x numbers will be completely sorted after applying this sorting network x times.
We can convert this sorting network into a neural network, using the procedure form Section 6.1. Using Equation 6, we can calculate that a neural network that can be used to sort \( x \) numbers using iterative (recurrent) processing, needs \( 4 \lceil 1.5x \rceil x \) parameters. This reduces the network size for our identity task to \( \approx 7.3 \) million parameters with and \( \approx 26.1 \) billion parameters without an attention mechanism.

Suppose we also allow for iterative processing of the list itself. In that case, the sorting problem can be solved by iteratively applying the same comparator on each pair of numbers of the list. This approach is in effect an implementation of bubble sort and can be solved with a single neural network with only 12 parameters.

This reduction in parameter counts does not only reduce the amount of resources such neural networks need, it also drastically reduces the amount of training data needed and leads to better generalizability of the networks.

In Section 7.1 we will show experimentally how useful iterative (recurrent) processing is for our proposed problem.

7. Experimental Evaluation

In the following sections, we will substantiate our theoretical findings with experimental results. We will show that detecting identical numbers in a list of numbers (which is, as we showed, the most efficient input the convolutional part of a CNN can give to the fully connected part for the identity task) becomes much easier, once the list of numbers is sorted. We will also show that recurrent networks perform much better at this task. Besides, we will highlight that an attentional mechanism makes the identity task much easier to solve.

7.1. The Importance Of Sorting

As we have shown in Section 5, the identity task boils down to the decision if at least half of the numbers of a list of numbers are identical.

To show that sorting this list decreases the difficulty of the problem in practice, we ran experiments where we used fully connected neural networks to solve this classification problem. The list of numbers was either sorted or unsorted before being given to the neural network for classification. Different architectures of up to ten layers with 100 neurons in each layer were tested for lists of ten numbers. The dataset consisted of 4,000 randomly generated lists for training and 1,000 for testing. Binary cross entropy was used as a loss, the networks were optimized using Ranger by Wright (2019) and ReLU by Hahnloser et al. (2000) was used as an activation function for all but the last layer (which used a logistic sigmoid function). Weights were initialized following the method proposed by He et al. (2015).

Since we were not able to get an accuracy above 0.8 with the unsorted list, we also tried to solve this task using 40,000 training samples. Looking at the results in Table 1, it is easy to see that the task is much easier, once the list is sorted. Even with one hidden layer and only ten neurons, we were able to achieve
Table 1. Accuracy of different fully connected neural network architectures, when trying to classify whether a list of ten numbers contains the same number at least five times. The list is sorted or unsorted and either a small or large training set is provided. Many more combinations were tested and only the more interesting combinations are shown.

| Input          | Hidden Layers | Layersize | Number of Parameters | Test Accuracy | Train Accuracy |
|----------------|---------------|-----------|----------------------|---------------|---------------|
| sorted small   | 1             | 10        | 132                  | 0.99          | 0.99          |
| sorted small   | 1             | 100       | 1302                 | 1.0           | 1.0           |
| unsorted small | 1             | 500       | 6502                 | 0.77          | 0.98          |
| unsorted small | 1             | 1000      | 13002                | 0.80          | 1.0           |
| unsorted small | 2             | 100       | 11402                | 0.79          | 1.0           |
| unsorted small | 2             | 500       | 257002               | 0.80          | 1.0           |
| unsorted small | 2             | 1000      | 1014002              | 0.79          | 1.0           |
| unsorted small | 10            | 100       | 92202                | 0.75          | 1.0           |
| unsorted large | 10            | 5         | 337                  | 0.76          | 0.77          |
| unsorted large | 10            | 10        | 1122                 | 0.84          | 0.85          |
| unsorted large | 10            | 50        | 23602                | 0.91          | 0.94          |

almost perfect accuracy for sorted lists. On the other hand, for the unsorted list, only a network with ten layers and 50 neurons for each layer and a training set of 50,000 samples was able to achieve an accuracy above 0.9. These results show that once the list of numbers is sorted, the task itself becomes very easy.

As previously mentioned, we hypothesize that iterative processing would solve many of the shortcomings of neural networks for these kinds of problems. To test this hypothesis, we also tested the previous dataset with a Long Short Term Memory (LSTM) architecture by [Hochreiter & Schmidhuber, 1997], which contains recurrent connections and can iteratively process data because of this. The results of this architecture for unsorted lists can be seen in Table 2. It is evident that a recurrent architecture has two advantages when solving this kind of problem: First, the networks need far fewer parameters and second, they can solve the problem using less training data because they generalize much better, which is a consequence of them being able to solve the problem with fewer parameters.

7.2. Training Neural Sorting Networks from Data

In Section 6.1 we have shown how sorting networks can be used to construct neural networks that can sort numbers. Since we expect our networks to learn the sorting operation as part of a more extensive network, we will experimentally determine how many parameters neural networks need to be able to learn the sorting operation from data alone.

As training data, the networks were provided with an unlimited supply of unsorted vectors containing numbers from 0 to 1 as input and the correctly sorted vectors as training targets. The networks consisted of a variable amount
Table 2: Accuracy of different LSTM neural network architectures, when trying to classify whether a list of ten numbers contains the same number at least five times. The lists were unsorted.

| Training Samples | Layers | Hidden State Size | Number of Parameters | Train Accuracy | Test Accuracy |
|------------------|--------|-------------------|----------------------|----------------|--------------|
| 4000             | 1      | 5                 | 152                  | 0.84           | 0.81         |
| 4000             | 1      | 10                | 502                  | 0.91           | 0.88         |
| 4000             | 1      | 15                | 1052                 | 0.92           | 0.88         |
| 4000             | 2      | 5                 | 374                  | 0.87           | 0.87         |
| 4000             | 2      | 10                | 1342                 | 0.91           | 0.89         |
| 4000             | 2      | 15                | 2912                 | 0.95           | 0.92         |
| 40000            | 1      | 5                 | 152                  | 0.86           | 0.86         |
| 40000            | 1      | 10                | 502                  | 0.93           | 0.93         |
| 40000            | 1      | 15                | 1052                 | 0.95           | 0.95         |
| 40000            | 2      | 5                 | 374                  | 0.87           | 0.87         |
| 40000            | 2      | 10                | 1342                 | 0.95           | 0.95         |
| 40000            | 2      | 15                | 2912                 | 0.96           | 0.95         |

of fully connected layers, all containing the same amount of neurons (except for the input and output layers). The networks were again optimized using Ranger by Wright (2019), using ReLU by Hahnloser et al. (2000) as the activation function. Weights were initialized following the method by He et al. (2015), and we used mean squared error as the loss function.

The number of layers and neurons per layer were systematically tested multiple times. We consider a network to have learned the sorting task if it reaches a loss smaller than $1 \cdot 10^{-5}$. We report the networks with the lowest number of parameters. Unfortunately, we were not able to meaningfully search for networks that sort more than five numbers. Sorting a list of numbers is in itself a surprisingly tricky problem to learn for networks and finding a solution is very sensitive to the weight initialization. Even with the reported smallest networks, we had to test the same architecture more than 50 times to be able to teach the network to sort, and this problem became exponentially more difficult with each new number. Thus, we were not able to perform a meaningful search for the smallest networks sorting six numbers and above.

Table 3 shows the smallest networks we could find through training. The configurations in the table show the number of neurons per layer. As can be seen, the number of parameters of trained networks is strictly greater than the number of parameters needed by networks that are constructed by the method presented in Section 6.1 and the difference grows rapidly with the size of the vector to be sorted.

7.3. The Influence of Attention

To show that attention is useful for solving the identity task, we trained a ResNet-18 architecture by He et al. (2016) (which was already pre-trained on
Table 3: Smallest neural networks for sorting that could be found by training, in comparison to the smallest networks that can be constructed using the method from Section 6.1.

| Numbers to Sort | Learned Layer Configurations | Parameters Learned | Parameters Constructed |
|-----------------|-----------------------------|-------------------|------------------------|
| 3               | 3 7 6 3                     | 97                | 72                     |
| 4               | 4 7 7 7 4                   | 235               | 144                    |
| 5               | 5 11 11 11 11 11 11 11 11 11 11 11 11 5 | 1446              | 340                    |

(a) Without attention (one channel)  
(b) With attention (three channels)

Figure 11: Example images for the two tested versions of the identity task. In a) all patches are combined into one channel. In b) the patches are presented to the system in different channels, which simulates pre-attention. In this case, a) is an example of the non-identity-class and b) of the identity class. The varying darkness of the images stems from the pre-processing for ResNet.

the ImageNet dataset) for two different variants of the task. In both cases, all images contain three patches with a size of $3 \times 3$. The images were generated at a resolution of $14 \times 14$ and scaled up afterwards to the typical resolution for ResNet of $224 \times 224$. In addition, the background of the image is kept grey, to make it easier for the system to detect which part belongs to a patch and which does not. The images can be classified in the two classes “identity” if two of the three patches are identical and “non-identity” if they are not.

For the first variant, the three patches are presented as a regular image to the neural network. The three patches are presented to the network in three separate channels in the second variant. See Figure 11 for example images of those two variants.

The separation for the second variant, in essence, pre-attends the input, an observation that [Kim et al. (2018)](2018) already made. The authors were also able to show that separating objects into separate channels vastly improves performance on the SVRT dataset by [Fleuret et al. (2011)](2011) and make the task very easy to solve for CNNs.

For our experiments, the amount of training data that was available to the system was not limited, and no care was taken to separate training from testing data. Although the task looks easy on the surface, the neural network seems to solve it mainly by memorization, since even restricting the training data to 50,000 images leads to overfitting on the training data from the start and an
accuracy on the validation set of about 0.6 (presumably because of some overlap of the training and validation set).

ResNet-18 was trained on both variants until an accuracy of 0.98 was achieved on the validation set. Graphs of the losses and validation accuracy can be seen in Figures 13 and 12.

Comparing the results with and without pre-attention shows that the task becomes much easier when pre-attention is used. With attention, around 500,000 images are needed for an accuracy above 0.98, while the network without attention needs around 3,000,000 images, which amounts to six times more training data and training time.

8. Conclusions and Discussion

As we have shown, the reason that CNNs perform poorly on the identity task is threefold:

First, the convolutional layers cannot perform much meaningful work, since the identity task is inherently global, but convolutions are inherently local. This mismatch leaves most of the work for the fully connected part of the network, which operates on a global level.

Second, since CNNs usually do not provide an effective attention mechanism, the number of features forwarded to the fully connected layer is unnecessarily high. At the level of the fully connected layers, the identity task can mostly be reduced to a sorting problem. We used sorting networks to propose a lower bound on the number of parameters a neural network needs to solve this task.
Figure 13: Loss and accuracy during training of a ResNet-18, trained on data without pre-attention.

Third, since most of the processing has to be done in fully connected layers which do not share weights, without recurrent connection or other forms of iterative processing, the resulting system is also using data very inefficiently. This inefficiency can be seen by the fact that even only looking at the fully connected part of the problem, solving the identity task for ten numbers needs 50,000 training samples to solve.

All those arguments do not necessarily mean that smaller networks can not solve the identity task, but they likely will not generalize well and only solve the problem with memorization.

In our opinion, two mechanisms are needed to solve comparison tasks efficiently (regarding the number of parameters of the networks as well as the amount of training data needed). First, an attention mechanism to reduce the number of entities that need to be compared to prevent a combinatorial explosion is needed. Second, the information extracted from these entities has to be processed iteratively using a recurrent architecture to reduce the number of parameters, as well as the amount of training data needed through weight sharing.

9. Bibliography

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