Boundary Chaos

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Scrambling in many-body quantum systems causes initially local observables to spread uniformly over the whole available Hilbert space under unitary dynamics, which in lattice systems causes exponential suppression of dynamical correlation functions with system size. Here, we present a perturbed free quantum circuit model, in which ergodicity is induced by an impurity interaction placed on the system’s boundary, that allows for demonstrating the underlying mechanism. This is achieved by mapping dynamical correlation functions of local operators acting at the boundary to a partition function with complex weights defined on a two dimensional lattice with a helical topology. We evaluate this partition function in terms of transfer matrices, which allow for numerically treating system sizes far beyond what is accessible by exact diagonalization and whose spectral properties determine the asymptotic scaling of correlations. Combining analytical arguments with numerical results we show that for impurities which remain unitary under partial transpose correlations are exponentially suppressed with system size in a particular scaling limit. In contrast for generic impurities or generic locations of the local operators correlations show persistent revivals with a period given by the system size.

Spatiotemporal correlation functions provide the key diagnostic tool for studying spatially extended complex quantum many-body systems, both theoretically \cite{1,2} and experimentally \cite{3–5,13}, and most recently also in quantum simulations \cite{6}. They give rise to a notion of ergodicity in quantum systems \cite{7,8} and allow for quantifying scrambling of local quantum information \cite{9–12}. Moreover, two-point correlation functions determine transport coefficient of conserved currents \cite{13} as well as the relaxation dynamics of local operators and their approach to thermal equilibrium \cite{14}. More precisely, the relaxation dynamics of auto-correlation functions encodes statistical properties of the operators’ matrix elements in the energy eigenbasis and hence yields a convenient tool to study the validity of the famous eigenstate thermalization hypothesis (ETH) \cite{14–16}. The latter conjectures universal statistics of these matrix elements. In order for ETH to hold true correlation functions of large finite systems are required to relax to a value which is exponentially suppressed with the system size.

The computation of spatiotemporal correlation functions is a challenging task in general due to entangling dynamics of generic interacting many-body quantum systems, which limits the applicability of numerical methods to relatively small systems. In contrast, recent progress in identifying exactly solvable chaotic many-body systems provided numerous examples where correlation functions can be computed exactly for all times in the thermodynamic limit or for extensive finite times in finite systems. Most notably in quantum circuit models \cite{7,8,17–21} which exhibit a duality between space and time, so-called dual unitarity \cite{7,22}, correlation functions are determined by single-site quantum channels as the rest of the system plays the role of a perfect markovian bath. These systems can be robust upon small perturbations which break space-time duality \cite{20} and hence give rise to an ideal testbed for studying properties of ergodic many-body systems.

Here we present a setup which can be thought of as a toy model for a non-interacting (aka free) system in which chaos and ergodicity, i.e. decay of correlations with time, are induced by a local impurity. Chaos and ergodicity have been observed in interacting integrable systems upon an integrability breaking local impurity \cite{23–29} but it is less clear, whether this is also the case if the unperturbed system is non-interacting. Conceptually, studying perturbed free systems might allow for exactly integrating out the trivial free dynamics in analogy to the concept of Poincaré sections used in classical billiard’s dynamics \cite{30} and the associated quantized Poincaré transfer operator \cite{31,32}, which reduces the effective dimension of phase space via often trivial integration between subsequent sections (boundary collisions in billiards). In the context of many-body systems the integrated free dynamics is in close analogy with the influence matrix approach \cite{33} as it describes the influence of the trivial bulk dynamics on the nontrivial part of the systems exactly even for finite systems.

In this paper we apply this strategy to generally chaotic Floquet quantum circuits in which two-point correlation functions between local operators can be computed without approximations for times up to a fixed number of multiples of system size for arbitrarily large systems. We can thus target a niche regime between the realm of exact diagonalization in finite systems and exact results in the thermodynamic limit. Specifically, we consider a free quantum circuit composed as a brickwork of swap gates in which ergodicity and scrambling is induced by perturbing the circuit with an impurity interaction placed on the system’s boundary. We map the correlation functions of local operators at the boundary subject to the Floquet dynamics at time $t$ induced by the circuit of size $L$ to a partition function defined on a quasi-one-dimensional $\sim t/L \times L$ lattice with a helical topology. The latter is evaluated in terms of transfer matrices which for the considered time regime are of much
smaller dimensionality than the actual Floquet operator. We derive the asymptotic behavior of correlations for large \(L\) and fixed \(t/L\) from the leading parts of the transfer matrices’ spectra and find exponential suppression of correlations in \(L\) for impurity interactions that remain unitary under partial transposition. Supported by the numerical computation of the leading eigenvalues for increasing \(t/L\) we conjecture correlations to be exponentially suppressed for all times \(t > L\) in this case. In contrast, for generic impurities or generic locations of the local operators we find persistent revivals of correlations with period \(L\), i.e. around integer \(t/L\).

I. BOUNDARY CHAOS

The models we consider are built as free brickwork quantum circuits composed of swap gates on a lattice of size \(L+1\) with open boundary conditions. To each lattice site \(x \in \{0,1,\ldots,L\}\) we assign a local Hilbert space \(\mathbb{C}^q\) of dimension \(q\) giving rise to a total Hilbert space \(\bigotimes_x \mathbb{C}^q \cong \mathbb{C}^N\), where \(N = q^{L+1}\). We argue that ergodicity and scrambling may be induced by replacing the swap gate acting on the first two lattice sites by an impurity interaction, i.e., a generic unitary \(U \in U(q^2)\). We refer to this setup as boundary chaos. Formally, we define the Floquet operator corresponding to this circuit layout as \(U = U_2 U_1 \in \text{End}(\mathbb{C}^N)\) with

\[
U_1 = \prod_{i=1}^{[L/2]} P_{2i-1,2i}, \quad U_2 = U_{0,1} \prod_{i=1}^{[(L-1)/2]} P_{2i,2i+1}, \quad (1)
\]

where \(P_{ij}\ (U_{ij})\) denotes unitary gate acting nontrivially as the swap \(P\ (interaction\ U)\) on sites \(i,j\), and trivially otherwise. The resulting circuit is found to exhibit quantum chaos in the spectral sense as its spectral statistics match random matrix theory for typical \(U\), see App. B. Here, however, we are interested in ergodicity and mixing in the sense of the decay of dynamical correlations

\[
C_{ab}(t) = \frac{1}{N} \text{tr} \left( U^{-t} a \rho U^t b \right) \quad (2)
\]

between local operators \(a_0\) and \(b_0\) acting as traceless Hermitian operators \(a\) and \(b\) \(\in \text{End}(\mathbb{C}^q)\) on lattice site 0, respectively, and trivially otherwise. We treat local operators acting nontrivial on arbitrary lattice sites in Sec. V. Note, that the normalized trace corresponds to taking averages with respect to the invariant infinite temperature state, which constitutes the natural, and in generic case of ergodicity, unique equilibrium state in Floquet systems.

We cast the Heisenberg time evolution of operators into a quantum circuit formulation with enlarged local Hilbert spaces allowing for a diagrammatic representation: The so called folded picture \([34, 35]\) which we introduce in the following. To this end we employ the unitary (w.r.t. the Hilbert-Schmidt scalar product on \(\text{End}(\mathbb{C}^q)\)) operator-to-state mapping by bilinear extension of \(\text{End}(\mathbb{C}^q) \ni |m\rangle \langle n| \mapsto |m\rangle \otimes |n\rangle \in \mathbb{C}^q \otimes \mathbb{C}^q \cong \mathbb{C}^{q^2}\). Here \(\{ |n\rangle |n = 1,\ldots,q\}\) is the canonical basis of \(\mathbb{C}^q\). We write \(|a\rangle\) for the image of an operator \(a\) and, specifically, \(|\circ\rangle\) for the image of \(q^{-1/2} q\). Via tensor multiplication the vectorization mapping extends to an isomorphism \(\text{End}(\mathbb{C}^q)^{\otimes (L+1)} \cong (\mathbb{C}^{q^2})^{\otimes (L+1)}\) while the Heisenberg time evolution is cast into a quantum circuit formulation. This yields a Floquet operator \(W = W_1 W_2\) acting on the lattice (of size \(L+1\)) of \(q^2\)-dits, with

\[
W_2 = W_{0,1} \prod_i S_{2i,2i+1}, \quad W_1 = \prod_i S_{2i-1,2i}, \quad (3)
\]

where the folded local gates are defined as \(S = P \otimes P\) and \(W = U^1 \otimes U^T\).

We obtain a diagrammatic representation \([36]\) for correlation functions \((2)\) by introducing the graphical notation \(S = \begin{tikzpicture} \draw (-0.5,0) node[draw,shape=rectangle,fill=black] {$S$} ; \end{tikzpicture}\) and \(W = \begin{tikzpicture} \draw (-0.5,0) node [draw,shape=rectangle] {$W$} ; \end{tikzpicture}\), for the local gates of the circuit as well as \(|a\rangle = |a_0\rangle\) and \(|\circ\rangle = |\circ_0\rangle\), for vectorized operators. Then the circuit formulation of Heisenberg time evolution is used to recast Eq. (2) as a particular tensor network with nontrivial boundary

\[
C_{ab}(t) = \langle b_0 | W^t | a_0 \rangle = \begin{tikzpicture} \draw (-0.5,0) node [draw,shape=rectangle,fill=black] {$C_{ab}$} ; \end{tikzpicture} \quad (4)
\]

II. MAPPING TO HELICAL CIRCUIT

In order to evaluate Eq. (4) we map the \((L+1) \times t\) tensor network representing the correlation function to a partition function on a \((\tau + 1) \times L\) lattice with the topology of a helix. The non-negative integer \(\tau\) is defined by writing time as \(t = L \tau + \delta\) with remainder \(\delta \in \{0,1,\ldots,L-1\}\). The partition function can then be expressed by transfer matrices \(T_\tau \in \text{End}((\mathbb{C}^{q^2})^{\otimes \tau})\) given as matrix product operators whereas the helix topology as well as the initial and final operators are encoded by a shift operator \(C_{ab,\tau} \in \text{End}((\mathbb{C}^{q^2})^{\otimes \tau})\) with fixed boundary conditions \(a\) and \(b\), respectively. Both \(T_\tau\) and \(C_{ab,\tau}\) are most conveniently defined using their diagrammatic representation (here for \(\tau = 4\)) as

\[
T_\tau = \begin{tikzpicture} \draw (-0.5,0) node [draw,shape=rectangle,fill=black] {$T_\tau$} ; \end{tikzpicture}, \quad \text{where} \quad \begin{tikzpicture} \draw (-0.5,0) node [draw,shape=rectangle,fill=black] {$V$} ; \end{tikzpicture} = \begin{tikzpicture} \draw (-0.5,0) node [draw,shape=rectangle,fill=black] {$V$} ; \end{tikzpicture}
\quad (5)
\]

\[
C_{ab,\tau} = \begin{tikzpicture} \draw (-0.5,0) node [draw,shape=rectangle,fill=black] {$C_{ab,\tau}$} ; \end{tikzpicture}, \quad \text{where} \quad \begin{tikzpicture} \draw (-0.5,0) node [draw,shape=rectangle,fill=black] {$\tau$} ; \end{tikzpicture} = \begin{tikzpicture} \draw (-0.5,0) node [draw,shape=rectangle,fill=black] {$\tau$} ; \end{tikzpicture}
\quad (6)
\]
Using this definitions we obtain for the dynamical correlation function (2):

$$C_{ab}(t) = \text{tr} \left( [T_{\tau}^{L-\delta} \otimes 1_{q^2}] T_{\tau+1} C_{ab,\tau+1} \right)$$

which can be verified by tracing the wires corresponding to the swap gates in the diagrammatic representation (4); see App. C for a formal derivation. Intuitively, the two different tensor network representations (4) and (8) are related as follows: The nontrivial local operator $a$ in the bottom left corner of the network (4) might be scattered into the bulk (swap) part of the network by the impurity at $t = 1$. Subsequently it travels freely forth and back through the system in time $t = L$ until the corresponding wire runs into an impurity interaction at the boundary again. In the helical network (8) this process corresponds to the operator travelling from left to right. Consequently, the transfer matrices $T_{\tau}$ describe the process of local operators freely traveling forth and back through the bulk of the network (4) $\tau$ times along the wires corresponding to the swap gates and being scattered back, whenever these wires hit the impurity at the boundary. Additionally, instead of being scattered into the bulk, the local operator $a$ might just travel along the boundary in the network (4). This corresponds to the operator travelling from bottom to top in the helical network.

From the computational complexity point of view, we replaced direct computation of correlation functions, which is linear in $t$ and exponential in $L$, by transfer matrix contraction of the tensor network (8), which is linear in $L$ and exponential in $\tau \approx t/L$. Hence Eqs. (7) and (8) allow to efficiently determine the initial dynamics of correlation functions up to times $t = \tau L$ for not too large fixed $\tau$ for system sizes $L$ much larger than what is accessible by direct methods. Figure 2 depicts a representative example for $L = 200$.

Moreover, the above Eqs. (7) and (8) suggest that the asymptotic scaling of $C_{ab}(\tau L + \delta)$ for both $L - \delta$ and $\delta$ being large is dominated by the leading eigenvalues of $T_{\tau}$ and $T_{\tau+1}$. Hence we describe the spectral properties of the transfer matrices $T_{\tau}$ in the following. $T_{\tau}$ is a vectorization of a quantum channel, a non-expanding map, with spectrum $\text{spec}(T_{\tau})$ contained in the complex unit disk [20]. Its eigenvalues are either real or come in complex conjugate pairs, and $T_{\tau}$ is in general not diagonalizable, but exhibits nontrivial Jordan blocks. Unitarity of interaction $U$ implying unitality of the folded gate $V$, i.e.,

$$\begin{align*}
\begin{array}{c}
\text{●} \ = \ \otimes \\
\text{□} \ = \ \otimes
\end{array}
\end{align*}$$

guarantees that there is always the trivial (left and right) eigenvector $|0\rangle^{\otimes\tau}$ with trivial eigenvalue $1 \in \text{spec}(T_{\tau})$. Moreover, unitarity of $V$ implies that the spectra of transfer matrices grow with $\tau$, i.e., $\text{spec}(T_{\tau}) \subseteq \text{spec}(T_{\tau+1})$.

### III. T-DUAL IMPURITIES

In order to be able to analyze the nontrivial eigenvectors as well we first assume folded gates $V$ to be dual-unitary [7]. More precisely, upon exchanging the role of space and time the folded gate $V$ remains unitary (unital), which might be expressed as

$$\begin{align*}
\begin{array}{c}
\text{●} \ = \ \otimes \\
\text{□} \ = \ \otimes
\end{array}
\end{align*}$$

Note, that dual unitarity of $V$ is equivalent to the impurity interaction $U$ being T-dual [8], i.e., the partial transpose with respect to the first (or equivalently the second) site of $U$ being unitary. Such gates can be parameterized as [7]

$$U = (u_+ \otimes u_-) \exp \left( iJ \sigma_{g^2-1} \otimes \sigma_{g^2-1} \right) (v_+ \otimes v_-).$$

with $\sigma_i$ the generalized Gell-Mann matrices, $J \in [0, \pi/4]$ and $u_+, v_+ \in U(q)$. This parameterization is exhaustive for $q = 2$ only.

For T-dual impurities and hence dual-unitary gates $V$ we observe that $T_{\tau}$ is generically diagonalizable. Moreover, the structure of nontrivial eigenvectors of $T_{\tau}$ can be described in some detail. For the right (left) eigenvector $|\lambda\rangle$ ($\langle \lambda|$) with eigenvalue $\lambda$, $\langle \lambda| T_{\tau} = \lambda \langle \lambda|$, $T_{\tau} |\lambda\rangle = |\lambda\rangle$, also the vector

$$|\lambda, s\rangle = |0\rangle^{\otimes s} \otimes |\lambda\rangle \otimes |0\rangle^{\otimes \rho - \tau - s}$$

(12)

and analogous expression for $\langle \lambda, s|$), with $s \in \{0, \ldots, \rho - \tau\}$ is a right (left) eigenvector of $T_{\tau}$ for $\rho > \tau$ corresponding to the same eigenvalue. Consequently, $\text{spec}(T_{\tau}) \subseteq \text{spec}(T_{\tau+1})$. For each eigenvalue $\lambda$ there is thus $\tau_s$ such that $\lambda \in \text{spec}(T_{\tau_s})$ but $\lambda \notin \text{spec}(T_{\tau_s-1})$. The corresponding eigenvector (eigenoperator) has full support on the lattice on which $T_{\tau_s}$ acts. We use the notation $|\lambda\rangle$ ($\langle \lambda|$) exclusively for the right (left) eigenvector of $T_{\tau_s}$ and write $|\lambda, s\rangle$ ($\langle \lambda, s|$) for the right (left) eigenvectors of $T_{\tau}$ for $\tau > \tau_s$. We call such eigenvalues with $\tau_s = \tau$ relevant at $\tau$ and denote the leading (largest) relevant eigenvalue by $\lambda_1$. Furthermore, we denote the leading nontrivial eigenvalue of $T_{\tau}$ by $\lambda_0$ giving $|\lambda_1| \leq |\lambda_0| \leq 1$, where $1$ is the trivial eigenvalue. Assuming no accidental degeneracies the eigenvalue $\lambda$ is $(\tau - \tau_s + 1)$-fold degenerate. The projection $\mathcal{P}_{\lambda\tau}$ for given $\tau$ onto the corresponding eigenspace can be constructed as follows: For each $\rho \leq \tau$ we can choose the left and right eigenvectors corresponding to fixed $\tau_s = \rho$ to be biorthogonal, i.e., $\langle \lambda, s| r_{\lambda}, s' \rangle = \delta_{s, s'} \delta_{\lambda, \lambda'}$. Hence we describe the spectral properties of the transfer matrices $T_{\tau}$ in the following. $T_{\tau}$ is a vectorization of a quantum channel, a non-expanding map, with spectrum $\text{spec}(T_{\tau})$ contained in the complex unit disk [20]. Its eigenvalues are either real or come in complex conjugate pairs, and $T_{\tau}$ is in general not diagonalizable, but exhibits nontrivial Jordan blocks. Unitarity of interaction $U$ implying unitality of the folded gate $V$, i.e.,

$$\begin{align*}
\begin{array}{c}
\text{●} \ = \ \otimes \\
\text{□} \ = \ \otimes
\end{align*}$$

guarantees that there is always the trivial (left and right) eigenvector $|0\rangle^{\otimes\tau}$ with trivial eigenvalue $1 \in \text{spec}(T_{\tau})$. Moreover, unitarity of $V$ implies that the spectra of transfer matrices grow with $\tau$, i.e., $\text{spec}(T_{\tau}) \subseteq \text{spec}(T_{\tau+1})$.
The projections onto the corresponding eigenspaces are given by

$$\mathcal{P}_{\lambda, \tau} = \sum_{s=0}^{\tau - \tau_0} |\lambda, s\rangle \langle \lambda, s|$$

(13)

for nontrivial eigenvalues and $\mathcal{P}_{1, \tau} = |\rangle \langle \rangle^{\otimes \tau}$. They form -- using the numerically observed fact that $\mathcal{T}_{\tau}$ is diagonalizable -- a resolution of identity $\sum_{\lambda} \mathcal{P}_{\lambda, \tau} = \mathbb{1}_{g^2 \tau}$.

Writing $\mathcal{T}_{\tau} = \sum_{\lambda} \lambda \mathcal{P}_{\lambda, \tau}$ and inserting into Eq. (7) we obtain for $t = \lambda T + \delta$

$$C_{ab}(t) = \sum_{\lambda, \sigma} \lambda^{L-\delta} \sigma^\delta \langle \langle \lambda | \otimes \langle b | \rangle \rangle \langle \sigma | \rangle \langle a | \otimes \langle \lambda | \rangle \rangle ,$$

(14)

where the sums run over all nontrivial eigenvalues $\lambda \in \text{spec}(\mathcal{T}_{\tau})$ and $\sigma \in \text{spec}(\mathcal{T}_{\tau+1})$ for which $\tau_\lambda = \tau$ and $\tau_\sigma = \tau + 1$. The latter restriction is due to the property that $\text{tr} (\mathcal{P}_{\lambda, \tau} \otimes \mathbb{1}_{g^2}) \mathcal{P}_{\sigma, \tau+1} = 0$ if $\tau_\lambda < \tau$ or $\tau_\sigma < \tau + 1$, essentially following from $\text{tr}(a) = \text{tr}(b) = 0$; see App. E for details. This justifies the notion of relevant eigenvalues, as only the eigenvalues relevant at $\tau$ and $\tau + 1$ contribute to the correlation functions (8). Their asymptotic scaling is hence determined by the leading relevant eigenvalues of $\mathcal{T}_{\tau}$ and $\mathcal{T}_{\tau+1}$, respectively. Assuming unique leading relevant eigenvalues $\lambda_1$ and $\sigma_1$, the correlations scale as

$$C_{ab}(t) \sim \lambda_1^{L-\delta} \sigma^\delta \langle \langle \lambda_1 | \otimes \langle b | \rangle \rangle \langle \sigma_1 | \rangle \langle a | \otimes \langle \lambda_1 | \rangle \rangle ,$$

(15)

if both $L - \delta$ and $\delta$ are large. Here the factors involving the scalar products of eigenvectors as well as initial and final operators depend on $\tau$ only but not on $L$ and $\delta$. Hence for fixed $\tau \geq 1$ and arbitrary $\delta$ correlations are bounded by $|C_{ab}(t)| \leq \text{const} \times (\max \{ |\lambda_1|, |\sigma_1| \})^T$, implying exponential suppression of correlations in large but finite systems for $t > L$, provided there is a spectral gap, i.e. $|\lambda_1| < 1$, $|\sigma_1| < 1$. By numerically computing both the leading nontrivial eigenvalue $\lambda_0$ and the leading relevant eigenvalue $\lambda_1$ for the respective largest accessible system sizes, we confirm that, with probability one, the leading (relevant) eigenvalues have modulus smaller than one. For this we choose more than 1000 realizations with fixed $J = 1/2$ and Haar random $u_\pm, v_\pm \in U(q)$, see Eq. (11). The corresponding probability densities $p(|\lambda|)$ are depicted in Fig. 1(a,b). For $q \in \{3, 4\}$ the probability density $p(|\lambda|)$ approaches zero for $|\lambda| \to 1$ for both the leading nontrivial and leading relevant eigenvalue. Although this is not the case for qubits, $q = 2$, we found no instance for which $|\lambda_1| = 1$. This difference between qubits, $q = 2$, and $q \geq 3$ might be related due the fact that T-dual impurity interactions cannot be maximally entangling as their entangling power is strictly smaller than the maximal possible value $[8, 37]$. In contrast, for larger $q$ the impurity interaction may exhibit the maximal possible entangling power. Further note that $p(|\lambda|)$ depends only weakly on $\tau$, i.e. the distributions depicted in Fig. 1 do not change significantly with $\tau$. We therefore conclude that the exponential suppression of correlation functions with system size is a generic feature of the boundary-chaos model in case of T-dual impurities.

This is also depicted in Fig. 2(a), where correlations for a representative example system are shown for system sizes between $L = 10$ and $L = 200$ clearly demonstrating the exponential dependence of correlations on $L$. In particular for $L = 200$ we find good qualitative agreement with the asymptotic result (15). Here the computationally maximal accessible system size is restricted by machine precision due to the exponential suppression. In particular for sufficiently small leading relevant eigenvalues correlations for much larger system sizes than $L = 200$ can be obtained. Time intervals in which correlations decay with $t$ correspond to $\tau$ for which $|\lambda_1| > |\sigma_1|$ while growing correlations correspond to $|\sigma_1| > |\lambda_1|$. Corrections to the asymptotic scaling are dominated by the next to leading relevant eigenvalues $\lambda_2$ and $\sigma_2$ of $\mathcal{T}_{\tau}$ and $\mathcal{T}_{\tau+1}$ and are of the order $(\lambda_2/\lambda_1)^{L-\delta} (\sigma_2/\sigma_1)^\delta$. Note that the gap between the leading and subleading eigenvalue will in general shrink as $\tau$ grows. In any case deviations to the asymptotic scaling are most prominent when $t \approx \tau L$, i.e., when either $\delta$ or $L - \delta$ are small as it is also seen in Fig. 2(a).

In order to establish exponential suppression of correlations with system size for all $\tau$ one needs to show that there is a spectral gap $\Delta_1 = \lim_{\tau \to \infty} \sup (1 - |\lambda_1(\tau)|) > 0$. This is possible for a specific system, where $\Delta_1 > 0$ and $|\lambda_{\tau+1}| < 1$ for $\tau > \tau_0$. The resulting exponential suppression then follows from the relation $\sum_{\lambda} \lambda \mathcal{P}_{\lambda, \tau} \approx \sum_{\lambda < |\lambda|} \lambda \mathcal{P}_{\lambda, \tau}$, meaning that the spectral gap $\Delta_1$ controls the exponential suppression of correlations.
0, where $\lambda_1(\tau)$ denotes the leading relevant eigenvalue at $\tau$. We are able to address this question only numerically by considering instead the leading nontrivial eigenvalue $\lambda_0(\tau)$ of $T_\tau$ and the corresponding spectral gap $\Delta_0 = \lim_{\tau \to \infty} (1 - |\lambda_0(\tau)|)$, as $\lambda_0(\tau)$ can be computed efficiently using Arnoldi iteration and the matrix-product structure of $T_\tau$. Note that $|\lambda_0(\tau)|$ grows monotonically with $\tau$. For T-dual impurities drawn from the ensemble defined above, i.e., fixed $J = 1/2$ and Haar random local unitaries, we find the probability $p(|\lambda_0(\tau + 1)| > |\lambda_0(\tau)|)$ for the leading eigenvalue to grow from $\tau$ to $\tau + 1$ to quickly decrease with $\tau$. This is illustrated in Fig. 3, where we depict $p(|\lambda_0(\tau + 1)| > |\lambda_0(\tau)|)$ as a function of $\tau$. Thus the leading eigenvalue will generically be constant from a fixed $\tau$ on. Combining this with the properties of the probability distribution for the leading eigenvalues we conclude that for generic choices of T-dual impurities there exists a finite spectral gap $\Delta_0 > 0$ and hence $\Delta_1 > 0$ implying exponential suppression of correlation functions (2) with system size for any $\tau = |t/L| \geq 1$. However, due to the overlaps between eigenstates and initial and final operators in Eq. (14) this might not be uniform in $\tau$.

![FIG. 2. Autocorrelation functions $C_{aa}(t)$ for (a) T-dual and (b) generic impurities for qubits, $q = 2$, and $a = \sigma_x$ via Eq. (7). Lines correspond to system sizes (top to bottom) $L = 10$ (blue), $L = 50$ (red), $L = 100$ (orange), $L = 150$ (green), and $L = 200$ (magenta). Blue circles denote correlation functions obtained from exact diagonalization at $L = 10$. Dashed lines depict the asymptotic scaling from (a) Eq. (15) and (b) $\sim \sigma_\delta^0$ with fitted prefactor.](image)

![FIG. 3. Probability $p(|\lambda_0(\tau + 1)| > |\lambda_0(\tau)|)$, i.e., for the largest nontrivial eigenvalue to grow when advancing from $\tau$ to $\tau + 1$ for various $q$ and for both the T-dual and the generic case. Here we use more than 2000 realizations and 500 for the largest accessible values of $\tau$, respectively from the same ensembles as used for Fig. 1. The maximum system size is given by $\tau + 1 = 11$, $\tau + 1 = 8$, and $\tau + 1 = 5$ for $q = 2, 3, 4$, respectively. In particular, for $q = 2, 3$ the probability is found to be zero, when there is no bar depicted. For $q = 4$ this holds only up to $\tau = 4$ as we compute the leading eigenvalue only up to $\tau + 1 = 5$.](image)

### IV. GENERIC UNITARY IMPURITIES

In the following we comment on the case of generic unitary impurity interactions $U \in U(q^2)$ for which correlation functions exhibit qualitatively different properties. This again can be understood in terms of the spectral properties of the transfer matrices $T_\tau$, which differ in some important points from the T-dual case. We list those differences obtained using both numerical and analytical arguments below. For more details we refer the reader to App. F. Firstly, $T_\tau$ fails to be diagonalizable but exhibits nontrivial Jordan blocks, which for eigenvalue $\lambda$ is of dimension $\tau - \tau_\lambda + 1$. Secondly, using the notation of Eq. (12), only $\langle \lambda, 0 \rangle$ and $|\tau, \rho - \tau\rangle$ is a left or, respectively, right eigenvector of $T_\rho$, implying $\text{spec}(T_\tau) \subseteq \text{spec}(T_{\tau + 1})$ also for generic impurity interactions. We have to consider the Jordan decomposition of the transfer matrices, for which the projection onto the Jordan block corresponding to $\lambda$ is not given by Eq. (13) but can only be constructed numerically for small $\tau$. Inserting the Jordan decomposition into Eq. (7) nevertheless yields the asymptotic scaling of correlations with system size. In contrast to the T-dual case now the notion of relevant eigenvalues breaks down as all pairs $\lambda, \sigma$ of eigenvalue $\lambda$ of $T_\tau$ and nontrivial eigenvalue $\sigma$ of $T_{\tau + 1}$ contribute to the correlation function. Note that the trivial eigenvalue $1 \in \text{spec}(T_\tau)$ is not excluded and in the absence of further eigenvalues with modulus one dominates the correlation function for $L - \delta$ being large. Replacing $T_\tau$ by the projection onto the trivial eigenspace...
|\rangle \langle \sigma|^\otimes \tau$ and reversing the mapping from the original tensor network (4) to the helical network (8) allows for some intuition on the role of the trivial eigenvector: The local operator $a$ cannot travel along the boundary for more than $\delta$ subsequent time steps before being scattered into the bulk, where it undergoes free dynamics. This minimizes the total number of scattering events at the impurity interaction from $t$ to $t \times \delta$ and hence gives rise to the dominant contribution to the correlation function. In contrast, the above process has a vanishing contribution to the correlation function in the T-dual case as a consequence of dual unitarity of the folded gates $V$. 

In order to estimate contributions from the remaining part of the spectrum we confirm that generically the leading nontrivial eigenvalue has modulus smaller than one by computing the leading nontrivial eigenvalue $\lambda_0$ for generic impurity interactions correlations are not exponential scaling of correlation decays for large $\tau$ and small (fixed) $\delta$ in terms of column transfer matrix (with leading nontrivial eigenvalue $\chi$, $|\chi| < 1$) instead of the row transfer matrices $\mathcal{T}_{\tau+1}$ we find the dominant contribution $\sim \chi^\tau$. This suggest exponential correlation decay on an extensive time scale $-L/\log |\chi|$, which is expected due to the $1/L$ density of impurities. In particular, this causes the amplitude of the revivals to decay exponentially with $\tau$ and the system will eventually thermalize.

V. CORRELATION FUNCTIONS FOR ARBITRARY LOCAL OPERATORS

The techniques to study correlations functions between local operators $a_0$ and $b_y$ acting on the first lattice site can be extended to local operators acting on arbitrary sites. To this end let $a_x$ and $b_y$ denote operators acting as traceless Hermitian operators $a$ and $b$ on lattice sites $x$ and $y$, respectively, and trivially otherwise. Again we are interested in correlation functions given by $\langle \mathcal{D}^{-1} \mathcal{U}^{-1} a_x \mathcal{U} b_y \rangle$. Due to unitality of the folded gates $W$ and the free dynamics in the bulk it is sufficient to consider only local operators $a_x$ acting on the lattice sites $x \in \{0, 1\}$ and $b_y$ acting on sites $y \in \{0, 2\}$. We denote their correlations by

$$D_{ab}(t) = \frac{1}{N} \text{tr} \left( \mathcal{D}^{-1} \mathcal{U}^{-1} a_x \mathcal{U} b_y \right).$$

Dynamical correlations for different lattice sites are just shifted in time by $\Delta t = \Delta t_x + \Delta t_y \leq 2(L-1)$ and are zero for times $t < \Delta t$. Here, $\Delta t_x = L-x/2$ if $x$ is even, $\Delta t_x = (x-1)/2$ if $x$ is odd, and $\Delta t_y = 0$ if $y = 0$, whereas $\Delta t_y = y/2$ if $y$ is odd, $\Delta t_y = L-(y+1)/2$ if $y$ is even, and $\Delta t_y = 0$ if $y = 0$. Hence we focus on the computation of $D_{ab}(t)$ and for the sake of concreteness consider only the case $x \neq 0 \neq y$ in the following. Writing again $t = \tau L + \delta$ we obtain a tensor network representation of $D_{ab}(t)$ as

$$D_{ab}(t) = \mathcal{D}^{-1} \mathcal{U}^{-1} a_x \mathcal{U} b_y.$$
VI. CONCLUSIONS

The concept of boundary chaos provides a minimal model of an interacting many-body systems, which despite exhibiting quantum chaos allows for a rigorous treatment via the mapping to the helical circuit. This effectively reduces the dimension of the relevant tensor networks $t \times L \rightarrow |t/L| \times L = t$. We use this mapping to efficiently compute correlation functions between local operators for system sizes much larger than what is accessible by direct computation for times up to a few multiples of system size. For impurity interactions fulfilling the T-duality property we establish exponential suppression of correlations with system size for all times and for local operators acting nontrivially at the first lattice site. For arbitrary locations of the operators this is still the case for almost all times as $L \rightarrow \infty$. In contrast, for generic impurity interactions, correlations show persistent revivals with a period given by the system size $L$ irrespective of the location of the operators. The amplitude of these revivals nevertheless decays exponentially with $\tau$ leading to thermalization on an extensive (in $L$) time scale. Our construction bears curious analogy with the method of Poincaré section in few-body dynamics [30], specifically with quantized Poincaré transfer operator [31, 32] as well as with the influence matrix approach to many-body quantum dynamics [33]. Our method ‘integrates out’ the free flights through swap circuits between subsequent

matrices $T_\tau$ and $T_{\tau+1}$ with the original boundary conditions given by $|\varnothing\rangle$. Replacing these transfer matrices by their spectral or Jordan decomposition respectively allows for studying the asymptotics of correlations.

Again, we first consider the case of T-dual impurity interactions. T-duality causes correlations to vanish for times $1 < t < L$, i.e. $\tau = 0$ and $\delta > 0$. For $\tau > 0$ the notion of relevant eigenvalues as defined for $x = y = 0$ breaks down. Instead, for $\delta \neq 1$, the pairs of nontrivial eigenvalues $\lambda$ of $T_\tau$ and $\sigma$ of $T_{\tau+1}$ with $\tau_{\sigma} = \tau_{\lambda} + 1$ contribute to the correlations. Each such pair gives a contribution $\sim \lambda^{L-\delta} \sigma^\delta$ to the correlation function. Assuming a spectral gap $\Delta_0 > 0$ this gives rise to exponential suppression of correlations. This is depicted in Fig. 4(a), where correlations clearly show exponential suppression for $\delta \neq 1$. The asymptotic scaling, however, at fixed $\tau$ is typically not given by a single pair of eigenvalues, as pairs for which $\sim \lambda^{L-\delta} \sigma^\delta$ decreases with $\delta$ compete with pairs for which this expression grows with $\delta$, leading to the switching between positive and negative slope of the correlations at fixed $\tau$. In contrast to the above, for $\delta = 1$ the correlations are approximately independent from system size leading to persistent $L$-periodic revivals. These revivals are due to the trivial eigenvalue of $T_\tau$, which gives a non-vanishing contribution to the correlation function for $\delta = 1$. Instead the correlations are determined by a two-qubit quantum channel $\mathcal{M}$ introduced in Ref. [7], where it governs correlations along light rays in dual-unitary brickwork quantum circuits. The asymptotic correlation functions then read $\langle b | \mathcal{M}^\tau | a \rangle$. Hence the amplitude of the revivals decays with $\tau$ as $\chi$ with $\chi$ the leading nontrivial eigenvalue of $\mathcal{M}$, implying thermalization on an extensive time scale $-L/|\log |\chi||$ for typical T-dual impurities. In contrast if either of the local operators $a_x$ or $b_y$ acts on lattice site $x = 0$ or $y = 0$, respectively, revivals of correlations do not appear and correlations are exponentially suppressed at all times.

For generic impurity interactions the analysis of correlation functions is up to minor details the same for $D_{ab}(t)$, Eq. (18), and $C_{ab}(t)$, Eq. (2). That is, for large $L$ the correlations are dominated by the trivial eigenvector of $T_\tau$ leading to persistent revivals of correlations with period $L$, which decay on an extensive time scale as discussed in the case of $x = y = 0$. For both $L$ and $\delta$ large, correlations scale as $\sigma_0^{\delta-2}$ with $\sigma_0$ the leading nontrivial eigenvalue of $T_{\tau+1}$. The correlations and their asymptotic scaling are depicted in Fig. 4(b) and show qualitatively the same behavior as in Fig. 2(b) for $x = y = 0$. 
impurity interactions and hence is able to exactly describe the influence of the trivial bulk on the nontrivial boundary. As swap gates represent a particular solution of Yang-Baxter equation, one may expect that our concept can be extended to boundary perturbations of generic integrable circuits where one would need to ‘integrate out’ interacting integrable dynamics between collisions. Aside from a natural extension of our analysis to operator entanglement and out-of-time-order correlations, it may allow for a rigorous proof of ETH once the existence of a spectral gap can be established. Moreover, one should attempt a rigorous analysis of the spectral form factor, which should reveal more structure than in uniform dual-unitary circuits \[8\], e.g. we observe a nontrivial Thouless time scale; see App. B.

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**Appendix A: Parameterization of T-dual Impurities**

A unitary matrix \( U \in U(q^2) \) acting on \( \mathbb{C}^q \otimes \mathbb{C}^q \) is called T-dual \[8\], if its partial transpose \( U^{T_1} \) (with respect to the first tensor factors) is unitary as well. More precisely, denoting the matrix elements with respect to the canonical product basis by \( U_{ij}^{T_1} \), the matrix elements of the partial transpose are given by

\[
(U^{T_1})^{ij}_{kl} = U^{kj}_{il}.
\]

(A1)

Equivalently, one might define T-duality by requiring that the partial transpose with respect to the second tensor factor is unitary, as unitarity under partial transpose with respect to one tensor factor implies unitarity under partial transpose with respect to the other tensor factor. A parametrization of a subset of T-dual gates can be obtained from a parametrization of dual-unitary gates \[18, 38\] (which is complete for \( q = 2 \) \[7\]):

\[
U_{du} = (u_+ \otimes u_-) \exp \left(iJ \sigma_{q^2-1} \otimes \sigma_{q^2-1}\right) P (v_- \otimes v_+) \in U(q^2)
\]

(A2)

with \( \sigma_i \) the generalized Gell-Mann matrices, \( J \in [0, \pi/4] \) and \( u_\pm, v_\pm \in U(q) \). Hence T-dual gates which are of the form \( U = U_{du}P \) can be parameterized as given by Eq. (11) This parametrization, however, is neither surjective nor injective for \( q > 2 \). For numerical investigations we fix \( J = 1/2 \) throughout this article, and either pick a fixed set of generic local unitaries \( u_\pm, v_\pm \) or sample them randomly and Haar distributed in \( U(q) \). Note, that the operator entanglement entropy \( E(U) = \sin^2(2J)/2 \) \[37\] as well as the entangling power \( e(U) = 2 \sin^2(2J)/3 \% \) for \( q = 2 \) is independent of the local unitaries. In particular the latter is strictly smaller than the maximal value of \( e(U) = 2/3 \) if \( J < \pi/4 \) and hence the ensembles of T-dual gates considered do not give rise to maximally entangling (chaotic) dynamics.

**Appendix B: Spectral Statistics**

In this section we present some numerical results on spectral properties of the quantum circuits studied in the main text in order to justify our hypothesis, that the impurity interaction on the circuit’s boundary is sufficient, despite non-interacting dynamics in the bulk, to generate random-matrix spectral statistics (and hence justify the name ‘boundary chaos’). Note that in a weaker context, namely considering local perturbation to integrable interacting spin chain, observation of quantum chaos has been reported in the literature before, see e.g. \[23, 27, 29\].

To this end we consider two measures of level statistics: the nearest neighbor level-spacing distribution \( p(s) \) and the spectral form factor \( K(t) \). We find in general good agreement with random matrix theory (RMT) with the notable exception of T-dual impurities for qubits \( q = 2 \). In Fig. 5(a) we show the level-spacing distribution for the cases for which we depict dynamical correlation functions in Fig. 2 in the main text, where \( q = 2 \). At the chosen system size of \( L = 14 \) the level-spacing distribution of the T-dual impurity interaction resembles Wigner-Dyson distribution corresponding to the circular orthogonal ensemble (COE), indicating existence of an antiunitary symmetry of the circuit. For the generic impurity interaction the distribution is close to that of the circular unitary ensemble (CUE) but is slightly shifted to the left. This seems to be reminiscent of a generic feature of the systems studied here: For small system size the distribution is shifted to the left but approaches the
corresponding random matrix result for large enough systems, possibly beyond what can be studied by numerical exact diagonalization. However, except for T-dual qubit (|q| = 2) circuits, we find random matrix spectral statistics even for moderate system sizes. This is indicated in Fig. 5(b) for ensembles of impurities and local Hilbert space dimensions |q| = 2 and |q| = 3. We computed spectral statistics also for |q| = 4 but do not show the corresponding data as they can not be distinguished from |q| = 3 on the shown scale. In order to improve statistics we computed level statistics for an ensemble of systems with randomly sampled impurity gates, while we checked that the histograms of level spacing distribution do not change significantly for fixed typical impurities (not shown). In the T-dual case the parameter J = 1/2 entering the parameterization (11) is kept fixed while the local units U± and v± are drawn independently from the Haar ensemble. In the generic case impurities are chosen Haar random. Except for qubits in the T-dual case the level spacing distribution is close to that of the CUE. For T-dual qubits the distribution is closer to the COE but it is shifted to the left. The RMT result might be approached for larger system sizes.

As a further spectral indicator of quantum chaos we compute the spectral form factor K(t), i.e., the Fourier transform of the connected two-point function of the spectral density. The spectral form factor is computed via $K(t) = \langle \text{tr} \left( \mathcal{U}^t \mathcal{U}^\dagger \right)^2 \rangle$, where the brackets denote the average over the ensembles of systems with varying impurity gates as described above. This averaging is now necessary as the spectral form factor is not self-averaging [39]. In Fig. 5(c) we again show data for both T-dual and generic impurities for local Hilbert space dimension |q| = 2 and |q| = 3. There, both the spectral form factor and the average are measured in units of Heisenberg time $t_H = q^{L+1}$. We again computed the spectral form factor also for |q| = 4 but omit showing the data as it yields nearly identical results as |q| = 3. For the sake of a clearer presentation we additionally perform a moving time average over a window of width $\Delta t = 20$. As it is the case for the level spacing distribution except for T-dual qubits we find good agreement with the spectral form factor of the CUE given by $K(t) = t$ for $t < t_H$ and $K(t) = t_H$ for $t > t_H$ after some initial non-universal regime corresponding to the so-called Thouless time $t_{\text{Th}}$. Remarkably, for generic impurity interactions and |q| $\geq$ 3 we find almost no deviation from RMT spectral form factor for all times, indicating that $t_{\text{Th}} = 0$ for large |q|. This curious observation certainly justifies a separate study. For T-dual qubits we find strong enhancement of $K(t)$ at times $t = \tau L$ as it is shown in the inset where no moving time average is performed. For the moving time average in the main panel we neglect these times and rescale the Heisenberg time by a factor $(L - 1)/L$ accordingly. This leads to a spectral form factor close to, but slightly below, the spectral form factor of the COE given by $K(t) = 2t - 2t^2/t_H + 2t^3/t_H^2 - \ldots$.
Appendix C: Boundary to Helix Mapping

In this section we provide the proof for the equality claimed in Eq. (7). In fact, we prove a stronger statement by providing a matrix product representation of integer powers of the evolution operator

\[ \mathcal{W} = S_2V_{0,1}S_1 \]

which is shown here for odd \( L \). To this end we rewrite \( \mathcal{W} \) in terms of the folded gate \( V \) instead of \( W \) and introduce the combined action of the swap gates \( S_{x,x+1} \) acting on neighboring lattice sites \( x \) and \( x + 1 \) in the first (last) layer as \( S_1 \) \( (S_2) \). Additionally, we indicate the labels of the lattice sites above. The matrix-product representation of \( \mathcal{W}^t \) is obtained by contracting the inner legs corresponding to the swap gates to a compact form and by translating the swap gates which connect to in- and out-going legs, respectively, into suitable permutations of local lattice sites. To make this more formal let us begin by introducing some notation. We fix the size \( L \) of the system and we denote the local Hilbert spaces obtained by the vectorization of local operators by \( K_x \sim \mathbb{C}^{q^2} \) for the lattice sites \( x \in \{0,1,\ldots,L\} \). That is, \( \mathcal{W} \) acts on \( \mathcal{K} := \bigotimes_{x=0}^{L} K_x \sim \mathbb{C}^{q^2(L+1)} \). Moreover we define \( [L] := \{1,2,\ldots,L\} \) and introduce the Hilbert space \( \mathcal{K}_{[L]} := \bigotimes_{x \in [L]} K_x \), i.e., \( \mathcal{K} = K_0 \otimes \mathcal{K}_{[L]} \). By fixing an orthonormal basis \( \{|i\} |i \in \{0,1,\ldots,q^2-1\} \) of \( \mathbb{C}^{q^2} \) we obtain a product basis both in \( \mathcal{K} \) and \( \mathcal{K}_{[L]} \) whose elements we denote by \( |i_0i_1\ldots i_L\rangle := |i_0\rangle \otimes |i_1\rangle \otimes \ldots \otimes |i_L\rangle \) and \( |i_1\ldots i_L\rangle \), respectively.

The symmetric group \( S_L \) on \( L \) elements acts on \( \mathcal{K}_{[L]} \) via the unitary representation \( P : S_L \rightarrow U(\mathcal{K}_{[L]}) \) which permutes tensor factors. More precisely, we identify elements of the product basis \( \{|i_1\ldots i_L\rangle \} \) \( |i_x \in \{0,\ldots,q^2-1\} \) with maps \( i : [L] \rightarrow \{|i\} |i \in \{0,\ldots,q^2-1\} \) via \( i \mapsto |i(1)\ldots i(L)\rangle = |i_1\ldots i_L\rangle \). A left action of \( S_L \) on the set of these maps is given by \( i \mapsto i \circ \sigma^{-1} \) for \( \sigma \in S_L \). This translates to a left action of \( S_L \) on the product basis which gives rise to the unitary operator \( P_{\sigma} \) defined by its action on basis vectors

\[ P_{\sigma} |i_1i_2\ldots i_L\rangle := |i_{\sigma^{-1}(1)}i_{\sigma^{-1}(2)}\ldots i_{\sigma^{-1}(L)}\rangle \]

and linear extension. Note, that with this definition one has

\[ P_{\rho}P_{\sigma} |i_1\ldots i_L\rangle = P_{\rho} |i_{\sigma^{-1}(1)}\ldots i_{\sigma^{-1}(L)}\rangle = |i_{\sigma^{-1}\rho^{-1}(1)}\ldots i_{\sigma^{-1}\rho^{-1}(L)}\rangle = |i_{(\rho\sigma)^{-1}(1)}\ldots i_{(\rho\sigma)^{-1}(L)}\rangle = P_{\rho\sigma} |i_1\ldots i_L\rangle \]

for any \( \sigma, \rho \in S_L \). We represent the unitary operator \( P_{\sigma} \) diagrammatically by a box as

\[ P_{\sigma} = \begin{array}{c} \sigma \
 \end{array} \]

for each permutation \( \sigma \in S_L \). Let us make a remark about the diagrammatic technique used here: As \( P_{\sigma} \) maps simple tensors to simple tensors with only its tensor factors permuted the box in the above definition might be thought of \( L \) wires each of which connects an in-going leg with an out-going leg. In a diagrammatic representation those wires might be arbitrarily continuously deformed without changing the operator as long as in-going and out-going legs are held fixed at the corresponding lattice site.

In preparation for our proof we consider specific permutations and study some of their relations in the following. To this end it is convenient to adapt the following convention: For any integer \( x \in \mathbb{Z} \) we take the representative of \( x \mod L \) in the set \([L]\), i.e., we identify \( L \equiv 0 \). Using this convention we define permutations \( \sigma_{\delta} \in S_L \), indexed by \( \delta \in \mathbb{Z} \), by

\[ \sigma_{\delta}(x) := \begin{cases} 2(x + \delta) \mod L & \text{if } (x + \delta) \mod L < \frac{L+1}{2}, \\ (-2(x + \delta) + 1) \mod L & \text{if } (x + \delta) \mod L \geq \frac{L+1}{2} \end{cases} \]

for \( x \in [L] \). A simple calculation shows that the inverse of \( \sigma_{\delta} \) is given by

\[ \sigma_{\delta}^{-1}(x) := \begin{cases} (\frac{x}{2} - \delta) \mod L & \text{if } x \text{ is even}, \\ (-\frac{x+1}{2} - \delta) \mod L & \text{if } x \text{ is odd.} \end{cases} \]
By straightforward computations one moreover obtains the following properties.

**Lemma 1:** Given $\alpha, \delta \in \mathbb{Z}$ one has

(i) $\sigma^{-1}_\alpha \sigma_{\delta+\alpha} = \eta_{\delta} \in S_L$, where $\eta_{\delta}$ denotes the periodic shift by $\delta$, i.e., $\eta_{\delta}(x) := (x + \delta) \mod L$ for $x \in [L]$ and

(ii) $\pi_{\delta} := \sigma_{\alpha} \sigma^{-1}_{\delta+\alpha} \in S_L$ is given by

$$\pi_{\delta}(x) = \begin{cases} 
(x - 2\delta) \mod L & \text{if } x \text{ is even and } \left\lfloor \frac{x}{2} - \delta \right\rfloor \mod L < \frac{L+1}{2}, \\
(-x + 2\delta + 1) \mod L & \text{if } x \text{ is even and } \left\lfloor \frac{x}{2} - \delta \right\rfloor \mod L \geq \frac{L+1}{2}, \\
(-x + 2\delta + 1) \mod L & \text{if } x \text{ is odd and } \left\lfloor \frac{x+1}{2} - \delta \right\rfloor \mod L < \frac{L+1}{2}, \\
(x + 2\delta) \mod L & \text{if } x \text{ is odd and } \left\lfloor \frac{x+1}{2} - \delta \right\rfloor \mod L \geq \frac{L+1}{2}
\end{cases} \quad (C7)$$

for $x \in [L]$ and independent of $\alpha$.

In case $\delta = 1$ the second part of Lemma 1 reduces to

$$\pi_1(x) = \begin{cases} 
x - 2 & \text{if } x \text{ is even and } x > 2 \\
1 & \text{if } x = 2 \\
L - 1 & \text{if } x \text{ is odd and } x = L \\
x + 2 & \text{if } x \text{ is odd and } x < 2
\end{cases} \quad (C8)$$

for $x \in [L]$.

As last preparatory step before formulating our claim we introduce operators $R_\delta$ for $\delta \in \{0, 1, \ldots, L\}$ acting on $K$ in the form of two-layer matrix-product operators intertwined by their boundary conditions which are most conveniently defined by their respective diagrammatic representation

$$R_\delta := \begin{array}{c}
\begin{array}{c}
\ldots \\
\ldots \\
0 1 2 \\
\ldots \\
L - \delta \\
\ldots \\
L - \delta + 2 \\
\ldots \\
\ldots \\
x \\
\end{array}
\end{array} \quad (C9)$$

Here all local tensors in the lower layer corresponding to lattice sites in $[L]$ as well as those in the upper layer corresponding to lattice sites in $\{1, 2, \ldots, L - \delta\}$ are given by swap gates. In contrast, the local tensors at the remaining lattice sites in the upper layer are given by the folded gates $V$ with the orientation indicated by the mark in the respective top corner. Note that $R_0 = 1_K$. One might also rewrite $R_\delta$ as a quantum circuit with $2L$ layers and
one nontrivial gate per layer as

\[ R^\prime_\delta = R^\prime_\delta (V_{L-\delta,L-\delta+1} \cdots V_{L-2,L} V_{L-1,L}) R^\prime_\delta. \] \hspace{1cm} (C10)

Here, we introduced \( R^\prime_\delta \) (\( R^\prime\prime_\delta \)) which combines the action of the swap gates in the first \( L \) (last \( L-\delta \)) layers. The above rewriting of \( R_\delta \) additionally demonstrates its unitarity.

We are now able to state our main claim as

**Proposition:** For any non-negative integer \( t \), written as \( t = \tau L + \delta \) with unique non-negative integer \( \tau \) and \( \delta \in \{0,1,\ldots,L-1\} \), one has

\[ W^t = (I_K \otimes P_{\sigma_0}) R_\delta R^\tau_L \left( I_K \otimes P_{\sigma_0}^{-1} \right). \] \hspace{1cm} (C11)

Intuitively, the first and the last factor describe the action of those swap gates which connect to out-going and in-going legs of a diagrammatic representation of \( W \) while the factor \( R_\delta R^\tau_L \) combines the effect of the \( t \) impurities at the systems boundary. We prove the proposition by induction on \( t \).

The case \( t = 0 \) is trivial as \( \tau = \delta = 0 \) and \( R_0 = I_K \) yielding

\[ (I_K \otimes P_{\sigma_0}) R_0 \left( I_K \otimes P_{\sigma_0}^{-1} \right) = \left( I_K \otimes P_{\sigma_0} P_{\sigma_0}^{-1} \right) = I_K = W^0. \] \hspace{1cm} (C12)

However, we additionally need to consider the case \( t = 1 \), i.e., \( \tau = 0 \) and \( \delta = 1 \), separately. It suffice to check that \( W |i_0 \cdots i_L \rangle = (I_K \otimes P_{\sigma_1}) R_1 \left( I_K \otimes P_{\sigma_0}^{-1} \right) |i_0 \cdots i_L \rangle \) for any basis state \(|i_0 \cdots i_L \rangle\). We restrict ourselves to odd \( L \), the case of even \( L \) works analogously. To this end, we use the factorization of \( W \) introduced in Eq. (C1) and obtain

\[ W |i_0 i_1 \cdots i_L \rangle = S_2 V_{0,1} S_1 |i_0 i_1 \cdots i_L \rangle \]
\[ = S_2 V_{0,1} |i_1 i_0 i_2 \cdots i_L i_{L-1} \rangle \]
\[ = \sum_{k,l} S_2 V^{kl}_{i_1 i_0} |k l i_3 i_2 \cdots i_L i_{L-1} \rangle \]
\[ = \sum_{k,l} V^{kl}_{i_1 i_0} |k i_3 l i_5 i_2 \cdots i_L i_{L-3} i_{L-1} \rangle, \] \hspace{1cm} (C16)
where $V^{kl} := \langle kl | V | ij \rangle$ denotes the matrix elements of the folded gate $V$. On the other hand we have

\begin{equation}
(I_{K_0} \otimes P_{\sigma_1}) \mathcal{R}_1 (I_{K_0} \otimes P_{\sigma_0}^{-1}) |i_0 \cdots i_L\rangle = (I_{K_0} \otimes P_{\sigma_1}) \mathcal{R}^{\prime \prime}_0 V_{L-1,L} \mathcal{R}^{\prime}_0 (I_{K_0} \otimes P_{\sigma_0}^{-1}) |i_0 \cdots i_L\rangle \tag{C17}
\end{equation}

\begin{equation}
= (I_{K_0} \otimes P_{\sigma_1}) \mathcal{R}^{\prime \prime}_0 V_{L-1,L} \mathcal{R}^{\prime}_0 |i_0 i_{\sigma_0(1)} \cdots i_{\sigma_0(L)}\rangle \tag{C18}
\end{equation}

\begin{equation}
= (I_{K_0} \otimes P_{\sigma_1}) \mathcal{R}^{\prime \prime}_0 V_{L-1,L} i_{\sigma_0(1)} i_{\sigma_0(2)} \cdots i_{\sigma_0(L)} i_0 \rangle \tag{C19}
\end{equation}

\begin{equation}
= \sum_{k,l} V^{kl}_{i_1 i_0} (I_{K_0} \otimes P_{\sigma_1}) \mathcal{R}^{\prime \prime}_0 |i_0 i_{\sigma_0(1)} i_{\sigma_0(2)} \cdots k l \rangle \tag{C20}
\end{equation}

\begin{equation}
= \sum_{k,l} V^{kl}_{i_1 i_0} (I_{K_0} \otimes P_{\sigma_1}) |k i_{\sigma_0(1)} i_{\sigma_0(2)} \cdots l \rangle \tag{C21}
\end{equation}

\begin{equation}
= \sum_{k,l} V^{kl}_{i_1 i_0} |k i_{\sigma_0^{-1}(1)} i_{\sigma_0^{-1}(2)} i_{\sigma_0^{-1}(3)} i_{\sigma_0^{-1}(4)} \cdots i_{\sigma_0^{-1}(L)} \rangle \tag{C22}
\end{equation}

\begin{equation}
= \sum_{k,l} V^{kl}_{i_1 i_0} |k i_{\pi_1(1)} i_{\pi_1(2)} i_{\pi_1(4)} \cdots i_{\pi_1(L)} \rangle, \tag{C23}
\end{equation}

where in line (C21) we used $\sigma_0(L) = 1$ and in line (C22) we used $\sigma_1(L) = 2$. In the last line we inserted the definition of $\pi_1$ from Lemma 1 (ii). Using the lemma (and Eq. (C8)) we see that the expressions (C16) and (C23) coincide thereby proving our claim for $t = 1$.

Now let us assume Eq. (C11) is true for any $t' \leq t$. This implies the claim for $t + 1$, corresponding to the step $\delta \rightarrow \delta + 1$, as we demonstrate in the following. Note, that all preceding arguments work also for the borderline cases $\delta = 0$ and $\delta = L - 1$. In any case we have

\begin{equation}
\mathcal{W}^{t+1} = \mathcal{W} \mathcal{W}^t = \mathcal{W} (I_{K_0} \otimes P_{\sigma_\delta}) \mathcal{R}_\delta \mathcal{R}^{\prime \prime}_L (I_{K_0} \otimes P_{\sigma_\delta}^{-1}) \tag{C24}
\end{equation}

\begin{equation}
= (I_{K_0} \otimes P_{\sigma_1}) \mathcal{R}_1 (I_{K_0} \otimes P_{\sigma_\delta}^{-1}) (I_{K_0} \otimes P_{\sigma_\delta}) \mathcal{R}_\delta \mathcal{R}^{\prime \prime}_L (I_{K_0} \otimes P_{\sigma_\delta}^{-1}) \tag{C25}
\end{equation}

\begin{equation}
= (I_{K_0} \otimes P_{\sigma_1}) \mathcal{R}_1 (I_{K_0} \otimes P_{\sigma_\delta}^{-1}) P_{\sigma_\delta} \mathcal{R}_\delta \mathcal{R}^{\prime \prime}_L (I_{K_0} \otimes P_{\sigma_\delta}^{-1}) \tag{C26}
\end{equation}

\begin{equation}
= (I_{K_0} \otimes P_{\sigma_1}) \mathcal{R}_1 (I_{K_0} \otimes P_{\eta_\delta}) \mathcal{R}_\delta \mathcal{R}^{\prime \prime}_L (I_{K_0} \otimes P_{\sigma_\delta}^{-1}), \tag{C27}
\end{equation}

where we used the assumption for $\mathcal{W}^t$ ($\mathcal{W}^1$) in the first (second) line. In the last line we used the fact, that $P$ is a homomorphism as well as Lemma 1 (i). In order to show that the right hand side of Eq. (C27) equals $(I_{K_0} \otimes P_{\sigma_{\delta+1}}) \mathcal{R}_{\delta+1} \mathcal{R}^{\prime \prime}_L (I_{K_0} \otimes P_{\sigma_\delta}^{-1})$ it suffices to show that

\begin{equation}
(I_{K_0} \otimes P_{\sigma_{\delta+1}}) \mathcal{R}_{\delta+1} = (I_{K_0} \otimes P_{\sigma_1}) \mathcal{R}_1 (I_{K_0} \otimes P_{\eta_\delta}) \mathcal{R}_\delta \tag{C28}
\end{equation}

\begin{equation}
\iff \mathcal{R}_{\delta+1} = (I_{K_0} \otimes P_{\sigma_1}^{-1}) (I_{K_0} \otimes P_{\sigma_1}) \mathcal{R}_1 (I_{K_0} \otimes P_{\eta_\delta}) \mathcal{R}_\delta \tag{C29}
\end{equation}

\begin{equation}
= (I_{K_0} \otimes P_{\sigma_1}^{-1}) \mathcal{R}_1 (I_{K_0} \otimes P_{\eta_\delta}) \mathcal{R}_\delta, \tag{C30}
\end{equation}

where in the last line we again used Lemma 1 (i). As $P_{\eta_\delta}$ implements a periodic shift on the lattice sites in $[L]$ conjugation of $\mathcal{R}_1$ by $(I_{K_0} \otimes P_{\eta_\delta})$ shifts the action of the only non-swap gate from lattice site $L$ to lattice site $L - \delta$, i.e.,

\begin{equation}
(I_{K_0} \otimes P_{\eta_\delta}^{-1}) \mathcal{R}_1 (I_{K_0} \otimes P_{\eta_\delta}) =
\end{equation}

\begin{equation}
= \eta_{-\delta} \tau_{\delta} \ldots
green\ldots
green\ldots
\end{equation}

which might also be checked on the basis vectors by using the representation (C10). Here we used $\eta_{-\delta} = \eta_{\delta}^{-1}$ in the diagrammatic representation. Using the above equation, the right hand side of Eq. (C30) can be evaluated
diagrammatically (or by again using a similar representation as in Eq (C10)) as

\[
(I_{K_0} \otimes P_{\eta_3}^{-1}) \mathcal{R}_1 (I_{K_0} \otimes P_{\eta_3}) \mathcal{R}_\delta = \begin{array}{c}
0 & 2 & \ldots & L - \delta & L \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 2 & \ldots & L - \delta & L
\end{array}
\]

\[
= \begin{array}{c}
0 & 2 & \ldots & L - \delta & L \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 2 & \ldots & L - \delta & L
\end{array}
\]

\[(C32)\]

\[
= \mathcal{R}_{\delta+1}.
\]

\[(C33)\]

\[(C34)\]

This proves the claim for \(t + 1\) and hence completes the proof of the proposition. As a final remark we note that the above construction might be repeated for the unfolded circuit \(U\) in a similar fashion as no argument depends on the origin of the local Hilbert spaces \(K_x\) from the vectorization of operators. However, one needs to take the placement of the impurity interaction in the second rather than the first layer of the circuit into account.

We now come back to the computation of correlation functions \(C_{ab}(t)\) for local operators \(a_0\) and \(b_0\) acting nontrivially only on the first site, i.e., site 0. In vectorized form \(a_0\) reads

\[
|a_0\rangle = |a\rangle \otimes |\circ\rangle^L
\]

and is obviously an eigenvector of \(I_{K_0} \otimes P_\sigma\) with eigenvalue 1 for any permutation \(\sigma \in S_L\). The same of course holds for \(b_0\). Hence the correlation function at time \(t = \tau L + \delta\) is given by

\[
C_{ab}(t) = \langle b_0 | W^t | a_0 \rangle = \langle b_0 | \mathcal{R}_\delta \mathcal{R}_L^\tau | a_0 \rangle
\]

\[(C35)\]

\[
(C36)\]

where in the last step we contracted the first layer of swap gates and added the top layer minding that the internal wires corresponding to the swap gates can undergo arbitrary continuous deformations without changing the value of \(C_{ab}(t)\). The above diagrammatic representation is, however, just a ninety-degree counterclockwise rotated version of Eq. (8). This concludes the proof of the equality claimed in Eqs. (7) and (8).

### Appendix D: Spectral Properties of Transfer Matrices

In this section we present some numerical results on the spectral properties of transfer matrices \(T_{\tau}\) for both T-dual and generic impurity interactions as well as various local Hilbert space dimensions (of the original spatial lattice)
$q \in \{2, 3, 4\}$. As $\mathcal{T}_r$ is real in the sense that it maps the vectorization of Hermitian operators to the vectorization of Hermitian operators its eigenvalues are either real or come in complex conjugate pairs. We depict the respective spectral densities of nontrivial eigenvalues of the transfer matrices for the ensembles introduced in the previous sections in Fig. 6. In any case we find the distribution to be a superposition of the distribution of real eigenvalues and the distribution of complex conjugates pairs of eigenvalues. The former is clearest seen for the T-dual case, where it is supported on the whole interval $[-1, 1]$, and is less pronounced and of strictly smaller support in the generic case. In the T-dual case the distribution of complex eigenvalues has highest weight on a ring with mean radius approximately $1/2$ for both $q = 3$ and $q = 4$, with $q = 2$ again showing different behavior. In contrast, for generic impurities the distribution is supported mainly within the disk of radius $1/q$. From our numerical experiments we find these distributions to be stable upon varying $\tau$ (not shown).

Therefore one might hope, that also the leading nontrivial eigenvalue $\lambda_0 = \lambda_0(\tau)$ does not grow when increasing $\tau$ at least when $\tau$ is sufficiently large. In Ref [20] it is demonstrated that this happens typically for $q = 2$. As $|\lambda_0|$ is a monotonically non-decreasing function of $\tau$ which is bounded by 1 the leading nontrivial eigenvalue converges as $\tau \to \infty$. In order to estimate whether this limit is smaller than 1 and hence whether there is a spectral gap $\Delta_0 > 0$ we compute the leading nontrivial eigenvalues for random realizations of the impurity interaction drawn from the ensembles introduced in the main text for various $\tau$. From this we estimate the probability $p = p(|\lambda_0(\tau + 1)| > |\lambda_0(\tau)|)$ for the leading nontrivial eigenvalue to grow when increasing $\tau$. In order to efficiently compute the leading nontrivial eigenvalue we use the following numerical protocol: We project a random initial vector, which is orthogonal to the trivial eigenvector, to the Jordan subspace (eigenspaces in the T-dual case) corresponding to the leading nontrivial eigenvalues. This is achieved by applying $\mathcal{T}_r^k$ to the initial state for sufficiently large $k$ using the matrix product structure of $\mathcal{T}_r$. Subsequently we construct the Krylov subspace containing the leading nontrivial eigenvalue’s Jordan subspace via Arnoldi iteration applied to the projected initial vector taking also the possibility of a complex conjugate pair of leading nontrivial eigenvalues into account. This allows to estimate the leading eigenvalue of $\mathcal{T}_r$ by diagonalizing $\mathcal{T}_r$ within the Krylov subspace. However, as a consequence of $\mathcal{T}_r$ in general exhibiting a nontrivial Jordan structure, the accuracy of the obtained eigenvalues is relatively low. Nevertheless, this procedure allows to estimate $p(|\lambda_0(\tau + 1)| > |\lambda_0(\tau)|)$ as it is shown in Fig. 3. The probability decreases fast with $\tau$ and we find no increase in the leading eigenvalue at the largest accessible system sizes, except for T-dual gates at $\tau = 4$. More precisely for $q = 2$ ($q = 3$) we compute the leading eigenvalue up to a maximum value of $\tau = 11$ ($8$) and find no increase in the leading eigenvalue for $\tau > 5$. For $q = 4$ we compute the leading eigenvalue only up to a maximum value of $\tau = 5$ and therefore Fig. 3 does not include data for $p(|\lambda_0(\tau + 1)| > |\lambda_0(\tau)|)$ for $\tau > 4$ if $q = 4$. We find no increase in the leading eigenvalue already for $\tau > 2$ for generic impurities and $q = 4$. In contrast, for T-Dual impurities we find a nonzero probability for the leading eigenvalue to increase even at the largest accessible $\tau$. Nevertheless $p(|\lambda_0(\tau + 1)| > |\lambda_0(\tau)|)$ is decreasing with $\tau$ for $\tau > 2$ and we can hope for it to approach zero as well for larger $\tau$ than what we can access numerically. Based on the above observations we conjecture, that generically the leading eigenvalue will become stationary for sufficiently large $\tau$ implying a nonzero spectral gap $\Delta_0$ and hence also a nonzero spectral gap for the leading relevant eigenvalue $\Delta_1 > 0$ in the T-dual case.

Appendix E: Eigenstates of transfer matrices in the T-dual case

In this section we motivate the notion of relevant eigenvectors (eigenoperators) in the case of T-dual impurities and derive the asymptotics of correlation functions given by Eq. (15). To this end we first construct the spectral decomposition

$$\mathcal{T}_r = \sum_\lambda \lambda \mathcal{P}_{\lambda, \tau}. \tag{E1}$$

More precisely, we demonstrate that the projections $\mathcal{P}_{\lambda, \tau}$ onto the eigenspaces corresponding to nontrivial eigenvalues $\lambda \in \text{spec}(\mathcal{T}_r)$ are given by Eq. (13). To this end, we show that the right eigenvectors $|r_\lambda, s\rangle$, Eq. (12), and the corresponding left eigenvectors $\langle l_\lambda, s|$ are biorthogonal. We initially restrict to the nontrivial eigenvalues and extend the notation from the main text by adding a subscript $\tau$ in the bra-ket notation, e.g., we write $|\phi\rangle_\tau$, $|\psi\rangle_\tau \in \mathbb{C}^q_\tau$ as well as $\langle \phi|\psi\rangle_\tau$ for the respective bra-vector in order to indicate the number $\tau$ of lattice sites. We moreover denote scalar products of such vectors by $\langle \phi|\psi\rangle_\tau$ and write $|\phi\rangle_\tau|\psi\rangle_\tau$ for rank-one operators. Additionally, we denote the right (left) eigenvectors for eigenvalue $\lambda$ of $\mathcal{T}_r$ with $\tau_\lambda = \tau$ by $|r_\lambda\rangle_{\tau_\lambda}$ ($\langle l_\lambda|$) and use the graphical representation

$$|r_\lambda\rangle_{\tau_\lambda} = \begin{array}{c} \lambda \end{array} \cdots \lambda \quad \text{and} \quad \tau_\lambda\langle l_\lambda| = \begin{array}{c} \lambda \end{array} \cdots \lambda.$$

$$\tag{E2}$$
where the number of out-going legs and, respectively, in-going legs is $\tau\lambda$. These eigenvectors obey

$$|r_\lambda, +\rangle_{\tau\lambda-1} := \ldots \underbrace{1}_{\lambda} \ldots = 0 = |r_\lambda, -\rangle_{\tau\lambda-1},$$  \hspace{1cm} (E3)

$$\tau\lambda-1|l_\lambda, +\rangle := \ldots \underbrace{1}_{\lambda} \ldots = 0 = \tau\lambda-1|l_\lambda, -\rangle.$$  \hspace{1cm} (E4)

Note, that contraction with the states $|\circ\rangle$ at a given subset of lattice sites corresponds to taking the partial trace with respect to the corresponding subsystem upon reversing the operator-state mapping. Moreover, this demonstrates that the corresponding operators have full support on the $\tau$ lattice.

Equations (E3) and (E4) can be obtained using unitality, Eq. (9), and dual unitality, Eq. (10), of the gate $V$. For instance we find

$$\mathcal{T}_{\tau\lambda-1} |r_\lambda, +\rangle_{\tau\lambda-1} = \lambda |r_\lambda, +\rangle_{\tau\lambda-1},$$  \hspace{1cm} (E5)

where the second equality holds due to unitality of $V$ (in backward time direction) and the third equality reflects the fact that $\mathcal{T}_{\tau\lambda} |r_\lambda\rangle_{\tau\lambda} = \lambda |r_\lambda\rangle_{\tau\lambda}$. Hence if $|r_\lambda, +\rangle_{\tau\lambda-1}$ was nonzero we would have $\lambda \in \text{spec}(\mathcal{T}_{\tau\lambda-1})$ in contrast to $\tau\lambda$ being minimal. This argument might be repeated for the remaining cases yielding all of Eqs. (E3) and (E4).

FIG. 6. Histogram of the nontrivial eigenvalues for $q = 2$, $\tau = 7$ (a, b), $q = 3$, $\tau = 4$ (c, d), and $q = 4$, $\tau = 3$ (e, f) for the T-dual case (a, c, e) as well as the generic case (b, d, f) for 500 realizations of the circuit for each case. The black circles have radii 1 and $1/q$, respectively. Each eigenvalue is weighted by the degree of its degeneracy in the T-dual case and by the dimension of the corresponding Jordan block in the generic case.
From this we prove the biorthogonality of the left and right eigenvectors of $T_{\tau}$ for fixed $\tau$. For $\lambda, \sigma \in \text{spec}(T_{\tau})$ with $\tau\lambda \neq \tau\sigma$ for all possible choices of $r$ and $s$ we find $\langle l_{\sigma}, s | r_{\lambda}, r \rangle = 0$ as at least one of the rightmost or leftmost legs of one of the states $\tau\langle l_{\sigma}, s \rangle$ or $| r_{\lambda}, r \rangle_{\tau}$ is contracted with $| \circ \rangle$. The same reasoning yields $\langle l_{\sigma}, s | r_{\lambda}, r \rangle = 0$ if $\tau\lambda = \tau\sigma$ but $r \neq s$. In the remaining case, $\tau\lambda = \tau\sigma$ and $r = s$ we have $\langle l_{\sigma}, s | r_{\lambda}, r \rangle = \langle l_{\sigma}|r_{\lambda}\rangle_{\tau} = \delta_{\tau\lambda}$ which vanishes if $\sigma \neq \lambda$ and which is one if $\sigma = \lambda$ as we can chose $| r_{\lambda} \rangle_{\tau}$ and $\langle l_{\sigma} | s \rangle$ biorthogonal. As by Eqs. (E3) and (E4) the trivial left and right eigenvectors are orthogonal to all other eigenvectors we conclude that the left and right eigenvectors given by Eq. (12) are biorthogonal and hence

$$P_{\lambda,\tau}P_{\sigma,\tau} = \sum_{s, r} | r_{\lambda}, s \rangle \langle l_{\lambda}, s | | r_{\sigma}, r \rangle \langle l_{\sigma}, l | = \sum_{s, r} | r_{\lambda}, s \rangle \langle l_{\sigma}, r |_{\tau} \delta_{r, s} \delta_{\tau, \lambda} = P_{\lambda,\tau} \delta_{\tau, \lambda} (E6)$$

This determines the spectral decomposition of the transfer matrix $T_{\tau}$ given by Eq. (E1) and allows for determining the eigenvalues which contribute to the correlation functions (2). Inserting the spectral decomposition in Eq. (7) we obtain

$$C_{ab}(t) = \sum_{\lambda, \sigma} \lambda^{k-\delta} \sigma^b c_{ab,\tau}(\lambda, \sigma), (E7)$$

where the double sum runs over all eigenvalues $\lambda \in \text{spec}(T_{\tau})$ and $\sigma \in \text{spec}(T_{\tau+1})$. We moreover defined

$$c_{ab,\tau}(\lambda, \sigma) := \text{tr} \left( (P_{\lambda,\tau} \otimes I_{q^2}) P_{\sigma,\tau+1} C_{ab,\tau+1} \right) = \sum_{s, r} \text{tr} \left( (| r_{\lambda}, s \rangle \langle l_{\lambda}, s |_{\tau} \otimes I_{q^2}) | r_{\sigma}, r \rangle \langle l_{\sigma}, r |_{\tau+1} C_{ab,\tau+1} \right) =: \sum_{s, r} c_{ab,\tau}^s(r, \lambda, \sigma) (E8)$$

which can be simplified using a diagrammatic representation. To this end we extend the definition (E2) as

$$| r_{\lambda}, s \rangle = \begin{array}{c} \lambda, s \\ \vdots \end{array} \quad \text{and} \quad \tau \langle l_{\lambda}, s | = \begin{array}{c} \lambda, s \\ \vdots \end{array}. (E11)$$

Inserting this into $c_{ab,\tau}^s(r, \lambda, \sigma)$ for fixed $r$ and $s$ we obtain

$$c_{ab,\tau}^s(r, \lambda, \sigma) = = \left( \tau \langle l_{\lambda}, s | \otimes 1(b) \right) | r_{\sigma}, r \rangle \langle l_{\sigma}, r |_{\tau+1} (| a \rangle_1 \otimes | r_{\lambda}, s \rangle_{\tau}). (E13)$$

We aim for showing that the latter expression is nonzero only if $\tau\lambda = \tau$ and $\tau\sigma = \tau + 1$ and hence the double sum over $r$ and $s$ in Eq. (E10) reduces to a single term. On the one hand the condition on $\tau\sigma$ is due to the local operators $a$ and $b$ being traceless. If the left (right) eigenvector corresponding to $\lambda$ has a factor $| \circ \rangle_1$ on the first (last) site this produces as factor $\langle \circ | a \rangle_1 = 0 (| b \rangle_1 = 0).$ On the other hand the condition on $\tau\lambda$ is a consequence of the properties (E3) and (E4). If the left (right) eigenvector corresponding to $\lambda$ has a factor $| \circ \rangle$ on the first (last) site contraction along the corresponding wire maps $| r_{\sigma}, r+1 \rangle_{\tau+1} (r_{\tau+1} | l_{\sigma} \rangle_{\tau})$ to $| r_{\sigma}, r-1 \rangle_{\tau} = 0 (r_{\sigma}, r+1) = 0).$ This argument might be formalized as follows: First, assume that $\tau\sigma < \tau + 1$ and thus $r \in \{0, 1, \ldots, \tau + 1 - r\}$. If $r > 0$ we might write $\tau+1| l_{\sigma}, r \rangle = \tau| \circ \rangle \otimes \tau \langle l_{\sigma}, r-1 |$ and hence the second factor in Eq. (E13) becomes

$$\tau+1| l_{\sigma}, r \rangle \langle l_{\sigma}, r |_{\tau+1} (| a \rangle_1 \otimes | r_{\lambda}, s \rangle_{\tau}) = \langle \circ | a \rangle_1 \langle l_{\sigma}, r-1 | r_{\lambda}, s \rangle_{\tau} = 0 (E14)$$

as we have chosen the local operator $a$ to be traceless and therefore Hilbert-Schmidt orthogonal to the identity which in vectorized form reads $\langle \circ | a \rangle_1 = 0$. If on the other hand $r = 0$ we might repeat the argument and write $| r_{\sigma}, r \rangle_{\tau+1} = | r_{\sigma}, r \rangle_{\tau} \otimes \circ_1$ for which the first factor becomes

$$\langle \tau \langle l_{\lambda}, s | \otimes 1(b) \rangle | r_{\sigma}, r \rangle_{\tau+1} (| a \rangle_1 \otimes | b \rangle_1) = 0 (E15)$$
as $b$ is traceless. This proofs that $c_{ab,\tau}(\lambda, \sigma) \neq 0$ only if $\tau_\sigma = \tau + 1$ and hence $|r_\sigma, r\rangle_{\tau+1}$ can be replaced by $|r_\sigma\rangle_{\tau+1}$. We continue by demonstrating that for $c_{ab,\tau}(\lambda, \sigma) \neq 0$ and hence $c_{ab,\tau}(\lambda, \sigma) \neq 0$ also $\tau_\lambda = \tau$ has to hold by similar reasoning. Assume, that this is not the case and therefore $s \in \{0, 1, \ldots, \tau - \tau_\lambda\}$. If $s > 0$ we write

$$r\langle l_\lambda, s | = |r\rangle \otimes \tau_{\lambda - 1}(l_\lambda, s - 1) \text{ and obtain}$$

$$r\langle l_\lambda, s | \otimes i(b) |r\rangle_{\tau+1} = (i| \otimes r_{\lambda - 1}(l_\lambda, s - 1) \otimes i(b)) |r\rangle_{\tau+1} = (\tau - r_\lambda)(l_\lambda, s - 1) \otimes i(b)) |r\rangle_{\tau+1} = r\langle l_\lambda, s \rangle = 0, \quad (E16)$$

where we used Eq. (E3). Similarly, for $s = 0$ we write $|r_\lambda, s\rangle = |r_\lambda, s\rangle_{\tau - 1} \otimes |0\rangle \otimes i(b) |r_\lambda, s\rangle_{\tau - 1} \otimes |0\rangle_1$ and obtain

$$r(\langle l_\lambda | (a)_{\tau} \otimes |r_\lambda, s\rangle_{\tau}) = r(\langle l_\lambda | (a)_{\tau} \otimes |r_\lambda, s\rangle_{\tau} \otimes 1 |0\rangle_1) = r(\langle l_\lambda | (a)_{\tau} \otimes |r_\lambda, s\rangle_{\tau} \otimes 1 |0\rangle_1) = 0. \quad (E17)$$

Therefore we conclude that $c_{ab,\tau}(\lambda, \sigma) \neq 0$ only if $\tau_\lambda = \tau$ and hence $|r_\lambda, s\rangle_{\tau}$ can be replaced by $|r_\lambda\rangle_{\tau}$ yielding

$$c_{ab,\tau}(\lambda, \sigma) = \delta_{\lambda,\tau} \delta_{\tau_\sigma,\tau + 1} (\tau\langle l_\lambda | (a)_{\tau} \otimes |r_\lambda\rangle_{\tau} (\langle l_\lambda | (a)_{\tau} \otimes |r_\lambda\rangle_{\tau}). \quad (E18)$$

Inserting the latter expression back into Eq. (E7) we finally obtain Eq. (14).

**Appendix F: Asymptotics of correlations for generic impurities**

In this section we derive the asymptotic expression (16) for correlation functions in the case of generic impurities. Our numerical experiments show that the transfer matrix $T_\tau$ in general fails to be diagonalizable if $\tau > 1$ and the nontrivial eigenvalues lead to nontrivial Jordan blocks. Hence we start by describing the Jordan decomposition of $T_\tau$ in the following. Using the notation from the T-dual case, Eq. (12), only $|r_\lambda, \tau - \tau_\lambda\rangle (\langle l_\lambda | 0\rangle)$ is a right (left) eigenvector with eigenvalue $\lambda$. The corresponding Jordan block, again assuming no accidental degeneracies, has dimension $d_\lambda = \tau - \tau_\lambda + 1$ and corresponds to the $T_\tau$ invariant subspace $\ker(N_{\lambda,\tau}^{d_\lambda})$, where $N_{\lambda,\tau} = T_{\tau} - L \lambda q_i^2$. Let us denote the projection onto this subspace by $P_{\lambda,\tau}$, which does not coincide with Eq. (13). Note that $N_{\lambda,\tau}^{d_\lambda} P_{\lambda,\tau} = 0$ and in particular for the trivial eigenvalue $N_1, P_{1,\tau} = 0$. The projections $P_{\lambda,\tau}$ again form a resolution of identity and we write

$$T_{\tau} = \sum_{\lambda \in \text{spec}(T_{\tau})} (\lambda L q_\tau + N_{\lambda,\tau}) P_{\lambda,\tau} = |\langle 0 | \langle 0 | \otimes |1_{q_\tau}\rangle T_{\tau+1}^{\delta} C_{ab,\tau+1} + \sum_{\lambda} (\lambda L q_\tau + N_{\lambda,\tau}) P_{\lambda,\tau}, \quad (F1)$$

where the sums on the right hand side and in what follows runs only over the nontrivial eigenvalues $\lambda \in \text{spec}(T_{\tau})$. As we are interested in the scaling of correlations with system size $L$ let us assume that $L - \delta \gg \tau$ and hence $L - \delta > d_\lambda$ for all $\lambda \in \text{spec}(T_{\tau})$ for fixed $\delta$. Upon inserting the Jordan decomposition into Eq. (7) we obtain

$$C_{ab}(t) = \text{tr} \left( |\langle 0 | \langle 0 | \otimes |1_{q_\tau}\rangle T_{\tau+1}^{\delta} C_{ab,\tau+1} \right) + \sum_{\lambda} \text{tr} \left( (\lambda L q_\tau + N_{\lambda,\tau})^{L - \delta} P_{\lambda,\tau} \otimes |1_{q_\tau}\rangle T_{\tau+1}^{\delta} C_{ab,\tau+1} \right) \quad (F2)$$

$$= |\langle 0 | \langle 0 | \otimes |\langle b | T_{\tau+1}^{\delta} |a\rangle \otimes |\langle 0 | \otimes + \sum_{\lambda} \lambda^{L - \delta - d_\lambda} L - \delta \sum_{n=0}^{L - \delta} \sum_{n=0}^{L - \delta} \lambda^{d_\lambda - n} \text{tr} \left( [N_{\lambda,\tau}^{d_\lambda} P_{\lambda,\tau} \otimes |1_{q_\tau}\rangle T_{\tau+1}^{\delta} C_{ab,\tau+1} \right) \right. \quad (F3)$$

Assuming a spectral gap, as suggested by our numerical experiments, the second term in the last line vanishes as $L^7 \lambda_0^L$ as $L$ tends to infinity, with the factor $L^7$ bounding the asymptotic growth of the binomial coefficients $(L^n - \delta)$. Hence asymptotically the second term is exponentially suppressed as $\lambda_0^L$ and the first term which coincides with Eq. (16) governs the correlation function. Let us further explore the $\delta$-dependence of the second term. To this end we assume $\delta \gg \tau + 1$, i.e., $\delta$ larger than the dimension of the largest Jordan block. By inserting the Jordan decomposition of $T_{\tau+1}$ into the second term of the right hand side in Eq. (F3) we obtain for the trace

$$\text{tr} \left( [N_{\lambda,\tau}^{d_\lambda} P_{\lambda,\tau} \otimes |1_{q_\tau}\rangle T_{\tau+1}^{\delta} C_{ab,\tau+1} \right) = \sum_{\sigma} \sigma^{d_\sigma - d_\lambda} \sum_{m=0}^{d_\lambda} \left( \delta \right)^m \sigma^{d_\sigma - m} \text{tr} \left( [N_{\lambda,\tau}^{d_\lambda} P_{\lambda,\tau} \otimes |1_{q_\tau}\rangle N_{\sigma,\tau+1}^{m} P_{\sigma,\tau+1} C_{ab,\tau+1} \right) \right. \quad (F4)$$

Here the sum over $\sigma$ runs again over nontrivial eigenvalues only as $|\langle 0 | \langle 0 | \otimes |1_{q_\tau}\rangle C_{ab,\tau+1} = 0$. Consequently, the above expression is dominated by the leading nontrivial eigenvalues $\sigma_0$ of $T_{\tau+1}$ for large $\delta$ and hence the second term in Eq. (F3) is suppressed at least as $|\sigma_0|^L$ for all $\delta$ since $|\lambda_0| \leq |\sigma_0|$. Again the binomial coefficient gives only a polynomial correction proportional to $\delta^7$. Hence the first term dominates the large $L$ asymptotics of correlation
functions. Inserting the Jordan decomposition of $T_{\tau+1}$ there and noting that $|\phi\rangle^{\otimes \tau+1}$ is orthogonal to $|\phi\rangle^{\otimes \tau} \otimes |a\rangle$ yields

$$
\langle \phi|^{\otimes \tau} \otimes \langle b| T_{\tau+1}^\delta |a\rangle \otimes |\phi\rangle^{\otimes \tau} = \sum_\sigma \sigma^{\delta - d_\sigma} \sum_{m=0}^{d_\sigma} \left( \begin{array}{c} \delta \\ m \end{array} \right) \sigma^{d_\sigma - m} \langle \phi|^{\otimes \tau} \otimes \langle b| N_{\sigma, \tau+1}\mathcal{P}_{\sigma, \tau+1} |a\rangle \otimes |\phi\rangle^{\otimes \tau}. 
$$

By the same reasoning as above this is dominated by the leading nontrivial eigenvalue $\sigma_0$ of $T_{\tau+1}$ for sufficiently large $\delta$. Hence one has the asymptotics $C_{ab}(t = \tau L + \delta) \sim \sigma_0(\tau)^\delta$ if both $\delta$ and $L - \delta$ are large. Here, we indicate the $\tau$ dependence of $\sigma_0$ in the notation and for simplicity assume a unique real leading eigenvalue. Note, that our numerical experiments on the spectral properties of the transfer matrices imply $|\sigma_0| \approx 1/q$. As indicated in the main text at times $t \approx \tau L$ non-universal corrections to the asymptotic scaling need to be taken into account.

Using the diagrammatic representation (17) of the leading contributions to the correlation function suggests that the correlations should exponentially decay with $\tau$. This can be seen by evaluating the diagram in terms of column transfer matrices instead of the (row) transfer matrices $T_{\tau+1}$. Both types of transfer matrices share similar spectral properties and hence the diagram gives a contribution scaling with the largest nontrivial eigenvalue of the column transfer matrix $\chi$ of $\chi_0$. The arguments outlined above, however, do not guarantee that the second term in Eq. (F3) gives rise only to subleading terms when $L$ is kept fixed and $\tau \to \infty$.

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