Non-existence of tight neighborly triangulated manifolds with $\beta_1 = 2$

Nitin Singh
Department of Mathematics, Indian Institute of Science, Bangalore 560012, India.
Email: nitin@math.iisc.ernet.in

To appear in ‘Advances in Geometry’

Abstract: All triangulated $d$-manifolds satisfy the inequality \((f_0 - d - 1) / 2 \geq (d+2) \beta_1\) for $d \geq 3$. A triangulated $d$-manifold is called tight neighborly if it attains equality in the bound. For each $d \geq 3$, a $(2d+3)$-vertex tight neighborly triangulation of the $S^{d-1}$-bundle over $S^1$ with $\beta_1 = 1$ was constructed by Kühnel in 1986. In this paper, it is shown that there does not exist a tight neighborly triangulated manifold with $\beta_1 = 2$. In other words, there is no tight neighborly triangulation of $(S^{d-1} \times S^1)^\#2$ or $(S^{d-1} \times S^1)^\#2$ for $d \geq 3$. A short proof of the uniqueness of Kühnel’s complexes for $d \geq 4$, under the assumption $\beta_1 \neq 0$ is also presented.

MSC 2000: 57Q15, 57R05.
Keywords: Stacked sphere; Triangulated manifolds; Tight neighborly triangulation.

Introduction

Tight neighborly triangulations were introduced by Lutz, Sulanke and Swartz in [11]. Using a result of Novik and Swartz [12], the authors in [11] obtained a lower bound on the minimum number of vertices in a triangulation of a $d$-manifold in terms of its $\beta_1$ coefficient (see Proposition 1.4). Triangulations that meet the lower bound are called tight neighborly. Thus tight neighborly triangulations are vertex minimal triangulations. Effenberger [8] showed that for $d \geq 4$, tight neighborly triangulated manifolds are $\mathbb{Z}_2$-tight. In conjunction with a recent result of Bagchi and Datta [3], this implies they are strongly minimal for $d \geq 4$. Apart from the following classes of vertex-minimal triangulations, namely

- the $(d + 2)$-vertex triangulation of the $d$-sphere $S^d$,
- Kühnel’s $(2d + 3)$-vertex triangulations [10] of $S^{d-1} \times S^1$ (for even $d$) and $S^{d-1} \times S^1$ (for odd $d$),

very few examples of tight neighborly triangulations are known. A first sporadic example, a 15-vertex triangulation of a 4-manifold with $\beta_1 = 3$, was obtained by Bagchi and Datta [2]. Recently in [4], we obtained tight neighborly triangulations of 4-manifolds with 21, 26 and 41 vertices. For $\beta_1 = 2$, the parameters (integer solutions of the tight neighborliness condition) for the first few possible tight neighborly triangulations are $(f_0, d) = (35, 13)$ and $(f_0, d) = (204, 83)$. The main result of this paper shows that such triangulations do not exist. In this article, unless the field is explicitly stated, we assume $\beta_1(X) = \beta_1(X; \mathbb{Z}_2)$. 
1 Preliminaries

All graphs considered here are simple (i.e., undirected with no loops or multiple edges). For the standard terminology on graphs, see [7, Chapter 1] for instance. For a graph $G$, $V(G)$ and $E(G)$ will denote its set of vertices and edges respectively. A graph $G$ is said to be $k$-connected if $|V(G)| \geq k + 1$ and $G - U$ is connected for all $U \subseteq V(G)$ with $|U| < k$. It is easily seen that for a $k$-connected graph $G$, $d_G(v) \geq k$ for all $v \in V(G)$.

All simplicial complexes considered here are finite and abstract. By a triangulated manifold/sphere/ball, we mean an abstract simplicial complex whose geometric carrier is a topological manifold/sphere/ball. We identify two complexes if they are isomorphic. A manifold/sphere/ball, we mean an abstract simplicial complex whose geometric carrier is a standard ball with facet $\sigma$. In this paper, by a ball, we shall mean a 2-neighborly complex.

A simplicial complex is called a weak pseudomanifold if for a $d$-dimensional weak pseudomanifold $X$, the boundary $\partial X$ of $X$ consists of a single vertex $x$, we write $x \star Y$ for $X \star Y$. For a face $\alpha$ of $X$, the link of $\alpha$ in $X$, denoted by $\text{lk}_X(\alpha)$, is the subcomplex of $X$ consisting of all faces $\beta$ such that $\alpha \cap \beta = \emptyset$ and $\alpha \cup \beta$ is a face of $X$. When $\alpha$ consists of a single vertex $v$, we write $\text{lk}_X(v)$ instead of $\text{lk}_X(\{v\})$. For a vertex $v$ of $X$, we define the star of $v$ in $X$, denoted by $\text{st}_X(v)$ as the cone $v \star \text{lk}_X(v)$. The subcomplex of $X$ consisting of faces of dimension at most $k$ is called the $k$-skeleton of $X$, and is denoted by $\text{skel}_k(X)$. By the edge graph of a simplicial complex $X$, we mean its 1-skeleton.

If $X$ and $Y$ are simplicial complexes with disjoint vertex sets, we define $X \star Y$ to be the simplicial complex whose faces are the (disjoint) unions of faces of $X$ with faces of $Y$. When $X$ consists of a single vertex $x$, we write $x \star Y$ for $X \star Y$. For a face $\alpha$ of $X$, the link of $\alpha$ in $X$, denoted by $\text{lk}_X(\alpha)$, is the subcomplex of $X$ consisting of all faces $\beta$ such that $\alpha \cap \beta = \emptyset$ and $\alpha \cup \beta$ is a face of $X$. When $\alpha$ consists of a single vertex $v$, we write $\text{lk}_X(v)$ instead of $\text{lk}_X(\{v\})$. For a vertex $v$ of $X$, we define the star of $v$ in $X$, denoted by $\text{st}_X(v)$ as the cone $v \star \text{lk}_X(v)$. The subcomplex of $X$ consisting of faces of dimension at most $k$ is called the $k$-skeleton of $X$, and is denoted by $\text{skel}_k(X)$. By the edge graph of a simplicial complex $X$, we mean its 1-skeleton.

If $X$ is a $d$-dimensional simplicial complex then, for $0 \leq j \leq d$, the number of its $j$-faces is denoted by $f_j = f_j(X)$. The vector $(f_0, \ldots, f_d)$ is called the face vector of $X$ and the number $\chi(X) := \sum_{i=0}^d (-1)^i f_i$ is called the Euler characteristic of $X$. As is well known, $\chi(X)$ is a topological invariant, i.e., it depends only on the homeomorphic type of $|X|$. A simplicial complex $X$ is said to be $l$-neighborly if any $l$ vertices of $X$ form a face of $X$. In this paper, by a neighborly complex, we shall mean a 2-neighborly complex.

A standard $d$-ball is a pure $d$-dimensional simplicial complex with one facet. The standard ball with facet $\sigma$ is denoted by $\overline{\sigma}$. A $d$-dimensional pure simplicial complex $X$ is called a stacked $d$-ball if there exists a sequence $B_1, \ldots, B_m$ of pure simplicial complexes such that $B_1$ is a standard $d$-ball, $B_m = X$ and, for $2 \leq i \leq m$, $B_i = B_{i-1} \cup \overline{\sigma_i}$ and $B_{i-1} \cap \overline{\sigma_i} = \overline{\sigma_i}$, where $\sigma_i$ is a $d$-face and $\tau_i$ is a $(d-1)$-face of $\sigma_i$. Clearly, a stacked ball is a pseudomanifold. A simplicial complex is called a stacked $d$-sphere if it is the boundary of a stacked $(d+1)$-ball. A trivial induction on $m$ shows that a stacked $d$-ball actually triangulates a topological $d$-ball, and hence a stacked $d$-sphere triangulates a topological $d$-sphere. If $X$ is a stacked ball then clearly $\Lambda(X)$ is a tree.

**Proposition 1.1** (Datta and Singh [6]). Let $X$ be a pure $d$-dimensional simplicial complex.

(a) If $\Lambda(X)$ is a tree then $f_0(X) \leq f_d(X) + d$.

(b) $\Lambda(X)$ is a tree and $f_0(X) = f_d(X) + d$ if and only if $X$ is a stacked ball.
Proposition 1.4. Theorem 4 in [11], we infer the following:

Moreover for $d \geq 4$, the members of $\mathcal{K}(d)$ can be obtained as boundary of members of $\overline{\mathcal{K}}(d + 1)$. For a simplicial complex $X$, let $V(X)$ denote its vertex set. For a set $S$, let $\left( \frac{S}{k} \right)$ denote the collection of subsets of $S$ of cardinality at most $k$. From the results in [4], we have:

**Proposition 1.2** (Bagchi and Datta). Let $d \geq 4$ and $M \in \mathcal{K}(d)$. Then $\overline{M}$ defined by

$$\overline{M} := \left\{ \alpha \subseteq V(M) : \left( \frac{\alpha}{\leq 3} \right) \subseteq M \right\}$$

is the unique member of $\overline{\mathcal{K}}(d + 1)$ such that $\partial \overline{M} = M$.

In the above construction, notice that $V(\overline{M}) = V(M)$ and $\overline{M}$ is neighborly if and only if $M$ is neighborly.

Let $\sigma_1, \sigma_2$ be two facets of a pure simplicial complex $X$. Let $\psi : \sigma_1 \to \sigma_2$ be a bijection such that $x$ and $\psi(x)$ have no common neighbor in the edge graph (1-skeleton) of $X$ for each $x \in \sigma_1$. Let $X^{\psi}$ denote the complex obtained by identifying $x$ with $\psi(x)$ in $X\setminus\{\sigma_1, \sigma_2\}$. Then $X^{\psi}$ is said to be obtained from $X$ by a combinatorial handle addition. We know the following:

**Proposition 1.3** (Kalai [9]). For $d \geq 4$, a connected simplicial complex $X$ is in $\mathcal{K}(d)$ if and only if $X$ is obtained from a stacked $d$-sphere by $\beta_1(X)$ combinatorial handle additions. In consequence, any such $X$ triangulates either $(S^{d-1} \times S^1)^{\# \beta_1}$ or $(S^{d-1} \times S^1)^{\# \beta_1}$ depending on whether $X$ is orientable or not. (Here $\beta_1 = \beta_1(X) = \beta_1(X; \mathbb{Z}_2)$.)

In the above, $S^{d-1} \times S^1$ denotes the (orientable) sphere product with a circle, while $S^{d-1} \times S^1$ denotes the (non-orientable) twisted sphere product with a circle. As usual $X^{\# k}$ denotes the connected sum of $k$ copies of the manifold $X$. From Proposition 1.2 and Theorem 4 in [11], we infer the following:

**Proposition 1.4.** Let $X$ be a connected triangulated $d$-manifold. Then $X$ satisfies

$$\left( f_0 - d - 1 \right) \geq \left( \frac{d + 2}{2} \right)^2 \beta_1.$$  

Moreover for $d \geq 4$, the equality holds if and only if $X$ is a neighborly member of $\mathcal{K}(d)$.

For $d \geq 3$, a triangulated $d$-manifold is called tight neighborly if it satisfies (2) with equality.

For a field $F$, a $d$-dimensional simplicial complex $X$ is called tight with respect to $F$ (or $F$-tight) if (i) $X$ is connected, and (ii) for all induced sub-complexes $Y$ of $X$ and for all $0 \leq j \leq d$, the morphism $H_j(Y; F) \to H_j(X; F)$ induced by the inclusion map $Y \hookrightarrow X$ is injective.

A $d$-dimensional simplicial complex $X$ is called strongly minimal if $f_i(X) \leq f_i(Y)$, $0 \leq i \leq d$, for every triangulation $Y$ of the geometric carrier $|X|$ of $X$.

Effenberger [8] proved that for $d \geq 4$, tight neighborly triangulated $d$-manifolds are $\mathbb{Z}_2$-tight. Bagchi and Datta [3] proved that for $d \geq 4$, $F$-tight members of Walkup’s class $\mathcal{K}(d)$ are strongly minimal. Thus for $d \geq 4$, tight neighborly triangulated $d$-manifolds are strongly minimal.
2 Non-existence of tight neighborly manifolds with $\beta_1 = 2$

The following is the main result of this article.

**Theorem 2.1.** For $d \geq 4$, there does not exist a tight neighborly triangulated $d$-manifold $M$ with $\beta_1(M) = 2$.

**Proof-Sketch:** Suppose there exists a tight neighborly $d$-manifold $M$ with $\beta_1(M) = 2$. Then by Proposition 1.4 $M$ is a neighborly member of $\mathcal{K}(d)$. By Proposition 1.2 there exists a neighborly member $\overline{M}$ of $\mathcal{K}(d+1)$ such that $\partial \overline{M} = M$ and $V(\overline{M}) = V(M)$. The proof rests on the following observation which follows from Corollary 2.10.

Let $T$ be the set of facets of $\overline{M}$ with degree three or more in $\Lambda(\overline{M})$. Then the facets in $T$ together contain all the vertices of $\overline{M}$.

Since $\overline{M}$ is $(d+1)$-dimensional, we have the obvious inequality $f_0(M) = f_0(\overline{M}) \leq |T|(d+2)$. Since $M$ is tight neighborly, we have $(f_0(M) - d - 1)(f_0(M) - d - 2) = \beta_1(M)(d+1)(d+2)$. For the case $\beta_1(M) = 2$, we shall see that the inequality and the equation cannot be simultaneously satisfied, thus proving the theorem.

The following are used in the proof of Theorem 2.1. In the results below we shall assume that $M \in \overline{\mathcal{K}}(d)$ is not the standard $d$-ball $B^d_{d+1}$.

**Lemma 2.2.** Let $M \in \overline{\mathcal{K}}(d)$. If $M$ is neighborly, then the dual graph of $M$ is 2-connected.

**Proof.** Let $\Lambda$ denote the dual graph of $M$. Suppose $\Lambda$ is not 2-connected. Then there exists $\sigma \in V(\Lambda)$ such that $\Lambda - \sigma$ is disconnected. Let $C_1, C_2$ be different components of $\Lambda - \sigma$. Choose $\sigma_1 \in C_1$ and $\sigma_2 \in C_2$. Now choose $x_1 \in \sigma_1$ and $x_2 \in \sigma_2$ such that $x_1, x_2 \notin \sigma$. Note that $\sigma, \sigma_1$ and $\sigma_2$ are facets of $M$, while $x_1$ and $x_2$ are vertices of $M$. Let $V_1$ denote the set of vertices of $\Lambda$ that contain $x_1$ and $V_2$ denote the set of vertices of $\Lambda$ containing $x_2$. Observe that the subgraph of $\Lambda$ induced by $V_1$ is precisely the dual graph of $\text{st}_M(x_1)$, i.e., $\Lambda(\text{lk}_M(x_1)) \cong \Lambda(\text{st}_M(x_1)) = \Lambda[V_1]$. Similarly $\Lambda(\text{lk}_M(x_2)) \cong \Lambda[V_2]$. Since $M \in \overline{\mathcal{K}}(d)$, the vertex-links are stacked balls, and hence, by Proposition 1.1 their dual graphs are trees. Thus $V_1$ and $V_2$ induce trees on $\Lambda$. Since $x_1, x_2 \notin \sigma$ we conclude that $V_1$ and $V_2$ induce trees on $\Lambda - \sigma$. Now since $\sigma_1 \in V_1$ and $\sigma_2 \in V_2$ we conclude that $\Lambda[V_1] \subseteq C_1$ and $\Lambda[V_2] \subseteq C_2$. Therefore $V_1 \cap V_2 = \emptyset$. Therefore $x_1x_2$ is not a simplex in $M$, contradicting the neighborliness of $M$. This proves the lemma.

**Lemma 2.3.** Let $M \in \overline{\mathcal{K}}(d)$ be neighborly. Then each $x \in V(M)$ is contained in $f_0(M) - d$ facets of $M$, which induce a tree on $\Lambda(M)$.

**Proof.** Let $L = \text{lk}_M(x)$ denote the link of $x$ in $M$. Then $L$ is a stacked $(d-1)$-ball. Since $M$ is neighborly, we have $f_0(L) = f_0(M) - 1$. Then from Proposition 1.1 we see that $f_{d-1}(L) = f_0(M) - d$ and that $\Lambda(L)$ is a tree. Finally we observe that $\Lambda(L) \cong \Lambda(M)[V_2]$, where $V_x$ is the set of facets of $\Lambda(M)$ containing $x$. This completes the proof.

**Lemma 2.4.** Let $M \in \overline{\mathcal{K}}(d)$ be neighborly. Then $\Lambda(M)$ has $n(n-d)/(d+1)$ vertices and $n(n-d-1)/d$ edges, where $n = f_0(M)$.

**Proof.** Let $\Lambda(M)$ have $v$ vertices and $e$ edges. We count the pairs $(x, \tau)$ where $x \in V(M)$, $\tau$ is a facet of $M$ containing $x$. Now $x \in \tau \Rightarrow \tau \setminus \{x\} \in \text{lk}_M(x)$. By Lemma 2.3 we conclude that each $x$ appears in $n - d$ facets of $M$. Thus the number of pairs is $n(n - d)$. However since each facet of $M$ is $d$-simplex, each facet occurs with $d+1$ values of $x$. Counting this
way gives us $\nu(d + 1)$ pairs. Equating the two we have $\nu = n(d - d)/(d + 1)$. To get the number of edges in the dual graph we count the pairs

$$(x, \sigma \tau)$$

where $x \in V(M), \sigma \tau \in E(\Lambda(M))$ and $x \in \sigma \cap \tau$. (3)

Let $S_x$ denote the star of $x$ in $M$. Then as seen previously $\Lambda(S_x)$ is an $(n - d)$-vertex induced tree of $\Lambda(M)$. Now note that (3) is equivalent to saying that $\sigma, \tau$ are facets in $S_x$, and moreover they form an edge in $\Lambda(S_x)$. Since $\Lambda(S_x)$ has $n - d - 1$ edges, we see that each $x$ contributes $n - d - 1$ pairs. Thus the number of pairs is $n(n - d - 1)$. However we can count differently. Consider the pair of facets $\{\sigma, \tau\}$ forming an edge in $\Lambda(M)$. Then $|\sigma \cap \tau| = d$ and hence each edge occurs with $d$ values of $x$. This gives the number of pairs as $d\varepsilon$. Equating the two values we get $\varepsilon = n(n - d - 1)/d$. \square

**Corollary 2.5.** Let $M \in \overline{K}(d)$ be neighborly. Then $f_0(M) \geq 2d + 1$ and the equality holds if and only if $\Lambda(M)$ is a cycle.

**Proof.** Let $\nu$ and $\varepsilon$ denote the number of vertices and edges of $\Lambda(M)$ respectively. Let $V := V(\Lambda(M))$ denote the vertex set of the graph $\Lambda(M)$. By Lemma 2.2 $\Lambda(M)$ is 2-connected. Thus all vertex degrees are at least two, i.e., $d_{\Lambda(M)}(\sigma) \geq 2$ for all $\sigma \in V$. Let $T = \{\sigma \in V : d_{\Lambda(M)}(\sigma) \geq 3\}$. Then $\Lambda(M)$ is a cycle if and only if $T = \emptyset$. Now we have, $2\nu \leq \sum_{\sigma \in V} d_{\Lambda(M)}(\sigma) = 2\varepsilon$, or $\varepsilon \geq \nu$. Clearly the equality occurs when $T = \emptyset$. Using Lemma 2.4 we have,

$$\frac{n(n - d - 1)}{d} \geq \frac{n(n - d)}{d + 1},$$

where $n = f_0(M)$

Thus $f_0(M) = n \geq 2d + 1$ and equality occurs only when $T = \emptyset$, or equivalently when $\Lambda(M)$ is a cycle. \square

**Lemma 2.6.** For $d \geq 4$, let $M \in \mathcal{K}(d)$ be neighborly and let $\overline{M} \in \overline{K}(d + 1)$ be such that $\partial \overline{M} = M$. If $\nu$ and $\varepsilon$ denote the number of vertices and edges of $\Lambda(\overline{M})$ respectively, then $\beta_1(M) = \varepsilon - \nu + 1$.

**Proof.** Since $M \in \mathcal{K}(d)$ is neighborly, by Proposition 1.11 we have $\beta_1(M) = \left(\frac{n - d - 1}{2}\right)/\left(\frac{d + 2}{2}\right)$. Since $\overline{M} \in \overline{K}(d + 1)$, by Lemma 2.4 we have $\nu = n(n - d - 1)/(d + 2), \varepsilon = n(n - d - 2)/(d + 1)$, where $n = f_0(M) = f_0(\overline{M})$. Then it follows that $\beta_1(M) = \varepsilon - \nu + 1$. \square

**Definition 2.7.** Let $M$ be a neighborly member of $\overline{K}(d)$. A set $S \subseteq V(\Lambda(M))$ is said to be critical in $M$ if each of the connected components of $\Lambda(M) - S$ contains fewer than $f_0(M) - d$ vertices. A set of facets is said to be a cover of $M$ if they together contain all the vertices.

Observe the following.

**Lemma 2.8.** Let $M \in \overline{K}(d)$ be neighborly. If $S \subseteq V(\Lambda(M))$ is critical in $M$, then $S$ is a cover of $M$.

**Proof.** Since $S$ is critical in $M$, each component of $\Lambda(M) - S$ is of size at most $f_0(M) - d - 1$. Let $x$ be an arbitrary vertex of $M$. Let $V_x$ be the set of facets of $M$ containing $x$. By Lemma 2.3 we know that $\Lambda(M)[V_x]$ is an induced tree with $f_0(M) - d$ vertices. Hence $V_x$ must intersect $S$, or equivalently a facet in $S$ contains $x$. Since $x$ was arbitrary, we conclude that the facets in $S$ contain all the vertices, and hence $S$ is a cover of $M$. \square
Lemma 2.9. Let $M \in \overline{K}(d)$ be neighborly with $f_0(M) > 2d + 1$. Let $u_0 u_1 \ldots u_r$ be a path in $\Lambda(M)$. Let $x_i$ be the unique element of $u_{i-1}\setminus u_i$ for $1 \leq i \leq r$. If all the internal vertices of the path have degree at most two in $\Lambda(M)$, then we have the following:

(a) $x_i \neq x_j$ for $i \neq j$.

(b) $x_i \in u_0$ for all $1 \leq i \leq r$.

(c) $r \leq d + 1$.

Proof. We first prove (a). If possible, let there exist $i, j$ with $i < j$ such that $x_i = x_j = x$. Then by definition $x \in u_{i-1}, u_{j-1}$ but $x \notin u_i, u_j$; hence $j > i + 1$. Since the set of facets containing $x$ must induce a tree, we conclude that there is a $u_{i-1}u_{j-1}$ path in $\Lambda(M) - \{u_i, u_j\}$. However we see that if all the internal vertices have degree at most two, this is not possible. This proves (a).

Suppose (b) is not true. Let $i$ be the minimum such that $x_i \notin u_0$. As $x_1 \in u_0$, we have $i > 1$. By minimality of $i$, we must have $\{x_1, \ldots, x_{i-1}\} \subseteq u_0$. Since $|u_0| = d + 1$, we see that $i \leq d + 2$. But then we have $x_i \notin u_{i-1}$, $x_i \notin u_i$, and $x_i \notin u_0$. Let $V_i$ denote the set of facets containing $x_i$. Since $\Lambda(M)[V_i]$ is a tree, it must be contained in a unique component of $\Lambda(M) - \{u_0, u_i\}$. Since $x_i \notin u_{i-1}$, $\Lambda(M)[V_i]$ is contained in the component of $\Lambda(M) - \{u_0, u_i\}$ containing $u_{i-1}$, which according to our assumptions is the path $u_1 \ldots u_{i-1}$.

Thus $V_i \subseteq \{u_1, \ldots, u_{i-1}\}$. But then $|V_i| \leq d + 1 < f_0(M) - d$, contradicting Lemma 2.8. This proves (b). Since $|u_0| = d + 1$, it is readily seen that (a) and (b) yield (c). This completes the proof of the lemma.

Corollary 2.10. For $d \geq 4$, let $M \in \overline{K}(d)$ be neighborly. Let $T = \{\sigma \in V(\Lambda(M)) : d_{\Lambda(M)}(\sigma) \geq 3\}$. If $f_0(M) > 2d + 1$, then $T$ is a cover of $M$.

Proof. From Corollary 2.5, $f_0(M) > 2d + 1$ implies $T \neq \emptyset$. Clearly each component of $\Lambda(M) - T$ is an induced path in $\Lambda(M)$. Then by Lemma 2.9, we see that each component of $\Lambda(M) - T$ is of size at most $d + 1 < f_0(M) - d$. Hence $T$ is critical in $M$. Therefore by Lemma 2.8, we conclude that $T$ is a cover of $M$.

We are now in a position to give a complete proof of Theorem 2.1.

Proof (Theorem 2.1): If possible let $M$ be a tight neighborly $d$-manifold for $d \geq 4$ with $\beta_1 = \beta_1(M) = 2$. Let $n = f_0(M)$. Then we have

$$(n-d-1)(n-d-2) = 2(d+2)(d+1)$$

(4)

By Proposition 1.4, $M$ must be a neighborly member of $\mathcal{K}(d)$. Then by Proposition 1.2, we have $|\partial M| = d_{\Lambda(M)}(\sigma)$. Further we know that $\overline{M}$ is neighborly and $V(\overline{M}) = V(M)$. By Corollary 2.5, we have $f_0(M) = n = f_0(\overline{M}) \geq 2(d+1) + 1 = 2d + 3$. For $n = 2d+3$, we see that $\Lambda(M)$ is a cycle, and hence by Lemma 2.6, $\beta_1(\sigma) = 1$. Thus we can assume $n > 2d + 3$. Let $\mathcal{V}$ and $\mathcal{E}$ denote the vertex and edge set of $\Lambda(M)$ respectively. Let $T \subseteq \mathcal{V}$ be the set of facets of $\overline{M}$ with degree three or more in $\Lambda(M)$. By Corollary 2.10, $T$ is a cover of $M$. Since $\overline{M}$ is $(d+1)$-dimensional, we have $n \leq |T|(d+2)$. We now estimate $|T|$. By Lemma 2.2, $\Lambda(M)$ is 2-connected. Hence $d_{\Lambda(M)}(\sigma) \geq 2$ for all $\sigma \in \mathcal{V}$. Now we have,

$$|T| \leq \sum_{\sigma \in \mathcal{V}} (d_{\Lambda(M)}(\sigma) - 2) = \sum_{\sigma \in \mathcal{V}} d_{\Lambda(M)}(\sigma) - 2|\mathcal{V}| = 2(|\mathcal{E}| - |\mathcal{V}|).$$
By Lemma\textsuperscript{[\textcolor{red}{2.6}]} we have $|E| - |V| = \beta_1 - 1 = 1$. Thus $|T| \leq 2$, and hence $n \leq 2(d+2) = 2d+4$. But then,

$$(n - d - 1)(n - d - 2) \leq (d + 2)(d + 3) < 2(d + 2)(d + 1)$$

which contradicts (4). This completes the proof. \hfill \square

**Remark 2.11.** Theorem\textsuperscript{[\textcolor{red}{2.7}]} shows that there does not exist a tight neighborly triangulation with $(\beta_1, d, f_0) = (2, 13, 35)$, which was one of the open cases in [8 Section 4]. The next few triples $(\beta_1, d, f_0)$ with $\beta_1 = 2$ are $(2, 83, 204)$ and $(2, 491, 1189)$, which also do not exist by Theorem\textsuperscript{[\textcolor{red}{2.7}]} Indeed in conjunction with Propositions\textsuperscript{[\textcolor{red}{1.3}]} and\textsuperscript{[\textcolor{red}{1.4}]} we get:

**Corollary 2.12.** If $X$ is an $n$-vertex triangulation of $(S^{d-1} \times S^1)^{\#2}$ or $(S^{d-1} \times S^1)^{\#2}$ and $d \geq 4$, then

$$\left(\frac{n - d - 1}{2}\right) \geq d^2 + 3d + 3.$$

## 3 Uniqueness of Kühnel’s Tori

For $d \geq 2$, $d$-dimensional Kühnel’s torus $K_{2d+3}^d$\textsuperscript{[\textcolor{red}{10}]} is defined as the boundary of the $(d + 1)$-dimensional pseudomanifold $K_{2d+3}^{d+1}$ on the vertex set $\{0, \ldots, 2d + 2\}$ with facets $\{(i, i + 1, \ldots, i + d + 1) : 0 \leq i \leq 2d + 2\}$, where the addition is modulo $2d + 3$. For even $d$, $K_{2d+3}^d$ triangulates the sphere product $S^{d-1} \times S^1$ and for odd $d$, it triangulates the twisted product $S^{d-1} \times S^1$. The following result was proved in [11].

**Proposition 3.1.** For $d \geq 3$, Kühnel’s torus $K_{2d+3}^d$ is the only non-simply connected $(2d + 3)$-vertex triangulated manifold of dimension $d$.

For $d \geq 4$, the uniqueness of $K_{2d+3}^d$ was also proved in [5] for the bigger class of homology $d$-manifolds, but with assumption $\beta_1 \neq 0$ and $\beta_2 = 0$. We prove the above result for $d \geq 4$, under the assumption $\beta_1 \neq 0$. More specifically we prove:

**Theorem 3.2.** For $d \geq 4$, let $M$ be a triangulated $d$-manifold with $2d + 3$ vertices and $\beta_1(M; \mathbb{Z}_2) \neq 0$. Then $M \cong K_{2d+3}^d$.

**Proof.** By Proposition\textsuperscript{[\textcolor{red}{1.4}]} we must have $\beta_1(M) = 1$ and $M$ must be tight neighborly. Therefore $M \in K(d)$. By Proposition\textsuperscript{[\textcolor{red}{1.2}]} there exists $\overline{M} \in \overline{K}(d + 1)$ such that $\partial \overline{M} = M$. Let $V(M) = V(\overline{M}) = \{0, 1, \ldots, 2d + 2\}$. Since $f_0(\overline{M}) = 2d + 3 = 2(d + 1) + 1$, by Corollary\textsuperscript{[\textcolor{red}{2.5}]}, we conclude that $\Lambda(\overline{M})$ is a cycle with $2d + 3$ vertices. Let $\sigma_0, \ldots, \sigma_{2d+2}$ be facets of $\overline{M}$ such that $\Lambda(\overline{M})$ is the cycle $\sigma_0 \sigma_1 \ldots \sigma_{2d+2} \sigma_0$. For $0 \leq i \leq 2d + 2$, let $V_i$ denote the set of facets of $\overline{M}$ containing the vertex $i$. By Lemma\textsuperscript{[\textcolor{red}{2.3}]} $V_i$ induces a $(2d + 3) - (d + 1) = d + 2$ vertex tree on $\Lambda(\overline{M})$. In other words, $V_i$ induces a $(d + 1)$-length path on $\Lambda(\overline{M})$. There are exactly $2d + 3$ induced paths of $\Lambda(\overline{M})$ with length $d + 1$, namely the paths $P_k = \sigma_{k-1} \sigma_{k-2} \ldots \sigma_0$ for $0 \leq k \leq 2d + 2$. Next we show that the sets $V_i$ are distinct for $i = 0, \ldots, 2d + 2$. If not, suppose $V_i = V_j$ for some $i \neq j$. Let $P = \Lambda(\overline{M})[V_i] = \Lambda(\overline{M})[V_j]$. Clearly, $P$ is an induced path. Let $\sigma$ be an end-vertex of $P$ and let $\tau$ be a neighbor of $\sigma$ in $\Lambda(\overline{M})$, not on $P$ (such a neighbor exists, as the degree of each vertex in $\Lambda(\overline{M})$ is at least two). But then $\{i, j\} \subseteq \sigma \cup \tau$, which is not possible, as $\sigma \tau$ is an edge of $\Lambda(\overline{M})$. Thus $V_i \neq V_j$ for $i \neq j$. Therefore, for each $0 \leq k \leq 2d + 2$, there exists a unique $l$ such that $\Lambda(\overline{M})[V_i] = P_k$. Let $\phi$ denote this association by $\phi$, i.e., $\phi(k) = l$ if $\Lambda(\overline{M})[V_i] = P_k$. Then we have $\sigma_i = \{\phi(i), \phi(i + 1), \ldots, \phi(i + d + 1)\}$. We notice that in this case $\phi$ is a simplicial
isomorphism from $\overline{K}_{2d+3}^{d+1}$ to $\overline{M}$. Therefore $K_{2d+3}^d = \partial \overline{K}_{2d+3}^{d+1} \cong \partial \overline{M} = M$. This completes the proof.

\textbf{Acknowledgement :} The author thanks the anonymous referee for useful comments regarding the presentation of the paper. The author also thanks Basudeb Datta and Bhaskar Bagchi for their valuable comments and suggestions. The author would also like to thank ‘IISc Mathematics Initiative’ and ‘UGC Centre for Advanced Study’ for support.

\textbf{References}

[1] B. Bagchi, B. Datta, Minimal triangulations of sphere bundles over the circle, \textit{J. Combin. Theory (A)} \textbf{115} (2008), 737–752.

[2] B. Bagchi, B. Datta, On Walkup’s class $\mathcal{K}(d)$ and a minimal triangulation of $(S^3 \times S^1)^\# 3$, \textit{Discrete Math.} \textbf{311} (2011), 989–995.

[3] B. Bagchi, B. Datta, On stellated spheres and a tightness criterion for combinatorial manifolds. [arXiv:1207.5599v1], 2012, 22 pages.

[4] B. Bagchi, B. Datta, On $k$-stellated and $k$-stacked spheres. [arXiv:1208.1289v1], 2012, 8 pages.

[5] J. Chestnut, J. Sapir, E. Swartz, Enumerative properties of triangulations of spherical bundles over $S^1$, \textit{Euro. J. Combin.} \textbf{29} (2008), 662–671.

[6] B. Datta, N. Singh, Tight triangulations of some 4-manifolds, [arXiv:1207.6182v2], 2012, 8 pages.

[7] R. Diestel, \textit{Graph Theory}, Springer-Verlag, Hiedelberg, 2006.

[8] F. Effenberger, Stacked polytopes and tight triangulations of manifolds, \textit{J. Combin. Theory (A)} \textbf{118} (2011), 1843-1862.

[9] G. Kalai, Rigidity and the lower bound theorem 1, \textit{Invent. math.} \textbf{88} (1987), 125–151.

[10] W. Kühnel, Higher dimensional analogues of Császár’s torus, \textit{Results in Mathematics} \textbf{9} (1986), 95–106.

[11] F. H. Lutz, T. Sulanke, E. Swartz, $f$-vector of 3-manifolds, \textit{Electron. J. Comb.} \textbf{16} (2009), #R13, 1–33.

[12] I. Novik, E. Swartz, Socles of Buchsbaum modules, complexes and posets, \textit{Adv. in Math.} \textbf{222} (2009), 2059–2084.

[13] D. W. Walkup, The lower bound conjecture for 3- and 4-manifolds, \textit{Acta Math.} \textbf{125} (1970), 75–107.