Nature of eigenstates in a mesoscopic ring coupled to a side branch

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Abstract

We formalize the parity effect of a multichannel ring coupled to a side branch using the tight binding model. We find that the tight binding model gives slightly different result from the continuum model. We also show that some states of the system can be nonmagnetic. In the multichannel ring coupled to a side branch, the persistent current in the ring does not change sign, over gigantic fluctuations in N (the number of particles in the ring).

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I Introduction

Prior to the experiments [1–3], Buttiker et al [4] had shown the possibility of observing equilibrium persistent currents in normal metal or semiconductor rings pierced by a magnetic flux $\phi$. The eigenstates of a quantum ring are flux dependent and as the flux is varied, one gets a $E$ versus $\phi$ dispersion curve. States whose $E$ versus $\phi$ dispersion curve have positive slope carry diamagnetic current and states whose $E$ versus $\phi$ dispersion curve have negative slope carry paramagnetic current. For free electrons in a single channel ring consecutive eigenenergies have opposite slope and the magnetic response of the ring is diamagnetic for an odd number of electrons and paramagnetic for an even number of electrons in the ring [5]. This is the essence of the parity effect. It was shown by Leggett that this effect is true for any one body and two body scattering and arises just because of the antisymmetric nature of the many body fermionic wavefunction [5]. With the addition of an extra electron in the ring, there is a statistical phase that shifts the $E$ versus $\phi$ dispersion curve by $\phi_0/2$ and hence the $N$ body and $N+1$ body states carry opposite persistent currents [6]. The parity effect is not destroyed by spin, finite temperature or disorder [5,7]. Interactions do not destroy the parity effect but interactions with spin can lead to the creation of a fractional Aharonov-Bohm effect, which has however not been observed experimentally yet [8–16]. The parity effect is also observed in multichannel simulations [17]. However, there the essence of parity effect is slightly modified. When there is complete rotational symmetry (clean multichannel ring), subband quantum number is a good quantum number and consecutive states belonging to the same subband have opposite slope. If the rotational symmetry is destroyed (say, by impurities), then different subbands can mix. But this only opens up a gap at points where states cross (i.e. it lifts degeneracy) and flattens the $E$ versus $\phi$ dispersion curves, but does not change the sign of the slope. As a result, what is seen in a multichannel ring is that the states change slope at the scale of Thouless energy. So the parity dominated current of a single ring has an $a$ priori random sign depending on the number of electrons present in it and this allows us to treat a collection of rings [1] as a statistical ensemble. Various models have been used to calculate the ensemble average [17,20], which is a nontrivial task. In one
such model, one assumes that the number of electrons $N$ in all the rings is the same (strong canonical ensemble) and averaging is done over impurity configurations.

The system of a ring coupled to a side branch has recently attracted attention \[21–24\]. So far it is studied using a 1D modeling. In this model the geometry consists of a finite wire or a side chain, of length $v$ being attached to a ring of length $u$, as shown in Fig (1). Ref. \[21\] points out some interesting aspects of the electronic states in a quantum ring weakly coupled to a side branch. Potential scattering at the junction of the ring and the side branch can cause such weak coupling of the states. On the other hand, ref \[22\] studies the states of the combined system of ring coupled to a side branch, from first principles without any additional potential scattering at the junction. In this strong coupling regime, one cannot distinguish the states of the ring and the side branch separately because they carry the same amount of persistent current. However, one can clearly observe the effect of the two length scales ($v$ and $u$) on the eigenstates of the system. It was shown that the states associated with the energy scale $v$ do not show a parity effect. If there are $n$ states associated with the length scale $v$ between two states associated with the length scale $u$ then all the $n$ states have the same slope. Ref \[24\] studies the effect of charge transfer by polarizing the side branch with respect to the ring. Under such charge transfer, one can measure the capacitance coefficients of the ring \[21, 24\]. The flux dependence of the eigenenergies determine the flux dependence of the capacitance coefficients. As magnetization measurements \[1–3\] are a difficult task, one can alternately measure the flux dependent capacitance coefficients of the ring, to probe it’s equilibrium properties. Ref \[25\] presents an interesting study of coupled quantum rings.

For the first time, we try to analyze the flux dependence of the eigenenergies of a multichannel ring coupled to a multichannel side branch. We restrict ourselves to the free electron case, but, following the argument of Leggett, electron-electron interaction will not change this parity effect observed for free electrons \[6, 23\] and this has been rigorously established in the framework of a Luttinger Liquid \[9, 10\]. Also it was proposed in 1D \[22\], that by fabricating such a geometry, with a large $\frac{v}{u}$ (say, 10) one can get rid of the parity effect and hence the tough problem of ensemble averaging in a many ring experiment. In fact, the
strong canonical ensemble may be a good approximation to estimate the disorder averaged persistent current, if for small fluctuations in $N$, the persistent current does not change sign. The effect that was shown in 1D should survive multichannel effects if it is to be of experimental relevance. Besides, the conductance across such a system (side branch coupled to a ring) was measured in a recent experiment \cite{26} in the Coulomb blockade regime and the conductance features are believed to depend on the parity of the eigenstates of the closed system \cite{27}.

II Theoretical Treatment

To study the multichannel situation, it is sufficient to consider a 2D geometry. One of the easiest ways of treating a multichannel ring is to have a ring made up of sites, the system being described by the tight-binding Hamiltonian. The system thus considered is shown in Fig (2). If $a$ is the lattice parameter, then the length of the ring is $L_r a$, width of the ring is $w_r a$, length of the side chain is $L_s a$ and width of the side chain is $w_s a$. The Hamiltonian describing the system can be written as,

$$H = -\sum_{jk} \left[ \delta t^\alpha_{jk} a^\dagger_{j,k} a_{j+1,k} + t^\beta_{jk} a^\dagger_{j,k} a_{j+1,k} + h.c. \right]$$

Here, $\delta = e^{i \frac{2\pi \phi}{\Phi_0}}$, where $\phi$ is the flux threading the 2D ring, $\Phi_0 = \frac{hc}{e}$, the elementary flux quantum. $a^\dagger_{jk}$ is the electron creation operator at the site $(j,k)$. The first term in $H$ represents hopping in the azimuthal direction which is the propagating direction ($\alpha$ being the index for it). The second term represents hopping in the transverse direction. We have considered $t^\alpha = t^\beta = t$ and only nearest neighbour hopping. We perform an exact diagonalization of the Hamiltonian to evaluate the single particle levels. The persistent current carried by a particular level $E_n$ is given by, (we have set $\hbar = 2 m = c = 1$)

$$I_n = -\frac{\partial E_n}{\partial \phi}$$

and the total persistent current is just the sum of the persistent currents of all the filled levels. In presence of spin each state will be filled by two electrons but the effect of spin is
not the subject of the present study. The probability that a level is filled is given by the Fermi factor. We restrict ourselves to finding the persistent current at zero temperature.

Using the continuum model in a multichannel ring is very hazardous. So far we know that the two models (continuum and discrete) give results that agree with each other. But we find that if the ring has a finite side chain attached to it, as is the case considered by us then the two models give qualitatively the same result as in 1D but there are some major quantitative differences. A free electron in an infinite 1D line behaves like an electron in the band of an infinite 1D periodic system. But the finite size effects in the two cases are always different, i.e. a free electron in a finite 1D line with some boundary conditions at the two ends of the line is expected to behave differently than an electron in a finite lattice. A ring has no boundaries and is effectively an infinite system. However, if we attach a finite side chain to the ring, then putting sites in it or not putting sites in it will always make a difference. As the electron is moving in a background of positively charged ions, it may feel a periodic potential and then the tight binding model may be the more appropriate description. On the other hand, if the number of sites is very large then it approaches the continuum limit. First we will establish that the two models give qualitatively the same result in the 1D regime. This will justify studying the multichannel regime using the tight binding model.

III Results and Discussions

To formalize the study of the parity effect for a tight binding ring we first consider a 1D ring described by tight binding Hamiltonian. In Fig (3) we show the typical spectrum of $E$ versus $\phi$ dispersion curve of a ring made up of 5 sites to which a side chain of 10 sites is attached. The slope of states change after every three states. The continuum analogue of this system is a stub of length $v$ attached to a ring of length $u$ such that $\frac{v}{u} = 2$. We calculate the $E$ versus $\phi$ dispersion curve of this system using the method of ref. [22] and the spectrum is shown in Fig (4). In this case the slopes change after every two states in contrast to three in the tight binding model. We have checked for various other cases and for large values of $v/u$.
and we find that the number of consecutive states with the same sign of slope in the tight binding model and continuum model at most differ by one. If there are 10 sites in the stub, then the stub contributes 10 states to the system and along with the 5 sites of the main wire we have 15 states in all. If the stub were absent, then, the system would have 5 states with consecutive states having opposite slope and so the slope would have changed 5 times in all. When the stub is attached, the 15 states are found to change their slope exactly 5 times. This makes the slope change after 3 consecutive states. Hence, the additional states created by the side chain in the tight binding model are like states associated with the length scale \( v \) and they all are parity violating states. So, the continuum model and the tight binding model give qualitatively identical results. Only, the number of states created by the tight binding side chain may not be the same as that of the continuum case.

When the ring is detached from the side branch, we can assign a definite quantum number ‘s’ to the states of the side branch. And similarly, the states of the ring can be assigned a definite quantum number ‘r’. The parity effect of the states of the ring can be understood according to Leggett’s conjecture from the antisymmetric property of the many body wavefunction. When the ring and the side branch are coupled, the states that leak into the ring from the side branch become magnetic. But states with quantum numbers ‘s’ and ‘s + 1’ do not have opposite slopes. However, this too can be understood from the antisymmetric property of the many body wavefunction. The statistical phase is neutralized by another phase \( \pi \) that originates from geometric scattering by the side branch \[23\]. The difference in phase acquired by an electron in two such consecutive states in going round the ring once does not depend on the kinetic energy difference of the electron in the two states alone but also on the special phase \( \pi \). Note that in ref \[16\], Haldane has shown that interactions can give an additional phase \( \pi \) that can neutralize the statistical phase.

An interesting quantum effect that has no classical analog sometimes arises if the total number of states in the combined system of the ring and the side branch, is not an integer times the number of states of the isolated ring. For example if the ring is made up of three sites and the side branch is made up of two sites then there are five states in all. The
eigenvalues of this system can be found analytically by diagonalizing the Hamiltonian and the 5 eigenvalues $e_1, e_2, e_3, e_4, e_5$ in the range $0 < \alpha = \frac{2\pi \phi}{\phi_0 L_r} < 2\pi$ are given below:

\[
e_1 = \frac{c_1 \exp(\alpha)}{x(\alpha)} + \frac{\exp(-\alpha) x(\alpha)}{18^{1/3}}
\]

\[
e_2 = -1
\]

\[
e_3 = i \frac{\exp(-\alpha) y(\alpha)}{2^{2/3} 3^{1/2} x(\alpha)}
\]

\[
e_4 = 1
\]

\[
e_5 = -e_3
\]

where, $x(\alpha) = (-9 - 9 \exp(6i\alpha) + \sqrt{3}(27 - 202 \exp(6i\alpha) + 27 \exp(12i\alpha))^{1/2})^{1/3}$,

\[
y(\alpha) = -24 \exp(2i\alpha) + 8i \sqrt{3} \exp(2i\alpha) + i 2^{1/3} 3^{1/6} x^2(\alpha) + 18^{1/3} x^2(\alpha),
\]

\[
c_1 = \left(\frac{128}{3}\right)^{1/3}.
\]

Note that $e_2$ and $e_4$ do not depend on $\phi$. Physically these two states remain completely localized inside the side branch although there is no potential restricting it from leaking out. If it leaks out then the breakdown of time reversal symmetry necessitates it to be magnetic (i.e. carry a persistent current) whereas the antisymmetric property of the many body wavefunction does not allow the five states to change their slope more than three times (the ring being made up of three sites), maintaining the symmetry of the tight binding band. So, an interplay of these symmetry principles, keep this state localized to the stub. The state forms a node at the junction of the ring and the side branch and the wavefunction vanishes inside the ring. These are the very states which give rise to a zero in the transmission if the ring is severed at a point, such that the system becomes the same as the T shaped stub of ref [28]. It was shown that the zero in the transmission was a consequence of unitarity [28]. We argue from symmetry principles, that such states will exist which vanish in the ring and therefore will be nonmagnetic.

Now, realistic systems are not 1D and there are other length scales apart from $L_r (\equiv u)$ and $L_s (\equiv v)$, like $w_r$ and $w_s$ that determine the number of states associated with the ring and the stub. Having justified the study of the parity effect using the tight binding model we proceed to explore the parity of the states associated with the length scale $w_s$ of a realistic multichannel ring, using the tight binding model. We consider the 2D geometry as shown
in Fig (2). It is difficult to infer about the parity effect of a multichannel ring by looking at E versus $\phi$ dispersion curves as in Fig (3). Things are more transparent if we look at the I(N) versus N curve, where I(N) is the total persistent current in the ring in units of ($\Gamma = \frac{2\pi c}{\hbar \phi_0}$) when there are N electrons in the ring. In Fig (5), we have chosen $L_r = 10$, $w_r = 5$, $L_s = 100$ and $w_s = 2$ and plotted $I(N)/\Gamma$ versus N. The persistent current does not change sign over long ranges of N. Most importantly, the number of oscillations or number of peaks in $I(N)/\Gamma$ versus N curve is equal to that of a $10 \times 5$ clean ring (ie. the stub detached from the system of Fig (5)). This shows that all states associated with the stub (or the states associated with the length scales $L_s$ and $w_s$) between two states of the multichannel ring have the same sign of slope. As the width of the stub ($w_s$) increases, the stub can accommodate more and more states within a certain interval of Fermi energy, say, in the interval of two states of the $10 \times 5$ multichannel ring. We proceed to study the parity of these states. Keeping $L_s$, $L_r$, $w_r$ unchanged (ie. $L_r = 10$, $w_r = 5$ and $L_s = 100$) we increase $w_s$ to 4. The $I(N)/\Gamma$ versus N plot is shown in Fig (6). Then we increase $w_s$ to 6 and the corresponding $I(N)/\Gamma$ versus N curve is shown in Fig (7). We find that as $w_s$ increases, over a larger range of N the persistent current does not change sign. Each curve from Fig (5) to Fig (7) looks similar but magnified. The fact that the number of oscillations in the $I(N)/\Gamma$ versus N curve does not increase signifies that in the multichannel situation the states created by the additional length scale $w_s$ of the stub are all parity violating states. All the states arising due to the length scale $w_s$ have the same sign of slope. To compare to what order of fluctuations in N, the persistent current changes sign in a multichannel ring we take a ring that has the same number of propagating modes and the same number of states as the system of Fig (5). This means that we take a ring whose width consists of 5 sites and whose length consists of 50 sites and plot $I(N)/\Gamma$ versus N in Fig (8). The order of oscillations in the curve compared to that in Fig (5), is prominent. Compared to Fig (8), in Fig 5, 6 and 7, there are very broad regions of N over which the $I(N)/\Gamma$ versus N curve does not change sign. The broadest region is the central region. We take the central region where say from $N = N_1$ to $N = N_1 + \delta N_1$ the persistent current does not change sign. Then
we plot $\delta N_1$ versus $w_s$ in Fig (9) and find a linear scaling between them. One can see that in the case of Fig (7), if the ring is at half filling, even if $N$ fluctuates by 29% of the total number of electrons in the ring, the sign of the response will not be a priori random. One should note that the first region where the persistent currents do not change sign is always diamagnetic but quite narrow. The third and fourth regions are quite broad and one can perform a many ring experiment populating the ring up to the third or fourth regions.

If we open the ring at a point then we can define a transmission amplitude $T$ across the two open ends. In the 1D regime, it is the transmission amplitude that relates the outgoing state to the incoming ones. In the multichannel regime, the outgoing channels are related to the incoming ones by $T_{mn}$ where $T_{mn}$ is the transmission amplitude between the $m$th subband on one side and the $n$th subband on the other side. In the multichannel situation, the discontinuities in $T_{mn}$ will destroy the parity effect just as those in $T$ do in 1D

These discontinuities will give the necessary phase $\pi$ in the electron wavefunction to neutralize the statistical phase. This gives rise to many more states with the same slope. The fact that in the multichannel situation, every state coming from the additional length scale $w_s$ of the stub, in between the states of the ring, is a parity violating state establish this.

IV Conclusion

So, for the first time we have analyzed in detail the nature of eigenstates in a ring coupled to a side branch in the multichannel situation. A number of earlier studies are restricted to the 1D situation. All states associated with the length scale $w_s$ (the width of the side branch) between two states of the ring have the same sign of slope. This is the main result of this work. As a result, even for 29% fluctuations in $N$, persistent current in the system may not change sign. This makes the system a much better candidate to perform many ring experiments to explore the finer details of persistent currents without encountering the problem of ensemble averaging. We also show that some states in the system can be non-magnetic due to purely quantum mechanical effects.

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FIGURE CAPTIONS

Fig. 1. A stub of length \( v \) attached to a ring of length \( u \).

Fig. 2. Schematic diagram of a 2D multichannel ring made up of sites.

Fig. 3. Energy versus \( \phi \) dispersion curve for a 1D ring made up of 5 sites and a 1D stub of 10 sites attached to it.

Fig. 4. Energy versus \( \phi \) dispersion curve of a 1D ring of length ‘\( u \)’ with a 1D stub of length ‘\( v \)’ attached to it for \( \frac{v}{u} = 2 \).

Fig. 5. Persistent Current \( \frac{I(N)}{I} \) in a multichannel ring with a multichannel stub attached to it, plotted as a function of the number of electrons (\( N \)). Here, \( \frac{L_r}{a} = 10 \), \( \frac{w_r}{a} = 5 \), \( \frac{L_s}{a} = 100 \), \( \frac{w_s}{a} = 2 \).

Fig. 6. Persistent Current \( \frac{I(N)}{I} \) in a multichannel ring with a multichannel stub attached to it, plotted as a function of the number of electrons (\( N \)). Here, \( \frac{L_r}{a} = 10 \), \( \frac{w_r}{a} = 5 \), \( \frac{L_s}{a} = 100 \), \( \frac{w_s}{a} = 4 \).

Fig. 7. Persistent Current \( \frac{I(N)}{I} \) in a multichannel ring with a multichannel stub attached to it, plotted as a function of the number of electrons (\( N \)). Here, \( \frac{L_r}{a} = 10 \), \( \frac{w_r}{a} = 5 \), \( \frac{L_s}{a} = 100 \), \( \frac{w_s}{a} = 6 \).

Fig. 8. Persistent current \( I(N) \) in a clean multichannel ring (no stub attached) plotted as a function of the number of electrons (\( N \)) in the ring. \( \frac{L_r}{a} = 50 \), \( \frac{w_r}{a} = 5 \).

Fig. 9. Plot of range (\( \delta N_1 \)) of the number of electrons over which persistent current does not change sign around half filling as a function of the stub width \( \frac{w_s}{a} \).