The geodesic motion near hypersurfaces in the warped products spacetime

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Abstract

In the framework of Lorentzian multiply warped products we study the Gibbons-Maeda-Garfinkle-Horowitz-Strominger (GMGHS) spacetime near hypersurfaces in the interior of the event horizon. We also investigate the geodesic motion in hypersurfaces.

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1 Introduction

The concept of a warped product manifold was introduced to provide a class of complete Riemannian manifolds with negative curvature everywhere [1], and was developed to point out that several of the well-known exact solutions to Einstein field equations are pseudo-Riemannian warped products [2]. Furthermore, certain causal and completeness properties of a spacetime can be determined by the presence of a warped product structure [3], and a general theory Lorentzian multiply warped were applied to discuss the Schwarzschild spacetime in the interior of the event horizon [4, 5, 6, 7]. The role of warped products in the study of exact solutions to Einstein’s equations are now firmly established to generate interest in other areas of geometry.

On the other hand, there were enormous interests in the spherically symmetric static charged black holes in the four-dimensional heterotic string theory, which have similarities as well as differences with the Reissner-Nordström black hole in general relativity [8]. By turning antisymmetric tensor gauge fields off, the static charged black hole solution was found by Gibbons, Maeda [9], and by Garfinkle, Horowitz, Strominger [10], independently. Recently, null geodesics and hidden symmetries in the Sen black hole was investigated by Hioki and Miyamoto [11], which is reduced to the GMGHS black hole in the nonrotating limit. Gad [12] also studied geodesic and geodesic deviation of the GMGHS black hole solution. Very recently, Fernando [13] fully investigated null geodesic motions of the same solution both in the Einstein and string frame. However, the studies of these null geodesics solutions are mainly based on the exterior region of the event horizon. In a Lorentzian multiply warped product spacetime, by exchanging timelike and spacelike coordinates, we are interest in the interior region of the event horizon.

In this paper we study the GMGHS interior spacetime of the framework of Lorentzian multiply warped products. We also investigate the geodesic motion near hypersurfaces of this spacetime. We shall use geometrized units, i.e., $G = c = 1$, for notational convenience.
2 GMGHS black hole in the framework of warped products

The four dimensional low energy GMGHS action obtained from heterotic string theory [8, 9, 10] is given by

$$ S = \frac{1}{16\pi} \int d^4x \sqrt{-g} \left[ R - 2(\nabla \Phi)^2 - e^{-2\Phi} F_{\mu\nu} F^{\mu\nu} \right], \quad (2.1) $$

where $\Phi$ is a dilaton field and $F_{\mu\nu}$ the Maxwell field strength tensor. The low energy limit of string theory includes a scalar dilaton field, which is massless in all finite orders of perturbation theory [14]. In particular, the dilaton is coupled with the Maxwell field and it affects the geometry of spacetime, making different from the Reissner-Nordström solution of the Einstein-Maxwell theory.

The GMGHS solution of the Einstein field equations in Einstein frame describes the geometry exterior to a spherically symmetric static charged black hole as

$$ ds^2 = -\left(1 - \frac{2m}{r}\right)dt^2 + \left(1 - \frac{2m}{r}\right)^{-1} dr^2 + r^2 \left(1 - \frac{\alpha}{r}\right)(d\theta^2 + \sin^2 \theta d\phi^2), \quad (2.2) $$

where

$$ e^{2\Phi} = 1 - \frac{\alpha}{r}, \quad F_{rt} = \frac{Q}{r^2} \quad (2.3) $$

with $\alpha = Q^2 / m$. The parameters $m$ and $Q$ are mass and charge respectively. Note that the metric in the $t-r$ plane is identical to the Schwarzschild case. As like in the Schwarzschild spacetime, the GMGHS has an event horizon at $r = 2m$. We also note that the area of the sphere of the GMGHS black hole, defined by $\int d\theta d\phi \sqrt{g_{\theta\theta} g_{\phi\phi}}$, is smaller than the Schwarzschild spacetime by an amount depending on the charge. In particular, the area of the sphere approaches zero as $r \to \alpha$, leading to a surface singularity as

$$ R = \frac{\alpha^2(r - 2m)}{2r^3(r - \alpha)^2}. \quad (2.4) $$

As far as the case of $\alpha \leq 2m$, the singular surface remains inside the event horizon so that the Penrose diagram is identical to the Schwarzschild spacetime. Also the case of $\alpha = 2m$, which implies $Q^2 = 2m^2$ is the extremal limit which the event horizon and the surface singularity meet.
It is also interesting to note that in the GMGHS solution (2.2), the limit of \( Q \to 0 \) makes the dilaton vanishing so that the only black hole solution is the Schwarzschild one. On the contrary, every solution with nonzero Maxwell fields must have a nonconstant dilaton which will alter the geometry [8, 9, 10]. This is the main difference from the Reissner-Nordström solution in which case can be embedded into the 5 dimensional 3-parameter class of charged solitons \textit{a la} Kaluza-Klein [15, 16, 17].

On the other hand, the line element for the GMGHS metric for the interior region \( r < 2m \) can be described by

\[
ds^2 = -\left(\frac{2m}{r} - 1\right)^{-1}dr^2 + \left(\frac{2m}{r} - 1\right)dt^2 + r^2\left(1 - \frac{\alpha}{r}\right)(d\theta^2 + \sin^2\theta d\phi^2), \tag{2.5}\]

where \( r \) and \( t \) are now new temporal and spacial variables, respectively. A multiply warped product manifold, denoted by \( M = (B \times F_1 \times ... \times F_n, g) \), consists of the Riemannian base manifold \( (B, g_B) \) and fibers \( (F_i, g_i) \) \((i = 1, ..., n)\) associated with the Lorentzian metric [4]. In particular, for the specific case of \( (B = R, \ g_B = -d\mu^2) \), the GMGHS metric (2.5) can be rewritten as a multiply warped products \( (a, b) \times f_1 R \times f_2 S^2 \) by making use of a lapse function

\[
N^2 = \frac{r_H - r}{r}, \tag{2.6}
\]

as well as warping functions given by \( f_1 \) and \( f_2 \) as follows

\[
f_1(\mu) = \left(\frac{2m}{F^{-1}(\mu)} - 1\right)^{1/2}, \]
\[
f_2(\mu) = \left(F^{-1}(\mu)^2 - \alpha F^{-1}(\mu)\right)^{1/2}. \tag{2.7}
\]

The lapse function (2.6) is well defined in the region \( r < r_H(=2m) \) to rewrite it as a multiply warped products spacetime by defining a new coordinate \( \mu \) as follows

\[
\mu = \int_0^r \frac{dx}{(r_H - x)^{1/2}} = F(r). \tag{2.8}
\]

Setting the integration constant zero as \( r \to 0 \), we have

\[
\mu = 2m \cos^{-1}\left(\frac{r_H - r}{r_H}\right) - [(r_H - r)r]^{1/2}, \tag{2.9}
\]
which has boundary conditions as follows
\[
\lim_{r \to r_H} F(r) = (2n - 1)m\pi, \quad \lim_{r \to 0} F(r) = 0,
\]
for positive integer \(n\), and \(dr/d\mu > 0\) implies that \(F^{-1}(\mu)\) is well-defined function. We can thus rewrite the GMGHS metric (2.5) with the lapse function (2.6)
\[
ds^2 = -d\mu^2 + \left(\frac{2m}{F^{-1}(\mu)} - 1\right)dr^2 + \left(F^{-1}(\mu)^2 - \alpha F^{-1}(\mu)\right)d\Omega^2
\]
by using the warping functions (2.7). Note that the GMGHS metric in the multiply warped product spacetime has the same form with the Kantowski-Sachs solution [18] given by
\[
ds^2 = -dt^2 + A^2(t)dr^2 + B^2(t)d\Omega^2,
\]
which represents homogeneous but anisotropically expanding (contracting) cosmology.

Thus, in the case of the interior region \(r < 2m\), the GMGHS metric has been rewritten as a multiply warped product spacetime having the warping functions in terms of \(f_1\) and \(f_2\). Moreover, we can write down the Ricci curvature on the multiply warped products as
\[
R_{\mu\mu} = -f''_1f''_2f_2 - 2f''_1f'_2f'_2 - f'1f'_1f'_2f'_2 + f'_2f''_2 + f''^2_2 + 1,
\]
\[
R_{tt} = f_1f'' + \frac{2f_1f'_1f'_2f'_2}{f_2},
\]
\[
R_{\theta\theta} = \frac{f'_1f_2f'_2}{f_1} + f'^2_2 + f_2f''_2 + 1,
\]
\[
R_{\phi\phi} = \left(\frac{f'_1f_2f'_2}{f_1} + f'^2_2 + f_2f''_2 + 1\right)\sin^2\theta,
\]
\[
R_{mn} = 0, \text{ for } m \neq n,
\]
which have the same form with the Ricci curvature of the multiply warped interior Schwarzschild metric [4]. The only difference from the Schwarzschild is the \(\alpha\) term in the warping function \(f_2\) in Eq. (2.7).
3 Geodesic motion near hypersurface

A full understanding of the GMGHS spacetime having an event horizon with an essential singularity at the center and a surface singularity at \( r = \alpha \), etc, was recently achieved only comparatively. Also, since the geodesics in the GMGHS spacetime illuminate some basic aspects of a universe within the event horizon, we shall include an account of them. In this section, we briefly revisit the GMGHS interior spacetime with two warping functions at a singular point \( r = \alpha \) in the hypersurfaces, and we are interested in investigating the geodesic curves of a static spherically symmetric GMGHS spacetime near hypersurfaces.

In local coordinates \( \{ x^i \} \) the line element corresponding to this metric (2.5) will be denoted by

\[
dS^2 = g_{ij} dx^i dx^j. \tag{3.1}
\]

Consider the equations of geodesics in the GMGHS spacetime with affine parameter \( \lambda \) given by

\[
\frac{dx^i}{d\lambda^2} + \Gamma^i_{jk} \frac{dx^j}{d\lambda} \frac{dx^k}{d\lambda} = 0. \tag{3.2}
\]

Let a geodesic \( \gamma(\tau) = (\mu(\tau), r(\tau), \theta(\tau), \phi(\tau)) \) of the interior GMGHS spacetime in the case of \( r < 2m \) from Eq. (2.5), then the orbits of the geodesics equation are given as follows

\[
\frac{d^2\mu}{d\tau^2} + f_1 \frac{df_1}{d\mu} \left( \frac{dr}{d\tau} \right)^2 + f_2 \frac{df_2}{d\mu} \left( \frac{d\theta}{d\tau} \right)^2 + f_2 \frac{df_2}{d\mu} \sin^2 \theta \left( \frac{d\phi}{d\tau} \right)^2 = 0, \tag{3.3}
\]

\[
\frac{d^2r}{d\tau^2} + \frac{2 df_1}{f_1} \frac{dr}{d\tau} = 0, \tag{3.4}
\]

\[
\frac{d^2\theta}{d\tau^2} + 2 \frac{df_2}{f_2} \frac{d\theta}{d\tau} - \sin \theta \cos \theta \left( \frac{d\phi}{d\tau} \right)^2 = 0, \tag{3.5}
\]

\[
\frac{d^2\phi}{d\tau^2} + \frac{2 df_2}{f_2} \frac{d\phi}{d\tau} + 2 \cot \theta \frac{d\theta}{d\tau} \frac{d\phi}{d\tau} = 0 \tag{3.6}
\]

with a following constraint along the geodesic

\[
- \left( \frac{d\mu}{d\tau} \right)^2 + f_1^2 \left( \frac{dr}{d\tau} \right)^2 + f_2^2 \left( \frac{d\theta}{d\tau} \right)^2 + f_2^2 \sin^2 \theta \left( \frac{d\phi}{d\tau} \right)^2 = \varepsilon. \tag{3.7}
\]
Note that a timelike (nulllike) geodesic is taken as $\varepsilon = -1$ ($\varepsilon = 0$).

Hereafter, without loss of generality, suppose the geodesic

$$\gamma(\tau_0) = (\mu(\tau_0), r(\tau_0), \theta(\tau_0), \phi(\tau_0))$$

(3.8)

for some $\tau_0$ and the equatorial plane of $\theta = \frac{\pi}{2}$, thus $\frac{d\theta}{d\tau} = 0$. Then, the geodesic equations are reduced to

$$\frac{d^2\mu}{d\tau^2} + f_1 \frac{df_1}{d\mu} \left( \frac{dr}{d\tau} \right)^2 + f_2 \frac{df_2}{d\mu} \left( \frac{d\phi}{d\tau} \right)^2 = 0,$$

(3.9)

$$\frac{d^2 r}{d\tau^2} + \frac{2}{f_1} \frac{df_1}{d\tau} \frac{dr}{d\tau} = 0,$$  

(3.10)

$$\frac{d^2 \theta}{d\tau^2} = 0,$$  

(3.11)

$$\frac{d^2 \phi}{d\tau^2} + \frac{2}{f_2} \frac{df_2}{d\tau} \frac{d\phi}{d\tau} = 0$$

(3.12)

with a constraint

$$- \left( \frac{d\mu}{d\tau} \right)^2 + f_1 \left( \frac{dr}{d\tau} \right)^2 + f_2 \left( \frac{d\phi}{d\tau} \right)^2 = \varepsilon.$$  

(3.13)

These geodesic equations can be simplified as follows

$$\frac{dr}{d\tau} = \frac{c_1}{f_1^2},$$  

(3.14)

$$\frac{d\phi}{d\tau} = \frac{c_2}{f_2^2},$$  

(3.15)

$$\frac{d^2 \theta}{d\tau^2} = 0$$  

(3.16)

with a constraint

$$- \left( \frac{d\mu}{d\tau} \right)^2 + \frac{c_1^2}{f_1^2} + \frac{c_2^2}{f_2^2} = \varepsilon.$$  

(3.18)

The constant $c_1$ represents the total energy per unit rest mass of a particle as measured by a static observer [12, 19, 20], and $c_2$ represents the angular
momentum in the GMGHS spacetimes. The equations for $r$ and $\phi$ are obtained from Eqs. (3.10) and (3.12), respectively. Making use of these $r$, $\phi$ equations, we can show that Eq. (3.9) is the exactly same with Eq. (3.13) when we take the integration constant as $-\varepsilon/2$.

Now, we recall the GMGHS spacetime $M = (a, b) \times f_1 R \times f_2 S^2$ in the framework of the Lorentzian multiply warped products. Let subspace $\Sigma_{x^i}$ of the GMGHS spacetime $M$ be a regularly embedded $x^i$-directed hypersurface having coordinate neighborhood $U(p)$ with local coordinates $(x_1, x_2, x_3, x_4)$ such that $\Sigma_{x^i} \cap U = \{(x_1, x_2, x_3, x_4) \in U \mid x^i = p\}$ for all $p \in \Sigma_{x^i}$. For convenience, we say that such a neighborhood $U$ is partitioned by $\Sigma_{x^i}$.

First of all, we consider the null geodesics in the $r$-direction, which is defined by the hypersurface $\Sigma_r$ by taking $d\theta = d\phi = 0$. Then, we have $c_2 = 0$ in Eq. (3.15). Two equations (3.14) and (3.18) are now reduced to give

$$dr = \frac{d\mu}{f_1(\mu)}, \quad \text{(3.19)}$$

which can be solved to

$$\mu = \sqrt{(2m-r)r + m \tan^{-1} \left( \frac{r-m}{\sqrt{(2m-r)r}} \right)} + \frac{m\pi}{2} \equiv h(r). \quad \text{(3.20)}$$

We draw the null geodesics in the $r$-direction in Fig. 1, with the mass parameters of $m = 1$ and $m = 2$. In this Figure, we see the radial coordinate $r = h^{-1}(\mu)$ is a monotonic function of $\mu$. Therefore, as the time flows between $0 < \mu < m\pi$, the radial coordinate increases monotonically as $0 < r < 2m$.

Next, let us consider the geodesic in the $\phi$-direction, which lies on the hypersurface $\Sigma_\phi$ at $\theta = \frac{\pi}{2}$ with $dr = 0$. Then, we have $c_1 = 0$ in Eq. (3.14). Two equations (3.15) and (3.18) are reduced to give

$$d\phi = \frac{d\mu}{f_2(\mu)}, \quad \text{(3.21)}$$

where $f_2(\mu)$ is given by Eq. (2.7). In Fig. 2, we have numerically drawn the azimuth angle. The left panel is drawn for $m = 1$, 2 with $\alpha = 0$, which corresponds to the zero charged pure Schwarzschild limit. On the other hand,
Figure 1: The null geodesic $r(\mu) = h^{-1}(\mu)$ with the mass parameters, $m = 1, 2$, on the $\Sigma_r$ hypersurface.

Figure 2: Geodesic curve $\phi(\mu)$ at the plane $\theta = \frac{\pi}{2}$ with $dr = 0$: Left panel is for $m = 1, 2$ with $\alpha = 0$, while right panel is for $\alpha = 0.5, 1$ with $m = 1$. Here we note that the azimuth angles start to appear at $\alpha = 0.5, 1$, respectively, for the first time.

Finally, let us find the geodesic in the $\mu$-direction, which is defined by the
hypersurface $\Sigma_\mu$, eliminating $\mu$ in Eqs. (3.19) and (3.21), leading to
\[
d\phi = \frac{1}{r} \sqrt{\frac{2m - r}{r - \alpha}} dr.
\] (3.22)

This has a solution as
\[
\phi(r) = \sqrt{\frac{2m}{r}} \cot^{-1} \left( \frac{2\sqrt{2m\alpha(2m - r)(r - \alpha)}}{2mr - 4m\alpha + r\alpha} \right) + \tan^{-1} \left( \frac{2(m - r) + \alpha}{2\sqrt{(2m - r)(r - \alpha)}} \right).
\] (3.23)

In Fig. 3, we have drawn the geodesic curve $\phi(r)$ for $\alpha = 0.5, 1$ on the hypersurface $\Sigma_\mu$.

4 Conclusions

In this paper, we have studied the GMGHS interior spacetime in associated with a multiply warped product manifold. In the multiply warped product manifold, the GMGHS spacetime has been characterized by two warping functions $f_1(\mu)$ and $f_2(\mu)$, compared with the Schwarzschild spacetime which has the one warping function of $f_1(\mu)$. 
We have also investigated the geodesic motions in the multiply warped product spacetime near the hypersurfaces for each directions. As results, in the multiply warped product spacetime, we have shown the $r$-directed geodesic motion $r = h^{-1}(\mu)$ monotonically increases as $\mu$, while $\phi$-directed geodesic motion has no azimuth angle when $r = F^{-1}(\mu)$ is smaller than $\alpha$ and starts to appear when $r \geq \alpha$. We have also obtained the most general geodesic curve $\phi(r)$ with $\alpha$ along the $r$ variation.

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