Symmetry in finite phase plane

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Abstract. The known symmetries in one-dimensional systems are inversion and translations. These symmetries persist in finite phase plane, but a novel symmetry arises in view of the discrete nature of the coordinate $x_i$ and the momentum $p_i$. Thus, if $x_i$ assumes $M$ discrete values, $i = 0, 1, 2, \ldots, M - 1$, a permutation will change the order of the set $x_0, x_1, \ldots, x_{M-1}$ into a new ordered set. Such a symmetry element does not exist for a continuous $x$-coordinate in an infinite phase plane. Thus, in a finite phase plane, translations can be replaced by permutations. This is also true for the inversion operator. The new permutation symmetry has been used for the construction of conjugate representations and for the splitting of the $M$-dimensional vector space into independent subspaces. This splitting is exhaustive in the sense that if $M = \prod M_i$ with $M_i$ being prime numbers, the $M$-dimensional space splits into $M_1, M_2, \ldots, M_n$-dimensional independent subspaces. It is shown that following this splitting one can design new potentials with appropriate constants of motion. A related problem is the Weyl-Heisenberg group in the $M$-dimensional space which turns into a direct product of its subgroups in the $M_i$-dimensional subspaces. As an example we consider the case of $M = 8$.

1. Introduction

In recent years it has turned out that quantum mechanics in finite phase plane is a fascinating subject which is closely related to some notions of elementary number theory. Here are some results that depend crucially on the nature of the number $M$, the dimension of the vector space in the finite phase plane. It was Schwinger who in his fundamental paper on finite phase space quantum mechanics in 1960 [1], first showed that if $M = M_1 M_2$ with $M_1$ and $M_2$ relatively prime, then the $M$-dimensional space can be subdivided into two independent subspaces of dimensions $M_1$ and $M_2$. Somewhat later Ivanovic, in 1981 [2] has proved that when $M$ is a prime number, one can construct $M + 1$ orthogonal bases with scalar products of all the vectors in different bases having the same absolute value of magnitude $\frac{1}{\sqrt{M}}$. These orthogonal bases were named mutually unbiased bases and the result was generalized in 1989 to cases when the dimension of the space $M$ is a power of a prime number [3]. Quantum mechanics in finite phase plane can also depend on whether $M$ is even or odd. Thus, the Wigner function for odd $M$ can directly be obtained by replacing the integration by summation over the discrete $x_i$-values. This is not the case for even $M$, and there is a vast literature on this subject with different derivations for Wigner functions in the finite phase plane [4]. One of the reasons that the results can differ in finite and infinite phase plane is connected with the replacing of the variables $x$ and $y$ by the

1 Work done with B. Simkhovich, A. Mann and M. Revzen
variables \(s\) and \(t\) in the following transformation
\[
x + y = s, \quad x - y = t,
\]
where all the variables are mod \(M\). It can be checked \([4]\) that when \(x\) and \(y\) assume all the different values from 1 till \(M\), \(s\) and \(t\) from Eq. (1) are either both even or both odd. To put it in other words, there are, for example, no integers \(x\) and \(y\) that solve Eq. (1) for \(s = 1, t = 2\). A related result is the action of the inversion operator on the wave function \(\psi(x)\) in the finite phase plane \([6]\). One can check that for odd \(M\) one can derive the Wigner function in the same way as in the infinite phase plane \([7]\). This does not apply, however, for \(M\) even. The intuition one has acquired in infinite phase plane is, in general, not applicable in finite plane. As an example let us mention the inversion operator \(I\), which in the infinite case is defined as follows
\[
Ix = -x.
\]
In the finite phase plane this assumes the form
\[
Ix_k = -x_k = x_{-k} = x_{-k+M} \pmod{M}.
\]
For \(k = M\), \(Ix_M = x_M\). In the infinite case this corresponds to \(Ix = x\), when \(x = 0\). As was pointed out above, for the inversion operator there is a difference between the cases of \(M\) even and \(M\) odd. Thus, when \(M\) is even, it follows from Eq. (3) that
\[
Ix_{M/2} = x_{M/2} \pmod{M}.
\]
This means that when \(M\) is even there are two points on the \(x\)-axis, \(x_{M/2}\) and \(x_M(x_0)\), which remain unchanged under inversion. For \(M\) odd only the point \(x_M(x_0)\) is unchanged under inversion, which is like point \(x = 0\) in the infinite case of Eq. (2). One of the conclusions of this analysis is that for \(M = 2\), the inversion operator \(I\) turns into the unit operator. Or in other words in the \(M = 2\) space there are only even functions.

In this paper, we follow closely our previous work \([8]\). In a finite phase plane the coordinate \(x\) and momentum \(p\) operators are replaced by the exponential operators \([8]\).
\[
\tau(\frac{2\pi}{Mc}) = \exp(i\frac{2\pi}{Mc}x) \quad \text{and} \quad T(c) = \exp(-i\frac{\hbar}{\Theta c}p),
\]
where \(Mc\) is the size of the phase plane in the \(x\)-direction and \(c\) is a constant used in the discretization of \(x\) and \(p\)
\[
x_s = sc, s = 0, 1, \ldots, M - 1; \quad p_t = \hbar \frac{2\pi}{Mc} t, t = 0, 1, \ldots, M - 1.
\]
The discrete nature of \(x\) and \(p\) in Eq. (6) is a consequence of the finiteness of the phase plane given by the following requirement on the operators in Eq. (5)
\[
\tau^M(\frac{2\pi}{Mc}) = T^M(c) = 1.
\]
In order to make contact with Schwinger’s paper \([1]\), we denote the operators in Eq. (5) by
\[
U(x) = \tau(\frac{2\pi}{Mc}) = e^{i\frac{2\pi}{Mc}x}, \quad V(p) = T(c) = e^{-i\frac{\hbar}{\Theta c}p}.
\]
We have
\[
UV = e^{i\frac{2\pi}{Mc}} V U.
\]
U and V are called, according to Dirac [9], mutually conjugate operators. In Ref. [1] it was proven that when $M = M_1 M_2$, with $M_1$ and $M_2$ mutually prime, the $M$-dimensional space can be split into $M_1$ and $M_2$-dimensional spaces, which are independent of one-another. In this paper we use a recently developed method [10] for carrying out such a splitting when $M = M_1 M_2$, but without the requirement for $M_1$ and $M_2$ to be mutually prime, as was the case in Ref. [1]. This method uses permutations of the discrete coordinates, that have previously been used in the Cooley-Tukey algorithm for the fast Fourier transform [11]. These permutations have been called stride permutations [12]. We show in this paper that when used in the splitting of the $M$-dimensional phase space, these permutations lead to completely new periodic potentials that can exist only in the finite phase plane. Related to these potentials are special translation symmetries which are new constants of motion. We present an explicit example for the case of $M = 8 = 2^3$.

The operators $U$ and $V$ in Eq. (6) are generators of the Weyl-Heisenberg group in the $M$-dimensional phase plane [13]. The splitting of this space into its subspaces of prime dimensions $M_i$, $i = 1, 2, \ldots, n$ is accompanied by the writing of the Weyl-Heisenberg group in the $M$-dimensional space into a direct product of its subgroups in the $M_i$-dimensional subspaces.

2. Symmetry in Finite Phase Plane

In a finite phase plane the coordinate $x$ and momentum $p$ assume discrete values, as given in Eq. (6). Like in the infinite case, it is meaningful to use also in the finite phase plane the $x$ or $p$-representations. However, in the finite case, this opens up the possibility of a new kind of symmetry. Consider, for example, the case of dimensionality $M = 4$. According to Eq. (6) the coordinate $x_s$ assumes 4 values. If a function is given, such that

$$\psi(x_1) = \psi(x_3)$$

then this function is invariant under the permutation of interchanging the values of the coordinates $x_1$ and $x_3$. In the $x$-representation the operator $U$ in Eq. (8) turns into a multiplicative function of $x_s$, $U(x_s)$. It is easy to see that for $M = 4$

$$U^2(x_0) = U^2(x_2) = 1 \text{ and } U^2(x_1) = U^2(x_3) = -1.$$  

This means that $U^2(x_s)$ is invariant under the permutation cycles

$$(x_0, x_2) \text{ and } (x_1, x_3).$$

At this point it is worthwhile to make the following remark. In quantum mechanics the group of permutations (the symmetric group) is used in the statistics of identical particles [14,15]. Here, instead of applying permutations to coordinates of identical particles, they are applied to different values of the discrete coordinate (or momentum). A general permutation $S$ in the $M$-dimensional configuration space is,

$$S = \begin{pmatrix}
    x_0 & x_1 & \ldots & x_{M-1} \\
    x_{s_0} & x_{s_1} & \ldots & x_{s_{M-1}}
\end{pmatrix} \equiv \begin{pmatrix}
    x_k \\
    x_{s_k}
\end{pmatrix}$$

which means that $x_k$ goes into $x_{s_k}$. This is the action of the permutation $S$ in the configuration space. The action of $S$ on a function $\psi(x_k)$ of $x_k$ will respectively be [15]

$$S\psi(x_k) = \psi(S^{-1}x_k),$$

(in Ref. [15], pages 105, 106, a detailed discussion is given of the action in Eq. (14)) where $S^{-1}$ is the inverse of $S$. As an example we consider the function in Fig. 1, and let $P$ be the permutation

$$P = \begin{pmatrix}
    x_0 & x_1 & x_2 & x_3 \\
    x_2 & x_3 & x_0 & x_1
\end{pmatrix} = (x_0, x_2)(x_1, x_3).$$

ψ(x_k) is a function in a 4-dimensional space. ψ(x_0) = ψ(x_2), but ψ(x_1) ≠ ψ(x_3)

Under a general permutation ψ(x_k) will go into a different function ϕ(x_k) according to the rule of Eq. (14)

ϕ(x_k) = Pψ(x_k) = ψ(P−1x_k).

For the example of the permutation in Eq. (15) we have

ϕ(x_0) = ψ(x_2),  ϕ(x_1) = ψ(x_3),  ϕ(x_2) = ψ(x_0),  ϕ(x_3) = ψ(x_1)

in Fig. 1. This means that ψ(x_k) is invariant under the transposition (x_0, x_2), but not under (x_1, x_3).

As we know, the translation operator V in Eq. (8) acts on the wave function ψ(x_k) in the following way

Vψ(x_k) = ψ(x_k−1).

By using Eqs. (14) and(18) we can define a permutation S_V in the function space which corresponds to the translation operator V

S_V = \begin{pmatrix} x_k \\ x_{k+1} \end{pmatrix}.

(19)

From Eqs. (14) and (19) it follows

S_Vψ(x_k) = ψ(x_{k−1}),

(20)

because S_V−1 = \begin{pmatrix} x_k \\ x_{k−1} \end{pmatrix}. We have therefore replaced the translation operator V by the permutation S_V in Eq. (19) [12].

As pointed out above, the operator U in the x-representation is a multiplicative operator U(x_k). The eigenfunctions (⟨x_k|x_l⟩) of this operator in the x representation are

⟨x_k|x_l⟩ = ψ_{x_l}^k(x_k) = Δ^{(M)}(x_k - x_l).

(21)

Here Δ^{(M)}(x) is a Kornecker delta-like function which equals 1 when x = 0, mod M, and zero otherwise [8]; x_k is the coordinate and x_l determines the eigenvalues of U(x)

exp(ik\frac{2\pi}{Mc}),

(22)
where \( x'_i \) (it assumes the same values as \( x_i \)) is given in Eq. (6). The functions in Eq. (21) form a basis in the \( M \)-dimensional space. By applying the permutation in Eq. (13) to the basis in Eq. (21), we get a new set of functions.

\[
S\psi_{x'_i}(x_k) = \Delta^{(M)}(S^{-1}x_k - x'_i) = \langle S^{-1}x_k|x'_i \rangle = \langle x_k|Sx'_i \rangle,
\]

where \( S \) is the permutation in Eq. (13). For writing the last equality the fact was used that \( S \) is a unitary operator. In the new basis functions, the operators \( U(x) \) and \( V(p) \) in Eq. (8) will become (we keep in mind that \( V \) and \( M \) are mutually prime. Here we remove this restriction on \( M \)).

\[
U_S(x_k) = SU(x_k)S^{-1} = \exp(i\frac{2\pi}{Mc}S^{-1}x_k)
\]

\[
V_S = SVS^{-1} = SS_V S^{-1}.
\]

Again \( S \) is the general permutation in Eq. (13). It is worthwhile to make the following remark concerning the operators \( U(x_k), V(p) \) and \( S \) above. As was pointed out, \( U(x_k) \) is a multiplicative operator in the \( x \) representation, while \( V(p) \) and \( S \) are defined in Eqs. (18) and (14), respectively. In the representation of the eigenfunctions of \( U \) in Eq. (21), the operators \( U(x_k) \) and \( V(p) \) assume the following matrix form (\( \omega = \exp(i\frac{2\pi}{M}) \))

\[
U_M = \begin{pmatrix}
1 & 0 & 0 & \ldots & 0 \\
0 & \omega & 0 & \ldots & 0 \\
0 & 0 & \omega^2 & \ldots & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & \ldots & 0 & \omega^{M-1} & 0
\end{pmatrix}
\]

\[
V_M = \begin{pmatrix}
0 & 0 & 0 & \ldots & 0 & 1 \\
1 & 0 & 0 & \ldots & 0 & 0 \\
0 & 1 & 0 & \ldots & 0 & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & 1 & 0
\end{pmatrix}
\]

\( U_M \) is diagonal, while \( V_M \) has 1 underneath the diagonal, 1 in the upper right corner and zero everywhere else. Similarly, one can represent the permutation \( S \) in Eq. (13) by the permutation matrix \( S_M \) which in the \( k \)-th column has 1 in the \( s_k \)-th row and zero otherwise [16]. As an example we write the permutation matrix \( P_M \) for the permutation \( P \) in Eq. (15)

\[
P_M = \begin{pmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0
\end{pmatrix}
\]

It is easy to check that \( V_S \) and \( U_S \) in Eqs. (24) and (25) satisfy the same commutation relation as the operators \( U \) or \( V \) in Eq. (9). Each of these operators \( U \) or \( V \) form a complete commuting set of operators. The same is true about \( U_S \) and \( V_S \). When \( M \) is a prime number this also holds for any power of \( U(U_S) \) or \( V(V_S) \). This is, however, not true, in general, when \( M \) can be factorized into 2 or more factors, as is shown in the next section.

3. Splitting of a \( M = M_1M_2 \)-dimensional space into \( M_1 \) and \( M_2 \)-dimensional spaces
Part of this section is a review of some results of Ref. [10]. Having established the action of permutations in the finite phase plane, we show in this section how to use the new permutation symmetry in carrying out the splitting of an \( M = M_1M_2 \)-dimensional space into \( M_1 \) and \( M_2 \)-dimensional subspaces. In Ref. [1] such a splitting was carried out for the case when \( M_1 \) and \( M_2 \) are mutually prime. Here we remove this restriction on \( M_1 \) and \( M_2 \) and show how to carry out the splitting for any \( M_1 \) and \( M_2 \). Obviously, this process can be continued if either \( M_1 \) or \( M_2 \) (or
both) are non-prime. Eventually, this will lead to an exhaustive splitting of the $M$-dimensional phase plane into a direct product of $M_1$-dimensional subspaces, where $M = M_1 M_2$ and $M_1$ and $M_2$ are the prime factors of the decomposition of $M$. When $M = M_1 M_2$, we can use powers of the operators $U$ and $V$ in Eq. (8) for defining operators of two $kq$-representations [17]

$$U^{M_2} = \exp(i \frac{2\pi}{a} x), \quad V^{M_1} = \exp(-\frac{i}{\hbar} p a)$$

$$U^{M_1} = \exp(i \frac{2\pi}{b} x), \quad V^{M_2} = \exp(-\frac{i}{\hbar} p b)$$

where $a = M_1 c$ and $b = M_2 c$. Either of the couples of the operators in Eq. (28) or Eq. (29) forms a complete set of commuting operators. As such, each forms a quantum-mechanical representation, which can replace the $x$ or $p$-representations. In a recent publication [8] it was shown that when $M_1$ and $M_2$ are relatively prime the couples of operators in Eq. (28) and (29) are mutually conjugate. This means that in the eigenstates of the operators of Eq. (28), all eigenvalues of the operators in Eq. (29) are equally probable, and vice versa, in the eigenstates of the operators of Eq. (29), all eigenvalues of the operators in Eq. (28) are equally probable. Let us use the following notation:

$$u_1 = U^{M_2}, \quad v_1 = V^{M_2}, \quad u_2 = U^{M_1}, \quad v_2 = V^{M_1}.$$  \hspace{1cm} (30)

In references [1,8] it was shown that under the condition that $M_1$ and $M_2$ are mutually prime, the operators $u_1$ and $v_1$ are mutually conjugate in the subspace of dimension $M_1$. Similarly, the operators $u_2$ and $v_2$ are mutually conjugate in the subspace of dimension $M_2$. It should be added that the operators $u_1$ and $v_1$ commute with the operators $u_2$ and $v_2$ (no matter whether or not $M_1$ and $M_2$ are mutually prime). The use of the operators $u_1$, $v_1$ and $u_2$, $v_2$ splits the $M$-dimensional space into $M_1$ and $M_2$-dimensional space when $M_1$ and $M_2$ are mutually prime.

In what follows we show how to use the above introduced permutations in the finite phase plane in order to split the $M = M_1 M_2$-dimensional space into subspaces of dimension $M_1$ and $M_2$ when the latter are not necessarily mutually prime. For this we write the eigenvectors $|k, q\rangle$ [8,17] of the operators in Eq. (28) by means of the eigenvectors $|x\rangle$ of $U$ [see Eq. (21)]

$$|k, q\rangle = \frac{1}{\sqrt{M_2}} \sum_{n=0}^{M_2-1} \exp(ikM_1 n)|q + nM_1 c\rangle$$

$$\hspace{1cm} \text{(31)}$$

where $k$ and $q$ are the quasimomentum and the quasicoordinate, respectively. They assume the following discrete values

$$k = \frac{2\pi}{M_0} f, \quad f = 0, 1, \ldots, M_2 - 1: \quad q = gc, \quad g = 0, 1, \ldots, M_1 - 1.$$  \hspace{1cm} (32)

In order to keep in mind that $k$ assumes $M_2$ values and $q$ assumes $M_1$ values, let us rewrite Eq. (31) in the following way (we assume $c = 1$)

$$|k_2, q_1\rangle = \frac{1}{\sqrt{M_2}} \sum_{n_2=0}^{M_2-1} \exp(i k_2 \frac{2\pi}{M_2} n_2)|q_1 + n_2 M_1\rangle$$

$$\hspace{1cm} \text{(33)}$$

where now $k_2$ is used instead of $k$, and also it replaces $f$ in Eq. (32), $n_2$ appears instead of $n$ and $q$ is replaced by $q_1$. Similarly, we write the eigenvectors $|k_1, q_2\rangle$ of the operators in Eq. (29)

$$|k_1, q_2\rangle = \frac{1}{\sqrt{M_1}} \sum_{m_1=0}^{M_1-1} \exp(i k_1 \frac{2\pi}{M_1} m_1)|q_2 + m_1 M_2\rangle.$$  \hspace{1cm} (34)
where
\[k_1 = 0, 1, \ldots, M_1 - 1, \quad q_2 = 0, 1, \ldots, M_2 - 1.\] (35)

In Ref. [8] it was shown that for \(M_1\) and \(M_2\) relatively prime (e.g. \(M = 6, M_1 = 2, M_2 = 3\)) we have
\[|\langle k_2, q_1 | k_1, q_2 \rangle| = \frac{1}{\sqrt{M}},\] (36)
which means that the bases \(|k_2, q_1\rangle\) and \(|k_1, q_2\rangle\) are mutually unbiased [2,3]. However, when \(M_1\) and \(M_2\) are not relatively prime, the states \(|k_2, q_1\rangle\) and \(|k_1, q_2\rangle\) in Eqs. (33) and (34) are not mutually unbiased.

We now carry out the following transformation on the states \(|k_2, q_1\rangle\) in Eq. (33) by means of the operator \(A\), such that
\[A|q_1 + n_2 M_1\rangle = |n_2 + q_1 M_2\rangle,\] (37)
where \(|q_1 + n_2 M_1\rangle\) and \(|n_2 + q_1 M_2\rangle\) are eigenstates \(|x\rangle\) of the operator \(U\) in Eq. (8). We keep in mind that \(q_1 = 0, 1, \ldots, M_1 - 1\) and \(n_2 = 0, 1, \ldots, M_2 - 1\). The definition of \(A\) in Eq. (37) is based on the fact that any \(x\) can be represented uniquely by [11,12]
\[x = q_1 + n_2 M_1 \quad \text{or} \quad x' = n_2 + q_1 M_2.\] (38)

The presentation of \(x\) in Eq. (38) plays an important role in the Cooley-Tukey algorithm for the fast Fourier transform [11]. As an example of the transformation \(A\), let us consider the case of \(M = 8, M_1 = 2, M_2 = 4\). The results for Eq. (38) are given in Table 1.

| Table 1. | \(x\) and \(x'\) in Eq. (38) for \(M = 8, M_1 = 2, M_2 = 4\). |
|---|---|
| \(q_1\) | 0 0 0 0 1 1 1 1 |
| \(n_2\) | 0 1 2 3 0 1 2 3 |
| \(x\) | 0 2 4 6 1 3 5 7 |
| \(x'\) | 0 1 2 3 4 5 6 7 |

The transformation \(A\) which transforms \(x\) into \(x'\), Eq. (38), \(x' = Ax\), can be written as a permutation
\[A = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 0 & 4 & 1 & 5 & 2 & 6 & 3 & 7 \end{pmatrix} = (0)(1,4,2)(3,5,6)(7).\] (39)

The corresponding permutation matrix \(A_M\) is [See the permutation matrix \(P_M\) in Eq. (27) for the permutation \(P\) in Eq. (15)]:
\[A_M = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \] (40)

where the rows and columns are labeled from 0 to 7. It is often called stride permutation matrix [12].
Under the transformation in Eq. (37) we have
\[ A|k_2, q_1 \rangle = |k_2, q_1 \rangle = \frac{1}{\sqrt{M_2}} \sum_{n_2=0}^{M_2-1} \exp(\frac{2\pi i}{M_2} n_2)|n_2 + q_1 M_2\rangle. \] (41)

We keep in mind that the eigenstates \(|x\rangle\) are periodic in \(x\) with the period \(M\), because of the periodicity of \(U\) in Eq. (8) with the same period. The new states \(|k_2, q_1\rangle\), as can be checked, are mutually unbiased to the states \(|k_1, q_2\rangle\) in Eq. (34)
\[ \langle k_2, q_1 | k_1, q_2 \rangle = \frac{1}{\sqrt{M}} \exp(i k_1 \frac{2\pi}{M_1} q_1 - i k_2 \frac{2\pi}{M_2} q_2). \] (42)

We should point out that this result holds, no matter whether or not \(M_1\) and \(M_2\) are relatively prime. When we take the absolute value of both sides in Eq. (42), we get the result like in Eq. (36).

With the definition in Eq. (41), it is easy to see that \(|k_2, q_1\rangle\) are eigenstates of the operators [See Eq. (28)]
\[ \bar{u}_1 = \bar{U}^{M_2} \equiv AV_{M_2} A^{-1}, \quad \bar{v}_2 = \bar{V}^{M_1} = AV_{M_1} A^{-1}, \] (43)

since \(|k_2, q_1\rangle\) are eigenstates of the operators \(u_1 = U^{M_2}, v_2 = V^{M_1}\) in Eq. (30).

One can check that the following commutation relations hold
\[ \bar{u}_1 v_1 = v_1 \bar{u}_1 \exp\left(\frac{2\pi i}{M_1}\right), \quad u_2 \bar{v}_2 = \bar{v}_2 u_2 \exp\left(\frac{2\pi i}{M_2}\right), \] (44)
where \(v_1\) and \(u_2\) are defined in Eq. (30) and \(\bar{u}_1\) and \(\bar{v}_2\) in Eq. (43). As it follows from Eq. (44), the pairs of operators \(\bar{u}_1, v_1\) and \(u_2, \bar{v}_2\) are mutually conjugate. In Ref. [8] it was shown that for \(M_1\) and \(M_2\) relatively prime, the pairs \(u_1, v_1\) and \(u_2, v_2\) in Eq. (30) are mutually conjugate. In Eq. (44), the \(M_1\)-power of the operators with the subscript 1 equals 1, and, similarly for the operators for the \(M_2\)-power with the subscript 2, one has
\[ \bar{u}_1^{M_1} = v_1^{M_1} = u_2^{M_2} = \bar{v}_2^{M_2} = 1. \] (45)

We have therefore achieved the following goal. We have started with an \(M\)-dimensional space and the operators \(U\) and \(V\), which are conjugate according to Eq. (9). By assuming that \(M = M_1 M_2\), we have defined the operators in Eqs. (28)-(30), and consequently the operators in Eq. (43). This brought us to the operators in Eq. (44) (satisfying Eq. (45)), which split the \(M\)-dimensional space into \(M_1\) and \(M_2\)-dimensional spaces. We keep in mind that the operators in Eq. (44) with subscript 1 commute with those with the subscript 2. Clearly, if either \(M_1\) or \(M_2\) (or both) are not prime, then this splitting process can be continued. Eventually, this process will come to an end, when we have split the \(M\)-dimensional space into \(M_1, M_2, \ldots, M_n\) - dimensional subspaces, where \(M_1, M_2, \ldots, M_n\) are the prime factors into which \(M\) splits.

A similar statement follows from this splitting with respect to the Weyl-Heisenberg group in the \(M\)-dimensional phase plane. As is known [13] this group has a single \(M\)-dimensional irreducible representation. When \(M = \cap_i M_i\) we denote the Weyl-Heisenberg group in the \(M\)-dimensional phase plane \(G_M\), and the corresponding subgroups in the \(M_i\) - dimensional spaces by \(G_{M_i}\). Then our results show that \(G_M\) is a direct product [18] of its subgroups \(G_{M_1}, G_{M_2}, \ldots, G_{M_n}\). The \(M\)-dimensional irreducible representation \(D^{(M)}\) of \(G_M\) is therefore a direct product
\[ D^{(M)} = D^{(M_1)} \times D^{(M_2)} \times \ldots \times D^{(M_n)} \] (46)
of the irreducible representations \(D^{M_i}\) of its subgroups. It is satisfying that this statement is consistent with the product \(M = \cap_i M_i\).
4. Example: $M = 8$
As a first step we choose $M_1 = 2$, $M_2 = 4$. They are not mutually prime, and therefore the results of Refs. [1,8] do not apply, and we shall turn to the operators in Eqs. (30) and (43) of this paper ($a = 2c$, $b = 4c$)

$$u_1 = U^4 = e^{i \frac{2\pi}{c} x}, \quad v_2 = V^2 = e^{-i \frac{\hbar}{p} 2c} = (0, 2, 4, 6) (1, 3, 5, 7)$$

$$\tilde{u}_1 = A u_1 A^{-1} = e^{i \frac{\pi}{A} 2c^{-1} x}, \quad \tilde{v}_2 = A v_2 A^{-1} = (0, 1, 2, 3) (4, 5, 6, 7)$$

where we have used Eqs. (30), (39) and (43). In Figs. 2 and 3 we plot the real parts of the operators $u_1$ and $\tilde{u}_1$.

**Figure 2.** $\text{Re} [u_1(x)]$ as a function of $x$.

**Figure 3.** $\text{Re} [\tilde{u}_1(x)]$ as a function of $x$.

There are a number of features one can see in these figures. $\text{Re} [u_1(x)]$ in Fig. 2 is a conventional periodic Bloch potential on a lattice with period $a = 2c$ for the discrete coordinate $x_s = sc$ $s = 0, 1, \ldots, 7$. One can see from this figure that the operator $v_2$, which is a shift by the period $2c$, commutes with the potential $\text{Re} [u_1(x)]$ (it also commutes with $u_1$ itself in Eq. (47)). A more interesting case is the potential $\text{Re} [\tilde{u}_1(x)]$ in Fig. 3. This seems to be a new kind of a potential which is invariant under the new kind of symmetry $\tilde{v}_2$ in Eq. (48). $\tilde{v}_2$ consists of a product of 2 independent permutations, each operating in a 4-dimensional space. From the graph in Fig. 3 one can see that $\tilde{v}_2$ commutes with $\text{Re} [\tilde{u}_1(x)]$ (it also commutes with $u_1(x)$ in Eq. (48)). According to Eq. (30) we have a pair of commuting operators

$$u_2 = e^{i \frac{2\pi}{4c} x}, \quad v_1 = e^{-i \frac{\hbar}{p} 4c}.$$  

(49)
Respectively, from Eq. (44) the following commutation relations hold for $M = 8$

$$\bar{u}_1 v_1 = v_1 \bar{u}_1 \exp(i \frac{2\pi}{2}), \quad u_2 \bar{v}_2 = \bar{v}_2 u_2 \exp(i \frac{2\pi}{4}),$$  \hspace{1cm} (50)

and the operators with index 1 commute with those of index 2.

Having in mind that $v_1$ in Eq. (49) is a shift by $4c$, we can read directly from Fig. 3 the first commutation relation in Eq. (50). We can see that any shift by $4c$ changes the sign of $\cos(\frac{\pi}{c} x - 1)$. In order to see the same for the second relation in Eq. (50), we need the graph for $u_2(x)$. In Fig. 4 we plot the real part, $\text{Re}[u_2(x)]$, of the operator $u_2(x)$.

First, we can easily see from the graph in Fig. 4 that $v_1$ in Eq. (49) commutes with $\text{Re}[u_2(x)]$. The second relation in Eq. (50) contains the permutation $\bar{v}_2$ in Eq. (48), which permutes the coordinates $(0, 1, 2, 3)$ and $(4, 5, 6, 7)$ among themselves. Such permutations change the phase by $\frac{\pi}{2}$, like in Eq. (50).

The commutation relations in Eq. (50) are the same as the one in Eq. (9). The only difference is that in Eq. (9) $M$ appears (in our example $M = 8$) while in the first relation of Eq. (50) we have $M = 2$, and in the second one $M = 4$. As was pointed out in Eq. (45), the operators in Eq. (50) are of the order 2 and 4 respectively

$$\bar{u}_1^2 = v_1^2 = u_2^4 = \bar{v}_2^4 = 1.$$ \hspace{1cm} (51)

Eqs. (50) and (51) therefore show that the two pairs of operators $\bar{u}_1, v_1$ and $u_2, \bar{v}_2$ split the 8-dimensional space for $M = 8$ into 2-dimensional and 4-dimensional spaces, respectively. The first commutation relation in Eq. (50) is in a 2-dimensional space and cannot be split any further. The second relation is in a 4-dimensional space. As was pointed out above, this relation is the same as the one in Eq. (9), for $M = 4$. We can now repeat the process and split the 4-dimensional space into two 2-dimensional spaces. This will complete the splitting of the 8-dimensional space into three 2-dimensional spaces.

We will not repeat this splitting and give directly the results. For this we write the matrices $U_m$ and $V_m$ in Eq. (26) for $M = 8$. For $U_8$ we list only its diagonal elements (the non-diagonal elements are all zero). $V_8$ we represent by its permutation in Eq. (19). We have ($w = \exp(i \frac{2\pi}{8})$)

$$U_8 = \begin{pmatrix} 1 & w & \cdots & w^7 \end{pmatrix}, \quad V_8 = (0, 1, \ldots, 7).$$ \hspace{1cm} (52)
For giving the results of the splitting of the $M = 8$-dimensional space into three independent 2-dimensional spaces we need the operators, which are 4th powers of the ones in Eq. (52)

$$U_8^4 = \begin{pmatrix}
1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\
1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 \\
1 & 1 & 1 & -1 & 1 & 1 & -1 & 1 \\
1 & 1 & 1 & 1 & -1 & 1 & 1 & 1
\end{pmatrix}, \quad V_8^4 = (0, 4)(1, 5)(2, 6)(3, 7). \quad (53)$$

The operators that carry out the splitting of the 8-dimensional space in 3 two-dimensional spaces are then

$$U_1 = U_8^4 = \begin{pmatrix}
1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\
1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 \\
1 & 1 & 1 & -1 & 1 & 1 & -1 & 1 \\
1 & 1 & 1 & 1 & -1 & 1 & 1 & 1
\end{pmatrix}, \quad V_1 = A^{-1}V_8^4A = (0, 1)(2, 3)(4, 5)(6, 7), \quad (54)$$

$$U_2 = AU_8^4A^{-1} = \begin{pmatrix}
1 & 1 & -1 & 1 & -1 & 1 & 1 & -1 \\
1 & 1 & 1 & -1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & -1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & -1 & 1 & 1
\end{pmatrix}, \quad V_2 = V_8^4 = (0, 4)(1, 5)(2, 6)(3, 7), \quad (55)$$

$$U_3 = BU_8^4B = \begin{pmatrix}
1 & 1 & -1 & 1 & -1 & 1 & 1 & -1 \\
1 & 1 & 1 & -1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & -1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & -1 & 1 & 1
\end{pmatrix}, \quad V_3 = CV_8^4C = (0, 2)(1, 3)(4, 6)(5, 7), \quad (56)$$

where $A$ is given in Eqs. (39), (40), and

$$B = B^{-1} = (0)(1, 2)(3)(4)(5, 6)(7), \quad C = C^{-1} = (0)(1)(2, 4)(3, 5)(6)(7). \quad (57)$$

The operators in Eqs. (54)-(56) satisfy the following commutation relations

$$U_iV_i = -V_iU_i, \quad i = 1, 2, 3. \quad (58)$$

This is the same commutation relation as in Eq. (9) for the operators $U$ and $V$ but for $M = 2$. Also, every pair of operators in one row of Eqs. (54)-(56) commutes with the 4 operators in the other 2 rows. In addition, the squares of all the operators in Eqs. (54)-(56) equal to 1, which is
in agreement with Eqs. (7) and (8) for the operators $U$ and $V$ for $M = 2$. The conclusion is that the operators in Eqs. (54)-(56) split the 8-dimensional space in three 2-dimensional independent spaces.

This result enables us to establish a one-to-one correspondence between the operators in Eqs. (54)-(56) and the spin half Pauli matrices by the following identification

$$
\sigma_{ix} = U_i, \quad \sigma_{iy} = V_i, \quad i = 1, 2, 3,
$$

where $\sigma_x$ and $\sigma_y$ are the Pauli matrices [24]

$$
\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}.
$$

With the identification in Eq. (59) we accommodate 3 spin half operators in the 8-dimensional phase plane. This accommodation can be extended to $n$ spin half particles in a $2^n$-dimensional phase plane.

The Weyl-Heisenberg group in 8 dimensions $G_8$ has $8^3$ elements (is of the order 512) [13]. The elements in each row of the Eqs. (54)-(56) are generators of the subgroups $G_2$ of $G_8$. We have 3 such subgroups which we denote by $G_2^i$, $i = 1, 2, 3$. These subgroups commute with one another, and therefore $G_8$ can be written as a direct product of 3 subgroups of order 8.

$$
G_8 = G_2^{(1)} \times G_2^{(2)} \times G_2^{(3)}.
$$

Correspondingly, the single irreducible representation of dimension 8, $D^{(8)}$ of $G^{(8)}$ can be written as a direct product in the following way

$$
D^{(8)} = D^{(2)} \times D^{(2)} \times D^{(2)},
$$

where each $D^{(2)}$ is a 2-dimensional representation of one of the subgroups $G^{(i)}$, $i = 1, 2, 3$ in Eq. (61). The splitting of the 8-dimensional phase space, $M = 8$, into 3 independent 2-dimensional spaces, is therefore accompanied by the elegant results in Eqs. (61) and (62) for the Weyl-Heisenberg group.

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