ON THE HOM-ASSOCIATIVE WEYL ALGEBRAS

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Abstract. The first (associative) Weyl algebra is formally rigid in the classical sense. In this paper, we show that it can however be formally deformed in a nontrivial way when considered as a so-called hom-associative algebra, and that said deformation preserves properties such as the commutator, while deforming others, such as the center, power associativity, the set of derivations, and some commutation relations. We then show that the deformation induce a formal deformation of the corresponding Lie algebra into what is known as a hom-Lie algebra, when using the commutator as bracket. We also prove that all homomorphisms between any two purely hom-associative Weyl algebras are in fact isomorphisms. In particular, all endomorphisms are automorphisms in this case, hence proving a hom-associative analogue of the Dixmier conjecture to hold true.

1. Introduction

Hom-associative algebras has its origins in hom-Lie algebras, the latter proposed by Hartwig, Larsson, and Silvestrov in [10] as a generic framework to describe deformations of Lie algebras obeying a generalized Jacobi identity, the latter now twisted by a homomorphism; hence the name. Hom-associative algebras, introduced by Makhlouf and Silvestrov in [15], now play the same role as associative algebras do for Lie algebras; equipping a hom-associative algebra with the commutator as bracket give rise to a hom-Lie algebra. Just as the Jacobi identity in the latter algebras is twisted, the same holds true for the associativity condition in the former. In particular may hom-associative algebras be seen to include associative algebras and general non-associative algebras in the following way: when the map twisting this condition is the identity map, one recovers the associativity condition, and when equal to the zero map, said condition becomes null. When the map is not a multiple of the identity, we call it the purely hom-associative case. (Note that a purely hom-associative algebra can happen to be associative as well.)

The first (associative) Weyl algebra may be seen as an Ore extension, or a non-commutative polynomial ring as Ore extensions were first named by Ore when he introduced them in 1933 [18]. Non-associative Ore extensions were later introduced by Nystedt, Öinert, and Richter in the unital case [17], and then generalized to the non-unital, hom-associative setting in [4] by Silvestrov and the authors. The authors further developed this theory in [3], introducing a Hilbert’s basis theorem for unital, hom-associative or non-associative Ore extensions.

The first (associative) Weyl algebra is formally rigid in the classical sense of Gerstenhaber, who introduced formal deformation theory for associative algebras and

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rings in the seminal paper [7]. However, as described above, any associative algebra may be seen as a hom-associative algebra with twisting map equal to the identity map, and we show in this paper that as such, it may be formally deformed in a non-trivial way. This results in the hom-associative Weyl algebras, first introduced in [4] along with the hom-associative quantum plane and a universal enveloping algebra of the two-dimensional non-abelian Lie algebra, the latter two which also turned out to be formal deformations of their associative counterparts [2]. We also show that the formal deformation of the first Weyl algebra into the hom-associative Weyl algebras induce a formal deformation of the corresponding Lie algebra into hom-Lie algebras, when using the commutator as bracket. More formally can they be described as one-parameter formal hom-associative deformations and one-parameter formal hom-Lie deformations, respectively, two notions introduced earlier by Makhlouf and Silvestrov [16]. Furthermore, we see that the former deformation preserves some properties, such as the commuter, while deforming others such as the center, power associativity, the set of derivations, and some commutation relations. The perhaps most interesting fact we are able to prove, however, is that all homomorphisms between any two purely hom-associative Weyl algebras are in fact isomorphisms. This particularly implies that all endomorphisms are automorphisms in this case, proving a hom-associative analogue of the Dixmier conjecture to hold true, the latter a still unsolved conjecture that has its origins in a question raised by Dixmier in [5] (cf. 11. Problèmes).

The paper is organized as follows:

Section 2 is concerned with preliminaries on general non-associative algebra (Subsection 2.1), hom-associative algebras and hom-Lie algebras (Subsection 2.2), non-unital, hom-associative Ore extensions (Subsection 2.3), the first Weyl algebra (Subsection 2.4), and the hom-associative Weyl algebras (Subsection 2.5).

Section 3 contains results on some of the standard properties of the hom-associative Weyl algebras: it is shown that they contain no zero divisors (Corollary 1), that their commuter is equal to the ground field (Proposition 6), that the first Weyl algebra is the only power associative hom-associative Weyl algebra (Proposition 7), and moreover are all their derivations described (Corollary 4). It is further shown that all homomorphisms between any two purely hom-associative Weyl algebras are isomorphisms of a certain type (Proposition 8), this implying that all endomorphisms are automorphisms in this case (Corollary 5).

Section 4 gives a brief review of one-parameter formal hom-associative deformations and one-parameter formal hom-Lie deformations. It is then shown that the hom-associative Weyl algebras can be seen as a one-parameter formal hom-associative deformation of the first Weyl algebra (Proposition 9), inducing also a one-parameter formal hom-Lie deformation of the Lie algebra when using the commutator as bracket (Proposition 10).

2. Preliminaries

Throughout this paper, we denote by \( \mathbb{N} \) the nonnegative integers.

2.1. General non-associative algebra. By a non-associative algebra \( A \) over an associative, commutative, and unital ring \( R \), we mean an \( R \)-algebra which is not necessarily associative. Furthermore, \( A \) is called unital if there exists an element \( 1_A \in A \) such that for all \( a \in A \), \( a \cdot 1_A = 1_A \cdot a = a \), and non-unital if there does not necessarily exist such an element. For a non-associative and non-unital
algebra $A$, the commutator $[\cdot, \cdot]: A \times A \to A$ is defined by $[a, b] := a \cdot b - b \cdot a$ for arbitrary $a, b \in A$, and the commutator of $A$, written $C(A)$, as the set $C(A) := \{a \in A: [a, b] = 0, b \in A\}$. The associator $(\cdot, \cdot, \cdot): A \times A \times A \to A$ is defined by $(a, b, c) = (a \cdot b) \cdot c - a \cdot (b \cdot c)$ for any elements $a, b, c \in A$, and the left, middle, and right nuclei of $A$, denoted by $N_l(A), N_m(A)$, and $N_r(A)$, respectively, as the sets $N_l(A) := \{a \in A: (a, b, c) = 0, b, c \in A\}$, $N_m(A) := \{b \in A: (a, b, c) = 0, a, c \in A\}$, and $N_r(A) := \{c \in A: (a, b, c) = 0, a, b \in A\}$. The nucleus of $A$, written $N(A)$, is defined as the set $N(A) := N_l(A) \cap N_m(A) \cap N_r(A)$. The center of $A$, denoted by $Z(A)$, is the intersection of the commutator and the nucleus, $Z(A) := C(A) \cap N(A)$.

If only the two-sided ideals in $A$ are the zero ideal and $A$ itself, $A$ is called simple. A way to measure the non-associativity of $A$ can be done by using the associator: $A$ is called power associative if $(a, a, a) = 0$, left alternative if $(a, a, b) = 0$, right alternative if $(b, a, a) = 0$, flexible if $(a, b, a) = 0$, and associative if $(a, b, c) = 0$ for all $a, b, c \in A$. A linear map $\delta: A \to A$ is called a derivation if for any $a, b \in A$, $\delta(a \cdot b) = \delta(a) \cdot b + a \cdot \delta(b)$. If $A$ is associative, then all maps of the form $[a, \cdot]: A \to A$ for an arbitrary $a \in A$ are derivations called inner derivations. If $A$ is not associative, such a map need not be a derivation, however. At last, recall that $A$ embeds into a non-associative and non-unital algebra $B$ if there is an injective morphism from $A$ to $B$, the idea being that $A$ then may be seen as a subalgebra of $B$.

2.2. Hom-associative algebras and hom-Lie algebras. This section is devoted to restating some basic definitions and general facts concerning hom-associative algebras and hom-Lie algebras. Though hom-associative algebras as first introduced in [15] and hom-Lie algebras in [10] were defined by starting from vector spaces, we take a slightly more general approach here, following the conventions in e.g. [2, 3, 4], starting from modules; most of the general theory still hold in this latter case, though.

**Definition 1** (Hom-associative algebra). A hom-associative algebra over an associative, commutative, and unital ring $R$, is a triple $(M, \alpha)$ consisting of an $R$-module $M$, a binary operation $\cdot: M \times M \to M$ linear over $R$ in both arguments, and an $R$-linear map $\alpha: M \to M$, satisfying, for all $a, b, c \in M$, $\alpha(a) \cdot \alpha(b \cdot c) = (a \cdot b) \cdot \alpha(c)$.

Since $\alpha$ twists the associativity, it is referred to as the twisting map of a hom-associative algebra. A multiplicative hom-associative algebra is one where the twisting map is multiplicative, i.e. a homomorphism.

**Remark 1.** A hom-associative algebra over $R$ is in particular a non-unital, non-associative $R$-algebra, and in case $\alpha = \text{id}_M$, a non-unital, associative $R$-algebra. In case $\alpha = 0_M$, the hom-associative condition becomes null, and thus hom-associative algebras can be seen as generalizations of both associative and non-associative algebras.

**Definition 2** (Hom-associative ring). A hom-associative ring is a hom-associative algebra over the integers.

**Definition 3** (Weakly unital hom-associative algebra). Let $A$ be a hom-associative algebra. If for all $a \in A$, $e_l \cdot a = \alpha(a)$ for some $e_l \in A$, we say that $A$ is weakly left unital with weak left unit $e_l$. In case $a \cdot e_r = \alpha(a)$ for some $e_r \in A$, $A$ is called...
weakly right unital with weak right unit $e_r$. If there is an $e \in A$ which is both a weak left and a right unit, $e$ is called a weak unit, and $A$ weakly unital.

Remark 2. The notion of a weak unit can thus be seen as a weakening of that of a unit. A weak unit, when it exists, need not be unique.

Proposition 1 ([21, 6]). Let $A$ be a unital, associative algebra with unit $1_A$, $\alpha$ an algebra endomorphism on $A$, and define $*: A \times A \to A$ by $a * b := \alpha(a \cdot b)$ for all $a, b \in A$. Then $(A, *, \alpha)$ is a weakly unital hom-associative algebra with weak unit $1_A$.

Note that we are abusing the notation in Definition 1 a bit here; $A$ in $(A, *, \alpha)$ does really denote the algebra and not only its module structure. From now on, we will always refer to this construction when writing $*$.

Definition 4 (Hom-Lie algebra). A hom-Lie algebra over an associative, commutative, and unital ring $R$ is a triple $(M, [\cdot, \cdot], \alpha)$ where $M$ is an $R$-module, $\alpha: M \to M$ a linear map called the twisting map, and $[\cdot, \cdot]: M \times M \to M$ a map called the hom-Lie bracket, satisfying the following axioms for all $a, b, c \in M$ and $r, s \in R$:

\[
[ra + sb, c] = r[a, c] + s[b, c], \quad [a, rb + sc] = r[a, b] + s[a, c], \quad (\text{bilinearity}),
\]

\[
[a, a] = 0, \quad (\text{alternativity}),
\]

\[
[\alpha(a), [b, c]] + [\alpha(c), [a, b]] + [\alpha(b), [c, a]] = 0, \quad (\text{hom-Jacobi identity}).
\]

As in the case of a Lie algebra, we immediately get anti-commutativity from the bilinearity and alternativity by calculating $0 = [a + b, a + b] = [a, a] + [a, b] + [b, a] + [b, b] = [a, b] + [b, a]$, so $[a, b] = -[b, a]$ holds for all $a$ and $b$ in a hom-Lie algebra as well. Unless $R$ has characteristic two, anti-commutativity also implies alternativity, since $[a, a] = -[a, a]$ for all $a$.

Remark 3. If $\alpha = \text{id}_M$ in Definition 4, we get the definition of a Lie algebra. Hence the notion of a hom-Lie algebra can be seen as a generalization of that of a Lie algebra.

Proposition 2 ([15]). Let $(M, \cdot, \alpha)$ be a hom-associative algebra with commutator $[\cdot, \cdot]$. Then $(M, [\cdot, \cdot], \alpha)$ is a hom-Lie algebra.

Note that when $\alpha$ is the identity map in the above proposition, one recovers the classical construction of a Lie algebra from an associative algebra. We refer to the above construction as the commutator construction.

2.3. Non-unital, hom-associative Ore extensions. Here, we give some preliminaries from the theory of non-unital, hom-associative Ore extensions, as introduced in [4]. First, if $R$ is a non-unital, non-associative ring, a map $\beta: R \to R$ is called left $R$-additive if for all $r, s, t \in R$, we have $r \cdot \beta(s + t) = r \cdot \beta(s) + r \cdot \beta(t)$. If given two left $R$-additive maps $\delta$ and $\sigma$ on a non-unital, non-associative ring $R$, by a non-unital, non-associative Ore extension of $R$, written $R[x; \sigma, \delta]$ we mean the set of formal sums $\sum_{i\in\mathbb{N}} r_i x^i$ where finitely many $r_i \in R$ are non-zero, equipped with the following addition:

\[
\sum_{i\in\mathbb{N}} r_i x^i + \sum_{i\in\mathbb{N}} s_i x^i = \sum_{i\in\mathbb{N}} (r_i + s_i) x^i, \quad r_i, s_i \in R,
\]
and the following multiplication, first defined on monomials $rx^m$ and $sx^n$ where $m, n \in \mathbb{N}$:

$$rx^m \cdot sx^n = \sum_{i \in \mathbb{N}} (r \cdot \pi_i^m(s))x^{i+n}$$

and then extended to arbitrary polynomials $\sum_{i \in \mathbb{N}} r_ix^i$ in $R[x;\sigma,\delta]$ by imposing distributivity. The function $\pi_i^m: R \to R$ is defined as the sum of all $\binom{m}{i}$ compositions of $i$ instances of $\sigma$ and $m-i$ instances of $\delta$, so that for example $\pi_2^2 = \sigma \circ \sigma \circ \delta + \sigma \circ \delta \circ \sigma + \delta \circ \sigma \circ \sigma$, and by definition, $\pi_0^m = \text{id}_R$. Whenever $i < 0$, or $i > m$, we put $\pi_i^m \equiv 0$.

That this really gives an extension of the ring $R$, as suggested by the name, can now be seen by the fact that $rx^0 \cdot sx^0 = \sum_{i \in \mathbb{N}} (r \cdot \pi_i^0(s))x^{i+0} = (r \cdot \pi_0^0(s))x^0 = (r \cdot s)x^0$, and similarly $rx^0 + sx^0 = (r+s)x^0$ for any $r, s \in R$. Hence the isomorphism $r \mapsto rx^0$ embeds $R$ into $R[x;\sigma,\delta]$. We shall only be concerned with the case $\sigma = \text{id}_R$ however, in which case (2) simplifies to

$$rx^m \cdot sx^n = \sum_{i \in \mathbb{N}}\binom{m}{i} (r \cdot \delta^{m-i}(s)) x^{i+n}.$$  

Starting with a non-unital, non-associative ring $R$ equipped with two left $R$-additive maps $\delta$ and $\sigma$ and some additive map $\alpha: R \to R$, we extend $\alpha$ homogeneously to $R[x;\sigma,\delta]$ by putting $\alpha(rx^m) = \alpha(r)x^m$ for all $rx^m \in R[x;\sigma,\delta]$, and then assuming additivity. If $\alpha$ is further assumed to be multiplicative and to commute with $\sigma$, we can turn a non-unital (unital), associative Ore extension into a non-unital (weakly unital), hom-associative Ore extension by using this extension, as the following proposition demonstrates:

**Proposition 3** ([4]). Let $R[x;\sigma,\delta]$ be a non-unital, associative Ore extension of a non-unital, associative ring $R$, and $\alpha: R \to R$ a ring endomorphism that commutes with $\delta$ and $\sigma$. Then $(R[x;\sigma,\delta],*,\alpha)$ is a multiplicative, non-unital, hom-associative Ore extension with $\alpha$ extended homogeneously to $R[x;\sigma,\delta]$.

**Remark 4.** Note that if $S := R[x;\sigma,\delta]$ in Proposition 3 is unital with unit $1_S$, then $(R[x;\sigma,\delta],*,\alpha)$ is weakly unital with weak unit $1_S$ by Proposition 1.

2.4. The first Weyl algebra. The first Weyl algebra $A_1$ over a field $K$ of characteristic zero, from now on just referred to as the Weyl algebra, is the free, associative and unital algebra on two letters $x$ and $y$, modulo the commutation relation $[x,y] := x \cdot y - y \cdot x = 1_{A_1}$, $1_{A_1} = 1_Ky^0x^0$ being the unit element in $A_1$. It may be exhibited as the associative and unital iterated Ore extension $K[y]/[x;\text{id}_K[y],d/dy]$ where $d/dy$ is the ordinary derivative on $K[y]$ (see e.g. [9], also giving a nice introduction to the theory of (associative) Ore extensions). Furthermore, $A_1$ is a classical example of a non-commutative domain, and as a vector space it has a basis $\{y^ix^j : i,j \in \mathbb{N}\}$. Littlewood [13] proved that $A_1$ is simple when $K = \mathbb{R}$ and $K = \mathbb{C}$, and Hirsch [11] then generalized this to when $K$ is an arbitrary field of characteristic zero, as well as for higher order Weyl algebras. Sridharan showed in [20] (cf. Remark 6.2 and Theorem 6.1) that the cohomology of $A_1$ is zero in all positive degrees (see also Theorem 5 in [8]). In particular, the vanishing of the cohomology in the first and second degree imply that all derivations are inner and that $A_1$ is formally rigid in the classical sense of Gerstenhaber [7]. It should be mentioned that there exists however a nontrivial so-called non-commutative deformation, due to Pinczon [19]. In said deformation, the deformation parameter no longer commutes...
Proposition 4.

Dixmier further undertook a thorough investigation of $A_1$ in [5], the same paper in which he asked whether all its endomorphisms are actually automorphisms? Although this question still remains unanswered, Dixmier managed to describe its automorphism group $\text{Aut}_K(A_1)$, which Makar-Limanov then gave a new proof of in [14], describing the generators of $\text{Aut}_K(A_1)$ as follows:

**Theorem 1 ([14]).** $\text{Aut}_K(A_1)$ is generated by linear automorphisms,

$$x \mapsto ax + by, \quad y \mapsto cx + dy, \quad \begin{vmatrix} a & c \\ b & d \end{vmatrix} = 1,$$

for $a, b, c, d \in K$, and triangular automorphisms,

$$x \mapsto x, \quad y \mapsto y + p(x), \quad p(x) \in K[x].$$

2.5. The hom-associative Weyl algebras. In [4], a family of hom-associative Weyl algebras $\{A^k_1\}_{k \in K}$ were constructed as generalizations of $A_1$ to the hom-associative setting, including $A_1$ in the member corresponding to $k = 0$: $A^0_1 = A_1$. Concretely, one finds that an endomorphism $\alpha_k$ commutes with $d/dy$ (and $\text{id}_{K[y]}$) on $K[y]$ if and only if it is of the form $\alpha_k(y) = y + k$, $\alpha_k(1_{A_1}) = 1_{A_1}$, for some arbitrary $k \in K$. Hence, in the light of Proposition 3, we have for each $k \in K$ a hom-associative Weyl algebra $A^k_1 = (A_1, *, \alpha_k)$, where $\alpha_k(x) = x$. The unit element $1_{A_1}$ of $A_1$ is now acting as a weak unit in the whole of $A^k_1$, where for instance $1_{A_1} * y := \alpha_k(1_{A_1} * y) = \alpha_k(y) = y + k$. Moreover, if we use the subscript $*$ whenever the multiplication is that defined in Proposition 1, so that $(\cdot, \cdot)_*$ is the associator and $[\cdot, \cdot]_*$ the commutator in $A^k_1$, we have $[x, y]_* = [x, y] = 1_{A_1}$. It was further shown in [4] that $A^k_1$ are all simple.

3. Morphisms, derivations, commutation and association relations

This section contains results of some of the standard properties of $A^k_1$.

** Lemma 1.** Surjective morphisms of hom-associative algebras preserve weak left (right) units.

**Proof.** Let $f : A \to B$ be a surjective morphism between two hom-associative algebras with twisting maps $\alpha_A$ and $\alpha_B$ respectively, and $e_A$ a weak left unit of $A$. We show the left case; the right case is analogous. For any element $b \in B$, there is an $a \in A$ such that $b = f(a)$, so $f(e_A) \cdot b = f(e_A) \cdot f(a) = f(e_A \cdot a) = f(\alpha_A(a)) = \alpha_B(f(a)) = \alpha_B(b)$.$\square$

**Proposition 4.** $K$ embeds as a subfield into $A^k_1$.

**Proof.** Let $f : A \to B$ be a surjective morphism between two hom-associative algebras with twisting maps $\alpha_A$ and $\alpha_B$ respectively, and $e_A$ a weak left unit of $A$. We show the left case; the right case is analogous. For any element $b \in B$, there is an $a \in A$ such that $b = f(a)$, so $f(e_A) \cdot b = f(e_A) \cdot f(a) = f(e_A \cdot a) = f(\alpha_A(a)) = \alpha_B(f(a)) = \alpha_B(b)$.$\square$

$K$ is embedded into the associative Weyl algebra $A_1 := K[y][x; \text{id}_{K[y]}, \frac{dx}{dy}]$ by the isomorphism $f : K \to K' := \{ay^0x^0 : a \in K\} \subseteq A_1$ defined by $f(a) = ay^0x^0$ for any $a \in K$. The same map embeds $K$ into $A^k_1$, i.e. it is also an isomorphism from the hom-associative subalgebra $K' \subseteq A^k_1$. First, note that the unit element $1_{A_1}$ of $A_1$ is $1_K y^0x^0$, and that $\alpha_k$ is linear over $K$. Since $\alpha_k$ is a $K$-automorphism on $A_1$, $\alpha_k(1_{A_1}) = 1_{A_1}$. Hence, for any $a \in K$, $\alpha_k(ax^0y^0) = ax^0y^0$, i.e. $\alpha_k|_{K'} = \text{id}_{K'}$. Thus, for any $b, c \in K$,

$f(b \cdot c) = (b \cdot c)y^0x^0 = \alpha_k((b \cdot c)y^0x^0) = \alpha_k(by^0x^0 \cdot cy^0x^0) = by^0x^0 \cdot cy^0x^0$
= f(b) * f(c), and moreover is f ◦ α_k|_{K'} = f ◦ id_{K'} = f = id_K ◦ f, so f is indeed an isomorphism of hom-associative algebras.

Just as in the associative case, the above proposition makes it possible to identify ay^0 x^0 with a for any a ∈ K, something we will do from now on.

**Lemma 2.** 1_{A_1} is a unique weak left and a unique weak right unit of A_1^k.

**Proof.** First note that α_k is an automorphism of A_1^k; it is an automorphism of A_1, and since the underlying set is the same for the two algebras, also a bijective map on A_1^k. Moreover it is both linear and multiplicative on A_1^k. Assume e_l ∈ A_1^k is a weak left unit. Then e_l * 1_{A_1} = α_k(1_{A_1}). Since 1_{A_1} is a weak right unit, e_l * 1_{A_1} = α_k(e_l), so α_k(e_l) = α_k(1_{A_1}). By the injectivity of α_k is then e_l = 1_{A_1}, and analogously for the right case.

If L is some linear operator on A_1^k such that for each p ∈ A_1^k, only finitely many elements L'p for i ∈ N are nonzero, then we can define e^L using the ordinary formal power series. The next proposition gives an example of that.

**Proposition 5** (Product and twisting map). α_k = e^{k \frac{d}{dy}}, so for all p, q ∈ A_1^k,

\[ p \ast q = e^{k \frac{d}{dy}}(p \cdot q). \]

**Proof.** Put \( p = \sum_{i \in \mathbb{N}} \sum_{j \in \mathbb{N}} p_{ij} y^i x^j \) for some \( p_{ij} \in K \). Then, by defining the exponential of the partial differential operator as its formal power series and putting 0y^i to be zero whenever \( i < 0 \),

\[
\alpha_k(p) = \sum_{i \in \mathbb{N}} \sum_{j \in \mathbb{N}} p_{ij} (y + k)^i x^j = \sum_{i \in \mathbb{N}} \sum_{j \in \mathbb{N}} \sum_{l=0}^{i} p_{ij} \binom{i}{l} k^l y^i x^j
\]

\[ = \sum_{i \in \mathbb{N}} \sum_{j \in \mathbb{N}} \sum_{l=0}^{i} p_{ij} \left( \left( k \frac{\partial}{\partial y} \right)^i \right) y^i x^j = \sum_{i \in \mathbb{N}} \sum_{j \in \mathbb{N}} \sum_{l=0}^{i} p_{ij} \left( \left( k \frac{\partial}{\partial y} \right)^i \right) y^i x^j
\]

\[ = \sum_{l \in \mathbb{N}} \left( \left( k \frac{\partial}{\partial y} \right)^l \right) \sum_{i \in \mathbb{N}} \sum_{j \in \mathbb{N}} p_{ij} y^i x^j = e^{k \frac{d}{dy}}p. \]

At last, \( p \ast q := \alpha_k(p \cdot q) \), and hence (4) holds for all \( p, q \in A_1^k \). □

**Remark 5.** The inverse of \( e^{k \frac{d}{dy}} \) is simply \( e^{-k \frac{d}{dy}} \), so by (4) is \( p \cdot q = e^{-k \frac{d}{dy}}(p \ast q) \).

**Corollary 1.** There are no zero divisors in A_1^k.

**Proof.** If \( p, q \in A_1^k \) are arbitrary, then \( p \ast q = 0 \iff (4) e^{k \frac{d}{dy}}(p \cdot q) = 0 \iff p \cdot q = e^{-k \frac{d}{dy}}0 = 0 \), using Remark 5. Since \( A_1 \) contains no zero divisors, \( p \cdot q = 0 \iff p = 0 \) or \( q = 0 \). □

**Corollary 2** (Commutation relations). For any polynomial \( p(x, y) \in A_1^k \),

\[
[x, p(x, y)]_a = e^{k \frac{d}{dy}}[x, p(x, y)] = \frac{\partial}{\partial y}p(x, y + k),
\]

\[
[p(x, y), y]_a = e^{k \frac{d}{dy}}[p(x, y), y] = \frac{\partial}{\partial x}p(x, y + k).
\]
Proof. In $A_1$, we have $[x, y^m x^n] = x \cdot y^m x^n - y^m x^n \cdot x = \sum_{i \in \mathbb{N}} (\frac{1}{i}) 2^{i-1} y^{m-1} x^{n+i} - y^m x^{n+1} = ny^m x^n$ for any $m, n \in \mathbb{N}$, defining $0y^{-1}$ to be zero. By linearity in the second argument, it follows that $[x, p(x, y)] = \frac{\partial}{\partial y} p(x, y)$. By using Proposition 5,
\[ [x, p(x, y)] = x \ast p(x, y) - p(x, y) \ast x = \alpha_k(x \cdot p(x, y)) - \alpha_k(p(x, y) \cdot x) \]
\[ = e^{k \frac{\partial}{\partial y}} (x \cdot p(x, y) - p(x, y) \cdot x) = e^{k \frac{\partial}{\partial y}} [x, p(x, y)] = e^{k \frac{\partial}{\partial y}} \frac{\partial}{\partial y} p(x, y) \]
\[ = \frac{\partial}{\partial y} e^{k \frac{\partial}{\partial y}} p(x, y) = \frac{\partial}{\partial y} p(y + k, x). \]
In $A_1$, we also have $[y^m x^n, y] = y^m x^n \cdot y - y \cdot y^m x^n = ny^m x^n$ for any $m, n \in \mathbb{N}$, defining $0x^{-1}$ to be zero. By linearity in the first argument, it follows that $[p(x, y), y] = \frac{\partial}{\partial x} p(x, y)$. Hence,
\[ [p(x, y), y] = e^{k \frac{\partial}{\partial y}} [p(x, y), y] = e^{k \frac{\partial}{\partial y}} \frac{\partial}{\partial x} p(x, y) = \frac{\partial}{\partial x} e^{k \frac{\partial}{\partial y}} p(x, y) = \frac{\partial}{\partial x} p(y + k, x). \]

Proposition 6 (The commutator). $C(A_k^1) = K$.

Proof. Let $a \in K$ and $q \in A_k^1$ be arbitrary. Then $[a, q] = \alpha_k ([a, q]) = \alpha_k (0) = 0$, so $K \subseteq C(A_k^1)$. For any $p \in C(A_k^1)$, $[x, p] = e^{k \frac{\partial}{\partial y}} [x, p] = 0$, which implies that $[x, p] = 0$. From Corollary 2, $[x, p] = \frac{\partial}{\partial x} p$, so $p \in K[x]$. Continuing, $[p, y] = e^{k \frac{\partial}{\partial y}} [p, y] = 0$, which implies that $[p, y] = 0$. Again, from Corollary 2, $[p, y] = \frac{\partial}{\partial x} p$, so $p \in K$. □

Corollary 3 (The center).

\[ Z(A_k^1) = \begin{cases} K & \text{if } k = 0, \\ \{0\} & \text{otherwise.} \end{cases} \]

Proof. Recall from Subsection 2.1 that $Z(A_k^1) = C(A_k^1) \cap N(A_k^1)$. When $k = 0$, $N(A_k^1) = A_k^1$, and hence $Z(A_k^1) = C(A_k^1) = K$. Assume instead that $k \neq 0$, and let $c \in K$ be arbitrary. Then a straightforward calculation yields $(c, y, y)_* = -2ck^2 - cky = 0 \iff c = 0$. On the other hand, $0 \in N(A_k^1)$, so $Z(A_k^1) = \{0\}$. □

Proposition 7 (Power associativity). $A_k^1$ is power associative if and only if $k = 0$.

Proof. If $k = 0$, then $A_k^1$ is associative and hence also power associative. On the other hand, one readily verifies that $(yx, yx, yx)_* = kx + 2k^2 x^2$, so if $A_k^1$ is power associative, then $k = 0$. □

Remark 6. Note that due to the proposition above, $A_k^1$ is not left alternative, right alternative, or flexible either, let alone associative.

In Subsection 2.1 we said that maps of the form $[a, \cdot] : A \to A$ for any $a$ in an associative algebra $A$ are derivations called inner derivations, and that such a map need not be a derivation if $A$ is not associative. For a concrete example of this latter fact, one can consider the map $[y^2, \cdot]_*$ in $A_k^1$, which is a derivation if and only if $k = 0$. The reason for this failure when $k = 0$ is due to the next lemma.

Lemma. $\delta$ is a derivation on $A_k^1$ if and only if $\delta$ is a derivation on $A_1$ that commutes with $e^{k \frac{\partial}{\partial y}}$. 
Proof. First note that linearity follows immediately in both directions as it is defined on the common underlying vector space of $A^k_1$ and $A_1$. Now, let $\delta$ be a derivation on $A^k_1$. We claim that $\delta(1_{A_1}) = 0$. First, $\delta(1_{A_1} \ast 1_{A_1}) = \delta(1_{A_1}) \ast 1_{A_1} + 1_{A_1} \ast \delta(1_{A_1}) = 2\alpha_k(\delta(1_{A_1})) = 2e^{k \frac{\partial}{\partial y}} \delta(1_{A_1})$, using that $\alpha_k = e^{k \frac{\partial}{\partial y}}$ from Proposition 5. On the other hand, $\delta(1_{A_1} \ast 1_{A_1}) = \delta(\alpha_k(1_{A_1})) = \delta(1_{A_1})$. The equality of the two expressions is then equivalent to the eigenvector problem $e^{k \frac{\partial}{\partial y}} p = \frac{1}{2} p$.

It turns out it has no solution, which may be seen from solving the equivalent PDE $p + 2 \left( k \frac{\partial}{\partial y} + \frac{k^2}{2l} \frac{\partial^2}{\partial y^2} + \cdots + \frac{k^m}{m! \frac{\partial^m}{\partial y^m}} \right) p = 0$. To see this, let us put $p := \delta(1_{A_1}) = \sum_{i=0}^m \sum_{j=0}^n p_{ij} y^i x^j$ for some $p_{ij} \in K$ and $m, n \in \mathbb{N}$. Then, by comparing coefficients, starting with $p_{00}$ for some arbitrary $j$ and working our way down to $0$, we have that $p_{ij} = 0$ for all $i, j \in \mathbb{N}$. Therefore $\delta(1_{A_1}) = 0$ as claimed, and for an arbitrary $q \in A^k_1$, $\delta \left( e^{k \frac{\partial}{\partial y}} q \right) = e^{k \frac{\partial}{\partial y}} \delta(q) = \delta(q \ast 1_{A_1}) = \delta(q) \ast 1_{A_1} + q \ast \delta(1_{A_1}) = \delta(q) \ast 1_{A_1} = \alpha_k(\delta(q)) = e^{k \frac{\partial}{\partial y}} \delta(q)$. Then $\alpha_k(\delta(r \ast s)) = e^{k \frac{\partial}{\partial y}} \delta(r \ast s) = \delta \left( e^{k \frac{\partial}{\partial y}} (r \ast s) \right)$ where $r, s \in A_1$ are arbitrary. By the injectivity of $\alpha_k$, $\delta(r \ast s) = \delta(r) \ast s + r \ast \delta(s)$. Assume now instead that $\delta$ is a derivation on $A_1$ that commutes with $e^{k \frac{\partial}{\partial y}}$, and that $r, s \in A^k_1$. Then, $\delta \left( e^{k \frac{\partial}{\partial y}} (r \ast s) \right) = e^{k \frac{\partial}{\partial y}} \delta(r \ast s) = \alpha_k(\delta(r) \ast s + r \ast \delta(s)) = \alpha_k(\delta(r) \ast s) + \alpha_k(r \ast \delta(s)) = \alpha_k(\delta(r) \ast s) + \alpha_k(r \ast \delta(s))$. □

Corollary 4 (Derivations). $\delta$ is a derivation on $A^k_1$ for $k$ nonzero if and only if $\delta = [cy + p(x), ·] = e^{-k \frac{\partial}{\partial y}} [cy + p(x), ·]$ for some $c \in K$ and $p(x) \in K[x]$.

Proof. Recall from Subsection 2.4 that all derivations on $A_1$ are inner, i.e. of the form $[q, ·]$ for some $q \in A_1$. From Lemma 3, there is a one-to-one correspondence between the derivations on $A^k_1$ and the derivations on $A_1$ that commute with $e^{k \frac{\partial}{\partial y}}$. Hence we are looking for $q \in A_1$ such that $e^{k \frac{\partial}{\partial y}} [q, x] = [q, e^{k \frac{\partial}{\partial y}} x] = [q, x]$ and $e^{k \frac{\partial}{\partial y}} [q, y] = [q, e^{k \frac{\partial}{\partial y}} y] = [q, y + k] = [q, y]$. We thus have two eigenvector problems of the form $e^{k \frac{\partial}{\partial y}} s = s$ with $s \in \{q, x, [q, y]\}$. This is equivalent to the PDE $\left( k \frac{\partial}{\partial y} + \frac{k^2}{2l} \frac{\partial^2}{\partial y^2} + \cdots + \frac{k^m}{m! \frac{\partial^m}{\partial y^m}} \right) s = 0$, and by putting $s = \sum_{i \in \mathbb{N}} \sum_{j \in \mathbb{N}} s_{ij} y^i x^j$ for some $s_{ij} \in K$, we see by comparing coefficients that $s = \sum_{j \in \mathbb{N}} s_{0j} x^j$. Now, using that $s \in K[x]$, $[q, x] = \frac{\partial}{\partial x} q$ and $[q, y] = \frac{\partial}{\partial y} q$ from Corollary 2, we get $\frac{\partial}{\partial x} q \in K[x]$ and $\frac{\partial}{\partial y} q \in K[x]$. If we put $q = \sum_{i \in \mathbb{N}} \sum_{j \in \mathbb{N}} q_{ij} y^i x^j$, then the former implies that $q = \sum_{j \in \mathbb{N}} (q_{0j} + q_{1j} y) x^j$, which upon plugging into the second yields $q = q_{10} y + \sum_{j \in \mathbb{N}} q_{0j} x^j$. We claim that it is also sufficient that $q$ is on this form in order that for any $u \in A_1$, $e^{k \frac{\partial}{\partial y}} [q, u] = [q, e^{k \frac{\partial}{\partial y}} u]$. First, $e^{k \frac{\partial}{\partial y}} q = k q_{10} + q$. Recalling that $e^{k \frac{\partial}{\partial y}}$ is an endomorphism on $A_1$, $e^{k \frac{\partial}{\partial y}} [q, u] = e^{k \frac{\partial}{\partial y}} q, e^{k \frac{\partial}{\partial y}} u] = [kq_{10} + q, e^{k \frac{\partial}{\partial y}} u] = [q, e^{k \frac{\partial}{\partial y}} u]$. If $q_{10} := c$ and $p(x) := \sum_{j \in \mathbb{N}} q_{0j} x^j$, then $q = cy + p(x)$. By Remark 5, $[cy + p(x), ·] = e^{-k \frac{\partial}{\partial y}} [cy + p(x), ·]$. □

Remark 7. For any endomorphism $f$ on $A_1$, $f(1_{A_1}) = 1_{A_1}$. This follows from the fact that $A_1$ is a domain, and hence $f(1_{A_1}) = f(1_{A_1}) \cdot f(1_{A_1}) \iff f(1_{A_1}) \cdot (1_{A_1} -$
Proposition 8

\[ e(f(1_{A_1})) = 0 \implies 1_{A_1} - f(1_{A_1}) = 0 \text{ since } f(1_{A_1}) \neq 0 \text{ due to injectivity and the fact that } f(0) = 0 \text{ by linearity.} \]

**Lemma 4.** \( f: A_k^1 \to A_k^1 \) is a homomorphism if and only if \( f \) is an endomorphism on \( A_1 \) such that \( e^{-\frac{d}{dt}} f(x) = f(x) \) and \( e^{\frac{d}{dt}} f(y) = f(y) + k. \)

**Proof.** Let \( f: A_k^1 \to A_k^1 \) be a homomorphism, i.e. a \( K \)-linear map such that \( f \circ \alpha_k = \alpha_k \circ f \) and \( f(a \star_k b) = f(a) \star_i f(b) \) for all \( a, b \in A_k^1 \). Since we may view the underlying vector space of \( A_k^1, A_k^1, \) and \( A_1 \) as the same, we only need to show that \( e^{\frac{d}{dt}} f(x) = f(x), e^{-\frac{d}{dt}} f(y) = f(y) + k, \) and \( f(a \cdot b) = f(a) \cdot f(b) \). The former follows from \( f \circ \alpha_k = \alpha_k \circ f \) with \( \alpha_k = e^{\frac{d}{dt}} \) from Proposition 5, together with Remark 7. The latter follows from the fact that \( f(a \star_k b) = f(\alpha_k(a \cdot b)) = \alpha_l(f(a \cdot b)), \) whereas \( f(a) \star_i f(b) = \alpha_i(f(a) \cdot f(b)), \) and since \( \alpha_i \) is injective, \( f(a \cdot b) = f(a) \cdot f(b) \), and therefore an isomorphism from \( A_k^1 \) to \( A_k^1 \) for all \( a, b \in A_k^1, \) we have \( f(a \star_k b) = f(\alpha_k(a \cdot b)) = \alpha_l(f(a \cdot b)) = \alpha_i(f(a) \cdot f(b)) = f(a) \star_i f(b). \)

Assume instead that \( f \) is an endomorphism on \( A_1 \) such that \( e^{\frac{d}{dt}} f(x) = f(x) \) and \( e^{-\frac{d}{dt}} f(y) = f(y) + k. \) Then, with \( \alpha_1 = e^{\frac{d}{dt}} \), for any \( y^m x^n \in A_1, \)

\[ \alpha_l(f(y^m x^n)) = \alpha_l(f(y^m(x))^n) = (\alpha_l(f(y)))^n = (\alpha_k(y))^n = f(\alpha_k(x))^n, \]

so \( \alpha_l \circ f = f \circ \alpha_k \). Moreover, for all \( a, b \in A_k^1, \) we have \( f(a \star_k b) = f(\alpha_k(a \cdot b)) = \alpha_l(f(a \cdot b)) = \alpha_i(f(a) \cdot f(b)) = f(a) \star_i f(b). \)

**Proposition 8 (Morphisms).** Any homomorphism \( f: A_k^1 \to A_k^1 \) for \( k, l \neq 0 \) is an isomorphism of the form \( f(x) = \frac{1}{k} x + c, f(y) = \frac{1}{l} y + p(x) \) for some \( c \in K \) and \( p(x) \in K[x]. \)

**Proof.** Let us try to find a homomorphism \( f: A_k^1 \to A_k^1 \) when \( k \) and \( l \) are nonzero. By Lemma 4 is this equivalent to finding an endomorphism \( f \) on \( A_1 \) such that \( e^{\frac{d}{dt}} f(x) = f(x) \) and \( e^{-\frac{d}{dt}} f(y) = f(y) + k. \) The former of the two conditions was considered in the proof of Corollary 4, and it turned out to be equivalent to \( f(x) \in K[x]. \)

If we put \( f(y) = \sum_{i=0}^n a_{ij} x^j \) for some \( a_{ij} \in K \) and \( i, j \in \mathbb{N}, \) then, by comparing coefficients, \( f(y) = \frac{k}{l} y + p(x) \) where \( p(x) := \sum_{j=0}^n a_{ij} x^j. \) Now, note that \( f \) is an endomorphism on \( A_1 \) only if \( f(x), f(y) = f([x,y]) = f(1_{A_1}) = 1_{A_1}. \)

Calculating the left-hand side, \( [f(x), \frac{k}{l} y + p(x)] = \frac{k}{l} [f(x), y] = \frac{k}{l} f(x) \)

which is equal to \( 1_{A_1} \), if and only if \( f(x) = \frac{k}{l} x + c \) for some \( c \in K. \) Let us introduce the following functions:

\[ g_1(x) := \frac{k}{l} x + y, \quad g_2(x) := x, \quad g_3(x) := x - \frac{k}{l} y, \quad g_4(x) := x, \]

\[ g_1(y) := \frac{k}{l} y, \quad g_2(y) := y + c, \quad g_3(y) := y, \quad g_4(y) := y - c + \frac{l}{k} p(x). \]

According to Theorem 1, these are all automorphisms on \( A_1, \) and moreover, \( f = g_4 \circ g_3 \circ g_2 \circ g_1 \) since \( g_4 \circ g_3 \circ g_2 \circ g_1(x) = \frac{k}{l} x + c = f(x) \) and \( g_4 \circ g_3 \circ g_2 \circ g_1(y) = \frac{k}{l} y + p(x) = f(y). \) Hence, \( f \) is an automorphism on \( A_1 \) such that \( e^{\frac{d}{dt}} f(x) = f(x) \)

and \( e^{-\frac{d}{dt}} f(y) = f(y) + k, \) and therefore an isomorphism from \( A_k^1 \) to \( A_k^1. \)

**Corollary 5 (Hom-Dixmier).** Any endomorphism \( f \) on \( A_k^1 \) for \( k \neq 0 \) is an automorphism of the form \( f(x) = x + c \) and \( f(y) = y + p(x) \) for some \( c \in K \) and \( p(x) \in K[x]. \)
Proof. This follows from Proposition 8 with \( k = l \). \( \square \)

4. One-parameter formal deformations

One-parameter formal hom-associative deformations and one-parameter formal hom-Lie deformations were first introduced by Makhlouf and Silvestrov in [16] together with an attempt at describing a compatible cohomology theory in lower degrees. In the multiplicative case, this was later expanded on by Ammar, Ejbehi and Makhlouf in [1], and then by Hurle and Makhlouf [12]. Only in this latter paper, treating the multiplicative, hom-associative case, did the cohomology theory include the twisting map \( \alpha \) in a natural way. This is indeed essential, as the idea behind these kinds of deformations is to deform not only the multiplication map or the Lie bracket, but also the twisting map \( \alpha \), resulting also in a deformation of the twisted associativity condition and the twisted Jacobi identity, respectively. In the special case when the deformations start from \( \alpha \) being the identity map and the multiplication being associative or the bracket being the Lie bracket, one gets a deformation of an associative algebra into a hom-associative algebra, and in the latter case a deformation of a Lie algebra into a hom-Lie algebra. Perhaps the main motivation for studying these kinds of deformations is that they provide a framework in which some algebras can now be deformed, which otherwise could not when considered as objects of the category of associative algebras or that of Lie algebras. The first Weyl algebra constitute such an example; in the classical sense, it is rigid (see e.g. [20, 8] for a proof of this fact). In this section, we show that the hom-associative Weyl algebras are one-parameter formal hom-associative deformations of the first Weyl algebra, and that they induce formal deformations of the corresponding Lie algebras into hom-Lie algebras. Here, we use a slightly more general approach than that given in [16], replacing vector spaces by modules; this follows our convention in the preliminaries and previous work (cf. [2, 3, 4]), with the advantage of e.g. being able to treat rings as algebras. First, if \( R \) is an associative, commutative, and unital ring, and \( M \) an \( R \)-module, we denote by \( R[[t]] \) the formal power series ring in the indeterminate \( t \), and by \( M[[t]] \) the \( R[[t]] \)-module of formal power series in the same indeterminate, but with coefficients in \( M \). By Definition 1, this allows us to define a hom-associative algebra \((M[[t]], \cdot_t, \alpha_t)\) over \( R[[t]] \).

**Definition 5** (One-parameter formal hom-associative deformation). A one-parameter formal hom-associative deformation of a hom-associative algebra \((M, \cdot_0, \alpha_0)\) over \( R \), is a hom-associative algebra \((M[[t]], \cdot_t, \alpha_t)\) over \( R[[t]] \), where

\[
\cdot_t = \sum_{i \in \mathbb{N}} \cdot_i t^i, \quad \alpha_t = \sum_{i \in \mathbb{N}} \alpha_i t^i,
\]

and for each \( i \in \mathbb{N} \), \( \cdot_i : M \times M \to M \) is a binary operation linear over \( R \) in both arguments, and \( \alpha_i : M \to M \) an \( R \)-linear map, extended homogeneously to a binary operation linear over \( R[[t]] \) in both arguments, \( \cdot_i : M[[t]] \times M[[t]] \to M[[t]] \), and an \( R[[t]] \)-linear map \( \alpha_i : M[[t]] \to M[[t]] \), respectively.

Here, and onwards, a homogeneous extension is defined analogously to that of an Ore extension in Subsection 2.3, so that for any \( r, s \in R \), \( a, b \in M \), and \( i, j, l \in \mathbb{N} \), we have \( \alpha_i(rat^j + sbt^l) = r\alpha_i(a)t^j + s\alpha_i(b)t^l \), and similarly for the product \( \cdot_i \).

**Proposition 9.** \( A_1^k \) is a one-parameter formal hom-associative deformation of \( A_1 \).
Proof. We put $t := k$, and regard $t$ as an indeterminate of the formal power series $K[[t]]$ and $A_1[[t]]$; this gives a deformation $(A_1[[t]], \cdot, \cdot_0)$ of $(A_1, \cdot_0, \text{id}_{A_1})$, where the latter simply is $A_1$ in the language of hom-associative algebras, $\cdot_0$ denoting the multiplication in $A_1$. Explicitly, with $\alpha_t = e^{\frac{t}{t!}}$ from Proposition 5, it is clear that $\alpha_t$ is a formal power series in $t$ by definition, and moreover is $\alpha_0 = \text{id}_{A_1}$. Next, we extend $\alpha_t$ linearly over $K[[t]]$ and homogeneously to all of $A_1[[t]]$. To define the multiplication $\cdot_0$ in $A_1[[t]]$, we first extend $\cdot_0 : A_1 \times A_1 \to A_1$ homogeneously to a binary operation $\cdot : A_1[[t]] \times A_1[[t]] \to A_1[[t]]$ linear over $K[[t]]$ in both arguments, and then simply compose $\alpha_t$ with $\cdot_0$, so that $\cdot := \alpha_t \circ \cdot_0 = e^{\frac{t}{t!}} \circ \cdot_0$. This is again a formal power series in $t$ by definition, and hom-associativity now follows from Proposition 3 as explained in Subsection 2.5.

\[\square\]

From now on, we refer to one-parameter formal hom-associative deformations as just deformations.

**Definition 6** (One-parameter formal hom-Lie deformation). A one-parameter formal hom-Lie deformation of a hom-Lie algebra $(M, [\cdot, \cdot]_0, \alpha_0)$ over $R$ is a hom-Lie algebra $(M[[t]], [\cdot, \cdot]_t, \alpha_t)$ over $R[[t]]$, where

\[\cdot_t = \sum_{i \in \mathbb{N}} [\cdot, \cdot]_i t^i, \quad \alpha_t = \sum_{i \in \mathbb{N}} \alpha_i t^i,\]

and for each $i \in \mathbb{N}$, $[\cdot, \cdot]_i : M \times M \to M$ is a binary operation linear over $R$ in both arguments, and $\alpha_i : M \to M$ an $R$-linear map, extended homogeneously to a binary operation linear over $R[[t]]$ in both arguments, $[\cdot, \cdot]_i : M[[t]] \times M[[t]] \to M[[t]]$, and an $R[[t]]$-linear map $\alpha_i : M[[t]] \to M[[t]]$, respectively.

**Remark 8.** Alternativity of $[\cdot, \cdot]_t$ is equivalent to alternativity of $[\cdot, \cdot]_i$ for all $i \in \mathbb{N}$.

**Proposition 10.** The deformation of $A_1$ into $A_1^k$ induces a one-parameter formal hom-Lie deformation of the Lie algebra of $A_1$ into the hom-Lie algebra of $A_1^k$, using the commutator as bracket.

**Proof.** Using the deformation of $A_1$ into $A_1^k$ in Proposition 9, we put $t := k$; this gives a deformation $(A_1[[t]], [\cdot, \cdot]_t, \alpha_t)$ of $(A_1, [\cdot, \cdot]_0, \text{id}_{A_1})$, where the latter is the Lie algebra of $A_1$ obtained from the commutator construction with $[\cdot, \cdot]_0$ as the commutator. To see this, we first note that by construction, $\alpha_t$ is the same map as defined in the proof of Proposition 9. Hence, we only need to show that $[\cdot, \cdot]_t$ is a deformation of the commutator $[\cdot, \cdot]_0$, and that the hom-Jacobi identity is satisfied. We first extend $[\cdot, \cdot]_0 : A_1 \times A_1 \to A_1$ homogeneously to a binary operation $[\cdot, \cdot]_0 : A_1[[t]] \times A_1[[t]] \to A_1[[t]]$ linear over $K[[t]]$ in both arguments. Next, we define $[\cdot, \cdot]_t : A_1[[t]] \times A_1[[t]] \to A_1[[t]]$ as $\alpha_t \circ [\cdot, \cdot]_0 = e^{\frac{t}{t!}} [\cdot, \cdot]_0$. The hom-Jacobi identity is now satisfied by Proposition 2 and the construction given in Subsection 2.5.

\[\square\]

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