The Coxeter element and the branching law for the finite subgroups of $SU(2)$

BERTRAM KOSTANT*

0. Introduction

0.1. Let $\Gamma$ be a finite subgroup of $SU(2)$. The question we will deal with in this paper is how an arbitrary (unitary) irreducible representation of $SU(2)$ decomposes under the action of $\Gamma$. The theory of McKay assigns to $\Gamma$ a complex simple Lie algebra $g$ of type $A-D-E$. The assignment is such that if $\tilde{\Gamma}$ is the unitary dual of $\Gamma$ we may parameterize $\tilde{\Gamma}$ by the nodes (or vertices) of the extended Coxeter-Dynkin diagram of $g$.

Let $\ell = rank g$ and let $I = \{1, \ldots, \ell\}$. Let $I_{ext} = I \cup \{0\}$. The nodes may be identified with a set of simple roots of the affine Kac-Moody Lie algebra associated to $g$ and are indexed by $I_{ext}$. We can then write $\Gamma = \{\gamma_i\}, i \in I_{ext}$. Let $\Pi = \{\alpha_i\}, i \in I$, be the set of simple roots of $g$ itself. One has $\gamma_0$ is the trivial 1 dimensional representation of $\Gamma$ and, for $i \in I$,

$$\text{dim } \gamma_i = d_i$$

(0.1)

where

$$\psi = \sum_{i \in I} d_i \alpha_i$$

(0.2)

is the highest root. For proofs and details about the McKay correspondence see e.g. [G-S,V], [M] and [St].

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0.2. The unitary dual of $SU(2)$ is indexed by the set $\mathbb{Z}_+$ of nonnegative integers and will be written as $\{\pi_n\}, n \in \mathbb{Z}_+$ where

$$\dim \pi_n = n + 1$$ (0.3)

Our problem is the determination of $m_{n,i}$ where $n \in \mathbb{Z}_+, i \in I_{\text{ext}}$ and

$$m_{n,i} = \text{multiplicity of } \gamma_i \text{ in } \pi_n|\Gamma$$

It is resolved with the determination of the formal power series

$$m(t)_i = \sum_{n=0}^{\infty} m_{n,i} t^n$$ (0.4)

To do this one readily notes that it suffices to consider only the case where $\Gamma = F^*$ and $F^*$ is the pullback to $SU(2)$ of a finite subgroup $F$ of $SO(3)$. This eliminates only the case where $\Gamma$ is a cyclic group of odd order and $\mathfrak{g}$ is of type $A_\ell$ where $\ell$ is even. For the remaining cases the Coxeter number $h$ of $\mathfrak{g}$ is even and we will put

$$g = h/2$$ (0.5)

Also for the remaining cases there is a special index $i_* \in I$. If $\mathfrak{g}$ is of type $D$ or $E$ then $\alpha_{i_*}$ is the branch point of the Coxeter-Dynkin diagram of $\mathfrak{g}$. If $\mathfrak{g}$ is of type $A_\ell$ then $\alpha_{i_*}$ is the midpoint of the diagram (recalling that $\ell$ is odd).

If $i = 0$ the determination of $m(t)_0$ is classical and is known from the theory of Kleinian singularities. In fact there exists positive integers $a < b$ such that

$$m(t)_0 = \frac{1 + t^h}{(1 - t^a)(1 - t^b)}$$ (0.6)

The numbers $a$ and $b$ in Lie theoretic terms is given in

**Theorem 0.1.** One has $a = 2d_{i_*}$ and $b$ is given by the condition that

$$ab = 2|F^*|$$

$$= 4|F|$$
It remains then to determine $m(t)_i$ for $i \in I$.

**Proposition 0.2.** If $i \in I$ there exists a polynomial $z(t)_i$ of degree less than $h$ such that

$$m(t)_i = \frac{z(t)_i}{(1 - t^a)(1 - t^b)} \quad (0.7)$$

The problem then is to determine the polynomial $z(t)_i$. This problem was solved in [K] using the orbits of a Coxeter element $\sigma$ on a set of roots $\Delta$ for $\mathfrak{g}$. In the present paper we will put the main result of [K] in a simplified form. See Theorem 1.13 in the present paper. Also the present paper makes explicit some results that are only implicit in [K]. For example introducing $\tilde{\Pi}$ (see (1.10)) and making the assertions in Remark 10 and Theorems 8, 9, 11 and 12.

The set $\Pi$ generates a system, $\Delta_+$, of positive roots. The highest root $\psi \in \Delta$ defines a certain subset $\Phi \subset \Delta_+$ of cardinality $2h - 3$. Because of its connection with a Heisenberg subalgebra of $\mathfrak{g}$ this subset is referred to as the Heisenberg subsystem of $\Delta_+$. The new formulation explicitly shows how the polynomials $z(t)_i$ arise from the intersection

$$(\text{orbits of the Coxeter element } \sigma) \cap (\text{the Heisenberg subsystem } \Phi) \quad (0.8)$$

The polynomials $z(t)_i$ are listed in [K]. The special case where $\mathfrak{g}$ is of type $E_8$ is also given in the present paper (see Example 1.7.). Unrelated to the Coxeter element the polynomials $z(t)_i$ are also determined in Springer, [Sp]. They also appear in another context in Lusztig, [L1] and [L2]. Recently, in a beautiful result, Rossmann, [R], relates the character of $\gamma_i$ to the polynomial $z(t)_i$.

1. **The main result - Theorem 1.13.**

1.1. Proofs of the main results stated here are in [K].
Let $F$ be a finite subgroup of $SO(3)$ and let
\begin{equation}
F^* \subset SU(2)
\end{equation}
be the pullback of the double covering
\[ SU(2) \rightarrow SO(3) \]
The unitary dual $\widehat{SU(2)}$ of $SU(2)$ is represented by the set $\{\pi_n\}, n \in \mathbb{Z}_+$, where if $S(\mathbb{C}^2)$ is the symmetric algebra then
\[ \pi_n : SU(2) \rightarrow S^n(\mathbb{C}^2) \]
is the $n+1$ dimensional representation defined by the natural action of $SU(2)$ on $\mathbb{C}^2$.

We are ultimately interested in determining how the restriction $\pi_n|F^*$ decomposes into irreducible representations of the finite subgroup $F^*$, for any $n$, and relating this determination to the structure of the simple Lie algebra corresponding to $F^*$ by the McKay correspondence. We now recall this correspondence.

Let $\mathfrak{g}$ be a complex simple Lie algebra and let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$. Let $\ell = rank \mathfrak{g}$, and if $\mathfrak{h}'$ is the dual space to $\mathfrak{h}$, let $\Delta \subset \mathfrak{h}'$ be the set of roots for the pair $(\mathfrak{g}, \mathfrak{h})$. Let $W$, operating in $\mathfrak{h}'$, be the Weyl group. Let $\Pi$ be the set of simple positive roots with respect to a choice, $\Delta_+$, of positive roots. If $I = \{1, \ldots, \ell\}$ we will write $\Pi = \{\alpha_i\}, i \in I$. We may regard $\Pi$ as the vertices (or nodes) of the Coxeter-Dynkin diagram associated to $\mathfrak{g}$. The extended Coxeter-Dynkin diagram has an additional node $\alpha_0$.

The McKay correspondence assigns to $F^*$ a complex simple Lie algebra $\mathfrak{g} = \mu(F^*)$ of type $A - D - E$. The assignment has a number of properties: (1), the unitary dual $\widehat{F^*}$ may be parameterized by indices of the nodes of the extended Coxeter-Dynkin diagram of $\mathfrak{g}$. In particular $card \, \widehat{F^*} = \ell + 1$ and we can write $\widehat{F^*} = \{\gamma_i\}, i \in I_{ext} = I \cup \{0\}$. Next (2), $\gamma_0$ is the trivial 1-dimensional representation, and if $i \in I$, then
\[ dim \gamma_i = d_i \]
where
\[ \psi = \sum_{i=1}^{\ell} d_i \alpha_i \]
is the highest root in \( \Delta \). In addition (3), if \( \gamma \) is the two-dimensional representation defined by (1.1) and \( A \) is the \((\ell + 1) \times (\ell + 1)\) matrix defined so that
\[ \gamma_i \otimes \gamma = \sum_{j=0}^{\ell} A_{ij} \gamma_j \] (1.2)
then \( C \) is the Cartan matrix of the extended Coxeter-Dynkin of \( g \) where
\[ C_{ij} = 2\delta_{ij} - A_{ij} \]

1.2. Returning to our main problem, for \( i \in I_{ext} \) and \( n \in \mathbb{Z}^+ \), let
\[ m_{n,i} = \text{multiplicity of } \gamma_i \text{ in } \pi_n|F^* \]
and introduce the generating formal power series
\[ m(t)_i = \sum_{n=0}^{\infty} m_{n,i} t^n \]
If \( i = 0 \), the determination of \( m(t)_i \) is classical and is known from the theory of Kleinian singularities. That is, in this case
\[ m_{n,0} = \text{dim} (S^n(\mathbb{C}))|F^* \]
In fact let \( h \) be the Coxeter number of \( g \) so that
\[ \ell(h + 1) = \text{dim} \ g \]
Then there exists positive integers \( a < b \) such that
\[ m(t)_0 = \frac{1 + t^h}{(1 - t^a)(1 - t^b)} \] (1.3)
To define the numbers $a$ and $b$ in Lie theoretic terms one notes that $\mu(F^*)$ is of type $D$, $E$ or $A_\ell$ where $\ell$ is odd. In any of these case there is a special index $i_* \in I$. If $\mu(F^*)$ is of type $D$ or $E$, then $\alpha_{i_*}$ is the branch point of the Coxeter-Dynkin diagram of $g$. If $\mu(F^*)$ is of type $A_\ell$, then $\alpha_{i_*}$ is the midpoint of the diagram (recall that $\ell$ is odd in this case).

**Theorem 1.1.** One has $a = 2d_{i_*}$ and $b$ is given by the condition that

$$ab = 2|F^*|$$

$$= 4|F|$$

(1.4)

See Lemma 5.14 in [K]. The cases under consideration are characterized by the condition that $h$ is even. We put $g = h/2$. The parity of $g$ will play a later role.

**Remark 1.2.** One proves (see Lemma 5.7 in [K]) that $b$ may also be given by

$$b = h + 2 - a$$

(1.5)

so that $b$, as well as $a$, is even.

The following table lists the various cases under consideration. In the table $\Delta_n$ is the dihedral group of order $2n$.

| $F$  | $g$   | $a$  | $b$  | $h$  | $g$  |
|------|-------|------|------|------|------|
| $\mathbb{Z}_n$ | $A_{2n-1}$ | 2    | 2$n$ | 2$n$ | $n$  |
| $\Delta_n$ | $D_{n+2}$ | 4    | 2$n$ | 2$n+2$ | $n+1$ |
| $Alt_4$  | $E_6$  | 6    | 8    | 12   | 6    |
| $Sym_4$  | $E_7$  | 8    | 12   | 18   | 9    |
| $Alt_5$  | $E_8$  | 12   | 20   | 30   | 15   |

**Proposition 1.3.** There exists a unique partition

$$\Pi = \Pi_1 \cup \Pi_2$$

(1.6)
such that if $k = 1, 2$ and $\alpha_i, \alpha_j, \in \Pi_k$ where $i \neq j$ then $\alpha_i$ is orthogonal to $\alpha_j$. Furthermore all the roots in $\Pi_2$ are orthogonal to the highest root $\psi$, or equivalently the root $\alpha_0$ is orthogonal to all the roots in $\Pi_2$.

One has the disjoint union $I = I_1 \cup I_2$ where, if $k \in \{1, 2\}$, $\Pi_k = \{\alpha_i \mid i \in I_k\}$.

**Remark 1.4.** It is immediate from (1.2) that if $A_{ij} \neq 0$ and $i \in I_k$ then $j$ is in the complement of $I_k$ in $I$. It then follows that $\gamma_i$ descends to a representation of $F$ (i.e., $\gamma_i(-1) = 1$) if and only if $k = 2$. In particular

$$m_{n,i} = 0 \text{ if } n \text{ and } k \text{ have opposite parities where } \alpha_i \in \Pi_k. \quad (1.7)$$

If $i \in I$ let $s_i \in W$ be the reflection defined by $\alpha_i$ so that $s_i$ commutes with $s_j$ if $i, j \in I_k$, $k \in \{1, 2\}$. Put $\tau_k = \prod_{i \in I_k} s_i$. Then

$$\tau_1^2 = \tau_2^2 = \text{identity}$$

One defines a Coxeter element $\sigma \in W$ by putting

$$\sigma = \tau_2 \tau_1 \quad (1.8)$$

**Remark 1.5.** Every element in $W$ is contained in a dihedral subgroup of $W$. Since, as one knows, the centralizer of a Coxeter element is the cyclic group (necessarily of order $h$) generated by the Coxeter element, a dihedral group containing the Coxeter element is unique. It is clear that $\tau_1$ and $\tau_2$ are in the dihedral group containing $\sigma$ and, in fact, are in the complementary coset of the cyclic group generated by $\sigma$.

As a extension of (1.3) one knows (see (5.7.2) in [K]) that for any $i \in I$ there exists a polynomial $z(t)_i$ of degree less than $h$ such that

$$m(t)_i = \frac{z(t)_i}{(1 - t^a)(1 - t^b)} \quad (1.9)$$
so that \( m(t)_i \) is known as soon as one knows the polynomial \( z(t)_i \).

**Remark 1.6.** Note that by (1.6) and evenness of \( a \) and \( b \) (Remark 1.2) one must have that the only powers of \( t \) which have a nonzero coefficient are odd if \( i \in I_1 \) and even if \( i \in I_2 \).

**Example 1.7.** Consider the case where \( F \) is the icosahedral group so that \( \mu(F^*) = E_8 \). In the listing of \( z(t)_i \) below we will replace the arbitrary index \( i \) by the more informative \( \{ d_i \} \). Since there exists in certain cases two distinct \( i,j \in I \) such that \( \dim \gamma_i = \dim \gamma_j \) we will write \( \{ d_j \} \) for \( j \) when the “distance” of \( \alpha_j \) to \( \alpha_0 \) is greater than the “distance” of \( \alpha_i \) to \( \alpha_0 \). Note that \( d_i = 6 \).

\[
\begin{align*}
    z(t)_{(2)} &= t + t^{11} + t^{19} + t^{29} \\
    z(t)_{(3)} &= t^2 + t^{10} + t^{12} + t^{18} + t^{20} + t^{28} \\
    z(t)_{(4)} &= t^3 + t^9 + t^{11} + t^{13} + t^{17} + t^{19} + t^{21} + t^{27} \\
    z(t)_{(5)} &= t^4 + t^8 + t^{10} + t^{12} + t^{14} + t^{16} + t^{18} + t^{20} + t^{22} + t^{26} \\
    z(t)_{(6)} &= t^5 + t^7 + t^9 + t^{11} + t^{13} + 2t^{15} + t^{17} + t^{19} + t^{21} + t^{23} + t^{25} \\
    z(t)_{(4)} &= t^6 + t^8 + t^{12} + t^{14} + t^{16} + t^{18} + t^{22} + t^{24} \\
    z(t)_{(2)} &= t^7 + t^{13} + t^{17} + t^{23} \\
    z(t)_{(3)} &= t^6 + t^{10} + t^{14} + t^{16} + t^{20} + t^{24}
\end{align*}
\]

We now modify II by defining

\[
    \tilde{\Pi} = \{ \beta_i \mid i \in I \} \quad \text{(1.10)}
\]

where \( \beta_i = \alpha_i \) if \( i \in I_1 \) and \( \beta_i = -\alpha_i \) if \( i \in I_2 \). Let \( Z \subset W \) be the cyclic group generated by the Coxeter element \( \sigma \). Recall \((h + 1)\ell \) so that

\[
    \text{card } \Delta = h \ell \quad \text{(1.11)}
\]
We have shown that \( \sigma \) has \( \ell \) orbits in \( \Delta \), each with \( h \)-elements, and that each orbit contains a unique element of \( \tilde{\Pi} \). That is, one has

**Theorem 1.8.** For any \( i \in I \) the \( \sigma \)-orbit \( Z \cdot \beta_i \) has \( h \) elements and one has the disjoint union

\[
\Delta = \bigsqcup_{i=1}^{\ell} Z \cdot \beta_i
\]  

(1.12)

This result is readily proved using (6.9.2) in [K].
For any \( i \in I \) let \((Z \cdot \beta_i)_+ = \Delta_+ \cap Z \cdot \beta_i\). One has (see (0.5))

\[
\Delta_+ = g \ell
\]  

(1.13)

**Theorem 1.9.** For any \( i \in I \) one has \( \text{card} (Z \cdot \beta_i)_+ = g \) and the disjoint union

\[
\Delta_+ = \bigsqcup_{i \in I} (Z \cdot \beta_i)_+
\]  

(1.14)

It follows from (5.6.2) in [K] that (see (0.5))

\[
\alpha_{i_*} \in \Pi_2 \text{ if } g \text{ is even and } \alpha_{i_*} \in \Pi_1 \text{ if } g \text{ is odd.}
\]  

(1.15)

Let \( \kappa \) be the long element of the Weyl group. One has (see Lemma 4.9 in [K]) the following result of Steinberg:

\[
\sigma^g = \kappa
\]  

(1.16)

so that \( \kappa \in Z \).

**Remark 1.10.** Recall that \( \psi \) is the highest root. It is a consequence of (5.6.2) in [K] that one has \( \psi \) and \( \beta_{i_*} \) are in the same \( \sigma \) orbit. In fact if \( g \) is odd then

\[
\sigma^{\frac{g-1}{2}} (\psi) = \beta_{i_*}
\]

\[
= \alpha_{i_*}
\]  

(1.17)
and if \( g \) is even then

\[
\sigma^\frac{g}{2}(\psi) = \beta_{i_*} \\
= -\alpha_{i_*}
\]  \hspace{1cm} (1.18)

One easily has that \( \sigma^g \) commutes with \( \tau_1 \) and \( \tau_2 \) so that, for \( k \in \{1, 2\} \),

\[
\sigma^g(\Pi_k) = -(\Pi_k) \hspace{1cm} (1.19)
\]

Furthermore since \( \kappa(\psi) = -\psi \) one has that

\[
\sigma^g(\alpha_{i_*}) = -\alpha_{i_*} \hspace{1cm} (1.20)
\]

so that in any case

\[
\psi \text{ and } \alpha_{i_*} \text{ lie in the same } \sigma\text{-orbit} \hspace{1cm} (1.21)
\]

1.3. We come now to the main result—the determination of \( z(t)_i \) in terms of the orbit structure of \( \sigma \) on \( \Delta \). For any \( \varphi \in \Delta_+ \) let \( i_\varphi \in I \) be defined so that (by Theorem 1.9)

\[
\varphi \in (Z \cdot \beta_{i_\varphi})_+ \hspace{1cm} (1.22)
\]

But then there exists \( k_\varphi \in \{1, 2\} \) such that

\[
i_\varphi \in I_{k_\varphi} \hspace{1cm} (1.23)
\]

The following result follows from (6.9.2) in [K].

**Theorem 1.11.** Let \( \varphi \in \Delta_+ \). Then there exists a unique positive integer \( n(\varphi) \) where \( 1 \leq n(\varphi) \leq h \) with the same parity as \( k_\varphi \) such that if \( k_\varphi = 1 \) then

\[
\sigma^{\frac{n(\varphi)-1}{2}}(\varphi) = \beta_{i_\varphi} \hspace{1cm} (1.24)
\]

If \( k_\varphi = 2 \) then

\[
\sigma^{\frac{n(\varphi)}{2}}(\varphi) = \beta_{i_\varphi} \hspace{1cm} (1.25)
\]
One also has (see Remark 6.10 in [K])

**Theorem 1.12.** For any $i \in I_1$ the map

$$(Z \cdot \beta_i)_+ \to \{0, 1, \ldots, g - 1\}, \quad \varphi \mapsto \frac{n(\varphi) - 1}{2} \quad (1.26)$$

is a bijection and for any $i \in I_2$ the map

$$(Z \cdot \beta_i)_+ \to \{1, \ldots, g\}, \quad \varphi \mapsto \frac{n(\varphi)}{2} \quad (1.27)$$

is a bijection.

Let $(\varphi, \varphi')$ be the restriction to $\Delta$ of the $W$-invariant bilinear form on $h'$ induced by the Killing form on $g$. Let $\Phi = \{\varphi \in \Delta \mid (\psi, \varphi) > 0\}$. One easily has that $\Phi \subset \Delta_+$. Obviously $\psi \in \Phi$. One has

$$\text{card } \Phi = 2h - 3 \quad (1.28)$$

Because of its connection with a Heisenberg subalgebra of $g$ we refer to $\Phi$ as the Heisenberg subsystem of $\Delta_+$. For $i \in I$ let $\Phi^i = \Phi \cap (Z \cdot \beta_i)_+$. Our main result is

**Theorem 1.13.** Let $i \in I - \{i_*\}$. Then

$$z(t)_i = \sum_{\varphi \in \Phi^i} t^{n(\varphi)} \quad (1.29)$$

Furthermore

$$\text{card } \Phi^i = 2d_i \quad (1.30)$$

In addition all the coefficients of $z(t)_i$ are either 1 or 0 so that

$$z(1)_i = 2d_i \quad (1.31)$$

For $i = i_*$ one has

$$z(t)_{i_*} = 2t^g + \sum_{\varphi \in \Phi^{i_*}, \varphi \neq \psi} t^{n(\varphi)} \quad (1.32)$$
In addition the coefficient of $t^g$ is 2 and all the other coefficients of $z(t)_{i_*}$ are either 0 or 1. One also has
\[
    z(1)_{i_*} = 2 d_{i_*}
\]
\[
    = a
\]
(1.33)

Finally
\[
    z(t)_{i_*} = t^{g-a+2} + t^{g-a+4} + \ldots + t^{g-2} + 2 t^g + t^{g+2} + \ldots + t^{g+a-4} + t^{g+a-2} \tag{1.34}
\]

Theorem 1.13 combines Theorem 6.6 and Lemma 6.14 in [K]. We note also that the expression (1.32) for $z(t)_{i_*}$ in Theorem 1.13 follows from the proof of Theorem 6.6 in [K] (see especially (5.8.1) in [K]).

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Bertram Kostant
Dept. of Math.
MIT
Cambridge, MA 02139

E-mail kostant@math.mit.edu