TOPOLOGICAL DYNAMICS OF FLOWS AND SEMIFLOWS
ASSOCIATED WITH GRAPHS

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Abstract. Finite directed graphs with no multiple edges give rise to semiflows on the power set of their nodes and to flows on the set of their bi-infinite paths. This paper studies the topological dynamics for both setups. For the flow we use the standard concepts of Morse decompositions, attractor-repeller sequences and chain recurrent components. For the semiflow, we adapt these concepts to obtain an equivalent characterisation under weaker assumptions.

1. Introduction

We start by presenting basic definitions and notations used along this work. Our first goal is to produce two simple graph decompositions motivated by concepts from dynamical systems. Here we stress the fact that vertex communication is not, in general, an equivalence relation. Then we define communicating sets and classes and, a partial order between communicating classes is presented. A necessary condition to meaningfully study asymptotic behaviour on graphs, via the concept of orbit, is introduced. Section 3 is devoted to analyse the topological dynamics by adapting the concept of Morse decomposition to finite directed graphs with no multiple edges between vertices. These graphs give rise to semiflows on the power set of their nodes. The standard metric defined on a discrete space leads to interesting dynamical phenomena. We give a full description of the global behaviour of the graph dynamics via the concept of Morse Decompositions. In Section 4 we consider well known ideas from symbolic dynamics. An extra condition on $G$ leads to the existence of a flow defined by the shift map on the standard symbolic dynamics space of a graph, $\Omega_G$. Then, via the concept of chain recurrence, we use the machinery developed in [2] to characterise the finest Morse decomposition for the flow defined on $\Omega_G$.

2. Decomposition of graphs

Throughout this note whenever we refer to a graph $G = (V, E)$ we mean a finite directed graph, when $V$ is the set of vertices in $G$ and, $E$ is the set of edges in $G$ such that no multiple edges between any two elements in $V$ are allowed.

2.1. Orbits and communicating classes. The communicating structure in graphs is a central concept in this work. In this section we introduce the concepts of communicating sets and communicating classes based on the idea of orbits.
Let $G$ be a graph. An edge from the vertex $i$ to the vertex $j$ is denoted $(i, j) \in E$. A path in $G$ corresponds to a sequence of vertices that agree with the incidence and direction in $G$. A path is denoted by $\langle i_0 i_1 \ldots i_n \rangle$. Sometimes we write $i \in \gamma$ to specify that the vertex $i$ belongs to the path $\gamma$. We define the set:

\[
\Gamma^n = \{ \gamma : \ell(\gamma) = n, \ n \in \mathbb{N} \}
\]

as the set of all paths of length $n$, and we set $\Gamma^0 = V$.

We can specify vertices in a path $\gamma$ in terms of the projection maps $\pi_p$ for $0 \leq p \leq n$:

\[
\pi_p : \Gamma^n \to V, \quad \pi_p(\gamma) = i_p
\]

where $i_p$ is the $p^{th}$ vertex in $\gamma$. In other words,

\[
\gamma = (\pi_0(\gamma) \ldots \pi_p(\gamma) \ldots \pi_n(\gamma)).
\]

A subpath $\gamma'$ of $\gamma$ is a subsequence of $\gamma$ of consecutive edges (or vertices) belonging to $\gamma$. In particular, any edge of a path is a subpath of length one. Composition of paths will play a role in many of our proofs.

**Definition 2.1.** For two paths $\gamma_1$ and $\gamma_2$ with $\ell(\gamma_1) = m$ and $\ell(\gamma_2) = n$ such that $\gamma_1 = (i \ldots j)$ and $\gamma_2 = (j \ldots k)$ we define the concatenation of the paths as

\[
\langle \gamma_1, \gamma_2 \rangle = (i \ldots j) \ast (j \ldots k) = (i \ldots k)
\]

with $\ell((i \ldots k)) = m + n$ and $\pi_m((i \ldots k)) = j$.

**Definition 2.2.** A vertex $i \in V$ has access to a vertex $j \in V$ if there exists a path of length $\geq 1$ from $i$ to $j$. We say that the vertices $i$ and $j$ communicate, written as $i \sim j$, if they have mutual access. A subset $U$ of $V$ is a communicating set if any two vertices of $U$ communicate.

**Proposition 2.3.** The vertex communication relation $\sim$ is symmetric and transitive but, in general, it lacks the reflexivity property.

**Proof.** Symmetry is obvious from the definition of mutual access. To see transitivity, take $i, j, k \in V$ with $i \sim j$ and $j \sim k$. By definition there exist paths $\gamma_1 = (i \ldots j)$, $\gamma_2 = (j \ldots i)$, $\gamma_3 = (j \ldots k)$ and $\gamma_4 = (k \ldots j)$. Now the concatenation of paths $\langle \gamma_1, \gamma_3 \rangle = (i \ldots k)$ links the vertices $i$ and $k$, and the path $\langle \gamma_4, \gamma_2 \rangle = (k \ldots i)$ links vertices $k$ and $i$, and therefore $i \sim k$, which completes the proof. Note that the relation $\sim$ is reflexive iff for all $i \in V$ there exists a path $\gamma_{ii} = (i \ldots i)$, a property that does not always hold. 

The lack of reflexivity of the communication relation $\sim$ means that $V/\sim$ may not determine a partition of $V$. We therefore define a smaller set on which this property holds. We denote the union of all communicating sets by:

\[
V_c = \{ i \in V : i \sim j \text{ for some } j \in V \}.
\]

For $i \in V_c$ we define $[i] = \{ j \in V, j \sim i \}$. Then $V_c/\sim = \{ [i], i \in V_c \}$ is a partition of $V_c$, that is, $[i] \cap [j] = \emptyset$ for $j \notin [i]$, and $\cup [i] = V_c$.

**Definition 2.4.** Let $G$ be a graph with communication relation $\sim$. Each set $[i]$ for $i \in V_c$ is called a communicating class of $G$. We denote the set $V_c/\sim$ of all communicating classes by $\mathcal{C}$.

Note that by definition, communicating classes are communicating sets. They are characterised by their maximality.
Proposition 2.5. Communicating classes are maximal communicating sets with respect to the set inclusion. Vice versa, maximal communicating sets are communicating classes.

Proof. Assume that \([j] \in \mathcal{C}\) is not maximal, then there exist \(i \in [j]\), and \(k \notin [j]\) with \(i \sim k\) i.e., \([j]\) is not maximal. Since \(i \in [j]\) there exist paths \(\gamma_1 = \langle i \ldots j \rangle\) and \(\gamma_2 = \langle j \ldots i \rangle\). Moreover, since \(i\) and \(k\) communicate we have paths \(\gamma_3 = \langle i \ldots k \rangle\) and \(\gamma_4 = \langle k \ldots i \rangle\). The concatenations \(\langle \gamma_4, \gamma_1 \rangle\) and \(\langle \gamma_2, \gamma_3 \rangle\) imply \(k \sim j\) and therefore \(k \in [j]\), which leads to a contradiction, proving the first claim of the proposition. The second part follows by definition of communicating classes. \(\square\)

Definition 2.8. The positive and negative orbit of a vertex \(i \in V\) are defined as:
\[
\mathcal{O}^+(i) = \{ j \in V : \exists n \geq 1, \exists \gamma \in \Gamma^n \text{ such that } \pi_0(\gamma) = i, \pi_n(\gamma) = j \}, \\
\mathcal{O}^-(i) = \{ j \in V : \exists n \geq 1, \exists \gamma \in \Gamma^n \text{ such that } \pi_0(\gamma) = j, \pi_n(\gamma) = i \},
\]
where \(\pi_0(\gamma)\) and \(\pi_n(\gamma)\) represent the initial and the final vertices in \(\gamma\).

Our first result shows that communicating classes can be characterised using orbits of vertices.

Theorem 2.7. Every communicating class \(C \in \mathcal{C}\) is of the form
\[C = \mathcal{O}^+(i) \cap \mathcal{O}^-(i)\]
for some \(i \in V\). Vice versa, if \(C = \mathcal{O}^+(i) \cap \mathcal{O}^-(i) \neq \emptyset\) for some \(i \in V\), then \(C\) is a communicating class.

Proof. Let \(C\) be a communicating class with \(i \in C\). Then since \(i \sim i\), we have that \(C\) contains a path \(\gamma = \langle i \ldots i \rangle\) and hence it follows that \(i \in \mathcal{O}^+(i) \cap \mathcal{O}^-(i)\), that is, \(C \subseteq \mathcal{O}^+(i) \cap \mathcal{O}^-(i)\). Now consider \(j \in \mathcal{O}^+(i) \cap \mathcal{O}^-(i)\) for some \(i \in V\). Then there exist a path \(\gamma\) and \(n \geq 1\) such that \(\pi_0(\gamma) = i\) and \(\pi_n(\gamma) = j\), as well as a path \(\gamma'\) such that \(\pi_0(\gamma') = j\) and \(\pi_n(\gamma') = i\), for some \(m \geq 1\). This immediately implies \(i \sim j\) for every element \(j \in \mathcal{O}^+(i) \cap \mathcal{O}^-(i)\), and therefore \(\mathcal{O}^+(i) \cap \mathcal{O}^-(i) \subseteq [i]\). Conversely, assume that \(\mathcal{O}^+(i) \cap \mathcal{O}^-(i) \neq \emptyset\) for some \(i \in V\). We have to show that \(\mathcal{O}^+(i) \cap \mathcal{O}^-(i)\) is a communicating class, i.e. \(\mathcal{O}^+(i) \cap \mathcal{O}^-(i) = [i]\). Take \(j \in \mathcal{O}^+(i) \cap \mathcal{O}^-(i)\), then we argue as before that \(j \sim i\) and hence \(j \in [i]\). On the other hand, if \(j \notin \mathcal{O}^+(i) \cap \mathcal{O}^-(i)\), then \(j \notin \mathcal{O}^+(i)\) or \(j \notin \mathcal{O}^-(i)\). In the first case there is no path from \(i\) to \(j\), in the second case there is no path from \(j\) to \(i\). Any of these two statements implies that \(i \sim j\), which completes the proof. \(\square\)

Definition 2.8. A transitory vertex is a vertex that does not belong to a communicating class.

Note that by Theorem 2.7 transitory vertices are exactly those vertices \(i \in V\) for which \(\mathcal{O}^+(i) \cap \mathcal{O}^-(i) = \emptyset\). This also means that \(\mathcal{O}^+(i) \cap \mathcal{O}^-(i) \neq \emptyset\) \iff \(i \in \mathcal{V}_c\), i.e. exactly these vertices anchor communicating classes. In addition, the statements in Theorem 2.7 take on this simple form because we have defined orbits in Definition 2.6 as starting with paths of length 1, not 0. If we include paths of length 0 in an orbit, then it always holds that \(i \in \mathcal{O}^+(i) \cap \mathcal{O}^-(i)\). This trivial situation then needs to be excluded in Theorem 2.7. Similarly, we have defined communicating classes in Definition 2.4 using mutual access, i.e. a vertex \(i \in V\) satisfies \(i \in \mathcal{V}_c\) if there exists a path of length \(\geq 1\) from \(i\) to \(i\). This avoids the triviality that each vertex communicates with itself. Note that for systems on continuous state spaces one needs separate non-triviality conditions, such as the existence of an infinite path.
within a 'communicating class' and a condition on the richness of the orbits, see the discussion in \cite{4}, Chapter 3 for control systems.

\subsection{Communicating sets in \textit{L}-graphs.} We analyse communicating structure in graphs that admit limit behaviour.

\textbf{Definition 2.9.} In a directed graph $G$, the out-degree of a vertex $i \in V$ is defined as the number of edges \textit{going out of the vertex} $i$. In other words, 

$$O(i) = \# \{(i,j): (i,j) \in E \text{ for some } j \in V\}.$$ 

The in-degree of a vertex $i$ corresponds to the number of edges \textit{coming into} $i$. In other words, 

$$I(i) = \# \{(j,i): (j,i) \in E \text{ for some } j \in V\}.$$ 

\textbf{Definition 2.10.} A graph is called an \textit{L}-graph if every vertex has positive out-degree.

From now on we only concentrate on a graph $G = (V,E)$ such that $O(i) \geq 1$ for all $i \in V$. \textit{L}-graphs are needed to ensure the existence of various objects, such as communicating classes. Definition 2.10 states a non-degeneracy condition on the orbits of a graph that will ensure the existence of communicating classes (with certain additional properties). This non-degeneracy condition plays the same role in our present context that is played by the accessibility condition for continuous control systems, compare \cite{4}, Chapter 3 and Appendix A.

\textbf{Lemma 2.11.} Each \textit{L}-graph has paths of arbitrary length.

\textit{Proof.} Let $G$ be an \textit{L}-graph and $n \geq 1$. Pick $i_0 \in V$, then by definition $O(i_0) \geq 1$. This ensures the existence of $i_1 \in V$ such that $(i_0,i_1)$ is an incidence in $G$. Using the same argument we see that there is a vertex $i_2 \in V$ and an incidence $(i_1,i_2)$. Continuing with this process up to step $n$ we infer the existence of a path 

$$\gamma = \{i_0,(i_0,i_1),i_1,(i_1,i_2),\ldots,i_{n-2},(i_{n-1},i_n),i_n\}$$

or equivalently, 

$$\gamma = \{i_0\ldots i_n\}$$

with $\ell(\gamma) = n$. \hfill $\Box$

\textbf{Definition 2.12.} A path $\gamma$ of length $\ell(\gamma) = n$, with $n \geq 1$, is said to be a loop if there exists a vertex $i \in \gamma$ such that $\pi_0(\gamma) = \pi_n(\gamma) = i$.

\textbf{Lemma 2.13.} In a \textit{graph} $G$ with $d$ vertices any \textit{path} $\gamma$ of length $\ell(\gamma) = n$, with $n \geq d$ contains a loop.

\textit{Proof.} We consider a path $\gamma$ in $G$ such that $\gamma = \{i_0,i_1\ldots i_d\}$ with $\ell(\gamma) = d$. Assume that the subpath $\{i_0\ldots i_{d-1}\}$ contains no loop. Then all the vertices of $\{i_0\ldots i_{d-1}\}$ are distinct and hence $\{i_0,\ldots,i_{d-1}\} = V$. Now $i_d \in V$ implies that there exists $\alpha \in \{0,\ldots,d-1\}$ with $i_d = i_\alpha$. Hence the subpath $\{i_\alpha\ldots i_d\}$ is a loop contained in $\gamma$. \hfill $\Box$

The next three lemmata explore the relationship between loops and communicating classes, leading to the existence of communicating classes in \textit{L}-graphs.

\textbf{Lemma 2.14.} If $G$ has a loop, then there exists a communicating class $C$ in $G$ such that the vertices in the loop are contained in $C$. 


Lemma 2.15. Let $G$ be a graph and $\gamma = \langle i_0 \ldots i_n \rangle$ a path in $G$. If there is a communicating class $C$ with $i_0 \in C$ in $G$, and if there is $\alpha \in \{1, \ldots, n\}$ with $i_\alpha \notin C$, then $i_\beta \notin C$ for all $\beta \in \{\alpha, \ldots, n\}$.

Proof. Using the notation of the statement of the lemma, assume, to the contrary, that there exists $\beta \geq \alpha$ with $i_\beta \in C$. Then there are paths $\gamma_1 = \langle i_0 \ldots i_\alpha \rangle$ and $\gamma_2 = \langle i_\alpha \ldots i_\beta \ldots i_0 \rangle$, showing that $i_0 \sim i_\alpha$ and hence $i_\alpha \in C$, which is a contradiction. □

Lemma 2.16. If a vertex $i$ belongs to a communicating class $C$ in $G$, then there exists a loop $\lambda$ in $C$ such that $i \in \lambda$.

Proof. Let $G$ be a graph and $C$ a communicating class in $G$. Then there is pairwise communication between the elements in $C$, i.e., given $i, j \in C$ there exists a path $\gamma_{ij} = \langle i \ldots j \rangle$. In particular, for $i = j$ we have the path $\gamma_ii = \langle i \ldots i \rangle$ with $\pi_0(\gamma_ii) = \pi_n(\gamma_ii) = i$ for some $n \geq 1$. Hence $\gamma$ is indeed a loop containing $i$. By Lemma 2.15 all components of this loop are in $C$. □

Proposition 2.17. An $L$-graph has at least one communicating class.

Proof. Consider an $L$-graph $G$ with $d$ vertices. By Lemma 2.14 there exists a path $\gamma$ such that $\ell(\gamma) = n$ with $n \geq d$. By Lemma 2.13 the path $\gamma$ contains a loop $\lambda$, and by Lemma 2.14 there exist a communicating class containing the vertices of $\lambda$. □

Remark 2.18. The proof of Proposition 2.17 actually shows the stronger statement: Let $G$ be an $L$-graph and $i \in V$. Then there exists at least one communicating class $C$ with $C \subseteq \mathcal{O}(i)$. Note that, in general, $i \in \mathcal{O}(i)$ may not hold.

As a final idea of this section we explore an order on the set of communicating classes, which will lead to a characterisation of so-called forward invariant classes.

Definition 2.19. Let $G$ be a graph with a family $\mathcal{C} = \{C_1, \ldots, C_k\}$ of communicating classes. We define a relation on $\mathcal{C}$ by

\[ C_\mu \preceq C_\nu \quad \text{if} \quad \exists \gamma \in \Gamma^n \text{ with } \pi_0(\gamma) \in C_\mu \text{ and } \pi_n(\gamma) \in C_\nu. \]

Lemma 2.20. Let $G$ be a graph with a family $\mathcal{C} = \{C_1, \ldots, C_k\}$ of communicating classes. The relation $\preceq$ defines a (partial) order on $\mathcal{C}$.

Proof. Reflexivity: Let $C_\mu$ be a communicating class in $G$. By Lemma 2.16 for any $i \in C_\mu$ there exists a loop $\lambda$ in $C_\mu$ such that $i \in \lambda$. Since $C_\mu$ is a communicating class, the loop $\lambda$ gives us a path form $C_\mu$ to itself, and therefore $C_\mu \preceq C_\mu$.

Antisymmetry: Assume that $C_\mu \preceq C_\nu$ and $C_\nu \preceq C_\mu$. From the first relation we get a path $\gamma_{\mu\nu} \in \Gamma^p$ for some $p \geq 1$ such that

\[ \pi_0(\gamma_{\mu\nu}) \in C_\mu \quad \text{and} \quad \pi_p(\gamma_{\mu\nu}) \in C_\nu. \]

By the second relation there exists a path $\gamma_{\nu\mu} \in \Gamma^q$ for some $q \geq 1$ with

\[ \pi_0(\gamma_{\nu\mu}) \in C_\nu \quad \text{and} \quad \pi_q(\gamma_{\nu\mu}) \in C_\mu. \]

Hence the concatenation $\langle \gamma_{\mu\nu}, \gamma_{\nu\mu} \rangle$ is a path in $C_\mu$ of length $p + q$. Since communicating classes are maximal, and any two vertices in a communicating class have mutual access, it holds that $C_\mu = C_\nu$. 

Transitivity. Let \( C_\mu, C_\nu, C_\xi \) in \( \mathcal{C} \) and suppose that \( C_\mu \preceq C_\nu \) and \( C_\nu \preceq C_\xi \) hold. Since \( C_\mu \preceq C_\nu \), there exists a path \( \gamma_1 \) such that \( \pi_0(\gamma_1) \in C_\mu \) and \( \pi_{n_1}(\gamma_1) \in C_\nu \) for some \( n_1 \in \mathbb{N} \). Moreover, since \( C_\nu \preceq C_\xi \), there exists a path \( \gamma_3 \) such that \( \pi_0(\gamma_3) \in C_\nu \) and \( \pi_{n_3}(\gamma_3) \in C_\xi \) for some \( n_3 \in \mathbb{N} \). Since \( C_\nu \) is a communicating class, there exists a path \( \gamma_2 \) in \( C_\nu \) such that \( \pi_0(\gamma_2) = \pi_{n_2}(\gamma_1) \) and \( \pi_{n_2}(\gamma_2) = \pi_0(\gamma_3) \) for some \( n_2 \in \mathbb{N} \). The path \( \gamma = (\gamma_1, \gamma_2, \gamma_3) \) is such that \( \pi_0(\gamma) \in C_\mu \) and \( \pi_{m}(\gamma) \in C_\xi \) for \( m = n_1 + n_2 + n_3 \). Therefore, \( C_\mu \preceq C_\xi \).

Definition 2.21. A set of vertices \( U \) of \( G \) is called forward invariant if,
\[
\mathcal{O}^+(U) \subset U.
\]
Similarly, \( U \) is called backward invariant if,
\[
\mathcal{O}^-(U) \subset U
\]
and invariant if,
\[
\mathcal{O}^+(U) \cup \mathcal{O}^-(U) \subset U.
\]

Remark 2.22. Note that, by definition, a forward invariant communicating class is maximal with respect to the order \( \preceq \) introduced in Definition 2.19.

Proposition 2.23. An \( L \)-graph \( G \) contains a forward invariant communicating class.

Proof. Consider an \( L \)-graph \( G \) with set of communicating classes \( \mathcal{C} = \{C_1, \ldots, C_k\} \). Since an order relation on a finite set has maximal element, denote by \( C_\mu \) a maximal element in \( (\mathcal{C}, \preceq) \). We show that \( C_\mu \) is forward invariant: Assume to the contrary that \( C_\mu \) is not forward invariant. Since then \( \mathcal{O}^+(C_\mu) \) is not contained in \( C_\mu \), there exists \( i,j_0 \in V \) with \( i \in C_\mu \) and \( j_0 \notin C_\mu \) such that \( (i,j_0) \in E \). Since \( O(j_0) \geq 1 \), there exists \( j_1 \in V \) such that \( (j_0,j_1) \in E \). Note that by Lemma 2.15, \( j_1 \) cannot belong to \( C_\mu \). By Remark 2.18, there exists a communicating class \( C \subset \mathcal{O}^+(j_0) \) and \( C \cap C_\mu = \emptyset \) by Lemma 2.15. It follows that, \( C_\mu \preceq C \), which contradicts maximality of \( C_\mu \).

Remark 2.24. Note that in the proof of Proposition 2.23, we actually showed the stronger statement: Let \( G \) be an \( L \)-graph and \( i \in V \). Then \( \mathcal{O}^+(i) \) contains a forward invariant communicating class.

Remark 2.25. Summarizing Remark 2.22 and Proposition 2.23, we see that for an \( L \)-graph the maximal elements of \( (\mathcal{C}, \preceq) \) are exactly the forward invariant communicating classes. It also follows directly from Definition 2.19 that backward invariant communicating classes are minimal in \( (\mathcal{C}, \preceq) \). Note, however, that minimal communicating classes in \( (\mathcal{C}, \preceq) \) need not be backward invariant. For this fact to hold we would need a backward nondegeneracy condition similar to the \( L \)-graph property, e.g., using the in-degree of vertices. For an analogue of this issue in the theory of control systems in discrete time compare [1].

2.3. Quotient graphs. There are at least two quotient structures associated with the idea of communicating sets in a graph. The first idea is to simply take the order graph given in Definition 2.19. Equivalently, this graph is obtained as the quotient \( V_\mathcal{C}/\sim \). The graph obtained from Definition 2.19 does not necessarily cover all the vertices of a given graph \( G \), and its edges may not be edges of \( G \).
For a given directed graph $G = (V, E)$ we define $V_Q := C \cup \{V \setminus C \}$ as a set of vertices. The set of edges is constructed as follows: For $A, B \in V_Q$ we set $(A, B) \in E_Q$ if there exist $i \in A$ and $j \in B$ with $(i, j) \in E$. (Note the abuse of notation: if $A \in V_Q$ is a (transitory) vertex of $G$ then "$i \in A$" is to be interpreted as "$i = A$".) The graph $G_Q = (V_Q, E_Q)$ is called the extended quotient graph of $G$. It is easily seen that the extended quotient graph of $G_Q$ is $G_Q$ itself.

3. The Semiflow of a Finite Directed Graph

Our second approach to study decompositions of graphs is based on an idea from the theory of dynamical systems. A Morse decomposition describes the global behaviour of a dynamical system, i.e. the limit sets of a system and the flow between these sets.

The following definition and basic ingredients are standard. These can be found for example in [2] or [4].

Definition 3.1. A Morse decomposition of a flow on a compact metric space $X$ is a finite collection $\{M_i, i = 1, ..., n\}$ of nonvoid, pairwise disjoint, and compact isolated invariant sets such that:

- For all $x \in X$ one has $\omega(x), \omega^*(x) \subset \bigcup_{i=1}^n M_i$.
- Suppose there are $M_{j_0}, M_{j_1}, ..., M_{j_l}$ and $x_1, ..., x_l \in X \setminus \bigcup_{i=1}^n M_i$ with $\omega^*(x_i) \subset M_{j_{i-1}}$ and $\omega(x_i) \subset M_{j_i}$ for $i = 1, ..., l$; then $M_{j_0} \neq M_{j_l}$.

The elements of a Morse decomposition are called Morse sets.

A Morse decomposition results in an order among the components, the Morse sets, of the decomposition. It can be constructed from attractors and repellers, and the behaviour of the system on the Morse sets is characterised by (chain) recurrence. Here we then construct an analogue for (discrete) systems defined by $L$-graphs. Unfortunately, this analogy is not complete since systems defined by these directed graphs only lead to semiflows, i.e. systems for which the time set is $\mathbb{N}$, and not all of $\mathbb{Z}$.

3.1. Semiflows associated with graphs. When adapting the idea of Morse decompositions and attractors-repellers to systems induced by directed graphs, one faces two main challenges: The first concerns the topology on discrete spaces that has some interesting consequences for limit sets, isolated invariant sets, etc. The second challenge stems from the fact that the out-degree of vertices can be $> 1$, resulting in set-valued systems. This is the reason why graphs define semiflows (on $\mathbb{N}$ instead of $\mathbb{Z}$).

Let $G = (V, E)$ be a directed graph. We denote by $\mathcal{P}(V)$ the power set of the vertex set. We consider the topology derived by the discrete metric on $\mathcal{P}(V)$.

The directed graph $G$ gives rise to two semiflows, one with the positive integers $\mathbb{N}$ as time set, and one with the negative integers $\mathbb{N}^- := \{-n, n \in \mathbb{N}\}$:

$$\Phi_G : \mathbb{N} \times \mathcal{P}(V) \to \mathcal{P}(V),$$

$$\Phi_G(n, A) = \left\{ j \in V : \exists i \in A \text{ and } \gamma \in \Gamma^n \text{ such that } \pi_0(\gamma) = i \text{ and } \pi_n(\gamma) = j \right\}. \quad (3.1)$$
Similarly, we define
\[ \Phi_G : \mathbb{N}^- \times \mathcal{P}(V) \to \mathcal{P}(V), \]
\[ \Phi_G(n, A) = \left\{ j \in V : \exists i \in A \text{ and } \gamma \in \Gamma_i \text{ such that } \pi_0(\gamma) = j \text{ and } \pi_{-n}(\gamma) = i \right\}. \] (3.2)

We note that the definition of the semiflows \( \Phi_G(n, A) \) and \( \Phi^-_G(n, A) \) only requires the basic ingredients of a graph: the sets of vertices and of edges.

The next proposition collects some properties of the maps defined in (3.1) and (3.2).

**Proposition 3.2.** Consider a graph \( G \) and the associated map \( \Phi_G \) defined in (3.1). This map is a semiflow, i.e. it has the properties:

- \( \Phi_G \) is continuous,
- \( \Phi_G(0, A) = A \) for all \( A \in \mathcal{P}(V) \),
- \( \Phi_G(n + m, A) = \Phi_G(n, \Phi_G(m, A)) \) for all \( A \in \mathcal{P}(V), m, n \in \mathbb{N} \).

The same properties hold for the negative semiflow \( \Phi^-_G \).

**Proof.** Note that in the discrete topology every function is continuous. The second item holds by definition of \( \Gamma_0 \) in (2.1): The vertices reached from \( A \) under the flow at time zero, are the elements in \( V \) that belong to \( A \). To show the third item we first assume that the three sets \( \Phi_G(m, A), \Phi_G(n + m, A), \) and \( \Phi_G(n, \Phi_G(m, A)) \) are nonempty. Consider \( j \in \Phi_G(n + m, A) \): There exist \( i \in A \) and a path \( \alpha \in \Gamma_i \) such that \( \pi_0(\alpha) = i \) and \( \pi_{n+m}(\alpha) = j \). We split \( \alpha \) as the concatenation of two paths \( \beta \) and \( \gamma \) with \( \ell(\beta) = m \) and \( \ell(\gamma) = n \), having, \( \pi_0(\beta) = i, \pi_m(\beta) = k \) and \( \pi_0(\gamma) = k, \pi_n(\gamma) = j \) for some \( k \in V \). Observe that by definition we have \( k \in \Phi_G(m, A) \) and hence \( j \in \Phi_G(n, \Phi_G(m, A)) \).

To finish the proof for \( \Phi_G \), we consider the case that (at least) one of the three sets \( \Phi_G(m, A), \Phi_G(n, \Phi_G(m, A)), \) and \( \Phi_G(n + m, A) \) is empty: Note first of all that by the definition of \( \Phi_G \) in (3.1) we have \( \Phi_G(n, \emptyset) = \emptyset \) for all \( n \in \mathbb{N} \). (i) If \( \Phi_G(m, A) = \emptyset \) then we have \( \Phi_G(n, \Phi_G(m, A)) = \emptyset \) by the preceding argument. Now if \( \Phi_G(n + m, A) = \emptyset \), then we can construct, as in the previous paragraph, a point \( k \in \Phi_G(m, A) \) by splitting a path \( \alpha \in \Gamma_i \) such that \( \pi_0(\alpha) = A \) and \( \pi_{n+m}(\alpha) \in \Phi_G(n + m, A) \). This contradicts \( \Phi_G(m, A) = \emptyset \) and therefore \( \Phi_G(n, \Phi_G(m, A)) = \emptyset \).

(ii) Assume next that \( \Phi_G(n, \Phi_G(m, A)) = \emptyset \). Using the same reasoning from the previous paragraph, we see that if \( j \in \Phi_G(n + m, A) \) then \( j \in \Phi_G(n, \Phi_G(m, A)) \). Hence it holds that \( \Phi_G(n + m, A) = \emptyset \). (iii) If \( \Phi_G(n + m, A) = \emptyset \) then there exists no path \( \gamma \in \Gamma_i \) such that \( \pi_0(\gamma) = A \). But by the reasoning in the previous paragraph, if \( j \in \Phi_G(n, \Phi_G(m, A)) \) then there exist \( i \in A \) and \( \gamma \in \Gamma_i \) such that \( \pi_0(\gamma) = i \) and \( \pi_{n+m}(\gamma) = j \), which cannot be true, and hence we see that \( \Phi_G(n, \Phi_G(m, A)) = \emptyset \).

The proof for \( \Phi^-_G \) follows the same lines. \( \square \)

**Remark 3.3.** If \( G \) is an \( L \)-graph, then \( \Phi_G(n, A) \neq \emptyset \) for \( A \neq \emptyset \) and \( n \in \mathbb{N} \). This observation may not hold for the negative semiflow \( \Phi^-_G \) without additional assumptions.

One might wonder if the semiflows \( \Phi_G \) defined in (3.1) and \( \Phi^-_G \) from (3.2) can be combined to a flow on \( Z \). It is not hard to see that in general this is not possible.
even if the graph \( G \) has additional properties (such as being an \( L \)-graph) or if one restricts oneself to graphs that are one communicating class.

**Remark 3.4.** The semiflow \( \Phi_G \) as defined in \( \text{[3.2]} \) can be interpreted as the positive semiflow of the graph \( G^T = (V, E^T) \), where \((i, j) \in E^T \) iff \((j, i) \in E \). \( \Phi_G \) is sometimes called the time-reverse semiflow of \( \Phi_G \). Under the corresponding assumptions, all statements for a positive semiflow also hold for its time-reverse counterpart.

### 3.2. Morse decompositions of semiflows

Note the state space \( \mathcal{P}(V) \) of the semiflow \( \Phi_G \) is finite with the discrete topology. It suffices to introduce the concepts for points \( A \in \mathcal{P}(V) \). To avoid trivial situations where \( \Phi_G(n, A) = \emptyset \) for some \( n \in \mathbb{N} \) and \( A \in \mathcal{P}(V), A \neq \emptyset \), we assume that all graphs are \( L \)-graphs, compare Remarks \( \text{[3.5]} \) above and \( \text{[3.8]} \) below.

Next we adapt the necessary ingredients for a meaningful Morse decomposition of a semiflow in an \( L \)-graph.

**Invariance:** A point \( A \in \mathcal{P}(V) \) is said to be (forward) invariant if \( \Phi_G(n, A) \subset A \) for all \( n \in \mathbb{N} \). Note that \( A \in \mathcal{P}(V) \) is invariant under \( \Phi_G \) iff \( A \) is a forward invariant set of the underlying graph \( G = (V, E) \), compare Definition \( \text{[2.21]} \). Hence invariance under \( \Phi_G \) is a fairly strong requirement of a set \( A \in \mathcal{P}(V) \). As we will see, a meaningful Morse decomposition of the semiflow \( \Phi_G \) only requires a weak form of invariance.

**Definition 3.5.** A point \( A \in \mathcal{P}(V) \) is said to be weakly invariant if for all \( n \in \mathbb{N} \) we have \( \Phi_G(n, A) \cap A = \emptyset \).

**Isolated invariance:** For a (forward) invariant set \( A \in \mathcal{P}(V) \) one could define *forward isolated invariant*. But because of the discrete topology, we could choose \( \mathcal{N}(A) = A \), and any forward invariant set then satisfies this property. As we will see, because of the discrete topology employed, a meaningful Morse decomposition of the semiflow \( \Phi_G \) does not require the property of isolated invariance.

**Limit sets:** To adapt the concept of a limit set from the standard definition to the semiflow \( \Phi_G \), note that a sequence converges in the discrete topology iff it is eventually constant. Hence limit sets can be defined in the following way:

**Definition 3.6.** The \( \omega \)-limit set of a point \( A \in \mathcal{P}(V) \) under \( \Phi_G \) is defined as

\[
\omega(A) = \left\{ y \in V, \text{ there are } t_k \to \infty \text{ such that } y \in \Phi(t_k, A) \right\} \in \mathcal{P}(V).
\]

**Remark 3.7.** Note that by definition of the discrete topology we have for \( A \in \mathcal{P}(V) \) the fact \( \omega(A) = \omega(\{i\}) \), \( i \in A \).

**Remark 3.8.** The existence of \( \omega \)-limit sets and the \( L \)-graph property are closely related: Let \( G \) be a graph and \( \Phi_G \) its associated semiflow. Then \( G \) is an \( L \)-graph iff \( \omega(A) \neq \emptyset \) for all \( A \in \mathcal{P}(V) \). This observation justifies our concentration on \( L \)-graphs in this section.

For continuous dynamical systems Morse decompositions are required to contain all of the \( \omega, \omega^* \)-limit sets of the system.

For semiflows induced by graphs a weaker condition of recurrence turns out to be appropriate:

**Definition 3.9.** Let \( G \) be an \( L \)-graph with associated semiflow \( \Phi_G \) on \( \mathcal{P}(V) \). A one-point set \( \{i\} \in \mathcal{P}(V) \) is called recurrent, if there exists a sequence \( n_i \in \mathbb{N} \),
\( n_l \to \infty \), such that \( \{i\} \subset \Phi_G(n_l, \{i\}) \). A set \( B \in \mathcal{P}(V) \) is called recurrent if for each \( i \in B \) the one-point set \( \{i\} \) is recurrent under \( \Phi_G \). The set \( \mathcal{R} := \{i \in V, \{i\} \) is recurrent} is called the recurrent set of \( \Phi_G \). If \( \mathcal{R} = V \) the semiflow \( \Phi_G \) is called recurrent.

Note that by Definition 3.9 it holds that \( \{i\} \in \mathcal{P}(V) \) is recurrent iff \( i \in \omega(\{i\}) \) iff \( i \in \lambda \) for some loop \( \lambda \) of \( G \).

**No-cycle condition:** For continuous dynamical systems the no-cycle property (second item in Definition 3.1) is essential for the characterization of a Morse decomposition via an order. For semiflows induced by graphs we can formulate an analogue of the no-cycle condition either using only the (forward) semiflow \( \Phi \) composition via an order. For semiflows induced by graphs we can formulate an analogue of the no-cycle condition either using only the (forward) semiflow \( \Phi \) or a combination of \( \Phi \) and \( \Phi_G \).

**Definition 3.10.** Consider the semiflow \( \Phi_G \) and a finite collection \( \mathcal{A} = \{A_1, ..., A_n\} \) of points in \( \mathcal{P}(V) \). \( \mathcal{A} \) is said to satisfy the no-cycle condition for \( \Phi_G \) if for any subcollection \( A_{j_0}, ..., A_{j_l} \) of \( \mathcal{A} \) with \( \omega(A_{j_0}) \cap A_{j_{l+1}} \neq \emptyset \) for \( \alpha = 0, ..., l-1 \) it holds that \( A_{j_0} \neq A_{j_l} \).

**Remark 3.11.** Alternatively, we can define \( A \in \mathcal{P}(V) \) to be a no-return set if for all one-point sets \( \{i\} \in \mathcal{P}(V) \) we have: If \( \omega^*(\{i\}) \cap A \neq \emptyset \) and \( \omega(\{i\}) \cap A \neq \emptyset \) then \( \{i\} \subset A \), where \( \omega^*(B) \) is the \( \omega \)-limit set for \( B \in \mathcal{P}(V) \) under the negative semiflow \( \Phi_G^- \).

With these preparations we can now introduce our concept of a Morse decomposition of the semiflow \( \Phi_G \).

**Definition 3.12.** Let \( G = (V, E) \) be an \( L \)-graph. A Morse decomposition of the semiflow \( \Phi_G \) on \( \mathcal{P}(V) \) is a finite collection of nonempty, pairwise disjoint and weakly invariant sets \( \{\mathcal{M}_{\mu} \in \mathcal{P}(V) : \mu = 1, ..., k\} \) such that:

- \( \mathcal{R} \subset \bigcup_{\mu=1}^{k} \mathcal{M}_{\mu} \)
- \( \{\mathcal{M}_{\mu} \in \mathcal{P}(V) : \mu = 1, ..., k\} \) satisfies the no-cycle condition from Definition 3.11.

The elements of a Morse decomposition are called Morse sets.

**Proposition 3.13.** Let \( G = (V, E) \) be an \( L \)-graph and let

\[ \mathcal{M} = \{\mathcal{M}_{\mu} \in \mathcal{P}(V) : \mu = 1, ..., k\} \]

be a finite collection of nonempty, pairwise disjoint and weakly invariant sets of the semiflow \( \Phi_G \) on \( \mathcal{P}(V) \). The collection \( \mathcal{M} \) is a Morse decomposition of \( \Phi_G \) iff the following properties hold:

- \( \mathcal{R} \subset \bigcup_{\mu=1}^{k} \mathcal{M}_{\mu} \).
- The relation “\( \leq \)” defined by

\[ \mathcal{M}_{\alpha} \leq \mathcal{M}_{\beta} \text{ if there are } \mathcal{M}_{j_{0}} = \mathcal{M}_{\alpha}, \mathcal{M}_{j_{1}}, ..., \mathcal{M}_{j_{l}} = \mathcal{M}_{\beta} \text{ in } \mathcal{M} \]

with \( \omega(\mathcal{M}_{j_i}) \cap \mathcal{M}_{j_{i+1}} \neq \emptyset \) for \( i = 0, ..., l-1 \)

is a (partial) order on \( \mathcal{M} \).

We use the indices \( \mu = 1, ..., k \) in such a way that they reflect this order, i.e. if \( \mathcal{M}_{\alpha} \leq \mathcal{M}_{\beta} \) then \( \alpha \leq \beta \).

**Proof.** Assume first that \( \mathcal{M} = \{\mathcal{M}_{\mu} \in \mathcal{P}(V) : \mu = 1, ..., k\} \) is a Morse decomposition, we need to show that the relation “\( \leq \)” is an order. Reflexivity: Let \( \mathcal{M}_{\mu} \in \mathcal{M} \), then \( \mathcal{M}_{\mu} \) is weakly invariant, i.e. \( \Phi_G(n, A) \cap A \neq \emptyset \) for all \( n \in \mathbb{N} \). Since \( \mathcal{M}_{\mu} \) consists of
finitely many elements there is at least one $i \in \mathcal{M}_\mu$ such that $i \in \Phi_G(n_k, \mathcal{M}_\mu) \cap \mathcal{M}_\mu \neq \emptyset$ for infinitely many $n_k \in \mathbb{N}$. Hence $i \in \omega(\mathcal{M}_\mu) \cap \mathcal{M}_\mu$ and therefore $\omega(\mathcal{M}_\mu) \cap \mathcal{M}_\mu \neq \emptyset$, which shows $\mathcal{M}_\mu \preceq \mathcal{M}_\beta$. Antisymmetry: Assume there are $\mathcal{M}_\alpha, \mathcal{M}_\beta \in \mathcal{M}$ with $\mathcal{M}_\alpha \preceq \mathcal{M}_\beta$ and $\mathcal{M}_\beta \preceq \mathcal{M}_\alpha$. This means there are $\mathcal{M}_{j_0} = \mathcal{M}_\alpha, \mathcal{M}_{j_1}, ..., \mathcal{M}_{j_l} = \mathcal{M}_\beta$ in $\mathcal{M}$ with $\omega(\mathcal{M}_{j_i}) \cap \mathcal{M}_{j_{i+1}} \neq \emptyset$ for $i = 0, ..., l - 1$ and $\mathcal{M}_{k_0} = \mathcal{M}_\beta, \mathcal{M}_{k_1}, ..., \mathcal{M}_{k_m} = \mathcal{M}_\alpha$ in $\mathcal{M}$ with $\omega(\mathcal{M}_{j_i}) \cap \mathcal{M}_{j_{i+1}} \neq \emptyset$ for $i = 0, ..., m - 1$. If there were two different sets in the collection $\mathcal{M}_{j_0} = \mathcal{M}_\alpha, \mathcal{M}_{j_1}, ..., \mathcal{M}_{j_l} = \mathcal{M}_\beta$, $\mathcal{M}_{k_0} = \mathcal{M}_\beta, \mathcal{M}_{k_1}, ..., \mathcal{M}_{k_m} = \mathcal{M}_\alpha$, then $\mathcal{M}_\alpha \neq \mathcal{M}_\beta$, which cannot hold. Hence all sets in this collection are the same, in particular $\mathcal{M}_\alpha = \mathcal{M}_\beta$. Transitivity: This follows directly from the definition of “$\preceq$”.

Assume now that $\mathcal{M} = \{\mathcal{M}_\mu \in \mathcal{P}(V) : \mu = 1, ..., k\}$ is a finite collection of nonempty, pairwise disjoint and weakly invariant sets such the relation “$\preceq$” is an order. We need to show that $\mathcal{M}$ satisfies the no-cycle condition: If $\mathcal{M}_{j_0}, ..., \mathcal{M}_{j_l}$ is a subcollection of $\mathcal{M}$ with $\omega(A_{j_0}) \cap A_{j_{l+1}} \neq \emptyset$ for $\alpha = 0, ..., l - 1$ then $\mathcal{M}_{j_0} \preceq \mathcal{M}_{j_l}$. If this subcollection is disjoint, then $\mathcal{M}_{j_0} \not\preceq \mathcal{M}_{j_l}$, in particular $\mathcal{M}_{j_0} \not\preceq \mathcal{M}_{j_l}$.

As in the case of continuous dynamical systems, Morse decompositions for semiflows induced by $L$-graphs need not be unique. For instance, the collection $\{V, \emptyset\}$ always is a Morse decomposition of any $\Phi_G$. As is the case of continuous dynamical systems we can use intersections of Morse decompositions to refine existing ones, compare page 8 in [2]. Since the sets $V$ and $\mathcal{P}(V)$ are finite, the semiflow $\Phi_G$ on $\mathcal{P}(V)$ admits a (unique) finest Morse decomposition for any $L$-graph $G$. The next result characterises the finest Morse decomposition.

**Theorem 3.14.** Let $G = (V, E)$ be an $L$-graph with associated semiflow $\Phi_G$ on $\mathcal{P}(V)$. For a finite collection of nonempty, pairwise disjoint sets

$$\mathcal{M} = \{\mathcal{M}_\mu \in \mathcal{P}(V) : \mu = 1, ..., k\}$$

the following statements are equivalent:

- $\mathcal{M}$ is the finest Morse decomposition of $\Phi_G$.
- $\mathcal{M} = \mathcal{C}$, the set of communicating classes of $G$, compare Lemma 2.20.

**Proof.** Assume first that $\mathcal{M} = \{\mathcal{M}_\mu \in \mathcal{P}(V) : \mu = 1, ..., k\}$ is the finest Morse decomposition of $\Phi_G$ and let $x, y \in \mathcal{M}_\mu$ for some $\mu = 1, ..., k$. We have to show that $x$ and $y$ communicate. Observe first that $x \in \mathcal{M}_\mu$ implies that there exists a loop $\gamma$ of the graph $G$ with $x \in \gamma$. If there is no such loop then $\{\mathcal{M}_\mu \setminus \{x\}, \mathcal{M}_\alpha : \alpha \neq \mu\}$ is still a Morse decomposition and hence $\mathcal{M}$ cannot be the finest one. If $x$ does not communicate with $y$, take the communicating classes $[x] \neq \emptyset$ and $[y] \neq \emptyset$, $[x] \cap [y] = \emptyset$ together with set $\mathcal{L} = \{\lambda : \lambda$ is a loop not in $[x] \cup [y]\}$, to form the new Morse decomposition $\mathcal{M}' = \{[x], [y], \mathcal{L}, \mathcal{M}_\alpha : \alpha \neq \mu\}$, which is finer than the given $\mathcal{M}$, leading to a contradiction.

To see the converse, let $\mathcal{C} = \{C_1, ..., C_k\}$ be the set of communicating classes of the graph $G$. The $C_\alpha$ are clearly nonempty, pairwise disjoint and weakly invariant for all $\alpha = 1, ..., k$. Recall that $\{i\} \in \mathcal{P}(V)$ is recurrent iff $i \in \lambda$ for some loop $\lambda$ in $G$, and therefore we have $\mathcal{R} \subset \bigcup_{\alpha=1}^{k} C_\alpha$. Finally, Lemma 2.20 shows that the relation “$\preceq$” defined in Proposition 3.13 is indeed an order relation. Hence, the two ordered sets $(\mathcal{C}, \preceq)$ and $(\mathcal{M}, \preceq)$ agree. □

**Remark 3.15.** The proofs of Proposition 3.13 and of Theorem 3.14 show the relationship between $\omega$-limit sets of $\Phi_G$ and loops of $G$: For each $A \in \mathcal{P}(V)$ the limit set $\omega(A)$ contains at least one loop. And vice versa, if $i \in \lambda$ is a vertex of a
loop $\lambda$ of $G$, then $i \in \omega(\{i\})$. This shows that for the finest Morse decomposition $\mathcal{M} = \{\mathcal{M}_\mu, \mu = 1, \ldots, k\}$ of $\Phi_G$ we have:

$$\cup_{\mu=1}^k \mathcal{M}_\mu = \{i \in \mathcal{N} : \lambda \text{ is a loop of } G\} \subset \mathcal{A} \cup \{\omega(A), A \in \mathcal{P}(V)\}.$$ 

This situation is different from the one for continuous dynamical systems, where Morse sets may contain points that are not contained in limit sets, see [2], Example 5.11. Indeed, it is not hard to see that for discrete semiflows not all points in $\omega$-limit sets need to be elements of a Morse set.

Remark 3.16. Consider an $L$-graph $G$ with associated semiflow $\Phi_G$. It follows from Theorem 3.14 that $i \in \cup\{\omega(A), A \in \mathcal{P}(V)\} \setminus \cup\{\mathcal{M}_\mu, \mathcal{M}_\mu \text{ is a finest Morse set}\}$ iff $i$ is a transitory vertex and there exists a (finest) Morse set $\mathcal{M}$ with $i \in \Phi_G(n, \mathcal{M})$ for some $n \geq 1$.

3.3. Attractors and recurrence in semiflows. Next we adapt the concept of an attractor to the semiflow on an $L$-graph and analyse the connection with Morse decompositions.

Definition 3.17. Let $G = (V, E)$ be an $L$-graph with associated semiflow $\Phi_G$ on $\mathcal{P}(V)$. A point $A \in \mathcal{P}(V)$ is called an attractor if there exists a set $N \subset V$ with $A \subset N$ such that $\omega(N) = A$.

A set $N$ as in Definition 3.17 is called an attractor neighborhood. Note that in the discrete topology $A$ is a neighborhood of itself and hence a point $A \in \mathcal{P}(V)$ is an attractor iff $\omega(A) = A$. We also allow the empty set as an attractor.

A definition of repellers for semiflows of graphs is not obvious, but the idea of complementary repellers from [2] carries over with an obvious modification for semiflows:

Definition 3.18. For an attractor $A \in \mathcal{P}(V)$, the set

$$A^* = \{i \in V, \omega(\{i\}) \setminus A \neq \emptyset\} \in \mathcal{P}(V)$$

is called the complementary repeller of $A$, and $(A, A^*)$ is called an attractor-repeller pair.

Morse decompositions of semiflows can be characterized by attractor-repellers pairs, in analogy to [2].

Theorem 3.19. Let $G = (V, E)$ be an $L$-graph with associated semiflow $\Phi_G$ on $\mathcal{P}(V)$. A finite collection of sets $\mathcal{M} = \{\mathcal{M}_\mu \in \mathcal{P}(V) : \mu = 1, \ldots, k\}$ defines a Morse decomposition of $\Phi_G$ if and only if there is a strictly increasing sequence of attractors

$$\emptyset = A_0 \subset A_1 \subset A_2 \subset \ldots \subset A_n \subset V,$$

such that

$$\mathcal{M}_{n-i} = A_{i+1} \cap A_i^* \text{ for } 0 \leq i \leq n - 1.$$

Proof. Recall the indexing convention for Morse sets from Proposition 3.13.

Let $\mathcal{M} = \{\mathcal{M}_\mu \in \mathcal{P}(V) : \mu = 1, \ldots, k\}$ be a Morse decomposition of $\Phi_G$. In analogy to the continuous time case we define the sets $A_k$ for $k = 1, \ldots, n$ as follows:

$$A_k = \{x \in V, \omega^*(x) \cap (\mathcal{M}_{n} \cup \ldots \cup \mathcal{M}_{n-k+1}) \neq \emptyset\}.$$ 

Note that for semiflows of graphs we have $A_k = \mathcal{O}^*(\mathcal{M}_{n} \cup \ldots \cup \mathcal{M}_{n-k+1})$. We first need to show that each $A_k$ is an attractor. The inclusion $\omega(A_k) \subset A_k$ follows directly from the characterization above of $A_k$ as a positive orbit. To see that $A_k \subset \omega(A_k)$
pick $x \in A_k$. Then there exists $\mu \in \{n-k+1, \ldots, n\}$ with $x \in \mathcal{O}^+ (M_\mu)$. According to Theorem 3.14 the set $M_\mu$ is a communicating class of the graph $G$ and hence every element $z \in M_\mu$ is in a loop $\gamma$ that is completely contained in $M_\mu$, compare Lemma 2.10. Therefore there is $z \in A_k$ with $x \in \Phi_G (n_i, \{z\})$ for as sequence $n_i \to \infty$, i.e. $A_k \subset \omega (A_k)$. Hence each $A_k$ is an attractor.

Next we show that $M_{n-i} = A_{i+1} \cap A_i^*$ for $0 \leq i \leq n - 1$. To see that $M_{n-i} \subset A_{i+1}$ pick $x \in M_{n-i}$. Since $M_{n-i}$ is a communicating class of the graph $G$ we have $\omega^+(x) \cap M_{n-i} \neq \emptyset$ and therefore $x \in A_{i+1}$. To see that $M_{n-i} \subset A_i^*$ assume that there exists $x \in M_{n-i}$ with $x \not\in A_i^*$, i.e. $\omega(x) \backslash A_i = \emptyset$ or $\omega(x) \subset A_i$. But $x \in M_{n-i}$ means $\omega(x) \cap M_{n-i} \neq \emptyset$, and by definition we have $M_{n-i} \cap A_i = \emptyset$, which is a contradiction.

This shows $M_{n-i} \subset A_{i+1} \cap A_i^*$ for $0 \leq i \leq n - 1$. To see the reverse inclusion, let $x \in A_{i+1} \cap A_i^*$, i.e. $x \in \mathcal{O}^+ (M_{n-i})$ and $\omega(x) \backslash A_i = \emptyset$. Recall that by Remark 3.15 $\omega(x)$ contains a loop $\gamma$ of the graph $G$, and by Lemma 2.10 and Theorem 3.14 each loop is contained in a Morse set. Hence $\omega(x) \cap (M_{n-i} \cup \ldots \cup M_{n-i}) \neq \emptyset$, which by definition of a Morse decomposition means that $x \in M_{n-i}$.

Let $M_{n-i} = A_{i+1} \cap A_i^*$ for $0 \leq i \leq n - 1$ be defined as in the statement of the theorem. We have to show that $\{M_1, \ldots, M_n\}$ form a Morse decomposition. We start by proving that the sets $M_{n-i} = A_{i+1} \cap A_i^*$ are nonempty. Note first of all that $A_1, \ldots, A_n \neq \emptyset$ by assumption. We have by definition of attractor-repeller pairs that $V = A_0^* \supset A_1^* \supset \ldots \supset A_{n-1}^* \supset A_n^*$. Now $A_{n-1}^* \neq \emptyset$ can be seen like this: If $A_{n-1}^* = \emptyset$ then for all $x \in V$ we have $\omega(x) \backslash A_{n-1} = \emptyset$, i.e. $\omega(x) \subset A_{n-1}$. Hence there is $m \in \mathbb{N}$ such that for all $\alpha \geq m$ we have $\Phi_G (\alpha, V \backslash A_{n-1}) \subset A_{n-1}$ and therefore $A_n$ cannot be an attractor. We conclude that $A_0^*, A_1^*, \ldots, A_{n-1}^* \neq \emptyset$. Now if $A_{i+1} \cap A_i^* = \emptyset$ then we have by the same reasoning as before: For all $x \in A_{i+1}$ it holds that $\omega(x) \backslash A_i = \emptyset$, i.e. $\omega(x) \subset A_i$ and $A_{i+1}$ cannot be an attractor.

The sets $M_i$ are pairwise disjoint: Let $\alpha < \beta$, then $M_{n-\alpha} \cap M_{n-\beta} = A_{\alpha+1} \cap A_\beta^* \cap A_{\beta+1} \cap A_\alpha^* = A_{\alpha+1} \cap A_\beta^* = \emptyset$.

The sets $M_i$ are weakly invariant: As above, it suffices to prove that $M_{n-i} = A_{i+1} \cap A_i^*$ contains a loop of the graph $G$. If there is no loop in $A_{i+1} \cap A_i^*$, then there exists $m \in \mathbb{N}$ such that for all $\alpha \geq m$ we have $\Phi_G (\alpha, A_{i+1} \cap A_i^*) \subset A_i$ and therefore $A_{i+1}$ cannot be an attractor.

The collection $\{M_1, \ldots, M_n\}$ satisfies the no-cycle condition: This is just a restatement of the assumption that $A_0 \subset A_1 \subset A_2 \subset \ldots \subset A_n$ is a strictly increasing sequence of attractors. We have shown so far that $\mathcal{M} = \{M_1, \ldots, M_n\}$ satisfies the conditions of a Morse decomposition, except for $\mathcal{R} \subset \cup_{\mu=1}^k M_\mu$. Now let $\mathcal{M}' = \{M'_1, \ldots, M'_k\}$ be the finest Morse decomposition of $\Phi_G$. Since the recurrence condition was not used in the first part of the proof of Theorem 3.14 and since $\{i\} \in \mathcal{P}(V)$ is recurrent iff $i \in \lambda$ for some loop $\lambda$ of $G$, we know by Lemma 2.14 that $\mathcal{R} \subset \cup_{\mu=1}^k M'_\mu \subset \cup_{\mu=1}^k M_\mu$. Altogether we see that $\mathcal{M} = \{M_1, \ldots, M_n\}$ is a Morse decomposition.

**Corollary 3.20.** Let $\mathcal{M} = \{M_\mu \in \mathcal{P}(V) : \mu = 1, \ldots, k\}$ be the finest Morse decomposition of a semiflow $\Phi_G$ on $\mathcal{P}(V)$, with order $\preceq$. Then the maximal (with respect to $\preceq$) Morse sets are attractors. Furthermore, the smallest (with respect to set inclusion) non-empty attractors are exactly the maximal (with respect to $\preceq$) Morse sets.

**Proof.** If $\mathcal{M}$ is a maximal Morse set of the semiflow $\Phi_G$, then, according to Theorem 3.14 and Proposition 3.13 $\mathcal{M}$ is a maximal communicating class of the graph $G$. 

Hence $M$ is forward invariant, $\omega(M) = M$ and $M$ does not contain any attractor, except for the empty set.

Vice versa, if $A$ is a smallest (with respect to set inclusion) non-empty attractor, then $A$ is a Morse set according to Theorem 3.19. If $A$ is not maximal (with respect to $\subseteq$), then $A$ is not forward invariant for the graph $G$ and hence there exists a point $x \in \mathcal{O}^*(A) \setminus A$ such that $\mathcal{O}^*(x) \cap A = \emptyset$ (by Lemma 2.15). According to Remark 2.24, $\mathcal{O}^*(x)$ contains a maximal communicating class, which is an attractor $A' \nsubseteq A$ and hence $A$ is not a smallest non-empty attractor. \hfill $\Box$

It remains to analyse the behaviour of the semiflow $\Phi_G$ on a Morse set. Definition 3.21 and Remark 3.15 already point at a recurrence property that holds for $\omega$-limit sets: Note that by Definition 3.9 it holds: \( \{ i \} \in \mathcal{P}(V) \) is recurrent if $i \in \omega(\{ i \})$ iff $i \in \lambda$ for some loop $\lambda$ of $G$. Hence we obtain from Remark 3.15 for the finest Morse decomposition $M = \{ M_\mu, \mu = 1, ..., k \}$ of $\Phi_G$

\[
\mathcal{R} = \bigcup_{\mu=1}^{k} M_\mu.
\]

The recurrent set is partitioned into the disjoint sets of the finest Morse decomposition under the following natural concept of connectedness.

**Definition 3.21.** A set $B \in \mathcal{P}(V)$ is called connected under $\Phi_G$ if for any $i,j \in B$ there exist $n \in \mathbb{N}$ and a map $p : \{0, ..., n\} \to B$ with the properties

- $p(0) = i$, $p(n) = j$
- $p(m+1) \in \Phi_G(1, \{p(m)\})$ for $m = 0, ..., n-1$.

The flow $\Phi_G$ is called strongly connected if the set of vertices $V$ is connected under $\Phi_G$.

The following result then characterises the behaviour of the semiflow $\Phi_G$ on its Morse sets, compare Theorem 6.4 in [2] for continuous dynamical systems.

**Theorem 3.22.** Let $G = (V,E)$ be an L-graph with associated semiflow $\Phi_G$ on $\mathcal{P}(V)$. The recurrent set $\mathcal{R}$ of $\Phi_G$ satisfies

\[
\mathcal{R} = \bigcap \{ A \cup A^*, A \text{ is an attractor} \}
\]

and the (finest) Morse sets of $\Phi_G$ coincide with the $\Phi_G$-connected components of $\mathcal{R}$.

**Proof.** Assume that $x \in \mathcal{R}$, then $x \in \gamma$ for some loop $\gamma$ of the graph $G$. Let $A$ be an attractor for $\Phi_G$, then if $x \in A$ we are done. Otherwise if $x \notin A$ then it holds that $\gamma \cap A = \emptyset$. But $\gamma \subseteq \omega(x)$ and therefore $\omega(x) \setminus A = \emptyset$, which means that $x \in A^*$. Conversely, if $x \in \cap \{ A \cup A^*, A \text{ is an attractor} \}$, then $x$ is in any attractor containing $\omega(x)$. Arguing as in the proof of Theorem 3.19 there exists a loop $\gamma$ of the graph such that $x \in \gamma$, which shows that $x \in \mathcal{R}$.

The second statement of the theorem follows directly from Definition 3.21 and (3.3). \hfill $\Box$

As discussed in the paragraph about invariance (see page 9), forward invariance under $\Phi_G$ is a fairly strong requirement for a set $A \subset V$, and thus it appears that there are few sets to which one can restrict the semiflow $\Phi_G$, namely (unions of positive) orbits.

**Definition 3.23.** Let $G = (V,E)$ be an L-graph with associated semiflow $\Phi_G$ on $\mathcal{P}(V)$. Let $G' = (V',E')$ be the subgraph of $G$ for a subset of vertices $V' \subset V$. The resulting semiflow $\Phi_{G'}$ on $\mathcal{P}(V')$ is called the semiflow $\Phi_G$ restricted to $V'$. 

Note that if $M \subset V$ is a Morse set of $\Phi_G$, then the induced graph $(V_M, E_M)$ is an $L$-graph. This observation allows us to prove the following fact about Morse sets and recurrence.

**Corollary 3.24.** Under the conditions of Theorem 3.22, the semiflow $\Phi_G$ restricted to any Morse set is recurrent.

**Proof.** The proof follows directly from Theorem 3.14 and Definition 3.9.

As we have seen, most of the concepts used to characterise the global behaviour of continuous dynamical systems can be adapted in a natural way to the positive semiflow of an $L$-graph, resulting in very similar characterisations. Indeed, the proofs for semiflows on a finite set are considerably simpler than the corresponding ones for continuous dynamical systems. What is missing in the context of semiflows is first of all the group property of a flow, and hence limit objects for $t \to -\infty$. This results in missing some of the invariance properties of crucial sets, such as limit sets, Morse sets, the (components of) the recurrent set, etc. And secondly, the use of the discrete topology implies that while all points in the (finest) Morse sets are limit points, not all $\omega$-limit points of the semiflow are contained in the (finest) Morse sets. But those exceptional limit points (and hence the set of all limit points) can be characterised, compare Remark 3.16.

### 4. The Flow of a Finite Directed Graph

In this section we assume an extra condition on $G$. The idea is to work in a more general setting in terms of asymptotic behaviour of orbits which initial conditions are single vertices in $G$. Recall that $G = (V,E)$ is a finite directed graph with no multiple edges between any two elements in $V$.

**Definition 4.1.** An $L$-graph is called a bi-directed if every vertex has positive in-degree.

In other words, a bi-directed graph satisfies the following two conditions:

\begin{align*}
O(i) &> 0 \quad \text{for all } i \in V \\
I(i) &> 0 \quad \text{for all } i \in V.
\end{align*}

Assumption (4.2) is required for bi-infinite sequences of edges to be defined. We consider the vertex set $V$ as an alphabet so that each vertex has associated the symbol given by the vertex notation. We define the standard space of symbolic dynamics over a bi-directed graph $G$ as:

$$
\Omega_G = \{ \omega \in V^Z, (\omega_k, \omega_{k+1}) \in E \text{ for all } k \in \mathbb{Z} \}.
$$

The shift map on $\Omega_G$ is given by:

$$
\phi : \Omega_G \to \Omega_G, \quad (\omega_k) \mapsto (\omega_{k+1})
$$

and gives rise to a dynamical system

$$
\Psi : \mathbb{Z} \times \Omega_G \to \Omega_G, \quad (n, \omega) \mapsto \phi^n(\omega).
$$

Associated with this set up are the evaluation maps:

$$
\Pi_k : \Omega_G \to V, \quad \omega \mapsto \omega_k \quad \text{for } k \in \mathbb{Z}
$$
Remark 4.6. \( k \Pi \)\n\begin{align*}
\text{Definition 4.5. (Communicating sets and communicating classes).} \\
\text{An element} \quad \text{Definition 4.3.} \\
\Omega \\
\text{Maximal elements in} \quad \text{I(} \\
\text{positive (negative) maximal elements using in such cases} \\
\text{Paths.} \\
\text{Denote again by} \quad \text{Γ} \\
\text{embedded} \\
\text{standard} \\
\text{every} \\
\text{and the ranges,} \\
R(\omega) = \{\Pi_j \omega, j \in \mathbb{Z}\} \subset V \\
R^{+}(\omega) = \{\Pi_{k+j} \omega, j \in \mathbb{Z}^+\} \subset V \\
R^{-}(\omega) = \{\Pi_{k+j} \omega, j \in \mathbb{Z}^-\} \subset V \\
\text{where} \quad k \in \mathbb{Z}, \ 1 \in \mathbb{Z}^+ = \{1, 2, \ldots\} \quad \text{and} \quad \mathbb{Z}^- = \{-1, -2, \ldots\}. \quad \text{We write} \ R^+ \text{ instead} \ R^+_{\omega}. \\
\text{Throughout this section we only deal with bi-directed graphs. Therefore, when we} \\
\text{refer to a graph} \ G \ \text{we mean a bi-directed graph} \ G. \\
\text{Remark 4.2.} \ \text{Recall that} \ \Omega \ \text{is called the full} \\
n\text{n-shift and it corresponds to the space} \\
of all possible bi-infinite sequences with \ n \ \text{symbols. It is clear that} \ \Omega_G \subset \Omega. \\
\text{4.1. (The communication structure of} \ G \ \text{in terms of the flow} \ \Psi. \ \text{In this} \\
\text{section we show that all the concepts related with the communication structure} 
of a finite directed graph can be interpreted using concepts on the level of the flow \ \Psi \\n\text{defined above. We use this parallelism to describe the dynamics on} \ \Omega_G. \\
\text{Paths.} \ \text{Denote again by} \ \Gamma^n \ \text{the set of all paths} \ \gamma = (i_0 \ ... \ i_n) \ \text{in} \ G. \ \text{Each path} \ \gamma \in \Gamma^n \\n\text{corresponds uniquely to an equivalence class} \ \mathcal{I}(\gamma) \ \text{of points in} \ \Omega_G \ \text{by requiring that} \\
\omega \in \mathcal{I}(\gamma) \ \text{if} \ \Pi_k \omega = i_k \ \text{for} \ k = 0, \ldots, n. \ \text{Note that} \ \mathcal{I}(\gamma) \neq \emptyset \ \text{because of assumptions} \\
\text{[4.1] and [4.2]. According to Lemma 2.13 for any bi-infinite sequence} \ \omega \in \Omega_G \ \text{there exist} \\
m, n \in \mathbb{Z} \ \text{with} \ \Pi_m \omega = \Pi_n \omega, \ \text{due to the finiteness of} \ V. \ \text{This means that each} \\
\omega \in \Omega_G \ \text{has embedded loops.} \\
\text{Orbits.} \ \text{The positive and negative orbits of vertices are easily identified using} 
\text{concepts on the level of the flow as follows:} \\
O^+(i) = \bigcup_{\Pi_0 \omega = i} R^+(\omega) \quad \text{for} \ i \in V \\
O^-(i) = \bigcup_{\Pi_0 \omega = i} R^-(\omega) \quad \text{for} \ i \in V. \\
\text{In general, we can define the orbit of a set} \ A \subset V \ \text{by:} \\
O^+(A) = \bigcup_{i \in A} \bigcup_{\Pi_0 \omega = i} R^+(\omega) \quad \text{for} \ A \subset V. \\
\text{Maximal elements in} \ \Omega_G. \\
\text{Definition 4.3.} \ \text{An element} \ \omega \in \Omega_G \ \text{is called maximal if for all} \ \omega' \in \Omega_G \ \text{with} \\
R(\omega) \cap R(\omega') \neq \emptyset \ \text{it holds that} \ R(\omega') \subset R(\omega). \ \text{A similar definition holds for} \\
\text{positive (negative) maximal elements using in such cases} \ R^+ \text{ and} \ (R^-) \text{ respectively.} \\
\text{Note that if a maximal element exists it may not be unique.} \\
\text{Remark 4.4. (On components and orbits)} \\
\bullet \ \text{An element} \ \omega \in \Omega_G \ \text{is maximal if and only if} \ R(\omega) \ \text{is a component of} \ G. \\
\bullet \ \text{If} \ \omega \in \Omega_G \ \text{is a positive maximal element then} \ R^+(\omega) = O^+(i) \ \text{for} \ i = \Pi_0 \omega. \\
\bullet \ \text{If} \ \omega \in \Omega_G \ \text{is a negative maximal element then} \ R^-(\omega) = O^+(i) \ \text{for} \ i = \Pi_0 \omega. \\
\text{Communicating sets and communicating classes.} \\
\text{Definition 4.5.} \ \text{An element} \ \omega \in \Omega_G \ \text{is called periodic if there exists} \ p \in \mathbb{Z}^+ \ \text{with} \\
\Pi_k \omega = \Pi_{k+p} \omega \ \text{for all} \ k \in \mathbb{Z}. \\
\text{Remark 4.6. (On communicating sets and communicating classes)}
A set $A \subset V$ is a communicating set (see Definition 2.2) if and only if $A = R(\omega)$ for some periodic $\omega \in \Omega_G$.

A set $C \subset V$ is a communicating class (see Definition 2.4) if and only if $C = R(\omega)$ for some maximal periodic $\omega \in \Omega_G$.

Let $\prec$ be the order between communicating classes (see Definition 2.19). Then $C_1 \prec C_2$ if there exists $\omega \in \Omega_G$ with $\Pi_0 \omega \in C_1$ and $R^+(\omega) \cap C_2 \neq \emptyset$.

$C \subset V$ is a maximal communicating class if and only if $R^+(\omega) \subset C$ for all $\omega \in \Omega_G$ with $\Pi_0 \omega \in C$.

The preceding remarks show, not surprisingly, that the orbit and communication structure of $G$ can be recovered in terms of $\Omega_G$ and the evaluation maps $\Pi_k$, $k \in \mathbb{Z}$, or, alternatively, in terms of $\Omega_G$, $\Psi$ and $\Pi_0$. Section 3 in this work related the orbit and communication structure of $G$ to the semiflow $\Phi_G : \mathbb{N} \times \mathcal{P}(V) \to \mathcal{P}(V)$ by exploring, for semiflows, appropriate concepts of Morse decompositions, attractor-repeller pairs, and chain recurrence. From now on, we concentrate on the relation between the topological dynamics of the flow $\Psi$ and the semiflow $\Phi_G$. By the remarks above and the characterisations from Section 3 it is sufficient to relate the topological dynamics of $\Psi$ to the communication structure of $G$ using the evaluation maps.

### 4.2. Topological Dynamics of $\Psi$.

The results exposed in the next paragraph are standard in the theory of symbolic dynamics. These can be found in [6], [7] and [8].

The space $\Omega_G$ can be made into a topological space. The metric:

$$\rho : \Omega_G \times \Omega_G \to \mathbb{R}_0^+$$

where

$$\rho(\omega, \omega') = \sum_{n \in \mathbb{Z}} \frac{\delta(\omega_n, \omega'_n)}{|\lambda|^n}$$

and

$$\delta(\omega_n, \omega'_n) = \begin{cases} 
0 & \text{if } \omega_n = \omega'_n \\
1 & \text{if } \omega_n \neq \omega'_n 
\end{cases}$$

induces a topology in $\Omega_G$ in the standard manner. In particular, we choose $\lambda = 4$ in equation (4.3) in order to make cylinders into open balls with respect this metric. It is well known that $\Omega_G$ is a compact metric space with respect to $\rho$. In addition, the map $\Psi$ is continuous and satisfies the standard flow properties. The space $\Omega_G$ is closed in the full $n$-shift $\Omega$ and the flow $\Psi$ leaves $\Omega_G$ invariant.

### 4.3. Structural connections between $G$ and $\Omega_G$.

We start by describing the connections between the structure of bi-directed graphs and the set of bi-infinite sequences having as alphabet the vertex set in $G$. Concepts as Morse decompositions and chain recurrence can be analysed in this context.

**Definition 4.7.** For a subgraph $F = (V_F, E_F) \subset G$. The lift generated by $F$ correspond to:

$$L(F) = \{ \omega \in \Omega_G : \omega_i \in V_F, (\omega_i, \omega_{i+1}) \in E_F \} \subset \Omega_G.$$
In this context the concept of lift lead us to a parallelism between the dynamics in
a graph and their counterpart in terms of bi-infinite sequences of symbols. Observe
that by definition lifts are invariant under $\Psi$.

**Proposition 4.8.** Lifts of communicating classes are compact sets.

*Proof.* Consider a communicating class $C$ in $G$ together with an arbitrary sequence
$\omega^{(n)}$ in $L(C)$. The idea is to construct a convergent subsequence $\omega^{(n_k)}$ of $\omega^{(n)}$. For
each $k \geq 0$ recursively find a decreasing sequence of infinite subsets $I_k$ of positive
integers (distinguished indices of $\omega^{(n)}$) such that for elements $(n_k)_i \in I_k$, the $2k+1$
central symbols are coincident i.e., $\omega_j^{(n_k)_i} = \omega_j^{(n_k)_2}$ for $|j| \leq k$ where $\omega^{(n_k)_1}, \omega^{(n_k)_2} \in$
$L(C)$ are elements of $\omega^{(n)}$. The system is provided just with a finite number of
symbols, so there is an infinite set $I_0$ of positive integers (indices $(n_0)_i$ of $\omega^{(n)}$)
such that for all the associated elements $\omega^{(n_0)}$, of $\omega^{(n)}$ the central symbol coincide.
Continuing with this procedure we see that for each $k \geq 0$ we find a set $I_{k+1} \subset I_k$
such that for every element in $I_k$ the $2k+1$ central symbols are identical. Let us
define $\omega$ (the limit point) to be the bi-infinite sequence such that its $2k+1$ central
symbols agreed with all elements $\omega^{(n_k)_i}$ of $\omega^{(n)}$, with $(n_k)_i \in I$ for all $k \geq 0$. Then
define $n_k$ as the smallest element of $I_k$ which exceeds $n_{k-1}$. □

4.4. **Characterising finest Morse decompositions of $\Psi$ on $\Omega_G$**. Next we link
concepts from the theory of dynamical systems with lifts of bi-directed graphs
by verifying finiteness on the number components of the chain recurrent set of $\mathcal{R}$. We conclude this section by giving a characterisation for the finest Morse
decompositions of $\Psi$ on $\Omega_G$.

The following example motivates Theorem 4.10 and shows that not all lifts are isolated invariant sets.

**Example 4.9.** Consider the following bi-directed graph $G = (V_G, E_G)$ with $V_G =\{1, 2, 3, 4\}$ and $E_G = \{(1, 1), (1, 2), (2, 3), (3, 3), (1, 4), (4, 3)\}$. Consider $F = (V_F, E_F) \subset$
$G$ with $V_F = \{1, 2, 3\}$ and $E_F = \{(1, 1), (1, 2), (2, 3), (3, 3)\}$. The set $L(F)$ contains
$\omega = (1)^\infty.(1)^m(2)(3)^\infty$ but does not contain $\nu = (1)^\infty.(1)^m(2)(4)(3)^\infty$.
However $\rho(\Psi(k, \nu), \omega) = \frac{1}{m}$, that is, for a given arbitrarily closed neighbourhood $N$ of $L(F)$
we have that $\Psi(k, \nu)$ belongs to it for $m$ sufficiently large. We conclude that not
every lift of $F \subset G$ is an isolated invariant set.

Next result give us an clue about the natural candidates to be Morse sets.

**Theorem 4.10.** Lifes of communicating classes are isolated invariant sets.

*Proof.* Consider $\omega \in \Omega_G$ not in the lift a communicating class $C$ in $G$. The last
means that every element $\nu$ of $L(C)$ differs at least in one symbol with $\omega$ which
implies the existence of an integer $n$ such that $\rho(\Psi(\omega, n), \nu) \geq 1$. Now consider
the neighbourhood $N(L(C)) = \{\omega \in \Omega_G : \rho(\omega, \nu) < \frac{1}{2}\}$ of $L(C)$. We see that $L(C)$ does
not admit a neighbourhood which contains the orbit of a element not in $L(C)$. □

**Remark 4.11.** Lifts of subgraphs of communicating classes are not necessarily iso-
lated invariant sets. Consider the bi-directed graph $G$ given by $V_G = \{1, 2, 3\}$ and
$E_G = \{(1, 2), (2, 1), (1, 3), (3, 3), (3, 2)\}$ we apply a similar argument as in the pre-
ceding example by considering $F$ as given by $V_F = \{1, 2\}$ with $E_F = \{(1, 2), (2, 1)\}$.
The set $L(F)$ does not contain the point $\omega = (21)^\infty.(3)(12)^\infty$ but as we can see for
an arbitrarily closed neighbourhood $N$ of $L(F)$ we have that $\Psi(\omega, n)$ belongs to
$N(L(F))$ for $n$ sufficiently large.
Definition 4.12. Define $\mathcal{L} = \{L(C) : C \in \mathcal{C}\}$. We say that $\omega \in \mathcal{L}$ if $R(\omega)$ is contained in one and only one communicating class. Otherwise $\omega$ is called a transient point.

Proposition 4.13. Choose $\omega \in \Omega_G$ and $k \in \mathbb{Z}$. The sets $R_k^l$ and $R_k^{-l}$ contain a communicating class.

Proof. Consider a directed graph $G = (V, E)$ with $V$ having $n$ vertices. Select an arbitrary $\omega \in \Omega_G$ and $k \in \mathbb{Z}$ and set

$$D = \{\Pi_{k+l} \omega, 0 \leq i \leq l\}$$

Note that $(\omega_{k+i}, \omega_{k+i+1}) \in E$. By lemma 2.14 every path of length greater or equal to $n$ contains a loop. Consider $l > n$, then by virtue of Lemma 2.15 there exists a communicating class $C$ of $G$ containing such a loop. Therefore $C \subset D$ for sufficiently large $l$, implying that $C \subset R_k^l \omega$ as desired. The proof for $R_k^{-l}$ follows identically. □

Lemma 4.14. Choose $\omega \in \Omega_G$ and $\omega_i, \omega_j \in R(\omega)$. There exists a path $\gamma$ such that

$$\pi_0(\gamma) = \omega_i, \quad \pi_n(\gamma) = \omega_j \quad \text{or} \quad \pi_0(\gamma) = \omega_j, \quad \pi_n(\gamma) = \omega_i$$

for some $n \in \mathbb{Z}^+$. □

Proof. Recall that $\omega \in \Omega_G$ means $(\omega_k, \omega_{k+1}) \in E_G$ for every integer $k$. If $i < j$ we have that $(\omega_{i+l}, \omega_{i+l+1}) \in E_G$ and $1 \leq l \leq j - i$. The path $\gamma = (\omega_i, \ldots, \omega_j)$ is such that $\pi_0(\gamma) = \omega_i, \pi_n(\gamma) = \omega_j$ for $n = j - i$. An analogous argument applies for $j < i$. □

Theorem 4.15. Given $\omega \in \Omega_G$ with set of communicating classes of $G$ given by $\mathcal{C} = \{C_1, \ldots, C_n\}$, such that each $C_i \subset R(\omega)$, for $1 \leq i \leq n$. Then $\mathcal{C}$ is a totally ordered.

Proof. Consider $\omega \in \Omega_G$ together with communicating classes $C_i, C_j \subset R(\omega)$. Suppose that neither $C_i < C_j$ nor $C_j < C_i$. By the definition of order in $\mathcal{C}$ (compare Definition 2.20) for $\omega_i \in C_i$ and $\omega_j \in C_j$ no paths $\gamma, \gamma'$ are such that:

$$\pi_0(\gamma) = \omega_i, \quad \pi_n(\gamma) = \omega_j \quad \text{and} \quad \pi_0(\gamma') = \omega_j, \quad \pi_m(\gamma') = \omega_i$$

$n, m \in \mathbb{Z}^+$, contradicting that $\omega_i, \omega_j \in R(\omega)$. □

Remark 4.16. Finite totally ordered sets admit unique maximal (minimal) elements. Given $\omega \in \Omega_G$ the set of all communicating classes in $R(\omega)$ admit unique maximal (minimal) elements; these will be denoted by $C_n$ and $C_1$ respectively. Here $n$ denotes the number of communicating classes in $R(\omega)$.

4.4.1. Chain recurrence behavior on $\Omega_G$. In contrast with the semiflow defined in $G$, from Section 4 we are dealing with a flow defined in $\Omega_G$. So, the standard results about Morse decompositions and their characterisations, as for example via attractor-repeller pairs or the order characterisation (compare [5], [2] and [3]), can be immediately applied to the flow in defined in $\Omega_G$. What is left though is the characterisation of the finest Morse decomposition. We analyse the chain recurrence of $\Psi$ in $\Omega_G$ and conclude that the chain recurrence behaviour under the flow $\Psi$ is found just on the lifts of communicating classes. For the sake of exposition we include the next standard definition. For a more general definition refer to Appendix B in [4].
Definition 4.17. For \( \omega, \nu \in \Omega_G \) and \( \epsilon, T > 0 \) an \((\epsilon, T)\)-chain from \( \omega \) to \( \nu \) is given by a natural number \( n \in \mathbb{N} \), together with points \( \omega_0 = \omega, \omega_1, \ldots, \omega^n = \nu \) and \( T_0, \ldots, T_{n-1} \geq T \) such that \( \rho(\Psi(T_i, \omega_i), \omega_{i+1}) < \epsilon \) for \( i = 0, \ldots, n - 1 \). A point \( \omega \in \Omega_G \) is called chain recruit if for all \( \epsilon, T > 0 \) there exists an \((\epsilon, T)\)-chain from \( \omega \) to \( \omega \). The chain recurrent set \( \mathcal{R} \) is the set of all chain recurrent points.

Theorem 4.18. Lifts of communicating classes are chain recurrent sets.

Proof. Consider a communicating class \( C \) in \( G \) and \( \omega, \omega' \in L(C) \) with \( \omega = (\ldots \omega_{-2} \omega_{-1} \omega_0 \ldots) \) and \( \omega' = (\ldots \omega'_{-1} \omega'_0 \omega'_1 \ldots) \). Next we will construct and \((\epsilon, T)\)-chain from \( \omega \) to \( \omega' \) via and auxiliary element \( \bar{\omega} \in L(C) \). Observe it is sufficient to consider \( T = 1 \) for the first step. Then \( \Psi(1, \omega) = (\ldots \omega_{-2} \omega_{-1} \omega_0 \ldots) \) and for this point we consider,

\[
\bar{\omega} = (\ldots \bar{\omega}_{-m-1} \bar{\omega}_{-m} \ldots \bar{\omega}_{-1} \bar{\omega}_0 \bar{\omega}_1 \ldots \bar{\omega}_m \bar{\omega}_{m+1} \ldots)
\]

where the \( 2m + 1 \) central symbols of \( \bar{\omega} \) are the same as the \( 2m + 1 \) central symbols of \( \Psi(1, \omega) \),

\[
\bar{\omega} = (\ldots \bar{\omega}_{-m-1} \bar{\omega}_{-m} \ldots \bar{\omega}_{-1} \bar{\omega}_0 \bar{\omega}_1 \ldots \bar{\omega}_m \bar{\omega}_{m+1} \ldots).
\]

In other words, for \( \Psi_j(1, \omega) \) (the \( j \)th component of \( \Psi(1, \omega) \)) we have that \( \Psi_j(1, \omega) = \bar{\omega}_j \) for \( |j| \leq m \). Therefore, given \( \epsilon > 0 \),

\[
\rho(\Psi(1, \omega), \bar{\omega}) = \sum_{j \in \mathbb{Z}} \frac{1}{|j|} \delta(\Psi_j(1, \omega), \bar{\omega}_j) \leq 2 \sum_{j=m+1}^{\infty} \frac{1}{|j|} < \epsilon
\]

for \( m \) sufficiently large. As second step, first note that \( R(\omega), R(\omega') \subset C \), and at the level of graphs, there exists a path \( \gamma \in \Gamma^l \) together with \( l \in \mathbb{Z}_0^+ \) such that:

\[
\pi_0(\gamma) = \bar{\omega}_m \text{ and } \pi_1(\gamma) = \omega'_n.
\]

With the later we complete the construction of \( \bar{\omega} \) for \( j > m \) by considering,

\[
\bar{\omega} = (\ldots \omega_{-m} \ldots \omega_{-1} \omega_0 \omega_1 \ldots \omega_m \bar{\omega}_{m+1} \ldots \omega_{m+l} \omega_{m+l+1} \ldots \omega_{m+l+n} \ldots)
\]

such that,

\[
\bar{\omega} = (\ldots \omega_{-m} \ldots \omega_{-1} \omega_0 \omega_1 \ldots \omega_m \bar{\omega}_{m+1} \ldots \omega_{m+l} \omega_{m+l+1} \ldots \omega_{m+l+n} \omega_{l+n+1} \ldots \omega'_0 \ldots).
\]

In other words, \( \bar{\omega}_{m+l+n} = \omega'_{l+n} \) for \( 0 \leq j \leq n \), that is, we continue \( \bar{\omega} \) from the symbol \( \omega_{m+l} \) forward via the sequence \( \omega' \) from the symbol \( \omega_n \) forward. Therefore we have that \( \rho(\Psi(m+l+|n|), \omega') < \epsilon \) for \( |n| \) sufficiently large as before. So, given two elements in the lift of a communicating class a two step \((\epsilon, T)\)-chain can always be constructed. \( \square \)

Theorem 4.19. Points \( \omega \in \Omega_G \) not in the lift of exactly one communicating class are not chain recurrent under \( \Psi \).

Proof. Consider \( \omega \in \Omega_G \) such that \( R(\omega) \) contains at least two distinct communicating classes. Since the restriction of \( \mathcal{C} \) to \( R(\omega) \) leads to a total order compare Theorem 4.13, we label the communicating classes \( C_1 < C_2 \). Without loss of generality, consider the point \( \omega \) such that \( \Pi_0 \omega \in C_1 \). By Lemma 4.14 there exists \( T_1 \in \mathbb{Z}^+ \) such that \( \Psi(T_1, \omega) \in C_2 \). Note that in virtue of Proposition 4.13, the sets \( R^-(\Psi(T_1, \omega)) \) and \( R^+(\Psi(T_1, \omega)) \) each contain at least one communicating class. We proceed to prove that an \((\epsilon, T)\)-chain from \( \omega \) to itself is not possible to be constructed. Let \( \epsilon < 1 \). When we apply the flow \( \Psi \) to \( \omega \) at time \( T_1 \) we obtain:

\[
\Psi(T_1, \omega) = (\ldots \omega_{-1} \omega_0 \omega_1 \ldots \omega_{T_1-1} \omega \omega_{T_1+1} \ldots)
\]
where

\[ \omega = (\ldots \omega_{-1}, \omega_0, \omega_1 \ldots). \]

Note that for each \( T \geq T_1 \), \( \rho(\Psi(T, \omega), \omega) \) is such that \( \delta(\omega_T, \omega_0) = 1 \) since \( C_1 \neq C_2 \). Therefore

\[ \rho(\Psi(T, \omega), \omega) \geq 1 > \epsilon. \]

So after the application of the flow to \( \omega \) for sufficiently large time \( T \), no element in an arbitrary close neighbourhood of \( \Psi(T, \omega) \) will be sufficiently close to \( \omega \) when \( R(\omega) \) contains more than one communicating class. Therefore recurrence for \( \omega \) is not possible concluding the proof. The case when \( \Pi_0 \omega \in C_2 \) has an analogous proof.

\[ \Box \]

Remark 4.20. A point \( \omega \in \Omega_G \) such that its range \( R(\omega) \) contains transient symbols cannot be a chain recurrent point. Hence the only chain recurrent points in \( \Omega_G \) are exactly those in the lifts of communicating classes. For a chain recurrent point its range is a communicating set contained in one and only one communicating class.

**Theorem 4.21.** The chain recurrent set \( \mathcal{R} \) coincides with the set of lifts of communicating classes \( \mathcal{L} \). In particular, \( \mathcal{R} \) has a finite number of components.

**Proof.** By virtue of Theorem 4.17 if \( \omega \in \mathcal{R} \) then \( \omega \in L(C) \) for a unique communicating class \( C \) in \( G \) and therefore \( \omega \in \mathcal{L} \). Conversely, consider a point \( \omega \in \mathcal{L} \) then \( \omega \in L(C) \) for some communicating class \( C \) in \( G \). Due to Theorem 4.18 we have \( L(C) \) is a chain recurrent set and therefore \( \omega \in \mathcal{R} \). Since \( C \) is finite so \( \mathcal{L} \) is finite then of course \( \mathcal{R} \) is also finite.

Next we focus on the characterisation of the Morse sets under the flow \( \Psi \) defined in \( \Omega_G \). For the sake of notation denote the \( \omega \)-limit set as \( \alpha^* \)-limit set and also denote the \( \omega^* \)-limit set as \( \alpha \)-limit set.

**Proposition 4.22.** If \( R(\omega) \) contains \( n \) communicating classes then:

- \( \alpha(\omega) \subset L(C_1) \).
- \( \alpha^*(\omega) \subset L(C_n) \).

**Proof.** Consider \( \omega \in \Omega_G \). If \( R(\omega) \) contains just one communicating class \( C \), then it cannot contain transient vertices, otherwise there exists at least two communicating classes. If \( n = 1 \) then \( V_G = C \). Note that \( \nu \in \alpha^*(\omega) \) with \( R(\nu) \not\subset C \) immediately contradicts the fact that \( C \) is forward invariant.\(^1\) Suppose now there exists \( n \) communicating classes in \( R(\omega) \). Since there exists a total order in \( R(\omega) \) given in Theorem 4.17 we list these communicating classes as \( C_1 < C_2 < \ldots < C_n \). By Lemma 4.13 and, due to the order relation, there exists \( \omega \in \Omega_G \) and \( m_i \in \mathbb{Z} \) such that \( \Pi_0 \Psi(m_i, \omega) \in C_i \) and \( \Pi_0 \Psi(m_{i+1}, \omega) \notin C_i \) here \( 1 \leq i \leq n-1 \). After a applying the shift a finite number of times we attain the maximal communicating class \( C_n \) which is forward invariant so we get \( R_k^*(\omega) = C_n \) for some sufficiently large integer \( k \). It is easy to see that the set \( \alpha^*(\omega) \) cannot contain other elements than the elements in \( C_n \). Therefore \( \alpha^*(\omega) \subset L(C_n) \). The proof of the first statement runs with analogous argument.

\[ \Box \]

Remark 4.23. Next we list the properties Morse sets satisfy and related them with our work. Lifts of communicating classes are:

\(^1\)A similar statement holds for \( \nu \in \alpha(\omega) \), contradicting the backward invariance condition.
Nonempty: We proved that every communicating class contains a loop.

Pairwise disjoint: Since communicating classes are maximal communicating sets we have that the elements in \( L \) are pairwise disjoint.

Compact: See Theorem 4.8

isolated invariant: See Theorem 4.10

Order and no-cycle condition: If \( R(\omega) \) contains more than one communicating class then trivially \( C_1 \neq C_n \). Also, no transient communicating class in \( R(\omega) \) can be a Morse set. The order in the set of all communicating classes induces in a natural way an order in \( L \) as follows: \( L(C) \leq L(D) \) if, and only if, \( C \preceq D \) for the communicating classes \( C \) and \( D \). In addition, for every \( \omega \in \Omega_G \) under the hypothesis in 4.15 the limit sets \( \alpha(\omega) \subset L(C_1) \) and \( \alpha^*(\omega) \subset L(C_n) \) of course \( L(C_1) \neq (C_n) \). Compare Theorem 4.22.

Theorem 4.24. Let \( G \) be a bi-directed graph with associated flow \( \Psi \) on \( \Omega_G \). For a finite collection of nonempty, pairwise disjoint sets
\[
\mathcal{M} = \{ \mathcal{M}_\mu \in \mathcal{P}(V) : \mu = 1, \ldots, k \}
\]
The following statements are equivalent:

- \( \mathcal{M} = \mathcal{L} \) the set of lifts of communicating classes of \( G \).
- \( \mathcal{M} \) is the finest Morse decomposition of \( \Psi \).

Proof. The result follows immediately due to the finite number of components in \( \mathcal{R} \), compare Theorem 4.21 and Theorem 6.4 in [2]. \( \square \)

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