Intermediate rings of complex-valued continuous functions

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\textbf{Abstract}

For a completely regular Hausdorff topological space $X$, let $C(X, \mathbb{C})$ be the ring of complex-valued continuous functions on $X$, let $C^*(X, \mathbb{C})$ be its subring of bounded functions, and let $\Sigma(X, \mathbb{C})$ denote the collection of all the rings that lie between $C^*(X, \mathbb{C})$ and $C(X, \mathbb{C})$. We show that there is a natural correlation between the absolutely convex ideals/prime ideals/maximal ideals/$\omega$-ideals/$\omega^*$-ideals in the rings $P(X, \mathbb{C})$ in $\Sigma(X, \mathbb{C})$ and in their real-valued counterparts $P(X, \mathbb{C}) \cap C(X)$. These correlations culminate to the fact that the structure space of any such $P(X, \mathbb{C})$ is $\beta X$. For any ideal $I$ in $C(X, \mathbb{C})$, we observe that $C^*(X, \mathbb{C}) + I$ is a member of $\Sigma(X, \mathbb{C})$, which is further isomorphic to a ring of the type $C(Y, \mathbb{C})$. Incidentally these are the only $C$-type intermediate rings in $\Sigma(X, \mathbb{C})$ if and only if $X$ is pseudocompact. We show that for any maximal ideal $M$ in $C(X, \mathbb{C})$, $C(X, \mathbb{C})/M$ is an algebraically closed field, which is furthermore the algebraic closure of $C(X)/M \cap C(X)$. We give a necessary and sufficient condition for the ideal $C_\mathcal{P}(X, \mathbb{C})$ of $C(X, \mathbb{C})$, which consists of all those functions whose support lie on an ideal $\mathcal{P}$ of closed sets in $X$, to be a prime ideal, and we examine a few special cases thereafter. At the end of the article, we find estimates for a few standard parameters concerning the zero-divisor graphs of a $P(X, \mathbb{C})$ in $\Sigma(X, \mathbb{C})$.

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1. Introduction

In what follows, $X$ stands for a completely regular Hausdorff topological space and $C(X, \mathbb{C})$ denotes the ring of all complex-valued continuous functions on $X$. $C^*(X, \mathbb{C})$ is the subring of $C(X, \mathbb{C})$ containing those functions which are bounded over $X$. As usual $C(X)$ designates the ring of all real-valued continuous functions on $X$ and $C^*(X)$ consists of those functions in $C(X)$ which are bounded over $X$. An intermediate ring of real-valued continuous functions on $X$ is a ring that lies between $C^*(X)$ and $C(X)$. Let $\Sigma(X)$ be the aggregate of all such rings. Likewise an intermediate ring of complex-valued continuous functions on $X$ is a ring lying between $C^*(X, \mathbb{C})$ and $C(X, \mathbb{C})$. Let $\Sigma(X, \mathbb{C})$ be the family of all such intermediate rings. It turns out that each member $P(X, \mathbb{C})$ of $\Sigma(X, \mathbb{C})$ is absolutely convex in the sense that $|f| \leq |g|, g \in P(X, \mathbb{C}), f \in C(X, \mathbb{C})$ implies $f \in P(X, \mathbb{C})$. It follows that each such $P(X, \mathbb{C})$ is conjugate-closed in the sense that if whenever $f + ig \in P(X, \mathbb{C})$ where $f, g \in C(X)$, then $f - ig \in P(X, \mathbb{C})$. It is realised that there is a natural correlation between the prime ideals/ maximal ideals/ $z$-ideals/ $z^0$-ideals in the rings $P(X, \mathbb{C})$ and the prime ideals/ maximal ideals/ $z$-ideals/ $z^0$-ideals in the ring $P(X, \mathbb{C}) \cap C(X)$. In the second and third sections of this article, we examine these correlations in some detail. Incidentally an interconnection between prime ideals in the two rings $C(X, \mathbb{C})$ and $C(X)$ is already observed in Corollary 1.2[7]. As a follow up of our investigations on the ideals in these two rings, we establish that the structure spaces of the two rings $P(X, \mathbb{C})$ and $P(X, \mathbb{C}) \cap C(X)$ are homeomorphic. The structure space of a commutative ring $R$ with unity stands for the set of all maximal ideals of $R$ equipped with the well-known hull-kernel topology. It was established in [21] and [22], independently that the structure space of all the intermediate rings of real-valued continuous functions on $X$ are one and the same viz the Stone-Cech compactification $\beta X$ of $X$. It follows therefore that the structure space of each intermediate ring of complex-valued continuous functions on $X$ is also $\beta X$. This is one of the main technical results in our article. We like to mention in this context that a special case of this result telling that the structure space of $C(X, \mathbb{C})$ is $\beta X$ is quite well known, see [19]. We call a ring $P(X, \mathbb{C})$ in the family $\Sigma(X, \mathbb{C})$ a $C$-type ring if it is isomorphic to a ring of the form $C(Y, \mathbb{C})$ for Tychonoff space $Y$. We establish that if $I$ is any ideal of $C(X, \mathbb{C})$, then the linear sum $C^*(X, \mathbb{C}) + I$ is a $C$-type ring. This is the complex analogue of the corresponding result in the intermediate rings of real-valued continuous functions on $X$ as proved in [16]. We further realise that these are the only $C$-type intermediate rings in the family $\Sigma(X, \mathbb{C})$ when and only when $X$ is pseudocompact i.e. $C(X, \mathbb{C}) = C^*(X, \mathbb{C})$.

It is well-known that if $M$ is a maximal ideal in $C(X)$, then the residue class field $C(X)/M$ is real closed in the sense that every positive element in this field is a square and each odd degree polynomial over this field has a root in the same field [17, Theorem 13.4]. The complex analogue of this result as we realise
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is that for a maximal ideal $M$ in $C(X, \mathbb{C})$, $C(X, \mathbb{C})/M$ is an algebraically closed field and furthermore this field is the algebraic closure of $C(X)/M \cap C(X)$.

In section 4 of this article, we deal with a few special problems originating from an ideal $\mathcal{P}$ of closed sets in $X$ and a certain class of ideals in the ring $C(X, \mathbb{C})$. A family $\mathcal{P}$ of closed sets in $X$ is called an ideal of closed sets in $X$ if for any two sets $A, B$ in $\mathcal{P}$, $A \cup B \in \mathcal{P}$ and for any closed set $C$ contained in $A, C$ is also a member of $\mathcal{P}$. We let $C_\mathcal{P}(X, \mathbb{C})$ be the set of all those functions $f$ in $C(X, \mathbb{C})$ whose support $\text{cl}_X(X \setminus Z(f))$ is a member of $\mathcal{P}$; here $Z(f) = \{x \in X : f(x) = 0\}$ is the zero set of $f$ in $X$. We determine a necessary and sufficient condition for $C_\mathcal{P}(X, \mathbb{C})$ to become a prime ideal in the ring $C(X, \mathbb{C})$ and examine a few special cases corresponding to some specific choices of the ideal $\mathcal{P}$. The ring $C_\infty(X, \mathbb{C}) = \{f \in C(X, \mathbb{C}) : f$ vanishes at infinity in the sense that for each $n \in \mathbb{N}, \{x \in X : |f(x)| \geq \frac{1}{n}\}$ is compact} is an ideal of $C^*(X, \mathbb{C})$ but not necessarily an ideal of $C(X, \mathbb{C})$. On the assumption that $X$ is locally compact, we determine a necessary and sufficient condition for $C_\infty(X, \mathbb{C})$ to become an ideal of $C(X, \mathbb{C})$.

The fifth section of this article is devoted to finding out the estimates of a few standard parameters concerning zero divisor graphs of a few rings of complex-valued continuous functions on $X$. Thus for instance we have checked that if $\Gamma(P(X, \mathbb{C}))$ is the zero divisor graph of an intermediate ring $P(X, \mathbb{C})$ belonging to the family $\Sigma(X, \mathbb{C})$, then each cycle of this graph has length 3, 4 or 6 and each edge is an edge of a cycle with length 3 or 4. These are the complex analogues of the corresponding results in the zero divisor graph of $C(X)$ as obtained in [9].

2. Ideals in intermediate rings

Notation: For any subset $A(X)$ of $C(X)$ such that $0 \in A(X)$, we set $[A(X)]_c = \{f + ig : f, g \in A(X)\}$ and call it the extension of $A(X)$. Then it is easy to see that $[A(X)]_c \cap C(X) = A(X) = [A(X)]_c \cap A(X)$. From now on, unless otherwise stated, we assume that $A(X)$ is an intermediate ring of real-valued continuous functions on $X$, i.e. $A(X)$ is a member of the family $\Sigma(X)$. It follows at once that $[A(X)]_c$ is an intermediate ring of complex-valued continuous functions and it is not hard to verify that $[A(X)]_c$ is the smallest intermediate ring in $\Sigma(X, \mathbb{C})$ which contains $A(X)$ and the constant function $i$. Furthermore $[A(X)]_c$ is conjugate-closed meaning that if $f + ig \in [A(X)]_c$, with $f, g \in A(X)$, then $f - ig \in [A(X)]_c$. The following result tells that intermediate rings in the family $\Sigma(X, \mathbb{C})$ are the extensions of intermediate rings in $\Sigma(X)$.

**Theorem 2.1.** Let $P(X, \mathbb{C})$ be an intermediate ring of $C(X, \mathbb{C})$. Then $P(X, \mathbb{C})$ is absolutely convex.

**Proof.** Let $|f| \leq |g|$, $f \in C(X, \mathbb{C}), g \in P(X, \mathbb{C})$. Then $f = \frac{f}{|f|}(1 + g^2) \in P(X, \mathbb{C})$. Hence $P(X, \mathbb{C})$ is absolutely convex. \qed

**Theorem 2.2.** An intermediate ring $P(X, \mathbb{C})$ of $C(X, \mathbb{C})$ is conjugate closed.
Proof. Let \( f + ig \in P(X, \mathbb{C}) \). We have \( |f| \leq |f + ig|, |g| \leq |f + ig| \) and \( f + ig \in P(X, \mathbb{C}) \). Since \( P(X, \mathbb{C}) \) is absolutely convex, then \( f, g \in P(X, \mathbb{C}) \). This implies \( f, ig \in P(X, \mathbb{C}) \) as \( i \in P(X, \mathbb{C}) \). Thus \( f - ig \in P(X, \mathbb{C}) \). Hence \( P(X, \mathbb{C}) \) is conjugate closed. \( \Box \)

**Theorem 2.3.** A ring \( P(X, \mathbb{C}) \) of complex valued continuous functions on \( X \) is a member of \( \Sigma(X, \mathbb{C}) \) if and only if there exists a ring \( A(X) \) in the family \( \Sigma(X) \) such that \( P(X, \mathbb{C}) = [A(X)]_c \).

**Proof.** Assume that \( P(X, \mathbb{C}) \in \Sigma(X, \mathbb{C}) \) and let \( A(X) = P(X, \mathbb{C}) \cap C(X) \). Then it is clear that \( A(X) \in \Sigma(X) \) and \( [A(X)]_c \subseteq P(X, \mathbb{C}) \).

To prove the reverse containment, let \( f + ig \in P(X, \mathbb{C}) \). Here \( f, g \in C(X) \). Since \( P(X, \mathbb{C}) \) is conjugate closed, \( f - ig \in P(X, \mathbb{C}) \), and hence \( 2f \) and \( 2ig \) both belong to \( P(X, \mathbb{C}) \). Since constant functions are bounded and hence in \( P(X, \mathbb{C}) \), both the constant functions \( \frac{1}{2} \) and \( \frac{1}{2i} \) are in \( P(X, \mathbb{C}) \). It follows that both \( f \) and \( g \) are in \( P(X, \mathbb{C}) \cap C(X) \), and hence in \( A(X) \). Consequently, \( f + ig \in [A(X)]_c \). Thus, \( P(X, \mathbb{C}) \subseteq [A(X)]_c \). \( \Box \)

The following facts involving convex sets will be useful. A subset \( S \) of \( C(X) \) is called absolutely convex if whenever \( |f| \leq |g| \) with \( g \in S \) and \( f \in C(X) \), then \( f \in S \).

**Theorem 2.4.** Let \( A(X) \in \Sigma(X) \). Then

(a) \( A(X) \) is an absolutely convex subring of \( C(X) \) (in the sense that if \( |f| \leq |g| \) with \( g \in A(X) \) and \( f \in C(X) \), then \( f \in A(X) \)) ([16, Proposition 3.3]).

(b) A prime ideal \( P \) in \( A(X) \) is an absolutely convex subset of \( A(X) \) ([13, Theorem 2.5]).

The following convenient formula for \( [A(X)]_c \) with \( A(X) \in \Sigma(X) \) will often be helpful to us.

**Theorem 2.5.** For any \( A(X) \in \Sigma(X) \), \( [A(X)]_c = \{ h \in C(X, \mathbb{C}) : |h| \in A(X) \} \).

**Proof.** First assume that \( h = f + ig \in [A(X)]_c \) with \( f, g \in A(X) \). Then \( |h| \leq |f| + |g| \). This implies, in view of Theorem 2.4(a), that \( h \in A(X) \) and also \( |h| \in A(X) \). Conversely, let \( h = f + ig \in C(X, \mathbb{C}) \) with \( f, g \in C(X) \), be such that \( |h| \in A(X) \). This means that \( (f^2 + g^2)^{\frac{1}{2}} \in A(X) \). Since \( |f| \leq (f^2 + g^2)^{\frac{1}{2}} \), this implies in view of Theorem 2.4(a) that \( f \in A(X) \). Analogously \( g \in A(X) \). Thus \( h \in [A(X)]_c \). \( \Box \)

**Theorem 2.6.** If \( I \) is an ideal in \( A(X) \in \Sigma(X) \), then \( I_c = \{ f + ig : f, g \in I \} \) is the smallest ideal in \( [A(X)]_c \) containing \( I \). Furthermore \( I_c \cap A(X) = I = I_c \cap C(X) \).

**Proof.** It is easy to show that \( I_c \) is an ideal in \( [A(X)]_c \) containing \( I \). Let \( K \) be an ideal of \( [A(X)]_c \) containing \( I \). To show \( I_c \subseteq K \). Let \( f + ig \in K \), where \( f, g \in I \). Since \( I \subseteq K \), then \( f, g \in K \). Now \( K \) is an ideal of \( [A(X)]_c \), \( f, g \in K \).
implies \( f + ig \in K \). Therefore \( I_c \subseteq K \). Hence \( I_c \) is the smallest ideal of \([A(X)]_c\)
containing \( I \).

Proof of the second part is trivial. \( \square \)

**Theorem 2.7.** If \( I \) and \( J \) are ideals in \( A(X) \in \Sigma(X) \), then \( I \subseteq J \) if and only if \( I_c \subseteq J_c \). Also \( I \nsubseteq J \) when and only when \( I_c \nsubseteq J_c \).

Proof. If \( I \subseteq J \), then clearly \( I_c \subseteq J_c \).

Conversely, let \( I_c \subseteq J_c \). Let \( f \in I \). Since \( I \subseteq I_c \), we have \( f \in I_c \subseteq J_c \). Now \( f = f + i0 \) and \( J_c = \{ g + ih : g, h \in J \} \). Therefore \( f \in J_c \). Hence \( I \subseteq J_c \).

For the second part we consider \( I \nsubseteq J_c \) and \( f \in J \setminus I \). Then \( f \in J_c \setminus I_c \). Thus \( I_c \nsubseteq J_c \).

Conversely, let \( I_c \subseteq J_c \) and \( f + ig \in J_c \setminus I_c \). Then either \( f \) or \( g \) is outside \( I \). Let \( f \notin I \). Then \( f \in J \setminus I \). Hence \( I_c \nsubseteq J \). This completes the proof. \( \square \)

We have the following convenient formula for \( I_c \) when \( I \) is an absolutely convex ideal of \( A(X) \).

**Theorem 2.8.** If \( I \) is an absolutely convex ideal of \( A(X) \) (in particular if \( I \) is a prime ideal or a maximal ideal of \( A(X) \)), then \( I_c = \{ h \in [A(X)]_c : |h| \in I \} \).

Proof. Let \( h = f + ig \in I_c \). Then \( f, g \in I \). Since \( |h| \leq |f| + |g| \), the absolute convexity of \( I \) implies that \( |h| \in I \). Conversely, let \( h = f + ig \in [A(X)]_c \) be such that \( |h| \in I \). Here \( f, g \in A(X) \). Since \( |f| \leq (f^2 + g^2)^{1/2} = |h| \), it follows from the absolute convexity of \( I \) that \( f \in I \). Analogously \( g \in I \). Hence \( h \in I_c \). \( \square \)

The above theorem prompts us to define the notion of an absolutely convex ideal in \( P(X, \mathbb{C}) \in \Sigma(X, \mathbb{C}) \) as follows:

**Definition 2.9.** An ideal \( J \) in \( P(X, \mathbb{C}) \in \Sigma(X, \mathbb{C}) \) is called absolutely convex if for \( g, h \) in \( C(X, \mathbb{C}) \) with \( |g| \leq |h| \) and \( h \in J \), it follows that \( g \in J \).

The first part of the following proposition is immediate, while the second part follows from Theorem 2.3 and Theorem 2.8.

**Theorem 2.10.** Let \( P(X, \mathbb{C}) \in \Sigma(X, \mathbb{C}) \).

(i) If \( J \) is an absolutely convex ideal of \( P(X, \mathbb{C}) \), then \( J \cap C(X) \) is an absolutely convex ideal of the intermediate ring \( P(X, \mathbb{C}) \cap C(X) \in \Sigma(X) \).

(ii) An ideal \( I \) in \( P(X, \mathbb{C}) \cap C(X) \) is absolutely convex in this ring if and only if \( I_c \) is an absolutely convex ideal of \( P(X, \mathbb{C}) \).

(iii) If \( J \) is an absolutely convex ideal of \( P(X, \mathbb{C}) \), then \( J = [J \cap C(X)]_c \).

Proof. (iii) It is trivial that \([J \cap C(X)]_c \subseteq J \). To prove the reverse implication let \( h = f + ig \in J \), with \( f, g \in C(X) \). The absolute convexity of \( J \) implies that \( |h| \in J \). Consequently \( |h| \in J \cap C(X) \). But since \( |f| \leq (f^2 + g^2)^{1/2} = |h| \), it follows again due to the absolute convexity of \( P(X, \mathbb{C}) \) as a subring of \( C(X, \mathbb{C}) \) that \( f \in P(X, \mathbb{C}) \). We further use absolute convexity of \( J \) in \( P(X, \mathbb{C}) \) to assert that \( f \in J \). Analogously \( g \in J \). Thus \( h = f + ig \in [J \cap C(X)]_c \). Therefore \( J \subseteq [J \cap C(X)]_c \). \( \square \)
Remark 2.11. For any $P(X, \mathbb{C}) \in \Sigma(X, \mathbb{C})$, the assignment $I \mapsto I_c$ provides a one-to-one correspondence between the absolutely convex ideals of $P(X, \mathbb{C}) \cap C(X)$ and those of $P(X, \mathbb{C})$.

The following theorem gives a one-to-one correspondence between the prime ideals of $P(X, \mathbb{C})$ and those of $P(X, \mathbb{C}) \cap C(X)$.

**Theorem 2.12.** Let $P(X, \mathbb{C})$ be member of $\Sigma(X, \mathbb{C})$. An ideal $J$ of $P(X, \mathbb{C})$ is prime if and only if there exists a prime ideal $Q$ in $P(X, \mathbb{C}) \cap C(X)$ such that $J = Q_c$.

**Proof.** Let $J$ be a prime ideal in $P(X, \mathbb{C})$ and let $Q = J \cap C(X)$ and $A(X) = P(X, \mathbb{C}) \cap C(X)$. Then $Q$ is a prime ideal in the ring $A(X)$. It is easy to see that $Q_c \subseteq J$. To prove the reverse containment, let $h = f + ig \in J$, where $f, g \in P(X, \mathbb{C})$. Note that $P(X, \mathbb{C}) = [A(X)]_c$ by Theorem 2.3. Hence $f, g \in A(X)$ and therefore $f - ig \in P(X, \mathbb{C})$. As $J$ is an ideal of $P(X, \mathbb{C})$, it follows that $(f + ig)(f - ig) \in J$ i.e., $f^2 + g^2 \in J \cap C(X) = Q$. Since $Q$ is a prime ideal in $A(X)$, we can apply Theorem 2.4(b), yielding $f^2 \in Q$ and hence $f \in Q$. Analogously $g \in Q$. Thus $h \in Q_c$. Therefore $J \subseteq Q_c$.

To prove the converse of this theorem, let $Q$ be a prime ideal in $A(X)$. It follows from Theorem 2.8 that $Q_c = \{h \in P(X, \mathbb{C}) : |h| \in Q\}$ and therefore $Q_c$ is a prime ideal in $P(X, \mathbb{C})$. Finally we note that $Q_c \cap C(X) = Q$. \qed

Remark 2.13. For any $P(X, \mathbb{C}) \in \Sigma(X, \mathbb{C})$, the collection of all prime ideals in $P(X, \mathbb{C})$ is precisely $\{Q_c : Q$ is a prime ideal in $P(X, \mathbb{C}) \cap C(X)\}$.

Remark 2.14. The collection of all minimal prime ideals in $P(X, \mathbb{C})$ is precisely $\{Q_c : Q$ is a minimal prime ideal in $P(X, \mathbb{C}) \cap C(X)\}$. [This follows from Remark 2.13 and Theorem 2.7].

**Theorem 2.15.** For any $P(X, \mathbb{C}) \in \Sigma(X, \mathbb{C})$, the collection of all maximal ideals in $P(X, \mathbb{C})$ is $\{M_c : M$ is a maximal ideal of $P(X, \mathbb{C}) \cap C(X)\}$.

**Proof.** Let $M$ be a maximal ideal in $P(X, \mathbb{C}) \cap C(X) = A(X)$. Then by Theorem 2.12, $M_c$ is a prime ideal in $P(X, \mathbb{C})$. Suppose that $M_c$ is not a maximal ideal in $P(X, \mathbb{C})$, then there exists a prime ideal $T$ in $P(X, \mathbb{C})$ such that $M_c \subseteq T$. By Remark 2.11, there exists a prime ideal $P$ in $A(X)$ such that $J = P_c$. So $M_c \subseteq P_c$. This implies in view of Theorem 2.5 that $M \subseteq P$, a contradiction to the maximality of $M$ in $A(X)$.

Conversely, let $J$ be a maximal ideal of $P(X, \mathbb{C})$. In particular $J$ is a prime ideal in this ring. By Remark 2.13, $J = Q_c$ for some prime ideal $Q$ in $A(X)$. We claim that $Q$ is a maximal ideal in $A(X)$. Suppose not; then $Q \subseteq K$ for some proper ideal $K$ in $A(X)$. Then by Theorem 2.7, $Q_c \subseteq K_c$ and $K_c$ a proper ideal in $P(X, \mathbb{C})$; this contradicts the maximality of $J = Q_c$. \qed

We next prove analogous of Remark 2.13 and Theorem 2.15 for two important classes of ideals viz $z$-ideals and $z^r$-ideals in $P(X, \mathbb{C}) \in \Sigma(X, \mathbb{C})$. These ideals are defined as follows.
2.4(b) that, for any maximal $I$ in $R$ is called a $z$-ideal (respectively $z^{\circ}$-ideal) if for each $a \in I$, $M_a \subseteq I$ (respectively $P_a \subseteq I$).

This notion of $z$-ideals is consistent with the notion of $z$-ideal in $C(X)$ (see [17, 4A5]). Since each prime ideal in an intermediate ring $A(X) \in \Sigma(X)$ is absolutely convex (Theorem 2.4(b)), it follows from Theorem 2.10(ii) and Remark 2.13 that each prime ideal in $P(X, \mathbb{C}) \in \Sigma(X, \mathbb{C})$ is absolutely convex. In particular each maximal ideal is absolutely convex. Now if $I$ is a $z$-ideal in $P(X, \mathbb{C}) \in \Sigma(X, \mathbb{C})$ and $|f| \leq |g|$, $g \in I, f \in P(X, \mathbb{C})$, then $M_g \subseteq I$. Let $M$ be a maximal ideal in $P(X, \mathbb{C})$ containing $g$. It follows due to the absolute convexity of $M$ that $f \in M$. Therefore $f \in M_g \subseteq I$. Thus each $z$-ideal in $P(X, \mathbb{C})$ is absolutely convex. Analogously it can be proved that each $z^{\circ}$-ideal in $P(X, \mathbb{C})$ is absolutely convex.

The following subsidiary result can be proved using routine arguments.

**Lemma 2.17.** For any family $\{I_\alpha : \alpha \in \Lambda\}$ of ideals in an intermediate ring $A(X) \in \Sigma(X)$, $(\bigcap_{\alpha \in \Lambda} I_\alpha)_{e} = \bigcap_{\alpha \in \Lambda} (I_\alpha)_{e}$.

**Theorem 2.18.** An ideal $J$ in a ring $P(X, \mathbb{C}) \in \Sigma(X, \mathbb{C})$ is a $z$-ideal in $P(X, \mathbb{C})$ if and only if there exists a $z$-ideal $I$ in $P(X, \mathbb{C}) \cap C(X)$ such that $J = I_e$.

**Proof.** First assume that $J$ is a $z$-ideal in $P(X, \mathbb{C})$. Let $I = J \cap C(X)$. Since $J$ is absolutely convex, it follows from Theorem 2.10(iii) that $J = I_e$. We show that $I$ is a $z$-ideal in $P(X, \mathbb{C}) \cap C(X)$. Choose $f \in I$. Suppose $\{M_\alpha : \alpha \in \Lambda\}$ is the set of all maximal ideals in the ring $P(X, \mathbb{C}) \cap C(X)$ which contain $f$. It follows from Theorem 2.15 that $\{(M_\alpha)_{e} : \alpha \in \Lambda\}$ is the set of all maximal ideals in $P(X, \mathbb{C})$ containing $f$. Since $f \in J$ and $J$ is a $z$-ideal in $P(X, \mathbb{C})$, it follows that $\bigcap_{\alpha \in \Lambda} (M_\alpha)_{e} \subseteq J$. This implies in the view of Lemma 2.17 that $(\bigcap_{\alpha \in \Lambda} M_\alpha)_{e} \cap C(X) \subseteq I$ if and only if $|f| = 1$. Thus it is proved that $I$ is a $z$-ideal in $P(X, \mathbb{C}) \cap C(X)$.

Conversely, let $I$ be a $z$-ideal in the ring $P(X, \mathbb{C}) \cap C(X)$. We shall prove that $I_e$ is a $z$-ideal in $P(X, \mathbb{C})$. We recall from Theorem 2.3 that $[P(X, \mathbb{C}) \cap C(X)]_{e} = P(X, \mathbb{C})$. Choose $f$ from $I_e$. From Theorem 2.8, it follows that (taking care of the fact that each $z$-ideal in $P(X, \mathbb{C})$ is absolutely convex) $|f| = 1$. Let $\{N_\beta : \beta \in \Lambda^{*}\}$ be the set of all maximal ideals in $P(X, \mathbb{C}) \cap C(X)$ which contain the function $|f|$. The hypothesis that $I$ is a $z$-ideal in $P(X, \mathbb{C}) \cap C(X)$ therefore implies that $\bigcap_{\beta \in \Lambda^{*}} N_\beta \subseteq I$. This further implies in view of Lemma 2.17 that $\bigcap_{\beta \in \Lambda^{*}} (N_\beta)_{e} \subseteq I_e$. Again it follows from Theorem 2.8 that, for any maximal ideal $M$ in $P(X, \mathbb{C}) \cap C(X)$ and any $g \in P(X, \mathbb{C})$, $g \in M_e$ if and only if $|g| \in M$. Thus for any $\beta \in \Lambda^{*}, |f| \in N_\beta$ if and only if $f \in (N_\beta)_{e}$. This means that $\{(N_\beta)_{e} : \beta \in \Lambda^{*}\}$ is the collection of maximal ideals in $P(X, \mathbb{C})$ which contain $f$, and we have already observed that $f \in \bigcap_{\beta \in \Lambda^{*}} (N_\beta)_{e} \subseteq I_e$. Consequently $I_e$ is a $z$-ideal in $P(X, \mathbb{C})$. \hfill \Box
If we use the result embodied in Remark 2.14 and take note of the fact that each minimal prime ideal in \( P(X, \mathbb{C}) \) is absolutely convex and argue as in the proof of Theorem 2.18, we get the following proposition:

**Theorem 2.19.** An ideal \( J \) in a ring \( P(X, \mathbb{C}) \in \Sigma(X, \mathbb{C}) \) is a \( z^0 \)-ideal in \( P(X, \mathbb{C}) \) if and only if there exists a \( z^0 \)-ideal \( I \) in \( P(X, \mathbb{C}) \cap C(X) \) such that \( J = I_c \).

An ideal \( J \) in \( P(X, \mathbb{C}) \in \Sigma(X, \mathbb{C}) \) is called fixed if \( \bigcap_{f \in J} Z(f) \neq \emptyset \). The following proposition is a straightforward consequence of Theorem 2.6.

**Theorem 2.20.** An ideal \( J \) in a ring \( P(X, \mathbb{C}) \in \Sigma(X, \mathbb{C}) \) is a fixed ideal in \( P(X, \mathbb{C}) \) if and only if \( J \cap C(X) \) is a fixed ideal in \( P(X, \mathbb{C}) \cap C(X) \).

We recall that a space \( X \) is called an almost \( P \) space if every non-empty \( G_\delta \) subset of \( X \) has non-empty interior. These spaces have been characterized via \( z \)-ideals and \( z^0 \)-ideals in the ring \( C(X) \) in [8]. We would like to mention that the same class of spaces have witnessed a very recent characterization in terms of fixed maximal ideals in a given intermediate ring \( A(X) \in \Sigma(X) \). We reproduce below these two results to make the paper self-contained.

**Theorem 2.21** ([8]). \( X \) is an almost \( P \) space if and only if each maximal ideal in \( C(X) \) is a \( z^0 \)-ideal if and only if each \( z \)-ideal in \( C(X) \) is a \( z^0 \)-ideal.

**Theorem 2.22** ([12]). Let \( A(X) \in \Sigma(X) \) be an intermediate ring of real-valued continuous functions on \( X \). Then \( X \) is an almost \( P \) space if and only if each fixed maximal ideal \( M^I_A = \{g \in A(X) : g(p) = 0\} \) of \( A(X) \) is a \( z^0 \)-ideal.

It is further realised in [12] that if \( X \) is an almost \( P \) space, then the statement of Theorem 2.21 cannot be improved by replacing \( C(X) \) by an intermediate ring \( A(X) \), different from \( C(X) \). Indeed it is shown in [12, Theorem 2.4] that if an intermediate ring \( A(X) \neq C(X) \), then there exists a maximal ideal in \( A(X) \) (which is incidentally also a \( z \)-ideal in \( A(X) \)), which is not a \( z^0 \)-ideal in \( A(X) \).

We record below the complex analogue of the above results.

**Theorem 2.23.** \( X \) is an almost \( P \) space if and only if each maximal ideal of \( C(X, \mathbb{C}) \) is a \( z^0 \)-ideal if and only if each \( z \)-ideal in \( C(X, \mathbb{C}) \) is a \( z^0 \)-ideal.

**Proof.** This follows from combining Theorems 2.15, 2.18, 2.19, and 2.21. \( \square \)

**Theorem 2.24.** Let \( P(X, \mathbb{C}) \in \Sigma(X, \mathbb{C}) \). Then \( X \) is almost \( P \) if and only if each fixed maximal ideal \( M^I_P = \{g \in P(X, \mathbb{C}) : g(p) = 0\} \) of \( P(X, \mathbb{C}) \) is a \( z^0 \)-ideal.

**Proof.** This follows from combining Theorems 2.15, 2.20, and 2.22. \( \square \)

**Theorem 2.25.** Let \( X \) be an almost \( P \) space and let \( P(X, \mathbb{C}) \) be a member of \( \Sigma(X, \mathbb{C}) \) such that \( P(X, \mathbb{C}) \subseteq C(X, \mathbb{C}) \). Then there exists a maximal ideal in \( P(X, \mathbb{C}) \), which is not a \( z^0 \)-ideal in \( P(X, \mathbb{C}) \).
Thus, within the class of almost $P$-spaces $X$, $C(X, \mathbb{C})$ is characterized amongst all the intermediate rings $P(X, \mathbb{C})$ of $\Sigma(X, \mathbb{C})$ by the requirement that $z$-ideals and $z^\circ$-ideals (equivalently maximal ideals and $z^\circ$-ideals) in $P(X, \mathbb{C})$ are one and the same.

**Proof.** This follows from combining Theorems 2.15, 2.18, and 2.19 of this article together with [12, Theorem 2.4].

We recall the classical result that $X$ is a $P$ space if and only if $C(X)$ is a von-Neumann regular ring meaning that each prime ideal in $C(X)$ is maximal. Incidentally the following fact was rather recently established:

**Theorem 2.26** ([3, 20, 12]). If $A(X) \in \Sigma(X)$ is different from $C(X)$, then $A(X)$ is never a regular ring.

Theorems 2.12, 2.15, and 2.26 yield in a straightforward manner the following result:

**Theorem 2.27.** If $P(X, \mathbb{C}) \in \Sigma(X, \mathbb{C})$ is a proper subring of $C(X, \mathbb{C})$, then $P(X, \mathbb{C})$ is not a von-Neumann regular ring.

It is well-known that if $P$ is a non maximal prime ideal in $C(X)$ and $M$ is the unique maximal ideal containing $P$, then the set of all prime ideals in $C(X)$ that lie between $P$ and $M$ makes a Dedekind complete chain containing no fewer than $2^{\aleph_1}$ many members (see [17, Theorem 14.19]). If we use this standard result and combine with Theorems 2.7, 2.12, and 2.15, we obtain the complex-version of this fact:

**Theorem 2.28.** Suppose $P$ is a non maximal prime ideal in the ring $C(X, \mathbb{C})$. Then there exists a unique maximal ideal $M$ containing $P$ in this ring. Furthermore, the collection of all prime ideals that are situated between $P$ and $M$ constitutes a Dedekind complete chain containing at least $2^{\aleph_1}$ many members.

Thus for all practical purposes (say for example when $X$ is not a $P$ space), $C(X, \mathbb{C})$ is far from being a Noetherian ring. Incidentally we shall decide the Noetherianness condition of $C(X, \mathbb{C})$ by deducing it from a result in Section 4; in particular, we show that $C(X, \mathbb{C})$ is Noetherian if and only if $X$ is a finite set.

### 3. Structure spaces of intermediate rings

We need to recall a few technicalities associated with the hull-kernel topology on the set of all maximal ideals $M(A)$ of a commutative ring $A$ with unity. If we set for any element $a$ of $A$, $M(A)_a = \{ M \in M(A) : a \in M \}$, then the family $\{ M(A)_a : a \in A \}$ constitutes a base for closed sets of the hull-kernel topology on $M(A)$. We may write $M_a$ for $M(A)_a$ when context is clear. The set $M(A)$ equipped with this hull-kernel topology is called the *structure space* of the ring $A$. 

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For any subset $M \subseteq \mathcal{M}(A)$, its closure $\overline{M}$ in this topology is given by:

\[ \overline{M} = \{ M \in \mathcal{M}(A) : M \supseteq \bigcap_{x \in X} M_{x} \}. \]

For further information on this topology, see [17, 7M].

Following the terminology of [14], by a (Hausdorff) compactification of a Tychonoff space we mean a pair $(\alpha, \alpha X)$, where $\alpha X$ is a compact Hausdorff space and $\alpha : X \to \alpha X$ a topological embedding with $\alpha(X)$ dense in $\alpha X$. For simplicity, we often designate such a pair by the notation $\alpha X$. Two compactifications $\alpha X$ and $\gamma X$ of $X$ are called topologically equivalent if there exists a homeomorphism $\psi : \alpha X \to \gamma X$ with the property $\psi \circ \alpha = \gamma$. A compactification $\alpha X$ of $X$ is said to possess the extension property if given a compact Hausdorff space $Y$ and a continuous map $f : X \to Y$, there exists a continuous map $f^\alpha : \alpha X \to Y$ with the property $f^\alpha \circ \alpha = f$. It is well known that the Stone–Čech compactification $\beta X$ of $X$ or more formally the pair $(\epsilon, \beta X)$, where $\epsilon$ is the evaluation map on $X$ induced by $C^*(X)$ defined by the formula:

\[ e(x) = (f(x) : f \in C^*(X)) \]

such that $\epsilon : X \to \mathbb{R}^{C^*(X)}$, enjoys the extension property. Furthermore this extension property characterizes $\beta X$ amongst all the compactifications of $X$ in the sense that whenever a compactification $\alpha X$ of $X$ has extension property, it is topologically equivalent to $\beta X$. For more information on these topics, see [14, Chapter 1].

The structure space $\mathcal{M}(A(X))$ of an arbitrary intermediate ring $A(X) \in \Sigma(X)$ has been proved to be homeomorphic to $\beta X$, independently by the authors in [21] and [22]. Nevertheless we offer yet another independent technique to establish a modified version of the same fact by using the above terminology of [14].

**Theorem 3.1.** Let $\eta_A : X \to \mathcal{M}(A(X))$ be the map defined by $\eta_A(x) = M^x = \{ g \in A(X) : g(x) = 0 \}$ (a fixed maximal ideal in $A(X)$). Then the pair $(\eta_A, \mathcal{M}(A(X)))$ is a (Hausdorff) compactification of $X$, which further satisfies the extension property. Hence the pair $(\eta_A, \mathcal{M}(A(X)))$ is topologically equivalent to the Stone–Čech compactification $\beta X$ of $X$.

**Proof.** Since $X$ is Tychonoff, $\eta_A$ is one-to-one. Also $d_M(\mathcal{M}(A(X))) = \{ M \in \mathcal{M}(A(X)) : M \supseteq \bigcap_{x \in X} M^x_A \} = \{ M \in \mathcal{M}(A(X)) : M \supseteq \{ 0 \} \} = \mathcal{M}(A(X))$. It follows from a result proved in Theorem 3.3 and Theorem 3.4 [23] that $\mathcal{M}(A(X))$ is a compact Hausdorff space and $\eta_A$ is an embedding. Thus $(\eta_A, \mathcal{M}(A(X)))$ is a compactification of $X$. To prove that this compactification of $X$ possesses the extension property we take a compact Hausdorff space $Y$ and a continuous map $f : X \to Y$. It suffices to define a continuous map $f^{\beta A} : \mathcal{M}(A(X)) \to Y$ with the property that $f^{\beta A} \circ \eta_A = f$. Let $M$ be any member of $\mathcal{M}(A(X))$ i.e. $M$ is a maximal ideal of the ring $A(X)$. Define $\hat{M} = \{ g \in C(Y) : g \circ f \in M \}$. Note that if $g \in C(Y)$ then $g \circ f \in C(X)$. Further note that since $Y$ is compact and $g \in C(Y)$, $g$ is bounded i.e. $g(Y)$ is a bounded subset of $\mathbb{R}$. It follows that $(g \circ f)(X)$ is a bounded subset of $\mathbb{R}$ and hence $g \circ f \in C^*(X)$. Consequently $g \circ f \in A(X)$. Thus the definition of $\hat{M}$ is without any ambiguity. It is easy to see that $\hat{M}$ is an ideal of $C(Y)$. It follows, since $M$ is a maximal ideal and therefore a prime ideal of $A(X)$, that
$\hat{M}$ is a prime ideal of $C(Y)$. Since $C(Y)$ is a Gelfand ring, $\hat{M}$ can be extended to a unique maximal ideal $N$ in $C(Y)$. Since $Y$ is compact, $N$ is fixed (see [17, Theorem 4.11]). Thus we can write: $N = N_y = \{g \in C(Y) : g(y) = 0\}$ for some $y \in Y$. We observe that $y \in \bigcap_{g \in \hat{M}} Z(g)$. Indeed $\bigcap_{g \in \hat{M}} Z(g) = \{y\}$ for if $y_1, y_2 \in \bigcap_{g \in \hat{M}} Z(g)$, for $y_1 \neq y_2$, then $\hat{M} \subseteq N_{y_1}$ and $\hat{M} \subseteq N_{y_2}$ which is impossible as $N_{y_1} \neq N_{y_2}$ and $C(Y)$ is a Gelfand ring. We then set $f^{\beta_A}(M) = y$. Note that $\{f^{\beta_A}(M)\} = \bigcap_{g \in \hat{M}} Z(g)$. Thus $f^{\beta_A} : \mathcal{M}(A(X)) \to Y$ is a well defined map. Now choose $x \in X$ and then $g \in \hat{M}_x$; then $g \circ f \in M_x^\circ$, which implies that $(g \circ f)(x) = 0$. Consequently $f(x) \in Z(g)$ for each $g \in M_x^\circ$. On the other hand $\{f^{\beta_A}(M_x^\circ)\} = \bigcap_{g \in \hat{M}_x} Z(g)$. This implies that $f^{\beta_A}(M_x^\circ) = f(x)$; in other words $(f^{\beta_A} \circ \eta_A)(x) = f(x)$ and this relation is true for each $x \in X$. Hence $f^{\beta_A} \circ \eta_A = f$.

Now towards the proof of the continuity of the map $f^{\beta_A}$, choose $M \in \mathcal{M}(A(X))$ and a neighbourhood $W$ of $f^{\beta_A}(M)$ in the space $Y$. In a Tychnoff space every neighbourhood of a point $x$ contains a zero set neighbourhood of $x$, which contains, a co-zero set neighbourhood of $x$. So there exist some $g_1, g_2 \in C(Y)$, such that $f^{\beta_A}(M) \in Y \setminus Z(g_1) \subseteq Z(g_2) \subseteq W$. It follows that $g_1 g_2 = 0$ as $Z(g_1) \cup Z(g_2) = Y$ which means that $Z(g_1 g_2) = Y$. Furthermore $f^{\beta_A}(M) \notin Z(g_1)$. Since $\{f^{\beta_A}(M)\} = \bigcap_{g \in \hat{M}} Z(g)$, as observed earlier, we then have $g_1 \notin \hat{M}$. This means that $g_1 \circ f \notin M$. In other words $M \in \mathcal{M}(A(X)) \setminus \mathcal{M}_{g_1 \circ f}$, which is an open neighbourhood of $M$ in $\mathcal{M}(A(X))$. We shall check that $f^{\beta_A}(\mathcal{M}(A(X)) \setminus \mathcal{M}_{g_1 \circ f}) \subseteq W$ and that settles the continuity of $f^{\beta_A}$ at $M$. Towards that end, choose a maximal ideal $N \in \mathcal{M}(A(X)) \setminus \mathcal{M}_{g_1 \circ f}$. This means that $N \notin \mathcal{M}_{g_1 \circ f}$, i.e. $g_1 \circ f \notin N$. Thus $g_1 \notin N$. But as $g_1 g_2 = 0$ and $\hat{N}$ is prime ideal in $C(Y)$, it must be that $g_2 \notin \hat{N}$. Since $\{f^{\beta_A}(N)\} = \bigcap_{g \in \hat{N}} Z(g)$, it follows that $f^{\beta_A}(N) \notin Z(g_2) \subseteq W$.

To achieve the complex analogue of the above mentioned theorem, we need to prove the following proposition, which is by itself a result of independent interest.

**Theorem 3.2.** Let $A(X) \in \Sigma(X)$. Then the map $\psi_A : \mathcal{M}([A(X)]_c) \to \mathcal{M}(A(X))$ mapping $M \to M \cap A(X)$ is a homeomorphism from the structure space of $[A(X)]_c$ onto the structure space of $A(X)$.

**Proof.** That the above map $\psi_A$ is a bijection between the structure spaces of $[A(X)]_c$ and $A(X)$ follows from Theorems 2.3, 2.6, 2.7, and 2.15. Recall (same notation as before) that $\mathcal{M}([A(X)]_c)_f$ is the set of maximal ideals in the ring $[A(X)]_c$ containing the function $f \in [A(X)]_c$. A typical basic closed set in the structure space $\mathcal{M}([A(X)]_c)$ is given by $\mathcal{M}([A(X)]_c)_h$ where $h \in [A(X)]_c$. Note that $\mathcal{M}([A(X)]_c)_h = \{J \in \mathcal{M}([A(X)]_c) : h \in J\}$. So for $h \in [A(X)]_c$, $J \in \mathcal{M}([A(X)]_c)_h$ if and only if $h \in J$, and this is true in view of Theorem 2.8 and the absolute convexity of maximal ideals (see Theorem 2.4(b) of the present article) if and only if $|h| \in J \cap A(X)$, and this holds when and only when $J \cap A(X) \in \mathcal{M}(A(X))|_h|$, which is a basic closed set in the structure space.
\[
\mathcal{M}(A(X)) \text{ of the ring } A(X). \text{ Thus } \\
\psi_A[M([A(X)]_c)_{|h|}] = \mathcal{M}(A(X))_{|h|} 
\]

Therefore \(\psi_A\) carries a basic closed set in the domain space onto a basic closed set in the range space. Now for a maximal ideal \(N\) in \(A(X)\) and a function \(g \in A(X)\), \(g\) belongs to \(N\) if and only if \(|g| \in N\), because of the absolutely convexity of a maximal ideal in an intermediate ring. Consequently \(\mathcal{M}(A(X))_g = \mathcal{M}(A(X))_{|g|}\) for any \(g \in A(X)\). Hence from relation (3.1), we get: \(\psi_A[M([A(X)]_c)_g] = \mathcal{M}(A(X))_g\) which implies that \(\psi_A^*[M(A(X))_g] = \mathcal{M}([A(X)]_c)_g\). Thus \(\psi_A^{-1}\) carries a basic closed set in the structure space \(\mathcal{M}(A(X))\) onto a basic closed in the structure space \(\mathcal{M}([A(X)]_c)\). Altogether \(\psi_A\) becomes a homeomorphism. □

For any \(x \in X\) and \(A(X) \in \Sigma(X)\), set \(M^*_A[C] = \{b \in [A(X)]_c : h(x) = 0\}\). It is easy to check by using standard arguments, such as those employed to prove the textbook theorem [17, Theorem 4.1], that \(M^*_A[C]\) is a fixed maximal in \([A(X)]_c\) and \(M^*_A[C] \cap A(X) = M^*_A = \{g \in A(X) : g(x) = 0\}\). Let \(\zeta_A : X \to \mathcal{M}([A(X)]_c)\) be the map defined by: \(\zeta_A(x) = M^*_A[C]\). Then we have the following results.

**Theorem 3.3.** \((\zeta_A, \mathcal{M}([A(X)]_c))\) is a Hausdorff compactification of \(X\). Furthermore \((\psi_A \circ \zeta_A)(x) = \eta_A(x)\) for all \(x \in X\). Hence \((\zeta_A, \mathcal{M}([A(X)]_c))\) is topologically equivalent to the Hausdorff compactification \((\eta_A, \mathcal{M}(A(X)))\) as considered in Theorem 3.1. Consequently \((\zeta_A, \mathcal{M}([A(X)]_c))\) turns out to be topologically equivalent to the Stone-Čech compactification \(\beta X\) of \(X\).

**Proof.** Since \(\mathcal{M}(A(X))\) is Hausdorff [23], it follows from Theorem 3.2 that \(\mathcal{M}([A(X)]_c)\) is a Hausdorff space. Now by following closely the arguments made at the very beginning of the proof of Theorem 3.1, one can easily see that \((\zeta_A, \mathcal{M}([A(X)]_c))\) is a Hausdorff compactification of \(X\). The second part of the theorem is already realised in Theorem 3.2. The third part of the present theorem also follows from Theorem 3.2. □

**Definition 3.4.** An intermediate ring \(A(X) \in \Sigma(X)\) is called \(C\)-type in [16], if it is isomorphic to \(C(Y)\) for some Tychonoff space \(Y\).

In [16], the authors have shown that if \(I\) is an ideal of the ring \(C(X)\), then the linear sum \(C^*(X) + I\) is a \(C\)-type ring and of course \(C^*(X) + I \in \Sigma(X)\). Recently the authors in [1] have realised that these are the only \(C\)-type intermediate rings of real-valued continuous functions on \(X\) if and only if \(X\) is pseudocompact. We now show that the complex analogues of all these results are also true. We reproduce the following result established in [15], which will be needed for this purpose.

**Theorem 3.5.** A ring \(A(X) \in \Sigma(X)\) is \(C\)-type if and only if \(A(X)\) is isomorphic to the ring \(C(v_A X)\), where \(v_A X = \{p \in \beta X : f^*(p) \not\in \mathbb{R} \text{ for each } f \in A(X)\}\) and \(f^* : \beta X \to \mathbb{R} \cup \{\infty\}\) is the Stone extension of the function \(f\).
We extend the notion of C-type ring to rings of complex-valued continuous functions: a ring \( P(X, \mathbb{C}) \in \Sigma(X, \mathbb{C}) \) is a C-type ring if it is isomorphic to a ring \( C(Y, \mathbb{C}) \) for some Tychonoff space \( Y \).

**Theorem 3.6.** Suppose \( A(X) \in \Sigma(X) \) is a C-type intermediate ring of real-valued continuous functions on \( X \). Then \( [A(X)]_c \) is a C-type intermediate ring of complex-valued continuous functions on \( X \).

**Proof.** Since \( A(X) \) is a C-type intermediate ring by Theorem 3.5, there exists an isomorphism \( \psi : A(X) \to C(v_A X) \). Let \( \hat{\psi} : [A(X)]_c \to C(v_A X, \mathbb{C}) \) be defined as follows: \( \hat{\psi}(f + ig) = \psi(f) + i\psi(g) \), where \( f, g \in A(X) \). It is not hard to check that \( \hat{\psi} \) is an isomorphism from \( [A(X)]_c \) onto \( C(v_A X, \mathbb{C}) \). \( \square \)

**Theorem 3.7.** Let \( I \) be a \( z \)-ideal in \( C(X, \mathbb{C}) \). Then \( C^*(X, \mathbb{C}) + I \) is a C-type intermediate ring of complex-valued continuous functions on \( X \). Furthermore these are the only C-type rings lying between \( C^*(X, \mathbb{C}) \) and \( C(X, \mathbb{C}) \) if and only if \( X \) is pseudocompact.

**Proof.** As mentioned above, it is proved in [16] that for any ideal \( J \) in \( C(X) \), \( C^*(X) + J \) is a C-type intermediate ring of real-valued continuous functions on \( X \). In light of this and Theorem 3.6, it is sufficient to prove for the first part of this theorem that \( C^*(X, \mathbb{C}) + I = [C^*(X)]_c + I \cap C(X) \). Towards proving that, let \( f, g \in C^*(X) + I \cap C(X) \). We can write \( g = g_1 + g_2 \) where \( g_1 \in C^*(X) \) and \( g_2 \in I \cap C(X) \). It follows that \( ig_1 \in C^*(X, \mathbb{C}) \) and \( ig_2 \in I \) and this implies that \( i(g_1 + g_2) \in C^*(X, \mathbb{C}) + I \). Thus \( f + ig \in C^*(X) + I \).

Hence \( [C^*(X) + I \cap C(X)]_c \subseteq C^*(X, \mathbb{C}) + I \). To prove the reverse inclusion relation, let \( h_1 + h_2 \in C^*(X, \mathbb{C}) + I \), where \( h_1 \in C^*(X, \mathbb{C}) \) and \( h_2 \in I \). We can write \( h_1 = f_1 + ig_1, \quad h_2 = f_2 + ig_2 \), where \( f_1, f_2, g_1, g_2 \in C(X) \). Since \( h_1 \in C^*(X, \mathbb{C}) \), it follows that \( f_1, g_1 \in C^*(X) \). Thus \( |f_2| \leq |h_2| \) and \( h_2 \in I \). This implies, because of the absolute convexity of the \( z \)-ideal \( I \) in \( C(X, \mathbb{C}) \), that \( f_2 \in I \). Analogously \( g_2 \in I \). It is now clear that \( f_1 + f_2 \in C^*(X) + I \cap C(X) \) and \( g_1 + g_2 \in C^*(X) + I \cap C(X) \). Thus \( h_1 + h_2 = (f_1 + f_2) + (g_1 + g_2) \in [C^*(X) + I \cap C(X)]_c \). Hence \( C^*(X, \mathbb{C}) + I \subseteq [C^*(X) + I \cap C(X)]_c \).

To prove the second part of the theorem, we first observe that if \( X \) is pseudocompact, then there is practically nothing to prove. Assume therefore that \( X \) is not pseudocompact. Hence by [1], there exists an \( A(X) \in \Sigma(X) \) such that \( A(X) \) is a C-type ring but \( A(X) \neq C^*(X) + J \) for any ideal \( J \) in \( C(X) \). It follows from Theorem 3.6 that \( [A(X)]_c \) is a C-type intermediate ring of complex-valued continuous functions belonging to the family \( \Sigma(X, \mathbb{C}) \). We assert that there does not exist any \( z \)-ideal \( I \) in \( C(X, \mathbb{C}) \) with the relation: \( C^*(X, \mathbb{C}) + I = [A(X)]_c \) and that finishes the present theorem. Suppose towards a contradiction, there exists a \( z \)-ideal \( I \) in \( C(X, \mathbb{C}) \) such that \( C^*(X, \mathbb{C}) + I = [A(X)]_c \). Now from the proof of the first part of this theorem, we have already settled that \( C^*(X, \mathbb{C}) + I = [C^*(X) + I \cap C(X)]_c \). Consequently \( [C^*(X) + I \cap C(X)]_c = [A(X)]_c \), which yields \( [C^*(X) + I \cap C(X)]_c \cap C(X) = [A(X)]_c \cap C(X) \), and hence \( C^*(X) + I \cap C(X) = A(X) \), a contradiction. \( \square \)
We shall conclude this section after incorporating a purely algebraic result pertaining to the residue class field of $C(X, \mathbb{C})$ modulo a maximal ideal in the same field.

For each $a = (a_1, a_2, \ldots, a_n) \in \mathbb{C}^n$ if $P_1 a, P_2 a, \ldots, P_n a$ are the zeroes of the polynomial $P_a(\lambda) = \lambda^n + a_1 \lambda^{n-1} + \cdots + a_n$, ordered so that $|P_1 a| \leq |P_2 a| \leq \cdots \leq |P_n a|$, then by following closely the arguments of [17, 13.3(a)], the following result can be obtained.

**Theorem 3.8.** For each $k$, the function $\mathcal{P}_k : \mathbb{C}^n \mapsto \mathbb{C}$, described above, is continuous.

By employing the main argument of [17, Theorem 13.4], we obtain the following proposition as a consequence of Theorem 3.8.

**Theorem 3.9.** For any maximal ideal $N$ in $C(X, \mathbb{C})$, the residue class field $C(X, \mathbb{C})/N$ is algebraically closed.

We recall from Theorem 2.15 that the assignment $M \mapsto M_\varepsilon$ establishes a one-to-one correspondence between maximal ideals in $C(X)$ and those in $C(X, \mathbb{C})$. Let $\phi : C(X)/M \mapsto C(X, \mathbb{C})/M_\varepsilon$ be the induced assignment between the corresponding residue class fields, explicitly $\phi(f + M) = f + M_\varepsilon$ for each $f \in C(X)$. It is easy to check that $\phi$ is a ring homomorphism and is one-to-one because if $f + M_\varepsilon = g + M_\varepsilon$, then $f - g \in M_\varepsilon \cap C(X) = M$ and hence $f + M = g + M$. Furthermore, if $f + M_\varepsilon$ is a root of the polynomial $\lambda^2 - 2(f + M_\varepsilon) + (f^2 + g^2 + M_\varepsilon)$ over the field $\phi(C(X)/M)$. Identifying $C(X)/M$ with $\phi(C(X)/M)$, and taking note of Theorem 3.9 we get the following result.

**Theorem 3.10.** For any maximal ideal $M$ in $C(X)$, the residue class field $C(X, \mathbb{C})/M_\varepsilon$ is the algebraic closure of $C(X)/M$.

4. IDEALS OF THE FORM $C_P(X, \mathbb{C})$ AND $C_P^X(X, \mathbb{C})$

Let $\mathcal{P}$ be an ideal of closed sets in $X$. We set $C_P(X, \mathbb{C}) = \{ f \in C(X, \mathbb{C}) : cl_X(X \setminus Z(f)) \in \mathcal{P} \}$ and $C_P^X(X, \mathbb{C}) = \{ f \in C(X, \mathbb{C}) : \text{ for each } \varepsilon > 0 \text{ in } \mathbb{R}, \{x \in X : |f(x)| \geq \varepsilon \} \in \mathcal{P} \}$. These are the complex analogues of the rings, $C_P(X) = \{ f \in C(X) : cl_X(X \setminus Z(f)) \in \mathcal{P} \}$ and $C_P^X(X) = \{ f \in C(X) : \text{ for each } \varepsilon > 0, \{x \in X : |f(x)| \geq \varepsilon \} \in \mathcal{P} \}$ already introduced in [4] and investigated subsequently in [5], [12]. As in the real case, it is easy to check that $C_P(X, \mathbb{C})$ is a $z$-ideal in $C(X, \mathbb{C})$ with $C_P^X(X, \mathbb{C})$ just a subring of $C(X, \mathbb{C})$. Plainly we have: $C_P(X, \mathbb{C}) \cap C(X) = C_P(X)$ and $C_P^X(X, \mathbb{C}) \cap C(X) = C_P^X(X)$.

The following results need only routine verifications.

**Theorem 4.1.** For any ideal $\mathcal{P}$ of closed sets in $X$, $[C_P(X)]_c = \{ f + ig : f, g \in C_P(X) \}$ and $[C_P^X(X)]_c = C_P^X(X, \mathbb{C})$.

**Theorem 4.2.**

a) If $I$ is an ideal of the ring $C_P(X)$, then $I_c = \{ f + ig : f, g \in I \}$ is an ideal of $C_P(X, \mathbb{C})$ and $I_c \cap C_P(X) = I$. 
b) If $I$ is an ideal of the ring $C^P(X)$, then $I_c$ is an ideal of $C^P(X, \mathbb{C})$ and $I_c \cap C^P_\infty(X) = I$. 

We record below the following consequence of the above theorem.

**Theorem 4.3.** If $I_1 \subseteq I_2 \subseteq \cdots$ is a strictly ascending sequence of ideals in $C_P(X)$ (respectively $C^P_\infty(X)$), then $I_1 \subseteq I_2 \subseteq \cdots$ becomes a strictly ascending sequence of ideals in $C_P(X, \mathbb{C})$ (respectively $C^P_\infty(X, \mathbb{C})$).

The analogous results for a strictly descending sequence of ideals in both the rings $C_P(X)$ and $C^P_\infty(X)$ are also valid.

**Definition 4.4.** A space $X$ is called locally $P$ if each point of $X$ has an open neighbourhood $W$ such that $cl_X W \in P$.

Observe that if $P$ is the ideal of all compact sets in $X$, then $X$ is locally $P$ if and only if $X$ is locally compact.

Towards finding a condition for which $C_P(X, \mathbb{C})$ and $C^P_\infty(X, \mathbb{C})$ are Noetherian ring/Artinian rings, we reproduce a special version of a fact proved in [6]:

**Theorem 4.5 (from [6, Theorem 1.1]).** Let $P$ be an ideal of closed sets in $X$ and suppose $X$ is locally $P$. Then the following statements are equivalent:

1) $C_P(X)$ is a Noetherian ring.
2) $C_P(X)$ is an Artinian ring.
3) $C^P_\infty(X)$ is a Noetherian ring.
4) $C^P_\infty(X)$ is an Artinian ring.
5) $X$ is finite set.

We also note the following standard result of Algebra.

**Theorem 4.6.** Let $\{R_1, R_2, \ldots, R_n\}$ be a finite family of commutative rings with identity. The ideals of the direct product $R_1 \times R_2 \times \cdots \times R_n$ are exactly of the form $I_1 \times I_2 \times \cdots \times I_n$, where for $k = 1, 2, \ldots, n$, $I_k$ is an ideal of $R_k$.

Now if $X$ is a finite set, with say $n$ elements, then as it is Tychonoff, it is discrete space. Furthermore if $X$ is locally $P$, then clearly $P$ is the power set of $X$. Consequently $C_P(X, \mathbb{C}) = C^P_\infty(X, \mathbb{C}) = C(X, \mathbb{C}) = \mathbb{C}^n$, which is equal to the direct product of $\mathbb{C}$ with itself ‘$n$’ times. Since $\mathbb{C}$ is a field, it has just 2 ideals, hence by Theorem 4.6 there are exactly $2^n$ many ideals in the ring $\mathbb{C}^n$. Hence $C_P(X, \mathbb{C})$ and $C^P_\infty(X, \mathbb{C})$ are both Noetherian rings and Artinian rings. On the other hand if $X$ is an infinite space and is locally $P$ space then it follows from the Theorem 4.3 and Theorem 4.5 that neither of the two rings $C_P(X, \mathbb{C})$ and $C^P_\infty(X, \mathbb{C})$ is either Noetherian or Artinian. This leads to the following proposition as the complex analogue of Theorem 4.5.

**Theorem 4.7.** Let $P$ be an ideal of closed sets in $X$ and suppose $X$ is locally $P$. Then the following statements are equivalent:

1) $C_P(X, \mathbb{C})$ is a Noetherian ring.
2) $C_P(X, \mathbb{C})$ is an Artinian ring.
3) $C^n_c(X, \mathbb{C})$ is a Noetherian ring.
4) $C^n_c(X, \mathbb{C})$ is an Artinian ring.
5) $X$ is finite set.

A special case of this theorem, choosing $\mathcal{P}$ to be the ideal of all closed sets in $X$ reads: $C(X, \mathbb{C})$ is a Noetherian ring if and only if $X$ is finite set.

The following gives a necessary and sufficient condition for the ideal $C_{\mathcal{P}}(X, \mathbb{C})$ in $C(X, \mathbb{C})$ to be prime.

**Theorem 4.8.** Let $\mathcal{P}$ be an ideal of closed sets in $X$ and suppose $X$ is locally $\mathcal{P}$. Then the following statements are equivalent:

1) $C_{\mathcal{P}}(X, \mathbb{C})$ is a prime ideal in $C(X, \mathbb{C})$.
2) $C_{\mathcal{P}}(X)$ is a prime ideal in $C(X)$.
3) $X \notin \mathcal{P}$ and for any two disjoint co-zero sets in $X$, one has its closure lying in $\mathcal{P}$.

**Proof.** The equivalence of (1) and (2) follows from Theorem 2.12 and Theorem 4.1. Towards the equivalence (2) and (3), assume that $C_{\mathcal{P}}(X)$ is a prime ideal in $C(X)$. If $X \in \mathcal{P}$, then for each $f \in C(X)$, $cl_X(X \setminus Z(f)) \in \mathcal{P}$ meaning that $f \in C_{\mathcal{P}}(X)$ and hence $C_{\mathcal{P}}(X) = C(X)$, a contradiction to the assumption that $C_{\mathcal{P}}(X)$ is a prime ideal and in particular a proper ideal of $C(X)$.

Thus $X \notin \mathcal{P}$. Now consider two disjoint co-zero sets $X \setminus Z(f)$ and $X \setminus Z(g)$ in $X$, with $f, g \in C(X)$. It follows that $Z(f) \cup Z(g) = X$, i.e. $fg = 0$. Since $C_{\mathcal{P}}(X)$ is prime, this implies that $f \in C_{\mathcal{P}}(X)$ or $g \in C_{\mathcal{P}}(X)$, i.e. $cl_X(X \setminus Z(f)) \in \mathcal{P}$ or $cl_X(X \setminus Z(g)) \in \mathcal{P}$.

Conversely let the statement (3) be true. Since a $z$-ideal $I$ in $C(X)$ is prime if and only if for each $f, g \in C(X)$, $fg = 0$ implies $f \in I$ or $g \in I$ (see [17, Theorem 2.9]) and since $C_{\mathcal{P}}(X)$ is a $z$-ideal in $C(X)$, it is sufficient to show that for each $f, g \in C(X)$, if $fg = 0$ then $f \in C_{\mathcal{P}}(X)$ or $g \in C_{\mathcal{P}}(X)$. Indeed $fg = 0$ implies that $X \setminus Z(f)$ and $X \setminus Z(g)$ are disjoint co-zero sets in $X$. Hence by supposition (3), either $cl_X(X \setminus Z(f)) \mathcal{P}$ or $cl_X(X \setminus Z(g)) \in \mathcal{P}$ meaning that $f \in C_{\mathcal{P}}(X)$ or $g \in C_{\mathcal{P}}(X)$. \[\Box\]

A special case of Theorem 4.8, with $\mathcal{P}$ equal to the ideal of all compact sets in $X$, is proved in [10]. We examine a second special case of Theorem 4.8.

A subset $Y$ of $X$ is called a bounded subset of $X$ if each $f \in C(X)$ is bounded on $Y$. Let $\beta$ denote the family of all closed bounded subsets of $X$. Then $\beta$ is an ideal of closed sets in $X$. It is plain that a pseudocompact subset of $X$ is bounded but a bounded subset of $X$ may not be pseudocompact. Here is a counterexample: the open interval $(0, 1)$ in $\mathbb{R}$ is a bounded subset of $\mathbb{R}$ without being a pseudocompact subset of $\mathbb{R}$. However for a certain class of subsets of $X$, the two notions of boundedness and pseudocompactness coincide. The following well-known proposition substantiates this fact:

**Theorem 4.9.** (Mandelkar [18]). A support of $X$, i.e. a subset of $X$ of the form $cl_X(X \setminus Z(f))$ for some $f \in C(X)$, is a bounded subset of $X$ if and only if it is a pseudocompact subset of $X$. 

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It is clear that the conclusion of Theorem 4.9 remains unchanged if we replace $C(X)$ by $C(X, \mathbb{C})$.

Let $C_\psi(X) = \{ f \in C(X) : f \text{ has pseudocompact support} \}$ and recall that $C_\beta(X) = \{ f \in C(X) : f \text{ has bounded support} \}$. We would like to mention here that the closed pseudocompact subsets of a pseudocompact space $X$ might not constitute an ideal of closed sets in $X$. Indeed a closed subset of a pseudocompact space may not be pseudocompact. The celebrated example of a Tychonoff plank in [17, 8.20]: $[0, \omega_1] \times [0, \omega] \setminus \{(\omega_1, \omega)\}$, where $\omega_1$ is the 1st uncountable ordinal and $\omega$ is the first infinite ordinal, demonstrates this fact. Nevertheless $C_\psi(X)$ is an ideal of the ring $C(X)$. Indeed it follows directly from Theorem 4.9 that $C_\psi(X) = C_\beta(X)$.

A Tychonoff space $X$ is called locally pseudocompact if each point on $X$ has an open neighbourhood with its closure pseudocompact. On the other hand, $X$ is called locally bounded (or locally $\beta$) if each point in $X$ has an open neighbourhood with its closure bounded. Since each open neighbourhood of a point $x$ in a Tychonoff space $X$ contains a co-zero set neighbourhood of $x$, it follows from Theorem 4.9 that $X$ is locally bounded if and only if $X$ is locally pseudocompact. This combined with Theorem 2.12 leads to the following special case of Theorem 4.8.

**Theorem 4.10.** Let $X$ be locally pseudocompact. Then the following statements are equivalent:

1. $C_\psi(X)$ is a prime ideal of $C(X)$.
2. $C_\psi(X, \mathbb{C}) = \{ f \in C(X, \mathbb{C}) : f \text{ has pseudocompact support} \}$ is a prime ideal of $C(X, \mathbb{C})$.
3. $X$ is not pseudocompact and for any two disjoint co-zero sets in $X$, the closure of one of them is pseudocompact.

Since for $f \in C(X, \mathbb{C})$, $f \in C_\infty(X, \mathbb{C})$ if and only if $|f| \in C_\infty(X)$, it follows that $C_\infty(X, \mathbb{C})$ is an ideal of $C(X, \mathbb{C})$ if and only if $C_\infty(X)$ is an ideal of $C(X)$. In general however $C_\infty(X)$ need not be an ideal of $C(X)$. If $X$ is assumed to be locally compact, then it is proved in [2] and [11] that $C_\infty(X)$ is an ideal of $C(X)$ when and only when $X$ is pseudocompact. Therefore the following theorem holds.

**Theorem 4.11.** Let $X$ be locally compact. Then the following three statements are equivalent:

1. $C_\infty(X, \mathbb{C})$ is an ideal of $C(X, \mathbb{C})$.
2. $C_\infty(X)$ is an ideal of $C(X)$.
3. $X$ is pseudocompact.

5. **Zero divisor graphs of rings in the family $\Sigma(X, \mathbb{C})$**

We fix any intermediate ring $P(X, \mathbb{C})$ in the family $\Sigma(X, \mathbb{C})$. Suppose $\mathcal{G} = \mathcal{G}(P(X, \mathbb{C}))$ designates the graph whose vertices are zero divisors of $P(X, \mathbb{C})$ and there is an edge between vertices $f$ and $g$ if and only if $fg = 0$. For any two vertices $f, g$ in $\mathcal{G}$, let $d(f, g)$ be the length of the shortest path between $f$ and
g and \( \text{Diam} \mathcal{G} = \sup \{ d(f, g) : f, g \in \mathcal{G} \} \). Suppose \( \text{Gr} \mathcal{G} \) designates the length of the shortest cycle in \( \mathcal{G} \), often called the girth of \( \mathcal{G} \). It is easy to check that a vertex \( f \) in \( \mathcal{G} \) is a divisor of zero in \( P(X, C) \) if and only if \( \text{Int}_X Z(f) \neq \emptyset \). This parallels the statement that a vertex \( f \) in the zero-divisor graph \( \Gamma \mathcal{C} \mathcal{C}(X) \) is a divisor of zero in \( C(X) \). It is easy to check that several facts related to the nature of the vertices and the length of the cycles related to \( \Gamma \mathcal{C} \mathcal{C}(X) \) have been established in \cite{9} by employing skillfully the last mentioned simple characterization of divisors of zero in \( C(X) \). It is expected that the analogous facts pertaining to the various parameters of the graph \( \mathcal{G}(P(X, C)) = \mathcal{G} \) should also hold. We therefore just record the following results related to the graph \( \mathcal{G} \), without any proof.

**Theorem 5.1.** Let \( f, g \) be vertices of the graph \( \mathcal{G} \). Then \( d(f, g) = 1 \) if and only if \( Z(f) \cup Z(g) = X \); \( d(f, g) = 2 \) if and only if \( Z(f) \cup Z(g) \subseteq X \) and \( \text{Int}_X Z(f) \cap \text{Int}_X Z(g) \neq \emptyset \); \( d(f, g) = 3 \) if and only if \( Z(f) \cup Z(g) \subseteq X \) and \( \text{Int}_X Z(f) \cap \text{Int}_X Z(g) = \emptyset \). Consequently on assuming that \( X \) contains at least 3 points, \( \text{Diam} \mathcal{G} \) and \( \text{Gr} \mathcal{G} \) are both equal to 3 (compare with \cite{9}, Corollary 1.3).

**Theorem 5.2.** Each cycle in \( \mathcal{G} \) has length 3, 4 or 6. Furthermore every edge of \( \mathcal{G} \) is an edge of a cycle with length 3 or 4 (compare with \cite{9}, Corollary 2.3).

**Theorem 5.3.** Suppose \( X \) contains at least 2 points. Then
\begin{enumerate}
\item Each vertex of \( \mathcal{G} \) is a 4 cycle vertex.
\item \( \mathcal{G} \) is a triangulated graph meaning that each vertex of \( \mathcal{G} \) is a vertex of a triangle if and only if \( X \) is devoid of any isolated point.
\item \( \mathcal{G} \) is a hypertriangulated graph in the sense that each edge of \( \mathcal{G} \) is edge of a triangle if and only if \( X \) is a connected middle \( P \) space (compare with the analogous facts in \cite{9}, Proposition 2.1).
\end{enumerate}

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