On the Implied Volatility of Asian Options Under Stochastic Volatility Models

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ABSTRACT
In this paper, we study the short-time behaviour of the at-the-money implied volatility for arithmetic Asian options with fixed strike price. The asset price is assumed to follow the Black–Scholes model with a general stochastic volatility process. Using techniques of the Malliavin calculus developed in Alòs, García-Lorite, and Muguruza [2022. On Smile Properties of Volatility Derivatives: Understanding the VIX Skew. SIAM Journal on Financial Mathematics. 13(1): 32–69. https://doi.org/10.1137/19M1269981], we give sufficient conditions on the stochastic volatility in order to compute the level of the implied volatility of the option when the maturity converges to zero. Then, we find a short maturity asymptotic formula for the skew slope of the implied volatility that depends on the correlation between prices and volatilities and the Hurst parameter of the volatility model. We apply our general results to the SABR and fractional Bergomi models, and provide numerical simulations that confirm the accuracy of the asymptotic formulas.

1. Introduction
This paper is devoted to the study of Asian call options with payoff of the form
\[
\left( \frac{1}{T} \int_0^T S_u du - K \right)_+, \tag{1}
\]
where \( T \) denotes the maturity, \( S \) the price of the underlying, and \( K \) the strike of the contract. Asian options of this type are extremely important in energy markets for different reasons. From one hand, typical energy transactions use to take place via multiple deliveries. Then, these transactions are priced on average and not only on a terminal price. Secondly, the payoff is less sensitive to extreme market fluctuations, which becomes interesting in non-liquid markets. Finally, these options tend to be cheaper than the corresponding European vanillas.

The aim of this paper is to study the short-time maturity behaviour of the at-the-money implied volatility (ATMIV) of arithmetic Asian options. The study of implied volatility is
useful in many ways. Firstly, it can be used to obtain volatilities for pricing OTC options (and other derivatives) with strikes and maturities that are different from the ones offered by option exchanges. Secondly, the shape of the implied volatility surface can be used to assess the adequacy of an option pricing model. If the option pricing model is adequate, then it should capture the main properties of the empirical implied volatility surface. In particular, one of the key characteristics of the implied volatility is its skew at the short end and one can easily filter the class of suitable models if the theoretical value of the skew is available for the models of interest. Finally, as one will see further in the paper, one can use implied volatility and its skew to efficiently approximate the option price. Last, but not least, due to the smile effect the hedge ratio has to be adjusted to take into account the market skew. As a result, availability of analytical values of the skew can improve the performance of hedging.

The behaviour of the implied volatility for vanilla options has been the object of many works (see for example Lee 2005 for a basic introduction to this topic). However, the case of exotic options and more specifically Asian options is less studied and the number of exact analytic results is more limited.

Yang and Ewald (2009) compute the implied volatility for OTC traded Asian options under Black-Scholes with constant volatility by combining Monte-Carlo techniques with the Newton method in order to solve nonlinear equations. Approximation methods for pricing Asian options under stochastic volatility models are studied by Forde and Jacquier (2010). Chatterjee et al. (2018) develop a Markov chain-based approximation method to price arithmetic Asian options for short maturities under the case of geometric Brownian motion. Fouque and Han (2003) generalize the dimension reduction technique of Vecer for pricing arithmetic Asian options. They use the fast mean-reverting stochastic volatility asymptotic analysis to derive an approximation to the option price which takes into account the skew of the implied volatility surface. This approximation is obtained by solving a pair of one-dimensional partial differential equations. The methodology requires the key parameters needed in the PDEs to be estimated from the historical stock prices and the implied volatility surface.

Asymptotics of arithmetic Asian implied volatilities have been studied by Pirjol and Zhu (2016) in the case of local volatility. In this paper, the authors make use of large deviations techniques to get accurate approximation formulas for the implied volatility, which are shown to be accurate when compared with the Monte-Carlo simulations. Arithmetic Asian options under the CEV (constant elasticity of variance) model are studied in Pirjol and Zhu (2019). The leading order short maturity limit of the Asian option prices under the CEV model is obtained in closed form. Authors propose an analytic approximation for the Asian options prices which reproduced the exact short maturity asymptotics. Alòs and León (2019) compute the short-time level and the skew of the implied volatility of floating strike arithmetic Asian options under the Black-Scholes model with constant volatility by the means of Malliavin calculus.

In this paper, we contribute to the existing literature in several ways. We extend the application of the Malliavin calculus developed in Alòs, García-Lorite, and Muguruza (2022) by giving general sufficient conditions on a general stochastic volatility model in order to obtain formulas for the short-time limit of the at-the-money level and skew of the implied volatility for Asian options. Moreover, we show how studying Asian option under stochastic volatility reduces to the study of European-type options where the underlying is
represented by a certain stochastic volatility model, with a modified volatility process which depends on $T$. This methodology developed in Alòs, Garcia-Lorite, and Muguruza (2022) allows to adapt the results on vanilla options to options on a non lognormal-type distribution and it can also be applied to other European-type exotic options. See for example Alòs, Garcia-Lorite, and Muguruza (2022) for an application of this technique to the analysis of the VIX skew. This method is very classical in mathematical finance, for instance, when working with stochastic rates. The difference in the present paper is that the volatility of the log-forward price (see equation (4)) depends on the filtration of the Brownian motion $W$, and not only on $W'$. This makes the problem more challenging in order to find general hypotheses on a general stochastic volatility model in order for the asymptotic formulas to hold (see Hypotheses 3 and 4). Typically these hypotheses are in expectation, see for example Alòs and Shiraya (2019) and Alòs and León (2017), but in our case they need to hold almost surely in order to check the hypotheses of Theorem 3 (see Section 4.2).

To sum up, we study the short-end behaviour of the ATMIV of Asian options for local, stochastic, and fractional volatilities. In particular, we show that

- The short-end limit of the ATMIV is equal to $\frac{\sigma_0}{\sqrt{3}}$, where $\sigma_0$ denotes the short-end limit of the spot volatility. See equation (11) in Theorem 2.1.
- We compute the the short-end skew of the ATMIV, which depends on the correlation between prices and stochastic volatility and on the Malliavin derivative of the volatility process, which in the case of fractional volatility models will depend on the Hurst parameter $H \in (0, 1)$. See equation (12) in Theorem 2.1. If prices and volatilities are uncorrelated, the short-end skew is equal to $\frac{\sqrt{3}\sigma_0}{10}$. In the case of rough volatilities, that is $H < \frac{1}{2}$, we observe in equations (31) and (32) a blow-up that is of the same order as the one we observe in vanilla options (see Alòs, León, and Vives 2007).
- We apply the preceding results to the constant volatility case, the SABR model, the fractional Bergomi model, and the local volatility, and perform numerical simulations that confirm the accurateness of the asymptotic formulas. See Section 5. In the case of local volatilities, we verify that our results fit the asymptotic analysis of Pirjol and Zhu (2016) and Pirjol and Zhu (2019).

The paper is organized as follows: in Section 2 we introduce the main elements of the Malliavin calculus needed through the paper, and the main problem, results and notations. In Section 3 we introduce some preliminary results needed for the proof of the main theorem. In Section 4 we give the proof the main results of the paper. Finally, Section 5 is devoted to the application of the main results to the constant volatility case, the SABR model, the fractional Bergomi model, and the local volatility model, together with some numerical simulations to confirm the accurateness of the asymptotic formulas. The Appendix contains some Malliavin derivatives computations needed through the paper.

2. Notations and Main Results

2.1. A Primer on Malliavin Calculus

We introduce the elementary notions of the Malliavin calculus used in this paper (see Nualart and Nualart 2018). Let us consider a standard Brownian motion $Z = (Z_t)_{t \in [0,T]}$
defined on a complete probability space \((\Omega, \mathcal{F}, \mathbb{P})\) and we denote by \(\mathcal{F}_t\) the filtration generated by \(Z_t\). Let \(S^Z\) be the set of random variables of the form

\[
F = f(Z(h_1), \ldots, Z(h_n)),
\]

with \(h_1, \ldots, h_n \in L^2([0, T])\), \(Z(h_i)\) denotes the Wiener integral of the function \(h_i\), for \(i = 1, \ldots, n\), and \(f \in C_0^\infty(\mathbb{R}^n)\) (i.e., \(f\) and all its partial derivatives are bounded). Then the Malliavin derivative of \(F\), \(D_ZF\), is defined as the stochastic process given by

\[
D_Z F = \sum_{j=1}^n \frac{\partial f}{\partial x_j} (Z(h_1), \ldots, Z(h_n)) h_j(s), \quad s \in [0, T].
\]

This operator is closable from \(L^p(\Omega)\) to \(L^p(\Omega; L^2([0, T]))\), for all \(p \geq 1\), and we denote by \(D_{Z,p}\) the closure of \(S^Z\) with respect to the norm

\[
||F||_{1,p} = \left( \mathbb{E} ||F||^p + \mathbb{E} ||D_Z F||_{L^2([0, T])}^p \right)^{1/p}.
\]

We also consider the iterated derivatives \(D^{Z,n}_{p}\) for all integers \(n > 1\) whose domains will be denoted by \(D^{n,p}_{Z}\), for all \(p \geq 1\). We will use the notation \(L^{n,p}_{Z} := L^p([0, T]; D^{n,p}_{Z})\).

### 2.2. Statement of the Problem and Main Results

We denote by \((V_t)_{t \in [0,T]}\) the value of a fixed strike arithmetic Asian call option where \(T\) is the maturity. Then, the payoff can be written as

\[
V_T = (A_T - K)_+, \quad A_T = \frac{1}{T} \int_0^T S_t dt,
\]

where \((S_t)_{t \in [0,T]}\) is the price of the underlying asset and \(K\) is the fixed strike price.

For the sake of simplicity, we assume that the interest rate is equal to zero and we consider the following general stochastic volatility model for the underlying asset price

\[
dS_t = \sigma_t S_t dW_t
\]

\[
W_t = \rho W_t' + \sqrt{1 - \rho^2} B_t,
\]

where \(S_0 > 0\) is fixed, \(W_t\), \(W_t'\), and \(B_t\) are three standard Brownian motions on \([0, T]\) defined on the same complete probability space \((\Omega, \mathcal{G}, \mathbb{P})\). We assume that \(W_t'\) and \(B_t\) are independent and \(\rho \in [-1,1]\) is the correlation coefficient between \(W_t\) and \(W_t'\).

We consider the following assumption on the stochastic volatility of the asset price.

**Hypothesis 2.1:** The process \(\sigma = (\sigma_t)_{t \in [0,T]}\) is adapted to the filtration generated by \(W'\), a.s. positive and continuous, and satisfies that for all \(t \in [0, T]\),

\[
c_1 \leq \sigma_t \leq c_2,
\]

for some positive constants \(c_1\) and \(c_2\).
Remark 2.1: Hypothesis 2.1 may seem too restrictive since it is not satisfied by the stochastic volatility models considered in Section 5. However, we will show that using a truncation argument, Theorem 2.1 is still true in all our examples.

We define the forward price as the martingale \( M_t = \mathbb{E}_t(A_T) \), where \( \mathbb{E}_t \) denotes the conditional expectation wrt to the filtration \( \mathcal{F}_t \) generated by \( W_t \). Applying the stochastic Fubini’s theorem we get that

\[
A_T = \frac{1}{T} \int_0^T S_t \, dt = \frac{1}{T} \int_0^T \left( S_0 + \int_0^t \sigma_u S_u \, dW_u \right) \, dt =
\]

\[
= S_0 + \frac{1}{T} \int_0^T \sigma u_S u \left( \int_u^T \, dt \right) \, dW_u =
\]

\[
= S_0 + \frac{1}{T} \int_0^T (T - u) \sigma_u S_u \, dW_u,
\]

which implies that

\[
dM_t = \frac{\sigma_t S_t(T - t)}{T} \, dW_t = \phi_t M_t \, dW_t,
\]

where

\[
\phi_t := \frac{\sigma_t S_t(T - t)}{TM_t}.
\]

Furthermore, the log-forward price \( X_t = \log(M_t) \) satisfies

\[
dX_t = \phi_t \, dW_t - \frac{1}{2} \phi_t^2 \, dt.
\]

Remark 2.2: One can easily check that Hypothesis 2.1 implies that \( \phi_t \) is positive a.s. and belongs to \( L^p([0, T] \times \Omega) \), for all \( p \geq 2 \). In fact, Hypothesis 2.1 implies that for all \( p \geq 2 \), \( S_t \) belongs to \( L^p([0, T] \times \Omega) \), \( A_T \) belongs to \( L^p(\Omega) \), and \( M_t^{-1} \) belongs to \( L^p([0, T] \times \Omega) \).

Remark 2.3: Notice that \( \phi_0 = \sigma_0 \). Moreover,

\[
M_t = \frac{1}{T} \left( \int_0^t S_u \, du + S_t(T - t) \right).
\]

Then, \( TM_t \geq S_t(T - t) \), and this implies that \( \phi_t \leq \sigma_t \) almost surely.

The goal of this paper is to study the implied volatility of the Asian call option \( V_t \) which is defined as follows. We denote by \( BS(t, x, k, \sigma) \) the classical Black-Scholes price of a European call with time to maturity \( T-t \), log-forward price \( x \), log-strike price \( k \) and volatility \( \sigma \). That is,

\[
BS(t, x, k, \sigma) = e^{x}N(d_+(k, \sigma)) - e^{k}N(d_-(k, \sigma)),
\]

\[
d_\pm(k, \sigma) = \frac{x - k}{\sigma \sqrt{T - t}} \pm \frac{\sigma}{2} \sqrt{T - t},
\]
where $N$ is the cumulative distribution function of the standard normal random variable.

Next, we observe that, as $BS(T, x, k, \sigma) = (e^x - e^k)_+$ for every $\sigma > 0$, the price of our Asian call option $V_t = E_t(e^{X_T} - e^k)_+$ can be written as

$$V_t = E_t(BS(T, X_T, k, v_T)), \quad v_t = \sqrt{\frac{1}{T - t} \int_t^T \phi_s^2 \, ds}. \quad (6)$$

In particular, $V_T = BS(T, X_T, k, v_T)$. Then, we define the implied volatility of the option as $I(t, k) = BS^{-1}(t, X_t, k, V_t)$, and we denote by $I(t, k^*_t)$, where $k^*_t = X_t$, the corresponding ATMIV which, in the case of zero interest rates, takes the form $BS^{-1}(t, X_t, X_t, V_t)$.

We apply the Malliavin calculus techniques developed in Alòs, García-Lorite, and Muguruza (2022) in order to obtain formulas for

$$\lim_{T \to 0} I(0, k^*_t) \quad \text{and} \quad \lim_{T \to 0} \partial_k I(0, k^*_t)$$

under the general stochastic volatility model (2), where we have set $k^*_t = k^*_0$ for the sake of simplicity.

In our setting, since we have three Brownian motions $W, W'$ and $B$, if $h$ is a random variable in $L^2([0, T])$, then we have in view of relation (2) that

$$W(h) = \rho W'(h) + \sqrt{(1 - \rho^2)} B(h).$$

Then, a random variable in $D^{1,2}_W \cap D^{1,2}_B$ is also in $D^{1,2}_W$. In fact, it is easy to see that if $X$ is a random variable in $S^W$, then

$$D^W X = \rho D^W X + \sqrt{1 - \rho^2} D^B X. \quad (7)$$

Thus, we deduce that for all $X \in D^{1,2}_W \cap D^{1,2}_B$,

$$D^W X = \rho D^W X + \sqrt{1 - \rho^2} D^B X. \quad (8)$$

We will need the following additional assumption on the Malliavin differentiability of the stochastic volatility process.

**Hypothesis 2.2:** $\sigma \in L^{2,p}_W$, for all $p \geq 2$.

**Remark 2.4:** Hypotheses 1 and 2 imply that $\phi_t$ belongs to $L^{2,p}_W$ and $A_T$ belong to $D^{2,p}_W$ for all $p \geq 2$. This hypothesis on $A_T$ corresponds to $(H1)$ in Alòs, García-Lorite, and Muguruza (2022).

In order to give the asymptotic skew of the implied volatility as a function of the roughness of the stochastic volatility process we consider the following assumption.
**Hypothesis 2.3:** There exists $H \in (0, 1)$ such that for all $0 \leq s \leq r \leq t \leq T$

$$|D^W_r \sigma_t| \leq M_{r,t}(t - r)^{H - \frac{1}{2}}$$

(9)

and

$$|D^W_s D^W_r \sigma_t| \leq N_{s,r,t}(t - r)^{\frac{1}{2}}(t - s)^{H - \frac{1}{2}},$$

(10)

where $M_{r,t}$ and $N_{s,r,t}$ are positive random variables satisfying for all $p \geq 1$,

$$\mathbb{E}(\sup_{0 \leq r \leq t \leq T} M^{p}_{r,t}) \leq c_1,$$

and

$$\mathbb{E}(\sup_{0 \leq s \leq r \leq t \leq T} N^{p}_{s,r,t}) \leq c_2,$$

for some positive constants $c_1$ and $c_2$.

Finally, we will need the following additional assumption on the continuity of the paths of the volatility process.

**Hypothesis 2.4:** There exists $\gamma \in (0, H)$ such that for all $0 \leq s \leq r \leq T \leq 1$

$$|\sigma_r - \sigma_s| \leq K_{r,s}(r - s)^{\gamma},$$

where $K_{r,s}$ is a positive random variable satisfying for all $p \geq 1$,

$$\mathbb{E}(\sup_{0 \leq r \leq t \leq T} K^{p}_{r,s}) \leq c$$

where $c > 0$ and $H$ is the Hurst parameter form Hypothesis 3.

We next provide the the main result of this paper, which is the short-time ATMIV level and skew of an Asian call option under the general volatility model (2).

**Theorem 2.1:** Assume Hypotheses 1–4. Then,

$$\lim_{T \to 0} I(0, k^*) = \frac{\sigma_0}{\sqrt{3}}$$

(11)

and

$$\lim_{T \to 0} T^{\max\left(\frac{1}{2} - H, 0\right)} \partial_k I(0, k^*)$$

$$= \lim_{T \to 0} T^{\max\left(\frac{1}{2} - H, 0\right)} \frac{3\sqrt{3} \rho}{\sigma_0 T^5} \int_0^T \left((T - r) \int_r^T (T - u)^2 \mathbb{E}(D^W_r σ_u) du\right) dr$$

$$+ \lim_{T \to 0} T^{\max\left(\frac{1}{2} - H, 0\right)} \frac{\sqrt{3} \sigma_0}{30},$$

(12)

and the limit on the right hand side of (12) is finite.
We observe that the level (11) is independent of the correlation $\rho$ and the Hurst parameter $H$, and coincides with the constant volatility case, see Pirjol and Zhu (2016) and Alòs and León (2019). Observe also that it coincides with the a.s. limit of $v_0$. In fact, by Hypothesis 1 and since $S_0 = M_0$ we have that a.s.

$$\lim_{T \to 0} v_0 = \lim_{T \to 0} \sqrt{\frac{1}{T^3} \int_0^T \frac{\sigma_s^2 S_s^2 (T - s)^2}{M_s^2} ds} = \frac{\sigma_0}{\sqrt{3}}. \quad (13)$$

The skew (12) depends on the correlation parameter $\rho$ and on the Hurst parameter $H$. When prices and volatilities are uncorrelated then the short-time skew equals $\sqrt{3} \sigma_0$, which again coincides with the constant volatility case, see Pirjol and Zhu (2016) and Alòs and León (2019). Observe also that if the term $\mathbb{E}(D_r W \sigma_u)$ is of order $(u - r)^{H - \frac{1}{2}}$ (see Hypothesis 3), then the limit of the right hand side of (12) will be 0 if $H > 1/2$ and it will converge to a constant when $H = \frac{1}{2}$. When $H < \frac{1}{2}$ we need to multiply by $T^{\frac{1}{2} - H}$ in order to obtain a finite limit. This is because when $H > 1/2$, the fractional Brownian motion is smoother than standard Brownian motion and the effect of the stochastic volatility on the short-time implied volatility will be the same as it was constant, while when $H < 1/2$, the fractional Brownian motion is rougher than standard Brownian motion and we obtain the same effect as in the case of vanilla options, see Alòs, León, and Vives (2007).

The results of Theorem 2.1 can be used in order to derive an approximation formula for the price of an Asian call option. By definition, the price of the Asian call option writes as

$$V_0 = BS(0, X_0, k, I(0, k))$$

Then, using Taylor’s formula we can use the approximation

$$I(0, k) \approx I(0, k^*) + \partial_k I(0, k^*)(k - k^*). \quad (14)$$

Of course, this approximation is only linear and can be expected to have a limited validity, restricted to a narrow region around the ATM point. One would expect to obtain better results if one has a short maturity asymptotic formula for the curvature $\partial^2_k I(0, k^*)$. The short-time maturity asymptotics for the ATM curvature of the implied volatility for European calls under general stochastic volatility models is computed in Alòs and León (2017). A second-order Taylor expansion for short maturity limit of the implied volatility for Asian options around the ATM point when the underlying asset follows a local volatility model is obtained in Proposition 19 of Pirjol and Zhu (2016). In the constant volatility case, the short maturity limit is known for all $k$ (see Proposition 8 of Pirjol and Zhu (2016)), and the short-maturity limit of the skew and the curvature for $k^*$ are known in closed from Pirjol (2023). In our setting, computing the curvature and a second-order approximation of the price around the ATM point is more challenging and we leave it for further work.

3. Preliminary Results

We start quoting the two main results obtained in Alòs, García-Lorite, and Muguruza (2022) that will be crucial for the proof of our main Theorem and use the general framework detailed in Section 2.2. The first result is Theorem 6 in Alòs, García-Lorite,
and Muguruza (2022) which shows that the short-time limit of the ATMIV equals the short-time limit of the future average of the volatility of the log forward price.

**Theorem 3.1:** Assume that for all $p > 1$, $A_T \in D^{2p}_W$, $M^{-1}_t \in L^p([0, T] \times \Omega)$, and

$$
\lim_{T \to 0} E \left( \int_0^T \frac{\Lambda_s}{v_s^2(T - s)} ds \right) = 0, \quad (15)
$$

$$
\lim_{T \to 0} \frac{1}{T^2} E \left( \frac{1}{v_0} \int_0^T \left( \int_s^T D_s^W \phi_s^2 dr \right)^2 ds \right) = 0, \quad (16)
$$

where $\Lambda_s = \phi_s \int_s^T D_s^W \phi_s^2 dr$. Then,

$$
\lim_{T \to 0} I(0, k^*) = \lim_{T \to 0} E(v_0).
$$

The second result is Theorem 8 of Alòs, García-Lorite, and Muguruza (2022) which gives an approximation formula for the short-time limit of the ATMIV skew.

**Theorem 3.2:** Assume that for all $p > 1$, $A_T \in D^{3p}_W$, $M^{-1}_t \in L^p([0, T] \times \Omega)$, hypotheses (15) and (16) are satisfied,

$$
\lim_{T \to 0} \frac{T^\max(\frac{1}{2} - H, 0)}{\sqrt{T}} E \left( \int_0^T (v_s^2(T - s))^{\frac{3}{2}} \Lambda_s \left( \int_s^T \Lambda_r dr \right) ds \right) = 0, \quad (17)
$$

$$
\lim_{T \to 0} \frac{T^\max(\frac{1}{2} - H, 0)}{\sqrt{T}} E \left( \int_0^T (v_s^2(T - s))^{\frac{5}{2}} \phi_s \left( \int_s^T D_s^W \Lambda_r dr \right) ds \right) = 0, \quad (18)
$$

and

$$
\lim_{T \to 0} \frac{T^{\max(\frac{1}{2} - H, 0)}}{T^2} E \left( \frac{1}{v_0^3} \int_0^T \Lambda_s ds \right) < \infty. \quad (19)
$$

Then,

$$
\lim_{T \to 0} T^{\max(\frac{1}{2} - H, 0)} \frac{\partial_k I(0, k^*)}{T^2} = \frac{1}{2} \lim_{T \to 0} \frac{T^{\max(\frac{1}{2} - H, 0)}}{T^2} E \left( \frac{1}{v_0^3} \int_0^T \Lambda_s ds \right).
$$

**Remark 3.1:** Observe that there are two typo in Theorem 8 of Alòs, García-Lorite, and Muguruza (2022). First a factor $T^{-\gamma}$ missing in their hypothesis (H5). Here we are taking $\gamma = \min(H - \frac{1}{2}, 0) \in (-\frac{1}{2}, 0]$. Moreover there is a square missing in the $u_s(T - s)$ and it should be $u_s^2(T - s)$, see for example Lemma 6.3.1 in Alòs and García-Lorite (2021).

We next present some technical lemmas that will be needed in order to check that the hypotheses of the preceding theorems are satisfied.

**Lemma 3.1:** Assume Hypothesis 1. Then, for every $p \geq 1$, there exists a constant $c_p > 0$ such that for all $0 \leq t < T \leq 1$,

$$
\left( E \left[ v_t^{-2p} \right] \right)^{1/p} \leq c_p \frac{T^2}{(T - t)^2}.
$$
**Proof:** We follow a similar idea used in Lemma 3 of Alòs and León (2019). We observe that by the definition of $\phi_t$ and equation (5), we get that

$$
\int_t^T \phi_r^2 dr = \int_t^T \left( \frac{\sigma_r S_r(T - r)}{\int_0^r S_u du + S_r(T - r)} \right)^2.
$$

Then, using Hypothesis 2.1 we get that

$$
\int_t^T \phi_r^2 dr \geq c_1^2 \exp \left( -4 \sup_{t \in [0,T]} \left| \int_0^t \sigma_s dW_s - \frac{1}{2} \int_0^t \sigma_s^2 ds \right| (T - t)^3 \right).
$$

Thus, using again Hypothesis 2.1,

$$
\left( \int_t^T \phi_r^2 dr \right)^{-p} \leq c_2^{-p} e^{2pTc_3^2} \exp \left( 4p \sup_{t \in [0,T]} \left| \int_0^t \sigma_s dW_s \right| \right) \frac{3pT^{2p}}{(T - t)^{3p}}.
$$

By Burkholder-Davis-Gundy inequality and Hypothesis 1, for any integer $n \geq 1$,

$$
\mathbb{E} \left( \sup_{t \in [0,T]} \left| \int_0^t \sigma_s dW_s \right|^n \right) \leq C n^{n/2} (cT)^{n/2},
$$

for some positive constants $c, C$. Therefore, for $T \leq 1$,

$$
\mathbb{E} \exp \left( 4p \sup_{t \in [0,T]} \left| \int_0^t \sigma_s dW_s \right| \right) \leq C \sum_{n=1}^{\infty} \frac{(cn)^{n/2}}{n!},
$$

which is a convergent series. This completes the proof.

**Lemma 3.2:** Assume Hypothesis 1. Then, for any $p \geq 1$ there exists a constant $c_p > 0$ such that for all $0 \leq t \leq T \leq 1$,

$$
\mathbb{E}(M_t^{-p}) \leq c_p.
$$

**Proof:** Using (5) and a similar argument as in the proof of Lemma 3.1, we get that

$$
\mathbb{E}(M_t^{-p}) \leq e^{pTc_3^2} \mathbb{E} \exp \left( p \sup_{t \in [0,T]} \left| \int_0^t \sigma_s dW_s \right| \right),
$$

and the result follows from (21).

Next, we obtain approximation formulas for $\phi$ and its Malliavin derivative.

**Lemma 3.3:** Under Hypotheses 1, 2, (9), and 4 the following holds for all $0 \leq s \leq r \leq T$,

$$
\phi_r = \frac{\sigma_0(T - r)}{T} + X_1^1,
$$

$$
\phi_r^2 = \frac{\sigma_0^2(T - r)^2}{T^2} + X_1^2,
$$

where $X_1$ and $X_2$ are defined in Wang (2019).
\[ D_s^W \phi_r = \frac{\rho(T - r)D_s^W \sigma_r}{T} + \frac{(T - r)\sigma_0^2}{T} - \frac{(T - r)(T - s)\sigma_0^2}{T^2} + X^3_{T,r,s}, \tag{24} \]
\[ D_s^W \phi_r^2 = \frac{2\sigma_0\rho(T - r)^2D_s^W \sigma_r}{T^2} + \frac{2(T - r)^2\sigma_0^3}{T^2} - \frac{2(T - r)^2(T - s)\sigma_0^3}{T^3} + X^4_{r,s}, \tag{25} \]

where \( X^i \) are random variables satisfying for all \( 0 \leq s \leq r \leq T \leq 1 \),
\[ |X^1_r| \leq Y^1_r \frac{(T - r)r'}{T}, \]
\[ |X^2_r| \leq Y^2_r \frac{(T - r)^2r'}{T^2}, \]
\[ |X^3_{r,s}| \leq Y^3_{r,s} \frac{(T - r)(r - s)H}{T}, \]
\[ |X^4_{r,s}| \leq Y^4_{r,s} \frac{(T - r)^2r' (r - s)H}{T^2}, \]

and \( Y^i \) are positive random variables satisfying for all \( p \geq 1 \)
\[ \mathbb{E}(\sup_{0 \leq s \leq r \leq T \leq 1} |Y^i_{r,s}|^p) \leq c_i \]
for some positive constants \( c_i \) only dependent on \( p \) and \( \gamma > 0 \) is from Hypothesis 4.

**Proof:** We start proving the decomposition for \( \phi_r \). We consider the function
\[ F(S_s, M_s) := \frac{\sigma_0 S_s(T - r)}{TM_s}, \quad 0 \leq s \leq r. \]

Observe that
\[ \phi_r = F(S_r, M_r) + \left( \sigma_r - \sigma_0 \right) \frac{S_r(T - r)}{TM_r}. \]

Then, using Itô’s lemma, we get that we get
\[ F(S_r, M_r) = \frac{\sigma_0(T - r)}{T} + \frac{(T - r)}{T} \]
\[ \times \left\{ \int_0^r \frac{\sigma_0}{M_s} dS_s - \int_0^r \frac{\sigma_0 S_s}{M_s^2} dM_s + \int_0^r \frac{\sigma_0 \sigma_0^2 S_s^3 (T - s)^2}{M_s^3} \frac{d}{T^2} ds \right\}. \]

Then, using Hypotheses 2.1 and 2.4 and Lemma 3.2, we conclude (22). Similarly, we can write
\[ \phi_r^2 = F^2(S_r, M_r) + \left( \sigma_r^2 - \sigma_0^2 \right) \frac{S_r^2(T - r)^2}{T^2 M_r^2}. \]
and applying Itô’s formula to the function \( F^2(S_r, M_r) \) to obtain (23).

We next prove (24). Using expression (A1) of the Malliavin derivatives computed in the Appendix, we see that the leading terms are equal to
\[ \frac{\rho(T - r)S_r D_s^W \sigma_r}{T M_r} + \frac{\rho^2(T - r)\sigma_r S_r \sigma_s}{T M_r} - \frac{\rho^2(T - r)S_r \sigma_r S_s (T - s)}{T^2 M_r^2} \]
\[ + \frac{(1 - \rho^2)(T - r)\sigma_r\sigma_s}{TM_r} T^2 M_r^2 \]
\[ \frac{(1 - \rho^2)(T - r)\sigma_r\sigma_s}{TM_r} T^2 M_r^2 = \frac{\rho(T - r)S_t D_s^W \sigma_r}{TM_r} + \frac{(T - r)\sigma_r\sigma_s}{TM_r} - \frac{(T - r)\sigma_r\sigma_s S_t(T - s)}{TM_r}. \]

Then, applying Itô’s formula to the functions \( F(S_t, M_s) = \frac{S_t}{M_s} \) and \( F(S_t, M_s) = \frac{S_t^2}{M_s^2} \) as above, we obtain (24).

Finally, in order to check (25) it suffice to use the formula \( D_s^W \phi_r^2 = 2\phi_r D_s^W \phi_r \) together with (22) and (24). This concludes the proof.

4. Proof of Theorem 2.1

4.1. Proof of (11) in Theorem 2.1: ATM Implied Volatility Level

By (13), it suffices to check that the Hypotheses of Theorem 3.1 hold true. It is easy to check that Hypotheses 1 and 2 imply the first two hypotheses of Theorem 3.1 (see Remarks 2.2 and 2.4).

We next check (15). Using equation (20) and Cauchy-Schwarz inequality we get that

\[ \mathbb{E} \int_0^T \frac{\Lambda_s}{v_s^2(T - s)} ds \leq \int_0^T \frac{T^2}{(T - s)^3} \mathbb{E}(X_T|\Lambda_s)| ds \]
\[ \leq \int_0^T \frac{T^2}{(T - s)^3} \left( \mathbb{E}(X_T^2) \right)^{1/2} \left( \mathbb{E}(\Lambda_s^2) \right)^{1/2} ds, \]

where \( X_T = 3c_1^{-2}e^{2Tc_2} \exp(4 \sup_{t \in [0, T]} |\int_0^t \sigma_s dW_s|) \).

Next, due to equation (21) we conclude that \( \mathbb{E}(X_T^2) \) is bounded as \( T \leq 1 \). Since \( \phi_t \leq \sigma_t \), Cauchy-Schwarz inequality, Lemma 3.3 and Hypothesis (9) imply that

\[ \mathbb{E}(\Lambda_s^2) \leq C(T - s) \int_s^T \mathbb{E}(|D_s^W \phi_r^2|^2) dr \]
\[ = O \left( (T - s) \int_s^T \frac{(T - r)^4}{T^4} (r - s)^{2H - 1} dr \right) \]
\[ = O \left( (T - s)^{5 + 2H} \frac{1}{T^4} \right). \]

Finally, we conclude that

\[ \mathbb{E} \int_0^T \frac{\Lambda_s}{v_s^2(T - s)} ds = O \left( \int_0^T (T - s)^{H - \frac{1}{2}} ds \right) = O \left( T^{H + \frac{1}{2}} \right), \]

which proves (15).

Similarly, in order to check (16), we use Lemma 3.3 together with Cauchy-Schwarz inequality, to get that

\[ \frac{1}{T^2} \mathbb{E} \left( \frac{1}{v_0} \int_0^T \left( \int_s^T D_s^W \phi_r^2 dr \right)^2 ds \right) \leq \frac{C}{T^2} \int_0^T (T - s) \int_s^T \mathbb{E} \left( (D_s^W \phi_r^2)^2 \right) dr ds \]
\[ O\left(\frac{1}{T^2} \int_0^T (T-s) \int_s^T \frac{(T-r)^4}{T^4} (r-s)^{2H-1} dr ds\right) \]
\[ = O\left(\int_0^T (T-s)^{5+2H} T^6 ds\right) \]
\[ = O(T^{2H}). \]

Thus, condition (16) also holds and the proof is completed.

### 4.2. Proof of (12) in Theorem 2.1: ATM Implied Volatility Skew

We will apply Theorem 3.2. We start checking hypothesis (17).

Using (20), Lemma 3.3 and Hypothesis (9), we get that

\[
\mathbb{E}\left( \int_0^T \left( \int_s^T \phi_r^2 dr \right)^{-3} \Lambda_s \left( \int_s^T \Lambda_r dr \right) ds \right) = O\left( \int_0^T \left( \frac{X_T T^6}{(T-s)^9} \right) \frac{\int_s^T (T-r)^2}{T^2} |D_s^W \sigma_r| dr \right)
\times \int_s^T \frac{T-r}{T} \int_r^T \left( \frac{T-u}{T^2} \right) |D_r^W \sigma_u| dudr) \right)
\]
\[ = O\left( \int_0^T \frac{1}{(T-s)^3} \int_s^T (r-s)^{H-\frac{1}{2}} dr \int_s^T (u-r)^{H-\frac{1}{2}} dudr \right) \]
\[ = O\left( \int_0^T (T-s)^{2H-1} \right) = O(T^{2H}). \]

Thus, as \( \lim_{T \to 0} \frac{T^{\max(\frac{1}{2}-H,0)}}{\sqrt{T}} T^{2H} = \lim_{T \to 0} T^{\max(H,2H-\frac{1}{2})} = 0 \) for all \( H \in (0,1) \), we get that (17) holds true for the leading terms of Lemma 3.3. The other terms can be treated similarly and we conclude that (17) holds true.

We next check (18). By the definition of \( \Lambda_s \), we have

\[
\int_s^T D_s^W \Lambda_r dr = \int_s^T D_s^W \left( \phi_r \int_r^T D_r^W \phi_u^2 du \right) dr
\]
\[ = \int_s^T \left( (D_s^W \phi_r) \int_r^T D_r^W \phi_u^2 du + \phi_r \int_r^T D_s^W D_r^W \phi_u^2 du \right) dr, \]

where \( D_s^W D_r^W \phi_u^2 = 2(D_s^W \phi_u D_r^W \phi_u + \phi_u D_s^W D_r^W \phi_u). \)

Next, using Lemma 3.3 and Hypothesis 2.3, we get that the leading terms are in expectation of order

\[
\int_s^T D_s^W \phi_r \int_r^T D_r^W \phi_u^2 dudr \]
\[
\begin{align*}
&= O\left(\int_s^T \frac{(T-r)(r-s)^{H-\frac{1}{2}}}{T} \int_r^T \frac{(T-u)^2(u-r)^{H-\frac{1}{2}}}{T^2} \mathrm{d}u \mathrm{d}r\right) \\
&= O\left(\frac{(T-s)^{4+2H}}{T^3}\right),
\end{align*}
\]

and
\[
\begin{align*}
&\int_s^T D_s^W \phi_r \int_r^T D_r^W \phi_u^2 \mathrm{d}u \mathrm{d}r \\
&= O\left(\int_s^T \frac{(T-r)}{T} \int_r^T \frac{(T-u)^2(u-r)^{H-\frac{1}{2}}(u-s)^{H-\frac{1}{2}}}{T^2} \mathrm{d}u \mathrm{d}r\right) \\
&= O\left(\frac{(T-s)^{4+2H}}{T^3}\right).
\end{align*}
\]

Then, we conclude that the leading terms satisfy that
\[
\begin{align*}
&= O\left(\int_0^T \left(\int_s^T \phi_r^2 \mathrm{d}r\right)^{-2} \phi_s \left(\int_s^T D_s^W \Lambda_r \mathrm{d}r\right) \mathrm{d}s\right) = O\left(\int_0^T \frac{(T^3)}{(T-s)^5} \frac{(T-s)^{4+2H}}{T^3} \mathrm{d}s\right) \\
&= O\left(\int_0^T (T-s)^{2H-1}\right) = O(T^{2H}).
\end{align*}
\]

Following as above this proves (18).

We are left to check hypothesis (19). Similarly as above, we have that
\[
\begin{align*}
\frac{T^{\max\left(\frac{1}{2}-H,0\right)}}{T^2} \mathbb{E}\left(\frac{1}{v_0^3} \int_0^T \Lambda_r \mathrm{d}s\right) = O(1),
\end{align*}
\]

and thus the limit is finite. Therefore, all the hypotheses of Theorem 3.2 are satisfied.

We finally compute the limit of (19) to check that it coincides with (12). Using Lemma 3.3, we obtain
\[
\begin{align*}
&\lim_{T \to 0} \frac{1}{T^2} \mathbb{E}\left(\frac{T^{\max\left(\frac{1}{2}-H,0\right)}}{v_0^3} \int_0^T \phi_s \left(\int_s^T D_s^W \phi_r^2 \mathrm{d}r\right)^2 \mathrm{d}s\right) \\
&= \lim_{T \to 0} \mathbb{E}\left(\frac{T^{\max\left(\frac{1}{2}-H,0\right)}}{T^2 v_0^3} \int_0^T \int_s^T \sigma_0(T-s) \frac{T}{T} \int_s^T \frac{2\sigma_0 \rho (T-r)^2 D_s^W \sigma_r}{T^2} \mathrm{d}r \mathrm{d}s\right) \\
&+ \frac{2(T-r)^2 \sigma_0^3}{T^2} - \frac{2(T-r)^2 (T-s) \sigma_0^3}{T^3} \int \mathrm{d}r \mathrm{d}s\right) \\
&= \lim_{T \to 0} \mathbb{E}\left(\frac{2\sigma_0^2 \rho T^{\max\left(\frac{1}{2}-H,0\right)}}{T^5 v_0^3} \int_0^T \left(T-s\right) \int_s^T (T-r)^2 D_s^W \sigma_r \mathrm{d}r \mathrm{d}s\right) \\
&+ \frac{2(T-r)^2 \sigma_0^3}{T^2} - \frac{2(T-r)^2 (T-s) \sigma_0^3}{T^3} \int \mathrm{d}r \mathrm{d}s\right) \\
&= \lim_{T \to 0} \mathbb{E}\left(\sigma_0^4 \frac{2\sigma_0^2 \rho T^{\max\left(\frac{1}{2}-H,0\right)}}{45v_0^3}\right).
\end{align*}
\]
Using (20) and dominated convergence we see that

\[
\lim_{T \to 0} T^{\max(\frac{1}{2} - H, 0)} \mathbb{E} \left( \frac{\sigma_0^4}{45 v_0^3} \right) = \lim_{T \to 0} T^{\max(\frac{1}{2} - H, 0)} \frac{\sqrt{3} \sigma_0}{60},
\]

since \( v_0^2 \) converges a.s. towards \( \frac{\sigma_0^2}{3} \) as \( T \to 0 \). In order to compute the remaining limit we write

\[
\lim_{T \to 0} \mathbb{E} \left( \frac{2\sigma_0^2 \rho T^{\max(\frac{1}{2} - H, 0)}}{T^5 v_0^3} \int_0^T \left( (T - s) \int_s^T (T - r)^2 D_{s}^\nu r \sigma_r \, dr \right) \, ds \right) = \lim_{T \to 0} \mathbb{E} \left( \left( \frac{1}{v_0^3} - \frac{3\sqrt{3}}{\sigma_0^3} \right) A_T \right) + \lim_{T \to 0} \frac{3\sqrt{3}}{\sigma_0^3} \mathbb{E} (A_T),
\]

where

\[
A_T = \frac{2\rho \sigma_0^2 T^{\max(\frac{1}{2} - H, 0)}}{T^5} \int_0^T \left( (T - s) \int_s^T (T - r)^2 D_{s}^\nu r \sigma_r \, dr \right) \, ds.
\]

By dominated convergence we see that

\[
\lim_{T \to 0} \mathbb{E} \left( \left( \frac{1}{v_0^3} - \frac{3\sqrt{3}}{\sigma_0^3} \right) A_T \right) = 0,
\]

which concludes the proof of (12).

## 5. Numerical Analysis

In this section we present numerical evidence of the adequacy of Theorems 2.1 in different settings.

### 5.1. The Black-Scholes Model Under Constant Volatility

We consider the Black-Scholes model (2) under constant volatility \( \sigma > 0 \), that is,

\[
dS_t = \sigma S_t dW_t, \quad S_t = S_0 e^{\sigma W_t - \frac{\sigma^2}{2} t}.
\]

Appealing to Theorem 2.1 with \( \rho = 0 \) and \( H = \frac{1}{2} \), we conclude that the level and the skew of the at-the-money implied volatility satisfy that

\[
\lim_{T \to 0} I(0, k^*) = \frac{\sigma}{\sqrt{3}} \quad \text{and} \quad \lim_{T \to 0} \partial_k I(0, k^*) = \frac{\sigma \sqrt{3}}{30},
\]

Notice that these results coincide with the ones obtained in Pirjol and Zhu (2016), see Section 5.4 below.
We next proceed with numerical simulations with parameters
\[ S_0 = 10, \quad T = \frac{1}{252}, \quad \sigma \in [0.1, 0.2, \ldots, 1.4]. \]

We use the control variates method in order to get estimates of an Asian call option price. As a control variate we use a geometric Asian call option whose price is given by
\[
BS_{\text{GeomAsian}} = e^{\frac{-1}{4} \sigma^2_G T} S_0 N(d_1) - K N(d_2),
\]
(26)

where
\[
d_1 = \frac{\log \frac{S_0}{K} + \frac{1}{4} \sigma^2_G T}{\sigma_G \sqrt{T}}, \quad d_2 = d_1 - \sigma_G \sqrt{T}, \quad \sigma_G = \frac{\sigma}{\sqrt{3}}.
\]

Then, the Asian call option price estimator has the following form
\[
\hat{BS}_{\text{Asian}} = \frac{1}{N} \sum_{i=1}^{N} V^i_T - e^* \frac{1}{N} \sum_{i=1}^{N} (\hat{BS}^i_{\text{Asian}} - BS_{\text{GeomAsian}}),
\]
(27)

where
\[
e^* = \sum_{i=1}^{N} (V^i_T - \frac{1}{N} \sum_{i=1}^{N} A^i_T) (\hat{BS}^i_{\text{Asian}} - BS_{\text{GeomAsian}}) \sum_{i=1}^{N} (\hat{BS}^i_{\text{Asian}} - BS_{\text{GeomAsian}})^2,
\]
and
\[
\hat{BS}^i_{\text{Asian}} = \max(\sqrt{S^i_0 S^i_1 \cdots S^i_m} - K, 0),
\]

where \( N = 2000000, \ m = 50, \ V^i_T = \max(A^i_T - K, 0) \) and the sub-index \( i \) indicates the quantity estimated from a realization of a path from Monte Carlo simulation.

In order to retrieve an estimation for the implied volatility \( \hat{\Gamma}(0, k^*) \) from the estimated Asian call price we use the algorithm presented in Jäckel (2015). For the estimation of the skew, we use the following finite difference approximation
\[
\hat{c}_k \hat{\Gamma}(0, k^*) = \frac{\hat{\Gamma}(0, k^* \log(1 + \Delta k)) - \hat{\Gamma}(0, \frac{k^*}{\log(1 + \Delta k)})}{2 \log(1 + \Delta k)},
\]
(28)

where \( \Delta k = 0.001 \).

The at-the-money level and the skew of the implied volatility are presented at Figure 1. We conclude that the results of the numerical simulation are in line with the presented theoretical formulas.

### 5.2. The SABR Model

In this section we consider the SABR stochastic volatility model with skewness parameter 1, which is the most common case from a practical point of view. This corresponds to equation (2), where \( S_t \) denotes the forward price of the underlying asset and
\[
d\sigma_t = \alpha \sigma_t dW'_t, \quad \sigma_t = \sigma_0 e^{\sigma^2 W_t - \frac{\sigma^2}{2} t}.
\]

where \( \alpha > 0 \) is the volatility of volatility.
Notice that this model does not satisfy Hypothesis 1, so a truncation argument similar as in Section 5 in Alòs and Shiraya (2019) is needed in order to check that Theorem 2.1 is true for this model. We define $\varphi(x) = \sigma_0 \exp(x)$. For every $n > 1$, we consider a function $\varphi_n \in C_b^2$ satisfying that $\varphi_n(x) = \varphi(x)$ for any $x \in [-n, n]$, $\varphi_n(x) \in [\varphi(-2n) \vee \varphi(x), \varphi(-n)]$ for $x \leq -n$, and $\varphi_n(x) \in [\varphi(n), \varphi(x) \wedge \varphi(2n)]$ for $x \geq n$. We set

$$\sigma^n_t = \varphi_n \left( \alpha W'_t - \frac{\alpha^2}{2} t \right).$$

It is easy to see that $\sigma^n_t$ satisfies Hypotheses 1, 2, (9), and 4. In fact, for $r \leq t$, we have that

$$D_r \sigma^n_t = \varphi'_n \left( \alpha W'_t - \frac{\alpha^2}{2} t \right) \alpha,$$

which implies that (9) holds with $H = \frac{1}{2}$ and Hypothesis 4 is satisfied with $\gamma < 1/2$. Therefore, appealing to Theorem 2.1 and using the fact that $\sigma^n_0 = \sigma_0$, we conclude that

$$\lim_{T \to 0} I^n(0, k^*) = \frac{\sigma_0}{\sqrt{3}}. \quad (29)$$

where $I^n$ denotes the implied volatility under the volatility process $\sigma^n_t$. We then write

$$I(0, k^*) = I^n(0, k^*) + I(0, k^*) - I^n(0, k^*).$$

By the mean value theorem,

$$I(0, k^*) - I^n(0, k^*) = \partial_{\sigma} (BS^{-1}(0, X_0, X_0, \xi))(V_0 - V^n_0)$$

$$= e^{-X_0 + \frac{\xi^2}{2} T + \frac{2\pi}{\sqrt{T}} (V_0 - V^n_0)},$$

for some $\xi \in (V_0, V^n_0)$, where $V^n_0$ is the option price under $\sigma^n$. Thus, for $T \leq 1$ and $n > \alpha^2$,

$$|I(0, k^*) - I^n(0, k^*)| \leq \frac{C_n}{\sqrt{T}} \mathbb{E} \left( \left| e^{X_T} - e^{X^n_T} \right| 1_{\sup_{s \in [0, T]} |\ln(\sigma_s/\sigma_0)| > n} \right)$$

$$\leq \frac{C_n}{\sqrt{T}} \mathbb{E} \left[ \left( |e^{X_T} + e^{X^n_T}|^2 \right) \right]^{1/2} \left[ \mathbb{P} \left( \sup_{s \in [0, T]} |\ln(\sigma_s/\sigma_0)| > n \right) \right]^{1/2}$$
\[
\frac{C_n}{\sqrt{T}} \left[ \mathbb{P} \left( \sup_{s \in [0,T]} |\alpha W_s' - \frac{\alpha^2 s}{2}| > n \right) \right]^{\frac{1}{2}} \\
\leq \frac{C_n}{\sqrt{T}} \left[ \mathbb{P} \left( \sup_{s \in [0,T]} |W_s| > \frac{n}{2\alpha} \right) \right]^{\frac{1}{2}},
\]
for some constant \(C_n > 0\) that changes from line to line. Then, Markov’s inequality implies that for all \(p > 2\),
\[
|I(0, k^*) - I^n(0, k^*)| \leq \frac{C_n}{\sqrt{T}} \left( \frac{2\alpha}{n} \right)^{p/2} \left[ \mathbb{E} \left( \sup_{s \in [0,T]} |W_s|^p \right) \right]^{1/2} \leq C_n T^{p/2 - 1/2},
\]
Thus, taking \(p > 4\), we conclude that
\[
\lim_{T \to 0} I(0, k^*) = \frac{\sigma_0}{\sqrt{3}}.
\]
On the other hand, for \(s \leq r \leq t\), we have
\[
D_s^t W_r W_t = \varphi_n''(\alpha W_t - \frac{\alpha^2}{2} t) \sigma_t^2,
\]
which implies that (10) holds with \(H = \frac{1}{2}\). Therefore, appealing to Theorem 2.1 we get that
\[
\lim_{T \to 0} \partial_k I^n(0, k^*) = \frac{\sqrt{3}\rho \alpha \varphi_n'(\sigma_0)}{5\sigma_0^2} + \frac{\sqrt{3} \rho \alpha^3}{30} = \frac{\sqrt{3} \rho \alpha}{5} + \frac{\sqrt{3} \sigma_0}{30}.
\]
Next, similarly as above we can write
\[
\partial_k I(0, k^*) = \partial_k I^n(0, k^*) + \partial_k (I(0, k^*) - I^n(0, k^*)).
\]
By the mean value theorem,
\[
\partial_k (I(0, k^*) - I^n(0, k^*)) = \partial_{\sigma} \partial_k (BS^{-1}(0, X_0, X_0, \xi))(V_0 - V_0^n)
\]
\[
= -e^{-X_0 + \frac{\sigma^2 t}{2}} \frac{\sqrt{2 \pi}}{2} \xi (V_0 - V_0^n),
\]
for some \(\xi \in (V_0, V_0^n)\). Thus, proceeding as above we conclude that
\[
\lim_{T \to 0} \partial_k I(0, k^*) = \frac{\sqrt{3} \rho \alpha}{5} + \frac{\sqrt{3} \sigma_0}{30}.
\]
We next proceed with some numerical simulations using the following parameters
\[
S_0 = 10, \quad T = \frac{1}{252}, \quad dt = \frac{T}{50}, \quad \alpha = 0.5, \quad \rho = -0.3, \quad \sigma_0 = (0.1, 0.2, \ldots, 1.4).
\]
In order to get estimates of an Asian call option we use antithetic variates. The estimate of the price is defined as follows
\[
\hat{V}_{sabr} = \frac{1}{N} \sum_{i=1}^{N} V_T^i + \frac{1}{N} \sum_{i=1}^{N} V_T^{i,A},
\]
where \(N = 2000000\) and the sub-index \(A\) denotes the value of an Asian call option computed on the antithetic trajectory of a Monte Carlo path.
We use equation (28) in order to get estimates of the skew. In Figure 2, we present the results of a Monte Carlo simulation which aims to evaluate numerically the level and the skew of the at-the-money implied volatility of an Asian call option under the SABR model. Again, the numerical results fit the theoretical ones.

Finally, we use the linear approximation (14) for higher maturities. We run a Monte Carlo simulation with model parameters $\rho = -0.3$, $\alpha = 0.2$ and $\sigma_0 = 0.4$. The result is presented in Figure 3. As one can see, taking into account the scale, the asymptotic formula performs well in the case of higher maturities and acts as an upper bound of the implied volatility. Note that discrepancy increases with the maturity of the option, as expected.

5.3. The Fractional Bergomi Model

The fractional Bergomi stochastic volatility model assumes equation (2) with

$$\sigma_t^2 = \sigma_0^2 e^{\nu \sqrt{2H} Z_t - \frac{1}{2} \nu^2 t^{2H}}, \quad Z_t = \int_0^t (t - s)^{H-\frac{1}{2}} dW'_s,$$

where $H \in (0, 1)$ and $\nu > 0$, see Example 2.5.1 in Alòs and García-Lorite (2021).
As for the SABR model, a truncation argument is needed in order to apply Theorem 2.1, as Hypothesis 1 is not satisfied. We define \( \varphi \) and \( \varphi_n \) as for the SABR model, and we set
\[
\sigma^n_t = \varphi_n \left( \frac{1}{2} v \sqrt{2HZ_t} - \frac{1}{4} v^2 t^{2H} \right).
\]
It is easy to see that \( \sigma^n_t \) satisfies Hypotheses 1, 2, (9), and 4. In fact, for \( r \leq t \), we have that
\[
D_r^{W_s} \sigma^n_t = \varphi'_n \left( \frac{1}{2} v \sqrt{2HZ_t} - \frac{1}{4} v^2 t^{2H} \right) \frac{1}{2} v \sqrt{2H(t - r)^{H - \frac{1}{2}}},
\]
which implies that Hypothesis (9) holds and Hypothesis 4 is satisfied with \( \gamma < H \). Moreover, for \( s \leq r \leq t \), we have
\[
D_s^{W_r} D_r^{W_s} \sigma^n_t = \varphi''_n \left( \frac{1}{2} v \sqrt{2HZ_t} - \frac{1}{4} v^2 t^{2H} \right) \frac{1}{4} v^2 \sqrt{4H^4(t - r)^{H - \frac{1}{2}}(t - s)^{H - \frac{1}{2}}},
\]
which implies that (10) holds. Therefore, by Theorem 2.1, we get that (29) holds. Concerning the short maturity limit of the skew, we observe that
\[
\mathbb{E}(D_r^{W_s} \sigma_u) = e^{-\frac{1}{8} v^2 u^{2H}} \frac{1}{2} \sigma_0 v \sqrt{2H(u - r)^{H - \frac{1}{2}}},
\]
which gives
\[
\lim_{T \to 0} \partial_k \mathcal{I}^n(0, k^*) = \begin{cases} \frac{\sqrt{3} \sigma_0}{30} & \text{if } H > \frac{1}{2} \\ \frac{\sqrt{3} \rho v}{10} + \frac{\sqrt{3} \sigma_0}{30} & \text{if } H = \frac{1}{2}, \end{cases} \quad (31)
\]
and for \( H < \frac{1}{2} \)
\[
\lim_{T \to 0} T^{\frac{1}{2} - H} \left( \partial_k \mathcal{I}^n(0, k^*) - \frac{\sqrt{3} \sigma_0}{30} \right) = \frac{3 \sqrt{6H \rho v}}{(1 + H - \frac{1}{2})(2 + H - \frac{1}{2})(3 + H - \frac{1}{2})(5 + H - \frac{1}{2})}. \quad (32)
\]
Finally, similarly as for the SABR model one can easily show that for \( n \) sufficiently large but fixed,
\[
\lim_{T \to 0} (I(0, k^*) - I^n(0, k^*)) = 0
\]
and
\[
\lim_{T \to 0} \partial_k (I(0, k^*) - I^n(0, k^*)) = 0,
\]
so (29), (31), and (32) are also true for \( I(0, k^*) \).

The parameters used for the Monte Carlo simulation are the following
\[
S_0 = 10, \; T = 0.001, \; dt = \frac{T}{50}, \; H = (0.4, 0.7),
\]
\[
v = 0.5, \; \rho = -0.3, \; \sigma_0 = (0.1, 0.2, \ldots, 1.4).
\]
In order to obtain an estimate of the price of an Asian call option under the fractional Bergomi model we use the combination of antithetic and control variates presented in equations (27) and (30). That is, we first sample the process from the Bergomi model and the antithetic analogue. We then average the payoffs calculated from both paths. Finally, use the geometric Asian as control variate assuming constant volatility model at level $\sigma_0$.

In Figure 4, we plot the estimates of the level of the ATM IV of the Asian call option and we observe that the result is independent of $H$ as stated in Theorem 2.1.

In Figure 5, we simulate the ATM IV skew of the Asian call option as a function of the maturity as well as its least squares fit in order to observe the blow up to $-\infty$ for the case $H = 0.4$.

We then plot the quantities $T^{\frac{1}{2}} H \hat{\partial}_k \hat{I}(0, k^*)$ for $H = 0.4$ and $\hat{\partial}_k \hat{I}(0, k^*)$ for $H = 0.7$ in Figure 6 as a function of $\sigma_0$. For $H = 0.4$, the line $-0.0243 + 0.032\sigma_0$ corresponds to the least square fit while formula (32) gives the line $-0.0286 + 0.029\sigma_0$. This difference is due to the numerical instability of the finite difference estimation at short maturity in the presence of rough noise and could be improved by increasing considerably the number of Monte Carlo samples or applying a variance reduction technique. For $H = 0.7$, we observe that formula (31) fits well the Monte Carlo estimates.
Figure 6. At-the-money IV skew as a function of $\sigma_0$ under fractional Bergomi model. (a) $H=0.4$ & (b) $H=0.7$.

Figure 7. Implied volatility approximation as a function of the strike. Rough Bergomi model. (a) 3 Months, (b) 6 Months, (b) 1 Year.

As for the SABR model, we use the approximation (14) for higher maturities. We run a Monte Carlo simulation with model parameters $\rho = -0.3$, $\nu = 0.2$, $h = 0.4$ and $\sigma_0 = 0.4$. The result is presented in Figure 7. We conclude that the approximation performs well since the scale is very small and the approximation acts as an upper bound for an actual implied volatility.

5.4. Local Volatility Model

The short-maturity limit of the ATMIV level and skew of an Asian option under local volatility has already been computed in Pirjol and Zhu (2016). The aim of this section is to check that our Theorem 2.1 provides the same asymptotics as the ones obtained in that paper. We consider the local volatility model

$$dS_t = \sigma(S_t)S_t dW_t,$$

(33)

where $\sigma(.)$ is a twice differentiable function. In Proposition 19 of Pirjol and Zhu (2016) they show that the following expansion holds for $x=\log(KS_0)$ around the ATM point

$$\lim_{T \to 0} I(0, k^*) = \frac{\sigma(S_0)}{\sqrt{3}} \left( 1 + \left( \frac{1}{10} + \frac{3 \sigma'(S_0)}{5 \sigma(S_0)} S_0 \right) x + O(x^2) \right).$$

(34)
Then, differentiating equation (34), we obtain that
\[
\lim_{T \to 0} \partial_k I(0, k^*) = \frac{1}{\sqrt{3}} \left( \frac{1}{10} \sigma(S_0) + \frac{3}{5} S_0 \sigma'(S_0) \right). 
\]  
(35)

We next apply Theorem 2.1 in the case of the local volatility model (33) with \( \sigma_t = \sigma(S_t) \) and \( \rho = 1 \) to verify that we obtain the same expressions as in (11) and (35). For the level, we directly see that when \( \sigma(S_t) \) equals \( \sigma_t \) and \( K = S_0 \), (34) coincides with the limit in (11). For the skew, we need to compute \( D_r \sigma(S_t) \). We have for \( r \leq u \),
\[
D_r \sigma(S_u) = \sigma'(S_u) D_r(S_u) = \sigma'(S_u) \left( \sigma(S_r) S_r + \int_r^u D_r(\sigma(S_s) S_s) dW_s \right).
\]
In particular,
\[
\mathbb{E} (D_r \sigma(S_u)) = \mathbb{E} \left( \sigma'(S_u) \sigma(S_r) S_r \right).
\]
This can be written as
\[
\mathbb{E} (D_r \sigma(S_u)) = \sigma'(S_0) \sigma(S_0) S_0 + \mathbb{E} \left( (\sigma'(S_u) - \sigma'(S_0)) \sigma(S_r) S_r \right)
+ \sigma'(S_0) \mathbb{E} \left( (\sigma(S_r) - \sigma(S_0)) S_r \right).
\]
Then, using the mean value theorem and the fact that \( S_t \) has Hölder continuous sample paths of any order \( \gamma < \frac{1}{2} \), we see that the last two terms of the last display will not contribute in the limit (12). Thus, (12) gives
\[
\lim_{T \to 0} \partial_k I(0, k^*) = \frac{\sqrt{3}}{5} S_0 \sigma'(S_0) + \frac{\sqrt{3} \sigma(S_0)}{30},
\]
which is the same as in (35). This serves as one more evidence of the validity of Theorem 2.1.

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Appendix. Computation of Malliavin derivatives

In this section, we provide the computations of the first and second Malliavin derivatives of the processes $S_t$, $M_t$ and $\phi_t$ defined in Section 2.

Using the fact that $\sigma_t$ is adapted to the filtration of $W^\prime$ and the formula for the derivative of a stochastic integral (see for example (3.6) in Nualart and Nualart 2018), we get that, for $0 \leq s \leq r \leq T$,

$$D^W_s S_r = S_r \left( \rho \sigma_s - \frac{1}{2} \int_s^r D^W_u \sigma^2_u du + \int_s^r D^W_u \sigma_u dW_u \right),$$

$$D^B_s S_r = S_r \sigma_s \sqrt{1 - \rho^2},$$
\[
D_s^W M_r = \frac{\rho \sigma_S(T - s)}{T} + \int_s^T \frac{(T - u)D_s^W (\sigma_u S_u)}{T} \, dW_u,
\]
\[
D_s^B M_r = \frac{\sqrt{1 - \rho^2} \sigma_S(T - s)}{T} + \int_s^T \frac{(T - u)\sigma_u D_s^B (S_u)}{T} \, dW_u.
\]

Moreover, appealing to (8), we find that
\[
D_s^W S_r = \rho S_r \left( -\frac{1}{2} \int_s^T D_s^W \sigma_u^2 \, du \right) + \rho \int_s^T \left( T - u \right) D_s^W (\sigma_u S_u) \, dW_u + D_s^W S_s,
\]
\[
D_s^W M_r = \frac{\sigma_S(T - s)}{T} + \rho \int_s^T \left( T - u \right) D_s^W (\sigma_u S_u) \, dW_u + \sqrt{1 - \rho^2} \int_s^T \left( T - u \right) D_s^B (\sigma_u S_u) \, dW_u.
\]

Finally, from the definition of \( \phi_t \), we conclude that
\[
D_t^W \phi_t = \frac{\rho(T - r)D_t^W (\sigma_r S_r)}{TM_t} - \frac{\rho(T - r)S_r \sigma_r D_t^W M_r}{TM_t^2} + \frac{\sqrt{1 - \rho^2}(T - r)D_t^B (\sigma_r S_r)}{TM_t^2} - \frac{\sqrt{1 - \rho^2}(T - r)S_r \sigma_r D_t^B M_r}{TM_t^2}.
\]

(A1)

We next compute the second Malliavin derivatives. Similarly as before, using the fact that we can differentiate Lebesgue integrals of stochastic processes (see for example Proposition 3.4.3 in Nualart and Nualart (2018)), we get that, for \( 0 \leq s \leq r \leq u \leq T \),
\[
D_s^B D_r^W S_u = S_u \sigma_s \sqrt{1 - \rho^2} \left( \rho \sigma_r - \frac{1}{2} \int_r^u D_r^W \sigma_v^2 \, dv + \int_r^u D_r^W \sigma_v \, dW_v \right),
\]
\[
D_s^W D_r^W S_u = S_u \left( \rho \sigma_r - \frac{1}{2} \int_r^u D_r^W \sigma_v^2 \, dv + \int_r^u D_r^W \sigma_v \, dW_v \right),
\]
\[
\times \left( \rho \sigma_s - \frac{1}{2} \int_r^u D_r^W \sigma_v^2 \, dv + \int_r^u D_r^W \sigma_v \, dW_v \right) + S_u \left( \rho D_s^W \sigma_r - \frac{1}{2} \int_r^u D_s^W D_r^W \sigma_v^2 \, dv + \int_r^u D_s^W D_r^W \sigma_v \, dW_v \right),
\]
\[
D_s^W D_r^B S_u = \sqrt{1 - \rho^2} S_u D_s^W \sigma_r + \sqrt{1 - \rho^2} S_u D_s^W S_u, \tag{A2}
\]
\[
D_s^W D_r^W M_u = \frac{\rho(T - r)D_s^W (\sigma_r S_r)}{T} + \int_r^u \left( T - v \right) D_s^W D_r^W \left( \sigma_r S_r \right) \, dW_v,
\]
\[
D_s^B D_r^W M_u = \frac{\rho(T - r)\sigma_r D_s^B S_r}{T} + \int_r^u \left( T - v \right) D_s^B D_r^W \left( \sigma_r S_r \right) \, dW_v,
\]
\[
D_s^W D_r^B M_u = \frac{\sqrt{1 - \rho^2}(T - r)D_s^W (\sigma_r S_r)}{T} + \int_r^u \left( T - v \right) D_s^B D_r^W \left( \sigma_r S_r \right) \, dW_v,
\]
\[
D_s^B D_r^B M_u = \frac{\sqrt{1 - \rho^2}(T - r)\sigma_r D_s^B S_r}{T} + \int_r^u \left( T - v \right) \sigma_r D_s^B D_r^W \left( S_r \right) \, dW_v,
\]
and
\[
D_s^W D_r^W \phi_u = \frac{\rho^2(T - u)D_s^W D_r^W (\sigma_u S_u)}{TM_u} - \frac{\rho^2(T - u)D_s^W (\sigma_u S_u)D_s^W M_u}{TM_u^2} - \frac{\rho^2(T - u)D_s^W \left( \sigma_u S_u D_s^W M_u \right)}{TM_u^2} + \frac{2 \rho^2(T - u)\sigma_u S_u D_s^W M_u D_s^W M_u}{TM_u^3}.
\]
\[
+ \frac{\rho \sqrt{1 - \rho^2} (T - u) D_1^W D_1^B (\sigma_u S_u)}{T M_u} - \frac{\rho \sqrt{1 - \rho^2} (T - u) D_1^B (\sigma_u S_u) D_1^W M_u}{T M_u^2} \\
- \frac{\rho \sqrt{1 - \rho^2} (T - u) D_1^W (\sigma_u S_u D_1^B M_u)}{T M_u^2} + \frac{2\rho \sqrt{1 - \rho^2} (T - u) \sigma_u S_u D_1^B M_u D_1^W M_u}{T M_u^3} \\
+ \frac{\rho \sqrt{1 - \rho^2} (T - u) D_1^B D_1^W (\sigma_u S_u)}{T M_u} - \frac{\rho \sqrt{1 - \rho^2} (T - u) D_1^W (\sigma_u S_u) D_1^B M_u}{T M_u^2} \\
- \frac{\rho \sqrt{1 - \rho^2} (T - u) D_1^B (\sigma_u S_u D_1^W M_u)}{T M_u^2} + \frac{\rho \sqrt{1 - \rho^2} (T - u) \sigma_u S_u D_1^B M_u D_1^W M_u}{T M_u^3} \\
+ \frac{(1 - \rho^2) (T - u) D_1^B D_1^B (\sigma_u S_u)}{T M_u} - \frac{(1 - \rho^2) (T - u) D_1^B (\sigma_u S_u) D_1^B M_u}{T M_u^2} \\
- \frac{(1 - \rho^2) (T - u) D_1^B (S_u \sigma_u D_1^B M_u)}{T M_u^2} + \frac{2(1 - \rho^2) (T - u) S_u \sigma_u D_1^B M_u D_1^B M_u}{T M_u^3}. \tag{A3}
\]