On the Implications of Lookahead Search in Game Playing

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Abstract. Lookahead search is perhaps the most natural and widely used game playing strategy. Given the practical importance of the method, the aim of this paper is to provide a theoretical performance examination of lookahead search in a wide variety of applications.

To determine a strategy play using lookahead search, each agent predicts multiple levels of possible re-actions to her move (via the use of a search tree), and then chooses the play that optimizes her future payoff accounting for these re-actions. There are several choices of optimization function the agents can choose, where the most appropriate choice of function will depend on the specifics of the actual game - we illustrate this in our examples. Furthermore, the type of search tree chosen by computationally-constrained agent can vary. We focus on the case where agents can evaluate only a bounded number, $k$, of moves into the future. That is, we use depth $k$ search trees and call this approach $k$-lookahead search.

We apply our method in five well-known settings: AdWord auctions; industrial organization (Cournot’s model); congestion games; valid-utility games and basic-utility games; cost-sharing network design games. We consider two questions. First, what is the expected social quality of outcome when agents apply lookahead search? Second, what interactive behaviours can be exhibited when players use lookahead search?

Myopic game playing (whose corresponding equilibria are Nash equilibria), where each player can only foresee the immediate effect of her own actions, is the special case of 1-lookahead search. Thus, for the first question, it is natural to ask whether social outcomes improve when players use more foresight than in myopic behaviour. The answer depends on the game played:

(i) In Adword auctions (or generalized second-price auctions), we show that 2-lookahead game playing results in outcomes that are always optimal to within a constant factor; in contrast, myopic game play can produce arbitrarily poor equilibrium outcomes.

(ii) For the Cournot game, applying 2-lookahead leads to a $12.5\%$ increase in output and a $5.5\%$ increase in social surplus compared with myopic competition. Similar bounds arise as the length $k$ of foresight increases.

(iii) For congestion games, as with myopic game playing, lookahead search leads to constant factor qualitative guarantees.

(iv) For basic-utility games, on the other hand, whilst myopic game playing always leads to constant factor approximations, additional foresight can lead to arbitrarily bad solutions!

(v) In a simple Shapley network design game, qualitative guarantees improve with the length of foresight.

Regarding the second question, a variety of interesting game playing characteristics also arise with lookahead search. Stackelberg leader-follower behaviours can be induced when the players have asymmetric computational power. For example, Stackelberg equilibria can be produced in the Cournot game. Lookahead search can also generate “uncoordinated” cooperative behaviour! An example of this is shown for the Shapely network design game.
1. Introduction

Our goal here is not to prescribe how games should be played. Rather, we wish to analyse how games actually are played. To wit we consider the strategy of lookahead search, described by Pearl in his classical book on heuristic search as being used by “almost all game-playing programs”. To understand the lookahead method and the reasons for its ubiquity in practice, consider an agent trying to decide upon a move in a game. Essentially, her task is to evaluate each of her possible moves (and then select the best one). Equivalently, if she know the values of each child node in the game tree then she can calculate the value of the current node. However, the values of the child nodes may also be unknown! Recall two prominent ways to deal with this. Firstly, crude estimates based upon local information could be used to assign values to the children; this is approach taken by best response dynamics. Secondly, the values of the children can be determined recursively by finding the values of the grandchildren. At its computation extreme, this latter approach in a finite game is Zermelo’s algorithm - assign values to the leaf nodes of the game tree and apply backwards induction to find the value of the current node.

Both these approaches are special cases of lookahead search: choose a local search tree \( T \) rooted at the current node in the game tree; valuations (or estimates thereof) are given to leaf nodes of \( T \); valuations for internal tree nodes are then derived using the values of a node’s immediate descendants via backwards induction; a move is then selected corresponding to the value assigned the root. For best response dynamics the search tree is simply the star graph consisting of the root node and its children. With unbounded computational power, the search tree becomes the complete (remaining) game tree used by Zermelo’s algorithm.

We remark that the actual shape of the search tree \( T \) is chosen dynamically. For example, if local information is sufficient to provide a reliable estimate for a current leaf node \( w \) then there is no need to grow \( T \) beyond \( w \). If not, longer branches rooted at \( w \) need to be added to \( T \). Thus, despite our description in terms of “backwards induction”, lookahead search is a very forward looking procedure. Subject to our computational abilities, we search further forward only if we think it will help evaluate a game node. Indeed, in our opinion, it is this forward looking aspect that makes lookahead search such a natural method, especially for humans and for dynamic (or repeated) games.

Interestingly, the lookahead method was formally proposed as long ago as 1950 by Shannon, who considered it a practical way for machines to tackle complex problems that require “general principles, something of the nature of judgement, and considerable trial and error, rather than a strict, unalterable computing process”. To illustrate the method, Shannon described in detail how it could be applied by a computer to play chess. The choice of chess as an example is not a surprise: as described the lookahead approach is particularly suited to game-playing. It should be emphasised again, however, that this approach is natural for all computationally constrained agents, not just for computers. Lookahead search is an instinctive strategic method utilised by human beings as well. For example, Shannon’s work was in part inspired by De Groot’s influential psychology thesis on human chess players. De Groot found that all players (of whatever standard) used essentially the same thought process - one based upon a lookahead heuristic. Stronger players were better at evaluating positions and at deciding how to grow (prune or extend) the search tree but the underlying approach was always the same.

Despite its widespread application, there has been little theoretical examination of the consequences of decision making determined by the use of local search trees. The goal of this paper is to begin such a theoretical analysis. Specifically, what are the quantitative outcomes and dynamics in various games when players use lookahead search?

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1. Often the values of the leaf nodes will be true values rather than estimates, for example when they correspond to end positions in a game.
2. In contrast, strategies that are prescribed by axiomatic principles, equilibrium constraints, or notions of regret are much less natural for dynamic game players.
1.1. Lookahead Search: The Model.

Having given an informal presentation, let’s now formally describe the lookahead method. Here we consider games with sequential moves that have complete information. These assumptions will help simplify some of the underlying issues, but the lookahead approach can easily be applied to games without these properties.

We have a strategic game $G(\mathcal{P}, \mathcal{S}, \{\alpha_i : i \in \mathcal{P}\})$. Here $\mathcal{P}$ is the set of $n$ players, $\mathcal{S}_i$ is the set of possible strategies for $i \in \mathcal{P}$, $\mathcal{S} = (\mathcal{S}_1 \times \mathcal{S}_2 \ldots \times \mathcal{S}_n)$ is the strategy space, and $\alpha_i : \mathcal{S} \rightarrow R$ is the payoff function for player $i \in \mathcal{P}$. A state $\bar{s} = (s_1, s_2, \ldots, s_n)$ is a vector of strategies $s_i \in \mathcal{S}_i$ for each player $i \in \mathcal{P}$.

Suppose player $i \in \mathcal{P}$ is about to decide upon a move. Recall, with lookahead search, she wishes to assign a value to her current state node $\bar{s} \in \mathcal{S}$ that corresponds to the highest value of a child node. To do this she selects a search tree $T_i$ over the set of states of the game rooted at $\bar{s}$. For each leaf node $\bar{l}$ in $T_i$, player $i$ then assigns a valuation $\Pi_{\bar{l}} = \alpha_j(\bar{l})$ for each player $j$. Valuations for internal nodes in $T_i$ are then calculated by induction as follows: if player $p$ is destined to move at game node $\bar{v}$ then his valuation of the node is given by

$$\Pi_{\bar{v}} = \max_{u \in C(\bar{v})} [r_{p, u} + \Pi_{\bar{u}}].$$

Here, $C(\bar{v})$ denotes the set of children of $\bar{v}$ in $T_i$, and $r_{p, u}$ is some additional payoff received by player $p$ at node $\bar{v}$. Should $p$ choose the child $\bar{u}^* \in C(\bar{v})$ then assume any non-moving player $j \neq p$ places a value of $\Pi_{\bar{u}} = r_{j, u} + \Pi_{\bar{u}}$ on node $\bar{v}$. Then given values for children of the root node $\bar{s}$ of $T_i$, player $i$ is thus able to compute the lookahead payoff $\Pi_{\bar{s}}$ which she uses to select a move to play at $\bar{s}$. [The method is defined in an analogous manner if players seek to minimise rather than maximise their ”payoffs”].

After $i$ has moved, suppose player $j$ is then called upon to move. He applies the same procedure but on a local search tree $T_j$ rooted at the new game node. Note that $j$’s move may not be the move anticipated by $i$ in her analysis. For example, suppose all the players use 2-lookahead search. Then player $i$ calculates on the basis that player $j$ will use a 1-lookahead search tree $T_j'$ when he moves – because for computational purposes it is necessary that $T_j' \subseteq T_i$. But when he moves player $j$ actually uses the 2-lookahead search tree $T_j$ and this tree goes beyond the limits of $T_i$.

1.2. Lookahead Search: The Practicalities.

Observe that there is still a great deal of flexibility in how the players implement the model. This versatility, we would argue, is a major strength (and another reason underlying its ubiquity) and not a weakness of the method. For example, it accords well with Simon’s belief, discussed in Section 1.4, that behaviours should be adaptable. We now give some examples of this adaptability and highlight those aspects that we analyse in this paper.

- **Dynamic Search Trees.** Recall that search trees may be constructed dynamically. Thus, the exact shape of the search tree utilized will be heavily influenced by the current game node, and the experience and learning abilities of the players. Whilst clearly important in determining gameplay and outcomes, these influences are a distraction from our focal point, namely, computation and dynamics in games in which players use lookahead search strategies. Therefore, we will simply assume here that each $T_i$ is a breadth first search tree of depth $k_i$. Implicitly, $k_i$ is dependent on the computational facilities of player $i$.

- **Evaluation Functions.** Different players may evaluate leaf nodes in different ways. To evaluate internal nodes, as described above, we make the standard assumption that they use a max (or min) function. This need not be the case. For example, a risk-averse player may give a higher value to a node (that it does not own) with many high value children than to a node with few high value children – we do not consider such players here.

- **Internal Rewards or Not: Path Model vs Leaf Model.** We distinguish between two broad classes of game that fit in this framework but are conceptually quite different. In the first category, payoffs are determined only by outcomes at the end of game. Valuations at leaf nodes in the local
search trees are then just estimates of the what the final outcome will be if the game reaches that point. Clearly chess falls into this category. In the second category, payoffs can be accumulated over time - thus different paths with the same endpoints may give different payoffs to each player. Repeated games, such as industrial games over multiple time periods, can be modelled as a single game in this category. The first category is modelled by setting all internal rewards \( r_{p,v} = 0 \). Thus what matters in decision making is simply the initial (estimated) valuations a player puts on the leaf nodes. We call this the leaf (payoff) model as an agent then strives to reach a leaf of \( T_i \) with as high a value as possible. The second category arises when the internal rewards, \( r_{p,v} \), can be non-zero. Each agent then wishes to traverse paths that allow for high rewards along the way. More specifically, in this model, called the path (payoff) model, the internal reward is \( r_{p,v} = \alpha_p(\bar{v}) \).

- **Order of Moves: Worst-Case vs Average-Case.** In multiplayer games, the order in which the players move may not be fixed. This adds additional complexity to the decision making process, as the local search tree will change depending upon the order in which players move. Here, we will examine two natural approaches a player may use in this situation: worst case lookahead and average case lookahead. In the former situation, when making a move, a risk-averse player will assume that the subsequent moves are made by different players chosen by an adversary to minimize that player's payoff. In the latter case, the player will assume that each subsequent move is made by a player chosen uniformly at random; we allow players to make consecutive moves. In both cases, to implement the method the player must perform calculations for multiple search trees. This is necessary to either find the worst-case or perform expectation calculations.

### 1.3. Techniques and Results

We want to understand the social quality of outcomes that arise when computationally-bounded agents use \( k \)-lookahead search to optimise their expected or worst-case payoff over the next \( k \) moves. Two natural ways we do this are via equilibria and via the study of game dynamics. To explain these approaches, consider the following definition. Given a lookahead payoff function, \( \Pi_{i,s} \), a lookahead best-response move for player \( i \), at a state \( \bar{s} \in \mathcal{S} \), is a strategy \( s_i \) maximising her lookahead payoff, that is, \( \forall s'_i \in S_i: \Pi_{i,\bar{s}} \geq \Pi_{i,(s_{-i},s'_i)} \). [A move \( s'_i \) for player \( i \), at a state \( \bar{s} \in \mathcal{S} \), is lookahead improving if \( \Pi_{i,\bar{s}} \leq \Pi_{i,(\bar{s}_{-i},s'_i)} \).] A lookahead equilibrium is then a collection of strategies such that each player is playing her lookahead best-response move for that collection of strategies. Our focus here is on pure strategies. Then, given a social value for each state, the coordination ratio (or price of anarchy) of lookahead equilibria is the worst possible ratio between the social value of a lookahead equilibrium and the optimal global social value.

To analyse the dynamics of lookahead best-response moves, we examine the expected social value of states on polynomial length random walks on the lookahead state graph, \( \mathcal{G} \). This graph has a node for each state \( s \in \mathcal{S} \) and an edge from \( \bar{s} \) to a state \( \bar{t} \) with a label \( i \in \mathcal{P} \) if the only difference between \( \bar{s} \) and \( \bar{t} \) is that player \( i \) changes strategy from \( s_i \) to \( t_i \), where \( t_i \) is the lookahead best response move at \( \bar{s} \). The coordination ratio of lookahead dynamics is the worst possible ratio between the expected social value of states on a polynomially long random walk on \( \mathcal{G} \) and the optimal global social value.

For practical reasons, we are usually more interested in the dynamics of lookahead best-response moves than in equilibria. For example, as with other equilibrium concepts, lookahead best-response moves may not lead to lookahead equilibria. Indeed, such equilibria may not even exist. Typically, though, the methods used to bound the coordination ratio for \( k \)-lookahead equilibria can be combined with other techniques to bound the coordination ratio for \( k \)-lookahead dynamics. We show how to do this for congestions games in Section \([\text{4}]\) see also Goemans et al. \([30]\) for several examples with respect to 1-lookahead dynamics. Consequently, for both simplicity and brevity, most of the results we give here concern the coordination ratio for lookahead equilibria. We are particularly interested in discovering when lookahead equilibria guarantee good social solutions, and how outcomes vary with different levels of foresight (\( k \)). We perform our analyses for an assortment of games including an AdWord auction game, the Cournot game, congestion games, valid-utility games, and a cost-sharing network design game.
We begin, in Section 2, by considering strategic bidding in an AdWord generalised second-price auction, and studying the social values of the allocations in the resulting equilibria. In particular, we show that 2-lookahead game playing results in the optimal outcome or a constant-factor approximate outcome under the leaf and path models, respectively. This is in contrast to 1-lookahead (myopic) game playing which can result in arbitrarily poor equilibrium outcomes, and shows that more forward-thinking bidders would produce efficient outcomes.

Second, in Section 3, we examine the Cournot duopoly game. Here two firms compete in producing a good consumed by a set of buyers via the choice of production quantities. We study equilibria of these simple games resulting from $k$-lookahead search. The equilibria of these simple games for myopic game playing, $k = 1$, is well-understood. For $k > 1$, however, firms produce over 10% more than if they were competing myopically; this is better for society as it leads to around a 5% increase in social surplus. Surprisingly, the optimal level of foresight for society is $k = 2$. Furthermore, we show that Stackelberg behaviours arise as a special case of lookahead search where the firms have asymmetric computational abilities.

Third, in Section 4, we examine congestion games with linear latency functions, and study the average of delay of players in those games. We show that 2-lookahead game playing results in constant-factor approximate solutions. In particular, the coordination ratio of lookahead dynamics is a constant. These guarantees are similar to those obtained via 1-lookahead.

Fourth, in Section 4.1, we consider two classes of resource sharing games, known as valid-utility and basic-utility games. For both of these games, we show that lookahead game playing may result in very poor solutions. For valid-utility games, we show $k$-lookahead can give a coordination ratio for lookahead dynamics of $\Theta(\sqrt{n})$. Myopic game play can also give very poor solutions [30], but additional foresight does not significantly improve outcomes in the worst case. For basic-utility games, however, myopic game dynamics give a constant coordination ratio [30] whereas we show that 2-lookahead game playing may result in $o(1)$-approximate social welfare with the leaf model. Thus, additional foresight in games need not lead to better outcomes, as is traditionally assumed in decision theory.

Finally, in Section 5, we present a simple example of a cost-sharing network design game that illustrates how the use of lookahead search can encourage cooperative behaviour (and better outcomes) without a coordination mechanism.

Observe that our results show that lookahead search has different effects depending upon the game. It would be interesting to study further which game structures lead to more beneficial outcomes when longer foresight is used, and which game structures lead to more detrimental outcomes.

1.4. Background and Related Work.
This work is best viewed within the setting of bounded rationality pioneered by Herb Simon. In Rational Choice Theory a rational agent (or economic man) makes decisions via utility maximisation. Whilst the non-existence of economic man is not in doubt, rationality remains a central assumption in economic thought. This is typically justified using an as if as expounded by Friedman [26]: whether people are actually rational or not is unimportant provided their actions can be viewed in a way that is consistent with rational decision making - that is, provided agents act as if they are rational. Friedman concluded that a model should be judged by it predictive value rather than by the realism of its assumptions. On this scale rationality often (but not always) does very well.

However, motivated by considerations of computational power and predictive ability, Simon [69] argued that “the task is to replace the global rationality of economic man with a kind of rational behaviour that is compatible with the access to information and the computational capacities that are actually possessed by organisms, including man, in the kinds of environments in which such organisms exist”. He argued that, instead of optimising, agents apply heuristics in decision making. An example of this being the satisficing heuristic: agents search for feasible solutions, stopping when then discover

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For example, a consumer whose purchasing strategy allocates fixed proportions of her budget to specific goods (regardless of price levels) can be viewed as rational consumer with a Cobb-Douglas utility function!
an outcome that achieves an aspired level of satisfaction. We remark that the use of a search phase provides a fundamental distinction between rational and boundedly rational agents. For rational agents the search is irrelevant as they will anyway make an optimal choice given the constraints of the problem. For agents of bounded rationality the form of the search can heavily influence decision making.

Interestingly, De Groot’s work on chess players also heavily influenced Simon’s general thinking on cognitive science. This is exemplified in his famous book with Newell on human problem solving, where humans are viewed as information processing systems.

The label bounded rationality is currently used in a number of disparate areas some of which actually go against the main thrust of Simon’s original ideas; see Selten and Rubenstein for some discussion on this point. Two schools of thought developed by psychologists, experimental economists, and behavioural economists are, however, well worth mentioning here. First, the Heuristics and Biases program espoused by Kahneman and Tversky and, second, the Fast and Frugal Heuristics program espoused by Gigerenzer. Whilst both programs agree that humans routinely use simple heuristics in decision making, their philosophical outlooks are very different. The former program primarily looks for outcomes (caused by the use of heuristics) in violation of subjective excepted utility theory, and views such biases as a sign of irrationality likely to lead to poor decision making. In contrast, the latter program views the use of heuristics as natural and, in principle, entirely compatible with good decision making. For example, simple heuristics may be more robust to environmental changes and actually outperform methods based upon subjective excepted utility maximisation. As with the work of Simon, for the fast and frugal heuristics school, the actual quality of an heuristic is assumed to be dependent upon the search - how to search and when to stop searching - and the choice of decision rule after the search is terminated. Clearly, the lookahead heuristic can be viewed in this light: there is a search (via a local search tree), there is a “stopping rule” (determined, for example, by computational constraints and by the expertise of the player), and there is a decision rule (backwards induction).

The value of lookahead search in decision-making has been examined by the artificial intelligence community; for examples in effective diagnostics and real-time planning see. Lookahead search is also related to the sequential thinking framework in game theory. However, compared to these works and the research carried out by the two schools above, our focus is more theoretical and less experimental and psychological. Specifically, we desire quantitative performance guarantees for our heuristics.

Our research is also related to works on the price of anarchy in a game, and convergence of game dynamics to approximately optimal solutions and to sink equilibria. Numerous articles study the convergence rate of best-response dynamics to approximately optimal solutions. For example, polynomial-time bounds have been proven for the speed of convergence to approximately optimal solutions for approximate Nash dynamics in a large class of potential games, and for learning-based regret-minimisation dynamics for valid-utility games. Our work differs from all the above as none of them capture lookahead dynamics. In another line of work, convergence of best-response dynamics to (approximate) equilibria and the complexity of game dynamics and sink equilibria have been studied, but our paper does not focus on these types of dynamics or convergence to equilibria.

Motivated by concerns of stability, convergence, and predictability of equilibria and game dynamics, various equilibrium concepts other than Nash equilibria have been studied in the economics literature. Among them are correlated equilibria, stable equilibria, stochastic adjustment models, strategy subsets closed under rational behaviour (CURB set), iterative elimination of dominated strategies, the set of undominated strategies, etc. Convergence and strategic stability of equilibria in evolutionary game theory is also an important subject of study. Many other game-theoretic models have been proposed to capture the self-interested behaviour of agents. As well as best-response dynamics...

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4Over time, and depending upon what is found in the search, this aspiration level may be changed.  
5In fact, Simon sent his student George Baylor to help translate De Groot’s work into English.
myopic behaviour can produce very poor outcomes. Our main results are that outcomes are provably good when agents use additional foresight; in contrast, should bid truthfully, that is \( v \) slots, each with a private valuation \( c \). Each player \( i \) makes a bid \( b^i \). Slots are then allocated via a generalised second price auction. Denote the \( j \)th highest bid in the descending bid sequence by \( b_j \), with corresponding valuation \( v_j \). The \( j \)th best slot, for \( j \leq T \), is assigned to the \( j \)th highest bidder who is charged a price equal to \( b_{j+1} \). The \( T \) highest bidders are called the “winners”. According to the pricing mechanism, if bidder \( i \) were to get slot \( t \) in the final assignment, then he would get utility \( u^i_t = (v^i_t - b_{t+1})c_t \). We denote a player \( i \)’s utility if he bids \( b^i \) by \( u^i(b^i) \) (the other players bids are implicit inputs for \( u^i )

This auction is used in the context of keyword ad auctions (e.g, Google AdWords) for sponsored search. Given the continuous nature of bids in the GSP auction, the best response of each bidder \( i \) for any vector of bids by other bidders corresponds to a range of bid values that will result in the same outcome from \( i \)’s perspective. Among these set of bid values, we focus on a specific bid value \( b^i \), called the balanced bid \( b^i \). The balanced bid \( b^i \) is a best-response bid that is as high as possible such that player \( i \) cannot be harmed by a player with a better slot undercutting him, i.e. bidding just below him. It is easy to calculate that for player \( i \) in slot \( t \), \( 1 \leq t < T \), the only balanced bid is

\[
b^i = (1 - \frac{c_t}{c_{t-1}}) v^i + \frac{c_t}{c_{t-1}} b_{t+1}.
\]

An important property of balanced bidding is that each “losing” player \( i \) (one not assigned a slot) should bid truthfully, that is \( b^i = v^i \). To see this add dummy slots with \( c_t = 0 \) if \( t > T \). The player who wins the top slot should also bid truthfully under balanced bidding. Balanced bidding is the most commonly used bidding strategy \[13, 18\]. For some intuition behind this, note that balanced bidding has several desirable properties. For a competitive firm, bidding high obviously increases the chance of obtaining a good slot. Within a slot this also has the benefit of pushing up the price a competitor pays without affecting the price paid by the firm. On the other hand, bidding high increases the upper bound on the price the firm may pay, leading to the possibility that the firm may end up paying a high price for one of the less desirable slots. Balanced bidding eliminates the possibility that a change in bid from a higher bidder can hurt the firm. (Clearly, it is impossible to obtain such a guarantee with respect to a lower bidder.) Thus, balanced bidding provides some of the benefits of high bidding at less risk. Balanced bidding naturally converges to Nash equilibria unlike other bidding strategies such as altruistic bidding or competitor busting \[13\]. Moreover, the other bidding strategies would require some discretization of players’ strategy space in order to analyse the best response dynamics \[13, 18\]. Consequently, balanced bidding is the most natural strategy choice for our analysis.

For this auction problem, we consider only the leaf model. The leaf model seems more natural than the path model for a single auction as players are interested in the final allocation output by the auction (there are no intermediary payoffs). We analyse both worst-case and average-case lookahead; depending upon the level of risk-aversion of the agents both cases seem natural in auction settings.
Let player \( i \)'s lookahead payoff (or utility) at bid \( b' \) with respect to player \( j \), denoted by \( u^{ij}(b') \), be player \( i \)'s payoff (or utility) after player \( j \) makes a best-response move. In the worst-case lookahead model, we define player \( i \)'s lookahead payoff for a vector \( \tilde{b} \) of bids as \( \Pi_{i, \tilde{b}} = \tilde{u}^i(\tilde{b}') = \min_j u^{ij}(\tilde{b}') \). In the average-case lookahead model, player \( i \)'s lookahead payoff \( \Pi_{i, \tilde{b}} \) for a bid vector \( \tilde{b} \) is \( \Pi_{i, \tilde{b}} = \bar{u}^i(\tilde{b}') = \frac{1}{n} \sum_j u^{ij}(\tilde{b}') \). Changing strategy from bid \( b' \) to bid \( \tilde{b} \) is a \emph{lookahead improving} move if lookahead utility increases, i.e., \( \bar{u}^i(\tilde{b}') > \tilde{u}^i(\tilde{b}') \). We are at a \emph{lookahead equilibrium} if no player has a lookahead improving move.

It is known that the social welfare of Nash equilibria for myopic game playing can be arbitrarily bad \cite{13} unless we disallow over-bidding \cite{10}. Here, we prove the advantage of additional foresight by showing that 2-lookahead equilibria have much better social welfare. In particular, we show that all such equilibria are optimal in the worst-case lookahead model, and all such equilibria are constant-factor approximate solutions in the average-case lookahead model.

### 2.1. Worst-Case Lookahead

Our proof for the worst-case lookahead model can be seen as a generalisation of the proof of \cite{12} for a slightly different model. We start by proving a useful lemma in this context.

**Lemma 2.1.** Consider the worst-case lookahead model with the leaf model. Label the players so that player \( i \) is in slot \( i \), and suppose there is a player \( t \) such that \( v^t < v^{t+1} \). Then player \( t \) myopically prefers slot \( t + 1 \) to slot \( t \).

**Proof.** Suppose not. Then, as player \( t \) does not myopically prefer slot \( t + 1 \) we have

\[
(v^t - b_{t+1})c_t \geq (v^t - b_{t+2})c_{t+1}
\]

By definition, \( b_{t+1} = v_{t+1} - \frac{c_{t+1}}{c_t}(v_{t+1} - b_{t+2}) \). Plugging this in gives

\[
(v^t - b_{t+2})c_{t+1} \leq \left( v^t - \frac{c_t}{c_t}v_{t+1} - \frac{c_{t+1}}{c_t}b_{t+2} \right) c_t < \left( \frac{c_{t+1}}{c_t}v^t - \frac{c_{t+1}}{c_t}b_{t+2} \right) c_t = (v^t - b_{t+2})c_{t+1}
\]

Thus we obtain our desired contradiction. Note that the strict inequality above follows directly from the fact that \( v^t < v^{t+1} \).

An equilibrium is \emph{output truthful} if the slots are assigned to the same bidders as they would be if bidders were to bid truthfully. It is easy to verify that an an allocation optimizes social welfare if and only if it is output truthful. Thus to prove 2-lookahead equilibria are socially optimal it suffices to show they are output truthful.

**Theorem 2.2.** For GSP auctions, any 2-lookahead equilibrium gives optimal social welfare in the worst-case, leaf model.

**Proof.** We proceed by contradiction. Consider a non-output-truthful 2-lookahead equilibrium. Again, label the players so that the player \( i \) is in slot \( i \). Amongst all the winning players, take the one with the lowest valuation, \( v_i \). First suppose that \( v_i \) is not amongst the \( T \) highest valuations. Then, there is a losing player with a higher value than \( v_i \). But this player is bidding his value, as a result of balanced bidding. Consequently, player \( i \)'s utility must be negative, a contradiction.

Thus, we may assume that \( v_i \) is amongst the \( T \) highest valuations; specifically it must have exactly the \( T \)th highest valuation. We will show that player \( i \) moving into slot \( T \) is a lookahead improving move. Notice that the lookahead value for player \( i \) staying in slot \( i \) is at most the myopic value of staying in that slot. This follows as the choice of a player two slots below \( i \) cannot improve the utility of player \( i \) (neither in terms of price nor slot position), but only could make it worse. Hence, it suffices to show that the lookahead value of changing slots is better than the myopic value of staying in slot \( i \).

By several applications of Lemma 2.1, we see that player \( i \) myopically prefers slot \( T \) to slot \( i \). However, in moving to slot \( T \), player \( i \) will still make a balanced bid. Thus, no other winning player may reduce \( i \)'s utility by undercutting him. Also, no losing player \( j \) wants to move to a winning slot as
they can only be left with negative utility - since \( j \) cannot then be amongst the \( T \) highest valuations. So moving to slot \( T \) is a lookahead improving move for player \( i \).

If player \( i \) were originally in slot \( T \), then the entire argument can be applied with regards to slots 1 to \( T - 1 \). Inductively, we then conclude that in any non-output-truthful equilibrium, there is a lookahead improving move, which is a contradiction. This gives us the desired result. \( \square \)

2.2. Average Case Lookahead.

Next, we consider the average-case lookahead model and show that the above theorem does not hold for this case.

**Theorem 2.3.** In GSP auctions, there exist 2-lookahead equilibria that are not output-truthful in the average-case, leaf model.

**Proof.** Consider the following example with \( n = T = 4 \). Let the click-through rates be \( c_1 = 35, c_2 = 26, c_3 = 25 \), and \( c_4 = 20 \). Let the valuations be \( v_1 = 82, v_2 = 83, v_3 = 100, v_4 = 93 \). Starting with the highest slot and working to the lowest, let bidder \( i \) bid the balanced bid for slot \( i \). It can be verified that this turns out to be a non-output-truthful equilibria. \( \square \)

Despite this negative result, 2-lookahead equilibria cannot have arbitrarily bad social welfare.

**Theorem 2.4.** In GSP auctions, the coordination ratio of 2-lookahead equilibria is constant in the average-case, leaf model.

**Proof.** Suppose that we are at an equilibrium. Let \( v_i^* \) be the \( i \)th highest valuation, let player \( i^* \) denote the corresponding player, let \( b_i^* \) denote their bid, and \( c_i^* \) the click through rate of the slot they currently occupy. We recall that \( v_i \) denotes the player in slot \( i \) and it has click through rate \( c_i \) and bid \( b_i \). The social utility of a set \( A \) of players is \( \sum_{i \in A} v_i c_i \). Thus, by the above definitions, the optimal social utility is \( \sum_i v_i^* c_i \).

Now, choose \( \alpha, \beta < 1 \) such that \( (1-\alpha)^2 > m \beta \). Let \( I \) be the set of indices \( i \) that satisfy both \( v_i < \alpha v_i^* \) and \( c_i^* < \beta c_i \). Note that for all \( i \notin I \) the pair of players \( v_i, v_i^* \) contribute at least \( \min\{\alpha, \beta\} v_i^* c_i \) to \( \text{OPT} \). So if \( I \) is empty, then we have achieved a constant coordination ratio. We may thus suppose \( I \) is not empty and choose \( i \in I \).

Consider \( c_{i^*-1} \). As we assume “balanced” bidding,

\[
b_i^* \geq (1 - \frac{c_{i^*}}{c_{i^*-1}}) v_i^* \]

Since \( b_i^* < b_i < v_i < \alpha v_i^* \) by assumption, we have \( c_{i^*-1} < \frac{1}{1-\alpha} c_{i^*} \). Choose \( m > 1 \). We first prove the following claim.

**Claim 2.5.** For all \( i \in I \), we have \( c_{i+1} \leq \frac{c_i}{m} \).

**Proof.** Suppose \( c_{i+1} > \frac{c_i}{m} \), for some \( i \in I \). We will show that player \( i^* \) moving into slot \( i \) is then lookahead improving. Consider his lookahead utility for staying put. Ignoring a repeat move for player \( i^* \), which occurs with probability \( \frac{1}{n} \), player \( i^* \)'s utility in every other circumstance is at most \( c_{i^*-1} v_i^* \), as other players can improve his position by at most one. On the other hand, if player \( i^* \) moves into slot \( i \) then his lookahead utility is at least \( c_{i+1}(v_i^* - b_i) \); he wins at least slot \( i + 1 \) and pays at most his bid. If player \( i \) is chosen to repeat his move then his utility is the same for both cases (as he will then simply play a best response move). Thus, it is enough for us to show that

\[
c_{i+1}(v_i^* - b_i) > c_{i^*-1} v_i^* \]
However $b_i < v_i < av_i^*$ and putting this together with the above inequalities gives

$$
c_i(v_i^* - b_i) > \frac{c_i}{m}(1 - \alpha)v_i^*
\geq \frac{\beta}{1 - \alpha}c_i v_i^*
\geq \frac{\beta}{1 - \alpha}c_i v_i^*
> \frac{c_i}{1 - \alpha}v_i^*
> c_{i^* - 1}v_i^*
$$

We are now done, by our choice of $\alpha$ and $\beta$, and have shown that player $i^*$ moving into slot $i$ is a lookahead improving move. This contradicts the fact we are at an equilibria.

Thus we have established that for all $i \in I$, $c_i < \frac{c_i}{m}$. Thus, we can bound the optimal social utility contributed by the slots $i \in I$ by $\frac{m}{m-1}c_{i_0}v_{i_0^*}$ where $i_0 = \min_{i \in I} i$.

Now if $1 \notin I$ then we have achieved our constant coordination ratio since then either $c_1v_1 > \alpha c_1v_1^*$ or $c_{i^*}v_{i^*} \geq \beta c_1v_1^*$. Hence, we are guaranteed at least $\min\{\alpha, \beta\}c_1v_1 \geq \min\{\alpha, \beta\}c_{i_0}v_{i_0^*}$, that is, a least a constant factor of the social utility from all the slots in $I$ in the optimal allocation. So we suppose $1 \in I$.

Choose $\alpha_1 = \frac{m}{m-1}\alpha$ and consider the player currently in slot 2. By this choice of $\alpha_1$, we ensure that this player does not have value more than $\alpha_1v_1^*$. To see this, recall the player is bidding in a balanced manner and so, by Claim 2.5, his bid $b_2$ satisfies

$$v_2 \geq b_2 \geq (1 - \frac{c_2}{c_1})v_2 \geq (1 - \frac{1}{m})v_2$$

On the other hand, as $1 \in I$ we have

$$b_1 = v_1 \leq \alpha v_1^*$$

Thus, we must have $v_2 \leq \frac{m}{m-1}\alpha v_1^* = \alpha_1v_1^*$, or the second player would win the first slot.

Now let $\Gamma$ be the set of players with value at least $\alpha_1v_1^*$. Choose some constant $\gamma$. If $|\Gamma| < \gamma n$, then player $1^*$’s lookahead utility for moving into slot one is at least $(1 - \gamma)(1 - \alpha)v_1^*$. If player $1^*$ stays put, ignoring a repeat move for player $1^*$, which occurs with probability $\frac{1}{n}$, player $i^*$’s utility in every other circumstance is at most

$$c_{i^* - 1}v_{i^*} < \frac{1}{1 - \alpha}c_{i^*}v_{i^*} < \frac{\beta}{1 - \alpha}c_1v_1^*$$

Since player $1^*$’s utility is the same for both cases when a repeated move occurs and since we can choose $\beta$ sufficiently small (i.e, $\beta < (1 - \gamma)(1 - \alpha)(1 - \alpha_1)$), player $1^*$ will improve by moving into slot 1 in this case, contradicting the fact that we are at an equilibrium.

Thus, we may suppose $|\Gamma| > \gamma n$. Let $i_1 = \max_{i \in I} i$. Then the players in $\Gamma$ contribute at least $\gamma n\alpha_1v_1^*c_{i_1}$ to the social utility. Take a constant $\delta$ and suppose that $c_{i_1} \geq \delta c_{i_1}$. Then the players in $\Gamma$ would contribute at least $\gamma \delta \alpha_1 c_1v_1^*$. Again, this a constant fraction of social utility that is contributed in the optimal allocation by player $1^*$ which, in turn, is a constant factor of the optimal social utility of the slots in $I$. Thus, we would achieve a constant factor of the optimal social utility.

So we may assume $c_{i_1} < \frac{\delta}{\gamma n^2}$. Consider player $i_1$. His lookahead utility for staying in place, ignoring the case of a repeated move, is at most

$$c_{i_1 - 1}v_{i_1} \leq \frac{1}{1 - \alpha}c_{i_1}v_{i_1} \leq \frac{1}{1 - \alpha}\frac{\delta}{n}c_{i_1}v_{i_1} \leq \frac{1}{1 - \alpha}\frac{\delta}{n}c_{i_1}v_{i_1^*}$$

We may assume that player $v_1 \leq (1 - \epsilon)\alpha_1v_1^*$, for some constant $\epsilon$, otherwise we are done. Therefore, if player $i_1$ moves to slot 1 then he will earn at least $\epsilon \alpha_1v_1^*$ provided that player 1 makes the next move. This occurs with probability $1/n$, and so his total lookahead utility, ignoring a repeated move,
3. INDUSTRIAL ORGANISATION: COURNOT COMPETITION

Next we consider the classical game theoretic topic of duopolistic competition. Economists have considered a number of alternative models for market competition [17], prominent amongst them is the Cournot model [17]. Our main result here is that the social surplus increases when firms are not myopic; surprisingly, social welfare is actually maximized when firms use 2-lookahead.

The Cournot model assumes players sell identical, nondifferentiated goods, and studies competition in terms of quantity (rather than price). Each player takes turns choosing some quantity of good to produce, $q_l$, and pays some marginal cost to produce it, $c$. The price for the good is then set as a function of the quantities produced by both players, $P(q_l + q_j) = (a - q_l - q_j)$, for some constant $a > c$. On turn $l$, each player $i$ makes profit: $\Pi^i_l(q_l, q_j) = q_l(a - q_l - q_j - c)$. In this form, the model then has only one equilibrium, called the Cournot equilibrium, where $q_l = (a - c)/3$ for each player. At equilibrium, each player make a profit of $\Pi^i_l(q_l, q_j) = q_l(1 - 2q_l)$. The consumer surplus is then $2q^2_l$ and the social surplus is then $2q^2_l(1 - q_l)$.

3.1. PRODUCTION UNDER LOOKAHEAD SEARCH.

We analyse this game when players apply $k$-lookahead search. In industrial settings it is natural to assume that payoffs are collected over time (as in a repeated game); thus, we focus upon the path model. We define this model inductively. In a $k$-step lookahead path model, each player $i$’s utility is the sum of his utilities in the current turn and the $k-1$ subsequent turns. He models the quantities chosen in the subsequent turns as though the player acting during those turns were playing the game with a smaller lookahead. More specifically, he assumes that the player acting in the $t^{th}$ subsequent turn chooses their quantity to maximise their utility under a $k-t$ lookahead model. In order to rewrite this rigorously, let $\pi_j^t$ be the contribution to his utility that player $i$ expects on the $l^{th}$ subsequent turn (and $\pi^0_j$ be the contribution to his utility that player $i$ expects on his current turn), let $\pi^t_j$ be the contribution to player $j$’s utility that player $i$ expects on the $l^{th}$ subsequent turn, and let $q^t_l$ (respectively, $q_l^j$) be the quantity that player $i$ expects to choose (respectively, expects his opponent to choose) under this model.

Then in the path model, player $i$’s expected utility function is $\Pi_l = \sum_{t=0}^{k-1} \pi^t_l$. Player $j$’s expected utility function on player $i$’s turn is $\Pi_j = \sum_{t=0}^{k-1} \pi^t_j$. Our aim now is to determine the quantities that player $i$ expects to be chosen by both players in the subsequent turns and, thereby, determine the quantity he chooses this turn and the utility he expects to garner. To facilitate the discussion, it should be noted that unless noted otherwise, any reference to a “turn” refers to a turn during player $i$’s calculation and not an actual game turn.

To simplify our analysis, we will define $q_l$ to be the quantity chosen on turn $l$ by whichever player is acting and $\Pi_l$ to be the expected utility that that player garners from turn $l$ to turn $k$. So $\Pi_0 = \Pi$, $\Pi_1 = \sum_{t=0}^{k-1} \pi^t_1$, etc. We define $\Pi_l$ to be the utility garnered from turn $l$ to turn $k$ by the player who does not act during turn $l$. So $\Pi_0 = \Pi$, $\Pi_1 = \sum_{t=1}^{k-1} \pi^t_1$, etc. It is clear that on each turn $l$, the active player is trying to maximise $\Pi_l$.

We are now ready to compute these quantities and utilities recursively. We may assume that $a = 1$ and $c = 0$. By our definition above, we have that $\Pi_k = q_k(1 - q_k - q_k - c)$ and $\Pi_k = q_k(1 - q_k c - q_k + c)$.

Our definition also gives us the recursive formula for $l < k$ that $\Pi_l = q_l(1 - q_l - q_{l+1} + c) + \Pi_{l+1}$ and $\Pi_l = q_{l-1}(1 - q_{l-1} - q_{l-1}) + \Pi_{l+1}$. Note that in each of these formulas, $\Pi_l$ and $\Pi_l$ are each functions of $q_l$ for $t \geq l$; $q_{l-1}$ is in fact fixed on the previous turn and is, therefore, not a variable in $\Pi_l$. It is now possible to calculate $q_l$ recursively.
Lemma 3.1. The form of \( q_l \) is \( \beta_l - \alpha_l q_{l-1} \), where \( \beta_k = \alpha_k = \beta_{k-1} = \frac{1}{2}, \alpha_{k-1} = \frac{1}{3} \) and, for \( l < k - 1 \),

\[
\beta_l = \frac{2 - \beta_{l+1} + \alpha_{l+1} \beta_{l+2} - \alpha_{l+1} \alpha_{l+2} \beta_{l+1}}{4 - 2 \alpha_{l+1} - \alpha_{l+1}^2 \alpha_{l+2}}, \quad \alpha_l = \frac{1}{4 - 2 \alpha_{l+1} - \alpha_{l+1}^2 \alpha_{l+2}}.
\]

Proof. We proceed by inducting down from \( q_k \). Consider \( q_k \) which is the active player’s choice on the final turn. As it is the final turn, he is acting myopically and so will choose \( q_k \) so as to maximise \( \Pi_k = q_k (1 - q_k - q_{k-1}) \). This parabola as a function of \( q_k \) is maximised when \( q_k = \frac{1}{2} - q_{k-1} \). Doing a similar calculation for \( \Pi_{k-1} = q_{k-1} (1 - q_{k-1} - q_{k-2}) + \Pi_k \) gives us the desired values for \( \beta_{k-1} \) and \( \alpha_{k-1} \). We now assume the lemma for all \( l > L \) and try to prove it for \( q_l \). Recall the recursive formula \( \Pi_L = q_L (1 - q_L - q_{L-1}) + \Pi_{L+1} \). Taking the derivative of this with respect to \( q_L \) and setting it all equal to zero gives us

\[
0 = (1 - 2q_L - q_{L-1}) + (1 - 2q_L - q_{L+1}) - \frac{\partial \Pi_{L+1}}{\partial q_L} q_L - \frac{\partial q_{L+1}}{\partial q_L} q_{L+2} + \frac{\partial \Pi_{L+2}}{\partial q_{L+2}} \frac{\partial q_{L+2}}{\partial q_L}.
\]

The last term of the above sum is zero, since \( q_{L+2} \) is chosen so that \( \frac{\partial \Pi_{L+2}}{q_{L+2}} = 0 \). Thus, if we plug in the inductive hypothesis into the above equation and simplify, we get

\[
2 - \beta_{L+1} + \alpha_{L+1} \beta_{L+2} - \alpha_{L+1} \beta_{L+2} - \alpha_{L+1} \alpha_{L+2} \beta_{L+1} = (4 - 2 \alpha_{L+1} - \alpha_{L+1}^2 \alpha_{L+2}) q_L - q_{L-1}
\]

This gives us the desired result. \( \square \)

Our goal is now to calculate \( q_0 \) as this will tell us the quantity that player \( i \) actually chooses on his turn. From the above lemma, we can calculate \( q_0 \) if we can determine \( \alpha_0 \) and \( \beta_0 \). Using numerical methods on the above recursive formula, we see that as \( k \to \infty \), \( \alpha_0 \) decreases towards a limit of 0.2955977... and \( \beta_0 \) approaches a limit of 0.4790699... These values also converge quite quickly; they both converge to within 0.0001 of the limiting value for \( k \geq 10 \). Thus, at a lookahead equilibrium, player \( i \) will choose \( q_i \approx 0.4790699 - 0.2955977 q_j \) and player \( j \), symmetrically, will choose \( q_j \approx 0.4790699 - 0.2955977 q_i \). So each player will choose a quantity \( q \approx 0.369767 \), which is more than in the myopic equilibrium. Indeed, it is easy to show that for every \( k \geq 2 \), each player will produce more than the myopic equilibrium. This is illustrated in Figure 1. Observe the quantity produced does not change monotonically with the length of foresight \( k \), but it does increase significantly if non-myopic lookahead is applied at all. Consequently, in the path model looking ahead is better for society overall but worse for each individual firm’s profitability (as the increase in sales is outweighed by the consequent reduction in price).

![Figure 1](image.png)

**Figure 1.** How output varies with foresight \( k \)
Theorem 3.2. For Cournot games under the path model, output at a k-lookahead equilibrium peaks at $k = 2$ with output 12.5% larger than at a myopic equilibrium ($k = 1$). As foresight increases, output is 10.9% larger in the limit. The associated rises in social surplus are 5.5% and 4.9%, respectively.

3.2. Stackelberg Behaviour.
We could also analyse this game under the leaf model, but this model is both less realistic here and trivial to analyse. However, it is interesting to note that for the leaf model with asymmetric lookahead, where player $i$ has 2-lookahead and player $j$ has 1-lookahead, we get the same equilibrium as the classic Stackelberg model for competition. Thus, the use of lookahead search can generate leader-follower behaviours.

4. Unsplittable Selfish Routing
Now consider the unsplittable selfish routing game. We show that any 2-lookahead equilibrium has a constant coordination ratio. We then show how to derive a similar result for 2-lookahead dynamics.

For this game we have a directed graph $G = (V,E)$ and a set of $n$ agents. Agent $i$ wants to route 1 unit of flow from a source $s_i$ to a destination $t_i$. Each agent $i$ chooses an $s_i - t_i$ path $P_i$ and these paths together generate a flow $f$. We assume that there is a linear latency function $\lambda_e(f_e) = a_e f_e + b_e$ on each edge edge $e \in E$. The total latency of a flow $f$ is denoted $l(f) = \sum_{e \in E} \lambda_e(f_e) = (a_e f_e + b_e)$. The latency of player $i$ is denoted $l_i(f) = \sum_{e \in P_i} a_e f_e + b_e$; observe that $l(f) = \sum_{i \in U} l_i(f)$. For this game, we consider 2-lookahead in both the leaf and path models, under the average-case lookahead model.

Recall, in the leaf model, a player $i$’s move from a flow $f$ to a flow $f'$ is lookahead improving if $E(l_i(f'')|f') > E(l_i(f'')|f)$ where $f''$ is the flow obtained after the next player (chosen uniformly at random amongst all the players) makes a (myopic) best response. In the path model a player $i$’s move from a flow $f$ to a flow $f'$ is lookahead improving if $\frac{1}{2} l_i(f') + \frac{1}{2} E(l_i(f'')|f') > \frac{1}{2} l_i(f) + \frac{1}{2} E(f''|f)$ where $f''$ is as above.

Theorem 4.1. In the average-case 2-lookahead leaf model, the coordination ratio for an equilibrium is at most $(1 + \sqrt{5})^2$.

Proof. This proof adapts the result in [3] to our setting. Let $f$ be any flow at a lookahead equilibrium and $f^*$ be an optimal flow. Suppose player $i$ is taking path $P_j$ in flow $f$ and path $P^*_j$ in flow $f^*$. Let $J(e)$ be the set of players using edge $e$ in the flow $f$ and let $J^*(e)$ be the same for $f^*$.

At a lookahead equilibrium, player $j$ doesn’t want to move from $P_j$ to $P^*_j$. This means that after a random/worst case next move, the strategy $P_j$ has a higher (expected) payoff than the strategy $P^*_j$. In particular, it must the case that the best possible outcome resulting from from choosing $P_j$ has a higher (expected) payoff than the worst possible outcome resulting from the strategy $P^*_j$. In the former case, the best possible outcome is that the next player had also been using the path $P_j$ but then moves completely off the path. Similarly, in the latter case, the worst possible outcome is that the next player had not been using any edge on the path $P^*_j$ but then changes strategy and also selects the path $P^*_j$ entirely. Thus we must have:

$$\sum_{e \in P^*_j} a_e (f_e + 2) + b_e \geq \sum_{e \in P_j} a_e f_e + b_e - \sum_{e \in P_j, f_e \geq 2} a_e$$
Summing over all players \( j \), we obtain

\[
\sum_j \sum_{e \in P_j^*} a_e(f_e + 2) + b_e \geq \sum_j \left( \sum_{e \in P_j} a_ef_e + b_e - \sum_{e \in P_j : f_e \geq 2} a_e \right)
\]

\[
= \sum_{e \in E} \left( a_e(f_e + 2) + b_e \right) - \sum_{j} \sum_{e \in P_j : f_e \geq 2} a_e f_e
\]

\[
\geq \sum_{e \in E} \left( a_e(f_e + b_e) - \sum_{e \in P_j : f_e \geq 2} \frac{1}{2} a_e f_e^2 \right)
\]

\[
\geq \sum_{e \in E} \frac{1}{2} (a_e(f_e + b_e) f_e)
\]

\[
= \frac{1}{2} \sum_{e \in E} \lambda_e(f_e)
\]

Rearranging gives and applying the Cauchy-Schwartz inequality\(^6\) produces

\[
\frac{1}{2} \sum_{e \in E} \lambda_e(f_e) \leq \sum_j \left( \sum_{e \in P_j^*} a_e(f_e + 2) + b_e \right)
\]

\[
= \sum_{e \in E} \left( a_e(f_e + 2) + b_e \right) f_e^*
\]

\[
\leq \sum_{e \in E} a_e f_e f_e^* + (2a_e + b_e) f_e^*
\]

\[
\leq \sum_{e \in E} a_e f_e f_e^* + 2\lambda_e(f_e^*)
\]

\[
\leq \sqrt{\sum_{e \in E} a_e f_e^2} \cdot \sqrt{\sum_{e \in E} \lambda_e(f_e^*)} + 2 \sum_{e \in E} \lambda_e(f_e^*)
\]

\[
\leq \sqrt{\sum_{e \in E} \lambda_e(f_e)} \cdot \sqrt{\sum_{e \in E} \lambda_e(f_e^*)} + \frac{2}{3} \sum_{e \in E} \lambda_e(f_e^*)
\]

Set \( \rho = \sqrt{\sum_{e \in E} \frac{\lambda_e(f_e)}{\lambda_e(f_e^*)}} \) and observe that \( \rho^2 \) is the coordination ratio, given we choose the worst lookahead equilibrium \( f \). Consequently, \( \frac{1}{2}\rho^2 \leq \rho + 2 \). Solving gives \( \rho \leq 1 + \sqrt{5} \) as desired. \( \square \)

Next we consider the lookahead dynamics and study coordination ratio for the lookahead dynamics.

**Theorem 4.2.** In the average-case 2-lookahead model, the coordination ratio for lookahead dynamics is a constant for the leaf model.

**Proof.** We follow a similar approach to Theorem 4.1 in [30] and start by proving some sub-lemmas.

**Lemma 4.3.** If player \( i \) makes a lookahead improving move from path \( P_i \) to \( P_i' \) which changes the flow from \( f \) to \( f_i' \) then \( l_i(f_i') \leq 2l_i(f) + \frac{1}{2} l(f) \).

---

\(^6\)For any two vectors \( x \) and \( y \), we have \( x^T y \leq \sqrt{x^T x} \cdot \sqrt{y^T y} \).
Proof. So player $i$’s lookahead cost with $f'_i$ is less than his cost with $f$. Moreover, we can lower bound the lookahead cost of $f'_i$ by the quantity

$$
\sum_{e \in P'_i} \frac{f_e}{n}(a_e f_e + b_e) + (1 - \frac{f_e}{n})(a_e(f_e + 1) + b_e)
= \sum_{e \in P'_i} a_e(f_e + 1) + b_e - \frac{f_e}{n}a_e
\geq \sum_{e \in P'_i} (1 - \frac{1}{n})a_e(f_e + 1) + b_e
\geq \sum_{e \in P'_i} (1 - \frac{1}{n})(a_e(f_e + 1) + b_e)
= (1 - \frac{1}{n})l_i(f'_i)
$$

This would be the cost incurred if the randomly selected next player $j$ avoids any edge $e$ that player $i$ is on (either by moving away from $e$ or not moving onto $e$). Using similar reasoning, we may upper bound the cost to player $i$ of sticking with $P_i$ by

$$
\sum_{e \in P_i} \frac{f_e}{n}(a_e f_e + b_e) + (1 - \frac{f_e}{n})(a_e(f_e + 1) + b_e)
= \sum_{e \in P_i} (a_e f_e + b_e) + (1 - \frac{f_e}{n})a_e
\leq \sum_{e \in P_i} a_e(f_e + 1) + b_e \leq \sum_{e \in P_i} 2a_e f_e + b_e
\leq 2l_i(f)
$$

Here we assumed the next player $j$ selects every edge $e$ that player $i$ is on (either by staying on $e$ or by moving onto $e$). Therefore, $l_i(f'_i) \leq 2(1 + \frac{1}{n-1})l_i(f)$ which implies the statement in the lemma. □

Applying Lemma 4.3 with Lemma 4.2 in [30], we get:

Lemma 4.4. If agent $i$ changes his path from $P_i$ to $P'_i$, changing the flow from $f$ to $f'_i$, then $l(f'_i) \leq l(f) + (d + 1)l_i(f'_i) - l_i(f)$. In particular, if agent $i$ makes a lookahead improving move then $l(f'_i) \leq (1 + \frac{1}{n})l(f) + 3l_i(f)$.

Now, applying Lemma 4.4 with Lemma 4.3 in [30].

Lemma 4.5. Let $f$ be the current flow. Suppose we chose a player at random and they make a lookahead best response resulting in flow $f'$. Then $E(l_i(f')|f) \leq (1 + \frac{4}{n})l(f)$.

Finally, we prove the following lemma which will imply the statement of the theorem.

Lemma 4.6. Let $f$ be the current flow. Suppose we chose a player at random and they make a lookahead best response resulting in flow $f'$. Then either $E(l(f')|f) \leq (1 - \frac{1}{2n})l(f)$ or $l(f) < (6 + \sqrt{37})OPT$.

Proof. Suppose player $i$ changes his path from $P_j$ to $P'_j$ resulting in the flow changing from $f$ to $f'_i$. Thus $E(l(f')|f) = \frac{1}{n} \sum_i l(f'_i)$. 

Case 1: $\sum_i 4l_i(f_i') \leq \sum_i l_i(f)$

$$E(l(f')|f) = \frac{1}{n} \sum_i l(f_i')$$

$$\leq \frac{1}{n} \sum_i l(f) + l_i(f_i') - l_i(f) + \sum_{e \in P_i - P_i} a_e f_i,e$$

$$\leq \frac{1}{n} \sum_i l(f) + 2l_i(f_i') - l_i(f)$$

$$\leq \frac{1}{n} \sum_i l(f) + \frac{1}{2} l_i(f) - l_i(f)$$

$$= (1 - \frac{1}{2n})l(f)$$

Case 2: $\sum_i 4l_i(f_i') > \sum_i l_i(f) = l(f)$

Let $f^*$ be the optimal flow and let $P_i^*$ be player $i$'s path in this flow. Let $J^*(e)$ be the set of players on edge $e$ in $f^*$. Since $P_i^*$ is a lookahead best response, we may apply Lemma 4.3 to see that $l_i(f_i') \leq 2l_i(f^*) + \frac{1}{n}l(f^*)$. Thus

$$l(f) < 4 \sum_i l_i(f_i') \leq 4 \sum_i 2l_i(f^*) + \frac{1}{n}l(f^*)$$

$$= 12 \sum_i l_i(f^*) = 12 \sum_i \sum_{e \in E} a_e f^*_e + b_e$$

$$\leq 12 \sum_{e \in E \cap J^*(e)} \sum_{e \in E} a_e (f_e + 1) + b_e$$

$$= 12 \sum_{e \in E} a_e f^*_e b_e f^*_e + a_e f^*_e$$

$$\leq 12 \sqrt{l(f)l(f^*)} + l(f^*)$$

where the last inequality follows from Cauchy-Schwartz. Thus, if we set $x = \sqrt{\frac{l(f)}{OPT}}$, the above can be transformed into the inequality $x^2 \leq 12x + 1$. \hfill \square

The remainder of the proof of Theorem 4.2 follows by applying the above lemmas as shown in \[30\]. \hfill \square

4.1. Valid Utility Games.

Here is a bad example for the path model (a slightly modified example applies to the leaf model). It applies for any number $t$ of lookahead moves. Take a Steiner Set System $S(2, k, n)$. For example, these exist with $n = q^2 + q + 1$ and $k = q + 1$. Let each subset in the system induce a "sub-game" - thus each pair of players are together in exactly one subgame. Consequently, each player is in $\frac{n-1}{2} = q + 1 = k$ subgames, and $n$ games in total. The strategy set of a player $i$ in subgame $g$ is $\{y_{i}^{g}, x_{i,1}^{g}, x_{i,2}^{g} \ldots , x_{i,k}^{g}\}$. It has one nice strategy and $k$ naughty strategies: player $i$ always gets one point for playing the nice strategy $y_{i}^{g}$, but gets two points for playing a naughty strategy $x_{i,li}^{g}$ provided $\sum_j l_j = i \mod k$, where the sum is over all players $j$ who are playing a strategy $x_{j,li}^{g}$ - we call $i$ the winner of subgame $g$ in the case.

Thus a player $i$ who moves next can guarantee $k$ points by playing $y$s but can guarantee $2k$ points by playing $x$'s to win all $k$ subgames it is in. Moreover, the player can lose at most one game in each subsequent time period. This follows as the next $t = k$ players share exactly one game each with player $i$. Thus the player, in the worst case receives $2k + 2(k - 1) + \cdots + 4 + 2 = k(k + 1)$ in the next $k$ moves. This is greater than the $k^2$ payoff from playing only $y$s.
Consider then the dynamics of this game under $k$-lookahead search. Over time, at any state of play, the total value of the game will be $2n$; in each of the $n$ subgames all the players are behaving naughtily. The optimal value however is $n(k + 1)$; in each subgame, $k - 1$ of the players are nice and one is naughty. So we have shown:

**Lemma 4.7.** For valid utility games, in the path model the coordination ratio of $k$-lookahead dynamics is at least $\frac{k+1}{2} = \frac{\kappa + 1}{2} \geq \frac{1}{2}\sqrt{n}$. \hfill \qed

4.2. Basic Utility Games.

For basic utility games, good guarantees can be obtained for the path model. More interestingly, for the leaf model lookahead equilibria can be extremely bad, even for 2-lookahead equilibria.

**Lemma 4.8.** In basic utility games, the coordination ratio of 2-lookahead equilibria can be arbitrarily bad in the leaf model.

**Proof.** Consider the following symmetric 2-player game. Let each player have a groundset $\{B, T, G\}$. A feasible strategy consists of playing at most one action in the groundset. We create a submodular social function using the table

|       | $\emptyset$ | $B$ | $T$ | $G$ |
|-------|-------------|-----|-----|-----|
| $B$   | 6           | 6   | 6   | 1   |
| $T$   | $\kappa-9$  | $\kappa-9$ | 7   | 4   |
| $G$   | $\kappa-5$  | $\kappa-10$ | 8   | 5   |

Set $\gamma(\emptyset, \emptyset) = 0$. Then let the $ij$th entry of the matrix, $\delta_{ij}$, be the marginal value of adding action $i$ when action $j$ is being played by the other player. For example, $\gamma(B, \emptyset) = \gamma(\emptyset, \emptyset) + \delta_{B, \emptyset} = 0 + 6 = 6$. Similarly, $\gamma(B, B) = 12$, $\gamma(T, \emptyset) = \kappa - 9$, $\gamma(G, \emptyset) = \kappa - 5$, $\gamma(B, G) = \kappa - 4$, $\gamma(B, T) = \kappa - 3$, $\gamma(T, T) = \kappa - 2$, $\gamma(T, G) = \kappa - 1$, $\gamma(G, G) = \kappa$.

We need to extend this definition to all subsets. Suppose that Player 1 is currently choosing $S_1$ and Player 2 is currently choosing $S_2$. To complete the definition of $\gamma$, we say that the marginal value of adding action $i$ to the subset $S = S_1 \cup S_2$, is $\delta_{i,S} = \min_{j \in S_1 \cup S_2} \delta_{ij}$.

Note that this is true if $i$ is added to $S_1$ and if $i$ is added to $S_2$. This processes produces a submodular social function. The payoff functions are then defined in accordance with the Vickrey condition.

Clearly, as the players are constrained to play singleton actions, the optimal solution $\Omega = \{G, G\}$ has value $\kappa$. We claim that $\{B, B\}$, with social value 12, is the only equilibrium in the leaf model. Thus, for any $\kappa$, we can be a factor $\Omega(\kappa)$ away from the optimal social value.

To prove this, first suppose that Player 1 plays $B$. According to the Vickrey condition, the best response of Player 2 is to play $T$ (she needs to choose $\ast$ maximize $\gamma(B, \ast)$). The payoff to player 1 is then $\gamma(B, T) - \gamma(\emptyset, T) = (\kappa - 3) - (\kappa - 9) = 6$. Second suppose that Player 1 plays $T$. According to the Vickrey condition, the best response of Player 2 is to play $G$ (she needs to maximize $\gamma(T, \ast)$). The payoff to player 1 is then $\gamma(T, G) - \gamma(\emptyset, G) = (\kappa - 1) - (\kappa - 5) = 4$. Finally suppose that Player 1 plays $G$. According to the Vickrey condition, the best response of Player 2 is to play $G$ - observe this must be the case as $\{G, G\}$ is the optimal solution. The payoff to player 1 is then $\gamma(G, G) - \gamma(\emptyset, G) = \kappa - (\kappa - 5) = 5$.

Thus, with 2-lookahead, Player 1 will always think it in his interest to play $B$. (Note that in the leaf model, it is irrelevant for Player 1 what strategy Player 2 is currently playing.) By a symmetric argument, Player 2 will always think it in her interest to play $B$. \hfill \qed

5. Shapley Network Design Games

For our final example we show that the use of lookahead search may allow for “uncoordinated” cooperative behaviours. By looking ahead, a player may select a cooperative move whose consequence can be to induce other players to also make cooperative moves. We give a very simple illustration of this behaviour. Consider the following Shapley network design game: Given a network, there is a single source $s$ and a single sink $t$. We have $n$ players, each wanting to route from $s$ to $t$. There are $N$
Theorem 5.1. The coordination ratio of $k$-lookahead dynamics for Shapley network design games in the leaf model is at most $n/k$.

Proof. We present the proof for the worst-case lookahead model. The proof for the average-case model uses the same idea. Assume the players are currently choosing the paths $\{P_1, P_2, \ldots, P_n\}$. Consider the depth $k$ tree when the players move in the order $1, 2, \ldots, k$. Take a decision node for player $k - 1$. This has $N$ children that are decision nodes for player $k$. Let the paths chosen by player $k$ at these nodes be $Q_1, Q_2, \ldots, Q_N$, respectively. Suppose that in response to this move, player $k - 1$ chooses the path $P_j$. We claim that $P_j = Q_j$.

Thus

$$c(Q_j, T') = c(Q_j, T) - \sum_{e \in Q_j/P_j} \left( \frac{c_e}{n_e} - \frac{c_e}{n_e + 1} \right)$$

Now since $c(Q_j, T) \leq c(P, T)$ we have

$$c(P, T') = c(P, T) - \sum_{e \in (P \cap Q_j)/P_j} \left( \frac{c_e}{n_e} - \frac{c_e}{n_e + 1} \right)$$

This proves the claim. Applying induction, we see that each player $1, \ldots, k$ will play the same strategy $P^*$, and thus, receive the same payoff. Let’s take the worst case choice for players 2, $\ldots, k$ from the point of view of player 1. If $P^* = P^{SP}$, the shortest $s - t$ path, then each of the $k$ chosen players will have a cost of at most
\[ \frac{c(P^{SP})}{k} \leq \frac{\text{OPT}}{k} \]

Thus, if \( P^* \neq P^{SP} \), then player 1 can guarantee himself a cost of at most \( \frac{\text{OPT}}{k} \). This argument applies for all players so, in an equilibrium, the total cost is at most, \( \frac{2}{k} \text{OPT} \). □

References

[1] H. Ackermann, H. Röglin, and B. Vöcking, “On the impact of combinatorial structure on congestion games”, *Journal of the ACM, 55*(6), 2008.

[2] R. Aumann, “Subjectivity and correlation in randomized strategies”, *Journal of Mathematical Economics, 1*, pp67-96, 1974.

[3] B. Awerbuch, Y. Azar and A. Epstein, “The price of routing unsplittable flow”, *Proceedings of the 37th Annual ACM Symposium on Theory of Computing (STOC)*, 2005.

[4] B. Awerbuch, Y. Azar, A. Epstein, V. Mirrokni, and A. Skopak, “Fast convergence to nearly-optimal solutions in potential games”, *Proceedings of the 9th ACM Conference on Electronic Commerce (EC)*, pp264–273, 2008.

[5] B. Awerbuch and R. Kleinberg, “Online linear optimization and adaptive routing”, *Journal of Computer and System Sciences, 74*(1), pp97-114, 2008.

[6] K. Basu and J. Weibull, “Strategy Subsets Closed Under Rational Behaviour”, Papers 479, Stockholm - International Economic Studies.

[7] P. Berenbrink, T. Friedetzky, L. Goldberg, P. Goldberg, Z. Hu, and R. Martin, “Distributed selfish load balancing”, *Proceedings of the 17th Annual ACM-SIAM Symposium on Discrete Algorithms (SODA)*, pp354-363, 2006.

[8] A. Blum, E. Even-Dar, and K. Ligett, “Routing without regret: on convergence to Nash equilibria of regret-minimizing algorithms in routing games”, *Proceedings of the 21st Annual ACM Symposium on Principles of Distributed Computing (PODC)*, pp45-52, 2006.

[9] A. Blum, M. Hajiaghayi, K. Ligett and A. Roth, “Regret minimization and the price of total anarchy”, *Proceedings of the 40th Annual ACM Symposium on Theory of Computing (STOC)*, pp373-382, 2008.

[10] A. Blum and Y. Mansour, “Learning, regret minimization and correlated equilibria”, In *Algorithmic Game Theory*, N. Nisan, T. Roughgarden, E. Tardos, V. V. Vazirani (eds.), pp79-102, Cambridge University Press, 2007.

[11] G. Brown, “Iterative solutions of games by fictitious play”, in *Activity Analysis of Production and Allocation*, T. Koopmans (ed.), pp374-376, Wiley, 1951.

[12] T. Bu, X. Deng, X., Q. Qi, “Forward looking Nash equilibrium for keyword auction”, *Information Processing Letters, 105*(2), pp41-46, 2008.

[13] M. Cary, A. Das, B. Edelman, I. Giotis, K. Heimerl, A. Karlin, C. Mathieu and M. Schwarz, “Greedy bidding strategies for keyword auctions”, *Proceedings of the ACM International Conference on Electronic Commerce (EC)*, 2007.

[14] S. Chien and A. Sinclair, “Convergence to approximate Nash equilibria in congestion games”, *Proceedings of the 18th Annual ACM Symposium on Discrete Algorithms (SODA)*, pp169-178, 2007.

[15] G. Christodoulou, V. Mirrokni, and A. Sidiropoulos, “Convergence and approximation in potential games”, *Proceedings of the 18th Annual Symposium on Theoretical Aspects of Computer Science (STACS)*, pp349-360, 2006.

[16] J. Conlisk, “Why Bounded Rationality?”, *Journal of Economic Literature, 34*(2), pp669-700, 1996.

[17] A. Cournot, *Recherches sur les Principes Mathématiques de la Théorie des Richesses*, Paris, 1838.

[18] B. Edelman, M. Ostrovsky and M. Schwarz, “Internet advertising and the generalised second-price auction: selling billions of dollars worth of keywords”, *American Economic Review, 97*(1), pp242-259, 2007.

[19] E. Even-Dar, Y. Mansour, and U. Nadav, “On the convergence of regret minimization dynamics in concave games”, *Proceedings of the 41st Annual ACM Symposium on Theory of Computing (STOC)*, 2009.

[20] G. Ellison, “Learning, Local Interaction, and Coordination”, Econometrica, 61, pp1047-1071, 1993.

[21] A. Fabrikant and C. Papadimitriou, “The complexity of game dynamics: BGP oscillations, sink equilibria, and beyond”, *Proceedings of the 19th Annual ACM-SIAM Symposium on Discrete Algorithms (SODA)*, pp844-853, 2008.

[22] A. Fabrikant, C. Papadimitriou, and K. Talwar, “The complexity of pure Nash equilibria”, *Proceedings of the 36th Annual ACM Symposium on Theory of Computing (STOC)*, pp604-612, 2004.

[23] A. Fanelli, M. Flammini, and L. Moscandelli, “The speed of convergence in congestion games under best-response dynamics”, *Proc. of the 35th International Colloquium on Automata, Languages and Programming (ICALP)*, pp796-807, 2008.

[24] L. Fortnow and R. Santhanam, “Bounding rationality by discounting time”, *Proceedings of The First Symposium on Innovations in Computer Science (ICS)*, 2010.

[25] D. Foster and R. Vohra, “Calibrated learning and correlated equilibrium”, *Games and Economic Behavior, 21*, pp40-55, 1997.
[63] E. Sefer and U. Kuter and D. Nau, “Real-time A* Search with Depth-k Lookahead”, Proceedings of the International Symposium on Combinatorial Search, 2009.
[64] R. Selten, “What is bounded rationality?”, in Bounded Rationality: the Adaptive Toolbox, G. Gigerenzer and R. Selten (eds), MIT Press, pp13-36, 2001.
[65] R. Selten, “Boundedly Rational Qualitative Reasoning on Comparative Statics”, in Advances in Understanding Strategic Behavior: Game Theory, Experiments and Bounded Rationality, Steffen Huck (ed.), Palgrave Macmillan, pp1-8, 2004.
[66] A. Sen, “Rational fools: a critique of the behavioral foundations of economic theory”, Philosophy and Public Affairs, 6(4), pp317-344, 1977.
[67] A. Sen, “Internal consistency of choice”, Econometrica, 61(3), pp495-521, 1993.
[68] C. Shannon, “Programming a computer for playing chess”, Philosophical Magazine, Series 7, 41(314), pp256-275, 1950.
[69] H. Simon, “A behavioral model of rational choice”, Psychological Review, 63, pp129-138, 1955.
[70] H. Simon, “Rational choice and the structure of the environment”, Psychological Review, 63, pp129-138, 1956.
[71] H. Simon, The Sciences of the Artificial, 3rd edition, MIT Press, 1996.
[72] A. Skopalk and B. Vöcking, “Inapproximability of pure Nash equilibria”, Proceedings of the 40th Annual ACM Symposium on Theory of Computing (STOC), pp355-364, 2008.
[73] D. Stahl and P. Wilson, “Experimental evidence on players’ models of other players”, Journal of Economic Behavior and Organization, 25(3), pp309-327, 1994.
[74] D. Stahl and P. Wilson, “On players’ models of other players: theory and experimental evidence, Games and Economic Behavior, 10(1), pp218-254, 1995.
[75] J. Tirole, The Theory of Industrial Organization, MIT Press, 1988.
[76] A. Tversky and D. Kahneman, “Judgement under uncertainty: heuristics and biases”, Science, 185(4157), pp1124-1131, 1974.
[77] H. Varian, “Position auctions”, International Journal of Industrial Organization, 25, pp1163-1178, 2007.
[78] A. Vetta, “Nash equilibria in competitive societies, with applications to facility location, traffic routing and auctions”, Proceedings of 43rd Symposium on Foundations of Computer Science (FOCS), pp416-425, 2002.
[79] H. Young, “The evolution of conventions”, Econometrica, 61, pp57-84, 1993.
[80] H. Young, Strategic learning and its limits, Oxford University Press, 2004.