Bjorken Sum Rule at low $Q^2$

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Abstract

A description of the generalized Gerasimov-Drell-Hearn sum rules for proton and neutron is suggested, using their relation to the Bjorken sum rule. The results support an earlier conjecture, that the structure function $g_T$ features a smooth $Q^2$—dependence, while the structure function $g_2$ is changing rapidly, due to the elastic contribution to the Burkhardt-Cottingham sum rule. A possible violation of this later sum rule is briefly discussed.
The Bjorken sum rule [1] is known to be one of the most fundamental constraints for our understanding of the nucleon spin structure. Its appearance in QCD relies on the Operator Product Expansion, describing the large $Q^2$ region, while at moderate $Q^2$ the perturbative [2] and power [3] corrections are of major importance. Moreover, the transition to the entirely non-perturbative $Q^2$ region is rather cumbersome.

Nevertheless, the limit for $Q^2 = 0$ is provided by taking the difference for proton and neutron of the Gerasimov-Drell-Hearn (GDH) sum rules [4, 5]. Contrary to the proton case, for this difference $p - n$, the asymptotic and real photon values are of the same sign which provides the opportunity for checking a possible smooth approximation [6]. In this paper we study this extrapolation in some detail.

The generalized ($Q^2$-dependent) GDH sum rules are just being tested experimentally with a high accuracy for both proton and neutron [7, 8, 9]. The striking feature of the proton data is the low ($\sim 200 - 250 MeV^2$) “crossover” point, which is in complete agreement with our prediction [10, 11], published almost 10 years ago. Our approach is making use of the relation to the Burkhardt-Cottingham sum rule for structure function $g_2$, whose elastic contribution is the main source of a strong $Q^2$-dependence, while the contribution of the other structure function, $g_T = g_1 + g_2$ is smooth.

However, the preliminary neutron data [9] are going well above the prediction, made in the similar manner [12]. We address this problem and show, that the reason for the discrepancy is the model for the neutron structure function $g_T^n$. In fact, the general arguments of [11] are not applicable for the neutron, and the model used in Ref.[12] appears to be more or less ad hoc.

To avoid this problem, we consider here the $Q^2$-dependence of the non-singlet combination $g_1^p - g_1^n$, whose asymptotic behaviour is described by the Bjorken sum rule. We apply the same method and describe its behaviour in the low $Q^2$ region, together with its proton and neutron components, being in a reasonable agreement with recent experimental data.

To recall our approach let us first note, that the presence of $g_2$ in the description of the longitudinal polarization is by no means surprising, as soon as one uses the language of invariant, rather than helicity amplitudes [11].

To define the spin-dependent structure functions one should express the antisymmetric part of the hadronic tensor $W^{\mu\nu}$ as a linear combination of all possible Lorentz-covariant tensors. These tensors should be orthogonal to the virtual photon momentum $q$, as required by gauge invariance, and they are linear in the nucleon covariant polarization $s$, from a general property of the density matrix. If the nucleon has momentum $p$, we have as usual, $s \cdot p = 0$ and $s^2 = -1$. There are only two such tensors: the first one arises already in the Born diagram

$$T_1^{\mu\nu} = \epsilon^{\mu\nu\alpha\beta} s_\alpha q_\beta$$

and the second tensor is just

$$T_2^{\mu\nu} = (s \cdot q) \epsilon^{\mu\nu\alpha\beta} p_\alpha q_\beta.$$  

The scalar coefficients of these tensors are specified in a well-known way, since we have

$$W_A^{\mu\nu} = \frac{-i \epsilon^{\mu\nu\alpha\beta}}{p \cdot q} q_\beta (g_1(x, Q^2) s_\alpha + g_2(x, Q^2) (s_\alpha - p_\alpha \frac{s \cdot q}{p \cdot q})) =$$
\[-\frac{i e^{\mu \nu \alpha \beta}}{p \cdot q} q_\beta ((g_1(x, Q^2) + g_2(x, Q^2)) s_\alpha - g_2(x, Q^2) p_\alpha \frac{s \cdot q}{p \cdot q}). \]  

This tells us that $g_2$, due to the factor $(s \cdot q)$, is making the difference between longitudinal and transverse polarizations, while $g_T = g_1 + g_2$ contributes equally in both cases. More exactly, $g_2$ provides this difference because its contribution is non-zero in the case of longitudinal polarization and is zero in the case of transverse polarization. It is just in this sense that we are speaking about the $g_2$ contribution to longitudinal polarization.

Let us consider the $Q^2$-dependent integral

$$I_1(Q^2) = \frac{2 M^2}{Q^2} \int_0^1 g_1(x, Q^2) dx.$$  

It is defined for all $Q^2$, and $g_1(x, Q^2)$ is the obvious generalization for all $Q^2$ of the standard scale-invariant $g_1(x)$. Note that the elastic contribution at $x = 1$ is not included in the above sum rule. Then one recovers at $Q^2 = 0$ the GDH sum rule

$$I_1(0) = -\frac{\mu_A^2}{4}$$  

where $\mu_A$ is the nucleon anomalous magnetic moment in nuclear magnetons. While $I_1(0)$ is always negative, its value at large $Q^2$ is determined by the $Q^2$ independent integral $\int_0^1 g_1(x) dx$, which is positive for the proton and negative for the neutron.

The separation of the contributions of $g_T$ and $g_2$ leads to the decomposition of $I_1(Q^2)$ as the difference between $I_T(Q^2)$ and $I_2(Q^2)$

$$I_1(Q^2) = I_T(Q^2) - I_2(Q^2),$$

where

$$I_T(Q^2) = \frac{2 M^2}{Q^2} \int_0^1 g_T(x, Q^2) dx, \quad I_2(Q^2) = \frac{2 M^2}{Q^2} \int_0^1 g_2(x, Q^2) dx.$$  

There are solid theoretical arguments to expect a strong $Q^2$-dependence of $I_2(Q^2)$. It is the well-known Burkhardt-Cottingham sum rule [13], derived independently by Schwinger [14], using a rather different method. It states that

$$I_2(Q^2) = \frac{1}{4} \mu G_M(Q^2) \frac{\mu G_M(Q^2) - G_E(Q^2)}{1 + \frac{Q^2}{4 M^2}},$$

where $\mu$ is the nucleon magnetic moment, $G$’s denoting the familiar Sachs form factors which are dimensionless and normalized to unity at $Q^2 = 0$. For large $Q^2$, as a consequence of the $Q^2$ behavior of the r.h.s. of (8), we get

$$\int_0^1 g_2(x, Q^2) dx = 0.$$  

In particular, from Eq.(9) it follows that

$$I_2(0) = \frac{\mu_A^2 + \mu_A e}{4},$$
$e$ being the nucleon charge in elementary units. To reproduce the GDH value (see Eq.(5)) one should have

$$I_T(0) = \frac{\mu_A e}{4},$$

(11)

which was indeed proved by Schwinger [14]. The importance of the $g_2$ contribution can be seen already, since the entire $\mu_A$-term for the GDH sum rule is provided by $I_2$.

Note that $I_T$ does not differ from $I_1$ for large $Q^2$ due to the BC sum rule, but it is positive in the proton case. It is possible to obtain a smooth interpolation for $I_T^p(Q^2)$ between large $Q^2$ and $Q^2 = 0$ [10].

$$I_T^p(Q^2) = \theta(Q^2 - Q^2_0)(\frac{\mu_A p}{4} - \frac{2M^2Q^2}{(Q_0^2)^2}\Gamma^p_1) + \theta(Q^2 - Q^2_0)^2\frac{2M^2}{Q^2}\Gamma^p_1,$$

(12)

where $\Gamma^p_1 = \int_0^1 g^p_1(x)dx$. The continuity of the function and of its derivative is guaranteed with the choice $Q^2_0 = (16M^2/\mu_A p)^2 \sim 1GeV^2$, where the integral is given by the world average proton data. It is quite reasonable to distinguish the perturbative and the non-perturbative regions. As a result one obtains a crossing point at $Q^2 \sim 0.2GeV^2$, below the resonance region [10], while the positive value at $Q^2 = 0.5GeV^2$ is in a good agreement with the E143 [7] data. A fair agreement with HERMES data has been also observed for larger $Q^2$ values [8].

This smooth interpolation seems to be very reasonable in the framework of the QCD sum rules method as well. Then one should choose some "dominant" tensor structure to study the $Q^2$-dependence of its scalar coefficient and $T_1$ appears to be a good candidate. This seems also promising from another point of view. It is not trivial to obtain, within the QCD sum rules approach, the GDH value at $Q^2 = 0$. Since the r.h.s. of Eq.(12) is linear in $\mu_A$, it may be possible to obtain it using the Ward identities, just like the normalization condition for the pion form factor [15]. This, in turn opens the possibility to apply the powerful tool of quark-hadron duality. The latter relies on the perturbative theory, and it is quite clear that it is much more plausible to describe linear (one-loop), than quadratic terms. One should recall here that while the sum rule for $I_T$ was checked in QED long ago [16], the GDH sum rule required much more work [17], since it gets non-trivial contributions only at two loops order, although the vanishing value at one loop level is also non-trivial [18, 19].

Note also that the large contribution of $g_2$ by no means contradicts the resonance approaches [20] and may be considered complimentary to them. In these cases, $\Delta(1232)$ plays a central role: it provides a significant amount of the GDH integral at $Q^2 = 0$ and gives a clear qualitative explanation of rapid $Q^2$-dependence [3]. The $\Delta$ photoproduction is dominated by the magnetic dipole form factor, leading to a positive $I_2$ and a vanishing $I_T$, so implying a negative $I_1$. The sign change is just related to the fast decrease of the $\Delta$ contribution [3].

To generalize our approach to the neutron case, one needs a similar smooth parametrization of $g_T$ for the neutron. Since the value at $Q^2 = 0$ is equal to zero, it is not sufficient to limit oneself to the simplest linear parameterization. One needs to add a term, quadratic in $Q^2$. A simple parametrization providing the continuity of the function and its derivative was suggested in [12], which however, leads to a result in contradiction with the data [14]. This does not seem to be fortuitous, bearing in mind the
argument presented above. Indeed, the general reason supporting the smoothness of interpolation for \( g_T \) is its linearity in \( \mu_A \). As soon as this term appears to be equal to zero for some special reason (which, in our case, is nothing but the neutron neutrality!), there is also no more reason to expect such a smoothness.

To bypass this difficulty, we use the difference between proton and neutron instead of the neutron itself. Although it is possible, in principle, to construct a smooth interpolation for the functions \( g_1 \) themselves\(^6\), it does not fit the suggested general argument, since \( I_T^{p-n}(0) \) is proportional to \( \mu_{A,n}^2 - \mu_{A,p}^2 \), which is quadratic and, moreover, has an additional suppression due to the smallness of isoscalar anomalous magnetic moment.

So we suggest the following parametrization for the isovector contribution of \( I_T(Q^2) \), namely \( I_T^{p-n}(Q^2) \)

\[
I_T^{p-n}(Q^2) = \theta(Q_1^2 - Q^2)(\frac{\mu_{A,p}}{4} - \frac{2M^2Q^2}{(Q_1^2)^2}\Gamma_1^{p-n}) + \theta(Q^2 - Q_1^2)\frac{2M^2}{Q^2}\Gamma_1^{p-n},
\]

where the transition value \( Q_1^2 \) may be determined by the continuity conditions in a similar way. We get the value \( Q_1^2 \sim 1.3GeV^2 \), which is of the same order as for the proton case.

The elastic contribution to the BC sum rule should be included for the neutron separately, so we need the neutron elastic form factors. While the electric one, might be neglected, the magnetic form factor is well described by the dipole formula \[^2]\]

\[
G_M(Q^2) = \frac{1}{(1 + Q^2/0.71)^2}
\]

The plot representing \( \Gamma_1^p(Q^2) \) is displayed on Fig.1.

One can see, that \( \Gamma_1^p(Q^2) \) remains quite close to its asymptotic value down to \( Q^2 \) values, as low as \( 3GeV^2 \). Moreover, the preliminary data \[^3\] might bear some aspects of the structure generated in our approach, by the interplay of the transitions values \( Q_0^2 \) and \( Q_1^2 \).

Now we have all the ingredients to show the behavior of \( I_T^{p-n}(Q^2) \), a quantity directly related to the Bjorken sum rule \( \Gamma^{p-n}(Q^2) \) (see Fig.2). It is clear that the smooth linear interpolation at low \( Q^2 \) to the value \( (\mu_{A,n}^2 - \mu_{A,p}^2)/4 \), would result in a much lower value for \( I_T^p(Q^2) \), inconsistent with the experimental data on \( \Gamma_1^p(Q^2) \) \[^3\].

Note that very accurate data from JLAB may require more elaborate models for \( g_T \). In particular, it is possible to take into account the perturbative and power corrections to \( I_1 \), and also the contribution of \( I_2 \) at the matching point \( Q_0^2 \), since in fact we have assumed that at this point \( I_T(Q_0^2) = I_1^{asymptotic}(Q_0^2) \). However, these effects are acting in opposite directions and should partially cancel each other. Indeed, in our model \( I_1(Q_0^2) = I_T(Q_0^2) - I_2(Q_0^2) = I_1^{asymptotic}(Q_0^2) - I_2(Q_0^2) \), and \( I_2 \) produces a negative contribution, just like the corrections to \( I_1 \).

Another interesting problem is a possible violation of the BC sum rule reported recently \[^22\]. In this connection, note that from a theoretical point of view, the only reason for a violation is the divergence of the integral. So to put it in a dramatic manner, "the BC integral is either zero or infinity". In this sense, the reported finite value \[^22\] requires an interpretation.
Figure 1: Our prediction for $\Gamma_1^n(Q^2)$.

Figure 2: Our prediction for $I_1^{p-n}(Q^2)$, directly related to the Bjorken sum rule.
One reason for the BC sum rule violation is the contribution of Regge cuts \[6\], which are located at very low \(x\) \[23\]. Therefore, one may ask if it is possible to separate this contribution

\[ g_2(x) = g_2^{BC}(x) + g_2^{\text{cut}}(x) \]  

(15)
in such a way that

\[ \int_0^1 g_2^{BC}(x)dx = 0. \]  

(16)

Then one could define the BC integral with a lower cutoff

\[ I_\delta = \int_{\delta}^1 g_2(x)dx, \]  

(17)
such that \(I_\delta\) may be sufficiently close to zero for small \(\delta\), where \(g_2^{\text{cut}}\) is still negligible. We have considered such a separation \[11\] and we have showed that the crossing point for proton is not sensitive to the cut contribution. Moreover, the details of \(I_1\) may be sensitive to the cut contribution and provide an additional way of its investigation.

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