TORSION FOR ABELIAN VARIETIES OF TYPE III

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Abstract. Let $A$ be an abelian variety defined over a number field $K$. The number of torsion points that are rational over a finite extension $L$ is bounded polynomially in terms of the degree $[L : K]$ of $L$ over $K$. Under the following three conditions, we compute the optimal exponent for this bound, in terms of the dimension of abelian subvarieties and their endomorphism rings. The three hypothesis are the following: (1) $A$ is geometrically isogenous to a product of simple abelian varieties of type I, II or III, according to the Albert classification; (2) $A$ is of “Lefschetz type”, that is, the Mumford–Tate group is the group of symplectic or orthogonal similitudes which commute with the endomorphism ring; (3) $A$ satisfies the Mumford–Tate conjecture. This result is unconditional for a product of simple abelian varieties of type I, II or III with specific relative dimensions. Further, building on work of Serre, Pink, Banaszak, Gajda and Krasoń, we also prove the Mumford–Tate conjecture for a few new cases of abelian varieties of Lefschetz type.

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1. Introduction

Mordell-Weil’s theorem states that, for an abelian variety $A$ defined over a number field $K$, the group of $L$-rational points is finitely generated for any finite extension $L$ over $K$. One can wonder if we can get a bound for the number of torsion points that are rational over a finite extension $L$, depending on the degree $[L : K]$ and the dimension of the abelian variety, when the abelian variety varies in a certain class or the number field $L$ varies among all finite extensions of $K$.

We will focus on the case where the abelian variety is fixed and the number field $L$ varies among all finite extensions of $K$; our main concern is to obtain an optimal bound which only
depends on the degree of the number field \([L : K]\). Hindry and Ratazzi have given several results in this direction concerning some classes of abelian varieties (see [HiRa10], [HiRa12] and [HiRa16]).

The aim of this paper is to present new results, which generalize the previous ones from Hindry and Ratazzi. They concern the class of abelian varieties which are isogenous to a product of simple abelian varieties of type I, II or III (in the sense of Albert’s classification) and such that each simple abelian variety is fully of Lefschetz type. This means that Mumford–Tate conjecture holds for each abelian variety and the Mumford–Tate group is the group of symplectic or orthogonal similitudes which commute with the endomorphism ring. Let us recall that the case where the abelian variety is simple of type III is very interesting because it is known that they carry exceptional Hodge classes, for further details we refer to [Mur84].

Let \(A\) be an abelian variety defined over a number field \(K\) of dimension \(g\). Let \(L\) be a finite extension of \(K\). We would like to give an upper bound of the order of the finite group \(A(L)_{\text{tors}}\). In order to do so, let us define the following invariant, introduced by Ratazzi (see [Rat07]):

**Definition 1.1.** The invariant \(\gamma(A)\) is defined as follows:

\[
\gamma(A) = \inf\{x > 0 \mid \forall L/K \text{ finite}, \ |A(L)_{\text{tors}}| \ll [L : K]^x \},
\]

where the notation \(\ll\) means that there exists a constant \(C\) (which only depends on the abelian variety \(A\) and \(x\)) such that

\[
|A(L)_{\text{tors}}| \leq C[L : K]^x.
\]

More precisely \(\gamma(A)\) is the optimal exponent: that means that it is the minimal value such that, for every positive \(\varepsilon\), one has, for every finite extension \(L/K\)

\[
|A(L)_{\text{tors}}| \ll [L : K]^{\gamma(A) + \varepsilon}.
\]

Masser had already established an upper bound, which is polynomial on the degree of the number field \(L\), see [Mas86]. More precisely, one has the following bound:

\[
\gamma(A) \leq g.
\]

**Remark 1.2.** Masser’s bound is not an optimal bound, except in the case where \(A\) is isogenous to a power of a CM elliptic curve.

In order to present an optimal bound, Ratazzi gave, in the case of abelian varieties of CM type, an explicit formula for the invariant \(\gamma(A)\) in terms of the characters of the Mumford–Tate group (see [Rat07, Théorème 1.10]). Let us recall that the Mumford–Tate group and the Hodge group of an abelian variety \(A\) are related by the following relation:

\[
\text{MT}(A) = \mathbb{G}_m \cdot \text{Hg}(A).
\]

Few months later, Hindry and Ratazzi focused their attention on abelian varieties which are isogenous to a product of elliptic curves: they equally established an explicit formula for the invariant \(\gamma(A)\) (see [HiRa10, Théorème 1.6]).

In 2010, the same authors gave an explicit formula of the invariant \(\gamma(A)\) in the case where the abelian variety \(A\) is isogenous to a product of simple abelian varieties of type GSp, that means that they are simple abelian varieties such that the Mumford–Tate group is isogenous
to the group of symplectic similitudes and such that Mumford–Tate conjecture holds for
these varieties (see [HiRa12, Théorème 1.6]).

In this last paper, Hindry and Ratazzi stated the following conjecture:

**Conjecture 1.3.** ([HiRa12 Conjecture 1.1]) Let $A$ be an abelian variety isogenous to a
product of abelian varieties $\prod_{i=1}^{d} A_i^{n_i}$ where the $A_i$ are simple abelian varieties no pairwise
isogenous. Then

$$
\gamma(A) = \max_{\emptyset \neq I \subset \{1, \ldots, n\}} \frac{2 \sum_{i \in I} n_i \dim A_i}{\dim \text{MT}(A)}.
$$

This last conjecture holds for abelian varieties of type I and II which are fully of Lefschetz
type, see [HiRa16, Théorème 1.6], and for abelian varieties which are isogenous to a product
of abelian varieties of type I or II (see [HiRa16, Théorème 1.14]).

Let us introduce some notations concerning the abelian variety $A$. Let us denote by
$D = \text{End}^\circ(A) := \text{End}(A) \otimes \mathbb{Q}$ the endomorphism algebra of $A$ and $E = \mathbb{Z}(D)$ the center
of $D$. Let us recall Albert’s classification

- If $A$ is of type I then, $D = E$ is a totally real field of degree $e$, more precisely $g = eh$;
- If $A$ is of type II then, $D$ is a totally indefinite quaternion algebra over a totally real
  field $E$ of degree $e$, more precisely $g = 2eh$;
- If $A$ is of type III then, $D$ is a totally definite quaternion algebra over a totally real
  field $E$ of degree $e$, more precisely $g = 2eh$;
- If $A$ is of type IV then, $D$ is a division algebra over a CM field $E$.

The integer $h$ introduced in each case corresponds to the following notation:

**Definition 1.4.** Let $A$ be an abelian variety of dimension $g$ of type I, II or III, the relative
dimension of $A$ is the following integer:

$$
h := \dim_{\text{rel}} X = \begin{cases} 
geq \frac{g}{e} & \text{type I,} \\
\frac{g}{2e} & \text{type II and III,} 
\end{cases}
$$

where $e = \lfloor E : \mathbb{Q} \rfloor$ and $d^2 = \lfloor D : E \rfloor$

Let $\phi$ be a polarization of the abelian variety $A$, it is known that $Hg(A) \subset \text{Sp}(D, \phi)$, where
$\text{Sp}(D, \phi)$ is the group of symplectic similitudes which commutes with the endomorphisms in
$D$ and preserves the polarization.

In the case where the abelian variety $A$ is of type I or II, we know that the Hodge group
is always contained in $\text{Res}_{E/Q} \text{Sp}_{2h}$.

Moreover, when the abelian variety $A$ is of type III, we know that the Hodge group is
always contained in $\text{Res}_{E/Q} \text{SO}_{2h}$. This result follows from [BGK10] and further details will
be found in section [1]. In some cases we have the equality. For instance,

- when $A$ is of type I or II, $Hg(A) = \text{Res}_{E/Q} \text{Sp}_{2h}$ when $h = 2$ or an odd integer (see
  [HiRa16 Théorème 3.7] and [BGK06]) and Mumford–Tate conjecture holds for $A$;
- when $A$ is of type III, $Hg(A) = \text{Res}_{E/Q} \text{SO}_{2h}$ when $h$ is in the set $\{2k + 1, k \in \mathbb{N}\} \setminus
  \left\{\frac{1}{2} \left(\frac{2m+2}{m+1}\right), m \in \mathbb{N}\right\}$ (see [BGK10] and corollary [11]) and Mumford–Tate conjecture
  holds for $A$. 

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Let \( \ell \) be a prime number, we denote \( T_\ell(A) = \lim_{\leftarrow} A[\ell^n] \) the \( \ell \)-adic Tate module of \( A \). Let us consider the action of the absolute Galois group \( G_K = Gal(\bar{K}/K) \) over the \( \ell \)-torsion points. Therefore we can consider the following \( \ell \)-adic representation:

\[
\rho_\ell : G_K \to GL(T_\ell(A))
\]

Let \( G_\ell \) be the \( \mathbb{Q}_\ell \)-algebraic group defined as the Zariski closure of the image of \( \rho_\ell \), it is known as the \( \ell \)-adic monodromy group of \( A \). Let \( V_\ell(A) = T_\ell(A) \otimes \mathbb{Q}_\ell \).

**Definition 1.5.** The abelian variety \( A \) is fully of Lefschetz type if

- \( MT(A) \subset L(A) \) is an equality,
- Mumford–Tate conjecture hold for \( A \), i.e. \( MT(A) \otimes \mathbb{Q}_\ell = G_\ell \).

Let us recall that the Lefschetz group \( L(A) \) of \( A \) is the group of symplectic similitudes which commute with the endomorphisms sometimes denoted \( GSp(D, \phi) \), for further details see [Mil99].

Moreover, thanks to [LaPi95, Theorem 4.3] we know that, if the Mumford–Tate conjecture holds for one prime number \( \ell \), then it holds for every prime number.

In this paper, we prove conjecture \([1.3]\) in the case of abelian varieties of type III which are fully of Lefschetz type.

**Theorem 1.6.** Let \( A \) be a simple abelian variety defined over a number field \( K \) of type III and dimension \( g \). We have \( e = [E : \mathbb{Q}] \), \( d = 2 \) and \( h \) the relative dimension. We suppose that \( A \) is fully of Lefschetz type.

Then the invariant \( \gamma(A) \) is equal to

\[
\gamma(A) = \frac{2deh}{1 + eh(2h - 1)} = \frac{2 \dim A}{\dim MT(A)}.
\]

The numerical value of \( \gamma(A) \) is clearly sharper than the bound given by Masser, which would be \( deh \). Also, our proof provides this value which is then identified with the conjecture value \( \frac{2 \dim A}{\dim MT(A)} \).

Finally, we can generalize conjecture \([1.3]\) to the case where \( A \) is isogenous to a product of simple abelian varieties of type I, II or III and fully of Lefschetz type. More precisely we present a new result where just the type IV, in the sense of Albert’s classification, is excluded.

**Theorem 1.7.** Let \( A \) be an abelian variety defined over a number field \( K \) isogenous over \( K \) to \( \prod_{i=1}^d A_i^{n_i} \) where the abelian varieties \( A_i \) are pairwise non isogenous over \( K \) of dimension \( g_i \) and \( n_i \geq 1 \). We suppose that the abelian varieties \( A_i \) are simple, not of type IV and fully of Lefschetz type.

For every non empty subset \( I \subset \{1, \ldots, d\} \) we denote \( A_I := \prod_{i \in I} A_i \). Moreover we define \( e_i = [E_i : \mathbb{Q}] \) where \( E_i = Z(\text{End}^0(A_i)) \), \( h_i = \text{dim}_{\mathbb{Q}} A_i \) and

\[
d_i = \begin{cases} 1 & \text{if } A_i \text{ is of type I,} \\ 2 & \text{if } A_i \text{ is of type II or III,} \\ \end{cases} \quad \eta_i = \begin{cases} 1 & \text{if } A_i \text{ is of type I or II,} \\ -1 & \text{if } A_i \text{ is of type III.} \\ \end{cases}
\]

Then one has:

\[
\gamma(A) = \max_I \left\{ \frac{2 \sum_{i \in I} n_i d_i e_i h_i}{1 + \sum_{i \in I} e_i (2h_i^2 + \eta_i h_i)} \right\} = \max_I \left\{ \frac{2 \sum_{i \in I} n_i \dim A_i}{1 + \dim Hg(\prod_{i \in I} A_i)} \right\}.
\]
Recently, Zywina proved conjecture 1.3 in the case of abelian varieties which verify Mumford–Tate conjecture (see [Zyw17, Theorem 1.1]). This result, proved independently, is more general than our theorem. Nevertheless let us remark that our method is more precise, in the sense that it gives more information about the dimension of the stabilizers involved. Moreover, our method allows us to obtain a better lower bound for the degree of the field extension generated by a torsion point and this kind of informations cannot be deduced from Zywina’s work.

**Theorem 1.8.** Let $A$ be a simple abelian variety defined over a number field $K$ of type III, with relative dimension $h$ and fully of Lefschetz type. Then there exists a constant $c_1 := c_1(A, K) > 0$ such that, for every torsion point $P$ of order $m$ in $A(K)$, one has:

$$[K(P) : K] \geq c_1^{\omega(m)} m^{2h}.$$  

In the case of a product $A \simeq \prod_{i=1}^d A_i^{n_i}$ we obtain the same results with $h = \min_i h_i$.

Let us remark that one of the main hypothesis of theorems 1.6 and 1.7 is that the abelian variety $A$ must be fully of Lefschetz type, in particular, Mumford–Tate conjecture must hold for $A$. In this direction the following two theorems give examples of abelian varieties that are fully of Lefschetz type.

The following theorem gives a correction of a subtle point of [BGK10, Theorem 5.11]:

**Theorem 1.9.** Let $A$ be a simple abelian variety of dimension $g$ and of type III. Let us recall that $g = 2eh$ where $e = [E : \mathbb{Q}]$ and $h$ is the relative dimension. We assume that

1. $h \in \{2k + 1, k \in \mathbb{N}\} \setminus \left\{\frac{1}{2}(2m+1), m \in \mathbb{N}\right\}$.

Then $A$ is fully of Lefschetz type.

This theorem is stated in [BGK10] with the hypothesis $h$ odd, nevertheless, it seems to be important to exclude the values of the form $\frac{1}{2}(2m+2)$ with $m \in \mathbb{N}$ (see section 6 for details).

Let us consider the following sets:

$$\Sigma := \left\{g \geq 1; \exists k \geq 3, \exists a \geq 1, g = 2^{k-1}a^k\right\} \cup \left\{g \geq 1; \exists k \geq 3, 2g = \left(\frac{2}{k}\right)^{2m+1}, m \in \mathbb{N}\right\}.  \tag{1.1}$$

$$\Sigma' := \bigcup_{s > 1} \left\{2^{(4k+3)s-1}, k \geq 0\right\} \cup \left\{2^{4ks-1}, k \geq 1\right\} \cup \left\{2^{2s(4k+1)-1}, k \geq 1\right\} \cup \left\{2^{2s-1}k^{2s}, k \geq 2\right\}$$

$$\cup \left\{\frac{1}{2}\left(\frac{4k+4}{2k+2}\right)^s, k \geq 0\right\} \cup \left\{\frac{1}{2}\left(\frac{4k+2}{2k+1}\right)^{2s}, k \geq 0\right\}.  \tag{1.2}$$

Then we can state the following theorem which generalizes results of [Pin98]:

**Theorem 1.10.** Let $A$ be a simple abelian variety of type III such that the center of $\text{End}^c(A)$ is $\mathbb{Q}$. Assume that the relative dimension $h$ is not in the set $\Sigma'$ then, the abelian variety is fully of Lefschetz type. More precisely, Mumford–Tate conjecture holds for the abelian variety $A$.

The following corollary gives a criterion for varieties of type III to be fully of Lefschetz type in terms of the relative dimension:
Corollary 1.11. Let $E$ be a totally real number field of degree $e = [E : \mathbb{Q}]$. Let $A$ be an abelian variety defined over a number field such that $A$ is simple of type III and the center of the endomorphisms algebra is $E$. Assume that one of the following hypothesis hold:

1. The relative dimension $h$ is in the set \( \{2k + 1, k \in \mathbb{N}\} \setminus \left\{ \frac{1}{2}(2m+1)^2, m \in \mathbb{N} \right\} \).
2. We have $e = 1$ and the relative dimension $h$ is not in $\Sigma'$.

Then $A$ if fully of Lefschetz type. In particular one has

$$\gamma(A) = \frac{2dhe}{1 + e(2h^2 - h)} = \frac{2 \dim A}{\dim Hg(A) + 1}.$$ 

Remark 1.12. In the first point of the previous corollary we are excluding the following odd integers $h$: $3, 35, 6435$ (when $h \leq 10^6$). In the second case, when the abelian variety is simple of type III such that $E = \mathbb{Q}$ the previous corollary is excluding the following 21 possible values of $g = 2h \leq 10^6$:

$$\Sigma' = \{4, 6, 8, 16, 36, 64, 70, 100, 128, 144, 196, 216, 256, 324, 400, 484, 512, 576, 676, 784, 900\}.$$

When $g \leq 10^6$ the set $\Sigma'$ has 513 elements.

Using the previous results and some results of Ichikawa [Ich91] and Lombardo [Lom16] one can prove that the product of simple abelian varieties of type I, II or III that are fully of Lefschetz type is also fully of Lefschetz type. We can therefore extend corollary 1.11 to the case of an abelian variety isogenous to a product of abelian varieties of type I, II or III.

Corollary 1.13. Let $A$ be an abelian variety defined over a number field $K$ isogenous over $\overline{K}$ to $\prod_{i=1}^d A_i^{n_i}$ where the abelian varieties $A_i$ are pairwise non isogenous over $\overline{K}$ of dimension $g_i$ and $n_i \geq 1$. We suppose that the abelian varieties $A_i$ are simple and not of type IV. Assume that one of the following hypothesis hold:

1. The relative dimensions $h_i$ is even or equal to 2 when the abelian variety $A_i$ is of type I or II and $h_i \in \{2k + 1, k \in \mathbb{N}\} \setminus \left\{ \frac{1}{2}(2m+1)^2, m \in \mathbb{N} \right\}$ when $A_i$ is of type III.
2. We have $e_i = 1$ (i.e. $E_i = \mathbb{Q}$) and $h_i$ is not in the set $\Sigma$ when $A_i$ is of type I or II and $h_i$ is not in the set $\Sigma'$ when $A_i$ is of type III.

Then $A$ if fully of Lefschetz type. In particular, with the notations of theorem 1.7 one has

$$\gamma(A) = \max_I \left\{ \frac{2 \sum_{i \in I} n_i d_i e_i h_i}{1 + \sum_{i \in I} e_i (2h_i^2 + \eta_i h_i)} \right\} = \max_I \left\{ \frac{2 \sum_{i \in I} n_i \dim A_i}{1 + \dim Hg(\prod_{i \in I} A_i)} \right\}.$$ 

This paper is organized as follows: in section 2 we present some lemmas of group theory. In section 3 we describe $\lambda$-adic pairings coming from the Weil pairing and some properties of the $\ell$-adic Galois representations like the so called “property $\mu$”. This property corresponds to the the study of the cyclotomic part of those Galois representations. In section 4 we present the proof of theorems 1.6 and 1.7. In section 5 we prove theorem 1.8 and in the last section we present the proof of the new cases of the Mumford–Tate conjecture, namely theorems 1.9 and 1.10 and corollaries 1.11 and 1.13.
2. Group lemmas

Let $V$ be a vector space endowed with a symplectic form (resp. quadratic), $G$ be the group of symplectic similitudes (resp. orthogonal), $\phi$ the bilinear form and $Q$ the quadratic form. The following calculation is perhaps already known but we have not been able to find it in the literature.

**Theorem 2.1.** Let $W$ be a vector subspace of $V$ of codimension $d$. Let us consider the stabilizer $G_W$ of $W$:

$$G_W = \{ g \in G, g_W = id_W \},$$

Then

$$\dim(G_W) = \frac{d(d+\varepsilon)}{2}, \quad \text{where} \quad \varepsilon = \begin{cases} 1 & \text{in the symplectic case}, \\ -1 & \text{in the orthogonal case}. \end{cases}$$

**Remark 2.2.** Let us point out that what we call stabilizer is sometimes call fixator.

**Proof.** We are going to proof this theorem in the orthogonal case. A similar computation of the codimension in the symplectic case can be found in [HiRa16, Paragraph 6, Remark 1]. Let $W$ be a vector subspace of $V$ of codimension $d$ and dimension $r$, we are going to split the proof into two parts. First, let us suppose that $W \cap W^\perp = \{0\}$, then

$$G_W = \left\{ \left( \begin{array}{cc} I_r & 0 \\ 0 & GO_d \end{array} \right) \right\}.$$  

Therefore, $G_W \simeq GO_{(W^\perp, \phi|_{W^\perp})}$, and his dimension is

$$\dim(G_W) = \frac{d(d-1)}{2}.$$  

Let us assume now that $W \cap W^\perp \neq \{0\}$, take $w_1 \in (W \cap W^\perp) \setminus \{0\}$ and let $w'_1 \in V \setminus W$ such that $\Pi := w_1, w'_1 >$ is a hyperbolic plane. One can prove by induction that $\dim(G_W) = \frac{d(d-1)}{2}$ using some well known results from algebraic geometry. Let $W' = W + \Pi$; it is a subspace of codimension $d' = d - 1$, hence, by induction, we have that $\dim(G_{W'}) = \frac{(d')(d'-1)}{2}$. We introduce the following variety

$$X := \left\{ y \in V, Q(y) = 0 \text{ and } \forall w \in W \phi(w, y) = \phi(w, w'_1) \right\}.$$  

Let us consider the surjective map $f : G_W \to X$ defined by $f(g) = g(w'_1)$. The dimension of $G_W$ is equal to the dimension of $X$ plus the dimension of the generic fiber, that is:

$$\dim G_W = \dim X + \dim G_{W+\Pi}.$$  

Then using the Jacobian criterion we obtain

$$\dim X = \dim T_g(X) = n - (r + 1) = d - 1.$$  

Finally

$$\dim G_W = d - 1 + \frac{(d - 1)(d - 2)}{2} = \frac{d(d - 1)}{2}.$$  

$\square$
Let $H$ be a finite subgroup of $A[ℓ^{∞}]$, then there exists a integer $n ∈ \mathbb{N}^*$, such that

$$H ⊂ A[ℓ^n].$$

By the theorem of structure of abelian $ℓ$-groups we know that one can decompose $H$ in the following way:

$$H = \prod_{i=1}^{t} (Z/ℓ^{m_i}Z)^{α_i} ⊂ A[ℓ^n],$$

where $m_1 < ... < m_t$ is a strictly increasing sequence of integers and $1 ≤ t ≤ 2g$.

For every $i ∈ [1, t]$ we can consider the following projection:

$$\pi_{m_i} : T_ℓ(A) → T_ℓ(A)/ℓ^{m_i}T_ℓ(A).$$

Therefor we can associate to $H ⊂ A[ℓ^n]$ a filtration of saturated submodules $W_i ⊂ \ldots ⊂ W_1$ of $T_ℓ(A)$ such that

$$\pi_{m_i}(W_i) = H[ℓ^{m_i}].$$

For instance, using the previous decomposition (2.1) of $H$, we have:

$$\pi_{m_1}(W_1) = H[ℓ^{m_1}] ∼ (Z/ℓ^{m_1}Z)^{α_1+\ldots+α_t} \quad \text{and} \quad \pi_{m_t}(W_t) = H[ℓ^{m_t}] ∼ (Z/ℓ^{m_t}Z)^{α_1}$$

We can associate to each submodule $W_i$ its stabilizer $G_{W_i}$ and we can describe it in the following way:

$$G_{W_i} = \{ g ∈ MT(A)(Z_ℓ), \forall x ∈ W_i, g(x) = x \}. \quad (2.4)$$

We notice that $G_{W_1} ⊂ \ldots ⊂ G_{W_t}.$

Let us denote by $G(H)$ the stabilizer of $H$ and let us remark that $ρ_ℓ(\text{Gal}(\overline{K}/K(H)))$ can be identify with $G(H).$ We can give the following description of $G(H)$ up to some finite index:

$$G(H) = \{ M ∈ \text{MT}(A)(Z_ℓ), M ∈ G_{W_i} \text{ mod } ℓ^{m_i}, i ∈ [1, t] \}. \quad (2.5)$$

This description of the stabilizer $G(H)$ allows us to introduce the following lemma. Actually, what it is important to remark is that the stabilizers $G_{W_i}$ are defined by some equations of bounded degree.

**Lemma 2.3.** Let $H$ be a finite subgroup of $A[ℓ^{∞}]$, $G$ be an algebraic subgroup over $\mathbb{Z}$ of $GL_{2g}$, $t$ be an integer and $G_{W_1} ⊂ \ldots ⊂ G_{W_t}$ be $t$ algebraic subgroups over $\mathbb{Z}_ℓ$ of $G_{\mathbb{Z}_ℓ}$ defined as above of codimension $d_i = \text{codim } G_{W_i}.$

For every prime number $ℓ$ we have the following equality:

$$(G(\mathbb{Z}_ℓ) : G(H)) \gg_A ℓ^{∑_{i=1}^{t} d_i(m_i-m_{i-1})}, \quad (2.6)$$

where $G(H)$ is the stabilizer of the subgroup $H$ and the notation $\gg_A$ means that we have the equality modulo some multiplicative constant which depends on $A.$

In order to proof lemma 2.3 we need to introduce the some results. First let us recall [Oes82, Théorème 1]:

**Theorem 2.4 (Oesterlé).** Let $Y$ be an algebraic variety of dimension $r$ defined by some equations of degree less or equal than $d$ such that $Y ⊂ \mathbb{A}^N.$ Then, for every $m ≥ 1$ one has

$$|Y_m| ≤ c(N, d, r)ℓ^{mr},$$

where $Y_m := im(\text{red} : Y → (\mathbb{Z}/ℓ^m\mathbb{Z})^N).$
Lemma 2.5. Let $H$ be a finite subgroup of $A[\ell^\infty]$ and denote $Y = G(H)$. Then, for every integer $m \geq 1$ there exists some constants $c_1$ and $c_2$, which depends on $\ell$, such that
\[ c_1 \ell^m \dim Y \leq |Y_m| \leq c_2 \ell^m \dim Y. \]

Proof. We know that $Y = G(H)$ is defined by a finite number of polynomials of $N = (2g)^2$ variables with coefficients in $\mathbb{Z}_\ell$. By theorem 2.4 one can get an upper bound for $|Y_m|$. We know that for every integer $m \geq 1$ there exists a constant $c_2$, which depends on $N$, on the degree of polynomials which define $Y$ and on the dimension of $Y$ denoted $r$ such that:
\[ |Y_m| \leq c_2 \ell^m \dim G(H). \tag{2.7} \]
Notice that the constant $c_2$ does not depend on the prime number $\ell$.

In order to give a lower bound of $|Y_m|$ we are going to prove the following inequalities for every integer $m \geq 0$:
\[ c(G, \ell) \ell^m \dim G \leq |G_m| \leq |G_m| \times |(G/Y)_m| \leq c(\deg(Y), G, N) \ell^m \dim(G/Y) \times |Y_m| \tag{2.8} \]
then, we will have
\[ c_1(g, r, \ell) \ell^m \dim Y \leq |Y_m|, \tag{2.9} \]
where $c_1(g, r, \ell) = \frac{c(G, \ell)}{c(\deg(Y), G, N)}$.

(1) In order to obtain the first inequality in (2.8) we use a weaker version of [Ser81, Theorem 9] that can be stated as follows:
\[ \forall m \geq 0 \text{ we have } c(G, \ell) \ell^m \dim G \leq |G_m| \tag{2.10} \]
where the constant $c(G, \ell)$ only depends on $G$ and on the prime number $\ell$.

Recall that for every finite subgroup $H \subset A[\ell^\infty]$ there exists a filtration of submodules $W_i \subset .. \subset W_1$ of $T_\ell(A)$ such that
\[ G(H) = \{ M \in MT(A)(\mathbb{Z}_\ell), M \in G_{W_i} \mod \ell^{m_i}, i \in [1, t]\}, \]
where the $G_{W_i}$ are the stabilizers of the submodules $W_i$. Let denote by $W = W_1 \otimes \mathbb{Q}_\ell$ the $\mathbb{Q}_\ell$-vector space of $V_\ell(A)$ of dimension $r$. From Chevalley’s theorem (see [Hum81, Théorème p. 80]) we know that there exists a vector space $U$ and a one-dimensional space $L \subset U$ such that there exists a rational representation $\rho : G \rightarrow GL(U)$ such that $G(W) = Stab_L$.

As a corollary of Chevalley’s theorem we know that there exists an action of $G$ over $\mathbb{P}(U)$
\[ G \times \mathbb{P}(U) \rightarrow \mathbb{P}(U), \]
such that the orbit of $[L]$ can be identified with $G/G(W)$ because $G(W) = Stab_L$.

Therefore the quotient $G/G(W)$ is a subvariety locally closed of $\mathbb{P}(U)$. Let us introduce the following map
\[ \psi : G \rightarrow Z \subset \mathbb{P}(U) \]
\[ g \mapsto g \cdot [L] \tag{2.11} \]
where $Z = \overline{\psi(G)}$ is closed. We have the following isomorphism:
\[ G/G(W) \simeq Z^0 \subset Z. \tag{2.12} \]

Let us remark that the variety $Z^0$ is actually a quasi-projective variety, nevertheless, in order to use Oesterlé’s theorem one need to work with affine varieties. Therefore, we can
cover the variety \( Z \) with some affine open subsets and then apply Oesterl’s theorem to these open subsets.

Let \( \ell \) be a prime number, then one have for every integer \( m \geq 1 \) the following morphism:

\[
\psi_{\ell^m} : G_m \to Z_m^o \tag{2.13}
\]

Thanks to the isomorphism (2.12) we know that

\[
(G/G(W))_m \simeq Z_m^o \tag{2.14}
\]

We can therefore determine the two last inequalities of (2.8).

(2) For every integer \( m \geq 1 \) we have the following equality:

\[
|G_m| = \sum_{x \in Z_m^o} |\psi_{\ell^m}^{-1}\{x\} \cap G_m| \tag{2.15}
\]

We know that \( \psi_{\ell^m}^{-1}\{x\} \cap G_m \) is a homogeneous space under \( G(W) \), therefore \( \psi_{\ell^m}^{-1}\{x\} \cap G_m \) is also an homogeneous space under \( Y_m \). Then we have the following surjection, such that when \( \psi_{\ell^m}^{-1}\{x\} \neq \emptyset \) we have \( y_0 \in \psi_{\ell^m}^{-1}\{x\} \cap G_m \)

\[
Y_m \to \psi_{\ell^m}^{-1}\{x\} \cap G_m
\]

\[
g \mapsto g \cdot y_0, \tag{2.16}
\]

then

\[
|\psi_{\ell^m}^{-1}\{x\} \cap G_m| \leq |Y_m|. \tag{2.17}
\]

From equations (2.15) and (2.17) one can deduce the following inequality

\[
|G_m| \leq |Z_m^o| \times |Y_m|, \tag{2.18}
\]

which is equivalent to the second inequality of (2.8).

(3) In order to obtain the last inequality of (2.8) one needs to prove that for every integer \( m \geq 1 \) we have

\[
|Z_m^o| \leq c(Y, G, N)\ell^m \dim Z^o. \tag{2.19}
\]

To do so we use theorem 2.4 and the fact that \( G/Y \simeq Z^o \subset \mathbb{P}(U) \) is entirely determined by some equations of bounded degree.

Therefore, with the following notations \( Y = G(H) \) and \( Y_m = im(Y \to GL_{2g}(\mathbb{Z}/\ell^m\mathbb{Z})) \) we proved that, for every submodule \( H \subset A[\ell^\infty] \) we have,

\[
\forall m \geq 0 \quad c_1(g, r, \ell)\ell^m \dim Y \leq |Y_m| \leq c_2(g, r, Y)\ell^m \dim Y.
\]

\[\square\]

Proof. (of lemma 2.3) When the prime number \( \ell \) is large enough, say \( \ell \geq \ell_0(A, K) \), Lombardo proves that the stabilizer \( G(H) \) is smooth over \( \mathbb{Z}_\ell \) (see [Lom16, Lemma 2.13]). Therefore one can use [HiRa12, Lemma 2.4] in order to obtain the following inequalities:

\[
c_1 \ell^\sum_{i=1}^t d_i(m_i - m_{i-1}) \leq (G(\mathbb{Z}_\ell) : G(H)) \leq c_2 \ell^\sum_{i=1}^t d_i(m_i - m_{i-1}), \tag{2.20}
\]

where \( c_1 \) and \( c_2 \) are two constants independent from \( m_i \) and \( \ell \) for \( \ell \geq \ell_0(A, K) \).

The main problem occurs when \( \ell < \ell_0(A, K) \), therefore, in order to obtain our result we are going to introduce some notations from [Ser81]. Let us denote \( Y = G(H) \), then

\[
Y \subset G(\mathbb{Z}_\ell) \subset GL_{2g}(\mathbb{Z}_\ell) \quad \text{and} \quad Y \subset \mathbb{Z}_\ell^N \quad \text{where} \quad N = 4g^2. \tag{2.21}
\]
Let $X_m := (\mathbb{Z}/\ell^m\mathbb{Z})^N$, we can define the following finite subgroup:

$$Y_m := im(\text{red} : Y \to X_m).$$

As it has been pointed up by Serre [Ser81, Remarque p. 346], one can use the methods, developed by Oesterlé and Robba (see [24]), in order to obtain a better estimation of the index $(G(\mathbb{Z}_\ell) : G(H))$ than the one given by [Ser81, Théorème 8].

By lemma 2.5 we know that, for $\ell < \ell_0(A, K)$, we have that for every integer $m \geq 1$ there exists two constants $c_1$ and $c_2$, which depend on $\ell$, such that:

$$c_1\ell^{\dim Y} \leq |Y_m| \leq c_2\ell^{\dim Y}. $$

Therefore there exists two constants $C_1$ and $C_2$, independent form the integers $m_i$, which possibly depends on $\ell$ for $\ell < \ell_0(A, K)$, such that

$$C_1\ell^{\sum_{i=1}^t d_i(m_i-m_i-1)} \leq (G(\mathbb{Z}_\ell) : G(H)) \leq C_2\ell^{\sum_{i=1}^t d_i(m_i-m_i-1)}. $$

Finally our result is independent of the prime number $\ell$ and we obtain the following equality, up to some multiplicatives constants depending on $A$:

$$(G(\mathbb{Z}_\ell) : G(H)) \ll_A \ell^{\sum_{i=1}^t d_i(m_i-m_i-1)}. $$

\[\square\]

3. Galois representations

3.1. Weil pairing. Let $A^\vee$ be the dual variety of $A$, then there exists a bilinear non-degenerate form over $T_\ell(A) \times T_\ell(A^\vee)$ which is Galois equivariant, called Weil pairing:

$$\langle \cdot, \cdot \rangle : T_\ell(A) \times T_\ell(A^\vee) \to \mathbb{Z}_\ell(1) = \lim_{\mu_\ell^n}. $$

Let $\phi : A \to A^\vee$ be a polarization of $A$, it induces a bilinear non-degenerate alternating pairing:

$$\phi_{\ell^n} : T_\ell(A) \times T_\ell(A) \xrightarrow{id \times \phi} T_\ell(A) \times T_\ell(A^\vee) \xrightarrow{\langle \cdot, \cdot \rangle} \mathbb{Z}_\ell(1) = \lim_{\mu_\ell^n}. $$

Recall that $D = \text{End}^\phi(A)$ is a quaternion algebra defined over the totally real field $E$, then we can define the following set:

**Definition 3.1.** Let denote $S$ the finite set of prime numbers $\ell$ such that $\ell$ is ramified in $O_E$ or divide the degree of the fixed polarization $\phi$ of $A$ or in the case of type II or III, the quaternion algebra $D$ does not decompose at some $\lambda|\ell$.

Let us recall that $O_{E_\ell} = \prod_{\lambda|\ell} O_\lambda$, therefore,

$$T_\ell(A) = \prod_{\lambda|\ell} T_\ell(A) \otimes_{\mathbb{Z}_\ell} O_\lambda,$$

where by definition $T_\lambda(A) := T_\ell(A) \otimes_{\mathbb{Z}_\ell} O_\lambda$. By [BGK10, Theorem 3.23] we know that, in the case where $A$ is an abelian variety of type II or III and $\ell \notin S$, $T_\lambda(A) = T_\lambda(A) \oplus T_\lambda(A)$ where $T_\lambda(A)$ is a free $O_\lambda$-module of rank $2h$. Moreover, for every $\lambda|\ell$ there exists a symmetric, non-degenerate pairing

$$\phi_{\lambda,\ell} : T_\lambda(A) \times T_\lambda(A) \to O_\lambda$$

compatible with the $G_K$-action, which is induced by $\phi_{\ell^n}$.  

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Therefore we know that the $\mathbb{Q}_\ell$-vector space $V_\ell$ decompose as follows: $V_\ell = \prod_{\lambda \mid \ell} V_\lambda \oplus V_\lambda$ and $Hg(A) \otimes \mathbb{Q}_\ell \subset \text{Sp}(D, \phi) \otimes \mathbb{Q}_\ell = \prod_{\lambda \mid \ell} \text{SO}(V_\lambda, \phi_\lambda)$.

3.2. Property $\mu$. Let us introduce the property called “propriété $\mu$” (see [HiRa12, Definition 6.3]). We say that the abelian variety $A$ satisfy the property $\mu$ if for every prime number $\ell$ and every subgroup $H \subset A[\ell]$ there exists an integer $m = m(H)$, such that, up to some finite index bounded independently of $\ell$ (notation $\asymp$), one has:

$$K(H) \cap K(\mu_{\ell m}) \asymp K(\mu_{m}) .$$

Let assume that the abelian variety satisfy Mumford-Tate conjecture. then in fig. 1 we have the following equalities up to some finite index bound independently of $\ell$:

$$\text{Gal}(K(A[\ell]) / K) \asymp \text{MT}(A)(\mathbb{Z}_\ell) ,$$
$$\text{Gal}(K(A[\ell]) / K(\mu_{\ell m})) \asymp Hg(A)(\mathbb{Z}_\ell) ,$$
$$\text{Gal}(K(\mu_{\ell m}) / K) \asymp \mathbb{G}_m(\mathbb{Z}_\ell) .$$

![Diagram](image.png)

**Figure 1.** General case : $H \subset A[\ell]$.

Let us introduce the following groups:

$$G_0(H) := \{ \sigma \in \text{MT}(A)(\mathbb{Z}_\ell), \sigma|_H = id|_H \} = \text{Gal}(K(A[\ell]) / K(H)) ,$$
$$G(H) := G_0(H) \cap Hg(A)(\mathbb{Z}_\ell) .$$

**Lemma 3.2.** Let $H \subset A[\ell]$ and let $\delta(H) := [K(\mu_{em}) : K]$. Then we have, up to some finite index bounded independently of the prime number $\ell$:

$$[K(H) : K] = (Hg(A)(\mathbb{Z}_\ell) : G(H)) \cdot \delta(H) .$$

**Proof.** Recall that $G(H)$ is the stabilizer of $H$ in $\text{MT}(A)(\mathbb{Z}_\ell)$ and that the abelian variety $A$ is simple of type III. The main key idea is that the morphism $\text{mult} : G(H) \to \mathbb{G}_m$ is surjective for every maximal isotropic subgroup $H$ of $A[\ell]$, one can see that this is true readapting [HiRa16, Proposition 5.5]. Then we will have $\text{mult}(G(H))(\mathbb{Z}_\ell) = \mathbb{Z}_\ell^\times$ and therefore $\delta(H) = [K(\mu_{em}) : K] = (\mathbb{Z}_\ell^\times : \text{mult}(G(H))(\mathbb{Z}_\ell)) = 1$. \(\square\)
Let us remark that in the case where $H \subset A[\ell^n]$ we also have the property $\mu$, then there exists an integer $m = m(H)$ such that, up to some finite index bound independently of $\ell$, one has:

$$K(H) \cap K(\mu_{\ell^n}) = K(H) \cap K(\mu_{\ell^\infty}) \cong K(\mu_{\ell^m}). \quad (3.1)$$

4. MAIN RESULTS AND PROOFS

In this section we are going to present a complete proof of the following theorem:

**Theorem 4.1.** Let $A$ be a simple abelian variety defined over a number field $K$ of type III and dimension $g$. We have $e = [E : \mathbb{Q}]$, $d = 2$ and $h$ the relative dimension. We suppose that $A$ is fully of Lefschetz type.

Then the invariant $\gamma(A)$ is equal to

$$\gamma(A) = \frac{2deh}{1 + eh(2h - 1)} = \frac{2 \dim A}{\dim \text{MT}(A)}. \quad (4.1)$$

Let us recall the definition of the invariant $\gamma(A)$:

$$\gamma(A) = \inf \{x > 0 \mid \forall L/K \text{ finite}, |A(L)_{\text{tors}}| \ll [L : K]^x\}. \quad (4.2)$$

A main result to compute this invariant is the criterion of the independence of $\ell$-adic representations introduced by Serre in [Ser13]. Roughly speaking, there exists a finite extension $K'/K$, such that for every finite subgroup $H_{\text{tors}}$ of $A(L)_{\text{tors}}$: if we write

$$H_{\text{tors}} = \prod_{\ell \in \mathcal{P}} H_{\ell},$$

where $H_{\ell}$ is a subgroup of $A[\ell^\infty]$ we have

$$[K'(H_{\text{tors}}) : K'] = \prod_{\ell \in \mathcal{P}} [K'(H_{\ell}) : K'].$$

Using this criterion, one can actually concentrate our attention to a finite subgroup $H$ of $A[\ell^\infty]$, therefore, we suppose that the number field $K$ is such that the $\ell$-adic representations are independent in the sense of Serre (see [Ser13]).

In order to compute our invariant, we use the following equivalence: for every finite subgroup $H$ of $A[\ell^\infty]$ we have

$$|H| \ll [K(H) : K]^{\gamma(A)} \iff \gamma(A) \geq \frac{\log \ell |H|}{\log \ell [K(H) : K]}.$$

One can notice that the determination of $\gamma(A)$ only depends on the order of the finite subgroup $H$ and the degree of the extension $K(H)$ over $K$.

After extending $K$ and replacing $A$ by an isogenous abelian variety, we may assume:

- The abelian variety $A/K$ is such that $\text{End}_K(A) = \text{End}_K(A)$ that we denote by $\text{End}(A)$.
- We suppose that $\text{End}(A)$ is a maximal order of $\text{End}(A) \otimes \mathbb{Q}$,
- The subgroup $H \subset A[\ell^\infty]$ is stable under $\text{End}(A)$,
- The number field $K$ is such that $G_\ell$ is connected and the $\ell$-adic representations are independent in the sense of Serre (see [Ser13]).
The notation $\cong$ means that it is an equality up to some finite index bound independently of the prime number $\ell$.

**Proof.** Recall that $A$ is a simple abelian variety defined over a number field $K$ of type III, dimension $g = 2eh$. Let $G_K$ be the absolute Galois group and let $\ell$ be a prime number not in $S$. Moreover we assume that the abelian variety $A$ is fully of Lefschetz type, more precisely one has the following equality

$$Hg(A)_{\mathbb{Q}_\ell} = \prod_{\lambda | \ell} \text{SO}_{2h}(O_{\lambda}) = \prod_{\lambda | \ell} \text{Res}_{E_\lambda/\mathbb{Q}_\ell} \text{SO}_{2h,E_\lambda}.$$  

(4.3)

We can therefore deduce that

$$\dim Hg(A)_{\mathbb{Q}_\ell} = \sum_{\lambda | \ell} [E_\lambda : \mathbb{Q}_\ell] \frac{2h(2h - 1)}{2} = e(2h^2 - h).$$

The proof of this theorem is going to be divided into two parts.

1. **Simple case :** $H \subset A[\ell]$.
   One has the following decomposition for $A[\ell] = T_{\ell}(A)/\ell T_{\ell}(A)$, more precisely, one has $A[\ell] = \prod_{\lambda | \ell} T_{\lambda}[\ell] \oplus T_{\lambda}[\ell]$ where $T_{\lambda}[\ell] = T_{\lambda}(A)/\ell T_{\lambda}(A)$.

   Therefore, for every subgroup $H$ of $A[\ell]$ one has the following decomposition

$$H = \prod_{\lambda | \ell} H_{\lambda} \otimes H_{\lambda},$$

where $H_{\lambda}$ is a subgroup of $T_{\lambda}[\ell]$ of dimension $r_{\lambda} \leq 2h$. One can therefore deduce the order of $H$:

$$|H| = \ell^{2 \sum_{\lambda | \ell} f(\lambda)r_{\lambda}}.$$  

(4.4)

Let us determine the degree of the extension $K(H)$ over $K$. To do so, we use the property $\mu$ (see section 3.2), therefore we know that there exists an integer $m = m(H)$, such that, up to some finite index bound independently of $\ell$, one has, as in the equality (3.1):

$$K(H) \cap K(\mu_{\ell m}) \cong K(\mu_{\ell m}).$$

Let us introduce the following groups as in section 3

$$G_0(H) := \{ \sigma \in \text{MT}(A)(\mathbb{F}_\ell), \sigma|_H = \text{id}_H \} = \text{Gal}(K(A[\ell])/K(H)),$$

$$G(H) := G_0(H) \cap \text{Hg}(A)(\mathbb{F}_\ell).$$

By lemma 3.2 we know that $\delta(H) := [K(\mu_{\ell m}) : K]$ and we have the following equality, up to some finite index bound independently of $\ell$,

$$[K(H) : K] = [K(H) : K(\mu_{\ell m})][K(\mu_{\ell m}) : K] = (\text{Hg}(A)(\mathbb{F}_\ell) : G(H)) \cdot \delta(H).$$  

(4.5)

In order to compute $[K(H) : K]$ one needs to determine $(\text{Hg}(A)(\mathbb{F}_\ell) : G(H))$. To do so we use theorem 2.1 and lemma 2.3.

Let us recall that for every subgroup $H \subset A[\ell]$ one has the following decomposition: $H = \prod_{\lambda | \ell} H_{\lambda} \oplus H_{\lambda}$. One can apply the property $\mu$ to each subgroup $H_{\lambda} \subset T_{\lambda}[\ell]$ of dimension $r_{\lambda}$. Therefore there exists an integer $m_{\lambda}$ such that

$$K(H_{\lambda}) \cap K(\mu_{\ell}) \cong K(\mu_{\ell m_{\lambda}}).$$
Let denote $m := \max_{\lambda} m_\lambda$. Moreover, we know that we have the following equality, up to some finite index bound independently of the prime number $\ell$:

$$\text{Gal}(K(A[\ell]) / K(\mu_\ell)) \cong \prod_{\lambda|\ell} \text{Gal}(K(T_\lambda[\ell]) / K(\mu_\ell)).$$

Therefore one has:

$$(Hg(A)(\mathbb{F}_\ell) : G(H)) = \prod_{\lambda|\ell} (Hg(A)(\mathbb{F}_\lambda) : G(H_\lambda)).$$

In order to compute $(Hg(A)(\mathbb{F}_\ell) : G(H))$ one needs to compute $(Hg(A)(\mathbb{F}_\lambda) : G(H_\lambda))$ for every $\lambda|\ell$. Using lemma 2.3 one obtain:

$$(Hg(A)(\mathbb{F}_\lambda) : G(H_\lambda)) \gg \ll (#\mathbb{F}_\lambda)^{\text{codim} G(H_\lambda)}.$$ 

Therefore

$$(Hg(A)(\mathbb{F}_\ell) : G(H)) \gg \ll \prod_{\lambda|\ell} (#\mathbb{F}_\lambda)^{\text{codim} G(H_\lambda)},$$

Let $\delta$ be the integer such that $\delta(H) = \ell^\delta$, then equation (4.5) become

$$[K(H) : K] \gg \ll \ell^{\sum_{\lambda|\ell} f(\lambda) \text{codim} G(H_\lambda) + \delta}. \quad (4.6)$$

where by theorem 2.1 one has $\text{codim} G(H_\lambda) = \frac{(4h-1)r_\lambda - r_\lambda^2}{2}$.

We know that for every subgroup $H \subset A[\ell^\infty]$ one has

$$\gamma(A) \geq \frac{\log_\ell |H|}{\log_\ell [K(H) : K]}.$$

Therefore one has

$$\gamma(A) = \max_{r_\lambda} \psi(r),$$

where $r = (r_\lambda)_{\lambda|\ell}$ and the function $\psi$ is define as follows:

$$\psi(r) = \frac{2 \sum_{\lambda|\ell} f(\lambda)r_\lambda}{\delta + \sum_{\lambda|\ell} f(\lambda) \text{codim} G_{H_\lambda}}.$$

The idea now is to study the maximum of this function when the integer $r_\lambda$ varies, to do so, we are are going to separate the study into to parts:

(a) First of all, we suppose that $H_\lambda$ is contained in a maximal isotropic space, therefore $0 \leq r_\lambda \leq h$ for every $\lambda|\ell$ and $\delta(H_\lambda) = 1$. Therefore $\delta = 0$.

(b) Then, we suppose that $H_\lambda$ is not contained in a maximal isotropic space then $\delta(H_\lambda) = \ell$ therefore $\delta = 1$.

Then, one can obtain the maximum of the function $\psi$ when we compare the maximum obtained in each case. We are going to study the following function:

$$\psi(r) = \frac{2 \sum_{\lambda|\ell} f(\lambda)r_\lambda}{\delta + \sum_{\lambda|\ell} f(\lambda)(4h-1)r_\lambda^2 - r_\lambda^2}.$$ 

In the case where $\delta = 0$ we have:
\[
\psi(\mathbb{Z}) = \frac{4 \sum_{\lambda \ell} f(\lambda)r_\lambda}{\sum_{\lambda \ell} f(\lambda)(-r_\lambda^2 + r_\lambda(4h - 1))}
\]

After the study of this function, we notice that \( \psi \) is an increasing function and his maximum is obtain when \( r_\lambda = h \). Then,
\[
\max_{r_\lambda \in [0,h]} \psi(r) = \frac{4}{3h - 1}.
\]

In the case where \( \delta = 1 \) we have:
\[
\psi(\mathbb{Z}) = \frac{4 \sum_{\lambda \ell} f(\lambda)r_\lambda}{2 + \sum_{\lambda \ell} f(\lambda)(r_\lambda(4h - 1) - r_\lambda^2)}
\]

After the study of this function, we notice that \( \psi \) is an increasing function and his maximum is obtain when \( r_\lambda = 2h \). Then,
\[
\max_{r_\lambda \in [0,2h]} \psi(r) = \frac{4eh}{1 + e(2h^2 - h)}.
\]

We can therefore conclude that the maximum of the function \( \psi \) is the following
\[
\max_{r_\lambda} \psi(r) = \max \left( \frac{4}{3h - 1}, \frac{4eh}{1 + e(2h^2 - h)} \right) = \frac{4eh}{1 + e(2h^2 - h)}
\]

Then because \( d = 2 \) one has
\[
\gamma(A) = \frac{2deh}{1 + e(2h^2 - h)} = \frac{2 \dim A}{1 + \dim \text{Res}_{E/Q} SO_{2h}} = \frac{2 \dim A}{\dim MT(A)}.
\]

(2) General case : \( H \subset A[\ell^\infty] \).

Let \( H \) be a subgroup of \( A[\ell^\infty] \), then, there exists \( n \in \mathbb{N}^* \) such that \( H \subset A[\ell^n] \). As in the previous case, \( H = \prod_{\lambda \ell} H_\lambda \otimes H_\lambda \) where \( H_\lambda \subset T_\lambda[\ell^n] \). One has the following decomposition as in (2.1):
\[
H_\lambda = \prod_{i=1}^{t_\lambda}(\mathbb{Z}/\ell m^i_\lambda \mathbb{Z})^{\alpha_\lambda,i}f(\lambda) \simeq \prod_{i=1}^{t_\lambda}(O_\lambda/\ell m^i_\lambda O_\lambda)^{\alpha_\lambda,i},
\]
where \( 1 \leq t_\lambda \leq 2h \) and \( m^1_\lambda < \ldots < m^t_\lambda \) is a strictly decreasing sequence of integers. Therefore one can deduce the order of \( H_\lambda \), \( |H_\lambda| = \ell^{f(\lambda) \sum_{i=1}^{t_\lambda} \alpha_\lambda,i m^i_\lambda} \). Then, the order of the subgroup \( H \) is
\[
|H| = \ell^{\sum_{\lambda \ell} f(\lambda) \sum_{i=1}^{t_\lambda} \alpha_\lambda,i m^i_\lambda}.
\]

Let us determine the degree of the extension \( K(H) \) over \( K \). As before, by lemma 3.2 we know that \( \delta(H) := [K(H, \mu_{\ell^n}) : K] \) and we have the following equality, up to some finite index bound independently of \( \ell \),
\[
[K(H) : K] = (\text{Hg}(A)(\mathbb{Z}_\ell) : G(H)) \cdot \delta(H).
\]
As before, one has
\[
(\text{Hg}(A)(\mathbb{Z}_\ell) : G(H)) = \prod_{\lambda \ell} (\text{Hg}(A)(O_\lambda) : G(H_\lambda)).
\]
Our goal is to estimate the value, for every $\lambda|\ell$, of
\[
(H_g(A)(O_\lambda) : G(H_\lambda)).
\] (4.11)
In order to do so, we use lemma 2.3 and we study more deeply the structure of each stabilizer $G(\lambda)$.

As in section 2 we introduce, for every $\lambda|\ell$ and every $i \in [1, t_\lambda]$, the filtration $W_{t_\lambda} \subset ... \subset W_1$ of saturated submodules of $T_\lambda$ associated to the subgroups $H_\lambda$ (see (2.2)).

We can therefore define
\[
r_i := \text{rk}_{O_\lambda} W_i = \sum_{k=1}^{t_\lambda-(i-1)} \alpha_{\lambda,k}.
\]

Let $G_{W_i}$ be the stabilizers of $W_i$, therefore $G(\lambda)$ can be describe as follows:
\[
G(\lambda) = \{ M \in MT(A)(O_\lambda), M \in G_{W_i} \mod \ell^t_{\lambda} \forall i \in [1, t_\lambda] \}.
\]

By theorem 2.1 we know that the codimension $d_i$ of $G_{W_i}$ is:
\[
d_i = \frac{r_i(4h - 1 - r_i)}{2}.
\]

By lemma 2.3 we know that for every prime number $\ell$, we have the following equality, up to some constants
\[
(H_g(A)(O_\lambda) : G(\lambda)) \gg \ell f(\lambda) \sum_{i=1}^{t_\lambda} d_i (m_{T_\lambda}^{t_\lambda-(i-1)} - m_{T_\lambda}^{t_\lambda-(i-1)+1}).
\]
Therefore the equality (4.10) became
\[
(H_g(A)(\mathbb{Z}_\ell) : G(\lambda)) \gg \ell \sum_{i=1}^{t_\lambda} d_i (m_{T_\lambda}^{t_\lambda-(i-1)} - m_{T_\lambda}^{t_\lambda-(i-1)+1}).
\] (4.12)

By convention $m_{T_\lambda}^{t_\lambda+1} = 0$.

Then the equality (4.9) became
\[
[K(H) : K] \gg \ell \sum_{i=1}^{t_\lambda} d_i (m_{T_\lambda}^{t_\lambda-(i-1)} - m_{T_\lambda}^{t_\lambda-(i-1)+1}) \cdot \delta(H). \]

Now the idea is to get a lower bound of $\delta(H)$. To do so, we introduce two integers, $m_H$ and $h_H$ defined as follow:

- Let us denote by $m_H$ the maximal integer $m_H \geq 1$ such that $\exists P, Q \in H$ of order $\ell^m$ and such that $\phi_0(\ell^{m}P, \ell^{-1}Q) \in \mu_\ell$. If such a $m_H$ does not exist, then we would say that $m_H = 0$;
- Let us denote by $h_H$ the minimal integer in $[1, t_\lambda]$ such that $m_{T_\lambda}^{h_H} \leq m_H$, when $m_H = 0$ we would say that $h_H = t_\lambda + 1$.

Let us recall that we have the following inclusions: $G_{W_1} \subset ... \subset G_{W_i}$. We can attached to each stabilizer the integer $\delta_{\lambda,i}$ for $i \in [1, t_\lambda]$ and $\lambda|\ell$ which takes value in $\{0, 1\}$ depending in the fact that $W_i$ is contained in a maximal isotropic space or not. Since $W_{t_\lambda} \subset ... \subset W_1$ we notice that from the moment when one of the $W_i$ is not contained in a maximal isotropic space, we will have $\delta_{\lambda,i} = ... = \delta_{\lambda,t_\lambda} = 1$. This idea can be translated in terms of the integer $h_H$ previously defined as follows:
\[
\delta_{\lambda,1} = ... = \delta_{\lambda,t_\lambda+1-h_H} = 1 \quad \text{and} \quad \delta_{\lambda,t_\lambda+1-(h_H-1)} = ... = \delta_{\lambda,t_\lambda} = 0.
\]
Actually the integers $m$ and $h$ have been introduced in order to have $\delta(H) \gg \ell^{m_h}$. Let us remark that we can write $m_{h}^{H}$ in terms of the $\delta_{\lambda, i}$ in the following way:

$$
\delta_{\lambda, i} = \max \sum_{i=1}^{t_{\lambda}} (m_{\lambda}^{t_{\lambda} - (i - 1)} - m_{\lambda}^{t_{\lambda} - (i - 1) + 1}) \delta_{\lambda, i}.
$$

Then, from lemma 3.2, one can deduce the following inequality, up to some finite index:

$$
[K(H) : K] \gg \ell^{\sum_{i=1}^{t_{\lambda}} f(\lambda) \sum_{i=1}^{t_{\lambda}} d_{i} (m_{\lambda}^{t_{\lambda} - (i - 1)} - m_{\lambda}^{t_{\lambda} - (i - 1) + 1})} \ell^{m_{h}^{H}},
$$

then

$$
[K(H) : K] \gg \ell^{\sum_{i=1}^{t_{\lambda}} f(\lambda) (d_{i} + \delta_{\lambda, i}) (m_{\lambda}^{t_{\lambda} - (i - 1)} - m_{\lambda}^{t_{\lambda} - (i - 1) + 1})}.
$$

We can therefore use some combinatorial arguments, as in [HiRa12 Paragraph 4.2]. We have the following equivalence:

$$
|H| = \ell^{\sum_{i=1}^{t_{\lambda}} f(\lambda) a_{\lambda, i} m_{\lambda}^{i}} \ll [K(H) : K]^{\gamma(A) \iff}
$$

$$
\gamma(A) \gg \max \left\{ \frac{\sum_{i=1}^{t_{\lambda}} f(\lambda) a_{\lambda, i} m_{\lambda}^{i}}{\sum_{i=1}^{t_{\lambda}} f(\lambda) (d_{i} + \delta_{\lambda, i}) (m_{\lambda}^{t_{\lambda} - (i - 1)} - m_{\lambda}^{t_{\lambda} - (i - 1) + 1})} \right\}.
$$

Let us recall that $a_{\lambda, i} = d a_{\lambda, i}$ where $d = 2$.

After some modifications of the denominator we obtain

$$
\gamma(A) \gg \max \left\{ \frac{\sum_{i=1}^{t_{\lambda}} f(\lambda) a_{\lambda, i} m_{\lambda}^{i}}{\sum_{i=1}^{t_{\lambda}} f(\lambda) m_{\lambda}^{i} (d_{t_{\lambda} + 1 - i} + \delta_{t_{\lambda} + 1 - i} + d_{t_{\lambda} + 2 - i} - d_{t_{\lambda} + 2 - i} - \delta_{t_{\lambda} + 2 - i})} \right\}.
$$

We take the maximum over $m_{\lambda}^{1} < \ldots < m_{\lambda}^{t_{\lambda}}$, let denote

$$
M = \max_{m_{\lambda}^{1} \geq \ldots \geq m_{\lambda}^{t_{\lambda}}} \left\{ \frac{\sum_{i=1}^{t_{\lambda}} f(\lambda) a_{\lambda, i} m_{\lambda}^{i}}{\sum_{i=1}^{t_{\lambda}} f(\lambda) m_{\lambda}^{i} (d_{t_{\lambda} + 1 - i} + \delta_{t_{\lambda} + 1 - i} + d_{t_{\lambda} + 2 - i} - d_{t_{\lambda} + 2 - i} - \delta_{t_{\lambda} + 2 - i})} \right\}.
$$

From [HiRa12 Lemma 2.7], one has

$$
M = \max_{1 \leq k \leq t_{\lambda}} \left\{ \frac{\sum_{i=1}^{k} f(\lambda) a_{\lambda, i}}{\sum_{i=1}^{k} f(\lambda) (d_{t_{\lambda} + 1 - i} + \delta_{t_{\lambda} + 1 - i} + d_{t_{\lambda} + 2 - i} - d_{t_{\lambda} + 2 - i} - \delta_{t_{\lambda} + 2 - i})} \right\}.
$$

By convention, one has $d_{t_{\lambda} + 1} = 0 = \delta_{t_{\lambda} + 1 + k}$. Let us remark that

$$
r_{t_{\lambda} + 1 - k} := r_{k} O_{\lambda} W_{t_{\lambda} + 1 - k} = \sum_{i=1}^{k} a_{\lambda, i}.
$$

After some simplification we obtain:

$$
M = \max_{1 \leq k \leq t_{\lambda}} \left\{ \frac{\sum_{i=1}^{k} f(\lambda) r_{t_{\lambda} + 1 - k}}{\sum_{i=1}^{k} f(\lambda) (d_{t_{\lambda} + 1 - k} + \delta_{t_{\lambda} + 1 - k})} \right\}.
$$

We can assume, without lose of generalities, that $\delta(H) = \delta(H_{\lambda'})$ for a fixed place $\lambda'$ over $\ell$. Then, $\forall \lambda | \ell$ such that $\lambda \neq \lambda'$ one has $\delta_{\lambda, t_{\lambda} + 1 - k} = 0$ for every $k \in [1, t_{\lambda}]$. Then,
\[ M = \max_{1 \leq k \leq \lambda} \left\{ \frac{\sum_{\ell} f(\lambda) r_{\lambda+1-k}}{\delta_{\lambda', t_{\lambda'+1-k}} + \sum_{\ell} f(\lambda) d_{t_{\lambda+1-k}}} \right\}. \]

The maximum will be taken according to the values of \( \delta_{\lambda', t_{\lambda'+1-k}} \). Two cases can occur:

- If \( 1 \leq k < h_H \) then \( t_{\lambda'} + 1 - k \in [t_{\lambda'} + 2 - h_H, t_{\lambda'}] \) and therefore \( \delta_{\lambda', t_{\lambda'+1-k}} = 0 \).
- If \( h_H \leq k \leq t_{\lambda'} \) then \( t_{\lambda'} + 1 - k \in [1, t_{\lambda'} + 1 - h_H] \) and therefore \( \delta_{\lambda', t_{\lambda'+1-k}} = 1 \).

Then we see that this maximum is

\[ M = \max \left\{ \max_{1 \leq k < h_H} \frac{\sum_{\ell} f(\lambda) r_{t_{\lambda}+1-k}}{\sum_{\ell} f(\lambda) d_{t_{\lambda}+1-k}}, \max_{h_H \leq k \leq t_{\lambda}} \frac{\sum_{\ell} f(\lambda) r_{t_{\lambda}+1-k}}{1 + \sum_{\ell} f(\lambda) d_{t_{\lambda}+1-k}} \right\}. \]

Let us recall that

\[ d_{t_{\lambda}+1-k} = \frac{r_{t_{\lambda}+1-k}(4h - 1 - r_{t_{\lambda}+1-k})}{2}. \]

Let introduce the following functions

\[ f_1(r_{t_{\lambda}+1-k}) = \frac{\sum_{\ell} f(\lambda) r_{t_{\lambda}+1-k}}{\sum_{\ell} f(\lambda) r_{t_{\lambda}+1-k}^{4h - 1 - r_{t_{\lambda}+1-k}}}, \]

and

\[ f_2(r_{t_{\lambda}+1-k}) = \frac{\sum_{\ell} f(\lambda) r_{t_{\lambda}+1-k}}{1 + \sum_{\ell} f(\lambda) r_{t_{\lambda}+1-k}^{4h - 1 - r_{t_{\lambda}+1-k}}}, \]

then,

\[ M = \max \left\{ \max_{1 \leq k < h_H} f_1(r_{t_{\lambda}+1-k}), \max_{h_H \leq k \leq t_{\lambda}} f_2(r_{t_{\lambda}+1-k}) \right\}. \]

The study of the functions, allows us to deduce that both functions \( f_1 \) and \( f_2 \) are increasing over their domain of definition. Therefore, they reach their maximum in \( r_{t_{\lambda}+1-k} = h \) and \( r_{t_{\lambda}+1-k} = 2h \) respectively. When one of the \( W_i \) is contain in a maximal isotropic space we remark that his rank is at most \( h \). The maximal case for \( f_2 \) occurs when \( W_1 = T_\ell(A) \). Then

\[ M = \max \left\{ \frac{2}{3h - 1}, \frac{2eh}{eh(2h - 1) + 1} \right\}. \]

Let us remark that the case \( H \subset A[\ell^\infty] \) can be reduce to the case \( H \subset A[\ell] \). We can conclude that

\[ \gamma(A) = \frac{2(2eh)}{eh(2h - 1) + 1} = \frac{2 \dim A}{\dim Hg(A) + 1}. \]

Finally, in order to conclude the proof, one need to prove this theorem in the case where the prime number \( \ell \) is in the finite set \( S \), to do so, one can follow exactly the different cases treated in [HiRa16, Paragraph 8]. \( \square \)

In order to prove theorem [14], one need to prove that a product of simple abelian varieties which are fully of Lefschetz type is an abelian variety fully of Lefschetz type. To do so we use [Ich91, Theorem 1A] and [Lom16, Theorem 4.1] which states respectively that the Hodge group (resp. \( \ell \)-adic monodromy group) of a product is the product of each Hodge group.
(resp. $\ell$-adic monodromy group). Knowing this previous results one can prove theorem \ref{thm:1.7} using \cite[Theorem 1.14]{HiRa16} and the same techniques developed in the proof of theorem \ref{thm:1.6} and the proof of \cite[Theorem 1.14]{HiRa16}.

5. Order of the extension generated by a torsion point

The following results are actually a consequence of the strategy of the proof of theorems \ref{thm:1.6} and \ref{thm:1.7}.

**Theorem 5.1.** Let $A$ be an abelian variety defined over a number field $K$, simple of type III, relative dimension $h$ and fully of Lefschetz type. There exists a constant $c_1 := c_1(A, K) > 0$ such that, for every torsion point $P$ of order $m$ in $A(K)$, one has:

$$[K(P) : K] \geq c_1^{\omega(m)} m^{2h}.$$  

**Theorem 5.2.** Let $A$ be an abelian variety defined over a number field $K$, isogenous over $\overline{K}$ to $\prod_{i=1}^{d} A_i^{n_i}$ where the abelian varieties $A_i$ are simple, not of type IV, relative dimension $h_i$ and fully of Lefschetz type. There exists a constant $c_1 := c_1(A, K) > 0$ such that, for every torsion point $P$ of order $m$ in $A(\overline{K})$, one has:

$$[K(P) : K] \geq c_1^{\omega(m)} m^{2h},$$

where $h = \min_i h_i$.

**Proof.** Since theorem \ref{thm:5.2} follows from theorem \ref{thm:5.1} we prove the latter and assume that the abelian variety $A$ is simple. Let $P$ be a torsion point of order $m$. Then $P = P_1 + ... + P_r$ where each $P_i$ is a point of order $\ell_i^{n_i}$ and $m = \prod_{i=1}^{r} \ell_i^{n_i}$. Let us denote $H_P$ the End($A$)-module generated by $P$, then, one has $K(P) = K(H_P)$. By the independence criterion of $\ell$-adic representations introduced by Serre, we know that, up to some multiplicative constants, uniform on $m$ and on $P$, we have the following inequality:

$$[K(P) : K] = [K(P_1, ..., P_r) : K] \gg \prod_{i=1}^{r} [K(P_i) : K],$$

Moreover, one has the following inequality, up to some multiplicative constants, uniform on $\ell_i$ and on $P_i$:

$$[K(P_i) : K] \gg \ell_i^{2hn_i}.$$  

Let $\omega(m)$ be the number of prime factors, then, there exists a positive constant $c_1 := c_1(A, K)$ such that

$$[K(P) : K] = [K(P_1, ..., P_r) : K] \gg \prod_{i=1}^{r} [K(P_i) : K] \gg \prod_{i=1}^{r} c_1^{2hn_i} \geq c_1^{\omega(m)} m^{2h}. \quad (5.1)$$  

\[\square\]

**Remark 5.3.** Using the following inequality

$$\omega(m) \leq c \cdot \frac{\log m}{\log \log m},$$

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one can rewrite the inequality (5.1) where $c_2$ is a constant:

$$[K(P) : K] \geq c_1 m^{2h} - \frac{c_2}{\log \log m}.$$  \hspace{1cm} (5.2)

### 6. New results concerning the Mumford–Tate conjecture

Let us remark that one of the main hypothesis of theorems 1.6 and 1.7 is that the abelian variety $A$ must be fully of Lefschetz type, in particular, Mumford–Tate conjecture must hold for $A$. In this direction, the following theorems give examples of abelian varieties that are fully of Lefschetz type and for which theorems 1.6 and 1.7 are unconditional.

Some results of Noot and Shimura show that there exists abelian varieties, defined over a number field $K$, of type III in the sense of Albert’s classification such that the Mumford–Tate conjecture holds for this abelian varieties. More precisely, one can state the following theorem:

**Theorem 6.1** (Noot, Shimura). Let $h$ be an integer $h \geq 1$, $E$ a totally real field and $D$ a quaternion algebra totally definite over $E$. Exclude the case $h = 1$ or $2$ when $D$ is of type III. One has the following results:

1. There exists a universal family of abelian varieties which contain $D$ in their endomorphism algebra. In particular, if $A$ is a general point then $\text{End}^\circ(A) = D$ and $\text{MT}(A) = \mathcal{L}(A)$ (Lefschetz group of $A$).
2. Moreover, there exists a number field $K$ and an abelian variety $A$, defined over $K$, such that $\text{End}^\circ(A) = D$, $\text{MT}(A) = \mathcal{L}(A)$ and the Mumford–Tate conjecture holds for $A$.

**Remark 6.2.** The first point is due to Shimura (see [BiLa04, Theorem 9.1]) and the second point is the [Noo95, Théorème 1.7].

The following theorem gives a correction of a subtle point in [BGK10, Theorem 5.11]:

**Theorem 6.3.** Let $A$ be a simple abelian variety of dimension $g$ and of type III. Let us recall that $g = 2eh$ where $e = [E : \mathbb{Q}]$ and $h$ is the relative dimension. We assume that

1. $h \in \{2k + 1, k \in \mathbb{N}\} \setminus \left\{\frac{1}{2} (2^{m+2}), m \in \mathbb{N}\right\}$.

Then $A$ is fully of Lefschetz type.

**Remark 6.4.** Theorem 6.3 is stated in [BGK10] with the hypothesis $h$ odd, nevertheless, it seems to be important to exclude the values of the form $\frac{1}{2} (2^{m+2})$ with $m \in \mathbb{N}$.

The proof of theorem 6.3 still almost the same as the one from [BGK10, Theorem 5.11]. Nevertheless it is important to point out that the fact that we need to assume that the relative dimension $h$ does not belong to $\left\{\frac{1}{2} (2^{m+2}), m \in \mathbb{N}\right\}$ has not been taken into consideration in [BGK10, Lemma 4.13].

With the notations of [BGK10, Paragraph 4] we consider an ideal $\lambda$ in $\mathcal{O}_E$ such that $\lambda | \ell$. Let us introduce the following $\lambda$-adic representation:

$$\rho_\lambda^\circ : G_K \to \text{GL}(V_\lambda).$$

We denote $G_\lambda$ the Zariski closure of $\rho_\lambda^\circ(G_K)$ and $g_\lambda$ the Lie algebra of $G_\lambda$. Let $\phi_\lambda^\infty$ be the $\lambda$-adic pairing defined over $V_\lambda$, with notations borrowed from [BGK10]. We can therefore state the following lemma:
Lemma 6.5. ([BGK10] Lemma 4.13) Under the hypothesis of theorem 6.3 we have the following equality:

\[ g_\lambda^{ss} = so_{V_\lambda \phi_\lambda^\infty}. \]

The proof of lemma 6.5 is almost the same as the one in page 175-176 of [BGK10]. Let us recall some notations: let \( V_\lambda := V_\lambda \otimes \mathbb{Q}_\ell \), we have the following decomposition:

\[ V_\lambda = E(\omega_1) \otimes ... \otimes E(\omega_t), \quad (6.1) \]

where for every \( i \in \llbracket 1, t \rrbracket \), \( E(\omega_i) \) is an irreducible module of the Lie algebra of maximal weight \( \omega_i \).

| Root system | Minuscule weight | Dimension | Duality properties |
|-------------|------------------|-----------|-------------------|
| \( A_\ell (\ell \geq 1) \) | \( \omega_r, 1 \leq r \leq \ell \) | \( (\ell+1) \choose r \) | \( (-1)^r \), if \( r = \frac{\ell+1}{2} \) \(< 0 \) otherwise |
| \( B_\ell (\ell \geq 2) \) | \( \omega_\ell \) | \( 2^\ell \) | +1, if \( \ell \equiv 3, 0 \mod 4 \) \(-1, \) if \( \ell \equiv 1, 2 \mod 4 \) |
| \( C_\ell (\ell \geq 2) \) | \( \omega_1 \) | \( 2\ell \) | -1 |
| \( D_\ell (\ell \geq 3) \) | \( \omega_{\ell-1}, \omega_\ell \) | \( 2^{\ell-1} \) | +1, if \( \ell \equiv 0 \mod 4 \) \(-1, \) if \( \ell \equiv 2 \mod 4 \) \(0, \) if \( \ell \equiv 1 \mod 2 \) |

Figure 2. Minuscule weight for the classical Lie algebras

The penultimate sentence of the proof of lemma 6.5 states that, since \( h \) is odd, the investigation of the tables of minuscule weights (see table 2 or [Bou06, Table 1 et 2, P. 213–214]) and the dimensions of associated representations shows that the tensor product (6.1) can contain only one factor which is orthogonal. Therefore one has two possibilities:

1. Either of type \( D_n \), minuscule weight \( w_1 \) and dimension \( 2n \).
2. Or type \( A_{4k+3} \), minuscule weight \( w_{2k+2} \) and dimension \( (4k+4) \choose (2k+2) \).

It seems that the last possibility has been overlooked in [BGK10] when the authors defined the class \( B \) of abelian varieties. Let us present this last point.

The representation \( A_{4k+3} \) is orthogonal indeed because \( r = 2k + 2 \) is an even number. Moreover we know that \( (4k+4) \choose (2k+2) = 2h_1 \) because there is only one orthogonal representation. We settle in the next lemma for which values of the integer \( k \) the number \( \frac{1}{2} (4k+4) \choose (2k+2) \) is odd.

Lemma 6.6. The integer \( \frac{1}{2} (4k+4) \choose (2k+2) \) is odd if and only if \( k+1 = 2^m \) for an integer \( m \geq 1 \).

Proof. First one need to study the parity of the integer \( (4k+4) \choose (2k+2) \), more precisely, one need to compute his 2-adic valuation. One has the following equalities:

\[ (4k + 4) \choose (2k + 2) = \frac{2^{k+1}}{(k+1)!} \times (\text{odd number}). \quad (6.2) \]

The equality above shows that the 2-adic valuation of \( (4k+4) \choose (2k+2) \) is related to the 2-adic valuation of \( (k+1)! \).

Let us consider the binary development of \( k+1 = \sum_{i=0}^m \varepsilon_i 2^i \), with \( m \geq 1 \) and \( \varepsilon_m = 1 \).
One knows that $v_2((k + 1)!) = \sum_{h \geq 1} \lfloor \frac{k+1}{2^h} \rfloor$.
From the equality (6.2), we find that

$$v_2\left(\frac{4k + 4}{2k + 2}\right) = k + 1 - v_2((k + 1)!) = \varepsilon_0 + \ldots + \varepsilon_m = \varepsilon_0 + \ldots + \varepsilon_{m-1} + 1.$$ 

Then

$$\frac{1}{2} \left(\frac{4k + 4}{2k + 2}\right) \text{ is odd } \iff \varepsilon_0 = \ldots = \varepsilon_{m-1} = 0 \iff k + 1 = 2^m \iff k = 2^m - 1$$

\[\square\]

The main goal of the next last paragraph is to extend Pink’s results in order to proof new cases of the Mumford–Tate conjecture. Concerning this last conjecture, Pink applies his results to the case where $\text{End}(A) = \mathbb{Z}$, nevertheless, as he pointed up ([Pin98, Remark p. 33]) his results can be generalized. Therefore, we are going to present new results concerning simple abelian varieties of type III. More precisely we are going to state an analogous result to [Pin98, Proposition 4.7] in the case of absolute irreducible representations which are orthogonal.

Let $A$ be a simple abelian variety of type III defined over a number field $K$ of dimension $\dim A = g$. Thanks to [BGK10, Theorem 3.23], we know that the associated representation of the vector space $V_\lambda := T_\lambda \otimes E_\lambda$ is absolutely irreducible and that $V_\ell = \bigoplus_{\lambda \ell} V_\lambda \oplus V_\lambda$. As in the previous section, we consider the $\lambda$-adic representation $\rho_{\lambda}^\circ$, the algebraic group $G_{\lambda}$ together with his associated Lie algebra $\mathfrak{g}_{\lambda}$.

In order to state an analogue of [Pin98 Proposition 4.7], we need to define Mumford–Tate pairs (see [Pin98, Definition 4.1]). Let $G$ be a reductive algebraic group and $\rho$ a faithful finite dimensional representation of $G$.

**Definition 6.7.** The pair $(G, \rho)$ is a weak Mumford–Tate pair of weight $\{0, 1\}$, if and only if there exist cocharacters $\mu_i : \mathbb{G}_{m, \overline{K}} \to G_{\overline{K}}$ for every $i \in \llbracket 1, k \rrbracket$ such that

- $G_{\overline{K}}$ is generated by the images of all $G(\overline{K})$–conjugates of all $\mu_i$, and
- the weights of each $\rho \circ \mu_i$ are in $\{0, 1\}$.

The pair $(G, \rho)$ is a strong Mumford–Tate pair of weight $\{0, 1\}$, if and only if the pair is a weak Mumford–Tate pair and the cocharacters are conjugate under $\text{Gal}(\overline{K}/K)$.

We therefore assume that the center of the endomorphism algebra $E = Z(\text{End}^0(A))$ is equal to $\mathbb{Q}$. Then, $[E : \mathbb{Q}] = 1$ and $g = 2eh = 2h$. Thanks to [Pin98] Fact 5.9 we know that, in this case, all weak Mumford–Tate pairs are actually strong Mumford–Tate pairs.

Let us recall the reductive algebraic group $G = \mathbb{G}_{m, \overline{K}} \cdot G_{\text{der}}$ where $G_{\text{der}}$ is the derived group of $G$. The representation $\rho$ can be decomposed into the following exterior tensor product:

$$\rho \cong \rho_0 \otimes \rho_1 \otimes \ldots \otimes \rho_s,$$

where every $\rho_i$ is an absolute irreducible representation, $\rho_0$ is the standard representation associated to $\mathbb{G}_{m, \overline{K}}$ and the $\rho_i$ (for $1 \leq i \leq s$) are the representations associated to each factor $G_i$.

**Proposition 6.8.** ([Pin98, Proposition 4.4]) If $(G, \rho)$ is a strong Mumford–Tate pair of weight $\{0, 1\}$ over $K$ such that the representation $\rho$ is absolutely irreducible, then $G_{\text{der}}$ is
either almost simple over $K$ or trivial. In particular, all simple tensor factors of $(G, \rho)$ over $\overline{K}$ have the same type in table 2 with the same integer $\ell$.

From table 2 one can obtain the following table of orthogonal representations.

| Root system | Minuscule weight | Dimension | Duality properties |
|-------------|------------------|-----------|--------------------|
| $A_\ell (\ell \geq 1)$ | $\omega_r, 1 \leq r \leq \ell$ | $\ell+1$ | $+1$ with $r = \frac{\ell+1}{2} \equiv 0 \mod 2$ |
| $B_\ell (\ell \geq 2)$ | $\omega_\ell$ | $2^\ell$ | $+1$, if $\ell \equiv 3, 0 \mod 4$ |
| $D_\ell (\ell \geq 3)$ | $\omega_1, \omega_\ell, \omega_{\ell-1}$ | $2\ell-1$ | $+1$, if $\ell \equiv 0 \mod 4$ |

Figure 3. Minuscule weight for orthogonal Lie algebras

Let us recall that we denote by $s$ the number of simple factors of $\rho|_{G^{\der}}$. The following proposition is analogous to [Pin98, Proposition 4.5]:

**Proposition 6.9.** Consider a strong Mumford–Tate pair $(G, \rho)$ of weight $\{0, 1\}$ over $K$ such that the representation $\rho|_{G^{\der}}$ is absolutely irreducible and orthogonal. Then all simple tensor factors of $(G, \rho)$ over $\overline{K}$ are either

- orthogonal and in any number ($s \geq 1$);
- or symplectic and in even number ($s = 2k$ for $k \geq 1$).

**Proof.** Since $\rho|_{G^{\der}}$ is absolutely irreducible and orthogonal, and thanks to proposition 6.8 we know that all simple factors of the tensor product of $(G, \rho)$ over $\overline{K}$ have the same type. Therefore one has two possible cases: either all simple factors are orthogonal in any number $s$, or all simple factors are symplectic and in even number. Actually, a tensor product of two symplectic representations is orthogonal. □

We can therefore state an analogous proposition of [Pin98, Proposition 4.7]:

**Proposition 6.10.** Consider a strong Mumford–Tate pair $(G, \rho)$ of weights $\{0, 1\}$ over $K$ such that $\rho|_{G^{\der}}$ is absolutely irreducible and orthogonal. Assume that $n := \dim \rho$ is greater than 1 and neither

- a power of 2;
- $(2a)^k$ with $k$ an even integer, $k \geq 2$ and $a \geq 2$;
- a power of $\binom{2k}{k}$ where $k$ is an even integer $k \geq 2$;
- an even power of $\binom{2k}{k}$ where $k$ is an odd integer $k \geq 1$.

Then we have $G = \mathbb{G}_m \cdot SO_{n,\overline{Q}}$.

**Remark 6.11.** Actually the proof gives a sharper set that the one we just announced in the proposition. Therefore, in the conclusions we can exclude a slightly smaller subset.

**Proof.** Let $T_\ell \in \{A_\ell, B_\ell, C_\ell, D_\ell\}$ be one of the four simple types in table 2. The main goal of this proof is going to be to exclude the possible values of $h$ for which the representation is not orthogonal. To do so, we are going to study every possible combination that can occur. Let $n = 2h = \dim \rho$.

The first case that need to be consider is when the representation associated to the type $T_\ell$ is orthogonal. Let us consider a more detailed table than the table 3.
Actually, we want to obtain the type $D_k$ associated to the standard orthogonal representation. Therefore one has to remove the following four cases that still remain:

- $D_{4k}$: of dimension $2^{(4k-1)s}$ where $k \geq 1$ and $s \geq 1$. If $h \neq 2^{(4k-1)s-1}$ for every $k \geq 1$ and $s \geq 1$, this case can be excluded.
- $B_{4k+3}$ of dimension $2^{(4k+3)s}$ where $k \geq 0$ and $s \geq 1$. If $h \neq 2^{(4k+3)s-1}$ for every $k \geq 0$ and $s \geq 1$, this case can be excluded.
- $B_{4k+3}$ of dimension $2^{(4k)s}$ where $k \geq 1$ and $s \geq 1$. If $h \neq 2^{4k-s-1}$ for every $k \geq 1$ and $s \geq 1$, this case can be excluded.
- $A_{4k+3}$ of dimension $(\frac{4k+4}{2k+2})^s$ where $k \geq 0$ and $s \geq 1$. If $h \neq \frac{1}{7}(\frac{4k+4}{2k+2})^s$ for every $k \geq 0$ and $s \geq 1$, this case can be excluded.

Let us remark that the two first cases give the same conditions for the integer $h$. We can therefore give the following set:

$$\Sigma_1 = \left\{ 2^{(4k+3)s-1}, k \geq 0, s \geq 1 \right\} \cup \left\{ 2^{4k-s-1}, k \geq 1, s \geq 1 \right\} \cup \left\{ \frac{1}{2}(\frac{4k+4}{2k+2})^s, k \geq 0, s \geq 1 \right\}$$

The second case that need to be consider is the one where the type $T_\ell$ is associated to a symplectic representation and the number is even. Let us consider the following table which corresponds to the symplectic representations (see table $\ref{table:symplectic}$):

Figure 5. Minuscule weights: symplectic

| Root system | Minuscule weight | Dimension | Duality properties |
|-------------|----------------|-----------|-------------------|
| $A_{4k+1} (k \geq 0)$ | $\omega_1, \ldots, \omega_{4k+1}$ | $\frac{4k+2}{2k+1}$ | $-1$ |
| $B_{4k+1} (k \geq 1)$ | $\omega_{4k+1}$ | $2^{4k+1}$ | $-1$ |
| $B_{4k+2} (k \geq 0)$ | $\omega_{4k+2}$ | $2^{4k+2}$ | $-1$ |
| $C_k (k \geq 2)$ | $\omega_1$ | $2k$ | $-1$ |
| $D_{4k+2} (k \geq 1)$ | $\omega_{4k+1}, \omega_{4k+2}$ | $2^{4k+1}$ | $-1$ |

- $D_{4k+2}$: de dimension $2^{(4k+1)2s}$ où $k \geq 1$ et $s \geq 1$. Si $h \neq 2^{2s(4k+1)-1}$ pour tout $k \geq 1$ et $s \geq 1$, this case can be excluded.
- $C_k$: of dimension $(2k)^{2s}$ where $k \geq 2$ and $s \geq 1$. If $h \neq 2^{2s-1}k^{2s}$ for every $k \geq 2$ and $s \geq 1$, this case can be excluded.
- $B_{4k+2}$ of dimension $2^{(4k+2)2s} = 2^{4s(2k+1)}$ where $k \geq 0$ and $s \geq 1$. If $h \neq 2^{4s\alpha-1}$ for every $\alpha \geq 1$ odd and $s \geq 1$, this case can be excluded.
Let us remark that the first and the fourth cases give the same conditions over the integer $h$. We can therefore give the following set:

$$\Sigma_2 = \left\{ 2^{2s(4k+1)-1}, k \geq 1, s \geq 1 \right\} \cup \left\{ 2^{4\alpha s-1}, \alpha \text{ odd}, s \geq 1 \right\}$$

$$\cup \left\{ 2^{2s-1}k^{2s}, k \geq 2, s \geq 1 \right\} \cup \left\{ \frac{1}{2}\left(\frac{4k+2}{2k+1}\right)^{2s}, k \geq 0, s \geq 1 \right\}. \quad (6.4)$$

Let us concluded by saying that if $h \notin \Sigma_1 \cup \Sigma_2$ the representation $\rho_{G^{\text{der}}}$ is absolutely irreducible, orthogonal and correspond to the standard representation.

Let us consider the following set:

$$\Sigma' := \Sigma_1 \cup \Sigma_2. \quad (6.5)$$

After some simplifications one can obtain the set introduce in the introduction (see (1.2)):

$$\Sigma' := \bigcup_{s>1} \left\{ 2^{(4k+3)s-1}, k \geq 0 \right\} \cup \left\{ 2^{4ks-1}, k \geq 1 \right\} \cup \left\{ 2^{2s(4k+1)-1}, k \geq 1 \right\} \cup \left\{ 2^{2s-1}k^{2s}, k \geq 2 \right\}$$

$$\cup \left\{ \frac{1}{2}\left(\frac{4k+4}{2k+2}\right)^s, k \geq 0 \right\} \cup \left\{ \frac{1}{2}\left(\frac{4k+2}{2k+1}\right)^{2s}, k \geq 0 \right\}. \quad (6.5)$$

Therefore we can state the following theorem which generalize results of [Pin98]:

**Theorem 6.12.** Let $A$ be a simple abelian variety of type III such that the center of $\text{End}^\circ(A)$ is $\mathbb{Q}$. Assume that the relative dimension $h$ is not in the set $\Sigma'$ then, the abelian variety is fully of Lefschetz type. More precisely, Mumford–Tate conjecture holds for the abelian variety $A$.

Using theorems 6.3, 6.12, some results of Ichikawa [Ich91] and Lombardo [Lom16] Théorème 4.1 and [HiRa16, Corollaires 1.8 and 1.15] one can prove that the product of simple abelian varieties of type I, II or III that are fully of Lefschetz type is also of Lefschetz type. This last results prove corollaries 1.11 and 1.13.

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