A Mirage of Market Allocation

YANNAI A. GONCZAROWSKI, The Hebrew University of Jerusalem and Microsoft Research
MOSHE TENNENHOLTZ, Technion — Israel Institute of Technology

Market Allocation — a situation where competitors agree to not compete with each other in specific markets, by dividing up geographic areas, types of products, or types of customers.

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Can noncooperative behaviour of merchants lead to a market split that prima facie seems anticompetitive? We introduce a model in which service providers, with internet service providers (ISPs) being the main example, aim at optimizing the number of customers who use their services, while customers aim at choosing service providers with low customer load (which translates to high effective bandwidth per subscriber, in the case of ISPs). Each service provider chooses between a variety of levels of service (latencies, in the case of ISPs), and as long as it does not lose customers, aims at minimizing its level of service; the minimum level of service required to satisfy a customer varies across customers. We consider a two-stage competition, in the first stage of which the service providers select their levels of service, and in the second stage — customers choose between the service providers. (We show via a novel construction that for any choice of strategies for the service providers, a unique distribution of the customers’ mass between them emerges from all Nash equilibria among the customers, showing the incentives of service providers in this two-stage game to be well defined.) In the two-stage game, we show that the competition among the service providers possesses a unique Nash equilibrium, which is moreover super-strong; we also show that all sequential better-response dynamics of service providers reach this equilibrium, with best-response dynamics doing so surprisingly fast. If service providers choose their levels of service according to this equilibrium, then the unique Nash equilibrium among customers in the second phase is essentially a split of the market between the service providers, based on the customers’ minimum acceptable quality of service; moreover, each service provider’s chosen level of service is the lowest acceptable by the entirety of the slice of the market that chooses it, seemingly making no attempt to attract any other customers. Our results show that this prima facie market allocation (collusive split of the market) arises as the unique and highly robust outcome of noncooperative (i.e. free from any form of collusion), even myopic, service-provider behaviour. The results of this paper are applicable to a wide variety of scenarios, from explaining phenomena observable in some food markets, to shedding a surprising light on aspects of location theory, such as the formation and structure of a city’s central business district.

Key Words and Phrases: Game Theory, Congestion Games, Location Theory, Two-Stage Competition

1. INTRODUCTION

1.1. Setting

1.1.1. Shopping for an Internet Connection. In today’s world, an internet connection has become a necessity in many households. Of the many parameters characterizing an internet connection, two have emerged as most important to the home user: the ever-popular bandwidth, and the latency. While the bandwidth measures the amount of data transmitted (equivalently, received) per second, the latency measures the time it takes a single packet of data to reach its destination. In the metaphorical highway

1Indeed, these are precisely the two parameters measured by the popular internet speed-testing website www.speedtest.net.
2Twice the latency is sometimes referred to as the ping time.
3We emphasize that high latency corresponds to bad quality of service.

Authors' addresses: Y. A. Gonczarowski, Einstein Institute of Mathematics, Rachel & Selim Benin School of Computer Science & Engineering, and Federmann Center for the Study of Rationality, The Hebrew University of Jerusalem, Israel; and Microsoft Research, Email: yannai@gonch.name; M. Tennenholtz, William Davidson Faculty of Industrial Engineering and Management, Technion — Israel Institute of Technology (work carried out while at Microsoft Research), Email: moshet@ie.technion.ac.il.

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of the internet, the bandwidth may be thought of as corresponding to the number of lanes, while the latency corresponds the the length of the highway. While some users may not be sensitive to latency (indeed, when streaming a 45-minute TV show from, say, Netflix or Hulu, most users would not mind waiting an extra second before the show begins; this is also the case when downloading content for future offline consumption), some other users may have very harsh latency limitations (indeed, when playing a multiplayer video game online, it is extremely important for each player that whenever she presses a button on her controller, the associated action happens as soon as possible; a delay longer that of the adversary by as little as one tenth of a second may be unacceptable).

We consider a stylized model, in which each customer is interested in precisely one internet connection, and is willing to tolerate a latency of at most \( d \) milliseconds (a customer-dependent real value). As long as this customer's latency demand is met, her sole consideration is that of maximizing her effective bandwidth (we think of subscription costs as low and similar, as is the case in real life). The effective bandwidth of each customer subscribed to a given internet service provider, or ISP, is the total bandwidth available to this ISP (a fixed known ISP-dependent value) divided by the number of customers subscribed to this ISP.\(^4\) A Nash equilibrium among the customers is therefore an assignment of ISPs to customers, s.t. for each customer with latency limit \( d \), no ISP with latency no greater than \( d \) has a subscriber pool smaller than that of the ISP assigned to this customer.

We consider a scenario with finitely many ISPs and continuously many customers, the distribution of \( d \) among whom is given by an arbitrary finite measure. Preparing the ground for the main results of this paper, which follow below, in Section 3 we use a novel construction to show the following result (similar in spirit to other results regarding congestion games and crowding games).

**Theorem 1.1** (Informal version of Theorems 3.10 and 3.11, Corollary 3.12, and Algorithm 1). Fix the characteristics (latency and total bandwidth) of \( n \) ISPs.

1. A Nash equilibrium among the customers exists. Furthermore, there exists such a Nash equilibrium for which the strategies can be computed efficiently.
2. The effective bandwidth of each customer, as well as the number of subscribers to each ISP, are the same across all Nash equilibria.

1.1.2. An ISP Game. Obviously, each ISP would like to offer a latency that maximizes its number of subscribers. (By Theorem 1.1, the number of subscribers is well defined given the latencies of all ISPs, assuming a Nash equilibrium among the customers.) That being said, as low-latency infrastructure is costlier to erect, each ISP would like to offer the highest latency possible, as long as this does not reduce the size of its subscriber pool. As we think of the number of subscribers as indicative of monthly income, and of the investment in infrastructure as a one-time expense (with infrastructure upkeep cost being independent of latency), we have that each ISP would like to offer a latency that first and foremost maximizes its number of subscribers, and only then (as a tie-breaking rule among latency values that yield the same number of subscribers) is as high as possible. In Section 4.2, and more generally in Section 5, we show the following — the first of our main results regarding this two-stage competition.

\(^4\)In Section 5, we deal with a generalized model, which accommodates also for, e.g. some ISPs purchasing more total bandwidth as their subscriber pool grows. Our main results surveyed in the introduction continue to hold even under such generalizations.
**Theorem 1.2** (*Informal version of Theorems 4.28 and 5.2*).

1. For every ordering \( \pi \) of the \( n \) ISPs, there exists a unique Nash equilibrium among them s.t. their latency levels are ordered according to \( \pi \).
2. This Nash equilibrium is super-strong.\(^5\)
3. Each ISP has the same number of subscribers in all Nash equilibria (regardless of the chosen ordering of ISPs \( \pi \)).

We further demonstrate the robustness of the Nash equilibrium defined in Theorem 1.2 by considering dynamics among ISPs. A sequential best-response dynamic is a process starting with arbitrary latency levels, and in which at each turn an arbitrary ISP changes its latency level to one that, ceteris paribus, maximizes its preferences (we show that such a latency level always exists for every possible measure on customers latency limits); we assume that each ISP is allowed to change its latency level infinitely often. A round in a best-response dynamic is a sequence of consecutive steps in which each ISP is allowed to change its latency level at least once. Finally, a sequential \( \delta \)-better-response dynamic is a sequential dynamic in which each change in latency need not necessarily maximize the ISP’s preferences, as long as it increases the size of its subscriber pool by at least \( \delta \) of the entire market size.\(^6\) In Section 4.2, we show the following main result.

**Theorem 1.3** (*Informal version of Corollaries 4.37 and 4.38*).

1. For every \( \delta > 0 \), every sequential \( \delta \)-better-response dynamic reaches a Nash equilibrium in finitely many steps, and remains constant from that point onward.
2. Sequential best-response dynamics reach a Nash equilibrium in a small number of rounds.

We also analyse dynamics in which several ISPs change their latency levels simultaneously. (See Theorems 4.15 and 4.22 and Corollary 4.35.) We emphasize that Theorem 1.3 does not stem from any “hidden” introduction of any exogenous costs on restructuring infrastructure; i.e. this theorem holds in very general settings, even when the utility of an ISP from a given number of subscribers and a given latency does not decrease with the number of latency changes in previous steps of the studied dynamics.

**1.1.3. Prima Facie Market Allocation.** Our study culminates with the analysis of the structure of the unique equilibrium from Theorem 1.2 and of the underlying equilibrium among customers, which turns out to be unique as well.

**Theorem 1.4** (*Informal version of Theorems 4.28, 4.31 and 5.2*). Fix a Nash equilibrium among the ISPs. Denote the number of subscribers to the first ISP (the one with lowest latency) by \( \ell_1 \), the number of subscribers to the second ISP (the one with second-lowest latency) by \( \ell_2 \), and so forth.

1. The \( \ell_1 \) customers with smallest latency limits all subscribe to the first ISP, whose latency is the highest that still accommodates all of these customers.
2. The next \( \ell_2 \) customers all subscribe to the second ISP, whose latency is the highest that still accommodates all of these customers.
3. etc.

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\(^5\)See Theorem 4.6 in Section 4.1 and the preceding discussion for the definition of super-strong equilibrium.

\(^6\)Since we consider continuously many customers, we demand a \( \delta \)-improvement in order to avoid improvements à la Zeno’s “Race Course” paradox.
Theorem 1.4 shows that the market is split among the various ISPs based on the willingness of a customer to accept a high latency level, and each ISP chooses the highest latency level acceptable by the entirety of its slice of the market, seemingly making no attempt to attract any other customers. Theorems 1.2 and 1.3 show that this *prima facie* market allocation (collusive split of the market) among the various ISPs arises as the unique possible outcome, not as a result of anticompetitive practices, but rather as a result of noncooperative dynamics, each ISP only looking to myopically maximize its preferences at every step; no signalling (via e.g. choice of latency level) or any other collusive or cooperative “trick” whatsoever is used in order to reach and maintain this market split.

1.2. Alternative Interpretations/Applications

It is worthwhile to point out that our framework captures far more than merely the ISP-competition scenario introduced above, by thinking of a latency level more generally as a quality of service (QoS) of sorts of an ISP: the lower the latency of an ISP, the better the quality of service that it provides. In Sections 6 and 7, we give two examples of other possible applications stemming from this insight. In each of these examples, QoS is given different meanings, which, in turn, result in different meanings of market split based on acceptable QoS. These examples provide insights into the breadth of meanings that can be captured by the idea of QoS and consequently by our model, and into the meaning of market split based on acceptable QoS. The example given in Section 6 gives an application to location theory, and derives results for an extended model with multiple types of goods. The example given in Section 7 offers real-world evidence supporting the applicability of our model to certain food markets.

For generality, we henceforth use the more generic term *producers* to refer to e.g. ISPs, *consumers* to refer to e.g. customers, and *QoS* to refer to e.g. latency.

1.3. Related Work

Our consumer games are a form of congestion games, and more specifically, of resource-selection games. Congestion games with finitely many players have been introduced by Rosenthal [1973]; in fact, the term has been coined in a paper by Monderer and Shapley [1996], titled “Potential Games”, where it is shown that a game has a potential iff it is a congestion game. While the discussion there refers to atomic games with finitely many players, work in computer science and game theory also deals with nonatomic games, in which there may be a continuum of players as in our model (see e.g. [Roughgarden and Tardos 2002] for work in CS that uses such games). While substantial parts of our introductory Theorem 1.1 can also be deduced from results by Schmeidler [1973] and by Beckmann et al. [1956], we emphasize that the novel constructive machinery that we introduce in order to prove it is simpler, allows for efficient calculation, provides for more general results, and provides for auxiliary results useful in the analysis of our producer game and in obtaining our main results. Holzman and Law-Yone [1997, 2003] look at restrictions on strategy sets of atomic congestion games; one way to view our consumer games is as a special form of restricted nonatomic congestion games defined for general measure functions on agents’ types, capturing their possible strategy sets. As it turns out, this set of games possess many desired game-theoretic properties.

The actual games that we study are in fact two-stage games, where the second stage is a congestion game as discussed above; the first stage can be viewed as a form of facility-location game among producers, with QoS playing the role of location (see Section 6), where the main aim of producers is to select a QoS to be selected by as many consumers as possible. This resembles the literature on location theory initiated by Hotelling [1929], although the utility function of the producers in our setting is differ-
ent, and allows for fine preferences based on distance from a location most preferred by consumers. Given the above, our model can be viewed as a novel combination of facility-location games among producers with congestion games among consumers.

Another type of related literature deals with scheduling and queuing with multiple machines, where the jobs choose among available services and the level of service they receive depends on the selections by other jobs. Recently, two-stage games in these contexts have been studied, consisting of a strategic selection by machines between queuing policies [Ashlagi et al. 2013] or scheduling policies [Ashlagi et al. 2010], followed by a strategic selection by jobs between the various selected policies. Our work introduces a novel type of a two-stage scenario, which may be considered as somewhat related. More remotely is the literature on competing mechanisms in the context of auctions, which employs such two-phase setting, but in a very different context of mechanism design with money (see e.g. [McAfee 1993]).

The proof of Theorem 1.1 draws its intuition from an analogy to a hydraulic system of communicating vessels (see Fig. 1). Kaminsky [2000] (see also [Aumann 2002]) uses an analogy to quite a different system of communicating vessels to solve rationing problems; his motivation is quite different, and involves extending bilateral rationing rules. While Kaminsky uses a set of two-way communicating vessels, we use a set of one-way communicating vessels. In this context, the problem of finding a Nash equilibrium among consumers may be regarded as a rationing problem with certain “reserves” for producers with high quality of service. Our treatment, especially in light of the discussion in Section 5 (see in particular Fig. 2), also sheds new light on rationing problems, as congestion games of sorts among a continuum of good-fragments.

2. NOTATION

Definition 2.1 (Notation).

— (Naturals). We denote the natural numbers by \( \mathbb{N} \triangleq \{0, 1, 2, \ldots\} \).

— (Nonnegative Reals). We denote the nonnegative reals by \( \mathbb{R}_+ \triangleq \{r \in \mathbb{R} \mid r \geq 0\} \).

— (Maximizing Arguments). Given a set \( S \) and a function \( f : S \to \mathbb{R} \) that attains a maximum value on \( S \), we denote the set of arguments in \( S \) maximizing \( f \) by \( \arg \max_{s \in S} f(s) \triangleq \{s \in S \mid f(s) = m\} \), where \( m \triangleq \max_{s \in S} f(s) \).

— (Simplex). For a finite set \( S \) and a nonempty subset \( S' \subseteq S \), we define

\[
\Delta^{S'} = \left\{ s \in [0, 1]^S \mid \sum_{j \in S'} s_j = 1 \ & \forall j \in S \setminus S' : s_j = 0 \right\}.
\]

(The set \( S \) will be clear from context.)

— For every \( n \in \mathbb{N} \), we define \( \mathbb{P}_n \triangleq \{0, 1, \ldots, n - 1\} \).

— Given a tuple \( t = (t_0, \ldots, t_{n-1}) \in S^{\mathbb{P}_n} \) for some set \( S \) and some \( n \in \mathbb{N} \), and given \( j \in \mathbb{P}_n \) and \( t' \in S \), we define \( (t_{-j}, t') \triangleq (t_0, \ldots, t_j, t_{j+1}, \ldots, t_{n-1}, t' ) \in S^{\mathbb{P}_n} \).

— For every \( n \in \mathbb{N} \), we denote the set of permutations on \( \mathbb{P}_n \) by \( \mathbb{P}_n! \).

3. PRELUDE: THE CONSUMER (CUSTOMER) GAME

Preparing the ground for the main results of this paper, in this section we define the congestion game among consumers, and use a novel construction\(^7\) to prove the existence of Nash equilibrium and the uniqueness of equilibrium loads, and to efficiently

\(^7\)See Gonczarowski and Tennenholtz [2014] for a significant, highly nontrivial, generalization of our treatment of only the consumer game (without the producer game) to arbitrary resource-selection games (in which the resources available to a player may be any subset of \( \mathbb{P}_n \) and not merely a “QoS-prefix” of \( \mathbb{P}_n \) to which the construction of this section is inherently tailored) and beyond.
calculate these loads.\footnote{As mentioned in Section 1.3, while the existence of Nash equilibrium and the uniqueness of equilibrium loads can also be derived from theorems by Schmeidler [1973] and by Beckmann et al. [1956], respectively, the novel constructive machinery that we introduce here is simpler, provides for more general uniqueness results, provides for auxiliary results useful in analysing dynamics of the producer game in Section 4, and allows for efficient calculation.} Full proofs and auxiliary results are provided in Appendices A.1 and A.2.

In this section and in Section 4, for ease of presentation, we present a model in which each consumer would like to consume from a least-loaded producer (i.e. in which all ISPs have the same total bandwidth); we remove this requirement in Section 5.

Definition 3.1 (Quality-of-Service Space). For ease of presentation, we use $T \triangleq [0, 1]$ as the type space in the consumer game (and later as the strategy space in the producer game). We consider lower values as indicating better qualities of service.

For the duration of this section, fix a finite measure $\mu$ on $T$, a natural $n \in \mathbb{N}$ and producer QoS levels (e.g. ISP latencies) $\ell = (t_0, \ldots, t_{n-1}) \in \mathbb{T}^n$. We consider the $n$-producers consumer game $(\mu; \ell) = (\mu; t_0, \ldots, t_{n-1})$, which we now define.

Definition 3.2 (Strategies). For every $d \in T$, we define the set of strategies available to a player with type (i.e. worst acceptable QoS) $d$ as $S_d \triangleq \{j \mid t_j \leq d\} \cup \{\neg\}$, where $\neg$ denotes not consuming from any producer. We define $S \triangleq \cup_{d \in T} S_d = \mathbb{P}_n \cup \{\neg\}$ — the set of pure strategies available to any player, and consider $S$ as a measurable space with the $\sigma$-algebra $2^S$ of all of its subsets.

Definition 3.3 (Pure-Consumption Profile/Nash Equilibrium).

1. A pure-consumption (strategy) profile in the $n$-producers consumer game $(\mu; \ell)$ is a measurable function $s : T \rightarrow S$ s.t. $s(d) \in S_d$ for every $d \in T$.
2. Given a pure-consumption profile $s$ in the $n$-producers consumer game $(\mu; \ell)$, we define $\ell_j^s \triangleq \mu(s^{-1}(j))$ for every $j \in S$ — the load on producer $j$. ($\ell_j^s$ is the measure of consumers not consuming from any producer.)
3. A pure-consumption Nash equilibrium in the $n$-producers consumer game $(\mu; \ell)$ is a pure-consumption profile $s$ s.t. for every $d \in T$, both the following hold.
   a. $s(d) = \neg$ only if $S_d = \{\neg\}$.
   b. $\ell_j^s(d) \leq \ell_j^b$ for every $j \in S_d \setminus \{\neg\}$.

We now turn to define mixed-consumption strategies. We think of such a strategy not as a probabilistic one, but rather as meaning “a certain fraction of the continuum of players with type $d$ have one strategy, while others have other strategies”.

Definition 3.4 (Mixed-Consumption Profile/Nash Equilibrium).

1. A mixed-consumption (strategy) profile in the $n$-producers consumer game $(\mu; \ell)$ is a measurable function $s : T \rightarrow [0, 1]^S$ s.t. $s(d) \in \Delta^{S_d}$ for every $d \in T$.
2. Given a mixed-consumption profile $s$ in the $n$-producers consumer game $(\mu; \ell)$, we define $\ell_j^s \triangleq \int_T s_j \, d\mu$ for every $j \in S$ — the load on producer $j$. ($\ell_j^s$ is the measure of consumers not consuming from any producer in this case as well.)
3. A mixed-consumption Nash equilibrium in the $n$-producers consumer game $(\mu; \ell)$ is a mixed-consumption profile $s$ s.t. for every $d \in T$, both of the following hold.

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strictly lower load of 0, since all consumers are of the same effective type, all would game (symmetric mixed-consumption Nash equilibrium exists in the consumer game, they result in the same load for both consumers and producers. Despite this, of course). We now show that while in general many Nash equilibria may exist in communicating vessels (nonetheless, the proofs given in Appendix A.1 are completely formal, of course). The intuition underlying this novel construction builds upon hydraulic systems of communicating vessels.

\[ \text{Check: } \exists \text{NASH EQUILIBRIUM.} \]

If \( \mu \) is atomless, then a pure-consumption Nash equilibrium exists in the \( n \)-producers consumer game \((\mu; t)\).

Example 3.6 (Necessity of Atomlessness Condition). Consider a nonzero measure \( \mu \) concentrated entirely on the atom \( d = 1 \in T \). For \( n > 1 \), no pure-consumption Nash equilibrium exists in any induced \( n \)-producers consumer game. Indeed, in any pure-consumption profile, all consumers with type \( d = 1 \) would consume from the same producer, leaving another producer with a strictly lower load of 0; as this producer is acceptable by all consumers with type \( d = 1 \), they would all rather deviate to it.

Definition 3.7 (Effective Type). We say that two types \( d_1, d_2 \in T \) are of the same effective type if \( S_{d_1} = S_{d_2} \).

The Nash equilibrium constructed in the proof of Theorem 3.5 is asymmetric in the sense that players with the same effective type may behave differently. As we momentarily below, this asymmetry cannot be avoided. Nonetheless, a reader who finds this asymmetry aesthetically unpleasing may instead consider a more-symmetric, yet mixed-consumption, Nash equilibrium, which in fact exists even when \( \mu \) is not atomless.

Definition 3.8 (Symmetric Strategy Profile). A strategy profile \( s \) is said to be symmetric if \( S_{d_1} = S_{d_2} \iff s(d_1) = s(d_2) \) for every \( d_1, d_2 \in T \), i.e. each player’s strategy depends only on the player’s effective type.

Example 3.9 (Nonexistence of a Symmetric Pure-Strategy Nash Equilibrium). Consider any nonzero measure \( \mu \). For \( n > 1 \), if \( t_j = 0 \) for every \( j \in P_n \), then all consumers are of the same effective type. Thus, no symmetric pure-consumption equilibrium exists in the induced \( n \)-producers consumer game. Indeed, in any symmetric pure-consumption profile, since all consumers are of the same effective type, all would consume from the same producer, leaving another producer (acceptable to all) with a strictly lower load of 0; therefore, all consumers would rather deviate to this producer.

Theorem 3.10 (\( \exists \text{SYMMETRIC MIXED-CONSUMPTION NASH EQUILIBRIUM.} \)) A symmetric mixed-consumption Nash equilibrium exists in the \( n \)-producers consumer game \((\mu; t)\). Furthermore, there exists such an equilibrium for which the strategies can be computed in \( O(n^2) \) time.

See Fig. 1 for an illustration of the constructive proof of Theorem 3.10; as illustrated, the intuition underlying this novel construction builds upon hydraulic systems of communicating vessels (nonetheless, the proofs given in Appendix A.1 are completely formal, of course). We now show that while in general many Nash equilibria may exist in the consumer game, they result in the same load for both consumers and producers.

Theorem 3.11 (Producers are Indifferent between Nash Equilibria). \( \ell_j^s = \ell_j^{s'} \) for every \( j \in P_n \) and every mixed-consumption Nash equilibria \( s, s' \) in \((\mu; t)\).

Corollary 3.12 (Consumers are Indifferent between Nash Equilibria). \( \ell_k^s = \ell_{k'}^{s'} \) for every \( k \in \supp(s(d)) \) and \( k' \in \supp(s'(d)) \), for every \( d \in T \) and every mixed-consumption Nash equilibria \( s, s' \) in \((\mu; t)\).

By Theorems 3.10 and 3.11, the following is well defined.
Fig. 1. Illustration of the construction in the proof of Theorem 3.10 for \( n = 5 \). E.g. as exactly 80% of the blue (i.e. darkest when viewed in b/w) liquid in Fig. 1(f) is in the third vessel and the remaining 20% is in the second one, the strategy for all consumer types \( d \in [t_2, t_3) \) in the symmetric mixed-consumption Nash equilibrium that we construct is 0.8 consumption from producer 2 and 0.2 consumption from producer 1.

\( (a) \) A set of 5 communicating vessels, corresponding, from left to right, to producers \( 0, \ldots, 4 \) respectively. Each pair of adjacent vessels is connected via a no-return valve, allowing the flow of liquids from right to left, but not the other way around.

\( (b) \) Pouring \( \mu([t_0, t_1)) \) liquid into the first vessel. The liquid does not penetrate the second vessel due to the no-return valve between these two vessels.

\( (c) \) Pouring \( \mu([t_1, t_2)) \) liquid into the second vessel.

\( (d) \) Pouring \( \mu([t_2, t_3)) \) liquid into the third vessel. Once the liquid surface level in the third vessel equals that in the second, any additional liquid causes spillage into the second vessel, maintaining even surface level among these two vessels.

\( (e) \) Pouring \( \mu([t_3, t_4)) \) liquid into the fourth vessel.

\( (f) \) Pouring \( \mu([t_4, t_5)) \) liquid into the fifth vessel. Some of the liquid penetrates into the fourth vessel, and later into the third and second vessels, so that no vessel has lower surface level than any vessel on its right.
Definition 3.13 (Producer Load). For every $j \in \mathbb{P}_n$, we define $\ell_j(\bar{t})$ to equal $\ell_j$ in any mixed-consumption Nash equilibrium $s$ in $(\mu; \bar{t})$.

By the proof of Theorem 3.10, we obtain Algorithm 1 — a simple algorithm for directly calculating $\ell_j(\bar{t})$ for all $j$, without the need to first calculate consumer’s strategies. While this algorithm runs in $O(n^2)$ time, i.e. has same worst-case asymptotic behaviour as explicitly computing a Nash equilibrium via Theorem 3.10 and then deducing all loads, it is considerably simpler, and also computes the loads sequentially, and so may be stopped mid-way, allowing to calculate the loads on the $j$ producers with lowest latency levels in $O(j \cdot n)$ time for any $j$.

**Algorithm 1** Direct computation of $\ell_j(\bar{t})$ for all $j \in \mathbb{P}_n$

1: procedure COMPUTE-$\ell(\mu; t_0, \ldots, t_{n-1})$  // Assumes $t_0 \leq t_1 \leq \cdots \leq t_{n-1}$.
2: \[ t_n \leftarrow 2 \]  // Any value $> 1$ will do here; assumes $\mu$ is defined on $[0, t_n]$, but has support $\mathcal{T}$.
3: \[ \ell^- \leftarrow \mu([0, t_0]) \]
4: \[ k \leftarrow 0 \]
5: while $k < n$ do
6: \[ k' \leftarrow \text{Max arg Max}_{k < k' \leq n} \frac{\mu([t_k, t_{k'}])}{t_{k'} - t_k} \]
7: \[ \ell \leftarrow \frac{\mu([t_k, t_{k'}])}{t_{k'} - t_k} \]
8: for all $k \leq j < k'$ do
9: \[ \ell_j \leftarrow \ell \]
10: end for
11: \[ k \leftarrow k' \]
12: end while
13: return $(\ell^-, \ell_0, \ldots, \ell_{n-1})$
14: end procedure

See Appendix A.2 for an analytic study of $\ell_j$, formalizing some main properties thereof, which we utilize in our proofs in the following sections. In particular, we show there that for every $j$, $\ell_j(\bar{t})$ is nonincreasing in $t_j$, weakly quasiconvex in $t_k$ for $k \neq j$, and Lipschitz (w.r.t. $\mu$) in each coordinate with Lipschitz constant 1.

4. THE PRODUCER (ISP) GAME

We now turn to the producer game, and to the main results of this paper. In this two-stage game, each producer chooses a strategy (i.e. QoS) in $\mathcal{T}$, and the utilities are determined according to the loads on producers in Nash equilibria in the induced consumer game. For the duration of this section, fix a natural $n \in \mathbb{N}$ and a finite measure $\mu$ on $\mathcal{T}$. Full proofs, as well as auxiliary results, are provided in Appendices A.3 and A.4.

In Section 4.1, we define a simplified version of the producer game; the definition of the (more intricate) producer game that is surveyed in the introduction is given in Section 4.2. While the simpler game defined in Section 4.1 has some trivialities that we point out, its analysis is nonetheless interesting, and the obtained results are useful when analysing the more-involved version in Section 4.2.

Recall that as in Section 3, for ease of presentation we present a model in which each consumer would like to consume from a least-loaded producer (i.e. in which all ISPs have the same total bandwidth); as noted above, we remove this requirement in Section 5.

4.1. Coarse Preferences (A Simplified Producer Game)

**Definition 4.1 (Producer Game with Coarse Preferences).** We define the producer game with coarse preferences $(n, \mu, \succeq_C)$ as the $n$-player game, with set of players (called producers) $\mathbb{P}_n$, in which the pure-strategy space available to each producer is $\mathcal{T}$, and
in which for each pure-strategy profile $\bar{t} \in \mathcal{T}^{\pi_n}$, the utility for each producer $j \in \mathbb{P}_n$ is strictly increasing in $t_j(\bar{t})$ (as defined in Definition 3.13).

4.1.1. Static Analysis. We begin with an analysis of domination in the producer game with coarse preferences, pointing out the trivialities in this simplified game, which will disappear in the more-involved version thereof that we analyse in Section 4.2.

**Definition 4.2 (Safe Alternative; Dominant Strategy).** Let $t$ be a strategy in the game $(n, \mu, \succeq_C)$.

- We say that $t$ is a safe alternative to some strategy $t'$ if for every strategy profile for all but one of the producers, playing $t$ gives the remaining producer utility at least as high a utility as playing $t'$.
- We say that $t$ is a dominant strategy if it is a safe alternative to all strategies.

**Theorem 4.3 (Dominant Strategies).** $t \in \mathcal{T}$ is a dominant strategy in $(n, \mu, \succeq_C)$ iff $\mu([0, t)) = 0$. Furthermore, each such dominant strategy guarantees a load of at least $\mu(T)/\pi$ on each producer playing it.

In particular, we have that every producer playing $0 \in \mathcal{T}$ constitutes a Nash equilibrium. (We emphasize that this is by far not the only Nash equilibrium — see Theorem 4.5 below.) This and other trivialities that result from domination (as well as the domination itself) disappear in Section 4.2, when we refine the order of preferences of the various producers. Before that, though, we continue to explore the consumer game with coarse preferences, obtaining results that aid our analysis of the consumer game with refined preferences in Section 4.2 below. Our next step is to not only characterize the Nash equilibrium loads (an immediate corollary of Theorem 4.3), but furthermore, show that every strategy profile inducing these loads is a Nash equilibrium.

**Theorem 4.4 (Nash Equilibrium Loads).** A pure-strategy profile $\bar{t} \in \mathcal{T}^{\pi_n}$ constitutes a Nash equilibrium in $(n, \mu, \succeq_C)$ iff $t_j(\bar{t}) = \frac{\mu(T)}{n}$ for every $j \in \mathbb{P}_n$.

We proceed to directly characterize the strategies played in Nash equilibria, in a way that does not necessitate solving the induced consumer game.

**Theorem 4.5 (Nash Equilibrium Characterization).** Let $t_0 \leq \cdots \leq t_{n-1} \in \mathcal{T}$. The pure-strategy profile $\bar{t} \triangleq (t_1, \ldots, t_{n-1})$ constitutes a Nash equilibrium in $(n, \mu, \succeq_C)$ iff $\mu([0, t_j)) \leq \frac{j}{n} \cdot \mu(T)$ for every $j \in \mathbb{P}_n$.

It should be emphasized that Theorem 4.5 does not imply that Nash equilibria are interchangeable (i.e. that the set of Nash equilibria is a Cartesian product of sets of strategies for the various producers). Consider, for example, $\mu = U(T)$ — the uniform measure on $\mathcal{T}$. In this case, by Theorem 4.5, $(0, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{n-1}{n})$ is a Nash equilibrium in $(n, \mu, \succeq_C)$, and so is any permutation thereof. Nonetheless, every player playing $\frac{n-1}{n} \in \mathcal{T}$ does not constitute a Nash equilibrium. We now move on to examine the stability of the Nash equilibria in $(n, \mu, \succeq_C)$ against group deviations.

The study of stability against group deviations was initiated by Aumann [1959], who considers deviations from which all deviators gain. Recently, the CS literature considers a considerably stronger solution concept, according to which a deviation is considered beneficial even if only some of the participants in the deviating coalition gain, as long as none of the participants lose (see e.g. [Rozenfeld and Tennenholtz 2006]). While stability against the classical all-gaining coalitional deviation is termed strong equilibrium, this more-demanding concept is referred to as super-strong equilibrium; there are very few results showing its existence in nontrivial settings.
THEOREM 4.6 (ALL NASH EQUILIBRIA ARE SUPER-HARD). Let $\bar{t} \in T^0_n$ be a pure-strategy Nash equilibrium in $(n, \mu, \succeq C)$. There exist no coalition $P \subseteq \mathbb{P}_n$, and strategies $\bar{p}' = (\bar{p}'_j)_{j \in P} \in T^0$ s.t. $\ell_j(\bar{p}_j(\bar{p}'_j), \bar{p}) \geq \ell_j(\bar{p}_j(\bar{p}), \bar{p})$ for every $j \in P$, with a strict inequality for at least one producer $j \in P$.

We conclude the static analysis of $(n, \mu, \succeq C)$ by deducing generalizations of Theorems 4.3 through 4.6 for mixed-strategy profiles, as well as showing that no mixed-strategy Nash equilibrium exhibits any ex-post regret.

THEOREM 4.7 (MIXED STRATEGIES). In $(n, \mu, \succeq C)$,

1. (DOMINANT STRATEGIES). Let $p$ be a mixed strategy. $p$ is a dominant strategy iff $\mu([0, \text{Max supp}(p)]) = 0$. Furthermore, each such dominant strategy guarantees a load of at least $\frac{\mu(T)}{n}$ with probability 1 on each producer playing it.

2. (NASH EQUILIBRIUM LOADS). A mixed-strategy profile $\bar{p} = (p_0, \ldots, p_{n-1})$ constitutes a Nash equilibrium iff $\ell_j(\bar{p}) = \frac{\mu(T)}{n}$ for every $j \in \mathbb{P}_n$ with probability 1.

3. (NASH EQUILIBRIUM CHARACTERIZATION). A mixed-strategy profile $\bar{p}$ constitutes a Nash equilibrium iff there exists a permutation on the producers $\pi \in \mathbb{P}_n$ such that $\mu([0, \text{Max supp}(p_{\pi(j)})]) \leq \frac{1}{n} \cdot \mu(T)$ for every $j \in \mathbb{P}_n$.

4. (ALL NASH EQUILIBRIA ARE SUPER-HARD). Let $\bar{p}$ be a mixed-strategy Nash equilibrium. There exist no coalition $P \subseteq \mathbb{P}_n$ and mixed strategies $\bar{p}' = (p'_j)_{j \in P}$ s.t. $E[\ell_j(\bar{p}', \bar{p})] \geq E[\ell_j(\bar{p}, \bar{p})]$ for every $j \in P$, with a strict inequality for at least one producer $j \in P$.

THEOREM 4.8 (NO EX-POST REGRET IN MIXED-STRATEGY NASH EQUILIBRIA). In any mixed-strategy Nash equilibrium in $(n, \mu, \succeq C)$, with probability 1 there exists no ex-post regret for any producer. In other words, a realization of a mixed-strategy Nash equilibrium is with probability 1 a pure-strategy Nash equilibrium.

4.1.2. Dynamics. When analysing dynamics henceforth, we assume that $\mu(T) > 0$.

Definition 4.9 (SCHEDULE; SEQUENTIAL SIMULTANEOUS SCHEDULE; ROUND).

1. A schedule is a sequence $(P_i)_{i=0}^\infty$ of nonempty subsets of $\mathbb{P}_n$, s.t. $j \in P_i$ for infinitely many values of $i \in \mathbb{N}$, for every $j \in \mathbb{P}_n$.

2. A schedule $(P_i)_{i=0}^\infty$ is said to be sequential if $|P_i| = 1$ for every $i \in \mathbb{N}$.

3. A schedule $(P_i)_{i=0}^\infty$ is said to be simultaneous if $P_i = \mathbb{P}_n$ for every $i \in \mathbb{N}$.

4. Let $i_1 \leq i_2 \in \mathbb{N}$. We say that $\{i \in \mathbb{N} | i_1 \leq i \leq i_2\}$ constitutes a round (in the schedule $(P_i)_{i=0}^\infty$) if $\bigcup_{i=i_1}^{i_2} P_i = \mathbb{P}_n$. (We emphasize that this union need not be a disjoint union.)

5. Let $i_1, i_2 \in \mathbb{N}$ and let $r \in \mathbb{N}$. We say that $i_2$ is reached from $i_1$ in $r$ rounds if $r - 1$ is the largest number of pairwise-disjoint rounds into which $\{i_1, i_1 + 1, \ldots, i_2 - 2\}$ can be partitioned. (Therefore, $\{i_1, i_1 + 1, \ldots, i_2 - 2\}$ cannot be partitioned into $r$ pairwise-disjoint rounds with a nonzero amount of "spare" trailing steps.)

Remark 4.10. In a simultaneous schedule, each step $\{i\}$ constitutes a round.

Definition 4.11 (WEAKLY-BETTER RESPONSE DYNAMIC; BEST-RESPONSE DYNAMICS; LAZY DYNAMICS).

1. A weakly-better-response dynamic in $(n, \mu, \succeq C)$ is a sequence $(\hat{t}_i, P_i)_{i=0}^\infty$ where $(P_i)_{i=0}^\infty$ is a schedule and $\hat{t}_i$ is a sequence of strategy profiles s.t. both of the

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10We consider a mixed strategy to be a random variable taking values in $T$.

11We emphasize that mixed-strategies of distinct producers are independent random variables.
following hold for every \( i \in \mathbb{N} \).

— For every \( j \in P_i \), \( t_{j}^{t+1} \) is a weakly better response than \( t_{j}^{t} \) to \( \bar{P}_{-j} \) (by \( j \)), i.e. \( \ell_{j}(\bar{P}_{-j}, t_{j}^{t+1}) \geq \ell_{j}(\bar{P}) \).

— For every \( j \notin P_i \), \( t_{j}^{t+1} = t_{j}^{t} \).

By slight abuse of notation, we sometimes write \((\bar{t}_{i})_{i=0}^{\infty}\) to refer to \((\bar{t}_{i}, P_{i})_{i=0}^{\infty}\), when the schedule is either inconsequential or clear from context.

(2) A weakly-better-response dynamic is said to be a best-response dynamic if for every \( i \in \mathbb{N} \) and \( j \in P_i \), \( t_{j}^{t+1} \) is a best response to \( \bar{P}_{-j} \), i.e. \( t_{j}^{t+1} \in \text{arg} \max_{t \in T} \ell_{j}(\bar{P}_{-j}, t) \).

(3) Let \( \delta > 0 \). A weakly-better-response dynamic is said to be a \( \delta \)-better-response dynamic if for every \( i \in \mathbb{N} \) and \( j \in P_i \), \( t_{j}^{t+1} \) is either a best response to \( \bar{P}_{-j} \), or a better response increasing \( j \)'s load by at least \( \delta \) compared to \( t_{j}^{t} \), i.e. \( \ell_{j}(\bar{P}_{-j}, t_{j}^{t+1}) \geq \ell_{j}(\bar{P}_{-j}, t_{j}^{t}) + \delta \).

(4) A weakly-better-response dynamic is said to be lazy if for every \( i \in \mathbb{N} \) and \( j \in P_i \), \( t_{j}^{t+1} = t_{j}^{t} \) whenever \( t_{j}^{t} \) is a best response to \( \bar{P}_{-j} \).

**Remark 4.12.** In \((n, \mu, \geq C)\),

— Every best-response dynamic is a \( \delta \)-better-response dynamic, for every \( \delta > 0 \).

— Every \( \delta \)-better-response dynamic is also a \( \delta' \)-better-response one, for every \( 0 < \delta' < \delta \).

— A weakly-better-response dynamic is a best-response dynamic iff it is a \( \delta \)-better-response dynamic for \( \delta = \mu(T) \).

**Remark 4.13 (A Best Response Always Exists).** Let \( j \in P \) and let \( \bar{t}_{-j} \in \mathcal{T}^{2n \setminus \{j\}} \). By Theorem 4.3, a best response (by \( j \)) to \( \bar{t}_{-j} \) exists in \((n, \mu, \geq C)\).

We commence with a negative result, showing that even best-response dynamics can go out of equilibrium.

**Example 4.14 (Nonsequential Nonlazy Best-Response Dynamics may Go Out of Equilibrium).** Let \( \mu = U(T) \). By Theorem 4.5, the (cyclically repeating) strategy-profile sequence \((0,0,\ldots,0),(\frac{n-1}{n},\frac{n-1}{n},\ldots,\frac{n-1}{n}),(0,0,\ldots,0),(\frac{n-1}{n},\frac{n-1}{n},\ldots,\frac{n-1}{n}),\ldots\) constitutes a (nonlazy) simultaneous best-response dynamic in \((n, \mu, \geq C)\) that visits nonequilibria infinitely often.

We continue by showing that the dynamic in Example 4.14 visiting Nash equilibria infinitely often is no coincidence.

**Theorem 4.15 (\( \delta \)-Better-Response Dynamics Visit Nash Equilibria Infinitely Often).** Let \( \delta > 0 \) and let \((\bar{t}_{i})_{i=0}^{\infty}\) be a \( \delta \)-better-response dynamic in \((n, \mu, \geq C)\). \( \bar{t} \) is a Nash equilibrium for infinitely many values of \( i \). Moreover, the first Nash equilibrium is reached (from \( 0 \)) in at most \( n \cdot \lceil \frac{\mu(T)}{\delta n} \rceil \) rounds, and from any later nonequilibrium, the next Nash equilibrium is reached in at most \( (n-1) \cdot \lceil \frac{\mu(T)}{\delta n} \rceil \) rounds.

**Remark 4.16.** In Theorem 4.15,

— if \((\bar{t}_{i})_{i=0}^{\infty}\) is simultaneous, then “rounds” may be replaced with “steps”.

— Finer analysis of similar nature may be used to show both that \( n \cdot \lceil \frac{\mu(T)}{\delta n} \rceil \) may be replaced with \( \sum_{h=1}^{n} \lceil \frac{\mu(T)}{\delta 2n} \rceil \approx \max\{\ln n \cdot \lceil \frac{\mu(T)}{\delta n} \rceil, n\} \), and that \((n-1) \cdot \lceil \frac{\mu(T)}{\delta n} \rceil \) may be replaced with \( \sum_{h=2}^{n} \lceil \frac{\mu(T)}{\delta 2n} \rceil \approx \max\{(\ln n - 1) \cdot \lceil \frac{\mu(T)}{\delta n} \rceil, n - 1\} \). We conjecture that considerably tighter bounds (esp. for small \( \delta \)) can be attained as well.

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12 Due to the continuous nature of strategies and loads, we require an improvement by at least \( \delta \), and not just any positive improvement, in order to avoid improvements à la Zeno’s “Race Course” paradox.
We now show that in a sense, Example 4.14 describes all the “issues” that might prevent best- and even $\delta$-better-response dynamics from remaining in Nash equilibria.

Remark 4.17 (Lazy Better-Response Dynamics Remain in Nash Equilibrium). Once a lazy weakly-better-response dynamic reaches a Nash equilibrium, it remains constant. (Directly by definition of Nash equilibrium and laziness.)

Theorem 4.18 (Sequential Better-Response Dynamics Remain in Nash Equilibria). Let $(P_t)_{t=0}^{\infty}$ be a schedule. If $(P_t)_{t=0}^{\infty}$ is sequential from some point, then once a $(P_t)_{t=0}^{\infty}$-scheduled weakly-better-response dynamic in $(n, \mu, \succeq_C)$ reaches a Nash equilibrium after that point, it never visits a nonequilibrium afterward.

Corollary 4.19 (Sequential/Lazy $\delta$-Better-Response Dynamics Reach Nash Equilibria and Remain). For every $\delta > 0$, every sequential or lazy $\delta$-better-response dynamic in $(n, \mu, \succeq_C)$ reaches a Nash equilibrium in a finite number of steps, and never visits a nonequilibrium after that point.

Proposition 4.20 (Every Nonsequential Schedule and Initial Profile have Nonlazy Best-Response Dynamics that Go Out of Equilibrium). If $\mu$ has no atom measuring $\frac{n-1}{n} \cdot \mu(T)$ or more and no tail of $(P_t)_{t=0}^{\infty}$ is sequential, then for every pure-strategy profile $v^0$ there exists a nonlazy best-response dynamic in $(n, \mu, \succeq_C)$ that is scheduled by $(P_t)_{t=0}^{\infty}$, starts at $v^0$ and visits nonequilibria infinitely often.

Remark 4.21. Analogues of Remark 4.17, Theorem 4.18, and Proposition 4.20 hold for mixed-strategy dynamics as well.

As every best-response dynamic is a $\delta$-better-response one for $\delta = \mu(T)$, we conclude from Theorem 4.15 that such a dynamic reaches a Nash equilibrium in at most $n \cdot \lceil \frac{\mu(T)}{\mu(T)n} \rceil = n$ rounds and afterward always “re-reaches” a Nash equilibrium in at most $(n-1) \cdot \lceil \frac{\mu(T)}{\mu(T)n} \rceil = n-1$ rounds. By applying some finer analysis, we can slightly improve this bound, and show that the new bound is tight.

Theorem 4.22 (Best-Response Time-to-Equilibrium and Time between Equilibria). Every best-response dynamic in $(n, \mu, \succeq_C)$ reaches a Nash equilibrium in at most $n-1$ rounds. Furthermore, if $n > 2$, then from any later nonequilibrium, the next Nash equilibrium is reached in at most $n-2$ rounds.

Remark 4.23. In Theorem 4.22, as in Theorem 4.15, if the dynamic in question is simultaneous, then “rounds” may be replaced with “steps”.

Example 4.24 (Tightness of Theorem 4.22). Let $\mu = U(T)$. The following is a (nonlazy) simultaneous best-response dynamic in $(n, \mu, \succeq_C)$, in which i) no two consecutive strategy profiles are both Nash equilibria, ii) the first Nash equilibrium is reached in precisely $n-1$ rounds (steps), and iii) from any nonequilibrium that follows a Nash equilibrium, the next Nash equilibrium is reached in precisely $n-2$ rounds (steps): $(1, 0, 0, \ldots, 0)$, $(\frac{n-1}{n}, \frac{n-2}{n}, 0, 0, \ldots, 0)$, $(\frac{n-2}{n}, \frac{n-3}{n}, 0, 0, \ldots, 0)$, $\ldots$, $(\frac{2}{n}, \frac{3}{n}, \ldots, \frac{2}{n}, 0, 0, \ldots, 0)$, $(\frac{n-1}{n}, \frac{n-2}{n}, \ldots, \frac{n-1}{n}, 0, 0, \ldots, 0)$, $(\frac{n-2}{n}, \frac{n-3}{n}, \ldots, \frac{n-2}{n}, 0, 0, \ldots, 0)$, $\ldots$ (cyclically repeating).

To summarize Theorems 4.18 and 4.22, Example 4.24, and Remark 4.17:

Corollary 4.25 (Sequential/Lazy Best-Response Dynamics Reach Nash Equilibria Fast and Remain). Every sequential or lazy best-response dynamic in $(n, \mu, \succeq_C)$ reaches a Nash equilibrium in at most a tight bound of $n-1$ rounds, and never visits a nonequilibrium after that point.
4.2. Fine Preferences

Definition 4.26 (Producer Game with Fine Preferences). We define the producer game with fine preferences \((n, \mu, \succeq_F)\) as the \(n\)-player game, with set of players (called producers) \(P_n\), in which the pure-strategy space available to each producer is \(T\), and in which for each pure-strategy profile \(t \in T^n\), the utility for each producer \(j \in P_n\) is strictly increasing in \(\ell_j(i)\) (as defined in Definition 3.13), with tie breaking (i.e. infinitesimal improvement) in favour of larger values of \(\ell_j\) over smaller ones.

4.2.1. Static Analysis. We define safe alternatives and dominant strategies in \((n, \mu, \succeq_F)\) as in Definition 4.2, only w.r.t. fine preferences. The following proposition shows that the tie-breaking refinement of the producers’ preferences into “fine preferences” indeed successfully removes the triviality captured by Theorem 4.3, in a strong sense.

Proposition 4.27 ((No) Dominant and (Few) Dominated Strategies). If \(\mu\) has no atom measuring \(\frac{n-1}{n} \cdot \mu(T)\) or more,\(^{13}\) then no strategies are dominant in \((n, \mu, \succeq_F)\). Moreover, at least \(\frac{n-1}{n}\) of the strategies in \(T\) (as measured by \(\mu\)) have no safe alternatives (other than themselves).

We now formally conclude the results captured informally in Theorem 1.2:

Theorem 4.28 (∃! Nash Equilibrium, and It is Super-Strong\(^{14}\)). A unique (up to permutations) pure-strategy Nash equilibrium exists in \((n, \mu, \succeq_F)\). The sorted Nash-equilibrium strategies \(t_0 \leq \cdots \leq t_{n-1} \in T\) are \(t_j \equiv \operatorname{Max}\{t \in T \mid \mu([0, t)) \leq \frac{j}{n} \cdot \mu(T)\}\) for every \(j \in P_n\). The load on each producer in this equilibrium is \(\frac{\mu(T)}{n}\). Furthermore, this equilibrium is super-strong.

Corollary 4.29 (Nash Equilibrium Characterization — Special Case). If the CDF of \(\mu\) is continuous (i.e. \(\mu\) is atomless) and strictly increasing, then for every \(j \in P_n\), the \(j^{th}\) sorted Nash-equilibrium strategy, \(t_j\), is the unique strategy satisfying \(\mu([0, t_j)) = \frac{j}{n} \cdot \mu(T)\).

Proposition 4.30 (Nash Equilibria Are in Pure Strategies). Every mixed-strategy Nash equilibria in \((n, \mu, \succeq_F)\) is in fact in pure strategies (and is thus given by Theorem 4.28/Corollary 4.29).

If \(\mu\) is atomless, then in the Nash equilibrium defined in Theorem 4.28 and Corollary 4.29, almost all (i.e. except for maybe an amount of measure zero) of the \(1/n\) of consumers (as measured by \(\mu\)) with numerically smallest types consume from producer 0, whose chosen strategy is the numerically largest one that accommodates almost all of this \(1/n\); almost all of the next \(1/n\) of consumers consume from producer 1, whose chosen strategy is the numerically largest one that accommodates almost all of this \(1/n\), and so forth. Essentially, the market is split between the various producers based on consumer types, and each producer chooses the numerically largest strategy that accommodates almost all of its slice of the market, seemingly making no attempt to attract any other consumers. We conclude the static analysis of \((n, \mu, \succeq_F)\) by formalizing these results.

Theorem 4.31 (Market Split). Let \(t_0 \leq \cdots \leq t_{n-1} \in T\) s.t. \(t_0\) constitutes a Nash equilibrium in \((n, \mu, \succeq_F)\), and let \(s\) be a mixed-consumption Nash equilibrium in the

\(^{13}\) The triviality in case of a very large atom should be compared to the triviality captured under the exact condition in Proposition 4.20. Indeed, both trivialities are possible exactly iff there exists \(t \in T\) s.t. \(\mu([0, t)) = 0\) and \(\mu([t', 1]) < \frac{\mu(T)}{n}\) for every \(t' > t\).

\(^{14}\) See Theorem 4.6 in Section 4.1 above, as well as the preceding discussion, for the definition of super-strong equilibrium, as well as a discussion regarding various group-deviation concepts.
induced consumer game $\langle \mu; \vec{t} \rangle$. If $\mu$ is atomless, then for every $j \in \mathbb{P}_n$, $s_j(d) = 1$ for almost all (w.r.t. $\mu$) consumer types $d \in T$ s.t. $\mu([0, d)) \in (\frac{j}{n} \cdot \mu(T), \frac{j+1}{n} \cdot \mu(T))$.

Remark 4.32 (Prima Facie Market Allocation). By Theorem 4.28, if $\mu$ is atomless, then the $j$th sorted Nash-equilibrium strategy, $t_j$, is the numerically largest strategy (i.e. worst QoS) acceptable by almost all consumer types $d \in T$ s.t. $\mu([0, d)) \in (\frac{j}{n} \cdot \mu(T), \frac{j+1}{n} \cdot \mu(T))$.

4.2.2. Dynamics. We define weakly-$\delta$-better/best-response dynamics in $(n, \mu, \succeq_F)$ as in Definition 4.11, only with best responses defined w.r.t. fine preferences. In particular, the definition of improvement by at least $\delta$ remains unchanged (i.e. it is defined solely w.r.t. the load). The analogue of the last part of Remark 4.12 is therefore:

Remark 4.33. In $(n, \mu, \succeq_F)$, a weakly-better-response dynamic is a best-response dynamic iff it is a $\delta$-better-response dynamic for some (equivalently, for all) $\delta > \mu(T)$.

We start by noting that best responses always exist — an observation that for general $\mu$ is considerably less trivial w.r.t. fine preferences than w.r.t. coarse ones.

Proposition 4.34 (A Unique Best Response Always Exists). Let $j \in \mathbb{P}_n$ and let $\vec{t}_j \in \mathcal{T}^\mathbb{P}_n \setminus \{j\}$. A unique best response (by $j$) to $\vec{t}_j$ exists in $(n, \mu, \succeq_F)$.

We give two proofs for Proposition 4.34: the first — quite-concise, and the second, while requiring more involved arguments, is constructive in the sense that in contrast to the first, it presents the best response in the form $\max\{t \in T \mid \mu([0, t]) \leq m\}$, for $m$ that can be explicitly calculated.

As in $(n, \mu, \succeq_F)$ no producer is ever indifferent between two strategies, all weakly-better-response dynamics in this game are lazy; therefore, if such a dynamic reaches a Nash equilibrium, it remains constant from that point on. We also note that Example 4.14 is also an example of a lazy best-response dynamic in $(n, \mu, \succeq_F)$ that never reaches a Nash equilibrium. Moreover, as best responses in $(n, \mu, \succeq_F)$ are unique, and as the strategies in each Nash equilibrium are distinct if $\mu$ is atomless, we have that no simultaneous best-response dynamic starting from a strategy profile with two or more identical strategies ever reaches a Nash equilibrium. It should be noted that this is not a boundary phenomenon; for example, if $\mu([0, t^0_j]) > 0$ for all $j \in \mathbb{P}_n$, then the best responses of all producers are identical (see Corollary A.29 in Appendix A.4). Many more such examples may be constructed. We now show that these phenomena are all avoided by sequential dynamics.

Corollary 4.35. Theorems 4.15 and 4.22 hold also regarding reaching a Nash equilibrium w.r.t. $(n, \mu, \succeq_C)$ by dynamics in the game $(n, \mu, \succeq_F)$.

Theorem 4.36 (Sequential $\delta$-Better-Response Dynamics Converge from Coarse-Preferences Nash Equilibrium). If $(\vec{P}_i)_{i=0}^\infty$ is sequential from some point, then for every $\delta > 0$, at most one round after a $\delta$-better-response dynamic in $(n, \mu, \succeq_F)$ reaches a Nash equilibrium w.r.t. $(n, \mu, \succeq_C)$ after that point, it reaches a Nash equilibrium w.r.t. $(n, \mu, \succeq_F)$, and remains constant from that point onward.

We hence formally conclude the results captured informally in Theorem 1.3:

Corollary 4.37 (Sequential $\delta$-Better-Response Dynamics Converge). For every $\delta > 0$, every sequential $\delta$-better-response dynamic in $(n, \mu, \succeq_F)$ reaches a Nash equilibrium in a finite number of steps, and remains constant from that point onward.
COROLLARY 4.38 (SEQUENTIAL BEST-RESPONSE DYNAMICS CONVERGE FAST).
Every sequential or lazy best-response dynamic in \((n, \mu, \succeq_F)\) reaches a Nash equilibrium in at most \(n\) rounds, and never visits a nonequilibrium after that point.

We conjecture than an even-tighter bound on convergence time than in Corollaries 4.35 and 4.38 is attainable. Corollaries 4.37 and 4.38 show that the prima facie market allocation (collusive split of the market) among the various producers shown in Theorem 4.31 and Remark 4.32 to be exhibited in every Nash equilibrium in \((n, \mu, \succeq_F)\) arises as the unique possible outcome, not as a result of anticompetitive practices, but rather as a result of noncooperative dynamics, each producer only looking to myopically maximize its preferences at every step; as the best response to any strategy profile is unique, no signalling or any other collusive or cooperative “trick” whatsoever is used in order to reach and maintain this market split.

5. HETEROGENEOUS PRODUCTS

We have so far (in Sections 3 and 4) assumed that each consumer wishes to consume from a producer with least load. More generally, however, as in the introduction, we may imagine that some ISPs have different total bandwidth than others, while some other ISPs may purchase more total bandwidth as their subscriber pool grows. In such a scenario, in order to surf with greatest speed, each consumer would no longer like to consume from a producer with least \(\ell_j\) (i.e. with as few subscribers as possible), but would rather consume from a producer with least \(f_j(\ell_j)\), where \(f_j\) is an increasing continuous function for every \(j \in \mathbb{P}_n\), possibly differing between producers (e.g. \(\ell_j/\delta_j\), where \(\delta_j\) is the total bandwidth of ISP \(j\)). The results of Sections 3 and 4 lend to generalization also to such a scenario via similar methods, with only quantitative rather than qualitative changes (the results regarding \(\delta\)-better-response dynamics require also that the functions \(f_j\) be Lipschitz); notably, the unique market-share division in both fine- and coarse-preferences Nash equilibria among producers is generally no longer of \(1/n\) of the market to each of the producers. E.g. Theorems 4.3 through 4.5 thus become:

**THEOREM 5.1 (HETEROGENEOUS PRODUCTS — COARSE PREFERENCES).** There exist amounts \(\tilde{\ell}_0, \tilde{\ell}_1, \ldots, \tilde{\ell}_{n-1} \in [0, \mu(T)]\) (for homogeneous products, \(\tilde{\ell}_j = \mu(T)/n\)) for every \(j \in \mathbb{P}_n\), s.t. all of the following hold.

1. (DOMINANT STRATEGIES). Each dominant strategy in \((n, \mu, \succeq_C)\) (the characterization of such strategies is unchanged from that given in the first part of Theorem 4.3), when played by a producer \(j \in \mathbb{P}_n\), guarantees a load of at least \(\tilde{\ell}_j\) on this producer.
2. (NASH EQUILIBRIUM LOADS). A pure-strategy profile \(t \in T^n\) constitutes a Nash equilibrium in \((n, \mu, \succeq_C)\) iff \(\ell_j(t) = \tilde{\ell}_j\) for every \(j \in \mathbb{P}_n\).
3. (NASH EQUILIBRIUM CHARACTERIZATION). Let \(t\) be a pure-strategy profile and let \(\pi \in \mathbb{P}_n\) be a permutation s.t. \(t_{\pi(0)} \leq t_{\pi(1)} \leq \cdots \leq t_{\pi(n-1)}\). \(t\) constitutes a Nash equilibrium in \((n, \mu, \succeq_C)\) iff \(\mu([0, t_{\pi(j)}]) \leq \sum_{k=0}^{j-1} \tilde{\ell}_{\pi(k)}\) for every \(j \in \mathbb{P}_n\).

Consequently, Theorem 4.28 and Corollary 4.29 become:

**THEOREM 5.2 (HETEROGENEOUS PRODUCTS — FINE PREFERENCES).**

1. (NASH EQUILIBRIUM, AND IT IS SUPER-STRONG). Let \(\pi \in \mathbb{P}_n\) be a permutation s.t. there do not exist \(j < k \in \mathbb{P}_n\) s.t. \(\ell_{\pi(j)} = 0\) while \(\ell_{\pi(k)} \neq 0\). A unique pure-strategy Nash equilibrium s.t. \(t_{\pi(0)} \leq \cdots \leq t_{\pi(n-1)}\) exists in \((n, \mu, \succeq_F)\). The strategies of this equilibrium are given by \(t_{\pi(j)} = \max\{t \in T \mid \mu([0, t]) \leq \sum_{k=0}^{j-1} \tilde{\ell}_{\pi(k)}\}\) for every \(j \in \mathbb{P}_n\).

The load on each producer \(j \in \mathbb{P}_n\) in this Nash equilibrium is \(\tilde{\ell}_j\). Furthermore, this equilibrium is super-strong. No other Nash equilibria exist in \((n, \mu, \succeq_F)\).

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similar “two-way” calculation among subsets of vessels generalizes Algorithm 1 to this scenario.

If one of these functions is not strictly increasing, e.g. if a sixth vessel identical to the fifth one is added in this figure, then Theorem 3.11 may no longer hold.) The producer-equilibrium loads \( \ell_0, \ldots, \ell_{n-1} \) can be found by pouring the entire \( \mu(T) \) of liquid into the rightmost vessel (i.e. computing the loads when each producer \( j \)'s strategy is the \( \ell_j \)-guaranteeing strategy \( 0 \in T \), or, equivalently, by simply removing the one-way valves (i.e. permitting liquid flow in both directions) and pouring \( \mu(T) \) liquid into the system (observe that either way, if all vessels are of the same shape, then we indeed obtain \( \ell_j = \frac{\mu(T)}{n} \) for every \( j \in P_n \), as in Section 4); a similar “two-way” calculation among subsets of vessels generalizes Algorithm 1 to this scenario.

(2) (Nash Equilibrium Characterization — Special Case). If the CDF of \( \mu \) is continuous (i.e. \( \mu \) is atomless) and strictly increasing, then for every \( j \in P_n \), in the Nash equilibrium corresponding to a permutation \( \pi \in P_n \! \) with the above properties, \( t_{\pi(j)} \) is the unique strategy satisfying \( \mu([0, t_{\pi(j)}]) = \sum_{k=0}^{j-1} \ell_{\pi(k)} \).

The remainder of the results of Sections 3 and 4, including those regarding dynamics, readily generalize to this scenario as well. So, we once again have that in a Nash equilibrium in \( (n, \mu, \succeq_P) \), the market is split between producers based on consumer types; if \( \mu \) is atomless and \( t_{\pi(0)} \leq \cdots \leq t_{\pi(n-1)} \), then almost all of the \( \ell_{\pi(0)} \) consumers with numerically smallest types consume from producer \( \pi(0) \) (who chooses the largest strategy acceptable by almost all of them, seemingly making no attempt to attract any other consumers), almost all of the next \( \ell_{\pi(1)} \) consumers consume from producer \( \pi(1) \) (who chooses the largest strategy acceptable by almost all of them, seemingly making no attempt to attract any others), and so forth. See Fig. 2 for an illustration regarding the adaptation of the results from Section 3 to this generalized model, and the calculation of \( \ell_0, \ldots, \ell_{n-1} \).

6. MULTIPLE PRODUCTS AND THE FORMATION OF MAIN STREET

We conclude the body of this paper with an aesthetically appealing corollary, obtained by extending our model to allow for multiple good types. (While we restrict ourselves in this section to the case of two good types, the results below readily generalize also

\[ \text{Fig. 2. A system of 5 one-way communicating vessels, corresponding to 5 heterogeneous ISPs (see the introduction) with the following characteristics, from left to right (i.e. from lowest latency/best QoS to highest latency/worst QoS): A “normal” ISP, an ISP with half the total bandwidth of a “normal” one, an ISP whose total bandwidth somewhat increases with its number of subscribers, an ISP whose total bandwidth somewhat decreases with its number of subscribers, and a “normal” ISP who buys additional bandwidth if needed, so that the bandwidth for a single subscriber never drops below some threshold. (After the surface of the liquid in the fifth vessel reaches the tube connecting this vessel to the container on its right, which we consider as part of the fifth vessel, any additional liquid poured into this vessel accumulates in the container on the right; assume that this container is large enough so as to never fill up.)} \]

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15See Gonczarowski and Tennenholtz [2014] for a formalization, as part of (as mentioned above) a significant, highly nontrivial, generalization of our treatment of only the consumer game (without the producer game) to arbitrary resource-selection games (in which the resources available to a player may be any subset of \( P_n \) and not merely a “QoS-prefix” of \( P_n \) to which the above construction is inherently tailored) and beyond.
to the case of more than two good types.) Consider the following alternative (non-ISP) interpretation of our model.

**Example 6.1 (Wine Market; QoS=Centrality of Location).** Consider the downtown area of the fictional city of Metropolis, the wine capital of the world. At its heart lies Metropolis Central Station. Every morning, shoppers (consumers) from throughout the Metropolis metropolitan area (and beyond) disembark the train at Metropolis Central, at the vicinity of which many wine shops (producers) are located, and go about their wine-shopping errands. Each shopper is interested in purchasing a single bottle of wine, and is willing to walk at most $d$ minutes (a shopper-dependant real value) in each direction in order to get it. All other wine characteristics being the same, each shopper would like the bottle of wine that she buys to be as exclusive as possible, i.e. she prefers to get her wine at the shop that sells the fewest bottles of wine throughout the day (so that it can be considered a “boutique wine”), as long as it is no more than $d$ minutes away from Metropolis Central, of course. As some wines may be known to be of superior types, are more extravagantly packaged, or have some other attractive quality, shoppers may be willing to compromise on “exclusivity” in favour of superior quality. Therefore, each shopper would like to minimize a wine-seller-dependent increasing function of the wine's circulation, e.g. shoppers may wish to maximize the quotient of quality and circulation.

Obviously, each wine seller would like to locate her store in a way that would maximize its sales volume. That being said, as real-estate prices rise the closer (in walking time) a shop is to Metropolis Central (we think of sales as indicative of daily income, and of real-estate cost as a one-time expense), each wine seller would like to place her store the farthest possible from the station, as long as this does not hurt sales.

Our results from the previous sections imply that in the scenario described in Example 6.1, the unique possible noncooperative outcome is once again for the market to be split between the various wine sellers based on the shoppers’ types, i.e. each shopper shopping at the store closest to Metropolis Central has a smaller walking-time limit than any of those shopping at the store second-closest to Metropolis Central, each of whom in turn having a smaller walking-time limit than all of those shopping at the third-closest store, etc., and each wine seller chooses the farthest location accessible by the entirety of its slice of the market, seemingly making no attempt to attract any other shoppers. While this characterizes the distance of each wine shop from Metropolis Central, the direction from Metropolis Central to each such shop can be arbitrary. Not for long, though.

Suppose now that merchants from the nearby town of Smallville, the extra-extra-extra-virgin-olive-oil capital of the world, wishing to widen the visibility of their product, have started moving their stores to Downtown Metropolis as well. Now that Metropolis has become both the wine- and the extra-extra-extra-virgin-olive-oil capital of the world, each shopper arriving at Metropolis Central would like to purchase not only a bottle of wine, but also a bottle of olive oil. Nonetheless, the walking-time limit of each shopper does not change — each shopper is still willing to walk at most $2d$ minutes in order to obtain both products. (This indeed introduces no change, as each shopper was previously willing to walk at most $d$ minutes in each direction.) As with wine, each shopper prefers to minimize a seller-dependent function of the circulation of the type of olive oil that she purchases, as long as her walking-time constraint is met. (One may again consider e.g. the case in which one would like to maximize the quotient of quality to circulation, optimizing some form of tradeoff between quality and “boutiqueness”.) Olive-oil merchants have preferences similar to those of wine sellers.

Formally, we have $n_1 \in \mathbb{N}$ producers of the first good (e.g. wine) and $n_2 \in \mathbb{N}$ producers of the second good (e.g. olive oil). The strategy of each producer is a point on the plane;
A Mirage of Market Allocation

a pure-consumption strategy of a consumer with type \( d \in T \) is a pair \((j, k) \in P_{n_1} \times P_{n_2}\), denoting consumption of the first good from producer \( j \) of this good, and of the second good — from producer \( k \) of that good; each consumer would like to minimize \( f_1^j(\ell_j) + f_2^k(\ell_k) \) (e.g. the sum of the quotients of the quality and circulation for each good), subject to the constraint that the circumference of the triangle, whose vertices are the origin (Metropolis Central Station) and the locations (strategies) of producer \( j \) of good 1 and of producer \( k \) of good 2, does not exceed \( 2d \) (the density of consumer types, as given by \( \mu \), remains unchanged). Each producer would like to first and foremost maximize its number of consumers, and only as a tie-breaker, maximize the norm of its strategy (i.e. its distance from the origin).

Under these conditions, roughly speaking, each producer would like to be located so that visiting it would never be too much of a detour on the way from the origin to a producer of the other good. Indeed, we now show that the unique stabllest outcome, in a precise sense, is for all shops to be placed on the same ray originating at Metropolis Central (with the distance of each store from Metropolis Central set as before, as if its good type were the only one on the market). (See Appendix A.5 for a proof, as well as a discussion regarding the necessity of the conditions below.)

**Theorem 6.2 (The Unique Super-Strong Equilibrium is a Main Street originating from the Origin).** Let \( \tilde{\ell}_0, \ldots, \tilde{\ell}_{n_1-1} \) be the producer-equilibrium loads when only the first good is on the market (i.e. as defined in Section 5 when the only producers are the \( n_1 \) producers of good 1) and let \( \tilde{\ell}^g_0, \ldots, \tilde{\ell}^g_{n_2-1} \) be the producer-equilibrium loads when only the second good is on the market. If no nonempty proper subset of the former loads and no nonempty proper subset of the latter loads have the same sum, and if \( \tilde{\ell}^g_j > 0 \) for all \( g \) and \( j \), then a producer strategy profile is a super-strong equilibrium iff the strategies of all producers of both products are on the same ray from the origin, with distances from the origin as in Theorem 5.2 (when computed separately for each good).

While most readers are likely to consider the formation of a main street as a fairly natural phenomenon due to its abundance in many cities, some readers may find it somewhat less natural for this main street, as deduced in Theorem 6.2, to originate from the city centre (e.g. Metropolis Central Station), rather than having the city centre in its middle. Such readers may compare this with the structure of many old European towns, at the heart of which lies the old stone-cobbled main street, on one end of which (as opposed to at the middle of which) lies the main town church.

**7. Discussion**

This paper shows, under quite general setting, that the appearance of collusion need not imply the actuality. While we believe a main strength of our model to lie in its theoretic generality and aesthetics (its novel combination of congestion and location games, its clean results despite complex nontrivial multistage game analysis, and its (surprising) qualitative lesson), the question of the applicability of our model to a real-life market is a valid one. Although our work is motivated mainly by internet monetization, and while we believe that its predictions will be confirmed with time, it is hard to validate its predictions on today’s home internet market for several reasons, the main of which being that in many countries, many customers are not yet educated enough regarding latency, which leads ISPs to differentiate themselves from their competitors using other traits. In this section, we offer real-world evidence supporting the applicability of our model to the food market in Israel.

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16 This sum may be replaced with any increasing continuous function of \( f_1^j \), \( f_2^k \), e.g. their weighted average.
The vast majority of groceries sold in Israel are Kosher. In fact, a nonnegligible part of the Jewish population in Israel, and in particular ultra-orthodox Jews, are only willing to buy food which is not only Kosher, but even more strictly monitored and restricted; we henceforth refer to such food as extra-monitored. As extra-monitoring can be certified only by a handful of third-party monitors, manufacturing the same food product from the same ingredients costs more when it is to be labelled as extra-monitored than when it is to be labelled as (“regular”) Kosher. Due to heavy lobbying on behalf of ultra-orthodox groups, though, even though producing an extra-monitored version of the same product costs more than producing a Kosher version of that product, both versions are sold by retailers for identical prices. (This holds in particular for products whose prices are regulated; no retailer would ever charge extra, beyond the regulated price, as compensation for extra-monitoring.) This property of the prices, together with the fact that a considerable amount of the population in Israel is primarily concerned with the monitoring level of their groceries, makes the food market in Israel fit squarely in our model, with retailers as providers, shoppers for “a week’s worth of groceries” as consumers, and the monitoring level of groceries as their QoS (there are in fact quite a few monitoring levels). Indeed, each shopper has a minimum required level of monitoring, beyond which she or he is indifferent (as it is physically the same product, at the exact same price), and it is quite reasonable that shoppers in a certain neighbourhood would therefore choose the least-crowded grocery store in the neighbourhood (no one likes to wait in line . . . ) out of those stores that meet their minimum required level of monitoring. From the retailers’ point of view, they would like to first and foremost maximize their number of shoppers, and as long as this number is not hurt, minimize the monitoring level of each of their products (the price difference for monitoring, while nonzero, is negligible relative to capturing more market share).

Our results from the previous sections predict that under these conditions, the unique possible noncooperative outcome is for the market to be split between the various retailers based on the shoppers’ minimum required monitoring level, i.e. each shopper shopping at the “minimum monitoring” retailer has a lower minimum required monitoring level than any of those shopping at the “second-lowest monitoring” retailer, each of whom in turn having a lower minimum required monitoring level than all of those shopping at the “third-lowest monitoring” retailer, etc., and each retailer chooses the minimum monitoring level that satisfies the entirety of its slice of the market, seemingly making no attempt to attract any other (stricter) shoppers. Indeed, in neighbourhoods with both nonnegligible ultra-orthodox population and nonnegligible orthodox populations, one notices that grocery stores label themselves by their specific monitoring level, which is applied to all products in the store. Our early study, to be further explored in a companion work, suggests that the number of stores of each monitoring level (when weighted by store size) roughly corresponds to the demand for this monitoring level as a minimum required level.

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APPENDIX
A. PROOFS AND AUXILIARY RESULTS
A.1. Proofs and Auxiliary Results for Section 3
We commence with a few lemmas used in the proofs of Theorems 3.5 and 3.10.

**Lemma A.1 (Load is Nonincreasing in Strategy).** Under the definitions of Section 3, if \( t_0 \leq t_1 \leq \cdots \leq t_{n-1} \), then for every mixed-consumption Nash equilibrium \( s \) in the \( n \)-producers consumer game \((\mu; t)\), we have \( \ell_s^0 \geq \ell_s^1 \geq \cdots \geq \ell_s^n \).

**Proof.** Let \( j \in \{0, \ldots, n-2\} \). If \( \ell^s_{j+1} = 0 \), then \( \ell^s_j \geq \ell^s_{j+1} \). Assume, therefore, that \( \ell^s_{j+1} > 0 \). Hence, there exists \( d \geq t_{j+1} \) s.t. \( s_{j+1}(d) > 0 \). By definition of \( s \), and as \( d \geq t_{j+1} \geq t_j \), we thus have \( \ell^s_{j+1} \leq \ell^s_j \), as required. \( \square \)

As mentioned above, the construction in the proof of Theorems 3.5 and 3.10 is illustrated in Fig. 1. In the context of that figure, the following lemma can be thought of as answering the following question: if the amount of liquid in each vessel \( j \in \mathbb{F}_n \) is \( \ell_j \), by
how much would the liquid in each vessel rise if we pour an additional amount \( m \) of liquid into vessel \( n - 1 \)? (The rise in the amount of liquid in vessel \( j \) is given by \( p_j \)).

**Lemma A.2.** Let \( \ell_0 \geq \ell_1 \geq \cdots \geq \ell_{n-1} \) be a finite nonincreasing sequence in \( \mathbb{R}_{\geq} \). For every \( m \in \mathbb{R}_{\geq} \), there exists \( p \in [0, m]^{\mathbb{R}^n} \), which may be computed in \( O(n) \) time, s.t. all of the following hold.

1. \( \sum_{j=0}^{n-1} p_j = m \).
2. \( \ell_0 + p_0 \geq \ell_1 + p_1 \geq \cdots \geq \ell_{n-1} + p_{n-1} \).
3. \( \ell_k + p_k = \min_{j \in \mathbb{R}^n} \{ \ell_j + p_j \} \) for every \( k \in \mathbb{R}^n \) s.t. \( p_k > 0 \).

**Proof.** We iteratively define a sequence \( p^n \leq p^{n-1} \leq \cdots \leq p^0 \in [0, m]^{\mathbb{R}^n} \) s.t. the following hold for every \( i \in \{0, \ldots, n\} \).

1. \( p^i_j = 0 \) for every \( j < i \).
2. \( \sum_{j=0}^{n-1} p^i_j \leq m \), with equality when \( i = 0 \).
3. There exists \( h_i \in \mathbb{R}_{\geq} \) s.t. all of the following hold.
   - If \( \sum_{j=0}^{n-1} p^{i+1}_j < m \), then \( \ell_j + p^i_j = h_i \) for every \( j \geq i \),
   - If \( \sum_{j=0}^{n-1} p^{i+1}_j < m \), then \( \ell_{i-1} + p^{i+1}_{i-1} = h_i \) as well.
   - \( \ell_j \geq h_i \) for every \( j < i \).

In the setting of Fig. 1, \( p^n \) describes the rise of liquid before we begin pouring the additional amount \( m \), while for every \( i \in \mathbb{R}^n \), \( p^i \) describes the rise of liquid at the last instant during the pouring process, in which no water has risen except in vessels \( i + 1, \ldots, n - 1 \). (This can be either the final rise in liquid if \( i = 0 \) or if the final rise does not involve a change in the amount of liquid in vessels \( j < i \), or alternatively the rise in liquid just before the liquid in vessel \( i - 1 \) begins to rise.)

For the base case, we define \( p^n = 0 \), and all parts trivially hold (with \( h_n \triangleq \ell_{n-1} \)). For the construction step, let \( i \in \{0, \ldots, n - 1\} \) and assume that \( p^{i+1} \) has been defined. Let \( c \triangleq \sum_{j=0}^{n-1} p^{i+1}_j \). By Property 2 for \( i + 1, c \leq m \). If \( i = 0 \), then we define \( r \triangleq m - c \geq 0 \); otherwise, we define \( r \triangleq \min\{\langle n - i \rangle \cdot (\ell_{i-1} - h_{i+1}), m - c \} \), and by Property 3, \( r \geq 0 \) in this case as well. We define \( p^i_j \triangleq 0 \) for every \( j < i \) (and so Property 1 holds for \( i \)), and \( p^i_j \triangleq p^{i+1}_j + \frac{r}{n-i} \geq p^{i+1}_j \) for every \( j \geq i \). Property 2 holds for \( i \) as \( \sum_{j=0}^{n-1} p^i_j = \sum_{j=0}^{n-1} p^{i+1}_j + r = c + r \leq m \), with equality when \( i = 0 \). Finally, we show that Property 3 holds for \( i \), with \( h_i \triangleq h_{i+1} + \frac{r}{n-i} \). If \( \sum_{j=0}^{n-1} p^{i+1}_j < m \), then as \( p^{i+2} \leq p^{i+1} \), we have that \( (i + 1 = n \text{ or } \sum_{j=0}^{n-1} p^{i+2}_j < m) \text{ as well. Therefore, by Property 3 for } i + 1 \), we have for every \( j \geq i \) that \( \ell_j + p^i_j = \ell_j + p^{i+1}_j + \frac{r}{n-i} = h_{i+1} + \frac{r}{n-i} = h_i \). If \( \sum_{j=0}^{n-1} p^i_j < m \), then \( r < m - c \) and so by definition, \( r = \langle n - i \rangle \cdot (\ell_{i-1} - h_{i+1}) \). Therefore, \( h_i = h_{i+1} + \frac{r}{n-i} = \ell_{i-1} = \ell_{i-1} + p^i_{i-1} \). Finally, for every \( j < i \), by \( \ell \) nonincreasing and by definition of \( r \) we have \( \ell_j \geq \ell_{i-1} \geq h_{i+1} + \frac{r}{n-i} = h_i \), and the proof of the construction is complete.

Let \( i \in \{0, \ldots, n\} \) be largest s.t. \( \sum_{j=0}^{n-1} p^i_j = m \); \( i \) is well defined by Property 2 for \( i = 0 \). We now show that \( p^i \triangleq p^5 \) meets the conditions of the lemma. By definition, \( \sum_{j=0}^{n-1} p^i_j = m \).

Let \( j \in \mathbb{R}^n \setminus \{n - 1\} \). If \( j < i - 1 \), then by Property 1 for \( i = i \), we have \( \ell_j + p_j = \ell_j + \ell_{j+1} + p_{j+1} \). If \( j = i - 1 \), then by Properties 1 and 3 for \( i = i \) and by definition of \( i \), we have \( \ell_j + p_j = \ell_j = \ell_{i-1} \geq h_i = \ell_i + p_i = \ell_{j+1} + p_{j+1} \). Otherwise, i.e. if \( j > i - 1 \), then by Property 3 for \( i = i \) and by definition of \( i \), we have \( \ell_j + p_j = h_i = \ell_{j+1} + p_{j+1} \).

We conclude that \( \min_{j \in \mathbb{R}^n} \{ \ell_j + p_j \} = \ell_{n-1} + p_{n-1} \). For every \( k \in \mathbb{R}^n \) s.t. \( p_k > 0 \), by...
Property 1 for \( i = \bar{i} \) we have \( k \geq \bar{i} \). Therefore, by Property 3 for \( i = \bar{i} \) and by definition of \( \bar{i} \), we have \( \ell_{\bar{i}} + p_{\bar{i}} = h_{\bar{i}} = \ell_{n-1} + p_{n-1} = \min_{j \in \mathbb{P}_n} \{ \ell_j + p_j \} \), as required.

Finally, although it may seem in first glance that \( O(n^2) \) time may be required to compute \( p \), we note that the sequence \((h_i)_{i=1}^n \) can be computed in \( O(n) \) time, that from this sequence \( i \) can be deduced in \( O(n) \) time as the largest s.t. \( h_i = h_0 \), and that from both, \( p \) can be calculated in \( O(n) \) time: \( p_j = 0 \) for \( j < \bar{i} \) by Property 1 for \( i = \bar{i} \), while \( p_j = h_{\bar{i}} - \ell_j \) for \( j \geq \bar{i} \) by Property 3 for \( i = \bar{i} \). \( \square \)

We now prove Theorem 3.10, and then deduce Theorem 3.5 therefrom. Alternatively, Theorem 3.5 can also be proven directly from Lemma A.2, similarly to the proof of Theorem 3.10.

**Definition A.3.** For a finite measure \( \mu \) on \( T \) and a measurable set \( E \subseteq T \), we denote by \( \mu\setminus \chi \) the finite measure on \( T \) given by \( \mu\setminus \chi (A) \triangleq \mu(A \cap E) \).

**Proof of Theorem 3.10.** We prove the claim by induction on \( n \). (Recall that the construction underlying this proof is illustrated by Fig. 1; also recall the explanation preceding the statement of Lemma A.2 regarding the meaning of that lemma in the context of that figure.)

**Base \((n = 0)\):** In this case, \( S = \{ -\} \), and so \( E \equiv \chi (\{ -\}) \) is a Nash equilibrium as required.

**Step \((n > 0)\):** Assume w.l.o.g. that \( t_0 \leq t_1 \leq \cdots \leq t_{n-1} \). By the induction hypothesis, there exists a symmetric mixed-consumption Nash equilibrium \( s' \) in the \((n - 1)-\)producers consumer game \((\mu|\cap [0, t_{n-1}]); t_0, \ldots, t_{n-2}) \). If \( \mu([t_{n-1}, 1]) = 0 \), then we define a mixed-consumption profile \( s \) in \((\mu; \ell)\) s.t. \( s|\cap [0, t_{n-1}] = s'|\cap [0, t_{n-1}] \), and \( s|\{t_{n-1}, 1\} \equiv \chi (\{ -\}) \). As \( s' \) is symmetric in \((\mu|\cap [0, t_{n-1}]); t_0, \ldots, t_{n-2}) \), so is \( s \) in \((\mu; \ell)\). As \( \ell_j \equiv \ell_j \) for every \( j \in \mathbb{P}_{n-1} \), by \( s' \) being a Nash equilibrium in \((\mu|\cap [0, t_{n-1}]); t_0, \ldots, t_{n-2}) \) we have that no player of any type \( d \in [0, t_{n-1}] \) has any incentive to unilaterally deviate from \( s \). As \( \mu([t_{n-1}, 1]) = 0 \), we have \( \ell_{n-1} \equiv 0 \), and so players of types \( d \in [t_{n-1}, 1] \) have no incentive to deviate from \( s \) either. Therefore, \( s \) is a symmetric Nash equilibrium as required, and the proof for this case is complete. Assume therefore, henceforth, that \( \mu([t_{n-1}, 1]) > 0 \).

Recall that \( \ell_0', \ell_1', \ldots, \ell_{n-2}' \) are the loads on producers in \( s' \), and by slight abuse of notation, define \( \ell_0'' \equiv 0 \leq \ell_1'' \leq \cdots \leq \ell_{n-2}'' \leq \ell_{n-1}'' \). Let \( p \) be as in Lemma A.2 for \( \ell_j \equiv \ell_j' \) for every \( j \in \mathbb{P}_n \), and for \( m = \mu([t_{n-1}, 1]) > 0 \). We define a mixed-consumption profile \( s \) in \((\mu; \ell)\) s.t. \( s|\cap [0, t_{n-1}] = s'|\cap [0, t_{n-1}] \), and \( s|\{t_{n-1}, 1\} \equiv \chi (\{ -\}) \) (by Lemma A.2(1), indeed \( \frac{\mu}{\mu ([t_{n-1}, 1])} \) \( \in \Delta^S_d \) for all \( d \geq t_{n-1} \)). Once again, as \( s' \) is symmetric, so is \( s \). It remains to show that \( s \) is indeed a Nash equilibrium as required.

By definition of \( s' \) and of \( s \), we have that \( \ell_j = \ell_j'' + p_j \) for every \( j \in \mathbb{P}_n \). Let \( d \in [0, t_{n-1}] \). As \( \ell_0', \ldots, \ell_{n-2}' \) and \( \ell_0'', \ldots, \ell_{n-2}'' \) are both nonincreasing (the former by Lemma A.1, and the latter — by Lemma A.2(2)), and as \( S_d \) is the same in both \((\mu|\cap [0, t_{n-1}]); t_0, \ldots, t_{n-2}) \) and \((\mu; \ell)\), we have that as no player of type \( d \) has any incentive to unilaterally deviate from \( s' \) in the former, neither does it from \( s \) in the latter. Let now \( d \in [t_{n-1}, 1] \). For every \( k \in \text{supp}(s(d)) \), we have by definition \( p_k > 0 \), and so, by Lemma A.2(3), \( \ell_k = \min_{j \in \mathbb{P}_n} \ell_j' \), and the proof is complete.

The complexity claim follows as each inductive step requires \( O(n) \) time — the time required to calculate \( p \), by Lemma A.2. \( \square \)

**Corollary A.4.** Let \( h \in \mathbb{P}_n \), let \( s' \) be a mixed-consumption Nash equilibrium in the \( h \)-producers consumer game \((\mu|\cap [0, t_h]); t_0, \ldots, t_{h-1}) \), and let \( s \) be the mixed-consumption Nash equilibrium in the \( n \)-producers consumer game \((\mu; \ell)\) constructed iteratively from
s' as in the proof of Theorem 3.10. For every 0 ≤ j < h, we have ℓ'_j ≥ ℓ'_j', with equality if ℓ'_{h−1} = ℓ'_{h}.

PROOF. By following the construction in the proof of Theorem 3.10, and by Lemma A.2(3). □

Theorem 3.5 follows from Theorem 3.10 and from the following lemma.

Lemma A.5 (Theorem 3.10 ⇒ Theorem 3.5). If a mixed-consumption Nash equilibrium exists in the n-producers consumer game \((μ; t)\), and if μ is atomless, then a pure-consumption Nash equilibrium exists in this game as well.

PROOF. Assume w.l.o.g. that \(t_0 ≤ t_1 ≤ \cdots ≤ t_{n−1}\). Let s be a mixed-consumption Nash equilibrium in the game \((μ; t)\). For every \(i \in \{0, \ldots, n−2\}\), set \(C^i \triangleq [t_i, t_{i+1})\), and set \(C^{−} \triangleq [0, t_0)\) and \(C^{n−1} \triangleq [t_{n−1}, 1]\); note that \(s_0 = \{−\}\) for all \(d \in C^{−}\), and that \(s_d = \{0, \ldots, i\} \cup \{−\}\) for all \(d \in C^i\), for every \(i \in \mathbb{P}_n\). For every \(i, j \in S\), define \(p^i_j \triangleq \int_{C^i} s_j \, dμ\); note that if \(p^i_j > 0\), then \(s_j(d) > 0\) for some \(d \in C^i\). Let \(i \in S\). We first consider the case in which either \(μ(C^i) > 0\) or \(C^i = \emptyset\). In this case, as \(μ\) is atomless, there exists a partition of \(C^i\) into \(n\) pairwise-disjoint measurable sets \((C^i_j)_{j∈S}\), s.t. \(μ(C^i_j) = p^i_j\) for all \(j \in S\), and s.t. \(C^i_j = \emptyset\) whenever \(p^i_j = 0\). Otherwise, i.e. if \(μ(C^i) = 0\) yet \(C^i ≠ \emptyset\), then let \(k ∈ S\) s.t. \(s_k(d) > 0\) for some \(d \in C^i\), and define \(C^i_k \triangleq C^i_1\), and \(C^i_k = \emptyset\) for every \(j \in S \setminus \{k\}\). Note that in this case we also have that \((C^i_j)_{j∈S}\) is a partition of \(C^i\) and \(μ(C^i_1) = 0\) = \(\int_{C^i} s_j \, dμ = p^i_j\) for all \(j \in S\).

We define a measurable function \(s' : T → S\) by \(s'|_{\bigcup_{j∈S} C^i_j} ≡ j\) for every \(j \in S\). For every \(j \in S\), we note that \(ℓ'_j = μ(\bigcup_{i∈S} C^i_j) = ∫_{C^i} s_j \, dμ = ∫_{C^i} s_j \, dμ = ℓ'_j\).

We conclude by showing that \(s'\) is indeed a pure-strategy profile, and moreover — a Nash equilibrium. Let \(d \in T\); by definition there exists \(i ∈ S\) s.t. \(d ∈ C^i_{s'(d)} \subseteq C^i\).

As \(C^i_{s'(d)} ≠ \emptyset\), by definition of \(C^i_{s'(d)}\), we have that \(s_{s'(d)}(d') > 0\) for some \(d' ∈ C^i\), and so \(s'(d) ∈ S_{d'}\). As by definition of \(C^i\) we have \(S_d = S_{d'}\), we obtain \(s'(d) ∈ S_{d'}\), and so \(s'\) is a pure-strategy profile. Furthermore, as \(s_{s'(d)}(d') > 0\), we obtain \(ℓ'_d = ℓ'_{s'(d)} = \min_{d' ∈ S_{d'}} ℓ'_d = \min_{d' ∈ S_{d'}} ℓ'_d = \min_{d' ∈ S_{d'}} ℓ'_d\), and the proof is complete. □

Proof of Theorem 3.11. Assume w.l.o.g. that \(t_0 ≤ t_1 ≤ \cdots ≤ t_{n−1}\). Let \(s, s'\) be two mixed-consumption Nash equilibria in the game \((μ; t)\). By definition of Nash equilibrium, we have \(s'_d ≡ 1_{[0, t_d)} = s_d\), and so \(\sum_{j=0}^{n−1} ℓ'_j = μ(\{t_0, 1\}) = \sum_{j=0}^{n−1} ℓ_j\). Assume for contradiction that there exists \(j ∈ \mathbb{P}_n\) s.t. \(ℓ'_j ≠ ℓ_j\); let \(j\) be minimal with this property, and assume w.l.o.g. that \(ℓ'_j > ℓ_j\).

Let \(j ≤ k < n\) be maximal s.t. \(ℓ'_k = ℓ'_j\). By Lemma A.1, for every \(j ≤ i ≤ k\), we have \(ℓ'_i = ℓ'_j > ℓ_j ≥ ℓ'_j\). Therefore, and as \(ℓ'_i = ℓ'_j\) for every \(0 ≤ i < j\), we have \(\sum_{j=0}^{k} ℓ'_j > \sum_{j=0}^{k} ℓ_j\). We thus obtain both that \(k < n−1\), and that \(\sum_{j=0}^{k} ℓ'_j > \sum_{j=0}^{k} ℓ_j ≥ μ(\{t_0, t_{k+1}\})\).

Therefore, \(\sum_{j=k+1}^{n−1} ℓ'_j < μ(\{t_{k+1}, 1\})\), and hence there exists \(d ∈ [t_{k+1}, 1]\) s.t. \(s'_d > 0\) for some \(0 ≤ i ≤ k\). As by Lemma A.1 we have \(ℓ'_i ≥ ℓ'_k > ℓ'_k\), we conclude that \(s'\) is not a Nash equilibrium — a contradiction. □

Proof of Corollary 3.12. Assume w.l.o.g. that \(t_0 ≤ t_1 ≤ \cdots ≤ t_{n−1}\). Let \(s\) be a mixed-consumption Nash equilibrium in the game \((μ; t)\), let \(d ∈ T\) and let \(k ∈ \text{supp}(s(d))\). If \(d < t_k\) for all \(j ∈ \mathbb{P}_n\), then \(k = −\) and so \(ℓ'_{k} = μ(\{0, t_0\})\). Otherwise, \(k ≠ −\) and so \(d ≥ t_0\); let \(i ∈ \mathbb{P}_n\) be largest s.t. \(t_i ≤ d\). By definition of \(s\) and by
Lemma A.1, we have \( \ell_k^* = \min\{\ell_k^s | t_j \leq d\} = \ell_k^s \). Either way (and by Theorem 3.11 when \( k \neq -\)), \( \ell_k^s \) does not depend on the choice of \( s \), as required.

From Algorithm 1, we obtain the following recursive identity for \( \ell_k(\ell) \).

**Corollary A.6.** If \( t_0 \leq t_1 \leq \cdots \leq t_{n-1} \), then defining \( t_n \triangleq 2 \), we have

\[
\ell_k(t_0, \ldots, t_{n-1}) = \max_{k < j \leq n} \mu((t_0, t_j)) - \sum_{i=0}^{k-1} \ell_i(\ell) = \max_{k < j \leq n} \mu((0, t_j)) - \sum_{i \in \mathbb{P}_k \cup \{\neg\}} \ell_i(\ell)
\]

(where by slight abuse of notation, \( \mu \) is treated as a measure on \([0, 2]\) with support \( \mathcal{T} \)) for every \( k \in \mathbb{P}_n \).

**Proof.** A direct corollary of Algorithm 1, by considering two cases: in the first, either \( k = 0 \) or \( \ell_k(\ell) < \ell_{k-1}(\ell) \) (and so the given value \( k \) is the value of the variable \( k \) in some iteration of Algorithm 1); in the second, \( k > 0 \) and \( \ell_k(\ell) = \ell_{k-1}(\ell) \) (and so Algorithm 1 calculates both \( \ell_k(\ell) \) and \( \ell_{k-1}(\ell) \) in the same iteration of the while loop, and therefore they are identical; it is straightforward to verify that the expression in the statement evaluates to the same value for both \( k - 1 \) and \( k \) in this case). \( \square \)

### A.2. Analysis of \( \ell \)

Before moving on to prove the results presented in Section 4, we now formalize three analytic properties of the function \( \ell \) (defined in Definition 3.13), which we later utilize when proving the results of Section 4. The first property is that the load on a producer does not decrease if the producer raises the offered quality of service (i.e., lowers its strategy).

**Lemma A.7 (\( \ell_j \) is nonincreasing in \( t_j \)).** For every \( j \in \mathbb{P}_n \) and for every \( \ell \in \mathcal{T}^\mathbb{P}_n \) and \( t_j' \in \mathcal{T} \), if \( t_j < t_j' \), then \( \ell_j(\ell_{-j}, t_j') \leq \ell_j(\ell) \).

**Proof.** Let \( \ell \in \mathcal{T}^\mathbb{P}_n \), \( k \in \mathbb{P}_n \) and \( t_k' \in (t_k, 1] \). Assume w.l.o.g. that \( t_0 \leq t_1 \leq \cdots \leq t_{n-1} \in \mathcal{T} \). If \( k \neq n - 1 \), then it is enough to consider the case \( t_k < t_k' \leq t_{k+1} \). Let \( s \) be a mixed-consumption Nash equilibrium in the induced consumer game \( (\mu; \ell) \). For every \( j \in \mathbb{P}_n \setminus \{k\} \), define \( t_j' \triangleq t_j \). Let \( s' \) be a mixed-consumption Nash equilibrium in \( (\mu; \ell') \), and assume for contradiction that \( \ell_k' > \ell_k \). Let \( i \in \mathbb{P}_n \) be maximal s.t. \( \ell_i' = \ell_i \); by definition, \( i \geq k \).

We claim that \( \ell_i' > \ell_i \) for every \( 0 \leq j < i \). Let \( 0 \leq h \leq i \) be minimal s.t. \( \ell_h' \leq \ell_i' \) (such \( h \) exists, and \( h \leq k \), as \( \ell_k' \leq \ell_k \)); we will show that \( \ell_j' \geq \ell_j \) separately for every \( 0 \leq j < h \) (if \( h > 0 \)) and for every \( h \leq j < i \). For every \( h \leq j < i \), by Lemma A.1, by definition of \( i \) and by definition of \( h \), we have \( \ell_j' \geq \ell_j' = \ell_j' \geq \ell_h' \), as required. We move on to show that \( \ell_j' \geq \ell_j' \) for every \( 0 \leq j < h \); assume that \( h > 0 \) (otherwise, there is nothing to show). Let \( \ell_0, \ldots, \ell_{h-1} \) be the loads on producers \( 0, \ldots, h - 1 \) in a Nash equilibrium in the game \( (\mu|_{[0,h]}; t_0, \ldots, t_{h-1}) \); similarly, let \( \ell_0', \ldots, \ell_{h-1}' \) be the loads on producers \( 0, \ldots, h - 1 \) in a Nash equilibrium in the game \( (\mu|_{[0,h]}; t_0', \ldots, t_{h-1}') \). As \( h > 0 \), by definition of \( h \) we have \( \ell_h' > \ell_h' \); therefore, by Corollary A.4 and Theorem 3.11, we have that \( \ell_j' = \ell_j' \) for every \( 0 \leq j < h \). By Corollary A.4 and Theorem 3.11, we obtain also that \( \ell_j' \geq \ell_j' \) for every \( 0 \leq j < h \). As \( k \geq h \), we have that \( t_j' = t_j \) for every \( 0 \leq j < h \) and that \( t_{h-1}' \geq t_{h-1} \); therefore, \( \ell_j' \geq \ell_j \) for every \( 0 \leq j < h \). (This follows by by tracing the construction in the proof of Theorem 3.10, as all inductive steps but the last are identical, while the last, examining Lemma A.2, increases each load by no less when computing \( \ell' \) than when computing \( \ell \); in the context of Fig. 1,
pouring a additional nonnegative amount of liquid into the rightmost vessel does not cause the liquid level in any vessel to fall. Alternatively, this can also be seen by tracing Algorithm 1, as each iteration when computing \( \ell' \) either computes the same load values for the producers as the corresponding iteration when computing \( \ell \), or is the last, thus computing loads that are not lower than those computed for \( \ell \).) Combining all of these, we have \( \ell'_j > \ell_j = \ell_j' \) for every \( 0 \leq j < h \), as required.

We conclude that \( \sum_{j=0}^{i} \ell'_j > \sum_{j=0}^{i} \ell_j \), as \( \ell'_j \geq \ell_j \) for every \( 0 \leq j \leq i \), with a strict inequality for \( j = k \). If \( i = n - 1 \), then \( \sum_{j=0}^{n-1} \ell'_j > \sum_{j=0}^{n-1} \ell_j = \mu([t_0, 1]) \geq \mu([0, 1]) \) — a contradiction; assume, therefore, that \( i < n - 1 \). Hence, \( \sum_{j=0}^{i} \ell'_j > \sum_{j=0}^{i} \ell_j \geq \mu([t_0, t_{i+1}]) \geq \mu([0, t_{i+1}]) \). Therefore, there exists \( d \geq t'_{i+1} \) s.t. \( s'(d) > 0 \) for some \( 0 \leq j \leq i \), but by Lemma A.1 and by definition of \( i \) we notice that \( \ell'_j \geq \ell'_i > \ell'_i \), so \( s' \) is not a Nash equilibrium in \((\mu; \ell')\) — a contradiction as well.

We note that an alternative proof may also be given via Algorithm 1 and Corollary A.6. \( \Box \)

The second property is that the load on producer \( j \) cannot increase as a result of other producers moving closer to \( j \)'s quality of service.

**Lemma A.8 (\( \ell_j \) is Weakly Quasiconvex in \( t_k \)).** For every \( j \in \mathbb{P}_n \cup \{-\} \) and \( j \neq k \in \mathbb{P}_n \) and for every \( t \in T^{[n]} \) and \( t'_k \in T \), if \( j \neq - \) and either \( t_k < t'_k \) or \( t_k < t_k' \), or if \( j = - \) and \( t'_k < t_k \), then \( \ell_j(t_{k-}, t_k) \leq \ell_j(t) \).

**Proof.** Assume w.l.o.g. that \( t_0 \leq t_1 \leq \cdots \leq t_{n-1} \). If \( k \neq n - 1 \), then it is enough to consider the case \( t_h < t'_h \leq t_{k-1} \). Let \( s \) and \( s' \) be mixed-consuming Nash equilibria in \((\mu; t)\) and \((\mu; \ell_{h-}, t'_{k})\), respectively.

Assume for contradiction that \( \ell'_s > \ell'_s \) for some \( k < i < n \), and let \( i \) be minimal with this property. Therefore (and by Lemma A.7 if \( i = k + 1 \)), \( \ell'_{i-1} \leq \ell'_{i-1} \). By Lemma A.1, we obtain \( \ell'_{i-1} \geq \ell'_{i-1} \geq \ell'_{i} \). Therefore, \( s_j([i, \infty]) \equiv 0 \) for every \( 0 \leq j < i \), and so \( \sum_{j=0}^{i-1} \ell'_j \leq \mu([t_i, 1]) \) for every \( 0 \leq j < i \), and \( \sum_{j=0}^{i-1} \ell'_j \geq \mu([0, t_{i+1}]) \). Hence, as \( \ell'_s > \ell'_s \), there exists \( i < h < n \) s.t. \( \ell'_h < \ell'_h \) — let \( h \) be minimal with this property. Therefore, \( \ell'_{h-1} \geq \ell'_{h-1} \), and by Lemma A.1, we obtain \( \ell'_{h-1} \geq \ell'_{h-1} \geq \ell'_{h} > \ell'_{h} \). By definition of \( h \), we have that \( \ell'_s > \ell'_s \) for every \( i \leq j < h \), with a strict inequality for \( j = i \) by definition of \( i \), and so \( \sum_{j=0}^{h-1} \ell'_j > \sum_{j=h}^{h-1} \ell'_j \geq \mu([t_h, 1]) \), with the last inequality since \( s_j([t_i, 1]) \equiv 0 \) for every \( 0 \leq j < i \). Therefore, there exists \( d \geq t_h \) s.t. \( s'(d) > 0 \) for some \( i \leq j < h \), but by Lemma A.1, \( \ell'_j \geq \ell'_{j-1} > \ell'_{j} \), so \( s' \) is not a Nash equilibrium in \((\mu; \ell_{h-}, t'_k)\) — a contradiction.

Assume now for contradiction that \( \ell'_s < \ell'_s \) for some \( 0 \leq i < k \), and let \( i \) be minimal with this property. As \( k > 0 \), we have \( \ell'_s = \mu([0, t_0]) = \ell'_s \). Therefore, \( \sum_{j=0}^{n-1} \ell'_j = \sum_{j=0}^{n-1} \ell'_j \) and by definition of \( h \) there exists \( h \in \mathbb{P}_n \) s.t. \( \ell'_h > \ell'_h \) — let \( h \) be minimal with this property. By Lemma A.7 and by the first part of this proof, \( h < k \). We now consider two cases: \( h < i \) and \( i < h \). We start with the case \( h < i \). In this case, by definition of \( i \) and by Lemma A.1, \( \ell'_{i-1} \geq \ell'_{i-1} \geq \ell'_{i} \) and \( s'_j([0, t_0]) \equiv 0 \) for every \( 0 \leq j < i \), and so, by Corollary A.4 and as \( i < k \), \( \ell'_h = \ell_h(\mu_{|0, t_h)}; t_0, \ldots, t_{i-1}) \leq \ell'_h \) — a contradiction. Similarly, if \( i < h \), then by definition of \( h \) and by Lemma A.1, \( \ell'_{h-1} \geq \ell'_{h-1} \geq \ell'_{h} \) and \( s'_j([0, t_h]) \equiv 0 \) for every \( 0 \leq j < h \), and so, by Corollary A.4 and as \( h < k \), \( \ell'_i = \ell_i(\mu_{|0, t_h)}; t_0, \ldots, t_{h-1}) \leq \ell'_i \) — a contradiction as well.

We conclude by examining the effect on \( \ell \). If \( k \neq 0 \), then let \( t'_0 = t_0 \). Regardless of the value of \( k \), we have \( t'_0 \geq t_0 \). By definition of \( s', s'' \), we have \( \ell''_s = \mu([0, t_0]) \leq \mu([0, t_0]) = \mu(0, t_0) \).
\[ \ell_i^p. \]

We note that an alternative proof may also be given via Algorithm 1 and Corollary A.6. □

Finally, the last property is that small perturbations in the producers’ strategies result in quantifiably small changes in the loads on the various producers.

**Lemma A.9** (\( \ell \) is Lipschitz in each coordinate with Lipschitz constant 1). For every \( j, k \in \mathbb{P}_n \) and for every \( i \in \mathbb{T}^n \) and \( t'_k \in \mathcal{T} \), if \( t_k < t'_k \), then \( |\ell_j(i_{-k}, t'_k) - \ell_j(i) - \ell_j(i_{-k})| \leq \mu((t_k, t'_k)) \).

**Proof.** Assume w.l.o.g. that \( t_0 \leq t_1 \leq \cdots \leq t_n-1 \). If \( k \neq n-1 \), then it is enough to consider the case \( t_k < t'_k \leq t_{k+1} \). Let \( s \) and \( s' \) be mixed-consumption Nash equilibria in \( (\mu; \ell) \) and \( (\mu; \ell_{-k}, t'_k) \), respectively. Define \( S_k \triangleq \mathbb{P}_k \cup \{-\} \). We start by showing that \( \sum_{j \in S_k} \ell_j - \sum_{j \in S_k} \ell_j' \leq \mu((t_k, t'_k)) \).

If \( k = 0 \), then this claim holds, as in this case \( S_k = \{-\} \), and by definition of \( s \) and \( s' \) we have \( \ell_j' - \ell_j = \mu((0, t'_k)) - \mu((0, t_k)) = \mu((t_k, t'_k)) \). Assume therefore that \( k > 0 \) and assume for contradiction that \( \sum_{j \in S_k} \ell_j - \sum_{j \in S_k} \ell_j' > \mu((t_k, t'_k)) \). As \( k > 0 \), we have \( \ell_j' = \mu((0, t_0)) = \ell_j' \), and so \( \sum_{j=0}^{k-1} \ell_j' - \sum_{j=0}^{k-1} \ell_j' > \mu((t_k, t'_k)) \) as well. Let \( i \in \mathbb{P}_n \) be maximal s.t. \( \ell_i' = \ell_{i-1}' \). By definition, \( i \geq k \). For every \( k \leq j \leq i \), by Lemmas A.1 and A.8, we have \( \ell_j' \geq \ell_i' = \ell_{i-1}' = \ell_t - \ell_i' > \ell_{t+1}' \). Therefore, we have \( \sum_{j=0}^{i-1} \ell_j' - \sum_{j=0}^{i-1} \ell_j' > \mu((t_k, t'_k)) \).

If \( i = n-1 \), then we obtain \( \sum_{j=0}^{n-1} \ell_j' > \sum_{j=0}^{n-1} \ell_j' = \mu([0, 1]) \), a contradiction; assume, therefore, that \( i < n-1 \). If \( i + 1 \neq k \), then let \( t_{k+1}' \triangleq t_{i+1} \). Hence,

\[
\sum_{j=0}^{i} \ell_j' > \sum_{j=0}^{i} \ell_j + \mu((t_k, t'_k)) \geq \mu((0, t_{i+1})) \geq \mu((0, t'_k)) \geq \mu((0, t_{i+1}))) \quad i + 1 = k
\]

Thus, there exists \( d \geq t_{i+1}' \) s.t. \( s_j'(d) > 0 \) for some \( 0 \leq j \leq i \), but by Lemma A.1 and by definition of \( i \) we notice that \( \ell_i' \geq \ell_i > \ell_{i+1} \), so \( s' \) is not a Nash equilibrium in \( (\mu; \ell_{-k}, t'_k) \) — a contradiction as well.

As \( \sum_{j \in S_k} \ell_j - \sum_{j \in S_k} \ell_j' \leq \mu((t_k, t'_k)) \) and as by Lemma A.8 \( \ell_j' \geq \ell_j \) for every \( j \in S_k \), we obtain \( 0 \leq \ell_j' - \ell_j \leq \mu((t_k, t'_k)) \) for every \( j \in S_k \). Similarly, As \( \sum_{j \in S_k} \ell_j - \sum_{j \in S_k} \ell_j' \leq \mu((t_k, t'_k)) \) and as \( \sum_{j \in S_k} \ell_j' = \mu(T) = \sum_{j \in S_k} \ell_j \), we have \( \sum_{j=k}^{n-1} \ell_j' - \sum_{j=k}^{n-1} \ell_j' \leq \mu((t_k, t'_k)) \), and as by Lemmas A.7 and A.8 \( \ell_j' \leq \ell_j \) for every \( k < j < n \), we obtain \( 0 \leq \ell_j' - \ell_j \leq \mu((t_k, t'_k)) \) for every \( k < j < n \), and the proof is complete.

We note that an alternative proof may also be given via Algorithm 1 / Corollary A.6. □

**A3. Proofs and Auxiliary Results for Section 4.1**

**A3.1. Proofs and Auxiliary Results for Section 4.1.1**

**Lemma A.10.** Let \( j \in \mathbb{P}_n \) and \( t_j \in \mathcal{T} \). For every \( i_{-j} \in \mathcal{T}^{j-1} \), we have \( \ell_j(i) \geq \frac{\mu([t_j, 1])}{n} \), which constitutes a tight bound.

**Proof.** Let \( s \) be mixed-consumption Nash equilibrium in \( (\mu; \ell) \). If \( \mu([t_j, 1]) = 0 \), then \( \ell_j = 0 \) and the claim trivially holds. Assume therefore that \( \mu([t_j, 1]) > 0 \). Let \( k \in \mathbb{P}_n \).
— We say that 

\[ t \] 

by definition of \( k \). Since \( \int_{[t,1]} s_k \, d\mu \geq \frac{\mu([t,1])}{n} \) by definition of \( k \). Since \( \int_{[t,1]} s_k \, d\mu \geq \frac{\mu([t,1])}{n} \) > 0, there exists \( d \geq t_j \) s.t. \( s_k(d) > 0 \). As \( j \in S_d \), by definition of Nash equilibrium we have \( \int_{[t,1]} s_j \, d\mu = \ell_j \geq \ell_k \geq \int_{[t,1]} s_k \, d\mu \geq \frac{\mu([t,1])}{n} \).

Alternatively, by Lemma A.8, \( \ell_j(t) \) is minimal given \( t_j \) when \( t_i = t_j \) for every \( i \in \mathbb{P}_n \setminus \{j\} \). By Theorem 3.11, by anonymity, and by definition of Nash equilibrium, the load on each producer in this case is exactly \( \frac{\mu([t,1])}{n} \).

**Corollary A.11.** Let \( i \in \mathcal{T}^p \). For every \( j \in \mathbb{P}_n \), if \( t_j = 0 \), then \( \ell_j(t) \geq \frac{\mu(T)}{n} \).

**Proof.** A direct corollary of Lemma A.10. \( \square \)

**Definition A.12 (Domination).** Let \( t, t' \) be strategies in \((n, \mu, \succeq_C)\).

— We say that \( t \) weakly dominates \( t' \) if \( t \) is a safe alternative to \( t' \) and moreover, there exists some strategy profile for all but one of the producers, s.t. playing \( t \) gives the remaining producer strictly higher utility than playing \( t' \).

— We say that \( t \) strongly dominates \( t' \) if for every strategy profile for all but one of the producers, playing \( t \) gives the remaining producer strictly higher utility than playing \( t' \).

**Lemma A.13 (Domination).** Let \( t < t' \in \mathcal{T} \) be strategies in \((n, \mu, \succeq_C)\).

1. \( t \) is a safe alternative to \( t' \).
2. \( t \) weakly dominates \( t' \) iff \( \mu([t, t']) > 0 \).
3. \( t \) strongly dominates \( t' \) iff \( \mu([t', 1]) < \frac{\mu([t, 1])}{n} \).

**Proof.** Part 1 follows from Lemma A.7. We move on to proving Part 2. \( \Rightarrow \): Assume that \( \mu([t, t']) = 0 \); by Algorithm 1 / Corollary A.6, \( t \) and \( t' \) are equivalent, and a fortiori \( t' \) is a safe alternative to \( t \). Nonetheless, we now also directly show that \( t \) and \( t' \) are equivalent. Let \( f_{t-0} \in \mathcal{T}_{\mathbb{P}_n \setminus \{0\}} \), and let \( s \) be a mixed-consumption Nash equilibrium in \((\mu; t_{-0}, t)\). Let \( s' \) be the mixed-consumption profile in \((\mu; t_{-0}, t')\) s.t. \( s'|_{\mathcal{T} \setminus \{t', t\}} = s|_{\mathcal{T} \setminus \{t', t\}} \) and s.t. for every \( d \in [t, t'] \), if \( s_d = \{\neg\} \) w.r.t. \((\mu; t_{-0}, t')\) then \( s'(d) = \mathbb{I}_{\{\neg\}} \), and otherwise \( s'(d) = \mathbb{I}_{\{j\}} \) for some \( j \in \arg\max\{2, \ell_0(t, t, \ldots, t)\} \). As \( s = s' \) almost everywhere w.r.t. \( \mu \), we have that \( \ell_j'(t) = \ell_j' \) for all \( j \in \mathbb{P}_n \). By definition of \( s \), theref, therefore no type \( d \in \mathcal{T} \setminus \{t, t'\} \) has any incentive to deviate from \( s' \) in the latter game. By Lemma A.1 (for \( s \)) and by definition of \( s' \), neither does any type \( d \in [t, t'] \) have any incentive to deviate from \( s' \) in the latter game. Therefore, \( s' \) is a Nash equilibrium in \((\mu; f_{t-0}, t)\). As in particular \( \ell_0' = \ell_0'', \) the proof of this direction is complete.

\( \Leftarrow \): Assume that \( \mu([t', t]) > 0 \); we will show that \( t' \) is not a safe alternative to \( t \). Define \( a \triangleq \mu([t, t']) > 0 \), \( b \triangleq \mu([t', 1]) \) and \( c \triangleq \mu(\{1\}) \). By Algorithm 1, we have \( \ell_0(t', 1, 1, \ldots, 1) = \max\{b, \frac{b+c}{n}\} < \max\{a+b, \frac{a+b+c}{n}\} = \ell_0(t, 1, 1, \ldots, 1) \), and the proof of this direction is complete as well.

We conclude by proving Part 3. \( \Rightarrow \): Assume that \( \mu([t', 1]) \geq \frac{\mu([t, 1])}{n} \). Therefore, \( \mu([t', t]) \leq \frac{n-1}{n} \cdot \mu([t, 1]) \), and hence \( \mu([t', t]) \leq \frac{\mu([t, 1])}{n} \cdot \frac{n-1}{n} \). By Theorem 3.11, by anonymity, and by definition of Nash equilibrium, the load on every producer, and in particular on producer \( 0 \), in a Nash equilibrium in the \( n \)-producer game \((\mu; t, \ldots, t)\) is \( \frac{\mu([t, 1])}{n} \). As \( \max\{\frac{\mu([t', t])}{n}, \frac{\mu([t, 1])}{n}\} = \frac{\mu([t, 1])}{n} \), by Algorithm 1 the load on every producer, and in particular on producer \( 0 \), in a Nash equilibrium in the game \((\mu|_{\mathcal{C}[0, t]}; t, t, \ldots, t)\) is \( \mu([t, 1]) \) as well, as required.
\[ \therefore \) Assume that \( \mu([t', 1]) < \frac{\mu(t, 1)}{n} \). Let \( \tilde{t}_{-0} \in \mathcal{T} \setminus \{0\} \). By Lemma A.10 and by definition of legal strategies in the consumer game, we obtain \( \ell_0(\tilde{t}_{-0}, t) \geq \frac{\mu(t, 1)}{n} > \mu([t', 1]) \geq \ell_0(\tilde{t}_{-0}, t') \). \]

**Proof of Theorem 4.3.** The first statement is a direct corollary of Lemma A.13, and the second — of Corollary A.11. \]

**Lemma A.14.** Let \( i \in \mathcal{T} \) and let \( s \) be a mixed-consumption profile in the consumer game \( (\mu, i) \). If \( t_j^s = \frac{\mu(t)}{n} \) for every \( j \in \mathcal{P}_n \), and if \( s_{-}(d) = 0 \) whenever \( S_d \neq \{-\} \), then \( s \) constitutes a Nash equilibrium in this game.

**Proof.** Directly from definition of mixed-consumption Nash equilibrium, no consumer has any incentive to unilaterally deviate from \( s \). \]

**Proof of Theorem 4.4.** The first part \( (\Rightarrow) \) follows directly from Corollary A.11. For the second part \( (\Leftarrow) \), let \( i \) be a pure-strategy profile in \( (n, \mu, \preceq) \) s.t. \( \ell_j(i) = \frac{\mu(t)}{n} \) for every \( j \in \mathcal{P}_n \). Assume w.l.o.g. that \( t_0 \leq \cdots \leq t_{n-1} \), and let \( s \) be a mixed-consumption Nash equilibrium in the induced consumer game \( (\mu, i) \); therefore, \( \ell_s^i = \ell_j(i) = \frac{\mu(T)}{n} \) for every \( j \in \mathcal{P}_n \). Let \( k \in \mathcal{P}_n \) and let \( t_k' \in \mathcal{T} \); we will show that producer \( k \) has no incentive to deviate to \( t_k' \) from \( t_k \). If \( t_k < t_k' \), then this follows directly from Lemma A.13(1). We therefore consider the case in which \( t_k' < t_k \). Let \( N \triangleq [t_k', t_0) \) (if \( t_0 \leq t_k' \), then \( N = \emptyset \)). By definition of \( s \), we have that \( \mu(N) = \mu([0, t_0)) = \ell_s^i = 0 \). We define a mixed-consumption profile \( s' \) in the consumer game \( (\mu, \tau_k, t_k') \) by \( s'|_N \equiv 1 \) and \( s'|_{\mathcal{T} \setminus N} = s|_{\mathcal{T} \setminus N} \) (if \( N = \emptyset \), then \( s' = s \)). This indeed is a mixed-consumption profile since \( t_k' < t_k \) and by definition of \( N \). As \( \mu(N) = 0 \), we have \( \ell_s^i = \ell_s^i = \frac{\mu(T)}{n} \) for every \( j \in \mathcal{P}_n \). Thus, by Lemma A.14 and by definition of \( s' \) via \( N \), we conclude that \( s \) is a Nash equilibrium in \( (\mu, \tau_k, t_k') \), and so \( k \) has no incentive to deviate to \( t_k' \) from \( t_k \) in this case either.

**Corollary A.15 (Least/Most Nash Equilibrium Load).** Let \( t_0 \leq \cdots \leq t_{n-1} \in \mathcal{T} \). The pure-strategy profile \( i \) constitutes a Nash equilibrium in \( (n, \mu, \preceq) \) iff either of the following equivalent conditions hold.

\[ \begin{align*}
&\ell_{n-1}(i) = \frac{\mu(T)}{n}, \\
&\mu([0, t_0)) = 0 \text{ and } \ell_0(i) = \frac{\mu(T)}{n}.
\end{align*} \]

**Proof.** A direct corollary of Theorem 4.4 and Lemma A.1. \]

**Proof of Theorem 4.5.** \( \Rightarrow \): Assume that \( i \) constitutes a pure-strategy Nash equilibrium in \( (n, \mu, \preceq) \). Let \( s \) be a mixed-consumption Nash equilibrium in the induced consumer game \( (\mu, i) \). For every \( j \in \mathcal{P}_n \), by Theorem 4.4, we obtain \( \mu([t_j, 1]) \geq \sum_{k=j}^{n-1} \ell_k = \frac{n-j}{n} \cdot \mu(T) \), and so \( \mu([0, t_j)) \leq \frac{j}{n} \cdot \mu(T) \).

\( \Leftarrow \): Assume that \( \mu([0, t_j)) \leq \frac{j}{n} \cdot \mu(T) \) for every \( j \in \mathcal{P}_n \); in particular, \( \mu([0, t_0)) = 0 \). Let \( s \) be a mixed-consumption profile in the induced consumer game \( (\mu, i) \). By Theorem 4.4, it is enough to show that \( \ell_s^i = \frac{\mu(T)}{n} \) for every \( j \in \mathcal{P}_n \). Assume for contradiction that this is not the case. Therefore, as \( \ell_s^i = \mu([0, t_0)) = 0 \), there exists \( k \in \mathcal{P}_n \) s.t. \( \ell_k^i > \frac{\mu(T)}{n} \); let \( k \) be maximal with this property. By Lemma A.1, we have \( \sum_{j=0}^{k} \ell_k^i \geq (k+1) \cdot \ell_k^i > \frac{(k+1) \cdot \mu(T)}{n} \). Therefore, there exists \( d \geq t_{k+1} \) s.t. \( s_j^d(d) > 0 \) for some \( 0 \leq j \leq k \), but by definition of \( k \) we notice that \( \ell_k^i > \frac{\mu(T)}{n} \geq \ell_k^i \), so \( s' \) is not a Nash equilibrium — a contradiction.

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We note that an alternative proof of the second direction (⇐) may also be given via Algorithm 1 / Corollary A.6. □

The second direction (⇐) of Theorem 4.5 can also be proven constructively. Such a proof is quite tedious in the general case, but simplifies greatly when µ is atomless. E.g., if µ = U(T) the uniform measure, then whenever t₀ ≤ ⋯ ≤ tₙ₋₁ meet the conditions of Theorem 4.5, then a Nash equilibrium can be formed by splitting the market as follows: every d ∈ [0, 1/ₙ) plays the pure strategy 0 ∈ Pₙ (the conditions of Theorem 4.5 guarantee that this is a legal strategy for all such d), every d ∈ [1/ₙ, 2/ₙ) plays the pure strategy 1 ∈ Pₙ (once again, the conditions of Theorem 4.5 guarantee that this is a legal strategy for all such d), and so on, until every d ∈ [n−1/n, 1], playing the pure strategy n − 1 ∈ Pₙ.

(We remark that if t_j = 1/ₙ for every j ∈ Pₙ, then this is in fact the unique Nash equilibrium among consumers, up to modifications of measure zero. More about this split and Nash equilibrium — in Theorem 4.31 and Remark 4.32 in Appendix A.4.) The load on each producer in this case is precisely 1/ₙ, and by Lemma A.14 and Theorem 4.4, the proof is complete.

PROOF OF THEOREM 4.6. Assume w.l.o.g. that t₀ ≤ t₁ ≤ ⋯ ≤ tₙ₋₁ and assume for contradiction that P ⊆ Pₙ and ℓ ∈ T' as in the statement exist. Let s be a mixed-consumption Nash equilibrium in (µ; t_p, ℓ'). As there exists j ∈ P s.t. ℓ_j > ℓ_j(t) = µ(T)/ₙ (with the last equality by Theorem 4.4), and as ∑ᵢ=₀ⁿ−₁ ℓ_j ≤ µ(T), there thus exists a producer k ∈ Pₙ s.t. ℓ_k < µ(T)/ₙ — let 0 ≠ K ⊆ Pₙ be the set of all such producers; by definition of P and by Theorem 4.4, we have K ⊆ Pₙ \ P. By Lemma A.1, K = {n−1, n−2, ..., n−|K|}. Therefore, and by Theorem 4.5, ∑ₖ=₀ⁿ−|K| ℓ_k < |K|µ(T)/ₙ ≤ µ([n−1|K|, 1]), and so there exists d ∈ [n−|K|, 1] s.t. s_j(d) > 0 for some 0 ≤ j < n−|K|, but by definition of K we notice that ℓ_j ≥ µ(T)/ₙ > ℓ_j([n−|K|, 1], and so (as n−|K| ⊈ P) we have that s is not a Nash equilibrium in (µ; t_p, ℓ') — a contradiction. □

PROOF OF THEOREMS 4.7 AND 4.8. Theorem 4.7(1) follows directly from Theorem 4.3. We move on to proving Theorem 4.8 and Theorem 4.7(2). Let p̄ be a mixed-strategy Nash equilibrium in (n, µ, ≥ C). By Corollary A.11, E[ℓ_j(p̄)] ≥ µ(T)/ₙ for every j ∈ Pₙ; since ∑ₖ=₀ⁿ−1 ℓ_j = µ(T), and by linearity of expectation, we obtain E[ℓ_j(p̄)] = µ(T)/ₙ for every j ∈ Pₙ. Let j ∈ Pₙ. By Lemma A.13(1), we have ℓ_j(p̄, 1, 1) ≥ ℓ_j(p̄). As j has no incentive to deviate from p_j to playing 0 ∈ T, we thus have that ℓ_j(p̄, 1, 1) = ℓ_j(p̄) with probability 1. By Corollary A.11, we have that ℓ_j(p̄, 1, 1) ≥ µ(T)/ₙ, and so ℓ_j(p̄) ≥ µ(T)/ₙ with probability 1. As E[ℓ_j(p̄)] = µ(T)/ₙ, we have that ℓ_j(p̄) = µ(T)/ₙ with probability 1.

Let now p̄ be a mixed-strategy profile in (n, µ, ≥ C), s.t. ℓ_j(p̄) = µ(T)/ₙ with probability 1 for every j ∈ Pₙ. By Theorem 4.4, the resulting realization is a pure-strategy Nash equilibrium with probability 1, and so with probability 1 no ex-post regret exists and a fortiori p̄ is a Nash equilibrium.

We move on to proving Theorem 4.7(3). Let p̄ be a mixed-strategy Nash equilibrium in (n, µ, ≥ C). We iteratively build a permutation π ∈ Pₙ! s.t. µ(0, Maxsupp(pₙπ(j))) ≤ 1/n for every j ∈ Pₙ. Let k ∈ Pₙ, and assume that π(j) has been defined for all 0 ≤ j < k. Define U ≜ Pₙ \ {π(0), ..., π(k−1)} and let tₘᵢᵣₖ be a random variable denoting the numerically smallest strategy realization of U, i.e. tₘᵢᵣₖ ≜ min_j∈U p_j. By Theorem 4.7(2), with probability 1 we have ∑ₖ∈U ℓ_j(p̄) = µ(T); therefore, with probability 1 we have µ((0, tₘᵢᵣₖ)) ≤ 1/n µ(T). Hence, by independence, it is not possible that P[µ((0, p_j))] > 

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For every \( j \in U \). Therefore, there exists \( \pi(k) \in U \) s.t. \( P[\mu([0, p_{\pi(k)}) \leq \frac{j}{n} \cdot \mu(T)] = 1 \), and so \( \mu([0, \text{Max supp}(p_{\pi(k)}))] \leq \frac{j}{n} \cdot \mu(T) \) and the construction is complete.

Let now \( \bar{\rho} \) be a mixed-strategy profile in \((n, \mu, \geq C)\) s.t. \( \mu([0, \text{Max supp}(p_{\pi(j)})]) \leq \frac{j}{n} \cdot \mu(T) \) for every \( j \in \mathbb{P}_n \). Therefore, \( \mu([0, p_{\pi(j)})] \leq \frac{j}{n} \cdot \mu(T) \) for every \( j \in \mathbb{P}_n \) with probability 1. By Theorem 4.5, the resulting realization is a pure-strategy Nash equilibrium with probability 1, and so with probability 1 no ex-post regret exists and a fortiori \( \bar{\rho} \) is a Nash equilibrium.

We conclude by proving Theorem 4.7(4). Let \( \bar{\rho} \) be a mixed-strategy Nash equilibrium in \((n, \mu, \geq C)\). By Theorems 4.6 and 4.8, a realization of \( \bar{\rho} \) is with probability 1 a super-strong equilibrium, and so a fortiori \( \bar{\rho} \) is a super-strong Nash equilibrium. \( \Box \)

A.3.2. Proofs and Auxiliary Results for Section 4.1.2

**Lemma A.16.** Let \( \bar{\iota} \in \mathcal{T}^{\mathbb{P}_n} \) be a pure-strategy profile, let \( h \in \arg\max_{j \in \mathbb{P}_n} t_j \) and let \( k \in \mathbb{P}_n \). If \( \bar{\iota} \) is not a Nash equilibrium in \((n, \mu, \geq C)\), but nonetheless \( t_k \) is a best response to \( \bar{\iota} - k \), then both \( \ell_h(\bar{\iota}) < \ell_k(\bar{\iota}) \) and \( \mu([t_k, t_h]) \geq \frac{\mu(T)}{n} \).

**Proof.** As \( t_k \) is a best response to \( \bar{\iota} - k \) and by Corollary A.11, \( \ell_k(\bar{\iota}) \geq \frac{\mu(T)}{n} \). As \( \bar{\iota} \) is not a Nash equilibrium, by Corollary A.15 we have \( \ell_h(\bar{\iota}) < \frac{\mu(T)}{n} \), and so \( \ell_h(\bar{\iota}) < \ell_k(\bar{\iota}) \). Therefore, in any mixed-consumption Nash equilibrium in \((\mu, \bar{\iota})\), no consumer with type \( d \geq t_h \) consumes a positive amount from producer \( k \), and so \( \mu([t_k, t_h]) \geq \ell_k(\bar{\iota}) \geq \frac{\mu(T)}{n} \), as required. \( \Box \)

**Definition A.17.** Let \( \bar{\iota} \in \mathcal{T}^{\mathbb{P}_n} \) be a pure-strategy profile in \((n, \mu, \geq C)\).

— For every \( q \in \{0, \ldots, n\} \), we define \( Q_q(\bar{\iota}) \triangleq \left\{ j \in \mathbb{P}_n \mid \mu([0, t_j]) \in \left( \frac{q-1}{n} \cdot \mu(T), \frac{q}{n} \cdot \mu(T) \right) \right\} \subseteq \mathbb{P}_n \).

— We define \( M(\bar{\iota}) \triangleq \max\{ q \in \{0, \ldots, n\} \mid Q_q(\bar{\iota}) \neq \emptyset \} \).

**Remark A.18.**

— \( (Q_q(\bar{\iota}))_{0 \leq q \leq n} \) is a partition of \( \mathbb{P}_n \).

— When the CDF of \( \mu \) is continuous and strictly increasing, then \( Q_0(\bar{\iota}) \) is the set of producers with strategy 0, while \( Q_q(\bar{\iota}) \) for \( 0 < q \leq n \) is the set of producers whose strategies lie in the \( q^{\text{th}} \) \( \frac{1}{n} \) of \( T \) (as measured by \( \mu \)), i.e. above the \( (q - 1) \) \( n \)-tile yet not above the \( q \) \( n \)-tile; for such a CDF, \( M(\bar{\iota}) \) is the index of the \( \frac{1}{n} \) of \( T \) containing the numerically largest strategy (or 0) if all strategies are 0), i.e. it is the index of the lowest \( n \)-tile above which no strategies lie.

**Lemma A.19.** Let \( \bar{\iota} \in \mathcal{T}^{\mathbb{P}_n} \) and let \( t'_k \) be a best response to \( \bar{\iota} - k \). If \( \bar{\iota} \) is not a Nash equilibrium, then \( \mu([0, t'_k]) \leq \frac{M(\bar{\iota}) - 1}{n} \cdot \mu(T) \).

**Proof.** Let \( h \in \arg\max_{j \in \mathbb{P}_n} t'_j \), where \( \bar{\iota} - k \triangleq \bar{\iota} - k \). We consider two cases. If \( (\bar{\iota} - k, t'_k) \) is not a Nash equilibrium, then by Lemma A.16 we have \( \mu([t'_k, t_h]) \geq \frac{\mu(T)}{n} \), and so \( \mu([0, t'_k]) \leq \mu([0, t_h]) - \frac{\mu(T)}{n} \leq \mu([0, t_h]) - \mu(T) \leq \frac{M(\bar{\iota}) - 1}{n} \cdot \mu(T) \), as required. Otherwise, \( (\bar{\iota} - k, t'_k) \) is a Nash equilibrium while \( \bar{\iota} \) is not. Therefore, by Theorem 4.5, there exists \( j \in \{0, \ldots, n - 1\} \) s.t. \( \mu([0, t'_k]) \leq \frac{\mu(T)}{n} \). As \( \mu([0, t_k]) \leq \frac{M(\bar{\iota})}{n} \cdot \mu(T) \), we thus have \( j < M(\bar{\iota}) \), and so \( \mu([0, t'_k]) \leq \frac{M(\bar{\iota}) - 1}{n} \cdot \mu(T) \), as required. \( \Box \)
Lemma A.20. Let $\delta > 0$, let $(\tilde{t}, P_i)_{i=0}^{\infty}$ be a $\delta$-better-response dynamic in $(n, \mu, \succeq_C)$ and let $i \in \mathbb{N}$. If $\tilde{t}$ is not a Nash equilibrium, then all of the following hold.

1. $M(\tilde{t}^{i+1}) \leq M(\tilde{t})$.
2. $Q_M(\tilde{t}^{i+1}) \subseteq Q_M(\tilde{t})$.
3. $\mu([t_j^{i+1}, t_j^i)) \geq \delta$, for every $j \in P_i \cap Q_M(\tilde{t}^{i+1})$.

Remark A.21. Finer analysis of similar nature may be used to show that $\delta$ may be replaced with $(n + 1 - M(\tilde{t})) \cdot \delta$ in Lemma A.20(3).

Proof of Lemma A.20. We commence by proving Part 1. Let $h \in \arg \max_{j \in P_n} t_j^i$. It is enough to show that $t_k^{i+1} < t_k^i$ for every $k \in P_i$. We consider two cases. If $t_k^i$ is not a best response to $\tilde{t}_{-k}$, then by definition of $\delta$-better-response dynamics and by Lemma A.13(1), $t_k^{i+1} < t_k^i$. Otherwise, $t_k^i$, and hence also $t_k^{i+1}$, are best responses to $\tilde{t}_{-k}$. Therefore, by Lemma A.16, $\ell_k(\tilde{t}) < \ell_k(\tilde{t}^{i+1})$; in particular, $k \neq h$. By anonymity and by Lemma A.8 (for $j = h$), we have $\ell_k(\tilde{t}_{-k}, t_k^i) = \ell_k(\tilde{t}_{-k}, t_k^{i+1}) \leq \ell_h(\tilde{t}) < \ell_k(\tilde{t}_{-k}, t_k^{i+1})$. Therefore, by Lemma A.7, $t_k^{i+1} < t_k^i$ in this case as well, as required.

We now proceed to prove Part 2. Let $k \in \mathbb{P}_n \setminus Q_M(\tilde{t})$; we must show that $k \notin Q_M(\tilde{t})$. It is enough to consider the scenario in which $k \in P_i$, and to show that under this condition, $\mu([0, t_k^{i+1}]) \leq M(\tilde{t}) \cdot \mu(T)$. Once again, we consider two cases. If $t_k^i$ is not a best response to $\tilde{t}_{-k}$, then by definition of $\delta$-better-response dynamics and by Lemma A.13(1), $t_k^{i+1} < t_k^i$ and so $\mu([0, t_k^{i+1}]) \leq M(\tilde{t}) \cdot \mu(T)$, as required. Otherwise, $t_k^i$, and hence also $t_k^{i+1}$, are best responses to $\tilde{t}_{-k}$. By Lemma A.19, in this case we have $\mu([0, t_k^{i+1}]) = M(\tilde{t}) \cdot \mu(T)$ as well, as required.

We conclude by proving Part 3. Let $k \in P_i \cap Q_M(\tilde{t})$. As $\mu([0, t_k^{i+1}]) > M(\tilde{t}) \cdot \mu(T)$, by Lemma A.19 we have that $\tilde{t}_{-k}$ is not a best response to $\tilde{t}_{-k}$. Therefore, by definition of $\delta$-better-response dynamics, we have that $\ell_k(\tilde{t}_{-k}, t_k^{i+1}) \geq \ell_k(\tilde{t}^{i+1}) + \delta$, and so by Lemmas A.7 and A.9, $\mu([t_k^{i+1}, t_k^i)) \geq \delta$, as required. □

Proof of Theorem 4.15. Let $\delta > 0$ and let $(\tilde{t}, P_i)_{i=0}^{\infty}$ be a $\delta$-better-response dynamic in $(n, \mu, \succeq_C)$. By definition, $M(\tilde{t}^{i+1}) \leq n$. By Lemma A.20, $M = M(\tilde{t})$ decreases by at least 1 in every $\lceil \frac{n(T)}{\delta n} \rceil$ rounds within which a Nash equilibrium is not reached. Therefore, if a Nash equilibrium is not reached in at most $n \cdot \lceil \frac{n(T)}{\delta n} \rceil$ rounds from 0, then $M(\tilde{t}) = 0$ and so $\tilde{t} \equiv 0$, which by Theorem 4.5 is a Nash equilibrium. If $\tilde{t}$ is a Nash equilibrium, then by Theorem 4.5, $M(\tilde{t}^{i+1}) \leq n - 1$, and so if a Nash equilibrium is not reached in at most $(n - 1) \cdot \lceil \frac{n(T)}{\delta n} \rceil$ rounds from $i + 1$, then we have $M = 0$ once more, and so a Nash equilibrium is reached again.

The tighter bounds described in Remark 4.16 may be shown in a similar manner, due to Remark A.21. □

Definition A.22 (k-Canonical Form). Let $k \in \mathbb{P}_n$ and let $\tilde{t} \in T^P_n$ be a pure-strategy Nash equilibrium in $(n, \mu, \succeq_C)$. We say that $\tilde{t}$ is in $k$-canonical form if all of the following hold.

1. $t_0 \leq t_1 \leq \cdots \leq t_{k-1} \leq t_{k+1} \leq \cdots \leq t_{n-1}$.
2. $\mu([0, t_j)) \leq \frac{k}{n} \cdot \mu(T)$ for every $j < k$.
3. Either $k = n - 1$, or $\mu([0, t_{k+1})) > \frac{k}{n} \cdot \mu(T)$. 

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By Theorem 4.5, we have that

\[ t_{\pi(0)} \leq t_{\pi(1)} \leq \cdots \leq t_{\pi(n-1)}. \]  

In particular, we have

\[ t_{\pi(0)} \leq t_{\pi(1)} \leq \cdots \leq t_{\pi^{-1}(k)} \leq t_{\pi^{-1}(k)+1} \leq \cdots \leq t_{\pi(n-1)}. \]  

(1)

By Theorem 4.5, we have that

\[ \mu([0, t_{\pi(j)}]) \leq \frac{j}{n} \cdot \mu(T) \]  

for every \( j \in \mathbb{P}_n \), and in particular for every \( j < \pi^{-1}(k) \). If \( \pi^{-1}(k) = n - 1 \) or \( \mu([0, t_{\pi^{-1}(k)+1})] > \frac{\pi^{-1}(k)}{n} \cdot \mu(T) \), then \( \pi \) is a permutation as required. Otherwise, we have

\[ \mu([0, t_{\pi^{-1}(k)+1})] \leq \frac{\pi^{-1}(k)}{n} \cdot \mu(T). \]  

(2)

In this case, we modify \( \pi \) to create a new permutation \( \pi' \in \mathbb{P}_n \) by incrementing \( \pi^{-1}(k) \), or more formally — by swapping the values of coordinates \( \pi^{-1}(k) \) and \( \pi^{-1}(k)+1 \) of \( \pi \). We note that Eq. (1) still holds w.r.t. \( \pi' \) (i.e. by substituting \( \pi' \) for \( \pi \)). By Eq. (2) w.r.t. \( \pi \) for all \( j < \pi^{-1}(k) \), we have that Eq. (2) holds w.r.t. \( \pi' \) for all \( j < \pi^{-1}(k) - 1 \); by Eq. (3) w.r.t. \( \pi' \), we have that Eq. (2) holds w.r.t. \( \pi' \) for \( j = \pi^{-1}(k) - 1 \) as well. Once again, if \( \pi^{-1}(k) = n - 1 \) or \( \mu([0, t_{\pi'((\pi^{-1}(k)+1)})] > \frac{\pi^{-1}(k)}{n} \cdot \mu(T) \), then \( \pi' \) is a permutation as required. Otherwise, Eq. (3) holds w.r.t. \( \pi' \), and we repeat the modification step. As \( \pi^{-1}(k) \) is incremented in each modification step, this process concludes in at most \( n - 1 \) steps, as it concludes if \( \pi^{-1}(k) \) reaches \( n - 1 \).

Lemma A.23. Let \( k \in \mathbb{P}_n \) and let \( \bar{t} \in \mathcal{T}_{\mathbb{P}_n} \) be a pure-strategy Nash equilibrium in \((n, \mu, \geq_C)\). There exists a permutation \( \pi \in \mathbb{P}_n \) s.t. \((t_{\pi(0)}, t_{\pi(1)}, \ldots, t_{\pi(n-1)}) \) is in \( \pi^{-1}(k) \) - canonical form.

Proof. We start by defining \( \pi \) such that \( t_{\pi(0)} \leq t_{\pi(1)} \leq \cdots \leq t_{\pi(n-1)} \). In particular, we have

\[ t_{\pi(0)} \leq t_{\pi(1)} \leq \cdots \leq t_{\pi^{-1}(k)} \leq t_{\pi^{-1}(k)+1} \leq \cdots \leq t_{\pi(n-1)}. \]  

Lemma A.24. Let \( k \in \mathbb{P}_n \) and let \( \bar{t} \in \mathcal{T}_{\mathbb{P}_n} \) be a pure-strategy Nash equilibrium in \((n, \mu, \geq_C)\) in \( k \)-canonical form. Both of the following hold.

1. \( \mu([0, t_j]) \leq \frac{j}{n} \) for every \( j \in \mathbb{P}_n \).
2. Either \( k = n - 1 \) or \( t_k < t_{k+1} \).
3. For every \( t \in \mathcal{T} \), \((\bar{t} - k, t)\) is a Nash equilibrium in \((n, \mu, \geq_C)\) iff \( \mu([0, t]) \leq \frac{k}{n} \).

Proof. By definition of \( k \)-canonical form, \( \mu([0, t_j]) \leq \frac{j}{n} \) for every \( j < k \). By definition of \( k \)-canonical form, we also have for every \( n > j > k \) that \( \mu([0, t_j]) \leq \mu([0, t_{k+1})] > \frac{k}{n} \cdot \mu(T) \). Therefore, by Theorem 4.5, \( \mu([0, t_j]) \leq \frac{k}{n} \cdot \mu(T) \) for every \( j \leq k \), and in particular \( \mu([0, t_k]) \leq \frac{k}{n} \cdot \mu(T) \). Furthermore, we obtain that \((t_{k+1}, t_{k+2}, \ldots, t_{n-1})\) are the \( n - k - 1 \) numerically largest strategies in \( \bar{t} \), and as they are sorted, by Theorem 4.5 we have that \( \mu([0, t_j]) \leq \frac{k}{n} \cdot \mu(T) \) for every \( j > k \) as well and the proof of Part 1 is complete.

Assume that \( k < n - 1 \). As we have shown that \( \mu([0, t_k]) \leq \frac{k}{n} \cdot \mu(T) \), but by definition of \( k \)-canonical form \( \mu([0, t_{k+1})] > \frac{k}{n} \cdot \mu(T) \), we have that \( t_k < t_{k+1} \) and Part 2 holds.

We conclude by proving Part 3; let \( t \in \mathcal{T} \). If \( \mu([0, t]) \leq \frac{k}{n} \), then by Part 1 and Theorem 4.5, \((\bar{t} - k, t)\) is a Nash equilibrium in \((n, \mu, \geq_C)\). Recall that \( \mu([0, t]) \leq \frac{k}{n} \cdot \mu(T) \) for every \( j > k \); therefore, if \( \mu([0, t]) \leq \frac{k}{n} \cdot \mu(T) \) as well, then by Theorem 4.5, \((\bar{t} - k, t)\) is not a Nash equilibrium in \((n, \mu, \geq_C)\).

Proof of Theorem 4.18. Let \( \bar{t} \in \mathcal{T}_{\mathbb{P}_n} \) be a Nash equilibrium in \((n, \mu, \geq_C)\) and let \( k \in \mathbb{P}_n \). By Lemma A.23, assume w.l.o.g. that \( \bar{t} \) is in \( k \)-canonical form. By Lemma A.24(3), it is enough to show that each \( t_k < t'_k \leq 1 \) s.t. \( \mu([0, t'_k]) \leq \frac{k}{n} \cdot \mu(T) \) is not a best response to \( \bar{t} - k \) in \((n, \mu, \geq_C)\). If \( k < n + 1 \), then by Lemma A.24(2),
Let \( s' \) and \( s'' \) be mixed-consumption Nash equilibria in the \( k \)-producer consumer game \((\mu\res{0,\tau_k'}, t_0, \ldots, t_{k-1})\) and in the (\( (n-k) \))-producer game \((\mu\res{0,\tau_n'}, t'_k, t_{k+1}, \ldots, t_{n-1})\), respectively, by abuse of notation, we think of \( s'' \) as \( s'' = (s''_k, s''_k, \ldots, s''_{n-1}) \) (\( s'' \equiv 0 \) by definition of \( s'' \)) and for each \( k \leq j < n \) define \( \ell''_j \equiv \int_T s''_j d(\mu\res{0,\tau_n'}(T)) \). For every \( 0 \leq j < k \), we have by definition of \( k \)-canonical form that

\[
\mu([0, t_j]) \leq \frac{j}{n} \cdot \mu(T) = \frac{j}{k} \cdot \mu(T) < \frac{j}{k} \cdot \mu([0, t'_k]) = \frac{j}{k} \cdot \mu\res{0,\tau_n'}(T).
\]

By Theorem 4.5, \((t_0, \ldots, t_{k-1})\) is therefore a Nash equilibrium in \((k, \mu\res{0,\tau_n'}, \geq c)\), and so by Theorem 4.4 we have \( \ell''_j = \frac{\mu\res{0,\tau_n'}(T)}{n-k} = \frac{\mu(\mu\res{0,\tau_n'}(T))}{n-k} \geq \frac{\mu(T)}{n} \) for every \( 0 \leq j < k \).

For every \( k < j < n \), we have by Lemma A.24(1) that

\[
\mu((t_j, 1)) \geq \frac{n-j}{n} \cdot \mu(T) > \frac{n-j}{n} \cdot \mu([t'_k, 1]) = \frac{n-j}{n} \cdot \mu([t'_k, 1]),
\]

and therefore

\[
\mu((t'_k, t_j)) = \mu([t'_k, 1]) - \mu([t_j, 1]) - \mu([t'_k, 1]) - \frac{n-j}{n-k} \cdot \mu([t'_k, 1]) = \frac{j-k}{n-k} \cdot \mu\res{0,\tau_n'}(T).
\]

Note that \( \mu((t'_k, t'_k)) = 0 = \frac{k-k}{n-k} \cdot \mu\res{0,\tau_n'}(T) \) trivially holds as well. By Theorem 4.5, \((t'_k, t_{k+1}, t_{k+2}, \ldots, t_{n-1})\) is therefore a Nash equilibrium in \((n-k, \mu\res{0,\tau_n'}, \geq c)\), and so by Theorem 4.4 we have that \( \ell''_j = \frac{\mu\res{0,\tau_n'}(T)}{n-k} = \frac{\mu(\mu\res{0,\tau_n'}(T))}{n-k} \leq \frac{\mu(T)}{n} \) for every \( k \leq j < n \).

Let \( s \) be the mixed-consumption profile defined by \( s_j\res{0,\tau_n'} = s''_j \) for every \( j \in \{-n, n, \ldots, n-1\} \) and \( s_j\res{0,\tau_n'} \equiv 0 \) for every \( k \leq j < n \), and by \( s_j\res{0,\tau_n'} = s''_j \) for every \( k \leq j < n \) and \( s_j\res{0,\tau_n'} \equiv 0 \) for every \( j \in \{-n, n, \ldots, n-1\} \). By definition of \( s' \) and of \( s'' \), we have that \( s \) is a legal mixed-consumption profile in \((\mu; \bar{t}_k, t'_k)\), and furthermore, that \( \ell''_j = \frac{\mu(\mu\res{0,\tau_n'}(T))}{n-k} \) for every \( j \in \{-n, n, \ldots, n-1\} \), and that \( \ell''_j = \frac{\mu(\mu\res{0,\tau_n'}(T))}{n-k} \) for every \( k \leq j < n \). By the former, and as \( s' \) is a Nash equilibrium, no type \( d \in [0, t'_k) \) has any incentive to deviate from \( s \) in \((\mu; \bar{t}_k, t'_k)\), and by the latter, as \( s'' \) is a Nash equilibrium and as \( \frac{\mu(\mu\res{0,\tau_n'}(T))}{n-k} < \frac{\mu(T)}{n} < \frac{\mu(\mu\res{0,\tau_n'}(T))}{k} \), we have that neither does any type \( d \in [0, t'_k) \). Therefore, \( s \) is a Nash equilibrium in \((\mu; \bar{t}_k, t'_k)\). As \( \ell_k(\bar{t}_k, t'_k) = \ell'_k = \frac{\mu(\mu\res{0,\tau_n'}(T))}{n-k} \), (with the last equality by Theorem 4.4, since \( \bar{t} \) is a Nash equilibrium in \((n, \mu, \geq c)\), we have that producer \( k \) strictly prefers \( t_k \) over \( t_k' \) given \( \bar{t}_k \), and so \( t'_k \) is not a best response to \( \bar{t}_k \) in \((n, \mu, \geq c)\), as required.

We note that an alternative proof may also be given via Algorithm 1 / Corollary A.6. □

**Proof of Corollary 4.19.** A direct corollary of Theorems 4.15 and 4.18 and Remark 4.17. □

**Proof of Proposition 4.20.** It is enough to show that some nonequilibrium can be reached in a finite number of steps from any Nash equilibrium. Let \( t \in \mathcal{T} \) s.t. \( 0 < \mu(0, t) \leq \frac{n-1}{n} \cdot \mu(T) \) (there must exist such \( t \) by definition of \( \mu \)) and let \( j \in \{1, \ldots, n-1\} \) be minimal s.t. \( \mu(0, t) \leq \frac{j}{n} \cdot \mu(T) \).

By Theorem 4.3, \( 0 \in \mathcal{T} \) is a best-response by any producer to any Nash equilibrium, and so a (nonlazy) best-response dynamic can reach \((0, 0, \ldots, 0)\) from any Nash equi-
librium in one round. Let \( i \in \mathbb{N} \) be more than one round into the future after reaching 
\((0, 0, \ldots, 0)\), s.t. \(|P_i| > 1\), and let \( k, h \in P_i \) s.t. \( k \neq h \). By definition of \( t \) and \( j \) and by 
Theorem 4.5, any strategy profile in which at most \( n - j \) producers play \( t \) and the rest 
play \( 0 \) is a Nash equilibrium. Therefore, within one round after reaching \((0, 0, \ldots, 0)\), a 
Nash equilibrium in which \( j + 1 \) producers, including \( k \) and \( h \), play \( 0 \) and the rest play 
\( t \), can be reached, and can be lazily maintained until the step \( i \). In step \( i \), all triggered 
producers may play \( t \in \mathcal{T} \), which is a best response for each of them since at least \( j 
\) producers playing \( 0 \) and the rest playing \( t \) is a Nash equilibrium. Therefore, and as \( k 
\) and \( h \) both switch from playing \( 0 \) to playing \( t \) at \( i \), at least \( n - j + 1 \) producers play \( t \) at 
\( i + 1 \), which, by definition of \( j \) and by Theorem 4.5, is a nonequilibrium. 

\[ \text{Lemma A.25.} \text{ Let } k \in \mathbb{P}_n, \text{ let } \tilde{t} - k \in \mathcal{T}^{n \setminus \{k\}}. \text{ If } \mu((0, t_j)) > 0 \text{ for all } j \in \mathbb{P}_n \setminus \{k\}, \text{ then } 
\tilde{t}'_k \in \mathcal{T} \text{ is a best response (by } k \text{) to } \tilde{t} - k \text{ in } (n, \mu, \succeq_C) \iff \mu((0, t'_k)) = 0. \]

**Proof.** By Lemma A.13, all strategies \( t \in \mathcal{T} \) for which \( \mu(0, t) = 0 \) are equivalent. 
As in particular, \( t = 0 \in \mathcal{T} \) since a strategy, it is therefore enough to show that \( k \) 
strictly prefers to play \( 0 \in \mathcal{T} \) over any \( t'_k \) s.t. \( \mu(0, t'_k) > 0 \). By Lemma A.13(1), it is 
possible to consider the case in which \( t'_k \leq t_j \) for all \( j \in \mathbb{P}_n \setminus \{k\} \). By Algorithm 1 (for 
\( t_n \) as defined there), \( \ell_k(\tilde{t} - k, 0) = \max_{0 < j \leq n} \frac{\mu((0, t'_j))}{j} > \max_{0 < j \leq n} \frac{\mu((0, t'_k))}{j} = \ell_k(\tilde{t} - k, t'_k) \), as 
required. □

**Proof of Theorem 4.22.** As in the proof of Theorem 4.15, and as a best-response 
dynamic is also a \( \mu(\mathcal{T}) \)-better-response dynamic, we have that \( M(\tilde{t}') \leq n \) and \( M \leq 
n - 1 \) on every step immediately following a Nash equilibrium, and that \( M \) decreases 
by at least 1 every round if a Nash equilibrium is not reached. Let \( i \in \mathbb{N} \) s.t. \( M(\tilde{t}^i) = 1, 
M(\tilde{t}^{i - 1}) = 2, \text{ and } \tilde{t}^{i - 1} \text{ is not a Nash equilibrium; it is enough to show that } \tilde{t} \text{ is a Nash } 
eq (0, t'_j)) = 0. As \( M(\tilde{t}'j) \neq M(\tilde{t}^{i - 1}) \), we have \( P_{i - 1} \neq \emptyset \). By Theorem 4.5, as \( M(\tilde{t}^{i - 1}) = 2 \) 
but \( \tilde{t}^{i - 1} \) is not a Nash equilibrium, there exists at most one producer \( k \in \mathbb{P}_n \) s.t. 
\( \mu((0, t'_k)) = 0 \). If there exists no such producer, then by Lemma A.25, we have 
\( \mu((0, t'_k)) = 0 \) for every \( j \in \mathbb{P}_n \), and as \( P_i \neq \emptyset \), the proof is complete. Otherwise, there 
exists a unique \( k \in \mathbb{P}_n \) s.t. \( \mu((0, t'_k)) = 0 \). If \( k \notin P_i \), then \( t'_k = t'_k^{-1} \) and the proof is 
complete. Otherwise, \( k \in P_i \) and by Lemma A.25, we have \( \mu((0, t'_k)) = 0 \), as required. □

**Proof of Corollary 4.25.** A direct corollary of Theorems 4.18 and 4.22, 
Remark 4.17, and Example 4.24. □

### A.4. Proofs and Auxiliary Results for Section 4.2

**A.4.1. Proofs and Auxiliary Results for Section 4.2.1**

**Lemma A.26 (Domination).** Let \( t \neq t' \in \mathcal{T} \) be strategies in 
\((n, \mu, \succeq_F)\), \( t \) is a safe 
alternative to \( t' \) iff either of the following hold. In either case, \( t \) strongly dominates \( t' \)

1. \( t > t' \) and \( \mu([t', t]) = 0. \)
2. \( \mu([t', 1]) < \frac{\mu([0, 1])}{n}. \)

**Proof.** \( t \) is a safe alternative to (alternatively, strongly dominates) \( t' \) iff either \( t \) 
always produces greater load than \( t' \), or \( t \) always produces at least as much load as \( t' \) and in addition \( t > t' \). By Lemma A.13(3), the former occurs iff \( \mu([t', 1]) < \frac{\mu([0, 1])}{n} \); by 
Lemma A.13(1,2), if \( t > t' \), then the latter occurs iff \( \mu([t', t]) = 0 \) □

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Lemma A.27. \(\mu\left(\bigcup\{[t,t'] \mid 0 \leq t < t' \& \mu([t,t']) \leq m\}\right) \leq m\), for every \(t' \in T\) and \(m \in \mathbb{R}_{\geq}\).

Proof. Define \(U \triangleq \bigcup\{[t,t'] \mid 0 \leq t < t' \& \mu([t,t']) \leq m\}\). If \(U = \emptyset\), then \(\mu(U) = 0 \leq m\) and the proof is complete; assume, therefore, that \(U \neq \emptyset\) and let \(u \triangleq \inf U \geq 0\). By definition, \(U\) is connected, and therefore by definition of \(U\) and \(u\), we have that either \(U = [u, t')\) or \(U = (u, t')\). If \(U = [u, t')\), then \(u \in U\), and by definition of \(U\), we obtain \(\mu(U) = \mu([u, t')) \leq m\), as required; assume therefore, that \(U = (u, t')\). In this case, \(U = \bigcup\{[t,t'] \mid t \in [0, t') \cap \mathbb{Q} \& \mu([t,t']) \leq m\}\), and by continuity of \(\mu\) from below, we obtain \(\mu(U) \leq m\), as required. \(\square\)

Proof of Proposition 4.27. Let \(t \in T\) be a dominant strategy in this game. By Lemma A.26, both \(\mu(\{0, t\}) = 0\) (as \(t\) is a safe alternative to 0) and \(\mu([t', 1]) < \frac{n\mu([t, 1])}{n}\) for every \(t' > t\) (as \(t\) is a safe alternative to every such \(t'\)). (Alternatively, the former holds as by definition, any strategy dominant w.r.t. fine preferences is also dominant w.r.t. coarse preferences, and by Theorem 4.3.) By the former, \(\mu([t, 1]) = \mu(T)\), and therefore and by the latter, \(\mu([t', 1]) < \frac{n\mu(T)}{n}\) for every \(t' \in (t, 1) \cap \mathbb{Q}\). Therefore, by continuity of \(\mu\) from below, \(\mu([t, 1]) \leq \frac{n\mu(T)}{n}\). Hence, \(\mu([t]) = \mu([0, t]) \geq \frac{n-1}{n} \cdot \mu(T)\). (Conversely, by Lemma A.26, it is easy to see that if there indeed exists \(t \in T\) s.t. \(\mu([0, t]) = 0\) and \(\mu([t', 1]) < \frac{n\mu(T)}{n}\) for every \(t' > t\), then it constitutes the unique dominant strategy.)

For the second statement, we note that by Lemma A.26, the set of dominated strategies is

\[
\left\{ t \in T \mid \mu([t, 1]) < \frac{n\mu(T)}{n} \right\} \cup \left\{ t \in T \mid \exists t < t' \leq 1 : \mu([t, t']) = 0 \right\} = \bigcup\{[t, 1] \mid t \in T \& \mu([t, 1]) < \frac{n\mu(T)}{n}\} \cup \bigcup\{[t,t'] \mid t \in T \cap \mathbb{Q} \& 0 \leq t' < t \& \mu([t', t']) = 0\}.
\]

By \(\sigma\)-additivity of \(\mu\) and by Lemma A.27 (applied twice),

\[
\mu\left(\left\{ t \in T \mid \mu([t, 1]) < \frac{n\mu(T)}{n} \right\} \cup \left\{ t \in T \mid \exists t < t' \leq 1 : \mu([t, t']) = 0 \right\}\right) \leq \mu\left(\bigcup\{[t, 1] \mid t \in T \& \mu([t, 1]) < \frac{n\mu(T)}{n}\}\right) + \sum_{\nu \in T \cap \mathbb{Q}} \mu\left(\bigcup\{[t,t'] \mid 0 \leq t < t' \& \mu([t, t']) = 0\}\right) \leq \frac{\mu(T)}{n} + 0 = \frac{\mu(T)}{n}.
\]

We note that this bound is attained if \(\mu\) is atomless, as in this case it is straightforward to verify that \(\mu(\bigcup\{[t, 1] \mid t \in T \& \mu([t, 1]) < \frac{n\mu(T)}{n}\}) = \frac{n\mu(T)}{n}\).

The second statement leads to an extremely concise, yet somewhat more obscure, proof for the first one. If a dominant strategy exists, then it is a safe alternative to all other strategies; in particular, all other strategies have safe alternatives (other than themselves). By the second statement, at least \(\frac{n-1}{n}\) of \(\mu\) is therefore concentrated on this dominated strategy, and the proof is complete. \(\square\)

Lemma A.28. \(\{ t \in T \mid \mu([t', t]) \leq m\}\) attains a maximum value for every \(t' \in T\) and \(m \in \mathbb{R}_{\geq}\).

Proof. Denote \(S \triangleq \{ t \in T \mid \mu([t', t]) \leq m\}\). We note that \(t' \in S\). If \(t' = \sup S\),
then \( t' = \text{Max} S \) and the proof is complete. Assume, therefore, that \( t' < \sup S \). Let 
\[ U = \bigcup \{ (t', t) | t \in T \ \& \ \mu(t', t) \leq m \} \]
As \( t' < \sup S \), we have that \( U \neq \emptyset \). Let 
\[ u = \sup U \leq 1 \]. By definition of \( U \) and of \( u \), we have \( U = [t', u] \). By definition of \( U \) and of \( S \), we have that \( u = \sup U = \sup S \), and so it is enough to show that \( u \in S \) i.e. that \( \mu([t', u]) \leq m \). As \( U = [t', u] \), this is equivalent to showing that \( \mu(U) \leq m \). Observe that
\[ U = \bigcup \{ (t', t) | t \in T \cap \mathbb{Q} \ \& \ \mu([t', t]) \leq m \} \].
By continuity of \( \mu \) from below, we thus obtain \( \mu(U) \leq m \), as required.

**Proof of Theorem 4.28.** For every \( j \in \mathbb{P}_n \), let \( t_j = \text{Max} \{ t \in T | \mu([0, t]) \leq \frac{j}{n} \cdot \mu(T) \} \). \( t_j \) is well-defined by Lemma A.28. By Theorem 4.5, \( t \) is a Nash equilibrium in \((n, \mu, \succeq_C)\), and so by Theorem 4.4, \( \ell_j(i) = \frac{\mu(T)}{\mu_n} \) for every \( j \in \mathbb{P}_n \). We now first show that no Nash equilibrium other than \( t \) (up to permutations) exists in \((n, \mu, \succeq_F)\), and then show that \( t \) (and hence all permutations thereof) is a super-strong Nash equilibrium in \((n, \mu, \succeq_F)\).

Let \( t_0' \leq \cdots \leq t_{n-1}' \in T \) s.t. \( t' \) is a Nash equilibrium in \((n, \mu, \succeq_F)\). We will show that \( t_j' = t_j \) for every \( j \in \mathbb{P}_n \). By definition of coarse and fine preferences, \( t' \) is also a Nash equilibrium in \((n, \mu, \succeq_C)\). Therefore, by Theorem 4.5 we have that \( \mu([0, t_j']) \leq \frac{j}{n} \cdot \mu(T) \) for every \( j \in \mathbb{P}_n \). Hence we have for every \( j \in \mathbb{P}_n \) both that \( t_j' \leq t_j \) and (by Theorem 4.5 again) that \( (t_j', t_j) \) is a Nash equilibrium in \((n, \mu, \succeq_C)\) as well. Therefore, by Theorem 4.4, \( \ell_j(t') = \frac{\mu(T)}{\mu_n} = \ell_j(t_{j-1}', t_j) \). As \( t' \) is a Nash equilibrium in \((n, \mu, \succeq_F)\), we therefore have that \( t_j' \geq t_j \), and so \( t_j' = t_j \), as required.

We now show that \( t \) is a super-strong Nash equilibrium in \((n, \mu, \succeq_F)\). Assume for contradiction that there exists a coalition \( P \subseteq \mathbb{P}_n \) and strategies \( \bar{t}' = (t_j')_{j \in P} \in T^P \) s.t. \( j \) weakly prefers \((t_{-j}, \bar{t}') \) over \( t \) w.r.t. fine preferences for every \( j \in \mathbb{P}_n \), with a strict preference for at least one producer \( j \in P \). For every \( j \in \mathbb{P}_n \), as \( j \) weakly prefers \((t_{-j}, \bar{t}') \) over \( t \), we have that \( \ell_j(t_{-j}, \bar{t}') \geq \ell_j(t) \). As \( t \) is a Nash equilibrium in \((n, \mu, \succeq_C)\), by Theorem 4.6, we therefore have \( \ell_j(t_{-j}, \bar{t}') = \ell_j(t) \) for every \( j \in \mathbb{P}_n \). Therefore, by definition of \( P \) and \( \bar{t}' \), we have \( t_j' \geq t_j \) for every \( j \in P \), with a strict inequality for at least one producer \( j \in P \) — let \( j \) be such a producer for which \( t_j' \) is greatest. Assume w.l.o.g. that either \( j = n - 1 \) or \( t_j < t_{j+1} \); therefore, \( t \) is in \( j \)-canonical form. As \( t \) is also a Nash equilibrium in \((n, \mu, \succeq_C)\), by Theorem 4.18, by Lemma A.24(3), by definition of \( t_j \), and as \( t_j' > t_j \), we have \( \ell_j(t) > \ell_j(t_{j-1}, t_j') \). By Lemma A.8 and by definition of \( j \), we have \( \ell_j(t_{j-1}, t_j') \geq \ell_j(t_{j-1}, \bar{t}') \). Therefore, \( \ell_j(t) > \ell_j(t_{j-1}, t_j') \geq \ell_j(t_{-j}, \bar{t}') \) — a contradiction.

**Proof of Corollary 4.29.** Since the CDF of \( \mu \) is continuous and strictly increasing, for every \( j \in \mathbb{P}_n \) there exists a unique strategy \( t_j \in T \) s.t. \( \mu([0, t_j]) = \frac{j}{n} \cdot \mu(T) \); hence, \( t_j = \text{Max} \{ t \in T | \mu([0, t]) \leq \frac{j}{n} \cdot \mu(T) \} \) and by Theorem 4.28 the proof is complete.

**Proof of Proposition 4.30.** Direct from definition of \((n, \mu, \succeq_F)\), as no player is ever indifferent between any two strategies, regardless of the information such player possesses regarding the strategies of the other players.

**Proof of Theorem 4.31.** By Theorem 4.28, for every \( j \in \mathbb{P}_n \), we have 
\[ \{ d \in T | \mu([0, d]) \in \left( \frac{j}{n}, \frac{j+1}{n}, \mu(T) \right) \} \subseteq [t_j, t_{j+1}] \].
Therefore, it is enough to show that \( s_j(d) = 1 \) for almost all \( d \in [t_j, t_{j+1}] \). By definition of \( s \), this is equivalent to showing that \( \int_{[t_j, t_{j+1}]} s_j d\mu = \mu([t_j, t_{j+1}]) \) for every \( j \in \mathbb{P}_n \), where \( t_n = 1 \). As \( \mu \) is atomless, by Theorem 4.28 and by definition of \( t_n \), we have \( \mu([t_j, t_{j+1}]) = \int_{[t_j, t_{j+1}]} s_j d\mu \).
from the definition of fine preferences. That this maximum value is a best response as required follows directly.

By Theorem 4.28 and definition of \( \ell_j(t) \) and of \( \ell^*_j \),

\[
\frac{\mu(T)}{n} = \ell_j(t) = \ell^*_j = \int_T s_j d\mu \geq \int_{[t_j, t_{j+1}]} s_j d\mu + \int_{[t_k, t_{k+1}]} s_j d\mu = \frac{\mu(T)}{n} + \int_{[t_k, t_{k+1}]} s_j d\mu,
\]

and so \( \int_{[t_k, t_{k+1}]} s_j d\mu = 0 \). By definition of Nash equilibrium, \( s_d(d) = 0 \) for every \( d \geq t_k \), and therefore \( \int_{[t_k, t_{k+1}]} s_d d\mu = 0 \) as well. Let \( S_k = \{ -n, 0, 1, 2, \ldots, k - 1 \} \); by definition of \( s \), we have that \( s(d) \in \Delta S_k \) for every \( d \in [t_k, t_{k+1}) \). Therefore,

\[
\int_{[t_k, t_{k+1}]} s_k d\mu = \mu([t_k, t_{k+1}]) = \sum_{j \in S_k} \int_{[t_k, t_{k+1}]} s_j d\mu = \frac{\mu(T)}{n} - 0 = \mu([t_k, t_{k+1}]),
\]

and the proof by induction is complete. \( \square \)

A.4.2. Proofs for Section 4.2.2

Proof of Proposition 4.34 (Nonconstructive). Let \( \tilde{\ell} \triangleq \ell_j(\tilde{f} - j, 0) \); by Theorem 4.3, \( \tilde{\ell} \) is the maximum load attainable by \( j \) given \( \tilde{f} - j \). Define \( S \triangleq \{ \mu([0, t]) \mid t \in T \} \& \ell_j(\tilde{f} - j, t) = \tilde{\ell} \). Observe that \( S \neq \emptyset \) as \( 0 \in S \) (given by \( t = 0 \)); let \( m \triangleq \sup S \). By Lemma A.13(1), every \( t \in T \) s.t. \( \mu([0, t]) < m \) maximizes \( \ell_j(\tilde{f} - j, t) \), while every \( t \in T \) s.t. \( \mu([0, t]) > m \) does not. Assume for contradiction that there exists \( t \in T \) s.t. \( \mu([0, t]) = m \) and \( \ell_j(\tilde{f} - j, t) < \tilde{\ell} \). Let \( \varepsilon \triangleq \tilde{\ell} - \ell_j(\tilde{f} - j, t) > 0 \). By definition of \( m \), there exists \( t' \) s.t. \( m \geq \mu([0, t']) > m - \varepsilon \) and \( \ell_j(\tilde{f} - j, t') = \tilde{\ell} = \ell_j(\tilde{f} - j, t) + \varepsilon \); by Lemma A.9, this is a contradiction. (We note that we have not shown (yet) that there exists \( t \in T \) s.t. \( \mu([0, t]) = m \), but rather that every such \( t \) maximizes the load on \( j \).) Therefore, we have that the set of load-maximizing strategies for \( j \) is precisely \( \{ t \in T \mid \mu([0, t]) \leq m \} \). By Lemma A.28, this set attains a maximum value. As by definition we have that a best response in \( (n, \mu, \succeq_F) \) is a numerically largest load-maximizing response, we obtain that this maximum value is a best response as required. Uniqueness follows directly from definition of fine preferences. \( \square \)

Before constructively proving Proposition 4.34, we first constructively prove it for two special cases.

Corollary A.29 (Proposition 4.34 — Special Case: Large Strategies). Let \( k \in \mathbb{P}_n \), and let \( \tilde{f} - k \in T^\setminus\{k\} \). If \( \mu([0, t_j]) > 0 \) for all \( j \in \mathbb{P}_n \setminus \{k\} \), then the unique best response (by \( k \)) to \( \tilde{f} - k \) in \( (n, \mu, \succeq_F) \) is \( \max \{ t \in T \mid \mu([0, t]) = 0 \} \).

Proof. A direct corollary of Lemma A.25, as a best response in \( (n, \mu, \succeq_F) \) is a numerically largest load-maximizing response; the specified strategy is well defined by Lemma A.28. \( \square \)

Lemma A.30 (Proposition 4.34 — Special Case: Coarse Equilibrium). Let \( \tilde{f} \in T^\mathbb{P}_n \) be a Nash equilibrium in \( (n, \mu, \succeq_C) \). For every \( j \in \mathbb{P}_n \), a best response (by \( j \)) to \( \tilde{f} - j \) exists in \( (n, \mu, \succeq_F) \).

Proof (Constructive). By Lemma A.23, assume w.l.o.g. that \( \tilde{f} \) is in \( j \)-canonical form. We will show that a best response as required is given by \( t'_j \triangleq \max \{ t \in T \mid \)
\( \mu((0, t)) \leq \frac{1}{k} \cdot \mu(T) \). \( t'_j \) is well defined by Lemma A.28.) By Lemma A.24(3) and by Theorem 4.18, a strategy \( t \in T \) maximizes \( \ell_j(T, t) \) iff \( \mu((0, t)) \leq \frac{1}{k} \cdot \mu(T) \). As by definition we have that a best response in \((n, \mu, \geq_F)\) is a numerically largest load-maximizing response, we obtain that \( t'_j \) is a best response as required. \( \square \)

**Proof of Proposition 4.34 (Constructive).** W.l.o.g. we prove the result for \( j = 0 \). Assume w.l.o.g. that \( t_1 \leq t_2 \leq \cdots \leq t_{n-1} \). Uniqueness follows directly from definition of fine preferences; it is therefore enough to show that a best response exists. If \( \mu((0, t_{j+1})) > 0 \), then by Corollary A.29 a best response exists as required. Assume therefore henceforth that \( \mu((0, t_{j+1})) = 0 \). If \((\bar{t}_0, 0)\) is a Nash equilibrium in \((n, \mu, \geq_C)\), then by Lemma A.30 a best response exists as required. Assume therefore henceforth that \((\bar{t}_0, 0)\) is not a Nash equilibrium in \((n, \mu, \geq_C)\).

Let \( k \in \mathbb{P}_n \) be minimal s.t. \( \ell_k(\bar{t}_0, 0) < \ell_0(\bar{t}_0, 0) \). (Such \( k \) exists by Corollary A.15, since \((\bar{t}_0, 0)\) is not a Nash equilibrium in \((n, \mu, \geq_C)\); by definition, \( k > 0 \).) By definition of \( k \) and by Corollary A.14 and Lemma A.14. \((0, t_1, t_2, \ldots, t_{k-1})\) is a Nash equilibrium in \((k, \mu|_{[0, t_{k-1})}, \geq_C)\). By Lemma A.30, there exists a best response \( t_0 \in T \) to \((t_1, t_2, \ldots, t_{k-1})\) in \((k, \mu|_{[0, t_{k-1})}, \geq_F)\). We claim that \( t_0 \) is a best response to \((\bar{t}_0, 0)\) in \((n, \mu, \geq_F)\) as well.

As \( \ell_k(\bar{t}_0, 0) < \ell_0(\bar{t}_0, 0) \), we have that \( \ell_0(\bar{t}_0, 0) \leq \mu((0, t_k)) \), and in particular \( \mu((0, t_k)) > 0 \). Therefore, by Lemma A.13(3) and by definition of \( t_0 \), we obtain \( t_0 < t_k \). By Theorem 4.18, \((t_0, t_1, \ldots, t_k)\) is also a Nash equilibrium in \((k, \mu|_{[0, t_{k-1})}, \geq_C)\). Therefore, by Theorem 4.4, \( \ell_j(\mu|_{[0, t_k]}), 0, t_1, t_2, \ldots, t_{k-1}) = \mu((0, t_k)) = \ell_j(\mu|_{[0, t_k]}), 0, t_1, \ldots, t_{k-1}) \) for every \( j \in \mathbb{P}_k \). By the construction in the proof of Theorem 3.10 and as \( t_0 < t_k \), therefore \( \ell_j(\bar{t}) = \ell_j(\bar{t}, 0) \) for every \( j \in \mathbb{P}_n \). By Corollary A.4 and by definition of \( k \), we have \( \ell_j(\bar{t}) = \ell_j(\bar{t}, 0) = \ell_j(\mu|_{[0, t_k]}), 0, t_1, \ldots, t_{k-1}) = \mu((0, t_k)) \) for every \( j \in \mathbb{P}_k \). As \( \ell_0(\bar{t}) = \ell_0(\bar{t}_0, 0) \), by Theorem 4.3 we have that \( t_0 \) maximizes the load on producer 0 in \((n, \mu, \geq_F)\).

Let \( h \in \mathbb{P}_k \) s.t. \( h < t_0 < t_{h+1} \). Such \( h > 0 \) exists as \( \mu((0, t_1)) \leq 0 \). and since \( t_0 \geq \max\{ t \in T \mid \mu((0, t)) = 0 \} \geq t_1 \) by Lemma A.13 (this maximum value is attained by Lemma A.28), and \( h < k \) since \( t_0 < t_k \). It remains to show that every \( t'_0 \in (0, 1) \) does not maximize the load on producer 0 in \((n, \mu, \geq_F)\). By Lemma A.13(1), it is enough to consider the case \( t_0 < t'_0 < t_{h+1} \). Note that \( t_k > t_0 \geq t_1 \) and so \( k > 1 \).

By definition of \( t'_0 \) and \( t_0 \), we have that \( t'_0 \) does not maximize the load on producer 0 in \((k, \mu|_{[0, t_{k-1})}, \geq_F)\). By Corollary A.4 and by Corollary A.15 (since \( \mu((0, t_{j+1})) = 0 \)), \( \ell_1(\bar{t}_0, t'_0) \geq \ell_1(\bar{t}_0, t_k) = \mu((0, t_k)) \) and so \( \ell_j(\bar{t}_0, t'_0) = \mu((0, t_k)) \) for every \( 0 < j < h \), and in particular \( j = h \). Therefore, and by Lemma A.13(1), \( \ell_h(\bar{t}_0, t'_0) = \ell_1(\bar{t}_0, t'_0) > \ell_1(\bar{t}_0, t'_0) = \ell_0(\bar{t}_0, t'_0) \). As \( \mu((0, t'_0)) = \mu((0, t_k)) \), we have \( \mu((0, t'_0)) = \mu((0, t_k)) \) for every \( j > h \). As \( \ell_h(\bar{t}_0, t'_0) > \ell_0(\bar{t}_0, t'_0) \), by definition of \( h \) and by Algorithm 1 and Corollary A.6, we obtain (for \( t_0 \) as defined there) that \( \ell_0(\bar{t}_0, t'_0) = \max_{h \leq j \leq n} \mu((0, t'_0)) h^{-1} \cdot \frac{1}{j} = \max_{h \leq j \leq n} \left( \frac{\mu((0, t_j))}{j} \cdot \frac{1}{j} \right) = \ell_0(\bar{t}, 0) \), and the proof is complete. \( \square \)

**Proof of Corollary 4.35.** A direct corollary of Theorems 4.15 and 4.22, as any weakly-\( \delta \)-better/best-response dynamic w.r.t. fine preferences is also a weakly-\( \delta \)-better/best-response dynamic w.r.t. coarse preferences. \( \square \)

**Proof of Theorem 4.36.** Let \( (P_j, F_j)_{j=0}^\infty \) be a sequential \( \delta \)-better-response dynamic in \((n, \mu, \geq_F)\) s.t. \( F^0 \) is a Nash equilibrium w.r.t. \((n, \mu, \geq_C)\). Let \( k \in \mathbb{P}_n \) and let
\[ \tilde{t}_i^1 \] be minimal s.t. \( k \in P_i \). It is enough to show that \( t_i^1 \) is constant for \( i > \tilde{i} \), as this implies that after one round \( \tilde{t} \) is constant regardless of \( P_i \), and is thus a Nash equilibrium in \((n, \mu, \succeq_f)\).

By Theorem 4.18, \( \tilde{t} \) is a Nash equilibrium w.r.t. \((n, \mu, \succeq_c)\) for every \( i \in \mathbb{N} \); therefore, by Theorem 4.4, the loads on all producers are constant throughout this dynamic. Therefore, by definition of \( \delta\)-better-response dynamics, we have both that \( (\tilde{t}^0, P_i)_{i=0}^\infty \) is a best-response dynamic in \((n, \mu, \succeq_f)\), and that \( (t_i^j)_{i=0}^\infty \) is monotone-nondecreasing for every \( j \in \mathbb{P}_n \).

Let \( i > \tilde{i} \) s.t. \( k \in P_i \). As \( t_i^{1+1} \geq t_k^{1+1} \), it is enough to show that \( t_k^{1+1} \leq t_i^{1+1} \). As \( \tilde{t}_i^{1+1} \geq (\tilde{t}_i^{1+1}, \tilde{t}_k^{1+1}) \) in every coordinate, by Theorem 4.5 and since \( \tilde{t}^{1+1} \) is a Nash equilibrium w.r.t. \((n, \mu, \succeq_c)\), so is \( (\tilde{t}_i^{1+1}, \tilde{t}_k^{1+1}) \), and so \( t_k^{1+1} \) maximizes the load on \( k \) given \( \tilde{t}_i^{1+1} \); therefore, and as \( t_k^{1+1} \) is a best response to \( t_i^{1+1} \) w.r.t. \((n, \mu, \succeq_f)\), we have that \( t_k^{1+1} \leq t_i^{1+1} \), and the proof is complete. \( \square \)

**Proof of Corollary 4.37.** A direct corollary of Corollary 4.35 (Theorem 4.15) and Theorem 4.36. \( \square \)

**Proof of Corollary 4.38.** A direct corollary of Corollary 4.35 (Theorem 4.22) and Theorem 4.36. \( \square \)

### A.5. Proof of Theorem 6.2

**Proof of Theorem 6.2.** The fact that each such strategy profile is a super-strong equilibrium with load \( \ell_j^g \) on each producer \( j \in \mathbb{P}_{n_g} \) of good \( g \in \{1, 2\} \) (and with the market split between the producers of each good as in the one-good scenario of Section 5) is an immediate consequence of Theorem 5.2; we therefore show that no other super-strong equilibrium exists. We give a proof for atomless \( \mu \); the proof for general \( \mu \) is similar and is left to the reader. Let \( \{(t_j^g, \theta_j^g)\}_{j \in \mathbb{P}_{n_g}} \) be a super-strong equilibrium among producers, given in polar coordinates. For every \( g \in \{1, 2\} \) and \( j \in \mathbb{P}_{n_g} \), let \( \ell_j^g \) be the load on producer \( j \) of good \( g \) in \( \{(t_j^g, \theta_j^g)\}_{j \in \mathbb{P}_{n_g}} \).

We begin by noting that a producer \( j \in \mathbb{P}_{n_g} \) of good \( g \in \{1, 2\} \) may still secure a load of at least \( \tilde{\ell}_j^g \) by choosing the origin as its location, and so \( \ell_j^g \geq \tilde{\ell}_j^g \) for every \( g \in \{1, 2\} \) and \( j \in \mathbb{P}_{n_g} \). As for every \( g \in \{1, 2\} \), we have \( \sum_{j \in \mathbb{P}_{n_g}} \ell_j^g \leq \mu(T) = \sum_{j \in \mathbb{P}_{n_g}} \tilde{\ell}_j^g \), we have that \( \ell_j^g = \tilde{\ell}_j^g \) for every \( j \in \mathbb{P}_{n_g} \), i.e. the load on every producer is as in the super-strong equilibria described in the statement of the theorem.

For every \( g \in \{1, 2\} \), let \( \pi_g \in \mathbb{P}_{n_g} \) be a permutation s.t. \( t_{\pi_g(0)}^g \leq t_{\pi_g(1)}^g \leq \cdots \leq t_{\pi_g(n_g-1)}^g \). For every \( g \in \{1, 2\} \) and \( j \in \mathbb{P}_{n_g} \), define \( m_j^g = \sum_{k=0}^{j-1} \tilde{\ell}_{\pi_g(k)}^g \); by the genericity assumption on partial sums of producer-equilibrium loads, and by positivity of equilibrium loads, we have \( m_j^g \neq m_k^g \) for every \( j \in \mathbb{P}_n \) and \( k \in \mathbb{P}_n \) s.t. either \( j > 0 \) or \( k > 0 \).

As for every \( j \in \mathbb{P}_{n_g} \), the distance \( t_{\pi_g(j)}^g \) is accessible by at least all consumer types consuming a positive amount from any of the producers \( \pi_g(j), \pi_g(j+1), \ldots, \pi_g(n_g-1) \) of good \( g \), we have that \( \mu([0, t_{\pi_g(j)}^g]) \leq \sum_{k=0}^{j-1} \tilde{\ell}_{\pi_g(k)}^g = m_j^g \) for every \( j \in \mathbb{P}_{n_g} \). Therefore, deviating to a super-strong equilibrium as in the statement of the theorem, while maintaining the order of distances from the origin among producers of the same good, harms no producer. If \( t_{\pi_g(j)}^g < \max\{t \in T \mid \mu([0, t]) \leq m_j^g\} \) for some \( g \in \{1, 2\} \) and \( j \in \mathbb{P}_{n_g} \), then producer \( \pi_g(j) \) of load \( g \) strictly benefits from such a deviation; therefore, \( t_{\pi_g(j)}^g = \max\{t \in T \mid \mu([0, t]) \leq m_j^g\} \) for every \( g \in \{1, 2\} \) and \( j \in \mathbb{P}_{n_g} \). Therefore, as \( \mu \)
is atomless, the market is split between the producers of each good as in the one-good scenario of Section 5.

Assume for contradiction that not all producer strategies in $((t^g_j, q^g_j))_{j \in \mathbb{P}_n}$ lie on the same ray from the origin. Therefore, w.l.o.g. there exist producers $j \in \mathbb{P}_{n_1}$ and $k \in \mathbb{P}_{n_2}$ whose strategies do not lie on the same ray, s.t. $t^1_j \leq t^2_k$ and either $j = \pi_1(n_1 - 1)$ or $t^2_k \leq t^1_{\pi_1(\pi^{-1}(j) + 1)}$. (The w.l.o.g. assumption refers to the part played by each good.)

If $j < \pi_1(n_1 - 1)$, then as $\mu$ is atomless, we have $\mu([0, t^2_k)) = \mu([1, t^1_j)) = m_1^{\pi_1(\pi^{-1}(j) + 1)} = \mu([0, t^1_{\pi_1(\pi^{-1}(j) + 1)}))$ and so $t^2_k < t^1_{\pi_1(\pi^{-1}(j) + 1)}$; otherwise, since by assumption $t^2_k > 0$ and as $\mu$ is atomless, we have $t^2_k < 1$. Either way, by market split there exists $\varepsilon > 0$ s.t. almost all (w.r.t. $\mu$) consumer types $d \in [t^2_k, t^2_k + \varepsilon)$ consume a positive amount both from producer $j$ of good 1 and from producer $k$ of good 2. Let $c$ be the circumference of the triangle whose vertices are the origin, $(t^1_j, \theta^1_j)$ and $(t^2_k, \theta^2_k)$; as the latter two do not lie on the same ray from the origin, $(t^1_j, \theta^1_j)$ is not a convex combination of the origin and $(t^2_k, \theta^2_k)$, and so by the triangle inequality we have $c > 2t^2_k$. By definition of $c$, no consumer with type $d \in [t^2_k, \frac{c}{2})$ can consume from both producer $j$ of good 1 and producer $k$ of good 2 without violating the consumer’s QoS limit. By combining these two, we have that for $\delta \eqdef \min\{\frac{c}{2} - t^2_k, \varepsilon\} > 0$, almost all consumer types $d \in [t^2_k, t^2_k + \delta)$ consume from both these producers, while no such consumer consumes from both of them — a contradiction, since by definition of $t^2_k$ we have that $\delta > 0$ implies $\mu([t^2_k, t^2_k + \delta)) > 0$. □

We note that the requirements in Theorem 6.2, both for every producer-equilibrium load to be positive and for the genericity of partial sums of producer-equilibrium loads, are required. Indeed, any producer with zero producer-equilibrium load can be moved to any ray without destabilizing the equilibrium. Furthermore, if there exist permutations $\pi_1 \in \mathbb{P}_{n_1}$! and $\pi_2 \in \mathbb{P}_{n_2}$! and producers $j \in \mathbb{P}_{n_2} \setminus \{0\}$ and $k \in \mathbb{P}_{n_2} \setminus \{0\}$ s.t. $\sum_{i=0}^{j-1} \pi^{-1}_1(i) = \sum_{i=0}^{k-1} \pi^{-1}_2(i)$, then moving all producers $j'$ of good 1 s.t. $\pi_1(j') \geq \pi_1(j)$ and all producers $k'$ of good 2 s.t. $\pi_2(k') \geq \pi_2(k)$ together to any ray does not destabilize the equilibrium either.