Testing QCD Sum Rules on the Light-Cone in $D \to (\pi, K)\ell\nu$ Decays

Patricia Ball

IPPP, Department of Physics, University of Durham, Durham DH1 3LE, UK

Abstract

We compare the predictions for the form factors $f_{+}^{D \to \pi, K}(0)$ from QCD sum rules on the light-cone with recent experimental results. We find $f_{+}^{D \to \pi}(0) = 0.63 \pm 0.11$, $f_{+}^{D \to K}(0) = 0.75 \pm 0.12$ and $f_{+}^{D \to \pi}(0)/f_{+}^{D \to K}(0) = 0.84 \pm 0.04$ in very good agreement with experiment. Although the uncertainties of the form factors themselves are larger than the current experimental errors and difficult to reduce, their ratio is determined much more accurately and with an accuracy that matches that of experiment.
1 Introduction

Exclusive semileptonic decays of $B$ and $D$ mesons are a favoured means of determining the weak interaction couplings of quarks within the Standard Model (SM) because of their relative abundance and, as compared to non-leptonic decays, simple theoretical treatment. The latter requires the calculation of form factors by non-perturbative techniques, the most precise of which, ultimately, will be lattice QCD simulations. Another, technically much less demanding, but also less rigorous approach is provided by QCD sum rules on the light-cone (LCSRs) [1]. While the main motivation for the calculation of $B \to \pi$ form factors is the determination of $|V_{ub}|$, see [2] for recent analyses, $D \to (\pi, K)$ form factors provide both the possibility to determine $|V_{cd}|$ and $|V_{cs}|$ from the semileptonic decays $D \to (\pi, K)\ell\nu$ and, due to the similarity of the calculation, a check of the validity of $B \to \pi$ form factor calculations. The impressive accumulation of data on the experimental side, with recent results from BaBar [3], Belle [4], BES [5], CLEO [6] and FOCUS [7], has been matched by lattice calculations [8], whereas the last comprehensive analysis from LCSRs dates back to 2000 [9]. In view of the recent developments in LCSRs, in particular the updates on the hadronic input, that is the light-cone distribution amplitudes (DAs) of $\pi$ and $K$ mesons of leading and higher twist, see Ref. [10], it is both timely and instructive to recalculate the corresponding form factors from LCSRs and confront the results with experimental data. This is the subject of this letter.

2 A Light-Cone Sum Rule for $f_{+}^{D\to\pi,K}(0)$

The key idea of light-cone sum rules is to consider a correlation function of the weak current and a current with the quantum numbers of the $D$ meson, sandwiched between the vacuum and a $\pi$ or $K$ state. For large (negative) virtualities of these currents, the correlation function is, in coordinate-space, dominated by distances close to the light-cone and can be discussed in the framework of light-cone expansion. In contrast to the short-distance expansion employed by conventional QCD sum rules à la SVZ [11], where non-perturbative effects are encoded in vacuum expectation values of local operators with vacuum quantum numbers, the condensates, LCSRs rely on the factorisation of the underlying correlation function into genuinely non-perturbative and universal hadron DAs $\phi$. The DAs are convoluted with process-dependent amplitudes $T_H$, which are the analogues of the Wilson coefficients in the short-distance expansion and can be calculated in perturbation theory. Schematically, one has

$$\text{correlation function} \sim \sum_n T_H^{(n)} \otimes \phi^{(n)}. \quad (1)$$

The expansion is ordered in terms of contributions of increasing twist $n$. The corresponding DAs have been studied in Refs. [12, 10], both for $\pi$ and $K$ mesons and including two- and three-particle Fock states up to twist 4. The light-cone expansion is matched to the description of the correlation function in terms of hadrons by analytic continuation into the physical regime and the application of a Borel transformation, which introduces the
Borel parameter $M^2$ and exponentially suppresses contributions from higher-mass states. In order to extract the contribution of the $D$ meson, one describes the contribution of other hadron states by a continuum model, which introduces a second model parameter, the continuum threshold $s_0$. The sum rule then yields the form factor in question, $f_+$, multiplied by the coupling of the $D$ meson to its interpolating field, i.e. the $D$ meson’s leptonic decay constant $f_D$.

LCSR calculations are available for the $D \rightarrow \pi, K$ form factor $f_+$ to $O(\alpha_s)$ accuracy for the twist-2 and part of the twist-3 contributions and at tree-level for higher-twist (3 and 4) contributions [13, 9, 14, 15]. Although these sum rules allow the prediction of $f_+$ as a function of $q^2$, the momentum transfer to the leptons, in this letter we only consider the case $q^2 = 0$. The reason is that, in contrast to $B$ decays, the range of $q^2$ accessible to LCSR calculations is rather limited in $D$ decays. Following Ref. [9], one can estimate this range as $q^2 < m^2_c - 2m_c \xi$, where $\xi$ is a hadronic scale independent of the flavour of the heavy quark.

In Ref. [9], $\chi \approx 0.5$ GeV was chosen, which translates into $q^2 < 0.6$ GeV$^2$. In Ref. [15], we chose $\chi \approx 1$ GeV for $B$ decays, which translates into $q^2 < -0.9$ GeV$^2$ for $D$ decays. This has to be compared with the kinematic range in $D$ decays: $0 \leq q^2 \leq (m_D - m_P)^2$, i.e. $q^2 < 3.0$ GeV$^2$ for $D \rightarrow \pi$ and $q^2 < 1.9$ GeV$^2$ for $D \rightarrow K$. That is: even in the optimistic scenario of Ref. [9], at most 30% of the available phase space can be accessed by direct LCSR calculations. The form factor for larger $q^2$ has then to be extrapolated, using, for instance, the modified two-pole formula by Becirevic and Kaidalov [16], which is also frequently used in experimental analyses. In view of this situation, and the converging experimental data on the shape in $q^2$, which allows a direct extraction of $f_+(0)$ from experiment, we decide to focus on the prediction of $f_+(0)$ only, whose theoretical uncertainty is smaller than that of the form factor for positive $q^2$. We compile the currently available experimental and theoretical results for $f_+(0)$ in Tab. 1.

Although, as mentioned before, the LCSR for $f_+$ has been investigated in quite a few publications, the actual formula turns out to be quite complicated and has never been given in a tangible form. In this letter, we present, for the first time, a compact formula for $f^D_+(0)$, $P = \pi, K$, to tree-level accuracy, which makes explicit the suppression factors for contributions of higher twist. At tree level, one has, to twist-4 accuracy:

$$\frac{m_D^2 f_D}{m_c} e^{-m_D^2/2M^2} f^{D\rightarrow P}(0) = f_pm_c \int_{u_0}^1 du e^{-m_c^2/(uM^2)} \left\{ \frac{\phi_{2,P}(u)}{2u} \right\}$$

$$+ \frac{m_D^2}{m_c(m_{q_1} + m_{q_2})} \left[ \frac{1}{2} \phi_{3,P}^0(u) + \frac{1}{12} \left( \frac{2}{u} - \frac{d}{du} \right) \phi_{3,P}^0(u) \right]$$

$$- \eta_{3P} \left( \frac{1}{u} + \frac{d}{du} \right) \int_0^u d\alpha_1 \int_0^u d\alpha_2 \frac{u - \alpha_1}{\alpha_3^2} \phi_{3,P}(\alpha) \right]$$

$$\frac{1}{m_c^2} \left[-\frac{1}{2} \frac{d}{du} \int_0^u d\alpha_1 \int_0^u d\alpha_2 \frac{1}{\alpha_3^2} \left( 2\Psi_{4,P}(\alpha) - \Phi_{4,P}(\alpha) + 2\tilde{\Psi}_{4,P}(\alpha) - \tilde{\Phi}_{4,P}(\alpha) \right) \right]$$

$^1$To be more precise, it is $|V_{cq}f_+(0)|$ that can be determined from experiment. Assuming, however, the SM to be correct, $|V_{cd}|$ and $|V_{cs}|$ are related to $\lambda$, the Wolfenstein parameter, and known with negligible uncertainty.
Experimental and theoretical values of $f^{D\to\pi,K}(0)$; LQCD = lattice QCD. All errors have been added in quadrature. BaBar has to date only published data on the shape of $f^{D\to\pi,K}(q^2)$, but not the absolute normalisation [3]. The LCSR value for $f^{D\to\pi,K}(0)$ corresponds to $m_s(2 \text{ GeV}) = (0.10 \pm 0.02) \text{ GeV}$ and has been obtained by an interpolation of the results given, in Ref. [9], for several values of $m_s$.

\[
\begin{array}{|c|ccc|}
\hline
 & f^{D\to\pi}(0) & f^{D\to\pi}(0) & f^{D\to\pi}(0)/f^{D\to K}(0) \\
\hline
\text{Belle [4]} & 0.695 \pm 0.023 & 0.624 \pm 0.036 & 0.898 \pm 0.045 \\
\text{BES [5]} & 0.78 \pm 0.05 & 0.73 \pm 0.15 & 0.93 \pm 0.20 \\
\text{CLEO [6]} & 0.760 \pm 0.012 & 0.670 \pm 0.031 & 0.882 \pm 0.050 \\
\text{FOCUS [7]} & - & - & 0.85 \pm 0.06 \\
\text{LCSR [9]} & 0.91 \pm 0.14 & 0.65 \pm 0.11 & 0.71 \pm 0.15 \\
\text{LQCD [8]} & 0.73 \pm 0.08 & 0.64 \pm 0.07 & 0.87 \pm 0.09 \\
\text{This Paper} & 0.75 \pm 0.12 & 0.63 \pm 0.11 & 0.84 \pm 0.04 \\
\hline
\end{array}
\]

**Table 1**: Experimental and theoretical values of $f^{D\to\pi,K}(0)$; LQCD = lattice QCD. All errors have been added in quadrature. BaBar has to date only published data on the shape of $f^{D\to\pi,K}(q^2)$, but not the absolute normalisation [3]. The LCSR value for $f^{D\to\pi,K}(0)$ corresponds to $m_s(2 \text{ GeV}) = (0.10 \pm 0.02) \text{ GeV}$ and has been obtained by an interpolation of the results given, in Ref. [9], for several values of $m_s$.

\[
-\frac{1}{8} \int_0^u du \frac{d^2}{du^2} \phi_{4, P}(u) - \frac{1}{2} \int_0^u dv \{ \psi_{4, P}(v) - m_P^2 \phi_{2, P}(v) \}
\]

\[
+\frac{1}{12} \frac{d}{du} \left[ m_P^2 u^2 \phi_{3, P}(u) \right] - \frac{d}{du} \left[ m_P^2 u^2 \phi_{2, P}(u) \right] + \frac{1}{8} \delta (1 - u) \frac{d}{du} \phi_{4, P}(u) \right]
\]

\[
\equiv f_m m_c \left[ R_1(u) + m_P \left\{ \frac{m_P^2}{m_c (m_{q_1} + m_{q_2})} R_2(u) + \frac{m_P^2}{m_c} R_3(u) + \frac{\delta_P}{m_c} R_4(u) \right\} \right] \]
split the $1/m_c^2$ corrections into two different terms is because, numerically, $\delta_0^2 \approx \delta_K^2$ [10], but $m_0^2 \ll m_K^2$.

It is evident from Eq. (3) that the respective weight of various contributions is controlled by powers of $1/m_c^2$; the next term in the light-cone expansion contains twist-3 and 5 DAs and is of order $1/m_c^3$. Nonetheless, (3) cannot be interpreted as $1/m_c$ expansion: for $m_c \to \infty$, the support of the integrals in $u$ also becomes of $O(1/m_c)$, as $1 - u_0 = 1 - m_c^2/s_0 \sim \omega_0/m_c$, with $\omega_0 \approx 1$ GeV a hadronic quantity [13]. In this case, the scaling of the various terms in $m_c$ is controlled by the behaviour of the DAs near the end-point $u = 1$. For finite $m_c$, however, the sum rules are not sensitive to the details of the end-point behaviour, see also Ref. [17]. Numerically, the expansion in terms of $1/m_c$ works very well for $B$ decays (with $m_c \to m_b$), whereas for $D$ decays the chirally enhanced term multiplying $R_2$ is $\sim 1.5$.

We shall come back to that point in Sec. 4.

It is possible to write down a similar sum rule also for $f_+^{D \to P}(0)$ and the ratio $f_+^{D \to \pi}(0)/f_+^{D \to K}(0)$; both values have been determined by several experiments, see Tab. 1. Whereas the ratio is largely independent of the precise values of the QCD sum rule parameters and can be determined with small uncertainty, very much like the form factor ratio for $B \to (\rho, K^{*})\gamma$ transitions [18], the value of $f_+^{D \to P}(0)$ itself also depends on $f_D$. This decay constant has recently been measured with impressive accuracy by CLEO, $f_D = (222.6 \pm 16.7^{+4.3}_{-4.4})$ MeV [19], which is the value we shall use in our calculation.

Compared with the analysis of Ref. [9], in this letter we implement the following improvements:

- updated values of twist-2 parameters, from both QCD sum rules and lattice calculations [20, 21, 22];
- two-loop evolution evolution of twist-2 parameters [23];
- updated values of light quark masses, leading to a significant reduction of the theoretical uncertainty [24, 25, 26];
- inclusion of $O(\alpha_s)$ corrections to the two-particle twist-3 contributions [14, 15];
- complete account for SU(3)-breaking in twist-3 and 4 DAs [10].

$^2$The $R_i$ themselves are independent of $m_c$.

$^3$Evidently, the $R_i(u)$ also become dependent on $q^2$. 
3 Hadronic Input

Let us now shortly discuss the hadronic input to Eq. (2). As for twist-2 DAs, the standard approach is to parametrise them in terms of a few parameters which are the leading-order terms in the conformal expansion

\[
\phi_{2,P}(u, \mu^2) = 6u(1-u) \left( 1 + \sum_{n=1}^{\infty} a_n^P(\mu^2) C_n^{3/2}(2u-1) \right). \tag{4}
\]

To leading-logarithmic accuracy the (non-perturbative) Gegenbauer moments \(a_n^P\) renormalize multiplicatively. This feature is due to the conformal symmetry of massless QCD at one-loop, the \(a_n^P\) start to mix only at next-to-leading order [23]. Although (4) is not an expansion in any obvious small parameter, the contribution of terms with large \(n\) to physical amplitudes is suppressed by the fact that the Gegenbauer polynomials oscillate rapidly and hence are “washed out” upon integration over \(u\) with a “smooth” (i.e. not too singular) perturbative hard-scattering kernel. One usually takes into account the terms with \(n = 1, 2\); the \(a_n^P\) are estimated from QCD sum rules, and, since very recently, lattice simulations. Both are expected to become less reliable for large-\(n\) moments which describe increasingly non-local characteristics of \(\phi_{2,P}\). As an alternative, one can build models for \(\phi_{2,P}\) based on an assumed fall-off behaviour of \(a_n^P\) for large \(n\). The model of Ball and Talbot (BT) [17], for instance, assumes that, at a certain reference scale, e.g. \(\mu = 1\) GeV, the even moments \(a_{2n}^P\) fall off as powers of \(n\):

\[
a_{2n}^P \propto \frac{1}{(n + 1)^p}. \tag{5}
\]

BT fix the absolute normalisation of the Gegenbauer moments by the first inverse moment:

\[
\int_0^1 \frac{du}{2u} \left( \phi_{2,P}(u) + \phi_{2,P}(1-u) \right) = 3\Delta = 3 \left( 1 + \sum_{n=1}^{\infty} a_{2n}^P \right),
\]

which can be viewed as a convolution with the singular hard-scattering kernel \(1/u\) and gives all \(a_{2n}^P\) the same (maximum) weight 1. The rationale of this model is that the DA is given in terms of only two parameters, \(p\) and \(\Delta\), and allows one to estimate the effect of higher order terms in the conformal expansion of observables. A similar model can be constructed for odd Gegenbauer moments. In this letter, we calculate the form factor using both conformal expansion, truncated after \(n = 2\), and the BT model, normalised by \(a_1^P\) and \(a_2^P\), respectively, and taking into account terms up to \(n = 9\). It turns out that the effect of terms with \(n > 2\) is very small.

As for the numerical values of the Gegenbauer moments, \(a_1^\pi\) vanishes by G-parity and \(a_1^K\) has been the subject of a certain controversy [27, 28], which has finally been decided in favour of the value

\[
a_1^K(1\text{ GeV}) = 0.06 \pm 0.03 \tag{6}
\]

obtained from QCD sum rules [20]. Very recently this value has been confirmed from lattice calculations:

\[
a_1^K(1\text{ GeV}) = 0.057(1) \quad \text{Ref. [21]},
\]
Figure 1: Left panel: contribution of $a_n^P$, $n \leq 5$, to $R_1$ as a function of $u$. Right panel: the same for $R_4$ in the renormalon model. The larger $n$, the more the contribution of $a_n^P$ diverges for $u \rightarrow 1$.

$$a_1^K(1 \text{ GeV}) = 0.068(6) \quad \text{Ref. [22],}$$

where we have rescaled the original value given in Ref. [21], at the scale $\mu = 2 \text{ GeV}$ by the next-to-leading order scaling factor 1.26 and the value of Ref. [22], given at 1.6 GeV, by the scaling factor 1.19. As for $a_2^\pi$, the situation as of spring 2006 is summarised in Ref. [10], with $a_2^\pi(1 \text{ GeV}) = 0.25 \pm 0.15$ averaged over all determinations and $0.28 \pm 0.08$ from QCD sum rules alone. In the meantime, a new lattice calculation has returned $a_2^\pi(2 \text{ GeV}) = 0.2 \pm 0.1$ [21], which translates into $a_2^\pi(1 \text{ GeV}) = 0.3 \pm 0.15$. For $a_2^K$, Ref. [28] quotes $0.27^{+0.37}_{-0.12}$ and Ref. [10] $0.30 \pm 0.15$, both QCD sum rule results at the scale 1 GeV. The first lattice determination of this quantity is $a_2^K(2 \text{ GeV}) = 0.18 \pm 0.05$ [21], which corresponds to $a_2^K(1 \text{ GeV}) = 0.26 \pm 0.07$. All these results are consistent with each other and indicate that the values of $a_2^\pi$ and $a_2^K$ are nearly equal. In this letter we use

$$a_2^\pi(1 \text{ GeV}) = 0.28 \pm 0.08 = a_2^K(1 \text{ GeV}).$$

As for twist-3 DAs, we use the expressions and parameters derived in Ref. [10]. For twist-4 DAs, i.e. the terms entering $R_4$ in (3), one can use expressions based on truncated conformal expansion or the renormalon model of Ref. [29]. The advantage of the latter is that the plethora of independent hadronic twist-4 parameters can all be expressed in terms of one genuine twist-4 parameter, $\delta_2^P$, and the twist-2 Gegenbauer moments $a_n^P$. One characteristic of the model is that the end-point behaviour of the DAs for $u \rightarrow 0, 1$ is more singular than that of the conformal expansion. While this is a small effect in $B$ decays because of the power-suppression $\sim 1/m_b^2$ of these contributions, it turns out to be rather problematic in $D$ decays where the suppression factor is much smaller, and results in a marked difference in numerics between using the conformally expanded twist-4 DAs and those based on the renormalon model. In addition, the conformal expansion converges only badly for the latter. We illustrate that in Fig. 1, whose left panel shows the weight factors with which the Gegenbauer moments $a_n^P$ enter $R_1$, whereas the right panel shows the corresponding weight factors for $R_4$. The divergence of the terms in $a_n^P$ for $u \rightarrow 1$ is rather striking. The result is a strong dependence of $R_4$, in the renormalon model, on the
order at which $\phi_{2P}$ is truncated. We hence decide to drop the renormalon model for the calculation of $D$ decays and only use the conformally expanded expression for $R_4$.

Other parameters that remain to be specified are the quark masses and $\alpha_s$. As for the charm quark mass, we use the one-loop pole mass $m_c = (1.40 \pm 0.05)$ GeV which can be obtained from the value for $\overline{m}_c^{\text{MS}}$ found, for instance, from inclusive $b \to c\ell\nu$ decays [30]. For the strange quark mass, we use $\overline{m}_s(2\text{ GeV}) = (0.10 \pm 0.02)$ GeV, which is in agreement with both lattice [24] and QCD sum rule results [25]. As for the light quark masses, we use the average mass $\overline{m}_q = (\overline{m}_u + \overline{m}_d)/2$ with $\overline{m}_s/\overline{m}_q = 24.6 \pm 1.2$ from chiral perturbation theory [26]. Concerning $\alpha_s$, we use two-loop running down from $\alpha_s(m_Z) = 0.1176 \pm 0.002$ [31], which results in $\alpha_s(1\text{ GeV}) = 0.497 \pm 0.005$.

## 4 Numerical Results

Equipped with the hadronic input parameters, we can now assess the respective size of the contributions of $R_i$ to the sum rule (3). In Fig. 2, the $R_i$ are plotted as functions of $u$. All $R_i$, or at least their integrals over $u$, are of order 1 and hence the parametric size of their contribution to the LCSR (3) is indeed set by the weight factors in (3). The central numerical values of these factors are given in Tab. 2, together with those for $B$ decay form factors. Whereas for $B$ decays twist-3 contributions are smaller than those of twist-2, this

### Table 2: Central values of the weight factors for $R_{2,3,4}$ in the LCSR (3); the weight factor for $R_1$ is 1. For comparison, the corresponding weights for $B$ decay form factors are also shown (based on $m_b = 4.8$ GeV).

| Decay | $\frac{m_P^2}{m_c(m_{q_1} + m_{q_2})}$ | $\frac{m_P^2}{m_c^2}$ | $\frac{\delta_P^2}{m_c^2}$ |
|-------|----------------------------------|----------------|----------------|
| $D \to \pi$ | 1.44 | 0.01 | 0.08 |
| $D \to K$ | 1.41 | 0.13 | 0.09 |
| $B \to \pi$ | 0.52 | < 0.001 | 0.006 |
| $B \to K$ | 0.50 | 0.01 | 0.007 |

Figure 2: $R_i$ as function of $u$. Solid line: $R_1$, long dashes: $R_2$, short dashes: $R_3$, dash-dotted line: $R_4$. 

![Figure 2](image_url)
is not the case for $D$ decays due to the chiral enhancement factor $m_p^2/(m_c(m_{q_1} + m_{q_2}))$. This feature was already noted in Ref. [9] and is a bit unfortunate from the point of view of using $D \to P$ decays to test LCSRs for $B \to P$: evidently $f_{+}^{D \to P}$ is more sensitive to the precise value of $1/(m_{q_1} + m_{q_2})$ than $f_{+}^{B \to P}$, but the LCSRs technique itself is of course completely independent of that parameter. In addition, $R_2$ is essentially independent of the Gegenbauer moments $a_n^{P}$, which enter $R_2$ only as quark-mass corrections in $m_{q_1} \pm m_{q_2}$, so that the sensitivity of $f_{+}^{D \to P}$ to $a_n^{P}$ is smaller than that of $f_{+}^{B \to P}$. Stated differently: a successful calculation of $f_{+}^{D \to \pi}(0)$ with a given set of Gegenbauer moments does not necessarily imply a correct prediction of $f_{+}^{B \to \pi}$ with the same moments. The light-cone expansion is also less convergent for $D$ than for $B$ decays, so that one may wonder about the size of the neglected twist-5 contributions $R_5$, which come with a weight factor $\epsilon_5 P m_p^2/(m_c^3(m_{q_1} + m_{q_2})) \sim 1.5 \epsilon_5^2 P/m_c^2$ with $\epsilon_5^2 P$ being a twist-5 hadronic matrix element.

Leaving these reservations aside, at least for the moment, we proceed to present results for $f_{+}(0)$. In Fig. 3 we plot $f_{+}^{D \to \pi,K}(0)$ as function of the Borel parameter $M^2$ for $s_0 = 6 \text{GeV}^2$ and central values of the hadronic input parameters; we also plot the twist-2, 3 and 4 contributions separately. Although the twist-3 contribution is larger than that of twist-2, due to the chiral enhancement factor, the hierarchy of higher-twist contributions is preserved and the total twist-4 contribution is much smaller than that of twist-2 and 3. Fig. 4 shows the ratio $f_{+}^{D \to \pi}(0)/f_{+}^{D \to K}(0)$ as function of $M^2$ and $s_0$, respectively. The dependence of the ratio on these parameters is remarkably small and causes it to vary in the very small interval $[0.83, 0.84]$ only. This is very similar to what we found in Ref. [18] for the ratio of form factors in $B \to (\rho, K^*)\gamma$ transitions and due to the fact that the Borel parameter $M^2$ controls the respective weights of contributions of different $u$; as these contributions are nearly equal in numerator and denominator of the ratio of form factors, except for moderately sized SU(3) breaking, it follows that the resulting dependence on $M^2$ is very small. The parameters to which the ratio is most sensitive are $R = m_s/m_q$ and $m_s$, and we show the corresponding curves in Fig. 5. Still, the dependence of $f_{+}^{D \to \pi}(0)/f_{+}^{D \to K}(0)$ on all these parameters is very moderate, which allows a very precise prediction of this quantity from LCSRs.

In order to obtain final results with a meaningful theoretical uncertainty, we take $M^2 = 4 \text{GeV}^2$ and $s_0 = 6 \text{GeV}^2$ as our central sum rule parameters and vary both $M^2$ and $s_0$ by $\pm 1 \text{GeV}^2$. We also vary all hadronic input parameters within their respective ranges as given above or in Ref. [10], including $\alpha_s(m_Z)$ and the factorisation scale $\mu^2$, whose central value is set to be $m_c^2 - m_c^2$. We also include the effect of switching from the BT model for $\phi_{2,P}$ to a conformal expansion truncated after the second Gegenbauer moment. Finally, we address the issue of possible chirally enhanced twist-5 contributions by varying the twist-4 contributions by a factor 3. When adding all these uncertainties in quadrature, we obtain the following results:

\[
\begin{align*}
  f_{+}^{D \to \pi}(0) & = 0.63 \pm 0.03 \pm 0.10 = 0.63 \pm 0.11, \\
  f_{+}^{D \to K}(0) & = 0.75 \pm 0.04 \pm 0.11 = 0.75 \pm 0.12. \\
\end{align*}
\]

Here the first uncertainty comes from the variation of the QCD sum rule parameters ($M^2$,
Figure 3: $f_{D \to \pi}^+(0)$ (left panel) and $f_{D \to K}^+(0)$ (right panel) as functions of the Borel parameter $M^2$ for central values of the input parameters. Solid lines: $f_+(0)$, long dashes: twist-2 contributions, short dashes: twist-3 contributions, dash-dotted lines: twist-4 contributions.

Figure 4: $f_{D \to \pi}^+(0)/f_{D \to K}^+(0)$ as function of the Borel parameter $M^2$ (left panel) and the continuum threshold $s_0$ (right panel), for central values of input parameters.

Figure 5: $f_{D \to \pi}^+(0)/f_{D \to K}^+(0)$ as function of $R = m_s/m_q$ (left panel) and $m_s(2 \text{ GeV})$ (right panel), for $M^2 = 4 \text{ GeV}^2$ and $s_0 = 6 \text{ GeV}^2$. 
and $s_0$), the second from the uncertainties of the hadronic input parameters which are dominated by $f_D$, $m_s$ and $R$. A slight reduction of the total uncertainty is possible, once more accurate determinations of these parameters will have become available in the future, but it will be difficult to get below $\pm 0.08$. Our result for $f_D^{D \rightarrow \pi}(0)$ nearly coincides with that obtained in Ref. [9]; this is, however, to a certain extent, an accident as quite a few parameters in Ref. [9] were chosen to have different values, notably $m_c$, $f_D$ and the chiral enhancement factor $m_s^2/(2m_q)$, which in Ref. [9] was tied to the value of the quark condensate. Our value for $f_D^{D \rightarrow K}(0)$ is significantly smaller than that of Ref. [9] as given in Tab. 1 for the same values of $m_s$ we use in this letter; this is partially due to the larger $f_D$ we use. The relative errors in (9) are also significantly larger than those quoted, in Ref. [15], for $f_{B \rightarrow P}(0)$. This is due to the fact that, for $D$ decays, some parametric uncertainties are larger than for $B$ decays: the uncertainty due to the light quark masses is three times larger (see the 2nd column in Tab. 2); there is a larger uncertainty due to neglected twist-5 contributions; the dependence of $f_+$ on $M^2$ and $s_0$ is larger; there is a larger uncertainty due to $f_D$ for which we use the experimental value instead of a QCD sum rule.

For the ratio of form factors we find, using the same procedure:

$$\frac{f_D^{D \rightarrow \pi}(0)}{f_D^{D \rightarrow K}(0)} = 0.84 \pm 0.04. \quad (10)$$

In this ratio, quite a few uncertainties cancel so that the total uncertainty is significantly smaller than that of both form factors separately. A reduction of this uncertainty will be very difficult and requires major progress for several quantities, including twist-5 contributions.

Our results are in perfect agreement with the lattice predictions given in Tab. 1, although our errors for the form factors are slightly larger. For the ratio of form factors, our error is by a factor 2 smaller than the lattice uncertainty quoted in Ref. [8]. Comparing with experiment, our results for the form factors are perfectly consistent with the experimental results, although the theoretical uncertainty is much larger than the experimental error quoted by Belle and CLEO. On the other hand, the theoretical uncertainty of the ratio of form factors is about the same size as its experimental counterpart and our predictions agree, within errors, to the experimental results.

5 Summary and Conclusions

The title of this letter is “Testing QCD Sum Rules on the Light-Cone in $D \rightarrow (\pi,K)\ell\nu$ Decays”. So what is the outcome of this test? We have found that the predictions of LCSRs for the form factors at zero momentum transfer, $f_D^{D \rightarrow \pi}(0)$ and $f_D^{D \rightarrow K}(0)$, do perfectly agree with both experiment and lattice calculations, although the errors are relatively large and not expected to be reduced in the near future. The ratio of both form factors, on the other hand, can be predicted with much better accuracy which matches that of current experimental data and surpasses that quoted by the Fermilab/MILC/HPQCD lattice collaboration [8]. Our result agrees within $1.5\sigma$ with all experimental determinations of that ratio, and within $1\sigma$ with the experimental average 0.88. This indicates that the
LCSR method works very well for these form factors and with the set of input parameters for \( \pi \) and \( K \) DAs given in Refs. [20, 10, 21, 22], and the light quark masses \( m_{q,s} \) obtained from lattice calculations [24], QCD sum rules [25] and chiral perturbation theory [26]. This success is certainly very encouraging for the LCSR method as such, but unfortunately can not be taken as proof that the results for \( f_{B \rightarrow \pi} \) and other \( B \) decay form factors with the same input parameters will be as successful. The main problem area is the larger weight given to \( a_P^B \) in \( B \) decays and the value of \( f_B \) which in Ref. [15] was taken from QCD sum rules. Although the \( B \) decay constant has been measured by Belle in early 2006 [32], the experimental uncertainty is yet too large for this measurement to be useful for phenomenology. Nonetheless, the overall result is that LCSRs have successfully passed their first serious experimental test in heavy flavour physics and remain a serious contender for predicting \( B \) decay form factors, alongside with and complementary to lattice calculations.

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