POINCARÉ–EINSTEIN HOLOGRAPHY FOR FORMS VIA CONFORMAL GEOMETRY IN THE BULK

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Abstract. We study higher form Proca equations on Einstein manifolds with boundary data along conformal infinity. We solve these Laplace-type boundary problems formally, and to all orders, by constructing an operator which projects arbitrary forms to solutions. We also develop a product formula for solving these asymptotic problems in general. The central tools of our approach are (i) the conformal geometry of differential forms and the associated exterior tractor calculus, and (ii) a generalised notion of scale which encodes the connection between the underlying geometry and its boundary. The latter also controls the breaking of conformal invariance in a very strict way by coupling conformally invariant equations to the scale tractor associated with the generalised scale. From this, we obtain a map from existing solutions to new ones that exchanges Dirichlet and Neumann boundary conditions. Together, the scale tractor and exterior structure extend the solution generating algebra of [31] to a conformally invariant, Poincaré–Einstein calculus on (tractor) differential forms. This calculus leads to explicit holographic formulæ for all the higher order conformal operators on weighted differential forms, differential complexes, and Q-operators of [9]. This complements the results of Aubry and Guillarmou [3] where associated conformal harmonic spaces parametrise smooth solutions.

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1. Introduction

The Poincaré model realises hyperbolic \((n+1)\)-space \(\mathbb{H}^{n+1}\) as the interior of a unit Euclidean ball, but equipped with a metric conformally related to the Euclidean metric in a way that places the boundary \(n\)-sphere \(S^n\) at infinity. This provides a concrete setting for identifying the isometry group of \(\mathbb{H}^{n+1}\) with the conformal group of \(S^n\) and so a geometric foundation for Poisson transforms linking representations of \(G = SO(n+1,1)\), as induced by its maximal parabolic, to those induced by its maximal compact subgroup [40]. We develop here new tools for this programme, however our main focus is its curved generalisation and our constructions are based in this setting. Such curved analogues of this “flat model” underlie striking new developments in mathematics and physics.

Let \(M\) be a \(d := n + 1\)-dimensional, compact manifold with boundary \(\Sigma = \partial M\) (all geometric structures assumed smooth). A pseudo-Riemannian metric \(g^\omega\) on the
interior $M^+$ of $M$ is said to be \textit{conformally compact} if it extends to $\Sigma$ by

$$g = r^2 g^0,$$

where $g$ is non-degenerate up to the boundary, and $r$ is a defining function for the boundary (i.e. $\Sigma$ is the zero locus of $r$, and $dr$ is non-vanishing along $\Sigma$). Assuming that $\Sigma$ does not contain null directions, the restriction of $g$ to $T\Sigma$ in $TM|\Sigma$ determines a conformal structure, and this is independent of the choice of defining function $r$; $\Sigma$ with this conformal structure is termed the \textit{conformal infinity} of $M^+$. In the case of Riemannian signature, if in addition

$$|dr|^2_g = 1$$

along $\Sigma$, then sectional curvatures approach $-1$ asymptotically and so the structure is said to be \textit{asymptotically hyperbolic (AH)}. On the other hand, if the interior is negative Einstein (without loss of generality the Ricci curvature satisfying $\text{Ric}^g = -ng^0$), then the structure is said to be \textit{Poincaré-Einstein (PE)}; PE restricts the asymptotic sectional curvatures and in the Riemannian case implies AH.

These structures provide a framework for relating conformal geometry and associated field theories of the boundary to the far field phenomena of the interior (pseudo-)Riemannian geometry of one higher dimension; the latter often termed the \textit{bulk}. Such problems may be viewed as analogues of the Poisson transform on the hyperbolic model structure. For the construction of local conformal invariants, a highly influential approach to PE structures was developed by Fefferman-Graham in [18]; this was partly inspired by related ideas from physics and general relativity [46]. On the global side critical aspects of the spectral theory for (Riemannian signature) conformally compact manifolds was developed by Mazzeo and Mazzeo-Melrose [49, 50, 51]. The local and global directions were brought together in [35] which develops a scattering matrix approach to PE structures.

In physics, there is a general notion of holography which strives to capture field/string theories and their geometries in terms of corresponding structures on a space of lower dimension [61, 59]; a hologram provides a visual analogue of this principle. Much of the mathematics of conformally compact structures may be viewed as a concrete realisation of this idea, and indeed these geometries are used as prototypes for holography and related renormalization ideas. In this Article, we shall use the term holography in reference to this programme in the setting of Poincaré-Einstein manifolds. Aside from the mathematics revealed, we view our work as contributing to the investigation of Maldacena’s conjectural AdS/CFT correspondence within quantum theory. This proposes to relate string theory on the bulk to a boundary conformal field theory, and has been the stimulus for much of the recent intense interest in the described directions [48, 1, 39, 45].

In a nutshell the current work is concerned with developing a comprehensive holographic treatment of differential forms, and conformally weighted differential forms; we study both bulk field equations and conformally invariant differential operators intrinsic to the boundary. Our approach uses heavily some new ideas surrounding a generalised notion of scale introduced in [23, 24, 26], that is closely linked to conformal tractor calculus [4, 11, 28]. Building on [31] we develop the new theory and calculus required to apply this to differential form problems.

The results we obtain draw together three recent developments. In [9] it is shown that differential forms host a particularly rich conformal theory. New conformal (detour) complexes and related gauge fixing operators found there lead to corresponding conformal cohomologies, a notion of conformal harmonics, as well as invariants and invariant operators that generalise Branson’s Q-curvature [6]. In a study of harmonic $k$-forms on PE
manifolds, Aubry and Guillarmou found a “holographic” meaning of these objects [3]; for example, each space of conformal harmonics as defined in [9], for the conformal infinity, is seen to fit into an exact sequence involving the spaces of harmonics and $L^2$-harmonics of the bulk. On the other hand, in what might at first appear to be an unrelated development, it was found in [30, 37] that massive, massless and partially massive particle theories [14, 15] are described simultaneously via a (tractor calculus mediated)-interaction of scale with conformal differential operators (this also underlies the recent “shadow field” approach of [53]). In fact, by design, the tools developed in [30] are entirely compatible with the boundary calculus of [31]. An extension of that machinery is used here to formulate a class of Proca systems for higher forms. We show that these are compatible with the forms problem treated in [3]. Here we solve formally Dirichlet and Neumann Proca problems, see Problem 5.9 and 5.17. We obtain explicit formulæ for complete solutions: in the massless case, we recover the conformal harmonic space from [9] as a condition for compatibility with smoothness of solutions, consistent with results of [3]. An important feature of this approach is that it provides explicit holographic formulæ for the detour and gauge operators defining these spaces. Here and throughout the Article, a holographic formula for an operator on the conformal infinity is a canonically obtained bulk operator which recovers the given boundary operator by restriction.

The scattering programme of [35] and others [38, 44, 43, 52, 63] surrounds natural boundary problems on conformally compact manifolds, see also the related Dirichlet problems [2, 58]; this programme is particularly rich for Poincaré-Einstein structures. In [31] it is shown quite generally that the asymptotics of such boundary problems can be solved to all orders using an algebra of geometric operators; there the problems treated include not only scalars but general twistings of such by tractor bundles; this laid the universal route to handling general tensor problems. A key point of that approach is that it is not only algebraically efficient, but it is also geometrically conceptual. A conformally compact manifold is the same as a (compact) conformal manifold with boundary equipped with a certain density (which should be interpreted as a generalised scale, termed later a defining scale) the zero locus of which defines the boundary. The point in [31] is that this structure canonically determines an $\mathfrak{sl}(2)$-generating triple of differential operators which not only yield the natural boundary problems (including those previously studied in the literature), but also a solution generating algebra which solves these. Here we develop a calculus of scale for differential forms which extends this idea in a way that solves higher form Proca systems.

Consider broadly the “holographic problem” for differential forms. Since the bulk has higher dimension than the bounding conformal infinity, we expect that freely specified boundary forms correspond to differential forms on the interior satisfying constraining equations. We investigate this idea on PE manifolds, and where the interior equation is the higher form Proca system

\begin{equation}
\delta dA - m^2 A = 0, \quad \delta A = 0,
\end{equation}

for a (differential) $k$-form $A$. Here $d$ is the exterior derivative, $\delta$ the negative of its formal adjoint, and the parameter $m^2$ is a constant over the manifold. Even though it will be seen that $m^2$ is quadratic in a natural spectral parameter, it can be negative; the historical notation is used because $m^2$ recovers the square of the rest mass in certain settings. Note also that $\delta dA - m^2 A = 0$ implies the divergence/transversality condition $\delta A = 0$, unless $m^2 = 0$ in which case $\delta A = 0$ is known as the Feynman gauge fixing relation. It implies that $\Delta A = m^2 A$, which for $m^2 = (\frac{n}{2} - k + \ell)(\frac{n}{2} - k - \ell)$ is the interior equation considered in [3] (for suitable integers $\ell$).
Let us summarise our main results:

- We show how to write the Proca system (1.1) using the tractor calculus machinery. In fact we obtain much more: We show that the Proca system (1.1) arises canonically from the PE structure, and the mass term is determined (quadratically) by the conformal weight. It is given by an obvious coupling of conformally invariant (or in physics terminology—Weyl invariant) tractor equations to the defining scale tractor, see Proposition 5.4.

- In contrast to (1.1), the tractor system of Proposition 5.4 is well-defined up to the boundary and so it is seen, via this use of conformal geometry and the tractor interpretation of a PE structure, that the interior system (1.1) canonically determines two compatible boundary problems and these are given in Problem 5.9 and the expression (5.18). Solutions of these problems are called, respectively, (Proca) solutions of the first type, and (Proca) solutions of the second type.

- Following [31, Proposition 5.10], we show that any solution of the Proca system (1.1) may be coupled with the defining scale of the PE structure to yield another solution, see e.g. Theorem 5.20 at the level of formal solutions this maps solutions of the first type to solutions of the second type, see Theorem 5.18 and Corollary 5.19. For solutions to the global boundary problem this scale duality map gives a new solution of the same interior equation with the rôles of the Dirichlet and Neumann data exchanged, see Remark 5.21.

- The Proca Boundary Problem 5.9 is solved formally to all orders in Theorem 5.16. See also Proposition 5.31 in Section 5.2 which shows that even when written directly in terms of (weighted) differential form boundary data, remarkably simple explicit formulæ are available for the solution. The solution to Problem 5.17 then follows by scale duality, and this is the content of Corollary 5.19. Other cases require log terms in the solutions. Within this context Problem 5.22 is generic and is solved in Theorem 5.23. The remaining exceptional weights are the important cases of true forms and their weight duals. These are the subject of Problem 5.25 and Problem 5.27 and are solved in, respectively, Theorem 5.26 and Theorem 5.28.

- In Section 2.8 we give a product form for these solutions expressed as a certain product of second order differential operators that projects arbitrary bulk forms to solutions. This provides an alternative solution to the tractor extension problem of [31] and was inspired by the Fefferman–Hirachi product solution [20] for an ambient, scalar, Goursat-type problem.

- In Section 6.3 we obtain holographic constructions and formulæ for the higher order, differential, Branson–Gover (BG) operators on forms from [9]. These yield holographic formulæ for the natural conformally invariant boundary operators $L_k$. In particular we obtain the detour operators $L_k$, and we also find holographic formulæ for gauge fixing operators $G_k$, the Q-operators $Q_k$, the factorisations

$$L_k = \delta Q_{k+1} d,$$

and hence differential detour complexes (see Theorem 6.12). These are seen to arise both from tangential operators along the boundary $\Sigma$ as well as the obstruction to smooth solutions to the Proca systems for true forms. The former
uses surprising holographic identities for powers of the Laplace–Robin operator (see Theorem 5.13).

The technique for solving all these problems is to construct a universal operator that projects arbitrary bulk forms to (formal) solutions. This is a composition of a projector which solves the problem of inserting forms in tractors (see Section 4.4) and a modification of the solution generating operator on tractors constructed in [31] which acts as a projector onto formal power series solutions. This extends the idea of curved translation, initiated in [17], which constructs new invariant differential operators from existing ones. Here we use a similar idea to obtain solutions of the Proca problem from solutions of a related universal tractor problem. Concretely this yields the following remarkably simple expression for solutions to the Proca system 1

\[ A = q^*:K^* q^0 A_0; \]

see Theorem 6.1 (This formula holds avoiding one family of distinguished weights indexed by form degree and requiring a separate treatment.) In the above, the differential form \( A_0 \) is an arbitrary smooth extension to the interior of Dirichlet boundary data along \( \Sigma \) and the operator \( q^W \) is a differential splitting operator mapping forms to tractors along the lines of [9] and developed in detail in Section 3.2 the bundle map \( q^* \) is a left inverse for this. The operator \( :K^* \) is a specially constructed variant of the solution generating operator of [31] adapted to forms and tuned to the Proca system

It is central to our approach to construct and solve the tractor version of the Proca system. Therefore we first develop a direct approach to exterior tractor calculus on general conformal structures. This avoids using the Fefferman–Graham ambient metric, yields explicit formulæ, and recovers and extends the identities found in [9]. Also, a complete algebra of differential splitting operators is obtained using this exterior tractor calculus (see Sections 3.1 and 3.2). In Section 3.3 classical conformally invariant equations on forms are surveyed and their origins from the cohomology of the nilpotent exterior Thomas D-operator are explained.

To describe the Proca system, it essential to couple tractor forms to the PE defining scale. This draws additional canonical operators into the tractor exterior calculus and determines natural boundary conditions for a canonical and universal class of extension problems that we call the tractor Proca equations:

- The solution generating algebra of [31] for the forms analog of the Laplace–Robin operator \( I \cdot D \) of [23, 24] is extended to include exterior and interior multiplication by the scale tractor \( I \). This is the basis for an exterior calculus of scale described in Section 4 and along the boundary gives an extrinsic and conformally invariant Robin-type operator \( \delta_R \) on forms.
- On PE structures, the Laplace–Robin operator has natural “square roots”

\[ \mathcal{I}^* \mathcal{D} + \mathcal{D} \mathcal{I}^* = I \cdot \mathcal{D} = \mathcal{I}^* \mathcal{D}^* + \mathcal{D}^* \mathcal{I}. \]

This supersymmetry is described in Section 4.1 and plays a critical rôle in the tractor description of the Proca system of Section 5
- We develop tangential versions of bulk tractor operators that allow us to follow the tractor Proca equations to the boundary in order to capture the required boundary conditions, see Sections 4.2 and 4.3.

1It has recently come to our attention that, in the context of completions of the enveloping algebra \( \mathcal{U}(\mathfrak{sl}(2)) \), the operator \( :K^* \) is an extremal projector [47, 62]. These are projections \( P \) such that \( P e_\alpha = 0 = e_\alpha P \) for all positive roots \( \alpha \) labeling Lie algebra generators \( e_\alpha \).
Solutions to the tractor Proca system are developed in Section 5.1. This relies on a holographic formula $\Pi$ for an operator that commutes with the solution generating operator $:K:$ and extends to the interior a projector onto boundary tractor sections with image isomorphic to boundary forms. Moreover, the operator $\Pi$ ensures the tractor analog of the Proca transversality condition $\delta A = 0$ and is compatible with the boundary conditions. The composition of $\Pi$ and $:K:$ projects arbitrary tractor forms $A_0$ onto formal solutions to the generic tractor Proca problem:

$$A = \Pi :K: A_0,$$

see Theorem 5.16.

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2. Bulk conformal geometry and extension problems

Here we lay out our conventions and some basic facts about conformal geometry (see Sections 2.1 and 2.2) before stating in Section 2.4 the general formulation of the extension problems solved in this Article. In Section 2.4, we solve a generalised divergence extension problem which is required later to handle the transversality condition of the Proca system. To solve the full Proca problem, tractor calculus plays an central rôle; the basic ingredients are stated in Section 2.5. In Section 2.6 we review the calculus of defining scales and construct the Laplace–Robin operator which controls the dynamics of the systems we solve. In Section 2.7 we give the solution generating algebra of [31]. This part of the Article is completed in Section 2.8 by giving a novel product form for the solution generating operator of [31]. For details and background on conformal geometry relied on here see [11, 28].

2.1. Riemannian conventions. The main background and notations in this Article are mostly those of [31]. Here we briefly highlight some main points. Throughout we focus on manifolds $M$ of dimension $d := n + 1$ at least three equipped with a metric, or a conformal equivalence class of Riemannian metrics. All structures will be assumed smooth. For a given metric with Levi-Civita connection $\nabla$, the Riemann curvature tensor $R$ is given by

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]} Z,$$

where $X, Y,$ and $Z$ are arbitrary vector fields. In an index notation $R$ is denoted by $R_{abcd}$, and $R(X, Y)Z$ is $X^a Y^b Z^c R_{abcd}$. This can be decomposed into the totally trace-free Weyl curvature $C_{abcd}$ and the symmetric Schouten tensor $P_{ab}$ according to

$$R_{abcd} = C_{abcd} + 2g_{c[a} P_{b]d} + 2g_{d[b} P_{a]c},$$

(2.1)

where $[\cdots]$ indicates antisymmetrisation over the enclosed indices. Thus $P_{ab}$ is a trace modification of the Ricci tensor $\text{Ric}_{ab} = R_{ca} e^c_b$.

$$\text{Ric}_{ab} = (n - 2) P_{ab} + J g_{ab}, \quad J := P^a_a.$$

2.2. Conformal and almost Riemannian geometry. Recall that a conformal geometry $(M, c)$ is a $d$-manifold $M$ equipped with an equivalence class of metrics $c$ such that $g, \hat{g} \in c$ obey

$$\hat{g} = \Omega^2 g, \quad \Omega := e^w,$$

for some $\omega \in C^\infty(M)$. Observe that a conformal structure can be viewed as a smooth ray subbundle $\mathcal{G} \subset S^2 T^* M$ whose fibre at a point $x \in M$ consists of the values of $g_x$ for all metrics $g \in c$. The principal bundle $\mathcal{G}$ has structure group $\mathbb{R}_+$ whose irreducible representations $\mathbb{R}_+ \ni t \mapsto t^{-w/2} \in \text{End}(\mathbb{C})$, are labeled by weights $w \in \mathbb{C}$. These induce “conformal density” line bundles $\mathcal{E} M[w]$. A section $\tau \in \Gamma \mathcal{E} M[w]$ is equivalent to a smooth function $\tau$ of $\mathcal{G}$ that satisfies the homogeneity property

$$\tau(x, \Omega^2 g) = \Omega^w \tau(x, g).$$

Each metric $g \in c$ determines a canonical, positive, section $\tau \in \Gamma \mathcal{E}_+ M[1]$, viz. the section with the property that $\tau(p, g) = 1$ for all $p \in M$. (The conformal density bundles are oriented, and a subscript plus indicates the $\mathbb{R}_+$-ray subbundle.) It follows that there is a tautological section of $S^2 T^* M \otimes \mathcal{E} M[2]$ that is termed the conformal metric, denoted $g$ with the property that any nowhere zero section $\tau \in \Gamma \mathcal{E} M[1]$ determines a metric $g \in c$ via $g := \tau^{-1} \tau$. Henceforth the conformal metric $g$ is the default object that will be used to identify $TM$ with $T^* M \cong T^* M \otimes \mathcal{E} M[2]$ (rather than a metric from the conformal class) and to form metric traces.
A Riemannian manifold can be treated as a conformal manifold \((M, c)\) equipped with a nowhere zero density \(\sigma \in \Gamma E M[1]\), since the metric is recovered by \(\sigma^{-2}g\). Following [26], we formulate an obvious generalised notion of this.

**Definition 2.1.** A defining scale is a section \(\sigma \in \Gamma E M[1]\) that is nowhere vanishing on an open dense subset of \(M\). A conformal manifold \((M, c)\) equipped with a defining scale \(\sigma \in \Gamma E M[1]\) is called an almost Riemannian structure \((M, c, \sigma)\); where \(\sigma\) is non-zero, \(\sigma^{-2}g\) defines a Riemannian metric.

In the case that \(\sigma\) is nowhere zero this is a true scale, so that \(\sigma^{-2}g\) is a metric everywhere.

This notion of almost Riemannian structures arises naturally in the context of defining densities: Consider a smooth, oriented hypersurface \(\Sigma\) given as the zero locus of some smooth (at least in a neighbourhood of \(\Sigma\)) defining function \(s\), with \(ds \neq 0\) along \(\Sigma\). Generally, to optimally employ the conformal structure \(c\), we will replace the defining function \(s\) by a defining density \(\sigma\) which is the unique conformal density \(\sigma \in \Gamma E M[1]\) that yields the defining function \(s\) in the trivialisation of \(E M[1]\) determined by some \(g \in c\). A preferred defining density is a special example of a defining scale.

This notion is especially natural and useful for conformally compact manifolds. A conformally compact manifold is an almost Riemannian manifold with boundary such that the zero locus \(Z(\sigma) = \partial M\). Thus, in the notation of the Introduction \(g^o = r^{-2}g = \sigma^{-2}g\) and \(r\) is the component function representing \(\sigma\) in the scale \(g\). In particular this applies to special case of PE structures where \(g^o\) is negative Einstein.

Since each \(g \in c\) determines a trivialisation of \(E M[w]\), it also defines a corresponding Levi-Civita connection \(\nabla\) (see e.g. [11]). Moreover, a metric \(g \in c\) canonically determines a true scale \(\tau \in \Gamma E M[1]\) by the requirement \(\tau(x, g) = 1\). We will write \(\nabla^{\tau}\) for the connection corresponding to this scale. Almost Riemannian structures \((M, c, \sigma)\) come equipped with the canonical Levi-Civita connection \(\nabla^{\sigma} := \nabla^{g^o}\) away from the zero locus \(Z(\sigma)\) of \(\sigma\). In the scale \(\tau\), acting on a density \(\mu \in \Gamma E M[w]\), on \(M \setminus Z(\sigma)\)

\[
\nabla^{\sigma} \mu = (\nabla^{\tau} - \omega^{-1}n)\mu, \quad n := \nabla^{\tau} \sigma.
\]

The operator

\[
\nabla := \sigma \nabla^{g^o} - \omega n
\]

extends \(\sigma \nabla^{g^o}\) to \(Z(\sigma)\). In the case where \(Z(\sigma) = \Sigma\) for some hypersurface \(\Sigma\), this reduces to \(\omega n\big|_{\Sigma}\) along \(\Sigma\). Thus, the Levi-Civita connection off \(\Sigma\) and (when \(w \neq 0\)) the normal to the hypersurface are smoothly and canonically incorporated in a single operator. Moreover, the Levi-Civita connection has the following important yet obvious property.

**Proposition 2.2.** Let \(\mu \in \Gamma E^\bullet M[w]\), where \(E^\bullet M[w]\) indicates any weight \(w\) tensor bundle. Then

\[
\nabla \sigma \mu = \sigma \nabla \mu.
\]

2.3. **Extension problems.** Our core extension problems are formulated as below.

**Problem 2.3.** Let \(y : \Gamma F \to \Gamma F'\) be a given operator acting on sections of some vector bundle \(F\) over \(M\) with codomain the section space of another vector bundle over \(M\). Then, for fixed “boundary data” \(f|_{\Sigma} \in \Gamma F|_{\Sigma}\), where \(\Sigma \subset M\) is a hypersurface in \(M\) (or \(\Sigma = \partial M\)), find \(f \in \Gamma F\) such that the extension \(\gamma f\) of \(f|_{\Sigma}\) obeys

\[
y f = 0.
\]
For the class of extension operators \( y \) studied here we are interested in explicit asymptotic solutions. Assuming that \( \sigma \) is a defining density for \( \Sigma = \mathcal{Z}(\sigma) \) the zero locus of \( \sigma \), these can be treated by an expansion in the density \( \sigma \). In the case that we view \( \sigma \) as a defining scale, so that it gives a preferred defining density, we obtain canonical coordinate independent expansions \( f(\ell) \) satisfying

\[
y f(\ell) = \sigma^\ell f_\ell,
\]

for some smooth \( f_\ell \in \Gamma(\mathcal{F} \otimes \mathcal{E}M[-\ell]) \). We will often abbreviate the right hand side of the above display by \( O(\sigma^\ell) \). In many cases we are able to find asymptotic solutions for arbitrarily high integers \( \ell \).

To begin with, we consider an asymptotic solution to a simple model problem coming from the transversality condition for massive form fields; this is an integrability condition of Proca’s equations. In the massless limit, this condition reduces to the Coulomb/Feynmann gauge condition for Maxwell’s equations.

### 2.4. The generalised divergence extension problem

Throughout, we will denote the tensor product \( \Lambda^k M \otimes \mathcal{E}M[w] := \mathcal{E}^k M[w] \) and refer to sections thereof as weighted forms, or simply forms. Sections of \( \Lambda^k M \) will be called true forms with section space denoted \( \Omega^k M = \mathcal{E}^k M[0] \). In a true scale \( \tau \), we will write \( d^\tau \) for the exterior action of the Levi-Civita connection \( d^\tau := \varepsilon(\nabla^\tau) : \Gamma\mathcal{E}^k M[w] \to \Gamma\mathcal{E}^{k+1} M[w] \). Similarly, for the codifferential, \( \delta \) will denote \( \delta^\tau := \iota(\nabla^\tau) : \Gamma\mathcal{E}^k M[w] \to \Gamma\mathcal{E}^{k-1} M[w-2] \). For almost Riemannian structures, the canonical Levi–Civita connection \( \nabla^\tau \) determines exterior and interior operators acting on \( \Gamma\mathcal{E}^k M[w] \)

\[
(2.2) \quad d^\tau := d - w\sigma^{-1}\varepsilon(n), \quad \delta^\tau := \delta - (d + w - 2k)\sigma^{-1}\iota(n).
\]

Here \( \varepsilon(.) \) and \( \iota(.) \) denote, respectively, exterior and interior products. We note the following obvious but highly useful identities.

**Lemma 2.4.** The operators

\[
d^\tau : \Gamma\mathcal{E}^k M[w] \to \Gamma\mathcal{E}^{k+1} M[w], \quad \delta^\tau : \Gamma\mathcal{E}^k M[w] \to \Gamma\mathcal{E}^{k-1} M[w-2]
\]

and

\[
\delta^\tau d^\tau + d^\tau \delta^\tau =: \Delta^\tau : \Gamma\mathcal{E}^k M[w] \to \Gamma\mathcal{E}^k M[w-2],
\]

defined away from \( \Sigma \), obey

\[
(d^\tau)^2 = 0 = (\delta^\tau)^2,
\]

\[
[\Delta^\tau, d^\tau] = 0 = [\delta^\tau, \Delta^\tau],
\]

\[
[d^\tau, \sigma] = 0 = [\sigma, \delta^\tau].
\]

**Proof.** All these results follow from standard ones for the exterior derivative, codifferential and form Laplacian calculating in the preferred interior choice of scale \( g^\tau \in c \) determined by \( \sigma \). They can also be obtained by explicit computation based on the following elementary commutator and anticommutator operator identities

\[
(2.3) \quad [d, \sigma] = \varepsilon(n), \quad [\delta, \sigma] = \iota(n), \quad [d, \varepsilon(n)] = 0, \quad [\delta, \iota(n)] = 0.
\]

More importantly, from \( d^\tau \) and \( \delta^\tau \) we can define operators that extend to the boundary \( \Sigma \).
Definition 2.5. Let $A \in \Gamma \mathcal{E}^k M[w]$. The operators

\[ \tilde{\varepsilon} : \Gamma \mathcal{E}^k M[w] \rightarrow \Gamma \mathcal{E}^{k+1} M[w+1] \quad \text{and} \quad \tilde{i} : \Gamma \mathcal{E}^k M[w] \rightarrow \Gamma \mathcal{E}^{k-1} M[w-1] \]

defined, for some $g \in c$ by

\[ \tilde{\varepsilon} A := [w\varepsilon(n) - \sigma d] A \quad \text{and} \quad \tilde{i} A := [(d + w - 2k)\iota(n) - \sigma \delta] A, \]

respectively extend $\tilde{\varepsilon}$ and $\tilde{i}$ to $\Sigma$.

Remark 2.6. The operators $\tilde{\varepsilon}$ and $\tilde{i}$ play a double rôle (see the discussion of $\nabla^o$ above): Away from $\Sigma$ in the preferred interior scale $g^o \in c$, they yield minus the codifferential $-d$ and exterior derivative $-\delta$. Along $\Sigma$ and avoiding weights $w = 0$ and $w = 2k - d$, respectively, they give interior and exterior multiplication by the normal covector $n$. In reference to this, we will often call them the holographic exterior and interior normals.

From Lemma 2.4 (and smoothness), we immediately deduce the algebra of the operators $\tilde{\varepsilon}$ and $\tilde{i}$.

Corollary 2.7. The holographic exterior normal $\tilde{\varepsilon}$ and holographic interior normal $\tilde{i}$ acting on weighted forms obey the relations

\[ \{ \tilde{i}, \tilde{\varepsilon} \} = \sigma^2 \Delta^o, \]

\[ \tilde{\varepsilon}^2 = 0 = \tilde{i}^2, \]

\[ [\Delta^o, \tilde{\varepsilon}] = 0 = [\tilde{i}, \Delta^o], \]

\[ [\tilde{\varepsilon}, \sigma] = 0 = [\sigma, \tilde{i}]. \]

To place our approach in a familiar context, we propose a (naïve) version of the extension Problem 2.3 for the case $f = A \in \mathcal{E}^k M[w+k]$ and $y = -\tilde{i}$. The weight $w+k$ is chosen for easier comparison with the tractor calculus analog of this problem.

Problem 2.8. Given $A|_\Sigma \in \Gamma \mathcal{E}^k M[w+k]|_\Sigma$ and an arbitrary extension $A_0$ of this, find $A_i \in \Gamma \mathcal{E}^k M[w+k-i]$, $i = 1, 2, \ldots$, so that

\[ A^{(\ell)} = A_0 + \sigma A_1 + \sigma^2 A_2 + \cdots + \sigma^{\ell'} A_{\ell'} + O(\sigma^{\ell'+1}) \]

solves

\[ \tilde{i} A^{(\ell)} = O(\sigma^{\ell}), \]

off $\Sigma$, for some integers $\ell'$ and $\ell \in \mathbb{N} \cup \infty$ as high as possible.

Remark 2.9. Away from $\Sigma$, $-\tilde{i} = \sigma \delta^o$ (see Definition 2.5). So, in the preferred scale $\sigma$ off $\Sigma$, the above Problem yields the divergence condition

\[ \delta A = 0, \]

often referred to as the Coulomb or, in a relativistic context, Feynmann gauge choice for massless fields. For the Proca equation, this is a necessary integrability condition. The point is that equation (2.4) naturally extends the condition (2.5) to $\Sigma$. Therefore we term the equation $\tilde{i} A = 0$ a generalised divergence condition.

An elementary approach to treating Problem 2.8 is to write out Equation (2.4) in some choice of scale $g \in c$ and solve iteratively: Using the first relation of (2.3), one immediately finds the recursion relation

\[ (i - d - w + k)\iota(n) A_i + \delta A_{i-1} = 0, \quad i = 0, 1, \ldots, \quad A_{-1} := 0. \]
When \( d + w - k \neq 0 \), the base case \( i = 0 \) gives the condition
\[
(2.7) \quad \iota(n)A_0 = O(\sigma).
\]

Note that we write \( O(\sigma) \) on the right hand side because this and higher order terms in \( \sigma \) can be removed by appropriately shifting \( A_1, A_2, \ldots \) in the ansatz.

Off \( \Sigma \), equation \((2.7)\) is a harmless algebraic condition on the choice of extension \( A_0 \) of \( A|\Sigma \). However, by the obvious isomorphism (used henceforth without comment) between \( T^*\Sigma \) and the annihilator of the normal in \( T^*M|\Sigma \), the condition \( \iota(n)A_0|\Sigma = 0 \) says that \( A_0|\Sigma \) is a differential form on \( \Sigma \). This shows that the problem cannot be solved in general, and we must make the restriction
\[
A|\Sigma = A_0|\Sigma = A_{\Sigma} \quad \text{for some} \quad A_{\Sigma} \in \Gamma\mathcal{E}^k\Sigma[w + k].
\]

Upon making this restriction on the data \( A|\Sigma \), the algebraic recursion clearly has solutions to all orders so long as the coefficient \( i - d - w + k \neq 0 \). When this does vanish the problem is potentially obstructed by \( \delta A_{i-1} \) (modulo \( O(\sigma) \)) where \( A_{i-1} \in \Gamma\mathcal{E}^kM[2k-n] \).

Observe, interestingly enough, the boundary codifferential problem is potentially obstructed by \( A \) equals \( 1 \) \( \delta \). However, by the obvious isomorphism (used henceforth without comment) between \( \delta \) and the annihilator of the normal in \( \Sigma \), the condition \( \delta \iota(n)A_0|\Sigma = 0 \) says that \( A_{\Sigma} \) is a differential form on \( \Sigma \). This shows that the problem cannot be solved in general, and we must make the restriction
\[
A_{\Sigma} = A_{0}\Sigma |\Sigma = A_{\Sigma} \quad \text{for some} \quad A_{\Sigma} = \Gamma\mathcal{E}^k\Sigma[w + k].
\]

When the structure \((M, c, \sigma)\) is \( \text{AH} \), the defining scale \( \sigma \) obeys \((g \in c)\)
\[
(2.8) \quad |\nabla \sigma|^2_g := |n|^2_g = 1 + O(\sigma),
\]
so that \( n_g \) is a unit conormal for any \( g \in c \). This condition is effectively no restriction; for any almost Riemannian structure \((M, g, \sigma)\) obeying \( |n|^2_g > 0 \) along \( \Sigma \), we can find a new scale \( \sigma' \) such that \((M, g, \sigma')\) is \( \text{AH} \) in an obvious way. For \( \text{AH} \) structures a simple product-type solution to Problem \((2.8)\) is available.

**Proposition 2.10.** For \( d + w - k \notin \mathbb{Z}_{\geq 1} \) and \( A|\Sigma \in \Gamma\mathcal{E}^k\Sigma[w + k] \), Problem \((2.8)\) can be solved to order \( \ell = \infty \). When \( d + w - k = 0 \), the restriction on \( A|\Sigma \) can be relaxed. For \( d + w - k = m \in \mathbb{Z}_{\geq 1} \), a solution exists to the order \( \ell = m \).

When the structure \((M, c, \sigma)\) is \( \text{AH} \), the defining scale \( \sigma \) obeys \((g \in c)\)
\[
(2.9) \quad A = \frac{1}{(w+k)(n+w-k)} \iota \varepsilon A_0.
\]

**Proposition 2.11.** Let \((M, c, \sigma)\) be an \( \text{AH} \) structure. Then, for any \( w \neq -k, k - n \), and \( A|\Sigma \in \Gamma\mathcal{E}^k\Sigma[w + k] \), Problem \((2.8)\) can be solved to order \( \ell = \infty \) by
\[
A = \frac{1}{(w+k)(n+w-k)} \iota \varepsilon A_0.
\]

**Proof.** Note \( \iota A = 0 \) identically by virtue of \( \iota \varepsilon = 0 \) as established in Corollary \((2.7)\). It remains to show that \((\frac{1}{(w+k)(n+w-k)} \iota \varepsilon A_0)|\Sigma = A|\Sigma \). Computing along \( \Sigma \) we have
\[
\frac{1}{(w+k)(n+w-k)} \iota \varepsilon A_0 = \iota(n)\varepsilon(n)A_0.
\]

Then, using that \( n \) is a unit conormal to \( \Sigma \) we note that \( \iota(n)\varepsilon(n) \) is a projector onto the subbundle \( \mathcal{E}^k\Sigma[w + k] \) of \( \mathcal{E}^kM[w + k]|\Sigma \). But since \( A_0|\Sigma = A|\Sigma \in \Gamma\mathcal{E}^k\Sigma[w + k] \) we have \( \iota(n)A_0 = 0 \) along \( \Sigma \). Thus, \((\{\iota(n), \varepsilon(n)\} = n^2)\) along \( \Sigma \) the above display equals \( A|\Sigma \).

**Remark 2.12.** Observe that the right hand side of \((2.9)\) actually provides a global solution to the \( \iota A = 0 \) problem with the given boundary data. The Proposition is an example of a more general holographic boundary projector technique that we shall develop in Section \((4.4)\). Here, this terminology is used to refer to a bulk operator that acts as a projector along \( \Sigma \) and solves a set of prescribed bulk equations (such as the one in Problem \((2.8)\)).
Remark 2.13. The series solution determined by (2.6) and the holographic boundary projector solution (2.9) are easily verified to be compatible, indeed the latter yields a series solution that terminates at $O(\sigma^2)$ because the “coefficient” at that order is coclosed. Together, Propositions 2.10 and 2.11 give an order $\ell = \infty$ solution to Problem 2.8 for any weight save $w$ subject to $n + w - k = 0$ or $w = -k \in \{-1, 0, \ldots, \lfloor \frac{d-1}{2} \rfloor \}$. These exceptional weights will be discussed in detail when we consider boundary conditions for higher form Proca systems in Section 4.3.

2.5. Conformal tractor calculus. For a conformal $d$-manifold $(M, c)$, a key tool will be the standard tractor bundle $T M$ and associated tractor calculus [4] for building conformally invariant differential operators. The tractor bundle is a rank $d + 2$ vector bundle equipped with a canonical tractor connection $\nabla^T$ (or simply $\nabla$ when the context is clear). A given metric $g \in c$ determines the isomorphism

$$T M \overset{g}{\cong} EM[1] \oplus T^* M[1] \oplus EM[-1].$$

We will often employ this isomorphism to express sections $T \in \Gamma T M$ as

$$T = g \begin{pmatrix} \nu \\ \mu_a \\ \rho \end{pmatrix} =: T^A.$$

Here, and throughout, we frequently employ an abstract index notation (cf. [54]) to denote sections of the various vector bundles encountered.

In the obvious way, the notation $\overset{g}{\cong}$ indicates calculations in a scale determined by $g \in c$. We will use this notation for emphasis if the scale is not clear by context. In terms of the above splitting, the tractor connection is given by

$$\nabla^T_a \begin{pmatrix} \sigma \\ \mu_b \\ \rho \end{pmatrix} := \begin{pmatrix} \nabla_a \nu - \mu_a \\ \nabla_a \mu_b + g_{ab} \rho + P_{ab} \nu \\ \nabla_a \rho - P_{ac} \mu_c \end{pmatrix}.$$  

Changing to a conformally related metric $\hat{g} = e^{2\omega} g$ gives a different isomorphism, which is related to the previous one by the transformation formula

$$\begin{pmatrix} \nu \\ \mu_b \\ \rho \end{pmatrix} = \begin{pmatrix} \nu \\ \mu_b + \nu \Upsilon_b - g_{cd} \Upsilon_c \mu_d - \frac{1}{2} \nu g_{cd} \Upsilon_c \Upsilon_d \\ \rho - g_{cd} \Upsilon_c \mu_d \end{pmatrix},$$

where $\Upsilon_a$ is the one-form $d\omega$. It is straightforward to verify that the right-hand-side of (2.11) also transforms in this way and this verifies the conformal invariance of $\nabla^T$.

In the above formulæ, we have denoted by $g_{ab}$ the [conformal metric] introduced in Section 2.2. From this, we can build the conformally invariant tractor metric $h$ on $T M$ given (as a quadratic form on $T^A$ as above) by

$$\begin{pmatrix} \nu \\ \mu \\ \rho \end{pmatrix} \mapsto g^{-1}(\mu, \mu) + 2\sigma \rho =: h(T, T) = h_{AB} T^A T^B;$$

it is preserved by the connection. We shall often write $T : T$ or $T^2$ as a shorthand for the right hand side of this display. Note that this has signature $(p + 1, q + 1)$ on a conformal manifold $(M, c)$ of signature $(p, q)$. The tractor metric $h_{AB}$ and its inverse $h^{AB}$ are used to identify $T M$ with its dual in the obvious way.

Tensor powers of the standard tractor bundle $T M$, and tensor products thereof, are vector bundles that are also termed tractor bundles. We shall denote an arbitrary tractor
bundle by $\mathcal{T}\Phi M$ and write $\mathcal{T}\Phi M[w]$ to mean $\mathcal{T}\Phi M \otimes \mathcal{E}M[w]$; $w$ is then said to be the weight of $\mathcal{T}\Phi[w]$. In the obvious way, we may introduce a weight operator $w$ on sections of weighted tractor bundles $\mathcal{T}\Phi M[w] \ni f$ by

$$ (2.12) \quad wf = wf. $$

Whereas the tractor connection maps sections of a weight 0 tractor bundle $\mathcal{T}\Phi M$, there is a conformally invariant operator which maps between sections of weighted tractor bundles. This is the Thomas D- (or tractor D-) operator

$$ D^A : \Gamma\mathcal{T}\Phi M[w] \mapsto \Gamma(\mathcal{T}^A M \otimes \mathcal{T}\Phi M[w - 1]), $$

given in a scale $g$ by

$$ (2.13) \quad D^AV = \left( \begin{array}{c} (d + 2w - 2)wV \\ (d + 2w - 2)\nabla_a V \\ - (\Delta + Jw)V \end{array} \right), $$

where $\Delta = g^{ab}\nabla_a \nabla_b$, $V \in \Gamma\mathcal{T}\Phi M[w]$ and $\nabla$ is the coupled Levi-Civita-tractor connection [4, 60].

A key point to emphasise here is that the Thomas D-operator is a fundamental object in conformal geometry. On a conformal manifold the tractor bundle is “as natural” as the tangent bundle. Moreover the Thomas D-operator acting on densities in $\Gamma\mathcal{E}M[1]$ basically defines the tractor bundle, see [10].

We will also make frequent use of the canonical conformally invariant operator

$$ X^A : \Gamma\mathcal{T}\Phi M[w] \mapsto \Gamma(\mathcal{T}^A M \otimes \mathcal{T}\Phi M[w + 1]), $$

defined by multiplication by the canonical tractor $X^A$. This derives from the canonical invariant map $\mathcal{E}M[-1] \to TM$ where $\rho \mapsto X^A\rho$. In a choice of splitting $g \in c$

$$ X^A \equiv \left( \begin{array}{c} 0 \\ 0 \\ 1 \end{array} \right). $$

We may also view $X^A$ as a canonical, null section of $\mathcal{T}^A M[1]$.

2.6. The calculus of scale. Almost Riemannian manifolds are equipped with a splitting of geometry into “conformal” and “scale” parts. This melds perfectly with the conformal tractor calculus. Together these yield a powerful calculus of scale [26, 24, 31] that we will further develop and exploit in the following Sections. Central to this is the object we now define.

**Definition 2.14.** Let $\sigma$ be a defining scale for an almost Riemannian structure $(M, g, \sigma)$. Then

$$ (2.14) \quad I^A := \frac{1}{d} D^A\sigma, $$

is called the scale tractor. Note that $\sigma = X^A I_A$.

In the metric $g^\sigma = \sigma^{-2} g$, $I^2 = - \frac{2I}{\sigma}$. Hence we use the terminology almost scalar constant [26] if an almost Riemannian geometry $(M, c, \sigma)$ obeys

$$ I^2 = \text{constant}. $$

Putting together the scale tractor and Thomas D-operator gives a canonical degenerate Laplace operator

$$ I \cdot D := I^A D_A : \Gamma\mathcal{T}\Phi[w] \to \mathcal{T}\Phi[w - 1]. $$
Recall we denote the zero locus \( \mathcal{Z}(\sigma) \) by \( \Sigma \) (which could possibly be empty). If we calculate in the metric \( g^\sigma = \sigma^{-2}g \) away from \( \Sigma \), and trivialise density bundles accordingly, we have

\[
I \cdot D g^\sigma = -\left( \Delta g^\sigma + \frac{2w(d + w - 1)}{d} J g^\sigma \right),
\]

where again \( \Delta = g^{ab} \nabla_a \nabla_b \) for the coupled Levi-Civita-tractor connection \( \nabla \). This allows a study of Laplacian eigen-equations.

If \( \sigma \) also satisfies the almost scalar constant condition with \( I^2 = 1 \), then along its zero locus

\[
I \cdot D = (d + 2w - 2) \delta_n,
\]

where \( \delta_n \equiv n^a \nabla_a - wH^g \) is the first order (tractor-twisted) conformal Robin operator [12, 8]. Here \( n^a \) is a unit normal and \( H^g \) is the mean curvature measured in the metric \( g \). Thus, on conformally compact manifolds, the operator \( I \cdot D \) unifies both the interior Laplace problem with boundary dynamics and hence we generally dub it the Laplace–Robin operator.

2.7. The Laplace–Robin solution generating algebra. Here and until further notice we work on an almost Riemannian geometry. In that setting the Laplace–Robin operator provides a distinguished choice for the extension operator \( y \). Moreover, together with the defining scale, it generates an algebra that facilitates the solution of extension problems [31]. To display this, we first define a triplet of canonical operators.

**Definition 2.15.** Let \( \sigma \in \Gamma \mathcal{E}M[1] \) be a defining scale with vanishing locus \( \Sigma \) and nowhere vanishing \( I^2 \). Then we define the triplet of operators \( \{x, h, y\} \)

\[
x : \Gamma \mathcal{T}^\Phi M[w] \xrightarrow{\psi} \Gamma \mathcal{T}^\Phi M[w + 1] \quad f \mapsto \psi \sigma f
\]

\[
h : \Gamma \mathcal{T}^\Phi M[w] \xrightarrow{\psi} \Gamma \mathcal{T}^\Phi M[w] \quad f \mapsto (d + 2w)f
\]

\[
y : \Gamma \mathcal{T}^\Phi M[w] \xrightarrow{\psi} \Gamma \mathcal{T}^\Phi M[w - 1] \quad f \mapsto -\frac{1}{2} I \cdot D f
\]

Following [31, Proposition 3.4] the following result holds.

**Proposition 2.16.** The operators \( \{x, h, y\} \) obey the \( \mathfrak{g} := \mathfrak{sl}(2) \) algebra

\[
[h, x] = 2x, \quad [x, y] = h, \quad [h, y] = -2y.
\]

This operator algebra was called a solution generating algebra in [31] for reasons that will rapidly become apparent.

**Remark 2.17.** As proven in [31] the above solution generating algebra holds for any defining scale \( \sigma \) such that its scale tractor \( I \) is nowhere null. Even in the case \( I^2 = 0 \), a contraction of the above algebra holds [31]. In the latter half of this Article we will draw further operators into the above algebra that act on tractor forms. That system has a proclivity towards almost Einstein structures.
In what follows we will often employ the quadratic Casimir
\[ c := xy + \frac{1}{4} h(h - 2), \]
which commutes with \( x, y \) and \( h \).

Our results rely heavily on identities in the enveloping algebra \( U(\mathfrak{g}) \), in particular, for any \( k \in \mathbb{Z}_{\geq 0} \)
\[ [y, x^k] = -k x^{k-1} (h + k - 1), \tag{2.16} \]
\[ [y^k, x] = -k y^{k-1} (h - k + 1). \tag{2.17} \]

It will also be highly advantageous to extend the enveloping algebra by the operator \( x^\alpha \), for any \( \alpha \in \mathbb{C} \), where
\[ x^\alpha : \Gamma T^\Phi M[w] \otimes \psi \rightarrow \Gamma T^\Phi M[w + \alpha] \]
\[ f \mapsto \sigma^\alpha f. \]

A straightforward calculation \[31\] shows that the identity \[2.16\] can be extended to arbitrary values of the exponent \( \alpha \in \mathbb{C} \) of \( x \)
\[ [y, x^\alpha] = -\alpha x^{\alpha-1} (h + \alpha - 1). \tag{2.18} \]

In Section 5.1.2 we will also need to draw \( \log x \) into our operator algebra. In \[31\] it was shown that
\[ [h, \log x] = 2, \quad [y, \log x] = -x^{-1} (h - 1). \tag{2.19} \]

Moreover, if \( \mu \in \Gamma \mathcal{E}_+[w] \) is any positive weighted conformal density, then \( \log \mu \) is a weighted log density, viz. a section of \( \mathcal{F}M[w_0] \), the log density bundle induced from the log representations of \( \mathbb{R}_+ \) \[31\]. It obeys
\[ [h, \log \mu] = 2w_0, \]
read as an operator relation acting on any section of a weighted tractor bundle. The relations in \[2.19\] hold on arbitrary sections of weighted tractor bundles as well as on log densities or tensor products of these with conformal densities, or more generally on a log density bundle in tensor product with any weighted tractor bundle.

2.8. **Product solutions.** In this Section we develop various operator identities valid in the universal enveloping algebra \( U(\mathfrak{g}) \) and completions thereof corresponding to power series in the generator \( x \). These are quite general in their validity and are intimately related to the theory of extremal projectors (see \[62\]). It is convenient to use \( x \) as a series variable in this algebraic context. Later, when we apply these results in a geometric setting, the Lie algebra generator \( x \) will be represented by the scale \( \sigma \).

2.8.1. **The first solution.** In \[31\], the following problem was posed and solved in terms of a certain solution generating operator based on a normal ordered representation for elements of \( U(\mathfrak{g}) \).

**Problem 2.18.** Given \( f|_\Sigma \), and an arbitrary extension \( f_0 \) of this to \( T^\Phi M[w_0] \) over \( M \), find \( f_i \in \mathcal{E}^\Phi[w_0 - i] \) (over \( M \), \( i = 1, 2, \cdots, \ell \)), so that
\[
 f^{(\ell)} := f_0 + x f_1 + x^2 f_2 + \cdots + x^{\ell} f_\ell + O(x^{\ell+1})
\]
solves
\[
 y f = O(x^{\ell}),
\]
off $\Sigma$, for $\ell \in \mathbb{N} \cup \infty$ as high as possible.

**Lemma 2.19.** If $f^{(\ell)}$ solves Problem 2.18 and $\ell \neq d + 2w_0 - 2$ to $O(x^{\ell})$, then

$$f^{(\ell+1)} = \frac{1}{(\ell + 1)(d + 2w_0 - 2 - \ell)} \left[ xy + (\ell + 1)(h - \ell - 2) \right] f^{(\ell)}$$

is a solution to $O(x^{\ell+1})$.

**Proof.** The proof relies on a direct computation utilizing the algebra (2.15) and (2.16). For brevity, we call $h_0 := d + 2w_0$ and in addition use that $y f^{(\ell)} = O(x^{\ell})$ means that

$$y \left[ xy + (\ell + 1)(h - \ell - 2) \right] f^{(\ell)} = \left[ xy + \ell(h - \ell - 1) \right] y f^{(\ell)}$$

$$= \left[ xy + \ell(h - \ell - 1) \right] x^{\ell} g_{\ell}$$

$$= x^{\ell+1} y g_{\ell} = O(x^{\ell+1}).$$

The normalization factor $\left[ (\ell + 1)(h_0 - \ell - 2) \right]^{-1}$ ensures that $f^{(\ell+1)}$ is again a series of the form $f_0 + x f_1 + \cdots$. $\square$

The above Lemma, suggests a solution to Problem 2.18 in terms of products of operators acting on the initial data $f_0$ on $\Sigma$. A key insight is to express elements in $U(g)$ in terms of the Casimir $c$ and Cartan generator $h$. In particular, the operator appearing in Lemma 2.19 is one of a sequence of operators defined for any $j \in \mathbb{Z}$

$$c_j := c - \frac{1}{4}(h - 2j)(h - 2j - 2).$$

Simple algebra shows that

$$c_j = xy + j(h - j - 1);$$

Lemma 2.19 utilises $c_{\ell+1}$. Moreover (and this fact will feature prominently in the following)

$$c_0 = xy.$$

Since $yc = cy$ and $yh = (h + 2)y$, it follows immediately that

$$y c_j = c_{j-1} y.$$

In turn we have

$$y c_1 c_2 \cdots c_{\ell} = c_0 c_1 \cdots c_{\ell-1} y$$

$$= xy c_1 c_2 \cdots c_{\ell-1} y$$

$$= x c_0 c_1 \cdots c_{\ell-2} y^2$$

$$\vdots$$

$$= x^{\ell} y^{\ell+1}.$$  

Hence we already see that

$$\mathcal{F} := c_1 c_2 \cdots c_{\ell} f_0$$

obeys $y \mathcal{F} = O(x^{\ell})$. To solve Problem 2.18, we still need to relate this product to a series expansion in $x$. To that end we recall the following identity from [31] (which also can be derived directly from equation (2.16))

$$y x^j y^j = j(j-1)x^{j-1} y^j - jx^{j-1} y^j(h - 2) + x^j y^{j+1}, \quad j \in \mathbb{Z}_{\geq 0}.$$
It is very convenient to elevate this relation to one for formal power series of elements in \( U(\mathfrak{g}) \) of the form
\[
:K(z) := \sum_j :z^j: \alpha_j(h), \quad K(z) \in \mathbb{C}[z].
\]

When we allow the coefficients \( \alpha_j \) to be polynomials in the Cartan generator \( h \), these are always ordered to the right. Moreover, for \( z := xy \) the normal ordering \( : \cdot : \) defines the operators
\[
:z^j: := x^{j}y^{j}, \quad :x^m z^j: := x^{m+j}y^{j}, \quad :z^j y^m: := x^j y^{j+m}.
\]

Then equation (2.23) can be restated as
\[
y : K(z) := (zK''(z) + K(z))y - :K''(z)y:(h-2).
\]

The following technical Lemma establishes the relationship between solutions in the product form (2.22) and the formal series solutions of [31].

**Lemma 2.20.** The product \( c_1 c_2 \cdots c_\ell \) equals the following polynomial in \( x \)
\[
(2.25) \quad c_1 c_2 \cdots c_\ell = \sum_{j=0}^{\ell} \frac{\ell!}{j!} x^j y^j (h-j-2)_{\ell-j} = :K^{(\ell)}(z),
\]
where the Pochhammer symbol denotes the product
\[
(h-j-2)_{\ell-j} := (h-j-2)(h-j-3)\cdots(h-\ell-1),
\]
(subject to \( m_0 := 1 \)).

**Proof.** We proceed by induction. The base case \( \ell = 1 \) holds by definition of \( c_1 \)
\[
:K^{(1)}(z) := c_1 = (h - 2) + xy.
\]
The induction step requires us to demonstrate that
\[
c_{\ell+1} \sum_{j=0}^{\ell} \frac{\ell!}{j!} x^j y^j (h-j-2)_{\ell-j} = \sum_{j=0}^{\ell+1} \frac{(\ell+1)!}{j!} x^j y^j (h-j-2)_{\ell-j+1}.
\]
Computing the left hand side using (2.24) and the fact that \( x : K(z) y : = z K(z) \) yields
\[
:z^2 \frac{d^2 K^{(\ell)}(z)}{dz^2} + z K^{(\ell)}(z) : = z \frac{d K^{(\ell)}(z)}{dz} : (h - 2) + (\ell + 1) : K^{(\ell)}(z) : (h - \ell - 2).
\]
By inspection, the last term reproduces all terms in the expression for \( :K^{(\ell+1)}(z)\); save for the last, \( j = \ell + 1 \), term in the sum on the right hand side of the induction step requirement above, namely \( x^{\ell+1} y^{\ell+1} \). The remaining terms produce exactly this missing term essentially because \( K^{(\ell)}(z) \) solves equation (2.24) to \( O(z^{\ell+1}) \). This is also easily explicitly verified by direct computation: Calling \( E := z \frac{d}{dz} \), we have
\[
:z^2 \frac{d^2 K^{(\ell)}(z)}{dz^2} + z K^{(\ell)}(z) : = z \frac{d K^{(\ell)}(z)}{dz} : (h - 2).
\]
\[ \frac{hf}{\ell!}(h_0 - 2) \ell : \left[ 1 + \frac{z}{h_0 - 2} + \frac{z^2}{2!(h_0 - 2)(h_0 - 3)} + \cdots + \frac{z^\ell}{\ell!(h_0 - 2)\ell} \right] : f_0. \]

Therefore we see that, away from values \( h_0 \in \mathbb{Z}_{\geq 2} \), \( \mathcal{F}/(\ell!(h_0 - 2)\ell) \) solves Problem 2.18 and thus have established the following result.

**Proposition 2.21.** The solution to Problem 2.18 for any \( \ell \in \mathbb{Z}_{\geq 1} \), when \( h_0 \notin \mathbb{Z}_{\geq 2} \), is

\[ f^{(\ell)} = \left[ \prod_{j=1}^{\ell} \frac{c_j}{j(h_0 - j - 1)} \right] f_0. \]

**Remark 2.22.** In the limit \( \ell \to \infty \), the polynomial in \( z \) in the squared brackets on the right-hand side of display (2.26) for the non-normalised solution, gives the series expansion of the Bessel function of the first kind

\[ K^{h_0}(z) := z^{\frac{h_0 - 1}{2}} \Gamma(2 - h_0) J_{1 - h_0}(2\sqrt{z}). \]

Once the Cartan generators \( h \) in the \( c_j \) of the product solution (2.27) are replaced by their eigenvalues \( h_0 \) when they act on \( f_0 \), then the infinite product from the \( \ell \to \infty \) limit can also be formally evaluated; use of the computer package Maple yields

\[ \prod_{j=1}^{\infty} \frac{c - \frac{1}{4}(h_0 - 2j)(h_0 - 2j - 2)}{j(h_0 - j - 1)} = \frac{-(c - \frac{1}{4}h_0(h_0 - 2))^{-1} \Gamma(2 - h_0)}{\Gamma \left( \frac{4 - h_0}{2} - \sqrt{c + \frac{1}{4}} \right) \Gamma \left( \frac{4 - h_0}{2} + \sqrt{c + \frac{1}{4}} \right)}. \]

Note that this ratio of gamma functions can be encoded by a single beta function and the left-hand side can also be expressed in terms of \( xy \) so

\[ \prod_{j=1}^{\infty} \frac{x y + j(h_0 - j - 1)}{j(h_0 - j - 1)} = - \left[ xy \beta \left( \frac{4 - h_0}{2} - \sqrt{c + \frac{1}{4}}, \frac{4 - h_0}{2} + \sqrt{c + \frac{1}{4}} \right) \right]^{-1}. \]

**Remark 2.23.** The formal square root of the Casimir \( \sqrt{c + \frac{1}{4}} \) appeared in [31, 42] where it was employed to extend the algebra \( \mathcal{U}(\mathfrak{g}) \) by an operator whose eigenvalues gave the so-called “depth” of states. In the language used here, this amounts to supposing that \( f \) is a simultaneous eigenfunction of the Casimir \( c \) and Cartan generator \( h \) and obeys \( y^j f \neq 0 = y^{j+1} f \). The integer \( j \) is the depth.
Lemma 2.24.

$$c_1 c_2 \ldots c_\ell x = x^{\ell+1} y^\ell.$$  

Proof. The proof is identical to the derivation of the identity \((2.21)\) for \(y\) acting from the left on a product of \(c_i\)'s save that one now uses the readily verified identity

$$c_j x = x c_{j-1}.$$  

\[\square\]

Remark 2.25. Lemma \(2.24\) shows that the solution \((2.27)\) does not depend on how \(f_x\) is extended off \(\Sigma\) to \(f_0 \in \Gamma \mathcal{T}^\Phi M[w_0]\) since shifting \(f_0 \to f_0 + x f_1\) for any \(f_1 \in \Gamma \mathcal{T}^\Phi M[w_0 - 1]\) we have \(c_1 c_2 \ldots c_\ell x f_1 = O(x^{\ell+1})\).

Remark 2.26. The poles at \(h_0 = 2, 3, \ldots\) in the solution given in Proposition \(2.21\) suggest that the simple product formula must be adjusted at these values. Indeed, observe that if \(h_0 \in \mathbb{Z}_{\geq 2}\) and \(f^{(h_0 - 2)}\) is an \(O(x^{h_0 - 2})\) solution to Problem \(2.18\) then Lemma \(2.19\) fails to provide a \(O(x^{h_0 - 1})\) solution because

$$c_{h_0 - 1} f^{(h_0 - 2)} = c_0 f^{(h_0 - 2)} = xy f^{(h_0 - 2)}$$

and therefore vanishes along \(\Sigma\).

Even when \(h_0 \in \mathbb{Z}_{\geq 2}\), we may still evaluate products of the form \((2.22)\) for \(\ell > h_0 - 1\)

$$\mathcal{F} = c_1 c_2 \ldots c_\ell f_0 = c_1 \ldots c_{h_0 - 2} x y c_{h_0} \ldots c_\ell f_0 = x^{h_0 - 1} y^{h_0 - 1} c_{h_0} \ldots c_\ell f_0 = x^{h_0 - 1} c_1 \ldots c_{\ell - h_0 + 1} f_0,$$

where \(f_0 = y^{h_0 - 1} f_0\).

Clearly \(y \mathcal{F} = O(x^{h_0 - 1 + \ell})\) but the behaviour of \(\mathcal{F}\) near \(\Sigma\) is no longer of the Dirichlet type stipulated in Problem \(2.18\). In fact, this is an example of a solution of the second kind.

2.8.2. The second solution. Since the equation \(y f = 0\) corresponds to a second order differential equation for a normal ordered solution generating operator as in \((2.24)\), we expect to find a second independent solution. In \([11]\), by extending the algebra \(\mathcal{U}(g)\) by \(x^\alpha\) for \(\alpha \in \mathbb{C}\), this was shown to be the case with precisely determined boundary behaviour encapsulated by the following.

Problem 2.27. Given \(f_0|\Sigma \in \Gamma \mathcal{T}^\Phi M[-d - w_0 + 1]|\Sigma\) and an arbitrary extension \(f_0\) of this to \(\Gamma \mathcal{T}^\Phi M[-d - w_0 + 1 - i]\), \(i = 1, 2, \ldots\), so that

$$f^{(\ell)} := x^{h_0 - 1} (f_0 + x f_1 + x^2 f_2 + \cdots + O(x^{\ell+1})), \quad h_0 := d + 2 w_0,$$

solves \(y f = O(x^{h_0 - 1 + \ell})\), off \(\Sigma\), for \(\ell \in \mathbb{N} \cup \infty\) as high as possible.

Remark 2.28. If the leading behaviour of \(f\) is relaxed to \(f = x^\alpha (f_0 + x f_1 + \cdots)\), one quickly learns (see \([11]\), Section 5.3) that the value \(\alpha = h_0 - 1\) is forced. An easy way to understand this is to note that for any \(g \in \Gamma \mathcal{T}^\Phi M[-d - w_0 + 1]\), by virtue of \((2.18)\), the following holds

$$y x^{h_0 - 1} g = x^{h_0 - 1} y g.$$  

This underlies a scale duality map on solutions which is implicit in the solution below and discussed in more detail in Section 5.1.1.
We now seek a product type solution of Problem 2.27. The above remark immediately shows us how to proceed. Since the operator \( y \) effectively commutes through the leading \( x^{h_0-1} \) behaviour, we are left with a version of the original Problem 2.18 but now at the weight \(-d - w_0 + 1\). This establishes the following result.

**Proposition 2.29.** The solution to Problem 2.27 for \( \ell \) arbitrarily high and any \( h_0 \notin \mathbb{Z}_{\leq 0} \) is

\[
(2.30) \quad \mathcal{J}^{(\ell)} = x^{h_0-1} \left[ \prod_{j=1}^{\ell} \frac{c_j}{j(1-j-h_0)} \right] \mathcal{J}_0.
\]

**Remark 2.30.** Again, we have a product form for the solution generating operator, defined independently of how \( \mathcal{J} \bigr|_{\Sigma} \) is extended off \( \Sigma \). Performing the infinite product, this is formally encoded by the operator

\[
- \frac{x^{h_0-1} \left( c - \frac{1}{4} h_0 (h_0 - 2) \right)^{-1} \Gamma(h_0)}{\Gamma \left( \frac{h_0+2}{2} - \sqrt{c + \frac{1}{4}} \right) \Gamma \left( \frac{h_0+2}{2} + \sqrt{c + \frac{1}{4}} \right)} : \Gamma T^\Phi M[-d - w_0 + 1]|_{\Sigma} \rightarrow \Gamma T^\Phi M[w_0].
\]

In addition, composing the above with the operator

\[
y^{h_0-1} : \Gamma T^\Phi M[w_0] \rightarrow \Gamma T^\Phi M[-d - w_0 + 1],
\]

yields

\[
- \frac{x^{h_0-1} \left( c - \frac{1}{4} h_0 (h_0 - 2) \right)^{-1} \Gamma(h_0) y^{h_0-1}}{\Gamma \left( \frac{h_0+2}{2} - \sqrt{c + \frac{1}{4}} \right) \Gamma \left( \frac{h_0+2}{2} + \sqrt{c + \frac{1}{4}} \right)} : \Gamma T^\Phi M[w_0]|_{\Sigma} \rightarrow \Gamma T^\Phi M[w_0].
\]

The image of both the above formal maps is in the kernel of the operator \( y : \Gamma T^\Phi M[w_0] \rightarrow \Gamma T^\Phi M[w_0] \).

Notice that for any \( h_0 - 1 \in \mathbb{Z}_{>0} \) (respectively \( \mathbb{Z}_{<0} \)), the \( \ell = \infty \) power series solutions of the first (respectively second) kind are obstructed. Moreover, when \( h_0 = 1 \), the solutions of the first and second kind coincide. As explained in Section 5.4 of [31] this presages the appearance of solutions with log terms. We will return to these in Section 5.1.2.

### 3. Tractor exterior calculus

Form bundles, by virtue of their exterior calculus, play a distinguished rôle among tensor bundles; the same holds for so-called tractor form bundles. This is perhaps not surprising on the basis of the close relationship between these and form bundles in the Fefferman–Graham ambient space. In fact, the latter connection is exploited heavily in the exposition on tractor forms given in [9]. Here we mainly eschew an ambient approach, and construct the main elements of a tractor exterior calculus directly from the viewpoint of the underlying conformal manifold \((M, c)\).

The natural tractor analog of a one-form belongs to the weight \(-1\), rank \(d + 2\) tractor bundle \( T^A M[-1] \) which for a given \( g \in c \) enjoys the isomorphism

\[
T^A M[-1] \overset{g}{\cong} \mathcal{E} M[0] \oplus \Lambda^1 M \oplus \mathcal{E} M[-2].
\]

The \( k \)-fold exterior product of this bundle yields a tractor bundle of weight \(-k\) which we denote \( T^k M[-k] \). It is isomorphic for a choice \( g \in c \) to the direct sum

\[
T^k M[-k] \overset{g}{\cong} \Lambda^{k-1} M \oplus \Lambda^k M \oplus \mathcal{E}^{k-2} M[-2] \oplus \mathcal{E}^{k-1} M[-2].
\]
(Recall that \(\mathcal{E}^k M[w]\) denotes the bundle \(\Lambda^k M \otimes \mathcal{E} M[w]\).) Tensoring with the weighted conformal density bundle \(\mathcal{E} M[w + k]\) we obtain a weight \(w\) bundle which we term the exterior (or form) tractor bundle \(\mathcal{T}^k M[w]\). For \(g \in c\) this obeys
\[
\mathcal{T}^k M[w] \cong \mathcal{E}^{k-1} M[w + k] \oplus \mathcal{E}^k M[w + k] \oplus \mathcal{E}^{k-2} M[w + k] \oplus \mathcal{E}^{k-1} M[w + k - 2].
\]
We may employ this isomorphism to express sections \(\mathcal{F} \in \Gamma \mathcal{T}^k M[w]\) as
\[
\mathcal{F} = \begin{pmatrix} F^+ & F & F^{++} & F^- \end{pmatrix}. 
\]
For ease of discourse, we will often refer to the various components of this splitting as the “northern”, “western”, “eastern” and “southern” slots, respectively, for the weighted forms \(F^+, F, F^{++}\) and \(F^-\). This terminology was devised in the presentation of [9] to reflect the composition series structure of the tractor form bundle. On the other hand, here we use a representation that makes operator compositions compatible with matrix multiplication.

Invariant sections of \(\mathcal{T}^k M[w]\) corresponding to conformally related choices of metric \(\hat{g} = e^{2\omega} g\), are related by the transformation formula

\[
\begin{pmatrix} F^+ \\ F \\ F^{++} \\ F^- \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \varepsilon(\Upsilon) & 1 & 0 & 0 \\ -\iota(\Upsilon) & 0 & 1 & 0 \\ \frac{1}{2}(\varepsilon(\Upsilon)\iota(\Upsilon) - \iota(\Upsilon)\varepsilon(\Upsilon)) & -\iota(\Upsilon) & -\varepsilon(\Upsilon) & 1 \end{pmatrix} \begin{pmatrix} F^+ \\ F \\ F^{++} \\ F^- \end{pmatrix}.
\]

(As usual we denote \(\Upsilon := d\omega\) and \(\iota, \varepsilon\) are the standard exterior and interior products on forms.) Moreover, in terms of the above splitting, the tractor connection is given by
\[
\nabla^\Upsilon_v \begin{pmatrix} F^+ \\ F \\ F^{++} \\ F^- \end{pmatrix} = \begin{pmatrix} \nabla_v & -\iota(v) & \varepsilon(v) & 0 \\ \varepsilon(\nabla_v) & \nabla_v & 0 & \varepsilon(v) \\ -\iota(\nabla_v) & 0 & \nabla_v & \iota(v) \\ 0 & -\iota(\nabla_v) & -\varepsilon(\nabla_v) & \nabla_v \end{pmatrix} \begin{pmatrix} F^+ \\ F \\ F^{++} \\ F^- \end{pmatrix},
\]

where \(v\) is an arbitrary section of \(TM\) and \(\mathcal{P}\) denotes the canonical endomorphism \(TM \to TM\) obtained from the Schouten tensor (in an index notation, \(\mathcal{P}_v := (P^a_b v^b)\)). It is easy to check that this formula enjoys the transformation property in (3.1). In what follows, we will often assume some choice of splitting and represent various operators on tractor forms by a \(4 \times 4\) matrix of operators as in the example above.

For completeness we record the exterior algebra and tractor Hodge star operator of [9] in the splitting notations above. The wedge product maps sections \(\mathcal{F} \mathcal{G}\) of \(\mathcal{T}^k M[w]\) and \(\mathcal{T}^k M[w]\), respectively, to a section \(\mathcal{F} \wedge \mathcal{G}\) of \(\mathcal{T}^{k+k'}[w+w']\) given in the above splitting as
\[
\mathcal{F} \wedge \mathcal{G} = \begin{pmatrix} F^+ & F \\ F^{++} \\ F^- \end{pmatrix} \wedge \begin{pmatrix} G^+ \\ G^{++} \\ G^- \end{pmatrix} = \begin{pmatrix} F^+ \wedge G + (-1)^k F \wedge G^+ \\ F^{++} \wedge G + (-1)^k F \wedge G^{++} \\ F^- \wedge G + (-1)^k F \wedge G^- \end{pmatrix}.
\]

The conformal Hodge star operator \(\ast : \mathcal{E}^k M[w] \xrightarrow{\sim} \mathcal{E}^{d-k} M[d + w - 2k]\). We shall denote the degree operator on forms by \(\mathcal{N}\), so that for \(A \in \Gamma \mathcal{E}^k M[w]\),
\[
\mathcal{N} A = k A.
\]
In these terms we have $\ast\ast = (-1)^{N(d-\mathcal{N})+q}$ (where $(-1)^q$ is the sign of the metric determinant in the metric signature $(p,q)$ and is unity for Riemannian metrics). From the conformal Hodge star we can build the tractor Hodge star

$$\star : \mathcal{T}^k M[w] \overset{\sim}{\rightarrow} \mathcal{T}^{d+2-k} M[w],$$

defined in a given splitting by

$$\star F \overset{\gamma}{=} \begin{pmatrix} (-1)^N & 0 & 0 & 0 \\ 0 & 0 & * & 0 \\ 0 & * & (-1) & 0 \\ 0 & 0 & 0 & * \end{pmatrix} \begin{pmatrix} F^+ \\ F \\ F^- \\ F^{++} \end{pmatrix}.$$

We call the degree operator on tractor forms $\mathcal{N}$ so that for $F \in \Gamma T^k M[w]$,

$$(3.2) \quad \mathcal{N} F = k F.$$

It follows directly that

$$\ast\ast = (-1)^{\mathcal{N}(d+2-\mathcal{N})+q+1},$$

in concordance with the relationship between tractor forms and forms in the Fefferman–Graham ambient space [9].

The standard inner product on $\Omega^k M$ provides an inner product on $\Gamma T^k M[w]$2 as follows. Firstly, $\Gamma T^{d+2} M[-d]$ is canonically isomorphic to $\Omega^d M$ and not only are elements of the latter conformally invariant, but they can be integrated over $M$ (oriented). For $B \in \Gamma T^{d+2} M[-d]$ we shall write $\int_M B$ assuming this above isomorphism. Since the tractor Hodge star $\star : \Gamma T^k M \left[ -\frac{d}{2} \right] \rightarrow \Gamma T^{d+2-k} M \left[ -\frac{d}{2} \right]$, given $\mathcal{A}, \mathcal{A}' \in \Gamma T^k M \left[ -\frac{d}{2} \right]$, we define the conformally invariant inner product $\langle \mathcal{A}, \mathcal{A}' \rangle = \int_M \mathcal{A} \wedge \star \mathcal{A}'$. Moreover, at arbitrary weights $\mathcal{A}, \mathcal{A}' \in \Gamma T^k M[w]$

$$\langle \mathcal{A}, \mathcal{A}' \rangle := \text{sgn} \left( \int_M \mathcal{A} \wedge \star \mathcal{A}' \right) \in \{ \pm 1, 0 \},$$

is conformally invariant. This pairing is useful for developing orthogonal decompositions.

3.1. Algebra of invariant operators. We now develop a conformally invariant exterior calculus of tractor forms. This efficiently compresses a large class of operators and accompanying identities.

The utility of differential forms relies on the exterior derivative operator

$$d : \Omega^k M \rightarrow \Omega^{k+1} M, \quad d^2 = 0,$$

its Hodge dual $(-1)^N \ast^{-1} d \ast$, viz. the codifferential

$$\delta : \Omega^k M \rightarrow \Omega^{k-1} M, \quad \delta^2 = 0,$$

and the supersymmetry algebra formed by these and the form Laplacian

$$\Delta = \delta d + d\delta =: \{ \delta, d \}, \quad [d, \Delta] = 0 = [\Delta, \delta].$$

An examination of the Thomas D-operator in (2.13), suggests there ought exist a conformally invariant operator on form tractors that unifies the exterior derivative, codifferential and form Laplacian. A natural candidate for such an operator would simply be the exterior, or skew, action of the Thomas D-operator $D\mathcal{A}$. Although partially true, this expectation is not fulfilled, essentially because what appears in the bottom slot of (2.13) is the Bochner–Laplacian, rather than its form counterpart. The solution to this problem was given in [9]; just as the difference between form and Bochner Laplacians is given
by a natural action of the curvature tensor, a similar modification $\mathcal{D}^A$ of the Thomas D-operator exists such that its exterior action on tractor forms is nilpotent.

It is straightforward to compute the exterior derivative-type operator $\varepsilon(\mathcal{D})$ of [9] in the $4 \times 4$ matrix notation for a given splitting as introduced above. We shall denote this operator by

$$\mathcal{D} : \Gamma^k M[w] \rightarrow \Gamma^{k+1} M[w-1],$$

and refer to it as the \textit{exterior tractor D-operator}. Our result for this computation is

$$D \varepsilon = \begin{pmatrix}
- (d+2w-2)d & (w+N)(d+2w-2) & 0 & 0 \\
0 & (d+2w-2)d & 0 & 0 \\
\Delta + (w+N-1)(J-2P) & -2\delta & (d+2w)d & (w+N-1)(d+2w) \\
[J-2P,d] & -\Delta - (w+N)(J-2P) & 0 & -(d+2w)d
\end{pmatrix}.$$

Here we have canonically extended $\mathcal{D}$ by linearity to act on forms of degree $k$. Nilpotency $D^2 = 0$, of $D$ follows from [9], but can also be readily verified by a simple matrix composition based on the above display.

The tractor analog of the codifferential is also given in [9] as the interior action of $\iota(\mathcal{D}) = (-1)^{N-1} \star^{-1} \mathcal{D} \star$. We will denote this operator by

$$D^* : \Gamma^k M[w] \rightarrow \Gamma^{k-1} M[w-1],$$

and dub it the \textit{interior tractor D-operator}. It acts in a given splitting according to

$$D^* g = \begin{pmatrix}
- (d+2w-2)\delta & 0 & -(d+2w-2)(d+w-N) & 0 \\
0 & -(d+2w-2)d & -2d & (d+2w)(d+w-N-1) \\
\Delta + (d+w-N-1)(J-2P) & (d+2w)\delta & 0 & (d+2w-2)\delta \\
[J-2P,\delta] & -\Delta + (d+w-N)(J-2P) & - (d+2w)\delta & 0
\end{pmatrix}.$$

Per [9], it is also nilpotent

$$D^* = 0,$$

and anticommutes with the exterior derivative

$$D^* D + D D^* = 0.$$

Altogether, including the weight operator $h := d + 2w$ (see (2.12)) and tractor form degree $\mathcal{N}$ (as in (3.2)) we have established the beginnings of an \textit{exterior tractor algebra}

$$\{\mathcal{D}, D^*\} = 0, \quad D^2 = 0 = D^{*2},$$

$$[\mathcal{N}, \mathcal{D}] = D, \quad [\mathcal{N}, D^*] = -D^*,$$

$$[h, \mathcal{D}] = -2D, \quad [h, D^*] = -2D^*.$$

We now augment these identities with exterior and interior multiplication by the canonical tractor $X^A$: Let us denote exterior multiplication by this with $\varepsilon(X)$, which we term the \textit{exterior canonical tractor}

$$\mathcal{X} : \mathcal{T}^k M[w] \rightarrow \mathcal{T}^{k+1} M[w+1].$$

In a choice of splitting this is represented by

$$\mathcal{X} \equiv \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0
\end{pmatrix}.$$
For later use, it is worth noting that this operator moves the northern and western slots, respectively, to the eastern and southern slot. Its adjoint, the interior canonical tractor $\mathcal{X}^* : T^k M[w] \to T^{k-1} M[w+1]$.  

In a choice of splitting this is represented by 

$$(3.6) \quad \mathcal{X}^* \equiv \begin{pmatrix} 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \end{pmatrix},$$

which moves the northern slot to the west and the eastern slot to the south.

Therefore, we may now add to our exterior tractor algebra the relations 

$$\{ \mathcal{X}, \mathcal{X}^* \} = 0, \quad \mathcal{X}^2 = 0 = \mathcal{X}^{**};$$

$$[\mathcal{X}, \mathcal{X}] = \mathcal{X}, \quad [\mathcal{X}, \mathcal{X}^*] = -\mathcal{X}^*;$$

$$[h, \mathcal{X}] = 2\mathcal{X}, \quad [h, \mathcal{X}^*] = 2\mathcal{X}^*.$$

To compute an algebra for products of the interior and exterior Thomas D- and canonical tractor operators amongst one another we rely on the following:

**Lemma 3.1.** Let $h^{AB}$ denote the tractor metric and $h := d + 2w$. Then the Thomas D- and canonical tractor operators obey the identity 

$$(3.8) \quad h X^A D^B - (h - 2) D^B X^A - 2 X^B D^A + h(h - 2) h^{AB} = 0.$$ 

The simplest proof of the above employs the ambient techniques of \cite{28, 11}, although this relation also follows from known tractor identities, see for example \cite{21, 22}. Proofs of this and other results relying on an ambient formulation are collected in Appendix A.

**Remark 3.2.** Suppose $W$ is any weight $-2$ tractor tensor in $\Gamma(\otimes^k \text{End}(T^k M))$ and let $\sharp$ denote the natural tensorial action of endomorphisms on tractor sections (to the right) so, for example, for $k = 1$ and acting on a rank one tractor $T^A$ 

$$W^\sharp T^A := T^B W_B^A.$$ 

Moreover, let us suppose that $W$ is orthogonal to the canonical tractor $X^A$, i.e., the contraction of $X^A$ with any index of $W$ vanishes. Hence 

$$(k \text{ times}) \quad W^{\sharp \cdots \sharp} X^A = 0.$$ 

It follows immediately that Lemma (3.1) still holds upon the replacement of the Thomas D-operator by 

$$D^A \mapsto D^A - X^A W^{\sharp \cdots \sharp}.$$ 

Now recall the definition of $D^A := D^A - X^A \Omega^\sharp$ of \cite{9}. Here $\Omega \in \Gamma(\otimes^2 \text{End}(T^k M))$ equals $1/(d-4)$ times the $W$-tractor of \cite{21} in dimensions other than 4, and is perpendicular to the canonical tractor $X^A$. It follows immediately from Lemma (3.1) and Remark (3.2) that 

$$(3.9) \quad h X^A D^B - (h - 2) D^B X^A - 2 X^B D^A + h(h - 2) h^{AB} = 0,$$

in dimensions $d \neq 4$. 

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Remark 3.3. As it stands, $\mathcal{D}$ is not defined in four dimensions. This restriction is inconsequential for our purposes. The first point is that the exterior version of the operator $\mathcal{D}$ given in (3.3) is well-defined in all dimensions $d \geq 3$, see [9]. (A quick way to see this is to observe $\mathcal{D} \Omega^w$ is well defined in any dimension.) On the other hand, beyond the exterior setting the above formula is useful for efficiently proving several results. In practice for the problems solved here, we may use the above formula in all dimensions because in the presence of an Einstein scale a version of it holds in four dimensions: It is possible to construct an invariant tractor $\tilde{W}/(d - 4)$ that equals $W/(d - 4)$ in dimensions other than four and is well defined when $d = 4$, although in that dimension (and only then) $\tilde{W}/(d - 4)$ depends on the scale [23, Section 4]; see also the proof of Proposition 4.5 below.

Proposition 3.4. The exterior and interior Thomas D- and canonical tractor operators $\mathcal{D}, \mathcal{D}^*, \mathcal{X}$ and $\mathcal{X}^*$, satisfy
\[
(h - 2) \mathcal{D} \mathcal{X} + (h + 2) \mathcal{X} \mathcal{D} = 0 = (h - 2) \mathcal{D}^* \mathcal{X}^* + (h + 2) \mathcal{X}^* \mathcal{D}^*,
\]
\[
h \mathcal{X} \mathcal{D}^* + (h - 2) \mathcal{D}^* \mathcal{X} + 2 \mathcal{X}^* \mathcal{D} - \left( \frac{d + h}{2} - N + 2 \right) h(h - 2) = 0,
\]
\[
h \mathcal{X}^* \mathcal{D} + (h - 2) \mathcal{D} \mathcal{X}^* + 2 \mathcal{X} \mathcal{D}^* + \left( \frac{d - h}{2} - N \right) h(h - 2) = 0.
\]

Proof. The proof amounts to acting with the left hand side of the identity (3.9) on an arbitrary tractor form and then taking appropriate irreducible parts. □

At the weight $w = -d/2$, the operators $\mathcal{D}, \mathcal{D}^*$ are the zero maps on $\ker \mathcal{X}, \ker \mathcal{X}^*$, respectively. We can, however, define a nontrivial analog of $\mathcal{D}$ and $\mathcal{D}^*$ at this weight on these spaces by considering the residues of $(d + 2w)^{-1} \mathcal{D}$ and $(d + 2w)^{-1} \mathcal{D}^*$ at $w = -d/2$. This motivates the following definitions.

Definition 3.5. Acting on $\mathcal{T}^k M[w]$ with $w \neq -\frac{d}{2}$, define the composition of operators
\[
\hat{\mathcal{D}} := \mathcal{D} \frac{1}{h}, \quad \hat{\mathcal{D}}^* := \mathcal{D}^* \frac{1}{h}.
\]

For $w = -\frac{d}{2}$ define conformally invariant operators
\[
\hat{\mathcal{D}} : \ker \mathcal{X} \subset \Gamma \mathcal{T}^k M[-\frac{d}{2}] \longrightarrow \ker \mathcal{X} \subset \Gamma \mathcal{T}^{k+1} M[-1 - \frac{d}{2}]
\]
and
\[
\hat{\mathcal{D}}^* : \ker \mathcal{X}^* \subset \Gamma \mathcal{T}^k M[-\frac{d}{2}] \longrightarrow \ker \mathcal{X}^* \subset \Gamma \mathcal{T}^{k-1} M[-1 - \frac{d}{2}]
\]
via their expressions for some $g \in c$ acting on $A \in \ker \mathcal{X} \subset \Gamma \mathcal{T}^k M[-\frac{d}{2}]$ and on $A \in \ker \mathcal{X}^* \subset \Gamma \mathcal{T}^k M[-\frac{d}{2}]$
\[
\hat{\mathcal{D}} A \overset{g}{=} \hat{\mathcal{D}} \begin{pmatrix} 0 \\ 0 \\ B \\ \phi \end{pmatrix} := \begin{pmatrix} 0 \\ 0 \\ dB + (k - \frac{d}{2} - 2) \phi \\ -d \phi \end{pmatrix}
\]
and on $A \in \ker \mathcal{X}^* \subset \Gamma \mathcal{T}^k M[-\frac{d}{2}]$
\[
\hat{\mathcal{D}}^* A \overset{g}{=} \hat{\mathcal{D}}^* \begin{pmatrix} A \\ 0 \\ \phi \end{pmatrix} := \begin{pmatrix} 0 \\ 0 \\ \delta A - (k - \frac{d}{2}) \phi \end{pmatrix}.
\]
Remark 3.6. The notation $\frac{1}{h}$ above makes sense because we can always work with weight eigenspaces.

The operators of the above Definition are intimately related to the exterior and interior actions of the first order differential operator $D^{AB} := X^B \tilde{D}^A - X^A \tilde{D}^B$ termed the double D-operator [21, 22]. In particular, the operators $\hat{D}^A$ and $\hat{D}^* A^*$ are given, for some $g \in c$, by

\begin{align*}
\mathcal{D}_2^A &:= \hat{D} A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -d & w + N & 0 & 0 \\ 0 & -d & 0 & 0 \end{pmatrix} \quad \text{and} \quad \mathcal{D}_2^* A^* := \hat{D}^* A^* = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \delta & 0 & d + w - N & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -\delta & 0 \end{pmatrix}.
\end{align*}

An important fact is that both the above, nilpotent, Grassmann even, first order operators obey the Leibnitz rule acting on products of tractor forms. We shall term $\mathcal{D}_2^A$ and $\mathcal{D}_2^* A^*$ the exterior, and interior double D-operators, respectively.

It is also useful to introduce a version of Definition 3.5 tailored to the cokernels of the operators $\mathcal{X}$ and $\mathcal{X}^*$. For a choice of $g \in c$, the cokernel of $\mathcal{X}$ is defined by the equivalence relation

$$\begin{pmatrix} \psi \\ A \\ B \\ \phi \end{pmatrix} \sim \begin{pmatrix} \psi \\ A \\ B + b \\ \phi + f \end{pmatrix} \in \text{coker} \mathcal{X},$$

(an analogous formula holds for $\text{coker} \mathcal{X}^*$) so we will employ the notations

$$\text{coker} \mathcal{X} \ni \begin{pmatrix} \psi \\ A \\ * \\ * \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \psi \\ * \\ B \\ * \end{pmatrix} \in \text{coker} \mathcal{X}^*,$$

for elements of coker $\mathcal{X}$ and coker $\mathcal{X}^*$, respectively.

Definition 3.7. Acting on $T^k M[w]$ with $w \neq 1 - \frac{d}{2}$, define the composition of operators

$$\tilde{\mathcal{D}} := \frac{1}{h} \mathcal{D}, \quad \tilde{\mathcal{D}}^* := \frac{1}{h} \mathcal{D}^*.$$ 

For $w = 1 - \frac{d}{2}$ define

$$\tilde{\mathcal{D}} : \text{coker} \left( \mathcal{X}, \Gamma T^k M[1 - \frac{d}{2}] \right) \longrightarrow \text{coker} \left( \mathcal{X}, \Gamma T^{k+1} M[-\frac{d}{2}] \right)$$

and

$$\tilde{\mathcal{D}}^* : \text{coker} \left( \mathcal{X}^*, \Gamma T^k M[1 - \frac{d}{2}] \right) \longrightarrow \text{coker} \left( \mathcal{X}^*, \Gamma T^{k+1} M[-\frac{d}{2}] \right)$$

via their expressions acting on $A \in \text{coker} (\mathcal{X}, \Gamma T^k M[1 - \frac{d}{2}])$ for some $g \in c$

$$\tilde{\mathcal{D}} A \equiv \tilde{\mathcal{D}} \begin{pmatrix} \psi \\ A \\ * \end{pmatrix} := \begin{pmatrix} -d \psi + (k - \frac{d}{2} + 1) A \\ dA \\ * \end{pmatrix}$$
and on $A \in \text{coker}\left(\mathcal{X}, \Gamma T^k M[1 - \frac{d}{2}]\right)$

$$\tilde{\mathcal{D}}^* A \overset{g}{=} \tilde{\mathcal{D}}^* \begin{pmatrix} \psi \\ B \end{pmatrix} := \begin{pmatrix} -\delta \psi + (k - \frac{d}{2} - 3)B \\ \delta B \end{pmatrix}.$$  

Observe that the operators $X \tilde{\mathcal{D}}$ and $X \mathcal{D}^*$ are well-defined acting on $\Gamma T^k M[w]$ at any weight. It is easy to relate them to the exterior and interior double D-operators using the first line of Proposition 3.4. The result of that computation is the following.

**Proposition 3.8.** On weighted tractor forms

$$\tilde{\mathcal{D}} \mathcal{X} + \mathcal{X} \tilde{\mathcal{D}} = 0 = \tilde{\mathcal{D}}^* \mathcal{X}^* + \mathcal{X}^* \tilde{\mathcal{D}}^*.$$  

To conclude this Section, we draw the exterior double D-operators into our algebra.

**Proposition 3.9.** On weighted tractor forms

$$[\mathcal{D}, \mathcal{X}] = -(\frac{h - d}{2} + N - 2) \mathcal{X}, \quad [\mathcal{D}^*, \mathcal{X}^*] = (\frac{h - d}{2} + N) \mathcal{D}^*,$$

$$[\mathcal{D}, \mathcal{D}^*] = (\frac{h + d}{2} - N + 2) \mathcal{D}, \quad [\mathcal{D}^*, \mathcal{D}^*] = -(\frac{h + d}{2} - N) \mathcal{D}^*.$$  

**Proof.** The most direct proof is a straightforward application of the matrix expressions for the exterior tractor operators as given in Equations (3.3), (3.4), (3.5), (3.7) and (3.10).  

3.2. **Algebra of differential splitting operators.** For many applications one begins with a differential form $A \in \Omega^\bullet M$, or perhaps more generally a form density $A \in \Gamma \mathcal{E}^\bullet[.]$ which one wishes to handle using the tractor machinery. Given a choice of $g \in c$ and the form $A$, there in fact exists a quartet of differential *insertion operators* $(q_N, q_E, q_S, q_W)$ which invariantly insert the form into a uniquely determined tractor form, respectively, whose northern, eastern, southern or western slot is given by $A$. We describe these in this Section.

Firstly, we have an isomorphism and its formal adjoint [9]

$q : \Gamma \mathcal{E}^k M[w + k] \overset{\cong}{\rightarrow} \text{coker}\left(\mathcal{X}, \Gamma T^k M[w]\right)$ and $q^* : \text{ker} \mathcal{X}^* \subset \Gamma T^k M[w] \overset{\cong}{\rightarrow} \Gamma \mathcal{E}^k M[w + k]$ via, for some $g \in c$

$$A \mapsto \begin{pmatrix} 0 \\ A \\ * \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 \\ A \\ 0 \\ \phi \end{pmatrix} \mapsto A.$$  

The following differential splitting operators have proved to be important [9, 16]. The first of these is in fact algebraic.

**Lemma 3.10 (South).** We have the isomorphism

$$q_S : \Gamma \mathcal{E}^{k-1} M[w + k - 2] \overset{\cong}{\rightarrow} \text{ker}(\mathcal{X}, \mathcal{X}^*) \subset \Gamma T^k M[w]$$

via

$$A \overset{g}{\mapsto} \begin{pmatrix} 0 \\ 0 \\ 0 \\ A \end{pmatrix}.$$
Figure 1. The insertion operators summarised by the points of a compass.

Remark 3.11. We will often refer to a tractor form in \( \ker(X, X^\ast) \) as a southern tractor, and employ the corresponding language for each of the insertion Lemmas below.

Proof. The proof follows immediately by computing the kernels of \( X \) and \( X^\ast \) by inspecting their matrix expressions in a given \( g \in c \) as displayed in equations (3.5) and (3.7).

Corollary 3.12. Let \( B \in \ker(\mathcal{D}, \mathcal{D}^\ast) \subset \Gamma T^k M[w] \) and \( w \neq -k + 2, k - d \). Then

\[
B = -\frac{1}{(w + k - 2)(d + w - k)} \mathcal{D}^\ast \mathcal{D} \mathcal{D}^\ast B,
\]

and when \( w = -k + 2, k - d \), \( \mathcal{D}^\ast \mathcal{D} \mathcal{D}^\ast B = 0 \) and the singularities there are removable.

In this Article we make heavy use of the western insertion operator that invariantly inserts a (weighted) degree \( k \)-form in a degree \( k \)-tractor form. It is described by the following result.

Lemma 3.13 (West). When \( w \neq k - d \) there is an isomorphism

\[
q_W : \Gamma E^k M[w + k] \xrightarrow{\cong} \ker(\mathcal{D}^\ast, \mathcal{D}^\ast) \subset \Gamma T^k M[w]
\]

via

\[
A \xrightarrow{g} \begin{pmatrix} 0 & A & 0 \\ A & 0 & \delta A \end{pmatrix}.
\]

At the critical weight \( w = k - d \), at any fixed scale \( g \in c \), there is a related isomorphism that we also denote \( q_W \)

(3.11)

\[
\ker(\delta, \delta) \subset (\Gamma E^k M[2k - d] \oplus \Gamma E^{k-1} M[2k - d - 2]) \xrightarrow{\cong} \ker(\mathcal{D}^\ast, \mathcal{D}^\ast) \subset \Gamma T^k M[k - d]
\]

via

\[
(A, \phi) \xrightarrow{g} \begin{pmatrix} 0 \\ A \\ 0 \end{pmatrix}.
\]

Proof. The proof proceeds as in the previous Lemma, but in addition requires a computation of the kernel of the interior tractor D-operator based on the display (3.4).
Remark 3.14. In Equation (3.11) it is natural to identify the domain objects \((A, \phi)\) with the components of the invariant tractor on the right in a scale \(g \in c\). Therefore, for some other \(\hat{g} \in c\), this pair is determined by Equation (3.1) to be \((\hat{A}, \hat{\phi})\) with \(\hat{A} = A\), \(\hat{\phi} = \phi - \iota(\Upsilon)A\) and \(\Upsilon\) defined as in Section 2.5. These are coclosed for any \(g \in c\).

Remark 3.15. The maps \(q\) and \(q^*\) are intimately related to \(q_W\) and \(q^{-1}_W\). Indeed, one has the operator identities
\[
\mathcal{D} q_W = \mathcal{D} q \quad \text{and} \quad q_W^{-1} \mathcal{D}^* = q^* \mathcal{D}^*.
\]

Obvious analogs of these relations also hold for the north and east insertion operators introduced below. The algebraic operators corresponding to \(q\) and \(q^*\) for those cases—defined on appropriate cokernel and kernels—will be denoted \(q(E)\), \(q(N)\) and \(q^*(E)\), \(q^*(N)\).

For consistency one should write \(q(W)\) for \(q\), but we shall use this operator so often that the latter notation is preferred.

The eastern insertion operator is related to its western counterpart by (tractor) Hodge duality. It is given by the following Lemma.

Lemma 3.16 (East). For \(w \neq -k + 2\)
\[
q_E : \Gamma \mathcal{E}^{k-2} M[w + k - 2] \xrightarrow{=} \ker(\widetilde{\mathcal{D}}, \mathcal{D}) \subset \Gamma \mathcal{T}^k M[w]
\]
by
\[
A \mapsto^g \begin{pmatrix} 0 \\ 0 \\ A \\ -\frac{1}{w+k-2} dA \end{pmatrix}.
\]

Remark 3.17. When \(w = -k + 2\) and at any fixed scale \(g \in c\) we have the isomorphism
\[
\ker(\mathcal{D}, \mathcal{D}^*) \subset (\Omega^{k-2} M \oplus \Omega^{k-1} M) \xrightarrow{=} \ker(\mathcal{D}, \mathcal{D}) \subset \Gamma \mathcal{T}^k M[-k + 2]
\]
with
\[
(A, F) \mapsto^g \begin{pmatrix} 0 \\ 0 \\ A \\ F \end{pmatrix}.
\]

For the northern insertion operator there are four classes of special weights to account for:
\[
w = \begin{cases} 1 - \frac{d}{2}, \\ -\frac{d}{2}, \\ -k, \\ -d - 2 + k, \end{cases} \quad k \in \{0, 1, \ldots, d + 2\}.
\]

Lemma 3.18 (North). For \(w \neq -\frac{d}{2}, -k, -d - 2 + k\)
\[
q_N : \Gamma \mathcal{E}^{k-1} M[w + k] \to \ker(\mathcal{D}, \mathcal{D}^*) \subset \Gamma \mathcal{T}^k M[w]
\]
To reach the west, write

\[ A \mapsto g \left( \begin{array}{c} A \\ \frac{1}{w+k} dA \\ - \frac{1}{d+w-k+2} dA \\ - \frac{1}{d+w-k+2} (\frac{1}{w+k} d - \frac{1}{d+w-k+2} d\delta + J - 2\delta) A \end{array} \right) \].

Moreover, the map \( q_N \) is an isomorphism whenever \( w \neq 1 - \frac{d}{2} \).

Although technically more involved, the proof uses exactly the same techniques as in the other insertion Lemmas.

**Corollary 3.19.** Let \( F \in \ker(\mathcal{D}, \mathcal{D}^*) \subset \Gamma T_k M[w] \) and \( w \neq 1 - \frac{d}{2}, -\frac{d}{2}, -k, k - d - 2 \), then

\[ F = -\frac{1}{(d+2w)(w+k)(d+w-k+2)} \mathcal{D}^* \mathcal{D} \mathcal{X}^* \mathcal{F}. \]

**Proof.** An elementary proof is to use the above explicit expression for \( \ker(\mathcal{D}, \mathcal{D}^*) \) and then calculate along the same lines as previously. Alternatively, Proposition 3.4 can be used to move the operators \( \mathcal{D} \) and \( \mathcal{D}^* \) to the right where they annihilate \( F \).

**Remark 3.20.** Note that at \( w = 1 - \frac{d}{2} \) the display of the above Lemma gives a solution to \( \mathcal{D} F = 0 = \mathcal{D}^* F \), but not the most general one.

**Remark 3.21.** It is clear how to project onto each slot and move from slot to slot. Starting from a weight \( w \), degree \( k \) northern tractor, and ignoring for this discussion distinguished weights, we have the projector

\[ A \mapsto 1 \]

which is the projector for weight \( w+1 \), degree \( k-1 \), western tractor. The same procedure holds for the south where \( F = \mathcal{D} \mathcal{D}^* \mathcal{B} \) and \( \mathcal{B} = -\frac{1}{(d+2w)(w+k)(d+w-k+2)} \mathcal{D}^* \mathcal{D} \mathcal{F} \) and the projector is given in Corollary 3.12.

The projector onto western tractors described in the above Remark is the one most often required in later developments. Hence we record it in the following.

**Proposition 3.22.** The operator \( \Pi_W : \Gamma T_k M[w] \rightarrow \Gamma T_k M[w] \), \( w \neq -k, k - d \), defined by

\[ \Pi_W := \frac{1}{(w+k)(d+w-k)} \mathcal{D}^* \mathcal{D} \]

obeys \( \Pi_W^2 = \Pi_W \). Moreover, if \( A \in \ker(\mathcal{D}^*, \mathcal{X}^*) \subset \Gamma T_k M[w] \), then \( \Pi_W A = A \).

We close this Section with a useful technical result which follows immediately from the machinery given in this Section.
Proposition 3.23. Let \( A \in \ker(\nabla^*, \nabla^*) \subset \Gamma T^k M[w] \). Then

\[
\nabla^* \nabla A = (w + k)A, \quad \nabla^* \nabla A = (d + w - k)A.
\]

Proof. For the second identity, observe that since \( \nabla^* A = 0 \), we may write \( A = \nabla^* B \) for some \( B \). Thus the left hand side is well defined for any weight \( w \). The result then follows by simple application of the algebra of Proposition 3.4. The first identity can be established the same way when \( w \neq 1 - \frac{d}{2} k - d \). The first weight is a simple removable singularity, as can be seen from Lemma 3.13. At the weight \( w = k - d \) there is no singularity but one needs to use Equation (3.11) of Lemma 3.13 to establish the result. \( \square \)

3.3. Conformally invariant equations and the cohomology of the Thomas \( D \) operator. For later developments, we need to understand the conformal differential operators on forms and their origins in the tractor calculus. At low orders, these follow from the basic equation

\[
\nabla F = 0,
\]

and various refinements thereof.

The simplest conformally invariant differential equation for a differential form is the closure condition

\[
dA = 0,
\]

which is conformally invariant for any \( A \in \Omega^k M \) with \( 0 \leq k \leq d \). For physical models, the form \( A \) can either be interpreted as a field strength or potential. In the former case, it is traditional to use the symbol \( F \). Imposing as well a divergence condition, we have the curvature version of the (higher form) Maxwell’s equations

\[
dF = 0 = \delta F.
\]

The divergence equation is conformally invariant in even dimensions when \( F \in \Omega^\frac{d}{2} M \).

The potential version of the higher form Maxwell’s equations are obtained by taking a divergence of the closure condition

\[
\delta dA = 0.
\]

This equation enjoys a gauge invariance

\[
A \sim A + d\alpha,
\]

for \( A \in \Omega^k M \) and \( \alpha \in \Omega^{k-1} M \) which is already evident from the conformally invariant de Rham complex

\[
d : \Omega^k M \xrightarrow{d} \Omega^{k+1} M.
\]

The potential form of Maxwell’s equations is conformally invariant in even dimensions for \( A \in \Omega^\frac{d}{2} M \).

Once weighted forms are considered, there are further conformally invariant equations: The higher form Branson–Deser–Nepomechie equation

\[
\Box_{\text{BDN}} A := \left( \frac{1}{k - \frac{d}{2} + 1} \delta d + \frac{1}{k - \frac{d}{2} - 1} d\delta + J - 2F \right) A = 0,
\]

is conformally invariant for \( A \in \Gamma \mathcal{E}^k M[k - \frac{d}{2} + 1] \) where the conformally invariant Branson–Deser–Nepomechie operator is a map \( \Gamma \mathcal{E}^k [k - \frac{d}{2} + 1] \to \Gamma \mathcal{E}^k [k - \frac{d}{2} - 1] \). (When it is unclear which degree forms \( \Box_{\text{BDN}} \) acts on, we will write \( \Box_{\text{BDN}}^{(k)} \).) This conformally invariant generalization of both Maxwell’s equations and the Yamabe equation seems to have been first uncovered in [5] and was then independently uncovered in a physical
context in [14]: The Yamabe equation appears at degree $k = 0$. The residues of the poles at $k = \frac{d}{2} \pm 1$ $(d \in 2\mathbb{N})$ give the higher form Maxwell’s equations in their standard potential form, or expressed in terms of a Hodge dual potential, respectively.

Finally, in even dimensions, Maxwell’s equations in their curvature form can be coupled conformally to a Proca-type equation to yield the conformally invariant \textit{coupled Proca–Maxwell} system of equations

\begin{equation}
* \delta F = (\Delta + 2(J - 2P))A, \quad dF = 0 = \delta A,
\end{equation}

where $A \in \Gamma E^{1+\frac{d}{2}}M[2]$ and $*F \in \Omega^d M$. Of course, all of the above systems of equations enjoy a dual formulation obtained by applying the isomorphism $\mathcal{D} F = 0$ equation for a definite choice of splitting $g \in c$ of $F \in \mathcal{T}^k M[w]$ (3.12)

\[
\begin{cases}
0 = (d + 2w - 2)(dF^+ - (w + k)F), \\
0 = (d + 2w - 2)dF, \\
0 = (\Delta + (w + k - 2)(J - 2P))F^+ - 2\delta F + (d + 2w)(dF^++(w + k - 2)F^-), \\
0 = [J - 2P, d]F^+ - (\Delta + (w + k)(J - 2P))F - (d + 2w)dF^-.
\end{cases}
\]

\textbf{Proposition 3.24.} Let $F \in \mathcal{T}^k M$. Then the system of equations

\begin{equation}
\mathcal{D} F = 0 = \mathcal{D}^* F = \mathcal{X}^* F
\end{equation}

describes

(i) the Branson–Deser–Nepomechie equation when $w = 1 - \frac{d}{2}$, $k \neq 1 + \frac{d}{2}$,

(ii) the coupled Proca–Maxwell system when $w = 1 - \frac{d}{2}$, $k = 1 + \frac{d}{2}$ and $d \in 2\mathbb{N}$.

The system of equations

\begin{equation}
\mathcal{D} F = \mathcal{D}^* F = \mathcal{X} F = \mathcal{X}^* F = 0
\end{equation}

describes the curvature version of the higher form Maxwell’s equations when $w = 1 - \frac{d}{2}$, $k = 1 + \frac{d}{2}$ $(d \in 2\mathbb{N})$.

\textbf{Proof.} Given the expressions for the exterior and interior tractor $D$- and canonical tractor operators for a choice of $g \in c$ explicated in Section [3.1], the proof is elementary. For expedience, note that the kernels of $\mathcal{X}^*$ and $\mathcal{D}^*$ are spelled out in Lemma [3.13]. To complete the isomorphism between the tractor equations and their differential form counterparts, note that we have called the western slot $A$. Also, for the Proca–Maxwell system, the Maxwell curvature $F$ is the Hodge dual of the southern slot. \hfill \square

\textbf{Remark 3.25.} For the case $k = 1$, the above characterization of the Branson–Deser–Nepomechie equation is exactly that given in [30].

It now remains only to explain the origin of the conformally invariant potential version of Maxwell’s equations. The key point here is the gauge invariance which generates a new solution $A + d\alpha$ from any given solution $A$. Since $d$ is nilpotent, this really amounts to a certain cohomology problem. Indeed, the system of equations (3.12) are also gauge invariant under

\begin{equation}
F \sim F + \mathcal{D} A.
\end{equation}
Hence the relevant problem is the cohomology of \( \mathcal{D} \) with \( \Gamma \mathcal{T}^\ast \cdot \cdot \cdot \) as the space of chains. This problem will also play an important rôle in our study of Proca equations in Section 5.

Our first result establishes that for generic weights this cohomology is in fact trivial.

**Proposition 3.26.** The cohomology of the differential complex

\[ \cdots \mathcal{D} \rightarrow \Gamma \mathcal{T}^{k+1} M[w+1] \mathcal{D} \rightarrow \Gamma \mathcal{T}^k M[w] \mathcal{D} \rightarrow \Gamma \mathcal{T}^k M[w-1] \mathcal{D} \rightarrow \cdots, \]

denoted \( \mathcal{H}^k M[w] \) at \( \Gamma \mathcal{T}^k M[w] \), is trivial whenever \( w \neq -\frac{d}{2}, 1 - \frac{d}{2}, -1 - \frac{d}{2}, -k, -k + 2 \).

**Proof.** Consider

\[ \frac{1}{(d + 2w)(w + k)} \mathcal{D} \mathcal{X}^\ast (1 - \frac{2}{(d + 2w + 2)(w + k - 2)}) \mathcal{X}^\ast \mathcal{D}^\ast \mathcal{F}. \]

We claim that if \( \mathcal{D} \mathcal{F} = 0 \), the above quantity identically equals \( \mathcal{F} \). The claim is easily verified using Proposition 3.4 to push \( \mathcal{D} \) to the right (assuming \( w \neq 1 - \frac{d}{2} \)) where it annihilates \( \mathcal{F} \). Therefore \( \mathcal{D} \mathcal{F} = 0 \Rightarrow \mathcal{F} = \mathcal{D} \mathcal{A} \) for some \( \mathcal{A} \).

To analyze further the cohomology of the exterior Thomas D-operator, observe that acting with \( \mathcal{D} \) we move on the \( (w, k) \) plane along lines \( w + k = \text{constant} \) as shown in Figure 2. Similarly, the interior tractor D-operator \( \mathcal{D}^\ast \) moves along lines \( w - k = \text{constant} \). In what follows we will not analyze the \( \mathcal{D}^\ast \) cohomology because it can easily be obtained from that of \( \mathcal{D} \) by Hodge duality.

Proposition 3.26 reveals that we may expect non-trivial cohomology only along the lines \( w + k = 0, 2 \) or when a \( w + k \) line meets the weights \( w = 1 - \frac{d}{2}, -\frac{d}{2}, -1 - \frac{d}{2} \). This can also be seen by examining the system of equations (3.12) which one has to solve for \( \mathcal{D} \) to be closed, as well as considering the same system after replacing \( w \rightarrow w + 1, k \rightarrow k - 1 \) in order to analyze when a \( \mathcal{D} \)-closed tractor form is \( \mathcal{D} \)-exact. First we give a result for general weights.

**Proposition 3.27.** When \( w + k \neq 0, 2 \) the cohomology \( \mathcal{H}^k M[w] \) of the operator \( \mathcal{D} \) can only be non-empty for \( w = 1 - \frac{d}{2}, -\frac{d}{2}, -1 - \frac{d}{2} \). Moreover:

(i) \( \mathcal{H}^k M[-\frac{d}{2}] \cong \ker \mathcal{H}^k \mathcal{D}, \)

(ii) \( \mathcal{H}^{k+1} M[-\frac{d}{2}] \cong \ker \mathcal{H}^k \mathcal{D} \oplus \coker \mathcal{H}^k \mathcal{D}, \)

(iii) \( \mathcal{H}^{k+2} M[-1 - \frac{d}{2}] \cong \coker \mathcal{H}^k \mathcal{D} \).

**Proof.** The method of proof for this and the other Proposition 3.28 for the cohomology of \( \mathcal{D} \) is the same: One examines the system of equations (3.12) first at the weight and degree \( (w, k) \) of interest in order to study the closure condition, and then again at \( (w + 1, k - 1) \) to control exactness. The general pattern for the closure conditions is that the first and third of these equations allow the western and southern slots \( F \) and \( F^- \) to be determined in terms of their northern and eastern counterparts, \( F^+, F^{-}, \) unless the weight and degree conspire to give them a zero coefficient. For exactness, at \( (w + 1, k - 1) \) the right hand sides of the system of equations 3.12 show that the northern and eastern slots of \( \mathcal{F} \in \ker \mathcal{D} \) are algebraically cohomologous to zero, again up to conspiracies between weights and degrees.

For the case at hand there are three conspiracies: (i) For \( \mathcal{H}^k [1 - \frac{d}{2}] \), the northern and eastern slots are still cohomologous to zero, but only the southern slot can be eliminated by the closure condition. (ii) At \( \mathcal{H}^{k+1} [-\frac{d}{2}] \) only the eastern slot is cohomologous to zero and only the western slot can be eliminated by the closure condition. (iii) For \( \mathcal{H}^{k+2} [-1 - \frac{d}{2}] \) the northern slot is again cohomologous to zero and both the western and
Figure 2. In this picture we represent the complex for the exterior and interior tractor D-operators; the lower and upper diagonal line of arrows describe the complexes along \( w + k = 0 \) and \( w + k = 2 \) of Proposition 3.28 which have non-empty cohomology. The central diagonal line corresponds to the case of generic values of \( w + k \) of Proposition 3.27, which only has non-trivial cohomology if the three red horizontal lines at the values \( w = 1 - \frac{d}{2}, -\frac{d}{2}, -1 - \frac{d}{2} \) are traversed.

Schematically then, a complex \( 0 \to W \to (N, S) \to E \to 0 \) results. It is not difficult to compute the differentials; the result is depicted below

\[
\begin{align*}
\Gamma \mathcal{E}^k M \left[ k - \frac{d}{2} + 1 \right] & \xrightarrow{\mathfrak{w}_{\text{HN}}} \Gamma \mathcal{E}^k M \left[ k - \frac{d}{2} + 1 \right] \oplus \Gamma \mathcal{E}^k M \left[ k - \frac{d}{2} - 1 \right] \\
& \xrightarrow{\mathfrak{w}_{\text{HN}}} \Gamma \mathcal{E}^k M \left[ k - \frac{d}{2} - 1 \right].
\end{align*}
\]

This leaves the cases \( w + k = 0, 2 \) which are closely related to both de Rham cohomology \( H^k M \) and the detour complexes that will appear later in the Article in a holographic context.

**Proposition 3.28.** When \( w + k = 0, 2 \) but \( w \neq 1 - \frac{d}{2}, -\frac{d}{2}, -1 - \frac{d}{2} \)

\[
\begin{align*}
\mathcal{H}^k M \left[ -k \right] & \cong H^{k-1} M \oplus H^k M, \quad w = -k, \\
\mathcal{H}^k M \left[ 2 - k \right] & \cong H^{k-2} M \oplus H^{k-1} M, \quad w = 2 - k.
\end{align*}
\]

Moreover, the differential complexes

\[
\mathfrak{D} \to \Gamma \mathcal{T}^{\frac{d}{2} - 1} M[-1 - \frac{d}{2}] \xrightarrow{\mathfrak{D}} \Gamma \mathcal{T}^{\frac{d}{2}} M[-\frac{d}{2}] \xrightarrow{\mathfrak{D}} \Gamma \mathcal{T}^{\frac{d}{2} + 1} M[-1 - \frac{d}{2}] \xrightarrow{\mathfrak{D}},
\]
and 
\[ \mathcal{D} : \Gamma^2 T^{\frac{d}{2}+1} M \rightarrow \Gamma^2 T^{\frac{d}{2}+2} M[2 - \frac{d}{2}] \rightarrow \Gamma^2 T^{\frac{d}{2}+3} M[-1 - \frac{d}{2}] \rightarrow, \]
corresponding to \( w = 1 - \frac{d}{2}, -\frac{d}{2}, -1 - \frac{d}{2} \) and \( w + k = 0, 2 \), respectively, are equivalent to the following differential complexes
\[
\begin{pmatrix}
-2d & 0 \\
0 & -2d
\end{pmatrix} \rightarrow \Omega^2 M \oplus \Omega^{\frac{d}{2}+1} M \rightarrow \Omega^{\frac{d}{2}+2} M \oplus \Gamma^2 T M \oplus \Gamma^2 T M[-2] \\
\begin{pmatrix}
\Delta - 2(1 - \frac{d}{2}) & 0 & 0 \\
0 & -2d & 0
\end{pmatrix} \rightarrow \Omega^{\frac{d}{2}+3} M \oplus \Gamma^2 T^2 M \oplus \Gamma^2 T^2 M[-2] \rightarrow \Gamma^2 T^2 M[0] \oplus \Omega^2 M \oplus \Omega^{\frac{d}{2}+1} M
\]
and
\[
\begin{pmatrix}
0 & 0 & 0 \\
4d & 0 & 0
\end{pmatrix} \rightarrow \Omega^{\frac{d}{2}+3} M \oplus \Omega^2 M \oplus \Omega^{\frac{d}{2}+1} M \rightarrow \Gamma^2 T^2 M[2] \oplus \Omega^2 M \oplus \Omega^{\frac{d}{2}+1} M \\
\begin{pmatrix}
\Delta + 2(1 - \frac{d}{2}) & 0 & 0 \\
0 & 2d & 0
\end{pmatrix} \rightarrow \Omega^{\frac{d}{2}+2} M \oplus \Gamma^2 T M \oplus \Gamma^2 T M[-2] \rightarrow \Gamma^2 T M[2] \oplus \Omega^2 M \oplus \Omega^{\frac{d}{2}+1} M \\
\begin{pmatrix}
4d & 0 & 0 \\
0 & -4d & 0
\end{pmatrix} \rightarrow \Omega^{\frac{d}{2}+1} M \oplus \Omega^{\frac{d}{2}+2} M \\
\]
Proof. Here the proof follows exactly the same methodology as for Proposition 3.27. \( \square \)

Remark 3.29. Although the above differential complexes appear novel, they are closely related to the systems studied already. For example, the second differential in the first diagram is a prolongation of the Maxwell operator \( \delta d \). The third differential is equivalent to the coupled Proca–Maxwell system (upon Hodge dualizing \( A \)).

To relate the cohomology of the exterior tractor D-operator \( \mathcal{D} \) to the conformally invariant Maxwell operator and its associated detour complex, we need to refine the space of chains. This is achieved via the following simple observation.

Lemma 3.30. There are well-defined maps
\[
\begin{align*}
\ker(\mathcal{D}^*, \mathcal{D}^*) \subset \Gamma^2 T^k M[-k] & \quad \rightarrow \quad \ker(\mathcal{D}^*, \mathcal{D}^*) \subset \Gamma^2 T^{k+1} M[-k-1], \quad k \neq 1 - \frac{d}{2}, -\frac{d}{2}, \\
\ker(\mathcal{D}^*, \mathcal{D}^*) \subset \Gamma^2 T^{\frac{d}{2}-1} M[1 - \frac{d}{2}] & \quad \rightarrow \quad \ker(\mathcal{D}^*, \mathcal{D}^*) \subset \Gamma^2 T^{\frac{d}{2}} M[-\frac{d}{2}], \quad d \in 2\mathbb{N}, \\
\ker(\mathcal{D}^*, \mathcal{D}^*) \subset \Gamma^2 T^2 M[-\frac{d}{2}] & \quad \rightarrow \quad \ker(\mathcal{D}^*, \mathcal{D}^*) \subset \Gamma^2 T^3 M[-1 - \frac{d}{2}], \quad d \in 2\mathbb{N},
\end{align*}
\]
given by 
\[ \mathcal{F} \mapsto \mathcal{D} \mathcal{F}. \]

Proof. To verify that at the weight \( w = -k \neq -\frac{d}{2} \) elements of the kernel of \((\mathcal{D}^*, \mathcal{D}^*)\) are mapped by \( \mathcal{D} \) again to \( \ker(\mathcal{D}^*, \mathcal{D}^*) \), it suffices to examine the third identity of Proposition 3.4. At \( w = -k = 1 - \frac{d}{2} \) the ker \( \mathcal{D}^* \) condition on the image of the map \( \mathcal{D} \) turns out to be an empty requirement but is augmented by \( \ker \mathcal{D}^* \) which follows from the first line of Proposition 3.4. To see that at \( w = -k = -\frac{d}{2} \), the map \( \mathcal{D} \) has image contained in \( \ker \mathcal{D}^* \) one uses the second line of Proposition 3.4 along with the fact that the domain is then taken to be \( \ker(\mathcal{D}^*, \mathcal{D}^*) \). \( \square \)

Remark 3.31. By the western Lemma 3.13, \( \ker(\mathcal{D}^*, \mathcal{D}^*) \subset \Gamma^2 T^k M[-k] \cong \Omega^k M \) so long as \( k \neq \frac{d}{2} \). Hence, \( \mathcal{D} \) induces a map \( \Omega^k M \rightarrow \Omega^{k+1} M \) when \( k \neq \frac{d}{2} - 1, \frac{d}{2} \). From the display (3.3), it follows that this map is the exterior derivative \( d \).
Proposition 3.32. Let \( d \in 2\mathbb{N} \). Then the differential complex

\[
\cdots \xrightarrow{\delta} \ker(D^*, \mathcal{F}^*) \subset \Gamma T^{\frac{d}{2}+1}M \begin{bmatrix}-1 - \frac{d}{2}\end{bmatrix} \xrightarrow{\delta} \ker(\mathcal{F}, \mathcal{F}^*) \subset \Gamma T^{\frac{d}{2}+2}M \begin{bmatrix}-2 - \frac{d}{2}\end{bmatrix} \xrightarrow{\delta} \cdots
\]

is equivalent to

\[
\xrightarrow{d} \Omega^d M \xrightarrow{d} \Omega^d M \xrightarrow{\delta d} \Gamma \mathcal{E}^{d-1}M[-2] \xrightarrow{0} \Omega^{d+1} M \xrightarrow{d} \Omega^{d+2} M \xrightarrow{d} \cdots
\]

Remark 3.33. In odd dimensions, the “detour” at \( w = -k = 1 - \frac{d}{2} \) is avoided and one simply has an equivalence between the cohomology of \( D \) acting on \( \ker(D^*, \mathcal{F}^*) \) for \( w + k = 0 \)

\[
\cdots \xrightarrow{\delta} \ker(D^*, \mathcal{F}^*) \subset \Gamma T^k M[-k] \xrightarrow{\delta} \cdots
\]

and de Rham cohomology

\[
\xrightarrow{d} \Omega^k M \xrightarrow{d} \cdots
\]

Proof. This follows directly from Lemma 3.30, its accompanying Remark 3.31 and a computation of \( D \) at weights \( w = -k = 1 - \frac{d}{2} \), \( \frac{d}{2} \) for some \( g \in c \) using (3.3). \( \square \)

Rather than connecting the de Rham complex via the detour operator \( \delta d \) followed by the zero map to the de Rham complex again, as in Proposition 3.32, a canonical maneuver is to continue on with the dual de Rham complex. Pictorially this gives the Maxwell detour complex

\[
\xrightarrow{d} \Omega^* M \xrightarrow{d} \cdots \xrightarrow{d} \Omega^d M \xrightarrow{\delta d} \Gamma \mathcal{E}^{d-1}M[-2] \xrightarrow{\delta} \cdots \xrightarrow{\delta} \Gamma \mathcal{E}^* M[\cdot] \xrightarrow{\delta},
\]

(where the chains in the outgoing dual de Rham complex belong to \( \Gamma \mathcal{E}^k M[2k-d] \)). This is an important but simplest case of a family of conformally invariant differential detour complexes \([3, 9, 3]\) whose study we take up again in Section 6.

We complete this Section by showing how the Maxwell detour complex arises in the current setting. First we rely on a Corollary of Proposition 3.30 and the southern and eastern Lemmas 3.10, 3.16.

Corollary 3.34. Let \( d \in 2\mathbb{N} \). Then there is a well-defined, conformally invariant, canonical “detour” map

\[
\ker(D^*, \mathcal{F}^*) \subset \Gamma T^{\frac{d}{2}} M \begin{bmatrix}1 - \frac{d}{2}\end{bmatrix} \rightarrow \ker(D, \mathcal{F}) \subset \Gamma T^{\frac{d}{2}+1} M \begin{bmatrix}1 - \frac{d}{2}\end{bmatrix},
\]

given by the composition of maps

\[
q_E \circ q_S^{-1} \circ D.
\]

Moreover,

\[
D^* \circ q_E \circ q_S^{-1} \circ D = 0.
\]

Proof. Only the statement that the composition of the interior tractor D-operator and the detour map vanishes requires further elaboration: The range of the differential \( \delta d \) in Proposition 3.32 at \( \Omega^d M \) is mapped to the eastern slot of a section of \( \Gamma T^{\frac{d}{2}+1} M \begin{bmatrix}1 - \frac{d}{2}\end{bmatrix} \). Then the eastern Lemma 3.16 implies that a dual version of Remark 3.31 holds: the action of \( D^* \) on \( \ker(D, \mathcal{F}) \subset \Gamma \mathcal{E}^k M[-d-2+k] \) induces the map

\[
\delta : \Gamma \mathcal{E}^k M[2k-d] \rightarrow \Gamma \mathcal{E}^{k-1} M[2k-d-2],
\]
so long as \( k \neq \frac{d}{2} + 2 \) (this is the point in the \((w, k)\)-plane dual to the one where the map \( q_E q_S^{-1} \) was needed to replace the \( \mathcal{D} \) operator). The proof is now complete since \( \delta^2 = 0 \).

**Remark 3.35.** Acting on weight \(-\frac{d}{2}\), degree \( \frac{d}{2} \) southern tractors (in even dimensions), the composition of operators \( q_E \circ q_S^{-1} = -2 \mathcal{Q} \), as can be easily verified by a direct computation.

We have therefore by now established the following result (depicted schematically in Figure 3).

**Proposition 3.36.** Let \( d \in 2\mathbb{N} \). Then the differential complex

\[
\cdots \xrightarrow{\mathcal{D}} \text{ker}(\mathcal{D}^*, \mathcal{X}^*) \subset \Gamma T^{d-2} M [2 - \frac{d}{2}] \xrightarrow{q_E q_S^{-1} \mathcal{D}} \text{ker}(\mathcal{D}, \mathcal{X}) \subset \Gamma T^{d+1} M [1 - \frac{d}{2}] \xrightarrow{q_E q_S^{-1} \mathcal{D}} \text{ker}(\mathcal{D}^*, \mathcal{X}^*) \subset \Gamma T^{d} M \xrightarrow{\mathcal{D}^*} \ker(\mathcal{D}, \mathcal{X}) \subset \Gamma T^{d-1} M [3 - \frac{d}{2}] \xrightarrow{\mathcal{D}^*} \cdots
\]

is equivalent to the Maxwell detour complex (of Equation (3.13)).
4. The exterior calculus of scale

We now come to a central point of our development, namely that there is a canonical way to introduce a (generalized) scale into the conformal calculus and algebra. Most of the structure is available in the general almost Riemannian setting, so we treat this first before refining to Poincaré–Einstein structures.

Let \((M, c, \sigma)\) be an almost Riemannian structure. The scale tractor defines canonical maps on \(\Gamma T^* M[.]\) by exterior and interior multiplication

\[
\mathcal{I} := \varepsilon(I) : \Gamma T^k M[w] \to T^{k+1} M[w], \quad \mathcal{I}^* := \iota(I) : \Gamma T^k M[w] \to T^{k-1} M[w].
\]

We may explicate these operators for a choice of \(g \in c\) by

\[
(4.1) \quad \mathcal{I} \equiv \begin{pmatrix}
-\varepsilon(n) & \sigma & 0 & 0 \\
0 & \varepsilon(n) & 0 & 0 \\
-\rho & 0 & \varepsilon(n) & \sigma \\
0 & \rho & 0 & -\varepsilon(n)
\end{pmatrix}, \quad \mathcal{I}^* \equiv \begin{pmatrix}
-\iota(n) & 0 & -\sigma & 0 \\
\rho & \iota(n) & 0 & \sigma \\
0 & 0 & \iota(n) & 0 \\
0 & 0 & \rho & -\iota(n)
\end{pmatrix},
\]

where \(I_A := (\rho, n, \sigma)\). From equations \((2.13)\) and \((2.14)\) we have

\[
n = \nabla \sigma, \quad \rho = -\frac{d}{\Delta} (\Delta g + J) \sigma.
\]

The following identities result trivially from standard properties of interior and exterior multiplication as well as the definition of the maps \(\mathcal{I}\) and \(\mathcal{I}^*\)

\[
\{\mathcal{I}, \mathcal{I}\} = 0 = \{\mathcal{I}^*, \mathcal{I}^*\},
\]

\[
\{\mathcal{I}^*, \mathcal{I}\} = \sigma = \{\mathcal{I}, \mathcal{I}^*\},
\]

\[
[N, \mathcal{I}] = \mathcal{I}, \quad [h, \mathcal{I}] = 0 = [h, \mathcal{I}^*], \quad [N, \mathcal{I}^*] = -\mathcal{I}^*.
\]

As always, we denote by \(x\) the map \(\Gamma T^k M[w] \to \Gamma T^k M[w-1]\) obtained by multiplying by the scale \(\sigma\); thus \(x = \{\mathcal{I}^*, \mathcal{I}\} = \{\mathcal{I}, \mathcal{I}^*\}\).

Finally we give identities that include the exterior and interior Thomas D-operators in our calculus.

**Proposition 4.1.** Let \(I^A\) be a scale tractor for an almost Riemannian conformal structure \((M, c, \sigma)\), and define the operator \(y := -I_A D^A\) that maps \(\Gamma T^k M[w] \to \Gamma T^k M[w-1]\). Then, for \(d \neq 4\), the following operator identities hold.

\[
[h, x] = 2x, \quad [x, y] = h, \quad [h, y] = -2y,
\]

\[
(h - 2) \mathcal{D}^* x - h x \mathcal{D}^* = 2 \mathcal{D}^* y + h(h - 2) \mathcal{I}^* ,
\]

\[
(h - 2) \mathcal{D} x - h x \mathcal{D} = 2 \mathcal{D} y + h(h - 2) \mathcal{I} ,
\]

\[
(h - 2) y \mathcal{X} - h \mathcal{X} y = 2 x \mathcal{D} - h(h - 2) \mathcal{I} ,
\]

\[
(h - 2) y \mathcal{X}^* - h \mathcal{X}^* y = 2 x \mathcal{D}^* - h(h - 2) \mathcal{I}^* .
\]

When \(d = 4\), the above identities hold for any almost Einstein structure.

**Proof.** The first three identities were proven already in [31]. The remaining identities are obtained contracting equation \((3.9)\) with the scale tractor on one of its indices and then performing either exterior or interior multiplication with the other. For the \(d = 4\) almost Einstein case, the same proof applies using also Remark [3.3] and the proof of Proposition [4.5].
4.1. **Poincaré–Einstein structures.** We now specialise to Poincaré–Einstein structures. For concreteness, we recall some basic definitions. A Riemannian metric \( g^o \) on the interior \( M^+ \) of a compact manifold \( M \) with boundary \( \Sigma := \partial M \) is said to be conformally compact if it extends to \( \Sigma \) by \( g = r^2 g^o \), with \( g \) non-degenerate up to \( \Sigma \) equaling the zero locus of a defining function \( r \); that is \( \Sigma \) is the zero locus \( Z(r) \) and \( dr|_\Sigma \neq 0 \). If the normal to \( \Sigma \) is nowhere null, then \( g \) determines a conformal structure \( c_\Sigma \). In this case \( (\Sigma, c_\Sigma) \) is called the conformal infinity of \( M^+ \). If the defining function obeys 
\[
|dr|^2_g = 1 ,
\]
along \( \Sigma \), the sectional curvatures of \( g^o \) tend to \( -1 \) at infinity and the structure is said to be asymptotically hyperbolic (AH) \([49]\).

Tractor calculus enables a treatment of any conformally compact structure \([26]\). A very strong indication that conformal geometries and their tractor treatment is fruitful for the study of physical models, is their strong predilection for Einstein metrics \([56]\), as partly captured by the following result \([27]\).

**Theorem 4.2.** On a conformal manifold \((M, c)\) there is a 1-1 correspondence between conformal scales \( \sigma \in \Gamma E M[1] \), such that \( g^o = \sigma^{-2} g \) is Einstein, and parallel standard tractors \( I \in \Gamma TM \) with the property that \( XAI^A \) is nowhere vanishing. The mapping from Einstein scales to parallel tractors is given by \( \sigma \mapsto 1_\sigma D^A \sigma \) while the inverse is \( I^A \mapsto X^A I_A \).

In the above, \( X^A \in \mathcal{T}^A M[1] \) is the canonical tractor—a distinguished invariant tractor; see Section 3.1 for further details. The statement of the Theorem is easily verified using \([21]\), or may be viewed as an easy consequence of the definition of the tractor connection from \([4]\).

For concreteness and later use, we explicate in tensor terms the parallel conditions 
\[
\nabla^T I^A = 0
\]
for the scale tractor for some \( g \in c \):
\[
\left\{ \begin{array}{l}
\nabla_a \sigma = n_a , \\
\nabla_a n_b = -\sigma P_{ab} - \rho g_{ab} , \\
\nabla_a \rho = P_{ab} n^b .
\end{array} \right.
\]

In light of the above Theorem and in line with \([24]\), we will say that a conformal manifold \((M, c)\), is almost Einstein if it is equipped with a non-zero parallel standard tractor \( I \). This notion slightly enlarges the standard Einstein condition. Indeed, from the Theorem above it follows that the defining scale \( \sigma \) is non-zero on an open dense set \([27]\). Moreover, if non-empty, the zero locus of the defining scale \( \sigma \) is a conformal infinity. I.e., the almost Einstein condition extends the standard Einstein one to describe manifolds with a conformal infinity. This boundary structure is precisely the one required to study a wide range of physical applications. Let us spell out some pertinent details:

Recall that an AH manifold which is Einstein in its interior is called Poincaré–Einstein. Following \([25]\) and \([24]\), this fits precisely in the almost Einstein picture and provides the first exterior identity specialized to this setting.

**Proposition 4.3.** A Poincaré–Einstein manifold is an almost Einstein manifold \((M, c, \sigma)\) with boundary \( \Sigma \) equaling the zero locus of \( \sigma \) such that \( I^2 := IAI^A = 1 \); thus 
\[
\{ \mathcal{F}^*, \mathcal{F} \} = 1 .
\]

In view of this observation, it makes sense to treat the almost Einstein setting generally.
4.1.1. Exterior calculus of almost Einstein scales. Here we develop exterior identities extending the solution generating algebra \( \{ I^A, I^B \} = 0 = \{ I^*, I^* \} \).

Using that \( I \) is parallel we come to the next exterior identities.

**Proposition 4.4.** If \( (M, c, \sigma) \) is almost Einstein, then

\[
\{ \mathcal{S}, \mathcal{D} \} = 0 = \{ \mathcal{S}^*, \mathcal{D}^* \}.
\]

**Proof.** This can easily be directly verified by anticommuting the matrix expressions for the exterior and interior scale tractors \( \mathcal{S} \) and \( \mathcal{S}^* \) in Equation (4.1), employing the almost Einstein identities [4.2]. A slicker argument is to note that since \( I \) is parallel

\[
[D^A, I^B] = 0,
\]

it immediately follows that \( \{ \mathcal{S}, \epsilon(D) \} = 0 = \{ \mathcal{S}^*, \epsilon(D) \} \) so it only remains to verify that \( \mathcal{S} \) and \( \mathcal{S}^* \) anticommute with \( \mathcal{D}^* \Omega^\mathcal{D} \mathcal{D} \) and \( \mathcal{S}^* \Omega^\mathcal{D} \mathcal{D} \), respectively. Away from dimension four, this holds trivially since for almost Einstein structures the \( W \)-tractor obeys \( I_A W^{ABCD} = 0 \). In dimension four, \( \mathcal{S} \) and \( \mathcal{S}^* \) still commute with the operator \( \frac{1}{d-4} W^{\mathcal{D} \mathcal{D}} \) of [23 Section 4] which is also discussed in the proof of Proposition 4.5 below. From this \( \mathcal{D}^A \) can still be defined, as commented upon in Remark 3.3 and we again obtain the result. \( \square \)

Next we consider anticommutators of \( \mathcal{S}^* \) and \( \mathcal{D} \) (or \( \mathcal{S} \) and \( \mathcal{D}^* \)).

**Proposition 4.5.** The map

\[
y : \Gamma^k M[w] \to \Gamma^k M[w - 1] \text{ where } y := -I_A \mathcal{D}^A
\]

obeys

\[
\{ \mathcal{S}, \mathcal{D} \} = -y = \{ \mathcal{S}, \mathcal{D}^* \}.
\]

Moreover

\[
[\mathcal{D}, y] = [\mathcal{S}, y] = 0 = [\mathcal{S}^*, y] = [\mathcal{D}^*, y].
\]

**Proof.** As usual, a rudimentary proof is to evaluate the matrix expression for each of the three operators for a given \( g \in c \). The result agrees for each of these, and since the operator \( y \) is a central player in this Article, we record the explicit result:

\[ -y = y = (d + 2w - 2) \delta R - \sigma \left( \Box_Y + \frac{d}{2} (d + 2w - 2) \right), \]

where \( \Box_Y \) is the identity (matrix), and along \( \Sigma, \delta R \) is a conformal Robin-type operator for \( \mathcal{D} \). It is given in general by

\[
\delta R := \begin{pmatrix}
\nabla_n + w \rho & -\epsilon(n) & \epsilon(n) & 0 \\
\epsilon(\nabla \rho) & \nabla_n + w \rho & 0 & \epsilon(n) \\
-\epsilon(\nabla \rho) & 0 & \nabla_n + w \rho & \epsilon(n) \\
0 & -\epsilon(\nabla \rho) & -\epsilon(\nabla \rho) & \nabla_n + w \rho
\end{pmatrix}.
\]

At weight \( w = 1 - d/2 \), \( \Box_Y \) is a conformally invariant Yamabe-type operator for forms. It is given at general weights by

\[
\Box_Y := \begin{pmatrix}
\Delta + (N - \frac{d}{2})(J - 2\mathcal{D}) & -2\delta & 2d & 2N - d \\
-J - 2\mathcal{D}, \mathcal{D} & \Delta + (N - \frac{d}{2} + 1)(J - 2\mathcal{D}) & 0 & 2d \\
-J - 2\mathcal{D}, \delta & 0 & \Delta + (N - \frac{d}{2} - 1)(J - 2\mathcal{D}) & 2\delta \\
-P^a_{\mathcal{D}} P^a_{\mathcal{D}} + 2 \text{ End}(\mathcal{D}, \mathcal{D}) + \frac{d}{2} \bar{\mathcal{D}} & -J - 2\mathcal{D}, \delta & [J - 2\mathcal{D}, \mathcal{D}] & \Delta + (N - \frac{d}{2})(J - 2\mathcal{D})
\end{pmatrix}.
\]
In this formula the slashed Bach endomorphism $\mathcal{B}$ is defined in any dimension in terms of the Cotton tensor by $(n^i(\nabla_c P^h_a - \nabla_a P^h_c))$. In dimensions not equal to four, on almost Einstein structures, it is related to the standard Bach tensor by

$$\mathcal{B} = \frac{\sigma E}{d - 4}.$$ 

Indeed, in dimensions other than four, the W-tractor divided by $d - 4$—from which one builds $\mathcal{B}$—is a sum of Weyl and Cotton tensor terms, plus the Bach tensor over $d - 4$. Replacing the Bach over $d - 4$ contribution by $\mathcal{B}/\sigma$, yields the tensor $\tilde{W}/(d - 4)$. It equals the standard $W/(d - 4)$ in any dimension not equal to four, but is also a well-defined tractor on four dimensional conformally Einstein manifolds [23]. In this way we have defined and computed the third expression $-I \cdot \mathcal{B}$ in arbitrary dimensions.

An alternative proof is to define $\mathcal{B}$ in this way, and then use that $[I^A, \mathcal{B}^B] = 0$. Thus the equalities $\{\mathcal{B}, \mathcal{B}^*\} = \{\mathcal{B}, \mathcal{B}^*\} = I \cdot \mathcal{B}$, follow immediately from the properties of exterior and interior multiplication. The remaining commutation relations quoted are also trivial, for example $[\mathcal{B}, y] = [(\mathcal{B}, \mathcal{B}^*), \mathcal{B}] = (\mathcal{B}^* \mathcal{B} + \mathcal{B}^* \mathcal{B}) \mathcal{B} = \mathcal{B} \mathcal{B}^* \mathcal{B} - \mathcal{B} \mathcal{B}^* \mathcal{B} = 0$, because $\mathcal{B}$ is nilpotent. \hfill $\square$

4.2. Boundary tractor. Let us first recall some general facts concerning hypersurface tractors [4, 25, 24] before specializing to Poincaré–Einstein structures and tractor forms in order to develop natural boundary conditions for the extension problems studied in the next Section. Assume, therefore, that $\Sigma$ is a boundary component of $(M, c)$ and the conformal structure extends smoothly to a collar neighborhood of $\Sigma$.

Assume $\Sigma$ is smooth and is the zero locus of a defining density $\sigma$ (see Section 2.2) which is also a defining scale for our structure. Let $n_a \in E_a[1]$ be a unit conormal so that, along $\Sigma$

$$g^{ab}n_an_b = 1.$$ 

In a scale $g \in c$, the mean curvature of $\Sigma$ is

$$H^g = \frac{1}{d - 1} \nabla^T_a n^a, \quad \nabla^T := \nabla - n\nabla_n.$$ 

From this data, we can build the normal tractor $N \in \Gamma TM|\Sigma$ of [4]

$$N^A g = \begin{pmatrix} 0 \\ n_a \\ -H^g \end{pmatrix}.$$ 

The normal tractor satisfies $N_A N^A = 1$ and is linked to the scale tractor when $(M, c, \sigma)$ is an almost scalar constant structure. By observing

$$I|_{\Sigma} g = \begin{pmatrix} 0 \\ \nabla_a \sigma \\ -\frac{1}{d} \Delta \sigma \end{pmatrix},$$

and using $I^2 = 1$ we have the following result [24 Proposition 3.5].

**Lemma 4.6.** If $(M, c)$ is an almost scalar constant structure with defining scale singularity set $\Sigma$ and scale tractor $I$, then the normal tractor of $\Sigma$ satisfies

$$N = I|_{\Sigma}.$$
The intrinsic tractor bundle $\mathcal{T}\Sigma$ of $(\Sigma, c_\Sigma)$, where the conformal structure $c_\Sigma$ is the one induced by $c$, is related to the standard tractor bundle $\mathcal{T}M$ along $\Sigma$ (i.e. $\mathcal{T}M|\Sigma$) \cite{36, 24, 57}. Indeed, there is a canonical, conformally invariant isomorphism between the canonical, rank $d + 1$ subbundle $N^\perp$ of $\mathcal{T}M|\Sigma$ orthogonal (with respect to the tractor metric) to the normal tractor $N$, and the intrinsic boundary tractor bundle

$$N^\perp \cong \mathcal{T}\Sigma.$$ 

We shall use this isomorphism to identify these spaces. The map between their respective section spaces, calculated in a scale $g \in c$ (and therefore $g_{\Sigma} = g|\Sigma$) is \[ \begin{pmatrix} \nu \\ \mu_a \\ \rho \end{pmatrix} \mapsto \begin{pmatrix} 1 & 0 & 0 \\ -H^g n_a & 1 & 0 \\ -\frac{1}{2}(H^g)^2 & H^g n_b & 1 \end{pmatrix} \begin{pmatrix} \nu \\ \mu_b \\ \rho \end{pmatrix}, \]

where $n^a \mu_a = H^g \nu$ because the left hand side is a section of $N^\perp$.

This boundary splitting isomorphism can be extended to tractor tensor bundles, in particular for tractor forms the section space map from the subbundle of $\mathcal{T}^*M[.]|\Sigma$ orthogonal to $M$ to $\mathcal{T}^*\Sigma[.]$ is

$$ (4.4) \quad \begin{pmatrix} F^+ \\ F \\ F^{+-} \end{pmatrix} \mapsto g \begin{pmatrix} 1 & 0 & 0 & 0 \\ -H^g \iota(n) & 1 & 0 & 0 \\ H^g \iota(n) & 0 & 1 & 0 \\ \frac{(H^g)^2}{2} [\iota(n), \iota(n)] & H^g \iota(n) & H^g \iota(n) & 1 \end{pmatrix} \begin{pmatrix} F^+ \\ F \\ F^{+-} \end{pmatrix}, $$

where here the orthogonal condition says (along $\Sigma$) that

$$ \iota(n) F^+ = 0 = \iota(n) F - H^g F^+ = \iota(n) F^{+-} = \iota(n) F^- + H^g F^{+-}. $$

In invariant terms, this can be expressed as $\iota(N) F = 0$ for $F \in \Gamma \mathcal{T}^*M[.]|\Sigma$.

Recall that for Riemannian geometries, the Gauß formula relates the interior and boundary covariant derivatives. In particular, if $u, v \in \Gamma TM$ are local extensions of $u_\Sigma, v_\Sigma \in \Gamma T\Sigma$, then the Levi–Civita connection $\nabla$ with respect to $g$ on $M$ and the Levi–Civita connection $\nabla^\Sigma$ on $\Sigma$ with respect to the metric $g_\Sigma$ induced by $g$ agree, in the sense

$$ (\nabla_u v)^T | \Sigma = \nabla^\Sigma_{u_\Sigma} v_\Sigma. $$

Here we denote the tangential component of $v \in T_x \Sigma_x M$ by $v^T$. Moving to the almost scalar constant setting, choosing a $g \in c$ and setting $n := \nabla \sigma$, in the above formula we can replace the connection $\nabla$ by $\nabla^T = \nabla - n \nabla_n$. In this way the independence on the left hand side from the choice of local extension is manifest because of the operator statement

$$ \nabla^T \sigma = O(\sigma). $$

There exists a natural and canonical tangential operator related to the Thomas D-operator that we shall only need in the Poincaré–Einstein setting. In fact, a main point we wish to emphasise here is that in that setting this operator is holographic formula for the boundary Thomas D-operator.

**Definition 4.7.** Let $(M, c, \sigma)$ be a Poincaré–Einstein manifold. Then we define the tangential Thomas D-operator defined acting on tractors in $\Gamma \mathcal{T}^9 M[w]$ with $w \neq 1 - \frac{d}{2}, 1 - \frac{n}{2}$ by

$$ D^T_A := D_A - I_A I - D + \frac{1}{(h-1)(h-2)} X_A (I - D)^2, $$
Remark 4.8. Along $\Sigma$, the tangential Thomas D-operator may be viewed as a tractor analog of the Gauss formula.

**Definition 4.9.** Suppose $\sigma$ is a defining density for a hypersurface $\Sigma$. Let $P$ be a differential operator acting on the section space of a vector bundle $\mathcal{F}$. We say $P$ acts **tangentially along** $\Sigma$ (or informally “$P$ is tangential”) if

$$P \circ \sigma = \sigma \circ \tilde{P},$$

for $\tilde{P}$ some smooth linear operator acting on $\Gamma(\mathcal{F} \otimes \mathcal{E}M[-1])$ in the same neighbourhood.

Remark 4.10. It is straightforward to verify that $D^T_A$ is tangential by employing the identity

$$(h - 2)D^A \sigma = h \sigma D^A - 2X^A I \cdot D + h(h - 2)I^A,$$

which follows, for Poincaré-Einstein structures, directly from equation (3.8).

Remark 4.11. It is useful to define $\Gamma\mathcal{F}|_{\Sigma}$ the space of equivalence classes of sections $A \sim A \in \mathcal{F}$. The space $\Gamma\mathcal{F}|_{\Sigma}$ is naturally isomorphic to $\Gamma\mathcal{F'}|_{\Sigma}$. Note that tangential operators $P : \Gamma\mathcal{F} \to \Gamma\mathcal{F}'$ act canonically on $\Gamma\mathcal{F}|_{\Sigma}$ by

$$\Gamma\mathcal{F}|_{\Sigma} \ni [A] \mapsto [PA] \in \Gamma\mathcal{F'}|_{\Sigma}.$$

The tangential Thomas D-operator and the (boundary) Thomas D-operator on the intrinsic tractor bundle of $(\Sigma, c_\Sigma)$ are related as follows.

**Proposition 4.12.** Let $(M, c)$ be Poincaré–Einstein with defining scale singularity set $\Sigma$ and let $U, V$ be local extensions of $U_\Sigma \in \Gamma T\Sigma$, $V_\Sigma \in \Gamma T\Sigma[w]$ with $w \neq 1 - \frac{d}{2}, 2 - \frac{d}{2}$ and subject to $I \cdot U = 0 = I \cdot V$. Then

$$(4.6) \quad D^T U \Sigma V_\Sigma = \left. \left( \frac{d + 2w - 3}{d + 2w - 2} D_U V + \frac{1}{(d + 2w - 2)(d + 2w - 4)} X \cdot U (I \cdot D)^2 V \right) \right|_{\Sigma}.$$  

The proof of Proposition 4.12 is given in Appendix A. There we demonstrate that $D^T_A = h(h - 1)^{-1} D^T h A$ along $\Sigma$ which suffices to establish the result. It can also be proven by a tedious explicit computation for a given $g \in c$.

Remark 4.13. Let us clarify the meaning of the formula (4.6):

- Because $I$ is (tractor) parallel, the right hand side of the above display is manifestly an element of $\Gamma(N^+ \otimes \mathcal{E}M[w])$ so equality is in the sense of the natural extension of the isomorphism $N^+ \cong T\Sigma$ (explained above) to weighted tractors. We will often use this isomorphism without further comment where appropriate.
- Given any extension $\hat{U}$ of $U_\Sigma$, we can construct another extension $U := \hat{U} - I \cdot \hat{U} I$ satisfying $I \cdot U = 0$.
- So long as $w \neq \frac{\ell}{2}, \ell = 2, 3, 4$, the tractor expression on the right hand side of the display in Proposition 4.12 is

$$\left. \frac{d + 2w - 3}{d + 2w - 2} U \cdot D^T V \right|_{\Sigma}.$$  

- At the boundary Yamabe weight $w = \frac{3}{2} - \frac{\ell}{2} = 1 - \frac{\ell}{2}$, the Proposition states

$$D^T U \Sigma V_\Sigma = -X \cdot U (I \cdot D)^2 V \big|_{\Sigma},$$

and so recovers the holographic formula for the Yamabe operator of $[31]$.  

• At the (excluded) bulk Yamabe weight $w = 1 - \frac{d^2}{2}$ there is a version of the Proposition applying to the double D-operator $D^{AB}$. We will develop this for the specialization of the above Proposition to tractor forms below.

• At the excluded weight $w = 2 - \frac{d^2}{2}$, the residue of the pole is the operator $(I.D)^2$ which is $\sigma^2$ times the bulk Paneitz operator [23], and therefore vanishes along $\Sigma$. In fact acting on tractor forms, this singularity is removable as we shall also show below.

For the purposes of this Article we need to develop a variant of Proposition 4.12 that uses the exterior tractor D-operator. To that end, on Poincaré–Einstein structures we introduce the **tangential exterior tractor D-operator**

$$ (4.7) \quad \mathcal{D}^T := \mathcal{D} + I \cdot y + \frac{1}{(h-1)(h-2)} \mathcal{D}^2 y^2. $$

Here $\mathcal{D}^T : \mathcal{T}^kM[w] \rightarrow \mathcal{T}^{k+1}M[w-1]$ and is defined whenever $w \neq \frac{3}{2} - \frac{d}{2}, 2 - \frac{d}{2}$. When $w = 2 - \frac{d}{2}$ we define

$$ (4.8) \quad \mathcal{D}^T := \mathcal{D} + I \cdot y + (I^* \mathcal{X} \mathcal{D} - \mathcal{X} \mathcal{D} I^*) y. $$

The tangential exterior tractor D-operator will play a central rôle in our later study of detours and gauge operators thanks to the following result.

**Theorem 4.14.** Let $(M, c, \sigma)$ be Poincaré–Einstein with defining scale singularity set $\Sigma$ and let $\mathcal{A} \in \Gamma \mathcal{T}^kM[w]$ be a local extension of $\mathcal{A}_\Sigma \in \Gamma \mathcal{T}^k\Sigma[w]$, with $w \neq \frac{3}{2} - \frac{d}{2}$ and subject to $I^* \mathcal{A} = 0$. Then

$$ (\mathcal{D}^T \mathcal{A})|_{\Sigma} = \frac{d + 2w - 2}{d + 2w - 3} \mathcal{D}_\Sigma \mathcal{A}_\Sigma, $$

where $\mathcal{D}_\Sigma$ is the exterior tractor D-operator of $\mathcal{T}^k\Sigma[w]$. When $w = \frac{3}{2} - \frac{d}{2}$ and all other preconditions as above hold, then

$$ (\mathcal{D}^T y^2 \mathcal{A})|_{\Sigma} = -\mathcal{D}_\Sigma \mathcal{A}_\Sigma. $$

The proof of this Theorem is given in Appendix A.

**Remark 4.15.** Similar remarks apply as for Proposition 4.12:

- The equalities are in the sense of the isomorphism between the orthogonal component of the bundle $\mathcal{T}^kM[..]$ along $\Sigma$ and $\mathcal{T}^k\Sigma[..]$. The condition $I^* \mathcal{A} = 0$ along $\Sigma$ implies $\iota(N)\mathcal{A}|_{\Sigma} = 0$. Moreover, given any extension of $\mathcal{A}$ of $\mathcal{A}_\Sigma$, we can always construct another extension in the kernel of $I^*$ by multiplying by the projector $I^* \mathcal{I}$.

- The **tangential exterior double D-operator** is defined by

$$ D^T_{[2]} := -\mathcal{D} \mathcal{D}^T. $$

With the same conditions as the Theorem and defining the $(\Sigma, c_\Sigma)$ version of $\mathcal{D}_\Sigma$ also using Definition 3.7, we have

$$ (\mathcal{D}^T_{[2]} \mathcal{A})|_{\Sigma} = -\mathcal{(X} \mathcal{D} \mathcal{D}^T \mathcal{A})|_{\Sigma} = -\mathcal{X} \mathcal{D}_\Sigma \mathcal{A}_\Sigma = D^T_{[2]} \mathcal{A}_\Sigma. $$

- The formula (4.8) for $\mathcal{D}^T$ at the bulk Paneitz weight $w = 2 - \frac{d}{2}$ can be understood by noting that away from $w = 2 - \frac{d}{2}$

$$ \mathcal{D}^T := \mathcal{D} + I \cdot y + \frac{1}{h-1} (I^* \mathcal{X} \mathcal{D} - \mathcal{X} \mathcal{D} I^*) y - \frac{1}{(h-1)(h-2)} x \mathcal{D} y, $$

...
where the singularity has been removed by discarding the last term which vanishes along $\Sigma$.

For later use, it is convenient to make the following definition.

**Definition 4.16.** Acting on $\Gamma \mathcal{T}^k M[w]$, we define the operator

$$\mathcal{D} := \begin{cases} (h - 1) \mathcal{D}_T, & w \neq 1 - d, 1 - n \frac{d}{2} \\ -\mathcal{X} y^2, & w = 1 - n \frac{d}{2} \end{cases}.$$ 

This is useful because $(\mathcal{D} A)|_\Sigma = \mathcal{D}_\Sigma A_\Sigma$ for $A \in \ker \mathcal{I}$ an extension of $A_\Sigma \in \Gamma \mathcal{T}^k \Sigma[w]$.

### 4.3. Boundary conditions for tractor forms.

An almost Einstein structure provides a simple construction of natural boundary conditions for differential forms. These can be classified by insertions of forms in tractors according to the analysis of Section 3.2. Here we focus on the “western case” needed later in the Article.

Suppose $A_\Sigma \in \Gamma \mathcal{E}^k \Sigma[w + k]$ is some boundary form which we shall view as the boundary data for our problem. The aim is to canonically extend this to an interior tractor $A \in \Gamma \mathcal{T}^k M[w]$. In Section 5.1 we will study an extension problem $\gamma A = 0$ (see also Problem 2.3) as far as possible uniquely determining an $A$ in the space of canonical extensions.

There are two obvious approaches to extending $A_\Sigma$ to a bulk form tractor: First extend to an interior differential form and thereafter embed in a tractor form, or secondly embed $A_\Sigma$ in a boundary tractor and then extend to an interior tractor form.

$$\begin{array}{ccc} \Gamma \mathcal{E}^k \Sigma[w + k] & \xrightarrow{\text{ext}} & \Gamma \mathcal{E}^k M[w + k] \\ \downarrow q_W & & \downarrow q_W \\ \Gamma \mathcal{T}^k \Sigma[w] & \xrightarrow{\text{Ext}} & \Gamma \mathcal{T}^k M[w] \end{array}$$

(4.9)

The key to writing tractor problems that precisely encode differential form problems is constructing extensions so this diagram commutes. For encoding boundary conditions we must find extensions so that this holds along $\Sigma$.

**Proposition 4.17.** Suppose $w \neq k - d, k - n$. Then any extensions $\text{ext}$ and $\text{Ext}$

$$\text{ext} : \Gamma \mathcal{E}^k \Sigma[w + k] \rightarrow \ker \iota \subset \Gamma \mathcal{E}^k M[w + k]$$

and

$$\text{Ext} : \Gamma \mathcal{T}^k \Sigma[w] \rightarrow \ker(\mathcal{I}^*, \mathcal{D}^*, \mathcal{X}^*) \subset \Gamma \mathcal{T}^k M[w],$$

obey

$$r \circ q_W \circ \text{ext} = r \circ \text{Ext} \circ q_W^\Sigma,$$

where $r$ denotes restriction to $\Sigma$.

**Proof.** We begin by setting up the structures involved and start with $\text{ext}$. Since $\Lambda^* \Sigma$ is naturally a subbundle of $\Lambda^* M|_\Sigma$, so too is $\mathcal{E}^k \Sigma[w]$ of $\mathcal{E}^k M[w]|_\Sigma$. Thus we choose any smooth weighted form $A \in \Gamma \mathcal{E}^k M[w + k]$ such that

$$A|_\Sigma = A_\Sigma.$$

Calculating in some choice of $g \in c$, this extension of $A_\Sigma$ obeys

$$\iota(n) A + \sigma \phi = 0,$$

for some $\phi \in \Gamma \mathcal{E}^{k-1} M[w + k - 2]$, since necessarily $(\iota(n) A)|_\Sigma = 0$. By continuity $\iota(n) \phi$ is zero everywhere.
The normal component of \( A \), i.e. \( \iota(n)A \), vanishes along \( \Sigma \), but its normal derivative does not and is encoded by \( \phi \) along \( \Sigma \) as follows from (4.10):

\[
\phi_\Sigma = -\left( \nabla_n [\iota(n)A] \right) |_\Sigma .
\] (4.11)

Looking ahead, we want to construct a west tractor. This suggests an ansatz for \( A \in \Gamma T^kM[w] \) by its expression in the scale \( g \in c \)

\[
\mathcal{A} := g \begin{pmatrix} 0 \\ A \\ 0 \\ \phi \end{pmatrix} ,
\] (4.12)

which satisfies \( \mathcal{D}^* \mathcal{A} = 0 \). At weights \( w \neq k - d \), \( \phi \) can be tied to \( A \) by imposing

\[
\mathcal{D} \mathcal{A} = 0.
\] (4.13)

We have arrived canonically at the generalised divergence equation of Problem 2.8 which was solved (to some order) for arbitrary boundary data \( A_\Sigma \) with weights \( w \neq k - n \) in Section 2.4. The compatibility with that problem is critical to subsequent developments.

To summarise, extending \( A_\Sigma \in \Gamma E^k\Sigma[w+k] \) to \( A \in \Gamma T^kM[w+k] \) subject to Equation (4.13) and then inserting this in a west tractor \( \mathcal{A} = qW A \in \Gamma T^kM[w] \) produces a solution to

\[
\mathcal{D}^* \mathcal{A} = \mathcal{D}^* \mathcal{A} = \mathcal{D}^* \mathcal{A} , \quad (\mathcal{D}^* \mathcal{A}) |_\Sigma = A_\Sigma .
\] (4.14)

To establish Proposition 4.17, it only remains to show equality of \( \mathcal{A}_\Sigma \) and \( qW A_\Sigma \). Along \( \Sigma \) Equation (4.13) implies \( A_\Sigma = A_\Sigma \in \Gamma E^k\Sigma[w+k] \) and so

\[
\mathcal{D} \mathcal{A} = \mathcal{D} \mathcal{A} = \mathcal{D} \mathcal{A} , \quad (\mathcal{D} \mathcal{A}) |_\Sigma = \mathcal{D} A_\Sigma .
\] (4.13)

which is a consequence of Lemma 4.18, which follows. This completes the proof since the inverse of the boundary splitting isomorphism, given in Equation (4.4), acts as the identity on such sections.

\[\square\]

Lemma 4.18. Let \( A \in \Gamma E^kM[w+k] \) be an extension of \( A_\Sigma \in \Gamma E^k\Sigma[w+k] \), subject to Equation (4.13). Then

\[
(n + w - k) \left( \mathcal{D} A \right) |_\Sigma = (d + w - k) \delta_\Sigma A_\Sigma .
\]
Remarkably the above display reduces to a west tractor along \( \phi \). That is Proposition 4.20. Given as data this result is encapsulated by the following version of Proposition 4.17. (4.15)

\[
(\delta A)|_\Sigma = (\delta (\iota(n)\varepsilon(n)) A)|_\Sigma = \delta_{\Sigma} A_\Sigma + \frac{1}{d+w-k} (\iota(n)\varepsilon(n)\delta A)|_\Sigma = \delta_{\Sigma} A_\Sigma + \frac{1}{d+w-k} (1 - \varepsilon(n)\iota(n))\delta A|_\Sigma.
\]

On the first line we inserted \( 1 = 2\rho\sigma + \iota(n)\varepsilon(n) + \varepsilon(n)\iota(n) \) and used that \( (\iota(n)A)|_\Sigma = 0 \). To obtain the second line we used that \( \iota(n)A = \frac{1}{d+w-k} \delta A \) as well as the relationship between bulk and boundary codifferentials

\[
(\delta \iota(n)\varepsilon(n)A)|_\Sigma = \delta_{\Sigma} A_\Sigma.
\]

Finally, \( \iota(n)\delta A = 0 \) as can be seen by acting upon Equation (4.13) with \( \iota(n) \) which completes the proof. \( \square \)

Remark 4.19. The weight \( w = k - n \) corresponds to a special case of the west Lemma 3.13 along the boundary. Indeed the above Lemma then imposes the condition \( \delta_{\Sigma} A_\Sigma = 0 \) which leads to a qualitatively different natural boundary problem.

Now we turn to distinguished weights. Consider first a tractor \( \mathcal{A} \) satisfying \( (4.14) \) at the value \( w = k - d \). Then from Lemma 3.13 we have

\[
\ker(\tilde{\mathcal{G}}^*, \mathcal{X}^*) \subset \mathcal{T}^k M[w] \ni \mathcal{A} := \begin{pmatrix} 0 \\ A \\ 0 \\ \phi \end{pmatrix}, \text{ such that } \delta A = 0 = \delta \phi.
\]

Remarkably the above display reduces to a west tractor along \( \Sigma \). To see this we need to show \( \phi|_{\Sigma} = \phi_{\Sigma} = \delta_{\Sigma} A_{\Sigma} \). This follows from a rapid computation using only \( \mathcal{I}^* \mathcal{A} = 0 \):

\[
0 = (\delta A)|_{\Sigma} = (\delta (2\rho\sigma + \iota(n)\varepsilon(n) + \varepsilon(n)\iota(n)))|_{\Sigma} = \delta_{\Sigma} A_\Sigma - (\delta \varepsilon(n)\sigma\phi)|_{\Sigma} = \delta_{\Sigma} A_\Sigma - \phi_{\Sigma}.
\]

(4.15)

This result is encapsulated by the following version of Proposition 4.17.

**Proposition 4.20.** Given as data \( A_{\Sigma} \in \Gamma \mathcal{E}^k \Sigma[2k - d] \), the following constructions of a tractor along \( \Sigma \) agree:

(i) Take any coclosed extension of \( A_{\Sigma} \) to \( A \in \Gamma \mathcal{E}^k M[2k - d] \) and pair it with a coclosed form \( \phi \in \Gamma \mathcal{E}^{k-1} M[2k - d - 2] \) by requiring \( \mathcal{A} : = q_W(A, \phi) \in \ker \mathcal{I}^* \), which is well-defined because

\[
(A, \phi) \in \ker(\delta, \delta) \subset (\Gamma \mathcal{E}^k M[2k - d] \oplus \Gamma \mathcal{E}^{k-1} M[2k - d - 2])
\]

Then take the restriction.

(ii) Map \( A_{\Sigma} \) to \( A_{\Sigma} \in \Gamma T^k \Sigma[k - d] \) with the boundary insertion operator \( q_{\Sigma}^W \). That is

\[
\mathcal{A}|_{\Sigma} = A_{\Sigma} \in \ker(\tilde{\mathcal{G}}^*, \mathcal{X}^*)
\].
Next, observe that the case \( w = k - n \) is also distinguished because it corresponds to a special case for the west Lemma along \( \Sigma \). Again the details are interesting and surprising. Suppose that \( A \in \ker(\mathcal{D}^*, \mathcal{X}^*, \mathcal{Y}^*) \) with \( w = k - n \), then for \( g \in c \)

\[
A = \begin{pmatrix} g \end{pmatrix}
\]

with \( \phi = -\delta A \) and \( \iota(n)A + \sigma\phi = 0 = \iota(n)\phi \).

Write \( A_\Sigma := A|_\Sigma \) and consider

\[
\delta_\Sigma A_\Sigma = \left( \delta \iota(n)\varepsilon(n)A \right)|_\Sigma = \left( \delta(-\varepsilon(n)\iota(n) + 1 - 2\rho\sigma)A \right)|_\Sigma = 0.
\]

This implies we can no longer use the boundary form \( A_\Sigma \) as Dirichlet data. Instead the extension \( A \) of \( A_\Sigma \) carries the independent Neumann data \( \phi_\Sigma \) according to (4.11). Moreover, \( \phi_\Sigma \) is also coclosed along \( \Sigma \) because

\[
\delta_\Sigma \phi_\Sigma = (\delta \iota(n)\varepsilon(n)\phi)|_\Sigma = (\delta(-\varepsilon(n)\iota(n) + 1 - 2\rho\sigma)\phi)|_\Sigma = 0,
\]

(using \( \delta\phi = -\delta^2 A = 0 \)). But exactly because \( w = k - n \), by virtue of Lemma 3.13, the coclosed boundary forms \( (A_\Sigma, \phi_\Sigma) \) are isomorphic to a boundary tractor \( A_\Sigma \in \ker(\mathcal{D}^*_\Sigma, \mathcal{X}^*_\Sigma) \).

Thus for this case we have a modified version of Proposition 4.17 giving a Dirichlet boundary condition for the pair of coclosed forms \( (A_\Sigma, \phi_\Sigma) \).

**Proposition 4.21.** Given the data

\[
(A_\Sigma, \phi_\Sigma) \in \ker(\delta_\Sigma, \delta_\Sigma) \subset \left( \Gamma\mathcal{E}^k\Sigma[2k - n] \oplus \Gamma\mathcal{E}^{k-1}\Sigma[2k - n - 2] \right),
\]

the following constructions of a tractor form along \( \Sigma \) agree:

(i) Take any extension of \( A_\Sigma \) to \( A \in \Gamma\mathcal{E}^k\Sigma[2k - n] \), where \( A_\Sigma = A|_\Sigma \) is Dirichlet data and \( \phi_\Sigma = - \left( \nabla_n [\iota(n)A] \right)|_\Sigma \) is Neumann data, and such that \( A := q_W A \in \Gamma\mathcal{T}^k\Sigma[k - n] \) satisfies \( \mathcal{J}^* A = 0 \). Then take the restriction.

(ii) Map \( (A_\Sigma, \phi_\Sigma) \) to \( A_\Sigma \in \Gamma\mathcal{T}^k\Sigma[k - n] \) with the boundary insertion operator \( q_W^\Sigma \).

That is

\[
A|_\Sigma = A_\Sigma \in \ker(\mathcal{D}^*_\Sigma, \mathcal{X}^*_\Sigma).
\]

### 4.4. Holographic boundary projectors.

Our aim at this stage is to construct projectors which solve the extension problem to be introduced in the next Section. A first step is to construct holographic formulae for projectors implementing the tractor boundary conditions introduced above, while at the same time solving the scale-transversality conditions. In equations, we seek a tractor \( A \) obeying

\[
\mathcal{J}^* A = \mathcal{D}^* A = \mathcal{X}^* A = 0
\]

subject to

\[
A|_\Sigma = A_\Sigma \in \ker(\mathcal{D}^*_\Sigma, \mathcal{X}^*_\Sigma).
\]

There are however three separate cases: \( w = k - n, w = -k \) and \( w \) generic. We begin with the last case.
4.4.1. Holographic projectors at generic weights. Recall from Section 3.2 the projector
\[ \Pi_k^\Sigma[w] \to \ker(\widehat{\mathcal{D}}_\Sigma^*, \mathcal{X}_\Sigma^* \mathcal{X}_\Sigma^*) \subset \Gamma^T \Sigma[w] \] for \( w \neq -k, k-n \) given by
\[ \Pi_k^\Sigma[w] := \frac{1}{(w+k)(n+w-k)} \mathcal{X}_\Sigma^* \mathcal{X}_\Sigma^* \mathcal{X}_\Sigma^* , \] satisfying \( (\Pi_k^\Sigma[w])^2 = \Pi_k^\Sigma[w] \).

There is a holographic formula for this boundary western projector. First we develop some new tools.

**Definition 4.22.** Let \((M, c, \sigma)\) be an almost Riemannian structure. The holographic interior and exterior triple \(D\)-operators are defined, respectively, by
\[ D[3] := \widehat{\mathcal{D}} \mathcal{X} \mathcal{I} \quad \text{and} \quad D^*[3] := \widehat{\mathcal{D}} \mathcal{X}^* \mathcal{I}^* , \]
where
\[ D[3] : \Gamma^T M[w] \to \Gamma^{T+3} M[w] \quad \text{and} \quad D^*[3] : \Gamma^T M[w] \to \Gamma^{T-3} M[w] . \]

For a choice of \( g \in c \) one has
\[ D[3] g \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\varepsilon(n)d & \bar{\varepsilon} & 0 & 0 \\ \varepsilon(n)d & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad D^*[3] g \begin{pmatrix} 0 & 0 & 0 & 0 \\ -\delta_i(n) & 0 & \bar{i} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \delta_i(n) & 0 \end{pmatrix} , \]
where
\[ \bar{\varepsilon} = \varepsilon(n)(w + N) - \sigma d \quad \text{and} \quad \bar{i} = \xi(n)(d + w - N) - \sigma \delta . \]

**Remark 4.23.** Like their double \(D\) ancestors, the holographic triple \(D\)-operators obey a graded Leibnitz rule. Furthermore, we view these as holographic formulae for the boundary double and (extrinsic) triple \(D\)-operators because along \( \Sigma \) they restrict to \( \varepsilon(N)\widehat{\mathcal{D}}_\Sigma \mathcal{X}_\Sigma^* \) and \( \widehat{\mathcal{D}}_\Sigma \mathcal{X}_\Sigma^* \mathcal{I}(N) \).

From their definitions, or the explicit expressions displayed for a choice of \( g \in c \), we immediately see that these operators are tangential because they commute with the scale operator \( x = \sigma \)
\[ [D[3], x] = 0 = [D^*[3], x] . \]
Moreover \( \text{ran} \ D[3] \subset \ker(\widehat{\mathcal{D}}, \mathcal{X}) \) and \( \text{ran} \ D^*[3] \subset \ker(\widehat{\mathcal{D}}^*, \mathcal{X}^*) \). On an almost Riemannian structure we may write \( D[3] = \mathcal{I} \widehat{\mathcal{D}} \mathcal{X} \) and \( D^*[3] = \mathcal{X}^* \widehat{\mathcal{D}}^* \mathcal{X}^* \). This is obvious for almost Einstein structures, but is also easily verified in the more general almost Riemannian setting. Thus
\[ \text{ran} \ D[3] \subset \ker(\mathcal{I}, \widehat{\mathcal{D}}, \mathcal{X}) \quad \text{and} \quad \text{ran} \ D^*[3] \subset \ker(\mathcal{X}^*, \widehat{\mathcal{D}}^*, \mathcal{X}^*) . \]

For most weights \( \Sigma \) may be replaced by equality, see Section 3.3.

**Definition 4.24.** Let \( w \neq -k, k-n \). We define the holographic boundary (west) projector
\[ \Pi : \Gamma^T M[w] \to \ker(\mathcal{I}, \widehat{\mathcal{D}}^*, \mathcal{X}^*) \subset \Gamma^T M[w] , \]
by
\[ \Pi := \frac{1}{w+k+n+w-k} \cdot D[3] D^*[3] . \]

Although we term \( \Pi \) the holographic boundary projector, it is only idempotent along \( \Sigma \); we view \( \Pi \) as a holographic formula for the boundary west projector \( \Pi^\Sigma \). In Lemma 4.20 we shall show that bulk idempotence holds on the kernel of the extension operator \( y = -I \cdot \mathcal{D} \).
**Proposition 4.25.** On a conformally compact manifold, the holographic boundary operator \( \Pi \) obeys
\[
I^\ast \Pi = 0 = \hat{I}^\ast \Pi = \hat{X}^\ast \Pi.
\]
If \( A \in \ker(I^\ast, \hat{I}^\ast, \hat{X}^\ast) \subset \Gamma T^kM[w], w \neq -k, k-n \), then
\[
(I \Lambda A)|_\Sigma = A_\Sigma.
\]
Moreover \( \Pi \) is tangential and if \( A \in \Gamma T^kM[w] \) with \( (I^\ast A)|_\Sigma = 0 \) then
\[
(I \Lambda A)|_\Sigma = \Pi^\ast w A_\Sigma, \quad A_\Sigma := A|_\Sigma.
\]
*Proof.* The first set of equalities are obvious using the discussion above the Proposition and nilpotency of the operators \((I^\ast, \hat{I}^\ast, \hat{X}^\ast)\). Tangentiality follows by construction because \( D^\ast[3] \) and \( D[3] \) are separately tangential. The remaining two claims are verified by choosing a scale \( g \in c \) and employing the explicit matrix expressions \((3.10)\) and \((4.17)\) for the double and triple D-operators. The last display also follows immediately from Remark 4.23 and the fact that the product \( \iota(N) \varepsilon(N) \) is unity on boundary tractors. \( \square \)

Later we will need finer information about the failure of the holographic boundary projector to be a linear projection in the bulk.

**Lemma 4.26.** Let \((M, c, \sigma)\) be almost Einstein. Then, acting on \( A \in \ker(I^\ast, \hat{I}^\ast, \hat{X}^\ast) \subset \Gamma T^kM[w], w \neq -k, k-n \), we have
\[
(I \Lambda A) = \left(1 + \frac{1}{(w+k)(n+w-k)} xy\right) A.
\]
More generally, at any weight \( w \)
\[
D^\ast[3] D[3] A = ((w+k)(n+w-k) + xy) A.
\]
*Proof.* In order to prove the second identity one has to push all the interior operators to the right where they annihilate \( A \). First we write
\[
D^\ast[3] D[3] A = D^\ast[2] I^\ast I D[2] A
\]
\[
= [D^\ast[2], I^\ast I] D[2] A + I^\ast I (w+k)(d+w-k) A
\]
\[
= [D^\ast[2], I^\ast I] D[2] A + (w+k)(d+w-k) A,
\]
where the second line employed Proposition 3.22 while the third used that \( I^\ast I A = A \).

We claim that the first term above equals \( (xy - (w+k)) A \). To demonstrate this we first use the algebra of Proposition 4.1 acting on generically weighted tractors, to find
\[
[D^\ast[2], I^\ast I] = I^\ast \left( \hat{D}^\ast x + \frac{1}{h} \hat{X}^\ast \right) = I^\ast \left( x \hat{D}^\ast + \frac{1}{h} y \hat{X}^\ast \right).
\]
Recall from Proposition 3.8 that the exterior double D-operator obeys \( D^\ast[2] \hat{D}^\ast \hat{X}^\ast = -\hat{X}^\ast \hat{D}^\ast \) so that \( \hat{D}^\ast D^\ast[2] = \hat{D} \hat{D}^\ast \hat{X}^\ast \) and \( D^\ast D^\ast[2] = X X^\ast \hat{D} \). This allows us to use Proposition 3.23 at generic weights \( w \), to obtain \( \hat{D}^\ast D^\ast[2] A = -(d+w-k) \hat{D} A \) and \( X^\ast D^\ast[2] A = (w+k) X A \). Thereafter using the obvious identities \( I^\ast \hat{D} A = -y A \), \( I^\ast \hat{X} A = x A \) and the fundamental identity \( [x, y] = h \) gives the result for generic weights. We need the result for all weights. In the process of the above computation, there are terms which could become singular. However, since the left hand side of the identity to be proved is manifestly a natural formula with coefficients polynomial in the weight, any singularities are removable. \( \square \)
Remark 4.27. It is interesting to explicate the holographic boundary projector $\Pi$ for some $g \in c$ to see how it manages to solve the scale-transversality conditions $\mathcal{I}^* \mathcal{A} = \mathcal{D}^* \mathcal{A} = \mathcal{X}^* \mathcal{A} = 0$. (The latter two are no mystery, since they are the content of the west Lemma 3.13) Taking $\mathcal{A} \in \Gamma T^k M[w]$ for $g \in c$ to be

$$\mathcal{A} := \begin{pmatrix} A + \frac{\psi}{w+k} d\psi \\ B \\ \phi \end{pmatrix},$$

an explicit computation (valid away from $w = -k, k - d, k - n$) gives

$$\Pi \mathcal{A} \overset{g}{=} \begin{pmatrix} 0 \\ \frac{A}{\psi} \\ 0 \\ -\frac{1}{d+w-k} \delta \tilde{A} \end{pmatrix} = q_{\mathcal{W}}(\tilde{A}) \in \ker(\mathcal{I}^*, \mathcal{D}^*, \mathcal{X}^*),$$

where

$$\tilde{A} = (\iota(n) - \frac{1}{n+w-k} \sigma \delta) (\varepsilon(n) - \frac{1}{w+k} \sigma d) A.$$

The operator appearing above is exactly the holographic projector solution to the Coulomb gauge extension problem given in Proposition 2.11.

4.4.2. True forms. When $w = -k$ the holographic boundary projector $\Pi$ can no longer be used to solve the scale-transversality conditions. So instead we solve a weaker problem that suffices for later purposes based around an operation we call $\hat{\Pi}$.

$$\Omega^k M \ni A \xrightarrow{\hat{\Pi}} -\frac{1}{n+2k} \mathcal{D}^*[\mathcal{I}] q_{(N)} A \in \Gamma T^k M[-k], \quad k \neq \frac{n}{2},$$

Here $q_{(N)} : \mathcal{E}^{k-1} M[w+k] \rightarrow \coker(\mathcal{I}^*, \mathcal{D}^*; \Gamma T^k M[w])$ via (cf. Remark 3.15)

$$q_{(N)} A \overset{g}{=} \begin{pmatrix} A \\ * \\ * \end{pmatrix}.$$

So the map (4.21) depends on the choice of a coset representative for the $(\mathcal{D}^*, \mathcal{D}^*)$ cokernel. However, upon composition with the canonical map

$$\pi_\Sigma : \Gamma T^k M[-k] \longrightarrow \Gamma T^k M[-k] \rvert_\Sigma,$$

this determines a well defined map $\pi_\Sigma \circ \hat{\Pi}$. (Recall that the notation $\bullet \rvert_\Sigma$ denotes equivalence classes of sections as in (4.5).) To show this we calculate on $A \in \Gamma \mathcal{E}^k M[-k], k \neq \frac{n}{2},$ for some $g \in c$ using the explicit expressions (4.17), (4.1) and (3.5) for the operators in $\hat{\Pi}$, and find

$$\hat{\Pi} A \overset{g}{=} \begin{pmatrix} 0 \\ \iota(n) \varepsilon(n) A \\ -\frac{1}{n+2k} \delta \iota(n) \varepsilon(n) A \end{pmatrix} \quad \text{along } \Sigma.$$

This is precisely $q_{\mathcal{W}}^\Sigma A_\Sigma$ with $A_\Sigma = A \rvert_\Sigma$. 

---
On the other hand, given a choice of cokernel representative determining \( q(N)A \), then \( \hat{\Pi}A \) gives a representative of \( \Gamma T^k M[-k] \), that obeys the analog of (4.18)
\[
\mathcal{S}^* \hat{\Pi} A = 0 = \hat{\mathcal{D}}^* \hat{\Pi} A = \mathcal{S}^* \hat{\Pi} A.
\]
So we have now established an analog of Proposition 4.25.

**Proposition 4.28.** Suppose \( k \neq \frac{n}{2} \). Then
\[
\pi_{\Sigma} \circ \hat{\Pi} : \Omega^k M \rightarrow \Gamma T^k M[-k],
\]
is a well defined map where
\[
\hat{\Pi} := -\frac{1}{n-2k} D_{[\eta]} \mathcal{S}^* q(N),
\]
obeys
\[
\mathcal{S}^* \hat{\Pi} = 0 = \hat{\mathcal{D}}^* \hat{\Pi} = \mathcal{S}^* \hat{\Pi}.
\]
Moreover \( \hat{\Pi} \) is tangential and if \( A \in \Omega^k M \) with \( A|_{\Sigma} \) is a well defined map where
\[
\hat{\Pi} := -\frac{1}{n-2k} D_{[\eta]} \mathcal{S}^* q(N),
\]
obeys
\[
\mathcal{S}^* \hat{\Pi} = 0 = \hat{\mathcal{D}}^* \hat{\Pi} = \mathcal{S}^* \hat{\Pi}.
\]
Moreover \( \hat{\Pi} \) is tangential and if \( A \in \Omega^k M \) with \( A|_{\Sigma} = A_\Sigma \in \Omega^k \Sigma \), then
\[
(\hat{\Pi} A)|_{\Sigma} = q_{\Sigma}^\tau A_\Sigma.
\]
Note that a true scale \( \tau \) determines a map \( \hat{\Pi}_{\tau} : \Omega^k M \rightarrow \Gamma T^k M[-k] \) with
\[
\hat{\Pi}_{\tau} := \frac{1}{n-2k} D_{[\eta]} \mathcal{S}^* q_{\tau}(N)
\]
where
\[
q_{\tau}(N) A \overset{\tau}{=} \begin{pmatrix} A \\ 0 \\ 0 \end{pmatrix}.
\]
The above Proposition applies to the map \( \hat{\Pi}_{\tau} \) *mutatis mutandis*.

4.4.3. *Dual weight true forms.* The remaining problem weight is \( n + w - k = 0 \) which is the special case of the west Lemma, see 3.13 as specialized to the boundary. Based on Proposition 4.21, at this weight one must consider coclosed boundary data \( (A_\Sigma, \phi_\Sigma) \).

Locally along \( \Sigma \) we may write
\[
A_\Sigma = \delta_{\Sigma} B_\Sigma \quad \text{and} \quad \phi_\Sigma = \delta_{\Sigma} \psi_\Sigma,
\]
where \([B_\Sigma] \in \text{coker}(\delta_{\Sigma}, \Gamma E^{k+1}[2k-n+2])\) and \([\psi_\Sigma] \in \text{coker}(\delta_{\Sigma}, \Gamma E^k[2k-n])\). To simplify the discussion, we will assume that this holds globally. So we take \([B_\Sigma], [\psi_\Sigma]\) as our boundary data. We proceed by working with representatives \( (B_\Sigma, \psi_\Sigma) \); our solutions to the Proca system of the next Section will not depend on this choice. We then extend these to bulk forms \( B \in \Gamma E^{k+1}[2k-n+2] \) and \( \psi \in \Gamma E^k M[2k-n] \) and insert them in a bulk tractor
\[
B \overset{q}{=} q_{(N)} B + q_{(E)} \psi \in \Gamma T^{k+2} M[k-n].
\]
(In fact, in the above, we really view \( (B_\Sigma, \psi_\Sigma) \) as components, in a scale \( g_\Sigma \in c_\Sigma \), of a representative section of \( \text{coker}(\mathcal{S}_{\Sigma}^*, \Gamma T^{k+2}[k-n]) \). Our final solutions will not depend on this choice of scale.) Now let us compute the following using (4.17)
\[
D_{[\eta]} \mathcal{S} B \overset{q}{=} \begin{pmatrix} \delta \iota(n) \varepsilon(n) B \\ 0 \\ \delta \iota(n) \varepsilon(n) \psi \end{pmatrix} + O(\sigma).
\]
Here we have used that, acting on $\Gamma^E_k M[2k - n + 1]$, the operator $\hat{i} = -\sigma \delta$. Moreover we have the identity $\delta \iota(n) B = O(\sigma)$. The latter holds because $B$ is a smooth extension of $B_\Sigma$, whence $\iota(n) B = \sigma C$ for some $C$ so $\delta \iota(n) B = \iota(n) C + O(\sigma)$, which vanishes along $\Sigma$ by continuity.

Along $\Sigma$, the operator $\delta \iota(n) \varepsilon(n) = \delta \Sigma$ so the above display equals $q_{\hat{W}}(A_\Sigma, \phi_\Sigma)$ there. Thus, we see that the above display, modulo $O(\sigma)$, is also independent of our choice of representatives $(B_\Sigma, \phi_\Sigma)$. Hence we have established the following result.

**Proposition 4.29.** Let $(A_\Sigma, \phi_\Sigma) \in \ker(\delta \Sigma, \delta \Sigma) \subset \Gamma^{E_k \Sigma}[2k - n] \oplus \Gamma^{E_{k-1} \Sigma}[2k - n - 2]$ and $A \in \Gamma^T M[k - n]$ be any extension of $q_{\hat{W}}(A_\Sigma, \phi_\Sigma)$. Then

$$A = D_{[\delta]} B + O(\sigma),$$

for some $B \in \Gamma^{T^{k+2} M}[k - n]$. Moreover, modulo $\sigma$, we may view $B$ as an element of coker $\mathcal{I}^*$.

## 5. Higher form Proca equations

The Proca equation \cite{55}

$$\delta dA - m^2 A = 0,$$

for a one-form $A$, first arose in the 1930’s as a relativistic extension of Maxwell’s equations to describe massive vector excitations. If the constant $m \neq 0$, then acting with the codifferential $\delta$ immediately yields a constraint

$$\delta A = 0.$$

Often it is convenient to consider, therefore, the equivalent system

$$(\Delta - m^2) A = 0 = \delta A.$$

At $m^2 = 0$ and in Lorentzian signature, these can be viewed as Maxwell’s equations in a Feynman gauge choice.

These equations generalise immediately to the case where $A$ is a form of arbitrary degree. They do not enjoy a conformal invariance, apart from Maxwell systems (meaning $m^2 = 0$ and arbitrary degree) in even dimensions with form degree $\frac{d}{2} - 1$. Remarkably, it is possible to unify these systems using the conformal tractor calculus by coupling to scale through the scale tractor. In \cite{30}, the authors considered a tractor vector $V^A \in T^A M[w]$ satisfying the equations

$$I_A F^{AB} = 0 = D_A V^A,$$

where $F^{AB} := D^A V^B - D^B V^A$ was called a tractor Maxwell field strength. Both equations enjoy the gauge invariance

$$V^A \sim V^A + D^A \alpha, \quad \alpha \in \mathcal{E}M[w + 1].$$

For generic weights and for Einstein structures, the above system of equations describes massive Proca excitations with masses dictated by the tractor weight $w$. At $w = 1 - \frac{d}{2}$ the Branson–Deser–Nepomechie equation arises while Maxwell’s equations appear at $w = -1$, see the work \cite{30}.

Here we study the generalization of the system \cite{55} to higher rank tractor forms on an almost Einstein structure. We start with a tractor field strength formulation of the higher form Proca systems by letting $F \in T^{k+1} M[w - 1]$. The natural higher form generalization of the field strength formulation of the vector model of \cite{30} is

$$\mathcal{D}^* F = 0 = \mathcal{D} F.$$
The coupling to scale is then achieved by generalizing the relation $I_A F^{AB} = 0$ to
\begin{equation}
\mathcal{J}^* F = 0.
\end{equation}
Since $\{\mathcal{J}^*, \mathcal{D}\} = -y$, an integrability condition follows.

**Proposition 5.1.** Equations (5.3) and (5.3) imply $y F = 0$.

By calculating away from distinguished weights, in the preferred interior scale with metric $g^o = \sigma^{-2} g$ (away from $\Sigma$), we can easily characterise the above system, $F \in \ker(\mathcal{D}, \mathcal{D}^*, \mathcal{J})$, as a Proca one. We record this in the following.

**Proposition 5.2.** Let $F \in \mathcal{T}^{k+1} M[w - 1]$ with $w \neq 2 - d, -k$, and $(M, c, \sigma)$ be almost Einstein. Then, away from $\Sigma$ in the Einstein scale $g^o$, the equations
\begin{equation}
\mathcal{D} F = D^* F = \mathcal{J}^* F = 0,
\end{equation}
capture precisely the Proca equation $\delta d A - m^2 A = 0$ with $\delta A = 0$ when $m = 0$, $A \in \Gamma \mathcal{E}^k M[w + k]$ and mass–Weyl weight relationship
\begin{equation}
m^2 = -\frac{2j}{d} (w + k)(n + w - k).
\end{equation}

**Proof.** For the preferred interior scale $g^o \in c$, we have
\begin{equation}
\mathcal{J}^* g^o = \begin{pmatrix}
0 & 0 & -1 & 0 \\
-\frac{j}{d} & 0 & 0 & 1 \\
0 & 0 & 0 & 0 \\
0 & 0 & -\frac{j}{d} & 0
\end{pmatrix}.
\end{equation}
Moreover, in the preferred scale the almost Einstein equations (4.2) tell us
\begin{equation}
\mathcal{P} g^o \equiv \frac{j}{d} N.
\end{equation}
Thus, together with the result for $\ker(\mathcal{D}, \mathcal{D}^*)$ given in the northern Lemma 3.18 (and calculating carefully the special cases not covered there) we find
\begin{equation}
\delta A = 0,
\end{equation}
and
\begin{equation}
(\Delta + \frac{2j}{d} (w + k)(n + w - k)) A = 0,
\end{equation}
which completes the proof. \[\square\]

**Remark 5.3.** Since the independent field content of the above field strength tractor formulation of the Proca system appears in the northern slot, we have depicted it and its Hodge dual model obtained by replacing $F \to \ast F$, at the northern point of the compass in Figure 4.

The main focus of the remainder of the Article is a potential formulation of the Equations of Proposition 5.2. Recall that the tractor Maxwell field strength $F \in \mathcal{T}^{k+1} M[w - 1]$ is subject to $\mathcal{D} F = 0$. Hence, by the cohomology result of Proposition 3.26, we can write
\begin{equation}
F = \mathcal{D} A,
\end{equation}
for some $A \in \Gamma \mathcal{T}^k M[w]$ so long as $w \neq 1 - \frac{d}{2}, 2 - \frac{d}{2}, -k, -k + 2$. Viewing the potential $A$ as the independent field content, our system of equations now becomes
\begin{equation}
\mathcal{D}^* \mathcal{D} A = 0 = \mathcal{J}^* \mathcal{D} A.
\end{equation}
Since $\mathcal{D}$ is nilpotent, solutions are only defined up to the gauge invariance
\begin{equation}
A \sim A + \mathcal{D}B,
\end{equation}
for some $B \in \Gamma\mathcal{T}^{k-1}M[w+1]$. Moreover, as the independent field content for the system was shown in Proposition 5.2 to be described by a degree $k$ differential form, this information must now reside in the western slot of the potential $A$; see Figure 4. To capture this precisely we firstly remove the gauge freedom (5.4) by requiring
\begin{equation}
X^\star A = 0.
\end{equation}
For generic weights, there always exists a suitable $B$ that achieves this, and the resulting $A$ is unique. Thus, for generic weights, the system $\mathcal{D}F = \mathcal{D}^\star F = \mathcal{I}^\star F = 0$ is equivalent to
\begin{equation}
yA = \mathcal{I}^\star A = \mathcal{D}^\star A = \mathcal{X}^\star A = 0,
\end{equation}
which we shall term the tractor Proca equations. Note that the first of these is a Laplace–Robin equation of the type solved for general tractors in \[31\] and Section 2.8. The latter three equations
\begin{equation}
\mathcal{I}^\star A = \mathcal{D}^\star A = \mathcal{X}^\star A = 0,
\end{equation}
are the scale-transversality conditions encountered in our study of boundary conditions for tractor forms in Section 4.3. They underly the transversality condition $\delta A = 0$. Further note that the conditions $\mathcal{D}^\star A = 0 = \mathcal{I}^\star A$ are exactly those of the western Lemma 3.13. This reflects Remark 3.21 which shows that a northern tractor $F$ can be written as the exterior tractor D-operator acting on a western tractor. Also, observe that $\mathcal{D}^\star$ is well defined at all weights acting on $A \in \ker \mathcal{X}^\star$.

We omit the proof of the above equivalence, because on the one hand the details are somewhat involved, and on the other we shall simply adopt (5.5) as the primary system
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of interest. As we shall see, it uniformly recovers the (higher form) Proca system

\[(\Delta - m^2)A = 0 = \delta A, \quad A \in \Gamma E^k M[w + k].\]

Thus the tractor Proca equations (5.5) will be the main focus of the latter part of this Article where we study solutions and obstructions. This is because they uniformly describe the Proca system (or its massless Maxwell limit), as we shall now chronicle.

**Proposition 5.4.** For \(A \in \Gamma T^k M[w]\) and \((M,c,\sigma)\) almost Einstein, away from \(\Sigma\) in the Einstein scale \(g^\sigma\), the tractor Proca equations \(yA = D^* A = \mathcal{I}^* A = 0\), precisely capture the Proca equations \(\delta dA - m^2 A = 0 = \delta A\) for \(A \in \Gamma E^k M[w + k]\) with a mass–Weyl weight relationship

\[
m^2 = -\frac{2J}{d} (w + k)(n + w - k).
\]

**Proof.** The proof is identical in method to that of Proposition 5.2 except that the western Lemma 3.13 is called upon instead of its northern counterpart. □

**Remark 5.5.** An important feature of the tractor Proca equations (5.5) is their conformal invariance in the interior. The two equations \(yA = 0 = \mathcal{I}^* A\) prescribe how the system couples to the defining scale; these arise from a canonical conformally invariant pairing of scale (mediated by the scale tractor) and derivatives of the fundamental fields. This is striking, since the Proca system, being massive, is traditionally viewed as a non-conformally invariant system.

Furthermore, the tractor system automatically includes massless and massive models: When \(w = -k\), Equation (5.8) gives \(m^2 = 0\) so \(\Delta A = 0 = \delta A\) which is the potential version of the higher form Maxwell system in a Feynman gauge choice.

**Remark 5.6.** Notice, that \(w = k - n\) also recovers the Maxwell system. This underlies an important duality of the tractor approach that we will utilize when studying solutions.

**Remark 5.7.** Corollary 3.19 of the northern Lemma 3.18 shows that, for generic weights, \(\mathcal{F} \in \Gamma T^{k+1} M[w - 1]\) subject to \(\mathcal{D} \mathcal{F} = 0 = \mathcal{D}^* \mathcal{F}\) can be written as \(\mathcal{F} = \mathcal{D}^* \mathcal{B}\) for some \(\mathcal{B} \in \Gamma T^{k+1} M[w + 1]\). Since \(\mathcal{I}^* \mathcal{F} = 0\), we obtain the model depicted at the southern compass point in Figure 4 with only a single equation of motion

\[
\mathcal{I}^* \mathcal{D}^* \mathcal{D} \mathcal{B} = 0.
\]

This model is strikingly similar to the model of \((p,q)\) form Kähler electromagnetism proposed in [13] where the Dolbeault operators \((\partial, \overline{\partial})\) play the rôle of \(\mathcal{D}, \mathcal{D}^*\) and the Kähler trace corresponds to \(\mathcal{I}^*\). Just as in that case, the model here has a pair of gauge invariances

\[
\mathcal{B} \sim \mathcal{B} + \mathcal{D} \mathcal{C} + \mathcal{D}^* \mathcal{C}',
\]

with \(\mathcal{C} \in \Gamma T^k M[w + 2]\) and \(\mathcal{C}' \in \Gamma T^{k+2} M[w + 2]\). The same Corollary also shows that we may generically fix this invariance by choosing \(\mathcal{D} \mathcal{B} = 0 = \mathcal{D}^* \mathcal{B}\). In that case the equations of motion are

\[
\mathcal{I}^* \mathcal{D}^* \mathcal{D} \mathcal{B} = \mathcal{D}^* \mathcal{B} = \mathcal{D} \mathcal{B} = 0.
\]

Then, in the Einstein scale \(g^\sigma\) and away from \(\Sigma\), one recovers the Proca system with mass–Weyl weight relationship (5.8).

**Remark 5.8.** To complete the classification of independent field content describing Proca systems according to east side of the diagram in Figure 4, one applies tractor Hodge duality.
Having surveyed formulations of the model we now turn to detailed solutions which are facilitated by the tractor Proca equations.

5.1. Solution generating operators for differential forms. We seek to solve the following problem on a Poincaré–Einstein structure \((M, c, \sigma)\).

**Problem 5.9.** Given \(A|_\Sigma = A_\Sigma \in \bar{\Gamma}^k \Sigma|w_0 + k\), \(w_0 \neq k - n\), and an arbitrary extension \(A_0 \in \bar{\Gamma}^k M|w_0 + k\) find \(A_\ell \in \bar{\Gamma}^k M|w_0 + k - 1\) such that

\[
A^{(\ell)} = A_0 + \sigma A_1 + \sigma^2 A_2 + \cdots + O(\sigma^{\ell+1})
\]

solves asymptotically the Proca system

\[
(\sigma^2 \delta^o d^o - (w_0 + k)(n + w_0 - k))A = 0 = \sigma \delta^o A,
\]

off \(\Sigma\), for \(\ell \in \mathbb{N} \cup \infty\) as high as possible.

Equation (5.9) is the higher form Proca system (5.7)

\[
(\delta d - (w_0 + k)(n + w_0 - k))A = 0 = \delta A,
\]

where \(\delta\) and \(d\) are given in the Einstein scale. Moreover recall that \(\delta^o\) and \(d^o\) are the usual interior and exterior differential operators on form densities determined by the Levi-Civita connection of the Poincaré–Einstein scale defined in (2.2). Although these recast \(\delta\) and \(d\) in terms of a scale that extends to the boundary, they are nevertheless singular along \(\Sigma\). However the operators \(\iota\) and \(\bar{\iota}\) of Definition 2.5 are well-defined everywhere and agree with \(-\sigma \delta^o\) and \(-\sigma d^o\) off \(\Sigma\). Therefore, Equations (5.9) extend to \(\Sigma\) as

\[
(\iota \bar{\iota} - (w_0 + k)(n + w_0 - k))A = 0 = \iota A.
\]

In Lemma 5.30 these are proven to be equivalent to the tractor Proca equations (5.5).

Evaluated along \(\Sigma\), these equations say

\[
(w_0 + k)(n + w_0 - k)[(\iota(n) \iota(n - 1) A) \big| \Sigma = 0 = (d + w_0 - k)(\iota(n) A) \big| \Sigma.
\]

This is consistent with the requirement that \(A_0\), and therefore \(A^{(\ell)}\), extends \(A_\Sigma \in \bar{\Gamma}^k \Sigma|w_0 + k\) to \(M\).

The normal derivatives of \(A\) along \(\Sigma\) are also determined; understanding the details is critical to linking Problem 5.9 to its equivalent tractor formulation in Problem 5.13 below.

**Proposition 5.10.** Solutions \(A \in \bar{\Gamma}^k M|w_0 + k\) to Problem 5.9 satisfy

\[
(\nabla_n [\iota(n) A]) \big| \Sigma = \frac{1}{n + w_0 - k} \delta_\Sigma A_\Sigma,
\]

and, when \(w_0 \neq 1 - \frac{d}{2}\),

\[
[ (\nabla_n - \iota(n)) \nabla_n \iota(n) - w_0 H^g) A ] \big| \Sigma = 0.
\]

**Proof.** In essence, equation (5.13) was already derived in Section 4.3; indeed at \(w_0 = k - d\) it follows by combining Equations (4.11) and (4.15). At all other weights \(w_0 \neq k - n\), by virtue of \(\iota A = 0\), we have

\[
(\nabla_n [\iota(n) A]) \big| \Sigma = \frac{1}{d + w_0 - k} (\delta A) \big| \Sigma = \frac{1}{n + w_0 - k} \delta_\Sigma A_\Sigma,
\]

where the last equality used Lemma 4.18.

To derive equation (5.14), we first use that \(\iota(n) A = -\sigma \phi\), for some \(\phi\), to rewrite \(\iota A = 0\) as

\[
\delta A = -(d + w_0 - k)\phi.
\]
Then using these facts as well as \((\nabla_n \sigma)\big|_{\Sigma} = 1\), we act on the first equation of \((5.9)\) with \(\nabla_n\) and evaluate the result along \(\Sigma\). Employing the operator identities
\[
\{\delta, \varepsilon(n)\} = \nabla_n - \sigma(\mathcal{P} - J) + \rho(\mathcal{N} - d),
\]
\[
\{\iota(n), d\} = \nabla_n - \sigma \mathcal{P} - \rho \mathcal{N},
\]
which are readily obtained from the parallel conditions \((1.2)\), plus \(\rho\big|_{\Sigma} = -H^g\), allows that equation to be written as
\[
(d + 2w_0 - 2)\left[\left(\nabla_n - w_0 H^g\right) A + \varepsilon(n) \phi\right] \big|_{\Sigma} = 0.
\]
In Section 4.3 we showed that \(\phi\big|_{\Sigma}\) was the southern slot of a boundary west tractor so that
\[
\phi\big|_{\Sigma} = -\frac{1}{n + w_0 - k} \delta_i A_e = -\frac{1}{n + w_0 - k} \left(\delta_i \varepsilon(n) \varepsilon(n) A\right) \big|_{\Sigma}.
\]
This allows us to compute the last term of Equation \((5.15)\) as follows
\[
\left(\varepsilon(n) \delta (n) \varepsilon(n) A\right) \big|_{\Sigma} = \left(\varepsilon(n) \left[\begin{array}{c} \delta - (\delta, \varepsilon(n)) \varepsilon(n) A \end{array}\right] \bigg|_{\Sigma} = \left(-\varepsilon(n) \left[(d + w_0 - k) \phi + \nabla_n \varepsilon(n) A\right]\right) \bigg|_{\Sigma}.
\]
The combination of the first operator identity above and Lemma 4.18 can be used to show
\[
\left[\varepsilon(n) \delta A_0\right] \big|_{\Sigma} = (d + w_0 - k) \left[\varepsilon(n) \nabla_n \varepsilon(n) A_0\right] \bigg|_{\Sigma}.
\]
In the first step we used \(\{\iota(n), \varepsilon(n)\} = 1 - 2\rho \sigma\) while for the second we used the first operator identity above as well as the formula given for \(\delta A\). Thereafter, elementary algebra gives the quoted result for Equation \((5.14)\).

**Remark 5.11.** By construction \(\nabla_n - \varepsilon(n) \nabla_n \iota(n) - w_0 H^g\) as an operator on forms in the kernel of \(\iota(n)\) is conformally invariant along \(\Sigma\) so gives a differential forms generalisation of the conformal Robin operator \([12, 8]\). Moreover, in the form given by Equation \((5.15)\) of the above proof, it can be written as
\[
(d + 2w_0 - 2)\delta_R A \bigg|_{\Sigma} = -yA \bigg|_{\Sigma} = 0,
\]
where \(A \in \ker \mathcal{H}^* \subset \Gamma^k \mathcal{M}[w_0]\) is a tractor given by the expression \((4.12)\) and \(\delta_R\) is the Robin operator on tractor forms given in Equation \((4.3)\). Note that the weight \(w_0 = 1 - \frac{d}{2}\) is the first in a series of exceptional weights that will be studied in detail.

**Remark 5.12.** In Equation \((5.10)\) the parameter \(m^2 = (w_0 + k)(n + w_0 - k)\). If we trivialise the density bundles with respect to the Einstein scale so that \(A\) is a differential form, then this interior system is unchanged under the replacement
\[
w_0 \mapsto -n - w_0.
\]
This weight duality leaves the parameter \(m^2\) invariant.

Although weight duality is seen to be a symmetry of the interior equations, it acts non-trivially on the boundary data. Shortly we will use this symmetry as a map between solutions with distinct boundary behaviours.

The case \(w_0 = k - n\) has been excluded from the statement of Problem 5.9 because the boundary problem is canonically of mixed Dirichlet-Neumann type, see Proposition 4.21. All weights are, however, handled uniformly in the tractor statement of our extension problem, to follow.
Problem 5.13. Given $A|\Sigma$ isomorphic to $A|\Sigma \in \ker(\hat{D}^*, \check{D}^*) \subset \Gamma T^k \Sigma M[w_0]$ and an arbitrary extension $A_0 \in \Gamma T^k M[w_0]$ of this subject to
\[ yA_0 = \mathcal{I}^* A_0 = \hat{D}^* A_0 = \mathcal{I}^* = O(\sigma), \]
find $A_i \in \Gamma T^k M[w_0 - i]$ such that
\[ A^{(\ell)} := A_0 + \sigma A_1 + \sigma^2 A_2 + \cdots + O(\sigma^{\ell+1}) \]
solves the tractor Proca equations
\[ yA = O(\sigma^{\ell}), \quad \mathcal{I}^* A = \hat{D}^* A = \mathcal{I}^* = 0, \]
off $\Sigma$, for $\ell \in \mathbb{N} \cup \infty$ as high as possible.

Regarding the equivalence of Problems 5.9 and 5.13, recall that Proposition 5.4 showed that the tractor Proca equations yield the Proca system uniformly for all weights in the interior. Moreover, the tractor Proca equations extend to the boundary and therefore determine boundary conditions naturally associated with the interior equations.

We are now well-positioned to solve these equations because the first Proca equation, $yA = 0$, is solved by the solution generating operator technique of [31] which is explained in Section 2. Moreover, the final three scale-transversality equations $\mathcal{I}^* A = \hat{D}^* A = \mathcal{I}^* = 0$, are solved by the holographic projector $\Pi$ of Proposition 4.25 for weights $w_0 \neq -k, k - n$. It remains to show compatibility of these results and handle the special weights. The latter encompass some of the most interesting features of the theory related to obstructions to smoothness, see Section 6.

A critical ingredient, especially for establishing the abovementioned compatibility, is the following result.

**Lemma 5.14.**
\[ [x, D^*[3]] = 0 = [y, D^*[3]], \]
\[ [x, D^*[3]] = 0 = [y, D^*[3]]. \]

**Hence**
\[ y \Pi = \Pi y. \]

**Proof.** As observed in Remark 4.23, the equalities on the left hand side hold, because the exterior and interior double D-operators obey the Leibnitz rule and their commutators with $x$ are proportional to $\mathcal{I}^* D^*$ and $D^* \mathcal{I}^*$, respectively.

The commutators on the right hand side only require a simple application of the algebra of Proposition 4.1. For example,
\[ y D^*[3] = -y \mathcal{I}^* \hat{D}^* D^* = \frac{h}{h-2} \mathcal{I}^* y \hat{D}^* D^* = -\frac{1}{h-2} \mathcal{I}^* y \mathcal{I}^* = -\mathcal{I}^* \hat{D}^* D^* y = D^*[3]y. \]

The apparent singularities above for exceptional weights are all removable.

**Remark 5.15.** The above Lemma is easily extended to demonstrate that the commutators of the interior and exterior triple D-operators with $x^\alpha$ (for any $\alpha \in C$) and the log density $\log x$ all vanish.

The above Lemma and pair of Remarks establish an all orders solution to the tractor Proca equations for generic weights.
Lemma (5.14)
Proof.

(5.18)
find a second homogeneous solution. We consider therefore a more general problem \( yA = 0 \)
from which it follows, if \( A_0 \) is of weight \( w_0 \), that
\[
\alpha = 0 \quad \text{or} \quad \alpha = h_0 - 1,
\]
in order that \( A_0 \) is \( O(\sigma^0) \). Since the case \( \alpha = 0 \) was solved above in Theorem 5.16 this
brings us to the following Problem.

Problem 5.17. Given \( A_0|_\Sigma \) isomorphic to \( A_0 \in \ker(\hat{\mathcal{R}}_\Sigma, \mathcal{R}_\Sigma^\ast) \subset \Gamma^k\Sigma[-w_0 - n] \) and an
arbitrary extension \( A_0 \in \Gamma^kM[-w_0 - n] \) of this subject to
\[
yA_0 = \mathcal{J}^\ast A_0 = \hat{\mathcal{J}}^\ast A_0 = \mathcal{R}^\ast A_0 = O(\sigma),
\]
find \( \tilde{A}_i \in \Gamma^kM[-w_0 - n - i] \) such that
\[
\tilde{A}^{(\ell)} := \sigma^h_{\alpha = 1}\left( A_0 + \sigma A_1 + \sigma^2 A_2 + \cdots + O(\sigma^{\ell+1}) \right) \in \Gamma^kM[w_0]
\]
solves the tractor Proca equations
\[
y\tilde{A} = O(\sigma^{\ell+h_0-1}), \quad \mathcal{J}^\ast \tilde{A} = \hat{\mathcal{J}}^\ast \tilde{A} = \mathcal{R}^\ast \tilde{A} = 0,
\]
off \( \Sigma_i \), for \( \ell \in \mathbb{N} \cup \infty \) as high as possible.

Problem 5.17 amounts to Problem 5.9 but instead with a solution of the form
\[
(5.18)
(\tilde{A}^{(\ell)} = \sigma^h_{\alpha = 1}\left( A_0 + \sigma A_1 + \sigma^2 A_2 + \cdots + O(\sigma^{\ell+1}) \right) ) .
\]
Solutions to this problem can be obtained from solutions to Problem 5.9 by what we
term the scale duality map, which is related to the weight symmetry alluded to above.
The scale duality map couples the defining scale \( \sigma \) with an existing solution to yield a
new solution of the same mass, as in the following result.

Theorem 5.18. \([\text{Scale Duality.}]\) Suppose \( A \in \Gamma^kM[-w_0 - n] \), \( w_0 \neq -k, k - n \), solves
\[
yA = O(\sigma^\ell), \quad \mathcal{J}^\ast A = \hat{\mathcal{J}}^\ast A = \mathcal{R}^\ast A = 0.
\]
Then
\[
\tilde{A} = \sigma^h_{\alpha = 1} \Pi A
\]
solves
\[
y\tilde{A} = O(\sigma^{\ell+h_0-1}), \quad \mathcal{J}^\ast \tilde{A} = \hat{\mathcal{J}}^\ast \tilde{A} = \mathcal{R}^\ast \tilde{A} = 0.
\]

Proof. Firstly from Equation (5.17), acting on \( \Pi A \), we have \([y, \sigma^h_{\alpha = 1}] = 0 \) and from
Lemma (5.14) \( y\Pi = \Pi y \) and \( y\Pi = \Pi x \) so
\[
y\tilde{A} = \sigma^h_{\alpha = 1} y\Pi A = \sigma^h_{\alpha = 1} y A = \sigma^h_{\alpha = 1} \Pi O(\sigma^\ell) = O(\sigma^{\ell+h_0-1}).
\]
Also we have \( \tilde{A} = \Pi \sigma^h_{\alpha = 1} A \), so automatically \( \mathcal{J}^\ast \tilde{A} = \hat{\mathcal{J}}^\ast \tilde{A} = \mathcal{R}^\ast \tilde{A} = 0. \)

Corollary 5.19. When \( w_0 \neq -k, k - n \), there is a bijection between solutions of the
weight \( -w_0 - n \) version of Problem 5.13 and solutions of Problem 5.17 given by
\[
A \mapsto \tilde{A} = \sigma^h_{\alpha = 1} \Pi A.
\]
Proof. It only remains to establish that the boundary conditions are correctly mapped from those of one Problem to the other. Observing that \( \mathcal{A}_0 = \Pi \mathcal{A}_0 \) (because \( \Pi x = x \Pi \)) the conditions \( \mathcal{J}^* \mathcal{A}_0 = \mathcal{D}^* \mathcal{A}_0 = \mathcal{K}^* \mathcal{A}_0 = O(\sigma) \) hold. Similarly \( y \mathcal{A}_0 = y \Pi \mathcal{A}_0 = \Pi y \mathcal{A}_0 = \Pi O(\sigma) = O(\sigma) \).

Remembering that solutions are defined up to \( O(\sigma^\ell) \), the inverse map is

\[
\mathcal{A} \mapsto \mathcal{A} = \sigma^{1-h_0} \Pi \mathcal{A},
\]

because

\[
(5.19) \quad \sigma^{1-h_0} \Pi \sigma^{h_0-1} \Pi \mathcal{A} = \Pi^2 \mathcal{A} = \left(1 + \frac{1}{(w+k)(w+k-n)} xy\right)^2 \mathcal{A} = \mathcal{A} + O(\sigma^{\ell+1}),
\]

where the second equality used Lemma 4.26. \( \square \)

Although not strictly needed for our subsequent discussion, it is worth noting that a strong global statement is available.

**Theorem 5.20.** [Global scale duality.] Given a global solution \( \mathcal{A} \) to the tractor Proca equations of weight \( w_0 \neq -k, -k-n \), then a solution of weight \(-w_0 - n \) is

\[
\mathcal{A} = \sigma^{1-h_0} \mathcal{A}.
\]

**Proof.** The equation \( y \mathcal{A} = 0 \) is again obvious from Equation (5.17). Moreover, by virtue of Lemma (4.26), \( \Pi \mathcal{A} = \mathcal{A} \) on global solutions. Thus \( \sigma^{1-h_0} \mathcal{A} = \sigma^{1-h_0} \Pi \mathcal{A} = \Pi \sigma^{1-h_0} \mathcal{A} \) so \( \mathcal{J}^* \mathcal{A} = \mathcal{D}^* \mathcal{A} = \mathcal{K}^* \mathcal{A} = 0 \). \( \square \)

**Remark 5.21.** Note the solutions \( \mathcal{A} \) and \( \mathcal{A} = \sigma^{1-h_0} \mathcal{A} \) in the Theorem have the same mass, in that they solve the Proca system (1.1) for the same fixed parameter value \( m^2 \), see Remark 5.12. It follows from the analysis in this work that on a Poincaré–Einstein manifold, for generic \( m^2 \) (and from the established theory for problems of this sort [51, 3]), one expects global solutions of the form

\[
\mathcal{A}_{\text{global}} = \mathcal{A} + \sigma^{h_0-1} \mathcal{A},
\]

where \( \mathcal{A}|_{\Sigma} \) may be viewed as the “Dirichlet data” and \( \mathcal{A}|_{\Sigma} \) the “Neumann data” of the solution \( \mathcal{A}_{\text{global}} \). Thus the scale duality map takes the weight \( w_0 \) solution \( \mathcal{A}_{\text{global}} \) of the Proca system (1.1) to the weight \(-w_0 - n \) solution

\[
\mathcal{A}'_{\text{global}} = \sigma^{1-h_0} \mathcal{A}_{\text{global}} = \mathcal{A} + \sigma^{1-h_0} \mathcal{A},
\]

and we see the that rôles of the Dirichlet and Neumann parts are swapped in the new solution (of the same interior Proca system (1.1)).

Note that away from the boundary we may work in the defining scale \( g^\sigma = \sigma^{-2} g \) and trivialise density bundles using \( \sigma \). Then, in this trivialisation, \( \sigma \) is represented by the constant function 1 and so in the interior the two solutions \( \mathcal{A}_{\text{global}} \) and \( \mathcal{A}'_{\text{global}} \) (which are sections of true form bundles) then appear indistinguishable. The solutions \( \mathcal{A}_{\text{global}} \) and \( \mathcal{A}'_{\text{global}} \) differ by their boundary behaviour, but this information is lost when we work on the interior in the “\( \sigma = 1 \)” scale.

### 5.1.2. Log solutions

To treat weights \( w_0 \) such that \( h_0 \in \mathbb{N} \) we need to draw log-type solutions into the picture. This is captured by the following Problem; see [31] for the definition of the log densities \( \log \sigma \) and \( \log \tau \).

**Problem 5.22.** Let \( h_0 = d + 2w_0 \in \mathbb{Z}_{\geq 2} \) and \( w_0 \neq -k, k-n \). Then, given \( \mathcal{A}|_{\Sigma} \) isomorphic to \( \mathcal{A}_{w_0} \in \ker(\mathcal{D}^*, \mathcal{K}^*) \subset \Gamma T^k \Sigma [w_0] \) and an extension \( \mathcal{A}_0 \in \Gamma T^k M[w_0] \) of this satisfying

\[
y \mathcal{A}_0 = \mathcal{J}^* \mathcal{A}_0 = \mathcal{D}^* \mathcal{A}_0 = \mathcal{K}^* \mathcal{A}_0 = O(\sigma),
\]
find $A_i \in \Gamma^T M[w_i - i]$ and $\overline{A}_i \in \Gamma^T M[-n - w_i - i]$ such that

$$\mathcal{A}^{(\ell)} := \left( A_0 + \sigma A_1 + \sigma^2 A_2 + \cdots \right) + \sigma_h^{-1} \left( \log \sigma - \log \tau \right) \left( \overline{A}_0 + \sigma \overline{A}_1 + \sigma^2 \overline{A}_2 + \cdots \right) + \mathcal{O}(\sigma^{\ell+1}) + \mathcal{O}(\sigma^{\ell+1} \log(\sigma/\tau))$$

solves the Proca equations

$$y A = \mathcal{O}(\sigma^0) + \mathcal{O}(\sigma^0 \log(\sigma/\tau)), \quad \mathcal{H}^* A = \mathcal{H}^* A = \mathcal{D}^* A = 0,$$

off $\Sigma$, for $\ell \in \mathbb{N} \cup \infty$ as high as possible.

When $h_0 = 1$, we set $A_0 = 0$ and take non-vanishing initial data $\overline{A}_0|_\Sigma$.

A key aspect of this Problem is solved in [31], which explains the set up and its special case $h_0 = 1$. Recall from there (see also Equation (2.24)) that the function $K^{h_0}(z)$ characterizing the solution generating operator $:K^{h_0}(z):$ can be obtained by solving the ordinary differential equation

$$z K''' - (h_0 - 2) K' + K = 0,$$

in a series expansion by the Frobenius method. However, for weights $h_0 = 2, 3, \ldots$ the power series solution breaks down. For $h_0 = 2, 4, \ldots$ the obstruction to a power series solution $[y^{h_0-1} A_0]|_\Sigma$ vanishes for almost Einstein structures [31]. When $h_0 = 3, 5, \ldots$, power series solutions are no longer possible, but solutions beyond the obstruction are obtainable by introducing a second, nowhere vanishing, scale $\tau \in \Gamma EM[1]$ and including terms $\sigma^k(\log \sigma - \log \tau)$ in the series expansion. Since $\tau$ is arbitrary, there is limited control over its algebra with differential operators $y, \mathcal{D}, \mathcal{D}^*$. In [31], this difficulty was circumvented by carefully ordering operators. The extension problem $y A = 0$ was solved via

$$A = \mathcal{O} A_0,$$

where the new solution generating operator $\mathcal{O}$ is given by

$$\mathcal{O} := :F_{h_0-2}(z): - \frac{z^{h_0} B(z)}{(h_0 - 1)!(h_0 - 2)!} - \frac{x^{h_0-1} \log x :K^{h_0}(z): y^{h_0-1} - x^{h_0-1} :K^{h_0}(z): (y^{h_0-1} \log \tau)_W}{(h_0 - 1)!(h_0 - 2)!}. \quad (5.20)$$

Explicit expressions for the order $h_0 - 2$ polynomial $F_{h_0-2}$ and power series $B(z)$ can be found in [31]. The notation $\left( \cdot \right)_W$ denotes the Weyl ordering $\frac{1}{2} \left\{ y^{h_0-1}, \log \tau \right\}$. In [31] it was shown that the above operator only depends on the log densities $\log \sigma$ and $\log \tau$ in the combination $\log \sigma - \log \tau = \log(\sigma/\tau)$ and $\sigma/\tau$ is a $C^\infty$ defining function for $\Sigma$. This implies that the operator $\mathcal{O}$ maps smooth sections of $\mathcal{T}^k M[w]$ to sections of $\mathcal{T}^k M[w]$ whose failure to be smooth is controlled by the term of order $(\sigma/\tau)^{h_0-1} \log(\sigma/\tau)$.

The key feature of $\mathcal{O}$ is that keeping terms up to order in $x^\ell$ and $x^\ell \log x$, and denoting this $\mathcal{O}^{(\ell)}$, one has the operator statement

$$y \mathcal{O}^{(\ell)} = \mathcal{O}(\sigma^\ell) + \mathcal{O}(\sigma^\ell \log \sigma).$$

(Note the log term is only present for $\ell \geq h_0 - 1$.) This machinery can now be combined with the tools developed above to handle log solutions for forms. We are first focussing on the cases $w_0 \neq -k, k - n$ so that a version of the holographic projector can still be employed.
Theorem 5.23. For weights $w_0 \neq -k, k - n$ and $h_0 = 2, 3, 4, \ldots$, Problem 5.22 has an $\ell = \infty$ solution given by

$$A = \frac{1}{(w+k)(\alpha+\omega-k)} \mathcal{D}_{[3]} \mathcal{O} \mathcal{D}_{[3]} A_0.$$  

Proof. To show $yA = O(\sigma^\ell) + O(\sigma^\ell \log \sigma)$, we employ Lemma 5.14 to write

$$y \mathcal{D}^*_{[3]} \mathcal{O}^\ell \mathcal{D}_{[3]} A_0 = \mathcal{D}^*_{[3]} y \mathcal{O}^\ell \mathcal{D}_{[3]} A_0 = \mathcal{D}^*_{[3]} (O(\sigma^\ell) + O(\sigma^\ell \log \sigma)) = O(\sigma^\ell) + O(\sigma^\ell \log \sigma).$$

By construction $A \in \ker(\mathcal{J}^*, \mathcal{D}^*, \mathcal{D}^*)$ so it remains only to verify that $A|_{\Sigma} = A_0|_{\Sigma}$. This is clear from Lemma 5.14 along with the form of $\mathcal{O}$ in Equation (5.20). □

Remark 5.24. As shown in [31], when $h_0 = 2, 4, \ldots$ and $(M, c, \sigma)$ is almost Einstein (as here), the coefficients of the log terms in $\mathcal{O}$ vanish. So the solutions are still of the type given by (5.20) without the terms displayed on the second line.

A solution of the form $A = \Pi \mathcal{O} A_0$ could also have been used in the above, but the expression given adapts easily to the exceptional case $w_0 = -k$ or $w_0 = k - n$. These solutions necessarily differ by some amount of a solution of the second kind which we now consider: The point here is that at the dual weight $-w_0 - n$, we have dual $h$-weight $1 - h_0$ which is necessarily negative. For those values, there is no obstruction to Dirichlet-type solutions. I.e., Problem 5.13 admits an $\ell = \infty$ solution at the dual weight. Hence Theorem 5.18 applies and generates an $\ell = \infty$ solution of the second kind.

It remains to discuss the case of weights $w_0$ such that $h_0 = 1$. This value of $h_0$ is invariant under the weight duality $w_0 \rightarrow -n - w_0$. As found in [31], there are two solutions. The one of the first kind has now as its leading behaviour $\log(\sigma/\tau)$. Thus, there is no interesting boundary operator appearing as an obstruction to smooth solutions. The solution of the second kind now has leading behaviour $\sigma^{h_0 - 1} = 1$ here, so is in fact Dirichlet. For the case of true forms $w_0 = -k$, (and their duals at $w_0 = k - n$) this weight corresponds to middle boundary forms of degree $-n/2$ for $n$ even.

5.1.3. True forms. We now treat the case where our Dirichlet boundary data is given by a true form $A_\Sigma \in \Omega^k \Sigma = \Gamma^\Sigma \Omega^k [0]$ so $k \in \{0, 1, \ldots, n\}$ which corresponds to a west tractor of weight $w_0 = -k$ and $h_0 \in \{n + 1, n, \ldots, 1 - n\}$. Thus the cases where the degree $k \in \{0, 1, \ldots, [\frac{d}{2}]\}$ potentially involve log terms. Since the boundary data is now given in terms of a true form, we modify our problem slightly.

Problem 5.25. Let $h_0 = d + 2w_0 \in \mathbb{Z}_{\geq 2}$ and $w_0 = -k$. Take $A_\Sigma := A|_{\Sigma} \in \Omega^k \Sigma$ and an extension $A_0 \in \Omega^k M$ of this. Find $A_i \in \Gamma^k M[w_0 - i]$ such that

$$A^{(\ell)} := (A_0 + \sigma A_1 + \sigma^2 A_2 + \cdots) + \sigma^{h_0 - 1}(\log \sigma - \log \tau)(\overline{A_0} + \sigma \overline{A_1} + \sigma^2 \overline{A_2} + \cdots) + O(\sigma^{\ell+1}) + O(\sigma^{\ell+1} \log(\sigma/\tau)),$$

where along $\Sigma$

$$A_0 = q_{\Sigma}(A_\Sigma),$$

and $A^{(\ell)}$ solves the tractor Proca equations

$$yA = O(\sigma^\ell) + O(\sigma^\ell \log(\sigma/\tau)), \quad \mathcal{J}^* A = \mathcal{D}^* A = \mathcal{D}^* A = 0,$$

off $\Sigma$, for $\ell \in \mathbb{N} \cup \infty$ as high as possible.
The main ingredients for solving this problem are again the solution generating operator $\mathcal{O}$ including log terms of above and the operator $\hat{\Pi}_r$ of Equation (4.22), the analog of the holographic boundary projector at these weights. Combining these gives our result.

**Theorem 5.26.** For weights $w_0 = -k \in \{0, -1, \ldots, \lfloor \frac{n-1}{2} \rfloor \}$, Problem 5.25 has an $\ell = \infty$ solution given by

$$\mathcal{A} = -\frac{1}{n-2k} \mathcal{D}_{[3]}^{\ast} \mathcal{I} \mathcal{I}^\ast q_{(N)}^\ast A_0.$$ 

**Proof.** Along $\Sigma$ we have

$$\mathcal{A} = \hat{\Pi}_r A_0,$$

so by Proposition 4.28, $\mathcal{A} |_{\Sigma} = q_{W}^\ast (A_\Sigma)$.  

Given the formula for $\mathcal{A}$, it only remains to check that $y \mathcal{A} = 0$. Using Proposition 5.14 twice we have

$$y \mathcal{D}_{[3]}^{\ast} \mathcal{I} \mathcal{I}^\ast q_{(N)}^\ast A_0 = \mathcal{D}_{[3]}^{\ast} y \mathcal{I} \mathcal{I}^\ast q_{(N)}^\ast A_0 = \mathcal{D}_{[3]}^{\ast} \left(O(\sigma^\ell) + O(\sigma^\ell \log(\sigma/\tau))\right) = O(\sigma^\ell) + O(\sigma^\ell \log(\sigma/\tau)).$$

This shows that $\mathcal{A}$ solves the tractor Proca equations to any given order. $\square$

For the remaining true form weights the argument simplifies considerably as there are no log terms, we may use Proposition 4.28, so the solution is simply $\mathcal{A} = :K^{h_0}(z):\hat{\Pi} A_0$.

An important feature of the log solution is the appearance of the second solution generating operator multiplied by $\lambda$, see Equation (5.20). In Section 5 we shall show that the operator $y^{h_0} A_0$ yields a holographic formula for the BG operators of $\mathcal{O}$. In particular, this means that $(y^{h_0} A_0) |_{\Sigma}$ consists of a pair of boundary forms in the range of the codifferential $\partial_{\Sigma}$ for any smooth western tractor $\mathcal{A}_0$.

### 5.1.4. Weight dual true forms

We now treat the case $w_0 = k - n$, in which case we take boundary data given by a pair of coclosed weighted forms $(A_\Sigma, \phi_\Sigma) \in \ker(\delta_{\Sigma}, \delta_{\Sigma}) \subset (\Gamma \mathcal{E}^k \Sigma[2k - n] \oplus \Gamma \mathcal{E}^{k-1} \Sigma[2k - n - 2])$. We focus on degrees $k \in \{0, 1, \ldots, \lfloor \frac{n-1}{2} \rfloor \}$, i.e., $h_0 = -n + 1, -n + 2, \ldots, 0$, to avoid the log solutions. The latter can be obtained by applying, in the obvious way, the weight/scale duality map to true form log solutions. On the other hand, the same map, applied to the solutions we derive in this Section, generates solutions of the second kind for the true form problem. The problem we solve is thus stated as follows.

**Problem 5.27.** Given $(A_\Sigma, \phi_\Sigma) \in \ker(\delta_{\Sigma}, \delta_{\Sigma}) \subset (\Gamma \mathcal{E}^k \Sigma[2k - n] \oplus \Gamma \mathcal{E}^{k-1} \Sigma[2k - n - 2])$, $k \in \{0, 1, \ldots, \lfloor \frac{n-1}{2} \rfloor \}$, consider an extension of these $A_0 \in \Gamma \mathcal{E}^k M[2k - n]$, satisfying the mixed Dirichlet–Neumann conditions

$$A_0 |_{\Sigma} = A_\Sigma \quad \text{and} \quad \left[\nabla_n (\iota(n) A_0)\right]_{\Sigma} = -\phi_\Sigma.$$ 

Let

$$A_0 := q_W (A_0) \in \Gamma T^k M[2k - n].$$

Find $A_i \in \Gamma T^k M[k - n - i]$ such that

$$\mathcal{A}^{(\ell)} := A_0 + \sigma A_1 + \sigma^2 A_2 + \cdots + O(\sigma^{\ell + 1})$$

solves the tractor Proca equations

$$y \mathcal{A} = O(\sigma^\ell), \quad \mathcal{I}^\ast \mathcal{A} = \mathcal{D}^\ast \mathcal{A} = \mathcal{D}^\ast \mathcal{A} = 0,$$

off $\Sigma$, for $\ell \in \mathbb{N} \cup \infty$ as high as possible.
Theorem 5.28. Problem \ref{5.27} has an \(\ell = \infty\) solution given by
\begin{equation}
A = :K^{h_0}(z):q_W A_0.
\end{equation}
where \(h_0 = 2k - n + 1\).

Proof. Firstly we calculate the southern slot of \(q_W A_0\) along \(\Sigma\) (see Lemma \ref{3.13})
\[-\delta A = -\delta(\alpha(n)\varepsilon(n)+\varepsilon(n)+2\rho\sigma) A_0 + \left(\delta\varepsilon(n)\sigma\phi\right)\Sigma = (\delta(n)\varepsilon(n)\phi)\Sigma = \phi_\Sigma.\]
This establishes that \((q_W A_0)_{\Sigma} = q_W^\Sigma(A_0; \phi_\Sigma)\). Thus we may employ Proposition \ref{4.29} from which we have \(A = :K^{h_0}(z):\mathcal{D}^*[\mathcal{I} A] = :K^{h_0}(z):\mathcal{I} B\) because \(K^{h_0}(z)\) annihilates terms \(O(\sigma)\). (Lemma \ref{5.14} was used for the last equality.) Thus, this expression is annihilated by \(y, \mathcal{I}^*, \mathcal{D}^*, \mathcal{D}^*\). Also by virtue of Proposition \ref{4.29} along \(\Sigma\) we have \(\mathcal{D}^*[\mathcal{I} A] = q_W^\Sigma(A_0; \phi_\Sigma)\). This agrees with \(q_W A_0\) thanks to Proposition \ref{4.21}. \(\square\)

One might wonder how general the simple form of the solution \ref{5.21} for \(w_0 = -k - n\) is. By construction, \(q_W A_0 \in \ker(\mathcal{D}^*[\mathcal{I} A])\) (when \(w_0 \neq k - d\)) so let us consider Equation \ref{5.21} at generic weights. In fact, for \(A_0\) an extension of \(A_\Sigma\), we have \(q_W A_0 + \sigma C \in \ker(\mathcal{D}^*[\mathcal{I} A])\) for some smooth tractor \(C\). Thus \(K^{h_0}(z):q_W A_0 = :K^{h_0}(z):\Pi q_W A_0\). So our generic solution \ref{5.16} amounts to \(A = :K^{h_0}(z):q_W A_0\). Hence, the solution to Problem \ref{5.9} is obtained by extracting the western slot of this expression
\[A = q^* :K^{h_0}(z):q_W A_0 \in \Gamma \mathcal{E}^k M[w_0 + k], \quad w_0 \neq -k, k - d, \quad h_0 \notin \mathbb{Z}_{\geq 1}.
\]

5.2. The product solution. In this Section, we show generically that there is a simple and explicit formula for solutions of the Proca system which can be stated without recourse to tractor formalism. Nevertheless, the tractor machinery plays the central rôle in obtaining this. These solutions use the product formula for the solution generating operator developed in Section 2.8 specialised to weighted tractor forms.

The main technical tool to translate between western tractors and weighted differential forms is as follows. Recall from the west Lemma \ref{3.13} that, for tractor weights \(w \neq k - d\), a solution \(A \in \Gamma \mathcal{T}^k M[w]\) to \(\mathcal{D}^* A = \mathcal{D}^* A = 0\) is isomorphic to a weighted differential form \(A \in \Gamma \mathcal{E}^k M[w + k]\) via
\[A = q_W(A).
\]

We now investigate how key operators between western tractors are intertwined by \(q_W\). Let us assume tractor weights \(w_0 \neq -k, k - n\) so that the scale-transversality equations
\[\mathcal{I}^* A = \mathcal{D}^* A = \mathcal{D}^* A
\]
are solved via
\[A = \Pi A_0, \quad A_0 \in \Gamma \mathcal{T}^k M[w_0].\]
Now we consider \(A_0 = q_W(A)\) so that
\[A = \Pi q_W(A),
\]
for some \(A \in \Gamma \mathcal{E}^k M[w + k]\). Putting together the result of Proposition \ref{2.11} and the computation given in Remark \ref{1.27} gives the following result.

Proposition 5.29. Suppose \(w \neq -k, k - d, k - n\) and \(A \in \Gamma \mathcal{E}^k M[w + k]\), then
\[\Pi q_W(A) = \frac{1}{(w + k)/(n + w - k)} q_W(\tilde{A} \tilde{\varepsilon} A).
\]
So the map
\[ A \mapsto \left( \frac{1}{(w+k)(n+w-k)} \right) \tilde{\varepsilon} A \]
solves the scale-transversality equations via the “intertwiner” \( q_W \).

It remains to solve the Laplace–Robin equation \( yA = 0 \). For that we use the following result capturing \( y \) in terms of an operator on weighted forms, which also demonstrates the equivalence of the tractor Proca equations (5.5) with the Proca system in Equations (5.9) and (5.11).

**Lemma 5.30.** Let \( A \in \Gamma E^k[w+k] \) with \( w \neq k - d \) and \( A = q_W(\tilde{\varepsilon} A) \). Then
\[ xyA = q_W \left[ (\tilde{\varepsilon} - (w+k)(n+w-k))\tilde{\varepsilon} A \right]. \]

**Proof.** Since \( \mathcal{F}^*A = 0 \), it follows that \( yA = -\mathcal{F}^*\mathcal{D} A \), so it is a simple matter of writing out \( \mathcal{F}^* \) and \( \mathcal{D} \) explicitly for some \( g \in c \) to verify the above. An alternate proof is to apply Lemma 4.26 to \( A = q_W(A) \) and then use Proposition 5.29 above. \( \square \)

Combining this result with the product solution of Proposition 2.21, we immediately have a product solution to the tractor Proca equations on weighted forms.

**Proposition 5.31.** For \( w_0 \neq -k, k - d, k - n \) Problem 5.13 has a solution to order \( \ell = \begin{cases} \infty, & h_0 \neq 2, 3, \ldots \\ h_0 - 2, & h_0 = 2, 3, \ldots \end{cases} \)
given by
\[ A^{(\ell)} = q_W(A^{(\ell)}), \]
with
\[ A^{(\ell)} = \tilde{\varepsilon} \left[ \prod_{j=1}^{\ell} \frac{\tilde{\varepsilon} - (w_0 + k - j)(n + w_0 - k - j)}{j(n + 2w_0 - j)} \right] A_0 . \]

Thus we have the following Theorem.

**Theorem 5.32.** The solution to the higher form Proca system of Problem 5.9 with the same weight and order conditions as in Proposition 5.31 is given by \( A^{(\ell)} \) as in (5.22).

**Proof.** We give an alternative proof of this Theorem that does not rely on tractors, and in particular without employing the intertwined version of the central relationship \([x, y] = \hbar \) of Proposition 2.16. Instead, the key ingredient in the commutativity of \( \sigma \) with the interior and exterior normals. This is as follows: Firstly we have an intertwined formula for the operators \( c_k \) of Equation 2.20
\[ c_j q_W(\tilde{\varepsilon} A_0) = q_W(\tilde{c}_j \tilde{\varepsilon} A_0), \]
where acting on \( \Gamma E^k M[w] \) we have
\[ \tilde{c}_j := \tilde{\varepsilon} - (w - j)(n + w - j - 2k). \]

Acting on \( \ker \tilde{\varepsilon} \), the operator \( \tilde{c}_0 \) is the intertwined version of \( xy \) and indeed, from Equation (5.12), \( \tilde{c}_0 \tilde{\varepsilon} A_0 = 0 \) along \( \Sigma \). Hence, on the kernel of \( \tilde{\varepsilon} \), the operator
\[ \tilde{\gamma} := \sigma^{-1} \tilde{c}_0 \]
extends smoothly to \( \Sigma \).

Now, since \( \sigma \) commutes with both \( \tilde{\varepsilon} \) and \( \tilde{\varepsilon} \) (see Corollary 2.7) and \( \sigma \) has weight 1, we immediately have the operator identity
\[ \sigma^\alpha \tilde{c}_j = \tilde{c}_{j+\alpha} \sigma^\alpha. \]
This is in fact valid everywhere in the interior of $M$ for any $\alpha \in \mathbb{C}$.

Since $\hat{\iota}^2 = 0$, the second part of Equation (5.9) holds trivially. To complete the proof, we need to establish Equation (2.21), namely

\[
\hat{y} \hat{c}_1 \hat{c}_2 \ldots \hat{c}_\ell = \sigma^{-1} \hat{c}_0 \hat{c}_1 \hat{c}_2 \ldots \hat{c}_\ell
\]

\[
\hat{c}_0 \hat{c}_1 \hat{c}_2 \ldots \hat{c}_\ell = \sigma \hat{y} \hat{c}_1 \ldots \hat{c}_{\ell-1} \hat{y}
\]

We have

\[
\hat{y} \hat{c}_1 \hat{c}_2 \ldots \hat{c}_\ell = \sigma^{-1} \hat{c}_0 \hat{c}_1 \hat{c}_2 \ldots \hat{c}_\ell
\]

\[
\sigma \hat{y} \hat{c}_1 \ldots \hat{c}_{\ell-1} \hat{y}
\]

\[
\hat{y} = \sigma \hat{c}_0 \hat{c}_1 \ldots \hat{c}_{\ell-1} \hat{y} = \sigma \hat{y} \hat{c}_1 \ldots \hat{c}_{\ell-1} \hat{y}
\]

Acting with $\hat{y}$ maps sections in $\ker \hat{\iota}$ to sections in $\ker \hat{\iota}$, so this identity extends to $\Sigma$. □

Let us end this Section with a rewriting of the product solution 5.22 which is useful because it makes direct contact with the Laplace operator acting on true forms. Firstly we need a pair of technical Lemmas.

**Lemma 5.33.** Let $P(z)$ be any polynomial and call

\[
\zeta := \hat{\iota} \hat{\varepsilon}
\]

and

\[
\mathcal{L} := \{\hat{\iota}, \hat{\varepsilon}\}.
\]

Then we have the operator identity

\[
\zeta P(\zeta) = \hat{\iota} P(\mathcal{L}) \hat{\varepsilon}.
\]

**Proof.** It suffices to prove the statement for a monomial $\zeta \zeta^\ell$ for some integer $\ell$. Recalling from Proposition 2.7 that $\hat{\iota}^2 = 0$ we have, in parallel to the usual exterior calculus of $\delta$ and $d$,

\[
\hat{\iota} \mathcal{L}^\ell \hat{\varepsilon} = \hat{\iota} (\hat{\iota} \hat{\varepsilon} + \hat{\varepsilon} \hat{\iota}) \mathcal{L}^{\ell-1} \hat{\varepsilon} = \cdots = \hat{\iota}(\hat{\varepsilon} \hat{\iota})^\ell \hat{\varepsilon} = \zeta \zeta^\ell.
\]

□

**Lemma 5.34.** Let $A \in \Gamma \mathcal{E}^k M[w]$. Then, away from $\Sigma$, we have the following results. First

\[
\hat{\iota} A = -\sigma^{w-d-k+1} \delta \sigma^{2k-w-d} A \quad \text{and} \quad \hat{\varepsilon} A = -\sigma w + 1 d \sigma^{-w} A.
\]

Moreover

\[
\mathcal{L} A := \{\hat{\iota}, \hat{\varepsilon}\} A = \sigma^w \hat{\mathcal{L}} \sigma^{-w} A,
\]

so that for a polynomial $P(z)$,

\[
P(\mathcal{L}) A = \sigma^w P(\hat{\mathcal{L}}) \sigma^{-w} A,
\]

where the operator $\hat{\mathcal{L}} : \Omega^k M \to \Omega^k M$ is given by

\[
\hat{\mathcal{L}} := \sigma^2 \Delta + (2k - d) \left[ \sigma \mathcal{L}_n + \hat{\varepsilon}(n) \hat{\iota}(n) \right] + 2\sigma \left[ \hat{\varepsilon}(n) \delta + \hat{\iota}(n) d \right].
\]

**Proof.** The first two identities follow immediately from first two relations in Equation (2.3). Equation (5.23) follows by a slightly more intricate application of these as well as Cartan’s magic formula for the Lie derivative

\[
\{d, \iota(n)\} = \mathcal{L}_n.
\]

□
Remark 5.35. In the formulæ above, the right hand sides are clearly not defined along \( \Sigma \), however, as is clear from Definition 2.5, the left hand sides are. When using expressions, such as those on the right hand sides, to express objects defined everywhere on \( M \) we will use the notation \( \lim \).

Applying these two Lemmas to the solution displayed in Equation (5.22) gives the following result.

Proposition 5.36. The solution to the higher form Proca system of Problem 5.9 with the same weight and order conditions as in Proposition 5.31, is given by

\[
A^{(\ell)} \equiv \sigma^{d+w_0-k} \delta \sigma^{2k-d+2} P^{(\ell)}(\widehat{L}) d \sigma^{-w_0-k} A_0, \tag{5.24}
\]

where the polynomial \( P^{(\ell)} \) is given by

\[
P^{(\ell)}(z) = \frac{1}{(w_0 + k)(n + w_0 - k)} \left[ \prod_{j=1}^{\ell} \frac{z - (w_0 + k - j)(n + w_0 - k - j)}{j(n + 2w_0 - j)} \right]
\]

and \( \widehat{L} \) is as displayed in Equation (5.23).

5.3. Solutions in Graham–Lee normal form. To make contact to a coordinate-based approach, let us explicate the solution generating operator in its product form in a choice of scale adapted to the Graham–Lee normal form for the interior metric \( g^o \). The aim is to present explicitly the operator of Lemma 2.19 that generates \( O(\sigma^{\ell+1}) \) solutions from \( O(\sigma^{\ell}) \) solutions.

Let \( \tau \in \mathcal{M}[1] \) be a scale that extends to the boundary which defines a metric \( g = \tau^{-2} g \in c \). Moreover, we take \( r = \sigma/\tau \) to be the function that gives \( \sigma \) in the scale \( \tau \).

In that case \( g^o = \frac{r^2}{\tau^2} \) where \( r \) is the defining function of \([34, 32]\). Working in terms of local coordinates in a collar neighbourhood \([0, \epsilon] \times \Sigma \) of the boundary with coordinate \( r \in [0, \epsilon] \), \( g \) extends to \( \Sigma \) with form

\[
g = dr^2 + h.
\]

Here \( h \) is a family of metrics on \( \Sigma \) parameterized by \( r \). In this choice of scale \( \varepsilon(n) = dr \wedge \) so differential forms \( A \in \Gamma \mathcal{E}^k \mathcal{M}[w] \) can be uniquely decomposed as

\[
A = A^\perp + dr \wedge A^\parallel, \quad (A^\perp, A^\parallel) \in \ker \iota(n).
\]

We will write this choice of splitting using a column vector notation

\[
A \equiv \begin{pmatrix} A^\perp \\ A^\parallel \end{pmatrix}.
\]

Tautologically then, in this splitting we have

\[
\iota(n) \equiv \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \varepsilon(n) \equiv \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.
\]

Defining

\[
\mathbf{d}^\perp := d - \varepsilon(n) \frac{\partial}{\partial r}, \quad \delta^\perp := \delta - \iota(n) \frac{\partial}{\partial r},
\]

the exterior derivative and codifferentials become

\[
\mathbf{d} \equiv \frac{\partial}{\partial r} \begin{pmatrix} \mathbf{d}^\perp & 0 \\ \partial_r & -\mathbf{d}^\perp \end{pmatrix}, \quad \delta \equiv \begin{pmatrix} \delta^\perp & \frac{\partial_r}{2(3-H)} \\ 0 & -\delta^\perp \end{pmatrix},
\]

where \( H \) is the scalar curvature.
and the form Laplacian is the anticommutator of these. Along \( \Sigma \), for \( A \in \ker i(n) \),
\[
(d^\perp A) \bigr|_{\Sigma} = d_{\Sigma} A_{\Sigma}, \quad (\delta^\perp A) \bigr|_{\Sigma} = \delta_{\Sigma} A_{\Sigma}.
\]
The operator \( \mathbb{H} \) is the natural endomorphism field on the subbundle of \( \ker i(n) \in \Gamma T^* M \) coming from \( h^{-1} \frac{\partial}{\partial r} \) in the local coordinates and then extended in the usual way to an endomorphism on forms in \( \ker i(n) \in \Omega^* M \). We have denoted \( \mathbb{H} := \text{tr} h^{-1} \frac{\partial}{\partial r} \). Note the relationship between bulk and boundary differentials and codifferentials
\[
(i(n) \varepsilon(n) dA) \bigr|_{\Sigma} = d_{\Sigma} A_{\Sigma}, \quad (\delta i(n) \varepsilon(n) A) \bigr|_{\Sigma} = \delta_{\Sigma} A_{\Sigma},
\]
for any extension \( A \) of \( A_{\Sigma} \in \Gamma \mathcal{E}^k \Sigma[w] \), is manifest in this splitting since
\[
i(n) \varepsilon(n) d \overset{\tau}{=} \left( \begin{array}{cc} d^\perp & 0 \\ 0 & 0 \end{array} \right), \quad \delta i(n) \varepsilon(n) \overset{\tau}{=} \left( \begin{array}{cc} \delta^\perp & 0 \\ 0 & 0 \end{array} \right).
\]
Moreover, the Lie derivative along \( n \) is simply
\[
\mathcal{L}_n = \{d, i(n)\} = \left( \begin{array}{cc} \frac{\partial}{\partial r} & 0 \\ 0 & \frac{\partial}{\partial \sigma} \end{array} \right).
\]
Inserting equations (5.26), (5.23) and the choice of scale \( \sigma = r \) in the product solution (5.24) of Proposition 5.36 gives an explicit formula for solutions in the Graham–Lee normal form for the interior metric. Moreover, we can combine these results with Lemma 2.19 to obtain the operator that increases the order of a solution.

**Theorem 5.37.** Let \( A^{(\ell)} \in \ker i \) solve Problem 5.3 to order \( \ell \) with \( w_0 \neq -k, k-d, k-n \) and \( \ell \neq d + 2w_0 - 2 \). Then an order \( \ell + 1 \) solution is given, in the scale \( \tau \) corresponding to the normal form (5.24), by
\[
A^{(\ell+1)} \overset{\lim}{=} \frac{LR - (w_0 + k - \ell - 1)(n + w_0 - k - \ell - 1)H}{(\ell + 1)(d + 2w_0 - 2 - \ell)} A^{(\ell)},
\]
where \( e := r \frac{\partial}{\partial \sigma} \),
\[
L := \begin{pmatrix} r^d & e - n - w_0 + k + \frac{\sigma}{2}(\mathbb{H} - 2\mathbb{H}) \\ 0 & -r^d \end{pmatrix}, \quad \text{and} \quad R := \begin{pmatrix} r \delta^\perp & 0 \\ e - w_0 - k & -r \delta^\perp \end{pmatrix}.
\]

**Proof.** First from \( A^{(\ell)} \) we construct the tractor \( \mathcal{A}^{(\ell)} = q_W(A^{(\ell)}) \). This obeys \( y_i \mathcal{A}^{(\ell)} = O(\sigma^\ell) \). Hence, by Lemma 2.19
\[
\mathcal{A}^{(\ell+1)} = \frac{1}{(\ell+1)(d + 2w_0 - 2 - \ell)} \left[ xy + (\ell + 1)(d + 2w_0 - 2 - \ell) \right] \mathcal{A}^{(\ell)}
\]
solves \( y_i \mathcal{A}^{(\ell+1)} = O(\sigma^{\ell+1}) \). Next we apply \( q^* \) to the above expression after re-expressing the operator \( xy \) in terms of \( i \bar{z} \) via Lemma 5.36 applied to \( A^{(\ell)} \) in place of \( i \bar{z} A \), which is legal because \( A^{(\ell)} \in \ker i \). Then \( q^* \mathcal{A}^{(\ell+1)} \propto (i \bar{z} - (w_0 + k - \ell - 1)(n + w_0 - k - \ell - 1)) \mathcal{A}^{(\ell)} \). This expression obeys the first equation of (5.9) by construction. Moreover, by virtue of the identity \( i^2 = 0 \), we have \( q^* \mathcal{A}^{(\ell+1)} \in \ker i \), so the second of those equations also holds. Thereafter we employ Lemma 5.34 to write the solution \( q^* \mathcal{A}^{(\ell+1)} \) compactly. The last step is to use Equation (5.26) for the exterior derivative and codifferential in the scale \( \sigma = r \).

**Remark 5.38.** The condition that the order \( \ell \) solution solves the transversality condition \( \iota A^{(\ell)} = 0 \) to all orders is not an essential restriction since this can always be achieved using the technology of Section 2.4.
6. Obstructions, Detours, Gauge Operators and $Q$-Curvature

Continuing in the Poincaré–Einstein setting, we now consider obstructions to smoothness of the solutions of Section 5. This is partly captured by the following result.

**Theorem 6.1.** Let $(M, c, \sigma)$ be Poincaré–Einstein. Then for any weight $w_0 \neq k - d$ the tractor Proca equations (5.5) have a smooth solution

$$A^{(\ell)} = :K^{(\ell)}(z): q_W(A_0) + O(\sigma^{\ell+1}),$$

to order

$$\ell = \left\{ \begin{array}{l} \infty, \quad h_0 \neq 3, 5, 7, \\ h_0 - 2, \quad h_0 = 3, 5, 7, \ldots \end{array} \right.$$ 

Here $A_0 \in \Gamma \Sigma[w_0 + k] = \text{an extension of } A_\Sigma \in \Gamma \Sigma[w_0 + k]$, and when $h_0 \neq 2, 4, 6, \ldots$ the solution generating operator $:K^{(\ell)}(z):$ is determined by $:K^{h_0}(z):$ as in (2.28). For the case $h_0 = 2, 4, 6, \ldots$, $:K^{(\ell)}(z):$ is determined by the solution generating operator in (5.20), omitting the log terms on the second line of that display.

**Remark 6.2.** Going beyond the space of smooth asymptotic solutions, for weights $h_0 = 3, 5, 7, \ldots$, the remainder in the expression (6.1), as computed in Section 5.1.2, is $O(\sigma^{\ell+1})$ log $\sigma$.

**Proof.** For weights $w_0 = k - n$, this is just a restatement of Theorem 5.28. For the other cases we use that the holographic triple D-operators $\mathcal{D}^*_{[3]}$ and $\mathcal{D}_{[3]}$ commute with the solution generating operator $:K^{(\ell)}(z):$. Moreover, when $w_0 \neq -k, k - n$ one has $\mathcal{D}^*_{[3]} \mathcal{D}_{[3]} A_0 = (w_0 + k)(n + w_0 - k) \Pi A_0$. The latter can be written as $q_W(A_0)$ for some $A_0$ (see Equation (4.19)). For $w_0 = -k$ we have $-\frac{1}{n-2k} \mathcal{D}^*_{[3]} \mathcal{D}_{[3]} q^\Sigma_{(N)} A_0 = q_W(A_0) + O(\sigma)$ which can be verified using the explicit expressions for $\mathcal{D}^*_{[3]}$ (see Equation (4.17)), $q^\Sigma_{(N)}$, $\mathcal{I}$ and $\mathcal{X}$ (see Equation (4.21) and Sections 3.1, 3.2). Thus the remaining results follow from Theorems 5.16, 5.23 and 5.26.

**Remark 6.3.** The restriction $w_0 \neq k - d$ in the above Theorem is an inessential one. For that weight, the map $q_W$ is not defined in the bulk. Instead the boundary data $A_\Sigma$ is mapped to a boundary tractor $q_W^\Sigma(A_\Sigma)$ which can be subsequently extended to a bulk tractor $A_0$. Thus $A^{(\ell)} = :K^{(\ell)}(z): \text{ext} \circ q_W^\Sigma(\Sigma) + O(\sigma^{\ell+1})$ and the remainder of this case and its proof follows mutatis mutandis.

Considering the cases $h_0 = 3, 5, 7, \ldots$, we could attempt to extend the above smooth solution to higher orders: Using that $\mathcal{D}^*_{[3]}$ commutes with the operators $x$ and $y$, it would suffice to solve the $y A = 0$ problem to solve all four tractor Proca equations. However, from 31, starting with any tractor boundary data $A_0$ and attempting to solve the corresponding $y A = 0$ extension problem at these weights, one encounters the obstruction $(y^{h_0-1} A_0)_{\Sigma}$; one succeed in obtaining a formal smooth solution if and only if this vanishes. In fact, again from 31, this is also the coefficient of the first log term $\sigma^{h_0-1} \log \sigma$ in solution generating operator (5.20). Specialising this result, we immediately have the following.

**Proposition 6.4.** For any weight $w_0 \neq k - d$ such that $h_0 = 3, 5, 7, \ldots$, consider the tractor Proca equations (5.5) with boundary data captured by $A_0 \in \Gamma \Sigma[w_0 + k]$ satisfying $(\iota(n) A_0)_{\Sigma} = 0$. The tractor field

$$y^{h_0-1} q_W(A_0),$$
is the coefficient of the first log term $\sigma^{\frac{n-1}{2}} \log \sigma$ in the solution generated by $\mathcal{O}$ in (5.20). The vanishing of this obstruction along $\Sigma$ is necessary and sufficient to obtain a smooth formal solution.

We now study the obstruction to smoothness given in expression (6.2).

6.1. Detour and gauge operators. From essentially the same argument as in the proof of Theorem 6.1 we see that the obstruction given in expression (6.2) is a west tractor along $\Sigma$ so consists of two pieces of weighted differential form data; the part residing in the west slot is conformally invariant. It can be extracted along $\Sigma$ using the operator $q^*$ of Section 3.2. To rewrite $q_W(A_0)$ in a more convenient format, we need the following result.

**Lemma 6.5.** Let $A_0 \in \Gamma \mathcal{E}^k M[w_0 + k]$ be an extension of $A_\Sigma \in \Gamma \mathcal{E}^k \Sigma[w_0 + k]$. Then, if $w_0 \neq k - d, k - n, -1 - \frac{n}{2}$, along $\Sigma$

$$q_W(A_0) = \frac{1}{(n + 2w_0 + 2)(n + w_0 - k)} \mathcal{D}^* \mathcal{D} q(A_0).$$

**Proof.** Since $A_0$ extends $A_\Sigma$ to $M$, it follows along $\Sigma$ that $\iota(n)A_0 = 0$ and (by continuity) $\iota(n)\delta A_0 = 0$. Thus, along $\Sigma$, we have $\mathcal{D}^* q_W(A_0) = 0$. Now, focussing on the case $w_0 \neq -k$, we can use Propositions 4.25 and 4.17 to write (along $\Sigma$)

$$q_W(A_0) = \Pi q_W(A_0) = \Pi_W^* (q_W(A_0)) |_{\Sigma} = \Pi_W^* q_W A_\Sigma = q_W^* A_\Sigma.$$

The third equality uses that $\Pi_W^*$ only sees the western slot of $(q_W(A_0)) |_{\Sigma}$. Noting the identity (again along $\Sigma$, using also Proposition 3.23)

$$q_W^* A_\Sigma = \frac{1}{(n + 2w_0 + 2)(n + w_0 - k)} \mathcal{D}^* \mathcal{D} q(A_0),$$

the result follows. For the case $w_0 = -k$, the proof is almost identical except that one replaces $\Pi_W q_W(A_0)$ with $\Pi q(A_0)$ and then relies upon Proposition 4.28 \qed

Writing $\ell = \frac{h_0 - 1}{2}$, this suggests the following definition.

**Definition 6.6.** We define $\mathcal{L}_k^\ell : \Gamma \mathcal{E}^k M[k + \ell - \frac{n}{2}] \rightarrow \Gamma \mathcal{T}^k M[-\ell - \frac{n}{2}]$, $\ell = 1, 2, 3, \ldots$ by the composition of tangential operators

$$\mathcal{L}_k^\ell := q^* \mathcal{L}_k^\ell.$$

Along $\Sigma$, we define

$$L_k^\ell := q^* \mathcal{L}_k^\ell.$$

As follows from the discussion above and as will be established in Theorem 6.13, the range of the map $\mathcal{L}_k^\ell$ lies in the range of $\mathcal{T}^k$ along $\Sigma$, hence the composition $L_k^\ell$ is well defined along $\Sigma$.

**Proposition 6.7.** The operator $L_k^\ell$ defines a conformally invariant differential operator

$$L_k^\ell : \Gamma \mathcal{E}^k \Sigma[k + \ell - \frac{n_1}{2}] \rightarrow \Gamma \mathcal{E}^k \Sigma[k - \ell - \frac{n_1}{2}],$$

and a necessary condition for extending $A_\Sigma \in \Gamma \mathcal{E}^k M[k + \ell - \frac{n_1}{2}]$ to a smooth solution to the tractor Proca equations is

$$L_k^\ell A_\Sigma = 0.$$

**Proof.** Tangentiality of $L_k^\ell$ has already been established and boundary forms canonically embed in bulk forms restricted to the boundary. The remainder of the Theorem is established by combining Lemma 6.5 and the expression Equation (6.2) for the obstruction of [8]. \qed
The operator $L_{\ell}^{\ell}$ is well-defined in the bulk, and in fact holographically extends the higher order conformally invariant operators on forms from [9].

**Proposition 6.8.** As an operator on boundary forms, the operator $L_{\ell}^{\ell}$ is the same as that defined in [9, Theorem 2.1].

**Proof.** First note that $y^{h_0-1}$ is the tractor form twisted GJMS operator as proved in [31]. On the other hand, $D^\star$ is a holographic formula for $D^\star \Sigma$ and so along $\Sigma$, Definition 6.6 coincides with the construction of $L_{\ell}^{\ell}$ given in [9]. □

Further to our observations, the importance of the $L_{\ell}^{\ell}$ is that generically they completely control the obstructions to smoothly solving the Proca problem.

**Theorem 6.9.** The Dirichlet Proca Problem 5.9 with boundary data $A_{\Sigma} \in \Gamma E_k \Sigma[w_0+k]$, for $w_0$ such that $h_0 = 3, 5, 7, \ldots$ and $w_0 \neq -k, k-n$, admits a formal smooth solution to all orders iff

$$L_{\ell}^{\ell} A_{\Sigma} = 0.$$ 

That is, for $w_0 \neq -k, k-n$, the space $\text{ker} L_{\ell}^{\ell}$ uniquely parameterises the smooth solution space of the Proca Problem 5.9 modulo the addition of second solutions.

**Proof.** Returning to display (6.2) we note that along $\Sigma$, $q\omega(A_0) = D_{[\Sigma]}^\star B$ for some $B$ and so since $y$ commutes with the holographic interior triple D-operator, $y^{h_0-1} q\omega(A_0)$ is a weight $-n-w_0$ boundary west tractor. Thus, provided $-n-w_0 \neq k-n$ (i.e., $w_0 \neq -k$), by the West Lemma 3.13 it is completely determined by its western slot.

Uniqueness of the parameterisation of the solution space follows by the iterative construction of the solution given in [31]. □

To understand fully smoothness, it remains now to study true forms.

### 6.2. True forms.

Here we focus on even boundary dimensions $n$ and weights $w_0 = -k \in \{0, -1, \ldots, 1 - \frac{n}{2}\}$ such that $h_0 = 3, 5, 7, \ldots$. We return to the full obstruction displayed in (6.2), which using Lemma 6.5 can be expressed as the restriction to $\Omega^k \Sigma$ of the tangential obstruction operator

$$y^{h_0-1} D^\star \omega q = L_k^{\frac{n}{2}-k} =: L_k.$$ 

Along $\Sigma$, the range of this operator consists of two parts; its western slot

$$q^\star y^{h_0-1} D^\star \omega q = L_k^{\frac{n}{2}-k} =: L_k,$$

dealt with above and its southern slot

$$q^\star a^\star y^{h_0-1} D^\star \omega q =: G_k.$$ 

(The operator $a^\star$ used here is defined in Equation (6.10) below.) We note that, in contrast to the other cases, $G_k$ is here not determined by $L_k$ since the obstruction operator displayed in (6.3) takes values in tractors of weight $k-n$ (see Lemma 3.13 applied to boundary tractors).

The operator $G_k$ depends on a second true scale $\tau \in \Gamma E M[1]$ (through $a^\star$) and for each such gives a canonical, tangential, linear differential operator, which upon restriction to $\Sigma$ is a map

$$G_k : \Omega^k \Sigma \longrightarrow \Gamma E^{k-1} \Sigma[2k-n-2].$$
As an operator on boundary forms, the operator $G_k$ of display (6.6) is the same as that defined in [9] Expression (3)]. Agreement follows immediately from the proof of Theorem 2.8 of [9].

By analogy with the proof of Theorem 6.9 we obtain the result characterising the obstruction to smooth solutions for true forms.

**Theorem 6.10.** In even boundary dimension $n$, the Dirichlet Proca Problem 5.9 with boundary data $A_x \in \Omega^k \Sigma$ with $k \in \{0, 1, \ldots, \frac{n}{2} - 1\}$ (so weights $w_0 = -k$ thus $h_0 \in \{3, 5, 7, \ldots\}$) admits a formal smooth solution to all orders iff

$$L_k A_x = 0 = G_k A_x.$$  

**Remark 6.11.** The differential operator $G_k$ is a gauge companion/fixing operator for the detour operator $L_k$, meaning that the pair $(L_k, G_k)$ is graded injectively elliptic and $H^0_k (\Sigma)$ is used to denote its kernel which is finite dimensional for compact boundary $\Sigma$ (see [2]).

Thus, we have an analog for true forms of the last part of Theorem 6.9, in that $H^0_k (\Sigma)$ uniquely parametrises, modulo second solutions, the formal smooth solutions to the gauge fixed higher form Maxwell system given by the formal Proca problem 5.9 with $w_0 = -k$.

For the global version of our problem, the second solutions are determined by the Dirichlet problem modulo the addition of solutions that vanish along $\Sigma$. More precisely, using results of [49], it was shown in [3] that the following sequence is exact

$$0 \longrightarrow H^k (M, \Sigma) \xrightarrow{i} K^k_\infty (M) \xrightarrow{r} H^k_\infty (\Sigma) \longrightarrow 0.$$  

Here, $H^k (M, \Sigma)$ is the relative de Rham cohomology of $M$ which was shown to be isomorphic to $\ker \Delta_{L^2}$ in [49]. The space $K^k_\infty (M) =: \ker \Delta$ is the space of smooth harmonic forms on $M$. Also, $i$ and $r$ denote, respectively, inclusion and restriction.

Displays (6.4) and (6.5) give holographic formulæ for the conformally invariant (detour operator, gauge companion) pair.

### 6.3. Fundamental holographic identities

A main aim of this Section is to show, from its holographic formula above, that there exists a factorisation of the boundary operator

$$L_k = \delta_x Q_{k+1} d_x : \Omega^k \Sigma \longrightarrow \Omega_k \Sigma,$$

where

$$\Omega_k \Sigma := \Gamma \varepsilon_k \Sigma [2k - n],$$

and the boundary $Q$-operator $Q_{k+1}$ of [9] Theorem 2.8 is the form analog of the Branson Q-curvature [9]. Thus we have a holographic proof of a theorem of [9].

**Theorem 6.12.** On $(\Sigma, c)$, for each $0 \leq k \leq n - 2$, there is a detour complex as follows

$$\Omega^0 \Sigma \xrightarrow{d_x} \ldots \xrightarrow{d_x} \Omega^k \Sigma \xrightarrow{L_k} \Omega_k \Sigma \xrightarrow{\delta_x} \Omega_{k-1} \Sigma \xrightarrow{\delta_x} \Omega_{k-2} \Sigma \xrightarrow{\delta_x} \ldots \xrightarrow{\delta_x} \Omega_0 \Sigma.$$  

This will require some holographic identities which hold in greater generality than strictly required for true forms. The first of these is the following.

**Theorem 6.13.** Let $A$ be a tractor $k$-form of weight $w_0$ and $F$ a tractor $k$-form of weight $w_0 - 1$, with $h_0 = 3, 5, 7, \ldots$. Then along $\Sigma$ the following identities hold.

$$\mathcal{D}^* y_{h_0 - 1} A = -(h_0 - 2)^2 y_{h_0 - 3} \mathcal{D}^* A, \quad \mathcal{D}^* y_{h_0 - 1} A = -(h_0 - 2)^2 y_{h_0 - 3} \mathcal{D}^* A,$$

and

$$y_{h_0 - 1} \mathcal{D}^* F = -(h_0 - 2)^2 \mathcal{D} y_{h_0 - 3} F, \quad y_{h_0 - 1} \mathcal{D}^* F = -(h_0 - 2)^2 \mathcal{D}^* y_{h_0 - 3} F.$$
Before proving this Theorem we establish a Lemma.

**Lemma 6.14.** Acting on $T^k M[w_0]$ for any $\ell \in \mathbb{Z}_{\geq 1}$ and weights $h_0 \neq 0, 2, 4, \ldots, 2\ell$, 
\[
\mathcal{X} y^\ell = \frac{1}{h + 2\ell - 2} \left[ (h - 2) y^\ell \mathcal{X} + \ell((\ell - 1)(h - 2) - 2 xy) y^{\ell-2} \frac{\partial}{\partial \mathcal{X}} \right] + \ell(h - 2) y^{\ell-1}.
\]

**Proof.** The proof is by induction. The base case is the identity on the fourth line of Proposition 4.1. Thereafter only that identity, the solution generating algebra 2.15 and the fact that $[y, \mathcal{D}] = 0$ from Proposition 4.5 is required to complete the induction. □

Armed with this Lemma we may prove Theorem 6.13.

**Proof.** The four identities are essentially equivalent. Since the algebra of $\mathcal{D}^*, \mathcal{X}^*$ and $\mathcal{I}^*$ satisfy the same identities as their unstarred counterparts, it suffices to prove one identity from each row as displayed. We will shortly prove the first of the four identities by simple application of known identities. To obtain the fourth identity from the first, observe that the operators above acting on $\mathcal{A}$ and $\mathcal{F}$ at quoted weights are all tangential. Moreover, so are their separate pieces $\mathcal{D}, \mathcal{D}^*, \mathcal{X}, \mathcal{X}^*$ and the given powers of $y$. Along the boundary the corresponding identities are in fact the formal adjoints of one another. Of course, one can also verify the fourth identity algebraically.

Turning to the first identity, note that from equation (4.7)
\[
\mathcal{X} y^2 = (h - 1)(h - 2)(\mathcal{D}^T - \mathcal{D} - \mathcal{I}) = (h - 2)(h\mathcal{D} - (h - 1)(\mathcal{D} + \mathcal{I} y))
\]
Therefore one writes $\mathcal{X} y^{h_0-1} = \mathcal{X} y^{h_0-3} y^2$ and firstly applies Lemma 6.14 at $\ell = h_0 - 3$.

The Theorem then follows by computing along $\Sigma$, using only $[\mathcal{D}, y] = 0$ and elementary algebra. □

Let us sketch how the factorization (6.7) of the long operator arises. Consider $\mathcal{F}$ satisfying the usual identities
\[
\mathcal{I}^* \mathcal{F} = 0 = \mathcal{D}^* \mathcal{F} = \mathcal{X}^* \mathcal{F},
\]
so it has entries in the west and southern slots, but otherwise is zero. Then $\mathcal{F} = \mathcal{X}^* \mathcal{A}$ for some $k + 1$ form tractor $\mathcal{A}$. Thus
\[
y^{h_0-1} \mathcal{F} = y^{h_0-3} \mathcal{X}^* \mathcal{A}.
\]
We assume that $h_0 = 3, 5, 7, \ldots$, and using the Theorem 6.13 we see that this takes the form
\[
\mathcal{D}^* y^{h_0-3} \mathcal{A},
\]
where we have dropped a non-zero overall constant. Now using Theorem 6.13 we see that the North slot of this vanishes (in fact it is annihilated by $\mathcal{X}^*$). Then it follows easily by inspecting the formula for $\mathcal{D}^*$ that, along $\Sigma$, the western slot takes the form $\delta \Sigma N$, for some $N$.

On the other hand the western slot of $y^{h_0-1} \mathcal{F}$ is moved to the southern slot by acting with $\mathcal{X}$. Thus we may study
\[
\mathcal{X} y^{h_0-1} \mathcal{F}.
\]
Using once again Theorem 6.13 and dropping the non-zero constant factor this becomes
\[
y^{h_0-3} \mathcal{D} \mathcal{F}.
\]
But now using that $\mathcal{F}$ has weight $-k$ and satisfies the system (6.9), it follows at once from the formula for $\mathcal{D}$ that this factors through $d_{\Sigma} F_{\Sigma}$ where $F_{\Sigma} = \left[ q^* \mathcal{F} \right]_{\Sigma}$. 
To follow this through carefully and show that the $Q$ operator arises as in (6.7) we need some preliminaries. Of course the operator $Q_{k+1}$ is not determined uniquely by the formula (6.7) for $L_k$. Thus, we use a special choice of true scale $\tau \in \Gamma E_+M[1]$ to pin down a preferred version.

Let us write
\begin{equation}
Y := \frac{1}{d - 2} D \log \tau \in \Gamma T^1 M[-1],
\end{equation}
where
\[(\delta_R \tau)|\Sigma = 0.\]
Note that this condition is easily solved given any $\tau|\Sigma$. Then
\[
\{X^*, Y\} = 1,
\]
and, along $\Sigma$,
\[
\{X^*, Y\} = 0.
\]
Now note the following Lemma for commutators of the tangential double $D$-operator (see Proposition 3.8 and Remark 4.15) with $Y$.

**Lemma 6.15.** Acting on $\Gamma T^k M[w]$ we have along $\Sigma$
\[
[\mathcal{X} \mathcal{D}^T, Y] = 0 = [\mathcal{X}^* \mathcal{D}^T, Y^*].
\]

**Proof.** The proof amounts to recalling that $\mathcal{X} \mathcal{D}^T = \mathcal{X} (\mathcal{D} - \mathcal{I} \{\mathcal{I}^*, \mathcal{D}\})$. Along $\Sigma$ the right hand side of this is $\mathcal{X} \mathcal{D} - \mathcal{I} \mathcal{I}^* \mathcal{X} \mathcal{D} + \mathcal{I} \mathcal{D} \mathcal{I} \mathcal{D}^*$. But the double $D$-operator $\mathcal{X} \mathcal{D}$ is Leibnizian so obviously commutes with $Y$. Moreover, $Y$ was chosen such that $\{\mathcal{I}^*, Y\} = 0$ along $\Sigma$. Note that $\mathcal{I} \mathcal{D} \mathcal{I} \mathcal{D}^* \{\mathcal{I}^*, Y\}$ vanishes along $\Sigma$ since an $x$ produced on the right by the anticommutator produces terms containing either a second $\mathcal{I}$ or $\mathcal{X}$ when pushed through to the left. $\square$

We shall also need the following which is obtained by an easy computation using the tangential double $D$-operator version of Theorem 4.14 given in Remark 4.15.

**Lemma 6.16.** Acting on $A \in \Omega^kM$ an extension of $A_{\Sigma} \in \Omega^k\Sigma$ we have
\[
[\mathcal{X} \mathcal{D}^T qA]_{\Sigma} = \mathcal{X}_{\Sigma} q_{\Sigma} d_{\Sigma} A_{\Sigma}.
\]

**Remark 6.17.** There is an obvious adjoint version of the above Lemma,
\[
q^* \mathcal{X}^* \mathcal{D}^*_{\Sigma} = -\delta_{\Sigma} q^* \mathcal{X}^*_{\Sigma}.
\]

We are now ready to give a holographic formula for the $Q$-operator in preparation for our main Theorem.

**Definition 6.18.** Let $n$ be even, fix a true scale $\tau \in \Gamma E_+M[1]$ and denote by $g \in c$ the corresponding metric (i.e., $g = \tau^{-2} g$). For each $k \in \{0, \ldots, \frac{n}{2}\}$. This determines the boundary differential $Q$-operator
\[
Q^b_{k} : \Omega^k \Sigma \longrightarrow \Omega_k \Sigma,
\]
determined by restriction of a bulk holographic formula, called the **holochronic $Q$-operator**
\[
Q^b_{k} := q^* Y^* X^* y^{n - 2k} X Y q,
\]
which acts on bulk true forms.

This enables a holographic proof of the following Theorem.
Theorem 6.19. On \((\Sigma, c_\Sigma)\) of even dimension \(n \geq 4\) and given any choice of \(g_\Sigma \in c_\Sigma\), the conformally invariant detour operators
\[ L_k : \Omega^k \Sigma \to \Omega_k \Sigma, \quad 0 \leq k \leq \frac{n}{2} - 1, \]
can be expressed as the composition
\[ L_k = \gamma_k \delta_x Q_{k+1}^\Sigma d_\Sigma, \]
where \(Q_{k+1}^\Sigma\) is the \(Q\)-operator from Definition 6.18 and
\[ \gamma_k = -(n - 2k)(n - 2k + 2)(n - 2k - 1)^2. \]

Proof. From Equation (6.4), \(L_k\) is the composition of operators
\[ L_k = q^* y^{h_0 - 1} \overrightarrow{\mathcal{D}}^* \mathcal{X} q, \]
evaluated along \(\Sigma\) with \(h_0 := d - 2k\).

This operator is tangential and throughout we will calculate along \(\Sigma\) without further comment. Now using that \(\{\mathcal{X}, \mathcal{Y}^*\} = 1\) and that the composition \(q^* \mathcal{X} \mathcal{Y}^* = 0\) (recall from Theorem 6.13 that \(y^{h_0 - 1} \overrightarrow{\mathcal{D}}^* \mathcal{X} q\) can be written as a composition of \(\mathcal{X}^*\) from the left) we come to
\[ L_k = q^* \mathcal{Y}^* \mathcal{X} y^{h_0 - 1} \overrightarrow{\mathcal{D}}^* \mathcal{X} q. \]
Now using Theorem 6.13 we have
\[ L_k = -(h_0 - 2)^2 q^* \mathcal{Y}^* y^{h_0 - 3} \overrightarrow{\mathcal{D}}^* \mathcal{X} q. \]
Since along \(\Sigma\) we have \(\overrightarrow{\mathcal{D}}^* \mathcal{Y} = -\overrightarrow{\mathcal{D}} \mathcal{Y}\) (because \(\{\mathcal{X}, \mathcal{Y}^*\} = 0\) and \(y^{h_0 - 3}\) acts tangentially here), thus
\[ L_k = -(h_0 - 2)^2 (h_0 + 1) q^* \mathcal{Y}^* y^{h_0 - 3} \overrightarrow{\mathcal{D}} \mathcal{X} q. \]
In the last step we used that along \(\Sigma\) one has
\[ \overrightarrow{\mathcal{D}} \mathcal{X} = \mathcal{X} \mathcal{Y} \mathcal{X} = -\frac{h_0 + 2}{h_0 - 2} \mathcal{X} \mathcal{Y} = -\frac{h + 1}{h - 3} \overrightarrow{\mathcal{D}} = -(h + 1) \overrightarrow{\mathcal{D}}^T. \]
Note that the singularity at \(h_0 = 3\) in the fourth expression above is a removable one as evidenced by the last equality.

We may now employ Lemma 6.16 to generate a boundary exterior derivative on the right (along \(\Sigma, \mathcal{X}\) and \(\mathcal{Y}\) agree):
\[ L_k = -(h_0 - 2)^2 (h_0 + 1) q^* \mathcal{Y}^* y^{h_0 - 3} \overrightarrow{\mathcal{D}} \mathcal{X} q d_\Sigma. \]
Using the second relation of Theorem 6.13 we then have
\[ L_k = (h_0 + 1) q^* \mathcal{Y}^* y^{h_0 - 1} \mathcal{X} q d_\Sigma. \]
Now we use that \(1 = \{\mathcal{X}^*, \mathcal{Y}\}\) and \(\mathcal{X} \mathcal{Y} \mathcal{X} q = 0\) to obtain
\[ L_k = -(h_0 + 1) q^* \mathcal{Y} \mathcal{X}^* y^{h_0 - 1} \mathcal{X} \mathcal{Y} q d_\Sigma. \]
Now the fourth relation in Theorem 6.13 gives
\[ L_k = (h_0 - 2)^2 (h_0 + 1) q^* \mathcal{Y} \mathcal{X}^* \overrightarrow{\mathcal{D}} y^{h_0 - 3} \mathcal{Y} \mathcal{Y} q d_\Sigma. \]
Now we use the second relation in Lemma 6.15 to move \(\mathcal{Y}^*\) to the right (being careful to replace \(\overrightarrow{\mathcal{D}}^* = (h - 1) \overrightarrow{\mathcal{D}}^T\))
\[ L_k = -(h_0 - 2)^2 (h_0 + 1)(h_0 - 1) q^* \mathcal{X} \overrightarrow{\mathcal{D}}^T \mathcal{Y} y^{h_0 - 3} \mathcal{Y} \mathcal{Y} q d_\Sigma. \]
The final step is to employ the adjoint version of Lemma 6.16 to extract a boundary codifferential on the left.

\[ L_k = -(h_0 - 2)^2(h_0 + 1)(h_0 - 1) \delta_x q^* \mathcal{R}^* g^{h_0 - 3} \mathcal{R}^g q \, d_x . \]

\[ \square \]

**Remark 6.20.** The way the above holographic proof employs the fundamental Theorem 6.13 mirrors the original proof of [9] Theorem 2.8 based on the ambient Fefferman–Graham metric. Similarly, consideration of \( Q_k, d_x \), and holographic arguments parallel to those in the proof above, recovers the conformal transformation formula of [9] Equation (1) for \( Q_k \) as an operator on closed \( k \)-forms. On the other hand, using again parallel arguments to above shows that \( \delta_x Q_k \) is proportional to \( G_k \) and recovers the conformal transformation law for \( G_k \) found in [9].

**Remark 6.21.** We observed in Remark 6.11 that \( \mathcal{H}_k^\Sigma \) parametrises the smooth, Dirichlet solution space to the gauge fixed Maxwell system (which amounts in the preferred interior scale to equations \( \Delta A = 0 = \delta A \)) up to the addition of second solutions. As a special case of this problem, one can begin with closed Dirichlet boundary data \( A_x \). It follows that solutions are then parametrised by \( A_x \) such that

\[ A_x \in \ker (d_x, G_k) =: \mathcal{H}_k^\Sigma . \]

This is exactly the conformal harmonic space for \( \Sigma \) defined in [9]. It is interesting to understand what this special boundary data means in the bulk. This is answered in [3, Theorem 1.3]: Essentially, this is captured by the space \( \ker (d, \delta) =: Z^k(M) \), more precisely it is there proven that there is an exact sequence

\[ 0 \rightarrow H^k(M, \Sigma) \rightarrow Z^k(M) \rightarrow H^k(\Sigma) \rightarrow H^{k+1}(M, \Sigma) , \]

where \( k < n \) and \( H^k(M, \Sigma) \equiv \ker_{L^2} \Delta \) is the relative cohomology of \( M \).

**APPENDIX A. THE AMBIENT MANIFOLD**

A detailed analysis of tractor forms using an ambient manifold approach has been given in [9]. We first sketch the ingredients needed to provide expedient proofs of Lemma 4.12 and Theorem 4.14.

A conformal structure is equivalent to the ray bundle \( \pi : \mathcal{G} \rightarrow M \) of conformally related metrics. Let us use \( \rho \) to denote the \( \mathbb{R}_+ \) action on \( \mathcal{G} \) given by \( \rho(t)(x, g_x) = (x, t^2 g_x) \). An ambient manifold is a smooth \( (d+2) \)-manifold \( \tilde{M} \) endowed with a free \( \mathbb{R}_+ \)-action \( \rho \) and an \( \mathbb{R}_+ \)-equivariant embedding \( i : \mathcal{G} \rightarrow \tilde{M} \). We write \( X \in \Gamma(T\tilde{M}) \) for the fundamental field generating the \( \mathbb{R}_+ \)-action. That is, for \( f \in C^\infty(\tilde{M}) \) and \( u \in \tilde{M} \), we have \( X f (u) = (d/ds) f (\rho (e^s) u) |_{s=0} \). For an ambient manifold \( \tilde{M} \), an ambient metric is a pseudo–Riemannian metric \( h \) of signature \( (d+1,1) \) on \( \tilde{M} \) satisfying the conditions: (i) \( L_X h = 2h \), where \( L_X \) denotes the Lie derivative by \( X \); (ii) for \( u = (x, g_x) \in \mathcal{G} \) and \( \xi, \eta \in T_u \mathcal{G} \), we have \( h(i_u \xi, i_u \eta) = g_x(\pi_x \xi, \pi_x \eta) \). In [18, 19] Fefferman and Graham considered formally the Gursat problem of obtaining \( \text{Ric}(h) = 0 \). They proved that, for the case of \( d = 2 \) and \( d \geq 3 \) odd, this may be achieved to all orders, while for \( d \geq 4 \) even, the problem is obstructed at finite order by a natural 2-tensor conformal invariant (this is the Bach tensor if \( d = 4 \), and is called the Fefferman–Graham obstruction tensor in higher even dimensions); for \( d \) even one may obtain \( \text{Ric}(h) = 0 \) up to the addition of terms vanishing to order \( d/2 - 1 \). See [19] for statements concerning uniqueness. For extracting results via tractors we do not need this, as discussed in e.g. [11, 28]. (In
fact, to obtain a correspondence with the normal tractor connection which is all that we require below, it suffices that the tangential components of Ricci vanish along $\mathcal{G}$.) We shall henceforth call any (approximately or otherwise) Ricci-flat metric on $\tilde{M}$ a Fefferman–Graham metric. In the subsequent discussion of ambient metrics all results can be assumed to hold formally to all orders.

In the following discussion we use bold symbols or tilded symbols for the objects on $\tilde{M}$. For example $\nabla$ denotes the Levi-Civita connection on $\tilde{M}$. Familiarity with the treatment of the Fefferman–Graham metric, as in e.g. [11,29] or [9], will be assumed. In particular, we shall use that suitably homogeneous tensor fields of $\tilde{M} |\mathcal{G}$ correspond to tractor fields. This correspondence is compatible with the Levi-Civita connection in that each weight zero tractor field $F$ on $\tilde{M}$ is identified with (the restriction to $\mathcal{G}$ of) a homogenous tensor field $\tilde{F}$ on $\tilde{M}$ with the property that it is parallel in the vertical direction, that is $\nabla_X F =: X^A \nabla_A F = 0$ along $\mathcal{G}$. The metric $h$ and its Levi-Civita connection $\nabla$ on $\tilde{M}$ determine a metric and connection on tractor bundles, and by [11, Theorem 2.5] this agrees with the normal tractor metric and connection. We use abstract indices in an obvious way on $M$ and these are lowered and raised using $h_{AB}$ and its inverse $h^{AB}$.

We shall say $F$ is homogeneous of weight $w_0$ if $\nabla_X F = w_0 F$, and this corresponds to a tractor field $F$ of weight $w_0$. We shall always take such fields to be extended off $\mathcal{G}$ smoothly and also such that $\nabla_X F = w_0 F$ on $\tilde{M}$. The operator $\nabla_X$ gives an ambient realisation of the weight operator, as applied to tensor fields of well defined weight along $\mathcal{G}$.

In this picture the operator $D^A = (d + 2w_0 - 2) \nabla^A - X^A \nabla^2$ on tensors homogeneous of weight $w_0$ corresponds to the tractor D-operator as applied to tractors of weight $w_0$. Thus we equivalently view this as a restriction of

$$D^A = \nabla^A (d + 2 \nabla_X - 2) + X^A \nabla^2.$$ 

Here $\nabla^2$ is the ambient Bochner Laplacian. The above operator acts tangentially along the submanifold $\mathcal{G}$ in $\tilde{M}$, [9] and [28].

This technology enables simple computations on $\tilde{M}$ of often complicated tractor quantities on $M$. For example the proof of Lemma 3.1 is:

$$h X^A D^B - (h - 2) D^B X^A - 2 X^B D^A + h(h - 2) h^{AB} = h X^A (h \nabla^B - X^B \nabla^2) - (h - 2) (h \nabla^B - X^B \nabla^2) X^A - 2 X^B (h \nabla^A - X^A \nabla^2) + h(h - 2) h^{AB} = h(h - 2) X^A \nabla^B - h X^A X^B \nabla^2 - h(h - 2) (X^A \nabla^B + h^{AB}) + (h - 2) X^B (X^A \nabla^2 + 2 \nabla^A) - 2(h - 2) X^B \nabla^A + 2 X^A X^B \nabla^2 + h(h - 2) h^{AB} = 0.$$

We will employ the same notations as used on $M$ for the natural operators on sections of $\Lambda^* \tilde{M}$. Namely $d$ for the ambient exterior derivative, $\delta$ the ambient codifferential and $\Delta = \{d, \delta\}$ for the ambient form Laplacian. No confusion should arise from this recycling of notation. As shown in [9], the ambient operator corresponding to the exterior Thomas D-operator is $\mathcal{D} = (d + 2w_0 - 2) d - \varepsilon(X) \Delta$ acting on ambient differential forms homogeneous of weight $w_0$. This, and its interior analog can both be equivalently viewed.
as restrictions of the operators
\[ D = h d - \varepsilon(X) \Delta, \quad D^\ast = h \delta - \iota(X) \Delta, \]
where
\[ h := d + 2\nabla_X. \]

Given a defining scale \( \sigma \in \Gamma \mathcal{E} M^1 \) on \( M \), we shall write \( \tilde{\sigma} \) for the corresponding homogeneous weight 1 function on \( \mathcal{G} \) with some homogeneous extension to \( \tilde{M} \). Then we introduce
\[ I^A := \frac{1}{d} D^A \tilde{\sigma} \]
and, again, we can define the interior and exterior operators
\[ I := \varepsilon(I), \quad I^\ast := \iota(I), \]
as well as
\[ \chi := \varepsilon(X), \quad \chi^\ast := \iota(X). \]

We now restrict our attention to the ambient analog of a Poincaré–Einstein structure with defining scale \( \sigma \). For simplicity, we begin by taking the dimension \( d \) odd and so may assume \([20]\) that \( I = \nabla \tilde{\sigma} \) is parallel with respect to the ambient Levi–Civita connection and has unit length \( I^2 = 1 \). Then a restriction of the differential operator
\[ -y := \{ I^\ast, D \} = \{ I, D^\ast \} = h \nabla_I - \tilde{\sigma} \Delta \]
(on \( \tilde{M} \)) lifts the operator \( I_A \partial^A \) on \( M \), enabling calculations on \( \tilde{M} \). Note now that the combination \( I^\ast I \) projects onto boundary objects, thus we call
\[ d^T := I^\ast I d \quad \text{and} \quad \delta^T := \delta I^\ast I. \]

We are ready now to prove Theorem 4.14 along \( \Sigma \) and acting on \( \ker I^\ast \) we find
\[
D^T = D + I y + \frac{1}{(h - 1)(h - 2)} \chi y^2
\begin{align*}
&= h (d^T + I I^\ast d) - \mathcal{X} (\delta(I I^\ast + I^\ast I) d + (I I^\ast + I^\ast I) d \delta I^\ast I) \\
&- \delta h \nabla_I + \frac{1}{(h - 1)(h - 2)} \chi h \nabla_I (h \nabla_I - \tilde{\sigma} \Delta) \\
&= h d^T - \delta \frac{h}{h - 1} \mathcal{X} (\Delta^T + \delta I \nabla_I + I \nabla_I \delta - \nabla_I^2) \\
&= \frac{h}{h - 1} (h - 1) d^T - \mathcal{X} \Delta^T = \frac{h}{h - 1} D^\Sigma.
\end{align*}
\]

In the second line we inserted \( 1 = \{ I, I^\ast \} \) in order to produce the operators \( d^T, \delta^T \) and \( \Delta^T := \{ d^T, \delta^T \} \) appearing in the third line. Also, to obtain the third line, we used, for example, \( \{ I^\ast, d \} = \nabla_I \). To obtain fourth line, we used \( \{ \delta, I \} = \nabla_I \) as well as \( [\nabla_I, \delta] = 0 \) to cancel all terms with a \( \nabla_I \) in the third line. In the last step we evaluated our result along the boundary using that \( d^T |_\Sigma = d_\Sigma \) and \( \Delta^T |_\Sigma = \Delta_\Sigma \) there.

Note that the special case of Paneitz weight where \( D^T \) acts on ambient forms of homogeneity \( w_0 = 2 - \frac{d}{2} \) is not covered by the above computation. However, it is easy to see how that case is proven from the above display: Focussing on the last term on the third line, we see that the potentially singular factor \( h - 2 \) in the denominator cancels because \( \chi h = (h - 2) \chi \). The definition of \( D^T \) exactly removes the last singular term on that line.
Finally, the last special case of boundary Yamabe weight can be handled by multiplying both sides of the above computation from the left by \( h - 1 \) and then specializing to homogeneity \( w_0 = 1 - \frac{n}{2} \).

\[ \square \]

The proof for the analogous formula for the Thomas D-operator acting on arbitrary tractors required for Proposition 4.12 is very similar. Using \( I \cdot D = h \nabla_I - \tilde{\sigma} \nabla^2 \) and

\[ \nabla^T_A := \nabla_A - I_A \nabla_I, \quad (\nabla^T)^2 = \nabla^2 - \nabla^2_I, \]

we have, calculating along \( \Sigma \)

\[
D^T_A = D_A - I_A I \cdot D + \frac{1}{(h - 1)(h - 2)} X_A (I \cdot D)^2 \\
= h(\nabla^T_A + I_A \nabla_I) - X_A \nabla^2 - I_A h \nabla_I + \frac{1}{(h - 1)(h - 2)} X_A h \nabla_I (h \nabla_I - \tilde{\sigma} \nabla^2) \\
= h\nabla^T_A - \frac{h}{h - 1} X_A (\nabla^T)^2 = \frac{h}{h - 1} D^\Sigma_A. \quad \square
\]

For \( d \geq 4 \) even, the harmonic extension problem for \( \tilde{\sigma} \) is potentially obstructed. However, we may still obtain that \( 0 = \Delta \tilde{\sigma} = \Delta^2 \tilde{\sigma} = \cdots = \Delta^{d/2} \tilde{\sigma} \) along \( G \) (see [33, 26]). This suffices for the above proofs of Theorem 4.14 and Proposition 4.12 to apply also in this dimension parity.


**APPENDIX B. LIST OF COMMON SYMBOLS**

- $d := n + 1$
- $\Lambda^* M$: Exterior bundle on $M$
- $\Omega^* M$: Sections of $\Lambda^* M$
- $d$: Exterior derivative
- $\delta$: Codifferential
- $\ast$: Hodge star
- $\Delta$: Form Laplacian
- $N$: Form degree
- $P$: Schouten endomorphism
- $\varepsilon$: Exterior multiplication
- $\iota$: Interior product
- $\tilde{\varepsilon}$: Holographic exterior normal
- $\tilde{\iota}$: Holographic interior normal
- $c$: Conformal structure
- $g$: Conformal metric
- $E M[.]$: Conformal density bundle on $M$
- $E\ast M[.]$: Exterior density bundle on $M$
- $\Gamma B$: Sections of bundle $B$
- $TM$: Standard tractor bundle on $M$
- $T^\phi M[.]$: Tractor bundle
- $\text{dim } M := \text{dim } \partial M + 1$
- $\Lambda^* M$: Exterior bundle on $M$
- $\Omega^* M$: Sections of $\Lambda^* M$
- $d$: Exterior derivative
- $\delta$: Codifferential
- $\ast$: Hodge star
- $\Delta$: Form Laplacian

**Tractor symbols**

- $h(\cdot, \cdot), h_{AB}$: Tractor metric
- $w$: Tractor weight operator
- $d + 2w$: Exterior tractor bundle on $M$
- $T^A M$: Thomas D-operator
- $D^A$: Tensorial endomorphism
- $\mathcal{D}, \varepsilon(\mathcal{D})$: Exterior tractor D-operator
- $\mathcal{D}^*, \iota(\mathcal{D})$: Interior tractor D-operator
- $\ast$: Tractor Hodge star
- $X^A$: Canonical tractor
- $\mathcal{X}, \varepsilon(X)$: Exterior canonical tractor
- $\mathcal{X}^*, \iota(X)$: Interior canonical tractor
- $N^N, q_E, q_S, q_W$: Tractor form degree
- $\mathcal{N}$: Insertion operators
- $\mathcal{I}^A$: Scale tractor
- $\mathcal{I}, \varepsilon(I)$: Exterior scale tractor
- $\mathcal{I}^*, \iota(I)$: Interior scale tractor
- $\gamma$: Extension operator
REFERENCES

[1] O. Aharony, S.S. Gubser, J.M. Maldacena et al., Large N field theories, string theory and gravity, Phys. Rept. 323, 183-386 (2000), arXiv:hep-th/9905111. Cited on page(s): 3

[2] M.T. Anderson, The Dirichlet problem at infinity for manifolds of negative curvature, J. Diff. Geom. 18, 701 (1983). Cited on page(s): 4

[3] E. Aubry, C. Guillarmou, Conformal harmonic forms, Branson-Gover operators and Dirichlet problem at infinity, arXiv:0808.0552. Cited on page(s): 4

[4] T.N. Bailey, M.G. Eastwood, and A.R. Gover, Thomas’s structure bundle for conformal, projective and related structures, Rocky Mountain J. Math. 24 (1994), 1191–1217. Cited on page(s): 3

[5] T.P. Branson, Conformally covariant equations on differential forms, Comm. Partial Differential Equations, 4 (1983), 393. Cited on page(s): 52

[6] T.P. Branson, Sharp inequalities, the functional determinant, and the complementary series, Trans. Amer. Math. Soc. 347 (1995), 3671–3742. Cited on page(s): 3

[7] T. Branson and A.R. Gover, Electromagnetism, metric deformations, ellipticity and gauge operators on conformal 4-manifolds, Differential Geom. Appl. 17 (2002), 229–249, arXiv:hep-th/0111003. Cited on page(s): 37

[8] T. Branson, A.R. Gover, Conformally invariant non-local operators, Pacific J. Math. 201 (2001), 19–60. Cited on page(s): 15, 43, 59

[9] T. Branson, A.R. Gover, Conformal invariants, differential forms, cohomology and a generalisation of Q-curvature, Comm. Partial Differential Equations 30 (2005), 1611–1669. Cited on page(s): 3

[10] A. Čap, and A.R. Gover, Tractor bundles for irreducible parabolic geometries. Global analysis and harmonic analysis, Sémin. Congr. 4, 129, Soc. Math. France 2000. Cited on page(s): 14

[11] A. Čap, and A.R. Gover, Standard tractors and the conformal ambient metric construction, Ann. Global Anal. Geom. 24 (2003), 231–295, arXiv:math/0207016. Cited on page(s): 3, 8, 9, 25, 78, 79

[12] P. Cherrier, Problèmes de Neumann non linéaires sur les variétés riemanniennes, J. Funct. Anal. 57 (1984), 154–206. Cited on page(s): 15, 59

[13] D. Cherney, E. Latini, A. Waldron, (p,q)-form Kähler Electromagnetism, Phys. Lett. B674, 316-318 (2009), arXiv:0901.3788. Cited on page(s): 57

[14] S. Deser, R. I. Nepomechie, Anomalous Propagation Of Gauge Fields In Conformally Flat Spaces, Phys. Lett. B132, 321 (1983); “Gauge Invariance Versus Masslessness In De Sitter Space, Annals Phys. 154, 396 (1984). Cited on page(s): 1, 33

[15] S. Deser and A. Waldron, Partial masslessness of higher spins in (A)dS, Nucl. Phys. B 607 (2001), 577-604, arXiv:hep-th/0103198. Cited on page(s): 3

[16] M.G. Eastwood, Notes on conformal differential geometry, Supp. Rend. Circ. Matem. Palermo, Ser. II, Suppl., 43 (1996), 57–76. Cited on page(s): 25

[17] M.G. Eastwood and J.W. Rice, Conformally invariant differential operators on Minkowski space and their curved analogues, Comm. Math. Phys. 109, 207 (1987); loc. cit. 144, 213, 1992. Cited on page(s): 6

[18] C. Fefferman, and C.R. Graham, Conformal invariants in: The mathematical heritage of Élie Cartan (Lyon, 1984). Astérisque 1985, Numero Hors Serie, 95–116. Cited on page(s): 3, 78

[19] C. Fefferman, and C.R. Graham, The Ambient Metric, Annals of Mathematics Studies, 178, Princeton University Press, arXiv:0710.0919. Cited on page(s): 78

[20] C. Fefferman and K. Hirachi, Ambient metric construction of Q-curvature in conformal and CR geometries, Math. Res. Lett. 10, (2003) 819–831, arXiv:math.DG/0303184. Cited on page(s): 5
[21] A.R. Gover, Aspects of parabolic invariant theory, Rend. Circ. Mat. Palermo (2) Suppl. No. 59 (1999), 25–47. Cited on page(s): 25–27

[22] A.R. Gover, Invariant theory and calculus for conformal geometries, Adv. Math. 163 (2001), 206–257. Cited on page(s): 25–27

[23] A.R. Gover, Laplacian operators and Q-curvature on conformally Einstein manifolds, Math. Ann. 336 (2006), 311–334, arXiv:math/0506037. Cited on page(s): 25–27

[24] A.R. Gover, Conformal Dirichlet-Neumann Maps and Poincaré-Einstein Manifolds, SIGMA Symmetry Integrability Geom. Methods Appl. 3 (2007), Paper 100, arXiv:0710.2585. Cited on page(s): 3, 8, 25, 27

[25] A.R. Gover, Almost conformally Einstein manifolds and obstructions, in Differential geometry and its applications, 247–260, Matfyzpress, Prague, 2005, arXiv:math/0412393. Cited on page(s): 3, 9, 14, 40, 42, 80

[26] A.R. Gover, Almost Einstein and Poincaré-Einstein manifolds in Riemannian signature, J. Geometry and Physics, 60 (2010), 182–204, arXiv:0803.3510. Cited on page(s): 3, 9, 14, 40, 42, 80

[27] A.R. Gover and P. Nurowski, Obstructions to conformally Einstein metrics in n dimensions, J. Geom. Phys. 56 (2006), 45–84, arXiv:math/0405304. Cited on page(s): 3, 9, 14, 40

[28] A.R. Gover, Conformally invariant powers of the Laplacian, Q-curvature, and tractor calculus, Comm. Math. Phys. 235 (2003), 339–378, arXiv:math-ph/0201030. Cited on page(s): 3, 8, 25, 78, 79

[29] A.R. Gover, L.J. Peterson, Lawrence, The ambient obstruction tensor and the conformal deformation complex, Pacific J. Math. 226 (2006), 309–351. Cited on page(s): 3, 8, 25, 78, 79

[30] A.R. Gover, A. Shankat and A. Waldron, Tractors, Mass and Weyl Invariance, Nucl. Phys. B 812 (2009), 291–326, arXiv:1104.4994. Cited on page(s): 4

[31] C.R. Graham, Volume and area renormalizations for conformally compact Einstein metrics, Rend. Circ. Mat. Palermo (2) Suppl. No. 63 (2000), 31–42. Cited on page(s): 69

[32] C.R. Graham, R. Jenne, L. Mason and G. Sparling, Conformally invariant powers of the Laplacian. I. Existence, J. London Math. Soc. (2) 46 (1992), 557–565. Cited on page(s): 69

[33] C.R. Graham, and J.M. Lee, Einstein metrics with prescribed conformal infinity on the ball, Adv. Math. 87 (1991), 186–225. Cited on page(s): 69

[34] C.R. Graham, and M. Zworski, Scattering matrix in conformal geometry, Invent. Math. 152 (2003), 89–118, arXiv:math/0109089. Cited on page(s): 3, 4

[35] M. Henningson, K. Skenderis, The Holographic Weyl anomaly, JHEP 9807, 023 (1998), arXiv:hep-th/9806087. Cited on page(s): 3, 4

[36] P. Hislop, P. Perry and S. Tang, CR-invariants and the scattering operator for complex manifolds with boundary, Anal. PDE 1 (2008), 197–227, arXiv:0709.1103. Cited on page(s): 3, 4

[37] M. Joshi, A. Sá Barreto, Inverse scattering on asymptotically hyperbolic manifolds, Acta Math. 184, 41 (2000), arXiv:math-ph/001118. Cited on page(s): 3

[38] A. Juhl, Families of conformally covariant differential operators, Q-curvature and holography, Progress in Mathematics, 275, Birkhäuser Verlag 2009. Cited on page(s): 3
[46] C.R. LeBrun, *H-space with a cosmological constant*, Proc. Roy. Soc. London Ser. A **380** (1982), 171–185. Cited on page(s): 3

[47] P. Löwdin, *Angular momentum wavefunctions constructed by projector operators*, Rev. Modern Phys. **36**, (1964) 966–976. Cited on page(s): 5

[48] J. Maldacena, *The large N limit of superconformal field theories and supergravity*, Adv. Theor. Math. Phys. **2** (1998), 231–252, arXiv:hep-th/9711200 Cited on page(s): 3

[49] R. Mazzeo, *The Hodge cohomology of a conformally compact metric*, J. Differential Geom., **28** (1988), 309–339. Cited on page(s): 3 40 73

[50] R. Mazzeo, *Unique continuation at infinity and embedded eigenvalues for asymptotically hyperbolic manifolds*, Amer. J. Math., **113** (1991), 25–45. Cited on page(s): 3

[51] R. Mazzeo and R. Melrose, *Meromorphic extension of the resolvent on complete spaces with asymptotically constant negative curvature*, J. Funct. Anal. **75** (1987), 260–310. Cited on page(s): 3 62

[52] R. Penrose, W. Rindler, *Spinors and space-time. Vol. 1. Two-spinor calculus and relativistic fields*, Cambridge Monographs on Mathematical Physics. Cambridge University Press, Cambridge, 1984. Cited on page(s): 53

[53] R. Proca, Sur la théorie ondulatoire des électrons positifs et négatifs, J. Phys. Radium 7, (1936) 347; Sur la théorie du positron, C. R. Acad. Sci. Paris **202**, (1936) 1366. Cited on page(s): 54

[54] S. Sasaki, *On the spaces with normal conformal connexions whose groups of holonomy fix a point or a hypersphere II*, Jap. J. Math. **18**, (1943) 623–633. Cited on page(s): 10

[55] R. Staff, *Tractor Calculus and Invariants for Conformal Sub-Manifolds*, M.Sc. Thesis, University of Auckland, 2005; www.math.auckland.ac.nz/mathwiki/images/c/cf/StaffordMSc.pdf. Cited on page(s): 43

[56] A. Vasy, *The wave equation on asymptotically de Sitter-like spaces*, Adv. Math., **223** (2010), 49–97, arXiv:0706.3669. Cited on page(s): 4

[57] D. Sullivan, *The Dirichlet problem at infinity for a negatively curved manifold*, J. Diff. Geom. **18**, 723 (1983). Cited on page(s): 4

[58] P. Lévy, *Le monde comme un hologramme*, J. Math. Phys. **36** (1995) 6377. arXiv:hep-th/9409089. Cited on page(s): 3

[59] T.Y. Thomas, *On conformal geometry*, Proc. Natl. Acad. Sci. USA **12**, 352–359 (1926). Cited on page(s): 14

[60] G. ’t Hooft, *Dimensional reduction in quantum gravity*, arXiv:gr-qc/9310026 Cited on page(s): 3

[61] V.N. Tolstoy, *Fortieth anniversary of extremal projector method for Lie symmetries in Noncommutative geometry and representation theory in mathematical physics*, Contemp. Math., **391**, 371–384, Amer. Math. Soc., Providence, RI, 2005. Cited on page(s): 6 16

[62] A. Vasy, *The wave equation on asymptotically de Sitter-like spaces*, Adv. Math., **223** (2010), 49–97, arXiv:0706.3669. Cited on page(s): 4