ENRICHED CHAIN POLYTOPES

HIDEFUMI OHSUGI AND AKIYOSHI TSUCHIYA

ABSTRACT. Stanley introduced a lattice polytope $C_P$ arising from a finite poset $P$, which is called the chain polytope of $P$. The geometric structure of $C_P$ has good relations with the combinatorial structure of $P$. In particular, the Ehrhart polynomial of $C_P$ is given by the order polynomial of $P$. In the present paper, associated to $P$, we introduce a lattice polytope $E_P$, which is called the enriched chain polytope of $P$, and investigate geometric and combinatorial properties of this polytope. By virtue of the algebraic technique on Gröbner bases, we see that $E_P$ is a reflexive polytope with a flag regular unimodular triangulation. Moreover, the $h^*$-polynomial of $E_P$ is equal to the $h$-polynomial of a flag triangulation of a sphere. On the other hand, by showing that the Ehrhart polynomial of $E_P$ coincides with the left enriched order polynomial of $P$, it follows from works of Stembridge and Petersen that the $h^*$-polynomial of $E_P$ is $\gamma$-positive. Stronger, we prove that the $\gamma$-polynomial of $E_P$ is equal to the $f$-polynomial of a flag simplicial complex.

INTRODUCTION

A lattice polytope $P \subset \mathbb{R}^n$ of dimension $n$ is a convex polytope all of whose vertices have integer coordinates. Given a positive integer $m$, we define

$$L_P(m) = |mP \cap \mathbb{Z}^n|.$$

The study on $L_P(m)$ originated in Ehrhart [5] who proved that $L_P(m)$ is a polynomial in $m$ of degree $n$ with the constant term 1. We say that $L_P(m)$ is the Ehrhart polynomial of $P$. The generating function of the lattice point enumerator, i.e., the formal power series

$$\text{Ehr}_P(x) = 1 + \sum_{k=1}^{\infty} L_P(k)x^k$$

is called the Ehrhart series of $P$. It is well known that it can be expressed as a rational function of the form

$$\text{Ehr}_P(x) = \frac{h^*(P,x)}{(1-x)^{n+1}}.$$

The polynomial $h^*(P,x)$ is a polynomial in $x$ of degree at most $n$ with nonnegative integer coefficients ([18]) and it is called the $h^*$-polynomial (or the $\delta$-polynomial) of $P$. Moreover, one has $\text{Vol}(P) = h^*(P,1)$, where $\text{Vol}(P)$ is the normalized volume of $P$.

In [19], Stanley introduced a class of lattice polytopes associated with finite partially ordered sets. Let $P = [n] := \{1,2,\ldots,n\}$ be a partially ordered set (poset, for short). An
antichain of $P$ is a subset of $P$ consisting of pairwise incomparable elements of $P$. Note that the empty set $\emptyset$ is an antichain of $P$. The chain polytope $\mathcal{C}_P$ of $P$ is the convex hull of

$$\{e_{i_1} + \cdots + e_{i_k} : \{i_1, \ldots, i_k\} \text{ is an antichain of } P\},$$

where $e_i$ is $i$-th unit coordinate vector of $\mathbb{R}^n$ and the empty set $\emptyset$ corresponds to the origin $0$ of $\mathbb{R}^n$. Then $\mathcal{C}_P$ is a lattice polytope of dimension $n$. There is a close interplay between the combinatorial structure of $P$ and the geometric structure of $\mathcal{C}_P$. For instance, it is known that the Ehrhart polynomial $L_{\mathcal{C}_P}(m)$ and the order polynomial $\Omega_P(m)$ are related by $\Omega_P(m+1) = L_{\mathcal{C}_P}(m)$. On the other hand, $\mathcal{C}_P$ has many interesting properties. In particular, the toric ring of $\mathcal{C}_P$ is an algebra with straightening laws, and thus the toric ideal possesses a squarefree quadratic initial ideal ([8]). Moreover, $\mathcal{C}_P$ is of interest in representation theory ([2]) and statistics ([24]).

Now, we introduce a new class of lattice polytopes associated with posets. The enriched chain polytope $\mathcal{E}_P$ is the convex hull of $\mathcal{E}(P) = \{\pm e_{i_1} \pm \cdots \pm e_{i_k} : \{i_1, \ldots, i_k\} \text{ is an antichain of } P\}$. Then $\dim \mathcal{E}_P = n$. It is easy to see that $\mathcal{E}_P$ is centrally symmetric (i.e., for any facet $\mathcal{F}$ of $\mathcal{E}_P$, $-\mathcal{F}$ is also a facet of $\mathcal{E}_P$), and the origin $0$ of $\mathbb{R}^n$ is the unique interior lattice point of $\mathcal{E}_P$. In the present paper, we investigate geometric and combinatorial properties of $\mathcal{E}_P$.

A lattice polytope $\mathcal{P} \subset \mathbb{R}^n$ of dimension $n$ is called reflexive if the origin of $\mathbb{R}^n$ is a unique lattice point belonging to the interior of $\mathcal{P}$ and its dual polytope $\mathcal{P}^\vee := \{y \in \mathbb{R}^n : \langle x, y \rangle \leq 1 \text{ for all } x \in \mathcal{P}\}$ is also a lattice polytope, where $\langle x, y \rangle$ is the usual inner product of $\mathbb{R}^n$. It is known that reflexive polytopes correspond to Gorenstein toric Fano varieties, and they are related to mirror symmetry (see, e.g., [14]). In each dimension there exist only finitely many reflexive polytopes up to unimodular equivalence ([15]) and all of them are known up to dimension 4 ([12]). Recently, there are several classes of reflexive polytopes are constructed by the virtue of the algebraic technique on Gröbner bases (c.f., [10, 11, 15]). By showing the toric ideal of $\mathcal{E}_P$ possesses a squarefree quadratic initial ideal (Theorem 1.3), in Section 1 we prove the following.

**Theorem 0.1.** Let $P = [n]$ be a poset. Then $\mathcal{E}_P$ is a reflexive polytope having a flag regular unimodular triangulation such that each maximal simplex contains the origin as a vertex.

We now turn to the discussion of the Ehrhart polynomial and the $h^*$-polynomial of $\mathcal{E}_P$. In fact, the Ehrhart polynomial of $\mathcal{E}_P$ is equal to a combinatorial polynomial associated to $P$. In Section 2 we prove the following.

**Theorem 0.2.** Let $P = [n]$ be a naturally labeled poset. Then one has

$$L_{\mathcal{E}_P}(m) = \Omega_P^{(\ell)}(m),$$

where $\Omega_P^{(\ell)}(m)$ is the left enriched order polynomial of $P$.

In [21], Stembridge developed the theory of enriched $(P, \omega)$-partitions, in analogy with Stanley’s theory of $(P, \omega)$-partitions. In the theory, enriched order polynomials were introduced. On the other hand, Petersen [17] introduced slightly different notion, left enriched
(\(P, \omega\))-partitions and left enriched order polynomials. Please refer to Section 2 for the details. Therefore, from Theorem 0.2 we call \(E_P\) the “enriched” chain polytope of \(P\).

Next, we discuss Gal’s Conjecture for enriched chain polytopes. Gal [6] conjectured that the \(h\)-polynomial of a flag triangulation of a sphere is \(\gamma\)-positive. On the other hand, Theorem 0.1 implies that \(h^*(E_P, x)\) coincides with the \(h\)-polynomial of a flag triangulation of a sphere (Corollary 2.1). Therefore, \(h^*(E_P, x)\) is expected to be \(\gamma\)-positive. From works of Stembridge [21] and Petersen [17] and Theorem 0.2, we can obtain the following.

**Theorem 0.3.** Let \(P = [n]\) be a naturally labeled poset. Then the \(h^*\)-polynomial of \(E_P\) is

\[
h^*(E_P, x) = (x + 1)^n \frac{4x}{(x + 1)^2},
\]

where \(W^f_P(x)\) is the left peak polynomial of \(P\). In particular, \(h^*(E_P, x)\) is \(\gamma\)-positive. Moreover, \(h^*(E_P, x)\) is real-rooted if and only if \(W^f_P(x)\) is real-rooted.

Note that \(W^f_P(x)\) is not necessarily real-rooted ([22]).

Finally, we discuss Nevo-Petersen’s Conjecture for enriched chain polytopes. In [14], Nevo and Petersen made a stronger conjecture than Gal’s Conjecture. They conjectured that the \(h\)-polynomial of a flag triangulation of a sphere is equal to the \(f\)-polynomial of a simplicial complex. In other words, the coefficients of the \(\gamma\)-polynomial of a flag triangulation of a sphere satisfy Kruskal-Katona inequalities. In Section 3 we construct explicit flag simplicial complexes whose \(f\)-polynomials are the \(\gamma\)-polynomials of enriched chain polytopes (Theorem 3.4).

**Acknowledgment.** The authors were partially supported by JSPS KAKENHI 18H01134 and 16J01549.

1. Squarefree Quadratic Gröbner Bases

In this section, we prove Theorem 0.1. First, we see the geometric structure of \(E_P\) of a finite poset \(P = [n]\). Given \(\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n) \in \{-1, 1\}^n\), let \(\mathcal{O}_\varepsilon\) denote the closed orthant \(\{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_i \varepsilon_i \geq 0 \text{ for all } i \in [n]\}\). Let \(\mathcal{L}(P)\) denote the set of linear extensions of \(P\). It is known [19, Corollary 4.2] that the normalized volume of the chain polytope \(E_P\) is \(|\mathcal{L}(P)|\).

**Lemma 1.1.** Work with the same notation as above. Then each \(E_P \cap \mathcal{O}_\varepsilon\) is the convex hull of the set \(E(P) \cap \mathcal{O}_\varepsilon\) and unimodularly equivalent to the chain polytope \(E_P\) of \(P\). In particular, the normalized volume of \(E_P\) is \(\text{Vol}(E_P) = 2^n \text{Vol}(E_P) = 2^n |\mathcal{L}(P)|\).

**Proof.** It is enough to show that \(E_P \cap \mathcal{O}_\varepsilon\subset \text{Conv}(E(P) \cap \mathcal{O}_\varepsilon)\). Let \(x = (x_1, \ldots, x_n) \in E_P \cap \mathcal{O}_\varepsilon\). Then \(x = \sum_{i=1}^n \lambda_i a_i\), where \(\lambda_i > 0\), \(\sum_{i=1}^n \lambda_i = 1\), and each \(a_i\) belongs to \(E(P)\). Suppose that \(k\)-th component of \(a_i\) is positive and \(k\)-th component of \(a_j\) is negative. Then we replace \(\lambda(a_i + a_j)\) in \(x = \sum_{i=1}^n \lambda_i a_i\) with \(\lambda((a_i - e_k) + (a_j + e_k))\), where \(\lambda = \min\{\lambda_i, \lambda_j\}\) and \(a_i - e_k, a_j + e_k \in E(P)\). Repeating this procedure finitely many times, we may assume that \(k\)-th component of each vector \(a_i\) is nonnegative (resp. nonpositive) if \(x_k \geq 0\) (resp. \(x_k \leq 0\)). Then each \(a_i\) belongs to \(E(P) \cap \mathcal{O}_\varepsilon\) and hence \(x \in \text{Conv}(E(P) \cap \mathcal{O}_\varepsilon)\).
In order to show that $\mathcal{C}_{P}$ is reflexive and has a flag regular unimodular triangulation, we use an algebraic technique on Gröbner bases. We recall basic materials and notation on toric ideals. Let $K[t_{1}^{\pm 1}, s] = K[t_{1}^{\pm 1}, \ldots, t_{n}^{\pm 1}, s]$ be the Laurent polynomial ring in $n + 1$ variables over a field $K$. If $\mathbf{a} = (a_{1}, \ldots, a_{n}) \in \mathbb{Z}^{n}$, then $t^{\mathbf{a}}s$ is the Laurent monomial $t_{1}^{a_{1}} \cdots t_{n}^{a_{n}}s \in K[t_{1}^{\pm 1}, s]$. Let $\mathcal{P} \subset \mathbb{R}^{n}$ be a lattice polytope and $\mathcal{P} \cap \mathbb{Z}^{n} = \{a_{1}, \ldots, a_{d}\}$. Then, the toric ring of $\mathcal{P}$ is the subalgebra $K[\mathcal{P}]$ of $K[t_{1}^{\pm 1}, s]$ generated by $\{t^{\mathbf{a}}s, \ldots, t^{\mathbf{a}_{d}}s\}$ over $K$. We regard $K[\mathcal{P}]$ as a homogeneous algebra by setting each $\deg t^{\mathbf{a}}s = 1$. Let $K[x] = K[x_{1}, \ldots, x_{d}]$ denote the polynomial ring in $d$ variables over $K$. The toric ideal $I_{\mathcal{P}}$ of $\mathcal{P}$ is the kernel of the surjective homomorphism $\pi : K[x] \to K[\mathcal{P}]$ defined by $\pi(x_{i}) = t^{\mathbf{a}_{i}}s$ for $1 \leq i \leq d$. It is known that $I_{\mathcal{P}}$ is generated by homogeneous binomials. See, e.g., [23]. The following lemma follows from the same argument in [9, Proof of Lemma 1.1].

Lemma 1.2. Let $\mathcal{P} \subset \mathbb{R}^{n}$ be a lattice polytope of dimension $n$ such that the origin of $\mathbb{R}^{n}$ is contained in its interior. Suppose that any lattice point in $\mathbb{Z}^{n+1}$ is a linear integer combination of the lattice points in $\mathcal{P} \times \{1\}$. If there exists a monomial order such that the initial ideal is generated by squarefree monomials which do not contain the variable corresponding to the origin, then $\mathcal{P}$ is reflexive and has a regular unimodular triangulation.

Let $P = [n]$ be a poset and let $R_{P}$ denote the polynomial ring

$$R_{P} = K[x_{i_{1}, \ldots, i_{k}}^{\varepsilon} : \{i_{1}, \ldots, i_{k}\} \text{ is an antichain of } P, \varepsilon \in \{-1, 1\}^{k}]$$

in $|\mathcal{C}_{P} \cap \mathbb{Z}^{n}|$ variables over a field $K$. In particular, the origin corresponds to the variable $x_{0}$. Then the toric ideal $I_{\mathcal{C}_{P}}$ of $\mathcal{C}_{P}$ is the kernel of a ring homomorphism $\pi : R_{P} \to K[t_{1}^{\pm 1}, s]$ defined by $\pi(x_{i_{1}, \ldots, i_{k}}^{(1, \ldots, 1)}) = t_{i_{1}}^{1} \cdots t_{i_{k}}^{1}s$. In addition,

$$I_{\mathcal{C}_{P}} \cap K[x_{i_{1}, \ldots, i_{k}}^{(1, \ldots, 1)} : \{i_{1}, \ldots, i_{k}\} \text{ is an antichain of } P]$$

is the toric ideal $I_{\mathcal{C}_{P}}$ of the chain polytope $\mathcal{C}_{P}$ of $P$. Hibi and Li essentially constructed a squarefree quadratic initial ideal of $I_{\mathcal{C}_{P}}$ in [8]. Let $\mathcal{J}(P)$ be the finite distributive lattice consisting of all poset ideals of $P$, ordered by inclusion. Given a subset $Z \subset P$, let $\max(Z)$ denote the set of all maximal elements of $Z$. Then $\max(Z)$ is an antichain of $P$. For a subset $Y$ of $P$, the poset ideal of $P$ generated by $Y$ is the smallest poset ideal of $P$ which contains $Y$. Given poset ideals $I, J \in \mathcal{J}(P)$, let $I \ast J$ denote the poset ideal of $P$ generated by $\max(I \cap J) \cup (\max(I) \cup \max(J))$. Then Hibi and Li proved in [8, Proof of Theorem 2.1] that the set of all binomials of the form

$$x_{\max(I)}^{(1, \ldots, 1)}x_{\max(J)}^{(1, \ldots, 1)} - x_{\max(I \cup J)}^{(1, \ldots, 1)}$$

is a Gröbner basis of $I_{\mathcal{C}_{P}}$ with respect to a monomial order $\prec$. Here the initial monomial of each binomial is the first monomial. It is known [23, Proposition 1.11] that there exists a nonnegative weight vector $\mathbf{w} \in \mathbb{R}^{|\mathcal{J}(P)|}$ such that $\text{in}_{\mathbf{w}}(I_{\mathcal{C}_{P}}) = \text{in}_{\mathbf{w}}(I_{\mathcal{C}_{P}})$. Then we define the weight vector $\mathbf{w}_{P}$ on $R_{P}$ such that the weight of each variable $x_{i_{1}, \ldots, i_{k}}^{\varepsilon}$ with respect to $\mathbf{w}_{P}$ is the weight of the variable $x_{i_{1}, \ldots, i_{k}}^{(1, \ldots, 1)}$ with respect to $\mathbf{w}$. In addition, let $\mathbf{w}_{\text{card}}$ be the weight vector on $R_{P}$ such that the weight of each variable $x_{i_{1}, \ldots, i_{k}}^{\varepsilon}$ with respect to $\mathbf{w}_{\text{card}}$ is
k. Fix any monomial order \( \prec \) on \( R_\prec \) as a tie-breaker. Let \( \prec \) be a monomial order on \( R_\prec \) such that \( u \prec v \) if and only if one of the following holds:

- The weight of \( u \) is less than that of \( v \) with respect to \( w_{\text{card}} \);
- The weight of \( u \) is the same as that of \( v \) with respect to \( w_{\text{card}} \), and the weight of \( u \) is less than that of \( v \) with respect to \( w_p \);
- The weight of \( u \) is the same as that of \( v \) with respect to \( w_{\text{card}} \) and \( w_p \), and \( u \prec v \).

**Theorem 1.3.** Work with the same notation as above. Let \( \mathcal{G} \) be the set of all binomials

\[
\begin{align*}
(1) & \quad x^{(e_1, \ldots, e_k)}_{i_1, \ldots, i_k} x^{(\mu_1, \ldots, \mu_l)}_{j_1, \ldots, j_l} - x^{(e_1, \ldots, e_p, e_{p+1}, \ldots, e_k)}_{i_1, \ldots, i_p, i_{p+1}, \ldots, i_k} x^{(\mu_1, \ldots, \mu_q, \ldots, \mu_l)}_{j_1, \ldots, j_q, \ldots, j_l}, \\
(2) & \quad x^{(e_1, \ldots, e_k)}_{i_1, \ldots, i_k} x^{(e_{k+1}, \ldots, e_{k+\ell})}_{i_1, \ldots, i_k} - x^{(\mu_1, \ldots, \mu_{\ell'})}_{j_1, \ldots, j_{\ell'}} x^{(\mu_{l'+1}, \ldots, \mu_{l'+\ell'})}_{j_{\ell'+1}, \ldots, j_{\ell'+\ell'}} \quad (\neq 0),
\end{align*}
\]

where \( i_p = j_q \) and \( e_p \neq \mu_q \).

**Proof.** It is easy to see that any binomial of type (1) belongs to \( I_{\mathcal{G}} \). The fact appearing in [8] guarantees that any binomial of type (2) belongs to \( I_{\mathcal{G}} \). Hence \( \mathcal{G} \) is a subset of \( I_{\mathcal{G}} \). For a binomial \( u - v \) of type (2), let \( \pi(u) = t_{1}^{a_1} \ldots t_{n}^{a_n} s^{2} \) and \( \pi(v) = t_{1}^{b_1} \ldots t_{n}^{b_n} s^{2} \). Since \( u - v \) satisfies conditions (a) and (b), we have \( k + \ell = \sum_{i=1}^{n} |a_i| = \sum_{i=1}^{n} |b_i| = k' + \ell' \). Hence the weight of \( u \) and \( v \) are the same with respect to \( w_{\text{card}} \). Thus the initial monomial of each binomial is the first monomial. In particular, the initial ideal is generated by squarefree quadratic monomials which do not contain the variable \( x_q \).

Then \( \mathcal{G} \) is a Gröbner basis of \( I_{\mathcal{G}} \) with respect to a monomial order \( \prec \). The initial monomial of each binomial is the first monomial. In particular, the initial ideal is generated by squarefree quadratic monomials which do not contain the variable \( x_q \).
with $I_1 \subset \cdots \subset I_r$ and $J_1 \subset \cdots \subset J_r$. Since $u$ and $v$ satisfy conditions (a) and (b) and since $f$ belongs to $I_{\leq i}^p$, it then follows that $x^{(\ell_{u-1} + \cdots + \ell_r)}_{\max(J_r)} = x^{(\ell_{i-1} + \cdots + \ell_r)}_{\max(J_r)}$. This contradicts the assumption that $f$ is irreducible. □

By the correspondence \cite[p. 8]{23} between a squarefree quadratic initial ideal of $I_{\leq m}^p$ and a flag regular unimodular triangulation of $\mathcal{E}_P$, Theorem \ref{thm0.1} follows from Lemma \ref{lem1.2} and Theorem \ref{thm1.3}.

2. $\gamma$-POSITIVITY AND REAL-ROOTEDNESS OF THE $h^*$-POLYNOMIAL OF $\mathcal{E}_P$

In this section, we discuss the Ehrhart polynomial and $h^*$-polynomial of $\mathcal{E}_P$ of a finite poset $P = [n]$. In particular, we prove Theorems \ref{thm0.2} and \ref{thm0.3}.

Let $(P, \omega)$ be a poset with $n$ elements and let $\Omega'_m = \{1, -1, 2, -2, \ldots, m, -m\}$ for $0 < m \in \mathbb{Z}$. A map $f : P \to \Omega'_m$ is called an enriched $(P, \omega)$-partition \cite{21} if, for all $x, y \in P$ with $x <_P y$, $f$ satisfies

- $|f(x)| \leq |f(y)|$;
- $|f(x)| = |f(y)| \Rightarrow f(x) \leq f(y)$;
- $f(x) = f(y) > 0 \Rightarrow \omega(x) < \omega(y)$;
- $f(x) = f(y) < 0 \Rightarrow \omega(x) > \omega(y)$.

In the present paper, we always assume that $(P, \omega)$ is naturally labeled. Then the above condition is equivalent to the following conditions:

- $|f(x)| \leq |f(y)|$;
- $|f(x)| = |f(y)| \Rightarrow f(y) > 0$.

For each $0 < m \in \mathbb{Z}$, let $\Omega'_m(m)$ denote the number of enriched $(P, \omega)$-partitions $f : P \to \Omega'_m$. Then $\Omega'_m(m)$ is a polynomial in $m$ and called the enriched order polynomial of $P$.

On the other hand, Petersen \cite{17} introduced slightly different notion “left enriched $(P, \omega)$-partitions” as follows. Let $\Omega_{m}^{(\ell)} = \{0, 1, -1, 2, -2, \ldots, m, -m\}$ for $0 < m \in \mathbb{Z}$. A map $f : P \to \Omega_{m}^{(\ell)}$ is called a left enriched $(P, \omega)$-partition if, for all $x, y \in P$ with $x <_P y$, $f$ satisfies the following conditions:

(i) $|f(x)| \leq |f(y)|$;
(ii) $|f(x)| = |f(y)| \Rightarrow f(y) \geq 0$.

For each $0 < m \in \mathbb{Z}$, let $\Omega_{m}^{(\ell)}(m)$ denote the number of left enriched $(P, \omega)$-partitions $f : P \to \Omega_{m}^{(\ell)}$. Then $\Omega_{m}^{(\ell)}(m)$ is a polynomial in $m$ and called the left enriched order polynomial of $P$. We can compute the left enriched order polynomial $\Omega_{m}^{(\ell)}(m)$ of $P$ from the enriched order polynomial $\Omega'_m(m)$ of $P$. In fact, it follows that

$$\Omega_{m}^{(\ell)}(m) = \frac{1}{2}(\Omega'_m(m + 1) - \Omega'_m(m)).$$

Now, we prove Theorem \ref{thm0.2}.

\textbf{Proof of Theorem} \ref{thm0.2} It is enough to construct a bijection from $m\mathcal{E}_P \cap \mathbb{Z}^n$ to the set $F(m)$ of all left enriched $(P, \omega)$-partitions $f : P \to \Omega_{m}^{(\ell)}$. Let $\varphi : F(m) \to m\mathcal{E}_P \cap \mathbb{Z}^n$ be a map
Thus the definition of

where

Moreover by the argument in [19, Proof of Theorem 3.2], we have

Suppose that belongs to . Conversely, let

for each . (This map arising from the map defined in [19, Theorem 3.2].)

Claim 1. ( is well-defined.) By condition (i) for a left enriched -partition , we have ( ) . Hence, by Lemma [1.1], belongs to .

Let be a map defined by for each , where

for each . Since is an extension of a map given in [19, Proof of Theorem 3.2], satisfies condition (i) in the definition of a left enriched partition. Suppose that , , and . Then . Since and , we have , a contradiction. Thus is a left enriched -partition.

Finally, we show that is a bijection. It is enough to show that is the inverse of . Let for . By conditions (i) and (ii) for together with the definition of and , we have

Thus the map satisfies

for any . Moreover by the argument in [19, Proof of Theorem 3.2], we have

Conversely, let for . By conditions (i) and (ii) for together with the definition of and , we have

Thus the map satisfies

Moreover by the argument in [19, Proof of Theorem 3.2], we have for . Thus is an identity map. Therefore is a bijection, as desired.
Let $f = \sum_{i=0}^{n} a_i x^i$ be a polynomial with real coefficients and $a_n \neq 0$. We now focus on the following properties.

(RR) We say that $f$ is real-rooted if all its roots are real.

(LC) We say that $f$ is log-concave if $a_i^2 \geq a_{i-1}a_{i+1}$ for all $i$.

(UN) We say that $f$ is unimodal if $a_0 \leq a_1 \leq \cdots \leq a_k \geq \cdots \geq a_n$ for some $k$.

If all its coefficients are nonnegative, then these properties satisfy the implications

$$(\text{RR}) \Rightarrow (\text{LC}) \Rightarrow (\text{UN}).$$

On the other hand, the polynomial $f$ is said to be palindromic if $f(x) = x^nf(x^{-1})$. It is $\gamma$-positive if $f$ is palindromic and there are $\gamma_0, \gamma_1, \ldots, \gamma_{\lfloor n/2 \rfloor} \geq 0$ such that $f(x) = \sum_{i \geq 0} \gamma_i x^i(1+x)^{n-2i}$. The polynomial $\sum_{i \geq 0} \gamma_i x^i$ is called $\gamma$-polynomial of $f$. We can see that a $\gamma$-positive polynomial is real-rooted if and only if its $\gamma$-polynomial is real-rooted. If $f$ is a palindromic and real-rooted, then it is $\gamma$-positive. Moreover, if $f$ is $\gamma$-positive, then it is unimodal.

In the rest of the present section, we discuss the $\gamma$-positivity and the real-rootedness on the $h^*$-polynomial of $\mathcal{P}$. It is known \cite{17} that the $h^*$-polynomial of a lattice polytope $\mathcal{P}$ with the interior lattice point $0$ is palindromic if and only if $\mathcal{P}$ is reflexive. Moreover, if a reflexive polytope $\mathcal{P}$ has a regular unimodular triangulation, then the $h^*$-polynomial is unimodal \cite{3}. On the other hand, if a reflexive polytope $\mathcal{P}$ has a flag regular unimodular triangulation such that each maximal simplex contains the origin as a vertex, then the $h^*$-polynomial coincides with the $h$-polynomial of a flag triangulation of a sphere. Hence from Theorem \cite{1,5}, we can show the following.

**Corollary 2.1.** Let $P = [n]$ be a poset. Then the $h^*$-polynomial of $\mathcal{P}$ is palindromic, unimodal, and coincides with the $h$-polynomial of a flag triangulation of a sphere.

Given a linear extension $\pi = \pi_1, \ldots, \pi_n$ of a poset $P = [n]$, a peak (resp. a left peak) of $\pi$ is an index $2 \leq i \leq n-1$ (resp. $1 \leq i \leq n-1$) such that $\pi_{i-1} < \pi_i > \pi_{i+1}$, where we set $\pi_0 = 0$. Let $pk(\pi)$ (resp. $pk^{(l)}(\pi)$) denote the number of peaks (resp. left peaks) of $\pi$. Then the peak polynomial $W_p(x)$ and the left peak polynomial $W_p^{(l)}(x)$ of $P$ are defined by

$$W_p(x) = \sum_{\pi \in \mathcal{L}(P)} x^{pk(\pi)}$$

and

$$W_p^{(l)}(x) = \sum_{\pi \in \mathcal{L}(P)} x^{pk^{(l)}(\pi)}.$$

Petersen \cite{17} computed the generating function for a left enriched order polynomial:

**Lemma 2.2** (\cite{17} Theorem 4.6). Let $P = [n]$ be a naturally labeled poset. Then we have the following generating function for the left enriched order polynomial of $P$:

$$\sum_{m \geq 0} \Omega_p^{(l)}(m)x^m = \frac{(x+1)^n}{(1-x)^{n+1}} W_p^{(l)} \left( \frac{4x}{(x+1)^2} \right).$$

Therefore, Theorem \cite{0,3} follows from Theorem \cite{0,2} and Lemma \cite{2,2}.
\textbf{Remark 2.3.} A poset $P$ is said to be narrow if the vertices of $P$ is partitioned into two chains. Stembridge [22] Proposition 1.1] essentially pointed out that, if $P$ is narrow, then $W_P^\ell(x)$ coincides with the $P$-Eulerian polynomial

$$W(P)(x) = \sum_{\pi \in \text{Dec}(P)} x^{\text{des}(\pi)},$$

where des($\pi$) is the number of descents of $\pi$. Thus, for a narrow poset $P$,

$$h^*(\ell_P, x) = (x + 1)^n W(P) \left( \frac{4x}{(x+1)^2} \right).$$

This fact coincides with the result in [16] for a bipartite permutation graph. Note that a naturally labeled narrow poset $P$ such that $W(P)(x)$ is not real-rooted is given in [22].

Given a poset $P = [n]$, the comparability graph $G(P)$ of $P$ is the graph on the vertex set $[n]$ with $i,j \in [n]$ adjacent if either $i <_P j \text{ or } j <_P i$. Then $\{i_1, \ldots, i_k\} \subset [n]$ is an antichain of $P$ if and only if $\{i_1, \ldots, i_k\}$ is a stable set (independent set) of $G(P)$. Hence $\ell_P = \ell_{P'}$ if $G(P) = G(P')$ for posets $P$ and $P'$. Thus we have the following immediately.

\textbf{Corollary 2.4.} Both the (left) enriched order polynomial of $P$ and the (left) peak polynomial of $P$ depend only on the comparability graph $G(P)$ of $P$.

3. The $\Gamma$-complexes

In [14], Nevo and Petersen conjectured the following.

\textbf{Conjecture 3.1 ([14] Conjecture 1.4]).} The $\gamma$-polynomial of any flag triangulation of a sphere is the $f$-polynomial of a simplicial complex.

Equivalently, the coefficients of the $\gamma$-polynomial satisfy the Kruskal–Katona inequalities. (See [20] Chapter II.2.) Clearly, Conjecture 3.1 is stronger than Gal’s Conjecture. Moreover, they gave the following problem.

\textbf{Problem 3.2 ([14] Problem 6.4]).} The $\gamma$-polynomial of any flag triangulation of a sphere is the $f$-polynomial of a flag simplicial complex.

In this section, we solve this problem for enriched chain polytopes.

Let $P = [n]$ be an antichain. Then $h^*(\ell_P, x)$ coincides with the Eulerian polynomial $B_n(x)$ of type B. See [17] Proposition 4.15. In this case, a flag simplicial complex $\Gamma(\text{Dec}_n)$ whose $f$-polynomial is the $\gamma$-polynomial of $h^*(\ell_P, x)$ is given in [14, Corollary 4.5 (2)] as follows. A \textit{decorated permutation} $w$ is a permutation $w \in \text{Dec}_n$ with bars colored in four colors $[0], 1, 2,$ and $3$ following the left peak positions. Let $\text{Dec}_n$ be the set of all decorated permutations. Given $w \in \text{Dec}_n$, there exist $4 \text{pk}^\ell(w)$ decorated permutations associated with $w$ in $\text{Dec}_n$. For example, $3^{224}1^{157}0^{689}$ belongs to $\text{Dec}_9$. Given

$$w = w_1 |^{c_1} \ldots |^{c_i-1} w_i |^{c_i} w_{i+1} |^{c_{i+1}} \ldots |^{c_{\ell-1}} w_\ell \in \text{Dec}_n,$$

let $w_i = \hat{w}_i | \hat{w}_i$ where $\hat{w}_i$ is the decreasing part of $w_i$ and $\hat{w}_i$ the increasing part of $w_i$. We say that $w \in \text{Dec}_n$ covers $u \in \text{Dec}_n$ if and only if $u$ is obtained from $w$ by removing a colored bar $|^{c_{i}}$ and reordering the word $w_i w_{i+1} = \hat{w}_i \hat{w}_i w_{i+1}$ as a word $\hat{w}_i \hat{a}$ where $\hat{a} = \text{sort}(\hat{w}_i w_{i+1})$. Then $(\text{Dec}_n, \leq )$ is a poset graded by number of bars.
We associate the set $\text{Dec}_n$ with the flag simplicial complex $\Gamma(\text{Dec}_n)$ on the vertex set $V = \{ w \in \text{Dec}_n : \text{pk}^\ell(w) = 1 \}$. In $\Gamma(\text{Dec}_n)$, two vertices $u = \hat{u}_1 |^\ell \hat{u}_2$ and $v = \hat{v}_1 |^\ell \hat{v}_2$ with $|\hat{u}_1| < |\hat{v}_1|$ are adjacent if and only if $w = \hat{u}_1 |^\ell \hat{u}_2 |^\ell \hat{v}_2$ belongs to $\text{Dec}_n$, where $\hat{a} = \text{sort}(\hat{u}_2 \cap \hat{v}_1)$. Then $\Gamma(\text{Dec}_n)$ is the collection of all subsets $F$ of $V$ such that every two distinct vertices in $F$ are adjacent. By definition, $\Gamma(\text{Dec}_n)$ is a flag simplicial complex. Let $\varphi : \text{Dec}_n \rightarrow \Gamma(\text{Dec}_n)$ be a map defined by

$$\varphi(w) = \{ w_1 |^{c_1} w_2 b_1, \ldots, a_i |^{c_{i+1}} w_{i+1} b_i, \ldots, a_{\ell-1} |^{c_{\ell-1}} w_{\ell} \}$$

for $w = w_1 |^{c_1} \ldots |^{c_{\ell-1}} w_{\ell} \in \text{Dec}_n$, where $a_i$ is the set of letters to the left of $w_{i+1}$ in $w$ written in increasing order and $b_i$ is the set of letters to the right of $w_{i+1}$ in $w$ written in increasing order. It was shown \cite{14} that $\varphi$ is an isomorphism of graded posets from $(\text{Dec}_n, \leq)$ to $(\Gamma(\text{Dec}_n), \subseteq)$. Thus we have the following ((\cite{14}, Corollary 4.5 (2))).

**Proposition 3.3.** Let $P = [n]$ be an antichain. Then the $\gamma$-polynomial of $h^*(\text{Dec}_P, x)$ is the $f$-polynomial of the flag simplicial complex $\Gamma(\text{Dec}_n)$.

Given a poset $P = [n]$, let $S_P = \{ w \in \text{Dec}_n : w \in \mathcal{L}(P) \}$. Then we have the following.

**Theorem 3.4.** Let $P = [n]$ be a poset. Then the image $\Gamma(S_P) := \varphi(S_P)$ is a flag simplicial subcomplex of $\Gamma(\text{Dec}_n)$ whose $f$-polynomial is the $\gamma$-polynomial of $h^*(\text{Dec}_P, x)$.

**Proof.** First, we show that $\Gamma(S_P)$ is a subcomplex of $\Gamma(\text{Dec}_n)$. Let $w$ of the form \(\Box\) be an element of $S_P$. If $u$ is obtained from $w$ by removing a colored bar $|^{c_i}$ and reordering the word $w_i w_{i+1} = \hat{w}_i \hat{w}_i w_{i+1}$ as a word $\hat{\hat{w}} \hat{a}$ where $\hat{a} = \text{sort}(\hat{w}_i w_{i+1})$, then $u \in \text{Dec}_n$ is obtained from $w \in \mathcal{L}(P)$ by sorting a consecutive part of $w$. Then $u \in \mathcal{L}(P)$, and hence $u \in S_P$. Thus $(S_P, \leq)$ is a lower ideal in $(\text{Dec}_n, \leq)$. Since $\varphi$ is an isomorphism of graded posets, it follows that $\Gamma(S_P)$ is a subcomplex of $\Gamma(\text{Dec}_n)$.

Second, we show that $\Gamma(S_P)$ is flag. Let $V_P = \{ w \in S_P : \text{pk}^\ell(w) = 1 \}$. Since $(S_P, \leq)$ is a lower ideal, $\varphi(w) \subseteq V_P$ if $w \in S_P$. Let $F = \{ u_1, \ldots, u_\ell \}$ be a pairwise adjacent vertices in $V_P$ ordered by increasing position of the bar in $u_\ell$. We show that $\varphi^{-1}(F)$ belongs to $S_P$. If $\ell = 1$, then it is trivial. Suppose by induction on $\ell$ that

$$\varphi^{-1}(\{ u_1, \ldots, u_{\ell-1} \}) = w = w_1 |^{c_1} \ldots |^{c_{\ell-1}} w_{\ell} \in S_P$$

with $u_{\ell-1} = \hat{u}_{\ell-1} |^{c_{\ell-1}} \hat{w}_{\ell} \hat{w}_{\ell}$. Then $\varphi^{-1}(F)$ is

$$w' = w_1 |^{c_1} \ldots |^{c_{\ell-1}} w_{\ell} \hat{w}_{\ell} \hat{w}_{\ell}$$

where $u_{\ell} = \hat{u}_{\ell,1} |^{c} \hat{u}_{\ell,2} \hat{w}_{\ell}$ and $\hat{a} = \text{sort}(\hat{w}_{\ell} \cap \hat{u}_{\ell,1})$. Since both $w$ and $u_\ell$ belong to $S_P$ and since $\hat{a} \cup \hat{u}_{\ell,2} \cup \hat{w}_{\ell,2} \subset \hat{w}_{\ell}$ and $\hat{u}_{\ell-1} \cup \hat{w}_{\ell} \cup \hat{a} \subset \hat{u}_{\ell,1}$, it follows that $w'$ belongs to $S_P$ as desired. \qed

**References**

[1] V. Batyrev, Dual polyhedra and mirror symmetry for Calabi-Yau hypersurfaces in toric varieties, *J. Algebraic Geom.*, 3 (1994), 493–535.

[2] R. Biswal and G. Fourier, Minuscule Schubert Varieties: Poset Polytopes, PBW-Degenerated Demazure Modules, and Kogan Faces, *Algebr. Represent. Theory*, 18 (2015), 1481–1503.

[3] W. Bruns and T. Römer, $h$-Vectors of Gorenstein polytopes, *J. Combin. Theory Ser. A* 114 (2007), 65–76.

[4] D. Cox, J. Little and H. Schenck, “Toric varieties”, Amer. Math. Soc., 2011.
[5] E. Ehrhart, “Polynômes Arithmétiques et Méthode des Polyédres en Combinatorie”, Birkhäuser, Boston/Basel/Stuttgart, 1977.
[6] S. R. Gal, Real Root Conjecture fails for five and higher dimensional spheres, Discrete Comput. Geom., 34 (2005), 269–284.
[7] T. Hibi, Dual polytopes of rational convex polytopes, Combinatorica 12 (1992), 237–240.
[8] T. Hibi and N. Li, Chain polytopes and algebras with straightening laws, Acta Math. Vietnam. 40 (2015), 447–452.
[9] T. Hibi, K. Matsuda, H. Ohsugi, and K. Shibata, Centrally symmetric configurations of order polytopes, J. Algebra 443 (2015), 469–478.
[10] T. Hibi and A. Tsuchiya, Facets and volume of Gorenstein Fano polytopes, Math. Nachr. 290 (2017), 2619–2628.
[11] T. Hibi and A. Tsuchiya, Reflexive polytopes arising from perfect graphs, J. Combin. Theory Ser. A 157 (2018), 233–246.
[12] M. Kreuzer and H. Skarke, Complete classification of reflexive polyhedra in four dimensions, Adv. Theor. Math. Phys. 4 (2000), 1209–1230.
[13] J. C. Lagarias and G. M. Ziegler, Bounds for lattice polytopes containing a fixed number of interior points in a sublattice, Canad. J. Math. 43 (1991), 1022–1035.
[14] E. Nevo and T. K. Petersen, On γ-vectors satisfying the Kruskal–Katona inequalities, Discrete Comput. Geom., 45 (2011), 503–521.
[15] H. Ohsugi and T. Hibi, Reverse lexicographic squarefree initial ideals and Gorenstein Fano polytopes, J. Commut. Alg. 10 (2018), 171–186.
[16] H. Ohsugi and A. Tsuchiya, Reflexive polytopes arising from bipartite graphs with γ-positivity associated to interior polynomials, arXiv:1810.12258
[17] T. K. Petersen, Enriched P-partitions and peak algebras, Adv. Math. 209 (2007) 561–610.
[18] R. P. Stanley, Decompositions of rational convex polytopes, Annals of Discrete Math. 6 (1980), 333–342.
[19] R. P. Stanley, Two poset polytopes, Disc. Comput. Geom. 1 (1986), 9–23.
[20] R. P. Stanley, Combinatorics and Commutative Algebra, 2nd edn, Birkhäuser, Boston (1996).
[21] J. R. Stembridge, Enriched P-partitions, Trans. Amer. Math. Soc. 349 (1997), 763–788.
[22] J. R. Stembridge, Counterexamples to the poset conjectures of Neggers, Stanley, and Stembridge, Trans. Amer. Math. Soc. 359 (2007), 1115–1128.
[23] B. Sturmfels, “Gröbner bases and convex polytopes,” Amer. Math. Soc., Providence, RI, 1996.
[24] S. Sullivant, Compressed polytopes and statistical disclosure limitation, Tohoku Math. J. 58 (2006), 433–445.

HIDEFUMI OHSUGI, DEPARTMENT OF MATHEMATICAL SCIENCES, SCHOOL OF SCIENCE AND TECHNOLOGY, KWANSEI GAKUIN UNIVERSITY, SANDA, HYOGO 669-1337, JAPAN
E-mail address: ohsugi@kwansei.ac.jp

AKIYOSHI TSUCHIYA, DEPARTMENT OF PURE AND APPLIED MATHEMATICS, GRADUATE SCHOOL OF INFORMATION SCIENCE AND TECHNOLOGY, OSAKA UNIVERSITY, SUITA, OSAKA 565-0871, JAPAN
E-mail address: a-tsuchiya@ist.osaka-u.ac.jp