Quadrupole dominance in the light Sn and in the Cd isotopes

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Background: The $B(E2; 2^+_1 \rightarrow 0^+_1)$ of the Sn isotopes for $N \leq 64$ exhibit enhancements hitherto unexplained. The same is true for the Cd isotopes.

Purpose: Describe the electromagnetic properties of the Sn and Cd isotopes.

Method: Shell model calculations with a minimally renormalized realistic interaction, supplemented by Quasi and Pseudo-SU3 symmetries and Nilsson-SU3 selfconsistent calculations.

Results for $N \leq 64$: Shell model calculations with the neutron effective charge as single free parameter describe well the $B(E2; 2^+_1 \rightarrow 0^+_1)$ and $B(E2; 4^+_1 \rightarrow 2^+_1)$ rates for $N \leq 64$ in the Cd and Sn isotopes. The former exhibit weak permanent deformation corroborating the prediction of a Pseudo-SU3 symmetry, which remains of heuristic value in the latter, where the pairing force erodes the quadrupole dominance. Calculations in $10^7$ and $10^{10}$-dimensional spaces exhibit almost identical patterns: A vindication of the shell model.

Results for $N \geq 64$: Nilsson-SU3 calculations describe $B(E2; 2^+_1 \rightarrow 0^+_1)$ patterns in $^{112–120}$Cd and $^{116–118}$Sn isotopes having sizable quadrupole moment of non-rotational origin denoted as q-vibrations. No calculations are proposed for the heavier species, for which the conventional seniority description is assumed for Sn, while in Cd the quadrupole moments change sign.

Conclusion: A radical reexamination of traditional interpretations in the region has been shown to be necessary, in which quadrupole dominance plays a major role. What emerges is a bumpy but coherent view.

I. INTRODUCTION

All nuclear species are equal, but some are more equal than others. The tin isotopes deserve pride of place, because $Z = 50$ is the most resilient of the magic numbers, because they are very numerous, and many of them stable, starting at $A = 112$. For these, accurate data have been available for a long time. As seen in Fig. 1, a parabola accounts very well for the $B(E2; 2^+_1 \rightarrow 0^+_1)$ trend, except at $^{112–114}$Sn. That these early results (Jonsson et al. [2]) truly signaled a change of regime became evident through work on the unstable isomers, starting with the measure in $^{108}$Sn by Banu et al. [3].

A flurry of activity followed [4–10], from which a new trend emerged in which the parabola—characteristic of a seniority scheme—gives way to a platform, predicted by a Pseudo-SU3 scheme (the squares). Here we are going a bit fast to follow the injunction of Montaigne: start at the end (‘‘Je veux qu’on commence par le dernier point’’ Essais II 10) [11, p. 298]. To slow down, we note that the idea to associate the plateau to Pseudo-SU3, originated in a study of the Cadmium isotopes, where quadrupole dominance is stronger and its consequences more cut. Therefore, it is convenient to study the Cd and Sn families together. Section II provides the necessary tools.

II. THEORETICAL FRAMEWORK

The basic idea is inspired by Elliott’s SU3 scheme [12, 13] and consists in building intrinsic determinantal states that maximize $g_0$, the expectation value of the quadrupole operator $g_0 = 2q_{20}$, i.e., $q_0 = (2q_{20})$ [14–16].

Fig. 2 implements the idea for $^{104}$Cd ($Z = 48$, $N = 56$). The single shell (S) contribution of the $g_{9/2} = g$ proton orbit (Sg) is given by Eq. (1) (with changed sign for hole states). For the neutron orbits, the Pseudo-SU3 scheme [16–18] (P generically, $Pr_p$ for specific cases) amounts to assimilate all the orbits of a major oscillator shell of principal quantum number $p$, except the largest (the $r_p$ set) to orbits in the $p – 1$ major shell. In our case the sdg shell has $p = 4$, and $r_4$ is assimilated to a pf shell. As

![Figure 1](image-url)

FIG. 1. The experimental $B(E2; 2^+_1 \rightarrow 0^+_1)$ for the Sn isotopes from compilations [1], compared with some arbitrary parabolic shape and Pseudo-SU3 results to be explained here (squares).

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the \( q_0 \) operator is diagonal in the oscillator quanta representation, maximum \((2q_{20})\) is obtained by orderly filling states \((n_u, n_y, n_z) = (300), (210), (201), . . . (012), (003),\)
with \( q_0 = (2q_{20}) = 2n_z - n_y - n_x = 6, 3, 3, 0, 0, . . . , -3, -3, \) as in Fig. 2. Using \( q(n) \) for the cumulated \( q_0 \) value (e.g. 24 for \(^{104}\text{Cd}\) in Fig. 2), the intrinsic quadrupole moment then follows as a sum of the single shell (S) and Pseudo-SU3 (P) contributions

\[
q_0(S) = 2(r^2 C_{20}) = \sum_m \left( p + 3/2 \right) j(j + 1) - 3m^2 / 2j(j + 1) \quad (1)
\]

\[
q_0(P) = q(n), \quad Q_0(SP) = \left[ (8e_\pi + q(n) e_\nu, b^2) \right] \text{ efm}^2 \quad (2)
\]

where we have introduced effective charges and recovered dimensions through \( Q = b^2 q \) with

\[
b^2 = \frac{4.1467/\hbar \omega}{\text{fm}^2} \quad \text{and} \quad \hbar \omega = \frac{45A^{-1/3} - 25A^{-2/3}}{A}. \quad (3)
\]

To adapt Eq. (2) to Sn, simply drop the S part (i.e., the \( 8e_\pi \) term).

To qualify as a Bohr Mottelson rotor, \( Q_0(SP) \) must coincide with the intrinsic spectroscopic \( Q_{0s} \) and transition \( Q_{0t} \) quadrupole moments, defined through (as, e.g. in Ref. [16])

\[
Q_{\text{spec}}(J) = <JJ|3z^2 - r^2|JJ>, \quad Q_{0s} = \frac{(J + 1)(2J + 3)}{3K^2 - J(J + 1)} Q_{\text{spec}}(J), \quad K \neq 1 \quad (3)
\]

\[
B(E2, J \rightarrow J - 2) = \frac{5}{16\pi} e^2 |(JK20|J - 2, K)|^2 Q_{0t}^2 \quad (4)
\]

\[
K \neq 1/2, 1, B(E2 : 2^+ \rightarrow 0^+) = Q_{0sp}^2 / 50.3 \text{ e}^2 \text{fm}^4 \quad (5)
\]

III. THE CD ISOTOPES

In Fig. 3 it is seen that \( utM = 000 \) and 101 give the same results provided \( e_\nu \) is properly chosen. There is little difference between \( utM = 111 \) and \( utM = 101 \) because as soon as neutrons are added they block the corresponding jumps, as mentioned above.
The calculation exhibits near perfect agreement with the Alaga rule: \( B(E2; 2^+_1 \rightarrow 2^+_1)/B(E2; 2^+_1 \rightarrow 0^+_1) \approx 1.43 \). In Fig. 3 it is shown for \( \nu = 101 \) but it holds as well for 000 and 111. The more stringent quadrupole quotient rule Eq. (6) yields an average \( Q_f/Q = 0.96 \) for \( ^{106-110} \text{Cd} \), corroborating the existence of a deformed region.

As announced immediately after Eq. (6), Eq. (5) can be checked directly by inspecting Fig. 2, as done in Table I describing the “back of an envelop” SP estimates.

Note that the naive form of P used so far (in \( q(n)_n \) and \( B20sp \)) is supplemented by the more accurate \( q(n)_f \) and \( B20SP \) using fully diagonalized values of \( 2q_{20} \). The remarkable property of the \( r^2_q \) space that produces four identical \( q(n)_f \) values for \( m = 6 - 12 \) has already been put to good use in Ref. [14] and Ref. [15, Fig. 38, TableVII]. In the present case it is seen to do equally well.

It follows that the very simple estimates suggested by Fig. 2 are quantitatively reliable and can be associated to stable deformation in Cd. In Sn, the same estimates will remain reliable but they cannot be associated to stable deformation. A paradox examined in section IV A.

### IV. THE SN ISOTOPES

The basic tenet of this paper is that quadrupole dominance is responsible for the \( B(E2; 2^+_1 \rightarrow 0^+_1) \) patterns in the light Cd and Sn isotopes. Which means that they should exhibit a Pseudo-SU3 symmetry. Hence, we expect the existence of an intrinsic state, implying the validity of the Alaga rule (Eq. (4)). The expectation is fulfilled in Cd (Fig. 3) but it fails in Sn, as seen in Fig. 4.

The \( B(E2; 2^+_1 \rightarrow 0^+_1) \) data from Jonsson et al. [2] for \( ^{112-114} \text{Sn} \), and from Siciliano et al. [29] for \( ^{106-108} \text{Sn} \) exhibit a Pseudo-SU3 validity, and are immune to details. The \( B(E2; 4^+_1 \rightarrow 2^+_1) \) rates are consistent with the single-particle energy displaced by \( \delta = 0.0, 0.4 \) and 0.8 MeV with respect to the GEMO value of 0.8 MeV.

### TABLE I

| \( A \) | \( n \) | \( 2q(n)_n \) | \( 2q(n)_f \) | \( B20_{\exp} \) | \( B20_{\text{SP}} \) |
|--------|--------|-------------|-------------|---------------|---------------|
| 100    | 12     | 14.8        | <560(4)     | 327           | 317           |
| 102    | 18     | 21.6        | 562(46)     | 536           | 511           |
| 104    | 24     | 29.5        | 779(80)     | 799           | 795           |
| 106    | 30     | 30.0        | 814(24)     | 808           | 804           |
| 108    | 36     | 29.6        | 838(28)     | 817           | 817           |
| 110    | 42     | 29.3        | 852(42)     | 824           | 818           |

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the first tentative description of the plateau. The more complete calculations of Togashi et al. [31, Fig. 2] demand $g$ excitations to achieve a satisfactory result, very close to ours in the upper Fig. 4, in spite of huge differences in the $g$ proton occupancies (spin and mass dependent in their case and nearly constant in ours). No $B(E2 : 4^+_1 \rightarrow 2^+_1)$ results are given in these references.

A. The $B(E2 : 4^+_1 \rightarrow 2^+_1)/B(E2 : 2^+_2 \rightarrow 0^+_2)$ anomaly and the pairing-quadrupole interplay

The $B(E2 : 4^+_1 \rightarrow 2^+_1)$ rates are seen to demand special scrutiny. To test how they are influenced by the single particle field, the energy of the $s_{1/2}$ orbit in $^{101}$Sn was displaced by 0.0, 0.4 and 0.8 MeV with respect to the present GEMO choice [21], called DZ (Duflo Zuker) in Ref. [22, Fig. 3.2.1] where an extrapolated value (EX) is given as reference. The position of the $s_{1/2}$ orbit for DZ and EX differ by 800 keV. In the calculations reported in Fig. 4, $I.3.4(0,0)$ and $I.3.4(0.8)$ correspond to DZ and EX respectively. The $B(E2 : 4^+_1 \rightarrow 2^+_1)$ differences are significant. Thanks to the recent $^{108}$Sn $B(E2 : 4^+_1 \rightarrow 2^+_1)$ measure of Siciliano et al. [29, Fig. 4b], the DZ choice is clearly favored.

The $B(E2 : 4^+_1 \rightarrow 2^+_1)/B(E2 : 2^+_2 \rightarrow 0^+_2) < 1$ anomaly had been detected in $^{114}$Xe [32], in $^{114}$Te [33] and more recently in $^{172}$Pt, Ref. [34], where it is stressed that no theoretical explanation is available. Here, the sensitivity to the pairing strength provides a clue. In Fig. 4, its decrease in going from I.3.4 to I.3.2 produces a substantial increase of $B(E2 : 4^+_1 \rightarrow 2^+_1)$. As can be gathered from Ref. [29, Figs. 4a and 4b], $B(E2 : 2^+_2 \rightarrow 0^+_2)$ is totally immune to pairing, while $B(E2 : 4^+_1 \rightarrow 2^+_1)$ is so sensitive that a sufficient decrease in strength could bring $B(E2 : 4^+_1 \rightarrow 2^+_1)/B(E2 : 2^+_2 \rightarrow 0^+_2)$ close to the Alaga rule. It appears that pairing is eroding the deformed band. Only the lowest $J = 0$ and 2 are spared, giving way to a pairing-quadrupole interplay, that will eventually end up in pairing dominance at $N \approx 70$. The transition region will be studied in section VI.

V. THE INTERACTION AND THE MODEL SPACES

There is a consequential result emerging from Fig. 4: the possibility to describe the $B(E2 : 2^+_2 \rightarrow 0^+_2)$ pattern through a neutron-only calculation (the 000 case). This is at variance with previous calculations [3, 6, 30] using the CDB (Charge Dependent Bonn) potential [35], renormalized following Ref. [36]. Which raises two questions: why the I.3.4 interaction succeeds where others fail? and why the neutron-only description is viable? They can be answered simultaneously and I start by explaining how severely truncated spaces may represent the exact results, by comparing the largest calculation available with smaller ones. In Reference [29, Table I], results are given for $^{106-108}$Sn in $utM = 444$ (m-dimensions 10$^{10}$) using the same interaction (called B in what follows) as in Banu et al. [3] (but omitting the $h_{11/2}$ orbit). In Fig. 5 it is shown as B444 (circles) and compared with B202 (squares, the same interaction in our standard space). The agreement is very good for the two points in $B(E2 : 2^+_2 \rightarrow 0^+_2)$ and $B(E2 : 4^+_1 \rightarrow 2^+_1)$. The result amounts to a splendid vindication of the shell model viewed as the possibility to describe in a small space the behavior of a large one. Although in general the reduction from large to small spaces demands renormalization of the operators involved, for our purpose only the effective charges are affected. A non trivial fact that invites further study.

For much of the region, discrepancies between I.3.4 and B can be traced to poor monopole behavior of the latter. If the interaction is made monopole free and supplemented by the GEMO single-particle field used in the I.p.q forces, the resulting BG202 in Fig. 5 produces $B(E2 : 2^+_2 \rightarrow 0^+_2)$ patterns identical to the ones for I.3.4-202, while for $B(E2 : 4^+_1 \rightarrow 2^+_1)/B(E2 : 2^+_2 \rightarrow 0^+_2)$ the pattern is close to I.3.0 (which is not shown, but can be guessed by extrapolation in Fig. 4 and from the analysis in Ref. [23] revealing the same $q \cdot q$ content in I.3.4 and B, and a much weaker pairing for the latter; so weak in fact, that the B results come close to the Alaga rule.

It follows that for I.3.0, say, the $Pr_q$ symmetry will
hold, at least partially. As the pairing force is switched on, the $J = 0_1, 2_1$ states are not affected, while $J = 4_1$ is. Which points to an unusual form of interplay between the two coupling schemes—pairing and quadrupole—traditionally associated to collectivity. In single fluid species, such as Sn, the seniority scheme can operate fully. It breaks down in the presence of two kinds of particles, which turns out to be the condition for quadrupole to operate successfully, as indicated by the Cd isotopes. What is unusual is the presence of quadrupole coherence in the light tins. It is shaky and challenged by pairing and we know that at about $A = 120$ the seniority scheme will prevail. For the transition nuclei $^{116–118}$Sn, my original guess was that a mixing of spherical and weakly oblate states would take place. I also expected that neutron-only calculations—in which the monopole field would play a crucial role—would be likely to shed light on these matters. The guess was totally wrong, as explained in section VI C. I mention this anecdote to stress that these two nuclei were the hardest to understand in the region.

Throughout this study, energy spectra have been ignored in favor of electromagnetic rates which are less sensitive to details, so I close by showing some spectra. In Fig. 6, some pairing state dependence has been allowed, but I.3.4 is seen to be the right overall choice.

VI. CD AND SN AT $N \geq 64$

In Table II the naive $Pr_4$ adimensional intrinsic quadrupole moments for prolate ($q_{0p}$) and oblate ($q_{0o}$) are compared. The former are the same as $q(n)_n$ in Table I. The latter are obtained by filling the platforms in reverse order (from the top). Up to $N = 56$ prolate dominates. From $N = 58$ to 62 there is oblate-prolate degeneracy. At $N = 64$, oblate dominates. In the absence of strong quadrupole dominance, these intrinsic values only indicate a trend in sign, respected by the calculated spectroscopic moments that opt for “oblate” shapes for $A > 108$. For $^{112–114}$Sn the shell model results are close—for the quadrupole moments—or agree—for the magnetic moments—with the measured values. Note: The magnetic moments are very sensitive to the anomalous $g_{lv}$.

### Table II. Intrinsic adimensional $q_0$ for prolate ($q_{0p}$), and oblate ($-q_{0o}$) states. Calculated spectroscopic Quadrupole moments and $g$-factor. $Q_2$, $Q_4$, $g$ for I.3.4 ($\epsilon_r$, $\epsilon_n$) = 0.72, 1.40; $g_{lv}$ = 0.75 with respect to bare values [15, Fig. 28].

| $N$ | $q_{0p}$ | $q_{0o}$ | $Q_2$ | $Q_4$ | $Q^{*}_2$ | $g^{*}$ | $g$ |
|-----|----------|----------|--------|--------|-----------|--------|-----|
| 52  | 12       | -16      | -24    |        |           |        | -0.157 |
| 54  | 18       | -21      | -21    |        |           |        | 0.012 |
| 56  | 24       | -16      | -17    |        |           |        | 0.103 |
| 58  | 24       | -5       | -2     |        |           |        | 0.142 |
| 60  | 24       | 3        | 10     |        |           |        | 0.142 |
| 62  | 24       | 14       | 26     | 4(9)   | 0.150(43) | 0.135 |
| 64  | 18       | 24       | 25     | 43     | 9(8)      | 0.138(63) | 0.106 |

By suggesting a very different behavior for the Sn and Cd families at $N=64$, that will be examined in what follows, Table II illustrates the heuristic value of relying on Pseudo-SU3, even when the symmetry does not hold in the strict sense.

So far the $sdg$ space has proven sufficient, as the effects of the $h_{11/2}$ orbit ($h$ for short) remain perturbative. For Sn we know from classic $(p,d)$ work [39], that the $h$ occupancy—very small up to $^{110}$Sn—increases at $^{112–114}$Sn, as borne out by calculations that indicate the need of a boost of some 10% in $B(E2 : 2^+_1 \rightarrow 0^+_1)$ [40] with respect to Fig. 4. Beyond $N = 64$, the explicit inclusion of the $h$ orbit becomes imperative but the situation is different for the two families. In Sn, we know that the traditional $hr_4$ space will eventually prove sufficient when the seniority scheme takes over at $N = 70$. For $^{116–118}$Sn, at this stage, nothing can be said. For Cadmium the calculations give systematically oblate values in line with Stone’s Q tables [41], but in $^{112}$Cd (excluded from both Table I and Fig. 3) they yield severe underestimates whose correction necessitates the introduction of a Quasi-SU3 mechanism (referred generically as Q in what follows, and Q$vh$ for the case I introduce next). It is illustrated in Fig. 7, where it is seen that at $N = 64$, promoting an extra particle to the $g_0 = -3$ platform reduces quadrupole coherence in Cd, while filling the seven upper platforms makes it possible for $^{114}$Sn to stay oblate. To obtain realistic estimates demands estimating the quadrupole moment in the presence of a central field. An economic way of doing so is through Nilsson-SU3 self-consistency [16], which is explained next.
Let us start by remembering that \( q_0 = (2q_{20}) \), decompose it, together with the corresponding operator \( \hat{q}_0 \), into the S, P and Q contributions, and introduce the normalized variant \( q_{0N} \). Then examine [16, Eq.(19)]

\[
\hat{q}_{0N} = \frac{\hat{q}_{PS}}{N_4} + \frac{\hat{q}_{PQ}}{N_5}, q_{0N} = \frac{q_{PS}}{N_4} + \frac{q_{PQ}}{N_5}
\]

\[H = H_{sp} - \frac{\hbar \omega}{3} \hat{q}_0 \equiv H_{sp} - \beta \hbar \omega \kappa \hat{q}_{0N} q_{0N}
\]

\[N^2 = \sum (2q_{20r})^2 = \sum \frac{p}{k=0} (k+1)(2p-3k)^2.
\]

(p is the principal quantum number). Eq. (8) compares the classic Nilsson problem to the left and the selfconsistent version to the right, which demands the solution of a linearized \( k \hbar \omega \hat{q}_0 \cdot \hat{q}_0 \cdot N^2 \) problem, taken to approximate Elliott’s quadrupole force, in its correct realistic normalized form, which involves the inclusion of the norm in Eq. (9), as demonstrated in Ref. [23]. The coupling constant \( \kappa = 3 \) is the same as in interactions L3.x, while \( H_{sp} \) is taken from GEMO [21]. The quantity we are after, \( q_0 \), is calculated while in the Nilsson case it is simply the parameter \( \delta \).

The selfconsistent solution of the problem is obtained by demanding that that input and output \( q_{0N} \) coincide. Calculations are done for each space separately. To ensure that the couplings involve the full \( q_0 \), a parameter \( \beta_X \) is introduced:

\[
\frac{\hat{q}_{VP}}{N_4^2} (q_{PS} + q_{VP}) \rightarrow \beta_X \frac{\hat{q}_{VP}}{N_4} q_{VP}
\]

\[
\frac{\hat{q}_{VQ}}{N_5^2} (q_{PS} + q_{VP}) \rightarrow \beta_Q \frac{\hat{q}_{VQ}}{N_5} q_{VQ}
\]

At each iteration a full spectrum of Nilsson-like energies \( \varepsilon(2k, i) \) is generated, from which quadrupole contributions \( q_0(2k, i) \) are extracted by subtracting the \( H_{sp} \) part. The full \( q_0 \) is the sum of all such contributions for a given \( A \), in the case of \(^{110}\text{Cd}\) in Fig. 7 it involves the six filled P-platforms. According to Eq. (10) this is the quantity to be extracted selfconsistently. However, it turns out that the results are little changed if it is replaced by the single lowest contribution. In other words, in \(^{110}\text{Cd}\) \( \beta_P \) may range from 1.3 for the full \( q_0 = q_{VP} \) to 2.8 for \( q_0(1, 1) \), for nearly identical final results \( q_0 = q_{VP} = 13.0(3) \), or 26.0(6) (for pair occupancy) to be compared with 24 and 29.3, the values in Table I in the absence of monopole field. Hence: the elementary SP arguments in Table I, the diagonalizations in Fig. 3, and the present selfconsistent results nearly coincide. A pleasant result.

At \( N = 64 \) and beyond, Eq. (11) applies. Since \( N^2 = 210 \) and 420 for \( p = 4 \) and 5 respectively, \( (q_{PS} + q_{VP})N_4 / N_4 = (4 + 13)\sqrt{2} \approx 24 \). The selfconsistent calculations in Tab. III are done using \( q_0 = (q_{01}, q_0(3, 1)) \) input-output values and \( \beta_q = 4 \), consistent with \( q_{0Q} = 8.06 \) in the table, as \((24 + 8)/8 = 4\).

In results so far, involving P and Q spaces (References [14] for the rare earth, [16] for \( N = Z \) nuclei and for \( N < 64 \) in the present study), the influence of \( H_{sp} \) is relatively minor, and \( B(E2 : 2^+_1 \rightarrow 0^+_1) \) rates remain close to their theoretical maxima represented by the \( q_0 \) diagrams. For the Q case, described in Tab. III, \( H_{sp} \) plays a major role, and I have chosen to present together the results for the strict \( Qhfp \) and selfconsistent cases \( (Qf \) and SC in what follows).

Under \( \phi^2_h \) I have listed the squared amplitude of the \( h = h_{11/2} \) components of the wave functions. They are on the average of about 56% for \( Qf \) and 98% for SC, leading to different filling6 patterns.

The filling patterns \((2ki)\) as a function of \( A \) are those of Fig. 7 for \( Qf \), but are dictated by the energies \( \varepsilon(2k, i) \)MeV for SC, in which case the \( q_0(2k, i) \) values are not necessarily the largest possible.

In particular, \( q_0(1, 2) \approx 5 \) is the largest, but it has a huge energy \( \varepsilon(1, 2) = 2.82 \) MeV, and \( \phi^2_h \approx 0 \). The (12) orbit will play two roles in what follows: as a purveyor of intruders and as signaling a transition between two deformed regimes.

The \( B(E2 : A) = B(E2 : 2^+_1 \rightarrow 0^+_1) \) rates, with \( e_{\nu}(A) \) from Tab. III, are calculated through

\[
B(E2 : 2 \rightarrow 0) = [2(q_{PS} e_{\nu} + (q_{VP} + q_{VQ})e_{\nu})\beta_X^2 / 50.3,
\]

Fig. 7. The \(^{110}\text{Cd} \) intrinsic state in the SPQ space. The schematic Pseudo-SU3 platforms \( Pr_{ks} \) are obtained by diagonalizing the \( \hat{q}_0 \) operator in the \( p = 3 \) space (full lines), while dashed lines correspond to the full \( q_0 \) in the \( rs \) space \((Pr_{rs})\). The Quasi-SU3 \((Qhfp)\) \( q_0 \) platforms (full lines) are obtained by diagonalizing the quasi-quadrupole operator in the \( hfp \) space i.e. the degenerate \( \Delta J = 2 \) sequence in \( pf/h \) shell: \( h_{11/2}, f_{7/2}, p_{3/2} \). Dashed lines \((Qhfp_{rs})\) are for the full quadrupole operator. The quasi-quadrupole operator is obtained by using the \( l \cdot s \) form of the \( \hat{q} \) matrix elements and then replacing \( l \) by \( j = l + 1/2 \) [16, Section IIIC].

A. Nilsson-SU3 selfconsistency in a SPQ context

\[
\hat{q}_{0N} = \frac{\hat{q}_{PS}}{N_4} + \frac{\hat{q}_{PQ}}{N_5}, q_{0N} = \frac{q_{PS}}{N_4} + \frac{q_{PQ}}{N_5}
\]

\[
H = H_{sp} - \frac{\hbar \omega}{3} \hat{q}_0 = H_{sp} - \beta \hbar \omega \kappa \hat{q}_{0N} q_{0N}
\]

\[
N^2 = \sum (2q_{20r})^2 = \sum (k+1)(2p-3k)^2.
\]
where \( q=r \), \( e_p = 1.4 \), and \( q_{p, p} = 14 \), rather than 13, calculated earlier, to account for rearrangements in the P space when Q pairs are added. The resulting agreement with data in Fig. 8 is satisfactory. To explain the \( e_\nu(A) \) choices, I start recalling that they are associated to 2h\( \omega \) and an 0h\( \omega \) contributions, described in the paragraphs preceding section III. Up to \( N = 64 \), the 0h\( \omega \) part is mediated by quadrupole jumps coupling \( r_4 \) neutrons to proton particle-hole excitations from the \( g \) orbit. For \( N \geq 64 \) the couplings are increasingly mediated by \( hfp \), i.e., \( p = 5 \) neutrons, leading to a suppression of quadrupole strength, due to the \( N_5/N_4 \) norm effect we have encountered earlier. As a consequence \( e_\nu \) is expected to decrease gradually as the \( r_4 \) orbits fill and reduce their contribution. The alternative is a constant, plausible for \( N > 64 \) but not at \( N = 64 \), which definitely demands a larger \( e_\nu \). Once the gradual decrease is accepted, the choice in Tab III is quite constrained and the good agreement in Fig. 8 follows naturally. The table and Eq.(12) contain all that is needed to explore alternatives, but they will hardly change the quality of the agreement. Independently of details, there are two indications that the calculations are on track: the drop at \( ^{120}\text{Cd} \), \( N = 72 \), and the strong underestimate at \( ^{122}\text{Cd} \) which signals the transition to oblate states detected through Stone’s Q tables [41]. Both indications are probably correlated and invite further study.

TABLE III. Quantities entering schematic and selfconsistent calculations for \(^{112-112}\text{Cd} \) (\( N = 64 - 74 \)). Single particle spectrum at \(^{114}\text{Sn} \) from GEMO [21]. \( \epsilon(h_{11/2}) = 0, \epsilon(f_{7/2}) = 3.6, \epsilon(p_{3/2}) = 5.4 \) MeV. See text for detailed explanations.

| \( q_{p, p} \) | \( q_0(2k, i) \) | \( q_{Q_{hfp}}(N) \) | \( q_{Q_{hfp}}(A) \) | \( q_{Q_{hfp}}(A) \) | \( B(E2, A) \) |
|-------------|----------------|----------------|----------------|----------------|----------------|
| 2k\( i \)  | 11 | 31 | 51 | 12 | 71 | 32 |
| 2k\( i \)  | 0.21 | 0.43 | 0.69 | 0.61 | 0.89 | 0.51 |
| 2k\( i \)  | 8.55 | 6.06 | 3.28 | 2.96 | 0.50 | 0.50 |
| 2k\( i \)  | 64 | 66 | 68 | 70 | 72 | 74 |
| 2k\( i \)  | 8.55 | 14.61 | 17.89 | 20.85 | 21.35 | 21.65 |
| 2k\( i \)  | 0.75 | 0.60 | 0.55 | 0.50 | 0.50 | 0.50 |
| 2k\( i \)  | 1016 | 1049 | 1095 | 1095 | 1130 | 1165 |

Self Consistent (SC)

\[ B(E2, A) \] values for \( \text{Cd} \) and \( \text{Sn} \), and the schematic and selfconsistent estimates for \( \text{Cd} \) at \( N = 64 \) and beyond. Note that the \( N \) values are boosted by a 1.9 factor.

FIG. 8. The observed \( B(E2 : 2^+ \rightarrow 0^+) \) values for \( \text{Cd} \) and \( \text{Sn} \), and the schematic and selfconsistent estimates for \( \text{Cd} \) at \( N = 64 \) and beyond. Note that the \( N \) values are boosted by a 1.9 factor.

B. CD PROSPECTS: VIBRATION, INTRUDERS, COEXISTENCE

Arguably, the most striking feature of the selfconsistent calculation is the overwhelming dominance of the \( h = h_{11/2} \) orbit, which makes it impossible to speak of a Quasi-SU3 symmetry, though one works in a Q space. The results of calculations in the full \( r_5 \) space, using \( H_{sp} \) from GEMO [21], are nearly identical to those in the Q space, which is vindicated as the correct choice. The transition between th Q and SC regimes can be followed through variations of the \( H_{sp} \) splittings. At \( \epsilon(h_{11/2}) = 0, \epsilon(f_{7/2}) = 2.0, \epsilon(p_{3/2}) = 3.0 \) MeV, \( \phi_h^2 = 0.95 \) and the SC filling pattern remains unchanged, but with a further reduction: \( \epsilon(h_{11/2}) = 0, \epsilon(f_{7/2}) = 1.0, \epsilon(p_{3/2}) = 2.0 \) MeV, \( \phi_h^2 = 0.66 \) the patterns change to Q\( f \), as the 2\( k_i = 12 \) orbit fills at \( N = 72 \): there is a change in regime from SC to Q\( f \). As to what is SC in Tab III, an answer is suggested in the tables [28]: the nuclei must be vibrational since \( B(E2 : 4^+_1 \rightarrow 2^+_1)/B(E2 : 2^+_1 \rightarrow 0^+_1) = 2 \). The problem is that vibrational nuclei are not supposed to be deformed, as pointed out by Tamura and Udagawa [42] after the observation of a large static quadrupole moment in \( ^{114}\text{Cd} \). After half a century, the question remains open [43, 44], though attempts have been made to modify the vibrational model so as to make it viable [45]. As of now I adopt a Gordan knot solution and call such states q-vibrational. The precise definition will be given in section VII D.

In introducing the SPQ spaces, I expected \(^{112-116}\text{Cd} \) to behave as weakly deformed states in analogy to their lighter counterparts, but something different is happening, as made clear in the calculations. In [43, Fig. 5], the contrast is clear: \(^{110}\text{Cd} \) follows the Alaga rule, while \( B(E2 : 4^+_1 \rightarrow 2^+_1)/B(E2 : 2^+_1 \rightarrow 0^+_1) = 2 \) for \(^{112-116}\text{Cd} \).

Recent BMF (Beyond Mean Field) calculations in \(^{110,112}\text{Cd} \) [46] are fairly successful at describing the numerous coexisting states. Referring to Fig. 7, it is easy to visualize how such states could be produced by promoting \( q_0 = 0 \) P pairs to the Q\( f \) space. Thus, promoting the 2\( k_i = 12 \) pair with \( q_0(1, 2) \approx 5 \) on top of \(^{112-114}\text{Cd} \) ground states leads to \( q_{Q_{hfp}} \approx 9 \) and 13 respectively, using...
$e_{\nu}$ from Tab. III, and then $B(E2 : 2^+_3 \rightarrow 0^+_2) = 47$ and 48 W.u. respectively, against the observed 51(12) and 65(9) W.u. The case of $^{110}$Cd does not demand calculations but a check: its $B(E2 : 2^+_3 \rightarrow 0^+_2) = 29(5)$ W.u. should be the same as $B(E2 : 2^+_5 \rightarrow 0^+_3) = 30.3(2)$ in $^{112}$Cd. They are. Experimental values from [28, 43, 46].

The calculations and estimates so far, are useful in describing $B(E2 : 2^+_5 \rightarrow 0^+_3)$ trends but the determinant challenge comes from the $B(E2 : 4^+_1 \rightarrow 2^+_1)/B(E2 : 2^+_1 \rightarrow 0^+_1)$ ratios. In the case of the light Cd isotopes, it was met by shell model calculations, but the irruption of the Q space puts them beyond reach for $N \geq 64$. The BMF spectra are of little help, as they yield ratios too large for Alaga in $^{110}$Cd (1.66), and below the vibrational limit in $^{112}$Cd (1.71) [46]. So, I propose to try something different.

1. Band Coupling

The idea is to prediagonalize the Hamiltonian in the SP and Q subspaces and couple the resulting bands to form a basis

$$J_{SP} \otimes J_Q.$$ 

In all probability, something similar has been proposed in the past, but I know of no successful implementation, probably because of the difficulty of defining the correct interaction to be used in the individual spaces, as can be understood by concentrating on the quadrupole force. The naive view is that, if we use $kq\cdot \hat{q}$ in the full space, we should use the same in each of the subspaces. What we have learned is that the correct choice is to change $k \rightarrow \beta k$. Though no calculations in the coupled basis will be attempted here, some runs were made in the $Q$ or SQ spaces, $hf_p^{2-10}$ or $(hf)^{(Q)}$ ($h_{10}$ for simplicity) with a large quadrupole force and with the realistic I interaction with the same quadrupole strength. The results with both interactions are very much coincided with those of the SC calculations for $B(E2 : 2^+_1 \rightarrow 0^+_1)$. Nothing close to $B(E2 : 4^+_1 \rightarrow 2^+_1)/B(E2 : 2^+_1 \rightarrow 0^+_1) \approx 2$ emerged. It is to be hoped that it will happen once the coupling to the SP space is implemented. Still, the rudimentary calculations confirmed that the quadrupole force determines the coupling scheme, and that, even with very large $H_{SP}$ splittings, and overwhelming $h_{11/2}$ dominance, prolate solutions prevailed, even for two particles. I propose to call such states $h^2$-prolate. They will prove important in what follows.

C. Sn prospects in $A=116$ and 118.

Following Montaigne again, I start with a spoiler: $^{116-118}$Sn are most probably q-vibrational nuclei. In spite of the spoiler, the story is of interest. At $N = 64$, the interaction favors oblate in Tab. II (consistent with data in Allmond et al.[38]), and the calculations do well, (Fig. 3). For Cd there is a change in regime marked by a jump in Fig. 8. For Sn, there is a smooth inflexion point, inviting the idea of a smooth transition. The natural assumption of an oblate $^{116}$Sn, obtained by adding an $h_{11/2}^2$ pair does not work. For two reasons. The first is that $B(E2 : 2^+_1 \rightarrow 0^+_1)$ of about 700-800 e$^2$fm$^4$ or 20-24 W.u. (adapting Eq. (12), eliminating protons an replacing $g_{qh}^{-}$ by $g_{qh}^{-} = -5$), double the observed value. The second is that $^{116}$Sn is prolate [41]. The way out is to use the prolate solution in Tab. II, i.e., $g_{QP} = 18$ rather than -24, and $h^2$-prolate i.e., $g_{Q} = 4.44$ from Tab III in Eq. (12) to obtain 12.1 W.u. for $e_{\nu} = 1.05$, against the observed 12.4(4) W.u.

All this may seem far fetched, but the corroborating evidence is strong: as $B(E2 : 4^+_1 \rightarrow 2^+_1) = 38(24)$ W.u., the $B(E2 : 4^+_1 \rightarrow 2^+_1)/B(E2 : 2^+_1 \rightarrow 0^+_1)$ quotient becomes vibrational-compatible. For $^{118}$Sn, the situation is similar.

D. q-vibrations

The total coincidence between Cd and Sn $B(E2 : 2^+_1 \rightarrow 0^+_1)$ patterns in Fig. 8 at $N < 64$ was challenged by the $B(E2 : 4^+_1 \rightarrow 2^+_1)/B(E2 : 2^+_1 \rightarrow 0^+_1)$ quotients, which ironically establish their similarity at $N \geq 64$ were they diverge abruptly. I have used the term q-vibrational for the latter region. It applies to states that fulfill two conditions: a) $E_2/E_2$ and $B(E2 : 4^+_1 \rightarrow 2^+_1)/B(E2 : 2^+_1 \rightarrow 0^+_1) \approx 2$, and b) sizable quadrupole moment of non-rotational origin, as calculated in the previous sections VI B and VI C. Condition b ensures a situation similar to that encountered at the beginning of this study: the encouraging SP suggestions for $B(E2 : 2^+_1 \rightarrow 0^+_1)$ were validated by further shell model work, still missing here, but expected to work equally well.

It could be objected that postulating q-vibrations is a blunt step that relies too heavily on data. Certainly, but to explain why $B(E2 : 4^+_1 \rightarrow 2^+_1)/B(E2 : 2^+_1 \rightarrow 0^+_1)$ is smaller than one in $^{114}$Sn and about two in $^{116}$Sn one has to be a bit blunt. And remember that the speculations rest on credible $B(E2 : 2^+_1 \rightarrow 0^+_1)$ estimates.

VII. CONCLUSIONS

In this paper calculations have oscillated between full rigor, heuristics and semi quantitative estimates, regions have moved from well developed deformed, to pairing-quadrupole coexistence, to the newly postulated q-vibrational. Cadmium and Tin come and go. “The world is but a perennial swing” (Le monde n’est qu’une branloire perenne. Essais III 2 [11, p. 584]).
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