On flavor mixing by an effective Light-Cone QCD-Hamiltonian

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Abstract

The $\uparrow \downarrow$-model has produced the physical masses of all flavor-asymmetric mesons like the $\pi^{\pm}$ or the $\rho^{\pm}$. Can the same model also account for the flavor-symmetric mesons like the $\pi^0$ or the $\rho^0$? — By adjusting one parameter in a $5 \times 5$-matrix, the $\pi$-$\eta$ degeneracy is lifted and the $\eta$–$\eta'$ splitting falls into the right ball park. The isospin triplets for pions and rhos are caused the mass-degeneracy of the up and down quark.

1 Introduction

In the fundamental hadronic theory, the gauge theory of quantum chromodynamics (QCD), isospin symmetry is not manifested in an obvious way. As part of an ongoing work on the light-cone approach, the present note contributes another facet to this problem [1].

As reviewed in [2], the light-cone approach is to diagonalize the light-cone Hamiltonian, $H_{LC}|\Psi\rangle = M^2|\Psi\rangle$, and to calculate the spectra and invariant mass-squared of physical particles. In particular the method addresses calculation of the associated wave functions $\Psi_n = \Psi_{q\bar{q}}, \Psi_{qg}, \ldots$, which are the Fock-space projections of the hadrons eigenstate. The total wave function for a meson is then $|\Psi_{\text{meson}}\rangle = \sum_i (\Psi_{q\bar{q}}(x_i, k_{\perp}, \lambda_i)|q\bar{q}\rangle + \Psi_{qg}(x_i, k_{\perp}, \lambda_i)|q\bar{q}g\rangle + \ldots)$. The problem has been solved thus far only for 1-space dimension by the method of Discretized Light-Cone Quantization, see [2], a method which in principle can be applied also to the physical 3-space and 1-time dimensions.

For 3-space dimensions, however, it is technically easier to resort to an effective interaction. One can reduce the full light-cone Hamiltonian with its complicated many-body aspects by the method of iterated resolvents [3] to an effective light-cone Hamiltonian $H_{\text{eff}}$, which yields the same eigenvalues and
which by definition acts only in the Fock space of a quark and an anti-quark

\[
H_{\text{eff}} |\Psi_{q\bar{q}}\rangle = M^2 |\Psi_{q\bar{q}}\rangle, \tag{1}
\]

with the effective Hamiltonian

\[
H_{\text{eff}} = T + U_{\text{OGE}} + U_{\text{TGA}} = T + V G_3 V + V G_3 V G_2 V G_3 V. \tag{2}
\]

The kinetic energy \( T \) is as usual a diagonal operator in momentum space while the effective interaction kernel \( U = U_{\text{OGE}} + U_{\text{TGA}} \) is off-diagonal. It has two contributions, an effective one-gluon-exchange interaction \( U_{\text{OGE}} = V G_3 V \) and an effective two-gluon-annihilation interaction \( U_{\text{TGA}} = V G_3 V G_2 V G_3 V \). The action of these two interactions is conceptually different, as illustrated in Fig. 1. In the upper left diagram, the vertex interaction \( V \) creates a virtual gluon which propagates with the two quarks in the \( q\bar{q}g \)-space by means of the propagator \( G_3(\omega) = G_{q\bar{q}g}(\omega) \), before it is annihilated by another action of \( V \). The momentum transfer causes a genuine \( q\bar{q} \)-interaction. If it is absorbed however on the same line by the quark, as illustrated by the diagram in the upper right of the figure, the momentum is unchanged, hence there is no interaction but a contribution to the effective quark mass. The vertex interaction \( V \) is known and tabulated in [2], as part of the total light-cone Hamiltonian \( H_{\text{LC}} \). A priori unknown is the propagator \( G_3(\omega) \), and much of the compact, and perhaps confusing, discussion in [3] deals with this question. A broader presentation is now available in [4]. There, it is also shown that the higher Fock-space wave functions like \( |\Psi_n\rangle = \Psi_{q\bar{q}} \ldots \), can be found by quadratures like \( |\Psi_n\rangle = GV \ldots GV |\Psi_{q\bar{q}}\rangle \), \( i.e. \) as functionals of \( \Psi_{q\bar{q}} \) without that an other eigenvalue problem has to be solved.

The second part of the interaction is even more complicated, as illustrated in
the lower diagram of Fig. 1. There, the second hit of \( V \) does not absorb the gluon but creates a second one. Together they propagate in the \( gg \)-space by means of the propagator \( G_2(\omega) = G_{gg}(\omega). \) Finally, a third and fourth action of \( V \) brings the system back to the \( q\bar{q} \)-space.— The graphs in the figure illustrate the difference of the two interactions: \( U_{OGE} \) conserves the flavor of each quark individually, as opposed to \( U_{TGA} \) which can act only if the quark and anti-quark have the same flavor and which scatters a pair \( f\bar{f} \) into the same or another pair \( f'\bar{f}' \).

The question I ask in this short note is: Does the excellent agreement between theory and experiment to be reported in section 2 gets spoiled when I include the flavor-changing interaction \( U_{TGA} \)?

For this purpose I summarize the main results for the flavor-conserving interaction \( U_{OGE} \) in section 2 and formulate a very simple model for including the flavor-changing interaction \( U_{TGA} \) in section 3. This will allow me to present numerical results in section 4, which are based on the numerical diagonalization of a \( 5 \times 5 \)-matrix. To understand these results analytically, I formulate a schematic model in section 4. I draw the conclusions in section 5.

### 2 The flavor-off-diagonal mesons

If one restricts to mesons whose valence quarks and anti-quarks have different flavor, one can solve the simpler equation

\[
H_{OGE} \left| \Psi_{f\bar{f}'} \right\rangle = (T + U_{OGE}) \left| \Psi_{f\bar{f}'} \right\rangle = M_{f\bar{f}'}^2 \left| \Psi_{f\bar{f}'} \right\rangle,
\]

(3)

to obtain physical masses \( M_{f\bar{f}'} \) and the associated wave functions \( \Psi_{f\bar{f}'} \). I refer to these mesons as flavor-off-diagonal.

The solution of this equation is far from trivial. One faces all the difficult questions of gauge field theory, among them introducing cut-offs and their removal by renormalization. By reducing the difficulties with the simple \( \uparrow\downarrow \)-model [5], it was possible to generate solutions with a rather modest effort. I see no point in quoting many details since they will not be needed here, except the fact that the model has the 7+1 canonical parameters of a gauge field theory: the strong coupling constant \( \alpha_s \), the six flavor quark masses \( m_f \), plus the one scale parameter \( \mu \), which arises due to renormalization. By adjusting them, all flavor-off-diagonal meson-masses are reproduced, see Tables 1 and 2, with the experimental data taken from [6].

Since the top quark is omitted here for simplicity, I deal with \( n_f = 5 \) flavors. Since the up and the down quark mass is put equal as usual, \( i.e. m_u = m_d = \)
Table 1
The calculated mass eigenvalues in MeV. Those for singlet-1s states are given in the lower, those for singlet-2s states in the upper triangle. Taken from [5].

|     | $\bar{u}$ | $\bar{d}$ | $\bar{s}$ | $\bar{c}$ | $\bar{b}$ |
|-----|-----------|-----------|-----------|-----------|-----------|
| $u$ | 768       | 871       | 2030      | 5418      |           |
| $d$ | 140       | 871       | 2030      | 5418      |           |
| $s$ | 494       | 494       | 2124      | 5510      |           |
| $c$ | 1865      | 1865      | 1929      | 6580      |           |
| $b$ | 5279      | 5279      | 5338      | 6114      |           |

Table 2
The empirical masses of the flavor-off-diagonal physical mesons in MeV [6]. The vector mesons are given in the upper, the scalar mesons in the lower triangle.

|     | $\bar{u}$ | $\bar{d}$ | $\bar{s}$ | $\bar{c}$ | $\bar{b}$ |
|-----|-----------|-----------|-----------|-----------|-----------|
| $u$ | 768       | 892       | 2007      | 5325      |           |
| $d$ | 140       | 896       | 2010      | 5325      |           |
| $s$ | 494       | 498       | 2110      | —         |           |
| $c$ | 1865      | 1869      | 1969      | —         |           |
| $b$ | 5278      | 5279      | 5375      | —         |           |

406 MeV, the model has 5 adjustable parameters which are determined by the underlined data in the table. There remain 12 data whose agreement with the available experimental values is remarkable, to say the least.

3 A crude model for flavor-diagonal mesons

When dealing with the flavor-diagonal mesons one has to include the effective two-gluon-annihilation interaction $U_{TGA} = V G_3 V G_2 V G_3 V$, but the explicit calculation of this operator is cumbersome and far from being trivial. I try to get around this work as long as possible, and for convenience define the ground-state-to-ground-state correlations

$$a_{f f'} \equiv \langle \Psi_f | U_{TGA} | \Psi_{f'} \rangle. \tag{4}$$

The matrix $a_{f f'}$ is symmetric and depends on the wave functions $\Psi$, which are solutions to Eq.(3). Whatever the structure of $V G_3 V G_2 V G_3 V$, the $a_{f f'}$ must obey

$$a_{uu} = a_{dd} = a_{ud} = a_{dd} \equiv a, \quad \text{and} \quad M_{d \bar{d}} = M_{u \bar{u}}, \tag{5}$$

since the up and the down quarks have equal masses.

Suppose for a moment that the would-be-$\pi^0$ is a pure $u \bar{u}$, and the would-be-$\eta$ a pure $d \bar{d}$-state. One could calculate the impact of $U_{TGA}$ by first order perturbation theory with the result that the $\pi^0$ and the $\pi^\pm$ can have very different masses for large $a$ and that the $\pi^0$ and $\eta$ are degenerate, i.e.

$$M_{\pi^0}^2 = M_{\pi^\pm}^2 + a, \quad \text{and} \quad M_{\pi^0} = M_{\eta}, \tag{6}$$
The kernel of the effective Hamiltonian is displayed as a block matrix to illustrate the flavor mixing in QCD. Diagonal blocks are \( D_i \equiv E_i + A_i \).

\[
\begin{array}{ccccccc}
\bar{u}u & \bar{d}d & \bar{s}s & \bar{c}c & \bar{b}b \\
D_1 & A_2 & A_4 & A_7 & A_{11} \\
D_3 & A_5 & A_8 & A_{12} \\
D_6 & A_9 & A_{13} \\
D_{10} & A_{14} \\
D_{15} & & & & & & \end{array}
\]

The wave function of physical neutral pseudo-scalar mesons in terms of the \( q\bar{q} \)-wave functions. The leading component is normalized to 10.

\[
\pi^0 \quad \eta \quad \eta' \quad \eta_c \quad \eta_b
\]

\[
\begin{array}{ccccccc}
\bar{u}u & u & 10.000 & -9.313 & 5.360 & 0.310 & 0.031 \\
\bar{d}d & dd & -10.000 & -9.313 & 5.360 & 0.310 & 0.031 \\
\bar{s}s & ss & -0.000 & 10.000 & 10.000 & 0.326 & 0.031 \\
\bar{c}c & cc & -0.000 & 0.251 & -0.658 & 10.000 & 0.034 \\
\bar{b}b & bb & 0.000 & 0.025 & -0.061 & -0.037 & 10.000 \\
\end{array}
\]

respectively. The first answer to the question above must be therefore: The model does not work for flavor-diagonal mesons.

Upon second thought one realizes that the flavor-changing interaction causes a flavor-mixing as illustrated in Table 3. The matrix shown in this table displays the kernel of the effective Hamiltonian as a matrix of block matrices [2]. Say, the effective Hamiltonian \( H_{\text{OGE}} \) contributes \( E_i \) to the diagonal blocks. The flavor-changing interaction \( U_{\text{TGA}} \) contributes \( A_i \), thus every diagonal block is \( D_i = E_i + A_i \). But \( U_{\text{TGA}} \) contributes also to the sector-dependend off-diagonal blocks which depend on the flavor masses.

The diagonalization of \( H_{\text{OGE}} \) and the generation of \( \Psi \) can be understood as a unitary transformation to pre-diagonalize the flavor-mixing matrix. Although

\[
\langle \Psi_{f'f}|U_{\text{TGA}}|\Psi_{f'f} \rangle = 0, \quad \text{for} \quad i \neq j,
\]  

(7)

is not true in general, one can expect that the off-diagonal matrix elements \( i \neq j \) are (much) smaller than the diagonal \( i = j \). Requiring Eq.(7) to be true, however, makes things very simple. Eq.(7) will be referred to as model assumption I. As model assumption II, I introduce

\[
a_{us} = a_{dc} = a_{ub} = \ldots = a_{bb} \equiv a,
\]

(8)

just to reduce their number. In principle, these correlations could be calculated from the wave functions, but here I adjust \( a \) as a free parameter to experiment. Note that \( a \) is different for pseudo-scalar and vector mesons, since their wave functions are different.

For \( n_f = 5 \) flavors, the model assumptions (7) and (8) reduce the problem to diagonalize a \( 5 \times 5 \)-matrix:

\[
(\langle f|H_{\text{M}}|f' \rangle = a + M_{ff'}^2 \delta_{f,f'}, \quad \text{for} \quad f, f' = 1, \ldots, n_f.
\]

(9)
The flavor-diagonal mass eigenvalues $M^2_{ij}$ are fixed by the $\uparrow\downarrow$-model and are tabulated below, in Tables 5 and 6. Adjusting the only free parameter $a$ to the mass of the $\eta'$ yields the eigenfunctions $\Phi$ as displayed in Table 4. The corresponding eigenvalues, the physical masses $M$, are given in Table 5.

The wave functions in Table 4 have a characteristic pattern which should be discussed to some further detail. Only the light flavors ($u\bar{u}, d\bar{d}, s\bar{s}$) are mixed significantly, while the heavy flavors ($c\bar{c}, b\bar{b}$) are essentially pure. The physical $\pi^0$ is a superposition of the $u\bar{u}$ and the $d\bar{d}$-state with no admixture of the $s\bar{s}$. The pattern $(1, -1, 0)$ is independent of $a$ and a consequence of the equal up and down quark mass, as to be understood by the considerations in Section 4. The wave functions of the $\eta$ and the $\eta'$ have the pattern $(-1, -1, 1)$ and $(1, 1, 2)$, respectively, in rough agreement with the SU(3)-pattern to be explained.

By adjusting one single parameter, one reproduces three empirical facts: (1) the mass of the $\pi^0$ is (roughly) degenerate with $\pi^\pm$ (isospin); (2) the unperturbed mass of the $\eta$ is lifted from the comparably small value of 140 MeV to the comparatively large value of 485 MeV; (3) the unperturbed mass of the $\eta'$ is lifted by roughly 50% to meet the experimental value due to the fit.

4 A schematic model for flavor-SU(2) and flavor-SU(3)

To understand better the pattern of the wave functions in Table 4, I select first to 2 flavors with equal masses $m_u = m_d$. The flavor-mixing matrix reduces to a 2 by 2 matrix, with

$$H_M = \begin{pmatrix} u\bar{u} & d\bar{d} \\ d\bar{d} & u\bar{u} + M^2_{u\bar{u}} \end{pmatrix}.$$  \hspace{1cm} (10)

The diagonalization of $H_M|\Phi_i\rangle = M^2_i|\Phi_i\rangle$ is easy. The two eigenstates,

$$|\Phi_1\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} |u\bar{u}\rangle \\ -|d\bar{d}\rangle \end{pmatrix}, \quad |\Phi_2\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} |u\bar{u}\rangle \\ |d\bar{d}\rangle \end{pmatrix},$$  \hspace{1cm} (11)

are associated with the eigenvalues

$$M^2_1 = M^2_{u\bar{u}}, \quad M^2_2 = M^2_{u\bar{u}} + 2a.$$  \hspace{1cm} (12)

The assumption of equal quark masses leads thus to $M_{ud} = M_{d\bar{u}} = M_1$. They can be arranged into a mass degenerate triplet of isospin 1, independent of
Table 5
Compilation for the neutral pseudoscalar mesons with $a = (491 \text{ MeV})^2$.
Masses are given in MeV.

| $M_{f\bar{f}}$ | $M$ | $M_{\text{exp}}$ |
|----------------|-----|-----------------|
| $\pi^0$       | 140 | 140             |
| $\eta$        | 140 | 485             |
| $\eta'$       | 661 | 958             |
| $\eta_c$      | 2870| 2915            |
| $\eta_b$      | 8922| 8935            |

Table 6
Compilation for the neutral pseudovector mesons with $a = (255 \text{ MeV})^2$.
Masses are given in MeV.

| $M_{f\bar{f}}$ | $M$ | $M_{\text{exp}}$ |
|----------------|-----|-----------------|
| $\rho^0$      | 768 | 768             |
| $\omega$      | 768 | 832             |
| $\Phi$        | 973 | 1019            |
| $J/\Psi$      | 3231| 3242            |
| $\Upsilon$    | 9822| 9825            |

the numerical value of $a$.

Next, consider 3 flavors. The flavor mixing matrix for the ground state becomes

$$H_M = \begin{pmatrix}
u u & d\bar{d} & s\bar{s} \\
 u + M_{u\bar{u}}^2 & a & a_{us} \\
 a_{us} & a + M_{u\bar{u}}^2 & a_{ds} \\
 s\bar{s} & a_{ds} & a_{ss} + M_{s\bar{s}}^2
\end{pmatrix}. \tag{13}$$

If one assumes $m_u = m_d = m_c = m$, thus $M_{s\bar{s}}^2 = M_{u\bar{u}}^2$, as above, $H_M$ can be diagonalized again in closed form. The three eigenstates

$$|\Phi_1\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} \nu u \\ 0 \end{pmatrix}, \quad |\Phi_2\rangle = \frac{1}{\sqrt{6}} \begin{pmatrix} -\nu u \\ -d\bar{d} \\ 2s\bar{s} \end{pmatrix}, \quad |\Phi_3\rangle = \frac{1}{\sqrt{3}} \begin{pmatrix} \nu u \\ d\bar{d} \\ s\bar{s} \end{pmatrix}, \tag{14}$$

are associated with the eigenvalues

$$M_1^2 = M_{u\bar{u}}^2, \quad M_2^2 = M_{u\bar{u}}^2, \quad M_3^2 = M_{u\bar{u}}^2 + 3a. \tag{15}$$

The coherent state picks up all the strength, again. The eigenvalues of the remaining two states coincide with the unperturbed ones. State $\Phi_1$ can again be interpreted as the eigenstate for the charge neutral $\pi^0$ and the mass of the coherent state $\Phi_3$ could be fitted with the $\eta'$. But then state $\Phi_2$ is degenerate with the $\pi^0$: Instead of a mass triplet, one has a mass quadruplet. In the calculation of Section 3, this degeneracy is broken by the quark mass differences.
5 Conclusions

In the present light-cone approach to gauge theory with an effective interaction isospin is not a dynamic symmetry, but a consequence of equal up and down mass. Flavor-SU(3) is an approximate symmetry. The approach explains the phenomenological observation that flavor-SU(3) symmetry works better than SU(4) or SU(5): the large mass of the heavy quarks dominates the flavor-mixing matrix so strongly that the symmetry induced by the annihilation interaction is destroyed. Despite the simple model, the present work contributes to the $\eta$-$\eta'$ puzzle [7] and has an accuracy comparable to state-of-art lattice-gauge calculations [8]. To the best of my knowledge no other model including the phenomenological ones [1] covers the whole range of flavored hadrons with the same set of parameters.

The present approach is in conflict, however, with other theoretical constructs. Zero modes are absent since one works with the light-cone gauge $A^+ = 0$ [2]. In consequence there are no chiral condensates which seem to be so important otherwise. They are not needed here since the parameter $a$ provides the additional mass scale. This fit-parameter is not on the same level as the physical parameters of the theory like the strong coupling constant $\alpha_s$ or the quark-flavor masses $m_f$. The present but still on-going work gives evidence why its calculation from theory might be worth the effort.

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