Constraints on $f(R)$ Cosmology in the Palatini Formalism

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In this work we derive the covariant and gauge invariant perturbation equations in general theories of $f(R)$ gravity in the Palatini formalism to linear order and calculate the cosmic microwave background (CMB) and matter power spectra for an extensively discussed model, $f(R) = R + \alpha (-R)^{\beta}$, which is a possible candidate for the late-time cosmic accelerating expansion found recently. These spectra are discussed and found to be sensitively dependent on the value of $\beta$. We are thus able to make stringent constraints on $\beta$ from cosmological data on CMB and matter power spectra: The three-year Wilkinson Microwave Anisotropy Probe (WMAP) data alone gives a constraint $|\beta| \lesssim O(10^{-3})$ while the joint WMAP, Supernova Legacy Survey (SNLS) and Sloan Digital Sky Survey (SDSS) data sets tightens this to $\beta \sim O(10^{-6})$, about an order of magnitude more stringent than the constraint from SDSS data alone, which makes this model practically indistinguishable from the standard $\Lambda$CDM paradigm.

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I. INTRODUCTION

It is observed that the universe is now undergoing accelerating expansion [1, 2, 3], which is also consistent with the three-year Wilkinson Microwave Anisotropy Probe (WMAP) data [4] and several other cosmological observations. The usual “explanation” for this involves a mysterious component, called the dark energy, which drives this accelerating expansion. However, this dark energy problem could also be attacked by modifying the theory of gravity so that it departs from the standard general relativity (GR) when the spacetime curvature becomes small. In one type of modified gravity theories, the Ricci scalar $R$ in the Einstein-Hilbert action is simply replaced by a function of $R$, commonly known as $f(R)$. Indeed, in [5, 6], it was shown that by adding correction terms, such as $R^2$, $R^\alpha R_{\alpha}$, and $R^\alpha b R_{\alpha b d}$, to the action, the late time accelerating cosmic expansion could be reproduced (see also [7] and references therein for related works). Another argument in favor of such generalizations is that the effective Lagrangian of the gravitational field generally will include higher order terms in the curvature invariants as a result of quantum corrections (see, e.g., [8]).

However, the conventional metric approach to $f(R)$ gravity leads to fourth order equations which may exhibit violent instabilities when matter is present in the weak gravity regime [9] (see however [10] for a discussion). On the other hand, in the Palatini variation of the action where the metric and connection are treated as independent dynamical variables [11], the resultant equations are second order, which are more tractable and concordant with field equations in other branches of physics. In particular, the typical form $f(R) = R + \alpha (-R)^{\beta}$ has been discussed extensively in the literature as an alternative dark energy model which fits rather well with the supernovae (SNe) Ia data, and it is also tested using cosmic microwave background (CMB) shift parameter and baryon acoustic oscillation (BAO) in [12, 13, 14, 15]. Possible constraints on other types of Palatini-$f(R)$ model have also been considered using big bang nucleosynthesis (BBN) and the requirements that the success of the inflationary paradigm is not spoiled [16]. As far as we know, there have been no attempts to confront Palatini $f(R)$ gravity models with CMB data to date.

In this work, we will concentrate on the model of $f(R) = R + \alpha (-R)^{\beta}$, where $\alpha$ is positive (so that it can reproduce the recent cosmic acceleration). We refer to it as the late $f(R)$ cosmological model because its corrections to GR dominate very lately, and we study both its CMB and matter power spectra. For this, we need the perturbation equations in the Palatini formalism, one set of which has been worked out in [18]. Here, however, we will derive a set of covariant and gauge invariant perturbation equations by the method of 3 + 1 decomposition (see Sec. II) for our calculations. Also we shall try to constrain the model parameters. Unlike previous works, we use the full three-year WMAP data set [4] instead of the CMB shift parameter only. This $f(R)$ gravity model will be constrained firstly by the WMAP CMB spectra data, and then jointly by the CMB spectra, SNe measurements from Supernova Legacy Survey (SNLS), plus the matter power spectrum data measured by the Sloan Digital Sky Survey (SDSS) [19].

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The paper is organized as follows. In Sec. II, we will first review briefly the theory of \( f(R) \) gravity in the Palatini formalism, and we then present the perturbation equations. Then in Sec. III the CMB and matter power spectra for different choices of parameters are displayed and discussed, and the constraints from various data sets will be presented. Finally, we conclude in Sec. IV.

There is also an appendix where we show that our perturbation equations are equivalent to those in the synchronous gauge under specified conditions. Throughout this work we will assume a flat universe filled with cold dark matter, photons, baryons, electrons and 3 species of neutrinos (all massless); the unit \( c = 1 \) is adopted. The metric convention used in this paper is \((+,-,-,-)\).

II. FIELD EQUATIONS IN THEORIES OF PALATINI - \( f(R) \) GRAVITY

In this section we shall first summarize the properties of the general theory of \( f(R) \) gravity in the Palatini formalism, and we then derive its perturbation equations. These equations could be found elsewhere [20], and here we list them just for completeness.

A. General theory of \( f(R) \) gravity in the Palatini approach

The starting point of our discussion on the Palatini-\( f(R) \) gravity is the modified Einstein-Hilbert action, which is given as

\[
S = \int d^4x \sqrt{-g} \left[ \frac{1}{2\kappa} f(R) + \mathcal{L}_m \right],
\]

(1)

where \( \kappa = 8\pi G \) (\( G \) is the Newton’s constant) and \( R = g^{ab}R_{ab}(\bar{\Gamma}) \) is the Ricci tensor and Ricci scalar calculated from \( g_{ab} \) as in GR, which instead are denoted by \( R_{ab} \) and \( R \) respectively in this work (\( R = g^{ab}R_{ab} \)). The matter Lagrangian density \( \mathcal{L}_m \), on the other hand, is assumed to be independent of \( \bar{\Gamma} \), which is the same as in GR.

The extremization of the action Eq. (1) with respect to the metric \( g_{ab} \) then gives the modified Einstein equations

\[
FR_{ab} - \frac{1}{2}g_{ab}f(R) = \kappa T_{ab},
\]

(3)

in which \( F = F(R) \equiv \frac{\partial f(R)}{\partial R} \) and \( T_{ab} \) is the energy-momentum tensor in the system discussed. The trace of Eq. (3) reads

\[
FR - 2f = \kappa T
\]

(4)

with \( T = \rho - 3p \) (\( \rho \) is the energy density and \( p \) the isotropic pressure) being the trace of the energy-momentum tensor. This is the so-called structural equation [21] which relates \( R \) directly to the energy components in the universe: given a specific form of \( f(R) \) and thus \( F(R) \), \( R \) can be obtained as a function of \( T \) by numerically or analytically solving this equation.

The variation of Eq. (4) with respect to the connection field \( \bar{\Gamma} \) leads to another equation

\[
\nabla_a[F(R)\sqrt{-g}g^{bc}] = 0,
\]

(5)

which indicates that the connection \( \bar{\Gamma} \) is compatible with a metric \( \gamma_{ab} \) that is conformal to \( g_{ab} \):

\[
\gamma_{ab} = F(R)g_{ab}.
\]

(6)

With the aid of Eq. (4) we could now easily obtain the relation between \( R_{ab} \) and \( \mathcal{R}_{ab} \) as

\[
R_{ab} = \mathcal{R}_{ab} + \frac{3D_aF_DbF - D_aD_bF}{2F} - \frac{g_{ab}D^cD_cF}{2F}.
\]

(7)

Note that in above we are using \( \mathcal{D} \) and \( \nabla \) to denote the covariant derivative operators which are compatible with \( g_{ab} \) and \( \gamma_{ab} \) respectively.

Since \( \mathcal{L}_m \) depends only on \( g_{ab} \) (and, of course, some matter fields) and the energy-momentum conservation law holds with respect to it, we shall treat this metric as the physical one. Consequently the difference between \( f(R) \) gravity and GR could be understood as a change in the manner in which the spacetime curvature and thus the physical Ricci tensor \( \mathcal{R}_{ab} \) responds to the distribution of matter (through the modified Einstein equations).

In order to make this point explicit, we can rewrite Eq. (4) by the use of Eq. (7) as

\[
\kappa T_{ab} = FR_{ab} - \frac{1}{2}g_{ab}f
+ \frac{3}{2F}D_aF_DbF - D_aD_bF - \frac{1}{2}g_{ab}D^cD_cF,
\]

(8)

in which \( F(T) \), \( f(T) \) are now simply functions of \( T \).

B. The Perturbation Equations

The perturbation equations in general theories of \( f(R) \) gravity have been derived in [18]. However, here we adopt a different, covariant and gauge invariant derivation which utilizes the method of 3 + 1 decomposition [22, 23, 24].

The main idea of 3 + 1 decomposition is to make spacetime splits of physical quantities with respect to the 4-velocity \( u^a \) of an observer. A projection tensor \( h_{ab} \) is then defined as \( h_{ab} = g_{ab} - u_a u_b \) which could be used to
obtain covariant tensors orthogonal to \( u \). For example, the covariant spatial derivative \( \hat{\mathcal{D}} \) of an arbitrary tensor field \( T^{\alpha \beta \gamma \delta}_{\nu \rho \sigma} \) (which, by definition, is orthogonal to \( u \)) is given as

\[
\hat{\mathcal{D}}_{\alpha \beta \gamma \delta}^{\nu \rho \sigma} = h_{\alpha}^{\alpha} h_{\beta}^{\beta} \cdots h_{\gamma}^{\gamma} h_{\delta}^{\delta} \hat{\mathcal{D}}^{\nu \rho \sigma}_{\gamma \delta \alpha \beta}.
\]  

(9)

The energy-momentum tensor and covariant derivative of the 4-velocity \( u \) could be decomposed respectively as

\[
\mathcal{T}_{\alpha \beta} = \pi_{\alpha \beta} + 2q_{\alpha \beta}(u) + \rho u_{\alpha} u_{\beta} - p h_{\alpha \beta},
\]

\[
\mathcal{D}_{\alpha \beta} u_{\gamma} = \sigma_{\alpha \beta} + \varpi_{\alpha \beta} + \frac{1}{3} \theta h_{\alpha \beta} + u_{\alpha} A_{\beta},
\]

(10)

(11)

In the above \( \pi_{\alpha \beta} \) is the projected symmetric trace free (PSTF) anisotropic stress, \( q_{\alpha \beta} \) the vector heat flux, \( \sigma_{\alpha \beta} \) the PSTF shear tensor, \( \varpi_{\alpha \beta} = \mathcal{D}_{\alpha \beta} u_{\gamma} \), \( \theta = \mathcal{D}^{\alpha} u_{\alpha} = 3 \dot{a} / a \) (a is the cosmic scale factor) the expansion scalar and \( A_{\alpha} = \dot{u}_{\alpha} \) the acceleration. The overdot expressed as \( \dot{\phi} = u^{\alpha} \mathcal{D}_{\alpha} \phi \) is the derivative with respect to the proper time of the comoving observer moving at velocity \( u \), and the square brackets denote antisymmetrization and parentheses symmetrization. The normalization is chosen to be \( u^2 = 1 \) in consistence with our metric convention.

Decomposing the Riemann tensor and making use of the modified Einstein equations with the general techniques used in GR, we obtain, after linearization, five constraint equations

\[
0 = \hat{\mathcal{D}}^{c}_{ab} (u^{cd} \varpi_{cd});
\]

\[
\frac{1}{F} \kappa q_{\alpha} = \frac{3 F \hat{\mathcal{D}}_{\alpha} F}{2 F^2} + \theta \hat{\mathcal{D}}_{\alpha} F - \hat{\mathcal{D}}_{\alpha} \hat{\mathcal{D}} F
\]

\[- \frac{2}{3} \hat{\mathcal{D}}_{\alpha} \theta + \hat{\mathcal{D}}^{b} \sigma_{\alpha b} + \hat{\mathcal{D}}^{b} \varpi_{\alpha b};
\]

\[
\mathcal{B}_{ab} = \left[ \hat{\mathcal{D}}^{c} \sigma_{d[a} + \hat{\mathcal{D}}^{c} \varpi_{d[a} \right] \epsilon_{b]ec} \epsilon^{e};
\]

\[
\hat{\mathcal{D}}^{b} \epsilon_{ab} = \frac{1}{2 F} \kappa \left[ \hat{\mathcal{D}}^{b} \pi_{ab} + \left( \frac{2}{3} + \frac{\dot{F}}{F} \right) q_{a} + \frac{2}{3} \hat{\mathcal{D}} F \right]
\]

\[- \frac{1}{2 F^2} \kappa (\rho + p) \hat{\mathcal{D}} F;
\]

\[
\hat{\mathcal{D}}^{b} \mathcal{B}_{ab} = \frac{1}{2 F} \kappa \left[ \hat{\mathcal{D}} q_{d} + (\rho + p) \varpi_{cd} \right] \epsilon_{ab} \epsilon^{d};
\]

(12)

(13)

(14)

(15)

(16)

Here \( \epsilon_{abcd} \) is the covariant permutation tensor, \( \epsilon_{ab} \) and \( \mathcal{B}_{ab} \) are respectively the electric and magnetic parts of the Weyl tensor \( \mathcal{W}_{abcd} \), given respectively by \( \epsilon_{ab} = u^{c} u^{d} \mathcal{W}_{abcd} \) and \( \mathcal{B}_{ab} = - \frac{1}{2} u^{c} u^{d} \epsilon_{acdf} \mathcal{W}_{fbd} \).

We also obtain seven propagation equations:

\[
\dot{\rho} + (\rho + p) \theta + \hat{\mathcal{D}}^{\alpha} q_{\alpha} = 0;
\]

\[
\dot{q}_{a} + \frac{4}{3} \theta q_{a} + (\rho + p) A_{a} - \hat{\mathcal{D}}^{\alpha} p_{\alpha} = 0;
\]

\[
\dot{\theta} + \frac{1}{3} \left[ \theta + \frac{3 \dot{F}}{2 F} \right] \theta - \hat{\mathcal{D}}^{\alpha} A_{a} = 0;
\]

\[
- \left[ \frac{3 \dot{F}^2}{2 F^2} - \frac{3 \dot{F}}{2 F} - \frac{\kappa \rho}{F} - \frac{f}{2 F} - \hat{\mathcal{D}}^{2} F \right] = 0;
\]

\[
\frac{1}{3} + \frac{\dot{F}}{2 F} = \frac{1}{6 F} \left[ \kappa (\rho + 3 p) - f \right].
\]

(17)

(18)

(19)

(20)

(21)

(22)

(23)

The angle brackets mean taking the trace free part of a quantity.

Besides the above equations, it would also be useful to express the projected Ricci scalar \( \hat{\mathcal{R}} \) in the hypersurfaces orthogonal to \( u^{\alpha} \) (the projected Riemann tensor, \( \hat{\mathcal{R}}_{abcd} \), is defined by \( [\mathcal{D}_{\alpha \beta} \mathcal{D}_{\gamma \delta}]_{\nu \rho \sigma}^{\nu \rho \sigma} = \hat{\mathcal{R}}_{abcd} u^{c} u^{d} \), similar to the definition of the full covariant Riemann tensor \( \hat{\mathcal{R}}_{abcd} \) but with a conventional opposite sign, and the calculations for the projected Ricci tensor \( \hat{\mathcal{R}}_{abcd} \) and projected Ricci scalar \( \hat{\mathcal{R}} \) just follow the same way as in GR) as

\[
\hat{\mathcal{R}} = \frac{(\rho + 3 p) - f}{F} - \frac{2}{3} \left[ \theta + \frac{3 \dot{F}}{2 F} \right]^2 - \frac{2 \hat{\mathcal{D}}^{2} F}{F}. \quad (24)
\]

The spatial derivative of this projected Ricci scalar, \( \eta_{a} = \frac{1}{a} \mathcal{D}_{a} \hat{\mathcal{R}} \), is then obtained after a lengthy calculation as

\[
\eta_{a} = \frac{a}{2 F} \kappa (\rho + 3 p) - \frac{a}{F} \left[ \frac{3}{2 F} \hat{\mathcal{D}} F - \theta \right] \mathcal{D}_{a} F
\]

\[- \frac{a}{2 F} \mathcal{D}_{a} f - \frac{a}{F} \mathcal{D}_{a} (\hat{\mathcal{D}}^{2} F) - \frac{2 a}{3} \frac{3}{2 F} \hat{\mathcal{D}} F \theta \mathcal{D}_{a} \theta
\]

\[+ \frac{a}{4 \theta} \left[ \frac{3}{2 F} \hat{\mathcal{D}} F + \theta \right] \mathcal{D}_{a} F, \quad (25)
\]

and its time evolution is governed by the propagation equation

\[
\eta_{a} + \frac{2 a}{3} \eta_{a} = \frac{a}{2 F} \left[ \frac{3}{2 F} \hat{\mathcal{D}} F - \frac{2}{3} \theta \right] \mathcal{D}_{a} \hat{\mathcal{D}}^{2} F - \frac{a}{F} \kappa \mathcal{D}_{a} \mathcal{D}_{b} q_{b}. \quad (26)
\]

As we are considering a spatially flat universe, its spatial curvature will vanish for large scales, meaning that \( \hat{\mathcal{R}} = 0 \). Thus from Eq. (24) we have

\[
\left[ \frac{1}{3} + \frac{\dot{F}}{2 F} \right]^2 = \frac{1}{6 F} \left[ \kappa (\rho + 3 p) - f \right]. \quad (27)
\]
This is just the modified (first) Friedmann equation in the \( f(R) \) version of gravitational theory, and the other modified background equations (the second Friedmann equation and the energy-conservation equation) could be obtained by taking the zeroth-order parts of Eqs. (17) and (19). It is easy to check that when \( f(R) = R \), we have \( \dot{F} = 1 \), and these equations just reduce to those in GR – in this case GR and the Palatini-\( f(R) \) theory lead to the same results. In the appendix we also show that this set of perturbation equations is equivalent to that derived in the synchronous gauge, which serves as a check for this work.

Remember that we have had \( f, \dot{F} \) and \( R \) as functions of \( \mathcal{T} \) at hand, it is then straightforward to calculate \( \dot{F}, \ddot{F}, \dddot{D}_a F, \dddot{D}_a \dot{F} \) etc. as functions of \( \mathcal{T} \equiv -(\rho_b + \rho_c)\theta \) and \( \ddot{D}_a \mathcal{T} = (1 - 3c_s^2)\ddot{D}_a \rho_b + \dddot{D}_a \rho_c, \) in which \( \rho_{ib}(c) \) is the energy density of baryons (cold dark matter) and \( c_s \) is the baryon sound speed. Note that in this work we choose to neglect the small baryon pressure except in the terms where its spatial derivative is involved, in which case they might be significant at small scales. The above equations could then be numerically propagated given the initial conditions, to obtain the evolutions of small density perturbations and the CMB and matter power spectra in theories of \( f(R) \) gravity. Finally the three-year WMAP data on CMB spectra, SNLS SN data and SDSS data on matter power spectrum could be used to constrain parameters in the \( f(R) \) models. These results will be given in the following section.

III. NUMERICAL RESULTS AND COSMOLOGICAL CONSTRAINTS ON THE MODEL

This section is devoted to numerical results and constraints of the present model. To this effect we will first very briefly summarize and explain the effects of the \( f(R) \) modifications to GR on the linear spectra; for more details see [20]. After that we shall employ the public Markov Chain Monte Carlo (MCMC) engine [28] to search the parameter space with the theoretical CMB and matter power spectra calculated by the modified CAMB code; the constraints are then summarized and discussed.

In Fig. 1 we have displayed the TT CMB spectra for the model with different choices of \( \beta \). It is obvious from this figure that, when \( \beta < 0 \), the spectrum gets a boost in the scales \( l \leq 100 \), which could be significant if \( |\beta| \) is large enough. This effect is due to a strong late-time integrated Sachs-Wolfe (ISW) effect [20, 25], which in turn originates from the unusually rapid late-time decay of the gravitational potential \( \phi \) of the present \( f(R) \) model compared with \( \Lambda \)CDM, as shown in the lower panel of Fig. 3 in [20] (see this reference for more details). We have not given the curves for \( \beta > 0 \) because in that case the spectrum generally blows up except for very small \( |\beta| \)'s (see below).

Another interesting feature in Fig. 1 is that for negative \( \beta \) the spectrum shifts towards the right-hand-side (larger \( l \)'s), likely due to the unusual angular-distance-redshift relation [26, 27]. Since the standard \( \Lambda \)CDM cosmology is expected to be valid in the early times when the correction to GR is negligible, the sound horizon and the thickness of the last scattering surface are the same as in \( \Lambda \)CDM. But at late times the Friedmann equation is modified (Eq. (27)), and so is the relation between redshift and conformal distance. This would cause the CMB spectrum to shift sideways. This shifting effect, however, is negligible for the constrained ranges of \( \beta \) obtained below.

The CMB EE polarization and cross correlation spectra show no additional interesting features and cannot be used to put strong constraint on the model parameters, and so we will not present and discuss them here.

We have also given in Fig. 2 the matter power spectra. As indicated in this figure, the matter power spectrum depends sensitively on the value of \( \beta \) and could differ from \( \Lambda \)CDM significantly even if \( |\beta| \) only deviates from 0 by a tiny amount \( \text{e.g., of order } 0(10^{-5}) \). This feature has been pointed out and discussed extensively in [17, 18, 20]. Basically, this is because of the sensitive response of the modified gravity to the spatial variations of matter distribution, which, at small enough scales would significantly affect the growth of density perturbations. To be explicit, for large enough \( \delta \)'s, the growth equation for the comoving energy density fluctuations \( \delta_m \) could be written as [17]

\[
\frac{d^2 \delta_m}{dN^2} = -\frac{k^2}{a^2 H^2} \frac{\dot{F}}{3F(2FH + \dot{F})} \delta_m,
\]

in which \( N \equiv \log(a) \) and \( \dot{F}/3F(2FH + \dot{F}) = c_{s, eff}^2 \) acts...
as an effective sound speed squared that vanishes in the ΛCDM model. For the present $f(R)$ model, we have $F = 1 - \alpha \beta (R)^{\beta - 1}$, in which $\alpha$ and $-R$ are positive and $-R$ decreases with time. So if $\beta < 0$, then $\dot{F} > 0$ and thus $c_{s,\text{eff}}^2 > 0$: this effective pressure term will restrict the growths of small-scale density perturbations and leads to oscillations (as shown in Fig. 2 for the case of $\beta = -0.00005$) of the spectrum in these scales. On the other hand, if $\beta > 0$ (as we hope to recover standard ΛCDM cosmology in earlier times, we shall also restrict ourselves to $\beta < 1$), then $\dot{F}$ and $c_{s,\text{eff}}^2$ will be negative; this will make the density fluctuations unstable and blow up, the same reason why the CMB spectrum depends so sensitively on positive $\beta$s.

Since from Figs. 1 and 2 we have seen that the linear spectra of our $f(R)$ model depend very sensitively on the model parameter $\beta$, it can be expected that the data on CMB and matter power spectra could place stringent constraints on $\beta$, as we will show now. As mentioned in Sec. I we shall firstly use the full three year WMAP data and then perform a joint constraint simultaneously using WMAP, SNLS and SDSS data to constrain the parameters.

Because the Hubble parameter $H_0$ is already measured to rather good precision by the HST Key Project, we shall use $H_0 = 72$ km/s/Mpc [29] in our calculations. Therefore we vary the following parameters: baryon density $\omega_b = \Omega_b h^2$, cold dark matter $\omega_c = \Omega_c h^2$, reionization redshift $z_{\text{re}}$, spectral index $n_s$, normalization amplitude $A_s$ and the model parameter $\beta$. In Fig. 3 the marginal distributions of $\Omega_m$ and $\beta$ are shown. The 95% confidence interval for $\Omega_m$ and $\beta$ are [0.233, 0.268] and $[-3.45 \times 10^{-3}, 3.07 \times 10^{-3}]$ respectively. We also present the contour plot of the joint distribution of $\Omega_m$ and $\beta$ in Fig. 4. From these figures we can see that the CMB spectra could constrain $|\beta|$ to $O(10^{-3})$, ~100 times more stringent than the constraint from the CMB shift parameter [12], which is of order 0.1. That the CMB spectra is much more powerful in constraining the parameters than the CMB shift parameter is expected because the former bears a lot more information than the latter. It looks from these figures that a slightly positive $\beta$ is preferred by the CMB data.

To tighten the bounds on the parameters, we perform a joint constraint making use of both the three-year WMAP and the SDSS data sets. In addition to the matter power spectrum, the SNe data from SNLS [30] is...
FIG. 5: The marginal distributions of the various model parameters, constrained simultaneously by the WMAP, SNLS and SDSS data sets. The distributions are normalized such that the maximum probability density is 1.

FIG. 6: (Color Online) The contour plot of joint distribution of $\Omega_m$ and $\beta$ under the constraints of the WMAP, SNLS and SDSS data sets. The inner and outer loops are the 68% and 95% confidence contours respectively.

also used in the joint constraint, though their effects are found to be negligible. For the SDSS data, we conservatively adopt the measurements for scales larger than $k = 0.2 h$ Mpc$^{-1}$ (where $h \equiv H_0/(100 \text{ km s}^{-1}\text{Mpc}^{-1})$) to avoid encountering the nonlinear effects in the measured matter power spectrum. The bias between galaxy power spectrum and matter power spectrum is assumed to be a scale independent constant; CosmoMC [28] assumes a flat prior on it and marginalizes analytically.

The calculation indicates that indeed the allowed range is shrunk, as indicated in Figs. 5 and 6 (for completeness in Fig. 7 we have also plotted the best-fitted curves with the observational data points from SNLS, WMAP and SDSS we use in the constraints). The 95% confidence intervals for $\Omega_m$ and $\beta$ now become [0.241, 0.274] and $[2.12 \times 10^{-6}, 5.98 \times 10^{-6}]$ respectively, and the 95% confidence interval for $\sigma_8$ is [0.85, 1.21]. The distribution of $\Omega_m$ does not change much since it is already well constrained by the WMAP, but the bound on $\beta$ is tightened to the order of $10^{-6}$ (and obviously future refined data could still further this constraint). What is more, these joint constraints also prefer a positive $\beta$ and actually have excluded the case of $\beta = 0$, i.e., the $\Lambda$CDM paradigm,
at the 95% confidence level. In Fig. 7 we could see that all the data sets are fitted very well. Furthermore, more stringent constraints on $\beta$ can be obtained if data points in the nonlinear regime are also used since the matter power spectrum will blow up in the small scales for positive $\beta$. Nonetheless, our stringent constraint on $\beta$ already makes the model nearly indistinguishable from $\Lambda$CDM for practical purposes, and without a natural motivation for such tiny values of $\beta$ this model should be more reasonably disfavored.

IV. DISCUSSION AND CONCLUSION

In conclusion, we have in this work derived the perturbation equations for general theories of Palatini-$f(R)$ gravity and applied them to a typical class of model $f(R) = R + \alpha(-R)^\beta$ which is proposed as an alternative to the cosmological constant to account for the late-time accelerating cosmic expansion and has been extensively studied. We then calculate the CMB and matter power spectra for this model using a modified CAMB code. It is shown that for negative $\beta$s the potential $\phi$ will see an unusually rapid decay at late times, leading to an enhancement of the ISW effect and thus a boost of the TT CMB spectrum at small $l$s. There also appears a positive effective pressure term in the equation governing the growth of density perturbations, which could be significant for small scales (large $k$s) and restricts the perturbation growths in these scales. For positive $\beta$s, however, the small-scale density fluctuations will become unstable and grow exponentially, resulting in blowing-ups of the matter power spectrum.

We have constrained the model parameters using the WMAP, SNLS and SDSS data. Because the CMB and matter power spectra are rather sensitive to the exact values of the parameter $\beta$, we are able to give much more stringent constraints on $\beta$ ($O(10^{-3})$ and $O(10^{-6})$ respectively) than those ($O(10^{-1})$) coming from the CMB shift parameter fitting or measurements on SNe [15]. Compared with the bound ($O(10^{-5})$) from SDSS data alone [17], our WMAP + SNLS + SDSS constraint is tighter because the allowed range of $\Omega_m$ is largely reduced here.

These constraints seem to make the present model (in its allowed parameter space) indistinguishable from the $\Lambda$CDM paradigm and raise a fine-tuning problem to the late $f(R)$ gravity theory. However, there still remains the interesting possibility that $f(R)$ modification of gravity enters at earlier times (or higher densities): can it survive the tests from WMAP and SDSS data? This topic is beyond the scope of this article and has been investigated in another work [20]; in that case the parameter space for $f(R)$ gravity is also highly limited.

Appendix

This appendix is devoted to showing the equivalence between the covariant perturbation equations derived here and the perturbation equations in the synchronous gauge [31]. In the latter case, the line element is expressed as (indices $i, j, k$ run over 1, 2, 3 hereafter)

$$ds^2 = a(\tau)^2 \left[ d\tau^2 - (\delta_{ij} + h_{ij}^S)dx^i dx^j \right], \quad (29)$$

in which $\tau$ is the conformal time given by $dt = a(\tau)d\tau$ and we have defined the scalar modes in Fourier space as
\[ h^{S}_{ij}(x, \tau) = \int d^3k \exp(ik \cdot x) \left[ \hat{k}_i \hat{k}_j h^S(k, \tau) + 6 \left( \hat{k}_i \hat{k}_j - \frac{1}{3} \delta_{ij} \right) \eta^S(k, \tau) \right], \tag{30} \]

where a superscript \('S'\) denotes quantities in the synchronous gauge and \( \hat{k} = k/k \) is the unit vector in the \( k \)-direction.

In order to relate the synchronous gauge variables \( h^S \) and \( \eta^S \) with those in the covariant perturbation method, let us do the harmonic expansion for the first-order quantities \( h_a = \bar{D}_a a \) and \( \eta_a = \sum k \delta Q^k_a \) and \( \eta_a = \sum k \delta Q^k_a \) (Here \( Q^k_a = \frac{1}{2} \bar{D}_a Q^k \) and \( Q^k \) is the eigenfunction of the generalized Helmholtz equation \( a^2 \bar{D}^2 Q^k = k^2 Q^k \). For more details see e.g., [23, 24]), to obtain the variables \( h \) and \( \eta \) relevant to specified \( k \)-modes. We shall choose the reference velocity \( u \) to be the 4-velocity of cold dark matter, which means that the cold dark matter heat flux \( q_\ell = \rho \nu_\ell \) and the acceleration \( A \) both vanish [23, 24]. Then using the relations \( \eta^S = -\eta/2 \) and \( h^S = 6h \) [32] (a prime here means taking derivative with respect to the conformal time \( \tau \) it is easy to show the equivalence between these two sets of perturbation equations. More explicitly: Eq. (26) is equal to the following first-order equation

\[ \bar{F} k^2 \eta^S \equiv \frac{k}{a^2} \left( \bar{F} \bar{\nu} + \bar{\bar{F}} \bar{\sigma} \right) \theta^S - \frac{1}{2} \bar{F}^2 + 3\bar{H} \frac{k^2 \delta F^*}{2} + \frac{1}{2} k^2 \delta F^*, \tag{31} \]

where \( \bar{H} \equiv a'/a \). Taking the spatial derivative of Eq. (19) leads to

\[ -\frac{\bar{F}}{2} \left[ h^{nS} + \bar{H} h^{tS} + \frac{\bar{F}'}{2F} h^{tS} \right] \equiv \kappa a^2 \delta \rho + \frac{a^2 \partial f(\bar{T})}{2} \delta T - \frac{3 \bar{F}'}{\bar{F} - \delta F} + 3 \delta F^* - \frac{3 \bar{F}'}{2} - \frac{3 \bar{H}^2}{2} \left[ \frac{3 a''}{a} + 6 \delta F + \frac{1}{2} \bar{F}^2 + \frac{k^2}{2} \right] \delta (32) \]

Then from Eqs. (25) and (32) we can obtain

\[ \bar{F} \left[ 4 k^2 \eta^S - \frac{5 \bar{H} h^{tS}}{2} - \frac{h^{nS}}{2} - \frac{5 \bar{F}'}{4F} h^{tS} \right] \equiv 3 a^2 \partial f(\bar{T}) \delta T - \kappa a^2 \delta T_i + 6 \bar{H} \delta F^* + \frac{3 \bar{F}'}{2} + \frac{3 \bar{H}^2}{2} \left[ \frac{3 a''}{a} + 6 \delta F^* + \frac{5 k^2}{2} \right] \delta (33) \]

and finally, taking time derivative of Eq. (26) and making use of Eqs. (19), (25), (27) and (32), we get

\[ \bar{F} \left[ \frac{2}{3} k^2 \eta - \frac{2}{3} \bar{H} (h^{tS} + 6 \eta^{nS}) - \frac{1}{3} (h^{nS} + 6 \eta^{nS}) \right] \equiv \kappa a^2 (\bar{\rho} + \bar{\bar{\rho}}) \sigma^S + \frac{2}{3} k^2 \delta F + \frac{1}{2} \bar{F}' (h^{tS} + 6 \eta^{nS}) \, . \tag{34} \]

In the above \( \delta \) means the spatial variation of a quantity for the discussed length scale (or \( k \)) and a bar its background value (up to first order all the barred quan-
ties in these equations could be equally replaced by their unbarred correspondences. The variables $\theta^S, \sigma^S$ are defined respectively as

$$\theta^S \equiv \frac{i}{3} \delta T^0_j$$

and

$$(\bar{\rho}+\bar{p})\sigma^S \equiv -(\delta k_i - \frac{1}{3}\delta(T_j^i - \frac{1}{3}\delta T^k_k)^i_j).$$

$T_{ab}$ is the energy-momentum tensor in the synchronous gauge (to be distinguished from the $T_{ab}$ used above) and $T$ its trace.

We have checked that Eqs. (31) - (34) are just the perturbation equations one has in the synchronous gauge for theories of Palatini-$f(R)$ gravity, as expected [23, 24].

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