Comment on “Fully covariant radiation force on a polarizable particle”

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Recently Pieplow and Henkel (PH) (NJP 15 (2013) 023027) presented a new fully covariant theory of the Casimir friction force acting on small neutral particle moving parallel to flat surface. We compare results of this theory with results which follow from a fully relativistic theory of friction in plate-plate configurations in the limit when one plate is considered as sufficiently rarefied. We show that there is an agreement between these theories.

I. INTRODUCTION

All bodies are surrounded by a fluctuating electromagnetic field due to the thermal and quantum fluctuations of the charge and current density inside the bodies. Outside the bodies this fluctuating electromagnetic field exists partly in the form of propagating electromagnetic waves and partly in the form of evanescent waves. The theory of the fluctuating electromagnetic field was developed by Rytov$^{8,9}$. A great variety of phenomena such as Casimir-Lifshitz forces$^{10}$, near-field radiative heat transfer$^{10}$, and friction forces$^{10}$ can be described using this theory.

In$^2$ we used the dynamical modification of the Lifshitz theory to calculate the friction force between two plane parallel surfaces in parallel relative motion with velocity $V$. The calculation of the van der Waals friction is more complicated than that of the Casimir-Lifshitz force and the radiative heat transfer because it requires the determination of the electromagnetic field between moving boundaries. The solution can be found by writing the boundary conditions on the surface of each body in the rest reference frame of this body. The relation between the electromagnetic fields in the different reference frames is determined by the Lorenz transformation. In$^2$ the electromagnetic field in the vacuum gap between the bodies was calculated to linear order in $V/c$, which gives the contribution to the friction force to order $(V/c)^2$. These relativistic corrections were neglected within the non-relativistic theory developed in$^2$. The same non-relativistic theory was used in$^2$ to calculate the frictional drag between quantum wells, and in$^{10,11}$ to calculate the friction force between flat parallel surfaces in normal relative motion. In Ref.$^{12}$ we presented a rigorous quantum mechanical calculation using the Kubo formula for the friction coefficient. This calculation confirmed the correctness of the approach based on the dynamical modification of the Lifshitz theory, at least to linear order in the sliding velocity $V$. For a review of the van der Waals friction see$^2$.

In Ref.$^2$ we developed a fully relativistic theory of the Casimir-Lifshitz forces and the radiative heat transfer at non-equilibrium conditions, when the interacting bodies are at different temperatures, and they move relative to each other with the arbitrary velocity $V$. In comparison with previous calculations$^{6,8,11}$, we did not make any approximation in the Lorentz transformation of the electromagnetic field. This allowed us to determine the field in one reference frame, knowing the same field in another reference frame. Thus, the solution of the electromagnetic problem was exact. Knowing the electromagnetic field we calculated the stress tensor and the Poynting vector which determined the Casimir-Lifshitz forces and the heat transfer, respectively. Taking the limit when one of the bodies is rarefied, it is possible to obtain the Casimir-Lifshitz force and friction, and the radiative heat transfer for a small particle-surface configuration. However, in this approach additional approximations were made which did not allow to make detailed comparison with other theories of friction for the particle-surface configuration in ultra relativistic case.

The problem of friction for a small neutral particle moving parallel to a solid surface (particle-surface configuration) was considered by number of authors (see$^{7,13,14}$, and references therein). At present the interest in this problem is increasing because it is linked to quantum Cherenkov radiation$^{15}$. Recently a fully covariant theory of friction in particle-surface configuration was proposed by Pieplow and Henkel (PH)$^{14}$ and comparison with results of previous authors was given. The theory presented by PH agrees with relativistic theory proposed by Dedkov and Kyasov (DK)$^{14}$. However, it is well known that the friction between a particle and solid surface, mediated by evanescent electromagnetic waves, can be extracted from friction acting between two plates assuming that one plate is sufficiently rarefied$^2$. A fully relativistic theory of friction between two plates in parallel relative motion (plate-plate configuration) was developed in$^2$. In the present Comment the friction in particle-plate configuration is calculated from the friction in plate-plate configuration assuming that one plate is sufficiently rarefied. We compare our results with the results of Ref.$^{14}$ and show that there is an agreement between these two theories.

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axis. Since the system is translational invariant in the $x^2$ plane parallel to body $xyz$. The upper solids moves parallel to other with velocity $V$. See Fig. 1. We introduce the two coordinate systems $K$ and $K'$ with coordinate axes $x'y'z'$. In the $K$ system body 1 is at rest while body 2 is moving with the velocity $V$ along the $x-$ axis (the $xy$ and $x'y'$ planes are in the surface of body 1, $x$ and $x'$- axes have the same direction, and the $z$ and $z'$- axes point toward body 2). In the $K'$ system body 2 is at rest while body 1 is moving with velocity $-V$ along the $x-$ axis. Since the system is translational invariant in the $x= (x, y)$ plane, the electromagnetic field can be represented by the Fourier integrals

$$E(x, z, t) = \int_{-\infty}^{\infty} d\omega \int \frac{d^2 q}{(2\pi)^2} e^{i\mathbf{q} \cdot \mathbf{x} - i\omega t} E(q, \omega, z),$$

$$B(x, z, t) = \int_{-\infty}^{\infty} d\omega \int \frac{d^2 q}{(2\pi)^2} e^{i\mathbf{q} \cdot \mathbf{x} - i\omega t} B(q, \omega, z),$$

where $E$ and $B$ are the electric and magnetic induction field, respectively, and $q$ is the two-dimensional wave vector in $xyp$- plane. After Fourier transformation it is convenient to decompose the electromagnetic field into $s$- and $p$-polarized components. For the $p$- and $s$-polarized electromagnetic waves the electric field $E(q, \omega, z)$ is in plane of incidence, and perpendicular to that plane, respectively. In the vacuum gap between the bodies the electric field $E(q, \omega, z)$, and the magnetic induction field $B(q, \omega, z)$ can be written in the form

$$E(q, \omega, z) = (v_s \hat{n}_s + v_p \hat{n}_p^+ e^{-k_z z} + (w_s \hat{n}_s + w_p \hat{n}_p^- e^{k_z z}$$

$$B(q, \omega, z) = (v_s \hat{n}_p^+ - v_p \hat{n}_s) e^{-k_z z} + (w_s \hat{n}_s - w_p \hat{n}_p^-) e^{k_z z}$$

where $k_z = ((q^2 - (\omega + i0^+)/c)^2)^{1/2}$, $\hat{n}_s = [\hat{z} \times \hat{q}] = (-q_y, q_x, 0)/q$, $\hat{n}_p^+ = [k^+ \times \hat{n}_s] = (\mp q_x ik_z, \mp q_y i k_z, q^2)/(kq)$, $k = \omega/c$, $k^\pm = (q \pm i\kappa k_z)/k$. At the surfaces of the bodies the amplitude of the outgoing electromagnetic wave must be equal to the amplitude of the reflected wave plus the amplitude of the radiated wave. Thus, the boundary conditions for the electromagnetic field at $z = 0$ in the $K$- reference frame can be written in the form

$$v_p(s) = R_{1p(s)}(\omega, q) w_p(s) + E_{1p(s)}^f(\omega, q)$$

where $R_{1p(s)}(\omega)$ is the reflection amplitude for surface 1 for the $p(s)$- polarized electromagnetic field, and where $E_{1p(s)}^f(\omega)$ is the amplitude of the fluctuating electric field radiated by body 1 for a $p(s)$-polarized wave.
In the $K'$- reference frame the electric field can be written in the form

$$
E'(q', \omega', z) = (v'_p \hat{n}'_s + v'_{p} \hat{n}'_{p}) e^{-k'z} + (w'_p \hat{n}'_s + w'_{p} \hat{n}'_{p}) e^{k'z}
$$

where $q' = (q'_x, q_y, 0), q'_s = (q_x - \beta k)\gamma, \omega' = (\omega - Vq_x)\gamma, \gamma = 1/\sqrt{1 - \beta^2}, \beta = V/c, \hat{n}'_s = (-q_y, q'_x, 0)/q', \hat{n}'_p = (q'_x k_z, +q_y k_z, q'^2)/(k'q')$.

The boundary conditions at $z = d$ in the $K'$- reference frame can be written in a form similar to Eq. (11):

$$
w'_{p(s)} = e^{-2kz_d} \rho_{2p(s)}(\omega', q') v'_{p(s)} + e^{-kz_d} E^{f}_{2p(s)}(\omega', q'),
$$

where $\rho_{2p(s)}(\omega)$ is the reflection amplitude for surface 2 for $p(s)$ - polarized electromagnetic field, and where $E^{f}_{2p(s)}(\omega)$ is the amplitude of the fluctuating electric field radiated by body 2 for a $p(s)$-polarized wave. A Lorentz transformation for the electric field gives

$$
E'_x = E_x, E'_y = (E_y - \beta B_z)\gamma, E'_z = (E_z + \beta B_y)\gamma
$$

Using Eqs. (3,4,6) and (8) we get

$$
v'_p = \frac{k'\gamma}{k q q} \left[ -i \beta k z q_y v_s + (q^2 - \beta k q_x) v_p \right],
$$

$$
w'_p = \frac{k'\gamma}{k q q} \left[ i \beta k z q_y w_s + (q^2 - \beta k q_x) w_p \right],
$$

$$
v'_s = \frac{k'\gamma}{k q q} \left[ i \beta k z q_y v_p + (q^2 - \beta k q_x) v_s \right],
$$

$$
w'_s = \frac{k'\gamma}{k q q} \left[ -i \beta k z q_y w_p + (q^2 - \beta k q_x) w_s \right].
$$

Substituting Eqs. (9,10,11) in Eq. (7) and using Eq. (5) we get

$$
(q^2 - \beta k q_x) \Delta_{pp} w_p + i \beta k z q_y \Delta_{sp} w_s
$$

$$
e^{-2kz_d} \rho'_{2p} \left[ (q^2 - \beta k q_x) E^{f}_{1p} - i \beta k z q_y E^{f}_{1s} \right] + \frac{k q q}{k'\gamma} e^{-kz_d} E^{f}_{2p},
$$

$$
(q^2 - \beta k q_x) \Delta_{sz} w_s - i \beta k z q_y \Delta_{ps} w_p
$$

$$
e^{-2kz_d} \rho'_{2s} \left[ (q^2 - \beta k q_x) E^{f}_{1s} + i \beta k z q_y E^{f}_{1p} \right] + \frac{k q q}{k'\gamma} e^{-kz_d} E^{f}_{2s},
$$

where

$$
\Delta_{pp} = 1 - e^{-2kz_d} R_{1p} R'_{2p}, \Delta_{ps} = 1 + e^{-2kz_d} R_{1p} R'_{2s},
$$

$$
\Delta_{ss} = \Delta_{pp}(p \leftrightarrow s), \Delta_{sp} = \Delta_{ps}(p \leftrightarrow s), R'_{2p(s)} = R_{2p(s)}(\omega', q'), \text{the symbol } (p \leftrightarrow s) \text{ means permutation of the indexes } p \text{ and } s. \text{ From Eqs. (13,14) and (5) we get}
$$

$$
w_p = \left\{ [(q^2 - \beta k q_x)^2 R_{2p} \Delta_{ss} + \beta^2 k^2 q_y^2 R_{2s} \Delta_{sp} ] E^{f}_{1p} e^{-2kz_d}ight\} + \left\{ [(q^2 - \beta k q_x)^2 R_{2p} \Delta_{ss} + \beta^2 k^2 q_y^2 R_{2s} \Delta_{sp} ] E^{f}_{1s} e^{-2kz_d}ight\} - \left\{ [(q^2 - \beta k q_x)^2 R_{2p} \Delta_{ss} + \beta^2 k^2 q_y^2 R_{2s} \Delta_{sp} ] E^{f}_{1p} e^{-2kz_d}ight\} - \left\{ [(q^2 - \beta k q_x)^2 R_{2p} \Delta_{ss} + \beta^2 k^2 q_y^2 R_{2s} \Delta_{sp} ] E^{f}_{1s} e^{-2kz_d}ight\}.
\[-i\beta k_z q_y (q^2 - \beta k q_x) (R_{2p}^2 + R_{2s}^2) E_1^f e^{-2k_z d}\]

\[+ \frac{k q q'}{k' \gamma} \left\{ (q^2 - \beta k q_x) \Delta_{ss} E_2^f e^{-ki_z d} - i\beta k_z q_y \Delta_{sp} E_2^f e^{-ki_z d} \right\} \Delta^{-1}, \quad (15)\]

\[v_p = \left\{ \left[ (q^2 - \beta k q_x)^2 \Delta_{ss} - \beta^2 k_z^2 q_y^2 \Delta_{sp} \right] E_1^f \right\} \Delta^{-1}, \]

\[+ i\beta k_z q_y (q^2 - \beta k q_x) R_{1p} (R_{2p}^2 + R_{2s}^2) e^{-2k_z d} E_1^f \]

\[w_s = \left\{ \left[ (q^2 - \beta k q_x)^2 D_{pp} \Delta_{pp} + \beta^2 k_z^2 q_y^2 R_{2p}^2 \Delta_{ps} \right] E_1^f e^{-2k_z d} \right\} \Delta^{-1}, \]

\[v_s = \left\{ \left[ (q^2 - \beta k q_x)^2 \Delta_{pp} - \beta^2 k_z^2 q_y^2 \Delta_{ps} \right] E_1^f \right\} \Delta^{-1}, \]

\[+ i\beta k_z q_y (q^2 - \beta k q_x) R_{1p} (R_{2p}^2 + R_{2s}^2) e^{-2k_z d} E_1^f \]

\[+ \frac{k q q'}{k' \gamma} R_{1s} \left\{ (q^2 - \beta k q_x) \Delta_{pp} E_2^f + i\beta k_z q_y \Delta_{ps} E_2^f e^{-ki_z d} \right\} \Delta^{-1}, \quad (17)\]

where

\[\Delta = (q^2 - \beta k q_x)^2 \Delta_{ss} \Delta_{pp} - \beta^2 k_z^2 q_y^2 \Delta_{ps} \Delta_{sp}.\]

The fundamental characteristic of the fluctuating electromagnetic field is the correlation function, determining the average product of amplitudes \(E_{p(s)}^f (q, \omega)\). According to the general theory of the fluctuating electromagnetic field (see for example [2]):

\[< |E_{p(s)}^f (q, \omega)|^2 > = \frac{\hbar \omega^2}{2e^2 |k_z|^2} \left( n(\omega) + \frac{1}{2} \right) [k_z - k_z^* (1 - |R_{p(s)}|^2)] \]

\[+ (k_z + k_z^*)(R_{p(s)}^* - R_{p(s)})], \quad (19)\]

where \(< ... >\) denote statistical average over the random field. We note that \(k_z\) is purely imaginary \((k_z = -i |k_z|)\) for \(q < \omega/c\) (propagating waves), and real for \(q > \omega/c\) (evanescent waves). The Bose-Einstein factor

\[n(\omega) = \frac{1}{e^{\hbar \omega/k_B T} - 1}.\]

Thus for \(q < \omega/c\) and \(q > \omega/c\) the correlation functions are determined by the first and the second terms in Eq. (19), respectively.
The force which acts on the surface of body 1 can be calculated from the Maxwell stress tensor $\sigma_{ij}$, evaluated at $z = 0$:

$$\sigma_{ij} = \frac{1}{4\pi} \int_0^\infty d\omega \int d^2q (2\pi)^2 \left[ <E_iE_j^* > + <E_i^*E_j > + <B_iB_j^* > + <B_i^*B_j > - \delta_{ij}(<E \cdot E^* > + <B \cdot B^* >) \right]_{z=0} \quad (20)$$

Using Eqs. (14) for the $x$-component of the force we get

$$\sigma_{xx} = \frac{i}{4\pi} \int_0^\infty d\omega \int d^2q q_x (2\pi)^2 \left[ (k_z - k_z^*) \left( |v_p|^2 + |v_s|^2 \right) - \langle |v_p|^2 \rangle - \langle |v_s|^2 \rangle + (k_z + k_z^*) \langle w_p v_p^* + w_s v_s^* - c.c. \rangle \right] \quad (21)$$

Substituting Eqs. (15-18) for the amplitudes of the electromagnetic field in Eq. (21), and performing averaging over the fluctuating electromagnetic field with the help of Eq. (17), we get the $x$-component of the force:

$$F_x = \sigma_{xx} = \frac{\hbar}{8\pi^3} \int_0^\infty d\omega \int_{q<\omega/c} d^2q q_x \frac{q_x}{|\Delta|^2} \left[ (q^2 - \beta k q_x)^2 - \beta^2 k^2 q_y^2 \right]$$

$$\times [(q^2 - \beta k q_x)^2(1 - |R_{1p}|^2)(1 - |R_{2p}'|^2)|\Delta_{s\bar{s}}|^2]$$

$$- \beta^2 k^2 q_y^2(1 - |R_{1p}|^2)(1 - |R_{2s}'|^2)|\Delta_{s\bar{s}}|^2$$

$$+ \frac{\hbar}{2\pi^3} \int_0^\infty d\omega \int_{q>\omega/c} d^2q q_x \frac{q_x}{|\Delta|^2} \left[ (q^2 - \beta k q_x)^2 - \beta^2 k^2 q_y^2 \right] e^{-2k_z d}$$

$$\times [(q^2 - \beta k q_x)^2 \text{Im} R_{1p} \text{Im} R_{2p}' |\Delta_{s\bar{s}}|^2]$$

$$+ \beta^2 k^2 q_y^2 \text{Im} R_{1p} \text{Im} R_{2s}' |\Delta_{s\bar{s}}|^2$$

$$+ (p \leftrightarrow s) \left( n_2(\omega') - n_1(\omega) \right). \quad (22)$$

The symbol $(p \leftrightarrow s)$ denotes the terms which can be obtained from the preceding terms by permutation of the indexes $p$ and $s$. The first term in Eq. (22) represents the contribution to the friction from propagating waves $(q < \omega/c)$, and the second term from the evanescent waves $(q > \omega/c)$.

III. A FULLY RELATIVISTIC THEORY OF THE CASIMIR FORCE AND FRICTION FORCE, AND RADIATED HEAT TRANSFER FOR A SMALL PARTICLE MOVING PARALLEL TO A FLAT SURFACE

According to Eq. (23) the contribution to the friction force from the evanescent waves is given by

$$F_x = \frac{\hbar}{2\pi^3} \int_0^\infty d\omega \int_{q>\omega/c} d^2q q_x \frac{q_x}{|\Delta|^2} \left[ (q^2 - \beta k q_x)^2 - \beta^2 k^2 q_y^2 \right] e^{-2k_z d}$$

$$\times \left( \text{Im} R_{1p} \text{Im} \Delta_p + (p \leftrightarrow s) \left( n_2(\omega') - n_1(\omega) \right) \right), \quad (23)$$

where

$$\Delta_p = (q^2 - \beta k q_x)^2 R_{2p}' |\Delta_{s\bar{s}}|^2$$

$$+ \beta^2 k^2 q_y^2 R_{2s}' |\Delta_{s\bar{s}}|^2.$$
\[ \Delta_s = \Delta_p (p \leftrightarrow s). \]  
If in Eq. (23) one neglects the terms of the order \( \beta^2 \) then the contributions from waves with \( p \)- and \( s \)-polarization will be separated. In this case Eq. (23) is reduced to the formula obtained in 6.

Thus, to the order \( \beta^2 \) the mixing of waves with different polarization can be neglected, what agrees with the results obtained in 6. At \( T = 0 \) K the propagating waves do not contribute to friction but the contribution from evanescent waves is not equal to zero. Taking into account that \( n(-\omega) = -1 - n(\omega) \) from Eq. (23) we get the friction mediated by the evanescent electromagnetic waves at zero temperature (in literature this type of friction is denoted as quantum friction 16).

\[ F_x = \frac{\hbar}{2\pi^3} \int_0^\infty \frac{d\omega}{q_x} \int_{q_x c}^{\pi q_x} d^2q_x e^{-2k_x d} \left( \frac{\text{Im} R_{1p} \text{Im} R_{2p}^{*}}{|\Delta_{pp}|^2} + \frac{\text{Im} R_{1s} \text{Im} R_{2s}^{*}}{|\Delta_{ss}|^2} \right) (n_2(\omega') - n_1(\omega)), \]  

(24)

The friction force acting on a small particle moving in parallel to a flat surface can be obtained from the friction between two semi-infinite bodies in the limit when one of the bodies is sufficiently rarefied. We will assume that the rarefied body consists of small particles which have electric dipole moments. We assume that the dielectric permittivity of this body, say body 2, is close to the unity, i.e. \( \varepsilon_2 - 1 \rightarrow 4\pi n\alpha \ll 1 \), where \( n \) is the concentration of particles in body 2 in the co-moving reference frame \( K' \), \( \alpha \) is their electric polarizability. To linear order in the concentration \( n \) the reflection amplitudes are

\[ R_{2p} = \frac{\varepsilon_2' k_z - \sqrt{k_z^2 - (\varepsilon_2' - 1)k_z^2}}{\varepsilon_2' k_z + \sqrt{k_z^2 - (\varepsilon_2' - 1)k_z^2}} \approx \frac{\varepsilon_2' - 1 q^2 + k_z^2}{4 k_z^2} \frac{n\pi q^2 + k_z^2}{k_z} \frac{1}{\alpha'}, \]

\[ R_{2s} = \frac{k_z - \sqrt{k_z^2 - (\varepsilon_2' - 1)k_z^2}}{k_z + \sqrt{k_z^2 - (\varepsilon_2' - 1)k_z^2}} \approx \frac{\varepsilon_2' - 1 q^2 - k_z^2}{4 k_z^2} \frac{n\pi q^2 - k_z^2}{k_z} \frac{1}{\alpha'}. \]

To linear order in the concentration \( n \) the functions \( \Delta_{pp}, \Delta_{ss}, \Delta_{sp} \) and \( \Delta_{ps} \) should be calculated at \( n = 0 \). Using that \( \Delta_{pp} = \Delta_{ss} = \Delta_{sp} = \Delta_{ps} = 1 \) for \( n = 0 \), we get

\[ \Delta = (q^2 - \beta k_{q_x})^2 - \beta^2 k_z^2 q_y^2 = \frac{(qq_y')^2}{\gamma^2}, \]

\[ \Delta_p = \left( q^2 (q^2 - \beta k_{q_x})^2 + \beta k_z^2 q_y^2 \right) + k_z^2 (q^2 - \beta k_{q_x})^2 - \beta^2 k_z^2 q_y^2 \frac{\pi n\alpha'}{k_z^2} \]

\[ = q^2 (q^2 [k_z^2 + (k - \beta k_{q_x})^2] + k_z^2 [q^2 - 2\beta^2 q_y^2]) \frac{\pi n\alpha'}{k_z^2} \]

\[ = q^2 (q^2 (k - \beta k_{q_x})^2 + 2k_z^2 [q^2 - \beta^2 q_y^2]) \frac{\pi n\alpha'}{k_z^2}, \]

\[ \Delta_s = \left( q^2 (q^2 - \beta k_{q_x})^2 + \beta k_z^2 q_y^2 \right) - k_z^2 [q^2 - \beta k_{q_x})^2 - \beta^2 k_z^2 q_y^2] \frac{\pi n\alpha'}{k_z^2} \]
The number of the particles in a slab with thickness $dz$ absorbed by plate in the system in the particle-plate configuration

where $\alpha' = \alpha(\omega')$.

The friction force acting on a particle moving parallel to a plane surface can be obtained as the ratio between the change of the frictional shear stress between two surfaces after displacement of body 2 by small distance $dz$, and the number of the particles in a slab with thickness $dz$:

$$f_{x}^{\text{part}} = \frac{dF_{x}(z)}{n'dz} \big|_{z=d} = \frac{\hbar}{\gamma \pi^{2}} \int_{0}^{\infty} d\omega \int_{q>\omega/c} d^{2}q_{x} \frac{e^{-2k_{z}d[\text{Im}R_{1p}(\omega)\phi_{p} + \text{Im}R_{1s}(\omega)\phi_{s}][\text{Im}\omega']}}{k_{z}^{2}} \omega/c \big\{\text{Im}R_{1p}(\omega)\phi_{p} + \text{Im}R_{1s}(\omega)\phi_{s}\big\}[\text{Im}\omega'] \big(n_{2}(\omega') - n_{1}(\omega)\big), \quad (27)$$

where $n' = \gamma n$ is the concentration of particles in body 2 in the reference frame $K$.

$$\phi_{p} = (\omega'/c)^{2} + 2\gamma^{2}(\omega^{2} - \beta^{2}q^{2})\frac{k_{z}^{2}}{q^{2}}$$

$$\phi_{s} = (\omega'/c)^{2} + 2\gamma^{2}\beta^{2}q^{2}\frac{k_{z}^{2}}{q^{2}}$$

At $T_{2} = T_{1} = 0$ K we get

$$f_{x}^{\text{part}} = -\frac{\hbar}{\gamma \pi^{2}} \int_{-\infty}^{\infty} dq_{y} \int_{0}^{\infty} dq_{x} \int_{0}^{q_{y}V} d\omega \frac{q_{x}}{k_{z}} e^{-2k_{z}d[\text{Im}R_{1p}(\omega)\phi_{p} + \text{Im}R_{1s}(\omega)\phi_{s}][\text{Im}\omega']} \big(n_{2}(\omega') - n_{1}(\omega)\big), \quad (28)$$

For $\beta^{2} \ll 1$ and $q \gg \omega/c$, Eq. (27) is reduced to the result of non-relativistic theory:

$$f_{x}^{\text{part}} = \frac{2\hbar}{\pi^{2}} \int_{0}^{\infty} d\omega \int_{q>\omega/c} d^{2}q_{x} q_{y} e^{-2q d[\text{Im}R_{1p}(\omega)\phi_{p} + \text{Im}R_{1s}(\omega)\phi_{s}][\text{Im}\omega']} \big(n_{2}(\omega') - n_{1}(\omega)\big), \quad (29)$$

The heat absorbed by the body 1 in the $K$ system in the plate-plate configuration is determined by the expression which is very similar to the expression for the friction force:

$$P_{1} = \frac{\hbar}{2\pi^{2}} \int_{0}^{\infty} d\omega \int_{q>\omega/c} d^{2}q \frac{\omega}{k_{z}} \big(|\omega| - \beta q_{x}\big)^{2} - \beta^{2}k_{z}^{2}q_{y}^{2} e^{-2k_{z}d[\text{Im}R_{1p}(\omega)\phi_{p} + \text{Im}R_{1s}(\omega)\phi_{s}][\text{Im}\omega']} \big(n_{2}(\omega') - n_{1}(\omega)\big), \quad (30)$$

Using result obtained for the friction in the particle-surface configuration from (30) and (27) we get the heat absorbed by plate in the $K$ system in the particle-plate configuration:

$$P_{1}^{\text{part}} = \frac{\hbar}{\gamma \pi^{2}} \int_{0}^{\infty} d\omega \int_{q>\omega/c} d^{2}q \frac{\omega}{k_{z}} e^{-2k_{z}d[\text{Im}R_{1p}(\omega)\phi_{p} + \text{Im}R_{1s}(\omega)\phi_{s}][\text{Im}\omega']} \big(n_{2}(\omega') - n_{1}(\omega)\big), \quad (31)$$

The heat absorbed by a particle in the $K'$ system ($P_{2}'$) can be obtained from the relation

$$f_{x}V = P_{1} + \frac{P_{2}'}{\gamma}, \quad (32)$$

which follows from the Lorentz transformation of the Poynting vector. From (32) we get

$$P_{2}' = \frac{\hbar}{\gamma \pi^{2}} \int_{0}^{\infty} d\omega \int_{q>\omega/c} d^{2}q \frac{\omega}{k_{z}} e^{-2k_{z}d[\text{Im}R_{1p}(\omega')\phi_{p} + \text{Im}R_{1s}(\omega')\phi_{s}][\text{Im}\omega]} \big(n_{1}(\omega') - n_{2}(\omega)\big), \quad (33)$$

where we transformed variables $\omega, q_{x}$ in the integrands (27) and (33) to $\omega', q_{x}'$ using the fact that the Lorentz transformation has unit Jacobian. After such changing we denoted “dummy” variable $\omega', q_{x}'$ as $\omega, q_{x}$.
The Casimir force between two moving plates mediated by the evanescent waves is given by

\[ F_z = \frac{\hbar}{4\pi^2} \text{Im} \int_0^\infty d\omega \int_{q>\omega/c} d^2q \frac{k_z}{\Delta} e^{-2k_zd} \left[ R_{1p}\Delta_{1p} + R_{1s}\Delta_{1s} \right] \left[ 1 + n_1(\omega) + n_2(\omega') \right] \]

\[ + \frac{\hbar}{4\pi^2} \int_0^\infty d\omega \int_{q>\omega/c} d^2q \frac{k_z}{\Delta} [ (q^2 - \beta k q_z)^2 - \beta^2 k_z^2 q_y^2 ] e^{-2k_zd} \]

\[ \times \{ \text{Im} R_{1p}\text{Re} \Delta_p - \text{Re} R_{1p}\text{Im} \Delta_p + (p \leftrightarrow s) \} \left( n_1(\omega) - n_2(\omega') \right). \]  

(34)

where

\[ \Delta_{1p} = (q^2 - \beta k q_z)^2 R_{2p} \Delta_{ss} + \beta^2 k_z^2 q_y^2 R_{2s} \Delta_{sp}, \]

\[ \Delta_{1s} = \Delta_{1p}(p \leftrightarrow s). \]  

In the limit \( n \to 0; \) \( \Delta_{1p(s)} = \Delta_{p(s)}. \) After similar calculations as above for the Casimir force acting on a small particle moving parallel to a flat surface we get

\[ F_{\text{part}}^z = \frac{\hbar}{2\gamma^2} \int_0^\infty d\omega \int_{q>\omega/c} d^2q e^{-2k_zd} \left\{ [\phi_p\text{Im} R_{1p} + \phi_s\text{Im} R_{1s}] \text{Re} \alpha' \coth \left( \frac{\hbar \omega}{k_B T_1} \right) \right\} \]

\[ + [\phi_p\text{Re} R_{1p} + \phi_s\text{Re} R_{1s}] \text{Im} \alpha' \coth \left( \frac{\hbar \omega'}{k_B T_2} \right) \]  

(35)

IV. COMPARISON WITH THE PREVIOUS RESULTS

Recently Pieplow and Henkel presented a fully covariant theory of the Casimir force and friction force acting on small neutral particle moving parallel to a flat surface. This theory is in agreement with relativistic theory presented by Dedkov and Kyasov. In this Comment we have shown that the results of PH and DK for contribution to friction from evanescent waves in the particle-plate configuration are determined by the first derivative of friction force in the plate-plate configuration assuming that one of the plate is sufficiently rarefied. However, inverse procedure is not possible. It is not possible to recover the whole function knowing only its first derivative. The contribution to friction from the propagating waves is more delicate. To make comparison between contributions to friction from propagating waves in the the particle-plate configuration and the plate-plate configuration it is necessary to consider slab with finite thickness and calculate the friction force acting on the both side of the slab. In contrast to the evanescent waves, which do not contribute to the friction force acting on the back side of the slab, the propagating waves contribute to the friction force acting on the both side of the slab. However, for large velocities (for example, above the Cherenkov threshold velocity) the friction is dominated by quantum friction determined by the evanescent waves.

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