On-shell recursion relations for gravity

Anthony Hall

Department of Physics and Astronomy, UCLA
Los Angeles, CA 90095–1547, USA
anthall@physics.ucla.edu

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Abstract

We extend the argument presented by Benincasa, Boucher-Veronneau, and Cachazo to show that graviton tree amplitudes are well behaved under large complex deformations of the momenta of a pair of like-helicity gravitons. This shows that BCFW recursion relations for gravity amplitudes can be constructed using such shifts, providing an alternative proof to the recent one by Arkani-Hamed and Kaplan. By using auxiliary recursion relations the cancellations which are hidden when using covariant Feynman diagrams become manifest.
I. INTRODUCTION

The standard Feynman diagram representation hides many remarkable properties of on-shell graviton amplitudes. For example, the Kawai-Lewellen-Tye relations from string theory indicate that tree-level gravity amplitudes are linear combinations of squared gauge-theory amplitudes \cite{1}. Thus the three- and four-point vertices of gauge theory generate the tree-level amplitudes derived from the Einstein-Hilbert action, with its infinite series of interactions. More recently Witten pointed out that scattering amplitudes in gauge and gravity theories have intriguing simplicity in twistor space \cite{2}. On-shell recursion relations \cite{3, 4, 5} which express an amplitude as products of physical on-shell amplitudes with fewer legs also exist in gravity theories \cite{6, 7, 8, 9}, although naive power counting based on Feynman diagrams indicates otherwise.

The recursion relations of Britto, Cachazo, Feng, and Witten (BCFW) produce very compact expressions for amplitudes and are obtained by shifting a pair of the external legs to complex momenta \cite{4}. The existence of BCFW recursion for any field theory hinges upon the deformed amplitudes vanishing when the shifted legs are taken to infinite momentum in special complex directions, and so, with the KLT relations in mind, it is natural to imagine that gravity may share these high-energy properties and satisfy on-shell recursion relations. The known examples of on-shell recursion relations for graviton tree amplitudes \cite{6, 7, 8, 9} shows that the high-energy behavior of gravity in special complex directions is better behaved than one expects from the power counting of momenta in individual Feynman diagrams. It is surprising that, for some complex deformations, gravity amplitudes are actually better behaved in these high-energy limits than gauge theory. Very interestingly, the existence of an on-shell recursive framework for the calculation of graviton tree amplitudes implies that all of perturbative gravity follows from the on-shell three vertex with complex momenta.

The surprising properties of graviton tree amplitudes have repercussions at loop level. Remarkable results for the maximally supersymmetric $\mathcal{N} = 8$ supergravity theory have inspired the “no-triangle hypothesis” which claims that the bubble and triangle graphs vanish along with additional rational terms, leaving only scalar box integrals contributing to one-loop $\mathcal{N} = 8$ supergravity amplitudes \cite{10, 11, 12, 13}. The maximally helicity-violating class of amplitudes \cite{10} provided the initial evidence, and recent developments have bolstered the hypothesis for $\mathcal{N} = 8$ supergravity. Explicit calculations have verified the conjecture for
\( \mathcal{N} = 8 \) supergravity at six points \cite{13}, seven-point amplitudes have infrared limits consistent with the hypothesis \cite{12, 13}, and beyond this the scaling and factorization properties of amplitudes provide evidence of the cancellations \cite{11, 13, 14}. The complete \( \mathcal{N} = 8 \) four-point amplitude at three loops gains improved ultraviolet behavior from these cancellations \cite{15}.

The observed cancellations do not appear to be due to supersymmetric constraints. One-loop calculations in pure gravity indicate that the requisite cancellations arise with the Einstein-Hilbert action alone, and that the high-energy improvements originate from the rather tame scaling behavior of graviton tree amplitudes \cite{14}. Thus one expects to understand the cancellations of ultraviolet divergences at loop level by studying the tree-level amplitudes.

Benincasa, Boucher-Veronneau, and Cachazo (BBC) \cite{9} proved that under the standard BCFW shift involving one positive and one negative helicity graviton the amplitudes are well behaved, confirming the existence of recursion relations for this case. In this paper we extend this proof to show that graviton tree amplitudes also scale nicely under BCFW shifts of like-helicity gravitons, proving the recursion relations for these cases as well. An auxiliary recursion is used to bring amplitudes into a form where the mild high-energy behavior is manifest. The upper bounds on the high-energy scaling are independent of the number of external gravitons being scattered.

Very recently Arkani-Hamed and Kaplan Ref. \cite{16} gave an elegant argument applicable to general \( D \)-dimensional field theories, using the Lagrangian in a particular gauge \cite{17, 18}, to find the high-energy scaling of gravity amplitudes under BCFW shifts of any helicity. The improved high energy behavior, relative to power-counting expectations, is attributed to an enhanced spin symmetry at infinite momentum. Our paper gives an alternative \( D = 4 \) proof that graviton amplitudes satisfy the scaling needed for BCFW recursion relations under the like-helicity BCFW shifts not handled in Refs. \cite{9, 14}.

We begin by reviewing the BCFW recursion relations. The BBC proof of the recursion formulas for pure gravity under the complex deformations of two opposite helicity gravitons are examined, particularly the use of specially designed auxiliary recursion formulas. We extend this argument to the like-helicity BCFW case and find that graviton amplitudes vanish when the shift parameter is taken to infinity.
II. REVIEW OF ON-SHELL RECURSION

We consider first the BCFW recursion relations for tree-level gravity amplitudes which are obtained by shifting a pair of spinors from the external gravitons $i$ and $j$,

$$
|i(z)\rangle = |i\rangle + z|j\rangle, \quad |j(z)\rangle = |j\rangle - z|i\rangle,
$$

(2.1)

where $|i\rangle$ and $|j\rangle$ are Weyl spinors of positive and negative helicity. The momenta of these two gravitons are still null but shifted to complex values,

$$
p_i(z) = |i\rangle[i] + z|j\rangle[i], \quad p_j(z) = |j\rangle[j] - z|j\rangle[i],
$$

(2.2)

while overall momentum remains conserved,

$$p_i(z) + p_j(z) = p_i + p_j.
$$

(2.3)

We use the shorthand "\langle ij\rangle" to denote this choice of shifted spinors.

The polarization tensors for gravitons of positive and negative helicity, written in terms of arbitrary reference spinors $|\mu\rangle$ and $|\mu\rangle$, are

$$
\epsilon_{++}^{ab,bb} = \frac{(|\mu\rangle_a[p]|\mu\rangle_b)(|\mu\rangle_b[p]|\mu\rangle_a)}{\langle \mu p \rangle^2},
$$

(2.4)

$$
\epsilon_{--}^{ab,bb} = \frac{(|p\rangle_a[\mu]|\mu\rangle_b)(|p\rangle_b[\mu]|\mu\rangle_a)}{|\mu p|^2}.
$$

If leg $p$ is deformed under the BCFW shift, then its polarization tensor also gains $z$ dependence. If, for example, leg $p$ is $i^+$ or $j^-$ we have $\epsilon_p(z) \sim \frac{1}{z^2}$.

After applying the BCFW shift, a gravity amplitude $M$ remains on shell but is now a meromorphic function $M(z)$ with simple poles in $z$ whenever a $z$-dependent propagator in its Feynman diagrams is on shell. At tree level a propagator partitions a Feynman diagram into two trees attached at the internal line, so the internal line is $z$-dependent for Feynman diagrams which partition legs $i$ and $j$ into opposite trees. Writing $P(z)$ for the sum of the external particles’ momenta at one tree which flows through the internal line, the condition for the propagator to yield a pole is

$$0 = P(z)^2 = P^2 + z\langle j|P|i\rangle,
$$

(2.5)

where $\langle j|P|i\rangle \equiv \langle j^-|P|i^-\rangle$. Provided that $M(z)$ vanishes as $z \to \infty$, we have $\oint M(z)/z = 0$. Applying Cauchy’s theorem to $\oint M(z)/z$ yields a residue for $z = 0$, the desired unshifted
amplitude, and residues for the values of $z$ for which $P(z)^2 = 0$. The universal factorization of amplitudes at these poles yields the BCFW formula \[4\],

\[ M_n(p_1, \ldots, p_n) = \sum_{\mathcal{I}, \mathcal{J}} \sum_{h=\pm} M_{\mathcal{I}}(p_{\mathcal{I}}(z), \ldots, -P_{\mathcal{I}}^h(z)) \times \frac{1}{P_{\mathcal{I}}^2} \times M_{\mathcal{J}}(P_{\mathcal{J}}^{-h}(z), \ldots, p_{\mathcal{J}}(z)). \tag{2.6} \]

Each term appearing in the recursion formula is one of the residues of $M(z)/z$. We use the index $\mathcal{J}$ to denote the external gravitons attached to the same subamplitude as the BCFW-shifted leg $j$, so that $j \in \mathcal{J}$, and $\mathcal{I}$ represents all the remaining external legs. The momentum $P_{\mathcal{I}}(z)$ is the shifted total momentum leaving from $M_{\mathcal{I}}$’s external legs, and the recursion formula sums over the helicity of the internal state with momentum $P_{\mathcal{I}}$. The subamplitudes $M_{\mathcal{I}, \mathcal{J}}$ are on shell, evaluated with the shifted spinors at the value of $z = z_{\mathcal{I}}$ for which $P_{\mathcal{I}}(z)^2 = 0$.

The internal on-shell momentum can be written explicitly as a two-spinor. From the identities

\[ P(z)[i] = P[i], \quad \langle j| P(z) = \langle j| P, \tag{2.7} \]

where the slash on the $P$ is implicit, the shifted spinors of an internal particle can be written as

\[ P(z) = |P(z)]|P(z)| = \frac{1}{\langle j|P[i]}(P[i]|\langle j| P). \tag{2.8} \]

The potential obstruction to constructing BCFW recursion relations for an arbitrary theory is that the amplitudes must vanish for large $z$. In general this condition is not straightforward to verify. For example, in gauge theory and gravity the Lagrangians provide vertices which scale like $z$ and $z^2$ at high energy, respectively. To tame this high-energy behavior of graviton scattering amplitudes, and in turn prove that the BCFW shift “$(i^+ j^-)$” indeed generates on-shell recursion, BBC used an auxiliary recursion \[9\].

The auxiliary recursion relations follow from a shift of all the positive-helicity gravitons $k^+$ and a single negative helicity leg $j$,

\[ \begin{aligned}
\{ |k(w)\} &= |k\} + w|j\} \quad \forall \ k \in k^+, \\
|j(w)\} &= |j\} - w \sum_{k \in k^+} |k\}.
\end{aligned} \tag{2.9} \]
The momentum of an internal propagator after applying the auxiliary shift is
\[ P_I(w) = P_I + w|j\rangle \sum_{k \in I^+} [k], \tag{2.10} \]
so that the values of \( w \) which yield an on-shell internal propagator are
\[ w_I = -\frac{P_I^2}{\sum_{k \in I^+} \langle j|P_I|k\rangle}. \tag{2.11} \]
We use the conventions of Ref. [9] where \( P_I = \sum_{k \in I} p_k \) and define \( I^+ \) to be the set of positive-helicity external gravitons in \( M_I \).

A non-vanishing contribution to the recursion can only be achieved when the on-shell internal propagator partitions the Feynman diagram such that at least one of the positive helicity gravitons is on the opposite side of the propagator from the others. Thus the auxiliary recursion generates diagrams which partition the external legs into \( I \) and \( J \) labeled such that \( j \in J \) in correlation with the BCFW shift, and at least one positive-helicity graviton lies in \( I \) and another in \( J \).

As in the BCFW case, the internal on-shell momentum can be written explicitly as a two-spinor. From the identities
\[ P_I(w)|\alpha\rangle = P_I|\alpha\rangle \text{ for } |\alpha\rangle = \sum_{k \in I^+} |k\rangle, \tag{2.12} \]
\[ \langle j|P_I(w) = \langle j|P_I, \]
the shifted spinors of an internal particle can be written as
\[ P_I(w) = |P_I(w)\rangle[P_I(w)| \tag{2.13} \]
\[ = \frac{1}{\langle j|P_I|\alpha\rangle(P_I|\alpha\rangle(\langle j|P_I). \]

To have the proper behavior to yield on-shell recursion for gravity amplitudes the auxiliary shift must make an \( n \)-point amplitude \( M_n(w) \) vanish as \( w \) is taken to infinity. The most divergent Feynman diagrams contributing to \( M_n(w) \) contain only cubic vertices, \( n - 2 \) of them, each contributing a factor of \( w^2 \). The \( p \) positive-helicity gravitons and the negative helicity leg \( j \) each yield a factor of \( 1/w^2 \) from their polarization tensors for a total of \( 1/w^{2p+2} \). Also the \( w \)-dependence in each of the \( n-3 \) propagators cannot cancel in general [9], so \( 1/w^{n-3} \) is gained from propagators. Overall the most divergent contributions to \( M_n(w) \) scale at large \( w \) as \( 1/w^{p-m+3} \), where we write \( p \) and \( m \) for the number of external gravitons of positive-
and negative-helicity, respectively. Thus the above auxiliary shift leads to on-shell recursion relations for gravity amplitudes with $p - m \geq -2$.

Likewise, gravity amplitudes with $m - p \geq -2$ vanish at large $w$ under a shift of all the negative-helicity gravitons $k^-$ and a single positive helicity leg $i$,

$$\left\{ |k(w)| = |k| - w|j]\right\} \forall k \in k^-,$$  
$$|i(w)| = |i| + w \sum_{k \in k^-} |k|.$$

The set of spinor deformations introduced by BBC give recursion relations for all graviton scattering amplitudes, and thus allow gravity amplitudes to be recursively calculated starting from on-shell, complex three vertices alone. However, these recursion relations are overly complicated compared to the relations obtained using the standard BCFW shift (2.1). They are, however, useful for proving that the standard shifts (2.1) lead to good recursion relations.

III. REVIEW OF THE BBC PROOF FOR “$\langle i^+ j^- \rangle$” SHIFTS

In order to obtain recursion relations for amplitudes under the BCFW shift we seek a representation of amplitudes where the $z$-dependence of polarization tensors will help tame divergent contributions from vertices. We first review the BBC proof of the large-$z$ behavior where leg $i$ is a positive-helicity graviton and leg $j$ is negative. As described in their paper, auxiliary recursions based on the shifts described in the previous section are very helpful for showing that the amplitude vanishes for large $z$ after a BCFW shift, eqn. (2.1). By applying the auxiliary shifts of eqns. (2.9) and (2.14) the associated recursion relations, valid for all graviton amplitudes, give $w$-dependence to each positive- or negative-helicity graviton, respectively. For each auxiliary recursion diagram, which partitions the amplitude into a pair of tree amplitudes connected by an internal graviton, Cauchy’s theorem freezes the value $w = w_I$ so that the internal graviton is on shell and thus yields a pole from its propagator.

Now consider applying the BCFW shift “$\langle i^+ j^- \rangle$” to the auxiliary diagrams. Gravitons $i$ and $j$ either lie on opposite sides of the internal graviton, or they are both part of the tree amplitude $M_J$. The two possibilities are illustrated in Fig. I.

If an auxiliary diagrams has $i$ and $j$ on opposite sides of the partition, then $P_I$ is shifted to $P_I(z) = P_I + z|j]]i]$ and the values of $w$ at the auxiliary poles for the deformation eqn. (2.9)
FIG. 1: After applying a BCFW shift to the auxiliary diagrams, legs $i$ and $j$ lie on either opposite sides of the internal propagator or both on the same tree amplitude. $M_J$ is labeled as the amplitude with the BCFW-shifted leg $j$, $M_I$ has the remaining external gravitons. The other external gravitons besides $i$ and $j$ are suppressed. The $z$-dependent gravitons and on-shell amplitudes are indicated in black.

are shifted to be just linear in $z$,

$$w_I(z) = -\frac{P_I^2 + z\langle j|P_I|i\rangle}{\sum_{k\in I^+}\langle j|P_I|k\rangle}$$

$$= w_I - z\frac{\langle j|P_I|i\rangle}{\sum_{k\in I^+}\langle j|P_I|k\rangle}.$$  

(3.1)

The internal momentum can be written as in eq. (2.13) so that after the BCFW shift with $z$ we have

$$P_I(w(z), z) = \frac{1}{\langle j|P_I|\alpha\rangle} \left( (P_I + z|j|i|\alpha) \right)(\langle j|P_I\rangle).$$  

(3.2)

The $z$-dependence of the internal momentum’s spinors is confined to $|P_I(w(z), z))$.

Provided that leg $i$ is not the only positive-helicity graviton on $M_I$, all the polarization tensors deformed by the auxiliary shift help to reduce the degree of divergence under the BCFW shift. The most divergent Feynman diagrams for $M_I(w(z), z)$ contain only three vertices which each contribute $z^2$. Let $m_I$ and $p_I$ denote the number of negative- and positive-helicity external gravitons for $M_I$. Including the internal graviton there are $m_I + p_I + 1$ particles in $M_I$ and thus would have $m_I + p_I - 1$ three vertices giving a total of $z^{2(m_I + p_I - 1)}$ from vertices. The $p_I$ positive helicity gravitons and the $|P(z)\rangle$ from the internal particle contribute $1/z^{2(p_I + h)}$ from polarization tensors. With all $m_I + p_I - 2$ propagators depending on $z$ as shown in Ref. [9] the propagators yield $1/z^{m_I + p_I - 2}$. Overall we have

$$M_I(w(z), z) \sim 1/z^{p_I - m_I + 2h}.$$  

(3.3)

The same counting for $M_J(w(z), z)$ gives an extra factor of $1/z^2$ from the polarization
tensor of leg $j$ and $1/z^{-2h}$ from the internal particle’s polarization. Overall we have

$$M_{\mathcal{J}}(w(z), z) \sim 1/z^{p_{\mathcal{J}}-m_{\mathcal{J}}-2h+2}. \quad (3.4)$$

Combining the contributions of $M_{\mathcal{I}}(w(z), z)$, $M_{\mathcal{J}}(w(z), z)$, and the internal propagator $1/P_{\mathcal{I}}(z)^2$ to the auxiliary diagrams with $i$ and $j$ on opposite trees, the leading $z$ behavior for this class of auxiliary diagrams is $1/z^{p-m+3}$ and similarly for auxiliary diagrams from the shifts of eqn. (2.14). The $z$ dependence of polarization tensors produced through the auxiliary recursion has thus tamed the divergent contributions from three vertices.

In case leg $i$ is the only external graviton with positive helicity on its partition, $\mathcal{I}^+ = \{i\}$, then $w(z) = w - z$ which gives

$$|i(w(z), z)\rangle = |i(z)\rangle + w(z)|j\rangle = |i\rangle + z|j\rangle + (w-z)|j\rangle = |i(w)\rangle, \quad (3.5)$$

$$P_{\mathcal{I}}(w(z), z) = \frac{1}{(j|P_{\mathcal{I}}i\rangle)}(\langle j| P_{\mathcal{I}}),$$

so that $M_{\mathcal{I}}$ is actually independent of $z$. The tree $M_{\mathcal{J}}$ has the deformed spinors

$$|j(w(z), z)\rangle = |j(z)\rangle - w(z) \sum_{k \in k^+} |k\rangle = |j\rangle - z|i\rangle - w \sum_{k \in k^+} |k\rangle + z \sum_{k \in k^+} |k\rangle \quad (3.6)$$

$$= |j(w)\rangle + z \sum_{k \in k^+} |k\rangle,$$

$$\left\{|k(w(z))\rangle = |k\rangle + w(z)|j\rangle = |k(w)\rangle - z|i\rangle \right\} \forall k \in \mathcal{J}^+.$$

Provided that the internal graviton leaving tree $\mathcal{J}$ has negative helicity, this is precisely the auxiliary shift of all the positive helicity gravitons again, where the parameter $w$ in eqn. (2.9) has been replaced by $-z$. The tree $\mathcal{J}$ contains a sufficient number of positive-helicity gravitons to ensure that $1/P_{\mathcal{I}}(z)^2 \times M_{\mathcal{J}}(z)$ vanishes at large $z$. If instead the internal graviton leaves tree $\mathcal{I}$ with negative helicity, then $M_{\mathcal{I}}$ is either a three vertex vanishing under the BCFW shift or trivially vanishes with too many negative-helicity legs. Similar arguments apply to the BCFW-shifted diagrams obtained from the auxiliary shifts of eqn. (2.14).

Auxiliary diagrams for which the BCFW-shifted legs $i$ and $j$ both lie on the same tree, $i \in \mathcal{J}$, give $w(z) = w$ and from eqn. (2.9) we have

$$|i(w, z)\rangle = |i(z)\rangle + w|j\rangle = |i(w)\rangle + z|j\rangle, \quad (3.7)$$

$$|j(w, z)\rangle = |j(z)\rangle - w \sum_{k \in k^+} |k\rangle = |j(w)\rangle - z|i\rangle.$$
Thus in this class of auxiliary diagrams, the net effect of both shifts is a BCFW deformation of the amplitude $M_{\mathcal{F}}(w)$. Similar statements are true for the auxiliary diagrams obtained from eqn. (2.14). By repeated application of the auxiliary recursion any graviton tree amplitude can be reduced to products of on-shell three vertices. In the process of reducing an amplitude into products of subamplitudes via the auxiliary recursion, one encounters partitions of the legs $i$ and $j$ which fall into one of the classes outlined above. The special class in which $i$ and $j$ both lie on $M_{\mathcal{F}}$ recursively yields a BCFW-shifted on-shell three vertex,

$$M(P^\pm, i^+(z), j^-(z)) \sim 1/z^2. \quad (3.8)$$

Thus graviton tree amplitudes under the “$\langle i^+j^- \rangle$” shift vanish at large $z$ and are dominated by the three vertices $M(P^\pm, i^+(z), j^-(z))$ scaling like $1/z^2$.

**IV. GRAVITY AMPLITUDES UNDER “$\langle i^-j^- \rangle$” SHIFTS**

Now we use the BBC auxiliary recursion to determine the scaling of graviton amplitudes under the like-helicity BCFW shifts, first considering the “$\langle i^-j^- \rangle$” case. Using the auxiliary recursion shown to vanish as $1/w^{p-m+3}$ in Ref. [9], the shifted external spinors are given by eqn. (2.9). This is the same as the auxiliary shift introduced in Ref. [9], but leg $i$ is no longer one of the positive helicity legs. The large-$w$ behavior is the same as Ref. [9] since we still have each of the $p$ positive-helicity polarization tensors contributing $1/w^2$ and the negative helicity leg $j$ gives another $1/w^2$ from $|j(w)|$ for a total of $1/w^{2p+2}$ from polarization tensors.

Thus we have a valid auxiliary recursion for $p-m \geq -2$. The shifted momentum of an internal propagator is given by eqn. (2.10) so that the values of $w$ in eqn. (2.11) yield an on-shell internal propagator.

For graviton amplitudes with $p-m \leq -2$ we use the following auxiliary shift. The negative-helicity spinors of all the negative helicity legs $k^-$ are shifted, except for leg $i^-$,

$$\left\{ |k(w)| = |k| - w|i| \right\} \quad \forall k \in k^- \setminus i,$n
$$|i(w)| = |i| + w \sum_{k \in k^- \setminus i} |k|.$$

Graviton amplitudes scale as $1/w^{m-p-1}$ at large $w$ under this auxiliary shift because the negative-helicity leg $i$ contributes a factor of $w^2$ from its polarization tensor. Thus for amplitudes with $p-m \leq -2$ the shift in eqn. (4.1) gives auxiliary recursion relations, and
if \( p - m \geq -2 \) then eqn. (2.9) is applicable. The combination of both auxiliary shifts can reduce any graviton amplitude to products of on-shell three vertices.

Consider applying the BCFW shift of two negative-helicity legs \( i \) and \( j \),

\[
|i(z)| = |i| + z|j|, \quad |j(z)| = |j| - z|i|, \quad (4.2)
\]
to an amplitude \( M(w) \). Auxiliary diagrams, as in Fig. 1 for which \( i \) and \( j \) lie on opposite sides of the propagator have \( w^I(z) \) linear in \( z \) as in eqn. (3.1), and as in Ref. 9 there are no accidental cancellations which remove the \( z \)-dependence of any of the Feynman diagram propagators. The most divergent contributions to \( M_I(w(z), z) \) and \( M_J(w(z), z) \) arise from Feynman diagrams with only three vertices that each scale as \( z^2 \). In the cases with \( p - m \geq -2 \), auxiliary diagrams with \( i \) and \( j \) on opposite sides of the propagator gain a factor of \( z^2 \) relative to the “\( i+j \)” case from \( |i(z)| \) in the polarization tensor of leg \( i \). This gives such auxiliary diagrams a \( z \)-dependence at large \( z \) which is \( 1/z^{p-m+1} \), vanishing for \( p \geq m \). In the cases with \( p - m \leq -2 \), \( |i(w(z), z)| \) gives the auxiliary diagrams with \( i \) and \( j \) on opposite sides of the propagator a factor of \( z^4 \) relative to the scaling of the “\( i+j \)” case. These auxiliary diagrams scale as \( 1/z^{m-p-1} \) for large \( z \), vanishing for \( m - p \geq 2 \). The special class of auxiliary diagrams for which \( J^- = \{j\} \), similar to the \( I^+ = \{i\} \) cases for the “\( i+j \)” argument, are relevant for the \( m - p \geq 2 \) auxiliary diagrams and also vanish at large \( z \).

Unlike the situation for “\( i+j \)” BCFW shifts, the auxiliary recursion relations do not directly give vanishing large-\( z \) behavior in the “\( i-j \)” cases where \( i \) and \( j \) lie on opposite sides of the propagator. Using the naive \( z^2 \) scaling for three vertices, the auxiliary diagrams scale like a constant at large \( z \) for the case \( p - m = -1 \). However, we have used the three vertex in de Donder gauge [19, 20] with two on-shell legs to find that the large-\( z \) scaling for the three vertex to which two external legs attaches is better than the naive scaling. We find that the large-\( z \) scaling of the vertex in Fig. 2 is improved by the factor \( 1/z \) for auxiliary diagrams with \( i \notin J \) under the auxiliary shift of eqn. (2.9). If a positive helicity leg \( q^+ \) is attached to this three vertex it has \( z \) dependence through \( w(z) \) under the shift of eqn. (2.9). The momentum \( q^- \) is independent of \( z \), but \( q^+ \) is shifted to

\[
q(w^I(z)) = q(w^I) - z \frac{\langle j | P^I | i \rangle}{\sum_{k \in I^+} \langle j | P^I | k \rangle} |j| |q|. \quad (4.3)
\]
The momentum \( p_j \) after applying the auxiliary and BCFW shifts is

\[
p_j(w^I(z), z) = p_j(w^I) - z|j| \left( \left| i \right| - \frac{\langle j | P^I | i \rangle}{\sum_{k \in I^+} \langle j | P^I | k \rangle} \sum_{k \in I^+} \langle k \rangle \right). \quad (4.4)
\]
FIG. 2: The vertices in $M_{\mathcal{J}}(w(z), z)$, with external gravitons $j$ and $q$ and internal line $P$, which scale at large $z$ better than expected from power counting.

We take the polarization vectors for the shifted legs $p_j^-$ and $q^\pm$ to be

$$\epsilon_j^- = \epsilon_q^+ = |j][q|, \quad \epsilon_q^- = |q][j|, \quad (4.5)$$

where $\epsilon_q^-$ need not shift under the deformations to remain orthogonal to $q^-$. The graviton polarization tensors are symmetric tensor products of these vectors, normalized as in eqn. (2.4). The possible vertices with two external $z$-dependent legs to consider are

$$V_{\mu\alpha, \nu\beta, \sigma\gamma}(p_j^-(w_I(z), z), q^-, P(z)), \quad (4.6)$$

$$V_{\mu\alpha, \nu\beta, \sigma\gamma}(p_j^+(w_I(z), z), q^+(w_I(z)), P(z)), \quad (4.7)$$

$$V_{\mu\alpha, \nu\beta, \sigma\gamma}(q_1^+(w_I(z)), q_2^+(w_I(z)), P(z)), \quad V_{\mu\alpha, \nu\beta, \sigma\gamma}(q_1^-, q_2^-(w_I(z)), P(z)).$$

The leading $z^2$ terms in these vertices cancel after contracting the vertex with polarization tensors for the external legs, $\epsilon_j^-$ and $\epsilon_q^{\pm}$. 

Above we assumed the vertex contributed $z^2$ with factors of $1/z^2$ from the polarization tensors $\epsilon_j^-$ and $\epsilon_q^{\pm}$, but the calculation from the three vertex shows that only the subleading terms proportional to $z$ contribute. This improves the estimate for the large-$z$ scaling of auxiliary diagrams with $i$ and $j$ on opposite sides of the propagator by a factor $1/z$, yielding $1/z^{p-m+2}$ behavior at large $z$ under “$(i^- j^-)$” for this type of auxiliary diagram when $p - m \geq -1$.

Auxiliary diagrams for which $i$ and $j$ lie on the same side of the propagator, $i \in \mathcal{J}$, give $w_I(z) = w_I$ and for the auxiliary shifts in eqns. (2.9) and (4.11) we have

$$|i(w, z)] = |i(w)] + z|j], \quad |j(w, z)] = |j(w)] - z|i], \quad (4.8)$$

As in the previous “$(i^+ j^-)$” case, we have $z$ dependence only in $M_{\mathcal{J}}(w)$ as a BCFW-shifted on-shell amplitude. To determine the large-$z$ scaling of this subamplitude, we apply the
auxiliary recursion to $M_J$ by deforming with the appropriate auxiliary shift in eqns. (2.9) and (4.1).

Applying the auxiliary recursion to $M_J$ yields auxiliary subdiagrams belonging to the class where either $i$ and $j$ are on opposite sides of the internal propagator or both $i$ and $j$ are on the same side. The situation after applying the auxiliary recursion relations to $M_J$ is depicted in Fig. 3.

After repeated application of the auxiliary shift to the subamplitudes in the latter subset of diagrams, eventually we reach an on-shell three vertex,

$$M(P^+, i^-(z), j^-(z)) = \left(\frac{\langle ij \rangle^3}{\langle jP \rangle \langle Pi(z) \rangle}\right)^2 \sim 1/z^2, \quad (4.9)$$

vanishing for large $z$. The auxiliary subdiagrams with $i$ and $j$ on opposite sides of the internal propagator have already been discussed above and vanish at large $z$. We conclude that all graviton amplitudes satisfy the “$\langle i^-, j^- \rangle$” BCFW recursion since all the auxiliary diagrams vanish at large $z$ under the BCFW shift.

V. GRAVITY AMPLITUDES UNDER “$\langle i^+ j^+ \rangle$” SHIFTS

Because we have shown that graviton scattering amplitudes vanish at large $z$ after applying the “$\langle i^- j^- \rangle$” BCFW shift, the same result for amplitudes under the “$\langle i^+ j^+ \rangle$” shift follows by applying a parity reflection. Alternatively, auxiliary recursion relations can be used to calculate the large-$z$ behavior for a “$\langle i^+ j^+ \rangle$” shift. The auxiliary shifts used are

$$\begin{align*}
|k(w)\rangle &= |k\rangle + w|j\rangle, \\
|j(w)\rangle &= |j\rangle - w \sum_{k \in k^+ \setminus j} |k\rangle,
\end{align*} \quad (5.1)$$

shifting the positive-helicity spinors of all the positive-helicity gravitons but $j$, and the shifts in eqn. (2.14). The analysis of the auxiliary diagrams after applying the “$\langle i^+ j^+ \rangle$” BCFW
shift is very similar to the above “\((i-j)\)” case.

VI. CONCLUSION

We have demonstrated that a set of auxiliary recursion relations bring graviton amplitudes \(M\) into a form where it is manifest that \(\lim_{z \to \infty} M(z) = 0\) under the BCFW shift of like-helicity gravitons, extending the argument of Benincasa, Boucher-Veronneau, and Cachazo for the “\((i^+j^-)\)” shift \(^9\). The vanishing of graviton amplitudes at large \(z\) agrees with the more general arguments of Ref. \(^{16}\), where the dominant contributions under large BCFW shifts are obtained from the Lagrangian in a special gauge. Our work gives an alternative proof of the vanishing large-\(z\) behavior when the shifted legs are of like helicity. We believe these results for high-energy graviton tree amplitudes will help shed light on the mysterious ultraviolet cancellations observed at loop level.

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