Complete Differentiable Semiclassical Spectral Asymptotics*, †

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Abstract

For an operator $A := A_h = A^0(hD) + V(x, hD)$, with a “potential” $V$ decaying as $|x| \to \infty$ we establish under certain assumptions the complete and differentiable with respect to $\tau$ asymptotics of $e_h(x, x, \tau)$ where $e_h(x, y, \tau)$ is the Schwartz kernel of the spectral projector.

1 Introduction

Consider a self-adjoint matrix operator

\begin{equation}
A := A_h = A^0(hD) + V(x, hD),
\end{equation}

where

\begin{equation}
|D^\beta \lambda^0(\xi)| \leq c_{\alpha \beta} (|\xi| + 1)^m \quad \forall \beta \forall \xi
\end{equation}

and

\begin{equation}
A^0(\xi) \geq c_0|\xi|^m - C_0 \quad \forall \xi.
\end{equation}
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We assume that $A^0(\xi)$ is $\xi$-microhyperbolic at energy level $\lambda$, i.e. for each $\xi$ there exists a direction $\ell(\xi)$ such that $|\ell(\xi)| \leq 1$ and

\[(\langle \ell(\xi), \nabla_\xi \rangle A^0(\xi) \nu, \nu) + |(A^0(\xi) - \lambda)\nu| \geq \epsilon_0 |\nu|^2 \quad \forall \nu. \tag{1.4}\]

Further, we assume that $V(x, \xi)$ is a real-valued function, satisfying

\[|D_\alpha \xi D_\beta x V(x, \xi)| \leq c_{\alpha, \beta}(|\xi| + 1)^m(|x| + 1)^{-\delta - |\beta|} \quad \forall \alpha, \beta \forall x, \xi \tag{1.5}\]

and

\[|D_\alpha \xi D_\beta x V(x, \xi)| \leq \epsilon \quad \forall \alpha, \beta: |\alpha| + |\beta| \leq 1 \forall x, \xi. \tag{1.6}\]

Our main theorem is

**Theorem 1.1.** Let conditions (1.2)–(1.4) and (1.6) with sufficiently small constant $\epsilon > 0$ be fulfilled. Then

(i) The complete spectral asymptotics holds for $\tau$: $|\tau - \lambda| \leq \epsilon$:

\[e_h(x, x, \tau) \sim \sum_{n \geq 0} \kappa_n(x, \tau) h^{-d+n} \tag{1.7}\]

where $e_h(x, y, \tau)$ is the Schwartz kernel of the spectral projector $\Theta(\tau - A_h)$ of $A_h$.

(ii) This asymptotics is infinitely differentiable with respect to $\tau$.

**Remark 1.2.** (i) Statement (i) was sketched under much more restrictive assumptions in Theorem 3.2 of [Ivr3]; however we provide here more detailed exposition.

(ii) In Theorem 2.8 we provide the dependence of the remainder on $|x|$.

(iii) This asymptotics is also infinitely differentiable with respect to $x$ but it is really easy.

Differentiability and completeness of the spectral asymptotics are really different. F.e. for operators with almost periodic with respect to $x$ perturbation $V(x, hD)$ the spectral asymptotics are complete (see [Ivr3] and references there) but in dimension 1 it is not necessarily differentiable even once due to spectral gaps. Furthermore, if we perturb an operator we study in this paper by an appropriate “negligible” operator (i. e. with $O(h^\infty)$
1. Introduction

norm), the absolutely continuous spectrum on the segment $[\lambda_-, \lambda_+]$ with $\lambda_\pm = \lambda + O(h^\infty)$ will be replaced by an eigenvalue of the infinite multiplicity and then the spectral asymptotics will complete albeit non-differentiable even once.

To establish spectral asymptotics we apply the “hyperbolic operator method”; namely, let us consider the Schwartz kernel of the propagator $e^{ih^{-1}tA_h}$:

$$u := u_h(x, y, t) = \int e^{ih^{-1}tr} d\tau e_h(x, y, \tau).$$

Then under ellipticity and microhyperbolicity conditions (1.3) and (1.4)

$$F_{t \to h^{-1}r}(t) \sim \sum_{n \geq 0} \kappa_n'(x, \tau)h^{1-d+n},$$

where here and below $\chi \in \mathcal{C}_0^\infty([-1, -\frac{1}{2}] \cup [\frac{1}{2}, 1]), \bar{\chi} \in \mathcal{C}_0^\infty([-1, 1]), \bar{\chi}(t) = 1$ on $[-\frac{1}{2}, \frac{1}{2}], \chi_\tau(t) = \chi(t/T)$ etc, $\kappa_n'(x, \tau) = \partial_r\kappa_n(x, \tau)$, and $T = T_\ast > 0$ is a small constant here.

Then, due to Tauberian theorem we arrive to the spectral asymptotics with the remainder estimate $O(h^{\infty})$. Next, under different assumptions one, using propagation of singularities technique, can prove that

$$|F_{t \to h^{-1}r}(t)u_h(x, x, t)| = O(h^{\infty})$$

for all $T \in [T_\ast, T^\ast]$. Then (1.9) holds with $T = T^\ast$ and again, due to Tauberian theorem, we arrive to the spectral asymptotics with the remainder estimate $O(T^{\ast-1}h^{1-d})$ (provided $T^\ast = O(h^{-K})$). In particular, if (1.10) holds provided $T^\ast = h^{-\infty}$, we arrive to complete spectral asymptotics. This happens i.e. in the framework of [Ivr3].

However we do not have Tauberian theorems for the derivatives (with respect to $\tau$) and we need to use an inverse Fourier transform and its derivatives

$$\partial^n_r e_h(x, x, \tau) = (2\pi h)^{-1} \int e^{-ih^{-1}rt}(-ih^{-1}t)^{n-1} u_h(x, x, t) dt$$

for $n \geq 1$. If we insert a factor $\bar{\chi}_\tau(t)$ into integral, we will get exactly $n$-th derivative of the right-hand expression of (1.7). However we need to estimate the remainder

$$\int e^{-ih^{-1}rt}(-ih^{-1}t)^{n-1}(1 - \bar{\chi}_\tau(t))u_h(x, x, t) dt$$
and to do this we need to properly estimate the left-hand expression of (1.10) for all $T \geq T_*$ (rather than for $T \in [T_*, T^*]$).

To achieve this we will use a more subtle propagation technique and prove that for $T \geq T_*(R)$ the left-hand expression of (1.10) is $O((h/T)\infty)$, provided $|x| \leq R$.

## 2 Proofs

### 2.1 Preliminary remarks

Observe that, due to assumptions (1.6) and (1.5) a propagation speed with respect to $\xi$ does not exceed $\min(\varepsilon, C(|x|+1)^{-1-\delta})$ and one can prove easily, that for for a generalized Hamiltonian trajectory\(^1\) $(x(t), \xi(t))$ on energy level $\tau \leq c$

\[(2.1) \quad \Sigma_\tau := \{(x, \xi): \text{Ker}(A(x, \xi) - \tau) \neq \{0\}\}
\]

with $A(x, \xi) = A^0(\xi) + V(x, \xi)$ we have $|\xi(t) - \xi(0)| \leq \varepsilon'$ for all $t$ with $\varepsilon' = \varepsilon'(\varepsilon) \to 0$ as $\varepsilon \to 0$ and therefore

\[(2.2) \quad \text{Let conditions (1.2)--(1.4), (1.5) and (1.6) with $\varepsilon = \varepsilon(\varepsilon') > 0$ with arbitrarily small $\varepsilon'$ be fulfilled. Then for a generalized Hamiltonian trajectory $(x(t), \xi(t))$ on $\Sigma_\tau$
\]

\[(2.3) \quad |\xi(t) - \xi(0)| \leq \varepsilon' \quad \text{and} \quad |x(t) - x(0)| \geq \varepsilon_2 |t| \quad \forall t \in \mathbb{R}.
\]

Then we conclude immediately that inequality

\[(2.4) \quad |F_{t\to T}u_T(t)h(x, x, t)| \leq C(T)h^s
\]

holds for arbitrarily constant $T > 0$.

Combining with (1.9) for small constant $T$ we conclude that

\[(2.5) \quad \text{Let conditions (1.2)--(1.4), (1.5) and (1.6) with sufficiently small constant $\varepsilon > 0$ be fulfilled. Then asymptotic decomposition (1.9) holds with an arbitrarily large constant $T$}.

\(^1\) For a definition of the generalized Hamiltonian trajectory see Definition 2.2.8 of [Ivr1].
2. Proofs

2.2 Propagation and local energy decay

First we have the finite speed with respect to $x$ propagation:

**Proposition 2.1.** For $\tau \leq c$ the following estimate holds

\[
|F_{t-h^{-1}\tau}(\chi_T(t)u(x, y, t))| \leq C'_h h^\tau R^{-s} \forall x, y : |x - y| \geq C_0 T, |x| + |y| \simeq R.
\]

**Proof.** In the zone $\{ x : |x| \simeq R \}$ we can apply scaling $x \mapsto x R^{-1}, t \mapsto t R^{-1}, h \mapsto h R^{-1}$ and apply the standard theory of Chapter 2 of [Ivr1]. The rest is trivial.

Next, we consider $R \leq \epsilon_1 T$ and apply energy estimate method to prove the local energy decay. Observe that one can select smooth $\ell(\xi)$ in condition (1.4). Consider operator $L^0(x, hF)$ with Weyl symbol $-\langle x, \ell(\xi) \rangle$ and $L(x, hD; t) = L^0 + \epsilon t$.

Then

\[
2h^{-1} \operatorname{Re} i((hD_t - A)v, L v)_{\Omega_T} = (Lv, v)_{t=0}^{t=T} - \operatorname{Re} ih^{-1}([hD_t - A, L]v, v)_{\Omega_T} = (Lv, v)_{t=0}^{t=T} - \epsilon \| v \|^2_{H^s_T} + \operatorname{Re} (ih^{-1}[A, L]v, v)_{\Omega_T},
\]

where $\| . \|_\Omega$ and $(., .)_\Omega$ are a norm and an inner product in $\mathcal{L}^2(\Omega)$ with $\Omega = \Omega_T = \mathbb{R}^d \times [0, T] \ni (x, t)$. Indeed, writing the left-hand expression as

\[
\operatorname{ih}^{-1} \left[ ((hD_t - A)v, L v)_{\Omega_T} - (L(hD_t - A)v)_{\Omega_T} \right] = \operatorname{ih}^{-1} \left[ (L(hD_t - A)v)_{\Omega_T} - ((hD_t - A)Lv)_{\Omega_T} \right] + (Lv, v)_{t=0}^{t=T}
\]

because $L^* = L$, we arrive to (2.7).

In virtue of (1.5) and (1.6) for sufficiently small constant $\epsilon$ and for $h \leq h_0(\epsilon_1)$ the operator norm of $h^{-1}[V, L]$ from $\mathcal{H}^m(\mathbb{R}^d)$ to $\mathcal{L}^2(\mathbb{R}^d)$ does not exceed $\epsilon_1$ with $\epsilon_1 = \epsilon_1(\epsilon) \to 0$ as $\epsilon \to 0$, and then due to the microhyperbolicity assumption we conclude that

\[
\operatorname{Re} (ih^{-1}[A, L]v, v) \geq (\epsilon_0 - 2\epsilon_1)\| v \|^2 - C \| (A - \tau) v \|^2
\]

for both $\mathbb{R}^d$ and $\Omega_T$. 

2. Proofs

Let us plug into (2.7) \( v = \varphi_\varepsilon(A - \tau)e^{ihA}\psi \) where \( \varphi \in \mathcal{C}_0^\infty([-1, 1]) \), 0 \( \leq \varphi \leq 1 \); then for sufficiently small constant \( \varepsilon > 0 \) we arrive to

\[
(2.9) \quad \varepsilon \|v\|^2_{\Omega} + (Lv, v)|_{t=T} \leq (Lv, v)|_{t=0}.
\]

On the other hand,

\[
(2.10) \quad \text{Re}(Lv, v)|_{t=T} \geq \varepsilon T\|v\|^2 - C\|x^{\frac{3}{2}}v\|^2
\]
with

\[
\|x^{\frac{3}{2}}v\|^2 = \|x^{\frac{3}{2}}v\|^2_{B(0, R)} + \|x^{\frac{3}{2}}v\|^2_{B(0, R') \setminus B(0, R)} + \|x^{\frac{3}{2}}v\|^2_{\mathbb{R}^d \setminus B(0, R')}
\]

with \( R' = C_0T \) and therefore for \( R \leq \varepsilon T \)

\[
(2.11) \quad \text{Re}(Lv, v)|_{t=T} \geq \varepsilon T\|v\|^2 - CR'\|v\|^2_{B(0, R') \setminus B(0, R)} - \|x^{\frac{3}{2}}v\|^2_{\mathbb{R}^d \setminus B(0, R')}
\]

Observe that

\[
|(Lv, v)|_{t=0} \leq C\left( \|v_0\|^2 + \|x^{\frac{3}{2}}v_0\|^2 \right)
\]
and then (2.9) and (2.10) imply that if \( R \leq \varepsilon T \) then

\[
(2.12) \quad \|v\|^2_{B(0, r)} \leq \sigma \|v_0\|^2 + C T^{-1}\left( \|x^{\frac{3}{2}}v\|^2_{\mathbb{R}^d \setminus B(0, R')} + \|v_0\|^2 + \|x^{\frac{3}{2}}v_0\|^2 \right)
\]

with \( \sigma < 1 \) and \( v|_{t=0} = v \); recall that \( \|v\| = \|v_0\| \).

Recall that \( v = e^{ih^{-1}TA}\varphi_\varepsilon(A - \tau)\psi_R(x)w \) where we plugged \( \psi_Rw \) instead of \( w, \psi \in \mathcal{C}_0^\infty(B(0, 1)) \), 0 \( \leq \psi \leq 1 \) and \( \psi = 1 \) in \( B(0, \frac{1}{2}) \).

One can prove easily that \( Q = \varphi_\varepsilon(A - \tau) \) is an operator with Weyl symbol \( Q(x, \xi) \), satisfying

\[
|D_\xi^\alpha D_\xi^\beta Q| \leq C_{\alpha\beta}e^{-|\alpha| - |\beta|}(1 + 1)^{-|\beta|}.
\]

Then \( \|v_0\|_{\mathbb{R}^d \setminus B(0, 2R)} \leq C(h/R)^s\|w\| \) and therefore \( \|x^{\frac{3}{2}}v_0\|^2 \leq 2R\|w\|^2 \). Further, then Proposition 2.1 implies that \( \|x^{\frac{3}{2}}v\|^2_{\mathbb{R}^d \setminus B(0, R')} \leq C(h/T)^s\|w\| \) provided or \( R' \geq C_0T \) with sufficiently large \( C_0 \) and we arrive to

**Proposition 2.2.** In the framework of Theorem 1.1

\[
(2.13) \quad \|\psi_Re^{ih^{-1}TA}\varphi_\varepsilon(A - \lambda)\psi_R\| < 1
\]
provided \( \varepsilon T \geq R \geq 1 \).
2. Proofs

While this statement looks weak, it will lead to much stronger one:

**Proposition 2.3.** *In the framework of Theorem 1.1*

\[ \| \psi_R e^{i \hbar^{-1} t A} \varphi_\varepsilon (A - \lambda) \psi_R \| \leq C_5 R^s T^{-s} \]

provided \( T \geq C_6 R, \ R \geq 1 \).

**Proof.** We want to prove by induction that

\[ \| \psi_R e^{i \hbar^{-1} T A} \varphi_\varepsilon (A - \lambda) \psi_R \| \leq C \nu^N + C_s n R^s T^{-s} \]

with \( \nu < 1 \).

Assuming that for \( n \) we have (2.15), we apply the previous arguments on the interval \([nt, (n+1) T]\) to \( v = e^{i \hbar^{-1} t A} \varphi_\varepsilon (A - \lambda) \psi_R/2 e^{i \hbar^{-1} n T A} \psi_R w \) and derive an estimate

\[ \| \psi_R e^{i \hbar^{-1} t A} \varphi_\varepsilon (A - \lambda) \psi_R/2 e^{i \hbar^{-1} n T A} \psi_R w \| \leq \nu \| \varphi_\varepsilon (A - \lambda) \psi_R e^{i \hbar^{-1} n T A} \psi_R w \| + C_K (h/R)^K \| w \| \]

with \( \nu < 1 \).

To make a step of induction we need to estimate the norm of

\[ \psi_R e^{i \hbar^{-1} t A} \varphi_\varepsilon (A - \lambda)(1 - \psi_R/2) e^{i \hbar^{-1} n T A} \psi_R w = \psi_R e^{i \hbar^{-1} t A} \varphi_\varepsilon (A - \lambda)(1 - \psi_R/2) Q^+ e^{i \hbar^{-1} n T A} \psi_R Q^+ + \psi_R e^{i \hbar^{-1} t A} \varphi_\varepsilon (A - \lambda)(1 - \psi_R/2) Q^- e^{i \hbar^{-1} n T A} \psi_R Q^- \]

with \( Q^\pm = Q^\pm (x, hD), \ Q^+ + Q^- = I \) to be selected to ensure that

\[ \text{Generalized Hamiltonian trajectories on } \Sigma, \text{ starting as } t = 0 \text{ from } \text{supp}(Q^\pm) \cap \text{supp}(1 - \psi_R/2) \text{ in the positive (negative) time direction, remain in the zone } \{|x| \geq \varepsilon_1 R + \varepsilon_2 |t|\}. \]

Then we show that

\[ \| \psi_R e^{i \hbar^{-1} t A} \varphi_\varepsilon (A - \lambda)(1 - \psi_R/2) Q^+ \| \leq C_5 (h/R)^s \]

and

\[ \| \varphi_\varepsilon (A - \lambda)(1 - \psi_R/2) Q^- e^{i \hbar^{-1} n T A} \psi_R \| \leq C_s (h/R)^s. \]
To achieve that consider $A^0(\xi)$ and for each $\xi$ in the narrow vicinity $\mathcal{W}$ of $\Sigma_\tau$ let $K^+(\xi) \subset \mathbb{R}^d \times \mathbb{R}^+$ be a forward propagation cone and $K^-(\xi) = -K^+(\xi)$ be a backward propagation cone. Let

\[(2.21) \quad \Omega^\pm = \{(x, \xi): x \notin \pi_x K^\pm(\xi)\}\]

where $\pi_x$ is $x$-projection.

Then $\Omega^\pm$ are open sets and since $\pi_x K^+(\xi) \cap \pi_x K^-(\xi) = \{0\}$ we conclude that $\Omega^+ \cup \Omega^- \supset \mathbb{S}^{d-1} \times \mathcal{W}$. We can then find smooth positively homogeneous of degree $\gamma$ with respect to $x$ symbols $q^\pm(x, \xi)$ supported in $\Omega^\pm$ such that $q^+ + q^- = 1$ on $\mathbb{S}^{d-1} \times \mathcal{W}$. Let $q^0 = 1 - (q^+ + q^-)$; then $(A^0 - \tau)$ is elliptic on $\text{supp}(q^0)$.

Finally, let $Q^\pm$ and $Q^0$ be operators with the symbols $q^\pm(x, \xi)\phi_R(x)$ and $q^0(x, \xi)\phi_R(x)$ correspondingly, where $\phi \in \mathcal{C}_0^\infty(\mathbb{R}^d \setminus 0)$, equal 1 as $c^{-1} \leq |x| \leq c$ with large enough constant $c$. Then

\[(2.22) \quad Q^+ + Q^- + Q^0 = \phi_R(x),\]

where (2.18) holds and $(A - \tau)$ is elliptic on the support of the symbol of $Q^0$.

Then Proposition 2.4 below implies that for $R \leq \varepsilon T$ with sufficiently small constant $\varepsilon$ both (2.19) and (2.20) hold. On the other hand, ellipticity of $(A - \tau)$ on $\text{supp}(Q^0)$ implies that

\[(2.23) \quad \|\varphi_\varepsilon(A - \lambda)(1 - \psi_{R/2})Q^0\| \leq C_s(h/R)^s.\]

Then we can make an induction step by $n$ and to prove (2.15). After this, let us replace in (2.15) $R$ and $T$ by $r$ and $t$. Next, for given $R$, $T$ such that $R \leq \varepsilon^3 T$ let us plug into (2.15) $n = (T/R)^{\frac{1}{2}}$, $t = T^{\frac{3}{2}}R^{\frac{3}{2}} = T/n$ and $r = T^{\frac{3}{2}}R^{\frac{3}{2}} = nR$ (obviously $r \leq \varepsilon t$). We arrive to (2.14) with a different but still arbitrarily large exponent $s$. \qed

As mentioned, we need the following proposition:

**Proposition 2.4.** Let conditions of Theorem 1.1 be fulfilled. Let $x \in \mathbb{R}^d \setminus 0$, $\xi \in \mathcal{W}$ and assume that $0 \notin x + \pi_x K^\pm(\xi)$. Let $K^\pm$ be a conical $\eta$-vicinity of $K^\pm(\xi)$ and $\mathcal{V}$ be $\eta R$-vicinity of $x$, $R = |x|$. Then

\[(2.24) \quad \|Q'e^{\pm i h^{-1}\tau A}Q\| \leq C_s(h/R)^s\]
provided $Q = Q(x, hD)$ and $Q' = Q'(x, hD)$ are operators with the symbols satisfying

$$|D_\xi^\alpha D_\zeta^\beta Q| \leq c_{\alpha\beta} r^{-|\beta|}$$

with $r = T + R$ and $r = R$ respectively, $R \geq R(\eta)$ and support of symbol of $Q$ does not intersect with $V + K^+|_{t=T}$, symbol of $Q'$ is supported in the sufficiently small vicinity of $(\bar{x}, \bar{\zeta})$.

**Proof.** Considering propagation in the zone $\{x: |x| \approx r\}$, we see that the propagation speed with respect to $\zeta$ does not exceed $Cr^{-\delta}$. To prove this we scale $x \mapsto xr^{-1}$, $t \mapsto tr^{-1}$, $h \mapsto h = hr^{-1}$ and apply the standard energy method (see Chapter 2 of [Ivr1]). We leave the easy details to the reader.

Therefore for time $t \approx r$ variation of $\xi$ does not exceed $Cr^{-\delta}$. Then, the propagation speed with respect to $\langle x, \ell(\bar{\zeta}) \rangle$ (which increases) is of magnitude 1 (as long as $\xi$ remains in the small vicinity of $\bar{\zeta}$). Again, to prove it we scale and apply the energy method (see Chapter 2 of [Ivr1]).

But then the contribution of the time interval $t \approx r$ to the variation of $\xi$ does not exceed $Cr^{-\delta}$ and therefore the variation of $\xi$ for a time interval $[\eta, T]$ with $T \geq 0$ does not exceed $CR^{-\delta} \leq \eta$ for $R \geq R(\eta)$. \qed

**Proposition 2.5.** In the framework of Theorem 1.1

$$\|\psi_R e^{ih^{-1}TA} \varphi_\varepsilon(A - \lambda)\psi_R\| \leq C_s h^s,$$

provided $T \geq C_0 R$, $R \geq 1$.

**Proof.** It follows immediately from Proposition 2.4 with the semiclassical parameter $hr^{-1}$ and with $r$ set to its minimal value along the cone of propagation, which is 1. \qed

Combining Propositions 2.3 and 2.5 we arrive to

**Corollary 2.6.** In the framework of Theorem 1.1

$$\|\psi_R e^{ih^{-1}TA} \varphi_\varepsilon(A - \lambda)\psi_R\| \leq C_s h^s R^s T^{-s}$$

provided $T \geq C_0 R$, $R \geq 1$. 
2.3 Traces and the end of the proof

**Proposition 2.7.** In the framework of Theorem 1.1 the following estimates hold for $T \geq 1$

\begin{align*}
(2.28) & \quad |F_{t\to \tau} \chi_T(t)u(x, x, t)| \leq C_s h^s (|x| + 1) + ((|x| + 1) + T)^{-s}, \\
(2.29) & \quad |F_{t\to \tau} \chi_T(t) \int \psi_R(x)u(x, x, t) \, dx| \leq C_s h^s T^{-s},
\end{align*}

provided $\psi \in C_0^\infty(B(0, 1))$. Further,

\begin{align*}
(2.30) & \quad |F_{t\to \tau} \chi_T(t) \int \psi_R(x)u(x, x, t) \, dx| \leq C_s h^s R^{-s} T^{-s},
\end{align*}

provided $\psi \in C_0^\infty(B(0, 1) \setminus B(0, \frac{1}{2}))$.

*Proof.* Estimate (2.28) follows immediately from (2.26). Estimate (2.30) follows from (2.26) and

\begin{align*}
|F_{t\to \tau} \chi_T(t) \int \psi_R(x)u(x, x, t) \, dx| \leq C_s h^s R^{-s} T^{-s},
\end{align*}

which holds because we can choose the time direction on the partition element (see Chapter 4 of [Ivr1]) and we chose the one in which $|x| \geq R$ (which is possible; see the part of proof of Proposition 2.3 dealing with $Q^\pm$ and $Q^0$).

Finally, estimate (2.29) follows from (2.30). \hfill \Box

Then we immediately arrive to the following theorem, which in turn implies Theorem 1.1:

**Theorem 2.8.** In the framework of Theorem 1.1 the following estimates hold

\begin{align*}
(2.31) & \quad |\partial^k_{\tau} \left(e(x, x, \tau) - \sum_{n \leq N-1} \kappa_n(x, \tau) h^{-d+n}\right)| \leq C_N h^{-d+N}(|x| + 1)^{-N} + C_s h^s (|x| + 1)^k
\end{align*}

and for $\psi \in C_0^\infty(B(0, 1))$

\begin{align*}
(2.32) & \quad |\partial^k_{\tau} \int \left(e(x, x, \tau) - \sum_{n \leq N-1} \kappa_n(x, \tau) h^{-d+n}\right) \psi_R(x) \, dx| \leq C_N h^{-d+N}.
\end{align*}
2.4 Discussion

**Corollary 2.9.** Let conditions of Theorem 1.1 be fulfilled. Assume that

\[ |D_\alpha^\nu V| \leq c_\alpha (|\xi| + 1)^m (|\nu| + 1)^{-d-|\alpha|-\delta}. \]

Then the asymptotics of the Birman-Schwinger spectral shift function

\[ N_h(\tau) := \int \left( e_h(x, x, \tau) - e_h^0(x, x, \tau) \right) dx \sim \sum_{n \geq 0} \kappa h^{-d+n} \]

is infinitely differentiable with respect to \( \tau \). Here \( e_h^0(x, x, \tau) = \kappa h^{-d} \) and \( e_h^0(x, y, \tau) \) is the Schwartz kernel of spectral projector for \( A^0(hD) \), and

\[ \kappa_n(\tau) = \int (\kappa_n(x, \tau) - \delta_n \kappa^0) \right) dx. \]

Indeed, condition (2.33) guarantees the absolute convergence of integrals in (2.35).

**Remark 2.10.** Our results could be easily generalized to non-semi-bounded elliptic \( A^0 \) (like in Subsection 3.1 of [Ivr3]). Then instead of \( e(x, y, \lambda) \) one needs to consider \( e(x, y, \lambda, \lambda') \) the Schwartz kernel of \( \theta(\lambda - A) - \theta(\lambda' - A) \) and either impose conditions for both \( \lambda \) and \( \lambda' \), or only for \( \lambda \) and mollify with respect to \( \lambda' \).

It looks strange that the last term in the remainder estimate (2.31) increases as \( |x| \) increases, but so far I cannot improve it to the uniform with respect to \( x \) in the general case, nor show by the counter-example that such improvement is impossible. However I hope to prove

**Conjecture 2.11.** Assume in addition that \( A^0 \) is a scalar operator and \( \Sigma^0_\lambda = \{ \xi : A^0(\xi) = \lambda \} \) is a strongly convex surface i.e.

\[ \pm \sum_{j,k} A^0_{\xi j} (\xi) \eta_j \eta_k \geq \epsilon |\eta|^2 \quad \forall \xi \in \Sigma^0_\lambda \quad \forall \eta : \sum_j A^0_{\xi j} (\xi) \eta_j = 0, \]

where the sign depends on the connected component of \( \Sigma_\lambda \), containing \( \xi \).

Then the last term in the right-hand expression of (2.31) could be replaced by \( C_s h^s (|x| + 1)^{k-d} \).
Rationale here is that only few Hamiltonian trajectories from $x$ with $|x| = R \gg 1$ pass close to the origin and even they do not spend much time there.

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