INEQUALITIES FOR MULTIVARIATE POLYNOMIALS

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Abstract

We summarize researches – in great deal jointly with my host Y. Saran-
topoulos and his PhD students V. Anagnostopoulos and A. Pappas –
started by a Marie Curie fellowship in 2001 and is still continuing.

The project was to study multivariate polynomial inequalities. In the
course of work we realized the role of the “generalized Minkowski func-
tional”, to which we devoted a throughout survey. Resulting from this,
infinite dimensional extensions of Chebyshev’s extremal problems were
tackled successfully. Investigating Bernstein-Markov constants for homo-
genous polynomials of real normed spaces led us to the application of
potential theory. Also we found at first unexpected connections of po-
larization constants of $\mathbb{R}^2$ and $\mathbb{C}^2$ to Chebyshev constants of $S^2$ and $S^3$,
respectively. In the study of polarization constants, a further application
of potential theory occurred. This led us to realize that the theory of ren-
dezvous numbers can be much better explained by potential theory, too.
Our methods for obtaining Bernstein type pointwise gradient estimates
for polynomials were compared in a recent case study to the yields of
pluripotential theoretic methods. The findings were that the two rather
different methods give exactly the same results, but the two currently
standing conjectures mutually exclude each other.

1 Polynomials In Higher Dimensions

In the whole paper $\mathbb{K}$ stands for either $\mathbb{R}$ or $\mathbb{C}$. If $X$ is a normed space,
$X^* = \mathcal{L}(X, \mathbb{K})$ is the usual dual space, and $S := S_X$, $S^* := S_{X^*}$, $B := B_X$
and $B^* := B_{X^*}$ are the unit spheres and (closed) unit balls of $X$ and $X^*$,
respectively. Moreover, $\mathcal{P} = \mathcal{P}(X)$ and $\mathcal{P}_n = \mathcal{P}_n(X)$ will denote the space of
continuous (i.e., bounded) polynomials of free degree and of degree $\leq n$, resp.,
from $X$ to $\mathbb{K}$.

There are several ways to introduce continuous polynomials over $X$, one
being the linear algebraic way of writing

$$\mathcal{P}_n := \mathcal{P}_0^* + \mathcal{P}_1^* + \cdots + \mathcal{P}_n^*, \quad \text{and} \quad \mathcal{P} := \bigcup_{n=0}^{\infty} \mathcal{P}_n \quad (1)$$

with $\mathcal{P}_k^* := \mathcal{P}(^kX; \mathbb{K})$ denoting the space of homogeneous (continuous) polyno-
mials of degree (exactly) $k \in \mathbb{N}$. That is, one considers bounded $k$-linear forms
\( L \in \mathcal{L}(X^k \to K) \) together with their “diagonal functions”
\[
\hat{L} : X \to K, \quad \hat{L}(x) := L(x, x, \ldots, x),
\]
(2)
and defines \( \mathcal{P}_n^* \) as the set of all \( \hat{L} \) for \( L \) running \( \mathcal{L}^s(n X) := \mathcal{L}(X^k \to K) \). In fact, it is sufficient to identify equivalent linear forms (having identical diagonal functions) by selecting the unique symmetric one among them: i.e., to let \( L \) run over \( \mathcal{L}^s(n X) \) denoting symmetric \( k \)-linear forms. Building up the notion of polynomials so is equivalent to
\[
\mathcal{P}_n := \{ p : X \to K : \|p\| < \infty, p|_Y \in \mathcal{P}_n(K) \}
\]
for all \( Y \subseteq X, \dim Y = 1, y \in X \},
\]
or to the definition arising from combining (1) and
\[
\mathcal{P}_n^* := \{ p : X \to K : \|p\| < \infty, p|_Y \in \mathcal{P}_n^*(K^2) \}
\]
for all \( Y \subseteq X, \dim Y = 2 \}.

Here and throughout the paper for any set \( K \subset X \) and function \( f : X \to K \) we denote, as usual,
\[
\|f\|_K := \sup_K |f| \quad \text{and} \quad \|f\| := \|f\|_B.
\]

For an introduction to polynomials over normed spaces see, e.g., [9, Chapter 1]. In particular, it is well-known that
\[
\|\hat{L}\| \leq \|L\| \leq C(n, X)\|\hat{L}\| \quad \text{for all } L \in \mathcal{L}^s(n X),
\]
(5)
and that \( C(n, X) \leq n^n/n! \), [9], while \( C(n, X) = 1 \) if \( X \) is a Hilbert space (Banach’s Theorem). Similarly to (5), one can consider special homogeneous polynomials which can be written as products of linear forms, that is \( L(x_1, \ldots, x_n) = \prod_{j=1}^n f_j(x_j) \) with \( f_j \in X^* \). Then \( \|L\| = \prod_{j=1}^n \|f_j\| \), i.e., the product of the norms, and one compares to the norm of the corresponding homogeneous polynomial, i.e., to \( \|\hat{L}\| = \|\prod_{j=1}^n f_j\| \). Note that here \( L \) is far from being symmetric, and this yields to an essentially different question, with the similarly defined polarization constants – the linear polarization constants – now ranging up to \( n^n \), see e.g. [6]. Polarization problems are typical, genuinely multivariate inequalities, as in dimension 1 they degenerate.

2 Linear Polarization Constants

Definition 1 (Benítez, Sarantopoulos, Tonge [6]). The \( n^{th} \) (linear) polarization constant of a normed space \( X \) is
\[
c_n(X) := \inf \{ M : \prod_{j=1}^n \|f_j\| \leq M \prod_{j=1}^n \|f_j\| (\forall f_j \in X^*) \}
\]
\[
= 1/ \inf_{f_1, \ldots, f_n \in \mathcal{S}_X} \sup_{\|x\|=1} |\prod_{j=1}^n f_j(x)|.
\]
(6)
Obviously \( c_n(X) \) is a nondecreasing sequence. Its growth, as \( n \to \infty \), is closely related to the structure of the space.

**Definition 2 (Révész, Sarantopoulos [28]).** The linear polarization constant of a normed space \( X \) is

\[
c(X) := \lim_{n \to \infty} c_n(X)^\frac{1}{n}.
\]  

(7)

One should have put only \( c(X) := \limsup_{n \to \infty} c_n(X)^\frac{1}{n} \) as a definition. However, we proved that the limit does exist, see [28, Proposition 4]. Note that \( c(X) \) can be infinity as well. More specifically, from [28, Theorem 12] we have

**Proposition 1 (Révész, Sarantopoulos [28]).** Let \( X \) be a normed space. Then \( c(X) = \infty \) iff \( \dim(X) = \infty \).

In the special case where \( X \) is a Hilbert space, it is easy to see that (writing \( Y \leq X \) for \( Y \) being a subspace of \( X \))

\[
c_n(X) = \sup \{ c_n(Y) : Y \leq X, \dim Y \leq n \}.
\]  

(8)

The Banach-Mazur distance \( d(X, Y) \) between two isomorphic Banach spaces \( X \) and \( Y \) can be used in comparing the \( n^{th} \) polarization constants of these spaces. Recall that

\[
d(X, Y) := \inf \{ \|T\| \cdot \|T^{-1}\| : T : X \leftrightarrow Y \text{ isomorphism} \}.
\]

**Proposition 2 (Benítez, Sarantopoulos, Tonge [6]).** If the normed spaces \( X \) and \( Y \) are isomorphic, then

\[
c_n(X) \leq d^n(X, Y)c_n(Y).
\]  

(9)

It seems very likely that \( c_n(X) \geq c_n(\ell_2^n) \) (\( \forall n \leq \dim X \)), but this is not known. However, we found

**Proposition 3 (Révész, Sarantopoulos [28]).** If \( X \) is an infinite dimensional normed space, then

\[
c_n(X) \geq c_n(\ell_2^n), \forall n \in \mathbb{N}.
\]  

(10)

As is well-known, for any \( n \)-dimensional Banach space \( X \)

\[
d(X, \ell_2^n) \leq \sqrt{n}.
\]  

(11)

Thus to determine \( c_n(\mathbb{R}^n) \) is interesting not only in the context of Hilbert space theory. For example, a combination of (8), (9), (11) and (10) yields the following result.

**Theorem 1 (Révész, Sarantopoulos [28]).** Let \( X \) be an infinite dimensional normed space and let \( H \) be the space \( \ell_2 \) over \( \mathbb{K} \). Consider \( \mathbb{K}^n \), i.e., the space \( \ell_2^n \) over \( \mathbb{K} \). For all \( n \in \mathbb{N} \) we have

\[
c_n(H) = c_n(\mathbb{K}^n) \leq c_n(X) \leq n^\frac{1}{2} c_n(\mathbb{K}^n) = n^\frac{1}{2} c_n(H).
\]
Note that determination of the linear polarization constant is closely connected to another famous problem, the Tarski plank problem, see, e.g., [3, 28]. Let us focus here on the problem of estimating the linear polarization constants of Hilbert spaces. Although this is a classical topic, there was a flourishing activity on this field even in the last ten years, and even recently, see, e.g., [3, 18, 22].

Ideally, one should look for the exact values of $c_n(\ell^2_2) = c_n(K^d)$, for any $d, n \in \mathbb{N}$, which, in view of (8), reduces to $d \leq n \in \mathbb{N}$. In fact, this question is posed in [17], attributed to the referee of the paper. In this direction a remarkable success is Arias-de-Reyna’s result.

Theorem A (Arias-de-Reyna [2]). $c_n(C^n) = n^{n/2}$.

An even more precise description was obtained recently by K. Ball [3]. Estimating $c_n(R^n)$ seems to be a harder task. In particular, the proof of the stronger result due to K. Ball relies heavily on complex function theoretic tools which are not valid in the case of $R^n$. Observe furthermore that Arias-de-Reyna’s entirely different technique, based on permanents, multilinear algebra and probability theory (particularly Gaussian random variables), strongly depends on the complex structure of $C^n$. As Arias-de-Reyna has mentioned in [2], his Theorem A leads to an upper estimate $c_n(R^n) \leq 2^n n^{n/2}$ even for the real case. This has been improved in [17] and [12], until a more refined approach was worked out in [21], using the natural complexification of a real Hilbert space. This has led to the currently best

Theorem 2 (Révész, Sarantopoulos [28]).

\[ n^{\frac{n}{2}} \leq c_n(R^n) \leq 2^{\frac{n}{2}} - 1 n^{\frac{n}{2}}. \]

Let us mention here the following conjecture, appearing already in [6] and [2] and formulated also in [28].

Conjecture 1.

\[ c_n(R^n) = n^{\frac{n}{2}}. \] (12)

We proved the conjecture for $n = 1, 2, 3, 4, 5$ in [23]. Moreover, in [23] we discussed a direct, real approach in detail. This approach seems to be interesting (even though the resulting exponential factor falls, unfortunately, only between $\sqrt{2}$ and 2), as it is independent of Theorem A.

In all, for a Hilbert space $H$ of infinite dimension we only know that $c(H) = \infty$ and $\log c_n(H) \sim \frac{1}{2} n \log n$. On the other hand, if $\dim H = d$ is fixed then by Theorem 1 $c(H)$ must be finite.

Trying to determine $c_n(K^d)$ for arbitrary $d \leq n \in \mathbb{N}$, by natural extrapolation one might have thought that $c_n(K^d) = d^{n/2}$. This was disproved first in [1].

Definition 3. The $n^{th}$ (metric) Chebyshev constant of a subset $F \subseteq X$ in a metric space $(X, \rho)$ is

\[ M_n(F) := \inf_{y_1, \ldots, y_n \in F} \sup_{y \in F} \rho(y, y_1) \cdots \rho(y, y_n). \] (13)
In particular, in a normed space $X$ with $\rho(x, y) = \|x - y\|$ and

$$M_n(F) := \inf_{y_1, \ldots, y_n \in F} \sup_{y \in F} \|y - y_1\| \cdots \|y - y_n\|.$$

Proposition 4 (Anagnostopoulos, Révész [1]). For the real space $\ell_2^2(\mathbb{R})$ we have

$$c_n(\ell_2^2(\mathbb{R})) = \frac{2^n}{M_n(S^1)} = 2^{n-1}, \quad \text{and so} \quad c(\ell_2^2(\mathbb{R})) = 2.$$ 

Furthermore, for the complex space $\ell_2^2(\mathbb{C})$ we have

$$c_n(\ell_2^2(\mathbb{C})) = \frac{2^n}{M_n(S^2)}, \quad \text{and also} \quad c(\ell_2^2(\mathbb{C})) = \sqrt{e}.$$

3 Potential Theory Emerges

We have seen above that, e.g., $c(\ell_2^2(\mathbb{C})) = \sqrt{e}$, disproving our initial guess of $c_n(K^d) = d^{n/2}$ (which then would have implied $c(K^d) = \sqrt{d}$). On the other hand, a surprising connection with the (metric) Chebyshev constants emerged from our study.

It turned out that for higher dimensions the connection breaks. However, in a more general setting we still plan to describe linear polarization constants by means of some more general Chebyshev constants. This is one of the intriguing questions we are occupied with recently.

But how the notion of Chebyshev constants, belonging classically to potential theory, can have a role here? Quite naturally. Following a potential-theory inspired approach, we could even describe polarization constants of all finite dimensional Hilbert spaces [23]. To that, let us start with a notation:

$$L(d, K) := \int_{S_k^d} \log \|\langle x, s \rangle\| d\sigma(x) \quad (< 0),$$

where $d\sigma(x)$ is the normalized surface Lebesgue measure of $S_k^d$ and $s \in S_k^d$ is arbitrary. Calculation of the explicit values of the constants $L(d, K)$ are standard.

Theorem B (García-Vázquez, Villa [12]). We have the equality $c(\mathbb{R}^d) = e^{-L(d, \mathbb{R})}$, with the constants $L(d, \mathbb{R})$ as in (15).

Theorem 3 (Pappas, Révész [23]). For the complex case we have $c(\mathbb{C}^d) = e^{-L(d, \mathbb{C})}$, with the constants $L(d, \mathbb{C})$ defined by (15).

Our proof in [23] is a unified, potential theory flavored approach. But the real case were already obtained a few years earlier by García-Vázquez and Villa, with no use of potential theory at all. Instead, they applied a nice theorem of O. Gross [13] on the existence of rendezvous numbers.
Gross’ Theorem states that in any compact, connected metric space \((X, \rho)\) there exists a unique rendezvous number \(r = r(X)\), such that for arbitrary choice of any set of points \(x_1, \ldots, x_n \in X\), there always exists some point \(x \in X\), such that its average distance from the \(x_j\) is equal to \(r\). This is a beautiful result, which somehow stood in itself for half a century: some authors even named \(r(X)\) “the magical number” of \(X\). Nevertheless, several dozens of extensions, applications and investigations in various contexts were published about rendezvous numbers.

But how come, that so different approaches can give the same results for polarization constants? In [10] we could satisfactorily describe the theory of rendezvous numbers by general (linear) potential theory as laid down by, e.g., [11]. So it turns out that the approach of García-Vázquez and Villa was not that much different, after all: also below the surface of Gross’ Theorem potential theory lies behind.

4 More Polynomial Extremal Problems

The further extremal problems come from natural extensions of classical, univariate approximation theory questions. Although we do not have space here to describe their widespread use and various applications, they indeed are rather important questions in particular in infinite dimensional holomorphy and in multivariate approximation.

One of the main groups of problems may be called Chebyshev type problems of polynomial growth. A typical, key example is determination of the quantity

\[ C_n(K, x) := \sup \{ p(x) : p \in P_n, \| p \|_K \leq 1 \}, \tag{16} \]

for arbitrary fixed \(x \in X\) and \(K \subset X\) a fixed convex body.

In dimension 1 the only convex body is the interval, and for the unit interval \(I := [-1, 1]\) the answer is given for all \(x \notin I\) by \(T_n(|x|)\), with \(T_n\) the Chebyshev polynomials

\[ T_n(x) := \frac{1}{2} \left\{ (x + \sqrt{x^2 - 1})^n + (x - \sqrt{x^2 - 1})^n \right\}. \]

Another group is derivative estimates. A fundamental property of polynomials is that having control over the size (of values) of a polynomial (say, on a convex body \(K\)) automatically provides some finite bound even on their derivatives. One can seek the best bounds at some given point \(x\), (usually within \(K\)), or some global, uniform bound on the whole of \(K\), say. Also, we can group these problems according to possible further restrictions on the type of polynomials, or the type of derivatives we consider: e.g., we can restrict to homogeneous polynomials, or we can consider tangential derivatives etc. A further direction of research is dealing with higher order derivatives.

First let us consider here the pointwise or Bernstein problem of maximizing \(\| \nabla p(x) \|\) for a given \(x \in \text{int}K\) and among all polynomials with \(\| p \|_K \leq 1\). In analogy to the best one dimensional estimates, we can normalize the extremal
quantity by the “Bernstein-Szegő factor” and thus define the \((n^{th})\) Bernstein constant as

\[
B_n(K, x) := \frac{1}{n} \sup_{\deg p \leq n, |p(x)| < ||p||_K} \frac{\|\nabla p(x)\|}{\sqrt{||p||_K^2 - p^2(x)}}.
\]

Next, let us consider the homogeneous Markov factor for arbitrary (say, centrally symmetric) convex bodies \(K\). As then \(K\) generates a corresponding norm \(\|\cdot\|_K\), we can equivalently seek for the estimation of derivatives of homogeneous polynomials in arbitrary given norms. So put

\[
M_n^{(k)}(K) := \sup_{\|x\|_K \leq 1, p \in P^*_n} \|\hat{D}^k p(x)\|,
\]

where for given \(x\) \(\|\hat{D}^k p(x)\|\) stands for the norm of the diagonal \(k\)-form – the \(k\)-homogeneous polynomial – of the \(k\)-linear mapping \(D^k p(x)\) of \(X^k\), with \(D^k\) being \(k\)-fold differentiation. Taking \(M_n^{(k)} := \sup_K M_n^{(k)}(K)\), already Harris [14] has shown that \(M_n^{(k)} = c_m^{(k)}\), where

\[
c_m^{(k)} := \max\{|p^{(k)}(0)}: p \in P, |p(t)| \leq (1 + |t|)^m (t \in \mathbb{R})\}.
\]

5 Generalized Minkowski Functional

If \(x\) lies in \(\mathbb{R}\) and we want to quantify its position with respect to the unit interval \(I := [-1, 1]\), say, then it suffices to take \(|x|\). In several dimensions a more sophisticated quantitative notion is needed. This was found essentially by Rivlin and Shapiro [30]; but it turned out, that the quantity is actually (equivalent to) an older geometric notion, going back to Minkowski [20] and contemporaries. We extended the notion even to topological vector spaces, and gave a throughout description of its properties, many equivalent definitions and some of the related interesting problems, see [26]. Here we recall just one definition. By convexity, \(K\) is the intersection of its “supporting halfspaces” \(X(K, v^*)\), and grouping opposite halfspaces to form layers \(L(K, v^*) := X(K, v^*) \cap X(K, -v^*)\) we get

\[
K = \bigcap_{v^* \in S^*} X(K, v^*) = \bigcap_{v^* \in S^*} L(K, v^*).
\]

Any layer can be homothetically dilated with quotient \(\lambda \geq 0\) at any of its symmetry centers lying on its central symmetry hyperplane to get \(L^\lambda\), and we can even define

\[
K^\lambda := \bigcap_{v^* \in S^*} L^\lambda(K, v^*).
\]

Using the convex, closed, bounded, increasing and (as easily seen, cf. [26, Proposition 3.3]) even absorbing set system \(\{K^\lambda\}_{\lambda \geq 0}\) the generalized Minkowski functional is

\[
\alpha(K, x) := \inf\{\lambda \geq 0: x \in K^\lambda\}.
\]
One advantage of this formulation is that it defines $\alpha$ in a unified way both for $x \in K$ and for $x \in X \setminus K$. For the Chebyshev problem we need it only for exterior points: for the Bernstein problem we are concerned with interior points only. The same generalized Minkowski functional occurs naturally in different extremal problems.

6 Inequalities For Polynomials

In what follows, denote $h(K, u^*) := \sup_K u^*$ the support functional and $w(K, u^*) := h(K, u^*) + h(K, -u^*)$ the width of $K$ in direction of $u^* \in S^*$. Also, take $w(K) := \min_{S^*} w(K, \cdot)$ to be the minimal width of $K$. The Chebyshev problem (16) has the following answer.

**Theorem 4** (Révész, Sarantopoulos [26]). For an arbitrary convex body $K \subset X$ and any point $x \notin K$ we have

$$C_n(K, x) = T_n(\alpha(K, x)).$$

Moreover, $C_n(K, x)$ is actually a maximum, attained by

$$P(x) := T_n\left(\frac{2(v^*, x) - h(K, v^*) + h(K, -v^*)}{w(K, v^*)}\right).$$

The result was obtained for finite dimensional spaces and strictly convex bodies by Rivlin and Shapiro [30]. Removing strict convexity is not too difficult, and can be done several ways, but to extend to infinite dimensions turned to be more delicate. The reason is that not only compactness is lost, but also the existence of parallel supporting hyperplanes of a certain configuration. Even though counterexamples show that we no longer have that configuration, a new proof goes through – by means of more careful analysis of $\alpha(K, x)$. For details see [26, 27, 25].

Even if this seems to settle the question satisfactorily, for many further applications, in particular for Bernstein and Markov problems, an extended answer for complex points $z \in Z$, where $Z := X + iX$ is a complexification, would be very interesting even for finite dimensional $X$.

Note that even for dimension 1 this is far from being obvious. Indeed, for complex points close to $(-1, 1)$, Chebyshev polynomials may have values arbitrarily close to 0, thus we cannot expect just one polynomial to be extremal for all $z \in \mathbb{C}$. On the other hand, results in this direction can be applied in the Bernstein and Markov problems, see, e.g., [7]. For the multivariate complex case we only have some vague ideas at the moment.

**Theorem 5** (Révész, Sarantopoulos [29]). With some absolute constant $c_1 > 0$ we have

$$c_1 m \log m \leq c_m^{(1)} \leq 3m \log m.$$

(23)
The upper estimate was already obtained by Harris [14] with some less precise constant – the more difficult lower estimate was new. By iterating the result for $M_n^{(k)}$, one even gets some upper estimate for all $M_n^{(k)} = c_n^{(k)}$. We also computed some better values for higher derivatives, see [29].

Finally let us consider the Bernstein problem. Here the first, and one of the still very successful methods – the method of inscribed ellipses – we introduced by Sarantopoulos in 1991 [32]. The key of all of the method is the Inscribed Ellipse Lemma:

**Lemma A (Sarantopoulos [32]).** Let $K$ be any subset in a vector space $X$. Suppose that $x \in K$ and the ellipse
\[
    r(t) = \cos a + b \sin t y + x - a \quad (t \in [-\pi, \pi])
\]
lies inside $K$. Then we have for any polynomial $p$ of degree at most $n$ the Bernstein type inequality
\[
    |\langle Dp(x), y \rangle| \leq \frac{n}{b} \sqrt{||p||_K^2 - p^2(x)}. \tag{25}
\]

The method was applied even to nonsymmetric convex bodies, but in this case our result is still not final.

**Theorem C (Kroó, Révész, Sarantopoulos [16], [26]).** Let $K$ be an arbitrary convex body, $x \in \text{int}K$ and $\|y\| = 1$, where $X$ can be an arbitrary normed space. Then we have
\[
    |\langle Dp(x), y \rangle| \leq \frac{2n \sqrt{||p||_K^2 - p^2(x)}}{\tau(K, y) \sqrt{1 - \alpha(K, x)}}, \tag{26}
\]
for any polynomial $p$ of degree at most $n$. Here $\tau(K, y) := \sup\{\lambda : \exists x \in K \text{ such that } x + \lambda y \in K\}$ stands for the “maximal” chord in direction $y$.

Since for $x \in K \alpha(K, x) \leq 1$, and for $\alpha \in (0, 1) \frac{1}{1-\alpha} \leq \frac{1+\alpha}{1-\alpha^2}$, we also have
\[
    B_n(K, x) \leq \frac{2\sqrt{2n}}{w(K) \sqrt{1 - \alpha^2(K, x)}}. \tag{27}
\]

Note that apart from the $\sqrt{1 + \alpha(K, x)} \leq \sqrt{2}$ factor, the estimate gives the sharp result even in dimension 1. Hence it was natural to conjecture

**Conjecture 2 (Révész, Sarantopoulos [26]).**
\[
    B_n(K, x) = \frac{2n}{w(K) \sqrt{1 - \alpha^2(K, x)}}. \tag{28}
\]

Note that (26) was only a – delicate, but not exact – estimate. Recently we showed [19], that not even in the case of the standard simplex can the method reach Conjecture 2.
7 Potential Theory Once More

We have already seen how potential theory plays a role in the linear polarization constant problem. It is also well-known, see, e.g., [31], that the theory of weighted approximation and univariate orthogonal polynomials with respect to weights can be analyzed via (weighted) potential theory of the complex plane. Because the original problem is translated to determine $c_m^{(k)}$ in (18), it is not so much surprising, that with a slight extension of the theory even our multivariate homogeneous polynomial Markov problem could be treated. That was our approach in [29].

However, there is a genuinely multivariate potential theoretic approach to multivariate polynomial inequalities: pluripotential theory. Just a few years after Sarantopoulos, M. Baran [4] obtained the same results as [32], with the method extending to other cases as well.

This theory is well described in, e.g., [8, 15], so here we summarize only very briefly. The starting point is the Zaharjuta–Siciak extremal function, which is defined, e.g., in $Y := \mathbb{C}^d$ with respect to a compact set $E \subset Y$ (or $E \subset X := \mathbb{R}^d$, say), as follows: $V_E$ vanishes on $E$, while outside $E$ we have the definition

$$V_E(z) := \sup \{ \frac{\log |p(z)|}{\deg p} : 0 \neq p \in \mathcal{P}(Y), \|p\|_E \leq 1 \}$$

(29)

For $E \subset X$ one can easily restrict even to $p \in \mathcal{P}(X)$. Note that $\log |p(z)|$ is a plurisubharmonic function (PSH, for short). In case $E$ is some nice set – e.g., if it is a convex body – then already $V_E$ is continuous. However, even in the general case the upper semicontinuous regularization $V_E^*$ is at least upper semicontinuous, hence locally bounded for non-pluripolar $E$, which we now assume.

The growth of $(1/\deg p) \log |p(z)|$ is at most $\log_+ |z| + O(1)$. So it is reasonable to consider the Lelong class:

$$\mathcal{L}(E) := \{ u \in \text{PSH} : u|_E \leq 0, u(z) \leq \log_+ |z| + O(1) \}$$

and to define

$$U_E(z) := \sup \{ u(z) : u \in \mathcal{L}(E) \} .$$

(30)

This function is named the pluricomplex Green function. The Zaharjuta–Siciak Theorem says that (30) and (29) are equal, at least as long as $E \subset \mathbb{C}^d$ is compact.

The extension of the Laplace- and Poisson equations is the so-called complex Monge–Ampère equation:

$$(\overline{\partial \partial u})^d := d! 4^d \det \frac{\partial^2 u}{\partial z_j \partial z_k}(z) \, dV(z),$$

(31)

where $dV(z)$ is just the usual volume element in $\mathbb{C}^d$. Due to the work of Bedford and Taylor, the operator extends, in the appropriate sense, even to the
whole set of locally bounded PSH functions (which includes $V_E^*$ for non-pluri-polar $E$). Therefore, it makes sense to consider $(\partial \partial^* V_E^*)^d$, which is then a compactly supported measure $\lambda_E$ and is called the complex equilibrium measure of the set $E$. The support of $\lambda_E$ lies in the polynomial convex hull $\hat{E}$ of $E$: in case $E$ is convex, $\hat{E} = E$ and $V_E^* = V_E$: moreover, it is also shown that $\lambda|_E(C) = \lambda|_{\hat{E}} = (2\pi)^d$. Observe that $V_E(z) := \sup_{n \in \mathbb{N}} \frac{1}{n} \log |C_n(E, z)|$ which gives rise for any convex body $K$ and $x \in X \setminus K$ to the formula $V_K(x) = \log \left( \frac{1}{\alpha(K, x) + \sqrt{\alpha(K, x)^2 - 1}} \right)$. However, in the Bernstein problem the values of $V_K$ are much more of interest for complex points $z = x + iy$, in particular for $x \in K$ and $y$ small and nonzero. More precisely, the important quantity is the normal (sub)derivative

\[ D^+ y V_E(x) := \liminf_{\epsilon \to 0} \frac{V_E(x + i\epsilon y)}{\epsilon}. \]  

(32)

\[ |\langle Dp(x), y \rangle| \leq n D^+ y V_E(x) \sqrt{\|p\|_E^2 - p(x)^2}. \]  

(33)

Also, one can use the inscribed ellipse method for the estimation of $D_y p(x)$. In the special case of the standard simplex the yield of both methods can be calculated explicitly. So one can compare.

Corollary 1 (Milev, Révész [19]). The estimate (33), calculated for the standard simplex $\Delta$ of $\mathbb{R}^d$ at any point $x \in \Delta$ and in any direction $y \in S^*$ gives exactly identical result to the yield of the inscribed ellipse method.

Much remains to explain in this striking coincidence.

There are further yields of the theory of PSH functions, when applied to the Bernstein problem. For more precise notation now we introduce (interpreting $0/0$ as $0$ here)

Definition 4.

\[ G(E, x) := \{ \frac{\nabla p(x)}{n \sqrt{\|p\|^2 - p(x)^2}} : 0 \neq p \in \mathcal{P}_n, n \in \mathbb{N} \}, \]  

(34)

and following Baran we consider also the convex hull

\[ \tilde{G}(E, x) := \text{con} \ G(E, x) . \]  

(35)

Clearly for any compact $E \subset \mathbb{R}^d$ $\sup_{n \in \mathbb{N}} B_n(E, x) = \sup_{u \in G(E, x)} \|u\|$ holds.

Theorem E (Baran, [5]). Let $E$ be a compact subset of $\mathbb{R}^d$ with nonempty interior. Then the equilibrium measure $\lambda|_E$ is absolutely continuous in the interior of $E$ with respect to the Lebesgue measure of $\mathbb{R}^d$. Denote its density function by
\( \lambda(x) \) for all \( x \in \text{int} \, E \). Then we have \( \frac{1}{d!} \lambda(x) \geq \text{vol} \, \tilde{G}(E, x) \) for a.a. \( x \in \text{int} \, E \). Moreover, if \( E \) is a symmetric convex domain of \( \mathbb{R}^d \), then here we have exact equality.

**Conjecture 3** (Baran, [5]). Even if \( E \) is a non-symmetric convex body in \( \mathbb{R}^d \) we have \( \frac{1}{d!} \lambda(x) = \text{vol} \, \tilde{G}(E, x) \).

However, in our recent analysis [24] we found that for dimension 2 \( \tilde{G}(\Delta, x) \subset E_x \) with some ellipsoid domain \( E_x \) of area \( \lambda(x)/2 \) and major axis larger than (28). So we close this paper with the following corollary.

**Corollary 2.** The two conjectures Conjecture 2 and Conjecture 3 cannot hold true simultaneously.

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