BLOW-UP, EXPONENTIAL GROWTH OF SOLUTION FOR A NONLINEAR PARABOLIC EQUATION WITH $p(x)$ – LAPLACIAN

AMAR OUAOUA* AND MESSAOUD MAOUNI

Laboratory of Applied Mathematics and History and Didactics of Mathematics (LAMAHIS)
University of 20 August 1955, Skikda, Algeria

*Corresponding author: a.ouaoua@univ-skikda.dz

Abstract. In this paper, we consider the following equation

$$u_t - \text{div} \left( |\nabla u|^{p(x)-2} \nabla u \right) + \omega |u|^{m(x)-2} u_t = b |u|^{r(x)-2} u.$$  

We prove a finite time blowup result for the solutions in the case $\omega = 0$ and exponential growth in the case $\omega > 0$, with the negative initial energy in the both case.

1. Introduction

We consider the following boundary problem:

$$\begin{cases}
u_t - \text{div} \left( |\nabla u|^{p(x)-2} \nabla u \right) + \omega |u|^{m(x)-2} u_t = b |u|^{r(x)-2} u & \text{in } \Omega \times (0,T), \\
u(x, t) = 0, & x \in \partial \Omega, \ t \geq 0, \\
u(x, 0) = \nu_0(x) & \text{in } \Omega.
\end{cases}$$

(1.1)

where $\Omega$ is a bounded domain in $\mathbb{R}^n, n \geq 1$ with smooth boundary $\partial \Omega$ and $b > 0$, $\omega \geq 0$ are constants, $p(\cdot), m(x)$ and $r(\cdot)$ are given measurable functions on $\Omega$ satisfying

$$2 \leq m_1 \leq m(x) \leq m_2 < p_1 \leq p(x) \leq p_2 < r_1 \leq r(x) \leq r_2 \leq p_*(x).$$

(1.2)

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\[ p_1 := \text{ess}_{x \in \Omega} \inf(p(x)), \quad p_2 := \text{ess}_{x \in \Omega} \sup(p(x)), \]
\[ r_1 := \text{ess}_{x \in \Omega} \inf(r(x)), \quad r_2 := \text{ess}_{x \in \Omega} \sup(r(x)), \]
\[ m_1 := \text{ess}_{x \in \Omega} \inf(m(x)), \quad m_2 := \text{ess}_{x \in \Omega} \sup(m(x)), \]

and

\[
p_*(x) = \begin{cases} \frac{n p(x)}{\text{ess}_{x \in \Omega} (n - p(x))} & \text{if } p_2 < n, \\ +\infty & \text{if } p_2 \geq n \end{cases}.
\]

We also assume that \( p(\cdot), m(\cdot) \) and \( r(\cdot) \) satisfy the log-Hölder continuity condition:

\[
|q(x) - q(y)| \leq -\frac{A}{\log |x - y|}, \quad \text{for a.e. } x, y \in \Omega, \text{ with } |x - y| < \delta,
\]

\( A > 0, 0 < \delta < 1. \)

Equation (1.1) can be viewed as a generalization of the evolutional \( p \)-Laplacian equation

\[
\frac{\partial u}{\partial t} - \text{div} \left( |\nabla u|^{p-2} \nabla u \right) + \omega |u|^{m-2} u \frac{\partial u}{\partial t} = b |u|^{r-2} u,
\]

with the constant exponent of nonlinearity \( p, m, r \in (2, \infty) \), which appears in various physical contexts. In particular, this equation arises from the mathematical description of the reaction-diffusion/ diffusion, heat transfer, population dynamics processes, and so on (see [11]) and references therein). Recently in [1], in the case \( \omega = 0 \), Agaki proved an existence and blow up result for the initial datum \( u_0 \in L^r(\cdot) \). Ōtani [17] studied the existence and the asymptotic behavior of solutions of (1.1) and overcome the difficulties caused by the use of nonmonotone perturbation theory. The quasilinear case, with \( p \neq 2 \), requires a strong restriction on the growth of the forcing term \( |u|^{r-2} u \), which is caused by the loss of the elliptic estimate for the \( p \)-Laplacian operator defined by \( \Delta_p u = \text{div}(|\nabla u|^{p-2} \nabla u) \) (see [2]).

Alaoui et al [12] considered the following nonlinear heat equation

\[
\begin{align*}
\left\{ \begin{array}{ll}
\frac{\partial u}{\partial t} - \text{div} \left( |\nabla u|^{p(x)-2} \nabla u \right) = |u|^{r(x)-2} u + f, & \text{in } \Omega \times (0, T), \\
u(x, t) = 0, & \text{in } \partial \Omega \times (0, T), \\
u(x, 0) = u_0(x) & \text{in } \Omega.
\end{array} \right.
\end{align*}
\]

(1.4)

Where \( \Omega \) is a bounded domain in \( \mathbb{R}^n \) with smooth boundary \( \partial \Omega \). Under suitable conditions on \( r \) and \( p \) and for \( f = 0 \), they showed that any solution with nontrivial initial datum blows up in finite time. In the absence of the diffusion term in equation (1.1) when \( p(x) = p \) and \( r(x) = r \) proved the existence and blow up results have been established by many authors (See [1 − 3, 9, 14, 17]).
We should also point out that Polat [18] established a blow-up result for the solution with vanishing initial energy of the following initial boundary value problem

\[ u_t - u_{xx} + |u|^{m-2} u_t = |u|^{p-2} u. \]  

(1.5)

Where \( m \) and \( p \) are real constants.

In recent years, much attention has been paid to the study of mathematical models of electro-theological fluids. This models include hyperbolic, parabolic or elliptic equations which are nonlinear with respect to the gradient of the thought solution with variable exponents of nonlinearity, (see [4, 5, 10, 15]).

Our objective in this paper is to study: In the section 3, the blow up of the solutions of the problem (1.1) in the case \( \omega = 0 \), in the section 4, exponential growth of solution when \( \omega > 0 \).

2. Preliminaries

We present in this section some Lemmas about the Lebesque and Sobolev space with variables components (See [6–8, 12, 13]). Let \( p : \Omega \to [1, +\infty] \) be a measurable function, where \( \Omega \) is a domain of \( \mathbb{R}^n \).

We define the Lebesque space with a variable exponent \( p(.) \) by

\[ L^{p(.)} (\Omega) := \left\{ v : \Omega \to \mathbb{R} : \text{measurable in} \ \Omega, \ A_{p(.)} (\lambda v) < +\infty, \ \text{for some} \ \lambda > 0 \right\}, \]

where \( A_{p(.)} (v) = \int_{\Omega} |v(x)|^{p(x)} \ dx. \)

The set \( L^{p(.)} (\Omega) \) equipped with the norm (Luxemburg’s norm)

\[ \|v\|_{p(.)} := \inf \left\{ \lambda > 0 : \int_{\Omega} \frac{|v(x)|^{p(x)}}{\lambda} \ dx \leq 1 \right\}, \]

\( L^{p(.)} (\Omega) \) is a Banach space [13].

We next, define the variable-exponent Sobolev space \( W^{1,p(.)} (\Omega) \) as follows:

\[ W^{1,p(.)} (\Omega) := \left\{ v \in L^{p(.)} (\Omega) \text{ such that} \ \nabla v \text{ exists and} \ |\nabla v| \in L^{p(.)} (\Omega) \right\}. \]

This is a Banach space with respect to the norm \( \|v\|_{W^{1,p(.)} (\Omega)} = \|v\|_{p(.)} + \|\nabla v\|_{p(.)}. \)

Furthermore, we set \( W^{1,p(.)} (\Omega) \) to be the closure of \( C_0^\infty (\Omega) \) in the space \( W^{1,p(.)}_0 (\Omega) \). Let us note that the space \( W^{1,p(.)} (\Omega) \) has a different definition in the case of variable exponents.

However, under condition (1.3), both definitions are equivalent [13]. The space \( W^{-1,p'(.)} (\Omega) \), dual of \( W^{1,p(.)}_0 (\Omega) \), is defined in the same way as the classical Sobolev spaces, where \( \frac{1}{p(.)} + \frac{1}{p'(.)} = 1. \)

Lemma 2.1. (Poincaré’s inequality) Let \( \Omega \subset \mathbb{R}^n \) be a bounded domain and suppose that \( p(.) \) satisfies (1.3), then

\[ \|v\|_{p(.)} \leq c \|\nabla v\|_{p(.)}, \text{ for all } v \in W^{1,p(.)}_0 (\Omega). \]
Where $c > 0$ is a constant which depends on $p_1$, $p_2$, and $\Omega$ only. In particular, $\|\nabla v\|_{p(\cdot)}$ define an equivalent norm on $W^{1,p(\cdot)}_0(\Omega)$.

**Lemma 2.2.** If $p(\cdot) \in C(\overline{\Omega})$ and $q : \Omega \to [1, +\infty)$ is a measurable function such that

$$
es \inf_{x \in \Omega} (p_*(x) - q(x)) > 0 \text{ with } p_*(x) = \begin{cases} \frac{np(x)}{\ess_{x \in \Omega} (n-p(x))} & \text{if } p_2 < n \\ +\infty & \text{if } p_2 \geq n. \end{cases}$$

Then the embedding $W^{1,p(\cdot)}_0(\Omega) \hookrightarrow L^{q(\cdot)}(\Omega)$ is continuous and compact.

**Lemma 2.3.** (Hölder’s Inequality) Suppose that $p$, $q$, $s \geq 1$ are measurable functions defined on $\Omega$ such that

$$\frac{1}{s}(y) = \frac{1}{p(y)} + \frac{1}{q(y)}, \text{ for a.e. } y \in \Omega.$$ 

If $u \in L^{p(\cdot)}(\Omega)$ and $v \in L^{q(\cdot)}(\Omega)$, then $uv \in L^{s(\cdot)}(\Omega)$, with

$$\|uv\|_{s(\cdot)} \leq 2 \|u\|_{p(\cdot)} \|v\|_{q(\cdot)}.$$ 

**Lemma 2.4.** If $p$ a measurable function on $\Omega$ satisfying (1.2), then we have

$$\min \left\{ \|u\|_{p_1(\cdot)}, \|u\|_{p_2(\cdot)} \right\} \leq A_{p(\cdot)}(u) \leq \max \left\{ \|u\|_{p_1(\cdot)}, \|u\|_{p_2(\cdot)} \right\},$$

for any $u \in L^{p(\cdot)}(\Omega)$.

### 3. Blow up

In this section, we prove that the solution of equation (1.1) blow up in finite time when $\omega = 0$. We recall that (1.1), becomes

$$
\begin{aligned}
&\begin{cases}
u_t - \div (|\nabla u|^{p(x)-2} \nabla u) = b |u|^{r(x)-2} u & \text{in } \Omega \times (0,T), \\
u(x,t) = 0, & x \in \partial\Omega, \ t \geq 0, \\
u(x,0) = u_0(x) & \text{in } \Omega.
\end{cases}
\end{aligned}
$$

(3.1)

We start with a local existence result for the problem (1.1), which is a direct result of the existence theorem by Agaki and Otani [2].

**Proposition 3.1.** For all $u_0 \in W^{1,p(\cdot)}_0(\Omega)$, there exists a number $T_0 \in (0,T]$ such that the problem (1.1) has a solution $u$ on $[0,T_0]$ satisfying:

$$u \in C_w([0,T_0]; W^{1,p(\cdot)}_0(\Omega)) \cap C([0,T_0], L^{r(\cdot)}(\Omega)) \cap W^{1,2}(0,T_0; L^2(\Omega)) .$$

We define the energy functional associated of the problem (1.1)

$$E(t) = \int_\Omega \frac{1}{p(x)} |\nabla u|^{p(x)} \, dx - b \int_\Omega \frac{1}{r(x)} |u|^{r(x)} \, dx .$$

(3.2)
Theorem 3.1. Let the assumptions of proposition 1, be satisfied and assume that

\[ E(0) < 0. \] (3.3)

Then the solution of the problem (3.1), blow up in finite time.

Now, we let

\[ H(t) := -E(t), \] (3.4)

and

\[ L(t) = \frac{1}{2} \int_{\Omega} u^2 dx. \] (3.5)

To prove our result, we first establish some Lemmas.

Lemma 3.1. Assume that (1.2) and (1.3), hold and \( E(0) < 0 \). Then

\[ A_{p(\cdot)} (\nabla u) < \frac{bp_0}{r_1} A_{r(\cdot)} (u), \] (3.6)

and

\[ \frac{r_1}{b} H(0) < A_{r(\cdot)} (u). \] (3.7)

Proof. We multiply the first equation of (3.1) by \( u_t \) and integrating over the domain \( \Omega \), we get

\[ \frac{d}{dt} \left( \int_{\Omega} \frac{1}{p(x)} |\nabla u|^{p(x)} dx - b \int_{\Omega} \frac{1}{r(x)} |u|^{r(x)} dx \right) = -\|u_t\|_2^2, \]

then

\[ E'(t) = -\|u_t\|_2^2 \leq 0. \] (3.8)

Integrating (3.8) over \((0, t)\), we obtain

\[ E(t) \leq E(0) < 0. \] (3.9)

By (3.2) and (3.9), we have

\[ \int_{\Omega} \frac{1}{p(x)} |\nabla u|^{p(x)} dx < b \int_{\Omega} \frac{1}{r(x)} |u|^{r(x)} dx, \]

so that

\[ \int_{\Omega} \frac{1}{p_2} |\nabla u|^{p(x)} dx < \int_{\Omega} \frac{b}{r_1} |u|^{r(x)} dx. \]

On the other hand, we have

\[ H(t) = -\int_{\Omega} \frac{1}{p(x)} |\nabla u|^{p(x)} dx + b \int_{\Omega} \frac{1}{r(x)} |u|^{r(x)} dx \leq b \int_{\Omega} \frac{1}{r(x)} |u|^{r(x)} dx. \] (3.10)
Then, by (3.10), (3.4) and (3.9), we obtain

\[ 0 < H(0) < H(t) < \frac{b}{r_1} A_{r(\cdot)}(u). \]

\[ \square \]

**Lemma 3.2.** [16] Assume that (1.2), (1.3) hold and \( E(0) < 0 \). Then the solution of (3.1), satisfies for some \( c > 0 \),

\[ A_{r(\cdot)}(u) \geq c \| u \|_{r_1}^{r_1}. \]  

(3.11)

**Proof of theorem 1.** We have

\[
 L'(t) = \int_{\Omega} uu_t dx \\
 = \int_{\Omega} u \left( \text{div} \left( |\nabla u|^{p(x)-2} \nabla u \right) + b |u|^{r(x)-2} u \right) dx \\
 = -A_{p(\cdot)}(\nabla u) + b A_{r(\cdot)}(u). \]  

(3.12)

Combining of (3.12), (3.11) and (3.6), leads to

\[ L'(t) \geq cb \left( 1 - \frac{p_2}{r_1} \right) \| u \|_{r_1}^{r_1}. \]  

(3.13)

Now, we estimate \( L^{\frac{n}{2}}(t) \), by the embedding of \( L^{r_1}(\Omega) \hookrightarrow L^2(\Omega) \), we get

\[ L^{\frac{n}{2}}(t) \leq \left( \frac{1}{2} \| u \|_{r_1}^{2} \right)^{\frac{n}{2}} \leq c \| u \|_{r_1}^{r_1}. \]  

(3.14)

By combining (3.14) and (3.13), we obtain

\[ L'(t) \geq \xi L^{\frac{n}{2}}(t). \]  

(3.15)

A direct integration of (3.15), then yields

\[ L^{\frac{n}{2} - 1}(t) \geq \frac{1}{L^{\frac{n}{2} - 1}(0) - \xi t}. \]

Therefore, \( L \) blow up in a time \( t^* \leq \frac{1}{L^{\frac{n}{2} - 1}(0)}. \)  

\[ \square \]

4. EXPONENTIAL GROWTH

In this section, we prove that the solution of equation (1.1) exponential growth when \( \omega > 0 \).

**Lemma 4.1.** Suppose that (1.2) holds and \( E(0) < 0 \). Then,

\[ \int_{\Omega} |u|^{m(x)} dx \leq c \left( \| u \|_{r_1}^{r_1} + H(t) \right). \]  

(4.1)
Proof.

\[
\int_{\Omega} |u|^{m(x)} \, dx = \int_{\Omega^{-}} |u|^{m(x)} \, dx + \int_{\Omega^{+}} |u|^{m(x)} \, dx,
\]

where

\[
\Omega^{+} = \{ x \in \Omega / \, |u(x, t)| \geq 1 \} \quad \text{and} \quad \Omega^{-} = \{ x \in \Omega / \, |u(x, t)| < 1 \}.
\]

So, we get

\[
\int_{\Omega} |u|^{m(x)} \, dx \leq c \left( \left( \int_{\Omega^{-}} |u|^{r_{1}} \, dx \right)^{\frac{m_{1}}{r_{1}}} + \left( \int_{\Omega^{+}} |u|^{r_{1}} \, dx \right)^{\frac{m_{2}}{r_{1}}} \right) \leq c (\|u\|_{r_{1}}^{m_{1}} + \|u\|_{r_{1}}^{m_{2}}).
\]

Exploiting the algebraic inequality

\[
z^{v} \leq (z + 1) \leq \left( 1 + \frac{1}{a} \right) (z + a), \, \forall z > 0, \, 0 < v \leq 1, \, a \geq 0,
\]

we have

\[
\|u\|_{r_{1}}^{m_{1}} \leq c (\|u\|_{r_{1}}^{r_{1}})^{\frac{m_{1}}{r_{1}}} \leq c \left( 1 + \frac{1}{H(0)} \right) (\|u\|_{r_{1}}^{r_{1}} + H(0)) \leq c \left( \|u\|_{r_{1}}^{r_{1}} + H(t) \right).
\]

Similarly,

\[
\|u\|_{r_{1}}^{m_{2}} \leq c (\|u\|_{r_{1}}^{r_{1}})^{\frac{m_{2}}{r_{1}}} \leq c \left( 1 + \frac{1}{H(0)} \right) (\|u\|_{r_{1}}^{r_{1}} + H(0)) \leq c \left( \|u\|_{r_{1}}^{r_{1}} + H(t) \right).
\]

This gives

\[
\int_{\Omega} |u|^{m(x)} \, dx \leq c (\|u\|_{r_{1}}^{r_{1}} + H(t)).
\]

\[\square\]

**Theorem 4.1.** Let the assumptions of proposition 1, be satisfied and assume that (3.3) holds. Then the solution of the problem (1.1), grows exponentially.

**Proof.** By the same procedure of the proof the Lemma 5, we get

\[
E'(t) = -\|u_{t}\|^{2} - \omega \int_{\Omega} |u|^{m(x)-2} u_{t}^{2} \leq 0,
\]

then, we have

\[
H'(t) = \|u_{t}\|^{2} + \omega \int_{\Omega} |u|^{m(x)-2} u_{t}^{2} \geq 0.
\]
We define
\[ G(t) = H(t) + \epsilon L(t). \] (4.4)
for \( \epsilon \) small to be chosen later.

The time derivative of (4.4), we obtain
\[ G'(t) = H'(t) + \epsilon \int_{\Omega} uu_t \, dx. \]

By using (1.1), we get
\[ G'(t) = H'(t) - \epsilon A_{p(.)} (\nabla u) + \epsilon b A_{r(.)} (u) - \epsilon \omega \int_{\Omega} |u|^{m(x)-2} u_t u \, dx. \] (4.5)

To estimate the last term in the right hand side of (4.5), by using the following Young’s Inequality
\[ XY \leq \delta X^2 + \delta^{-1} Y^2, \quad X, Y \geq 0, \quad \delta > 0. \]

\[ \int_{\Omega} |u|^{m(x)-2} u_t u \, dx = \int_{\Omega} |u|^{m(x)-2} u_t |u|^{m(x)-2} u \, dx \leq \delta \int_{\Omega} |u|^{m(x)-2} u_t^2 \, dx + \delta^{-1} \int_{\Omega} |u|^{m(x)} \, dx. \]

We conclude
\[ G'(t) \geq (1 - \epsilon \delta) \int_{\Omega} |u|^{m(x)-2} u_t^2 \, dx + \|u_t\|^2_2 - \epsilon A_{p(.)} (\nabla u) \]
\[ + \epsilon b A_{r(.)} (u) - \epsilon \omega \delta^{-1} \int_{\Omega} |u|^{m(x)} \, dx. \] (4.6)

Then
\[ G'(t) \geq (1 - \epsilon \delta) \int_{\Omega} |u|^{m(x)-2} u_t^2 \, dx + \|u_t\|^2_2 - \epsilon \omega \delta^{-1} \int_{\Omega} |u|^{m(x)} \, dx \]
\[ + \epsilon (1 - \mu) r_1 H(t) + \epsilon b \mu A_{r(.)} (u) + \epsilon \left(1 - \mu \right) \frac{r_1}{P_2} - 1 A_{p(.)} (\nabla u), \]

where \( \mu \) is a constant such that \( 0 < \mu \leq 1 - \frac{P_2}{r_1} \).

Also, by using (3.6), we obtain
\[ G'(t) \geq (1 - \epsilon \delta) \int_{\Omega} |u|^{m(x)-2} u_t^2 \, dx + \|u_t\|^2_2 - \epsilon \omega \delta^{-1} \int_{\Omega} |u|^{m(x)} \, dx \]
\[ + \epsilon (1 - \mu) r_1 H(t) + \epsilon \left(b \mu + 1 - \mu - \frac{P_2}{r_1}\right) A_{r(.)} (u). \] (4.7)
Then, by Lemma 7 and (3.11), (4.7) becomes

$$G'(t) \geq (1 - \epsilon \delta) \int_{\Omega} |u|^{m(x)-2} u^2_t \, dx + \|u_t\|_2^2 - c \epsilon \omega \delta^{-1} \left(\|u\|_{r_1}^{r_1} + H(t)\right)$$

$$+ \epsilon (1 - \mu) r_1 H(t) + c \left(b \mu + 1 - \mu - p \frac{r_2}{r_1}\right) \|u\|_{r_1}^{r_1}.$$ 

(4.8)

So that

$$G'(t) \geq (1 - \epsilon \delta) \int_{\Omega} |u|^{m(x)-2} u^2_t \, dx + \|u_t\|_2^2 + \epsilon \left((1 - \mu) r_1 - c \omega \delta^{-1}\right) H(t)$$

$$+ \epsilon \left(c \left(b \mu + 1 - \mu - p \frac{r_2}{r_1}\right) - c \omega \delta^{-1}\right) \|u\|_{r_1}^{r_1}.$$ 

(4.9)

So, we chosen $\delta$ large sufficient and $\epsilon$ small enough for that we can find $\lambda_1$, $\lambda_2 > 0$, such that

$$G'(t) \geq \lambda_1 H(t) + \lambda_2 \|u\|_{r_1}^{r_1} \geq K_1 \left(H(t) + \|u\|_{r_1}^{r_1}\right),$$

(4.10)

and

$$G(0) = H(0) + \epsilon L(0) > 0.$$ 

Similarly in (4.7), we have

$$\|u\|_2^2 \leq c \left(H(t) + \|u\|_{r_1}^{r_1}\right).$$ 

(4.11)

On the other hand, by (4.11), we get

$$G(t) \leq K_2 \left(H(t) + \|u\|_{r_1}^{r_1}\right).$$ 

(4.12)

Combining with (4.12) and (4.10), we arrive at

$$G'(t) \geq \eta G(t).$$ 

(4.13)

Finally, a simple integration of (4.13) gives

$$G(t) \geq G(0) e^{\eta t}, \quad \forall t \geq 0.$$ 

(4.14)

Thus completes the proof. \(\square\)

References

[1] G. Akagi, Local existence of solutions to some degenerate parabolic equation associated with the p-Laplacian, J. Differential Equations 241 (2007), 359–385.

[2] G. Akagi and M. Ôtani, Evolutions inclusions governed by subdifferentials in reflexive Banach spaces, J. Evol. Equ. 4 (2004), 519–541.

[3] G. Akagi and M. Ôtani, Evolutions inclusions governed by the difference of two subdifferentials in reflexive Banach spaces, J. Differential Equations 209 (2005), 392–415.

[4] S.N. Antontsev and V. Zhikov, Higher integrability for parabolic equations of p(x, t)-Laplacian type. Adv. Differ. Equ. 10 (2005), 1053-1080.
[5] Y. Chen, S. Levine and M. Rao, Variable exponent, linear growth functions in image restoration. SIAM J. Appl. Math. 66 (2006), 1383-1406.

[6] D. Edmunds and J. Rakosnik, Sobolev embeddings with variable exponent, Stud. Math. 143 (3) (2000), 267–293.

[7] D. Edmunds and J. Rakosnik Sobolev embeddings with variable exponent. II, Math. Nachr. 246 (1) (2002), 53–67.

[8] X. Fan and D. Zhao, On the spaces $L^{p(x)}(\Omega)$ and $W^{m,p(x)}(\Omega)$, J. Math. Anal. Appl. 263 (2) (2001), 424–446.

[9] H. Fujita, On the blowing up solutions of the Cauchy problem for $u_t = \Delta u + u^{1+\alpha}$, J. Fac. Sci. Univ. Tokyo Sect. A.Math. 16 (1966), 105–113.

[10] Y. Gao, B. Guo and W.Gao, Weak solutions for a high-order pseudo-parabolic equation with variable exponents. Appl. Anal. 93 (2) (2014), 322-338.

[11] Z. Jiang, S. Zheng, and X. Song, Blow-up analysis for a nonlinear diffusion equation with nonlinear boundary conditions, Appl. Math. Lett. 17 (2) (2004), 193–199.

[12] A.M. Kbiri, S.A. Messaoudi and H.B. Khenous, A blow-up result for nonlinear generalized heat equation, Comput. Math. Appl. 68 (12) (2014), 1723–1732.

[13] D. Lars, P. Harjulehto, P. Hasto and M. Ruzicka, Lebesgue and Sobolev spaces with variable exponents, in: Lecture Notes in Mathematics, Springer, 2011.

[14] J. Leray and J.L.Lions, Quelques r´esultats de Visick sur les problémes elliptiques non linéaires pour les méthodes de Minty–Browder, Bull. Soc. Math. France 93 (1965), 97–107.

[15] S.Z. Lian, W.J. Gao, CL. Cao and HJ. Yuan, Study of the solutions to a model porous medium equation with variable exponents of nonlinearity. J. Math. Anal. Appl. 342 (2008), 27-38.

[16] S.A. Messaoudi and A. Talahmeh, Blowup in solutions of a quasilinear wave equation with variable-exponent nonlinearities, Math. Methods Appl. Sci. 40 (18) (2017), 6976-6986.

[17] M. Ötani, Nonmonotone perturbations for nonlinear parabolic equations associated with subdifferential operators, Cauchy problems, J. Differential Equations 46 (1982), 268–299.

[18] N. Polat, Blow up of solution for a nonlinear reaction diffusion equation with multiple nonlinearities, Int. J. Sci. Technol. 2 (2) (2007), 123–128.