ON THE IDEAL STRUCTURE OF THE FOURIER-STIELTJES ALGEBRA OF CERTAIN GROUPS

TIMO SIEBENAND

Abstract. We determine the structure of the weak*-closed $G$-invariant ideals in the Fourier-Stieltjes algebra $B(G)$ of certain groups $G$ by means of a $K$-theoretical obstruction. The groups to which this applies are groups whose only irreducible unitary representations that are not weakly contained in the left regular representation are class-one representations. In particular, this is the case for the groups $\text{SL}(2, \mathbb{R})$ and $\text{SL}(2, \mathbb{C})$, which we consider as explicit examples.

1. Introduction

Let $G$ be a locally compact group and $p \in [1, \infty]$. A unitary group representation $\pi : G \to U(V)$ on a Hilbert space $V$ is called $L^p$-representation if there exists a dense subspace $V_0 \subseteq V$ such that for all $v, w \in V_0$, the matrix coefficient function $\pi_{v,w} : G \to \mathbb{C}$, $s \mapsto \langle \pi(s)v, w \rangle$ lies in $L^p(G)$. The representation $\pi$ is called $L^p^+$-representation if $\pi$ is an $L^{p'}$-representation for all $p' > p$.

A group C*-algebra $C^*_p(G)$ is a completion of $C_c(G)$ with respect to a C*-norm $\| \cdot \|_\mu$ on $C_c(G)$ such that

$$\|f\|_r \leq \|f\|_\mu \leq \|f\|_u$$

for all $f \in C_c(G)$, where $\| \cdot \|_r$ and $\| \cdot \|_u$ denote the reduced and the universal C*-norm on $C_c(G)$, respectively.

The group C*-algebra $C^*_\mu(G)$ is called exotic if $\| \cdot \|_\mu$ is not equal to the reduced and not equal to the universal C*-norm. It follows immediately that the identity map on $C_c(G)$ extends to surjective *-homomorphisms $q : C^*_\mu(G) \to C^*_\mu(G)$ and $s : C^*_\mu(G) \to C^*_r(G)$, where $C^*_\mu(G)$ and $C^*_r(G)$ denote the universal and reduced group C*-algebra of $G$, respectively.

In this article, we are mainly interested in the potentially exotic group C*-algebras

$$C^*_{L^p}(G) \text{ and } C^*_{L^p^+}(G),$$

defined as the completion of the *-algebra $C_c(G)$ with respect to the C*-norms

$$\|f\|_{C^*_{L^p}} = \sup\{\|\pi(f)\| \mid \pi \text{ is an } L^p\text{-representation}\} \text{ and}$$

$$\|f\|_{C^*_{L^p^+}} = \sup\{\|\pi(f)\| \mid \pi \text{ is an } L^{p^+}\text{-representation}\},$$

respectively. The construction of these algebras essentially goes back to [BG13] (see also [SW18]).

Another natural way to describe group C*-algebras is by means of certain subspaces of the (Banach) dual space of the universal group C*-algebra. This
was studied systematically by Kaliszewski, Landstad and Quigg [KLQ13].

The Fourier-Stieltjes algebra \( B(G) \) of \( G \), consisting of all matrix coefficients of the unitary group representations of \( G \), is a subalgebra of the algebra of bounded continuous complex-valued functions on \( G \). It can be canonically identified with the (Banach) dual space \( C^*_\mu(G)' \) of the universal group C*-algebra \( C^*_\mu(G) \) via the pairing induced by

\[
\langle \varphi, f \rangle = \int_G \varphi f \, d\mu_G
\]

for \( \varphi \in B(G) \) and \( f \in C_c(G) \subseteq C^*_\mu(G) \), where \( \mu_G \) denotes a Haar measure on \( G \).

Besides the algebra structure, \( B(G) \) admits a canonical left and right \( G \)-action (see e.g. [KLQ13, Section 3]). A subspace \( E \subseteq B(G) \) is called \( G \)-module if \( E \) is closed under this left and right action. A \( G \)-module \( E \subseteq B(G) \) is called \( G \)-ideal if \( E \) is an ideal in \( B(G) \). It is proved in [KLQ13] that the weak*-closure of a \( G \)-module (resp. \( G \)-ideal) is again a \( G \)-module (resp. \( G \)-ideal). An example of a weak*-closed \( G \)-ideal is \( B_r(G) = C^*_r(G)' \). A weak*-closed \( G \)-module \( E \subseteq B(G) \) is said to be admissible if \( B_r(G) \subseteq E \). Every non-trivial weak*-closed \( G \)-ideal is admissible (see [RW16]). More generally, the Banach dual space \( C^*_\mu(G)' \subseteq B(G) \) of a group C*-algebra \( C^*_\mu(G) \) of \( G \) is an example of an admissible weak*-closed \( G \)-module, and it is proven in [KLQ13] that this correspondence from group C*-algebras of \( G \) to admissible weak*-closed \( G \)-submodules of \( B(G) \) is one-to-one.

Group C*-algebras corresponding to \( G \)-ideals are of particular importance. This can be explained by the fact that these group C*-algebras are stable under certain invariants. An illustration of this fact, and one of the major tools in this article, is given by the following result.

**Theorem 1.1** ([BEW18]). Let \( G \) be a K-amenable second countable locally compact group and \( C^*_\mu(G) \) a group C*-algebra such that \( C^*_\mu(G)' \) is an ideal in \( B(G) \). Then the canonical quotient maps \( q : C^*_\mu(G) \to C^*_\mu(G) \) and \( s : C^*_\mu(G) \to C^*_\mu(G) \) induce KK-equivalences.

Recall that a second countable locally compact group \( G \) is K-amenable if the identity in the \( G \)-equivariant KK-ring \( KK^G(\mathbb{C}, \mathbb{C}) \) is represented by a \( G \)-equivariant Kasparov module such that the \( G \)-representation is weakly contained in the left regular representation of \( G \) [Cun82], [JV84].

Since Theorem 1.1 and its proof are the main motivation for this article, we will outline the strategy of the proof in the following:

The theorem is a special case of a more general theorem (see [BEW18, Theorem 6.6]) on correspondence crossed product functors and generalizes a result of Julg and Valette (see [JV84, Proposition 3.4]). The major point in the proof of Julg and Valette’s result is the existence of the descent homomorphism (a functor from the category \( KK^G \) to the category \( KK \)) constructed by Kasparov in [Kas88, Theorem 3.11] for the universal crossed product functor. Buss, Echterhoff and Willett introduced in [BEW18] the so-called correspondence crossed product functors and generalized Kasparov’s construction (see [BEW18, Proposition 6.1]) for these correspondence crossed product functors, which enabled them to prove a result analogous to [JV84, Proposition 3.4] (see [BEW18, Theorem 6.6]).
In order to finally conclude the theorem as it is formulated above, it must be noted that for every group C*-algebra $C^*_\mu(G)$ corresponding to a $G$-ideal in the Fourier-Stieltjes algebra, there is a correspondence crossed product functor $- \rtimes G$ with $\mathbb{C} \rtimes G = C^*_\mu(G)$ \cite{BEW18}. On the other hand, it is actually a necessary condition for a correspondence crossed product functor $- \rtimes G$ that $(\mathbb{C} \rtimes G)'$ is an $G$-ideal in the Fourier-Stieltjes algebra \cite[Corollary 5.7]{BEW18}.

The proof of Theorem 1.1 besides its elegance, is of a general nature and leaves open to a certain extent the necessity of the ideal assumption of $C^*_\mu(G)'$ in the Fourier-Stieltjes algebra $B(G)$. Although it is not difficult to find counterexamples that justify a certain necessity of this assumption (see e.g. \cite[Example 4.12]{BEW17}), I hope to further emphasize this aspect. The main result of this article is the following theorem.

**Theorem A.** Let $G = \text{SL}(2,F)$ for $F = \mathbb{R}, \mathbb{C}$, and let $C^*_\mu(G)$ be a group C*-algebra of $G$ such that $C^*_\mu(G)' \subseteq B(G)$ is a $G$-ideal in $B(G)$. Then there exists a unique element $p \in [2, \infty]$ such that $B_{L^p}(G) = C^*_\mu(G)'$, where $B_{L^p}(G) := C^*_{L^p}(G)'$.

**Remark 1.2.** Theorem A was proved for $\text{SL}(2,\mathbb{R})$ in \cite[Theorem 7.3]{Wie15} by means of an explicit representation theoretic argument. This argument can be modified to work for $\text{SL}(2,\mathbb{C})$ (see \cite{Dab19}). The proof presented in this article uses less about the representation theory of the groups. Also, the proof provides a more general strategy, which can be applied to groups whose non-tempered irreducible unitary representations are class-one (see Section 4 for the definition of a non-tempered representation and Section 2 for the definition of a class-one representation). Exotic group C*-algebras of more general Lie groups have been studied in \cite{SW18} and \cite{dLS19}. The results presented here complement these, in the sense that they provide, for certain groups, a full description of the weak*-closed $G$-ideals in the Fourier-Stieltjes algebra of the group.

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2. **Preliminaries**

Let $G$ be a locally compact group and $K \leq G$ a compact subgroup. In this section, we assume $(G, K)$ to be a Gelfand pair, i.e.

$$C_c(K \setminus G/K) = \{ f \in C_c(G) \mid f(ksk') = f(s) \forall s \in G, \forall k, k' \in K \}$$

is a commutative involutive subalgebra of the involutive group algebra $C_c(G)$ of $G$. 

Recall that a function \( \varphi \in C(K \backslash G/K) \) with \( \varphi \neq 0 \) is called spherical if it satisfies
\[
\varphi(s)\varphi(t) = \int_K \varphi(skt) \, d\mu_K(k)
\]
for all \( s, t \in G \), where \( \mu_K \) denotes the normalized Haar measure on \( K \). We denote the set of positive definite spherical functions by \( S(K \backslash G/K) \).

An irreducible unitary representation \( \pi : G \to U(V) \) is called a class-one representation if the linear subspace \( V^K \) of \( K \)-invariant vectors is non-trivial. In that case the dimension of \( V^K \) is equal to 1 and for a normalized \( K \)-invariant vector \( v \in V^K \), the function \( \langle \pi(\cdot)v, v \rangle \) is a positive definite spherical function. On the other hand, if \( \varphi \in S(K \backslash G/K) \) is a positive definite spherical function on \( G \), then the GNS-construction \((L^2(G, \varphi), \pi_\varphi)\) of \( \varphi \) defines a class-one representation of \( G \). For more details we refer the reader to \cite{vD09}.

### 3. Groups whose non-tempered irreducible unitary representations are class-one

Let \((G, K)\) be a Gelfand pair consisting of a locally compact group \( G \) and a compact subgroup \( K \) of \( G \). The spherical unitary representation theory generally contains much information about the group \( G \). In the following, however, we will focus on the, from the point of view of representation theory, rather pathological assumption that the space \( \hat{G} \backslash \hat{G}_r \) consists of class-one representations, where \( \hat{G} \) is the unitary dual of \( G \) and \( \hat{G}_r \) is the subset of \( \hat{G} \) consisting of all irreducible representations weakly contained in the left regular representation of \( G \). For this reason we start the section with some examples.

Motivated by the representation theory of semisimple Lie groups, a unitary representation of \( G \) is said to be tempered if it is weakly contained in the left-regular representation \( \lambda_G \). Thus, the non-tempered irreducible unitary representations of \( G \) are the elements of \( \hat{G} \backslash \hat{G}_r \).

**Example 3.1.** The first and for our purpose most important examples of groups \( G \) which satisfy the property that \( \hat{G} \backslash \hat{G}_r \) consists of class-one representations are the groups \( \text{SL}(2, \mathbb{R}) \) and \( \text{SL}(2, \mathbb{C}) \). The property follows in these examples directly by the well-known classification of the unitary dual of these groups, as can be found for example in \cite[Chapter XVI §1]{Kna86}.

Further examples are given by certain subgroups of automorphism groups \( \text{Aut}(T) \) of a locally finite homogeneous tree \( T \). In the case of a closed subgroup \( G \) of \( \text{Aut}(T) \) which has the independence property and acts transitively on the boundary of \( T \), it can be shown that certain vector states of irreducible representations of \( G \), which are not class-one, have compact support \cite[Lemma 19]{Ama}, \cite[Chapter III, Proposition 3.2]{FTN91} (see also \cite{Ol'77}). This implies in particular that \( \hat{G} \backslash \hat{G}_r \) consists of \( L^2 \)-representations.

The assumption that the non-tempered irreducible unitary representations of \( G \) consists of class-one representations makes it particularly easy to determine the K-theory of group C*-algebras of \( G \) relative to \( C^*_r(G) \). In this context, the commutative C*-algebra \( \mathcal{H}(K \backslash G/K) \), defined as the closure of
its commutative subalgebra $C_c(K \backslash G/K)$ in the universal group C*-algebra $C^*(G)$ of $G$, is of particular interest. More generally, we make the following definition.

**Definition 3.2.** For a general group C*-algebra $C^*_\mu(G)$ we write $\mathcal{H}_\mu(K \backslash G/K)$ for the commutative C*-algebra $C_c(K \backslash G/K) \subseteq C^*_\mu(G)$.

The following observation, and the main result of this section, specifies the relation between the K-theory of a group C*-algebra $C^*_\mu(G)$ of $G$ and of its commutative subalgebra $\mathcal{H}_\mu(K \backslash G/K)$.

**Lemma 3.3.** Let $(G, K)$ be a Gelfand pair consisting of a second countable locally compact group $G$ and a compact subgroup $K$ of $G$. Suppose that $\hat{G} \backslash \hat{G}_r$ consists of class-one representations. Let $C^*_\mu(G)$ be a group C*-algebra and $\mathcal{H}_\mu(K \backslash G/K)$ as above. Then the canonical quotient map $s : C^*_\mu(G) \to C^*_\mu(G)$ induces an isomorphism in K-theory if and only if the canonical quotient map $\psi : \mathcal{H}_\mu(K \backslash G/K) \to \mathcal{H}_\mu(K \backslash G/K)$, i.e. the restriction of $s$ to $\mathcal{H}_\mu(K \backslash G/K)$ and $\mathcal{H}_\mu(K \backslash G/K)$, does.

**Proof.** Let $\iota_G : G \to \mathcal{U}\mathcal{M}(C^*(G))$ be the universal unitary representation of $G$, let $q : C^*(G) \to C^*_\mu(G)$ be the canonical quotient map, and $\iota_{G,\mu} = \overline{\iota_G} \circ q$, where $\overline{\iota_G} : \mathcal{M}(C^*(\hat{G})) \to \mathcal{M}(C^*_\mu(G))$ is the unique extension of $q$ to a *-homomorphism. Furthermore, let $p_K = \iota_{G,\mu}(\mu_K) \in \mathcal{M}(C^*_\mu(G))$ be the orthogonal projection given by $p_K(x) = \int_K \iota_{G,\mu}(k)x d\mu_K(k)$ for $x \in C^*_\mu(G)$.

First of all note that the equality of $p_K C_c(G)p_K$ and $C_c(K \backslash G/K)$, together with the continuity of the map $C^*_\mu(G) \to C^*_\mu(G)$, $b \mapsto p_K bp_K$, implies the identity $\mathcal{H}_\mu(K \backslash G/K) = p_K C^*_\mu(G)p_K$.

Therefore, the right ideal $X_\mu = p_K C^*_\mu(G)$ in $C^*_\mu(G)$ defines, in a canonical way, a partial $\mathcal{H}_\mu(K \backslash G/K)-C^*_\mu(G)$-imprimitivity bimodule that restricts to an $\mathcal{H}_\mu(K \backslash G/K)-C^*_\mu(G)p_K C^*_\mu(G)$-imprimitivity bimodule. The spectrum of $C^*_\mu(G)p_K C^*_\mu(G)$ identifies with the open subset of $C^*_\mu(G)$ consisting of all irreducible representations of $C^*_\mu(G)$ that do not vanish on the ideal $C^*_\mu(G)p_K C^*_\mu(G)$. These representations are exactly the class-one representations of $(G, K)$ that integrate to $C^*_\mu(G)$.

The assumption that $\hat{G} \backslash \hat{G}_r$, consists of class-one representations now implies that the bimodule $X_\mu$ restricts to a $\ker s$-ker $s$ imprimitivity bimodule $(X_\mu)_{\ker s} = X_\mu/\ker s$. Lemma 3.3 is therefore an immediate consequence of the six-term-exact-sequence in K-theory and the fact that $(X_\mu)_{\ker s}$ induces an isomorphism in K-theory. \qed

The following proposition is well known. For the convenience of the reader we provide the proof.

**Proposition 3.4.** Let $\Delta(\mathcal{H}(K \backslash G/K))$ be the Gelfand spectrum of the commutative C*-algebra $\mathcal{H}(K \backslash G/K)$.

The map from $\mathcal{S}(K \backslash G/K)$ to $\Delta(\mathcal{H}(K \backslash G/K))$ sending a positive definite spherical function $\varphi$ to the character $\chi_\varphi$ given by

$$\chi_\varphi(f) = \int f(s) \varphi(s^{-1}) d\mu_G(s)$$
for \( f \in C_c(K \backslash G / K) \) establishes a bijection, and, after equipping \( S(K \backslash G / K) \) with the topology of uniform convergence on compact subsets of \( G \), even a homeomorphism.

**Proof.** We adopt some of the notations from the previous proof. So let \( Y = C^*(G)p^K_1 \) be the partial \( C^*(G) \)-\( H(K \backslash G / K) \)-imprimitivity bimodule. First of all, we will describe the inverse of the above mapping. Note that for a character \( \chi \in \mathcal{H}(K \backslash G / K) \), \( Y - \text{Ind} \chi = Y \otimes \chi \mathbb{C} \) is an irreducible representation \((V, \pi)\) of \( C^*(G) \), where \( V = Y \otimes \chi \mathbb{C} \) denotes the balanced tensor product of the right Hilbert \( \mathcal{H}(K \backslash G / K) \)-module \( Y \) and the *-representation \((\mathbb{C}, \chi)\) of \( \mathcal{H}(K \backslash G / K) \), a Hilbert space with a canonical left multiplication \( \pi^* : C^*(G) \to \mathcal{L}(V) \) by \( C^*(G) \). Let \((V, \pi)\) denotes the corresponding unitary representation of \((V, \pi)\). The restriction \((V|_{\mathcal{H}(K \backslash G / K)}, \pi|_{\mathcal{H}(K \backslash G / K)})\) of \((V, \pi)\) to \( \mathcal{H}(K \backslash G / K) \) is unitary equivalent to the character \((\mathbb{C}, \chi)\), where \( V|_{\mathcal{H}(K \backslash G / K)} \) denotes the one-dimensional Hilbert space \( \pi(\mathcal{H}(K \backslash G / K))V \).

Let \( v \in V|_{\mathcal{H}(K \backslash G / K)} \) be a vector of length one. Then \( \varphi_\chi = \langle \pi(v)|v \rangle \) is a positive definite spherical function. The map \( \mathcal{H}(K \backslash G / K) \to S(K \backslash G / K), \chi \mapsto \varphi_\chi \) defines the inverse map of \( S(K \backslash G / K) \to \Delta(\mathcal{H}(K \backslash G / K)), \varphi \mapsto \chi_\varphi \).

The continuity of \( S(K \backslash G / K) \to \Delta(\mathcal{H}(K \backslash G / K)), \varphi \mapsto \chi_\varphi \) is easy to check. So it remains to show that the inverse map \( \Delta(\mathcal{H}(K \backslash G / K)) \to S(K \backslash G / K), \chi \mapsto \varphi_\chi \) is continuous. In order to prove this, let \((\chi_i)_{i \in I} \in \mathcal{H}(K \backslash G / K)^I \) be a convergent net with \( \chi \in \mathcal{H}(K \backslash G / K) \) as limit point. For all \( f \in C_c(G) \) we have

\[
\int_G f(s)\varphi_\chi(s^{-1})d\mu_G(s) = \int_G (p_Kf p_K)(s)\varphi_\chi(s^{-1})d\mu_G(s) \\
= \int_G (p_Kf p_K)(s)\varphi_\chi(s^{-1})d\mu_G(s) \\
= \int_G f(s)\varphi_\chi(s^{-1})d\mu_G(s)
\]

as \( i \to \infty \). Hence, \( \varphi_{\chi_i} \to \varphi_\chi \) with respect to \( \sigma(L^\infty(G), L^1(G)) \). The topology of uniform convergence on compact subsets of \( G \) and the topology induced by \( \sigma(L^\infty(G), L^1(G)) \) coincide on \( S(K \backslash G / K) \) (see [14D09 Proposition 6.4.2]). This proves the continuity of \( \mathcal{H}(K \backslash G / K) \to S(K \backslash G / K), \chi \mapsto \varphi_\chi \). \[\square\]

### 4. The examples \( \text{SL}(2, \mathbb{R}) \) and \( \text{SL}(2, \mathbb{C}) \)

Let \( \mathbb{F} \) be the real or complex number field. As indicated in Example 3.1, \( \text{SL}(2, \mathbb{F}) \setminus \text{SL}(2, \mathbb{F})_c \) consists of class-one representations. The special linear group \( \text{SL}(2, \mathbb{F}) \) is a double cover of the identity component of the Lorentz group \( \text{SO}_0(1, n) \) (with \( n = 2 \) in the case of \( \mathbb{F} = \mathbb{R} \) and \( n = 3 \) in the case of \( \mathbb{F} = \mathbb{C} \)). Hence it is, by [14LS19 Lemma 5.1], in order to prove Theorem A, equivalent to consider the groups \( \text{SO}_0(1, n) \) with \( n = 2, 3 \) instead of \( \text{SL}(2, \mathbb{F}) \). Let us therefore assume from now on \( G \) to be \( \text{SO}_0(1, n) \) with \( n = 2, 3 \).

We write

\[
K = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & k \end{pmatrix} \in \text{SO}_0(1, n) \mid k \in \text{SO}(n) \right\} \leq G
\]
for the canonical maximal compact subgroup,
\[ A = \left\{ a_t = \begin{pmatrix} \cosh t & 0 & \sinh t \\ 0 & 1 & 0 \\ \sinh t & 0 & \cosh t \end{pmatrix} \mid t \in \mathbb{R} \right\}, \]
and
\[ \mathcal{A}^+ = \{ a_t \in A \mid t \geq 0 \}. \]

With regard to the polar decomposition \( G = K \mathcal{A}^+ K \), the space \( \mathcal{S}(K\backslash G/K) \) of positive definite spherical functions can, as was proved in [15], be described by the parameter space \( \mathcal{P} = (0, \rho) \cup i\mathbb{R} \), where \( \rho = \frac{n-1}{2} \). More precisely, there is a one-to-one map from \( \mathcal{P} = (0, \rho) \cup i\mathbb{R} \) to \( \mathcal{S}(K\backslash G/K) \) sending an element \( \lambda \in \mathcal{P} \) to the function \( \varphi_\lambda \) uniquely determined by
\[ \varphi_\lambda(a_t) = \exp (t\lambda) \]
for \( a_t \in \mathcal{A}^+ \) (see e.g. [15, Lemma 5.3]).

**Proposition 4.1.** The map \( \Phi : \mathcal{P} \to \mathcal{S}(K\backslash G/K), \lambda \mapsto \varphi_\lambda \) is an homeomorphism.

**Proof.** The continuity of \( \Phi \) is directly seen by the identification from Proposition 3.4 and the theorem of the dominated convergence. In order to show the continuity of the inverse map of \( \Phi \), note that for all \( s \in G \) the evaluation map \( \text{ev}_s : C(G) \to \mathbb{C} \) is continuous with respect to the topology of uniform convergence on compact subsets. The identity \( \Phi^{-1} - \rho = \log \circ \text{ev}_a \) then implies the continuity of \( \Phi^{-1} \).

It is well known that \( G \) has the Haagerup property. Since every second countable locally compact group with the Haagerup property is K-amenable (see [17], Theorem 1.1) is applicable. Hence, the canonical quotient map \( s : C_{L^p}^+(G) \to C^*_r(G) \), and therefore, by Lemma 3.3 \( s : \mathcal{H}_{L^p}(K\backslash G/K) \to \mathcal{H}_r(K\backslash G/K) \) induces an isomorphism in K-theory for every \( p \in [2, \infty) \). In particular, the kernel \( \ker s \) of \( s \) has trivial K-theory.

The following proposition is a reformulation of [15, Proposition 5.4] and describes the commutative subalgebra \( \mathcal{H}_{L^p}(K\backslash G/K) \) of the group C*-algebra \( C_{L^p}^+(G) \) of \( G \) in terms of their Gelfand spectrum.

**Proposition 4.2.** Let \( p \in [2, \infty) \). Then
\[ \Delta(\mathcal{H}_{L^p}(K\backslash G/K)) = \left\{ \varphi_\lambda \mid \lambda \in \left(0, \frac{p-2}{p} \rho\right] \cup i\mathbb{R} \right\}, \]
and
\[ \Delta(\ker s) = \left\{ \varphi_\lambda \mid \lambda \in \left(0, \frac{p-2}{p} \rho\right] \right\}. \]

Now we are ready to prove Theorem 1.

**Proof of Theorem 1.** Without loss of generality, we can assume \( C^*_\mu(G) \subseteq B(G) \) to be a proper ideal in \( B(G) \). By Proposition 3.3 the canonical quotient map \( s : \mathcal{H}_\mu(K\backslash G/K) \to \mathcal{H}_r(K\backslash G/K) \) induces an isomorphism in K-theory. By Proposition 1.1 and 3.2 there is a largest \( \lambda_0 \in (0, \rho) \) such that \( \varphi_{\lambda_0} \in \Delta(\mathcal{H}_\mu(K\backslash G/K)) \) but \( \varphi_\lambda \notin \Delta(\mathcal{H}_\mu(K\backslash G/K)) \) for all \( \lambda \in (\lambda_0, \rho] \). If \( \Delta(\ker s) \neq \left\{ \varphi_\lambda \mid \lambda \in (0, \lambda_0) \right\} \) then \( \Delta(\ker s) \) contains an open compact subset, and
therefore $K_0(\ker s) \neq 0$, which is a contradiction. So

$$\Delta(\ker s) = \{\varphi_\lambda \mid \lambda \in (0, \lambda_0]\},$$

which implies that $\mathcal{H}_\mu(K \backslash G/K) = \mathcal{H}_{L^p}(K \backslash G/K)$ for some $p \in [2, \infty)$. It is easily seen that this implies that $C^*_\mu(G) = C^*_{L^p}(G)$. □

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Timo Siebenand
Westfälische Wilhelms-Universität Münster, Mathematisches Institut
Einsteinstrasse 62, 48149 Münster, Germany
E-mail address: timo.siebenand@uni-muenster.de