Quantum criticality of fermion velocities and critical temperature nearby a putative quantum phase transition in the $d$-wave superconductors

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Quantum critical behaviors induced by a putative quantum phase transition are vigilantly investigated, which separates a $d$-wave superconducting state and $d$-wave superconducting $+X$ state below the superconducting dome of the $d$-wave superconductors with tuning the non-thermal doping variable. Within the framework of the renormalization group approach, we start with a phenomenological effective theory originated from the Landau-Ginzburg-Wilson theory and practice one-loop calculations to construct a set of coupled flows of all interaction parameters. After extracting related physical information from these coupled evolutions, we address that both fermion velocities and critical temperatures exhibit critical behaviors, which are robust enough against the initial conditions due to strong quantum fluctuations. At first, the evolution of Yukawa coupling between $X$-state order parameter and nodal fermions in tandem with quantum fluctuations heavily renormalize fermion velocities and generally drive them into certain finite anisotropic fixed point at the lowest-energy limit, whose concrete value relies upon the very quantum phase transition. In addition, these unique properties of fermion velocities largely reshape the fate of superfluid density, giving rise to either an enhancement or a dip of critical temperature. Moreover, we find that fermion-fermion interactions bring non-ignorable quantitative corrections to quantum critical behaviors despite they are subordinate to quantum fluctuations of order parameters.

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I. INTRODUCTION

A plethora of both theoretical and experimental research efforts have been devoted to the $d$-wave cuprate superconductors in the last three decades owing to their unique pairing mechanisms and anomalous properties in the normal states.\cite{1-27} Compared to their $s$-wave counterparts,\cite{5,6,27} it is noteworthy that such superconductors own a $d_{x^2-y^2}$ superconducting gap\cite{16,28-32}, which vanishes at four nodes ($\pm \pi/2, \pm \pi/2$) in the first Brillouin zone.\cite{21,16,25} This indicates that the gapless nodal quasi-particles (QPs) can be excited from these nodal points and present even at the lowest energy in the superconducting phase.\cite{10,14,16,31} Generally, these nodal QPs are nearly non-interacting.\cite{21} However, this feature can be significantly changed once the nodal fermions interact with certain critical bosonic mode accompanied by a quantum phase transition (QPT)\cite{33-35}, around which the quantum fluctuations couple strongly to the nodal fermions, giving rise to severe fermion damping\cite{11,13,19,36} and other striking properties.\cite{17,20,22,27,39} It is therefore reasonably expected that these nodal QPs together with quantum critical degrees of freedom would be responsible for unusual behaviors around the QPT.\cite{16,23,25,31,39}

On the basis of diversity and complexity of realistic systems, a series of stimulated frameworks are proposed to explore and unravel the intimate connection between quantum criticality and unusual properties associated with nodal QPs in the $d$-wave superconductors. One of the most well-known pioneering scenarios was put forward by Vojta et al. in 2000.\cite{11-13} Within their strategy, a putative quantum critical point (QCP) exists somewhere in the superconducting dome accompanied by certain QPT from a $d_{x^2-y^2}$ superconducting state to another $d_{x^2-y^2} + X$ superconducting state as schematically presented in Fig. 1 due to the topological changes of nodal positions.\cite{11-13} Hereby, the $X$ state is developed by the $C_4$ symmetry breaking of nodal positions and owns seven potential candidates based upon the group-theory analysis.\cite{11-13} These states can be effectively reduced to four distinct types, which are associated with the QPTs denominated by Type-$\tau_{0, x, y, z}$ in this work. In particular, the Type-$\tau_x$ QCP dubbed the nematic QCP has been suggested and indirectly detected below the superconducting dome of $d$-wave high-$T_c$ superconductor, which is expected to be associated with several non-Fermi-liquid behaviors.\cite{17,20,22} This accordingly stimulates us to systematically investigate the critical consequences and differences of these distinct kinds of QCPs on the physics of related quantum critical regions owing to the combination of ferocious quantum fluctuations of order parameters and their interplay with other degrees of freedom, which as far as we know have not been yet sufficiently studied.

Specifically, the quantum fluctuation of $X$ order parameter nearby a QCP strongly couples to gapless nodal QPs, which then leads to nontrivial critical effects on two fermion velocities of nodal QPs consisting of the Fermi velocity $v_F$ and the gap velocity $v_{\Delta}$.\cite{45,50} Principally, their ratio $v_{\Delta}/v_F$ plays an important role in pinning down the low-energy fates of physical quantities in that it always enters into a number of important observable quantities including the superfluid density and critical temperature as well as electric and thermal conductivities.\cite{45,50,52,53} This signals any unusual renor-
nalization of this velocity ratio will give rise to certain enhancement or suppression of these observable quantities. It is therefore of considerable necessity to explore the low-energy tendency of \( v_{\Delta}/v_F \). Stimulated by this, Huh and Sachdev\(^{12}\) carefully examined the Type-\( \tau_x \) QPT, which is so-called nematic QPT with spontaneously breaking \( C_2 \) symmetry down to \( C_2 \) symmetry of the system,\(^{14,17,20,35,37,49,53-62} \) and obtain a fixed point \( v_{\Delta}/v_F \to 0 \) at the lowest-energy limit. In addition, Wang et al.\(^{35,64} \) addressed two distinct fixed points corresponding to \( v_{\Delta}/v_F \to 1 \) and \( v_{\Delta}/v_F \to \infty \) for Type-\( \tau_y \) and Type-\( \tau_z \) QPTs, respectively. Further, the consequences of these fixed points on the physical implications are subsequently investigated in Refs.\(^{48,63,64} \).

Despite of these considerable progresses on the behaviors of fermion velocities nearby the QPTs,\(^{17,20,22,48,63,64} \) several quantum critical degrees of freedom are insufficiently taken into account, which may be essential to dictate the low-energy behaviors of the system. On one hand, the Yukawa coupling between nodal QPs and certain order parameter is fixed as an energy-independent constant to approximately collect the physical ingredients nearby the QPTs in these works.\(^{17,20,22,48,63,64} \) Going beyond this fixed-coupling assumption, much more physical information would be captured and hence the low-energy fates of fermion velocities may be partially or heavily modified by the coupled entanglements of all interaction parameters due to quantum criticality. On the other hand, although the nodal fermions are always excited, they own a long lifetime and can coexist with the superconducting states.\(^{31} \) This implies the fermion-fermion interactions can be safely neglected away from the QCP. However, quantum criticality would coax these nodal QPs to mutually intertwine with each other and influence fermion velocities plus Yukawa coupling. Consequently, fermion-fermion interactions may play important roles in determining critical behaviors around certain QCPs.\(^{17,20,22,48,63,64} \) Consequently, one can expect that uncovering the contributions from these two quantum critical ingredients may well improve our understandings on the quantum criticality of certain QCP in the \( d \)-wave superconductor.

In order to encapsulate more physical information driven by the QCP, it is therefore imperative to systematically investigate the effects of fermion-order parameter couplings and fermion-fermion interactions as well as their interplay on the low-energy fates of fermion velocities and related observable quantities. To this purpose, we within this work employ the momentum-shell renormalization group (RG) approach\(^{100,102} \) to unbiasedly treat all these critical physical degrees of freedom nearby a putative QPT from the \( d \)-wave superconducting to \( d \)-wave superconducting+\( X \) state as illustrated in Fig. I. After collecting all one-loop corrections, a set of coupled RG flows of all interaction parameters are derived to characterize the quantum criticality nearby all four types of potential QPTs dubbed Type-\( \tau_{0,x,y,z} \) that are explicitly clarified in Sec. II.A.

Decoding the physical information contained in the coupled RG equations yields a number of quantum critical properties in the vicinity of all QCPs. At first, we find that the fermion velocities exhibit several interesting fixed points. With respect to the Type-\( \tau_0 \) QPT, the Yukawa interplay designated as \( \lambda \) between nodal QPs and related order parameter is marginal to one-loop level and the ratio of fermion velocities flows towards either fixed point \( (v_{\Delta}/v_F)^* \approx 0.3478 \) or \( (v_{\Delta}/v_F)^* \approx 0.0942 \) at the low-energy limit caused by the quantum criticality. Concerning Type-\( \tau_{x,y,z} \) QPTs, the evolution of Yukawa coupling \( \lambda \) and quantum fluctuations heavily reshape three fixed points \( v_{\Delta}/v_F \to 0, 1, \infty \) for Type-\( \tau_{x,y,z} \) under the fixed-coupling assumptions\(^{17,48} \) to evolve towards finite anisotropies as approaching the QPTs. To be specific, the extreme anisotropies of fermion velocities are changed to finite anisotropies for both Type-\( \tau_{x,y,z} \) QPTs but instead the isotropic fermion velocities for Type-\( \tau_0 \) QPT are broken and attracted by a finite anisotropic fixed point. In addition, we notice that the unusual behaviors of fermion velocities considerably modify the fates of superfluid density and critical temperature around the underlying four types of QPTs. As approaching the Type-\( \tau_{x,y} \) QPTs, the critical temperatures are largely suppressed. Conversely, both Type-\( \tau_y \) and Type-\( \tau_z \) QPTs are in favor of the superconductivity. Furthermore, the roles of fermion-fermion interactions that have not yet been adequately considered are also inspected in quantum criticality. We realize that they can give rise to quantitative contributions to quantum critical behaviors in the vicinity of all putative QPTs. Last but not the least important, it is worth pointing out that our qualitative results are considerably robust enough with the variation of initial conditions.

The rest of paper is organized as follows. In Sec. II we establish our low-energy effective field theory that includes the most of key physical ingredients to describe the main physics around the QPT. On the basis of the effective theory, we within Sec. III perform one-loop momentum-shell RG analysis to deliver the coupled RG equations of all interaction parameters. After combining both the tentatively analytical discussions and vigilant numerical calculations, Sec. IV and Sec. V are followed to present the critical behaviors of fermion velocities and superfluid density together with critical temperature nearby the QCP, respectively. Finally, we provide a brief summary in Sec. VI.

## II. EFFECTIVE THEORY

In this work, our focus is put on a putative QPT in the \( d \)-wave superconductor as schematically displayed in Fig. I as well as the associated critical behaviors of fermion velocities and physical quantities. To begin with, we within this section are going to construct the low-energy effective field theory around the QPT and defer the one-loop RG analysis to the next section III.
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S 
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wave superconductor and the field \( \phi \) characterizes the order 
 parmament of X state, which depends upon the specific sym-
metry breaking accompanied by the QPT. In addition, the 
very value \( r_c \) that roughly locates at the optimal doping is 
the so-called quantum critical point (QCP), which separates 
the disordered \( (\langle \phi \rangle = 0) \) and ordered \( (\langle \phi \rangle \neq 0) \) X phases at 
\( T = 0 \). As to the finite-temperature region around the QPT, 
critical behaviors are expected to be induced in the quantum 
critical region (QCR) due to the strong quantum fluctuations. 
The fate of critical temperature circled by the dashed line will be 
elicly addressed in Sec. \( \text{VI} \). 

A. Phenomenological model 

As the \( d \)-wave superconductor is pushed closer to the 
QCP depicted in Fig. I, the possible quantum critical-
ity in the quantum critical region (QCR) can be prin-
cipally ascribed to three major distinct types of physical 
ements that are gapless fermionic quasiparticles (QP) excited from the nodal points and quantum 
tration for order parameter \( \phi \) of X state in tandem with their intimate interplay. It is of importance to address that the quantum fluctuations are so ferocious that the QCR presented in Fig. I inherits the strong fluctuations of QCP and hence the quantum fluctuations dominate over the thermal fluctuations within such region. This indicates that the quantum fluctuations are in charge of the singular physical behaviors and henceforth the thermal fluctuations can be ignored. 

Without loss of generality, we within this work put 
our focus on the QCR. The phenomenological model is 
therefore introduced to capture the physical information 
early the QCP, 

\[
S = S_{\Psi} + S_{\phi_0} + S_{\Psi\phi_0}, 
\]

where \( S_{\Psi} \), \( S_{\phi_0} \), and \( S_{\Psi\phi_0} \) serve as the degrees of fermionic QPs, order parameter, and their couplings, respectively. To be concrete, the gapless fermions with linear disper-

\[
S_{\Psi} = \int \frac{d^2k}{(2\pi)^2} \frac{d\omega}{2\pi} \Psi_{1a}^\dagger (-i\omega + v_F k_x \tau^2 + v_\Delta k_y \tau^x) \Psi_{1a} + \int \frac{d^2k}{(2\pi)^2} \frac{d\omega}{2\pi} \Psi_{2a}^\dagger (-i\omega + v_F k_y \tau^x + v_\Delta k_x \tau^x) \Psi_{2a},
\]

with \( \tau^x, \tau^y, \tau^z \) denoting the Pauli matrices. Hereby, the spinors \( \Psi_{1a} \) and \( \Psi_{2a} \) with the repeated spin index \( a \) being summed from 1 to the number of fermion flavor \( N \) are employed to specify the fermionic QPs stemming from nodal points at \( (\pm \frac{\pi}{2}, \frac{\pi}{2}) \) plus \( (\pm \frac{\pi}{2}, \frac{\pi}{2}) \) respectively. Moreover, \( k_{x,y} \) describe the momenta with relative to the nodal points and \( v_F, v_\Delta \) correspondingly serve as the Fermi velocity and gap velocity.

With respect to the order-parameter part, there in all exist seven different sorts of order parameters for the state \( X \) in Fig. I, which are solely determined by the specific symmetry breaking of nodal positions as collected in Fig. I. It is of particular significance to point out that the couplings between gapless QPs and order parameters are heavily dependent on the symmetry breaking. As a result, it is convenient to bring about the Yukawa couplings before presenting the \( S_{\phi_0} \), which are written as,

\[
S_{\Psi\phi_0} = \int d^2x d\tau |\lambda_0 \phi_0 (\Psi_{1}^\dagger M_1 \Psi_1 + \Psi_{2}^\dagger M_2 \Psi_2)|, 
\]

where the matrices \( M_{1,2} \) are directly associated with the distinct types of order parameters with \( \lambda_0 \) designating the coupling strength, which are explicitly classified as follows: Case-I with \( M_1 = \tau_y, M_2 = \tau_y \), Case-II with \( M_1 = \tau_y, M_2 = -\tau_y \), Case-III with \( \lambda_0 = 0 \) (such situation is trivial and not discussed further), Case-IV with \( M_1 = \tau_x, M_2 = \tau_x \), Case-V with \( M_1 = \tau_z, M_2 = -\tau_z \), and Case-VI with \( M_1 = \tau_x, M_2 = -\tau_x \). In distinction to such six sorts, two real components \( \phi_{0A} \) and \( \phi_{0B} \) constitute the order parameter of Case-VII, which respectively interact with \( \Psi_1 \) and \( \Psi_2 \), yielding to

\[
S_{\Psi\phi_0} = \int d^2x d\tau |\lambda_0 (\phi_{0A} \Psi_{1}^\dagger M_1 \Psi_1 + \phi_{0B} \Psi_{2}^\dagger M_2 \Psi_2)|, 
\]

with \( M_1 = M_2 = \tau_0 \).

To proceed, one can figure out that the matrices \( M_1 \) and \( M_2 \) always appear in pairs during the calculations of one-loop corrections and henceforth the results are insensitive to their sign. As a corollary, these seven types of Yukawa couplings can be reduced to another four simplified categories of phase transitions, which are accordingly designated as Type-\( \tau_x \) with \( M_{1,2} = \tau_x \), Type-\( \tau_y \) with \( M_{1,2} = \tau_y \), Type-\( \tau_z \) with \( M_{1,2} = \tau_z \), and Type-\( \tau_0 \) with \( M_{1,2} = \tau_0 \), respectively. In order to be consistent with such version of classification, the corresponding order-parameter part \( S_{\phi_0} \) can be cast as

\[
S_{\phi_0} = \frac{1}{2} \int \frac{d^2q}{(2\pi)^2} \left[ -2(r - r_c) + q^2 \right] \phi_0^2. 
\]
for Type-$\tau_x$, Type-$\tau_y$, and Type-$\tau_z$. In comparison, one needs to replace the $\phi_0^2$ in Eq. (8) with $(\phi_0^{2A} + \phi_0^{2B})$ to obtain their Type-$\tau_0$ counterpart. For convenience, we hereby neglect the order-parameter self-interaction terms ($\phi_0^4$) in that they are irrelevant with the support of the power counting. Before proceeding further, it is of particular importance to point out that the free order-parameter action via involving two significant quantities including the polarization term that substantially modifies the action reformulates the quadratic part of the free order-parameter action (9) to replace the bare one.

B. Renormalized order-parameter action and effective theory

Before proceeding further, it is of particular importance to point out that the free order-parameter action would be qualitatively renormalized by one-loop corrections due to switching on the Yukawa couplings between nodal QPs and order parameter. In order to evaluate such effects, we are forced to compute the polarization function of order parameter depicted in Fig. 1(a), which can be formally expressed as

$$\Pi(q, \epsilon) = \int \frac{d^2k}{(2\pi)^2} \frac{d\omega}{2\pi} \text{Tr}[M G_0 \phi(k, \omega) M G_0^\dagger(k + q, \omega + \epsilon)], (6)$$

with the vertex matrix $M$ being designated in Eq. (4).

Hereby, the free fermionic propagator can be forwardly derived from Eq. (6). Specifically, it reads

$$G_0^\phi(k, \omega) = \frac{1}{-i\omega + v_F k_x \tau^x + v_\Delta k_y \tau^z}, (7)$$

for nodal QPs $\Psi_1$ and its $\Psi_2$ counterpart would be analogously obtained via exchanging the positions of momenta $k_x$ and $k_y$ in Eq. (7).

After performing long but straightforward calculations, we are left with one-loop polarization functions for four different types of phase transitions as follows, namely

$$\Pi^{\tau_x}(q, \epsilon) = \frac{1}{16v_F v_\Delta} \frac{\epsilon^2 + v_F q_x^2 + v_\Delta q_y^2}{\sqrt{\epsilon^2 + v_F q_x^2 + v_\Delta q_y^2}} (q_x \rightarrow q_y), (8)$$

for Type-$\tau_x$, Type-$\tau_y$, and Type-$\tau_z$, in tandem with Type-$\tau_0$ that are listed by

$$\Pi^{\tau_0}_A(q_x, q_y, \epsilon) = -\frac{1}{16v_F v_\Delta} \sqrt{\epsilon^2 + v_F q_x^2 + v_\Delta q_y^2}, (9)$$

$$\Pi^{\tau_0}_B(q_x, q_y, \epsilon) = -\frac{1}{16v_F v_\Delta} \sqrt{\epsilon^2 + v_F q_x^2 + v_\Delta q_y^2}. (10)$$

Inserting these polarization functions into the free order-parameter action (5) by virtue of Dyson equation reformulates the quadratic part of $S_{\phi 0}$ into

$$[-2(r - r_c) + q^2] \phi_0^2 \rightarrow [-2(r - r_c) + q^2 + \Pi^{\tau}(q)] \phi_0^2. (13)$$

At the low-energy regime, one can realize that the term $\Pi^{\tau}(q)$ is proportional to $q$ and such additional linear-$q$ term dominates over the $q^2$ term which henceforth can be neglected. This manifestly indicates that the incorporation of $q$ term qualitatively alters the dynamical nature of the order-parameter action. Additionally, it is the polarization term that substantially modifies the action via involving two significant quantities including the nodal QPs’ Fermi velocity $v_F$ and the gap velocity $v_\Delta$. In other words, it is now suitable to designate a renormalized order-parameter field $\phi$ to replace the bare one. As a result, we are left with the following renormalized

| Type of symmetry | Position of nodal points | Symmetry breakings |
|------------------|-------------------------|--------------------|
| $\tau = 0$       |                         |                    |
| $\tau_x$         |                         |                    |
| $\tau_y$         |                         |                    |
| $\tau_z$         |                         |                    |
| $\tau_0$         | no nodal points         | broken $\mathcal{T}$ |

FIG. 2: (Color online) Illustrations for the momenta $(k_x, k_y)$ and fermion velocities $(v_F, v_\Delta)$ of two pairs of nodal QPs excited from four nodal points of the $d$-wave superconductor.
In addition, and also advocated in many previous efforts, based upon the strong phenomenological model, we here would like to address measuring their coupling strengths.

order-parameter action

\[
S_\phi = \frac{1}{2} \int \frac{d^3q}{(2\pi)^3} \left[ -2(r-r_c) + \Pi^M(q) \right] \phi^2, \quad \mathcal{M} = \tau_j, 
\]

(14)

\[
S_\phi = \frac{1}{2} \sum_{a=A,B} \int \frac{d^3q}{(2\pi)^3} \left[ -2(r-r_c) + \Pi^M_a(q) \right] \phi_a^2, \quad \mathcal{M} = \tau_0. 
\]

(15)

And the order-parameter propagator for Type-IIA is then given by\textsuperscript{11,17,18,48}

\[
G^M_\phi (q, \epsilon) = \frac{1}{\Pi^M(q, \epsilon)}, 
\]

(16)

as approaching the QCP shown in Fig. 1, where \(\mathcal{M} = \tau_{0,x,y,z}\) correspond to the four potential distinct types of phase transitions defined in Sec. II A. In addition, the Yukawa coupling (3)-(4) between the nodal QPs and order parameter would be accordingly reshaped as

\[
S_\psi = \frac{1}{2} \int d^2x d\tau \left[ c_1 \lambda \phi(\Psi_1^\dagger M \Psi_1 + \Psi_2^\dagger M \Psi_2) + c_2 \lambda \phi^2 \right], \quad \mathcal{M} = \tau_j, 
\]

(17)

\[
S_\psi = \frac{1}{2} \int d^2x d\tau \left[ c_3 \lambda \phi(\Psi_1^\dagger A \Psi_1 + \Psi_2^\dagger B M \Psi_2) + c_4 \lambda \phi^2 \right] M = \tau_0, 
\]

(18)

with the coupling strength \(\lambda_0\) being also adjusted to \(\lambda\) for consistence.

In order to capture more critical information influenced by the QCP shown in Fig. 1, we hereafter reformulate the nodal QPs’ part \(S_\phi\) (2) via supplementing the interaction between nodal QPs themselves dubbed by \(S_\phi\) in conjunction with the renormalized order-parameter action (14)-(15) and Yukawa couplings (17)-(18) to establish our effective action as follows,

\[
S_{\text{eff}} = S_\psi + S_\phi + S_{\psi\phi} + S_\phi, 
\]

(19)

where the fermion-fermion interactions \(S_\phi\) can be expressed as\textsuperscript{65,66}

\[
S_\phi = \sum_{i=0}^3 u_i \int d^2x [\Psi_i^\dagger(x) \tau_i \Psi_i(x)]^2, 
\]

(20)

with the indexes \(i = 0, 1, 2, 3\) corresponding to four types of fermion-fermion interactions and the parameter \(u_i\) measuring their coupling strengths.

Compared to other physical ingredients involved in the phenomenological model, we here would like to address more comments on the fermion-fermion couplings, which as far as we know have not yet been seriously investigated. In the absence of a potential QCP or with the focus on the regions far away from the QCP, it is in principle sensible to drop the fermion-fermion interactions in the superconducting dome of \(d\)-wave superconductors as the nodal QPs are known to coexist harmoniously with the SC state.\textsuperscript{31} In a sharp contrast, as it concerns the question on the critical behaviors neighboring the QCP shown in Fig. 1, we therefore ought to take discreetly into account the contributions from the interplay between the nodal QPs. On one hand, these nodal QPs themselves may mutually intertwine with each other owing to the strong fluctuations and become one of the major elements at the lowest-energy limit.\textsuperscript{65-80,82-99} On the other hand, fermionic couplings can also impact other interaction parameters including fermion velocities \(v_{F,\Delta}\) and Yukawa coupling via participating in the coupled RG evolutions, which will be established in Sec. II B based upon the strong quantum fluctuations connecting various types of degrees of freedom. In this sense, they can indirectly influence and may play important roles in determining the critical behaviors induced by the QCP.

Before going further, it is of necessity to highlight that the nodal QPs are always assumed to be well-defined in the QCR within \(T < T_c\) as aforementioned in Sec. II A and also advocated in many previous efforts.\textsuperscript{11,13,16,17,19-23,27} This implies that our effective theory can capture the core physics of quantum criticality nearby the QCP although the nodal QPs may survive not very long.\textsuperscript{13,30,103} Afterwards, we adopt the effective action (19) as our starting point to derive the coupled flow equations of all associated parameters in the frame of one-loop RG approach\textsuperscript{100-102} and explore the physical behaviors of fermion velocities as well as their effects on superfluid densities and critical temperatures nearby all four types of QCPs illustrated in Fig. 1.

\section{III. \textbf{RG Analysis}}

To proceed, we within this section endeavor to perform the one-loop RG analysis\textsuperscript{100-102} for our effective action (19) to obtain the coupled RG equations of all
interaction parameters, from which the singular properties induced by the QCP are expected to be extracted. To this end, we from now on put our focus on the QCP, namely assuming \( r \to r_c \) in Eq. (19), and then compute all one-loop Feynman diagrams to carry out the standard momentum-shell RG procedures from the field theory perspective.

### A. One-loop corrections

We commence with the one-loop corrections to fermionic propagator. As depicted in Fig. 4(b), the free fermionic propagator would receive one-loop correction \( \Sigma^M \), which originates from the Yukawa coupling between the nodal QPs and Type-M order parameter with \( M = \tau_{0,x,y,z} \). After paralleling the strategy put forward in Refs. 100–102, we integrate out the momentum shell within \( hA - \Lambda \), where \( \Lambda \) is associated with the lattice constant to characterize the cutoff of energy scale and the variable parameter is designated as \( b = e^{-l} \) with \( l > 0 \) being a running energy scale 47, 19, 31, 22, 48, 68, 69, 75, 80, and eventually obtain

\[
\Sigma^{T_x}(k,\omega) = \lambda^2 \left[ A_1(-i\omega) + A_2 v_F k_x \tau^x + A_3 v_F k_y \tau^y \right] \tau_z l, \tag{21}
\]

for Type-\( \tau_x \) order parameter. As to the other three types with \( M = \tau_y, \tau_z, \tau_0 \), the structures of their results are analogous to Eq. (21) but the coefficients \( A_{1,2,3} \) are substituted respectively by \( B_{1,2,3}, C_{1,2,3}, \) and \( D_{1,2,3}, \) whose expressions are presented in Eqs. (A1)–(A4) of Appendix A and Appendix B. Accordingly, this gives rise to the renormalized fermionic propagator with the help of the Dyson equation 17, 18, 48.

\[
G^{-1}_\omega(k,\omega) = -i\omega + v_F k_x \tau^x + v_F k_y \tau^y - \Sigma^M(k,\omega), \tag{22}
\]

where \( \Sigma^M(k,\omega) \) with \( M = \tau_{0,x,y,z} \) specifies the self-energy owing to the Type-M QPT, which will be one of the critical factors to derive the RG equations.

Next, we take into account the one-loop corrections to the Yukawa coupling and fermion-fermion interactions. At first, we consider the former, which is marginal at the tree level. It is therefore of particular importance to examine its fate after including the one-loop corrections. To this end, we read off Fig. 3 and realize there exist two sorts of contributions, namely

\[
\Xi^M = \Xi^M_Y + \Xi^M_{ff}, \tag{23}
\]

with the following results,

\[
\Xi^M_Y = \begin{cases} 
-A_3 \lambda^2 \gamma_x l, & M = \tau_x, \tag{24} \\
(B_1 + B_2 + B_3) \lambda^2 \gamma_y l, & M = \tau_y, \tag{25} \\
-C_2 \lambda^2 \gamma_z l, & M = \tau_z, \tag{26} \\
-D_1 A^B \lambda^2 \gamma_0 l, & M = \tau_0, \tag{27}
\end{cases}
\]

for the order-parameter part and

\[
\Xi^M_{ff} = \begin{cases} 
\left(\frac{u_2^2 + u_2^2 - u_0^2 - u_3^2}{8\pi v_F \Delta}\right) \lambda \gamma_x l, & M = \tau_x, \tag{28} \\
\left(\frac{u_2^2 - u_2^2 - u_0^2 - u_3^2}{4\pi v_F \Delta}\right) \lambda \gamma_y l, & M = \tau_y, \tag{29} \\
\left(\frac{u_2^2 + u_2^2 - u_0^2 - u_3^2}{8\pi v_F \Delta}\right) \lambda \gamma_z l, & M = \tau_z, \tag{30}
\end{cases}
\]

for fermion-fermion part, respectively.

Then, we turn our focus to the fermion-fermion couplings. In analogy to the Yukawa vertex, both Yukawa couplings and fermion-fermion interactions can contribute to the fermion-fermion vertex dubbed by \( \Gamma \), which leads to

\[
\Gamma^M = \Gamma^M_Y + \Gamma^M_{ff}, \tag{31}
\]

where the indexes \( M, Y, \) and \( ff \) share the same meanings with the notations appearing in Eq. (23). We again parallel the approaches adopted in Refs. 47, 68, 69, 80, 90, 91 and arrive at the final results, which are attached in Appendix A for convenience.

### B. Coupled RG equations

With one-loop corrections in hand, we are now in a suitable position to derive the coupled RG flows of all interaction parameters that dictate the critical behaviors around the QCP. In the spirit of momentum-shell RG 100–102, we select the quadratic terms of effective action \( \phi \) as the “free fixed point” to deliver the RG rescaling transformations of momenta, energy, and fields in the following 17, 22, 48, 68, 69, 75, 80:

\[
k \to k' e^{-l}, \tag{32}
\]

\[
\omega \to \omega' e^{-l}, \tag{33}
\]

\[
\Psi_{1,2}(k,\omega) \to \Psi'_{1,2}(k',\omega') e^{\frac{l}{2} \int_0^1 (4 - \eta_l) dl}, \tag{34}
\]

\[
\phi(k,\omega) \to \phi'(k',\omega') e^{\frac{l}{2} \int_0^1 (4 - \eta_l) dl}, \tag{35}
\]

where the variable parameter \( l > 0 \) is adopted to specify a running energy scale and delimit the momentum-shell for every RG transformation, which is confined to \( hA - \Lambda \) with \( \Lambda \) specifying a cutoff of energy scale 17, 18, 48. The anomalous dimensions \( \eta_f \) and \( \eta_Y \) are determined by the one-loop corrections in Sec. III A. To one-loop level, we figure out that \( \eta_Y = 0 \) and \( \eta_f = -\lambda^2 Z \) which is inherited from Eq. (22) with \( Z = A, B, C, D \) for four distinct types of phase transitions classified in Sec. III A.
Subsequently, we gather all one-loop corrections to interaction parameters in Sec. III A and the RG transformation scalings \([32, 33, 35, 48]\) together to derive the coupled RG equations by carrying out the standard procedures of RG approach\([100, 102]\). In the following, we list the coupled RG equations for Type-\(\tau_2\) phase transition, consisting of energy-dependent evolutions of fermion velocities and Yukawa coupling,

\[
\frac{dv_\Delta}{dl} = \lambda^2 (A_1 - A_3)v_\Delta, \tag{36}
\]

\[
\frac{du_0}{dl} = \left\{-1 + 2A_1 - \frac{(u_1 u_2 + u_2 u_3)}{4\pi v_F v_\Delta u_0} + \frac{2\lambda^2}{3} \left[ \frac{u_2}{u_0} (A_3 - A_1) - 4A_1 \right] \right\} u_0, \tag{40}
\]

\[
\frac{dv_0}{dl} = \left\{-1 + 2A_1 + \frac{1}{4\pi v_F v_\Delta} \left[ (u_0 - u_1 - u_2 - 2u_3) + \frac{2u_2 u_3}{u_1} \right] + \frac{2\lambda^2}{3} \left[ \frac{u_3}{u_1} (A_3 - A_1) - 4A_1 \right] \right\} v_0, \tag{41}
\]

\[
\frac{dv_1}{dl} = \left\{-1 + 2A_1 + \frac{1}{4\pi v_F v_\Delta} \left[ (u_0 - u_1 - u_2 - 2u_3) + \frac{2u_2 u_3}{u_1} \right] + \frac{2\lambda^2}{3} \left[ \frac{u_3}{u_1} (A_3 - A_1) - 4A_1 \right] \right\} v_1, \tag{40}
\]

\[
\frac{dv_2}{dl} = \left\{-1 + 2A_1 + \frac{1}{4\pi v_F v_\Delta} \left[ (u_0 - u_1 - u_2 - 2u_3) + \frac{2u_2 u_3}{u_1} \right] + \frac{2\lambda^2}{3} \left[ \frac{u_3}{u_1} (A_3 - A_1) - 4A_1 \right] \right\} v_2, \tag{41}
\]

\[
\frac{dv_3}{dl} = \left\{-1 + 2A_1 + \frac{1}{4\pi v_F v_\Delta} \left[ (u_0 - u_1 - u_2 - 2u_3) + \frac{2u_2 u_3}{u_1} \right] + \frac{2\lambda^2}{3} \left[ \frac{u_3}{u_1} (A_3 - A_1) - 4A_1 \right] \right\} v_3. \tag{43}
\]

In order to make our presentations more compact, we collect and present the related coupled RG equations in Appendix B with respect to the rest three types of phase transitions. Specifically, Eqs. (B1)-B8 correspond to Type-\(\tau_0\), Eqs. (B9)-(B16) to Type-\(\tau_z\), and Eqs. (B17)-(B20) to Type-\(\tau_\theta\), respectively.

In the scenario of RG framework, these evolutions encode intimate entanglements of all interaction parameters\([33, 35, 102]\), which usually enter into the physical implications, and henceforth are expected to be of particular relevance and significance to dictate the low-energy rates of critical properties in the vicinity of certain QCP in Fig. 1. We are about to attentively investigate and address the physical consequences of them in the two looming sections.

IV. LOW-ENERGY BEHAVIORS OF FERMION VELOCITIES

Considering gapless nodal QPs intimately couple with quantum critical fluctuations around certain QPT shown in Fig. 1, two very fermion velocities \(v_F\) and \(v_\Delta\) as well as their ratio \(v_\Delta/v_F\) would be substantially renormalized. Given the behaviors of fermion velocities are of close relevance to the low-energy physical observables, they are henceforth expected to play an important role in determining the low-energy rates of \(d\)-wave superconductors\([16, 23, 25, 31, 35]\). Accordingly, it is of particular importance to inspect the critical behaviors of fermion velocities triggered by the QCP. To this end, we within this section are going to study the energy-dependent coupled RG flow equations addressed in Sec. III B which are assumed to contain the critical information of certain QCP.

A. In the absence of fermion-fermion interactions

Despite both fermion-fermion interactions and quantum fluctuations of order parameters are involved in our RG equations in Sec. III we at first switch off the fermion-fermion interactions to explicitly investigate the effects of all four sorts of order parameters with the reduced coupled RG equations of \(v_F\), \(v_\Delta\), plus \(\lambda\), and then defer the contribution from fermion couplings to next subsection IV B.

1. Fixed Yukawa coupling

As aforementioned in Sec. II there exist four reduced types of QPTs at \(r_c\), which are schematically shown in Fig. 1 on the basis of the group theory analysis\([11, 13]\). It is worth pointing out that the critical behaviors of fermion velocities with approaching the Type-\(\tau_x\), -\(\tau_y\), or -\(\tau_\theta\) QCP were carefully studied by several researchers in the absence of fermion-fermion interactions\([12, 20, 23, 38, 48]\). In order to simplify the analysis, the Yukawa coupling between nodal QPs and order parameter is regarded as a fixed constant and consequently three distinct fixed points are driven by the quantum criticality, namely \((v_\Delta/v_F)^* \rightarrow 0\), \((v_\Delta/v_F)^* \rightarrow 1\), and \((v_F/v_\Delta)^* \rightarrow 0\) for Type-\(\tau_x\), -\(\tau_y\) and -\(\tau_\theta\), respectively.
As to Type-\(\tau_0\) QPT, it has not yet been seriously investigated to the best of our knowledge. For the sake of completeness, we hereby examine the fate of fermion velocities for such QPT. Learning from the RG analysis in Sec. IIIB, it is of particular interest to figure out that the Yukawa coupling \(\lambda\) is marginal to one-loop level. In other words, this is equivalent to the situation of fixed Yukawa coupling. Performing numerical evaluation of coupled RG equations for Type-\(\tau_0\) QPT \cite{17-20,22,38,48,63,64} gives rise to the main results delineated in Fig. 6. It manifestly shows that the trajectories of \(v_\Delta/v_F\) with variation of initial values eventually converge to the same finite value at the lowest-energy limit. To be specific, with lowering the energy scale, \(v_\Delta/v_F\) is attracted by either fixed point \((v_\Delta/v_F)^* \approx 0.0942\) or \((v_\Delta/v_F)^* \approx 0.3478\), which corresponds to Type-\(\tau_{0A}\) or Type-\(\tau_{0B}\) component and is insensitive to initial conditions. As a consequence, Type-\(\tau_0\) QPT, in marked contrast to extreme anisotropies caused by its Type-\(\tau_{x,z}\) counterparts, prefers to induce some finite anisotropy of fermion velocities.

2. Flooding Yukawa coupling

Compared to the fixed-coupling assumption\cite{17,20,22,38,48,63,64}, much more physical information would be captured after seriously taking into account the potential evolution of Yukawa coupling \(\lambda\) appearing in Eq. 4. As apparently exhibited in Sec. IIIB, the coupled RG equations are jointly dictated by both the flow of \(\lambda\) and its entanglement with other interaction parameters. In this context, one can expect that the low-energy properties of fermion velocities may be partially or heavily modified by the evolution of coupling \(\lambda\) around the putative QCP. In order to clarify these intriguing and significant issues, we hereby place our focus on whether and how the tendencies of fermion velocities can be reshaped for all types of QPTs. With respect to the Type-\(\tau_0\) QPT, it is worthwhile to highlight that the coupling \(\lambda\) is marginal as depicted in Eq. \cite{1220}, indicating an effective fixed-coupling case which is studied in Sec. IV A.1 As to the other three types of QPTs, we subsequently address one by one in the following.

At the outset, we inspect how fermion velocities behave as approaching the Type-\(\tau_\sigma\) QPT. The corresponding coupled RG evolutions are provided in Eqs. \cite{36-39}, which are indicative of the close interplay between parameter \(\lambda\) and fermion velocities \(v_F\) and \(v_\Delta\). After choosing several representative initial conditions to perform numerical calculations, we realize that flowing \(\lambda\) plays an important role in the energy-dependent tendencies of

![FIG. 6: (Color online) Flows of \(v_\Delta/v_F\) with decreasing the energy scales (enlarging \(l\)) under three representative initial values \(v_{\Delta0}/v_{F0} = 0.05, 0.1, 0.5\) for (a) Type-\(\tau_{0A}\) and (b) Type-\(\tau_{0B}\) components of Type-\(\tau_0\) QPT at a fixed Yukawa coupling \(\lambda = 1\).

![FIG. 7: (Color online) Energy-dependent flows of interaction parameters for both the fixed (bare curves) and flowing (arrowed curves) Yukawa couplings nearby the Type-\(\tau_\sigma\) QPT (the critical energy scale \(l^*\) is designated as the saturated point for \(\lambda = 1\) case): (a) evolutions of \(\lambda, v_\Delta/v_{\Delta0}, v_F/v_{F0}\) at a representative initial value \(v_{\Delta0}/v_{F0} = 0.1\) and (b) fates of \(v_\Delta/v_F\) with Inset A displaying the low-energy limit at \(l > l^*\) for the running-coupling case and Inset B presenting its flows at three representative initial values.](image-url)
FIG. 8: (Color online) Relationships between the coefficients in RG equations and Yukawa coupling as well as fermion velocities for the potential QPTs. The left column serves as the Type-τ$_x$ QPT: (a1) the $v_{\Delta}/v_F$ dependence of functions $A_1 - A_3$ and $A_2 - A_3$ in Eq. (53)-(55); (a2) three underlying fates of $\lambda^2(l)$ and $\frac{\lambda^2}{v_F}(l)$, which are named as Case-I with $\frac{\lambda^2}{v_F} = 0$ and Case-II with $\lambda^2 = \frac{\lambda^2}{v_F} = 0$ as well as Case-III with $\lambda^2 = 0$; and (a3) the evolution of $\lambda^2(l)$-$\frac{\lambda^2}{v_F}(l)$ extracted from the coupled RG equations. The middle column denotes the Type-τ$_y$ QPT: (b1) the $v_{\Delta}/v_F$ dependence of functions $2B_1 + B_2 + B_3$ and $B_2 - B_3$ in Eq. (53)-(55); (b2) three underlying fates of $\lambda^2(l)$ and $\frac{\lambda^2}{v_F}(l)$, which are dubbed Case-I with $\frac{\lambda^2}{v_F} > 1$, Case-II with $\frac{\lambda^2}{v_F} = 1$ plus Case-III with $\frac{\lambda^2}{v_F} < 1$; and (b3) the evolution of $\lambda^2(l)$-$\frac{\lambda^2}{v_F}(l)$ extracted from the coupled RG equations. The right column corresponds to the Type-τ$_z$ QPT: (c1) the $v_F/v_{\Delta}$ dependence of functions $C_1 - C_2$ and $C_3 - C_2$ in Eq. (53)-(55); (c2) three underlying fates of $\lambda^2(l)$ and $\frac{\lambda^2}{v_F}(l)$, which are classified as Case-I with $\frac{\lambda^2}{v_F} = 0$, Case-II with $\lambda^2 = \frac{\lambda^2}{v_F} = 0$ and Case-III with $\lambda^2 = 0$; and (c3) the evolution of $\lambda^2(l)$-$\frac{\lambda^2}{v_F}(l)$ extracted from the coupled RG equations.

all related parameters as displayed in Fig. 8. Before going further, two helpful points need to be clarified. For the sake of comparison, we from now on would also supplement the corresponding results of fixed-coupling cases with $\lambda = 1$ for simplicity in our numerical results. Additionally, a critical energy scale denoted by $l^*$ is designated to serve as the saturated point for $\lambda = 1$ case. Learning from Fig. 7, we find that both $v_{\Delta}$ and $v_F$ for case $\lambda = 1$ rapidly decrease upon lowering the energy scales. In particular, $v_{\Delta}$ falls down more quickly than $v_F$ and thus their ratio $v_{\Delta}/v_F$ goes towards zero at the lowest-energy limit. This implies that $v_{\Delta}$ vanishes but instead $v_F$ still acquires a finite value at $l \geq l^*$. In striking comparison, once the Yukawa coupling is involved in the coupled RG evolutions as well, $\lambda$ itself gradually descends and evolves towards zero at the lowest-energy limit. As a result, the downtrends of fermion velocities are much slower than their $\lambda = 1$ counterparts. Specifically, both $v_{\Delta}$ and $v_F$ gently decrease and evolve towards finite values at $l^*$. Concerning their ratio $v_{\Delta}/v_F$, Fig. 7(b) shows...
that the extreme anisotropy is broken and replaced by a finite anisotropy at $l^*$ under the influence of the running coupling $\lambda$. Therefore, we come to a conclusion that the low-energy fates of fermion velocities are heavily affected by the participation of energy-dependent $\lambda$. In particular, the destruction of extreme anisotropy of $v_\Delta/v_F$ would impose a direct or indirect impact on the physical quantities nearby the QCP.

Prior to investigating the Type-$\tau_y$ QPT, we endeavor to examine the stability of $v_\Delta/v_F$ nearby the Type-$\tau_x$ QPT at $l = l^*$ and pinpoint its final fate at $l > l^*$. It is manifestly shown in Fig. 7(b) that $v_\Delta/v_F$ in the fixed-coupling case is nearly saturated with the extreme anisotropy at the critical energy scale. This is apparent to the running-coupling situation, in which the ratio reduces to certain finite value at $l = l^*$. With an aim to explore the tendency of $v_\Delta/v_F$ in the lowest-energy limit, we enlarge the variable $l > l^*$ to obtain the inset A of Fig. 7(b), displaying that the ratio is still unsaturated. In light of the technical deficiency of numerical evaluation, it seems unrealistic to determine whether $v_\Delta/v_F$ vanishes or reaches a finite value at $l \to \infty$. As a consequence, we resort to tentatively analytical analysis of RG equations in association with the numerical results. In principle, the behavior of $v_\Delta/v_F$ is directly determined by the coefficients in Eq. 58 including $\lambda^2$ and $A_2 - A_3$. Each of them going towards zero hints to the stop of RG flow with a stable $v_\Delta/v_F$. Once the coupling $\lambda$ goes towards zero before the latter, the ratio $v_\Delta/v_F$ can either be a finite value or zero. However, it is worth emphasizing that the vanishment of $A_2 - A_3$ is tantamount to $v_\Delta/v_F = 0$ as delineated in Fig. 8(a1). Accordingly, as illustrated in Fig. 8(a2), there are three distinct circumstances in all for the ratio of fermion velocities at the lowest-energy limit, which correspond to Case-I with $\lambda^2 \neq 0$ and Case-II with $\lambda^2 = v_\Delta/v_F = 0$ as well as Case-III with $v_\Delta/v_F \neq 0$. With the help of numerical results, Fig. 8(a3) recapitulates the tendencies of $\lambda^2$ and $v_\Delta/v_F$ with decreasing the energy scale. On the basis of these, we figure out that Case-III is selected by the coupled RG evolutions and hence $v_\Delta/v_F$ for Type-$\tau_x$ QPT is eventually attracted by a finite fixed point at the lowest-energy limit. This is consistent with the previous analysis at $l \leq l^*$ and therefore hints to the destruction of extreme anisotropy of fermion velocities due to the evolution of Yukawa coupling.

Subsequently, we move to examine the low-energy fates of fermion velocities as accessing the Type-$\tau_y$ QPT. Ini-
tially, let us aim at the regime \( l \in [0, l^*] \) in which the numerical results of the associated RG equations are provided in Fig. 9. As for the fixed-coupling \( \lambda = 17 – 20, 22, 38, 48, 63, 64 \), it can be seen from Fig. 9(a) that \( v_\Delta \) quickly climbs up and then keeps decreasing until it vanishes at \( l = l^* \). Meanwhile, \( v_F \) monotonically falls down to zero. These make the fermion velocities isotropic with \( (v_\Delta/v_F)^* = 1 \) at \( l = l^* \), which is insensitive to the starting condition as displayed in Fig. 9(b). In sharp contrast, the coupled RG equations (B.11-B.13) force the coupling \( \lambda \) to interact with other parameters and descend with lowering the energy scale. As a result, fermion velocities present distinct behaviors compared to \( \lambda = 1 \) as approaching the critical energy scale. In Fig. 9(a), \( v_\Delta \) gradually goes up to certain finite values, but instead \( v_F \) decreases to nonzero values. Accordingly, Fig. 9(b) shows that \( v_\Delta/v_F \) slowly climbs up and flows towards a finite value, which is smaller than \( (v_\Delta/v_F)^* = 1 \) and susceptible to the initial conditions. This suggests that the evolution of coupling \( \lambda \) prevents fermion velocities being isotropic but rather results in weak anisotropy as accessing the critical energy scale. Next, we go to judge the final fate at \( l > l^* \) under the influence of energy-dependent Yukawa coupling in that \( v_\Delta/v_F \) is not saturated at \( l = l^* \) and even a much larger \( l \) as shown in Inset A of Fig. 9(b). In analogy with Type-\( \tau_x \) case, the final fate of \( v_\Delta/v_F \) for the Type-\( \tau_y \) QPT depends upon which one of two coefficients \( B_2 – B_3 \) and \( \lambda^2 \) in Eq. (B.3) is driven to the fixed point more quickly. To respond to this, we realize the fate of \( B_2 – B_3 = 0 \) amounts to \( v_\Delta/v_F = 1 \) and then parallel the strategy for Type-\( \tau_x \) QPT to present three potential circumstances for \( \lambda^2 \) and \( v_\Delta/v_F \) in Fig. 8(b2), consisting of Case-I with \( v_\Delta/v_F > 1 \), Case-II with \( v_\Delta/v_F = 1 \) plus Case-III with \( v_\Delta/v_F < 1 \). The related numerical analysis of RG equations in Fig. 8(b3) exhibits Case-III is the dominant situation. This henceforth corroborates the results at \( l = l^* \) that fermion velocities are forced to a weak anisotropy due to the contribution from the running Yukawa coupling.

At last, we go to investigate the low-energy behaviors of fermion velocities by virtue of the coupled RG flows (B.2), (B.12) nearby Type-\( \tau_z \) QPT. The major results are presented in Fig. 10 in which the distinctions between fixed-coupled and energy-dependent cases are clearly exhibited. Studying from Fig. 10(a), \( v_F \) rapidly drops down and vanishes at \( l \approx l^* \) with a fixed \( \lambda = 1 \), but rather \( v_\Delta \) progressively descends and tends to a finite value \( 18, 63, 64 \). While the Yukawa coupling \( \lambda \) enters into the RG equations, it becomes energy-dependent and quickly climbs down with lowering the energy scale. This brings significant effects to fermion velocities, making \( v_\Delta \) drop much more than that of \( v_F \) despite both of them smoothly decrease as the energy scale is decreased. With respect to the ratio of fermion velocities at \( l \geq l^* \) in Fig. 10(b), we figure out that \( v_F/v_\Delta \) bears similarities to \( v_\Delta/v_F \) approaching the Type-\( \tau_x \) QPT illustrated in Fig. 10(a). In other words, the extreme anisotropy with \( v_F/v_\Delta \rightarrow 0 \) at \( \lambda = 18, 63, 64 \) is sabotaged and replaced with a finite anisotropy by the evolution of coupling \( \lambda \). By the same token, \( v_F/v_\Delta \) for Type-\( \tau_y \) QPT hereby does not saturate at \( l = l^* \) as shown in Inset A of Fig. 10(b). In this sense, we follow the previous tactic to identify its final fate, which heavily hinges upon the coefficients \( \lambda^2 \) and \( C_3 – C_2 \) in Eqs. (B.9)-(B.12). In resemblance to the analysis for \( v_\Delta/v_F \), \( C_3 – C_2 = 0 \) points to \( v_F/v_\Delta = 0 \) and then three distinct fates are diagrammatically illustrated in Fig. 8(c2) including Case-I with \( \lambda^2 \neq 0 \), Case-II with \( v_F/v_\Delta = \lambda^2 = 0 \) and Case-III with \( v_F/v_\Delta \neq 0 \), respectively. In the assistance of numerical evaluation, Fig. 8(c3) shows us that Case-III wins the competition with \( v_F/v_\Delta \) being governed by a finite value. It therefore signals that the evolution of coupling \( \lambda \) drives the extreme anisotropy \( v_F/v_\Delta \rightarrow 0 \) into a finite anisotropy at the lowest-energy limit.

B. In the presence of fermion-fermion interactions

As aforementioned in Sec. III, fermion-fermion interactions enter into the coupled RG equations and then may play an important role in the low-energy regime via intimately interacting with quantum fluctuations of order parameters and fermion velocities. Based upon the results in the absence of fermion-fermion interactions, we are now in a suitable position within this subsection to investigate how fermion-fermion interactions impact the behaviors of fermion velocities upon approaching the putative QCPs, which are insufficiently taken into account in previous efforts.\( 17, 20, 22, 38, 48, 63, 64 \).

To achieve this goal, we have to study the coupled RG equations, which consist of \( v_F, v_\Delta, \) and \( \lambda \) as well as fermion-fermion interactions characterized by \( u_i \) with \( i = 0, 1, 2, 3 \). To proceed, we at first consider the Yukawa fixed-coupling case \( 17, 20, 48 \). Learning from the RG equations of fermion velocities \( 35, 38 \), we can infer that the fermion-fermion interactions \( u_i \) cannot directly affect \( v_F \) and \( v_\Delta \), but rather only indirectly modify them via entangling with the interaction parameter \( \lambda \). In other words, the fermion velocities would receive the contributions from fermion-fermion interactions once the coupling \( \lambda \) flows under the RG equations. This implies that the low-energy properties of fermion velocities for the fixed-coupling situation are adequately robust against fermion-fermion interactions. Next, our focus is moved to the situation with the energy-dependent evolution of Yukawa-coupling \( \lambda \). After carrying out the numerical analysis of coupled RG equations \( 35, 38, 48 \), we present the comparison between the absence and presence of fermion-fermion interactions in Fig. 11(a) as approaching the Type-\( \tau_x \) QPT. It can be seen from Fig. 11(a) that the ratio of fermion velocities \( v_F/v_\Delta \) under fermion-fermion interactions shares the same downtrend with its behaviors in the absence of fermion-fermion interactions. However, one can unambiguously realize that fermion-fermion interactions do bring considerable quantitative effects, which are in favor of retarding the \( v_\Delta/v_F \)’s decrease as the energy
by some finite anisotropy. As to the Type-$\tau_0$ QPT, the evolution of coupling $\lambda$ drives the isotropic system into a finite anisotropic fixed point. On the other hand, we find that the fates of fermion velocities are principally robust against the fermion-fermion interactions although certain quantitative effects are generated to retard the tendencies flowing towards potential fixed points. Subsequently, it is ready to examine the consequences of these unusual behaviors of fermion velocities on the quantum criticality of physical observables.

V. SUPERFLUID DENSITY AND CRITICAL TEMPERATURE

Quantum criticality of fermion velocities around a putative QCP is carefully studied and detailedly addressed in the previous section [V] after simultaneously collecting the quantum fluctuations of order parameters and fermion-fermion interactions. In order to present these unique behaviors of fermion velocities that are inconvenient to be detected directly, one can resort to examining the low-energy physical observables in that the fermion velocities plus their ratio $v_\Delta/v_F$ usually enter into the physical quantities and play an important role in the low-energy regime. This therefore provides us a useful routine to study the distinctions among different QPTs and the very positions of QCPs.

For this purpose, we within this section concentrate on the properties of superfluid density and critical temperature upon accessing the QCPs, which are two of the most key quantities of superconductors. In order to simplify our analysis, the effects of fermion-fermion interactions are hereafter not considered since they are always subordinate to the Yukawa couplings between nodal QPs and order parameters and hence cannot alter the basic results caused by the quantum fluctuations as discussed in Sec. [V A]. Rather, we primarily try to examine how these two quantities behave under distinct fates of fermion velocities with approaching the assumed QCPs which are induced by the fermion-order parameter couplings and explicitly presented in Sec. [IV A].

A. Superfluid density and critical temperature nearby the QCP

Generally, the zero-temperature superfluid density of $d$-wave superconductor in underdoped region depends linearly on doping concentration $x$ and can be written as

$$\rho^s(0) = \frac{x}{a^2},$$

where $a$ stands for the lattice spacing constant. To proceed, it is inevitable that a certain amount of normal nodal QPs would be thermally excited out from the

![FIG. 11: (Color online) Effects of fermion-fermion interactions for the Type-$\tau_x$ QPT on (a) the behaviors of $v_\Delta/v_F$ at an initial value $v_{\Delta 0}/v_F = 0.1$ ($\lambda_F$ and $\lambda_F+$$f_f$ correspond to the absence and presence of fermion-fermion interactions, respectively), and (b) the energy-dependent evolutions of fermion-fermion interaction strengths $\nu_0, 1, 2, 3$ (the basic tendencies for Type-$\tau_{y, z}$ QPTs are similar and hence not shown here).](image)
SC condensate at a finite temperature, which can efficiently deplete the superfluid density. As a result, the temperature-dependent superfluid density can be expressed as

\[ \rho^s(T) = \rho^s(0) - \rho^s(T), \]

where \( \rho^s(T) \) and \( \rho^s(0) \) serve as the superfluid density and normal QPs density at \( T > 0 \), respectively. In the non-interacting situation, the normal QPs density exhibits a linear temperature dependence and takes the form of

\[ \rho^s(T) = m \frac{2 \ln 2}{\pi} \frac{v_F}{\nu_\Delta} T, \]

with the parameter \( m \) being the mass of nodal QP.

Hereby, it is of interest to address Božović et al. recently reported that the dependence of the zero-temperature superfluid density on the critical superconducting temperature for the overdoped region can change from linear to parabolic as the critical temperature is below a very value about 12 K. However, we emphasize that above formula (45) are believed to capture the crucial information of the underdoped and optimal regions as displayed in Fig. 1. In such scenario, the superfluid density decreases as the temperature is lifted and thus the critical temperature can be explicitly derived via assuming \( \rho^s(T) = 0 \) at \( T_c \),

\[ T_c = \frac{1}{2 \ln 2} \frac{v_\Delta}{\nu_\Delta} \frac{x}{m a^2}, \]

which is well consistent with the Uemura plot. This indicates \( T_c \) is readily obtained for the region away from the QCPs, in which the ratio of fermion velocities for noninteracting nodal QPs takes a constant, for instance \( v_\Delta/v_F \approx 0.1 \) for YBa\(_2\)Cu\(_3\)O\(_6+y\). In comparison, the involved physics is much more complicated but rather interesting in the vicinity of certain QCP depicted in Fig. 1. As systematically addressed in Sec. IV the fermion velocities \( v_F \) and \( v_\Delta \) as well as other interaction parameters in the effective theory with approaching the QCPs are heavily renormalized by ferocious quantum fluctuations and become energy-dependent under the control of the coupled RG equations in Sec. III B. It is worth emphasizing that the ratio of fermion velocities, which is directly related to the superfluid density as delineated in Eq. (45) exhibits a cornucopia of energy-dependent behaviors and flows towards several fixed points at the lowest-energy limit. With these respects, in order to capture the effects of quantum criticality, we take into account the renormalized fermion velocities and follow the approach in Refs. 45, 46, 48, 50, 51, 105 to construct the following renormalized normal QPs density,

\[ \rho^s_R(T) = \frac{4m}{k_B T} \int d^2 k \left[ \frac{v_F^2(k) e^{\sqrt{\nu_\Delta^2(k)^2 + \nu^2_F(k)^2}}}{\left( 1 + e^{\sqrt{\nu_\Delta^2(k)^2 + \nu^2_F(k)^2}} \right)^2} \right]. \]

\[ v_F^2(k) = \frac{\nu^2_F(k)^2}{\nu^2_F(k)^2 + \nu_\Delta^2(k)^2} \]

from which the renormalized critical temperature can be derived via taking \( \rho^s_R(T) = 0 \) at \( T = T_c \).

As a consequence, Eq. (46) together with Eq. (49) signal that both superfluid density and critical temperature are intimately associated with the energy-dependent fermion velocities, which are governed by the coupled RG evolutions in Sec. III B and display many peculiar properties for all four sorts of QPTs as presented in Sec. IV In the rest of this section, we are about to pin down the fates of superfluid density and critical temperature at the lowest-energy limit for all types of QPTs in Fig. 1.

\[ \rho^s_R(T) = \rho^s(0) - \rho^s_R(T), \]

where \( k_B \) denotes the Boltzmann constant and \( v_\Delta, v_F \) are dictated by associated RG equations in Sec. III B. This henceforth yields to the renormalized superfluid density

\[ \rho^s_R(T) = \rho^s(0) - \rho^s_R(T), \]

from which the renormalized critical temperature can be derived via taking \( \rho^s_R(T) = 0 \) at \( T = T_c \).

As a consequence, Eq. (46) together with Eq. (49) signal that both superfluid density and critical temperature are intimately associated with the energy-dependent fermion velocities, which are governed by the coupled RG equations in Sec. III B and display many peculiar properties for all four sorts of QPTs as presented in Sec. IV. In the rest of this section, we are about to pin down the fates of superfluid density and critical temperature at the lowest-energy limit for all types of QPTs in Fig. 1.

B. Fates at \( v_\Delta/v_F = 0.1 \)

On the basis of analysis in Sec. III A, \( v_\Delta/v_F \) plays a central role in determining the fates of both the superfluid density and critical temperature as explicitly displayed in Eq. (48) and Eq. (49). In order to obtain the energy-dependent \( v_\Delta/v_F \), it enables us to fix its initial value and then carry out the numerical evaluation of coupled RG equations in Sec. III B. Without loss of generality, we hereby place our primary focus on the initial condition with \( v_\Delta/v_F = 0.1 \) as such ratio appears in most of high-\( T_c \) superconductors, \( v_\Delta/v_F = 0.1 \), and then discuss the stability of basic results against the initial conditions in the following subsection.

To proceed, performing the numerical calculations of coupled RG equations for all types of QPTs with such starting condition and inserting them into Eqs. (48)–(49).
Type-\(\tau_y\) and Type-\(\tau_z\) QPTs, in which the basic tendencies of critical temperatures are analogous owing to the quantum fluctuations. At a fixed coupling \(\lambda = 1\), fermion velocities are driven to the isotropic situation for Type-\(\tau_y\) QPT\(^{13}\) but another extreme anisotropy with \(v_F/v_\Delta \to 0\) for Type-\(\tau_z\) QPT\(^{14}\). Accordingly, Fig. (12) presents that these cause a little promotion for superconductivity with \(T_c < T_y^c < T_z^c\). In comparison, Sec. (IV A 2) shows that both the isotropic and extremely anisotropic fermion velocities are destroyed by the evolution of coupling \(\lambda\) but instead \(v_\Delta/v_F\) are attracted by some finite values at the lowest-energy limit. As a consequence, Fig. (12) displays that critical temperatures of both cases reduce. Although the relationship of \(T_c < T_y^c < T_z^c\) is preserved, the critical temperature of Type-\(\tau_y\) QPT falls a little more than that of Type-\(\tau_z\) QPT as illustrated in Fig. (13b) and Fig. (13c).

To recapitulate, we come to a conclusion that the superconductivity is enhanced nearby the Type-\(\tau_y\) and Type-\(\tau_z\) QPTs and conversely suppressed in the proximity of the Type-\(\tau_0\), Type-\(\tau_B\) and Type-\(\tau_x\) QPTs. The renormalized critical temperatures under these QPTs in Fig. (11) are followed by \(T_c > T_y^c > T_c > T_B^c > T_B^c > T_x^c\), which are schematically summarized in Fig. (13).

C. Stability of \(T_c\) against \(v_\Delta/v_F\)

For the sake of completeness, we are now in a suitable position to inspect the stability of conclusions concerning the critical temperatures in Sec. (IV B) under the variation of initial condition \(v_\Delta/v_F\) as approaching distinct types of QPTs. There exist two points behind this issue as follows. Although the final fixed points are considerably insensitive to the starting values of fermion velocities as studied in Sec. (IV) one can learn from Eq. (18) and Eq. (19) that critical temperature depends not only upon the contributions from the fixed point but also upon the whole low-energy regime. In addition, although the \(v_\Delta/v_F\) approximately equals 0.1 in most of high-\(T_c\) superconductors\(^{16,31,111}\), this initial value is inevitable to be affected by various uncontrollable and unexpected factors in real materials.

To proceed, we select three representative initial values \(v_\Delta/v_F = 0.05, 0.1, 0.2\) to examine whether and how the fates of critical temperatures nearby distinct types of QPTs are renormalized by the initial conditions. In order to achieve this end, we parallel the analogous procedures in Sec. (IV B) with the help of associated RG equations and then obtain the main results collected in Fig. (14). At the first sight, we figure out that the effects caused by the variation of \(v_\Delta/v_F\) on a fixed-coupling \(\lambda = 1\) circumstance share the qualitative results with that of the evolution of \(\lambda\) case. To be concrete, the critical temperatures around Type-\(\tau_x\) and Type-\(\tau_0\) QPTs are susceptible to the initial values of \(v_\Delta/v_F\) and present a little downturns upon the increase of \(v_\Delta/v_F\) albeit the stability for suppression of superconductivity displayed...
in Fig. 14 (a) and Fig. 14 (d). In comparison, we can learn from Fig. 13 (b) and Fig. 13 (c) for Type-$\tau_y$ and Type-$\tau_z$ QPTs that both of the critical temperatures receives a certain mount of enhancements with tuning up the value of $v_{\Delta 0}/v_{F0}$. Meanwhile, the basic restriction between the bare and renormalized critical temperature with $\tilde{T}_c > T_0 > T_c$ is insensitive to the initial condition. This suggests that both the initial values of fermion velocities and evolution of Yukawa coupling are subordinate to unusual behaviors of the fermion velocities which are crucial to pin down the critical temperatures around the QPT.

To be brief, the fates of critical temperatures are of particular robustness against the initial values of fermion velocities in the proximity of the putative QCPs. In other words, the low-energy properties of the fermion velocities that are the external expressions of quantum criticality triggered by the QPTs play a more significant role than initial condition in determining the superfluid density and critical temperature. Accordingly, these distinct fates of critical temperatures schematically shown in Fig. 13 are closely associated with different sorts of QPTs and henceforth provide a helpful clue to experimentally detect the very QPT and fix its location.

VI. SUMMARY

In summary, we study the low-energy fates of fermion velocities and behaviors of superfluid density as well as critical temperature nearby the putative QPTs in $d$-wave superconductors, which stem from the topological change of nodal points. In order to facilitate the analysis, seven candidates of potential QPTs shown in Fig. 12 cluster into four effective categories, which are designated as the Type-$\tau_0$, Type-$\tau_x$, Type-$\tau_y$ and Type-$\tau_z$ QPTs in Sec. 14. By means of the momentum-shell RG approach, all primary physical ingredients including quantum fluctuations of order parameters, the couplings between order parameters and nodal QPs, and fermion-fermion interactions can be equally captured and encoded in a set of coupled RG equations after taking into account one-loop corrections in Sec. 15. On the basis of these RG equations, we, with the help of both analytical and numerical evaluations, achieve the main results concerning the fixed points of fermion velocities and related...
physical quantities in the vicinity of QPTs.

To be concrete, the fermion velocities exhibit a number of critical properties caused by the effects of quantum fluctuations and fermion-fermion interactions, which are expected to be in charge of the low-energy fates around the underlying four types of QPTs. At first, the focus is put on the fermion velocities. Besides three distinct fixed points of fermion velocities obtained at the fixed Yukawa coupling \( \lambda = 1 \), including \( v_A/v_F \rightarrow 0 \) for Type-\( \tau_0 \), \( v_\Delta/v_F \rightarrow 1 \) for Type-\( \tau_x \), \( v_\tau/v_\Delta \rightarrow 0 \) for Type-\( \tau_y \), it is of particular interest to point out that a series of new fixed points are generated due to the interplay between evolution of Yukawa coupling \( \lambda \) together with other interaction parameters. As for the Type-\( \tau_0 \) QPT, we notice that the ratio of fermion velocities is attracted by either fixed point \( (v_\Delta/v_F)^* \approx 0.0942 \) or \( (v_\Delta/v_F)^* \approx 0.3478 \), which corresponds to Type-\( \tau_0A \) or Type-\( \tau_0B \) component and is insensitive to initial conditions. In comparison, the fixed points for Type-\( \tau_x,y,z \) QPTs at a fixed-coupling \( \lambda = 117.48 \) are manifestly reshaped. Two kinds of extreme anisotropies of fermion velocities for Type-\( \tau_x,y \) QPTs are both broken and replaced by finite anisotropies. Meanwhile, the isotropic fermion velocities for Type-\( \tau_y \) QPT are driven to a finite anisotropic fixed point by the evolution of coupling \( \lambda \). This indicates that fermion velocities prefer to flow towards a finite anisotropy as approaching a putative QPT. In addition to these results caused by the evolution of Yukawa coupling \( \lambda \), we examine the effects of fermion-fermion interactions on fermion velocities as well, which have not yet been considered seriously. Despite they are subordinate to the quantum fluctuations of order parameters, it is noteworthy that the fermion-fermion interactions as shown in Fig. 11 can bring non-ignorable quantitative contributions to the fermion velocities in the vicinity of a putative QPT. Next, both of superfluid density and critical temperature, which are two of the most important observables for superconductors, are carefully investigated under the unconventional behaviors of fermion velocities around all potential QPTs. Concretely, after combining the expressions of these two observables that are dependent upon the fermion velocities and the coupled RG equations of all interaction parameters, we notice that the renormalized critical temperatures are restricted by \( T_c^x > T_c^y > T_c^z \) and \( T_c^{0B} > T_c^{0A} > T_c^x \) as schematically illustrated in Fig. 12 for all types of QPTs. In other words, both Type-\( \tau_0 \) and Type-\( \tau_x \) QPTs are in favor of the superconductivity but rather the critical temperature is suppressed by Type-\( \tau_0A \), Type-\( \tau_0B \) and Type-\( \tau_x \) QPTs. In addition, we check that the fates of critical temperatures are primarily determined by unusual behaviors of the fermion velocities and considerably robust against the initial values of fermion velocities.

Our results systematically account for the quantum criticality of both fermion velocities and critical temperatures under the competition among quantum fluctuations and interplay between nodal QPs and order parameters as well as the fermion-fermion interactions near all potential QPTs in \( d \)-wave superconductors. In particular, an underlying strategy is provided to experimentally seek the putative QPTs and locate their very positions by virtue of qualitatively distinct behaviors nearby different types of QPTs. Additionally, this may offer an operable strategy to classify the superconducting materials with distinct behaviors of critical temperatures. What is more, these theoretical results may stimulate experimental scientists to check and seek other potential critical physics nearby these QCPs, as well as further explore the possible relationships between anomalous properties in the normal state with \( T > T_c \) and the quantum fluctuations. To recapitulate, we anticipate that these instructive results would be helpful to improve our understandings of the quantum criticality and structure of phase diagram in the \( d \)-wave superconductors.

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AUTHOR CONTRIBUTIONS

J. W. initiated and supervised the project as well as performed the numerical analysis and wrote the manuscript with the assistance of the other two authors. X.Y. R. carried out the analytical calculations and plotted figures. Y. H. Z participated in the discussions and provided several useful suggestions.

ADDITIONAL INFORMATION

Competing interests: The authors declare no Competing Financial or Non-Financial Interests.

Appendix A: One-loop corrections for Type-\( \tau_y \), \( \tau_z \), and \( \tau_0 \)

1. Self energy and vertex

One-loop self energy as shown in Fig. 11(b) receives the corrections from the interplay between the nodal QPs and Type-\( M \) order parameter with \( M = \tau_{0,x,y,z} \) illustrated in Fig. 11. To be compact, we have just presented Type-\( \tau_z \) in Eq. 21 of Sec. III A. In the following, the rest three types are collected after integrating out the momentum shell within \( b \Lambda - \Lambda \).

\[
\Sigma^{\tau_y}(k, \omega) = \lambda^2 [B_1(-i\omega) + 2B_2v_Fk_x\tau^z + B_3v_\Delta k_y\tau^y]l, \quad \text{(A1)}
\]

\[
\Sigma^{\tau_z}(k, \omega) = \lambda^2 [C_1(-i\omega) + C_2v_Fk_x\tau^z + C_3v_\Delta k_y\tau^y]l, \quad \text{(A2)}
\]
where the indexes $A$ and $B$ denote the two components for Type-$\tau_0$.

With respect to the fermion-fermion interactions

\[
\begin{align*}
\Sigma_A^{\tau}(k,\omega) &= \lambda^{\tau} D_A^{\tau}(-i\omega) + D_A^{\tau} v_F k_x \tau^2 + D_A^{\tau} v_D k_y \tau^3, \quad (A3) \\
\Sigma_B^{\tau}(k,\omega) &= \lambda^{\tau} D_B^{\tau}(-i\omega) + D_B^{\tau} v_F k_x \tau^2 + D_B^{\tau} v_D k_y \tau^3, \quad (A4)
\end{align*}
\]

renormalized by one-loop corrections as depicted in Fig. [15] we only provide the formal expression in Eq. (31) of Sec. IIIA. To remedy this, the details of the one-loop contributions are listed as follows after practicing the strategy in Refs. [63,75,77,79,80,82,84,86,90-93].
FIG. 15: One-loop corrections to the fermion-fermion interactions due to fermion-fermion interactions (i)-(v) and Yukawa coupling between nodal fermion and order parameter (vi)-(x). The solid, dashed and wavy lines represent the fermion propagator, four-fermion interaction and order parameter, respectively.

\[
\times \left[ \frac{1}{4\pi v_F v_{\Delta}} (u_0 - u_1 - u_2 - 2u_3 + \frac{2u_2u_3}{u_1}) + \frac{2\lambda^2}{3} \left( \frac{u_2}{u_1} (C_3 - C_2) - C_1 - 5C_3 \right) \right] l, \tag{A14}
\]

\[
\Gamma_{u_2}^{\tau_x} = u_2 \int_{-\infty}^{\infty} \frac{d\omega_1 d\omega_2 d\omega_3}{(2\pi)^3} \int_{-\infty}^{\infty} \frac{d^2k_1 d^2k_2 d^2k_3}{(2\pi)^6} \Psi_\alpha^\dagger(\omega_1, k_1) \tau_2 \Psi_\sigma(\omega_2, k_2) \Psi_\sigma^\dagger(\omega_3, k_3) \tau_2 \Psi_\sigma^\dagger(\omega_1 + \omega_2 - \omega_3, k_1 + k_2 - k_3) \times \left\{ \frac{1}{4\pi v_F v_{\Delta}} \left[ (2u_0 - 3u_1 - 2u_2 - 3u_3) + \frac{2u_1u_3}{u_2} \right] + \frac{2\lambda^2}{3} \left[ 4C_2 - 5C_1 - 5C_3 + \frac{u_1}{u_2} (C_3 - C_2) \right] \right\} l, \tag{A15}
\]

\[
\Gamma_{u_3}^{\tau_x} = u_3 \int_{-\infty}^{\infty} \frac{d\omega_1 d\omega_2 d\omega_3}{(2\pi)^3} \int_{-\infty}^{\infty} \frac{d^2k_1 d^2k_2 d^2k_3}{(2\pi)^6} \Psi_\alpha^\dagger(\omega_1, k_1) \tau_3 \Psi_\sigma(\omega_2, k_2) \Psi_\sigma^\dagger(\omega_3, k_3) \tau_3 \Psi_\sigma^\dagger(\omega_1 + \omega_2 - \omega_3, k_1 + k_2 - k_3) \times \left\{ \frac{1}{4\pi v_F v_{\Delta}} \left[ (u_0 - u_3 - u_1 - 2u_2) + \frac{2(u_1u_2)}{u_3} \right] + \frac{2\lambda^2}{3} \left[ \frac{u_1}{u_3} (C_2 - C_1 - 4C_3) \right] \right\} l, \tag{A16}
\]

where Eqs. (A3)–(A6) are linked to Type-\(\tau_x\), Eqs. (A7)–(A12) to Type-\(\tau_y\) and Eqs. (A13)–(A16) to \(\tau_z\), respectively.

2. Designated coefficients

All related coefficients appearing in both above equations and elsewhere are designated by

\[
A_1 = \frac{2(v_{\Delta}/v_F)}{N_f \pi^3} \int_{-\infty}^{\infty} dx \int_{0}^{2\pi} d\theta \frac{x^2 - \cos^2 \theta - (v_{\Delta}/v_F)^2 \sin^2 \theta}{[x^2 + \cos^2 \theta + (v_{\Delta}/v_F)^2 \sin^2 \theta]^2} G_{11}(x, \theta), \tag{A17}
\]

\[
A_2 = \frac{2(v_{\Delta}/v_F)}{N_f \pi^3} \int_{-\infty}^{\infty} dx \int_{0}^{2\pi} d\theta \frac{-x^2 + \cos^2 \theta - (v_{\Delta}/v_F)^2 \sin^2 \theta}{[x^2 + \cos^2 \theta + (v_{\Delta}/v_F)^2 \sin^2 \theta]^2} G_{11}(x, \theta), \tag{A18}
\]

\[
A_3 = \frac{2(v_{\Delta}/v_F)}{N_f \pi^3} \int_{-\infty}^{\infty} dx \int_{0}^{2\pi} d\theta \frac{x^2 + \cos^2 \theta - (v_{\Delta}/v_F)^2 \sin^2 \theta}{[x^2 + \cos^2 \theta + (v_{\Delta}/v_F)^2 \sin^2 \theta]^2} G_{11}(x, \theta), \tag{A19}
\]

\[
B_1 = \frac{2(v_{\Delta}/v_F)}{N_f \pi^3} \int_{-\infty}^{\infty} dx \int_{0}^{2\pi} d\theta \frac{x^2 - \cos^2 \theta - (v_{\Delta}/v_F)^2 \sin^2 \theta}{[x^2 + \cos^2 \theta + (v_{\Delta}/v_F)^2 \sin^2 \theta]^2} G_{11}(x, \theta), \tag{A20}
\]

\[
B_2 = \frac{2(v_{\Delta}/v_F)}{N_f \pi^3} \int_{-\infty}^{\infty} dx \int_{0}^{2\pi} d\theta \frac{-x^2 - \cos^2 \theta - (v_{\Delta}/v_F)^2 \sin^2 \theta}{[x^2 + \cos^2 \theta + (v_{\Delta}/v_F)^2 \sin^2 \theta]^2} G_{11}(x, \theta), \tag{A21}
\]

\[
B_3 = \frac{2(v_{\Delta}/v_F)}{N_f \pi^3} \int_{-\infty}^{\infty} dx \int_{0}^{2\pi} d\theta \frac{x^2 - \cos^2 \theta - (v_{\Delta}/v_F)^2 \sin^2 \theta}{[x^2 + \cos^2 \theta + (v_{\Delta}/v_F)^2 \sin^2 \theta]^2} G_{11}(x, \theta), \tag{A22}
\]

\[
C_1 = \frac{2(v_{\Delta}/v_F)}{N_f \pi^3} \int_{-\infty}^{\infty} dx \int_{0}^{2\pi} d\theta \frac{x^2 - \cos^2 \theta - (v_{\Delta}/v_F)^2 \sin^2 \theta}{[x^2 + \cos^2 \theta + (v_{\Delta}/v_F)^2 \sin^2 \theta]^2} G_{11}(x, \theta), \tag{A23}
\]

\[
C_2 = \frac{2(v_{\Delta}/v_F)}{N_f \pi^3} \int_{-\infty}^{\infty} dx \int_{0}^{2\pi} d\theta \frac{x^2 - \cos^2 \theta - (v_{\Delta}/v_F)^2 \sin^2 \theta}{[x^2 + \cos^2 \theta + (v_{\Delta}/v_F)^2 \sin^2 \theta]^2} G_{11}(x, \theta), \tag{A24}
\]
\[ C_3 = \frac{2(v_F/v_\Delta)}{N_f \pi^3} \int_{-\infty}^{\infty} dx \int_0^{2\pi} d\theta \left[ -x^2 - \cos^2 \theta + (v_\Delta/v_F)^2 \sin^2 \theta \right] \frac{[x^2 + \cos^2 \theta + (v_\Delta/v_F)^2 \sin^2 \theta]}{[x^2 + \cos^2 \theta + (v_\Delta/v_F)^2 \sin^2 \theta]^2} G_{III}(x, \theta), \]
\[ D_1^A = \frac{2(v_\Delta/v_F)}{N_f \pi^3} \int_{-\infty}^{\infty} dx \int_0^{2\pi} d\theta \left[ x^2 - \cos^2 \theta - (v_\Delta/v_F)^2 \sin^2 \theta \right] \frac{[x^2 + \cos^2 \theta + (v_\Delta/v_F)^2 \sin^2 \theta]}{[x^2 + \cos^2 \theta + (v_\Delta/v_F)^2 \sin^2 \theta]^2} G_{IVA}(x, \theta), \]
\[ D_2^A = \frac{2(v_\Delta/v_F)}{N_f \pi^3} \int_{-\infty}^{\infty} dx \int_0^{2\pi} d\theta \left[ x^2 - \cos^2 \theta + (v_\Delta/v_F)^2 \sin^2 \theta \right] \frac{[x^2 + \cos^2 \theta + (v_\Delta/v_F)^2 \sin^2 \theta]}{[x^2 + \cos^2 \theta + (v_\Delta/v_F)^2 \sin^2 \theta]^2} G_{IVA}(x, \theta), \]
\[ D_3^A = \frac{2(v_\Delta/v_F)}{N_f \pi^3} \int_{-\infty}^{\infty} dx \int_0^{2\pi} d\theta \left[ x^2 + \cos^2 \theta - (v_\Delta/v_F)^2 \sin^2 \theta \right] \frac{[x^2 + \cos^2 \theta + (v_\Delta/v_F)^2 \sin^2 \theta]}{[x^2 + \cos^2 \theta + (v_\Delta/v_F)^2 \sin^2 \theta]^2} G_{IVA}(x, \theta), \]
\[ D_1^B = \frac{2(v_\Delta/v_F)}{N_f \pi^3} \int_{-\infty}^{\infty} dx \int_0^{2\pi} d\theta \left[ x^2 - \sin^2 \theta - (v_\Delta/v_F)^2 \cos^2 \theta \right] \frac{[x^2 + \sin^2 \theta + (v_\Delta/v_F)^2 \cos^2 \theta]}{[x^2 + \sin^2 \theta + (v_\Delta/v_F)^2 \cos^2 \theta]^2} G_{IVB}(x, \theta), \]
\[ D_2^B = \frac{2(v_\Delta/v_F)}{N_f \pi^3} \int_{-\infty}^{\infty} dx \int_0^{2\pi} d\theta \left[ x^2 - \sin^2 \theta + (v_\Delta/v_F)^2 \cos^2 \theta \right] \frac{[x^2 + \sin^2 \theta + (v_\Delta/v_F)^2 \cos^2 \theta]}{[x^2 + \sin^2 \theta + (v_\Delta/v_F)^2 \cos^2 \theta]^2} G_{IVB}(x, \theta), \]
\[ D_3^B = \frac{2(v_\Delta/v_F)}{N_f \pi^3} \int_{-\infty}^{\infty} dx \int_0^{2\pi} d\theta \left[ x^2 + \sin^2 \theta - (v_\Delta/v_F)^2 \cos^2 \theta \right] \frac{[x^2 + \sin^2 \theta + (v_\Delta/v_F)^2 \cos^2 \theta]}{[x^2 + \sin^2 \theta + (v_\Delta/v_F)^2 \cos^2 \theta]^2} G_{IVB}(x, \theta), \]

where the associated functions \( G_1 \), \( G_{III} \), \( G_{IVB} \), and \( G_{IVA} \) are nominated as

\[ G_1^{-1} = \frac{x^2 + \cos^2 \theta}{\sqrt{x^2 + \cos^2 \theta + (v_\Delta/v_F)^2 \sin^2 \theta}} + \frac{x^2 + \sin^2 \theta}{\sqrt{x^2 + \sin^2 \theta + (v_\Delta/v_F)^2 \cos^2 \theta}}, \]
\[ G_{III}^{-1} = \sqrt{x^2 + \cos^2 \theta + (v_\Delta/v_F)^2 \sin^2 \theta} + \sqrt{x^2 + \sin^2 \theta + (v_\Delta/v_F)^2 \cos^2 \theta}, \]
\[ G_{IVA}^{-1} = \frac{x^2 + (v_\Delta/v_F)^2 \sin^2 \theta}{\sqrt{x^2 + \cos^2 \theta + (v_\Delta/v_F)^2 \sin^2 \theta}} + \frac{x^2 + (v_\Delta/v_F)^2 \cos^2 \theta}{\sqrt{x^2 + \sin^2 \theta + (v_\Delta/v_F)^2 \cos^2 \theta}}, \]
\[ G_{IVB}^{-1} = -\sqrt{x^2 + \cos^2 \theta + (v_\Delta/v_F)^2 \sin^2 \theta}, \]
\[ G_{IVB}^{-1} = -\sqrt{x^2 + \sin^2 \theta + (v_\Delta/v_F)^2 \cos^2 \theta}. \]

Appendix B: Coupled RG equations for Type-\( \tau_\nu \), \( \tau_3 \), and \( \tau_0 \)

Besides the coupled RG equations for Type-\( \tau_\nu \) phase transition exhibited in Eqs. (39)-(43), we perform the standard procedures of momentum-shell RG approach\(^{100-106}\) and then deliver the corresponding RG evolutions for other types as follows,

\[ \frac{d v_F}{d l} = \lambda^2 (B_1 - B_2)v_F, \]
\[ \frac{d v_\Delta}{d l} = \lambda^2 (B_1 - B_3)v_\Delta, \]
\[ \frac{d v_F}{d l} = \lambda^2 (B_2 - B_3)\frac{v_\Delta}{v_F}, \]
\[ \frac{d \lambda}{d l} = \left[ 2B_1 + B_2 + B_3 + \frac{[(u_1)^2 + (u_3)^2 - (u_0)^2 - (u_2)^2]}{4\pi v_F v_\Delta} \right] \lambda^3, \]
\[ \frac{d u_0}{d l} = \left\{ -1 + 2B_1 - \frac{(u_1 u_2 + u_2 u_3)}{4\pi v_F v_\Delta u_0} + \frac{2\lambda^2}{3} \left[ -4B_1 - \frac{u_1}{u_0} (B_1 + B_3) - \frac{u_3}{u_0} (B_1 + B_2) \right] \right\} u_0, \]
\[ \frac{d u_1}{d l} = \left\{ -1 + 2B_1 + \frac{1}{4\pi v_F v_\Delta} \left[ (u_0 - u_1 - u_2 - 2u_3 + 2u_1 u_3) + \frac{2\lambda^2}{3} \left[ \frac{u_3}{u_1} (B_2 + B_3) - 4B_3 \right] \right] \right\} u_1, \]
\[ \frac{d u_2}{d l} = \left\{ -1 + 2B_1 + \frac{1}{4\pi v_F v_\Delta} \left[ (2u_0 - 3u_1 - 2u_2 - 3u_3) + \frac{2u_1 u_3}{u_2} \right] + \frac{2\lambda^2}{3} \left[ 4(B_1 + B_2 + B_3) - \frac{u_1}{u_2} (B_1 + B_2) \right] - \frac{u_3}{u_2} (B_1 + B_3) \right\} u_2, \]
\[ \frac{du_3}{dl} = \left\{-1 + 2B_1 + \frac{1}{4\pi v_F v_\Delta} \right\} \left[ (u_0 - u_3 - u_1 - 2u_2) + \frac{2u_1 u_2}{u_3} \right] + \frac{2\lambda^2}{3} \left[ \frac{u_1}{u_3} (B_2 + B_3 - 4B_2) \right] \right\} u_3, \tag{B8} \]

for Type-\(\tau_y\),

\[ \frac{dv_F}{dl} = \lambda^2 (C_1 - C_2) v_F, \tag{B9} \]

\[ \frac{dv_\Delta}{dl} = \lambda^2 (C_1 - C_3) v_\Delta, \tag{B10} \]

\[ \frac{dv_F^2}{dl} = \lambda^2 (C_1 - C_2)^2 v_F, \tag{B11} \]

\[ \frac{d\lambda}{dl} = \left[ C_1 - C_2 + \frac{\left[ (u_1)^2 + (u_2)^2 - (u_0)^2 - (u_3)^2 \right]}{8\pi v_F v_\Delta} \right] \lambda^3, \tag{B12} \]

\[ \frac{du_0}{dl} = \left\{-1 + 2C_1 - \frac{u_1 u_2 + u_2 u_3}{4\pi v_F v_\Delta u_0} + \frac{2\lambda^2}{3} \left[ \frac{u_2}{u_0} (C_2 - C_1) - 4C_1 \right] \right\} u_0, \tag{B13} \]

\[ \frac{du_1}{dl} = \left\{-1 + 2C_1 + \frac{1}{4\pi v_F v_\Delta} \left[ u_0 - u_1 - 2u_2 - 3u_3 \right] + \frac{2\lambda^2}{3} \left[ \frac{u_2}{u_1} (C_3 - C_2) - C_1 - 5C_3 \right] \right\} u_1, \tag{B14} \]

\[ \frac{du_2}{dl} = \left\{-1 + 2C_1 + \frac{1}{4\pi v_F v_\Delta} \left[ 2u_0 - 3u_1 - 2u_2 - 3u_3 \right] + \frac{2\lambda^2}{3} \left[ 4C_2 - 5C_1 - 5C_3 \right] + \frac{u_1}{u_2} (C_3 - C_2) \right\} u_2, \tag{B15} \]

\[ \frac{du_3}{dl} = \left\{-1 + 2C_1 + \frac{1}{4\pi v_F v_\Delta} \left[ u_0 - u_3 - u_1 - 2u_2 \right] + \frac{2u_1 u_2}{u_3} \right] + \frac{2\lambda^2}{3} \left[ \frac{u_0}{u_3} (C_2 - C_1) - 4C_2 \right] \right\} u_3, \tag{B16} \]

pointing out that the Yukawa coupling \(\lambda\) is still marginal to the one-loop level and hence does not flow with the decrease of energy scale. Given the fermion-fermion interactions can only indirectly influence the fermion velocities and accompanied physical implications via modifying such Yukawa coupling, we henceforth can safely skip the effects caused by fermion-fermion interactions, in other words neglecting the one-loop RG equations of the fermion-fermion interactions.

for Type-\(\tau_z\), and

\[ \frac{dv_F}{dl} = \lambda^2 (D_1^{A,B} - D_2^{A,B}) v_F, \tag{B17} \]

\[ \frac{dv_\Delta}{dl} = \lambda^2 (D_1^{A,B} - D_3^{A,B}) v_\Delta, \tag{B18} \]

\[ \frac{dv_F^2}{dl} = \lambda^2 (D_2^{A,B} - D_3^{A,B}) v_F, \tag{B19} \]

\[ \frac{d\lambda}{dl} = (D_1^{A,B} - D_3^{A,B}) \lambda^3 = 0, \tag{B20} \]

for Type-\(\tau_0\) phase transitions, respectively. It is worth

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