EQUIVALENT NOTIONS OF NORMAL QUANTUM
SUBGROUPS, COMPACT QUANTUM GROUPS WITH
PROPERTIES $F$ AND $FD$, AND OTHER APPLICATIONS

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Abstract. The notion of normal quantum subgroup introduced in al-
gebraic context by Parshall and Wang when applied to compact quan-
tum groups is shown to be equivalent to the notion of normal quantum
subgroup introduced by the author. As applications, a quantum analog
of the third fundamental isomorphism theorem for groups is obtained,
which is used along with the equivalence theorem to obtain results on
structure of quantum groups with property $F$ and quantum groups with
property $FD$. Other results on normal quantum subgroups for tensor
products, free products and crossed products are also proved.

1. Introduction

The notion of normal quantum subgroup is an important and subtle con-
cept in the theory of quantum groups. In purely algebraic context of Hopf
algebras, B. Parshall and J. Wang [17] defined a notion of normal quantum
subgroup using left and right adjoint coactions of the Hopf algebra on itself,
which was further studied by other authors such as Schneider [20], Takeuchi
[24], and Andruskiewitsch and Devoto [1]. Parshall and Wang noted that
left normal quantum groups may not be right normal in general, and given a
normal quantum subgroup in their sense, it is not known whether there exists
an associated exact sequence, and if an exact sequence exists, it may not be
unique. These difficulties are peculiar phenomena of general Hopf algebras
in purely algebraic context distinguishing Hopf algebras from groups. Other
complications related to the notion of normal quantum groups in purely
algebraic context are included in [20]. In $C^*$-algebraic context, the author
introduced [28] a notion of normal quantum subgroup of compact quantum

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groups using analytical properties of representation theory of compact quantum groups. It was not known whether these two notions of normality are equivalent when they are applied to the canonical dense Hopf $*$-algebras of quantum representative functions of compact quantum groups. In [33], the author’s notion of normal quantum groups was used in an essential way to define the notion of simple compact quantum groups. It was also announced in [33] without proof that the above two notions of normality are equivalent for compact quantum groups (see remark (b) after Lemma 4.4 in [33]).

As consequences, left normal and right normal defined in algebraic context by Parshall and Wang are also equivalent for compact quantum groups, and their normal quantum subgroups always give rise to a unique exact sequence. That is, the complications mentioned above in purely algebraic setting do not present themselves in the world of compact quantum groups. Such properties might be useful for formulating an appropriate notion of quantum groups in algebraic setting, which is still an open problem.

Other facts announced in [33] without proofs include general results on structure of compact quantum groups with property $F$ (resp. property $FD$), where, roughly speaking, a compact quantum group $G$ is said to have property $F$ if its quantum function algebra $A_G$ has the same property with respect to quotients by normal quantum subgroups as the function algebra of a compact group, and it is said to have property $FD$ if its quantum function algebra has the same property with respect to quotients by normal quantum subgroups as the quantum function algebra of the dual of a discrete group. See Definition 4.2 below for precise definitions of these concepts and notation used above. Compact quantum groups with property $F$ include all quantum groups obtained from compact Lie groups by deformation method, such as compact real form of the Drinfeld-Jimbo quantum groups and Rieffel’s deformation, as well as most of the universal quantum groups constructed by the author except the universal unitary quantum groups $A_u(Q)$ (also called the free unitary quantum groups), cf. [33].

The purposes of this paper are to give complete proof of the equivalence of the two notions of normality mentioned above and give the following applications of this Equivalence Theorem on the structure of compact quantum groups.

(1) We establish a complete quantum analog of the Third Fundamental Isomorphism Theorem. This is the only one among the three fundamental isomorphism theorems that has a complete quantum analog without added
conditions or restrictions. On the contrary, a surjection of compact quantum groups (i.e. inclusion of Woronowicz C*-algebras) does not always give rise to a quantum analog of the First Fundamental Isomorphism Theorem, except in the special case where an exact sequence can be constructed, cf. [17, 20, 24, 1] for this and other subtleties. Taking the example of the group C*-algebra $A_G := C^*(F_2)$ of the free group $F_2$ on two generators, a Woronowicz C*-subalgebra of $A_G$ does not give rise to an exact sequence unless it is the group C*-algebra of a normal subgroup of $F_2$. In addition, it is not clear at the moment how a quantum analog of the second fundamental isomorphism theorem can be formulated.

(2) Using the Equivalence Theorem and the quantum analog of the Third Fundamental Isomorphism Theorem, we show that quotient quantum groups of a compact quantum group with property $F$ also have property $F$, and quantum subgroups of a compact quantum group with property $FD$ also have property $FD$. We show that quotient quantum groups of a compact quantum group with property $FD$ also have property $FD$ provided $G$ has the pullback property. The pullback property is the quantum group version of the group situation in which every subgroup of $G/N$ is of the form $H/N$ for some subgroup $H$ of $G$ containing $N$. We give an example to show not all compact quantum groups have the pullback property.

(3) We prove results on normal quantum subgroups for tensor products, free products and crossed products. Note that the free product construction has no place in the classical world of compact groups. It is a total quantum phenomenon.

An outline of the paper is as follows. In Section 2, we recall the algebraic notion of normal quantum subgroups in [17] and the analytical notion of normal quantum subgroups in [28] respectively. In Section 3, the equivalence of these two notions of normality is proved. In Section 4, as applications of the Equivalence Theorem, we prove the quantum analog of the Third Fundamental Isomorphism Theorem, and results on structure of compact quantum groups with property $F$ and property $FD$. In Section 5, as further applications, properties of normal quantum subgroups for free products, tensor products and crossed products are given.

We note that most results in this paper, such as Theorem 2.7 and those in Sections 4 and 5, are also valid for cosemisimple Hopf algebras when they are appropriately re-formulated. For instance, one simply replaces the statement in (1) of Theorem 2.7 with the equality in (3)' of Proposition 3.2 for such a reformulation. The existence of the Haar integral/measure shared
by both compact quantum groups and cosemisimple Hopf algebras is a key element in the proofs of these results.

Besides the general abstract theory on compact quantum groups developed by Woronowicz and general constructions of particular classes of compact quantum groups, there seem to be few general results on the structure of infinite compact quantum groups in the literature with the possible exception of [10], a situation contrary to finite quantum groups for which there is much literature on their structure and classification. The results in sections 4 and 5 are a modest attempt at developing theory on structure of infinite compact quantum groups. It is expected that such results will be useful in the program [33] of classification of simple compact quantum groups and further study of the structure of compact quantum groups.

Convention: We use the notation and terminology in [28]. For a compact quantum group $G$, $A_G$ denotes the underlying Woronowicz $\mathcal{C}^*$-algebra and $A_G$ the associated canonical dense Hopf $\mathcal{C}^*$-algebra of quantum representative functions on $G$. Sometimes it is convenient to abuse the notation by calling $A_G$ a compact quantum group, referring to $G$. As was pointed out on p.533 of [30], morphisms between quantum groups are meaningful only for full Woronowicz $\mathcal{C}^*$-algebras $A_G$ (i.e. restriction of the norm $\| \cdot \|$ of $A_G$ to the $\mathcal{C}^*$-algebra $A_G$ is the maximum of all possible $\mathcal{C}^*$-norm on $A_G$), although one can define morphisms between arbitrary Woronowicz $\mathcal{C}^*$-algebras (cf. 2.3 in [28]). Unless otherwise explicitly stated, we assume that all Woronowicz $\mathcal{C}^*$-algebras considered in this paper to be full. We also use standard notation in Hopf algebras, including Sweedler’s summation convention [21 16], and $\Delta$, $\varepsilon$, $S$ for coproduct, counit and antipode, respectively. Relevant basic information on compact quantum groups and Hopf algebras can also be found in [14].

2. Two Notions of Normal Quantum Subgroups and Their Equivalence

We recall the two notions of normal quantum subgroups defined by the author in [28] analytically and by Parshall and Wang in [17] algebraically.

Definition 2.1. (cf. 2.3 and 2.13 in [28]) A quantum subgroup of a compact quantum group $G$ in the sense of [28] is a pair $(N, \pi)$, where $A_N$ is a Woronowicz $\mathcal{C}^*$-algebra and $\pi : A_G \longrightarrow A_N$ is a surjection of $\mathcal{C}^*$-algebras that satisfies

$$ (\pi \otimes \pi) \Delta_G = \Delta_N \pi, \quad (2.1) $$
where $\Delta_G$ and $\Delta_N$ are the coproducts of $A_G$ and $A_N$ respectively.

It can be shown (cf. 2.9 and 2.11 in [28] or 1.3.9 and 1.3.9 in [27]) that $(N, \pi)$ is a quantum subgroup of $G$ if and only if the kernel $\ker(\pi)$ (denoted by $I$) of $\pi$ is a Woronowicz $C^*$-ideal of $A_G$ in the sense that it is a $C^*$-ideal of $A_G$ that satisfies

$$\Delta_G(I) \subset \ker(\pi \otimes \pi). \quad (2.2)$$

When there is possible confusion, such as when referring to kernel of the morphism in Lemma 3.3 and when comparing analytically defined quantum subgroup with its algebraically defined counterpart, we use $\hat{\pi} : A_G \to A_N$ to denote the restriction of $\pi$ that maps the dense algebra $A_G$ of quantum representative functions on $G$ onto that $A_N$ on $N$. The quantum group $(N, \pi)$ should be more precisely called a closed quantum subgroup, but we will omit the word closed since we do not consider non-closed quantum subgroups.

For convenience of readers who are familiar with the language of comodules of Hopf algebras but less so with the notion of a finite dimensional representation of a compact quantum group $G$, we recall the definition of the latter. For definition of infinite dimensional (unitary) representations, see 2.4 in [28] or Section 3 in [36].

**Definition 2.2.** (cf. 2.1 in [35]) A representation of dimensional $d$ of a compact quantum group $G$ is an invertible element $v$ of the algebra $M_d(A_G)$ of $d \times d$ matrices with entries in $A_G$ such that

$$\Delta_G(v_{ij}) = \sum_{k=1}^{d} v_{ik} \otimes v_{kj}. \quad (2.3)$$

As shown in Proposition 13 on p30 in [14] and Proposition 3.2 in [35], finite dimensional representations of $G$ and comodules of the Hopf algebra $A_G$ are in natural one to one correspondence. Note that representation in the sense above is called non-degenerate representation in [35].

The algebra $M_d(A_G)$ is also written as $M_d(\mathbb{C}) \otimes A$, where $M_d(\mathbb{C})$ is the algebra of $d \times d$ matrices with entries in complex numbers $\mathbb{C}$. For this reason, for a linear map $\tau$ from $A_G$ to another vector space, the matrix $(\tau(v_{ij}))_{i,j=1}^{d}$ is often written in three different ways interchangeably with slight abuse of notation:

$$(\tau(v_{ij}))_{i,j=1}^{d} = (id \otimes \tau)(v) = \tau(v).$$
Definition 2.3. (cf. 2.13 in [28]) A quantum subgroup \((N, \pi)\) of \(G\) is called normal if for every irreducible representation \(u^\lambda\) of \(G\), the multiplicity of the trivial representation of \(N\) in \(\pi(u^\lambda)\) is equal to either zero or the dimension of \(u^\lambda\).

Let \(h_N\) be the Haar measure (also called Haar state or Haar integral) on \(N\). Then it is clear that \(N\) is normal if and only if for every irreducible representation \(u^\lambda\) of \(G\),

\[
either h_N \pi(u^\lambda) = I_{d_\lambda}, \text{ or } h_N \pi(u^\lambda) = 0,
\]

where \(d_\lambda\) is the dimension of \(u^\lambda\), and \(I_{d_\lambda}\) is the \(d_\lambda \times d_\lambda\) identity matrix.

We recall the definition of normal quantum subgroup in Parshall and Wang [17] adapted to Hopf *-algebras of the form \(A_G\) where \(G\) is a compact quantum group, though their definition applies to more general Hopf algebras.

Definition 2.4. (cf. 1.4 in [17]) An algebraic quantum subgroup of a compact group \(G\) is a pair \((N, \eta)\) where \(N\) is a compact quantum \(N\) and \(\eta : A_G \rightarrow A_N\) is a surjection of *-algebras that satisfies

\[
(\eta \otimes \eta) \Delta_G = \Delta_N \eta, \tag{2.4}
\]
\[
\varepsilon_G = \varepsilon_N \eta, \quad \eta S_G = S_N \eta, \tag{2.5}
\]

where \(\eta_G\) and \(\eta_N\) (resp. \(S_G\) and \(S_N\)) are the counits (resp. antipodes) of \(A_G\) and \(A_N\) respectively.

It is clear that \((N, \eta)\) is an algebraic quantum subgroup of \(G\) if and only if the kernel \(\text{ker}(\eta)\) (denoted by \(\mathcal{I}\)) of \(\eta\) is a Hopf *-ideal of \(A_G\) in the sense that it is a *-ideal of \(A_G\) that satisfies (cf. 1.4 in [17])

\[
\Delta_G(\mathcal{I}) \subset A_G \otimes \mathcal{I} + \mathcal{I} \otimes A_G, \tag{2.6}
\]
\[
\varepsilon_G(\mathcal{I}) = 0, \quad S_G(\mathcal{I}) \subset \mathcal{I}. \tag{2.7}
\]

In [17], the morphism \(\eta\) is not required to preserve the *-algebra structure and the ideal \(\mathcal{I}\) is not required to be a *-ideal. Since we restrict attention to compact quantum groups, we need to require both.

Using Woronowicz’s Peter-Weyl theory for compact quantum groups, one can easily show (cf. 2.10 of [28] and details in 1.2.16 of [27]) that if \((N, \pi)\) is a quantum subgroup of \(G\) in the sense of Definition 2.1 then \((N, \hat{\pi})\) is an algebraic quantum subgroup of \(G\). In particular, the counits and antipodes of the associated dense Hopf subalgebras are automatically preserved, i.e. conditions in (2.5) (resp. 2.7) automatically follow from (2.4) (resp. (2.6)),
which are postulated in 1.4 of Parshall and Wang [17]. However it must be cautioned that if \((N, \eta)\) is an algebraic quantum subgroup of \(G\) in the original sense of [17] without \(\eta\) preserving the \(*\)-structures of \(A_G\) and \(A_N\) and both \(G\) and \(N\) are compact, there is no morphism of compact quantum groups \(\pi : A_G \longrightarrow A_N\) with \(\hat{\pi} = \eta\).

The precise correspondence between analytical quantum subgroups in Definition 2.1 and algebraic quantum subgroups of \(G\) in Definition 2.4 is given by the following theorem (see 4.3.(2) in [33]), which essentially says that the two notions are equivalent. It is the first step that reduces the \(C^*\)-setting to algebraic setting for the proof of the equivalence theorem on normality:

**Theorem 2.5.** The map \(f(I) = \overline{I}\) is a bijection from the set of Hopf \(*\)-ideals \(\{I\}\) of \(A_G\) onto the set of Woronowicz \(C^*\)-ideals \(\{I\}\) of \(A_G\) with full quotient Woronowicz \(C^*\)-algebra \(A_G/I\). The inverse \(g\) of \(f\) is given by \(g(I) = I \cap A_G\).

**Remarks:** A detailed proof of the above theorem is given in [33]. We note that its proof is a nice interplay between the algebraic and analytical properties of compact quantum groups. We also note that many concrete constructions in the analytical \(C^*\)-algebraic context also have purely algebraic formulation, such as the quantum permutation groups in [31] and their algebraic counterpart in Bichon [3, 4]. The theory of compact quantum groups is a rich ground where algebraic aspects and analytical aspects pleasantly interplay with each other.

Thanks to Theorem 2.5, we can now focus on the algebraic object \(A_G\). Let \(a \in A_G\). The left and right adjoint coactions are defined respectively by

\[
\text{ad}_l(a) := \sum a_{(2)} \otimes a_{(1)} S(a_{(3)}), \quad \text{ad}_r(a) := \sum a_{(2)} \otimes S(a_{(1)}) a_{(3)},
\]

where \(S\) is the antipode of the Hopf algebra \(A_G\) and Sweedler’s notation [21] is used:

\[
(\Delta \otimes id)\Delta(a) = (id \otimes \Delta)\Delta(a) = \sum a_{(1)} \otimes a_{(2)} \otimes a_{(3)}.
\]

**Definition 2.6.** (cf. Definition 1.5 in [17]) An algebraic quantum subgroup \((N, \eta)\) of \(G\) is called **a-normal** if \(\ker(\eta)\) is a normal Hopf ideal of \(A_G\) in the sense that the following two conditions are satisfied,

\[
\text{ad}_l(a) \in \ker(\eta) \otimes A_G, \quad \text{and} \quad \text{ad}_r(a) \in \ker(\eta) \otimes A_G
\]

for all \(a \in \ker(\eta)\).
To compare with our Definition 2.3 for the time being we use the term \textit{a-normal} instead of normal for the situation considered by Parshall and Wang [17]. Following their paper we call $(N, \eta)$ \textbf{left a-normal} (resp. \textbf{right a-normal}) if the first (resp. second) condition in Definition 2.6 above is satisfied. In Schneider [20], a morphism such as $\eta$ used in Definition 2.6 above is also called a conormal morphism. In 1.1.7 of Andruskiewitsch and Devoto [1], $A_N$ is said be a right quotient $A_G$ comodule if the second condition above is satisfied, because the comodule structure $ad_r$ can then be induced to the quotient $A_G/\ker(\eta)$, which is $A_N$.

Our first goal in this paper is to prove the following Equivalence Theorem.

\textbf{Theorem 2.7.} Let $(N, \pi)$ be a quantum subgroup of a compact quantum group $G$. Then the following conditions are equivalent.

1. $(N, \pi)$ is normal.
2. $(N, \hat{\pi})$ is a-normal.
3. $(N, \hat{\pi})$ is left a-normal.
4. $(N, \hat{\pi})$ is right a-normal.

The proof is given in the next section.

3. Proof of Theorem 2.7

For convenience of the reader, we recall the notations to be used below. Define

$$A_{G/N} = \{ a \in A_G | (id \otimes \pi)\Delta(a) = a \otimes 1_N \},$$
$$A_{N \backslash G} = \{ a \in A_G | (\pi \otimes id)\Delta(a) = 1_N \otimes a \},$$

where $\Delta$ is the coproduct on $A_G$, $id$ is the identity map on $A_G$, and $1_N$ is the unit of the algebra $A_N$, which will simply be denoted by 1 when the context is clear. Similarly, we define

$$A_{G/N} = A_G \cap A_{G/N}, \quad A_{N \backslash G} = A_G \cap A_{N \backslash G}.$$

Note that $G/N$ and $N \backslash G$ should be denoted more precisely by $G/(N, \pi)$ and $(N, \pi) \backslash G$ respectively if there is a possible confusion. Let $h_N$ be the Haar measure on $N$. Let

$$E_{G/N} = (id \otimes h_N \pi)\Delta, \quad E_{N \backslash G} = (h_N \pi \otimes id)\Delta.$$

Then $E_{G/N}$ and $E_{N \backslash G}$ are projections of norm one (completely positive and completely bounded conditional expectations) from $A_G$ onto $A_{N \backslash G}$ and
The proposition below follows immediately from the above considerations.

**Proposition 3.1.** The *-subalgebras $A_{N\setminus G}$ and $A_{G/N}$ are dense in $A_{N\setminus G}$ and $A_{G/N}$ respectively under the norm of $A_G$.

From Proposition 3.1 we have following slight reformulation of Proposition 2.1 in [33]:

**Proposition 3.2.** Let $N$ be a quantum subgroup of a compact quantum group $G$. Then the following conditions are equivalent:

1. $A_{N\setminus G}$ is a Woronowicz $C^*$-subalgebra of $A_G$.
1'. $A_{N\setminus G}$ is a Hopf *-subalgebra of $A_G$.
2. $A_{G/N}$ is a Woronowicz $C^*$-subalgebra of $A_G$.
2'. $A_{G/N}$ is a Hopf *-subalgebra of $A_G$.
3. $A_{G/N} = A_{N\setminus G}$.
3'. $A_{G/N} = A_{N\setminus G}$.
4. $N$ is normal.

Because of Theorem 2.5, Proposition 3.1 and Proposition 3.2 we may (and will) work exclusively with the dense Hopf *-algebras of Woronowicz $C^*$-algebras from now on unless otherwise specified. As remarked after the proof of Proposition 2.1 in [33], the counit of $A_{G/N}$ is equal to the restriction morphism $\pi|_{A_{G/N}}$.

As usual, if $H$ is a Hopf algebra, $H^+$ denotes the augmentation ideal (i.e. $H^+$ is kernel of the counit $\varepsilon$ of $H$). Assume $N$ is a normal quantum subgroup of a compact quantum group $G$. Then we have a Hopf *-algebra $A_{G/N}$ and its augmentation ideal $A_{G/N}^+$.

Lemma 3.3 below is a key ingredient in the proof of Theorem 2.7 and it plays an important role in [33] and Theorem 4.3 below. In the case of an ordinary compact group $G$, its geometric meaning is the trivial fact that a normal subgroup $N$ of $G$ is the inverse image of the identity element in $G/N$ under the quotient map. However, in the case of quantum groups using the Hopf algebra language, it is non-trivial to prove, especially because of related complications concerning the notion of normality for arbitrary Hopf algebras such as Example 1.2 in Schneider [20]. Lemma 3.3 is a consequence of Takeuchi’s Theorem 2 in [23], as pointed out to us by the referee, because short exact sequences of comodules over cosemisimple Hopf algebras are
always split (cf. Theorem 3.1.5 of [11]), from which one immediately sees
using for instance 1.2.11(b) of [1] that $A_G$ is faithfully coflat over $A_N$ when
$N$ is normal quantum subgroup of a compact quantum group $G$, thus the
condition of Theorem 2 in [23] is fulfilled. We note that cosemisimple Hopf
algebras in general are not faithfully coflat over its quotient Hopf algebras
if the latter is not cosemisimple, as Chirvasitu shows by an example in [6].
Our quotient Hopf algebra $A_N$ is, however, cosemisimple, and the pathology
in Chirvasitu’s example does not occur.

Without using of faithfully coflatness and the above references, an outline
of another proof of Lemma 3.3 is sketched in Lemma 4.4 in [33]. Because
of its usefulness and for the convenience of the reader, we include here a
detailed and self-contained proof following the lines in [33] (cf. 16.0.2 in
Sweedler [21] and (4.21) in Childs [9] for finite dimensional case).

Lemma 3.3. (Reconstruction of $N$ from identity in $G/N$)
Let $(N, \pi)$ be a normal quantum subgroup of a compact quantum group $G$.
Let $\hat{\pi} = \pi|_{A_G}$ be the associated morphism from $A_G$ to $A_N$. Then,

$$\ker(\hat{\pi}) = A_G^+A_G = A_GA_G^+ = A_GA_G^+ = A_GA_G^+ = A_GA_G^+.$$

Proof. It suffices to prove $\ker(\hat{\pi}) = A_G^+A_G$, as we will have equality
$\ker(\hat{\pi}) = A_GA_G^+$ by the same method, and these will imply that

$$A_GA_G^+A_G = A_G^+A_GA_G = A_G^+A_G = \ker(\hat{\pi}) = A_GA_G^+.$$

Consider the right $A_N$-comodule structures $\alpha$ and $\beta$ on $A_G$ and $A_N$ defined respectively by

$$\alpha = (id \otimes \hat{\pi})\Delta_G : A_G \rightarrow A_G \otimes A_N,$$

$$\beta = \Delta_N : A_N \rightarrow A_N \otimes A_N,$$

where $\Delta_G$ and $\Delta_N$ are respectively the coproducts of the Hopf algebras
$A_G$ and $A_N$. Since $\hat{\pi}$ is compatible with the coproducts, one verifies that
$(\hat{\pi} \otimes id)\alpha = \beta \hat{\pi}$. That is, the surjection $\hat{\pi}$ is a morphism of $A_N$-comodules
from $A_G$ to $A_N$. The Hopf algebra $A_N$ is cosemisimple by the fundamental
work of Woronowicz [35] (see remarks in 2.2 of [28] which assures work in
[35] is valid for all compact quantum groups without separability assumption
on the underlying $C^*$-algebra $A_G$ because of Van Daele’s theorem [25] on
the Haar measure based on [35, 36]). Therefore it follows from Theorem
3.1.5 of [11] that every $A_N$-comodule is projective. Hence $\hat{\pi}$ has a comodule
splitting $s : A_N \rightarrow A_G$ with $\hat{\pi}s = id_{A_N}$. 

Let $x \in A_{G/N}^+$. It is straightforward to verify that $\pi|_{A_{G/N}}$ is the counit of $A_{G/N}$ (cf. remark (a) following Definition 2.2 in [33]). Hence $\hat{\pi}(x) = 0$, and therefore $A_{G/N}^+ A \subset \ker(\hat{\pi})$. It remains to show that $\ker(\hat{\pi}) \subset A_{G/N}^+ A_G$.

Define a linear map $\phi$ on $A_G$ by $\phi = (s\hat{\pi}) * S = m(s\hat{\pi} \otimes S)\Delta_G$, where $m$ and $S$ are respectively the multiplication map and antipodal map of $A_G$. We show that $\phi(A_G) \subset A_{G/N}$. To see this, let $a \in A_G$. Using the fact that $s$ is a comodule morphism, i.e. $\alpha s = (s \otimes id)\beta$ or $(id \otimes \hat{\pi})\Delta_G s = (s \otimes id)\Delta_N$, along with properties of Hopf algebras morphisms, we obtain

$$(id \otimes \pi)\Delta_G(\phi(a)) = (id \otimes \pi)\Delta_G(\sum s\pi(a(1))S(a(2)))$$

$$= (id \otimes \pi)(\sum \Delta_G(s\pi(a(1)))(S(a(3)) \otimes S(a(2))))$$

$$= \sum (s \otimes id)(\Delta_N(\pi(a(1)))(S(a(3)) \otimes \pi S(a(2))))$$

$$= \sum [s\pi(a(1)) \otimes \pi(a(2))][S(a(4)) \otimes \pi S(a(3))]$$

$$= \sum s\pi(a(1))S(a(3)) \otimes \varepsilon(a(2))$$

$$= \sum s\pi(a(1))S(\varepsilon(a(2))a(3)) \otimes 1 = \phi(a) \otimes 1,$$

which means that $\phi(A_G) \subset A_{G/N}$, where $\varepsilon$ is the co-unit on $A_G$.

Next we observe that

$$s\hat{\pi} = (s\hat{\pi}) * \varepsilon = (s\hat{\pi}) * (S * id) = ((s\hat{\pi}) * S) * id = \phi * id,$$

and

$$\varepsilon \phi(a) = \varepsilon(\sum s\pi(a(1))S(a(2))) = \sum \varepsilon(s\pi(a(1))\varepsilon(S(a(2))))$$

$$= \sum \varepsilon[s\pi(a(1))\varepsilon(a(2)))] = \varepsilon(s\pi(a)) = (\varepsilon_N \pi)(s\pi(a))$$

$$= \varepsilon_N(\pi(a)) = \varepsilon(a),$$

where $\hat{\pi}s = id_{A_N}$ is used. The above means $\varepsilon \phi = \varepsilon$. From these we obtain

$$id - s\hat{\pi} = \varepsilon * id - \phi * id = (\varepsilon - \phi) * id$$

$$= (\varepsilon \phi - \phi) * id = [(\varepsilon - id)\phi] * id.$$

Furthermore $(id - s\hat{\pi})(a) = a$ if $a \in \ker(\hat{\pi})$, we have $\ker(\hat{\pi}) \subset \text{Im}(id - s\hat{\pi})$. Therefore to show $\ker(\hat{\pi}) \subset A_{G/N}^+ A_G$, it suffices to show that $\text{Im}(id - s\hat{\pi}) \subset A_{G/N}^+ A_G$. Since $(\varepsilon - id)\phi(A_G) \subset A_{G/N}^+$ (because $\phi(A_G) \subset A_{G/N}$), the later follows from the identity

$$id - s\hat{\pi} = [(\varepsilon - id)\phi] * id = m((\varepsilon - id)\phi \otimes id)\Delta_G.$$

This proves Lemma [33].
Remarks. Let $\Phi$ and $\Psi$ be the notations of Schneider [20],

$$\Phi(A_{G/N}) := A_G A_{G/N}^+, \quad \text{and} \quad \Psi(\ker(\hat{\pi})) := A_{G/N}.$$  

Lemma 3.3 above can be restated as saying that for a compact quantum group $G$, the map $\Phi$ is the left inverse of $\Psi$, i.e., $\Phi$ is a surjection from the set of normal Hopf subalgebras of $A_G$ onto the set of its normal Hopf ideals. In addition, Chirvasitu recently showed in [6] that the Hopf algebra $A_G$ is faithfully flat over its Hopf subalgebras, as the author had conjectured in an earlier version of this paper and in [33] (cf. first part of Conjecture 1 on p3329 there). Chirvasitu’s result can be used along with those of Schneider [20] to conclude that the map $\Phi$ is also the right inverse of $\Psi$, i.e., $\Phi$ is also an injection, complementing Lemma 3.3. Therefore, the maps $\Phi$ and $\Psi$ are inverses to each other. The first result of this kind is due to Takeuchi [22] for commutative Hopf algebras, using which he gave a purely Hopf algebraic proof of the fundamental theorem of affine algebraic group schemes [12].

In the language of Andruskiewitsch et al [1], Lemma 3.3 implies that the sequence

$$1 \to N \to G \to G/N \to 1,$$

or the sequence

$$\mathbb{C} \to A_{G/N} \to A_G \to A_N \to \mathbb{C},$$

is exact, where the one dimensional Hopf algebra $\mathbb{C}$ is the “zero object” in the category of Hopf algebras. Note that in the purely algebraic situation of Parshall and Wang [17], for a given normal quantum subgroup in their sense (i.e. a-normal as defined in our paper here), the existence of an exact sequence is not known and the uniqueness does not hold in general (cf. 1.6 and 6.3 loc. cit.). Lemma 3.3 above shows that such complications do not present themselves in the world of compact quantum groups: when we have a normal quantum group, we always have a unique exact sequence. This property might help to formulate an appropriate notion of quantum groups in algebraic setting.

Note also that the notion of exact sequence of quantum groups in [20] is equivalent to the notion of strictly exact sequence in [1] under faithful (co)flat conditions, which are fulfilled for cosemisimple Hopf algebras and therefore for compact quantum groups thanks to the theorem of Chirvasitu [6] on faithfully flatness and the remarks before Lemma 3.3 on faithfully coflatness. In general, an arbitrary Hopf algebra need not be faithfully flat.
over its Hopf subalgebras, according to counter examples of Schauenburg [19] constructed in response to Question 3.5.4 of Montgomery [16].

We now prove the first main result of this paper.

**Proof of Theorem 2.7** (cf. Schneider [20]).

(1) ⇒ (2) and therefore (1) ⇒ (3) and (1) ⇒ (4): Let \((N, \hat{\pi})\) be normal, we show that \((N, \hat{\pi})\) is a-normal.

Let \(a \in \ker(\hat{\pi})\). We show that \(\text{ad}_l(a) \in \ker(\hat{\pi}) \otimes A_G\), where

\[
\text{ad}_l(a) = \sum a_{(2)} \otimes a_{(1)} S(a_{(3)}).
\]

By Lemma 3.3

\[
\ker(\hat{\pi}) = A_{G/N}^+ A_G = A_G A_{G/N}^+ = A_G A_{G/N}^+ A_G.
\]

We assume without loss of generality \(a = bc\) for \(b \in A_G\) and \(c \in A_{G/N}^+\). Then modulo \(\ker(\hat{\pi}) \otimes A_G\) we have

\[
\text{ad}_l(a) = \sum b_{(2)} c_{(2)} \otimes b_{(1)} c_{(1)} S(c_{(3)}) S(b_{(3)})
\]

\[
= \sum b_{(2)} \varepsilon(c_{(2)}) \otimes b_{(1)} c_{(1)} S(c_{(3)}) S(b_{(3)})
\]

\[
= \sum b_{(2)} \otimes b_{(1)} \sum (c_{(1)} S(c_{(2)}) S(b_{(3)})
\]

\[
= \sum b_{(2)} \otimes b_{(1)} \varepsilon(c) S(b_{(3)}) = 0,
\]

i.e., \(\text{ad}_l(a) \in \ker(\hat{\pi}) \otimes A_G\), where the property

\[
(\varepsilon - id)(A_{G/N}) \subset A_{G/N}^+ \subset \ker(\hat{\pi})
\]

is used, as well as the counital and antipodal properties.

Similarly, \(\text{ad}_r(a) \in \ker(\hat{\pi}) \otimes A_G\) by assuming \(a = cb\) for \(b \in A_G\) and \(c \in A_{G/N}^+\), where

\[
\text{ad}_r(a) = \sum a_{(2)} \otimes S(a_{(1)}) a_{(3)}.
\]

Hence \(\text{ad}_l(\ker(\hat{\pi})) \subset \ker(\hat{\pi}) \otimes A_G\) and \(\text{ad}_r(\ker(\hat{\pi})) \subset \ker(\hat{\pi}) \otimes A_G\), and \(N\) is a-normal.

(3) ⇒ (1): Assume \((N, \hat{\pi})\) is left a-normal. We prove the equality \(A_{G/N} = A_{N \setminus G}\), hence by equivalence of (3)' and (4) in Proposition 3.2, \(N\) is normal. Another proof of this is in 1.1.7 of [1], so readers familiar with it may skip the proof below. Note that our proof of this equality is for general Hopf algebras with bijective antipode not necessarily associated with compact quantum groups, just as 1.1.7 of [1].

Let \(a \in A_{N \setminus G}\), i.e.,

\[
\sum a_{(1)} \otimes a_{(2)} - 1 \otimes a \in \ker(\hat{\pi}) \otimes A_G.
\]
Applying \((ad_l \otimes id)\) to this and using the condition in the definition of left a-normal, we obtain

\[
\sum a_{(2)} \otimes a_{(1)} S(a_{(3)}) S(a_{(4)}) - 1 \otimes 1 \otimes a \in \ker(\hat{\pi}) \otimes A_G \otimes A_G.
\]

Multiplying the second and the third factors, we obtain

\[
\sum a_{(2)} \otimes a_{(1)} S(a_{(3)}) a_{(4)} - 1 \otimes a \in \ker(\hat{\pi}) \otimes A_G.
\]

By the antipodal and counital properties,

\[
\sum a_{(2)} \otimes a_{(1)} S(a_{(3)}) a_{(4)} = \sum a_{(2)} \otimes a_{(1)} \varepsilon(a_{(3)}) = \sum a_{(2)} \otimes a_{(1)}.
\]

Hence

\[
\sum a_{(2)} \otimes a_{(1)} - 1 \otimes a \in \ker(\hat{\pi}) \otimes A_G, \text{ and therefore}
\]

\[
\sum a_{(1)} \otimes a_{(2)} - a \otimes 1 \in A_G \otimes \ker(\hat{\pi}), \text{ i.e., } a \in A_{G/N}.
\]

That is, we have an inclusion \(A_{N \backslash G} \subset A_{G/N}\). Using definition of \(N \backslash G\) and \(G/N\) and properties of the antipode \(S\) we immediately have that \(S(A_{N \backslash G}) = A_{G/N}\) and \(S(A_{G/N}) = A_{N \backslash G}\). Using this and applying \(S\) to the above inclusion we obtain \(A_{G/N} = S(A_{N \backslash G}) \subset S(A_{G/N}) = A_{N \backslash G}\), i.e., we also have \(A_{G/N} \subset A_{N \backslash G}\). Hence \(A_{G/N} = A_{N \backslash G}\).

(4) ⇒ (1): The proof is similar to (3) ⇒ (1) above.

(2) ⇒ (1): This follows from either (4) ⇒ (1) or (3) ⇒ (1).

\(\square\)

**Remarks.** Although for general Hopf algebra not necessarily associated with compact quantum groups, \(A_{G/N} = A_{N \backslash G}\) follows from *either* \((N, \hat{\pi})\) being left a-normal or right a-normal, if the assertion in Lemma 3.3 is not valid for such Hopf algebra, which is a key ingredient in the proof of the implication (1) ⇒ (2) above, we probably cannot expect left a-normal or right a-normal to follow from \(A_{G/N} = A_{N \backslash G}\). We note that no example seems to be known of an algebraic quantum group that is left a-normal but not right a-normal, or vice versa. We suspect such an example may come from a Hopf algebra not faithfully coflat over its quotient Hopf algebra. We are not aware if an example of the latter has been produced in Hopf algebra literature, which should exist in view of counter examples on faithfully flatness (cf. [19]).

The Equivalence Theorem 2.7 enables results on normality in Hopf algebra literature be applicable to normal quantum subgroups of compact quantum groups in our sense, such as those in [17, 20, 24, 1], noting that conormal in [20] and a-normal are the same concept.
4. Third Fundamental Isomorphism Theorem for Quantum Groups and Properties $F$ and $FD$

In this section, we give several applications of Theorem 2.7. Because of this theorem and other results in Sections 2 and 3, we mostly focus on the dense Hopf $*$-subalgebras associated to compact quantum groups in this section. For ease of notation, we now use undecorated $\pi$ for Hopf algebra morphism $A_G \to A_N$, omitting the hat in $\hat{\pi}$ whenever no confusion arises.

The three fundamental isomorphism theorems in the theory of groups are foundational results on structure of groups. One way naturally expect their analogs to be valid in the theory of quantum groups. Unfortunately, quantum analog of the first fundamental isomorphism theorem is not always true for epimorphism of quantum groups (i.e. injection of Hopf algebras) except for the situation where exact sequence can be constructed, cf. [17, 20, 24, 1]. For instance, not every Woronowicz $C^*$-subalgebra of $A_G$ is of the form $A_G/N$ with $N$ a normal quantum subgroup of $G$. This already fails when $A_G$ is the group $C^*$-algebra $C^*(F_2)$ of the free group $F_2$ on two generators, because a Woronowicz $C^*$-subalgebra of $A_G$ is not of the form $A_G/N$ unless it is the group $C^*$-algebra $C^*(\Gamma)$ of a normal subgroup $\Gamma$ of $F_2$. Finally, it is not clear how a quantum analog of the second fundamental isomorphism theorem can be formulated.

However, on the bright side, as an application of Theorem 2.7, we have the following complete analog of the Third Fundamental Isomorphism Theorem for compact quantum groups.

**Theorem 4.1.** (Third Fundamental Isomorphism Theorem) Let $(N, \pi)$ be a normal quantum subgroup of $G$. Let $(H, \theta)$ be a quantum subgroup of $G$ that contains $(N, \pi)$, i.e., there is a morphism $\pi_1$ from $A_H$ to $A_N$ such that $(N, \pi_1)$ is a quantum subgroup of $H$ with $\pi = \pi_1 \theta$. Then $(N, \pi_1)$ is normal in $H$. If furthermore $H$ is normal in $G$ and letting $\theta' = \theta|_{A_G/N}$, the restriction of $\theta$ to $A_G/N$, then $(H/N, \theta')$ is normal in $G/N$ and

$$A_{(G/N)/(H/N)} = A_{G/H} = A_H \setminus G = A_{(N \setminus H)/(N \setminus G)}.$$

**Proof.** Let $z \in A_H$ be such that $\pi_1(z) = 0$. Assume $z = \theta(x)$ for some $x \in A_G$. Then $\pi(x) = 0$. Since $N$ is a-normal in $G$ by Theorem 2.7, we have

$$\sum \pi(x(2)) \otimes S_G(x(1))x(3) = 0, \quad \sum \pi(x(2)) \otimes x(1)S_G(x(3)) = 0.$$

Hence

$$\sum \pi_1 \theta(x(2)) \otimes \theta(S_G(x(1))x(3)) = 0, \quad \sum \pi_1 \theta(x(2)) \otimes \theta(x(1)S_G(x(3))) = 0.$$
It follows that
\[ \sum \pi_1(z_{(2)}) \otimes S_H(z_{(1)}) z_{(3)} = 0, \quad \sum \pi_1(z_{(2)}) \otimes z_{(1)} S_H(z_{(3)}) = 0. \]
This means that \((\pi_1, N)\) is a-normal and is therefore normal in \(H\) by Theorem 2.7.

Let \(a \in A_{G/N}\). Then it is immediate to verify that \(\theta(a) \in A_{H/N}\), and therefore \(\theta(A_{G/N})\) is contained in \(A_{H/N}\). Conversely, let \(b \in A_{H/N}\). Assume \(b = \theta(a)\) for some \(a \in A_G\). Put
\[ \tilde{a} = E_{G/N}(a) = (id \otimes h_N \pi) \Delta_G(a). \]
Then \(\tilde{a} \in A_{G/N}\) and
\[ b = E_{H/N}(b) = (id \otimes h_{N \pi_1}) \Delta_H(b) \]
by the remarks before Proposition 3.1 applied to \(G/N\) and \(H/N\) respectively. Moreover, we have
\[ \theta(\tilde{a}) = (\theta \otimes h_N \pi) \Delta_G(a) = (\theta \otimes h_{N \pi_1} \theta) \Delta_G(a) = (id \otimes h_N \pi) \Delta_H(\theta(a)) = (id \otimes h_{N \pi_1}) \Delta_H(b) = b. \]
That is \(\theta(\tilde{a}) = b\). Therefore \(\theta(A_{G/N}) = A_{H/N}\).

For ease of notation, let \(G' = G/N\) and \(H' = H/N\). The above shows that \((H', \theta')\) is a quantum subgroup of \(G'\).

Now assume \((H, \theta)\) is normal. If \(a \in A_{G'/H'}\), that is, \((id \otimes \theta') \Delta(a) = a \otimes 1_{H'}\), then it is clear that \(a\) is in \(A_{G/H}\) since \(\theta' = \theta\mid_{A_{G/N}}\) and \(1_{H'} = 1_H\). Conversely, if \(a \in A_{G/H}\), that is \((id \otimes \theta) \Delta(a) = a \otimes 1_H\), then \((id \otimes \pi_1 \theta) \Delta(a) = a \otimes 1_N\). This means that \(a \in A_{G'} = A_{G/N}\). Since the coproduct for the quantum group \(G' = G/N\) is a restriction of \(\Delta\), we have
\[ (id \otimes \theta') \Delta(a) = (id \otimes \theta) \Delta(a) = a \otimes 1_H = a \otimes 1_{H'}. \]
Hence \(a \in A_{G'/H'}\) and \(A_{G'/H'} = A_{G/H}\).

The result is completely proved.

Remark: Instead of an isomorphism such as in \((G/N)/(H/N) \cong G/H\) in group theory, we have exact equalities of quantum function algebras in Theorem 4.1 above.

The proof of Theorem 4.1 actually yields the following stronger result without assuming \((H, \theta)\) to be normal, which should be useful in harmonic analysis on homogeneous spaces.

**Theorem 4.1.** Let \((N, \pi)\) be a normal quantum subgroup of \(G\). Let \((H, \theta)\)
be a (not necessarily normal) quantum subgroup of \( G \) that contains \((N, \pi)\), i.e., there is a morphism \( \pi_1 \) such that \((N, \pi_1)\) is a quantum subgroup of \( H \) with \( \pi = \pi_1 \theta \). Let \( \theta' = \theta|_{A_{G/N}} \), the restriction of \( \theta \) to \( A_{G/N} \). Then \((N, \pi_1)\) is normal in \( H \), \((H/N, \theta')\) is a quantum subgroup of \( G/N \), and
\[ A_{(G/N)/(H/N)} = A_{G/H}, \quad A_{(N\setminus H)\setminus (N\setminus G)} = A_{H\setminus G}. \]

For other applications of Theorem 2.7, we first recall the following properties of compact quantum groups (cf. [33]).

**Definition 4.2.** A compact quantum group \( G \) is said to have **property F** if each Woronowicz \( C^* \)-subalgebra of \( A_G \) is of the form \( A_{G/N} \) for some normal quantum subgroup \( N \) of \( G \); \( G \) is said to have **property FD** if each quantum subgroup of \( G \) is normal.

These notions are motivated by the following facts (see Propositions 2.3 and 2.4 in [33]): if \( G \) is a compact group, then its function algebra \( C(G) = A_G \) has property \( F \); if \( G \) is the dual of a discrete group \( \Gamma \), then its quantum function algebra \( C^*(\Gamma) \) has property \( FD \). Therefore compact quantum groups with property \( F \) are closest to compact groups, while compact quantum groups with property \( FD \) are closest to the compact quantum group dual of discrete groups.

We note that the notions property \( F \) and property \( FD \) above can be defined almost verbatim for all Hopf algebras – one only needs to replace the words “compact quantum group” (resp. “Woronowicz \( C^* \)-subalgebra”) with the words “Hopf algebra” (resp. “Hopf subalgebra”) in the above definition. Moreover, because of Theorem 2.7 and remarks following Lemma 3.3 in Hopf algebra language, \( G \) has property \( F \) if each Hopf subalgebra of \( A_G \) is normal in the sense of 3.4.1 in [16]; it has \( FD \) if each Hopf \( * \)-ideal of \( A_G \) is normal in the sense of 3.4.5 in [16]. It follows from discussions after 3.4.5 in [16] that if \( G \) is a finite quantum group, i.e. \( A_G \) is finite dimensional, \( G \) has property \( F \) if and only if its Pontryagin dual \( G_d \) has property \( FD \), where the Pontryagin dual \( G_d \) is the finite quantum group with quantum function algebra equal to the dual Hopf algebra \( A'_G \) of \( A_G \).

In Franz et al. [13], in different terminology, compact quantum groups with property \( FD \) are called **hamiltonian**, of which quantum groups in the \( DS \) family are a special case. An example of noncommutative and noncommutative quantum group in the \( DS \) family (and therefore an example with property \( FD \)) is given in section 6 of [13].

In our earlier work [33], it is shown that all the quantum groups obtained by deformation of compact Lie groups, such as the compact real forms of
Drinfeld-Jimbo quantum group and Rieffel’s deformation of compact Lie groups, and all universal quantum groups [28, 26, 31] (except $A_u(Q)$) have property $F$. These are non-trivial and natural examples of compact quantum groups that have property $F$. Despite this multitude of examples, as discussed in [33, 34], much more is to be explored and it would be important to develop a classification theory for simple quantum groups with property $F$.

Furthermore, we have following result on the structure of compact quantum group with property $F$.

**Theorem 4.3.** Quotient group $G/N$ by a normal quantum subgroup $(N, \pi)$ has property $F$ if $G$ is a compact quantum group with property $F$.

**Proof.** Let $(N, \pi)$ be a normal quantum subgroup of $G$ and let $C \subset \mathcal{A}_{G/N}$ be a Hopf subalgebra of $\mathcal{A}_{G/N}$. We show that there is a normal quantum subgroup $(K, \theta')$ of $G/N$ such that $C = \mathcal{A}_{(G/N)/K}$. The proof below is suggested by the referee, replacing our original longer and direct proof without using Theorem 4.1 and Lemma 3.3.

As $C \subset \mathcal{A}_{G/N}$ is also a Hopf subalgebra of $\mathcal{A}_{G}$, by property $F$ of $G$, let $(H, \theta)$ be a normal quantum subgroup of $G$ such that $C = \mathcal{A}_{G/H}$. By Theorem 4.1, it is enough to show that $(H, \theta)$ contains $(N, \pi)$, i.e. $\ker(\theta) \subset \ker(\pi)$, because then the normal quantum subgroup $(K, \theta')$ of $G/N$ we need is simply $(H/N, \theta')$ in Theorem 4.1 $C = \mathcal{A}_{G/H} = \mathcal{A}_{(G/N)/(H/N)}$.

By Lemma 3.3 (noting that we have omitted the hats to simplify notation) we have

$$\ker(\theta) = \mathcal{A}_{G} \mathcal{A}_{G/H}^+ \mathcal{A}_{G}, \quad \ker(\pi) = \mathcal{A}_{G} \mathcal{A}_{G/N}^+ \mathcal{A}_{G}.$$

Since $\mathcal{A}_{G/H} \subset \mathcal{A}_{G/N}$, we have $\ker(\theta) \subset \ker(\pi)$. □

**Remarks.**

(1) By considering formal dual to Theorem 4.8 (2) below, a quantum subgroup $(H, \theta)$ of a compact quantum group $G$ with property $F$ probably does not always have property $F$, though it does if it is assumed that inverse images of Hopf subalgebras of $\mathcal{A}_H$ under $\theta$ are Hopf subalgebras of $\mathcal{A}_G$. This assumption turns out to be trivial in the sense that $(H, \theta)$ is either the identity group or the full group $G$, as kindly pointed out to us by the referee. Relevant parts of Theorem 3.7 in [33] and Theorem 5.6 in [34] on the same result need to be modified in the form stated in Theorem 4.3 above.

(2) We note that though there are more Woronowicz $C^*$-ideals in $\mathcal{A}_G$ than Hopf $*$-ideals in $\mathcal{A}_G$ in the correspondence of Theorem 2.5 (see Remark
(a) after Lemma 4.3 in [33], Woronowicz \(C^*\)-subalgebras of \(A_G\) and Hopf subalgebras \(B\) of \(A_G\) are in bijective correspondence: every Woronowicz \(C^*\)-subalgebra \(B\) of \(A_G\) uniquely corresponds to its canonical dense Hopf \(*\)-subalgebra \(B\) of \(B\). In connection with this, it would be of interest to answer the following question, which seems to have an affirmative answer according to our preliminary investigation for several special cases.

**Question 4.4.** Is a Woronowicz \(C^*\)-subalgebras of a full Woronowicz \(C^*\)-algebra necessarily full?

A related question is the following one on the relation between a Woronowicz \(C^*\)-ideal of a Woronowicz \(C^*\)-subalgebra and the ideal it generates in the original Woronowicz \(C^*\)-algebra. We formulate two equivalent versions:

**Question 4.5.** Assume \(A\) is a Woronowicz \(C^*\)-algebra and \(A_0\) a Woronowicz \(C^*\)-subalgebra with Hopf \(*\)-subalgebras \(A\) and \(A_0\) respectively.

(a) Let \(I_0\) be a Woronowicz \(C^*\)-ideal of \(A_0\). Let \(I := AI_0A\) be the Woronowicz \(C^*\)-ideal of \(A\) generated by \(I_0\). Is the identity \(I_0 = I \cap A_0\) always true?

(b) Let \(I_0\) be a Hopf \(*\)-ideal of \(A_0\). Let \(I := AI_0A\) be the Hopf \(*\)-ideal of \(A\) generated by \(I_0\). Is the identity \(I_0 = I \cap A_0\) always true?

Parts (a) and (b) of Question 4.5 are equivalent by Theorem 2.5.

It turns out the answer to the above question is affirmative for some quantum groups but negative for others. Before turning to examples and counterexamples kindly provided to us by the referee, we recall the following closely related fact pointed out to us also by the referee.

**Proposition 4.6.** Let \(A_0\) be a subalgebra of an algebra \(A\) over the complex numbers. Then the following are equivalent.

1. \(A\) is left faithfully flat over \(A_0\).
2. \(A\) is left flat over \(A_0\) and for every left ideal \(I_0\) in \(A_0\), \(I_0 = AI_0 \cap A_0\).

See p33 of Bourbaki [5] for a proof. It is clear that a similar result relating right faithfully flatness and right ideal is also valid.

**Example.** By Chirvasitu’s theorem [6], a cosemisimple Hopf algebra is both left and right faithfully flat over its Hopf subalgebras. It follows then from Proposition 4.6 that Question 4.5 has an affirmative answer for compact groups because the relevant algebras are commutative and left ideals are

\footnote{Note that Woronowicz’s fundamental work [33, 35] implies that a Hopf subalgebra of \(A_G\) is automatically closed under the \(*\)-operation since it is invariant under the antipode.}
two sided ideals. In the notation of the question above, here \( A = C(G) \), the continuous function algebra on a compact group \( G \), \( A_0 \) is the algebra of functions on a quotient group \( G/N \) by a normal subgroup, \( I_0 \) defines a subgroup \( H/N \) of the quotient group, and \( I \) defines the pullback \( H \) in \( G \) of \( H/N \) under the quotient map from \( G \) to \( G/N \).

In view of the discussions above, the following definition is natural.

**Definition 4.7.** We say a compact quantum group \( G \) has the **pullback property** if the answer for Question 4.5 is affirmative for \( A = A_G \), equivalently for \( A = A_G \).

Note the above pullback property can be defined for inclusion of rings and (\( C^* \))-algebras not necessarily associated with quantum groups when the words “Hopf \( * \)-ideal” or “Woronowicz \( C^* \)-ideal” in Question 4.5 are substituted by “ideal”. The following example shows that, unlike function algebra over compact groups in the example above, group (\( C^* \))-algebras do not have pullback property in general.

**Counter Example.** Let \( A = A = C \Sigma_3 \) be the group (\( C^* \))-algebra of the symmetric group \( S_3 \) on three symbols, and \( A_0 = A_0 = C \Gamma \cong C \mathbb{Z}/3\mathbb{Z} \) the group (\( C^* \))-algebra of the alternating subgroup \( \Gamma \) of \( S_3 \) generated by the three cycle \((123)\). We claim that there exists an ideal \( I_0 \) in \( A_0 \) for which the answer to Question 4.5 is negative.

To see this, using (non-commutative) Fourier transforms, identify \( A \) with \( \mathbb{C} \oplus \mathbb{C} \oplus B \), and \( A_0 \) with \( \mathbb{C} \oplus B_0 \), where \( B = M_2(\mathbb{C}) \) is the \( 2 \times 2 \) matrix algebra, corresponding to the (unique) two dimensional irreducible representation of \( S_3 \), and \( B_0 = \mathbb{C} \oplus \mathbb{C} \), corresponding to the two non-trivial irreducible one dimensional representations of \( \Gamma \). Since the restriction of the two dimensional irreducible representation of \( S_3 \) to \( \Gamma \) is a direct sum of the two non-trivial irreducible one dimensional representations of \( \Gamma \) (see e.g. [15], p150), we see that under the above Fourier transforms, \( B_0 \) is included as the subalgebra of diagonal matrices in \( B \) (note that \( B \) is not a Hopf algebra). The inclusion \( B_0 \subset B \) does not have pullback property because \( B \) has no nonzero proper ideal and for the nonzero proper ideal \( I_0 := \mathbb{C} \oplus 0 \) in \( B_0 \), \( BI_0B \cap B_0 = B_0 \neq I_0 \). Since \( I_0 \) is also an ideal in \( A_0 \), we have \( AI_0A \cap A_0 = BI_0B \cap B_0 = B_0 \neq I_0 \).

We are ready for the following result on property FD.

**Theorem 4.8.** Let \( G \) be a compact quantum group with property FD. Then

1. quantum subgroup \((H, \theta)\) of \( G \) also has property FD;
(2) quotient group $G/N$ by a normal quantum subgroup $(N, \pi)$ also has property $FD$ provided $G$ has pullback property.

Proof. (1) Let $(H_1, \pi_1)$ be a quantum subgroup of $H$. Then $(H_1, \pi_1\theta)$ is a quantum subgroup of $G$ that is contained in $(H, \theta)$. By property $FD$, $(H_1, \pi_1\theta)$ is normal in $G$. By Theorem 4.1, $(H_1, \pi_1)$ is normal in $H$. This shows that $H$ has property $FD$.

(2) Assume $G$ is a compact quantum group with the pullback property in addition to property $FD$.

Let $(N, \pi)$ be a normal quantum subgroup of $G$ and let $(K, \pi_0)$ be a quantum subgroup of $G/N$. Let $\mathcal{I}_0$ be the kernel of $\pi_0$ in $\mathcal{A}_{G/N}$ and $\mathcal{I}_N$ the kernel of $\pi$ in $\mathcal{A}_G$. Identify $\mathcal{A}_K$ with $(\mathcal{A}_{G/N})/\mathcal{I}_0$. Put $\mathcal{I} := \mathcal{A}\mathcal{I}_0\mathcal{A}$. Then $\mathcal{I}$ is a Hopf $(\ast)$-ideal in $\mathcal{A}_G$ and defines a quantum subgroup $(H, \theta)$ of $G$, where

$$\theta : \mathcal{A}_G \rightarrow \mathcal{A}_{G/\mathcal{I}}, \quad \mathcal{A}_H := \mathcal{A}_{G/\mathcal{I}}.$$

By the co-unital property $\varepsilon_{G/N} = \varepsilon_K\varepsilon_0$ and the fact that $\varepsilon_{G/N}$ is the restriction of $\pi$ to $\mathcal{A}_{G/N}$, we have $\pi = \varepsilon_K\varepsilon_0$ on $\mathcal{A}_{G/N}$. It follows that $\mathcal{I}_0 \subset \mathcal{I}_N$ and therefore we have an inclusion $\mathcal{I} \subset \mathcal{I}_N$. This means that $(H, \theta)$ is a quantum subgroup of $G$ containing $(N, \pi)$ and $(N, \pi_1)$ is normal in $H$ as shown in Theorem 4.1, where $\pi_1$ is defined by $\pi_1(\theta(a)) := \pi(a)$, for $a \in \mathcal{A}_G$. On the other hand, every element in $\mathcal{A}_K$ is of the form $\pi_0(a)$ for some $a \in \mathcal{A}_{G/N}$. If $\pi_0(a) = 0$, then $\theta(a) = 0$ because $\mathcal{I}_0$ is contained in $\mathcal{I} = \mathcal{A}\mathcal{I}_0\mathcal{A}$. Hence $\rho(\pi_0(a)) := \theta(a)$ gives a well defined morphism from $\mathcal{A}_K$ to $\mathcal{A}_H$, where $\pi_0(a) \in \mathcal{A}_K$. It can be checked that $\rho$ is a morphism of Hopf algebras.

We summarize all the morphisms in the following commutative diagram, where horizontal sequences are exact in the sense of [1].

\[
\begin{array}{cccccc}
\mathbb{C} & \longrightarrow & \mathcal{A}_{G/N} & \longrightarrow & \mathcal{A}_G & \longrightarrow & \mathcal{A}_N & \longrightarrow & \mathbb{C} \\
\downarrow{\pi_0} & & \downarrow{\pi} & & \downarrow{\varepsilon_N} & & \\
\mathbb{C} & \longrightarrow & \mathcal{A}_K & \longrightarrow & \mathcal{A}_H & \longrightarrow & \mathcal{A}_N & \longrightarrow & \mathbb{C}
\end{array}
\]

Moreover, the image of $\rho$ is in $\mathcal{A}_{H/N}$ because by property of Hopf algebra morphisms, for $a \in \mathcal{A}_{G/N}$ we have

\[
(id_H \otimes \pi_1)\Delta_H(\rho(\pi_0(a))) = (id_H \otimes \pi_1)\Delta_H(\theta(a))
\]
\[
= (\theta \otimes \pi_1\theta)\Delta_G(a) = (\theta \otimes id_N)(id_G \otimes \pi)\Delta_G(a)
\]
\[
= (\theta \otimes id_N)(a \otimes 1_N) = \theta(a) \otimes 1_N = \rho(\pi_0(a)) \otimes 1_N,
\]
which means $\rho(\pi_0(a)) \in A_{H/N}$.

In addition, if $\theta(a) = 0$ for some $a$ in $A_{G/N}$, then $a$ is $\mathcal{I} \cap A_{G/N}$, which is $\mathcal{I}_0$ by assumption, and $\pi_0(a) = 0$. Hence $\ker(\rho) = 0$ and $\rho$ is an injection. As in the proof of Theorem 4.1, $\theta(A_{G/N}) = A_{H/N}$. Therefore, $\rho$ is also a surjection onto $A_{H/N}$. Hence we have an identity (and isomorphism) $\rho(A_K) = A_{H/N}$.

Since $H$ is normal in $G$ by property FD of $G$, Theorem 4.1 guarantees that $K$ (which is $H/N$) is normal in $G/N$ and the proof is complete. \hfill \square

Remark. An examination of the proofs of Theorem 2.7 and results in this and the next section shows that they are also valid for cosemisimple Hopf algebras when the statements are appropriately modified.

5. Other Properties of Normal Quantum Subgroups

Tensor products of $C^*$-algebras is the analog of product of locally compact spaces. Unlike the classical situation of spaces, the algebraic tensor product $A_1 \otimes_{alg} A_2$ of two $C^*$-algebras $A_1$ and $A_2$ may have more than one $C^*$-norm. It has two canonical $C^*$-norms that may not agree, the maximal one and the minimal one. Its $C^*$-algebraic completions under these two norms are denoted respectively by $A_1 \otimes_{max} A_2$ and $A_1 \otimes_{min} A_2$. The maximal tensor product $A_1 \otimes_{max} A_2$ of two full Woronowicz $C^*$-algebras $A_1 = A_{G_1}$ and $A_2 = A_{G_2}$ is also full [29]. We use $G_1 \times G_2$ to denote the corresponding compact quantum group.

Proposition 5.1. Let $G_1, G_2$ be compact quantum groups with normal quantum subgroups $(N_1, \pi_1)$ and $(N_2, \pi_2)$. Then $N := (N_1 \times N_2, \pi_1 \otimes \pi_2)$ is a normal quantum subgroup of $G := G_1 \times G_2$ and $G/N \cong G_1/N_1 \times G_2/N_2$, where $\pi_1 \otimes \pi_2$ denotes the corresponding tensor product morphism

$$\pi_1 \otimes \pi_2 : A_{G_1} \otimes_{max} A_{G_2} \longrightarrow A_{N_1} \otimes_{max} A_{N_2}.$$ 

Proof. According to §2 and §3 and using the formula of the Haar measure on $G_1 \times G_2$ and formula for the coproduct of the tensor product in [29], one obtains immediately

$$A_{G/N} = E_{G/N}(A_{G_1} \otimes A_{G_2}) = A_{G_1/N_1} \otimes A_{G_2/N_2}.$$ 

This is a Hopf subalgebra of $A_{G_1} \otimes A_{G_2}$. Hence the proposition follows from the equivalence of (3)' and (4) in Proposition 3.2. \hfill \square

An appropriate reformulation of Proposition 5.1 for minimal tensor product is also valid when one of the $C^*$-algebras $A_{G_1}$, $A_{G_2}$, $A_{N_1}$ and $A_{N_2}$ is
exact. We will not elaborate on this except recalling that a $C^*$-algebra $A$ is called **exact** if the functor $A \otimes_{\min} \cdot$ preserves short exact sequences.

For free product [28], the situation is quite different from tensor product. The naive analog of Proposition 5.1 is false, even for the simple case with $\pi_1 = id_1$ and $\pi_2 = \varepsilon_2$, as described in the following proposition.

**Proposition 5.2.** Let $G_1, G_2$ be compact quantum groups (not necessarily duals of discrete groups). Let $\hat{G} = \hat{G}_1 \ast \hat{G}_2$ be the free product compact quantum group underlying $A_{G_1} \ast A_{G_2}$. Let $\pi$ be the natural embedding of $G_1$ into $\hat{G}$ defined by the surjection $\pi : A_{G_1} \ast A_{G_2} \twoheadrightarrow A_{G_1}$, $\pi = id_1 \ast \varepsilon_2$.

If $G_1$ has at least one irreducible representation of dimension greater than one, then $(G_1, \pi)$ is **not** a normal quantum subgroup of $\hat{G}_1 \ast \hat{G}_2$. Otherwise, $(G_1, \pi)$ is normal in $\hat{G}_1 \ast \hat{G}_2$.

The hat in the symbol $\ast$ above signifies the “Fourier transform” of $\ast$.

**Proof.** Let 

$$u = \sum_{ij} e_{ij}^u \otimes u_{ij}, \quad v = \sum_{kl} e_{kl}^v \otimes v_{kl}$$

be irreducible representations of $G_1, G_2$ respectively. Assume that the dimension of $u$ is greater than one. Let $\bar{u} = \sum_{ij} e_{ij}^u \otimes u_{ij}^*$ denote the conjugate representation of $u$. Then by [28], the interior tensor product representation

$$u \otimes_{in} v \otimes_{in} \bar{u} = \sum_{ijklrs} e_{ij}^u \otimes e_{kl}^v \otimes e_{rs}^u \otimes u_{ij} v_{kl} u_{rs}^*$$

is irreducible. Let $h_1$ be the Haar state of $G_1$. Then

$$h_1 \pi(u \otimes_{in} v \otimes_{in} \bar{u}) = \sum_{ijrs} e_{ij}^u \otimes I_v \otimes e_{rs}^u h_1(u_{ij} u_{rs}^*),$$

where $I_v$ is the identity matrix acting on the Hilbert space of $v$. Since $u$ is of dimension greater than one, $u \otimes \bar{u}$ properly contains the trivial representation of $G_1$ (with multiplicity one). Hence

$$h_1(u \otimes_{in} \bar{u}) = \sum_{ijrs} e_{ij}^u \otimes e_{rs}^u h_1(u_{ij} u_{rs}^*)$$

is neither the identity matrix, nor the zero matrix on $H_u \otimes H_{\bar{u}}$. (See also Theorem 5.7 of Woronowicz [35].) Therefore $h_1 \pi(u \otimes_{in} v \otimes_{in} \bar{u})$ is neither the identity matrix, nor the zero matrix on $H_u \otimes H_v \otimes H_{\bar{u}}$. By Definition 2.3, $G_1$ is not normal.
If $G_1$ has no non-trivial irreducible representations of dimension greater than one, then by [35], $G_1$ is the compact quantum group dual of a discrete group $\Gamma$, i.e., $A_{G_1} = C^*(\Gamma)$. By [28], every irreducible representation of $G$ is of the form

$$w^{\lambda_1} \otimes w^{\lambda_2} \otimes \cdots \otimes w^{\lambda_n},$$

where $w^{\lambda_i}$ is a non-trivial representation belonging to either the set $\Gamma$ or the set $\hat{G}_2$, $w^{\lambda_i}$ and $w^{\lambda_{i+1}}$ being in different sets. It is clear that

$$\pi(w^{\lambda_1} \otimes w^{\lambda_2} \otimes \cdots \otimes w^{\lambda_n})$$

is a constant diagonal matrix, with the constant diagonal entry equal to the product of those $w^{\lambda_i}$’s that belong to $\Gamma$. Use $[w^{\lambda_1} \otimes w^{\lambda_2} \otimes \cdots \otimes w^{\lambda_n}]$ to denote this diagonal entry. Since $h_{G_1}(\gamma) = 0$ if $\gamma$ is not the neutral element of $\Gamma$, one sees that Definition 2.3 is fulfilled. That is, $(G_1, \pi)$ is normal in $G = G_1 \ast G_2$ and $A_{G_1}$ is equal to the closure of the linear span of entries of the matrix $w^{\lambda_1} \otimes w^{\lambda_2} \otimes \cdots \otimes w^{\lambda_n}$ such that $[w^{\lambda_1} \otimes w^{\lambda_2} \otimes \cdots \otimes w^{\lambda_n}]$ is the neutral element of $\Gamma$. □

The above result suggests that the free product $G = G_1 \ast G_2$ rarely has normal quantum groups. This lead the author to conjecture that free product of simple quantum groups is simple, cf. Problem 4.3 of [34]. Recently, Chirvăsitu [8] gave the following remarkable solution of this conjecture for simple compact quantum groups without center, generalizing his earlier results [7] on simplicity of the quotient quantum group of the universal unitary quantum group by its center:

If $G_1$ and $G_2$ are simple compact quantum groups with trivial center, then the free product $G_1 \ast G_2$ is also simple.

In [8], Chirvăsitu also proved that the quantum reflection group of Banica and Vergnioux [2] is simple. We refer the reader to his paper for details on these and other interesting results.

Next we study normal quantum subgroups associated with crossed products. Recall [29] that a discrete Woronowicz $C^*$-dynamical system is a triple $(A, \Gamma, \alpha)$, where $A$ is a Woronowicz $C^*$-algebra, $\Gamma$ is a discrete group, and $\alpha$ is a homomorphism from $\Gamma$ to the automorphism group the Woronowicz $C^*$-algebra $A$, i.e., automorphisms of the $C^*$-algebra $A$ that preserves the coproduct on $A$. For such a dynamical system, it is shown in [29] that the crossed product $C^*$-algebra $A \rtimes_{\alpha} \Gamma$ is also a Woronowicz $C^*$-algebra, i.e., a compact quantum group, to abuse terminology. The dense Hopf subalgebra of $A \rtimes_{\alpha} \Gamma$ is $A \rtimes_{\alpha} \Gamma$. 
Similar to the second situation in Proposition 5.2, we have

Proposition 5.3. Consider a crossed product $A \rtimes_\alpha \Gamma$ of a compact quantum group $A$ by a discrete group $\Gamma$. Then $C^*(\Gamma)$ is a normal quantum subgroup of $A \rtimes_\alpha \Gamma$ with quotient $A$ via the morphism

$$\pi : A \rtimes_\alpha \Gamma \to C^*(\Gamma), \quad \pi(a\gamma) = \varepsilon(a)\gamma,$$

where $\varepsilon$ is the counit of $A$ and $a \in A$, $\gamma \in \Gamma$. In the language of [1], we have an exact sequence of Hopf algebras

$$\mathbb{C} \to A \to A \rtimes_\alpha \Gamma \to C \Gamma \to \mathbb{C}.$$

More generally, we have the following result that reduces to the above proposition by taking $I = 0$ and $K$ the one element group (we use the notation in Theorem 2.3 for correspondence of ideals).

Proposition 5.4. Let $(A, \Gamma, \alpha)$ be a discrete Woronowicz $C^*$-dynamical system. Let $I \triangleleft A$ be an $\alpha$-invariant Woronowicz $C^*$-ideal so that $A/I$ is a normal quantum subgroup of $A$ with quotient quantum group $B$. Let $K$ be a subgroup of the kernel of $\alpha$. Let $\tilde{\alpha}$ be the evident action of $\Gamma/K$ on $A/I$ obtained from $\alpha$. Then $A/I \rtimes_{\tilde{\alpha}} \Gamma/K$ is a normal quantum subgroup of $A \rtimes_\alpha \Gamma$ with quotient $B \rtimes_\alpha K$. In the language of [1], we have an exact sequence of Hopf algebras

$$\mathbb{C} \to B \rtimes_{\alpha} K \to A \rtimes_{\alpha} \Gamma \to A/I \rtimes_{\tilde{\alpha}} \Gamma/K \to \mathbb{C}.$$

Proof. Consider the morphisms

$$\pi : A \to A/I \rtimes_{\tilde{\alpha}} \Gamma/K, \quad U : \Gamma \to A/I \rtimes_{\tilde{\alpha}} \Gamma/K,$$

defined by $\pi(a) = \tilde{a}$, $U_\gamma = \tilde{\gamma}$. Here $\tilde{a}$ is the image in $A/I$ of an element $a \in A$ viewed as an element of $A/I \rtimes_{\tilde{\alpha}} \Gamma/K$ via inclusion; $\tilde{\gamma}$ is the image in $\Gamma/K$ of an element $\gamma \in \Gamma$ viewed as an element of $A/I \rtimes_{\tilde{\alpha}} \Gamma/K$ via inclusion. One can verify that $(\pi, U)$ is a covariant representation, i.e.,

$$U_\gamma \pi(a) U^{-1}_\gamma = \pi(\alpha_\gamma(a)).$$

Hence there is a surjection

$$\pi \times U : A \rtimes_\alpha \Gamma \to A/I \rtimes_{\tilde{\alpha}} \Gamma/K,$$

extending $\pi$ and $U$. This morphism preserves the coproducts because its restrictions to $A$ and $C^*(\Gamma)$ do. This shows that $A/I \rtimes_{\tilde{\alpha}} \Gamma/K$ is a quantum subgroup of $A \rtimes_\alpha \Gamma$ (true under only the assumption that $A/I$ is a quantum subgroup of $A$).
Let $A = A_G$ and assume that $A/I = A_N$ is a normal quantum subgroup of $A$ with quotient $B = A_{G/N}$. Let $\pi_0$ be the morphism from $A_G$ to $A_N$. Let $\theta$ denote the surjection $\pi \times U$ found above. Let $h = h_{A/I} \rtimes \hat{\alpha} h_{\Gamma}/K$ be the Haar state on $A/I \rtimes \hat{\alpha} \Gamma/K$ (cf. [29]). Then every irreducible representation of the quantum group $A \rtimes \alpha \Gamma$ is of the form $(\hat{u}^\lambda_{ij})$, where $(\hat{u}^\lambda_{ij})$ is an irreducible representation of dimension $d_{\lambda}$ of the quantum group $A$ and $\gamma \in \Gamma$ (cf. [29] as well). Let $S(N)$ be the set of $\lambda$'s such that $h_N \pi_0(\hat{u}^\lambda)$ is $Id_{\lambda}$, as in the proof of (4)$\Rightarrow$(3) in of Proposition 2.1 in [33]. Then

$$(h\theta(\hat{u}^\lambda_{ij} \gamma)) = (h(\hat{u}^\lambda_{ij})h(\gamma)) = \begin{cases} I_{d_{\lambda}} & \text{if } \lambda \in S(N), \gamma \in K, \\ 0 & \text{otherwise.} \end{cases}$$

Hence by Proposition 2.1 in [33] and its proof which asserts in part that $A_N\backslash G = A_{G/N} = \bigoplus \{ \mathbb{C}u^\lambda_{ij} : \lambda \in S(N), i,j = 1,\cdots,d_{\lambda} \}$, we conclude that $A/I \rtimes \hat{\alpha} \Gamma/K$ is a normal quantum subgroup of $A \rtimes \alpha \Gamma$ with quotient $B \rtimes \alpha K$.

The last statement on exact sequence of Hopf algebras now follows from Theorem [2.5] and remarks after Lemma 3.3. □

Note that $A$ is not a normal quantum subgroup of $A \rtimes \alpha \Gamma$ but rather a quotient quantum group, unlike the semi-direct product of groups.

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