Egocentric Effective Conductance Centrality and General Degree

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We study the popular centrality measure known as effective conductance or in some circles as information centrality. After reinterpreting this measure in terms of modulus (energy) of families of walks on the network, we introduce a new measure called shell modulus centrality, that relies on the egocentric structure of the graph. Egonetworks are networks formed around a focal node (ego) with a specific order of neighborhoods. We then propose efficient analytical and approximate methods for computing these measures on both undirected and directed networks. Finally, we describe a simple method inspired by shell modulus centrality, called general degree, which improves simple degree centrality and could prove to be a useful tool for network science.

The concept of information centrality was first introduced in \cite{1} and was later reinterpreted in terms of electrical conductance in \cite{2}. Given a network $G = (V, E)$ and a node $a \in V$, the effective conductance centrality of $a$ is defined as

$$C_{\text{eff}}(a) := \sum_{b \neq a} \frac{1}{R_{\text{eff}}(a, b)},$$

where $R_{\text{eff}}(a, b)$ is effective resistance distance between $a$ and $b$ in resistance networks. Note that this measure considers every possible path that electrical current flow might take from $a$ to an arbitrary sink $b$.

The situation can be clarified by introducing the notion of modulus of families of walks. This is a way of measuring the richness of certain families of walks on a network (and beyond, see \cite{3, 4}). Given two nodes $a$ and $b$ we may consider the connecting family $\Gamma(a, b)$ of all walks $\gamma$ from $a$ to $b$. Then, given edge density $\rho : E \to \mathbb{R}$ for $p \in [1, \infty]$, we define the $p$-modulus of $\Gamma$ to be

$$\text{Mod}_p(\Gamma) \triangleq \min_{\ell_p(\Gamma) \geq 1} \text{Energy}_p(\rho)$$

Namely, we minimize the energy of candidate edge-densities $\rho$ subject to the $p$-length of every walk in $\Gamma$ being greater than or equal one, i.e., $\ell_p(\Gamma) \geq 1$. These densities can be interpreted as costs of using the given edge. The energy we consider is

$$\text{Energy}_p(\rho) = \sum_{e \in E} \rho(e)^p,$$

thus modulus is a constrained convex optimization problem that has a unique extremal density $\rho^*$ when $1 < p < \infty$. This point of view allows for much more flexibility, because it can be applied to a variety of different families of objects: walks, cycles, tress, etc, and also works when the underlying network is directed or weighted. Moreover, modulus has very useful properties of $\Gamma$-monotonicity and countable subadditivity.

Furthermore, in the special case of connecting families, by varying the parameter $p$, we see that $\text{Mod}_p(\Gamma(a, b))$ generalizes classical measures such as shortest path, effective resistance and min cut \cite{5, 6}. For instance, when the network is undirected and $p = 2$, $\text{Mod}_2(\Gamma(a, b))$ is exactly the effective conductance between $a$ and $b$. In particular, effective conductance can be written as (see Appendix A)

$$C_{\text{eff}}(a) = \sum_{b \in V \setminus a} \text{Mod}_2(\Gamma(a, b))$$

For the rest of this paper, we consider $p = 2$ due to its physical interpretation as effective conductance as well as computational advantages, for instance, in this case \cite{2} is a quadratic program. Moreover, the right-hand side also makes sense on directed networks.

As mentioned above, $C_{\text{eff}}(a)$ is sociocentric in the sense that it considers all walks from $a$ to an arbitrary node in $G$. However, in practice, it can be prohibitive to scale socio-centric methods to very large networks. Moreover, in real-world situations it is not feasible to have access to the entire network. Rather, one can at best know local information up to a few neighborhood levels. For instance, when data is anonymized to protect privacy of network entities, identifying the sociocentric picture is impossible, e.g., sexual networks may be limited to the number of contacts of individuals.

An alternative approach is to consider measures that are adapted to egonetworks (also known as neighborhood networks). An ego network $G^0(r)$ around a node $a$ is constructed by collecting data (nodes and edges) starting from the ego $a$ and searching $G$ out to a predefined order of neighborhood $r \in \{1, \cdots \epsilon(a)\}$; where $\epsilon(a)$ is the eccentricity of node $a$ or the maximum distance from $a$ to nodes in $G$.

Egonetworks are often preferred because they support more flexible data collection methods \cite{7} and often involve less expensive computation costs. In this paper, we
focus on centrality measures that are adapted to the ego-centric paradigm as substitutes for sociocentric methods, with a focus on the scalability issue. These measures are more stable against network sampling and reliable (less sensitivity) with measurement errors. We concentrate on unweighted (binary) networks to simplify calculations, and we let \( d(a, b) \) denote the shortest-path distance between two nodes (smallest number of hops). The neighborhood structure around an ego \( a \) is described by the shells of order \( k \):

\[
S(a, k) := \{ y \in V : d(a, y) = k \},
\]

and the corresponding families of walks \( \Gamma(v, S(a, k)) \), consisting of simple walks that begin at ego \( v \in V \) and reach \( S(a, k) \) for the first time. Modulus allows a quantification of the richness of the family of walks, i.e., a family with many short walks has a larger modulus than a family with fewer and longer walks. Here we consider shell modulus \( \text{Mod}_2(v, S(a, k)) \) which quantifies the capacity of walks emanating from the ego up to the shell \( S(a, k) \) without having to account the data outside \( G^a(k) \). In particular, we propose the following egocentric version of \( C_{\text{eff}}(a) \):

\[
C_{\text{shell}}(a, r) := \sum_{k=1}^{r} \text{Mod}_2(v, S(a, k)) \quad (5)
\]

which we call shell modulus centrality and follows the same logic as \((4)\) but only requires the egocentric network data.

In Figure 1 centrality of nodes in three small networks are computed, where we consider the entire network and also \( C_{\text{shell}}(v, r) \) and \( C_{\text{eff}}(a) \) are different, they provide same ranking for centralities of nodes.

In Figure 1(a-c), node sizes are scaled with their \( C_{\text{shell}}(v, r) \) values and the computed centralities are, as expected, highly correlated with \( C_{\text{eff}}(a) \) with Spearman rank correlation 1, 0.94, and 0.99 respectively for Figures 1(a-1(c) meaning they are measuring a similar quantity.

For undirected networks, we can calculate both \( C_{\text{eff}}(a) \) and \( C_{\text{shell}}(v, r) \) analytically without going through the optimization problem in \((2)\) (Appendix A1 and Appendix A2).

In general, \((4)\) requires \(|V| \) modulus computations in all of \( G \), while \((5) \) only needs \( r \) modulus computations in \( G^a(r) \).

Shell modulus centrality can handle fairly large networks, e.g. 100,000 edges. The algorithm used here computes \((2)\) using an active set dual method quadratic programming \((13)\). We have shown that it’s theoretically enough to consider at most \(|E| \) active constraints \((13)\). Violated (active) constraints are found using Dijkstra’s algorithm and constraint matrix updating is done using the Cholesky decomposition.

In the following, we focus on approximating \((5)\) efficiently, while incorporating most of the benefits of shell modulus in a scalable framework.

First, we provide an upper bound that is known in the complex analysis literature as Ahlfors estimate \((14)\) Chapter 4, Equations 4-6], and in the context of electrical networks goes under the name of Nash-Williams inequality \((17)\). Given an egonetwork \( G^a(r) \), we consider the set of edges that connect a shell \( S(a, k-1) \) to the next shell \( S(a, k) \), for \( k \in \{1, \ldots, r\} \):

\[
E(a, k) := \{ e = \{ y, x \} \in E \mid x \in S(a, k-1), \ y \in S(a, k) \}.
\]

We call the sets \( E(a, k) \) shell connecting sets. Since \( \text{Mod}_2(v, S(a, r)) \) is a minimization problem \((2)\), we get an upper bound simply by choosing an appropriate admissible density \( \bar{\rho} \). Here, we pick the best admissible density that is constant for all edges in each shell connecting set.

After computing the minimized energy of this density, we obtain (see Appendix A3 for proof):

\[
\text{Mod}_2(a, S(a, r)) \leq \frac{1}{\sum_{k=1}^{r} \frac{1}{|E(a, k)|}}. \quad (6)
\]

To provide a lower bound for shell modulus, we focus on geodesic paths (shortest walks). These are usually the most important pathways of influence between the ego and other nodes. Classical measures of centrality, such as closeness centrality and betweenness centrality, are based uniquely on shortest paths \((18)\).

When collecting the egocentric data around an ego \( a \), one can take care to avoid forming cycles, and the resulting egonetwork becomes a tree. So assuming \( T^a(r) \) is a tree contained in \( G^a(r) \), we can use \( \Gamma \)-monotonicity to get a lower bound. Moreover, if we write \( \text{Mod}_2(T^a(r)) \) for the shell modulus of all walks in \( T^a(r) \) starting at the root \( a \) and reaching depth-level \( r \), this can be analytically calculated by the following recursive formula (proof in Appendix A5).

\[
\text{Mod}_2(T^a(k)) = \sum_{c \in C(a)} \frac{\text{Mod}_2(T_{c,k-1})}{1 + \text{Mod}_2(T_{c,k-1})} \quad (7)
\]

where \( C(a) := \{ c_1, c_2, \ldots, c_m \} \subseteq V \) are the children of \( a \) and \( T_{c,k-1} \) represents the subtree formed from \( T_n \) by keeping only \( c \) and its descendants.
Equation (7) computes \( \text{Mod}_2(T^a(k)) \) recursively. For each leaf node \( l_k \), set \( \text{Mod}_2(T_{l_k,a}) = \infty \). Then (7) will propagate the modulus to the ego. For example, to compute \( \text{Mod}_2(T_{a,2}) \) in the graph in Figure 2(b), we start by assigning \( \infty \) for modulus of the leaves \( e \) and \( f \). Then, by (7), each contributes 1 to node \( b \), and \( \text{Mod}_2(T_{b,1}) = 2 \). Thus \( \text{Mod}_2(T^a(2)) = \frac{\text{Mod}_2(T_{a,1})}{1 + \text{Mod}_2(T_{a,2})} = \frac{2}{3} \).

In conclusion, Ahlfors’ upper bound (4) considers all edges in the shell connecting sets even if they are not on the shortest paths, such as edge \( a-d \) in Figure 2(a). On the other hand, when using the ego-tree approximation, we inevitably lose valuable information hidden in the edges that where discarded. For example in Figure 2(b-c), to form a tree we need to solve the child custody problem between parents \( b \) and \( c \) and child \( f \). In particular, the lower bound calculation will discard at least one edge. Moreover, this leads to multiple possible lower bounds, e.g., \( \text{Mod}_2(T_{a,r}) = \frac{2}{3} \) in Figure 2(b) and \( \text{Mod}_2(T_{a,r}) = 1 \) for Figure 2(d).

As a compromise between the Ahlfors upper bound and the tree modulus lower bound, we propose a measure we call general degree. Fix a depth \( i = 1, 2, 3, \ldots, r \) and consider a tree rooted at the ego \( a \), whose leaves are all contained in the shell \( S(a, i) \), and such that the geodesics from the root to \( S(a, i) \) take exactly \( i \) hops. Let \( H(a, i) = (V_i, E_i) \) be the union of all such trees found by breadth first search. For instance, in Figure 2(d) we show \( H(a, 2) \) in that case. Note that we discarded nodes that are not on the geodesics paths from \( a \) to \( S(a, 2) \).

Since, in general, we cannot use the recursion (7) on \( H(a, r) \), we instead compute the upper bound (6). Namely, we consider the shell connecting sets \( E_i(a, k) \) for \( H(a, i) \) and define general degree to be the following expression:

\[
g\text{Deg}(a) := \sum_{i=1}^{r} \sum_{k=1}^{\frac{1}{|E_i(a,k)|}} \frac{1}{\frac{1}{2} + \frac{1}{4}} = 3 + 6/5 = 4.2.
\]

Observe that the first summand of 8 is the ordinary degree of the ego and thus our formula acts as a generalization of degree which takes into account information about the shells around the ego. For example, we have \( E_1(a,1) = 3, E_2(a, 1) = 2, E_2(a, 2) = 3 \) in Figure 2(d). For \( r = 2 \), \( g\text{Deg}(a) = 3 + \frac{1}{\frac{1}{2} + \frac{1}{4}} = 3 + 6/5 = 4.2 \). For small depths \( r \) the computation can be done by hand with keeping track of ancestral relations from the ego to nodes in each shell resulting in \( O(n_a) \) complexity for an ego network \( G^a(r) \) with \( n_a \) nodes. The pseudocode for computing general degree is in Appendix A.8. together with comparisons with shell modulus and the various bounds.

We illustrate the performance of general degree compared to the Ahlfors upper bound and the Tree modulus lower bound for conventional random network models such as Erdős-Rényi networks, scale-free (Barabasi-Albert model 19), Spatial (geometric model in the unit square 20), and small world (Watts-Strogatz model 21). Figure 3 shows that general degree gives a better approximation for \( C_{\text{shell}}(a, r) \) than the Ahlfors and Tree modulus estimates.

We see that for egocentric network data with medium sizes and order of neighborhood, general degree performs extremely well. However, it is possible to produce pathological network examples for which all of the estimates for shell modulus get worse as \( n, r \rightarrow \infty \), see Appendix A.11 for more details.

In summary, we analyzed egocentric network measures based on modulus of family of walks connecting ego to its neighborhood nodes. We compared the proposed measures with the sociocentric counterparts and illustrate the advantages of our methods. For undirected networks, shell modulus can be computed by solving a Laplacian system similar to 22. Moreover, for directed, multi-edge networks we propose approximations that carry the same benefits of the original definition while being easy and scalable. Finally, we introduced a generalization of degree called general degree. Applications of our methods are discussed in the supplemental materials and illustrate the advantages of the proposed measures, for instance to guide epidemic mitigation strategies (Appendix A.10).

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FIG. 3. Comparing the value of the Ahlfors upper bound, Tree modulus lower bound, General degree, and Shell modulus in simulated random network models (a) Erdős-Rényi networks with $p = 2 \log n/n$, (b) Scale free network by Barabasi and Albert model [19] with 6 edges preferential attachment. (c) Spatial network (random geometric network [20]) with distance threshold value $r = \sqrt{2 \log n/n}$ and small world network by Watts-Strogatz model with initial degree of $2 \log n$ and rewiring probability 0.3. General degree is providing a fair estimate of shell modulus in these networks.

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Appendix A: Supplementary Material

1. Proof for $C_{\text{eff}}(a) = \sum_{b \in V \setminus a} \text{Mod}_2(\Gamma(a, b))$ and formula for undirected networks

Let $\mathcal{F}$ be the set of all unit flows $\mathcal{f} : E \to \mathbb{R}$ that satisfy Kirchhoff’s node law and pass through a network $G$ from $a$ to $b$. Namely for $v \in V$

$$\nabla \cdot \mathcal{f} = \begin{cases} 1 & v = a \\ -1 & v = b \\ 0 & \text{otherwise} \end{cases}$$

corresponds to the injected currents at each node. The energy of $\mathcal{f}$ is

$$\text{Energy}(\mathcal{f}) \triangleq \sum_{e \in E} R(e) |\mathcal{f}(e)|^2$$

where $R(e) = \frac{1}{w(e)}$ is the resistance of edge $e$. A unit current flow $i \in \mathcal{F}$ is a unit flow that also satisfies Ohm’s law, i.e., there is a function $\mathbb{V} : V \to \mathbb{R}$ (called a potential) such that for every edge $(a, b)$:

$$R(a, b) i(a, b) = \mathbb{V}(b) - \mathbb{V}(a).$$

Let $\mathbb{U} : V \to \mathbb{R}$ be a vertex potential function. We can redefine the densities as the gradient of $\mathbb{U}$, i.e., for the edge $e = \{v, w\}$

$$\rho_{\mathbb{U}}(e) = |\mathbb{U}_v - \mathbb{U}_w|$$

Thus the admissibility condition for walks from $a$ to $b$ converts to $\mathbb{U}(a) = 0$, $\mathbb{U}(b) = 1$, and the 2-energy defined in $\mathbb{U}$ with $\rho_{\mathbb{U}}(e)$ is

$$\text{Energy}(\rho_{\mathbb{U}}) = \sum_{e \in E} \rho_{\mathbb{U}}(e)^2.$$
assuming each edge has a unit resistance and substituting $U$ by $V + C$, where $V$ is the electric potential when a unit current flow $i \in F$ is passing through the network with source $a$ and sink $b$ and the effective resistance between $a$ and $b$ is $R_{\text{eff}}$. By Thompson’s principle, $i \in F$ is the minimizer of the energy function on all unit flows, i.e.,

$$\sum_{e \in E} i(e)^2 = \min_{f \in F} \sum_{e \in E} f(e)^2$$

Therefore,

$$\text{Mod}_2(a, b) = \min_{U_a=0, U_b=1} \frac{\rho_T^U}{\rho_U} = \frac{1}{R_{\text{eff}}(a, b)} = C_{\text{eff}}(a, b). \quad (A1)$$

By Kirchhoff’s law of current conservation:

$$\sum_j A_{i,j}(V_i - V_j) = (\nabla \cdot i)(i)$$

where $A = [a_{ij}] \in \mathbb{R}^{N \times N}$ is the adjacency matrix of $G$, with $a_{ij} = 1$ if and only if $i, j \in E$. In matrix form:

$$L V = \mathbb{I} \quad (A2)$$

where $L$ is the Laplacian matrix of $G$ and $\mathbb{I} = \nabla \cdot i$. Because $V$ is defined up to an additive and the nullspace of $L$ is along the constant vector, we ground an arbitrary node $k$ and thus reduce $L$ by removing $k$th row and column denoted by $L_k$. Now we can find solve $L_k V = \mathbb{I}$:

$$k V = (k L)^{-1} k \mathbb{I}.$$ 

we denote $(k L)^{-1}$ by $G$ (reduced conductance matrix) and obtain effective resistance between nodes $a$ and $b$ is

$$R_{\text{eff}}(a, b) = k V_a - k V_b$$

$$= G_{a,a} + G_{b,b} - 2 G_{a,b} \quad (A3)$$

and from $(A1)$:

$$\text{Mod}_2(a, b) = (G_{a,a} + G_{b,b} - 2 G_{a,b})^{-1} \quad (A4)$$

2. Formula for $C_{\text{shell}}(a, r)$ in undirected networks

Similar to Section A1 to find Mod$(a, S(a, r))$ in $G^a(r)$, we solve Kirchhoff’s law of currents

$$L^v_{(r)} V = \mathbb{I} \quad (A5)$$

where $L^v_{(r)}$ is the Laplacian matrix of $G^a(r)$ and $\mathbb{I}$ is the applied external current vector with values 1 at ego and for nodes in $S(a, r)$

$$\mathbb{I}^T \mathbb{I}_S = -1 \quad (A6)$$

and zero for other nodes (see Figure 1). Nodes in $S(a, r)$ have similar electric potential $c$.

The above problem has a unique harmonic solution for $V$ up to a constant, we ground the potential at ego, i.e., $V_a = 0$ and find other nodes potentials by

$$V = G \mathbb{I}$$

where $G = (a L^v_{(r)})^{-1}$ is the reduced conductance matrix. Combining $(A5)$ and $(A6)$

$$\begin{bmatrix} V_2 \\ \vdots \\ c \\ c \\ \vdots \\ \vdots \\ c \end{bmatrix} = G \begin{bmatrix} 0 \\ \vdots \\ \mathbb{I}_S \mathbb{I}_S \\ \vdots \\ \vdots \\ \mathbb{I}_S \end{bmatrix} \rightarrow \begin{bmatrix} V_2 \\ \vdots \\ c \\ \frac{c}{s_1} \\ \vdots \\ \vdots \\ \frac{1}{1-s_1} \end{bmatrix} = G \begin{bmatrix} 0 \\ \vdots \\ 1 \\ 1 \\ \vdots \\ \vdots \\ 1 \end{bmatrix} = x$$

(A7)
![Diagram of electrical network with nodes and resistors](image)

**FIG. 4.** Interpreting $\text{Mod}(a, S(a, r))$ as finding effective conductance between grounded node $a$ and nodes with the same potential $c$ in $S(a, r)$ in an electrical network. Solution follows from the corresponding Laplacian system.

where $x_i = \sum_{j=S_1}^{S_s-1} G_{ij}$. If $|S| = s$ and for $i \in \{S_1, \ldots, S_s-1\}$

$$I_i = \frac{c}{x_i}$$

From (A7):

$$\frac{c}{-1 - c \sum_{j=S_1}^{S_s-1} \frac{1}{x_i}} = x_{S_s}$$

$$c = \frac{-x_s}{1 + x_s \sum_{j=S_1}^{S_s-1} \frac{1}{x_i}}$$

and the effective resistance between $a$ and $S(a, r)$:

$$R_{a,S(a,r)} = \frac{x_s}{1 + x_s \sum_{j=S_1}^{S_s-1} \frac{1}{x_i}}$$

and since $V_a = 0$ (grounded):

$$\text{Mod}(a, S(a, r)) = \frac{1 + x_s \sum_{j=S_1}^{S_s-1} \frac{1}{x_i}}{x_s}.$$  

### 3. Derivation of Ahlfors estimate

Since (2) is a minimization problem, an upper bound for the shell modulus $\text{Mod}_2(a, S(a, r))$ can be found by picking an appropriate density $\bar{\rho}$. Here we will restrict ourselves to densities that are constant on the shell connecting sets $E(a, k)$. Let

$$\bar{\rho}(e) := x_k \quad \text{if } e \in E(a, k).$$

Then we solve the following minimization problem:

$$\begin{aligned}
\text{minimize} & \quad \sum_{k=1}^{r} \theta_k x_k^2 \\
\text{subject to} & \quad \sum_{k=1}^{r} x_k = 1
\end{aligned}$$  

(A8)
where \( \theta_k := |E(a,k)| \). By Cauchy-Schwarz inequality

\[
1 \leq \left( \sum_{k=1}^{r} x_k \right)^2 \leq \left( \sum_{k=1}^{r} \frac{1}{\theta_k} \sqrt{\theta_k x_k} \right)^2 \leq \sum_{k=1}^{r} \frac{1}{\theta_k} \sum_{k=1}^{r} \theta_k x_k^2
\]

and thus the minimum in \( A8 \) is greater than \( \left( \sum_{k=1}^{r} \frac{1}{\theta_k} \right)^{-1} \). However, when \( x \) takes the form:

\[
x_k = \frac{C}{\theta_k},
\]

the minimum is achieved for

\[
C = \frac{1}{\sum_{k=1}^{r} \frac{1}{\theta_k}}.
\]

4. Ahlfors upper bound for Erdős-Rényi

We want to estimate the expected Ahlfors upper bound in Erdős-Rényi in the connected regime:

\[
p(N - 1) = 2 \log N.
\]

a. Concavity of the Ahlfors bound

We can use concavity and get

\[
\mathbb{E} \left( \sum_{i=1}^{r} \frac{1}{\sum_{k=1}^{i} \frac{1}{\theta_k}} \right) \leq \sum_{i=1}^{r} \frac{1}{\sum_{k=1}^{i} \frac{1}{\theta_k}}
\]

So we would like to estimate \( \mathbb{E}(\theta_k) \).

- First note that \( \theta_1 \) is Binomial\((N - 1, p)\). So:
  \[
  \mathbb{E}(\theta_1) = p(N - 1),
  \]
  from the binomial distribution.

- Now, given \( \theta_1 \) we must toss \( \theta_1 \) variables distribute as Binomial\((N - 1 - \theta_1, p)\), because the ego and the first shell are now out of consideration. So
  \[
  \mathbb{E}(\theta_2 | \theta_1) = \theta_1 p(N - 1 - \theta_1).
  \]

Therefore, computing the second moment of \( \theta_1 \) we get:

\[
\mathbb{E}\theta_2 = \mathbb{E}(\mathbb{E}(\theta_2 | \theta_1)) = \mathbb{E}(\theta_1)p(N - 1) - p\mathbb{E}(\theta_1^2) = p^2(1 - p)(N - 1)(N - 2).
\]

- Given \( \theta_1 \) and \( \theta_2 \) we must toss a certain number \( s \) of Binomial\((N - 1 - \theta_1, p)\) random variables, where \( s \) is the number of nodes in the second shell. However, this number \( s \) is not easy to calculate because it depends on the interaction at the previous step. For instance, if all the binomial variables in the previous step are equal to zero, then \( s = 0 \). But for higher values of \( s \) it becomes quite complicated.

In particular, we will have

\[
\mathbb{E}\theta_1 = \log N \quad \text{and} \quad \mathbb{E}\theta_2 \approx (\log N)^2.
\]
b. Lower bound for $E(\theta_k)$

First we will estimate $E\theta_k$ from below. Given an ego $a$, Spielman \[23\] sets

$$r(a) := \max \left\{ r : |B(r,a)| \leq \frac{N}{12 \log N} \right\}$$

and then shows that for $k \leq r(a)$,

$$\mathbb{P} \left[ |S(a, k + 1)| \leq \frac{1}{5} \log N |S(a, k)| \right] \leq N^{-1.2|S(a,k)|}.$$

He first finds that

$$E [|S(a, k + 1)| \mid G^a(k)] \geq \frac{5}{3} |S(a, k)| \log N,$$  \hspace{1cm} (A9)

and then applies the theory of Chernoff bounds. Note that by simply taking the expectation in (A9) we get

$$E|S(a, k + 1)| \geq \frac{5}{3} (\log N) E|S(a, k)|.$$

This gives geometric growth for $k \leq r(a)$:

$$E|S(a, k)| \geq (\log N)^k. \hspace{1cm} (A10)$$

In our case, since every $c \notin B(a, k)$ must toss $|S(a, k)|$ biased coins, we get

$$E[\theta_{k+1} \mid G^a(k)] = |S(a, k)| p(N - |B(a, k)|) \geq \frac{11}{12} |S(a, k)| pN = \frac{11}{6} (\log N) |S(a, k)|.$$  

Again we can take expectations and get

$$E\theta_{k+1} \geq \frac{11}{6} (\log N) E|S(a, k)|.$$

Using (A10), we get

$$E\theta_k \geq (\log N)^k.$$

c. Upper bound for $E\theta_k$

To get an upper bound we can compare the growth in the Erdős-Rényi graph with the growth for a Galton-Watson branching process with offspring distribution $X = \text{Binomial}(N - 1, p)$. This will be larger because there are no collisions and we always toss the maximum number of coins. If $Z_k$ is the population at time $k$, then

$$EZ_k = \mu^k$$

where $\mu = E(X) = p(N - 1) = 2 \log(N)$. So we get that

$$E\theta_k \leq (2 \log N)^k.$$
FIG. 5. The tree $T_a$ and its subtrees. Each child $c_i$ of $a$ can induce two subtrees—if it has descendants until depth $r - 1$. $T_{c_i,r}$ (outlined with a dashed line for $i = 3$ in the figure) is the subtree rooted at $a$ formed by removing all other children and their descendants from $T_r$. $T_{c_i,r-1}$ is the subtree rooted at $c_i$ formed by removing $a$ from $T_{c_i,r}$.

d. Upper bound for the Ahlfors estimate

We can apply this to our estimate of the average Ahlfors upper bound and get that:

$$
\mathbb{E} \left( \sum_{k=1}^{r} \frac{1}{\sum_{j=1}^{k} \frac{1}{\theta_j}} \right) \leq \sum_{k=1}^{r} \frac{1}{\sum_{j=1}^{k} \frac{1}{\theta_j}} \\
\leq \sum_{k=1}^{r} \frac{1}{\sum_{j=1}^{k} \frac{1}{(2\log N)^j}} \\
= (2\log N - 1) \sum_{k=1}^{r} \left[ 1 + \frac{1}{(2\log N)^k - 1} \right] \\
\approx (2\log N - 1) \left[ r + \sum_{k=1}^{r} \frac{1}{(2\log N)^k} \right] \\
= (2\log N - 1) \left[ r + \frac{1}{2\log N} - \left( \frac{1}{2\log N} \right)^{r} \right] \\
= (2\log N - 1) \left[ r + 1 - \frac{1}{(2\log N)^r((2\log N) - 1)} \right] \\
\approx (2\log N - 1)(r + 1)
$$

5. Proof of Tree modulus

Let $T_a$ be a rooted shortest tree at $a$ with vertex set $V$, and edge set $E$. Every density $\rho : E \to [0, \infty)$ gives a weighted distance on the tree defined by

$$
d_\rho(x,y) = \sum_{e \in \gamma(x,y)} \rho(e)
$$

We define the set of admissible densities $\text{Adm}(T_a^k)$, for walks starting from root $a$ (ego) to leaves at depth $k$, denoted by $l_k$

$$
\text{Adm}(T_a^k) := \{ \rho : E \to [0, \infty) : \ell_\rho(a, l_k) \geq 1 \}.
$$

with modulus

$$
\text{Mod}_2(T_a^k) := \inf_{\rho \in \text{Adm}(T_a,k)} \sum_{e \in E} \rho(e)^2
$$
Assuming \( a \) has at least one child, let \( C(a) := \{c_1, c_2, \ldots \} \subseteq V \) be the children. Each child \( c \) induces two rooted subtrees (Figure 5). Let \( T^a_c \) represent the subtree (still rooted at \( a \)) formed from \( T_v \) by pruning all of its children other than \( c \) along with their descendants, and let \( T_c \) represent the subtree (now rooted at \( c \)) formed by removing \( a \) from \( T^a_c \).

The following lemma is an immediate consequence of the parallel rule of modulus, i.e., given two families \( \Gamma_1 \) and \( \Gamma_2 \), where for every \( e \in E \) and \( \gamma_1 \in \Gamma_1 \) and \( \gamma_2 \in \Gamma_2 \) we have \( N(\gamma_1, e)N(\gamma_2, e) = 0 \). Thus \( \text{Mod}_2(\Gamma_1 \cup \Gamma_2) = \text{Mod}_2(\Gamma_1) + \text{Mod}_2(\Gamma_2) \).

**Lemma A.1.** The modulus of \( T^a_k \) is related to the moduli of the \( T^a_{c, k} \) as follows.

\[
\text{Mod}_2(T^a_k) = \sum_{i=1}^{m} \text{Mod}_2(T^a_{c, k}).
\]

By Lemma [A.1] we may restrict ourselves to the case that \( a \) has a single child \( c \). In this case, serial rule for modulus allows us to reduce the problem to finding the modulus of \( T_{c, k-1} \). This is explained in the following lemma.

**Lemma A.2.** The modulus of \( T^a_{c, k} \) is related to the modulus of \( T_{c, k-1} \) as follows.

\[
\text{Mod}_2(T^a_{c, k}) = \frac{\text{Mod}_2(T_{c, k-1})}{1 + \text{Mod}_2(T_{c, k-1})}
\]  

(A11)

**Proof.** If \( c \) is a leaf of \( T^a_k \), then \( \rho(a, c) = 1 \) is the minimizer for the modulus. Otherwise, by considering the density, \( \rho(v, c) \), on the edge from \( a \) to \( c \), the optimization effectively decouples. In order for \( \rho \) to be admissible, it is necessary that \( d_{\rho}(c, l) \geq 1 - \rho(a, c) \) for every leaf \( l_{k-1} \) of \( T_{c, k-1} \) at depth \( k - 1 \). For \( 0 \leq \ell \leq 1 \), define the parameterized set of admissible densities, for every leaf \( l_{k-1} \)

\[
\text{Adm}(T_{c, k-1}; \ell) := \{ \rho : E \to [0, \infty) : d(c, l_{k-1}) \leq \ell \}
\]

and the parameterized modulus problem

\[
\text{Mod}_2(T_{c, k-1}; \ell) = \inf_{\rho \in \text{Adm}(T_{c, k-1}; \ell)} \sum_{e \in E(T_c)} \rho(e)^2
\]

where \( E(T_{c, k-1}) \) represents the set of edges in the subtree \( T_{c, k-1} \). It is straightforward to verify that

\[
\text{Mod}_2(T_{c, k-1}; \ell) = \ell^2 \text{Mod}_2(T_c)
\]

and, thus

\[
\text{Mod}_2(T^a_{c, k}) = \inf_{0 \leq \rho(v, c) \leq 1} \{ \rho(a, c)^2 + \text{Mod}_2(T_{c, k-1} : 1 - \rho(v, c)) \}
\]

(A12)

\[
= \inf_{0 \leq \rho(a, c) \leq 1} \{ \rho(a, c)^2 + (1 - \rho(k, c))^2 \text{Mod}_2(T_{c, k-1}) \}
\]

The infimum, given by (A11), is attained when

\[
\rho(a, c) = \frac{\text{Mod}_2(T_{c, k-1})}{1 + \text{Mod}_2(T_{c, k-1})}
\]

\[
\square
\]

Lemmas [A.1] and [A.2] combined prove the following theorem.

**Theorem A.3.** The modulus \( \text{Mod}_2(T^a(k)) \) can be found by the formula

\[
\text{Mod}_2(T^a_k) = \sum_{c \in C(a)} \frac{\text{Mod}_2(T_{c, k-1})}{1 + \text{Mod}_2(T_{c, k-1})}
\]  

(A13)
| Quantity | $k = 1$ | $k = 2$ | $k = 3$ | total |
|----------|---------|---------|---------|-------|
| $\text{Mod}(a, S_k)$ | 3 | 1.26 | 0.44 | 4.71 |
| Lowerbound | 3 | 0.66 | 0.4 | 4.06 |
| Upperbound | 3 | 1.5 | 0.85 | 5.35 |
| General Degree | 3 | 1.2 | 0.4 | 4.6 |

**Algorithm 1** Algorithm for computing summands in (8).

1: $D \leftarrow$ set of all descendants for each ancestor
2: $r \leftarrow$ neighborhood order
3: $k \leftarrow 1$
4: for nodes in $\{ S^*(a, k), k \leq r \}$ do
5: Update $D$ with nodes as new descendants
6: Removing ancestors that do not have any descendants in nodes
7: $k \leftarrow k + 1$
8: end for
9: return harmonic means of number of ancestral relations in each $k$

6. Comparisons of shell modulus approximations and an algorithm for general degree

We illustrate the differences between the proposed method with (5), (6), and (7) in Table I for a small example egonetwork.

We can compute the summands in (8) with Algorithm 1. General degree, behaves similar to degree and no normalization is needed which is critical when comparing centrality of different egos, when there is no information about connections between their ego-networks.

In short, we keep track of ancestral relations from the ego to nodes in each shells, and discard nodes that do not have any descendants in shell $r$; leading to required information about $H(a, r)$ and thus we can find summands in (8). The overall time complexity of calculating (8) depends on the graph search in step 4 of Algorithm 1 and keeping the information of ancestral relationships, i.e, for an ego network $G^*(r)$ size $n_a$, algorithm performance is in $O(rn_a)$. 

![Graph](image-url)
TABLE II. Network attributes.

| Network                                      | |V| | |E| |
|---------------------------------------------|---|---|---|
| Sexual contact network of 40 homosexual men | 40 | 41 |
| Bottlenose dolphins social network          | 62 | 59 |
| Jazz musician collaboration                 | 198 | 2,742 |
| Davis southern club women                   | 32 | 89 |
| Power grid of western united states         | 4,941 | 6,594 |
| Users interaction network of Pretty Good Privacy (PGP) | 10,680 | 24,316 |
| Facebook friendship network of Princeton University | 13,081 | 88,266 |

FIG. 6. Correlation of values of gDeg for networks PGP and power grid computed in different cutoffs.

7. Applications

Computing node centrality in networks has numerous practical applications, for example, finding a set of influential nodes in immunization strategies. We evaluate the proposed measures with comparison to other existing popular centralities. Although, comparing egocentric measures to sociocentric ones is in favor of the latter, we can use the results to evaluate their performance, later when the entire network data is unavailable.

Because \( C_{eff} \) is a widely accepted measure and efficient algorithms are available for medare size undirected networks [24], we focus on benchmarking our egocentric measures with this sociocentric counterpart (4) in the subsequent discussions. In this section, we will consider empirical networks with properties shown in Table II.

8. Effects of neighborhood order \( r \) on general degree

We examine the correlation of the computed general degree when considering increasing order of neighborhood cutoff \( r \) in (8) with the sociocentric data (entire network). Our studies show that for most of the networks \( r = 3 \) can gives over \( \%90 \) correlation with respect to having the entire data in the measure. In Figure A 8, we illustrate two examples of PGP network and US power grid.

9. Ranking of nodes

In the seminal paper of Stephenson and Zelen [11], authors study a network of 40 homosexual men [25] (Figure 7(a)) and they show the advantages of information centrality \( C_{eff} \) and then rank the nodes based on this measure represents the node overall structural importance in the network. The resultant ranking is useful in detecting individuals that transfer HIV virus easily. We redo the experiment and compare the centrality of nodes with \( C_{eff} \) with degree and general degree in Table III.

Top three central nodes are similar in both \( C_{eff} \) and general degree. General degree distinguishes between importance of peripheral nodes (with degree 1) such as 14 and 15 that are connected to the most central node 16. Moreover,
| Rank | $C_{eff}$ | Degree | gDegree |
|------|----------|--------|--------|
| 1    | 16 (0.0104) | 16 (8) | 16 (14.096) |
| 2    | 22 (0.0097) | 5 (5)  | 26 (10.935) |
| 3    | 26 (0.0096) | 26 (5) | 22 (9.102)  |
| 4    | 20 (0.0089) | 22 (4) | 5 (8.352)   |
| 5    | 11 (0.0087) | 8 (3)  | 11 (8.350)  |
| 6    | 28 (0.0087) | 11 (3) | 28 (8.180)  |
| 7    | 19 (0.0083) | 20 (3) | 20 (7.808)  |
| 8    | 31 (0.0077) | 28 (3) | 31 (7.579)  |
| 9    | 14 (0.0075) | 31 (3) | 8 (6.360)   |
| 10   | 12 (0.0074) | 32 (3) | 19 (6.252)  |
| 11   | 15 (0.0074) | 34 (3) | 32 (6.158)  |
| 12   | 17 (0.0074) | 26 (3) | 38 (5.991)  |
| 13   | 21 (0.0074) | 2 (2)  | 54 (5.323)  |
| 14   | 38 (0.0072) | 9 (2)  | 14 (5.184)  |
| 15   | 23 (0.0072) | 14 (2) | 29 (5.064)  |
| 16   | 25 (0.0071) | 19 (2) | 33 (4.823)  |
| 17   | 27 (0.0070) | 23 (2) | 36 (4.766)  |
| 18   | 5 (0.0070)  | 29 (2) | 23 (4.749)  |
| 19   | 8 (0.0068)  | 33 (2) | 2 (4.717)   |
| 20   | 18 (0.0066) | 36 (2) | 9 (4.587)   |
| 21   | 29 (0.0066) | 1 (1)  | 12 (4.234)  |
| 22   | 32 (0.0062) | 3 (1)  | 15 (4.234)  |
| 23   | 36 (0.0060) | 4 (1)  | 17 (4.234)  |
| 24   | 13 (0.0058) | 6 (1)  | 21 (4.234)  |
| 25   | 39 (0.0057) | 7 (1)  | 18 (4.078)  |
| 26   | 40 (0.0057) | 10 (1) | 27 (4.048)  |
| 27   | 24 (0.0056) | 12 (1) | 3 (3.863)   |
| 28   | 2 (0.0056)  | 13 (1) | 4 (3.863)   |
| 29   | 3 (0.0055)  | 15 (1) | 6 (3.863)   |
| 30   | 4 (0.0055)  | 17 (1) | 25 (3.826)  |
| 31   | 6 (0.0055)  | 18 (1) | 7 (3.732)   |
| 32   | 9 (0.0054)  | 21 (1) | 30 (3.565)  |
| 33   | 7 (0.0054)  | 24 (1) | 39 (3.559)  |
| 34   | 34 (0.0054) | 25 (1) | 40 (3.559)  |
| 35   | 33 (0.0054) | 27 (1) | 13 (3.478)  |
| 36   | 30 (0.0052) | 30 (1) | 1 (3.416)   |
| 37   | 37 (0.0049) | 35 (1) | 35 (3.415)  |
| 38   | 1 (0.0046)  | 37 (1) | 37 (3.388)  |
| 39   | 10 (0.0045) | 39 (1) | 10 (3.373)  |
| 40   | 35 (0.0044) | 40 (1) | 24 (3.246)  |

Lowest central peripheral node 35 in $C_{eff}$ is also a low central node in general degree. The general degree shows a close correlation to $C_{eff}$ compared to degree. Although, degree cannot distinguish between nodes with the same degree while they are different in the other two measures.

10. Application in immunization strategies

Targeted immunizations in computer networks and human populations can greatly impact the overall outcome of spreading processes. Mitigating an epidemic with random immunization of nodes, requires vaccinating over 80% of the population and thus identifying a good set of target nodes has attracted much attention. However, most of the methods for finding proper sets of nodes for immunization requires global knowledge about the network, making it impossible to use in practical situations. Therefore, scientists prefer algorithms that are agnostic to the global structure of the network, for example, acquaintance immunization that chooses random neighbors of randomly picked nodes. In what follows, we illustrate the immunization performance of general degree when $r = 3$, i.e., knowledge of neighbors together with neighbors of neighbors, compared to other popular methods, such as acquaintance, effective conductance, and betweenness and eigenvector centrality.

We consider the epidemic model susceptible, infected, recovered (SIR) that represents infectious processes that are
FIG. 7. (a) Network of 40 homosexual men [25] (b) Pearson $r$ and Spearman $\rho$ correlation of general degree with $C_{\text{eff}}$ are 0.82 and 0.73 respectively. (c) Pearson $r$ and Spearman $\rho$ correlation of degree with $C_{\text{eff}}$ are 0.68 and 0.54 respectively.

FIG. 8. Schematic of the SIR model.

not reversible. Susceptible nodes in the network become infected I (proportional to infectious severity $\beta$ rate and the number of infected neighbors) and eventually rest in R state (immune) after a recovery period $1/\delta$ days (see Figure 8). We assume a constant $\delta = 0.1$, i.e., nodes stay in I state in average for 10 days. To model widespread diseases such as Flu that caused by close contacts, the infectious rate $\beta$ is chosen to have reproduction number $R_0 \sim \frac{\beta}{\delta \langle k \rangle} = 3$, where $\langle k \rangle$ is the average degree of the network [32].

We investigate the vaccination strategies that choose different fractions of population to immunize. After updating the contact networks with the immunized nodes, we assess the performance of each strategy. In our experiments, all nodes are initially susceptible and the infectious process starts from a randomly chosen patient zero. The algorithm performances are monitored by measuring the epidemic final size, i.e., number of nodes in R state after there is no more I state nodes.

We simulate the process 2000 times to get more insights into the underlying spreading nature in the newly obtained contact networks with different immunization strategies. The simulations are done with GEMFSim, that employs event-based exact stochastic simulation [34]. In Figure 9 we compare immunization performance of effective conductance, acquaintance, and betweenness and eigenvector centrality to general degree with $r = 3$. We test the significance of comparisons of the obtained results by the nonparametric Mann-Whitney test [35].

In addition to US power grid and PGP networks, we consider the friendship network for Princeton University and University of North Carolina at Chapel Hill (UNC) extracted from Facebook [27]. To assume potential physical networks, Salathe et. al. [32] suggest considering interactions of individuals in the same dormitory or same year and major. This makes the networks extremely modular and poses a big challenge for centrality measures that emphasize on closeness of a node to others. As it is shown in Figure 9 up to $\%15$ fraction of immunization betweenness centrality measure is delivering better choice of immunization, but with increasing the immunization coverage, our egocentric measure stats to perform better.

Effective conductance and betweenness centrality performs better than general degree in small immunization coverages. One explanation is these centrality measures are computed for the initial networks and with removing nodes, networks are changing and the central nodes will differ consequently. Therefore, with increasing immunization coverage, General degree performs better (or similarly) compared to other methods. General degree is performing better than both eigenvector centrality and acquaintance immunization. The latter is considering less information than general degree.
FIG. 9. Comparing different immunization strategies with effective conductance, acquaintance, eigenvector centrality, and betweenness centrality with general degree ($r = 3$). The immunization coverage varies from %1 to %30 of the highest central nodes. Bars show the difference of final size of epidemic outbreak. Negative differences shows general degree performs better in the immunization compared to the other policy. By increasing the coverage, general degree outperforms other methods. Results are inferred by 2000 simulations of SIR epidemic model and statistically nonsignificant results are shown by shaded bars. Empirical networks are US power grid (Grid) [21], PGP network (PGP) [26], Facebook friendship network of Princeton university (PR) [27]. Statistically insignificant differences are shown by shaded colors.

11. Behavior of shell modulus estimates when $n, r \to \infty$

Modulus on the complete graph

Verifying that a metric $\rho$ is extremal for $p$-modulus can be done using Beurling’s criterion (proof in [3]).

**Theorem A.4** (Beurling’s Criterion for Extremality). Let $G$ be a simple graph, $\Gamma$ a family of walks on $G$, and $1 < p < \infty$. Then, a density $\rho \in \text{Adm}(\Gamma)$ is extremal for $\text{Mod}_p(\Gamma)$, if there is a subfamily $\tilde{\Gamma} \subset \Gamma$ with $\ell_\rho(\gamma) = 1$ for all $\gamma \in \tilde{\Gamma}$, such that for all $h \in \mathbb{R}^E$:

$$\sum_{e \in E} \mathcal{N}(\gamma, e) h(e) \geq 0, \quad \text{for all } \gamma \in \tilde{\Gamma} \quad \implies \quad \sum_{e \in E} h(e) \rho^{\frac{p-1}{p}}(e) \geq 0. \quad \text{(A14)}$$

The complete graph $K_N$ is a simple graph on $N$ nodes, where every node is connected to each other, see Figure 10.
Figure 11 depicts the extremal density $\rho^*$ for $\Gamma(a,b)$ in $K_N$.

In formulas, $\rho^*(a,x) = 1/2 = \rho^*(b,x)$ for every $x \neq a,b$, and $\rho^*(a,b) = 1$, otherwise $\rho^*$ is zero. To verify Beurling’s criterion, consider the subfamily $\tilde{\Gamma}$ of simple paths consisting of $a \rightarrow b$ and $a \rightarrow x \rightarrow b$ for any $x \neq a,b$. We get that

$$\text{Mod}_p(\Gamma(a,b)) = 1 + 2(N-2) \cdot \frac{1}{2p},$$

$$\text{Mod}_2(\Gamma(a,b)) = \frac{N}{2}.$$

Take $n$ complete graphs $K_1, \ldots, K_n$.

**Modulus on a chain of complete graphs**

a. **Constant sizes** For $j = 1, \ldots, n$, assume that $|V(K_j)| = N$, and pick a pair of distinct nodes $x_{j-1}, y_j \in V(K_j)$. Then, for $j = 1, \ldots, n-1$, glue $y_j \in V(K_j)$ to $x_j \in V(K_{j+1})$. We denote the resulting graph by $G(N,n)$.

For convenience, for $j = 1, \ldots, n$, we write $A_j := V(K_j) \setminus \{x_{j-1}, y_j\}$, so that the shell at level $j$ is $S_j = V(K_j) \setminus \{x_{j-1}\} = A_j \cup \{y_j\}$. Then, fix $m = 1, \ldots, n$, and for $j = 1, \ldots, m-1$, define the following density on $e \in E(K_j)$:

$$\rho^*(e) := \begin{cases} 
\frac{1}{m} & \text{if } e = \{x_{j-1}, y_j\} \\
\frac{1}{2m} & \text{if } e = \{x_{j-1}, a\} \text{ or } e = \{y_j, a\} \text{ for some } a \in A_j \\
0 & \text{otherwise}
\end{cases}$$

For $j = m$, and $e \in E(K_m)$, set

$$\rho^*(e) := \begin{cases} 
\frac{1}{m} & \text{if } e = \{x_{m-1}, a\} \text{ for some } a \in A_m \cup \{y_m\} \\
0 & \text{otherwise}
\end{cases}$$
Observe that the support of $\rho^*$ can be decomposed as the disjoint union of $N - 1$ paths. To see this, enumerate each $A_j = \{a_j,k\}_{k=1}^{N-2}$. Then, for $k = 1, \ldots, N - 2$, let

$$\gamma_{m,k} := x_0 a_{1,k} x_1 a_{2,k} \cdots x_{m-1} a_{m,k}.$$  

Finally set

$$\gamma_{m,0} := x_0 y_1 \cdots x_{m-1} y_m.$$  

One can check that $\tilde{\Gamma} = \{\gamma_{m,k}\}_{k=0}^{N-2}$ is a Beurling subfamily for the shell modulus $\text{Mod}_2(x_0, S_m)$. So

$$\text{Mod}_2(x_0, S_m) = \frac{1}{m} + (N - 2) \left[ \frac{2m - 2}{4m^2} + \frac{1}{m^2} \right] = \frac{N}{2m} \left( 1 + \frac{1}{m} \right) - \frac{1}{m^2},$$

which is roughly $N/(2m)$. Also note that for $m = 1$ we recover the degree of $x_0$. If we sum we get

$$\sum_{m=1}^{n} \text{Mod}_2(x_0, S_m) \simeq \frac{N}{2} \sum_{m=1}^{n} \frac{1}{m} \simeq \frac{N}{2} \log n.$$  

The Ahlfors upper bound gives

$$\sum_{m=1}^{n} \frac{1}{\sum_{j=1}^{m} \frac{1}{N-1}} = (N - 1) \sum_{m=1}^{n} \frac{1}{m} \simeq (N - 1) \log n.$$  

The generalized degree, gives

$$\sum_{m=1}^{n} \frac{1}{m - 1 + \frac{1}{N-1}} \simeq N + \log n.$$  

b. Increasing sizes  
Now we repeat the construction above, but this time, setting $k_j := |V(K_j)|$, we have $k_1 = \alpha_1 + 2$ and, for $j = 2, \ldots, n$, we assume that $k_j = \alpha_j(k_{j-1} - 2) + 2$, for an increasing sequence of positive integers $\{\alpha_j\}_{j=2}^{n}$.  

Then, fix $m = 1, \ldots, n$, and for $j = 1, \ldots, m - 1$, define the following density on $E(K_j)$:

$$\rho^*(e) := \begin{cases} 
\prod_{k=1}^{m} \alpha_k & \text{if } e = \{x_{j-1}, y_j\} \\
2^{-1} \prod_{k=1}^{m} \alpha_k & \text{if } e = \{x_{j-1}, a\} \text{ or } e = \{y_j, a\} \text{ for some } a \in A_j \\
0 & \text{otherwise}
\end{cases}$$

For $j = m$, and $e \in E(K_m)$, set

$$\rho^*(e) := \begin{cases} 
\prod_{k=1}^{m} \alpha_k & \text{if } e = \{x_{m-1}, a\} \text{ for some } a \in A_m \cup \{y_m\} \\
0 & \text{otherwise}
\end{cases}$$

Now form $k_m - 1$ paths. Set

$$\gamma_{m,0} := x_0 y_1 \cdots x_{m-1} y_m.$$  

As before, enumerate each $A_j = \{a_{j,k}\}_{k=1}^{k_j-2}$. Now, $k_m - 2 = \alpha_m(k_m - 2)$, so we can group the $k_m - 2$ edges $\{x_{m-1}, a\}$ for $a \in A_m$ into $k_m - 2$ groups of $\alpha_m$ edges. Each such group will then flow through a different node in $A_m$, and then we repeat. The claim is that this gives rise to a Beurling family of paths $\tilde{\Gamma}$. By construction, they all have $\rho^*$ length equal to 1. We only need to check Beurling’s criterion. So suppose $h \in \mathbb{R}^E$ satisfies

$$\ell_h(\gamma) \geq 0 \quad \text{for all } \gamma \in \tilde{\Gamma}.$$  

Then $\sum_{e \in E} \rho^*(e) h(e)$ is equal to:

$$\sum_{j=1}^{m} (\rho^* h)(x_{j-1}, y_j) + \sum_{j=1}^{m-1} \sum_{i=1}^{k_{j-2}} [(\rho^* h)(x_{j-1}, a_{j,k}) + (\rho^* h)(a_{j,k}, y_j)] + \sum_{i=1}^{k_{m-2}} (\rho^* h)(x_{m-1}, a_{m,k}).$$
And if we write \( \alpha := 1 + \sum_{j=1}^{m} \prod_{k=j+1}^{m} \alpha_k \), and collect terms, this equals

\[
\alpha^{-1} \left( \alpha \sum_{j=1}^{m} h(x_{j-1}, y_j) + \sum_{j=1}^{m-1} \left( \prod_{k=j+1}^{m} \alpha_k \right) \sum_{i=1}^{k_j-2} \left[ h(x_{j-1}, a_{j,k}) + h(a_{j,k}, y_j) \right] + \sum_{i=1}^{k_m-2} h(x_{m-1}, a_{m,k}) \right).
\]

which is \( \geq 0 \), because for every \( j = 1, \ldots, m-1 \)

\[
(k_j - 2) \prod_{k=j+1}^{m} \alpha_k = k_m - 2
\]

So we get

\[
\text{Mod}_2(x_0, S_m) = \alpha^{-2} \left( 1 + \frac{3}{2} (k_m - 2) \sum_{j=1}^{m} \prod_{k=j+1}^{m} \alpha_k + (k_m - 2) \right)
\]

Now choose \( \alpha_j \equiv 2 \). Then

\[
\alpha = 1 + 2 + 4 + \cdots + 2^{m-1} = 2^m - 1.
\]

Also

\[
k_m - 2 = 2^{m-1} \alpha_1
\]

So

\[
\text{Mod}_2(x_0, S_m) \simeq \alpha_1.
\]

And

\[
\sum_{m=1}^{n} \text{Mod}_2(x_0, S_m) \simeq \alpha_1 n.
\]

On the other hand the generalized degree is

\[
\sum_{m=1}^{n} \frac{1}{m-1 + \frac{1}{k_m-1}} \simeq \log n.
\]