POST-NEWTONIAN FRAMES OF REFERENCE

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Abstract

A general theory of frames of reference proposed in a preceding
publication is considered here in the framework of the post-Newtonian
approximation, assuming that the frame of reference is centered on a
time-like geodesic. The problem of taking into account the rotation
of the frame of reference, which is usually ignored or incorrectly over-
simplified, is here discussed in detail and solved.

1 Introduction

The problem of defining Post-Newtonian reference frames and developing
the corresponding theory of adapted coordinate transformations has been
discussed in two main papers: [1], and [2] in the framework of covariant
metric theories of gravity and related fields, and by [3] in one of the so-called
PPN frameworks. None of these papers is based on a general theory and
they all suffer of a long-lasting confusion between frames of reference and
systems of coordinates. As a consequence they rely more on improvisations
suggested on the way by details of the approximate calculations than on a
well-defined logical line. The first paper considers frames of reference with
rotation but it does it in an oversimplified and insufficiently accurate way.
The two other papers ignore the rotating frames of reference and are for this
reason of restricted interest as far as the theory of such frames is concerned..
In [4] and [5] a general theory of frames reference and a particular post-
Newtonian application was considered. This tentative theory has been sub-
sequently modified and a few other applications have been reviewed in [4].
This paper deals with an explicit description of this renewed theory in the
framework of the post-Newtonian approximation.

The second section contains a short account of the main ingredients built
in the concept of a frame of reference, namely the definition of a meta-rigid
motion and a chorodesic synchronization. It includes also the definition of
a particular type of frames of reference: the quo-inertial ones. The third
section describes the general framework of the post-Newtonian approxima-
tion, slightly enlarged to include all types of inertial fields at the Newtonian
approximation. The fourth and fifth sections deal with the main object of
this paper: to use the general definition of Sect. 2 to develop a restricted
but explicit theory of frames of reference as far as the post-Newtonian
approximation allows it. The Appendix contains a summary of a few basic
definitions.

2 Frames of Reference

A frame of reference is a pair of geometrical objects intrinsically defined in
the space-time being considered:

- A time-like congruence $\mathcal{R}$ of a particular type, that we shall call the
  \textit{Meta-rigid motion} of the frame of reference,

- A space-like foliation of a particular type $\mathcal{S}$, that we shall call the
  \textit{Chorodesic synchronization} of the frame of reference.

2.1 Meta-rigid motions

A meta-rigid motion $\mathcal{R}$ has to possess at least three general properties
which are meant to make this concept to inherit as much as possible of the
formal properties of rigid motions in classical physics:

i) Their local characterization must be intrinsic and have a meaning in-
dependently of the space-time being considered.
ii) The knowledge of any open sub-bundle of the congruence must be sufficient to characterize the whole congruence.

iii) Its definition does not have to distinguish any particular world-line of the congruence.

To implement these three properties the most natural approach is to require the vector field $u^\alpha$ of the meta-rigid motions to satisfy a set of differential conditions. Obvious candidates which satisfy this requirement are the differential equations defining the Killing congruences

$$
\Sigma_{\alpha\beta} = 0, \quad \partial_\beta \Lambda_\alpha - \partial_\alpha \Lambda_\beta = 0.
$$

(1)

Other congruences to be acceptable as meta-rigid motions are the Born congruences, which are a generalization of the Killing congruences and are defined by the single group of conditions:

$$
\Sigma_{\alpha\beta} = 0.
$$

(2)

We shall continue to call these congruences rigid congruences, as usual. Meta-rigid ones will be an appropriate generalization of them.

The Born conditions, and a fortiori the Killing conditions, are very restrictive in any space-time including Minkowski’s one ([7], [8]), and the question may be raised of generalizing the concept of meta-rigid motion in such a way that the family of congruences $\mathcal{R}$ generalize Born congruences.

When facing this problem confronted with a physical application which requires the sublimated modelisation of the behavior of a rigid body, many authors resort to use Fermi congruences. By this we mean the time-like congruences which accept adapted Fermi coordinates based on a distinguished world-line seed. Because of this distinction Fermi congruences are not in general acceptable candidates to the concept of a meta-rigid motion.

Harmonic congruences can be defined as those time-like congruences, with unit tangent vector $u^\alpha$, for which there exist three space-like functions $f^a(x^\alpha)$ satisfying the following equations:

$$
\Box f^a = 0, \quad u^\alpha \partial_\alpha f^a = 0, \quad a, b, c = 1, 2, 3
$$

(3)

where $\Box$ is the d’Alembertian operator corresponding to the space-time metric. Harmonic congruences, with the corresponding harmonic coordinates,

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1Definitions in the Appendix.
have been used extensively in the literature for several reasons, including mainly the fact that they simplify many calculations. Harmonic congruences could in principle be considered good representatives of meta-rigid motions. In fact they are not. They are acceptable generalizations of Killing congruences but they are not generalizations of the Born congruences. More precisely it has been proved [11] that irrotational, and not Killing, Born congruences are never harmonic congruences.

To solve the problem just mentioned concerning the harmonic congruences it has been proposed to identify the meta-rigid motions with appropriately selected Quo-harmonic congruences [12], [9], [10]). These being those congruences for which there exist three independent space-like solutions \( f^a(x^a) \) satisfying the following equations:

\[
\hat{\Delta} f^a \equiv (\Box + \Lambda^a \partial_a) f^a = 0, \quad u^a \partial_a f^a = 0, \quad a, b, c = 1, 2, 3 \quad (4)
\]
or equivalently, using adapted coordinates and the notations of the Appendix:

\[
\hat{\Delta} f^a = \hat{\gamma}^{ij} (\hat{\partial}_i \hat{\partial}_j - \hat{\Gamma}^k_{ij} \hat{\partial}_k) f^a = 0, \quad f^a = f^a(x^i) \quad (5)
\]

These are proper generalizations of Born congruences in the sense that any Born congruence is also a quo-harmonic one. In fact when using adapted coordinates the operator \( \hat{\Delta} \) in the Eq. above becomes the usual Laplacian operator corresponding to a 3-dimensional Riemannian metric.

Besides the above mentioned general local conditions to be imposed to the concept of Meta-rigid motions global specific conditions will be necessary depending on the particular global structure of the space-time being considered. Also particular but important will be the case, where the congruence contains a time-like geodesic and it make sense, physically, to implement the principle of geodesic equivalence.

Notice that quo-harmonic congruences can be described and studied using any system of coordinates. But quo-harmonic coordinates, i.e. any system of three independent functions \( f^a \) of Eq. [4], are in general the most convenient ones to use.

2.2 Chorodesic synchronizations
The choice of a foliation to define the synchronization time of a frame of reference is to some extent less essential than the choice of a quo-harmonic congruence to define its meta-rigid motion. Besides, time is a much better understood concept in General relativity than the concept of space. The role of a synchronization is to be a first step towards a convenient definition of a universal scale of time, and the properties that are to be required from a synchronization will depend on the use for which it is intended. Atomic and sidereal time are two scales of common use which correspond to different synchronizations of the frame of reference co-moving with the Earth. Not to mention many other scales of time used in astronomy. On the contrary, for instance, it would not make sense to use different types of congruences to describe the motion of the Earth, at the approximation which considers it as rigid.

We consider here briefly how the International Atomic Time (IAT) scale is defined. It is based on the definition of the second in the international system of units SI as a duration derived from the frequency of a particular atomic transition. This definition is universal, in the sense that any physicist is able to use a well defined second, but at the same time is local because the identification of atomic time with proper-time duration implies that this definition of the second can be only used to define a scale of time along the world-line of the clock being used as a standard reference.

To define a scale of time on Earth to be used on navigation systems like the GPS for instance requires a different approach. In this case the second is defined as above with some precisions added. The resulting definition of IAT is:

\[
\text{IAT is a coordinate time scale defined in a geocentric reference frame with an SI second realized on the rotating geoid.}\]

It follows from this that the operational definition of the second is relative to some particular locations, i.e. world-lines of a particular congruence, which in this case is the rotating Killing congruence corresponding to the frame of reference co-moving with the Earth. A second at some other location is then defined as to nullify the relativistic red-shift between any pair of clocks of reference co-moving with the Earth. Therefore the second at any other

\[\text{See for instance}\]
\[\text{The geoid is a level surface of constant geo-potential (gravity and centrifugal potential) at “mean sea level”, extended below the continents.}\]
\[\text{If the influence of the Sun and the Moon is neglected}\]
location, not in the geoid, is then the interval of time separating the arrival
of two light signals sent, one second apart, from a standard clock located on
the geoid. The red-shift formulas of General relativity allow then to compare
the rhythm of any two clocks that can be joined with light signals.

Let \( \mathcal{R} \) be any time-like congruence. A Chorodesic \( C \) of \( \mathcal{R} \) is by definition, a space-like line such that its tangent vector \( p^\alpha \) satisfy the following equations:

\[
\frac{\nabla p^\alpha}{d\lambda} = \frac{1}{2} u^\alpha \Sigma_{\mu\nu} p^\mu p^\nu, \quad p^\alpha = \frac{dx^\alpha}{d\lambda} \tag{6}
\]

where \( \lambda \) is the proper length along \( C \). Obviously if \( \Sigma_{\mu\nu} = 0 \), i.e. if \( \mathcal{R} \) is a Born congruence then any chorodesic of \( \mathcal{R} \) is also a geodesic of the space-time.

Chorodesics of a congruence are important mainly because of the follow-
ing result:

If a space-like chorodesic \( C \) is orthogonal to a world-line of a congruence
\( \mathcal{R} \) then it is orthogonal to all the world-lines of the congruence \( \mathcal{R} \) that it
crosses.

This follows from differentiating the scalar product \( p^\rho u_\rho \) along \( C \). We
thus get:

\[
\frac{d}{d\lambda} (p^\rho u_\rho) = -\frac{1}{2} \Sigma_{\mu\nu} p^\mu p^\nu + \frac{1}{2} p^\rho p^\sigma (\nabla_\rho u_\sigma + \nabla_\sigma u_\rho) \tag{7}
\]

and using the definition of \( \Sigma_{\mu\nu} \) this is equivalent to:

\[
\frac{d}{d\lambda} (p^\rho u_\rho) = -2(p^\rho u_\rho)(p^\rho \Lambda_\rho) \tag{8}
\]

from where the statement above follows.

Particular foliations associated with a congruence \( \mathcal{R} \) are the one param-
eter family of hyper-surfaces generated by the chorodesics orthogonal to any
particular world-line of the congruence. We call these foliations Chorodesic
synchronizations. In general a chorodesic synchronization depends on a
world-line seed, except if the congruence is integrable in which case all its
chorodesic synchronizations coincide with the family of hyper-surfaces or-
thogonal to the congruence.

Let \( \mathcal{S} \) be the chorodesic synchronization orthogonal to some world-line
\( W \) of a frame of reference \( \mathcal{R} \). Then the associated Atomic (or Proper) Time
Coordination (ATC) scale is by definition a time coordinate which value at
any event $E$ is the proper time interval between some arbitrary event on $W$ and the intersection of $W$ with the leave of $S$ which contains $E$.

Notice that, even when the congruence of reference $\mathcal{R}$ is hypersurface orthogonal, i.e. there exists a single chorodesic synchronization, the ATC may depend on the particular world-line $W$ which is used to identify the coordinate time with proper-time along $W$.

Switching from a world-line seed, $W$, of an ATC to another $W'$ gives rise to a sub-group of adapted coordinate transformations. A complete definition of the ATC requires therefore to choose a particular world-line $S$, or eventually a bunch of equivalent ones. On Earth the ATC, i.e. the IAT, is associated to the bunch of world-lines $S$ of locations on the geoid.

The interest of the consideration of chorodesic synchronizations stems from the fact that they generalize the hyper-surfaces orthogonal to integrable congruences, as well as being a particular case of synchronizations of equal cyclic adapted time for both static and stationary space-times.

3 Inertial and gravitational fields: The Post-Newtonian framework

Our general framework will consist of the following items:

- A domain $D$ of $\mathbb{R}^4$ to be called the primary domain, or global domain if global conditions are required.

- A four dimensional space-time metric with signature $+2$, $g_{\alpha\beta}$, which can be expanded in power of $1/c$ as follows:

\begin{align*}
g_{00} &= -1 + \frac{1}{c^2} f_{00} + \frac{1}{c^4} h_{00} \\
g_{0j} &= \frac{1}{c} f_{0j} + \frac{1}{c^3} h_{0j} \\
g_{ij} &= \delta_{ij} + \frac{1}{c^2} h_{ij}
\end{align*}

\footnote{An analytical example of such sub-groups can be seen in [12]}


The corresponding developments of $\xi$, $\varphi_i$, and the Fermat quo-tensor $\hat{g}_{ij}$ are:

\begin{align*}
\xi &= 1 - \frac{1}{2c^2} f_{00} - \frac{1}{2c^4} (h_{00} + \frac{1}{4} f_{00}^2) \tag{12} \\
\varphi_i &= \frac{1}{c} f_{0i} + \frac{1}{c^3} (h_{0i} + f_{0i} f_{00}) \tag{13} \\
\hat{g}_{ij} &= \delta_{ij} + \frac{1}{c^2} \alpha_{ij} \quad \alpha_{ij} = h_{ij} + f_{0i} f_{0j} \tag{14}
\end{align*}

Every further term will be systematically neglected.

Our general requirements on both $D$ and $g_{\alpha\beta}$ are the following:

- The time-like congruence $\mathcal{R}$ defined by the parametric equations $x^i = const.$ is, at the required approximation, a quo-harmonic congruence and the $x^i$ are the space quo-harmonic coordinates adapted to $\mathcal{R}$. That is to say we assume that we have:

\[ \partial_i \alpha^i_j - \frac{1}{2} \partial_j \alpha = 0, \quad \alpha = \delta^{ij} \alpha_{ij} \tag{15} \]

This follows from making explicit that the functions $x^i$ themselves are solutions of Eqs. 5.

- The foliation $\mathcal{S}$ defined by the hyper-surfaces $t = const.$ is the chorodesic synchronization of $\mathcal{R}$ with seed on a world-line $W$ with parametric equations $x^i = x^i_0 = const.$ To see what this means at the required approximation let us follow the steps below:

  - Since by definition (See Sect. 2) on $W$ $t$ is its proper time we have:

\[ g_{00} \sim 1 \tag{16} \]

where here and in the remaining of this paragraph $\sim$ means that the corresponding relation is satisfied on $W$. 
Since by construction the chorodesics at any event $P$ of $W$ remain on a hypersurface $x^0 = \text{const}$, their tangent vector have a zero time-component, $p^0 = 0$. On the other hand the unit tangent vector at $P$ to $W$, with components $u^0 = 1, u^i = 0$, is orthogonal to every chorodesic through $P$ therefore we shall have:

$$g_{0i} \sim 0$$  \hspace{1cm} (17)$$

From $p^0 = 0$ all along the path of every chorodesic passing through $P$ and from it it follows also that:

$$\frac{d}{d\lambda} p^0 \sim 0 \quad \text{or} \quad \Gamma^0_{ij} p^i p^j \sim 0$$  \hspace{1cm} (18)$$

Using the explicit expressions of $\Gamma^0_{ij}$ and $\Sigma_{ij}$ as functionals of $g_{\alpha\beta}$, and taking into account that the relations above have to hold whatever the values of $p^i$, we readily find that:

$$\partial_i g_{0j} + \partial_j g_{0i} \sim 0$$  \hspace{1cm} (19)$$

This process could be continued indefinitely to prove that whatever the number of derivatives one has:

$$\partial (ijk\ldots g_{s})_{0} \sim 0$$  \hspace{1cm} (20)$$

the round brackets meaning complete symmetrization. But to implement this construction at the post-Newtonian level only the three first steps are necessary.

- $f_{0i}$ in will be required to be a linear function of $x^i$ so that the rotation rate of $R$ at the order $1/c$ be always a function of time only. This will allows to include in our framework the general theory of Newtonian frames of reference.

$$\partial_i f_{0j} - \partial_j f_{0i} = \omega_{ij}(t)$$  \hspace{1cm} (21)$$

This general framework will be called \textit{The post-Newtonian formalism}. It can be further restricted to deal with more specific situations corresponding to two broad categories:
• The first one is that for which the Riemann tensor of the space-time metric is zero at the appropriate approximation, this meaning here that:

\[ R_{0\alpha j} = O(c^{-6}), \quad R_{0ijk} = O(c^{-5}), \quad R_{ijkl} = O(c^{-4}) \]  \hspace{1cm} (22)

in which case \[ \mathbb{R} \] is an approximation to an inertial field.

• the second broad category includes all cases for which 22 does not hold, and among them all metric theories of gravitation, alone or coupled to other fields. No use whatsoever of specific field equations will be made in this paper that will cover also therefore any appropriate parameterized formalism.

Two particular cases have to be mentioned because the general framework starts from them. They are the following:

• Inertial and gravitational fields at the Newtonian approximation can be described assuming that:

\[ h_{\alpha\beta} = 0 \]  \hspace{1cm} (23)

• An extended Newtonian theory of gravitation that we discussed in \[ [14] \] can be described assuming that:

\[ h_{00} = 0, \quad h_{ij} = 0 \]  \hspace{1cm} (24)

Because in both cases \( h_{ij} = 0 \) these two approximate frameworks can be said, in a generalized sense, to be invariant under the group of rigid motions:

\[ t = t, \quad x^{i} = s^{i}(t) + R_{j}^{i}(t)y^{j} \]  \hspace{1cm} (25)

where \( R_{j}^{i}(t) \) is a time-dependent rotation matrix. It is this family of transformations that we generalize in the two next sections.

At the post-Newtonian approximation the Newtonian field is:

In \[ [14] \] both theories where described as exact. Newton’s theory as an affine theory and the extended theory as a semi-metric one.
\[ \Lambda_i = \lambda_i + \frac{1}{c^2} \Xi_i \]  
(26)

\[ \lambda_i = -\frac{1}{2} \partial_i f_{00} - \partial_t f_{0i} \]  
(27)

\[ \Xi_i = -\left( \frac{1}{2} \partial_i h_{00} + \frac{3}{2} \partial_t f_{00} f_{0i} + \partial_t h_{0i} - f_{00} \lambda_i \right) \]  
(28)

The Coriolis, or rate of rotation field, is:

\[ \Omega_{ij} = \omega_{ij} + \frac{1}{c^2} \Psi_{ij} \]  
(29)

\[ \omega_{ij} = \partial_i f_{0j} - \partial_j f_{0i} \]  
(30)

\[ \Psi_{ij} = \partial_i h_{0j} - \partial_j h_{0i} + \frac{1}{2} f_{00} \omega_{ij} \] 
\[ -(2\lambda_i + 3\partial_t f_{0i}) f_{0j} + (2\lambda_j + 3\partial_t f_{0j}) f_{0i} \]  
(31)

And the so-called rate of deformation field is:

\[ \Sigma_{ij} = \frac{1}{c^2} \partial_t \alpha_{ij} \]  
(32)

4 Post-Newtonian quo-harmonic coordinate transformations

We assume in this section that the potentials \( \mathcal{F} \) considered in Sect. 2 are defined in the appropriate global domain and are referred to a frame of reference of a particular type that we shall call quo-inertial. By definition this type is defined as equivalent to the class of frames of reference satisfying the following conditions:

- The quo-harmonic congruence defining the meta-rigid motion of the frame of reference contains one or many time-like geodesics.

- The restricted Principle of geodesic equivalence can be implemented on at least one of these geodesics, say \( \mathcal{C} \), to be called the center of motion of the frame of reference. This meaning that a system of global adapted
quo-harmonic coordinates can be found such that on this geodesic the Fermat tensor is the unit tensor, and the Zel’manov-Cattaneo connection is zero.

\[ \hat{g}_{ij}(x_0^s) = \delta_{ij}, \quad \hat{\Gamma}^i_{jk}(x_0^s) = 0 \]  

\( x_0^i \) being the coordinates of \( C \). As we shall see this is locally always possible. It is an assumption to say that a globally well-defined quo-harmonic congruence has this property.

We consider below the construction of any other quo-inertial frame of reference as a second order approximate development around its center world-line. This construction can be split in two steps:

- The first step defines the meta-rigid motion by giving the parameterized equations of a quo-harmonic congruence referring this congruence to a system of quo-harmonic coordinates.

- The second step constructs the chorodesic synchronization seeded on the center geodesic of the quo-inertial frame of reference.

As already mentioned in Sect. 2, these two steps have very different physical meanings and geometrical interpretations. The first step has to do with a modeling of an idealized rigid body with its motion being referred to the global frame of reference. The second step in making a particular choice of a possible time-distribution protocol. Both are necessary to be able to define physically meaningful tensor transformations between the two frames of reference.

Let us consider a geodesic with parameterized equations:

\[ x^i = s^i(t) \]  

To implement the construction of a quo-inertial meta-rigid motion centered on this geodesic we consider the congruence defined by equations:

\[ t = t \]  

\[ x^i = s^i(t) + R^i_p(t)y^p + \frac{1}{c^2} R^i_p(t)\delta_2y^p \]  

\[ \delta_2y^p(t) = L^p_r(t)y^r + \frac{1}{2} Q^p_{rk}(t)y^r y^k + \frac{1}{3!} C^p_{rkl}(t)y^r y^k y^l \]
where $R^i_p(t)$ is a time dependent rotation matrix and $y^i$ can equivalently be considered either as parameters or as a system of adapted coordinates of the new congruence. Obviously the zero order choice in these expressions has been chosen to guarantee the correct Newtonian limit. Also to simplify the writing we have chosen the coordinates $y^i$ such that the parametric equations of the geodesic $C$ of the new congruence be $y^i = 0$.

Differentiating the expressions above we obtain:

$$dt = dt, \quad dx^i = W^i dt + R^i_j dy^j + \frac{1}{c^2}(A^i_k dy^k + B^i dt),$$

where:

$$W^i = v^i + \dot{R}^i_p y^p$$

$$A^i_j = R^i_p (L^p_j + Q^p_k y^k + \frac{1}{2} C^p_{jk} y^k y^l)$$

$$B^i = R^i_p (\dot{L}^p_j y^j + \frac{1}{2} \dot{Q}^p_k y^k y^l + \frac{1}{3!} \dot{C}^p_{jk} y^k y^l y^l)$$

where $v^i = \dot{s}^i$, the over-head dot meaning a derivative with respect to $t$.

A straightforward calculation gives the transformed basic metric quantities:

$$\bar{g}_{00} = -1 + \frac{1}{c^2} \bar{f}_{00} + \frac{1}{c^2} \bar{h}_{00}$$

$$\bar{f}_{00} = f_{00} + 2 f_{0i} W^i + W^2$$

$$\bar{h}_{00} = h_{00} + 2 h_{0i} W^i + 2(f_{0i} + W_i) B^i + h_{ij} W^i W^j + h^*_0$$

$$h^*_0 = + (\partial_k f_{00} + \omega_{ki} R^k_p W^i) \delta_2 y^p$$

$$\bar{g}_{0i} = \frac{1}{c} \bar{f}_{0i} + \frac{1}{c^3} \bar{h}_{0i}$$

$$\bar{f}_{0i} = (f_{0s} + W_s) R^s_i$$

$$\bar{h}_{0i} = (h_{0s} + B_s + h_{sk} W^k) R^s_i + (f_{0k} + W_k) A^k_i + h^*_0$$

$$h^*_0 = \frac{1}{2} \omega_{rs} R^r_k R^s_i \delta_2 y^k$$

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\[ \bar{g}_{ij} = \delta_{ij} + \frac{1}{c^2} \bar{h}_{ij} \quad (50) \]

\[ \bar{h}_{ij} = h_{rs} R^r_i R^s_j + \delta_{rs} (A^r_i R^s_j + A^s_j R^r_i), \quad (51) \]

\[ \hat{g}_{ij} = \delta_{ij} + \frac{1}{c^2} \bar{\alpha}_{ij} \quad (52) \]

\[ \bar{\alpha}_{ij} = \bar{h}_{ij} + \bar{f}_{0i} \bar{f}_{0j} \quad (53) \]

The starred terms in 42 and 46 coming from substituting \( x^i \) in \( \bar{f}_{00} \) and \( \bar{f}_{0i} \) according to 35.

The congruence defined by the parametric Eqs. 35 will be a quo-harmonic one if we choose the free functions of time \( L^i_j(t), Q^p_{rk}(t) \) and \( C^p_{rkl}(t) \) in such way that:

\[ \frac{\partial}{\partial y^i} \bar{\alpha}^i_j - \frac{1}{2} \frac{\partial}{\partial y^j} \bar{\alpha} = 0 \quad (54) \]

i.e., in such a way that the parameters \( y^i \) are quo-harmonic coordinates.

The conditions stated in 33 are further restrictions on these conditions. We shall rewrite them at the appropriate approximation as:

\[ \alpha_{ij} \sim 0, \quad \frac{\partial}{\partial y^k} \bar{\alpha}_{ij} \sim 0 \quad (55) \]

where from now on the symbol \( \sim \) will mean that the coordinates \( x^i \) have been replaced by \( s^i(t) \) and \( y^k \) by 0.

The antisymmetric part of \( L_{ij} \) can be chosen as to give an unambiguous physical meaning to the rotation matrix \( R^i_j \).

Taking into account 46 - 52 the first group of conditions require that:

\[ L_{(ij)} \sim -\frac{1}{2} (h_{rs} + (f_{0s} + v_s)(f_{0r} + v_r)) R^s_i R^r_j \quad (56) \]

where:

\[ L_{ij} = \delta_{ik} L^k_j, \quad L_{(ij)} = \frac{1}{2} (L_{ij} + L_{ji}) \quad (57) \]

We shall see at the end of this section that the antisymmetric part of \( L_{ij} \) can be chosen as to give an unambiguous physical meaning to the rotation matrix \( R^i_j \).
The second group of conditions 33 leads to:

\[ Q_{j,ik} + Q_{i,jk} + X_{k,ij} \sim 0 \quad (58) \]

or:

\[ Q_{i,jk} = \frac{1}{2}(X_{j,ki} + X_{k,ji} - X_{i,jk}) \quad (59) \]

where:

\[ X_{k,ij} = \left[ \partial_l h_{rs} + \frac{1}{2}(\omega_{ls}(f_{or} + v_r) + (f_{os} + v_s)\omega_{lr})R^l_k R^r_i R^s_j \right] + \frac{1}{2}(\Delta_{ki}(f_{or} + v_r)R^r_j + \Delta_{kj}(f_{os} + v_s)R^s_i) \quad (60) \]

and:

\[ \Delta_{ij} = 2\tilde{\tilde{R}}^s_i R_{sj} \quad (62) \]

Notice that the conditions 53 could have been implemented on any world-line. But it is only because we have assumed that the world-line \( x^i = s^i(t) \) is a geodesic that these conditions are physically justified and required by the principle of geodesic equivalence.

Consider now the next step in the approximation. Using again 46 - 52 we have:

\[ \frac{\partial^2}{\partial y^k y^l} \tilde{\bar{\alpha}}_{ij} \sim Y_{ij,kl} + C_{i,jkl} + C_{j,ikl} \quad (63) \]

where:

\[ Y_{ij,kl} = \partial_m^2 \alpha_{rs} R^m_i R^r_j R^s_l \quad (64) \]

\[ + \frac{1}{4}(\tilde{\bar{\omega}}_{ki}\tilde{\bar{\omega}}_{lj} + \tilde{\bar{\omega}}_{kj}\tilde{\bar{\omega}}_{li}) - \frac{1}{4}(\tilde{\bar{\omega}}_{ki}\tilde{\bar{\omega}}_{lj} + \tilde{\bar{\omega}}_{kj}\tilde{\bar{\omega}}_{li}) \quad (65) \]

where:

\[ \tilde{\bar{\omega}}_{ij} = \tilde{\bar{\omega}}_{ij} + \Delta_{ij}, \quad \tilde{\bar{\omega}}_{ij} = \omega_{rs} R^r_i R^s_j \quad (66) \]
At the order $1/c^2$ we have:
\[
\hat{R}_{ijkl} \sim \hat{R}_{ijkl} + \frac{3}{4}(\bar{\omega}_{ij}\bar{\omega}_{kl} - \bar{\omega}_{ij}\bar{\omega}_{kl}) \tag{67}
\]
where:
\[
\hat{R}_{ijkl} = \frac{1}{2}\left(\partial^2_{mn}\alpha_{rs} + \partial^2_{rs}\alpha_{mn} - \partial^2_{mr}\alpha_{ns} - \partial^2_{ns}\alpha_{mr}\right) R^r_i R^s_j R^m_k R^n_l \tag{68}
\]
\[
\hat{R}_{ijkl} = \frac{1}{2}\left(\frac{\partial}{\partial y^k}\bar{\alpha}_{jl} + \frac{\partial}{\partial y^l}\bar{\alpha}_{jk} - \frac{\partial}{\partial y^j}\bar{\alpha}_{lk} - \frac{\partial}{\partial y^l}\bar{\alpha}_{kj}\right) \tag{69}
\]
Since (67) is independent of $C^i_{jkl}$ and $\Delta_{ij}$ is an arbitrary skew-symmetric quotient tensor, it is not in general possible to nullify the second derivatives of the Fermat tensor on the center of a quasi-inertial motion.

On the contrary it is always possible to implement the second order quasi-harmonic condition:
\[
\frac{\partial}{\partial y^k}\left(\frac{\partial}{\partial y^l}\bar{\alpha}_{ij} - \frac{1}{2}\frac{\partial}{\partial y^l}\bar{\alpha}_{ij}\right) \sim 0 \tag{70}
\]
In fact from (63) and (64) it follows that the preceding condition is equivalent to:
\[
C_{ijkl}\delta^{kl} \sim \frac{1}{2}(\bar{\omega}_{ik}\bar{\omega}_{jl} - \bar{\omega}_{ik}\bar{\omega}_{jl})\delta^{kl} \tag{71}
\]
The solution of this equation compatible with the symmetry of the problem is:
\[
C^s_{ijk} = \frac{1}{4}(\bar{\Theta}^s_i\bar{\delta}_{jk} + \bar{\Theta}^s_j\bar{\delta}_{ki} + \bar{\Theta}^s_k\bar{\delta}_{ij}) - \frac{1}{4}(\bar{\Theta}^s_i\tilde{\delta}_{jk} + \bar{\Theta}^s_j\tilde{\delta}_{ki} + \bar{\Theta}^s_k\tilde{\delta}_{ij}) \tag{72}
\]
where:
\[
\bar{\Theta}_{ij} = \frac{1}{2}\epsilon^k\bar{\omega}_{ik}\bar{\omega}_{kj}, \quad \bar{\tilde{\Theta}}_{ij} = \frac{1}{2}\epsilon^k\tilde{\omega}_{ik}\tilde{\omega}_{kj}, \quad \epsilon^k = 1 \tag{73}
\]
\[
\bar{\delta}_{jk} = -2\bar{\Theta}_{jk}/(\bar{\omega}_{rs}\bar{\omega}^{rs}) \tag{74}
\]
\[
\tilde{\delta}_{jk} = -2\bar{\tilde{\Theta}}_{jk}/(\tilde{\omega}_{rs}\tilde{\omega}^{rs}) \tag{75}
\]
To understand the meaning of this solution let us look at the coordinate transformations as the composition of two transformations:

\[ \begin{align*}
  x^i &= s^p + R^i_p (y^p + \frac{1}{c^2} (L^p_r(t)y^r + \frac{1}{2} Q^p_r(t)y^r y^k) + \frac{1}{3!} C^p_{jkl} y^j y^k y^l)) \quad (76) \\
  y^i &= R^i_p (y^p + \frac{1}{3!c^2} C^p_{jkl} y^j y^k y^l) \quad (77)
\end{align*} \]

with:

\[ R^i_k R^m_n = \delta^i_j \quad (78) \]

which leads to the composition law:

\[ C^i_{jkl} = C^i_{jkl} + R^i_p C^p_{mn} R^m_k R^l_n R^r_i \quad (79) \]

If we choose \( R^i_k \) such that:

\[ 2 \tilde{R}^i_k = -\omega_{mk} R^m_i \quad (80) \]

i.e. if choose \( R^i_k \) such that, according to 62 and 66, the rotation rate of the congruence defined by the first group of equations 76 is zero, then from 72 we have:

\[ C^i_{jkl} = -\frac{1}{4} (\Theta^i_{rs} \delta^r_{jk} + \Theta^i_{js} \delta^r_{ki} + \Theta^i_{ks} \delta^r_{ij}) \quad (81) \]

with:

\[ \Theta^i_{rs} = \frac{1}{2} \epsilon^k_{ik} \omega^r_{kj}, \quad \delta^r_{jk} = -2 \Theta^r_{jk} / (\omega^r_{rs} \omega^{rs}), \quad \omega^r_{rs} = \omega_{ij} R^i_r R^j_s \quad (82) \]

this being the unique acceptable solution of 71 if \( \tilde{\omega}^i_{ij} = 0 \), because one has to take into account the complete symmetry of \( C^i_{jkl} \) with respect to its covariant indices and the facts that \( \Theta^i_{ij} \) is symmetric and that the dual of \( \omega^r_{ij} \) is an eigen-vector with eigen-value zero.

Using again 71 and 72 with \( \tilde{\omega}^i_{ij} = 0 \), we get:

\[ C^i_{ijk} = \frac{1}{4} (\Theta^i_{rs} \delta^r_{jk} + \Theta^i_{js} \delta^r_{ki} + \Theta^i_{ks} \delta^r_{ij}) \quad (83) \]
Combining 78, 79, 82 and 84 we uncover the meaning of $R_{ij}^{\prime}$ and $R_{ij}^{\prime\prime}$. The first is the rotation leading to the non rotating congruence centered on the geodesic of the transformed congruence; the second is the rotation of the latter congruence as measured from the non rotating reference defined by the first one.

To determine the skew-symmetric part of $L_{ij}$ we shall demand on the center geodesic $C$ of the new congruence that:

$$\bar{\Psi}_{ij} \sim 0$$  \hspace{1cm} (84)

i.e. that the post-Newtonian correction to $\omega_{ij}$ on $C$ be zero. This requirement does not have to be considered as an additional demand to the quasi-harmonic condition but as a precision made to the meaning of the matrix $R_{ij}^{\prime}$ in the transformations [33]. Equivalently we could say that the real parameter in these transformations is not $R_{ij}^{\prime}$ but the rotation rate $\bar{\omega}_{ij}$ on $C$, of the new congruence. In fact considering the latter as a free parameter it does not make sense to consider separately its classical and relativistic correction. It has to be considered as a whole and have the meaning that it has at the classical approximation.

From 23 and from 33 - 50 it follows that $\bar{\Psi}_{ij}$ can be written as:

$$\bar{\Psi}_{ij} \sim \dot{L}_{[ij]} + \bar{\omega}_{ik}L_{j}^{k} - \bar{\omega}_{jk}L_{i}^{k} + Z_{ik}, \hspace{1cm} L_{[ij]} = \frac{1}{2}(L_{ij} - L_{ji})$$  \hspace{1cm} (85)

where both the symmetric part of $L_{ij}$, as given by 50, and the skew-symmetric object $Z_{ij}$ depend only on quantities that have been already calculated. It follows that the requirement [34] is equivalent to a differential equation for $L_{[ij]}$ whose general solution will depend on its value at a given instant. These constants can be considered to be zero without any loss of generality because whatever their value they will all lead to the same congruence.

### 5 Post-Newtonian chorodesic synchronizations

We consider in this section a chorodesic foliation $\mathcal{F}$ orthogonal to the center geodesic $C$ of the meta-rigid motion we are considering, i.e. the family of hyper-surfaces spanned by the chorodesics orthogonal to this geodesic. The corresponding IAT will be a time coordinate $t' = t'(t, y')$ such that $t' = \text{const}$ be the local equations of the chorodesic leaves of the foliation.
Let $t'$ be defined as the inverse of the following time transformation:

\[ t = t' + \frac{1}{c^2} \delta_2 t' + \frac{1}{c^4} \delta_4 t' \]  
(86)

\[ \delta_2 t' = I(t') + L_i(t') y^i + \frac{1}{2} Q_{ij}(t') y^i y^j \]  
(87)

\[ \delta_4 t' = H(t') + K_i(t') y^i + \frac{1}{2} P_{ij}(t') y^i y^j \]  
(88)

Differentiating this expression we get:

\[ dt = dt' + \frac{1}{c^2} (D dt' + C_i dy^i) + \frac{1}{c^4} (F dt' + D_i dy^i) \]  
(89)

where:

\[ D = \dot{I}' + \dot{L}_i y^i + \frac{1}{2} \dot{Q}_{ij} y^i y^j \]  
(90)

\[ C_i = \dot{L}_i + Q_{ij} y^j \]  
(91)

\[ F = \dot{H}' + \dot{K}_i y^i + \frac{1}{2} \dot{P}_{ij} y^i y^j \]  
(92)

\[ E_i = K_i + P_{ij} y^j \]  
(93)

where a dotted and primed quantity means that it has been differentiated with respect to $t'$. With these notations the new potentials are:

\[ g'_{00} = -1 + \frac{1}{c^2} f'_{00} + \frac{1}{c^4} h'_{00} \]  
(94)

\[ f'_{00} = \bar{f}_{00} - 2D \]  
(95)

\[ h'_{00} = \bar{h}_{00} + 2 \bar{f}_{00} D - 2F - D^2 + h'_{00}^* \]  
(96)

\[ h'_{0i} = \partial_i \bar{f}_{00} \delta_2 t' \]  
(97)

\[ g'_{0i} = \frac{1}{c} f'_{0i} + \frac{1}{c^3} h'_{0i} \]  
(98)

\[ f'_{0i} = \bar{f}_{0i} - C_i \]  
(99)

\[ h'_{0i} = \bar{h}_{0i} + \bar{f}_{00} C_i + \bar{f}_{0i} D - E_i + h'_{0i}^* \]  
(100)

\[ h'_{0i}^* = \partial_i \bar{f}_{00} \delta_2 t' \]  
(101)
the starred terms in $h_{00}'$ and $h_{0i}'$ coming from using 86 in the lower order terms $f_{00}'$ and $f_{0i}'$. And also:

\[
g_{ij}' = \delta_{ij} + \frac{1}{c^2}h_{ij}'
\]

\[
h_{ij}' = \bar{h}_{ij} + \bar{f}_{0i}C_j + \bar{f}_{0j}C_i - C_iC_j
\]

and:

\[
\hat{g}_{ij}' = \delta_{ij} + \frac{1}{c^2}\alpha_{ij}', \quad \alpha_{ij}' = \bar{\alpha}_{ij}
\]

this last result following directly from the fact that the Fermat object, $\hat{g}_{ij}$, is a quo-tensor, i.e. a three dimensional tensor under coordinate transformations that leave invariant the congruence.

The conditions to be demanded to 86 were derived in Sect. 3. Taking into account 94, 16 requires that:

\[
\dot{I}' \sim \frac{1}{2}\bar{f}_{00}
\]

\[
\dot{H}' \sim \frac{1}{2}\bar{h}_{00} + \frac{3}{4}\bar{f}_{00}^2
\]

The functions $I$ and $H$ have to be obtained integrating these differential equations and each will depend on an arbitrary constant. The second group of conditions 17 and 98 require that:

\[
L_i \sim \bar{f}_{0i}
\]

\[
K_i \sim \bar{h}_{0i} + \frac{1}{2}\bar{f}_{00}(f_{0i} + L_i)
\]

Finally the third group of conditions 19 and the relations obtained differentiating 98 with respect to $y^i$ yield:

\[
Q_{ij} \sim \frac{1}{2}(\bar{\partial}_i\bar{f}_{0j} + \bar{\partial}_j\bar{f}_{0i})
\]

\[
P_{ij} \sim \frac{1}{2}(N_{ij} + N_{ji})
\]
with:

\[
N_{ij} = \bar{\partial}_i \bar{h}_{0j} + \bar{\partial}_j \bar{f}_{0i} - \frac{1}{2} \bar{f}_{00} Q_{ij} + \dot{L}_i (\bar{f}_{0j} - L_j)
\]  

(111)

This concludes our modelisation of the chorodesic atomic time distribution protocol at the post-Newtonian approximation.

**Appendix: Some definitions and notations**

Given any space-time with line element:

\[
ds^2 = g_{\alpha\beta}(x^\rho)dx^\alpha dx^\beta, \quad \alpha, \beta, \ldots = 0, 1, 2, 3, \quad x^0 = ct,
\]

let us consider a time-like congruence \( \mathcal{R} \), \( u^\alpha \) being its unit tangent vector field \((u_\alpha u^\alpha = -1)\). The *Projector* into the plane orthogonal to \( u^\alpha \) is:

\[
\hat{g}_{\alpha\beta} = g_{\alpha\beta} + u_\alpha u_\beta
\]

By definition the *Newtonian field* is the opposite to the acceleration field:

\[
\Lambda_\alpha = -u^\rho \nabla_\rho u_\alpha, \quad \Lambda_\alpha u^\alpha = 0.
\]

The *Coriolis field*, or the rotation rate field, is the skew-symmetric 2-rank tensor orthogonal to \( u^\alpha \):

\[
\Omega_{\alpha\beta} = \hat{\nabla}_\alpha u_\beta - \hat{\nabla}_\beta u_\alpha, \quad \Omega_{\alpha\beta} u^\alpha = 0.
\]

where:

\[
\hat{\nabla}_\alpha u_\beta \equiv \hat{g}_\rho^\alpha \hat{g}_\beta^\sigma \nabla_\alpha u_\beta.
\]

And *Born's deformation rate field* is the symmetric 2-rank tensor orthogonal to \( u^\alpha \):

\[
\Sigma_{\alpha\beta} = \hat{\nabla}_\alpha u_\beta + \hat{\nabla}_\beta u_\alpha, \quad \Sigma_{\alpha\beta} u^\alpha = 0,
\]

Let \( x^\alpha \) be a system of adapted coordinates, i.e., such that:

\[
u^i = 0, \quad i, j, \ldots = 1, 2, 3
\]

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We use the following notations:

\[ \xi = \sqrt{-g_{00}}, \quad \varphi_i = \xi^{-2} g_{0i}, \]

and:

\[ \hat{g}_{ij} = g_{ij} + \xi^2 \varphi_i \varphi_j, \quad \hat{g}^{ij} = g^{ij} \]

which we shall call the *Fermat* quo-tensor of the congruence. Here and below quo-tensor refers to an object, well defined on the quotient manifold \( \mathcal{V}_3 = \mathcal{V}_4 / R \), whose covariant components are the space components of a tensor of \( \mathcal{V}_4 \) orthogonal to \( u^\alpha \).

The *Newtonian field* is then the quo-vector:

\[ \Lambda_i = -c^2 (\hat{\partial}_i \ln \xi + \frac{1}{c} \hat{\partial}_i \varphi_i), \quad \hat{\partial}_i \equiv \partial_i + \frac{1}{c} \varphi_i \partial_t. \]

The *Coriolis field*, or Rotation rate field, is the skew-symmetric quo-tensor:

\[ \Omega_{ij} = c \xi (\hat{\partial}_i \varphi_j - \hat{\partial}_j \varphi_i) \]

and the *Born’s deformation rate field* is the symmetric quo-tensor:

\[ \Sigma_{ij} = \hat{\partial}_t \hat{g}_{ij}, \quad \hat{\partial}_t = \xi^{-1} \partial_t \]

To these familiar geometrical objects it is necessary to add the following ones, [13], [14], [17]:

\[ \hat{\Gamma}^i_{jk} = \frac{1}{2} \hat{g}^{is} (\hat{\partial}_j \hat{g}_{ks} + \hat{\partial}_k \hat{g}_{js} - \hat{\partial}_s \hat{g}_{jk}), \]

which are the *Zel’manov-Cattaneo symbols*, and:

\[ \hat{R}^i_{jkl} = \hat{\partial}_k \hat{\Gamma}^i_{jl} - \hat{\partial}_l \hat{\Gamma}^i_{jk} + \hat{\Gamma}^i_{sk} \hat{\Gamma}^s_{jl} - \hat{\Gamma}^i_{sl} \hat{\Gamma}^s_{jk}, \]

which is the *Zel’manov-Cattaneo quo-tensor*. 

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