Abstract

In this paper we study the Spectral Decomposition Theorem and translate it to the quantum language through the Wigner Transform. We obtain a Quantum Version of The Spectral Decomposition Theorem that allows us to three things: First, to rank the levels of the Quantum Ergodic Hierarchy. Second, to analyze the classical limit in quantum ergodic systems and quantum mixing systems. And third, and maybe the most important feature, to find a relevant and simple connection between the first three levels of the quantum ergodic hierarchy (ergodic, exact and mixing) and the type of spectrum of the quantum systems in the classical limit.

Key words: QSDT-Quantum Ergodic Hierarchy-classical limit-weak limit-Wigner transform

1 Introduction

As in previous works in this paper we study the quantum chaos from the view point of the Berry’s definition: “a quantum system is chaotic if its classical limit exhibits chaos”. This particular approach of quantum chaos deviates from the standard approaches such as the complexity (see [6], [7]), the exponential divergence trajectories (see [8], [9]), the threatment of chaos based on the introduction of non-linear terms in the Schrodinger equation (see [10]) and the non-unitary evolution of a quantum system as an indicator of quantum chaos (see [11]). However, as mentioned in [11] pag. 247, 248 the study of chaos based on the quantum ergodic hierarchy takes into account the real conflict that the classical limit of a system does not exhibit a chaotic behavior, which would be a threat to the correspondence principle. The actual quantum version of the Spectral Decomposition Theorem gives a direct connection with the Quantum Ergodic Hierarchy and the classical limit, and also the degree of generality of it presents could be used as an attempt towards a general theory of quantum chaos. All these aspects in accordance with the conceptual foundations in the context of Belot-Earman program’s. The
main goal of this paper is to obtain the first two levels of the quantum ergodic hierarchy from a
Quantum Version of the Spectral Decomposition Theorem [5] and establish a simple connection
between these ergodic levels and the type of spectrum that presents a quantum closed system
in the classical limit. In section 2 we begin introducing a brief review of the minimal notions of
the theory of densities necessary for the development of the following sections.

2 Theory of densities and Markov Operators

Historically, the concept of density has only recently appeared in order to unify the descriptions
of the phenomena of statistical nature. Clear examples of this were the Maxwell velocity distrib-
ution and the quantum mechanics both as attempts to unify the theory of gases and to justify
the derivation of the Planck distribution of radiation from the black body respectively. Further
development of modern physics demonstrated the usefulness of the densities for the description
of large systems with a large number of degrees of freedom which have an uncertainty by igno-
rance. In this section we introduce a brief review of the concepts of the theory of densities and
the Markov Operators based on the formalism of dynamical systems [5].

2.1 Densities functions and Dynamical Systems

We start by recalling the fundamental mathematical elements of the theory of dynamical systems
([2], [5]). Given a set \( X \), \( \Sigma \) is a \( \sigma \)-algebra of subsets of \( X \) if it satisfies:

(I) \( X \in \Sigma \)

(II) \( A, B \in \Sigma \implies A \Delta B \in \Sigma \)

(III) \( (B_i) \in \Sigma \implies \bigcup_i B_i \in \Sigma \)

A function \( \mu \) on \( \Sigma \) is a probability measure if it satisfies:

(I) \( \mu : \Sigma \to [0, 1] \) and \( \mu(X) = 1 \)

(II) For all family of pairwise disjoint subsets \( (B_i) \in \Sigma \implies \mu(\bigcup_i B_i) = \sum_i \mu(B_i) \)

A measure space is any terns of the form \( (X, \Sigma, \mu) \). Given a measure space \( (X, \Sigma, \mu) \), a measurable preserving transformation or automorphism \( T \) is a biyective function \( T : X \to X \) which satisfies:

\[
\forall A \in \Sigma : \mu(T^{-1}A) = \mu(A)
\]

We say then that the family of automorphisms \( \tau := \{T_t\}_{t \in I} \) is a group of measure-preserving automorphisms and we call it a dynamical system \( \tau \). With these definitions, we say that the quaternary \( (X, \Sigma, \mu, \tau) \) is a dynamical law \( \tau \). The central notion of the dynamic system concept is the definition of density: given a dynamical system \( (X, \Sigma, \mu, \tau) \) and \( D(X, \Sigma, \mu) = \{f \in L^1(X, \Sigma, \mu) : f \geq 0 ; \|f\| = 1\} \) then any function \( f \in D(X, \Sigma, \mu, \tau) \) is called a density. For a dynamical system \( S \) in the context of classical mechanics it is usual to take \( X = \mathcal{M} \) the phase space, \( \Sigma = \mathcal{P}(\mathcal{M}) \) the subsets of the phase space, \( \mu \) the Lebesgue measure and \( T_t \) the temporal evolution. This context will be clarified in the next sections. Besides the concept of dynamical system, the other fundamental component in order to describe the temporal evolution of a classical density is the notion of Markov operator. This important class of operators is presented below.
2.2 Markov Operators

Given a classical system $S$ with an initial state given by a density $f_0$ we know that its temporal evolution will be determined by the Liouville equation. Except in simple cases we know that this equation has no exact solution and therefore we are forced to use another strategy to study the evolution of the system. In this context Markov operator are very useful because their properties allow us to know the asymptotic behavior of the densities and general behavior of the densities can be well developed in both dynamical systems and stochastic systems. Markov operators contain global information of the densities when $t \to \infty$ and under certain hypotheses on these gives certain conditions for the existence of an equilibrium density $f^*$ which physically corresponds to the arrival of the system at the equilibrium. This approach at the equilibrium when $t \to \infty$ through the global properties of Markov operators will be the connection between the classical limit and the Quantum Spectral Decomposition Theorem. This will be considered in the section 4. We present a brief review of the concepts necessary for the development of the present paper beginning with the following definition (see [5], pag. 32).

**Definition 2.1.** (Markov Operator) Given a measure space $(X, \Sigma, \mu)$, a linear operator $P : L^1 \to L^1$ is called a Markov operator if it satisfies:

(a) $Pf \geq 0$

(b) $\|Pf\| = \|f\|

for all $f \in L^1$, $f \geq 0$

From the condition (b) of the definition of Markov operator follows that $P$ is monotonic, that is if $f, g \in L^1$ with $f \geq g$ then $Pf \geq Pg$.

Markov operators satisfy the following important properties that will be crucial in order to obtain the quantum version of the spectral decomposition theorem (see [5], pag. 33):

**Theorem 1.** Let $(X, \Sigma, \mu)$ be a $\sigma$-algebra and let $f \in L^1$. If $P$ is a Markov operator then:

(I) $\|Pf\| \leq \|f\|$ (contractive property)

(II) $|Pf(x)| \geq P|f(x)|$

The following concept of a fixed point of a Markov operator $P$ is crucial for establishing the arrival of a density $f$ at the equilibrium (see [5], pag. 35).

**Definition 2.2.** (Fixed Point) Let $P$ be a Markov operator. If $f \in L^1$ with $Pf = f$ then $f$ is called a fixed point of $P$. In a more general way, any $f \in D(X, \Sigma, \mu)$ that satisfies $Pf = f$ is called a stationary density of $P$.

A family of automorphisms $\{T_t\}_{t \in I}$ which represent the temporal evolution of any dynamical system are a special class of Markov operators called Frobenius-Perron operators. They are defined as follows (see [5], pag. 36).

**Definition 2.3.** (Frobenius-Perron Operator) Given a measure space $(X, \Sigma, \mu)$ and $T : X \to X$ a non singular automorphism (e.g. $\mu(T^{-1}(A)) = 0$ for all $A \in \Sigma$ such that $\mu(A) = 0$) the unique operator $P : L^1 \to L^1$ defined for all $A \in \Sigma$ by the equation

$$\int_A Pf(x)\mu(dx) = \int_{T^{-1}(A)} f(x)\mu(dx) \quad (2)$$

is called the Frobenius-Perron operator corresponding to $T$. 

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From the equation (2) it follows that the Frobenius-Perron operator is lineal and satisfies the following properties (see [5], pag. 37):

**Theorem 2.** Let $T$ be an automorphism. If $P$ and $P_n$ are the Frobenius-Perron operators corresponding to $T$ and $T^n$ respectively. Then we have that

(I) $\int_X Pf(x)\mu(dx) = \int_X f(x)\mu(dx)$

(II) $P_n = P^n$

The adjoint operator to the Frobenius-Perron operator is defined as follows (see [5], pag. 42).

**Definition 2.4.** (Koopman Operator) Given a measure space $(X, \Sigma, \mu)$ and $T : X \to X$ a non-singular automorphism, the unique operator $U : L^\infty \to L^\infty$ defined for all $f \in L^\infty$ by the equation

$$Uf(x) = f(T(x))$$

(3)

which is called the Koopman operator corresponding to $T$.

From the equation (3) it follows that the Koopman operator is lineal, and satisfies the following properties (see [5], pag. 43):

**Theorem 3.** Let $T$ be an automorphism. If $U$ and $U_n$ are the Koopman operators corresponding to $T$ and $T^n$ respectively. Then

(I) $\|Uf\|_{L^\infty} \leq \|f\|_{L^\infty}$

(II) $U_n = U^n$

(III) $\langle P_n f, g \rangle = \langle f, U_n g \rangle$ for every $f \in L^1, g \in L^\infty, n \in \mathbb{N}_0$

where $\langle f, g \rangle = \int_X f(x)g(x)d(\mu x)$ for all $f \in L^1, g \in L^\infty$.

We now recall the *Ergodic Hierarchy* for dynamical systems (see [5], pag. 68):

**Theorem 4.** (Ergodic, Mixing and Exact) Let $(X, \Sigma, \mu)$ be a normalized measure space, $T : X \to X$ an automorphism and $P, U$ the Frobenius-Perron and Koopman operators corresponding to $T$. Then:

(a) $T$ is ergodic $\iff \lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} \langle P^k f, g \rangle = \lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} \langle f, U_k g \rangle = \langle f, 1 \rangle \langle 1, g \rangle$

for all $f \in L^1, g \in L^\infty$

(b) $T$ is mixing $\iff \lim_{n \to \infty} \langle P^n f, g \rangle = \lim_{n \to \infty} \langle f, U_n g \rangle = \langle f, 1 \rangle \langle 1, g \rangle$

for all $f \in L^1, g \in L^\infty$

(b) $T$ is exact $\iff \lim_{n \to \infty} \|P^n f - \langle f, 1 \rangle\| = 0$ for all $f \in L^1$

From these definitions it follows that

$$MIXING \subseteq EXACT \subseteq ERGODIC$$

(4)

these are the inclusions of the Ergodic Hierarchy levels, and these inclusions are strict.

To complete this section we introduce the notion of constrictive operator allowing us to guarantee the existence of an equilibrium density (see [5], pag. 87).
Definition 2.5. (constrictive operator) A Markov operator $P$ will be called constrictive if there exist a precompact set $F \subseteq L^1$ such that for all $f \in D(X, \Sigma, \mu)$:

$$\lim_{n \to \infty} d(P^n f, F) = \lim_{n \to \infty} \inf_{g \in F} \|P^n f - g\| = 0$$ (5)

An important result is that every Markov constrictive operator has an equilibrium density (see [5], pag. 87).

Theorem 5. Let $(X, \Sigma, \mu)$ be a normalized measure space, and $P : L^1 \to L^1$ a constrictive Markov operator. Then $P$ has a stationary density, i.e there is a $f_* \in L^1$ such that $P f_* = f_*$. 

The existence of an equilibrium density $f_*$ is fundamental in order to obtain a framework for quantum chaos and it can be translated to quantum language through the Wigner transform in a way that the equilibrium state of any quantum closed system of continuous spectrum is represented by the weak limit $\rho_*$ (see [12], pag. 889 eq. (3.28)).

3 The Spectral Decomposition Theorem of Dynamical Systems

With all the mathematical background of the previous sections we are able to present one of the main results of the theory of dynamical systems which is called the Spectral Decomposition Theorem (see [5], pag. 88):

Theorem 6. (The Spectral Decomposition Theorem (version I)) Let $P$ be a Markov constrictive operator. Then there is an integer $r$, two sequences of nonnegative functions $g_i \in D(X, \Sigma, \mu)$, $k_i \in L^\infty$, $i = 1, ..., r$, and an operator $Q : L^1 \to L^1$ such that for all $f \in L^1$, $P f$ may be written as

$$P f(x) = \sum_{i=1}^{r} \lambda_i(f) g_i(x) + Q f(x)$$ (6)

where

$$\lambda_i(f) = \int_X f(x) k_i(x) \mu(dx) = \langle f(x), k_i(x) \rangle$$ (7)

The functions $g_i$ and the operator $Q$ have the following properties:

(I) $g_i(x) g_j(x) = 0$ for all $i \neq j$, so that functions $g_i$ have disjoint supports.

(II) For each integer $i$ exists a unique integer $\alpha(i)$ such that $P g_i = g_{\alpha(i)}$. Further $\alpha(i) \neq \alpha(j)$ for $i \neq j$ and thus the operator $P$ just allows to permute the functions $g_i$.

(III) $\|P^n Q f\| \to 0$ as $n \to \infty$ for every $f \in L^1$.

Basically, this theorem describes the temporal evolution of any density in a simple manner with an initial term which oscillates between each of the densities $g_i$ under the assumption that this evolution has a constrictive Perron-Frobenius operator. And with a remaining term $Q f$ goes to zero, which is the expression of a process of relaxation to equilibrium which we will analyze in the section 5. By the property (2) of the theorem, it follows that

$$P^n f(x) = \sum_{i=1}^{r} \lambda_i(f) g_{\alpha^n(i)}(x) + P^{n-1} Q f(x) = \sum_{i=1}^{r} \lambda_{\alpha^{-n}(i)}(f) g_i(x) + Q_{\alpha} f(x)$$ (8)
where $Q_n f(x) = P^{n-1} Q f(x)$ and $\{\alpha^{-n}(i)\}$ is the inverse permutation of $\{\alpha^n(i)\}$.

In the case that the measurable space is normalized and the Markov operator $P$ has a constant stationary density $f^*$ (e.g. if $P$ is an Frobenius-Perron operator this is equivalent to $\mu f$ being invariant (see [5], pag. 46) the Spectral Decomposition Theorem takes the following compact form (see [5], pag. 90):

**Theorem 7.** (The Spectral Decomposition Theorem (version II)) Let $(X, \Sigma, \mu)$ be a normalized measure space and $P : L^1 \to L^1$ a constrictive Markov operator. If $P$ has an stationary density, then the representation of $P^n f$ takes the simple form for all $f \in L^1$

$$P^n f(x) = \sum_{i=1}^{r} \lambda_{\alpha^{-n}(i)}(f) \mathcal{T}_{A_i}(x) + Q_n f(x)$$  \hspace{1cm} (9)

where

$$\mathcal{T}_{A_i}(x) = [1/\mu(A_i)]1_{A_i}$$  \hspace{1cm} (10)

$$\bigcup_iA_i = X \hspace{0.5cm} \text{with} \hspace{0.5cm} A_i \cap A_j = \emptyset \hspace{0.5cm} \text{for} \hspace{0.5cm} i \neq j$$  \hspace{1cm} (11)

and

$$\mu(A_i) = \mu(A_j) \hspace{0.5cm} \text{if} \hspace{0.5cm} j = \alpha^n(i) \hspace{0.5cm} \text{for some} \hspace{0.5cm} n.$$  \hspace{1cm} (12)

The last version of the Spectral Decomposition Theorem allows to characterize the Ergodic Hierarchy through the permutation $\{\alpha^n(i)\}$ (see [5], pag. 92,93 and 94).

**Theorem 8.** (The Spectral Decomposition Theorem and The Ergodic Hierarchy) Let $(X, \Sigma, \mu)$ be a normalized measure space and $P : L^1 \to L^1$ a constrictive Markov operator. Then

(I) $P$ is ergodic $\iff$ the permutation $\{\alpha(1),...,\alpha(r)\}$ of the sequence $\{1,...,r\}$ is cyclical (that is, for which there no is invariant subset).

(II) If $r = 1$ in the representation of equation (6) $\implies P$ is exact.

(III) If $P$ is mixing $\implies r = 1$ in the representation of equation (6).

The version 2 of the Spectral Decomposition Theorem says that if the classical Hamiltonian is such that Frobenius-Perron operator $P$ associated to the temporal evolution $T$ admits an equilibrium density $f_e$ then for very long times $(n \to \infty)$ the state of the system $U_n f(x) = f(T_n(x))$ will oscillate between the characteristic functions $\mathcal{T}_{A_i}(x)$ with a remanent term $Q_n f(x)$ going to zero. In the next section we will see that this decomposition in two terms contains relevant information with each term associated to the type of spectrum. We have seen that the hypothesis of constrictiveness of the Markov operator $P$ and normalization of the measure space are sufficient to ensure the existence of a stationary density and to obtain a representation of the temporal evolution of any density through The Spectral Decomposition Theorem.
4 The Quantum Version of The Spectral Decomposition Theorem (QSDT)

The aim of this paper is to obtain a quantum version of The Spectral Decomposition Theorem that can be useful to study quantum systems in the classical limit and can give a general framework to a theory describing the quantum chaos according to the Berry’s definition. In this section we begin by defining the mathematical elements necessary for this purpose using the observables as central object of quantum treatment and functional states as living in the dual space of those. Let $\hat{A}$ be the characteristic algebra of the quantum system, that is, $\hat{A}$ is an algebra whose self-adjoint elements $\hat{O} = \hat{O}^{\dagger}$ are the observables belonging to the space $\mathcal{O}$. The space of states is the positive cone $A = \{ \hat{\rho} \in \mathcal{O} : \hat{\rho} \hat{1} = 1, \quad \hat{\rho}^{\dagger} = \hat{\rho}, \quad \hat{\rho}(a\hat{a}^{\dagger}) \geq 0 \quad \text{for all} \quad \hat{a} \in \mathcal{O} \} \quad (13)$

where the action $\hat{\rho}(\hat{O})$ of the functional $\hat{\rho} \in \mathcal{O}$ on the observable $\hat{O} \in \mathcal{O}$ is denoted by $(\hat{\rho}\hat{O})$, and in the case that $\hat{O} = \hat{1}$ this action is the trace of $\hat{\rho}$ equal to $\text{tr}(\hat{\rho}) = \hat{\rho}(\hat{1}) = (\hat{\rho}\hat{1}) = 1$.

In this approach the state $\hat{\rho}$ is unknown and we study the expectation values $(\hat{\rho}(t)\hat{O})$ when $t \to \infty$. If there exists a unique $\hat{\rho}_{\ast} \in \mathcal{O}$ for all $\hat{\rho} \in \mathcal{O}$ such that

$$\lim_{t \to \infty} (\hat{\rho}(t)\hat{O}) = \lim_{t \to \infty} (\hat{U}_{t}\hat{\rho}\hat{U}_{t}^{\dagger}\hat{O}) = (\hat{\rho}_{\ast}\hat{O}) \quad (14)$$

we say that the evolution $\hat{U}_{t}$ has weak-limit $\hat{\rho}_{\ast}$ (see [1] pag. 248). This functional $\hat{\rho}_{\ast}$ is interpreted as the average value that would result if the state $\hat{\rho}(t)$ had a limit $\hat{\rho}_{\ast}$ for large times, i.e. it is a weak limit and it is not a limit in the (strong) usual sense. In other words, $\hat{\rho}_{\ast}$ is the equilibrium state in the weak sense that the system reaches in the relaxation.

In this framework the fundamental element to realize the classical limit of a quantum system, which mathematically consists in to obtain a classical algebra from a quantum algebra via an algebra “deformation” in the limit $\hbar \to 0$, is the Wigner transform. This transform allows us to obtain a function $f(\phi)$ defined over the phase space $\Gamma$ from a quantum state $\hat{\rho}$ where the function $f(\phi)$ can be interpreted such a distribution probability analogous to the statistic mechanics density $\rho(p,q)$ governed by the Liouville equation. Now we present a brief review of the more important Wigner transform properties.

Let $\Gamma = M_{2(N+1)} \equiv \mathbb{R}^{2(N+1)}$ be the phase space. The Wigner transformation $\text{symb} : \hat{A} \to A_{q}$ that sends the quantum algebra $\hat{A}$ to “classical like” $A_{q}$ algebra is given by (see [14], [15], [16]).

$$\text{symb}(\hat{f}) = f(\phi) = \int (q + \Delta |\hat{f}|q - \Delta)e^{i\frac{\hbar q}{2}}d\Delta + 1 \quad (15)$$

where $f(\phi) \in A_{q}$ are the functions over the space $\Gamma$ phase with coordinates $\phi = (q^{1}, ..., q^{N+1}, p_{q}^{1}, ..., p_{q}^{N+1})$.

The star product between two operators $\hat{f}, \hat{g} \in \hat{A}$ is given by (see [39])

$$\text{symb}(\hat{f}\hat{g}) = \text{symb}(\hat{f}) \ast \text{symb}(\hat{g}) = (f \ast g)(\phi) \quad (16)$$

and the Moyal bracket is

$$\{ f, g \}_{\text{mb}} = \frac{1}{i\hbar}(\text{symb}(\hat{f}) \ast \text{symb}(\hat{g}) - \text{symb}(\hat{g}) \ast \text{symb}(\hat{f})) = \text{symb}(\frac{1}{i\hbar}[\hat{f}, \hat{g}]) \quad (17)$$

Two important properties are (see [36])

$$(f \ast g)(\phi) = f(\phi)g(\phi) + 0(\hbar), \quad \{ f, g \}_{\text{mb}} = \{ f, g \}_{\text{pb}} + 0(\hbar^{2}) \quad (18)$$
The **symmetrical** or Weyl ordering prescription is used to define the inverse \( \text{symb}^{-1} \), that is
\[
\text{symb}^{-1}[q^{i}(\phi), p^{j}(\phi)] = \frac{1}{2}(\hat{q}^{i}\hat{p}^{j} + \hat{p}^{j}\hat{q}^{i})
\]  
(19)

Therefore, with \( \text{symb} \) and \( \text{symb}^{-1} \) is defined an isomorphism between the algebras \( \hat{A} \) and \( A_{q} \),
\[
\text{symb} : \hat{A} \rightarrow A_{q} \quad \text{and} \quad \text{symb}^{-1} : A_{q} \rightarrow \hat{A}
\]  
(20)

On the other hand the Wigner transformation for states is
\[
\rho(\phi) = (2\pi\hbar)^{-(N+1)}\text{symb}(\hat{\rho})
\]  
(21)

and the fundamental property of the Wigner transformation used in this work is the preservation of the inner product between states \( \hat{\rho} \in N \) and observables \( \hat{O} \in \hat{A} \) which physically represents the preservation of the expectation values that gives the same result to be calculated in both \( \hat{A} \) and \( A_{q} \), that is,
\[
\hat{\rho}(\hat{O}) = \langle \hat{O} \rangle_{\hat{\rho}} = \langle \rho(\phi), O(\phi) \rangle = \int d\phi 2^{(N+1)}\rho(\phi)O(\phi)
\]  
(22)

### 4.1 The Quantum Spectral Decomposition Theorem (QSDT)

In the previous section we have establish the framework based on the classical limit and the Wigner transform. Now we can write the spectral decomposition theorem in quantum language. First, we assume that

- In the classical limit \( h \rightarrow 0 \) the quantum system has a classical evolution \( T \) defined over the phase space \( \Gamma \) with an associated Frobenius-Perron operator \( P \) constreictive.

- There exists a stationary density \( f_{s} \), that is, \( Pf_{s} = f_{s} \).

Under these hypothesis we have the following quantum version of the Spectral Decomposition Theorem.

**Theorem 9.** (The Quantum Spectral Decomposition Theorem(QSDT)) Let \( \hat{\rho} \in N \) and let \( \hat{O} \) be an observable. Then there exists pure states \( \hat{\rho}_{1}, \hat{\rho}_{2}, \ldots, \hat{\rho}_{r} \); observables \( \hat{O}_{1}, \hat{O}_{2}, \ldots, \hat{O}_{r} \); a permutation \( \alpha : \{1, \ldots, r\} \rightarrow \{1, \ldots, r\} \) and \( \tilde{\rho}_{0} \in O' \) such that
\[
(\hat{\rho}(n)|\hat{O}) = \sum_{i=1}^{r} \lambda_{\alpha^{-1}(i)}(\hat{\rho}_{i}|\hat{O}) + (\tilde{\rho}_{0}(n-1)|\hat{O})
\]  
(23)

where
\[
\lambda_{i}(\hat{\rho}) = (\hat{\rho}|\hat{O}_{i})
\]  
(24)

The states \( \hat{\rho}_{i} \) and \( \tilde{\rho}_{0} \) have the following properties:

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\(^{1}\)Of course, we make the natural choice of \( T \) as \( T : \Gamma \rightarrow \Gamma \) with \( T(\phi = (q,p)) = \phi(1) = (q(1), p(1)) \), that is, \( T \) is the classical evolution operator determined by the Hamilton equations.
Therefore, the equation (29) reads as

\[ \hat{\rho}_i \hat{\rho}_j = 0(h) \text{ for all } i \neq j \text{ and } \hat{\rho}_i^2 = \hat{\rho}_i + 0(h). \]  

So that the states \( \hat{\rho}_i \) are projectors in the classical limit \( (h \to 0) \). Moreover, we have a descomposition of the identity:

\[ \hat{1} = \sum_i \alpha_i \hat{\rho}_i \text{ with } \alpha_i \geq 0, \sum_i \alpha_i = 1 \]  

(II) For each integer \( i \) exists a unique integer \( \alpha(i) \) such that \((\hat{U}_i \hat{\rho}_i \hat{U}_i^\dagger | \hat{O}) = (\hat{\rho}_{\alpha(i)} | \hat{O})\). Further \( \alpha(i) \neq \alpha(j) \) for \( i \neq j \) an thus the evolution operator \( \hat{U} = e^{-\frac{i}{\hbar} \hat{H}} \) (that is, \( \hat{U}_t = e^{-\frac{i}{\hbar} \hat{H}t} \) with \( t = 1 \)) is used to permute the states \( \hat{\rho}_i \).

(III) \( (\tilde{\rho}_{0(n-1)} | \hat{O}) \to 0 \text{ as } n \to \infty. \)

**Proof.** Let \( \hat{\rho} \in \mathcal{N} \) and let \( \hat{O} \) be an observable. If we define \( f = \text{symb}(\hat{\rho}) \) and \( g = \text{symb}(\hat{O}) \) then multiplying the equation 9 by \( g \) and integrating over all space we have

\[ \int_X P^n f(x)g(x)dx = \sum_{i=1}^r \lambda_{\alpha-n(i)}(f) \int_X T_{A_i}(x)g(x)dx + \int_X Q_n f(x)g(x)dx \]  

(26)

Equivalently,

\[ \langle P^n f, g \rangle = \sum_{i=1}^r \lambda_{\alpha-n(i)}(f) \langle T_{A_i}, g \rangle + \langle P^{n-1} Q f, g \rangle \]  

(27)

Since \( \langle P^n f, g \rangle = \langle f, U^n g \rangle \) and \( \langle P^{n-1} Q f, g \rangle = \langle Q f, U^{n-1} g \rangle \) (the Koopman operator is the dual operator of the Frobenius-Perron operator) and thus

\[ \langle f, U^n g \rangle = \sum_{i=1}^r \lambda_{\alpha-n(i)}(f) \langle T_{A_i}, g \rangle + \langle Q f, U^{n-1} g \rangle \]  

(28)

Now if we call \( \hat{\rho}_i = \text{symb}^{-1}(T_{A_i}), \tilde{\rho}_0 = \text{symb}^{-1}(Q f) \) and we use the fact that \( U^n g = g(n) = \text{symb}(\hat{O}(n)), U^{n-1} g = g(n-1) = \text{symb}(\hat{O}(n-1)) \) (see equation 3) then

\[ \langle \text{symb}(\hat{\rho}), \text{symb}(\hat{O}(n)) \rangle = \sum_{i=1}^r \lambda_{\alpha-n(i)}(f) \langle \text{symb}(\hat{\rho}_i), \text{symb}(\hat{O}) \rangle + \langle \text{symb}(\tilde{\rho}_0), \text{symb}(\hat{O}(n-1)) \rangle \]  

(29)

If we call \( k_i = \text{symb}(\hat{O}_i) \) and using equation 7 then the coefficient \( \lambda_{\alpha-n(i)}(f) \) can be written as

\[ \lambda_{\alpha-n(i)}(f) = \int_X f(x) k_{\alpha-n(i)}(x) dx = \langle f, k_{\alpha-n(i)} \rangle = \langle \text{symb}(\hat{\rho}), \text{symb}(\hat{\rho}_{\alpha-n(i)}) \rangle = \lambda_{\alpha-n(i)}(\hat{\rho}) \]  

(30)

Therefore, the equation (29) reads as
\begin{equation}
\langle \text{symb}(\hat{\rho}), \text{symb}(\hat{O}(n)) \rangle = \sum_{i=1}^{r} \lambda_{\alpha-n(i)}(\hat{\rho}) \langle \text{symb}(\hat{\rho}_i), \text{symb}(\hat{O}) \rangle + \langle \text{symb}(\tilde{\rho}_0), \text{symb}(\hat{O}(n-1)) \rangle
\end{equation}

Finally we can use the property of the Weyl symbol (see [12], pag 251 eq.(24)), that is

\begin{equation}
\forall \hat{O} \in \hat{A}, \forall \hat{\rho} \in \hat{A}': (\hat{\rho} | \hat{O}) = \langle \text{symb}(\hat{\rho}), \text{symb}(\hat{O}) \rangle = \int \rho(\phi) O(\phi) \phi
\end{equation}

Using this property the equation (31) can be expressed in quantum language as

\begin{equation}
(\hat{\rho}(n) | \hat{O}) = \sum_{i=1}^{r} \lambda_{\alpha-n(i)}(\hat{\rho}) (\hat{\rho}_i | \hat{O}) + (\tilde{\rho}_0(n-1) | \hat{O})
\end{equation}

and therefore we have proved the equation (23).

(I): On one hand we have that

\begin{equation}
tr(\hat{\rho}_i) = (\hat{\rho}_i | \hat{1}) = \langle \text{symb}(\hat{\rho}_i), \text{symb}(\hat{1}) \rangle = \langle \mathbf{T}_{A_i}, 1_X \rangle = \int_{X} \mathbf{T}_{A_i} dx = \int_{X} [1/\mu(A_i)] 1_{A_i} dx = [1/\mu(A_i)] \int_{X} 1_{A_i} dx = [1/\mu(A_i)] \mu(A_i) = 1
\end{equation}

where we have used the definition of \( \mathbf{T}_{A_i} \) (equation (10)), the property of the Weyl’s symbol (equation (32)) and \( \text{symb}(\hat{1}) = 1_X \). Therefore \( tr(\hat{\rho}_i) = 1 \), that is, \( \hat{\rho}_i \in \mathcal{N} \) for all \( i \). On the other hand if we use equations (10) and (17) applied to \( f = \mathbf{T}_{A_i} \) and \( g = \mathbf{T}_{A_j} \) we have

\begin{equation}
\text{symb}(\hat{\rho}_i \hat{\rho}_j) = \mathbf{T}_{A_i}(x) \mathbf{T}_{A_j}(x) = 0(h) \text{ if } i \neq j
\end{equation}

Now applying the inverse \( \text{symb}^{-1} \) to both sides of the equation (36) we have

\begin{equation}
\hat{\rho}_i \hat{\rho}_j = 0(h) \text{ if } i \neq j
\end{equation}

On the other hand from the equations (10), (11) we have that \( 1_X = \sum_i \mu(A_i) \mathbf{T}_{A_i} \) and therefore

\begin{equation}
\text{symb}^{-1}(1_X) = \text{symb}^{-1}(\sum_i \mu(A_i) \mathbf{T}_{A_i}) = \sum_i \mu(A_i) \text{symb}^{-1}(\mathbf{T}_{A_i}) \text{ where we have used that } \text{symb}^{-1}
\end{equation}
is a linear map. Now if we call \( \alpha_i = \mu(A_i) \) since \( \text{symb}^{-1}(1_X) = \hat{1} \) and \( \hat{\rho}_i = \text{symb}^{-1}(\hat{T}_{A_i}) \) then we deduce the equation (25).

(II): Due to part (II) of the Spectral Decomposition Theorem version I and considering that we are in the hypothesis of version II of the Spectral Decomposition Theorem we have

\[
P\hat{T}_{A_i} = \hat{T}_{A_{\alpha(i)}}
\]

where \( \alpha : \{1, ..., r\} \to \{1, ..., r\} \) is a permutation which satisfies \( \alpha(i) \neq \alpha(j) \) for \( i \neq j \) and thus the operator \( P \) is used to permute the functions \( \hat{T}_{A_i} \). Let \( \hat{O} \) be an observable and \( g = \text{symb}(\hat{O}) \).

Then from equation (38) it follows that

\[
\langle P\hat{T}_{A_i}, g \rangle = \langle \hat{T}_{A_{\alpha(i)}}, g \rangle
\]

and noting that

\[
\begin{align*}
\langle P\hat{T}_{A_i}, g \rangle &= \langle \hat{T}_{A_i}, U_g \rangle = \langle \text{symb}(\hat{\rho_i}), \text{symb}(\hat{O}(1)) \rangle = \langle \hat{\rho_i}|\hat{O}(1) \rangle = (\hat{U}\hat{\rho}_i\hat{U}^\dagger|\hat{O}) \\
\langle \hat{T}_{A_{\alpha(i)}}, g \rangle &= \langle \text{symb}(\hat{\rho}_{\alpha(i)}), \text{symb}(\hat{O}) \rangle = \langle \hat{\rho}_{\alpha(i)}|\hat{O} \rangle
\end{align*}
\]

then from (38) we have that

\[
(\hat{U}\hat{\rho}_i\hat{U}^\dagger|\hat{O}) = (\hat{\rho}_{\alpha(i)}|\hat{O}).
\]

(III): Let \( \hat{O} \) be an observable and \( \epsilon > 0 \). Then by the condition (III) of the Spectral Decomposition Theorem version I we have

\[
\|P^{n-1}Qf\| = \|Q_nf\| < \frac{\epsilon}{\max_{\phi \in X} |O(\phi)|} = \frac{\epsilon}{\|O\|_\infty}\text{ with } O = \text{symb}(\hat{O})
\]

Then

\[
(\hat{\rho}_0(n-1)|\hat{O}) = \langle \text{symb}(\hat{\rho}_0(n-1)), \text{symb}(\hat{O}) \rangle = \langle Q_nf, O \rangle \leq \|Q_nf\|\|O\|_\infty < \epsilon
\]

Therefore, from the equation (42) it follows that \( (\hat{\rho}_0(n-1)|\hat{O}) \to 0. \)

4.2 The Levels of The Quantum Ergodic Hierarchy: Ergodic, Mixing and Exact

The theorem 8 and the Spectral Decomposition Theorem (theorem 7) can be used simultaneously to determine the ergodicity, the mixing or the exactness just looking the terms that there are in the sum of (6) or (9). And this also holds for the Quantum Spectral Decomposition Theorem (QSDT, theorem 9) simply because this is a quantum language translation of the spectral theorem for dynamical systems. Therefore we apply the theorem 8 and QSDT to obtain the following theorem which classifies the ergodic, exact and mixing levels of the Quantum Ergodic Hierarchy (see [1] theorems 1,2 pages 261, 263).

**Theorem 10.** (The levels of The Quantum Ergodic Hierarchy: Ergodic, Exact and Mixing)

Let \( S \) be a quantum system. Let \( \hat{\rho} \in \mathcal{N} \) and let \( \hat{O} \) be an observable. Then

(I) \( S \) is ergodic \( \iff \) the permutation \( \{\alpha(1), ..., \alpha(r)\} \) of the sequence \( \{1, ..., r\} \) is cyclical (that is, for which there no is invariant subset).

(II) If \( r = 1 \) in the representation of (23) \( \Rightarrow S \) is exact.
(III) If $S$ is mixing (see [1] theorem 1 pag. 261) $\implies r = 1$ in the representation of (23).

Proof. It is enough to apply the theorem 8 and the theorem 9. □

In the following subsections we examine the ergodic hierarchy established by the theorem 10 and their consequences in more detail.

4.2.1 A consequence of QSDT: homogenization of the mixing level

If the system is mixing then from theorem 10 and the equation (23) it follows that

$$(\hat{\rho}(n)\mid \hat{O}) = (\hat{\rho}\mid \hat{O}_1)(\hat{\rho}_1\mid \hat{O}) + (\hat{\rho}_0(n-1)\mid \hat{O})$$

(43)

where $\hat{\rho}_1$ is a pure state (in the classical limit) and $\hat{O}_1$ is an observable which does not depend on the observable $\hat{O}$. Further, since the system is mixing then it has a weak limit $\hat{\rho}_s$ such that

$$\lim_{n \to \infty} (\hat{\rho}(n)\mid \hat{O}) = (\hat{\rho}_s\mid \hat{O})$$

From this limit and the equation (44) we obtain

$$(\hat{\rho}_s\mid \hat{O}) = \lim_{n \to \infty} (\hat{\rho}(n)\mid \hat{O}_1)(\hat{\rho}_1\mid \hat{O}) + \lim_{n \to \infty} (\hat{\rho}_0(n-1)\mid \hat{O}) = (\hat{\rho}\mid \hat{O}_1)(\hat{\rho}_1\mid \hat{O})$$

(44)

since $\lim_{n \to \infty} (\hat{\rho}_0(n-1)\mid \hat{O}) = 0$ (see Theorem 9 (III) subsection 4.1). Now if we make $O = 1$ in the equation (44) since $\langle \hat{\rho}_s\mid \hat{1} \rangle = tr(\hat{\rho}_s) = 1$ and $\langle \hat{\rho}_1\mid \hat{1} \rangle = tr(\hat{\rho}_1) = 1$ we conclude that $(\hat{\rho}\mid \hat{O}_1) = 1$ then $(\hat{\rho}_s\mid \hat{O}) = (\hat{\rho}_1\mid \hat{O})$ for all $\hat{O}$ observable and therefore $\hat{\rho}_1 = \hat{\rho}_s$. That is, physically, the mixing level is responsible for “homogenize” the initial state $\hat{\rho}$ to take it to the weak limit $\hat{\rho}_1$ which is a pure state. In this sense QSDT gives a physical interpretation of the mixing level.

4.2.2 The exact level

Let $\hat{\rho}$ be an state and $\hat{O}$ an observable. Then by QSDT we have the representation of $(\hat{\rho}\mid \hat{O})$ given by the equation (23). If $r = 1$ then by the theorem 10 the system is exact. That is, for the exact case the theorem 10 only gives a sufficient condition.

4.2.3 The ergodic level: Oscillation of the mean values

From theorem 10 we can deduce a necessary and sufficient condition for ergodicity. The system is ergodic if and only if the permutation $\alpha$ of

$$(\hat{\rho}(n)\mid \hat{O}) = \sum_{i=1}^{r} (\hat{\rho}\mid \hat{O}_{\alpha^{-n}(i)})(\hat{\rho}_i\mid \hat{O}) + (\hat{\rho}_0(n-1)\mid \hat{O})$$

(45)

is cyclical and $(\hat{\rho}_0(n-1)\mid \hat{O}) \to 0$. Since $\alpha$ is cyclical there is an integer $N > 0$ such that $\alpha^N(i) = i$ and $\alpha^{-N}(i) = i$ for $i = 1, ..., r$. That is, the inverse permutation $\alpha^{-1}$ operates on the indices $i = 1, ..., r$ as

$$(1, ..., r) \rightarrow \alpha^{-1}(2, 3, ..., r-2, r-1, r, 1) \rightarrow \alpha^{-2}(3, 4, ..., r-1, r, 1, 2) ... \rightarrow \alpha^{-k} \rightarrow \alpha^{-N}(1, ..., r)$$

(46)

After $N$ successive steps we are back to the original cycle $(1, ..., r)$. This behavior indicates that the sum of (45) will also return to its original value after $N$ successive time instants. Then the sum of (45) is periodic with a period equal to $N$, i.e. with the same period as the cycle $\alpha^{-1}$. Indeed, we have
\( \langle \hat{\rho}(0) | \hat{O} \rangle = (\hat{\rho}|\hat{O}_1)(\hat{\rho}_1|\hat{O}) + (\hat{\rho}|\hat{O}_2)(\hat{\rho}_2|\hat{O}) + ... + (\hat{\rho}|\hat{O}_{F-1})(\hat{\rho}_{F-1}|\hat{O}) + (\hat{\rho}|\hat{O}_{F})(\hat{\rho}_F|\hat{O}) + (\hat{\rho}_0(-1)|\hat{O}) \)

\( \langle \hat{\rho}(1) | \hat{O} \rangle = (\hat{\rho}|\hat{O}_2)(\hat{\rho}_1|\hat{O}) + (\hat{\rho}|\hat{O}_3)(\hat{\rho}_2|\hat{O}) + ... + (\hat{\rho}|\hat{O}_{F})(\hat{\rho}_{F-1}|\hat{O}) + (\hat{\rho}|\hat{O}_1)(\hat{\rho}_F|\hat{O}) + (\hat{\rho}_0(0)|\hat{O}) \)

\( \langle \hat{\rho}(2) | \hat{O} \rangle = (\hat{\rho}|\hat{O}_3)(\hat{\rho}_1|\hat{O}) + (\hat{\rho}|\hat{O}_4)(\hat{\rho}_2|\hat{O}) + ... + (\hat{\rho}|\hat{O}_1)(\hat{\rho}_{F-1}|\hat{O}) + (\hat{\rho}|\hat{O}_2)(\hat{\rho}_F|\hat{O}) + (\hat{\rho}_0(1)|\hat{O}) \)

\( \langle \hat{\rho}(N) | \hat{O} \rangle = (\hat{\rho}|\hat{O}_1)(\hat{\rho}_1|\hat{O}) + (\hat{\rho}|\hat{O}_2)(\hat{\rho}_2|\hat{O}) + ... + (\hat{\rho}|\hat{O}_{F-1})(\hat{\rho}_{F-1}|\hat{O}) + (\hat{\rho}|\hat{O}_F)(\hat{\rho}_F|\hat{O}) + (\hat{\rho}_0(N-1)|\hat{O}) \)

\( \langle \hat{\rho}(N+1) | \hat{O} \rangle = (\hat{\rho}|\hat{O}_2)(\hat{\rho}_1|\hat{O}) + (\hat{\rho}|\hat{O}_3)(\hat{\rho}_2|\hat{O}) + ... + (\hat{\rho}|\hat{O}_F)(\hat{\rho}_{F-1}|\hat{O}) + (\hat{\rho}|\hat{O}_1)(\hat{\rho}_F|\hat{O}) + (\hat{\rho}_0(N)|\hat{O}) \)

Then \( \sum_{i=1}^{r} (\hat{\rho}|\hat{O}_{\alpha-(N+1)(i)})(\hat{\rho}_i|\hat{O}) = \sum_{i=1}^{r} (\hat{\rho}|\hat{O}_{\alpha-(1)(i)})(\hat{\rho}_i|\hat{O}) \).

That is, if we call \( F(n) = \sum_{i=1}^{r} (\hat{\rho}|\hat{O}_{\alpha-(1)(i)})(\hat{\rho}_i|\hat{O}) \) then

\[ F(N+1) = F(1) \]

and therefore this sum \( F(n) \) is periodic with a period equal to \( N \). Therefore, we see that in the ergodic case the mean values consist of an oscillating part plus a term which tends to zero for large times \( (n \text{ goes to } \infty) \). The physical interpretation of this behavior is studied in the next section and it is the key to find the type of spectrum of the ergodic systems.

## 5 QSDT and the quantum spectrum

The theorem QSDT besides characterizing the mean values of the ergodic hierarchy also gives us a connection between ergodic hierarchy levels and the spectra. In this section we extend this connection to the cases of the discrete spectra, the continuous spectra and for a more general case where both spectra are simultaneously present.

### 5.1 Discrete spectrum

Let \( \hat{\rho} \in \mathcal{N} \) an state and let \( \hat{O} \) be an observable. We assume that the spectrum is discrete and \( E_1, E_2, ... \) are the energies of the system with \( \omega_1 = \frac{E_1}{\hbar}, \omega_2 = \frac{E_2}{\hbar}, ... \) the natural frequencies of each energy level. Then the mean value of \( \hat{O} \) in the state \( \hat{\rho} \) in the instant \( t = n \) is

\[ \langle \hat{\rho}(n) | \hat{O} \rangle = \sum_{j} (\hat{\xi}_j|\hat{O}) e^{-\frac{j\omega}{\hbar}n} \]

where \( \hat{\xi}_j = |j\rangle\langle j| \in \mathcal{N} \) and \( |j\rangle \) is the eigenstate with energy \( E_j \). Furthermore, by the QSDT theorem we have

\[ \langle \hat{\rho}(n) | \hat{O} \rangle = \sum_{i=1}^{r} \lambda_{\alpha-(1)(i)}(\hat{\rho}_i|\hat{O}) + (\hat{\rho}_0(n-1)|\hat{O}) \]

Then,
\[
\sum_{j} (\xi_{j} | \hat{O}) e^{-j x_{j} n} = \sum_{i=1}^{r} \lambda_{\alpha-i} \langle \hat{r}_{i} | \hat{O} \rangle + \langle \tilde{\rho}_{0} (n-1) | \hat{O} \rangle \tag{51}
\]

Since \(\sum_{j} (\xi_{j} | \hat{O}) e^{-j x_{j} n} \) and \(\sum_{i=1}^{r} \lambda_{\alpha-i} \langle \hat{r}_{i} | \hat{O} \rangle \) are periodic functions and \(\langle \tilde{\rho}_{0} (n-1) | \hat{O} \rangle \to 0\) when \(n \to \infty\) it follows that \(\tilde{\rho}_{0}\) is equal to zero. Then we conclude that the discrete spectrum corresponds to the ergodic level. That is, from the QSDT theorem it follows that in the classical limit the quantum systems of spectrum discrete correspond to the ergodic level.

### 5.2 Continuous spectrum

Now we assume that the spectrum is continuous with \(\omega \in [0, \infty)\) and \(|\omega|\) which are the energies and the eigenvectors of the system. Let \(\hat{r} \in \mathcal{N}\) an state and let \(\hat{O}\) be an observable. In order to obtain an equilibrium arrival of the system we restrict the space of observables only considering the Van Hove observables. This restriction does not lose generality because the observables that do not belong to Van Hove space are not experimentally accessible (see [17] for a complete argument). The components of a Van Hove observable \(\hat{O}_{R}\) are \(O_{R}(\omega, \omega') = O(\omega)\delta(\omega - \omega') + O(\omega, \omega')\). Then we can expand \(\hat{O}\) in the basis \(|\omega\rangle \langle \omega|, |\omega\rangle \langle \omega'|\) as

\[
\hat{O} = \int_{0}^{\infty} O(\omega)|\omega\rangle \langle \omega| + \int_{0}^{\infty} \int_{0}^{\infty} O(\omega, \omega') \langle \omega \rangle \langle \omega'| \tag{52}
\]

Therefore, the mean value of \(\hat{O}\) in the state \(\hat{r}\) for the instant \(t = n\) is

\[
\langle \hat{r}(n) | \hat{O} \rangle = \int_{0}^{\infty} \rho(\omega) O(\omega) d\omega + \int_{0}^{\infty} \int_{0}^{\infty} \rho(\omega, \omega') O(\omega, \omega') e^{-i \frac{(\omega - \omega')}{\hbar} n} d\omega d\omega' \tag{53}
\]

On the other hand, by QSDT theorem we have

\[
\langle \hat{r}(n) | \hat{O} \rangle = \sum_{i=1}^{r} \langle \hat{r} | \hat{O}_{\alpha-i} \rangle \langle \hat{r}_{i} | \hat{O} \rangle + \langle \tilde{\rho}_{0} (n-1) | \hat{O} \rangle = \int_{0}^{\infty} \rho(\omega) O(\omega) d\omega + \int_{0}^{\infty} \int_{0}^{\infty} \rho(\omega, \omega') O(\omega, \omega') e^{-i \frac{(\omega - \omega')}{\hbar} n} d\omega d\omega' \tag{54}
\]

If we suppose that \(\rho(\omega, \omega') O(\omega, \omega') \in L^{1}([0, \infty) \times [0, \infty))\) then by the Riemann-Lebesgue Lemma it is

\[
\int_{0}^{\infty} \int_{0}^{\infty} \rho(\omega, \omega') O(\omega, \omega') e^{-i \frac{(\omega - \omega')}{\hbar} n} d\omega d\omega' \to 0 \tag{55}
\]

when \(n \to \infty\). That is, under the restriction of the observable space (Van Hove algebra) and under the assumption that \(\rho(\omega, \omega') O(\omega, \omega') \in L^{1}([0, \infty) \times [0, \infty))\) then the system is mixing.

Moreover, from (53) and (55) it follows that \(\langle \hat{r}(n) | \hat{O} \rangle \to \langle \hat{r}_{*} | \hat{O} \rangle = \int_{0}^{\infty} \rho(\omega) O(\omega) d\omega\) as \(n \to \infty\) with \(\hat{r}_{*} = \int_{0}^{\infty} \rho(\omega) |\omega\rangle \langle \omega| d\omega\).

Therefore, \(\hat{r}\) has weak limit equal to \(\hat{r}_{*}\). Now since the system is mixing then by the theorem 10 it follows that \(r = 1\) in the sum of (51). Then we have

\[
\langle \hat{r}(n) | \hat{O} \rangle = \langle \hat{r} | \hat{O}_{1} \rangle \langle \hat{r}_{1} | \hat{O} \rangle + \langle \tilde{\rho}_{0} (n-1) | \hat{O} \rangle = \int_{0}^{\infty} \rho(\omega) O(\omega) d\omega + \int_{0}^{\infty} \int_{0}^{\infty} \rho(\omega, \omega') O(\omega, \omega') e^{-i \frac{(\omega - \omega')}{\hbar} n} d\omega d\omega' \tag{56}
\]
Since \(\langle \hat{\rho} | \hat{O}_1 \rangle (\hat{\rho}_1 | \hat{O})\) and \(\int_0^\infty \rho(\omega)^* O(\omega) d\omega\) are constants and
\[
\int_0^\infty \int_0^\infty \rho(\omega, \omega')^* O(\omega, \omega') e^{-i \frac{(\omega - \omega')n}{\hbar}} d\omega d\omega', (\tilde{\rho}_0(n - 1) | \hat{O}) \rightarrow 0 \text{ then by (56)} \text{ we have that}
\]
\[
(\hat{\rho} | \hat{O}_1) (\hat{\rho}_1 | \hat{O}) = \int_0^\infty \rho(\omega)^* O(\omega) d\omega
\]
\[
(\tilde{\rho}_0(n - 1) | \hat{O}) = \int_0^\infty \int_0^\infty \rho(\omega, \omega')^* O(\omega, \omega') e^{-i \frac{(\omega - \omega')n}{\hbar}} d\omega d\omega'.
\]

From (57) we see the physical interpretation of the term \((\tilde{\rho}_0(n - 1) | \hat{O})\) in the decomposition (23), that is, the term \((\tilde{\rho}_0(n - 1) | \hat{O})\) is the manifestation of the Riemann-Lebesgue Lemma in closed quantum systems of continuous spectrum [12]. Therefore, a consequence of QSDT is that the mixing systems which have continuous spectrum are those with only one term \((\hat{\rho} | \hat{O}_1) (\hat{\rho}_1 | \hat{O})\) in the decomposition of the mean value \((\hat{\rho}(n) | \hat{O})\) given by the equation (23), and with the Riemann-Lebesgue Lemma contained in the term \((\tilde{\rho}_0(n - 1) | \hat{O})\) that goes to zero as \(n \rightarrow \infty\).

### 5.3 The General Case: Discrete and Continuous Spectrum

If the discrete and continuous spectrum are simultaneously present in accordance with (49) and (53) we have
\[
(\hat{\rho}(n) | \hat{O}) = \sum_j (\hat{\xi}_j | \hat{O}) e^{-j \frac{\omega}{\hbar} n} + \int_0^\infty \rho(\omega)^* O(\omega) d\omega + \int_0^\infty \int_0^\infty \rho(\omega, \omega')^* O(\omega, \omega') e^{-i \frac{(\omega - \omega')n}{\hbar}} d\omega d\omega' \quad (58)
\]
where the sum is the contribution to the mean value of the discrete spectrum and integrals are the contributions from the continuum. Now if we call \(A = \max_{1 \leq n \leq N} \{ \sum_{i=1}^r \langle \hat{\rho} | \hat{O}_{\alpha-n(i)} \rangle (\hat{\rho}_1 | \hat{O}) \}\) and \(B = \min_{1 \leq n \leq N} \{ \sum_{i=1}^r \langle \hat{\rho} | \hat{O}_{\alpha-n(i)} \rangle (\hat{\rho}_1 | \hat{O}) \}\) we can express the decomposition given by the equation (23) as
\[
(\hat{\rho}(n) | \hat{O}) = \{ \sum_{i=1}^r \langle \hat{\rho} | \hat{O}_{\alpha-n(i)} \rangle (\hat{\rho}_1 | \hat{O}) - \frac{A + B}{2} \} + \frac{A + B}{2} + (\tilde{\rho}_0(n - 1) | \hat{O}) \quad (59)
\]
where we have used the equation (24) for the coefficients \(\lambda_{\alpha-n(i)}\). We see the usefulness of this rewriting of \((\hat{\rho}(n) | \hat{O})\) below. Now if we compare the three terms of (58) and (59) we conclude that
\[
\sum_j (\hat{\xi}_j | \hat{O}) e^{-j \frac{\omega}{\hbar} n} = \{ \sum_{i=1}^r \langle \hat{\rho} | \hat{O}_{\alpha-n(i)} \rangle (\hat{\rho}_1 | \hat{O}) - \frac{A + B}{2} \}
\]
\[
\int_0^\infty \rho(\omega)^* O(\omega) d\omega = \frac{A + B}{2} \quad (60)
\]
\[
\int_0^\infty \int_0^\infty \rho(\omega, \omega')^* O(\omega, \omega') e^{-i \frac{(\omega - \omega')n}{\hbar}} d\omega d\omega' = (\tilde{\rho}_0(n - 1) | \hat{O})
\]
Therefore, from (60) we see that the QSDT theorem can express the mean value of any observable and also at the same time it indicates to which level of the ergodic hierarchy it belongs. This
relation that QSDT establishes between the type of spectrum and the ergodic hierarchy levels provides a formal framework for the study of the possible connections between the classical limit and the quantum chaos.

6 An Application of QSDT: Two-Level System

We conclude this paper by applying QSDT on the two-level system to make clear the consequences described in section 4.2.3. Despite being one of the simplest quantum models, the two-level system is one of the most used for testing formalisms, for teaching purposes, etc. The Hamiltonian of the two-level system is given by

$$\hat{H} = E_1|1\rangle\langle 1| + E_2|2\rangle\langle 2|$$  \hfill (61)

where \(|1\rangle\), \(|2\rangle\) are the eigenstates whose energies are \(E_1\) and \(E_2\) respectively. We consider an observable \(\hat{O}\) and a state \(\hat{\rho}\) given by

$$\hat{\rho} = \rho_{11}|1\rangle\langle 1| + \rho_{22}|2\rangle\langle 2| + \rho_{12}|1\rangle\langle 2| + \rho_{21}|2\rangle\langle 1|$$ \hfill (62)

where \(\rho_{ij} = \langle i|\hat{\rho}|j\rangle\) and \(\rho_{ii} \geq 0\), \(\rho_{ij} = \bar{\rho}_{ij}\) with \(i, j = 1, 2\). The state \(\rho\) in the instant \(t = n\) is

$$\hat{\rho}(n) = \rho_{11}|1\rangle\langle 1| + \rho_{22}|2\rangle\langle 2| + \rho_{12}e^{-\frac{i}{\hbar}(E_1-E_2)n}|1\rangle\langle 2| + \rho_{21}e^{-\frac{i}{\hbar}(E_2-E_1)n}|2\rangle\langle 1|$$ \hfill (63)

Then the mean value of \(\hat{O}\) in the instant \(t = n\) is

$$\langle \hat{\rho}(n)|\hat{O}\rangle = tr(\hat{\rho}(n)\hat{O}) = \langle 1|\hat{\rho}(n)\hat{O}|1\rangle + \langle 2|\hat{\rho}(n)\hat{O}|2\rangle =$$

$$\rho_{11}O_{11} + \rho_{22}O_{22} + \rho_{12}e^{-\frac{i}{\hbar}(E_1-E_2)n}O_{12} + \rho_{21}e^{-\frac{i}{\hbar}(E_2-E_1)n}O_{21}$$ \hfill (64)

where \(\langle j|\hat{O}|i\rangle = O_{ij}\) with \(i, j = 1, 2\). Now by QSDT we have

$$\sum_{i=1}^{r}(\hat{\rho}_\psi|\hat{O}_\text{\alpha^{-n}(i)})(\hat{\rho}_i|\hat{O}) + (\hat{\rho}_{0}(n - 1)|\hat{O}) =$$

$$\rho_{11}O_{11} + \rho_{22}O_{22} + \rho_{12}e^{-\frac{i}{\hbar}(E_1-E_2)n}O_{12} + \rho_{21}e^{-\frac{i}{\hbar}(E_2-E_1)n}O_{21}$$ \hfill (65)

Since the right member of (65) is oscillatory because there are imaginary exponentials, then the left member of (65) must be oscillatory. From this fact it follows that \(\rho_{0} = 0\) and the permutation \(\alpha\) is cyclical. Therefore, from the theorem 10 (I) it follows that the two-level system is ergodic. Moreover, we can obtain the Césaro limit (see \[3\] corollary 4.4.1. (a)):

$$\lim_{M \to \infty} \frac{1}{M} \sum_{n=0}^{M-1} (\hat{\rho}(n)|\hat{O}) =$$

$$= \rho_{11}O_{11} + \rho_{22}O_{22} + \rho_{12}O_{12}\{\lim_{M \to \infty} \frac{1}{M} \sum_{n=0}^{M-1} e^{-\frac{i}{\hbar}(E_1-E_2)n}\} + \rho_{21}O_{21}\{\lim_{M \to \infty} \frac{1}{M} \sum_{n=0}^{M-1} e^{-\frac{i}{\hbar}(E_2-E_1)n}\} =$$

$$= \rho_{11}O_{11} + \rho_{22}O_{22} + \rho_{12}O_{12}\{\lim_{M \to \infty} \sigma_M \} + \rho_{21}O_{21}\{\lim_{M \to \infty} \sigma_M \}^* =$$

$$= \rho_{11}O_{11} + \rho_{22}O_{22} + \rho_{12}O_{12}\left(\frac{1}{1 - z}\right) + \rho_{21}O_{21}\left(\frac{1}{1 - z}\right)^*$$ \hfill (66)
with $\sigma_M = \frac{1}{M} \sum_{n=0}^{M-1} e^{-\frac{i}{\hbar}(E_1-E_2)n}$ and $z = e^{-\frac{i}{\hbar}(E_1-E_2)}$. From (66) we have that the Césaro limit $\hat{\rho}_c$ of $\hat{\rho}$ is

$$\hat{\rho}_c = \rho_{11}|1\rangle\langle 1| + \rho_{22}|2\rangle\langle 2| + \rho_{12}(\frac{1}{1-z})|1\rangle\langle 2| + \rho_{21}(\frac{1}{1-z})^*|2\rangle\langle 1|$$  \hspace{1cm} (67)

This is so because

$$(\hat{\rho}_c|\hat{O}) = (1|\hat{\rho}_c\hat{O}|1) + (2|\hat{\rho}_c\hat{O}|2) =
\rho_{11}O_{11} + \rho_{22}O_{22} + \rho_{12}O_{12}\left(\frac{1}{1-z}\right) + \rho_{21}O_{21}\left(\frac{1}{1-z}\right)^*$$  \hspace{1cm} (68)

and therefore, from (66) and (68) it follows that

$$\lim_{M \to \infty} \frac{1}{M} \sum_{n=0}^{M-1} (\hat{\rho}(n)\hat{O}) = (\hat{\rho}_c|\hat{O})$$  \hspace{1cm} (69)

7 Conclusions

Under the assumption that in the classical limit the quantum system has an analogue classical system whose evolution $T$ has an associated Frobenius-Perron operator $P$ constrictive we have been able to translate The Spectral Decomposition Theorem of dynamical systems (theorems 6,7) to quantum language (QSDT) which gives a representation of the expectation value of an observable $\hat{O}$ (equation (23)) in a state $\hat{\rho}$.

The relevant feature of the QSDT representation is that it can express in a simple way in the classical limit ($\hbar \to 0$) the oscillatory nature of the expectation values in the case of discrete spectrum (equation (47)) and the manifestation of the Riemann-Lebesgue lemma for the Van Hove observables in the case of continuous spectrum (equation (57)).

QSDT theorem also connects the lower levels of the hierarchy ergodic with spectral type (equations (51),(57)) obtaining that the discrete spectrum corresponds to the ergodic level. For the mixing case, QSDT provides a physical interpretation of the “homogenization” (equations (43) and (44)) that is represented by the presence of the pure state $\hat{\rho}_1$ as the weak limit for any initial state $\hat{\rho}$.

When both spectra are present QSDT also gives a representation of the expectation value where the sum represents the discrete spectrum and the constant term plus the term that goes to zero represents the continuous spectrum (equation (60)).

In Section 6 the application of QSDT allowed us to classify the two-level system in the ergodic level and this was verified by performing the corresponding Cesaro limit (equations (66), (67) and (68)). We hope that all the features and connections provided by the QSDT among the classical limit, the Quantum Ergodic Hierarchy and the type of spectrum can be extended in future studies through more examples or theoretical essays.

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