Revised neutron drip density of magnetized matter

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Abstract

We study the onset of neutron drip in high density matter in the presence of magnetic field. It has been found that for systems having only protons and electrons in the presence of magnetic field $\gtrsim 10^{15}$G, the drip occurs at a density which is at least an order of magnitude higher compared to that in a nonmagnetic system. In a system with heavier ions, the effect of magnetic field, however, starts arising at a much higher field, $\gtrsim 10^{17}$G. These results may have important implications in high magnetic neutron stars and white dwarfs and, in general, nuclear astrophysics when the system is embedded with high magnetic field.
1 Introduction

The neutron stars are believed to have surface magnetic field as large as $\sim 10^{15}$G (magnetar model) [1] and hence their interior field ($B_{\text{int}}$) could even be a few orders of magnitude higher, say $\sim 10^{18}$G [2, 3] where the density is also higher. On the other hand, for white dwarfs with typical radius ($R \sim 5000$km), the maximum $B_{\text{int}}$ could be restricted to $\sim 10^{12}$G from the scalar virial theorem. However, for the recently proposed high density white dwarfs with highly tangled/fluctuating magnetic field and radius, e.g., $R \sim 70$km, $B_{\text{int}} \gtrsim 10^{17}$G [4]. Such smaller white dwarfs’ central density ($\rho_c$) is $\sim 10^{13}$gm/cc and hence quite above the non-magnetic threshold density of neutronization. All these motivate us to study the effect of (high) magnetic field on neutronisation for the degenerate fermions at high densities. This will be helpful to interpret the detailed spectra of cooling neutron stars, radio emission and $\gamma$–ray burst from neutron stars. For such high density regions of neutron stars and white dwarfs, the mean Fermi energy and the cyclotron energy of an electron exceed its rest-mass energy; for the latter condition to hold, the magnetic field needs to be sufficiently high, making the electrons relativistic. Further, we know that inverse $\beta$–decay can occur when the energy of electrons becomes higher than the difference between the rest mass energies of neutron and proton. Such a condition is expected to be modified in the presence of magnetic field which we plan to explore here at zero temperature.

The plan of the paper is the following. In the next section, we discuss the neutron drip density in a neutron-proton-electron ($n-p-e$) gas. Subsequently, in §3 we derive the drip density in a system with heavier ions. Finally, we end with a summary in §4.

2 Neutron drip in a neutron-proton-electron gas

In this system, the generic condition for the initiation of inverse $\beta$–decay and hence neutronization is chemical equilibrium given by

$$E_c^F + E_p^F = E_n^F,$$

(1)
where $E_{e,p,n}^F$ are the Fermi energies for electron, proton and neutron given by

$$E_{e,p}^F = \sqrt{p_{F}^2c^2 + \Delta_{2,1}c^4} \quad \text{with} \quad \Delta_{2,1} = m_{e,p}^2(1 + 2\nu_{e,p}B_{D}^{e,p})$$

and

$$E_n^F = \sqrt{p_{n}^2c^2 + m_n^2c^4},$$

where $m_{e,p,n}$ are the masses of electron, proton and neutron respectively, $B_{D}^{e,p}$ are the magnetic fields in the units of respective critical fields $B_{c}^{e} = 4.414 \times 10^{13} \text{G}$ and $B_{c}^{p} = 1.364 \times 10^{20} \text{G}$, $\nu_{e,p}$ are the occupied Landau levels in the respective energies, $p^F$ is the Fermi momentum of electrons and protons, $p_n^F$ is the Fermi momentum of neutrons and $c$ is the speed of light. The drip density for this system was investigated by previous authors [2]. Nevertheless, here we discuss the results categorically. Let us consider three possible cases, some of them may turn out to be absurd.

**Case I:** $p_{e}^2/c^2 >> \Delta_{1,2}$: This happens when both protons and electrons are highly relativistic. At the beginning of formation of neutrons (and hence the corresponding momentum is zero), from eqn. (1) we obtain

$$p^F/c = m_n/2,$$

which however violates initial choice as $m_p \sim m_n$. Hence, this situation is unphysical to reveal the neutron drip.

**Case II:** $p_{e}^2/c^2 \sim \Delta_1 >> \Delta_2$: This happens when electrons are highly relativistic but protons are just becoming relativistic. From eqn. (1) the initiation of drip occurs at

$$p^F/c = \frac{m_n^2 - \Delta_1}{2m_n},$$

which, however, turns out to be negative unless $\Delta_1 \sim m_p^2$. However, this corresponds to $E_p^F \sim m_pc^2$ which cannot reveal relativistic protons. Hence, this case is unphysical.
Case III: $p_F^2/c^2 \sim \Delta_2 \ll \Delta_1$: This corresponds to relativistic electrons but nonrelativistic protons, which reads eqn. (1) as

$$\Delta_1^{1/2} + \sqrt{\left(\frac{p_F}{c}\right)^2 + \Delta_2} = m_n$$

at the drip, revealing further

$$p_F/c = m_n\sqrt{\left(1 - \frac{m_p}{m_n}\sqrt{1 + 2\nu_pB_D}\right)^2 - \left(\frac{m_e}{m_n}\right)^2(1 + 2\nu_eB_D^e)}.$$  

(7)

When the Fermi energy and magnetic field of the system are such that the electrons lie in the ground Landau level only (with $\nu_e = 0$) and protons are not affected by magnetic field (with $E_{Fp_{max}} \sim m_pc^2$), the neutron drip density

$$\rho_{drip} = \frac{2eB_D^eB_c^e}{h^2c}p^F(m_e + m_p) = 0.343B_D^e10^7 \text{ gm/cc},$$

(8)

when $e$ is the charge of electrons and $h$ the Planck’s constant.

However, beyond drip, the chemical equilibrium eqn. (1) gives

$$\sqrt{p^F + m_e^2c^2} + \sqrt{p^F + m_p^2c^2} = \sqrt{p_n^F + m_n^2c^2},$$

(9)

where we have assumed that all the electrons reside in the ground Landau level and for this to be true, $B_D > 2.77$ (see [2]).

Therefore, the density beyond drip is given by

$$\rho_{bd} \approx m_p n_e + \epsilon_n, \quad \text{where} \quad n_e = \frac{2B_D}{(2\pi)^2}\frac{\lambda_e^3}{p_F^F},$$

(10)

when

$$\epsilon_n = \frac{m_n c^2}{\lambda_n^3} \chi(x_n^F), \quad x_n^F = p_n^F/m_n c,$$

where

$$\chi(x_n^F) = \frac{1}{8\pi^2} \left[ x_n^F(1 + 2x_n^F)^2 \sqrt{1 + x_n^F^2} - \log(x_n^F + \sqrt{1 + x_n^F^2}) \right].$$

(11)

Here $n_e$ is the number density of electrons (which is also the number density of protons owing to charge neutrality in the system) and $\epsilon_n$ is the energy density of neutrons. Now eqn. (9) expresses $p_F^F$ in terms of $p_n^{F_F}$. Hence $\rho_{bd}$ varies as a function of $x_n^F$ for a given $B_D$, shown in Fig. 1.
Figure 1: Density in gm/cc beyond the drip in an $n - p - e$ system as a function of Fermi momentum of neutrons.
3 Neutron drip in a gas with heavy ions

We adopt the Harrison-Wheeler formalism [5] in describing the equilibrium composition of nuclear matter. We assume that given enough time following nuclear burning, cold catalyzed material will achieve complete thermodynamic equilibrium. The matter composition and equation of state will then be determined by the lowest possible energy state of the matter. The constituents of the matter are assumed to be ions, free electrons and free neutrons and the drip density is obtained by finding the density of the equilibrium composition when the neutron number density is zero (neutrons just start appearing). We further assume the strong magnetic field present in the matter is constant throughout. We then obtain the equilibrium density by minimizing the energy density of the mixture with respect to the number density of the ions, electrons and neutrons. Due to the presence of the magnetic field, the electrons are Landau quantized but the field of present interest (say, upto that within a neutron star) is small enough to affect the ions. The total energy of the system can be written as

$$\epsilon_{\text{total}} = n_{\text{ion}} M(A, Z) + \epsilon_e'(n_e) + \epsilon_n(n_n).$$

(12)

Here $M(A, Z)$ is the energy of a single ion $(A, Z)$, including the rest mass energy, where $A$ and $Z$ are atomic mass and atomic number respectively. It is conventional to include the rest mass of the electrons in the energy of the ions (see, e.g., [6] for details of non-magnetic results). Therefore, $\epsilon_e'$ is the energy density of the electrons after subtracting $n_e m_e c^2$ from the total energy density $\epsilon_e$ and $\epsilon_n$ is the total energy density of neutrons. The number densities $n_{\text{ion}}$, $n_e$ and $n_n$, of ions, electrons and neutrons respectively, and the total baryon number density $n$ are related as

$$n = n_{\text{ion}} A + n_n, \quad n_e = n_{\text{ion}} Z.$$  

(13)

These can be written in terms of mean numbers per baryon as

$$Y_{\text{ion}} + Y_n = 1, \quad Y_{\text{ion}} Z = Y_e.$$  

(14)

where $Y_i = n_i/n$. Hence at $T = 0$, we can write $\epsilon_{\text{total}}$ as a function of either $(n, A, Z, Y_n)$ or $(n, Y_{\text{ion}}, Y_e, Y_n)$. The equilibrium composition is obtained by minimizing $\epsilon_{\text{total}}$ with respect to $A$, $Z$ and $Y_n$ for a constant $n$. 
We use the semi-empirical mass formula of Green [7] based on the liquid drop nuclear model

\[
M(A, Z) = m_u c^2 \left[ b_1 A + b_2 A^{2/3} - b_3 Z + \frac{b_4}{A} \left( \frac{A}{2} - Z \right)^2 + \frac{b_5 Z^2}{A^{1/3}} \right],
\]

(15)

where

\[b_1 = 0.991749, \quad b_2 = 0.01911, \quad b_3 = 0.000840, \quad b_4 = 0.10175, \quad b_5 = 0.000763\]

and \(m_u = 1.66 \times 10^{-24}\text{gm}\) (1 amu). Hence eqn. (12) becomes

\[\epsilon_{\text{total}} = \frac{n(1 - Y_n)}{A} M(A, Z) + \epsilon_e'(n_e) + \epsilon_n(n_n).\]

(16)

Moreover,

\[\frac{d\epsilon_e'}{dn_e} = E_e^F - m_e c^2, \quad \frac{d\epsilon_n}{dn_n} = E_n^F.\]

(17)

Now we approximate \(A\) and \(Z\) as continuous variables and equations for equilibrium compositions can be found from eqns. (16) and (17) as

\[\frac{\partial \epsilon_{\text{total}}}{\partial Z} = 0, \quad \text{and hence} \quad \frac{\partial M}{\partial Z} = -(E_e^F - m_e c^2),\]

(18)

\[\frac{\partial \epsilon_{\text{total}}}{\partial A} = 0, \quad \text{and hence} \quad A^2 \frac{\partial}{\partial A} \left( \frac{M}{A} \right) = Z \left( E_e^F - m_e c^2 \right),\]

(19)

and finally

\[\frac{\partial \epsilon_{\text{total}}}{\partial Y_n} = 0, \quad \text{and hence using eqn. (19)} \quad \frac{\partial M}{\partial A} = E_n^F.\]

(20)

Now combining eqns. (18) and (19), we obtain

\[Z = \left( \frac{b_2}{2b_5} \right) A^{1/2}.\]

(21)

Eqns. (18) and (19) also give

\[\left[ b_3 + b_1 \left( 1 - 2Z \frac{A}{A} \right) - 2b_5 \frac{Z}{A^{1/3}} \right] m_u c^2 = E_e^F - m_e c^2,\]

(22)
\[
\left[ b_1 + \frac{2}{3} b_2 A^{-1/3} + b_4 \left( \frac{1}{4} - \frac{Z^2}{A^2} \right) - \frac{b_5 Z^2}{3 A^{4/3}} \right] m_u c^2 = E_n^F, \tag{23}
\]

where \( E_n^F = \sqrt{(1 + x_n^F)^2} m_n c^2 \), and at the onset of neutron drip, \( x_n = 0 \) when the number density of neutron is just zero. Therefore, at the onset of drip, eqn. (23) can be written as

\[
b_1 + \frac{2}{3} b_2 A^{-1/3} + b_4 \left( \frac{1}{4} - \frac{Z^2}{A^2} \right) - \frac{b_5 Z^2}{3 A^{4/3}} = \frac{m_n}{m_u}. \tag{24}
\]

Solving eqns. (21) and (24), we obtain \( (A, Z) \approx (122, 39.1) \) for the onset of drip. Using these values for \( A \) and \( Z \), we can obtain the electron Fermi energy at the drip from eqn. (22) as

\[
\frac{E_{ed}^F}{m_e c^2} = 47.2367. \tag{25}
\]

Figure 2 shows, combining eqns. (21) and (22), that beyond drip, \( E_e^F \) first increases with the increase of \( A \) and subsequently saturates to a value \( \sim 128.3 \).

The Fermi energy of a Landau quantized electron with magnetic field strength \( B \) confined in \( \nu_e \) levels can be rewritten as

\[
E_{e}^F = \left[ c^2 p_{ze}(\nu_e)^2 + m_e c^2 \left( 1 + \frac{2 \nu_e B}{B_e} \right) \right]^{1/2},
\]

where \( p_{ze} \) is \( z \)-component of the Fermi momentum of the electron. Therefore, from eqn. (25)

\[
x_e^F(\nu_e) = \frac{p_{ze}}{m_e c} = \left( \frac{2230.31 - 2 \nu_e B}{B_e} \right)^{1/2}. \tag{26}
\]

Let us now determine the number of states occupied by the Landau quantized electrons [2]. This is given by

\[
n_e = \int d\nu_e = \sum_{\nu_e=0}^{\nu_m} \frac{e B}{\hbar^2 c} g_{\nu_e} \int dp_z = \frac{2 B_D^e}{(2\pi)^2 \lambda_e^3} \sum_{\nu_e=0}^{\nu_m} g_{\nu_e} x_e^F(\nu_e), \tag{27}
\]

where \( \lambda_e = \hbar/m_e c \), is the Compton wavelength of the electron, \( g_{\nu_e} \) the degeneracy factor which is \( 2 - \delta_{\nu_e,0} \). The upper limit \( \nu_m \) in the summation is obtained from the condition that \( p_{ze}^F(\nu_e) \geq 0 \), which gives

\[
\nu_e \leq \frac{(E_{e}^F/m_e c^2)^2 - 1}{2 B_D^e} \quad \text{and hence} \quad \nu_m = \frac{(E_{e\text{max}}^F/m_e c^2)^2 - 1}{2 B_D^e}. \tag{28}
\]
Figure 2: Electron Fermi energy in $m_e c^2$ beyond drip as a function of atomic mass.
The electron energy density at zero temperature is then [2]

\[ \epsilon_e = \frac{2B_e}{(2\pi)^2 \lambda_\epsilon^3} \sum_{\nu_e} g_{\nu_e} \int_0^{x_F(\nu_e)} E_e^F \, d \left( \frac{p_{ze}}{m_e c} \right) \]

\[ = m_e c^2 \frac{2B_e^2}{(2\pi)^2 \lambda_\epsilon^3 \nu_e} \sum_{\nu_e} g_{\nu_e} (1 + 2\nu_e B_e^D) \Psi \left( \frac{x_F(\nu_e)}{(1 + 2\nu_e B_e^D)^{1/2}} \right), \quad (29) \]

where

\[ \Psi(z) = \frac{1}{2} z \sqrt{1 + z^2} + \frac{1}{2} \ln(z + \sqrt{1 + z^2}). \]

The density of the mixture of ions and electrons at the onset of drip is therefore

\[ \rho_{\text{drip}} = \frac{\epsilon_{\text{total}}}{c^2} = \frac{n_e M(A, Z)}{Z} + \epsilon_e - n_e m_e c^2. \quad (30) \]

Now eqn. (26) gives \( x_e \) as a function of \( B_e^D \). Therefore, the above equation gives us a relationship between \( \rho_{\text{drip}} \) and \( B_e^D \). Figure 3 shows that for \( B \gtrsim 10^{16} \)G, drip density increases linearly with magnetic field.

### 4 Summary

We have studied the effect of magnetic field on the onset of neutron drip for \( n - p - e \) and ion-electron gas systems. We have found that beyond a certain value of magnetic field, the drip density of either of the systems increases linearly with the magnetic field. This value of field for the former is \( B_e^D \sim 2.7 \) and for the latter \( \sim 1115 \). The density becomes linear due to the fact that beyond these values of \( B_e^D \), the Landau quantized electrons reside in the ground state energy level. We also notice that before the onset of linearity, the drip density oscillates about the non-magnetic result, with small amplitudes. There is a significant change in the drip density only in the linear regime. For instance, the drip density for the ionic system increases by an order, compared to that of the nonmagnetic gas, for a field \( \sim 6.4 \times 10^{17} \)G.

This may have interesting consequences to the recently proposed high density, high magnetic field white dwarfs [4, 8, 9, 10]. The inner region of white dwarfs would have been neutronized if the drip density had not been changed with the field. Even if the white dwarfs with very large field consist of central region with density larger than the modified drip density, the present finding
Figure 3: Drip density in gm/cc in a gas of electrons and ions as a function of magnetic field.
will help in putting a constraint on such models of white dwarfs with pure electron degenerate matter.

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