Laplace and Schrödinger operators on regular metric trees: the discrete spectrum case

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Dedicated to Prof. Hans Triebel on the occasion of his 65th birthday

1 Introduction

Spectral theory of differential operators on metric trees is an interesting branch of such theory on general metric graphs. Among the trees, the so-called regular trees are of particular interest due to their very special geometry.

Let $\Gamma$ be a tree rooted at some vertex $o$ and having infinitely many edges. Below $|x|$ stands for the distance between a point $x \in \Gamma$ and the root $o$. For a vertex $x$, its generation is the number of vertices lying between $o$ and $x$ (including $x$ but excluding $o$). We say that a tree $\Gamma$ is regular if for any vertex $x$ the quantity $|x|$ and the number of edges emanating from $x$ depend only on the generation of $x$; see Definition 2.1 for the more detailed description.

The regular trees are highly symmetric. This allows one to construct an orthogonal decomposition of the space $L^2(\Gamma)$ which reduces the Schrödinger operator $A_V = -\Delta + V$ with any symmetric (i.e. depending on $|x|$) potential $V$. We call this decomposition the basic decomposition of $L^2(\Gamma)$. The parts of $A_V$ in the components of the basic decomposition are denoted by $A_{V,k}$. Each operator $A_{V,k}$ appears with a multiplicity which rapidly grows as $k \to \infty$. The study of the spectrum $\sigma(A_V)$ of the operator $A_V$ is thus reduced to the study of the spectra of the parts $A_{V,k}$. Each part can be identified with a differential operator acting in a weighted space $L^2((t_k,R), g_k)$, $k = 0, 1, \ldots$ where the intervals $(t_k, R)$ and the weight functions $g_k$ are determined by the geometry of $\Gamma$. In particular, the quantity $R = R(\Gamma) = \sup\{|x| : x \in \Gamma\} \leq \infty$ is the radius of the tree.

According to general spectral theory,

$$\sigma(A_V) = \bigcup_{k=0}^{\infty} \sigma(A_{V,k}).$$

However, the quantitative characteristics of $\sigma(A_V)$ cannot be obtained automatically from the corresponding characteristics for the operators $A_{V,k}$, due to the growing multiplicities of $A_{V,k}$ as parts of $A_V$.

The main purpose of this paper is to study the situations when the spectrum of the Schrödinger operator $A_V$ is discrete. More exactly, we consider two typical cases: for the regular trees of finite radius we show that the classical Weyl asymptotic law holds for the eigenvalues of each operator $A_{V,k}$ with any bounded potential $V$, including the basic case $V = 0$. If the tree $\Gamma$ has finite total length, we show that the Weyl formula holds for the whole operator $A_V$.

Another case is the operator $A_V$ with growing potential on a tree of infinite radius. Under certain assumptions about the geometry of the tree and the behaviour of the potential, we find a version of the Weyl formula for such operators.

To make the general picture more complete, we also present, without proofs, some known results concerning the structure of $\sigma(A_V)$ in the case when the spectrum is not discrete.
There are several papers, devoted to differential operators on regular metric trees. In [4], the case of the homogeneous trees $\Gamma_b$ was considered. A regular tree is called homogeneous if all its edges have equal length (say 1) and all vertices have the same number of edges (say $b$) emanating from them. In [4] the potential $V$ was supposed to be periodic and even. It was shown that the spectrum $\sigma(A_{0V})$ has the band-gap structure, with no more than one eigenvalue in each gap. Actually, this eigenvalue is of infinite multiplicity, but the multiplicities were not discussed in [4].

In [10] the weighted spectral problems of the form $-\Delta f = \lambda V f$ on general (not necessarily regular) trees were investigated. The estimates and, under some additional assumptions, the asymptotic behaviour of the eigenvalues were found. For the regular trees, the basic decomposition of $L^2(\Gamma)$ was discovered in [10], and much more advanced results for such trees were obtained with the help of this decomposition.

In the paper [5] the basic decomposition was re-discovered, and a detailed spectral analysis of the operators $A_{V,k}$ for the regular trees $\Gamma$ with finite radius was given. In particular, it was shown that if $R(\Gamma) < \infty$ but the total length of $\Gamma$ is infinite, the operators $A_{V,k}$ do not require the boundary condition at $t = R$. It was also shown that each operator $A_{V,k}$ has compact resolvent, and the corresponding eigenvalue distribution function $N(\lambda; A_{V,k})$ grows not faster than $O(\lambda^{1/2+\epsilon})$ for any $\epsilon > 0$. Our Theorem 5.3 considerably refines this result.

The main topic of [5] is the Hardy-type inequalities on regular trees. As a consequence, a necessary and sufficient condition of the positive definiteness of the Laplacian was established, see Theorem 5.6.

In [12] operators $A_{V}$ with decaying symmetric potentials on the homogeneous trees $\Gamma_b$ were investigated. For $V = 0$ (that is, for the free Laplacian) the spectrum was explicitly calculated. This result is presented here as Theorem 5.7. In accordance with the results of [11], the spectrum has infinitely many gaps. Perturbation of the operator $A_0$ by a decaying potential may create eigenvalues in each gap, and in [12] their behaviour was investigated in detail both for positive and negative perturbations.

It is necessary to mention here also the papers [6] and [7], though formally they do not deal with the Laplacian on trees. In [6] the Hardy-type integral operators on trees were introduced in connection with the spectral analysis of the Neumann Laplacian in certain irregular domains. For these “ridged” domains, there exists a tree that serves as a “ridge”, or a “skeleton”, for the domain. There is a close relation between the approximation numbers of the Hardy-type integral operators and the eigenvalues of the problem $-\Delta f = \lambda V f$ on the tree. It was the paper [6] which attracted the author’s attention to operators on trees.

In [7] the behaviour of the approximation numbers of the Hardy-type integral operators on trees was studied in detail, not only in $L^2$-case but also in the general $L^p$-case. When applied to the Laplacian, the estimates obtained substantially refine some results of [11], Sect. 4. The asymptotic formulae found in [7] may provide an alternative approach to the proof of our Theorem 5.3 (ii).

The structure of the present paper is as follows. The short sections 2 – 4 contain the necessary preliminary material: the definitions of regular and homogeneous trees and of the Laplace and Schrödinger operators on them, and the description of the basic orthogonal decomposition of $L^2(\Gamma)$. Sect. 5 contains formulations of the main results, the proofs are given in Sect. 6. In the final Sect. 7 we discuss the extension of the presented results to the case of regular trees without boundary.
2  Regular rooted trees

2.1  Geometry of a tree

Let $\Gamma$ be a rooted tree with the root $o$, the set of vertices $V = V(\Gamma)$ and the set of edges $E = E(\Gamma)$. We suppose that $\#V = \#E = \infty$. Unlike the combinatorial trees, whose edges are just pairs of vertices, each edge $e$ of a metric tree is viewed as a non-degenerate line segment. The distance $\rho(x, y)$ between any two points $x, y \in \Gamma$ (and thus the metric topology on $\Gamma$) is introduced in a natural way. As it was already said in Introduction, $|x|$ stands for $\rho(x, o)$.

We write $y \preceq z$ if $|z| = |y| + \rho(y, z)$, and $y \prec z$ means that $y \preceq z$ and $z \neq y$.

The relation $\prec$ defines on $\Gamma$ a partial ordering. If $y \prec z$, we denote $\langle y, z \rangle := \{x \in V : y \preceq x \preceq z\}$.

For any vertex $y$ its generation $Gen(y)$ is defined as $Gen(y) = \#\{x \in V : o \prec x \preceq y\}$.

We assume that $Gen(y) < \infty$ for any vertex $y$. For an edge $e$ we define $Gen(e)$ as the generation of its initial point.

The branching number $b(y)$ of a vertex $y$ is defined as the number of edges emanating from $y$. We assume that $b(o) = 1$ and $b(y) > 1$ for $y \neq o$. We denote by $e^-_y$ the only edge which terminates at a vertex $y \neq o$, and by $e^1_y, \ldots, e^{b(y)}_y$ the edges emanating from any vertex $y \in V$.

\textbf{Definition 2.1} We call a tree $\Gamma$ regular if all the vertices of the same generation have equal branching numbers, and all the edges of the same generation are of the same length.

In this paper we consider only the regular trees. Evidently, any such tree is fully determined by specifying two number sequences, $\{b_k\}$ and $\{t_k\}, k = 0, 1, \ldots$ such that $b(y) = b_{Gen(y)}, |y| = t_{Gen(y)}$ for each $y \in V(\Gamma)$.

According to our assumptions, $b_0 = 1$ and $b_k \geq 2$ for any $k > 0$. It is clear that $t_0 = 0$ and the sequence $\{t_k\}$ is strictly increasing, and we denote $R = R(\Gamma) = \lim_{k \to \infty} t_k = \sup_{x \in \Gamma} |x|$. We call $R(\Gamma)$ the radius of the tree. Another important characteristic of a tree is its total length (in other terminology, volume)

$|\Gamma| = \sum_{e \in E(\Gamma)} |e|.$

The natural measure $dx$ on $\Gamma$ is induced by the Lebesgue measure on the edges. Below $L^2(\Gamma) = L^2(\Gamma, dx)$.

2.2  Homogeneous trees

A rooted tree is called homogeneous if its edges are all of the same length (for definiteness, of the length one) and all the vertices $y \neq o$ have the same branching number $b$. A homogeneous tree is evidently regular. It is fully determined by specifying the parameter $b$ and we use for it the notation $\Gamma_b$. For the tree $\Gamma_b$ one has $t_k = k, k = 0, 1, \ldots$, and $b_k = b, k = 1, 2, \ldots$. 

3 The Laplace and the Schrödinger operators on a regular tree

The notion of differential operator on any metric graph, in particular on a tree, is well known. Still, for the sake of completeness we present here the variational definitions of the Laplacian and of the Schrödinger operator on a tree.

We say that a scalar-valued function \( f \) on \( \Gamma \) belongs to the Sobolev space \( \mathcal{H}^1 = \mathcal{H}^1(\Gamma) \) if \( f \) is continuous, \( f \upharpoonright e \in \mathcal{H}^1(e) \) for each edge \( e \), and

\[
\|f\|_{\mathcal{H}^1}^2 := \int_{\Gamma} (|f'(x)|^2 + |f(x)|^2)dx < \infty. \tag{1}
\]

The derivative of a function \( f \upharpoonright e \) at an interior point \( x \in e \) is always taken in the direction compatible with the partial ordering on \( \Gamma \). This agreement is indifferent for the definition \( \|f\|_{\mathcal{H}^1}^2 \) but we shall use it later.

The set \( \mathcal{H}^1_0 \) of all boundedly supported functions \( u \in \mathcal{H}^1 \) is dense in \( \mathcal{H}^1 \). Indeed, for any number \( L > 0 \) let \( \varphi_L(t) \) be the continuous function on \( \mathbb{R}_+ \), which is 1 for \( t \leq L \), is 0 for \( t \geq L + 1 \) and is linear on \( [L, L + 1] \). Given a function \( f \in \mathcal{H}^1(\Gamma) \), denote \( f_L(x) = \varphi_L(|x|)f(x) \). Then \( f_L \in \mathcal{H}^1_0 \) and an elementary calculation shows that \( f_L \to f \) in \( \mathcal{H}^1(\Gamma) \) as \( L \to \infty \).

Along with \( \mathcal{H}^1(\Gamma) \), let us introduce also its subspace of codimension one:

\[
\mathcal{H}^{1,0} := \mathcal{H}^{1,0}(\Gamma) = \{f \in \mathcal{H}^1(\Gamma) : f(o) = 0\}.
\]

We define the Dirichlet Laplacian \(-\Delta\) on \( \Gamma \) as the self-adjoint operator in \( L^2(\Gamma) \), associated with the quadratic form \( \int_\Gamma |f'|^2dx \) considered on the form domain \( \text{Quad}(-\Delta) = \mathcal{H}^{1,0}(\Gamma) \). It is easy to describe the operator domain \( \text{Dom}(\Delta) \) and the action of \( \Delta \). Evidently \( f \in \text{Dom}(\Delta) \) \( \Rightarrow \) \( f \upharpoonright e \in \mathcal{H}^2(e) \) for each edge \( e \) and the Euler–Lagrange equation reduces on \( e \) to \( \Delta f = f'' \). At the root we have the boundary condition \( f(o) = 0 \), since \( \text{Dom}(\Delta) \subset \mathcal{H}^{1,0}(\Gamma) \). At each vertex \( y \neq o \) the functions \( f \in \text{Dom}(\Delta) \) satisfy certain matching conditions. In order to describe them, denote by \( f_- \) the restriction \( f \upharpoonright e^-_o \) and by \( f_j \), \( j = 1, \ldots, b(y) \) the restrictions \( f \upharpoonright e^j_y \). The matching conditions at \( y \neq o \) are

\[
f_-(y) = f_1(y) = \ldots = f_b(y); \quad f'_1(y) + \ldots + f'_b(y) = f'_-(y). \tag{2}
\]

The first condition comes from the requirement \( f \in \mathcal{H}^1(\Gamma) \) which includes continuity of \( f \), and the second appears as the natural condition in the sense of Calculus of Variations. It is easy to check that the conditions listed are also sufficient for \( f \in \text{Dom}(\Delta) \).

Let \( V \) be a measurable, real-valued, bounded from below and symmetric (that is, depending only on \(|x|\)) function on \( \Gamma \). Along with the Laplacian we shall be interested also in the Schrödinger operator with the potential \( V(|x|) \):

\[
\mathcal{A}_V f := -\Delta f + V(|x|)f. \tag{3}
\]

The operator \( \mathcal{A}_V \) is defined via its quadratic form

\[
\mathcal{a}_V[f] := \int_\Gamma (|f'(x)|^2 + V(|x|)|f(x)|^2)dx
\]

considered on the natural domain \( \text{Dom}(\mathcal{a}_V) = \text{Quad}(\mathcal{A}_V) = \mathcal{H}^{1,0}(\Gamma) \cap L^2(\Gamma) \). On this domain the quadratic form \( \mathcal{a}_V \) is bounded from below and closed in \( L^2(\Gamma) \), and the corresponding self-adjoint operator is taken as the realization of the operator \( \mathcal{A}_V \). We do not need the precise description of the domain and the action of \( \mathcal{A}_V \).
4 The basic decomposition of \( L^2(\Gamma) \)

Consider two types of subtrees \( T \subset \Gamma \). Namely, for any vertex \( y \) and for any edge \( e = (z, w) \) we set

\[
T_y = \{ x \in \Gamma : x \geq y \}, \quad T_e = e \cup T_w.
\]

Evidently \( T_y = \Gamma \).

For any subtree \( T = T_y \) or \( T = T_e \) its branching function \( g_T(t) \) is defined as

\[
g_T(t) = \#\{ x \in T : |x| = t \}.
\]

If \( T = T_e \) and \( \text{Gen}(e) = k \geq 0 \), then \( g_T(t) = g_k(t) \) where

\[
g_k(t) = \begin{cases} 0, & t < t_k, \\ 1, & t_k \leq t \leq t_{k+1}, \\ b_{k+1} \ldots b_n, & t_n < t \leq t_{n+1}, \; n > k. \end{cases}
\]

In particular, \( g_0(t) = g_1(t) = 1 \) for \( 0 \leq t \leq t_1 \) and

\[
g_T(t) = b_1 \ldots b_n, \quad t_n < t \leq t_{n+1}, \; n \geq 1.
\]

So we see that

\[
g_k(t) = (b_1 \ldots b_k)^{-1} g_{t_k}(t), \quad t > t_k, \; k > 0. \tag{4}
\]

Note that

\[
\int_{\Gamma} g_T(t) dt = |\Gamma|. \tag{5}
\]

Given a subtree \( T \subset \Gamma \), we say that a function \( f \in L^2(\Gamma) \) belongs to the set \( \mathcal{F}_T \) if and only if \( f = 0 \) outside \( T \) and

\[
f(x) = f(y) \quad \text{if } x, y \in T \text{ and } |x| = |y|.
\]

Evidently \( \mathcal{F}_T \) is a closed subspace of \( L^2(\Gamma) \). Any function \( f \in \mathcal{F}_{T_e} \), \( \text{Gen}(e) = k \geq 0 \) can be naturally identified with the function \( u := J_e f \) on \( (t_k, R) \), such that \( f(x) = u(|x|) \) for each \( x \in T_e \) and \( f(x) = 0 \) outside \( T_e \). We have

\[
\int_{\Gamma} |f(x)|^2 dx = \| u \|^2_{L^2((t_k, R); g_k)} := \int_{t_k}^{t_R} |u(t)|^2 g_k(t) dt; \tag{6}
\]

\[
f \in \mathcal{F}_{T_e}, \; u = J_e f.
\]

This shows that the operator \( J_e \) defines an isometry of the subspace \( \mathcal{F}_{T_e} \) onto the weighted space \( L^2((t_k, R); g_k) \). Along with (3), we have

\[
\int_{\Gamma} |f'(x)|^2 dx = a_k[u] := \int_{t_k}^{t_R} |u'(t)|^2 g_k(t) dt; \tag{7}
\]

\[
f \in \mathcal{F}_{T_e} \cap H^{1,0}(\Gamma), \; u = J_e f.
\]

For the sake of brevity, below we use the notations \( \mathcal{F}_y \) for \( \mathcal{F}_{T_y} \) and \( \mathcal{F}^j_y \) for \( \mathcal{F}_{T^j_y}, \; j = 1, \ldots, b = b(y) \). It is clear that the subspaces \( \mathcal{F}^1_y, \ldots, \mathcal{F}^b_y \) are mutually orthogonal and their orthogonal sum

\[
\mathcal{F}'_y = \mathcal{F}^1_y \oplus \ldots \oplus \mathcal{F}^b_y
\]

contains \( \mathcal{F}_y \). Denote

\[
\mathcal{F}'_y = \mathcal{F}'_y \cap \mathcal{F}_y
\]
Theorem 4.1  Let $\Gamma$ be a regular tree.

(i) The subspaces $F'_y$, $y \in \mathcal{V}(\Gamma)$ are mutually orthogonal and orthogonal to $F_{\Gamma}$.
Moreover,

$$L^2(\Gamma) = F_{\Gamma} \oplus \sum_{y \in \mathcal{V}(\Gamma)} \oplus F'_y.$$  \hspace{1cm} (8)

(ii) Let $V(t)$ be a real, measurable and bounded below function on $\mathbb{R}_+$. Then the decomposition (8) reduces the Schrödinger operator (2).

According to Theorem 4.1, description of the spectrum $\sigma(A_{V})$ reduces to the similar problem for the parts of $A_{V}$ in the components of the decomposition (8). These parts can be described in terms of auxiliary differential operators $A_{V,k}$, $k = 0, 1, \ldots$ acting in the spaces $L^2((t_k, R), g_k)$.

For $f \in \text{Dom}(a) \cap F_{\Gamma}$, the quadratic form $a_V$, cf. (3), transforms as follows:

$$a_V[f] = a_{V,k}[u] := \int_{t_k}^R \left(|u'(t)|^2 + V(t)|u(t)|^2\right)g_k(t)dt,$$  \hspace{1cm} (9)

For $V \equiv 0$ the quadratic form $a_{V,k}[u]$ turns into the quadratic form $a_k[u]$ defined in (3). We define $A_{V,k}$ as the self-adjoint operator in $L^2((t_k, T), g_k)$, associated with the quadratic form $a_{V,k}$. We drop the subindex $V$ in these notation when dealing with the free Laplacian $-\Delta = A_0$.

The following result was actually proved in [4] and [11]; minor distinctions in the formulations are unessential.

Theorem 4.2  Let $\Gamma$ be a regular tree and $V$ be a bounded from below, real-valued symmetric potential on $\Gamma$. Then the part of $A_{V}$ in the subspace $F_{\Gamma}$ is unitarily equivalent to the operator $A_{V,0}$ and the part of $A_{V}$ in each subspace $F'_y$, Gen$(z) = k > 0$, is unitary equivalent to the orthogonal sum of $(b_k - 1)$ copies of $A_{V,k}$.

The next theorem is an immediate consequence of Theorems 4.1 and 4.2. Below $A^{[r]}$ stands for the orthogonal sum of $r$ copies of a self-adjoint operator $A$. The symbol “$\sim$” means unitary equivalence.

Theorem 4.3  Under the assumptions of Theorem 4.2 the operator $A_{V,\Gamma}$ is unitary equivalent to the orthogonal sum of the operators $A_{V,k}$, with growing multiplicities:

$$A_{V,\Gamma} \sim A_{V,0} \oplus \sum_{k=1}^{\infty} \oplus A_{V,k}^{[b_1 \ldots b_k-1(b_k-1)]}.$$  \hspace{1cm} (10)

5 Main results

5.1 The eigenvalue counting functions

For a self-adjoint, bounded from below operator $A$ with discrete spectrum, we denote by $N(\lambda; A)$ the distribution function of its eigenvalues $\lambda_j(A)$ (counted according to their multiplicities),

$$N(\lambda; A) = \#\{j : \lambda_j(A) < \lambda\}, \quad \lambda \in \mathbb{R}.$$  \hspace{1cm} (11)

We start with the following simple but useful statement.
Theorem 5.1 Let $\Gamma$ be a regular tree and $V$ be a symmetric measurable real-valued function, bounded below. The spectrum $\sigma(\mathcal{A}_V)$ is discrete if and only if the spectrum of the operator $\mathcal{A}_{V,0}$ is discrete. If this is the case, then

$$N(\lambda; \mathcal{A}_V) = N(\lambda; \mathcal{A}_{V,0}) + \sum_{k=1}^{\infty} b_1 \ldots b_{k-1}(b_{k} - 1)N(\lambda; \mathcal{A}_{V,k}), \quad \lambda \in \mathbb{R}. \quad (11)$$

Proof. Consider the Rayleigh quotient $a_{V,k}[u]/\|u\|_{L^2(t_k,R),g_k}^2$, cf. (6) and (9). Due to the equality (4), this ratio does not change if we replace in its numerator and denominator the weight function $g_k(t)$ by $g_\Gamma(t)$. Now it follows from the variational principle that the spectrum of each operator $\mathcal{A}_{V,k}$, $k = 1, 2, \ldots$ is discrete provided this is true for $k = 0$. Moreover, we see that

$$N(\lambda; \mathcal{A}_{V,k_2}) \leq N(\lambda; \mathcal{A}_{V,k_1}), \quad k_1 < k_2, \quad \lambda > 0.$$  

The discreteness of $\sigma(\mathcal{A}_{V,k})$ for all $k$ implies the same property of $\sigma(\mathcal{A}_V)$. The converse is evident. The equality (11) is an immediate consequence of the relation (10).

The detailed study of the function $N(\lambda; \mathcal{A}_V)$ is hampered by the presence of the rapidly growing factors $b_1 \ldots b_{k-1}(b_{k} - 1)$. These factors reflect geometry of the tree and do not depend on the potential $V$. For this reason, sometimes we consider another counting function (introduced in [12]):

$$\tilde{N}(\lambda; \mathcal{A}_V) := \sum_{k=0}^{\infty} N(\lambda; \mathcal{A}_{V,k}). \quad (12)$$

5.2 The spectrum of the Laplacian

The spectrum of the Laplacian on a regular tree depends on the behaviour of the sequences $\{t_k\}$ and $\{b_k\}$ and can be quite different. We present here several results in this direction. The proofs of those which are new are given in the next section. In other cases we give the relevant references.

Our first result is quite elementary and its proof is standard. The result applies to arbitrary metric graphs rather than to trees only.

Theorem 5.2 Let $\Gamma$ be a metric graph such that $\sup_{e \in E(\Gamma)} |e| = \infty$. Then the spectrum of the Laplacian $-\Delta$ on $\Gamma$ coincides with $[0, \infty)$.

Other results concern the regular trees. We start with the trees of finite radius, for which the information provided is rather complete.

Theorem 5.3 Let $\Gamma$ be a regular tree and $R(\Gamma) < \infty$.

(i) The spectrum of the Laplacian $-\Delta$ on $\Gamma$ is discrete. For each operator $\mathcal{A}_k$ its eigenvalues behave according to the Weyl law,

$$\pi N(\lambda; \mathcal{A}_k) = \sqrt{\lambda}(R - t_k) + o(\sqrt{\lambda}), \quad \lambda \to \infty. \quad (13)$$

(ii) If $|\Gamma| < \infty$, then the Weyl asymptotic law holds for the operator $-\Delta$:

$$\pi N(\lambda; -\Delta) = \sqrt{\lambda}|\Gamma| + o(\sqrt{\lambda}), \quad \lambda \to \infty. \quad (14)$$
(iii) If
\[ \widetilde{R}(\Gamma) := \sum_{k=0}^{\infty} (R - t_k) < \infty, \]
then
\[ \pi \widetilde{N}(\lambda; -\Delta) = \sqrt{\lambda} \widetilde{R}(\Gamma) + o(\sqrt{\lambda}), \quad \lambda \to \infty. \]

Theorem 5.3 refines an earlier result of [5]. The case of general (i.e., not necessarily regular) trees was analyzed in [10], Theorem 4.1. For the trees with \(|\Gamma| < \infty\), satisfying some additional assumptions, the Weyl asymptotics (14) follows from this theorem. However, the result of [10] does not cover the case of arbitrary regular trees of finite total length.

The next statement can be derived from Theorem 5.3 by means of the elementary variational arguments. We present it without proof.

**Corollary 5.4** Let \( \Gamma \) be a regular tree, \( R(\Gamma) < \infty \), and the potential \( V(x) \) (not necessarily symmetric) be bounded. Then the spectrum \( \sigma(-\Delta + V) \) is discrete. If in addition \(|\Gamma| < \infty\), then the asymptotic formula (14) holds for its eigenvalues.

If in addition, the potential is symmetric, then the asymptotic formula (13) holds for each operator \( A_{V,k} \).

If for a regular tree \( \Gamma \) one has \( R(\Gamma) < \infty \) but \(|\Gamma| = \infty\), then the asymptotic behaviour of \( N(\lambda; -\Delta) \) can be rather exotic. The following example can serve as an illustration.

Fix the numbers \( q \in (0, 1) \) and \( b \in \mathbb{N} \). Consider the regular tree \( \Gamma = \Gamma_{q,b} \) defined by the sequences \( t_k = 1 - q^k \), \( k = 0, 1, \ldots \) and \( b_k = b \), \( k = 1, \ldots \). Then \( R(\Gamma) = 1 \), so that the spectrum of the Laplacian on \( \Gamma \) is always discrete. Further, \( g_0(t) = b^k \) for \( t_k < t \leq t_{k+1} \). The total length of \( \Gamma \) is
\[ |\Gamma| = 1 - q + \sum_{k=1}^{\infty} b^k (q^k - q^{k+1}) = (1 - q) \sum_{k=0}^{\infty} (bq)^k. \]
Hence, \(|\Gamma| = \frac{1 - q}{1 - bq} < \infty \) if \( bq < 1 \) and \(|\Gamma| = \infty \) otherwise. In the first case, Theorem 5.3 (ii) shows that the Weyl law (14) holds for the eigenvalues of \(-\Delta\). Besides, \( \tilde{R}(\Gamma_{q,b}) = (1 - q)^{-1} < \infty \), and by Theorem 5.3 (iii) the asymptotic formula (15) holds for any \( q < 1 \) and any \( b \).

Below we present the results for the function \( N(\lambda; -\Delta) \), for \( bq \geq 1 \). The case \( bq > 1 \) was analyzed in [10], Example 8.2 (where one should take \( \alpha = 0 \)). The result of [10] for \( bq = 1 \) was not complete.

**Theorem 5.5** Let \( \Gamma = \Gamma_{q,b} \).

(i) If \( bq > 1 \), then there exists a bounded and bounded away from zero periodic function \( \psi \) of period \( \ln(q^{-2}) \) such that
\[ \pi N(\lambda; -\Delta) = \lambda^{\beta/2} (\psi(\ln \lambda) + o(1)), \quad \lambda \to \infty \quad (16) \]
where \( \beta = - \log_q b > 1 \).

(ii) If \( bq = 1 \), then
\[ \pi N(\lambda; -\Delta) = \frac{1 - q}{2 \ln b} \sqrt{\lambda} (\ln \lambda + O(1)), \quad \lambda \to \infty. \quad (17) \]
The proof given in the next section covers both cases. For \( bq > 1 \), it reproduces the argument from \([10]\).

The results for the trees with \( R(\Gamma) = \infty \) are much less exhaustive. We start with a criterion of positive definiteness of the Laplacian on a regular tree, proven in \([11]\).

**Theorem 5.6** Let \( \Gamma \) be a regular tree and \( R(\Gamma) = \infty \). Then the Laplacian on \( \Gamma \) is positive definite in \( L^2(\Gamma) \) if and only if

\[
\sup_{t > 0} \left( \int_0^t g_1(s) \, ds \cdot \int_t^\infty \frac{ds}{g_1(s)} \right) < \infty. \tag{18}
\]

The condition (18) is satisfied, in particular, for the homogeneous trees \( \Gamma_b \). For them the spectrum can be described completely. The next result is proven in \([12]\), Theorem 3.3. Introduce the number

\[
\theta = \arccos \frac{2}{b^{1/2} + b^{-1/2}}.
\]

**Theorem 5.7** The spectrum of the operator \(-\Delta\) on the tree \( \Gamma_b \) is of infinite multiplicity and consists of the bands \( [(\pi(l-1)+\theta)^2, (\pi l - \theta)^2] \), \( l \in \mathbb{N} \) and the eigenvalues \( \lambda_l = (\pi l)^2 \).

So, in this case the spectrum has the band-gap structure which is typical for periodic problems. An analogue of Theorem 5.7 can be proved for regular trees for which the sequences \( t_{k+1} - t_k \) and \( b_k \) are not necessarily constant, as for \( \Gamma_b \), but periodic.

Suppose now that for a regular tree all the branching numbers are equal, \( b_1 = b_2 = \ldots = b \), but the edge lengths \( l_k = t_k - t_{k-1} \) are identically distributed random variables.

More precisely, let \([L_1, L_2]\) be a finite segment, \( L_1 > 0 \). Suppose that \( \mu \) is a Borelian probability measure on \([L_1, L_2]\). Denote by \( \mu^\infty \) the product of infinitely many copies of \( \mu \); this is a measure on the space of all sequences \( \{l_k\}_{k \in \mathbb{N}} \) taking their values in \([L_1, L_2]\).

**Theorem 5.8** Let the measure \( \mu \) be absolute continuous. Suppose that for each \( k = 1, 2, \ldots \) the lengths \( l_k \) are independent random variables with distribution \( \mu \). Then almost surely with respect to the measure \( \mu^\infty \), the spectrum of the operator \(-\Delta\) on \( \Gamma \) contains no absolute continuous component.

The proof, which we do not present in this paper, was obtained in cooperation with G.Berkolaiko, K.Naimark, and U.Smilansky. Its starting point is the equality \([11]\). Then the spectrum of each operator \( A_k \) is analyzed with the help of Fürstenberg’s Theorem on the product of random matrices.

Later the author had an opportunity to discuss this result with I.Goldsheid. Here is the information provided by him.

1. The result (for the components \( A_k \)) was known to him and to S.Molchanov before.

2. Moreover, the spectrum \( \sigma(A_k) \) is almost surely pure point and the eigenfunctions exponentially decay as \( t \to \infty \) (the property which is called Anderson localization).
5.3 Operators $-\Delta + V$ with growing potential

**Theorem 5.9** Let $\Gamma$ be a regular tree and $R(\Gamma) = \infty$. Denote by $\Psi$ the counting function for the sequence $\{t_k\}$,

$$\Psi(\lambda) = \# \{k : t_k < \lambda\}, \quad \lambda > 0.$$  

Let $V(t)$ be a non-negative, strictly monotonically increasing, unbounded continuous function on $\mathbb{R}_+$. Let $Q$ stand for its inverse. Suppose that the functions $Q$ and $\Psi \circ Q$ satisfy the $\Delta_2$-condition

$$Q(2\lambda) \leq CQ(\lambda), \quad \lambda \geq \lambda_0; \quad (19)$$

$$\Psi(Q(2\lambda)) \leq C\Psi(Q(\lambda)), \quad \lambda \geq \lambda_0 \quad (20)$$

and that

$$\Psi(t) = o(t\sqrt{V(t)}), \quad t \to \infty. \quad (21)$$

Then for the counting function $\tilde{N}(\lambda; A_V)$, cf. (12), the asymptotic formula is valid:

$$\pi \tilde{N}(\lambda; A_V) = (1 + o(1)) \sum_{k=0}^{\infty} \int_{t_k}^{\infty} (\lambda - V(t))^{1/2} dt, \quad \lambda \to \infty. \quad (22)$$

The asymptotic formula (22) looks quite natural. The condition (19) is standard for this class of problems. Two other conditions are rather restrictive and we do not know whether they are sharp. Note that the condition (21) is automatically satisfied if we suppose that the function $\Psi$ itself satisfies the $\Delta_2$-condition. Note also that for $t_k = k^r, \ r > 0$ and $V(t) = t^\gamma, \ \gamma > 0$ the assumption (21) reduces to $r^{-1} < 1 + \gamma/2$.

6 Proofs

6.1 Proof of Theorem 5.2

It is enough to show that for any $r > 0$ the point $\lambda = r^2$ belongs to $\sigma(-\Delta)$. For this purpose we fix a non-negative function $\varphi \in C_0^\infty(-1, 1)$ such that $\varphi(t) = 1$ on $(-1/2, 1/2)$. Further, choose an edge $e \in \mathcal{E}(\Gamma)$. In an appropriate coordinate system, $e$ can be identified with the interval $(-l, l)$ where $l = |e|/2$. The function $f$ on $\Gamma$,

$$f(t) = \varphi(t/l) \sin rt \text{ on } e, \quad f(t) = 0 \text{ otherwise},$$

belongs to $\text{Dom}(\Delta)$. An elementary calculation shows that

$$\|\Delta f + r^2 f\| \leq \varepsilon(l)\|f\|, \quad \varepsilon(l) \to 0 \text{ as } l \to \infty.$$ 

Choosing a sequence of edges $e$ such that $|e| \to \infty$, we obtain a Weyl sequence for the operator $-\Delta$ and the point $\lambda = r^2$. This implies that $\lambda \in \sigma((\Delta))$.  

10
6.2 Auxiliary material

We shall use the variational techniques, in the spirit of the book [3]. We present the material we need in the form, convenient for the applications to the operators $A_k$.

Let $w(t)$ be a monotonically growing function on a finite interval $[a,b)$. In our applications we shall take $[a,b) = [t_k,R)$ and $w(t) = g_k(t)$, which explains the nature of our assumptions about the function $w$. We suppose that $w(t) \geq 1$ and that the points $t_k$ of discontinuity of $w$ may accumulate at the point $b$ only. Consider the Hilbert space $H^1 \cdot ((a,b),w)$ whose elements are the functions $u$ on $[a,b)$, such that $u \in H^1((a,b) - \epsilon)$ for any $\epsilon > 0$, $u(a) = 0$, and

$$
\|u\|^2_{H^1 \cdot ((a,b),w)} := \int_a^b |u'(t)|^2 w(t)dt < \infty,
$$

cf. (7). We write $H^1 \cdot (a,b)$ instead of $H^1 \cdot ((a,b),1)$. The weighted Sobolev space with the weight $w$ is defined as

$$
H^1 \cdot ((a,b),w) = H^1 \cdot ((a,b),w) \cap L^2((a,b),w).
$$

Let us change the variables, taking

$$
s = s(t) = \int_a^t \frac{d\tau}{w(\tau)}. \tag{23}\nonumber
$$

The variable $s$ runs over the interval $[0,L)$ where

$$
L = \int_a^b \frac{d\tau}{w(\tau)}.
$$

Since $w(t) \geq 1$, we have $L \leq b - a < \infty$. Below $t(s)$ stands for the function on $[0,L)$, inverse to $s(t)$. The derivative $t'(s) = w(t(s))$ exists everywhere, except for the points $s_k = s(t_k)$.

Let $y(s) = u(t(s))$, then

$$
\|u\|^2_{H^1 \cdot ((a,b),w)} = \int_0^L |y'(s)|^2 ds \quad \tag{24}\nonumber
$$

and

$$
\|u\|^2_{L^2((a,b),w)} = \int_0^L W(s)|y(s)|^2 ds, \quad W(s) = w^2(t(s)).
$$

The function $W(s)$ is monotone, and

$$
\int_0^L W(s)ds = \int_0^L w(t(s))t'(s)ds = \int_a^b w(t)dt; \quad \tag{25}\nonumber
$$

$$
\int_0^L \sqrt{W(s)}ds = \int_0^L t'(s)ds = b - a. \quad \tag{26}\nonumber
$$

In the course of the proofs of Theorems 5.3 and 5.5 we make use of the following result. Its most important part (i) was obtained in [3], see Theorem 3.1 and, especially, Remark 3.1 there. See also an exposition in [2], Corollary 6.3. The part (ii) is new and we present it with proof.
Theorem 6.1 (i) Let $L \leq \infty$ and let $W \in L^{1/2}(0, L)$ be a monotone, non-negative function. Then the inequality holds
\[ \int_0^L W(s) |y(s)|^2 ds \leq C(W) \int_0^L |y'(s)|^2 ds, \quad y \in \mathcal{H}^{1/2}(0, L), \quad (27) \]
and therefore the quadratic form in the left-hand side generates in $\mathcal{H}^{1/2}(0, L)$ a bounded self-adjoint operator, say $T_W$. Moreover, the operator $T_W$ is compact and for its eigenvalues $\mu_j(T_W)$ the following estimate holds, with a constant factor which does not depend on $L$ and on $W$:
\[ \# \{ j : \mu_j(T_W) > \lambda^{-1} \} \leq C\sqrt{\lambda} \int_0^L \sqrt{W(s)} ds, \quad \lambda > 0. \quad (28) \]
Also, the asymptotic formula is valid:
\[ \# \{ j : \mu_j(T_W) > \lambda^{-1} \} = \frac{\sqrt{\lambda}}{\pi} \int_0^L \sqrt{W(s)} ds + o(\sqrt{\lambda}), \quad \lambda \to \infty. \quad (29) \]

(ii) Suppose in addition that the function $W$ satisfies the estimate
\[ W(s) \leq C(L - s)^{-r} \quad (30) \]
with some $r \in (0, 2)$. Then the following, uniform in $\lambda$ remainder estimate in the asymptotic formula \((23)\) is satisfied:
\[ \left| \# \{ j : \mu_j(T_W) > \lambda^{-1} \} - \frac{\sqrt{\lambda}}{\pi} \int_0^L \sqrt{W(s)} ds \right| \leq C(L) (\lambda^{1/(4-r)} + 1), \quad \lambda > 0. \quad (31) \]

Proof of (ii). For definiteness, we assume the function $W$ to be increasing.
Suppose at first that $W$ is bounded. Then we use the standard variational reasoning: divide $[0, L]$ into $n$ equal parts, on each part $(s_k, s_{k+1})$ replace $W(s)$ by its inf and sup and solve the resulting eigenvalue problem under the Dirichlet or the Neumann boundary conditions. We obtain, denoting $h = L/n$:
\[ \sum_{k=0}^{n-1} \left[ \frac{h}{\pi} \sqrt{\lambda W(s_{k+})} \right] \leq \# \{ j : \mu_j(T_W) > \lambda^{-1} \} \leq n + \sum_{k=1}^{n} \left[ \frac{h}{\pi} \sqrt{\lambda W(s_{k-})} \right]. \]
Roughening this inequality, we obtain:
\[ - n + \frac{h\sqrt{\lambda}}{\pi} \sum_{k=0}^{n-1} \sqrt{W(s_{k+})} \leq \# \{ j : \mu_j(T_W) > \lambda^{-1} \} \]
\[ \leq n + \frac{h\sqrt{\lambda}}{\pi} \sum_{k=1}^{n} \sqrt{W(s_{k-})}. \]
We also have, due to the monotonicity of $W$:
\[ h \sum_{k=0}^{n-1} \sqrt{W(s_{k+})} \leq \int_0^L \sqrt{W(s)} ds \leq h \sum_{k=1}^{n} \sqrt{W(s_{k-})}. \]
This yields
\[ \left| \# \{ j : \mu_j(T_W) > \lambda^{-1} \} - \frac{\sqrt{\lambda}}{\pi} \int_0^L \sqrt{W(t)} dt \right| \leq C(\sqrt{\lambda} \frac{L \sqrt{W(L)}}{n} + n). \quad (32) \]
Suppose now that $W(s)$ is unbounded. Then we choose a point $S < L$, insert the condition $y(S) = 0$, apply the inequality (32) on $(0, S)$ and use the estimate (28) on $(S, L)$. We obtain
\[
\left| \pi\#\{j : \mu_j(TW) > \lambda^{-1}\} - \sqrt{\lambda} \int_0^L \sqrt{W(s)} ds \right| 
\leq C \left( \sqrt{\lambda} \left( \frac{L \sqrt{W(S)}}{n} + \int_S^L \sqrt{W(s)} ds \right) + n + 1 \right).
\]
Now we use the inequality (30) and then minimize the right-hand side over $S \in (0, L)$. This gives
\[
\left| \pi\#\{j : \mu_j(TW) > \lambda^{-1}\} - \sqrt{\lambda} \int_0^L \sqrt{W(s)} ds \right| 
\leq C \left( \sqrt{\lambda} \left( L/n \right)^{1 - (r/2)} + n + 1 \right).
\]
We arrive at (31), taking here $n = \left[ \lambda^{1/(4 - r)} \right] + 1$.

6.3 Proof of Theorem 5.3

(i) Fix $k = 0, 1, \ldots$ and apply the construction in the beginning of Subsection 6.2 to the interval $[a, b] = [t_k, R]$ and the weight function $w(t) = g_k(t)$. Let $s_k(t)$ stands for the corresponding function (23) and $t_k(s)$ stands for its inverse. The assumptions of Theorem 6.1 are satisfied for the function $W_k(s) = g_k^2(t_k(s))$. According to (26), the relations (28) and (29) turn into
\[
\#\{j : \mu_j(TW_k) > \lambda^{-1}\} \leq C \sqrt{\lambda} \left( R - t_k \right) \lambda > 0, \quad k = 0, 1, \ldots
\]
where the constant $C$ does not depend on $k$, and
\[
\#\{j : \mu_j(TW_k) > \lambda^{-1}\} = \frac{\sqrt{\lambda}}{\pi} (R - t_k) + o(\sqrt{\lambda}), \quad \lambda \to \infty.
\]
Substituting $u(t) = y(s_k(t))$ in (27) (with $L = L_k = \int_{t_k}^R (g_k(\tau))^{-1} d\tau$ and $W = W_k$), we come to the inequality
\[
\int_{t_k}^R |u(t)|^2 g_k(t) dt \leq C a_k[u], \quad u \in \mathcal{H}^{1,1}((t_k, R), g_k)
\]
where $a_k[u]$ is the quadratic form defined in (3). This shows that the operator $A_k$ has bounded inverse and that the spectrum of $A_k^{-1}$ coincides with one of the operator $T_{W_k}$. The inequality (33) turns into
\[
N(\lambda; A_k) \leq C \sqrt{\lambda} \left( R - t_k \right), \quad \lambda > 0, k = 0, 1, \ldots
\]
and the asymptotic formula (34) turns into the formula (13).

(ii) By (3), we have
\[
\lambda^{-1/2} N(\lambda; -\Delta) = \left( \lambda^{-1/2} N(\lambda; A_0) \right) + \sum_{k=1}^{\infty} b_1 \ldots b_{k-1} (b_k - 1) \left( \lambda^{-1/2} N(\lambda; A_k) \right).
\]
As $\lambda \to \infty$, each term in big parentheses tends to $\pi^{-1}(R - t_k)$. Besides, by (34) the series is dominated by

$$C\left(R + \sum_{k=1}^{\infty} b_1 \ldots b_{k-1}(b_k - 1)(R - t_k)\right) = C|\Gamma|.$$ 

Now (34) follows from the Lebesgue Theorem on the dominated convergence.

(iii) The proof of (35) is the same and we skip it.

**Remark.** It follows from (24) that for any $u \in H^1((t_k, R), g_k)$ its image $y$ has a finite limit at $s = L_k$. Therefore, the same is true for the function $u$ at the point $t = R$. The equalities (25) and (5) imply that necessarily $u(R-) = 0$ for any function $u \in \text{Dom}(a_k)$, provided $|\Gamma| = \infty$. If $|\Gamma| < \infty$, then various boundary conditions at $t = R$ for functions are possible. This is consistent with the result of [5], Theorem 5.2 where the boundary value problems for the differential equations on regular trees were studied from a different point of view.

### 6.4 Proof of Theorem 5.5

For the tree $\Gamma_{q,b}$ we have for $k = 0, 1, \ldots$, cf. (3) and (6):

$$\|u\|^2_{L^2((t_k, R); g_k)} = \int_{1-q^k}^{1} |u(t)|^2 g_k(t)dt$$

and

$$a_k[u] = \int_{1-q^k}^{1} |u'(t)|^2 g_k(t)dt.$$ 

Substituting $t = 1 - q^k(1 - s)$, $u(t) = v(s)$, and taking (6) into account, we obtain:

$$\|u\|^2_{L^2((t_k, R); g_k)} = q^{-k}b^k\|v\|^2_{L^2((0, R); g_l)}; \quad a_k[u] = q^k b^k a_0[v].$$

This implies that for any $k$ the operator $A_k$ is unitarily equivalent to $q^{-2k}A_0$ and therefore,

$$N(\lambda; A_k) = N(\lambda q^{2k}; A_0), \quad \lambda > 0, \; k = 1, 2, \ldots$$ (36)

This property of self-similarity is the key observation which allows us to handle the problem.

It follows from Theorem 4.3 and the formula (36) that

$$N(\lambda; -\Delta) = N(\lambda; A_0) + (1 - b^{-1}) \sum_{k=1}^{\infty} b^k N(\lambda q^{2k}; A_0).$$ (37)

Now we are in a position to complete the proof of the statement (i). The function $N(\lambda; A_0)$ satisfies the inequality

$$N(\lambda; A_0) \leq C\sqrt{\lambda}, \quad \lambda > 0,$$ (38)

cf. (35). Denote $\mu = \ln \lambda, \; \eta = -2 \ln q, \; \Phi(\mu) = \lambda^{-\beta/2} N(\lambda; -\Delta)$ and $\varphi(\mu) = \lambda^{-\beta/2} N(\lambda; A_0)$, then the equality (37) turns into

$$\Phi(\mu) = \varphi(\mu) + (1 - b^{-1}) \sum_{k=1}^{\infty} \varphi(\mu - k\eta).$$
This yields
\[ \Phi(\mu) - \Phi(\mu - \eta) = \varphi(\mu) - b^{-1}\varphi(\mu - \eta). \] (39)

This is a particular case of the **Renewal Equation**, well known in probability. The function in the right-hand side of (39) is zero at \(-\infty\) (since \(N(\lambda; A_0) = 0\) for small \(\lambda > 0\)) and exponentially decays at \(+\infty\) (since \(N(\lambda; A_0)\) satisfies (38) and \(\beta > 1\)). Therefore, the Renewal Theorem applies, see e.g. [8], Chapter XI.1, or a modern exposition in [5]. The equation (39) involves the single shift (by \(\eta\)), hence this is the so-called lattice case. According to the Renewal Theorem, there exists an \(\eta\)-periodic function \(\psi(\mu)\) which is bounded and bounded away from zero, such that
\[ \Phi(\mu) = \psi(\mu) + o(1), \quad \mu \to \infty. \]

This immediately leads to (16).

(ii) As in the proof of Theorem 5.3, we use the scheme presented in the beginning of Subsect. 6.2. For the tree \(\Gamma = \Gamma_{b^{-1}, b}\) and \(k = 0\) we have \([a, b) = [0, 1)\) and
\[ w(t) = g_{\Gamma}(t) = b^j \text{ for } 1 - b^{-j} < t \leq 1 - b^{-j-1}, \quad j = 0, 1, \ldots. \]

It follows that
\[ w(t) \leq C(1 - t)^{-1}. \] (40)

Besides,
\[
L = \int_0^1 \frac{dt}{w(t)} = \sum_{j=0}^{\infty} \frac{b^{-j} - b^{-j-1}}{b^j} = \frac{b}{b + 1}.
\]

For the function \(s = s(t)\) defined by (23), we have
\[ s(t_k) = \sum_{j=0}^{k-1} \frac{b^{-j} - b^{-j-1}}{b^j} = L - \frac{b^{-2k}}{b + 1}, \]

or
\[ L - s(t_k) = \frac{(1 - t_k)^2}{b + 1}. \]

Since \(s(t)\) is monotone, this implies
\[ L - s(t) \geq c(1 - t)^2, \quad c > 0, \quad t \in (0, 1) \]
and therefore, \(1 - t(s) \leq c^{-1/2}(L - s)^{1/2}\). It follows from here and from (40) that
\[ W(s) = w^2(t(s)) \leq C(L - s)^{-1}, \]
so that the inequality (31) is satisfied with \(r = 1\). Correspondingly, the estimate (31) takes the form
\[ |\pi N(\lambda; A_0) - \lambda^{1/2}| \leq C(\lambda^{1/3} + 1), \quad \lambda > 0. \] (41)

Let us return to the equality (37) (where now \(q = b^{-1}\)). According to the estimate (35), with \(R = 1\) and \(k = 0\), we find that
\[ N(\lambda; A_0) = 0 \quad \text{if } C^2 \lambda < 1. \]
Therefore, summation in (17) is actually taken over such \( k \) that \( b^{2k} \leq C^2 \lambda \). Using for all such \( k \) the estimate (11), we obtain:

\[
\left| \pi N(\lambda; A_0) - \left( 1 + (1 - q)\#\{k > 1 : b^{2k} \leq C^2 \lambda \} \right) \lambda^{1/2} \right| 
\leq C \sum_{k : b^{2k} \leq C^2 \lambda} (\lambda^{1/3} b^{k/3} + 1).
\]

The sum in the right-hand side is of order \( O(\lambda^{1/2}) \), and the factor in front of \( \lambda^{1/2} \) in the left-hand side differs from \( \frac{(1-q)\ln \lambda}{2 \ln b} \) by \( O(1) \). This completes the proof of (17).

### 6.5 Proof of Theorem 5.9

We split the proof into several steps.

1. Denote

\[
J(\lambda; V) = \int_0^\infty (\lambda - V(t))^{1/2} dt = \int_0^{Q(\lambda)} (\lambda - V(t))^{1/2} dt.
\]

Under the assumption (19) one has

\[
J(\lambda; q) \approx Q(\lambda) \sqrt{\lambda}, \quad \lambda \to \infty
\]

where the symbol \( \approx \) means a two-sided estimate. Indeed, evidently \( J(\lambda; V) \leq Q(\lambda) \sqrt{\lambda} \), and for \( \lambda \geq 2 \lambda_0 \) one has

\[
J(\lambda; V) = \frac{1}{2} \int_{V(0)}^\lambda \frac{Q(s) ds}{(\lambda - s)^{1/2}} \geq \frac{Q(\lambda/2)}{2} \int_{\lambda/2}^\lambda \frac{ds}{(\lambda - s)^{1/2}} \geq c Q(\lambda) \sqrt{\lambda}.
\]

Later we shall need also the inequality

\[
\int_{r}^\infty (\lambda - V(t))^{1/2} dt \geq c Q(\lambda) \sqrt{\lambda}, \quad r \leq Q(\lambda/2), \quad \lambda \geq 2 \lambda_0.
\]

(42)

Its proof, and also the value of \( c \), are the same as in the preceding inequality.

2. Consider the Schrödinger operator \( K_V y = -y'' + V y \), \( u(y) = 0 \) in \( L^2(\mathbb{R}^+) \). Fix \( \lambda > V(0) \) and compare the values of \( N(\lambda; K_V) \) and \( \#\{j : \mu_j(T_W) > 1\} \) for the operator \( T_W \) introduced in Theorem 6.1, with \( L = Q(\lambda) \) and \( W(s) = \lambda - V(s) \). It follows from the decoupling principle and from the Birman – Schwinger principle that these two numbers differ no more than by 2. For estimating the number \( \#\{j : \mu_j(T_W) > 1\} \), we use the inequality (32) (with \( \mu = 1 \)). The only difference with (32) is that now the function \( W(s) \) is decreasing, and for this reason the term \( W(L-\) in the right-hand side must be replaced by \( W(0) \). As a result, we obtain (replacing in the result \( \lambda \) by \( \lambda \)):

\[
\left| \pi N(\lambda; K_V) - J(\lambda; V) \right| \leq C \left( \frac{Q(\lambda) \sqrt{\lambda - V(0)}}{n} + n + 1 \right).
\]

Let us stress that the factor \( C \) does not depend on \( \lambda \) and \( n \). Minimizing the right-hand side of the last inequality over \( n \), we come to the estimate

\[
\left| \pi N(\lambda; K_V) - J(\lambda; V) \right| \leq C \left( \frac{Q(\lambda) \sqrt{\lambda}}{n} + n + 1 \right).
\]

(43)
where $C$ is an absolute constant.

3. Consider the quadratic forms $a_{V,k}$ defined in (14). The corresponding operators $A_{V,k}$ act in the weighted spaces $L^2((t_k, \infty), g_k)$ (recall that in our case $R = \infty$). For us it is more convenient to deal with the operators acting in the “usual” $L^2$. For this purpose we make a substitution which we describe for $k$ needed in the case $k > 0$.

Therefore, since $g$ and $\Gamma$ differ by a subspace of dimension $2\Psi(Q)$ of (45). We derive from (42):

$$
\int_{\mathbb{R}_+} |u(t)|^2 g(t) dt = \int_{\mathbb{R}_+} |y(t)|^2 dt
$$

and

$$
a_{V,0}[u] = a_{V,0}[y] := \sum_{k=0}^\infty \int_{t_k}^{t_{k+1}} \left( |y(t)|^2 + V(t)|y(t)|^2 \right) dt. \quad (44)
$$

The domain $\text{Dom}(a_{V,0})$ consists of all functions $y(t)$, such that $y \upharpoonright (t_k, t_{k+1}) \in H^1(t_k, t_{k+1})$ for each $k$, the sum in the last side of (44) is finite, $y(0) = 0$ and the matching conditions at the points $t_k$ are fulfilled:

$$
y(t_k+) = \lim_{t \to t_k^+} y(t) = \lim_{t \to t_k^-} y(t) = \lim_{t \to t_k^+} y(t), \quad k = 1, 2, \ldots
$$

It is not difficult to derive from (43) a similar estimate for the operator $A_{V,0}$. Indeed, again the problem reduces to the interval $(0, Q(\lambda))$. The linear spaces $\text{Dom}(a_{V,0})$ and $\text{Quad}(K_V)$, restricted to the set of functions supported by this interval, differ by a subspace of dimension $2\Psi(Q(\lambda))$. Therefore, (43) implies

$$
|\pi N(\lambda; A_{V,0}) - J(\lambda; V)| \leq C \left( (Q(\lambda)\sqrt{\lambda})^{1/2} + \Psi(Q(\lambda)) \right).
$$

The term $1$ appearing in (43) can be dropped, since $\Psi(Q(\lambda)) \geq 1$ for any $\lambda > V(0)$. Quite similarly,

$$
|\pi N(\lambda; A_{V,k}) - \int_{t_k}^{\infty} (\lambda - V(t))^{1/2} dt| \leq C \left( (Q(\lambda)\sqrt{\lambda})^{1/2} + \Psi(Q(\lambda)) \right).
$$

4. Now we are in a position to complete the proof. We have

$$
\widetilde{N}(\lambda; A_V) = \sum_{k=0}^\infty N(\lambda; A_{V,k}) = \sum_{k:0 \leq t_k < Q(\lambda)} N(\lambda; A_{V,k}).
$$

Then, therefore,

$$
\left| \widetilde{N}(\lambda; A_V) - \sum_{k=0}^\infty \int_{t_k}^{\infty} (\lambda - V(t))^{1/2} dt \right|
$$

$$
\leq C \Psi(Q(\lambda)) \left( (Q(\lambda)\sqrt{\lambda})^{1/2} + \Psi(Q(\lambda)) \right). \quad (45)
$$

Our next task is to estimate from below the sum appearing in the left-hand side of (45). We derive from (42):

$$
\sum_{k=0}^\infty \int_{t_k}^{\infty} (\lambda - V(t))^{1/2} dt \geq \sum_{k: t_k \leq Q(\lambda)/2} \int_{t_k}^{\infty} (\lambda - V(t))^{1/2} dt
$$

$$
\geq c \Psi(Q(\lambda/2)) Q(\lambda) \sqrt{\lambda} \geq c' \Psi(Q(\lambda)) Q(\lambda) \sqrt{\lambda}.
$$
The latter inequality is implied by the $\Delta_2$-condition (20). It follows from the assumption (21) that
\[
\Psi(Q(\lambda)) \left( (Q(\lambda)\sqrt{\lambda})^{1/2} + \Psi(Q(\lambda)) \right) = 0 \left( \Psi(Q(\lambda))Q(\lambda)\sqrt{\lambda} \right).
\]
The desired asymptotic formula (22) immediately follows.

7 Regular trees without boundary

In conclusion, let us discuss the case when the tree $\Gamma$ has no boundary. Let $\Gamma$ be a general metric tree. Choose a vertex $o \in E(\Gamma)$ and suppose that there are $d$ edges of $\Gamma$ adjacent to $o$. Then $\Gamma$ can be split into $d$ rooted subtrees $\Gamma^1, \ldots, \Gamma^d$ having the common root $o$. We say that the tree $\Gamma$ is regular if and only if all the subtrees $\Gamma^j$ are regular in the sense of Definition 2.1 and the corresponding sequences $\{t_k\}$ and $\{b_k\}$ are the same for all $j = 1, \ldots, d$. Note that this definition is not invariant with respect to the choice of the vertex $o$.

Suppose now that all the subtrees $\{\Gamma^j\}$ are homogeneous, $\Gamma^1 = \ldots = \Gamma^d = \Gamma_b$ and $d = b + 1$. Then we say that the tree $\Gamma$ is homogeneous. Unlike the case of arbitrary regular trees, this definition is invariant with respect to the choice of $o$.

The definitions of the Laplacian and of the Schrödinger operator extend to the trees without boundary in a natural way. The only difference is that now we have no boundary condition at $o$. Instead, the functions from the quadratic domain of the operator are required to be continuous at $o$.

Replacement of this continuity condition by the Dirichlet boundary condition $u(o) = 0$ means the passage to a subspace of codimension 1 of $\text{Quad}(A_V)$. Therefore, the character of the spectrum is not affected, and moreover, the eigenvalue distribution function $N(\lambda; A_V)$ can change no more than by one. The new operator splits into the orthogonal sum of $d$ copies of the operator studied in the main part of this paper. This allows one to immediately reformulate all the results for this new situation. It is unnecessary to present their precise formulations.

Note that the papers [4] and [5] deal with the operators on trees without boundary.

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