Complexity vs Stability in Small-World Networks

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Abstract

According to the May-Wigner stability theorem, increasing the complexity of a network inevitably leads to its destabilization, such that a small perturbation will be able to disrupt the entire system. One of the principal arguments against this observation is that it is valid only for random networks, and therefore does not apply to real-world networks, which presumably are structured. Here we examine how the introduction of small-world topological structure into networks affect their stability. Our results indicate that, in structured networks, the parameter values at which the stability-instability transition occurs with increasing complexity is identical to that predicted by the May-Wigner criteria. However, the nature of the transition, as measured by the finite-size scaling exponent, appears to change as the network topology transforms from regular to random, with the small-world regime as the cross-over region. This behavior is related to the localization of the largest eigenvalues along the real axis in the eigenvalue plain with increasing regularity in the network.

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1 Introduction

The issue of whether increasing the complexity of a network contributes to its dynamical instability has long been debated. This ‘complexity vs stability’ debate is especially acute in the field of ecology [1], as it relates to the importance of diversity for the long-term survival of ecosystems. However, understanding the relation between the network structure and its stability (with respect to dynamical perturbations) is crucial, as it is related to the robustness of systems as ubiquitous as power grids, financial markets, and even complex societies and civilizations[2]. Pioneering studies on the stability of networks, both theoretical[3] and numerical[4], suggested that increasing the network complexity, as measured by its size ($N$), density of connections ($C$) and the
strength of interactions between coupled elements (σ), almost inevitably leads to the destabilization of any arbitrary equilibrium state of the system. This result, known as the May-Wigner stability theorem, seemed to fly in the face of conventional wisdom that higher diversity makes a system more capable of surviving perturbations and has since led to much research on the connection between network complexity and stability[5].

The May-Wigner argument [3] confines itself to analyzing the local stability of an arbitrarily chosen equilibrium point of the network dynamics. Under such constraints, the explicit dynamics at the nodes can be ignored and the stability is governed by the leading eigenvalue of the linear stability matrix \( J \). As a first approximation, one can consider the network elements to be coupled randomly with each other. If the connection weights between linked nodes follow a Gaussian distribution (with mean 0 and variance \( \sigma^2 \)), then it follows that \( J \) is a random matrix. Therefore, existing rigorous results on the eigenvalue distribution of random matrices can be applied, which allows one to make the assertion that the network is almost certainly stable if \( \sqrt{NC\sigma^2} < 1 \), and almost certainly unstable otherwise.

Objections to the May-Wigner argument have often revolved around the assumption of a randomly connected network. As pointed out by many ecologists, most networks occurring in nature are not random, and seem to have structures such as trophic levels in the predator-prey relations between different species. Some early studies seemed to suggest that introducing a hierarchical organization (e.g., by partitioning the adjacency matrix of the network into blocks [6] or by having tree structures [7]) can increase the stability of a network under certain conditions. However, no general consensus on this issue has yet been achieved.

The introduction of “small-world” connection topology[8] has allowed the possibility of having different kinds of structures in a network, other than a straightforward hierarchy of levels. Small-world networks have the global properties of a random network (short average path length between the elements) while at the local level they resemble regular networks with a high degree of clustering among neighbors. In fact, several empirically obtained food web networks have been analyzed by different research groups looking for evidence of small-world structure. Initial reports of small-world ecological networks based on the analysis of 4 food webs [9] have been challenged by a study based on 7 food webs [10], and, more recently, by a comprehensive analysis of 16 food webs covering a wide variety of habitats[11]. The latter studies did not see significantly high clustering in most of these systems, compared to a random network.

In light of this, it is inevitable to ask oneself whether the introduction of small world connectivity confer any advantage to the network. If the occurrence of higher than average clustering has no functional significance, then the occurrence of small-world structures in a few networks are probably due to chance alone. In particular, we can ask whether introducing such structures in a net-
work can make it more stable, and therefore, able to survive perturbations compared to its random counterpart.

In this paper we strive to answer the above question. Although there have been previous studies on the eigenvalue distribution of small-world networks\cite{12,13}, the issue of stability has not been looked at in any depth. In the next section, we have described the basic model used to study the stability of the network as its structure is changed from regular through small-world to random. The results of extensive numerical studies is reported in Section 3, which suggests that the stability-instability transition occurs at the same critical value independent of the network structure; but the nature of this transition (as measured by the finite-size scaling exponent) changes with the topology. Finally, we conclude with a brief discussion of the implications of our results.

2 The Randomly Weighted Network Model

To observe the stability of networks at the small-world regime we follow the basic Watts-Strogatz construction \cite{8}. A ring consisting of \( N \) nodes, with each node connected to \( 2k \) neighbors (i.e., neighborhood size is \( k \)), is rewired with probability \( p \). In other words, a fraction \( p \) of the links among the nodes in the lattice are broken and then randomly reconnected, subject to the condition that the total number of links does not change and that two nodes are not connected by more than one directed link. As outlined in Ref.\cite{8}, increasing \( p \) decreases both the average path length and the clustering between nodes. In addition, we introduce randomly distributed weights to each of the links. Following May \cite{3}, we generate the corresponding linear stability matrix \( J \), such that, the non-zero entries are chosen from a Gaussian distribution with mean 0 and variance \( \sigma^2 \). In addition, to ensure that the nodes are individually stable in the absence of connections, the diagonal elements of \( J \) are chosen to be \(-1\). For a given \( p \), the stability of the resulting network is then examined by observing the sign of the leading eigenvalue \( \lambda_{\text{max}} \) of \( J \) as a function of size \( (N) \), connectance \( (C \approx \frac{2k+1}{N}) \) and strength of connectivity \( (\sigma) \). If \( \lambda_{\text{max}} > 0 \), the network is considered unstable. The corresponding probability of stability \( P_{\text{stability}} \) is calculated by carrying out a large number of network realizations. While this analysis does not explicitly consider dynamics, recent studies\cite{14,15} indicate that including the dynamics of the nodes does not qualitatively change the results obtained using the above technique.

It is possible that a sparsely connected network can be broken up into disconnected clusters by the rewiring procedure. For this reason, in the simulations reported here we have used \( k \gg \ln(N) \), which ensures that the entire network remains connected. Note that, unlike another study on the stability of small-world networks \cite{16}, we are analyzing the stability of an asymmetric sparse matrix whose non-zero entries are normally distributed.
Fig. 1. Finite-size scaling of the stability-instability transition for different network topologies \((C \approx 0.1)\). The tuning parameter is \(x = \sqrt{NC\sigma^2}\) \((x_c \rightarrow 1\) as \(N \rightarrow \infty\), as predicted by the May-Wigner theorem\) and the order parameter is \(P_{\text{stability}}\), the probability that a network is stable \((\lambda_{\text{max}} < 0)\). The scaling exponent \(\nu \approx 2\) for \(p = 0\) (left), \(\nu \approx 1.72\) for \(p = 0.01\) (center) and \(\nu \approx 1.5\) for \(p = 1\) (right). Data shown for \(N = 200\) (circles), \(N = 400\) (squares), \(N = 800\) (diamonds) and \(N = 1000\) (triangles). 1000 realizations were performed for each data point.

3 Results

As mentioned in the previous section, we observe the order parameter \(P_{\text{stability}}\) against the different network complexity parameters. For instance, keeping \(N\) and \(C\) fixed, as we increase \(\sigma\), \(P_{\text{stability}}\) decreases from 1 to 0, i.e., the network shows a stability-instability transition. The critical parameter value, \(\sigma_c\), at which the transition occurs remains unchanged as we vary the connection topology from regular \((p = 0)\) through small-world to random \((p = 1)\). This implies that changing the connection topology does not affect the stability of a network. However, the transition appears to get sharper as the network becomes more random.

To quantitatively measure the increase in steepness with randomness we used finite-size scaling analysis. The sharpness of the stability-instability transition increases with \(N\) for all topologies; finite-size scaling allows us to measure how the relative width of the transition region decreases with increasing \(N\). It was noted by May\cite{May3} that for random networks, this scales as \(N^{-2/3}\). We have carried out this analysis for networks with different values of \(p\), where the width of the transition region scales as \(N^{-1/\nu}\), and we observe the variation of \(\nu\) with \(p\). As shown in Fig. 1, \(\nu \approx 2\) for regular networks, and gradually
Fig. 2. The eigenvalue plane for regular (left) and random networks (right) with $N = 1000$, $C = 0.021$ and $\sigma = 0.206$. The data shown is for 20 realizations of each kind of network. Note the tails along the real axis for $p = 0$.

Fig. 3. The eigenvalue distributions for networks with $p = 1$ (squares), $p = 0.1$ (crosses), $p = 0.01$ (diamonds) and $p = 0$ (circles). $(N = 800$, $C = 0.021$ and $\sigma = 0.206)$. The data points are obtained after averaging over 1000 realizations.

decreases with $p$, ultimately becoming $\frac{3}{2}$ for $p = 1$ (as expected at the random network limit).

To understand why the nature of the stability-instability transition is affected by the network topology, we look at the eigenvalue plain of the regular and random networks (Fig. 2). The eigenvalues of the latter are bounded by a circle centered at $-1$ and having a radius of $\sqrt{NC\sigma^2}$. However, for regular networks, there are extensions from this circle along the real axis. The largest eigenvalues
Fig. 4. The largest eigenvalue $\lambda_{\text{max}}$ (left) and the fraction of real eigenvalues $f_{\text{real}}(\lambda)$ (right) plotted against network size $N$ for regular ($p = 0$) and random networks ($p = 1$) with $k = 10$ and $\sigma = 0.206$. The data points are obtained by averaging over 1000 realizations.

are located on this ‘tail’. The extended tails of the eigenvalue distribution for $p << 1$ are shown in greater detail in Fig. 3. For $p = 1$, the distribution is bounded, as predicted by Wigner’s semicircle theorem. However, in the presence of clustering (i.e., as $p \to 0$), the distribution extends out of the limits predicted by the semicircle distribution. The stability of the network is governed by the maximal eigenvalue. For the regular network, this is found at the tail of the eigenvalue distribution where the relative variance of $\lambda_{\text{max}}$ is much larger than if it was located in the bulk (as is the case for $p = 1$). This results in a smoother transition from stability to instability for regular networks.

It has been pointed out in Ref. [13] that in real-world networks, links are ‘expensive’. Therefore, we also looked at the case where $k$ is fixed as the system size increases (so that $C$ decreases with $N$). For low values of $k$ (relative to fixed $N$) we observe that the eigenvalue distribution shows a peak at the center, presumably due to contributions from small isolated clusters as mentioned in Ref. [13]. At higher values of $k$, we observe distributions similar to the ones obtained for the constant $C$ case reported before. However, a major difference was the relation between system size ($N$) and the largest eigenvalue, $\lambda_{\text{max}}$, as well as the fraction of real eigenvalues, $f_{\text{real}}(\lambda)$ (Fig. 4). For random networks, $\lambda_{\text{max}}$ attains a constant value for large values of $N$. Further, as pointed out in Ref. [17], the excess density of real eigenvalues decreases with $N$ roughly as $N^{-1/2}$. However, for regular networks, $\lambda_{\text{max}}$ grows with $N$ as $\log(N^\beta)$ (in Fig. 4, $\beta \sim 0.07$) and the excess density of real eigenvalues becomes constant for large $N$. This implies that as the regular network increases in size, the tail of the eigenvalue distribution gets longer (while the bulk remains fixed in size, similar to random networks). Also, more and more eigenvalues migrate from
the complex plane to the real axis, keeping its density constant even though the system size (and hence, the total number of eigenvalues) is increasing. These results imply that for the case of fixed number of links, regular networks are likely to be more unstable than random networks as the system size is increased.

4 Conclusion

Based on the results reported above we conclude that the introduction of small-world structure, i.e., a high degree of clustering among the nodes of a network, does not increase the network stability. On the contrary, in certain conditions, such a structure might make the network more unstable than its random counterpart. However, it was established quantitatively (using finite-size scaling) that the nature of the stability-instability transition with increasing complexity appears to change with the connection topology. In particular, networks with higher degree of regularity destabilize more smoothly compared to the abrupt transition to instability for random networks. This may have implications for the occurrence of small-world structure in certain food webs. Although unable to make the network more stable, such clustering structures may avoid the disastrous consequences of instability by making the deterioration of the network more gradual, compared to a random network.

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