Doubly periodic wave structures of the (2+1)-dimensional Nizhnik-Novikov-Veselov equation

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Abstract: A mapping approach is suggested to solve the (2+1)-dimensional Nizhnik-Novikov-Veselov equation, three families of solutions (including solitary wave solutions, periodic wave solutions and rational function solutions) with two arbitrary functions are obtained, among which two sets of wave packets are expressed as rational functions of the Jacobi elliptic functions.

1. Introduction
In recent years, the dynamics of localized structures has become a fascinating and important subject in nonlinear science. Many exotic one-dimensional localized excitations, such as kinks, breathers, instantons, and peakons, have been extensively studied, however, studies on higher spatial dimensions are rare and incomplete though much progress has been made on discovery and understanding of (2+1)-dimensional nonlinear evolution equations (NEEs) in recent years [1,2]. In one spatial dimension, a periodic wave can usually be regarded as a superposition of an infinite array of equally spaced identical solitons [3]. The corresponding situation for NEEs in higher spatial dimensions is much less well understood. Though some types of doubly periodic solutions have been obtained, however, their analysis has either not been performed or not been completed yet. In some special cases, these doubly periodic patterns can be regarded as the generalization of a two-solitoff, a singly periodic perturbed line soliton or a single straightline kink soliton [4]. Since the dromion is the fundamental coherent structure in (2+1)-dimensions, it would be natural to investigate if doubly periodic wave patterns can be regarded as a two-dimensional superposition of arrays of dromions.

The (2+1)-dimensional Nizhnik-Novikov-Veselov (NNV) equation is chosen as an illustrative example here for several reasons. First, the dromion has been studied intensively for this system. Second, a special procedure, named here as the mapping approach via the projective Riccati equations [5-7], is established and will lead to exact solutions for the (2+1)-dimensional NNV equation. By choosing elementary functions as the building blocks in this algorithm, various localized solutions can be found. In the present approach, the classical Jacobi elliptic functions will be employed as the building blocks, resulting in doubly periodic wave patterns for the NNV equation.

The celebrated (2+1)-dimensional NNV equation for three functions $u \equiv u(x,y,t), v \equiv v(x,y,t), w \equiv w(x,y,t)$ is defined by

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This model is an isotropic Lax extension of the classical (1+1)-dimensional shallow water wave KdV system. In recent years, many authors studied the NNV equation. For example, Boiti et al. [8] solved the NNV equation via the inverse scattering transformation. Tagami [9] obtained the soliton-like solutions of the NNV equation while Hu [10] gave out its nonlinear superposition formula. Some special types of the multi-dromion solutions and certain interesting localized excitations of Eq. (1) were given in [11,12], respectively. Zhang obtained its variable separation solution via a homogeneous balance approach [13]. The localized structures such as chaos and fractals were derived by Zheng et al. [14] based on a multiple linear variable separation method. However, to the best of our knowledge, its excitations obtained here with the aid of the projective Riccati equations approach were not reported in the preceding literature.

It is shown in Section 2 that the exact solutions, including multiple soliton solutions, periodic soliton solutions and rational function solutions for the NNV equation, are derived by a special mapping transformation procedure. Section 3 deals with the nonlinear coherent structures of the NNV equation. Doubly periodic (periodic in both directions) and semi-localized (decaying in one direction and periodic in the other) structures of Eq. (1) are investigated. The last section consists of a short summary and discussion.

2. New solutions of the (2+1)-dimensional NNV equation

Letting \( f \equiv f(\xi(X)), g \equiv g(\xi(X)) \) [where \( \xi \equiv \xi(X) \) is a still undetermined function of the independent variables \( X \equiv (x_0 = t, x_1, x_2, \cdots, x_m) \)], the projective Riccati equations [5-7] are defined by

\[
f'(\xi) = pf(\xi)g(\xi),
\]

\[
g'(\xi) = q + pg^2(\xi) - rf(\xi),
\]

where \( p^2 = 1, q \) and \( r \) are two real constants. When \( \delta = \pm 1 \), and the relation between \( f \) and \( g \) satisfies

\[
g^2 = \frac{1}{p} \left[ q - 2rf + \frac{r^2 + \delta}{q} f^2 \right],
\]

Eqs.(2) and (3) have ever been discussed in [5-7].

We introduce the mapping approach via the above projective Riccati equations. The basic idea of the algorithm is: Considering a nonlinear partial differential equation (NPDE) with independent variables \( X \equiv (x_0 = t, x_1, x_2, \cdots, x_m) \), and the dependent variable \( u \equiv u(X) \),

\[
P(u, u_t, u_{x_1}, u_{x_2}, \cdots) = 0,
\]

where \( P \) is a polynomial function of its arguments and the subscripts denote the partial derivatives, we assume that its solution is written as the standard truncated Painlevé expansion, namely

\[
u = A_0(X) + \sum_{i=1}^{n} (A_i(X)f(\xi(X)) + B_i(X)g(\xi(X))) f^{i-1}(\xi(X)).
\]


Here $A_i(X), A_j(X), B_k(X) \ (i=1, 2, \cdots, n)$ are arbitrary functions to be determined, and $f, g$ satisfy the projective Riccati equations (2) and (3).

To determine $u$ explicitly, one proceeds as follows: First, similar to the usual mapping approach, we can determine $n$ by balancing the highest-order nonlinear term with the highest-order partial derivative term in Eq. (5). Second, substituting Eq. (6) with Eqs. (2), (3) and (4) into the given NPDE, collecting the coefficients of the polynomials of $f^i g^j \ (i=0, 1, j=0, 1)$ and eliminating each of them, we can derive a set of partial differential equations of $A_i(X), A_j(X), B_k(X) \ (i=1, 2, \cdots, n)$ and $\xi(X)$. Third, to calculate $A_i(X), A_j(X), B_k(X) \ (i=1, 2, \cdots, n)$ and $\xi(X)$, we solve these partial differential equations. Finally, substituting $A_i(X), A_j(X), B_k(X) \ (i=1, 2, \cdots, n)$, $\xi(X)$ and the solutions of Eqs. (2) and (3) into Eq. (6), one obtains solutions of the given NPDE.

Now, we apply the above mapping approach to the NNV equation. According to the balancing procedure, Eq. (6) becomes

$$u = A + B f(\xi) + C f^2(\xi) + E g(\xi) + F f(\xi) g(\xi),$$

$$v = a + b f(\xi) + c f^2(\xi) + d g(\xi) + e f(\xi) g(\xi),$$

$$w = H + J f(\xi) + K f^2(\xi) + L g(\xi) + M f(\xi) g(\xi),$$

where $A, B, C, E, F, a, b, c, d, e, H, J, K, L, M$ and $\xi$ are arbitrary functions of $\{x, y, t\}$ to be determined, $f \equiv f(\xi(X)), g \equiv g(\xi(X))$ satisfy the projective Riccati equations (2) and (3). Substituting Eq. (7) with Eqs. (2) - (4) into Eq. (1), collecting the coefficients of the polynomial of $f^i g^j \ (i=0, 1, 3, 4, 5, j=0, 1)$ and setting each of the coefficients to zero, we can derive a set of partial differential equations for $A, B, C, E, F, a, b, c, d, e, H, J, K, L, M$ and $\xi$. It is difficult to obtain the general solution of these algebraic equations based on the solutions of Eqs. (2) and (3). Fortunately, in the special case that $\xi = \chi(x, t) + \eta(y, t)$, where $\chi \equiv \chi(x, t), \eta \equiv \eta(y, t)$ are two arbitrary variable separated functions of $(x, t)$ and $(y, t)$, respectively, we can obtain solutions of Eq. (1).

**Theorem** For the (2+1)-dimensional NNV equation (1), there are three couples of variable separated solutions, related to the projective Riccati equations (2) and (3).

(a) For $h^2 + r^2 > s^2$ and $pq < 0$, a couple of solitary solutions is

$$u = pr \chi, \eta, f(\xi) - \frac{p(h^2 + r^2 - s^2) \chi, \eta, f^2(\xi)}{q} \pm \sqrt{- \frac{p(h^2 + r^2 - s^2)}{q} \chi, \eta, f(\xi) g(\xi)},$$

$$v = \frac{X_s + X_{ss} - pq \chi_s^3}{3X_s} + \left(pr \chi_s^2 + \sqrt{- \frac{p(h^2 + r^2 - s^2)}{q} \chi, \eta, f^2(\xi)} - \frac{p(h^2 + r^2 - s^2)}{q} \chi, \eta, f(\xi) g(\xi)\right) \pm \sqrt{\frac{p(h^2 + r^2 - s^2) \chi, \eta, f^2(\xi)}{q}}$$

$$+ pr \chi_s g(\xi) \pm \sqrt{\frac{-p(h^2 + r^2 - s^2) \chi_s f(\xi) g(\xi)}{q}},$$

$$w = \frac{\eta_s + \eta_{ss} - pq \eta_s^3}{3\eta_s} + \left(p \eta_s^2 + \sqrt{- \frac{p(h^2 + r^2 - s^2)}{q} \eta_s f(\xi)} - \frac{p(h^2 + r^2 - s^2)}{q} \eta_s f^2(\xi)\right) \pm \sqrt{\frac{p(h^2 + r^2 - s^2) \eta_s f^2(\xi)}{q}}$$

$$+ p \eta_s g(\xi) \pm \sqrt{\frac{-p(h^2 + r^2 - s^2) \eta_s f(\xi) g(\xi)}{q}},$$

where $p = \pm 1, s$ and $h$ are arbitrary constants, $f$ and $g$ are expressed by
\[ f = \frac{q}{r + s \cosh(\sqrt{-pq} \xi) + h \sinh(\sqrt{-pq} \xi)} , \]
\[ g = -\frac{\sqrt{-pq}}{p} \frac{s \sinh(\sqrt{-pq} \xi) + h \cosh(\sqrt{-pq} \xi)}{r + s \cosh(\sqrt{-pq} \xi) + h \sinh(\sqrt{-pq} \xi)} . \]

(b) For \( r^2 < h^2 + s^2 \) and \( pq > 0 \), the trigonometric function solutions are
\[ u = pr \chi_s \eta_y f(\xi) - \frac{p(-h^2 + r^2 - s^2)}{q} \chi_s \eta_y f^2(\xi) \pm p \sqrt{\frac{p(-h^2 + r^2 - s^2)}{q} \chi_s \eta_y f(\xi) - \frac{p(-h^2 + r^2 - s^2)}{q} \chi_s f^2(\xi)} , \]
\[ v = \frac{\chi_s + \chi_{xx} - pq \chi^2}{3 \chi_s} + (pr \chi_s^2 \pm p(\chi_s^2 \mp \frac{p(-h^2 + r^2 - s^2)}{q} \eta_y) f(\xi) - \frac{p(-h^2 + r^2 - s^2)}{q} \eta_y f^2(\xi)} , \]
\[ + p \chi_{xx} g(\xi) \pm p \sqrt{\frac{p(-h^2 + r^2 - s^2)}{q} \chi_s f(\xi) g(\xi)} , \]
\[ w = \frac{\eta_s + \eta_{yy} - pq \eta^2}{3 \eta_s} + (pr \eta_s^2 \pm \frac{p(-h^2 + r^2 - s^2)}{q} \eta_{yy}) f(\xi) - \frac{p(-h^2 + r^2 - s^2)}{q} \eta_s f^2(\xi) , \]
\[ + p \eta_{yy} g(\xi) \pm p \sqrt{\frac{p(-h^2 + r^2 - s^2)}{q} \eta_s f(\xi) g(\xi)} , \]
where \( p = \pm 1 \), \( s \) and \( h \) are arbitrary constants, \( f \) and \( g \) are expressed by
\[ f = \frac{q}{r + s \cos(\sqrt{pq} \xi) + h \sin(\sqrt{pq} \xi)} , \]
\[ g = \frac{\sqrt{pq}}{p} \frac{s \sin(\sqrt{pq} \xi) - h \cos(\sqrt{pq} \xi)}{r + s \cos(\sqrt{pq} \xi) + h \sin(\sqrt{pq} \xi)} . \]

(c) For \( C_1^2 + 4C_2 pr > 0 \) and \( q = 0 \), a pair of rational solutions is
\[ u = pr \chi_s \eta_y f(\xi) + \frac{1}{4} p^2 (C_1^2 + 4C_2 pr) \chi_s \eta_y f^2(\xi) \pm \frac{1}{2} p^2 \sqrt{C_1^2 + 4C_2 pr} \chi_s \eta_y f(\xi) g(\xi) , \]
\[ v = \frac{\chi_s + \chi_{xx} - pq \chi^2}{3 \chi_s} + (pr \chi_s^2 \pm \frac{1}{2} \sqrt{C_1^2 + 4C_2 pr} \chi_{xx}) f(\xi) + \frac{1}{4} p^2 (C_1^2 + 4C_2 pr) \chi_s f^2(\xi) , \]
\[ + p \chi_{xx} g(\xi) \pm \frac{1}{2} p^2 \sqrt{C_1^2 + 4C_2 pr} \chi_s f(\xi) g(\xi) , \]
\[ w = \frac{\eta_s + \eta_{yy} - pq \eta^2}{3 \eta_s} + (pr \eta_s^2 \pm \frac{1}{2} \sqrt{C_1^2 + 4C_2 pr} \eta_{yy}) f(\xi) + \frac{1}{4} p^2 (C_1^2 + 4C_2 pr) \eta_s f^2(\xi) , \]
\[ + p \eta_{yy} g(\xi) \pm \frac{1}{2} p^2 \sqrt{C_1^2 + 4C_2 pr} \eta_s f(\xi) g(\xi) , \]
where \( C_1, C_2 \) and \( r \) are arbitrary constants, \( p = \pm 1 \), \( f \) and \( g \) are expressed by

\[
f = \frac{2}{pr\xi^2 + C_1\xi - C_2}, \quad g = -\frac{2pr\xi + C_1}{(pr\xi^2 + C_1\xi - C_2)p}.
\]  

(13)

\[\text{Fig.1. (a) Structure of the physical quantity } U \text{ expressed by Eq. (14) with the condition (15), the parameters are chosen as } \alpha = \beta = 1, \eta_1 = \eta_2 = 0.1 \text{ and the time } t = 0. \text{ (b) Density of the doubly periodic structure related to (a). (c) Semi-localized structure of the physical quantity } U \text{ expressed by Eq. (14) for the condition (15), but the parameter } n, \text{ now chosen as } n_1 = 1, \text{ with the other constants the same as those in (a). The structures in (a) and (c) are continuing periodically in both } x \text{ and } y \text{ directions [for (a)] or in } y \text{ direction only [for (c)].}

3. Nonlinear coherent structures of the (2+1)-dimensional NNV equation

3.1. Doubly periodic waves

It is known that for a nonlinear system, the periodic (and the doubly periodic) wave solutions can usually be expressed by means of the Jacobi elliptic functions with constant moduli. For instance, if we take \( p = 1, q = -3, r = 2, s = 3 \) and \( h = 1 \), then the absolute value \( U \) of the function \( u \) in the solution (8) becomes

\[
U \equiv |u| = \left| \frac{6(4 + (1 - 3i)\sinh(\sqrt{3}(\chi + \eta)) + (3 - i)\cosh(\sqrt{3}(\chi + \eta)))}{(2 + \sinh(\sqrt{3}(\chi + \eta)) + 3\cosh(\sqrt{3}(\chi + \eta))^2)^2} \right|_{\chi, \eta},
\]

(14)
where $i$ denotes the imaginary unit of a complex number. If the two arbitrary functions $\chi$ and $\eta$ are chosen as

$$\chi = \alpha^{-1} \arcsin(sn(\alpha(x,t), n_1), \eta = \beta^{-1} \arcsin(sn(\beta(y+t), n_2)),$$

where $\alpha, \beta$ are arbitrary constants, $n_1, n_2$ are constant moduli of the Jacobi elliptic functions which satisfy $\chi = \tanh(\alpha(x+t), n_1), \eta = \tanh(\beta(y+t), n_2)$, the physical quantity $U$ defined in Eq. (14) shows a special type of doubly periodic pattern. Fig. 1(a) displays $U$ expressed by Eq. (14) with the condition (15), where the parameters are chosen as $\alpha = \beta = 1, n_1 = n_2 = 0.1$ and the time $t = 0$. Fig. 1(b) shows the density of this pattern. From there, one can see that the doubly periodic structure presents the shape of rhombus. However, by taking one of the moduli to be unity, patterns periodic in one direction but localized in the other are obtained. Fig. 1(c) illustrates one of these scenarios when the modulus $n_1$ is allowed to tend to 1 (for this limit, the Jacobi elliptic function $sn$ degenerates into the hyperbolic tanh function and is no longer periodic), but the other parameters are the same as those in Fig. 1(a).

Fig. 2. (a) A doubly quasi-periodic wave structure of the physical quantity $U$ expressed by Eq. (14) with the condition (16) and the parameters are chosen as Eq. (17) at time $t = 0$. (b) Density of the doubly quasi-periodic structure related to (a). (c) A special quasi-periodic line solitary wave of the physical quantity $U$ expressed by Eq. (14) for the condition (18), but the parameters are chosen as $\alpha = \beta = 1, a = 0.1, k = 0.25$ at time $t = 0$. 
3.2. Doubly quasi-periodic waves

In this subsection, we give doubly quasi-periodic (quasi-periodic in both \( x \) and \( y \) directions) wave solutions by selecting the arbitrary functions as Jacobi elliptic functions with variable moduli. For instance, when choosing \( \chi \) and \( \eta \) as

\[
\chi = \alpha^{-1} \arcsin(sn(\alpha(x+t), n_1)), \eta = \beta^{-1} \arcsin(sn(\beta(y+t), n_2)),
\]

where \( n_1 = a \tanh^2(k_1 x + l_1 t), n_2 = b \tanh^2(k_2 y + l_2 t), \alpha, \beta, k_1, l_1, k_2, l_2 \) are arbitrary constants, then Eq. (14) denotes a doubly quasi-periodic wave solution for \(|a| < 1 \) and \(|b| < 1\). Fig. 2(a) exhibits the structure of the doubly quasi-periodic wave solution with the condition (16) and the parameter selections as

\[
\alpha = \beta = 1, a = b = 0.1, k_1 = k_2 = 0.25
\]

at time \( t = 0 \). If one of the functions \( \chi \) and \( \eta \) is selected as a localized function such as the hyperbolic \( \tanh \) function or the hyperbolic \( \sech \) function and another is selected as the Jacobi elliptic function with variable modulus, then Eq. (14) becomes a quasi-periodic line solitary wave solution. Fig. 2(c) displays a special quasi-periodic line solitary wave expressed by Eq. (14) with the condition

\[
\chi = \alpha^{-1} \arcsin(tanh(\alpha(x+t))), \eta = \beta^{-1} \arcsin(sn(\beta(y+t), a \tanh^2(k y + l t))),
\]

and the parameter selections as \( \alpha = \beta = 1, a = 0.1, k = 0.25 \) at time \( t = 0 \).

4. Summary

First, with the use of the projective Riccati equations, we have obtained three types of variable separated solutions for the (2+1)-dimensional Nizhnik-Novikov-Veselov (NNV) equation, including solitary wave solutions, periodic wave solutions and rational function solutions. Second, the periodic wave solutions have been expressed by the Jacobi elliptic functions with constant moduli while the quasi-periodic wave solutions are written down by virtue of the Jacobi elliptic functions with space-time dependent moduli. Selecting the variable moduli suitably, we have found exact doubly quasi-periodic wave solutions, quasi-periodic line solitary waves with several parameters are varied.

Because of the wide application and the complexity of the doubly periodic structures related to the Jacobi elliptic functions, more details about these types of exact solutions deserve further study.

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