Mass and Charge of the Quantum Vortex in the (2 + 1)-d $O(2)$ Scalar Field Theory

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Using numerical simulations, a vortex is studied in the broken phase of the (2 + 1)-d $O(2)$-symmetric scalar field theory in the vicinity of the Wilson-Fisher fixed point. The vortex is an infraparticle that is surrounded by a cloud of Goldstone bosons. The $L$-dependence of the vortex mass in a finite $C$-periodic volume $L^2$ leads to the determination of the renormalized vortex charge.

Vortices are topological excitations that arise in superfluids, superconductors, and Bose-Einstein condensates [1]. In three spatial dimensions vortices are line-defects, including cosmic strings [2], that sweep out a world-sheet during their time-evolution, while in two dimensions vortices are point-defects. In the 2-d classical XY model, they drive the Berezinskii-Kosterlitz-Thouless phase transition [3, 4]. Popov was first to note that vortices and phonons in a (2 + 1)-d superfluid are dual to charged particles and photons in scalar QED [5]. He concluded that the mass of a vortex corresponds to its rest energy divided by the square of the speed of sound. Duan found the mass to diverge logarithmically with the volume, but attributed a finite mass to the vortex core [6]. According to Baym and Chandler, the core-mass corresponds to the mass of the superfluid within the core [7]. An equivalent concept, the Kopnin-mass exists for superconductors and fermionic superfluids [8–11]. Thouless and Anglin studied the reaction of a vortex to an external force by a pinning potential by means of the Gross-Pitaevskii equation [12]. They confirmed that the vortex mass is indeed logarithmically divergent. A finite vortex mass in (2 + 1)-d is also inconsistent with a fully controlled Monte Carlo calculation which shows that the vortex mass is finite in the infinite volume limit [13], thus contradicting the previous results.

A C-periodic vortex field is illustrated in Fig.1. Vortices arise as classical solutions in the broken phase of the (2+1)-d $O(2)$ symmetric field theory for a complex scalar field $\Phi(x) \in \mathbb{C}$ with the Lagrangian

$$\mathcal{L} = \frac{1}{2} \partial_{\mu} \Phi^* \partial^{\mu} \Phi - V(\Phi), \quad V(\Phi) = \frac{\lambda}{4!} (|\Phi|^2 - v^2)^2. \quad (1)$$

Static classical vortex solutions have the form $\Phi(r, \varphi) = f(r) \exp(i\varphi)$ and obey the Euler-Lagrange equation

$$\Delta \Phi = \frac{\lambda}{6} (|\Phi|^2 - v^2) \Phi \Rightarrow \left( \partial_r^2 + \frac{1}{r} \partial_r - \frac{1}{r^2} \right) f = \frac{\lambda}{6} (f^2 - v^2) f. \quad (2)$$

At large distances $\Phi$ approaches the vacuum value $v$ of the scalar field as $f(r) \sim v - 3/(\lambda rv^2)$. The energy density $\mathcal{H} = \frac{1}{2} \nabla \Phi^* \cdot \nabla \Phi + V(\Phi)$ of the static vortex is

$$\mathcal{H}(r) = \frac{1}{2} \left( (\partial_r f)^2 + \frac{f^2}{r^2} \right) + \frac{\lambda}{4!} (f^2 - v^2)^2 \sim \frac{v^2}{2r^2}. \quad (3)$$

This leads to an infrared logarithmic divergence of the vortex mass. Integrating the energy density over a disc of radius $R$ one obtains

$$E(R) = 2\pi \int_0^R dr r \mathcal{H}(r) \Rightarrow E(R) \sim \pi v^2 \log \frac{R}{R_0}. \quad (4)$$

The logarithmic divergence is due to a cloud of massless Goldstone bosons that surrounds the vortex. As we will see, the quantum vortex is dual to a charged particle.
in (2 + 1)-d scalar QED, with the Goldstone boson being the dual photon. The divergence of the vortex mass arises because the logarithmic Coulomb potential in two spatial dimensions is confining. In this way, the prefactor $e^2/(4\pi) = \pi v^2$ of the logarithm is associated with the dual electric charge $e$ of the vortex.

The semi-classical treatment of vortices is limited to the quantization of their collective degrees of freedom. Here, for the first time, we present a completely controlled, fully non-perturbative, translation invariant calculation of the quantum vortex in the continuum limit of the (2 + 1)-d O(2) scalar field theory, approaching the Wilson-Fisher fixed point from the broken phase. For this purpose, we regularize the theory on a 3-d cubic Euclidean space-time lattice. The complex scalar field is represented by a unit-vector $(\cos(\varphi_x), \sin(\varphi_x))$ associated with the lattice sites $x$. The resulting (2 + 1)-d XY model is defined by the partition function

$$Z = \prod_x \frac{1}{2\pi} \int_{-\pi}^{\pi} d\varphi_x \exp(-S[\varphi]), \quad S[\varphi] = \sum_{xy} s(\varphi_{xy}). \quad (5)$$

Here $x$ and $y$ are nearest-neighbor lattice points and $\varphi_{xy} = \varphi_x - \varphi_y$. The standard action has $s(\varphi_{xy}) = \frac{1}{g^2}(1 - \cos\varphi_{xy})$, while the Villain action [22] is given by

$$\exp(-s(\varphi_{xy})) = \sum_{n_{xy} \in \mathbb{Z}} \exp \left( -\frac{1}{2g^2} (\varphi_{xy} - 2\pi n_{xy})^2 \right). \quad (6)$$

The XY model can be dualized exactly to a (2 + 1)-d Abelian gauge theory with integer-valued non-compact vector potentials $A_l \in 2\pi\mathbb{Z}$ associated with the links $l$ of the dual lattice. The dual partition function is

$$Z = \prod_l \sum_{A_l \in 2\pi\mathbb{Z}} \exp(-S[A]), \quad S[A] = \sum_\square s(F_\square). \quad (7)$$

Here the field strength $F_\square = dA_l = A_{l1} + A_{l2} - A_{l3} - A_{l4} = 2\pi \mathbb{Z}$ is the lattice curl of the vector potentials associated with the four links $l_1, l_2, l_3, l_4$ that encircle a dual plaquette $\square$. The dual Boltzmann weight is given by

$$\exp(-s(F_\square)) = \frac{1}{2\pi} \int_{-\pi}^{\pi} d\varphi_{xy} \exp(-s(\varphi_{xy}) + i n_\square \varphi_{xy}), \quad (8)$$

where $\square$ is the plaquette dual to the original nearest-neighbor link $(xy)$. The dual standard action is given by the modified Bessel function $\exp(-s(F_\square)) = I_{n_\square}(1/g^2)$, while the dual Villain action is simply given by $s(F_\square) = \frac{1}{4g^2} F_\square^2$. Here $e$ is the dual charge that obeys the Dirac quantization condition $eg = 2\pi$.

Next, we relate the dual integer gauge theory to (2 + 1)-d QED with a charged scalar field $\chi_{\tilde{x}} \in U(1)$ that represents the vortex in the dual description. Here $\tilde{x}$ is a dual lattice site (the center of a cube on the original lattice). The corresponding lattice action is given by

$$S[A, \chi] = \frac{1}{2e^2} \sum_\square F_\square^2 - \kappa \sum_l \text{Re} \left( \chi_{\tilde{x}}^* \exp(iA_l) \chi_{\tilde{x}} \right) \Rightarrow S[A, \chi = 1] = \frac{1}{2e^2} \sum_\square F_\square^2 - \kappa \sum_l \cos A_l. \quad (9)$$

Here $l$ is the link that connect the dual nearest-neighbor sites $\tilde{x}$ and $\tilde{y}$. The action is gauge invariant against

$$A'_l = A_l - d\alpha_{\tilde{x}} = A_l + \alpha_{\tilde{x}} - \alpha_{\tilde{y}}, \quad \chi'_{\tilde{x}} = \exp(i\alpha_{\tilde{x}}) \chi_{\tilde{x}}. \quad (10)$$

Here $d$ is the lattice distance. In the unitary gauge, $\chi_{\tilde{x}} = 1$, the action reduces to the second line of eq.(9). Taking the limit $\kappa \to \infty$ leads to the constraint $A_l \in 2\pi\mathbb{Z}$, which corresponds to the integer gauge theory that is dual to the (2 + 1)-d XY model. Scalar QED exists in two phases. The Coulomb phase at large values of $e$ is dual to the broken phase of the (2 + 1)-d XY model at small values of $g$, with the Goldstone boson being the dual photon. The charged particle in the Coulomb phase of (2 + 1)-d scalar QED is dual to the vortex in the (2 + 1)-d XY model, with $e$ being the bare vortex charge. At small values of $e$, the theory exists in a Higgs phase in which the photon picks up a mass. This phase, in which vortices condense, is dual to the massive symmetric phase of the (2 + 1)-d XY model at large values of $g$.

Here we concentrate on the Coulomb phase in which vortices are dual to charged scalar particles with bare charge $e$. A charged particle is surrounded by a cloud of massless photons, which extends to infinity. The resulting non-local object is known as an infraparticle, which, due to the soft photon cloud, does not simultaneously have a well-defined charge and a well-defined mass [18]. The operator that creates the infraparticle is given by

$$\chi_{\tilde{x}}^C = \exp \left( i\alpha_{\tilde{x}}^C \right) \chi_{\tilde{x}} = \exp \left( i\Delta^{-1} \delta A_l \right) \chi_{\tilde{x}}. \quad (11)$$

It leads from the vacuum into the charge 1 superselection sector. The operator is non-local because it contains not only $\chi_{\tilde{x}}$, but also the non-local Coulomb cloud surrounding the charge, which is represented by $\alpha_{\tilde{x}}^C = \Delta^{-1} \delta A_l$. Here $\Delta$ is the 2-d spatial Laplacian, and $\delta A_l = A_{l1} + A_{l2} - A_{l3} - A_{l4}$ is the 2-d lattice divergence of $A_l$, which is constructed from the links $l_1$ and $l_2$ that exit the dual site $\tilde{x}$ in the positive 1- and 2-direction, and the links $l_3$ and $l_4$ that enter $\tilde{x}$ from the negative 1- and 2-direction. In fact, $\alpha_{\tilde{x}}^C$ is the gauge transformation that turns the gauge field $A_l$ into the Coulomb gauge

$$\delta A'_l = \delta (A_l - d\alpha_{\tilde{x}}^C) = \delta A_l - \Delta \alpha_{\tilde{x}}^C = 0 \Rightarrow \alpha_{\tilde{x}}^C = \Delta^{-1} \delta A_l. \quad (12)$$

Here we have used $\delta d\alpha_{\tilde{x}}^C = \Delta \alpha_{\tilde{x}}^C$. The gauge transformation $\exp(i\alpha_{\tilde{x}}^C)$ endows the charged particle with its surrounding Coulomb field. Under gauge transformations

$$\alpha_{\tilde{x}}^C = \Delta^{-1} \delta A_l = \Delta^{-1} \delta A_l - \Delta^{-1} \delta d\alpha_{\tilde{x}} = \alpha_{\tilde{x}}^C - \alpha_{\tilde{x}}, \quad (13)$$

such that the scalar field $\chi_{\tilde{x}}^C$ in the Coulomb gauge is gauge invariant and represents the non-local charged particle, which is just dual to the vortex surrounded by a cloud of massless Goldstone bosons.
The physical charged field from above was first constructed by Dirac [23] and was also used by Fröhlich and Marchetti in their construction of monopole superselection sectors in the Coulomb phase of (3 + 1)-d compact $U(1)$ lattice gauge theory [24]. In that case, the monopoles are dual to massive charged infrared particles in the Coulomb phase of (3 + 1)-d scalar QED. Fröhlich and Marchetti have also provided fully non-perturbative constructions of soliton sectors in a wide variety of systems [25]. This includes vortices in the Higgs phase of 3-d scalar QED which become anyons in the presence of a Chern-Simons term [26, 27]. The vortices considered here are dual to the charged particles in the Coulomb phase of (2 + 1)-d scalar QED. Because in two spatial dimensions the Coulomb potential is confining, as we already discussed at the classical level, the mass of an isolated vortex diverges logarithmically in the infrared.

While the vortex sectors in the physical Hilbert space are removed to infinite energy in the infinite volume limit, it is most interesting to construct them in a finite volume. A recent study that enforced a vortex by boundary conditions and fitted a vortex profile to numerical data concluded that the vortex mass has a finite infinite volume limit [13]. This contradicts our results. For an infrared sensitive infrared particle, imposing fixed boundary conditions (which break translation invariance) can be problematical. Periodic boundary conditions maintain translation invariance, but, as a consequence of Gauss' law, they do not allow the existence of charged states. For this reason, C-periodic boundary conditions were introduced both for Abelian [21] and for non-Abelian gauge theories [28–30]. When shifted by a distance $L$, a C-periodic field is replaced by its charge-conjugate. C-periodic boundary conditions leave translation invariance intact and allow the existence of charged states. However, charge and anti-charge states are mixed to form charge conjugation eigenstates. C-periodic boundary conditions are also used in lattice simulations of monopoles [21, 31] and of QCD coupled to QED [32]. C-periodic boundary conditions in scalar QED are discussed in detail in the supplementary material.

We consider (2 + 1)-d scalar QED in the unitary gauge, $\chi_{\vec{x}} = 1$, at $\varphi = \infty$ such that $A_t \in 2\pi\mathbb{Z}$. We work on a cubic space-time lattice with periodic temporal and C-periodic spatial boundary conditions, $A_{l'} = C A_l = -A_l$. The link $l'$ is shifted relative to $l$ by a distance $L$ in the spatial 1- or 2-direction. C-periodic Abelian gauge fields are anti-periodic, because charge conjugation changes the sign of the vector potential. In the unitary gauge the vortex field of eq.(11) takes the form $\chi_{\vec{x}} = \exp (i \Delta^{-1} \delta A_{l})$. With C-periodic boundary conditions the spatial Laplacian $\Delta$ has no zero-modes and $\Delta^{-1}\delta A_{l}$ is well-defined.

There is a $\mathbb{Z}(2)$ symmetry that changes the sign of the vortex field $\chi_{\vec{x}}$, which characterizes a non-trivial superselection sector and guarantees the stability of the vortex in a C-periodic volume. This symmetry results from a constant gauge transformation $\alpha_{z} = \pi$, such that $\exp(i\alpha_{z}) = -1$. This is the only global gauge transformation that is consistent with C-periodic boundary conditions. We denote it as vortex field reflection.

In a C-periodic volume the real-part $\text{Re} \chi_{\vec{x}}$ (which is C-even) obeys periodic while the imaginary part $\text{Im} \chi_{\vec{x}}$ (which is C-odd) obeys anti-periodic boundary conditions. This implies that the C-even and C-odd components of the charged vortex state necessarily have different spatial momenta and thus different energies. Since C-periodic boundary conditions maintain translation invariance, we construct vortex fields with definite spatial momenta $(p_1, p_2)$ at fixed Euclidean time $\bar{x}_3$

$$\chi^+(p_1, p_2, \bar{x}_3) = \sum_{\bar{x}_1, \bar{x}_2} \exp(ip_1 \bar{x}_1 + ip_2 \bar{x}_2) \text{Re} \chi_{\vec{x}},$$

$$\chi^-(p_1, p_2, \bar{x}_3) = \sum_{\bar{x}_1, \bar{x}_2} \exp(ip_1 \bar{x}_1 + ip_2 \bar{x}_2) \text{Im} \chi_{\vec{x}}.$$  

(14)

The momenta of the periodic C-even component $\chi^+(p_1, p_2, \bar{x}_3)$ are quantized in integer units, $p_i = 2\pi n_i/L$, $n_i \in \mathbb{Z}$, while the momenta of the anti-periodic C-odd component $\chi^-(p_1, p_2, \bar{x}_3)$ are quantized in half-odd-integer units, $p_i = 2\pi (n_i + \frac{1}{2})/L$. Specifically, we consider $\chi^+(0, 0, \bar{x}_3)$ at zero momentum and $\chi^-(p_1, p_2, \bar{x}_3)$ at the smallest possible momenta $p_i = \pm \pi/L$. At large Euclidean time separation, the corresponding correlation functions then decay exponentially

$$\langle \chi^+(0, 0, 0) \chi^+(0, 0, \bar{x}_3) \rangle \sim \exp(-m\bar{x}_3),$$

$$\langle \chi^-(p_1, p_2, 0) \chi^-(p_1, p_2, \bar{x}_3)^* \rangle \sim \exp(-E\bar{x}_3).$$  

(15)

Here $m$ is the rest mass of the C-even component of a vortex, while $E$ is the energy of the C-odd component that moves with minimal momentum $p_i = \pm \pi/L$. Both components are odd under vortex field reflection.

First, we have simulated the (2+1)-d lattice XY model in the broken phase using the Wolff cluster algorithm [33], applied both to the standard and Villain action. The model then has massless Goldstone bosons that are described by a low-energy effective field theory with the Euclidean action $S[\varphi] = \int d^2 x \varphi \partial_\mu \varphi \partial^\mu \varphi$. We have performed numerical simulations in various space-time volumes in the vicinity of the critical point, in order to determine the spin stiffness $\rho$ which we use to set the energy scale. Near the critical point, physical quantities $O$ with mass dimension 1 (including $\rho$, $m$, and $E$) scale as

$$O(1/g^2) = A_O(1/g_c^2 - 1/g^2)^\nu \left[1 + a_O(1/g_c^2 - 1/g^2)^\omega\right].$$  

(16)

Here $\nu = 0.67169(7)$ is the very accurately known critical exponent associated with the correlation length, and $\omega = 0.789(4)$ is a universal exponent that controls corrections to scaling [34]. The amplitude $A_O$ and the coefficient $a_O$ are observable-specific and not universal. However, the amplitude ratios $A_{\mu}/A_{\rho}$ and $A_{\rho}/A_{\rho}$ are universal and yield the continuum limit results approaching
the O(2) symmetric Wilson-Fisher fixed point from the broken phase. Our results for ρ are consistent with those of [36]. The critical coupling for the standard action, $1/g_0^2 = 0.4541652(11)$, [35] and for the Villain action, $1/g_0^2 = 3.00239(6)$ [36] are known very accurately. 

Next, we have simulated the vortex correlation functions of eq.(15) in the dual (2 + 1)-d Z gauge theory using a Metropolis algorithm. The quantities $m$ and $E$ are extracted from cosh-fits of the correlation functions at large Euclidean time separations $\tilde{x}_3$, as illustrated for the Villain action in Fig.2. In order to take the continuum limit, we approach the critical point and increase the number of lattice points while keeping the physical size $L_\rho$ fixed. We then take the continuum limit of the dimensionless ratios $m/\rho$ and $E/\rho$ by identifying the universal amplitude ratios $A_m/A_\rho$ and $A_E/A_\rho$ from fits to the scaling form of eq.(16), as illustrated for the Villain action in Fig.3. The continuum limit extrapolations for the standard action are consistent within statistical errors, thus confirming universality when approaching the fixed point.

The final continuum limit results for $m/\rho = A_m/A_\rho$ and $E/\rho = A_E/A_\rho$ are shown in Fig.4 as a function of the spatial size in physical units $\rho L$. For large $L$, the mass of the quantum vortex diverges logarithmically,

$$m \sim \frac{e_r^2}{4\pi} \log(L/r_0), \quad e_r = 2\sqrt{3.58(8)}\pi \rho, \quad (17)$$

thus confirming the expected Coulombic confinement that we already encountered for the classical vortex. The prefactor of the logarithm determines the renormalized vortex charge $e_r$, which is another universal feature of the Wilson-Fisher fixed point. As expected, in the large volume limit, the energy $E$ of the (then more and more slowly moving) C-odd component approaches the mass $m$ of the C-even component (which has zero momentum).

The energy $E$ differs from the relativistic expression $\sqrt{p^2 + m^2}$, since an infraparticle breaks Lorentz invariance spontaneously [18], in addition to the explicit breaking due to the finite volume. We define the kinetic finite-volume mass $m_k$ of the vortex as

$$E = m + \frac{p^2}{2m_k}, \quad p_1, p_2 = \pm \frac{\pi}{L} \Rightarrow m_k = \frac{\pi^2}{L^2(E - m)}. \quad (18)$$

For $L_\rho = 1.43(2)$ and 2.14(3), we obtain $m_k/m = 0.71(3)$ and 0.55(4), indicating significant differences between the kinetic and the rest mass. The typical size of the vortex core can be characterized by $r_0 = 0.64(3)/\rho$ in eq.(17), or by its universal charge radius (yet to be determined).

Taking the infraparticle nature of the vortex into account sheds light on the subtle concepts of its rest and kinetic mass. It would be most interesting to investigate the universal properties of the quantum vortex in experiments with superfluid films or Bose-Einstein condensates tuned to the vicinity of the Wilson-Fisher fixed point, e.g., by extracting the universal vortex charge $e_r$ from the Coulomb interactions of vortices or anti-vortices.
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C-PERIODIC BOUNDARY CONDITIONS

In scalar QED in the continuum C-periodic boundary conditions imposed on a square-shaped spatial volume $L \times L$ take the form

$$A_{\mu}(x + Le_i) = C A_{\mu}(x) - \partial_\mu \varphi_i(x),$$

$$\chi(x + Le_i) = \exp(i \varphi_i(x)) C \chi(x) = \exp(i \varphi_i(x)) \chi(x)^*, \quad (1)$$

where $e_i$ is the unit-vector that points in the spatial $i$-direction, and $\varphi_i(x)$ is a transition function that represents a gauge transformation at the boundary. In the Euclidean time 3-direction, periodic boundary conditions are imposed over an extent $\beta$ that sets the inverse temperature

$$A_{\mu}(x + \beta e_3) = A_{\mu}(x) - \partial_\mu \varphi_3(x),$$

$$\chi(x + \beta e_3) = \exp(i \varphi_3(x)) \chi(x). \quad (2)$$

The transition functions $\varphi_{\mu}(x)$ are physical degrees of freedom of the gauge field which ensure that the gauge variant fields $A_{\mu}(x)$ and $\chi(x)$ are periodic in Euclidean time and C-periodic in space only up to gauge transformations. Combining shifts in various space-time dimension, one derives the following cocycle consistency conditions on the transition functions

$$\varphi_j(x + Le_i) - \varphi_i(x) = \varphi_i(x + Le_i) - \varphi_j(x) + 2 \pi n_{ij},$$

$$\varphi_3(x + Le_i) + \varphi_i(x) = \varphi_i(x + \beta e_3) - \varphi_3(x) + 2 \pi n_{i3}. \quad (3)$$

Here $n_{\mu\nu} = -n_{\nu\mu} \in \mathbb{Z}$ with $n_{i3} = n_{3j}$ is known as the twist tensor. Performing a general (not necessarily periodic or C-periodic) gauge transformation, $A_{\mu}(x)' = A_{\mu}(x) - \partial_\mu \alpha(x)$, one obtains the gauge transformation behavior of the transition functions ($n_{\mu} \in \mathbb{Z}$).

$$\varphi_i(x)' = \varphi_i(x) + \alpha(x + Le_i) + \alpha(x) - 2 \pi n_i,$$

$$\varphi_3(x)' = \varphi_3(x) + \alpha(x + \beta e_3) - \alpha(x) - 2 \pi n_3. \quad (4)$$

The twist tensor then transforms as

$$n_{ij}' = n_{ij} + 2n_i - 2n_j, \quad n_{i3}' = n_{i3} - 2n_3. \quad (5)$$

By gauge transformations, the twist tensor can be restricted to $n_{\mu\nu} \in \{0, 1\}$.

DUALITY IN A FINITE VOLUME WITH TWISTED C-PERIODIC BOUNDARY CONDITIONS

The duality transformation that relates the vortex to a dual charged particle is typically discussed in the infinite volume. It is, however, possible to perform it in a finite volume with twisted C-periodic boundary conditions in space and periodic boundary conditions in Euclidean time. Analogous to the fields and their duals, the boundary conditions of the original and the dual theory are then related by a Fourier transform.

We start the dualization by recalling the partition function of the dual integer gauge theory,

$$Z = \left( \prod \sum_{A_l \in 2\pi \mathbb{Z}} \right) \prod_{\Box} \exp(-\tilde{s}(F_{\Box})). \quad (6)$$

In the first step of the dualization, the dual weights $\exp(-\tilde{s}(F_{\Box}))$ are expressed by their Fourier representations as

$$Z = \prod_{l} \sum_{A_l \in 2\pi \mathbb{Z}} \prod_{\Box} \int_{-\pi}^{\pi} d\eta_{\Box} \times \prod_{\Box} \exp(-s(\eta_{\Box})) \exp(-i\eta_{\Box} F_{\Box}). \quad (7)$$

The newly introduced auxiliary field $\eta$ is attached to the plaquettes of the dual lattice, or equivalently to the links $(xy)$ of the original lattice. Next a partial integration is performed. Here the boundary conditions need to be taken into account. A somewhat lengthy but straightforward calculation yields

$$\sum_{\Box} \eta_{\Box} F_{\Box} = \sum_{l} \delta_{l\Box} A_l + P_1 n_{23} + P_2 n_{31} - P_3 n_{12}. \quad (8)$$

Here $\delta_{l\Box}$ is the lattice curl of $\eta$ evaluated on the original lattice. It is attached to the links $l$ of the dual lattice or, equivalently, to the plaquettes of the original lattice. Furthermore, $n_{\mu\nu}$ is the twist tensor characterizing the boundary conditions of the dual gauge theory. The $P_\mu$ correspond to the Polyakov loops of $\eta$,

$$P_{\mu} = \sum_{\Box \in \sigma_{\mu}} \eta_{\Box}. \quad (9)$$

Here $\sigma_{\mu}$ is a set of plaquettes dual to links that form a closed loop on the original lattice, which wraps around
with spatial boundary twists $\theta_1, \theta_2$ that obey $\theta_1 - \theta_2 = P_1 - P_2$ and a temporal boundary twist of $\theta_3 = P_3$. The sets of links $\gamma_i$ and $\gamma'_i$ correspond to the paths illustrated in Figure 1. By construction it follows that $\eta_{xy} = \varphi_{xy} = \varphi_y - \varphi_x$. Furthermore, the constraint $d\eta_{xy} = 0$ ensures that the three definitions of $\varphi_x$ coincide for all $x$. The same constraint also implies that $P_1 - P_2 = 0 \mod \pi$ and $P_3 = 0 \mod \pi$.

This construction is not unique, and there is a remaining global $O(2)$ symmetry. Consider $\varphi_x$ and $\varphi'_x$, such that $\varphi_{xy} = \varphi'_{xy} = \eta_{xy}$. On the links within the volume, that do not cross a boundary, this implies that they differ by a constant, $\varphi_x - \varphi'_x = c$. For links that cross a twisted $C$-periodic boundary, this implies that $\theta_i - \theta'_i = 2c$ and for a link that crosses the (possibly twisted) periodic temporal boundary, it implies that $\theta_3 - \theta'_3 = 0$. The construction of $\varphi_x$ is therefore unique only up to the following global $O(2)$ symmetry,

$$\varphi'_x = \varphi_x + \delta, \quad \theta'_i = \theta_i + \delta, \quad \theta'_3 = \theta_3. \quad (12)$$

We can therefore replace the constrained integration over the auxiliary link variables $\eta_{xy}$ by an unconstrained integration over the spin variables $\varphi_x$. The final form of the partition function reads

$$Z = \frac{1}{2\pi} \left( \prod_x \int_{-\pi}^{\pi} d\varphi_x \int_{-\pi}^{\pi} d\theta_1 \sum_{\theta_2 \in \{0, \pi\}} \sum_{\theta_3 \in \{0, \pi\}} \exp(-i(\theta_1 - \theta_2)\eta_{31} - i\theta_3 n_{12}) \right) \times \prod_{\langle xy \rangle} \exp(-s(\varphi_{xy})) \exp(-i(\theta_1 - \theta_2)\eta_{31} - i\theta_3 n_{12}). \quad (13)$$

The division by $2\pi$ is due to the additional redundancy because of the global $O(2)$ symmetry. It is now apparent that the gauge invariant characteristic of the boundary conditions of the dual gauge theory, the twist tensor, is related to the boundary conditions of the original spin model by a Fourier transform.